Delocalised Spinors

N. S. Manton* and A. F. Schunck†

Department of Applied Mathematics and Theoretical Physics
University of Cambridge
Wilberforce Road, Cambridge CB3 0WA, England

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Abstract

Solutions to the four-dimensional Euclidean Weyl equation in the background of a general JNR $N$-instanton are known to be normalisable and regular throughout four-space. We show that these solutions are asymptotically given by a linear combination of simple singular solutions to the free Weyl equation, which can be interpreted as localised spinors. The ‘spinorial’ data parameterising the asymptotics of the delocalised solutions to the Weyl equation in the presence of the instanton almost determines the background instanton, yet not completely. However, it captures the geometry and symmetry of the underlying instanton configuration.

1 Introduction

It is a well-known fact that in the case of massless particles the Weyl equation, instead of the Dirac equation, provides an adequate description of particles with negative helicity, ignoring positive helicity particles, or conversely. For negative helicity particles one finds that in two-dimensional Euclidean space-time the Weyl equation corresponds to the Cauchy-Riemann equations. Thus the class of solutions is given by the holomorphic functions. To interpret the solutions as localised classical particles, they must approach zero at infinity. If one also insists on their analyticity in the whole of Euclidean two-space, the only possible solutions are the constants (Liouville’s theorem). Hence, any interesting solutions, that tend to zero at infinity, must exhibit a singularity. In the simplest case these solutions are therefore just given by $f(z) = a/(z - b)$, characterised by two complex constants, the residue $a$ of the function and the position parameter $b$. However, the
singularity makes the interpretation of these solutions as well-behaved particles again difficult.

In Euclidean four-space we find a similar situation. Here the Weyl equation for right-handed particles corresponds to the so-called Cauchy-Riemann-Fueter equation, the quaternionic analogue of the Cauchy-Riemann equations. Just as holomorphic functions satisfy the latter, quaternionic regular functions satisfy the former. This definition of regular for quaternionic functions is the basis of quaternionic analysis, which provides us with quaternionic analogues of e.g. Cauchy’s theorem, Cauchy’s integral formula and the Laurent expansion \([1]\). Since the proof of Liouville’s theorem depends only on Cauchy’s integral formula, there exists also the quaternionic analogue of this theorem. Hence, again any interesting regular function, that approaches zero at infinity, has to be singular in at least one point and thus, cannot easily be interpreted as a classical particle.

However, if one considers the Weyl equation in the background of an instanton, i.e. a topologically non-trivial solution to the four-dimensional Euclidean Yang-Mills equation, one finds that the instanton ‘washes out’ the singularity, leaving as a solution an everywhere nicely behaved particle. By that we mean that the asymptotic behaviour of the solution to the free Weyl equation and the solution in the instanton background can be shown to agree to appropriate order, i.e. up to and including terms of \(O(r^{-4})\). Since the solution to the free equation and the ‘instanton solution’ are topologically inequivalent, a comparison of the two solutions depends on a careful choice of gauge. It can be seen from the quaternionic Laurent series, that the first term of the expansion, which is of order \(O(r^{-3})\), is similar to the familiar dipole term in three space dimensions, whereas the second is the analogue of a quadrupole term. The dipole term is thus characterised by one quaternionic constant, which is the quaternionic analogue of a residue. In contrast the quadrupole term is characterised by three quaternionic constants, the interpretation of which is less clear. However, they can be shown to reflect the symmetry of the underlying instanton configuration.

In the next section we will briefly explain the concept of an instanton \([2]\), followed by the description of a special class of instantons, the so called Jackiw-Nohl-Rebbi (JNR) instantons \([3]\). We then go on to describe the solutions to the four-dimensional Euclidean Dirac equation in the background field of an arbitrary JNR instanton, as investigated by Grossman \([4]\). Before we start describing our own results we will then briefly introduce the theorems of quaternionic analysis \([1]\), that are most important for us.

## 2 Euclidean Yang-Mills configurations

In four-dimensional Euclidean space-time the Yang-Mills field equations possess localised solutions having a finite Euclidean action. These solutions are referred to as instantons. Topologically distinct from the trivial absolute minimum of
the action functional, instantons do not have a vanishing field strength $G_{\mu\nu}$. However, $G_{\mu\nu}$ has to vanish on the boundary of Euclidean four-space, $S^3_\infty$, which implies that the corresponding fields $A_\mu$ tend asymptotically to a pure gauge, thus defining a mapping from the surface at infinity into the group space of the gauge group. Instantons are therefore characterised by a topological invariant $N$, the so-called Pontryagin index, which takes integer values.

In the case of an SU(2) non-Abelian gauge theory this index is essentially the 'winding number' of $S^3_\infty$ into the group space of $SU(2) \cong S^3$. Therefore it is equal to the homotopy index, which characterises the discrete infinity of homotopy classes of the mapping $S^3 \to S^3$. Within each class $N$, the action is bounded from below by a constant multiple of $|N|$ and the absolute minimum value is in each class attained when the field strength satisfies $G_{\mu\nu} = \pm *G_{\mu\nu}$, where $*G_{\mu\nu}$ is the dual field tensor given by $*G_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G_{\rho\sigma}$.

Strictly speaking, the notion instanton only refers to self-dual solutions of the Yang-Mills equation whereas anti-self-dual solutions are referred to as anti-instantons. The fundamental difference between instantons and anti-instantons is given by the sign of the Pontryagin index, which is positive for instantons but negative for anti-instantons. For definiteness we will in the following only consider instantons though most results will be applicable to anti-instantons as well.

3 JNR instantons

The general $N$-instanton solution has been constructed implicitly by Atiyah et al. in 1978 [5] for an arbitrary compact classical group. In the case of $SU(2)$ as the gauge group, $8N - 3$ parameters characterise the solutions. A subset of these solutions can be constructed explicitly. These special instantons exhibit only $5N + 4$ parameters and were investigated by Jackiw, Nohl and Rebbi already in 1977 [3]. However, in the case of an $N = 2$ instanton this JNR solution to the self-duality equation is the most general one since an $N = 2$ instanton is fully described by 13 physical parameters. The fourteenth parameter occurring in the JNR formula corresponds to a gauge transformation. An even more restrictive case is the one of a 't Hooft instanton [6] which is characterised by $5N$ parameters, thus giving the most general $N = 1$ instanton. The $N = 1$ 't Hooft instanton is related to the JNR $N = 1$ instanton via a gauge transformation [7].

We begin by briefly summarising the construction of the JNR $N$-instanton solution as described in [3, 2]. To simplify notation we will consider a matrix representation of the three vector fields $A_\mu^a$, $a = 1, 2, 3$, taking values in the Lie Algebra of $SU(2)$

$$A_\mu = A_\mu^a \frac{\sigma_a}{2i}.$$  \hspace{1cm} (3.1)
Here $\sigma_a$ are the familiar Pauli matrices, the three generators of the group $SU(2)$. The field strength is given by

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \quad (3.2)$$

It is useful to define the following matrices

$$\alpha_0 = \mathbb{1}_2, \quad \alpha_i = -i\sigma_i, \quad \bar{\alpha}_0 = \mathbb{1}_2, \quad \bar{\alpha}_i = i\sigma_i \quad (3.3)$$

and

$$\bar{\sigma}_{\mu\nu} = \frac{1}{4i} (\bar{\alpha}_\mu \alpha_\nu - \bar{\alpha}_\nu \alpha_\mu). \quad (3.4)$$

Notice that $\bar{\sigma}_{\mu\nu}$ is both antisymmetric and anti-self-dual in its indices. In order to solve the self-duality equation for the field strength $G_{\mu\nu}$

$$G_{\mu\nu} = *G_{\mu\nu} \quad (3.5)$$

one makes an ansatz of the form

$$A_\mu = i\bar{\sigma}_{\mu\nu} a_\nu \quad \text{with} \quad a_\nu = \partial_\nu \ln \rho \quad (3.6)$$

where $\rho(x)$ is a scalar potential to be obtained, which must satisfy $\Delta \rho(x) = 0$. Here $\Delta$ is the four-dimensional Euclidean Laplace operator. Notice that the anti-self-dual symbols $\bar{\sigma}_{\mu\nu}$ in the fields $A_\mu$ lead to a self-dual field strength. One finds for $\rho(x)^1$

$$\rho(x) = \sum_{k=1}^{N+1} \frac{\lambda^2_{(k)}}{(x - x_{(k)})^2}. \quad (3.7)$$

This solution when inserted in Eqn. (3.6) will yield the $N$-instanton solution with homotopy index $N$. Note that although $\rho$ has a singularity at each point $x_{(k)}$, the instanton field has no singularity.

It is important to note that the scalar potential $\rho$ depends on $N + 1$ position parameters $x_{(k)}$. Thus there is no simple relation between these parameters and the positions of the $N$ instantons. In contrast, a ’t Hooft instanton has a potential depending on $N$ position parameters indeed corresponding to the positions of the instantons. The other $N$ JNR parameters — a common rescaling of the $\lambda_{(k)}$’s does not affect the fields $A_\mu$ — the weights, correspond to the size of the instantons.

As shown by Jackiw, Nohl and Rebbi \[3\] the ansatz in Eqn. (3.6) completely fixes the gauge, as long as the $N$ points $x_{(k)}$ do not lie on a circle (or on a straight line, which is a circle through the point at infinity). However, three points always lie on a circle and hence in the case of an $N = 2$ instanton one of the 14 parameters corresponds to a gauge transformation which moves the $x_{(k)}$ around the circle.

\[1\] ’t Hooft’s solution takes the form $\rho(x) = 1 + \sum_{k=1}^{N} \frac{\lambda^2_{(k)}}{(x - x_{(k)})^2}$
4 Solutions to the massless Dirac equation in an $N$-instanton background

Solutions to the massless Dirac equation in the background of an $N$-instanton have been explicitly constructed by Grossman \[4\] with the gauge group being SU(2). Later on Corrigan et al. \[8\] and independently Osborn \[9\] derived these solutions for an SU($n$) or Sp($k$) gauge group, respectively. However the latter solutions are only implicitly known. In the following we will briefly explain Grossman’s construction of which we shall make extensive use in due course\(^2\).

