THE ABELIANIZATION OF THE JOHNSON KERNEL

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\textbf{Abstract.} We prove that the first complex homology of the Johnson subgroup of the Torelli group $T_g$ is a non-trivial, unipotent $T_g$-module for all $g \geq 4$ and give an explicit presentation of it as a $\text{Sym}_n H_1(T_g, \mathbb{C})$-module when $g \geq 6$. We do this by proving that, for a finitely generated group $G$ satisfying an assumption close to formality, the triviality of the restricted characteristic variety implies that the first homology of its Johnson kernel $K$ is a nilpotent module over the corresponding Laurent polynomial ring, isomorphic to the infinitesimal Alexander invariant of the associated graded Lie algebra of $G$. In this setup, we also obtain a precise nilpotence test.

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1. Introduction

Fix a closed oriented surface $\Sigma$ of genus $g \geq 2$. The genus $g$ mapping class group $\Gamma_g$ is defined to be the group of isotopy classes of orientation preserving diffeomorphisms of $\Sigma$. For a commutative ring $R$, denote $H_1(\Sigma, R)$ by $H_R$. The intersection pairing $\theta : H_R \otimes H_R \to R$ is a unimodular, skew-symmetric bilinear form. Set $\text{Sp}(H_R) = \text{Aut}(H_R, \theta)$. The action of $\Gamma_g$ on $\Sigma$ induces a surjective

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homomorphism \( r : \Gamma_g \to \text{Sp}(H_Z) \). The Torelli group \( T_g \) is defined to be the kernel of \( r \). One thus has the extension

\[
1 \longrightarrow T_g \longrightarrow \Gamma_g \overset{r}{\longrightarrow} \text{Sp}(H_Z) \longrightarrow 1.
\]

Dennis Johnson [12] proved that \( T_g \) is finitely generated when \( g \geq 3 \).

The intersection form \( \theta \) spans a copy of the trivial representation in \( \wedge^2 H_R \). One therefore has the \( \text{Sp}(H_R) \)-module

\[
V_R := (\wedge^3 H_R)/(\theta \wedge H_R)
\]

which is torsion free as an \( R \)-module for all \( R \).

Johnson [11] constructed a surjective morphism (the “Johnson homomorphism”) \( \tau : T_g \to V_Z \) and proved in [14] that it induces an \( \text{Sp}(H_Z) \)-module isomorphism

\[
\bar{\tau} : H_1(T_g)/(2\text{-torsion}) \to V_Z.
\]

The \textit{Johnson group} \( K_g \) is the kernel of \( \tau \). By a fundamental result of Johnson [13], it is the subgroup of \( \Gamma_g \) generated by Dehn twists on separating simple closed curves.

The goal of this paper is to describe the \( \Gamma_g/K_g \)-module \( H_1(K_g, C) \). The first and third authors [4] proved that \( H_1(K_g, C) \) is finite dimensional whenever \( g \geq 4 \). Our first result is:

**Theorem A.** If \( g \geq 4 \), then \( H_1(K_g, C) \) is a non-trivial, unipotent \( H_1(T_g) \)-module and \( H_1(T_g, C_\rho) \) vanishes for all non-trivial characters \( \rho \) in the identity component \( \text{Hom}_Z(V_Z, C^*) \) of \( H^1(T_g, C^*) \).

When \( g \geq 6 \) we find a presentation of \( H_1(K_g, C) \) as a \( \Gamma_g/K_g \)-module. Describing this module structure requires some preparation.

Suppose that \( g \geq 3 \). Denote the highest weight summand of the second symmetric power of the \( \text{Sp}(H_C) \)-module \( \wedge^2 H_C \) by \( Q \).\(^1\) There is a unique \( \text{Sp}(H_C) \)-module projection (up to multiplication by a non-zero scalar), \( \pi : \wedge^2 V_C \to Q \).

Define a left \( \text{Sym}_*(V_C) \)-module homomorphism

\[
q : \text{Sym}_*(V_C) \otimes \wedge^3 V_C \to \text{Sym}_*(V_C) \otimes Q
\]

by

\[
q(f \otimes (a_0 \wedge a_1 \wedge a_2)) = \sum_{i \in \mathbb{Z}/3} f \cdot a_i \otimes \pi(a_{i+1} \wedge a_{i+2}).
\]

The map \( q \) is \( \text{Sp}(H_C) \)-equivariant. Thus, the cokernel of \( q \) is both a \( \text{Sp}(H_C) \)-module and a graded \( \text{Sym}_*(V_C) \)-module. We show in Section 4 that \( \text{coker}(q) \) is finite-dimensional, when \( g \geq 6 \). It follows that \( \text{coker}(q) \) is a \( (\text{Sp}(H_C) \ltimes V_C) \)-module, where \( v \in V_C \) acts via its exponential \( \exp v \). One therefore has the \( (\text{Sp}(H_C) \ltimes V_C) \)-module

\[
M := C \oplus \text{coker}(q),
\]

\(^1\)If \( \lambda_1, \ldots, \lambda_g \) is a set of fundamental weights of \( \text{Sp}(H_C) \), then \( Q \) is the irreducible module with highest weight \( 2\lambda_2 \). Alternatively, it is the irreducible module corresponding to the partition \([2, 2] \).
where \( C \) denotes the trivial module.

To relate the \((\text{Sp}(H_C) \ltimes V_C)\)-action on \( M \) to the \( \Gamma_g/K_g \)-action on \( H_1(K_g, \mathbb{C}) \), we recall that Morita [16] has shown that there is a Zariski dense embedding \( \Gamma_g/K_g \hookrightarrow \text{Sp}(H_C) \ltimes V_C \), unique up to conjugation by an element of \( V_C \), such that the diagram

\[
1 \rightarrow T_g/K_g \rightarrow \Gamma_g/K_g \rightarrow \text{Sp}(H_C) \rightarrow 1
\]

commutes.

**Theorem B.** If \( g \geq 6 \), then there is an isomorphism \( H_1(K_g, \mathbb{C}) \cong M \) which is equivariant with respect to a suitable choice of the Zariski dense homomorphism \( \Gamma_g/K_g \rightarrow \text{Sp}(H_C) \ltimes V_C \) described above.

### 1.1. Relative completion

These results are proved using the *infinitesimal Alexander invariant* introduced in [17] and the *relative completion* of mapping class groups from [8]. Alexander invariants occur as \( K_g \) contains the commutator subgroup \( T'_g \) of \( T_g \) and \( K_g/T'_g \) is a finite vector space over \( \mathbb{Z}/2\mathbb{Z} \). So one would expect \( H_1(K_g, \mathbb{C}) \) to be closely related to the complexified Alexander invariant \( H_1(T'_g, \mathbb{C}) \) of \( T_g \). A second step is to replace \( T_g \) by its Malcev (i.e., unipotent) completion, and \( K_g \) by the derived subgroup of the unipotent completion of \( T_g \). These groups, in turn, are replaced by their Lie algebras. The resulting module is the infinitesimal Alexander invariant of \( T_g \).

The role of relative completion of mapping class groups is that it allows, via Hodge theory, to identify filtered invariants, such as \( H_1(T'_g, \mathbb{C}) \), with their associated graded modules which are, in general, more amenable to computation. For example, the lower central series of \( T_g \) induces via conjugation a filtration on \( H_1(T'_g, \mathbb{C}) \), whose first graded piece is identified in [8] with \( V(2\lambda_2) \oplus \mathbb{C} \), over \( \text{Sp}(H_C) \).

### 1.2. Alexander invariants

The classical Alexander invariant of a group \( G \) is the abelianization \( G^{\prime \prime}_{ab} \) of its derived subgroup \( G^{\prime} := [G, G] \). Conjugation by \( G \) endows it with the structure of a module over the integral group ring \( \mathbb{Z}G_{ab} \) of the abelianization of \( G \). More generally, if \( N \) is a normal subgroup of \( G \) that contains \( G^{\prime} \), then one has the \( \mathbb{C}G_{ab} \)-module \( N_{ab} \otimes \mathbb{C} \). Our primary example is where \( G = T_g \) and \( N \) is its Johnson subgroup \( K_g \).

