Lévy flights in a box

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Abstract

It is shown that a quantum Lévy process in a box leads to a problem involving topological constraints in space, and its treatment in the framework of the path integral formalism with the Lévy measure is suggested. The eigenvalue problem for the infinite potential well is properly defined and solved. An analytical expression for the evolution operator is obtained in the path integral presentation, and the path integral takes the correct limit of the local quantum mechanics with topological constraints. An example of the Lévy process in oscillating walls is also considered in the adiabatic approximation.

Keywords: Fractional integral, Lévy flight, Topological constraints, Path integral

1. Introduction

The introduction of a fractional concept in quantum mechanics with motivating new implementations of non-local physics leads to many technical questions and often needs special care. A typical example is the “quantum Lévy flights” of a particle in an infinite potential well, suggested in [1]. This “simple” example has evoked an active discussion in the literature [2, 3, 4, 5, 6, 7] on how a non-local operator, defined on the infinite scale, acts in a finite-size range, such as a quantum box. The aim of the present research, related to this question, is an exploration of the Lévy flights in boundary value problems in a finite-size area. Among many possible applications of the problem, of special interest is

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the first-passage analysis \cite{8}, which is important for the investigation of the transport properties of Lévy glasses \cite{9}.

In quantum mechanics, the fractional concept has been introduced by means of the Feynman propagator for non-relativistic quantum mechanics as for Brownian path integrals \cite{1,10}. Equivalence between the Wiener and the Feynman path integrals \cite{11}, established by Kac \cite{12}, is natural, since both are Markov processes. As shown in the seminal papers \cite{1,10}, the appearance of the space fractional derivatives in the Schrödinger equation is natural and relates to the path integrals approach. The path integral approach for Lévy stable processes, considered for the fractional diffusion equation, in particular, for the fractional Fokker-Planck equation (FFPE), has been extended to a quantum Feynman-Lévy measure that leads to the space fractional Schrödinger equation (FSE) \cite{1,10}.

The introduction of the Lévy measure in quantum mechanics is based on the generalization of the self-consistency condition, known as the Bachelor-Smoluchowski-Kolmogorov chain equation (or the Einstein-Smoluchowski-Kolmogorov-Chapman equation, see e.g., \cite{13}), established for the Wiener process for the conditional probability \( W(x,t|x_0,t_0) \)

\[
W(x,t|x_0,t_0) = \int_{-\infty}^{\infty} W(x,t'|x_0,t_0) W(x',t|t_0,t_0) \, dx'.
\]  
(1)

In the case of the translational symmetry, it reads \( W(x,t|x_0,t_0) = W(x-x_0,t-t_0) \). Straightforward generalization of this expression by the Lévy process is expressed through the Fourier transform

\[
W(x,t|x_0,t_0) = \int_{-\infty}^{\infty} e^{ip(x-x_0)} e^{-K_\alpha t|p|^\alpha} \, dp,
\]  
(2)

where \( 0 < \alpha \leq 2 \) and \( K_\alpha \) is a generalized diffusion coefficient \cite{14}. Using Eqs. \cite{11} and \cite{14}, one defines the integration of function \( F[x(\tau)] \) over the generalized measure

\[
\int F[x(\tau)]d_L x(\tau) = \lim_{N \to \infty} \frac{1}{(2\pi \hbar)^n} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_N \int_{-\infty}^{\infty} dp_1 \cdots \int_{-\infty}^{\infty} dp_N \times F(x_1,\ldots,x_N) e^{-\frac{1}{\hbar}(x_1-x_0)p_1 + \cdots + (x_N-x_N)p_N} e^{-K_\alpha t|p_1|^\alpha \Delta t + \cdots + |p_N|^\alpha \Delta t}. \]  
(3)
Here, $\Delta t = t/N$. The Lévy distribution is defined by the Fox function and in the r.h.s. of Eq. (3) it is presented by means of the Fourier integral with a stretched exponential kernel. When $F[x(\tau)] = \exp\{-\int_0^\tau V[x(\tau)]d\tau\}$, with $V(x)$ being a potential, expression (3) is a generalized Feynman-Kac formula.

