A generalization of the Jensen divergence: The chord gap divergence

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Abstract

We introduce a novel family of distances, called the chord gap divergences, that generalizes the Jensen divergences (also called the Burbea-Rao distances), and study its properties. It follows a generalization of the celebrated statistical Bhattacharyya distance that is frequently met in applications. We report an iterative concave-convex procedure for computing centroids, and analyze the performance of the $k$-means++ clustering with respect to that new dissimilarity measure by introducing the Taylor-Lagrange remainder form of the skew Jensen divergences.

Key words Jensen divergence, Burbea-Rao divergence, Bregman divergence, Jensen-Bregman divergence, Bhattacharyya distance, Kullback-Leibler divergence, centroid, $k$-means++.

1 Introduction

In many applications, one faces the crucial dilemma of choosing an appropriate distance $D(\cdot, \cdot)$ between data elements. In some cases, those distances can be picked up a priori from well-grounded principles (e.g., Kullback-Leibler distance in statistical estimation [1]). In other cases, one is rather left at testing several distances [2], and choose a posteriori the distance that yielded the best performance. For the latter cases, it is judicious to consider a family of parametric distances $D_\alpha(\cdot, \cdot)$, and learn the hyperparameter $\alpha$ according to the application at hand and potentially the dataset (distance selection). Thus it is interesting to consider parametric generalizations of common distances [4] to improve performance in applications.

Some distances can be designed from inequality gaps [5, 6]. For example, the Jensen divergence $J_F(p, q)$ (also called the Burbea-Rao divergence [5, 7]) is designed from the inequality gap of Jensen inequality

$$F\left(\frac{p+q}{2}\right) \leq \frac{F(p) + F(q)}{2},$$

(1)

that holds for a strictly real-valued convex function $F$:

$$J_F(p, q) = \frac{F(p) + F(q)}{2} - F\left(\frac{p+q}{2}\right).$$


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We can extend the Jensen divergence to a parametric family of skew Jensen divergences $J^\alpha_F$ (with $\alpha \in (0, 1)$) built on the convex inequality gap:

$$F((1 - \alpha)p + \alpha q) \leq (1 - \alpha)F(p) + \alpha F(q).$$  \hspace{1cm} (3)

The skew Jensen divergences $J^\alpha_F$ are defined by:

$$J^\alpha_F(p : q) = (1 - \alpha)F(p) + \alpha F(q) - F((1 - \alpha)p + \alpha q),$$  \hspace{1cm} (4)

satisfying $J^\alpha_F(q : p) = J^{1-\alpha}_F(p : q)$ and $= J^\frac{1}{2}_F(p : q) = J_F(p, q)$. Here the ‘:’ notation emphasizes the fact that the distance is potentially asymmetric: $J^\alpha_F(p : q) \neq J^\alpha_F(q : p)$. The term divergence is used in information geometry [8] to refer to the smoothness property of the distance function that yields an information-geometric structure of the space induced by the divergence.

In applications, it is rather the relative comparisons of distances rather than their absolute values that is important. Thus we may multiply a distance by any positive scaling factor and include it in the class of that distance. When $F$ is strictly convex and differentiable, the class of Jensen divergences include in the limit cases the Bregman divergences [7, 9, 10]:

$$\lim_{\alpha \to 0^+} \frac{J^\alpha_F(p : q)}{\alpha} = B_F(q : p),$$  \hspace{1cm} (6)

$$\lim_{\alpha \to 1^-} \frac{J^\alpha_F(p : q)}{1 - \alpha} = B_F(p : q),$$  \hspace{1cm} (7)

where

$$B_F(p : q) = F(p) - F(q) - (p - q)^\top \nabla F(q),$$  \hspace{1cm} (8)

is the Bregman divergence [11]. Overall, one may define the smooth parametric family of scaled skew Jensen divergences:

$$sJ^\alpha_F(p : q) = \frac{1}{\alpha(1 - \alpha)} J_F(p : q),$$  \hspace{1cm} (9)

that encompasses the Bregman divergence $B_F(p : q)$ and the reverse Bregman divergence $B_F(q : p)$ in limit cases (with $\alpha \in \mathbb{R}$).

