ON THE INTERPRETATION OF $[D, a]$ AS AN INFINITE MATRIX

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Abstract. For a self-adjoint operator $D$ on a Hilbert space $H$ and a bounded operator $a$ on $H$ we define the commutator $[D, a]$ as an infinite matrix of bounded operators. This interpretation puts $[D, a]$ into an algebraic setting where products and higher derivatives like $[D, [D, \ldots [D, a] \ldots ]]$ will make sense as infinite matrices of bounded operators. We show that $[D, a]$ is the matrix of a bounded operator, if and only the function $t \to e^{itD}ae^{-itD}$ is Lipschitz continuous at $t = 0$. The set of those operators is a Banach $^*$-algebra under the norm $\|a\| + \|[D, a]\|$. By an elementary example we show that the function $(e^{itD}ae^{-itD} - a)/t$ may be bounded and without a norm limit for $t \to 0$.

1. Introduction

In mathematical physics and especially in quantum mechanics commutators between operators on Hilbert space have played a fundamental role for about 100 years. An expression like $i[H, a]$, where $H$ is an unbounded self-adjoint operator and $a$ is an operator representing an observable appears in the study of the time evolution of the observable.

Since the appearance of the papers [K] and [S] by Kadison and Sakai respectively, the research on bounded and unbounded derivations on C*-algebras [KR] took off, and an impressing amount of research have been published. As far as we know the article [PS], was the first which dealt with unbounded derivations on C*-algebras.

It is desirable to be able to perform commutators like $[D, a]$ inside the mathematical discipline noncommutative geometry [AC]. The reason is that in Connes' set up of noncommutative geometry the space is replaced by an algebra of operators on a Hilbert space. In the commutative case these operators may be multiplication operators induced by smooth functions on a manifold. In the general case they are just operators and the geometry is encoded via an unbounded self-adjoint operator $D$, called the Dirac operator. The smooth bounded functions are then modeled as those operators for which the commutator

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\([D, a]\) is densely defined and bounded, and also for the numerical value \(|D| = (D^2)^{1/2}\) all the commutators \([|D|, [|D|, \ldots, [|D|, a|\ldots]]]\) yield densely defined bounded operators.

The main idea in this article is to provide a framework inside which a commutator \([D, a]\) between an unbounded self-adjoint operator \(D\) on a Hilbert space \(H\) and any bounded operator \(a\) on \(H\) will make sense. In the ordinary setting one wants \([D, a]\) to be an operator, but then the domain of definition is quite often very difficult to control, and it gets worse if one wants to add two commutators of this type. Barry Simon suggests in [BS] that one should look at forms instead. In our case this point of view means, that we may replace the operator theoretic commutator \([D, a]\) by the sesquilinear form, say \(S([D, a])\) defined on the domain \(\text{dom}(D)\) of definition for \(D\) via the following formula

\[
\forall a \in B(H) \forall \xi, \eta \in \text{dom}(D) : S([D, a])(\xi, \eta) := \langle a\xi, D\eta \rangle - \langle aD\xi, \eta \rangle.
\]

This interpretation has the drawback that we do not know how to multiply a sesquilinear form with a bounded operator, so it is not clear how we shall understand expressions like \([D, a]b\) and \(a[D, b]\). Also commutators of the type \([D, [D, [\ldots, [D, a] \ldots]]]\) do not make sense as densely defined sesquilinear forms in an obvious way.

On the other hand it is well known that for a bounded operator \(a\) on \(H\) the operator valued function \(\alpha_t(a) := \text{Ad}(e^{itD})(a) = e^{itD}ae^{-itD}\) may be differentiable in norm, [WR] Theorem 13.35. Let \(\delta\) denote the generator of the one parameter group of automorphisms \(\alpha_t\) on \(B(H)\), then for any operator \(a\) in the domain of definition for \(\delta\) we have the norm limit

\[
\delta(a) = \lim_{t \to 0} (\alpha_t(a) - a)/t = \text{the closure of } i[D, a].
\]

This gives a set of differentiable operators which we will call strongly \(D\)--differentiable. If instead, we take a pair of vectors \(\xi, \eta\) in the domain of definition for \(D\) we see that the limit below exists for any bounded operator \(a\) and also fulfils the identities below.

\[
\lim_{t \to 0} ((\alpha_t(a) - a)\xi, \eta)/(it) = \lim_{t \to 0} (\langle ae^{-itD}\xi, (e^{-itD}\eta - \eta) \rangle + \langle a(e^{-itD}\xi - \xi), \eta \rangle)/(it) = \langle a\xi, D\eta \rangle - \langle aD\xi, \eta \rangle = S([D, a])(\xi, \eta).
\]

If the sesquilinear form \(S([D, a])(\xi, \eta)\) is bounded we say that \(a\) is weakly differentiable with respect to \(D\), and we will use the symbol \(wD(a)\) to denote the bounded operator on \(H\) which implements
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We show by an example that the set of weakly $D$–differentiable operators may be strictly larger than the set of strongly $D$–differentiable operators.

