THE LIFESPAN OF SOLUTIONS TO SEMILINEAR DAMPED WAVE EQUATIONS IN ONE SPACE DIMENSION

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Abstract. In the present paper, we consider the initial value problem for semilinear damped wave equations in one space dimension. Wakasugi [7] have obtained an upper bound of the lifespan for the problem only in the subcritical case. On the other hand, D’Abbicco & Lucente & Reissig [3] showed a blow-up result in the critical case. The aim of this paper is to give an estimate of the upper bound of the lifespan in the critical case, and show the optimality of the upper bound. Also, we derive an estimate of the lower bound of the lifespan in the subcritical case which shows the optimality of the upper bound in [7]. Moreover, we show that the critical exponent changes when the initial data are odd functions.

1. Introduction. In this paper we consider the initial value problem for semilinear damped wave equations:

\[
\begin{cases}
v_{tt} - v_{xx} + \frac{2}{1 + t} v_t = F(v), & \text{in } \mathbb{R} \times [0, \infty), \\
v(x, 0) = \varepsilon f(x), \quad v_t(x, 0) = \varepsilon g(x), & x \in \mathbb{R},
\end{cases}
\] (1)

where \( F(v) = |v|^p \) or \( F(v) = |v|^{p-1}v \) with \( p > 1 \), \( f, g \) have appropriate regularity, and \( \varepsilon > 0 \) is a “small” parameter.

When \( F(v) = |v|^p \) and \( 1 < p < 3 \), Wakasugi [7] showed that an upper bound of the lifespan which is the maximal existence time of solutions of (1) is \( C \varepsilon^{-(p-1)/(3-p)} \) for some positive initial data, where \( C \) is a positive constant independent of \( \varepsilon \). The proof of [7] is based on the “test function method”. (See 165p. in [7].) To be more precise, the upper bound of the lifespan in general spatial dimensions \( (n \geq 1) \) is given by a constant times \( \varepsilon^{-(p-1)/(2-n(p-1))} \) for \( 1 < p < p_F(n) := 1 + 2/n \), where \( p_F(n) \) (the Fujita exponent) is the critical exponent for the semilinear heat equation.

In the case of \( F(v) = |v|^p \) with \( p = 3 \), D’Abbicco & Lucente & Reissig [3] showed that the solution of (1) blows up in finite time if the initial data satisfy some positivity conditions, and have a compact support. However, they did not discuss about the estimates of the lifespan. We note that D’Abbicco [2] has obtained a global existence result for (1) in the case of \( p > 3 \) if \( \varepsilon \) is “small” enough.

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In the proof of [3], they reduced the problem to the following semilinear wave equations:
\[
\begin{align*}
\begin{cases}
    u_{tt} - u_{xx} = (1 + t)^{-(p-1)} F(u), & \text{in } \mathbb{R} \times [0, \infty), \\
    u(x, 0) = \varepsilon f(x), u_t(x, 0) = \varepsilon \{f(x) + g(x)\}, & x \in \mathbb{R},
\end{cases}
\end{align*}
\]
(2)
by setting \( u(x, t) = (1 + t)v(x, t) \), where \( F(u) = |u|^p \). Actually in [3], (2) is studied in general spatial dimensions:
\[
\begin{align*}
\begin{cases}
    u_{tt} - \Delta u = (1 + t)^{-(p-1)} F(u), & \text{in } \mathbb{R}^n \times [0, \infty), \\
    u(x, 0) = \varepsilon f(x), u_t(x, 0) = \varepsilon \{f(x) + g(x)\}, & x \in \mathbb{R}^n,
\end{cases}
\end{align*}
\]
(3)
where \( n \geq 1 \). Let
\[
p_c(n) = \max \{pF(n), p_0(n + 2)\},
\]
(4)
where \( p_0(n) \) is the Strauss exponent, that is the positive root of the quadratic equation \((n-1)p^2 - (n+1)p - 2 = 0\). When \( n = 2, 3 \), they showed that the problem (3) has a global in-time solution for \( p > p_c(n) \) if \( \varepsilon \) is “small” enough and \((f, g)\) have a compact support. On the other hand, the solution of (3) blows up in finite time if \( 1 < p \leq p_c(n) \) in the case of \( n \geq 1 \). Since \( p_0(3) = 1 + \sqrt{3} < 3 = p_F(1) \), we see that \( p_c(1) = 3 \).

To state our result, we define the lifespan \( T_\varepsilon \) of the \( C^2 \)-solution of (2) by
\[
T_\varepsilon \equiv T_\varepsilon(f, g) := \sup\{T \in [0, \infty) : \text{There exists a unique solution } u \in C^2(\mathbb{R} \times [0, T]) \text{ of (2).}\}
\]
for arbitrarily fixed \((f, g) \in C^2(\mathbb{R}) \times C^4(\mathbb{R})\). Our purpose in the present paper is to show the followings for the problem (2). The first one is to derive an estimate of the upper bound of the lifespan in the case of \( p = 3 \), and show the optimality of the upper bound. Namely we give an estimate of the lifespan from below which has the same order with respect to \( \varepsilon \) as the upper bound. The second one is to show the optimality of the result of [7] in the case of \( 1 < p < 3 \). The third one is to give an alternative proof of the estimates of [7]. Finally, the forth one is to show the critical exponent changes to \( 1 + \sqrt{3} \) from 3 when the initial data are odd functions. Our proof is based on the iteration argument which was introduced by John [4]. For the critical case in Theorem 1.2 and Theorem 1.4, we apply the “slicing method” which was introduced by Agemi & Kurokawa & Takamura [1].

The following theorem shows that the optimality of the upper bound of [7] in the subcritical case \( 1 < p < 3 \).

**Theorem 1.1.** Let \( F(u) = |u|^p \) or \( F(u) = |u|^{p-1}u \) with \( 1 < p \leq 3 \) in (2). Assume that both \( f \in C^2(\mathbb{R}) \) and \( g \in C^4(\mathbb{R}) \) have compact support contained in \( \{x \in \mathbb{R} : |x| \leq 1\} \). Then, there exists a positive constant \( c = c(f, g, p) \) such that
\[
T_\varepsilon \geq \begin{cases}
    c\varepsilon^{-(p-1)/(3-p)} & \text{if } 1 < p < 3, \\
    \exp(c\varepsilon^{-2}) & \text{if } p = 3,
\end{cases}
\]
(5)
holds for \( \varepsilon > 0 \).

To derive a blow-up result, we require the following assumptions on the data:

Let \( f \equiv 0 \) and \( g \in C^4(\mathbb{R}) \) does not vanish identically.

Assume \( g(x) \geq 0 \) for all \( x \in \mathbb{R} \) and \( \int_{-1}^1 g(y)dy > 0 \).

Then, we have the following.
Theorem 1.2. Let $F(u) = |u|^p$ or $F(u) = |u|^{p-1}u$ with $1 < p \leq 3$ in (2). Assume (6). Then, there exist positive constants $\varepsilon_0 = \varepsilon_0(g, p)$ and $C = C(g, p)$ such that

$$T_\varepsilon \leq \begin{cases} C\varepsilon^{-(p-1)/3} & \text{if } 1 < p < 3, \\ \exp(C\varepsilon^{-2}) & \text{if } p = 3, \end{cases}$$

holds for any $\varepsilon$ with $0 < \varepsilon \leq \varepsilon_0$.

Remark 1.1. Due to (7), the lower bound in the case of $p = 3$ in (5) is optimal. However, if the initial data are odd functions, we obtain different estimates of the lifespan and different critical exponent.

The following theorems show that the global existence and blow-up results when the initial data are odd functions. We define

$$Y_\kappa = \{(f, g) \in C^2(R) \times C^1(R) : \|(f, g)\|_{Y_\kappa} < \infty\},$$

$$\|(f, g)\|_{Y_\kappa} = \sup_{x \geq 0} \left\{ (1 + |x|)^{1+\kappa} \left( \sum_{j=0}^1 |f^{(j)}(x)| + |g(x)| \right) \right\}.$$  

Then we have the following.

Theorem 1.3. Let $F(u) = |u|^{p-1}u$ with $p > 2$ in (2). Suppose $p > 2$, $(f, g) \in Y_\kappa$ with $\kappa > \max\{1/p, (3-p)/(p-1)\}$ and $f$, $g$ are odd functions. Then, there exist positive constants $\varepsilon_0 = \varepsilon_0(f, g, p, \kappa)$ and $c = c(f, g, p, \kappa)$ such that

$$T_\varepsilon = \infty \quad \text{if} \quad p > 1 + \sqrt{2},$$

$$T_\varepsilon \geq \begin{cases} \varepsilon^{-(p-1)/(1+2p-p^2)} & \text{if } 2 < p < 1 + \sqrt{2}, \\ \exp(\varepsilon^{-(p-1)}) & \text{if } p = 1 + \sqrt{2}, \end{cases}$$

holds for any $\varepsilon$ with $0 < \varepsilon \leq \varepsilon_0$.

