QUASI-BIALGEBRAS AND DYNAMICAL $r$-MATRICES

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Abstract. We study the relationship between general dynamical Poisson groupoids and Lie quasi-bialgebras. For a class of Lie quasi-bialgebras $G$ naturally compatible with a reductive decomposition, we extend the description of the moduli space of classical dynamical $r$-matrices of Etingof and Schiffmann. We construct, in each gauge orbit, an explicit analytic representative $r_{\text{can}}$. We translate the notion of duality for dynamical Poisson groupoids into a duality for Lie quasi-bialgebras. It is shown that duality maps the dynamical Poisson groupoid for $l^{\text{can}}$ and $G$ to the dynamical Poisson groupoid for $l^{\text{can}}$ and the dual quasi-bialgebra $G^*$. 

1. Introduction

The classical dynamical Yang–Baxter equation (CDYBE) for a pair $(\mathfrak{g}, l \subset \mathfrak{g})$ of Lie algebras first appeared in [1, 11]. In [10], extending Drinfel’d’s classical work [6], this equation, supplemented by a condition of $l$-equivariance, was shown to coincide with the Jacobi identity for a natural Poisson bracket on the trivial groupoid $U \times G \times U$. Here, $G$ is a Lie group with $\text{Lie}(G) = \mathfrak{g}$ and $U \subset \mathfrak{g}^*$ is an $l$-invariant open set.

In [9], working on the formal disk $\mathbb{D}$, Etingof and Schiffmann have shown that the moduli space $\mathcal{M}(\mathbb{D}; l^*, \Omega)$ of formal solutions of (CDYBE) for reductive triples $\mathfrak{g} = l \oplus \mathfrak{m}$, $[l, \mathfrak{m}] \subset \mathfrak{m}$, where $\Omega \in \mathfrak{h} \otimes \mathfrak{h} \oplus \mathfrak{m} \otimes \mathfrak{m}$ is a symmetric $\mathfrak{g}$-invariant, is isomorphic to the algebraic variety $M_{\Omega} = \{t \in (\wedge^3 \mathfrak{m})^l \mid \langle t, t \rangle + \langle \Omega, \Omega \rangle \equiv 0 \mod l\}$. Here, the moduli space (first considered for $\Omega = 0$ by Xu in [23]) is the orbit space of solutions of (CDYBE) for the action of the group of equivariant maps $\text{Map}_0(\mathbb{D}, G)^l$ induced by formal base preserving, $l$-equivariant groupoid automorphisms of $\mathbb{D} \times G \times \mathbb{D}$. Their proof relies on formal induction arguments and the equivariant Poincaré lemma together with the existence of a canonical solution $r^{\text{AM}}$ of (CDYBE) on $l$, discovered by Alekseev and Meinrenken in [2].

In [16], Poisson groupoid structures on $U \times G \times U$ compatible with the natural inclusion of the Hamiltonian unit $L \times U$ (the so-called dynamical Poisson groupoids of [11]) were described. These brackets are given by pairs $(l, \varpi)$ where $l: U \to \mathcal{L}(\mathfrak{g}^*, \mathfrak{g})$ is a skew symmetric smooth map and $\varpi$ is a $\mathfrak{g}$-1-cocycle, and their Jacobi identity turns out to be equivalent to

- There exists a $\varphi \in (\wedge^3 \mathfrak{g})$ such that for all $\xi, \eta, \zeta \in \mathfrak{g}^*$,
  \[ \langle \xi \otimes \eta \otimes \zeta, \text{ad}^{(3)}_x \varphi \rangle = \bigotimes_{(\xi, \eta, \zeta)} \langle \xi, \varpi_{\mathfrak{x}, \eta, \zeta} \rangle \]
  \[ \text{(A)} \]
  and
  \[ \bigotimes_{(\xi, \eta, \zeta)} \left( \langle \zeta, [l_p, \xi], l_p \eta \rangle - \langle \xi, \mathfrak{h}, \eta \rangle \right) = \langle \xi \otimes \eta \otimes \zeta, \varphi \rangle \]
  \[ \text{(B)} \]
  for all $\xi, \eta, \zeta \in \mathfrak{g}^*$, $x \in \mathfrak{g}$, $p \in U$.

- together with the $l$-equivariance
  \[ d_p l(\text{ad}^*_p p) + \varpi_{\mathfrak{ix}} + \text{ad}_{\mathfrak{ix}} l_p + l_p \text{ad}^*_{\mathfrak{ix}} = 0, \quad \forall x \in \mathfrak{l}, \ p \in \mathbb{U} \]
  \[ \text{(C)} \]
which, naturally, reduce to (CDYBE) for vanishing cocycle \( \varpi \) and \( \mathfrak{g} \)-invariant \( \varphi = \langle \Omega, \Omega \rangle \).

The main purpose of the present work is to relate the above structures to Lie quasi-bialgebras. Note that, for the case \( I = \mathfrak{g}, \varpi = 0 \), and \( \varphi \in (\wedge^3 \mathfrak{g})^0 \), such a link was also observed in [8]. First of all, we extend, with natural assumptions, the Etingof-Schiffmann description of the moduli space in [9] to the equations \( (A), (B) \) and \( (C) \) with, as principal new result, the construction, in terms of the associated Lie quasi-bialgebra data, of a canonical analytic solution \( l^{can} \) providing an explicit representative in each formal gauge class. Secondly, we translate the notion of duality for Poisson groupoids \([22],[20]\) in terms of a more algebraic duality for Lie quasi-bialgebras, which up to (as yet formal) Poisson groupoid automorphisms provides an explicit description of the Poisson groupoid dual of the dynamical groupoid \( U \times G \times U \).

Our analysis begins with the observation (see Proposition \( \text{[8.3]} \)) that conditions \( (A), (B) \) and \( (C) \) above imply that the quadruple \( \mathcal{G} = (\mathfrak{g}, [\cdot, \cdot], \varpi, \varphi) \) is a Lie quasi-bialgebra with \( \varpi \) exact. We then proceed to describe the moduli space \( \mathcal{M} \) of solutions of \( (B) \) and \( (C) \) for fixed \( \varpi, \varphi \) satisfying \( (A) \).

To begin with, in close analogy with [9], we obtain (Corollary \( \text{[4.8]} \)) an embedding of \( \mathcal{M} \) into the algebraic variety

\[
\mathcal{M}_{\mathcal{G}, I, m} = \left\{ t \in (\wedge^2 m)^I \mid \varphi^t \equiv 0 \mod I \right\},
\]

where \( \varphi^t \) is the Drinfel’d twist of the associator \( \varphi \in \wedge^3 \mathfrak{g} \). Secondly for a class of Lie quasi-bialgebras \( \mathcal{G} \) canonically compatible (see Definition \( \text{[4.1]} \)) with a reductive decomposition \( \mathfrak{g} = I \oplus m \), we construct the analytic solution \( l^{can} \) (see theorem \( \text{[7.5]} \)) in terms of a generalization of \( r^{AM} \) on \( I \) (also considered in [8]), and the adjoint action of the double \( \mathfrak{d} \) of the underlying Lie quasi-bialgebra. The use of \( l^{can} \) together with Drinfel’d twists, then shows (see Corollary \( \text{[6.7]} \)) that, for Lie quasi-bialgebras compatible with the reductive decomposition \( \mathfrak{g} = I \oplus m \) the embedding above is a bijection, providing, in particular, the explicit analytic representative in each formal gauge orbit.

Note that our compatibility hypothesis on Lie quasi-bialgebras includes the class considered in [9], for which the canonical solution \( l^{can} \) coincides, up to a twist, with the formal representative they constructed.

For a contractible base \( U \) (and, for large classes of examples, up to covering, for arbitrary \( U \) as well), it was shown in [14] that Poisson groupoid duality preserves the class of pairs \( (U \times G \times U, I) \) where \( I : L \times U \to U \times G \times U \) is a morphism of the Hamiltonian unit. However, the explicit expression of the Poisson bracket of the dual pair \( (U \times G' \times U, I') \) relies on the knowledge of a non-canonical isomorphism (a so-called trivialization) of the algebroid dual \( A(U \times G \times U)^* \simeq A(U \times G' \times U) \).

Our approach to duality begins with the construction of an explicit trivialization of the algebroid dual for the canonical solution \( l^{can} \) (see Propositions \( \text{[7.1],[7.2]} \) and Theorem \( \text{[7.3]} \)). It turns out that such an isomorphism may be expressed solely in terms of the Drinfel’d isomorphism relating the doubles of the twisted pairs of Lie quasi-bialgebras \( \mathcal{G} \) and \( \mathcal{G}^{l^{can}} \) together with the adjoint action of the double \( \mathfrak{d} \) of \( \mathcal{G} \).

Duality for Lie quasi-bialgebras is then defined (see Definition \( \text{[7.4]} \)) as follows: let \( \mathfrak{g} = I \oplus m \) be a reductive decomposition. If \( \mathcal{G} = (\mathfrak{g}, [\cdot, \cdot], \varpi, \varphi) \) is a Lie quasi-bialgebra such that \( \varpi_0 = 0 \) and \( \varphi \equiv 0 \mod I \), then the dual \( \mathcal{G}^* \) (up to relative signs) is the Lie quasi-bialgebra associated with the Manin quasi-triple \( (\mathfrak{d}, I \oplus I^\perp, m \oplus m^\perp) \).

Our main duality assertion (see Theorem \( \text{[7.5]} \)) then states that the dual Poisson groupoid of the dynamical Poisson groupoid associated with \( l^{can} \) for \( \mathcal{G} \) is (isomorphic to) the source-connected, simply-connected covering of the dynamical Poisson groupoid associated with \( l^{can} \) for \( \mathcal{G}^* \). Note that this is tantamount to saying that (up to covering) the dual of any dynamical Poisson groupoid is dynamical if and only if the vertex algebra \( \mathfrak{g} \) admits a reductive decomposition \( \mathfrak{g} = I \oplus m \). This however will be postponed to another publication \([21]\).

The paper is organized as follows. In section 2, we recall some basic facts about Lie quasi-bialgebras with some additional material needed for the rest of the paper. In section 3, we establish
the relationship between dynamical Poisson groupoids and Lie quasi-bialgebras. In section 3 we adapt the analysis of the moduli space in [9] to the study of solutions of (A), (B) and (C). The brief section 5 provides a formulation of the dynamical $r$-matrix $r^{AM}$ in terms of the Lie quasi-bialgebra $G = (g, [\cdot,\cdot], 0, \varphi)$. In section 6 we construct the analytic representative $l^{an}$ for Lie quasi-bialgebras canonically compatible with a reductive decomposition. The last section 7 is devoted to duality statements together with some examples and a brief discussion on the link with duality of symmetric spaces. We have collected some technical lemmas and proofs in the Appendices.

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2. Lie quasi-bialgebras

In this section, we recall some basic facts about Lie quasi-bialgebras (see [7], see also [1]).

2.1. Notations. When $E$ and $F$ are finite dimensional vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we use the following notations:
- $E^*$ for the dual of $E$, and $\langle , \rangle$ for the canonical pairing between $E$ and $E^*$,
- $\mathcal{L}(E,F)$ for the set of linear maps from $E$ to $F$,
- $f^* \in \mathcal{L}(F^*,E^*)$ for the adjoint of $f \in \mathcal{L}(E,F)$,
- $\mathcal{A}(E^*,E)$ for the set of skew-symmetric linear maps from $E^*$ to $E$.

Let $g$ be a Lie algebra and let $G$ be a Lie group with $\text{Lie}(G) = g$. In the sequel, Lie algebra and Lie group cocycles will always take value in $\mathcal{A}(g^*,g)$ equipped with the adjoint action. Thus, a linear map $\varpi: g \to \mathcal{A}(g^*,g)$ is a Lie algebra 1-cocycle if it satisfies the following identity:
\[
\varpi_{[x,g]} = \text{ad}_x \varpi_y + \varpi_y \text{ad}_x^* - \text{ad}_y \varpi_x - \varpi_x \text{ad}_y^* \quad (2.1)
\]
for all $x, y \in g$, and exact 1-cocycles read as $\varpi_x = \text{ad}_x t + t \text{ad}_x^*$, for some $t \in \mathcal{A}(g^*,g)$. While a smooth map $\pi: G \to \mathcal{A}(g^*,g)$ is a Lie group 1-cocycle if it satisfies the following identity:
\[
\pi_{gh} = \pi_g + \pi_h \text{Ad}_g \text{Ad}_h^* \quad (2.2)
\]
for all $g, h \in G$, and exact 1-cocycles read as $\pi_g = \text{Ad}_g t \text{Ad}_g^* - t$, where $t \in \mathcal{A}(g^*,g)$.

Recall that $\varpi$ defined by $\varpi = T_1 \pi$ is a Lie algebra 1-cocycle. Moreover, if $G$ is connected and simply connected, then Van Est’s theorem (see e.g., [11]) ensures that any Lie algebra 1-cocycle $\varpi$ may be uniquely lifted to a Lie group 1-cocycle such that $\varpi = T_1 \pi$.

The symbol $\sum_{(a_1,\ldots,a_n)}$ means “sum over cycling permutations of $(a_1,\ldots,a_n)$”.

2.2. Lie quasi-bialgebras.

Definition 2.1. Let $(g, [\cdot,\cdot], \varpi, \varphi)$ be a Lie algebra, $\varpi: g \to \mathcal{A}(g^*,g)$ a Lie algebra 1-cocycle and $\varphi \in \wedge^3 g$. We say that the quadruple $G = (g, [\cdot,\cdot], \varpi, \varphi)$ is a Lie quasi-bialgebra if $\mathfrak{d} = g \oplus g^*$ together with the bracket $[\cdot,\cdot]_\mathfrak{d}$
\[
[x, y]_\mathfrak{d} = [x, y] \quad (2.3)
\]
\[
[x, \xi]_\mathfrak{d} = \varpi_x \xi - \text{ad}_x^* \xi \quad (2.4)
\]
\[
[\xi, \eta]_\mathfrak{d} = \langle \eta, \varpi_x \xi \rangle + \langle \xi \otimes \eta \otimes 1, \varphi \rangle \quad (2.5)
\]
for $x, y \in g$ and $\xi, \eta \in g^*$, is a Lie algebra. When $\varpi = 0$, the Lie quasi-bialgebra $(g, [\cdot,\cdot], 0, \varphi)$ is said to be cocommutative. The Lie algebra $(\mathfrak{d}, [\cdot,\cdot], 0, \varphi)$ is called the canonical double of the Lie quasi-bialgebra $(g, [\cdot,\cdot], \varpi, \varphi)$.

Remark 2.2. Note that our sign convention for the associator $\varphi$ differs from that of Drinfel’d in [7].
The double \( \mathfrak{d} \) comes equipped with a non-degenerate invariant symmetric bilinear form:

\[
(x + \xi, y + \eta)_0 = \langle \xi, y \rangle + \langle \eta, x \rangle
\]  

(2.6)

for \( x, y \in \mathfrak{g} \) and \( \xi, \eta \in \mathfrak{g}^* \), for which \( (\mathfrak{g}, [\, , \,]) \) is a lagrangian (that is maximal isotropic) subalgebra of \( (\mathfrak{d}, [\, ,\,]_\mathfrak{d}) \).

In practice, we will need the following:

**Proposition 2.3.** Let \( (\mathfrak{g}, [\, ,\,]) \) be a Lie algebra, let \( \varpi : \mathfrak{g} \to \mathcal{A}(\mathfrak{g}^*, \mathfrak{g}) \) be a Lie algebra 1-cocycle, and \( \varphi \in \wedge^3 \mathfrak{g} \). The quadruple \( (\mathfrak{g}, [\, ,\,], \varpi, \varphi) \) is a Lie quasi-bialgebra if and only if the following two equations hold:

\[
\langle \xi \otimes \eta \otimes \zeta, \text{ad}^\varpi_x(\varphi) - \varpi(\langle \xi, \text{ad}_x \eta \zeta \rangle) \rangle = 0
\]

(2.7)

\[
\varpi(\langle \eta, \text{ad}_x \xi \otimes \zeta \otimes \vartheta, \varphi \rangle + \langle \xi \otimes \eta \otimes \vartheta, \zeta \rangle) \varphi = 0
\]

(2.8)

for all \( \xi, \eta, \zeta, \vartheta \in \mathfrak{g}^* \) and \( x \in \mathfrak{g} \). In particular, the quadruple \( (\mathfrak{g}, [\, ,\,], 0, \varphi) \) is a Lie quasi-bialgebra if and only if \( \varphi \) lies in \( (\wedge^3 \mathfrak{g})^0 \).

**2.3. Manin pairs, Manin quasi-triples.**

**Definition 2.4.** Let \( (\mathfrak{d}, [\, ,\,]) \) be a Lie algebra together with a non-degenerate invariant symmetric bilinear form \( (\, ,\,)_{\mathfrak{d}} \). We say that a pair \( (\mathfrak{d}, \mathfrak{g}) \) is a Manin pair if \( \mathfrak{g} \) is a lagrangian subalgebra of \( \mathfrak{d} \).

We say that a triple \( (\mathfrak{d}, \mathfrak{g}, \mathfrak{h}) \) is a Manin quasi-triple if the pair \( (\mathfrak{d}, \mathfrak{g}) \) is a Manin pair, and if \( \mathfrak{h} \) is an isotropic complement of \( \mathfrak{g} \) in \( \mathfrak{d} \).

Hence, if \( (\mathfrak{g}, [\, ,\,], \varpi, \varphi) \) is a Lie quasi-bialgebra with canonical double \( \mathfrak{d} \), the double \( (\mathfrak{d}, \mathfrak{g}) \) is a Manin pair, and the triple \( (\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*) \) is a Manin quasi-triple.

Conversely, let \( (\mathfrak{d}, \mathfrak{g}, \mathfrak{h}) \) be a Manin quasi-triple. Identifying \( \mathfrak{h} \) with \( \mathfrak{g}^* \) by means of \( (\, ,\,)_{\mathfrak{d}} \) provides a Lie quasi-bialgebra structure on \( \mathfrak{g} \) denoted by \( \mathcal{G}(\mathfrak{d}, \mathfrak{g}, \mathfrak{h}) \). Its cocycle \( \varpi \) and associator \( \varphi \) are explicitly given by

\[
\varpi_x \xi = p_{\mathfrak{g}}[x, \Omega^{-1} \xi]_0
\]

(2.9)

\[
\langle \xi \otimes \eta \otimes \zeta, \varphi \rangle = \langle \Omega^{-1} \xi, [\Omega^{-1} \eta, \Omega^{-1} \zeta]_0 \rangle
\]

(2.10)

where \( \Omega \) is the identification \( \Omega : \mathfrak{h} \to \mathfrak{g}^* \) given by \( (\, ,\,)_{\mathfrak{d}} \).

