Electromagnetic and anisotropic extension of a plethora of well-known solutions describing relativistic compact objects

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Abstract We demonstrate a technique to generate new class of exact solutions to the Einstein-Maxwell system describing a static spherically symmetric relativistic star with anisotropic matter distribution. An interesting feature of the new class of solutions is that one can easily switch off the electric and/or anisotropic effects in this formulation. Consequently, we show that a plethora of well known stellar solutions can be identified as sub-class of our class of solutions. We demonstrate that it is possible to express our class of solutions in a simple closed form so as to examine its physical viability for the studies of relativistic compact stars.

Keywords Einstein-Maxwell system · Exact solution · Relativistic star · Anisotropy

1 Introduction

Exact solutions to Einstein-Maxwell system play a major role in the studies of relativistic compact objects. While the Reissner-Nordström solution uniquely describes the exterior gravitational field of a static spherically isolated object in the presence of an electromagnetic field, a large class of interior solutions are available in the literature which are regular, well behaved and physically meaningful. In the uncharged case, a large class of such exact solutions and their physical viability have been examined by Delgaty and Lake (1998). In the charged case, Ivanov (2002) has compiled different class of exact solutions.

In the recent past many new exact solutions have been developed some of which are, in fact, generalizations of many of the well-known solutions. Most of the extensions have generally been done either by incorporating an electromagnetic field or anisotropy or both into the system. The generalized models allow us to study the impacts of charge and/or anisotropy on the gross physical behaviour of a compact star. A prime motivating factor for such a generalization in most of our previous works was to fine-tune the stellar observables like mass and radius.

Local anisotropy, as indicated by many investigators in the past, plays a significant role in the studies of relativistic stellar objects (Ruderman 1972; Bowers and Liang 1974; Herrera and Santos 1997). In a recent article, it has been argued that pressure anisotropy cannot be ignored in the studies of relativistic compact stars as it is usually expected to develop by the physical processes inside such ultra-compact stars (Herrera 2020). Incorporation of an electromagnetic field in the studies of astrophysical objects is also well-motivated and many pioneering works have been done in this field in the past which includes the pioneering works of Majumdar (1947), Papapetrou (1947), de la Coopercosto and Cruz (1977) and Bekenstein (1971), amongst others. Consequently, different stellar models have been developed by relaxing the pressure isotropy condition as well as by incorporating a net charge into the system. Sharma et al. (2001) have generalized the widely used Vaidya and Tikekar (1982) stellar model by assuming a particular form of the electric field. The investigation shows a wide range of causal behaviour in the presence of the electric field. The Vaidya and Tikekar model was generalized by Kar-
markar et al. (2007) to analyze the impact of anisotropy on the maximum mass of a compact star. An anisotropic generalization of the Vaidya and Tikekar stellar model has been made by Thirukkanesh et al. (2019) recently. Earlier, Thirukkanesh et al. (2018) developed an algorithm to generalize a plethora of well-known exact solutions to Einstein field equations corresponding to a static spherically symmetric star by relaxing the pressure isotropy condition. Komathiraj and Sharma (2018) have developed a formalism to generate a new class of interior solutions corresponding to the exterior R-N metric which contained many previously found solutions. Komathiraj et al. (2019) also made an electromagnetic generalization of the Durgapal and Fuloria (1985) stellar solution. By relaxing the pressure isotropy condition, Sharma et al. (2017) generalized the Finch and Skea (1989) stellar model. For a specific charge distribution, Ratanpal et al. (2017) also made a generalization of the Finch and Skea stellar model to analyze the impact of the charge on the mass-radius relationship of a compact star, in particular. The relativistic stellar model of Mak and Harko (2004) was extended by Komathiraj and Maharaj (2007a) to include charge into the system. Maharaj et al. (2014) made a further generalization of the model by considering the system to be anisotropic as well. It is noteworthy that many of the static spherically symmetric anisotropic and/or charged stellar solutions available in the literature do not possess isotropic and/or charge neutral limits.

In this paper, we intend to generate new class of exact solutions corresponding to a static spherically anisotropic star possessing a net charge. The idea is that once the anisotropy and/or charge are/is switched off we should be able to regain some of the well-behaved, physically interesting stellar solutions found earlier. Such a generalization would allow us to investigate the impacts of anisotropy and charge on the physical features of a compact object in a neat manner. Moreover, physical acceptability of the generalized solutions can be ensured by suitable choice of the anisotropic and/or charge parameters as their isotropic and uncharged counterparts have already been found to be regular, well-behaved and physically meaningful.

The paper is organized as follows: In Sect. 2, we lay down the Einstein-Maxwell equations for an anisotropic fluid distribution. In Sect. 3, we propose a technique to generate solutions to the system of equations. We show how a large class of well known solutions can be regained from our general class of solutions. In Sect. 4, we express our solution in a closed form to analyze its features and physical viability. We conclude by discussing the key results of our investigation in Sect. 5.

