Position Vectors of Curves Generalizing General Helices and Slant Helices in Euclidean 3-Space

Abderrazak El Haimi, Malika Izid and Amina Ouazzani Chahdi

Abstract. In this paper, we give a new characterization of a k-slant helix which is a generalization of general helix and slant helix. Thereafter, we construct a vector differential equation of the third order to determine the parametric representation of a k-slant helix according to standard frame in Euclidean 3-space. Finally, we apply this method to find the position vector of some examples of 2-slant helix by means of intrinsic equations.

1 Introduction

In differential geometry, a curve called general helix is defined by the property that its tangent vector field makes a constant angle with a fixed straight line which is the axis of the general helix in Euclidean 3-space. A classical result stated by M.A. Lancret in 1802 and first proved by B. Saint Venant in 1845 (see [6, 11] for details) says that: A necessary and sufficient condition that a curve be a general helix is that the ratio

$$\sigma_0 = \frac{\tau}{\kappa},$$

is constant along the curve, where $\kappa$ and $\tau$ denote the curvature and the torsion of the curve, respectively. If both $\kappa$ and $\tau$ are non-zero constants, then the curve is called a circular helix.

Izumiya and Takeuchi [13] have introduced the concept of the slant helix by saying that the principal normal lines make a constant angle with a fixed straight line and they characterize a slant helix if and only if the geodesic curvature

$$\sigma_1 = \frac{\sigma'_0}{\kappa(1+\sigma_0^2)^{\frac{3}{2}}},$$

of principal image of the principal normal indicatrix is a constant function.

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Corresponding author: Malika Izid.
In [5], the authors investigate a curve whose spherical images (the tangent indicatrix and binormal indicatrix) are slant helices and called it as a $C-$slant helix where $C = \frac{N'}{||N||}$ ($N$ the principal normal of the curve). They characterize a $C-$slant helix if and only if the geodesic curvature

$$\sigma_2 = \frac{\sigma_1'}{\kappa \sqrt{1+\sigma_0^2(1+\sigma_1^2)^2}},$$

of the principal image of the vector field $C$ indicatrix is a constant function.

The determining of the position vector of some different curves according to the intrinsic equations $\kappa = \kappa(s)$ and $\tau = \tau(s)$ (where $\kappa$ and $\tau$ are the curvature and torsion of the curve) is considered as a one of important subjects. Recently, the parametric representation of general helices and slant helices as an important special curves in Euclidean space $E^3$ are deduced by Ali [1, 2].

The purpose of this paper is to determine the position vector of $k$-slant helices (see [3, 4]) which a generalization of general helices and slant helices. Firstly, we give a new characterization of $k-$slant helices and construct a vector differential equation of the third order to determine the parametric representation of $k-$slant helices. By applying this method, we present some examples of $2-$slant helix.

## 2 Preliminaries

In Euclidean space $E^3$, we known that each unit speed curve has at least four continuous derivatives, one can associate three orthogonal unit vector fields $T, N$ and $B$ are the tangent, the principal normal and the binormal vector fields respectively [7].

Let $\psi : I \subset \mathbb{R} \rightarrow E^3, \psi = \psi(s)$, be an arbitrary curve in $E^3$. Recall that the curve $\psi$ is said to be unit speed or parametrized by the arc-length if $\langle \psi'(s), \psi'(s) \rangle = 1$ for any $s \in I$. Thus, we will assume throughout this work that $\psi$ is a unit speed curve, where $\langle , \rangle$ denotes the standard inner product given by:

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2.$$

Let $(T(s), N(s), B(s))$ be the Frenet moving frame along $\psi$. The Frenet equations for $\psi$ are given by [6]:

$$\begin{bmatrix}
T'(s) \\
N'(s) \\
B'(s)
\end{bmatrix} = \begin{bmatrix}
0 & \kappa(s) & 0 \\
-\kappa(s) & 0 & \tau(s) \\
0 & -\tau(s) & 0
\end{bmatrix} \begin{bmatrix}
T(s) \\
N(s) \\
B(s)
\end{bmatrix},$$

