ON PARTIAL GALOIS ABELIAN EXTENSIONS

ANDRÉS CAÑAS, VÍCTOR MARÍN, AND HÉCTOR PINEDO

ABSTRACT. In this article we construct the inverse semigroup of equivalence classes of partial Galois abelian extensions of a commutative ring $R$ with same group $G$, called the Harrison partial inverse semigroup.

1. Introduction

In the 1960’s, M. Auslander and O. Goldman introduced in [1] the notion of Galois extension for commutative rings. After that, S. U. Chase, D. K. Harrison and A. Rosenberg developed in [5] the Galois theory for commutative rings extending the classical theory over fields. One of the main results of [5] is Theorem 2.3 which has two parts. Let $R \subset S$ be a Galois extension of commutative rings with Galois group $G$. The first part established a bijective correspondence between subgroups of $G$ and $R$-subrings of $S$ which are $G$-strong. In the second part, it was shown that if $H$ is a normal subgroup of $G$ the subring $S^H$ of $S$ of invariants by the action of $H$ is a Galois extension of $R$ with Galois group $G/H$.

A Galois extension of commutative rings $R \subset S$ with Galois group $G$ is called abelian when $G$ is an abelian group. For a fixed abelian group $G$ and a fixed commutative ring $R$, D. K. Harrison constructed in [7] a group $\mathcal{H}(G,R)$, which is called the Harrison group. The elements of $\mathcal{H}(G,R)$ are the classes of $G$-isomorphism of abelian extensions of $R$ with group $G$. To define a binary operation in $\mathcal{H}(G,R)$, D. K. Harrison used the second part of Theorem 2.3 of [5]; see details in [7].

M. Dokuchaev, M. Ferrero and A. Paques developed in [3] the Galois theory for commutative rings when the group acts partially. The results proved in [3] generalizes many results of [5]. For instance, Theorem 5.1 of [3] generalizes the first part of Theorem 2.3 of [5]. However, there is no generalization of the second part of Theorem 2.3 of [5] to the context of partial actions.

Given a finite abelian group $G$ and a commutative ring $R$, we consider the set $\mathcal{H}_{\text{par}}(G,R)$ of equivalence classes of partial Galois extensions of $R$ with group $G$. In particular, we have $\mathcal{H}(G,R) \subset \mathcal{H}_{\text{par}}(G,R)$. The main purpose of this paper is to provide a structure of inverse semigroup for $\mathcal{H}_{\text{par}}(G,R)$. For such, we will generalize the second part of Theorem 2.3 of [5] to the context of partial actions and we will follow similar ideas to those used in [7].
The paper is organized as follows. The basic notions and results that we used throughout the paper are presented in Section 2. In Section 3, we prove the second part of Theorem 2.3 of [5] for partial actions. Following ideas from [3], we study in Section 4 (partial) Galois extensions $G$-isomorphic. Finally, in Section 5, we prove that $\mathcal{H}_{par}(G, R)$ is an inverse semigroup and we illustrate the binary operation of $\mathcal{H}_{par}(G, R)$ with a concrete example.

Conventions. Throughout the paper, rings are always considered commutative with identity element. Each ring homomorphism is unitary, that is, it sends identity element in identity element. The extensions of rings have same identity element. If $S$ and $T$ are extensions of a same ring $R$ then $S \otimes T$ means $S \otimes_R T$. Moreover, $k$ will denote an associative and commutative ring with unity and $G$ a group. The identity element of $G$ will be denoted by 1.

2. Preliminaries

In this section we present the background about partial actions and globalization that will be used in the paper.

2.1. Partial action of groups. A partial action of a group $G$ on a $k$-algebra $S$ is a pair $\alpha = (S_g, \alpha_g)_{g \in G}$ that satisfies:

(P1) for each $g \in G$, $S_g$ is an ideal of $S$ and $\alpha_g : S_{g^{-1}} \to S_g$ is a $k$-algebra isomorphism,
(P2) $S_1 = S$ and $\alpha_1 = \text{id}_S$,
(P3) $\alpha_g(S_{g^{-1}} \cap S_h) = S_g \cap S_{gh}$, for all $g, h \in G$,
(P4) $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$, for all $x \in S_{h^{-1}} \cap S_{(gh)^{-1}}$ and $g, h \in G$.

The partial action $\alpha$ is called unital if every ideal $S_g$ is unital, that is, there exists $1_g \in S$ such that $S_g = 1_g S$. Notice that the conditions (P3) and (P4) imply that $\alpha_{gh}$ is an extension of $\alpha_g \circ \alpha_h$, for every $g, h \in G$.

A partial action $\alpha$ is said global if $S_g = S$, for all $g \in G$. Global actions of a group $G$ on a $k$-algebra $T$ induce, by restriction, partial actions on any ideal $S$ of $T$. Indeed, given a global action $\beta = (T_g, \beta_g)_{g \in G}$ of $G$ on $T$ we consider the ideals $S_g = S \cap \beta_g(S)$ of $S$ and the $k$-algebra isomorphisms $\alpha_g = \beta_g|_{S_{g^{-1}}}$. Then $\alpha = (S_g, \alpha_g)$ is a partial action of $G$ on $S$. Partial actions obtained in this way are called globalizable. The precise definition is given below.

Let $\alpha = (S_g, \alpha_g)_{g \in G}$ be a partial action of a group $G$ on a $k$-algebra $S$. A globalization of $\alpha$ is a global action $\beta = (T_g, \beta_g)_{g \in G}$ of $G$ on a $k$-algebra $T$ that satisfies:

(G1) $T$ is an ideal of $T$,
(G2) $S_g = S \cap \beta_g(S)$, for all $g \in G$,
(G3) $\beta_g(x) = \alpha_g(x)$, for all $x \in S_{g^{-1}}$,
(G4) $T = \sum_{g \in G} \beta_g(S)$.

If $\alpha$ admits a globalization we say that $\alpha$ is globalizable. The globalization of $\alpha$ is unique, up to isomorphism, and will be denote by $(T, \beta)$. Also, Theorem 4.5 of [2] implies that unital partial actions admit globalizations. More details related to globalization can be seen in [2].
From now on we assume that $G$ is a finite group, $\alpha = (S_g, \alpha_g)_{g \in G}$ is a unital partial action of $G$ on a $k$-algebra $S$ such that $S_g = S1_g$, for all $g \in G$, and $\beta = (T_g, \beta_g)_{g \in G}$ is a global action of $G$ on a $k$-algebra $T$ which is the globalization of $\alpha$. Notice that $1_S$ is a central idempotent element of $T$ and $S = T1_S$. Moreover, it was proved in [3, p.79] that

1. $1_g = \beta_g(1_S)1_S$, $\alpha_g(s1_{g^{-1}}) = \beta_g(s)1_S$, $\alpha_g(\alpha_h(s1_{h^{-1}})1_{g^{-1}}) = \alpha_{gh}(s1_{(gh)^{-1}})1_g$, for all $g, h \in G$ and $s \in S$. Let $H = \{h_1 = 1, h_2, \ldots, h_m\}$ be a subgroup of $G$. It was defined in [3, p.79] the map $\psi_H : T \to T$ by

$$
\psi_H(t) = \sum_{1 \leq i \leq m} \sum_{i_1 < \cdots < i_l} (-1)^{l+1} \beta_{h_{i_1}}(1_S)\beta_{h_{i_2}}(1_S) \cdots \beta_{h_{i_l}}(1_S)\beta_{h_i}(t), \quad t \in T.
$$

This map can be rewritten as

$$
\psi_H(t) = \sum_{i=1}^{m} \beta_{h_i}(t)e_i, \quad \text{for all } t \in T,
$$

where $e_i$’s are the following idempotents of $T$:

3. $e_1 = 1_S$, $e_i = (1_T - 1_S)(1_T - \beta_{h_2}(1_S)) \cdots (1_T - \beta_{h_{i-1}}(1_S))\beta_{h_i}(1_S), \quad 2 \leq i \leq m.$

Since $1_S$ is a central idempotent of $T$, given $2 \leq i \leq m$ we have

$$
\beta_g(e_1)1_S = (1_T - \beta_g(1_S))\beta_g(1_S) \cdots (1_T - \beta_{g_{h_{i-1}}}(1_S))\beta_{g_i}(1_S)1_S
$$

$$
= (1_S - \beta_g(1_S)1_S)\beta_{gh_2}(1_S) \cdots (1_S - \beta_{gh_{i-1}}(1_S)1_S)\beta_{gh_i}(1_S)1_S
$$

$$
= (1_S - 1_g)(1_S - 1_{gh_2}) \cdots (1_S - 1_{gh_{i-1}})1_{gh_i}.
$$

Thus

4. $\beta_g(e_1)1_S = 1_g$ and $\beta_g(e_i)1_S = \prod_{j=2}^{i} (1_S - 1_{gh_{i-1}})1_{gh_i}, \quad 2 \leq i \leq m.$

We recall from [3] that $S^\alpha := \{s \in S : \alpha_g(s1_{g^{-1}}) = s1_g, \text{ for all } g \in G\}$ is called the subalgebra of invariants of $S$ under $\alpha$. If $\alpha$ is global then $S^\alpha$ is the classical subalgebra of invariants, i.e. $S^\alpha = S^G = \{x \in S : \alpha_g(x) = x, \text{ for all } g \in G\}$.

For a subgroup $H$ of $G$, we shall denote by $\alpha_H$ the partial action of $H$ on $S$ obtained by restriction of $\alpha$, i.e. $\alpha_H = (S_h, \alpha_h)_{h \in H}$. Some properties of the map $\psi_H$ are given in the next.