Remark 1.2. One can also obtain a similar estimate to (9) in the case of $1 < p \leq 2$. See Remark 5.1.

Remark 1.3. We note that the estimates of (9) in the case of $2 < p < 1 + \sqrt{2}$ holds when $f$, $g$ are odd functions, and satisfy the same assumptions in Theorem 1.1. Hence, the estimates of (9) is an improvement of (5) for small $\varepsilon$, because

$$\frac{p-1}{3-p} < \frac{p(p-1)}{1+2p-p^2}$$

is equivalent to $p > 1$.

To derive a blow-up result when the initial data are odd functions, we require the following assumptions on the data:

Let $f \in C^1(R)$, $g \in C^2(R)$ are odd functions. Assume $f(x) > 0$, $g(x) > 0$ for all $x \in (0, \infty)$ and $f'(0) > 0$.  

Then, we have the following.

Theorem 1.4. Let $F(u) = |u|^{p-1}u$ with $1 < p \leq 1 + \sqrt{2}$ in (2). Assume (11). Then, there exist positive constants $\varepsilon_0 = \varepsilon_0(f, g, p)$ and $C = C(f, g, p)$ such that

$$T_\varepsilon \leq \begin{cases} C\varepsilon^{-(p-1)/(1+2p-p^2)} & \text{if} \quad 1 < p < 1 + \sqrt{2}, \\ \exp(C\varepsilon^{-p-1}) & \text{if} \quad p = 1 + \sqrt{2}, \end{cases}$$

holds for any $\varepsilon$ with $0 < \varepsilon \leq \varepsilon_0$. 


Remark 1.4. We note that Theorem 1.3 and Theorem 1.4 do not hold for $F(u) = |u|^p$, because $|u(x, t)|^p$ is not an odd function with respect to $x$.

Remark 1.5. There exists some initial data which satisfies the assumptions in Theorem 1.1 and Theorem 1.4. However, the estimates (5) does not contradict to (12) for small $\varepsilon$, because of (10).

Remark 1.6. Making use of the iteration argument in [4] and [1], it would be able to get the optimal estimates of the lifespan in the case of two and three space dimensions.

This paper is organized as follows. In the next section, we prepare some definitions and lemmas. The proofs of Theorem 1.1 and Theorem 1.2 shall be discussed in Section 3 and Section 4, respectively. The proofs of Theorem 1.3 and Theorem 1.4 are obtained in Section 5 and Section 6, respectively.

2. Preliminaries. In this section, we give some definitions and useful lemmas.

We define

$$u^0(x, t) = \frac{1}{2} \{ f(x + t) + f(x - t) \} + \frac{1}{2} \int_{x-t}^{x+t} \{ f(y) + g(y) \} dy$$

(13)

for $(f, g) \in C^2(\mathbb{R}) \times C^1(\mathbb{R})$, and

$$L(\Phi)(x, t) = \frac{1}{2} \int \int_{D(x, t)} \frac{\Phi(y, s)}{(1 + s)^{p-1}} dy ds$$

(14)

for $\Phi \in C(\mathbb{R} \times [0, \infty))$, where

$$D(x, t) = \{(y, s) \in \mathbb{R} \times [0, \infty) : 0 \leq s \leq t, x - t + s \leq y \leq x + t - s \}.$$  

For $(f, g) \in C^2(\mathbb{R}) \times C^1(\mathbb{R})$, if $u \in C(\mathbb{R} \times [0, \infty))$ is a solution of

$$u(x, t) = \varepsilon u^0(x, t) + L(F(u))(x, t), \quad (x, t) \in \mathbb{R} \times [0, \infty),$$

(15)

then $u \in C^2(\mathbb{R} \times [0, \infty))$ is the solution to the initial value problem (2) by $F(u) \in C(\mathbb{R} \times [0, \infty))$ and $\partial_t F(u) \in C(\mathbb{R} \times [0, \infty))$.

When $F(u) = |u|^{p-1}u$ and $(f, g)$ are odd functions, if $u$ is the $C^2$-solution of (2) then $-u(-x, t)$ is the solution to the problem (2). Making use of the uniqueness of the $C^2$-solution to (2), we see that $u(x, t)$ is odd function with respect to $x$. Therefore, in that case, we see that it is sufficient to consider the following integral equation:

$$u(x, t) = \varepsilon u^0(x, t) + \tilde{L}(|u|^{p-1}u)(x, t), \quad (x, t) \in [0, \infty)^2,$$

(16)

where we set

$$\tilde{L}(\Phi)(x, t) = \frac{1}{2} \int \int_{\tilde{D}(x, t)} \frac{\Phi(y, s)}{(1 + s)^{p-1}} dy ds$$

(17)

for $\Phi \in C([0, \infty)^2)$. Here $\tilde{D}(x, t)$ is defined by

$$\tilde{D}(x, t) = \{(y, s) \in [0, \infty]^2 : 0 \leq s \leq t, |x - t + s| \leq y \leq x + t - s \}.$$  

Next, we prepare some useful lemmas for proving Theorem 1.3.

Lemma 2.1. Let $\nu > 0$. Then there exists a positive constant $C_\nu$ such that

$$\int_{|x-t|}^{x+t} \frac{dy}{(1 + y)^{1+\nu}} \leq \frac{C_\nu \min\{x, t\}}{(1 + x + t)(1 + |x - t|)^\nu}$$

(18)

for $(x, t) \in [0, \infty)^2$. 

For the proof, see e.g. Lemma 2.1 in Kubo & Osaka & Yazici [5].

**Lemma 2.2.** Let $p > 2$ and $\sigma > 0$, and let $E_\sigma(\tau)$ be a function defined by

\[
E_\sigma(\tau) = \begin{cases} 
1 & \text{if } \sigma > 1, \\
\log(3 + \tau) & \text{if } \sigma = 1, \\
(1 + \tau)^{1-\sigma} & \text{if } \sigma < 1
\end{cases}
\]  

for $\tau \geq 0$. Then there exists a positive constant $C_{p,\sigma}$ such that

\[
\iint_{D(x,t)} \frac{(1 + s)dyds}{(1 + s + y)^p(1 + |s - y|)^\sigma} \leq C_{p,\sigma}E_\sigma(T)\frac{1 + t}{(1 + x + t)(1 + |x - t|)^{p-2}}
\]  

for $(x, t) \in [0, \infty) \times [0, T]$.

**Proof.** We set

\[
I(x,t) = \int_0^t \int_{|x-t+s|}^{x+t-s} \frac{(1 + s)dyds}{(1 + s + y)^p(1 + |s - y|)^\sigma}
\]

which is the left-hand side of (20). We shall estimate $I(x,t)$ on the following two domains:

$D_1(x,t) = \{(x,t) \in [0, \infty) \times [0, T] : x \geq 2t\}$,

$D_2(x,t) = \{(x,t) \in [0, \infty) \times [0, T] : x \leq 2t\}$.

(i) **Estimation in $D_1(x,t)$**. Noticing that

\[
1 + |s - y| = 1 + y - s \geq 1 + x - t \geq 1 + t \geq 1 + s
\]

for $y \geq x - t + s, x \geq 2t$ and $0 \leq s \leq t$, we get

\[
I(x,t) \leq \int_0^t \frac{(1 + s)ds}{(1 + s)^\sigma} \int_{x-t+s}^{x+t-s} \frac{dy}{(1 + s + y)^p}
\]

\[
\leq (1 + t) \int_0^t \frac{ds}{(1 + s)^\sigma} \int_{x-t+s}^{x+t-s} \frac{dy}{(1 + s + y)^p}
\]

for $(x,t) \in D_1(x,t)$. Since

\[
1 + s + y \geq 1 + x - t \geq \frac{x + t + 1}{3}
\]

for $y \geq x - t + s, s \geq 0$ and $x \geq 2t$, we have

\[
I(x,t) \leq C \frac{1 + t}{(1 + x + t)^p} \int_0^t \frac{(t-s)ds}{(1 + s)^\sigma} \leq C_{p,\sigma} \frac{1 + t}{(1 + x + t)^{p-1}} \cdot E_\sigma(T)
\]

\[
\leq C_{p,\sigma} \frac{1 + t}{(1 + x - t)^{p-2}(1 + x + t)} \cdot E_\sigma(T)
\]

for $(x,t) \in D_1(x,t)$.