One should recognize that the general form of the path integral in Eq. (3) does not correctly describe systems with a so-called topological constraint. In this case integrations over coordinates $x_1, x_2, \ldots, x_N$ have finite limits. As an example of such system, we consider fractional quantum mechanics in an infinite well potential. We show that the condition of the restriction of the integration in both the formulation of the problem in Eq. (3) and correspondingly, in the fractional Riesz derivative must be taken into account. Otherwise, the FSE for the infinite well potential cannot be obtained.

2. Lévy quantum mechanics in potential well

In complete analogy with the FFPE, fractional quantum mechanics can be constructed from the Feynman-Kac formula and $V(x)$ is the potential. Following [1], we determine a wave function at the moment $t + \Delta t$ by means of Eq. (3)

$$\psi(x, t + \Delta t) = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} e^{-iD_{\alpha}\Delta t|p|^\alpha} \int_{-\infty}^{\infty} dy e^{-\frac{i}{\hbar}p(x-y)} e^{-\frac{i}{\hbar}V(y)\Delta t}\psi(y, t).$$  \hspace{1cm} (4)

In the limit $\Delta t \to 0$, one obtains

$$i\hbar \partial_t \psi(x, t) = \frac{D_{\alpha}}{2\pi\hbar} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |p|^\alpha e^{-\frac{i}{\hbar}p(x-y)}\psi(y, t)dp + V(x)\psi(x, t).$$  \hspace{1cm} (5)

If one introduces the fractional Laplacian $(-\Delta)^{\alpha/2}$ by means of the Riesz fractional derivative [17]

$$(-i\hbar \partial_x)^{\alpha} \psi \equiv \hbar^\alpha(-\Delta)^{\alpha/2}\psi = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{-ipx} |p|^\alpha \phi(p)dp,$$  \hspace{1cm} (6)

where $\phi(p) = \int_{-\infty}^{\infty} e^{ipy}\psi(y)dy$ is the Fourier image, the fractional Schrödinger equation (FSE) is obtained from Eq. (3)

$$i\hbar \partial_t \psi = D_{\alpha}(-i\hbar)^{\alpha} \partial_x^\alpha \psi + V(x)\psi.$$  \hspace{1cm} (7)

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One sees that for $\alpha = 2$, FSE (7) reduces to a conventional Hamiltonian mechanics with $D_2 = 1/2m$ being a half inverse mass of a particle. It should be noted that this equation is valid only for the smooth potentials $V(x)$. In particular, for the infinite potential well

$$V(x) = \begin{cases} 
0 & \text{if } |x| \leq L \\
\infty & \text{if } |x| > L.
\end{cases}$$

(8)

one cannot obtain the Schrödinger equation (7) from Eqs. (4) and (5). A simple explanation of this problem relates to the absence of the expansion over $V\Delta t$ in Eq. (4), since $\exp\left[-\frac{i}{\hbar}\Delta tV(y)\right]$ is an infinitely fast oscillating function for any arbitrary small but finite $\Delta t$. Note that one first performs this expansion and then takes the limit $\Delta t = 0$. To overcome this obstacle, the cutoff of the limits of integration over $y$ is performed. Moreover, this is a correct formulation of the problem with the topological constraints, as the infinite potential well is. This notion is also relevant for conventional local quantum mechanics, see e.g., [16]. Therefore, FSEs (5) and (7) are reduced to the well defined problem of a free particle in a finite-size range with the FSE

$$i\hbar\partial_t \psi = \frac{D_2}{2\pi\hbar} \int_{-L}^{L} dy \int_{-\infty}^{\infty} |p|^\alpha e^{-\frac{\pi}{\hbar} p(x-y)} \psi(y,t) dp,$$

(9)

which is furnished with the boundary conditions $\psi(x = \pm L) = 0$. Obviously, the Riesz fractional derivative (6) can no longer be used for the potential well, and in addition the Fourier image $\phi(p)$ is not appropriately defined.

