Dynamics of nonlinear interacting dark energy models

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We investigate the cosmological dynamics of interacting dark energy models in which the interaction function is a nonlinear in terms of the energy densities. Considering explicitly the interaction between a pressureless dark matter and a scalar field, minimally coupled to Einstein gravity, we explore the dynamics of the spatially flat FLRW universe for the exponential potential of the scalar field. We perform the stability analysis for the three nonlinear interaction models of our consideration through the analysis of critical points and we investigate the cosmological parameters and we discuss the physical behaviour at the critical points. From the analysis of the critical points we find a number of possibilities that include the stable late time accelerated solution, $w_{CDM}$-like solution, radiation-like solution and moreover the unstable inflationary solution as well.

PACS numbers: 98.80.-k, 95.35.+d, 95.36.+x

1. INTRODUCTION

The dark sector of our universe, according to a series of past and latest observational evidences, is composed by two heavy dark fluids, namely a pressureless dark matter and a dark energy fluid. The former fluid is responsible for the structure formation of the universe while the latter fluid drives the present day accelerating phase of the universe. In addition to that, the observational evidences also estimate that nearly 96% of the total energy density is coming from this joint dark sector where in particular, the dark energy contributes around 68% of the total energy budget of the universe while the 28% of the total energy density of the universe is from dark matter. However, the evolution, origin and the nature of these dark fluids of the dark energy are not clearly understood yet. Although from indirect gravitational effects, the nature of dark matter seems to be partially known, however, the dark energy has remained to be extremely mysterious. As a consequence, a number of cosmological models have been introduced and investigated in the last couple of years. The simplest cosmological consideration is the non-interacting models of the universe where dark matter and dark energy are conserved separately, leading to two independent evolutions of these dark fluids. While on the other hand, a more generalized version of the cosmological models is available in which dark matter and dark energy are allowed to interact with other. In the present work we shall consider the interacting cosmological models.

The interaction between dark matter and dark energy is a potential mechanism to explain the cosmic coincidence problem, although its origin was motivated to explain the discrepancy in the cosmological constant. An earlier investigation by Wetterich shows that an interaction between a scalar field and gravity could lead to an effective cosmological constant which is dynamical in nature and asymptotically approaches toward a tiny value and thus the mismatched value in the cosmological constant gains a plausible explanation. Thus, the interaction in the dark sector started its beginning following these two motivations.

The dark sector interaction is a phenomenological consideration because there is no such fundamental principle that could derive it, however, from the theoretical ground, precisely from the particle physics theory, any two matter fields (here dark matter and dark energy fields) can interact with each other. Interestingly, this specific phenomenological theory has gained a massive interest in the cosmological community for several potential outcomes. It has been found that the allowance of an interaction can take the dark energy equation of state from quintessence to phantom regime, that means an effective quintom type of nature is imposed in the dark energy state parameter. The phantom crossing available in scalar field models with negative kinetic correction (known as phantom scalar field models) lead to instabilities at the classical and quantum levels. Secondly, the interaction has been found to be very efficient to address

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the mismatched value of the Hubble constant \( H_0 \) from the global (LCDM based Planck) and local measurements. Some other investigators have also found that the interaction might also be able to solve the tension on \( \sigma_8 \).

On the other hand, from the recent observational evidences, it has been already pointed out that irrespective of the dark energy equation of state \( (w_x = -1, w_x \) is constant but \( \neq -1 \), or \( w_x \) is dynamical), the interaction in the dark sector is allowed, although the strength of the interaction is mild, but it is not ruled out though \[16 \ 24\]. Thus, based on the above observational predictions, one can assume that the interaction in the dark sector might be a potential theory for further investigations. For a general overview of different interaction models and their cosmological consequences, we refer to a number of past \[10 \ 14 \ 25 \ 32\] and recent works \[16 \ 24 \ 33 \ 46\] containing some interesting observations.

Now concerning the choices of the interaction functions, as there is no such governing rule, thus, in principle a number of linear and nonlinear functions can be chosen. However, the models with nonlinear interactions are rare in the literature \[48 \ 50\], since the dynamics of the interacting fluids becomes much complicated compared to the dynamics for linear interaction. Nevertheless, it is always fascinating to explore the dynamics in the context of nonlinear interaction functions in order to see if we can extract more information out of that. Thus, being motivated, in the present work we consider an interacting scenario between a scalar field and the pressureless dark matter where the interaction functions are nonlinear in nature. We have performed the stability analysis of each interaction model in order to investigate their cosmological viabilities. The work has been organized as follows.

In Section 2 we present the gravitational equations for an interacting universe. In Section 3 we describe the formation of the dynamical system for the interacting scenarios. In section 4 we study the critical points for all the interacting scenarios and we present the stability analysis. Finally, our discussion and conclusions are given in Section 5.

### 2. FIELD EQUATIONS

In the large scale, our universe is homogeneous, isotropic and almost flat. Such a geometrical configuration is well described by the spatially flat FRW universe which is characterized by the following line element

\[
    ds^2 = -dt^2 + a^2(t) \left(dx^2 + dy^2 + dz^2\right),
\]

where \( a(t) \) is the expansion scale factor of the universe. In this space-time we consider that the main constituents of our universe are a pressureless matter and a non-canonical scalar field where the matter sector and the scalar field are interacting with each other through a non-gravitational interaction.

