A solution for tensor reduction of one-loop $N$-point functions with $N \geq 6$

J. Fleischer $^a$, T. Riemann $^b$, $^*$

$^a$ Fakultät für Physik, Universität Bielefeld, Universitätsstr. 25, 33615 Bielefeld, Germany
$^b$ Deutsches Elektronen-Synchrotron, DESY, Platanenallee 6, 15738 Zeuthen, Germany

**Abstract**

Collisions at the LHC produce many-particle final states, and for precise predictions the one-loop $N$-point corrections are needed. We study here the tensor reduction for Feynman integrals with $N \geq 6$. A general, recursive solution by Binoth et al. expresses $N$-point Feynman integrals of rank $R$ in terms of $(N - 1)$-point Feynman integrals of rank $(R - 1)$ (for $N \geq 6$). We show that the coefficients can be obtained analytically from suitable representations of the metric tensor. Contractions of the tensor integrals with external momenta can be efficiently expressed as well. We consider our approach particularly well suited for automatization.

© 2011 Published by Elsevier B.V.

1. Introduction

In a recent article [1], we have worked out an algebraic method to present one-loop tensor 5-point functions in terms of scalar one-loop 1-point to 4-point functions. The tensor integrals are defined as

$$ I_{N}^{\mu_1, \ldots, \mu_N} = \int \frac{d^4k}{i \pi^{d/2}} \prod_{i=1}^{N} \Pi^{R}_{r=1} k_i^{\mu_r} c_j, $$

with denominators $c_j$, having chords $q_j$,

$$ c_j = (k - q_j)^2 - m_j^2 - i \varepsilon. $$

In a subsequent article [2] we have calculated contractions of the $(N = 5)$-point tensor Feynman integrals with external momenta, resulting in the analytic evaluation of sums over products of scalar products of the chords and signed minors, yielding compact expressions for them. The present article is based on the observation that those sums are valid for arbitrary $N$. This allows us to extend our formalism to $N$-point tensor integrals with $N \geq 6$.

Following ideas presented in [3], an iterative approach has been systematically worked out in [4]. The $N$-point tensor integrals are represented there in terms of $(N - 1)$-point tensor integrals with smaller rank for arbitrary $N \geq 6$ as

$$ I_{N}^{\mu_1, \ldots, \mu_N} = - \sum_{r=1}^{N} C_j^{\mu_r} I_{N-1}^{\mu_1, \ldots, \mu_{r-1}, \mu_{r+1}, \ldots, \mu_N}, $$

where $r$ indicates the line scratched from $I_{N}^{\mu_1, \ldots, \mu_N}$. Eq. (61) of [4] will be our starting point; it contains an implicit solution for the coefficients $C_j^{\mu}$:

$$ \sum_{j=1}^{N} C_j^{\mu} q_j^\nu = \frac{1}{2} g^{\mu \nu}. $$

The subscript [4], indicating explicitly the 4-dimensional metric tensor, will be skipped in the following. We stress, however, that soft and collinear divergences will be regulated by dimensional regularization as described in [4]. Further, we will set $q_N = 0$ throughout this article, in notations of [4] $r_N = 0$, and $\Delta_j^\nu N = - q_j^\nu$. In the present work we develop a procedure to solve (1.4) analytically for arbitrary $N \geq 6$. Explicit examples will be given for $N = 7$ and $N = 8$.

Assume a representation of the metric tensor in the form

$$ \frac{1}{2} g^{\mu \nu} = \sum_{i,j=1}^{N-1} G_{ij} q_i^\mu q_j^\nu $$

is available. Then, necessarily, the vector

$$ C_j^{\mu} = \sum_{i=1}^{N-1} G_{ij} q_i^\mu $$

is a solution of (1.4). An additional requirement according to Eq. (62) in [4] has to be fulfilled for this vector:

$$ \sum_{j=1}^{N} C_j^{\mu} = 0, $$

which we will verify for the solutions we obtain.

---

$^*$ Corresponding author.

E-mail addresses: fleischer@physik.uni-bielefeld.de (J. Fleischer), TorRiemann@desy.de (T. Riemann).

