Relativistic Classical and Quantum Nonlinear Phenomena in the Induced Processes on Free Electrons

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Abstract

With the appearance of superpower laser sources of relativistic/ultrarelativistic intensities in the last decade, the laser-QED-vacuum-matter interaction physics has entered a new phase that makes real the observation of many nonlinear quantum electrodynamic (QED) and classical-quantum-mechanical phenomena revealed more than fore-five decades ago, serious advance in new generation of laser-plasma accelerators of ultrahigh energies, nuclear fusion etc. Hence, the present review article will help explorers-experimentalists in this field to attract attention on the fundamental properties and peculiarities of the dynamics of induced free-free transitions at high and superhigh intensities of stimulated radiation fields. In this connection it is of special interest the induced Cherenkov, Compton, undulator/wiggler coherent processes, as well as cyclotron resonance in a medium – possessing with nonlinear resonances of threshold nature and leading to many important nonlinear effects or applications, which are considered in this review.
I. INTRODUCTION

The unprecedented development of laser technologies in recent two decades, mainly due to chirped pulse amplification technique \[1\], has allowed increase of intensity of laser sources in the first stage a six orders of magnitude \[2–6\] reaching the relativistic intensities \(10^{16} \text{ W/cm}^2\) – \(10^{18} \text{ W/cm}^2\) in the infrared and optical domains, respectively, then exceeding these values up to \(10^{22} \text{ W/cm}^2\) for current superpower laser beams of ultrarelativistic intensities \[7, 8\]. In many laboratories \[9–12\] compact lasers can deliver \(1–10\) petawatt short pulses, with focused intensities as high as \(10^{22} \text{ W cm}^{-2}\). Next generation multipetawatt and exawatt optical laser systems \[13–15\] will be capable to deliver ultrahigh intensities exceeding \(10^{25} \text{ W cm}^{-2}\), which are well above the ultrarelativistic regime of electron-laser interaction. At such intensities the radiation-matter interaction enters the phase of ultrafast dynamics in supershort time scales, with formation of such important fields in Condensed-Matter-Physics that the Attoscience \[16–21\], and Relativistic Optics \[8, 22–25\] are.

The appearance of laser sources with ultrarelativistic intensities has opened real possibilities for revelation of many nonlinear electrodynamic phenomena at the interaction of superpower radiation fields with high brightness accelerator beams \[26\], relativistic plasma targets of solid densities \[27–32\], and QED vacuum \[33\]. Laser sources of such enormous intensities predetermine, in particular, the future of laser accelerators \[34–39\] of superperhigh energies, including laser-plasma accelerators \[27, 40, 47\]. Thus, currently available optical lasers provide \(\xi\) up to 100, meanwhile with the next generation of laser systems one can manipulate with beams at \(\xi > 1000\). In such ultrastrong laser fields, electrons can reach to ultrarelativistic energies. However to obtain ultrarelativistic net energies for field-free electrons we need the third body for satisfaction of conservation laws for energy-momentum of a free electron (after the interaction) with absorption/radiation of photons in the field of a plane monochromatic wave \[26\], or take advantage of nonplane character of tightly focused supershort laser pulses \[48\]. Nevertheless, the actual way to get net energy exchange of relativistic electrons is using of coherent processes of laser-particle interaction with the additional resonances. Among those the induced Cherenkov, Compton, and undulator processes are especially of interest \[26\]. These are coherent induced processes where a critical intensity of stimulated wave exists (because of coherent character of corresponding spontaneous process) above which the induced process proceeds only in one direction - coherent
radiation (wave amplification), or absorption (particle acceleration) depending on the particle initial velocity with respect to resonance value (satisfying the condition of coherency) of each process. The latter is the most important feature of nonlinear resonance inherent in these processes to get high net energy change of particles, specifically for laser acceleration [33], or free electron laser (FEL) [49, 50], as well as to generate relativistic electron beams of low energy spreads and emittances due to the threshold nature of nonlinear resonance in these induced coherent processes.

In principle, the realization of acceleration or inverse problem of FEL in any laser-induced process requires the possible largest coherent length of particle-laser stimulated interaction at which the accumulation of particle energy in the acceleration regime, or coherent radiation - in the wave amplification regime take place on the whole coherent length of certain process. In other cases, the existence of an additional resonance (e.g., in a static magnetic field [51–56]) is necessary where the particle-wave interaction cross section enhances by many orders in magnitude. These are the linear resonances well enough investigated during the past decades. Here we will consider laser-induced processes on free electrons with the large effective gain of interaction due to nonlinear resonance achieved in the given field of a strong laser radiation. In the induced Cherenkov, Compton, and undulator processes such resonance is adequate to fulfilment of condition of coherency between the particle and wave achieved in the field at the aforementioned critical intensity of the total wave-field. In the result of such interaction, at the laser intensities larger than critical value a nonlinear threshold phenomenon of particles “reflection” or capture take place -accelerating or decelerating electrons on the shortest interaction lengths, even shorter than a laser wavelength [57–59].

In contrast to induced Cherenkov process [57] where the intensities of applied laser fields are confined by ionization threshold of dielectric media [60, 61] in vacuum processes – in the induced Compton [58] and undulator [59] ones there is no restriction on the electromagnetic (EM) field strengths that allows to apply current superpower laser beams of ultrarelativistic intensities [8] for acceleration of particles up to superhigh energies. The other practical difference between these induced processes is in the relativistic character of initial resonance width of particles to reach the threshold value of corresponding nonlinear resonance in the field for “reflection”/capture phenomenon. Thus, in contrast to Cherenkov and undulator processes where practically initial relativistic/ultrarelativistic particles are needed for realization of nonlinear Cherenkov resonance in gaseous media [62], or nonlinear resonance
a magnetic undulator with the step of the order of a few cm \(^{63}\), induced Compton mechanism for particles “reflection”/capture and, consequently, acceleration regime practically may be realized for arbitrary initial energies of the particles \(^{64}\). In particular, using laser pulses of near frequencies one can accelerate initially nonrelativistic beams, or even particles in rest. Second, at the same frequencies of counterpropagating laser pulses in the induced Compton process one can achieve the cancellation of particles transverse momenta acquired in the field to obtain quasi-collinear particles bunches from the emerging relativistic electron targets (elimination of a bunch angular divergence), as it has been proposed in the paper \(^{65}\) for generation of uniform relativistic electron layers at the coherent Compton backscattering on the ultra-thin solid foils (see, also, \(^{66, 67}\)).

A promising way for achieving of laser-driven electron acceleration is the use of a plasma medium \(^{68}\). However, the laser-plasma accelerator schemes face problems connected with the inherent instabilities in laser-plasma interaction processes. On the other hand, the spectrum of direct acceleration mechanisms of charged particles by a single laser pulse is very restricted, since one should use the laser beams focused to subwavelength waist radii, or use subcycle laser pulses, or use radially polarized lasers, violating the plane character of a coherent laser pulse for particles acceleration by a single wave-pulse in vacuum without the third body. All these scenarios with different field configurations have been investigated both theoretically and experimentally in the works \(^{38, 69–85}\). Note that beside the focusing of a laser pulse to subwavelength waist radii we can get nonplane laser beam by terminating the field, either by reflection, absorption, or diffraction \(^{86}\). The proof of principle experiment of this type has been reported in Ref. \(^{87}\). Here it is used initially relativistic electron beam and a moderately strong laser pulse. To obtain dense enough electron bunches it is reasonable to consider electrons acceleration from nanoscale solid plasma-targets \(^{65, 88–90}\) at intensities high enough to separate all electrons from ions \(^{91}\). Thus, combining these two schemes one can obtain ultrarelativistic solid density electron bunches \(^{92–99}\). Such bunches can be used to obtain high-flux of positrons, \(\gamma\)-quanta \(^{100, 101}\) with possible applications in material science, medicine, and nuclear physics.

The recent achievements in the laser technology related to implementation of supershort–femtosecond laser sources and subsycle pulses of relativistic intensities exceeding the intra-atomic fields open real opportunities for realization of laser-plasma accelerators of ultrahigh energies \(^{27, 40, 47}\). These schemes are based on the laser-plasma-wake-field and laser-beat-
wave mechanisms of laser-assisted acceleration in plasma. The interaction of such powerful radiation fields with the plasma has led to generation of quasi-monoenergetic relativistic electron beams in laser-driven plasma accelerators [77–85]. Concerning the interaction of ultrashort laser pulses of relativistic intensities with the free electrons in vacuum, i.e. the implementation of laser accelerators, one should note that experiments in this area are gathering power at present.

One of the first laser accelerator concepts is the Inverse Cherenkov Accelerator (ICA). The ICA scheme has been investigated since the advent of lasers [see, for example, [102]]. The appearance of powerful laser sources already in the next decade after the first lasers has stimulated comprehensive theoretical [57, 103–112] and experimental [113–121] investigations towards the implementation of both induced Cherenkov problems –ICA and Cherenkov laser realization– in diverse schemes of multiphoton interaction that makes a phase velocity matching of EM wave with a particle by controlling the refraction index of gaseous medium so as to be a wave phase velocity in a dielectric medium less than light speed in vacuum. Various interaction schemes of electrons with stimulating radiation have been considered as with ultrarelativistic electron beams in a gaseous medium for the optical region [103, 106, 107, 110, 112], as well as with mildly relativistic beams in dielectric waveguides for the microwave region [122–131]. In the latter case, the electron beam passes over the dielectric in the vacuum –surface Cherenkov process– that enables one to avoid the impeding factors of the medium (the multiple scattering of electrons on atoms, the ionization losses in the medium etc.). Besides, the usage of solid-state dielectric as a slow-wave structure (with rather high dielectric constant $\varepsilon, \varepsilon - 1 \sim 1$) enables the achievement of electron-wave synchronism by mildly or moderately relativistic electron beams.

It is important to note that the effect of critical field exists even in a very week wave field if the interaction occurs in the region close to the initial Cherenkov resonance. Therefore one can not escape these peculiarities restricting only the intensity of the external wave field and solve the problem in the scope of the linear theory even for very week wave fields [132]. Note that regarding the induced Cherenkov process a series of systematical wrong works have been made during the last two decades of past century, which were discussed in special review [132] devoted to consideration of those results. In the mentioned review the all principal mistakes have been shown in detail, therefore in the current review article we will not repeat the analysis of those papers which reader can get in the review [132].
The particle “reflection” phenomenon from a slowed plane wave pulse has been applied also for cases where the group velocity of a wave pulse due to the dispersion can be less than light speed in vacuum \cite{133–136}: in Ref. \cite{133, 134} –for the case of a plasma, in Ref. \cite{135, 136} –at the focusing of an ultrashort laser pulse in vacuum, in the so-called by authors “dispersion-dominated propagation regime”.

Nonlinear resonance of threshold nature requiring shortest acceleration length also takes place at the charged particle interaction with strong laser radiation and static magnetic field along the wave propagation direction (when the resonant effect of the wave on the particle rotational motion in the uniform magnetic field is possible) in a medium \cite{53}. In vacuum, as a result of the interaction of a charged particle with a monochromatic EM wave and uniform magnetic field the resonance created at the initial moment for the free-particle velocity automatically holds throughout the whole interaction process due to the equal Doppler shifts of the Larmor and wave frequencies in the field. This phenomenon is known as “Autoresonance” \cite{51, 52}. This property of cyclotron resonance in vacuum makes possible as particles acceleration \cite{137}, as well as the creation of a generator of coherent microwave radiation/radioemission by an electron beam, namely, a cyclotron resonance maser (CRM) \cite{138–145}.

From the point of view of quantum theory the relativistic nonequidistant Landau levels of the particle in the wave field become equidistant in the autoresonance with the quantum recoil at the absorption/emission of photons by the particle. In addition, the dynamic Stark effect of the wave electric field on the transversal bound states of the particle does not violate the equidistance of Landau levels in the autoresonance. Then the inverse process, that is, multiphoton resonant excitation of Landau levels by strong EM wave and, consequently, the particle acceleration in vacuum due to cyclotron resonance, in principle, is possible.

In a medium with arbitrary refraction properties (dielectric or plasma) because of the different Doppler shifts of the Larmor and wave frequencies in the interaction process the autoresonance is violated. However, the threshold (by the wave intensity) phenomenon of electron hysteresis in a medium due to the nonlinear cyclotron resonance in the field of strong monochromatic EM wave takes place \cite{53} (this process has been treated in the earlier paper \cite{54}, where, however, only an oscillating solution has been obtained). In contrast to autoresonance, the nonlinear cyclotron resonance in a medium proceeds with a large enough resonant width. This so-called phenomenon of electron hysteresis leads to significant
acceleration of particles, especially in the plasmalike media where the superstrong laser fields of ultrarelativistic intensities can be applied \[33\]. Note that the use of dielectriclike (gaseous) media in this process makes it possible to realize cyclotron resonance in the optical domain (with laser radiation) due to the possibility of arbitrarily small Doppler shift of a wave frequency close to the Cherenkov cone, in contrast to the vacuum case where the cyclotron resonance for the existing maximal powerful static magnetic fields is possible only in the microwave domain \[51, 52\].

The particle “reflection” phenomenon has a significant property: at the above-threshold interaction the induced process occurs only in one direction–either direct (laser scheme) or inverse (accelerator scheme) \[26, 33\] meanwhile at the below-threshold interaction the both direct and inverse processes take place simultaneously that reduces in general the net gain for each problem. Therefore this phenomenon gives in principal new opportunity for implementation of nonlinear schemes of FEL or laser accelerators in the induced coherent processes at the above-threshold interaction regime. Due to this feature the “reflection” phenomenon may also be used for the monochromatization of particle beams \[146–149\]. On the other hand, because of “reflection” phenomenon the nonlinear Compton effect in media with \(\varepsilon > 1\) can proceed only at the wave intensities below the critical one \[106\].

For the nonplane EM wave pulses of small space sizes (tightly focused) and of short duration, such as current superpower laser pulses are, it is impossible to solve the problem analytically. Hence, in the papers \[62–64\] the “reflection” and capture phenomena in case of actual nonplane-short laser pulses has been investigated by numerical integration of relativistic equations of motion.

The phenomenon of particle “reflection” and capture by EM wave being of pure classical nature, nevertheless leads to quantum effects of probability density modulation of a free particle state on the hard x-ray frequencies because of interference of incident and reflected (inelastic) de Broglie waves after the reflection from a wave barrier \[104\] and of zone structure of particle states in the wave potential well–capture regime like the particle states in a crystal lattice \[150–152\]. On the other hand, it is evident that the role of particle spin in this process is important since in dielectriclike media the wave periodic EM field in the intrinsic frame of reference becomes a static magnetic field and spin interaction with such a field should resemble the Zeeman effect. Besides, the critical intensity depends on the particle spin projection and, consequently, it is differ for different initial spin orientations. Choosing the
certain value for critical intensity corresponding to the certain spin orientation of particles in the beam, with the appropriate laser intensity (above this critical value) one can achieve the particle beam polarization. These are the quantum effects in the above critical regime of induced Cherenkov process. Below the critical intensity the coherent effects of particles classical bunching on optical and shorter wavelengths, quantum effects of modulation of particle probability density on the wave harmonics, and inelastic diffraction scattering of the particles on the traveling EM wave in a dielectric medium takes place like the Kapitza–Dirac effect on a standing wave lattice or Bragg diffraction on a crystal lattice.

