Multiple Positive Solutions for a Class of Nonlinear Elliptic Eigenvalue Problems with a Sign-Changing Nonlinearity

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Abstract

In 2009 Loc and Schmitt established a result on sufficient conditions for multiplicity of solutions of a class of nonlinear eigenvalue problems for the p-Laplace operator under Dirichlet boundary conditions, extending an earlier result of 1981 by Peter Hess for the Laplacian. Results on necessary conditions for existence were also established. In the present paper the authors extend the main results by Loc and Schmitt to the Φ-Laplacian. To overcome the difficulties with this much more general operator it was necessary to employ regularity results by Lieberman, a strong maximum principle by Pucci and Serrin and a general result on lower and upper solutions by Le [10].

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1 Introduction

We study the nonlinear eigenvalue problem

\[
\begin{cases}
-\text{div} (\phi(|\nabla u|) \nabla u) = \lambda f(u) \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,
\end{cases}
\]

(1.1)

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where \( \Omega \subset \mathbb{R}^N \) is bounded domain with smooth boundary \( \partial \Omega \), \( \lambda > 0 \) is a parameter and \( \phi : (0, \infty) \rightarrow (0, \infty) \) is a \( C^1 \)-function satisfying

\[
(\phi_1) \quad \text{(i) } t \phi(t) \rightarrow 0 \text{ as } t \rightarrow 0,
\]
\[
(\phi_1) \quad \text{(ii) } t \phi(t) \rightarrow \infty \text{ as } t \rightarrow \infty,
\]
\[
(\phi_2) \quad t \phi(t) \text{ is strictly increasing in } (0, \infty),
\]

while \( f : [0, \infty) \rightarrow [0, \infty) \) is a continuous function satisfying

\[
(f_1) \quad f(0) \geq 0,
\]
\[
(f_2) \quad \text{there exist positive numbers } a_k, b_k, k = 1, \ldots, m \text{ such that}
\]
\[
0 < a_1 < b_1 < a_2 < b_2 < \cdots < b_{m-1} < a_m,
\]
\[
f(s) \leq 0 \quad \text{if } s \in (a_k, b_k),
\]
\[
f(s) \geq 0 \quad \text{if } s \in (b_k, a_{k+1}),
\]
\[
(f_3) \quad \int_{a_k}^{a_{k+1}} f(s) ds > 0, \quad k = 1, \ldots, m - 1.
\]

**Remark 1.1** We extend \( t \mapsto t \phi(t) \) to the whole of \( \mathbb{R} \) as an odd function.

In [9], by means of variational and topological methods and arguments with lower and upper solutions, Hess proved a result on existence of multiple solutions of (1.1) for the case of the Laplacian operator which means taking \( \phi(t) \equiv 1 \) in problem (1.1).

According to Hess, the results in [9] were motivated by Brown & Budin [3, 4] which in turn were motivated by the literature on nonlinear heat generation.

In [13], Loc & Schmitt extended the result by Hess to the \( p \)-Laplacian operator by means of taking \( \phi(t) = t^{p-2} \) with \( 1 < p < \infty \) in (1.1). Actually, in [13] the authors showed (see also Dancer & Schmitt [5]) that \((f_1) - (f_3)\) are sufficient conditions for the existence of \( m - 1 \) solutions of

\[-\Delta_p u = \lambda f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,\]

for \( \lambda \) large, while if \((f_1) - (f_2)\) hold, and \( u \) is a solution of the problem above with \( a_k < \|u\|_\infty \leq a_{k+1} \) then \((f_3)\) holds.
In the present work we were able to adapt the techniques in Hess [9] and in Loc & Schmitt [13] to prove a result extending the main theorem of [13] to the more general operator

$$\Delta_{\Phi} u := \text{div}(\phi(|\nabla u|) \nabla u)$$

to prove a result extending the main theorem of [13] to

This named $\Phi$-Laplacian, where $\Phi$ is the even function defined by

$$\Phi(t) = \int_0^t s\phi(s)ds, \; t \in \mathbb{R}.$$ 

and the function $\phi$ satisfies $(\phi_1) - (\phi_2)$ and the following further conditions

$$(\phi_3) \quad \sum_{i,j=1}^N \frac{\partial \alpha_j(\eta)}{\partial \eta_i} \xi_i \xi_j \geq \Gamma_1 \phi(|\eta|)|\xi|^2;$$

$$(\phi_4) \quad |\sum_{i,j=1}^N \frac{\partial \alpha_j(\eta)}{\partial \eta_i}| \leq \Gamma_2 \phi(|\eta|),$$

where $\Gamma_1, \Gamma_2 > 0$ are constants,

$$\xi = (\xi_1, \cdots, \xi_N), \quad \eta = (\eta_1, \cdots, \eta_N),$$

$$\alpha_j(\eta) = \phi(|\eta|)\eta_j, \quad j = 1, \cdots, N.$$ 

It is well known that the $p$-Laplacian is included in this class of operators. Moreover, our result includes a broader class of operators, for example $\Delta_{\Phi}$ with

$$\Phi(t) = (1 + t^2)^\gamma - 1 \quad \text{where} \quad \gamma > \frac{1}{2} \quad (1.2)$$

and

$$\Phi(t) = t^p \log(1 + t) \quad \text{where} \quad p \geq 1. \quad (1.3)$$

See the Appendix for further comments on these examples.