(2.1)

where $\kappa$ and $\tau$ are the curvature and the torsion of the curve $\psi$ in terms of Frenet frame, respectively.
Denote by \( \{N, C, W = N \wedge C\} \) the alternative moving frame along the curve \( \psi \) in Euclidean 3-space. Note that \( N, C = \frac{N'}{\|N'\|} \) and \( W = \frac{T + \kappa B}{\sqrt{T^2 + \kappa^2}} \) are the unit principal normal, the derivative of principal normal vector and the Darboux vector, respectively. For the derivatives of the alternative moving frame, we have:

\[
\begin{bmatrix}
N'(s) \\
C'(s) \\
W'(s)
\end{bmatrix} =
\begin{bmatrix}
0 & f_1(s) & 0 \\
-f_1(s) & 0 & g_1(s) \\
0 & -g_1(s) & 0
\end{bmatrix}
\begin{bmatrix}
N(s) \\
C(s) \\
W(s)
\end{bmatrix},
\]

(2.2)

where \( f_1 = \sqrt{\tau^2 + \kappa^2} = \kappa \sqrt{1 + \sigma_0^2} \) and \( g_1 = \sigma_1 f_1 \), are curvatures of the curve \( \psi \) in terms of the alternative moving frame.

### 3 k-slant helix and its characterizations

Let \( \psi = \psi(s) \) a natural representation of a unit speed curve in Euclidean 3-space, and let \((T(s), N(s), B(s))\) denotes the Frenet frame of \( \psi \) with \( \kappa(s), \tau(s) \) the curvature and the torsion of the curve \( \psi \), respectively.

We denote by \( C_0 = \psi(s) \),

\[
C_k(s) = \frac{C_{k-1}'(s)}{\|C_{k-1}'(s)\|} \quad \text{and} \quad W_{k+1}(s) = C_k(s) \wedge C_{k+1}(s), \quad k \in \{1, 2, \ldots\}.
\]

Therefore, we can see that \((C_k, C_{k+1}, W_{k+1})\) is the Frenet frame of \( s \to C_{k-1}(s) \). Then the derivative formulae of Frenet frame are given by:

\[
\begin{bmatrix}
C_k'(s) \\
C_{k+1}'(s) \\
W_{k+1}'(s)
\end{bmatrix} =
\begin{bmatrix}
0 & f_{k-1}(s) & 0 \\
-f_{k-1}(s) & 0 & g_{k-1}(s) \\
0 & -g_{k-1}(s) & 0
\end{bmatrix}
\begin{bmatrix}
C_k(s) \\
C_{k+1}(s) \\
W_{k+1}(s)
\end{bmatrix},
\]

(3.1)

where \( f_{k-1} \) and \( g_{k-1} \) are the curvatures of the curve \( C_{k-1} \) in terms of \((C_k, C_{k+1}, W_{k+1})\) moving frame. We can easily see that \( f_0 = \kappa \) and \( g_0 = \tau \).

If we write this curve in the other parametric representation \( C_{k-1} = C_{k-1}(t) \) where \( t = \int f_{k-1}(s) \, ds \), we have the new Frenet equations as follows:

\[
\begin{bmatrix}
C_k'(t) \\
C_{k+1}'(t) \\
W_{k+1}'(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & \sigma_{k-1}(t) \\
0 & -\sigma_{k-1}(t) & 0
\end{bmatrix}
\begin{bmatrix}
C_k(t) \\
C_{k+1}(t) \\
W_{k+1}(t)
\end{bmatrix},
\]

(3.2)

where \( \sigma_{k-1} = \frac{g_{k-1}}{f_{k-1}} \).
**Definition 1.** Let $\psi : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space. A curve $\psi$ is called a $k-$slant helix if the unit vector field $C_{k+1}$ makes a constant angle $V$ with a fixed direction $U$, that is,

$$\langle C_{k+1}, U \rangle = \cos V, \ V = \text{constant}$$

along the curve $\psi$.

