Local dependencies in random fields via a Bonferroni-type inequality

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Abstract. We provide an inequality which is a useful tool in studying both large deviation results and limit theorems for sums of random fields with "negligible" small values. In particular, the inequality covers cases of stable limits for random variables with heavy tails and compound Poisson limits of $0 - 1$ random variables.

1. Bonferroni-type inequalities in limit theorems for sums of stationary sequences

The simplest Bonferroni-type inequality can be formulated in the following way (see inequality I.17, p. 16, [GS96]):

\begin{equation}
0 \leq \sum_{i=1}^{n} P(A_i) - P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{1 \leq i < j \leq n} P(A_i \cap A_j),
\end{equation}

where $A_1, A_2, \ldots, A_n$ are events in some probability space.

In general this inequality gives very bad estimate for the difference $\sum_{i=1}^{n} P(A_i) - P(\bigcup_{i=1}^{n} A_i)$ (see p. 19, [GS96] for discussion of typical examples). However, when properly used, it brings essential simplification in many areas. Perhaps the most known (and the simplest) is the limit theory for order statistics of stationary sequences, as presented in [LLR83] or [G78]. It may be instructive to provide the reader with a brief outline of the reasoning leading to the basic result of this theory (Theorem 3.4.1, Chapter 3, [LLR83]).

Let $X_1, X_2, \ldots,$ be a stationary sequence and let $M_n = \max_{1 \leq i \leq n} X_i$ be partial maxima for this sequence. Given a sequence $\{u_n\}$ of numbers we want to calculate the limit for $P(M_n \leq u_n)$. For a large class of stationary sequences (satisfying so called condition $D(u_n)$), we can asymptotically replace $P(M_n \leq u_n)$ with

$$P(M_{[n/k_n]} \leq u_n)^{k_n},$$
with some $k_n \to \infty$. This in turn is asymptotically the same as

$$\exp(-k_n P(M_{[n/k_n]} > u_n)).$$

For fixed $n$, set $A_i = \{ X_i > u_n \}$ and observe that by

$$k_n P(M_{[n/k_n]} > u_n) - n P(X_1 > u_n) \leq k_n \sum_{1 \leq i < j \leq [n/k_n]} P(X_i > u_n, X_j > u_n).$$

If so called condition $D'(u_n)$ is also satisfied, then the last expression above tends to zero as $n \to \infty$ and we can calculate the limit for $P(M_n \leq u_n)$ as if the random variables $X_i$ were independent, i.e.

$$\lim_{n \to \infty} P(M_n \leq u_n) = \exp(- \lim_{n \to \infty} n P(X_1 > u_n)).$$

Condition $D(u_n)$ represents here “mixing” or “weak dependence” properties of the sequence in the form proper for maxima, while condition $D'(u_n)$ asserts that in the sequence $\{ X_i \}$ there are no local (within intervals of length $[n/k_n]$) clusters of values exceeding levels $u_n$. Since independent random variables satisfy condition $D'(u_n)$ for sequences $\{ u_i \}$ of interest, one can also say that the sequence essentially has no “local dependencies” between random variables. The latter terminology is even more convincing when one realizes that condition $D'(u_n)$ cannot hold for 1-dependent random variables $X_i = Y_{i-1} \vee Y_i$, where $Y_i$ is a sequence of independent and identically distributed random variables and $u_n$ is such that $\lim \inf_n n P(Y_1 > u_n) > 0$. Clearly, such $X_i$’s exhibit “local dependencies” and admit “local clusters” of values exceeding levels $u_n$.