Below we will introduce a linear *-space $\mathcal{M}$ whose elements are infinite matrices of bounded operators. In this space all products may not always be defined, but sometimes they are, and it turns out that there exists an injective *-homomorphism $m : B(H) \to \mathcal{M}$ and also a representative $m(D)$ for $D$ in $\mathcal{M}$, such that the commutator $[m(D), m(a)]$ is well defined inside $\mathcal{M}$. We show that this commutator represents a bounded operator exactly when $a$ is weakly $D$–differentiable and then $m(wD(a)) = [m(D), m(a)]$. The advantage, as we see it, of this representation of the weak $D$–derivatives is that algebraic operations such as $[m(D), m(a)]m(b)$ may be performed even if they make no sense as forms or as operators. Based on this we can define an operator $a$ to be $n$–times weakly differentiable, if all the commutators $[m(D), m(a)]$, $[m(D), [m(D), m(a)]], \ldots [m(D), [m(D), \ldots [m(D), m(a)] \ldots ]$ up to the order $n$ represent bounded operators. We then prove that the space consisting of all $n$–times weakly differentiable operators is a Banach *-algebra under the norm

$$\|\|a\|\|_n := \sum_{k=0}^n \frac{1}{k!}\|wD^k(a)\|.$$  

We link the concept of weak $D$–differentiability to a Lipschitz concept, which has been much studied by Rieffel [MR], and we will show that an operator $a$ in $B(H)$ is weakly $D$–differentiable if and only if, it satisfies the following Lipschitz condition

$$\exists c > 0 \forall t : \|\alpha_t(a) - a\| \leq c|t|.$$  

If $a$ is weakly $D$–differentiable then then the smallest possible constant $c$ equals $\|wD(a)\|$.

In order to understand the difference between weak and strong $D$–differentiability we provide a characterization of those weakly $D$–differentiable operators that are strongly $D$–differentiable, and that result implies that for $n > 1$ weak $D$–differentiability of order $n$ implies strong $D$–differentiability of order $n - 1$.

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2. Results

Given a bounded or unbounded self-adjoint operator $D$ on a Hilbert space $H$, then we will define a sequence of pairwise orthogonal projections $(e_n)_{n \in \mathbb{Z}}$ with strong operator sum $I_{B(H)}$ by letting $e_n$ be the spectral projection for $D$ corresponding to the interval $[n-1,n]$. We will let $\mathcal{M}$ denote the linear space consisting of all infinite matrices $y = (y_{rc})_{r,c \in \mathbb{Z}}$ where $y_{rc}$ is an operator in $B(e_c H, e_r H)$. In this way it is possible to define a linear mapping $m : B(H) \to \mathcal{M}$ by $m(a)_{rc} := e_r a | e_c H$. We will also equip $\mathcal{M}$ with the matrix product, whenever it may be defined and give a bounded operator at all entries. Also $D$ has an interpretation as an element $m(D)$ in $\mathcal{M}$. In order to give this definition we define for each $n$ in $\mathbb{Z}$ a bounded self-adjoint operator $d_n$ in $B(e_n H)$ by $d_n := D | e_n H$. Based on this we can define the representative $m(D)$ for $D$ in $\mathcal{M}$ by

$$m(D)_{rc} = \begin{cases} 0 & \text{if } r \neq c \\ d_c & \text{if } r = c \end{cases}.$$ 

It is quite obvious that for any matrix $y = (y_{rc})$ in $\mathcal{M}$, the commutator $[m(D), y]$ has a meaning, and it is the matrix given by

$$[m(D), y]_{rc} := d_r y_{rc} - y_{rc} d_c.$$ 

Hence for any operator $a$ in $B(H)$ we can define the commutator $[D, a]$ as the infinite matrix of bounded operators given by

$$[D, a]_{rc} = d_r (e_r a | e_c H) - e_r a e_c d_c.$$ 

The space $\mathcal{M}$ will be equipped with a *-operation which is an extension of the classical one defined on bounded matrices, so we define for an element $y = (y_{rc})$ of $\mathcal{M}$ the element $y^*$ as the matrix $(y^*)_{rc} = (y_{cr})^*$. Then it is clear that if $y^*$ is a matrix of a bounded operator, $y$ is the matrix of the adjoint of that bounded operator, so in particular $y$ is also the matrix of a bounded operator.