The interest to Kapitza–Dirac effect has been increased especially after the successful realization of experiment with high-intensity and strongly coherent laser beams-gratings. It follows to note that observation of diffraction effect during the next 70 years after its prediction had not been succeeded. For the acquaintance with the experimental situation in this area since the advent of this effect, or for detailed references on earlier work we refer the reader to review papers. The significance of Kapitza–Dirac effect, apart from its quantum-mechanical meaning as a best example of demonstration of electron matter wave diffracted by light and, moreover, as an unique sample of a diffraction system with reversed properties of the matter and light, is also conditioned by important applications, since electron beams diffracted from highly coherent laser gratings are coherent with each other. Hence, the Kapitza–Dirac effect is a very convenient, even maybe an irreplaceable means to realize coherent electron beams. Such beams can serve as a basis for construction of new important tools of diverse species, e.g., coherent beam splitters, new type electron interferometers which would operate at rather low electron energies (typical for existing now in atomic physics) etc.

Kapitza–Dirac effect on standing wave lattice is a particular case of the induced Compton process in the field of two counterpropagating EM waves of the same frequencies at which electron moves in perpendicular direction to wavevectors of counterpropagating waves (classical condition of resonance), to exclude the Doppler shifting of waves’ frequencies because of longitudinal component of electron velocity. At the quantum condition of resonance - taking into account the quantum recoil as well- the exact condition of resonance is satisfied at the small angle to perpendicular direction that corresponds to elastic Bragg diffraction on the phase lattice of a standing wave. Nevertheless, the phase matching between
the electrons and counterpropagating waves in the induced Compton process can also be fulfilled in general case of bichromatic EM waves of different frequencies if electron moves at the certain angle with respect to wavevectors of counterpropagating waves at which the condition of coherency between the electron and waves of different frequencies is satisfied \[156\]. However, in contrast to Kapitza–Dirac effect, diffraction of electrons in this case is inelastic. Thus, due to the induced Compton effect in the two wave fields an electron absorbs \(s\) photons from the one wave and coherently radiates \(s\) photons into the other wave of different frequency and vice versa. This is the condition of coherency in the induced Compton process corresponding to the resonance between the Doppler-shifted frequencies in the intrinsic frame of reference of an electron in the bichromatic counterpropagating waves at which the conservation of the number of photons in the induced Compton process takes place \[156, 167, 168\] (see, also the paper \[169\] where theoretical analysis of the scattering of electrons by a strong standing wave with a slowly varying amplitude has been made), in contrast to spontaneous Compton effect in the strong wave field where after the multiphoton absorption a single photon is emitted \[170\].

So, the scope of the Kapitza–Dirac effect has been extended since 1975 in the works \[108, 156, 157\] for inelastic diffraction scattering of electrons on strong travelling wave in the induced Compton, Cherenkov, and undulator/wiggler processes, taking into account the mentioned peculiarity in these processes. Note that the term "strong wave" here is relative, since the mentioned nonlinear resonance takes place even in the very weak wavefields if the electrons initially are close to the resonance state, i.e. the electrons' initial longitudinal velocity is close to the phase velocity of the slowed wave, at which the critical field is also very small and, respectively, very low wave intensities may be above critical \[132\]. At this background, the paper \[171\] is completely misunderstanding where authors, citing the papers \[108, 156, 157\] but bypassing the existence of critical field with aforementioned peculiarity, they obtain the same formula of the paper \[108\] for multiphoton probability of diffraction scattering on the travelling wave in a dielectric medium (by other way -on the base of the Helmholtz–Kirchhoff diffraction theory- following to papers \[172, 173\]), and after four decades claim about the possibility of diffraction effect of electrons on the travelling wave in a dielectric medium.

The organization of the paper is as follows. In Sec. II the exact nonlinear theory of induced Cherenkov process in the strong laser fields is presented. The peculiarity inherent
to nonlinear resonance in coherent processes of threshold character and existence of critical intensity leading to particle “reflection” and capture phenomenon is described in Subsec. II 1. In Subsec. II 2. laser acceleration of charged particles in gaseous media – Cherenkov accelerator with the variable refraction index of a gaseous medium is considered. In Subsec. II 3. the nonlinear cyclotron resonance of threshold nature in a medium – electron hysteresis phenomenon is presented. In Subsec. II 4. the theory of coherent radiation of charged particle beams in capture regime – Cherenkov amplifier is presented.

In Sec. III. the quantum theory of nonlinear induced Cherenkov process is considered. In Subsec. III 2. the spin-effects at the electron “reflection” phenomenon and the possibility to achieve the polarized particle beams is considered. In Subsec. III 3. quantum coherent effect of electrons reflection from the traveling-wave-phase-lattice in the exact resonance is investigated. In Subsec. III 4. the quantum regime of spinor particles capture by slowed traveling wave at the exact Bragg resonance is considered. In Subsec. III 5. the quantum modulation effect of particle state on the optical harmonics at the pump wave intensities below the critical value is presented.

In Sec. IV the vacuum versions of induced coherent processes – induced Compton and undulator/wiggler processes are investigated revealing the particle “reflection” nonlinear threshold phenomena in vacuum, as well as inelastic diffraction effect of electrons like to Kapitza–Dirac effect in vacuum are considered. In Subsec. IV 1. induced Compton process in the field of two bichromatic counterpropagating waves is investigated. In Subsec. IV 2. the “reflection” phenomenon in the undulator/wiggler is considered. In Subsec. IV 3. the general theory of inelastic diffraction effect of electrons on the slowing travelling wave is presented. In Subsec. IV 4. diffraction regime of electron coherent scattering on the traveling wave phase-lattice is considered. In Subsec. IV 5. the Bragg regime of exact resonance on travelling wave lattice is presented. Finally, conclusions are given in Sec. V.
II. INDUCED CHERENKOV PROCESS WITH STRONG LASER FIELDS

A. Critical Intensity in the Induced Cherenkov Process. Nonlinear Threshold Phenomenon of Particle “Reflection” and Capture

Relativistic classical equations of motion for a charged particle energy in the field of a plane EM wave give the exact solution 57:

\[
\mathcal{E}(\tau) = \frac{\mathcal{E}_0}{n^2 - 1} \left\{ n^2 \left( 1 - \frac{v_0}{cn} \cos \vartheta \right) \mp \left[ \left( 1 - n \frac{v_0}{c} \cos \vartheta \right)^2 - (n^2 - 1) \right] \times \left( \frac{mc^2}{\mathcal{E}_0} \right)^2 \left[ \xi^2(\tau) \cos^2 \omega \tau - 2 \frac{p_0 \sin \vartheta}{mc} \xi(\tau) \cos \omega \tau \right]^{1/2} \right\}
\]

where \( n = \sqrt{\varepsilon \mu} \) is the refraction index of a medium (\( \varepsilon \) and \( \mu \) are the dielectric and magnetic permittivities of the medium, respectively), \( m, \mathcal{E}_0, p_0, v_0 \) are respectively particle mass, initial energy, momentum and velocity -directed at the angle \( \vartheta \) to the wave propagation direction \( \nu \ (|\nu| = 1) \), \( c \) is the light speed in vacuum, and \( \xi(\tau) \) denotes the dimensionless relativistic invariant intensity parameter of the wave: \( \xi(\tau) \equiv eA(\tau)/mc^2 = eE(\tau)/mc\omega \) with the vector potential \( A(t, r) = A(t - n\nu r/c); \tau = t - n\nu r/c \) is the retarding wave coordinate of the plane EM wave in a medium, and \( A(\tau), E(\tau) \) – slowly varying amplitudes of the vector potential and electric field of a quasimonochromatic wave amplitudes (the wave is linearly polarized along the axis \( OY \): \( A_y = A(\tau) \cos \omega \tau \)).

Hereafter we will assume that the wave frequency \( \omega \) is far from the main resonance transitions between the atomic levels of the medium to prohibit the wave absorption and nonlinear optical effects in the medium and consequently \( n = n(\omega) \) will correspond to the linear refraction index of the medium.

As is seen from Eq. (1) the expression determining the particle energy in the wave field is, first, not single-valued and, second, may become imaginary depending on particle and wave parameters. The peculiarity arising in the induced Cherenkov process because of particle–strong wave nonlinear interaction is connected with this fact. Hence, treatment of the particle dynamics in this process should start by clarification of these questions.

To consider the behavior of a particle upon nonlinear interaction with a strong wave in a medium on the basis of Eq. (1) we will analyze the case where the initial velocity of the particle at \( \tau = -\infty \) (\( A(\tau) |_{\tau=-\infty} = 0 \)) is directed along the wave propagation direction for
which the picture of the particle nonlinear dynamics is physically more evident. In this case Eq. (1) becomes:

\[ E = E_0 n^2 - 1 \left[ n^2 \left( 1 - \frac{v_0}{cn} \right) \mp \left( 1 - n \frac{v_0}{c} \right) \sqrt{1 - \frac{\xi^2(\tau)}{\xi_{cr}^2}} \right], \]  

\[ \xi_{cr} \equiv \frac{E_0}{mc^2} \frac{|1 - n \frac{v_0}{c}|}{\sqrt{n^2 - 1}}. \]  

As is seen, Eq. (2) is twovalence and, at first, we shall provide the unique definition of the particle energy in accordance with the initial condition. In the case of plasma \((n < 1)\) or vacuum \((n = 1)\) the term under the root is always positive, hence, in these cases one has to take before the root only the upper sign \((-\)\) to satisfy the initial condition \(E(\tau) = E_0\) when \(\xi(\tau) = 0\). In the case \(n = 1\), Eq. (2) yields known vacuum result. Further investigation is devoted to the case of a medium with refraction index \(n > 1\). In this case the nature of the particle motion essentially depends on the initial conditions and the value of the parameter \(\xi(\tau)\) as far as the expression under the root in Eq. (2) may become negative. To solve this problem one needs to pass the complex plane were Eq. (2) has the form of known inverse Jukowski function.

If \(\xi_{max} < \xi_{cr}\) \((\xi_{max} \text{ is the maximum value of the parameter } \xi(\tau))\) the expression under the root in Eq. (2) is always positive and in front of the root one has to take the upper sign \((-\)\) according to the initial condition. Then \(E = E_0\) after the interaction \((\xi(\tau) \to 0)\) and the particle energy remains unchanged.

If \(\xi_{max} > \xi_{cr}\) the particle is unable to penetrate into the wave, i.e., into the region \(\xi > \xi_{cr}\) since at \(\xi > \xi_{cr}\) the root in Eq. (2) becomes a complex one. This complexity now is bypassed via continuously passing from one Riemann sheet to another, which corresponds to changing the inverse Jukowski function from “-“ to “+“ before the root. Hence, the upper sign \((-\)\) in this case stands up to the value of the wave intensity \(\xi(\tau) < \xi_{cr}\), then at \(\xi(\tau) = \xi_{cr}\) the root changes its sign from “-“ to “+“, providing continuous value for the particle energy in the field. The intensity value \(\xi(\tau) = \xi_{cr}\) of the wave is a turn point for the particle motion, so that we call it the critical value.

Thus, when the maximum value of the wave intensity exceeds the critical value a transverse plane EM wave in the medium becomes a potential barrier and the “reflection” of the particle from the wave envelope \((\xi(\tau))\) takes place. If now \(\xi(\tau) \to 0\), we obtain after the
“reflection” for the particle energy

\[ \mathcal{E} = \mathcal{E}_0 \left[ 1 + 2 \frac{1 - n \frac{v_0}{c} \cos \vartheta}{n^2 - 1} \right]. \]  

(4)

If the initial conditions are such that the wave pulse overtakes the particle \( v_0 < c/n \), then after the “reflection” \( \mathcal{E} > \mathcal{E}_0 \) and the particle is accelerated. But if the particle overtakes the wave \( v_0 > c/n \), then \( \mathcal{E} < \mathcal{E}_0 \) and particle deceleration takes place.

This threshold phenomenon of the particle “reflection” can be more clearly presented in the frame of reference connected with the wave. In this frame the electric field of the wave vanishes (\( \mathbf{E}' \equiv 0 \)) and there is only the static magnetic field (\( |\mathbf{H}'| = |\mathbf{H}| \sqrt{n^2 - 1/n} \)).

The phenomenon of charged particle “reflection” from a plane EM wave may also be used for the monochromatization of particle beams. The fact that above the critical intensity value the induced Cherenkov process occurs in only one direction – either emission or absorption – and for the initial Cherenkov velocity \( v_{0x} = c/n \) the energy of the particle after the “reflection” does not change, in principle enables conversion of the energetic or angular spreads of charged particle beams. In this case the energy of the particle is given by Eq. (1), which at the actual values of the parameters for induced Cherenkov process the second term under the root is much smaller than the third one, that is, \( 2p_0 |\sin \vartheta| /mc \gg \xi_{\text{max}} \) and for the critical field in this case (at the wave linear polarization - electric field strength along the axis \( OY \)) we have

\[ \xi_{\text{cr}}(\vartheta) = \frac{c}{2v_0 mc} \frac{\mathcal{E}_0 \left[ 1 - n \frac{v_0}{c} \cos \vartheta \right]^2}{(n^2 - 1) |\sin \vartheta|}; \quad \vartheta \neq 0 \]  

(5)

(in the case \( \vartheta = 0 \), \( \xi_{\text{cr}} \) is determined by Eq. (3)).

If the maximal value of the wave intensity \( \xi_{\text{max}} > \xi_{\text{cr}}(\vartheta) \), then the particle energy after the “reflection” is

\[ \mathcal{E}(\vartheta) = \mathcal{E}_0 \left[ 1 + \frac{2 \left( 1 - n \frac{v_0}{c} \cos \vartheta \right)}{n^2 - 1} \right]. \]  

(6)

Let the charged particle beam with an initial energetic \( \Delta_0 \) and angular \( \delta_0 \) spread interact with a plane transverse EM wave of intensity \( \xi_{\text{max}} > \xi_{\text{cr}}(\vartheta) \) in a gaseous medium.

To keep the mean energy \( \overline{\mathcal{E}}_0 \) of the beam unchanged after the interaction (at the adiabatic turning on and turning off of the wave) the axis of the beam with mean velocity \( \overline{v}_0 \) must be pointed at the Cherenkov angle \( \vartheta_0 \) to the laser beam, i.e., \( n(\overline{v}_0/c) \cos \vartheta_0 = 1 \). Under this condition the particles with velocities \( v_0 \cos \vartheta < c/n \) will acquire an energy and the
other particles for which the longitudinal velocities exceed the phase velocity of the wave \( (v_0 \cos \vartheta > c/n) \) will loss an energy according to Eq. (6). As a result the energies of the particles \( E(\vartheta) \) will approach close to the mean energy \( E_0 \) of the beam \( (E(\vartheta) \rightarrow E_0) \) and the final energetic width of the beam will become less than the initial one. As there is one free parameter (for a specified velocity \( v_0 \) the parameters \( \vartheta_0 \) and \( n \) are related by Cherenkov condition) it is possible to use it to control the exchange in the energy of the particles after the “reflection” (6) and to reach the minimal final energy spread of the beam \( \Delta \ll \Delta_0 \) – monochromatization. Depending on the relation between the initial energetic and angular spreads and mean energy of the beam, the opposite process may occur, namely angular narrowing of the beam. Physically it is clear that with the monochromatization the angular divergence of the beam will increase and the opposite – the angular narrowing of the beam – leads to demonochromatization (in accordance with Liouville’s theorem). More detailed consideration of this effect with the quantitative results can be found in the papers [146, 149] (for suppression of a beam spread by vacuum versions of “reflection” phenomenon see the papers [147, 148]).