**Definition 1.1** By a solution of (1.1) we mean a function $u \in C^1_0(\Omega)$ satisfying

$$\int_{\Omega} \phi(|\nabla u|) \nabla u \cdot \nabla vdx = \lambda \int_{\Omega} f(u)vdx, \quad v \in C^1_0(\Omega),$$

where

$$C^1_0(\Omega) = \{u \in C^1(\Omega) \mid u = 0 \text{ on } \partial \Omega\}.$$
Our main result below extends Theorem 1.1 by Loc & Schmitt in [13] to the more general operator $\Delta_\Phi$.

**Theorem 1.1** Assume $(\phi_1) - (\phi_4)$. Then

(i) if $(f_1) - (f_3)$ hold, there is $\lambda > 0$, such that for each $\lambda > \lambda$, (1.1) admits at least $m - 1$ solutions say $u_1, ..., u_{m-1}$ such that

$$a_1 < \|u_1\|_\infty \leq a_2 < \|u_2\|_\infty \leq \cdots \leq a_{m-1} < \|u_{m-1}\|_\infty \leq a_m,$$

(ii) if $u$ is a solution of (1.1) with $a_k < \|u\|_\infty \leq a_{k+1}$ and $(f_1) - (f_2)$ hold then $(f_3)$ also holds.

Due to the more general nature of $\Delta_\Phi$, in our proof of theorem 1.1 above it was necessary to get into the framework of Orlicz-Sobolev spaces. It was also necessary to employ regularity results by Lieberman [12, 11], a strong maximum principle due to Pucci & Serrin [14] which holds in our setting as well as a more general result on lower and upper solutions due to Le [10].

**2 Notations and Auxiliary Results**

Consider the family of problems associated to (1.1)

$$\begin{cases} -\Delta_\Phi u = \lambda f_k(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

where for $k = 2, \cdots, m$, $f_k : \mathbb{R} \rightarrow \mathbb{R}$ is the continuous function

$$f_k(s) = \begin{cases} f(0) & \text{if } s \leq 0, \\ f(s) & \text{if } 0 \leq s \leq a_k, \\ 0 & \text{if } s > a_k. \end{cases}$$

In this work

$$W^{1,\Phi}(\Omega) = \left\{ u \in L_\Phi(\Omega) \mid \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega), \ i = 1, ..., N \right\},$$

is the Orlicz-Sobolev space, where $L_\Phi(\Omega)$ is the Orlicz space defined through the $N$-function $\Phi$, endowed with the (Luxembourg) norm

$$\|u\|_\Phi = \inf \left\{ \lambda > 0 \mid \int_\Omega \Phi \left( \frac{u(x)}{\lambda} \right) \, dx \leq 1 \right\},$$
while \( W_0^{1,\Phi}(\Omega) \) denotes the closure of \( C_0^\infty(\Omega) \) with respect to the usual norm of \( W^{1,\Phi}(\Omega) \). We refer the reader to Adams [1], concerning Orlicz-Sobolev spaces.

**Remark 2.1** The reader is referred to [1], [7] for the basic results below:

(i) if \( \phi \) satisfies \((\phi_1)-(\phi_2)\) it is an easy matter to check that \( \Phi \) is an \( N \)-function (or Young function),

(ii) if \( \phi \) satisfies \((\phi_3)-(\phi_4)\) then, (cf. proposition 5.1 in the Appendix),

\[
\Gamma_1 \leq \frac{(t\phi(t))'}{\phi(t)} \leq \Gamma_2, \quad t > 0, \tag{2.2}
\]

(iii) if \((2.2)\) holds then (cf. remark 5.1 in the Appendix) there exist constants \( \gamma_1, \gamma_2 > 1 \) such that

\[
\gamma_1 \leq \frac{t\Phi'(t)}{\Phi(t)} \leq \gamma_2, \quad t > 0, \tag{2.3}
\]

(iv) It follows by [7, pg 542] and [1] thm 8.20 pg 274] that \( L_\Phi(\Omega) \) is reflexive if condition \((2.3)\) holds true.

(v) As a consequence of the remarks (i)-(iv) above, the spaces \( L_\Phi(\Omega) \) and \( W^{1,\Phi}(\Omega) \) are reflexive if \( \phi \) satisfies \((\phi_1)-(\phi_4)\).

The energy functional associated to \((2.1)\) is

\[
I_k(\lambda, u) = \int_\Omega \Phi(|\nabla u|)dx - \lambda \int_\Omega F_k(u)dx, \quad u \in W_0^{1,\Phi}(\Omega),
\]

where

\[
F_k(s) = \int_0^s f_k(t)dt.
\]

It is known that \( I_k(\lambda, \cdot) : W_0^{1,\Phi}(\Omega) \to \mathbb{R} \) is a \( C^1 \)-functional and

\[
\langle I_k(\lambda, u), v \rangle = \int_\Omega \phi(|\nabla u|)\nabla u \cdot \nabla vdx - \lambda \int_\Omega f_k(u)vdx, \quad v \in W_0^{1,\Phi}(\Omega).
\]

Thus, a critical point \( u \in W_0^{1,\Phi}(\Omega) \) of \( I_k(\lambda, \cdot) \) is a weak solution of \((2.1)\), in the sense that

\[
\int_\Omega \phi(|\nabla u|)\nabla u \cdot \nabla vdx = \lambda \int_\Omega f_k(u)vdx, \quad v \in W_0^{1,\Phi}(\Omega).
\]
Remark 2.2 If \( u \) is a weak solution of (2.1) then, since \( f_k \) is bounded and continuous, \( f_k(u) \in L^\infty(\Omega) \). It follows by Lieberman [12, theorem 1.7] that \( u \in C^{1,\alpha}(\Omega) \) where \( \alpha \in (0, 1) \) and so \( u \) is a solution of (2.1) in the sense of definition 1.1.