The zero-modes of the massless Dirac equation can be chosen to be chiral eigenfunctions of $\gamma_5$. According to Grossman it is possible to construct $N$ zero-energy modes of negative helicity, when the field strength is chosen to be self-dual\(^3\). A version of the Atiyah-Singer index theorem for isospinor fermion fields asserts that there are no solutions of positive helicity, since the difference between negative and positive helicity solutions must equal in absolute value the Pontryagin index $N$.

The covariant massless Dirac equation in the background of an SU(2) vector potential $A_\mu = A_\mu^a (\sigma_a/2i)$ is given by

$$ (\partial_\mu + A_\mu) \gamma_\mu \psi = 0. \quad (4.1) $$

We will choose a representation of the $\gamma$ matrices in which the equations for positive and negative helicity fields decouple, i.e. a representation in which $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$ is diagonal. This is given by

$$ \gamma_0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1_2 & 0 \\ 0 & 1_2 \end{pmatrix}. \quad (4.2) $$

Setting $\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$ such that $\psi_R$ and $\psi_L$ are $2 \times 2$ matrices in spin and isospin we get decoupled equations for the right-handed and left-handed spinor:

$$ (\partial_\mu + A_\mu) \bar{\alpha}_\mu \psi_R = 0 \quad (4.3) $$
$$ (\partial_\mu + A_\mu) \alpha_\mu \psi_L = 0. \quad (4.4) $$

It will be convenient to slightly rewrite Eqn. (4.3). We have, preserving all indices for the moment,

$$ (\partial_\mu \delta_{ij} + (A_\mu)_{ij}) (\bar{\alpha}_\mu)_{\beta \gamma} \psi_{\gamma j} = 0. \quad (4.5) $$

\(^2\)For later convenience we will, however, use a slightly different notation.

\(^3\)Grossman’s actual derivation holds for positive helicity particles. However, as pointed out by Grossman himself, the analysis will likewise lead to $N$ negative helicity solutions when considering self-dual fields $G_{\mu\nu}$, rather than anti-self-dual fields.

5
5. Quaternionic analysis

Using the fact that $\sigma_k^4 = \epsilon \sigma_k \epsilon$, where $\epsilon$ is the usual totally antisymmetric tensor with $\epsilon_{12} = 1$, and redefining the field $\psi_R' = \psi_R \epsilon$, such that $\psi_R'$ is a $2 \times 2$ matrix in isospin and spin, with components $\psi_{\alpha \beta}^j$ (here $j = 1, 2$ is the isotopic index and $\alpha = 1, 2$ is the Lorentz index), Eqn. (4.5) becomes

$$ (\partial_\mu \delta_{ij} + (A_\mu)_{ij}) \psi_{\alpha \beta}^j = 0, \quad (4.6) $$

or in index-free notation

$$ (\partial_\mu + A_\mu) \psi_R^\prime \alpha = 0. \quad (4.7) $$

Note that all products occurring in Eqn. (4.7) are now to be interpreted as matrix products. Defining

$$ \phi^{(k)}(x) = \lambda^2 \frac{2}{(x - x^{(k)})^2}, \quad k = 1, \ldots, N + 1 \quad (4.8) $$

and

$$ M^{(k)}_{\mu} = \rho^{1/2} \partial_\mu \left( \frac{\phi^{(k)}}{\rho} \right), \quad (4.9) $$

with $\rho$ given by Eqn. (3.7), one finds $N + 1$ solutions of the form

$$ \psi^{(k)}_R = M^{(k)}_{\beta} \bar{\alpha}_\beta. \quad (4.10) $$

However, since

$$ \sum_{k=1}^{N+1} M^{(k)}_{\mu} = 0 \quad (4.11) $$

there are only $N$ linearly independent solutions.

In the following we will briefly describe some major results of the theory of quaternionic regular functions, as they characterise the class of solutions to the four-dimensional Euclidean Dirac equation. In our description we shall follow the lines of Sudbery’s work \[1\], which gives a self-contained and rigorous account of the theory of quaternionic analysis.

5 Quaternionic analysis

The theory of quaternionic analysis was developed by Fueter and his collaborators in the years following 1935, when Fueter proposed the definition of regular for quaternionic functions \[10\]. By means of an analogue of the Cauchy-Riemann equations this led to a theory of regular functions similar to the theory of holomorphic functions.

A generic quaternion can be written as

$$ q = 1x_0 + ix_1 + jx_2 + kx_3 \quad \text{with} \quad x_\mu \in \mathbb{R}, \quad \mu = 0, \ldots, 3, \quad (5.1) $$
where 1, \(i\), \(j\) and \(k\) shall denote the elements of the standard basis for \(\mathbb{R}^4\). One defines the quaternionic product on \(\mathbb{R}^4\) as the \(\mathbb{R}\)-bilinear product
\[
\mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4; \quad (q_1, q_2) \mapsto q_1 q_2 \quad (5.2)
\]
with unit element 1, such that
\[
i^2 = j^2 = k^2 = -1 \quad (5.3)
\]
\[
i j = k = -ji, \quad j k = i = -kj, \quad k i = j = -ik. \quad (5.4)
\]
Here we will identify the subfield spanned by 1 with \(\mathbb{R}\). Notice that the quaternionic product is not commutative. The linear space \(\mathbb{R}^4\) with the quaternionic product defines the real associative algebra \(\mathbb{H}\) of the quaternions (see, for example, \([11]\)).

We will sometimes use \(e_i, i = 1, 2, 3\), to denote the basic quaternions \(i, j\) and \(k\). Then Eqn. (5.1) becomes
\[
q = x_0 + e_i x_i, \quad (5.5)
\]
or setting \(e_0 = 1\)
\[
q = e_\mu x_\mu, \quad (5.6)
\]
where summation over repeated indices is implied.

Identifying the subfield spanned by 1 and \(k\) with the complex field \(\mathbb{C}\) we will sometimes write
\[
q = y + jz \quad (5.7)
\]
where \(y = x_0 + k x_3\) and \(z = x_2 + k x_1\). The conjugate of \(q\), denoted by \(\bar{q}\), is given by
\[
\bar{q} = x_0 - i x_1 - j x_2 - k x_3 \quad (5.8)
\]
and the modulus by
\[
|q| = \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} \in \mathbb{R}. \quad (5.9)
\]
We have
\[
\bar{q}_1 \bar{q}_2 = \bar{q}_2 \bar{q}_1 \quad \text{and} \quad q^{-1} = \frac{\bar{q}}{|q|^2}. \quad (5.10)
\]

The following definition of a regular function is the most convenient for our purposes (Sudbery gives a different definition, but shows that it is equivalent to this).
5. Quaternionic analysis

**Definition**1 (the Cauchy-Riemann-Fueter equations)
A real-differentiable, quaternion-valued function \( f : \mathbb{R}^4 \rightarrow \mathbb{H} \) is right-regular at \( q \) if and only if
\[
\frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1} i + \frac{\partial f}{\partial x_2} j + \frac{\partial f}{\partial x_3} k = 0.
\] (5.12)
It is left-regular at \( q \) if and only if
\[
\frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2} + k \frac{\partial f}{\partial x_3} = 0.
\] (5.13)

The analysis of regular quaternionic functions is comparable to that of holomorphic functions, since the quaternionic analogues of Cauchy’s theorem, Cauchy’s integral theorem and the Laurent series exist for regular functions. As many of the standard theorems of complex analysis depend only on Cauchy’s theorem, they also hold for quaternionic regular functions. An example of this is Liouville’s theorem. Hence, a quaternionic function, which is regular in the whole of \( \mathbb{R}^4 \) and that tends to zero as \( |q| \rightarrow \infty \) must necessarily reduce to a constant.

We will in the following consider only right-regular functions which we shall call simply regular\(^4\).

We may introduce the following notation for the differential operators
\[
\bar{\partial}_r f = \frac{1}{2} \left( \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_i} e_i \right),
\] (5.14)
\[
\partial_r f = \frac{1}{2} \left( \frac{\partial f}{\partial x_0} - \frac{\partial f}{\partial x_i} e_i \right),
\] (5.15)
\[
\bar{\partial}_l f = \frac{1}{2} \left( \frac{\partial f}{\partial x_0} + e_i \frac{\partial f}{\partial x_i} \right),
\] (5.16)
\[
\partial_l f = \frac{1}{2} \left( \frac{\partial f}{\partial x_0} - e_i \frac{\partial f}{\partial x_i} \right).
\] (5.17)

Notice that \( \bar{\partial}_r, \partial_r, \bar{\partial}_l \) and \( \partial_l \) all commute and that we have
\[
\triangle = 4 \partial_r \bar{\partial}_r = 4 \partial_l \bar{\partial}_l.
\] (5.18)
Thus, a function \( f \) is regular if and only if \( \bar{\partial}_r f = 0 \).

As shown by Sudbery all regular functions are necessarily infinitely differentiable. It follows then from the above theorem, together with Eqn. (5.18), that all regular functions are harmonic.

Setting \( q = y + jz \), as in Eqn. (5.17), and \( f = h + gj \), where \( g \) and \( h \) are complex-valued functions, we find that Eqn. (5.12) is equivalent to the pair of complex equations
\[
- \partial_y g + \partial_z h = 0
\]
\[
\partial_y g + \partial_z h = 0.
\] (5.19)

\(^4\)Sudbery chooses to develop the theory of quaternionic analysis for left-regular functions. However, his results are easily rewritten for right-regular functions.
In the absence of a background field these complex equations correspond to Eqn. (4.7), which is the Weyl equation for right-handed spinors. Note that it is permitted to multiply $f$ with an arbitrary constant from the left. Thus $-jf = \bar{g} - \bar{h}j$ will be a solution to Eqn. (5.12), too.

For our purposes the last section of Sudbery’s paper, which addresses the theory of regular power series, is especially important. Sudbery introduces the following differential operator

$$
\partial_\nu = \frac{\partial^n}{\partial x_{i_1} \ldots \partial x_{i_n}} = \frac{\partial^n}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}}.
$$

(5.20)

This calls for some explanation. $\nu$ is an unordered set of $n$ integers $\{i_1, \ldots, i_n\}$ with $1 \leq i_r \leq 3$. The number of 1’s in $\nu$ is given by $n_1$, the number of 2’s by $n_2$ and the number of 3’s by $n_3$ such that $n_1 + n_2 + n_3 = n$. Hence there are $\frac{1}{2}(n+1)(n+2)$ such sets $\nu$ and we will denote the collection of these sets as $s_n$. They are to be used as labels. If $n = 0$, i.e. $\nu = \emptyset$, we use the suffix 0 instead of $\emptyset$.

