Compressing compound states

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Quantum compression can be thought of not only as compression of a signal, but also as a form of cooling. In this view, one is interested not in the signal, but in obtaining purity. In compound systems, one may be interested to cool the system to obtain local purity by use of local operations and classical communication [Oppenheim et al. Phys. Rev. Lett. 89, 180402 (2002)]. Here we compare it with usual compression and find that it can be represented as compression with suitably restricted means.

I. INTRODUCTION

The technique of quantum date compression (1) (cf. 2, 3), can be used to transform a signal from a quantum source onto a smaller number of qubits in order to obtain more efficient storage or transmission. In this scenario, one is interested in preserving the signal, and one discards the redundant qubits. These redundant qubits are then thought of as containing no information, and can be regarded as being in some known (or standard) state. One might also be interested in applying this technique in another situation, namely, in the case where one treats the signal as noise, and instead, one is interested in obtaining pure states in a standard state. Such a situation was considered in 4 where it was applied to produce more pure initial states for a NMR quantum computer. In this case, “the signal” is actually noise and one compresses and discards this noise to isolate the pure states (see 2 in this context).

Such a technique is also useful from the point of view of a paradigm introduced in 5. There, one considers parties in distant labs who share a state \( \rho_{AB} \), and who wish to distill local pure states. This can be thought of as the complementary procedure to the usual situation, where they attempt to distill entanglement. It was found that this paradigm allows one to understand nature of correlations in shared quantum states. The insight came from thermodynamics, where pure states in a known state can be treated as a resource and used to extract work from a single heat bath 6. A more information theoretic analysis was done in 7 and 11 where the techniques of quantum information, including compression where applied to such a paradigm.

Here, we expand on the ideas introduced in 8 and show in greater detail how to apply techniques developed by Rains 11 for entanglement theory, to the paradigm of distilling pure product states. This allows to express the problem of distilling local information in terms of compression with restricted means.

II. CONCENTRATION OF SUBJECTIVE AND OBJECTIVE INFORMATION

In this section we will discuss the dual pictures of compression of quantum information. We will follow the approach of Ref. 10. Let us first calculate the rate of compression in the general case where one is free to perform all operations.

Given a state \( \varrho^\otimes n \), with \( \varrho \) acting on Hilbert space \( C^d \) one can ask for the smallest Hilbert space that still carries most of the weight of the state. Such a Hilbert space is called the typical subspace. In other words we would like to know the minimal dimension of the projector \( P \) satisfying

\[
\text{Tr} \varrho^\otimes n P \geq 1 - \epsilon.
\]

The interesting characteristic is the number of qubits of the typical subspace, per input copy \( \varrho \)

\[
R_\epsilon = \lim_{n \to \infty} \frac{1}{n} \log \text{Tr} P.
\]

where \( P \) is chosen to have minimal dimension under the restriction (1). It turns out (2) that for \( \epsilon \in (0, 1) \), \( R_\epsilon \) does not depend on \( \epsilon \) and is equal to the von Neumann entropy of \( \varrho \)

\[
R_\epsilon \equiv R = S(\varrho).
\]

The rate \( R \) can be interpreted as the amount of qubits needed to carry the signal produced by the source, for which the density matrix of \( n \) messages is \( \varrho^\otimes n \). The actual scheme of transmission of the information from the source is as follows. One makes a measurement given by the identity resolution \( \{ P, I - P \} \) and with high probability the outcome corresponding to \( P \) is obtained. Then the resulting state collapses onto the typical subspace (the subspace of the projector \( P \)). Thus the support of the state has now dimension \( 2^n S \). The total system can be now unitarily transformed into the state \( \varrho' \otimes |\psi\rangle\langle\psi| \). The state \( \varrho' \) carries the information, while the state \( |\psi\rangle\langle\psi| \) represents redundancies. The state resides on the tensor product of two spaces: the “signal space” \( H_s \) consisting of \( nS \) qubits and the “redundancy space” \( H_r \) consisting of \( n(\log d - S) \) qubits.
After we have perform concentration of information we have two opposite schemes. In the "subjective" scenario information is randomness, and it is represented by the signal part. One then keeps this part, while discarding redundancies. This is the usual interpretation of Schumacher compression. The signal contains the information because it contains something which we have to read in order to learn about it. The more random something is, the more information it has. We are surprised by its contents.

In the opposite scheme – the "objective" scenario – the information is purity. A known pure state contains information because we know what the state it. A standard pure state is something which is known to us, hence we consider it to be information. Here, the "signal part" is treated as noise, and is rejected, while the redundancy part represents a valuable resource - qubits in pure states. The former approach is in spirit of Shannon, while the latter one is of thermodynamic origin and is in the spirit of Brillouin and Szilard.

