Asymptotic Nets and Discrete Affine Surfaces with Indefinite Metric

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Abstract. Asymptotic net is an important concept in discrete differential geometry. In this paper, we show that we can associate affine discrete geometric concepts to an arbitrary non-degenerate asymptotic net. These concepts include discrete affine area, mean curvature, normal and co-normal vector fields and cubic form, and they are related by structural and compatibility equations. We consider also the particular cases of affine minimal surfaces and affine spheres.

Keywords: Asymptotic nets, Discrete affine surfaces, Discrete affine indefinite metric.

1 Introduction

The expansion of computer graphics and applications in mathematical physics have recently given a great impulse to discrete differential geometry. In this discrete context, surfaces with indefinite metric are generally modelled as asymptotic nets. In particular, there are several types of discrete affine surfaces with indefinite metric modelled as asymptotic nets: Affine spheres ([3]), improper affine spheres ([7], [6]) and minimal surfaces ([5]). In this work we define a general class of discrete affine surfaces that includes these types as particular cases.

Beginning with an arbitrary asymptotic net, with the mild hypothesis of non-degeneracy, we define a discrete affine invariant structure on it. The discrete geometric concepts defined are affine metric, normal vector field, co-normal vector field, cubic form and mean curvature, and they are related by equations comparable with the correspondent smooth equations. The asymptotic net is called minimal when its mean curvature is zero, while the asymptotic net is an called an affine sphere when certain discrete derivatives described in section 4 vanishes. Asymptotic discretizations of the one-sheet hyperboloid are basic examples of discrete surfaces that satisfy our definition of affine spheres.

The main characteristic of our definition of the discrete affine geometric concepts is that they are related by discrete equations that closely resembles the smooth equations of affine differential geometry of surfaces. The simplicity of these equations are somewhat surprising, since the construction is very general. Structural equations describe the discrete immersion in terms of the affine metric, cubic form and mean curvature, and these concepts must satisfy compatibility equations. When the discrete affine metric, cubic form and mean curvature satisfy the compatibility equations, one can define an asymptotic net, unique up to equi-affine transformations of $\mathbb{R}^3$, that satisfies the structural equations.

The paper is organized as follows: Section 2 reviews the basic facts of smooth affine differential geometry with asymptotic parameters. Section 3 contains the main results: It shows how you can define the affine metric, normal and co-normal vector fields, cubic form and mean curvature from a given asymptotic net. It also relate this work with the discrete affine minimal surfaces of [5] and give an example of a “non-minimal” asymptotic net. Section 4 describes the structural equations and propose a new definition of affine spheres that is more general than the one proposed in [3]. Section 5 describes the compatibility equations and shows that an affine surface is defined, up to equi-affine transformations of $\mathbb{R}^3$, by its metric, cubic form and mean curvature.

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2 Smooth affine surface with indefinite metric

In this section we review the structure of smooth surfaces with indefinite Berwald-Blashcke metric. The concepts and equations of this section are the inspiration for the discrete concepts and equations of the rest of the paper. For details and proofs of this section, see [4].
2.1 Concepts and equations

Notation. Given two vectors \( V_1, V_2 \in \mathbb{R}^3 \), we denote by \( V_1 \times V_2 \) the cross product and by \( V_1 \cdot V_2 \) the dot product between them. Given three vectors \( V_1, V_2, V_3 \in \mathbb{R}^3 \), we denote by \( [V_1, V_2, V_3] = (V_1 \times V_2) \cdot V_3 \) their determinant.

Consider a parameterized smooth surface \( q : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \), where \( U \) is an open subset of the plane and denote

\[
L(u, v) = [q_u, q_v, q_{uu}],
M(u, v) = [q_u, q_v, q_{uv}],
N(u, v) = [q_u, q_v, q_{vv}].
\]

The surface is non-degenerate if \( LN - M^2 \neq 0 \), and, in this case, the Berwald-Blaschke metric is defined by

\[
ds^2 = \frac{1}{|LN - M^2|^{1/4}} \left( Ldud^2 + 2Mdudv + Ndv^2 \right).
\]

If \( LN - M^2 > 0 \), the metric is **definite** while if \( LN - M^2 < 0 \), the metric is **indefinite**. In this paper, we shall restrict ourselves to surfaces with indefinite metric. We say that the coordinates \( (u, v) \) are **asymptotic** if \( L = N = 0 \). In this case, the metric takes the form

\[
ds^2 = 2\Omega dudv,
\]

where \( \Omega^2 = M \).

The vector field \( \nu(u, v) = \frac{q_u \times q_v}{\Omega} \) is called the co-normal vector field. It satisfies Lelievre’s equations

\[
q_u = \nu \times q_u,
q_v = -\nu \times q_v.
\]

And the vector field \( \xi(u, v) = \frac{q_{uv}}{\Omega} \) is called the affine normal vector field. One can easily verify that \( \nu(u, v) \cdot \xi(u, v) = 1 \).

The functions

\[
A(u, v) = [q_u, q_{uu}, \xi],
B(u, v) = [q_v, q_{vv}, \xi]
\]

are the coefficients of the affine cubic form \( Adu^3 + Bdv^3 \) (see [8]). We can write

\[
q_{uu} = \frac{1}{\Omega} (\Omega_u q_u + Aq_v),
q_{vv} = \frac{1}{\Omega} (Bq_u + \Omega_v q_v). \tag{1}
\]

The function

\[
H(u, v) = \frac{\Omega_u \Omega_v - \Omega \Omega_{uu} - AB}{\Omega^3}
\]

is called the **affine mean curvature**. One can write

\[
\xi_u = -H q_u + \frac{A}{\Omega^2} q_v, \tag{3}
\]

\[
\xi_v = \frac{B}{\Omega^2} q_u - H q_v. \tag{4}
\]

Equations (1), (2), (3) and (4) are the structural equations of the surface. For a given surface, the quadratic form \( 2\Omega dudv \), the cubic form \( Adu^3 + Bdv^3 \) and the affine mean curvature \( H \) should satisfy the following compatibility equations:

\[
H_u = \frac{ABu}{\Omega^3} - \frac{1}{\Omega} \left( \frac{A}{\Omega} \right)_v, \tag{5}
H_v = \frac{BA_v}{\Omega^3} - \frac{1}{\Omega} \left( \frac{B}{\Omega} \right)_u. \tag{6}
\]

Conversely, given \( \Omega, A, B \) and \( H \) satisfying equations (5) and (6), there exists a parameterization \( q(u, v) \) of a surface with quadratic form \( 2\Omega dudv \), cubic form \( Adu^3 + Bdv^3 \) and affine mean curvature \( H \).
2.2 Example

Example 1 Consider the asymptotic parameterization of the one-sheet hyperboloid \( y^2 + z^2 - x^2 = c^2 \),

\[
q(u, v) = \frac{c}{\sinh(u + v)}(-\cosh(u - v), -\sinh(u - v), \cosh(u + v)).
\]

Taking derivatives

\[
q_u = \frac{c}{\sinh^2(u + v)}(\cosh(2v), -\sinh(2v), -1),
\]

\[
q_v = \frac{c}{\sinh^2(u + v)}(\cosh(2u), \sinh(2u), -1).
\]

Straightforward calculations shows that

\[
\Omega = \frac{2c^{3/2}}{\sinh^2(u + v)},
\]

\[
\nu(u, v) = c^{1/2} \frac{1}{\sinh(u + v)}(\cosh(u - v), \sinh(v - u), \cosh(u + v))
\]

and

\[
\xi(u, v) = c^{-1/2} \frac{1}{\sinh(u + v)}(-\cosh(u - v), \sinh(v - u), \cosh(u + v)).
\]

One can also verify that \( A(u, v) = B(u, v) = 0 \) and that \( H = c^{-3/2} \) is the affine mean curvature.