**Lemma 3.1.** Let $\psi : I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve in Euclidean 3-space. Then the curve $\psi$ is a $k-$slant helix if and only if the geodesic curvature

$$\sigma_k = \frac{\sigma'_{k-1}}{\kappa \sqrt{1+\sigma_0^2 \sqrt{1+\sigma_1^2 \ldots \sqrt{1+\sigma_{k-2}^2 (1+\sigma_{k-1}^2)}}}}$$

of the principal image of the vector field $C_{k+1}$ indicatrix is a constant function.

**Proof.** If $\psi$ is a $k-$slant helix, we can see that the curve $C_{k-1}$ is a slant helix. Then

$$\sigma_k = \frac{\sigma'_{k-1}}{f_{k-1}(1+\sigma_{k-1}^2)^{\frac{3}{2}}}, \quad (3.3)$$

is a constant function. Since

$$f_{k-1} = \sqrt{f_{k-2}^2 + g_{k-2}^2}$$

$$f_{k-1} = f_{k-2} \sqrt{1 + \sigma_{k-2}^2} \quad (3.4)$$

if we put (6) in (5), we obtain

$$\sigma_k = \frac{\sigma'_{k-1}}{f_{k-2} \sqrt{1+\sigma_{k-2}^2 (1+\sigma_{k-1}^2)}^{\frac{3}{2}}},$$

By continuing this process k-times, we get

$$\sigma_k = \frac{\sigma'_{k-1}}{\kappa \sqrt{1+\sigma_0^2 \sqrt{1+\sigma_1^2 \ldots \sqrt{1+\sigma_{k-2}^2 (1+\sigma_{k-1}^2)}}}}.$$

\(\blacksquare\)

The following lemma gives a new characterization for $k-$slant helices in $E^3$.

**Lemma 3.2.** Let $\psi : I \rightarrow E^3$ be a curve that is parameterized by arclength. The curve is a $k-$slant helix (its vector fields $C_{k+1}$ make a constant angle, with a fixed unit direction $U$ in $E^3$) if and only if

$$\sigma_{k-1}(s) = \pm \frac{m \int f_{k-1} ds}{\sqrt{1-m^2(\int f_{k-1} ds)^2}}, \quad (3.5)$$