0370-2693/$ – see front matter © 2011 Published by Elsevier B.V.
doi:10.1016/j.physletb.2011.12.050
Our approach consists in finding an object which, contracted with chords $q_\mu$ and $q_\nu$, results in $(q_\mu \cdot q_\nu)$. This yields a representation of the metric tensor requested in (1.5), from which the coefficients $C_5^{\mu \nu}$ can be obtained. For an $N$-point function with $N$ external momenta one has to do with $(N-1)$ vectors, $q_N=0$. For $N=5$ one has in general 4 independent vectors from which one can construct uniquely the metric tensor in 4 dimensions. For $N>5$ one has $(N-5)$ “scratched” vectors which are not supposed to enter the construction of the metric tensor. Nevertheless the contraction of the metric tensor with any pair of the available $(N-1)$ vectors must result in their scalar product. Given the above mentioned sums this is directly seen.

It might be interesting here to recall the corresponding relation for 5-point functions [5], which can be considered as an inhomogeneous analogue to (1.3):

$$I_5^{\mu_1 \cdots \mu_r} = I_5^{\mu_1 \cdots \mu_{r-1} \epsilon} - \sum_{s=1}^{5} I_4^{\mu_1 \cdots \mu_{r-1} \epsilon} \cdot Q_0^{\mu_s}.$$  

(1.8)

reducing 5-point functions to 4-point functions of lower rank. Here the vectors

$$Q_\sigma = \sum_{i=1}^{5} q_i^{\sigma \mu} (\epsilon)_{\mu 5}, \quad \sigma = 0, \ldots, 5,$$

(1.9)

$$Q_s^{\tau \mu} = \sum_{i=1}^{5} q_i^{\tau \mu} (\epsilon)_{\mu 5}, \quad s, t = 1, \ldots, 5$$

(1.10)

have been introduced and $(\cdots)_{5}$ are Cayley determinants with elements

$$Y_{ij} = -(q_i \cdot q_j)^2 + m_i^2 + m_j^2.$$  

(1.11)

In the following, setting up sums over scalar products multiplied with Cayley determinants, we will implicitly use the relation

$$(q_i \cdot q_j) = \frac{1}{2} [Y_{ij} - Y_{in} - Y_{nj} + Y_{mn}].$$  

(1.12)

which is valid if $q_N=0$. Therefore we have to assume this from the very beginning. Nevertheless, if $q_N \neq 0$ one has to consider the same sum for $N \rightarrow N+1$; putting $q_{N+1}=0$ and scratching line and column $N+1$ in the Cayley’s one still has (1.12) for $i, j \leq N$.

More on recursion relations among tensor integrals can be found in [6–8].

2. 6-point functions

6-point functions of arbitrary tensor rank are well known, see e.g. [9,10]. Eq. (1.3) reads in this case

$$I_6^{\mu_1 \mu_2 \cdots \mu_r} = -\sum_{r=1}^{6} C_r^{\mu_1 \mu_2 \cdots \mu_r \mu_r}.$$  

(2.1)

In (4.6) of [10], the $C_r^{\mu \nu}$ is given:

$$C_r^{\sigma \mu} = \sum_{i=1}^{5} q_i^{\sigma \mu} (\epsilon)_{\mu 6}, \quad \sigma = 0, \ldots, 6.$$  

(2.2)

Here the index $\sigma$ indicates a certain redundancy, i.e. the vector $C_r$ is not unique. This reflects the property of Eq. (58) of [4] to have only a “pseudo-inverse”. We will now verify this equation in order to demonstrate our approach. Due to (1.3), we have to find a solution of (1.4). To be systematic, we first collect the following set of sums, with $s = 1, \ldots, n$:

$$\sum_{i,j=1}^{n-1} (q_i \cdot q_j) (q_b \cdot q_j) (Q_i)_{n}^{0j} = \frac{1}{2} (q_b \cdot q_{0}) (0_{n}) + \frac{1}{4} (Y_{an} \cdot Y_{mn}) (Y_{bn} \cdot Y_{mn}),$$

