dS/CFT Duality on the Brane with a Topological Twist

by

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ABSTRACT

We consider a brane universe in an asymptotically de Sitter background spacetime of arbitrary dimensionality. In particular, the bulk spacetime is described by a “topological de Sitter” solution, which has recently been investigated by Cai, Myung and Zhang. In the current study, we begin by showing that the brane evolution is described by Friedmann-like equations for radiative matter. Next, on the basis of the dS/CFT correspondence, we identify the thermodynamic properties of the brane universe. We then demonstrate that many (if not all) of the holographic aspects of analogous AdS-bulk scenarios persist. These include a (generalized) Cardy-Verlinde form for the CFT entropy and various coincidences when the brane crosses the cosmological horizon.
1 Introduction

It is not uncommon to find the same physical system being described by two or more seemingly unrelated pictures. Nowhere is this ambiguity more apparent than in recent attempts at describing the universe itself. For instance, we have seen 11-dimensional membrane theory give rise to an abundance of dualities when its various manifestations are appropriately compactified [1]. On the other hand, we have it, on pretty good authority, that the physical universe can be effectively described by merely four spacetime dimensions. Such duality between theories of distinct dimensionality may, in fact, be a consequence of a more fundamental concept; namely, the “holographic principle” [2, 3].

The underlying premise of “holography” is that the maximal entropy within any given volume will be determined by the largest black hole that fits inside of that volume [4]. Since the entropy of a black hole is (up to a constant factor) given by its horizon surface area [5, 6], it follows that the relevant degrees of freedom of a black hole must, in some sense, “live” on the horizon. Moreover, given the holographic premise, it follows that the relevant degrees of any system must live on a surface that bounds the volume of that system.

The holographic principle has played its (perhaps) most prominent role in establishing a duality that seems to exist between any anti-de Sitter (AdS) spacetime[1] and a lower-dimensional conformal field theory (CFT) [2, 3, 4]. More specifically, it has been convincingly argued that the horizon thermodynamics of an $n+2$-dimensional AdS black hole can be identified with a certain $n+1$-dimensional (strongly-coupled) CFT. Significantly to these arguments, the dual CFT is assumed to live on a timelike surface that can be identified as an asymptotic boundary of the AdS spacetime.

In analogy to this well-accepted AdS/CFT duality, a de Sitter(dS)/CFT correspondence has similarly been conjectured [10]. (For earlier works in this regard, see Refs.[11]-[16].) Although dS space is obtained from AdS with a seemingly trivial sign change in the cosmological constant, there turns out to be quite severe implications. As a consequence, one finds that establishing the dS/CFT duality is a much more difficult challenge than in the AdS case.

1Note that anti-de Sitter denotes gravity with a negative cosmological constant and de Sitter, a positive cosmological constant. Typically, the gravity is described by Einstein theory, but not exclusively so.
For example, dS space lacks a globally timelike Killing vector and a spatial infinity (making it difficult to define conserved charges), while the black hole horizon (and its thermodynamic properties) have an ambiguous observer dependence.\(^2\) It is also problematic that dS solutions are conspicuously absent in string theories (and other quantum gravity theories); thus impeding any rigorous testing of the proposed duality.

In spite of these inherent complications, there has still been significant progress towards a holographic understanding of dS spacetimes \(^3\)\(^4\). With regard to the conjectured correspondence, the dS cosmological horizon is used in place of the (inner-lying) black hole horizon. Furthermore, the dual CFT is regarded as a Euclidean one that lives on a spacelike asymptotic boundary. Essential to these identifications is a renormalization group flow (between Euclidean CFTs at past and future infinity) that happens to be dual with time evolution in the dS bulk \(^3\).

Let us return our attentions, for the moment, to the AdS/CFT correspondence. In a relevant paper \(^4\), Verlinde directly applied this holographic duality to a radiation-dominated Friedmann-Robertson-Walker (FRW) universe (in \(n+1\) dimensions).\(^3\) This paper had a wide scope, but two observations are of particular interest. (i) The AdS/CFT correspondence leads to a CFT entropy that can be expressed in terms of a generalized Cardy formula \(^5\); with the Cardy “central charge” being a direct manifestation of the Casimir energy.\(^4\) (ii) When the Casimir entropy saturates a certain bound (namely, the Bekenstein-Hawking entropy \(^3\) of a universal-size black hole), then the cosmological evolution or Friedmann equations coincide with the generalized Cardy formula. We can express this point more eloquently: the CFT and FRW equations merge at a holographic saturation point, which implies that both sets of equations arise from some fundamental, underlying theory.