There is a nice relationship between the Jensen divergences operating on parameters (e.g., vectors, matrices) and a class of statistical distances between probability distributions (see Figure 1): Let $\{p(x; \theta)\}_\theta$ be an exponential family [10] (includes the Gaussian family and the finite discrete “multinoulli” family) with convex cumulant function $F(\theta)$. Then the skew Bhattacharyya distance [12]:

$$\text{Bhat}_\alpha(p : q) = - \log \int p(x)^{1-\alpha} q(x)^\alpha dx,$$  \hspace{1cm} (10)

$$\text{Bhat}_\alpha(p : q) = - \log \int p(x) \left( \frac{q(x)}{p(x)} \right)^\alpha dx,$$  \hspace{1cm} (11)
Bhat_α(p : q) = −∫ p(x)^{1−α} q(x)^α dx

J_α^F(θ_p : θ_q) = (F(θ_p)F(θ_q))^{α} − F((θ_pθ_q)^{α})

BF(θ_q : θ_p) = F(θ_q) − F(θ_p) − (θ_q − θ_p)^T ∇F(θ_q)

KL(p : q) = ∫ p(x) log p(x)/q(x) dx

Figure 1: Links between the statistical skew Bhattacharyya distances and parametric skew Jensen divergences when distributions belong to the same exponential family.

The proof of the skewed Bhattacharyya distance converging to the Kullback-Leibler divergence \[1\] proceeds as follows: We have:

\[
\left( \frac{q(x)}{p(x)} \right)^{α} = \exp \left( α \log \frac{q(x)}{p(x)} \right) \overset{α→0}{≈} 1 + α \log \frac{q(x)}{p(x)}.
\]

Thus we get:

\[
\log p(x)^{1−α} q(x)^α \overset{α→0}{≈} \log \int \left( p(x) + α \log \frac{q(x)}{p(x)} \right) dx = \log(1 − αKL(p : q)),
\]

and therefore

\[
\lim_{α→0^+} \frac{1}{α} Bhat_α(p : q) = KL(p : q),
\]

since \(\log(1 + x) ≈ x\) when \(x → 0\).

In statistical signal processing, information fusion and machine learning, one often considers the skew Bhattacharyya distance \[13, 14, 15\] or the Chernoff distance \[16, 17, 18\] for exponential families (e.g., Gaussian/multinoulli): This highlights the important role in disguise of the equivalent skew Jensen divergences (see Eq. \[12\]).
Figure 2: The triparametric chord gap divergence: The vertical distance between the upper chord $U$ and the lower chord $L$ is non-negative and zero iff. $p = q$.

The paper is organized as follows: Section 2 introduces the novel triparametric family of chord gap divergences that generalize the skew Jensen divergences (§2.1), describe several properties (§2.2), and deduce a generalization of the statistical Bhattacharyya distance (§2.3). Section 3 considers the calculation of the centroid (§3.1) for the chord gap divergences, and report probabilistic guarantee of the k-means++ seeding (§3.2) by highlighting the Taylor-Lagrange remainder forms of those divergences.

2 The chord gap divergence

2.1 Definition

Let $F : \mathcal{X} \to \mathbb{R}$ be a strictly convex function. For $\alpha, \beta \in (0, 1)$ with $\alpha \neq \beta$ and $\alpha \beta < 1$, the chord

$$L = \left[ \left((pq)_{\alpha}, F((pq)_{\alpha})\right), \left((pq)_{\beta}, F((pq)_{\beta})\right) \right]$$

(19)
is below the distinct chord

$$U = \left[ \left(p, F(p)\right), \left(q, F(q)\right) \right].$$

(20)

Thus we can define a divergence as the vertical gap between these two U/L chords for a given coordinate $x \in \left((pq)_{\alpha}, (pq)_{\beta}\right)$:

$$J_F^{\alpha, \beta, \gamma}(p : q) = \left(F(p)F(q)\right)_{\gamma} - \left(F((pq)_{\alpha})F((pq)_{\beta})\right)_{\lambda}$$

(21)
such that $\left((pq)_{\alpha}(pq)_{\beta}\right)_{\lambda} = (pq)_{\gamma}$ with $\gamma \in (\alpha, \beta)$ (see Figure 2). A calculation shows that:

$$\lambda = \lambda(\alpha, \beta, \gamma) = \frac{\gamma - \alpha}{\beta - \alpha},$$