## 2 Spacetime metric and field equations

We write the interior of a static spherically symmetric star by the line element

$$ds^2 = -e^{2v(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

(1)

in coordinates $(x^a) = (t, r, \theta, \phi)$, where $v(r)$ and $\lambda(r)$ are two unknown functions. For an anisotropic fluid in the presence of an electromagnetic field, we assume the energy momentum tensor in the form

$$T^i_j = \text{diag}(-\rho - \frac{1}{2}E^2, p_r - \frac{1}{2}E^2, pr + \frac{1}{2}E^2, p_t + \frac{1}{2}E^2),$$

(2)

where $\rho$ is the energy density, $p_r$ is the radial pressure and $p_t$ is the tangential pressure; measured relative to the comoving fluid 4-velocity $u^\nu = e^{-v}g_0^\nu$.

For the line element (1) and matter distribution (2), the Einstein field equations are obtained as

$$\frac{1}{r^2}(1 - e^{-2\lambda}) + \frac{2\lambda'}{r}e^{-2\lambda} = \rho + \frac{1}{2}E^2,$$

(3)

$$\frac{1}{r^2}(1 - e^{-2\lambda}) + \frac{2v'}{r}e^{-2\lambda} = p_r - \frac{1}{2}E^2,$$

(4)

$$e^{-2\lambda}\left(v'' + v'^2 + \frac{v'}{r} - v'\lambda' - \frac{\lambda''}{r}\right) = pt + \frac{1}{2}E^2,$$

(5)

$$\Delta = pt - p_r,$$

(6)

$$\sigma = \frac{1}{r^2}e^{-\lambda}(r^2E)^{\prime},$$

(7)

where a prime ($'$) denotes derivative with respect to the radial coordinate $r$. In the above, $E$ is the electric field, $\sigma$ is the charge density and $\Delta$ is the measure of anisotropy or anisotropic factor. We shall use units having $8\pi G = 1 = c$.

The mass of the gravitating object within a stellar radius $r$ is defined as

$$m(r) = \frac{1}{2} \int_0^r \tilde{r}^2 \rho(\tilde{r})d\tilde{r}.$$

(8)

Solutions of the above equations determine the physical behaviour of the anisotropic fluid distribution. The system (3)-(7) comprises five equations in eight unknowns namely, $v, \lambda, \rho, p_r, p_t, E, \Delta$ and $\sigma$. Therefore, it is necessary to choose any three of these variables involved in the integration process to solve the system.

## 3 Generating new solutions

Solutions to the system can be obtained by making physically reasonable choices for any three of the independent
where $a$ and $b$ are nonzero arbitrary constants and $R$ is the boundary of the star. A similar form of the metric potential was earlier used by Nasheeha et al. (2020) for the modeling of a neutron star and also by Komathiraj and Sharma (2018) for a superdense charged star. For particular choices of the parameters $a$ and $b$, it is possible to identify the metric ansatz with the following solutions: (i) charged stellar model of Komathiraj and Maharaj (2007b) for $b = 1$; (ii) stellar model of developed by Maharaj and Leach (1996) for $b = 1$; (iii) superdense stellar model developed by Tikekar (1990) for $a = -7$, $b = 1$; (iv) Vaidya and Tikekar superdense stellar model (Vaidya and Tikekar 1982) for $a = -2$, $b = 1$ and (v) Durgapal and Bannerji neutron star model (Durgapal and Bannerji 1983) for $a = 1$, $b = 1/2$. In other words, the general form (9) contains a large class of metric potentials which have been used to develop physically acceptable relativistic stellar models.

Using (4) and (5), for the particular choice (9), we obtain

$$\begin{align*}
\left(1 - \frac{a r^2}{R^2}\right) \left(1 - \frac{b r^2}{R^2}\right) (\nu'' + \nu^2 - \frac{\nu'}{r}) \\
- (b - a) \left(\frac{r}{R^2}\right) (\nu' + 1) + \frac{b - a}{R^2} \left(1 - \frac{a r^2}{R^2}\right) \\
- \left(1 - \frac{a r^2}{R^2}\right)^2 E^2 = \left(1 - \frac{a r^2}{R^2}\right)^2 \Delta,
\end{align*}$$

which is a highly non-linear differential equation with more than one unknowns. To make the equation tractable, it is convenient at this stage to introduce the following transformation

$$e^{\nu(r)} = \psi(x), \quad x^2 = 1 - \frac{b r^2}{R^2}.$$  \hfill (11)

The above transformation helps us to simplify the integration procedure as demonstrated by Maharaj and Leach (1996) and Komathiraj and Maharaj (2007b, 2010), amongst others.