Figure 1: Smooth hyperboloid with \( 1 \leq u \leq 3, 1 \leq v \leq 3 \).

3 Geometric concepts from an asymptotic net

In this section, we define the following quantities associated with a given non-degenerate asymptotic net: Affine metric, normal and co-normal vector fields, mean curvature and cubic form.

3.1 Discrete affine concepts

Notation. For a discrete real or vector function \( f \) defined on a domain \( D \subset \mathbb{Z}^2 \), we denote the discrete partial derivatives with respect to \( u \) or \( v \) by

\[
f_1(u + \frac{1}{2}, v) = f(u + 1, v) - f(u, v),
\]

\[
f_2(u, v + \frac{1}{2}) = f(u, v + 1) - f(u, v).
\]
The second order partial derivatives are defined by

\[ f_{11}(u, v) = f(u + 1, v) - 2f(u, v) + f(u - 1, v) \]
\[ f_{22}(u, v) = f(u, v + 1) - 2f(u, v) + f(u, v - 1) \]
\[ f_{12}(u, v) = f(u + 1, v + 1) + f(u, v) - f(u + 1, v) - f(u, v + 1). \]

Non-degenerate asymptotic nets

The asymptotic net can be described by a vector function \( q : D \subset \mathbb{Z}^2 \rightarrow \mathbb{R}^3 \), called the affine immersion, such that \( q_1(u + \frac{1}{2}, v), q_1(u - \frac{1}{2}, v), q_2(u, v + \frac{1}{2}) \) and \( q_2(u, v - \frac{1}{2}) \) are co-planar. For each quadrangle, let

\[ M(u + \frac{1}{2}, v + \frac{1}{2}) = [q_1(u + \frac{1}{2}, v), q_2(u, v + \frac{1}{2}), q_2(u + 1, v + \frac{1}{2})]. \]

We say that the asymptotic net is non-degenerate if \( M \) does not change sign. We shall assume throughout the paper that \( M(u + \frac{1}{2}, v + \frac{1}{2}) > 0 \), for any \((u, v) \in D\).

Affine metric

The affine metric \( \Omega \) at a quadrangle \((u + \frac{1}{2}, v + \frac{1}{2})\) is defined as

\[ \Omega(u + \frac{1}{2}, v + \frac{1}{2}) = \sqrt{M(u + \frac{1}{2}, v + \frac{1}{2})}. \]

Co-normal vector field

The co-normal \( \nu \) is defined at each vertex \((u, v)\) and is orthogonal to the plane containing \( q_1(u + \frac{1}{2}, v), q_1(u - \frac{1}{2}, v), q_2(u, v + \frac{1}{2}) \) and \( q_2(u, v - \frac{1}{2}) \). The length of \( \nu(u, v) \) is defined by the following lemma:

**Proposition 2** Consider an initial quadrangle \((u_0 + \frac{1}{2}, v_0 + \frac{1}{2})\) and fix \( \gamma(u_0 + \frac{1}{2}, v_0 + \frac{1}{2}) \) > 0. Then there exist unique scalars \( \gamma(u + \frac{1}{2}, v + \frac{1}{2}) > 0 \) such that the following formulas for \( \nu(u, v) \) coincide:

\[
\nu(u, v) = \frac{\gamma^{-1}(u + \frac{1}{2}, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})}(q_1(u + \frac{1}{2}, v) \times q_2(u, v + \frac{1}{2})),
\]
\[
\nu(u, v) = \frac{\gamma(u - \frac{1}{2}, v + \frac{1}{2})}{\Omega(u - \frac{1}{2}, v + \frac{1}{2})}(q_1(u - \frac{1}{2}, v) \times q_2(u, v + \frac{1}{2})),
\]
\[
\nu(u, v) = \frac{\gamma^{-1}(u - \frac{1}{2}, v - \frac{1}{2})}{\Omega(u - \frac{1}{2}, v - \frac{1}{2})}(q_1(u - \frac{1}{2}, v) \times q_2(u, v - \frac{1}{2})),
\]
\[
\nu(u, v) = \frac{\gamma(u + \frac{1}{2}, v - \frac{1}{2})}{\Omega(u + \frac{1}{2}, v - \frac{1}{2})}(q_1(u + \frac{1}{2}, v) \times q_2(u, v - \frac{1}{2})).
\]

Moreover, the co-normals satisfy the discrete Lelièvre’s equations

\[
\nu(u, v) \times \nu(u + 1, v) = q_1(u + \frac{1}{2}, v),
\]
\[
\nu(u, v) \times \nu(u, v + 1) = -q_2(u, v + \frac{1}{2}).
\]

The products \( p(u, v + \frac{1}{2}) = \gamma(u + \frac{1}{2}, v + \frac{1}{2})\gamma(u - \frac{1}{2}, v + \frac{1}{2}) \) and \( p(u + \frac{1}{2}, v) = \gamma(u + \frac{1}{2}, v + \frac{1}{2})\gamma(u + \frac{1}{2}, v - \frac{1}{2}) \) do not depend on the choice of \( \gamma(u_0 + \frac{1}{2}, v_0 + \frac{1}{2}) \).

**Proof.** We begin by fixing an initial value \( \gamma(u_0 + \frac{1}{2}, v_0 + \frac{1}{2}) \). The coincidence of \( \nu \) at \((u_0, v_0)\) determines \( \gamma(u_0 + \frac{1}{2}, v_0 - \frac{1}{2}) \). Observe that

\[
\nu(u, v) \times \nu(u + 1, v) = \frac{1}{\Omega^2(u + \frac{1}{2}, v + \frac{1}{2})}(q_1(u + \frac{1}{2}, v) \times q_2(u, v + \frac{1}{2})) \times (q_1(u + \frac{1}{2}, v) \times q_2(u + 1, v + \frac{1}{2}))
\]
\[
= q_1(u + \frac{1}{2}, v),
\]
and so \( \nu \) coincidences also at \((u_0 + 1, v_0)\). Similarly, define \( \gamma(u_0 - \frac{1}{2}, v_0 + \frac{1}{2}) \) by coincidence of \( \nu \) at \((u_0, v_0)\) and formula

\[ \nu(u, v) \times \nu(u, v + 1) = -q_2(u, v + \frac{1}{2}) \]

implies coincidence of \( \nu \) at \((u_0, v_0 + 1)\). Finally define \( \gamma(u_0 - \frac{1}{2}, v_0 - \frac{1}{2}) \) by coincidence of \( \nu \) at \((u_0, v_0)\) and the above formulas guarantee coincidence of \( \nu \) at \((u_0 - 1, v_0)\) and \((u_0, v_0 - 1)\). By repeating this procedure, we can define \( \gamma \) in all domain satisfying the above formulas.

