Construction of a one-parameter family of isospectral potential to study resonances in weakly bound halo nuclei

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Abstract. In this paper, a novel theoretical scheme is presented to investigate resonant levels in weakly bound nuclear systems by the use of isospectral potentials. In this scheme, a new potential is constructed which is strictly isospectral with the original shallow-well potential and has properties which are desirable to make the calculation of resonances more accurate and easier. To demonstrate the effectiveness of the method, the first 0⁺ resonance in the neutron-rich ²²C is calculated assuming a three-body (²⁰C+n+n) cluster model.

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1. Introduction
One of the most striking events in physics is the discovery of halo nuclei near the drip-lines after the advancements radioactive ion beam (RIB) facilities. Some of the observed neutron halo nuclei are ¹¹Li, ⁶Be, ¹¹Be, ¹⁴Be, ¹⁷B [1-4] while proton halo structure is found in ⁸B [5]. Experimental investigation by Audi et al [6] on ¹⁸−²²C inferred a 2n-halo structure of ²²C. Structure of halo nuclei is characterized by a stable dense core surrounded by one or more loosely bound nucleon(s) giving rise to an extended tail in the density distribution. This low density tail accounts for the unusually large r.m.s. matter radii (larger than the liquid-drop model prediction of $R_A \propto A^{1/3}$) of such exotic nuclei [7-8]. Further, the low-density tail is supposed to be the consequence of quantum mechanical tunneling of the last nucleon(s) through the combination of weak binding and short-range interaction. Halo nuclei rarely support excited bound states, they merely support one bound state with binding energy typically less than 1 MeV. However, these systems may have one or more resonance states above the ground state. Calculations for resonance states generally involve a large degree of computational error. Here we propose an effective technique for calculation of resonances together with ground state using supersymmetric quantum mechanics.

Literature survey reveals that there are three main theoretical approaches to study the structure of 2n-halo nuclei. First one is the microscopic model approach which assumes the valence neutrons to move around the conglomerate of other nucleons (protons and neutrons) without having any stable core. The second one is the cluster or three-body model in which the valence nucleons move around the structureless inert core. And the third one is the microscopic
cluster model in which the valence nucleons move around the excited or deformed core [9-11]. In most of the models, Jacobi coordinates are used to separate the center of mass motion and the system is described in terms of the relative coordinates.

Study of resonances in few-body system is of particular interest in many areas of physics and especially in weakly bound systems, where only a few bound states are possible. For the calculation of resonant states a number of methods are in use, viz., the positive energy solution of the Faddeev equation [12], complex coordinate rotation (CCR) [13-14], hyperspherical harmonics method (HHM) for scattering states [15], the analytic computation of bound state energies [16], the algebraic version of resonating group method (RGM) [17], continuum-discretized coupled-channels (CDCC) method in a club with cluster-orbital shell model (COSM) [18] etc.

In this work, we present a novel theoretical scheme for calculation of resonant states in weakly bound nuclei. As a prerequisite, it requires the existence of a bound ground state and can handle both of the bound and resonant states on the same footing. The scheme is based on the fact that, for any given potential (say, \( \omega \)), one can construct a family of isospectral potentials (\( \hat{\omega} \)), which depends on a free parameter (\( \delta \)). Whilst the original potential has a skin-deep well followed by a low and excessively wide barrier (poorly supporting the resonant state), the parameter \( \delta \) can be chosen judiciously to increase the depth of the well and height of the barrier in \( \hat{\omega} \) simultaneously. This deep well and high barrier combination in \( \hat{\omega} \) facilitates the calculation of resonant state accurately at the same energy as that in \( \omega \), since \( \hat{\omega} \) and \( \omega \) are strictly isospectral.

To check the validity of the scheme, we apply it to investigate the first \( 0^+ \) resonant state of the weakly bound \( ^{22}_{\text{C}} \) nucleus [18]. We assume a three-body cluster model for \( ^{22}_{\text{C}} \) (\( 2n+^{20}_{\text{C}} \)) in which the valence neutrons are assumed to move around the relatively stable and inert \( ^{20}_{\text{C}} \) core. The three-body effective potential shows a shallow well with a wide skinny barrier causing a large resonance width. In principle, one can find quasi-bound states in such a shallow potential, but it poses a difficult numerical task. For a finite barrier height, a particle can temporarily be trapped inside the shallow well when its energy is close to the resonance energy. However, there is a finite probability that the particle may tunnel out through the barrier. Thus, an accurate calculation of resonance energy is easily masked by the large resonance width resulting from a large tunneling probability due to a small barrier height. Hence, a straightforward calculation of resonance energy of these systems fails to produce accurate results.