where $m = \frac{n}{\sqrt{1-n^2}}$, and $\langle C_{k+1}, U \rangle = n$. 
Proof. (⇒) Let \( U \) be a unit fixed vector satisfying
\[
< C_{k+1}, U > = n. \tag{3.6}
\]
Differentiating the Eq. (8) with respect to the variable \( t = \int f_{k-1} (s) \, ds \) and using the derivative formulae (4), we get
\[
< -C_k(t) + \sigma_{k-1}(t)W_{k+1}(t), U > = 0. \tag{3.7}
\]
Therefore,
\[
< C_k(t), U > = \sigma_{k-1}(t) < W_{k+1}(t), U >.
\]
If we put \( < W_{k+1}(t), U > = b \), we can write
\[
U = b(t)\sigma_{k-1}(t)C_k(t) + nC_{k+1}(t) + b(t)W_{k+1}(t).
\]
From the unitary of the vector \( U \) we get
\[
b = \pm \sqrt{\frac{1-n^2}{1+\sigma_{k-1}^2}}. \tag{3.8}
\]
Therefore, the vector \( U \) can be written as
\[
U = \pm \sigma_{k-1}(t)\sqrt{\frac{1-n^2}{1+\sigma_{k-1}^2(t)}}C_k(t) + nC_{k+1}(t) \pm \sqrt{\frac{1-n^2}{1+\sigma_{k-1}^2(t)}}W_{k+1}(t). \tag{3.9}
\]
Differentiating the Eq. (9), we obtain
\[
< \sigma_{k-1}'(t)W_{k+1}(t) - (1 + \sigma_{k-1}^2)C_{k+1}(t), U > = 0. \tag{3.10}
\]
By Eqs. (12), (10) and (8), we get the following differential equation
\[
m = \pm \frac{\sigma_{k-1}'}{(1+\sigma_{k-1}^2)^{3/2}},
\]
where \( m = \frac{n}{\sqrt{1-n^2}} \). Integration the above equation, we obtain
\[
\frac{\sigma_{k-1}}{\sqrt{1+\sigma_{k-1}^2}} = \pm m \left( t + c_1 \right), \tag{3.11}
\]
where \( c_1 \) is an integration constant. The integration constant can disappear with a parameter change \( t \rightarrow t - c_1 \). Solving the Eq. (13) with \( \sigma_{k-1} \) as unknown we have
\[
\sigma_{k-1} = \pm \frac{mt}{\sqrt{1-m^2t^2}}, \tag{3.12}
\]
we obtain the result as desired.
Suppose that
\[ g_{k-1}(s) = \pm \frac{mf_{k-1}(s) \int f_{k-1}(s) ds}{\sqrt{1-m^2(\int f_{k-1}(s) ds)^2}}. \]
The function \( \sigma_{k-1} \) can be written as
\[ \sigma_{k-1}(t) = \pm \frac{mt}{\sqrt{1-m^2t^2}} \] and let us consider the vector
\[ U(t) = n \left( tC_k + C_{k+1} \pm \frac{\sqrt{1-m^2t^2}}{m} W_{k+1} \right). \]
Differentiating the vector \( U \) bye using the derivative formula (4),
\[ \frac{dU}{dt} = n \left( C_k + tC_{k+1} - C_k + \sigma_{k-1}W_{k+1} \mp \frac{mt}{\sqrt{1-m^2t^2}} W_{k+1} \mp \frac{\sigma_{k-1}\sqrt{1-m^2t^2}}{m} C_{k+1} \right) = 0. \]
Therefore, the vector \( U \) is constant and \( < C_{k+1}, U > = n \), which completes the proof.

4 Position vectors of \( k \)-slant helices

To determine the parametric representation of the position vector of a space curve called a \( k \)-slant helix (its vector fields \( C_{k+1} \) make a constant angle with a fixed direction), we firstly establish that for any arbitrary curve, the vector \( C_{k+1} \) satisfies a vector differential equation of the third order as follows:

**Theorem 4.1.** Let \( \psi = \psi(s) \) be a unit speed curve in Euclidean 3-space. Suppose \( \psi = \psi(t) \) is another parametric representation of this curve by the parameter \( t = \int f_{k-1} ds \). Then, the vector \( C_{k+1} \) satisfies a vector differential equation of the third order as follows:

\[ \frac{1}{\sigma_{k-1}(t)} \left[ \frac{1}{\sigma_{k-1}'(t)} \left( C_{k+1}''(t) + (1 + \sigma_{k-1}^2(t)) C_{k+1}'(t) \right) \right]' + C_{k+1}(t) = 0, \] (4.1)

where \( \sigma_{k-1}(t) = \frac{g_{k-1}(t)}{f_{k-1}(t)}. \)

**Proof.** If we differentiate the second equation of the derivative formulae (4) and using the first and third equations of (4), we get
\[ W_{k+1}(t) = \frac{1}{\sigma_{k-1}} \left[ C_{k+1}''(t) + (1 + \sigma_{k-1}^2(t)) C_{k+1}'(t) \right]. \] (4.2)

Differentiating the equation (16) and using the third equation from (4), we obtain a vector differential equation of the third order (15) as desired.