It is interesting to observe that the above co-normal vector field defines a Moutard net, i.e.,

\[ \gamma^2(u + \frac{1}{2}, v + \frac{1}{2})\nu(u, v) + \nu(u + 1, v + 1) = \nu(u, v) + \nu(u + 1, v) \]

(see [2]). And, in terms of the co-normals, the affine metric is given as

\[ \Omega(u + \frac{1}{2}, v + \frac{1}{2}) = \gamma^{-1}(u + \frac{1}{2}, v + \frac{1}{2}) [\nu(u, v), \nu(u + 1, v), \nu(u + 1, v)] . \]

The normal vector field

The affine normal vector field \( \xi \) is defined at each quadrangle \((u + \frac{1}{2}, v + \frac{1}{2})\) by

\[ \xi(u + \frac{1}{2}, v + \frac{1}{2}) = \frac{q_2(u + \frac{1}{2}, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})} \]

It satisfies the following equations:

\[ \nu(u, v) \cdot \xi(u + \frac{1}{2}, v + \frac{1}{2}) = \gamma^{-1}(u + \frac{1}{2}, v + \frac{1}{2}), \]
\[ \nu(u + 1, v) \cdot \xi(u + \frac{1}{2}, v + \frac{1}{2}) = \gamma(u + \frac{1}{2}, v + \frac{1}{2}), \]
\[ \nu(u, v + 1) \cdot \xi(u + \frac{1}{2}, v + \frac{1}{2}) = \gamma(u + \frac{1}{2}, v + \frac{1}{2}), \]
\[ \nu(u + 1, v + 1) \cdot \xi(u + \frac{1}{2}, v + \frac{1}{2}) = \gamma^{-1}(u + \frac{1}{2}, v + \frac{1}{2}). \]

The cubic form

Define the discrete cubic form as \( A(u, v)\delta u^3 + B(u, v)\delta v^3 \), where

\[ A(u, v) = [q_1(u - \frac{1}{2}, v), q_1(u + \frac{1}{2}, v), \gamma(u - \frac{1}{2}, v + \frac{1}{2})\xi(u + \frac{1}{2}, v + \frac{1}{2})], \]
\[ = [q_1(u + \frac{1}{2}, v), q_1(u - \frac{1}{2}, v), \gamma^{-1}(u + \frac{1}{2}, v + \frac{1}{2})\xi(u - \frac{1}{2}, v + \frac{1}{2})], \]
\[ = [q_1(u - \frac{1}{2}, v), q_1(u + \frac{1}{2}, v), \gamma^{-1}(u - \frac{1}{2}, v + \frac{1}{2})\xi(u - \frac{1}{2}, v + \frac{1}{2})], \]
\[ = [q_1(u - \frac{1}{2}, v), q_1(u + \frac{1}{2}, v), \gamma(u - \frac{1}{2}, v - \frac{1}{2})\xi(u - \frac{1}{2}, v - \frac{1}{2})], \]

and

\[ B(u, v) = [q_2(u, v - \frac{1}{2}), q_2(u, v + \frac{1}{2}), \gamma(u + \frac{1}{2}, v + \frac{1}{2})\xi(u + \frac{1}{2}, v + \frac{1}{2})], \]
\[ = [q_2(u, v - \frac{1}{2}), q_2(u, v + \frac{1}{2}), \gamma^{-1}(u + \frac{1}{2}, v - \frac{1}{2})\xi(u + \frac{1}{2}, v - \frac{1}{2})], \]
\[ = [q_2(u, v - \frac{1}{2}), q_2(u, v + \frac{1}{2}), \gamma^{-1}(u - \frac{1}{2}, v + \frac{1}{2})\xi(u - \frac{1}{2}, v + \frac{1}{2})], \]
\[ = [q_2(u, v - \frac{1}{2}), q_2(u, v + \frac{1}{2}), \gamma(u - \frac{1}{2}, v - \frac{1}{2})\xi(u - \frac{1}{2}, v - \frac{1}{2})]. \]

Since we are interested only in the coefficients \( A(u, v) \) and \( B(u, v) \) of the discrete cubic form, we shall not discuss in this paper the meaning of the symbols \( \delta u^3 \) and \( \delta v^3 \).

Mean curvature

We shall define mean curvature at each edge of the asymptotic net. At edges \((u, v + \frac{1}{2})\) and \((u + \frac{1}{2}, v)\) define

\[ h(u, v + \frac{1}{2}) = p(u, v + \frac{1}{2}) - p^{-1}(u, v + \frac{1}{2}), \]
\[ h(u + \frac{1}{2}, v) = p(u + \frac{1}{2}, v) - p^{-1}(u + \frac{1}{2}, v). \]
The mean curvature \( H(u, v + \frac{1}{2}) \) and \( H(u + \frac{1}{2}, v) \) are defined as

\[
H(u, v + \frac{1}{2}) = \frac{h(u, v + \frac{1}{2})}{\sqrt{\Omega(u - \frac{1}{2}, v + \frac{1}{2})\Omega(u + \frac{1}{2}, v + \frac{1}{2})}}
\]

\[
H(u + \frac{1}{2}, v) = \frac{h(u + \frac{1}{2}, v)}{\sqrt{\Omega(u + \frac{1}{2}, v - \frac{1}{2})\Omega(u + \frac{1}{2}, v + \frac{1}{2})}}.
\]

The reason for this definition will be clear later in subsection 4.2, when we obtain the equations for the derivatives of \( \xi \).

### 3.2 The case of minimal surfaces

Examples of asymptotic nets can be obtained as follows: Begin with a smooth 3-dimensional vector field \( \nu(u, v) \) satisfying \( \nu_{uv} = 0 \) and consider a sampling \( \nu(m\Delta u, n\Delta v) \). The discrete immersion is then obtained by integrating the discrete Lelievre’s equations.

This type of asymptotic net can be characterized by the constancy of the parameter \( \gamma \). More precisely, for these nets we can choose \( \gamma(u + \frac{1}{2}, v + \frac{1}{2}) = 1 \), for any \( (u, v) \), which implies that the discrete affine mean curvature defined above is zero. This nets were studied in [5], where they were called discrete affine minimal surfaces.

### 3.3 An example with non-vanishing mean curvature

**Example 3** Since the asymptotic curves the hyperboloid of example 1 are straight lines, sampling it in the domain of asymptotic parameters generate an asymptotic net. Denote by \( \Delta u \) and \( \Delta v \) the distance between samples in \( u \) and \( v \) directions, respectively.

