Ergodic actions of $S_\mu U(2)$ on $C^*$–algebras from $II_1$ subfactors

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Abstract

To a proper inclusion $N \subset M$ of $II_1$ factors of finite Jones index $[M : N]$, we associate an ergodic $C^*$–action of the quantum group $S_\mu U(2)$ (or more generally of certain groups $A_o(F)$). The higher relative commutant $N' \cap M_{r-1}$ can be identified with the spectral space of the $r$-th tensor power $u^\otimes r$ of the defining representation of the quantum group. The index and the deformation parameter are related by $-1 \leq \mu < 0$ and $[M : N] = |\mu + \mu^{-1}|$.

This ergodic action may be thought of as a virtual subgroup of $S_\mu U(2)$ in the sense of Mackey arising from the tensor category generated by the $N$–bimodule $N M_N$. $\mu$ is negative as $N M_N$ is a real bimodule.

Keywords: compact quantum groups, ergodic actions, $II_1$ subfactors

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Subj. Class.: quantum groups, noncommutative topology and geometry, dynamical systems

1 Introduction

Ergodic theory of compact quantum groups on unital $C^*$–algebras is the noncommutative analogue of the theory of homogeneous spaces over compact groups. Several examples are known, showing that this theory exhibits new aspects with respect to the classical case.

The first interesting examples, the quantum spheres, have been constructed by Podleś [33], and the theory has been continued by Boca [8]. While in the commutative case, homogeneous spaces are quotients by closed subgroups, some of Podleś examples show that is no longer true in the noncommutative situation. Hence, an ergodic $C^*$–action of a compact quantum group may be regarded as a virtual subgroup, introduced by Mackey in classical ergodic theory [22, 23], but in a compact and noncommutative setting.
Whereas the invariant state of a $C^*$-algebra carrying an ergodic action of a compact group is necessarily tracial \[15\], this is not the case of the ergodic actions of the van Daele-Wang quantum groups on type $III_\lambda$ factors constructed by \[42\]. Further examples of ergodic actions have been given in \[5\] starting from tensor equivalences of representation categories of compact quantum groups.

An axiomatization of the spectral functor associated to an ergodic $C^*$-action has been given in \[30\], where such functors were called quasitensor.

The aim of this paper is to establish a connection between the ergodic theory of compact quantum groups on unital $C^*$-algebras and the Jones theory of subfactors.

In his pioneering work, Jones showed that the theory of $II_1$ subfactors with finite index can be understood as a generalized group theory, exhibiting beautiful representations of the infinite braid group factoring through the Temperley-Lieb relations. A well known consequence is that Ocneanu’s bimodule category associated with the inclusion, contains, roughly speaking, a deformation of the representation category of $SU(2)$ generated by the flip map \[17\], \[18\], \[13\]. But as this deformed category is not associated with representations of the symmetric groups, this bimodule category is not described by a group, nor even by a quantum group, in general.

Our main point is that a finite index inclusion of $II_1$ factors always gives rise to an ergodic action of $SU(2)$ for suitable negative values of $\mu$. To understand why the values are negative note that the defining representation of $SU(2)$ is selfconjugate and pseudoreal whilst $M$, as a $N$-bimodule, is selfconjugate and real. However, the defining representation of the quantum group $S_\mu U(2)$ of Woronowicz \[51\] is real provided the deformation parameter is negative (see, e.g., \[29\]).

More precisely, we canonically associate to any proper inclusion of $II_1$ factors with finite index, an ergodic action of the van Daele–Wang quantum groups $A_\alpha(F)$ on unital $C^*$-algebras, where the spectral spaces of the action are the higher relative commutants $N' \cap M_r$. $F$ is subject to the conditions $FF^* = I$ and $\text{Tr}(F^*F) = [M : N]$. By a result of Banica \[2\] (see also \[30\]), whilst the quantum group is not unique, its representation category is determined by the above conditions up to a tensor isomorphism. In particular, choosing $F$ of minimal rank, yields ergodic actions of $S_\mu U(2)$ where the index and the deformation parameter are related by $-1 \leq \mu < 0$ and $|\mu + \mu^{-1}| = [M : N]$. On the other hand, when the index is an integer, the identity matrix of rank $[M : N]$ is a natural choice, providing an ergodic action of the Kac type quantum group $A_\alpha([M : N])$. In this case, we show that the invariant state of the associated $C^*$-algebra is tracial.

Taking Wenzl’s work \[47\], \[48\], \[49\] on constructing subfactors from algebraic quantum groups into account and passing through subfactors we connect quantum groups at roots of unity to ergodic actions of compact quantum groups whose quantum dimension depends on roots of unity. Thus quantum groups at roots of unity would seem to be virtual quantum subgroups of compact quantum groups.

However, the examples of Asaeda–Haagerup \[1\] show that not all subfac-
tors are associated with quantum groups. Correspondingly, we give examples of ergodic actions of $S_\mu U(2)$, for $\mu < 0$, not arising from quantum group constructions. This is a novelty compared to the group case, as Wassermann has shown that all ergodic actions of $SU(2)$ arise from closed subgroups and their irreducible projective representations via induction constructions [45].

Our approach uses the duality theorem of [30]. We start from the remark that the most important axiom of that theorem (see (2.4)) is in analogy with the commuting square condition of [35], [26], which plays a key role in Jones’s theory of subfactors. This analogy raises the question of whether this theory yields quasitensor functors and hence ergodic $C^*$-actions of compact quantum groups.

The first step of our construction is to show that if $\mu$ is suitably chosen, the full tensor $C^*$-subcategory of $\text{Rep}(S_\mu U(2))$ generated by the fundamental representation embeds into the $SU(2)$-like category contained in the bimodule tensor category generated by $NM_N$.

In particular, from the pivotal result of Jones restricting the values of the index [17], only the values $4 \cos^2 \pi/m$, $m \geq 4$ and $\geq 4$ of the quantum dimension of $S_\mu U(2)$ can possibly arise in our examples.

Because of finiteness of the factors in question, the quantum multiplicity of $u^\otimes r$ in the ergodic action, where $u$ is the defining representation of $A_o(F)$, takes its lower bound, the integral multiplicity, in turn given by the dimension of the higher relative commutant $N' \cap M_{r-1}$. Moreover the spectral space corresponding to $u^\otimes r$ with its Hilbert space structure can be identified with the higher relative commutant $N' \cap M_{r-1}$, with inner product coming from the Markov trace.

Furthermore these ergodic actions are not, in general, embedable into the translation action. We shall show this for non-integral values of the index of an extremal and amenable inclusion. The proof relies on Popa’s work [36].

We show in [31] that the results of this paper have analogues when subfactors are replaced by tensor $C^*$-categories with conjugation. The groups involved are $A_o(F)$ and $A_o(F)$. Such categories, arise, in particular, in the algebraic approach to QFT where they are in addition endowed with a unitary braided symmetry, (see, e.g. [14]).

In a subsequent paper, we will adopt Mackey’s point of view that a non-transitive ergodic action can be viewed as a virtual subgroup which should thus exhibit typical properties of a closed subgroup. We will develop a theory of induction and restriction for representations of these virtual subgroups [32].

The paper is organized as follows. In section 2 we recall the main invariants of ergodic $C^*$-actions and the duality theorem of [30].

In section 3 we recall Ocneanu’s bimodules associated with an inclusion of $II_1$ factors with finite Jones index and we show that extremal and amenable inclusions in the sense of [26], [36] give rise to categories which, for non-integral values of the index, are not embedable into the category of Hilbert spaces. When they are, only quantum groups with coinvolutive coinverses appear. Hence Ocneanu’s categories associated to amenable inclusions are, in this respect, rather
different from the representation categories of compact quantum groups, which are embedded in Hilbert spaces by construction, but often give rise to non-amenable inclusions.

In section 4 we state our main results: the inclusion gives rise to ergodic $C^*$-actions of the van Daele-Wang compact quantum group $A_o(F)$ associated with an invertible matrix $F \in M_n(\mathbb{C})$.

Section 5 is devoted to the proofs of the results. We conclude the paper with the necessary computations to yield a presentation of the dense $*$-subalgebra of spectral elements by generators and relations, in terms of the higher relative commutants $N' \cap M_r$.

2 Preliminaries on ergodic $C^*$-actions

Consider a unital $C^*$-algebra $\mathcal{C}$ and a compact quantum group $G = (\mathcal{Q}, \Delta)$, with coproduct $\Delta : \mathcal{Q} \to \mathcal{Q} \otimes \mathcal{Q}$ [52]. An action of $G$ on $\mathcal{C}$ is a unital $*$-homomorphism $\delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{Q}$ satisfying the group representation property:

$$\iota \otimes \Delta \circ \delta = \delta \otimes \iota \circ \delta,$$

and the nondegeneracy property requiring $\delta(\mathcal{C})I \otimes \mathcal{Q}$ to be dense in $\mathcal{C} \otimes \mathcal{Q}$. The spectrum of $\delta$, sp($\delta$), is defined to be the set of all representations $u$ of $G$ for which there is a faithful linear map $T : H_u \to \mathcal{C}$ intertwining the representation $u$ with the action $\delta$:

$$\delta \circ T = T \otimes \iota \circ u.$$

In other words, if $u_{ij}$ are the coefficients of $u$ corresponding to some orthonormal basis of $H_u$, we are requiring the existence of linearly independent elements $c_1, \ldots, c_d \in \mathbb{C}$, with $d$ the dimension of $u$, transforming like $u$ under the action:

$$\delta(c_i) := \sum_j c_j \otimes u_{ji}.$$ The linear span of all the $c_i$'s, denoted $\mathcal{C}_{sp}$, as $u$ varies in the spectrum, is a dense $*$-subalgebra of $\mathcal{C}$ [54].