(22)
or that
\[
\gamma = \lambda(\beta - \alpha) + \alpha, \tag{23}
\]
for \( \lambda \in [0, 1] \) (\( \gamma \in [\alpha, \beta] \)) when \( \alpha \neq \beta \), so that we get:
\[
J_{F}^{\alpha,\beta,\gamma}(p : q) = (F(p)F(q))\gamma - (F((pq)_\alpha)F((pq)_\beta))\frac{\lambda - \alpha}{\beta - \alpha}, \quad \gamma \in [\alpha, \beta] \tag{24}
\]
\[
= (F(p)F(q))\lambda(\beta - \alpha) + (F((pq)_\alpha)F((pq)_\beta))\lambda, \quad \lambda \in [0, 1]. \tag{25}
\]

2.2 Properties of the chord gap divergence and subfamilies

We have:
\[
J_{F}^{\alpha,\alpha,\alpha}(p : q) = J_{F}^{\alpha}(p : q), \tag{26}
\]
\[
J_{F}^{\alpha,1,\gamma}(p : q) = J_{F}(p : q), \tag{27}
\]
\[
J_{F}^{\alpha,\beta,\gamma}(q : p) = J_{F}^{1-\alpha,1-\beta,1-\gamma}(p : q), \tag{28}
\]

since \( \lambda(1 - \alpha, 1 - \beta, 1 - \gamma) = \frac{\gamma - \alpha}{\beta - \alpha} = \lambda(\alpha, \beta, \gamma) \) using the fact that \((ab)_{1-\delta} = (ba)_\delta\) for \( \delta \in [0, 1] \). Thus we also have:
\[
J_{F}^{1-\alpha,1-\alpha,1-\alpha}(p : q) = J_{F}^{\alpha}(q : p). \tag{29}
\]

More importantly, we can express the chord gap divergence as the difference of two skew Jensen divergences (Figure 2):
\[
J_{F}^{\alpha,\beta,\gamma}(p : q) = J_{F}^{\gamma}(p : q) - J_{F}^{\alpha}(pq)_\alpha : (pq)_\beta), \tag{30}
\]
with \( \lambda = \frac{\gamma - \alpha}{\beta - \alpha} \) or \( \gamma = \lambda(\beta - \alpha) + \alpha \) for \( \lambda \in [0, 1] \) and \( \gamma \in [\alpha, \beta] \). Thus the chord gap divergence can be interpreted as a truncated skew Jensen divergence.

A biparametric subfamily \( J_{F}^{\beta,\gamma} \) of \( J_{F}^{\alpha,\beta,\gamma} \) is obtained by setting \( \alpha = 0 \) so that \((pq)_\alpha = p\), so that the two upper/lower chords \( L \) and \( U \) coincide at extremity \( p \):
\[
J_{F}^{\beta,\gamma}(p : q) = (F(p)F(q))\gamma - (F(p)F((pq)_\beta))\frac{\gamma}{\beta}, \tag{31}
\]
\[
= \left(\frac{\gamma}{\beta} - \gamma\right)F(p) + \gamma F(q) - \frac{\gamma}{\beta} F((pq)_\beta), \tag{32}
\]
\[
= \gamma \left( F(p) + F(q) - \frac{1}{\beta} F((pq)_\beta) \right). \tag{33}
\]

When \( \beta = \frac{1}{2} \), we find that \( J_{F}^{\gamma}(p : q) = 2\gamma J_{F}(p : q) \):
\[
J_{F}^{\gamma}(p : q) = 2\gamma \left( \frac{F(p) + F(q)}{2} - F\left( \frac{p + q}{2} \right) \right) \tag{34}
\]
\[
is the ordinary (\( \gamma \)-scaled) Jensen divergence. When \( \beta \to 0 \), we have \( \lim_{\beta \to 0} \frac{1}{\beta} J_{F}^{\beta,\gamma}(p : q) = B_{F}(q : p) \) (with \( \gamma \in (0, \beta) \)) since \( -\frac{1}{\beta} F((pq)_\beta) \simeq -\frac{1}{\beta} - (q - p)^\top \nabla F(p) \) using a first-order Taylor expansion.
We may also consider $\beta = 1 - \alpha$, and define the biparametric subfamily:

$$J^\alpha_{\beta,\gamma}(p : q) = (F(p)F(q))_\gamma - (F((pq)_\alpha)F(((pq)_{1-\alpha}))_\gamma - \alpha, \quad \gamma \in [\alpha, 1 - \alpha], \quad \lambda \in [0, 1]. \quad (36)$$

Chord gap divergences operating on matrix arguments can be obtained by taking strictly convex matrix generators [11] (e.g., $F$ with $X$ symmetric positive definite matrices belong to the natural parameter space, and holds for distributions belonging to the same exponential families since (Note that we need the integral to converge properly in order to define $\Gamma$).