There is an infinitesimal analog of the Alexander invariant. It is obtained by replacing the group \( G \) by its (complex) Malcev completion \( \mathcal{G}(G) \) (also known as its unipotent completion). The Malcev completion of \( G \) is a prounipotent group, and is thus determined by its Lie algebra \( \mathfrak{g}(G) \) via the exponential mapping \( \exp : \mathfrak{g}(G) \rightarrow \mathcal{G}(G) \), which is a bijection. (Cf. [21, Appendix A].) The first version of the infinitesimal Alexander invariant of \( G \) is the abelianization \( \mathcal{B}(G) := \mathcal{G}(G)_{\text{ab}} \) of the
derived subgroup $G'(G) = [G(G), G(G)]$ of $G(G)$. One also has the abelianization $b(G)$ of the derived subalgebra $g'(G) = [g(G), g(G)]$ of $g(G)$. The exponential mapping induces an isomorphism $\exp : b(G) \to B(G)$. When $N/G'$ is finite, one has the diagram

\[ N_{ab} \otimes \mathbb{C} \longrightarrow B(G) \xrightarrow{\cong} b(G) \]

where the left most map is induced by the homomorphism $N \to G(G)'$.

The next step is to replace the Alexander invariant of $g(G)$ by a graded module by means of the lower central series. Recall the lower central series of a group $G$,

\[ G = G^1 \supseteq G^2 \supseteq G^3 \supseteq \cdots \]

It is defined by $G^{q+1} = [G^q, G^q]$. One also has the lower central series $\{g(G)^q\}_{q \geq 1}$ of its Malcev Lie algebra. There is a natural graded Lie algebra isomorphism

\[ g_\bullet(G) := \bigoplus_{q \geq 1} (G^q/G^{q+1}) \otimes \mathbb{C} \xrightarrow{\cong} \bigoplus_{q \geq 1} g(G)^q/g(G)^{q+1}, \]

where the bracket of the left-hand side is induced by the commutator of $G$.

The infinitesimal Alexander invariant $b_\bullet(G)$ of $G$, as introduced in [17], is the Alexander invariant of this graded Lie algebra, with a degree shift by 2:\footnote{That is, as a graded vector space, $b_q(G) = g_{q+2}(G)'_{ab}$, for $q \geq 0$.}

\[ b_\bullet(G) := g_\bullet(G)'_{ab}[2]. \]

The adjoint action induces an action of the abelian Lie algebra $g_\bullet(G)_{ab} = G_{ab} \otimes \mathbb{C}$ on $b_\bullet(G)$, and this makes the latter a (graded) module over the polynomial ring $\text{Sym}_\bullet(G_{ab} \otimes \mathbb{C})$. One reason for considering $b_\bullet(G)$ is that, in general, it is easier to compute than $b(G)$.

The invariant $b_\bullet(G)$ is most useful when $G$ is a group whose Malcev Lie algebra $g(G)$ is isomorphic to the degree completion $\widehat{g_\bullet(G)}$ of its associated graded Lie algebra. Groups that satisfy this condition include the Torelli group $T_g$ when $g \geq 3$, which is proved in [8], and 1-formal groups\footnote{In the sense of Dennis Sullivan [24]. Note that $T_g$ is 1-formal when $g \geq 6$, but is not when $g = 3$.} (such as Kähler groups). An isomorphism of $g(G)$ with $\widehat{g_\bullet(G)}$ induces an isomorphism of the infinitesimal Alexander invariant $B(G)$ with the degree completion of $b_\bullet(G)$.

When $G$ is finitely generated, $N/G'$ is finite and $H_1(N, \mathbb{C})$ is a finite dimensional nilpotent $\mathbb{C}G_{ab}$-module, it follows from Proposition 2.4 that all maps in (1.3) are isomorphisms.
1.3. Main general result. To emphasize the key features, it is useful to abstract
the situation. Define the Johnson kernel $K_G$ of a group $G$ to be the kernel of the
natural projection, $G \twoheadrightarrow G_{ab}$, where $G_{ab}$ denotes the maximal
torsion-free abelian quotient of $G$. Assume from now on that $G$ is finitely generated. For example, when
$g \geq 3$, the Torelli group $G = T_g$ is finitely generated, $G_{abf} = V_2$ and $K_G = K_g$.

Under additional assumptions, we want to relate the $\mathbb{C}G_{ab}$-module $H_1(K_G, \mathbb{C})$
to the graded $\text{Sym}_\bullet(G_{ab} \otimes \mathbb{C})$-module $\mathfrak{b}_\bullet(G)$. The first issue is that the rings $\mathbb{C}G_{ab}$
and $\text{Sym}_\bullet(G_{ab} \otimes \mathbb{C})$ are different. This is not serious as it is well-known that they
become isomorphic, after completion. Specifically, denote the augmentation ideal
of $\mathbb{C}G_{ab}$ by $I_{G_{ab}}$ and the $I_{G_{ab}}$-adic completion of $\mathbb{C}G_{ab}$ by $\hat{\mathbb{C}}G_{ab}$. The exponential
mapping induces a filtered ring isomorphism,

$$\exp : \hat{\mathbb{C}}G_{ab} \xrightarrow{\sim} \text{Sym}_\bullet(\hat{\mathbb{C}}G_{ab} \otimes \mathbb{C}),$$

with the degree completion of $\text{Sym}_\bullet(G_{ab} \otimes \mathbb{C})$.

Recall that a $\mathbb{C}G_{ab}$-module is nilpotent if it is annihilated by $I_{G_{ab}}^q$, for some $q$, and
trivial if it is annihilated by $I_{G_{ab}}$. When $H_1(K_G, \mathbb{C})$ is nilpotent, it has a natural
structure of $\text{Sym}_\bullet(G_{ab} \otimes \mathbb{C})$-module. Indeed, $H_1(K_G, \mathbb{C}) = H_1(K_G, \hat{\mathbb{C}})$
by nilpotence, so we may restrict via (1.4) the canonical $\text{Sym}_\bullet(G_{ab} \otimes \mathbb{C})$-module structure
of $H_1(K_G, \mathbb{C})$ to $\text{Sym}_\bullet(G_{ab} \otimes \mathbb{C})$. We may now state our main general result.

**Theorem C.** Suppose that $G$ is a finitely generated group whose Malcev Lie algebra
$\mathfrak{g}(G)$ is isomorphic to the degree completion of its associated graded Lie algebra
$\mathfrak{g}_\bullet(G)$. If $H_1(G, \mathbb{C}_\rho)$ vanishes for every non-trivial character $\rho : G \to \mathbb{C}^*$ that
factors through $G_{abf}$, then $H_1(K_G, \mathbb{C})$ is a finite-dimensional nilpotent $\mathbb{C}G_{ab}$-module
and there is a $\text{Sym}_\bullet(G_{ab} \otimes \mathbb{C})$-module isomorphism $H_1(K_G, \mathbb{C}) \cong \mathfrak{b}_\bullet(G)$. Moreover,
$I_{G_{ab}}^q$ annihilates $H_1(K_G, \mathbb{C})$ if and only if $\mathfrak{b}_q(G) = 0$.

The vanishing of $H_1(G, \mathbb{C}_\rho)$ above can be expressed geometrically in terms of the
character group $\mathbb{T}(G) = \text{Hom}(G_{abf}, \mathbb{C}^*)$ of $G$. Since $G$ is finitely generated, this is
an algebraic torus. Its identity component $\mathbb{T}^0(G)$ is the subtorus $\text{Hom}(G_{abf}, \mathbb{C}^*)$.
The restricted characteristic variety $\mathcal{V}(G)$ is the set of those $\rho \in \mathbb{T}^0(G)$ for which
$H_1(G, \mathbb{C}_\rho) \neq 0$. It is known that $\mathcal{V}(G)$ is a Zariski closed subset of $\mathbb{T}^0(G)$. (This
follows for instance by work from E. Hironaka in [10].) The vanishing hypothesis in
Theorem C simply means that $\mathcal{V}(G)$ is trivial, i.e., $\mathcal{V}(G) \subseteq \{1\}$.

Work by Dwyer and Fried [5] (as refined in [18]) implies that $\mathcal{V}(G)$ is finite precisely when $H_1(K_G, \mathbb{C})$ is finite-dimensional. This approach led in [4] to the conclusion
that $\dim_{\mathbb{C}} H_1(K_g, \mathbb{C}) < \infty$, for $g \geq 4$. Further analysis (carried out in Section
3) reveals that $\mathcal{V}(G)$ is trivial if and only if $H_1(K_G, \mathbb{C})$ is nilpotent over $\mathbb{C}G_{ab}$.

We show in Section 2 that the $I_{G_{ab}}$-adic completions of $H_1(G', \mathbb{C})$ and $H_1(K_G, \mathbb{C})$
are isomorphic. The triviality of $\mathcal{V}(G)$ implies that the finite-dimensional vector
space $H_1(K_G, \mathbb{C})$ is isomorphic to its completion. On the other hand, the first
hypothesis of Theorem C implies, via a result from [3], that the degree completion of the infinitesimal Alexander invariant $b_\bullet(G)$ is isomorphic to the $I_{G_{ab}}$-adic completion of $H_1(G', \mathbb{C})$. The details appear in Section 4.