It was R.A. Davis who first observed that similar results hold also for sums of stationary sequences with heavy tails. Using the technique of extreme value theory as well as the series representation for stable laws due to LePage, Woodroofe and Zinn, Davis [D83] proved that asymptotics of sums of “weakly dependent” stationary random variables with heavy tails and without local dependencies is essentially the same as if they were independent. Subsequent papers [JK89, DH95, K95] showed that the essence of Davis’ method was representing sums as integrals with respect to point processes on $\mathbb{R}^1 \setminus \{0\}$ built upon the sequence $X_1$. If one defines $N_n(A) = \sum_{i=1}^n I(X_i/B_n, A)$, then

$$\frac{X_1 + X_2 + \ldots + X_n}{B_n} = \int_{\mathbb{R}^1 \setminus \{0\}} x N_n(dx),$$

and weak convergence of $N_n$’s implies weak convergence of $S_n/B_n$. In particular, results for sums of dependent sequences with heavy tails can be obtained in a similar way as results for sums of independent sequences were derived in [R86] (this analogy is not applicable for functional convergence).

The difference between weakly dependent and independent case is that in the absence of conditions excluding clusters of “big” values (like $D'(u_n)$ in the theory for extremes), the parameters of the limiting stable law are determined by local dependence properties. Davis and Hsing [DH95] provide a probabilistic representation for these parameters. In some cases (e.g. for $m$-dependent random variables) another, much simpler representation is available [JK89], which is valid also for generalized Poisson limits [K95]. Comparing to stable limit theorems for $m$-dependent random vectors obtained by purely analytical methods by L. Heinrich in [H82, H85], probabilistic reasoning gave both deeper insight into the structure
of the limiting stable laws and allowed avoiding many of technicalities in formulation of results. When specialized to sums of \( m \)-dependent 0–1 random variables, the point processes method provides sufficient and necessary conditions for convergence to compound Poisson distribution \([\textup{K95}]\), contrary to the earlier methods based on Poisson approximations via the Chen-Stein method (see e.g. \([\textup{AGG90}]\)), where only sufficient conditions are given.

It is interesting that most of the above results can be obtained without employing point processes techniques and using the following Bonferroni-type inequality.

**Theorem 1.1 (Lemma 3.2, \([\textup{J97}]\)).** Let \( Z_1, Z_2, \ldots \) be stationary random vectors taking values in a linear space \((E, \mathcal{B}_E)\). Set \( S_0 = 0, S_k = \sum_{j=1}^{k} Z_j, k \in \mathbb{N} \).

If \( U \in \mathcal{B}_E \) is such that \( 0 \not\in U \), then for every \( n \in \mathbb{N} \) and every \( m, 0 \leq m \leq n \), the following inequality holds:

\[
|P(S_n \in U) - n \left( P(S_{m+1} \in U) - P(S_m \in U) \right)| \\
\leq 2mP(Z_1 \neq 0) + 2 \sum_{1 \leq i < j \leq n \atop j-i > m} P(Z_i \neq 0, Z_j \neq 0).
\]

(1.2)

Although inequality (1.2) does not fit the formal definition of the Bonferroni-type inequality given on p. 10 in \([\textup{GS96}]\), we call it Bonferroni-type for the following reasons.

(1) When \( m = 0 \) we obtain from (1.2)

\[
|P(S_n \in U) - nP(Z_1 \in U)| \leq 2 \sum_{1 \leq i < j \leq n} P(Z_i \neq 0, Z_j \neq 0),
\]

what is formally similar to (1.1). Notice that the constant 2 above is sharp.

(2) The inequality becomes interesting only if we deal with at least “weak dependence”, that is under purely probabilistic assumption.

(3) The inequality is proved by integrating its pointwise version and in this sense its proof is similar to proofs of the Bonferroni-type inequalities obtained by the “indicator method” (see \([\textup{GS96}]\)).

(4) In Section 3 we provide a unifying framework for both inequalities (1.1) and (1.2).

The inequality looks very restrictive and may seem applicable only to 0-1 stationary random variables \( Z_j = I_{A_j} \), in which case it reads as follows.

\[
\left| P\left( \sum_{j=1}^{n} I_{A_j} = k \right) - n \left( P\left( \sum_{j=1}^{m+1} I_{A_j} = k \right) - P\left( \sum_{j=1}^{m} I_{A_j} = k \right) \right) \right| \\
\leq 2mP(A_1) + 2 \sum_{1 \leq i < j \leq n \atop j-i > m} P(A_i \cap A_j).
\]

The above inequality can be directly applied to give an alternative (and much simpler!) proof of results due to Kobus \([\textup{K95}]\) for \( m \)-dependent 0–1 random variables.