We will furthermore need the following two functions:

$$
G_\nu(q) = \partial_\nu G(q) \quad \text{with} \quad G_0(q) \equiv G(q) = \frac{q^{-1}}{|q|^2},
$$

(5.21)

and

$$
P_\nu(q) = \frac{1}{n!} \sum (x_0e_{i_1} - x_{i_1}) \ldots (x_0e_{i_n} - x_{i_n}) \quad \text{with} \quad P_0(q) = 1
$$

(5.22)

where the sum is over all $n!/(n_1!n_2!n_3!)$ different orderings of $n_1$ 1’s, $n_2$ 2’s and $n_3$ 3’s. Note that the $P_\nu$ are both right- and left-regular. We have

$$
\frac{1}{2\pi^2} \int_S P_\mu(q) Dq G_\nu(q) = \delta_{\mu\nu}
$$

(5.23)

and, since the $P_\nu$ are also left-regular, we have as well

$$
\frac{1}{2\pi^2} \int_S G_\mu(q) Dq P_\nu(q) = \delta_{\mu\nu}.
$$

(5.24)

Here $S$ is any three-sphere surrounding the origin and the measure $Dq$ is the quaternion valued three form

$$
Dq = dx_1 \wedge dx_2 \wedge dx_3 - \epsilon_{ijk} e_i dx_0 \wedge dx_j \wedge dx_k.
$$

(5.25)

Sudbery then proves the following theorem:

**Theorem 1** (the Laurent series)

Suppose $f$ is regular in an open set $U$ except possibly at $q_0 \in U$. Then there is a
neighbourhood $N$ of $q_0$ such that if $q \in N$ and $q \neq q_0$, $f(q)$ can be represented by a series

$$f(q) = \sum_{n=0}^{\infty} \sum_{\nu \in s_n} (a_\nu P_\nu(q - q_0) + b_\nu G_\nu(q - q_0))$$

(5.26)

which converges uniformly in any hollow ball

$$\{q : r \leq |q - q_0| \leq R\}, \text{ with } r > 0, \text{ which lies inside } N.$$

The coefficients $a_\nu$ and $b_\nu$ are given by

$$a_\nu = \frac{1}{2\pi^2} \int_C f(q) DqG_\nu(q - q_0)$$

(5.27)

$$b_\nu = \frac{1}{2\pi^2} \int_C f(q) DqP_\nu(q - q_0),$$

(5.28)

where $C$ is any closed 3-chain in $U - \{q_0\}$ which is homologous to $\partial B$ for some ball $B$ with $q_0 \in B \subset U$ (so that $C$ has wrapping number 1 about $q_0$).\(^5\)

6 The Dirac Equation in a quaternionic notation

As remarked in the previous section, the Weyl equation for right-handed spinors in the absence of an external field\(^6\)

$$\partial_\mu \psi' \alpha_\mu = 0$$

(6.1)

corresponds to the complex version of the Cauchy-Riemann-Fueter equation, when using the complex variables $y = x_0 + ix_3$, $z = x_2 + ix_1$

$$-\partial_z \psi'_1 + \partial_y \psi'_2 = 0$$

$$\partial_y \psi'_1 + \partial_z \psi'_2 = 0.$$

Here $(\psi'_1, \psi'_2)$ are the two complex components of $\psi'$.

The correspondence of the Cauchy-Riemann-Fueter equation to the free Weyl equation can be seen more directly. Since the group of quaternions with absolute value 1, namely the symplectic group $\text{Sp}(1)$, is isomorphic to $S^3$ and thus to $\text{SU}(2)$ we can rewrite Eqn. (6.1) identifying $-i\sigma_1$ with the quaternionic $i$, $-i\sigma_2$ with $j$ and $-i\sigma_3$ with $k$. This yields directly the Cauchy-Riemann-Fueter equation, Eqn. (5.12), in terms of quaternionic variables

$$(\partial_0 + \partial_1 i_r + \partial_2 j_r + \partial_3 k_r) \psi'_q = 0.$$  

(6.2)

\(^5\)For the exact definition of the wrapping number see [1].

\(^6\)We will drop the subscript $R$ from now on.
Here the subscript \( r \) shall denote action from the right. Note that \( \psi' \) is a single quaternionic function. It is readily seen that \( \psi'_q \) is in terms of the complex functions \( \psi'_1 \) and \( \psi'_2 \) given by

\[
\psi'_q = \psi'_1 j + \psi'_2. \tag{6.3}
\]

We now turn to the discussion of the Dirac equation in the background field of an SU(2) vector potential given by \( A_\mu = -\frac{1}{2} A^a_\mu \sigma_a \). The Weyl equation for negative helicity particles is given by Eqn. (4.7)

\[
(\partial_\mu + A_\mu)\psi'^1 = 0, \tag{6.4}
\]

where \( \psi' \) is now a 2 \( \times \) 2 matrix in isospin and spin. Here, again, we can use the isomorphism \( \text{Sp}(1) \cong \text{SU}(2) \) to rewrite Eqn. (6.4). Applying this to both the Lorentz group and the gauge group will not be possible in general, since it would result in combining the four complex components of \( \psi' \) into a single quaternionic function. Thus it would lead to a reduction of the four complex parameters to two complex parameters. However, for our purpose it will be convenient to impose the following SU(2) gauge invariant conditions on the four complex components of \( \psi' \), which will allow us to rewrite Eqn. (6.4) in a purely quaternionic notation:

\[
\psi' = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} = \begin{pmatrix} \psi_{11} & \psi_{12} \\ -\bar{\psi}_{12} & \bar{\psi}_{11} \end{pmatrix}. \tag{6.5}
\]

Note that these conditions can be viewed as a Majorana condition. If we set

\[
\Psi_M = \begin{pmatrix} \psi_{11} \\ \psi_{12} \\ -\bar{\psi}_{12} \\ \bar{\psi}_{11} \end{pmatrix} \quad \text{with} \quad \Psi^C_M = C\Psi^t_M. \tag{6.6}
\]

we find

\[
\Psi_M = \Psi^C_M, \quad \text{with} \quad \Psi^C_M = C\Psi^t_M. \tag{6.7}
\]

Here \( C \) is the charge conjugation operator given by \( C = \gamma_0 \gamma_2 \) and \( \bar{\Psi} = \Psi^t \gamma_0 \) is the usual Dirac conjugate spinor.

Setting \( A^a_\mu = \frac{1}{2} \epsilon_{aA} A^a_\mu \) we find for Eqn. (6.4)

\[
(\partial_\mu + A_\mu^a)\psi'^q = 0, \tag{6.8}
\]

where \( \psi'^q \) is now a single quaternionic object, given by

\[
\psi'^q = \bar{\psi}_{11} - \bar{\psi}_{12}j = \bar{\psi}_{22} + \psi_{21}j. \tag{6.9}
\]

For the gauge transformation of \( \psi'^q \) and \( A^a_\mu \) we have

\[
\begin{align*}
\psi'^q &\rightarrow q\psi'^q, \\
A^a_\mu &\rightarrow qA^a_\mu q^{-1} - (\partial_\mu q)q^{-1},
\end{align*} \tag{6.10, 6.11}
\]
with $q$ a unit quaternion, i.e. $|q| = 1$.

As the simplest singular solution to the (complex) Eqn. (6.4) in the absence of a background field we find

$$
\psi' = \frac{1}{(y\bar{y} + z\bar{z})^2} \begin{pmatrix}
ay + bz \\
-az + by \\
-\bar{a}y + \bar{b} \bar{z} \\
a\bar{y} + b \bar{z}
\end{pmatrix},
$$

(6.12)

where we have already imposed the conditions Eqn. (6.5) on the four complex components of $\psi'$. If we write $a = c_0 + ic_3$, $b = c_2 + ic_1$, where we assume $c_\mu$ to be the components of some real vector $c$, we may write instead

$$
\psi' = \frac{(c.\alpha)(\bar{x.\bar{\alpha}})}{r^4},
$$

(6.13)

with $r = \sqrt{x^2}$ the distance from the origin.

In quaternionic notation we have (identifying as usual the quaternionic $k$ with the complex $i$)

$$
\psi'_q = \frac{1}{(y\bar{y} + z\bar{z})^2}(a + jb)(\bar{y} - \bar{z} j).
$$

(6.14)

Setting $q = y + jz$ as in Eqn. (5.7) and defining $c_q = a + jb$ we have

$$
\psi'_q = c_q |q|^4 = c_q q^{-1} = c_q G(q),
$$

(6.15)

with $G(q)$ as defined in Eqn. (5.21). Note that this is the general ‘dipole’ term of the quaternionic Laurent expansion, Eqn. (5.26). Thus, $\psi'_q$ is regular except at the origin.

As we shall show in the following, the solutions to Eqn. (6.4) in the presence of a JNR $N$ instanton can be written up to quadrupole order, i.e. up to and including terms of $O(r^{-4})$, as a linear combination of shifted dipoles. However, this depends on a careful choice of gauge, as will be shown below.

