Quantum Phenomenology for the Disoriented Chiral Condensate

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ABSTRACT

We consider the quantum state describing the Disoriented Chiral Condensate (DCC), which may be produced in high energy collisions. We show how a mean field treatment of the quantum equations corresponding to the classical linear sigma model leads to a squeezed state description of the pions emerging from the DCC. We examine various squeezed and coherent state descriptions of those pions with particular attention to charge and number fluctuations. We also study the phenomenology of multiple DCC domains.

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1 Introduction.

The possibility of Disoriented Chiral Condensate (DCC) production in high energy hadronic or heavy ion collisions is a subject of much current activity. After the first papers [1]-[4], this area has attracted much interest in the last 2 years and many different aspect have been discussed in numerous papers [5]-[19]. The aim of this paper is to discuss the possible quantum descriptions of the disoriented chiral condensate, paying particular attention to its decay into pions.

It is well known that the QCD Lagrangian is invariant (only approximately if nonzero masses for the light $N_f$ quarks are taken into account) under global chiral $SU(N_f)_L \times SU(N_f)_R$, where $N_f$ is the number of the light flavors. This symmetry is spontaneously broken to vector $SU(N_f)_V$, which leads to $N_f^2 - 1$ (quasi)goldstone bosons - pions (if $N_f = 2$) or pions, kaons and $\eta$ mesons (if $N_f = 3$). The order parameter for this breaking is the vacuum expectation value of the quark condensate $<\bar{\psi}\psi>$. In our normal vacuum, that condensate is an isotopic spin zero condensate (the $\sigma$). However one can imagine that under some special conditions, for example after high-energy collision, there is a region in which the condensate points in some other, arbitrary direction in isospin space. This “cool” region of disoriented chiral condensate, (DCC), would be surrounded by a “hot” relatively thin expanding shell, which separates the internal region from ordinary space. This picture was suggested by Bjorken, Kowalski and Taylor [6] and is called the ”Baked Alaska” scenario. After hadronization the interior disoriented vacuum will collapse, decaying into pions. The interesting signature of this process could be the coherent production of either nearly all charged or nearly all neutral pions. Such remarkable charge fluctuations have been reported in the Centauro cosmic ray events [20]. These events may be a manifestation of the DCC. Much of the theoretical impetus for discussions of the DCC have been based on classical models of QCD and of the classical pion field. But the pions that emerge from the decay of the DCC are quanta of that pion field, and hence we need a quantum discussion of the field emerging from the DCC. The principle results of this paper concern the possible forms of that quantum description.

There are two important quantum aspects of the DCC. The first involves the formation and evolution of the DCC and the second its decay into pions. The evolution of the DCC has been discussed in terms of a simple, classical, toy model, the linear sigma model. We will show that a quantum treatment of that model, in the mean field or Hartree approximation, suggest that the pions emerge from the decay of the DCC in a squeezed
state. Other quantum states for the pions coming from DCC decay have been proposed.
The coherent state is the first that comes to mind for quantizing a classical field. It has
been objected that a coherent state leads to large charge fluctuations. We will discuss
various coherent and squeezed states and show that a natural outgrowth of a real, classical
field theory of the pions is a real cartesian (in isospin) coherent or squeezed state and that
these states do not have large charge fluctuations. We also discuss briefly how a few
independent domains of DCC would affect the phenomenology.

In Section 2 we briefly review the classical linear sigma model treatment of the DCC.
This helps to establish our notation and connect our work to the DCC literature. In
Section 3 we show how mean field quantization of the sigma model leads to squeezed
states for the pions. Section 4 is a discussion of the various coherent state and squeezed
state pictures that can be used to describe DCC decay. We discuss treatments that
can be found in the literature and some new forms, with particular emphasis on solving
the problem of charge fluctuations. Section 5 briefly discusses the phenomenology of
independent domains of DCC and Section 6 presents a summary and conclusions. A
discussion of the neutral fraction probability for classical but not spherically symmetric
isospin distributions is given in Appendix A. Some technical material on squeezed states
and their relationship to coherent states is presented in Appendix B.

2 Simple Classical Model of the DCC

Instead of considering the real hadron world with all its complexities, most discussions of
the DCC begin with a very simple, toy, classical model describing the chiral dynamics, the
linear sigma-model. This expresses the dynamics in terms of a four component classical
field, \( \phi^a = (\sigma, \vec{\pi}) \), where \( \sigma \) and \( \vec{\pi} \) are isoscalar and isovector fields, respectively. The action
is given by

\[
S = \int d^4x \left[ \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{\lambda}{4} (\phi^a \phi^a - v^2)^2 + H\sigma \right]
\]  

(1)

where \( H \sim m_q \) describes the small explicit chiral symmetry breaking due to the quark
masses. The pion mass is \( m^2_\pi = H/f_\pi = \lambda(f^2_\pi - v^2) \), where \( f_\pi = < \sigma > \). The sigma
meson mass is \( m^2_\sigma = 3\lambda f^2_\pi - \lambda v^2 \approx 2\lambda f^2_\pi \). With \( m_\pi = 135MeV, m_\sigma = 600MeV \) and
\( f_\pi = 92.5MeV \) one has \( \lambda = 20 \) and \( v = 87.4MeV \).
The fields $\phi^a = (\sigma, \vec{\pi})$ parameterize the quark condensate

$$< \bar{\psi}^i \psi^j > \sim U^i_j = \sigma + i\vec{\pi} \tau = \begin{pmatrix} \sigma + i\pi^0 & i\pi^+ \\ -i\pi^- & \sigma - i\pi^0 \end{pmatrix}$$

(2)

Let us note that if one fixes the “absolute value” of the quark condensate, i.e. the determinant of the matrix $U^i_j$, $\det U = \sigma^2 + \vec{\pi}^2 = f^2_\pi$ one can get another representation, which is used usually for non-linear sigma-models:

$$U^i_j = f_\pi \exp \left( \frac{i\vec{\pi} \tau}{f_\pi} \right) = f_\pi \cos \frac{\pi}{f_\pi} + if_\pi \frac{\vec{\pi} \tau}{f_\pi} \sin \frac{\pi}{f_\pi}$$

(3)

where $\pi = |\vec{\pi}|$. In case of small $\pi << f_\pi = < \sigma >$ one can easily see the equivalence of these two parameterizations.