Integrating \( q_u \) and \( q_v \) one can show that

\[
q_1(u + \frac{\Delta u}{2}, v) = \frac{c \sinh(\Delta u)}{\sinh(u + v) \sinh(u + v + \Delta u)} (\cosh(2v), -\sinh(2v), -1),
\]

\[
q_2(u, v + \frac{\Delta v}{2}) = \frac{c \sinh(\Delta v)}{\sinh(u + v) \sinh(u + v + \Delta v)} (\cosh(2u), \sinh(2u), -1).
\]

Denoting by \( \nu(u, v) \) the co-normal vector of the smooth hyperboloid, one can verify that

\[
\nu(u, v) \times \nu(u + \Delta u, v) = q_1(u + \frac{\Delta u}{2}, v),
\]

\[
\nu(u, v) \times \nu(u, v + \Delta v) = q_2(u, v + \frac{\Delta v}{2}).
\]

Thus, since Lelievre’s formulas hold, we can consider \( \nu(u, v) \) as the co-normal of the discrete surface as well. Straightforward calculations shows that

\[
\Omega(u + \frac{\Delta u}{2}, v + \frac{\Delta v}{2}) = \frac{2c^{3/2} \sinh(\Delta u) \sinh(\Delta v)}{\sqrt{\sinh(u + v + \Delta u + \Delta v) \sinh(u + v + \Delta u) \sinh(u + v + \Delta v) \sinh(u + v)}}
\]

and

\[
\gamma(u + \frac{\Delta u}{2}, v + \frac{\Delta v}{2}) = \sqrt{\frac{\sinh(u + v + \Delta u + \Delta v) \sinh(u + v)}{\sinh(u + v + \Delta u) \sinh(u + v + \Delta v)}}.
\]

The affine normal \( \xi(u + \frac{\Delta u}{2}, v + \frac{\Delta v}{2}) = (\xi_x, \xi_y, \xi_z) \) is given by

\[
2c^{1/2} \xi_x = \frac{-\cosh(\Delta u) \sinh(2v + \Delta v) - \cosh(\Delta v) \sinh(2u + \Delta u)}{\sqrt{\sinh(u + v + \Delta u + \Delta v) \sinh(u + v + \Delta u + \Delta v) \sinh(u + v) \sinh(u + v + \Delta u)}}
\]

\[
2c^{1/2} \xi_y = \frac{-\cosh(\Delta u) \cosh(2v + \Delta v) - \cosh(\Delta v) \cosh(2u + \Delta u)}{\sqrt{\sinh(u + v + \Delta u + \Delta v) \sinh(u + v + \Delta u + \Delta v) \sinh(u + v + \Delta u)}}
\]

\[
2c^{1/2} \xi_z = \frac{\sinh(2u + 2v + \Delta u + \Delta v)}{\sqrt{\sinh(u + v + \Delta u + \Delta v) \sinh(u + v + \Delta u + \Delta v) \sinh(u + v + \Delta u)}}
\]
Also
\[ p(u, v + \Delta v) = \sqrt{\frac{\sinh(u + v - \Delta u) \sinh(u + v + \Delta u + \Delta v)}{\sinh(u + v - \Delta u + \Delta v) \sinh(u + v + \Delta v)}} \]

and
\[ h(u, v + \Delta v) = \frac{-2 \sinh(\Delta v) \sinh(\Delta u) \cosh(\Delta u)}{\sqrt{\sinh(u + v - \Delta u) \sinh(u + v + \Delta u) \sinh(u + v - \Delta u + \Delta v) \sinh(u + v + \Delta u + \Delta v)}}. \]

The affine mean curvature is thus
\[ H(u, v + \Delta v) = \frac{c^{-3/2} \cosh(\Delta u) (\sinh(u + v + \Delta v) \sinh(u + v))^{1/2}}{(\sinh(u + v - \Delta u) \sinh(u + v + \Delta u) \sinh(u + v - \Delta u + \Delta v) \sinh(u + v + \Delta u + \Delta v))^{1/4}}. \]

Figure 2: Discrete hyperboloid with \(1 \leq u \leq 3, 1 \leq v \leq 3, \Delta u = 0.1\) and \(\Delta v = 0.2\).

4 Structural Equations
In this section we show the equations relating the affine immersion with the geometric concepts defined in section 3.1, that we shall call structural equations.

4.1 Equations for the second derivatives of the parameterization
To obtain equations for the second derivative of the affine immersion, we begin by defining derivatives of the area element of the Berwald-Blaschke metric:

\[ \Omega_1^{-}(u, v + \frac{1}{2}) = p(u, v + \frac{1}{2})\Omega(u + \frac{1}{2}, v + \frac{1}{2}) - \Omega(u - \frac{1}{2}, v + \frac{1}{2}), \]
\[ \Omega_1^{+}(u, v + \frac{1}{2}) = \Omega(u + \frac{1}{2}, v + \frac{1}{2}) - p(u, v + \frac{1}{2})\Omega(u - \frac{1}{2}, v + \frac{1}{2}), \]
\[ \Omega_2^{-}(u + \frac{1}{2}, v) = p(u + \frac{1}{2}, v)\Omega(u + \frac{1}{2}, v + \frac{1}{2}) - \Omega(u + \frac{1}{2}, v - \frac{1}{2}), \]
\[ \Omega_2^{+}(u + \frac{1}{2}, v) = \Omega(u + \frac{1}{2}, v + \frac{1}{2}) - p(u + \frac{1}{2}, v)\Omega(u + \frac{1}{2}, v - \frac{1}{2}). \]
Proposition 4  The second derivatives of the affine immersion are given by

\[
p(u, v + \frac{1}{2})q_{11}(u, v) = \frac{\Omega^- (u, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})} q_1(u + \frac{1}{2}, v) + \frac{\gamma(u - \frac{1}{2}, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})} A(u, v) q_2(u, v + \frac{1}{2}),
\]

\[
q_{11}(u, v) = \frac{\Omega^+ (u, v - \frac{1}{2})}{\Omega(u + \frac{1}{2}, v - \frac{1}{2})} q_1(u + \frac{1}{2}, v) + \frac{\gamma(u + \frac{1}{2}, v - \frac{1}{2})}{\Omega(u + \frac{1}{2}, v - \frac{1}{2})} A(u, v) q_2(u, v - \frac{1}{2}),
\]

\[
q_{11}(u, v) = \frac{\Omega^- (u, v + \frac{1}{2})}{\Omega(u - \frac{1}{2}, v + \frac{1}{2})} q_1(u - \frac{1}{2}, v) + \frac{\gamma(u - \frac{1}{2}, v + \frac{1}{2})}{\Omega(u - \frac{1}{2}, v + \frac{1}{2})} A(u, v) q_2(u, v + \frac{1}{2}),
\]

\[
p(u, v - \frac{1}{2})q_{11}(u, v) = \frac{\Omega^+ (u, v - \frac{1}{2})}{\Omega(u - \frac{1}{2}, v - \frac{1}{2})} q_1(u - \frac{1}{2}, v) + \frac{\gamma(u + \frac{1}{2}, v - \frac{1}{2})}{\Omega(u - \frac{1}{2}, v - \frac{1}{2})} A(u, v) q_2(u, v - \frac{1}{2})
\]