In Ref.\(^2\), Savonije and Verlinde have extended the prior work to an intriguing scenario: a Randall-Sundrum brane world \(^7\)\(^8\) in the background of an AdS-Schwarzschild (black hole) geometry. In this context, the \(n+1\)-dimensional CFT is regarded as living on the brane, which serves as a suitable boundary for the \(n+2\)-dimensional AdS bulk spacetime. With ap-
appropriately chosen boundary conditions, Savonije and Verlinde have shown that the brane world corresponds to a FRW universe and the brane dynamics are described by the Friedmann equations for radiative matter. Moreover, it was shown that the CFT thermodynamic relations coincide with the Friedmann equations at a special cosmological point: when the brane intersects the black hole horizon.

Here, we note that many aspects of the Verlinde-Savonije program have since been extended and generalized. The relevant studies (for an AdS scenario) can be found in Refs.\[59\]-\[78\].

Most recently, the Verlinde-Savonije treatment \[41, 56\] has been extended to a dS/CFT holographic picture \[35\]-\[39\]. These studies were, for the most part, successful in generalizing the pertinent features of Refs.\[41, 56\] to dS scenarios. However, there were some bothersome issues that can be directly attributed to the inherent complexities of dS spacetimes. These issues include negative energy densities on the CFT boundary (also see Refs.\[32, 33\]), the total CFT entropy being bounded from above by the Casimir contribution (also see Ref.\[14\]), the CFT-based universe being inaccessible to a strongly self-gravitating regime (especially see Ref.\[39\]), and an inability to incorporate the thermodynamics of the relevant black hole horizons into the proposed duality.

A preliminary analysis by Cai \[38\] suggests that many (if not all) of these issues can be resolved by revising the duality to incorporate a certain brand of asymptotically dS geometries. These “topological de Sitter” (TdS) spacetimes were originally proposed in Ref.\[34\]. However, there is a “cost” to be extracted if one is to proceed along these lines. Such TdS spacetimes have no black hole horizon, and so a naked singularity is an inevitable consequence.\[5\] On the other hand, the existence of a well-defined CFT that can describe this singularity does not seem inconceivable. Given this possibility, it seems worth pursuing if the pertinent outcomes of Ref.\[56\] hold up under a TdS-bulk scenario. Just such an investigation is the focus of the current paper.

The rest of the paper is organized as follows. In Section 2, we identify the thermodynamics properties of the TdS cosmological horizon. We also formu-\[5\]Such studies may be of particular importance, given recent empirical evidence of a positive cosmological constant for our universe \[79\].

\[6\]In fact, the original motivation for considering TdS spacetimes \[34\] was to test an earlier conjecture on cosmological singularities \[15\].
late the brane dynamics, which are shown to be described by Friedmann-like equations. In Section 3, we apply the dS/CFT correspondence and identify the brane (or CFT) thermodynamic properties. Also, the generalized Friedmann equations are re-expressed so that their connection with CFT thermodynamics is manifest. In Section 4, we demonstrate that the CFT thermodynamic and Friedmann equations coincide when the brane crosses the horizon. In addition, the CFT entropy is shown to be expressible in a Cardy-Verlinde-like form \cite{55,41}. Section 5 considers the holographic entropy bounds in the context of this model. Finally, Section 6 ends with a summary and brief discussion.

2 Bulk Thermodynamics and Brane Cosmology

We begin the analysis by formulating the scenario of interest. Namely, a \(n+1\)-dimensional brane of constant tension in an \(n+2\)-dimensional topological de Sitter (TdS) background. In a suitably static gauge, the bulk solution can be written as follows \cite{34,38}

\[
ds^2_{n+2} = -h(a)dt^2 + \frac{1}{h(a)}da^2 + a^2d\Omega^2_n, \tag{1}
\]

\[
h(a) = k - \frac{a^2}{L^2} + \frac{\omega_{n+1}M}{a^{n-1}}, \tag{2}
\]

\[
\omega_{n+1} = \frac{16\pi G_{n+2}}{nV_n}. \tag{3}
\]

Here, \(L\) is the curvature radius of the dS background, \(d\Omega^2_n\) denotes the line element of an \(n\)-dimensional (constant-curvature) hypersurface with volume \(V_n\), \(G_{n+2}\) is the \(n+2\)-dimensional Newton constant, and \(M\) and \(k\) are constants of integration. \(M\) is roughly associated with the mass of the solution and will be regarded as non-negative. (Note the sign reversal in this term relative to the usual Schwarzschild-dS case.) Meanwhile, without loss of generality, \(k\) can be set to +1, 0 or -1. These choices describe a (cosmological) horizon geometry that is respectively elliptic, flat or hyperbolic.