### 7 $1/r$ expansion of Grossman’s solution

In the case of a JNR $N$ instanton as the background field there exist $N + 1$ (linearly dependent) solutions to the Weyl equation for right-handed spinors, which are given by Eqns. (4.8) – (4.10)

$$
\psi'_k = M^{(k)}_\beta \bar{\alpha}_\beta, \quad k = 1, \ldots, N + 1.
$$

Calculating these explicitly, we find for $M^{(k)}_\mu$

$$
M^{(k)}_\mu = \frac{-2\lambda^2}{\rho^{3/2}} \prod_{i=1}^{N+1} (x - x_{(i)})^4 \left[ \sum_{i \neq k} \prod_{j \neq i, k} (x - x_{(j)})^4 \right] \lambda_{(i)}^2 \left( (x_\mu - x_{(k)\mu})(x - x_{(i)})^2 - (x_\mu - x_{(i)\mu})(x - x_{(k)})^2 \right),
$$

(7.1)
with \( \rho(x) \) given as before by

\[
\rho(x) = \sum_{i=1}^{N+1} \frac{\lambda_{(i)}^2}{(x - x_{(i)})^2}.
\]

In order to be able to compare this solution to the solution of the free Weyl equation discussed in the previous section, one has to investigate its asymptotic behaviour, i.e. its behaviour far away from the instanton. Thus we have to expand \( M_{(k)}^{(k)} \) in powers of \( 1/r \). We shall write \( M_{(k)}^{(k)} \) as the sum of a ‘dipole’ and a ‘quadrupole term’, \( m_{(k)}^D \) and \( m_{(k)}^Q \) respectively, plus higher order contributions. It turns out that the first term of this expansion, the dipole term, is of \( \mathcal{O}(r^{-3}) \), the quadrupole term of \( \mathcal{O}(r^{-4}) \). Therefore

\[
M_{(k)}^{(k)} = m_{(k)}^D + m_{(k)}^Q + \mathcal{O}(\frac{1}{r^5}).
\]  

(7.2)

Defining \( \Lambda \) as the sum of the weights \( \lambda_{(i)}^2 \) and \( X \) as the ‘centre of mass’ of the instanton

\[
\Lambda = \sum_{i=1}^{N+1} \lambda_{(i)}^2 \text{ and } X = \frac{\sum_{i=1}^{N+1} \lambda_{(i)} x_{(i)}}{\Lambda}
\]

(7.3)

the dipole contribution is given by

\[
m_{(k)}^D = -\frac{1}{r^3} \cdot \frac{2\lambda_{(k)}^2}{\Lambda^{1/2}} \left[ (X_{(k)} - x_{(k)}) - 2(\hat{x} \cdot X - \hat{x} \cdot x_{(k)}) \hat{x}_{(k)} \right].
\]  

(7.4)

If we define

\[
\hat{n}_{(i)} = 2\hat{x} \cdot x_{(i)} \hat{x}_{(i)} - x_{(i)} \text{ and similarly } \hat{n}^{(X)} = 2\hat{x} \cdot X \hat{x}_{(k)} - X
\]

(7.5)

we can rewrite Eqn. (7.4) as

\[
m_{(k)}^D = -\frac{1}{r^3} \cdot \frac{2\lambda_{(k)}^2}{\Lambda^{1/2}} \left[ \hat{n}_{(k)} - \hat{n}^{(X)} \right].
\]  

(7.6)

We find for the quadrupole term

\[
m_{(k)}^Q = -\frac{1}{r^4} \cdot \frac{2\lambda_{(k)}^2}{\Lambda^{1/2}} \left[ \hat{x} \left( \frac{\sum_i \lambda_{(i)}^2 x_{(i)}^2}{\Lambda} - x_{(k)}^2 \right) + 2 \left( \hat{x} \cdot X \hat{n}_{(k)} - \hat{x} \cdot x_{(k)} \hat{n}^{(X)} \right) \right.
\]

\[
+ 3\hat{x} \cdot X (\hat{n}_{(k)}^{(X)} - \hat{n}_{(k)}^{(X)}) + 4 \left( \hat{x} \cdot x_{(k)} \hat{n}_{(k)}^{(X)} - \frac{\sum_i \lambda_{(i)}^2 \hat{x} \cdot x_{(i)} \hat{n}_{(i)}^{(X)}}{\Lambda} \right) \right].
\]  

(7.7)

If one now wants to compare this solution asymptotically to the solution of the Weyl equation in the absence of an instanton, one encounters the difficulty
that the fields \( A_\mu \) tend asymptotically to a pure gauge rather than to the vacuum, which is topologically different.

It would be desirable to find a gauge in which the fields \( A_\mu \) tend to zero at infinity more rapidly than the pure gauge does. Because of the topologically different nature of the vacuum and the instanton, there exists no gauge transformation that is non-singular at all \( x \), relating the vacuum and the instanton. However, it turns out that one can find a singular gauge such that \( A_\mu \) is (at most) of \( O(1/r^3) \). An easy estimate then shows that in this singular gauge \( \psi'_k \) will be — up to and including quadrupole order — not only a solution to the Weyl equation in the instanton background but also to the free Weyl equation. Note that since the free Weyl equation is homogeneous, i.e., does not mix powers of \( r \), the dipole term and the quadrupole term will be independently solutions to the free equation. The next task will obviously be to find the desired gauge transformations.

### 8 Singular gauge transformation

Under a gauge transformation, \( A_\mu \) transforms as usual as

\[
A_\mu \rightarrow A'_\mu = U A_\mu U^{-1} - (\partial_\mu U) U^{-1},
\]

(8.1)

Expanding \( A_\mu \) of Eqn. (3.6) in powers of \( 1/r \) we find

\[
A_\mu = -2i\bar{\sigma}_{\mu\nu} \left[ \frac{1}{r} \hat{x}_\nu + \frac{1}{r^2} \hat{n}_\nu(x) \right] + O(1/r^3).
\]

(8.2)

If we define \( U \in SU(2) \) as

\[
U = \hat{x}_\nu \alpha_\nu
\]

(8.3)

we have

\[
U^{-1} \partial_\mu U = \frac{-2i\bar{\sigma}_{\mu\nu}\hat{x}_\nu}{r},
\]

(8.4)

where \( U^{-1} \) is given by \( U^{-1} = \hat{x}_\nu \alpha_\nu \). Thus gauge transforming \( A_\mu \) with \( U \) as defined in Eqn. (8.3) cancels out the \( 1/r \) contribution to \( A_\mu \). The additional gauge transformation required to cancel out the \( 1/r^2 \) contribution can be found as follows. We shall write \( U_{\text{add}} = 1_2 + \epsilon \) for the additionally required \( SU(2) \) gauge transformation, where we assume \( \epsilon \) to be of \( O(r^{-1}) \). The inverse gauge transformation is \( U^{-1}_{\text{add}} = 1_2 - \epsilon \). Gauge transforming again we have

\[
A''_\mu \rightarrow A''_\mu = A'_\mu + \epsilon A'_\mu - A'_\mu \epsilon + \epsilon A'_\mu \epsilon - \partial_\mu \epsilon (1_2 - \epsilon),
\]

(8.5)

where \( A'_\mu \) denotes the gauge potential gauge transformed by \( U \). Thus, we can read off, that the \( O(r^{-2}) \) contribution to \( A'_\mu \) will be cancelled by \( \partial_\mu \epsilon \). Hence, if we are able to write the \( O(r^{-2}) \) contribution to \( A'_\mu \) as the derivative of some
function $\epsilon$ we will have obtained the required additional gauge transformation. Using the fact that

$$\delta_{\mu\nu} = \frac{1}{2}(\alpha_\mu \bar{\alpha}_\nu + \alpha_\nu \bar{\alpha}_\mu)$$

we find for the $O(r^{-2})$ contribution to $A'_\mu$

$$- 2i U \tilde{\sigma}_{\mu\nu} U^{-1} \frac{n^{(X)}_{\nu}}{r^2} = \frac{1}{r^2} \left[ -n^{(X)}_{\mu} + X.\alpha(2\hat{x}_\mu \hat{x}_\nu \bar{\alpha}_\nu - \bar{\alpha}_\mu) \right]$$

$$= \partial_\mu \left[ \frac{\dot{x}.X - (X.\alpha)(\dot{x}.\bar{\alpha})}{r} \right].$$

(8.7)

Thus we find that the additional gauge transformation is given by

$$U_{\text{add}} = 1 + \frac{\dot{x}.X - (X.\alpha)(\dot{x}.\bar{\alpha})}{r}. \quad (8.8)$$

Defining the self-dual symbols $\sigma_{\mu\nu}$ analogously to the anti-self-dual symbols $\bar{\sigma}_{\mu\nu}$

$$\sigma_{\mu\nu} = \frac{1}{4i} (\alpha_\mu \bar{\alpha}_\nu - \alpha_\nu \bar{\alpha}_\mu) \quad (8.9)$$

it is easy to see that $U_{\text{add}}$ is indeed an SU(2) gauge transformation, since we may write

$$\dot{x}.X - (X.\alpha)(\dot{x}.\bar{\alpha}) = -2i\sigma_{\mu\nu} X_\mu \dot{x}_\nu, \quad (8.10)$$

where we have used

$$\alpha_\mu \bar{\alpha}_\nu = 2i\sigma_{\mu\nu} + \delta_{\mu\nu}. \quad (8.11)$$

Note that the $\sigma_{\mu\nu}$’s take their values in the Lie algebra of SU(2) as do the $\bar{\sigma}_{\mu\nu}$’s.

It is remarkable that the additional gauge transformation depends only on the centre of mass $X$ of the instanton. In the centre of mass frame, where $X = 0$, the additional gauge transformation is the identity. Thus in this frame the quadrupole term is a solution to the free equation even without an additional gauge transformation.

The basic quantity $\hat{n}^{(k)}_\mu \bar{\alpha}_\mu$, defined in Eqn. (7.5), gauge transformed by $U$, is given by

$$\hat{\mathcal{N}}^{(k)}_s \equiv \hat{x}_\mu \alpha_\mu \hat{n}^{(k)}_\nu \bar{\alpha}_\nu$$

$$= 2 \hat{x} \cdot x^{(k)}_s - (\hat{x} \cdot \alpha)(\hat{x} \cdot \bar{\alpha})$$

$$= \alpha(\hat{x} \cdot \alpha)(\hat{x} \cdot \bar{\alpha}). \quad (8.12)$$

Comparing this to the singular solution discussed previously (Eqn. (6.13)) we find that $\hat{\mathcal{N}}^{(k)}_s$ is precisely the matrix occurring there, with the components of the vector $c$ determined by the components of the position parameter $x^{(k)}_s$ of the instanton.
8. Singular gauge transformation

We find for the dipole contribution \( \psi^D_{(k)} \) to the fully gauge transformed \( \psi'_{(k)} \)

\[
\psi^D_{(k)} = \hat{x} \cdot \alpha m^D_{(k)\mu} \bar{\tilde{\alpha}}_{\mu} = -\frac{1}{r^3} \cdot \frac{2\lambda^2_{(k)}}{\Lambda^{1/2}} \cdot \left[ \hat{N}_s^{(k)} - \hat{\bar{N}}^{(X)}_s \right].
\]  

(8.13)

Note that the dipole contribution \( \psi^D_{(k)} \) will be unaffected by the additional gauge transformation \( U_{\text{add}} \). However \( \psi^D_{(k)} \) yields a contribution to the quadrupole term, when one performs the additional gauge transformation \( U_{\text{add}} \).