Let us perform the Fourier inversion in Eq. (9) over $k = p/h$. This yields

$$\frac{(\hbar)^\alpha}{2\pi} \int_{-\infty}^{\infty} |k|^\alpha e^{-i\hbar kz} dk = \frac{(-i\hbar)^\alpha}{2\pi} (-\partial_x)^2 \int_{-\infty}^{\infty} |k|^{\alpha-2} e^{-i\hbar kz} dk.$$

(10)

One obtains this expression with the double differentiation for $1 < \alpha < 2$, while for $0 < \alpha < 1$ one differentiates only once. Therefore, we have the integration

$$\int_{-\infty}^{\infty} |k|^{-\nu} e^{-i\hbar kz} dk = 2 \int_{0}^{\infty} |k|^{-\nu} \cos kz = \frac{2\pi |z|^{\nu-1}}{2\Gamma(\nu) \cos(\nu \pi/2)},$$

(11)

where $\nu = 2 - \alpha$ for $1 < \alpha \leq 2$ and $\nu = 1 - \alpha$ for $0 < \alpha \leq 1$, and $\Gamma(\nu) = (\nu - 1)!$ is a gamma function. To be specific, we consider $1 < \alpha \leq 2$ in the following
analysis, and the integration in Eqs. (10) and (11) yields the fractional Laplace operator $\hat{L}$ for the FSE (9) in the form of the Riemann-Liouville fractional derivative \[17\]

$$\hat{L}^\alpha \psi(x) = \frac{(\hbar)^\alpha D^\alpha}{2\Gamma(2 - \alpha) \cos \frac{\alpha \pi}{2}} \int_{-L}^{L} |x - y|^{1 - \alpha} \psi(y) \, dy. \quad (12)$$

Note that \[
\frac{1}{2\Gamma(\nu) \cos \frac{\nu \pi}{2}} \int_a^b \frac{\phi(z) \, dz}{|x - z|^{1 - \nu}}
\]
is the Riesz fractional integral on the finite interval $[a,b]$ \[17\] with $a \leq x \leq b$ and $0 < \nu < 1$. It can be presented as the sum of the left and right Riemann-Liouville fractional integrals

$$\int_a^x \frac{\phi(z) \, dz}{(x - z)^{1 - \nu}} + \int_x^b \frac{\phi(z) \, dz}{(z - x)^{1 - \nu}}.$$

2.1. Eigenvalue problem for the fractional Laplace operator

Let us consider the eigenvalue problem

$$\hat{L}^\alpha \Psi = E \Psi \quad (13)$$

with boundary conditions $\Psi_E(x = \pm L) = 0$ that yields the solution of FSE (9) $\psi(x,t) = \sum_E a_E e^{-iEt} \Psi_E$, where coefficients $a_E$ are defined by the initial conditions $\psi_0(x) = \psi(x,t = 0)$. We rewrite the fractional Laplace operator in the form FSE (9), which is convenient in the following analysis,

$$\hat{L}^\alpha f(y) = D^\alpha \int_{-L}^{L} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} |k|^{\alpha - 2} ke^{-ik(x-y)} \, dk \right] f(y) \, dy. \quad (14)$$

Here we use again that on the real axis $|k|^2 = k^2$. One easily finds that the antisymmetric (odd) eigenfunction $\Psi_E^{\text{odd}}$ can be found in the form (see Appendix)

$$\Psi_E^{\text{odd}}(x) = \Psi_m^{\text{odd}}(x) = \frac{1}{\sqrt{L}} \sin \frac{m\pi x}{L}, \quad m = 1, 2, \ldots, \quad (15)$$

which satisfies the boundary condition $\Psi_m^{\text{odd}}(x = \pm L) = 0$. Substituting solution (15) with $f(y) = \Psi_E(y)$ in Eq. (14) and performing integration straightforwardly, one obtains

$$\hat{L}^\alpha \sin \frac{m\pi x}{L} = D^\alpha \left( \frac{hm\pi}{L} \right)^\alpha \sin \frac{m\pi x}{L}.$$
Therefore, $\Psi^{\text{odd}}_m(x)$ is the eigenfunction with corresponding eigenvalue

$$E^{\text{odd}}_m = D_\alpha \left( \frac{\hbar m \pi}{L} \right)^\alpha.$$  

(16)