\footnote{More precisely, \(M\) measures an excitation in gravitational energy relative to the pure \((M = 0)\) dS spacetime.}
Clearly, the above solution is asymptotically dS. However, the $M \geq 0$ condition leads to some distinguishing features. For instance, there is no black hole horizon (although a cosmological one). Moreover, there exists a naked singularity at $a = 0$ for any $M > 0$. However, we will assume that this singularity can somehow be described (in a non-singular fashion) by the dual CFT of interest and proceed on this basis.

For any asymptotically dS space, there exists a well-defined cosmological horizon having similar thermodynamic properties to that of a black hole horizon \cite{17}. For the above solution, this cosmological horizon ($a = a_H$) corresponds to the positive root of $h(a) = 0$. Thus, the following useful relation can be obtained:

$$k - \frac{a_H^2}{L^2} + \frac{\omega_{n+1} M}{a_H^{n-1}} = 0.$$  \hspace{1cm} (4)

In analogy with black hole thermodynamics \cite{80}, the cosmological horizon has an associated temperature and entropy that are respectively given as follows:

$$T_{dS} = \frac{(n + 1) a_H^2 - (n - 1) L^2 k}{4 \pi L^2 a_H},$$  \hspace{1cm} (5)

$$S_{dS} = \frac{a_H^n V_n}{4 G_{n+2}}.$$  \hspace{1cm} (6)

The premise of the dS/CFT correspondence is that the above thermodynamics can be identified, up to a conformal factor, with a Euclidean CFT that lives on a spacelike boundary (for the bulk) at temporal infinity. We will re-introduce and exploit this duality at an appropriate interval.

Let us now consider the brane, which can be regarded as a dynamical boundary of the TdS geometry. To describe these brane dynamics, we will presume a boundary action of the following form:

$$I_b = \frac{1}{8 \pi G_{n+2}} \int_{\partial M} \sqrt{|g^{ind}|} \mathcal{K} + \frac{\sigma}{8 \pi G_{n+2}} \int_{\partial M} \sqrt{|g^{ind}|},$$  \hspace{1cm} (7)

where $g_{ij}^{ind}$ is the induced metric on the boundary ($\partial M$), $\mathcal{K} \equiv \mathcal{K}_1^i$ is the trace of the extrinsic curvature and $\sigma$ is a parameter measuring the brane tension.

\footnote{In particular, the inverse temperature can be identified with the periodicity of Euclidean time and the entropy, with one quarter of the horizon surface area \cite{80}.}
By varying this action with respect to the induced metric (and assuming a one-sided brane scenario), we obtain an equation of motion as follows:

\[ K_{ij} = \frac{\sigma}{n} d_{ij}^{ind}. \]  

(8)

In analogy with Ref.[56], we can clarify the brane dynamics by introducing a new (cosmological) time parameter, \( \tau \); whereby \( a = a(\tau) \), \( t = t(\tau) \) and:

\[
\frac{1}{h(a)} \left( \frac{da}{d\tau} \right)^2 - h(a) \left( \frac{dt}{d\tau} \right)^2 = 1. \]  

(9)

Unlike in Ref.[56], \( \tau \) has been defined here so as to yield a spacelike line element. This choice naturally reflects the duality that (presumably) exists between an asymptotically dS spacetime and a Euclidean CFT [10].

Substituting the above condition into Eq.(1), we find that the induced brane metric adopts a Euclidean FRW form. More specifically:

\[
ds_{n+1}^2 = d\tau^2 + a^2(\tau)d\Omega_n^2. \]  

(10)

Keep in mind that the radial distance, \( a = a(\tau) \), is really just the size of the \( n+1 \)-dimensional brane universe.

