An EKF-SLAM algorithm with consistency properties

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Abstract

In this paper we address the inconsistency of the EKF-based SLAM algorithm that stems from non-observability of the origin and orientation of the global reference frame. We prove on the non-linear two-dimensional problem with point landmarks observed that this type of inconsistency is remedied using the Invariant EKF, a recently introduced variant of the EKF meant to account for the symmetries of the state space. Extensive Monte-Carlo runs illustrate the theoretical results.

1 Introduction

The problem of simultaneous localization and mapping (SLAM) has a rich history over the past two decades, which is too broad to cover here, see e.g. [18, 19]. The extended Kalman filter (EKF) based SLAM (the EKF-SLAM) has played an important historical role, and is still used, notably for its ability to close loops thanks to the maintenance of correlations between remote landmarks.

The fact that the EKF-SLAM is inconsistent (that is, it returns a covariance matrix that is too optimistic, see e.g., [3], leading to inaccurate estimates) was early noticed [28] and has since been explained in various papers [14, 2, 24, 25, 27, 23]. In the present paper we consider the inconsistency issues that stem from the fact that, as only relative measurements are available, the origin and orientation of the earth-fixed frame can never be correctly estimated, but the EKF-SLAM tends to “think” it can estimate them as its output covariance matrix reflects an information gain in those directions of the state space. This lack of observability, and the poor ability of the EKF to handle it, is notably regarded as the root cause of inconsistency in [26, 23] (see also references therein). In the present paper we advocate the use of the Invariant (I)-EKF to prevent covariance reduction in directions of the state space where no information is available.

The Invariant extended Kalman filter (IEKF) is a novel methodology introduced in [10, 11] that consists in slightly modifying the EKF equations to have them respect the
geometrical structure of the problem. Reserved to systems defined on Lie groups, it
has been mainly driven by applications to localization and guidance, where it appears
as a slight modification of the multiplicative EKF (MEKF), widely known and used in
the world of aeronautics. It has been proved to possess theoretical local convergence
properties the EKF lacks in [8], to be an improvement over the EKF in practice (see
e.g., [5,4,17,31] and more recently [8] where the EKF is outperformed), and has been
successfully implemented in industrial applications to navigation (see the patent [7]).

In the present paper, we slightly generalize the IEKF framework, to make it ca-
pable to handle very general observations (such as range and bearing or bearing only
observations), and we show how the derived IEKF-SLAM, a simple variant of the
EKF-SLAM, allows remedying the inconsistency of EKF-SLAM stemming from the
non-observability of the orientation and origin of the global frame.

1.1 Links and differences with previous literature
The issue of EKF-SLAM inconsistency has been the object of many papers, see [28,
[4,2,27] to cite a few, where empirical evidence (through Monte-Carlo simulations)
and theoretical explanations in various particular situations have been accumulated.
In particular, the insights of [2,27] have been that the orientation uncertainty is a
key feature in the inconsistency. The article [27], in line with [28,14,32,2], also
underlines the importance of the linearization process, as linearizing about the true
trajectory solves the inconsistency issues, but is impossible to implement in practice as
the true state is unknown. It derives a relationship that should hold between various
Jacobians appearing in the EKF equations when they are evaluated at the current state
estimate to ensure consistency.

A little later, the works of G.P. Huang, A.I. Mourikis, and S. I. Roumeliotis [24,
[26,23] have provided a sound theoretical analysis of the EKF-SLAM inconsistency as
carried by the EKF inability to correctly reflect the three unobservable degrees of free-
dom (as an overall rotation and translation of the global reference frame leave all the
measurements unchanged). Indeed, the filter tends to erroneously acquire information
along the directions spanned by those unobservable transformations. To remedy this
problem, the above mentioned authors have proposed various solutions, the most ad-
vanced being the Observability Constrained (OC)-EKF. The idea is to pick a lineariza-
tion point that is such that the unobservable subspace “seen” by the EKF system model
is of appropriate dimension, while minimizing the expected errors of the linearization
points.

Our approach, that relies on the IEKF, provides an interesting alternative to the
OC-EKF, based on a quite different route. Indeed, the rationale is to apply the EKF
methodology, but using alternative estimation errors to the standard linear difference
between the estimate and the true state. Any non-linear error that reflects a discrepancy
between the true state and the estimate, necessarily defines a local frame around any
point, and the idea underlying the IEKF amounts to write the Kalman Jacobians and
covariances in this frame. We notice and prove here that an alternative nonlinear error
defines a local frame where the unobservable subspace is everywhere spanned by the
same vectors. Using this local frame at the current estimate to express Kalman’s co-
variance matrix will be shown to ensure the unobservable subspace “seen” by the EKF
We thus obtain an EKF variant which automatically comes with consistency properties. Moreover, we relate unobservability to the inverse of the covariance matrix (called information matrix) rather than on the covariance matrix itself, and we derive guarantees of information decrease over unobservable directions. Contrarily to the OC-EKF, and as in the standard EKF, we use here the latest, and thus best, state estimate as the linearization point to compute the filter Jacobians.

In a nutshell, whereas the key fact for the analysis of [26] is that the choice of the linearization point affects the observability properties of the linearized state error system of the EKF, the key fact for our analysis is that the choice of the error variable has similar consequences. Theoretical results and simulations underline the relevance of the proposed approach.

Robot-centric formulations such as [15], and later [22, 30] are promising attempts to tackle unobservability, but they unfortunately lack convenience as the position of all the landmarks must be revised during the propagation step, so that the landmarks’ estimated position becomes in turn sensitive to the motion sensor’s noise. They do not provably solve the observability issues considered in the present paper, and it can be noted the OC-EKF has demonstrated better experimental performance than the robot-centric mapping filter, in [26]. In particular, the very recent papers [22, 30] propose to write the equations of the SLAM in the robot’s frame under a constant velocity assumption. Using an output injection technique, those equations become linear, allowing to prove global asymptotic convergence of any linear observer for the corresponding deterministic linear model. This is fundamentally a deterministic approach and property, and as the matrices appearing in the obtained linear model are functions of the observations, the behavior of the filter is not easy to anticipate in a noisy context: The observation noise thus corrupts the very propagation step of the filter.

Some recent papers also propose to improve consistency through local map joining, see [39] and references therein. Although appealing, this approach is rather oriented towards large-scale maps, and requires the existence of local submaps. But when using submap joining algorithm, “inconsistency in even one of the submaps, leads to an inconsistent global map” [25]. This approach may thus prove complementary, if the IEKF SLAM proposed in the present paper is used to build consistent submaps. Note that, the IEKF SLAM can also be readily combined with other measurements such as the GPS, whereas the submap approach is tailored for pure SLAM.

From a methodology viewpoint, it is worth noting our approach does not bring to bear estimation errors written in a robot frame, as [15, 22, 30, 39]. Although based on symmetries as well, the estimation errors we use are slightly more complicated.

Finally, nonlinear optimization techniques have become popular for SLAM recently, see e.g., [10] as one of the first papers. Links between our approach, and those novel methods are discussed in the paper’s conclusion.

1.2 Paper’s organization

The paper is organized as follows. In Section 2 the standard EKF equations and EKF-SLAM algorithm are reviewed. In Section 3 we recall the problem that neither the
origin nor the orientation of the global frame are observable, but the EKF-SLAM systematically tends to “think” it observes them, which leads to inconsistency. In Section 4 we introduce the IEKF-SLAM algorithm. In Section 5 we show how the linearized model of the IEKF always correctly captures the considered unobservable directions. In Section 6 we derive a property of the covariance matrix output by the filter that can be interpreted in terms of Fisher information. In Section 7 simulations support the theoretical results and illustrate the benefits of the proposed algorithm. Finally, the IEKF theory of [8] is briefly recapped in the appendix, and the IEKF SLAM shown to be an application of this theory indeed. The equations of the IEKF SLAM in 3D are then also derived applying the general theory.

## 2 The EKF-SLAM algorithm

### 2.1 Statement of the general standard EKF equations

Consider a general dynamical system in discrete time with state $X_n \in \mathbb{R}^N$ associated to a sequence of observations $(Y_n)_{n \geq 0} \in \mathbb{R}^p$. The equations are as follows:

$$X_n = f(X_{n-1}, u_n, w_n),$$  (1)

$$Y_n = h(X_n) + V_n,$$  (2)

where $f$ is the function encoding the evolution of the system, $w_n$ is the process noise, $u_n$ an input, $h$ the observation function and $V_n$ the measurement noise.

The EKF propagates the estimate $\hat{X}_{n-1|n-1}$ obtained after the observation $Y_{n-1}$, through the deterministic part of (1):

$$\hat{X}_{n|n-1} = f(\hat{X}_{n-1|n-1}, u_n, 0)$$  (3)

The update of $\hat{X}_{n|n-1}$ using the new observation $Y_n$ is based on the first-order approximation of the non-linear system (1), (2) around the estimate $\hat{X}_n$, with respect to the estimation errors $e_{n-1|n-1}$, $e_{n|n-1}$ defined as:

$$e_{n-1|n-1} = X_{n-1} - \hat{X}_{n-1|n-1}, \quad e_{n|n-1} = X_n - \hat{X}_{n|n-1}$$  (4)

Using the Jacobians $F_n = \frac{\partial f}{\partial X} (\hat{X}_{n-1|n-1}, u_n, 0)$, $G_n = \frac{\partial f}{\partial w} (\hat{X}_{n-1|n-1}, u_n, 0)$, and $H_n = \frac{\partial h}{\partial X} (\hat{X}_{n|n-1})$, the combination of equations (1), (2) and (3) yields the following first-order expansion of the error system

$$e_{n|n-1} = F_n e_{n-1|n-1} + G_n w_n,$$  (5)

$$Y_n - h(\hat{X}_{n|n-1}) = H_n e_{n|n-1} + V_n,$$  (6)

where the second order terms, that is, terms of order $O (\|e\|^2, \|w\|^2, \|e\||w|)$ have been removed according to the standard way the EKF handles non-additive noises in the model (see e.g., [35] p. 386). Using the linear Kalman equations with $F_n, G_n, H_n$, the gain $K_n$ is computed, and letting $z_n = Y_n - h(\hat{X}_{n|n-1})$, an estimate $e_{n|n} = K_n z_n$ of
the error $X_n - \hat{X}_{n|n-1}$ accounting for the observation $Y_n$ is computed, along with its covariance matrix $P_{n|n}$. The state is updated accordingly:

$$\hat{X}_{n|n} = \hat{X}_{n|n-1} + K_n z_n$$  \hfill (7)

The detailed equations are recalled in Algorithm 1. The assumption underlying the EKF is that through first-order approximations of the state error evolution, the linear Kalman equations allow computing a Gaussian approximation of the error $e_{n|n} \sim \mathcal{N}(0, P_{n|n})$ after each measurement, yielding an approximation of the sought density $P(X_n|Y_1, \cdots, Y_n) \approx \mathcal{N}(\hat{X}_{n|n}, P_{n|n})$. However, the linearizations involved induce inevitable approximations that may lead the filter to inconsistencies and sometimes even divergence.