The quadrupole contribution \( \psi^Q_{(k)} \) to \( \psi'_{(k)} \), fully gauge transformed by \( U \) and \( U_{\text{add}} \), is given by

\[
\psi^Q_{(k)} = \hat{x} \cdot \alpha m^Q_{(k)\mu} \bar{\tilde{\alpha}}_{\mu} + \frac{\hat{x} \cdot X - (X \cdot \alpha)(\hat{x} \cdot \bar{\alpha})}{r} \psi^D_{(k)}
\]  

(8.14)

\[
= -\frac{1}{r^4} \cdot \frac{2\lambda^2_{(k)}}{\Lambda^{1/2}} \left[ 4 \left( \hat{x} \cdot X \hat{N}_s^{(X)} - \hat{x} \cdot x_{(k)} \hat{N}_s^{(X)} \right) + \hat{x} \cdot x_{(k)} \hat{N}_s^{(k)} \right.
\]

\[
\left. \left. - \frac{1}{\Lambda} \sum_i \lambda^2_{(i)} \hat{x} \cdot x_{(i)} \hat{N}_s^{(i)} \alpha \right) \right] + \left( -X^2 + (X \cdot \alpha)(x_{(k)} \cdot \bar{\alpha}) - x_{(k)}^2 + \frac{\sum_i \lambda^2_{(i)} x_{(i)}^2}{\Lambda} \right). \]

To derive this we have used

\[
X \cdot \alpha \hat{N}_{\mu}^{(k)} \bar{\tilde{\alpha}}_{\mu} = 2 \hat{x} \cdot x_{(k)} \hat{N}_s^{(X)} - (X \cdot \alpha)(x_{(k)} \cdot \bar{\alpha})
\]  

(8.15)

and

\[
(X \cdot \alpha)(X \cdot \bar{\alpha}) = X^2.
\]  

(8.16)

If we write \( \tilde{\psi}'_{(k)} \) for the fully gauge transformed \( \psi'_{(k)} \) we have

\[
\tilde{\psi}'_{(k)} = \psi^D_{(k)} + \psi^Q_{(k)} + \mathcal{O}\left(\frac{1}{r^5}\right),
\]  

(8.17)

where \( \psi^D_{(k)} + \psi^Q_{(k)} \) should be a solution not only to the Dirac equation in the instanton background but also to the free equation. That this is indeed the case may be verified by considering the solution Eqn. (6.13) with its pole shifted by \( x_{(j)} \) and with \( c = x_{(j)} \), which is obviously still a solution to the free equation. This shifted solution has the expansion

\[
\left[ (x_{(j)} \cdot \alpha)(x \cdot \bar{\alpha}) - (x_{(j)} \cdot \alpha)(x_{(j)} \cdot \bar{\alpha}) \right]
\]

\[
(x - x_{(j)})^4
\]

\[
= \frac{1}{r^3} \hat{N}_s^{(i)} + \frac{1}{r^4} \left[ 4 \hat{x} \cdot x_{(j)} \hat{N}_s^{(i)} - (x_{(j)} \cdot \alpha)(x_{(j)} \cdot \bar{\alpha}) \right] + \mathcal{O}\left(\frac{1}{r^5}\right).
\]  

(8.18)

Note that as usual the dipole and the quadrupole contribution are separately solutions to the free equation. One now easily verifies that both the dipole term \( \psi^D_{(k)} \) and the quadrupole term \( \psi^Q_{(k)} \) solve the free equation, since one finds \( \psi^D_{(k)} \)
9. $N = 1$ instanton in singular gauge

to be a linear combination of terms of the form $r^{-3}\hat{N}_s^{(i)}$ and $\psi_Q^{(k)}$ to be a linear combination of terms of the form $r^{-4} \left[ 4\hat{x} \cdot x(j)\hat{N}_s^{(i)} - (x(i) \cdot \alpha)(x(j) \cdot \bar{\alpha}) \right]$.

Thus we find that the solution to the Dirac equation in the instanton background in this singular gauge coincides to appropriate order with a linear combination of singular solutions to the free equation as given by Eqn. (8.18). In this sense the instanton ‘washes out’ the singularity of this latter solution yielding an everywhere nicely behaved particle.

In the last section we will compare the quaternionic version of Eqn. (8.18), given by

$$\frac{q(i)(\bar{q} - \bar{q}(j))}{|q - q(j)|^4} = q(i) \left[ \frac{\hat{q}}{|q|^3} + \frac{4\hat{x} \cdot x(j)\hat{q} - \hat{q}(j)}{|q|^4} \right] + O\left(\frac{1}{|q|^5}\right), \quad (8.19)$$

where $q(i) = x(i)\mu \epsilon_{\mu}$ and similarly $\hat{q} = \hat{x}_\mu \epsilon_{\mu}$, as well as the asymptotic fields $\tilde{\psi}_Q^{(k)}$ in quaternionic notation, to the general quaternionic Laurent expansion. Investigating some special $N = 2$ instantons we will be able to show that the constants $b_\mu$ occurring in the quaternionic Laurent series reflect the symmetry of the underlying instanton configuration. First we will however briefly discuss the case of an $N = 1$ instanton.

9 $N = 1$ instanton in singular gauge

In the case of an $N = 1$ instanton there exist two linearly dependent solutions to the Weyl equation, the sum of which is equal to zero. Thus the two solutions differ only in sign. However, the JNR formula for an $N = 1$ instanton is largely redundant, whereas the ‘t Hooft formula yields already the most general $N = 1$ instanton. As stated previously a JNR $N = 1$ instanton is related to a ‘t Hooft $N = 1$ instanton via a gauge transformation. Explicitly (see [7]), consider a JNR instanton with potential

$$\rho = \frac{\lambda_{(1)}^2}{(x - x(1))^2} + \frac{\lambda_{(2)}^2}{(x - x(2))^2} \quad (9.1)$$

and a ‘t Hooft instanton with potential

$$\rho_{\text{tHooft}} = 1 + \frac{\lambda_{(0)}^2}{(x - x(0))^2} \quad (9.2)$$

where

$$x(0) = \frac{\lambda_{(1)}^2 x(2) + \lambda_{(2)}^2 x(1)}{\Lambda} \quad (9.3)$$

and

$$\lambda_{(0)}^2 = \frac{\lambda_{(1)}^2 \lambda_{(2)}^2 |x(2) - x(1)|^2}{\Lambda^2}. \quad (9.4)$$
The JNR $N = 1$ instanton may then be obtained from the 't Hooft $N = 1$ instanton by conjugating with the SU(2) matrix

$$U_0 = \frac{x^{(2)\mu} - x^{(1)\mu}}{|x^{(2)} - x^{(1)}|} \alpha_\mu, \quad (9.5)$$

or vice versa by conjugating with the inverse of $U_0$. This can be verified by calculating the gauge transform of the asymptotic field $\tilde{\psi}'(k), k = 1, 2$, given by Eqn. (8.17), with the gauge transformation given by $U_0^{-1}$. Thus one finds the solution to the Weyl equation in the background of the JNR $N = 1$ instanton to be gauge equivalent to the solution to the Weyl equation in the background of the 't Hooft $N = 1$ instanton. The latter solution is given by

$$\psi'_{\text{tHooft}} = \pm 2\lambda^2(0) \left( x_\mu - x_{(0)\mu} \right) \bar{\alpha}_\mu \left| x - x_{(0)} \right| \left( |x - x_{(0)}|^2 + \lambda^2(0) \right)^{3/2}. \quad (9.6)$$

However, it is a well-known fact [3] that the JNR $N = 1$ instanton and the 't Hooft $N = 1$ instanton are not only related by a gauge transformation but also by a limiting process. Taking the JNR scalar potential

$$\rho(x) = \sum_{i=1}^{N+1} \frac{\lambda^2(i)}{|x - x_{(i)}|^2}$$

the 't Hooft scalar potential

$$\rho(x) = 1 + \sum_{i=1}^{N} \frac{\lambda^2(i)}{(x - x_{(i)})^2}$$

may be regained in the limit $|x_{(N+1)}| \to \infty$, $\lambda^2(N+1) \to \infty$ with $\lambda^2(N+1)/|x_{(N+1)}| \to 1$. Similarly one may obtain 't Hooft’s solution to the Weyl equation in the presence of an $N = 1$ instanton, Eqn. (9.6), via the same limiting process from Grossman’s solution, given by Eqn. (4.10), to the Weyl equation in the background of a JNR $N = 1$ instanton. Thus the formula

$$\psi'^{(k)}_{\text{tHooft}} = \rho^{1/2}_{\text{tHooft}} \partial_\mu \left( \frac{\phi'^{(k)}_{\text{tHooft}}}{\rho_{\text{tHooft}}} \right) \bar{\alpha}_\mu \quad k = 1, 2 \quad (9.7)$$

with

$$\phi'^{(1)}_{\text{tHooft}} = \frac{\lambda^2(0)}{(x - x_{(0)})^2}, \quad \phi'^{(2)}_{\text{tHooft}} = 1, \quad (9.8)$$

and with $\rho_{\text{tHooft}}$ given by Eqn. (9.22), will yield precisely 't Hooft’s solution as stated in Eqn. (9.6). Expanding Eqn. (9.6) we have

$$\psi'^{(k)}_{\text{tHooft}} = \pm 2\lambda^2(0) \left( \frac{\dot{x}_\mu \bar{\alpha}_\mu}{r^3} + \frac{4\dot{x}.x_{(0)} \dot{x}_\mu \bar{\alpha}_\mu - x_{(0)\mu} \bar{\alpha}_\mu}{r^4} \right) + \mathcal{O}(\frac{1}{r^5}). \quad (9.9)$$
Comparing this to Eqn. (8.18), one finds that the expansion of 't Hooft’s solution agrees with the expansion of the singular solution with shifted pole apart from a constant gauge transformation. Note that if one includes this rigid gauge transformation in the solution, the 't Hooft $N = 1$ instanton is characterised by eight rather than by five parameters. By comparison, if we ignore an overall multiplicative real constant, the number of parameters in Eqn. (8.18) is seven in total, when the additional rigid gauge transformation is included. Thus the dipole and quadrupole term together almost determine the background $N = 1$ instanton but not completely.

