New representation results for planar graphs

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Abstract

A universal representation theorem is derived that shows any graph is the intersection graph of one chordal graph, a number of co-bipartite graphs, and one unit interval graph. Central to the the result is the notion of the clique cover width which is a generalization of the bandwidth parameter. Specifically, we show that any planar graph is the intersection graph of one chordal graph, four co-bipartite graphs, and one unit interval graph. Equivalently, any planar graph is the intersection graph of a chordal graph and a graph that has clique cover width of at most seven. We further describe the extensions of the results to graphs drawn on surfaces and graphs excluding a minor of crossing number of at most one.

1 Introduction and Summary

Graph theory, geometry, and topology stem from the same roots. Representing graphs as the intersection graphs of geometric or combinatorial objects is highly desired in certain branches of combinatorics, discrete and computational geometry, graph drawing and information visualization, and the design of geographic information systems (GIS). A suitable intersection model not only provides a better understanding of the underlying graph, but it can also lead to computational advances. A remarkable result in this area is Koebe’s (also Thurston’s) theorem, asserting that every planar graph is the touching graph of planar disks. A similar result is due to Thomassen [12] who showed that every planar graph is the intersection graph of axis parallel boxes in $R^3$. Another noteworthy result is due to Gavril [7] who proved that every chordal graph (a graph with no chordless cycles) is the intersection graph of a collection of subtrees of a tree.

Any (strict) partially ordered set [14] $(S, <)$ has a directed acyclic graph $G$ associated with it in a natural way: $V(G) = S$, and $ab \in E(G)$ if and only if $a < b$. The comparability graph associated with $(S, <)$ is the undirected graph which is obtained by dropping the orientation on edges of $G$. The complement of a comparability graph is an incomparability graph. Incomparability graphs are well studied due to their rich structures and are known to be the intersection graph of planar curves [8]. A interesting result in this area is due to Pach and Töröcsik [9] who showed, given a set of straight line segments in the plane, there are four incomparability graphs whose edge intersections give rise to the intersections of the segments. Moreover, recent work in combinatorial geometry has shown the connections between the intersection patterns of arbitrary planar curves and properties of incomparability graphs [2], [5].

An an interval graph is the intersection graph of a set of intervals on the real line [13]. It is easily seen that an interval graph is an incomparability graph. A unit interval graph is the intersection graph of a set of unit intervals.

Throughout this paper, $G = (V(G), E(G))$ denotes a connected undirected graph. Let $d \geq 1$, be an integer, and for $i = 1, 2, ..., d$ let $H_i$ be a graph with $V(H_i) = V$, and let $G$ be a graph with $V(G) = V$ and $E(G) = \cap_{i=1}^d E(G_i)$. Then we say $G$ is the intersection graph of $H_1, H_2, ..., H_d$, and write $G = \cap_{i=1}^d H_i$. A clique cover $C$ in $G$ is a partition of $V(G)$ into cliques. Throughout this paper, we will write $C = \{C_0, C_1, ..., C_t\}$ to indicate that $C$ is an ordered set of cliques. For a clique cover $C = \{C_0, C_1, ..., C_t\}$, in $G$, let the width of $C$, denoted by $W(C)$, denote $\max\{|j-i| : xy \in E(G), x \in C_j, y \in C_i, C_j \cap C_i \neq \emptyset\}$. The clique cover width of $G$ denoted by $CCW(G)$ is the smallest width all ordered clique covers in $G$. Note that $CCW(G) \leq BW(G)$, where $BW(G)$ denotes the bandwidth of $G$. A co-bipartite graph is the complement of a bipartite graph. Clearly, any co-bipartite graph is an incomparability graph.

1.1 Our Results

We recently proved the following result [11].

Theorem 1 Let $C$ be a clique cover in $G$ with $0 < W(C) \leq w, w \geq 1$. Then, there are $[\log(w)] + 1$ co-bipartite graphs $H_i, i = 1, 2, ..., [\log(w)] + 1$, and a unit interval graph $H_{[\log(w)]} + 2$, so that $G = \cap_{i=1}^{[\log(w)]} H_i$.

The main result in this paper is Theorem 5 which asserts any planar graph is the intersection graph of a chordal graph and a graph whose clique cover width is bounded by seven. The application of Theorem 5 then, gives another version of the result as stated

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the abstract. Theorem 5 is obtained using the Universal Representation Theorem, or Theorem 2 which is interesting on its own, and asserts that any graph is intersection graph of a chordal graph and a graph whose clique cover width is bounded. Nonetheless, the upper bound on the clique cover width of the second graph depends on the properties of the tree decompositions of the original graph. Theorem 5 is further extended to graphs drawn on surfaces, and graphs excluding a minor with the crossing number of at most one.