The action $\delta$ is called ergodic if the fixed point algebra

$$\mathcal{C}^\delta = \{ c \in \mathcal{C} : \delta(c) = c \otimes I \}$$

reduces to the complex numbers: $\mathcal{C}^\delta = \mathbb{C}I$. The simplest example of an ergodic action is the translation action of $G$ on $\mathcal{C} = \mathbb{C}$ with $\delta = \Delta$.

If an action $\delta$ is ergodic, spectral multiplets can be organized to form Hilbert spaces. In fact, for any representation $u$, consider the space

$$L_u := \{ T : H_u \to \mathcal{C}, \delta \circ T = T \otimes \iota \circ u \}.$$ If $S, T \in L_u$, $\langle S, T \rangle := \sum_i T(\psi_i)S(\psi_i)^*$, with $(\psi_i)$ an orthonormal basis of $H_u$, is an element of the fixed point algebra $\mathcal{C}^\delta$, and therefore a complex number. $L_u$ is known to be finite dimensional, and therefore a Hilbert space with the above inner product. This Hilbert space is nonzero precisely when $u$ contains a subrepresentation $v \in \text{sp}(\delta)$. In particular, for an irreducible $u$, the
conditions \( u \in \text{sp}(\delta) \) and \( L_u \neq 0 \) are equivalent. The dimension of \( L_u \) is called the \textit{multiplicity} of \( u \) and denoted \( \text{mult}(u) \).

The complex conjugate vector space \( L_u \), endowed with the conjugate inner product

\[
< S, T > := \sum_i S(\psi_i)T(\psi_i)^*,
\]

is called the \textit{spectral space} associated with \( u \).

If for example \( \delta \) is the translation action on \( Q \), any \( \psi \in H_u \) defines an element of \( L_u \) by

\[
T \psi(\psi') := \psi^* \otimes u(\psi').
\]

Hence the spectral space \( L_u \) can be identified with \( H_u \) through the unitary map \( \psi \in H_u \rightarrow T \psi \in L_u \).

If \( j : H_u \rightarrow H_u \) is an antilinear invertible defining a conjugate (unitary) representation \( u \) of \( u \) by \( u_{\phi,j \psi} = (u_{j, \phi, \psi})^* \), then there is an associated antilinear \( J : L_u \rightarrow L_u \) by \( J(T)(\phi) := T(j^{-1}(\phi))^* \) with inverse \( J^{-1} : L_u \rightarrow L_u \) given by \( J^{-1}(S)(\psi) = S(j(\psi))^* \). If \( u \) is irreducible, the \textit{quantum multiplicity} \( m(u) \) of \( u \) is defined by

\[
m(u)^2 := \text{Trace}(JJ^*) \text{Trace}((JJ^*)^{-1}) \]

One has:

\[
\text{mult}(u) \leq m(u) \leq d(u),
\]

an inequality which strengthens the inequality \( \text{mult}(u) \leq d(u) \), with \( d(u) \) the quantum dimension of \( u \), previously obtained by Boca \[8\], in turn generalizing HLS theorem \[15\] \( \text{mult}(u) \leq \dim(u) \) in the group case. If \( u \) is reducible, we define \( m(u) \) as the infimum of all the above trace values, when \( j \) ranges over all possible solutions of the conjugate equations. Then the inequality

\[
\dim(L_u) \leq m(u) \leq d(u)
\]

now holds for all representations \( u \). Notice that \( m(u) \) takes the smallest possible value \( \dim(L_u) \) precisely when for some \( j \) the associated \( J \) is a scalar multiple of an antiunitary. The actions we shall construct in this paper satisfy this property. Examples of ergodic actions of \( S_n U(2) \) where \( \dim(u) < \text{mult}(u) < m(u) = d(u) \) have been constructed in \[9\].

The study of categorical aspects of ergodic \( C^* \)-actions of compact quantum groups has been developed in generality in \[30\], where a spectral characterization has been obtained. It has been shown that the \textit{spectral functor} of an ergodic \( G \)-action is a dual object, in the sense that the \( G \)-action on the maximal completion of \( \mathcal{C}_\text{sp} \) can be reconstructed from it. Furthermore, spectral functors of ergodic \( C^* \)-actions of compact quantum groups are characterized among all \( * \)-functors from \( \text{Rep}(G) \) to the category of Hilbert spaces, by the property of being \textit{quasitensor}. More in detail, the spectral functor

\[
\mathcal{T} : \text{Rep}(G) \rightarrow \mathcal{H}
\]
associated with an ergodic $G$–action is the functor from the category of representations of $G$ to the category $\mathcal{H}$ of Hilbert spaces. This functor is defined as $u \rightarrow \overline{L_u}$ on objects and as follows on arrows.

If $A \in (u, v)$ and $T \in L_u$ then $T \circ A : H_u \rightarrow \mathcal{C}$ lies in $L_u$. Hence if we identify $\overline{L_u}$ canonically with the dual vector space of $L_u$, any arrow $A \in (u, v)$ in $\text{Rep}(G)$ induces a linear map $\overline{L_A} \in (\overline{L_u}, \overline{L_v})$ by

$$\overline{L_A} : \varphi \in \overline{L_u} \rightarrow (T \in L_v \rightarrow \varphi(T \circ A)) \in \overline{L_v}.$$ 

Taking into account the tensor $C^*$–category structure of $\text{Rep}(G)$ and $\mathcal{H}$ one can see that $\overline{L}$ becomes a $^*$–functor, but not a tensor $^*$–functor, in general.

As far as the tensor structure of $\overline{L}$ is concerned, for $u, v \in \text{Rep}(G)$, the tensor product Hilbert space $\overline{L_u \otimes L_v}$ is in general just a subspace of $\overline{L_u} \otimes \overline{L_v}$, in the sense that there is a natural isometric inclusion

$$\lambda_{u,v} : \overline{L_u} \otimes \overline{L_v} \rightarrow \overline{L_u \otimes L_v}$$

identifying any simple tensor $S \otimes T$ with the complex conjugate of the element of $\overline{L_u \otimes L_v}$ defined by

$$\psi \otimes \phi \in H_u \otimes H_v \rightarrow S(\psi)T(\phi).$$

The dual of the action is the pair $(\overline{L}, \lambda)$ consisting of the functor $\overline{L}$ and all the inclusions $\lambda_{u,v}$.

The maximal $C^*$–completion of $\mathcal{C}_\mathbb{P}$, together with the extended $G$–action, can be reconstructed from the dual $(\overline{L}, \lambda)$.

The main result of [30] is an axiomatization of the set of all duals of ergodic actions $(\overline{L}, \lambda)$ among all $^*$–functors

$$\tau : \text{Rep}(G) \rightarrow \mathcal{H}$$

endowed with isometries $\tilde{\tau}_{u,v} : \tau_u \otimes \tau_v \rightarrow \tau_{u \otimes v}$. All pairs $(\tau, \tilde{\tau})$ satisfying properties (3.1)–(3.6) in [30] have been shown to arise as the dual of an ergodic $G$–action. Such functors were called quasitensor. In [31] the following equivalent simpler axiomatization has been derived:

$$\tau_\iota = \iota, \quad (2.2)$$

$$\tilde{\tau}_{u,t} = \tilde{\tau}_{t,u} = 1_{\tau_u}, \quad (2.3)$$

$$\tilde{\tau}_{u,v \otimes w} \circ \tilde{\tau}_{u \otimes v,w} = 1_{\tau_u \otimes \tau_v \otimes \tau_w}, \quad (2.4)$$

$$\tau(S \otimes T) \circ \tilde{\tau}_{u,v} = \tilde{\tau}_{u',v'} \circ \tau(S) \otimes \tau(T), \quad (2.5)$$

for any other pair of objects $u'$, $v'$ and arrows $S \in (u, u')$, $T \in (v, v')$. In particular, a tensor functor $\tau$ is quasitensor with $\tilde{\tau}_{u,v} := 1_{\tau_u \otimes \tau_v}$. An ergodic $C^*$–action of $G$ on a unital $C^*$–algebra $\mathcal{C}$ can be constructed by duality from a quasitensor $^*$–functor $(\tau, \tilde{\tau}) : \text{Rep}(G) \rightarrow \mathcal{H}$. Once the ergodic action has been constructed, the pair $(\tau, \tilde{\tau})$ can be identified with the dual object of that action.
3 Ocneanu’s category from a \( II_1 \) inclusion

Consider an inclusion of \( II_1 \) factors \( N \subset M \) of finite Jones index \([M : N]\) \[17\] and denote the trace-preserving conditional expectation by \( E : M \to N \). Let

\[
N \subset M \subset M_1 \subset M_2 \ldots
\]

be the Jones tower of \( II_1 \) factors. We denote the \( r \)-th Jones projection derived from the trace-preserving conditional expectation \( E_{r-1} : M_{r-1} \to M_{r-2} \) by \( e_r \in L^2(M_{r-1}) \), where we have set \( M_0 = M \), \( M_{-1} = N \), \( E_0 = E \). Recall that \( M_r := M_{r-1} e_r M_{r-1} \) and that \( E_r(e_r) = [M : N]^{-1} \). Also recall that the algebras \( N' \cap M_r \), usually called the higher relative commutants, are finite dimensional and

\[
\dim(N' \cap M_{r-1}) \leq [M : N]^r, \quad r \geq 0.
\]

