ABSTRACT

We explain how all information about ambient component field spin assignments in higher-dimensional off-shell supersymmetry is accessibly coded in one-dimensional restrictions, known as shadows. We also explain how to determine whether the components of a given one-dimensional supermultiplet may assemble into representations of $\text{spin}(1, D - 1)$ and, if so, how to specifically determine those representations.

In a recent paper [1] we discussed how the representation theory of supersymmetry in diverse dimensions lies encoded within the representation theory of one-dimensional supersymmetry. This hinges on the recognition that each supersymmetric field theory has a unique analog, called its shadow, obtained by restricting to a zero-brane. In our earlier paper, we exhibited an algebraic “litmus test” which determines whether or not a given one-dimensional theory represents a shadow of another theory in a larger specified number of “ambient” dimensions. We call the (re)construction of a higher-dimensional field theory based on information encoded as a one-dimensional shadow mechanics “dimensional enhancement”. This describes the reverse of dimensional reduction. In our earlier work we used our new technology to reproduce known results about the representations of four-dimensional $N = 1$ supersymmetry using reasoning exclusively contextualized to one-dimension. In this letter we extend this process, explaining how to resolve the organization of multiplet component fields into specific representations of $\text{spin}(1, D - 1)$ based only on one-dimensional multiplet structures.

Our approach to this problem is enabled by casting the algebraic requirements of supersymmetry in terms of certain “linkage matrices”, or their graphical equivalents known as Adinkra diagrams [2]. This approach is complementary to traditional
methods based on Salam-Strathdee superfields, and enables new forms of analysis. The linkage matrices are generalizations of the “garden matrices” introduced in [4], while the feasibility, realized in [1], that generic supersymmetries might be universally encoded by one-dimensional superalgebras was conjectured in [5].

The physical motivation which enabled us to derive our enhancement condition was the observation, described in [1], that the linkgage matrices describing any supersymmetric field theory must be invariant under spin transformations. This is analogous to the statement that Dirac Gamma matrices should be Lorentz invariant, in the context of generic (not necessarily supersymmetric) field theories with spinors. Indeed, it is easy to show that Gamma matrices are invariant objects; the proof relies on the structure of the relevant Clifford algebras. Our method generalizes this observation to the case of supersymmetric theories. In this case Clifford algebras also play a central role. The reason for this is that the linkage matrices satisfy conditions, equivalent to the algebraic statements of supersymmetry, which generalize the “garden algebras” introduced in [4], which are themselves variants of Clifford algebras.

In [1] we presented a proof that spin invariance correlates all space-like linkage matrices in terms of time-like linkage matrices, and we also showed how spin invariance implies constraints on the spin(1, D − 1)-representations spanned by the multiplet component fields. In that paper we focussed on the implications of the former observation, and we did not fully resolve the implications concerning the specific organization of component fields into spin representations. In the bulk of this letter we do precisely that.

Supersymmetry is generated by first-order linear differential operators which are also matrices acting on the vector spaces spanned by the component bosons \( \phi_i \) and the component fermions \( \psi_\hat{i} \). Specifically, the supercharges are written as

\[
\left( Q_A \right)_i^i = \left( u_A \right)_i^i + \left( \Delta_A^\mu \right)_i^i \partial_\mu \\
\left( \tilde{Q}_A \right)_i^i = i \left( \tilde{u}_A \right)_i^i + i \left( \tilde{\Delta}_A^\mu \right)_i^i \partial_\mu, \tag{1}
\]

where \( u_A, \tilde{u}_A, \Delta_A^\mu, \) and \( \tilde{\Delta}_A^\mu \) are real valued “linkage matrices”, and the index \( A \) is an ambient spinor index. We consider here the important case where the linkage matrices are “Adinkraic”, which means that each of these has only one non-vanishing entry in each row and one non-vanishing entry in each column. Moreover, each non-vanishing

\footnote{\textsuperscript{1} Other novel approaches to this problem have also been developed, \textit{e.g.}, as explained in [3].} \\
\footnote{\textsuperscript{2} A more mathematically-precise discussion complementary to much of our analysis is provided in [6].}
entry assumes one of the two values ±1. As we explain in [1], these cases correspond to all known supermultiplets in dimensions \( D > 1 \), and is arguably non-restrictive. An explanation of how these matrix structures may be generically achieved by linear restructuring of the components is explained mathematically in [7].