Let us now return our attention to Eq.(8); that is, the boundary equation of motion. One can readily calculate the extrinsic curvature (see, for instance, Ref.[81]) and then express this result in terms of the functions \( a(\tau) \) and \( t(\tau) \). For any of the “angular components” of the induced metric (i.e., components with respect to the constant-curvature hypersurface), the described process yields:

\[
\frac{dt}{d\tau} = \frac{\sigma a}{nh(a)}. \]

(11)

Next, we define the Hubble parameter, \( H \equiv \dot{a} / a \), in the usual way. With this definition, Eq.(9) can be re-expressed in the following form:

\[
H^2 = \frac{k}{a^2} - \frac{1}{L^2} + \frac{\sigma^2 n^2}{a^{n+1} M} + \frac{\omega_{n+1} M}{a^{n+1}}, \]

(12)

where we have also applied Eqs.(2,11).

\footnote{Dots will always denote differentiation with respect to \( \tau \).}
In this model, the brane tension ($\sigma$) is a free parameter that can be conveniently fine tuned. Here, we choose $\sigma^2 = n^2/L^2$ and thus cancel off the $a$-independent terms in Eq.(12). This choice yields a (first) Friedmann-like equation:

$$H^2 = \frac{k}{a^2} + \frac{\omega_{n+1} M}{a^{n+1}}.$$  (13)

Furthermore, we can take the $\tau$ derivative of the above equation, which leads to the associated second Friedmann equation:

$$\dot{H} = -\frac{k}{a^2} - \frac{(n + 1)\omega_{n+1} M}{2a^{n+1}}.$$  (14)

Note that the TdS bulk effectively induces radiative matter ($\sim M/a^{n+1}$) in the brane universe.

### 3 Euclidean CFT on the Brane

Before proceeding, let us clarify the underlying premise of the dS/CFT correspondence. It has been conjectured that the thermodynamics of a dS cosmological horizon can be directly associated with the thermodynamics of a dual CFT. Significantly, this CFT should be a Euclidean one and living on an asymptotic boundary (in this case, the brane). On the basis of such considerations, we will identify the brane (CFT) thermodynamics by suitably adapting the AdS analysis of Ref.[56].

We begin here by noting the following observation: the metric for a boundary CFT can only be determined up to a conformal factor [8, 9]. Keeping this in mind, let us consider the asymptotic form of the TdS metric:

$$\lim_{a \to \infty} \left[ \frac{L^2}{a^2} ds_{n+2}^2 \right] = dt^2 + L^2 d\Omega_n^2,$$  (15)

which can also be identified with the Euclidean metric for the relevant CFT. Evidently, if the radius of the spatial sphere is to be set equal to $a$, the Euclidean CFT time must be rescaled by a factor of $a/L$. It follows that the same factor ($a/L$) will turn up when the thermodynamic properties of the dual spacetimes are related. (With one notable exception: the relation between the entropies [9].)
In view of the above discussion, the thermodynamic properties of the CFT can be expressed as follows [56]:

\[ E \equiv E_{CFT} = \frac{LM}{a}, \quad (16) \]

\[ T \equiv T_{CFT} = \frac{L}{a} T_{dS} = \frac{1}{4\pi a} \left[ \frac{(n+1)a_H}{L} - \frac{(n-1)Lk}{a_H} \right], \quad (17) \]

\[ S \equiv S_{CFT} = S_{dS} = \frac{a^n V_n}{4G_{n+2}}. \quad (18) \]

In relevance to the above, let us note the following. The gravitational energy associated with this type of asymptotically dS geometry is always greater than that of the pure (i.e., \( M = 0 \)) dS spacetime [34, 38]. This is a reversal from the case of Schwarzschild (and Reissner-Nordstrom) dS geometries, where a positive-mass black hole leads to an excitation of negative gravitational energy [32, 33]. For this reason, the CFT energy (16) has been defined here as a positive quantity; in direct contrast to a previous study [39].

At this point, it is helpful to define an energy density (\( \rho \equiv E/V \)) and pressure (\( p \equiv \rho/n \)) [40] where \( V = a^n V_n \) is the volume of the brane universe.

With the above definitions, the first and second Friedmann-like equations (13,14) can be re-expressed in the following form:

\[ H^2 = \frac{16\pi G}{n(n-1)} \rho + \frac{k}{a^2}, \quad (19) \]

\[ \dot{H} = -\frac{8\pi G}{(n-1)} [\rho + p] - \frac{k}{a^2}. \quad (20) \]

\[ \text{It should be further noted that, in Eq.(16), we have omitted the energy associated with the pure (} M = 0 \text{) dS background. This is consistent with the convention initiated in Refs. [11, 56].} \]

\[ \text{Note that } p = \rho/n \text{ is the standard equation of state for radiative matter.} \]
Here, we have used:

\[ G = \frac{(n-1)}{L} G_{n+2}, \] (21)

where \( G \) is the effective Newton constant on the brane.\(^{12}\) Significantly, the cosmological evolution can now be directly attributed to the energy density and pressure of radiative matter.