In the usual vacuum one has $< \sigma > = f_\pi$, $< \vec{\pi} > = 0$ and since $\sigma$ is an isoscalar there is an isoscalar condensate $< \bar{\psi}^i \psi^j > \sim \delta^i_j$ only. However one can consider another configuration - $< \sigma > = f_\pi \cos \theta$ and $< \vec{\pi} > = f_\pi \vec{n} \sin \theta$, here $\vec{n}$ is some unit vector in isospace. This condensate corresponds to the DCC, i.e. some classical pion field configuration that points in an arbitrary direction in isospin space, and which is metastable. Rajagopal and Wilczek have pointed out that the long wave length modes of the condensate grow exponentially fast after a quench \[5\]. After some time the metastable disoriented condensate decays into the normal vacuum by emitting pions. All directions in isospin space are equally probable, thus there is reasonable probability that the decaying DCC will lead to an event with an unusually large number of either neutral ($\pi^0$) or charged ($\pi^\pm$) pions. That would be a dramatic experimental signal for the DCC, a signal that may already have been seen in the Centauro events \[20\]. Using the classical picture of DCC, which has equal probability for all isotopic orientations, one finds \[1\] - \[4\] the probability $P(f) = 1/(2\sqrt{f})$, $\int_0^1 P(f) df = 1$ for the neutral pion fraction, $f = N_{\pi^0}/(N_{\pi^+} + N_{\pi^-} + N_{\pi^0})$. Thus this picture leads to a far higher probability for finding nearly all charged pions, $f \sim 0$ or nearly all neutral pions $f \sim 1$ in the decay of the DCC, than the very small probability for such distributions that would be found in a purely statistical decay of an excited pionic region.

The simple, classical, linear sigma model helps to motivate the DCC and its remarkable prediction for the distribution of pion charges in its decay. However the model is purely classical while the decay products of the DCC are pions, the quanta of the field. It is therefore important to study the quantum evolution of the DCC and its wavefunction for decay into pions. The usual approach to quantizing classical fields is to use coherent states \[21\]. We will discuss the various forms of coherent states that might be appropriate.
to the decay of the DCC and some of their problems, advantages and phenomenological consequences in Section 4, but before turning to a description we examine models of the quantum evolution of the DCC. We will see in the next Section that a quantum theory of the evolution of the DCC treated in the Hartree approximation leads to a picture of the decaying DCC in terms of two-pion coherent states, the so called “squeezed states.”

3 Quantum Evolution of the DCC and Squeezed States

As has been shown in [19] the most natural quantum description of the DCC is given in terms of squeezed states and the pion production in this state is a nice analog of an amplification of oscillations in a quantum parametric oscillator. To get the squeezed quantum state for the DCC let us consider the mechanism for amplification of the long wavelength pion modes, suggested by Rajagopal and Wilczek in paper [5], where the the dynamics of the $O(4)$ linear sigma model after quenching was considered. The amplification of the long wavelength pion modes was found in the period immediately after quenching. This amplification leads to the coherent pion oscillations, i.e. to the creation of the DCC. Such a behavior can be understood if one considers the equation of motion for the pion field

$$\frac{\partial^2}{\partial t^2} \pi(\vec{k}, t) + [\vec{k}^2 + \lambda(<\phi^2>(t) - v^2)]\pi(\vec{k}, t) = 0$$

(4)

where we substituted (as in [5]) the $\phi^a\phi^a$ in the nonlinear term in (4) by its spatial average $<\phi^2>(t)$. This is nothing but the Hartree or mean field approximation. If in the initial conditions one has $<\phi^2> < v^2$ then the long wavelength modes of the pion field with $\vec{k}^2 < \lambda(v^2 - <\phi^2>)$ start growing exponentially. The $<\phi^2>(t)$ starts to oscillate near the vacuum expectation value $<\sigma>$ and after some time the oscillations will be damped enough so that all the modes will be stable. Thus we see that at the classical level, each long wavelength mode is described by the equation for a parametrically excited oscillator and one gets the DCC as a result of the amplification of the zero-point quantum fluctuations of the pion field.

This picture is similar to one which was discussed in [22], where relic graviton production from zero-point quantum fluctuations during cosmological expansion was considered. For graviton mode with momentum $n$ the equation $y'' + [n^2 - (R''/R)]y = 0$ was obtained, where $R(\eta)$ is the scale factor of the metric $ds^2 = R^2(\eta)(d\eta^2 - d\vec{x}^2)$ and a prime represents $d/d\eta$. One can see that this equation is equivalent to the pion equation (4) if the scale factor $R$ is connected with $<\phi^2>(t)$ as $\lambda(v^2 - <\phi^2>(t)) = R''/R$. 

5
Our problem now is to present the quantum mechanical formulation in terms of pion creation and annihilation operators and to get the wave function of the DCC. In the mean field approximation the wave function \(|\Psi> = \prod_{i, \vec{k}} |\psi>_{i, \vec{k}}\) is the product of the wave functions for each mode with momentum \(\vec{k}\) and isotopic spin index \(i\). Later we shall omit \(i\). The equation of motion (4) for each mode \(\pi(\vec{k}, t)\) can be derived from the Lagrangian

\[
L_{k} = \frac{1}{2} \dot{\pi}^2(\vec{k}, t) - \frac{1}{2} \Omega^2(\vec{k}, t) \pi^2(\vec{k}, t)
\]

\[
\Omega^2(\vec{k}, t) = \vec{k}^2 + \lambda(<\phi^2>(t) - v^2)
\]

The wave function \(|\psi>_{\vec{k}}\) obeys the Schrödinger equation

\[
i \frac{\partial}{\partial t} |\psi>_{\vec{k}} = H_{k}(t) |\psi>_{\vec{k}} = \left[ \frac{1}{2} \mathcal{P}_{\pi}^2 + \frac{1}{2} \Omega^2(\vec{k}, t) \pi^2(\vec{k}) \right] |\psi>_{\vec{k}}
\]

where \(\pi(\vec{k})\) and \(\mathcal{P}_{\pi} = -id/d\pi(\vec{k})\) are the quantum-mechanical coordinate and momentum for the mode with the spatial momentum \(\vec{k}\).

After finding the wave function \(|\psi>_{\vec{k}}\) and constructing the full Hartree function \(|\Psi>\) one can get the value of the mean field \(<\phi^2>(t) = <\Psi | \sigma^2 + \pi^2 | \Psi>\) and then obtain an equation for the mean field evolution. The last part of this selfconsistent approximation will not be discussed in this paper, our aim is to analyze the general features of a Hartree wave function, taking the mean field \(<\phi^2>(t)\) as granted.

One can rewrite the Hamiltonian in (3) in terms of creation and annihilation operators which make it diagonal at any given moment. It is evident that we are interested in getting the wave function in terms of creation and annihilation operators of ordinary pions, so we must diagonalize the Hamiltonian at \(t \to \infty\), when the oscillation of the \(<\phi^2>(t)\) will be damped. Thus we define

\[
a(\vec{k}) = \frac{\omega(\vec{k}) \pi(\vec{k}) + i \mathcal{P}_{\pi}}{\sqrt{2 \omega(\vec{k})}}, \quad a^{\dagger}(\vec{k}) = \frac{\omega(\vec{k}) \pi(\vec{k}) - i \mathcal{P}_{\pi}}{\sqrt{2 \omega(\vec{k})}}
\]

where \(\omega(\vec{k}) = \Omega(\vec{k}, \infty) = \sqrt{\vec{k}^2 + m_{\pi}^2}\). It is easy to see that the Hamiltonian (up to a constant) is

\[
H_{k} = \frac{1}{2} \omega(\vec{k})[1 + \frac{\Omega^2(\vec{k}, t)}{\omega^2(\vec{k})}]a^{\dagger}(\vec{k})a(\vec{k}) - \frac{\omega^2(\vec{k}) - \Omega^2(\vec{k}, t)}{4 \omega(\vec{k})}[a^2(\vec{k}) + a^{12}(\vec{k})]
\]

It is the \(a^2(\vec{k})\) and \(a^{12}(\vec{k})\) terms in the Hamiltonian that transform the initial vacuum \(|0>\) into a squeezed state \(S(r, \phi)|0>\). The squeezed state is given by (24)

\[
|\xi> = S(\xi)|0> = \exp \left[ \frac{1}{2} \left( \xi a^2 - (a^{\dagger})^2 \right) \right] |0>
\]
and we are writing $\xi = r \exp(i\phi)$. More generally, one can consider the squeezed coherent state $S(\xi) \exp(\alpha a^\dagger)|0\rangle$. Some technical details about squeezed states are presented in Appendix B.