**2.4. Twists.** Let \( \mathcal{G} = (\mathfrak{g}, [\, ,\,], \varpi, \varphi) \) be a Lie quasi-bialgebra, with canonical double \( \mathfrak{d} \). For an isotropic complement \( \mathfrak{h} \) of \( \mathfrak{g} \) in \( \mathfrak{d} \), there exists a skew-symmetric linear map \( t : \mathfrak{g}^* \to \mathfrak{g} \) such that

\[
\mathfrak{h} = \{ t \xi + \xi | \xi \in \mathfrak{g}^* \}
\]

The Lie quasi-bialgebra induced by the Manin quasi-triple \( (\mathfrak{d}, \mathfrak{g}, \mathfrak{h}) \) is the quadruple \( (\mathfrak{g}, [\, ,\,), \varpi^t, \varphi^t) \) where:

\[
\varpi_x^t = \varpi_x + \text{ad}_x^t t + t \text{ad}_x^t
\]

(2.11)

\[
\langle \xi \otimes \eta \otimes \zeta, \varphi^t \rangle = \langle \xi \otimes \eta \otimes \zeta, \varphi \rangle + \langle \xi, [t \xi, t \eta] + \varpi_t \xi \eta \rangle
\]

(2.12)

for \( x \in \mathfrak{g}, \xi, \eta, \zeta \in \mathfrak{g}^* \). The Lie quasi-bialgebra \( (\mathfrak{g}, [\, ,\,), \varpi^t, \varphi^t) \) is called the twist of the Lie quasi-bialgebra \( \mathcal{G} \) via \( t \), and is denoted by \( \mathcal{G}^t \). Note that \( \mathcal{G} \) is a Lie quasi-bialgebra if and only if \( \mathcal{G}^t \) is for any twist \( t \in \mathcal{A}(\mathfrak{g}^*, \mathfrak{g}) \). Let \( \mathfrak{d}^t \) be the double of \( \mathfrak{g}^t \). The following isomorphism of Drinfel’d [7]

\[
\tau_t : \mathfrak{d}^t \rightarrow \mathfrak{d}
\]

(2.13)

will play a crucial role in the sequel. Note that \( \tau_t \) preserves the bilinear forms of \( \mathfrak{d} \) and \( \mathfrak{d}^t \). The inverse \( \tau_t^{-1} \) of \( \tau_t \) is given by \( \tau_t^{-1} = \tau_{-t} \), and, if \( t' \in \mathcal{A}(\mathfrak{g}^*, \mathfrak{g}) \), then \( (\mathcal{G}^t)^{t'} = \mathcal{G}^{t+t'} \).
We will use the following observation:

**Proposition 2.5.** Let \( g \) be a Lie algebra, \( \varpi: g \to A(g^*, g) \) a Lie algebra 1-cocycle, and \( \varphi \in \wedge^3 g \). For \( t \in A(g^*, g) \), let \( \varpi^t \) and \( \varphi^t \) be as in equations (2.11) and (2.12). Then the following equation holds:

\[
\langle \xi \otimes \eta \otimes \zeta, \text{ad}^3_x (\varphi^t) \rangle - \bigotimes \langle \xi, \varpi^t \eta \zeta \rangle = \langle \xi \otimes \eta \otimes \zeta, \text{ad}^3_x (\varphi^t) \rangle - \bigotimes \langle \xi, \varpi^t \eta \zeta \rangle
\]

for all \( x \in g \) and \( \xi, \eta, \zeta \in g^* \).

2.5. Lie quasi-bialgebra morphisms. Let \( G_j = (g_j, [\, , ]_j, \varpi^j, \varphi^j) \), \( j = 1, 2 \) be two Lie quasi-bialgebras, and \( \psi: g_1 \to g_2 \) a Lie algebra morphism. We say that \( \psi \) is a Lie quasi-bialgebra morphism from \( G_1 \) to \( G_2 \) if the following two conditions hold:

\[
\psi \psi^1 \varphi^* = \varphi^2 \psi^* \quad \forall x \in g_1
\]

\[
\psi(\varphi^t) = \varphi^2
\]

The effect of twisting \( G_1 \) via some \( t \in A(g^*_1, g_1) \) is given in the following proposition:

**Proposition 2.6.** Let \( \psi: g_1 \to g_2 \) be a Lie algebra morphism and let \( t \in A(g^*_1, g_1) \). Set \( t' = \psi \psi^* \). Then the morphism \( \psi \) is a Lie quasi-bialgebra morphism from \( G_1 \) to \( G_2 \) if and only if \( \psi \) is a Lie quasi-bialgebra morphism from \( G_1' \) to \( G_2' \).

We also have the following lemma (for a proof, see appendix B):

**Lemma 2.7.** Let \( G_j = (g_j, [\, , ]_j, \varpi^j, \varphi^j) \), \( j = 1, 2 \) be two Lie quasi-bialgebras with double \( \varpi^j \), and \( \psi \) a Lie quasi-bialgebra morphism from \( G_1 \) to \( G_2 \). Then the relations

\[
\psi p_{\varphi} \left( \text{ad}^1_{\varpi^1 \xi} \right)^n u = p_{\varphi^2} \left( \text{ad}^2_{\varphi^t} \right)^n \psi u
\]

\[
p_{\varphi} \left( \text{ad}^1_{\varpi^1 \xi} \right)^n \psi^* \varphi = p_{\varphi^2} \left( \text{ad}^2_{\varphi^t} \right)^n \psi^* \varphi
\]

\[
p_{\varphi} \left( \text{ad}^1_{\varpi^1 \xi} \right)^n u = p_{\varphi^2} \left( \text{ad}^2_{\varphi^t} \right)^n \psi^* \varphi
\]

\[
\psi p_{\varphi} \left( \text{ad}^1_{\varpi^1 \xi} \right)^n \psi^* \varphi = p_{\varphi^2} \left( \text{ad}^2_{\varphi^t} \right)^n \psi^* \varphi
\]

hold for all \( n \in \mathbb{N} \) and for all \( u \in g_1, \xi, \eta \in g_2^* \).

2.6. Lie quasi-bialgebras obtained from one another. Let \( G = (g, [\, , ] , \varpi, \varphi) \) be a Lie quasi-bialgebra. Let \( \psi \) be an automorphism of the Lie algebra \( (g, [\, , ]) \), and set

\[
\varpi^{\psi} = \varpi \psi_{\varpi^{-1}} \psi^*
\]

\[
\varphi^{\psi} = \psi (3) \varphi
\]

Then \( G^{\varphi} = (g, [\, , ], \varpi^{\psi}, \varphi^{\psi}) \) is a Lie quasi-bialgebra, and \( \psi \) is a Lie quasi-bialgebra isomorphism from \( G \) to \( G^{\varphi} \). More generally, let \( w \) be an automorphism of the vector space \( g \), equip \( g \) with the bracket

\[
[x, y]^w = w^{-1}[wx,wy]
\]

and set \( \varphi^w = \varphi^{(3)} \varphi \) and \( \varpi^{w} = \psi [\, , ]^w \) \( \psi^* \) for \( x \in g \). Then, the quadruple \( G^w = (g, [\, , ]^w , \varpi^w , \varphi^w) \) is a Lie quasi-bialgebra such that \( w \) is a Lie quasi-bialgebra isomorphism between \( G \) and \( G^w \).

The Lie quasi-bialgebra \( G^- = (g, [\, , ]^- , \varpi^- , \varphi^-) \) is called the inversion of the Lie quasi-bialgebra \( G \). Obviously, \( (G^-)^- = G \). If we denote by \( \ker \) and \( \ker^- \) the double of \( G \) and \( G^- \) respectively, then the map \( J: \ker \to \ker^- \) defined by \( J(x + \xi) = x - \xi \) for all \( x \in g \) and \( \xi \in \ker^* \) is a Lie algebra isomorphism.
2.7. The adjoint action. Let $G = (\mathfrak{g}, [\cdot, \cdot], \varpi, \phi)$ be a Lie quasi-bialgebra with canonical double $\delta$, let $D$ be the connected, simply-connected Lie group with Lie algebra $\delta$, let $\mathcal{G}$ be the connected Lie subgroup of $D$ with Lie algebra $\mathfrak{g}$, and let $\pi: G \to \mathcal{A}(\mathfrak{g}^*, \mathfrak{g})$ be the Lie group 1-cocycle integrating the Lie algebra 1-cocycle $\varpi$. Denote by $\text{Ad}^D$ the adjoint action of $D$ on its Lie algebra $\delta$. For any $x \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$ and $g \in G$, one has:

$$\text{Ad}^D_g(x + \xi) = \text{Ad}_g x + \pi_g \text{Ad}^D_{g^{-1}}(\xi) + \text{Ad}^D_g \xi$$

(2.24)

Indeed, it is easy to show that $\text{Ad}^D_g \text{Ad}^D_g = \text{Ad}^D_g$ for all $g, g' \in G$, and that $\frac{d}{dt}|_{t=0} \text{Ad}^D_{e^{tu}}(x + \xi) = \text{ad}_u^\mathfrak{g}(x + \xi)$ for all $u \in \mathfrak{g}$ and $x \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$.

3. Dynamical Poisson groupoids and Lie quasi-bialgebras

For further informations on dynamical Poisson groupoids, see [10] and [16].

3.1. Lie quasi-bialgebra associated with a trivial Poisson groupoid. Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. For any point $x \in G$, we denote by $D_x f \in \mathfrak{g}^*$ and $D'_x f \in \mathfrak{g}^*$ the right and left derivatives at $x$:

$$D_x f(u) = \frac{d}{dt}|_{t=0} f(e^{tu} x)$$

(3.1)

$$D'_x f(u) = \frac{d}{dt}|_{t=0} f(x e^{tu})$$

(3.2)

for all $u \in \mathfrak{g}$. Let $L$ be a connected Lie subgroup of $G$ with Lie algebra $\mathfrak{l}$, and $U$ an $\text{Ad}^\mathfrak{l}_x$-invariant open subset in $G^*$. We will denote the inclusion by $i: \mathfrak{l} \to \mathfrak{g}$. Consider the trivial Lie groupoid $\mathcal{G} = U \times G \times U$ with multiplication:

$$(p, x, q)(q, y, r) = (p, xy, r)$$

(3.3)

We say that a multiplicative Poisson bracket on $\mathcal{G}$ is dynamical if it is of the form:

$$\{f, g\}_{(p,x,q)} = \langle p, [\delta f, \delta g] \rangle - \langle q, [\delta' f, \delta' g] \rangle$$

$$- \langle D_g, i\delta f \rangle - \langle D'_g, i\delta' f \rangle$$

$$+ \langle D_f, i\delta g \rangle + \langle D'_f, i\delta' g \rangle$$

$$- \langle D_f, l_p D g \rangle + \langle D_f, \pi_x D g \rangle + \langle D'_f, l_q D' g \rangle$$

(3.4)

where $l: U \to \mathcal{A}(\mathfrak{g}^*, \mathfrak{g})$ is a smooth map, and $\pi: G \to \mathcal{A}(\mathfrak{g}^*, \mathfrak{g})$ is a group 1-cocycle. In this equation, $\delta f$ and $\delta' f$ denote the derivatives of $f$ with respect to the first and second $U$ factors, $D f$ and $D' f$ denote the right and left derivatives of $f$ with respect to the $G$ factor, and all derivatives are evaluated at $(p, x, q)$. Denote by $\varpi = T_1 \pi$ the Lie algebra 1-cocycle associated with $\pi$.

The map $P$ from $\mathcal{G}$ to $\mathcal{A}(\mathfrak{g}^*, \mathfrak{g})$ defined by:

$$P_{(p,x,q)} = -l_p + \pi_x + \text{Ad}_x l_q \text{Ad}^*_x$$

(3.5)

is called the groupoid cocycle associated with the dynamical Poisson bracket [3.24].

Using theorem 2.2.5. of [10], it may be shown that the Jacobi identity for a bracket of this type is equivalent to the following two conditions:

- There exists a $\varphi \in (\wedge^3 \mathfrak{g})$ such that for all $\xi, \eta, \zeta \in \mathfrak{g}^*$:

$$\langle \xi \otimes \eta \otimes \zeta, \text{ad}_x^{(3)} \varphi \rangle = \bigotimes_{(\xi, \eta, \zeta)} \langle \xi, \varpi_{x,y} \varphi \rangle$$

(3.6)

and for all $p \in U$ and $\xi, \eta, \zeta \in \mathfrak{g}^*$:

$$\bigotimes_{(\xi, \eta, \zeta)} \left( \langle \xi, d_p l(\pi^* \xi) \eta \rangle - \langle \zeta, [l_p \xi, l_p \eta] \rangle - \langle \zeta, \varpi_{l_p \xi} \varphi \rangle \right) = \langle \xi \otimes \eta \otimes \zeta, \varphi \rangle$$

(3.7)
• For all \( p \in U \) and \( z \in \mathfrak{g} \):
\[
d_p l(ad_z^* p) + \varpi_{iz} + ad_{iz} l_p + l_p ad_{iz}^* = 0 \tag{3.8}
\]

Equation (3.7) can be seen as a generalization of the modified classical dynamical Yang–Baxter equation to which it reduces when \( \varpi = 0 \). Equation (3.8) is exactly equation (2.7), and equation (3.8) is a generalization of the \( l \)-equivariance of the map \( l \).

**Remark 3.1.** Equation (3.7) may also be written:
\[
d_p l(i^* \xi) \eta - d_p l(i^* \eta) \xi - i d_p \langle \xi, l_\bullet \eta \rangle - [l_p \xi, l_p \eta] - l_p ad_{l_p \xi} \eta + l_p ad_{l_p \eta} \xi
\]
\[
- \varpi_{l_p \xi} \eta + \varpi_{l_p \eta} \xi + \langle \xi, \varpi_{l_p \bullet} \eta \rangle = \langle \xi \otimes \eta \otimes 1, \varphi \rangle \tag{3.9}
\]
for all \( \xi, \eta \in \mathfrak{g}^* \).

Now, by a result from [16], we know that, for a contractible base \( U \), the dual (see section 7 below, for more explicit information about duality) of the Poisson groupoid \( G \) with Poisson bracket (3.4) is still a trivial Poisson groupoid (not necessarily dynamical, though). Its vertex Lie group \( G_{q_0}^* \) is the connected, simply connected Lie group with Lie algebra (isomorphic to) the vector space
\[
\mathfrak{g}_{q_0} = \{ i(z) + \xi \in i(l) \oplus \mathfrak{g}^* \mid i^* \xi = ad_{i^*}^* q_0 \} \subset \mathfrak{g} \oplus \mathfrak{g}^*
\tag{3.10}
\]
for some \( q_0 \in U \), together with the Lie bracket:
\[
[i(z) + \xi, i(z') + \xi']_{q_0} = (i([z,z'])) + \varpi_{i(z)} \xi' + ad_{i(z)} l_{q_0} \xi' + l_{q_0} ad_{i(z)}^* \xi'
\]
\[
- \varpi_{i(z')} \xi - ad_{i(z')} l_{q_0} \xi - l_{q_0} ad_{i(z')}^* \xi
\]
\[
+ [l_{q_0} \xi, l_{q_0} \xi'] + l_{q_0} ad_{l_{q_0} \xi} \xi' - l_{q_0} ad_{l_{q_0} \xi}^* \xi
\]
\[
+ \varpi_{l_{q_0} \xi} \xi' - l_{q_0} ad_{l_{q_0} \xi} \xi - \langle \xi, \varpi_{l_{q_0} \bullet} \xi \rangle + \langle \xi \otimes \xi' \otimes 1, \varphi \rangle,
\]
\[
- ad_{i(z)}^* \xi' + ad_{i(z')}^* \xi - \langle \xi, \varpi_{i(z')} \xi \rangle - ad_{i(z')}^* \xi' + ad_{i(z)}^* \xi
\]
for all \( i(z) + \xi, i(z') + \xi' \in \mathfrak{g}_{q_0}^* \). Note that the Lie algebras \( \{ \mathfrak{g}_{q_0}^*, [\cdot, \cdot]^* \} \) are all isomorphic when \( q_0 \) ranges over \( U \).

We start with a lemma which relates a solution \( l \) of equation (3.7) to its translation \( l' = l - t \) by an element \( -t \in A(\mathfrak{g}^*, \mathfrak{g}) \):

**Lemma 3.2.** Let \( t \in A(\mathfrak{g}^*, \mathfrak{g}) \). Set \( l'_p = l_p - t \) for any \( p \in U \). Then equation (3.7) is satisfied for all \( \xi, \eta, \zeta \in \mathfrak{g}^* \) and \( p \in U \) if and only if the following equation is satisfied:
\[
\bigodot_{(\xi, \eta, \zeta)} \left( \langle \xi, d_p l'(i^* \xi) \eta \rangle - \langle \zeta, [l'_{p \xi}, l'_{p \eta}] \rangle - \langle \xi, \varpi z \xi \rangle \right) = \langle \xi \otimes \eta \otimes \eta, \varphi \rangle \tag{3.12}
\]
for all \( \xi, \eta, \zeta \in \mathfrak{g}^* \) and \( p \in U \), where \( \varpi^* \) and \( \varphi^* \) are defined by equations (2.11) and (2.12) (even though we don’t know yet that \( (\mathfrak{g}, [\cdot, \cdot], \varpi, \varphi) \) is a Lie quasi-bialgebra).

**Proof.** Straightforward computation using equations (2.11) and (2.12). \( \square \)

As an immediate consequence we can write equation (3.7) as
\[
\langle \xi \otimes \eta \otimes \zeta, \varphi \rangle = \bigodot_{(\xi, \eta, \zeta)} \langle \xi, d_p l(i^* \xi) \eta \rangle \tag{3.13}
\]
for all \( p \in U \) and \( \xi, \eta, \zeta \in \mathfrak{g}^* \). Thus, if \( l \) satisfies equation (3.7) on \( U \), then \( \varphi^p \equiv 0 \mod 1, \forall p \in U \).

We now come to a proposition which is basic for our subsequent analysis.

**Proposition 3.3.** Let \( q_0 \) be a point in \( U \). If equations (3.6), (3.7), and (3.8) are satisfied, then the quadruple \( \mathcal{G}^{q_0} = (\mathfrak{g}, [\cdot, \cdot], \varpi z, \varphi z) \) is a Lie quasi-bialgebra. Moreover, the Lie algebra \( \mathfrak{g}_{q_0}^* \) defined by equations (3.10) and (3.11) is a Lagrangian subalgebra of the canonical double \( \mathfrak{d}^{q_0} \) of \( \mathcal{G}^{q_0} \).
Before proving proposition 3.3, we state the following auxiliary result:

**Lemma 3.4.** Let \( l \) be a solution of (3.14) and (3.8) on \( U \) and set \( l' = l - l_{q_0} \). Then for all \( \xi, \eta, \zeta, \theta \in \mathfrak{g}^* \), the two following equations hold:

\[
\langle \theta, d_{q_0} l' (i^* \langle \eta, \varpi^{l_{q_0}} \xi \rangle) \rangle \xi = \langle \xi, d_{q_0} l' (i^* \langle \theta, \varpi^{l_{q_0}} \zeta \rangle) \rangle \eta \tag{3.14}
\]

\[
\circ \langle \xi, d_{q_0} l' (i^* \xi, i^* \theta) \rangle \eta = \circ \langle \xi, \varpi^{l_{q_0}} (i^* \xi) \rangle \eta \tag{3.15}
\]

**Proof.** Equation (3.14) is a consequence of equation (3.8), and equation (3.15) is the derivative of equation (3.12) in the direction \( i^* \theta \), evaluated at \( q_0 \). \( \square \)

**Proof of proposition 3.3.** According to proposition 2.5, the first condition of proposition 2.3 is satisfied, since equation (3.6) holds, so it only remains to show the second condition of proposition 2.3.