To illustrate the typical picture of nonlinear interaction of a charged particle with a strong EM wave in a medium we present the graphics of numerical solutions for the laser pulse of finite duration, showing the behavior of particle dynamics below and above critical intensity, with the effect of acceleration. At first we will not take into account the dependence of the slowly varying intensity envelope of a laser beam from the transversal coordinates. Thus, a laser beam may be modeled as

\[
E_x = 0, \quad E_z = 0, \quad E_y = \frac{E_0}{\cosh \left( \frac{\delta \tau}{\vartheta} \right)} \cos \omega \tau
\]

where \( \delta \tau \) characterizes the pulse duration. The particle initial energy is taken to be \( E_0 = 40 \) MeV and the initial velocity is directed at the angle \( \vartheta = 9 \times 10^{-3} \) rad to the wave propagation direction \( (p_{0x} = 0) \). The refraction index of the gaseous medium for this calculation has been chosen to be \( n - 1 = 10^{-4} \). Figure 1 illustrates the evolution of the particle energy: the energy versus the position \( x \) is plotted for a neodymium laser \( (\hbar \omega \simeq 1.17 \) eV) with electric field strength \( E_0 = 3 \times 10^8 \) V/cm and \( \delta \tau = 4T \) \( (T \) is the wave period). For these parameter values the wave intensity is above the critical point and, as we see from this figure, the particle energy is abruptly changed corresponding to the “reflection” phenomenon. Figure 2a illustrates the evolution of the energies of particles with different initial interaction angles.
The initial energies for all particles are $E_0 \approx 40 \text{ MeV}$. Figure 2b illustrates the role of initial conditions: the final energy versus the interaction angle is plotted. As follows from Eq. (5) the critical intensity, as well as the final energy (6), depend on the initial interaction angle and as a consequence we have this picture. Note that the acceleration rate neither depends on the field magnitude (only should be above threshold field) nor on the interaction length.

To demonstrate the dependence of the considered process on transversal profile of the laser intensity for actual beams in Fig. 3 the evolution of the energies of particles with various initial phases (with initial energies $E_0 \approx 40 \text{ MeV}$) is illustrated. The laser beam transversal profile is modeled by the Gaussian function

$$E_y = E_0 \exp \left( -\frac{4}{d^2} (y^2 + z^2) \right) \frac{\cos \omega \tau}{\cosh \left( \frac{x}{\delta \tau} \right)} \tag{8}$$

with $d = 10^3 \lambda$, $\delta \tau = 50T$. As we see from this figure the acceleration picture is essentially changed depending on the entrance coordinates of the particles. This is the manifestation of the threshold nature of the “reflection” phenomenon.

If for the intensity exceeding the critical value a plane EM wave becomes a potential barrier for the external particle (with respect to the wave), then for the particle initially situated in the wave it may become a potential well and particle capture by the wave will take place [26].

![Graph](image)

FIG. 1: “Reflection” of the particle. The energy versus the position $x$ is plotted when the wave intensity is above the critical point.
FIG. 2: “Reflection” of the particles with different initial interaction angles. Panel (a) displays the evolution of the energies of particles. In (b) the final energy versus the interaction angle is plotted.

B. Laser Acceleration in Gaseous Media. Cherenkov Accelerator

The phenomenon of charged particle “reflection” and capture by a transverse EM wave can be used for particle acceleration in laser fields. As the application of large intensities in this process is restricted because of the medium ionization the acceleration owing to “reflection” in the medium with refraction index \( n = \text{const} \) – single “reflection” – is relatively small. However, if the refraction index decreases along the wave propagation direction in such a way that the condition of particle synchronous motion with the wave \( v_x(x) = c/n(x) \) takes place continuously, the phase velocity of the wave will increase all the time and the particle being in front of the wave barrier (at \( \xi > \xi_{cr} \)) will continuously be “reflected”, i.e., continuously accelerated. The law \( n = n(x) \) must have an adiabatic character not to allow the particle to leave the wave after the single “reflection”. Such variation law of the refraction index can be realized in a gaseous medium adiabatically decreasing the pressure.
FIG. 3: The evolution of the energies of particles with various initial phases are shown for the laser beam with transversal intensity profile for the various entrance coordinates: (a) \( z = 0 \), (b) \( z = d/4 \), and (c) \( z = d/2 \).

For particle acceleration one can also use the capture regime. In this case in the medium with \( n = \text{const} \) the particle energy does not change on average (particle makes stable oscillations around the equilibrium phases in the wave moving with average velocity \( <v_x> = c/n \)). However, if one decreases the refraction index along the propagation direction of the wave, so that the particle does not leave the equilibrium phases, then the wave will continuously
accelerate the particle. Then, to realize the capture regime one needs \( p_{0y}/mc\xi < 2 \). For not very strong fields this is sufficiently strict confinement on the transverse momentum of the particle. For particle acceleration by laser fields one can use the capture regime corresponding to large transverse momenta of the particle \( p_{0y}/mc\xi > 2 \). So, we will consider the general case of particle capture with arbitrary initial momentum \( p_0 \) and laser acceleration in gaseous medium with varying refraction index \( n(x) \). The variation law of the medium refraction index for acceleration is determined by equation:

\[
\frac{dn(x)}{dx} = -\frac{\omega c\xi p_{ys}}{mc} \cos \phi_s \left( \frac{mc^2}{E_s(x)} \right)^2 n^2(x) \left[ n^2(x) - 1 \right].
\]

(9)

It is seen from this equation that for the particle acceleration in the capture regime via decreasing refraction index of the medium \((dn(x)/dx < 0)\) one needs \( p_{ys} \cos \phi_s > 0 \) (equilibrium transverse momentum of the particle must be directed along the vector of the wave electric field). In the opposite case the continuous deceleration of the particle will take place accompanied by induced Cherenkov radiation (regime of continuous amplification of the wave by the particle beam at \(dn(x)/dx > 0\)).

The energy of equilibrium particle acquired on the distance \( x \) is defined by

\[
E_s^2(x) = \frac{n^2(x)}{n^2(x) - 1} \left( m^2c^4 + c^2p_{ys}^2 \right).
\]

(10)

Equation (10) in the general case defines the particle acceleration in the capture regime when the medium refraction index falls along the wave propagation. It defines the longitudinal dimension of such “Cherenkov accelerator” as well. The transverse dimension of the latter is defined by

\[
E_s(y) = E_s(0) + mc\omega \xi (y - y_0) \cos \phi_s.
\]

(11)

Here \( E_s(0) \) and \( y_0 \) are the initial equilibrium values of the energy and transverse coordinate of the particle \((y - y_0) \) is the transverse dimension of “Cherenkov accelerator”). As is seen from Eq. (11) the particle acceleration takes place if \((y - y_0) \cos \phi_s > 0 \), and in the opposite case deceleration occurs \((E_s(y) < E_s(0))\) in accordance with what was mentioned above. For relativistic particles, when \( n(x) \sim 1 \) and \( n(x) - 1 \ll n(0) - 1 \), from Eq. (9) we have

\[
n(x) - 1 \simeq \frac{m^2c^4 + c^2p_{ys}^2}{4mc^2\omega p_{ys} \cos \phi_s} \frac{1}{x}.
\]

(12)

For energy of the equilibrium particle at large acceleration we have

\[
E_s(x) \simeq \sqrt{2mc^2\omega |p_{ys} \cos \phi_s| x}; \quad E_s(x) \gg E_s(0).
\]

(13)
The estimations show that, for example, at electric field strengths of laser radiation \( E \sim 10^8 \) V/cm an electron with initial energy \( E_s(0) \sim 5 \) MeV acquires energy \( E_s(x) \sim 50 \) MeV already at the distance \( x \sim 1 \) cm. The transverse dimension of acceleration \( y - y_0 \) is of the order of a few millimeters and the longitudinal dimension of the system is of the order of the transverse one (a few times larger). At the distance \( x \sim 1 \) m the particle energy gain is of the order of 1 GeV. Note that because of multiple scattering on the atoms of the medium the particles can leave the regime of stable motion as a result of change of \( p_{ys} \). The analysis shows that the multiple scattering essentially falls in the above-mentioned gaseous media for laser field strengths \( E > 10^7 \) V/cm.

To illustrate the particle acceleration in the capture regime we will represent the results of numerical solution of relativistic equations of motion in the field of an actual laser beam with the electric field strength

\[
E_y = E_0 \exp \left( -\frac{4}{d^2} (y^2 + z^2) \right) \frac{\cos \left( \frac{\omega}{c} \int n(x)dx - \omega t + \varphi_0 \right)}{\cosh \left( \frac{1}{\delta\tau} \int n(x)dx - t + \varphi_0 / \omega \right)},
\]

where \( \delta\tau \) characterizes the pulse duration and \( \varphi_0 \) is the initial phase. Simulations have been made for neodymium laser (\( \hbar \omega \simeq 1.17 \) eV) with electric field strength \( E_0 = 3 \times 10^8 \) V/cm and \( \delta\tau = 1000T, d = 5 \times 10^3\lambda \). The variation law for the refraction index of the medium is defined in self-consistent manner that may be approximated by the function

\[
n(x) = \frac{n_0 + n_f}{2} + \frac{(n_f - n_0)}{2} \tanh (\kappa x),
\]

where \( n_0, n_f \) are the initial and final values of the refraction index and \( \kappa \) characterizes the decreasing rate.

Figure 4 illustrates the evolution of the particle energy in the capture regime. The particle initial energy is taken to be \( E_0 = 50.5 \) MeV and the initial velocity is directed at the angle \( \vartheta = 9 \times 10^{-3} \) rad to the wave propagation direction (\( \varphi_0 = 0 \)). The initial value of the refraction index has been chosen to be \( n - 1 \simeq 10^{-4} \). As we see in the capture regime with variable refraction index, one can achieve considerable acceleration.

To show the role of initial conditions in Fig. 5a the evolution of the energies of particles with the same initial energies \( E_0 = 50.5 \) MeV (\( \vartheta = 9 \times 10^{-3} \) rad) and various initial phases is illustrated. The initial entrance coordinate is \( z = 0 \). Figure 5b displays the role of initial
FIG. 4: The evolution of the particle energy in the capture regime with variable refractive index.

conditions: the final energy versus the initial phase is plotted. In Fig. 6 the parameters are
the same as in Fig. 5a except the initial entrance coordinate, which is taken to be \( z = 0.25 \) mm. As we see from these figures the captured particles are accelerated, while the particles
situated in the unstable phases (or if the conditions for capture are not fulfilled) after the
interaction remain with the initial energy.

C. Cyclotron Resonance in a Medium. Nonlinear Threshold Phenomenon of
“Electron Hysteresis”

Consider now the dynamics of a charged particle in the field of a strong EM wave in a
medium at the presence of a static uniform magnetic field \( \mathbf{H}_0 \) along the wave propagation
direction – nonlinear cyclotron resonance in a medium. In this case the problem can be
solved analytically only for the circular polarization of monochromatic wave and if the
initial velocity of the particle is directed along the axis of the wave propagation.

As far as the equation for the particle longitudinal momentum is not changed in the
presence of a uniform magnetic field with respect to equation in the field of a plane EM
wave in a medium, and the equation for the particle energy change in the field remains
unchanged, then one can represent the particle longitudinal velocity \( v_x \) and energy

\[
\mathcal{E} = \frac{\mathcal{E}_0}{n^2 - 1} \left\{ n^2 \left( 1 - \frac{v_0}{cn} \right) - \left( 1 - n \frac{v_0}{c} \right) \left[ 1 \mp \frac{p_\perp^2(\tau)}{(mc\zeta)^2} \right]^{1/2} \right\}
\]  

(16)

via the transversal momentum \( p_\perp(\tau) = \{0, p_y(\tau), p_z(\tau)\} \) in the field. Here the parameter \( \zeta \)
FIG. 5: Acceleration of the particles in the capture regime. Panel (a) displays the evolution of the energies of particles with various initial phases. The initial entrance coordinate is $z = 0$. In (b) the final energy versus the initial phase is plotted.

is

$$
\zeta \equiv \frac{\mathcal{E}_0}{mc^2} \sqrt{\frac{1 - n \frac{\mathcal{E}}{\mathcal{E}_0}}{|n^2 - 1|}}. \tag{17}
$$

Note that $\zeta$ is the critical value of the wave intensity (3) (at $n > 1$) for the particle “reflection” phenomenon in the absence of a static magnetic field ($H_0 = 0$). The sign “−” under the root in (16) corresponds to the case of the interaction in dielectriclike media with $n > 1$ and the sign “+”, plasmalike media with $n < 1$. Note that in contrast to the case $H_0 = 0$ (induced Cherenkov process) in Eq. (16) before the root, only the sign “−” is taken (in accordance with the initial conditions $v_x = v_0$ and $\mathcal{E} = \mathcal{E}_0$ of the free particle) since, as will be shown below, in this case the expression under the root is always positive and consequently the root cannot change its sign.

Now the considered problem reduces to definition of the particle transversal momentum
FIG. 6: Acceleration of the particles in the capture regime. Panel (a) displays the evolution of the energies of particles with various initial phases. The initial entrance coordinate is \( z = 0.25 \text{ mm} \).

In (b) the final energy versus the initial phase is plotted.

The transversal momentum \( \mathbf{p}_\perp(\tau) \) which in the wave coordinate \( \tau = t - nx/c \) obey the following equation:

\[
\frac{dZ(\tau)}{d\tau} = \frac{eE(\tau)}{mc} - i\frac{\Omega}{(1 - n\frac{\omega}{c})\left[1 + \frac{|Z(\tau)|^2}{\zeta^2}\right]^{1/2}}Z(\tau),
\]

where the complex quantity \( Z(\tau) \) related to the dimensionless parameter of the particle transversal momentum

\[
Z(\tau) = \frac{p_y(\tau) + ip_z(\tau)}{mc}.
\]

\( E(\tau) = E_y(\tau) + iE_z(\tau) \) and \( \Omega = ecH_0/E_0 \) is the Larmor frequency for the initial velocity of the particle.

For an arbitrary plane EM wave Eq. (18) is a nonlinear equation the exact solution of which cannot be found. However, for the monochromatic wave of circular polarization when \( E(\tau) = E_0 \exp(-i\omega\tau) \) one can find the exact solution of Eq. (18). The latter is sought in
the form \( Z(\tau) = Z_0 \exp(-i\omega \tau) \) and for the transversal momentum of the particle we obtain
the following algebraic equation:

\[
\left( 1 - \frac{\Omega}{\omega (1 - n_\nu^\infty)} \sqrt{1 \mp \beta^2} \right) \beta = X,
\]

where the quantities \( E_0, Z_0 \) are expressed in the scale of the parameter \( \zeta \):

\[
\frac{Z_0}{\zeta} = i\beta; \quad \frac{eE_0}{mc\omega\zeta} = \frac{\xi_0}{\zeta} = X.
\]

We will not represent here the exact solution of Eq. (20) for \( \beta \). An interesting nonlinear
phenomenon exists in this process \[53\], which can be found out through the graphical solution
of Eq. (20). Thus, depending on the ratio of the Larmor and wave frequencies as well as on
the initial velocity of the particle (in the case of dielectric-like medium where \( v_0 \leq c/n \)) the
solution of Eq. (20) is a single-valued or multivalent that essentially changes the interaction
behavior of the particle with a strong EM wave at the nonlinear cyclotron resonance in a
medium. Hence, we will consider separately the cases \( \Omega \geq \omega' \) and \( \Omega < \omega' \) at \( v_0 < c/n \) where

\[
\omega' = \omega \left( 1 - n \frac{v_0}{c} \right)
\]

is the Doppler-shifted frequency of the wave for the initial velocity of the particle. If \( v_0 > c/n \)
the effects considered here will take place with the opposite circular polarization of the wave
(\( \omega \to -\omega \)) or in the opposite direction of the uniform magnetic field (\( H_0 \to -H_0 \)).