**Proposition 2.1.** Let $H = \{h_1 = 1, h_2, \ldots, h_m\}$ be a subgroup of $G$ and $\psi_H : T \to T$ the map defined in (2). Then:

(i) $\psi_H$ is a $k$-algebra homomorphism,

(ii) $\psi_H(1_S) = 1_T$ if and only if $H = G$,

(iii) $\psi_H$ is left and right $T^H$-linear map,

(iv) the restriction $\psi_H|_S$ to $S$ is injective,

(v) $e_H := \psi_H(1_S)$ is a central idempotent of $T$,

(vi) $\psi_H(S^\alpha_H) \subset T^H$, 


(vii) the restriction of \( \psi_H \) to \( S^{\alpha_H} \) is a \( k \)-algebra isomorphism from \( S^{\alpha_H} \) onto \( T^H e_H \) whose inverse is the multiplication by \( 1_S \). In particular \( T^H 1_S = S^{\alpha_H} \).

**Proof.** Notice that (i) is immediate because \( \beta_g, g \in G \), is a \( k \)-algebra homomorphism and \( \{e_i : 1 \leq i \leq m\} \) is a set of orthogonal idempotents of \( T \). For (ii), observe that \( \beta_{h_i}(1_S)e_i = e_i \), for all \( 1 \leq i \leq m \). Hence, \( \psi_H(1_S) = e_1 + \ldots + e_m \). Since \( T = \sum_{g \in G} \beta_g(S) \) by (G4), it follows that \( \psi_G(1_S) = 1_T \). Conversely, if \( \psi_H(1_S) = 1_T \) then \( 1_T \) belongs to the ideal \( \sum_{i=1}^{m} \beta_{h_i}(S)e_i \) of \( T \). Hence \( \sum_{i=1}^{m} \beta_{h_i}(S)e_i = T = \sum_{g \in G} \beta_g(S) \), which implies \( H = G \).

The item (iii) is clear. For (iv), take \( s \in S \) and notice that

\[
\psi_H(s)1_S = \sum_{i=1}^{m} \beta_{h_i}(s)e_i 1_S = \beta_1(s)1_S = s
\]

because \( e_i 1_S = 0 \), for all \( 2 \leq i \leq m \). Thus, (iv) follows. Since \( \{e_i : 1 \leq i \leq m\} \) is a set of central orthogonal idempotents of \( T \) and \( \psi_H(1_S) = e_1 + \ldots + e_m \), the item (v) follows.

For (vi), we need to show that \( \beta_h(\psi_H(s)) = \psi_H(s) \), for all \( s \in S^{\alpha_H} \) and \( h \in H \). Thus, it is enough to check that \( \beta_h(\psi_H(s)) \) gives us a permutation in the elements that appear in the sum of \( \psi_H(s) \) given in (2). First, observe that

\[
\beta_{h_i}(1_S)\beta_{h_j}(s) = \beta_{h_j}\left(\beta_{h_j}^{-1}h_i(1_S)s\right) = \beta_{h_j}\left(\beta_{h_j}^{-1}h_i(1_S)s1_S\right)
\]

\[
= \beta_{h_j}(s)\beta_{h_j}(s) \quad (**) = \beta_{h_j}(1_S)\beta_{h_j}(s),
\]

where (**) follows because \( \beta_{h_j}(s)1_S \overset{(1)}{=} \alpha_h(s1_{h^{-1}}) = s1_h \) and (**) follows using that \( \beta_{h_j}(1_S) \) is a central element of \( T \), for all \( h \in H \). Now, consider \( 1 \leq l \leq n \) and \( i_1 < \cdots < i_l \) and note that

\[
\beta_h(\beta_{h_{i_1}}(1_S)\cdots\beta_{h_{i_l-1}}(1_S)\beta_{h_{i_l}}(s)) = \beta_{h_{i_1}}(1_S)\cdots\beta_{h_{i_l-1}}(1_S)\beta_{h_{i_l}}(s)
\]

\[
= \beta_{h_{i_1}}(1_S)\cdots\beta_{h_{i_l-1}}(1_S)\beta_{h_{i_l}}(s),
\]

where \( h_{ik} = hh_{ik} \) for all \( 1 \leq k \leq l \). Since \( \beta_{h_i}(1_S), h \in H \), are central elements of \( T \), we can rearrange \( \beta_{h_{i_{j}}} (1_S), \ldots, \beta_{h_{i_{j-1}}} (1_S) \) such that \( j_1 < \ldots < j_{i-1} \). If \( j_{i-1} < j_i \), there is nothing to do. Otherwise, as we showed above, \( \beta_{h_{i-1}}(1_S)\beta_{h_{i}}(s) = \beta_{h_{i}}(1_S)\beta_{h_{i-1}}(s) \). Thus, \( \beta_{h_{i}}(1_S) \cdots \beta_{h_{i-1}}(1_S)\beta_{h_{i}}(s) \) appears in the sum of \( \psi_H(s) \) and the result follows.

Finally to (vii), notice that by (v) and (vi) we have \( \psi_H(S^{\alpha_H}) = \psi_H(S^{\alpha_H}1_S) \subset T^H e_H \). Hence, by (1), we conclude that \( \psi_H : S^{\alpha_H} \to T^H e_H \) is a \( k \)-algebra homomorphism. Also, if \( x \in T^H \) then

\[
\alpha_h((x1_S)1_{h^{-1}}) = \beta_h((x1_S)1_{h^{-1}}) \overset{(1)}{=} \beta_h(x1_S\beta_{h^{-1}}(1_S))
\]

\[
= \beta_h(x1_S)1_S = \beta_h(x)\beta_h(1_S)1_S \overset{(1)}{=} x1_h
\]

\[
= (x1_S)1_h,
\]
Moreover, if $T$ is a Galois extension, then $m : T^H \to S^{\alpha_H}$ is a $k$-algebra homomorphism well-defined, as $e_H 1_s = 1_s$. It is clear that $m(\psi_H(s)) = s$ and $\psi_H(m(x e_H)) = xe_H$, for all $s \in S^{\alpha_H}$ and $x \in T^H$.

\[ \sum_{i=1}^{m} x_i \alpha_g(y_i 1_{g^{-1}}) = \delta_{1,g}, \text{ for each } g \in G. \]

The elements $x_i, y_i$ are called partial Galois coordinates of $S$ over $S^\alpha$.

If $S$ is a $\alpha$-partial Galois extension of $S^\alpha$, then [3, Theorem 5.2] states that for every subgroup $H$ of $G$ one has that $\alpha_H$-partial Galois extension of $S^{\alpha_H}$. Now we give an addendum of this result.

**Theorem 3.1.** Let $S$ be a ring, $G$ a finite group and $\alpha$ a partial action of $G$ on $S$ such that $S \supseteq S^\alpha$ is a partial Galois extension, if for some $m \in \mathbb{N}$ there exist elements $x_i, y_i \in S$, $1 \leq i \leq m$, such that

\[ \sum_{i=1}^{m} x_i \alpha_g(y_i 1_{g^{-1}}) = \delta_{1,g}, \text{ for each } g \in G. \]

From now on, we assume that $H$ is a normal subgroup of $G$.

The proof of Theorem 3.1 will be obtained as consequence of several results which we state and prove below. By [5, Theorem 2.3], the global action $\beta$ of $G$ on $T$ induces a partial action $\alpha_{G/H}$ of $G/H$ on $T^H$ in the following way:

\[ \beta_{G/H} : G/H \to \text{Aut}(T^H), \quad gH \mapsto \beta_g|_{T^H}. \]

Furthermore, if $T$ is a $G$-Galois extension of $T^G$, then $T^H$ is a $G/H$-Galois extension of $T^G = (T^H)^{G/H}$.

On the other hand by (v) of Proposition 2.1 $T^H e_H$ is an ideal of $T^H$, then the action $\beta_{G/H}$ of $G/H$ on $T^H$ induces a partial action $\gamma_{G/H}$ of $G/H$ on $T^H e_H$ in a canonical way, that is, $\lambda_{G/H} = (D_{gH}, \gamma_{gH})_{gH \in G/H}$ is given by

\begin{align*}
(5) & \quad D_{gH} = (T^H e_H) \cap \beta_g(T^H e_H) = T^H e_{gH}, \text{ where } e_{gH} = e_H \beta_g(e_H), \\
(6) & \quad \gamma_{gH} = \beta_g|_{D_{g^{-1}H}} : D_{g^{-1}H} \to D_{gH}, \text{ for each } gH \in G/H.
\end{align*}

**Proposition 3.2.** With the notations above we have that $(T^H, \beta_{G/H})$ is an enveloping action of $(T^H e_H, \alpha_{G/H})$.