To prove Theorem A we need to check that $\mathcal{V}(T_g) \subseteq \{1\}$. This is achieved in two steps. Firstly, we improve one of the main results from [4], by showing that $\mathcal{V}(T_g)$ is not just finite, but consists only of torsion characters. This is done in a broader context, in Theorem 3.1. In this theorem, the symplectic symmetry plays a key role: the $\text{Sp}(H_Z)$-module $(T_g)_{ab}$ gives a canonical action of $\text{Sp}(H_Z)$ on the algebraic group $T^0(T_g)$. We know from [4] that this action leaves the restricted characteristic variety $\mathcal{V}(T_g)$ invariant. The second step is to infer that actually $\mathcal{V}(T_g) = \{1\}$. We prove this by using a key result due to Putman, who showed in [20] that all finite index subgroups of $T_g$ that contain $K_g$ have the same first Betti number when $g \geq 3$.

A basic result from [8], valid for $g \geq 3$, guarantees that $T_g$ satisfies the assumption on the Malcev Lie algebra in Theorem C. Theorem A follows. Again by [8], the group $T_g$ is 1-formal, when $g \geq 6$; equivalently, the graded Lie algebra $\mathfrak{g}_\bullet(T_g)$ has a quadratic presentation. Theorem B follows from a general result in [17], that associates to a quadratic presentation of the Lie algebra $\mathfrak{g}_\bullet(G)$ a finite $\text{Sym}_\bullet(G_{ab} \otimes \mathbb{C})$-presentation for the infinitesimal Alexander invariant $b_\bullet(G)$. When $g \geq 6$, we use the quadratic presentation of $\mathfrak{g}_\bullet(T_g)$ obtained in [8].

2. Completion

We start in this section by establishing several general results, related to $I$-adic completions of Alexander-type invariants. We refer the reader to the books by Eisenbud [6, Chapter 7] and Matsumura [15, Chapter 9], for background on completion techniques in commutative algebra. Throughout the paper, we work with $\mathbb{C}$-coefficients, unless otherwise specified. The augmentation ideal of a group $G$, $I_G$, is the kernel of the $\mathbb{C}$-algebra homomorphism, $\mathbb{C}G \to \mathbb{C}$, that sends each group element to 1.

Let $N$ be a normal subgroup of $G$. Note that $G$-conjugation endows $H_\bullet N$ with a natural structure of (left) module over the group algebra $\mathbb{C}(G/N)$, and similarly for cohomology. If $N$ contains the derived subgroup $G'$, both $H_\bullet N$ and $H^\bullet N$ may be viewed as $\mathbb{C}(G/G')$-modules, by restricting the scalars via the ring epimorphism $\mathbb{C}(G/G') \to \mathbb{C}(G/N)$. When $G$ is finitely generated, $H_1 N$ is a finitely generated module over the commutative Noetherian ring $\mathbb{C}(G/N)$.

An important particular case arises when $N = G'$. Denoting abelianization by $G_{ab} := G/G'$, set $B(G) := H_1 G' = G'_{ab} \otimes \mathbb{C} = (G'/G'') \otimes \mathbb{C}$, and call $B(G)$ the Alexander invariant of $G$. These constructions are functorial, in the following sense. Given a group homomorphism, $\varphi : \widehat{G} \to G$, it induces a $\mathbb{C}$-linear map, $B(\varphi) : \widehat{B(G)} \to B(G)$. The details appear in Section 4.

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$B(G) \to B(G)$, and a ring homomorphism $\mathbb{C}\varphi : \mathbb{C}G_{\text{ab}} \to \mathbb{C}G_{\text{ab}}$. Moreover, $B(\varphi)$ is $\mathbb{C}\varphi$-equivariant, i.e., $B(\varphi)(\bar{a} \cdot \bar{x}) = \mathbb{C}\varphi(\bar{a}) \cdot B(\varphi)(\bar{x})$, for $\bar{a} \in \mathbb{C}G_{\text{ab}}$ and $\bar{x} \in B(G)$.

The $I$-adic filtration of the $\mathbb{C}G_{\text{ab}}$-module $B(G)$, $\{I^n_{G_{\text{ab}}} \cdot B(G)\}_{q \geq 0}$, gives rise to the completion map $B(G) \to \hat{B}(G)$, and to the $I$-adic associated graded, $\text{gr}_I B(G)$.

By $\mathbb{C}\varphi$-equivariance, $B(\varphi)$ respects the $I$-adic filtrations. Consequently, there is an induced filtered map, $\hat{B}(\varphi) : \hat{B}(G) \to \hat{B}(G)$, compatible with the completion maps. One knows that $\hat{B}(\varphi)$ is a filtered isomorphism if and only if $\text{gr}_I (B\varphi) : \text{gr}_I B(G) \to \text{gr}_I B(G)$ is an isomorphism.

A useful related construction (see [22]) involves the lower central series of a group $G$. The (complex) associated graded Lie algebra

$$\mathfrak{g}_\bullet (G) := \bigoplus_{q \geq 1} \left(G^q / G^{q+1}\right) \otimes \mathbb{C}$$

is generated as a Lie algebra by $\mathfrak{g}_1 (G) = H_1 G$. Each group homomorphism $\varphi : \bar{G} \to G$ gives rise to a graded Lie algebra homomorphism, $\text{gr}_\bullet (\varphi) : \mathfrak{g}_\bullet (G) \to \mathfrak{g}_\bullet (G)$.

Malcev completion (over $\mathbb{C}$), as defined by Quillen [21, Appendix A], is a useful tool. It associates to a group $G$ a complex pronilpotent group $\mathcal{G}(G)$, and a homomorphism $G \to \mathcal{G}(G)$. The Malcev Lie algebra of $G$ is the Lie algebra $\mathfrak{g}(G)$ of $\mathcal{G}(G)$. It is pronilpotent. The exponential mapping $\exp : \mathfrak{g}(G) \to \mathcal{G}(G)$ is a bijection.

The lower central series filtrations

$$G = G^1 \supseteq G^2 \supseteq G^3 \supseteq \cdots$$
$$\mathcal{G}(G) = \mathcal{G}(G)^1 \supseteq \mathcal{G}(G)^2 \supseteq \mathcal{G}(G)^3 \supseteq \cdots$$
$$\mathfrak{g}(G) = \mathfrak{g}(G)^1 \supseteq \mathfrak{g}(G)^2 \supseteq \mathfrak{g}(G)^3 \supseteq \cdots$$

of $G$, $\mathcal{G}(G)$ and $\mathfrak{g}(G)$ are preserved by the canonical homomorphism $G \to \mathcal{G}(G)$ and the exponential mapping $\exp : \mathfrak{g}(G) \to \mathcal{G}(G)$. They induce Lie algebra isomorphisms of the associated graded objects:

$$\text{gr}_\bullet (G) \otimes \mathbb{C} \xrightarrow{\sim} \text{gr}_\bullet \mathcal{G}(G) \xleftarrow{\sim} \text{gr}_\bullet \mathfrak{g}(G)$$

(cf. [21, Appendix A]).

We will need the following basic fact, which is a straightforward generalization of a result [23] of Stallings: if a group homomorphism $\varphi : \bar{G} \to G$ induces an isomorphism $\varphi^1 : H^1 G \xrightarrow{\sim} H^1 \bar{G}$ and a monomorphism $\varphi^2 : H^2 G \hookrightarrow H^2 \bar{G}$, then

\begin{equation}
\varphi : \mathfrak{g}(\varphi) : \mathfrak{g}(\bar{G}) \xrightarrow{\sim} \mathfrak{g}(G)
\end{equation}

is a filtered Lie isomorphism. A proof can be found in [9, Corollary 3.2].

With these preliminaries, we may now state and prove our first result.
Proposition 2.1. Suppose that $G$ is a finite index subgroup of a finitely generated group $G$. If $\varphi_1 : H_1G \to H_1G$ is an isomorphism, then $\widehat{B}(\varphi) : \widehat{B}(G) \to \widehat{B}(G)$ is a filtered isomorphism, where $\varphi : G \hookrightarrow G$ is the inclusion map.

Proof. Since $[G : G]$ is finite, $\varphi^* : H^*G \to H^*G$ is a monomorphism. So, $\varphi^1$ is an isomorphism and $\varphi^2$ is injective. Hence, the filtered Lie isomorphism (2.1) holds.