Under the transformation (11), equation (10) becomes

$$\begin{align*}
(b - a + ax^2) \ddot{\psi} - ax \dot{\psi} \\
+ \left[\frac{a(a - b)}{b} + \frac{R^2(b - a + ax^2)^2}{b^2(x^2 - 1)}(E^2 + \Delta)\right] \psi = 0,
\end{align*}$$

in terms of the new dependent and independent variables $\psi$ and $x$ where a dot (.) denotes differentiation with respect to $x$. Under the transformation, the system (3) takes the following equivalent form

$$\begin{align*}
\rho &= \frac{b(b - a)}{R^2} \left(\frac{b - a + ax^2}{b^2(x^2 - 1)}\right)^2 - \frac{1}{2} E^2, \\
p_r &= \frac{b}{R^2(b - a + ax^2)} \left(-2bx \frac{\dot{\psi}}{\psi} + a - b\right) + \frac{1}{2} E^2, \\
p_t &= p_r + \Delta, \\
\sigma^2 &= \frac{b^2(2x E - (1 - x^2)\dot{E})^2}{R^2(1 - x^2)(b - a + ax^2)}.
\end{align*}$$

The mass function (8) becomes

$$m(x) = -\frac{R^3}{2b^2} \int_1^x \sqrt{1 - x^2} \rho(x) dx,$$

in terms of the new variable $x$.

Thus, we have essentially reduced the solution of the field equations to integrating equation (12). The differential equation (12) may be integrated once the electric field $E^2$ and the anisotropic factor $\Delta$ are known. Since we have the freedom to choose two more variables, we assume particular forms of the electric field $E^2$ and anisotropic parameter $\Delta$ at this stage. It is noteworthy that though a variety of choices are possible, the choices must ensure that they are regular, well behaved and can generate physically plausible stellar models. Keeping this in mind, we choose

$$E^2(x) = \frac{a b \alpha}{R^2(b - a + ax^2)^2},$$

$$\Delta(x) = \frac{b \beta \alpha}{R^2(b - a + ax^2)^2},$$

where $\alpha$ and $\beta$ are real constants. It should be pointed out that both the electric field and the anisotropic factor given in (18) and (19) are regular at the centre of the star. Our plan is to make use of these assumptions together with the potential (9) to generate new class of solutions describing a stellar configuration with desirable physical features. Using (18) and (19), we express (12) in the form

$$(b - a + ax^2) \ddot{\psi} - ax \dot{\psi} + \frac{a}{b} [(1 + \beta)(a - b) + \alpha] \psi = 0,$$

which is the master equation of the system and has to be integrated to find an exact model for a charged sphere with anisotropic pressure. Note that an uncharged and isotropic stellar solution can be regained simply by switching off the charge parameter $\alpha = 0$ and the anisotropic parameter $\beta = 0$ in (20). We intend to find new class of solutions for $\alpha \neq 0$ and $\beta \neq 0$. 
3.1 New class of charged anisotropic stellar solutions

In this section, we provide systematically a rich family of solutions to Einstein-Maxwell system describing an anisotropic charged superradeneed star in line with some of the previous treatments (Sharma et al. 2017; Thirukkanesh and Maharaj 2009; Maharaj and Thirukkanesh 2009; Maharaj and Komathiraj 2007; Komathiraj and Maharaj 2009; Maharaj and Thirukkanesh 2009c). We note that the point $x = 0$ is a regular singular point of the differential equation (20). Therefore, the solution of the differential equation (20) can be written in the form of an infinite series by the method of Frobenius:

$$
\psi(x) = \sum_{i=0}^{\infty} c_i x^i,
$$

(21)

where $c_i$ are the coefficients of the series.

To complete the solution we need to find the coefficients $c_i$ explicitly. Substituting (21) in (20), we obtain the recurrence relation

$$
a \left[ \frac{1}{b} [(1 + \beta) (a - b) + \alpha] + (i - 2) (i - 4) \right] c_{i-2}
+ (b - a) i(i - 1) c_i = 0, \ i \geq 2
$$

(22)

which governs the structure of the solution. With the help of (22), we express the general form for the even coefficients and odd coefficients in terms of the leading coefficient $c_0$ and $c_1$ respectively as:

$$
c_{2i} = \left( \frac{a}{a - b} \right)^i \frac{1}{(2i)!} \prod_{k=1}^{i} \left[ \frac{1}{b} [(a - b)(1 + \beta) + \alpha] \right] + (2k - 2)(2k - 4) c_0
$$

(23)

$$
c_{2i+1} = \left( \frac{a}{a - b} \right)^i \frac{1}{(2i+1)!} \prod_{k=1}^{i} \left[ \frac{1}{b} [(a - b)(1 + \beta) + \alpha] \right] + (2k - 1)(2k - 3) c_1.
$$

(24)

It is possible to verify the results (23) and (24) by using mathematical induction.