Then Eq.(15) is not easy to solve in the general case. If one solves this equation, we get the following lemma:
Lemma 4.1. The position vector of an arbitrary space curve can be determined as follows:

\[ \psi(s) = \int \left( \int \int f_0 \left( \int f_{k-1} C_{k+1} ds \right) ds \right) ds. \]  

(4.3)

Proof. Let \( \psi = \psi(s) \) a natural representation of an arbitrary curve. By using the first equation of formula (3), we have

\[ C_{k+1} = \frac{1}{f_{k-1}} \frac{dC_k}{ds}. \]  

(4.4)

For \( k \geq 1 \), we get

\[ C_k = \frac{1}{f_{k-2}} \frac{dC_{k-1}}{ds}. \]  

(4.5)

Substituting (19) in (18), we obtain

\[ C_{k+1} = \frac{1}{f_{k-1}} \frac{d}{ds} \left( \frac{1}{f_{k-2}} \frac{dC_{k-1}}{ds} \right). \]

By continuing this process \( k \)-times, we get

\[ C_{k+1} = \frac{1}{f_{k-1}} \frac{d}{ds} \left( \frac{1}{f_{k-2}} \frac{d}{ds} \left( \frac{1}{f_{k-3}} \frac{d}{ds} \left( \cdots \frac{1}{f_0} \frac{dC_1}{ds} \cdots \right) \right) \right), \]

where \( \frac{dC_1}{ds} = \frac{dT}{ds} = \frac{d^2 \psi}{ds^2} \).

We can solve the Eq.(15) in the case of a \( k \)-slant helix.

Lemma 4.2. Let \( \psi = \psi(s) \) a natural representation of a \( k \)-slant helix (its vector fields \( C_{k+1} \) make a constant angle \( V \) with a fixed direction). Suppose \( \psi = \psi(t) \) is another parametric representation of this curve by the parameter \( t = \int f_{k-1} ds \). Then the vector \( C_{k+1} \) satisfies a vector differential equation of the third order:

\[ \left( 1 - m^2 t^2 \right) C'''_{k+1} - 3m^2 t C''_{k+1} + C'_{k+1} = 0, \]

where \( m = \frac{n}{\sqrt{1-n^2}} \) and \( n = \cos(V) \).

Proof. If \( \psi \) is a k-slant helix, we can write

\[ \sigma_{k-1} = \pm \frac{m \int f_{k-1} ds}{\sqrt{1-m^2(\int f_{k-1} ds)^2}} = \pm \frac{mt}{\sqrt{1-m^2 t^2}}. \]

By differentiating the last formula, we obtain

\[ \sigma'_{k-1} = \pm m \left( 1 - m^2 t^2 \right)^{-3} \]  

(4.6)

and

\[ \sigma''_{k-1} = \pm 3m^3 t \left( 1 - m^2 t^2 \right)^{-5}. \]  

(4.7)
Therefore the equation (15) becomes
\[
\frac{-\sigma''_{k-1}C''_{k+1}}{\sigma_{k-1}\sigma''_{k-1}} + \frac{C''_{k+1}}{\sigma_{k-1}\sigma''_{k-1}} + \frac{(1+\sigma^2_{k-1})C'_{k+1}}{\sigma_{k-1}\sigma'_{k-1}} - \frac{\sigma''_{k-1}(1+\sigma^2_{k-1})C_{k+1}}{\sigma_{k-1}\sigma'_{k-1}} + 3C_{k+1} = 0. \tag{4.8}
\]
Substituting (20) and (21) in (22), we obtain the formula as desired. \hfill \Box

**Theorem 4.2.** The position vector \( \psi = (\psi_1, \psi_2, \psi_3) \) of a \( k \)-slant helix is computed in the natural representation form as follows
\[
\begin{align*}
\psi_1(s) &= \frac{n}{m} \int \left( \int f_0 \left( \int f_1 \left( \int f_{k-1} \cos \left[ \frac{1}{n} \arcsin \left( m \int f_{k-1}(s) ds \right) \right] ds \right) ds \right) ds \right), \\
\psi_2(s) &= \frac{n}{m} \int \left( \int f_0 \left( \int f_1 \left( \int f_{k-1} \sin \left[ \frac{1}{n} \arcsin \left( m \int f_{k-1}(s) ds \right) \right] ds \right) ds \right) ds \right), \\
\psi_3(s) &= n \int \left( \int f_0 \left( \int f_1 \left( \int f_{k-1} ds \right) ds \right) ds \right),
\end{align*}
\tag{4.9}
\]
where \( m = \frac{n}{\sqrt{1-n^2}} \), \( n = \cos(V) \) and \( V \) is the angle between the fixed straight line (axis of a \( k \)-slant helix) and the vector \( C_{k+1} \) of the curve.