We will now associate a sesquilinear form to each element in $\mathcal{M}$. This is done in an almost obvious way, and we say so because some simple experiments show that $\text{dom} D$ will not be the right space. Instead we introduce a core $E$ for $D$ which is defined as follows

For any natural number $n$, we define $E_n$ as the orthogonal projection $E_n := \sum_{j=1-n}^{n} e_j$, i.e. it is the spectral projection for $D$ corresponding to the interval $[-n,n]$. Further we define the core $E$ for $D$ by $E := \mathcal{U}_{n=1}^{\infty} E_n H$. The fact that the closure of the restriction $D|E$ equals $D$ follows easily from the next lemma which is trivial, but nice to have in an explicit form.
Lemma 2.1. For any \( \xi \) in \( \text{dom}(D) \):
\[
\lim_{n \to \infty} (\| D\xi - DE_n\xi \| + \| \xi - E_n\xi \|) = 0.
\]

Proof. The projections \( E_n \) converge strongly to \( I_{B(H)} \) so for any vector \( \eta \) in \( H \) we have \( \| \eta - E_n\eta \| \to 0 \) for \( n \to \infty \). Since \( E_n \) is a spectral projection for \( D \) it commutes with \( D \), which here means that \( E_nD \subseteq DE_n \), but \( \xi \) and \( E_n\xi \) are both in the domain of definition for \( D \), so we have \( DE_n\xi = E_nD\xi \) and then \( \| D\xi - DE_n\xi \| = \| D\xi - E_nD\xi \| \to 0 \) for \( n \to \infty \).

□

To define the sesquilinear form for a matrix \( y \) in \( \mathcal{M} \) we will let \( \mathcal{M}_0 \) denote the subset of \( \mathcal{M} \) consisting of matrices \( y = (y_{rc}) \) in \( \mathcal{M} \) where only finitely many entries are non vanishing. It is easy to see that any \( y \) in \( \mathcal{M}_0 \) is the matrix of a bounded operator say \( b \) in \( B(H) \) and for such a \( b \) there exists a natural number \( m \) such that for any natural number \( n \geq m \) we have \( b = E_n b E_n \). Based on this we make the following definition.

Definition 2.2. For any natural number \( n \) and any \( y = (y_{rc}) \) in \( \mathcal{M} \) let \( \pi_n(y) \) denote the bounded operator in \( E_n \) \( B(H) \) \( E_n \) which satisfies
\[
m((\pi_n(y))) = m(\pi(y)) y m(E_n).
\]

Lemma 2.3. Let \( y \) be in \( \mathcal{M} \) and \( \xi, \eta \) be vectors in \( \mathcal{E} \) then there exists a natural number \( m \) such that for any natural number \( n \geq m \) we have
\[
\langle \pi_n(y)\xi, \eta \rangle = \langle \pi_m(y)\xi, \eta \rangle.
\]

Proof. Since \( \xi \) and \( \eta \) both belong to \( \mathcal{E} \) there exists a natural number \( m \) such that \( \xi = E_m\xi \) and \( \eta = E_m\eta \). Hence for any \( n \geq m \) also \( \xi = E_n\xi \) and \( \eta = E_n\eta \), and then
\[
\langle \pi_n(y)\xi, \eta \rangle = \langle \pi_n(y)E_m\xi, E_m\eta \rangle = \langle \pi_m(y)E_m\xi, E_m\eta \rangle = \langle \pi_m(y)\xi, \eta \rangle.
\]

□

Based on the lemma above we define.

Definition 2.4. Let \( y \) be in \( \mathcal{M} \) and \( \xi, \eta \) in \( \mathcal{E} \). The sesquilinear form \( S(y) \) on \( \mathcal{E} \) is defined by
\[
S(y)(\xi, \eta) = \lim_{n \to \infty} \langle \pi_n(y)\xi, \eta \rangle.
\]

If the sesquilinear form is bounded we will let \( \pi(y) \) denote the bounded operator on \( B(H) \) which satisfies
\[
S(y)(\xi, \eta) = \langle \pi(y)\xi, \eta \rangle.
\]

The following theorem connects the weak \( D \)-differentiability of a bounded operator \( a \) in \( B(H) \) with the boundedness of the sesquilinear form \( S([m(D), m(a)]) \).
Theorem 2.5. Let $a$ be in $B(H)$ then $S([m(D), m(a)])$ is bounded if and only if $a$ is weakly $D$–differentiable. If $S([m(D), m(a)])$ is bounded then

$$\forall \xi, \eta \in \mathcal{E}: S([m(D), m(a)])(\xi, \eta) = S([D, m])(\xi, \eta) = \langle wD(a)\xi, \eta \rangle$$

and $\pi([m(D), m(a)]) = wD(a)$.