To calculate the squeezing and phase parameters $r$ and $\phi$ let us consider the solution of the Schrödinger equation (6) in the coordinate (here it is $\pi$) representation (we omit the mode label $\vec{k}$ for a moment)

$$<\pi|\psi>(t) = C(t) \exp(-B(t)\pi^2 + D(t)\pi)$$

(10)

where we also take into account the possibility of the time-dependent expectation value of the $\pi$ field $<\pi(t)> \approx <\psi(t)|\pi|\psi(t)> \sim D(t)/2B(t)$. This state corresponds to the squeezed coherent state. For $B = \omega/2$ this wave function describes the usual coherent state and, if $D = 0$, it is the vacuum state. For all other values this wave function describes the squeezed state (9) where the parameters $r$ and $\phi$ are connected to $B$ by the relation [24] (see also [26] - [22])

$$B = \frac{\omega \cosh r + \exp(2i\phi) \sinh r}{2 \cosh r - \exp(2i\phi) \sinh r}; \quad \cosh 2r = \frac{\omega^2 + 4|B|^2}{4\omega R e B}; \quad \sin 2\phi = \frac{1}{\sinh 2r} \frac{I m B}{R e B}$$

(11)

Substituting (11) into (6) one gets equations for $B(t), C(t)$ and $D(t)$:

$$i\dot{B}(t) = 2B^2(t) - \frac{1}{2} \Omega^2(t), \quad i\dot{D}(t) = 2B(t)D(t), \quad i\dot{C}(t)/C(t) = B(t) - \frac{1}{2} D^2(t)$$

(12)

which means that $B(t)$ is related to the solution of the classical equation (11)

$$B(t) = -\frac{i}{2} \frac{\dot{\psi}(t)}{\psi(t)}; \quad \ddot{\psi}(t) + \Omega^2(t)\psi(t) = 0$$

(13)

in terms of which $D(t)$ can also be obtained

$$D(t) = D(0) \exp(-2i \int_0^t B(\tau)d\tau) = D(0) \exp\left(- \int_0^t \frac{\dot{\psi}(\tau)}{\psi(\tau)} d\tau\right) = D(0) \frac{\psi(0)}{\psi(t)}$$

(14)

and $C(t) = C(0) \exp\left(i \int_0^t (B(\tau) - D^2(\tau)/2)d\tau\right)$ is an evident phase factor. The equation (13) can be viewed as a Schrödinger equation describing the wave function $\psi(t)$ of a “particle” with mass $m = 1/2$ on a line $t$ having energy $\omega^2(k) = k^2 + m^2_\pi$ and moving through the potential barrier $V(t) = -\lambda(<\phi^2>(t) - f_\pi^2)$$

$$-\frac{d^2\psi(t)}{dt^2} + \lambda(f_\pi^2 - <\phi^2>(t))\psi(t) = \omega^2(k)\psi(t)$$

(15)
Far from the barrier, i.e. at \( t \to \pm \infty \) one has \( V(\pm \infty) = 0 \) and the general solution of the Schrödinger equation at \( t \to \pm \infty \) is a superposition of left and right moving waves

\[
\psi^+(t) = S_R^+ e^{-i\omega(k)t} + S_L^+ e^{+i\omega(k)t}, \quad t \to +\infty
\]

\[
\psi(t) = S_R^- e^{-i\omega(k)t} + S_L^- e^{+i\omega(k)t}, \quad t \to -\infty
\]  \( (16) \)

Due to unitarity the total fluxes at \( t \to \pm \infty \) must be equal

\[
|S_L^+|^2 - |S_R^+|^2 = |S_L^-|^2 - |S_R^-|^2
\]

and one can find

\[
S_R^+ = \cosh rS_R^- - e^{2i\theta} \sinh rS_L^-
\]

\[
S_L^+ = -e^{-2i\theta} \sinh rS_R^- + \cosh rS_L^-
\]  \( (17) \)

where \( \theta \) is the scattering phase and the factor \( r \) is defined by the probability of transition through the barrier.

Let us remember that if we are starting from the vacuum at \( t \to -\infty \), i.e. from \( B = \omega/2 \), one must have \( S_R^- = 0 \). Even if we assume that the initial state is not the vacuum, but some coherent state with nonzero \( D(0) \) we still have to put \( B = \omega/2 \) initially. This means that at the left (large negative \( t \)) we have only a left moving outgoing wave \( S_L^- e^{+i\omega(k)t} \). At the right (large positive \( t \)) one has both left and right moving waves, i.e. the incoming \( S_L^+ e^{+i\omega(k)t} \) and reflected \( S_R^- e^{-i\omega(k)t} \) waves. The transition coefficient can be obtained from (17) by putting \( S_R^- = 0 \)

\[
\frac{|S_L^-|^2}{|S_L^+|^2} = \frac{1}{\cosh^2 r}
\]  \( (18) \)

Now let us calculate \( B(t) = -(i/2)(\psi/\dot{\psi}) \) at large positive \( t \). Using (17) one can find after simple calculation (restoring the \( k \) dependence) :

\[
B(k) = \frac{\omega(k) \cosh r(k) + \exp[2i(\theta - \omega(k)t)] \sinh r(k)}{2 \cosh r(k) - \exp[2i(\theta - \omega(k)t)] \sinh r(k)}
\]  \( (19) \)

in complete agreement with (11)), with the phase factor \( \phi(t) \) depending on time as \( \phi = \theta - \omega(k)t \).

The squeezing parameter \( r(k) \) depends on the absolute value, \( k \), of the mode spatial momentum \( \vec{k} \) and is determined by the probability of tunneling through the potential barrier \( V(t) = \lambda[f_x^2 - <\phi^2>(t)] \). Let us note that tunneling takes place precisely when \( \omega^2(k) - V(t) < 0 \), i.e. when the classical long wavelength modes are exponentially amplified and we see once more that squeezing is ultimately connected with the exponential growth
of the classical long wavelength modes and the squeezing for each mode \( k \) is determined by the function \( \langle \phi^2 \rangle (t) \). This is the only input information we must know to calculate the DCC wave function.