**Algorithm 1 Extended Kalman Filter (EKF)**

Choose an initial uncertainty matrix $P_0$ and estimate $\hat{X}_0$.

```
loop
    Define $F_n$, $G_n$ and $H_n$ through (5) and (6).
    Define $Q_n$ as Cov($w_n$) and $R_n$ as Cov($V_n$).

    Propagation
    $\hat{X}_{n|n-1} = f(\hat{X}_{n-1|n-1}, u_n, 0)$
    $P_{n|n-1} = F_n P_{n-1|n-1} F_n^T + G_n Q_n G_n^T$

    Update
    $z_n = Y_n - h(\hat{X}_{n|n-1})$
    $S_n = H_n P_{n|n-1} H_n^T + R_n$
    $K_n = P_{n|n-1} H_n^T S_n^{-1}$
    $P_{n|n} = [I - K_n H_n] P_{n|n-1}$
    $\hat{X}_{n|n} = \hat{X}_{n|n-1} + K_n z_n$
```

end loop

2.2 The considered SLAM problem

For simplicity’s sake let us focus on the standard “steered” bicycle (or unicycle) model [20]. The state is defined as:

$$X_n = (\theta_n, x_n, p_1^n, \ldots, p_K^n),$$  \hfill (8)

where $\theta_n \in \mathbb{R}$ denotes the heading, $x_n \in \mathbb{R}^2$ the 2D position of the robot/vehicle, $p_j^n \in \mathbb{R}^2$ the position of unknown landmark $j$ (landmarks or synonymously features, constitute the map). The equations of the model are:

$$\theta_n = \theta_{n-1} + \omega_n + w_\theta^n,$$
$$x_n = x_{n-1} + R(\theta_{n-1}) (v_n + w_v^n),$$
$$p_j^n = p_j^{n-1}, \quad 1 \leq j \leq K$$  \hfill (9)
where \( \omega_n \in \mathbb{R} \) denotes the odometry-based estimate of the heading variation of the vehicle, \( v_n \in \mathbb{R}^2 \) the odometry-based indication of relative shift, \( w_{\omega n} \) and \( w_{v n} \) their associated noises, and \( R(\theta) \) is the matrix encoding a rotation of angle \( \theta \):

\[
R(\theta) = \begin{pmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{pmatrix}.
\]

Note that a forward Euler discretization of the continuous time well-known unicycle equations leads to \( v_n \in \mathbb{R}^2 \) having its second entry null. More sophisticated integration methods or models including side slip may yet lead to non-zero values of both entries of \( v_n \) so we opt for a more general model with \( v_n \in \mathbb{R}^2 \). The covariance matrix of the noises will be denoted by

\[
Q_n = \text{Cov}
\begin{pmatrix}
w_{\omega n} \\
w_{v n} \\
0_{2K \times 1}
\end{pmatrix}
= \mathbb{E}
\begin{pmatrix}
w_{\omega n} \\
w_{v n} \\
0_{2K \times 1}
\end{pmatrix}
\begin{pmatrix}
w_{\omega n}^T \\
w_{v n}^T \\
0_{2K \times 1}^T
\end{pmatrix}
\in \mathbb{R}^{l \times l}
\tag{10}
\]

with \( l = 3 + 2K \). A general landmark observation in the robot’s frame reads:

\[
Y_n = \begin{pmatrix}
\tilde{h} [R(\theta_n)^T (p^1 - x_n)] + V^1_n \\
\vdots \\
\tilde{h} [R(\theta_n)^T (p^K - x_n)] + V^K_n
\end{pmatrix}
\tag{11}
\]

where \( Y_n \in \mathbb{R}^{2K} \) (or \( \mathbb{R}^K \) for monocular visual SLAM) is the observation of the features at time step \( n \), and \( V_n \) the observation noise, and \( \tilde{h} \) is any function.

**Remark 1.** Only a subset of the features is actually observed at time \( n \). However, to simplify the exposure of the filters’ equations, we systematically assume in the sequel that all features are observed.

We let the output noise covariance matrix be

\[
R_n = \text{Cov}
\begin{pmatrix}
V^1_n \\
\vdots \\
V^K_n
\end{pmatrix}
\tag{12}
\]

**Remark 2.** Note that, the observation model (11) encompasses the usual range and bearing observations used in the SLAM problem by letting \( \tilde{h} \left( \frac{y_1}{y_2} \right) = \left( \sqrt{y_1^2 + y_2^2}, \arctan2(y_2, y_1) \right) \).

If we choose instead the one dimensional observation \( \tilde{h} \left( \frac{y_1}{y_2} \right) = \arctan2(y_2, y_1) \) we recover the 2D monocular SLAM measurement. Note also we do not provide any specific form for the noise in the output: this is because the properties we are about to prove are related to the observability and thus only depend on the deterministic part of the system, so they are in fact totally insensitive to the way the noise enters the system.
2.3 The EKF-SLAM algorithm

We merely apply here the methodology of EKF to the SLAM problem described in Section 2.2. The first-order expansions (5), (6) applied to equations (9), (11) yield:

\[
F_n = \begin{pmatrix}
1 & R(\dot{\theta}_{n-1|n-1}) J v^T_n & 0_{1,2} & 0_{1,2K} \\
0 & I_2 & 0_{2,2K} & I_2K
\end{pmatrix},
\]

\[
G_n = \begin{pmatrix}
1 & 0_{1,2} & 0_{1,2K} & 0_{1,2K} \\
0 & R(\dot{\theta}_{n-1|n-1}) & 0_{2,2K} & 0_{2,2K}
\end{pmatrix},
\]

\[
H_n = \begin{pmatrix}
\nabla h^1 \cdot H^1_n \\
\vdots \\
\nabla h^K \cdot H^K_n
\end{pmatrix}
\]

with \( \dot{\theta} = \dot{\theta}_{n|n-1}, \) \( J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and \( \nabla h^k \) denotes the Jacobian of \( h \) computed at \( R(\dot{\theta}_{n|n-1}) \begin{pmatrix} \hat{p}^k_{n|n-1} - \hat{x}_{n|n-1} \\ \end{pmatrix} \in \mathbb{R}^2. \) The obtained EKF-SLAM algorithm is recaped in Algorithm 2.

Algorithm 2 EKF SLAM
Choose an initial uncertainty matrix \( P_0 \) and estimate \( \hat{X}_0 \)

loop
Define \( F_n, G_n \) and \( H_n \) as in (13).
Define \( Q_n, R_n \) as in (10) and (12).

Propagation
\( \hat{\theta}_{n|n-1} = \dot{\theta}_{n-1|n-1} + \omega_n \)
\( \hat{x}_{n|n-1} = \hat{x}_{n-1|n-1} + R(\dot{\theta}_{n-1|n-1}) v_n \)
\( \hat{p}^j_{n|n-1} = \hat{p}^j_{n-1|n-1} \) for all \( 1 \leq j \leq K \)
\( P_{n|n-1} = F_n P_{n-1|n-1} F_n^T + G_n Q_n G_n^T \)

Update
\[
z_n = Y_n - \begin{pmatrix}
R(\dot{\theta}_{n|n-1})^T \left( \hat{p}^1_{n|n-1} - \hat{x}_{n|n-1} \right) \\
\vdots \\
R(\dot{\theta}_{n|n-1})^T \left( \hat{p}^K_{n|n-1} - \hat{x}_{n|n-1} \right)
\end{pmatrix}
\]

\[
S_n = H_n P_{n|n-1} H_n^T + R_n,
\]

\[
K_n = P_{n|n-1} H_n^T S_n^{-1}
\]

\[
P_{n|n} = [I - K_n H_n] P_{n|n-1}
\]

\[
\hat{X}_{n|n} = \hat{X}_{n|n-1} + K_n z_n
\]

end loop

3 Observability issues and consistency of the EKF

In this section we come back to the general framework (1), (2). The standard issue of observability (21) is fundamentally a deterministic notion so the noise is systematically
Definition 1 (Unobservable transformation). We say a transformation \( \phi : \mathbb{R}^N \rightarrow \mathbb{R}^N \) is unobservable if for any initial conditions \( X_1^0 \in \mathbb{R}^N \) and \( X_0^2 = \phi (X_1^0) \) the induced solutions of the dynamics (1) with noise turned off, i.e., \( X_n = f(X_{n-1}, u_n, 0) \) yield the same output at each time step \( n \geq 0 \), that is:

\[
h(X_n^1) = h(X_n^2).
\]

It concretely means that (with all noises turned off) if the transformation is applied to the initial state then none of the observations \( Y_n \) are going to be affected. As a consequence, there is no way to know this transformation has been applied. In line with \[29, 24, 26\] we will focus here on the observability properties of the linearized system.

To that end we define the notion of non-observable (or unobservable) shift which is an infinitesimal counterpart to Definition 1, and is strongly related to the infinitesimal observability \[21\]:

Definition 2 (Unobservable shift). Let \( (X_n)_{n \geq 0} \) denote a solution of (1) with noise turned off. A vector \( \delta X_0 \in \mathbb{R}^N \) is said to be an unobservable shift of (1)–(2) around \( X_0 \) if:

\[
\forall n \geq 0, \quad H_n \delta X_n = 0,
\]

where \( H_n \) is the linearization of \( h \) at \( X_n \) and where \( \delta X_n \) is the solution at \( n \) of the linearized system \( \delta X_n = F_n \delta X_{n-1} \) initialized at \( \delta X_0 \), with \( F_n \) denoting the Jacobian matrix of \( f(\cdot, u_n, 0) \) computed at \( X_{n-1} \).

In other words (see e.g. \[26\]), for all \( n \geq 0 \), \( \delta X_n \) lies in the kernel of the observability matrix between steps 0 and \( n \) associated to the linearized error-state system model, i.e., \( \delta X_0^T \left[ H_0^T ; (H_1 F_1)^T ; \cdots ; (H_n F_n \cdots F_1)^T \right] = 0 \).

The interpretation is as follows: consider another initial state shifted from \( X_0 \) to \( X_0 + \delta X_0 \). Saying that \( \delta X_0 \) is unobservable means no difference on the sequence of observations up to the first order could be detected between both trajectories. Formally, this condition reads: \( h(X_n + \delta X_n) = h(X_n) + o \left( \delta X_n \right) \), i.e., \( H_n \delta X_n = 0 \). An estimation method conveying its own estimation uncertainty as the EKF, albeit based on linearizations, should be able to detect such directions and to reflect that accurate estimates along such directions are beyond reach.

3.1 Considered unobservable shifts

In the present paper we consider unobservability corresponding to the impossibility to observe the position and orientation of the global frame \[29, 24\]. The corresponding shifts have already been derived in the literature.

Proposition 1. \[26\] Let \( \hat{X} = (\hat{\theta}, \hat{x}, \hat{p}) \) be an estimate of the state. Only one feature is considered, the generalization of the proposition to several features is trivial. The first-order perturbation of the estimate corresponding to an infinitesimal rotation of angle \( \delta \alpha \) of the global frame consists of the shift

\[
\begin{pmatrix} 1 \\ J\hat{x} \\ J\hat{p} \end{pmatrix} \delta \alpha, \text{ with } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

In the
same way, the first-order perturbation of the estimate corresponding to an infinitesimal translation of the global frame of vector \( \delta u \in \mathbb{R}^3 \) consists of the shift \( \begin{pmatrix} 0 \\ \delta u \\ \delta u \end{pmatrix} \).