Proposition 5.4 from [3] guarantees that the filtered vector space $\widehat{B}(G)$ is functorially determined by the filtered Lie algebra $g(G)$. This completes the proof. \qed

Consider now a group extension

\begin{equation}
1 \to N \xrightarrow{\psi} \pi \to Q \to 1.
\end{equation}

Denote by $p_\pi : H_\pi N \to (H_\pi N)_Q$ the canonical projection onto the co-invariants. Clearly, $\psi : H_\pi N \to H_\pi \pi$ factors through $p_\pi$, giving rise to a map

\begin{equation}
q_\pi : (H_\pi N)_Q \to H_\pi \pi.
\end{equation}

When $Q$ is finite, $q_\pi$ is an isomorphism; see Brown’s book [1, Chapter III.10].

Given a $\mathbb{C}\pi$-module $M$, note that $I_N \cdot M$ is a $\mathbb{C}\pi$-submodule of $M$; see [1, Chapter II.2]. Consequently, the natural projection onto the $N$-co-invariants, $p : M \to M_N$, is $\mathbb{C}\pi$-linear and induces a filtered map, $\hat{p} : \hat{M} \to \hat{M}_N$, between $I_\pi$-adic completions.

We will need the following probably known result. For the reader’s convenience, we sketch a proof.

Lemma 2.2. Suppose that $N$ is a finite subgroup of a finitely generated abelian group $\pi$. If $M$ is a finitely generated $\mathbb{C}\pi$-module, then $\hat{p} : \hat{M} \to \hat{M}_N$ is a filtered isomorphism.

Proof. We start with a simple remark: if $R$ is a finitely generated commutative $\mathbb{C}$-algebra and $I \subseteq R$ is a maximal ideal, then the roots of unity $u$ from $1 + I$ act as the identity on $M/I^q \cdot M$, for all $q$, when $M$ is a finitely generated $R$-module. Indeed, $u - 1$ annihilates $I^s \cdot M/I^{s+1} \cdot M$ for all $s$, so the $u$-action on the finite-dimensional $\mathbb{C}$-vector space $M/I^q \cdot M$ is both unipotent and semisimple, hence trivial.

Now, consider the exact sequence of finitely generated $R$-modules,

\[ 0 \to I_N \cdot M \to M \to M_N \to 0, \]

where $R = \mathbb{C}\pi$. Tensoring it with $R/I^q$, we infer that our claim is equivalent to $I_N \cdot M \subseteq \cap_q I^q \cdot M$. This in turn follows from the remark. \qed

Remark 2.3. Let $M$ be a module over a group ring $\mathbb{C}\pi$, and $\pi \to \pi$ a group epimorphism, giving $M$ a structure of $\mathbb{C}\pi$-module, by restriction via $\mathbb{C}\pi \to \mathbb{C}\pi$. Plainly, $I_\pi^q \cdot M = I_{\pi^q}^q \cdot M$, for all $q$. In particular, the $I_\pi$-adic and $I_{\pi}$-adic completions of $M$ are filtered isomorphic, and $M$ is nilpotent (or trivial) over $\mathbb{C}\pi$ if and only if this happens over $\mathbb{C}\pi$. 

Given a group $G$, set $G_{ab f} := G_{ab}/(\text{torsion})$. The Johnson kernel, $K_G$, is the kernel of the canonical projection $G \twoheadrightarrow G_{ab f}$. When $G = T_g$ and $g \geq 3$, Johnson’s fundamental results from [11, 14] show that $K_G = K_g$, whence our terminology.

More generally, consider an extension
\begin{equation}
1 \to G' \xrightarrow{\psi} K \to F \to 1,
\end{equation}
with $F$ finite. Plainly, $\psi_* : H_*G' \to H_*K$ is $\mathbb{C}G_{ab}$-linear. Let $\hat{\psi}_*$ be the induced map on $I_{G_{ab}}$-adic completions. (When $K = K_G$, note that $H_*K_G$ is actually a $\mathbb{C}G_{ab}$-module, with $\mathbb{C}G_{ab}$-module structure induced by restriction, via $\mathbb{C}G_{ab} \to \mathbb{C}G_{ab f}$. By Remark 2.3, its $I_{G_{ab}}$-adic and $I_{G_{ab f}}$-adic completions coincide.) Here is our second main result in this section.

**Proposition 2.4.** If $G$ is a finitely generated group and $K$ is a subgroup like in (2.4), then $\hat{\psi}_*: H_1G' \to H_1K$ is a filtered isomorphism.

**Proof.** We apply Lemma 2.2 to $F \subseteq G_{ab}$ and $M = H_1G'$, to obtain a filtered isomorphism $\hat{\rho}: H_1G' \xrightarrow{\sim} (H_1G')_F$ between $I_{G_{ab}}$-adic completions. We conclude by noting that the isomorphism (2.3) coming from (2.4), $q : (H_1G')_F \xrightarrow{\sim} H_1K$, is $\mathbb{C}G_{ab}$-linear. The last claim is easy to check: plainly, the $\mathbb{C}F$-module structure on $H_1G'$ coming from (2.4) is the restriction to $\mathbb{C}F$ of the canonical $\mathbb{C}G_{ab}$-structure. □

### 3. Characteristic varieties

We show that the (restricted) characteristic variety of $T_g$ is trivial for all $g \geq 4$, as stated in Theorem A, thus improving one of the main results in [4]. Fix a symplectic basis of the first homology $H_2$ of the reference surface $\Sigma$. This gives an identification of $\text{Sp}(H_R)$ with $\text{Sp}_g(R)$ for all rings $R$.

We start by reviewing a couple of definitions and relevant facts. Let $G$ be a finitely generated group. The **character torus** $\mathbb{T}(G) = \text{Hom}(G_{ab}, \mathbb{C}^*)$ is a linear algebraic group with coordinate ring $\mathbb{C}G_{ab}$. The connected component of $1 \in \mathbb{T}(G)$ is denoted $\mathbb{T}^0(G) = \text{Hom}(G_{ab f}, \mathbb{C}^*)$ and has coordinate ring $\mathbb{C}G_{ab f}$.

The **characteristic varieties** of $G$ are defined for (degree) $i \geq 0$, (depth) $k \geq 1$ by
\begin{equation}
\mathcal{V}^i_k(G) = \{ \rho \in \mathbb{T}(G) \mid \dim_{\mathbb{C}} H_i(G, \mathbb{C}_\rho) \geq k \}.
\end{equation}
Here $\mathbb{C}_\rho$ denotes the $\mathbb{C}G$-module $\mathbb{C}$ given by the change of rings $\mathbb{C}G \to \mathbb{C}$ corresponding to $\rho$. Their restricted versions are the intersections $\mathcal{V}^i_k(G) \cap \mathbb{T}^0(G)$. The restricted characteristic variety $\mathcal{V}^i_k(G) \cap \mathbb{T}^0(G)$ is denoted $\mathcal{V}(G)$. As explained in [4, Section 6], it follows from results in [10] about finitely presented groups that both $\mathcal{V}^i_k(G)$ and $\mathcal{V}^i_k(G) \cap \mathbb{T}^0(G)$ are Zariski closed subsets, for all $k$.

When $G = T_g$ and $g \geq 3$, these constructions acquire an important symplectic symmetry; see [4]. We recall that the linear algebraic group $\text{Sp}_g(\mathbb{C})$ is defined over $\mathbb{Q}$, simple, with positive $\mathbb{Q}$-rank, and contains $\text{Sp}_g(\mathbb{Z})$ as an arithmetic subgroup.
The $\Gamma_g$-conjugation in the defining extension (1.1) for $T_g$ induces representations of $\Sp_g(\mathbb{Z})$ in the finitely generated abelian groups $(T_g)_{\ab}$ and $(T_g)_{\abf}$. They give rise to natural $\Sp_g(\mathbb{Z})$-representations in the algebraic groups $T(T_g)$ and $\mathbb{T}_0(T_g)$, for which the inclusion $\mathbb{T}_0(T_g) \subseteq T(T_g)$ becomes $\Sp_g(\mathbb{Z})$-equivariant. Furthermore, $\mathcal{V}(T_g) \subseteq \mathbb{T}_0(T_g)$ is $\Sp_g(\mathbb{Z})$-invariant.

By Johnson’s work [11, 14], we also know that the $\Sp_g(\mathbb{Z})$-action on $(T_g)_{\abf}$ extends to a rational, irreducible and non-trivial $\Sp_g(\mathbb{C})$-representation in $(T_g)_{\abf} \otimes \mathbb{C}$.