The component fields assemble into representations of \( \text{spin}(1, D-1) \), whereby the generating transformations act as 
\[
\delta \phi_i = \frac{1}{2} \theta^{\mu \nu} ( T_{\mu \nu} )_{i}^{\ j} \phi_j \quad \text{and} \quad \delta \psi_{\hat{i}} = \frac{1}{2} \theta^{\mu \nu} ( \tilde{T}_{\mu \nu} )_{\hat{i}}^{\ \hat{j}} \psi_{\hat{j}},
\]
where \( \theta^{0a} \) parameterizes a boost in the \( a \)-th spatial direction and \( \theta^{ab} \) parameterizes a rotation in the \( ab \)-plane. Our goal is to explain how these representations are determined from the linkage matrices described above.

In Appendix A of [1] we presented a simple derivation that \( \text{spin}(1, D-1) \)-invariance of the bosonic linkage matrices \( \Delta^0_A \) and \( u_a \) translates into the following six constraints,
\[
\Delta^a_A = -\frac{1}{2} ( \Gamma^0 \Gamma^a )_A^B \Delta^0_B - T^{0a} \Delta^0_A + \Delta^0_A \tilde{T}^{0a} \\
\frac{1}{2} ( \Gamma_{ab} )_A^B \Delta^0_B = \Delta^a_A \tilde{T}_{ab} - T_{0b} \Delta^0_A \\
\delta_b^a \Delta^0_A = \frac{1}{2} ( \Gamma_0 \Gamma_b )_A^B \Delta^0_B + T_{0b} \Delta^0_A - \Delta^0_A \tilde{T}_{0b} \\
\eta^{[ab]} \Delta^0_A + \frac{1}{2} ( \Gamma^{bc} )_A^B \Delta^0_B = \frac{1}{2} \Delta^a_A \tilde{T}^{bc} - \frac{1}{2} T^{bc} \Delta^0_A \\
\frac{1}{2} ( \Gamma_0 \Gamma_a )_A^B u_B = u_A \tilde{T}_{0a} - T_{0a} u_A \\
\frac{1}{2} ( \Gamma_{ab} )_A^B u_B = u_A \tilde{T}_{ab} - T_{ab} u_A.
\]
These are a direct translation of equations (A.4), (A.5), and (A.7) from that paper, where the component index structures such as \( (\cdot)_i^i \) have been suppressed, and where the boost and rotation operators have been separated. The requirement that the fermion linkage matrices \( \tilde{\Delta}^a_A \) and \( \tilde{u}_A \) be \( \text{spin}(1, D-1) \)-invariant imposes a set of constraints similar to (2). These “fermionic analogs” are obtained from (2) by adding tildes to the linkage matrices and by exchanging the boson and fermion spin generators, via \( T_{\mu \nu} \leftrightarrow \tilde{T}_{\mu \nu} \). There is significant redundancy in the equations (2) and the corresponding fermionic analogs. One of our goals in this letter is to distill these statements to their essence.

In this letter we focus on multiplets in a “standard” configuration, according to which \( \Delta^0_A = \tilde{u}^T_A \) and \( \tilde{\Delta}^0_A = u^T_A \). All non-gauge matter multiplets may be construed, using judicious linear re-organization of components, so that the linkage matrices have this feature. In terms of Adinkras, these identities say that every Adinkra edge codifies both an “upward directed” supersymmetry transformation and a corresponding “downward directed” supersymmetry transformation. The presence of gauge structures modifies the situation in a way which is tractable, but which adds distracting
technical details. In the interest of keeping things as simple as possible, but main-
taining generality for the wide class of interesting multiplets described by non-gauge
matter, we utilize this feature. We are developing the general case, which includes
gauge structures, in on-going work. We use also the features that boost generators
are described by symmetric matrices, $T_{0a} = T_{0a}^T$, and that rotation generators are
described by anti-symmetric matrices, $T_{ab} = -T_{ab}^T$.

We now address the matter of removing redundancies from the list of constraints
given in (2). First, the matrix transposes of the fifth equation in (2) and its fermionic
analog tell us

$$\frac{1}{2}(\Gamma_0 \Gamma_a)_{AB} \Delta_A^0 = \tilde{T}_{0a} \tilde{\Delta}_A^0 - \tilde{\Delta}_A^0 \tilde{T}_{0a},$$

$$\frac{1}{2}(\Gamma_0 \Gamma_a)_{AB} \Delta_A^0 = T_{0a} \Delta_A^0 - \Delta_A^0 \tilde{T}_{0a}, \quad (3)$$

where we have used the features $\Delta_A^0 = \tilde{u}_A^T$ and $\tilde{\Delta}_A^0 = u_A^T$ described above. (These
identities allow us to cast the entirety of the spin invariance requirements in terms of
the “down” matrices $\Delta_A^\mu$ and $\tilde{\Delta}_A^\mu$.)