**Proof.** When rotating the global frame the heading becomes:

\[ \hat{\theta} \rightarrow \hat{\theta} + \delta \alpha. \]

The position of the robot becomes:

\[
\begin{pmatrix} \hat{x} \\ \hat{\theta} \end{pmatrix} \rightarrow \begin{pmatrix} \cos(\delta \alpha) & -\sin(\delta \alpha) \\ \sin(\delta \alpha) & \cos(\delta \alpha) \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} \hat{x} \end{pmatrix}. 
\]

The position of the feature becomes:

\[
\begin{pmatrix} \hat{p} \\ \hat{\theta} \end{pmatrix} \rightarrow \begin{pmatrix} \cos(\delta \alpha) & -\sin(\delta \alpha) \\ \sin(\delta \alpha) & \cos(\delta \alpha) \end{pmatrix} \begin{pmatrix} \hat{p} \\ \hat{} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} \hat{p} \end{pmatrix}. 
\]

Stacking these results we obtain the first-order variation of the full state vector (regarding the rotation only, the effect of infinitesimal translation being trivial to derive):

\[
\begin{pmatrix} \hat{\theta} \\ \hat{x} \\ \hat{p} \end{pmatrix} \rightarrow \begin{pmatrix} \hat{\theta} \\ \hat{x} \\ \hat{p} \end{pmatrix} + \begin{pmatrix} 1 \\ J_{\hat{x}} \\ J_{\hat{p}} \end{pmatrix} \delta \alpha. 
\]

(14)

**Proposition 2.** [26] The shifts of Proposition 2 that correspond to infinitesimal rotations, are unobservable shifts of (9)–(11) in the sense of Definition 2.

The intuitive explanation is clear [26]: “if the robot and landmark positions are shifted equally along those vectors, it will not be possible to distinguish the shifted position from the original one through the measurements.”

### 3.2 Inconsistency of the EKF

This section recalls using the notations of the present paper, a result of [24]. It shows the infinitesimal rotations defined in Proposition 2 are not, in general, unobservable shifts of the system linearized about the trajectory estimated by the EKF. Indeed, applying Definition 2 to (13) in the case of a single feature (the generalization being straightforward) with \( \delta X_0 = (1, J_{\hat{x}_0} \bar{\hat{p}}_{0(0)})^T \) and \( \delta X_n = F_n \cdots F_1 \delta X_0 \) yields the condition for an infinitesimal rotation of the initial state to be unobservable for the linearized system. This condition writes \( H_n \delta X_n \equiv 0 \) and boils down to have for any \( n > 0 \) (see [24]):

\[
\nabla h_n \cdot J \cdot R(\theta_{n|n-1})^T \left[ - (\bar{\hat{p}}_{n|n-1} - \bar{\hat{p}}_{0(0)}) + \sum_{i=1}^{n-1} (\hat{x}_{i|n} - \hat{x}_{i|n-1}) \right] = 0
\]

\[
\nabla h_n \cdot J \cdot R(\theta_{n|n-1})^T \left[ - (\bar{\hat{p}}_{n|n-1} - \bar{\hat{p}}_{0(0)}) + \sum_{i=1}^{n-1} (\hat{x}_{i|n} - \hat{x}_{i|n-1}) \right] = 0
\]
where $\nabla h_n$ is the Jacobian of $\tilde{h}$ computed at $R(\hat{\theta}_{n|n-1})^T(\hat{p}_{n|n-1} - \hat{x}_{n|n-1})$. For example, if $\tilde{h}$ is invertible the condition boils down to

$$\forall n > 0, \left[ - (\hat{p}_{n|n-1} - \hat{p}_{0|0}) + \sum_{i=1}^{n-1} (\hat{x}_{i|i} - \hat{x}_{i|i-1}) \right] = 0. \tag{15}$$

We see the quantities involved are the updates of the state. As they depend on the noise, there is a null probability for the condition to be respected, and it is always violated in practice.

But the point of the present paper is to show that the problem is related to the (arbitrary in a non-linear context) choice to represent the estimation error as the linear difference $X - \hat{X}$, not to an inconsistency issue inherent to EKF-like methods applied to SLAM. By devising an EKF-SLAM based on another estimation error variable, which in some sense amounts to change coordinates, the false observability problem can be corrected. The qualitative reason why this is sufficient is related to the basic cause of false observability: a given fixed shift may or may not be observable depending on the linearization point $\hat{X}$, as proved by Proposition 2. It turns out that the latter property is not inherently related to the SLAM problem: it is in fact a mere consequence of the errors’ definition (4). Defining those errors otherwise can dramatically modify the condition (15). This is the object of the remainder of this article.

4 A novel EKF-SLAM algorithm

Building upon the theory of the Invariant (I)EKF on matrix Lie groups, as described and studied in [8], we introduce in this section a novel IEKF for SLAM. In Appendix A.1-A.2 the general theory of the IEKF is recalled and slightly extended to account for the very general form of output (11), and the algorithm derived herein is shown to be a direct application of the theory. To spare the reader a study of the Lie group based theory, we attempt to explain in simple terms theIEKF methodology on the particular SLAM example throughout the present section.

Consider the model equations (9) with state $X_n$ given by (8). Exactly as the EKF, the IEKF propagates the estimated state obtained after the observation $Y_{n-1}$ of (11) through the deterministic part of (9) i.e., $\hat{\theta}_{n|n-1} = \hat{\theta}_{n-1|n-1} + \omega_n$, $\hat{x}_{n|n-1} = \hat{x}_{n-1|n-1} + R(\hat{\theta}_{n-1|n-1})v_n$, $\hat{p}_{n|n-1} = \hat{p}_{n-1|n-1}$ for all $1 \leq j \leq K$. To update the predicted state $\hat{X}_{n|n-1}$ using the observation $Y_n$ we use a first order Taylor expansion of the error system. But, instead of considering the usual state error $X - \hat{X}$, we rather use the (linearized) estimation error defined as follows

$$\tilde{x}_{n|n-1} = \begin{pmatrix} \theta_n - \hat{\theta}_{n|n-1} \\ x_n - \hat{x}_{n|n-1} - (\theta_n - \hat{\theta}_{n|n-1})J\tilde{x}_n \\ p_n^1 - \hat{p}_{n|n-1}^1 - (\theta_n - \hat{\theta}_{n|n-1})J\tilde{p}_{n|n-1}^1 \\ \vdots \\ p_n^K - \hat{p}_{n|n-1}^K - (\theta_n - \hat{\theta}_{n|n-1})J\tilde{p}_{n|n-1}^K \end{pmatrix} \tag{16}$$
and \( \xi_{n|n} \) is analogously defined. For close-by \( X, \hat{X} \), this represents an error variable in the usual sense indeed, as \( \xi = 0 \) if and only if \( \hat{X} = X \). As in the standard EKF methodology, let us see how this alternative estimation error is propagated through a first-order approximation of the error system. Using the propagation equations of the filter, and \( \Xi \), we find

\[
\xi_{n|n-1} = \xi_{n|n-1} + \begin{pmatrix}
-w_n^{\omega} J \xi_{n-1|n-1} + R(\hat{\theta}_{n-1|n-1}) w_n \\
-w_n^{\omega} J \hat{p}_{n-1|n-1} \\
\vdots \\
-w_n^{\omega} J \hat{p}_{K-1|n-1}
\end{pmatrix}
\]

(17)

where terms of order \( O(\|\xi_{n-1|n-1}\|^2) \), \( O(\|w_n^{\omega}\|\|\xi_{n-1|n-1}\|) \), and \( O(\|w_n^{\omega}\|^2) \) have been neglected as in the standard the EKF handles non-additive noises \( \Xi \). To derive (17) we have used the equalities \( \forall \theta, \hat{\theta} \in \mathbb{R}, w \in \mathbb{R}^2 \):

\[
R(\theta)w = R(\hat{\theta})w + O(\|\hat{\theta} - \theta\|\|w\|)
\]

(18)

\[
R(\theta) - R(\hat{\theta}) - (\theta - \hat{\theta})JR(\hat{\theta}) = O(\|\theta - \hat{\theta}\|^2)
\]

(19)

Note that, the odometer outputs \( \omega_n, v_n \) have miraculously vanished. This is in fact a characteristics - and a key feature - of the IEKF approach.

Let us now compute the first-order approximation of the observation error, using the alternative state error (16). Define \( H_n \) as the matrix, depending on \( \hat{x}_{n|n-1} \) only, such that for all \( \xi_{n|n-1} \in \mathbb{R}^{2K+3} \) defined by (16), the innovation term

\[
\begin{pmatrix}
\hat{h} \left[ R(\theta_n)^T (p^1 - x_n) \right] \\
\vdots \\
\hat{h} \left[ R(\theta_n)^T (p^K - x_n) \right]
\end{pmatrix} - \begin{pmatrix}
\hat{h} \left[ R(\hat{\theta}_{n|n-1})^T (\hat{p}^1_{n|n-1} - \xi_{n|n-1}) \right] \\
\vdots \\
\hat{h} \left[ R(\hat{\theta}_{n|n-1})^T (\hat{p}^K_{n|n-1} - \xi_{n|n-1}) \right]
\end{pmatrix}
\]

is equal to \( H_n \xi_{n|n-1} + O(\|\xi_{n|n-1}\|^2) \). Using that \( R(\theta)^T (p - x) - R(\hat{\theta})^T (\hat{p} - \hat{x}) = R(\theta)^T ((p - x) - R(\theta - \hat{\theta})(\hat{p} - \hat{x})) \), and \( R(\theta)^T \xi = R(\hat{\theta})^T \xi + O(\|\xi\|^2) \) from (18), we see that \( H_n \) is defined as in (21) below. Thus the linearized (first-order) system model with respect to alternative error (16) writes

\[
\begin{align*}
\xi_{n|n-1} &= F_n \xi_{n|n-1} + G_n w_n, \\
Y_n - h(\hat{x}_{n|n-1}) &= H_n \xi_{n|n-1} + V_n
\end{align*}
\]

(20)
with \( w_n^T = (w_n^0, w_n^v)^T, 0_{1 \times 2K}^T \), and
\[
F_n = I_{2K+3}, \quad G_n = \begin{pmatrix}
1 & 0_{1,2} & 0_{1,2K} \\
-J \hat{x}_{n-1|n-1} & R(\hat{\theta}_{n-1|n-1}) & 0_{2,2K} \\
-J \hat{\theta}_{n-1|n-1}^1 & 0_2 & 0_{2,2K} \\
\vdots & \vdots & \vdots \\
-J \hat{\theta}_{n-1|n-1}^{K-1} & 0_2 & 0_{2,2K}
\end{pmatrix},
\]
\[
H_n = \begin{pmatrix}
\nabla \tilde{h}_1 \cdot R(\hat{\theta}_{n|n-1})^T (0_{2,1} & -I_2 & I_2 & 0_{2,2(K-1)}) \\
\nabla \tilde{h}_2 \cdot R(\hat{\theta}_{n|n-1})^T (0_{2,1} & -I_2 & 0_{2,2} & I_2 & 0_{2,2(K-2)}) \\
\vdots & \vdots & \vdots \\
\nabla \tilde{h}_K \cdot R(\hat{\theta}_{n|n-1})^T (0_{2,1} & -I_2 & 0_{2,2(K-1)} & I_2)
\end{pmatrix},
\]
where \( \nabla \tilde{h}^k \) is the Jacobian of \( \tilde{h} \) computed at \( R(\hat{\theta}_{n|n-1})^T (\hat{\theta}_{n|n-1} - \hat{x}_{n|n-1}) \). As in the standard EKF methodology, the matrices \( F_n, G_n, H_n \) allow to compute the Kalman gain \( K_n \) and covariance \( P_n \). Letting \( z_n \) be the standardly defined innovation (see Algorithm 3 just after “Update”), \( \tilde{z}_{n|n} = K_n z_n \) is an estimate of the linearized error \( \xi_{n|n-1} \) accounting for the observation \( Y_n \), and \( P_{n|n} \) is supposed to encode the dispersion \( \mathbb{E}(\xi_{n|n}^T \xi_{n|n}^T) \).