**Proof.** By construction, (G1), (G2) and (G3) of § 2.1 are satisfied. In order to prove (G4) it is enough to check that $T^H = \sum_{i=1}^{l} \beta_{g_i}(T^H e_H)$, where $T = \{g_1 = 1, g_2, \ldots, g_l\}$ is a transversal of $H$ on $G$. Let $H = \{h_1 = 1, h_2, \ldots, h_m\}$, and write the elements of $G$ in the following order

\[ G = \{1, h_2, \ldots, h_m, g_2, g_2 h_2, \ldots, g_2 h_m \ldots g_1, g_1 h_2, \ldots, g_1 h_m\}. \]
Corollary 3.3. Let \( \beta_g(T^H) = T^H \) for all \( g \in G \), and
\[
I = \sum_{i=1}^{l} \beta_{g_i}(T^He_H) = \sum_{i=1}^{l} T^H \beta_{g_i}(e_H)
\]
is an ideal of \( T^H \), we shall check that \( 1_T \in I \). Write \( f_H = \prod_{i=1}^{m} (1_T - \beta_{h_i}(1_S)) \) then \( f_H \in T^H \) and for \( g \in G \) we conclude that \( \beta_g(f_H) \in T^H \). Now denote by \( e_{(i,j)} \in T \) the idempotent in (3) corresponding to the element \( g_ih_j \in G \), by [3, P. 79] we have that
\[
1_T = \sum_{i=1}^{m} e_{(1,i)} + \sum_{i=1}^{m} e_{(2,i)} + \cdots + \sum_{i=1}^{m} e_{(l,i)}.
\]
But \( \sum_{i=1}^{m} e_{(1,i)} = e_H \in I \), \( e_{(2,1)} = f_H \beta_{g_2}(1_S) = f_H \beta_{g_2}(e_{(1,1)}) \) and for \( 2 \leq i \leq m \)
\[
e_{(2,i)} = f_H(1_T - \beta_{g_2}(1_S))(1_T - \beta_{g_2h_2}(1_S)) \cdots (1_T - \beta_{g_2h_2 \cdots h_i}(1_S)) \beta_{g_i}(1_S)
\]
\[
= f_H \beta_{g_2}(1_T - (1_S))(1_T - \beta_{h_2}(1_S)) \cdots (1_T - \beta_{h_{i-1}}(1_S)) \beta_{h_i}(1_S)
\]
\[
= f_H^{\beta_{g_2}}(e_{(1,i)}),
\]
from this we get \( \sum_{i=1}^{m} e_{(2,i)} = f_H^{\beta_{g}}(e_H) \in I \), analogously \( e_{(3,i)} = f_H^{\beta_{g_2}}(f_H)\beta_{g_3}(e_{(1,i)}) \), for \( 1 \leq i \leq m \), and \( \sum_{i=1}^{m} e_{(3,i)} = f_H^{\beta_{g_2}}(f_H)\beta_{g_3}(e_H) \), using this argument and the construction of the idempotents \( e_{(i,j)} \) it is straightforward to see that
\[
\sum_{i=1}^{m} e_{(k,i)} = f_H^{\beta_{g_2}}(f_H) \cdots \beta_{g_{k-1}}(f_H)\beta_{g_k}(e_H) \in I, \quad 2 \leq k \leq l.
\]
Using equations (7) and (8) we get \( 1_T \in I \), as desired. \( \square \)

Corollary 3.3. Let \( \gamma_{G/H} \) be the partial action of \( G/H \) on \( T^He_H \) given by (5) and (6). Then:

(i) \( \psi_{G/H} : T^H \rightarrow T^H \) is a \( T^G \)-linear homomorphism of \( k \)-algebras whose restriction to \( T^He_H \) is injective and such that \( \psi_{G/H}(e_H) = 1_T = \psi_G(1_S) \).

(ii) \( \psi_{G/H} \left( (T^He_H)^{\gamma_{G/H}} \right) \subset (T^H)^{G/H} = T^G \).

(iii) The restriction of \( \psi_{G/H} \) to \( (T^He_H)^{\gamma_{G/H}} \) is a \( k \)-algebra isomorphism of \( (T^He_H)^{\alpha_{G/H}} \) onto \( T^G = (T^H)^{G/H} \) with inverse given by the multiplication by \( e_H \). Particularly, \( T^Ge_H = (T^He_H)^{\gamma_{G/H}} \).

(iv) \( (T^He_H)^{\gamma_{G/H}} \subset T^He_H \) is a partial Galois extension with Galois group \( G/H \) if and only if \( T^G = (T^H)^{G/H} \subset T^H \) is a Galois extension with Galois group \( G/H \).

Proof. Items (i), (ii) and (iii) follow directly from Proposition 2.1, while item (iv) follows from Lemma 3.2 and [3, Theorem 3.3]. \( \square \)
We shall see that the partial action \( \gamma_{G/H} \) on \( \mathcal{T}^H_{e_H} \) induces a partial action \( \alpha_{G/H} \) of \( G/H \) on \( S^{\alpha_H} \) via multiplication by \( 1_S \). Indeed, we define

\[
\begin{align*}
\tilde{1}_{gH} &:= e_{gH}1_S = e_H\beta_g(e_H)1_S = \beta_g(e_H)1_S, \\
\tilde{D}_{gH} &:= D_{gH}1_S = \mathcal{T}^H_{e_{gH}}1_S = S^{\alpha_H}\tilde{1}_g, \\
\alpha_{gH} &:= (m_{1_S} \circ \gamma_{gH} \circ \psi_H)\tilde{D}_{g^{-1}H},
\end{align*}
\]

where \( m_{1_S} : \mathcal{T}^H_{e_H} \to S^{\alpha_H} \) is the multiplication map by \( 1_S \).

**Proposition 3.4.** Let \( H \) be a normal subgroup of \( G \), \( \tilde{D}_{gH} \) and \( \alpha_{gH} \) given respectively by (10) and (11). Then

(i) \( \alpha_{G/H} = (\tilde{D}_{gH}, \alpha_{gH})_{gH \in G/H} \) is a partial action of \( G/H \) on \( S^{\alpha_H} \),

(ii) \( S^{\alpha_H} \alpha_{G/H} = S^{\alpha} \),

(iii) \( (\mathcal{T}^H, \beta_{G/H}) \) is the globalization of \( \alpha_{G/H} \),

(iv) \( S^\alpha \subset S^{\alpha_H} \) is a partial Galois extension with Galois group \( G/H \) if and only if \( T^G \subset \mathcal{T}^H \) is a Galois extension with Galois group \( G/H \).

**Proof.** (i) By (9) we know that \( \tilde{1}_{gH} \) is a central idempotent of \( T \). Now we check that \( \tilde{1}_{gH} \in S^{\alpha_H} \). Let \( h \in H \), then taking into account (vi) of Proposition 2.1 and the fact that \( e_H = \psi_H(1_S) \) we have

\[
\alpha_h(\tilde{1}_{gH}1_{h^{-1}}) = \alpha_h(\beta_g(e_H)1_{h^{-1}}) = \beta_{g_h}(e_H)1_h = \beta_{g_h}(e_H)1_h = \beta_g(e_H)1_S1_h = \tilde{1}_{gH}1_h,
\]

then \( \tilde{D}_{gH} \) is an ideal of \( S^{\alpha_H} \). Furthermore, Now by (i) and (iii) of Proposition 2.1 we get

\[
\psi_H(S^{\alpha_H}\tilde{1}_g) = \mathcal{T}^H_{e_{g^{-1}H}},
\]

which is the domain of \( \gamma_{gH} \), then

\[
\alpha_{gH}(\tilde{D}_{g^{-1}H}) \overset{(12)}{=} m_{1_S} \circ \gamma_{gH}(\mathcal{T}^H_{e_{g^{-1}H}}) = \mathcal{T}^H_{e_{gH}}1_S = S^{\alpha_H}\beta_g(e_H)1_S \equiv \tilde{D}_{gH}.
\]

Thus \( \alpha_{gH} : \tilde{D}_{g^{-1}H} \to \tilde{D}_{gH} \) is well defined and is a ring isomorphism as it is composition of isomorphisms and (P1) of §2.1. is satisfied, now condition (P2) is clear. To check (P3) take \( gH, lH \in G/H \), then

\[
\alpha_{gH}(\tilde{D}_{g^{-1}H} \cap \tilde{D}_{lH}) = (m_{1_S} \circ \gamma_{gH} \circ \psi_H)(S^{\alpha_H}\tilde{1}_{g^{-1}H}1_{lH}) \overset{(12)}{=} (m_{1_S} \circ \gamma_{gH})(\mathcal{T}^H_{e^{-1}H}1_{lH}) = \mathcal{T}^H_{e_{gH}}1_S\mathcal{T}^H_{e_{lH}}1_S = \tilde{D}_{gH} \cap \tilde{D}_{lH}.
\]
Finally, to check (P4) take \( x \in \tilde{D}_{t^{-1}H} \cap \tilde{D}_{(gl)^{-1}H} \). Then,
\[
\alpha_{gH} \circ \alpha_{lH}(x) = (m_1s \circ \gamma_{gH} \circ \psi_H) \circ (m_1s \circ \gamma_{lH} \circ \psi_H)(x) \\
= (m_1s \circ (\gamma_{gH} \circ \gamma_{lH}) \circ \psi_H)(x) \\
\overset{(12)}{=} (m_1s \circ \alpha_{glH} \circ \psi_H)(x) \\
= \alpha_{gLH}(x).
\]

(ii) First of all observe that
\[
(\psi_H \circ \alpha_{gH} = \gamma_{gH} \circ \psi_H \quad \text{on} \quad \tilde{D}_{g_{H^{-1}}} \quad \text{for all} \quad g \in H.)
\]
Moreover since \( \tilde{I}_{g^{-1}H} \subseteq S^\alpha_H \), then \( \psi_H(\tilde{I}_{g^{-1}H}) = \beta_{g^{-1}}(e_H) = \beta_{g^{-1}}(e_H) e_H = e_{g^{-1}H} \).
This implies \( \psi_H((S^\alpha_H)^{\alpha_G/H}) = \psi_H(S^\alpha_H)^{\gamma_{G/H}} \).
\[\text{Indeed, let} \ s \in S^\alpha_H \ \text{such that} \ \psi_H(s) \in \psi_H((S^\alpha_H)^{\gamma_{G/H}}) \ \text{Then,}
\]
\[
\alpha_{gH}(s \tilde{I}_{g^{-1}H}) = \psi_H(\alpha_{gH}(s \tilde{I}_{g^{-1}H}))1_s = \gamma_{gH} \circ \psi_H(s \tilde{I}_{g^{-1}H})1_s \\
\overset{(12)}{=} \gamma_{gH}(\psi_H(s) e_{g^{-1}H})1_s = \psi_H(s) e_{gH}1_s = s \tilde{I}_{gH},
\]
and \( s \in S^\alpha_G/H \), thus \( \psi_H(S^\alpha_H)^{\gamma_{G/H}} \subseteq \psi_H((S^\alpha_H)^{\alpha_G/H}) \), in an analogous way one shows the other inclusion. Then
\[
(S^\alpha_H)^{\alpha_G/H} \overset{\text{(vi)Prop.2.1}}{=} \psi_H((S^\alpha_H)^{\alpha_G/H})1_s = \psi_H(S^\alpha_H)^{\gamma_{G/H}}1_s \\
= (T^H e_H)^{\gamma_{G/H}}1_s = (T^G e_H)1_s = T^G 1_s = S^\alpha_H,
\]
as desired.