Consider first the case of a medium with refractive index \( n > 1 \) (sign “−” under the root) in Eq. (20). We will turn on the EM wave adiabatically and draw the graphic of dependence
of the particle transversal momentum on the wave intensity \( \beta(X) \). For the case \( \Omega \geq \omega' \) the
latter is illustrated in Fig. 7a. As is seen from this graphic with the increase of the wave
intensity the transversal momentum of the particle increases in the field (consequently the
energy as well) and vice versa: with the decrease of the wave intensity it decreases in the
field and after the passing of the wave (\( X = 0 \)) the transversal momentum becomes zero
(\( \beta = 0 \)), i.e., the particle momentum-energy remain unchanged: \( p = p_0 \) and \( E = E_0 \).

With the increase of the transversal momentum the longitudinal velocity of the particle
increases as well, but in contrast to the case \( H_0 = 0 \) it always remains smaller than the
wave phase velocity if initially the wave overtakes the particle (\( v_0 < c/n \)) and larger if the
particle overtakes the wave (\( v_0 > c/n \)). For this reason the particle “reflection” phenomenon
vanishes in the presence of a uniform magnetic field. Indeed, as is seen from Eq. (20) for
an arbitrary finite value of $X$ we have $\beta < 1$ and longitudinal velocity of the particle in the field $v_x < c/n$ if $v_0 < c/n$ and $v_x > c/n$ if $v_0 > c/n$. The value $\beta = 1$ may be reached only at $X = \infty$ when the root in Eq. (20) becomes zero and $v_x = c/n$. So, the expression under the root in Eq. (20) cannot become zero for finite intensities of the EM wave and, consequently, the root cannot change its sign.

Consider now the case $\Omega < \omega'$. The graphic of dependence of the particle transversal momentum on the wave intensity $\beta(X)$ in this case is illustrated in Fig. 7b. As is seen from this graphic $\beta(X)$ is already a multivalent function: for wave intensities smaller than the value corresponding to the maximum point of the curve $\beta(X)$ three values of the particle transversal momentum exist for each value of the wave intensity. At the maximum point, which will be called a critical one, the wave intensity has the value

$$X_{cr} = \left[1 - \left(\frac{\Omega}{\omega'}\right)^{2/3}\right]^{3/2}.$$  

(23)
There are two values $\beta'_{cr}$ and $\beta''_{cr}$ which correspond to critical intensity (23). The first one, $\beta'_{cr}$, is the value of the parameter $\beta$ corresponding to particle transversal momentum at the maximum point of the curve $\beta(X)$. From the extremum condition of Eq. (20) for $\beta'_{cr}$ we have

$$\beta'_{cr} = \left[1 - \left(\frac{\Omega}{\omega'}\right)^{2/3}\right]^{1/2}. \tag{24}$$

The second critical value for the parameter $\beta$ corresponding to critical intensity $X_{cr}$ is situated on the left-hand side branch of the curve $\beta(X)$. To determine its value one needs the analytic solution $\beta = \beta(X)$ of Eq. (20), but there is no necessity here to present the bulk expression for $\beta''_{cr}$.

We shall decide on that branch of the curve $\beta(X)$ which corresponds to real motion of the particle. Up to the critical point the particle transversal momentum can be changed on that branch which corresponds to initial condition $\beta = 0$ at $X = 0$. On this branch the particle momentum increases with the increase of the wave intensity and vice versa. It is evident that with further increase of the field the particle cannot be situated on the right-hand side from the critical point. Hence, it should pass to the left-hand side branch of the curve $\beta(X)$. Indeed, it is easy to see that the critical point is an unstable state for the particle, while all states on the left-hand side branch of the curve $\beta(X)$ are stable and at the critical point the particle changes instantaneously its transversal momentum and passes by jumping to that branch. The further variation of the particle transversal momentum occurs already on this branch. Note that the instantaneity here is related to the fact that the solution of Eq. (18) has been found for the monochromatic wave. It is clear that the momentum change actually occurs during finite time. This jump variation of the particle momentum (energy) is due to the induced resonant absorption of energy from the wave at the critical point because of which the particle state at this point becomes unstable and it leaves the resonance point for a stable state that corresponds to the transversal momentum $\beta''_{cr}$ on the left-hand side branch of the curve $\beta(X)$. Indeed, if one draws a graphic of the dependence of the particle transversal momentum on the ratio of the Larmor and wave frequencies $\Omega/\omega'$ for a certain intensity of the wave (Fig. 8), then it will be seen from the graphic $\beta(\Omega/\omega')$ that the cyclotron resonance in the strong EM wave field takes place at the critical point with the satisfaction of the condition $\Omega < \omega'$. The latter means that to reach the cyclotron resonance in a medium, in contrast to vacuum autoresonance it is necessary to be initially
under the resonance condition, since due to the effect of the strong wave field in a medium with refractive index \( n > 1 \) the Larmor frequency increases in the field and then reaches the resonance value. In vacuum the cyclotron resonance proceeds at \( \Omega = \omega' \) which survives infinitely, because of which the energy of the particle turns to infinity. Thus, from Eq. (20) in this case \( (n = 1) \) for the particle transversal momentum we have

\[
\beta = \frac{X}{1 - \Omega / \omega'},
\]

which diverges (consequently the energy as well) at \( \Omega = \omega' \). As is seen from Fig. 8 this divergence vanishes in a medium.

With the further increase of the field \( (X > X_{cr}) \) the transversal momentum of the particle will continuously increase on the left-hand side branch of the curve \( \beta(X) \) and tend to value \(-1\) at \( X \to \infty \). With the decrease of the field the transversal momentum decreases on this branch and at \( X = X_{cr} \) already has only the value \( \beta_{cr}' \) since the value \( \beta_{cr}' \) corresponds to the unstable state at the resonance point and now there is no reason for inverse transition from the stable state to the unstable one. With the further decrease of the field the transversal momentum decreases, but as is seen from Fig. 7 after the interaction \( (X = 0) \) the particle does not return to the initial state \( (\beta = 0 \text{ at } X = 0) \) and remains with the final transversal momentum

\[
\beta_F = -\left[ 1 - \left( \frac{\Omega}{\omega'} \right)^2 \right]^{1/2}.
\]

This is a nonlinear phenomenon of charged particle hysteresis in the cyclotron resonance with a strong EM wave in a medium at intensities exceeding the threshold value.
The energy acquired by the particle due to hysteresis is given by

$$E = E_0 \left[ 1 + \frac{(1 - n v_0/c)}{n^2 - 1} \right].$$  \hspace{1cm} (27)

If the wave intensity is smaller than the critical value (23) the energy of the particle oscillates in the field and after the interaction remains unchanged (such solution only has been obtained in the paper [54]).

Equation (27) determines the particle acceleration due to a strong transversal EM wave at the cyclotron resonance with the powerful static magnetic field in a gaseous medium \((n - 1 \ll 1)\). Because of the latter one can achieve the cyclotron resonance using optical (laser) radiation in a medium with the refractive index \(n > 1\), since the Doppler shift for a wave frequency \(1 - n v_0/c\) (see Eq. (22)) in this case may be arbitrarily small in contrast to vacuum, where the cyclotron resonance for the existing powerful static magnetic fields is possible only in the radio-frequency domain. On the other hand, the application of powerful laser radiation for large acceleration of the particles in gaseous media is confined by the ionization threshold of the medium.

Consider now the case of a plasmous medium \((n < 1)\). In Eq. (20) this case should take the sign “+”under the root at which the confinement for the particle transversal momentum, existing in a dielectriclike medium, vanishes. In addition, the above-considered behavior of the cyclotron resonance in a plasmous medium takes place with the inverse relation between the initial Larmor and wave frequencies \(\Omega/\omega'\). Thus, at \(\Omega \leq \omega'\) with the increase of the wave intensity the transversal momentum of the particle increases in the field and vice versa: with the decrease of the wave intensity it decreases in the field and after the passing of the wave \((X = 0)\) the transversal momentum becomes zero \((\beta = 0)\), i.e., the particle momentum-energy remain unchanged: \(p = p_0\) and \(E = E_0\). The nonlinear phenomenon of particle hysteresis in a plasmous medium takes place at \(\Omega > \omega'\), since in a medium with refractive index \(n < 1\) the Larmor frequency decreases in the field and then becomes equal to the resonance value. The graphic of dependence of the particle transversal momentum on the wave intensity \(\beta(X)\) in this case is illustrated in Fig. 9. As is seen from this graphic, in contrast to the case of dielectriclike media the parameter \(\beta\) in the plasmas increases with no limit at the increase of the field. The latter allows the large acceleration of the particles achieved by the current superstrong laser fields of ultrarelativistic intensities \((\xi \gg 1)\) due to this phenomenon of hysteresis in the plasmas. The final transversal momentum of the
FIG. 9: Dependence of normalized transversal momentum $\beta$ on the normalized EM wave amplitude $X$ at $n_0 < 1$.

particle as a result of the hysteresis in this case is

$$\beta_F = \left[ \left( \frac{\Omega}{\omega'} \right)^2 - 1 \right]^{1/2},$$

(28)

the final energy of which will be determined by the same equation (27) since both the numerator and denominator of the fraction in the expression analogous to Eq. (27) for the particle energy in a plasma change sign.

Note an interesting effect at the cyclotron resonance in a medium as well. At $\Omega = \omega'$ no matter how weak the EM wave field is $- \xi_0 \ll \zeta$ (that is, $\xi_0 \ll 1$ even for $\zeta \sim 1$) – from Eq. (20) it follows that

$$|\beta| \approx \left( \frac{2\xi_0}{\zeta} \right)^{1/3},$$

(29)

that is, an essential nonlinearity ($\sim \xi_0^{1/3} \gg \xi_0$) arises in a case where one would expect a
linear dependence on the field according to linear theory. It is the consequence of nonlinear cyclotron resonance the width of which is large enough in this case:

\[ \Delta \omega \simeq 2^{-1/3} \left( \frac{\xi_0}{\zeta} \right)^{2/3} \omega'. \]

(30)

D. Coherent Radiation of Charged Particles in Capture Regime. Cherenkov Amplifier

The particle capture phenomenon may in principle serve as a FEL mechanism (Cherenkov amplifier). For the latter one needs to solve the self-consistent problem on the basis of the set of Maxwell–Vlasov equations. This problem can be solved analytically if the particle initial velocity in the capture regime is directed along the wave propagation and has a value close to the Cherenkov one: \( v_0 = v_{0x} = (c/n_0)(1 + \mu) \); parameter \( \mu \ll 1 \).

The amplitude of the initial wave field should be a slowly varying function of the space-time coordinates \((x, t)\) with respect to the phase. The problem is sensitive to the wave polarization, therefore it will be investigated for both circular and linear polarizations. Consider at first the wave Cherenkov amplification in the capture regime for the circular polarization of stimulated wave with the boundary conditions \( E_y(0, t) = E_0 \cos \omega_0 t, \quad E_z(0, t) = -E_0 \sin \omega_0 t \). Related to particle we assume that it crosses the boundary of the medium \( x = 0 \) at the moment \( t = t_0 \). To define the electric current of the particle stream we assume that the space is continuously filled with the charged particles. Then at the moment \( t_0 \) in the point \( x \) will be situated only the particles for which \( t_0 = t - n_0x/c \) (with accuracy of a small parameter \( \mu \ll 1 \)). Then for the self-consistent field we obtain the equation

Equation for self-consistent field has a simpler form over wave coordinates \( \tau = t - n_0x/c, \quad \eta = x \). For the field amplitude in these coordinates \( E(t, x) = f(\tau, \eta) \) we have

\[ \frac{\partial}{\partial \eta} f(\tau, \eta) = \frac{2\pi e \rho_0}{n_0(n_0^2 - 1)^{1/2} \mu} \sin \left[ \frac{e(n_0^2 - 1)}{mc^2} \int_{\eta_0}^{\eta} f(\tau, \eta')d\eta' \right]. \]

(31)

where \( \rho_0 \) is the mean density of the particles in the initial stream, which will be assumed constant (since \( \mu \ll 1 \) the variation \( \rho_0 \) is small and can be neglected). The simple analytic solution can be received at the incident monochromatic wave: \( f(\tau, 0) = E_0 \). In this case \( f(\tau, \eta) \) does not depend on \( \tau \), i.e., \( f(\tau, \eta) = f(\eta) \), and for the quantity

\[ \varphi = \frac{e(n_0^2 - 1)}{mc^2} \int_{\eta_0}^{\eta} f(\eta')d\eta' \]

(32)
we have the nonlinear equation of anharmonic oscillator
\[ \varphi'' = \frac{2\pi e^2 \rho_0 (n_0^2 - 1)^{1/2}}{mc^2 n_0^2} \mu \sin \varphi, \] (33)
the general solution of which is the incomplete elliptic integral of the first kind
\[
\frac{1}{2} (n_0^2 - 1) \frac{eE_0 x}{mc^2} = \int_0^{\varphi/2} \frac{dz}{\sqrt{1 + \zeta^2 \sin^2 z}},
\]
\[ \zeta^2 = \frac{8\pi \mu}{n_0 (n_0^2 - 1)^{3/2}} \frac{mc^2 \rho_0}{E_0^2}. \] (34)

In the linear case when \( \varphi \ll 1 \) from Eq. (34) we have \[ \text{103} \]
\[ E(x) = E_0 \begin{cases} \cosh \left( \frac{x}{l_c} \right), & \mu > 0, \\ \cos \left( \frac{x}{l_c} \right), & \mu < 0. \end{cases} \] (35)

Hence, for \( \mu > 0 \), which corresponds to particles’ initial velocity \( v_0 > c/n_0 \), exponential amplification of the incident wave occurs. For \( \mu < 0 \), that is, \( v_0 < c/n_0 \), the amplification vanishes on average. The quantity in Eq. (35)
\[ l_c = \left( \frac{mc^2 n_0}{2\pi e^2 \mu \rho_0 (n_0^2 - 1)^{1/2}} \right)^{1/2} \] (36)
is the coherent length of amplification. Equation (31) is an analogue of the equation of the quantum amplifier. The role of inverse population in atomic systems here performs detuning of the Cherenkov resonance \( v_0 - c/n_0 \) (parameter \( \mu \)).