**Proof.** The curve \( \psi \) is a \( k \)-slant helix, i.e. the vector \( C_{k+1} \) makes a constant angle, \( V = \arccos(n) \), with the constant vector called the axis of the \( k \)-slant helix. Then the vector \( C_{k+1} \) satisfies a vector differential equation:
\[
(1 - m^2 t^2) C''_{k+1} - 3m^2 t C''_{k+1} + C'_{k+1} = 0. \tag{4.10}
\]
So, without loss of generality, we can take the axis of the \( k \)-slant helix parallel to \( e_3 \), where \( (e_1, e_2, e_3) \) is an orthonormal frame in \( E_3 \), then
\[
C_{k+1}(t) = C_{k+11}(t)e_1 + C_{k+12}(t)e_2 + ne_3. \tag{4.11}
\]
From the unitary of the vector \( C_{k+1} \), we get
\[
C_{k+1}^2 + C_{k+12}^2 = 1 - n^2 = \frac{n^2}{m^2}. \tag{4.12}
\]
The solution of Eq.(26) is given as follows:
\[
\begin{align*}
C_{k+1}(t) &= \frac{n}{m} \cos(\lambda(t)), \\
C_{k+12}(t) &= \frac{n}{m} \sin(\lambda(t)),
\end{align*}
\tag{4.13}
\]
where \( \lambda \) is an arbitrary function of \( t \). Every component of the vector \( C_{k+1} \) satisfies the Eq.(24). So, substituting the components \( C_{k+11}(t) \) and \( C_{k+12}(t) \) in the Eq.(24), we have the following differential equations of the function \( \lambda(t) \)
\[
3\lambda(t) \left[ m^2 t \lambda'(t) - (1 - m^2 t^2) \right] \cos(\lambda(t)) \\
- \left[ \lambda'(t) - 3m^2 t \lambda''(t) - (1 - m^2 t^2) (\lambda'^3(t) - \lambda''(t)) \right] \sin(\lambda(t)) = 0, \tag{4.14}
\]
\[3\lambda'(t) \left[ m^2 t \lambda'(t) - (1 - m^2 t^2) \right] \sin(\lambda(t)) + \left[ \lambda'(t) - 3m^2 t \lambda''(t) - (1 - m^2 t^2) \left( \lambda'^3(t) - \lambda''(t) \right) \right] \cos(\lambda(t)) = 0. \tag{4.15}\]

It is easy to prove that the above two equations lead to the following two equations:

\[m^2 t \lambda'(t) - (1 - m^2 t^2) \lambda''(t) = 0, \tag{4.16}\]

\[\lambda'(t) - 3m^2 t \lambda''(t) - (1 - m^2 t^2) \left( \lambda'^3(t) - \lambda''(t) \right) = 0. \tag{4.17}\]

The general solution of Eq. (30) is

\[\lambda(t) = c_1 \arcsin(mt) + c_2, \tag{4.18}\]

where \(c_1\) and \(c_2\) are constants of integration. The constant \(c_2\) can be disappear if we change the parameter \(\lambda \rightarrow \lambda + c_2\). Substituting the solution (32) in the Eq. (31), we obtain the following condition:

\[c_1 m \left( 1 + m^2 (1 - c_1) \right) = 0,\]

which leads to \(c_1 = \frac{\sqrt{1 + m^2}}{m} = \frac{1}{n}\), where \(m \neq 0\) and \(c_1 \neq 0\).

