Entropy signature of the running cosmological constant

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Abstract. The renormalization group (RG) improved cosmologies based upon a RG trajectory of quantum Einstein gravity (QEG) with realistic parameter values are investigated using a system of cosmological evolution equations which allows for an unrestricted energy exchange between the vacuum and the matter sector. It is demonstrated that the scale dependence of the gravitational parameters, the cosmological constant in particular, leads to an entropy production in the matter system. The picture emerges that the Universe started out from a state of vanishing entropy, and that the radiation entropy observed today is essentially due to the coarse graining (RG flow) in the quantum gravity sector which is related to the expansion of the Universe. Furthermore, the RG improved field equations are shown to possess solutions with an epoch of power law inflation immediately after the initial singularity. The inflation is driven by the cosmological constant and ends automatically once the RG running has reduced the vacuum energy to the level of the matter energy density.

Keywords: inflation, quantum gravity phenomenology, cosmology of theories beyond the SM, physics of the early universe

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1. Introduction

After the introduction of a functional renormalization group for gravity [1] detailed investigations of the nonperturbative renormalization group (RG) behaviour of quantum Einstein gravity have become possible [1]–[16]. The exact RG equation underlying this approach defines a Wilsonian RG flow on a theory space which consists of all diffeomorphism invariant functionals of the metric $g_{\mu\nu}$. The approach turned out to be an ideal setting for investigating the asymptotic safety scenario in gravity [17,18] and, in fact, substantial evidence was found for the nonperturbative renormalizability of quantum Einstein gravity. The theory emerging from this construction (sometimes denoted ‘QEG’) is not a quantization of classical general relativity. Instead, its bare action corresponds to a nontrivial fixed point of the RG flow and is a prediction therefore, and not as usually in quantum field theory an ad hoc assumption defining some ‘model’. Independent support for the asymptotic safety conjecture came from a two-dimensional symmetry reduction of the gravitational path integral [19]. The approach of [1] employs the effective
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average action [20]–[22] which has crucial advantages as compared to other continuum implementations of the Wilson RG [23]. In particular it is closely related to the standard effective action and defines a family of effective field theories \( \{ \Gamma_k[g_{\mu\nu}], 0 \leq k < \infty \} \) labelled by the coarse graining scale \( k \). The latter property opens the door to a rather direct extraction of physical information from the RG flow, at least in single-scale cases: if the physical process or phenomenon under consideration involves only a single typical momentum scale \( p_0 \) it can be described by a tree-level evaluation of \( \Gamma_k[g_{\mu\nu}] \), with \( k = p_0 \). The precision which can be achieved by this effective field theory description depends on the size of the fluctuations relative to mean values. If they are large, or if more than one scale is involved, it might be necessary to go beyond the tree analysis.

The effective field theory techniques proved useful for an understanding of the scale-dependent geometry of the effective QEG spacetimes [24, 26, 27]. In particular it has been shown [3, 5, 24] that these spacetimes have fractal properties, with a fractal dimension of 2 at small, and 4 at large, distances. The same dynamical dimensional reduction was also observed in numerical studies of Lorentzian dynamical triangulations [25, 28, 29] and in [30] Connes et al speculated about its possible relevance to the noncommutative geometry of the standard model.

The RG flow of the effective average action, obtained by different truncations of theory space, has been the basis of various investigations of ‘RG improved’ black hole and cosmological spacetimes [31]–[41]. We shall discuss some aspects of this method below.

A special class of RG trajectories obtained from QEG in the Einstein–Hilbert approximation [1], namely those of the ‘Type IIIa’ [4], possess all the qualitative properties one would expect from the RG trajectory describing gravitational phenomena in the real Universe we live in. In particular they can have a long classical regime and a small, positive cosmological constant in the infrared (IR). Determining its parameters from observations, one finds [39] that, according to this particular QEG trajectory, the running cosmological constant \( \Lambda(k) \) changes by about 120 orders of magnitude between \( k \) values of the order of the Planck mass and macroscopic scales, while the running Newton constant \( G(k) \) has no strong \( k \) dependence in this regime. For \( k > m_{\text{Pl}} \), the non-Gaussian fixed point (NGFP) which is responsible for the renormalizability of QEG controls their scale dependence. In the deep-ultraviolet \( (k \to \infty) \), \( \Lambda(k) \) diverges and \( G(k) \) approaches zero.