For later convenience, we point out that the Friedmann equations (19, 20) can alternatively be expressed as follows:

\[ S_H = 2 \pi a \frac{n}{N} \sqrt{E_{BH} [2E + kE_{BH}]}, \] (22)

\[ -kE_{BH} = n [E + pV - T_H S_H], \] (23)

where we have defined:

\[ S_H \equiv (n-1) \frac{HV}{4G}, \] (24)

\[ E_{BH} \equiv n(n-1) \frac{V}{8 \pi Ga^2}, \] (25)

\[ T_H \equiv -\frac{\dot{H}}{2 \pi H}. \] (26)

The first Friedmann equation (Eq. (19) or (22)) can also be expressed in the following suggestive manner:

\[ S_H^2 = 2 S_B S_{BH} + k S_{BH}^2, \] (27)

where:

\[ S_B \equiv \frac{2 \pi a}{n} E, \] (28)

\[ S_{BH} \equiv \frac{(n-1)}{4Ga} V. \] (29)

The parameters of Eqs. (24-26, 28-29) are identical to those defined in Ref. [41]. For an AdS bulk, each of these parameters plays a significant role with regard to holographic bounds. (See Refs. [41, 56] for a complete discussion.) At this point, we have introduced the parameters for illustrative convenience and remind the reader that their respective roles do not necessarily translate over to a dS holographic theory. We consider this issue further in Section 5.\(^{12}\) This relation between bulk and brane gravitational constants is the usual one for a Randall-Sundrum brane world (generalized to arbitrary dimensionality) [57].
4 Thermodynamics at the Horizon and the Cardy-Verlinde Entropy

One of the remarkable outcomes of Ref. [56] was the coincidence of two distinct theories at a special moment in the evolution of the brane (in an AdS bulk). In particular, it was demonstrated that the CFT thermodynamic relations coincide with the cosmological (i.e., Friedmann) equations when the brane crosses the black hole horizon. Our current interest is to ascertain if the same behavior occurs at the cosmological horizon of a TdS bulk spacetime.

We begin here by comparing Eq. (13) for $H^2$ with the equation for the cosmological horizon (4). One can easily observe that the Hubble constant must obey:

$$ H = \pm \frac{1}{L} \quad \text{at} \quad a = a_H. \quad (30) $$

The $+$ sign indicates an expanding brane universe, while the $-$ sign describes a brane universe that is contracting. For illustrative purposes, we will subsequently focus on the expanding case.

Next, let us reconsider Eq. (18) for the CFT entropy. As one might anticipate (given the second law of thermodynamics), this total entropy remains constant as the system temporally evolves. However, this is not true of the entropy density:

$$ s \equiv \frac{S}{V} = \frac{(n-1)a_H^n}{4GLa^n}, \quad (31) $$

which certainly evolves along with the radial size of the brane.

When the brane crosses the horizon, this entropy density is given by:

$$ s = \frac{(n-1)H}{4G} \quad \text{at} \quad a = a_H. \quad (32) $$

It directly follows that (cf. Eq. (24)):

$$ S = S_H \quad \text{at} \quad a = a_H. \quad (33) $$

It is of similar interest to consider the CFT temperature (17) when the brane and horizon meet up. By applying Eq. (20) for $\dot{H}$, along with Eqs. (4, 26, 30), we find that:

$$ T = -\frac{\dot{H}}{2\pi H} = T_H \quad \text{at} \quad a = a_H. \quad (34) $$
Hence, when the brane crosses the horizon, the CFT entropy and temperature can be simply expressed in terms of the Hubble parameter and its derivative. These expressions are universal insofar as they do not depend explicitly on $M$ or $k$ (i.e., the parameters describing the TdS geometry).

Let us now introduce a quantity that can be readily identified with the Casimir energy of the brane universe $^{11, 50}$:

$$E_C \equiv n [E + pV - TS].$$

Given that $T = T_H$ and $S = S_H$ at $a = a_H$, we can further deduce that (cf. Eq. (23)):

$$E_C = n [E + pV - T_H S_H] = -k E_{BH} \quad \text{at} \quad a = a_H. \quad (36)$$

We will elaborate on the significance of the Casimir energy below.