After calculating \( B(t) \) one can study the effect of a displacement \( D(t) \). All we need to know is the ratio \( \psi(t)/\psi(0) \), where “0” is an arbitrary initial time and we shall put it to \( t_\rightarrow \rightarrow -\infty \). It is easy to see that (we put the scattering phase \( \theta = 0 \))

\[
D(t_+ \rightarrow \infty) = D(t_- \rightarrow -\infty) \frac{S_L^- \exp(+i\omega(k)t_-)}{S_R^+ \exp(-i\omega(k)t_+)} + S_L^+ \exp(+i\omega(k)t_+) =
\]

\[
e^{-i\omega(k)(t_+-t_-)} \frac{1}{\cosh r (1 - \exp(-2i\omega(k)t_+) \tanh r)}
\]

and one can see that at large \( t_+ \), on average, \( D(t_+) \) is small due to the suppression factor \( 1/\cosh r \). Only at moments \( t = \pi n/\omega \) during the short intervals \( \delta t \sim e^{-2r}/\omega \) does one get large \( D(t) \). However these pulses may be an artefact of the approximation, with the average \( D(t) \) being of order \( e^{-r} \). Thus we see that for those modes for which the squeezing is large, the initial displacement, i.e. the coherent part of the wave function, disappears at late times independently of how large it was in the beginning. The mean value of the pion field \( \langle \pi(t) \rangle \sim D(t)/B(t) \) and one gets

\[
\langle \pi(t) \rangle \sim e^{-i\omega(k)(t_+-t_-)} \frac{1}{\cosh r (1 + \exp(-2i\omega(k)t_+) \tanh r)}
\]

and on average, \( \langle \pi(t) \rangle \) is also suppressed by a factor \( 1/\cosh r \). Only at moments \( t = (\pi n + \pi/2)/\omega \) during the short intervals \( \delta t \sim e^{-2r}/\omega \) does one get large \( \langle \pi(t) \rangle \). However these pulses may be also an artefact of the approximation we use as the pulses for \( D(t_+) \) and on average the classical pion field is small. This is a very important point demonstrating that for low momentum pion modes, the classical pion field, even if it were produced at some intermediate step, will ultimately be negligible. This is for the displacement field. The average of the square of the pion field, \( \langle (\pi(t))^2 \rangle \) is, of course, not small but is the exponentially growing condensate of the DCC. This is the part that leads to the squeezed state.

In the quasiclassical approximation it is easy to calculate the squeezing parameter \( r(k) \):

\[
r(k) = 2Re \int dt \sqrt{\lambda[v^2 - \langle \phi^2 \rangle (t)] - k^2}
\]

which is valid for small \( k \) when \( r(k) \gg 1 \) The average number of particles in each mode is \( < N_k > = \sinh^2 r(k) \) and we see that \( < N_k > \) sharply decreases with the increase of \( k \).
To make a rough estimate let us consider a simple model for $<\phi^2>(t)$. We assume that $<\phi^2>(t) = 0$ in the interval $t \in (0, \tau)$ and is equal to its usual v.e.v. $<\phi^2>(t) = f_\pi^2$ outside this interval. Such behavior for $<\phi^2>(t)$ is a very rough approximation to a more realistic behavior obtained in [5] in numerical experiments, nevertheless one can use it for some preliminary estimates. The transmission coefficient in this case is well known (see for example [28]) and in our notation takes the form

$$\frac{1}{\cosh^2 r(k)} = \frac{4\omega^2(k)(m_\sigma^2/2 - \omega^2(k))}{4\omega^2(k)(m_\sigma^2/2 - \omega^2(k)) + (m_\sigma^2/2)^2 \sinh^2 \tau \sqrt{m_\sigma^2/2 - \omega^2(k)}}$$

(23)

where we used the relation $m_\sigma^2 = 2\lambda f_\pi^2$. Then the average number of particles with momentum $k$ is

$$<N(k)> = \sinh^2 r(k) = \left(\frac{m_\sigma^2}{2}\right)^2 \sinh^2 \tau \sqrt{m_\sigma^2/2 - \omega^2(k)}$$

$$\frac{4\omega^2(k)(m_\sigma^2/2 - \omega^2(k))}{4\omega^2(k)(m_\sigma^2/2 - \omega^2(k))}$$

(24)

One can estimate the average number of particles $<N(k)>$ for different $k$. For $\tau$ we shall use an estimate $\tau = 3 - 6 m_\sigma^{-1}$ which comes from the results of [5] $\tau = 1 - 2 (200MeV)^{-1}$. Then for $k = 0$ one gets $<N(0)> \approx 10^2 - 10^3$, at $k = m_\pi$ the $<N(m_\pi)>$ is approximately twice as small but at $k = 3m_\pi \approx 400MeV$ it is two orders of magnitude smaller $<N(3m_\pi)> \approx 1 - 10$. Of course this is the very crude estimate, however it demonstrates the qualitative features of the phenomenon - the sharp exponential dependence on $k$ and large amplification factor in $r$ which is of order of $m_\sigma \tau$, where $\tau$ is the characteristic time of damping of $<\phi^2>(t)$ oscillations.

In modeling a realistic “Baked Alaska” scenario one should include the effect of the expansion. In [5] it was suggested that the expansion be described by including the term $\dot{a}\pi/a$ in the equations of motion, where $a(t)$ is a scale factor for the expanding plasma. This is the same as considering the problem in a space-time with metric $ds^2 = dt^2 - a^2(t) d\vec{x}^2$. It is easy to show that choosing the new time $\tilde{t}(t) = \int dt/a^3(t)$ one gets the same Schrödinger equation as in (6) but with new $\tilde{\Omega}^2(k, \tilde{t}) = a^6(\tilde{t}) \Omega^2(k, \tilde{t})$. Thus taking into account the expansion one gets squeezed state again but with parameters which depend on the details of an expansion - the scale factor $a(t)$. In a case of anisotropic 1+1 expansion, which seems more realistic at early stages of the DCC formation (if any) and which was considered in [5], [8] and later in [18] the convenient coordinates parameterizing the interval are proper time $\tau$ and rapidity $\eta$

$$\tau = \sqrt{t^2 - x^2}, \quad \eta = \frac{1}{2} \ln \left(\frac{t - x}{t + x}\right)$$

(25)
and $ds^2 = d\tau^2 - \tau^2 d\eta^2 - dx_1^2$. In this case one gets the Schrödinger equation for evolution in $\tau$ with the $\Omega^2(k_\perp, k_\eta, \tau)$ where momentum $k_\eta$ appears in the combination $k_\eta^2/\tau^2$, which means that at small $\tau$ one has amplification only at very low $k_\eta$. One can get analogs of (24). In this case it will be $\mathcal{N}(k_\eta, k_\perp)$ and one can get the distribution in rapidities $\eta$ after taking the Fourier transform in $k_\eta$.

Thus we see that quantizing the linear sigma model for the DCC and treating the non-linear equation in the Hartree or mean field approximation leads to a simple picture of the quantum DCC that suggest a squeezed state treatment of the emerging pion waves. We also find the exponential amplification of the long wavelength modes that is seen classically leading, in a simple model, to a very large number of low energy coherent pions. We now turn to various treatments for the wave function of those emerging pions.