The main relations are the following. For \( r \geq -1 \),

\[
[M_r, e_{r+2}] = 0, \quad (3.1)
\]

\[
e_{r+1} m e_{r+1} = E_r(m) e_{r+1}, \quad m \in M_r, \quad (3.2)
\]

implying the Jones projection relations:

\[
e_i e_j = e_j e_i, \quad |i - j| \geq 2, \quad (3.3)
\]

\[
e_i e_{j \pm 1} e_i = [M : N]^{-1} e_i. \quad (3.4)
\]

We review the well known construction of the Jones tower, \( M_r, r \geq 0 \), in terms of Ocneanu’s bimodules \([25, 26, 27, 7]\). Regard \( M \) as a right Hilbert \( N \)-module with \( N \)-valued inner product

\[
\langle m, m' \rangle := E(m^* m'), \quad m, m' \in M.
\]

Since the index is finite, \( M \) is finitely generated over \( N \). Left multiplication on \( M \) by elements of \( N \) makes \( M \) into a Hilbert bimodule in the sense considered in \([28]\). Therefore we can take tensor powers of \( M \) over \( N \) and get further Hilbert bimodules. When no confusion arises, this tensor product will be simply denoted by \( \otimes \). As \( N \)-bimodules:

\[
M \otimes M \simeq M_1.
\]

Therefore iteratively, for \( r = 1, 2, \ldots \),

\[
M_{r-1} \simeq M^\otimes r. \quad (3.5)
\]

Consider the category \( \mathcal{T}_M \) with objects the \( N \)-bimodules \( M^\otimes r, r \geq 0 \), and arrows the bimodule mappings. This is a tensor \( C^* \)-category in the sense of \([11]\), with tensor product structure on arrows naturally induced by the tensor product of Hilbert bimodules. The arrow space \((i, M^\otimes r)\) can be identified, as a vector space, with \( N' \cap M_{r-1} \). In particular, \((i, M) \simeq N' \cap M \neq 0\) and is one-dimensional precisely when the inclusion is irreducible. These observations
allow to show that \( \mathcal{T}_M \), as a tensor \( C^* \)-category, is determined by an isometry \( S \in (\iota, M) \) and its subcategory of arrow spaces \( (M^{\otimes r}, M^{\otimes r}) \) with same source and range objects.

Identifying the \( N \)-bimodules \( M^{\otimes r} \simeq M_{r-1} \), the algebra \( \mathcal{L}_N(M^{\otimes r}) \) of right \( N \)-module maps can be identified with Jones’s basic construction associated with the inclusion \( N \subset M_{r-1} \), which, in turn, can be identified with \( M_{2r-1} \). The Jones projection of this inclusion is given by

\[
f_{r-1} := [M : N]^{(r-1)/2}(e_r \ldots e_1)(e_{r+1} \ldots e_2) \ldots (e_{2r-1} \ldots e_r)
\]

Therefore as \( C^* \)-algebras,

\[
(M^{\otimes r}, M^{\otimes r}) \simeq N' \cap M_{2r-1}.
\]

Only terms of the Jones tower with odd indices appear as we started with the bimodule \( N_M \) rather than \( \sigma := M_M \) or \( \sigma := N_M \). One has \( N_M \simeq \sigma \otimes_M \sigma \). It is well known and easy to check that tensoring on the right by \( 1_M \), namely \( T \in (M^{\otimes r}, M^{\otimes r}) \to T \otimes 1_M \in (M^{\otimes r+1}, M^{\otimes r+1}) \) corresponds to the natural inclusion \( N' \cap M_{2r-1} \subset N' \cap M_{2r+1} \), whereas tensoring on the left by \( 1_M \) corresponds to Ocneanu’s canonical shift \( \Gamma : N' \cap M_{2r-1} \to N' \cap M_{2r+1} \).

We next give a result showing that inclusions of factors often provide examples of categories \( \mathcal{T}_M \) that cannot arise from representations of compact quantum groups.

In nongeneric cases, where \( \mathcal{T}_M \) is embedable, it must generate the category of representations of a compact quantum group with involutive coinverse.

**3.1 Proposition** Let \( N \subset M \) be a finite index, extremal and amenable inclusion of \( II_1 \) factors. If \( [M : N] \) is not an integer then the tensor \( C^* \)-category \( \mathcal{T}_M \) cannot be embedded into the category of Hilbert spaces. Conversely, if \( [M : N] \) is an integer and if \( \mathcal{T}_M \) is embedable then the Hilbert space corresponding to \( \mathcal{T}_M \) is isomorphic to the representation category of a compact quantum subgroup of the compact Kac quantum group \( A_A(I_{[M:N]}) \) where \( I_{[M:N]} \) is the identity matrix of size \( [M : N] \).

**Proof** Popa shows that, under the amenability assumption,

\[
[M : N] = \lim_r \dim(N' \cap M_r)^{1/r}
\]

(see Theorem 4.4.1(3) in [36]). Therefore if \( [M : N] \) is not an integer, for sufficiently large \( r \),

\[
\dim(\iota, M^{\otimes r}) = \dim(N' \cap M_{r-1}) > i^r
\]

where \( i \) is the integral part of the index. If \( \mathcal{T}_M \) were embedable and if \( n \) denotes the dimension of the Hilbert space corresponding to \( M \) then we must have \( n \leq [M : N] \). Hence \( n \leq i \). Furthermore for all \( r \) we should also have

\[
\dim(\iota, M^{\otimes r}) \leq n^r \leq i^r,
\]

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a contradiction. Conversely, assume that $[M : N] = i$ and $\mathcal{T}_M$ embedable. If the Hilbert space corresponding to $M$ had dimension $n < i$ then

$$i = \lim_r \dim(\iota, M^{\otimes r})^{1/r} \leq n < i,$$

again a contradiction. Therefore the dimension of the Hilbert space corresponding to $M$ is uniquely determined by the index. The intertwiner $R$ needs then to correspond to an element $R' \in H \otimes H$ with $\|R'\|^2 = [M : N] = \dim(H)$.

We will show later that $M$ is a real object in the sense recalled at the beginning of Sect. 5 (see Theorem 5.2), hence $R'$ makes $H$ into a real object with $\|R'\|^2 = \dim(H)$. This equality is possible only if $R' = \sum_i e_i \otimes e_i$ for some orthonormal basis of $H$, and the proof is complete.

**Remark** A notion of amenability for an object $\rho$ of a tensor $C^*$–category is introduced in [21]. It implies $d(\rho) = \lim_r \dim(\rho^r, \rho^r)^{1/2r}$. As here, it is shown that $\mathcal{T}_\rho$ is not embedable unless $d(\rho)$ is an integer and that when it is then $d(\rho) = \dim(H)$.

We illustrate the previous proposition with some known examples.

a) **Fixed point and crossed products inclusions.** The basic examples of inclusions with integer index are those arising from an outer action of a finite group $G$ on a $II_1$ factor via fixed point algebras or crossed products. The index is $|G|$.

In the fixed point algebra case, $N = M^G \subset M$, it is well known that $M_1$ can be identified with the crossed product $M \rtimes G$ and $N' \cap M_1$ with $CG$ [17].

This inclusion is irreducible and has depth 2, the higher terms of the chain $N' \cap N \subset N' \cap M \subset N' \cap M_1 \subset \ldots$ are determined as in 4.7 a), [13]. It is well known that the category $\mathcal{T}_M$ is described by the representation theory of $G$. It is also well known that this generalizes to finite dimensional Kac algebras and that, by a result of Ocneanu, any irreducible finite index depth 2 inclusion arises in this way (see [38]).

b) **Bisch–Haagerup inclusions.** Recall that these inclusions are obtained composing fixed point subfactors with crossed products subfactors: $N = P^H \subset P \rtimes K = M$ where $H$ and $K$ are finite groups acting properly and outerly on the $II_1$ factor $P$. Recall from [6] the following results: $N \subset M$ is always extremal, with integer index given by $|H||K|$, the associated graph is amenable if and only if the group $G$ generated by $H$ and $K$ in $\text{Out}(P)$ is amenable, $N \subset M$ is of finite depth if and only if $G$ is finite. Irreducible depth 2 inclusions correspond to matched pair of groups: $G = H K$, $H \cap K = \{e\}$. The corresponding Kac algebras have been identified in [10]. Ocneanu’s duality has been generalized by Nikshych and Vainerman to reducible depth 2 inclusions. They proved that in this case finite dimensional weak Hopf algebras (or quantum groupoids) in the sense of [9] replace Kac algebras [24]. Vallin then proved that relative matched pairs (i.e. $G = H \ltimes K$ but $H \cap K$ is not required to be trivial) give rise to such inclusions, and hence to weak Hopf algebras [40].

c) Known classes of extremal and strongly amenable [36] (hence amenable) inclusions are given by: subfactors $N$ of the hyperfinite $II_1$ factor $R$ with $[R : N] \leq 4$
or of finite index and finite depth, e.g. the Jones subfactors \[13\] and Wenzl subfactors arising from representations of the Hecke algebras of type \(A, B, C, D\) at roots of unity \[47, 48\]. These provide non-embedable categories as the index is not integral

c) Compact groups. Subfactors arising from actions of compact groups \(G\) on \(R\) and finite dimensional unitary irreducible \(G\)-representations have been considered in \[13, 46\]. These, on the contrary, provide amenable embedable categories with integer values of the index.

d) Compact quantum groups. Compact quantum groups give rise to examples of extremal non-amenable subfactors with embedable standard invariant, as shown by Banica in \[4\]. More precisely, he shows that, given a unitary f.d. representation \(u\) of a compact quantum group, the selfintertwiners of the iterated tensor products of \(u\) with \(u\), gives rise to a standard \(\lambda\)-lattice in the sense of Popa, and hence, by the main result of \[37\], to an extremal inclusion of factors \(N \subset M\) with \(\lambda^{-1} = [M : N] = d(u)^2\), where \(d(u)\) is the quantum dimension of \(u\). By construction, the corresponding lattice is embedded in the full lattice associated with the Hilbert space of \(u\). However, by Theorem D of the same paper, this lattice is non-amenable if the quantum dimension of \(u\) differs from its Hilbert space dimension, in agreement with Prop. 3.1.

e) Asaeda–Haagerup subfactors. As is well known, the finite depth subfactors of indices \((5 + \sqrt{13})/2\) and \((5 + \sqrt{17})/2\) of \[1\] are not associated to classical groups or quantum groups. These provide examples of tensor categories \(\mathcal{T}_M\) that can not be embedded into Hilbert spaces.