The Action integral of such a cosmological scenario is given by

\[
    S = \int d^4x \sqrt{-g} \left[ R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] + \mathcal{L}_m
\]

where \( \mathcal{L}_m \) denotes the Lagrangian for the matter field terms, that means pressureless matter and the scalar field. In the background \[1\], the energy density and the pressure for the scalar field take the forms

\[
    \rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi),
\]

\[
    p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi),
\]

from where the equation of state parameter \( w_\phi \) for the scalar field is defined to be the ratio of its pressure to the energy density, that means,

\[
    w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}.
\]

Moreover, we assume that \( \rho_m \) and \( p_m \) are respectively the energy density and pressure of the matter sector. Since we assume the pressureless matter, thus, we have \( p_m = 0 \), and consequently, the equation of state parameter for this matter sector \( w_m = 0 \).

The field equations can be obtained by varying the action with respect to the metric coefficients \( g_{\mu\nu} \) of the space-time as (in the units \( k^2 = 8\pi G = c = 1 \))

\[
    H^2 = \frac{1}{3} (\rho_m + \rho_\phi),
\]

\[
    \dot{H} = -\frac{1}{2} (\rho_m + \rho_\phi + p_\phi),
\]
where an ‘overdot’ represents the cosmic time differentiation, and $H = \dot{a}/a$ is the Hubble parameter of the FRLW universe. Furthermore, from the Bianchi identity we have that $\text{eff} T^{ab}_{;b} = 0$, where $\text{eff} T^{ab} = \phi T^{ab} + m T^{ab}$. However, because we consider interaction between the scalar field and the dust fluid, the Bianchi identity gives the following equations

$$\phi T^{ab}_{;b} + m T^{ab}_{;b} = 0,$$

or equivalently,

$$\dot{\rho}_m + 3H\rho_m = Q,$$

$$\dot{\rho}_\phi + 3H(1 + w_\phi)\rho_\phi = -Q,$$

where we have introduced the quantity $Q$ which indicates the rate of energy exchange between the dark sector. Positive value of $Q$ indicates that there is an energy transfer from the scalar field $\rho_\phi$ to the cold dark matter $\rho_m$, while for $Q < 0$, the reverse scenario happens, that means energy transfer from the cold dark matter to scalar field.

An equivalent way to write the set of equations (9), (10) is with the use of the ratio $r(t) = \rho_m/\rho_\phi$, where the equation (9) becomes

$$\dot{\rho}_\phi - Q(1 + r) - 3H\rho_\phi r = 0.$$

The nature of the interaction function $Q$ is purely unknown and until now we don’t have any device available to derive this function from some fundamental physical principle. There are various phenomenological approaches in the literature which have shown that the existence of the interaction can explain the values of various cosmological parameters during the late-time acceleration phase of our universe.

Some interaction models which have been proposed in the literature and much well known to the interaction theory are the linear models, such as $Q_1 = \alpha_m H \rho_m$ [27], $Q_2 = \alpha_\phi H \rho_\phi$ [47] and $Q_3 = \alpha (H \rho_m + H \rho_\phi)$ [29]. While only a few nonlinear models have been proposed and investigated in the literature [48–50].

In this work we shall study the evolution of the field equations (6), (7), (9) and (10) for some nonlinear interacting models of the form $Q(\rho_m, \rho_\phi)$. Such an analysis provides us with the information for different phases of the universe provided by the field equations and also the stability of these phases as well. In order to perform our analysis we prefer to work with the dimensionless variables, and more specifically, we select to work with the so-called $H$–normalization as followed in [51].

3. CONSTRUCTION OF THE DYNAMICAL SYSTEM

We continue with the introduction of the dimensionless variables [51]

$$x = \frac{\dot{\phi}}{\sqrt{6} H}, \quad y = \frac{\sqrt{V(\phi)}}{\sqrt{3} H},$$

in the $H$–normalization. Moreover, we assume that the new independent variable is the lapse time $N = \ln a$, which is also called the e-folding parameter.

In the new variables the gravitational field equations (6), (7), (9) and (10) reduce to the following autonomous system of algebraic-differential system

$$\frac{dx}{dN} = -3x + \sqrt{\frac{3}{2}} \lambda y^2 + \frac{3x}{2} ((2 + r)x^2 + ry^2) - (6x)^{-1} \hat{Q},$$

$$\frac{dy}{dN} = -3\sqrt{2} \lambda xy + 3\sqrt{3} y ((2 + r)x^2 + ry^2),$$

$$\frac{d\lambda}{dN} = -\sqrt{6} x \lambda^2 (\Gamma (\lambda) - 1),$$

$$\frac{dr}{dN} = 3 \left( x^2 + y^2 \right)^{-1} \left( (1 + r) \hat{Q} + 9r (x^2 - y^2) \right),$$

with the algebraic constraint

$$1 - (1 + r) \left( x^2 + y^2 \right) = 0,$$
in which
\[ \lambda = -\frac{V_\phi}{V}, \quad \bar{Q} = \frac{Q}{H^3}, \quad \text{and} \quad \Gamma = \frac{VV_{,\phi\phi}}{(V_\phi)^2}. \] (18)

Furthermore, the first Friedmann equation (15) in the dimensionless variables becomes
\[ \Omega_m = 1 - x^2 - y^2, \] (19)
where \( \Omega_m = \frac{1}{3} \rho_m H^{-2} \), is the dimensionless density parameter for the matter sector. The cosmological parameters for the scalar field, namely the equation of state parameter \( w_\phi \) and the dimensionless density parameter \( \Omega_\phi \) in the new variables can be written as follows
\[ w_\phi = \frac{x^2 - y^2}{x^2 + y^2}, \quad \Omega_\phi = \frac{\rho_\phi}{3H^2} = x^2 + y^2 \] (20)
while the total equation of state parameter is
\[ w_{tot} = \frac{\rho_\phi}{\rho_m + \rho_\phi} = x^2 - y^2. \] (21)

Finally, one can calculate the deceleration parameter that assumes the following expression
\[ q = -1 - \frac{\ddot{H}}{H^2} = \frac{1}{2} (1 + 3 \left( x^2 - y^2 \right)). \] (22)

As far as concerns, for the solution of the scale factor, at any point \((x_0, y_0, \lambda)\), we have that \( w_{tot} = \text{const.}; \) hence from (17) it follows that \( a(t) \propto t^{\frac{2}{1 + 3 w_{tot}}} \) for \( w_{tot} \neq -1 \) and \( a(t) = a_0 e^{H_0 t} \) for \( w_{tot} = -1 \), where the latter corresponds to the de Sitter points.