(2.3)

$$\sum_{i,j=1}^{n-1} (q_i \cdot q_b) (q_b \cdot q_j) (Q_i)_{n}^{sj} = \frac{1}{2} (q_b \cdot q_{0}) (\delta_{s 1} + \delta_{s 2} + \delta_{s 3} + \delta_{s 4}) (Y_{bn} \cdot Y_{mn}),$$

(2.4)

$$\sum_{i,j=1}^{n-1} (q_i \cdot q_b) (q_b \cdot q_j) (Q_i)_{n}^{sj} = \frac{1}{2} (q_b \cdot q_{0}) (\delta_{s 1} + \delta_{s 2} + \delta_{s 3} + \delta_{s 4}) Y_{bn} + Y_{mn},$$

(2.5)

In fact (2.4) and (2.5) have been written in [2] for $n=5$, but as mentioned above it turns out that all the sums written in [2] are as well valid for any $n$. For $n = 6$ it is $(\epsilon)_6 = 0$. Indeed equations (2.3)–(2.5) have to be considered as identities in terms of the $Y_{ij}$, and the property $(\epsilon)_6 = 0$ has to be taken into account as special property for the $Y_{ij}$ in (1.11). Thus we can finally write the metric tensor as

$$\frac{1}{2} g^{\mu \nu} = \sum_{i,j=1}^{5} \frac{1}{6} (0_{i 6}) (0_{j 6}) q_i^{\mu \nu} q_j^{\mu \nu},$$

(2.6)

$$\frac{1}{2} g^{\mu \nu} = \sum_{i,j=1}^{5} (0_{i 6}) (s_{j 6}) q_i^{\mu \nu} q_j^{\mu \nu},$$

(2.7)

$$\frac{1}{2} g^{\mu \nu} = \sum_{i,j=1}^{5} (s_{i 6}) (s_{j 6}) q_i^{\mu \nu} q_j^{\mu \nu},$$

(2.8)

and due to (1.6). Eqs. (2.6) and (2.7) yield the solution (2.2) for $\sigma = 0$ and $\sigma = s \neq 0$, respectively, while (2.8) yields another option; see also (75) in [11]. In this case we have

$$C_r^{s,\mu} = \sum_{i=1}^{5} q_i^{s \mu} (\delta)_{6}, \quad s = 1, \ldots, 6,$$

(2.9)

in the notation of (1.10). We remark that $(\epsilon)_6$ in (2.9) is a Gram determinant of a 5-point function, which may become small in certain domains of phase space. Here, we have a certain choice, $s = 1, \ldots, 6$, so that one presumably will find an s for which $(\epsilon)_6$ is not small.

It is interesting to note that the $C_r$ in (2.2) for $\sigma = 0$ and $\sigma > 0$ agree. One may see this, starting from the identity [see also (A.11) of [10] or (A.13) of [12]]

$$C_r^{s,\mu} = \sum_{i=1}^{5} q_i^{s \mu} (\delta)_{6}, \quad s = 1, \ldots, 6,$$

(2.9)

or

$$\sum_{i,j=1}^{5} (Q_i)_{6}^{0j} = \sum_{i,j=1}^{5} (Q_i)_{6}^{0j}$$

(2.10)
Multiplying with $q_i^\mu$ and summing over $i$ proves our statement since
\begin{equation}
\sum_{i=1}^{6} q_i^\mu {0 \choose i}_6 = 0.
\end{equation}
(2.12)
The last identity follows from the vanishing of all scalar products of (2.12) with any non-vanishing chord; see also (A.2) of [2]:
\begin{equation}
\sum_{i=1}^{5} (q_a \cdot q_i) {0 \choose i}_6 = -\frac{1}{2} (Y_{a6} - Y_{66}) \cdot {0 \choose 6} = 0,
\end{equation}
(2.13)
a = 1, \ldots , 5.

A similar calculation shows that (2.9) indeed differs from (2.2).
The reason why different representations are of interest is a possible optimization of the numerics: Some representations may have small determinants in the denominator, while others don’t.