4 Statement of the results

4.1 Theorem Let \(N \subset M\) be a proper inclusion of \(II_1\) factors with finite index. For any integer \(2 \leq n \leq [M : N]\), let \(F \in M_n(\mathbb{C})\) be an invertible matrix satisfying \(FF^* = I, \quad \text{Trace}(F^*F) = [M : N]\). Then

a) there is an ergodic action of \(A_o(F)\) on a unital \(C^*\)-algebra \(\mathcal{C}\) with spectral spaces \(\mathcal{L}_{n^\otimes r} = N' \cap M_{r-1}\), \(r \geq 0\), with inner product defined by the restriction of the normalized trace on \(M_{r-1}\). One has:

\[
m(u^{\otimes r}) = \dim(N' \cap M_{r-1}),
\]

where \(u\) is the defining representation of \(A_o(F)\).

b) In particular, if we choose \(n = 2\) we get an action of \(S_{\mu}U(2)\) with \(0 < \mu \leq 1\) determined by \(\mu + \mu^{-1} = [M : N]\).

c) If \([M : N]\) is an integer, say \(p\), then we get an ergodic \(C^*\)-action of the Kac compact quantum group \(A_o(I_p)\). In this case the unique invariant state is a trace.
Remark As is well known, a large class of subfactors have been constructed from compact groups [13], [46], quantum groups, first at roots of unity [47], [48], [49] and later even from compact quantum groups [4]. Now the modular theory of the Haar state of compact quantum groups is not trivial unless the coinverse is involutive [50]. Furthermore, in contrast to the classical results on group actions, where an ergodic action implies a finite trace [15], or, for SU(2), an algebra of type I [43], [44], [45], Wang showed that $A_u(Q)$ acts ergodically both on the type III$_\lambda$ factors of Powers, and on the Cuntz algebras, while $A_u(I_n)$ acts ergodically on the hyperfinite II$_1$ factor [42]. Thus we cannot expect our C*-algebra, $C$, to be finite, in general, and we do not know how the type of $C$ is related to properties of the spectral functor of the action.

The following result is related to Prop. 3.1. For simplicity, we give a direct proof.

4.2 Corollary If the inclusion $N \subset M$ is extremal and amenable in the sense of [36] and if $[M : N]$ is not an integer then the above ergodic action of $A_o(F)$ as is not embedable into the translation action on $A_o(F)$.

Proof By [36],

$$\lim \dim((L_u \otimes r))^{1/r} = \lim \dim(N' \cap M_{r-1})^{1/r} = [M : N].$$  \hspace{1cm} (4.1)

Hence if $[M : N]$ is not an integer, $\dim((L_u \otimes r)) > n^r$ for $r$ large enough. Thus the action cannot be embedable in the translation action, as this would imply $L_u \otimes r \subset H_u \otimes r$ for all $r$.

Let us discuss some examples.

Irreducible depth 2 inclusions. We compare our construction with Ocneanu’s duality recalled in a), which reconstructs an outer action of a f.d. Kac algebra $G$ from a depth 2 irreducible finite index inclusion. The fixed point subfactor gives rise to a category $\mathcal{F}_M$ isomorphic to the category generated by the tensor powers of the regular representation $\lambda$ of $G$. Since $[M : M^G] = \dim(\ell^2(G)) = n$, $F := I_n$, and hence $A_o(n)$, is a natural choice. Hence the spectral space $L_u \otimes r$ of the resulting ergodic action of $A_o(n)$, is given by the space of fixed vectors of $\ell^2(G)^\otimes r$ under the $r$-th tensor power of $\lambda$. On the other hand the regular representation $\lambda$ of $G$ is a real object of intrinsic dimension $n$. Hence $G$, regularly represented, may be regarded as a quantum subgroup of $A_o(n)$. Our construction thus gives the quantum quotient space $G \backslash A_o(n)$.

The case $[M : N] = 2$. In this case $N = M^Z_2$ by Goldman’s theorem [12]. $A_o(2)$ is the only possible quantum group arising in our framework, up to similarity between compact matrix quantum groups. In particular, if $u = (u_{ij})$ and $v = (v_{ij})$ denote the fundamental representations of $A_o(2)$ and $S_{-1}U(2)$ respectively, the map $\phi : C(S_{-1}U(2)) \to A_o(2)$ defined by $i \otimes \phi(v) = VuV^{-1}$, with $V = (V_{ij})$ the scalar valued matrix $V_{11} = -V_{22} = i, V_{12} = -V_{21} = 1$, is a natural similarity [3]. We next identify the corresponding quotient space $Z_2 \backslash S_{-1}U(2)$. It is clear from the work of [39], based on previous results of [44], that $Z_2$ gives rise to
two different quotient $S_\mu U(2)$-spaces, one corresponding to the usual diagonal embedding of $\mathbb{Z}_2$, and another one. We show that our ergodic action identifies with the latter. By an argument of [39], it suffices to show that the corresponding restriction map $r: C(S_{-1}U(2)) \to A_o(2) \to C(\mathbb{Z}_2)$ satisfies $r(v_{11})(g) = 0$. We have, up to a scalar, $\phi(v_{11}) = u_{11} - iu_{21} + iu_{12} + u_{22}$. The restriction of $u$ to the subgroup $\lambda(\mathbb{Z}_2)$ is its fundamental representation $\hat{u} = (\hat{u}_{ij})$ as a subgroup of $U(\ell^2(\mathbb{Z}_2))$, which is given by convolution on $\ell^2(\mathbb{Z}_2)$. Hence $\hat{u}_{11}(g) = \hat{u}_{22}(g) = 0$, $\hat{u}_{21}(g) = \hat{u}_{12}(g) = 1$, and the proof is complete.

**Bisch–Haagerup subfactors for relative matched pairs.** Let us consider two finite subgroups $H$, $K$ of $\text{Out}(R)$ forming a relative matched pair, as recalled in b) of the previous section. Our construction realizes Vallin’s quantum groupoid [40] associated to $R^H \subset R \rtimes K$ as a virtual subgroup of $A_o(n)$, with $n = |H||K|$.

**Jones subfactor.** Consider the Jones subfactor $R_\beta \subset R$ of the hyperfinite $II_1$ factor with index $\beta = 4\cos^2 \frac{\pi}{m}$, with $m \geq 4$. Here $R_{\beta_1} \cap R_{r-1}$ is the algebra $B_{\beta_1,r}$ generated by the Jones projections $e_1, \ldots, e_{r-1}$. It carries a unitary representation of the braid group $\mathbb{B}_r$ as we are in the case $\beta < 4$, see, [17] or [13] (hence the tensor $C^*$-category generated by the Hilbert bimodule $N_M^N$ has a unitary braiding, as we find such representations in the $C^*$-algebras $(M_{\beta_1} \otimes \otimes, M_{\beta_1} \otimes \otimes) \simeq R_{\beta_1} \cap R_{2r-1}$). If we apply the previous theorem, we deduce that $B_{\beta_1,r}$, regarded as a Hilbert space with inner product defined by its Markov trace, does arise as the spectral space $\overline{\text{Tr}}_{\mu \otimes r}$ of an ergodic $C^*$-action of $S_{-\mu}U(2)$, with $\mu + \mu^{-1} = \beta$.

**Wenzl subfactors.** In more generality, let $N \subset M$ be the $II_1$ subfactor arising from quantum groups at roots of unity as in [49]. The higher relative commutants $N' \cap M_{r-1}$ are there shown to correspond to the arrow spaces of the fusion tensor category of the quantum group (Theorem 4.4 in [49]). These spaces, by the previous theorem, again arise as spectral spaces of ergodic actions of compact quantum groups. Furthermore the quantum dimension of the compact quantum group in question depends on the roots of unity. Thus these algebraic quantum groups seem to be virtual quantum subgroups of compact quantum groups.