(iii) By (vii) of Proposition 2.1, there is a ring monomorphism \( \psi_H : S^\alpha_H \to T^H \), such that \( \psi_H(S^\alpha_H) = T^H e_H \) is an ideal of \( T^H \).
Moreover

(i) (G2) The equality \( \psi_H(\tilde{D}_{gH}) = \psi_H(S^\alpha_H) \cap \beta_{g}(\psi_H(S^\alpha_H)) \), follows from (12).

(ii) (G3) It follows by (6) and (13) that \( \psi_H \circ \alpha_{gH} = \beta_{gH} \circ \psi_H \) on \( \tilde{D}_{gH} \).

(iii) (G4) Finally using Proposition (3.2) that
\[
T^H = \sum_{gH \in G/H \cap T} \beta_{gH}(T^H e_H) = \sum_{gH \in G} \beta_{gH}(\psi_H(S^\alpha_H)),
\]
and the result follows using \[2, \text{Definition 4.2}\].

(iv) This follows from (iii) and \[3, \text{Theorem 3.3}\]. \( \square \)

\textbf{Proof of Theorem 3.1.} This is a consequence of iv) of Proposition 3.4 and item (ii) of [5, Theorem 2.3].

4. ENVELOPING ACTIONS AND PARTIAL G-ISOMORPHISMS

In this section we will fix a commutative ring \( R \) and work with partial Galois extensions of \( R \) with Galois group \( G \).

The trace map plays an important role when having partial actions of finite groups on algebras, and for a partial action \((S, \alpha)\) is defined as follows \( tr_{S/R}(s) = \sum_{g \in G} \alpha_g(s1_{g^{-1}}) \),
for all \( s \in S \), by [3, Lemma 2.1] \( tr_\alpha : S \to R \) is a \( R \)-linear map.
Definition 4.1. We say that two partial Galois extensions \((S, \alpha)\) and \((S', \alpha')\) of \(R\) are called partially \(G\)-isomorphic, and denoted \((S, \alpha) \simpar (S', \alpha')\), if there is a \(R\)-algebra isomorphism \(f : S \to S'\) such that for all \(g \in G\):

(i) \(f(S_g) \subseteq S'_g\),

(ii) \((f \circ \alpha_g)|_{S_{g-1}} = (\alpha'_g \circ f)|_{S_{g-1}}\).

The relation \(\simpar\) defined above is an equivalence relation. We denote by \([S, \alpha]\) the equivalence class of \((S, \alpha)\) relative to \(\simpar\). Next result implies that we may only require that the map \(f\) to be a \(R\)-algebra homomorphism.

Proposition 4.2. Let \((S, \alpha)\) and \((S', \alpha')\) be partial Galois extension of \(R\). Then \((S, \alpha)\) and \((S', \alpha')\) are partially \(G\)-isomorphic, if and only if, there is an \(R\)-algebra homomorphism \(f : S \to S'\) satisfying (i) and (ii) above.

Proof. Note that \(1_S = 1_R = 1_{S'}\). Let \(x_i, y_i, 1 \leq i \leq n\) be partial Galois coordinates of \(S\) over \(R\) relative to \(\alpha\). Take \(s' \in S'\) then:

\[
f \left( \sum_{i=1}^n x_i \text{tr}_{\alpha'}(f(y_i1_{g-1})s') \right) = \sum_{i=1}^n f(x_i) \text{tr}_{\alpha'}(f(y_i1_{g-1})s')
\]

\[
= \sum_{i=1}^n f(x_i) \sum_{g \in G} \alpha'_g(f(y_i1_{g-1}))\alpha'_g(s'1'_{g-1})
\]

\[
= \sum_{i=1}^n f(x_i) \sum_{g \in G} (\alpha'_g \circ f)(y_i1_{g-1})\alpha'_g(s'1'_{g-1})
\]

\[
= \sum_{i=1}^n f(x_i) \sum_{g \in G} (f \circ \alpha_g)(y_i1_{g-1})\alpha'_g(s'1'_{g-1})
\]

\[
= \sum_{g \in G} f \left( \sum_{i=1}^n x_i \alpha_g(y_i1_{g-1}) \right) \alpha'_g(s'1'_{g-1})
\]

\[
= \sum_{g \in G} \delta_{1,g}1_S \alpha'_g(s'1'_{g-1}) = s'1_s = s',
\]

and \(f\) is surjective.

To prove that \(f\) is injective take \(s \in S\) be such that \(f(s) = 0\), we have

\[
f(\alpha_g(y_i1_{g-1})) = (f \circ \alpha_g)(y_i1_{g-1}) = (\alpha'_g \circ f)(y_i1_{g-1})
\]

\[
= \alpha'_g(f(y_i)f(s)f(1_{g-1})) = 0,
\]

for all \(g \in G\) and \(\text{tr}_\alpha(y_i) = f(\text{tr}_\alpha(y_i)1_S) = 0\), for all \(1 \leq i \leq n\). Then,

\[
0 = \sum_{i=1}^n x_i \text{tr}_\alpha(y_i) = \sum_{g \in G} \left( \sum_{i=1}^n x_i \alpha_g(y_i1_{g-1}) \right) \alpha_g(s1_{g-1})
\]

\[
= \sum_{g \in G} \delta_{1,g} \alpha_g(s1_{g-1}) = s1_s = s,
\]
and \( f \) is injective.

Given two (global) actions \((T^1, \beta^1)\) and \((T^2, \beta^2)\), with \((T^1)^G = A = (T^2)^G\) we recall from [7] that \((T^1, \beta^1)\) and \((T^2, \beta^2)\) are \(G\)-isomorphic if there is an \(A\)-algebra homomorphism \(f: T^1 \to T^2\) such that \(f \circ \beta = \beta' \circ f\).

For our purposes, the concept of (global) \(G\)-isomorphism for enveloping actions of partial action needs one more condition than the classical concept of \(G\)-isomorphisms defined in [7].

**Definition 4.3.** Let \((T, \beta)\) and \((T', \beta')\) be enveloping actions of \((S, \alpha)\) and \((S', \alpha')\) respectively, suppose that \((T^G) = A = (T'^G)\). We say that \((T, \beta)\) and \((T', \beta')\) are globally \(G\)-isomorphic, and we denote \((T, \beta) \cong (T', \beta')\), if they are \(G\)-isomorphic and the map \(f: T \to T'\) giving the \(G\)-isomorphism satisfies \(f(1_S) = 1_{S'}\).

**Remark 4.4.** The relation \(\cong\) defined above is an equivalence relation, and the equivalence class of \((T, \beta)\) is denoted by \([T, \beta]\).

Now we give the main result of this section.

**Theorem 4.5.** Assume that \((S, \alpha)\), \((S', \alpha')\) are partial Galois extension of \(R\) and, \((T, \beta)\) and \((T', \beta')\) be their enveloping actions, respectively. Let \(H\) be a normal subgroup of \(G\), if \((T, \beta)\) and \((T', \beta')\) are globally \(G\)-isomorphic, then \((S^G, \alpha_{G/H})\) and \((S'^G, \alpha'_{G/H})\) are partially \(G/H\)-isomorphic. In particular, \((S, \alpha)\) and \((S', \alpha')\) are partially \(G\)-isomorphic.

**Proof.** Let \(H = \{h_1 = 1, h_2, \ldots, h_m\}\) and \(T = \{g_1 = 1, g_2, \ldots, g_l\}\) be a transversal of \(H\) in \(G\). Consider an \(A\)-algebra isomorphism \(f: T \to T'\) such that

\[
\begin{align*}
&f(1_S) = 1_S, \\
f \circ \beta_g = \beta'_g \circ f, \quad \text{for all} \ g \in G.
\end{align*}
\]

It is clear that \(f\) induces (by restriction to \(B^H\)) an isomorphism between the \(A\)-algebras \(T^H\) and \(T'^H\). By Proposition 3.2 \((T'^H, \beta'_{G/H})\) is an enveloping action of \((T^H e_H, \alpha_{G/H})\), where \(e_H = \psi_H(1_S) = \sum_{j=1}^m \beta_{h_j}(1_S) e_j\), and the family \(\{e_j\}_{1 \leq j \leq m}\) is given by (3).

In the same way \((T'^H, \beta'_{G/H})\) is an enveloping action of \((T^H e'_H, \theta_{G/H})\), where \(e'_H = \sum_{j=1}^m \beta'_{h_j}(1_S) e'_j\), and

\[
e'_1 = 1_S \quad \text{and} \quad e'_j = (1_{B'} - 1_S) \cdots (1_{B'} - \beta'_{h_{j-1}}(1_S)) \beta'_{h_j}(1_S), \quad 1 \leq j \leq n.
\]

By (14) we have that \(f(e_j) = e'_j\), for all \(1 \leq j \leq n\), and consequently \(f(T^H e_H) = T'^H e'_H\), in particular, \(f(e_H) = e'_H\).