Originally however inequality (1.2) was designed to manipulate with probabilities of large deviation for sums of random variables with heavy tails. An extensive
discussion of such results as well as their meaning for stable limit theorems (essential part of necessary and sufficient conditions) can be found in [J93], [J97] and [JNZ97] (for necessary results on stable laws we refer to [JW94] and [ST94]). Here let us sketch basic ideas only.

Let $X_1, X_2, \ldots$ be a stationary sequence, $S_n = X_1 + X_2 + \ldots + X_n$, $B_n \to \infty$ be a $1/p$-regularly varying sequence, where $0 < p < 2$ and let $x_n \to \infty$. We are interested in asymptotic behavior of large deviation probabilities $P(S_n/B_n > x_n)$. More precisely, we want to prove that under relatively mild assumptions

\begin{equation}
 x_n^p P(S_n/B_n > x_n) \to c_+,
\end{equation}

where the constant $0 < c_+ < \infty$ can be identified. The first step consists in proving that as $n \to \infty$

\begin{equation}
 x_n^p \left( P(S_n/B_n > x_n) - P\left( \sum_{j=1}^{n} Z_{n,j}^\delta > x_n \right) \right) \to 0,
\end{equation}

where

\[
 Z_{n,j}^\delta = \begin{cases} 
 0 & \text{if } |X_j| < B_n \cdot x_n \cdot \delta_n, \\
 X_j/B_n & \text{otherwise}. 
\end{cases}
\]

This requires some polynomial domination condition on tail probabilities of $X_j$’s and, if $1 \leq p < 2$, some assumptions on the size of variances of random variables $T_n^\delta = S_n/B_n - \sum_{j=1}^{n} Z_{n,j}^\delta$. In the next, essential step, we apply inequality (1.2) to random variables $Z_{n,j}^\delta$, $j = 1, 2, \ldots, n$, and $U = (x_n, \infty)$. Careful control of the size of $x_n$ and $\delta_n$ plus information on dependence (e.g. $m$-dependence) plus return to original random variables allow reducing (1.3) to

\begin{equation}
 x_n^p \left( P(X_1 + X_2 + \ldots + X_{m+1} > x_n B_n) - P(X_1 + X_2 + \ldots + X_m > x_n B_n) \right) \to c_+.
\end{equation}

This shows that the limiting parameter $c_+$ can be calculated using only finite dimensional (of size $m + 1$) distributions of the sequence $X_1, X_2, \ldots$, and that its value depends on local dependence structures, as desired.

Clearly, variants of the above reasoning with $m$ varying are also workable.

In the present paper we are going to prove an analog of (1.2) for random fields and in nonstationary case. Following the line of [J97] it allows deriving results for $m$-dependent random fields, similar to stable limit theorems of Heinrich [H86], [H87] or results on convergence to compound Poisson distributions [AGG90]. We leave their extensive discussion to other place.

2. A Bonferroni-type inequality for random fields

In what follows we choose and fix two integer numbers:

d - the dimension of the lattice $\mathbb{Z}^d$ indexing random fields $\{Z_t\}_{t \in \mathbb{Z}^d}$;
m - the admissible size of local clusters, $m \geq 0$.

If $\{Z_t\}$ is a random field and $\Lambda \subset \mathbb{Z}^d$ is a finite set, we define

\begin{equation}
 S_\Lambda = \sum_{t \in \Lambda} Z_t, \quad S_\emptyset = 0.
\end{equation}

Let

$$ B = \{0, 1, \ldots, m\}^d, $$
and $B_t = B + t$, $t \in \mathbb{Z}^d$. Further, let

$$
\mathcal{E} = \{0, 1\}^d = \{ \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d) : \varepsilon_j = 0 \text{ or } 1 \}
$$

and let

$$
B_\varepsilon^t = B_t \cap B_{t+\varepsilon}, \quad \varepsilon \in \mathcal{E}.
$$