According to equality (3.13), the second condition of proposition 2.3 for \( (\mathfrak{g}, [\, , \,], \varpi^{l_{q_0}}, \varphi^{l_{q_0}}) \) reads:

\[
\circ \langle \eta, \varpi^{l_{q_0}} \xi \rangle \otimes \xi \otimes \theta, \varphi^{l_{q_0}} \rangle + \langle \xi \otimes \eta \otimes \theta, \varpi^{l_{q_0}} \zeta \rangle, \varphi^{l_{q_0}} \rangle
\]

\[
\circ \langle \eta, \varpi^{l_{q_0}} l_{q_0} (i^* \xi) \rangle \xi + \langle \xi, \varpi^{l_{q_0}} l_{q_0} (i^* \theta) \rangle \zeta + \langle \theta, d_{q_0} l' (i^* \eta, \varpi^{l_{q_0}} \xi) \rangle \xi
\]

which vanishes by Lemma 3.2 and Schwarz’ lemma. Hence, the quadruple \( \mathcal{G}^{l_{q_0}} = (\mathfrak{g}, [\, , \,], \varpi^{l_{q_0}}, \varphi^{l_{q_0}}) \) is a Lie quasi-bialgebra.

It is clear from equation (3.11) that \( \mathfrak{g}^{l_{q_0}} \) is a Lie subalgebra of the canonical double \( \mathfrak{d}^{l_{q_0}} \) of \( \mathcal{G}^{l_{q_0}} \), and a simple verification shows that it is Lagrangian. \( \square \)

As mentioned above, beware that the dual Poisson groupoid of a dynamical Poisson groupoid is not dynamical in general, and even if it is, the Lie quasi-bialgebra on \( \mathfrak{g}^{l_{q_0}} \), which the dual Poisson groupoid is associated with, is not directly obtained from the Manin pair \( (\mathfrak{d}, \mathfrak{g}^{l_{q_0}}) \).

We now introduce the following definitions:

**Definition 3.5.** Let \( \mathcal{G} = (\mathfrak{g}, [\, , \,], \varpi, \varphi) \) be a Lie quasi-bialgebra, \( \mathfrak{l} \) a Lie subalgebra of \( \mathfrak{g} \), and \( U \subset \mathfrak{l}^* \) an \( \text{Ad}_{L}^{\text{r}} \)-invariant open subset.

1. We say that a smooth map \( l: U \to A(\mathfrak{g}^*, \mathfrak{g}) \) is a dynamical \( \ell \)-matrix on \( U \) associated with the Lie quasi-bialgebra \( \mathcal{G} \) if it satisfies equations (3.7) and (3.8). A dynamical \( \ell \)-matrix associated with a cocommutative Lie quasi-bialgebra is called a dynamical \( \ell \)-matrix.

2. Let \( q \in \mathfrak{l}^* \), and let \( \mathcal{D}_q \subset \mathfrak{l}^* \) be the formal neighborhood of \( q \). We say that a (formal) map \( \mathfrak{l}: \mathcal{D}_q \to A(\mathfrak{g}^*, \mathfrak{g}) \) is a formal dynamical \( \ell \)-matrix at \( q \) associated with the Lie quasi-bialgebra \( \mathcal{G} \) if it satisfies equations (3.7) and (3.8) formally.

3. We denote by \( \text{Dynl}(U, \mathcal{G}) \) the set of dynamical \( \ell \)-matrices on \( U \) associated with the Lie quasi-bialgebra \( \mathcal{G} \), and by \( \text{Dynl}(\mathcal{D}_q, \mathcal{G}) \) the set of formal dynamical \( \ell \)-matrices at \( q \) associated with the Lie quasi-bialgebra \( \mathcal{G} \).

Lemma 3.2 has the following interpretation:

**Proposition 3.6.** With these notations, \( \forall t \in A(\mathfrak{g}^*, \mathfrak{g}), \)

\[
\text{Dynl}(U, \mathcal{G}^t) = \text{Dynl}(U, \mathcal{G}) - t
\]

\[
\text{Dynl}(\mathcal{D}_q, \mathcal{G}^t) = \text{Dynl}(\mathcal{D}_q, \mathcal{G}) - t
\]
3.8 that introduced in [10] for dynamical the scheme of [9].

Using proposition 3.6 we get:

\[ \mathcal{D}yn(U, \mathcal{G}) = \bigcup_{t \in \mathcal{A}(g^*, g)} \mathcal{D}yn(U, \mathcal{G}^t) + t \]
\[ \mathcal{D}yn(D, \mathcal{G}) = \bigcup_{t \in \mathcal{A}(g^*, g)} \mathcal{D}yn(D, \mathcal{G}^t) + t \]

Remark 3.7. For \( \mathcal{D}yn(U, \mathcal{G}) \) to be non empty, it is necessary that there exists a twist \( t \in \mathcal{A}(g^*, g) \) such that \( \varpi_t = 0 \) and that \( \varphi_t \equiv 0 \mod 1 \). Indeed, if \( l \in \mathcal{D}yn(U, \mathcal{G}) \), then \( l' = l - l_0 \in \mathcal{D}yn(U, \mathcal{G}^{l_0}) \), as shown by proposition 3.6. Now lemma 3.2 implies that \( \varphi^{l_0} \equiv 0 \mod 1 \) and equation 3.8 implies that \( \varpi_{l_0} = 0 \). Obviously, the same holds for \( \mathcal{D}yn(D, \mathcal{G}) \).

Remark 3.8. Let \( q_0 \in l^* \) such that \( \text{ad}_{\varphi}^* q_0 = 0 \) for all \( z \in l \), and let \( l \in \mathcal{D}yn(U, \mathcal{G}) \). Then the map \( l': U + q_0 \to \mathcal{A}(g^*, g) \) defined by \( l'_p = l_p - q_0 \) lies in \( \mathcal{D}yn(U + q_0, \mathcal{G}) \).

4. Gauge transformations

In this section we recall the action of the gauge group on dynamical \( \ell \)-matrices which was introduced in [10] for dynamical \( r \)-matrices. Also, we describe the associated moduli space, following the scheme of [9].

For any subset \( A \) of the vector space \( E \), we denote by \( A^\perp \) the orthogonal space to \( A \):

\[ A^\perp = \{ v \in E^* \mid \langle v, a \rangle = 0, \forall a \in A \}. \]

We shall denote by \( \ell_x \) and \( r_x \) the left and right action of a Lie group \( G \) on its tangent bundle associated with the left and right multiplications of \( G \) on itself.

4.1. Trivial groupoid morphisms. Let \( \mathcal{G}_1 = U \times G_1 \times U \) and \( \mathcal{G}_2 = U \times G_2 \times U \) be two trivial Lie groupoids over the same base \( U \) which is assumed to contain 0. Let \( \Psi: \mathcal{G}_1 \to \mathcal{G}_2 \) be a base preserving groupoid morphism.

Proposition 4.1. The morphism \( \Psi \) has the form:

\[ \Psi(p, x, q) = (p, \sigma_p \psi(x) \sigma_q^{-1}, q) \] (4.1)

for all \( (p, x, q) \in \mathcal{G}_1 \), where \( \sigma: U \to G_2 \) is a smooth map satisfying \( \sigma_0 = 1 \), and \( \psi: G_1 \to G_2 \) is a Lie group morphism.

Proof. The most general form for a base preserving map \( \Psi: \mathcal{G}_1 \to \mathcal{G}_2 \) is:

\[ \Psi(p, x, q) = (p, \psi_{p, q}(x), q) \] (4.2)

where \( \psi_{p, q}: G_1 \to G_2 \). We set \( \psi = \psi_{0, 0} \) and \( \sigma_p = \psi_{p, 0}(1) \). If \( \Psi \) is a groupoid morphism, then \( \psi_{p, q}(x) = \psi_{p, 0}(1) \psi_{0, 0}(x) \psi_{0, q}(1) \) and \( \psi_{0, q}(x) = \psi_{0, 0}(x^{-1})^{-1} \) for all \( x \in G \). Thus, \( \psi \) is a Lie group morphism, \( \psi_{p, q}(x) = \sigma_p \psi(x) \sigma_q^{-1} \) and \( \sigma_0 = 1 \).

Let \( L \) be a connected Lie subgroup of both \( G_1 \) and \( G_2 \), with Lie algebra \( l \). If \( U \) is an \( \text{Ad}_l^* \)-invariant subset of \( l^* \), there are two actions of \( L \) on \( \mathcal{G}_k \), \( k = 1, 2 \), namely:

- A left action: \( h \cdot (p, x, q) = (\text{Ad}_l^* \rho \cdot h, x, q) \),
- and a right action: \( (p, x, q) \cdot h = (p, x, \text{Ad}_l^* \rho \cdot h) \),
By definition, the groupoid morphism $\Psi$ is said to be $L$-biequivariant if and only if it is equivariant for both left and right actions of $L$, that is if and only if
\[
\sigma (\operatorname{Ad}_{h^{-1}}^* p) = h \sigma_p \psi(h)^{-1}
\] (4.3)
for all $h \in L$ and $p \in U$. Since $L$ is connected and $0 \in U$, this condition is also equivalent to its infinitesimal version:
\[
\psi z = z
\] (4.4)
\[
r_{\sigma_p^{-1}}(T_p \sigma) \operatorname{ad}^*_{\sigma_p} z = \operatorname{Ad}_{\sigma_p} z - z
\] (4.5)
for all $z \in \mathfrak{g}$ and $p \in U$, where $\psi = T_1 \psi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is the Lie algebra morphism associated to the Lie group morphism $\psi$.

Now, let $P^1$ and $P^2$ be groupoid cocycles on $\mathcal{G}_1$ and $\mathcal{G}_2$ associated with some dynamical Poisson brackets on $\mathcal{G}_1$ and $\mathcal{G}_2$. For $f \in C^\infty(\mathcal{G}_2)$ and $X = (p, x, q)$ a point in $\mathcal{G}_1$, a direct calculation yields:
\[
\begin{align*}
\delta_X (f \circ \Psi) &= \delta_{\Psi(X)} f + (T_p \sigma)^* r_{\sigma_p}^* D_{\Psi(X)} f \\
D_X (f \circ \Psi) &= \psi^* \operatorname{Ad}_{\sigma_p}^* D_{\Psi(X)} f \\
D'_X (f \circ \Psi) &= \psi^* \operatorname{Ad}_{\sigma_p}^* D'_{\Psi(X)} f \\
\delta_X' (f \circ \Psi) &= \delta'_{\Psi(X)} f - (T_q \sigma)^* r_{\sigma_q}^* D'_{\Psi(X)} f
\end{align*}
\] (4.6-4.9)
Using these equations and equations (4.14) and (4.5), one can show the following:

**Proposition 4.2.** The groupoid morphism $\Psi$ is a Poisson groupoid morphism if and only if the two following conditions hold:

- The groupoid morphism $\Psi$ is $L$-biequivariant,
- The equation
\[
\operatorname{Ad}_{\sigma_p} \psi \cdot P_X^1 \psi^* \operatorname{Ad}_{\sigma_p}^* + \Theta_{\Psi(X)}^\Psi = P_{\Psi(X)}^2
\] (4.10)
is satisfied for all $X = (p, x, q) \in \mathcal{G}_1$, where
\[
\Theta_{(p, y, q)}^\Psi = \operatorname{Ad}_p \theta_p^\Psi \operatorname{Ad}_{\sigma_p}^* - \theta_p^\Psi
\] (4.11)
\[
\theta_p^\Psi = r_{\sigma_p}^{-1}(T_p \sigma)^* \operatorname{Ad}_{\sigma_p}^* - (T_q \sigma)^* r_{\sigma_q}^{-1}
\] (4.12)
for all $(p, y, q) \in \mathcal{G}_2$.

Notice that $\theta^\Psi$, as defined by equation (4.14), is skew-symmetric (use equation (4.5)), and that $\Theta^\Psi$ is an exact groupoid 1-cocycle.

For $j = 1, 2$, write $P_{(p, x, q)}^j = -l_p^j + \pi x_j + \operatorname{Ad}_{x_j} l_q^j \operatorname{Ad}_{x_j}^*$ for all $(p, x, q) \in \mathcal{G}_j$. Without loss of generality, we may assume that $l_0^j = \theta_0^\Psi + \psi \psi^* \psi^*$ (this is done by translating $l^j$, while adding an exact group-cocycle to $\pi^j$). Denote by $\mathcal{G}_j = (\mathfrak{g}_j, [\cdot, \cdot], \frac{\omega}{\pi^j}, \varphi^j)$ the Lie quasi-bialgebras associated with $l^j$ and the Poisson bracket on $\mathcal{G}_j$.

**Proposition 4.3.** With this convention, the following equation holds for all $p \in U$:
\[
l_p^2 = \operatorname{Ad}_{\sigma_p} \psi \pi_x^1 \psi^* \operatorname{Ad}_{\sigma_p}^* + \theta_p^\Psi + \pi_{\sigma_p}^2
\] (4.13)
and the Lie algebra morphism $\psi$ is a Lie quasi-bialgebra morphism from $\mathcal{G}_1$ to $\mathcal{G}_2$.

**Proof.** Evaluating equation (4.10) at $X = (0, 1, p)$ for some $p \in U$ yields equation (4.13), and evaluating equation (4.10) at $X = (0, x, 0)$ for $x \in \mathcal{G}_1$ yields:
\[
\psi \pi_x^1 \psi^* = \pi_{\sigma_p x}^2
\] (4.14)
There exists an open subset $U'$ of $U$, with $0 \in U'$, and a smooth map $\Sigma: U' \rightarrow \mathfrak{g}$ satisfying $\Sigma_0 = 0$ such that $\sigma_p = e^{\Sigma_p}$ for all $p \in U'$. Set $A = d_0 \Sigma \in \mathcal{L}(\mathfrak{g}, \mathfrak{g}_2)$. Notice that $A^*$ takes values in $\mathfrak{g}$ and
that $A$ is $l$-equivariant, since $\sigma$ is. Using the differential of the exponential map (see section A.2 in the appendix), one computes (here and below, $i^*$ stands for $i_\Sigma^*$):

$$\theta^\psi_p = \frac{\text{Ad}^\psi_{\Sigma_p} - 1}{\text{ad}_{\Sigma_p}} (d_p \Sigma) i^* \text{Ad}^\Sigma_{\Sigma_p} - (d_p \Sigma)^* \frac{\text{Ad}^\psi_{\Sigma_p} - 1}{\text{ad}_{\Sigma_p}}$$

(4.15)

so that

$$\theta^\psi_0 = A i^* - A^*$$

(4.16)

$$\langle \eta, d_0 \theta^\psi (\alpha) \xi \rangle = \frac{1}{2} \langle \eta, [A \alpha, A i^* \xi] \rangle - \frac{1}{2} \langle \xi, [A \alpha, A i^* \eta] \rangle + \langle \eta, A i^* \text{ad}_{A \alpha} \xi \rangle$$

$$+ \langle \eta, d_0^2 \Sigma (\alpha, i^* \xi) \rangle - \langle \xi, d_0^2 \Sigma (\alpha, i^* \eta) \rangle$$

(4.17)

for all $\alpha \in l^*$, and $\xi, \eta \in g^*$. Now, by equation (3.7), we have for all $\xi, \eta, \zeta \in g^*_2$:

$$\langle \xi \otimes \eta \otimes \zeta, \varphi^2 \rangle = \bigcirc_{(\xi, \eta, \zeta)} \langle \xi, d_0 l^2 (i^* \eta) \zeta - \overline{\imath_0 l^2 \zeta} - \overline{\omega^2_{\xi \eta} \zeta} \rangle$$

(4.18)

Using equations (4.16) and (4.17) and the fact that $\psi z = z$ for all $z \in l$, we compute the three terms in the right hand side of equation (4.18):

$$\bigcirc_{(\xi, \eta, \zeta)} \langle \xi, d_0 l^2 (i^* \eta) \zeta \rangle = \bigcirc_{(\xi, \eta, \zeta)} \langle \xi, \psi d_0 l^2 (i^* \eta) \psi^* \zeta + [A i^* \eta, \psi d_0 l^2 \psi^* \zeta] + \psi d_0 l^2 \psi^* \eta, A i^* \zeta \rangle$$

$$+ [A i^* \eta, A i^* \zeta] - [A^* \eta, A i^* \zeta] + \overline{\omega^2_{A i^* \eta} \zeta}$$

(4.19)

$$\bigcirc_{(\xi, \eta, \zeta)} \langle \xi, \overline{\imath_0 l^2 \zeta} \rangle = \bigcirc_{(\xi, \eta, \zeta)} \langle \xi, \psi \overline{\imath_0 l^2 \psi^* \eta + A i^* \eta - A^* \eta, \psi \overline{\imath_0 l^2 \psi^* \zeta + A i^* \zeta - A^* \zeta} \rangle$$

(4.20)

$$\bigcirc_{(\xi, \eta, \zeta)} \langle \xi, \overline{\omega^2_{\xi \eta} \zeta} \rangle = \bigcirc_{(\xi, \eta, \zeta)} \langle \xi, \overline{\omega^2_{\psi \overline{\imath_0 l^2 \psi^* \eta + A i^* \eta - A^* \eta} \psi^* \zeta} \rangle$$

(4.21)

Now, using equations (3.7), (3.8), (4.19), (4.20), (4.21), and the $l$-equivariance of $A$, one shows that equation (4.18) reads:

$$\langle \xi \otimes \eta \otimes \zeta, \varphi^2 \rangle = \langle \xi \otimes \eta \otimes \zeta, \varphi^{(3)} \rangle$$

(4.22)

$\xi, \eta, \zeta \in g^*_2$. Proposition 4.3 is thus proved.

4.2. Gauge transformations. Let $G = U \times G \times U$ be a dynamical Poisson groupoid over $U \subset l^*$ with associated groupoid cocycle $P$, and let $\Psi: G \to G$ be a base preserving groupoid automorphism. Using $\Psi$, we can transform the Poisson structure on $G$ into another Poisson structure, which is dynamical if and only if $\Psi$ is $L$-biequivariant. Such a transformation on the dynamical Poisson structures of $G$ is called gauge transformation.