It proves helpful that the boost generators $T_{0a}$ and $\tilde{T}_{0a}$ appear in (3) in the same
combination as in the first equation in (2). Accordingly, we can use (3) to eliminate
these terms, to derive a constraint independent of the component spin representation
assignments. Specifically, if we substitute (3) into the first equations of (2) and its
fermionic analog, we easily determine

$$\Delta_A^a = - (\Gamma_0 \Gamma_a)_{AB} \Delta_B^0,$$

$$\tilde{\Delta}_A^a = - (\Gamma_0 \Gamma_a)_{AB} \tilde{\Delta}_B^0. \quad (4)$$

This was discussed in [1] where the equation (1) enabled the first important “sieve”
in the enhancement problem. This is an interesting result because this demonstrates
that the space-like linkage matrices are completely determined by the time-like linkage
matrices, and also because this result is disconnected from the spin representation
assignments of the component fields.

It is easy to verify that only the first, second, and fifth equation in (2) describe
independent conditions, and that the third, fourth, and sixth equations are satisfied
automatically given these, along with the “standard” conditions $\tilde{\Delta}_A^0 = u_A^T$ and $\Delta_A^0 = \tilde{u}_A^T$.
Specifically, the transpose of the fermionic analog of the sixth equation in (2) is identical to the second equation in (2). Next, if we substitute equations (3) and

---

3 The symmetry properties of the spin generators were addressed in footnote 20 in [1], and
can be proved using standard representation theory.
into the third equation in (2), we find this is satisfied automatically. Finally, if we substitute the second equation in (2) and equation (4) into the fourth equation in (2), we find that this is satisfied automatically. We have already replaced the first equation in (2) with (4). This leaves only the second and the fifth equations in (2) as independent conditions, the latter of which has already been replaced with (3).

For purposes of being helpfully specific, we restrict our focus to a four-dimensional ambient space

We also streamline our notation as follows. We designate $B_a := T_{0a}$ and $\tilde{B}_a := \tilde{T}_{0a}$ as the generators of boosts in the $a$-th spatial direction as realized on the bosons and fermions, respectively. And we designate $R^a := \frac{1}{2} \varepsilon^{abc} T_{bc}$ and $\tilde{R}^a = \frac{1}{2} \varepsilon^{abc} \tilde{T}_{bc}$ as the generators of rotations about the $a$-th coordinate axis as realized on the bosons and fermions, respectively. Moreover, the boost generator acting on ambient spinors is $B_a = \Delta^0_A$ and $\tilde{B}_a = \tilde{\Delta}^0_A$, whereby the “standard” symmetry properties become $u_A = \tilde{d}^T_A$ and $\tilde{u}_A = d^T_A$.

Using the notational refinements described in the previous paragraph, we can rewrite (3) and its fermionic analog as

$$
(B_a)_A B (d_B)_i^i = (d_A)_i^j (B_a)_j^i - (B_a)_i^j (d_A)_j^i
$$

$$
(B_a)_A B (u_B)_i^i = -(u_A)_i^j (B_a)_j^i + (B_a)_i^j (u_A)_j^i.
$$

Note that the second equation in (5) may be obtained from the first by toggling the placement of tildes, taking a matrix transpose, using the standard relationship $\tilde{d}^T_A = u^T_A$, and the property that the boost generators are symmetric matrices.

Similarly, we can rewrite the second equation in (2) and its fermionic analog as

$$
(R_a)_A B (d_B)_i^i = (d_A)_i^j (R_a)_j^i - (R_a)_i^j (d_A)_j^i
$$

$$
(R_a)_A B (u_B)_i^i = (u_A)_i^j (R_a)_j^i - (R_a)_i^j (u_A)_j^i.
$$

Note that the second equation in (6) may be obtained from the first by toggling the placement of tildes, taking a matrix transpose, using the standard relationship $\tilde{d}^T_A = u^T_A$, and the property that the rotation generators are anti-symmetric matrices.