The same procedure is performed to find symmetric (even) eigenfunctions

$$\Psi^{\text{even}}_E = \Psi^{\text{even}}_{2m+1} = \frac{1}{\sqrt{L}} \cos \left( \frac{(2m+1)\pi}{2L} \right) x$$  

(17)

with the eigenvalue

$$E^{\text{even}}_m = D_\alpha \left[ \frac{\hbar (2m+1) \pi}{2L} \right]^\alpha.$$  

(18)

2.1.1. A comment on the ground state

This result coincides with the solutions obtained in Ref. [1], but the crucial difference here is that solutions (15) - (18) belong to a completely different fractional operator (12), which is defined on the finite-size range of the potential well. Note also that in this case a deficiency with the ground state $\frac{1}{\sqrt{L}} \cos \frac{\pi x}{2L}$, correctly stated in Refs. [2, 5], no longer exists, since it is obtained by the appropriately determined operator.

3. Fractional path integral: fractional quantum mechanics with topological constraints

One should bear in mind that the Feynman-Kac formula in Eq. (4) and FSE (5) are introduced in unbounded Euclidian (configuration) space. In contrast, the configuration space of a particle in the infinite potential well is contracted to the finite size of the potential well. This leads to the quantum mechanics in a space with topological constraints. Even for the conventional local quantum mechanics, the path integral presentation is not an easy task, as stated in [16]. Fortunately, since eigenvalue solutions (13) - (17) are known, the path integral for the Lévy process in the finite area can be constructed. Therefore, one can obtain Eq. (7) with infinite limits of integration for the Riesz operator. Let us define for convenience $|x\rangle \equiv \Psi^{\text{odd}}_m(x)$. Then, Eq. (4) can be rewritten:

$$\psi(x, \Delta t) \equiv \langle x|\psi(\Delta t)\rangle = \langle x|e^{-i\hat{L}^\alpha \Delta t/\hbar}|0\rangle = \int_{-L}^{L} dx_1 \langle x|e^{-i\hat{L}^\alpha \Delta t/\hbar}|x_1\rangle \langle x_1|0\rangle.$$  

(19)
We focus on the evolution Green’s function

\[ G(x, \Delta t; x_1) = \langle x|e^{-i\hat{L}^\alpha \Delta t/\hbar}|x_1\rangle = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-i|k|^\alpha D_\alpha \Delta t} \int_{-L}^{L} dy e^{-ik(x_1-y)} \langle x|y\rangle. \]  

Taking into account that

\[ \langle x|y\rangle = \sum_{n=0}^{\infty} \sin(k_n x) \sin(k_n y) \quad k_n \equiv \frac{\pi n}{L} \]  

and the Poisson summation formula

\[ \sum_{l=-\infty}^{\infty} e^{2\pi i x l} = \sum_{m=-\infty}^{\infty} \delta(x - m), \]  

one obtains (see also [16])

\[ G(x, \Delta t; x_1) = \frac{1}{2} \sum_{z=\pm x} \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-i|k|^\alpha D_\alpha \Delta t} \left[ e^{ik(x_1+2L l)} - e^{ik(x_1+2L (z-x_1))} \right] \]

\[ \times \exp\left\{ ik(z-x_1+2L l) + i\pi[\theta(-z)-\theta(-x_1)]\right\}, \]

where the \( \theta \) functions in the exponential provide the correct signs for \( z = \pm x \).

Now, we take into account

\[ \frac{1}{2} \sum_{z=\pm x} \sum_{l=-\infty}^{\infty} \int_{-\infty}^{L} dx_1 \leftrightarrow \int_{-\infty}^{\infty} dx_1, \]

which returns one to the fractional integration with infinite limits, as in Eq. (4).