The final step of the standard EKF methodology is to update the estimated state \( \hat{x}_{n|n-1} \) thanks to the estimated linearized error \( \tilde{z}_{n|n} = K_n z_n \). There is a small catch, though: \( \xi \) being not anymore defined as a mere difference \( X - \hat{X} \), simply adding \( \tilde{z}_{n|n} \) to \( \hat{x}_{n|n} \) would not be appropriate. The most natural counterpart to (7) in our setting would be to choose for \( \hat{x}_{n|n} \), the values of \( (\hat{\theta}_n, x_n, \ldots, p_n^1, p_n^K) \) making the right member of (16) equal to the just computed \( \tilde{z}_{n|n} \). However, the IEKF theory recalled in Appendix A.2 suggests an update that amounts to the latter to the first order, but whose non-linear structure ensures better properties \[8\]. Thus, the state is updated as follows
\[
\hat{x}_{n|n} = \varphi(\tilde{z}_{n|n}, \hat{x}_{n|n-1}) = \varphi(K_n z_n, \hat{x}_{n|n-1}),
\]
with \( \varphi \) defined by
\[
\varphi(\begin{pmatrix}
\delta \theta \\
\delta \hat{x} \\
\delta p^1 \\
\vdots \\
\delta p^K
\end{pmatrix}) = \begin{pmatrix}
\hat{\theta} + \delta \theta \\
R(\delta \theta) \hat{\hat{x}} + B(\delta \theta) \delta \hat{x} \\
R(\delta \theta) \hat{p^1} + B(\delta \theta) \delta p^1 \\
\vdots \\
R(\delta \theta) \hat{p^K} + B(\delta \theta) \delta p^K
\end{pmatrix}
\]
where \( B(\alpha) = \begin{pmatrix}
\sin(\alpha) & -1 - \cos(\alpha) \\
\alpha & \sin(\alpha)
\end{pmatrix} \). Algorithm 3 recapacs the various steps of the IEKF SLAM.

5 Remedying EKF SLAM consistency

In this section we show the infinitesimal rotations and translations of the global frame are unobservable shifts in the sense of Definition 2 regardless of the linearization points
Algorithm 3 IEKF SLAM

The state is defined by \( X = (\theta, x^T, (p^1)^T, \cdots, (p^K)^T) \in \mathbb{R}^{3+2K} \). Pick an initial uncertainty matrix \( P_0 \) and estimate \( \hat{X}_0 \).

**loop**
- Define \( F_n, G_n \) and \( H_n \) as in (21).
- Define \( Q_n, R_n \) as in (10) and (12).

**Propagation**
\[
\hat{\theta}_{n|n-1} = \hat{\theta}_{n-1|n-1} + \omega_n
\]
\[
\hat{x}_{n|n-1} = \hat{x}_{n-1|n-1} + R (\hat{\theta}_{n-1|n-1}) v_n
\]
\[
\hat{p}_{n|n-1}^j = \hat{p}_{n-1|n-1}^j \quad \text{for all} \quad 1 \leq j \leq K
\]
\[
P_{n|n-1} = F_n P_{n-1|n-1} F_n^T + G_n Q_n G_n^T
\]

**Update**
\[
z_n = Y_n - \begin{bmatrix}
\hat{h} \left[ R(\hat{\theta}_{n|n-1})^T \left( \hat{p}_{n|n-1}^1 - \hat{x}_{n|n-1} \right) \right] \\
\vdots \\
\hat{h} \left[ R(\hat{\theta}_{n|n-1})^T \left( \hat{p}_{n|n-1}^K - \hat{x}_{n|n-1} \right) \right]
\end{bmatrix}
\]
\[
S_n = H_n P_{n|n-1} H_n^T + R_n,
\]
\[
K_n = P_{n|n-1} H_n^T S_n^{-1}
\]
\[
P_{n|n} = [I - K_n H_n] P_{n|n-1}
\]
- Use (22) to compute \( \hat{X}_{n|n} = \varphi(K_n z_n, \hat{X}_{n|n-1}) \).

end loop

used to compute the matrices \( F_n \) and \( H_n \) of eq. (21), a feature in sharp contrast with the usual restricting condition (15) on the linearization points. In other words we show that infinitesimal rotations and translations of the global frame are always unobservable shifts of the system model linearized with respect to error (16) regardless of the linearization point, a feature in sharp contrast with previous results (see Section 3.2 and references therein).

5.1 Main result

We can consider only one feature \( (K = 1) \) without loss of generality. The expression of the linearized system model has become much simpler, as the linearized error has the remarkable property to remain constant during the propagation step in the absence of noise, since \( F_n = I_{3+2K} \) in (20)-(21). First, let us derive the impact of first-order variations stemming from rotations and translations of the global frame on the error as defined by (16), that is, an error of the following form

\[
\xi = \begin{pmatrix}
(\theta - \hat{\theta}) \\
\hat{x} - \hat{x} - (\theta - \hat{\theta}) J \hat{\gamma} \\
p - \hat{p} - (\theta - \hat{\theta}) J \hat{p}
\end{pmatrix},
\]  

(23)

**Proposition 3.** Let \( \hat{X} = (\hat{\theta}, \hat{x}, \hat{p})^T \) be an estimate of the state. The first-order perturbation of the linearized estimation error defined by (23) around 0, corresponding to an
infinitesimal rotation of angle \(\delta\alpha\) of the global frame, reads
\[
\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
2 & 0 & 1
\end{pmatrix}
\delta\alpha.
\]
In the same way, an infinitesimal translation of the global frame with vector \(\delta u \in \mathbb{R}^2\) implies a first-order perturbation of the error system \eqref{eq:23} of the form \((0, \delta u, \delta u)^T\).

**Proof.** According to Proposition \(1\) an infinitesimal rotation by an angle \(\delta\alpha \ll 1\) of the true state corresponds to the transformation \(\theta \to \theta + \delta\alpha\), \(x \to x + \delta\alpha Jx\) and \(p \to p + \delta\alpha Jp\). Regarding \(\xi\) of eq \eqref{eq:23} it corresponds to the variation
\[
\xi \to \xi + \begin{pmatrix}
\delta\alpha \\
0 \\
2 \\
1 \\
0
\end{pmatrix} + O(\delta\alpha \|\xi\|) + O(\delta\alpha^2).
\]
This direction of the state space is “seen” by the linearized error system as the vector \((0, \delta u, \delta u)^T\). Similarly, a translation of vector \(\delta u\) of the global frame yields the transformation \(\theta \to \theta, x \to x + \delta u, p \to p + \delta u\). The effect on the linearized error \(\xi\) of \eqref{eq:23} is obviously the perturbation \((0, \delta u, \delta u)^T\) neglecting terms of order \(\delta u\|\xi\|\).

We can now prove the first major result of the present article: the infinitesimal transformations stemming from rotations and translations of the global frame are unobservable shifts for the IEKF linearized model.

**Theorem 1.** Consider the SLAM problem defined by equations \(9\) and \(11\), and the IEKF-SLAM algorithm \(3\). Let \(\delta X_0\) denote a linear combination of infinitesimal rotations and translations \(\delta X_0^R, \delta X_0^1, \delta X_0^2\) of the whole system defined as follows
\[
\delta X_0^R = \begin{pmatrix} 1 \\ 0_{2,1} \\ 0_{2,1} \end{pmatrix}, \quad \delta X_0^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \delta X_0^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]
Then \(\delta X_0\) is an unobservable shift of the linearized system model \eqref{eq:20} – \eqref{eq:21} of the IEKF SLAM in the sense of Definition \(2\) and this whatever the sequence of true states and estimates \((X_n, \hat{X}_{n|n}, \hat{X}_{n|n-1})\): the very structure of the IEKF is consistent with the considered unobservability.

**Proof.** Note that Definition \(2\) involves a propagated perturbation \(\delta X_n\), but as here \(F_n\) is \(I_5\); we have \(\forall n \geq 0, \delta X_n = \delta X_0\). Thus, the only point to check is: \(H_n(\delta X_0) = 0\), i.e.,
\[
\nabla \hat{h} \cdot R \left(\hat{\theta}_{n|n-1}\right)^T \begin{pmatrix} 0_{2,1} & -I_2 & -I_2 \end{pmatrix} \delta X_0 = 0.
\]
This is straightforward replacing \(\delta X_0\) with alternatively \(\delta X_0^1, \delta X_0^2, \delta X_0^R\).

We obtained the consistency property we were pursuing: the linearized model correctly captures the unobservability of global rotations and translations. As a byproduct, the unobservable seen by the filter is automatically of appropriate dimension.
5.2 Interpretation and discussion

The standard EKF is tuned to reduce the state estimation error $\hat{X} - X$ defined through the original state variables $X, \hat{X}$ of the problem. Albeit perfectly suited to the linear case, the latter state error has in fact absolutely no fundamental reason to rule the linearization process in a non-linear setting. The basic difference when analyzing the EKF and the IEKF is that

- In the standard EKF, there is a trivial correspondence between a small variation of the true state and a small variation of the estimation error (4). But the global rotations of the frame make the error vary in a non-trivial way as recalled in Section 3.

- In the IEKF approach, the effect of a small rotation of the state on the variation of the estimation error (16) becomes trivial as ensured by Proposition 3. But the error is non-trivially related to the state, as its definition explicitly depends on the linearization point $\hat{X}$. Many consistency issues of the EKF stem from the fact that the updated covariance matrix $P_{n|n}$ is computed before the update, namely at the predicted state $\hat{X}_{n|n-1}$, and thus does not account for the updated state’s value $\hat{X}_{n|n}$, albeit supposed to reflect the covariance of the updated error. This is why the OC-EKF typically seeks to avoid linearizing at the latest, albeit best, state estimate, in order to find a close-by state such that the covariance matrix resulting from linearization preserves the observability subspace dimension. The IEKF approach is wholly different: the updated covariance $P_{n|n}$ is computed at the latest estimate $\hat{X}_{n|n-1}$, which is akin to the standard EKF methodology. But it is then indirectly adapted to the updated state, since it is interpreted as the covariance of the error $\xi_{n|n}$. More intuitively, we can say the confidence ellipsoids encoded in $P_{n|n}$ are attached to a basis that undergoes a transformation when moved over from $\hat{X}_{n|n-1}$ to $\hat{X}_{n|n}$, this transformation being tied to the unobservable directions. This prevents spurious reduction of the covariance over unobservable shifts, which are not identical at $\hat{X}_{n|n-1}$ and $\hat{X}_{n|n}$.

Finally, note the alternative error (23) is all but artificial: it naturally stems from the Lie group structure of the problem. This is logical as the considered unobservability actually pertains to an invariance of the model (9)-(11), that is the SLAM problem, to global translations and rotations. Thus it comes as no surprise the Invariant approach, that brings to bear invariant state errors that encode the very symmetries of the problem, prove fruitful (see the appendix for more details).