On the other hand, it follows from (10) that \(\bar{D}_{gH} = T^H e_H \beta_g(e_H) 1_S\) and \(\bar{D}'_{gH} = T'^H e'_H \beta'_g(e'_H) 1_{S'}\), and the partial isomorphisms are

\[
\begin{align*}
\alpha_{gH} &= (m_{1_S} \circ \gamma_{gH} \circ \psi_H) |_{\bar{D}_{g-1_H}} \quad \text{and} \quad \alpha'_{gH} = (m_{1_S} \circ \gamma'_{gH} \circ \psi'_H) |_{\bar{D}'_{g-1_H}},
\end{align*}
\]

respectively, for all \(g \in G\). Now the map \(\varphi = m_{1_S} \circ f \circ \psi_H : S \to S'\) is a ring homomorphism, we shall check that it is \(R\)-linear. By item (ii) of Proposition 2.1 we have
A1_S = R. Then, for \( r = a1_S \in R \) and \( s \in S \) we have that
\[
\varphi(rs) = (m_1S \circ f \circ \psi_H)(a1_Ss) = (m_1S \circ f)(a\psi_H(1_S)\psi_H(s))
\]
\[
= (m_1S \circ f)(ae_H\psi_H(s)) = 1_S(af(1_B)f(e_H)f(\psi_H(s)))
\]
\[
= 1_S(a1_B'e_H'f(\psi_H(s))) = (a1_S)(1se_H')f(\psi_H(a)) = r(1sf(\psi_H(s)))
\]
\[
= r(m_1S \circ f \circ \psi_H)(s) = r\varphi(s).
\]

Now, we check that \( \varphi \) satisfies conditions (i) and (ii) of Definition 4.1. Let \( g \in G \).

Thus condition (i) is satisfied. Now we check (ii). Let \( x \in \tilde{D}_{g^{-1}} \). Then
\[
(\varphi \circ \alpha_{gH})(x) = (m_1S \circ (f \circ \gamma_{gH} \circ \psi_H))(x)
\]
\[
= [(m_1S \circ (\gamma_{gH} \circ f) \circ \psi_H)](x)
\]
\[
= [(m_1S \circ (\gamma_{gH} \circ \alpha'_{gH} \circ \psi_H))(x)
\]
\[
= [(m_1S \circ \alpha'_{gH} \circ \psi_H) \circ (m_1S \circ f \circ \psi_H)](x)
\]
\[
= (a'_{gH} \circ \varphi)(x).
\]

Finally, by Proposition 4.2 \( (S^{\alpha_H}, \alpha_{G/H}) \) and \( (S^{\alpha'_H}, \alpha'_{G/H}) \) are partially \( G/H \)-isomorphic. For the last affirmation, it is enough to take \( H = \{1\} \).

\[ \square \]

5. The Construction of \( H_{\text{par}}(G, R) \)

From now on in this work, \( G \) will denote a finite abelian group.

Let \( H_{\text{par}}(G, R) \) the set of equivalence classes of partial abelian extensions of \( R \) with group \( G \), that is partial Galois actions \( (S, \alpha) \) of \( R \) with group \( G \). In this section we construct a product in \( H_{\text{par}}(G, R) \) which turns it into a commutative inverse semigroup. First we recall the classical construction of the Harrison group (see [7]).
5.1. The Harrison group \( \mathcal{H}(G, A) \). Let \( A \) be a commutative ring with unit. In [7] the author introduced and studied the abelian group \( \mathcal{H}(G, A) \) whose elements are equivalence classes of (global) \( G \)-equivalent Galois extensions of \( A \). We denote the equivalence class of \((T, \beta)\) by \( \text{cl}(T, \beta) \) the Multiplication in \( \mathcal{H}(G, A) \) is defined in the following way:

Let \( \text{cl}(B, \beta), \text{cl}(B', \beta') \in \mathcal{H}(G, A) \). It is well known that the tensor product \( B \otimes_A B' \) is an abelian extension of \( A \) with Galois group \( G \times G \). By [5, Theorem 2.2] we have that \( (T \otimes T')^{\delta G} \supseteq A \) is a Galois extension with Galois group \( G = G \times G/\delta G \), where \( \delta G = \{(g, g^{-1}) \mid g \in G\} \). The group \( G \) acts on \( (T \otimes T')^{\delta G} \) via

\[
\Lambda_{(G \times G)/\delta G} : \left( g, \sum_i t_i \otimes t'_i \right) \mapsto \sum_i \beta_g(t_i) \otimes t'_i = \sum_i b_i \otimes \beta'_g(t'_i),
\]

for all \( t_i \in B, t'_i \in B' \) and \( g \in G \). We set

\( \text{cl}(B, \beta)\text{cl}(B', \beta') = \text{cl}\left( (B \otimes B')^{\delta G}, \Lambda_{(G \times G)/\delta G} \right) \).

The identity element of \( \text{cl}(E, \rho) \) containing the ring \( E_G(A) \) constructed in the following way. We choose symbols \( \{e_g\}_{g \in G} \) and we consider the free \( A \)-module \( E_G(A) = \bigoplus_{g \in G} Ae_g \), with basis \( \{e_g\}_{g \in G} \). For the basis elements we define the product \( e_ge_h = \delta_{g,h}e_g \). Then \( E_G(A) \) is an \( A \)-algebra. The action of \( \rho \) of \( G \) on \( A \) is given by the \( A \)-linear extension of \( \rho_g \cdot e_h = e_{gh} \), for all \( g, h \in G \).

Now we indicate the construction of the inverse element in \( \mathcal{H}(G, A) \). Let \( \text{cl}(T, \beta) \in \mathcal{H}(G, A) \), the element \( \text{cl}(T, \beta)^{-1} \) is represented by \( T \) with action of \( G \) defined by \( \beta'_g(t) = \beta_{g^{-1}}(t) \), for all \( g \in G \) and \( t \in T \).

It is shown in [7] that the study of the group \( \mathcal{H}(G, A) \) reduces to the case of cyclic Galois groups.

5.2. The group \( \mathcal{H}_{gl}(G, A) \). From Definition 4.3 and Remark 4.4 we can to consider the group \( \mathcal{H}_{gl}(G, A) \) consists of all equivalence classes \([T, \beta] \) of the relation \( \equiv^g \) defined on the set of enveloping actions, with fixed part \( A \), of partial Galois extensions \( S \supseteq R \). We define a product \(*_{gl} \) in \( \mathcal{H}_{gl}(G, A) \) by the formula (15), that is

\[
[T, \beta] *_{gl} [T', \beta'] = [(T \otimes T')^{\delta G}, \Lambda_{(G \times G)/\delta G}],
\]

for all \([T, \beta], [T', \beta'] \in \mathcal{H}_{gl}(G, A)\). Then we have the following.

**Proposition 5.1.** The set \( \mathcal{H}_{gl}(G, A) \) is a group, and the map \( \omega : \mathcal{H}_{gl}(G, A) \ni [T, \beta]_{gl} \mapsto \text{cl}(T, \beta) \in \mathcal{H}(G, A) \) is a group homomorphism.

**Proof.** To prove that \([E_G(A), \rho] \in \mathcal{H}_{gl}(G, A) \) we notice that \((E_G(A), \rho)\) is an enveloping action of \((E_G(R), m_{1s} \circ \rho)\). Indeed, by (vii) of Proposition 2.1 the map

\[
\Psi_G : E_G(R) \ni \sum_{g \in G} xe_g \mapsto \sum_{g \in G} \psi_G(x)e_g \in E_G(A),
\]

is a \( k \)-algebra isomorphism. Now for \( g, h \in G \) and \( xe_h \in E_G(R) \) we have

\[
\rho_g \circ \Psi_G(xe_h) = \rho_g(\psi_G(x)e_h) = \psi_G(x)e_{gh} = \Psi_G \circ (m_{1s} \circ \rho_g)(xe_h).
\]
Moreover, it is clear that that \( \sum_{g \in G} \rho_g(\Psi_G(E_G(R))) = E_G(A) \), and thus \([E_G(A), \rho]\) is the identity element of \( \mathcal{H}_{gl}(G, A) \).

To check that \( \mathcal{H}_{gl}(G, A) \) is closed under products, take \([T, \beta], [T', \beta'] \in \mathcal{H}_{gl}(G, A)\), where \((T, \beta)\) and \((T', \beta')\) are the enveloping actions of the partial Galois extensions \((S, \alpha)\) and \((S', \alpha')\) of \(R\). By Proposition 3.4, \((S \otimes S')^{\alpha \circ \alpha'}_{G \times \delta G} \) is a \((\alpha \circ \alpha')_{(G \times G)/\delta G}\) partial abelian extension of \(R\). Since \((g, h) = (g, 1) \text{ (mod } \delta G\) \) for all \((g, h) \in G \times G\), one has by Proposition 3.4 that \(((B \otimes B')^{\delta G}, \beta \circ \beta')_{(G \times G)/\delta G}\) is an enveloping action of \(((S \otimes S')^{\alpha \circ \alpha'}_{G \times \delta G}, \alpha \circ \alpha')_{(G \times G)/\delta G}\), where

\[
D_{(g, 1)\delta G} = (T \otimes T')^{\delta G}(\beta_g \otimes \beta'_g |_{\delta G})1_S \otimes 1_S,
\]

\[
(\alpha \circ \alpha')_{G \times \delta G} = (D_{(g, 1)\delta G}, (\gamma \otimes \gamma')_{(g, 1)\delta G})_{g \in G},
\]

\[
\alpha \circ \alpha'_{(g, 1)} = (m_1 S \otimes 1_S \circ \alpha \circ \alpha'_{(g, 1)\delta G} \circ \psi_{|\delta G}) \circ D_{(g^{-1}, 1)\delta G},
\]

then \([T \otimes T']^{\delta G}, (\beta \circ \beta')_{(G \times G)/\delta G} \) \(\in \mathcal{H}_{gl}(G, A)\) and \(\mathcal{H}_{gl}(G, A)\) is closed under products.