We adopt Jacobi coordinates to separate the centre of mass motion and the hyperspherical harmonics expansion (HHE) method [19] to solve the three-body Schrödinger equation in relative coordinates. The three-body relative wavefunction is expanded in a complete set of hyperspherical harmonics (HH), substitution of which in the Schrödinger equation and use of orthonormality of HH leads to set of coupled differential equations (CDE). The HHE method adopted here is an essentially exact, as it involves no other approximation than an eventual truncation of the expansion basis. This truncation of the expansion basis can be done to achieve any desired precision in energy finally restricted by the capacity of the available computer. Hyperspherical convergence theorem [20] permits extrapolation of the truncated set of results to predict the fully converged solution. The theorem found useful in yielding high precision results in atomic systems [21-22] and in nuclear systems [23]. Convergence of HH being generally slow, one needs a large number of CDE’s to be solved. Hence, for further simplification of the numerical task, we adopt the hyperspherical adiabatic approximation (HAA) [24] to solve the resulting single differential equation (SDE) involving a three-body effective potential \( \omega_0(\rho) \) for the ground state energy, \( E_0 \) and corresponding wavefunction \( \psi_0(\rho) \) [25].

We next construct the isospectral potential \( \hat{\omega}_0(\delta; \rho) \) following prescription of the SSQM [26-28]. Finally we solve the SDE for \( \hat{\omega}_0(\delta; \rho) \) for different positive energies. Thereafter we calculate probability of finding the particle trapped in the deep narrow well surrounded by an amplified barrier. A plot of this probability as a function of the energy shows a sharp peak at the resonance energy. The actual width of the resonance is obtained by back-transforming the wave function.
where \( \omega_0(\delta; \rho) \) to \( \psi_0 \) of \( \omega_0(\rho) \).

The paper is organized as follows. In sections 2 we briefly review the HHE method and in section 3 we present very short details of the SSQM procedure to construct the one-parameter family of isospectral potential. Results of present calculation are discussed in section 4 and conclusions are drawn in section 5.

2. Hyperspherical Harmonics Expansion Method

The nucleus \(^{22}\text{C}\) is treated as a three-body system [18] where the core \(^{20}\text{C}\) is labeled as particle 1 and the two valence neutrons as particles 2 and 3 respectively. A set of Jacobi coordinates for a given partition \( i \) (in which particle \( i \) is a spectator) is defined as:

\[
\xi_i = a_i(r_j - r_k); \quad \eta_i = \frac{1}{a_i}\left(r_i - \frac{m_j r_j + m_k r_k}{m_j + m_k}\right); \quad \vec{R} = \frac{(m_i r_i + m_j r_j + m_k r_k)}{M}
\]

(1)

where \( a_i = \left[\frac{m_j m_k}{m_i(m_j + m_k)}\right]^{\frac{1}{2}} \); \( r_i \) are the mass and position of the \( i \)th particle and \( M = m_i + m_j + m_k \), \( \vec{R} \) are those of the centre of mass (CM) of the system. Since the interactions among constituent particles depend on relative separations only, the centre of mass motion separates out and the relative motion of the three-body system is described by:

\[
\left[-\frac{\hbar^2}{2\mu}(\nabla_{\xi_i}^2 + \nabla_{\eta_i}^2) + V_{jk}(\xi_i, \eta_i) + V_{kl}(\xi_i, \eta_i) + V_{ij}(\xi_i, \eta_i) - E\right] \Psi(\xi_i, \eta_i) = 0
\]

(2)

where \( \mu = \left[\frac{m_j m_k}{M}\right]^{\frac{1}{2}} \rightarrow \) effective mass parameter, \( V_{ij} \rightarrow \) interaction potential between particles \( i \) and \( j \), \( \xi_i = \rho \cos \phi_i; \ \eta_i = \rho \sin \phi_i; \ \phi_i = \tan^{-1}(\frac{\eta_i}{\xi_i}); \ \rho = \sqrt{\xi_i^2 + \eta_i^2} \). The hyperradius \( \rho \), an invariant under three dimensional rotations and permutations of particle indices together with the five angular variables \( \Omega_i \rightarrow \{\phi_i, \theta_{x_i}, \theta_{y_i}, \phi_{y_i}\} \) constitute hyperspherical variables of the system. It is to be noted that hyperangles \( \Omega_i \) depend on the choice of the particular partition \( i \). In terms of hyperspherical variables \( (\rho, \Omega_i) \) the three-body Schrödinger equation becomes:

\[
\left[-\frac{\hbar^2}{2\mu}\left(\frac{1}{\rho^5} \frac{\partial}{\partial \rho} (\rho^5 \frac{\partial}{\partial \rho}) - \frac{\vec{N}^2(\Omega_i)}{\rho^2}\right)\right] + V(\rho, \Omega_i) - E\right] \Psi(\rho, \Omega_i) = 0
\]