(ii) **Estimation in $D_2(x,t)$**. Since $1 + s \leq 1 + s + y$ for $y \geq 0$, we have

\[
I(x,t) \leq \int_0^t \int_{|x-t+s|}^{x+t-s} \frac{dyds}{(1 + s + y)^p(1 + |s - y|)^\sigma}
\]

Replacing the domain of integration by

$(\tilde{D}(x,t) \subset \{(y,s) \in [0, \infty) \times \mathbb{R} : -(t + x) \leq s - y \leq t - x, |t - x| \leq s + y \leq t + x\})$

and changing the variables in the above integral by

\[
\alpha = s + y, \ \beta = s - y,
\]

(21)
we get
\[ I(x,t) \leq C \int_{-(t+x)}^{t-x} \frac{d\beta}{(1 + |\beta|)^\sigma} \int_{|x-t|}^{x+t} \frac{d\alpha}{(1 + \alpha)^{p-1}}. \]

Making use of Lemma 2.1 with setting \( \nu = p - 2 > 0 \), we have
\[ I(x,t) \leq C \frac{1 + t}{(1 + x^2)(1 + |x-t|)^{p-2}} \int_{(t+x)}^{t-x} \frac{d\beta}{(1 + |\beta|)^\sigma}. \]

Then, the \( \beta \)-integral is dominated by
\[ \frac{1 + t}{(1 + x^2)(1 + |x-t|)^{p-2}} \int_{(t+x)}^{t-x} \frac{d\beta}{(1 + |\beta|)^\sigma}. \]

for \( 0 \leq x \leq 2t \), hence we get
\[ I(x,t) \leq C_{\sigma} \epsilon(T) \frac{1 + t}{(1 + x^2)} \int_{(t+x)}^{t-x} \frac{d\beta}{(1 + |\beta|)^\sigma}. \]

for \( (x,t) \in D_2(x,t) \). This completes the proof. \( \square \)

3. Proof of Theorem 1.1. In this section, we prove Theorem 1.1. First of all, we introduce a Banach space
\[ X = \{ u \in C(\mathbb{R} \times [0,T]) : \|u\|_{L^\infty(\mathbb{R} \times [0,T])} < \infty, \text{ supp } u(x,t) \subset \{ |x| \leq t + 1 \} \}, \] (22)
which is equipped with a norm
\[ \|u\|_{L^\infty(\mathbb{R} \times [0,T])} = \sup_{(x,t) \in \mathbb{R} \times [0,T]} |u(x,t)|. \] (23)

We shall construct a solution of the integral equation (15) in \( X \) under suitable assumption on \( T \) such as (27) below. Define a sequence of functions \( \{u_n\}_{n \in \mathbb{N}} \subset X \) by
\[ u_n = u_0 + L(F(u_{n-1})), \quad u_0 = \epsilon u_0^0, \] (24)
where \( F(u) = |u|^p \) or \( F(u) = |u|^{p-1}u \), \( L \) and \( u^0 \) are given by (14) and (13), respectively. It follows that
\[ \|u_0\|_{L^\infty(\mathbb{R} \times [0,T])} \leq M \epsilon, \]
where \( M = \|f\|_{L^\infty(\mathbb{R})} + \|f + g\|_{L^1(\mathbb{R})} \). Since \( (f,g) \) has a compact support, \( M \) is a finite number, so that \( u_0 \in X \).

The following \textit{a priori} estimate plays a key role in the proof of Theorem 1.1.

Lemma 3.1. Let \( V \in X \), \( 1 < p \leq 3 \), and let \( D(\tau) \) be a function defined by
\[ D(\tau) = \begin{cases} (1 + \tau)^{3-p} & \text{if} \quad 1 < p < 3, \\ \log(1 + \tau) & \text{if} \quad p = 3 \end{cases} \] (25)
for \( \tau \geq 0 \). Then, there exists a positive constant \( C_p \) such that
\[ \|L(V)\|_{L^\infty(\mathbb{R} \times [0,T])} \leq C_p D(T) \|V\|_{L^\infty(\mathbb{R} \times [0,T])}. \] (26)
Proof. We divide the proof into two cases, 1 < \( p < 2 \) and \( 2 \leq p \leq 3 \).

(i) Estimation in the case of \( 1 < p < 2 \). The left-hand side in (26) is dominated by

\[
\frac{\|V\|_{L^\infty(R \times [0,T])}}{2} \int_0^T \int_{D(x,t)} \frac{dy ds}{(1+s)^{p-1}} = \|V\|_{L^\infty(R \times [0,T])} \int_0^T \frac{(t-s)ds}{(1+s)^{p-1}}
\]

\[
\leq \|V\|_{L^\infty(R \times [0,T])} \cdot t \int_0^T \frac{ds}{(1+s)^{p-1}}
\]

for \( (x,t) \in R \times [0,T] \). Therefore, we get (26) in the case of \( 1 < p < 2 \).

(ii) Estimation in the case of \( 2 \leq p \leq 3 \). Making use of the support condition for \( V \in X \), we have

\[
\int_R |V(x,t)|dx = \int_{|x| \leq t+1} |V(x,t)|dx
\]

\[
\leq \|V\|_{L^\infty(R \times [0,T])} \int_{|x| \leq t+1} dx = 2(1+t)\|V\|_{L^\infty(R \times [0,T])}
\]

for \( t \in [0,T] \). It follows that

\[
|L(V)(x,t)| \leq \frac{1}{2} \int_0^t \frac{ds}{(1+s)^{p-1}} \int_R |V(y,s)|dy
\]

\[
\leq \|V\|_{L^\infty(R \times [0,T])} \int_0^t \frac{ds}{(1+s)^{p-2}} \leq C_p \|V\|_{L^\infty(R \times [0,T])} D(T)
\]

for \( (x,t) \in R \times [0,T] \). Therefore, we get (26) in the case of \( 2 \leq p \leq 3 \). This completes the proof of Lemma 3.1. \( \square \)

Now, we move on to the proof of Theorem 1.1. First of all, we take \( T > 0 \) such that

\[
2^{p+1}C_p D(T)M^{p-1}e^{p-1} \leq 1,
\]

where \( C_p \) is the one in Lemma 3.1.

Analogously to the proof of Theorem 1.2 in [6] (see p.16 in [6]), we see from Lemma 3.1 that \( \{u_n\}_{n \in N} \) is a Cauchy sequence in \( X \), provided (27) holds. Since \( X \) is complete, there exists \( u \in X \) such that \( u_n \) converges to \( u \) in \( X \). Therefore \( u \) satisfies the integral equation (15), so that \( u \) is the \( C^2 \)-solution of (2). Hence, the proof of Theorem 1.1 is completed. \( \square \)

4. Proof of Theorem 1.2. In this section, we prove Theorem 1.2. We show that the solution to the following integral equation blows up in finite time:

\[
u(x,t) = \varepsilon \int_{x-t}^{x+t} g(y)dy + \frac{1}{2} \int_0^T \int_{D(x,t)} \frac{|u(y,s)|^p}{(1+s)^{p-1}}dy ds
\]

(28)

for \( (x,t) \in R \times [0, \infty) \). Because, if \( u \in C(R \times [0, \infty) \) is a solution of (28), then \( u \) satisfies \( u(x,t) \geq 0 \) for \( (x,t) \in R \times [0, \infty) \) by \( g(x) \geq 0 \) for all \( x \in R \). Therefore, this \( u \) must solve the equation (15) with \( F(u) = |u|^{p-1}u \) by the uniqueness of solutions to (2).

Before proving Theorem 1.2, we prepare some definitions and lemmas. For \( T > 0 \), we define the following domains:

\[
\Sigma_0 = \{(x,t) \in [0, \infty) \times [0,T] : t-x \geq 1\},
\]

\[
\Sigma_j = \{(x,t) \in [0, \infty) \times [0,T] : t-x \geq l_j\} \ (j = 1, 2, \ldots),
\]

\[
\Sigma_\infty = \{(x,t) \in [0, \infty) \times [0,T] : t-x \geq 2\},
\]

(29)
where
\[ l_j = 1 + \sum_{k=1}^{j} 2^{-k} = 2 \left( 1 - \frac{1}{2^{j+1}} \right) \quad \text{for } j \geq 1. \] (30)

**Lemma 4.1.** Let \( p > 1, c_0 > 0 \), and let us define a sequence \( \{C_{p,j}\}_{j=1}^{\infty} \) by
\[
\begin{align*}
C_{p,j} &= \exp\left\{ p^{j-1} \left( \log(C_{p,1} F_p^{-S_j} E_p^{1/(p-1)}) - \log E_p^{1/(p-1)} \right) \right\} \quad (j \geq 2), \\
C_{p,1} &= c_0^p k_p e^p,
\end{align*}
\] (31)

where
\[
\begin{align*}
E_p &= \begin{cases} 
\frac{(p-1)^2}{2^{p+3} p^2}, & \text{if } 1 < p < 3, \\
\left(2^p \cdot 3\right)^{-1}, & \text{if } p = 3,
\end{cases} \\
F_p &= \begin{cases} 
p^2, & \text{if } 1 < p < 3, \\
6 & \text{if } p = 3,
\end{cases} \\
k_p &= \begin{cases} 
2^{-(p+2)}, & \text{if } 1 < p < 3, \\
\left(2^p \cdot 3\right)^{-1}, & \text{if } p = 3,
\end{cases}
\] (32) (33) (34)

and
\[ S_j = \sum_{i=1}^{j-1} \frac{i}{p^i}. \] (35)

Then, we have the following relation:
\[
C_{p,j+1} = \frac{C_{p,j} E_p^p}{F_p^j} \quad (j \in \mathbb{N}).
\] (36)

Since this lemma follows from Lemma 3.1 in [6], if \( C_{a,j}, F_{p,a}, E_{p,a} \) and \( k_a \) are replaced by \( C_{p,j}, F_p, E_p, \) and \( k_p, \) respectively, we omit the proof.