To check that \( \mathcal{H}_{par}(G, A) \) is closed under inverse elements, consider \(\alpha^\ast\) the partial action of \(G\) on \(S^\ast = S\), with ideals \(S^\ast_g = S_{g^{-1}}\) and partial isomorphisms \(\alpha^\ast_g: S^\ast_{g^{-1}} \to S^\ast_g\) given by \(\alpha^\ast_g = \alpha_{g^{-1}}\) for all \(g \in G\). We denote by \(\beta^\ast\) the global action of \(G\) on \(T^\ast = T\) given by \(\beta^\ast_g = \beta_g^{-1}\) for all \(g \in G\). Note that \((S^\ast, \alpha^\ast)\) is a partial abelian extension of \(R\), and \((B^\ast, \beta^\ast)\) is an enveloping action for \((S^\ast, \alpha^\ast)\), with \((T^\ast)^G = A\). Then \([T^\ast, \beta^\ast] = [T, \beta]^{-1}\) in \(\mathcal{H}_{gl}(G, A)\). Finally it is clear that \(w\) is a group homomorphism. \(\square\)

5.3. The inverse semigroup \( \mathcal{H}_{par}(G, R) \). Consider \([S, \alpha], [S', \alpha'] \in \mathcal{H}_{par}(G, R)\), then by \([4, \text{Proposition 2.9}]\) we have that \(S \otimes S'\) is a \(\alpha \circ \alpha'\)-partial abelian extension of \(R \otimes R = R\). We define on \( \mathcal{H}_{par}(G, R) \) the operation \(*_{par}\) by

\[
[(S, \alpha)] *_{par} [(S', \alpha')] = [(S \otimes S')^{\alpha \circ \alpha'}_{G \times \delta G}, (\alpha \circ \alpha')_{(G \times G)/\delta G}],
\]

where \((\alpha \circ \alpha')_{(G \times G)/\delta G}\) is given by \((9), (10)\) and \((11)\).

Before proving that the product \((16)\) is well defined we present a description of the idempotents \(\hat{I}_{gH} = \beta_g(e_H)1_S, g \in G\) given in \((9)\) and the maps \(\alpha_{gH}\) in \((11)\). Let \(H = \{h_1, h_2, \ldots, h_m\}\). Then

\[
\hat{I}_{gH} = \sum_{i=1}^{m} \beta_{gh_i}(1_S)\beta_g(e_i)1_S = \sum_{i=1}^{m} \beta_{gh_i}(1_S)\beta_g(e_i)1_S = \sum_{i=1}^{m} 1_{gh_i}\beta_g(e_i)1_S \equiv 1_g + \sum_{i=2}^{m} 1_{gh_i} \prod_{j=2}^{i}(1_S - 1_{gh_{j-1}})1_{gh_i} = 1_g + \sum_{i=2}^{m} \prod_{j=2}^{i}(1_S - 1_{gh_{j-1}})1_{gh_i}.
\]

Then

\[
\hat{I}_{gH} = 1_g + \sum_{i=2}^{m} \prod_{j=2}^{i}(1_S - 1_{gh_{j-1}})1_{gh_i}.
\]
Now by (11) we have for \( g \in G \) and \( x \in \tilde{D}_{g^{-1}H} \) that

\[
\alpha_{gH}(x) = (m_1 \circ \gamma_{gH} \circ \psi_H)(x) = \sum_{i=1}^{m} \beta_{gh_i}(x)\beta_g(e_i)1_S
\]

\[
= \sum_{i=1}^{m} \alpha_{gh_i}(x1_{(gh_i)^{-1}})\beta_g(e_i)1_S
\]

\[
= \alpha_g(x1_{g^{-1}}) + \sum_{i=2}^{m} \alpha_{gh_i}(x1_{(gh_i)^{-1}}) \prod_{j=2}^{i-1}(1_S - 1_{gh_j^{-1}}).
\]

That is

\[
(18) \quad \alpha_{gH}(x) = \alpha_g(x1_{g^{-1}}) + \sum_{i=2}^{m} \prod_{j=1}^{i-1}(1_S - 1_{gh_j}) \alpha_{gh_i}(x1_{(gh_i)^{-1}}).
\]

**Lemma 5.2.** Let \((S, \alpha)\) and \((S', \alpha')\) partial Galois extension of \(R\) with the same group \(G\). If \(H\) is a subgrou of \(G\) then \(S^H, \alpha_{G/H}\) and \(S'^H, \alpha'_{G/H}\) are \(G/H\)-equivalent.

**Proof.** Let \(f: S \to S'\) be the \(G\)-isomorphism. Then it is clear that \(f\) restricts to a \(R\)-algebra isomorphism \(f|_S: S^H \to S'^H\) and the equality \(f(\tilde{1}_gH) = \tilde{1}'_{gH}\) follows by (17), then \(f(\tilde{D}_{gH}) = \tilde{D}'_{gH}\) thanks to (10). Finally using the fact that \(f\) is a \(G\)-isomorphism and (18) we get that \(f|_S\) is a \(G/H\)-isomorphism which implies the result. \(\square\)

**Proposition 5.3.** The set \(H_{par}(G, R)\) with product \(*_{par}\) is a commutative semigroup.

**Proof.** First of all by Lemma 5.2 the product \(*_{par}\) is well defined and the fact that \(H_{par}(G, R)\) is closed under product follows from Theorem 3.1. Finally it is clear that with this product \(H_{par}(G, R)\) is commutative and associative. \(\square\)

We recall that an inverse semigroup \(\mathcal{I}\) is a semigroup in which the following conditions hold.

- \(\mathcal{I}\) is regular, that is, given \(x \in \mathcal{I}\) there is an element \(x^* \in \mathcal{I}\) such that \(xx^*x = x\) and \(x^*xx^* = x^*\).
- The idempotents of \(\mathcal{I}\) commute.

It is well known that the two conditions above are equivalent to the fact that, for any \(x \in \mathcal{I}\) there exists a unique \(x^* \in \mathcal{I}\) such that \(xx^*x = x\) and \(x^*xx^* = x^*\). The element \(x^*\) is called the inverse of \(x\). One can check that

\[
E(\mathcal{I}) = \{xx^* \mid x \in \mathcal{I}\}
\]

is the set of idempotents of \(\mathcal{I}\), and \(E(\mathcal{I})\) is a meet semilattice with respect to the partial ordering \(e \leq f\), if \(ef = e\). (For more details on inverse semigroups the interested reader may consult [9]).

We have the following.

**Theorem 5.4.** \(H_{par}(G, R)\) is an abelian inverse semigroup, which contains \(H(G, R)\).
Proof. By Proposition 5.3 is a $\mathcal{H}_{par}(G, R)$ is a commutative semigroup. Thus we only need to show that $\mathcal{H}_{par}(G, R)$ is regular. Let $(S, \alpha)$ be a partial abelian extension of $R$ with enveloping action $(B, \beta)$. We denote by $\alpha^*$ the partial action of $G$ on $S^* = S$ defined in the proof of Proposition 5.1 and $(B^*, \beta^*)$ an enveloping action $(S^*, \alpha^*)$. We shall check that $[S^*, \alpha^*] = [S, \alpha]^*$. Note that $[B^*, \beta^*] = [B, \beta]^{-1} \in \mathcal{H}_{par}(G, A)$, thus $\mathcal{H}_{par}(G, A)$ is a par of $\mathcal{H}_{par}(G, A)$.

We give the following.

Proposition 5.5. Let $\alpha = \{\alpha_g: S_{g-1} \to S_g\}_{g \in G}$ be a unital partial action of $G$ on $S$ such that $S/R$ is a partial Galois extension. Then

(i) The family $\hat{\alpha} =$ $\left\{ \hat{\alpha}_{(l,t)}: \left( \prod_{g \in G} S_g \right)_{(l,t)-1} \to \left( \prod_{g \in G} S_g \right)_{(l,t)} \right\}_{(l,t) \in G \times G}$, where

$$
\left( \prod_{g \in G} S_g \right)_{(l,t)} = \prod_{g \in G} S_{gS_lS_{l-1}g} = \prod_{g \in G} S_g \theta_{l-1}^{-1}g,
$$

and

$$
\hat{\alpha}_{(l,t)}(d_g1_{l-1}g)_{g \in G} = (\alpha_l(d_g1_{l-1}g)_{g \in G},
$$

with $d_g \in S_g$ for all $g \in G$, is a unital partial action of $G \times G$ on $S$. With $d_g \in S_g$ for all $g \in G$, is a unital partial action of $G \times G$ on $S$.

(ii) The partial action $\alpha \otimes \alpha^* = \{(\alpha \otimes \alpha^*)(_{(l,t)}: S_{l-1} \otimes S_l \to S_{l} \otimes S_{l-1} \left\}_{(l,t) \in G \times G}$, where $\alpha \otimes \alpha^*_{(l,t)} = \alpha_l \otimes \alpha_{l-1}$, for all $(l, t) \in G \times G$ is partially $G \times G$-isomorphic to $\hat{\alpha}$.