4 Quantum Wavefunctions for Decay of the DCC

There are many ways to go from a classical pion field theory to a quantum state, and many of these have appeared in the literature of DCC and other places. In this Section, rather that only give the squeezed state description we developed in Section 3, we present and contrast these various treatments. Thus this section does not depend on the squeezes state scenario, but rather discusses a range of possible quantum treatments of the classical decaying DCC.

The simplest quantum description of a classical field is given in terms of a coherent state \cite{21}

$$|\pi> = \exp\left(-\frac{\bar{N}}{2}\right) \exp\left(\int d^3\vec{k}\pi(\vec{k}) \cdot a^\dagger(\vec{k})\right) |0>$$  \hspace{1cm} (26)

where $\pi$ is the isotriplet pion field and and $a^\dagger$ are corresponding creation operators. The mean number of pions in the state is given by

$$\bar{N} = \int d^3\vec{k}\pi \cdot \pi^*$$  \hspace{1cm} (27)

which number also comes into the normalization of the coherent state. The root mean square number fluctuation in this state is $\sqrt{\bar{N}}$, and hence the fractional number fluctuation decreases for large $\bar{N}$. If one defines position and momentum like variables for each mode, $\vec{k}$, by

$$\vec{r} = \frac{1}{\sqrt{2}}(a^\dagger + a)$$  \hspace{1cm} (28)
and
\[ \vec{p} = -i \frac{1}{\sqrt{2}} (a^\dagger - a) \] (29)
one also finds that the coherent state is a minimal uncertainty state. It is these two
properties, minimal uncertainty and small number fluctuation in the large \( \bar{N} \) limit, that
make the coherent state attractive as the quantum representative of the classical field.

However this coherent state wave function has been criticized in [4] because, for arbi-
trary \( \pi(\vec{k}) \) such a description leads to large charge fluctuations in the pion system. In the
remainder of this section we will discuss ways of addressing this criticism while maintain-
ing as much as possible of the advantages of the coherent state, minimum uncertainty and
small number fluctuations, and also maintaining the relatively large probability of finding
a neutral pion fraction near zero or one. We will divide our discussion in two parts, states
with identically zero charge and charge fluctuations, and states with only average charge
zero.

Let us begin with states with average charge zero but still having charge fluctuations.
Consider the coherent state defined in (26). Let us use a cartesian rather than a spherical
tensor representation for the isospin of the pions, both for the creation and annihilation
operators and for the classical field, \( \pi(\vec{k}) \). Since the classical field equations are real, we
might expect that classical field configuration to be real as well. Hence a real cartesian
quantum description is a natural outgrowth of the classical beginnings of the DCC. We
will now show that this leads to zero expected charge in the coherent state and small
fractional charge fluctuations. In terms of cartesian pions, the number operator for \( \pi^+ \) is
given by
\[ n_+ = \frac{1}{2} (a^\dagger_x a_x + a^\dagger_y a_y) + \frac{i}{2} (a^\dagger_y a_x - a^\dagger_x a_y) \] (30)
and for \( \pi^- \) by
\[ n_- = \frac{1}{2} (a^\dagger_x a_x + a^\dagger_y a_y) - \frac{i}{2} (a^\dagger_y a_x - a^\dagger_x a_y) \] (31)
so that the charge, \( Q = n_+ - n_- \) is
\[ Q = i (a^\dagger_y a_x - a^\dagger_x a_y) \] (32)
This is for each mode, \( \vec{k} \), the total charge is the integral over \( \vec{k} \). It is easy to see that, for
real cartesian pion field, the expected value of the charge in the coherent state is zero.
The expected value of the charge squared is
\[ < Q^2 > = \int d^3 \vec{k} (\pi_x^2 + \pi_y^2) \] (33)
which is a number of order $\bar{N}$. Hence the fractional root mean square charge fluctuations are of order $1/\sqrt{\bar{N}}$ and therefore small in the large $\bar{N}$ limit. It is clear that the real cartesian coherent state also continues to be a minimal uncertainty state.

We would expect that in the DCC the classical pion field will point in some fixed direction in isospin space for all $\vec{k}$. Suppose we label that direction with the usual polar angles, $\theta$, $\phi$, and continue to use the real cartesian coherent state. Then it is easy to see that the average number of $\pi^0$'s is given by $\bar{N}\cos^2 \theta$. The neutral pion fraction is thus given by $f = \cos^2 \theta$, and corresponds to a probability of $P(f) = 1/(2\sqrt{f})$, as before. The root mean square pion charge fluctuation is given by $\sqrt{\bar{N}}\sin \theta$ and the fractional charge fluctuation by $\sin \theta/\sqrt{\bar{N}}$, which is small in the large $\bar{N}$ limit. Hence the real cartesian coherent state has many desirable features for describing the radiation of the DCC. Its major faults are that it does not have identically zero charge and that it is not well linked to a simple quantum theory of the DCC dynamics itself, which suggests squeezed states. We now turn to these issues.

Let us begin with the application of squeezed states to the problem of decay of the DCC. We first review some features of squeezed states. For a single mode created by the operator $b^\dagger$, the normalized squeezed state may be written

$$|\alpha> = (1 - |\alpha|^2)^{1/2} \exp\left(\frac{\alpha}{2} (b^\dagger)^2\right) |0>$$

where $\alpha$ is a complex number with modulus less than one and is related to the parameters of (9). (See Appendix B for more details.) The squeezed state is a minimum uncertainty state, for position and momenta defined in terms of the operators $b$ as in (28) and (29). In the case of the coherent state, that minimum uncertainty is achieved by having each of $\Delta x$ and $\Delta p$ take on their minimum values. For the squeezed state, only their product need be minimum, and one can be large while the other is “squeezed” to keep the product fixed. An important difference between the squeezed state and the coherent state comes in the number fluctuations. For the squeezed state we find

$$<\alpha|(b^\dagger b)^2 - (\bar{N})^2|\alpha> = 2\bar{N}(\bar{N} + 1)$$

which means that the fractional root mean square fluctuations are of order 1, in the large $\bar{N}$ limit rather than of order $1/\sqrt{\bar{N}}$ as they are in the coherent state case.

In spite of the large number fluctuations in the squeezed state, let us look at its application to the DCC. Suppose we construct the operator $b^\dagger$ from a real isospin rotation of the cartesian pion operators

$$b^\dagger = \cos \theta a^\dagger_z + \sin \theta \cos \phi a^\dagger_x + \sin \theta \sin \phi a^\dagger_y$$

(36)
where $\theta$ and $\phi$ are the usual polar angles. We may now label our squeezed state $|\alpha, \theta, \phi\rangle$. It is clear that the mean number of pions in the state is still $\bar{N} = \frac{|\alpha|^2}{1-|\alpha|^2}$. The expectation of the total charge in this state is easily seen to be zero. The expected value of $\pi^0$'s is given by

$$<\alpha, \theta, \phi|a^+_z a_z|\alpha, \theta, \phi> = \bar{N}(\cos \theta)^2$$  \hspace{1cm} (37)

The neutral pion fraction is $f = \cos^2 \theta$ and its probability is again $P(f) = \frac{1}{2\sqrt{f}}$. Hence the squeezed state with arbitrary real cartesian isospin direction has zero average charge, minimum uncertainty, and a large probability for neutral fraction near zero or near one. However it has large (of order $\bar{N}$) charge fluctuations.