**Banica subfactors.** Banica’s subfactors associated to a unitary representation $v$ of a compact quantum group have relative commutant $N'_r \cap M_r$ given by the selfintertwiners of the tensor product representation $v \otimes \mathcal{V} \otimes v \ldots (r + 1$ factors) [4]. By Frobenius reciprocity, this space is linearly isomorphic to the space of invariant vectors of the representation $(v \otimes \mathcal{V})^{r+1}$. Hence this space may be regarded as the spectral space $\overline{\text{Tr}}_{\nu \otimes r+1}$ of an ergodic action of $A_o(F)$.

For convenience, we next give a presentation by generators and relations of the ergodic $C^*$-algebra $\mathcal{C}$ in terms of the higher relative commutants. This will be deduced from a simpler presentation in terms of natural relations in Ocneanu’s tensor $C^*$-category of Hilbert bimodules that will appear in the course of the proof in the next section, (cf. relations (5.1)–(5.3) and Theorem 5.2).
Before stating the result we need some notation. Set \( H := C^{\times n} \), \( j := Fc \), where \( c : H \rightarrow H \) is the antiunitary fixing the canonical basis of \( H \), and
\[
R_u := \sum \psi_k \otimes j\psi_k,
\]
where \((\psi_k)\) is any orthonormal basis of \( H \). We introduce certain reduced words in the algebra generated by the Jones projections. Set \( \lambda := [M : N]^{1/2} \) and for nonnegative integers \( k, r, s \), define elements \( p_{r,s}^{(k)} \in M_{k+r+s-1} \) by
\[
p_{0,s}^{(k)} = p_{r,0}^{(k)} := I,
\]
and, for \( r, s \geq 1 \),
\[
p_{r,s}^{(k)} := \lambda^r e_{r+k} e_{r+k-1} \ldots e_{1+k} \ldots e_{r+k+s-1} e_{r+k+s-2} \ldots e_{s+k}.
\]
We shall simply write \( p_{r,s} \) for \( p_{r,s}^{(0)} \). By \( [27] \), \( p_{r,s} \) reduces to a scalar multiple of the Jones projection associated to \( N \subset M_{r-1} \).

In the next result, in order to avoid confusion with the tensor products, we shall denote the \( r \)-th tensor powers of \( H \) and \( u \) by \( H^r \) and \( u^r \) respectively.

4.3 Theorem The \( C^* \)-algebra \( \mathcal{C} \) is obtained by completing the *-subalgebra with generators \( \overline{T} \otimes \xi \), \( T \in N' \cap M_{r-1}, \xi \in H^r \), \( r = 0, 1, 2, \ldots \), and relations, where \( T' \in N' \cap M_{s-1}, \xi' \in H^s, \xi_1, \ldots, \xi_r \in H, \eta \in H^{r+s}, \eta' \in H^{r+s} \),
\[
a) \ (T \otimes \xi)(T' \otimes \xi') = Tp_{r,s}^{(k)} \otimes \xi',
\]
\[
b) \ (\overline{T} \otimes \xi_1 \ldots \xi_r)^* = \overline{T^r} \otimes j\xi_r \ldots j\xi_1,
\]
for \( r \geq s \):
\[
c) \ \overline{T} \otimes (1_{u^r} \otimes R_u^* \otimes 1_{u^r}\eta) = \lambda S\overline{p_{r-s,2}^{(2r)}} \otimes \eta,
\]
\[
c') \ \overline{S} \otimes (1_{u^r} \otimes R_u \otimes 1_{u^r}\eta') = \lambda E_{r+s} E_{r+s+1}(S'p_{r-s,2}^{(2s)})^* \otimes \eta',
\]
for \( r < s \):
\[
d) \ \overline{S} \otimes (1_{u^r} \otimes R_u^* \otimes 1_{u^r}\eta) = \lambda \overline{p_{2r-s,2}^{(2s)}} S \otimes \eta,
\]
\[
d') \ \overline{S} \otimes (1_{u^r} \otimes R_u \otimes 1_{u^r}\eta') = \lambda E_{r+s} E_{r+s+1}(p_{2r-s,2}^{(2s)})^* S' \otimes \eta',
\]
in the maximal \( C^* \)-norm. The \( A_o(F) \)-action \( \beta \) is uniquely defined by
\[
\beta(T \otimes \xi) = \overline{T} \otimes u^{\otimes r}(\xi), \quad T \in N' \cap M_{r-1}, \xi \in H^r,
\]
where \( u \) is the defining representation of \( A_o(F) \) on \( H \).
5 Proof of the results

We shall refer to [11] for the definition of an abstract tensor $C^*$–category $\mathcal{J}$. The tensor unit object will be denoted $\iota$. We shall always assume $(\iota, \iota) = \mathbb{C}$. An object $\rho$ will be called real (or pseudoreal) if there is an $R \in (\iota, \rho^2)$ satisfying $R^* \otimes 1_\rho \circ 1_\rho \otimes R = 1_\rho$ (or $R^* \otimes 1_\rho \circ 1_\rho \otimes R = -1_\rho$).

5.1 Theorem Let $\rho$ be a real or pseudoreal object of $\mathcal{J}$ defined by $R \in (\iota, \rho^2)$ with $\|R\|^2 \geq 2$. For any integer $2 \leq n \leq \|R\|^2$ let $F \in M_n(\mathbb{C})$ be an invertible matrix such that $FFT = \pm I$ and $\text{Trace}(F^*F) = \|R\|^2$. Then there is an ergodic action of $A_\rho(F)$ on a unital $C^*$–algebra $\mathcal{C}$ with spectral functor $L$, where $L_{\iota^\otimes r} = (\iota, \rho^*)$ and $L_{\sum_k \psi_k \otimes F\psi_k}$ is left compositon by $R$. In particular, for $n = 2$ we get an action of $\bar{S}_\rho U(2)$ for a nonzero $-1 \leq \mu \leq 1$ determined by $|\mu + \mu^{-1}| = \|R\|^2$ and $\mu > 0$ if $\rho$ is pseudoreal and negative otherwise.

The above theorem was proved in [30]. We shall outline the proof for convenience.

Outline of proof. One first shows that the tensor $^*$–subcategory generated by an arrow $R$ making $\rho$ real (or pseudoreal) is uniquely determined by the quantum dimension $d = \|R\|^2$. If $F$ is chosen as indicated, it gives rise to a realization of this category in the category of Hilbert spaces which generates the representation category of $A_\rho(F)$. Thus we have a tensor $^*$–functor, still denoted by $\rho$, from the full tensor subcategory of $\text{Rep}(A_\rho(F))$ generated by the fundamental representation $u$ to $\mathcal{J}_\rho$ taking $u$ to $\rho$ and the basic intertwiner $\sum_k \psi_k \otimes F\psi_k$ to $R$. We next apply the duality theorem in [30] (cf. Sect. 2) to the quasitensor functor $u^\otimes r \to (\iota, \rho^r)$. That theorem yields a presentation by generators and relations of the dense linear space (in fact a $^*$–subalgebra) $\mathcal{C}_{sp}$ generated by the spectral elements $T^* \otimes \xi$ with relations:

\begin{align}
(T^* \otimes \xi)(T'^* \otimes \xi') &= T^* \otimes T'^* \otimes \xi \xi', \\
(T^* \otimes \xi)^* &= \rho(R_{u^r}) \circ 1_{\rho^r} \otimes T \otimes j_{u^r} \xi, \\
T^* \rho(A) \otimes \xi' &= T^* \otimes A \xi',
\end{align}

where $T \in (\iota, \rho^r)$, $T' \in (\iota, \rho^s)$, $\xi \in H^r$, $\xi' \in H^s$, $A \in (u^s, u^r)$, and $j_{u^r}$ defines a solution $R_{u^r}$, $\overline{R}_{u^r}$ of the conjugate equations for the $r$–th tensor power $u^r$ of the defining representation $u$.

Finiteness of the Jones index implies that $\mathcal{J}_M$ has conjugates. We start by recalling some facts on module bases for $M_N$ from [20]. A basis for $M$, as a right Hilbert module, is a finite set of elements $(u_i)$ in $M$ such that $\sum_i u_i E(u_i^* m) = m$ for all $m \in M$. We shall refer to $(u_i)$ as a Pimsner-Popa basis. Such bases exist as $M_N$ is finitely generated. If $(u_i)$ is a Pimsner-Popa basis then $\sum_i u_i e_{11}^* u_i^* = I$ in $M_1$. We have: $\sum_i u_i u_i^* = [M : N]$. The following result has been shown in [20] in more generality.
5.2 Theorem $N M_N$ is a real object of the category $\mathcal{T}_M$. A solution of the conjugate equations for $M$ is given by

$$R = R = \sum_i u_i \otimes u_i^*, \quad (5.4)$$

where $(u_i)$ is a Pimsner-Popa basis for $M$. One has $\|R\|^2 = [M : N]$.