The essential difference here is the appearance of the topological phase due the \( \theta \) functions in Green’s function (23). Continuing this procedure by repeating it \( N \) times (\( t = N\Delta t \)), one arrives at the correct analogy of Eq. (3) for the infinite well potential

\[ G(x, t; x_0) = \sum_{l=-\infty}^{\infty} \sum_{z=\pm x+2L l} \prod_{j=1}^{N+1} \int_{-\infty}^{\infty} dx_j \prod_{j=1}^{\infty} \frac{dk_j}{2\pi} \]

\[ \times \exp\{ i \sum_{j=1}^{N+1} [k_j(x_j - x_{j-1}) + \pi \theta(-x_j) - \pi \theta(-x_{j-1}) - \hbar^{\alpha - 1}|k_j|^\alpha D_\alpha \Delta t]\}. \]
where \( x_{N+1} \equiv z \). The topological term is a purely boundary expression\(^1\), and therefore, integration over all \( x_j \) in Eq. (25) yields a product of \( \delta \) functions \( \prod_{j=1}^{N} \delta(k_j-k_{j+1}) \), which finally yields the Fourier inversion of the Green function in the form of Fox’s \( H_{2,2}^{1,1} \) function \(^{18}\) in its Fourier form

\[
G(x, t; x_0) = \sum_{l=-\infty}^{\infty} \frac{1}{4\pi} \int_{-\infty}^{\infty} dk e^{-ik\alpha^{-1}|k|^\alpha} D_{\alpha} t \left[ e^{ik(x-x_0+2L)} - e^{-ik(x+x_0+2L)} \right].
\]

Because of the properties of \( H_{2,2}^{1,1} \), Fox’s function, at \( \alpha = 2 \), the Green function (26) reduces to the free particle in the box with the Green function \(^{16}\)

\[
\langle x, t | x_0, 0 \rangle_{\text{box}} = \sum_{l=-\infty}^{\infty} \frac{1}{\sqrt{8i\pi \hbar t}} \left[ e^{im\pi [x-x_0+2L]^2 4t} - e^{-im\pi [x+x_0+2L]^2 4t} \right].
\]

4. Example: Adiabatic approximation in moving walls

Handling the path integral expression in such a simple form as Eq. (26), one can consider a system of a Lévy particle inside moving walls. For \( \alpha = 2 \), this problem corresponds to the so-called Fermi acceleration \(^{19}\), where chaotic dynamics can be realized because of the interaction of nonlinear resonances. However, for \( 1 < \alpha < 2 \), the physical implementation of the classical limit of the problem is vague, since a particle is not free. Therefore, the interaction of nonlinear resonances is not considered. Here, the problem is treated in the adiabatic approximation. In this case, Eq. (26) can be used for further quantum mechanical analysis. Let the infinite walls at \( x = \pm L \) move periodically, such that the boundary conditions for the wave function are \( \psi(x = \pm L(t)) = 0 \), where \( L(t) = L + \varepsilon \sin(\nu t) \) with \( \varepsilon/L \ll 1 \). To calculate the density of states, we