To deal with the charge fluctuations, it was suggested in [4] that the quantum state that describes the DCC be an isosinglet. A particular isospin zero wave function was considered a long time ago by Horn and Silver [23]. Consider the operators $A$ and $A^\dagger$, with

$$A^\dagger = 2a^+_x a^-_x - (a^+_0)^2 = -((a^+_x)^2 + (a^+_y)^2 + (a^+_z)^2)$$  \hspace{1cm} (38)

which is clearly an isoscalar operator. Any function of $A^\dagger$ operating on the vacuum will create a state with isospin zero and thus an eigenstate of charge with zero eigenvalue. Such a state will have not just zero average charge, but zero charge fluctuations as well. We note, in passing, that $A$, $A^\dagger$ and $N = \sum_i a^+_i a_i$ are the generators of an $sl(2)$ algebra

$$[A, A^\dagger] = 4N + 6, \quad [N, A] = -2A, \quad [N, A^\dagger] = 2A^\dagger$$  \hspace{1cm} (39)

A particularly simple state of isospin zero and exactly $2N$ pions has been suggested by a number of authors, cf. [4]. It is of the form

$$|\Psi> = \frac{1}{\sqrt{(2N+1)!}}(A^\dagger)^N|0>$$  \hspace{1cm} (40)

Using Stirling’s formula for the asymptotics of $n! \sim \sqrt{2n \exp(n \ln(n/e))}$ it is easy to see that the probability of having $2n$ neutral pions in the state $|\Psi>$ is [23], [4]

$$P(n, N) = \frac{(N!)^2 2^{2N} (2n)!}{(2N + 1)! (n!2n)^2} \sim \sqrt{N/n}, \quad n, N >> 1$$  \hspace{1cm} (41)

and corresponds to the $1/\sqrt{f}$ distribution of the classical picture.

The same distribution would be obtained if an arbitrary relative phase factor were inserted between the charged and neutral creation operators $\left(2a^+_x a^-_x - \exp(i\theta)(a^+_0)^2\right)^N$
This might seem to imply that the zero isospin condition is not so important, and that one can introduce any arbitrary factor. This is not the case. With any arbitrary factor in front of \((a_0^\dagger)^2\), the states created will still have the third component of isospin equal to zero and hence be zero eigenstates of charge, but the operator \(A^\dagger\) will now be a combination of isoscalar and second rank iso-tensor. All directions in iso-space will not be equivalent and we will no longer have a \(1/\sqrt{f}\) neutral fraction distribution. For example if the arbitrary factor is zero, there will be no neutral fraction.

The state \(|\Psi\rangle\) considered above is quite restrictive. It has exactly \(2N\) pions. A quantum state arising from a classical field is not expected to have a sharp number of quanta. We now consider two isoscalar states that are closer in spirit to the coherent state, and have an indefinite number of pions. First consider the squeezed state made by exponentiating \(A^\dagger\). We write

\[
|\Psi, \alpha\rangle = (1 - |\alpha|^2)^\frac{3}{4} \exp\left(\frac{\alpha}{2} A^\dagger\right) |0\rangle
\]  

(42)

To understand the normalization factor note that this state can be written in terms of cartesian pions as a product of three simple squeezed states,

\[
|\Psi, \alpha\rangle = (1 - |\alpha|^2)^\frac{3}{4} \exp\left(-\frac{\alpha}{2} (a_x^\dagger)^2 + (a_y^\dagger)^2 + (a_z^\dagger)^2\right) |0\rangle
\]  

(43)

The probability of finding \(2n\) neutral pions in this state is

\[
P(n) = \sum_{N=0}^{\infty} \frac{1}{(N!)^2} P(n, N) \sim \frac{1}{\sqrt{n}}
\]  

(44)

- again the same distribution. Although this state is a minimum uncertainty state, and there are no charge fluctuations in it, there are large particle number fluctuations. The number operator can be written in terms of the cartesian number operators as \(n_x + n_y + n_z\). Its expectation value in the state \(|\Psi, \alpha\rangle\) we call \(\bar{N}\). Using the product nature of the state we can then calculate

\[
\sigma^2 = \langle \Psi, \alpha | (n_x + n_y + n_z)^2 | \Psi, \alpha \rangle = -\bar{N}^2
\]  

(45)

using the results for a simple squeezed state. We find

\[
\sigma^2 = \frac{\bar{N}^2}{3} + 2\bar{N}
\]  

(46)

This shows that the number fluctuations grow like \(\bar{N}\), or the fractional fluctuations like 1 rather than like \(1/\sqrt{\bar{N}}\).
As an alternative for the iso-scalar state one can consider the iso-spin projected coherent state first introduced by Botke, Scalapino and Sugar, \cite{30} and recently used to describe the coherent pions emerging from nucleon-antinucleon annihilation. \cite{31}, \cite{32}

The $I = 0$ projected coherent state is

$$|\lambda, 0> = \mathcal{N} \int d\hat{T} e^{\lambda(k) \hat{d}^\dagger \hat{T}} |0> Y_{0,0}^\ast (\hat{T})$$  \hspace{1cm} (47)

where the normalization is given by

$$\mathcal{N} = (4\pi j_0(-i\bar{N}))^{-1/2}$$  \hspace{1cm} (48)

and the mean number of particles, $\bar{N}$ is given by

$$\bar{N} = \int d^3 k \lambda(k) \lambda^\ast(k)$$  \hspace{1cm} (49)

This is an iso-scalar state and hence has zero average charge and zero charge fluctuation. It is also easy to show that for large $\bar{N}$, the variance, $\sigma$ grows like $\sqrt{\bar{N}}$ so that the fractional particle number fluctuations decrease for large $\bar{N}$. However the state $|\lambda, 0>$, is not a minimum uncertainty state. To study the uncertainty in the state let us look at the uncertainty in the third component of isospin (all components are equivalent since we are in an $I = 0$ state). Define

$$x_0 = \frac{1}{\sqrt{2}} (a_0 + a_0^\dagger)$$  \hspace{1cm} (50)

and

$$p_0 = -i \frac{1}{\sqrt{2}} (a_0 - a_0^\dagger)$$  \hspace{1cm} (51)

where the subscript zero refers to the third component of isospin. It is easy to see that

$$<\lambda, 0|x_0|\lambda, 0> = <\lambda, 0|p_0|\lambda, 0> = 0$$  \hspace{1cm} (52)

This is also a feature of the $I = 0$ squeezed state. We need to look at $x^2$ and $p^2$. They are given by

$$x_0^2 = \frac{1}{2} (a_0^2 + (a_0^\dagger)^2) + a_0^\dagger a_0 + 1/2$$  \hspace{1cm} (53)

and

$$p_0^2 = -\frac{1}{2} (a_0^2 + (a_0^\dagger)^2) + a_0^\dagger a_0 + 1/2$$  \hspace{1cm} (54)

One finds (taking $\lambda(k)$ as real)

$$<\lambda, 0|a_0^2|\lambda, 0> = <\lambda, 0|(a_0^\dagger)^2|\lambda, 0> = \frac{\bar{N}}{3}$$  \hspace{1cm} (55)
and

\[ \langle \lambda, 0|a_0^\dagger a_0|\lambda, 0 \rangle = \frac{\bar{N} i j_1(-i\bar{N})}{3 j_0(-i\bar{N})} \]  

Finally in the large \( \bar{N} \) limit one obtains

\[ \Delta x \Delta p = \frac{\bar{N}}{3} \]  

Hence the isospin projected coherent state is not a minimal uncertainty state.