Proof. The proof is just a matter of book keeping based on the fact that for vectors $\xi, \eta$ in $E_n H$ we know that

$$S([m(D), m(a)])(\xi, \eta) = \langle \pi_n([m(D), m(a)])\xi, \eta \rangle = \langle [DE_n, a]\xi, \eta \rangle$$

$$= \langle a\xi, D\eta \rangle - \langle aD\xi, \eta \rangle = S([D, a])(\xi, \eta).$$

So if $a$ is weakly differentiable, the operator $wD(a)$ implements the bounded form $S([D, a])(\xi, \eta)$, and $\pi([m(D), m(a)]) = wD(a)$. If the form $S([D, a])(\xi, \eta)$ is bounded then the operators $\pi_n([m(D), m(a)])$ form a bounded sequence of operators, which converges strongly to $wD(a)$. □

Remark 2.6. In the rest of this article we will simply assume that $B(H)$ is a sub-algebra of $M$ and then write $[D, a]$ instead of $[m(D), m(a)]$.

We will now introduce a concept, the definition of which is inspired by the concept called Lipschitz continuity of real functions.

Definition 2.7. An operator $a$ in $B(H)$ is said to be $D$–Lipschitz, if

$$\lim_{s \to 0+} \inf_{0 < t < s} \|\alpha_t(a) - a\| / |t| < \infty.$$ 

If the limit exists in $\mathbb{R}$ then we will let $\|a\|_{\text{Lip}}$ denote this number.

We can then link this Lip-norm to our previous result.

Theorem 2.8. An operator $a$ in $B(H)$ is weakly $D$–differentiable if and only if it is $D$–Lipschitz. If it is $D$–Lipschitz then

$$\|wD(a)\| = \|a\|_{\text{Lip}}, \text{ and } \forall t : \|\alpha_t(a) - a\| \leq \|wD(a)\||t|.$$ 

Proof. Suppose $a$ is $D$–Lipschitz, then we can find a decreasing sequence $(t_n)$ of positive reals such that $\lim \|\alpha_{t_n}(a) - a\| / t_n = \|a\|_{\text{Lip}}$. Then the sequence $((\alpha_{t_n}(a) - a) / (it_n))$ is a bounded sequence of operators, so it has an ultraweakly convergent subnet

$$((\alpha_{t_n(\gamma)}(a) - a) / (it_n(\gamma)))_{\gamma \in \Gamma}$$

with an ultraweak limit point, which we denote $b$, and we have $\|b\| \leq \|a\|_{\text{Lip}}$. We will now compute the matrix entries $b_{rc}$ for $b$, so let $\xi$ be in
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Let $\xi, \eta$ in $e_rH$, then

$$\langle b\xi, \eta \rangle = \lim_{\gamma \in \Gamma} \langle (\alpha_{t_n(\gamma)}(a) - a)\xi, \eta \rangle / (it_n(\gamma))$$

$$= \lim_{\gamma \in \Gamma} \langle a(e^{-it_n(\gamma)}D - I)\xi, e^{-it_n(\gamma)}D\eta \rangle / (it_n(\gamma))$$

$$+ \lim_{\gamma \in \Gamma} \langle a\xi, e^{-it_n(\gamma)}D - I)\eta \rangle / (it_n(\gamma))$$

$$= - \langle aD\xi, \eta \rangle + \langle a\xi, D\eta \rangle$$

$$= S([D, a])(\xi, \eta).$$

Hence $S([D, a])$ is bounded, $a$ is weakly $D-$differentiable with $wD(a) = b$ and $\|wD(a)\| = \|b\| \leq \|a\|_{\text{Lip}}$.

Let us now suppose that $a$ is weakly $D-$differentiable, then for $\xi$ and $\eta$ in $E$ we define a function $g(t) := -i\langle (\alpha_t(a) - a)\xi, \eta \rangle$. We know it is differentiable at $t = 0$ with derivative $g'(0) = \langle wD(a)\xi, \eta \rangle$. Since $e^{-itD}\xi$ and $e^{-itD}\eta$ are in $E$ we get that $g(t)$ is differentiable at any $t$ with derivative $g'(t) = \langle wD(a)e^{-itD}\xi, e^{-itD}\eta \rangle$. Since $g(0) = 0$ we get

$$|g(t)| = |\int_0^t g'(s)ds| \leq |t|\|wD(a)\|\||\xi||\|\eta||,$$

and we may conclude that $\|\alpha_t(a) - a\| \leq |t|\|wD(a)\|$. $\square$

The proof of the theorem yields an immediate corollary, which justifies the term weakly $D-$differentiable.

**Corollary 2.9.** An operator $a$ in $B(H)$ is weakly $D-$differentiable if and only if for any pair of vectors $\mu, \nu$ in $H$, the function $f_{\mu\nu}(t) := \langle \alpha_t(a)\mu, \nu \rangle$ is differentiable for $t = 0$.