We will need the following refinement of a basic result on propagation of irreducibility, proved by Dimca and Papadima in [4]. This refinement is closely related to an open question formulated in [19, Section 10], on outer automorphism groups of free groups.

**Theorem 3.1.** Let $L$ be a $D$-module which is finitely generated and free as an abelian group. Assume that $D$ is an arithmetic subgroup of a simple $\mathbb{C}$-linear algebraic group $S$ defined over $\mathbb{Q}$, with $\text{rank}_\mathbb{Q}(S) \geq 1$. Suppose also that the $D$-action on $L$ extends to an irreducible, non-trivial, rational $S$-representation in $L \otimes \mathbb{C}$. Let $W \subset \mathbb{T}(L)$ be a $D$-invariant, Zariski closed, proper subset of $\mathbb{T}(L)$. Then $W$ is a finite set of torsion elements in $\mathbb{T}(L)$.

**Proof.** According to one of the main results from [4] (which needs no non-triviality assumption on the $S$-representation $L \otimes \mathbb{C}$), $W$ must be finite. We have to show that any $t \in W$ is a torsion point of $\mathbb{T} = \mathbb{T}(L)$. We know that the stabilizer of $t$, $D_t$, has finite index in $D$. By Borel’s density theorem, $D_t$ is Zariski dense in $S$.

Suppose that $t \in W$ has infinite order, and let $\mathbb{T}_t \subseteq \mathbb{T}$ be the Zariski closure of the subgroup generated by $t$. By our assumption, the closed subgroup $\mathbb{T}_t$ is positive-dimensional. Since $\mathbb{T}_t$ is fixed by $D_t$, the Lie algebra $T_1\mathbb{T}_t \subseteq T_1\mathbb{T} = \text{Hom}(L \otimes \mathbb{C}, \mathbb{C})$ is $D_t$-fixed as well. By Zariski density, $T_1\mathbb{T}_t$, is then a non-zero, $S$-fixed subspace of $T_1\mathbb{T}$, contradicting the non-triviality hypothesis on $L \otimes \mathbb{C}$. \hfill \Box

We know from [4] that $\mathcal{V}(T_g)$ is finite, for $g \geq 4$. We may apply Theorem 3.1 to $D = \Sp_g(\mathbb{Z})$ acting on $L = (T_g)_{\abf}$, $S = \Sp_g(\mathbb{C})$ and $W = \mathcal{V}(T_g)$. We infer that $\mathcal{V}(T_g)$ consists of $m$-torsion elements in $\mathbb{T}_0(T_g)$, for some $m \geq 1$.

To derive the triviality of $\mathcal{V}(T_g)$ from this fact, we will use another standard tool from commutative algebra. For an affine $\mathbb{C}$-algebra $A$, let $\text{Specm}(A)$ be its maximal spectrum. For a $\mathbb{C}$-algebra map between affine algebras, $f : A \to B$, $f^* : \text{Spec}(B) \to \text{Spec}(A)$ stands for the induced map, that sends $\text{Specm}(B)$ into $\text{Specm}(A)$. For a finitely generated $A$-module $M$, the support $\text{supp}_A(M)$ is the Zariski closed subset of $\text{Spec}(A)$ $V(\text{ann}(M)) = V(E_0(M))$, where $E_0(M)$ is the ideal generated by the codimension zero minors of a finite $A$-presentation for $M$; see [6, Chapter 20].

There is a close relationship between characteristic varieties and supports of Alexander-type invariants. Let $N$ be a normal subgroup of a finitely generated
group $G$, with abelian quotient. Denote by $\nu : G \to G/N$ the canonical projection, and let $\nu^* : \mathbb{T}(G/N) \to \mathbb{T}(G)$ be the induced map on maximal spectra of corresponding abelian group algebras. It follows for instance from Theorem 3.6 in [18] that $\nu^*$ restricts to an identification (away from 1)

$$(3.2) \quad \text{Specm}(\mathbb{C}(G/N)) \cap \text{supp}_{\mathbb{C}(G/N)}(H_1 N) \equiv \text{im}(\nu^*) \cap \mathbb{V}^1(G).$$

We will need the following (presumably well-known) result on supports. For the sake of completeness, we include a proof.

**Lemma 3.2.** If $f : A \to B$ is an integral extension of affine $\mathbb{C}$-algebras and $M$ is a finitely generated $B$-module, then $M$ is finitely generated over $A$ and $\text{supp}_A(M) = f^*(\text{supp}_B(M))$, $\text{Specm}(A) \cap \text{supp}_A(M) = f^*(\text{Specm}(B) \cap \text{supp}_B(M))$.

**Proof.** The reader may consult [6, Chapter 4], for background on integral extensions. Clearly, $\text{ann}_A(M) = A \cap \text{ann}_B(M)$, and the extension $\bar{f} : A/\text{ann}_A(M) \to B/\text{ann}_B(M)$ is again integral. The inclusion $f^*(\text{supp}_B(M)) \subseteq \text{supp}_A(M)$ follows from the definitions. For the other inclusion, pick any prime ideal $p$ containing $\text{ann}_A(M)$. Since $\bar{f}$ induces a surjection on prime spectra, there is a prime ideal $q$ containing $\text{ann}_B(M)$, such that $f^*(q) = p$, and $q$ is maximal, if $p$ is maximal. \hfill $\square$

The interpretation (3.2) for the closed points of the support of Alexander-type invariants leads to the following key nilpotence test.

**Lemma 3.3.** Let $\nu : G \to H$ be a group epimorphism with finitely generated source, abelian image and kernel $N$. Then the following are equivalent.

1. The $\mathbb{C}H$-module $H_1 N$ is nilpotent.
2. The inclusion $\text{Specm}(\mathbb{C}H) \cap \text{supp}_{\mathbb{C}H}(H_1 N) \subseteq \{1\}$ holds.
3. The intersection $\nu^* (\mathbb{T}(H)) \cap \mathbb{V}^1(G)$ is contained in $\{1\}$.

**Proof.** Note that $1 \in \mathbb{T}(H)$ corresponds to the maximal ideal $I_H \subseteq \mathbb{C}H$. With this remark, the equivalence $(1) \iff (2)$ becomes an easy consequence of the Hilbert Nullstellensatz. The equivalence $(2) \iff (3)$ follows directly from (3.2). \hfill $\square$

For a group $G$, we denote by $p_G : G \to G_{\text{abf}}$ the canonical projection. Assume $G$ is finitely generated and fix an integer $m \geq 1$. Denoting by $\iota_m : G_{\text{abf}} \hookrightarrow G_{\text{abf}}$ the multiplication by $m$, note that its extension to group algebras, $\mathbb{C}\iota_m : \mathbb{C}G_{\text{abf}} \hookrightarrow \mathbb{C}G_{\text{abf}}$, is integral, and the associated map on maximal spectra, $\mathbb{C}\iota^*_m : \mathbb{T}(G_{\text{abf}}) \to \mathbb{T}(G_{\text{abf}})$, is the $m$-power map of the character group $\mathbb{T}(G_{\text{abf}})$.

Let $p_m : G(m) \to G_{\text{abf}}$ be the pull-back of $p_G$ via $\iota_m$. Clearly, $G(m)$ is a normal, finite index subgroup of $G$ containing the Johnson kernel $K_G$, with inclusion denoted $\varphi_m : G(m) \hookrightarrow G$, and

$$(3.3) \quad p_G \circ \varphi_m = \iota_m \circ p_m.$$
Lemma 3.4. Assume that all finite index subgroups of \( G \) containing \( K_G \) have the same first Betti number. Then the following hold.

\[ \begin{align*}
(1) & \text{ The map induced by } \varphi_m \text{ on } I\text{-adic completions, } B(\varphi_m) : B(G(m)) \to \hat{B}(G), \text{ is a filtered isomorphism.} \\
(2) & \text{ The inclusion } K_{G(m)} \subseteq K_G \text{ is actually an equality.}
\end{align*} \]

Proof. Our assumption implies that \( \varphi_m \) induces an isomorphism \( H_1 G(m) \to \hat{H}_1 G \). Property (1) follows then from Proposition 2.1. The second claim is a consequence of the fact that \( p_m \) may be identified with \( p_{G(m)} \). To obtain this identification, we apply to (3.3) the functor \( abf \). By construction, \( abf(p_G) \) is an isomorphism and \( abf(t_m) = \iota_m \) is a rational isomorphism. We also know that \( abf(\varphi_m) \otimes \mathbb{Q} : H_1(G(m), \mathbb{Q}) \to \hat{H}_1(G, \mathbb{Q}) \) is an isomorphism. We infer that \( abf(p_m) \) is a rational isomorphism, hence an isomorphism. \( \square \)