3 Technical Lemmata

The result below is crucial in this paper, it was proved by Loc & Schmitt for Sobolev spaces and its proof in our case is similar. We leave its proof to the end of the section.

Lemma 3.1 Let \( g : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( g(s) \geq 0 \) for \( s \in (-\infty, 0) \) and assume that there is some \( s_0 \geq 0 \) such that \( g(s) \leq 0 \) for \( s \geq s_0 \). Let \( u \in W^{1,\Phi}_0(\Omega) \) be a weak solution of

\[
\begin{cases}
-\Delta \Phi u = g(u) \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega.
\end{cases}
\] (3.1)

Then \( 0 \leq u \leq s_0 \) a.e. in \( \Omega \).

Lemma 3.2 Let \( \lambda > 0 \). Then there is \( v_k \equiv v_k(\lambda) \in W^{1,\Phi}_0(\Omega) \) such that

\[
I_k(\lambda, v_k) = \min_{u \in W^{1,\Phi}_0(\Omega)} I_k(\lambda, u).
\]

Proof. It is enough to show that \( I_k(\lambda, \cdot) \) is both coercive and weakly sequentially lower semicontinuous, (w.s.l.s.c. for short).

To show the coerciveness, by lemma [5,1] (cf. Appendix), the continuous embedding \( W^{1,\Phi}_0(\Omega) \hookrightarrow L^1(\Omega) \) and the Poincaré inequality, (cf [S]), we have

\[
I_k(\lambda, u) \geq \min \{ \|\nabla u\|_{\gamma_1}, \|\nabla u\|_{\gamma_2} \} - \lambda C \|u\|_1
\]

\[
\geq \min \{ \|\nabla u\|_{\gamma_1}, \|\nabla u\|_{\gamma_2} \} - \lambda C \|\nabla u\|,
\]

which shows that \( I_k(\lambda, \cdot) : W^{1,\Phi}_0(\Omega) \to \mathbb{R} \) is coercive.

To show that \( I_k(\lambda, \cdot) \) is w.s.l.s.c, at first notice that

\[
u \in W^{1,\Phi}_0(\Omega) \mapsto \int_\Omega \Phi(|\nabla u|)dx
\]
is continuous and convex. Take \((u_n)\) such that \(u_n \rightharpoonup u \) in \(W^{1,\Phi}_0(\Omega)\). Using the embedding (cf. Adams [1]),

\[ W^{1,\Phi}_0(\Omega) \hookrightarrow \text{cpt} L^\Phi(\Omega), \]

and arguments with the convexity of \(\Phi\), there is \(h \in L^\Phi(\Omega)\) such that

\[ u_n \rightharpoonup u \text{ in } W^{1,\Phi}_0(\Omega), \quad u_n \to u \text{ and } |u_n| \leq h \text{ a.e. in } \Omega. \]

By Lebesgue’s Theorem,

\[ \int_\Omega F_k(u_n)dx \to \int_\Omega F_k(u)dx. \]

Since \(\Phi\) is continuous and convex,

\[ \int_\Omega \Phi(|\nabla u|)dx \leq \lim \inf \int_\Omega \Phi(|\nabla u_n|)dx. \]

It follows that

\[ I_k(\lambda, u) \leq \lim \inf I_k(\lambda, u_n). \]

As a consequence, there is minimum \(v_k \equiv v_k(\lambda)\) of \(I_k(\lambda, \cdot)\). \(\square\)

**Lemma 3.3** There is \(\lambda_k > 0\) such that

\[ a_{k-1} < \|v_k\|_\infty \leq a_k \]

for each minimum \(v_k \equiv v_k(\lambda)\) of \(I_k(\lambda, \cdot)\) with \(\lambda > \lambda_k\).

**Proof of Lemma 3.3** The proof is similar to the ones in [9] [13]. So we will just sketch the main steps. Take \(\delta > 0\) and consider the open set

\[ \Omega_\delta = \{x \in \Omega \mid \text{dist}(x, \partial \Omega) < \delta\}. \]

Set

\[ \tilde{\alpha}_k := F(a_k) - \max \{F(s) \mid 0 \leq s \leq a_{k-1}\} \]

and note that by \((f_3)\), \(\tilde{\alpha}_k > 0\). Choose \(w_\delta \in C^\infty_0(\Omega)\) such that

\[ 0 \leq w_\delta \leq a_k \text{ and } w_\delta(x) = a_k, \quad x \in \Omega \setminus \Omega_\delta. \]
Writing $\Omega = \Omega_\delta \cup (\Omega \setminus \Omega_\delta)$ and setting $C_k = \max \{|F(s)| \mid 0 \leq s \leq a_k\}$ we get to,

$$\int_{\Omega} F(w_\delta) dx \geq \int_{\Omega} F(a_k) dx - 2C_k |\Omega_\delta|.$$ 

Let $u \in W^{1,\Phi}_0(\Omega)$ such that $0 \leq u \leq a_{k-1}$. By the inequality above we have

$$\int_{\Omega} F(w_\delta) dx - \int_{\Omega} F(u) dx \geq \tilde{\alpha}_k |\Omega| - 2C_k |\Omega_\delta|.$$ 

Since $|\Omega_\delta| \to 0$ as $\delta \to 0$ there is $\delta > 0$ such that

$$\eta_k := \tilde{\alpha}_k |\Omega| - 2C_k |\Omega_\delta| > 0.$$ 

Set $w = w_\delta$ and pick $u \in W^{1,\Phi}_0(\Omega)$ with $0 \leq u \leq a_{k-1}$. Choosing $\lambda_k > 0$ large enough, taking $\lambda \geq \lambda_k$ and making use of the expressions of $I_k(\lambda, w), I_{k-1}(\lambda, u)$ and the inequality just above we infer that

$$I_k(\lambda, w) - I_{k-1}(\lambda, u) \leq \int_{\Omega} \Phi(|\nabla w|) \nabla u \nabla u - dx - \lambda \eta_k < 0 \quad (3.2)$$

and hence

$$I_k(\lambda, w) < I_{k-1}(\lambda, u) \quad \text{for} \quad \lambda \geq \lambda_k. \quad (3.3)$$