Note that $\hat{\alpha}_{(l,t)}(\cdot \otimes \cdot)_{(l,t)} = \alpha_l \otimes \alpha_{l-1}$, for all $(l, t) \in G \times G$ is partially $G \times G$-isomorphic to $\hat{\alpha}$.
(iii) Let \( E(S, \alpha) = \left( \prod_{g \in G} S_g \right)^{\delta G} \). Then
\[
E(S, \alpha) = \left\{ (d_g)_{g \in G} \in \prod_{g \in G} S_g : \alpha_t(d_g 1_{t-1}) 1_g = d_g 1_l 1_g, \forall l \in G \right\},
\]
and \([E(\alpha), \hat{\alpha} \times G/\delta G]\) is an idempotent in \( H_{par}(G, R) \), where
\[
\hat{\alpha}(1_{(l,t)\delta G})(d_g)_{g \in G} = (d_g 1_l g)_{g \in G} + \sum_{i=2}^{m} \prod_{j=1}^{i-1} (1_g - 1_{h_i} 1_{h_j} g)_{g \in G}(d_g 1_{h_i} 1_{h_j} 1_l g)_{g \in G}.
\]

Proof. 1) First of all notice that for all \((l, t) \in G \times G\) we have
\[
\alpha'_{(l,t)}[(d_g 1_{l-1} 1_{gt})_{g \in G}] = (\alpha_t(d_g 1_{l-1}) 1_g 1_{ltg})_{g \in G} = (\alpha_t(d_g 1_{l-1}) 1_l 1_{l^{-1}} \lambda)_{\lambda \in G},
\]
and \(\hat{\alpha}_{(l,t)}\) is a well defined isomorphism whose inverse is \(\hat{\alpha}_{(l^{-1}, t^{-1})}\). Moreover it is not difficult to check conditions (P1)-(P4) in § 2.1.

2) Let \(\varphi : S \otimes S \rightarrow \prod_{g \in G} S_g\) defined by \(\varphi(x \otimes y) = (x \alpha_g(y 1_{g^{-1}}))_{g \in G}\), then by [3, Theorem 4.1, iv] \(\varphi\) is a \(R\)-algebra isomorphism. We check that \(\varphi\) satisfies conditions (i)-(ii) of Definition 4.1.

(i) For \((l, t) \in G \times G\), we have
\[
\varphi[S_l \otimes S_{t^{-1}}] = \prod_{g \in G} S_l \alpha_g(S_{t^{-1}} 1_{g^{-1}}) = \prod_{g \in G} S_g S_l S_{gt^{-1}} = \left( \prod_{g \in G} S_g \right)^{(l,t)}.
\]

(ii) Take \(x, y \in S\), then
\[
(\varphi \circ (\alpha \otimes \alpha^*)(l,t))(x 1_{l^{-1}} \otimes y 1_l) = \varphi[\alpha_t(x 1_{l-1}) \otimes \alpha_{t^{-1}}(y 1_l)] = (\alpha_t(x 1_{l^{-1}}) \alpha_{t^{-1}}(y 1_{g^{-1}}) 1_{g})_{g \in G} = (\alpha_t(x \alpha_g(y 1_{g^{-1}}) 1_{l^{-1}}) 1_{gt})_{\lambda \in G} = (\alpha_t(x \alpha_g(y 1_{l^{-1}}) 1_{l^{-1}}))_{\lambda \in G},
\]
on the other hand
\[
(\alpha'_{(l,t)} \circ \varphi)(x 1_{l^{-1}} \otimes y 1_l) = \alpha'_{(l,t)}[(x 1_{l^{-1}} \alpha_g(y 1_{g^{-1}}) 1_{gt})_{g \in G}] = \alpha'_{(l,t)}[(x \alpha_g(y 1_{g^{-1}}) 1_{l^{-1}} 1_{gt})_{g \in G}] = (\alpha(x \alpha_g(y 1_{l^{-1}}) 1_{l^{-1}} 1_{gt}))_{g \in G},
\]
and thus \(\varphi \circ (\alpha \otimes \alpha^*)(l,t) = \alpha'_{(l,t)} \circ \varphi\) in \(S_{t^{-1}} \otimes S_t\), for all \(l, t \in G\). We conclude that \(\alpha \otimes \alpha^*\) and \(\alpha'_{(l,t)}\) are partially \((G \times G)\)-isomorphic.

3) To check (20) notice that \((d_g)_{g \in G} \in \left( \prod_{g \in G} S_g \right)^{\delta G}\), if and only if, for all \(l \in G\)
\[
(d_g 1_l 1_{lg})_{g \in G} = \hat{\alpha}_{(l,l^{-1})}[(d_g 1_{l^{-1}} 1_{l^{-1}})_{g \in G}] = (\alpha(l)(d_g 1_{l^{-1}} 1_l)_{g \in G}.
\]
Now since \((S \otimes S, \alpha \otimes \alpha^*)\) and \(\left( \prod_{g \in G} S_g, \hat{\alpha} \right)\) are \(G \times G\)-isomorphic, then by Proposition 5.2 the partial actions \(((S \otimes S)^{G,G}, (\alpha \otimes \alpha^*))_{G \times G/\delta G}\) and \(\langle E(S, \alpha), \hat{\alpha}_{G \times G/\delta G} \rangle\) are partially \(G\)-isomorphic, thus \([\langle E(S, \alpha), \hat{\alpha}_{G \times G/\delta G} \rangle]\) is an idempotent in \(\mathcal{H}_{\text{par}}(G, R)\).

Finally to prove (21) note that \(\hat{\alpha}_{(1,i)G}((d_g)_{g \in G})\) equals

\[
\alpha_{(1,i)}(d_g1_{(1,l^{-1})})_{g \in G} + \sum_{i=2}^{m} \prod_{j=1}^{i-1} ((1_g)_{g \in G} - 1_{(h_j,h_j^{-1})}) \alpha_{(h_i,h_i^{-1})}(d_g1_{(h_i^{-1},l^{-1}h_i)})_{g \in G} \quad \text{(19)}
\]

\[
(d_g1_{lg})_{g \in G} + \sum_{i=2}^{m} \prod_{j=1}^{i-1} ((1_g)_{g \in G} - 1_{(h_j1_{h_jg})}) \alpha_{h_i}(d_g1_{h_i^{-1})1_{lg})_{g \in G} \quad \text{(20)}
\]

\[
(d_g1_{lg})_{g \in G} + \sum_{i=2}^{m} \prod_{j=1}^{i-1} ((1_g)_{g \in G} - 1_{(h_j1_{h_jg})}) \alpha_{h_i}(d_g1_{h_i1_{h_i}1_{lg})_{g \in G},
\]

as desired. \(\square\)

**Remark 5.6.** Let \(E(S, \alpha)_g\) the projection of \(E(S, \alpha)\) onto the \(g\)th coordinate. Then \(R = E(S, \alpha)_e\) and \(R_{1,g} \subseteq E(S, \alpha)_g\) for \(g \neq e\).

Let \(E\) be the meet semilattice of idempotents of \(\mathcal{H}_{\text{par}}(G, R)\), by Proposition 5.5, the element \(E(\alpha)\) is such that its equivalence class belongs to \(E\). From this we have that \(\mathcal{H}_{\text{par}}(G, R) = \bigcup_{E(S, \alpha)} \mathcal{H}_{\text{par}, E(S, \alpha)}(G, R)\), where \(\mathcal{H}_{\text{par}, E(S, \alpha)}(G, R)\) is the subgroup of \(\mathcal{H}_{\text{par}}(G, R)\) whose identity element is the class \([E(S, \alpha), \hat{\alpha}_{G \times G/\delta G}]\). Then the following is clear.

\[
\mathcal{H}_{\text{par}, E(\alpha)}(G, R) = \{ [S', \alpha'] : [E(S', \alpha'), \hat{\alpha}'_{G \times G/\delta G}] = [E(S, \alpha), \hat{\alpha}_{G \times G/\delta G}] \}.
\]

Using (21) we get the following.

**Proposition 5.7.** Let \(R \subseteq S'\) be a partial Galois extension with partial action \(\alpha'\), and \(S'_g = S_{1,g}^{'}\) for all \(g \in G\). Then \([S', \alpha'] \in \mathcal{H}_{\text{par}, E(S, \alpha)}(G, R)\), if and only if, there is a \(R\)-algebra homomorphism \(f : E(S, \alpha) \rightarrow E(S', \alpha')\) such that \(f((1_g)_{g \in G}) = (1_{g})_{g \in G}\).