Let us now reconsider the scenario of a generically positioned brane radius. As one might expect, the CFT thermodynamic properties can be shown to satisfy the first law of thermodynamics. That is:

$$TdS = dE + PdV. \quad (37)$$

It is indeed more revealing if the first law is reformulated in terms of densities. This expression takes on the following form:

$$T ds = d\rho + n [\rho + p - T s] \frac{da}{a}, \quad (38)$$

where we have applied the equation of state ($p = \rho/n$) and $dV = nV da/a$ (since $V \sim a^n$).

As expressed above, the square-bracket combination represents the sub-extensive contribution to the thermodynamic system. Such a contribution should effectively describe the Casimir energy, which notably agrees with our prior definition (35). Next, we will obtain a more explicit form of this Casimir contribution.

As an initial step in this process, it is helpful if the CFT energy density is re-expressed (by way of Eqs. (14,16)) as follows:

$$\rho = \frac{ML}{a^{n+1}V_n} = \frac{na_H^2}{16\pi G_n+2a^{n+1}} \left[ \frac{a_H}{L} - \frac{kL}{a_H} \right]. \quad (39)$$
Next, we incorporate \( p = \rho/n \), Eq.(31) for \( s \) and Eq.(17) for \( T \) into the above expression. This procedure ultimately yields:

\[
n [\rho + p - T s] = -\frac{2k\gamma}{a^2}, \tag{40}
\]

where we have defined:

\[
\gamma \equiv \frac{n(n-1)a_H^{n-1}}{16\pi Ga^n}. \tag{41}
\]

Comparing with Eq.(35), which defines the Casimir energy, we have:

\[
E_C = -\frac{2kV\gamma}{a^2} = -k\frac{n(n-1)V_n a_H^{n-1}}{8\pi Ga}. \tag{42}
\]

Notably, this expression does not depend explicitly on the mass parameter, \( M \); although it does depend on the TdS geometrical parameter, \( k \).

The above formalism can be used to relate the entropy density (31) and the “Casimir quantity” (i.e., \( \gamma \)). After some straightforward manipulations, we obtain:

\[
s^2 = \left( \frac{4\pi}{n} \right)^2 \gamma \left[ \rho + \frac{k\gamma}{a^2} \right]. \tag{43}
\]

Significantly, this entropy formula has a Cardy-like form [55]. Moreover, the Casimir-related quantity (\( \gamma \)) assumes the role of the Cardy “central charge” [55].

Let us now reconsider the special cosmological moment; that is, when the brane passes through the cosmological horizon. At this coincidence point, Eq.(13) leads directly to the first Friedmann-like equation (19). Similarly, the second Friedmann-like equation (20) follows when \( a = a_H \) is imposed on Eq.(19). Hence, we have extended the key results of Ref.[56] for the case of a TdS bulk.

5 Cosmological Considerations

In this section, we examine some of the cosmological implications of the prior results. First, it is useful to re-express the generalized Cardy-Verlinde

\[\text{In Cardy’s formalism [55], the central charge describes the multiplicity of massless particle species. It is clear that such a quantity should be directly related to the Casimir energy density, as we have found.}\]
formula \((43)\) in the following equivalent form:

\[
S = \sqrt{\frac{2\pi a}{n} S_C [2E - E_C]},
\]

(44)

where we have suitably defined the following Casimir entropy (in analogy with Refs.\([41, 74]\)):

\[
S_C \equiv \left. \frac{2\pi a}{n} E_{BH} \right|_{a=a_H} = \frac{(n-1)V_n a_H^{n-1}}{4G}.
\]

(45)

Note that \(S_C\) is strictly non-negative and independent of \(k\). This is in stark contrast to the Casimir energy. In fact, the two quantities are related as follows (cf. Eq. \((42)\)):

\[
E_C = -k \frac{n}{2\pi a} S_C.
\]

(46)

Given that \(S_C\) has no explicit dependence on \(k\) (which can be +1, 0 or −1), we prefer to think of the Casimir entropy as the “fundamental” quantity, from which \(E_C\) can be obtained via the above “definition” \((46)\). It just so happens that this definition for \(E_C\) coincides precisely with the prior one \((35)\).