Note that in the first scenario, one has to be careful that the operation does not damage the information of the signal part. Thus the compression is attained by a highly degenerated measurement, so that the pure states - individual signals - will not lose their quantum coherences. However if we are interested in the pure part, then we can make a measurement that is non-degenerate. In particular it can be a complete measurement, with one dimensional projectors. Then, if the noise were actually some "quantum information", it would get totally destroyed. But the pure part would remain untouched (if the measurement is chosen in such a way that |ψ⟩ corresponds to possible outcomes).

In this paper we are interested in the "objective" scenario, and information for us is "purity". The rate of obtaining pure qubits is given by

\[ R^{conc} = \log d - S = \log d - \lim_{n} \frac{1}{n} \log \text{Tr} P \rho^\otimes n \]  

(recall that \( P \) depends on \( n \)). This is equal to the information \( I \), and is a unique measure of information\(^{[10]}\). One can generalize the picture, by replacing the projectors with positive operators\(^{[11]}\). One is then interested in the positive operator \( A \) of minimal trace, satisfying

\[ \text{Tr}(\rho^\otimes n A) \geq 1 - \epsilon \]  

It turns out that the rate is the same as in the case of projectors. Thus one can express the rate of concentration of information as follows

\[ R^{conc} = \log d - \lim_{n} \frac{1}{n} \log \text{Tr} A \rho^\otimes n \]  

where \( A \) is chosen to have minimal trace and satisfy the condition\(^{[8]}\). In the following section we will see that the concentration of information in the distn lab paradigm is connected with a similar question, however the operator \( A \) will be suitably constrained.

### III. NLOCC Maps and Dual Maps

In entanglement theory, the paradigm is based on considering Local Operations and Classical Communication (LOCC maps). Here we will define a map which we call NLOCC maps\(^{[9]}\) for Noisy Local Operations and Classical Communication. The motivation for using NLOCC maps, as opposed to the usual LOCC maps is that here we are interested not in distilling singlets, but in distilling pure product states, and so, care must be taken to properly account for all pure states, including local ancillas, involved in any transformation. We will show that the dual map of an NLOCC map is (up to a factor) also an NLOCC map. We will need this to represent distillation to product states as compression with restricted means.

#### A. NLOCC Maps

NLOCC maps are any maps that can be composed of the following maps:

1. local unitary transformations
2. adding quantum system in a maximally mixed state
3. discarding local subsystems (local partial trace)
4. sending subsystems down completely decohering (dephasing) channels

The latter channel is of the form

\[ \varrho_{in} \rightarrow \varrho_{out} = \sum_{i} P_{i} \varrho_{in} P_{i} \]  

where \( P_{i} \) are one-dimensional projectors. For a qubit system, it acts as

\[ \varrho_{in} = \begin{bmatrix} \varrho_{11} & \varrho_{12} \\ \varrho_{21} & \varrho_{22} \end{bmatrix} \rightarrow \varrho_{out} = \begin{bmatrix} \varrho_{11} & 0 \\ 0 & \varrho_{22} \end{bmatrix} \]  

i.e. the state becomes classical, as if it has been measured. Here, \( \varrho_{in} \) is at the sender’s site, while \( \varrho_{out} \) is at the receiver’s site. The operation\(^{[11]}\) can be disassembled into two parts: (i) local dephasing (at say, the sender’s site) and (ii) sending qubits intact (through a noiseless quantum channel) to the receiver. Thus suppose that Alice and Bob share a state \( \varrho_{AB} \equiv \varrho_{A^\prime A^\prime^\prime B} \), and Alice decides to send subsystem \( A^\prime \) to Bob, down the dephasing channel. The following action will have the same effect: Alice dephases locally the subsystem \( A^\prime \)

\[ \varrho_{A^\prime A^\prime^\prime B} \rightarrow \sum_{i} (P_{i}^{A^\prime^\prime} \otimes I_{A^\prime B}) \varrho_{A^\prime A^\prime^\prime B} (P_{i}^{A^\prime^\prime} \otimes I_{A^\prime B}) \]  

The state is now of the form

\[ \varrho_{A^\prime A^\prime^\prime B}^{out} = \sum_{i} P_{i} A^{i} \otimes \varrho_{i}^{AB} \]
Thus the part $A''$ is classically correlated with the rest of the system (this is a stronger statement than to say that the state is separable with respect to $A'' : A'B$). Now Alice sends the system $A''$ to Bob through an ideal channel. Thus the final state differs from the state $\psi_{A'' : A'B}$ only in that the system $A''$ is at Bob site. It follows that the operation $\Gamma$ can be replaced by the following two operations

4a) Local dephasing

4a) Sending completely dephased subsystem.