It remains to verify (1.7). For (2.2) with $\sigma = 0$ we use
\begin{equation}
\sum_{r=1}^{6} {0 \choose r}_6 = - {0 \choose 1}_6
\end{equation}
(2.14)
such that
\begin{equation}
\sum_{r=1}^{6} c_r^{0,\mu} = \sum_{r=1}^{6} \sum_{i=1}^{5} q_i^\mu {0 \choose i}_6 = - \frac{1}{2} \sum_{r=1}^{5} q_i^\mu {0 \choose i}_6 = 0,
\end{equation}
(2.15)
due to (2.12). For (2.9) the proof is simpler since all we need is
\begin{equation}
\sum_{r=1}^{6} {s \choose r}_6 = 0, \quad s = 1, \ldots , 6.
\end{equation}
(2.16)

3. 7-point functions

For the 7-point functions we first investigate the corresponding representation of (2.2) for $\sigma = 0$. Eq. (A.9) of [2] can be written for arbitrary $n$ and $s = 1, \ldots , n$ as an identity in terms of the $Y_{ij}$:
\begin{equation}
\sum_{i,j=1}^{n-1} (q_a \cdot q_i)(q_b \cdot q_j) {0si \choose 0sj}_n = \frac{1}{2} (q_a \cdot q_b) {0s \choose 0}_n
- \frac{1}{4} \left\{ \left( {s \choose s}_n (Y_{an} - Y_{mn}) + {s \choose 0}_n (\delta_{as} - \delta_{sn}) \right) (Y_{bn} - Y_{mn})
+ \left( {s \choose 0}_n (Y_{an} - Y_{mn}) + {0 \choose 0}_n (\delta_{as} - \delta_{ns}) \right) (\delta_{bs} - \delta_{ns}) \right\}.
\end{equation}
(3.1)
As was observed for the 6-point function in the discussion of (2.3)–(2.5), the vanishing of certain determinants simplifies the result considerably. Quite generally with dimension 4 of the chords, all $(s)_n, n \geq 7$, have rank 6, i.e. any (signed) minor of order 7 vanishes [12]. The $(s)_7$ is of order 8 and thus also the $(0)_7, (s)_7$, and $(s)_7$ vanish, and therefore the whole curly bracket in (3.1) vanishes with the result
\begin{equation}
\sum_{i,j=1}^{n-1} (q_a \cdot q_i)(q_b \cdot q_j) {0si \choose 0sj}_7 = \frac{1}{2} (q_a \cdot q_b) {0s \choose 0}_7.
\end{equation}
(3.2)
In general, it is $(0)_7 \neq 0$ and one can write
\begin{equation}
\frac{1}{2} b^\tau_{\nu} = \sum_{i,j=1}^{6} \frac{1}{6} {0si \choose 0sj}_7 q_i^\mu q_j^\nu,
\end{equation}
(3.3)
from which we read off
\begin{equation}
c_r^{0,\mu} = \sum_{i=1}^{6} {0si \choose 0}_7 q_i^\mu.
\end{equation}
(3.4)

This is exactly the result as in (2.2) ($\sigma = 0$), only that now a line and a column of the $(0)_7$ is scratched in addition – a very natural result. Even more, we also find the correspondence of (2.2) for $\sigma = s > 0$ in the form
\begin{equation}
c_r^{s,\mu} = \sum_{i=1}^{6} {0si \choose 0}_7 (qs)_7 q_i^\mu.
\end{equation}
(3.5)

In order to show that (3.4) and (3.5) are equal, we proceed as for the 6-point function, starting from an extensional\(^4\) of (2.10),
\begin{equation}
{0s \choose ts}_7 (0sj)_7 = {0s \choose 0}_7 (0sj)_7 + {0s \choose is}_7 (0sj)_7
\end{equation}
(3.6)
or
\begin{equation}
{0si \choose 0}_7 = {0si \choose 0}_7 + {0si \choose 0}_7.
\end{equation}
(3.7)
Multiplying with $q_i^\mu$ and summing over $i$ proves the statement since
\begin{equation}
\sum_{i=1}^{6} q_i^\mu {0s \choose is}_7 = 0.
\end{equation}
(3.8)
Eq. (3.8) follows again from the vanishing of all scalar products of (3.8) with any non-vanishing chords as given in (A.6) of [2]:
\begin{equation}
\Sigma_n^{2,s} = \sum_{i=1}^{n-1} (q_a \cdot q_i) {0s \choose is}_n
= \frac{1}{2} \left\{ {s \choose s}_n (Y_{an} - Y_{mn}) + {s \choose 0}_n (\delta_{as} - \delta_{ns}) \right\}.
\end{equation}
(3.9)
For $n = 7$ the order of the determinants on the right-hand side of (3.9) is 7, but their rank is 6 and thus they all vanish.