It will be generally convenient to include the three parameters arising from rigid gauge transformations into the number of parameters characterising an instanton, though they are not of physical significance. One then finds, that the general AHDM $N$-instanton is characterised by $8N$ parameters, the JNR $N$ instanton by $5N + 7$ and the 't Hooft $N$ instanton by $5N + 3$.

10. $N = 2$ instantons and interpretation of constants

We shall now investigate the dipole and quadrupole term of the quaternionic Laurent expansion a bit further. As stated previously, the dipole term of this series, given by Eqn. (6.15) is characterised by one constant $c_q$ which is the quaternionic analogue of a residue. However, the next term of the expansion, the quadrupole term, is instead characterised by three constants, the interpretation of which is less clear. The quadrupole term of the Laurent series is given by

$$ Q \equiv \sum_{i=1}^{3} b_i \partial_i G(q) $$

with the $b_i$ given by Eqn. (5.28). Calculating the derivative of $G(q)$ with respect to $\partial_i$ explicitly we find

$$ \frac{\partial}{\partial x_i} G(q) = - \left[ \frac{e_i}{|q|^4} + \frac{4\hat{q} \cdot \hat{x}_i}{|q|^4 |\hat{x}_i|^2} \right]. $$

(10.2)

Notice that only derivatives with respect to the spatial coordinates $x_i$ occur, since the derivative of $G(q)$ with respect to the time coordinate $x_0$ is determined via the Cauchy-Riemann-Fueter equation, once the spatial derivatives are known. However, this explicitly breaks the symmetry between the time coordinate and the spatial coordinates. Setting

$$ \tilde{b}_0 e_i - \tilde{b}_i \equiv b_i \quad \text{with} \quad \tilde{b}_\mu \in \mathbb{H} $$

(10.3)
10. \( N = 2 \) instantons and interpretation of constants

we may restore this symmetry in the equations. We then find for the quadrupole term

\[
Q = \sum_{i=1}^{3} -b_i \left[ e_i \frac{q}{|q|^4} + \frac{4\hat{q}}{|q|^4} \hat{x}_i \right]
\]

\[
= -\left[ \tilde{b}_0 - \sum_{i=1}^{3} \tilde{b}_i e_i \right] \frac{q}{|q|^4} + 4 \left[ \sum_{\mu=0}^{3} \hat{x}_\mu \tilde{b}_\mu \right] \frac{\hat{q}}{|q|^4}
\]

(10.4)

where we have used \( \sum_{i=1}^{3} e_i \hat{x}_i \cdot \hat{q} = 1 - \hat{x}_0 \hat{q} \) to derive the last line. We may simplify notation further assuming the \( \tilde{b}_\mu \)'s to be the components of some quaternionic valued vector \( \tilde{b} \) and writing \( \tilde{b}_q = \tilde{b}_0 + \sum_{i=1}^{3} \tilde{b}_i e_i \) and \( \tilde{b}_q = \tilde{b}_0 - \sum_{i=1}^{3} \tilde{b}_i e_i \), respectively. We then have

\[
Q = -\frac{\tilde{b}_q}{|q|^4} + 4 \hat{x} \tilde{b} \frac{\hat{q}}{|q|^4}.
\]

(10.5)

However, one should note that this notation is somewhat misleading, since the components \( \tilde{b}_\mu \) of \( \tilde{b}_q \) are themselves quaternions. Therefore \( \tilde{b}_q \) will in general not be the quaternionic conjugate of \( \tilde{b}_q \). Nevertheless this notation proves to be useful, as can be seen directly when comparing Eqn. (10.5) to the \( \mathcal{O}(r^{-4}) \) contribution to Eqn. (8.15). These two expressions have a functionally similar form, which will allow us to read off the constants \( \tilde{b}_\mu \) for the quadrupole term of some specific instanton configuration.

Sometimes we will however use a slightly different notation. Since the \( \tilde{b}_\mu \) are themselves quaternions we may write

\[
\tilde{b}_\mu = \tilde{b}_{\mu\nu} e_\nu \quad \text{with} \quad \tilde{b}_{\mu\nu} \in \mathbb{R}.
\]

(10.6)

We then have for Eqn. (10.5)

\[
Q = -\frac{\tilde{e}_\mu \tilde{b}_{\mu\nu} e_\nu}{|q|^4} + 4 \hat{x}_\mu \tilde{b}_{\mu\nu} e_\nu \frac{\hat{q}}{|q|^4}.
\]

(10.7)

This notation will prove especially useful when investigating the symmetry properties of the solution to the Dirac equation in the instanton background.

Note that the number of parameters occurring in the Laurent series up to quadrupole order is 15, ignoring an overall multiplicative real constant, but including rigid gauge transformations. This should be compared with 16 parameters describing an \( N = 2 \) instanton if one includes rigid gauge transformations. Hence, as in the \( N = 1 \) instanton case, the dipole and the quadrupole term together almost determine the background instanton but not completely. In the remainder we will show that the parameters \( \tilde{b}_\mu \) are related to the symmetry of the underlying two-instanton configuration.
10. $N = 2$ instantons and interpretation of constants

Figure 1: The circle and the ellipse associated with a $N = 2$ instanton in $\mathbb{R}^4$ and one member of the porism of triangles with vertices $x_{(1)}$, $x_{(2)}$ and $x_{(3)}$ on the circle and tangent to the ellipse at $a_{(1)}$, $a_{(2)}$ and $a_{(3)}$.

With each $N = 2$ instanton in $\mathbb{R}^4$ there is an associated pair of a circle and an ellipse, which satisfies the Poncelet condition, which means that there is a porism (or one-parameter family) of triangles with vertices on the circle and tangent to the ellipse, see Fig. 1. In fact, if one such triangle exists then there is automatically a porism of triangles and any point on the circle may be a vertex. From this geometrical data one may reconstruct the instanton fields in JNR form (see [7]), by choosing one triangle of the porism with vertices $x_{(1)}$, $x_{(2)}$ and $x_{(3)}$ on the circle, and tangent to the ellipse at $a_{(1)}$, $a_{(2)}$ and $a_{(3)}$. Then define weights $\lambda^2_{(1)}$, $\lambda^2_{(2)}$ and $\lambda^2_{(3)}$ up to a common multiple, by

$$\frac{\lambda^2_{(1)}}{\lambda^2_{(2)}} = \frac{x_{(1)} a_{(3)}}{a_{(3)} x_{(2)}}, \quad \text{etc.} \quad (10.8)$$

The JNR potential is then given by

$$\rho = \sum_{i=1}^{3} \frac{\lambda^2_{(i)}}{(x - x_{(i)})^2}. \quad (10.9)$$

If another triangle of the porism is chosen, then a different expression for the scalar potential $\rho$ is obtained, but the instanton will change only by a gauge transformation. This corresponds to the gauge invariance noted by Jackiw, Nohl and Rebbi, that moving the vertices around the circle will change the instanton only by a gauge transformation.

As the first example we will consider the limiting case of an $N = 2$ instanton, where $x_{(1)}$, $x_{(2)}$ and $x_{(3)}$ are collinear, with $x_{(1)} = 0$ and $x_{(2)} = -x_{(3)}$, as shown in Fig. 2. Notice that in this limiting case the associated ellipse becomes degenerate. The scale parameters $\lambda^2_{(i)}$ are given by $\lambda^2_{(1)} = \lambda^2$ and $\lambda^2_{(2)} = \lambda^2_{(3)} = \mu^2$. Thus the centre of mass is $X = 0$, and the sum of the weights is given by $\Lambda = \lambda^2 + 2\mu^2$. 
10. \( N = 2 \) instantons and interpretation of constants

\[
\begin{align*}
\mathbf{x}(2) &= -\mathbf{x}(3) & \mathbf{x}(1) &= 0 & \mathbf{x}(3) \\
\mu^2 & & \lambda^2 & & \mu^2
\end{align*}
\]

Figure 2: The three vertices associated with an \( N = 2 \) instanton in \( \mathbb{R}^4 \) are chosen to be collinear. This is the limiting case of a general \( N = 2 \) instanton in \( \mathbb{R}^4 \), for which the three vertices lie on a circle.

There are three solutions \( \tilde{\psi}'(k) \), whose dipole and the quadrupole terms are given by Eqn. (8.13) and Eqn. (8.14), respectively. However, since \( \tilde{\psi}'(1) + \tilde{\psi}'(2) + \tilde{\psi}'(3) = 0 \), these three solutions are linearly dependent. Thus it is convenient to consider suitable linear combinations, which we will choose to reflect the symmetry of the instanton. These will be given by

\[
\tilde{\psi}'(a) = \tilde{\psi}'(2) - \tilde{\psi}'(3) \quad \text{and} \quad \tilde{\psi}'(b) = \tilde{\psi}'(1). \tag{10.10}
\]

We then find for the expansion of \( \tilde{\psi}'(a) \)

\[
\tilde{\psi}'(a) = -4\mu^2 \frac{1}{\Lambda^{1/2}} \cdot \frac{1}{r^3} \hat{\mathcal{N}}^{(2)}_s + \mathcal{O} \left( \frac{1}{r^5} \right). \tag{10.11}
\]

Here we have used that since \( \hat{\mathcal{N}}^{(k)}_s = (\mathbf{x}(k) \cdot \alpha)(\hat{x} \cdot \bar{\alpha}) \) and \( x(3) = -x(2) \), therefore \( \hat{\mathcal{N}}^{(3)}_s = -\hat{\mathcal{N}}^{(2)}_s \). Notice that the quadrupole contribution to \( \tilde{\psi}'(a) \) vanishes. For \( \tilde{\psi}'(b) \) we find instead

\[
\tilde{\psi}'(b) = -\frac{1}{r^4} \cdot \frac{2\lambda^2 \mu^2}{\Lambda^{3/2}} \left[ -8\hat{x} \cdot x(2) \hat{\mathcal{N}}^{(2)}_s + 2x^2(2) \right] + \mathcal{O} \left( \frac{1}{r^6} \right). \tag{10.12}
\]

Here the dipole contribution is equal to zero. Notice that \( \tilde{\psi}'(b) \) can be written as the sum of two shifted dipoles, namely

\[
\tilde{\psi}'(b) \approx -\frac{2\lambda^2 \mu^2}{\Lambda^{3/2}} \left[ \frac{(x(2) \cdot \alpha)(x \cdot \bar{\alpha} - x(2) \cdot \bar{\alpha})}{(x - x(2))^4} + \frac{(x(3) \cdot \alpha)(x \cdot \bar{\alpha} - x(3) \cdot \bar{\alpha})}{(x - x(3))^4} \right]. \tag{10.13}
\]