\(^1\)For \( \alpha = 2 \), this expression coincides with the results in Ref. \(^{16}\). For the self-contained presentation, we shall adjust some formulae thereof. Namely, taking the continuous time limit, one arrives at the path integral \( G(x, t; x_0) = \frac{1}{2} \sum_{l=-\infty}^{\infty} \int_{x_0}^{x} D\xi(t) \int \frac{Dp(t)}{2\pi} \exp \left\{ \frac{i}{\hbar} S[z] \right\} \), where the action \( S[z] = \int_{0}^{t} d\tau \{ p \dot{x} - H(p) - \hbar \pi \dot{z} \delta(z) \} \) contains the new topological term \( S_{\text{top}}[z] = -\hbar \pi \int_{0}^{t} d\tau [ \dot{x} \delta(x) ] = \hbar \pi [ \theta(-z) - \theta(-x_0) ] \), which is a purely boundary expression.
are interested in the trace of the Green function
\[ g(t) = \int_{-L}^{L} dx_0 G(x_0, t; x_0) \]  
(28)
where \( G(x = x_0, t; x_0) \) can be obtained from Eq. 26 in the form
\[ G(x_0, t; x_0) = \sum_{l=-\infty}^{\infty} \frac{1}{4\pi} \int_{-\infty}^{\infty} dk e^{-ik|\alpha|} D_\alpha t \left[ e^{ik(2L)} - e^{-ik(2x_0 + 2L)} \right]. \]  
(29)
Performing summation over \( l \) by the Poisson summation formula
\[ \sum_{l=-\infty}^{\infty} e^{\pm 2iklL} = 1 + 2 \sum_{l=1}^{\infty} e^{2iklL} = \frac{\pi}{L} \sum_{m=-\infty}^{\infty} \delta \left( k - \frac{\pi m}{L} \right), \]  
(30)
one can perform integration over \( k \), which leads to summation over the spectrum \( k = k_m = \pi m/L \). Then, performing integration over \( x_0 \), one obtains for the trace
\[ g_0(t) = \sum_{m=1}^{\infty} e^{-i|\pi m/L|\alpha} D_\alpha t/\hbar. \]  
(31)
One also obtains this result from the expression for the Green function \( G(x_0, t; x_0) = \sum_n \exp(-iE_{\text{odd}} n t/\hbar) \Psi_n(x_0)\Psi_n^*(x_0) \). The Fourier transform \( \tilde{g}_0(E) = \mathcal{F}[g_0(t)] \) yields the density of states \( \rho_0(E) = -\frac{1}{\pi} \Im \tilde{g}(E) \)
\[ \rho_0(E) = \hbar \sum_{m} \delta \left( E - D_\alpha |\pi m/L| \right), \]  
(32)
where \( E \equiv E_{\text{odd}} \), which satisfies the eigenvalue problem, discussed in Sec. 2.1.

In the case of the moving walls, at the replacement \( L \rightarrow L(t) \), the adiabatic approximation makes it possible to obtain the trace of the Green’s function in the form
\[ g_{\epsilon}(t) = e^{(\epsilon \sin \nu t) \frac{\pi}{2\pi}} g_0(t) = \sum_{m} e^{-i(\pi m/L(t))\alpha} D_\alpha t, \]  
(33)

where we take \( \hbar \equiv 1 \). Let us present a shift operator in the form
\[ e^{(\epsilon \sin \nu t) \frac{\pi}{2\pi}} = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\xi e^{\epsilon \sin \xi \frac{\pi}{2\pi}} e^{i n \nu t}. \]  
(34)
Then, the Fourier transform over time can be easily performed, which yields the following density of states
\[ \rho_{\epsilon}(E) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\xi e^{(\epsilon \sin \xi \frac{\pi}{2\pi}} e^{i n \nu t} 2L \sum_{m} \delta \left( E - n\nu - D_\alpha |\pi m/L| \right). \]  
(35)
Parameter $\xi$ determines the quasi-energy spectrum, namely, its band. Therefore, for the moving walls the discrete spectrum \((16)\) changes essentially:

$$E_m \to E_{m,n}(\xi) = n\nu + D_\alpha \left| \frac{\pi m}{L + \varepsilon \sin \xi} \right|^\alpha \approx n\nu + D_\alpha \left| \frac{\pi m}{L} \right|^\alpha - \varepsilon \alpha D_\alpha \left| \frac{\pi m}{L} \right|^\alpha \sin \xi.$$  \hspace{1cm} (36)

This solution corresponds to the reconstruction of the energy spectrum of the stationary problem \((13)\) to the quasi-energy spectrum with a narrow band structure.