(2.2) Define, for $U \in \mathcal{B}_E$ and $t \in \mathbb{Z}^d$,

$$
\Delta_t(U) = \sum_{\varepsilon \in \mathcal{E}} (-1)^{|\varepsilon|} P(S_{B_\varepsilon^t} \in U)
$$

where

$$
|\varepsilon| = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_d.
$$

Put $1 = (1, \ldots, 1) \in \mathcal{E}$. Define the “boundary” of a set $\Lambda \subset \mathbb{Z}^d$ by

$$
\partial \Lambda = \{ s \notin \Lambda : \exists t \in \Lambda s \in B_t \} \cup \{ t \in \Lambda : \exists s \in \Lambda^c t \in B_s \bsetminus B_s+1 \}
$$

(2.3) Notice that the second part of $\partial \Lambda$, consisting of points from $\Lambda$, is empty when $d = 1$.

**Theorem 2.1.** Let $Z_t$, $t \in \mathbb{Z}^d$ be a random field with values in a linear space $(\mathcal{E}, \mathcal{B}_E)$. If $U \in \mathcal{B}_E$ and $0 \notin U$ then

$$
|P(S_\Lambda \in U) - \sum_{t \in \Lambda} \Delta_t(U)| \leq c_1(d, m) \sum_{s \in \partial \Lambda} P(Z_s \neq 0) + c_2(d, m) \sum_{s,t \in \Lambda} P(Z_s \neq 0, Z_t \neq 0),
$$

(2.4) where $c_1(d, m) = 2^d((m+1)^d - 1)$, and $c_2(d, m) = 2^{-1}(1 + 2^d(2m + 1)^d)$.

**Proof.** Let

$$
\delta_t(U) = \sum_{\varepsilon \in \mathcal{E}} (-1)^{|\varepsilon|} I(S_{B_\varepsilon^t} \in U).
$$

(2.5) Since $\Delta_t(U) = E\delta_t(U)$, it is enough to establish a “pointwise” version of (2.4), i.e.

$$
|I(S_\Lambda \in U) - \sum_{t \in \Lambda} \delta_t(U)| \leq c_1(d, m) \sum_{s \in \partial \Lambda} I(Z_s \neq 0) + c_2(d, m) \sum_{s,t \in \Lambda} I(Z_s \neq 0, Z_t \neq 0),
$$

(2.6) where $c_1(d, m) = 2^d((m+1)^d - 1)$, and $c_2(d, m) = 2^{-1}(1 + 2^d(2m + 1)^d)$.

We shall deal with a modification of $Z_t$ which vanishes outside our set $\Lambda$:

$$
Z_t' = \begin{cases} 
Z_t & \text{if } t \in \Lambda, \\
0 & \text{if } t \notin \Lambda.
\end{cases}
$$

Let $S_\Lambda'$ and $\delta_t'(U)$ denote quantities defined by replacement of $Z_t$ with $Z_t'$ in formulas (2.1) and (2.3), respectively. Then by the very definition we have