**Proof.** Suppose $a$ is weakly $D-$differentiable then the proof above shows that

$$\lim_{t \to 0} (1/t)(\alpha_t(a) - a)$$

exists in the ultraweak topology and equals $wD(a)$. In particular

$$\lim_{t \to 0} (1/t)\langle (\alpha_t(a) - a)\mu, \nu \rangle = \langle wD(a)\mu, \nu \rangle$$

and $f_{\mu\nu}(t)$ is differentiable for $t = 0$.

If the function $f'_{\mu\nu}(t) = \langle \alpha_t(a)\mu, \nu \rangle$ is differentiable at 0, then there exists a $\delta > 0$ such that for $0 < |t| < \delta$ we have

$$|f'_{\mu\nu}(0) - (1/t)(f_{\mu\nu}(t) - f_{\mu\nu}(0))| < 1.$$
For $|t| \geq \delta$ we have $|(1/t)(f_{\mu\nu}(t) - f_{\mu\nu}(0))| \leq (1/\delta)2\|a\|\|\mu\|\|\nu\|$, so for each pair $\mu, \nu$ we have

$$\sup_{0 < |t|} |(1/t)(\alpha_t(a) - a)\mu, \nu| < \infty.$$  

Successive applications of the uniform boundedness principle show that

$$\sup_{0 < |t|} \|\alpha_t(a) - a\| < \infty,$$

so $a$ is $D$–Lipschitz and hence weakly $D$–differentiable. □

The result that a weakly $D$-differentiable operator satisfies $\|\alpha_t(a) - a\| \leq |t|\|wD(a)\|$ makes it possible to obtain yet another characterization of weak $D$–differentiability. We will start with a lemma, we expect to be well known, but for which we have no reference for a proof.

**Lemma 2.10.** Let $\xi$ be a vector in $H$. If $\lim_{t \to 0^+, 0 < s < t} \|(e^{isD}\xi - \xi)/s\| < \infty$ then $\xi$ is in $\text{dom}(D)$.  

Proof. The assumption shows that there exists a decreasing sequence $(t_k)$ of positive reals with limit 0, such that the sequence $\eta_k := (e^{it_kD}\xi - \xi)/(it_k)$ is a bounded sequence in $H$. Since $H$ is a Hilbert space this sequence will have a weakly convergent subnet $(\eta_{k(\gamma)})_{\gamma \in \Gamma}$ with weak limit point $\eta$.

Since the projections $E_n$ are spectral projections for $D$ corresponding to bounded intervals we have

$$E_n\eta = \text{w-lim}_{\gamma \in \Gamma} E_{n\eta_{k(\gamma)}} = \text{norm-lim}_{k \in \mathbb{N}} (e^{it_kD}E_n\xi - E_n\xi)/(it_k) = DE_n\xi.$$

Since $E_n\eta$ converges towards $\eta$ and $E_n\xi$ converges towards $\xi$ in norm, we see that $(\xi, \eta)$ is in the closure of the graph of $D$, so $\xi$ is in $\text{dom}(D)$. □

**Theorem 2.11.** If an operator $a$ in $B(H)$ is weakly $D$–differentiable then $\text{dom}D$ is invariant under $a$, and the operator $Da - aD$ is bounded. If a bounded operator $a$ has the property that the domain of definition for $Da - aD$ is a core for $D$ and $Da - aD$ is bounded, then $a$ is weakly $D$–differentiable.

Proof. Suppose $a$ is weakly $D$–differentiable and let $\xi$ be a vector in $\text{dom}(D)$. We will then use the previous theorem and lemma to show that $a\xi$ is in $\text{dom}(D)$.

$$\frac{1}{it}(e^{itD}a - a)\xi = \frac{1}{it}(e^{itD}ae^{-itD} - a)\xi + \frac{1}{it}e^{itD}a(I - e^{-itD})\xi$$
Since $a$ is weakly $D-$differentiable and $\xi$ is in $\text{dom}(D)$ we get
\[
\left\| \frac{1}{it}(e^{itD}ae^{-itD}\xi - a)\xi \right\| \leq \|wD(a)\|\|\xi\|
\]
\[
\lim_{t \to 0^+} \left\| \frac{1}{it}e^{itD}a(I - e^{-itD})\xi \right\| = \|aD\xi\|
\]
By the lemma we see that $a\xi$ is in $\text{dom}(D)$. The previous results show that $wD(a)$ implements the form $S([D,a])(\xi, \eta)$ so we have for $\xi, \eta$ in $\text{dom}(D)$ that
\[
\langle(Da - aD)\xi, \eta \rangle = \langle a\xi, D\eta \rangle - \langle aD\xi, \eta \rangle = S([D,a])(\xi, \eta) = \langle wD(a)\xi, \eta \rangle.
\]
Hence $Da - aD$ is bounded, defined on $\text{dom}(D)$, and its closure is $wD(a)$.