Proof As we have a tensor product over $N$ it is easily checked that $R$ is independent of the choice of the Pimsner–Popa basis. On the other hand for any unitary $u \in N$, $(u u_i)$ is another Pimsner–Popa basis, hence $u Ru^* = R$, showing that $R$ is an intertwiner in $\mathcal{T}_M$. For $m \in M$,

$$R^* \otimes 1_M \otimes 1_M \otimes R(m) = \sum_i R^* \otimes 1_M (m \otimes u_i \otimes u_i^*) =$$

$$\sum_{i,j} <u_j \otimes u_j^*, m \otimes u_i > u_i^* = \sum_{i,j} <u_j^*, E(u_j^* m) u_i > u_i^* =$$

$$\sum_{i,j} E(u_j E(u_j^* m) u_i) u_i^* = \sum_{i} E(m u_i) u_i^* = m.$$

We also have

$$\|R\|^2 = R^* R = \sum_{i,j} <u_i \otimes u_i^*, u_j \otimes u_j^*> =$$

$$\sum_{i,j} E(u_i E(u_i^* u_j) u_j^*) = \sum_{j} E(u_j u_j^*) = [M : N].$$

Applying Theorem 5.1 to Ocneanu’s category $\mathcal{T}_M$ and the intertwiner $R = \sum u_i \otimes u_i^*$, we get the desired ergodic action of $A_o(F)$ on a unital $C^*$–algebra with spectral spaces $\mathcal{T}_{u r} = (u, M^{\otimes r}) \simeq N' \cap M_{r - 1}$. It remains to identify the inner products on the spaces $(u, M^{\otimes r})$ arising from the category $\mathcal{T}_M$ and the algebraic presentation of $C_{sp}$ in terms of the higher relative commutants $N' \cap M_{r - 1}$.

Now the inner product of $(u, M^{\otimes r})$ is the restriction of the $N$–valued inner product of $M^{\otimes r}$ which, through the unitary identification $M^{\otimes r} \simeq L^2(M_{r - 1})$ as $N$–$N$–correspondences in the sense of Connes [10], arises from the normalized trace of $M_{r - 1}$ (see, e.g., Prop. 3.1 in [7]).

For later use, we shall need explicit Hilbert $N$–bimodule unitaries from $M^{\otimes r}$ to $M_{r - 1}$. Consider the $N$–bimodule isomorphism between $M_{r+1} := M_r e_{r+1} M_r$ and $M_r \otimes_{M_{r-1}} M_r$, given by

$$m \otimes m' \rightarrow \lambda m e_{r+1} m', \quad m, m' \in M_r.$$

Regard $M_r$ as a $M_{r-1}$–Hilbert bimodule with inner product defined by the normalized conditional expectation $E_r : M_r \rightarrow M_{r-1}$. Then the tensor product of the $M_{r-1}$–valued inner products on $M_r \otimes_{M_{r-1}} M_r$, corresponds, under the above
isomorphism $U$ to the inner product induced by the (normalized) conditional expectation $E_r E_{r+1} : M_{r+1} \to M_r$ by
\[ < S, T > = E_r E_{r+1}(S^*T), \quad S, T \in M_{r+1}. \]

Iterating this procedure (recall (3.5)), leads to canonical isomorphisms of $N$–bimodules $\Gamma_r : M^{\otimes r} \to M_{r-1}$ transforming the tensor product of $N$–valued inner products into the inner product induced by the conditional expectation
\[ E_{(r)} := EE_1 \cdots E_{r-2} E_{r-1} : M_{r-1} \to N. \quad (5.5) \]

$\Gamma_r : M^{\otimes r} \simeq M_{r-1}$ is obtained in the following way. First replace each factor $M$ occurring in 2nd to $r$–th position in $M^{\otimes r}$ by $M \otimes_M M$, giving $2r - 2$ factors tensored alternately over $N$ and $M$. Thus
\[ M^{\otimes r} = (M \otimes_N M)^{\otimes_{M_{r-1}}}. \]

Finally use the isomorphism $M \otimes_N M \simeq M_1$ giving by iteration
\[ M^{\otimes r} \simeq M^{\otimes_{M_{r-1}}} \simeq M^{(2^{M_{r-1}}-2)} \simeq \cdots \simeq M_{r-2} \otimes M_{r-3} M_{r-2} \simeq M_{r-1}. \]
The resulting isomorphism $\Gamma_r$ is
\[ m_1 \otimes \cdots \otimes m_r \to \lambda^{<r>} \sum_{\iota \in \mathcal{I}} m_{\iota_1} m_{\iota_2} \cdots m_{\iota_{r-1}} e_{\iota_{r-1}} \cdots e_1 m_r, \quad (5.6) \]
where $< r > := (r-1) + (r-2) + \cdots + 1 = r(r-1)/2$. Summarizing, one has the following result.

**5.3 Proposition** Under the $N$–bimodule isomorphisms $\Gamma_r : M^{\otimes r} \to M_{r-1}$, the $N$–valued tensor power inner product on $M^{\otimes r}$ corresponds to the inner product on $M_{r-1}$ defined by the trace-preserving conditional expectation $E_{(r)} : M_{r-1} \to N$:
\[ < S, T > := E_{(r)}(S^*T), \quad S, T \in M_{r-1}. \]

In particular, the Hilbert space structure of $(\iota, M^{\otimes r})$ defined by the category $\mathcal{J}_M$ corresponds, under the restriction of $\Gamma_r$, to the inner product of $N' \cap M_{r-1}$ defined by the restriction of the normalized trace on $M_{r-1}$.

We need to establish the algebraic presentation of the dense spectral $^*$–subalgebra $\mathcal{C}_R$. From [BM] (cf. Theorem 5.1 and its proof) applied to $\mathcal{J} = \mathcal{J}_M$, Ocneanu’s tensor $C^*$–category associated with the inclusion $N \subset M$ as in section 3, and to the tensor $^*$–functor from the full tensor $C^*$–subcategory of $\text{Rep}(A_u(F))$ generated by the defining representation $u$ to $\mathcal{J}_M$ and taking $u$ to $N M_N$ and the basic intertwiner $\sum_k \psi_k \otimes F \psi_k$ to $R = \sum u_i \otimes u_i^\flat$, we know that an algebraic presentation of $\mathcal{C}_R$ is given by generators $T^\star \otimes \xi$ with $T \in (\iota, M^{\otimes r})$, $\xi \in H'$ subject to the relations (5.1)–(5.3).

We start by computing the $^*$–involution. Starting from the antilinear intertwiner $j_u := j = Fc$, corresponding to $R_u = \sum \psi_i \otimes j \psi_i$, for $r = 1$, we can form its tensor power $j_{u^r}(\xi_1 \cdots \xi_r) := j(\xi_1) \cdots j(\xi_r)$. With this choice we have
\[ R_{u^r} = 1_{u^{r-1}} \otimes R_u \otimes 1_{u^{r-1}} \otimes R_{u^{r-1}}. \]
By tensoriality of the inclusion $\rho$ of the full tensor $C^*$–subcategory of $\text{Rep}(A_o(F))$ in $\mathcal{T}_M$,
\[
\rho(R_{ur}) = 1_{M \otimes r} \otimes \rho(R_u) \otimes 1_{M \otimes r} \circ \rho(R_{ur-1}) = \\
\sum_{i_1, \ldots, i_r} u_{i_1} \otimes \ldots \otimes u_{i_r} \otimes u_{i_r}^* \otimes \ldots \otimes u_{i_1}^* \in M^{\otimes 2r}.
\]

On the other hand, since $u^r$ is real, it is selfconjugate, hence $\overline{u^r} = u^r$. Therefore, for $T \in (i, M^{\otimes r})$,
\[
(T \otimes \xi_1 \ldots \xi_r)^* = 1_{M^{\otimes r}} \otimes T^* \circ \rho(R_{ur}) \otimes j\xi_r \ldots j\xi_1 = \\
\sum_{i_1, \ldots, i_r} u_{i_1} \otimes \ldots \otimes u_{i_r} < T, u_{i_r}^* \otimes \ldots \otimes u_{i_1}^* > \otimes j\xi_r \ldots j\xi_1.
\]

We thus need to compute $\sum u_{i_1} \otimes \ldots \otimes u_{i_r} < T, u_{i_r}^* \otimes \ldots \otimes u_{i_1}^* >$ for $T \in (i, M^{\otimes r})$ under the identification $(i, M^{\otimes r}) \cong N' \cap M_{r-1}$. Now it is easy to see, using the trace norm of $N$, that $E(SX) = E(XS)$ for $S \in N' \cap M$ and $X \in M$, hence, for $r = 1$, and $T \in N' \cap M$,
\[
\sum_i u_i < T, u_i^* > = \sum_i u_i E(T^* u_i^*) = \sum_i u_i E(u_i^* T^*) = T^*.
\]

In the general case, if $T \in N' \cap M_{r-1}$, identifying $M^{\otimes r}$ and $M_{r-1}$ as right Hilbert $N$–modules, $u_{i_1} \otimes \ldots \otimes u_{i_r}$ corresponds to a Pimsner–Popa basis for the inclusion $N \subset M_{r-1}$, and the above argument for $r = 1$ shows that $\sum u_{i_1} \otimes \ldots \otimes u_{i_r} < T, u_{i_r}^* \otimes \ldots \otimes u_{i_1}^* >$ corresponds to the adjoint $T^*$ of $T$ in $N' \cap M_{r-1}$.

Summarizing:

**5.4 Proposition** Under the restriction of the $N$–bimodule unitary $\Gamma_r : M^{\otimes r} \to M_{r-1}$, the antilinear map
\[
J_r : T \in (i, M^{\otimes r}) \to 1_{M^{\otimes r}} \otimes T^* \circ \rho(R_{ur}) \in (i, M^{\otimes r})
\]

corresponds to the antiunitary $^* $–involution
\[
T \in N' \cap M_{r-1} \to T^* \in N' \cap M_{r-1}.
\]

Therefore for the resulting ergodic $C^*$–action of $A_o(F)$ we have:
\[
m(u^r) = \dim(N' \cap M_{r-1}),
\]
where $m$ is the quantum multiplicity.

Hence formula b) in Theorem 4.3 has been established.