(3)

where \( V(\rho, \Omega_i) = V_{jk} + V_{kl} + V_{ij} \) is the total interaction potential in the partition \( i \) and \( \vec{N}^2(\Omega_i) \) is the square of hyper angular momentum operator satisfying the eigenvalue equation:

\[
\vec{N}^2(\Omega_i) \mathcal{Y}_{N\alpha_i}(\Omega_i) = N(N + 4) \mathcal{Y}_{N\alpha_i}(\Omega_i)
\]

(4)

The quantum number \( N \) represents the hyperangular momentum and \( \alpha_i \equiv \{l_{x_i}, l_{y_i}, L, M\} \) is a short hand notation, where \( L \) and \( M \) stand for the total orbital angular momentum and its projection respectively. The partition dependent normalized eigenfunctions \( \mathcal{Y}_{N\alpha_i}(\Omega_i) \) are called the hyperspherical harmonics (HH), a closed analytic expressions for which can be seen in ref. [12].

In the HHE method, the wavefunction \( \Psi(\rho, \Omega_i) \) is expanded in the complete set of hyperspherical harmonics associated with a given partition "\( i \)"

\[
\Psi(\rho, \Omega_i) = \sum_{N\alpha_i} \frac{U_{N\alpha_i}(\rho)}{\rho^{\frac{5}{2}}} \mathcal{Y}_{N\alpha_i}(\Omega_i)
\]

(5)
Substitution of Eq.(5) in Eq.(3) and use of orthonormality of HH, give rise to a set of coupled differential equations (CDE) in $\rho$:

$$\left\{ -\frac{\hbar^2}{2\mu} \frac{d^2}{d\rho^2} + \frac{(N+\frac{1}{2})(N+\frac{3}{2})\hbar^2}{2\mu\rho^2} - E \right\} U_{N\alpha_i}(\rho) + \sum_{N',\alpha_i'} <N\alpha_i| V(\rho, \Omega_i) |N'\alpha_i'> U_{N'\alpha_i'}(\rho) = 0. \right\}$$

in which

$$<N\alpha_i| V| N'\alpha_i'> = \int \mathcal{Y}_{N\alpha_i}^*(\Omega_i) V(\rho, \Omega_i) \mathcal{Y}_{N'\alpha_i'}(\Omega_i) d\Omega_i$$

The set of CDE’s, Eq.(6), is in principle an infinite one which arises due to an infinite number of basis states. For practical purposes, the HH expansion in Eq.(5) is truncated by retaining all $N$ values up to a maximum of $N_{max}$. For a given $N$, all allowed values of $\alpha_i$ are included. The number of basis states is further restricted by symmetry requirements and associated conserved quantum numbers. The reduced set of CDE’s are solved adopting hyperspherical adiabatic approximation (HAA) [24]. In HAA the CDE’s are approximated by a single differential equation assuming that the hyperradial motion is much slower compared to hyperangular motion. Hence the angular motion is first solved for a fixed value of $\rho$. This corresponds to diagonalization of the potential matrix (including the hyper centrifugal repulsion term) for each $\rho$-mesh point and choosing the lowest eigenvalue $\omega_0(\rho)$ as the lowest eigen potential [25]. Then the energy of the system is obtained by solving the hyperradial motion for the chosen lowest eigen potential ($\omega_0(\rho)$), which is the effective potential for the hyperradial motion:

$$\left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{d\rho^2} + \omega_0(\rho) - E \right] \psi_0(\rho) = 0$$

Renormalized Numerov method algorithm subject to appropriate boundary conditions in the limit $\rho \rightarrow 0$ and $\rho \rightarrow \infty$ is then applied to solve Eq.(8) for $E \leq E_0$. The hyper-partial wave $U_{N\alpha_i}(\rho)$ is given by

$$U_{N\alpha_i}(\rho) = \psi_0(\rho) \chi_{N\alpha_i,0}$$

where $\chi_{N\alpha_i,0}(\rho)$ is the $(N\alpha_i)^{th}$ element of the eigenvector, corresponding to the lowest eigen potential $\omega_0(\rho)$.