2 Main Results

Definition 1 A tree decomposition \( T \) of a graph \( G \) is a pair \( (X, T) \) where \( T \) is a tree, and \( X = \{ X_i \mid i \in V(T) \} \) is a family of subsets of \( V(G) \), each called a bag, so that the following hold:
- \( X \cup_{i \in V(T)} X_i = V(G) \)
- for any \( uv \in E(G) \), there is an \( i \in V(T) \) so that \( v \in X_i \) and \( u \in X_i \).
- for any \( i,j,k \in V(T) \), if \( j \) is on the path from \( i \) to \( k \) in \( T \), then \( X_i \cap X_k \subseteq X_j \).

Theorem 2 (Universal Representation Theorem) Let \( G \) be a graph and let \( L = \{ L_1, L_2, ..., L_k \} \) be a partition of vertices, so that for any \( xy \in E(G) \), either \( x, y \in L_i \) where \( 1 \leq i \leq k \), or, \( x \in L_i, y \in L_{i+1} \), where \( 1 \leq i \leq k-1 \). Let \( (X, T) \) be a tree decomposition of \( G \). Let \( t^* = \max_{i=1,2,...,k} \{|L_i \cap X_i| \mid j \in V(T)\} \). (Thus, \( t^* \) is the largest number of vertices in any element of \( L \) that appears in any bag of \( T \)). Then, there is a graph \( G_1 \) with \( \text{CCW}(G_1) \leq 2t^* - 1 \) and a chordal graph \( G_2 \) so that \( G = G_1 \cap G_2 \).

Proof. For any \( v \in V(G) \), let \( X_v \) be the set of bags in \( X \) that contains vertex \( v \), and let \( T_v \) be the subtree of \( T \) on the vertex set \( X_v \). Let \( G_2 \) be the intersection graph of these subtrees. Thus, for any \( v, w \in V(G) \), \( vw \in E(G_2) \), if \( X_v \cap X_w \neq \emptyset \). It is well known that \( G_2 \) is chordal. See work of Gavril [7]. Now let \( \omega \) be the largest clique in \( G_2 \) among all cliques whose vertices are entirely in \( L_i \), for some \( i = 1, 2, ..., k \). It follows from established properties on the tree decomposition that all vertices in \( \omega \) should appear in one bag \( B \) in \( X \). Consequently, \( |\omega| \leq |B \cap L_i| \leq t^* \). Next observe that for \( i = 1, 2, ..., k \), \( G_2[L_i] \) is chordal and hence perfect, and thus there must be at most \( t^* \) disjoint independent sets \( L_i^j, j = 1, 2, ..., t^* \) whose union is \( L_i \). Now construct \( G_1, V(G_1) = V(G) \), as follows: \( E(G_1) = E(G) \cup E' \), where \( E' \) is obtained by placing an edge between any vertex pair in each independent set \( L_i^j \) for \( i = 1, 2, ..., k, j = 1, 2, ..., t^* \). Clearly, \( G = G_1 \cap G_2 \). In addition, for \( i = 1, 2, ..., k \), \( G_1[L_i] \) is covered with at most \( t^* \) disjoint cliques, hence any ordering of these cliques will give rise to a clique cover \( C \) of \( G_1 \) with \( W(C) \leq 2t^* - 1 \), since any edge \( e \in E(G_1) \) either has both ends in one previously prescribed clique in \( G_1[L_i] \), or must have end points in two consecutive elements in \( L \).

The following definitions are from [2].

Definition 2 A maximal spanning forest of \( G \) is a spanning forest \( T \) that contains a spanning tree from each component of \( G \). Thus, when \( G \) is connected, any spanning tree of \( G \) is also a maximal spanning forest. Let \( T \) be a maximal spanning tree of \( G \), and let \( ab \in E(G) - E(T) \); The detour of \( ab \) in \( T \) is the unique \( ab \)-path in \( T \). Let \( e \in E(T) \). The edge remember number of \( e \), denoted by \( er(e, T, G) \), is the number of edges in \( E(G) - E(T) \) whose detour contains \( e \); Equivalently, \( er(e, T, G) \) is the number of fundamental cycles in \( G \) relative to \( T \), that contain \( e \). Similarly, for \( v \in V(G) \), the vertex remember number \( v \) denoted by \( vr(v, T, G) \), is the number of edges in \( E(G) - E(T) \) whose detour, or the fundamental cycle associated with it, contains \( v \). To remedy technical issues, for any \( e \in E(G) - E(T) \), we define \( er(e, T, G) = 0 \). The edge remember number and vertex remember number of \( G \) in \( T \), denoted by \( er(G, T) \) and \( vr(G, T) \), are the largest remember numbers overall edges in \( E(T) \) and vertices in \( V(T) \), respectively.