Due to the energy condition \( 0 \leq \Omega_m \leq 1 \), we have two different conditions which depends on the nature of the scalar field. If the action integral for the scalar field is that of the quintessence, i.e. \( \varepsilon = 1 \), it follows that \( x^2 + y^2 \leq 1 \). Which means that the parameters \( \{x, y, z\} \) take values inside a unit sphere. However from the definition of the dimensionless variables (12) we have \( y \in [0, 1] \) and \( \{x\} \in [-1, 1] \).

However, because we shall work with interactions of the form \( \bar{Q}(r, x, y) \), the limit where \( \Omega_\phi = x^2 + y^2 \to 0 \), corresponds to \( r \to +\infty \), consequently to the matter dominated era.

As far as concerns the nonlinear interaction models of our analysis we consider the following
\[ \bar{Q}_A = Q_0 \frac{r}{x^2}, \quad \bar{Q}_B = Q_0 \frac{x^2}{r}, \quad \bar{Q}_C = Q_0 \frac{y^2}{x^2 + y^2 r^2} \] (23)
where \( Q_0 \) is a constant and it is the interaction parameter or the coupling parameter. The parameter could describe the strength of interaction (from its magnitude) and the direction of energy flow between the dark sectors (through its sign).

In terms of the energy densities \( \rho_m, \rho_\phi \) the interactions in (23) are expressed as
\[ \bar{Q}_A \simeq \frac{\phi^2 \rho_m}{\rho_\phi}, \quad \bar{Q}_B \simeq \frac{\phi^2 \rho_\phi}{\rho_m}, \quad \bar{Q} \simeq V(\phi) \left( \frac{\rho_m}{\rho_\phi} \right)^2 \] (24)

With the use of the constraint (17) we can reduce the dynamical system (13)-(16) to a three dimensional system by replacing \( r = (x^2 + y^2)^{-1} - 1 \); while when \( \lambda = \text{const.} \), the dynamical system is reduced to a two dimensional system because equation (13) is identically satisfied. The only potential function \( V(\phi) \) where \( \lambda = \text{const.} \), always, is the exponential potential \( V(\phi) = V_0 e^{-\lambda \phi} \) if.

4. CRITICAL POINTS AND STABILITY

Before we proceed with the specific interactions let us consider the most general scenario \( \bar{Q} = \bar{Q}(r, x, y) \). By replacing this interaction into equation (13), the critical points for the two-dimensional dynamical system (13), (14) in which
\[ r = \frac{1 - x^2 - y^2}{x^2 + y^2}, \] (25)
can be categorized into two families. Family (A) consists of the points with \( y = 0 \), and Family (B) consists of the points with \( y \neq 0 \); more specifically,

\[
y^2 = 1 + x^2 - \sqrt{\frac{2}{3}} \lambda x. \tag{26}
\]

These two families follow from the solution of the algebraic equation

\[
\left[(2 + r) x^2 + ry^2 - \sqrt{\frac{2}{3}} \lambda xy\right] y = 0. \tag{27}
\]

As far as concerns, the total number of points corresponding to each family, depends on the prescribed interaction function \( \bar{Q}(r, x, y) \), and also on how many real solutions are admitted by the algebraic equation

\[
-3x + \sqrt{\frac{3}{2}} \lambda y^2 + \frac{3x}{2} \left((2 + r) x^2 + ry^2\right) - (6x)^{-1} \bar{Q}(r, x, y) = 0, \tag{28}
\]

for \( x \in [-1, 1] \). Moreover, for the points of Family A the physical parameters are simplified as

\[
\Omega_m(x_A) = 1 - x_A^2, \quad w_\phi(A) = 1, \quad w_{\text{tot}} = x_A^2, \tag{29}
\]

which means that the scalar field acts as a stiff fluid, while because of the interaction \( x_A \) can be different from zero, that means the dust fluid can contribute to the universe. For the points of Family B, the physical parameters are written in terms of \( x_B \) as follows

\[
\Omega_m(x_B, \lambda) = \frac{\sqrt{6}}{3} x_B \left(\lambda - \sqrt{6} x_B\right), \tag{30}
\]

\[
w_\phi(x_B, \lambda) = \frac{\sqrt{6} \lambda x_B - 3}{3 + \sqrt{6} x_B (\lambda - \sqrt{6} x)}, \tag{31}
\]

and

\[
w_{\text{tot}}(x_B, \lambda) = -1 + \sqrt{\frac{2}{3}} \lambda x_B. \tag{32}
\]

At this point, from (30) one can find the constraint on \( \lambda \) (recall that the constraint on \( \Omega_m \) is, \( 0 \leq \Omega_m \leq 1 \)) as

\[
\sqrt{6} |x_B| \leq \lambda \leq \frac{3 + 6 |x_B|^2}{\sqrt{6} |x_B|}. \tag{33}
\]

In Fig. 1 the area in the space \( \{x_B, \lambda\} \) is presented where the points of Family B exist.