When interpolated distribution by generalizing the notion of Bhattacharya distance, as expected.

It follows that:

$$\text{Bhat}_{\alpha,\beta,\gamma}(\theta_p : \theta_q) = \text{Bhat}_\gamma(\theta_p : \theta_q) - \text{Bhat}_\lambda((\theta_p\theta_q)_\alpha : (\theta_p\theta_q)_\beta). \quad (38)$$

Note that when $\alpha = \beta$, we have:

$$p(x; (\theta_p\theta_q)_\alpha)^{1-\lambda}p(x; (\theta_p\theta_q)_\beta) = p(x; (\theta_p\theta_q)_\alpha), \quad (40)$$

and therefore the denominator becomes $\int p(x; (\theta_p\theta_q)_\alpha)dx = 1$, and we recover the skew Bhattacharyya distance, as expected.

We shall extend the generalized Bhattacharyya divergence of Eq. 39 to arbitrary distributions by generalizing the notion of interpolated distribution:

$$p(x; (\theta_p\theta_q)_\delta) = \Gamma_\delta(p(x; \theta_p), p(x; \theta_q)). \quad (41)$$

When $\delta$ ranges from 0 to 1, we obtain a Bhattacharyya arc linking distribution $p(x; \theta_p)$ to distribution $p(x; \theta_q)$ (the arc is called an exponential or $e$-geodesic in information geometry [8]).

We define:

$$\Gamma_\delta(p(x), q(x)) = \frac{p(x)^{1-\delta}q(x)^{\delta}}{Z_\delta(p(x) : q(x))}, \quad (42)$$

with

$$Z_\delta(p(x) : q(x)) = \int p(x)^{1-\delta}q(x)^{\delta}d\nu(x). \quad (43)$$

Note that we need the integral to converge properly in order to define $\Gamma_\delta(p(x), q(x))$. This always holds for distributions belonging to the same exponential families since $(\theta_p\theta_q)_\delta$ is guaranteed to belong to the natural parameter space, and

$$Z_\delta(p(x; \theta_p) : p(x; \theta_q)) = \exp(-J^\delta_{\beta, \gamma}(\theta_p : \theta_q)). \quad (44)$$
By extension, the triparametric Bhattacharryya distance can be defined by:

\[
\text{Bhat}^\alpha,\beta,\gamma(p(x) : q(x)) = -\log \left( \frac{\int p(x)^1-q(x)^\gamma d\nu(x)}{\Gamma_\alpha(p(x), q(x))^{1-\gamma} \Gamma_\beta(p(x), q(x))} \right). 
\]

Thus we explicitly define the generalized Bhattacharyya distance by:

\[
\text{Bhat}^\alpha,\beta,\gamma(p(x) : q(x)) = -\log \left( \frac{\int p(x)^1-q(x)^\gamma d\nu(x)}{\int \left( \frac{p(x)^1-q(x)^\gamma d\nu(x)}{\int p(x)^1-q(x)^\gamma d\nu(x)} \right)^{1-\lambda} \left( \frac{p(x)^1-q(x)^\gamma d\nu(x)}{\int p(x)^1-q(x)^\gamma d\nu(x)} \right)^\lambda d\nu(x)} \right). 
\]

Notice that when \( \alpha = \beta \), for any \( \lambda \in [0, 1] \), the denominator collapses to one, and we find that \( \text{Bhat}_{\alpha,\beta,\gamma}(p(x) : q(x)) = \text{Bhat}_\alpha(p(x) : q(x)) \), as expected.