Next, we derive a lower bound of the solution to (28) which is a first step of our iteration argument (for the proof, see e.g. Lemma 3.2 in [6]).

**Lemma 4.2.** Suppose that the assumptions in Theorem 1.2 are fulfilled. Let \( u \in C(\mathbb{R} \times [0,T]) \) be the solution of (28). Then, \( u \) satisfies
\[
u(x,t) \geq \varepsilon c_0 \quad \text{for } (x,t) \in \Sigma_0,
\] (37)

where \( c_0 = \frac{1}{2} \int_{-1}^{1} g(y) dy > 0. \)

Our iteration argument will be done by using the following estimates.

**Proposition 4.1.** Suppose that the assumptions in Theorem 1.2 are fulfilled. Let \( j \in \mathbb{N} \) and let \( u \in C(\mathbb{R} \times [0,T]) \) be the solution of (28). Then, \( u \) satisfies
\[
u(x,t) \geq C_{p,j} \left\{ (t-x)^{-(p-1)}(t-x-1)^2 \right\}^{a_j} \quad \text{if } 1 < p < 3
\] (38)

for \( (x,t) \in \Sigma_0, \) and
\[
u(x,t) \geq C_{3,j} \left\{ \log \left( \frac{t-x}{l_j} \right) \right\}^{a_j} \quad \text{if } p = 3
\] (39)

for \( (x,t) \in \Sigma_j, \) where \( \Sigma_0 \) and \( \Sigma_j \) are defined in (29). Here \( C_{p,j} \) is the one in (31) with \( c_0 = \frac{1}{2} \int_{-1}^{1} g(y) dy > 0 \) and \( a_j \) is defined by
\[
a_j = \frac{p^j - 1}{p - 1} \quad (j \in \mathbb{N}).
\] (40)
Proof. We shall show (38) and (39) by induction. From (28), we get

\[ u(x, t) \geq \frac{1}{2} \int D(x, t) \frac{|u(y, s)|}{(1 + s)^p} dy ds \in \mathbb{R} \times [0, \infty). \]  

**Proof of (38).** Let \((x, t) \in \Sigma_0\). Define

\[ D_0(x, t) = \{(y, s) \in D(x, t) : 1 \leq s - y \leq t - x, \ 0 \leq y \leq t - x - s\}. \]

Replacing the domain of integration by \(D_0(x, t)\) in the integral of (41), and changing the variables by (21), we get

\[ u(x, t) \geq \frac{1}{4} \int_1^{t-x} d\beta \int_{\beta}^{t-x} \frac{|u(y, s)|}{\{1 + (\alpha + \beta)/2\}^{p-1}} d\alpha \in \Sigma_0. \]  

Making use of (37) and \(D_0(x, t) \subset \Sigma_0\) for \((x, t) \in \Sigma_0\), we have

\[ u(x, t) \geq \frac{C_{p, 0}^p}{4} \int_1^{t-x} d\beta \int_{\beta}^{t-x} \frac{d\alpha}{\{1 + (\alpha + \beta)/2\}^{p-1}} \in \Sigma_0. \]  

(43)

It follows from

\[ 1 + \frac{\alpha + \beta}{2} \leq 1 + t - x \leq 2(t - x) \]

for \(\alpha \leq t - x, \ \beta \leq t - x\) and \(t - x \geq 1\), that

\[ u(x, t) \geq \frac{C_{p, 0}^p}{2^{p+1}(t - x)^{p-1}} \int_1^{t-x} (t - x - \beta) d\beta = C_{p, 1}(t - x - 1)^2/(t - x)^{p-1} \in \Sigma_0. \]

Therefore, (38) holds for \(j = 1\).

Assume that (38) holds for some \(j \in \mathbb{N}\). Noticing that \(D_0(x, t) \subset \Sigma_0\) for \((x, t) \in \Sigma_0\) and putting (38) into (42), we have

\[ u(x, t) \geq \frac{C_{p, j}^p}{4} \int_1^{t-x} \frac{(\beta - 1)^2 p_{a_j}}{\beta^{p(p-1)a_j}} d\beta \int_{\beta}^{t-x} \frac{d\alpha}{\{1 + (\alpha + \beta)/2\}^{p-1}} \in \Sigma_0. \]

Analogously to the case of \(j = 1\), we get

\[ u(x, t) \geq \frac{C_{p, j}^p}{2^{p+1}(t - x)^{p-1}(p_{a_j} + 1)} \int_1^{t-x} (\beta - 1)^2 p_{a_j} (t - x - \beta) d\beta \]

\[ \geq \frac{C_{p, j}^p}{2^{p+1}(p_{a_j} + 1)^2(t - x)^{p-1}(p_{a_j} + 1)} \]

in \(\Sigma_0\). Recalling the definition of \(a_j\), we have

\[ p a_j + 1 = a_{j+1} \leq \frac{p_{j+1}}{p - 1}. \]  

(44)

Therefore, making use of (36), we get (38) for all \(j \in \mathbb{N}\).

**Proof of (39).** Let \((x, t) \in \Sigma_1\). Noticing that \(\Sigma_1 \subset \Sigma_0\), we get

\[ u(x, t) \geq \frac{C_{p, 0}^3}{4} \int_1^{t-x} d\beta \int_{\beta}^{t-x} \frac{d\alpha}{\{1 + (\alpha + \beta)/2\}^2} \in \Sigma_1 \]

by (43). Since

\[ 1 + \frac{\alpha + \beta}{2} \leq 1 + \alpha \]  

for \(\beta \leq \alpha\)

we get

\[ u(x, t) \geq \frac{C_{p, 0}^3}{4} \int_1^{t-x} d\beta \int_{\beta}^{t-x} \frac{d\alpha}{(1 + \alpha)^2} \in \Sigma_1. \]
It follows from
\[
\int_{\beta}^{t-x} \frac{d\alpha}{(1+\alpha)^2} = \frac{t-x-\beta}{(1+\beta)(1+t-x)} \geq \frac{1}{2^2} \cdot \frac{t-x-\beta}{(t-x)\beta}
\]
for \( \beta \geq 1 \) and \( t-x \geq l_1 = 3/2 > 1 \), that
\[
u(x,t) \geq c_3^3 \frac{e^{-3}}{2^4(t-x)} \int_{l_1}^{t-x} t-x-\beta \beta d\beta = c_3^3 \frac{e^{-3}}{2^4(t-x)} \int_{l_1}^{t-x} \log \beta d\beta \quad \text{in} \quad \Sigma_1.
\]
Since \( 1 \leq (t-x)/l_1 \) for \( \Sigma_1 \), the \( \beta \)-integral is estimated as follows:
\[
\int_{l_1}^{t-x} \log \beta d\beta \geq \int_{(t-x)/l_1}^{t-x} \log \beta d\beta \geq \left(1 - \frac{1}{l_1}\right) (t-x) \log \left(\frac{t-x}{l_1}\right) = \frac{t-x}{3} \log \left(\frac{t-x}{l_1}\right)
\]
in \( \Sigma_1 \). Hence from (36) we get
\[
u(x,t) \geq C_{3,1} \log \left(\frac{t-x}{l_1}\right) \quad \text{in} \quad \Sigma_1.
\]
Therefore, (39) holds for \( j = 1 \).