$$
I(S_\Lambda \in U) = I(S_\Lambda' \in U).
$$
Further, $\delta_t(U) \neq \delta_t(U)$ implies that there exists $s \in B_t \cap \Lambda^c$ such that $Z_s \neq 0$. Hence we can estimate
\begin{align*}
|I(S_{\Lambda} \in U) - \sum_{t \in \Lambda} \delta_t(U) - (I(S'_{\Lambda} \in U) - \sum_{t \in \Lambda} \delta_t'(U))| &\leq \sum_{t \in \Lambda} 2^d I(\exists s \in B_t \cap \Lambda^c Z_s \neq 0) \\
&\leq 2^d \sum_{t \in \Lambda} \sum_{s \in B_t \cap \Lambda^c} I(Z_s \neq 0) =: R_1,
\end{align*}
(2.7)
where the factor $2^d$ comes from the cardinality of $E$. Furthermore, if $\partial_1 \Lambda$ denotes the first part of the boundary (2.3) consisting of points from $\Lambda^c$, then
\begin{align*}
R_1 &\leq 2^d \sum_{t \in \Lambda} \sum_{s \in B_t \cap \Lambda^c} \sum_{t \in \Lambda} I(B_t \cap \Lambda^c(s))I(Z_s \neq 0) \\
&\leq 2^d (m + 1)^d - 1 \sum_{s \in \partial_1 \Lambda} I(Z_s \neq 0).
\end{align*}
(2.8)
Hence it suffices to prove (2.6) under the assumption that
\begin{equation}
Z_t = 0 \text{ for } t \notin \Lambda.
\end{equation}
(2.9)
In this case the first sum on the right hand side of (2.6) will be over the second part of the boundary (2.3) consisting of points from $\Lambda$. Now define a random set
$$
\Lambda_0 = \{ s \in \mathbb{Z}^d : Z_t \neq 0 \}
$$
and let
$$
diam(\Lambda_0) = \sup\{ \|s - u\|_{\infty} : s, u \in \Lambda_0 \}.
$$
Notice that (2.9) gives
\begin{equation}
\Lambda_0 \subset \Lambda
\end{equation}
(2.10)
so that $diam(\Lambda_0)$ is a bounded random variable. For a fixed point $\omega$ in the probability space we will consider three particular cases of $diam(\Lambda_0(\omega))$:

**Case 1.** $diam(\Lambda_0) \leq m$.
This assumption implies that $\Lambda_0 \subset B_{t_0}$ for some $t_0 \in \mathbb{Z}^d$. Hence
\begin{equation}
I(S_{\Lambda} \in U) = I(S_{\Lambda_0} \in U) = I(S_{B_{t_0}} \in U).
\end{equation}
(2.11)
The first observation is that if $t \notin B_{t_0}$, then $\delta_t(U) = 0$ and consequently
\begin{equation}
\sum_{t \in \Lambda} \delta_t(U) = \sum_{t \in B_{t_0} \cap \Lambda} \delta_t(U).
\end{equation}
(2.12)
Indeed, for $t = (t_1, t_2, \ldots, t_d) \notin B_{t_0}$ we have
$$
\delta_t(U) = \sum_{\varepsilon \in E} (-1)^{\varepsilon} I(S_{B_t^{\varepsilon}} \in U) = \sum_{\varepsilon \in E} (-1)^{\varepsilon} I(S_{B_t^{\varepsilon} \cap B_{t_0}^{\varepsilon} \cap B_{t_0}} \in U).
$$
If \( t_k \geq t_k^0 \), \( k = 1, 2, \ldots, d \), where \( t_0 = (t_0^1, t_0^2, \ldots, t_0^d) \), then either \( t \in B_{t_0} \) or \( B_t \cap B_{t_0} = \emptyset \). The former possibility has been excluded and by the latter \( \delta_t(U) = 0 \). So assume that \( t_k < t_k^0 \) for some \( k \), \( 1 \leq k \leq d \). Let \( \varepsilon' = (\varepsilon_1, \ldots, \varepsilon_{k-1}, 0, \varepsilon_{k+1}, \ldots, \varepsilon_d) \) and \( \varepsilon'' = (\varepsilon_1, \ldots, \varepsilon_{k-1}, 1, \varepsilon_{k+1}, \ldots, \varepsilon_d) \). Then

\[
S_{B_t \cap B_{t+k+} \cap B_{t_0}} = S_{B_t \cap B_{t+k+} \cap B_{t_0}},
\]

hence

\[
(−1)^{|\varepsilon'|} I(S_{B_t \cap B_{t+k+} \cap B_{t_0}} \in U) + (−1)^{|\varepsilon''|} I(S_{B_t \cap B_{t+k+} \cap B_{t_0}} \in U) = 0,
\]

and so \( \delta_t(U) = 0 \). Thus (2.12) follows and now we will prove that

\[
(2.13) \quad I(S_{B_{t_0}} \in U) = \sum_{t \in B_{t_0}} \delta_t(U).
\]