We continue our analysis by presenting the physical parameters for the critical points of Family A and Family B, and also the region where the critical points are stable. Special cases of study are given.

### 4.1. Interaction \( \bar{Q}_A \)

For the nonlinear interaction \( \bar{Q}_A \), the critical points are

\[
P_A(\bar{Q}_A) = \left(\pm \left(-\frac{Q_0}{9}\right)^{\frac{1}{6}}, 0\right) \tag{34}
\]

\[
P_B(\bar{Q}_A) = \left(x_B, \sqrt{1 + x_B^2 - \sqrt{\frac{2}{3}} \lambda x_B}\right) \tag{35}
\]

where \( x_B \) is a solution of the fourth-order algebraic equation

\[
Q_0 = 3x_B^2 \left(3 - 2x_B \left(\sqrt{6} \lambda (1 + x_B^2) - (3 + \lambda^2) x_B\right)\right) \tag{36}
\]

From the algebraic expressions of \( P_A(\bar{Q}_A) \) and \( P_B(\bar{Q}_B) \), one can infer that Family A consists of two critical points, while Family B is composed by 0, 2 or 4 critical points.
4.1.1. Critical Points of Family A

Points $P_A(\bar{Q}_A)$ are real only when $Q_0 < 0$, while exits for values of $|Q_0|$ in the range
\[
0 < |Q_0| \leq 9,
\]
while the total equation of state parameter $w_{\text{tot}}$ is constrained by
\[
0 < w_{\text{tot}} \leq 1.
\]
As far as concerns the stability of the critical points $P_A(\bar{Q}_A)$ we derive the eigenvalues
\[
e_1(P_A(\bar{Q}_A)) = 6, \quad e_2(P_A(\bar{Q}_A)) = 0
\]
from which one can easily infer that the critical points $P_A(\bar{Q}_A)$ are unstable in nature.

4.1.2. Critical Points of Family B

Because of the nonlinearity of the algebraic equation (36) we are unable to get the exact expressions for the physical parameters and also the stability of the points as well. Thus, we proceed with the numerical solution of (36) in order to investigate the cosmological parameters. Following this, in Fig. 2 the contour plots for the physical parameters $\Omega_m(P_B(\bar{Q}_A))$, $w_\phi(P_B(\bar{Q}_A))$ and $w_{\text{tot}}(P_B(\bar{Q}_A))$ are presented, and also the specific values of the surface (36) in the plane $x_B - \lambda$, using the constraint in (33). In addition, the shaded areas in Fig. 2 define the region of the parameter $\{x_B, \lambda\}$, where the critical points are stable.

From the plots in Fig. 2 we can infer that there exist regions of the parameters $\{x_B, \lambda\}$ where the stable critical points cannot describe an accelerated universe, since we always have $-\frac{1}{3} < w_{\text{tot}}$, and there we have $\Omega_m(x_B, \lambda) \neq 0$. On the other hand, the de Sitter universe can be described only as an unstable solution for the current interaction model.

We now proceed by considering a specific value for the parameter $\lambda$.

a. Special case $\lambda = 2$ : Let us now consider a specific value of $\lambda = 2$. One can clearly see that the points $\bar{P}_A(Q_A)$ do not change since there is no $\lambda$ dependence, so we focus on points $P_B(Q_B)$. For that specific value of the parameter $\lambda$, the algebraic equation (36) is simplified as follows
\[
Q_0 = 3x_B^2 \left(3 - 2x_B \left(2\sqrt{6}(1 + x_B^2) - 7x_B\right)\right)
\]
while from (33) parameter $x_B$ is constrained to be $x_B \in [0, \sqrt{2/3}]$. In that range, from Fig. 3 it is clear that there exists two real points which correspond to the Family B.

In Fig. 4 the qualitative evolution of the physical parameters $w_\phi$, $w_{\text{tot}}$ and $\Omega_m$ in terms of $Q_0$ are given, and also the evolution of the eigenvalues of the linearized system near the critical points. We conclude that for that specific value of $\lambda = 2$, the critical points are stable for $\frac{\sqrt{2}}{6} < x_B < \sqrt{\frac{2}{3}}$, which means
\[
-2 < Q_0 < \frac{1}{3}
\]
FIG. 2: Region plot in the space \( \{x_B, \lambda\} \) for the interaction constant \( Q_0 \), and also the physical parameters \( \Omega_m, w_\phi, w_{tot} \) for the points which belong to Family B of the interaction \( Q_A \). The shaded regions define the areas where the critical points are stable.

FIG. 3: The figure describes the numerical solution for the polynomial equation (40).
FIG. 4: Evolution of the physical parameters \( w_\phi(Q_0) \), \( w_{\text{tot}}(Q_0) \) and \( \Omega_m(Q_0) \) for the interaction model \( Q_A \) and \( \lambda = 2 \) at the critical points of Family B. The evolution of the eigenvalues of the linearized system is given where we observe that the points are stable for \(-2 < Q_0 < \frac{1}{3}\)

while in that space of variables

\[ -\frac{1}{3} < w_{\text{tot}} < \frac{1}{3}, \quad -\frac{1}{2} < w_\phi < \frac{1}{3} \]  

(42)

and

\[ 0 < \Omega_m < \frac{1}{3} \]  

(43)

That means the present accelerated solution is not admissible for this case.