Proof of Theorem A (except for the non-triviality assertion). We first prove that \( \mathcal{V}(T_g) = \{1\} \). According to a recent result of Putman [20], the group \( G = T_g \) satisfies for \( g \geq 3 \) the hypothesis of Lemma 3.4. Denote by \( \psi : G' \to K_G \) and \( \psi_m : G(m)' \to K_{G(m)} = K_G \) the inclusions. According to Proposition 2.4, they induce filtered isomorphisms, \( \hat{B}(G) \to \hat{H}_1 K_G \) and \( \hat{B}(G(m)) \to \hat{H}_1 K_{G(m)} \), between the corresponding \( I\)-adic completions. In these isomorphisms, the \( \mathbb{C}G_{abf}\)-module structure of \( M = \hat{H}_1 K_G \) comes from the group extension associated to \( p_G \), respectively \( p_m \). Denote the second \( \mathbb{C}G_{abf}\)-module by \( m^m M \), and note that \( m^m M \) is obtained from \( M \) by restriction of scalars, via \( \mathcal{C}_{t_m} : \mathbb{C}G_{abf} \to \mathbb{C}G_{abf} \).

Taking into account the isomorphism from Lemma 3.4(1), it follows that \( id_M : m^m M \to M \), viewed as a \( \mathcal{C}_{t_m}\)-equivariant map, induces an isomorphism between the corresponding \( I\)-adic completions.

Let \( \rho \in \mathbb{T}(G_{abf}) \) be a closed point of \( \text{supp}_{\mathbb{C}G_{abf}}(M) \). By (3.2), applied to \( N = K_G \), \( \mathcal{C}_{t_m}^* (\rho) = 1 \), since \( \mathcal{V}(G) \) consists of \( m \)-torsion points, for \( g \geq 4 \). We infer from Lemma 3.2, applied to \( f = \mathcal{C}_{t_m} : \mathbb{C}G_{abf} \to \mathbb{C}G_{abf} \), that \( \text{Specm}(\mathbb{C}G_{abf}) \cap \text{supp}_{\mathbb{C}G_{abf}}(m^m M) \subseteq \{1\} \). Take \( \nu = p_m : G(m) \to G_{abf} \) in Lemma 3.3, whose kernel is \( K_{G(m)} = K_G \). We deduce that \( m^m M \) is nilpotent over \( \mathbb{C}G_{abf} \), that is, \( I^q \cdot m^m M = 0 \) for some \( q \), where \( I \subseteq \mathbb{C}G_{abf} \) is the augmentation ideal.

Denote by \( \kappa_m : M \to \hat{m^m M} \) and \( \kappa : M \to \hat{M} \) the completion maps, with kernels \( \cap_{r \geq 0} I^r \cdot m^m M \) and \( \cap_{r \geq 0} I^r \cdot M \). It follows from naturality of completion that \( \kappa \) is injective, since \( \kappa_m \) is injective and \( \text{id}_{\hat{M}} : \hat{m^m M} \to \hat{M} \) is an isomorphism.

We also know from [4] that \( \dim \bar{M} < \infty \). It follows that the \( I\)-adic filtration of \( M \) stabilizes to \( I^q \cdot M = \cap_{r \geq 0} I^r \cdot M = 0 \), for \( q \) big enough. Applying Lemma 3.3 to \( \nu = p_G : G \to G_{abf} \), with kernel \( K_G \), we infer that \( \mathcal{V}(G) = \{1\} \).
We extract from the preceding argument the following corollary. Together with the triviality of $V(T_g)$, this completes the proof of Theorem A (except for the non-triviality assertion) via Remark 2.3.

**Corollary 3.5.** If $g \geq 4$, then $H_1(K_g, \mathbb{C})$ is a nilpotent module, over both $\mathbb{C}(T_g)_{ab}$ and $\mathbb{C}(T_g)_{abf}$.

### 4. Infinitesimal Alexander Invariant

Our next task is to prove Theorem B and the non-triviality assertion of Theorem A. These follow from general results about infinitesimal Alexander invariants.

Let $h_\bullet$ be a positively graded Lie algebra. Consider the exact sequence of graded Lie algebras

\[
0 \to h_\bullet'/h_\bullet'' \to h_\bullet/h_\bullet'' \to h_\bullet/h_\bullet' \to 0.
\]

The universal enveloping algebra of the abelian Lie algebra $h_\bullet/h_\bullet'$ is the graded polynomial algebra $\text{Sym}_\bullet(h_{ab})$. (When the Lie algebra $h_\bullet$ is generated by $h_1$, $\text{Sym}_\bullet(h_{ab}) = \text{Sym}_\bullet(h_1)$, with the usual grading.) The adjoint action in (4.1) yields a natural graded $\text{Sym}_\bullet(h_{ab})$-module structure on $h_\bullet''$. It will be convenient to shift degrees and define the *infinitesimal Alexander invariant* $b_\bullet(h) := h_\bullet''[2]$, by analogy with the Alexander invariant of a group. The graded vector space $b_\bullet(h) = \oplus_{q \geq 0} b_q(h)$, where $b_q(h) = h_{q+2}/h_{q+2}'$, becomes in this way a graded module over $\text{Sym}_\bullet(h_{ab})$.

When $h_\bullet = g_\bullet(G)$, we denote the graded $\text{Sym}_\bullet(G_{ab} \otimes \mathbb{C})$-module $b_\bullet(h)$ by $b_\bullet(G)$. Note that the degree filtration of $b_\bullet(G)$, $\{b_{\geq q}(G)\}_{q \geq 0}$, coincides with its $(G_{ab} \otimes \mathbb{C})$-adic filtration, where $(G_{ab} \otimes \mathbb{C})$ is the ideal of $\text{Sym}(G_{ab} \otimes \mathbb{C})$ generated by $G_{ab} \otimes \mathbb{C}$.

The infinitesimal Alexander invariant, introduced and studied in [17], is functorial in the following sense. A graded Lie map $f : h \to \mathfrak{g}$ obviously induces a degree zero map $b_\bullet(f) : b_\bullet(h) \to b_\bullet(\mathfrak{g})$, equivariant with respect to the graded algebra map $\text{Sym}(f_{ab}) : \text{Sym}(h_{ab}) \to \text{Sym}(\mathfrak{g}_{ab})$.

Let $\mathbb{L}_\bullet(V)$ be the free graded Lie algebra on a finite-dimensional vector space $V$, graded by bracket length. Use the Lie bracket to identify $\mathbb{L}_2(V)$ and $\wedge^2 V$. For a sub-space $R \subseteq \wedge^2 V$, consider the (quadratic) graded Lie algebra $g = \mathbb{L}(V)/\text{ideal } (R)$, with grading inherited from $\mathbb{L}_\bullet(V)$. Denote by $\iota : R \hookrightarrow \wedge^2 V$ the inclusion.

Theorem 6.2 from [17] provides the following finite, free $\text{Sym}_\bullet(V)$-presentation for the infinitesimal Alexander invariant: $b_\bullet(\mathfrak{g}) = \text{coker}(\nabla)$, where

\[
\nabla := \text{id} \otimes \iota + \delta_3 : \text{Sym}_\bullet(V) \otimes (R \oplus \wedge^3 V) \to \text{Sym}_\bullet(V) \otimes \wedge^2 V,
\]

$R$, $\wedge^3 V$ and $\wedge^2 V$ have degree zero, and the $\text{Sym}(V)$-linear map $\delta_3$ is given by $\delta_3(a \wedge b \wedge c) = a \otimes b \wedge c + b \otimes c \wedge a + c \otimes a \wedge b$, for $a, b, c \in V$.

We begin by simplifying the presentation (4.2). To this end, let $\beta : \wedge^2 \mathfrak{g}_1 \to \mathfrak{g}_2$ be the Lie bracket.
Lemma 4.1. For any quadratic graded Lie algebra $\mathfrak{g}$, $b_*(\mathfrak{g}) = \text{coker}(\nabla)$, as graded Sym(V)-modules, where the Sym(V)-linear map

$$\nabla : \text{Sym}(V) \otimes \wedge^3 \mathfrak{g}_1 \to \text{Sym}(V) \otimes \mathfrak{g}_2$$

is defined by $\nabla = (\text{id} \otimes \beta) \circ \delta_3$.

Proof. It is straightforward to check that the degree zero Sym(V)-linear map $\text{id} \otimes \beta$ induces an isomorphism $\text{coker}(\nabla) \simeq \text{coker}(\nabla)$.