### 3. Isospectral Potential

In this section we present a brief review of the scheme of construction of one parameter family of isospectral potentials. In 1-D supersymmetric quantum mechanics, one defines a superpotential for a system in terms of its ground state wave function ($\psi_0$) [26] as

$$W(\rho) = -\frac{\hbar}{\sqrt{2m}} \frac{\psi'_0(\rho)}{\psi_0(\rho)}.$$  

The energy scale is next shifted by the ground state energy ($E_0$) of the potential $\omega_0(\rho)$, so that in this shifted energy scale the new potential becomes

$$\omega_1(\rho) = \omega_0(\rho) - E_0 = \frac{\hbar^2}{2m} \frac{\psi'_0}{\psi_0}$$

having its ground state at zero energy (i.e. $E_0^{(1)} = 0$). It can easily be checked that $\omega_1(\rho)$ is expressible in terms of the superpotential via the Riccati equation

$$\omega_1(\rho) = W^2(\rho) - \frac{\hbar}{\sqrt{2m}} W'(\rho).$$
and the Hamiltonian factorizes as

\[ H_1 = -\frac{\hbar^2}{2m} \frac{d^2}{d\rho^2} + \omega_1(\rho) = A^\dagger A \]  

(13)

where

\[ A^\dagger = W(\rho) - \frac{\hbar}{\sqrt{2m}} \frac{d}{d\rho} \]
\[ A = W(\rho) + \frac{\hbar}{\sqrt{2m}} \frac{d}{d\rho} \]  

(14)

which are the operators for creation and annihilation of nodes in the wave function. Next we introduce a partner Hamiltonian \( H_2 \equiv AA^\dagger \), corresponding to a new partner potential \( \omega_2(\rho) \):

\[ H_2 = -\frac{\hbar^2}{2m} \frac{d^2}{d\rho^2} + \omega_2(\rho) = AA^\dagger; \]  

(15)

\[ \omega_2(\rho) = W^2(\rho) + \frac{\hbar}{\sqrt{2m}} W'(\rho) \]  

(16)

In the above, functions \( \omega_1(\rho) \) and \( \omega_2(\rho) \) are known as supersymmetric partner potentials.

Now we shall see that the energy eigenvalues and the wave functions of \( H_1 \) and \( H_2 \) are related. It can be noticed that the energy eigenvalues of both \( H_1 \) and \( H_2 \) are positive semidefinite (\( E_n^{(1,2)} \geq 0 \)). For \( n > 0 \), the Schrödinger equation for \( H_1 \)

\[ H_1 \psi_n^{(1)} = A^\dagger A \psi_n^{(1)} = E_n^{(1)} \psi_n^{(1)} \]
\[ \text{and, } H_2 (A \psi_n^{(1)}) = AA^\dagger A \psi_n^{(1)} = E_n^{(1)} (A \psi_n^{(1)}) \]  

(17)

Similarly, the Schrödinger equation for \( H_2 \)

\[ H_2 \psi_n^{(2)} = AA^\dagger \psi_n^{(2)} = E_n^{(2)} \psi_n^{(2)} \]
\[ \text{and, } H_1 (A^\dagger \psi_n^{(2)}) = A^\dagger AA^\dagger \psi_n^{(2)} = E_n^{(2)} (A^\dagger \psi_n^{(2)}) \]  

(18)

From Eqs. (17)-(18) and the fact that \( E_0^{(1)} = 0 \), it is easy understood that the eigenvalues and eigenfunctions of the partner Hamiltonians \( H_1 \) and \( H_2 \) are related by \((n = 0, 1, 2, 3, \ldots)\)

\[ E_n^{(2)} = E_{n+1}^{(1)}, \quad E_0^{(1)} = 0, \]  

(19)

\[ \psi_n^{(2)} = \frac{1}{\sqrt{E_{n+1}^{(1)}}} (A \psi_n^{(1)}), \]  

(20)

\[ \psi_{n+1}^{(1)} = \frac{1}{\sqrt{E_n^{(2)}}} (A^\dagger \psi_n^{(2)}). \]  

(21)