\footnote{Cases involving diverse dimensions can be dealt with similarly. However, the general case requires notational wizardry which we think detracts from the elegance of our result, at least for this introductory presentation.}

\footnote{The boost operator $B_a$ should not be confused with the magnetic field components assigned a similar name in $\Pi$.}
Note that equations (5) and (6) describe the full implication implied by the Lorentz invariance of the linkage matrices aside from the requirement (4). Since (4) does not involve the component spin representation matrices, it follows that (5) and (6) represent the distillation we had sought.

For a given Adinkra the matrices $d_A$ and $u_A$ are readily determined. Moreover, as explained in [11], we lose no generality by selecting an ambient spin basis, whereby the matrices $(B_a)_A^B$ and $(R_a)_A^B$ are also determined. This reduces (5) and (6) into a linear algebra problem cast as a system of matrix equations in terms of the 12 yet-undetermined matrices $B_a$, $R_a$, $\tilde{B}_a$, and $\tilde{R}_a$. The linear algebra problem posed by (5) and (6) is tractable in all cases. By this we mean that if solutions exist, these may be readily obtained via routine systematic algorithmic methods. But in certain important cases, this problem actually admits a closed-form solution. It is interesting to explain one such circumstance in detail, before commenting on the general case.

Consider the case where the bosons are Lorentz scalars, whereby $B_a = 0$ and $R_a = 0$. Under this circumstance, we can derive a closed-form solution for the component spin generators $\tilde{B}_a$ and $\tilde{R}_a$, which act on the fermion fields. To do this we reorganize (5) and (6) into equivalent versions obtained by taking sums and differences of the first and second equations in each case. In this way we derive

$$
(B_a)_A^B (d_B \pm u_B)_{i\dot{i}} = (d_A \mp u_A)_{i\dot{j}} (\tilde{B}_a)_{j\dot{i}}
$$

$$
(R_a)_A^B (d_B \pm u_B)_{i\dot{i}} = (d_A \mp u_A)_{i\dot{j}} (\tilde{R}_a)_{j\dot{i}}.
$$

By considering separately the upper and lower choices for the signs, these describe four groups of equations, each group with three equations — one for each choice of the spatial coordinate index $a$ — for each of the $N$ choices of the spinor index $A$. Thus, these describe $2 \times 3 \times 4 \times N = 24N$ matrix equations.

As it turns out, for standard Adinkras, the matrices $d_A \pm u_A$ are in all cases non-singular. Moreover, these combinations describe Clifford actions, and each such combination describes a matrix which squares to plus or minus the identity. These assertions are easy to prove using the algebra defined by (2.7) and (2.8) in [11].

---

6 This counting is specific to the case of four-dimensions, of course.
7 The four-dimensional $N = 1$ Chiral multiplet is a characteristic example. Other examples with more supersymmetries, corresponding to analogs of hypermultiplets without central charges, were identified in [8].
8 Note that $N$ counts the number of one-dimensional supersymmetries.
9 These combinations also describe garden matrices [14].
| \( u_1 \) | \( d_1 \) | \( u_2 \) | \( d_2 \) | \( u_3 \) | \( d_3 \) | \( u_4 \) | \( d_4 \) |
|---|---|---|---|---|---|---|---|
| \[
\begin{pmatrix}
1 & 1 \\
1 & 0 \\
0 & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & 0 \\
0 & 1 \\
1 & 1
\end{pmatrix}
\] | \[
\begin{pmatrix}
-1 & 1 \\
-1 & 0 \\
0 & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & 0 \\
0 & -1 \\
1 & 1
\end{pmatrix}
\] | \[
\begin{pmatrix}
-1 & -1 \\
-1 & -1 \\
0 & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & 0 \\
1 & 1 \\
1 & 1
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & -1 \\
0 & 0 \\
0 & 1
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & -1 \\
-1 & 0 \\
1 & 1
\end{pmatrix}
\] |

**Table 1**: Linkage matrices corresponding to the shadow of the four-dimensional \( N = 1 \) Chiral multiplet.
The fact that \( d_A \pm u_A \) are non-singular allows us to “solve” (7) by writing

\[
\begin{align*}
(\tilde{B}_a)_{i}^{\dot{j}} &= \frac{1}{N} (B_a)_{A}^{B} \left( (d^A \pm u^A)^{-1} (d_B \mp u_B) \right)_{i}^{\dot{j}} \\
(\tilde{R}_a)_{i}^{\dot{j}} &= \frac{1}{N} (R_a)_{A}^{B} \left( (d^A \pm u^A)^{-1} (d_B \pm u_B) \right)_{i}^{\dot{j}}.
\end{align*}
\] (8)