Suppose $Da - aD$ is bounded and defined on a core $\mathcal{D}$ for $D$. Let $b$ denote the bounded operator, which is the closure of $Da - aD$. For any $\xi$ in $\text{dom}(D)$ there exists a sequence $(\xi_n)$ in $\mathcal{D}$ such that the sequence $(\xi_n, D\xi_n)$ converges in norm towards $(\xi, D\xi)$, then
\[
(Da - aD)\xi_n = b\xi_n \to b\xi \text{ for } n \to \infty
\]
\[
da\xi_n \to aD\xi \text{ for } n \to \infty, \text{ so}
\]
\[
(a\xi_n, Da\xi_n) \to (a\xi, b\xi + aD\xi) \text{ for } n \to \infty.
\]
Hence $a\xi$ is in $\text{dom}(D)$ and $(Da - aD)\xi = b\xi$. This implies that the form $S([D,a])(\xi, \eta)$ is bounded on $\text{dom}(D)$ and $a$ is weakly $D-$differentiable with $wD(a) = b$. \hfill $\square$

**Example 2.12.** We will show that a bounded operator $a$ may be weakly $D-$differentiable, but not strongly $D-$differentiable.

We let $\mathbb{T}$ denote the unit circle with normalized Lebesgue measure $(1/(2\pi))m$, $H$ the Hilbert space $L^2(\mathbb{T}, m/(2\pi))$ and $D = (1/i)\frac{d}{d\theta}$ the differentiation with respect to arc length.

We will look at the multiplication operators $M_f$ where $f(\theta)$ is an essentially bounded measurable function, and we assume that it is well known that $\|M_f\| = \|f\|_{\infty}$. The operator $D$ has the standard orthonormal basis $u_n(\theta) = e^{i\theta}$ as its sequence of eigenvectors. The projections $e_n$ becomes the projections onto $\mathbb{C}u_n$, and if we write the commutator $[D,a]$ it has the matrix $[D,a]_{rc} = (r - c)a_{rc}$. Let us then look at the function
\[
|x|(\theta) := \begin{cases} -\theta & \text{for } -\pi \leq \theta < 0 \\ \theta & \text{for } 0 \leq \theta < \pi. \end{cases}
\]
Its derivative except at $-\pi, 0$ and $\pi$ is the sign of $\theta$ and we define a function $\text{sign}(x)(\theta)$ on the circle by

$$\text{sign}(x)(\theta) := \begin{cases} -1 & \text{for } -\pi < \theta < 0 \\ 0 & \text{for } \theta \in \{-\pi, 0, \pi\} \\ 1 & \text{for } 0 < \theta < \pi. \end{cases}$$

The Fourier series for these 2 functions are as follows

$$|x|(\theta) \sim \frac{\pi}{2} + \sum_{k \in \mathbb{Z}} -\frac{2}{\pi(2k+1)^2}u_{2k+1}(\theta)$$
$$\text{sign}(x)(\theta) \sim \sum_{k \in \mathbb{Z}} -\frac{2i}{\pi(2k+1)}u_{2k+1}(\theta)$$

Simple computations show that $M|x|$ is weakly $D-$differentiable with $\text{wD}(M|x|) = (1/i)M_{\text{sign}(x)}$. Since $\alpha_t(M|x|)$ is the multiplication operator given by the function $|x|_t(\theta) = |x|(\theta + t)$ it follows easily that $\|\alpha_t(M|x|) - M|x|| = |t|$, so $M|x|$ is $D-$Lipschitz with $\|M|x||_{\text{Lip}} = 1$. On the other hand it is quite obvious that the limit $\lim_{t \to 0}(|x|_t - |x|)/t$ does not exist in the uniform norm, since the limit function will not be continuous at the points $p\pi$. This means that $M|x|$ is not strongly $D-$differentiable. In connection with the next proposition it is worth to notice that the function $t \to \alpha_t(\text{sign}(x))$ is not norm continuous. A simple computation shows that for $0 < t < \pi$ we have $\|\alpha_t(\text{sign}(x)) - \text{sign}(x)|| = 2$.

Our next result provides a characterization of those weakly $D-$differentiable operators that are strongly $D-$differentiable.

**Theorem 2.13.** Let $a$ be a bounded operator on $H$ then $a$ is strongly $D-$differentiable if and only if it is weakly $D-$differentiable and the function $t \to \alpha_t(\text{wD}(a))$ is norm continuous.