This will give the correct expansion of \( \tilde{\psi}'(b) \) at least up to quadrupole order. In contrast, \( \tilde{\psi}'(a) \) can be interpreted as a pure dipole at \( x(1) = 0 \). We have for \( \tilde{\psi}'(a), \tilde{\psi}'(b) \), respectively, in quaternionic notation

\[
\begin{align*}
\tilde{\psi}'(a) &= -\frac{4\mu^2}{\Lambda^{1/2}} \cdot \frac{\hat{q}}{|\hat{q}|^3} \\
\tilde{\psi}'(b) &= -\frac{4\lambda^2 \mu^2}{\Lambda^{3/2}} \cdot \frac{-4\hat{x} \cdot x(2) \hat{\bar{q}} + \bar{q}(2)}{|\hat{q}|^4} \tag{10.14}
\end{align*}
\]
Now we are able to read off the constants $\tilde{b}_\mu$ characterising the quadrupole term of the quaternionic Laurent series. For $\tilde{\psi}'_{(b)}$,

$$\tilde{b}_\mu = \frac{4\lambda^2 \mu^2}{\Lambda^{3/2}} \cdot q(2) \cdot x(2)_\mu$$  \hspace{1cm} (10.16)

or in terms of $\tilde{b}_{\mu\nu}$

$$\tilde{b}_{\mu\nu} = \frac{4\lambda^2 \mu^2}{\Lambda^{3/2}} \cdot x(2)_\mu x(2)_\nu.$$  \hspace{1cm} (10.17)

One easily checks that this gives also the correct constant contribution to the quadrupole term. For $\tilde{\psi}'_{(a)}$ the $\tilde{b}_\mu$ are all zero as there is no quadrupole term.

The constant $q(2)$ can be interpreted as the residue of $\tilde{\psi}'_{(a)}$. Notice that, if one chooses the vertices to lie on some particular axis, say the $x_\mu$-axis, only one of the parameters is not equal to zero, namely $\tilde{b}_\mu$, which is proportional to $e_\mu$.

This nicely reflects the symmetry of the configuration of vertices, if one assumes a generic quaternion $q = x_0 + x_i e_i$ to be invariant under the following (generalised) rotations of $\mathbb{R}^3$. Consider for example a rotation of $\mathbb{R}^3$, which means $x_i \mapsto R_{ij}x_j$ with $R \in SO(3)$. Note that the quaternionic basis vectors $e_i$ are in a sense arbitrary, since one may take linear combinations of them leading to an equivalent description of $\mathbb{H}$. If we identify the space spanned by $e_i$ with the space spanned by $R_{ij}^t e_i$, the map of $\mathbb{R}^3$ into the space of the pure quaternions with $x_0 = 0$ is spherically symmetric under the orthogonal transformation $R$, since then $q = x_i e_i \mapsto q = R_{ij}x_j \cdot e_i = x_j \cdot R_{ji}^t e_i = x_j e_j \equiv x_j e_j$. In other words, under such transformations the generic quaternion $q$ will be invariant.

For example, the general dipole term of the Laurent expansion $L_D$ will under a rotation $x_i \mapsto R_{ij}x_j$ with $R \in SO(3)$ transform as

$$L_D = (c_0 + c_i e_i) \cdot \frac{\bar{q}}{|q|^4} \mapsto (c_0 + c_i R_{ji} \bar{e}_j) \cdot \frac{x_0 - x_i \bar{e}_i}{|q|^4} \equiv (c_0 + c_i R_{ji} \bar{e}_j) \cdot \frac{\bar{q}}{|q|^4}. \hspace{1cm} (10.18)$$

If we write the product of two basic quaternions

$$e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k$$  \hspace{1cm} (10.19)

in terms of the new basis we find (since $\det(R) = 1$)

$$e_i e_j = R_{mi} \bar{e}_m R_{nj} \bar{e}_n = -\delta_{ij} + \det(R) \epsilon_{ijn} R_{mn} \bar{e}_m = -\delta_{ij} + \epsilon_{ijn} e_n. \hspace{1cm} (10.20)$$

We are now in the position to investigate the symmetry properties under rotations of solutions $\tilde{\psi}'_{(a)}$ and $\tilde{\psi}'_{(b)}$, Eqn. (10.14) and Eqn. (10.15), respectively. Choosing for example the vertices in the above example to lie on the $x_0$-axis, the background instanton is spherically symmetric in the spatial directions. But under a spatial rotation in the above sense also $\tilde{\psi}'_{(a)}$ remains invariant as does $\tilde{\psi}'_{(b)}$. Thus $\tilde{\psi}'_{(a)}$ and $\tilde{\psi}'_{(b)}$ show the same symmetry properties as the background.
As the next example we will consider an instanton, where $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ lie at the vertices of an equilateral triangle, such that $\sum \mathbf{x}_i = \mathbf{0}$, $\mathbf{x}_i^2 = 1$ and where the weights are all equal, $\lambda^2 = \lambda^2$. For definiteness we shall assume that the triangle lies in the $(x_1, x_2)$ plane, see Fig. 3. We have

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ \cos \alpha \\ \sin \alpha \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ -\frac{1}{2}(\cos \alpha + \sqrt{3} \sin \alpha) \\ -\frac{1}{2}(\sin \alpha - \sqrt{3} \cos \alpha) \end{pmatrix}, \quad \mathbf{x}_3 = -(\mathbf{x}_1 + \mathbf{x}_2).$$

(10.21)

Here we have explicitly accounted for the fact that moving the vertices around the circle corresponds to a gauge transformation. Thus all configurations arising from different values of $\alpha$ in Eqn. (10.21) are related via gauge transformations.

Here again we shall consider suitable linear combinations of the three linearly dependent solutions $\tilde{\psi}'_k$, respectively

$$\tilde{\psi}'_1 = -\frac{2\lambda \left[ q(1) \right]}{\sqrt{3} \left| q \right|^3} + \frac{4}{3|q|^4} \left( -2 \hat{x} \cdot \mathbf{x}_2 q(2) + \hat{x} \cdot \mathbf{x}_1 q(1) - \hat{x} \cdot \mathbf{x}_1 q(2) - \hat{x} \cdot \mathbf{x}_2 q(1) \right) \hat{q} + O\left( \frac{1}{|q|^5} \right)$$

(10.22)

$$\tilde{\psi}'_2 = -\frac{2\lambda \left[ q(2) \right]}{\sqrt{3} \left| q \right|^3} + \frac{4}{3|q|^4} \left( -2 \hat{x} \cdot \mathbf{x}_1 q(1) + \hat{x} \cdot \mathbf{x}_2 q(2) - \hat{x} \cdot \mathbf{x}_1 q(2) - \hat{x} \cdot \mathbf{x}_2 q(1) \right) \hat{q} + O\left( \frac{1}{|q|^5} \right)$$

(10.23)

$$\tilde{\psi}'_3 = -\frac{2\lambda \left[ q(2) + q(1) \right]}{\sqrt{3} \left| q \right|^3} + \frac{4}{3|q|^4} \left( \hat{x} \cdot \mathbf{x}_1 q(1) + \hat{x} \cdot \mathbf{x}_2 q(2) + 2(\hat{x} \cdot \mathbf{x}_1 q(2) + \hat{x} \cdot \mathbf{x}_2 q(1)) \right) \hat{q} + O\left( \frac{1}{|q|^5} \right).$$

(10.24)

These are

$$\psi_{(a)} = \tilde{\psi}'_1 + \omega \tilde{\psi}'_2 + \omega^2 \tilde{\psi}'_3$$

(10.25)

$$\psi_{(b)} = \tilde{\psi}'_1 + \omega^2 \tilde{\psi}'_2 + \omega \tilde{\psi}'_3$$

(10.26)

with $\omega = \exp\left( \frac{2\pi k}{3} \right) = -\frac{1}{2}(1 - \sqrt{3}k)$. To calculate these linear combinations it is useful to notice that

$$q_1 = i \exp(-k\alpha), \quad q_2 = \omega q_1, \quad q_3 = \omega^2 q_1.$$

(10.27)

We then find

$$\psi_{(a)} = \frac{4\sqrt{3}\lambda}{|q|^4} \cdot \exp(2k\alpha) \cdot (-i\hat{x}_1 + j\hat{x}_2) \cdot \hat{q} + O\left( \frac{1}{|q|^5} \right)$$

(10.28)

$$\psi_{(b)} = -\frac{2\sqrt{3}\lambda}{|q|^3} \cdot \exp(k\alpha) \cdot i \cdot \hat{q} + O\left( \frac{1}{|q|^4} \right).$$

(10.29)
10. $N = 2$ instantons and interpretation of constants

\[ x_2 \]
\[ x_1 \]
\[ x_{(2)}, \lambda^2 \]
\[ x_{(3)}, \lambda^2 \]
\[ x_{(1)}, \lambda^2 \]

Figure 3: The Poncelet pair of two concentric circles associated with the circularly symmetric $N = 2$ instanton and one member of the porism of triangles with vertices on the outer circle and tangent to the inner circle.

$\psi_{(a)}$ is a pure quadrupole and $\psi_{(b)}$ is a pure dipole with no quadrupole. The left-multiplicative factors $\exp (2k\alpha)$ and $\exp (k\alpha)$, respectively, nicely reflect the fact that moving the vertices around the circle corresponds to a gauge transformation. We can now read off the constants $\tilde{b}_\mu$ from $\psi_{(a)}$. Setting $\alpha = 0$ from now on, we have

\[ \tilde{b} = \sqrt{3}\lambda \begin{pmatrix} 0 \\ -i \\ j \\ 0 \end{pmatrix}, \quad (10.30) \]

or in terms of $\tilde{b}_{\mu\nu}$

\[ (\tilde{b}_{\mu\nu}) = \sqrt{3}\lambda \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (10.31) \]

Since the gauge invariant data of this instanton configuration is given by the Poncelet pair of two concentric circles [12] as shown in Fig. 3 we have circular symmetry in the $(x_1, x_2)$ plane. As in our previous example the solutions $\psi_{(a)}$ and $\psi_{(b)}$, Eqn. (10.28) and Eqn. (10.29), respectively, exhibit the same symmetry properties as the background. Thus if

\[ x_i \mapsto R_{ij} x_j \quad \text{with} \quad R = \begin{pmatrix} \cos \beta & -\sin \beta & 0 \\ \sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (10.32) \]

we find that

\[ \psi_{(a)} \mapsto \exp (-2k\beta) \cdot \psi_{(a)} \quad (10.33) \]
\[ \psi_{(b)} \mapsto \exp (-k\beta) \cdot \psi_{(b)}. \quad (10.34) \]
10. \( N = 2 \) instantons and interpretation of constants

Hence, under rotations \( \psi(a) \) and \( \psi(b) \) are invariant up to a gauge transformation.