In the above equations \( E_n^{(i)} \) represents the energy of the \( n^{th} \) excited state of \( H_i \) \((i=1,2)\). It is to be noted that if \( \psi_{n+1}^{(1)}(\psi_n^{(2)}) \) of \( H_1(H_2) \) is normalized then the wave function \( \psi_n^{(2)}(\psi_{n+1}^{(1)}) \) in Eqs. (20) and (21) is also normalized. Further, the operator \( A(A^\dagger) \) not only converts an eigenfunction of \( H_1(H_2) \) into an eigenfunction of \( H_2(H_1) \) with the same energy, but it also destroys (creates) an extra node in the eigenfunction. Since the ground state wave function of \( H_1 \) is annihilated by the operator \( A \), this state has no SUSY partner. Thus the scenario we get is that knowing all the eigenfunctions of \( H_1 \) we can determine the eigenfunctions of \( H_2 \) using the operator \( A \), and vice versa using \( A^\dagger \) we can reconstruct all the eigenfunctions of \( H_1 \) from those of \( H_2 \) except for the ground state.
Figure 1. Schematic representation of the energy levels of the supersymmetric partner potentials $\omega_1$ and $\omega_2$ and the action of the operators $A$ and $A^\dagger$ on the corresponding eigenfunctions. The spectrum of $\omega_1$ and $\omega_2$ are identical except that $\omega_1$ has an extra state at zero energy $E_0^{(1)}=0$.

As evident from Eq.(19), $H_1$ and $H_2$ have exactly same spectra except for the fact that $H_2$ has one less bound state (which is the partner state of $H_2$ corresponding to the ground state of $H_1$). Hence, from Eqs. (19)-(21) it can be concluded that the Hamiltonians $H_1$ and $H_2$ have identical spectra, except that the partner state of $H_2$ corresponding to the ground state of $H_1$ is missing in the spectrum of $H_2$ [26]. Thus, the supersymmetric partner potentials $\omega_1$ and $\omega_2$ are not strictly isospectral.

However one can construct a one parameter (say, $\delta$) family of strictly isospectral potentials $\tilde{\omega}_1(\delta; \rho)$ taking advantage of the nonlinear nature of Riccati equation (Eqs. (12) and (16)) and the fact that for a given $\omega_2(\rho); \omega_1(\rho)$ and the superpotential $W(\rho)$ are not unique. Following [26-28], it can be shown that the most general superpotential satisfying the Riccati equation for $\omega_2(\rho)$ (Eq. (16)) is given by

$$\tilde{W}(\rho) = W(\rho) + \frac{\hbar}{\sqrt{2m}} \frac{d}{d\rho} \ln[I_0(\rho) + \delta]$$

(22)

where $\delta$ is a constant of integration and

$$I_0(\rho) = \int_{\rho'=0}^{\rho} [\psi_0(\rho')]^2 d\rho'$$

(23)
in which $\psi_0(\rho)$ is the normalized ground state wave function of $\omega_0(\rho)$. Then the potential

\[
\hat{\omega}_1(\delta; \rho) = \hat{W}^2(\rho) - \frac{\hbar}{\sqrt{2m}} \hat{W}'(\rho) = \omega_1(\rho) - 2\frac{\hbar^2}{2m} \frac{d^2}{d\rho^2} \ln[I_0(\rho) + \delta]
\]

has the same SUSY partner potential $\omega_2(\rho)$ and the ground state at zero energy. Hence $\hat{\omega}_1(\delta; \rho)$ is strictly isospectral with $\omega_1(\rho)$. The parameter $\delta$ is arbitrary in the intervals $-\infty < \delta < -1$ and $0 < \delta < \infty$. Since $I_0(\rho)$ lies between 0 and 1, the interval $-1 \leq \delta \leq 0$ is forbidden, for which $\hat{\omega}_1(\delta; \rho)$ will have singularities. One can easily see that for $\delta \to \pm\infty$, $\hat{\omega} \to \omega$ and for $\delta \to 0+$, $\hat{\omega}_1$ develops a narrow and deep attractive well near the origin. This well-barrier combination effectively traps the particle giving rise to a sharp resonance. This method has been tested successfully for a three-dimensional finite square well [29] with parameters such that the well can support one or more resonance state(s) in addition to one bound state.