4.2. Interaction \( \bar{Q}_B \)

Interaction \( \bar{Q}_B \) is the reverse function of interaction \( \bar{Q}_A \), that is, \( \bar{Q}_B \simeq (\bar{Q}_A)^{-1} \). For that interaction we find the following real critical points

\[
P_A (\bar{Q}_B) = \left( \pm \sqrt{\frac{18 + Q_0 - \sqrt{Q_0(36 + Q_0)}}{3\sqrt{2}}}, 0 \right) \]

(44)

\[
P_B (\bar{Q}_B) = \left( x_B, \sqrt{1 + x_B^2 - \sqrt{\frac{2}{3}} \lambda x_B} \right) \]

(45)

where \( x_B \) is given by the following algebraic equation

\[
Q_0 = 6 \left( 3 - \sqrt{6} \lambda x_B + \lambda^2 - 9 \left( 3 + 6 x_B^2 - \sqrt{6} \lambda x_B \right)^{-1} \right) .
\]

(46)

Hence, Family A admits two critical points, while Family B admits 1 or 3 real critical points.
4.2.1. Critical Points of Family A

The energy density for the dust fluid for the points of Family A is calculated to be

\[ \Omega_m \left( P_A \left( \tilde{Q}_B \right) \right) = \frac{1}{18} \left( \sqrt{Q_0 \left( 36 + Q_0 \right)} - Q_0 \right) \]  

(47)

from where we can infer that \( Q_0 \geq 0 \) in order \( \Omega_m \) to be bounded.

Moreover, for points \( P_A \left( \tilde{Q}_B \right) \) we calculate the two eigenvalues of the linearized system given by

\[ e_1 \left( P_A \left( \tilde{Q}_B \right) \right) = \frac{1}{6} \left( 36 + Q_0 - \sqrt{Q_0 \left( 36 + Q_0 \right)} \right) \]  

and \( e_2 \left( P_A \left( \tilde{Q}_B \right) \right) = 0. \) 

(48)

Therefore, for \( Q_0 > 0 \) we find that always \( \text{Re} \left[ e_1 \left( P_A \left( \tilde{Q}_B \right) \right) \right] > 0 \), consequently we conclude that points \( P_A \left( \tilde{Q}_B \right) \) are unstable.

4.2.2. Critical Points of Family B

The critical points which correspond to the second family can be one or three depending on the number of real solutions that the polynomial equation (46) admits. We calculate the eigenvalues of the linearized system and in Fig. 5 we present the contour diagrams in the space \( \{ x_B, \lambda \} \) for the cosmological parameters \( \Omega_m \left( P_B \left( \tilde{Q}_B \right) \right), \ w_\phi \left( P_B \left( \tilde{Q}_B \right) \right) \) and \( w_{\text{tot}} \left( P_B \left( \tilde{Q}_B \right) \right) \) in which the shaded areas mark the surfaces where both eigenvalues are negative meaning that the critical points are stable. Moreover, the contour plot of parameter \( Q_0 \) is presented.

From Fig. 5 we can infer that a de Sitter universe can be seen as a future attractor and the model can describe the late-time acceleration phase of the universe. However, unstable accelerated eras are not provided which means that this interacting model cannot explain the early acceleration phase of the universe.

We now follow a similar fashion as done with earlier interaction model, that means we aim to investigate the critical points of Family B for a specific value of the parameter \( \lambda \).

a. Special case \( \lambda = 2 \) : Consider now that \( \lambda = 2 \). Because we have a fixed value for one of the parameters we can make plots of \( \Omega_m \left( P_B \left( \tilde{Q}_B \right) \right), \ w_\phi \left( P_B \left( \tilde{Q}_B \right) \right) \) and \( w_{\text{tot}} \left( P_B \left( \tilde{Q}_B \right) \right) \) in terms of the interaction parameter \( Q_0 \). Furthermore, as before, for \( \lambda = 2 \), points \( P_B \left( \tilde{Q}_B \right) \) exists when \( x_B \in (0, \sqrt{2/3}] \).

For \( \lambda = 2 \), the algebraic equation (46) becomes

\[ Q_0 = 6 \left( 7 - 2\sqrt{6x_B} - 9 \left( 3 + 6x_B^2 - 2\sqrt{6x_B} \right)^{-1} \right), \]  

(49)

where from Fig. 6 it is clear that for a specific value of \( Q_0 \), only one point \( P_B \left( \tilde{Q}_B \right) \) exists and this becomes true if and only if \( Q_0 > 0 \).

The evolution of the physical parameters in terms of \( Q_0 \) is presented in Fig. 7. It is clear that for large values of \( Q_0 \) we reach the de Sitter point, \( w_{\text{tot}} \left( Q_0 \right) \to -1 \), while an inflationary scenario where \( \Omega_m \neq 0 \) and \( w_{\text{tot}} < -\frac{1}{3} \) is supported by the specific model.

Last but not least, we observe that for small values of \( Q_0 \), i.e. \( Q_0 \to 0 \), there exists a radiation-like solution, where the scalar field can mimic the radiation fluid \( w_\phi \simeq \frac{1}{3} \), however, such solution is unstable.

4.3. Interaction \( Q_C \)

For the interaction \( Q_C \) we find that the dynamical system admits critical points which belong only to the Family B. More specifically, the critical points are

\[ P_B \left( \tilde{Q}_C \right) = \left( x_B, \sqrt{1 + x_B^2 - \sqrt{\frac{2}{3}\lambda x_B}} \right) \]  

(50)

where now \( x_B \) satisfies the polynomial equation

\[ 2x_B A(x_B) Q_0 = 3B(x_B), \]  

(51)
FIG. 5: Region plot in the space \( \{x_B, \lambda\} \) for the interaction constant \( Q_0 \), and the physical parameters \( \Omega_m \), \( w_\phi \) and \( w_{\text{tot}} \) for the points which belong to Family B of the interaction function \( Q_B \). The shaded areas define the areas where the critical points are stable.