Proof of Theorem C. In Lemma 3.3, let $\nu$ be the canonical projection $G \twoheadrightarrow G_{abf}$, with kernel $K_G$. By our hypothesis on $\mathcal{V}(G)$ and Remark 2.3, we infer that the module $H_1 K_G$ is nilpotent, over both $\mathbb{C}G_{abf}$ and $\mathbb{C}G_{ab}$. Therefore, $\dim_{\mathbb{C}} H_1 K_G < \infty$ (since $H_1 K_G$ is finitely generated over $\mathbb{C}G_{abf}$) and the $I_{G_{ab}}$-adic completion map

$$H_1 K_G \xrightarrow{\cong} \hat{H}_1 K_G$$

is a filtered isomorphism. Proposition 2.4 provides another filtered isomorphism,

$$\hat{B}(G) \xrightarrow{\cong} \hat{H}_1 K_G,$$

between $I_{G_{ab}}$-adic completions. A third filtered isomorphism is a consequence of our assumption on $\mathfrak{g}(G)$:

$$\hat{B}(G) \xrightarrow{\cong} \hat{b}_*(G),$$

where the completion of $b_*(G)$ is taken with respect to the degree filtration; see [3, Proposition 5.4]. Since $\dim_{\mathbb{C}} b_*(G) < \infty$, we deduce that $\dim_{\mathbb{C}} b_*(G) < \infty$. Hence, the degree filtration is finite, and the completion map

$$b_*(G) \xrightarrow{\cong} \hat{b}_*(G)$$

is a filtered isomorphism.

By construction, the isomorphism (4.3) is equivariant with respect to the $I_{G_{ab}}$-adic completion homomorphism, $\mathbb{C}G_{ab} \rightarrow \hat{\mathbb{C}G_{ab}}$. Again by construction, the isomorphism (4.4) is $\hat{\mathbb{C}G_{ab}}$-linear. By Proposition 5.4 from [3], the isomorphism (4.5) is $\exp$-equivariant, where $\exp : \hat{\mathbb{C}G_{ab}} \rightarrow \text{Sym}(\mathbb{G}_{ab} \otimes \mathbb{C})$ is the identification (1.4). Since the degree filtration of $b_*(G)$ coincides with its $(\mathbb{G}_{ab} \otimes \mathbb{C})$-adic filtration, as noted earlier, the isomorphism (4.6) is plainly equivariant with respect to the $(\mathbb{G}_{ab} \otimes \mathbb{C})$-adic completion homomorphism, $\text{Sym}(\mathbb{G}_{ab} \otimes \mathbb{C}) \rightarrow \hat{\text{Sym}(\mathbb{G}_{ab} \otimes \mathbb{C})}$. Putting these facts together, we deduce from (4.3)-(4.6) that the natural $\hat{\text{Sym}(\mathbb{G}_{ab} \otimes \mathbb{C})}$-module structure of the nilpotent $\mathbb{C}G_{ab}$-module $H_1 K_G$, explained in the Introduction, is isomorphic to $b_*(G)$ over $\text{Sym}(\mathbb{G}_{ab} \otimes \mathbb{C})$, as stated in Theorem C.

To finish the proof of Theorem C, we have to show that $I_{G_{ab}}^q \cdot H_1 K_G = 0$ if and only if $b_q(G) = 0$, for any $q \geq 0$. This assertion will follow from the easily checked remark that, given a vector space $M$ endowed with a decreasing Hausdorff filtration
\{F_r\}_{r \geq 0}$ (i.e., $\cap_r F_r = 0$), $F_q = 0$ if and only if $\text{gr}_{\geq q}(M) = \oplus_{r \geq q} F_r/F_{r+1} = 0$.

Plainly, all maps (4.3)-(4.6) induce isomorphisms at the associated graded level, and all filtrations are Hausdorff. We deduce that $I_{G_{ab}}^q \cdot H_1 K_G = 0$ if and only if $b_r(G) = 0$ for $r \geq q$. Since $b_*(G)$ is generated in degree zero over $\text{Sym}(G_{ab} \otimes \mathbb{C})$, this is equivalent to $b_q(G) = 0$. The proof of Theorem C is complete. \hfill \Box

**Proof of the non-triviality assertion of Theorem A.** The group $G = T_g$ satisfies the hypotheses of Theorem C when $g \geq 4$. Consequently, if $H_1 K_g$ is a trivial $\mathbb{C}(T_g)_{ab}$-module, then $b_1(T_g) = g_3(T_g) = 0$. This implies that $g_{\geq 3}(T_g) = 0$, since the Lie algebra $g_*(T_g)$ is generated in degree 1. In particular, $\dim_{\mathbb{C}} g_*(T_g) < \infty$, which contradicts Proposition 9.5 from [8]. \hfill \Box

For the proof of Theorem B, we need to recall the main result of Hain from [8], that gives an explicit presentation of the graded Lie algebra $g_*(T_g)$, for $g \geq 6$, in representation-theoretic terms. For representation theory, we follow the conventions from Fulton and Harris [7], like in [8].

The conjugation action in (1.1) induces an action of $\text{Sp}_g(\mathbb{Z})$ on $g_*(T_g)$, by graded Lie algebra automorphisms. By Johnson’s work, the $\text{Sp}_g(\mathbb{Z})$-action on $g_1(T_g)$ extends to an irreducible rational representation of $\text{Sp}_g(\mathbb{C})$. It follows that the $\text{Sp}_g(\mathbb{Z})$-action on $g_*(T_g)$ extends to a degree-wise rational representation of $\text{Sp}_g(\mathbb{C})$, by graded Lie algebra automorphisms. By naturality, the symplectic Lie algebra $\mathfrak{sp}_g(\mathbb{C})$ acts on $b_*(T_g)$.

The fundamental weights of $\mathfrak{sp}_g(\mathbb{C})$ are denoted $\lambda_1, \ldots, \lambda_g$. The irreducible finite-dimensional representation with highest weight $\lambda = \sum_{i=1}^g n_i \lambda_i$ is denoted $V(\lambda)$. By Johnson’s work, $g_1(T_g) = V(\lambda_3) := V$. The irreducible decomposition of the $\mathfrak{sp}_g(\mathbb{C})$-module $\wedge^2 V(\lambda_3)$ is of the form $\wedge^2 V(\lambda_3) = R \oplus V(2\lambda_2) \oplus V(0)$, with all multiplicities equal to 1. For $g \geq 6$, $g_* := g_*(T_g) = \mathbb{L}_* (V)/\text{ideal (R)}$, as graded Lie algebras with $\mathfrak{sp}_g(\mathbb{C})$-action. In particular, $\beta : \wedge^2 g_1 \rightarrow g_2$ is identified with the canonical $\mathfrak{sp}_g(\mathbb{C})$-equivariant projection $\wedge^2 V(\lambda_3) \rightarrow V(2\lambda_2) \oplus V(0)$.

Set $V(0) = \mathbb{C} \cdot z$, $R = \mathbb{R} + \mathbb{C} \cdot z$, and denote by $\pi : \wedge^2 V(\lambda_3) \rightarrow V(2\lambda_2)$ the canonical $\mathfrak{sp}_g(\mathbb{C})$-equivariant projection. Note that both $\text{id} \otimes \pi : \text{Sym}(V(\lambda_3)) \otimes \wedge^2 V(\lambda_3) \rightarrow \text{Sym}(V(\lambda_3)) \otimes V(2\lambda_2)$ and the map $\delta_3 : \text{Sym}(V(\lambda_3)) \otimes \wedge^3 V(\lambda_3) \rightarrow \text{Sym}(V(\lambda_3)) \otimes \wedge^2 V(\lambda_3)$ from (4.2) are $\mathfrak{sp}_g(\mathbb{C})$-linear. Consequently,

$$\nabla := (\text{id} \otimes \pi) \circ \delta_3 : \text{Sym}(V(\lambda_3)) \otimes \wedge^3 V(\lambda_3) \rightarrow \text{Sym}(V(\lambda_3)) \otimes V(2\lambda_2)$$

is both $\text{Sym}(V(\lambda_3))$-linear and $\mathfrak{sp}_g(\mathbb{C})$-equivariant. We are going to view the $\mathfrak{sp}_g(\mathbb{C})$-trivial module $\mathbb{C} \cdot z$ as a trivial $\text{Sym}_*(V(\lambda_3))$-module concentrated in degree zero, and assign degree 0 to both $\wedge^3 V(\lambda_3)$ and $V(2\lambda_2)$.