To finish, assume, on the contrary, that there is a minimum $v_k(\lambda)$ of $I_k(\lambda, \cdot)$ such that $v_k(\lambda) \leq a_{k-1}$. It follows by (3.3) and lemma 3.1 that

$$I_k(\lambda, w) < I_{k-1}(\lambda, v_k(\lambda)).$$

On the other hand, since $v_k(\lambda)$ is a minimum of $I_k(\lambda, \cdot)$ we have

$$I_k(\lambda, v_k) \leq I_k(\lambda, w)$$

The definitions of $I_k(\lambda, \cdot)$ and $I_{k-1}(\lambda, \cdot)$ and the inequalities just above lead to a contradiction. This ends the proof of lemma 3.3. \hfill \Box

**Proof of Lemma 3.1** Let $u \in W^{1,\Phi}_0(\Omega)$ be a weak solution of (3.1). Recall that (even for Orlicz-Sobolev spaces) $u^- = \max\{-u, 0\} \in W^{1,\Phi}_0(\Omega)$. We have

$$\int_{\Omega} \phi(|\nabla u|) \nabla u \nabla u^- dx = \int_{\Omega} g(u) u^- dx = - \int_{u<0} g(u) u dx \geq 0. \quad (3.4)$$
Moreover (also for Orlicz-Sobolev spaces) one has
\[ \nabla u^- = -\nabla u \chi_{(u<0)} \text{ a.e. in } \Omega. \]
Using this in (3.4) we find that
\[ \int_{\Omega} \phi(|\nabla u|)|\nabla u|^2 \chi_{(u<0)} dx = 0 \]
which shows that \( u \geq 0 \) in \( \Omega \).

Let \((u-s_0)^+ = \max\{u-s_0, 0\}\). By using an argument as the one above, we infer that \( u \leq s_0 \). This proves lemma 3.1.

\[ \square \]

4 Proof of Theorem 1.1

The proof is based on Loc & Schmitt [13]. However, we will get into details taking into account the Orlicz-Sobolev spaces framework. In this sense we will make use of a general result on lower and upper solutions by Le [10, theorem 3.2] and a general strong maximum principle by Pucci & Serrin [14, theorem 1.1] which hold in our setting.

The proof of (i) is easier. For the proof of (ii) we will need the two lemmas below whose proofs are left to the end of this section.

**Lemma 4.1** Let \( u \) be a solution of (1.1) in the sense of definition 1.1. If \( u \geq 0 \) and \( f(0) > 0 \) then \( u > 0 \) in \( \Omega \).

The proof of this lemma, which is left to the end of this section, strongly uses a general form of the maximum principle due to Pucci & Serrin [14].

In order to state the second lemma take an open ball \( B \) centered at 0 with radius \( R \) containing \( \Omega \). Consider the functions \( \alpha, \beta : \overline{B} \to \mathbb{R} \) defined as follows:
\[
\alpha(x) = \begin{cases} 
  u(x), & x \in \overline{\Omega} \\
  0, & x \in \overline{B} \setminus \Omega,
\end{cases} \quad \beta(x) = a_{k+1}, \ x \in \overline{B}.
\]
Since \( u \in W_{0}^{1,\Phi}(\Omega) \) there is a sequence \( \{u_n\} \subseteq C_0^{\infty}((\Omega) \) such that \( u_n \to u \) in the norm of \( W_{0}^{1,\Phi}(\Omega) \). Extending each \( u_n \) to \( B \) as zero outside \( \text{supp}(u_n) \subseteq \Omega \) it follows that \( \{u_n\} \subseteq C_0^{\infty}(B) \) and \( u_n \to \alpha \) in the norm of \( W_{0}^{1,\Phi}(B) \) so that \( \alpha \in W_{0}^{1,\Phi}(B) \).
Lemma 4.2 $\beta$ and $\alpha$ are respectively upper and lower solutions of
\[\begin{cases} -\Delta \Phi u = \lambda f(u) \text{ in } B, \\ u \in W_0^{1, \Phi}(B). \end{cases}\] (4.1)

The proof of this lemma, which is left to the end of this section, uses a general theorem on lower and upper solutions due to Le [10].

Proof of (i) of theorem 1.1 Take $\lambda > 0$. By lemma 3.2 for each $k = 2, \cdots, m$ there is a minimum $v_k \equiv v_k(\lambda)$ of $I_k(\lambda)$, which is actually a weak solution of problem (2.1). By remark 2.2, $v_k \in C^1(\Omega)$ and by lemma 3.1 $0 \leq v_k \leq a_k$ a.e. in $\Omega$.

By lemma 3.3 there is $\lambda > \max_{2 \leq k \leq m} \{\lambda_k\}$ such that for $\lambda > \lambda$, $v_2, \cdots, v_m$ are solutions of problem (1.1) satisfying
\[a_1 < \|v_2\|_\infty \leq a_2 < \|v_3\|_\infty \leq \cdots \leq a_{m-1} < \|v_m\|_\infty \leq a_m\]

We set $u_{k-1} \equiv v_k(\lambda)$, $k = 2, \cdots, m$. This ends the proof of the first part of theorem 1.1.