Consider now the case of linear polarization of incident wave \( E_y = E(x, t) \cos (\omega_0 n_0 x/c - \omega_0 t) \). By analogy with the previous case for the slowly varying amplitude of the self-consistent field we have the equation
\[
2i\omega_0 \left( \frac{n_0 \partial E_s}{c \partial x} + \frac{n_s^2 \partial E_s}{c^2 \partial t} \right) + \frac{s^2 \omega_0^2}{c^2} (n_s^2 - n_0^2) E_s
= i^s \frac{4\pi e \rho_0 s \omega_0}{c (n_0^2 - 1)^{1/2}} \mu J_s(\alpha), \] (37)
according to which all harmonics are radiated in contrast to circular polarization of the wave.

Consider Eq. (37) with regard to the presence and absence of synchronism. In the last case, when \( n_s \neq n_0 \) taking into account the slow variation of the field amplitude from Eq. (37) we obtain \[ \text{103} \]
\[ E_s = i^s \mu \frac{4\pi e \rho_0}{c (n_0^2 - 1)^{1/2}} \frac{1}{s \omega_0} \frac{1}{n_s^2 - n_0^2} J_s(\alpha). \] (38)
As is seen from this formula in the absence of synchronism, there is a weak dependence of radiation field on harmonics’ number.

In the case of synchronism \((n_s = n_0)\), Eq. \((37)\) becomes

\[
\frac{\partial E_s}{\partial x} + \frac{n_0}{c} \frac{\partial E_s}{\partial t} = i^{s-1} \mu \frac{2\pi e \rho_0}{n_0 (n_0^2 - 1)^{1/2}} J_s(\alpha). \tag{39}
\]

For the first harmonic (fundamental coherent radiation) the results repeat almost exactly the case of wave circular polarization (Eqs. \((34)–(36)\)), the only difference being that the coherence length in this case is \(\sqrt{2}l_c\).

To determine the radiation on the other harmonics in the case of synchronism consider the problem in the given field. Then, for large \(x\) when

\[
\frac{e (n_0^2 - 1) E_0 x}{mc^2} \gg 1
\]

for the harmonics’ amplitudes we have

\[
E_s = i^{s-1} \mu \frac{2\pi mc^2 \rho_0}{n_0 (n_0^2 - 1)^{3/2}} \frac{1}{E_0}. \tag{40}
\]

Hence, the radiation intensity on the harmonics

\[
I_s = \frac{c}{8\pi} |E_s|^2 \simeq c^2 e \left(\frac{\lambda_0^3 \rho_0}{\lambda_0^4} \right)^2 \left( \frac{\mathcal{E}_0}{mc^2} \right)^2. \tag{41}
\]

As in the linear regime the coherence length increases as energy squared, and the losses of the particles in the medium depend on energy logarithmically, then the energy increase for amplification of weak signals does not give an essential advantage. The optimal energy is \(\mathcal{E}_0 \sim mc^2\). Then \(l_c \sim (r_0 \lambda_0 \rho_0)^{-1}\), where \(r_0 = e^2/mc^2\) is the electron classical radius. The estimations show that for the amplification of optical radiation in the capture regime with \(n_0 = \text{const}\), electron beams of large densities are necessary. The situation considerably will be improved if media with varying refraction index \(n_0(x)\) are used. Then along the direction of increase of \(n_0(x)\) the particles will be continuously decelerated, and the wave continuously amplified (a regime inverse to Cherenkov accelerator).
III. QUANTUM EFFECTS IN INDUCED CHERENKOV PROCESS

A. Quantum Description of “Reflection” Phenomenon. Particle Beam Quantum Modulation at X-Ray Frequencies

Though the phenomenon of particle “reflection” from the front of a plane EM wave is of classical nature, which means that quantum effects of above-barrier reflection and tunnel passage should be small enough, nevertheless the quantum consideration of this phenomenon is worthy of note in relation to the appearance of an important coherent quantum effect as a result of classical “reflection” of particles. The influence of spin interaction is not essential here; on the other hand, it is quantitatively small enough in the induced Cherenkov process (for optical frequencies) and may be neglected. The qualitative aspect of spin effects in the induced Cherenkov process will be considered below. Neglecting the spin interaction, the Dirac equation in quadratic form becomes the Klein–Gordon equation, the exact solution of which can be obtained when the particle initial velocity is parallel to the wave propagation direction ($p_{\perp 0} = 0$) and the latter is monochromatic of circular polarization ($A^2(\tau) = \text{const}$). To calculate the probability of reflection from the wave barrier one needs to consider an EM pulse with the envelope of intensity damped asymptotically at infinity. For a laser pulse of form

$$\xi^2(\tau) = \frac{\xi_0^2}{\cosh^2 \frac{\tau}{\tau_0}},$$

($\xi_0^2$ is the maximal value of intensity and $\tau_0$ is the half-width of the pulse) at $\xi_0 > \xi_{cr}$ for the coefficient of reflection we have

$$R = \frac{\exp \left[ \pi \tilde{\Omega} \tau_0 \left( \frac{\xi_0}{\xi_{cr}} - 1 \right) \right]}{1 + \exp \left[ \pi \tilde{\Omega} \tau_0 \left( \frac{\xi_0}{\xi_{cr}} - 1 \right) \right]},$$

(43)

where

$$\tilde{\Omega} = 2 \frac{\xi_0}{\hbar (n^2 - 1)} \left| 1 - n \frac{v_0}{c} \right|$$

(44)

is the quantum frequency corresponding to particle classical energy change due to “reflection” (see Eq. (41)) and $\xi_{cr}$ is the classical value of critical intensity (3). The major quantity $\tilde{\Omega}\tau_0$ in Eq. (43) $\tilde{\Omega}\tau_0 \gg 1$ (for actual parameters of electron and laser beams in a medium with refractive index $n - 1 \sim 10^{-4}$ the parameter $\tilde{\Omega}\tau_0 \sim 10^{15} \div 10^{11}$ for laser pulse duration $\tau_0 \sim 10^{-8} \div 10^{-12}$ s), hence Eq. (43) corresponds to above reflection regime $\xi_0 > \xi_{cr}$. 
This equation shows that \( R = 1 \) with great accuracy (the coefficient of tunnel passage in this case is of the order \( \exp\left[\left(-10^{15}\right)\div\left(-10^{11}\right)\right] \)). If \( \xi_0 < \xi_{cr} \) then the coefficient of reflection \( R = 0 \) with the same accuracy, i.e. the above barrier reflection is negligibly small in this case. Thus, the quantum effects of tunnel passage and above barrier reflection do not impact on the classical phenomenon of particle “reflection” from the plane EM wave. This is physically clear since the Compton wavelength of a particle (electron) is much smaller than the space size of actual EM pulses. Nevertheless, due to the particle quantum feature as a result of classical reflection the coherent effect of quantum modulation of the free particle probability density and, consequently, electric current density occurs because of superposition of an incident and reflected particle’s matter waves.

The density of electric current of the particle beam is modulated at frequency \( \tilde{\Omega} \)

\[
J(x, t) = J_0 \left\{ 1 + \cos \left[ \tilde{\Omega} \left( t - \frac{x}{c} \right) - \varphi_0 \right] \right\},
\]

where \( J_0 = \text{const} \) is the electric current density of the initially homogeneous and monochromatic particle beam. The modulation frequency \( \tilde{\Omega} \) in actual cases lies in the X-ray domain as follows from the estimation of particle classical energy change \( \Delta E \) due to “reflection” \( (\tilde{\Omega} = \Delta E /\hbar) \). Note that quantum modulation in contrast to classical modulation is exceptionally the feature of a single particle and so is conserved after the interaction.

### B. Spin Effects in Induced Cherenkov Process

Consider now the role of spin effects in the nonlinear quantum dynamics of a spinor particle in the field of a plane monochromatic EM wave in a medium, i.e., in the “reflection” and capture phenomenon of an electron. The exact solution of the Dirac equation for external electron (with respect to wave) can be found for the above-considered case when the particle initial velocity is parallel to the wave propagation direction and the latter is of circular polarization \( (A(\tau) = \{0, A_0 \sin \omega \tau, A_0 \cos \omega \tau\}) \). In the result of spin interaction two critical values of the wave intensity appear corresponding to different initial spin projections \( \sigma_x = \mp 1 \) along the direction of particle motion:

\[
\xi_{cr1,2}^2 = \left( \frac{\xi_0}{mc^2} \right)^2 \left[ 1 - n \frac{\nu_0}{c} \pm \frac{\hbar \omega}{2\xi_0 (n^2 - 1)} \right]^2.
\]
and for particles energies we have respectively:

$$\mathcal{E}_{1,2} = \mathcal{E}_0 + \frac{\mathcal{E}_0}{n^2 - 1} \left( 1 - n \frac{v_0}{c} \pm \frac{\hbar \omega}{2\mathcal{E}_0} (n^2 - 1) \right) \left( 1 - \sqrt{1 - \frac{\xi_0^2}{\xi_{cr1}^2}} \right).$$  \hspace{1cm} (47)$$

For the reflected particles energies with the spin projections $\sigma_x = +1$ and $\sigma_x = -1$, respectively, we have:

$$\mathcal{E}_{3,4} = \mp \hbar \omega + \mathcal{E}_0 + \frac{\mathcal{E}_0}{n^2 - 1} \left( 1 - n \frac{v_0}{c} \pm \frac{\hbar \omega}{2\mathcal{E}_0} (n^2 - 1) \right) \times \left( 1 - \sqrt{1 - \frac{\xi_0^2}{\xi_{cr1,2}^2}} \right).$$  \hspace{1cm} (48)$$

In particular, from this equation it follows that in Eq. (46) $\xi_{cr2}$ corresponds to a particle with the spin directed along the axis $OX$, while $\xi_{cr1}$ corresponds to the opposite one.

The expressions of particle wave functions show that the degener ation of particle states over the spin projection that takes place in vacuum (Volkov states) vanishes in a dielectric-like medium. In that case the wave function $\Psi_1$ corresponds to superposition state with energies $\mathcal{E}_1$ and $\mathcal{E}_1 - \hbar \omega$, while $\Psi_2$ corresponds to energies $\mathcal{E}_2$ and $\mathcal{E}_2 + \hbar \omega$. The removal of degeneration of Volkov states is related to the fact that in a medium with refractive index $n > 1$ in the intrinsic frame of reference of the wave there is only a static magnetic field and the spin interaction with such a field results in the splitting of the particle states as by the Zeeman effect. The splitting value is: $\Delta \mathcal{E} = |\mathcal{E}_1 - \mathcal{E}_2| = |\mathcal{E}_4 - \mathcal{E}_3|$. In vacuum this splitting vanishes as it follows from Eqs. (46), (47) and Dirac wave function in a medium passes to the Volkov wave function (26).

The spin interaction in a medium within the nonlinear threshold phenomenon of particle “reflection” may lead to particle beam polarization since the critical intensity depends on spin projection along the direction of particle motion. Thus, if the condition $\xi_{cr2}^2 < \xi^2 < \xi_{cr1}^2$ holds, then only the particles with certain direction of the spin (along the axis $OX$) will be reflected. Since the velocities of reflected particles are different from the nonreflected ones, then by separating the particles after the interaction a polarized beam may be obtained.

C. Reflection of Electron From the Phase Lattice of Slowed EM Wave

As was shown above, the exact solution of the Dirac equation can be achieved only for the particular case when the particle initial velocity is parallel to the wave propagation direction, which is monochromatic and is of circular polarization. In other cases the quantum
equations of motions (both nonrelativistic and relativistic) are reduced to ordinary differential equations of the second order of Hill or Mathieu type, the exact solution of which are unknown. In these cases one needs to develop adequate approximations for the quantum description of particle–wave nonlinear interaction at the intensities close to critical value (indeed below the threshold of classical “reflection” phenomenon) when the probabilities of multiphoton absorption/emission become maximal \[150–152\]. The expected quantum effect in this case is electron reflection from the phase lattice of slowed EM wave in a dielectric medium \[150\]. Note that on the base of this effect a nonlinear scheme of coherent x-ray source (induced FEL) has been proposed \[174\].

To solve Dirac equation it is more straightforward to pass to the frame of reference of the rest of the wave (\(R\) frame moving with velocity \(V = c/n\)). In the \(R\) frame there is only the static magnetic field that will be described by the following vector potential \(A_R = \{0, A_0(x') \cos k'x', 0\}\), where the wavenumber in this frame \(k' = (\omega/c) \sqrt{n^2 - 1}\). The wave function of a particle in the \(R\) frame is connected with the wave function in the laboratory frame \(L\) by the Lorentz transformation of the bispinors \(\Psi = \hat{S}(\vartheta)\Psi_R, \hat{S}(\vartheta) = \text{ch} \vartheta/2 + \alpha_x \text{sh} \vartheta/2\) (\(\text{th} \vartheta = V/c = 1/n\)). Since the interaction Hamiltonian does not depend on the time and transverse coordinates the eigenvalues of the Hamiltonian \(\hat{H}'\) and momentum operators \(\hat{p}'_y, \hat{p}'_z\) are conserved: \(E' = \text{const}, p'_y = \text{const}, p'_z = \text{const}\), the solution of Dirac equation for wave function \(\Psi_R\) can be represented in the form of a linear combination of free solutions of the Dirac equation \(\Psi_i^{(0)}\) with amplitudes \(a_i(x')\) depending only on \(x'\):

\[
\Psi_R(x', t') = \sum_{i=1}^{4} a_i(x') \Psi_i^{(0)}. \tag{50}
\]

The solution in the form \(50\) corresponds to the expansion of the wave function in a complete set of the wave functions of a particle with certain energy and transverse momentum \(p'_y\) (with longitudinal momenta \(\pm(\sqrt{E'^2/c^2 - p'^2_y - m^2c^2})^{1/2}\) and spin projections \(S_z = \pm 1/2\)). The latter are normalized to one particle per unit volume. Since there is symmetry with respect to the direction \(A_R\) (the \(OY\) axis), we have taken, without loss of generality, the vector \(p'\) in the \(XY\) plane (\(p'_z = 0\)).

According to expansion \(50\) the induced Cherenkov effect in the \(R\) frame corresponds to elastic scattering process by which the reflection of the particle from the wave field occurs:
However, in contrast to classical reflection when the periodic wave field becomes a potential barrier for the particle at the intensity $\xi > \xi_{cr}$, this quantum above-barrier reflection takes place regardless of how weak the wave field is. Hence, the probability of multiphoton absorption/radiation of the incident wave photons by the particle in the $L$ frame, that is, induced Cherenkov effect, will be determined by the probability of particle elastic reflection in the $R$ frame.

Substituting Eq. (50) into Dirac equation and then multiplying by the Hermitian conjugate functions we obtain a set of differential equations for the unknown functions $a_i(x')$. For simplicity we shall assume that before the interaction there are only particles with specified longitudinal momentum and spin state, i.e.,

\begin{align}
|a_1(-\infty)|^2 &= 1, \\
|a_3(+\infty)|^2 &= 0, \\
|a_2(-\infty)|^2 &= 0, \\
|a_4(+\infty)|^2 &= 0. 
\end{align}

(51)

From the condition of conservation of the norm we have

\begin{align}
|a_1(x')|^2 - |a_3(x')|^2 &= \text{const} 
\end{align}

(52)
and the probability of reflection is $|a_{3,4}(-\infty)|^2$.