Using (21), (23) and (24), we can now generate the general solutions to (20), for the choice of the electric field (18) and the anisotropic factor (19), as

$$
\psi(x) = c_0 \psi_1(x) + c_1 \psi_2(x),
$$

(25)

where we have set

$$
1 + \sum_{i=1}^{\infty} \left( \frac{a}{a - b} \right)^i \frac{1}{(2i)!} \prod_{k=1}^{i} \left[ \frac{1}{b} [(a - b)(1 + \beta) + \alpha] \right] + (2k - 2)(2k - 4) x^{2i} = \psi_1(x)
$$

(26)

$$
x + \sum_{i=1}^{\infty} \left( \frac{a}{a - b} \right)^i \frac{1}{(2i+1)!} \prod_{k=1}^{i} \left[ \frac{1}{b} [(a - b)(1 + \beta) + \alpha] \right] + (2k - 1)(2k - 3) x^{2i+1} = \psi_2(x).
$$

(27)

The general solution (25) can be expressed in terms of elementary functions which is a more desirable form for the physical description of a charged anisotropic relativistic star. This is possible, in general, because the series (26) and (27) terminate for restricted values of the parameters $a, b, \alpha$ and $\beta$ so that elementary functions are possible.

In our work, we develop two sets of general solutions in terms of elementary functions by imposing the specific restrictions on the quantity $\frac{1}{b} [(a - b)(1 + \beta) + \alpha]$ for a terminating series. The elementary functions, obtained using this method, can be written as polynomials and polynomials with algebraic functions.

We express the first category of solutions to (20) as

$$
\psi(x) = A \sum_{i=0}^{n} \gamma^i \frac{(n + i - 2)!}{(n - i)! (2i)!} x^{2i} + B (b - a + ax^2)^{3/2} \sum_{i=0}^{n-2} \gamma^i \frac{(n + i)!}{(n - i - 2)! (2i + 1)!} x^{2i+1},
$$

(28)

where

$$
\gamma = \frac{4a [4n(1 - n) + 1 + \beta]}{4an(n - 1) + \alpha},
$$

$$
4n(1 - n) = \frac{1}{b} [(a - b)(1 + \beta) + \alpha],
$$

$$
x^2 = 1 - \frac{a(1 + \beta) + \alpha}{4n(1 - n) + 1 + \beta} \frac{r^2}{R^2}.
$$

The second category of solution is obtained as

$$
\psi(x) = A \sum_{i=0}^{n} \mu^i \frac{(n + i - 1)!}{(n - i)! (2i + 1)!} x^{2i+1} + B (b - a + ax^2)^{3/2} \sum_{i=0}^{n-1} \mu^i \frac{(n + i)!}{(n - i - 1)! (2i)!} x^{2i},
$$

(29)

where

$$
\mu = \frac{4a (2 - 4n^2 + \beta)}{a(4n^2 - 1) + \alpha},
$$

$$
1 - 4n^2 = \frac{1}{b} [(a - b)(1 + \beta) + \alpha],
$$

$$
x^2 = 1 - \frac{a(1 + \beta) + \alpha}{2 - 4n^2 + \beta} \frac{r^2}{R^2}.
$$

In the above, $A$ and $B$ are arbitrary constants.
Therefore, we generate two new class of solutions (28) and (29) in terms of elementary functions from the infinite series solution (25). It should be stressed that the new class of solutions holds good for isotropic as well as anisotropic; charged as well as uncharged cases. In the following, we demonstrate how our class of solutions can be used to regain a wide variety of previously developed well known stellar solutions:

### 3.2 Sub-class of solutions

It is not difficult to show that a large class of previously developed stellar models are actually sub-classes of our general class of solutions. The known solutions can either be explicitly regained from the general series solution (25) or from the elementary functions (28) and (29). This is illustrated by generating the following stellar models:

#### 3.2.1 The anisotropic and uncharged stellar model of Nasheeha et al. (2020)

**Class: I**

We set \(a = -\tilde{b}, \ b = (\tilde{b} - k)/2, \ \alpha = 0, \) and \(\beta = 2k/(k - 3\tilde{b})\) \((n = 1)\) in (29) so that \(\mu = 8\tilde{b}/(k - 3\tilde{b})\). Equation (29) then takes the form

\[
\psi(x) = A_1 \sqrt{2 + (k - \tilde{b})\tilde{x}} (5\tilde{b} - 3k + 2\tilde{b}(\tilde{b} - k)\tilde{x}) + B_1 (1 + \tilde{b}\tilde{x})^{3/2},
\]

where, \(x^2 = 1 - \frac{a(1 + \beta)}{\beta - 2} \frac{r^2}{\tilde{R}^2} = 1 + \frac{(k - \tilde{b})}{2} \tilde{x}, \ \tilde{x} = C r^2 (= \frac{r^2}{\tilde{R}^2}),\) and \(A_1, B_1\) are constants. The solution (30) was the first of its class solutions found by Nasheeha et al. (2020). The exact solution (30) has been comprehensively studied (Nasheeha et al. 2020) and it has been shown the solution corresponds to an uncharged and anisotropic fluid sphere satisfying all the necessary conditions of a physically acceptable stellar model.