FIG. 6: Numerical solution for the polynomial equation (49) in which \( A(x_B) \) and \( B(x_B) \) are given by

\[
A(x_B) = \left( 6x_B^3 \left( 3x - 2\sqrt{6}\lambda \right) + 3\lambda^2 - \sqrt{6}\lambda \left( 6 + \lambda^2 \right) x_B + 3 \left( 6 + 5\lambda^2 \right) x_B^2 \right),
\]

\[
B(x_B) = \left( 4\sqrt{6}\lambda^5 x_B^4 - 54x_B \left( 2x_B^2 + 1 \right)^3 + 12\sqrt{6}\lambda^3 x_B^2 \left( 12x_B^4 + 13x_B^2 + 3 \right) + 9\sqrt{6}\lambda \left( 2x_B^2 + 1 \right)^2 \left( 4x_B^4 + 10x_B^2 + 1 \right) + 2x_B^2 + 1 - 36\lambda^2 x_B \left( 16x_B^6 + 30x_B^4 + 15x_B^2 + 2 \right) \right),
\]

and it is of order eight in terms of \( x_B \). That means that the number of critical points could be 0, 2, 4, 6 or 8, depending on the values of the free parameters \( Q_0 \) and \( \lambda \).

In Fig. 8 the numerical solution of the polynomial equation (51) is presented and the qualitative evolution for different cosmological parameters namely \( w_{\text{tot}} \), \( \Omega_m \) and \( w_\phi \) are also shown. The shaded areas in Fig. 8 indicate that
FIG. 7: Evolution of the physical parameters $w_\phi(Q_0)$, $w_{tot}(Q_0)$ and $\Omega_m(Q_0)$ for the interaction model $Q_B$ and $\lambda = 2$ at the critical points of Family B. The evolution of the eigenvalues of the linearized system is given where we observe that the points are stable for $-Q_0 \gtrsim 0.01$. An unstable radiation-like solution is provided for values of $Q_0 < 0.01$

the real parts of the eigenvalues of the linearized system near to the critical points are negative, which means that the critical points are stable. It is straightforward to observe that the present model provides stable accelerated eras, and also the stable critical points where $\Omega_m \neq 0$ and $w_\phi < -\frac{1}{3}$. In order to explain this better we proceed with the specific case $\lambda = \sqrt{6}$.

a. Special case $\lambda = \sqrt{6}$: We assume that $\lambda = \sqrt{6}$ and now all of our physical parameters depend only on $Q_0$, which means that we can study the one dimensional parameters $\Omega_m(P_B(Q_C))$, $w_\phi(P_B(Q_C))$ and $w_{tot}(P_B(Q_C))$. However, firstly, we need to determine the number of possible critical points. Indeed from Fig. 9 we determine that real critical points $P_B(Q_C)$ exist only when $Q_0 \gtrsim 68$, while the number of critical points is two.

As presented in Fig. 10 the critical points provide that $\Omega_m(P_B(Q_C)) \neq 0$ while $w_\phi(P_B(Q_C)) \lesssim -0.7$. In particular, there are two branches of cosmological solutions with different cosmological parameters. The one branch is always stable, while the other branch is stable only for $Q_0 \gg 300$. Consequently, for smaller values of that range we can have one stable accelerated solution which approaches the de Sitter universe and an unstable solution where the scalar field has an equation of state parameter $w_\phi(P_B(Q_C)) \lesssim -0.7$ and the dust fluid contributes to the universe.

The latter observation is important, because that kind of models can provide $w_{CDM(-like)}$ universes. It is important to mention that the limits given in the Fig. 10 depend on the value of $\lambda$, hence, other values of $\lambda$ provide another values for the physical parameters.

5. CONCLUSIONS

Interaction in the dark sector mainly between the dark matter and dark energy is a possible approach to explain the dynamical features of the universe. Observational data from various astrophysical sources data have shown that a mild interaction in the dark sector is allowed. The allowance of an interaction in the dark sector has been found to explain the tensions between the cosmological parameters. Additionally, interaction has a well motivated origin in the cosmological regime. Usually the theory of interaction rests on the choices of the interaction rates that could be either linear or nonlinear in nature. The linear interaction rates are relatively easy compared to the nonlinear models...
FIG. 8: Region plot in the space \( \{x_B, \lambda\} \) for the interaction constant \( Q_0 \), and the physical parameters \( \Omega_m, w_\phi \) and \( w_{\text{tot}} \) for the points which belong to Family B of the interaction function \( Q_C \). The shaded regions define the areas where the critical points \( P_B(Q_C) \) are stable.

FIG. 9: Numerical solution for the polynomial equation (51) for \( \lambda = \sqrt{6} \).

for interaction just because of their construction and as a consequence this particular kind of interaction models have got much attention. Since the exact interaction rate between the dark sectors is not yet known, thus, the cosmological models allowing nonlinear interaction rates are equally welcome to understand the dynamics of the universe. The studies on nonlinear interaction rates are also important to understand its necessity in the cosmological regime.