As the quantum multiplicity necessarily takes on its minimal value, $II_1$ subfactors do not yield all canonical ergodic actions of $A_o(F)$.

Before exploiting the operator product (5.5) in $\mathcal{C}$, we complete the proof of Theorem 4.1.

**Proof of Theorem 4.1 c)**
We need to show that the unique invariant state $h$ is a trace. In the special case that we are considering, $A_v(I_p)$ has an involutive coinverse $[11]$, hence $j_w$ can be chosen antiunitary for all representations $w$. In particular the solution $R_w, \overline{R_w}$ of the conjugate equations is standard. By Cor. A.10 of [11], the solution $\hat{R}_{u^r}, \overline{R_{u^r}}$ is also standard. We close $\mathcal{T}_M$ under subobjects and then extend the inclusion functor $\rho$ to a relaxed tensor *-functor from $\text{Rep}(A_v(F))$ to this closure, so that $\rho_v$ is now defined for any irreducible subrepresentation $v$ of some $u^r$. It follows that $J_r \mid_{(\iota, \rho_v)}$ coincides with an antilinear invertible $J_v$ constructed from a normalized solution of the conjugate equations for $v$. Since $J_r$ is antiunitary, $J_v$ is antiunitary as well. Let $\hat{\rho}$ be a complete set of irreducible representations of $A_v(F)$. If $v, w \in \hat{\rho}, a = \overline{S} \otimes \psi, b = \overline{T} \otimes \phi, \psi \in H_v, \phi \in H_w, S \in (\iota, \rho_v), T \in (\iota, \rho_w)$ then $v \otimes w$ contains the trivial representation $\iota$ if and only if $w = \psi$, and the multiplicity of $\iota$ is 1. Hence $S = ||R_v||^{-1} R_v \in (\iota, v \otimes \psi)$ is an isometry. It follows that

$$h(ab) = \delta_{w,\psi} ||R_v||^{-2} \rho(\overline{R_v}) S \otimes T \otimes R_w \psi \otimes \phi =$$

$$\delta_{w,\psi} ||R_v||^{-2} <j_v S, T > <j_v \psi, \phi > .$$

Similarly,

$$h(ba) = \delta_{w,\psi} ||R_w||^{-2} <j_v T, S > <j_v \phi, \psi >$$

which in turn equals

$$\delta_{w,\psi} ||R_v||^{-2} <j_v^{-1} T, S > <j_v^{-1} \phi, \psi > =$$

$$\delta_{w,\psi} ||R_v||^{-2} <j_v S, T > <j_v \psi, \phi > = h(ab).$$

We next spell out the tensor product operation between arrows in Ocneanu’s category in terms of the higher relative commutants.

In detail, the isomorphism $\Gamma_r$ allows us to write down the tensor product $\xi \otimes \eta$ of elements $\xi \in M^{\otimes r}$ and $\eta \in M^{\otimes s}$ in terms of a bilinear map

$$M_{r-1} \times M_{s-1} \to M_{r+s-1}.$$ 

If, e.g., $\xi = \xi_1 \otimes \xi_2, \eta = \eta_1 \otimes \eta_2 \otimes \eta_3,$

$$\Gamma_5(\xi \otimes \eta) = \lambda^{10} \xi_1 e_1 \xi_2 e_2 e_1 \eta_1 e_3 e_2 e_1 \eta_2 e_4 e_2 e_1 \eta_3 =$$

$$\lambda^9 \Gamma_2(\xi) e_2 e_1 \eta_1 e_3 e_2 e_1 \eta_2 e_4 e_2 e_1 \eta_3 =$$

$$\lambda^9 \Gamma_2(\xi) e_2 e_1 e_3 e_2 e_4 e_3 (\eta_1 e_1 e_2 e_1 \eta_3) =$$

$$\lambda^6 \Gamma_2(\xi) e_2 e_1 e_3 e_2 e_4 e_3 \Gamma_3(\eta).$$

In general, if $\xi \in M^{\otimes r}, \eta \in M^{\otimes s}$ then

$$\Gamma_{r+s}(\xi \otimes \eta) = \lambda^{r+s-1} (\xi \otimes \eta) \Gamma_r(\xi) (e_{r+1} \ldots e_1)(e_{r+1} \ldots e_2) \ldots (e_{r+s-1} \ldots e_s) \Gamma_s(\eta) =$$

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\[ \lambda^s \Gamma_r(\xi)(e_r \ldots e_1)(e_{r+1} \ldots e_2) \ldots (e_{r+s-1} \ldots e_s) \Gamma_s(\eta) = \Gamma_r(\xi)p_{r,s} \Gamma_s(\eta), \]

with \( p_{r,s} = p_{r,s}^{(0)} \).

5.5 Proposition Under the \( N \)-bimodule unitary \( \Gamma_r : M \otimes r \to M_{r-1} \) the tensor product

\[ \xi \in (\iota, M \otimes r), \eta \in (\iota, M \otimes s) \to \xi \otimes \eta \in (\iota, M \otimes r+s) \]  \hspace{1cm} (5.10)

defined in \( \mathcal{T}_M \) corresponds to the map

\[ S \in N' \cap M_{r-1}, T \in N' \cap M_{s-1} \to S p_{r,s} T \in N' \cap M_{r+s-1}. \]  \hspace{1cm} (5.11)

We are left to account for the \( \text{Rep}(A_o(F)) \)-bimodule structure (5.3) used to define \( \mathcal{E}_p \). Now the full tensor \( * \)-subcategory of \( \text{Rep}(A_o(F)) \) generated by \( u \) is generated, as a linear category, by the arrows \( 1_{M \otimes r} \otimes \lambda \otimes 1_{M \otimes s} \), \( r, s \geq 0 \), and by their adjoints. Therefore we are led to compute the composition of arrows of the form \( 1_{M \otimes r} \otimes \lambda \otimes 1_{M \otimes s} \circ T \) and \( 1_{M \otimes r} \otimes \lambda^* \otimes 1_{M \otimes s} \circ S \) with \( T \in (\iota, M \otimes r+s) \), \( S \in (\iota, M \otimes s+s+2) \), \( r, s \geq 0 \), in terms of the higher relative commutants. For \( \xi \in M \otimes r, \eta \in M \otimes s \),

\[ 1_{M \otimes r} \otimes \lambda \otimes 1_{M \otimes s} : \xi \otimes \eta = \xi \otimes \eta, \]

hence

\[ \Gamma_{r+2+s}(1_{M \otimes r} \otimes \lambda \otimes 1_{M \otimes s} (\xi \otimes \eta)) = \]

\[ \Gamma_{r+2}(\xi \otimes \lambda) p_{r+2,s} \Gamma_s(\eta) = \]

\[ \Gamma_r(\xi) p_{r+2} \Gamma(\lambda) p_{r+2,s} \Gamma_s(\eta). \]

Now

\[ \Gamma_2(\lambda) = \sum_{i} \Gamma_2(u_i \otimes u_i^*) = \lambda \sum_{i} u_i e_i u_i^* = \lambda. \]

Therefore

\[ \Gamma_{r+2+s}(1_{M \otimes r} \otimes \lambda \otimes 1_{M \otimes s} (\xi \otimes \eta)) = \lambda \Gamma_r(\xi) p_{r+2} p_{r+2,s} \Gamma_s(\eta). \]  \hspace{1cm} (5.12)

We therefore need to write \( p_{r+2} p_{r+2,s} \) as a reduced word in the algebra generated by the Jones projections \( e_1, \ldots, e_{r+s+1} \). It is known that elements of the form

\[ (e_{j_1} e_{j_1-1} \ldots e_{i_1})(e_{j_2} \ldots e_{j_2-1} \ldots e_{i_2}) \ldots (e_{j_p} e_{j_p-1} \ldots e_{i_p}) \]

are in reduced form if \( j_1 < j_2 < \cdots < j_p \) and \( i_1 < i_2 < \cdots < i_p \). The following lemma is useful.

5.6 Lemma For \( p \leq j \leq r < s \) we have

\[ (e_r e_{r-1} \ldots e_j)(e_s e_{s-1} \ldots e_p) = \lambda^{-2}(e_r \ldots e_p) (e_s \ldots e_{j+2}) \] for \( s > j + 1 \),

\[ (e_r e_{r-1} \ldots e_j)(e_s e_{s-1} \ldots e_p) = \lambda^{-2}(e_r \ldots e_p) \] for \( s = j + 1 \).
Proof We do the computation only in the case \( s > j + 1 \).

\[
(e_r e_{r-1} \ldots e_j)(e_s \ldots e_{j+2} e_{j+1} e_j \ldots e_p) = (e_r e_{r-1} \ldots e_{j+1})(e_s \ldots e_{j+2} e_{j+1} e_j \ldots e_p) = \\
\lambda^{-2}(e_r e_{r-1} \ldots e_{j+1})(e_s \ldots e_{j+2} e_{j+1} e_j \ldots e_p)(e_s \ldots e_{j+2}).
\]

### 5.7 Lemma

For \( s \geq 0 \) and \( r \geq s \) we have:

\[
p_{r,2(p+2,s)} = p_{r,s}p_{r-s,2}^{(2s)}.\]

**Proof** The formula is obvious for \( s = 0 \). We can then assume \( s > 0 \). We write down the left hand side explicitly:

\[
p_{r,2(p+2,s)} = \lambda^{2r+2s+rs}(e_r \ldots e_1)(e_{r+1} \ldots e_2)(e_{r+2} \ldots e_1) \ldots (e_{r+s+1} \ldots e_s).
\]