**Proof.** Let us first assume that $a$ is strongly $D-$differentiable, then it is clearly weakly $D-$differentiable. Since $\alpha_t(a)$ is norm differentiable there exist a function $\varepsilon(t)$ with values in $B(H)$ which satisfies $\varepsilon(0) = 0$, is continuous at $0$ and

$$\alpha_t(a) = a + (it)\text{wD}(a) + (it)\varepsilon(t).$$

If we take this identity at $-t$ and apply $\alpha_t$ on both sides we get

$$a = \alpha_t(a) - (it)\alpha_t(\text{wD}(a)) - (it)\alpha_t(\varepsilon(-t)),$$

By adding these equations, rearranging and dividing by $it$ we get

$$\alpha_t(\text{wD}(a)) = \text{wD}(a) = \varepsilon(t) - \alpha_t(\varepsilon(-t))$$
so the function is norm continuous at \( t = 0 \). At a point, say \( s \) in \( \mathbb{R} \), the identity \( \alpha_t(a) = a + (it)wD(a) + (it)\varepsilon(t) \) may be translated to \( s \) by \( \alpha_s \) and it becomes

\[
\alpha_t(\alpha_s(a)) = \alpha_s(a) + (it)\alpha_s(wD(a)) + (it)\alpha_s(\varepsilon(t)),
\]

and we can repeat the argument and show continuity at the point \( s \) also.

Now assume that \( a \) is a weakly \( D \)-differentiable operator satisfying the condition that \( t \to \alpha_t(wD(a)) \) is norm continuous. Define a function \( f : \mathbb{R} \to B(H) \) by

\[
f(t) := a + i \int_0^t \alpha_s(wD(a))ds.
\]

Then \( f(t) \) is norm differentiable everywhere with \( f'(t) = i\alpha_t(wD(a)) \). For any pair of vectors \( \xi, \eta \) from \( \text{dom}(D) \) the scalar valued function \( g(t) := \langle f(t)\xi, \eta \rangle \) is then differentiable everywhere and we have \( g'(t) = i\langle \alpha_t(wD(a))\xi, \eta \rangle \). On the other hand the weak differentiability of \( a \) implies that the function \( h(t) := \langle \alpha_t(a)\xi, \eta \rangle \) is weakly differentiable with derivative \( h'(t) = i\langle \alpha_t(wD(a))\xi, \eta \rangle \). Hence \( g'(t) = h'(t) \) and since we also have \( g(0) = h(0) = i\langle wD(a)\xi, \eta \rangle \) we have \( g(t) = h(t) \) and then \( f(t) = \alpha_t(a) \), so \( a \) is strongly \( D \)-differentiable. \( \square \)

**Corollary 2.14.** If \( a \) in \( B(H) \) is \( n \)-times weakly \( D \)-differentiable, then it is \( (n-1) \)-times strongly \( D \)-differentiable.

**Proof.** Let \( k \) in \( \mathbb{N} \) and \( n \geq k > 1 \), then \( wD^{(k-1)}(a) \) is weakly \( D \)-differentiable and then \( D \)-Lipschitz, so the function \( \alpha_t(wD^{(k-1)}(a)) \) is Lipschitz continuous, and by the proposition \( wD^{(k-1)}(a) \) is strongly \( D \)-differentiable. \( \square \)

We will now introduce a \( C^* \)-algebra \( A \) on \( H \) in order to be able to formulate our results in a frame which may be applied to noncommutative geometry. A \( C^* \)-algebra is a self-adjoint norm closed subalgebra of \( B(H) \) and it may be that \( A \) is just \( B(H) \).

**Definition 2.15.** Let \( A \) be a \( C^* \)-algebra on \( H \) and \( n \) a non negative integer then \( A_{wD}^n \) denotes the linear space consisting of all operators in \( A \) which are \( n \)-times weakly \( D \)-differentiable and equipped with the norm \( \| \cdot \|_n \).

**Theorem 2.16.** For any non-negative integer \( n \), \( A_{wD}^n \) is a Banach*-algebra.
Proof. For \( n = 0 \), we have \( A^0_{wD} = A \). Let us then consider the case \( n = 1 \). It is clear that \( A^1_{wD} \) is a linear subspace and it is easy to check that for any weakly \( D - \)differentiable element \( a \) the infinite matrices 
\[
[D, a] \quad \text{and} \quad [D, a^*]
\]
satisfy 
\[
[D, a]^* = -[D, a^*].
\]
Hence \( a^* \) is in \( A^1_{wD} \) and \( wD(a^*) = -wD(a)^* \), so \( A^1_{wD} \) is a self-adjoint subspace of \( A \).

For a pair of operators \( a \) and \( b \) from \( A^1_{wD} \) we can show that \( ab \) is weakly \( D - \)differentiable by showing that \( ab \) is \( D - \)Lipschitz,
\[
\|\alpha_t(ab) - ab\| \leq \|(\alpha_t(a) - a)\alpha_t(b)\| + \|a(\alpha_t(b) - b)\| \leq |t|\|wD(a)\|\|b\| + |t|\|a\|\|wD(b)\|.
\]
Hence \( ab \) is weakly \( D - \)differentiable with \( \|wD(ab)\| \leq \|wD(a)\|\|b\| + \|a\|\|wD(b)\| \). The last inequality may be strengthened by showing that the Leibniz rule holds. Since \( ab \) is weakly \( D - \)differentiable and \( \text{dom}(D) \) is invariant under \( a \), \( b \) and \( ab \) we may perform the manipulations below
\[
\forall \xi \in \text{dom}(D) : wD(ab)\xi = (Dab - abD)\xi = (Da - aD)b\xi + a(Db - bD)\xi = wD(a)b\xi + awD(b)\xi,
\]
and the Leibniz rule follows.