In the present paper we are going to ask whether there is any experimental or observational evidence that would hint at this enormous scale dependence of the gravitational parameters, the cosmological constant in particular. Clearly the natural place to search for such phenomena is cosmology. Even though it is always difficult to give a precise physical interpretation to the RG scale \( k \) is fairly certain that any sensible identification of \( k \) in terms of cosmological quantities will lead to a \( k \) which decreases during the expansion of the Universe. As a consequence, \( \Lambda(k) \) will also decrease as the Universe expands. Already the purely qualitative assumption of a positive and decreasing cosmological constant supplies an interesting hint as to which phenomena might reflect a possible \( \Lambda \)-running.

To make the argument as simple as possible, let us first consider a Universe without matter, but with a positive \( \Lambda \). Assuming maximal symmetry, this is nothing but de Sitter space, of course. In static coordinates its metric is

\[
\begin{align*}
\text{d}s^2 &= -\left(1 + 2\Phi_N(r)\right)\text{d}t^2 + \left(1 + 2\Phi_N(r)\right)^{-1}\text{d}r^2 + r^2(\text{d}\theta^2 + \sin^2\theta \text{d}\phi^2)
\end{align*}
\] (1.1)

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Figure 1. The quasi-Newtonian potential corresponding to de Sitter space. The curve moves upward as the cosmological constant decreases.

\[ \Phi_N(r) = -\frac{1}{6}\Lambda r^2. \]  

with

In the weak field and slow motion limit \( \Phi_N \) has the interpretation of a Newtonian potential, with a correspondingly simple physical interpretation. Figure 1 shows \( \Phi_N \) as a function of \( r \); for \( \Lambda > 0 \) it is an upside-down parabola. Point particles in this spacetime, symbolized by the black dot in figure 1, ‘roll down the hill’ and are rapidly driven away from the origin and from any other particle. Now assume that the magnitude of \( \Lambda \) is slowly (‘adiabatically’) decreased. This will cause the potential \( \Phi_N(r) \) to move upward as a whole, its slope decreasing. So the change in \( \Lambda \) increases the particle’s potential energy. This is the simplest way of understanding that a positive decreasing cosmological constant has the effect of ‘pumping’ energy into the matter degrees of freedom. More realistically one will describe the matter system in a hydrodynamics or quantum field theory language and one will include its backreaction onto the metric. But the basic conclusion, namely that a slow decrease of a positive \( \Lambda \) transfers energy into the matter system, will remain true.

We are thus led to the suspicion that, because of the decreasing cosmological constant, there is a continuous inflow of energy into the cosmological fluid contained in an expanding Universe. It will ‘heat up’ the fluid or, more exactly, lead to a slower decrease of the temperature than in standard cosmology. Furthermore, by elementary thermodynamics, it will increase the entropy of the fluid. If during the time \( dt \) an amount of heat \( dQ > 0 \) is transferred into a volume \( V \) at the temperature \( T \) the entropy changes by an amount \( dS = dQ/T > 0 \). To be as conservative (i.e. close to standard cosmology) as possible, we assume that this process is reversible. If not, \( dS \) is even larger.

In standard Friedmann–Robertson–Walker (FRW) cosmology the expansion is adiabatic and the entropy (within a comoving volume) is constant. It has always been somewhat puzzling therefore where the huge amount of entropy contained in the present Universe comes from. Presumably it is dominated by the CMBR photons which contribute an amount of about \( 10^{88} \) to the entropy within the present Hubble sphere. (We use units such that \( k_B = 1 \).) In fact, if it is really true that no entropy is produced during the expansion then the Universe would have had an entropy of at least \( 10^{88} \) immediately after
the initial singularity which, for various reasons, seems quite unnatural [42]. In scenarios which invoke a ‘tunnelling from nothing’, for instance, spacetime was ‘born’ in a pure quantum state, so the very early Universe is expected to have essentially no entropy [43]. Usually it is argued that the entropy present today is the result of some sort of ‘coarse graining’ which, however, typically is not considered an active part of the cosmological dynamics in the sense that it would have an impact on the time evolution of the metric, say.