and

\[
p(u + \frac{1}{2}, v)q_{22}(u, v) = \frac{\gamma(u + \frac{1}{2}, v - \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})} q_1(u + \frac{1}{2}, v) + \frac{\Omega^- (u + \frac{1}{2}, v)}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})} q_2(u, v + \frac{1}{2}),
\]

\[
q_{22}(u, v) = \frac{\gamma(u - \frac{1}{2}, v - \frac{1}{2})}{\Omega(u + \frac{1}{2}, v - \frac{1}{2})} q_1(u - \frac{1}{2}, v) + \frac{\Omega^- (u - \frac{1}{2}, v)}{\Omega(u + \frac{1}{2}, v - \frac{1}{2})} q_2(u, v - \frac{1}{2}),
\]

\[
p(u - \frac{1}{2}, v)q_{22}(u, v) = \frac{\gamma(u - \frac{1}{2}, v + \frac{1}{2})}{\Omega(u - \frac{1}{2}, v + \frac{1}{2})} q_1(u - \frac{1}{2}, v) + \frac{\Omega^+ (u - \frac{1}{2}, v)}{\Omega(u - \frac{1}{2}, v + \frac{1}{2})} q_2(u, v + \frac{1}{2}).
\]

Proof. We prove the first formula, the others being similar. Since crosses are planar, \( q_{11}(u, v) \) is a linear combination of \( q_1(u + \frac{1}{2}, v) \) and \( q_2(u, v + \frac{1}{2}) \). The coefficient of \( q_2(u, v + \frac{1}{2}) \) is

\[
\frac{[q_1(u - \frac{1}{2}, v), q_1(u + \frac{1}{2}, v), \xi(u + \frac{1}{2}, v, \frac{1}{2})]}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})} = \frac{\gamma^{-1}(u + \frac{1}{2}, v + \frac{1}{2}) A(u, v)}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})},
\]

while the coefficient \( a \) of \( q_2(u, v + \frac{1}{2}) \) satisfies

\[
(q_1(u + \frac{1}{2}, v) - q_1(u - \frac{1}{2}, v)) \times q_2(u, v + \frac{1}{2}) = a \gamma(u + \frac{1}{2}, v + \frac{1}{2}) \Omega(u + \frac{1}{2}, v + \frac{1}{2}) v(u, v).
\]

Thus

\[
a = \frac{\gamma(u + \frac{1}{2}, v + \frac{1}{2}) \Omega(u + \frac{1}{2}, v + \frac{1}{2}) - \gamma^{-1}(u - \frac{1}{2}, v + \frac{1}{2}) \Omega(u - \frac{1}{2}, v + \frac{1}{2})}{\gamma(u + \frac{1}{2}, v + \frac{1}{2}) \Omega(u + \frac{1}{2}, v + \frac{1}{2}) - \gamma(u - \frac{1}{2}, v + \frac{1}{2}) \Omega(u - \frac{1}{2}, v + \frac{1}{2})}
\]

\[
= \frac{p^{-1}(u, v + \frac{1}{2}) \left( p(u, v + \frac{1}{2}) \Omega(u + \frac{1}{2}, v + \frac{1}{2}) - \Omega(u - \frac{1}{2}, v + \frac{1}{2}) \right)}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})}.
\]

\[\blacksquare\]

4.2 Equations for the derivatives of the affine normal

To obtain equations for the derivatives of the normals, we begin by defining derivatives of the coefficients of the cubic form:

\[
A_1^+ (u, v + \frac{1}{2}) = \gamma(u + \frac{1}{2}, v + \frac{1}{2}) A(u, v + 1) - \frac{A(u, v)}{\gamma(u + \frac{1}{2}, v + \frac{1}{2})},
\]

\[
A_1^- (u, v + \frac{1}{2}) = \frac{A(u, v + 1)}{\gamma(u - \frac{1}{2}, v + \frac{1}{2})} - \gamma(u - \frac{1}{2}, v + \frac{1}{2}) A(u, v),
\]

\[
B_1^+ (u + \frac{1}{2}, v) = \gamma(u + \frac{1}{2}, v + \frac{1}{2}) B(u + 1, v) - \frac{B(u, v)}{\gamma(u + \frac{1}{2}, v + \frac{1}{2})},
\]

\[
B_1^- (u + \frac{1}{2}, v) = \frac{B(u + 1, v)}{\gamma(u + \frac{1}{2}, v - \frac{1}{2})} - \gamma(u + \frac{1}{2}, v - \frac{1}{2}) B(u, v).
\]
Observe now that
\[
\xi_2^-(u, v + \frac{1}{2}) = p(u, v + \frac{1}{2})\xi(u + \frac{1}{2}, v + \frac{1}{2}) - \xi(u - \frac{1}{2}, v + \frac{1}{2}),
\]
\[
\xi_1^+(u, v + \frac{1}{2}) = \xi(u + \frac{1}{2}, v + \frac{1}{2}) - p(u, v + \frac{1}{2})\xi(u - \frac{1}{2}, v + \frac{1}{2}),
\]
are orthogonal to \(\nu(u, v)\) and \(\nu(u, v + 1)\), respectively. Similarly,
\[
\xi_2^-(u + \frac{1}{2}, v) = p(u + \frac{1}{2}, v)\xi(u + \frac{1}{2}, v + \frac{1}{2}) - \xi(u + \frac{1}{2}, v - \frac{1}{2}),
\]
\[
\xi_1^+(u + \frac{1}{2}, v) = \xi(u + \frac{1}{2}, v + \frac{1}{2}) - p(u + \frac{1}{2}, v)\xi(u + \frac{1}{2}, v - \frac{1}{2}),
\]
are orthogonal to \(\nu(u, v)\) and \(\nu(u + 1, v)\), respectively.