To calculate the resonance energy accurately, we compute the probability of finding the system within the well region of the potential $\hat{\omega}_1(\delta; \rho)$,

\[
P(E) = \int_{\rho_B}^{\rho_B} |\psi_E(\rho', \delta)|^2 d\rho'
\]

where $\rho_B$ is the position of the top of the barrier component of the potential $\hat{\omega}_1(\delta; \rho)$ for a chosen $\delta$. Here $\psi_E(\rho, \delta)$, the solution for the potential $\hat{\omega}_1(\rho; \delta)$, corresponding to a positive energy $E$ is normalized to have a constant amplitude in the asymptotic region. Plot of the quantity $P(E)$ as a function of $E$ ($E > 0$) shows a peak at the resonance energy $E = E_R$. Here, choice of $\delta$ to be made judiciously in order to avoid numerical errors creeping in the wavefunction in the extremely narrow well for $\delta \to 0+$.

The width ($\Gamma$) of the resonance is obtained from the mean life of the state using the energy-time uncertainty relation. The mean life is the reciprocal of the decay constant, which is expressed as a product of the frequency of hit per unit time on the barrier and the probability of tunneling through the barrier. $\Gamma$ is obtained using the relation

\[
\Gamma = 2\sqrt{\frac{\hbar^2}{2\mu} \exp(-2 \int_0^{\infty} d\rho' \frac{2\mu}{\hbar^2} (\hat{\omega}(\delta; \rho) - E_R))}
\]

where $a$ and $b$ are the classical turning points of the nuclear well, while $b$ and $c$ are the classical turning points of the barrier in $\hat{\omega}_1(\delta; \rho)$, corresponding to the resonance energy $E_R$.

4. Results and Discussions

Ground state of $^{22}\text{C}$ is a $T = 1, J^\pi = 0^+$ state and there exists a resonance state of the same $J^\pi$. Thus, the forgoing procedure starting from the ground state of $^{22}\text{C}$ will give $T = 1, J^\pi = 0^+$ resonance(s). Standard supersoft core GPT potential [30] is chosen for the n-n pair while for the core-nucleon ($^{20}\text{C-n}$) pair choose four component SBB [31] Gaussian type potential. The s-wave component of SBB potential is kept repulsive to simulate Pauli principle between the core nucleons and the extra-core neutrons. Here, the parameters of the SBB core-n potential is choosen following our previous work [23] to reproduce the observed values of the ground state of $^{22}\text{C}$. The SBB potential has the form

\[
V_{\text{Core-n}}(r) = \sum_l V_{\text{Core-n}}^{(l)} \exp(-\frac{\mu}{b_{\text{Core-n}}^{(l)}} r^2) + V_{\text{S}}(l, s) \exp(-\frac{\mu}{b_{\text{Core-n}}^{(l)}} r^2)
\]

where $V_{\text{Core-n}}^{(l)}$ is $+50$ MeV, $-47.32$ MeV, $-23.0$ MeV for $l=0, 1$ and 2 respectively and $V_{\text{S}} = -5.855$ MeV and $b_{\text{Core-n}} = 2.32 fm$ for all the four components. As the original core-n potential
gives overbinding we adjusted the range of core-n potential by about 10\% to get the resonant $0^+$ state of $^{22}$C at about 1.27 MeV above the ground ($0^+$) state. With these potentials, we first solve Eq.(6) for the ground state using HAA. The two-neutron separation energy ($B$) and the RMS matter radius ($R_A$) for gradually increasing $N_{\text{max}}$ are presented in Columns 2 and 4 of Table 1. The results indicate a clear convergence trend in $B$, but even at $N_{\text{max}} = 24$ calculated $B$ is far away from convergence. For this reason, we applied the hyperspherical extrapolation technique \[21-23\], based on the hyperspherical convergence theorem \[20\] to get the energy at sufficiently large $N_{\text{max}}$. The extrapolated binding energies with increasing $N_{\text{max}}$ is presented in Table 2. Contribution of different partial waves are also presented in columns 5-9 of Table 1 which indicate larger contribution to the ground state energy from the even orbital angular momentum values. The final extrapolated value of energy ($=0.4224$ MeV) shown in column 3 of Table is in excellent agreement with the experimental value of $0.42 \pm 0.94$ MeV \[8\]. The calculated R.M.S. matter radius also agrees fairly with the observations \[8, 32-33\].

We next utilize the property of the isospectral potential to find the resonant state. The three-body effective potential $\omega_0(\rho)$ when plotted against $\rho$ presents a shallow well followed by a broad and low barrier as shown in Figure 1. This low well-barrier combination can give rise to resonant states. However, since the well is very shallow and the barrier is not sufficiently high, the resonance width is very large and a numerical calculation of the resonant state is quite cumbersome.