Using proposition 4.2, the transformation of $P$ by $\Psi$ is given by:

$$P^\Psi_{(p, x, q)} = l^\Psi_p + \pi^\Psi_x + \text{Ad}_x l^\Psi_q \text{Ad}_x^*$$

(4.23)

where

$$l^\Psi_p = \text{Ad}_{\sigma_p} \psi l^\psi_p \psi^* \text{Ad}_{\sigma_p} + \theta^\psi_p + \psi \pi_{\psi^{-1} \sigma_p} \psi^*$$

(4.24)

$$\pi^\Psi_x = \psi \pi_{\psi^{-1} x} \psi^*$$

(4.25)

The map $l^\Psi$ satisfies equation (3.7) for the cocycle $\omega^\Psi = \omega^\psi$ defined by equation (2.21), and for the 3-tensor $\varphi^\Psi = \varphi^\psi$ defined by equation (2.22). Thus,

$$\text{Dynl}(U, G)^\Psi \subset \text{Dynl}(U, G^\Psi)$$

(4.26)

for all $L$-biequivariant groupoid morphism $\Psi$. 

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4.3. Gauge group and moduli space of dynamical \( \ell \)-matrices. The group \( \text{Map}_0(U,G)^1 \) of \( \ell \)-equivariant smooth maps \( \sigma : U \to G \) satisfying \( \sigma_0 = 1 \), with pointwise multiplication acts on \( \text{Dynl}(U,G) \) by setting
\[
l_p^\sigma = \text{Ad}_{\sigma_p} l_p \text{Ad}_{\sigma_p}^* + \theta_p^\sigma + \pi_{\sigma_p},
\]
for all \( \sigma \in \text{Map}_0(U,G)^1 \), where \( \theta_p^\sigma \) is defined as the right hand side of equation (4.12). It can be checked directly that this action is a left action:
\[
(l^\sigma)^{\sigma'} = l^{\sigma'\sigma}
\]
This fact can also be proved at a groupoid level, by considering the action of \( \text{Map}_0(U,G)^1 \) on the groupoid cocycles.

The subgroup \( \text{Map}_0^{(2)}(U,G)^1 \) of \( \text{Map}_0(U,G)^1 \) consisting of smooth maps \( \sigma \in \text{Map}_0(U,G) \) such that \( T_0 \sigma = 0 \) acts on \( \text{Dynl}_0(U,G) \).

There is an equivalent at the formal level: the group \( \text{Map}_0^{(2)}(\mathbb{D},G)^1 \) of \( \ell \)-equivariant formal maps \( \sigma : \mathbb{D} \to G \) satisfying \( \sigma_0 = 1 \), with (formal) pointwise multiplication acts on the space of formal dynamical \( \ell \)-matrices on \( \mathbb{D} \) by the same formula (4.24) (here, use Van Est’s formula (14) to give a meaning to the term \( \pi_{\sigma_p} \)). While the subgroup \( \text{Map}_0^{(2)}(\mathbb{D},G)^1 \) of \( \text{Map}_0(\mathbb{D},G)^1 \) consisting of formal maps \( \sigma \in \text{Map}_0(\mathbb{D},G) \) such that \( T_0 \sigma = 0 \) acts on \( \text{Dynl}_0(\mathbb{D},G) \).

In the terminology of [10], the group \( \text{Map}_0(U,G)^1 \) (resp., \( \text{Map}_0(\mathbb{D},G)^1 \)) is called the gauge group of dynamical \( \ell \)-matrices (resp., formal dynamical \( \ell \)-matrices).

Set \( M(G,1) = \text{Dynl}(\mathbb{D},G)/\text{Map}_0(\mathbb{D},G)^1 \). In the terminology of [23], the space \( M(G,1) \) is called the moduli space of formal dynamical \( \ell \)-matrices associated with the Lie quasi-bialgebra \( G \).

4.4. Reductive decomposition and Lie quasi-bialgebra compatible with a reductive decomposition. The statements of this section are generalisations of those of Etingof and Schiffmann in [9]. Eventhough the proofs are analogous, we shall repeat the short arguments in the body of the text for the convenience of the reader. The proof of theorem 4.6 below, which is identical to that of [9] is reproduced in an appendix.

Recall that a reductive decomposition of the Lie algebra \( (g, [, ]) \) is a vector space decomposition \( g = l \oplus m \) such that \( l \) is a subalgebra of \( g \) and \( [l,m] \subset m \). Let \( G = (g, [, ], \varpi, \varphi) \) be a Lie quasibialgebra, \( l \) a Lie subalgebra of \( g \) and let \( U \subseteq \ell^* \) be an Ad\( \ell^* \)-invariant subset of \( \ell^* \) containing 0. If \( \text{Dynl}(U,G) \) is to be nonempty, then \( \varpi \) must vanish on \( l \). Let \( G = U \times X \times U \) be the dynamical Poisson groupoid associated with some \( l \in \text{Dynl}_0(U,G) \). It then follows from equation 3.11 that \( g^*_0 = l^\perp \oplus l^\perp \) is a reductive decomposition. Therefore, if we want the Poisson groupoid dual to \( G \) to be dynamical too, then \( g \) must admit a reductive decomposition \( g = l \oplus m \). It is natural to ask the Lie quasi-bialgebra to be compatible in some sense with this additional structure on \( g \). This is the purpose of definition 4.10 below. From now on we make use of the following notation: we denote by \( p_1 : g \to l \) the projection of \( g \) on \( l \) along \( m \), and by \( s = (p_1)^* : \ell^* \to g^* \) its dual. Notice that the image of \( s \) is \( m^\perp \) and that \( s \) is \( l \)-equivariant.

We recall the equivariant Poincaré lemma (for a proof, see e.g., [2]).

Lemma 4.4 (equivariant Poincaré lemma). Let \( l \) be a finite dimensional Lie algebra, \( V \) a finite-dimensional \( l \)-module, and \( \omega \) an \( l \)-equivariant closed \( k \)-form on \( \mathbb{D} \) with values in \( V \) for some \( k \geq 1 \). Then, there exists an \( l \)-equivariant \((k-1)\)-form \( \nu \) with values in \( V \) such that \( d^{\text{Rham}} \nu = \omega \), where \( d^{\text{Rham}} \) is the de Rham differential operator of differential forms with values in \( V \).

Proposition 4.5. Let \( G = (g, [, ], \varpi, \varphi) \) be a Lie quasi-bialgebra such that \( \varpi_l = 0 \), and assume that \( g \) admits a reductive decomposition \( g = l \oplus m \). Then every formal dynamical \( \ell \)-matrix \( l \in \text{Dynl}(\mathbb{D},G) \) is gauge-equivalent to a dynamical \( \ell \)-matrix \( l' \in \text{Dynl}(\mathbb{D},G) \) such that \( l'^*_0 m^\perp = 0 \) and \( l'^*_0 l^\perp \subset m \).
Proof. Define \( \Sigma : l^* \to g \) by \( \Sigma_\alpha = -\frac{1}{2} D l_0 \sigma_\alpha - p_m l_0 \sigma_\alpha \), and set \( \sigma_p = e^{2\pi}. \) Since \( l_0 \) is \( l \)-equivariant (because \( \varpi_1 = 0 \)), since \( g = l \oplus m \) is a reductive decomposition, and since \( \Sigma_0 = 0, \sigma \) lies in \( \text{Map}_l(D, G)^l \), and thus defines a gauge transformation on \( \text{Dyn}_l(D, G) \). Now, equations (4.24) and (4.16) imply:

\[
l_0^\nu \sigma_\alpha = l_0 \sigma_\alpha - p_m l_0 \sigma_\alpha - p_l l_0 \sigma_\alpha = 0 \tag{4.29}
\]

\[
p_l l_0^\nu \xi = p_l l_0 \xi - p_l l_0 \xi = 0 \tag{4.30}
\]

for all \( \alpha \in l^* \) and \( \xi \in l^1 \).

\[\square\]

**Theorem 4.6.** Let \( \mathcal{G} = (\mathfrak{g}, [\cdot, \cdot], \varpi, \varphi) \) be a Lie quasi-bialgebra, assume that \( \mathfrak{g} = l \oplus m \) is a reductive decomposition of the Lie algebra \( \mathfrak{g} \), and that \( \varpi_1 = 0 \). Let \( l \) and \( l' \) be two formal dynamical \( l \)-matrices in \( \text{Dyn}_l(D, \mathcal{G}) \) such that \( l_0 = l'_0 \). Then there exists a gauge transformation which transforms \( l \) into \( l' \).

**Proof.** The proof is that of [3]. See appendix [C].

As a corollary to theorem 4.6, we have:

**Corollary 4.7.** If the Lie algebra \( \mathfrak{g} \) possesses a reductive decomposition \( \mathfrak{g} = l \oplus m \) and if \( \varpi_1 = 0 \) and \( \varphi \equiv 0 \mod l \) then the quotient space \( \text{Dyn}_l(U, \mathcal{G})/\text{Map}_l^l(U, G)^l \) consists of at most one point.

Consider the following algebraic variety

\[
\mathcal{M}_{G, l, m} = \left\{ t \in (\wedge^2 m)^l \left| \varphi^t = 0 \mod l \right. \right\} \tag{4.31}
\]

Here, \( (\wedge^2 m)^l \) denotes the set of \( l \)-equivariant elements \( t \) of \( \mathcal{A}(\mathfrak{g}^*, \mathfrak{g}) \) satisfying \( t m^\perp = 0 \) and \( t l^\perp \subset m \).

It is immediate from equation (4.27) that if \( l \) and \( l' \) are gauge equivalent, and if \( l_0 m^\perp = 0, l_0 l^\perp \subset m, l_0^\prime m^\perp = 0, \) and \( l_0^\prime l^\perp \subset m \), then \( l_0 = l'_0 \).

As a corollary to proposition 4.3 and theorem 4.6, we have:

**Corollary 4.8.** Let \( \mathfrak{g} = l \oplus m \) be a reductive decomposition of the Lie algebra, and let \( \mathcal{G} = (\mathfrak{g}, [\cdot, \cdot], \varpi, \varphi) \) be a Lie quasi-bialgebra such that \( \varpi_1 = 0 \). Then, there is a well-defined embedding

\[
\mathcal{M}(\mathcal{G}, l) \longrightarrow \mathcal{M}_{G, l, m} \tag{4.32}
\]

which sends a class \( C \) to \( l_0 \), where \( l \in C \) is any representative such that \( l_0 m^\perp = 0 \) and \( l_0 l^\perp \subset m \).

**Proposition 4.9.** Let \( \mathcal{G} = (\mathfrak{g}, [\cdot, \cdot], \varpi, \varphi) \) be a Lie quasi-bialgebra, assume that \( \mathfrak{g} = l \oplus m \) is a reductive decomposition of the Lie algebra \( \mathfrak{g} \), that \( (m^\perp, \varpi_1 m^\perp) = 0, \varpi_1 = 0, \) and that \( \varphi \in \text{Alt}(l \otimes l \otimes l \otimes m \otimes m) \). Then any formal dynamical \( l \)-matrix \( l \in \text{Dyn}_l(D, \mathcal{G}) \) is gauge equivalent to a formal dynamical \( l \)-matrix \( l' \in \text{Dyn}_l(D, \mathcal{G}) \) such that \( l' m^\perp \subset l \) and \( l' l^\perp \subset m \).

**Proof.** This proposition can be proved using theorems 4.6 and 6.1 but we give here a direct proof. Let \( k \geq 1 \) and assume that \( l m^\perp \subset l \) and \( l l^\perp \subset m \) modulo terms of degree \( \geq k \), i.e., that \( \langle \xi, l s_0 \rangle = 0 \) modulo terms of degree \( \geq k \) for all \( \alpha \in l^* \) and \( \xi \in l^1 \).

If \( \Sigma : D \to \mathfrak{g} \) is an \( l \)-equivariant homogeneous map of degree \( k + 1 \), set \( \sigma = e^{\Sigma} \). Then \( \sigma \) is \( l \)-equivariant too, and \( l^s = l + d \Sigma i^s - (d \Sigma)^s \) modulo terms of degree \( \geq k + 1 \). Let \( \nu \) be the homogeneous term of degree \( k \) of \( -p_m l s_0 \). Then \( \nu \) is an \( l \)-equivariant 1-form on \( D \) with values in \( \mathfrak{g} \), since \( \varpi_1 = 0 \), and since \( \mathfrak{g} = l \oplus m \) is a reductive decomposition. We now show that \( \nu \) is a closed 1-form: let \( \alpha, \beta \in l^* \). Then, by equation (3.0), and by the assumption that \( p_m l s_0 = 0 \) modulo terms of degree \( \geq k \), we obtain:

\[
d \nu(\alpha) \beta = -[p_m d l(\alpha) \beta]_{k-1} = p_m [-d l(\beta) \alpha + \langle s_0, \varpi, s_\beta \rangle - \langle s_0 \otimes s_\beta \otimes 1, \varphi \rangle]_{k-1} \tag{4.33}
\]

Now, by the assumptions \( (m^\perp, \varpi_1 m^\perp) = 0 \) and \( \varphi \in \text{Alt}(l \otimes l \otimes l \otimes m \otimes m) \), equation (4.33) reads \( d \nu(\alpha) \beta = d \nu(\beta) \alpha \). Thus \( \nu \) is an \( l \)-equivariant closed 1-form on \( D \) with values in \( \mathfrak{g} \), and hence, by the equivariant Poincaré lemma, there exists a homogeneous \( l \)-equivariant map \( \Sigma : D \to \mathfrak{g} \) of degree
k + 1 such that d Σ = ν. Now, it is easy to see that setting σ = e^x yields \langle ξ, l^α sα \rangle = 0 modulo terms of degree ≥ k + 1 for all α ∈ I^r and ξ ∈ I^1.

The proof follows by induction as in the proof of theorem 4.6 reproduced in appendix C. □

This proposition motivates the following

Definition 4.10. Let g = I ⊕ m be a reductive decomposition of the Lie algebra g, and let G = (g, [, ], w, ϕ) be a Lie quasi-bialgebra.

1. A dynamical I-matrix I satisfying Im ⊂ I and I \ m ⊂ m is said to be compatible with the reductive decomposition g = I ⊕ m.

2. The Lie quasi-bialgebra G is said to be compatible with the reductive decomposition g = I ⊕ m if the following three conditions hold:
   \[ w_ I = 0 \]
   \[ \langle m^⊥, w m^⊥ \rangle = 0 \]
   \[ ϕ ∈ Alt(I ⊗ I ⊗ I ⊕ I ⊗ m ⊗ m ⊗ m ⊗ m ⊗ m) \]

3. A Lie quasi-bialgebra G = (g, [, ], w, ϕ) is said to be canonically compatible with the reductive decomposition g = I ⊕ m if G is compatible with the reductive decomposition g = I ⊕ m and if ϕ ≡ 0 mod I.

4. Assume that g is compatible with the reductive decomposition g = I ⊕ m. Let g’ be a Lie algebra and g’ = I ⊕ m’ a reductive decomposition of g’, and let G’ = (g’, [, ’], w’, ϕ’) be a Lie quasi-bialgebra compatible with the reductive decomposition g’ = I ⊕ m’. A morphism of Lie quasi-bialgebra compatible with a reductive decomposition on I from G to G’ is a Lie quasi-bialgebra morphism ψ: G → G’ such that ψ(z) = z for all z ∈ I and ψm ⊂ m’.

Let G = (g, [, ], w, ϕ) be a Lie quasi-bialgebra compatible with the reductive decomposition g = I ⊕ m. First, notice that for all t ∈ M_{(g,l,m)}, the Lie quasi-bialgebra G^t is canonically compatible with the reductive decomposition g = I ⊕ m, and that g_0^t = I ⊕ m^⊥ is a lagrangian subalgebra of the canonical double d^t of the twisted Lie quasi-bialgebra G^t. Also notice that the quadruple (I, [, ], t, 0, p^{(3)} t) is a cocommutative Lie quasi-bialgebra.

5. Alekseev–Meinrenken dynamical r-matrix associated with a cocommutative Lie quasi-bialgebra

In this section we study the special case I = g. We give a dynamical r-matrix associated with a cocommutative Lie quasi-bialgebra in a neighbourhood of 0 ∈ g^*, which generalizes the Alekseev–Meinrenken dynamical r-matrix r^{AM} discovered in [2] for the compact case, and then adapted to the quadratic case (see e.g., [9, 13]). It was first observed in [8] that r^{AM} only depends on the associator. We give a simplified proof of this fact, obtained as a corollary to the theorems of [4, 13].

If E is a vector space, and f ∈ L(E, E), we denote by Sp_E(f) ⊂ C the spectrum of the endomorphism f of E.

Consider the meromorphic function on C:

\[ F(z) = \coth(z) - \frac{1}{z} \]  

Observe that F is analytic around 0. We recall the Alekseev–Meinrenken theorem (for a proof, see [2] for the compact case, and [9, 13] for the general case):

Theorem 5.1 (Alekseev–Meinrenken). Let (g, [, ]) be a Lie algebra, and Ω ∈ (S^2g)^θ a non-degenerate invariant symmetric 2-tensor. Then the meromorphic map

\[ R: g → \mathcal{A}(g, g) \]

\[ p → F(\text{ad}_p) \]  

(5.2)
satisfies the identity:

\[ d_p R(X)Y - d_p R(Y)X - \Omega^p(X, d_p R(\cdot)Y) - [R_p(X), R_p(Y)] + R_p([R_p(X), Y] + [X, R_p(Y)]) = [X, Y] \quad (5.3) \]

for all \( X, Y \in \mathfrak{g} \) and \( p \) such that \( \text{Sp}_\mathfrak{g}(\text{ad}_p) \cap i\pi \mathbb{Z}^* = \emptyset \).

In equation (5.3), the term \( \Omega^p(X, d_p R(\cdot)Y) \) is the element \( u \) of \( \mathfrak{g} \) such that \( (u, v) = (X, d_p R(v)Y) \) for all \( v \in \mathfrak{g} \), where \((\ , \)\) is the non-degenerate invariant symmetric bilinear form on \( \mathfrak{g} \) associated with \( \Omega \).

As a corollary, we have:

**Corollary 5.2.** Let \( \mathcal{G} = (\mathfrak{g}, [\ , \], 0, \varphi) \) be a cocommutative Lie quasi-bialgebra with double \( \mathfrak{d} \). We consider the \( \text{Ad}^*_G \)-invariant open subset of \( \mathfrak{g}^* \) containing 0:

\[ U = \{ p \in \mathfrak{g}^* \mid \text{Sp}_\mathfrak{g}(\text{ad}_p) \cap i\pi \mathbb{Z}^* = \emptyset \} \quad (5.4) \]

Then, the map:

\[ r^{AM}: U \longrightarrow \mathcal{A}(\mathfrak{g}^*, \mathfrak{g}) \]

\[ p \longmapsto F(\text{ad}^0_p) \quad (5.5) \]

is a dynamical \( r \)-matrix over \( U \) associated with the Lie quasi-bialgebra \( \mathcal{G} \), where \( \text{ad}^0 \) is the adjoint action of the Lie algebra \( \mathfrak{d} \).