We have \( 2 + s \) factors between parentheses. Let us apply the previous lemma iteratively between the second and the third factor. If \( r + 2 > 3 \), i.e. \( r > 1 \), we have,

\[
(e_{r+1} \ldots e_2)(e_{r+2} \ldots e_1) = \lambda^{-2}(e_{r+1} \ldots e_1)(e_{r+2} \ldots e_2+2).
\]

We proceed to apply the lemma to the new third and old fourth factors: if \( r + 3 > 5 \), i.e. \( r > 2 \),

\[
(e_{r+2} \ldots e_2+2)(e_{r+3} \ldots e_2) = \lambda^{-2}(e_{r+2} \ldots e_2)(e_{r+3} \ldots e_2+4).
\]

If \( n > s \) after \( s \) iterations of the lemma we still find a product of \( 2 + s \) factors:

\[
p_{r,2(p+2,s)} = \lambda^{2r+rs}(e_r \ldots e_1)(e_{r+1} \ldots e_1) \ldots (e_{r+s} \ldots e_s)(e_{r+s+1} \ldots e_{2+2s}).
\]

If \( r = s \) instead, the computation goes through but the last application of the lemma requires the second formula for the reduced word. Hence the last factor needs to be replaced by the identity:

\[
p_{r,2(p+2,r)} = \lambda^{2r+r^2}(e_r \ldots e_1)(e_{r+1} \ldots e_1)(e_{r+2} \ldots e_2) \ldots (e_{2r} \ldots e_r).
\]

If \( r > s \) we apply the lemma iteratively, but now only \( s \) times (in spite of the \( s + 2 \) factors) to the first two factors, the second and third factor and so on. We get

\[
p_{r,2(p+2,s)} = \lambda^{2r+rs-2s}(e_r \ldots e_1) \ldots (e_{r+s-1} \ldots e_s)(e_{r+s} \ldots e_{2s+1})(e_{r+s+1} \ldots e_{2+2s}) = \\
\lambda^{2(r-s)}p_{r,s}(e_{r+s} \ldots e_{1+2s})(e_{r+s+1} \ldots e_{2+2s}) = p_{r,s}p_{r-s,2}^{(2s)},
\]

and the formula is proved. Now assume \( r = s \). Then the right hand side of the desired formula reduces to \( p_{r,r} \). The same computation goes through except at the last \( s \)-th iteration, where

\[
p_{r,2(p+2,r)} = \lambda^{r^2}(e_r \ldots e_1) \ldots (e_{2r-1} \ldots e_r) = p_{r,r}.
\]
The proof is now complete. It is now not difficult to interpret left tensoring by the translates of \( R \) in terms of the Jones tower.

5.8 Proposition For \( r, s \geq 0 \) and \( \zeta \in M^{\otimes r+s} \), we have:

a) for \( r > s \), \( \Gamma_{r+2,s}(1_{M^{\otimes r}} \otimes R \otimes 1_{M^{\otimes s}} \circ \zeta) = \lambda \Gamma_{r,s+2}(\zeta)p^{(2s)}_{r-s,2} \),

b) for \( r = s \), \( \Gamma_{2r+2}(1_{M^{\otimes r}} \otimes R \otimes 1_{M^{\otimes r}} \circ \zeta) = \lambda \Gamma_{2r}(\zeta) \),

c) for \( r < s \), \( \Gamma_{r+2,s}(1_{M^{\otimes r}} \otimes R \otimes 1_{M^{\otimes s}} \circ \zeta) = \lambda p^{(2r)}_{2,s-r} \Gamma_{r+s}(\zeta) \).

Proof For \( r = 0 \) or \( s = 0 \) the formula can be deduced from \( \Gamma_2(R) = \lambda \) and the computation of the tensor product given in Prop. 5.5. We can then assume \( r, s > 0 \). In this case a) and b) follow from the previous lemma, property (5.12) and the fact that the \( e_k \) commute with \( M_{s-1} \) for \( k \geq 2s + 1 \) (recall (3.1)). We prove c). Let us choose \( \zeta \) of the form \( \zeta = \xi \otimes \eta_1 \otimes \eta_2 \) with \( \xi, \eta_1, \eta_2 \in M^{\otimes r} \), \( \eta_2 \in M^{\otimes k} \), where \( s = r + k \). Then

\[
\Gamma_{r+2,s}(1_{M^{\otimes r}} \otimes R \otimes 1_{M^{\otimes s}} \circ \zeta) = \Gamma_{r,s+2}(\xi \otimes R \otimes \eta_1 \otimes \eta_2) = \Gamma_{2r+2}(\xi \otimes \eta_1)p_{2r+2,k}\Gamma_k(\eta_2) = \lambda \Gamma_{2r}(\xi \otimes \eta_1)p_{2r+2,k}\Gamma_k(\eta_2).
\]

On the other hand

\[
p_{2r+2,k} = \lambda^{k(2r+2)}((e_{2r+2}e_{2r+1})e_{2r} \ldots e_1) \ldots ((e_{2r+k+1}e_{2r+k})e_{2r+k-1} \ldots e_k) = \lambda^{2k}(e_{2r+2}e_{2r+1})(e_{2r+3}e_{2r+2}) \ldots (e_{r+s+1}e_{r+s})p_{2r,k},
\]

which implies

\[
\Gamma_{r+2,s}(1_{M^{\otimes r}} \otimes R \otimes 1_{M^{\otimes s}} \circ \zeta) = \lambda^{1+2(s-r)}\Gamma_{2r}(\xi \otimes \eta_1)(e_{2r+2}e_{2r+1})(e_{2r+3}e_{2r+2}) \ldots (e_{r+s+1}e_{r+s})p_{2r,k}\Gamma_k(\eta_2) = \lambda^{1+2(s-r)}(e_{2r+2}e_{2r+1})(e_{2r+3}e_{2r+2}) \ldots (e_{r+s+1}e_{r+s})\Gamma_{r+s}(\zeta) = \lambda p^{(2r)}_{2,s-r} \Gamma_{r+s}(\zeta).
\]

We next compute the operators in the Jones tower corresponding to tensoring on the left by \( 1_{M^{\otimes r}} \otimes R^* \otimes 1_{M^{\otimes r}} \).

5.9 Proposition For \( r, s \geq 0 \), \( \zeta \in M^{\otimes r+s+2} \) we have:

a) for \( r > s \), \( \Gamma_{r+s}(1_{M^{\otimes r}} \otimes R^* \otimes 1_{M^{\otimes s}} \circ \zeta) = \lambda E_{r+s}E_{r+s+1}(\Gamma_{r,s+2}(\zeta)(p^{(2s)}_{r-s,2})^*) \),

b) for \( r = s \), \( \Gamma_{2r}(1_{M^{\otimes r}} \otimes R^* \otimes 1_{M^{\otimes r}} \circ \zeta) = \lambda E_{r+s}E_{r+s+1}(\Gamma_{2r+2}(\zeta)) \),

c) for \( r < s \), \( \Gamma_{r+s}(1_{M^{\otimes r}} \otimes R^* \otimes 1_{M^{\otimes s}} \circ \zeta) = \lambda E_{r+s}E_{r+s+1}(p^{(2r)}_{2,s-r} \Gamma_{r+s+2}(\zeta)) \).

Proof With respect to the inner products the \( N \)-bimodule operators \( 1_{M^{\otimes r}} \otimes R \otimes 1_{M^{\otimes s}} : M^{\otimes r+s+2} \to M^{\otimes r+s+2} \) and \( 1_{M^{\otimes s}} \otimes R^* \otimes 1_{M^{\otimes r}} : M^{\otimes r+s+2} \to M^{\otimes r+s+2} \) are adjoints of each other. Recall that, by Prop. 10.5, \( \Gamma_p : M^{\otimes p} \to M_{p-1} \) is a \( N \)-bimodule unitary if \( M_{p-1} \) is regarded as a \( N \)-bimodule with inner product

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defined by the conditional expectation \( E(p) = EE_1 \ldots E_{p-1} : M_{p-1} \to N. \) Therefore we need to compute
\[
\Gamma_{r+s}1_{M^{\otimes r}} \otimes R^* \otimes 1_{M^{\otimes s}} \Gamma_{r+s+2}^* = (\Gamma_{r+s+2}1_{M^{\otimes r}} \otimes R\Gamma_{r+s}^*)^*.
\]
Now by the previous lemma, for \( r > s \), \((\Gamma_{r+s+2}1_{M^{\otimes r}} \otimes R\Gamma_{r+s}) : M_{r+s-1} \to M_{r+s+1}\) is right multiplication by the element \( A = \lambda_p^{2s} \in M_{r+s+1} \). Hence its adjoint, \( r_A^* \), is
\[
<X, r_A^* Y> = <X, A Y> = E_{(r+s+2)}(A^* X^* Y) = E_{(r+s+2)}(X^* Y A^*)
\]
where \( X \in M_{r+s-1}, Y \in M_{r+s+1} \), as \( A \) commutes with \( N \). Hence
\[
<X, r_A^* Y> = E_{(r+s)}(X^* E_{r+s} E_{r+s+1}(Y A^*)),
\]
as \( X \in M_{r+s-1} \), and this shows that \( r_A^* Y = E_{r+s} E_{r+s+1}(Y A^*) \). The remaining cases follow similarly.

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