**Class: II**

We set \(a = -\tilde{b}, \ b = (\tilde{b} - k)/7, \ \alpha = 0, \) and \(\beta = 7k/(k - 8\tilde{b})\) \((n = 2)\) in (28), so that \(\gamma = 28\tilde{b}/(k - 3\tilde{b})\). Subsequently, equation (28) becomes

\[
\psi(x) = A_2 [3(k - 8\tilde{b})^2 + 12\tilde{b}(k - 8\tilde{b})(7 + (k - \tilde{b})\tilde{x}) + 8\tilde{b}^2(7 + (k - \tilde{b})\tilde{x})^2] + B_2 (1 + \tilde{b}\tilde{x})^{3/2} \sqrt{7\tilde{b} - \tilde{b}(\tilde{b} - k)\tilde{x}},
\]

where, \(x^2 = 1 - \frac{a(1 + \beta)}{\beta - 2} \frac{r^2}{\tilde{R}^2} = 1 + \frac{(k - \tilde{b})}{2} \tilde{x}, \ \tilde{x} = C r^2 (= \frac{r^2}{\tilde{R}^2}),\) and \(A_2, B_2\) are constants. The solution (31) was the second class of solutions obtained by Nasheeha et al. (2020). Note that our solution (31) corrects a minor misprint in the result obtained by (Nasheeha et al. 2020).

#### 3.2.2 Charged stellar model of Komathiraj and Maharaj (2007b)

If we set \(\beta = 0\), equation (28) yields

\[
\psi(x) = A \sum_{i=0}^{n} (-\gamma)^i \frac{(n + i - 2)!}{(n - i)!} x^{2i} + B(b - a + ax^2)^{3/2} \times \sum_{i=0}^{a-2} (-\gamma)^i \frac{(n + i)!}{(n - i - 2)!(2i + 1)!} x^{2i+1},
\]

where \(\gamma = 4 - \frac{4b}{3b a(n-1) + a}, \ a + \alpha = b[2 - (2n - 1)^2], \ x^2 = 1 - \frac{br^2}{\tilde{R}^2}.
\)

If we set \(\beta = 0\), equation (29) yields

\[
\psi(x) = A \sum_{i=0}^{n} (-\mu)^i \frac{(n + i - 1)!}{(n - i)!} x^{2i+1} + B(b - a + ax^2)^{3/2} \times \sum_{i=0}^{n-1} (-\mu)^i \frac{(n + i)!}{(n - i - 1)!(2i)!} x^{2i},
\]

where \(\mu = 4 - \frac{4b}{b(4n^2-1)+a}, \ a + \alpha = 2b(1 - 2n^2), \ x^2 = 1 - \frac{br^2}{\tilde{R}^2}.
\)

Solutions (32) and (33) correspond to the isotropic charged stellar model of Komathiraj and Maharaj (2010). These solutions reduce to Komathiraj and Maharaj (2007b) model if we set \(b = 1\).

#### 3.2.3 Superdense stellar model of Maharaj and Leach (1996)

If we set \(b = 1, \ \alpha = 0\) and \(\beta = 0\), then equation (28) yields

\[
\psi(x) = A \sum_{i=0}^{n} (-\gamma)^i \frac{(n + i - 2)!}{(n - i)!} x^{2i} + B(1 - a + ax^2)^{3/2} \times \sum_{i=0}^{a-2} (-\gamma)^i \frac{(n + i)!}{(n - i - 2)!(2i + 1)!} x^{2i+1},
\]

where \(\gamma = 4 - \frac{4}{3n(n-1)}, \ a = [2 - (2n - 1)^2], \ x^2 = 1 - \frac{r^2}{\tilde{R}^2}.
\)

If we set \(b = 1, \ \alpha = 0\) and \(\beta = 0\), then (29) yields

\[
\psi(x) = A \sum_{i=0}^{n} (-\mu)^i \frac{(n + i - 1)!}{(n - i)!(2i)!} x^{2i+1} + B(1 - a + ax^2)^{3/2} \sum_{i=0}^{n-1} (-\mu)^i \frac{(n + i)!}{(n - i - 1)!(2i)!} x^{2i},
\]
where \( \mu = 4 - \frac{4}{(4n-1)}, \alpha = 2(1-2n^2), x^2 = 1 - \frac{r^2}{R^2} \).