Note that the Weyl equation is not invariant under reflections. Yet the background instanton is invariant under reflections in the \((x_1, x_2)\) plane, and also under the reflection \( x_3 \mapsto -x_3 \). In order to show that the spinor solutions respect this symmetry, we will make use of the fact that two successive reflections correspond to a rotation. We will therefore combine a reflection, say \( x_1 \mapsto -x_1 \), with the reflection \( x_3 \mapsto -x_3 \). The combination of these is the 180° rotation \( x_i \mapsto R_{ij} x_j \), with

\[
R = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\]

We find for the transformation of \( \psi(a) \) and \( \psi(b) \), respectively,

\[
\psi(a) \mapsto \psi(a) \quad \text{(10.36)}
\]

\[
\psi(b) \mapsto -\psi(b) \quad \text{(10.37)}
\]

Hence, under this combination of reflections we find \( \psi(a) \) and \( \psi(b) \) to be invariant up to a gauge transformation. Similarly, there is invariance for any reflection axis in the \((x_1, x_2)\) plane.

So far we have only investigated two highly symmetric \( N = 2 \) instantons. As the final example we want to discuss the case where the circle and ellipse, the gauge invariant data of an \( N = 2 \) instanton, are concentric, see Fig. 4. Note that the moduli space associated with a general \( N = 2 \) instanton is — excluding rigid gauge transformations — a 13-dimensional manifold whereas the orbits of the conformal group are 12-dimensional. After quotienting out the latter a general \( N = 2 \) instanton is described by only one free parameter. Indeed it may be proved that any \( N = 2 \) instanton is conformally related to an instanton of which the gauge invariant data is given by the Poncelet pair of a concentric circle and ellipse, the one free parameter being the eccentricity \( a/b \) of the ellipse. In this sense an \( N = 2 \) instanton with associated concentric circle and ellipse is the most general one.

Suppose, in Fig. 4 the radius of the circle is \( R = 1 \). A porism of triangles with vertices on the circle and tangent to the ellipse exists if and only if \( a + b = R \). For simplicity we will explicitly choose one member of the porism of triangles, such that \( x_{(1)} \), \( x_{(2)} \) and \( x_{(3)} \) are given by

\[
x_{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad x_{(2)} = \begin{pmatrix} 0 \\ -a \\ \sqrt{1 - a^2} \\ 0 \end{pmatrix}, \quad x_{(3)} = \begin{pmatrix} 0 \\ -a \\ -\sqrt{1 - a^2} \\ 0 \end{pmatrix}.
\]

The weights associated with the vertices may be calculated using Eqn. (10.8).
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Scaling the weights such that $\lambda_1^2 = \lambda^2 = 1$, we find for $\lambda_2^2$, $\lambda_3^2$, respectively

$$\lambda_2^2 = \lambda_3^2 = \mu^2 = \frac{a}{b}.$$  \hfill (10.39)

Again we find suitable linear combinations of the three dependent solutions $\tilde{\psi}'_{(k)}$, $k = 1, 2, 3$, such that the two resulting solutions form a natural basis of the solution space. The expressions for $\tilde{\psi}'_{(k)}$ are similar to Eqns. (10.22), (10.23), (10.24) but a bit more elaborate. The linear combinations are

$$\psi_{(A)} = \tilde{\psi}'_{(1)} + k \frac{1}{\sqrt{\Lambda}}(\tilde{\psi}'_{(2)} - \tilde{\psi}'_{(3)})$$ \hfill (10.40)

$$\psi_{(B)} = \tilde{\psi}'_{(1)} - k \frac{1}{\sqrt{\Lambda}}(\tilde{\psi}'_{(2)} - \tilde{\psi}'_{(3)})$$ \hfill (10.41)

where $\Lambda = \frac{1-a}{1+a}$. We find that $\psi_{(A)}$ is again a pure quadrupole, and $\psi_{(B)}$ is a pure dipole. They are, respectively

$$\psi_{(A)} = \frac{1}{|q|^4} \cdot \frac{8a}{\sqrt{\Lambda}} \cdot \left[ 4(-ia\hat{x}_1 + jb\hat{x}_2) \cdot \hat{q} + (b - a) \right] + O\left(\frac{1}{|q|^5}\right)$$ \hfill (10.42)

$$\psi_{(B)} = -\frac{1}{|q|^3} \cdot \frac{8a}{\sqrt{\Lambda}} \cdot i \cdot \hat{q} + O\left(\frac{1}{|q|^4}\right).$$ \hfill (10.43)

It is now easy to read off that the constants $\tilde{b}_\mu$ associated with $\psi_{(A)}$ are

$$\tilde{b} = \frac{8a}{\Lambda^{1/2}} \begin{pmatrix} 0 \\ -ia \\ jb \\ 0 \end{pmatrix}$$ \hfill (10.44)

Figure 4: The Poncelet pair of a concentric circle and ellipse associated with the general $N = 2$ instanton and one member of the porism of triangles with vertices on the circle and tangent to the ellipse.
and thus
\[
\tilde{b}_{\mu\nu} = \frac{8a}{\Lambda^{1/2}} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -a & 0 & 0 \\
0 & 0 & b & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\] (10.45)

In this example the background field exhibits just a 180° rotational symmetry in the \((x_1, x_2)\) plane and two reflection symmetries, namely under the transformations \(x_1 \mapsto -x_1\) and \(x_2 \mapsto -x_2\), respectively. Under the 180° rotation in the \((x_1, x_2)\) plane we find for the transformation of \(\psi(A)\) and \(\psi(B)\), respectively
\[
\psi(A) \mapsto \psi(A) \quad \text{and} \quad \psi(B) \mapsto -\psi(B).
\] (10.46)\hspace{1cm} (10.47)

Investigating the symmetry properties of the two solutions under reflections we encounter the same problem as in the case of the circularly symmetric instanton, namely that the Weyl equation itself is not invariant under reflections. However we may tackle this problem in exactly the same way as before. We will therefore consider the transformation of \(\psi(A)\) and \(\psi(B)\) under the following two rotations, which are each combinations of two reflections:
\[
x_i \mapsto R_{ij} x_j \quad \text{with} \quad R = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix},
\] (10.48)

and
\[
x_i \mapsto R_{ij} x_j \quad \text{with} \quad R = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\] (10.49)

Thus, the first rotation consists of the combined reflections \(x_1 \mapsto -x_1\) and \(x_3 \mapsto -x_3\), whereas the second consists of the combined reflections \(x_2 \mapsto -x_2\) and \(x_3 \mapsto -x_3\). We find for the transformation of \(\psi(A)\) and \(\psi(B)\) under the rotation given by Eqn. (10.48)
\[
\psi(A) \mapsto \psi(A) \quad \text{and} \quad \psi(B) \mapsto -\psi(B).
\] (10.50)\hspace{1cm} (10.51)

and under the rotation given by Eqn. (10.49)
\[
\psi(A) \mapsto \psi(A) \quad \text{and} \quad \psi(B) \mapsto \psi(B).
\] (10.52)\hspace{1cm} (10.53)

Note that again the two solutions respect the symmetry properties of the background instanton.
11 Conclusion

We have seen that a non-trivial solution to the free Weyl equation in Euclidean four-space, which is bounded at infinity, necessarily exhibits a singularity at one point at least. In the simplest case such a solution is in quaternionic notation given by

$$\Psi_q = a_q \cdot \frac{\bar{q} - \bar{b}_q}{|q - b_q|^4}$$  \hspace{1cm} (11.1)

and thus parameterised by two quaternionic constants, $a_q$ and $b_q$, respectively. Here $a_q$ is the quaternionic analogue of a residue and as such it may be interpreted as the ‘spinorial charge’ of the spinor wave function. The spinor carrying this spinorial charge is clearly localised around $b_q$, $b_q$ being the position parameter of $\Psi_q$.

Solutions to the Weyl equation in the background of an arbitrary JNR $N$-instanton, however, are normalisable and regular in the whole of Euclidean four-space $\mathbb{R}^4$. Comparing the asymptotic behaviour of these solutions with the asymptotic behaviour of the singular solution to the free Weyl equation we found that the former solutions can be written up to order $O(|q|^{-4})$ as linear combinations of the latter. In this sense the introduction of the instanton gauge field results in a delocalisation of the spinor. Investigating some special $N = 2$ instantons we were able to show that the parameters describing this delocalised spinor reflect the geometry of the underlying instanton configuration.

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References

[1] A. Sudbery, Math. Proc. Camb. Phil. Soc. 85, 199 (1979).
[2] R. Rajaraman, Solitons and Instantons (Elsevier Science, Amsterdam, 1982), chap. 4, pp. 84 – 111.
[3] R. Jackiw, C. Nohl, and C. Rebbi, Phys. Rev. D15, 1642 (1977).
[4] B. Grossman, Phys. Lett. 61A, 86 (1977).
[5] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld, and Y. I. Manin, Phys. Lett. 65A, 185 (1978).
[6] G. 't Hooft, unpublished.
[7] M. F. Atiyah and N. S. Manton, Commun. Math. Phys. 152, 391 (1993).
[8] E. F. Corrigan, D. B. Fairlie, S. Templeton, and P. Goddard, Nucl. Phys. B140, 31 (1978).

[9] H. Osborn, Nucl. Phys. B140, 45 (1978).

[10] R. Fueter, Comment. Math. Helv. 7, 307 (1935).

[11] I. R. Porteous, Clifford algebras and the classical groups (Cambridge University Press, Cambridge, 1995), chap. 8, pp. 57 – 66.

[12] R. Hartshorne, Commun. Math.Phys. 59, 1 (1978).