Proof of (ii) of theorem 1.1 We distinguish between two cases.

Case 1 $f(0) > 0$.

This case is more difficult. In order to address it we state and prove the lemma below.

Lemma 4.3 Assume $(\phi_1) - (\phi_4)$, $(f_1) - (f_2)$ and $f(0) > 0$. If $u$ is a non-negative weak solution of (1.1) such that $a_{k-1} < \|u\|_\infty \leq a_k$ then
\[\int_{a_k}^{a_{k+1}} f(s) > 0.\]

Proof Let us think of $k = 2$, for a while. Take the lower and upper solutions respectively $\alpha$ and $a_2$ of (4.1).

Applying theorem 3.2 of [10] there is a maximal solution say $\overline{u}$ of (4.1) such that $\alpha(x) \leq \overline{u}(x) \leq a_2$ for $x \in B$.

By remark 2.2, $\overline{u} \in C^1(\Omega)$ and by lemma 4.1, $\overline{u} > 0$ in $B$.

Claim 4.1 $\overline{u}$ is radially symmetric, e.g. $\overline{u}(x_1) = \overline{u}(x_2), \ x_1, x_2 \in B, \ |x_1| = |x_2|.$
Indeed, assume on the contrary that
\[ u(x_1) < u(x_2) \] for some \( x_1, x_2 \in B \) with \( |x_1| = |x_2| \).

Choose a rotation matrix \( P \) such that \( x_2 = Px_1 \). Recall that \( P^\top P = I \) and \( |\det P| = 1 \).

Set \( u_1(x) = \overline{u}(Px) \). Since \( \nabla u_1(x) = P\nabla \overline{u}(Px) \), \( x \in \Omega \), it follows that, \( P \) is an isometry,
\[ |\nabla u_1(x)| = |\nabla \overline{u}(Px)|. \]

We contend that \( u_1 \) is a weak solution of (4.1). Indeed, let \( \varphi \in W^{1,\Phi}_0(B) \) and set \( \psi(x) = \varphi(P^\top x) \in W^{1,\Phi}_0(B) \). We have by easy computation,
\[
\int_B \phi(|\nabla u_1(x)|) \nabla u_1(x) \nabla \varphi(x) dx = \int_B \phi(|\nabla \overline{u}(P^\top x)|) \nabla \overline{u}(P^\top x) \nabla \varphi(x) dx \\
= \int_B \phi(|\nabla \overline{u}(y)|) \nabla \overline{u}(y) \nabla \psi(y) |\det(P)| dy \\
= \lambda \int_B f(\overline{u}(y)) \psi(y) dy \\
= \lambda \int_B f(\overline{u}(P^\top x)) \psi(P^\top x) |\det P^\top| dx \\
= \lambda \int_B f(u_1(x)) \varphi(x) dx,
\]
showing that \( u_1 \) is a solution of (4.1).

Of course, \( u_1 \) is a subsolution of (4.1). Therefore (4.1) has two subsolutions namely \( \alpha \) and \( u_1 \). By [10, theorem 3.4], (4.1) has a further solution say \( u_2 \), satisfying
\[ \max\{\alpha, u_1\} \leq u_2 \leq \beta. \]

By the maximality of \( \overline{u} \), we infer that
\[ \overline{u}(x_1) < \overline{u}(x_2) = u_1(x_1) \leq u_2(x_1) \leq \overline{u}(x_1), \]
a contradiction. Thus Claim 4.1 holds true.
We set \( u(r) = \overline{u}(x) \) where \( r = |x| \) and \( x \in B \).

and notice that 
\[
 u \geq 0, \quad u \neq 0, \quad u'(0) = u(R) = 0.
\]

Now, let \( r \in (0, R) \) and pick \( \epsilon > 0 \) small such that \( r + \epsilon < R \). Remember that \( \overline{u} \in C_0^1(\overline{B}) \) and
\[
\int_B \phi(|\nabla \overline{u}|) \nabla \overline{u} \nabla v dx = \lambda \int_B f(\overline{u})v dx, \quad v \in W_0^1,\Phi(B). \tag{4.2}
\]

Adapting an argument employed in [2], consider the radially symmetric cut-off function \( v_{r,\epsilon}(x) = v_{r,\epsilon}(r) \), where
\[
v_{r,\epsilon}(t) := \begin{cases} 
1 & \text{if } 0 \leq t \leq r, \\
\text{linear} & \text{if } r \leq t \leq r + \epsilon, \\
0 & \text{if } r + \epsilon \leq t \leq R.
\end{cases}
\]
and notice that \( v_{r,\epsilon} \in W_0^1,\Phi(B) \cap \text{Lip}(\overline{B}) \). Setting \( v = v_{r,\epsilon} \) in (4.2) and using the radial symmetry we get to
\[
-\frac{1}{\epsilon} \int_r^{r+\epsilon} t^{N-1} \phi(|u'|)u' \, dt = \int_0^r t^{N-1} \lambda f(u) \, dt + \int_r^{r+\epsilon} t^{N-1} \lambda f(u) v \, dt.
\]
Making \( \epsilon \to 0 \) gives
\[
-r^{N-1} \phi(|u'(r)|)u'(r) = \int_0^r \lambda f(u) t^{N-1} \, dt, \quad 0 < r < R. \tag{4.3}
\]

Set
\[
\|u\|_{\infty} = \max \{ u(r) \mid r \in [0, R] \},
\]
and choose numbers \( r_0, r_1 \in [0, R] \) with \( r_1 \in (r_0, R) \) such that
\[
u(r_0) = \|u\|_{\infty} \quad \text{and} \quad u(r_1) = a_1.
\]