We construct the one-parameter family of isospectral potentials $\hat{\omega}(\delta; \rho)$ according to Eq.(24) by judicious choice of an appropriate value of $\delta$, that gives rise to a narrow and deep well followed by a high barrier. Some of them are demonstrated in Figure 1 as representative cases. This well-barrier combination effectively traps the particles to form a strong resonant state. Calculated parameters of the isospectral potential for a number of $\delta$ values, along with the original one $\omega(\rho)$ are presented in columns 2-5 of Table 3. From this, we can see that when $\delta$ decreases from 0.1 to 0.001, the depth of the well increases from -22.1 MeV at 2.67 fm to -117.2 MeV at 1.65 fm while the height of the barrier jumps from 4.2 MeV at 5.1 fm to 51.5 MeV at 2.56 fm. This
demonstrates the dramatic effect on the lowest eigen potential \( \omega_1(\delta; \rho) \) as \( \delta \) approaches 0+. Still smaller positive values of \( \delta \) are not desirable since the well will be too narrow to calculate the wave function accurately by a numerical procedure.

We calculated the probability of \( P(E) \) of the system to be trapped within the well-barrier combination for a range of energies \( E \). A representative plot of \( P(E) \) versus \( E \) shown in Figure 3, exhibits resonance peak at the energy \( E_R \approx 1.27 \) MeV.

The calculated widths of resonance for different \( \delta \to 0^+ \) indicates dramatic enhancement in trapping effect of \( \omega_0(\delta; \rho) \) for smaller values of delta. From column 6 of Table 3, it can be seen that the width of the resonance decreases significantly with decrease in \( \delta \) values. Hence for the original potential \( \omega_0(\rho) \) (which corresponds to \( \delta \to +\infty \)), the exact value of resonance energy becomes uncertain. But the uncertainty in \( E_R \) decreases as \( \delta \) decreases towards 0+.

The enhancement of accuracy in determination of \( E_R \) is the principal advantage of using Supersymmetric formalism. Since \( \omega_1(\delta; \rho) \) is strictly isospectral with \( \omega_1(\rho) \), any value of \( \delta \) is admissible in principle. However, a judicious choice of \( \delta \) is necessary for accurate determination of the resonance energy. Results of the present calculations are summarized in Table 4 and compared with other works found in literature.

5. Summary and Conclusions

In this work have employed hyperspherical harmonics expansion method to investigate the bound state properties of the loosely bound nucleus \( ^{22}\text{C} \) in the framework of three-body \( (^{20}\text{C}+n+n) \) system. Standard GPT [30] potential is used for the \( n-n \) pair while for the remaining pairs we employed SBB potential [31] with a slightly modified range parameter so as to reproduce the experimental excitation energy of the 0\(^{+}\) state. The lowest-eigen potential \( \omega_0(\rho) \) for the ground state \( (J^\pi = 0^+) \) exhibits shallow well followed by a skinny wide barrier (Figure 2). Computation of resonances in such a shallow well and wide barrier combination presents a very difficult numerical task. However, in our scheme we managed to circumvent this difficulty by using the isospectral potential technique by judicious choice of the free parameter \( \delta \), which gives rise to a sufficiently deep and narrow well surrounded by relatively high and thin barrier. This narrow well-barrier combination facilitates trapping effect thereby resulting in a prominent resonance. Thus, we may conclude that use of isospectral potential technique is an essential one for the accurate determination determination of resonances in weakly bound exotic systems which support only one bound state. Even, the technique can also be extended to systems which do not have any bound state by invoking suitable modification in it so as to fit for bound state in continuum.

6. Tables

| \( N_{\text{max}} \) | Energy, -E (MeV) | \( \chi \) | \( R_A \) (fm) | \( -E_{l_\xi}(MeV) \) |
|----------------|----------------|--------|----------------|-----------------|
|                |                |        |                | \( l_\xi = 0 \) | \( l_\xi = 1 \) | \( l_\xi = 2 \) | \( l_\xi = 3 \) | \( l_\xi = 4 \) |
| 12             | 0.2419         | 0.2032 | 3.4259         | 0.21449         | 0.00678         | 0.02309         | 0.00016         | 0.00211         |
| 16             | 0.3035         | 0.1185 | 3.4217         | 0.27247         | 0.00753         | 0.02469         | 0.00018         | 0.00231         |
| 20             | 0.3443         | 0.0719 | 3.4187         | 0.31169         | 0.00819         | 0.02536         | 0.00020         | 0.00237         |
| 24             | 0.3710         | 0.34163 | 0.33735 | 0.00875 | 0.02550 | 0.00022 | 0.00241 |
Table 2. Extrapolated energy ($B_{\text{Extrapolated}}$) and the corresponding relative convergence trend ($\chi$) with increasing $N_{\text{max}}$ for the ground ($J^\pi = 0^+$) state of $^{22}$C.