For multivariate gaussians/normals belonging to the family \( \mathcal{N}(\mu, \Sigma) : \mu \in \mathbb{R}^d, \Sigma \in \mathbb{P}_{++}^d \), we have the natural parameter \( \theta = (v, M) = (\Sigma^{-1}\mu, -\frac{1}{2}\Sigma^{-1}) \), and the cumulant function:

\[
F(v, M) = \frac{d}{2} \log 2\pi - \frac{1}{2} \log | - 2M | - \frac{1}{4} v^\top M^{-1} v,
\]

that can also be expressed in the usual parameters:

\[
F(\mu, \Sigma) = \frac{1}{2} \log(2\pi)^{d/2} | \Sigma | + \frac{1}{2} \mu^\top \Sigma^{-1} \mu.
\]

We have:

\[
(\theta_p\theta_q)_{\delta} = ((1 - \delta)\Sigma_p^{-1}\mu_p + \delta\Sigma_q^{-1}\mu_q, -\frac{1}{2}\Sigma_p^{-1} - \frac{1}{2}\Sigma_q^{-1}),
\]

so that we get \( \text{Bhat}^\alpha,\beta,\gamma \) for multivariate Gaussians. See \( [21] \) for applications clustering multivariate normals.

### 3 Centroid-based clustering

Bhattacharyya clustering is often used in statistical signal processing, information fusion, and machine learning (see \( [13, 22, 23, 24, 25, 26] \) for some illustrative examples). A popular type of clustering algorithms are center-based clustering, where each cluster stores a prototype, and data are assigned to the cluster with the closest prototype wrt. a distance function. The cluster prototypes are then updated, and the algorithm iterates until (local) convergence. This scheme includes the \( k \)-means and the \( k \)-medians \( [27] \). Lloyd \( k \)-means heuristic updates the prototype \( c \) of a cluster \( X \) by choosing its center of mass \( c = \frac{1}{|X|} \sum_{x \in X} x \) that minimizes the cluster variance: \( \min c \sum_{x \in X} \| x - c \|^2 \) (this holds for any Bregman divergence too \( [10] \)).
3.1 Chord gap divergence centroid

We extend \( k \)-means for a weighted point set:

\[
P = \{(w_1, p_2), \ldots, (w_n, p_n)\},
\]

with \( w_i > 0 \) and \( \sum_i w_i = 1 \), using the chord gap divergence by solving the following minimization problem:

\[
\min_x E(x) = \sum_{i=1}^{n} w_i J^{\alpha, \beta, \gamma}_F(p_i : x).
\]

By expanding the chord gap divergence formula and removing all terms independent of \( x \), we obtain an equivalent minimization problem as a difference of convex function programming [28]:

\[
\min_x E(x) = \min_x A(x) - B(x),
\]

with

\[
A(x) = \sum_{i=1}^{n} (F(p_i) F(x))\gamma,
\]

\[
B(x) = \sum_{i=1}^{n} (F((p_i x)_{\alpha}) F((p_i x)_{\beta}))\lambda,
\]

both strictly convex functions. It follows a concave-convex procedure [29] (CCCP) solving locally \( \min_x A(x) - B(x) \): initialize \( x_0 = p_1 \) and then iteratively update as follows:

\[
\nabla A(x_{t+1}) = \nabla B(x_t).
\]

When the reciprocal gradient \( \nabla A^{-1} \) (such that \( \nabla A^{-1}(\nabla A(x)) = x \)) is available in closed form, we end up with the following update:

\[
x_{t+1} = \nabla A^{-1}(\nabla B(x_t)).
\]

Since we have

\[
\nabla A(x) = n\gamma \nabla F(x),
\]

\[
\nabla B(x) = \sum_i (1 - \lambda)\alpha \nabla F((p_i x)_{\alpha}) + \lambda \beta \nabla F((p_i x)_{\beta}),
\]

the update rule is

\[
x_{t+1} = \nabla F^{-1}\left(\frac{1}{\gamma} \sum_i w_i((1 - \lambda)\alpha \nabla F((p_i x_t)_{\alpha}) + \lambda \beta \nabla F((p_i x_t)_{\beta}))\right).
\]

When \( \alpha = \beta = \gamma \), we find the simplified update rule:

\[
x_{t+1} = \nabla F^{-1}\left(\sum_i w_i \nabla F((p_i x_t)_{\alpha})\right),
\]

corresponding to the skew Jensen divergences [7]. Note that it is enough to improve iteratively the prototypes to get a variational Lloyd’s \( k \)-means [30] that guarantees monotone convergence to a (local) optimum.
3.2 Performance analysis of k-means++

For high-performance clustering, one may use k-means++ [31] that is a guaranteed probabilistic initialization of the cluster prototypes. To get an expected competitive ratio of $2U^2(1 + V)(2 + \log k)$ [30], we need to upper bound:

- $U$ such that the divergence $D = J^\alpha_F(p : q)$ satisfies the $U$-triangular inequality $D(x : z) \leq U(D(x : y) + D(y : z))$, and
- $V$ such that the divergence satisfies the symmetric inequality $D(y : x) \leq V D(x : y)$.