Assume that (39) holds for some \( j \in \mathbb{N} \). Let \((x,t) \in \Sigma_{j+1}\). Define
\[
D_j(x,t) = \{(y,s) \in D(x,t) : l_j \leq s-y \leq t-x, \ 0 \leq y \leq t-x-s\}
\]
for \( j \geq 1 \). Replacing the domain of integration in (41) by \( D_j(x,t) \), and making use of (21), we have
\[
u(x,t) \geq \frac{1}{4} \int_{l_j}^{t-x} d\beta \int_{\beta}^{t-x} \frac{|u(y,s)|^3 d\alpha}{\{1+(\alpha+\beta)/2\}^2} \quad \text{in} \quad \Sigma_{j+1}.
\]
Noticing that \( D_j(x,t) \subset \Sigma_j \) for \((x,t) \in \Sigma_{j+1}\) and putting (39) into the integral above, we have
\[
u(x,t) \geq \frac{C_{3,j}^3}{4} \int_{l_j}^{t-x} \left\{ \log \left(\frac{\beta}{l_j}\right) \right\}^{3a_j} d\beta \int_{\beta}^{t-x} \frac{d\alpha}{\{1+(\alpha+\beta)/2\}^2} \quad \text{in} \quad \Sigma_{j+1}.
\]
Analogously to the case of \( j = 1 \), we get
\[
u(x,t) \geq \frac{C_{3,j}^3}{2^4(t-x)} \int_{l_j}^{t-x} \left\{ \log \left(\frac{\beta}{l_j}\right) \right\}^{3a_j} \frac{(t-x-\beta)}{\beta} d\beta \quad \text{in} \quad \Sigma_{j+1}.
\]
Making use of integration by parts in the integral above, we obtain
\[
\int_{l_j}^{t-x} \left\{ \log \left(\frac{\beta}{l_j}\right) \right\}^{3a_j} \frac{(t-x-\beta)}{\beta} d\beta \geq \frac{1}{3a_j+1} \int_{l_j}^{t-x} \left\{ \log \left(\frac{\beta}{l_j}\right) \right\}^{3a_{j+1}} d\beta \\
= \frac{1}{3a_j+1} \int_{(t-x)l_j/l_{j+1}}^{t-x} \left\{ \log \left(\frac{\beta}{l_j}\right) \right\}^{3a_{j+1}} d\beta \\
\geq \frac{1}{3a_j+1} \left(1 - \frac{l_j}{l_{j+1}}\right) (t-x) \left\{ \log \left(\frac{t-x}{l_{j+1}}\right) \right\}^{3a_{j+1}}
\]
in $\Sigma_{j+1}$. By (44) and recalling the definition of $l_j$, given by and (30), we have
\[ 3a_j + 1 = a_{j+1} \leq \frac{3^{j+1}}{2}, \]
\[ 1 - \frac{l_j}{l_{j+1}} = \frac{l_{j+1} - l_j}{l_{j+1}} = \frac{2^{-(j+1)}}{l_{j+1}} \geq 2^{-(j+2)}. \]
Making use of (36), we get
\[ u(x, t) \geq \frac{C_{3,j}^3}{2^3 \cdot 3 \cdot 6^j} \left\{ \log \left( \frac{t - x}{l_{j+1}} \right) \right\}^{a_{j+1}} = C_{3,j}^{a_{j+1}} \left\{ \log \left( \frac{t - x}{l_{j+1}} \right) \right\}^{a_{j+1}} \]
in $\Sigma_{j+1}$. Therefore, (39) holds for all $j \in \mathbb{N}$.

The proof of Proposition 4.1 is now completed. \qed

End of the proof of Theorem 1.2. Let $u \in C([0, T])$ be the solution of the integral equation (28). Setting $S = \lim_{j \to \infty} S_j$, we see from (35) that $S_j \leq S$ for all $j \in \mathbb{N}$. Therefore, (31) yields
\[ C_{p,j} \geq \exp\left\{ p^{j-1} \left( \log(C_{p,1}F_p^{-S}E_p^{1/(p-1)}) \right) - \log(E_p^{1/(p-1)}) \right\} \]
\[ = E_p^{-1/(p-1)} \exp\left\{ p^{j-1} \left( \log(C_{p,1}F_p^{-S}E_p^{1/(p-1)}) \right) \right\}. \] (45)

(i) Upper bound of the lifespan in the case of $1 < p < 3$.

We take $\epsilon_1 = \epsilon_1(g, p) > 0$ so small that
\[ B_1 \epsilon_1^{-(p-1)/(3-p)} \geq 4, \]
where we set
\[ B_1 = (k_pC_0^{p(p-5)/(p-1)}F_p^{-S}E_p^{1/(p-1)})^{(p-1)/(p(3-p))} > 0. \]

Next, for a fixed $\epsilon \in (0, \epsilon_1]$, we suppose that $T$ satisfies
\[ T > B_1 \epsilon^{-(p-1)/(3-p)} (\geq 4). \] (46)

Combining (45) with (38), we have
\[ u(x, t) \geq E_p^{-1/(p-1)} \exp\left\{ p^{j-1} \left( \log(C_{p,1}F_p^{-S}E_p^{1/(p-1)}) \right) \right\} \]
\[ \times \left\{ \frac{(t - x - 1)^2}{(t - x)^{(p-1)}} \right\}^{(p-1)/(p-1)} \]
in $\Sigma_0$. Let $(x, t) = (t/2, t)$ for $t \in [4, T]$. Then $(x, t) \in \Sigma_0$ and $t - x - 1 \geq (t - x)/2$.

Hence we get
\[ u(t/2, t) \geq (2^{p-5}E_p)^{-1/(p-1)} \exp\left\{ p^{j-1} \left( \log(C_{p,1}F_p^{-S}E_p^{1/(p-1)}) \right) \right\} \]
\[ \times t^{(3-p)/(p-1)} \]
\[ = (2^{p-5}E_p)^{-1/(p-1)} \exp\left\{ p^{j-1}H_1(t) t^{-(3-p)/(p-1)} \right\} \]
for $t \in [4, T]$, where we set
\[ H_1(t) = \log \left( \epsilon^{p}k_pC_0^{p(p-5)/(p-1)}F_p^{-S}E_p^{1/(p-1)}t^{p(3-p)/(p-1)} \right). \]

By (46) and the definition of $B_1$, we have $H_1(T) > 0$. Therefore, we get
\[ u(T/2, T) \to \infty \text{ as } j \to \infty. \]

Hence, (46) implies that $T_\epsilon \leq B_1 \epsilon^{-(p-1)/(3-p)}$ for $0 < \epsilon \leq \epsilon_1$.

(ii) Upper bound of the lifespan in the case of $p = 3$.

We take $\epsilon_2 = \epsilon_2(g) > 0$ so small that
\[ B_2 \epsilon_2^{-2} \geq \log 4, \]
where we set
\[ B_2 = (c_0^3 k_3 e^{-S} E_3^{1/2})^{-2/3} > 0. \]

Next, for a fixed \( \varepsilon \in (0, \varepsilon_2] \), we suppose that \( T \) satisfies
\[ T > \exp\{2B_2 \varepsilon^{-2}\} (> 4). \]
Combining (45) with (39), we have
\[ u(x, t) \geq E_3^{-1/2} \exp\{3^{j-1} \{\log(\varepsilon^3 c_0^3 k_3 e^{-S} E_3^{1/2})\}\} \left\{ \log \left( \frac{t-x}{j} \right) \right\}^{(3j-1)/2} \]
in \( \Sigma_j \). Now, note that \( (t/2, t) \in \Sigma_\infty \) for \( t \in [4, T] \), where \( \Sigma_\infty \) is defined in (29). Since \( l_j < 2 \) for \( j \geq 1 \), we get \( (t-x)/l_j > (t-x)/2 \). Hence we obtain
\[ u(t/2, t) \geq E_3^{-1/2} \exp\{3^{j-1} \{\log(\varepsilon^3 c_0^3 k_3 e^{-S} E_3^{1/2})\}\} \left\{ \log \left( \frac{t}{4} \right) \right\}^{(3j-1)/2} \]
for \( t \in [4, T] \), where we set
\[ H_2(t) = \log \left\{ e^3 c_0^3 k_3 e^{-S} E_3^{1/2} \left\{ \log \left( \frac{t}{4} \right) \right\}^{3/2} \right\}. \]
By (47) and the definition of \( B_2 \), we have \( H_2(T) > 0 \). Therefore, we get \( u(T/2, T) \to \infty \) as \( j \to \infty \). Hence, (47) implies that \( T_\varepsilon \leq \exp\{2B_2 \varepsilon^{-2}\} \) for \( 0 < \varepsilon \leq \varepsilon_2 \). Therefore, the proof of Theorem 1.2 is now completed.  