With regard to the Casimir entropy, it is particularly significant that:

\[
S_C = S_{BH} \quad \text{at} \quad a = a_H,
\]

(47)

where \(S_{BH}\) is the Bekenstein-Hawking entropy of Eq. \((29)\). Recall that a similar (but not exact) coincidence was found between the Casimir energy and \(E_{BH}\); cf. Eq. \((36)\).

It can be readily shown that, when \(a = a_H\), the total entropy \((S)\) actually coincides with the “Hubble entropy” \((S_H)\) of Eq. \((24)\). To illustrate this occurrence, let us first consider the following equivalent form of Eq. \((44)\):

\[
S^2 = 2S_BS_C + kS_C^2,
\]

(48)

where \(S_B\) is the “Bekenstein entropy” of Eq. \((28)\). Comparing Eq. \((48)\) for \(S\) with Eq. \((27)\) for \(S_H\), we clearly observe the equivalence of these two entropies when the brane crosses the horizon.
Given the outcomes of the seminal studies [41, 56], one might wonder if such entropic coincidences (at $a = a_H$) actually represent the saturation points of holographic bounds. It turns out that this is indeed the case, provided that Verlinde’s conjectured upper bound on the Casimir entropy [41]:

$$S_C \leq S_{BH}$$

(49)

continues to hold. Verlinde originally proposed this universal bound on the premise of a holographic upper limit on the degrees of freedom of the CFT as measured by the Casimir entropy. There seems no reason that such a bound would fail to persist in our model given the following points. (i) $S_C$ and $S_{BH}$ are equivalent when the brane crosses the horizon. (ii) The Casimir entropy has no explicit dependence on $k$ or $M$ and, hence, is not sensitive to the details of the TdS geometry. It is interesting to note that this bound implies (cf. Eqs.(29,45)) that $a \geq a_H$; that is, the brane must remain outside of (or at) the horizon.

Again taking our cue from Verlinde, let us now make the distinction between a strongly and weakly self-gravitating brane universe. In the prior work [41], a strongly (weakly) self-gravitating regime was defined by the condition: $Ha \geq 1$ ($Ha \leq 1$). With this definition, Verlinde was able to deduce the following [41]:

$$S_B \geq S_{BH} \quad \text{and} \quad E \geq E_{BH} \quad \text{for} \quad Ha \geq 1,$$  

(50)

$$S_B \leq S_{BH} \quad \text{and} \quad E \leq E_{BH} \quad \text{for} \quad Ha \leq 1.$$  

(51)

As it so happens, virtually the same set of criteria are obtainable for the TdS-bulk model, with only a minor modification. Incorporating Eq.(19) for $H^2$ into the appropriate defining relations (25,28,29), we find:

$$S_B \geq S_{BH} \quad \text{and} \quad E \geq E_{BH} \quad \text{for} \quad Ha \geq \sqrt{2+k},$$  

(52)

$$S_B \leq S_{BH} \quad \text{and} \quad E \leq E_{BH} \quad \text{for} \quad Ha \leq \sqrt{2+k}.$$  

(53)

That is, the definition of a strongly (weakly) self-gravitating universe must now be revised to incorporate the value of $k$, but the general formalism otherwise persists. Note that it is the $k = -1$ (hyperbolic) case that exactly reproduces the original Schwarzschild-AdS criteria.
Let us now investigate the possibility of holographic bounds on the CFT total entropy, $S$. First, we consider a strongly self-gravitating regime, for which it follows that (cf. Eqs.(49,52)):

$$S_B \geq S_{BH} \geq S_C \quad \text{for} \quad Ha \geq \sqrt{2+k}. \quad (54)$$

It is clear from Eq.(48), that $S$ is monotonically increasing in $S_C$ (for any allowed $k$) as long as $S_C \leq S_B$. Also in evidence, $S$ will reach its maximum value (for this range) when $S_C = S_B$. This means that, for a strongly self-gravitating universe, $S$ will reach its maximum value when $S_C = S_{BH} = S_B$. Comparing Eq.(48) with Eq.(27) for the Hubble entropy ($S_H$), we now see that this maximum value of $S$ coincides precisely with $S_H$. So, for any of the prescribed values of $k\dagger$, we can establish the following bound:

$$S \leq S_H \quad \text{for} \quad Ha \geq \sqrt{2+k}. \quad (55)$$

Hence, the $a = a_H$ coincidence of $S$ and $S_H$ can also be viewed as the saturation point of a holographic bound.