Definition 3 Let \( T \) be a maximal spanning tree of \( G \), and let \( \hat{T} \) be a forest that is obtained by inserting vertices of degree two to the edges of \( T \). Thus, \( \hat{T} = (V(T) \cup E(T), E(\hat{T})) \). Now, for any \( v \in V(T) \) place \( v \) in \( X_v \), and for any \( e = ab \in E(T) \) place \( a \) and \( b \) in \( X_v \). Next, for any \( e = ab \in E(G) - E(T) \), take one of \( a \) or \( b \), say \( a \), and place it in \( X_a \), for any \( v \) which is on the unique \( ab \)-detour in \( T \); Similarly, place \( a \) in \( X_a \) for any edge \( e \) which is on the unique \( ab \)-detour in \( T \). Finally, define \( X = \{ X_i \mid i \in V(T) \cup E(T) \} \).

Bodlaender [2] showed the following.

Theorem 3 Let \( T \) be a maximal spanning tree of \( G \), and let \( \hat{T} \) and \( \hat{X} \) be as defined above. Then, \( (\hat{T}, \hat{X}) \) is a tree decomposition of \( G \) whose width is at most \( \max\{vr(G, T), er(G, T) + 1\} \).

In light of the above result, we will refer to \( (\hat{T}, \hat{X}) \) (in definition 3) as a tree decomposition of \( G \) relative to \( T \). Note that the construction in definition 3 would allow the same vertex to appear in \( X_v \) or \( X_a \) more than once, where each appearance is associated with an end point of an edge \( e \in E(G) - E(T) \), representing a distinct fundamental cycle containing \( v \), or, \( e \). With that in mind, we have \( |X_v| = vr(v, T, G) + 1 \) and \( |X_a| = er(e, T, G) + 2 \). However, when viewing \( |X_v| \) and \( |X_a| \) as sets, the duplicate members would be removed, thereby, \( = \) would become \( \leq \).

The following Lemma is extended from [2]. The notations and claims are slightly perturbed to exhibit
additional properties of the construction of Bodlaender, that we will use later.

Lemma 4 Let $G$ be a plane graph, let $O$ be the set of all vertices in the outer boundary of $G$, let $H, V(H) = V(G)$ be a graph obtained by removing all edges in the outer boundary of $G$. Let $T'$ be a maximal spanning forest of $H$ and let $(X', T')$ be a tree decomposition of $H$ relative to $T'$.

(i) $T'$ can be extended to a maximum spanning forest $T$ of $G$ so that $vr(v, T, G) = vr(v, T', H) + \Delta(G)$ and $\epsilon\epsilon(e, T, G) \leq \epsilon\epsilon(e, T', H) + 2$, for all $v \in V(G)$ and $e \in E(T)$.

(ii) $(X', T')$ can be extended to a tree decomposition $(X, T)$ of $G$ relative to $T$ so that $|X_v \cap O| \leq |X_v \cap O'| + \Delta(G)$ and $|X_v \cup O| \leq |X_v \cup O'| + 2$ for all $v \in V(G)$ and $e \in E(T)$.

Proof. For (i), let $K$ be graph with $V(K) = V(G)$ and $E(K) = E(T') \cup (E(G) - E(H))$, and note that the external face of $K$ is the same as external face of $G$. Extend $T'$ to a maximal spanning tree $T$ of $K$ by adding edges from $E(G) - E(H)$. Note that for any $e = xy \in E(K) - E(T)$, then $x$ and $y$ must be on the boundary of $K$. Thus, the associated $x$-y detour $p$ in $T$ plus $e$ must form the boundary of a non-external face in $K$. Since any edge in $T$ is common to at most 2 non-external faces, and each vertex in $T$ is common to at most $\Delta(G)$ many non-external faces, in $K$, it follows that for any $e \in E(T)$ and any $v \in V(G)$, $\epsilon\epsilon(e, T, K) \leq 2$ and $vr(v, K, T) \leq \Delta(G)$. As $T$ is also a maximal spanning tree of $G$ and each fundamental cycle in $G$ is either a fundamental cycle of $K$ relative to $T$, or a fundamental cycle of $H$ relative to $T'$, we must have $\epsilon\epsilon(e, T, G) \leq \epsilon\epsilon(e, T', H) + \epsilon\epsilon(e, T, K) \leq \epsilon\epsilon(e, T', H) + 2$, and $vr(v, G, T) \leq vr(v, T', H) + vr(v, T, K) \leq vr(v, T, H) + \Delta(G)$.