5. Proof of Theorem 1.3. In this section, we prove Theorem 1.3. First of all, we define the following weighted \( L^\infty \) space. For \( \gamma > 0 \) and \( 0 < T \leq \infty \), we define
\[ X_\gamma := \{ U \in C([0, \infty) \times [0, T]) : \| U \|_{X_\gamma} < \infty \}, \]
\[ \| U \|_{X_\gamma} = \sup_{(x,t) \in [0,\infty) \times [0,T)} w_\gamma(x,t) |U(x,t)|, \]
where
\[ w_\gamma(x,t) = \frac{(1+x+t)(1+|x-t|)^\gamma}{1+t} \]
for \( (x,t) \in [0,\infty)^2 \).

Next we prepare the following lemmas which play a key role in the proof of Theorem 1.3.

**Lemma 5.1.** Let \( \kappa > 0 \). Assume that \( (f,g) \in Y_\kappa \) and \( f, g \) are odd functions. Then there exists a positive constant \( C_\kappa \) such that
\[ \| u^0 \|_{X_\kappa} \leq C_\kappa \| (f,g) \|_{Y_\kappa}, \]
where \( u^0 \) is defined in (13).

**Proof.** Noticing that \( f \) and \( g \) are odd functions, and making use of Lemma 2.1, we get
\[ \left| \int_{x-t}^{x+t} \{ f(y) + g(y) \} dy \right| \leq \int_{|x-t|}^{x+t} \| (f,g) \|_{Y_\kappa} dy \]
\[ \leq C_\kappa w_\kappa(x,t)^{-1} \| (f,g) \|_{Y_\kappa} \]
for \( (x,t) \in [0,\infty)^2 \).
Lemma 5.2. Suppose $H$ is an odd functions, we have

$$f(x+t) + f(x-t) = f(x+t) - f(t-x) = \int_{t-x}^{t+x} f'(y)dy.$$  

Analogously to the above, we get

$$|f(x+t) + f(x-t)| \leq C_\kappa \|(f,0)\|_{Y_\kappa w_\kappa(x,t)}^{-1}.$$  

Next we consider the case of $0 \leq t \leq x$. Then we have

$$\frac{1}{1+x-t} \leq \frac{2(1+t)}{1+x+t}.$$  

Hence, we get

$$|f(x+t) + f(x-t)| \leq (1 + x + t)^{-(1+\kappa)}\|(f,0)\|_{Y_\kappa} + (1 + x - t)^{-(1+\kappa)}\|(f,0)\|_{Y_\kappa}.$$  

Thus we find (50) is valid via (13).  

**Lemma 5.2.** Suppose $p > 2$, $(f,g) \in Y_\kappa$ with $\kappa > \max\{1/p,(3-p)/(p-1)\}$, $f$, $g$ are odd functions and $U \in X_{p-2}$. Then there exist positive constants $C_{p,\kappa}$ and $C_p$ such that

$$\|\tilde{L}(u_0^p)\|_{X_{p-2}} \leq C_{p,\kappa}\|u_0^p\|_{X_\kappa},$$  \hspace{1cm} (51)

$$\|\tilde{L}(u_0^p - U)\|_{X_{p-2}} \leq C_{p,\kappa}\|\kappa_\kappa^p - U\|_{X_{p-2}},$$  \hspace{1cm} (52)

$$\|\tilde{L}(U^p)\|_{X_{p-2}} \leq C_pE_p(T)\|U\|_{X_{p-2}},$$  \hspace{1cm} (53)

where $E_p(T)$ is the one in (19), $\tilde{L}$ is the one in (17), and $u^0$ is the one in (13).  

**Proof.** We note that $1/p > (3-p)/(p-1)$ if $p > 1 + \sqrt{2}$, and $(3-p)/(p-1) \geq 1/p$ if $p \leq 1 + \sqrt{2}$. First, we consider (51). It follows from (49) that

$$|\tilde{L}(u_0^p(x,t))| \leq C_p\|u_0^p\|_{X_\kappa} \int_{D(x,t)} \int_0^{(1+s)dyds} (1+s)^p(1+|s-y|)^{\kappa}$$

for $(x,t) \in [0,\infty)^2$. Making use of Lemma 2.2 with $\sigma = pk > 1$, we have

$$|\tilde{L}(u_0^p(x,t))| \leq C_{p,\kappa}\|u_0^p\|_{X_{p-2}} w_{p-2}(x,t)^{-1}$$

for $(x,t) \in [0,\infty)^2$. This gives us (51).  

Next, we shall show (52). It follows from (49) that

$$|\tilde{L}(u_0^p - U)(x,t)| \leq C_{p,\kappa}\|u_0^p - U\|_{X_{p-2}} \int_{D(x,t)} \int_0^{(1+s)dyds} (1+s)^p(1+|s-y|)^{\kappa(p-1)+p-2}$$

for $(x,t) \in [0,\infty)^2$. Making use of Lemma 2.2 with $\sigma = \kappa(p-1) + p - 2 > 1$, we have

$$|\tilde{L}(u_0^p - U)(x,t)| \leq C_{p,\kappa}\|u_0^p - U\|_{X_{p-2}} w_{p-2}(x,t)^{-1}$$

for $(x,t) \in [0,\infty)^2$. This gives us (52).  

Finally, we shall show (53). Analogously to the above, we get

$$|\tilde{L}(U^p)| \leq C_p\|U\|_{X_{p-2}} \int_{D(x,t)} \int_0^{(1+s)dyds} (1+s)^p(1+|s-y|)^{p(p-2)}$$

for $(x,t) \in [0,\infty)^2$. Making use of Lemma 2.2 with $\sigma = p(p-2) > 0$, we obtain (53). This completes the proof.  

\qed
Now, we move on to the proof of Theorem 1.3. In what follows, we consider the following integral equation:

\[ U = \tilde{L}\{(U + U^0)^p(U + U^0)\} \quad \text{in } [0, \infty) \times [0, T), \]

where we set \( U^0 = \epsilon u^0 \), with \( u^0 \) being the one in (13). If \( U \in C([0, \infty) \times [0, T)) \) is the solution of (54), then \( u := U + U^0 \) satisfies (16). Since \( U^0 \) exists globally in time, it suffices to examine the lifespan of \( U \).

We shall construct a solution of the integral equation (54) in \( X_{p-2} \) with \( p > 2 \) under suitable assumption on \( T \) such as (57) below. Define a sequence of functions \( \{U_n\}_{n \in \mathbb{N}} \subset X_{p-2} \) by

\[ U_n = \tilde{L}\{(U_{n-1} + U_0)^p(U_{n-1} + U_0)\}, \quad U_0 = U^0 = \epsilon u^0, \quad (55) \]

We take \( \epsilon_0 = \varepsilon_0(f, g, p, \kappa) > 0 \) so small that

\[ p2^{p+3}C_p\kappa^{-1}p^{-1}\|f|g\|^p_{X_p} \epsilon_0 \leq 1. \quad (56) \]

For a fixed \( \varepsilon \in (0, \varepsilon_0) \), we take \( T > 0 \) such that

\[ 2^{p+2+p}C_pC_{p, \kappa}^{-1}C_{p, \kappa}^{-1}p^{-1}\|f|g\|^p_{X_p} \varepsilon E_{p(p-2)}(T) \varepsilon^{p(p-1)} \leq 1. \quad (57) \]

Analogously to the proof of Theorem 1.2 in [6] (see p.16 in [6]), we see from Lemma 5.1 and Lemma 5.2 that \( \{U_n\}_{n \geq 2} \) is a Cauchy sequence in \( X_{p-2} \), provided (56) and (57) hold. Since \( X_{p-2} \) is complete, there exists \( U \in X \) such that \( \{U_n\}_{n \in \mathbb{N}} \) converges to \( U \) in \( X_{p-2} \). Therefore \( U \) satisfies the integral equation (54). Hence, the proof of Theorem 1.3 is completed.

\[ \square \]

**Remark 5.1.** Analogously to the proof of Theorem 1.3, one can get a lower bound of the lifespan in the case of \( 1 < p \leq 2 \). In fact, we have only to change the weight function by

\[
w(x, t) = \begin{cases} (1 + x + t)^{p-1} & \text{if } 1 < p < 2, \\ 1 + t & \text{if } p = 2, \end{cases}
\]

for \( (x, t) \in [0, \infty)^2 \).