It is interesting to note that, by virtue of Eqs.(48,54), the condition $S_C \leq S$ follows automatically for a strongly self-gravitating universe (for any allowed $k$). This bound is intuitively expected, given that a massive TdS solution induces a positive energy excitation on the brane. This is a pleasant reversal from an analogous study with regard to “conventional” black hole-dS solutions [39]. In this prior work, it was found that the total entropy is always bounded from above by the Casimir contribution.

Next, let us see what can be deduced for a weakly self-gravitating universe. It is instructive to begin with the $k = -1$ case, for which Eq.(48) takes on the form:

$$S^2 + (S_B - S_C)^2 = S_B^2. \quad (56)$$

If we accept the intuitive bound of $S_C \leq S$ to be universally valid (see above), then the above relation further implies that $S_B \geq S_C$. It seems reasonable to assume that this bound continues to hold for any allowed value of $k$, and we will proceed on this basis. Using this assumption, we know from above that (for any allowed $k$) $S$ is monotonically increasing in $S_C$ and reaches its

\dagger\text{In fact, hypothetically speaking, this bound would remain valid for any } k \geq -2. \text{ For } k < -2, \text{ not only is the bound no longer valid, but the “litmus test” (} Ha \geq \sqrt{2+k} \text{ versus } Ha \leq \sqrt{2+k} \text{) clearly breaks down.}
maximum value when $S_C = S_B$. Hence, Eq.(48) implies the following bound for a weakly self-gravitating universe:

$$S \leq \sqrt{2 + k} S_B \quad \text{for} \quad Ha \leq \sqrt{2 + k}. \quad (57)$$

We again point out the necessity for an assumption in establishing this bound. Hence, it is on a somewhat weaker footing than the rigorously confirmed bound of Eq.(53).

6 Conclusion

In the preceding paper, we have considered a brane universe in a topological de Sitter background spacetime. To begin the analysis, we identified the thermodynamic properties of the TdS cosmological horizon. Brane dynamics were subsequently examined, and it was demonstrated that (with a suitable choice of brane tension) the evolution equations take on a Friedman-like form.

After these initial considerations, we applied the dS/CFT correspondence and deduced the thermodynamics of a Euclidean CFT that lives on the brane. We were then able to demonstrate that the CFT thermodynamic properties coincide with the Friedmann-like equations when the brane crosses the cosmological horizon. Moreover, it was shown that the CFT entropy can be expressed in terms of a generalized Cardy-Verlinde formula [55, 41]. In this context, the Casimir energy (i.e., the sub-extensive energy contribution) adopts the role of the Cardy central charge.

Finally, some of the cosmological implications of our results were considered. For instance, we found that the Casimir entropy coincides with the Bekenstein-Hawking entropy when the brane crosses the horizon. A similar coincidence was found between the total entropy and the so-called Hubble entropy. On the basis of these results (and other considerations), we have conjectured that an upper bound on the Casimir entropy persists even for the exotic topology of our model. (Such a bound was originally proposed by Verlinde [11], for an AdS spacetime, as a universal consequence of the holographic principle [2, 3].) With this conjecture, it thus follows (either directly or indirectly) that the observed entropic coincidences actually represent the saturation points of their respective holographic bounds.
It is interesting to compare this TdS bulk scenario with the case of a Schwarzschild-dS background spacetime. In a recent study on the latter \cite{39} we identified several troublesome issues: a negative energy density on the brane, the total brane entropy being bounded from above by the Casimir contribution, and the brane universe being constrained to a weakly self-gravitating regime. Furthermore, there remains the open question of how to incorporate the thermodynamics of the black hole horizon into the proposed dS/CFT duality (which, so far, only seems to probe the cosmological horizon). However, as we have now shown, all of these issues can be circumvented by reversing the sign of the mass term (while maintaining a non-negative mass).

Given the apparent resolution of the noted issues, the results of the current paper seem to strengthen the status of the dS/CFT correspondence. And yet, such TdS solutions have the unfortunate side-effect of a naked cosmological singularity. It remains a possibility, however, that there exists a well-defined CFT which contains some appropriate description of the TdS singularity. In this event, the singular behavior in the bulk would not be problematic from the perspective of a brane observer. Clearly, this point will require further investigation. Thus, for the time being, the outcomes of this paper should be regarded as speculative.

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