With respect to the completeness of \( A^1_{wD} \) under the norm \( \|\cdot\|_1 \), we see that a Cauchy sequence \( (a_k)_{k \in \mathbb{N}} \) with respect to the norm \( \|\cdot\|_1 \) is a Cauchy sequence with respect to the operator norm and hence converges in operator norm towards an element \( a \) in \( A \). Since the sequence \( (wD(a_k))_{k \in \mathbb{N}} \) also is a Cauchy sequence in the operator norm it converges towards a bounded operator \( d \) in \( B(H) \). It is quite easy to check that for each pair \( r, c \) of integers we have
\[
e_r d|e_c H = \lim_{k \to \infty} e_r wD(a_k)|e_c H = \lim_{k \to \infty} (d_r a_k - e_r a_k d_c)|e_c H = (d_r a - e_r a d_c)|e_c H = [D, a]|e_c H.
\]
Hence \( [D, a] \) is the matrix of the bounded operator \( d \) and \( a \) is differentiable with \( wD(a) = d \), so \( A^1_{wD} \) is a complete Banach space.

The formula \( wD(ab) = awD(b) + wD(a)b \) shows that \( \|ab\|_1 \leq \|a\|_1 \|b\|_1 \). Finally if \( A \) is unital then the unit \( I \) is weakly \( D - \)differentiable and \( wD(I) = 0 \), so \( \|I\|_1 = 1 \).

For \( n > 1 \) the arguments showing completeness run in a similar way by induction. It is a well known fact from classical analysis, but also easy to check, that when the Leibniz rule works the norm \( \|\cdot\|_n \) becomes sub-multiplicative, so also for \( n > 1 \) the algebra \( A^n_{wD} \) becomes a Banach*-algebra. \( \square \)
Given a $C^*$-algebra $A$ on $H$, then inside noncommutative geometry one wants to study higher commutators of an operator $a$ with the numerical value of $D$, which is defined as $|D| = (D^2)^{1/2}$. Hence we define for each natural number $n$ an $ncg$ norm on a certain $*$-subalgebra of $A$.

**Definition 2.17.** Let $A$ be a $C^*$-algebra on $H$, $D$ a self-adjoint operator and $n$ a natural number. The normed $*$-algebra $A_{ncg}^n$ is defined by

$$A_{ncg}^n := \{ a \in A \mid a \in A_{a}D^1 \text{ and } a \in A_{w[D]} \},$$

with the norm

$$\|a\|_{ncg}^n := \|wD(a)\| + \|a\|_{w[D]}^n.$$

Then we have an immediate corollary to the theorem.

**Corollary 2.18.** The normed algebra $A_{ncg}^n$ is a Banach $*$-algebra.

**Proof.** The theorem above and its proof yield that the normed space $A_{ncg}^n$ is a self-adjoint complete algebra, so we only have to prove that its norm is sub-multiplicative. For $a, b$ in $A_{ncg}^n$ we have by the Leibniz rule and the theorem above that

$$\|ab\|_{ncg}^n \leq \|a\| \|wD(b)\| + \|wD(a)\| \|b\| + \|a\|_{w[D]}^n \|b\|_{w[D]}^n,$$

$$\leq \|a\|_{w[D]}^n \|wD(b)\| + \|wD(a)\| \|b\|_{w[D]}^n + \|a\|_{w[D]}^n \|b\|_{w[D]}^n,$$

$$\leq \|a\|_{ncg}^n \|b\|_{ncg}^n.$$

We can repeat the theorem above with respect to strong $D$—differentiability and we define.

**Definition 2.19.** The set of all $n$—times strongly $D$—differentiable operators in $A$ is denoted $A_{sD}^n$.

The generator $\delta$ of the one parameter automorphism group $\alpha_t$ of $B(H)$ is known to be a closed operator $[WR]$, and for a pair of bounded operators $a$ and $b$ in the domain of $\delta$ the equality

$$\alpha_t(ab) - ab = (\alpha_t(a) - a)\alpha_t(b) + a(\alpha_t(b) - b),$$

shows that the following well known result holds.

**Theorem 2.20.** For any non-negative integer $n$, $A_{sD}^n$ is a Banach $*$-algebra.

**Corollary 2.21.** For any non-negative integer $n$, $A_{ncg}^n$ is a Banach $*$-algebra.
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