**Proposition 5** The derivatives of the normal vector field are given by
\[
\xi_1^- (u, v + \frac{1}{2}) = -\frac{h(u, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})} q_1(u + \frac{1}{2}, v) + \frac{A_1^+(u, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})\Omega(u - \frac{1}{2}, v + \frac{1}{2})} q_2(u, v + \frac{1}{2}),
\]
\[
\xi_2^- (u + \frac{1}{2}, v) = -\frac{h(u, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v)} q_1(u + \frac{1}{2}, v) + \frac{A_1^-(u, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v)\Omega(u - \frac{1}{2}, v + \frac{1}{2})} q_2(u + 1, v),
\]
\[
\xi_1^+ (u, v + \frac{1}{2}) = -\frac{h(u, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})} q_1(u, v + \frac{1}{2}) + \frac{A_1^+(u, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})\Omega(u - \frac{1}{2}, v + \frac{1}{2})} q_2(u + \frac{1}{2}, v),
\]
\[
\xi_2^+ (u + \frac{1}{2}, v) = -\frac{h(u, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v)} q_1(u + \frac{1}{2}, v) + \frac{A_1^-(u, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v)\Omega(u - \frac{1}{2}, v + \frac{1}{2})} q_2(u + 1, v),
\]
and
\[
\xi_2^- (u + \frac{1}{2}, v) = \frac{B_1^- (u, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v)} q_1(u, v + \frac{1}{2}) - \frac{h(u + \frac{1}{2}, v)}{\Omega(u + \frac{1}{2}, v)} q_2(u, v + \frac{1}{2}),
\]
\[
\xi_1^- (u + \frac{1}{2}, v) = \frac{B_1^+ (u, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})\Omega(u + \frac{1}{2}, v - \frac{1}{2})} q_1(u, v + \frac{1}{2}) - \frac{h(u + \frac{1}{2}, v)}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})\Omega(u + \frac{1}{2}, v - \frac{1}{2})} q_2(u, v + \frac{1}{2}),
\]
\[
\xi_2^+ (u, v + \frac{1}{2}) = \frac{B_1^- (u + \frac{1}{2}, v)}{\Omega(u + \frac{1}{2}, v - \frac{1}{2})\Omega(u + \frac{1}{2}, v + \frac{1}{2})} q_1(u, v + \frac{1}{2}) - \frac{h(u + \frac{1}{2}, v)}{\Omega(u + \frac{1}{2}, v - \frac{1}{2})\Omega(u + \frac{1}{2}, v + \frac{1}{2})} q_2(u + 1, v),
\]
\[
\xi_1^- (u, v + \frac{1}{2}) = \frac{B_1^- (u + \frac{1}{2}, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})\Omega(u + \frac{1}{2}, v - \frac{1}{2})} q_1(u + \frac{1}{2}, v) - \frac{h(u + \frac{1}{2}, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})\Omega(u + \frac{1}{2}, v - \frac{1}{2})} q_2(u + 1, v + \frac{1}{2}).
\]

**Proof.** We prove the formulas for the derivatives of \(\xi\) with respect to \(u\). The proof of the formulas for the derivatives with respect to \(v\) is similar. Since \(\xi_1^- (u, v + \frac{1}{2})\) is orthogonal to \(\nu(u, v)\) and \(\xi_1^+ (u, v + \frac{1}{2})\) is orthogonal to \(\nu(u, v + 1)\), we can write
\[
\Omega\left(u + \frac{1}{2}, v + \frac{1}{2}\right) \xi_1^- (u, v + \frac{1}{2}) = + \left[ q_2(u, v + \frac{1}{2}), \xi(u - \frac{1}{2}, v + \frac{1}{2}), \xi(u + \frac{1}{2}, v + \frac{1}{2}) \right] q_1(u, v + \frac{1}{2})
\]
\[
\quad - \left[ q_1(u + \frac{1}{2}, v), \xi(u - \frac{1}{2}, v + \frac{1}{2}), \xi(u + \frac{1}{2}, v + \frac{1}{2}) \right] q_2(u, v + \frac{1}{2}),
\]
\[
\Omega\left(u + \frac{1}{2}, v + \frac{1}{2}\right) \xi_1^- (u, v + \frac{1}{2}) = + \left[ q_2(u + \frac{1}{2}, v), \xi(u - \frac{1}{2}, v + \frac{1}{2}), \xi(u + \frac{1}{2}, v + \frac{1}{2}) \right] q_1(u + \frac{1}{2}, v)
\]
\[
\quad - \left[ q_1(u + \frac{1}{2}, v), \xi(u - \frac{1}{2}, v + \frac{1}{2}), \xi(u + \frac{1}{2}, v + \frac{1}{2}) \right] q_2(u, v + \frac{1}{2}),
\]
\[
\Omega\left(u + \frac{1}{2}, v + \frac{1}{2}\right) \xi_1^+ (u, v + \frac{1}{2}) = + \left[ q_2(u + \frac{1}{2}, v), \xi(u - \frac{1}{2}, v + \frac{1}{2}), \xi(u + \frac{1}{2}, v + \frac{1}{2}) \right] q_1(u + \frac{1}{2}, v + 1)
\]
\[
\quad - \left[ q_1(u + \frac{1}{2}, v + 1), \xi(u - \frac{1}{2}, v + \frac{1}{2}), \xi(u + \frac{1}{2}, v + \frac{1}{2}) \right] q_2(u, v + \frac{1}{2}),
\]
\[
\Omega\left(u + \frac{1}{2}, v + \frac{1}{2}\right) \xi_1^+ (u, v + \frac{1}{2}) = + \left[ q_2(u + \frac{1}{2}, v), \xi(u - \frac{1}{2}, v + \frac{1}{2}), \xi(u + \frac{1}{2}, v + \frac{1}{2}) \right] q_1(u, v + \frac{1}{2})
\]
\[
\quad - \left[ q_1(u, v + \frac{1}{2}), \xi(u - \frac{1}{2}, v + \frac{1}{2}), \xi(u + \frac{1}{2}, v + \frac{1}{2}) \right] q_2(u + 1, v + \frac{1}{2}).
\]
Now the following relations hold:

\[
[q_1(u + \frac{1}{2}, v), \xi(u + \frac{1}{2}, v + \frac{1}{2}), \xi(u - \frac{1}{2}, v + \frac{1}{2})] = \frac{A_2^+(u, v + \frac{1}{2})}{\Omega(u - \frac{1}{2}, v + \frac{1}{2})},
\]

\[
[q_1(u + \frac{1}{2}, v + 1), \xi(u + \frac{1}{2}, v + \frac{1}{2}), \xi(u - \frac{1}{2}, v + \frac{1}{2})] = \frac{A_2^+(u, v + \frac{1}{2})}{\Omega(u - \frac{1}{2}, v + \frac{1}{2})},
\]

\[
[q_1(u - \frac{1}{2}, v), \xi(u + \frac{1}{2}, v + \frac{1}{2}), \xi(u - \frac{1}{2}, v + \frac{1}{2})] = \frac{A_2^+(u, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})},
\]

\[
[q_1(u - \frac{1}{2}, v + 1), \xi(u + \frac{1}{2}, v + \frac{1}{2}), \xi(u - \frac{1}{2}, v + \frac{1}{2})] = \frac{A_2^+(u, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})}.
\]

We prove the first one, the others being similar.

\[
[q_1(u + \frac{1}{2}, v), \xi(u + \frac{1}{2}, v + \frac{1}{2}), \xi(u - \frac{1}{2}, v + \frac{1}{2})] = \frac{[q_1(u + \frac{1}{2}, v), \xi(u + \frac{1}{2}, v + \frac{1}{2}), \xi(u - \frac{1}{2}, v + \frac{1}{2})] - q_1(u - \frac{1}{2}, v)}{\Omega(u - \frac{1}{2}, v + \frac{1}{2})} = \frac{\gamma(u + \frac{1}{2}, v + \frac{1}{2})A(u, v + 1) - \gamma^{-1}(u + \frac{1}{2}, v + \frac{1}{2})A(u, v)}{\Omega(u - \frac{1}{2}, v + \frac{1}{2})}.
\]