Note that
\[
u(r_0) > u(r_1) \quad \text{and} \quad 0 \leq r_0 < r_1 < R.
\]

Claim 4.2 \( \|u\|_{\infty} > b_1 \).
Indeed, assume on the contrary that, \( u(r_0) \leq b_1 \). Take \( \delta > 0 \) small such that
\[
a_1 < u(r) \leq u(r_0), \quad r_0 \leq r \leq r_0 + \delta.
\]
We have by (4.3)
\[
- r_0^{N-1} \phi(|u'(r_0)|)u'(r_0) = \int_{0}^{r_0} \lambda f(u) t^{N-1} \, dt, \tag{4.4}
\]
\[
- r^{N-1} \phi(|u'(r)|)u'(r) = \int_{0}^{r} \lambda f(u) t^{N-1} \, dt. \tag{4.5}
\]
Subtracting (4.5) minus (4.4) term by term and recalling that \( u'(r_0) = 0 \),
\[
- r^{N-1} \phi(|u'(r)|)u'(r) = \int_{r_0}^{r} \lambda f(u) t^{N-1} \, dt, \quad r_0 \leq r \leq r_0 + \delta.
\]
Since \( f \leq 0 \) on \([a_1, b_1]\),
\[
r^{N-1} \phi(|u'(r)|)u'(r) \geq 0, \quad r_0 \leq r \leq r_0 + \delta.
\]
It follows that \( u'(r) \geq 0 \) for \( r_0 \leq r \leq r_0 + \delta \). But, since \( r_0 \) is a global maximum on \([0, R]\), it follows that \( u' = 0 \) on \([r_0, r_0 + \delta]\). By a continuation argument we get \( u' = 0 \) on \([r_0, r_1]\) so that \( u = \|u\|_\infty \) on \([r_0, r_1]\), contradicting \( u(r_0) > a_1 \). As a consequence, \( \|u\|_\infty > b_1 \), proving Claim 4.2.

**Claim 4.3** \( u \in C^2(\mathcal{O}) \) where \( \mathcal{O} := \{ r \in (0, R) \mid u'(r) \neq 0 \} \).

Of course \( \mathcal{O} \) is an open set. Motivated by the left hand side of (4.3) consider
\[
G(z) = \phi(z) z, \quad z \in \mathbb{R},
\]
where \( z \) is set to play the role of \( u' \). Recall that
\[
G \text{ is odd, } \quad G'(z) = (\phi(z) z)' > 0 \quad \text{for } z > 0
\]
and
\[
G(z) = \phi(|u'(r)|)u'(r) = -\frac{1}{r^{N-1}} \int_{r_0}^{r} \lambda f(u) t^{N-1} \, dt.
\]
Since \( \phi(z) z \in C^1 \) and \( (\phi(z) z)' \neq 0 \) for \( z \neq 0 \), we get by applying the Inverse Function Theorem in \( \mathcal{O} \) that \( z = z(r, u) \) is a \( C^1 \)-function of \( r \). Since \( z = u' \), the claim is proved.
Claim 4.4 \( \int_{a_1}^{\|u\|_\infty} f(s)ds > 0. \)

Differentiating in (4.3) and multiplying by \( u' \) we get
\[
(t^{N-1} \phi(|u'(t)|)u'(t))'u'(t) = -\lambda f(u(t))u'(t)t^{N-1},
\]
and hence
\[
[(N-1)t^{N-2}\phi(|u'(t)|)u'(t) + t^{N-1}(\phi(|u'(t)|u'(t)' u'(t)) = -\lambda f(u(t))u'(t)t^{N-1},
\]
which gives
\[
\frac{(N-1)}{t} \phi(|u'|)(u')^2 + (\phi(|u'|)u')'u' = -\lambda f(u)u'.
\]

Integrating from \( r_0 \) to \( r_1 \) we have
\[
-\left[ \int_{r_0}^{r_1} \frac{(N-1)}{t} \phi(|u'|)(u')^2dt + \int_{r_0}^{r_1} [\phi(|u'|)u']'u'dt \right] = \int_{r_0}^{r_1} \lambda f(u)u'dt. \quad (4.6)
\]
Computing the second integral in the left hand side of (4.6) we get
\[
\int_{r_0}^{r_1} [\phi(|u'(t)|)u'(t)]'u'(t)dt = \int_{r_0}^{r_1} [\phi(u')u' + \phi(u')(u')^2]u''dt \text{ if } u' > 0,
\]
and
\[
\int_{r_0}^{r_1} [\phi(|u'(t)|)u'(t)]'u'(t)dt = \int_{r_0}^{r_1} [\phi(-u')u' - \phi(-u')(u')^2]u''dt \text{ if } u' < 0.
\]
Making the change of variables \( s = u'(t) \) above we get
\[
\int_{r_0}^{r_1} [\phi(|u'(t)|)u'(t)]'u'(t)dt = \int_{0}^{u'(r_1)} [s\phi(|s|)]'sds.
\]
Taking into (4.6) we get
\[
\int_{\|u\|_\infty}^{a_1} \lambda f(s)ds = -\int_{r_0}^{r_1} \frac{(N-1)}{t} \phi(|u'|)(u')^2dt - \int_{0}^{u'(r_1)} [s\phi(|s|)]'sds < 0.
\]

On the other hand \( u' \) is not identically zero on \([r_0, r_1]\) because otherwise we would have \( u(t) = u(r_0) = u(r_1) \) contradicting \( a_1 < \|u\|_\infty \leq a_2. \) Therefore
\[
\int_{a_1}^{\|u\|_\infty} f(s)ds > 0,
\]
proving Claim 4.4

Since \( f \geq 0 \) on \((b_1, a_2)\) and \( \|u\|_\infty > b_1 \) it follows that

\[
\int_{a_1}^{a_2} f(s)ds > 0,
\]

ending the proof of lemma 4.3.