6. **Proof of Theorem 1.4.** In this section, we prove Theorem 1.4. We show that the solution of (16) blows up in finite time. First of all, we prepare some definitions and lemmas. For \( T > 0 \) and \( \delta \in (0, 1) \), we define the following domains:

\[
\Gamma = \{ (x, t) \in [0, \infty)^2 : 0 \leq x - t \leq \delta/2, \ x + t \geq \delta \}, \\
\sum_2 = \{ (x, t) \in [0, \infty) \times [0, T] : x \geq t - x \geq \tilde{l}_j \} \ (j = 1, 2, \ldots), \\
\sum_\infty = \{ (x, t) \in [0, \infty) \times [0, T] : x \geq t - x \geq 2 \},
\]

where

\[
\begin{aligned}
\tilde{l}_1 &= 1, \\
\tilde{l}_j &= \tilde{l}_1 + \sum_{k=1}^{j-1} 2^{-k} = 2 - \frac{1}{2^{j-1}} & \text{for } j \geq 2.
\end{aligned}
\]

Let \( p > 1, \ c_1 > 0 \), and let us define a sequence \( \{C_{p, j}\}_{j=1}^\infty \) by

\[
\begin{aligned}
C_{p, j} &= \exp\{p^{-1}(\log(C_{p, 1} F_p - S_j E_p^{-1/(p-1)})) - \log E_p^{1/(p-1)}\} \ (j \geq 2), \\
C_{p, 1} &= c_1 F_p c_p,
\end{aligned}
\]
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where $S_j$ is the one in (35), and we put

$$
\widetilde{E}_p = \begin{cases} 
\frac{(p-1)^2}{2^{2p-1}(p+1)^2}, & \text{if } 1 < p < 1 + \sqrt{2}, \\
\frac{p-1}{2^{3p-1}}, & \text{if } p = 1 + \sqrt{2}, 
\end{cases}
$$

(61)

$$
\widetilde{F}_p = \begin{cases} 
p^2, & \text{if } 1 < p < 1 + \sqrt{2}, \\
2p, & \text{if } p = 1 + \sqrt{2}, 
\end{cases}
$$

(62)

$$
\widetilde{k}_p = \delta \cdot 2^{p-45^{-p-1}}.
$$

(63)

We note that $\{\tilde{C}_{p,j}\}_{j=1}^{\infty}$ satisfies the following relation:

$$
\tilde{C}_{p,j+1} = \frac{\tilde{C}_{p,j} \cdot \widetilde{E}_p \cdot \tilde{F}_j}{\tilde{F}_j} (j \in \mathbb{N}).
$$

(64)

The following lemma shows the positivity of the solution of (16).

**Lemma 6.1.** Suppose that the assumptions in Theorem 1.4 are fulfilled. Let $u \in C([0, \infty)^2)$ be the solution of (16). Then, $u$ satisfies

$$
u(x, t) > 0 \quad \text{for } (x, t) \in (0, \infty)^2.
$$

**Proof.** Fix an arbitrarily point $(x_0, t_0) \in (0, \infty)^2$. If we set

$$
t_1 = \inf \{ t > 0 : \ u(x, t) = 0, \ \text{with } (x, t) \in \overline{D}(x_0, t_0) \},
$$

then we have $t_1 > 0$ by $u_1(x, 0) = \varepsilon \{ f(x) + g(x) \} > 0$ for all $x \in (0, \infty)$, $u_2(0, 0) = f'(0) > 0$ and the continuity of $f$, $f'$, $g$ and $u$. Assume that there exists a point $(x_1, t_1) \in \overline{D}(x_0, t_0)$ with $x_1 > 0$ such that $u(x_1, t_1) = 0$. By the definition of $t_1$, we get $u > 0$ in $\overline{D}(x_1, t_1)$. Hence, we have

$$
u(x_1, t_1) \geq \varepsilon u^0(x_1, t_1) > 0.
$$

However this is a contradiction to the definition of $t_1$. Therefore we have $u > 0$ in $\overline{D}(x_0, t_0)$, so that we get $u > 0$ in $(0, \infty)^2$.

Next, we derive a lower bound of the solution to (16) which is a first step of our iteration argument.

**Lemma 6.2.** Suppose that the assumptions in Theorem 1.4 are fulfilled. Let $u \in C([0, \infty) \times [0, T])$ be the solution of (16). Then, $u$ satisfies

$$
u(x, t) \geq \varepsilon c_1 \quad \text{for } (x, t) \in \Gamma,
$$

(65)

where $c_1 = \frac{1}{2} \int_{t/2}^{t} \{ f(y) + g(y) \} dy > 0$.

**Proof.** Making use of Lemma 6.1 and (11), we get

$$
u(x, t) \geq \frac{\varepsilon}{2} \int_{x-t}^{x+t} \{ f(y) + g(y) \} dy \geq \varepsilon c_1 \quad \text{for } (x, t) \in \Gamma.
$$

Hence, we obtain (65). This completes the proof.

Our iteration argument will be done by using the following estimates.
Proposition 6.1. Suppose that the assumptions in Theorem 1.4 are fulfilled. Let \( j \in \mathbb{N} \) and let \( u \in C([0, \infty) \times [0, T]) \) be the solution of (16). Then, \( u \) satisfies

\[
\begin{align*}
\alpha & \geq C_{p,j} \frac{(t-x-1)^{b_j}}{(t-x)^{d_j}} & \text{if } & 1 < p < 1 + \sqrt{2} \tag{66}
\end{align*}
\]

for \((x, t) \in \overline{\Sigma}_1\), and

\[
\begin{align*}
\alpha & \geq \frac{C_{p,j}}{(t-x)^{p-2}} \left\{ \log \left( \frac{t-x}{l_j} \right) \right\} f_j & \text{if } & p = 1 + \sqrt{2} \tag{67}
\end{align*}
\]

for \((x, t) \in \overline{\Sigma}_j\), where \( \overline{\Sigma}_j \) is defined in (58). Here \( C_{p,j} \) is the one in (60) with \( c_1 = \frac{1}{2} \int_{\delta/2}^{\delta} (f(y) + g(y)) \, dy > 0 \) and \( b_j, d_j \) and \( f_j \) are defined by

\[
\begin{align*}
b_j & = \frac{\{p^{j-1}(p+1) - 2\}}{p-1} & (j \in \mathbb{N}), \\
d_j & = p^j - 1 & (j \in \mathbb{N}), \\
f_j & = \frac{p^{j-1} - 1}{p-1} & (j \in \mathbb{N}).
\end{align*}
\]

Proof. From (16), (11) and Lemma 6.1, we get

\[
\begin{align*}
u(x,t) & \geq \frac{1}{2} \int \frac{u(y,s)^p}{(1+s)^{p-1}} \, dyds & \text{in } [0, \infty)^2.
\end{align*}
\]

First, we show (66) and (67) in the case of \( j = 1 \). Let \((x, t) \in \overline{\Sigma}_1\). Define

\[
\bar{\Gamma}(x,t) = \{(y,s) \in \bar{D}(x,t) : -\delta/2 \leq s-y \leq 0, \ t-x \leq y \leq 3(t-x) - s\}.
\]

Replacing the domain of integration by \( \bar{\Gamma}(x,t) \) and making use of (21), we get

\[
\begin{align*}
u(x,t) & \geq \frac{1}{4} \int_{-\delta/2}^{0} d\beta \int_{2(t-x)+\beta}^{3(t-x)} \frac{u(y,s)^p}{(1+(\alpha + \beta)/2)^{p-1}} \, dy \, d\alpha & \text{in } \overline{\Sigma}_1.
\end{align*}
\]

Since \( \bar{\Gamma}(x,t) \subset \Gamma \) for \((x, t) \in \overline{\Sigma}_1\), we can use (65) and get

\[
\begin{align*}
u(x,t) & \geq \frac{c_1^p \epsilon^p}{4} \int_{-\delta/2}^{0} d\beta \int_{2(t-x)+\beta}^{3(t-x)} \frac{d\alpha}{(1+(\alpha + \beta)/2)^{p-1}} & \text{in } \overline{\Sigma}_1.
\end{align*}
\]

It follows from

\[
1 + \frac{\alpha + \beta}{2} \leq 1 + \frac{3(t-x)}{2} \leq \frac{5(t-x)}{2}
\]

for \( \alpha \leq 3(t-x), \, \beta \leq 0 \) and \( t-x \geq 1 \), that

\[
u(x,t) \geq \frac{c_1^p \epsilon^p}{2^{p-2} 5^{p-3}(t-x)^{p-1}} \int_{-\delta/2}^{0} (t-x-\beta) d\beta \geq \frac{C_{p,1}(t-x)}{(t-x)^{p-1}} & \text{in } \overline{\Sigma}_1.
\]

Hence, we see that (66) and (67) hold for \( j = 1 \).

(i) Proof of (66).