The proof follows the proof reported in [30] for total Jensen divergences once we can express the divergences in their Taylor-Lagrange remainder forms:

\[ D(p : q) = (p - q)^\top H_D(p : q)(p - q), \tag{61} \]

where $H_D(p : q) \succ 0$. For example, the Taylor-Lagrange remainder form of the Bregman divergence [32] is obtained from a first-order Taylor expansion with the exact Lagrange remainder:

\[ B_F(p : q) = \frac{1}{2} (p - q)^\top \nabla^2 F(x)(p - q), \tag{62} \]

for some $\xi \in [p, q]$. This expression can be interpreted as a squared Mahalanobis distance:

\[ M_Q(p, q) = (p - q)^\top Q(p - q), \tag{63} \]

with precision matrix $Q = \frac{1}{2} \nabla^2 F(x) \succ 0$ depending on $p$ and $q$. Any squared Mahalanobis distance satisfies $U = 2$ (see [33]) and $V = 1$, and can be interpreted as a squared norm-induced distance:

\[ M_Q(p, q) = \|Q^{\frac{1}{2}}(p - q)\|^2. \tag{64} \]

We report the Taylor-Lagrange remainder form of the skew Jensen divergences: There exists $\xi_1, \xi_2 \in [p, q]$, such that the skew Jensen divergence can be expressed as $J^\alpha_F(p : q) = (p - q)^\top H_F^\alpha(p : q)(p - q)$, with

\[ H_F^\alpha(p : q) = \frac{1}{2} \alpha(1 - \alpha)(\alpha \nabla^2 F(\xi_1) + (1 - \alpha) \nabla^2 F(\xi_2)). \tag{65} \]

The proof relies on introducing the skew Jensen-Bregman (JB) divergence [7] defined by

\[ JB^\alpha_F(p : q) = (1 - \alpha)B_F(p : (pq)_\alpha) + \alpha B_F(q : (pq)_\alpha), \tag{66} \]

and observing the $JB^\alpha_F(p : q) = J^\alpha_F(p : q)$ since $p - (pq)_\alpha = \alpha(p - q)$ and $q - (pq)_\alpha = (1 - \alpha)(q - p)$ (and therefore the $\nabla F((pq)_\alpha)$-terms cancel out). Then we apply the Taylor-Lagrange remainder form of Bregman divergences of Eq. 62 to get the result. Notice that when $\alpha \to 0$ or $\alpha \to 1$, the scaled skew Jensen difference tend to Bregman divergences, and we have $\lim_{\alpha \to 0} \frac{J^\alpha_F(p : q)}{\alpha(1 - \alpha)} = \frac{1}{2} (p - q)^\top \nabla^2 F(\xi_1)(p - q) = B_F(p : q)$ for $\xi_1 \in [p, q]$, and $\lim_{\alpha \to 1} \frac{J^\alpha_F(p : q)}{\alpha(1 - \alpha)} = \frac{1}{2} (p - q)^\top \nabla^2 F(\xi_2)(p - q) = B_F(q : p)$ for $\xi_2 \in [p, q]$, as expected.
Using expression of Eq. 30 for the chord gap divergence, and the fact that $(pq)_\alpha - (pq)_\beta = (\alpha - \beta)(q - p)$, we get the Taylor-Lagrange form of the chord gap divergence $J_{F}^{\alpha,\beta,\gamma} = (p - q)^\top H_{F}^{\alpha,\beta,\gamma}(p : q)(p - q)$ with

$$
H_{F}^{\alpha,\beta,\gamma}(p : q) = \frac{1}{2}\gamma(1 - \gamma)\nabla^2 F(\xi') - \frac{1}{2}\lambda(1 - \lambda)(\alpha - \beta)^2\nabla^2 F(\xi''),
$$

(67)

$$
= \frac{1}{2}(\gamma(1 - \gamma)\nabla^2 F(\xi') - (\gamma - \alpha)(\gamma - \beta)\nabla^2 F(\xi'')),
$$

(68)

for $\xi', \xi'' \in \mathcal{X}$.