Similarly we proceed for the analogue of (2.9), which was proven for the 6-point function. We start from a relation like (3.1), which was not directly needed in [2], but occurred there in an intermediate step:
\begin{equation}
\sum_{i,j=1}^{n-1} (q_a \cdot q_i)(q_b \cdot q_j) {0si \choose 0sj}_n = \frac{1}{2} (q_a \cdot q_b) {0s \choose st}_n
- \frac{1}{4} \left\{ \left( {s \choose s}_n (\delta_{at} - \delta_{nt}) - {s \choose t}_n (\delta_{as} - \delta_{ns}) \right) (\delta_{bs} - \delta_{ns})
+ \left( {s \choose 0}_n (\delta_{at} - \delta_{nt}) - {s \choose t}_n (\delta_{as} - \delta_{ns}) \right) (\delta_{bs} - \delta_{ns}) \right\}.
\end{equation}
(3.10)
with $s, t = 1, \ldots , n$. According to the above discussion, it is $(s)_7 = (t)_7 = (s)_7 = (t)_7 = 0$, such that again the curly bracket in (3.10) vanishes. Since $(s)_7 \neq 0$ in general – it is a 5-point Gram determinant – we can write
\[\text{See also [11].}\]
\[\text{4 See [12] for extensionals.}\]
From which we read off
\[ C_{r}^{st,\mu} = \sum_{i=1}^{6} \frac{1}{\delta(\mu)_{n}} \left( \begin{array}{c} sti \\ stj \end{array} \right) \gamma_{\mu}^{ij} q_{\mu}^{ij}, \]
\[ \left( \begin{array}{c} st \end{array} \right) \left( \begin{array}{c} t \end{array} \right) \left( \begin{array}{c} u \end{array} \right) \gamma_{\mu}^{ij} q_{\mu}^{ij}, \]
\[ \frac{1}{2} g_{\mu\nu} = \sum_{i,j=1}^{6} \frac{1}{\delta(\mu)_{n}} \left( \begin{array}{c} sti \\ stj \end{array} \right) q_{\mu}^{ij}, \]
which from (2.9), only that another line and column are scratched in the notation of (1.9) and (1.10). Here again the redundancy in the determination of this vector. They can be given any values within \( s, t = 1, \ldots, 7 \). In fact this vector differs again from (3.4) and (3.5). Similarly, to the case of the 6-point function one proves (17) for \( C_{r}^{8,\mu} \).

### 4. 8-point functions

For the 8-point functions – and analogously for \( N > 8 \) – the calculation follows the same lines as for the 7-point function. Again, at first we investigate the representation corresponding to (2.2) for \( \sigma = 0 \). Eq. (A.13) of [2] can be written for \( s, t = 1, \ldots, 8 \) as
\[ \sum_{i,j=1}^{8} (qa \cdot qb)(q_{a} \cdot q_{b}) \left( \begin{array}{c} st \end{array} \right) \left( \begin{array}{c} tu \end{array} \right) \gamma_{\mu}^{ij} q_{\mu}^{ij}, \]
which from (4.1) vanishes. Again, the curly bracket in (4.4) vanishes. \( l_{f}^{\mu} \) is also the Gram determinant of a 5-point function, obtained from the 8-point function under consideration by scratching lines and columns \( s, t \) and \( u \). It does not vanish in general. Thus we obtain
\[ \sum_{i,j=1}^{8} \frac{1}{\delta(\mu)_{n}} \left( \begin{array}{c} st \end{array} \right) \left( \begin{array}{c} tu \end{array} \right) \gamma_{\mu}^{ij} q_{\mu}^{ij}, \]
from which we read off
\[ C_{r}^{sttu,\mu} = \sum_{i=1}^{7} \frac{1}{\delta(\mu)_{n}} \left( \begin{array}{c} sti \\ str \end{array} \right) q_{\mu}^{ij} q_{\mu}^{ij}, \]
in the notation of (1.9) and (1.10). Here again the upper indices \( s, t \) and \( u \) stand for the redundancy of the vector and can be freely chosen.