Define

\[
q_t = I(S_{B_t \cap B_{t_0}} \in U)
\]

and for \( A \subset \mathbb{Z}^d \)

\[
Q(A) = \sum_{t \in A \cap B_{t_0}} q_t.
\]

Notice that for some \( t \in B_{t_0} \) the points \( t + \varepsilon \) may lie outside of \( B_{t_0} \), but then \( q_{t+\varepsilon} = 0 \) and so

\[
Q(B_{\varepsilon}) = \sum_{t \in B_{t_0}} q_{t+\varepsilon}.
\]

By the assumption currently in force

\[
(2.14) \quad \sum_{t \in B_{t_0}} \delta_t(U) = \sum_{t \in B_{t_0}} \sum_{\varepsilon \in \mathcal{E}} (−1)^{|\varepsilon|} I(S_{B_t \cap B_{t+k+} \cap B_{t_0}} \in U)
\]

\[
= \sum_{t \in B_{t_0}} \sum_{\varepsilon \in \mathcal{E}} (−1)^{|\varepsilon|} q_{t+\varepsilon}
\]

\[
= \sum_{\varepsilon \in \mathcal{E}} (−1)^{|\varepsilon|} \sum_{t \in B_{t_0}} q_{t+\varepsilon}
\]

\[
= \sum_{\varepsilon \in \mathcal{E}} (−1)^{|\varepsilon|} Q(B_{\varepsilon})
\]

The function \( Q \) is additive, hence by the inclusion-exclusion formula

\[
(2.15) \quad Q(\bigcup_{k=1}^{d} B_{e_k}) = \sum_{\varepsilon \in \mathcal{E}} (−1)^{|\varepsilon|-1} Q(\bigcap_{k: \varepsilon_k = 1} B_{e_k}) = \sum_{\varepsilon \in \mathcal{E}} (−1)^{|\varepsilon|-1} Q(B_{\varepsilon}),
\]

where \( e_k \) are the standard unit vectors in \( \mathbb{Z}^d \). Combining (2.14) and (2.15) we obtain

\[
\sum_{t \in B_{t_0}} \delta_t(U) = Q(B_0) - Q(\bigcup_{k=1}^{d} B_{e_k}) = q_0 = I(S_{B_{t_0}} \in U).
\]

Hence, in view of (2.11)–(2.12),

\[
I(S_{\Lambda} \in U) - \sum_{t \in \Lambda} \delta_t(U) = \sum_{\varepsilon \in B_{t_0} \cap \Lambda^c} \delta_\varepsilon(U).
\]
Observe that $\delta_s(U) \neq 0$ implies that there is $t \in B_s \setminus B_{s+1}$ such that $Z_t \neq 0$. Indeed, if this is not the case then $S_{B_t^s} = S_{B_t^1}$ for every $\varepsilon \in \mathcal{E}$, and

$$\delta_s(U) = \left(\sum_{\varepsilon \in \mathcal{E}} (-1)^{|\varepsilon|} I(S_{B_t^s} \in U)\right) = 0. \tag{2.16}$$

Consequently,

$$|I(S_\Lambda \in U) - \sum_{t \in \Lambda} \delta_t(U)| \leq 2^d \sum_{s \in \Lambda^c} \sum_{t \in \Lambda} I(t \in B_s \setminus B_{s+1}) I(Z_t \neq 0) \leq 2^d ((m+1)^d - m^d - 1) \sum_{t \notin \partial \Lambda} I(Z_t \neq 0)$$

which is clearly dominated by the right hand side of (2.6).

**Case 2.** $m < \text{diam}(\Lambda_0) \leq 2m$.

In this case there exist $t_0, s_0 \in \Lambda$ such that $Z_{t_0} \neq 0, Z_{s_0} \neq 0$, and $\|t_0 - s_0\|_\infty > m$. Hence

$$2^{-1} \sum_{s,t \in \Lambda, \|t - s\|_\infty > m} I(Z_s \neq 0, Z_t \neq 0) \geq 1 \tag{2.17}$$

and trivially,

$$I(S_\Lambda \in U) \leq 2^{-1} \sum_{s,t \in \Lambda, \|t - s\|_\infty > m} I(Z_s \neq 0, Z_t \neq 0). \tag{2.18}$$