Also,

\[
[q_2(u, v + \frac{1}{2}), \xi(u - \frac{1}{2}, v + \frac{1}{2}), \xi(u + \frac{1}{2}, v + \frac{1}{2})] = -h(u, v + \frac{1}{2}).
\]

To prove this formula, we observe that, since crosses are planar,

\[
\xi(u - \frac{1}{2}, v + \frac{1}{2}) \times \xi(u + \frac{1}{2}, v + \frac{1}{2}) = \frac{(q_1(u - \frac{1}{2}, v + 1) - q_1(u - \frac{1}{2}, v)) \times (q_1(u + \frac{1}{2}, v + 1) - q_1(u + \frac{1}{2}, v))}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})\Omega(u - \frac{1}{2}, v + \frac{1}{2})} = \frac{q_1(u + \frac{1}{2}, v) \times q_1(u + \frac{1}{2}, v + 1) + q_1(u - \frac{1}{2}, v + 1) \times q_1(u - \frac{1}{2}, v)}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})\Omega(u - \frac{1}{2}, v + \frac{1}{2})}.
\]

But

\[
[q_2(u, v + \frac{1}{2}), q_1(u + \frac{1}{2}, v), q_1(u - \frac{1}{2}, v + 1)] = \frac{p(u, v + \frac{1}{2})}{\Omega(u - \frac{1}{2}, v + \frac{1}{2})} [q_2(u, v + \frac{1}{2}), q_1(u - \frac{1}{2}, v), q_1(u - \frac{1}{2}, v + 1)] = \frac{p(u, v + \frac{1}{2})}{\Omega(u - \frac{1}{2}, v + \frac{1}{2})},
\]

and similarly

\[
[q_2(u, v + \frac{1}{2}), q_1(u + \frac{1}{2}, v + 1), q_1(u - \frac{1}{2}, v)] = \frac{[q_2(u, v + \frac{1}{2}), q_1(u + \frac{1}{2}, v + 1), q_1(u + \frac{1}{2}, v)]}{p(u, v + \frac{1}{2})\Omega(u + \frac{1}{2}, v + \frac{1}{2})} = \frac{\Omega(u + \frac{1}{2}, v + \frac{1}{2})}{p(u, v + \frac{1}{2})},
\]

thus completing the proof of the proposition.

\[\blacksquare\]

### 4.3 Affine Spheres

Which property of smooth affine spheres should we consider to define discrete affine spheres? This question has not a unique answer.

In [3], a definition of discrete affine spheres is proposed preserving the duality between $q$ and $v$. But this class does not include the hyperboloid of example 3 with a general sampling pair $(\Delta u, \Delta v)$. This hyperboloid satisfies the definition of [3] if and only if $\Delta u = \Delta v$.

We propose here a new definition of discrete affine sphere that includes the definition of [3] as a particular case. With this new definition, the hyperboloid of example 3 with any sampling pair $(\Delta u, \Delta v)$ becomes an affine sphere. On the other hand, we lose the duality property.
We say that the asymptotic net is an affine sphere if
\[
\frac{A(u, v + 1)}{\gamma(u + \frac{1}{2}, v + \frac{1}{2})} = \frac{A(u, v)}{\gamma(u + \frac{1}{2}, v + \frac{1}{2})},
\]
\[
\frac{B(u + 1, v)}{\gamma(u + \frac{1}{2}, v + \frac{1}{2})} = \frac{B(u, v)}{\gamma(u + \frac{1}{2}, v + \frac{1}{2})}.
\]
These conditions imply that
\[
\xi_1^-(u, v + \frac{1}{2}) = \frac{-h(u, v + \frac{1}{2})p(u, v + \frac{1}{2})}{1 + p(u, v + \frac{1}{2})}, \quad \xi_1^+(u, v + \frac{1}{2}) = \frac{-h(u, v + \frac{1}{2})p(u, v + \frac{1}{2})}{1 + p(u, v + \frac{1}{2})},
\]
\[
\xi_2^-(u + \frac{1}{2}, v) = \frac{-h(u + \frac{1}{2}, v)p(u + \frac{1}{2}, v)}{1 + p(u + \frac{1}{2}, v)}, \quad \xi_2^+(u + \frac{1}{2}, v) = \frac{-h(u + \frac{1}{2}, v)p(u + \frac{1}{2}, v)}{1 + p(u + \frac{1}{2}, v)}.
\]

We remark that an affine sphere as defined above is also an affine sphere as defined in [3] if and only if
\[
c\Omega(u + \frac{1}{2}, v + \frac{1}{2})\gamma(u + \frac{1}{2}, v + \frac{1}{2}) = 1 - \gamma^2(u + \frac{1}{2}, v + \frac{1}{2}),
\]
for some constant \(c\).

5 Compatibility Equations
The first compatibility equation is given by
\[
\frac{\Omega(u - \frac{1}{2}, v + \frac{1}{2})p^{-1}(u, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})} = \frac{p(u, v - \frac{1}{2})\Omega(u - \frac{1}{2}, v - \frac{1}{2})}{\Omega(u + \frac{1}{2}, v - \frac{1}{2})} + \frac{A(u, v)B(u, v)\gamma(u + \frac{1}{2}, v - \frac{1}{2})\gamma^{-1}(u + \frac{1}{2}, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})\Omega(u + \frac{1}{2}, v - \frac{1}{2})}.
\]

There are two other compatibility equations, as follows:
\[
\frac{\gamma(u - \frac{1}{2}, v - \frac{1}{2})h(u, v - \frac{1}{2})}{\Omega(u + \frac{1}{2}, v - \frac{1}{2})} = \frac{h(u, v + \frac{1}{2})\gamma^{-1}(u - \frac{1}{2}, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})} = \frac{B_1^+(u + \frac{1}{2}, v)\gamma^{-1}(u + \frac{1}{2}, v - \frac{1}{2})}{\Omega(u + \frac{1}{2}, v - \frac{1}{2})\Omega(u + \frac{1}{2}, v + \frac{1}{2})} - \frac{\gamma(u - \frac{1}{2}, v - \frac{1}{2})B_1^+(u - \frac{1}{2}, v)}{\Omega(u - \frac{1}{2}, v - \frac{1}{2})\Omega(u - \frac{1}{2}, v + \frac{1}{2})}
\]
and
\[
\frac{\gamma(u - \frac{1}{2}, v - \frac{1}{2})h(u - \frac{1}{2}, v)}{\Omega(u + \frac{1}{2}, v - \frac{1}{2})} = \frac{h(u + \frac{1}{2}, v)\gamma^{-1}(u - \frac{1}{2}, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})} = \frac{A_1^+(u, v + \frac{1}{2})\gamma^{-1}(u - \frac{1}{2}, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v - \frac{1}{2})\Omega(u - \frac{1}{2}, v + \frac{1}{2})} - \frac{\gamma(u - \frac{1}{2}, v + \frac{1}{2})A_1^+(u, v - \frac{1}{2})}{\Omega(u + \frac{1}{2}, v - \frac{1}{2})\Omega(u - \frac{1}{2}, v - \frac{1}{2})}
\]

The proofs of these equations are left to the appendix.