### 5. Contractions of tensor integrals with external momenta

In [2] and [13] we have advertised the contraction of tensor integrals with external momenta for the calculation of Feynman diagrams. This led us to a systematic study of sums over products of contracted chords and signed minors. These sums have found in the present work a generalization by exploiting the fact that they are valid not only for the specific value \( n = 5 \) as was assumed for the 5-point functions. Indeed, they are correct for any \( n \). In the following, we demonstrate that due to this property we cannot only derive as above specific representations of the metric tensor, but also the contraction with external momenta can be performed in the same way for \( N \)-point functions with \( N > 6 \), quite similarly as it was done for 5-point functions. Just for the purpose of demonstration we confine ourselves to tensors up to rank 3 of the 7-point function.

For the vector integral (1.3) yields
\[ l_{r}^{\mu} = \sum_{i=1}^{7} C_{r}^{sttu,\mu} l_{i}, \]
where \( C_{r}^{sttu,\mu} \) can be chosen, e.g., from (3.12), and the \( l_{i}^{\mu} \) is the scalar 6-point function obtained from the scalar 7-point function by scratching line \( r \). For the contraction of \( C_{r}^{sttu,\mu} \) with a chord \( q_{a}^{ij} \) we need the generalization of (A.15) of [2] for \( n = 7 \):
\[
(q_a \cdot C^s_t)^n = \sum_{i=1}^{n-1} \left( \frac{\sum_{j=1}^{n} (q_a \cdot q_i)}{\sum_{j=1}^{n} (q_s \cdot q_i)} \right) (st)_n \\
= \frac{1}{2} \sum_{i=1}^{n} \left( (st)_n (\delta_{ar} - \delta_{at}) - (st)_n (\delta_{ar} - \delta_{at}) \right) \\
- \left( \frac{st}{tr} \right)_n (\delta_{as} - \delta_{at}),
\]
(5.2)

with \( s, t, r = 1, \ldots, n \). Eq. (5.2) has a surprising consequence. According to the construction in [4], the original tensor remains unchanged no matter how the vector \( C^s_t \) is chosen, as long as conditions (1.5) and (1.7) are fulfilled. Thus, contracting \( C \) with some chord \( q_a \), one still may select the redundancy indices \( s, t \). The optimal choice is apparently \( s, t \neq a, N \). In this case only the first term in the square bracket of (5.2) remains and moreover \( (st)_n \) cancels, i.e. the redundancy disappears and formally we can write

\[
(q_a \cdot C^s_t) = \frac{1}{2} (\delta_{ar} - \delta_{at}).
\]
(5.3)

As a result, only 2 terms remain in the sum (1.3) after the contraction. Since \( C \) carries the first index \( \mu_1 \) of tensors of any rank this scalar product will occur in all applications.

For the tensor of rank 2, Eq. (1.3) yields

\[
I^u_7 = -\sum_{r=1}^{7} C^u_r I^v_6 = -\sum_{r=1}^{7} C^u_r \left[ -\sum_{i=1}^{6} q^i_r (v^u_{ij})_N \cdot i^v_{st} \right].
\]
(5.4)

where the square bracket is the 6-point vector according to (2.1). For \( C^u_r \) we have taken for convenience the form resulting from (2.9). Here again (5.2) with \( n = 7 \) can be used for the projection of the 6-point vector. The only freedom we have now, however, is the choice of \( s \). Contracting with another vector \( q_b \) the choice \( s \neq r, b, N \) is optimal with the result (\( N = 7 \))

\[
q_b \cdot \sum_{j=1}^{N-1} q^j_r (v^u_{ij})_N \cdot \left( \frac{1}{2} (\delta_{bt} - \delta_{bt}) - \frac{1}{2} (tr)_{st} (\delta_{bt} - \delta_{bt}) \right).
\]
(5.5)

Only the sums over the basic functions \( I^u_7 \) survive. The reason for the appearance of the second term in (5.5) is that for \( r = b \) the 6-point function as a scratched 7-point function is contracted with a vector, which is not among the vectors defining the 6-point function, and for \( r = N \) all 6-point vectors are non-vanishing. Thus there remains the redundancy index \( s \).