Now, the vector \(C_{k+1}\) take the following form:

\[
\begin{cases}
C_{k+1_1}(t) = \frac{n}{m} \cos \left( \frac{1}{n} \arcsin mt \right), \\
C_{k+1_2}(t) = \frac{n}{m} \sin \left( \frac{1}{n} \arcsin mt \right), \\
C_{k+1_3}(t) = n.
\end{cases} \tag{4.19}
\]

If we substitute the Eq. (33) in the Eq. (17), we have the Eq. (23), which completes the proof. \(\square\)

5 Applications

In this section, we introduce the position vectors of some 2−slant helices, by using new parametric representations.

**Corollary 5.1.** The position vector \(\psi = (\psi_1, \psi_2, \psi_3)\) of a 2−slant helix whose the vector \(C_3 = \frac{C_2}{||C_2||} = \frac{N'}{||N'||}\) makes a constant angle with a fixed straight line in the space is expressed in the natural representation form as follows:

\[
\begin{align*}
\psi_1(s) &= \frac{n}{m} \int \left[ \int f_0(s) \left[ \int f_1(s) \cos \left( \frac{1}{n} \arcsin \left( m \int f_1(s) ds \right) \right) \right] ds \right] ds, \\
\psi_2(s) &= \frac{n}{m} \int \left[ \int f_0(s) \left[ \int f_1(s) \sin \left( \frac{1}{n} \arcsin \left( m \int f_1(s) ds \right) \right) \right] ds \right] ds, \\
\psi_3(s) &= n \int \left[ \int f_0(s) \left[ \int f_1(s) ds \right] ds \right] ds.
\end{align*} \tag{5.1}
\]
with \( f_1(s) = \sqrt{g_0^2(s) + f_0^2(s)} \), \( m = \frac{n}{\sqrt{1 - n^2}} \) and \( n = \cos(V) \), or in the parametric form

\[
\begin{align*}
\psi_1(\theta) &= \frac{n}{m} \int_1 f_{1i}(\theta) \left[ \int f_{0i}(\theta) \left[ \int \cos \left( \frac{1}{n} \arcsin (m \theta) \right) dt \theta \right] \right] d\theta, \\
\psi_2(\theta) &= \frac{n}{m} \int_1 f_{1i}(\theta) \left[ \int f_{0i}(\theta) \left[ \int \sin \left( \frac{1}{n} \arcsin (m \theta) \right) dt \theta \right] \right] d\theta, \\
\psi_3(\theta) &= n \int_1 f_{1i}(\theta) \left[ \int f_{0i}(\theta) dt \theta \right] d\theta,
\end{align*}
\]

or in the useful parametric form

\[
\begin{align*}
\psi_1(t) &= \frac{n^4}{m^4} \int \cos(nt) f_{1i}(\theta) \left[ \int f_{0i}(\theta) \cos(nt) \left[ \int \cos(t) \cos(nt) dt \right] dt \right] dt, \\
\psi_2(t) &= \frac{n^4}{m^4} \int \cos(nt) f_{1i}(\theta) \left[ \int f_{0i}(\theta) \cos(nt) \left[ \int \sin(t) \cos(nt) dt \right] dt \right] dt, \\
\psi_3(t) &= \frac{n^3}{m^3} \int \cos(nt) f_{1i}(\theta) \left[ \int f_{0i}(\theta) \cos(nt) \sin(nt) dt \right] dt,
\end{align*}
\]

where \( \theta = \int f_1(s) \, ds \), \( t = \frac{1}{n} \arcsin (m \theta) \), \( m = \frac{n}{\sqrt{1 - n^2}} \), \( n = \cos(V) \) and \( V \) is the angle between the fixed straight line (axis of a 2–slant helix) and the vector \( C_3 \) of the curve.

Now, we take several choices for the curvature \( f_0 \) and torsion \( g_0 \) of a regular curve. We check that the curve is a 2–slant helix, and next, we apply corollary 5.1.