**Proof.** Since \( \text{ad}^0_p \mathfrak{g} \subset \mathfrak{g}^* \) and \( \text{ad}^0_p \mathfrak{g}^* \subset \mathfrak{g} \) for all \( p \in \mathfrak{g}^* \), and since \( F \) is an odd function, it is clear that \( F(\text{ad}^0_p) \mathfrak{g}^* \subset \mathfrak{g} \). Also, by the invariance of the (canonical) bilinear form on \( \mathfrak{d} \), and the oddness of the function \( F \), we have \( F(\text{ad}^0_p)i_{\mathfrak{g}^*} \in \mathcal{A}(\mathfrak{g}^*, \mathfrak{g}) \), so that \( r^{AM} \) is well-defined.

Now, if \( g \in G \) and \( p \in U \), using equation (2.24), one finds:

\[ \text{Ad}_g r^{AM}_p \text{Ad}_g^* = \text{Ad}^*_g F(\text{ad}^0_p) \text{Ad}_g^* = F(\text{ad}^0_{\text{Ad}^*_g p}) \text{Ad}^*_g \text{Ad}_g^* = r^{AM}_{\text{Ad}^*_g p} \]

so that \( r^{AM} \) is \( \mathfrak{g} \)-equivariant.

It only remains to show that \( r^{AM} \) satisfies the classical dynamical Yang-Baxter equation. To do so, apply theorem 5.1 to the Lie algebra \((\mathfrak{d}, [\ , \], \varphi)\) with its canonical invariant non-degenerate symmetric bilinear form.

**Corollary 5.3.** Let \( \mathcal{G} = (\mathfrak{g}, [\ , \], \varpi, \varphi) \) be a Lie quasi-bialgebra, and \( \mathcal{D} \) the formal neighborhood of \( 0 \in \mathfrak{g}^* \). Then \( \text{Dynl}(\mathcal{D}, \mathcal{G}) \neq \emptyset \) if and only if \( \varpi \) is an exact 1-cocycle. In this case, the moduli space \( \mathcal{M}(\mathcal{G}, \mathfrak{g}) \) consists of one point.

6. **Dynamical \( \ell \)-matrix compatible with a reductive decomposition**

In this section, we present the most important result of this article, namely, we give an analytic formula for an \( \ell \)-matrix associated with a Lie quasi-bialgebra canonically compatible with a reductive decomposition of a Lie algebra. As a corollary, we show that the embedding of the moduli space of Corollary 4.8 is an isomorphism.

**Theorem 6.1.** Let \( \mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m} \) be a reductive decomposition of the Lie algebra \((\mathfrak{g}, [\ , \])\), and let \( \mathcal{G} = (\mathfrak{g}, [\ , \], \varpi, \varphi) \) be a Lie quasi-bialgebra canonically compatible with the reductive decomposition \( \mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m} \) with canonical double \( \mathfrak{d} \). Let \( U \) be the \( \text{Ad}^*_L \)-invariant open subset of \( \mathfrak{l}^* \) containing 0 defined by:

\[ U = \{ p \in \mathfrak{l}^* \mid \text{Sp}_{\mathfrak{g}\oplus \mathfrak{m}}(\text{ad}^0_{\mathfrak{m} \oplus \mathfrak{l}}) \cap i\pi \mathbb{Z}^* = \emptyset \text{ and } 0 \not\in \text{Sp}_\mathfrak{g}(p_{\mathfrak{g}} \text{Ad}^0_{\mathfrak{e} \oplus \mathfrak{m}} i_{\mathfrak{g}}) \} \quad (6.1) \]

For any \( p \in U \), define:

\[ r^{AM}_p(\alpha + \xi) = r^{AM}_p \alpha - (p_{\mathfrak{g}} \text{Ad}^0_{\mathfrak{e} \oplus \mathfrak{m}} i_{\mathfrak{g}})^{-1} p_{\mathfrak{g}} \text{Ad}^0_{\mathfrak{e} \oplus \mathfrak{m}} \xi \quad (6.2) \]
for \( \alpha \in \mathfrak{t}^* \) and \( \xi \in \mathfrak{l}^\perp \subset \mathfrak{g}^* \), where \( r^{AM} \) is the Alekseev–Meinrenken \( r \)-matrix associated with the Lie quasi-bialgebra \( (\mathfrak{l}, [\cdot, \cdot], \iota, 0, \mathfrak{p}^{(3)}_{\mathfrak{l}} \varphi) \). Then \( l^{can} \) lies in \( \text{Dyn}_0(U, \mathfrak{g}) \) and is compatible with the reductive decomposition \( \mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m} \).

In the sequel, we drop the suffix \( \mathfrak{d} \) of the adjoint action, since no confusion is possible.

Before proving theorem \ref{thm:main_theorem}, we state three lemmas.

**Lemma 6.2.** For all \( p \in U \) and for all \( n \in \mathbb{N} \), we have the following inclusions:

\[
\begin{align*}
\text{ad}^{2n}_{\mathfrak{sp}} \mathfrak{m}^\perp & \subset \mathfrak{m}^\perp, \\
\text{ad}^{2n+1}_{\mathfrak{sp}} \mathfrak{m}^\perp & \subset \mathfrak{m}^\perp, \\
\text{ad}^n_{\mathfrak{sp}} (\mathfrak{m} \oplus \mathfrak{l}^\perp) & \subset \mathfrak{m} \oplus \mathfrak{l}^\perp
\end{align*}
\]

In particular, \( l_p^{can} \), as defined by formula \eqref{eq:lp_can}, satisfies \( l_p^{can} \mathfrak{m}^\perp \subset \mathfrak{l} \) and \( l_p^{can} \mathfrak{l}^\perp \subset \mathfrak{m} \) for all \( p \in U \).

**Proof.** The proof is straightforward using the assumptions made on \( \varpi \) and \( \varphi. \)

**Lemma 6.3.** \( l_p^{can} \), as defined by formula \eqref{eq:lp_can}, is skew-symmetric for all \( p \in U \).

**Proof.** Applying lemma \ref{lem:skew-symmetry} to the vector space decomposition \( \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^* \) and to the automorphism \( \text{Ad}_{a_{\mathfrak{sp}}} \) of \( \mathfrak{d} \) yields:

\[
l_p^{can} \xi = p_{\mathfrak{g}} \text{Ad}_{e_{\mathfrak{sp}}} (p_{\mathfrak{g}} \text{Ad}_{e_{\mathfrak{sp}}} i_{\mathfrak{g}^*})^{-1} \xi (6.3)
\]

for all \( \xi \in \mathfrak{l}^\perp \). Since \( \text{Ad}_{a_{\mathfrak{sp}}}^* = \text{Ad}_{a_{\mathfrak{sp}}} \) and since \( l_p^{can} \xi \in \mathfrak{m} \) by lemma \ref{lem:inclusion}, we have \( l_p^{can} \xi = -(l_p^{can})^* \xi. \)

Obviously, if \( \alpha \in \mathfrak{t}^* \), then \( (l_p^{can})^* s \alpha = -(l_p^{can}) s \alpha. \) Lemma \ref{lem:skew-symmetry} is thus proved.

**Lemma 6.4.** For all \( p \in U \) and \( \xi \in \mathfrak{l}^\perp \),

\[
\text{Ad}_{e_{\mathfrak{sp}}} (l_p^{can} \xi + \xi) \in \mathfrak{g}^* (6.4)
\]

More precisely, one has:

\[
\text{Ad}_{e_{\mathfrak{sp}}} (l_p^{can} \xi + \xi) = (p_{\mathfrak{g}^*} \text{Ad}_{e_{\mathfrak{sp}}} i_{\mathfrak{g}^*})^{-1} \xi (6.5)
\]

for all \( \xi \in \mathfrak{l}^\perp \).

**Proof.** Write \( \text{Ad}_{a_{\mathfrak{sp}}} X = p_{\mathfrak{g}} \text{Ad}_{e_{\mathfrak{sp}}} X + p_{\mathfrak{g}^*} \text{Ad}_{e_{\mathfrak{sp}}} X \) for all \( X \in \mathfrak{d}. \) Then,

\[
\begin{align*}
\text{Ad}_{e_{\mathfrak{sp}}} (l_p^{can} \xi + \xi) & = -p_{\mathfrak{g}} \text{Ad}_{a_{\mathfrak{sp}}} \xi - p_{\mathfrak{g}^*} \text{Ad}_{e_{\mathfrak{sp}}} (p_{\mathfrak{g}} \text{Ad}_{e_{\mathfrak{sp}}} i_{\mathfrak{g}^*})^{-1} p_{\mathfrak{g}} \text{Ad}_{e_{\mathfrak{sp}}} \xi + \text{Ad}_{a_{\mathfrak{sp}}} \xi \\
& = p_{\mathfrak{g}^*} \text{Ad}_{e_{\mathfrak{sp}}} \xi - p_{\mathfrak{g}^*} \text{Ad}_{a_{\mathfrak{sp}}} (p_{\mathfrak{g}} \text{Ad}_{e_{\mathfrak{sp}}} i_{\mathfrak{g}^*})^{-1} p_{\mathfrak{g}} \text{Ad}_{e_{\mathfrak{sp}}} \xi \\
& = p_{\mathfrak{g}^*} \text{Ad}_{e_{\mathfrak{sp}}}(l_p^{can} \xi + \xi)
\end{align*}
\]

Now, by lemma \ref{lem:skew-symmetry}, one has:

\[
\begin{align*}
\text{Ad}_{e_{\mathfrak{sp}}} (l_p^{can} \xi + \xi) & = p_{\mathfrak{g}^*} \text{Ad}_{e_{\mathfrak{sp}}} (p_{\mathfrak{g}^*} \text{Ad}_{e_{\mathfrak{sp}}} (p_{\mathfrak{g}^*} \text{Ad}_{e_{\mathfrak{sp}}} i_{\mathfrak{g}^*})^{-1} \xi + \xi) \\
& = p_{\mathfrak{g}^*} \text{Ad}_{e_{\mathfrak{sp}}} \text{Ad}_{e_{\mathfrak{sp}}} (p_{\mathfrak{g}^*} \text{Ad}_{e_{\mathfrak{sp}}} i_{\mathfrak{g}^*})^{-1} \xi \\
& = (p_{\mathfrak{g}^*} \text{Ad}_{e_{\mathfrak{sp}}} i_{\mathfrak{g}^*})^{-1} \xi
\end{align*}
\]

Lemma \ref{lem:skew-symmetry} is thus proved.

**Proof of theorem 6.7.** We write \( l \) in place of \( l^{can}. \)

We show that the map \( l \) is \( l \)-equivariant: let \( h \in L \). First, notice that \( s \text{Ad}_{\eta_{-1}}^* p = \text{Ad}_{\eta} \mathfrak{p} \) and that \( \text{Ad}_{h} p_{\mathfrak{g}} = p_{\mathfrak{g}} \text{Ad}_{h}, \) since \( \pi_{h} = 0 \) (see equation \eqref{eq:2.24}). Clearly, if \( \alpha \in \mathfrak{t}^* \), then \( l_{\text{Ad}_{h_{-1}}^* p \alpha} \)
\(\text{Ad}_h l_p \text{Ad}_h^* s\alpha\). If \(\xi \in \mathfrak{l}^1\), then
\[
\left[ \text{Ad}_h^{-1} p \xi \right] = \left[ \text{Ad}_h p \xi \right] = -(p_g \text{Ad}_{h e^{-sp} h^{-1}} i_g)^{-1} p_g \text{Ad}_{h e^{-sp} h^{-1}} \xi = -(p_g \text{Ad}_h \text{Ad}_{e^{-sp} h^{-1}} i_g)^{-1} p_g \text{Ad}_h \text{Ad}_{e^{-sp}} \text{Ad}_h^{-1} \xi = -\text{Ad}_h (p_g \text{Ad}_{e^{-sp} i_g})^{-1} p_g \text{Ad}_{e^{-sp}} \text{Ad}_h^* \xi = \text{Ad}_h l_p \text{Ad}_h^* \xi
\]

Thus, \(l\) is \(t\)-equivariant.

Clearly, \(l_0 = 0\), and \(l\) is compatible with the reductive decomposition \(g = \mathfrak{l} \oplus \mathfrak{m}\). Thus, it only remains to show that \(l\) satisfies the generalization of the classical dynamical Yang–Baxter equation (3.3). Notice that we can write this equation as:

\[
d_p l(i^* \xi) \eta - d_p l(i^* \eta) \xi - i d_p (\xi, l_\eta) - [l_p \xi, l_p \eta] + l_p p_g^* [l_p \xi, \eta] + l_p p_g^* [\xi, l_p \eta]
\]

\[
- p_g [l_p \xi, \eta] - p_g [\xi, l_p \eta] + l_p p_g^* [\xi, \eta] = p_g [\xi, \eta]
\]

(6.6)

where the bracket is that of the Lie algebra \(\mathfrak{d}\). We now prove that equation (6.6) is satisfied for all \(\xi, \eta \in \mathfrak{g}^*\):

1. If \(\xi, \eta \in \mathfrak{m}^1\), equation (6.6) holds, by corollary 5.2 and because \(l\) is compatible with the reductive decomposition \(g = \mathfrak{l} \oplus \mathfrak{m}\).
2. If \(\xi = s\alpha\) for some \(\alpha \in \mathfrak{l}^*\) and \(\eta \in \mathfrak{l}^1\), equation (6.6) becomes:

\[
d_p l(\alpha) \eta - [l_p s\alpha, l_p \eta] + l_p [l_p s\alpha, \eta] + l_p p_g^* [s\alpha, l_p \eta] - p_g [s\alpha, l_p \eta] + l_p p_g^* [s\alpha, \eta] = p_g [s\alpha, \eta]
\]

(6.7)

Now, using the differential of the exponential map (see section A.2 in the appendix), the fact that \(\text{Ad}_{e^{-sp}}\) is an automorphism of the Lie algebra \(\mathfrak{d}\), and the property \(p_g[l, \mathfrak{g}^*] = 0\), we obtain:

2a) \(d_p l(\alpha) \eta = (p_g \text{Ad}_{e^{-sp} i_g})^{-1} p_g \left( \left[ \text{sh ad}_{\mathfrak{d}^*} s\alpha, p_g \text{Ad}_{e^{-sp}} (l_p \eta + \eta) \right] \right)\)

2b) \([l_p s\alpha, l_p \eta] = -(p_g \text{Ad}_{e^{-sp} i_g})^{-1} p_g \left( [\text{Ad}_{e^{-sp}} l_p s\alpha, p_g \text{Ad}_{e^{-sp}} \eta] + \left[ \text{ch ad}_{\mathfrak{d}^*} - \frac{s\alpha}{\text{ad}_{\mathfrak{d}^*}} \right] \right) s\alpha, p_g \text{Ad}_{e^{-sp}} l_p \eta)\)

2c) \(l_p [l_p s\alpha, \eta] = (p_g \text{Ad}_{e^{-sp} i_g})^{-1} p_g \left( -[\text{Ad}_{e^{-sp}} l_p s\alpha, p_g \text{Ad}_{e^{-sp}} \eta] + \left[ \text{ch ad}_{\mathfrak{d}^*} - \frac{s\alpha}{\text{ad}_{\mathfrak{d}^*}} \right] \right) s\alpha, p_g \text{Ad}_{e^{-sp}} l_p \eta)\)

2d) \(l_p p_g^* [s\alpha, l_p \eta] = p_g [s\alpha, l_p \eta] + (p_g \text{Ad}_{e^{-sp} i_g})^{-1} p_g \left( [\text{Ad}_{e^{-sp}} s\alpha, p_g \text{Ad}_{e^{-sp}} \eta] - \left[ \text{ch ad}_{\mathfrak{d}^*} s\alpha, p_g \text{Ad}_{e^{-sp}} l_p \eta \right] \right)\)

2e) \(l_p p_g^* [s\alpha, \eta] = p_g [s\alpha, \eta] - (p_g \text{Ad}_{e^{-sp} i_g})^{-1} p_g \text{Ad}_{e^{-sp}} [s\alpha, \eta]\)

Assembling these terms shows that equation (6.7) is satisfied.

3. If \(\xi, \eta \in \mathfrak{l}^1\), equation (6.6) becomes:

\[-i d_p (\xi, l_\eta) - [l_p \xi, l_p \eta] + l_p p_g^* [l_p \xi, \eta] + l_p p_g^* [l_p \xi, \eta] = p_g [l_p \xi, \eta] = p_g [\xi, \eta]\]

(6.8)

By the previous item, we know that the projection on \(\mathfrak{l}\) of this equation is satisfied. So it only remains to consider the projection on \(\mathfrak{m}\), namely:

\[-[l_p \xi, l_p \eta] + l_p p_g^* [l_p \xi, \eta] + l_p p_g^* [l_p \xi, \eta] - p_g [l_p \xi, \eta] - p_g [\xi, \eta] = p_g [\xi, \eta] \in \mathfrak{l}\]

(6.9)
As in the preceding item, one obtains:

(a) \[ [p\xi, lp\eta] = (p_0 \operatorname{Ad}_{\operatorname{sp}} i_0)^{-1} p_0 [p_0 \operatorname{Ad}_{\operatorname{sp}} \xi - p_0^* \operatorname{Ad}_{\operatorname{sp}} lp\xi, p_0 \operatorname{Ad}_{\operatorname{sp}} \eta - p_0^* \operatorname{Ad}_{\operatorname{sp}} lp\eta] \]

(b) \[ lp_0 p_0^* [lp\xi, lp\eta] = p_0 [lp\xi, lp\eta] + (p_0 \operatorname{Ad}_{\operatorname{sp}} i_0)^{-1} p_0 [p_0 \operatorname{Ad}_{\operatorname{sp}} \xi, p_0 \operatorname{Ad}_{\operatorname{sp}} \eta] + [p_0 \operatorname{Ad}_{\operatorname{sp}} \xi, p_0^* \operatorname{Ad}_{\operatorname{sp}} \eta] - [p_0^* \operatorname{Ad}_{\operatorname{sp}} lp\xi, \operatorname{Ad}_{\operatorname{sp}} \eta] \]

(c) \[ lp_0 p_0^* [lp\xi, lp\eta] = p_0 [lp\xi, lp\eta] + (p_0 \operatorname{Ad}_{\operatorname{sp}} i_0)^{-1} p_0 [p_0 [p_0 \operatorname{Ad}_{\operatorname{sp}} \xi, p_0 \operatorname{Ad}_{\operatorname{sp}} \eta] + [p_0^* \operatorname{Ad}_{\operatorname{sp}} lp\xi, p_0 \operatorname{Ad}_{\operatorname{sp}} \eta] - [\operatorname{Ad}_{\operatorname{sp}} \xi, p_0^* \operatorname{Ad}_{\operatorname{sp}} lp\eta] \]

(d) \[ lp_0 p_0^* [lp\xi, lp\eta] = p_0 [lp\xi, lp\eta] - (p_0 \operatorname{Ad}_{\operatorname{sp}} i_0)^{-1} p_0 [\operatorname{Ad}_{\operatorname{sp}} \xi, \operatorname{Ad}_{\operatorname{sp}} \eta] \]

Using \( p_0 [l^\perp, l^\perp] \subset l \), then implies that (6.9) holds.