6 IEKF consistency and information

Our approach can be related to the previous work [26]. Indeed, according to the latter article, failing to capture the right dimension of the observability subspace in the linearized model leads to “spurious information gain along directions of the state space where no information is actually available” and results in “unjustified reduction of the
covariance estimates, a primary cause of filter inconsistency”. Theorem 1 proves that
infinitesimal rotations and translations of the global frame, which are unobservable in
the SLAM problem, are always “seen” by the IEKF linearized model as unobservable
directions indeed, so this filter does not suffer from “false observability” issues. This
is our major theoretical result.

That said, the results of the latter section concern the system with noise turned off,
and pertain to an automatic control approach to the notion of observability as in [26].
The present section is rather concerned with the estimation theoretic consequences of
Theorem 1. We prove indeed, that the IEKF’s output covariance matrix correctly re-
fects an absence of “information gain” along the unobservable directions, as mentioned
above, but where the information is now to be understood in the sense of Fisher infor-
mation. As a by-product, this allows relating our results to a slightly different approach
to SLAM consistency, that rather focuses on the Fisher information matrix than on the
observability matrix, see in particular [38, 1].

6.1 The general Bayesian Fisher Information Matrix

The exposure of the present section is based on the seminal article [36]. See also [38, 1]
for related ideas applied to SLAM. Consider the system (1) with output (2). Define the
collection of state vectors and observations up to time \( t_n \):

\[
\tilde{X}_n = (X_0^T, \cdots, X_n^T)^T, \quad \tilde{Y}_n = (Y_0^T, \cdots, Y_n^T)^T,
\]

The joint probability distribution of the \((n+1)N\) vector \( \tilde{X}_n \) and of the \(np\) vector \( \tilde{Y}_n \) is

\[
p(\tilde{Y}_n, \tilde{X}_n) = p(X_0) \prod_{i=1}^{n} p(Y_i | X_i) p(X_i | X_{i-1})
\]

The Bayesian Fisher Information Matrix (BIFM) is defined as the following
\(Nk \times Nk\) matrix based upon the dyad of the gradient of the log-likelihood:

\[
J(\tilde{X}_n) = E(\nabla_{\tilde{X}_n} \log p(\tilde{Y}_n, \tilde{X}_n) | \nabla_{\tilde{X}_n} \log p(\tilde{Y}_n, \tilde{X}_n)^T)
\]

and note that, for the SLAM problem it boils down to the matrix of [26]. This matrix is
of interest to us as it yields a lower bound on the accuracy achievable by any estimator
used to attack the filtering problem (1)-(2). Indeed let \( J_n \) be defined as the inverse
of the \(N \times N\) right-lower block of \([J(\tilde{X}_n)]^{-1}\). This matrix provides a lower bound on the
mean square error of estimating \(X_n\) from past and present measurements \(\tilde{Y}_n\) and prior
\(p(X_0)\). Indeed, for any unbiased estimator \(T: \mathbb{R}^{np} \rightarrow \mathbb{R}^N\):

\[
\mathbb{E}([T(\tilde{Y}_n)][T(\tilde{Y}_n)]^T) \geq J_n^{-1}
\]

where \(A \succeq B\) means \(A - B\) is positive semi-definite. \(J_n^{-1}\) is called the Bayesian or
Posterior Cramér-Rao lower bound for the filtering problem [36]. Most interestingly,
in the case where \(f\) and \(h\) are linear, the prior distribution is Gaussian, and the noises
are additive and Gaussian, we have

\[
J_n = P_n^{-1}
\]
where $P_{n|n}$ is the covariance matrix output by the Kalman filter. Thus, in the linear Gaussian case, $P_{n|n}^{-1}$ reflects the statistical information available at time $n$ on the state $X_n$. By extension in the SLAM literature $P_{n|n}^{-1}$ is often simply referred to as the information matrix, in non-linear contexts also, e.g. when using extended information filters [37].

6.2 Application to IEKF-SLAM consistency

In the last section, we have recalled that in the linear Gaussian case, the inverse of the covariance matrix output by the Kalman filter is the Fisher information available to the filter (this is also stated in [3] p. 304). In the light of those results, it is natural to expect from any EKF variant, that the inverse of the output covariance matrix $P_{n|n}^{-1}$ reflects an absence of information gain along unobservable directions indeed. If the filter fails to do so, the output covariance matrix will be too optimistic, that is, inconsistent, and wrong covariances yield wrong gains [3]. The following theorem shows the linearized system model of the IEKF allows ensuring the desired property of the covariance matrix. It is our second major result.

**Theorem 2.** Consider the SLAM problem defined by equations (9) - (11) and the IEKF-SLAM Algorithm 3. Let $\delta X_0$ denote a linear combination of infinitesimal rotations and translations $\delta X_0^R, \delta X_0^1, \delta X_0^2$ of the whole system, as defined in Theorem 1. $\delta X_0$ is thus an unobservable shift. If the matrix $P_{n|n}$ output by the IEKF remains invertible, we have at all times:

$$\left(\delta X_0\right)^T P_{n|n}^{-1} \delta X_0 \leq \left(\delta X_0\right)^T P_{n-1|n-1}^{-1} \delta X_0.$$  \hspace{1cm} (24)

**Proof.** As $F_n = I_{2K+1}$ in (21), the unobservable shifts remain fixed i.e. $\delta X_n = \delta X_0$. At the propagation step we have:

$$\delta X_0^T P_{n|n-1}^{-1} \delta X_0 = \delta X_0^T \left( P_{n-1|n-1} + G_n Q_n G_n^T \right)^{-1} \delta X_0 \leq \delta X_0^T P_{n-1|n-1}^{-1} \delta X_0,$$

as $G_n Q_n G_n^T$ is positive semi-definite. And at the update step (see the Kalman Information Filter form in [3]) we have:

$$\delta X_0^T P_{n|n}^{-1} \delta X_0 = \delta X_0^T \left[ P_{n|n-1}^{-1} + H_n^T P_{n}^{-1} H_n \right] \delta X_0 = \delta X_0^T P_{n|n-1}^{-1} \delta X_0$$

as $H_n \delta X_0 = 0$ as shown in the proof of Theorem 1. Thus $(\delta X_0)^T P_{n|n}^{-1}(\delta X_0)$ is non-increasing over time $n$. Note that, the proof evidences that if $P_{n|n}$ is not invertible, the results of the theorem still hold, writing the IEKF in information form. \hfill $\square$

Our result essentially means the linearized model of the IEKF has a structure which guarantees that the covariance matrix at all times reflects an absence of “spurious” (Bayesian Fisher) information gain over directions that correspond to the unobservable rotations and translations of the global frame.
Figure 1: Simulated trajectory: the displayed loop is driven 10 times by a robot able to measure the relative position of the landmarks lying in a range of 5 m around him. Velocity is constant (1 m/s) as well as angular velocity (9 deg/s).

7 Simulation results

In this section we verify in simulation the claimed properties on the one hand, and on the other we illustrate the striking consistency improvement achieved by the IEKF SLAM. To that end, we propose to consider a similar numerical experiment as in the sound work [26] dedicated to the inconsistency of EKF and the benefits of the OC-EKF. The IEKF is compared here to the standard EKF, the UKF, the OC-EKF, and the ideal EKF, which is the - impossible to implement - variant of the EKF where the state is linearized about the true trajectory.
Figure 2: Evaluation of the consistency of the five filters using the NEES indicator (for the entire 3-DoF pose) over 50 runs, in the experimental setting described in Section 7.1. We see the indicator stays around 1 for IEKF SLAM and OC-EKF SLAM over the whole time interval, as expected from a consistent estimation method. The “ideal” EKF, where the system is linearized on the true value of the state, yields similar results. To the opposite, we see the EKF is inconsistent, and the UKF also.

7.1 Simulation setting

The simulation setting we chose is (deliberately) similar to the one used in [26] (Section 6.2). The vehicle (or robot) drives a 15m-diameter loop ten times in the 2D plane, finding on its path 20 unknown features as displayed on Figure 1. The velocity and angular velocity are constant (1 m/s and 9 deg/s respectively). The relative position of the features in the reference frame of the vehicle is observed once every second (where $\tilde{h}$ of (11) is the identity). The standard deviation $\sigma$ of the velocity measurement on each wheel of the vehicle is 2% of the velocity. This yields the standard deviation $\sigma_v$ of the resulting linear velocity and $\sigma_\omega$ of the rotational velocity: $\sigma_v = (\sqrt{2}/2)\sigma$ and $\sigma_\omega = (\sqrt{2}/a)\sigma$, where $a = 0.5m$ is the distance between the drive wheels (see [26]). The features are visible if they lie within a sensing range of 5 m, in which case they are observed with an isotropic noise of standard deviation 10 cm. The initial uncertainty over position and heading is zero - which will prove a condition not sufficient to prevent failure of the EKF. Each time a landmark is seen for the first time, its position...
Figure 3: Evaluation of the performance of the proposed IEKF SLAM algorithm, in terms of RMS of the vehicle heading and position error (in rad and m respectively). We see the results are very similar to those of the OC-EKF and "ideal" EKF, the latter being impossible to implement for real (the system is linearized at the true state). These results are much better than those of the UKF and classical EKF.
Figure 4: Inconsistency illustrated on a single run. The plotted EKF and UKF heading errors (top plots) do not remain in the 99% uncertainty envelope computed by the filter. Filters with theoretical properties regarding non-observable directions (IEKF, OC-EKF and ideal EKF) remedy this problem.
is initialized in the earth frame using the current estimated pose of the robot, the associated uncertainty is set to a very high value compared to the size of the map, then a Kalman update is performed to correlate the position of the new feature with the other variables. Each second, all visible landmarks (i.e., those in a range of 5m) are processed simultaneously in a stacked observation vector.

Five algorithms are compared:

1. The classical EKF, described in Algorithm 2.

2. The proposed IEKF SLAM algorithm described in Algorithm 3.

3. The ideal EKF as defined in [26], i.e., a classical EKF where the Riccati equation is computed at the true trajectory of the system instead of the estimated trajectory. Although not usable in practice, the latter is a good reference to compare with, as it is supposed to be an EKF with consistent behavior.

4. The OC-EKF described in [26], which is so far the only method that guarantees the non-observable subspace has appropriate dimension.

5. The Unscented Kalman Filter (UKF), known to better deal with the non-linearities than the EKF.

Before going further, the next subsection introduces the NEES indicator used in the simulations to measure the consistency of these methods.

7.2 The NEES indicator

Classical criteria used to evaluate the performance of an estimation method, like Root Mean Squared (RMS) error do not inform about consistency as they do not take into account the uncertainty returned by the filter. This point is addressed by the Normalized Estimation Error Squared (NEES), which computes the average squared value of the error, normalized by the covariance matrix of the EKF. For a sample \( \{X_i\}_{i=1}^p \) of error values having dimension \( d \), each of them with a covariance matrix \( P_i \) of size \( d \times d \), the NEES is defined by:

\[
\text{NEES} = \frac{1}{p \times d} \sum_{i=1}^p X_i^T P_i^{-1} X_i.
\]

If each \( X_i \) is a zero mean Gaussian with covariance matrix \( P_i \), then for large \( p \) we have \( \text{NEES} \approx 1 \). The case \( \text{NEES} > 1 \) reveals an inconsistency issue: the actual uncertainty is higher than the computed uncertainty. This situation typically occurs when the filter is optimistic as it believes to have gained information over a non-observable direction. The NEES indicator will be used, along with the usual RMS, to illustrate our solution to SLAM inconsistency in the sequel.