Theorem 6 Given function \(\Omega\), \(H\), \(A\) and \(B\) satisfying the compatibility equations, there exists an asymptotic net \(q\) with quadratic form \(\Omega\), mean curvature \(H\) and cubic form \(A\partial u^3 + B\partial v^3\). Moreover, two asymptotic nets with the same quadratic and cubic forms are affine equivalent.

Proof. We begin by choosing four points \(q(0, 0), q(1, 0), q(0, 1)\) and \(q(1, 1)\) satisfying
\[
[q(1, 0) - q(0, 0), q(0, 1) - q(0, 0), q(1, 1) - q(0, 0)] = \Omega^2(u, v).
\]
This four points are determined up to an affine transformation of \(\mathbb{R}^3\).

From a quadrangle \((u - \frac{1}{2}, v - \frac{1}{2})\), one can extend the definition of \(q\) to the quadrangles \((u + \frac{1}{2}, v - \frac{1}{2})\) and \((u - \frac{1}{2}, v + \frac{1}{2})\) by the formulas of subsection 4.1. With these extensions, we can calculate \(\xi(u + \frac{1}{2}, v - \frac{1}{2})\) and \(\xi(u - \frac{1}{2}, v + \frac{1}{2})\). It is clear

\[
\gamma(u, v + 1) = \frac{\gamma(u + \frac{1}{2}, v + \frac{1}{2})}{\gamma(u + \frac{1}{2}, v + \frac{1}{2})} = \frac{A(u, v)}{\gamma(u + \frac{1}{2}, v + \frac{1}{2})},
\]
\[
\gamma(u + 1, v) = \frac{\gamma(u, v + \frac{1}{2})}{\gamma(u, v + \frac{1}{2})} = \frac{B(u, v)}{\gamma(u, v + \frac{1}{2})}.
\]
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Appendix: Proofs of Compatibility Equations

To prove the first compatibility equation, we calculate \( q_{112}(u, v + \frac{1}{2}) \) in two different ways. Since \( q_{12}(u + \frac{1}{2}, v + \frac{1}{2}) = \Omega(u + \frac{1}{2}, v + \frac{1}{2})\xi(u + \frac{1}{2}, v + \frac{1}{2}), \) we write

\[
q_{112}(u, v + \frac{1}{2}) = (\Omega(u + \frac{1}{2}, v + \frac{1}{2}) - p(u, v + \frac{1}{2})\Omega(u - \frac{1}{2}, v + \frac{1}{2})) \xi(u + \frac{1}{2}, v + \frac{1}{2}) + \Omega(u + \frac{1}{2}, v + \frac{1}{2})\xi_1(u, v + \frac{1}{2}).
\]

From equations of subsection 4.2 we conclude that the coefficient of \( q_1(u + \frac{1}{2}, v) \) of the expansion of \( q_{112}(u, v + \frac{1}{2}) \) in the basis \( \{q_1(u + \frac{1}{2}, v), q_2(u, v + \frac{1}{2}), \xi(u + \frac{1}{2}, v + \frac{1}{2})\} \) is

\[
- h(u, v + \frac{1}{2}) \frac{\Omega(u - \frac{1}{2}, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})}.
\]

On the other hand, \( q_{112}(u, v + \frac{1}{2}) = q_{11}(u, v + 1) - q_{11}(u, v) \), with

\[
q_{11}(u, v + 1) = \left( 1 - \frac{p(u, v + \frac{1}{2})\Omega(u - \frac{1}{2}, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})} \right) q_1(u + \frac{1}{2}, v + 1) + \frac{\gamma(u + \frac{1}{2}, v + \frac{1}{2})A(u, v + 1)}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})} q_2(u, v + \frac{1}{2})
\]

\[
q_{11}(u, v) = \left( 1 - \frac{p(u, v - \frac{1}{2})\Omega(u - \frac{1}{2}, v - \frac{1}{2})}{\Omega(u + \frac{1}{2}, v - \frac{1}{2})} \right) q_1(u + \frac{1}{2}, v) + \frac{\gamma(u + \frac{1}{2}, v - \frac{1}{2})A(u, v)}{\Omega(u + \frac{1}{2}, v - \frac{1}{2})} q_2(u, v - \frac{1}{2}).
\]

Since

\[
q_1(u + \frac{1}{2}, v + 1) = q_1(u + \frac{1}{2}, v) + \Omega(u + \frac{1}{2}, v + \frac{1}{2})\xi(u + \frac{1}{2}, v + \frac{1}{2})
\]

and

\[
q_2(u, v - \frac{1}{2}) = - \gamma^{-1}(u + \frac{1}{2}, v + \frac{1}{2})B(u, v) \xi(u + \frac{1}{2}, v + \frac{1}{2}) + \frac{p^{-1}(u, v + \frac{1}{2})\Omega(u - \frac{1}{2}, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})} q_2(u, v + \frac{1}{2}),
\]
the coefficient of \( q_1(\frac{u + \frac{1}{3}}{2}, v) \) of the expansion of \( q_{112}(u, v + \frac{1}{2}) \) in the basis
\[ \{ q_1(\frac{u + \frac{1}{3}}{2}, v), q_2(u, v + \frac{1}{2}), \xi(u + \frac{1}{3}, v + \frac{1}{2}) \} \] is
\[
- \frac{p(u, v + \frac{1}{2})\Omega(u - \frac{1}{2}, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})} + \frac{p(u, v - \frac{1}{2})\Omega(u - \frac{1}{2}, v - \frac{1}{2})}{\Omega(u + \frac{1}{2}, v - \frac{1}{2})} + \frac{\gamma^{-1}(u + \frac{1}{2}, v + \frac{1}{2})\gamma(u + \frac{1}{2}, v - \frac{1}{2})A(u, v)B(u, v)}{\Omega(u + \frac{1}{2}, v - \frac{1}{2})\Omega(u + \frac{1}{2}, v + \frac{1}{2})} \tag{8}
\]
The first compatibility equation follows directly from (7) and (8).

To prove the second and third compatibility equation, we observe that
\[
\frac{\xi_1^-(u, v + \frac{1}{2})}{\gamma(u - \frac{1}{2}, v + \frac{1}{2})} - \frac{\xi_1^+(u, v - \frac{1}{2})}{\gamma(u + \frac{1}{2}, v - \frac{1}{2})} = \frac{\xi_2^-(u + \frac{1}{2}, v)}{\gamma(u + \frac{1}{2}, v - \frac{1}{2})} - \frac{\xi_2^+(u - \frac{1}{2}, v)}{\gamma(u - \frac{1}{2}, v + \frac{1}{2})}.
\]
Using the formulas of subsection 4.2 and comparing the coefficients of \( q_1(u + \frac{1}{3}, v) \), we obtain the second compatibility equation. The third compatibility equation is obtained by comparing the coefficients of \( q_2(u, v + \frac{1}{2}) \).