To obtain this result we matrix multiply both sides of (7) from the left by the matrix inverse of \( d_A \pm u_A \) for any particular choice of the index \( A \). Since we must obtain the same result for \( (\tilde{B}_a)_{i}^{\dot{j}} \) and \( (\tilde{R}_a)_{i}^{\dot{j}} \) for each separate choice of the index \( A \), it follows that we can add a sum over \( A \) and then divide by \( N \), since the index \( A \) takes \( N \) values. But this implies several checks on the ability to consistently define the matrices \( \tilde{B}_a \) and \( \tilde{R}_a \). In particular, each of the \( N \) terms in the \( A \)-sums in (8) needs to describe the same matrix for each choice of the ambiguous sign and for each choice of the index \( a \). In total, this imposes \( 3(2N - 1) \) conditions for each of the two equations in (8), for a total of \( 6(2N - 1) \) extra “spin-consistency” conditions.

In [1] we derived consistency conditions describing a “first sieve” which distinguishes whether postulate enhancements of one-dimensional supermultiplets properly close the higher-dimensional supersymmetry algebra. In that paper we also described a “second sieve”, corresponding to the spin-statistics theorem, according to which fermions must assemble as spinors and bosons as tensors. The consistency conditions described above would seemingly impose a third sieve, corresponding to the requirement that the component fields resolve into representations of \( \text{spin}(1, D - 1) \).

As a helpful example, we consider the particular case of the shadow of the four-dimensional \( \mathcal{N} = 4 \) Chiral multiplet. In [1] we showed that this corresponds to a particular 2-4-2 Adinkra equivalent to the specific linkage matrices shown in Table [1]. Presently, we use these linkage matrices as a starting point, and proceed to use the above formalism to resolve the spin representation assignments of the fermions, using the assumption that the bosons enhance to scalars. We will show that our formalism properly predicts how the fermion vertices assemble as an ambient spacetime spinor. In this case we know that the one-dimensional multiplet enhances because we obtained this also by dimensional reduction, as explained in Appendix C of [1]. But our analysis here is blind to this fact, so that our analysis provides a built-in consistency check on the very formalism that we are developing. With this in mind, it is gratifying that the following analysis properly accounts for the four-dimensional spin structure.

We use the particular 4D spin structure implied by the Majorana basis described in Appendix A of [1]. In this basis, the ambient spinor boost and rotation generators

\footnote{This counting is specific to a four-dimensional ambient space.}
\[
\begin{align*}
\mathcal{B}^1 &= \frac{1}{2} \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}, & \mathcal{R}_1 &= \frac{1}{2} \begin{pmatrix}
1 & -1 \\
1 & -1
\end{pmatrix} \\
\mathcal{B}^2 &= \frac{1}{2} \begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}, & \mathcal{R}_2 &= \frac{1}{2} \begin{pmatrix}
1 & 1 \\
-1 & -1
\end{pmatrix} \\
\mathcal{B}^3 &= \frac{1}{2} \begin{pmatrix}
1 & 1 \\
-1 & -1
\end{pmatrix}, & \mathcal{R}_3 &= \frac{1}{2} \begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}
\end{align*}
\]

Table 2: Lorentz boost and rotation operators in the spinor representation using a Majorana basis.

\(\mathcal{B}_a\) and \(\mathcal{R}_a\) are exhibited in Table 2. Using these matrices, and using the linkage matrices in Table 1 we can apply (8) to address our spin-consistency condition, \(i.e.,\) the third sieve, and then use (9) to actually compute the component spin structure.