(ii) follows from (i). In particular, note that additional 2 or $\Delta(G)$ fundamental edges that contribute to $vr(v, G, T)$ and $\epsilon\epsilon(e, G, T)$, respectively, are those edges in $E(G) - E(T)$ that have both end points in $O$. Now obtain a tree decomposition of $G$ relative to $T$, by extending each bag of $T'$, to a bag of $T$ by the possible addition of one end point of such a fundamental edge, as described in definition.

By a plane graph we mean an embedding of a planar graph in the plane. A plane graph is 1-outer planar, if it is outer planar. For $k \geq 2$, a plane graph $G$ is $k$-outer planar, if after removal of all vertices (and edges incident to these vertices) in the external face of $G$, a $k - 1$ outer planar graph is obtained.

Theorem 5 Let $G$ be a planar graph, then, there is a graph $G_1$ with $CCW(G_1) \leq 7$ and a chordal graph $G_2$ so that $G = G_1 \cup G_2$.

1 In (i) and (ii) we follow the assumption that $\epsilon\epsilon(e, T', G) = 0$ and $X_v' = \emptyset$, for $e \in E(T') - E(T)$.

Proof. Assume $G$ is $k$-outer planar. Thus, there are graphs $G = G_1, G_2, ..., G_k$ so that for $i = 1, 2, ..., k$, $G_i$ is $(k - i + 1)$-outer planar, and $G_{i+1}$ is obtained by removing the vertices in the outer face of $G_i$. For $i = 1, 2, ..., k$, let $O_i$ denote the set of vertices on the outer face of $G_i$. Note that for $i = 1, 2, ..., k$, one can replace any vertex $v$ of degree $d \geq 4$ in the outer face of $O_i$ by a path $p_v$ of $d - 2$ vertices of degree 3, so that $G$ is transformed to another $k$-outer planar graph $G'$. Specifically, for $i = 1, 2, ..., k$, let $O'_i$ denote the set of vertices corresponding to $O_i$, after this transformation. Note that $G'$ is $k$-outer planar and has maximum degree 3, let $G'_1 = G'$, and for $i = 2, ..., k + 1$, let $G_i'$ denote the graph that is obtained after removing all edges in the outer face of $G'_i - 1$, and note that $G'_i$ is $(k - i + 1)$-outer planar and of maximum degree 3. Note that $G_{k+1}$ is acyclic and let $T_{k+1} = T_{k+1}$. Clearly, $vr(v, T_{k+1}, T_{k+1}) = 0, \epsilon\epsilon(e, T_{k+1}, T_{k+1}) = 0$, for any $v \in V(G)$, and any $e \in E(T_{k+1})$. Thus, for the tree decomposition $(X_{k+1}, T_{k+1})$ of $G_{k+1}$ relative to $T_{k+1}$, and bags $X_v, X_e, v \in V(G), e = ab \in E(T_{k+1})$, we have $|X_v| = 1$ (since $X_v = \{v\}$), and $|X_e| = 2$ (since $X_e = \{a, b\}$), respectively. Next, for $j = k, k - 1, 1$, let $T_j$ and $(X_j, T_j)$ be a maximal spanning forest and a tree decomposition $G_j'$ relative to $T_j$, that are obtained by the application of Part (i) and Part (ii) of Lemma 4 to $T_{j+1}$ and $(X_{j+1}, T_{j+1})$, respectively. Thus, $(X_1, T_1)$ is a tree decomposition of $G'$. Then, one can show (by induction) that for any $j, i = k, k - 1, 1$, and any $X'_j, X'_i \in X_j$ with $v \in V(G), e \in E(T_j)$

$|X'_j \cap O'_i| = |X'_j \cap O'_i| - 1 + \Delta(G') \leq 1 + 4 = 5$ if $i \neq j$, whereas, $|X'_j \cap O'_i| \leq 1 + \Delta(G'_i) \leq 1 + 4 = 5$ if $i = j$.

$|X'_j \cap O'_i| \leq 2 + 4 = 6$ if $i = j$.