5. Conclusion

The introduction of the quantum Lévy process in a box leads to the need to account for the topological constraints in the space. This problem can be treated in the framework of the path integral formalism with the Lévy measure. A correct path integral consideration is possible when the eigenvalue problem \((13)\) is appropriately defined for the infinite well potential \((8)\), and the eigenfunctions are known, or can be obtained for the fractional (nonlocal) operator \((12)\). An analytical expression for the evolution operator is obtained in the path integral presentation. For $\alpha = 2$, the path integral has the limit of the correct expression, which is well known in local quantum mechanics with topological constraints \([16]\). Although the results of the eigenvalue problem coincide with the solutions obtained in Ref. \([1]\), the solutions \((15) - (18)\) belong to the fractional operator \((12)\), which differ from those obtained in \([1]\), since these are defined on the finite-size range of the potential well and correspond to the Lévy walks. In this case, the ground state $\frac{1}{\sqrt{L}} \cos \frac{\pi x}{L}$ is correctly defined as well, and to some extent, this ends the discussion on the ground states (see Refs. \([2, 5]\)).

An important point of any “fractional quantum mechanics” is the classical limit $\hbar \to 0$, where a serious question emerges about the physical meaning of its classical counterpart. Obviously, a quantum Lévy particle is not a free particle in the classical limit. As an example of such a possible realization, the Lévy process is considered for oscillating walls. For $\alpha = 2$, this problem corresponds to the so-called Fermi acceleration \([19]\) leading to classical and quantum chaos. But
for $1 < \alpha < 2$, the physical implementation of the classical limit of the problem is vague, since a particle is not free. Therefore, the interaction of nonlinear resonances is not considered here. The problem is treated in the adiabatic perturbative approach. This question can be an interesting task for future studies of classical and quantum chaos of the Fermi acceleration mechanism in the nonlocal quantum mechanics.

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Appendix A. Eigenvalue problem

Let us consider the eigenvalue problem for the fractional Laplace operator and find the antisymmetric (odd) eigenfunctions $\Psi_{E}^{\text{odd}}$ in the form

$$\Psi_{E}^{\text{odd}}(x) = \Psi_{m}^{\text{odd}}(x) = \frac{1}{\sqrt{L}} \sin \frac{m\pi x}{L}, \quad m = 1, 2, \ldots, \quad (A. 1)$$

which satisfies the boundary condition $\Psi_{m}^{\text{odd}}(x = \pm L) = 0$. We rewrite the fractional Laplace operator in the form FSE (9), which is convenient in the analysis,

$$\hat{L}^{\alpha} \Psi_{m}^{\text{odd}}(x) = D_{\alpha} h^{\alpha}(i\partial_{x}) \int_{-L}^{L} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} |k|^{\alpha} e^{-ik(x-y)} dk \right] \Psi_{m}^{\text{odd}}(y) dy. \quad (A. 2)$$

Let us first perform integration over $y$. This yields

$$\frac{1}{2i} \int_{-L}^{L} \left[ e^{iky+iz} - e^{iky-iz} \right] = (-1)^{m} \frac{2\pi m}{iL} \frac{\sin(kL)}{(k+z)(k-z)}, \quad (A. 3)$$

where $z = \pi my/L$. The next step is integration over $k$. From Eqs. (A. 2) and (A. 3), we have integrals

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{k|k|^{\alpha-2}}{(k+z)(k-z)} \left[ e^{ik(L-x)} - e^{-i(L+x)} \right] \equiv I^{(+)} - I^{(-)}, \quad (A. 4)$$

where sign $(+)$ corresponds to the analytical continuation in the upper half plain, while $(−)$ corresponds to the analytical continuation in the lower half plain. Using the Residue theorem, one obtains that the integration yields

$$I^{(+)} - I^{(-)} = i(-1)^{m} \left( \frac{m\pi}{L} \right)^{\alpha-2} \cos \left( \frac{m\pi}{L} x \right). \quad (A. 5)$$
Acting on this result by the rest part of the operator, which reads \((-1)^m D_\alpha h_\alpha \left( \frac{m\pi}{L} \right)(i\partial_x)\), one obtains

\[
\mathcal{L}_m^\alpha \Psi_m^{\text{odd}}(x) = D_\alpha h_\alpha \left( \frac{m\pi}{L} \right)^\alpha \Psi_m^{\text{odd}}(x) \equiv E_m^{\text{odd}} \Psi_m^{\text{odd}}(x). \tag{A. 6}
\]

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