The present work thus investigates the cosmological dynamics in presence of various nonlinear interaction rates between dark matter and dark energy in the background of a spatially flat FLRW universe where the gravitational sector is described by the Einstein gravity. The dark matter is considered to be pressureless and dark energy is a minimally coupled scalar field to gravity and additionally the potential of the scalar field has been assumed to be exponential: \( V(\phi) = V_0 \exp(-\lambda \phi) \). In particular, we have considered three distinct nonlinear interaction rates \( Q(r, x, y) \) given in (23) where \( x, y \) are the dimensionless variables defined in (12) and \( r = \frac{\rho_m}{\rho_\phi} \), is the coincidence parameter. All interaction models have only one free parameter \( Q_0 \), known as interaction parameter or the coupling parameter that describes the strength of the interaction and the direction of energy transfer. We then perform the
FIG. 10: Evolution of the physical parameters $w_\phi(Q_0)$, $w_{\text{tot}}(Q_0)$ and $\Omega_m(Q_0)$ for the interaction model $Q_C$ and $\lambda = \sqrt{6}$ at the critical points of Family B. The evolution of the eigenvalues of the linearized system is given.

dynamical analysis through the analysis of critical points and stability. Due to nonlinearity in the models the dynamics becomes complicated and hence we perform numerical simulation. Finally we have considered two distinct analysis of the critical points, one when $y = 0$ (Family A) and the one with $y \neq 0$ (Family B) (see section 4).

For the first nonlinear interaction $\bar{Q}_A$, our observations are as follows. The real critical points of Family A are unstable in nature while for Family B, we could have stable and unstable real critical points. The stable critical points cannot describe the present accelerated expansion (see Fig. 2) while the de Sitter universe can be obtained as an unstable solution of this interaction model.

The second nonlinear interaction $\bar{Q}_B$ is just the reverse of $\bar{Q}_A$. Concerning the dynamics of this interaction model, we find that the critical points belonging to Family A are unstable. Now, for the critical points of Family B we have some interesting outcomes. For the critical points of Family B we find that a stable late time accelerated expansion is possible (see Fig. 3) while usually an inflationary solution is not obtained. However, for specific values of $\lambda$, inflationary solution is possible. Additionally, for small values of the interaction parameter $Q_0$, radiation-like solution is also admitted.

For the last nonlinear interaction model in this series, namely $\bar{Q}_C$, we find that the critical points only belong to Family B. Depending on the free parameters the number of critical points could be one of $\{0, 2, 4, 6, 8\}$. From the numerical simulation (shown in Fig. 5), we find that the model could allow stable critical points where the accelerated expansion of the universe is possible. The model has been closely examined for a specific value of $\lambda = \sqrt{6}$ (see Figs. 2 and 4), from which one can see that one stable accelerated expansion approaching toward the de Sitter universe and one unstable accelerated expansion are possible. In addition, the model realizes the $w$CDM like universe. Thus, this interaction model is quite interesting in the perspective of cosmological dynamics.

Thus, one can see that the nonlinear interaction models are quite appealing in the context of universe’s evolution by offering many possibilities. Since the interacting dynamics allows us to consider some alternative models, hence, one may explore the dynamical features with some other models as well. The existence of stable de Sitter solution is an interesting outcome of the present nonlinear interaction models.
Acknowledgments

AP acknowledges financial support of FONDECYT grant no. 3160121. SP acknowledges the financial support through the Faculty Research and Professional Development Fund (FRPDF) Scheme of Presidency University, Kolkata, India. WY thanks the financial support through the National Natural Science Foundation of China under Grants No. 11705079 and No. 11647153.