These two categories of solutions (34) and (35) correspond to Maharaj and Leach (1996) model describing a relativistic compact star. The Maharaj and Leach solution has a simple form in terms of elementary functions and provides a physically reasonable model for neutron stars.

3.2.4 Tikekar (1990) superdense stellar model

If we set \( a = -7, b = 1, \alpha = 0, \) and \( \beta = 0 \) \((n = 2)\) in (28), then we get \( \gamma = -\frac{4}{7} \) and subsequently equation (28) yields

\[
\psi(x) = A_3 \left(1 - \frac{7}{2} x^2 + \frac{49}{24} x^4\right) + B_3x \left(1 - \frac{7}{8} x^2\right)^{3/2},
\]

(36)

where, \( x^2 = 1 - \frac{r^2}{R^2} \), \( A_3 \) and \( B_3 \) are constants. The solution (36) was found by Tikekar (1990) for the description of compact stars like neutron stars.

3.2.5 Vaidya and Tikekar (1982) compact stellar model

If we set \( a = -2, b = 1, \alpha = 0, \) and \( \beta = 0 \) \((n = 1)\) in (29), then we have \( \mu = -\frac{8}{7} \) and equation (29) becomes

\[
\psi(x) = A_4x \left(1 - \frac{4}{9} x^2\right) + B_4 \left(1 - \frac{2}{3} x^2\right)^{3/2},
\]

(37)

where, \( x^2 = 1 - \frac{r^2}{R^2} \), \( A_4 \) and \( B_4 \) are constants. The exact solution (37), developed by Vaidya and Tikekar (1982), has been widely used for the studies of relativistic compact stars.

3.2.6 Durgapal and Bannerji (1983) relativistic stellar model

If we set \( a = -1, b = \frac{1}{2}, \alpha = 0, \) and \( \beta = 0 \) \((n = 1)\) in (29), then we have \( \mu = -\frac{8}{7} \) and equation (29) yields

\[
\psi(x) = A_3(2 - \tilde{x})^{1/2}(5 + 2\tilde{x}) + B_4(1 + \tilde{x})^{3/2},
\]

(38)

where, \( x^2 = 1 - \frac{1}{2} \frac{r^2}{R^2} = 1 - \frac{1}{2} \tilde{x}, \tilde{x} = Cr^2 (= \frac{r^2}{R^2}) \), and \( A_3 \) and \( B_4 \) are constants which is the Durgapal and Bannerji (1983) stellar model. The model has been shown to satisfy all the physical requirements of a realistic star and has got widespread attention for the modeling of relativistic stellar configurations.

4 New closed-form solutions and its physical features

In the previous section, it has been shown how a large class of previously reported solutions can be regained from our general class of solutions. It is interesting to note that the solutions can also be obtained in simple analytic forms which facilitates its physical analysis. This is demonstrated as follows.

We set

\[
b = \frac{a(1 + \beta) + \alpha}{\beta - 2} \quad (n = 1)
\]

in (29) so that we have

\[
\mu = \frac{4a(\beta - 2)}{3a + \alpha}.
\]

Equation (29) then yields

\[
\psi(x) = Ax \left(1 + \frac{2a(\beta - 2)}{3(3a + \alpha)} x^2\right) + \frac{3a + \alpha}{\beta - 2 + a}\left(\frac{\beta - 2}{\beta - 2 + a}\right)^{3/2},
\]

(39)

where, \( x^2 = 1 - \frac{a(1+\beta)+a}{\beta-2} \frac{r^2}{R^2} \). Using (9), (11), (13)-(16), (18) and (19), we obtain

\[
e^{2\psi} = \frac{\alpha + 3a + a(\beta - 2)x^2}{(\alpha + a\beta)x^2},
\]

(40)

\[
e^{2\nu} = \psi^2,
\]

(41)

\[
\rho = \frac{(3a + \alpha)(a + \alpha + a\beta)[3a + a(5 + 2\beta + (\beta - 2)x^2)]}{R^2(\beta - 2)[3a + \alpha + a(\beta - 2)x^2]^2} - \frac{\alpha a(\beta - 2)(a + \alpha + a\beta)(x^2 - 1)}{2R^2[3a + \alpha + a(\beta - 2)x^2]^2},
\]

(42)

\[
\rho_r = -\frac{(a + \alpha + a\beta)}{R^2(\beta - 2)[3a + \alpha + a(\beta - 2)x^2]} \times \left[3a + \alpha + 2(a + \alpha + a\beta)\frac{1}{\psi}\right]^\frac{1}{\psi} + \frac{\alpha a(\beta - 2)(a + \alpha + a\beta)(x^2 - 1)}{2R^2[3a + \alpha + a(\beta - 2)x^2]^2},
\]