The equations for amplitudes $a_1$, $a_3$ and $a_2$, $a_4$ are separated and after unitarian transformation

\begin{align}
a_{1,3}(x') &= b_{1,3}(x') \exp \left[ \pm \left( \frac{i p'_y}{\hbar p'_x} \right) \int_{-\infty}^{x'} A_y(\eta) d\eta + i \frac{\vartheta'}{2} \right] 
\end{align}

(53)
the problem simplifies and we obtain set of equations for the amplitudes $b_1(x')$ and $b_3(x')$: 

\begin{align}
\frac{db_{1,3}(x')}{dx'} &= - \sum_{N=-\infty}^{\infty} f_N \exp \left[ \mp \frac{i}{\hbar} \left( 2p'_x - N\hbar k' \right) x' \right] b_{3,1}(x'), 
\end{align}

(54)
In unitarian transformation $\vartheta'$ is the angle between the particle momentum and the direction of the wave propagation in the $R$ frame. The new amplitudes $b_1(x')$ and $b_3(x')$ satisfy the same initial conditions: $|b_1(-\infty)|^2 = 1$, $|b_3(+\infty)|^2 = 0$, according to Eq. (51). A similar set of equations is also obtained for the amplitudes $a_2(x')$ and $a_4(x')$, then after unitarian transformation -for the amplitudes $b_2(x')$ and $b_4(x')$.

The new amplitudes $b_1(x')$ and $b_3(x')$ satisfy the same initial conditions: $|b_1(-\infty)|^2 = 1$, $|b_3(+\infty)|^2 = 0$, according to Eq. (51) and

\begin{align}
f_N = \frac{p'_y}{2 p'_y} N k' J_N \left( 2 \frac{mc}{p'_x} \frac{p'_y}{\hbar k'} \right). 
\end{align}

(55)
Because of conservation of particle energy and transverse momentum (in \( R \) frame) the real transitions in the field will occur from a \( p'_x \) state to the \(-p'_x\) one and, consequently, the probabilities of multiphoton scattering will have maximal values for the resonant transitions

\[
2p'_x = s\hbar k' \quad (s = \pm 1, \pm 2, \ldots).
\]

The latter expresses the condition of exact resonance between the particle de Broglie wave and the incident "wave lattice". In the \( L \) frame the inelastic scattering of the particle on the moving phase lattice takes place and Eq. (56) corresponds to the known Cherenkov conservation law

\[
\frac{2\mathcal{E}_0(1 - n\frac{\omega}{c}\cos \vartheta)}{(n^2 - 1)} = s\hbar \omega,
\]

where \( \vartheta \) is the angle between the particle momentum and the wave propagation direction (the Cherenkov angle), and \( v_0 \) and \( \mathcal{E}_0 \) are the particle initial velocity and energy in the \( L \) frame.

At the exact resonance \((\delta_s = 0)\), according to the boundary conditions (52) for the reflection coefficient we have

\[
R^{(s)} = \left| b_3^{(s)}(-\infty) \right|^2 = \tanh^2 \left[ f_s \triangle x' \right],
\]

where \( \triangle x' \) is the coherent interaction length. The reflection coefficient in the laboratory frame of reference is the probability of absorption at \( v_0 < c/n \) or emission at \( v_0 > c/n \). The latter can be obtained expressing the quantities \( f_s \) and \( \triangle x' \) by the quantities in this frame since the reflection coefficient is Lorentz invariant. So

\[
R^{(s)} = \tanh^2 \left[ F_s \triangle \tau \right],
\]

where

\[
F_s = \left[ \frac{(1 - n\frac{\omega}{c}\cos \vartheta)^2}{n^2 - 1} + \frac{v_0^2}{c^2} \sin^2 \vartheta \right]^{1/2}
\times \frac{s\omega c}{2v_0 \sin \vartheta} J_s \left( \frac{2mv_0 c \sin \vartheta}{\hbar \omega(1 - n\frac{\omega}{c}\cos \vartheta)} \right)
\]

and \( \triangle \tau \) for actual cases is the laser pulse duration in the \( L \) frame. The condition of applicability of resonant approximation is equivalent to the condition \(|F_s| \ll \omega\) which restricts the intensity of the wave as well as the Cherenkov angle. Besides, we must take into account the very sensitivity of the parameter \( F_s \) toward the argument of Bessel function, according
to Eq. (60). For the wave intensities when $F_s \Delta \tau \gtrsim 1$ the reflection coefficient is of the order of one that can occur for a large number of photons $s \gg 1$ for the argument of the Bessel function $\alpha \sim s \gg 1$ in Eq. (60) (according to the asymptotic behavior of Bessel function $J_s(\alpha)$ at $\alpha \simeq s \gg 1$).

For the off resonant solution, when $\delta_s \neq 0$, but $f_s^2 > \delta_s^2 / 4$ we have the following expression for the reflection coefficient:

$$R^{(s)} = \frac{f_s^2 \sinh^2[\Omega_s \Delta x']}{\Omega_s^2 1 + \frac{f_s^2}{\Omega_s^2} \sinh^2[\Omega_s \Delta x']}, \quad \Omega_s = \sqrt{f_s^2 - \delta_s^2 / 4}, \quad (61)$$

which has the same behavior as in the case of exact resonance. In the opposite case when $f_s^2 \leq \delta_s^2 / 4$ the reflection coefficient is an oscillating function of interaction length.

**D. Quantum Description of Capture Phenomenon in Induced Cherenkov Process**

The multiphoton induced Cherenkov interaction in the capture regime corresponding to transitions between the particle bound states occurs at the nonzero initial angles of particle motion with respect to the wave propagation direction, at which, as mentioned above, the Dirac or Klein–Gordon equations are of Hill or Mathieu type and unable to solve it exactly. However, as was shown in the quantum description of “reflection” phenomenon (free–free transitions), the interaction at the arbitrary initial angle resonantly connects two states of the particle (in the intrinsic frame of reference of the wave the states with longitudinal momenta $p_x$ of the incident particle and $p_x + shk$ of the scattered particle; $s$ is the number of absorbed or radiated photons with a wave vector $k$), which makes available the application of resonant approximation to determine the multiphoton probabilities of free–free transitions in induced nonlinear Cherenkov process. Concerning the quantum description of the particle’s bound states in the capture regime one must take into account the degeneration of initial states of free particles in the “longitudinal momentum”. Therefore, regardless of how weak the field of the wave is, the usual perturbation theory in stimulated Cherenkov process is not applicable because of such degeneration of the states, and the interaction near the resonance is needed for description by the secular equation. The latter, in particular, reveals the zone structure of the particle states in the field of a transverse EM wave in a dielectriclike medium. Note that in contrast to the zone structure for the energy of electron states in a crystal lattice,
the zone structure in this process holds for the conserved quantity

\[ p_\eta = \frac{1}{2} \left( \frac{c}{n} p_x - E \right) = \text{const}, \quad (62) \]

as the energy could not be quantum characteristic of the state in the nonstationary field of the wave [150–152].

At first we will consider the case of scalar particles. According to Floquet’s theorem the solution of Klein–Gordon equation in the wave coordinate \( \tau \) may be represented in the form

\[ U(\tau) = e^{i \frac{p_\tau}{\hbar} \tau} \sum_{s=-\infty}^{\infty} \Phi_s e^{-i s \omega \tau}, \quad (63) \]

where

\[ p_\tau^2 \equiv \frac{\xi_0^2}{(n^2 - 1)^2} \left[ \left( 1 - \frac{v_0}{c} \cos \vartheta \right)^2 - (n^2 - 1) \left( \frac{mc^2}{\xi_0^2} \right)^2 \xi_0^2 \right], \quad (64) \]

is the major quantity in the induced nonlinear Cherenkov process, which is the renormalized (because of intensity effect) generalized momentum of the particle in the laboratory frame conjugate to wave coordinate \( \tau \). It connects the “width of initial Cherenkov resonance” \( 1 - nv_0/c \) and wave intensity \( (\xi_0^2) \) as the main relation between the physical quantities of this process determining also the condition of nonlinear resonance \( v_x(\xi) \, |_{\xi=\xi_r} = c/n \). In the intrinsic frame of reference of the wave \( p_\tau \) corresponds to longitudinal momentum \( p_x \) of the particle on which the degeneration exists. The recurrent equation for the coefficients \( \Phi_s \) can be solved in approximation of the perturbation theory by the wave function which is valid at the satisfaction of condition

\[ \left| s^2 \hbar^2 \omega^2 - 2 s \hbar \omega p_\tau \right| \gg \left| \frac{mc^3}{(n^2 - 1)^2} p_0 \xi_0 \sin \vartheta \right|. \quad (65) \]

Regarding those values \( p_\tau \) for which condition (65) does not hold, the usual perturbation theory is already not applicable. In particular, if the expression on the left-hand side of this condition is zero, i.e., at \( s = 0 \) and \( s = \ell \) \( (\ell = 1, 2, 3, ... ) \), when

\[ 2 \frac{p_\tau}{\hbar} = \ell \omega, \quad (66) \]

it is evident that we already have two states \( \Phi_0 \) and \( \Phi_\ell \), which are degenerated in the “longitudinal momentum” \( p_\tau \), since \( p_\tau^2 = (p_\tau - \ell \hbar \omega)^2 \). Because of this double degeneration in the state parameter \( p_\tau \) for the definite \( p_\eta \) of the initial unperturbed system it is necessary to use perturbation theory for the degenerated states on the basis of the secular equation.
Thus, within secular perturbation theory from the compatibility of equations for $\Phi_0$ and $\Phi_\ell$ we have $\Delta_\tau = \pm \alpha_1$, where $\Delta_\tau$ is the correction to the value $p_\tau^2$ at the fulfillment of condition (66) for $\ell = 1$:

$$\Delta_\tau \equiv \frac{8n^2 p_\eta^{(0)}}{(n^2 - 1)^2 p_\eta^{(1)}}, \quad \alpha_1 \equiv \frac{mc^3 p_0 \xi_0 \sin \vartheta}{(n^2 - 1)}. \tag{67}$$

By the standard method from Eq. (67) one can obtain the following set of equations for the amplitudes $\Phi_0$ and $\Phi_1$. The signs “+” and “−” relate to $p_\tau^2 > \hbar^2 \omega^2 / 4$ and $0 < p_\tau^2 < \hbar^2 \omega^2 / 4$, respectively. Thus, at the fulfillment of condition (66) we have a jump in the value of $p_\tau^2$, which is equal to $2\alpha_1$, i.e.,

$$\frac{E_0^2}{(n^2 - 1)^2} \left\{ \left(1 - \frac{n v_0}{c} \cos \vartheta \right)^2 - (n^2 - 1) \left( \frac{mc^2}{E_0} \right)^2 \xi_0^2 \right\}$$

$$\geq \frac{\hbar^2 \omega^2}{4} + \alpha_1,$$

$$0 \leq \frac{E_0^2}{(n^2 - 1)^2} \left\{ \left(1 - \frac{n v_0}{c} \cos \vartheta \right)^2 - (n^2 - 1) \left( \frac{mc^2}{E_0} \right)^2 \xi_0^2 \right\}$$

$$\leq \frac{\hbar^2 \omega^2}{4} - \alpha_1. \tag{68}$$

For $\ell = 1$ the matrix element of transition from state $\Phi_0$ to state $\Phi_1$ (here we note the state without a phase) is equal to $\alpha_1$. For large $\ell$ ($\ell \geq 2$) the matrix element of transition $\Phi_0 \leftrightarrow \Phi_\ell$ is equal to zero in the first order of perturbation theory. In this case it makes sense to take into account the transitions to the states with other energies in higher order. For example, for $\ell = 2$ it is necessary to consider the transitions $\Phi_0 \rightarrow \Phi_1$ and $\Phi_0 \rightarrow \Phi_2$. For arbitrary $\ell$ the matrix element of transition is defined by

$$\alpha_\ell = \frac{\alpha_1^\ell}{((\ell - 1)!)^2 (\hbar \omega)^{2\ell - 1}}. \tag{69}$$

It should be noted that here it is also necessary to take into account the corrections to the energy eigenvalue of state $\Phi_0$ in the appropriate order, however, the latter are only of quantitative character, unlike the qualitative corrections (69), and will be omitted.

As is seen from Eq. (68), the permitted and forbidden zones arise for the particle states in the wave. The widths of permitted zones in the general case of $\ell$-photon resonance are defined from the condition

$$\frac{\ell^2 \hbar^2 \omega^2}{4} + \alpha_\ell$$
\[
\leq \frac{E_0^2}{(n^2 - 1)^2} \left\{ \left( 1 - n \frac{v_0}{c} \cos \nu \right)^2 - \left( \frac{mc^2}{E_0} \right)^2 \xi_0^2 \right\}
\]

\[
\leq \frac{(\ell + 1)^2 \hbar^2 \omega^2}{4} - \alpha_{\ell+1}.
\] (70)

Such zone structure for the particle states in the wave arises in dielectriclike media because of particle capture by the wave and periodic character of the field – quantum influence of infinite “potential” wells on the particle states similar to zone structure of electron states in a crystal lattice \[159\].

Consider now the case of spinor particles. Proceeding from the Dirac equation for the components of the spinor \(V\):

\[
V_1(\tau) = e^{i\frac{p}{h} \tau} \sum_{s=-\infty}^{\infty} K_s e^{-i\omega_s \tau}.
\] (71)

Repeating the procedure as in the case of scalar particles, we obtain the Bragg condition \[60\], at which it is necessary to use the secular perturbation theory for degenerated states. At \(\ell = 1\) we obtain the following system of equations for coefficients \(K_0\) and \(K_1\):

\[
\begin{align*}
\Delta \tau K_0 + \left( -i\alpha_1 + \frac{1}{2}\mu \right) K_1 &= 0, \\
\left( i\alpha_1 + \frac{1}{2}\mu \right) K_0 + \Delta \tau K_1 &= 0,
\end{align*}
\] (72)

where

\[
\mu = \frac{\hbar c H}{\sqrt{n^2 - 1}}.
\] (73)

From Eq. (72) for the correction to \(p^2_\tau\) we obtain

\[
\Delta \tau = \pm \left( \frac{1}{4} \mu^2 + \alpha_1^2 \right)^{\frac{1}{2}}.
\] (74)

It is easy to see that \(K_1 = \mp K_0 e^{i\varphi}\), where \(tg \varphi = 2\alpha_1/\mu\). Hence, each spinor component of particle wave function has two values corresponding to the top and bottom borders of the first forbidden zone:

\[
V_1^\pm(\tau) = K_0 \left( e^{i\frac{\varphi}{2}} \mp e^{-i\frac{\varphi}{2}} \right).
\] (75)

For \(V_2(\tau)\) we have the same expressions as (75), where it is only necessary to replace \(\varphi\) by \(-\varphi\).

At \(\ell = 2\) we have already two channels for the transition from state \(K_0\) to state \(K_2\). The first is the result of the interaction described by a term quadratic in the field (\(\sim A^2\),
the matrix element of which at $\ell = 2$ is equal to $(mc^2)^2 \xi_0^2 / 4\hbar^2(n^2 - 1)$, and the second channel proceeds both in the case of scalar particles via transitions $K_0 \rightarrow K_1$ and $K_0 \rightarrow K_2$, stipulated by the charge interaction $\sim pA$, as well as for the spin interaction, the matrix elements of which at each transition are equal to $-i\alpha_1$ and $\mu/2$, respectively. Therefore, for two-photon transition for the main parameter – correction to $p^2\tau$ we obtain

$$\Delta_\tau = \pm \frac{1}{\hbar^2 \omega^2} \left[ \left( \frac{1}{4} \mu^2 - \alpha_1^2 + \frac{\hbar^2 \omega^2 (mc^2)^2 \xi_0^2}{4(n^2 - 1)} \right)^2 + \alpha_1^2 \mu^2 \right]^{\frac{1}{2}}$$

(76)

The obtained results for spinor particles are valid at the fulfillment of the condition $|\Delta_\tau| \ll \hbar^2 \omega^2 / 4$.