| $N_{\text{max}}$ | $B_{\text{Extrapolated}}$ | $\chi$ | $N_{\text{max}}$ | $B_{\text{Extrapolated}}$ | $\chi$ |
|-----------------|--------------------------|------|-----------------|--------------------------|------|
| 24              | 0.3710825                | 0.072043208 | 96              | 0.4224221                | 0.000033559 |
| 28              | 0.3886654                | 0.045239185 | 100             | 0.4224314                | 0.000022071 |
| 32              | 0.4002297                | 0.028894056 | 104             | 0.4224375                | 0.000014516 |
| 36              | 0.4078355                | 0.018649189 | 108             | 0.4224416                | 0.000009547 |
| 40              | 0.4128378                | 0.012116931 | 112             | 0.4224442                | 0.000006279 |
| 44              | 0.4161279                | 0.007906285 | 116             | 0.4224459                | 0.000004129 |
| 48              | 0.4182917                | 0.005173055 | 120             | 0.4224471                | 0.000002716 |
| 52              | 0.4197149                | 0.003390777 | 124             | 0.4224479                | 0.000001786 |
| 56              | 0.4206509                | 0.002225148 | 128             | 0.4224484                | 0.000001175 |
| 60              | 0.4212665                | 0.001461339 | 132             | 0.4224487                | 0.000000773 |
| 64              | 0.4216714                | 0.000960198 | 136             | 0.4224489                | 0.000000508 |
| 68              | 0.4219377                | 0.000631123 | 140             | 0.4224491                | 0.000000334 |
| 72              | 0.4221128                | 0.000414916 | 144             | 0.4224492                | 0.000000219 |
| 76              | 0.4222820                | 0.000272816 | 148             | 0.4224492                | 0.000000145 |
| 80              | 0.4223038                | 0.000179398 | 152             | 0.4224492                | 0.000000095 |
| 84              | 0.4223536                | 0.000117976 | 156             | 0.4224492                | 0.000000063 |
| 88              | 0.4223864                | 0.000077587 | 160             | 0.4224492                | 0.000000041 |
| 92              | 0.4224079                | 0.000051026 | 164             | 0.4224493                | 0.000000027 |

Table 3. Parameters of the isospectral potential, $\hat{\omega}_0(\delta; \rho)$ as the parameter $\delta$ decreases from $+\infty$ (for the original potential $\omega_0(\rho)$) towards $0^+$.

| $\delta$ | Potential well | Potential Barrier |
|----------|----------------|--------------------|
|          | Depth(MeV)     | Position(fm) | Height(MeV) | Position(fm) |
| 1000000  | -8.298         | 3.068          | 0.623       | 6.579        |
| 100      | -8.321         | 3.068          | 0.627       | 6.576        |
| 50       | -8.343         | 3.067          | 0.630       | 6.573        |
| 1        | -10.404        | 3.008          | 1.041       | 6.372        | 18937344.7486 |
| 0.1      | -22.098        | 2.674          | 4.181       | 5.101        | 6.242        |
| 0.001    | -117.207       | 1.648          | 51.541      | 2.555        | 0.134        |
| 0.0001   | -185.743       | 1.211          | 103.451     | 1.993        | 1.543E-03    |
| 0.00001  | -337.228       | 0.879          | 162.829     | 1.456        | 2.504E-06    |
| 0.000001 | -596.238       | 0.632          | 300.474     | 1.076        | 3.788E-10    |
Table 4. Comparison of the calculated results with experimental data and other reference works found in the literature for $^{22}$C.

| State | Observables | Present work | Others work | Reference |
|-------|-------------|--------------|-------------|-----------|
| $0^+$ | E           | -0.422 MeV   | -0.420 ± 0.940 MeV | [8,18]    |
|       | $R_A$       | 3.42 fm      | 5.4±0.9 fm   |           |
|       |             |              | 3.03 – 3.34 fm | [32]      |
|       |             |              | 3.58 – 3.74 fm | [33]      |
| $0^+_1$ | $E_R$   | 1.27 MeV     | 1.02 MeV    | [18]      |
|       | $\Gamma$   | 0.51 MeV     | 0.52 MeV    |           |

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