Hence, for $i = 1, 2, ..., k$, and $X'_1, X'_i \in X_1$, with $v \in V(G)$ and $e \in E(T_1)$, we have, $|X'_i \cap O'_i| \leq 4$ and $|X'_i \cap O'_i| \leq 4$. Next, for any $v \in V(G)$, contract all the vertices in $p_v$ to $v$, thereby, for $i = 1, 2, ..., k$ contracting $O'_i$ to $O_i$. For any bag $X'_1 \in X_1$ with $t \in V(G) \cup E(T_1)$, let $Y_t = (X_1 - p_v) \cup \{v\}$. Now let $Y = \{Y_t | t \in V(G) \cup E(T_1)\}$. Since $G$ is a minor of $G'$, it follows that $(Y, T')$ is a tree decomposition of $G$ with the property that for any $Y_i \in Y$ with $t \in V(T_1) \cup E(T_1)$, and any $i = 1, 2, ..., k$, we have $|Y_i \cap O_i| \leq 4$. Now the result follows from Theorem 2 by taking $L = \{O_1, O_2, ..., O_k\}$.

Combining Theorems 5 and 6 we obtain the following.

Theorem 6 Let $G$ be a planar graph, then, there are co-bipartite graphs $G_1, G_2, G_3, G_4$, an unit interval graph $G_5$, and a chordal graph $G_6$ so that $G = \bigcup_{i=1}^{6} G_i$.
2.1 Extensions

The result for planar graphs give rise to the following.

Theorem 7 Let $G$ be a graph of genus $g$. Then, there is an integer $c = O(\log(g))$, co-bipartite graphs $G_i, i = 1, 2, ..., c$, a unit interval graph $G_{c+1}$, and a chordal graph $G_{c+2}$ so that $G = \bigcap_{i=1}^{c+2} G_i$.

Proof Sketch. One can show the claim by induction on $g$, where Theorems 5 and 6 establish the base of the induction. □

Theorem 8 Let $G$ be a graph that does not have a minor, a graph $H$ whose crossing number is at most one. Then there is an integer $c = O(\log(g))$, co-bipartite graphs $G_i, i = 1, 2, ..., c$, a unit interval graph $G_{c+1}$, and a chordal graph $G_{c+2}$ so that $G = \bigcap_{i=1}^{c+2} G_i$. Here, $C_H = 20^2(2|V(H)|+4|E(H)|)^5$.

Proof Sketch. It is known that any graph that does not have a minor $H$ of crossing number of at most one, can be obtained by taking the clique sum of a finite set of graphs, where each graph is either planar, or has a tree width of at most $C_H$. So $G = H_1 \bigoplus H_2 \bigoplus \bigoplus H_k$, where $\bigoplus$ stands for the clique sum operation, and for $i = 1, 2, ..., k$, each $H_i$ is either planar, or has a tree width of at most $C_H$. We prove the claim by induction on $k$. When $k = 1$ the result follows from Theorems 5 and 6 and the definition of $C_H$. Now assume that the claim is true for $k = t - 1$, let $k = t \geq 2$, and set $F = H_1 \bigoplus H_2 \bigoplus \bigoplus H_{t-1}$. Then, $G = F \bigoplus H_k$. By induction, $F = F_1 \bigcap F_2$, where $F_2$ is chordal and $CCW(F_1) \leq C_H$. Moreover, since $H_{t-1}$ is either planar, or has a tree width of at most $C_H$, by Theorem 5 we have $G_{t-1} = F_3 \bigcap F_4$, where $CCW(F_3) \leq 2C_H$ and $F_4$ is chordal. Now let $G_1 = F_1 \bigoplus F_3$, and $G_2 = F_2 \bigoplus F_4$, then, it is easy to verify that $G_2$ is chordal. To finish the proof, one can verify using properties of the clique cover width that, $CCW(G_2) \leq 2C_H$. Now the claim follows from Theorem 1. □

3 Computational Aspects

All constructions provided here can be done in polynomial time, with the exception of Theorem 6.

In [11] we have shown that if $G$ is the intersection graph of a chordal graph and a graph whose clique cover width is bounded by a constant, then $G$ can be separated with a splitting ratio of $1/3 - 2/3$, for a variety of measures, where the measure associated with the separator is “small”. Consequently, the planar separator theorem 8 and its extensions follow from the representation results in this paper.

We highly suspect that the computation of the clique cover width is an NP-hard problem, due to its connection with the bandwidth problem.

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