Thus, the quantum picture of induced Cherenkov interaction for charged spinor particles does not differ qualitatively from the case of scalar particles, i.e., the spin interaction results only in quantitative corrections to the quantities describing the process. However, in the absence of charge interaction ($pA = 0$) in the first order in the field, i.e., for one-photon interaction, the first forbidden zone ($\ell = 1$) does not exist for scalar particles, but exists for spinor particles due to the spin interaction.

### E. Quantum Modulation of Charged Particles at Optical Harmonics

Coherent interaction of charged particles with a plane EM wave of intensity smaller than the critical one in the induced Cherenkov process leads to quantum modulation of the particle probability density and, consequently, current density after the interaction at the wave fundamental frequency and its harmonics. In contrast to classical modulation of particles’ current density proceeding in the free drift region after the interaction (see, e.g. [113, 114]) and conserving for short distances, the quantum modulation, being quantum feature of a single particle, is conserved after the interaction unlimitedly long [110]. To reveal this quantum coherent effect it is necessary to take into account the quantum character of particle–wave interaction entirely, i.e. the quantum recoil as well, in contrast to the eikonal approximation for particle wave function at the description of multiphoton interaction with strong EM wave. The mathematical point of view requires taking into account in quantum equations of motion the second-order derivatives of the wave function as well, which are neglected in the eikonal approximation (see, below the description of the diffraction effect).
To describe the effect of particle quantum modulation with regard to the wave harmonics we will solve the Klein–Gordon equation by perturbation theory in the field of monochromatic wave \( \mathbf{A}(\tau) = \{0, A_0 \cos \omega \tau, A_0 \sin \omega \tau\} \) of intensity \( \xi_0 \ll \xi_{cr} \ll 1 \) at which one can neglect again the constant term \( \sim A_0^2 \). Then looking for the solution of Klein–Gordon equation in the form which will be solved in the approximation of perturbation theory by wave function. Thus, the amplitude of the particle wave function corresponding to \( s \)-photon induced radiation \((s > 0)\) and absorption, respectively, reads

\[
\Psi_{\pm s} = \frac{(\pm 1)^s}{s!} \frac{b^s}{(\mu \pm \Delta_h)(\mu \pm 2\Delta_h) \cdots (\mu \pm s\Delta_h)}, \quad (77)
\]

Here the dimensionless parameter of one-photon interaction

\[
b = \frac{eA_0 v_0}{2 \hbar \omega c} \sin \vartheta_0 \quad (78)
\]

is the small parameter of perturbation theory: \(|b| \ll 1\) and

\[
\mu = 1 - n \frac{v_0}{c} \cos \vartheta_0; \quad \Delta_h = \left( n^2 - 1 \right) \frac{\hbar \omega}{2 \xi_0} \quad (79)
\]

are the dimensionless Cherenkov resonance width and quantum recoil parameter, respectively.

The current density of the particles after the interaction is given by the formula:

\[
\mathbf{j}(t, x) = \mathbf{j}_0 \left\{ 1 + 2 \sum_{s=1}^{\infty} \frac{b^s}{s!} \frac{1}{(\mu + \Delta_h) \cdots (\mu + s\Delta_h)} \right. \\
+ \left. \frac{(-1)^s}{(\mu - \Delta_h) \cdots (\mu - s\Delta_h)} \right\} \cos s \omega (t - nx/c) \\
+ 2 \sum_{s=1}^{\infty} \sum_{s'=1}^{\infty} (-1)^{s'} \frac{b^{s+s'} s! s'}{s! s'} \cos [(s + s') \omega (t - nx/c)] \\
\times \frac{1}{(\mu + \Delta_h) \cdots (\mu + s\Delta_h) \cdot (\mu - \Delta_h) \cdots (\mu - s\Delta_h)} \right\}, \quad (80)
\]

where \( \mathbf{j}_0 = \text{const} \) is the current density of initially uniform particle beam. As is seen from Eq. \(80\) as a result of direct and inverse induced Cherenkov effect the current density of initially uniform particle beam is modulated at the wave fundamental frequency and its harmonics. This is a result of coherent superposition of particle states with various energy and momentum due to absorbed and emitted photons in the radiation field that remains after the interaction unlimitedly long (for a monochromatic beam).
The denominators in Eq. (80) becomes zero at the fulfillment of exact quantum conservation law for multiphoton Cherenkov process (57). In this case perturbation theory is not applicable and the consideration in the scope of above-developed secular perturbation is required. However, in actual cases because of nonmonochromaticity of particle beams the width of Cherenkov resonance is rather larger than quantum recoil ($\Delta \hbar \ll \mu$) and one can neglect the latter in Eq. (80). Then the modulation depth ($B$) at the wave fundamental frequency is expressed via critical intensity (5): $B = \xi/2\xi_{cr}(\vartheta)$. The latter shows that the effect of quantum modulation at the stimulating wave harmonics proceeds at intensities smaller than the critical one when the induced Cherenkov interaction of the particles with the periodic wave field (photons) occurs. In the opposite case the interaction proceeds with the potential barrier, and as was shown above the quantum modulation of the particles due to “reflection” phenomenon occurs because of interference of the incident and reflected electron’s matter waves. For this reason the modulation frequency (actually x-ray) corresponds to particle’s energy exchange in the result of interaction with the moving barrier. On the other hand, it is clear that the modulated by anyway particle beam is a coherent source of EM radiation.

IV. VACUUM VERSIONS OF “REFLECTION” AND CAPTURE PHENOMENON

A. Induced Nonlinear Compton Process

Consider now the induced Compton and undulator processes in vacuum, at which the restriction on the wave intensity taking place in dielectric media for induced Cherenkov process vanishes and one can use laser pulses of extremely large intensities to achieve laser acceleration of particles of superhigh energies by considering phenomena, as well as realization of many nonlinear QED phenomena from vacuum. At first let investigate the classical dynamics of a charged particle at the interaction with two bichromatic counterpropagating (along the axis $O.X$) plane EM waves (with frequencies $\omega_1$ and $\omega_2$) in vacuum. Relativistic equations of motion allow exact solution in case of monochromatic waves of circular polarization and if the particle initial momentum is directed along the axis of waves’ propagation (initial transverse momentum $P_{0\perp} = 0$). Then for the particle energy in the field we have
The parameter $n_1$ included in Eq. (81) is

$$n_1 = \frac{\omega_1 + \omega_2}{|\omega_1 - \omega_2|}$$

and the parameters $\xi_{1,2} \equiv eE_{1,2}/mc\omega_{1,2}$ (the waves are turned on and turned off adiabatically at $t \to \mp\infty$).

As is seen from Eq. (81) due to the effective interaction of the particle with the counterpropagating waves a slowed traveling wave in vacuum arises. The parameter $n_1$ denotes the refractive index of this interference wave and since $n_1 > 1$ (see Eq. (82)) the phase velocity of the effective traveling wave $v_{ph} = c/n_1 < c$. Then the expression under the root in Eq. (81) evidences the peculiarity in the interaction dynamics like the induced Cherenkov one that causes the analogous threshold phenomena of particle “reflection” and capture by the interference wave in the induced Compton process. Hence, omitting the same procedure related to bypass of the multivalence and complexity of Eq. (81), which has been made in detail for the analogous expression in the Cherenkov process, we will present the final results for particle “reflection” and capture by the effective interference wave in the induced Compton process. The threshold value of the “reflection” phenomenon or the critical field for nonlinear Compton resonance is

$$\xi_{cr} (\omega_{1,2}) \equiv (\xi_1 + \xi_2)_{cr} = \frac{\mathcal{E}_0}{mc^2} \left| \frac{\omega_1 (1 - \frac{v_x}{c}) - \omega_2 (1 + \frac{v_x}{c})}{2\sqrt{\omega_1 \omega_2}} \right|.$$ 

(83)

If one knows the longitudinal velocity $v_x$ of the particle in the field, then it is easy to see that $\xi_{cr} (\omega_{1,2})$ is the value of the total intensity of counterpropagating waves at which $v_x$ becomes equal to the phase velocity of the effective interference wave: $v_x = v_{ph} = c/n_1$ irrespective of the magnitude of particle initial velocity $v_0$. The latter is the condition of coherency of induced Compton process

$$\omega_1 \left(1 - \frac{v_x}{c}\right) = \omega_2 \left(1 + \frac{v_x}{c}\right).$$

(84)
Under condition (84) the nonlinear resonance in the field of counterpropagating waves of different frequencies occurs and because of induced Compton radiation/absorption the particle velocity becomes smaller or larger than the phase velocity of the interference wave and the particle leaves the slowed effective wave. In the rest frame of the latter the particle swoops on the motionless barrier \(\xi_1 + \xi_2 > \xi_{cr}(\omega_{1,2})\) and the elastic reflection occurs. In the laboratory frame it corresponds to inelastic “reflection” and from Eq. (81) for particle energy after the “reflection” \(\xi_{1,2} \to 0\) adiabatically at \(t \to +\infty\) we have
\[
E = E_0 \omega_1^2 \left(1 - \frac{v_0}{c}\right) + \omega_2^2 \left(1 + \frac{v_0}{c}\right) \frac{2\omega_1\omega_2}{2\omega_1\omega_2}.
\] (85)

From this equation it follows that the energy of the particle with the initial velocity \(v_0 = c |\omega_1 - \omega_2| / (\omega_1 + \omega_2)\) corresponding to the resonance value of the induced Compton process does not change after the interaction \((E = E_0)\). For such particle \(\xi_{cr}(\omega_{1,2}) = 0\), i.e., it cannot enter the field: \(\xi_1 = \xi_2 = 0\). The particle with the initial velocity \(v_0 > c |\omega_1 - \omega_2| / (\omega_1 + \omega_2)\) after the “reflection” is decelerated, while at \(v_0 < c |\omega_1 - \omega_2| / (\omega_1 + \omega_2)\) it is accelerated because of direct and inverse induced Compton processes. At the acceleration the particle absorbs photons from the wave of frequency \(\omega_1\) and coherently radiates into the wave of frequency \(\omega_2\) if \(\omega_1 > \omega_2\) and at the deceleration the inverse process takes place. Hence, at the particle acceleration the amplification of the wave of a smaller frequency holds, while at the deceleration the wave of a larger frequency is amplified.

In the case of \(\omega_1 = \omega_2 \equiv \omega\) the refractive index of the interference wave \(n_1 = \infty\) and nonlinear interaction of the particle with the strong standing wave occurs. It is evident that in this case the process is elastic: \(E = E_0 = \text{const}\) (see Eq. (85)) and for the longitudinal momentum of the particle in the field we have
\[
p_x = \pm \sqrt{p_0^2 - m^2c^2 \left(\xi_1^2 + \xi_2^2 + 2\xi_1\xi_2 \cos \frac{2\omega}{c}x\right)}.
\] (86)

From this equation it is seen that at \(\xi_1 + \xi_2 > \xi_{cr}(\omega) = |p_0| / mc\) the standing wave becomes a potential barrier for the particle and elastic reflection occurs: the root changes its sign and \(p_x = -p_0\) (if \(\xi_1 + \xi_2 < \xi_{cr}(\omega)\) we have \(p_x = p_0\)).

Because of limitation on the volume we will not consider here the capture phenomenon in vacuum processes. The reader interested in those may find it in Refs. [26, 33].
B. Induced Nonlinear Undulator/Wiggler Process

Electrons "reflection" or capture phenomenon is also possible by a plane EM wave propagating in electric or magnetic undulator/wiggler [59]. As far as descriptions of electron dynamics in a plane monochromatic wave in the electric and magnetic undulators are coincide in many features, here we will consider only more important case of magnetic undulator/wiggler, which is currently the most perspective coherent tool with extremely large length of coherency, specifically due to which the x-ray free electron laser has been realized in the wiggler [49, 50].

Consider the nonlinear dynamics of a charged particle at the interaction with a strong EM wave in a magnetic undulator/wiggler. Let a particle with an initial velocity $v_0 = v_{0x}$ enters into a magnetic undulator with circularly polarized field

$$
H(x) = \begin{cases} 
0, & -H \cos \frac{2\pi}{l} x, H \sin \frac{2\pi}{l} x 
\end{cases}
$$

($l$ is the space period or step of an undulator) along the axis of which propagates a plane monochromatic EM wave of circular polarization. Relativistic equations of motion in this case allow exact solution and for the particle energy we have [59]:

$$
E = E_0 \frac{n^2 - 1}{n^2 - 1} \left\{ \frac{1}{n^2} \left( 1 - \frac{v_0}{cn_2} \right) \mp \left[ \left( 1 - n_2 \frac{v_0}{c} \right)^2 - \left( n_2^2 - 1 \right) \left( m_0 c^2 \right)^2 \right]^{1/2} \right\}
$$

where relativistic invariant dimensionless interaction parameter

$$
\xi_H = \frac{eH}{2\pi mc^2}
$$

(for large magnitudes of undulator field strength $H$ and space period $l$ when $\xi_H > 1$ such undulator is called a wiggler).

From Eq. (88) it follows that at the particle–wave nonlinear resonance interaction in the undulator an effective slowed traveling wave is formed as in the induced Compton process. The parameter

$$
n_2 = 1 + \frac{\lambda}{l}
$$

is the refractive index of this slowed wave, which causes the analogous threshold phenomenon of particle “reflection” in the induced undulator process. The effective critical field at which
the nonlinear resonance and then the particle “reflection” take place in the undulator, is

$$\xi_{cr} \left( \frac{\lambda}{l} \right) \equiv (\xi_0 + \xi_H)_{cr} = \frac{1 - (1 + \frac{\lambda}{l}) \frac{v_0}{c}}{\sqrt{2\frac{\lambda}{l} (1 + \frac{\lambda}{2l})}} \frac{E_0}{mc^2}. \quad (91)$$

At this value of the resulting field the longitudinal velocity of the particle $v_x$ reaches the resonant value in the field at which the condition of coherency in the undulator

$$\frac{2\pi}{l} v_x = \omega \left( 1 - \frac{v_x}{c} \right) \quad (92)$$
is satisfied. The latter has a simple physical explanation in the intrinsic frame of the particle. In this frame of reference the static magnetic field (87) becomes a traveling EM wave with the frequency

$$\omega' = \frac{2\pi v_x}{l} \sqrt{1 - \frac{v_x^2}{c^2}}$$

and phase velocity $v_{ph} = v_x$. For coherent interaction process this frequency must coincide with the Doppler-shifted frequency of stimulated wave.

The energy of the particle after the “reflection” (in Eq. (88) $\xi_0 = \xi_H = 0$ at the sign “+” before the root) is

$$E = E_0 \left[ 1 + \frac{1 - (v_0/c) (1 + \lambda/l)}{(\lambda/l) (1 + \lambda/2l)} \right] \quad (93)$$

From this equation it follows that the particle with the initial velocity $v_0 < c/(1 + \lambda/l)$ after the “reflection” accelerates, while at $v_0 > c/(1 + \lambda/l)$ it decelerates because of induced undulator radiation.