Note that the definition of NLOCC maps differ from LOCC only in that one cannot add local ancilla in any state, but only in maximally mixed one. One allows POVM’s which involves adding a pure state ancilla, but we do so by including the ancilla as part of the initial state $\rho$. This difference causes us to consider in more detail the classical communication. Indeed, since information becomes the resource, we have to take care also about the bits that carry information between the labs. The information represented by the carriers must be counted. Otherwise, given the initial state, we could supply an infinite amount of local information, just by multiplying the number of bits communicated.

The reader may have the impression that unlike in LOCC, we have disallowed making measurement, and that this is another big difference between our maps and LOCC. As a matter of fact, we do allow measurements, however we care not only about the system to be measured but also the measuring apparatus. This is natural approach when one counts pure states as a resource, since a measuring device must initially be in a pure state to be effective, and one must therefore take care to properly account for this. This is also natural from the thermodynamical point of view: Maxwell’s demon was exorcised just by including itself into the description [12].

**B. Dual maps**

Let us now describe the dual maps to the elementary NLOCC maps of the previous subsection, and show that they are also NLOCC maps up to a factor. Consider a map

$$\Lambda : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$$

The dual map

$$\Lambda^\dagger : \mathcal{B}(\mathcal{H}_2) \rightarrow \mathcal{B}(\mathcal{H}_1)$$

is defined with respect to the Hilbert-Schmidt scalar product

$$\langle A | B \rangle = \text{Tr}(A^\dagger B)$$

by the following relation

$$\text{Tr}(A^\dagger \Lambda(B)) = \text{Tr}(\Lambda^\dagger(A^\dagger)B)$$

for any operators $A \in \mathcal{B}(\mathcal{H}_1)$, $B \in \mathcal{B}(\mathcal{H}_2)$. If the map is completely positive i.e. it is of the form

$$\Lambda(\varrho) = \sum_i V_i \varrho V_i^\dagger$$

its dual is of the form

$$\Lambda^\dagger(\varrho) = \sum_i V_i^\dagger \varrho V_i$$

i.e. it is still completely positive, though perhaps not trace preserving. If the output dimensions are equal to each other, then the dual is trace preserving iff the map preserves the identity (is bistochastic). Out of our elementary maps, the maps with equal in and out dimensions are (1),(4a) and (4b). They also preserve the identity, hence their duals are valid physical operations. Specifically, the dual to a local unitary (the inverse one), and the local dephasing is self-dual, i.e. is equal to its dual. This can be seen simply by inspecting (15) and (16).

A dual to the map (4b) is simply sending a qubit in the converse direction. Consider now the maps (2) and (3). They are not trace preserving, but the only reason is that the in and out dimensions are not equal, hence, as trace preserving maps, they cannot preserve identity. However our maps (2) and (3) preserve the maximally mixed state. As a result the duals will be trace preserving up to a factor. For a state $\varrho$ we have

$$\text{Tr}(\Lambda^\dagger(\varrho)) = \text{Tr}(\Lambda^\dagger(\varrho)(I_{out})) = \text{Tr}(\varrho(I_{out})) = \text{Tr}(\varrho) = d_{out} \text{Tr}(\varrho) / d_{in}$$

where $d_{in}$ and $d_{out}$ are the dimensions of input and output Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Consequently the map $\Gamma = d_{out} / d_{in} \Lambda^\dagger$ is a valid physical operation.

One then easily finds that up to a factor the map (2) of adding a maximally mixed state is dual to map (3) of local partial trace and vice versa. More precisely if

$$\Lambda(\varrho_{A'':A'B}) = \varrho_{A''B}$$

is (local) partial trace then the dual map is given by

$$\Lambda^\dagger(\varrho_{A'B}) = \varrho_{A'B} \otimes I_{A''} = \dim_{A''} \left[ \varrho_{A'B} \otimes \frac{I_{A''}}{\dim_{A''}} \right]$$

**IV. DISTILLING PURE PRODUCT STATES AND COMPRESSION**

In this section we will represent distillation of product states by NLOCC as compression with restricted means. First of all it is clear that in comparison with the usual compression situation described in section III we have a restricted class of operations. As we shall see, this will lead us to optimize over suitably constrained positive operators $A$. We will follow Rains approach [11] (see also
make a finer measurement that will commute with the above \( P \). This is okay in this paradigm, since the pure states remain pure (the signal is damaged, but we do not care about that in this “objective” compression scheme). If the measurement is finer only in the part \( P \), so that it is of the form \( \{P_1, \ldots, P_n, I - P\} \), then the concentration is done as well by NLOCC as by means of NO: the pure state is untouched, even though the signal state is cut into incoherent pieces. However if it is finer in the second part \( I - P \), then this may damage the state \( \psi \).