(43)

\[
p_t = \frac{\beta a(3a + \alpha)(a + \alpha + a\beta)(1 - x^2)}{R^2[3a + \alpha + a(\beta - 2)x^2]^2},
\]

(44)

\[
E_r^2 = \frac{\alpha a(\beta - 2)(a + \alpha + a\beta)(x^2 - 1)}{R^2[3a + \alpha + a(\beta - 2)x^2]^2},
\]

(45)

\[
\Delta = \frac{\beta a(3a + \alpha)(a + \alpha + a\beta)(1 - x^2)}{R^2[3a + \alpha + a(\beta - 2)x^2]^2},
\]

(46)

\[
\sigma^2 = -\alpha x^2(a + \alpha + a\beta)^3 \times \left[3a + a(5 + 2\beta + (\beta - 2)x^2)\right]^2 \frac{1}{R^4[3a + a(\beta - 2)x^2]^5},
\]

(47)

where \( \psi \) is given in (39).
The mass function (17) takes the form
\[
m(x) = -\frac{3\alpha R}{8a} \tanh^{-1} \left[ \sqrt{\frac{a(\beta - 2)(1 - x^2)}{(a + \alpha + a\beta)}} \right] - \frac{R\sqrt{\beta - 2)(1 - x^2)}{8a\sqrt{a + \alpha + a\beta}[3a + \alpha + a(\beta - 2)x^2]} \times [12a^2(x^2 - 1) - 3\alpha^2 - a\alpha(11 + \beta + 2(\beta - 4)x^2)].
\]

(48)

The simple closed-form nature of the above solution facilitates its physical analysis as discussed below.

We note from (40) that
\[
e^{2k}(r = 0) = 1, \quad (e^{2k})'(r = 0) = 0
\]
and from (41) we have
\[
e^{2\nu}(r = 0) = \left[ A\left(1 + \frac{2a(\beta - 2)}{3(\alpha + \alpha)}\right) + B\left(\frac{3a + \alpha}{\beta - 2} + a\right)\right]^{3/2},
\]
\[
(e^{2\nu})'(r = 0) = 0.
\]

Obviously, the gravitational potentials are regular at the origin.

From (42), we obtain the central density as
\[
\rho_0 = \frac{3(\alpha + \alpha)}{R^2(\beta - 2)},
\]
which implies that we must have
\[
\frac{(3a + \alpha)}{(\beta - 2)} > 0. \quad \text{(49)}
\]

Using (43) and (44), we obtain the radial and tangential pressures at \(r = 0\) as
\[
p_r(r = 0) = p_t(r = 0) = -\frac{3\alpha + \alpha + 2(a + \alpha + a\beta)\left[\frac{\psi}{\varphi}\right]_{r=0}}{R^2(\beta - 2)},
\]
where \(\psi\) is given in (39). That density and pressure should be positive puts the following bound on the model parameters
\[
0 > \frac{3a + \alpha}{2 - \beta} > \frac{2(a + \alpha + a\beta)}{(\beta - 2)\left(\frac{5a + 3a + 2a\beta}{3(3a + \alpha)} + B\left(\frac{a + a + a\beta}{\beta - 2}\right)^{2}\right)} \times \left(\frac{A\alpha - a + 2a\beta}{3a + \alpha} + 3Ba\sqrt{\frac{a + a + a\beta}{\beta - 2}}\right). \quad \text{(50)}
\]

At the boundary of the star \((r = R)\), the radial pressure must vanish, i.e.
\[
p_r(r = R) = p_t(r = \sqrt{1 - \frac{a(1 + \beta)}{\beta - 2}}) = 0,
\]
which yields
\[
B = \sqrt{\frac{2 + a + \alpha - \beta(1 - a)}{2 - \beta}} \left(\frac{\beta - 2}{a + a + a\beta}\right)^{2} \times \frac{s_1}{s_2}, \quad \text{(51)}
\]

where \(s_1 = 12(1 - a)(a + \alpha + \rho\beta)\left[-\alpha + a(1 + 2a - 2\beta + 2a(1 + \beta))\right] - [2\alpha + a(6 - 6a + a(\beta - 4))]\left[3\alpha + a(5 - 2a + 2\beta - 2a(1 + \beta))\right]\) and \(s_2 = 3(a - 1)(3a + \alpha)\left[-2\alpha + 2a(9 + 9a + 8a) + a(12 + 12a - \alpha)\right]\).