Thus \( l \) is a dynamical \( \ell \)-matrix, and theorem 6.1 is proved. \( \square \)

Remark 6.5. Theorem 6.1 improves the result of [9] as it provides, on the one hand (see below), explicit analytic dynamical \( r \)-matrices in each formal gauge orbit of [9], and, on the other, explicit \( \ell \)-matrices for certain classes of Lie quasi-bialgebras which are not necessarily quasi-triangular nor cocommutative.

Definition 6.6. Let \( \mathcal{G} = (\mathfrak{g}, [\cdot, \cdot], \varpi, \varphi) \) be a Lie quasi-bialgebra canonically compatible with a reductive decomposition \( \mathfrak{g} = l \oplus m \) of the Lie algebra \( \mathfrak{g} \).

- The dynamical \( \ell \)-matrix provided by theorem 6.1 is called the canonical dynamical \( \ell \)-matrix on \( l \) associated with the Lie quasi-bialgebra \( \mathcal{G} \) compatible with the reductive decomposition \( \mathfrak{g} = l \oplus m \), and will be denoted by \( \ell_{\text{can}}(\mathcal{G}, l, m) \), or simply \( \ell_{\text{can}} \) when no confusion is possible.

- The dynamical Poisson groupoid \( \mathcal{G} = U \times G \times U \) associated with the canonical \( \ell \)-matrix \( \ell_{\text{can}}(\mathcal{G}, l, m) \) on \( U \) is called the canonical dynamical Poisson groupoid associated with the Lie quasi-bialgebra \( \mathcal{G} \) compatible with the reductive decomposition \( \mathfrak{g} = l \oplus m \), and will be denoted by \((\mathcal{G}, \mathcal{G}, l, m)_{\text{can}} \).

As a corollary to theorem 6.1, we obtain:

Corollary 6.7. Let \( \mathfrak{g} = l \oplus m \) be a reductive decomposition of the Lie algebra \( \mathfrak{g} \), and let \( \mathcal{G} = (\mathfrak{g}, [\cdot, \cdot], \varpi, \varphi) \) be a quasi-Lie bialgebra compatible with the reductive decomposition of \( \mathfrak{g} \). Then the embedding \( \mathcal{M}(\mathcal{G}, l) \to \mathcal{M}_{\mathcal{G}, l, m} \) of corollary 4.5 is an isomorphism.

Proof. We only need to show that this map is onto. Let \( \rho \in \mathcal{M}_{\mathcal{G}, l, m} \). It is easy to check that the Lie quasi-bialgebra \( \mathcal{G}^\rho \) is still compatible with the reductive decomposition of \( \mathfrak{g} \), and since \( \varphi^\rho \equiv 0 \) mod \( l \), theorem 6.1 provides a dynamical \( \ell \)-matrix \( \ell_{\text{can}} \in \text{Dynl}_0(D, \mathcal{G}^\rho) \). Now, using proposition 3.6, \( \ell' = \ell_{\text{can}} + \rho \) lies in \( \text{Dynl}(D, \mathcal{G}) \) and is mapped to \( \rho \). \( \square \)

Example 6.8 \( (\ell_{\text{can}} \) for the cocommutative case of [9], [8], see also [3] [12]). (a) Let \( \mathfrak{g} = l \oplus m \) with \( [l, m] \subset m \), and consider the Lie quasi-bialgebra \( \mathcal{G} = (\mathfrak{g}, [\cdot, \cdot], \varpi, \varphi) \) with \( \varphi \in \text{Alt}(l \otimes l \otimes l \otimes m \otimes m) \). We have

\[ p_0(\operatorname{Ad}_{\operatorname{sp}} \xi) = -s \operatorname{ad}_{\operatorname{sp}} \xi \in m, \quad \forall \xi \in l^\perp \quad (6.10) \]
\[ p_0(\operatorname{Ad}_{\operatorname{sp}} u) = \operatorname{ch}_{\operatorname{sp}} u \in m, \quad \forall u \in m \quad (6.11) \]

Therefore,

\[ \ell_{\text{can}}(s \alpha + \xi) = \left( \coth \operatorname{ad}_{\operatorname{sp}} - \frac{1}{\operatorname{ad}_{\operatorname{sp}}} \right) s \alpha + \theta \operatorname{ad}_{\operatorname{sp}} \xi \quad (6.12) \]

(b) Note that (3) applies to the following classical situation: assume that the quadratic Lie algebra \((\mathfrak{g}, B_0)\) admits a splitting

\[ \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-, \quad [\mathfrak{g}_\pm, \mathfrak{g}_\pm] \subset \mathfrak{g}_+, \quad [\mathfrak{g}_+, \mathfrak{g}_-] \subset \mathfrak{g}_- \quad (6.13) \]
Proposition 6.9. Clearly, for all $p \oplus l$ we obtain:

$$g_{\text{quasi-bialgebra structures on } g \xi l \text{ with a reductive decomposition on } g \text{ by equation (5.1) is skew-symmetric). Now, let } \psi \text{ Proof. Let }$$

Proposition 6.10. Proposition 6.9 is thus proved. □

We end this section with two propositions:

\begin{proposition}
Let $g = l \oplus m$ be a reductive decomposition of the Lie algebra $g$, let $\mathcal{G} = (g, [\cdot, \cdot], \varphi)$ be a Lie quasi-bialgebra canonically compatible with the reductive decomposition $g = l \oplus m$. Then,

$$l_{p \text{can}}^{\mathcal{G}^{-}, lm} = -l_{-p \text{can}}^{\mathcal{G}, lm}$$

(6.16)

for all $p \in U$.

\begin{proof}
Clearly, for all $\alpha \in l^{*}$, $l_{p \text{can}}^{\mathcal{G}^{-}, lm} \Omega \alpha = l_{p \text{can}}^{\mathcal{G}, lm} \Omega \alpha = -l_{-p \text{can}}^{\mathcal{G}, lm} \Omega \alpha$ (since the map $F$ defined by equation (5.11) is skew-symmetric). Now, let $\xi \in l^{*}$. Then, using the fact that the map $J$, as defined in section 2.6, is an isomorphism, we have:

$$l_{p \text{can}}^{\mathcal{G}^{-}, lm} \xi = - (p_{g} \text{Ad}_{e^{-p} l}^{\alpha} \xi) = -1_{p \text{can}}^{\mathcal{G}, lm} \xi$$

(6.17)

Proposition 6.9 is thus proved. □

\begin{proposition}
Let $g_{1} = l \oplus m_{1}$ and $g_{2} = l \oplus m_{2}$ be reductive decompositions of the Lie algebras $(g_{1}, [\cdot, \cdot], 1)$ and $(g_{2}, [\cdot, \cdot], 2)$, and let $\mathcal{G}_{1} = (g_{1}, [\cdot, \cdot], 1, \varphi^{1})$ and $\mathcal{G}_{2} = (g_{2}, [\cdot, \cdot], 2, \varphi^{2})$ be two Lie quasi-bialgebra structures on $g_{1}$ and $g_{2}$ canonically compatible with the reductive decompositions $g_{1} = l \oplus m_{1}$ and $g_{2} = l \oplus m_{2}$. Let $\psi: \mathcal{G}_{1} \rightarrow \mathcal{G}_{2}$ be a morphism of Lie quasi-bialgebras compatible with a reductive decomposition on $l$. Then we have:

$$\psi l_{p \text{can}}^{\mathcal{G}_{1}, lm} \psi^{*} = l_{p \text{can}}^{\mathcal{G}_{2}, lm}$$

(6.17)

\begin{proof}
For $j = 1, 2$, we denote by $s_{j}: l^{*} \rightarrow g_{j}^{*}$, the dual of the projection of $g_{j}$ on $l$, along $m_{j}$. Since $\psi z = z$ for all $z \in l$ and since $\psi m_{1} \subset m_{2}$, then $\psi^{*} s_{2} = s_{1} p$ for all $p \in U$. Now, using lemma 2.7, we obtain: $\psi p_{g_{1}} \text{Ad}_{e^{-p} l}^{1} \psi_{g_{1}} = p_{g_{2}} \text{Ad}_{e^{-p} l}^{2} \psi_{g_{2}}$. Thus,

$$\psi p_{g_{1}} \text{Ad}_{e^{-p} l}^{1} \psi_{g_{1}} = (p_{g_{2}} \text{Ad}_{e^{-p} l}^{2})^{-1} \psi$$

(6.18)

By using lemma 2.7 again, we obtain for all $\xi \in l^{*} \subset g_{2}^{*}$:

$$\psi p_{g_{1}} \text{Ad}_{e^{-p} l}^{1} \psi^{*} \xi = p_{g_{2}} \text{Ad}_{e^{-p} l}^{2} \xi$$

(6.19)
Assembling equations (6.18) and (6.19) proves proposition 6.10.

7. TRIVIALIZATION AND DUALITY

7.1. Trivial Lie algebroids. Let \((g, [\cdot, \cdot]_g)\) be a Lie algebra and \(M\) a manifold. Recall (see [17]) that the trivial Lie algebroid on \(M\) with vertex algebra \(g\) is the vector bundle \(A = TM \times g\) over \(M\), where the anchor is the projection of \(A\) on \(TM\) along \(g\) and the bracket is defined as follows:

let \(\sigma\) and \(\sigma'\) be two sections of the vector bundle \(A\), say \(\sigma = (X, x)\) and \(\sigma' = (X', x')\) where \(X\) and \(X'\) are two vector fields on \(M\) and \(x, x' : M \to g\), and set

\[
[\sigma, \sigma']_A = ([X, X'], X \cdot x' - X' \cdot x + [x, x']_g)
\]

(7.1)
The bracket in the first component of the right hand side of equation (7.1) is the bracket of vector fields on \(M\), and \(X \cdot x'\) denotes the derivative of \(x'\) in the direction of \(X\).

7.2. Duality for Lie bialgebroids. Recall that a Lie bialgebroid on a base \(M\) is a pair \((A, A')\) of algebroids \(A\) and \(A'\) over \(M\), together with a non-degenerate pairing between \(A\) and \(A'\), and a supplementary compatibility condition (see [19] for the explicit statement). Now let \((G, \{\cdot, \cdot\})\) be a Poisson groupoid, denote by \(A(G)\) its associated Lie algebroid, and by \(N_G\) the conormal bundle of the unit in \(G\). Note that \(N_G\) is canonically isomorphic to \(A(G)^*\), the dual vector bundle of \(A(G)\).

It follows from Weinstein’s coisotropic calculus (see [22]) that \(N_G\) carries a Lie algebroid structure induced by the Poisson bracket \{ , \} on \(G\) such that the pair \((A(G), N_G)\) is a Lie bialgebroid.

Now, let \((A, A')\) be a Lie bialgebroid over a base \(M\) and denote by \(a\) and \(a'\) the anchors of \(A\) and \(A'\) respectively. One can show (see [19]) the following assertions:

1. the pair \((A', A)\) is a Lie bialgebroid,
2. the map \(a' \circ a^*\) from \(T^* M\) to \(TM\) defines a Poisson bivector on \(M\),
3. \(a \circ (a')^* = -a' \circ a^*\).

By definition (see [18]), the dual Lie bialgebroid of the Lie bialgebroid \((A, A')\) is the Lie bialgebroid \((A', -A)\), where \(-A\) is the Lie algebroid obtained by changing the sign of both anchor and bracket of \(A\). The sign “−” appears in the duality to keep the same induced Poisson structure on the base \(M\), as justified by point 3 above. By definition, a Poisson groupoid dual to a Poisson groupoid \((G, \{\cdot, \cdot\})\) is any (connected, source-simply-connected) Poisson groupoid \((G^*, \{\cdot, \cdot\}^*)\) such that the Lie bialgebroid \((A(G)^*, N_{G^*})\) is the Lie bialgebroid dual to the Lie bialgebroid \((A(G), N_G)\). The dual is unique up to isomorphism, but may not exist (globally) in general.

7.3. Trivialization. Let \(G\) be a connected, simply connected Lie group with Lie algebra \(g\), \(L\) a Lie subgroup of \(G\) with Lie algebra \(l\) and \(U\) an \(L\)-invariant open subset in \(G\). Consider the trivial groupoid \(G = U \times G \times U\), with Poisson structure given by a dynamical \(\ell\)-matrix \(l\) associated with a Lie quasi-bialgebra \(G = (g, [\cdot, \cdot], \varpi, \varphi)\). Recall (see [16], see also [5] for the case \(\varpi = 0\)) that the Lie algebroid of the Poisson groupoid dual to \(G\) is the vector bundle \(N(U) = U \times l \times g^*\) over \(U\) together with the bracket on its sections:

\[
[(z, \xi), (z', \xi')]^N(U) \equiv \left( \begin{array}{l}
  d_p z' (a^N_p(z', \xi')) - d_p z (a^N_p(z, \xi')) - [z, z']_g + \langle \xi, d_p l(z) \cdot \xi' \rangle, \\
  d_p \xi' (a^N_p(z', \xi')) - d_p \xi (a^N_p(z, \xi')) - \text{ad}^*_{z_p} \xi'_p - \text{ad}^*_{z_p} \xi_p \\
  + \langle \xi_p, \varpi \cdot \xi'_p \rangle + \text{ad}^*_{z_p} \xi'_p - \text{ad}^*_{z_p} \xi_p \end{array} \right)
\]

(7.2)

and the anchor:

\[
a^N_p(z, \xi) = i^* \xi - \text{ad}^*_{z_p} p
\]

(7.3)

Since the anchor is a submersion onto \(U\), a theorem of Mackenzie (see [17]) shows that for a contractible base \(U\) this algebroid is trivializable (that is, it is isomorphic to the trivial Lie algebroid \(U \times \mathfrak{h}^* \times \text{Ker} a^N_p\) for any \(p \in U\)).
In this section, we give an explicit trivialization for the dynamical $\ell$-matrix of theorem 6.1 (note that, here, $U$ is not in general contractible). First of all, we need a Lie algebra isomorphism $\phi_p: g_p^* \to g_0^*$, given by the following proposition:

**Proposition 7.1.** The hypotheses and notations are those of theorem 6.1. For all $p \in U$, the map:

\[ \phi_p: g_p^* \to g_0^* \]

is a Lie algebra isomorphism. Thus, the bundle map

\[ \psi: U \times g_0^* \to \text{Ker } a^{N(U)} \subset U \times (l \oplus g^*) \]

\[ (p, X) \mapsto (p, -\phi_p^{-1} X) \]

is a Lie algebra bundle isomorphism.

As in section 6, we drop the suffix $d$ in the adjoint actions.

**Proof.** Let $p \in U$. For any $z \in l$, since $s \text{ad}_z^* p = \text{ad}_{sp} z$, a computation yields (see lemmas 6.2 and 6.3):

\[ \text{Ad}_{e^{-sp}} \tau_p (z + s \text{ad}_z^* p) = \frac{\text{ad}_{sp}}{\text{sh ad}_{sp}} z \in l \]

and for any $\xi \in l^\perp$,

\[ \text{Ad}_{e^{-sp}} \tau_p \xi = (p g^* \text{Ad}_{e^{sp}} i_{g^*})^{-1} \xi \in l^\perp \]

thus $\phi_p$ is well-defined. It is clearly a Lie algebra isomorphism (since $\text{Ad}_{e^{-sp}}$ and $\tau_p$ are Lie algebra isomorphisms). \qed

To complete the trivialization, we need a flat connection:

**Proposition 7.2.** Under the hypotheses and notations of theorem 6.1 and proposition 7.1, the bundle map

\[ \theta: U \times l^* \to N(U) = U \times l \times g^* \]

\[ (p, \alpha) \mapsto \left( p, \frac{\text{sh ad}_{sp} - \text{ad}_{sp}}{\text{ad}_{sp}^2} s\alpha, \frac{\text{sh ad}_{sp} - \text{ad}_{sp}}{\text{ad}_{sp}} s\alpha \right) \]

is a flat connection satisfying:

\[ [\theta \alpha, \psi X]^{N(U)} = \psi (d X(\alpha)) \]

for any smooth section $\alpha \in \Gamma(U \times l^*)$ and $X \in \Gamma(U \times g_0^*)$.

**Proof.** First, notice that

\[ \frac{\text{sh ad}_{sp} - \text{ad}_{sp}}{\text{ad}_{sp}^2} s\alpha = \frac{\text{ch ad}_{sp} - 1}{\text{ad}_{sp}} s\alpha - l_p \frac{\text{sh ad}_{sp}}{\text{ad}_{sp}} s\alpha \]

for all $p \in U$ and $\alpha \in l^*$. For $p \in U$ and $\alpha \in l$, an easy computation shows that $a_p^{N(U)}(\theta(p, \alpha)) = \alpha$. Thus, $\theta$ is a flat connection if and only if:

\[ [\theta(\alpha), \theta(\beta)]^{N(U)} = 0 \]
for all constant sections \( \alpha, \beta \). For \( \alpha, \beta \in \mathfrak{l}^* \) seen as constant sections of the vector bundle \( U \times \mathfrak{l}^* \),

\[
[\theta(\alpha), \theta(\beta)]_{p}^{N(U)} = \left( d_{p} \operatorname{sh} \operatorname{ad}_s - \operatorname{ad}_s \frac{d_{p} \operatorname{sh} \operatorname{ad}_s - \operatorname{ad}_s}{\operatorname{ad}_s^2} (\alpha) s \beta - d_{p} \operatorname{sh} \operatorname{ad}_s - \operatorname{ad}_s \frac{d_{p} \operatorname{sh} \operatorname{ad}_s - \operatorname{ad}_s}{\operatorname{ad}_s^2} (\beta) s \alpha \right.
\]
\[ - \left[ \frac{\operatorname{sh} \operatorname{ad}_s - \operatorname{ad}_s}{\operatorname{ad}_s^2} s \alpha, \frac{\operatorname{sh} \operatorname{ad}_s - \operatorname{ad}_s}{\operatorname{ad}_s^2} s \beta \right] - \left[ \frac{\operatorname{sh} \operatorname{ad}_s - \operatorname{ad}_s}{\operatorname{ad}_s^2} s \beta, \frac{\operatorname{sh} \operatorname{ad}_s - \operatorname{ad}_s}{\operatorname{ad}_s^2} s \alpha \right] \left. \right) ,
\]

\[ d_{p} \frac{\operatorname{sh} \operatorname{ad}_s - \operatorname{ad}_s}{\operatorname{ad}_s^2} (\alpha) s \beta - d_{p} \frac{\operatorname{sh} \operatorname{ad}_s - \operatorname{ad}_s}{\operatorname{ad}_s^2} (\beta) s \alpha
\]

\[ - \left[ \frac{\operatorname{sh} \operatorname{ad}_s - \operatorname{ad}_s}{\operatorname{ad}_s^2} s \alpha, \frac{\operatorname{sh} \operatorname{ad}_s - \operatorname{ad}_s}{\operatorname{ad}_s^2} s \beta \right] - \left[ \frac{\operatorname{sh} \operatorname{ad}_s - \operatorname{ad}_s}{\operatorname{ad}_s^2} s \beta, \frac{\operatorname{sh} \operatorname{ad}_s - \operatorname{ad}_s}{\operatorname{ad}_s^2} s \alpha \right] \left. \right) .
\]

(7.12)

The first equation of lemma \( \text{A}.2 \) shows that the second component of the right hand side of equation (7.12) vanishes. To prove that the first component vanishes too, use equation (7.10), the generalization of the classical dynamical Yang–Baxter equation (3.9), lemma \( \text{A}.2 \) and the \( \mathfrak{l} \)-equivariance equation (3.8). Thus \( \theta \) is a flat connection.