For example, using the matrix \((\mathcal{B}_1)_A^B\) specified in Table 2 the first equation in (9) implies

\[
\tilde{B}_1 = \frac{1}{8} (d^1 - u^1)^{-1} (d^3 + u^3) + \frac{1}{8} (d^3 - u^3)^{-1} (d^1 + u_1) \\
+ \frac{1}{8} (d^2 - u^2)^{-1} (d_4 + u_4) + \frac{1}{8} (d^4 - u^4)^{-1} (d_2 + u_2).
\]

It is easy to check using the specific linkage matrices in Table 1 that each of the four terms in (9) are equal. This verifies three of the 42 spin-consistency conditions in this case. It is similarly easy to check that we get exactly the same matrix for each of the four terms when we compute using the opposite relative sign in each of the matrix sums in (9). (The reader should find this an instructive exercise.) This adds four more checks, so that we have thus verified seven of the 42 spin-consistency conditions. If we do a similar thing for each of the six Lorentz generators specified by (8) we readily verify all \(6 \times 7 = 42\) requirements of our third sieve; direct computation using the matrices in Table 2 and Table 1 does show that our spin consistency test is affirmative in each case. This is an interesting non-trivial check corroborating the ability of the linkage matrices to enhance to four-dimensions.
We may now use (9) to determine the component spin representation matrices. Direct computation shows that \((\tilde{B}_1)_{i}^{j}\) has the same \(4 \times 4\) matrix form as \((B_1)_{A}^{B}\) specified in Table 1. The index structures differ, however, because the fields \(\psi_i\) and the ambient supercharges \(Q_A\) span a priori distinct vector spaces. After we repeat a similar analysis for each of the remaining five equations described by (8), we find that all three boost matrices \(B_a\) and all three rotation matrices \(R_a\) are determined to take the same \(4 \times 4\) matrix form as the respective matrices \(B_a\) and \(R_a\). Since the latter collectively generate a spinor representation, it follows that the \(B_a\) and \(R_a\) generate collectively the same spinor representation. (This is because they are described by the same specific matrices.) In this way we have shown that the vector space spanned by the supercharges \(Q_A\) and the ostensibly distinct vector space spanned by the fermion components \(\psi_i\) are isomorphic; i.e., that the fermions assemble into a spinor.

Now, we can generalize our analysis to cases where both the bosons and the fermions have potentially non-trivial spin assignments. If we allow for non-vanishing \(B_a\) and non-vanishing \(R_a\), then equation (7) generalizes to

\[
(B_a)_A^B (d_B \pm u_B) = (d_A \mp u_A) \tilde{B}_a - B_a (d_A \mp u_A) \\
(R_a)_A^B (d_B \pm u_B) = (d_A \pm u_A) \tilde{R}_a - R_a (d_A \pm u_A),
\]

(10)

where the component indices have been suppressed. Note that equation (7) is obtained from equation (10) by setting \(B_a = 0\) and \(R_a = 0\). This reduces the problem of identifying the generating matrices \(B_a, R_a, \tilde{B}_a,\) and \(\tilde{R}_a\) to straightforward linear algebra.

Since the matrices \(d_A \pm u_A\) are non-singular, we can use elementary matrix algebra to re-organize (10) into an equivalent set of six equations which each involve exactly one of the unknown Lorentz generators. For example, the fermion boost generators are determined by the equation

\[
[M_{AB}, \tilde{B}_a]_{i}^{j} = (N_{aAB})_{i}^{j},
\]

(11)

where \(M_{AB}\) and \(N_{aAB}\) are sets of specific matrices determined by the linkage matrices codifying a given one-dimensional supermultiplet, as

\[
M_{AB} = (d_A - u_A)^{-1}(d_B + u_B) \\
N_{aAB} = (B_a)_B^C (d_A - u_A)^{-1}(d_C - u_C) \\
\quad + (B_a)_A^C (d_A - u_A)^{-1}(d_C + u_C) (d_A - u_A)^{-1}(d_B + u_B),
\]

(12)

where a sum is implied over the repeated \(C\)-index but not over the repeated \(A\)-indices. Each of the matrices specified in (12) are straightforward to compute using
a given set of linkage matrices and a selected four-dimensional spin structure. Each of the matrix equations (11) are then well-posed linear algebra problems. For each choice of the $a$-index, (11) supplies a separate matrix equation for each choice of the spinor indices $A$ and $B$. The requirement that each of these equations provides an identical solution for $\tilde{B}_a$ provides the generalized statement of the spin-consistency “third sieve” requirement discussed above in the simpler setting involving de-facto scalar bosons.

The way to solve (11) is determined by elementary linear algebra. Each map $X_i^j \to [M_{AB}, X]_i^j$ describes a linear transformation that can be written as a new matrix, known as the adjoint transformation for $(M_{AB})_i^j$, called $(\text{ad} M_{AB})_i^j l^k$. The solutions to (11) are obtained by solving the new matrix equation

$$(\text{ad} M_{AB})_i^j l^k (\tilde{B}_a)_l^k = (N_{aAB})_i^j$$

(13)

by row reduction. This provides an algorithm readily implementable by computer. Part of the question of whether a compatible Lorentz structure exists becomes the question of whether (13) has a solution, providing a “fourth sieve”.