7.3 Numerical results

Figure [2] displays the NEES indicator of the vehicle pose estimate (heading and position) over time, computed for 50 Monte-Carlo runs of the experiment described in
Section 7.1 As expected, the profile of the NEES for classical EKF, ideal EKF and OC-EKF is the same as in the previous paper [26] which inspired this experimental section. Note that we used here a normalized version of the NEES, making its swing value equal to 1. We see also that the result is similar for OC-EKF SLAM and ideal EKF SLAM: the NEES varies between 1 and 1.7, in contrast to the EKF SLAM and UKF SLAM which exhibit large inconsistencies over the robot pose. We see here that the IEKF remedies inconsistency, with a NEES value that remains close to 1. Note that, it performs here even better than OC-EKF and ideal EKF (whose results are very close to each other), in terms of consistency. The basic difference between IEKF and these filters lies in using or not the current estimate as a linearization point. Uncertainty directions being very dependent from the estimate, what Figure 2 suggests is that they may not be correctly captured if computed on a different point. The other aspect of the evaluation of an EKF-like method is performance: regardless of the relevance of the covariance matrix returned by the filter (i.e. consistency), pure performance can be evaluated through RMS of the heading and position error, whose values over time are displayed in Figure 3. They confirm an expectable result: solving consistency issues improves the accuracy of the estimate as a byproduct, as wrong covariances yield wrong gains [3].

Selecting a single run, we can also illustrate the inconsistency issue in terms of covariance and information. Figure 4 displays the heading error for EKF, UKF, IEKF, OC-EKF and ideal EKF SLAM, and the 99% envelope returned by each filter. This illustrates both the false observability issue and the resulting inconsistency of EKF and UKF: the heading uncertainty is reduced over time while the estimation error goes outside the 99% envelope. To the opposite, the behavior of IEKF, OC-EKF and ideal EKF is sound. Figure 5 shows the map and the landmarks 99% uncertainty ellipsoids: Similarly, both EKF and UKF fail capturing the true landmarks’ positions within the 99% ellipsoids whereas the three over filters succeed to do so. Finally, Figure 6 displays the evolution of the information over a shift corresponding to infinitesimal rotations as defined in Theorem 2, that is the evolution over time of the quantity \( \delta X_n^T P_n^{-1} \delta X_n \). The theorem is successfully illustrated: the latter quantity is always decreasing for the IEKF, ideal EKF, OC-EKF but not for the EKF and the (slightly better to this respect) UKF.

8 Conclusion

This work evidences that the EKF algorithm for SLAM is not inherently inconsistent - at least regarding inconsistency related to unobservable transformations of the global frame - but the choice of the right coordinates for the linearization process is pivotal. We showed that applying the recent theory of the IEKF - an EKF (slight) variant - leads to provable properties regarding observability and consistency. Extensive Monte-Carlo simulations have illustrated the consistency of the new method and the striking improvement over EKF, UKF, OC-EKF, and more remarkably over the ideal EKF also, which is the - impossible to implement - variant of the EKF where the system is linearized about the true trajectory.

Note that, the IEKF approach may prove relevant beyond SLAM to some other
problems in robotics as well, such as autonomous navigation (see [4]), and in combination with controllers, notably for motion planning purposes, see [17]. In [9] the IEKF has proved to possess global asymptotic convergence properties on a simple localization problem of a wheeled robot, which is a strong property. The IEKF has also been patented for navigation with inertial sensors [7].

Nowadays nonlinear optimization based SLAM algorithms are becoming popular as compared with EKF SLAM, see e.g. [16] for one of the first papers on the subject. We yet anticipate a simple EKF SLAM with consistency properties will prove useful to the research community, the EKF SLAM having been abandoned in part due to its inconsistency. The general EKF has proved useful in numerous industrial applications, especially in the field of guidance and navigation. It has the benefits of being 1-recursive, avoiding to store the whole trajectory and 2-suited to on-line real-time applications. Moreover the aerospace and defense industry has developed a corpus of experience for its industrial implementation and validation. And the IEKF is a variant that, being in every respect similar to EKF, retains all its advantages, but which possesses additional guaranteed properties. Note also that, all the improvements of the EKF for SLAM such as e.g., the SLAM of [33] and sparse extended information filters [37], can virtually be turned into their invariant counterpart.

The high dimensional optimization formulation of the SLAM problem being prone to local minima, having an accurate initial value (i.e. a small initial estimation error) is very critical [39]. The IEKF SLAM algorithm proposed in the present paper may thus be advantageously used to initialize those methods in challenging situations.

Besides, we anticipate our approach based on symmetries could help improve (at least first order) optimization techniques for SLAM. To understand why, assume by simplicity the sensors to be noise free. Then, moving a candidate trajectory along unobservable directions will not change the cost function, and an efficient optimization algorithm should account for this. And when a gradient descent algorithm is used, only a first-order expansion of the cost function is considered. Our Lie group approach will allow defining steepest descent directions in a alternative geometric way, that will “stick” to the unobservable directions, and the corresponding update will move along the (Lie group) state space in a non-linear yet relevant way. This issue is left for future work, but a thorough understanding of the interest of the invariant approach for the EKF, is a first step in this direction.

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A IEKF theory, applications to 2D and 3D SLAM

In this section we provide more details on the IEKF theory on matrix Lie groups, and show how the underlying Lie group structure of the SLAM problem has been used indeed to build the IEKF SLAM Algorithm [8]. We also provide the IEKF equations for 3D SLAM. For more information on the IEKF see [8] and references therein.
A.1 Primer on matrix Lie groups

A matrix Lie group $G$ is a subset of square invertible $N \times N$ matrices $\mathcal{M}_N(\mathbb{R})$ verifying the following properties:

$$I_G \in G, \quad \forall g \in G, g^{-1} \in G, \quad \forall a, b \in G, ab \in G$$

where $I_G$ is the identity matrix of $\mathbb{R}^N$. If $\gamma(t)$ is a curve over $G$ with $\gamma(0) = I_G$, then its derivative at $t = 0$ necessarily lies in a subset $\mathfrak{g}$ of $\mathcal{M}_N(\mathbb{R})$. $\mathfrak{g}$ is a vector space and it is called the Lie algebra of $G$. It has same dimension $\text{dim} \ g$ as $G$. Thanks to a linear invertible map denoted by $\mathcal{L}_\theta : \mathbb{R}^{\text{dim} \ g} \to g$, one can advantageously identify $\mathfrak{g}$ to $\mathbb{R}^{\text{dim} \ g}$. Besides, the vector space $\mathfrak{g}$ can be mapped to the matrix Lie group $G$ through the classical matrix exponential $\exp_m$. Thus, $\mathbb{R}^{\text{dim} \ g}$ can be mapped to $G$ through the Lie exponential map defined by $\exp(\xi) := \exp_m(\mathcal{L}_\theta(\xi))$ for $\xi \in \mathbb{R}^{\text{dim} \ g}$. This map is invertible for small $\xi$, and we have $(\exp(\xi))^{-1} = \exp(-\xi)$. The well-known Baker-Campbell-Hausdorff (BCH) formula gives a series expansion for the product $\exp(\xi) \exp(\zeta)$. In particular it ensures $\exp(\xi) \exp(\zeta) = \exp(\xi + \zeta + T)$, where $T$ is of the order $O(\|\xi\|^2, \|\zeta\|^2, \|\xi\| \|\zeta\|)$. For any $g \in G$, the adjoint matrix $\text{Ad}_g \in \mathbb{R}^{\text{dim} \ g \times \text{dim} \ g}$ is defined by $g \exp(\zeta) g^{-1} = \exp(\text{Ad}_g \zeta)$ for all $\zeta \in \mathfrak{g}$. We now give explicit formulas for two groups of particular interest for the SLAM problem.

A.1.1 Group of direct planar isometries $SE(2)$

This famous group in robotics can be defined using homogeneous matrices, i.e., $G = SE(2) := \{ g = \begin{pmatrix} R(\theta) & x \\ 0_{1,2} & 1 \end{pmatrix} : \theta \in \mathbb{R}, x \in \mathbb{R}^2 \}$. Let $\zeta = \begin{pmatrix} \alpha \\ u \end{pmatrix}$, then $\mathcal{L}_{SE(2)}(\zeta) = \begin{pmatrix} \alpha J & u \\ 0_{1,2} & 0 \end{pmatrix}$, where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We have $\text{Ad}_g = \begin{pmatrix} 1 & 0_{1 \times 2} \\ -Jx & R(\theta) \end{pmatrix}$.

A.1.2 Group of multiple direct spatial isometries $SE_{K+1}(2)$

We now introduce a simple extension of $SE(2)$, inspired from preliminary remarks in [13][8]. For $\theta \in \mathbb{R}$ and $x, p^1, \cdots, p^K \in \mathbb{R}^2$, consider the map $\Psi : \mathbb{R}^{2K+3} \to \mathcal{M}_{K+3}(\mathbb{R})$ defined by

$$\Psi : (\theta, x, p^1, \cdots, p^K) \mapsto \begin{pmatrix} R(\theta) & x & p^1 & \cdots & p^K \\ 0_{1,2} & \vdots & \vdots & \cdots & \vdots \\ 0_{1,2} & \cdots & \cdots & \cdots & \cdots \\ 0_{1,2} & \cdots & \cdots & \cdots & I_{K+1} \end{pmatrix}$$

(25)

and let $G \subset \mathbb{R}^{(K+3) \times (K+3)}$ be defined by $G = \{ \Psi(\theta, x, p^1, \cdots, p^K) : \theta \in \mathbb{R}, x, p^1, \cdots, p^K \in \mathbb{R}^2 \}$.
This section is a summary of the IEKF methodology of [8, 6]. Let $\chi$ denote the general dynamical system.

### A.2 Statement of the general IEKF equations

Consider a general dynamical system $\chi$ and denote it by $G = SE_{K+1}(2)$. Note that, we recover $SE(2)$ for $K = 1$, i.e., $SE(2) = SE_2(2)$. Letting $\alpha \in \mathbb{R}$ and $\zeta = \begin{pmatrix} \alpha \\ x \\ p^i \\ \vdots \\ p^k \end{pmatrix}$, yields $\mathcal{L}_{SE_{K+1}(2)}(\zeta) = \begin{pmatrix} \alpha I \\ x \\ p^1 \\ \vdots \\ p^n \end{pmatrix}$.

Let $g = se_{K+1}(2) = \{\mathcal{L}_{SE_{K+1}(2)}(\zeta); \zeta \in \mathbb{R}^{2K+3}\}$. It turns out, by extension of the $SE(2)$ results, that there exists a closed form for the Lie exponential $\exp = \exp_m \circ \mathcal{L}_{SE_{K+1}(2)}$ that writes $\exp(\zeta) = \Psi(\alpha, B(\alpha)x, B(\alpha)p^1, \ldots, B(\alpha)p^K)$ with $B(\alpha) = \begin{pmatrix} \sin(\alpha)/\alpha \\ 1 - \cos(\alpha)/\alpha \\ -1/\sin(\alpha) \end{pmatrix}$.