Similarly we proceed for the tensor of degree 3:

\[
I^u_7 = -\sum_{r=1}^{7} C^u_r I^v_6 = -\sum_{r=1}^{7} C^u_r \left[ -\sum_{i=1}^{6} q^i_r (v^u_{ij})_N \cdot i^v_{st} \right].
\]
(5.6)

The vector of the 5-point function can be written as \[1\]

\[
I^u_5 = -\sum_{i=1}^{6} q^i_r \sum_{u=1}^{7} \frac{q^u_r}{(q^u_r)} (0stu)_n.
\]
(5.7)

According to (5.7), we see that the only new sum needed for the projection is 5

\[5 \text{ See also (A.17) of [2].}\]
Acknowledgements

The authors are grateful to G. Heinrich and S. Dittmaier for communication. J.F. thanks DESY for kind hospitality. Work is supported in part by Sonderforschungsbereich/Transregio SFB/TRR 9 of DFG “Computergestützte Theoretische Teilchenphysik” and European Initial Training Network LHCPHENOnet PITN-GA-2010-264564.

References

[1] J. Fleischer, T. Riemann, Phys. Rev. D 83 (2011) 073004, doi:10.1103/PhysRevD.83.073004, arXiv:1009.4436.
[2] J. Fleischer, T. Riemann, Phys. Lett. B 701 (2011) 646, doi:10.1016/j.physletb.2011.06.033, arXiv:1104.4067.
[3] Z. Bern, L.J. Dixon, D.A. Kosower, Nucl. Phys. B 412 (1994) 751, doi:10.1016/0550-3213(94)90398-0, arXiv:hep-ph/9306240.
[4] T. Binoth, J. Guillet, G. Heinrich, E. Pilon, C. Schubert, JHEP 0510 (2005) 015, doi:10.1088/1126-6708/2005/10/015, arXiv:hep-ph/0504267.
[5] T. Diakonidis, J. Fleischer, T. Riemann, J.B. Tausk, Phys. Lett. B 683 (2010) 69, doi:10.1016/j.physletb.2009.11.049, arXiv:0907.2115.
[6] F. del Aguila, R. Pittau, JHEP 0407 (2004) 017, doi:10.1088/1126-6708/2004/07/017, arXiv:hep-ph/0404120. Erratum added online, Feb/4/2005.
[7] A. van Hameren, JHEP 0907 (2009) 088, doi:10.1088/1126-6708/2009/07/088, arXiv:0905.1005 [hep-ph].
[8] F. Cascioli, P. Maierhofer, S. Pozzorini, Scattering amplitudes with open loops, arXiv:1111.5206 [hep-ph].
[9] A. Denner, S. Dittmaier, Nucl. Phys. B 734 (2006) 62, doi:10.1016/j.nuclphysb.2005.11.007, arXiv:hep-ph/0509141.
[10] T. Diakonidis, J. Fleischer, J. Gluza, K. Kajda, T. Riemann, J. Tausk, Phys. Rev. D 80 (2009) 036003, doi:10.1103/PhysRevD.80.036003, arXiv:0812.2134.
[11] J. Fleischer, F. Jegerlehner, O. Tarasov, Nucl. Phys. B 566 (2000) 423, doi:10.1016/S0550-3213(99)00678-1, arXiv:hep-ph/9907327.
[12] D.B. Melrose, Nuovo Cim. 40 (1965) 181, doi:10.1007/BF028329.
[13] J. Fleischer, T. Riemann, Acta Phys. Polon. B 42 (2011) 2371, doi:10.5506/APhysPolB.42.2371, arXiv:1111.4153.