Prime cordial labeling of n-chain vertex merged staircase graphs

M Antony Arockiasamy\(^1\)* and A Manivasagam\(^2\)
\(^1\),\(^2\)Department of Mathematics, Sacred Heart College ( Autonomous), Tirupattur, Tamil nadu, India
E-mail:*arockiaanto2008@gmail.com.

Abstract.
A prime cordial labeling of a graph \(G\) with vertex set \(V\) is a bijection \(h: V \rightarrow \{1,2,3,\ldots,|V|\}\) such that, for each edge \(u,v\), the label 1 is assigned, if \(g.c.d(h(u),h(v)) = 1\), otherwise 0, and the total number of edges labeled as 0, and the total number of edges labeled as 1, differ atmost by 1. In this paper, we prove that n-chain vertex merged staircase graphs are prime cordial.

Key words: Prime cordial labeling, Triple vertex merged staircase graphs, Four chain vertex merged staircase graph, n-Chain vertex merged Staircase graph.

1. Introduction
The dominant feature of Graph theory lies in the vast and extensive applications of graph labeling in almost all spheres of technology and day to day life. Graph labeling gives a lot of scope in several fields of technology such as Networking, Web designing, Structural designing, Communication Engineering, Online Banking, etc. Graceful labeling, Cordial labeling, Elegant labeling, Prime cordial labeling, Product cordial labeling are some of the leading labeling techniques applied in the fields mentioned above.

Graph labeling traces its root in 1967, to Rosa [5] who introduced the \(\beta -\)valuation which was later called as "Graceful" by Golomb. In 1987, Cahit [4] diluted the graceful labeling. He coined cordial labeling as a weaker version of graceful labeling. In 2005, the prime cordial labeling was introduced by Sundaram and Ponraj[6]. In this paper, we investigate the prime cordial labeling for a new family of staircase graphs introduced by Solairaju and Antony Arockiasamy [7]. In section 2, main results related to vertex merged staircase graphs are given. In section 3, some openings for future research are listed.

Definition 1.1[4] Consider a graph \(G = (V,E)\), a vertex labeling function \(h: V \rightarrow \{0,1\}\) induces an edge labeling function \(h^*: E(G) \rightarrow \{0,1\}\) defined as \(h(xy) = |h(x) - h(y)|\). Then \(h\) is called a cordial labeling of graph \(G\), if

\[
|v_{h}(0) - v_{h}(1)| \leq 1 \quad \text{and} \quad |e_{h^*}(0) - e_{h^*}(1)| \leq 1.
\]

where \(v_{h}(i)\) is the number of vertices of \(G\) having label \(i\) under \(h\) and \(e_{h^*}(i)\) is the number of edges of \(G\) having label \(i\) under \(h^*\) for \(i = 0,1\). A graph \(G\) is said to be cordial graph, if it permits a cordial labeling.

Definition 1.2[6] A graph \(G(V,E)\) with vertex set \(V(G)\) is said to have a prime cordial labeling, if its vertices are labeled with distinct integers \(\{1,2,3,\ldots,|V|\}\) such that for every \(uv \in E\) the labels assigned to \(u\) and \(v\) are relatively prime. Throughout this paper, we denote Prime Cordial Labeling as PCL.

Definition 1.3[7] \(\ell C_4\) is a graph consisting of \(\ell\) copies of \(C_4\) with a vertex common in between any two consecutive copies of \(C_4\) except first and last copies of \(C_4\) as shown in the Figure 1.
Figure 1: $\ell^*C_4$

**Definition 1.4** A staircase graph $G$ is a graph with order $\ell$ ($\ell \geq 2, \ell \in \mathbb{Z}$) it can be obtained by merging $(2\ell - 2)$ edges of $\ell^*C_4$ with $(2\ell - 2)$ edges in the shortest path $Vc_1-Vc_3$ of $(\ell - 1)^{th}$ staircase as given in the following Figure 2.

Figure 2: $G(S, \ell)$

**Definition 1.5** Vertex merged twin staircase graph is a graph which consist of two staircase graphs with any one of the corner vertices of a staircase graph joined with another corner vertex of the another staircase graph. It is denoted by $G(2S_v, \ell)$, as given in the following Figure 3. In same way we can represent a triple and four chain vertex merged staircase graphs as $G(3S_v, \ell)$, $G(4S_v, \ell)$ respectively as given in the following Figure 4.