By the present assumption there exists $t_0$ such that

$$\Lambda_0 \subset \{0, \ldots, 2m\}^d + t_0 := K_{t_0}.$$ 

Similarly as in Case 1 we argue that $\delta_t(U) = 0$ for $t \notin K_{t_0}$. Since each such term is bounded by $2^{d-1}$, we get

$$\sum_{t \in \Lambda} |\delta_t(U)| = \sum_{t \notin \Lambda \setminus K_{t_0}} |\delta_t(U)| \leq 2^{d-1}(2m+1)^d 2^{-1} \sum_{s,t \in \Lambda, \|t - s\|_\infty > m} I(Z_s \neq 0, Z_t \neq 0). \tag{2.19}$$

Now (2.18) and (2.19) together imply (2.6).

**Case 3.** $\text{diam}(\Lambda_0) > 2m$.

The assumption implies (2.17), hence (2.15). Moreover, for every $u \in \mathbb{Z}^d$ there exists $s \in \Lambda_0$ such that $\|u - s\|_\infty > m$. From the proof of Case 1 (see (2.16)) we know that $\delta_t(U) \neq 0$ implies that $Z_u \neq 0$ for some $u \in B_t \setminus B_{t+1}$ and, under the present assumption, for such an $u$ there is $s$ such that $Z_s \neq 0$ and $\|u - s\|_\infty > m$. 
Hence we have the following estimates
\[
\sum_{t \in \Lambda} \delta_t(U) \leq 2^{d-1} \sum_{t \in \Lambda} \sum_{u \in (B_t \setminus B_{t+1}) \cap \Lambda} \sum_{s \in \Lambda \atop \|s-u\|_\infty > m} I(Z_s \neq 0, Z_u \neq 0)
\]
\[
= 2^{d-1} \sum_{s,u \in \Lambda \atop \|s-u\|_\infty > m} \left[ \sum_{t \in \Lambda} I(u \in B_t \setminus B_{t+1}) \right] I(Z_s \neq 0, Z_u \neq 0)
\]
\[
\leq 2^{d-1} (m+1)^d - m^d \sum_{s,u \in \Lambda \atop \|s-u\|_\infty > m} I(Z_s \neq 0, Z_u \neq 0)
\]
\[
\leq 2^d (2m + 1)^d 2^{-1} \sum_{s,u \in \Lambda \atop \|s-u\|_\infty > m} I(Z_s \neq 0, Z_u \neq 0).
\]

The proof of Theorem 2.1 is complete. \(\square\)

3. An abstract form of the Bonferroni–type inequality

Consider a family of events \(A = \{A_T\}\) indexed by finite subsets \(T\) of \(\mathbb{Z}^d\). A family of events \(C = \{C_t\}\) indexed by points \(t \in \mathbb{Z}^d\) is said to be a complete cover of \(A\) if for every finite sets \(T, T_1, T_2 \subset \mathbb{Z}^d, A_T \subset \bigcup_{t \in T} C_t\) and

\[
A_{T_1} \triangle A_{T_2} \subset \bigcup_{t \in T_1 \triangle T_2} C_t.
\]

Define
\[
\Delta_t = \sum_{\varepsilon \in E} (-1)^{|\varepsilon|} P(A_{B_t^\varepsilon}),
\]

where \(B_t^\varepsilon\) is given by (2.2). Following the steps of the proof of Theorem 2.1 we can prove the following “abstract form” of the Bonferroni–type inequality.

**Theorem 3.1.** Let \(A\) and \(C\) be families satisfying (3.1). Then for every finite set \(\Lambda \subset \mathbb{Z}^d\)

\[
|P(A_\Lambda) - \sum_{t \in \Lambda} \Delta_t| \leq c_1(d, m) \sum_{s \in B_\Lambda} P(C_s)
\]

\[
+ c_2(d, m) \sum_{s,t \in \Lambda \atop \|s-t\|_\infty > m} P(C_s \cap C_t)
\]

where constants \(c_1\) and \(c_2\) are the same as in Theorem 2.1.