Consider the canonical, $\mathfrak{sp}_g(\mathbb{C})$-equivariant graded Lie epimorphism

$$f : g_* = \mathbb{L}_* (V)/\text{ideal (R)} \twoheadrightarrow \mathbb{L}_* (V)/\text{ideal (R)} = \mathfrak{t}_*.$$
Lemma 4.2. The induced $\text{Sym}(V)$-linear, $\mathfrak{sp}_g(\mathbb{C})$-equivariant map $\mathfrak{b}_*(f)$ is onto, with 1-dimensional kernel $\mathbb{C} \cdot z$.

Proof. The first three claims are obvious. It is equally clear that $\mathfrak{b}_0(f)$ has kernel $\mathbb{C} \cdot z$. To prove injectivity in degree $q \geq 1$, start with the class $\bar{x}$ of an arbitrary element $x \in \mathbb{L}_{q+2}(V)$. If $b(f)(\bar{x}) = 0$, then $x$ is equal, modulo $\mathbb{L}(V)$, with a linear combination of Lie monomials of the form $\text{ad}_{v_1} \cdots \text{ad}_{v_q}(\bar{r})$, with $\bar{r} \in \bar{R}$.

Therefore, $\bar{x}$ belongs to the $\mathbb{C}$-span of elements of the form $\text{ad}_{v_1} \cdots \text{ad}_{v_q}(z)$. As shown in [8], the class of $z$ is a central element of the Lie algebra $\mathbb{L}(V)/\text{ideal}(R)$, and so we are done, since $q \geq 1$.

Proof of Theorem B. By Theorem C, $H_1 K_g = \mathfrak{b}(g)$, over $\text{Sym}(V)$, with $g$ as in (4.8). By Lemma 4.2, $\mathfrak{b}(g) = \mathfrak{b}(t) \oplus \mathbb{C} \cdot z$, as graded $\text{Sym}(V)$-modules, where $\mathbb{C} \cdot z$ is $\text{Sym}(V)$-trivial (since $z$ is central in $g$), with degree 0.

By Lemma 4.1, the graded $\text{Sym}(V)$-module $\mathfrak{b}_*(g)$ has presentation (1.2); see (4.7). Note also that the identification $\mathfrak{b}(g) = \mathfrak{b}(t) \oplus V(0)$ is compatible with the natural $\mathfrak{sp}_g(\mathbb{C})$-symmetry of $\mathfrak{b}(g) = \mathfrak{b}(T_g)$.

It remains to prove the assertion about the action of $\Gamma_g/K_g$ on $H_1(K_g, \mathbb{C})$. For this we use the theory of relative completion of mapping class groups developed and studied in [8]. Denote the completion of the mapping class group with respect to the standard homomorphism $\Gamma_g \rightarrow \text{Sp}_g(\mathbb{C})$ by $\mathcal{R}(\Gamma_g)$. Right exactness of relative completion implies that there is an exact sequence

$$\mathcal{G}(T_g) \rightarrow \mathcal{R}(\Gamma_g) \rightarrow \text{Sp}_g(\mathbb{C}) \rightarrow 1$$

such that the diagram

$$\begin{array}{cccccc}
1 & \rightarrow & T_g & \rightarrow & \Gamma_g & \rightarrow & \text{Sp}_g(\mathbb{Z}) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{G}(T_g) & \rightarrow & \mathcal{R}(\Gamma_g) & \rightarrow & \text{Sp}_g(\mathbb{C}) & \rightarrow & 1
\end{array}$$

commutes, where $\mathcal{G}(T_g)$ denotes the Malcev completion of $T_g$.

The conjugation action of $\Gamma_g$ on $T_g$ induces an action of $\Gamma_g$ on the Malcev Lie algebra $\mathfrak{g}(T_g)$ of the Torelli group. Basic properties of relative completion imply that this action factors through the natural homomorphism $\Gamma_g \rightarrow \mathcal{R}(\Gamma_g)$. This action descends to an action of $\mathcal{R}(\Gamma_g)$ on the Alexander invariant $b(T_g)$ of $\mathfrak{g}(T_g)$. Its kernel contains the image of $\mathcal{G}(T_g)'$ in $\mathcal{R}(\Gamma_g)$. Basic facts about the Lie algebra of $\mathcal{R}(\Gamma_g)$ given in [8] imply that, when $g \geq 3$, $\mathcal{R}(\Gamma_g)/\text{im} \mathcal{G}(T_g)'$ is an extension

$$1 \rightarrow V \rightarrow \mathcal{R}(\Gamma_g)/\text{im} \mathcal{G}(T_g)' \rightarrow \text{Sp}_g(\mathbb{C}) \rightarrow 1.$$

Levi’s theorem implies that this sequence is split. However, we have to choose compatible splittings of the lower central series of $\mathfrak{g}(T_g)$ and this sequence. The existence
of such compatible splittings is a consequence of the existence of the mixed Hodge structures on \( g(T_g) \) and on the Lie algebra of \( \mathcal{R}(\Gamma_g) \), and the fact that the weight filtration of \( g(T_g) \) (suitably renumbered) is its lower central series. Such compatible mixed Hodge structures are determined by the choice of a complex structure on the reference surface \( \Sigma \). With such compatible splittings, one obtains a commutative diagram

\[
\begin{array}{ccc}
\Gamma_g/K_g & \longrightarrow & \mathcal{R}(\Gamma_g)/\text{im} \mathcal{G}(T_g)' \\
\downarrow & & \downarrow \\
\text{Aut}(\mathfrak{b}(T_g)) & \longrightarrow & \text{Aut}(\mathfrak{b}_*(T_g))
\end{array}
\]

when \( g \geq 4 \). This completes the proof of Theorem B. \( \square \)

**Example 4.3.** Let us examine the simple case when \( G = F_n \), the non-abelian free group on \( n \) generators. In this case, \( H_1(K_G, \mathbb{C}) = B(G) \otimes \mathbb{C} \). Since \( G \) is 1-formal, Theorem 5.6 from [3] identifies the \( I \)-adic completion \( \hat{H}_1 K_G \) with the degree completion \( \hat{b}_*(g) \), where \( g = g_*(G) = L_* (V) \), and \( V = H_1(F_n, \mathbb{C}) = \mathbb{C}^n \).

On the other hand, \( V_1^1 (G) = V_1^1 (G) \cap \mathbb{T}^0 (G) = (\mathbb{C}^*)^n \) is infinite, in contrast with the setup from Theorem C. It follows from Corollary 6.2 in [18] that \( \dim \hat{b}_*(g) = \infty \).

It is also well-known that \( \dim \hat{b}_*(g) = \infty \) when \( n > 1 \). This non-finiteness property of \( \hat{b}_*(g) \) can be seen concretely by using the exact Koszul complex, \( \{ \delta_i : P_* \otimes \wedge^i V \rightarrow P_* \otimes \wedge^{i-1} V \} \), where \( P_* = \text{Sym}(V) \). Indeed, we infer from (4.2) that, for every \( q \geq 0 \),

\[
\hat{b}_q (g) = \text{coker} (\delta : P_{q-1} \otimes \wedge^3 V \rightarrow P_q \otimes \wedge^2 V) \cong \ker (\delta : P_{q+1} \otimes V \rightarrow P_{q+2})
\]

has dimension \( (q + n) / (q + 2) (q + 1) \), a computation that goes back to Chen’s thesis [2]. Note also that each \( \hat{b}_q (g) \) is an \( \mathfrak{sl}_n (\mathbb{C}) \)-module. It turns out that these modules are irreducible, as we now explain.

Let \( \{ \lambda_1, \ldots, \lambda_{n-1} \} \) be the set of fundamental weights of \( \mathfrak{sl}_n (\mathbb{C}) \) associated to the ordered basis \( e_1, \ldots, e_n \) of \( V \), as in [7]. One can easily check that, for each \( q \geq 0 \), the image \( v \) of the vector

\[
u = e_1^q \otimes (e_1 \wedge e_2) \in P_q \otimes \wedge^2 V
\]

in \( \hat{b}_q (g) \) is non-zero. Since \( u \) is a highest weight vector of weight \( q \lambda_1 + \lambda_2 \), it follows that \( v \) generates a copy of the irreducible \( \mathfrak{sl}_n (\mathbb{C}) \)-module \( V(q \lambda_1 + \lambda_2) \) in \( \hat{b}_q (g) \). Since \( \dim V(q \lambda_1 + \lambda_2) = \dim \hat{b}_q (g) \), we conclude that

\[
\hat{b}_q (g) = V(q \lambda_1 + \lambda_2).
\]
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