Now, we show that equation (7.9) is satisfied. As in the first part of this proof, we only need to show that equation (7.9) is satisfied for all \( \alpha \in \mathfrak{l}^* \) considered as a constant section:

1. If \( X_{p} = z_{p} \in \mathfrak{l} \) for all \( p \in U \),

\[ [\theta(\alpha), \psi(z)]_{p}^{N(U)} = -[\theta(\alpha), \theta(i^{\ast} \operatorname{ad}_s z)]_{p}^{N(U)} - [\theta(\alpha), (z, 0)]_{p}^{N(U)} \]

\[ = -\theta(\alpha, i^{\ast} \operatorname{ad}_s z)_{p} - [\theta(\alpha), (z, 0)]_{p}^{N(U)} , \text{ since } \theta \text{ is a flat connection} \]

\[ = \psi_{p}(d_{p} z(\alpha)) \]

by lemma \( \text{A}.3 \).

2. A direct computation (using the fact that \( \operatorname{ad}_z p_{\mathfrak{g}^*} = p_{\mathfrak{g}^*} \operatorname{ad}_z \) for all \( z \in \mathfrak{l} \)) shows that equation (7.9) is satisfied for \( X_{p} = \xi_{p} \in \mathfrak{l}^{\perp} \) for all \( p \in U \).

Proposition 7.2 is thus proved.

Assembling proposition 7.1 and proposition 7.2 we get:

**Theorem 7.3.** Under the hypotheses and notations of theorem 6.1, the bundle map:

\[ T: U \times \mathfrak{l}^* \times \mathfrak{g}^*_0 \longrightarrow N(U) = U \times \mathfrak{l} \times \mathfrak{g}^* \]

(7.13)

given by

\[ T_{p}(\alpha, z + \xi) = \left( \frac{\operatorname{sh} \operatorname{ad}_s - \operatorname{ad}_s}{\operatorname{ad}_s^2} s \alpha - \frac{\operatorname{sh} \operatorname{ad}_s - \operatorname{ad}_s}{\operatorname{ad}_s^2} s \beta - \frac{\operatorname{sh} \operatorname{ad}_s - \operatorname{ad}_s}{\operatorname{ad}_s^2} s \alpha - \operatorname{sh} \operatorname{ad}_s z - p_{\mathfrak{g}^*} \operatorname{Ad}_{e^{\xi}} \right) \]

(7.14)

is a Lie algebroid isomorphism.

The inverse \( T^{-1} \) of \( T \) is given by

\[ T_{p}^{-1}(z, s \alpha + \xi) = \left( \alpha - \operatorname{ad}_s z, \frac{\operatorname{sh} \operatorname{ad}_s - \operatorname{ad}_s}{\operatorname{ad}_s \operatorname{sh} \operatorname{ad}_s} s \alpha - \operatorname{sh} \operatorname{ad}_s z - (p_{\mathfrak{g}^*} \operatorname{Ad}_{e^{\xi}} i_{\mathfrak{g}^*})^{-1} \xi \right) \]

(7.15)

for \( z \in \mathfrak{l}, \alpha \in \mathfrak{l}^* \) and \( \xi \in \mathfrak{l}^{\perp} \).
7.4. Duality. We start with a definition of a duality for Lie quasi-bialgebras:

**Definition 7.4.** Let $\mathcal{G} = (\mathfrak{g}, [\cdot, \cdot], \varpi, \varphi)$ be a Lie quasi-bialgebra with canonical double $\mathfrak{d}$, and assume that $\mathfrak{l}$ is a Lie subalgebra of $\mathfrak{g}$ such that $\varpi_l = 0$ and $\varphi \equiv 0 \mod \mathfrak{l}$. The Lie quasi-bialgebra

$$
\mathcal{G}^* = \left( \mathcal{G}(\mathfrak{d} \oplus \mathfrak{l}^\perp, m \oplus m) \right)^-
$$

is called the dual over $\mathfrak{l}$ of the Lie quasi-bialgebra $\mathcal{G}$.

Observe that if $\varpi_l = 0$ and $\varphi \equiv 0 \mod \mathfrak{l}$ then $\mathfrak{g}^* = \mathfrak{l} \oplus \mathfrak{l}^\perp$ is indeed a Lagrangian subalgebra of $\mathfrak{d}$, so that the dual over $\mathfrak{l}$ is well-defined. Also observe that if a Lie quasi-bialgebra is canonically compatible with a reductive decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$ (see definition 1.10), then its dual over $\mathfrak{l}$ is also canonically compatible with the reductive decomposition $\mathfrak{g}^* = \mathfrak{l} \oplus \mathfrak{l}^\perp$.

Let $\text{op}: \mathfrak{g} \to \mathfrak{g}$ be the standard involution associated to the canonical double involutive (involutive) isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$, the Lie algebra $\mathfrak{g}$ is isomorphic to the Lie subalgebra $\mathfrak{g}^{\text{op}} = \mathfrak{l} \oplus \mathfrak{m}$ of $\mathfrak{d}^*$. Second, observe that under the canonical identification $\mathfrak{d}^* \simeq \mathfrak{d}$, then $(\mathcal{G}^*)^* \neq \mathcal{G}$, but rather $(\mathcal{G}^*)^* = \mathcal{G}^{\text{op}}$, which is isomorphic to $\mathcal{G}$.

We now turn to our main duality statement which provides the dual Poisson groupoid of a Poisson groupoid associated with a canonical $\ell$-matrix:

**Theorem 7.5.** Under the hypotheses and notations of theorem 6.1, the dual Poisson groupoid of the dynamical Poisson groupoid associated with the canonical $\ell$-matrix $\ell^{\text{can}}$ is (isomorphic to) the connected, source-simply-connected covering of the dynamical Poisson groupoid $U \times \mathcal{G}^* \times U$ with the Poisson structure associated with the canonical $\ell$-matrix on $U$ for the Lie quasi-bialgebra $\mathcal{G}^*$, where $G^*$ is the connected, simply connected Lie group with Lie algebra $\mathfrak{g}^*$.

**Proof.** The Poisson bracket on the dual groupoid $G^*$ is uniquely determined (up to automorphism) by the requirement that the trivialization map $T$ of theorem 4.3 is a Lie bialgebroid isomorphism, that is, by the condition that the map $-T^*$ is a Lie bialgebroid isomorphism from the Lie algebroid $A(G)$ of $G$ to the Lie algebroid $N_G^*({U})$ (the conormal bundle of the unit of the Poisson groupoid $G^*$). We compute $-T^*: U \times I^* \times \mathfrak{g} \to U \times I \times (\mathfrak{g}^*)^* \simeq U \times I \times (I^* \oplus \mathfrak{m})$:

$$
-T^*_p(\alpha, z + u) = \left( \frac{\text{sh} \text{ad}_{sp} - \text{ad}_{sp}}{\text{ad}_{sp}^2} s\alpha - \frac{\text{sh} \text{ad}_{sp}}{\text{ad}_{sp}} z, \frac{\text{sh} \text{ad}_{sp}}{\text{ad}_{sp}} s\alpha - \text{sh ad}_{sp} z + p_g \text{Ad}_{e-sp} u \right)
$$

(7.16)

where $\alpha \in I^*$, $z \in I$ and $u \in \mathfrak{m}$. Here, the adjoint action is that of the double $\mathfrak{d}$ of $G$ on itself.

Let $G^{\text{op}}$ be the trivial groupoid $U \times G^{\text{op}} \times U$, and denote by $T^*: A(G^{\text{op}}) \to U \times I \times (\mathfrak{g}^*)^*$ the trivialization associated with the data $G^*$ and $\mathfrak{g}^* = I \oplus I^\perp$ given by theorem 4.3. The Lie algebroid isomorphism $T^*$ is given by:

$$
T^*_p(\alpha, z + u) = \left( \frac{\text{sh} \text{ad}_{sp} - \text{ad}_{sp}}{(\text{ad}_{sp})^2} s\alpha - \frac{\text{sh} \text{ad}_{sp}}{\text{ad}_{sp}} z, \frac{\text{sh} \text{ad}_{sp}}{\text{ad}_{sp}} s\alpha - \text{sh ad}_{sp} z - p(\varphi^*), \text{Ad}_{e^{sp}} u \right)
$$

(7.17)

Here, $\text{ad}^*$ denotes the adjoint action of the double $\mathfrak{d}$ of $G^*$ on itself. Denote by $J^*$ the linear isomorphism from $\mathfrak{d}$ to $\mathfrak{g}^*$ defined by $J^*(z + \xi) = z + \xi$ for all $z + \xi \in \mathfrak{g}^*$ and $J^*(s\alpha + u) = -s\alpha - u$ for all $\alpha \in I^*$ and $u \in \mathfrak{m}$ (we use the canonical vector space identification $\mathfrak{d}^* \simeq \mathfrak{d}$). Recall that $J^*$
is a Lie algebra isomorphism from $\mathfrak{d}^*$ to $\mathfrak{d}$ (see section 2.6). Using $J^*$, equation (7.17) reads:

\[
T_p^* (\alpha, z + u) = \left( J^* \frac{\text{sh ad}_{sp}^* - \text{ad}_{sp}^*}{(\text{ad}_{sp}^*)^2} \text{sox} - J^* \frac{\text{sh ad}_{sp}^* z - \text{sh ad}_{sp}^* \text{sox} + J^* \text{sh ad}_{sp}^* z + p_g J^* \text{Ad}_{e^p}^* u} {\text{ad}_{sp}^*} \right)
\]

Now, denote by $\partial \mathcal{P}: U \times \mathfrak{g} \rightarrow U \times \mathfrak{g}^\text{op}$ the trivial Lie algebroid isomorphism given by: $\partial \mathcal{P}(\alpha, z + u) = (\alpha, z - u)$. Clearly, $-T^* \circ \partial \mathcal{P} = T^*$. Thus, $-T^*$ is a Lie algebroid isomorphism.

Example 7.6 (Dual Lie quasi-bialgebras for Etingof–Varchenko dynamical r-matrices). Let $\mathfrak{g}$ be a complex simple Lie algebra with Killing form $B_\mathfrak{g}$, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, $\Delta$ (resp., $\Delta^\pm$) the set of roots (resp., simple roots). Denote by $\langle \Gamma \rangle \subset \Delta$ the root span of a fixed subset $\Gamma \subset \Delta^\pm$, and set $\Gamma^\pm = \Delta^\pm \setminus \langle \Gamma \rangle^\pm$, where $\Delta^\pm$ denotes positive and negative roots. Let

\[
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha
\]

be the root space decomposition of $\mathfrak{g}$. Denote by $(x_i)_{1 \leq i \leq \text{rank } \mathfrak{g}}$ an orthonormal basis of $\mathfrak{h}$ and choose root vectors $(e_\alpha)_{\alpha \in \Delta}$ such that $B_\mathfrak{g}(e_\alpha, e_{-\alpha}) = 1$.

In [10], Etingof and Varchenko have shown that, up to gauging, analytic dynamical r-matrices at 0 associated with the Lie quasi-bialgebra $\mathcal{G} = (\mathfrak{g}, [\cdot, \cdot], 0, \frac{1}{4}(\Omega, \Omega))$ are given by

\[
R^{\text{EV}}_q (\eta) = \sum_{i,j} C_{ij}(q) \langle x_j, \eta \rangle x_i + \sum_{\alpha \in \Delta} \phi_\alpha(q)(e_{-\alpha}, \eta) e_{\alpha}
\]

for all $\eta \in \mathfrak{g}^*$, where

\[
\phi_\alpha(q) = \frac{1}{2} \coth \left( \frac{\alpha, q - \mu}{2} \right), \quad \forall \alpha \in \langle \Gamma \rangle,
\]

\[
\phi_\alpha(q) = \pm \frac{1}{2}, \quad \forall \alpha \in \Gamma^\pm.
\]

Here, $\sum_{i,j} C_{ij} \, d x_i \otimes d x_j$ is an arbitrary closed analytic 1-form on $\mathfrak{h}^*$ vanishing at 0, and $\mu \in \mathfrak{h}^*$ lies in the complement of the singular hyperplanes $\langle \alpha, \mu \rangle = 0$, $\alpha \in \langle \Gamma \rangle$.

Let $\rho = R^{\text{EV}}_0$. Note that $R^{\text{EV}}_0$ lies in the algebraic variety $M^{\text{ev}, \text{im}}_{\mathcal{G}, 1, \text{m}}$ of $\mathfrak{g}$. By corollary 6.7, $R^{\text{EV}}$ is (formally) gauge equivalent to $l^{\text{can}} + \rho$, where $l^{\text{can}}$ is the canonical $\ell$-matrix associated with the twisted Lie quasi-bialgebra $\mathcal{G}^\rho = (\mathfrak{g}, [\cdot, \cdot], \partial \rho, \varphi^\rho)$, and hence, the Poisson groupoids associated with $l^{\text{can}}$ and $R^{\text{EV}}$ are (formally) isomorphic. In particular, the dual Poisson groupoid is given (up to a formal isomorphism) by Theorem 6.2.

We now give the pair of dual Lie quasi-bialgebras $(\mathcal{G}^\rho, \mathcal{G}^\ast)$. In the formulae below, we use the identifications $\mathfrak{g}^* \simeq \mathfrak{g}$, $\mathfrak{h} \oplus \mathfrak{h}^\perp \simeq \mathfrak{h} \oplus \mathfrak{n}$ and $\mathfrak{m}^\perp \oplus \mathfrak{m} \simeq \mathfrak{h} \oplus \mathfrak{n}$ induced by the Killing form.

For $\mathcal{G}^\rho$, we have, for all $x, y \in \mathfrak{g}$,

\[
\partial \rho_{xy} = [x, \rho(y)]_\mathfrak{g} - \rho([x, y])_\mathfrak{g}
\]

\[
(x \otimes y \otimes 1, \varphi^\rho) = \frac{1}{4} [x, y]_\mathfrak{g} + [\rho(x), \rho(y)]_\mathfrak{g} - \rho([\rho(x), y])_\mathfrak{g} + [x, \rho(y)]_\mathfrak{g}
\]

The structural data for $\mathcal{G}^\ast = (\mathfrak{g}^* = \mathfrak{h} \oplus \mathfrak{h}^\perp, [\cdot, \cdot], \varphi^\ast, \varphi^*)$ are as follows: the Lie bracket $[\cdot, \cdot]$ of $\mathfrak{h} \oplus \mathfrak{h}^\perp \simeq \mathfrak{h} \oplus \mathfrak{n}$ is that of the double $\mathfrak{d}$ for $\mathcal{G}^\rho$:

\[
[z, z']^* = 0
\]

\[
[z, u]^* = [z, u]_\mathfrak{g}
\]

\[
[u, u']^* = (u \otimes u' \otimes 1, \varphi^\rho) + (u', \partial \rho u)
\]
for $z, z' \in h$ and $u, u' \in n$.

Put $\phi_\alpha = \phi_\alpha(0)$. The bracket $[\cdot, \cdot]^{\ast}_{[n \times n]}$ then reads as

\[
[e_\alpha, e_\beta]^\ast = (\phi_\alpha + \phi_\beta)[e_\alpha, e_\beta]_g
\]

\[
[e_\alpha, e_{-\alpha}]^\ast = (\frac{1}{4} - \phi_\alpha^2)[e_\alpha, e_{-\alpha}]_g
\]

for all $\alpha, \beta \in \Delta$ such that $\alpha + \beta \neq 0$, which implies that $\mathfrak{g}^\ast$ is isomorphic to the semi-direct product

$I_\Gamma \ltimes (n_1^+ \otimes n_1^-)$

where

\[
I_\Gamma = h \oplus \bigoplus_{\alpha \in \Gamma} \mathfrak{g}^\alpha
\]

is the Levi factor and $n_1^\pm$ are the corresponding nilpotent radicals (a fact already observed in [15]).

The cocycle $\varpi^\ast$ is given by $-p_{b \oplus h^\perp}[z + u, z' + u']_b$, for $z + u \in h \oplus h^\perp$, and $z' + u' \in m^\perp \oplus m$:

\[
\varpi^\ast_{z+u, z'+ u'} = p_h \partial \rho_{u'} u - p_h (u \otimes z' \otimes 1, \varphi^0) - (z', \partial \rho u) + p_n[u, u]_g
\]

(7.22)

which explicitly reads as

\[
\varpi^\ast_{z, z'} = 0
\]

\[
\varpi^\ast_{e_\alpha, z} = \phi_\alpha [z, e_\alpha]_g
\]

\[
\varpi^\ast_{e_\beta, e_\alpha} = -[e_\alpha, e_\beta]_g
\]

\[
\varpi^\ast_{e_{-\alpha}, e_\alpha} = -\phi_\alpha [e_\alpha, e_{-\alpha}]_g
\]

for $z \in h$, and for all $\alpha, \beta \in \Delta$ such that $\alpha + \beta \neq 0$.

The associator $\varphi^\ast$ is given by $p_{b \oplus h^\perp}[z + u, z' + u']_b$, for $z + u, z' + u' \in m^\perp \oplus m$:

\[
(u \otimes u' \otimes 1, \varphi^\ast) = p_h[u, u']_g
\]

\[
(u \otimes z' \otimes 1, \varphi^\ast) = p_h \partial \rho_{u'} z' + [u, z']_g = [u, z']_g
\]

\[
(z \otimes z' \otimes 1, \varphi^\ast) = p_h (z \otimes z' \otimes 1, \varphi^0) + (z', \partial \rho z) = 0
\]

One checks that, when $\Gamma \neq \Delta$, the cocycle $\varpi^\ast : \mathfrak{g}^\ast \rightarrow \mathcal{L}(m^\perp \oplus m, \mathfrak{g}^\ast)$ is not exact, thus providing a genuine example of non-exact Lie quasi-bialgebra which is compatible with a reductive decomposition.