In this paper we prove that any triple, four and n-chain vertex merged staircase graph admit prime cordial labeling.
2 Main Results

**Theorem 2.1** Any staircase graph of order \( \ell \) admits prime cordial labeling [2].

**Theorem 2.2** Any twin vertex merged staircase graph of order \( \ell \) admits prime cordial labeling [2].

**Theorem 2.3** Any triple vertex merged staircase graph of order \( \ell \) admits PCL.

**Proof.**

The PCL is applicable for both odd and even order of twin vertex merged staircase graph.

Let \( G(3S_\nu, \ell) \) be the triple vertex merged staircase graph of order \( \ell \), it contains three staircase graphs \( G_1(S_\nu, \ell), G_2(S_\nu, \ell), G_3(S_\nu, \ell) \).

It contains \( p = \left( \frac{11\ell^2 + 15\ell + 6}{2} \right) - 2 \) vertices and \( (3\ell^2 + 9\ell) \) edges.

A pattern of labeling for odd and even order of \( G(3S_\nu, \ell) \) are given in Figures 05 and 06.

Let \( h : V(G) \to |V| \) be the vertex labeling function.

Induced edge labeling function \( h^* : E(G) \to \{0, 1\} \) defined as

\[
h^*(e = uv) = \begin{cases} 1; & \text{if } g . c . d \left( h(u), h(v) \right) = 1 \\ 0 & \text{otherwise} \end{cases}
\]

the resultant number of edges labeled as 0 and the number of edges labeled as 1 differ atmost by 1.

Therefore, the triple vertex merged staircase graph \( G(3S_\nu, \ell) \) that admits prime cordial labeling.
Figure 5: PCL of $G(3S_v, \ell')$ of odd order

Figure 6: PCL of $G(3S_v, \ell')$ of even order
Example 2.3.1

Figure 7: PCL of $G(3S_v, 3)$

Example 2.3.2

Figure 8: PCL of $G(3S_v, 4)$
**Theorem 2.4.** Any four chain vertex merged staircase graph of order ℓ admit prime cordial labeling.

**Proof.**
The prime cordial labeling is applicable for both odd and even order of four chain vertex merged staircase graph.

Let G(4S_v, ℓ) be the four chain vertex merged staircase graph of order ℓ, it contains three staircase graphs G_1(S_1, ℓ) and G_2(S_2, ℓ), G_3(S_3, ℓ) and G_4(S_4, ℓ).

It contains \( p = 2\ell^2 + 10\ell + 1 \) vertices and \( (4\ell^2 + 12\ell) \) edges.

A pattern of labeling for odd and even order of G(4S_v, ℓ) are given in Figures 09 and 10.

Let \( h: V(G) \rightarrow \{0, 1\} \) be the vertex labeling function.

An edge labeling function \( h^*: E(G) \rightarrow \{0, 1\} \) defined as \( h^*(e = uv) = \begin{cases} 1 & \text{if } \gcd(h(u), h(v)) = 1 \\ 0 & \text{otherwise} \end{cases} \)
gives the total number of edges labeled as 0 and the total number of edges labeled as 1 differ at most by 1.

Therefore, the four chain vertex merged staircase graph G(4S_v, ℓ) that admits prime cordial labeling.

\[ p = 2\ell^2 + 10\ell + 1 \]
\[ i = \ell^2 + 7\ell \]
\[ j = \ell^2 + 9\ell \]
\[ k = \ell^2 + 5\ell \]

**Figure 9:** PCL of G(4S_v, ℓ) of odd order
Figure 10: PCL of $G(4S_v, \ell')$ of even order

\[ p = 2\ell^2 + 10\ell + 1 \]
\[ i = \ell^2 + 7\ell \]
\[ j = \ell^2 + 9\ell \]
\[ k = \ell^2 + 5\ell \]
Example 2.4.1

![Diagram](example2.4.1Diagram)

Figure 11: PCL of $G(4S_3, 3)$

Example 2.4.2

![Diagram](example2.4.2Diagram)

Figure 12: PCL of $G(4S_3, 4)$
Theorem 2.5. Any n-chain vertex merged staircase graph of order \( \ell \) admit prime cordial labeling.

**Proof.**

The n-chain vertex merged staircase Graph consists of n staircase graphs. For \( n=\text{odd} \), the pattern of labeling is similar to \( G(3S_v, \ell) \) and for \( n=\text{even} \), the pattern of labeling is similar to \( G(4S_v, \ell) \). Hence for any finite positive integer \( n \) the n-chain vertex merged staircase graphs is prime cordial.

3 Future Directions

Staircase graph of different order can be combined to create twin, triple vertex merged staircase graphs. There are many labeling methods not tested on the existing set of staircase graphs and the new ones to be generated. These staircase graph gives a lot of future opportunities of research.

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