Thus we see that there are many classes of trial wave functions that can reproduce some of the phenomenologically desirable features of DCC decay. It is natural to ask the question what is the most natural class of these functions and what are the dynamical mechanisms leading to the generation of these functions. In the previous section we argued that simple quantum dynamics for the evolution of the DCC suggests the squeezed state as being most natural. In this section we have presented a number of quantum state candidates for discussing the pion state from the decay of the DCC. We have examined them from the point of view of charge fluctuations, number fluctuations, minimal uncertainty and neutral fraction probability. None of the candidates pass all criteria, but the cartesian pion squeezed state and the cartesian pion coherent state seem the most attractive. The first has the better dynamic credentials, but it also has much larger number fluctuations than the second.

5 Domains of DCC

We have seen that most reasonable quantum treatments of the pions from decay of the DCC lead to a probability \( P(f) \) of neutral pion fraction \( f \) of \( P(f) = 1/(2\sqrt{f}) \). This means that the probability of finding nearly all charged pions or nearly all neutral pions is quite large compared with what it would be for a purely random collection of \( N \) (\( N \) large) pions. The probability of finding \( f \) within \( \epsilon \) of zero (corresponding to nearly all charged pions) is \( \sqrt{\epsilon} \), while the probability of finding \( f \) within \( \epsilon \) of one (nearly all neutral pions) is \( \frac{1}{2} \epsilon \) for small \( \epsilon \). It is these surprisingly large probabilities that has fueled much of the interest in DCC, since they imply a strong and unambiguous experimental signal for the DCC. But in reactions in which a DCC can be created it is possible that not a single domain of DCC will be created but that perhaps a number of disconnected domains will be created each with a separate random direction in isospin space for the condensate. What then will be the probability of an overall neutral pion fraction \( f \)?
Suppose there are a number of regions, domains, or bubbles of DCC and that the isospin condensate direction in each is arbitrary. If, of course, there are many many such domains the average of $f$ over the domains will be $1/3$ with very small variance. In particular the chance of finding $f$ near zero would be very very small. But a more likely scenario is that there may be a few domains, more perhaps than one, but not many many. Then the probability of finding small $f$ is not prohibitively small. For example, suppose there are two domains, and we average the pion observations over both. Then, the probability of finding neutral fraction $f$ is $\frac{\pi}{2}$ for $f < 1/2$ and $\frac{\pi}{2} - 2 \arccos\left(\frac{1}{\sqrt{2}}\right)$ for $1/2 < f < 1$. Similar forms can be easily obtained for $n$ domains. In particular one sees that the probability of finding neutral fraction less that $\epsilon$ is $\sqrt{\epsilon}$ for the one domain case, and $(\pi/2)\epsilon$ in the two domain case. For $n$ domains the probability of finding neutral fraction less than $\epsilon$ goes like $\epsilon^{\frac{1}{2n}}$. For reasonably small numbers of domains this is not ridiculously small.

Hence one might build a phenomenological model of how the DCC manifests itself in terms of a small number of statistically independent domains of DCC. That might be closer to what happens in heavy ion collisions, say, and yet holds out hope of showing an interesting signal.

6 Summary and Conclusions

The idea that a Disordered Chiral Condensate (DCC) might occur in a rapid quench after a high energy collision is an attractive one. The signature of a DCC would be the emission of a large number of pions either nearly all neutral or nearly all charged. Such a remarkable process may have already been seen in the Centauro events.[20] Most discussions of the DCC concentrate on how it arises in a simple classical field theory, the linear sigma model, and how the long wave length modes grow until they encounter the hot boundary of the quenched region ("Baked Alaska"). But the particles detected from the DCC are pions, the quanta of the field, and hence a quantum discussion both of the growth of the DCC and of its deacy is needed. In this paper we have attempted to provide the beginnings of such a discussion. By quantizing the linear sigma model in the mean field or Hartree approximation, we have shown how the quantum DCC grows and how it leads to a squeezed state of the pions. More generally we have examined quantum wave functions for the pions from DCC decay, both of the coherent state and squeezed type. We have shown that the classical starting point leads naturally to a quantum description of the
pions in terms of real cartesian (in isospin) pions, and that that description avoids many of
the problem of charge fluctuations in the final pion state. We have also tried to emphasize
the phenomenological differences among the various quantum descriptions. We have also
discussed how multiple independent domains of DCC, might manifest themselves. These
are likely to occur in heavy ion collisions.

Much remains to be done in this growing subject. Better classical and corresponding
quantum models are needed, with a better understanding of how the classical models
connect to particular quantum descriptions. The notion of solving the difficult strong
interaction dynamics classically and then tying the solution onto a quantum state is a
very promising one and has already shown some success in the description of nucleon
antinucleon annihilation.\[31],\[32] However the real test of the DCC concept must come
from experiment, and therefore more phenomenological as well as purely theoretical work
is need as we approach the time of data.

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**Appendix A: Ellipsoidal Isospin Distributions**

All discussions of the neutral fraction proceed from the assumption of a spherically sym-
metric a priori distribution in isospin for the DCC condensate. For completeness we
investigate in this appendix what happens if we replace this with an ellipsoidal distribu-
tion. We continue to assume that the neutral pion fraction, \( f \), is given by \( \cos^2 \theta \), but now
ask for the probability of finding a given \( f \) over an ellipse rather than a sphere. To do this
take ordinary cartesian coordinates in isospin space. Then \( \cos^2 \theta = \frac{x^2}{x^2+y^2+z^2} \). We must
take an ellipse that is symmetric in \( x \) and \( y \) to preserve zero charge, hence we take for
our ellipse

\[
\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1
\]  

The probability of neutral fraction \( f \), \( P(f) \) is then

\[
P(f) = N \int \delta(1 - \frac{x^2 + y^2}{a^2} - \frac{z^2}{b^2})\delta(f - \frac{z^2}{x^2 + y^2 + z^2})dxdydz
\]  

where the normalization is given by

\[
1/N = \int \delta(1 - \frac{x^2 + y^2}{a^2} - \frac{z^2}{b^2})dxdydz
\]
It is clear that the probability is normalized, \( \int P(f) df = 1 \). Since \( f \) and \( P(f) \) are dimensionless, the result for \( P(f) \) can only depend on \( \rho = a/b \). We find

\[
P(f) = \frac{\rho}{2\sqrt{f}} \frac{1}{(a + f(\rho^2 - 1))^\frac{3}{2}}
\]  

(61)

This is seen to reduce to the usual spherical answer for \( \rho = 1 \). It is also easy to see that for \( \rho \) near 1, it is quite close to the usual answer. For \( \rho \) very large, only very small \( f \) dominates (nearly all charged) and for \( \rho \) very small, \( f \) near one dominates (nearly all neutral).