Case 2 \( f(0) = 0 \).

Let \( u \) be a solution of (1.1) with \( a_1 < \|u\|_\infty \leq a_2 \). Consider a continuous function \( \tilde{f} \) such that \( \tilde{f}(0) > 0 \), \( \tilde{f}(s) \geq f(s) \) if \( 0 \leq s \leq a_1 \) and \( \tilde{f}(s) = f(s) \) if \( a_1 \leq s < \infty \). It follows that \( u \) is a subsolution of

\[
\begin{cases}
-\Delta \Phi u = \lambda \tilde{f}(u) & \text{in } B \\
 0 \geq u, & u \in W^{1,\Phi}_0(B).
\end{cases}
\] (4.7)

As in Case 1 we use \( \beta(x) = a_2 \) as a supersolution of (4.7).

Hence (1.7) has a solution \( \tilde{u} \) satisfying \( u \leq \tilde{u} \leq a_2 \). We now proceed as in the first part of the proof with \( \tilde{f} \) in place of \( f \) to obtain

\[
\int_{a_1}^{a_2} f(s)ds = \int_{a_1}^{a_2} \tilde{f}(s)ds > 0.
\]

The proof for \( a_k, k > 2 \) follows the same lines. Theorem 1.1 is proved.

It remains to proof lemmas 4.1 and 4.2.

Proof of lemma 4.1 At first, using the facts that \( f(0) > 0 \) and \((s\Phi(s))\) is strictly increasing for \( s > 0 \) there is a constant \( c > 0 \) such that

\[
\lambda f(s) + c(s\Phi(s))' \geq 0, \quad s \in [0, a_k]
\] (4.8)

and remember that

\[
\int_{\Omega} \phi(|\nabla u|)\nabla u \nabla vdx = \lambda \int_{\Omega} f(u)vdx, \quad v \in W^{1,\Phi}_0.
\] (4.9)

Adding

\[
c \int_{\Omega} (u\Phi(u))'vdx
\]
to both sides of (4.9), taking \( v \geq 0, \ 0 \leq u \leq a_k \) and using (4.8) we have
\[
\int_{\Omega} \phi(|\nabla u|) \nabla u \nabla v \, dx + c \int_{\Omega} (u \Phi(u))' v \, dx = \lambda \int_{\Omega} (f(u) + c(u \Phi(u))') v \, dx \geq 0.
\]

At this point, we will use Theorem 1.1 of [14]. In order to verify condition (1.6) of [14] set \( H(s) = s \Phi'(s) - \Phi(s) \) for \( s \geq 0 \) and \( F(s) = cs\Phi(s) \). Note that
\[
\frac{cs\Phi(s)}{H(s)} = \frac{cs}{\frac{s\Phi'(s)}{\Phi(s)}} - 1 \leq \frac{cs}{\gamma_1 - 1},
\]
where in the last inequality we used (2.3).

By the inequality above choose \( \delta > 0 \) such that \( \frac{cs\Phi(s)}{H(s)} \leq 1 \) for \( s \in (0, \delta) \). Since \( H^{-1} \) is strictly increasing, we infer that \( H^{-1}(cs\Phi(s)) \leq s \) for \( s \in (0, \delta) \), from which condition (1.6) of [14] follows.

This ends the proof of lemma 4.1.

**Proof of lemma 4.2** Of course \( \beta \) is an upper-solution of (4.1). To deal with \( \alpha \) define
\[
v_n(x) = n \min \{u(x), \frac{1}{n}\} \text{ for } x \in \Omega, \ n \geq 1 \text{ is an integer.}
\]

Notice that \( \nabla u \cdot \nabla v_n \geq 0 \) and by the very definition, \( v_n \) converges to 1, pointwisely in \( \Omega \). Take \( w \geq 0, w \in C_0^\infty(B) \) and note that \( wv_n \in \tilde{W}_0^{1, \Phi}(\Omega) \). This implies
\[
\int_{\Omega} \phi(|\nabla u|) \nabla u \nabla (wv_n) \, dx = \lambda \int_{\Omega} f(u)wv_n \, dx
\]
which gives
\[
\int_{\Omega} w\phi(|\nabla u|) \nabla u \nabla v_n \, dx + \int_{\Omega} v_n \phi(|\nabla u|) \nabla u \nabla w \, dx = \lambda \int_{\Omega} f(u)wv_n \, dx
\]
We observe that
\[
0 \leq w\phi(|\nabla u|) \nabla u \nabla v_n \leq w\phi(|\nabla u|)|\nabla u|^2 \text{ and } 0 \leq v_n \leq 1.
\]
By the Lebesgue Theorem we infer that
\[
\int_B \phi(|\nabla \alpha|) \nabla \alpha \nabla w \, dx = \int_\Omega \phi(|\nabla u|) \nabla u \nabla w \, dx \\
= \lim_{\Omega} \int v_n \phi(|\nabla u|) \nabla u \nabla v_n \, dx \\
= \lim_{\Omega} \left( \lambda f(u) w v_n - w \phi(|\nabla u|) \nabla u \nabla v_n \right) \, dx \\
\leq \lambda \int_\Omega f(u) \, w \, dx \\
\leq \lambda \int_B f(\alpha) \, w \, dx.
\]

This ends the proof of lemma 4.2. \( \Box \)

5 Appendix

In this section we present for the sake of completeness several rather simple results and remarks which were employed in the body of the paper.