Assume that (66) holds for some \( j \in \mathbb{N} \). Define

\[
\bar{D}_1(x,t) = \{(y,s) \in \bar{D}(x,t) : 1 \leq s-y \leq t-x, \ t-x \leq y \leq 3(t-x) - s\}.
\]
Replacing the domain of integration in (71) by \( \tilde{D}_1(x, t) \) and making use of (21), we have

\[
\begin{align*}
u(x, t) & \geq \frac{1}{4} \int_{\Sigma_1}^{t-x} \frac{u(y, s)^p d\alpha}{\beta} \int_{2(t-x)+\beta}^{3(t-x)} \{1 + \frac{\alpha + \beta}{2}\}^{p-1} \quad \text{in } \tilde{\Sigma}_1. 
\end{align*}
\]

(73)

Noticing that \( \tilde{D}_1(x, t) \subset \tilde{\Sigma}_1 \) for \( (x, t) \in \tilde{\Sigma}_1 \) and putting (66) into (73), we have

\[
u(x, t) \geq \frac{C_{p,j}}{4} \int_{\Sigma_1}^{t-x} \frac{(\beta - 1)^{pb_j} (t - x - \beta) d\beta}{\beta^{p\cdot j}} \int_{2(t-x)+\beta}^{3(t-x)} \{1 + \frac{\alpha + \beta}{2}\}^{p-1} \quad \text{in } \tilde{\Sigma}_1.
\]

It follows from

\[
1 + \frac{\alpha + \beta}{2} \leq 1 + 2(t - x) \leq 3(t - x)
\]

for \( \alpha \leq 3(t - x), \beta \leq t - x \) and \( t - x \geq 1 \), that

\[
u(x, t) \geq \frac{C_{p,j}}{3^{p-1}4(t-x)^{p\cdot j}+p-1} \int_{\Sigma_1}^{t-x} (\beta - 1)^{pb_j} (t - x - \beta) d\beta
\]

\[
= \frac{C_{p,j}}{3^{p-1}4(pb_j + 2)^2(t-x)^{p\cdot j}+p-1}
\]

in \( \tilde{\Sigma}_1 \). Recalling the definitions of \( b_j \) and \( d_j \), we have

\[
pb_j + 2 = b_{j+1} \leq \frac{p(p + 1)}{p - 1}, \quad d_{j+1} = pd_j + p - 1.
\]

(75)

Making use of (64) and the definition of \( d_j \), we get

\[
u(x, t) \geq \frac{C_{p,j}}{2^23^{p-1}(p + 1)^2p^{2j}} \frac{(t - x - 1)^{b_{j+1}}}{(t - x)^{d_{j+1}}} = \frac{C_{p,j+1}}{(t - x)^{d_{j+1}}}
\]

in \( \tilde{\Sigma}_1 \). Therefore, (66) holds for all \( j \in \mathbb{N} \).

(ii) Proof of (67). Assume that (67) holds for some \( j \in \mathbb{N} \). Let \( (x, t) \in \tilde{\Sigma}_j \). Define

\[
\tilde{D}_j(x, t) = \{(y, s) \in D(x, t) : \tilde{l}_j \leq s - y \leq t - x, \ t - x \leq y \leq 3(t - x) - s \}
\]

for \( j \geq 1 \). Replacing the domain of integration in (71) by \( \tilde{D}_j(x, t) \) and making use of (21), we have

\[
u(x, t) \geq \frac{1}{4} \int_{\tilde{I}_j}^{t-x} d\beta \int_{2(t-x)+\beta}^{3(t-x)} \frac{u(y, s)^p d\alpha}{\beta^{p\cdot j} \{1 + \frac{\alpha + \beta}{2}\}^{p-1}} \quad \text{in } \tilde{\Sigma}_j.
\]

Noticing that \( \tilde{D}_j(x, t) \subset \tilde{\Sigma}_j \) for \( (x, t) \in \tilde{\Sigma}_j \) and putting (67) into the integral above, we have

\[
u(x, t) \geq \frac{C_{p,j}}{4} \int_{\tilde{I}_j}^{t-x} \left\{ \log \left( \frac{\beta}{l_j} \right) \right\}^{pb_j} \beta^{-p(p-2)} d\beta \int_{2(t-x)+\beta}^{3(t-x)} \frac{d\alpha}{\{1 + \frac{\alpha + \beta}{2}\}^{p-1}}
\]

in \( \tilde{\Sigma}_{j+1} \). Noticing that \( p(p - 2) = 1 \) and (74), we get

\[
u(x, t) \geq \frac{C_{p,j}}{3^{p-1}4(t-x)^{p-1}} \int_{\tilde{I}_j}^{t-x} \left\{ \log \left( \frac{\beta}{l_j} \right) \right\}^{pb_j} \frac{(t - x - \beta)}{\beta} d\beta
\]
in $\Sigma_{j+1}$. Making use of integration by parts in the integral above, we obtain
\[
\begin{align*}
\int_{l_j}^{t-x} \left\{ \log \left( \frac{\beta}{l_j} \right) \right\}^{p f_j} \frac{(t-x-\beta)}{\beta} d\beta \\
= \frac{1}{p f_j + 1} \int_{l_j}^{t-x} \left\{ \log \left( \frac{\beta}{l_j} \right) \right\}^{p f_j + 1} d\beta \\
\geq \frac{1}{p f_j + 1} \int_{(t-x)l_j/l_{j+1}}^{t-x} \left\{ \log \left( \frac{\beta}{l_j} \right) \right\}^{p f_j + 1} d\beta \\
\geq \frac{1}{p f_j + 1} \left( 1 - \frac{l_j}{l_{j+1}} \right) (t-x) \left\{ \log \left( \frac{t-x}{l_{j+1}} \right) \right\}^{p f_j + 1}
\end{align*}
\]
in $\Sigma_{j+1}$. Recalling the definition of $f_j$ and $l_j$, given by (70) and (59), we have
\[
p f_j + 1 = f_{j+1} \leq \frac{p^j - 1}{p - 1},
\]
\[
1 - \frac{l_j}{l_{j+1}} = \frac{l_{j+1} - l_j}{l_{j+1}} = 2^{-j} \geq 2^{-(j+1)}.
\]
Hence, we get
\[
u(x, t) \geq \frac{C_{p,j}^p (p-1)}{2^j \cdot 3^{p-1} \cdot (2p) \cdot (t-x)^{p-2}} \left\{ \log \left( \frac{t-x}{l_{j+1}} \right) \right\}^{f_{j+1}} \\
= \frac{C_{p,j+1}^{p+1}}{(t-x)^{p-2}} \left\{ \log \left( \frac{t-x}{l_{j+1}} \right) \right\}^{f_{j+1}}
\]
in $\Sigma_{j+1}$, by (64). Therefore, (67) holds for all $j \in \mathbb{N}$. The proof of Proposition 6.1 is now completed.

End of the proof of Theorem 1.4. Let $u \in C([0, \infty) \times [0, T])$ be the solution of the integral equation (16).

(i) Upper bound of the lifespan in the case of $1 < p < 1 + \sqrt{2}$.

We take $\varepsilon_3 = \varepsilon_3(f, g, p) > 0$ so small that
\[
B_3 \varepsilon_3^{-(p-1)/(1+2p-p^2)} \geq 4,
\]
where we set
\[
B_3 = \left( k_p \cdot \sigma_2 \cdot (2+3p-p^2)/(p-1) \cdot \frac{S^{-1/(p-1)}}{F_p} \cdot E_p \right)^{-1/(p-1)}/(1+2p-p^2) > 0.
\]
Next, for a fixed $\varepsilon \in (0, \varepsilon_3]$, we suppose that $T$ satisfies
\[
T > B_3 \varepsilon^{-(p-1)/(1+2p-p^2)} (\geq 4).
\]
Analogously to the proof of Theorem 1.2, we get $u(T/2, T) \to \infty$ as $j \to \infty$ by (76) and the definition of $B_3$. Hence, (76) implies that $T_\varepsilon \leq B_3 \varepsilon^{-(p-1)/(1+2p-p^2)}$ for $0 < \varepsilon \leq \varepsilon_3$.

(ii) Upper bound of the lifespan in the case of $p = 1 + \sqrt{2}$.

We take $\varepsilon_4 = \varepsilon_4(f, g, p) > 0$ so small that
\[
B_4 \varepsilon_4^{-(p-1)} \geq \log 4,
\]
where we set

\[ B_4 = (e^{p_k \tilde{F}_p} \tilde{E}_1)^{1/(p-1)} > 0. \]

Next, for a fixed \( \varepsilon \in (0, \varepsilon_4] \), we suppose that \( T \) satisfies

\[ T > \exp\{2B_4 \varepsilon^{-p(p-1)}\} (> 4). \]

Analogously to the proof of Theorem 1.2, we get \( u(T/2, T) \to \infty \) as \( j \to \infty \) by (77) and the definition of \( B_4 \). Hence, (77) implies that \( T \leq \exp\{2B_4 \varepsilon^{-p(p-1)}\} \) for \( 0 < \varepsilon \leq \varepsilon_4 \). Therefore, the proof of Theorem 1.4 is now completed.

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