An alternative proof considers the Taylor first-order expansion of $F$ with exact Lagrange remainder:

$$
F(x) = F(a) + (x - a)^\top \nabla F(a) + \frac{1}{2}(x - a)^\top \nabla^2 F(\xi)(x - a), \quad \xi \in [a, x].
$$

(69)

Therefore we get the following Taylor expansions with exact Lagrange remainders:

$$
F(p) = F((pq)_\alpha) + \alpha(p - q)^\top \nabla F((pq)_\alpha) + \frac{1}{2}\alpha^2(p - q)^\top \nabla^2 F(\xi_1)(p - q), \quad \xi_1 \in (p, (pq)_\alpha),
$$

(70)

$$
F(q) = F((pq)_\alpha) + (1 - \alpha)(q - p)^\top \nabla F((pq)_\alpha) + \frac{1}{2}(1 - \alpha)^2(q - p)^\top \nabla^2 F(\xi_2)(q - p), \quad \xi_2 \in (q, (pq)_\alpha).
$$

Multiplying the first equation by $1 - \alpha$ and the second equation by $\alpha$ and summing up, we obtain:

$$(1 - \alpha)F(p) + \alpha F(q) = F((pq)_\alpha) + \frac{1}{2}\alpha^2(1 - \alpha)(p - q)^\top \nabla^2 F(\xi_1)(p - q) + \frac{1}{2}(1 - \alpha)^2\alpha \nabla^2 F(\xi_2)(p - q),
$$

(72)

since the gradient terms cancel out, and we get:

$$
J_{F}^{(\alpha)}(p : q) = (1 - \alpha)F(p) + \alpha F(q) - F((pq)_\alpha),
$$

(73)

$$
= \frac{1}{2}\alpha(1 - \alpha)(p - q)^\top (\alpha \nabla^2 F(\xi_1) + (1 - \alpha)\nabla^2 F(\xi_2))(p - q).
$$

(74)

Thus it follows the Taylor-Lagrange remainder form of skew Jensen divergences:

$$
J_{F}^{(\alpha)}(p : q) = \frac{\alpha(1 - \alpha)}{2}(p - q)^\top (\alpha \nabla^2 F(\xi_1) + (1 - \alpha)\nabla^2 F(\xi_2))(p - q).
$$

(75)

When dealing with a finite (weighted) point set $\mathcal{P}$, let

$$
\rho = \frac{\sup_{\xi', \xi'', p, q \in \text{co}(\mathcal{P})} \| (\nabla^2 F(\xi'))^{\frac{1}{2}}(p - q) \|}{\inf_{\xi', \xi'', p, q \in \text{co}(\mathcal{P})} \| (\nabla^2 F(\xi''))^{\frac{1}{2}}(p - q) \|} < \infty,
$$

(76)

where $\text{co}(\mathcal{P})$ denotes the convex closure of $\mathcal{P}$. Then it comes that $U = O_\rho(1)$ and $V = O_\rho(1)$ so that $k$-means++ probabilistic seeding is $O_\rho(\log k)$ competitive for the chord gap divergence.
4 Concluding remarks

We introduced the chord gap divergence as a generalization of the skew Jensen divergences \cite{7,11}, studied its properties and obtained a generalization of the skew Bhattacharrya divergences. We showed that the chord gap divergence centroid can be obtained using a convex-concave iterative procedure \cite{7}, and analyzed the $k$-means++ \cite{31} performance by giving the Taylor-Lagrange forms of the skew Jensen and chord gap divergences. We expect our contributions to be useful for the signal processing, information fusion and machine learning communities where the Bhattacharya \cite{34,35} or Chernoff information \cite{2,16} is often used. In practice, the triparametric chord gap divergence shall be tuned according to the application at hand (and the dataset for supervised tasks using cross-validation for example).

Public Java\textsuperscript{TM} source code is available for reproducible research: 

\begin{verbatim}
http://www.lix.polytechnique.fr/~nielsen/CGD/
\end{verbatim}

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