7.5. **Link with the duality of symmetric space.** In this section, we show that the duality of symmetric spaces (see [15]), which relies on duality for orthogonal symmetric Lie algebras, is related to the duality of quasi-bialgebra introduced in definition [1.3]. We first recall the definition of orthogonal symmetric Lie algebras (see [15]).

**Definition 7.7.** An orthogonal symmetric Lie algebra is a pair $(\mathfrak{g}, \sigma)$ where $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$, $\sigma$ is an involutive automorphism of $\mathfrak{g}$, and $l$, the set of fixed points of $\sigma$ is a compactly imbedded subalgebra of $\mathfrak{g}$.

When $(\mathfrak{g}, \sigma)$ is an orthogonal symmetric Lie algebra, we consider the reductive splitting $\mathfrak{g} = l \oplus m$ of $\mathfrak{g}$ into the eigenspaces of $\sigma$ for the eigenvalue $+1$ and $-1$ respectively as in example [6.8].

There is a notion of duality for orthogonal symmetric Lie algebras which goes as follows: let $(\mathfrak{g}, \sigma)$ be an orthogonal symmetric Lie algebra. Let $\mathfrak{g}^\ast$ denote the subset $l + im$ of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$. Notice that $\mathfrak{g}^\ast$ is a (real) Lie subalgebra of $\mathfrak{g}^C$, since $[m, m] \subset l$. Now, the mapping $\sigma^\ast : z + iu \mapsto z - iu$ is an involutive automorphism of $\mathfrak{g}^\ast$, and the pair $(\mathfrak{g}^\ast, \sigma^\ast)$ is again an orthogonal symmetric Lie algebra, called the dual of the orthogonal symmetric Lie algebra $(\mathfrak{g}, \sigma)$.

We can translate this duality when $\mathfrak{g}$ is semi-simple: let $(\mathfrak{g}, \sigma)$ be an orthogonal symmetric Lie algebra, and assume that $\mathfrak{g}$ is a semi-simple Lie algebra. We consider the complexified Lie algebra $\mathfrak{g} = \mathfrak{g}^C$, which we view as a real Lie algebra. We consider the bilinear, symmetric, invariant,
and non-degenerate bilinear form on \( \mathfrak{d} \): \( (\ , \)_0 = \Im B \), where \( B \) is the Killing form on \( \mathfrak{g} \). Then \( \mathfrak{g} \) is a lagrangian Lie subalgebra of \( \mathfrak{d} \), and \( i\mathfrak{g} \) is an isotropic complement. We set \( \mathcal{G} = \mathcal{G}((\mathfrak{g},\mathfrak{g},\mathfrak{i})) \).

Clearly, the Lie quasi-bialgebra \( \mathcal{G} \) is cocommutative, and its associator \( \varphi \) is given by \( \varphi = -(\Omega,\Omega) \), where \( \Omega \in (\mathfrak{g}^*\mathfrak{g})^0 \) is the Casimir element of \( \mathfrak{g} \), and \( (\ , \) \) is Drinfel’d’s bracket. Now, since \( \Omega \) lies in \( \mathfrak{i} \otimes \mathfrak{l} \oplus \mathfrak{m} \otimes \mathfrak{m} \), and since \([\mathfrak{m},\mathfrak{m}] \subset \mathfrak{l} \), the Lie quasi-bialgebra \( \mathcal{G} \) is canonically compatible with the reductive decomposition \( \mathfrak{g} = \mathfrak{i} \oplus \mathfrak{m} \). The dual of the orthogonal symmetric Lie algebra \((\mathfrak{g},\sigma)\) is \((\mathfrak{g}^*,\sigma^*)\), where \( \mathfrak{g}^*_0 \) is the underlying Lie algebra of \( \mathcal{G}^* \), the dual of the quasi-bialgebra \( \mathcal{G} \), and where \( \sigma^* \) is the standard involution associated to the reductive decomposition on \( \mathfrak{g}^* \). We can note that \( \mathfrak{g}^*_0 \) is still semi-simple (use Cartan’s criterion), but that \( \mathfrak{g} \) and \( \mathfrak{g}^* \) are not isomorphic in general: this can be seen by comparing the signature of the Killing form on \( \mathfrak{g} \) and on \( \mathfrak{g}^*_0 \). Thus, the Lie quasi-bialgebra \( \mathcal{G} \) is not self-dual, nor is the dynamical Poisson groupoid \( \mathcal{G} \) associated to \( \mathcal{G} \).

**Appendix A. Complements**

**A.1. A linear algebra result.** To prove that the map defined by equation (5.2) is skew-symmetric, we use the following lemma:

**Lemma A.1.** Let \( E \) be a vector space, \( F \) and \( F' \) two subspaces such that \( E = F \oplus F' \), and denote by \( p \) and \( p' \) the projections on \( F \) and \( F' \) along \( F' \) and \( F \) and by \( i \) and \( i' \) the inclusions of \( F \) and \( F' \) into \( E \). Let \( f \) be an automorphism of \( E \), and assume that \( pf_i \) and \( p'f^{-1}i' \) are automorphisms of \( F \) and \( F' \). Then:

\[
(pf_i)^{-1}pf'i' = -pf^{-1}i'(p'f^{-1}i')^{-1}
\]

**Proof.** We start with the relation:

\[
\mathcal{L}(F',F) \ni 0 = pf f^{-1}i' = (pf_i)pf^{-1}i' + pf'i'(p'f^{-1}i')^{-1}
\]

(A.2)

Since \( pf_i \) and \( p'f^{-1}i' \) are automorphisms of \( F \) and \( F' \), one obtains equation (A.1) by multiplying \( (pf_i)^{-1} \) and \( (p'f^{-1}i')^{-1} \) on the left and on the right of equation (A.2).

**A.2. Differential of the exponential map.** For the convenience of the reader we recall the expression of the differential of the exponential map (for a proof see e.g., [15]):

Let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \), and denote by \( \exp: \mathfrak{g} \to G \) the exponential map. For any \( x \in \mathfrak{g} \), the differential of \( \exp \) at \( x \) is given by

\[
T_x \exp(u) = \ell_{\exp,[x]} \frac{1 - e^{-\text{ad}_x}}{\text{ad}_x} u
\]

(A.3)

for all \( u \in \mathfrak{g} \). In particular, one has:

\[
d_x \text{Ad}_{e^x}(u) = \text{Ad}_{e^x} \frac{\text{ad}_{1 - \text{Ad}_{e^{-x}}}}{\text{ad}_x} u
\]

(A.4)

for all \( u \in \mathfrak{g} \).

**A.3. More differential identities.** We prove some differential identities which are used to prove proposition (7.2):

**Lemma A.2.** For all \( p \in U \) and \( \alpha, \beta \in \mathfrak{l}' \), the following three equations hold:

\[
d_x \text{ad}^n(u)v = \sum_{i=0}^{n-1} \binom{n}{i+1} [\text{ad}_x u, \text{ad}_x^{n-i-1} v]
\]

(A.5)

\[
d_p \text{sh}_{\text{ad}_x}(\alpha)s\beta - d_p \text{sh}_{\text{ad}_x}(\beta)s\alpha =
\]

\[
\left[ \frac{\text{ch}_{\text{ad}_p}}{\text{ad}_p} s\alpha, \frac{\text{sh}_{\text{ad}_p}}{\text{ad}_p} s\beta \right] + \left[ \frac{\text{sh}_{\text{ad}_p}}{\text{ad}_p} s\alpha, \frac{\text{ch}_{\text{ad}_p}}{\text{ad}_p} s\beta \right]
\]

(A.6)
\[
\frac{d_p}{\text{ad}_s} (\alpha) s\beta - \frac{d_p}{\text{ad}_s} (\beta) s\alpha = \left[ \frac{\text{sh ad}_sp}{\text{ad}_sp} s\alpha, \frac{\text{sh ad}_sp}{\text{ad}_sp} s\beta \right] + \left[ \frac{\text{ch ad}_sp - 1}{\text{ad}_sp} s\alpha, \frac{\text{ch ad}_sp - 1}{\text{ad}_sp} s\beta \right]
\]  

(A.7)

Proof. To prove equation \((A.5)\), write:

\[
d_x \text{ad}^n(u)v = \sum_{i=0}^{n-1} \text{ad}_x^n[u, \text{ad}_x^{n-i-1} v]
\]  

(A.8)

and use Leibniz’ relation:

\[
\text{ad}_x^n[u, v] = \sum_{i=0}^{n} \binom{n}{i} [\text{ad}_x^i, \text{ad}_x^{n-i} v]
\]  

(A.9)

and the identity

\[
\sum_{i=0}^{n-k} \binom{i+k}{k} = \binom{n+1}{k+1}
\]  

(A.10)

Recall that the expressions \(\frac{\text{sh ad}_sp}{\text{ad}_sp}\) and \(\frac{\text{ch ad}_sp - 1}{\text{ad}_sp}\) are respectively the even and odd parts of \(\frac{\text{Ad}_sp - 1}{\text{ad}_sp}\). Using lemma \(A.2\) we compute:

\[
d_p \frac{\text{Ad}_sp - 1}{\text{ad}_sp} (\alpha) s\beta - \frac{d_p}{\text{ad}_s} \frac{\text{Ad}_sp - 1}{\text{ad}_sp} (\beta) s\alpha = \sum_{n \geq 0} \frac{1}{(n+2)!} \left( \frac{d_p}{\text{ad}_s} \text{ad}_s^{n+1}(\alpha) s\beta - \frac{d_p}{\text{ad}_s} \text{ad}_s^{n+1}(\beta) s\alpha \right)
\]

\[
= \sum_{n \geq 0} \sum_{i=0}^{n} \frac{1}{(n+2)!} \binom{n+2}{i+1} [\text{ad}_s^i s\alpha, \text{ad}_s^{n-i} s\beta]
\]

\[
= \left[ \frac{\text{Ad}_sp - 1}{\text{ad}_sp} s\alpha, \frac{\text{Ad}_sp - 1}{\text{ad}_sp} s\beta \right]
\]

Now, selecting respectively odd and even parts of this expression yields relations \((A.6)\) and \((A.7)\).

The following lemma is used to prove relation \((7.9)\) of proposition \(7.2\).

**Lemma A.3.** For all entire functions \(f\), the identity

\[
\frac{d}{dt} \bigg|_{t=0} f(\text{ad}_s t \text{ad}_sp z) = f(\text{ad}_sp) \text{ad}_s z - \text{ad}_s f(\text{ad}_sp)
\]

(A.11)

holds for all \(z \in \mathfrak{g}\) and for all \(p \in U\).

Proof. Let \(f\) be an entire function, say \(f(x) = \sum_{n \geq 0} f_n x^n\). Then, for \(v \in \mathfrak{g}\) and \(z \in \mathfrak{g}\), using lemma \(A.2\) one computes:

\[
d_{\text{ad}_sp} f(\text{ad}_sp z)v = \sum_{n \geq 1} f_n \text{ad}_sp^n(\text{ad}_sp z)v
\]

\[
= \sum_{n \geq 1} f_n \sum_{i=0}^{n-1} \binom{n}{i} [\text{ad}_sp^i z, \text{ad}_sp^{n-i} v]
\]

\[
= \sum_{n \geq 0} f_n (\text{ad}_sp^n[z, v] - [z, \text{ad}_sp^n v])
\]

Lemma \(A.3\) is thus proved.  

□
Appendix B. Proof of Lemma 2.7

Observe that equation (2.18) is the dual of equation (2.17), so the two are equivalent. We prove lemma 2.7 by induction on $n$: for $n = 1$, the relations (2.17), (2.19) and (2.20) hold, by definition of a quasi-bialgebra morphism. Assume that these relations hold for some $n \in \mathbb{N}$. Then,

$$
\psi p_{g_1} \left( (ad^{1}_{\psi \ast \xi})^{n+1} \right) u
= \psi p_{g_1} \left( (ad^{1}_{\psi \ast \xi})^{n} \right) u + \psi p_{g_1} \left( (ad^{1}_{\psi \ast \xi})^{n} \right) u
= p_{g_2} \left( ad^{2}_{\psi} \right) p_{g_2} \left( (ad^{2}_{\psi})^{n} \right) u + p_{g_2} \left( ad^{2}_{\psi} \right) p_{g_2} \left( (ad^{2}_{\psi})^{n} \right) u
$$

which proves (2.17) at rank $n + 1$,

$$
p_{g_1} \left( (ad^{1}_{\psi \ast \xi})^{n+1} \right) \psi^* \eta
= p_{g_1} \left( (ad^{1}_{\psi \ast \xi})^{n} \right) \psi^* \eta + p_{g_1} \left( (ad^{1}_{\psi \ast \xi})^{n} \right) \psi^* \eta
= p_{g_2} \left( ad^{2}_{\psi} \right) p_{g_2} \left( (ad^{2}_{\psi})^{n} \right) \eta + p_{g_2} \left( ad^{2}_{\psi} \right) p_{g_2} \left( (ad^{2}_{\psi})^{n} \right) \eta
$$

which proves (2.19) at rank $n + 1$, and

$$
\psi p_{g_1} \left( (ad^{1}_{\psi \ast \xi})^{n+1} \right) \psi^* \eta
= \psi p_{g_1} \left( (ad^{1}_{\psi \ast \xi})^{n} \right) \psi^* \eta + \psi p_{g_1} \left( (ad^{1}_{\psi \ast \xi})^{n} \right) \psi^* \eta
= p_{g_2} \left( ad^{2}_{\psi} \right) p_{g_2} \left( (ad^{2}_{\psi})^{n} \right) \eta
$$

which proves (2.20) at rank $n + 1$. Lemma 2.7 is thus proved.

Appendix C. Proof of Theorem 3.6

Let $k \geq 1$, and assume that $l = l'$ modulo terms of degree $\geq k$. We show that there exists $\sigma \in \text{Map}_{\mathbb{D}}(\mathbb{D}, G)^1$ such that $l^\sigma = l'$ modulo terms of degree $\geq k + 1$: for all $\xi \in \mathbb{D}^1$, one has $p_{m l} \xi = p_{m l'} \xi$ modulo terms of degree $\geq k + 1$. Indeed, equation (3.7) shows that the term of degree $k - 1$ of $(\xi, d l(\alpha) \eta)$ only depends on the terms of degree $\leq k - 1$ of $l$. Thus, for all $\alpha \in \mathbb{I}^*$ and $\xi, \eta \in \mathbb{I}^1$, one has $\langle \xi, d l(\alpha) \eta \rangle = \langle \xi, d l'(\alpha) \eta \rangle$ modulo terms of degree $\geq k$, and since $l_0 = l'_0$, the equality $p_{m l} \xi = p_{m l'} \xi$ holds modulo terms of degree $\geq k + 1$. Now, if $\Sigma: \mathbb{D} \rightarrow g$ is an $l$-equivariant homogeneous map of degree $k + 1$, set $\sigma = \Sigma^\mathbb{D}$. Then $\sigma$ is $l$-equivariant, and one checks that $l^\sigma = l + d \Sigma^* - (d \Sigma)^*$ modulo terms of degree $\geq k + 1$. We show that there exists such a map $\Sigma$ such that $l^\sigma = l'$ modulo terms of degree $\geq k + 1$: we define a 2-form $\mu$ on $\mathbb{D}$ (with values in the ground field) by setting $(\mu, \alpha \wedge \beta) = (s_{\alpha} \cdot l(l - l) s_{\beta}) k$, for $\alpha, \beta \in \mathbb{I}^*$, and we define a 1-form $\nu$ on $\mathbb{D}$ with values in $g$ by setting $(\nu, \alpha) = p_{m l}(l(l - l)) k s_{\alpha}$, for $\alpha \in \mathbb{I}^*$. Here, the bracket $\cdot|_{k}$ means to select the homogeneous term of degree $k$. Since $\Sigma^m = 0$ and $g = l \oplus m$ is a reductive decomposition, both $\mu$ and $\nu$ are $l$-equivariant forms (the scalar field is seen as a trivial $l$-module). Using equation (5.7).
and the assumption that $l = l'$ modulo terms of degree $\geq k$, we check that they are closed forms:
\[
\begin{align*}
\langle l, \alpha \rangle (\gamma) &= [\langle s\alpha, d(l' - l)(\gamma) s\beta \rangle]_{k-1} \\
 &= [\langle s\alpha, d(l' - l)(\gamma) s\beta \rangle]_{k-1} + [\langle s\gamma, d(l' - l)(\alpha) s\beta \rangle]_{k-1} \\
 &= \langle l, \alpha \rangle (\gamma) + \langle l, \alpha \rangle (\gamma) \\
\end{align*}
\]
Thus, $\mu$ is an $l$-equivariant closed 2-form on $D$ and $\nu$ is an $l$-equivariant closed 1-form on $D$ with values in $\mathfrak{g}$. Thus, by the equivariant Poincaré lemma, there exists a homogeneous $l$-equivariant 1-form $\chi$ on $D$ of degree $k + 1$ such that $d^{Rh} \chi = \mu$, and a homogeneous $l$-equivariant map $\lambda : D \to \mathfrak{g}$ of degree $k + 1$ such that $d \lambda = \nu$. The 1-form $\chi$ may be seen as an $l$-equivariant map from $D$ to $l$, which will be denoted by $\overline{\chi}$. Now, set $\Sigma_p = \overline{\chi}_p + \lambda_p$. We check that setting $\sigma = e^{\Sigma}$ yields $l^\sigma = l'$ modulo terms of degree $\geq k + 1$: let $\alpha, \beta \in l^\ast$. Clearly, $\langle l, \alpha \rangle (\gamma) = [\langle s\alpha, d(l' - l)(\gamma) s\beta \rangle]_{k-1}$, so that $\langle l, \alpha \rangle (\gamma) = 0$ modulo terms of degree $\geq k + 1$. Let $\alpha \in l^\ast$ and $\xi \in l^\perp$. Then, $\langle l, \alpha \rangle (\xi) = \langle l, \lambda (\alpha) \rangle = \langle l, (l' - l) s\alpha \rangle$, so that $\langle l, (l' - l) s\alpha \rangle = 0$ modulo terms of degree $\geq k + 1$. Therefore, $l^\sigma = l'$ modulo terms of degree $\geq k + 1$.

It is clear that $l = l'$ modulo terms of degree $\geq 1$. Thus, by induction we construct a sequence $\sigma^{(k)} \in \text{Map}_0(D, G)^l$ of homogeneous map of degree $k$ such that $l^{\sigma^{(k)} \cdot \sigma^{(k-1)}} = l'$ modulo terms of degree $\geq k + 1$. Clearly, the sequence $\sigma^{(k)} \cdot \sigma^{(k-1)}$ converges in $\text{Map}_0(D, G)^l$ to a map $\sigma \in \text{Map}_0(D, G)^l$ such that $l^\sigma = l'$. Theorem 4.6 is thus proved.

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