Let us now go back to discussion of the constraints (i) and (ii) above that \( \Pi \) must satisfy. The first one is the same as the one of eq. (6) in the usual compression scheme, with positive operators instead of projectors. Thus we should consider the second constraint that was not present in the usual compression case. First of all, the operator \( \Pi \) must be, up to a factor, a separable state. Thus it must be a mixture of product positive operators. This is because the map \( \Lambda^j \) is up to a factor, an NLOCC operation, hence it can not transform a product state \( P_{(00)}^{m} \) into an entangled state.

This suggests considering the maximal rate possible with operators \( \Pi \) being separable ones. This is certainly an upper bound for the optimal rate of concentration of information. It closely resembles compression of an unknown source which we now describe roughly [14]. Suppose we have a source \( \Phi^{n} \) but think it is \( \Phi^{n} \). We then take projectors \( P \), which are suitable for \( \sigma \). Of course it will not work for our source. However, we can try to take projectors of greater dimension built in an analogous way to \( P \). The question is how large should the subspace be, to be good also for the state \( \Phi^{n} \) ? The answer is that the compression rate will now be worse. The penalty will be \( S(\rho|\sigma) = \text{Tr}_P \log \rho - \text{Tr}_P \log \sigma \). Thus the rate of “objective” information concentration will be \( \log d - S(\rho) - S(\rho|\sigma) \). We thus get the bound for the rate under NLOCC

\[
r \leq \log d - S(\rho) - \inf_{\sigma \in \text{sep}} S(\rho|\sigma)
\]

with \( \sigma \) taken from the set of separable states. In view of the discussion above, this scheme of concentration of information to a local form is a sort of compression while imagining that the state is separable, or at least while using “separable tools”.

If such a bound were achievable, then we would essentially have that the penalty one pays under NLOCC for compression, is the relative entropy of entanglement [15]. However, it seems that instead, the penalty is given not only by the entanglement, but also, by so-called non-locality without entanglement [16]. Whether this is the case, amounts to the question: can we produce by use of the map \( \Gamma = \frac{m}{m} \Lambda^j \), where \( \Lambda \) is NLOCC, all possible separable states of the input system out of \( P_{(00)}^{m} \) ? The answer seems to be negative: Namely there are additional constraints. Since the map \( \Gamma \) is NLOCC, it cannot increase the amount of information. The initial information is \( 2m \) bits (this is actually the final information
in the real scenario, yet here we deal with the dual scenario. Thus one can obtain only such separable states, that can be prepared by NLOCC from $2m$ bits of information. The question of information of formation was considered in Ref. [17]. It was pointed out there, that the local information needed to create a separable state may be greater than $n - S$. This is because, a separable state may be a mixture of product states, that are locally orthogonal. Then to create such states, one will be forced to use some irreversible operations. In particular, one expects density matrices whose eigenbasis are the “sausage states” of Ref. [16] to not be preparable with only $n - S$ bits. Thus the fact that we deal with NLOCC operations rather than LOCC ones enter the scene twice. First, it restricts the trace of operator II making the problem nontrivial. Second, it restricts the class of separable states, that corresponds to II up to a factor.

\[ r = \log d - S(\rho) - \inf_{\sigma \in \text{IPB}} S(\rho | \sigma) \]  

(26)

where IPB is the set of states whose eigenbasis were orthogonal projectors implementable under NLOCC. i.e. projectors which can be achieved from the standard projector $P^{\otimes m}$. It would be very valuable to prove such a conjecture.

Here, we have explored an “objective” scheme of compression of information, where instead of protecting a signal, one instead wishes to obtain pure states. It also may be interesting to explore the “subjective” scheme of compression in the distant labs paradigm. Such scenarios might lead one to consider various quantum versions of the Slepian-Wolf theorem [18] (c.f. [19, 20, 21]).

In Slepian-Wolf encoding, two people in distant labs attempt to compress a signal. They then meet at some later time to decompress it. In a quantum version of such a scenario, the compression rate would depend heavily on the ensemble. If the signal states were entangled, then very little compression could take place, but if the signal states were classical states, then the entire state should be compressable. The rate of compression would therefore not be a universal function of the density matrix, but would instead depend on the ensemble. Further exploration of a quantum Slepian-Wolf theorem may be of interest.

\[ \text{Acknowledgments} \]

This work is supported by EU grants RESQ, contract No. IST-2001-37559, and QUPRODIS, contract No. IST-2001-38877. J.O. acknowledges the support of the Lady Davis Fellowship Trust, and grant No. 129/00-1 of the Israel Science Foundation.

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