The $Ad$ is also easily derived by extension of $SE(2)$, but to save space, we only display it once: $Ad_{\Psi(\theta, \xi, \rho^1, \ldots, \rho^K)}$ is defined as the matrix $G_n$ of eq (21).

#### A.2 Statement of the general IEKF equations

This section is a summary of the IEKF methodology of [8, 6]. Let $G$ be a matrix Lie group. Consider a general dynamical system $\chi_n \in G \subset \mathbb{R}^{N \times N}$ on the group, associated to a sequence of observations $(Y_n)_{n \geq 0} \in \mathbb{R}^p$, with equations as follows:

\[
\chi_n = \chi_{n-1} \exp(w_n) \Gamma_n \tag{26}
\]

\[
Y_n = h(\chi_n) + V_n, \tag{27}
\]

where $\Gamma_n \in G$ is an input matrix which encodes the displacement according to the evolution model, $w_n \in \mathbb{R}^{\dim \theta}$ is a vector encoding the model noise, $h : G \to \mathbb{R}^p$ is the observation function and $V_n \in \mathbb{R}^p$ the measurement noise.

The IEKF propagates an estimate obtained after the previous observation $Y_{n-1}$ through the deterministic part of (26):

\[
\hat{\chi}_{n|n-1} = \hat{\chi}_{n-1}|n-1 \Gamma_n \tag{28}
\]

To update $\hat{\chi}_{n|n-1}$ using the new observation $Y_n$, one has to consider an estimation error that is well-defined on the group. In this paper we will use the following right-invariant errors

\[
\eta_{n-1|n-1} := \chi_{n-1} \hat{\chi}_{n-1|n-1}^{-1}, \quad \eta_{n|n-1} := \chi_n \hat{\chi}_{n|n-1}^{-1} \tag{29}
\]

which are equal to $I_G$ when $\hat{\chi} = \chi$. The terminology stems from the fact they are invariant to right multiplications, that is, transformations of the form $\chi_{n|n-1} \mapsto \chi_{n|n-1} g, \chi_n \mapsto \chi_n g$ with $g \in G$. Note that, one could alternatively consider left-invariant errors but it turns out to be less fruitful for SLAM.
A.2.1 Linearized error equations over the group $G$

The IEKF update is based upon a first-order expansion of the non-linear system associated to the errors (29) around $I_G$. First, compute the full error’s evolution

$$\eta_{n|n-1} = \chi_n \hat{x}_{n|n-1}^{-1}$$

$$= \chi_{n-1} \exp(w_n) \Gamma_n \hat{x}_{n|n-1}^{-1}$$

$$= \chi_{n-1} \exp(w_n) \hat{x}_{n-1|n-1}^{-1}$$

$$= \chi_{n-1} \hat{x}_{n-1|n-1}^{-1} \exp(w_n) \hat{x}_{n-1|n-1}^{-1}$$

$$= \eta_{n-1|n-1} \exp(Ad_{\hat{x}_{n|n-1}^{-1}} w_n)$$

Note that the term $\Gamma$ has disappeared! This is a key property for the successes of the invariant filtering approach [12][8]. To linearize this equation we define $\xi_{n-1|n-1}$, $\xi_{n|n-1} \in \mathbb{R}^{\dim \mathcal{G}}$ around $I_G$ through

$$\eta_{n-1|n-1} = \exp(\xi_{n-1|n-1}), \quad \eta_{n|n-1} = \exp(\xi_{n|n-1}).$$

(30)

As in the standard non-additive noise EKF methodology [35] all terms of order $\| \xi_{n-1|n-1} \|$, $\| \chi_{n-1} \|$, $\| w_n \|$, $\| \xi_{n-1|n-1} \|$ are assumed small and are neglected. Using the BCH formula, and neglecting the latter terms, we get

$$\exp(\xi_{n|n-1}) = \exp(\xi_{n-1|n-1} + Ad_{\hat{x}_{n|n-1}^{-1}} w_n)$$

Using the local invertibility of $\exp$ around $0_{\dim \mathcal{G} \times 1}$, we get the following linearized error evolution in $\mathbb{R}^{\dim \mathcal{G}}$

$$\xi_{n|n-1} = F_n \xi_{n-1|n-1} + G_n w_n,$$

(31)

where $F_n = I_{\dim \mathcal{G}}$ and $G_n = Ad_{\hat{x}_{n|n-1}^{-1}}$.

To linearize the output error, we now slightly adapt the IEKF theory [8] to account for the general form of output (11). Note that, $Y_n - h(\hat{x}_{n|n-1}) = h(\chi_n) - h(\hat{x}_{n|n-1}) + V_n = h(\exp(\xi_{n|n-1} \hat{x}_{n|n-1}^{-1}) - h(\hat{x}_{n|n-1}^{-1}) + V_n$. As $\xi_{n|n-1}$ is assumed small, and as $\exp(0_{\dim \mathcal{G} \times 1}) = I_G$, a first-order Taylor expansion in $\xi \in \mathbb{R}^{\dim \mathcal{G}}$ arbitrary, allows defining $H_n$ as follows

$$h(\exp(\xi) \hat{x}_{n|n-1}^{-1}) = h(\hat{x}_{n|n-1}^{-1}) := H_n \xi + O(\| \xi \|^2)$$

(32)

A.2.2 Computing the Kalman gain

The gain is computed as for the standard EKF [35], but where the linearized error system to be considered is (31)-(32).

A.2.3 Update

As in the standard theory, the Kalman gain matrix $K_n$ allows computing an estimate of the linearized error after the observation $Y_n$ through $\hat{\xi}_{n|n} = K_n z_n$, where $z_n = Y_n - h(\hat{x}_{n|n-1})$. Recall the state estimation errors defined by (29)-(30) are of the form $\exp(\xi) = \chi \hat{x}^{-1}$, that is, $\chi = \exp(\xi) \hat{x}$. Thus an estimate of $\chi_n$ after observation $Y_n$
which is consistent with \((29)-(30)\), is obtained through the following Lie group counterpart of the linear update \((7)\)

\[
\hat{\chi}_n^{|n} = \exp(K_n z_n) \hat{\chi}_n^{|n-1}
\]

The equations of the filter are detailed in Algorithm 4.

**Algorithm 4 Invariant Extended Kalman Filter (IEKF)**

Choose initial \(\hat{\chi}_{0|0} \in G\) and \(P_{0|0} \in \mathbb{R}^{\dim g \times \dim g} = \text{Cov}(\xi_0)\)

**loop**
- Define \(H_n\) as in \((32)\) and let \(F_n = I_{\dim g}\) and \(G_n = \text{Ad}_{\hat{\chi}_{n|n-1}}\).
- Define \(Q_n\) as \(\text{Cov}(w_n)\) and \(R_n\) as \(\text{Cov}(V_n)\).

**Propagation**
- \(\hat{\chi}_{n|n-1} = \hat{\chi}_{n-1|n-1} \exp(u_n)\)
- \(P_{n|n-1} = F_n P_{n|n-1} F_n^{-1} + G_n Q_n G_n^T\)

**Update**
- \(z_n = Y_n - h(\hat{\chi}_{n|n-1})\)
- \(S_n = H_n P_{n|n-1} H_n^T + R_n\)
- \(K_n = P_{n|n-1} H_n^T S_n^{-1}\)
- \(P_{n|n} = [I - K_n H_n] P_{n|n-1}\)
- \(\hat{\chi}_{n|n} = \exp(K_n z_n) \hat{\chi}_{n|n-1}\)

**end loop**

### A.3 Lie group based derivation of the 2D IEKF-SLAM

In this section we show step by step the IEKF-SLAM Algorithm is a strict application of Algorithm 4.

#### A.3.1 Underlying Lie group

The Lie group that underlies the SLAM problem, is \(G = SE_{K+1}(2)\) introduced in Appendix A.1.2. Let us apply the general theory of the IEKF to this group. To define the Lie group counterpart \(\chi_n\) of the state \(X_n\) defined by \((8)\), we let \(\chi_n = \Psi(X_n)\). The model equations \((9)\) write

\[
\chi_n = \chi_{n-1} \tilde{\Gamma}_n
\]

with \(\tilde{\Gamma}_n = \Psi(\omega_n + w_n^\omega, (v_n + w_n^v)\), \(0_{1\times 2} \cdots 0_{1\times 2})^T\). At the propagation step, the IEKF propagates the estimate through the corresponding deterministic equations

\[
\hat{\chi}_{n|n-1} = \hat{\chi}_{n-1|n-1} \Gamma_n
\]

with \(\Gamma_n = \Psi(\omega_n, v_n, 0_{1\times 2} \cdots 0_{1\times 2})^T\).
A.3.2 Right-invariant error (29)
A simple matrix multiplication shows that
\[ \eta_n|n-1 = \chi_n|n-1 = \Psi(e_n|n-1), \]
where
\[ e_n|n-1 = \begin{pmatrix}
\theta_n - \hat{\theta}_n|n-1 \\
x_n - [R(\theta_n - \hat{\theta}_n|n-1)\hat{x}_n|n-1] \\
p_n^1 - [R(\theta_n - \hat{\theta}_n|n-1)\hat{p}_n^1] \\
\vdots \\
p_n^K - [R(\theta_n - \hat{\theta}_n|n-1)\hat{p}_n^K]
\end{pmatrix} \]
(34)
and where \(e_n|n\) is defined analogously.

A.3.3 Linearized error
The linearized error is defined by (30), that is,
\[ \eta_n|n-1 = \exp(\xi_n|n-1). \]
As terms of order \(O(\|\xi_n|n-1\|^2)\) are to be neglected in the linearized equations, it suffices to compute a first-order approximation of \(\xi_n|n-1\). First note that \(\xi_n|n-1\) defined by (16) is a linear approximation to \(e_n|n-1\), that is,
\[ e_n|n-1 = \xi_n|n-1 + O(\|\xi_n|n-1\|^2) \]
This readily implies
\[ I_{2K+3} + \mathcal{L}_{\rho_{SE_{K+1}}(\xi)}(\xi) + O(\|\xi\|^2) = \exp(\xi) \]
Recalling \(I_{2K+3} + \mathcal{L}_{\rho_{SE_{K+1}}(\xi)}(\xi) + O(\|\xi\|^2) = \exp(\rho_{SE_{K+1}}(\xi)) := \exp(\xi)\) we see (16) is a first approximation of \(\xi_n|n-1\) as defined in (30) indeed.

A.3.4 Linearized error propagation
The way the linearized error (16) propagates has already been computed and consists of (17), which is the same as the first equation of (20). It exactly matches what can be expected from the general theory, that is, equation (31), recalling that \(G_n\) of (21) is the map \(Ad_{\chi_n|n-1}\) of the group \(G = SE_{K+1}(2)\) indeed.

A.3.5 Linearized output map
Note the general definition of \(H_n\) of (32) here boils down to the one of Section 4, as here \(\dot{\xi} = \Psi(\dot{X})\), \(\exp(\xi)\dot{X} = \chi = \Psi(X_n)\) and \(\xi\) is given by (16).

A.3.6 Estimate update
Referring to Appendix A.1.2 and the definition of exponential map of \(G = SE_{K+1}(2)\), a simple matrix multiplication shows that \(\exp(\xi)\Psi(X) = \phi(\xi, \dot{X})\) with \(\phi\) defined by (22) indeed.