The “reflection” phenomenon of charged particles from a plane EM wave, as was shown in the induced Cherenkov process, may be used for monochromatization of the particle beams. Note that the considered vacuum versions of this phenomenon are more preferable for this goal taking into account the influence of negative effects of the multiple scattering and ionization losses in a medium. On the other hand, the refractive index of the effective slowed waves in vacuum $n_1$ or $n_2$ in corresponding induced Compton and undulator processes may be varied choosing the appropriate frequencies of counterpropagating waves or wiggler step. In particular, for monochromatization of particle beams with moderate or low energies via the induced Cherenkov process one needs a refractive index of a medium $n - 1 \sim 1$ that corresponds to solid states. Meanwhile, such values of effective refractive index may be reached in the induced Compton process at the frequencies $\omega_1 \sim \omega_2$ of the counterpropagating waves. However, we will not consider here the possibility of particle beam monochromatization on
the basis of the vacuum versions of “reflection” phenomenon since the principle of conversion of energetic or angular spreads is the same. To study the subject in more detail we refer the reader to original papers \[147–149\].

Considered above quantum effects in the induced Cherenkov process as in the above critical regime, as well as below the critical point also take place in described vacuum versions of particle “reflection” and capture phenomena at the induced Compton and undulator/wiggler nonlinear interaction. However, because of similar physical picture and investigation methods we will not repeat here the consideration of these effects.

C. Inelastic Diffraction of Electron on a Traveling EM Wave

As was mentioned in Introduction, at the wave intensities below the threshold of particle “reflection” and capture phenomena, the inelastic diffraction scattering of electron matter wave – de Broglie wave on a traveling light phase-lattice in the induced Cherenkov, Compton, and undulator processes is possible. For efficient probabilities of multiphoton diffraction effect on such wave-gratings the intensities of slowed traveling wave should be close to the critical value: \(\xi \lesssim \xi_{cr}\) to be near the nonlinear resonance of particle-wave interaction in each induced process.

We will proceed the solution of diffraction effect in general form for Cherenkov and vacuum processes. As the coordinates \(r_\perp = \{y, z\}\) are cyclic, then the corresponding components of generalized momentum \(p_\perp\) are conserved. Hence, according to Floquet’s theorem the solution of Klein–Gordon equation may be sought in the form

\[
\Psi = e^{\frac{i}{\hbar}p_\perp\cdot r} \sum_s C_s (t) e^{\frac{i}{\hbar}(p_s + s\hbar k)x} e^{-\frac{i}{\hbar}(E + s\hbar k v_{ph})t}.
\]

where \(E = \sqrt{c^2 p^2 + m^2 c^4}\) and we will assume that \(C_s (t)\) are slowly varying functions: \(|\partial C_s / \partial t| \ll E |C_s| / \hbar\), and \(E \ll s\hbar k v_{ph}\) (this condition is always satisfied for optical photons). From the Klein–Gordon equation with Eq. (94) for the coefficients \(C_s (t)\) we obtain the set of equations

\[
i \frac{\partial C_s (t)}{\partial t} + \Gamma_s C_s (t) = \frac{W_{eff}}{4\hbar E} (C_{s-1} (t) + C_{s+1} (t)),
\]

where

\[
\Gamma_s = \frac{2E s\hbar k (v_{ph} - v_x) + (v_{ph}^2 - c^2) (s\hbar k)^2}{2\hbar E}
\]
is the resonance width. The wave function (94) at the initial condition $C_s(0) = \delta_{s,0}$ describes inelastic scattering of the electron on the slowed traveling wave. The electron energy and momentum after the scattering are:

$$
\mathcal{E}' = \mathcal{E} + \hbar k v_{ph}, \quad p'_x = p_x + \hbar k,
$$

$$
p_{\perp} = \text{const}; \quad s = 0, \pm 1, \ldots \quad (96)
$$

The probability of these processes are determined by the coefficients $C_s(t)$:

$$
W_s = |C_s(t)|^2. \quad (97)
$$

We will represent the solution of equations Eqs. (95) in the two interaction regimes: diffraction – at the classical conditions of coherency (84), (92) and Bragg resonance – at the exact quantum conditions of coherency with the quantum recoil of electron due to photon absorption/radiation.

D. Diffraction Regime of Electron Coherent Scattering on the Traveling Wave Phase-Lattice

For the finite electron-wave interaction time the interaction energy is uncertain by the quantity $\delta \mathcal{E} \approx \hbar / t_{int}$. The diffraction regime of electron coherent scattering corresponds to the short interaction times and intense wave fields. Thus, at the satisfaction of the conditions

$$
\delta \mathcal{E} \gg \hbar |\Gamma_s|, \quad \frac{W_{eff}}{4\hbar \mathcal{E}} \gg |\Gamma_s|
$$

one can neglect the term $\sim \Gamma_s C_s(t)$ in Eqs. (95) and the dynamics of electron in the wavefields will be described by the following equation (175):

$$
\frac{\partial C_s(t)}{\partial t} = \frac{W_{eff}}{4\hbar \mathcal{E}} (C_{s-1}(t) + C_{s+1}(t)). \quad (98)
$$

for which at the initial condition $C_s(0) = \delta_{s,0}$ one can obtain the following solution:

$$
C_s(t) = J_s \left( \frac{1}{2\hbar \mathcal{E}} \int_0^t W_{eff} dt' \right) e^{-i s \frac{\pi}{2}}.
$$

Consequently, the probability of this process according to Eq. (97) will be done by the formula:

$$
W_s = J_s^2 \left[ \frac{1}{2\hbar \mathcal{E}} \int_0^t W_{eff} dt' \right]. \quad (99)
$$
In the case of a monochromatic wave from Eq. (99) we have
\[
W_s = J^2_s \left( \frac{W_{\text{eff}} t_{\text{int}}}{2\hbar E} \right),
\]
where \( t_{\text{int}} \) is the time-duration of the particle motion in the wavefield. As is seen from Eq. (100) in the diffraction regime symmetric diffraction \( (J^2_s = J^2_{-s}) \) into many momentum states is possible. The process dynamics is defined by the argument of the Bessel function
\[
\alpha = W_{\text{eff}} t_{\text{int}} / (2\hbar E).
\]
For \( \alpha \lesssim 1 \) only few diffraction maxima are possible. For the values \( \alpha \gg 1 \), the process is essentially multiphoton. The most probable number of absorbed/emitted Cherenkov photons is \( s \simeq \alpha \). The width of the main diffraction maximums \( \Delta(s) \simeq s^{1/3} h k \) and since \( s \gg 1 \) then \( \Delta(s) \ll |p'_x - p_x| \).

For the concreteness let us explicitly write probability for the Cherenkov diffraction and Kapitza–Dirac effects. In case of Cherenkov diffraction of electron on a traveling wave in a dielectric medium, from Eq. (100) we have
\[
W^{(\text{Cherenkov})}_s = J^2_s \left( \frac{\xi m c^2}{\hbar} \frac{cp \sin \vartheta_{ch}}{E} t_{\text{int}} \right).
\]
For the actual values of the parameters \( \alpha \gg 1 \), that is, the process is essentially multiphoton. The most probable number of absorbed/emitted Cherenkov photons is
\[
\bar{s} \simeq \frac{\xi m c^2 v}{\hbar} \sin \vartheta_{ch} \cdot t_{\text{int}}.
\]
The scattering angles of the \( s \)-photon Cherenkov diffraction are determined by the formula:
\[
\tan \vartheta_s = \frac{s n \hbar \omega \sin \vartheta_{ch}}{c p + s n \hbar \omega \cos \vartheta_{ch}}.
\]
From Eq. (103) it follows that at the inelastic diffraction there is an asymmetry in the angular distribution of the scattered particle: \(|\vartheta_{-s}| > \vartheta_s\), i.e., the main diffraction maxima are situated at different angles with respect to the direction of particle initial motion. However, in accordance with the condition \(|s| n \hbar \omega / c \ll p\) of the eikonal approximation this asymmetry is negligibly small and for the scattering angles of the main diffraction maxima from Eq. (103) we have \( \vartheta_{-s} \simeq -\vartheta_s \). Hence, the main diffraction maxima will be situated at the angles
\[
\vartheta_{\pm s} = \pm \frac{s n \hbar \omega}{c p} \sin \vartheta_{ch}
\]
with respect to the direction of the particle initial motion.
For the Kapitza–Dirac effect, which is the particular case of induced Compton effect at \( \omega_1 = \omega_2 \equiv \omega, E_1 = E_2 = E \), from Eqs. (94) and (100) we can write

\[
\Psi = e^{-\frac{i}{\hbar}E t} e^{\frac{i}{\hbar}p \cdot \mathbf{r}} \sum_s e^{-i \frac{s}{2} \pi} J_s \left( \frac{e^2 c^2 E^2 t_{\text{int}}}{2 \hbar \omega^2} \right) e^{\frac{i}{\hbar} \left( p_x + 2s \hbar \omega \right) x}.
\]  

(105)

Hence, the probability of \( s \)-photons Kapitza–Dirac diffraction effect on the strong standing EM wave in vacuum is determined by the formula:

\[
W_s^{(\text{Kapitza–Dirac})} = J_s^2 \left( \frac{e^2 c^2 E^2 t_{\text{int}}}{2 \hbar \omega^2} \right).
\]  

(106)

Note that formula (106) for the nonrelativistic case \( E \approx \frac{mc^2}{\sqrt{1-v^2}} \) coincides with analogous formula (7) of the paper [165] up to a factor of 1/2 (in the paper [165]) the factor 1/2 has been missed perhaps at the transformation of recurrent relation of Bessel functions \( J_s(x) \).

In general case of inelastic Compton diffraction on counterpropagating waves of different frequencies and electron diffraction on a travelling EM wave in an undulator/wiggler the probabilities of \( s \)-photon diffraction effect may be written by analogy of formulas (101) and (106) with corresponding parameters.

Note that the general formula (101) for Cherenkov diffraction that expresses the probabilities of multiphoton absorption/radiation processes by electron at the induced Cherenkov effect has been applied in the paper [109] for explanation of the experiment on energetic widening of an electron beam at the induced Cherenkov process in a gaseous medium, implemented in SLAC [113] (see, also the next experiment of this group [115], made in the same conditions).

Formula (101) for Cherenkov diffraction in a dielectric medium, which had been received forty years ago [108] and included in the monographs [26] and [33], recently has been considered in the paper [175] because of reproduction of electron diffraction effect on a travelling EM wave in a dielectric medium by the authors of the paper [171], nevertheless represented it as a new phenomenon predicted in the paper [171]. Repeating the results of the paper [108] with the main formula (101) in other way of derivation -within the Helmholtz–Kirchhoff diffraction theory (as was mentioned in Introduction, by the method developed in the paper [172]), authors report in the Abstract that they showed on the possibility of electrons diffraction effect on a travelling EM wave in a dielectric medium. Apart from such misunderstanding resulted to serious confusion in scientific literature, it is interesting the claim of the authors even in the Abstract of the paper [171] (and this is the only difference from
the paper \cite{108} that explains diffraction effect by the "group velocity of light" (and this is repeated also in the text -and for a monochromatic wave!). This is interesting fact since repeating the results, at the same time authors of the paper \cite{171} quote to original papers \cite{108, 156, 157} without a comment. While, as it has been shown above, diffraction effect is thoroughly the result of the phase relations and is conditioned exceptionally by the phase velocity of light that must be smaller than \(c\) – just the physical reason which causes electron diffraction on a light phase lattice due to phase matching between the particle and wave phase velocity. Beside this rough mistake, authors of the paper \cite{171} ignored the existence of critical field in this process and influence of considered phenomenon of particle "reflection" or capture on the diffraction effect. Meanwhile, one of the authors \cite{171} is also a co-author of both inelastic diffraction effect \cite{157} and "reflection" phenomenon in the undulators \cite{59}.

E. Bragg Regime of Electron Coherent Scattering on the Traveling Wave Phase-Lattice

For the diffraction effect with sufficiently long interaction time one can fulfill resonance condition with quantum recoil for the concrete \(s_0\): \(\Gamma_{s_0} = 0\). The latter can be written as

\[
v_{ph} - v_x = \left( c^2 - v_{ph}^2 \right) \frac{s_0 \hbar k}{2\mathcal{E}}.\]

(107)

The condition (107) has transparent physical interpretation in the intrinsic frame of reference of the slowed wave. In this frame, due to the conservation of particle energy and transverse momentum the real transitions in this strongly quantum regime occur from a \(p'_x\) state to the \(-p'_x\) one and we reach the Bragg diffraction effect on a slowed traveling wave at the fulfilment of the condition:

\[
2p'_x = -s_0 \hbar k' \quad (s_0 = \pm 1; \pm 2 ...).
\]

(108)

The latter expresses the condition of exact resonance between the particle de Broglie wave and the "wave motionless lattice". In particular, in this case when the above mentioned particle capture regime by the slowed traveling wave \cite{26} takes place, we have the quantum effect of zone structure of particle states like the particle states in a crystal lattice, and at the condition (107) the diffraction maxima take place see Fig. 1. Here we just write the solution for the resonant case \(\Gamma_1 = 0\). At the condition

\[
\delta\mathcal{E} \ll \hbar |\Gamma_s|, \frac{W_{eff}}{4\hbar\mathcal{E}} \ll |\Gamma_s|; s \neq 0, 1
\]

(109)
from the set of equations (95) one can keep only resonant ones for $C_0$ and $C_1$:

$$i \frac{\partial C_{01}(t)}{\partial t} = \frac{W_{\text{eff}}}{4\hbar \mathcal{E}} C_{10}(t),$$

with the solution

$$C_0(t) = \cos \left( \frac{1}{4\hbar \mathcal{E}} \int_0^t W_{\text{eff}} dt' \right)$$

$$C_1(t) = -i \sin \left( \frac{1}{4\hbar \mathcal{E}} \int_0^t W_{\text{eff}} dt' \right) \quad (110)$$

For the concreteness let us explicitly write probability for Kapitza–Dirac effect - $\omega_1 = \omega_2 \equiv \omega$, $E_1 = E_2 = E_0$, $v_{ph} = 0$, $k = 2\omega/c$). From Eq. (107) at $s_0 = 1$ we obtain resonant initial momentum $p_x = -\hbar \frac{\omega}{c}$ and wave function can be written as

$$\Psi = e^{\frac{i}{\hbar} p_x \hat{r} - \frac{i}{\hbar} \mathcal{E} t} \left[ C_0(t) e^{-i \frac{\omega}{c} x} + C_1(t) e^{i \frac{\omega}{c} x} \right]. \quad (111)$$

with the probabilities

$$W_0 = \cos^2 \left( \frac{e^2 c^2 E_0^2 t_{\text{int}}}{4 \mathcal{E} \hbar \omega^2} \right),$$

$$W_1 = \sin^2 \left( \frac{e^2 c^2 E_0^2 t_{\text{int}}}{4 \mathcal{E} \hbar \omega^2} \right). \quad (112)$$

For the nonrelativistic case $\mathcal{E} \simeq mc^2$ Eqs. (112) coincide with the results in Ref. [165].

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