A procedure similar to the one we have outlined for determining $\tilde{B}_a$ may be brought to bear on each of the remaining Lorentz generators, $\tilde{R}_a$, $B_a$, and $R_a$. The relevant matrix equation analogous to (11) is obtained by straightforward algebraic manipulation of (11). In each case, this provides a multiplicity of linear algebra problems analogous to those codified by the different possible choices of the spinor indices $A$ and $B$ in (11). By demanding consistency among the solutions for each case we generalize the spin-consistency problem described above. For consistent cases, the relevant spin generators are determined by the same technique as described above in the case of $\tilde{B}_a$.

There is another layer of subtlety to our algorithm. In fact, equations such as (13) typically have more than one solution. So, in principle, we need to keep track of all solutions for each of the six boson spin generators ($B_a, R_a$) and each of the six fermion spin generators ($\tilde{B}_a, \tilde{R}_a$). Then we need to enforce that the spin algebra is satisfied on each set. In the general case, this would require systematic checking of the various possible combinations. We are developing software to attend to this task, and are eager to see how this plays out in specific examples. We expect to have more to say about this in the future.

---

11 Some of our sieve-like consistency requirements described in [1] and in this paper may be redundant. We leave this as future work to sort out a minimalist statement of spin-consistency.
In conclusion, we have explained a practicable algorithm for resolving the full implications implied by the imposing Lorentz invariance on postulate enhancements of one-dimensional linkage matrices. This provides a specific way to test whether spin structures can be consistently assigned to enhancements of one-dimensional supermultiplets. In cases where our spin-consistency conditions are satisfied, the method described above allows one to actually compute spin representation assignments of the enhanced component fields. We have shown also how this technique simplifies into tidy closed-form expressions for the component fermion spin generators in cases where the bosons are postulated to be scalars. This presentation adds an interesting ingredient to the paradigm described in [1], and explains how information about ambient spin assignments are accessibly coded in one-dimensional shadows.

Acknowledgements

The authors are grateful to Kevin Iga for beneficial discussions, and also to Charles Doran, Tristan Hübsch, and S. J. Gates, Jr. for collaborative work which precipitated this paper. M.F. is thankful to the Slovak Institute for Basic Research (SIBR), in Podvažie Slovakia, where much of this work was completed, for providing hospitality, peace, love, and halušky.

References

[1] M. G. Faux, K. M. Iga, and G. D. Landweber: *Dimensional Enhancement via Supersymmetry*, arXiv:hep-th/0907.3605;

[2] M. Faux and S. J. Gates, Jr.: *Adinkras: A Graphical Technology for Supersymmetric Representation Theory*, Phys. Rev. D71 (2005), 065002;

[3] Z. Kuznetsova, M. Rojas, F. Toppan, *Classification of irreps and invariants of the N-extended supersymmetric quantum mechanics*, JHEP 03 (2006) 098;

[4] S.J. Gates, Jr. and L. Rana: *A Theory of Spinning Particles for Large N Extended Supersymmetry*, Phys. Lett. B352 (1995) 50-58, [hep-th/9504025]; S.J. Gates, Jr. and L. Rana: “A Theory of Spinning Particles for Large N Extended Supersymmetry (II)”, Phys. Lett. B369 (1996) 262-268, [hep-th/9510151];

[5] S. J. Gates, Jr., W.D. Linch, III and J. Phillips: *When Superspace is Not Enough*, [hep-th/0211034];

[6] C. Doran, M. Faux, S. J. Gates, Jr., T. Hübsch, K. Iga, G. Landweber: *Off-Shell supersymmetry and filtered Clifford supermodules*, arXiv:math-ph/0603012v2 (in progress);
[7] C. Doran, M. Faux, S. J. Gates, Jr., T. Hübsch, K. Iga, G. Landweber: *On Graph Theoretic Identifications of Adinkras, Supersymmetry Representations and Superfields*, Int. J. Mod. Phys. A22 (2007) 869-930;

[8] C. Doran, M. Faux, S. J. Gates, Jr., T. Hübsch, K. Iga, G. Landweber: *On the matter of N = 2 matter*, Phys. Lett. B659 (2008) 441-446.