**Proof.** The part \((\Rightarrow)\) is clear. Conversely, suppose that there is a \(f : E(S, \alpha) \rightarrow E(S', \alpha')\) such that \(f((1_g)_{g \in G}) = (1_{g})_{g \in G}\), then by (17) we see that \(f(E(S, \alpha)1_{(g,1)\delta G}) \subseteq E(S', \alpha')\), for all \(g \in G\), moreover by (21) we get that \(f\) satisfies ii) of Definition 4.1. Then the result follows from Proposition 4.2. \(\square\)

### 5.4. Examples and remarks

The study of partial Galois extension of finite abelian groups reduces to the study of partial Galois extension of cyclic groups. Indeed let \(G\) be a finite abelian group, then there are cyclic groups \(G_1, G_2, \cdots, G_n\) such that \(G = G_1 \times G_2 \times \cdots \times G_n\). Further, for each \(1 \leq i \leq n\), let \(S_i/R\) be a partial Galois extension with group \(G_i\) and partial action \(\alpha_i\). Consider \(S = S_1 \otimes_R S_2 \otimes_R \cdots \otimes_R S_n\), then by [4, Proposition 2.9] we have that \(S/R\) is a partial Galois extension with group \(G\) and partial
action \( \alpha = \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n \). Thus, we have a map
\[
\prod_{i=1}^n \mathcal{H}_{\text{par}}(G_i, R) \ni ([S_1, \alpha_1], [S_2, \alpha_2], \cdots, [S_n, \alpha_n]) \mapsto [S, \alpha] \in \mathcal{H}_{\text{par}}(G, R).
\]

To construct its inverse, let \( S/R \) be a partial Galois extension of \( G \) on \( S \) with partial action \( \alpha \), for each \( 1 \leq i \leq n \), consider \( H_i = G_1 \times G_2 \times \cdots \times G_{i-1} \times \{ e \} \times G_{i+1} \times \cdots \times G_n \), then \( H_i \) acts partially on \( S \). Write \( S_i = S^{\alpha_{H_i}} \), then there is a group isomorphism \( G/H_i \simeq G_i \) and by Proposition 3.4 \( S_i/R \) is a partial Galois extension with group \( G_i \), and we have a map
\[
\mathcal{H}_{\text{par}}(G, R) \ni [S, \alpha] \mapsto ([S_1, \alpha_1], [S_2, \alpha_2], \cdots, [S_n, \alpha_n]) \in \prod_{i=1}^n \mathcal{H}_{\text{par}}(G_i, R).
\]

It is not difficult to check that the map \( \phi \) is a bijection with inverse \( \varphi \).

**Remark 5.8.** Note that for \( \sum_i x_i \otimes y_i \in (A \otimes B)^{\delta G} \) we have
\[
\sum_i x_i 1_{g^{-1}} \otimes y_i 1_g = \sum_i x_i 1_g \otimes y_i 1_{g^{-1}}.
\]

Hence,
\[
(\alpha_{g^{-1}} \otimes \alpha_1)(\sum_i x_i 1_g \otimes y_i 1_{g^{-1}}) = (\alpha_{g^{-1}} \otimes \alpha)(\alpha_g \otimes \alpha_{g^{-1}})(\sum_i x_i 1_g \otimes y_i 1_g)
\]
\[
= \sum_i x_i 1_{g^{-1}} \otimes \alpha_{g^{-1}}(y_i 1_g)
\]
\[
= (\alpha_1 \otimes \alpha_{g^{-1}})(\sum_i x_i 1_{g^{-1}} \otimes y_i 1_g).
\]

Now we give some examples to illustrate our results.

From now on \( G \) will denote the cyclic group \( G = \langle g \mid g^4 = 1 \rangle \) of order 4.

**Example 5.9.** Let \( R \) be a commutative ring and set \( S = R e_1 \oplus R e_2 \oplus R e_3 \), where \( e_1, e_2, e_3 \) are non-zero orthogonal idempotents whose sum is one and let and the partial action of \( G \) on \( S \) given by [3, Example 6.1]. That is \( S_g = R e_1 \oplus R e_2, S_{g^2} = R e_1 \oplus R e_3, S_{g^3} = R e_2 \oplus R e_3 \), and

\[
\alpha_g : S_{g^3} \to S_g; \alpha_g(e_2) = e_1, \alpha_g(e_3) = e_2,
\]
\[
\alpha_{g^2} : S_{g^2} \to S_{g^2}; \alpha_{g^2}(e_1) = e_3, \alpha_{g^2}(e_3) = e_1 \quad \text{and}
\]
\[
\alpha_{g^3} : S_g \to S_{g^3}; \alpha_{g^3}(e_1) = e_2, \alpha_{g^3}(e_2) = e_3.
\]

Hence, \( S \) is an \( \alpha \)-partial Galois extension of \( R \). Let \( H = \{ 1, g^2 \} \leq G \) and \( T = \{ 1, g \} \) a transversal of \( H \) in \( G \). Then \( S^{\alpha_H} = R(e_1 + e_3) \oplus R e_2 \) and the family \( \alpha_{G/H} = (D_g, \alpha_{g})_{g \in T} \) is a \( G/H \)-partial Galois extension of \( S^\alpha = R \), where by (??) we have \( \tilde{1}_H = 1_S = \tilde{1}_{gH} \) and by equations (11), (18) we have
\[
\tilde{D}_H = \tilde{D}_{gH} = [R(e_1 + e_3) \oplus R e_2](e_1 + e_2) = S^{\alpha_H}.
\]
\[\alpha_H = \text{id}_{S^{\alpha_H}} \text{ and } \alpha_g(x) = \alpha_g(x1_g) + \alpha_g(x1_g)(1_S - 1_g), \text{ for all } x \in S^{\alpha_H}.\]

It follows by Example 5.9 that in general if \(\alpha_{G/H}\) is global, then the partial action \(\alpha\) is not necessarily global.

**Example 5.10.** Consider the ring \(S' = Re_1' \oplus Re_2'\), where \(e_1', e_2'\) are non-zero orthogonal idempotents whose sum is one. We define a partial action \(\theta\) of \(G\) on \(S'\) by taking \(S'_1 = S'\), \(S'_g = Re_2'\), \(S'_{g^2} = \{0\}\), \(S'_{g^3} = Re_1'\), and setting \(\theta_1 = \text{id}_{S'}\),

\[\theta_g : S'_{g^3} \to S'_{g^2}; \theta_g(e_1') = e_2', \text{ and } \theta_g : S'_{g^2} \to S'_{g^3}; \theta_g(e_2') = e_1'.\]

Note that, \(S'^\theta = R\) and \(\{x_1 = y_1 = e_1', x_2 = y_2 = e_2'\}\) are partial Galois coordinates of \(S'\) over \(R\). We calculate the product \([S'(S, \theta^*)]_{\text{par}} \circ\circ ([S', \theta])\)

First we get that

\[\theta_g : S'_{g^3} \to S'_{g^2}; \theta_g(e_1') = e_2', \text{ and } \theta_g : S'_{g^2} \to S'_{g^3}; \theta_g(e_2') = e_1'.\]

Now we use equation (17) to find the idempotents \(1_{(x, 1)}\delta_G\), with \(x \in G\). Let \(h_i = g^{-1}, 1 \leq i \leq 4\). Then

\[1_{(x, 1)}\delta_G = 1_{x}^* \otimes 1_S + \sum_{i=2}^{4} \prod_{j=1}^{i-1} (1_S \otimes 1_S - 1_{xh_j}^* \otimes 1_{h_j^{-1}})(1_{xh_i}^* \otimes 1_{h_i^{-1}})\]

\[= 1_{x^{-1}} \otimes 1_S + \sum_{i=2}^{4} \prod_{j=1}^{i-1} (1_S \otimes 1_S - 1_{(xh_j)^{-1}} \otimes 1_{h_j}^{-1})(1_{(xh_i)^{-1}} \otimes 1_{h_i}^{-1})\]

Then

- \(\bar{1}_{(g, 1)}\delta_G = e_1' \otimes 1_S + e_2' \otimes e_2';\)
- \(\bar{1}_{(g^2, 1)}\delta_G = e_2' \otimes e_1' + e_1' \otimes e_2';\)
- \(\bar{1}_{(g^3, 1)}\delta_G = e_2' \otimes 1_S + e_1' \otimes e_1'.\)

Then

\[1'_{(g^3, 1)} = e_2' \otimes 1_S + e_1' \otimes e_1'.\]

From this we get,

\[\bar{D}_{(g, 1)}\delta_G = (S \otimes S')^{\delta_G}1_{(g, 1)}\delta_G\]

\[= [R(e_1' \otimes e_1' + e_2' \otimes e_2') \oplus R(e_1' \otimes e_2') \oplus R(e_2' \otimes e_1')][e_1' \otimes 1_S + e_2' \otimes e_2']\]

\[= R(e_1' \otimes e_1' + e_2' \otimes e_2') \oplus R(e_1' \otimes e_2'),\]

\[\bar{D}_{(g^3, 1)}\delta_G = (S \otimes S')^{\delta_G}1_{(g^3, 1)}\delta_G\]

\[= [R(e_1' \otimes e_1' + e_2' \otimes e_2') \oplus R(e_1' \otimes e_2') \oplus R(e_2' \otimes e_1')][e_2' \otimes 1_S + e_1' \otimes e_1']\]

\[= R(e_1' \otimes e_1' + e_2' \otimes e_2') \oplus R(e_2' \otimes e_1').\]
and
\[
\hat{D}_{(g^2,1)^G} = (S \otimes S')^G 1_{(g^2,1)^G} \\
= [R(e'_1 \otimes e'_1 + e'_2 \otimes e'_2) \oplus R(e_1 \otimes e_2) \oplus R(e'_2 \otimes e'_1)(e'_1 \otimes e'_2 + e'_2 \otimes e'_1) \\
= R(e'_1 \otimes e'_2) \oplus R(e'_2 \otimes e'_1).
\]

Now, using equation (18) we find \(\hat{\lambda}_{(l,1)^G}\) with \(l \in G\), where \(\lambda = \theta \otimes \theta^*\). We get
\begin{itemize}
  \item \(\hat{\lambda}_{(g,1)^G}(r(e'_1 \otimes e'_1 + e'_2 \otimes e'_2) \oplus s(e'_2 \otimes e'_1)) = r(e'_1 \otimes e'_1 + e'_2 \otimes e'_2) + s(e'_1 \otimes e'_2)\).
  \item \(\hat{\lambda}_{(g^2,1)^G}(r(e'_1 \otimes e'_2) + s(e'_2 \otimes e'_1)) = s(e'_1 \otimes e'_2) + r(e'_2 \otimes e'_1)\).
  \item \(\hat{\lambda}_{(g^3,1)^G} = \hat{\lambda}_{(g,1)^G}^{-1}\).
\end{itemize}

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