Proposition 5.1 If \( \phi \) satisfies (\( \phi_3 \)) \(-\phi_4 \) then (2.2) holds true.

Proof Indeed by (\( \phi_3 \)),
\[
\sum_{i,j=1}^{N} \frac{\partial \alpha_j}{\partial \eta_i}(\eta) \delta_{i,j} \geq \Gamma_1 \phi(|\eta|) \xi^2,
\]
where \( \delta_{i,j} \) is the Kronecker symbol. Thus
\[
\sum_{i,j=1}^{N} \frac{\partial \phi(|\eta| \eta_j)}{\partial \eta_i} \delta_{i,j} \geq \Gamma_1 \phi(|\eta|) \xi^2.
\]

Take \( t > 0, \eta = (t, 0, ..., 0) \) and \( \xi = (1, 0, ..., 0) \). Then
\[
\sum_{i,j=1}^{N} \frac{\partial \phi(|\eta| \eta_j)}{\partial \eta_i} \delta_{i,j} = \frac{d(t \phi(t))}{dt} \\
\geq \Gamma_1 \phi(t).
\]
Therefore
\[
\frac{(t\phi(t))'}{\phi(t)} \geq \Gamma_1, \ t > 0.
\]
On the other hand, assuming \((\phi_4)\) we have
\[
\sum_{i,j=1}^{N} \frac{\partial}{\partial \eta_i} [\phi(|\eta|) \eta_j] \delta_{i,j} \leq \Gamma_2 \phi(|\eta|).
\]
Take \(t > 0, \eta = (t, 0, \ldots, 0)\) and \(\xi = (1, 0, \ldots, 0)\). Arguing as above we find
\[
\frac{(t\phi(t))'}{\phi(t)} \leq \Gamma_2, \ t > 0.
\]
This ends the proof of proposition 5.1. \(\square\)

Remark 5.1 Verification of (iii) in remark (2.1).

By (2.2) we have
\[
\Gamma_1 \phi(s) \leq (s\phi(s))' \leq \Gamma_2 \phi(s), \ s > 0.
\]
Multiplying by \(s\) and integrating from 0 to \(t\) we have
\[
\Gamma_1 \Phi(t) \leq t^2 \phi(t) - \Phi(t) \leq \Gamma_2 \Phi(t), \ t > 0.
\]
As a consequence,
\[
(\Gamma_1 + 1)\Phi(t) \leq t\Phi'(t) \leq (\Gamma_2 + 1)\Phi(t), \ t > 0,
\]
showing (2.3).

Remark 5.2 (On example 1.2).

Let \(\phi(t) = 2\gamma(1 + t^2)^{\gamma-1}\) with \(\gamma > \frac{1}{2}\). Then \(\Phi(t) = (1 + t^2)^{\gamma} - 1\).

Differentiating in the expression of \(\phi\) we get
\[
\phi'(t) = 4\gamma(\gamma - 1)(1 + t^2)^{\gamma-2}t.
\]
It follows that
\[
\frac{(t\phi(t))'}{\phi(t)} = 1 + 2(\gamma - 1) \frac{t^2}{1 + t^2}.
\]
and so
\[
\min\{1, 2\gamma - 1\} \leq \frac{(t\phi(t))'}{\phi(t)} \leq \max\{1, 2\gamma - 1\}.
\]
By proposition 5.1, \(\phi\) satisfies \((\phi_3) - (\phi_4)\). It follows that \(\phi\) satisfies \((\phi_i), \ i = 1, \ldots, 4\). \(\square\)
Remark 5.3 (On example 1.3).

Consider
\[ \phi(t) = \frac{pt^{p-2}(1 + t)\ln(1 + t) + t^{p-1}}{1 + t}, \quad t > 0. \]

Then
\[ \Phi(t) = t^p \ln(1 + t). \]

By computing, we get
\[ (t\phi(t))' = t^{p-2}\left[p(p - 1)\ln(1 + t) + \frac{2pt}{1 + t} - \frac{t^2}{(1 + t)^2}\right] \]
so that
\[ \frac{(t\phi(t))'}{\phi(t)} = \frac{2p(1 + t)t - t^2 + p(p - 1)(1 + t)^2 \ln(1 + t)}{(1 + t)(t + p(1 + t)\ln(1 + t))}, \]
which is a decreasing function. Moreover,
\[ \lim_{t \to \infty} \frac{(t\phi(t))'}{\phi(t)} = p - 1 \quad \text{and} \quad \lim_{t \to 0} \frac{(t\phi(t))'}{\phi(t)} = p \]
and so
\[ p - 1 \leq \frac{(t\phi(t))'}{\phi(t)} \leq p, \]

By proposition 5.1, \( \phi \) satisfies \((\phi_3) - (\phi_4)\).

We refer the reader to [7] and references therein for the lemma below whose proof is elementary.

Lemma 5.1 Assume that \( \phi \) satisfies \((\phi_1) - (\phi_3)\). Set
\[ \zeta_0(t) = \min\{t^{\gamma_1}, t^{\gamma_2}\}, \quad \zeta_1(t) = \max\{t^{\gamma_1}, t^{\gamma_2}\}, \quad t \geq 0. \]

Then \( \Phi \) satisfies
\[ \zeta_0(t)\Phi(\rho) \leq \Phi(\rho t) \leq \zeta_1(t)\Phi(\rho), \quad \rho, t > 0, \]
\[ \zeta_0(\|u\|_\Phi) \leq \int_\Omega \Phi(u)dx \leq \zeta_1(\|u\|_\Phi), \quad u \in L_\Phi(\Omega). \]
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