**Proof.** We will only indicate the main steps of the proof. Since \(\Delta_t = E\delta_t\), where

\[
\delta_t = \sum_{\varepsilon \in E} (-1)^{|\varepsilon|} I(A_{B_t^\varepsilon}),
\]
it is enough to prove that
\[
|I(A_\Lambda) - \sum_{t \in \Lambda} \delta_t| \leq c_1(d, m) \sum_{s \in \partial \Lambda} I(C_s)
\]
\[
+ c_2(d, m) \sum_{s, t \in \Lambda \mid |t - s|_{\infty} > m} I(C_s \cap C_t)
\]
holds everywhere on the probability space. First we will show that it suffices to prove (3.4) for the modifications \(A' = \{A'_T\}\) and \(C' = \{C'_t\}\) defined as follows
\[
A'_T = A_{T \cap \Lambda}
\]
and
\[
C'_t = \begin{cases} 
C_t & \text{if } t \in \Lambda, \\
\emptyset & \text{if } t \notin \Lambda.
\end{cases}
\]
Note that \(C'\) is a complete cover of \(A'\). Define \(\delta'_t\) by replacing in \(3.3\) \(A_B\) with \(A'_{B'_t}\). If \(\delta_t \neq \delta'_t\), then (3.1) yields
\[
|\delta_t - \delta'_t| \leq \sum_{s \in \mathcal{E}} |I(A_{B'_t}) - I(A'_{B'_t})|
\]
\[
\leq \sum_{s \in \mathcal{E}} \sum_{s \in B'_{t} \cap \Lambda^c} I(C_s)
\]
\[
\leq 2^d \sum_{s \in B'_{t} \cap \Lambda^c} I(C_s)
\]
which makes the reduction from \(A, C\) to \(A', C'\) possible, analogously to the first part of the proof of Theorem 2.1, (2.7)–(2.8). Now we can assume that \(C_t = \emptyset\) for \(t \notin \Lambda\). Define a random set
\[
\Lambda_0 = \{s \in \mathbb{Z}^d : I(C_s) = 1\}.
\]
(3.4) can now be established by considering the three cases of \(\text{diam}(\Lambda_0)\), exactly as in the proof of Theorem 2.1. \(\square\)

Theorem 3.1 gives Bonferroni-type inequalities in a variety of important cases. We mention below some of them.

Examples.

(i) \(C\) is an arbitrary family of events and
\[
A_T = \bigcup_{t \in T} C_t.
\]
In this case Theorem 3.1 generalizes the classical Bonferroni inequality \(1.1\).

(ii) \(A_T = \{\max_{t \in T} Z_t > \lambda\}\) and \(C_t = \{Z_t > \lambda\}\), where \(\{Z_t\}\) is a real-valued random field. This is a special important case of (i).

(iii) \(A_T = \{\sum_{t \in T} Z_t \in U\}\), where \(Z_t\) and \(U\) are as in Section 2, and \(C_t = \{Z_t \neq 0\}\). This shows that Theorem 2.1 is a special case of Theorem 3.1.

(iv) \(A_T = \{\prod_{t \in T} Z_t \in U\}\) and \(C_t = \{Z_t \neq 1\}\), where \(\{Z_t\}\) is a complex-valued random field and \(U\) is a Borel subset of the complex plane such that \(1 \notin U\).
(v) Let \( \{Z_t\} \) be a random field taking values in a measurable semigroup \( G \) with the neutral element \( I \). Fix a linear order in \( \mathbb{Z}^d \) to avoid the ambiguity in the definition of \( \Pi_T = \prod_{t \in T} Z_t \) in the case when \( G \) is non-Abelean (for instance, the lexicographical order). Then \( A_T = \{\Pi_T \in U\} \) and \( C_I = \{Z_t \neq I\} \) satisfy the assumptions of Theorem 3.1 provided \( I \not\in U \). In particular, Theorem 3.1 gives the Bonferroni-type inequality for products of random matrices.

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