**Example 1.** The case of a 2–slant helix with

\[
f_0 = \frac{\mu}{m} \cos(\mu s) \cos \left( \frac{1}{m} \cos(\mu s) \right) \quad \text{and} \quad g_0 = \frac{-\mu}{m} \cos(\mu s) \sin \left( \frac{1}{m} \cos(\mu s) \right).
\]

Therefore \( f_1 = \frac{\mu}{m} \cos(\mu s) \) and \( g_1 = \frac{\mu}{m} \sin(\mu s) \), so we have \( \sigma_2 = m \). Substituting \( f_0 \) and \( f_1 \) in the Eq.(34), we have the explicit parametric representation of such curve as follows:

\[
\begin{align*}
\psi_1(s) &= \frac{n^2 \mu}{2m^3} \int \left[ \int \cos(\mu s) \cos \left( \frac{1}{m} \cos(\mu s) \right) \left[ \cos \left( \frac{n+1}{n} \mu s \right) + \cos \left( \frac{n-1}{n} \mu s \right) \right] \right] ds, \\
\psi_2(s) &= \frac{-n^2 \mu}{2m^3} \int \left[ \int \cos(\mu s) \cos \left( \frac{1}{m} \cos(\mu s) \right) \left[ \cos \left( \frac{n+1}{n} \mu s \right) + \cos \left( \frac{n-1}{n} \mu s \right) \right] \right] ds, \\
\psi_3(s) &= \frac{-m}{m} \int \cos(\mu s) \cos \left( \frac{1}{m} \cos(\mu s) \right) + m \cos \left( \frac{1}{m} \cos(\mu s) \right) ds.
\end{align*}
\]

**Example 2.** The case of a slant–slant helix with

\[
f_0 = \frac{ms}{\sqrt{1 - m^2 s^2}} \cos(s) \quad \text{and} \quad g_0 = \frac{ms}{\sqrt{1 - m^2 s^2}} \sin(s).
\]
Therefore $f_1 = \frac{ms}{\sqrt{1 - m^2 s^2}}$ and $g_1 = 1$, we have $\sigma_2 = -m$. Substituting $f_0 = \frac{ms}{\sqrt{1 - m^2 s^2}} \cos(s) = \frac{\cos(nt)}{\sin(nt)} \cos\left(\frac{1}{m} \cos(nt)\right)$, and $f_1 = \frac{ms}{\sqrt{1 - m^2 s^2}} = \frac{\cos(nt)}{\sin(nt)}$ in the Eq.(36), we have the explicit parametric representation of such curve as follows:

$$\psi_1(t) = \frac{n^4}{2m^4} \int \sin(nt) \left[ \int \cos(nt) \cos\left(\frac{1}{m} \cos(nt)\right) \left(\frac{\sin((n+1)t)}{n+1} + \frac{\sin((n-1)t)}{n-1}\right) \right] dt,$$

$$\psi_2(t) = -\frac{n^4}{2m^4} \int \sin(nt) \left[ \int \cos(nt) \cos\left(\frac{1}{m} \cos(nt)\right) \left(\frac{\cos((n+1)t)}{n+1} + \frac{\cos((1-n)t)}{1-n}\right) \right] dt,$$

$$\psi_3(t) = \frac{n}{m} \left( \cos(nt) \cos\left(\frac{1}{m} \cos(nt)\right) - 2m \sin\left(\frac{1}{m} \cos(nt)\right) \right),$$

where $\theta = \frac{-1}{m} \sqrt{1 - m^2 s^2}$ and $t = \frac{1}{n} \arcsin(m \theta)$.

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**Abderrazak El Haimi**  Departement of Mathematics and Informatic, Faculty of Sciences Ben M’Sik, University of Hassan II- Casablanca, Morocco.

E-mail: abderrazzakelhaimi@gmail.com

**Malika Izid**  Departement of Mathematics and Informatic, Faculty of Sciences Ben M’Sik, University of Hassan II- Casablanca, Morocco.

E-mail: izid.malika@gmail.com

**Amina Ouazzani Chahdi**  Departement of Mathematics and Informatic, Faculty of Sciences Ben M’Sik, University of Hassan II- Casablanca, Morocco.

E-mail: ouazzaniamina@hotmail.com