1. A. G. Riess et al. [Supernova Search Team], Astron. J. 116, 1009 (1998) [astro-ph/9805201].
2. S. Perlmutter et al. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999) [astro-ph/9812133].
3. D. N. Spergel et al. [WMAP Collaboration], Astrophys. J. Suppl. 148, 175 (2003) [astro-ph/0302209].
4. M. Tegmark et al. [SDSS Collaboration], Phys. Rev. D 74, 123507 (2006) [astro-ph/0608632].
5. D. N. Spergel et al. [WMAP Collaboration], Astrophys. J. Suppl. 170, 377 (2007) [astro-ph/0603449].
6. P. A. R. Ade et al. [Planck Collaboration], Astron. Astrophys. 594, A13 (2016) [arXiv:1502.01589 [astro-ph.CO]].
7. L. Amendola et al., Living Rev. Rel. 21, no. 1, 2 (2018) [arXiv:1606.00180 [astro-ph.CO]].
8. N. Aghanim et al. [Planck Collaboration], [arXiv:1807.06209 [astro-ph.CO]].
9. T. M. C. Abbott et al. [DES Collaboration], [arXiv:1811.02374 [astro-ph.CO]].
10. L. Amendola, Phys. Rev. D 62, 043511 (2000) [arXiv:astro-ph/9908023].
11. L. Amendola and C. Quercellini, Phys. Rev. D 68, 023514 (2003) [arXiv:astro-ph/0303228].
12. D. Pavón and W. Zimdahl, Phys. Lett. B 628, 206 (2005) [arXiv:gr-qc/0505020].
13. S. del Campo, R. Herrera and D. Pavón, Phys. Rev. D 78, 021302 (2008) [arXiv:0806.2110 [astro-ph]].
14. S. del Campo, R. Herrera and D. Pavón, J. Cosmol. Astropart. Phys. 0901, 020 (2009) [arXiv:0812.2210 [gr-qc]].
15. C. Wetterich, Astron. Astrophys. 301, 321 (1995) [arXiv:hep-th/9408025].
16. W. Yang and L. Xu, Phys. Rev. D 89, no. 8, 083517 (2014) [arXiv:1401.1286 [astro-ph.CO]].
17. V. Salvatelli, N. Said, M. Brunni, A. Melchiorri and D. Wands, Phys. Rev. Lett. 113, no. 18, 181301 (2014) [arXiv:1406.7297 [astro-ph.CO]].
18. W. Yang and L. Xu, Phys. Rev. D 90, no. 8, 083532 (2014) [arXiv:1409.5533 [astro-ph.CO]].
19. R. C. Nunes, S. Pan and E. N. Saridakis, Phys. Rev. D 94, 023508 (2016) [arXiv:1605.01712 [astro-ph.CO]].
20. S. Kumar and R. C. Nunes, Phys. Rev. D 94, 123511 (2016) [arXiv:1608.02454 [astro-ph.CO]].
21. W. Yang, H. Li, Y. Wu and J. Lu, JCAP 1610, no. 10, 007 (2016) [arXiv:1608.07039 [astro-ph.CO]].
22. W. Yang and L. Xu, Phys. Rev. D 96, no. 10, 103511 (2017) [arXiv:1702.02143 [astro-ph.CO]].
23. E. Di Valentino, A. Melchiorri and O. Mena, [arXiv:1704.08342 [astro-ph.CO]].
24. W. Yang, N. Banerjee and S. Pan, Phys. Rev. D 95, 123527 (2017) [arXiv:1705.09275 [astro-ph.CO]].
25. A. P. Billyard and A. A. Coley, Phys. Rev. D 61, 083503 (2000) [arXiv:astro-ph/9908224].
26. J. D. Barrow and T. Clifton, Phys. Rev. D 73, 103520 (2006) [gr-qc/0604063].
27. L. Amendola, G. Camargo Campos and R. Rosenfeld, Phys. Rev. D 75, 083506 (2007) [astro-ph/0610806].
28. G. Caldera-Cabral, R. Maartens, L.A. Ureña-López, Phys. Rev. D 79, 063518 (2009) [arXiv:0812.1827 [gr-qc]].
29. M. Quartin, M. O. Calvao, S. E. Joras, R. R. Reis and I. Waga, JCAP 0805, no. 007 (2008), [arXiv:0802.0546 [astro-ph]].
30. J. Väliiviita, R. Maartens and E. Majerotto, Mon. Not. Roy. Astron. Soc. 420, 2355 (2010), [arXiv:0907.4987 [astro-ph.CO]].
31. T. Clemson, K. Koyama, G. B. Zhao, R. Maartens and J. Valiviita, Phys. Rev. D 85, 043007 (2012), [arXiv:1109.6234 [astro-ph.CO]].
32. M. Thorsrud, D. F. Mota and S. Hervik, JHEP 1210, 066 (2012), [arXiv:1205.0261 [hep-th]].
33. S. Pan, S. Bhattacharya and S. Chakraborty, Mon. Not. Roy. Astron. Soc. 452, 3038 (2015), [arXiv:1210.0326 [gr-qc]].
34. S. Pan and G. S. Sharov, Mon. Not. Roy. Astron. Soc. 472, 4736 (2017), [arXiv:1610.02827 [gr-qc]].
35. G. S. Sharov, S. Bhattacharya, S. Pan, R. C. Nunes and S. Chakraborty, Mon. Not. Roy. Astron. Soc. 466, no. 3, 3497 (2017), [arXiv:1701.00780 [astro-ph.CO]].
36. S. Pan, A. Mukherjee and N. Banerjee, Mon. Not. Roy. Astron. Soc. 477, 1189 (2018), [arXiv:1710.03725 [astro-ph.CO]].
37. W. Yang, S. Pan, E. D. Valentino, R. C. Nunes, S. Vagnozzi and D. F. Mota, JCAP 1809, no. 09, 019 (2018), [arXiv:1805.08252 [astro-ph.CO]].
38. W. Yang, S. Pan and P. A. Billyard, Mon. Not. Roy. Astron. Soc. 482, 1007 (2019), [arXiv:1804.08555 [gr-qc]].
39. W. Yang, S. Pan, L. Xu and D. F. Mota, Mon. Not. Roy. Astron. Soc. 482, 1858 (2019), [arXiv:1804.08455 [astro-ph.CO]].
40. B. Wang, E. Abdalla, F. Atrio-Barandela and D. Pavon, Prog. Theor. Phys. 79, 096901 (2016).
41. E. Majerotto, J. Valiviita and R. Maartens, Nucl. Phys. B 194, 260 (2006).
42. H. Zonunmavia, W. Khyllep, N. Roy, J. Dutta and N. Tamanini, Phys. Rev. D 96, 083527 (2017).
43. G. Papagiannopoulos, S. Basilakos, A. Paliathanasis, S. Savvidou and P.C. Stavrinos, Class. Quant. Grav. 34, 225008 (2017).
44. G. Leon, A. Paliathanasis and J.L. Morales-Martinez, Eur. Phys. J. C 78, 753 (2018).
45. S.Kr. Biswas, W. Khyllep, J. Dutta and S. Chakraborty, Phys. Rev. D 95, 103009 (2017).
46. P.Tsiapi and S. Basilakos, Mon. Not. Roy. Astron. Soc. 10.1093/mnras/stz540 [arXiv:1810.12902].
47. D. Pavón and B. Wang, Gen. Rel. Grav. 41, 1 (2009) [arXiv:0712.0565 [gr-qc]].
48. L. P. Chimento, Phys. Rev. D 81, 043525 (2010) [arXiv:0911.5687 [astro-ph.CO]].
[49] F. Arevalo, A. P. R. Bacalhau and W. Zimdahl, Class. Quant. Grav. 29, 235001 (2012) [arXiv:1112.5095 [astro-ph.CO]].
[50] W. Yang, S. Pan and J. D. Barrow, Phys. Rev. D 97, no. 4, 043529 (2018) [arXiv:1706.04953 [astro-ph.CO]].
[51] E. J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006), [hep-th/0603057].
[52] E. J. Copeland, A. R. Liddle and D. Wands, Phys. Rev. D 57, 4686 (1998) [gr-qc/9711068].