The exterior solution to the Einstein-Maxwell system for \(r > R\) is given by the Reissner-Nordström line element
\[
ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2\sin^2\theta d\phi^2, \quad \text{(52)}
\]

where, \(M\) and \(Q\) are the total mass and charge, respectively. Matching of the line element (1) and (52), across the boundary \(r = R\), we have
\[
\left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{-1} = \frac{(a-1)(\beta-2)}{3a+3a+3a\beta},
\]
\[
\left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{1/2} = A\sqrt{\frac{2+\alpha+\alpha-a(1-1/\beta)}{2-\beta}} \times \frac{\left[\frac{3a+3a+3a\alpha+2a\beta-2a(1+\beta)}{3(3a+\alpha)}\right]}{B\left(\frac{1-a(\alpha+\alpha+b)}{\beta-2}\right)^{1/2}}. \quad \text{(54)}
\]

These matching conditions and (51) help us to determine the constants \(A\) and \(B\) explicitly in terms of the model parameters \(a\), \(\alpha\) and \(\beta\) as follows:
\[
A = \frac{f(a, \alpha, \beta)}{g(a, \alpha, \beta)}, \quad \text{(55)}
\]
\[
B = \frac{h(a, \alpha, \beta)}{i(a, \alpha, \beta)}, \quad \text{(56)}
\]

where,
\[
f(a, \alpha, \beta) = -\sqrt{\frac{3a+\alpha}{1-a(\beta-2)})} \times [2\alpha - 2a(9 + 8a) + a\beta(12 + \alpha) - 6a^2(3 + 2\beta)],
\]
\[
g(a, \alpha, \beta) = 4(3a + \alpha) \sqrt{\frac{2 + a + \alpha - (1 - a)\beta}{2 - \beta}}.
\]
To analyze the behavior of the physical variables, for a star of radius $R = 1$ (with $8\pi G = c = 1$), we set $a = -1$, $\alpha = 0.5$ and $\beta = -0.1$, which are consistent with the bounds (49)-(51). Using these values in (55) and (56), we determine the constants as $A = 0.809637$ and $B = 1.01215$. Making use of these values, we analyze the physical features of the model. Figures 1 and 2 show that the gravitational potentials $\epsilon^{2\lambda}$ and $\epsilon^{2\nu}$ are continuous, regular and well-behaved at the interior of the star. Figure 3 shows that the energy density $\rho$ is positive, finite and monotonically decreases radially outward from its maximum value at the centre. The behaviour of radial pressure $p_r$ and the tangential pressure $p_t$ are plotted in Figs. 4 and 5 respectively which show that both the pressures are positive and decreasing monotonically while the radial pressure vanishes at the boundary. In
Fig. 6  Radial variation of electric field $E^2$

Fig. 7  Radial dependence of charge density $\sigma$

Fig. 8  Radial variation of anisotropy $\Delta$

Fig. 9  Fulfillment of energy conditions

Fig. 10  Sound speeds within the stellar interior

Fig. 6, we show that the electric field intensity $E$ is continuous throughout the interior and increases from the centre to the boundary, which is physically reasonable. The radial variation of the charge density is shown in Fig. 7. In Figure 8, radial variation of the anisotropic factor $\Delta$ is shown. We note that $\Delta$ is positive and monotonically increases from the centre until it attains a maximum value at the boundary of the star. This profile is similar to that obtained byaurya and Maharaj (2017) and Komathiraj et al. (2019). Figure 9 illustrates that the energy conditions $\rho + p_r + 2p_t > 0$ and $\rho - p_r - 2p_t > 0$ are satisfied throughout the stellar configuration. In Fig. 10, the sound speed in radial and transverse directions, i.e. $v_r^2 = \frac{d p_r}{d \rho}$ and $v_t^2 = \frac{d p_t}{d \rho}$ are shown which confirms that the causality condition is not violated in this model, a desirable feature for the modelling of a stellar structure as pointed out by Delgaty and Lake (1998). For an anisotropic fluid sphere, a potentially stable configuration is ensured if we have $-1 \leq v_t^2 - v_r^2 \leq 0$ (Herera 1992; Abreu et al. 2007). This condition is also satisfied in our model as shown in Fig. 11. Figure 12 shows the mass function profile within the stellar interior which is regular at the centre. Thus, we show that there exists particular sets of model parameters for which solution (40) satisfies all the requirements of a realistic star.
5 Discussion

Through our investigation, we have provided a general class of charged anisotropic relativistic stellar solutions which is regular and well-behaved. The most interesting feature of the class of solutions is that many well known stellar solutions can be regained simply by switching off the parameters specifying the anisotropy and/or charge distribution in this formulation.

It is to be stressed that for physical analysis, we have generated one particular closed form solution by suitably fixing the model parameters. It will be interesting to probe what other combinations of the model parameters can provide new solutions in simple analytic forms. This will be taken up elsewhere.

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