**Appendix B: Squeezed and Coherent States**

Squeezed states have been known for a some time in quantum optics and measurement theory (for a review of squeezed states see, for example \([24]\)−\([27]\)). The simplest one-mode squeezed state is parametrized by the two parameters \( r \) and \( \phi \) and can be obtained by acting on the vacuum with the unitary squeezing operator \( S(\xi) \)

\[
|\xi\rangle = S(\xi)|0\rangle = \exp\left[\frac{1}{2} (\xi a^2 - \xi (a^\dagger)^2)\right]
\]  

(62)

where \( \xi = r \exp(i\phi) \) is the squeezing parameter. The mean number of quanta in the squeezed state is \( \bar{N} = \sinh^2 r \).

To see this let us note that using the squeezing operator \( S \) one can make the Bogolubov transformation \( b = SaS^\dagger, \ b^\dagger = S a^\dagger S^\dagger \) and after some algebra one gets:

\[
b = \cosh ra + \exp(i\phi) \sinh ra^\dagger, \quad b^\dagger = \exp(-i\phi) \sinh ra + \cosh ra^\dagger
\]

\[
a = \cosh rb - \exp(i\phi) \sinh rb^\dagger, \quad a^\dagger = -\exp(-i\phi) \sinh ra + \cosh ra^\dagger
\]  

(63)

The new operator \( b \) is the annihilation operator for the squeezed state

\[
b|\xi\rangle = bS|0\rangle = S a S^\dagger |0\rangle = S a|0\rangle = 0
\]  

(64)

Then it is easy to see that

\[
\bar{N} = \langle \xi | a^\dagger a | \xi \rangle = \sinh^2 r < \xi | b b^\dagger | \xi \rangle = \sinh^2 r
\]  

(65)

Let us note that one can rewrite \([33]\)

\[
\exp\left[\frac{1}{2} (\xi a^2 - \xi (a^\dagger)^2)\right] = \exp\left(\alpha(\xi) a^2\right) \exp\left(\beta(\xi) a^\dagger a\right) \exp\left(\gamma(\xi)(a^\dagger a + \frac{1}{2})\right)
\]  

(66)
because $K_+ = a^+a^+ / 2$, $K_- = aa / 2$ and $K_0 = 1 / 2(a^+a + 1 / 2)$ are the generators of the $SU(1, 1)$ algebra:

$$[K_-, K_+] = 2K_0, \quad [K_0, K_\pm] = \pm K_\pm \tag{67}$$

As a result, the squeezed state can be rewritten as

$$|\xi > = \exp (\gamma / 2) \exp \left( \alpha(\xi)a^+{}^2 \right) |0 > \tag{68}$$

To find $\alpha$ and $\gamma$ one has to calculate the $\lambda$ dependence of the overlap between a squeezed and a coherent state $< \lambda|\xi > = < 0|e^{\lambda a}|\xi >$. Then

$$\exp (\gamma / 2) = < 0|\xi >, \quad \alpha = \frac{1}{2} < 0|a^2 e^{\alpha(\xi)a^+{}^2} |0 > = \frac{1}{2} \exp (-\gamma / 2) \frac{d^2 < \lambda|\xi >}{d\lambda^2} |_{\lambda=0} \tag{69}$$

Using (63) twice, and the fact that $b|\xi > = 0$, $< 0|a^+ = 0$ and $[\exp(\lambda a), a^+] = \lambda \exp(\lambda a)$ one gets after straightforward calculations

$$\frac{d < \lambda|\xi >}{d\lambda} = -\lambda e^{i\phi} \tanh r < \lambda|\xi > \tag{70}$$

The solution of this differential equation is

$$< \lambda|\xi > = \exp \left( -\frac{\lambda^2}{2} e^{i\phi} \tanh r \right) < 0|\xi > \tag{71}$$

To find $< 0|\xi >$ one has to consider the derivative $d < 0|\xi > /dr$ and again using (63) twice and $b|\xi > = 0$, $< 0|a^+ = 0$. We find

$$\frac{d}{dr} < 0|\xi > = \frac{d}{dr} < 0| \exp \left[ \frac{r}{2} \left( \exp(-i\phi)a^2 - \exp(i\phi)(a^+){}^2 \right) \right] |0 > = \frac{1}{2} \exp(-i\phi) < 0|a^2|\xi > = \frac{1}{2} \exp(+i\phi) \sinh^2 r < 0|b^+{}^2|\xi > = \frac{1}{2} \cosh r \sinh r < 0|\xi > \tag{72}$$

$$\sinh^4 r \frac{d}{dr} < 0|\xi > = \frac{1}{2} \cosh r \sinh r (\sinh^2 r - 1) < 0|\xi >$$

and finally

$$\frac{d}{dr} < 0|\xi > = -\frac{1}{2} \tanh r < 0|\xi > \tag{73}$$

The solution of this differential equation using the boundary condition, $< 0|0 > = 1$ is

$$< 0|\xi > = \frac{1}{\sqrt{\cosh r}} \tag{74}$$

The overlap between the squeezed and coherent state is useful for expressing the squeezed state as a superposition of coherent states, recalling that the coherent states are complete.
After a simple calculation we get

\[ \alpha = - \tanh r \exp(i\phi), \quad \exp(\gamma) = \frac{1}{\cosh r} \]  

(75)

Introducing new variable \( \alpha = - \tanh r \exp(i\phi) \) one can rewrite the normalized squeezed state as

\[ |\alpha> = (1 - |\alpha|^2)^{1/4} \exp \left( \frac{\alpha}{\sqrt{2}} (a^\dagger)^2 \right) |0> \]  

(76)

where \( \alpha \) is a complex number with modulus less than one. The mean number of quanta in this state, \( \bar{N} \) is given by

\[ < \alpha | a^\dagger a | \alpha > = \frac{|\alpha|^2}{1 - |\alpha|^2} \]  

(77)

The squeezed state is a minimum uncertainty state, for position and momenta defined in terms of the operators \( a: \Delta x \Delta p = 1/2 \), where \( a(a^\dagger) = (x \mp ip)/\sqrt{2} \). In the case of the coherent state, that minimum uncertainty is achieved by having each of \( \Delta x \) and \( \Delta p \) take on their minimum values. Coherent states have minimal quantum noise \( \Delta x = \Delta p = 1/\sqrt{2} \). For the squeezed state, only their product need be minimum, and one can be large while the other is “squeezed” to keep the product fixed. For example for \( \phi = 0 \) one has \( \Delta x = \exp(-r), \Delta p = \exp(r) \).

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