A.4 Equations of the IEKF-SLAM in 3D
Extending the group \(SE_{K+1}(2)\) to the 3D case, and applying the general IEKF theory of Section A.2, we derive in the present section an IEKF for 3D SLAM. Due to space
limitations as it is not the primary object of the present paper. We pursue extremeness of exposure. See also [6, 8]. Note that, although the 3D SLAM equations make use of rotation matrices, they are in fact totally intrinsic: When using quaternions (recommended) or Euler angles (not recommended) they write the same as the group \( SE_{K+1}(3) \) we introduce does in fact not depend on a specific representation of rotations.

### A.4.1 3D SLAM model

The equations of the robot in 3D and in continuous time write:

\[
\frac{d}{dt} R_t = R_t (\omega_t + w^\omega_t) \times, \quad \frac{d}{dt} x_t = R_t (v_t + w^v_t),
\]

where \( R_t \in SO(3) \) is a rotation matrix that represents the robot’s orientation at time \( t \), \( \omega_t \in \mathbb{R}^3 \) denotes the angular velocity of the robot measured by a gyrometer or by odometry (in combination with a unicycle model for a terrestrial vehicle), \( v_t \in \mathbb{R}^3 \) the velocity in the robot’s frame, and \( p^j_t \in \mathbb{R}^3 \) is the position of landmark \( j \), and where \((b)_\times \) for \( b \in \mathbb{R}^3 \) denotes the skew symmetric matrix of \( \mathbb{R}^{3 \times 3} \) such that for any \( x \in \mathbb{R}^3 \) we have \((b)_\times x = b \times x \). Finally \( w^\omega_t \) and \( w^v_t \) denote (resp.) the noise on angular and linear velocities. Although the theory of IEKF could very well be applied directly to this continuous time dynamics as in [8], we apply it here to a discretized model, to be consistent with the rest of the article. Although exact discretization of the noisy model on the group is beyond reach [6], letting \( \Delta t \) be the time step, the following first-order integration scheme is widely used:

\[
\begin{align*}
R_n &= R_{n-1} \exp_n[(w^\omega_n)_\times] \Omega_n, \quad x_n = x_{n-1} + R_{n-1}(v_n + w^v_n), \\
p^j_n &= p^j_{n-1}, \quad 1 \leq j \leq K
\end{align*}
\]

where the increments \( \Omega_n, v_n \) are obtained solving the noise-free initial conditions during the \( n \)-th time step with initial condition \( R = I_3, x = 0 \), and where the following discrete noise

\[
w^T_n := ((w^\omega_n)^T, (w^v_n)^T, 0_{1 \times 3K})^T, \quad Q_n = \mathbb{E}(w_n w_n^T)
\]

is obtained by integration of the corresponding white noises. Note that, this scheme is accurate to first-order terms in \( \Delta t \). A general landmark observation in the car’s frame reads:

\[
Y_n = \begin{pmatrix}
\tilde{h} \left[ R_n^T (p^1 - x_n) \right] + V_n^1 \\
\vdots \\
\tilde{h} \left[ R_n^T (p^K - x_n) \right] + V_n^K
\end{pmatrix}
\]

where \( Y_n \in \mathbb{R}^{3K} \) (or \( \mathbb{R}^{2K} \) for monocular visual SLAM) is the observation of the features at time step \( n \), and \( V_n \) the observation noise. We let the output noise covariance matrix be \( \tilde{R}_n \in \mathbb{R}^{3K \times 3K} \) (not to be confused with the rotation \( R_n \)).
A.4.2 Underlying Lie group

The Lie group that underlies the problem is the group $G = SE_{K+1}(3)$ that we introduce as follows. For $R \in SO(3)$ and $x, p^1, \ldots, p^K \in \mathbb{R}^3$ let

$$\Psi : (R, x, p^1, \ldots, p^K) \mapsto \begin{pmatrix} R & x & p^1 & \cdots & p^n \\ 0_{1,3} & \vdots & \vdots & \ddots & \vdots \\ 0_{1,3} & I_{K+1} \end{pmatrix}$$

(39)

and let $G \subset \mathbb{R}^{(K+4) \times (K+4)}$ be defined as

$$G = \{ \Psi(R, x, p^1, \ldots, p^K); R \in SO(3), x, p^1, \ldots, p^K \in \mathbb{R}^3 \}$$

and denote it by $G = SE_{K+1}(3)$. We then have $\mathcal{L}_{se_{K+1}}(2) \begin{pmatrix} \omega \\ x \\ p^1 \\ \vdots \\ p^K \end{pmatrix} = \begin{pmatrix} (\omega) \times & x & p^1 & \cdots & p^n \\ 0_{1,3} & \vdots & \vdots & \ddots & \vdots \\ 0_{1,3} & 0_{(K+1) \times (K+1)} \end{pmatrix}$

and $g = se_{K+1}(3) = \{ \mathcal{L}_{se_{K+1}}(3) \zeta; \zeta \in \mathbb{R}^{3K+6} \}$. For $\zeta \in \mathbb{R}^{3K+6}$, by extension of the $SE(3)$ results, we have the closed form:

$$\exp(\zeta) = I_{K+4} + S + \frac{1 - \cos(||\zeta||)}{||\zeta||^2} S^2 + \frac{||\zeta|| - \sin(||\zeta||)}{||\zeta||^3} S^3$$

(40)

where $S = \mathcal{L}_{se_{K+1}}(3) \zeta$. As easily seen by analogy with $SE(3)$

$$Ad_{\Psi(R, x, p^1, \ldots, p^K)} = \begin{pmatrix} R & 0_{3 \times 3} & \cdots & 0_{3 \times 3} \\ (x) \times R & \vdots & \ddots & \vdots \\ (p^K) \times R & \cdots & \ddots & R \end{pmatrix}$$

(41)

A.4.3 Link with the dynamical model

Let the state be $X = (R, x, p^1, \ldots, p^K)$, and let $\chi \in G$ be $\chi = \Psi(X)$, and let $\hat{X}$ and $\hat{\chi}$ be their estimated counterparts. It is easily seen that up to terms that will disappear in the linearization process anyway, the model (36) for the state is mapped through $\Psi$ defined at (39), to a model of the form (26).

A.4.4 Right-invariant error (29)

It writes $\eta := \chi^T \hat{\chi}^{-1} = \Psi(R \hat{R}^T, x - R \hat{R}^T \hat{x}, p^K - R \hat{R}^T \hat{p}^K)$. 
A.4.5 Linearized error

Using the matrix logarithm, define $\tilde{\omega} \in \mathbb{R}^3$ as the solution of $\exp_m[(\tilde{\omega})_\times] = R \hat{R}^T$. Neglecting terms of order $O(\|\tilde{\omega}\|^2)$, we have $x - R\hat{R}^T \hat{x} = x - \tilde{x} - \tilde{\omega} \times \hat{x}$. A first order identification as in Appendix A.3.3 thus yields $\xi = (\tilde{\omega}, x - R\hat{R}^T \hat{x}, \cdots, p_{K} - R\hat{R}^T \hat{p}_K - \tilde{\omega} \times R\hat{R}^T \hat{p}_K)$ as a vector that satisfies the definition $\exp(\xi) = \eta$ up to terms of order $O(\|\xi\|^2)$.

A.4.6 Linearized error propagation

As in the standard EKF theory, the IEKF propagates an estimate obtained after the previous observation $Y_{n-1}$ through the deterministic part of (36), or equivalently (26) in matrix form. Thus, the propagation equation is given by (31) where $F_n = I_3K + 6$ and $G_n w_n = Ad_{\hat{R}_{n|n-1}}(w_n)$ as a direct application of the theory.

A.4.7 Linearized output map

To apply Definition (32), simply note that $R^T p - \hat{R}^T \hat{p} \approx \hat{R}^T (p - \hat{R}^T \hat{p})$. Thus

$$H_n = \begin{bmatrix} \nabla \tilde{h}^1 \cdot R (\hat{\theta}_{n|n-1})^T (0_{3,1} \begin{bmatrix} -I_3 \ I_3 \ 0_{3,3(K-1)} \end{bmatrix}) \cr \vdots \cr \nabla \tilde{h}^K \cdot R (\hat{\theta}_{n|n-1})^T (0_{3,1} \begin{bmatrix} -I_3 \ 0_{3,3(K-1)} \ I_3 \end{bmatrix}) \end{bmatrix},$$

(42)

where $\nabla \tilde{h}^k$ is the Jacobian of $\tilde{h}$ computed at $\hat{R}_{n|n-1}^T \hat{p}_{n|n-1} - \hat{x}_{n|n-1}$.

The various steps are gathered in Algorithm 5.

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Algorithm 5 IEKF SLAM: the 3D case

The state is defined by $X = (R, x, p^1, \cdots, p^K) \in SO(3) \times \mathbb{R}^{3+3K}$ and the model is (36)–(38). Pick an initial estimate $\hat{X}_0$ with covariance matrix $P_0$.

loop

Let $F_n = I_{3K+6}$, $H_n$ as in (42), $G_n = Ad_{\hat{X}_{n-1}}$ using (41)

Define $Q_n$ by (37). $\tilde{R}_n$ is the observation noise cov. matrix.

Propagation

$\hat{R}_{n|n-1} = \hat{R}_{n-1|n-1} \Omega_n$

$\hat{x}_{n|n-1} = \hat{x}_{n-1|n-1} + \hat{R}_{n-1|n-1} v_n$

$\hat{p}^j_{n|n-1} = \hat{p}^j_{n-1|n-1}$ for all $1 \leq j \leq K$

$P_{n|n-1} = F_n P_{n-1|n-1} F^T_n + G_n Q_n G^T_n$

Update

$z_n = Y_n - \left[ \tilde{h} \begin{bmatrix} \tilde{R}^T_{n|n-1} (\hat{p}^1_{n|n-1} - \hat{x}_{n|n-1}) \\ \vdots \\ \tilde{h} \begin{bmatrix} \tilde{R}^T_{n|n-1} (\hat{p}^K_{n|n-1} - \hat{x}_{n|n-1}) \end{bmatrix} \end{bmatrix} \right]$

$S_n = H_n P_{n|n-1} H^T_n + \tilde{R}_n$

$K_n = P_{n|n-1} H^T_n S_n^{-1}$

$P_{n|n} = [I - K_n H_n] P_{n|n-1}$

Compute $\hat{x}_{n|n} = \exp(K_n z_n) \Psi(\hat{x}_{n|n-1})$ using (40) and let $\hat{X}_{n|n} = \Psi^{-1}(\hat{x}_{n|n})$.

end loop
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Figure 5: Trajectory and final covariance ellipsoids returned by the implemented filters. We see EKF and UKF are not consistent, mainly because the elongation of ellipsoids, that is related to heading uncertainty, is underestimated. It is not the case with ideal EKF (idEKF), OC-EKF and IEKF.
Figure 6: Illustration of Theorem 2. Bottom plot is a zoom of the first time steps. The information over an infinitesimal perturbation corresponding to a rotation of the whole system is decreasing for the IEKF SLAM, which is a consistent behavior as this perturbation is unobservable. Ideal EKF and OCEKF give similar results, but EKF and UKF do not. The plot also confirms EKF and UKF SLAM tend to acquire spurious information over this unobservable direction.