In the present paper we are going to argue that in principle the entire entropy of the massless fields in the present Universe can be understood as arising from the mechanism described above, the ‘heating’ of matter by a decreasing cosmological constant. If energy can be exchanged freely between the cosmological constant and the matter degrees of freedom, the entropy observed today is obtained precisely if the initial entropy at the ‘big bang’ vanishes.

The assumption that the matter system must allow for an unhindered energy exchange with Λ is nontrivial. In [32] and [33], henceforth referred as [I] and [II], respectively, ‘RG improved’ cosmologies were studied which, too, are based upon the RG trajectories of QEG. In these investigations it has been assumed, however, that there is no injection of energy into the matter system due to the time dependence of Λ, and that the evolution is adiabatic therefore. In the present paper we explore the opposite situation of a completely unobstructed energy transfer. Technically this amounts to dropping the so-called ‘consistency condition’ imposed in [I] and [II]. Which one of the two cases is more realistic depends on the cosmological epoch and on properties of the matter model (particle masses, couplings, etc.).

As in [I] and [II] the computational setting of the present paper is the RG improved Einstein equations: by means of a suitable cutoff identification $k = k(t)$ we turn the scale dependence of $G(k)$ and $Λ(k)$ into a time dependence, and then substitute the resulting $G(t) \equiv G(k(t))$ and $Λ(t) \equiv Λ(k(t))$ into the Einstein equations. We shall obtain the RG trajectory by solving the flow equation for the Einstein–Hilbert truncation with a sharp cutoff [1,4]. We then construct quantum corrected cosmologies by (numerically) solving the RG improved cosmological evolution equations.

We model the matter in the early Universe by a gas with $n_b$ bosonic and $n_f$ fermionic massless degrees of freedom, all at the same temperature. In equilibrium its energy density, pressure and entropy density are given by the usual relations ($n_{\text{eff}} = n_b + \frac{7}{8} n_f$)

$$\rho = 3p = \frac{\pi^2}{30} n_{\text{eff}} T^4$$  \hspace{1cm} (1.3a)

$$s = \frac{2\pi^2}{45} n_{\text{eff}} T^3$$  \hspace{1cm} (1.3b)

so that, in terms of $U \equiv \rho V$ and $S \equiv sV$,

$$T \, dS = dU + p \, dV.$$  \hspace{1cm} (1.3c)

In an out-of-equilibrium process of entropy generation the question arises how the various thermodynamical quantities are related then. To be as conservative as possible, we make the assumption that the irreversible inflow of energy destroys thermal equilibrium as little as possible in the sense that the equilibrium relation (1.3) continues to be (approximately) valid.
This kind of thermodynamics in an FRW-type cosmology with a decaying cosmological constant has been analysed in detail by Lima [44], see also [45]. It was shown that if the process of matter creation \( \Lambda(t) \) gives rise to is such that the specific entropy per particle is constant, the relations of equilibrium thermodynamics are preserved. This means that no finite thermalization time is required since the particles originating from the decaying vacuum are created in equilibrium with the already existing ones. Under these conditions it is also possible to derive a generalized black-body spectrum which is conserved under time evolution. Such minimally non-adiabatic processes were termed ‘adiabatic’ (with the quotation marks) in [44, 45].

In section 3 of the present paper we shall discuss the ‘adiabatic’ generation of entropy in the framework of the RG improved cosmology.

There is another, more direct potential consequence of a decreasing positive cosmological constant which we shall also explore in this paper, namely a period of automatic inflation during the very first stages of the cosmological evolution. It is not surprising, of course, that a positive \( \Lambda \) can cause an accelerated expansion, but in the classical context the problem with a \( \Lambda \)-driven inflation is that it would never terminate once it has started. In popular models of scalar driven inflation [50] this problem is circumvented by designing the inflaton potential in such a way that it gives rise to a vanishing vacuum energy after a period of ‘slow roll’.

In this paper we shall see that generic RG cosmologies based upon the QEG trajectories have an era of \( \Lambda \)-driven inflation immediately after the big bang which ends automatically as a consequence of the RG running of \( \Lambda(t) \). Once the scale \( k \) drops significantly below \( m_{\text{Pl}} \), the accelerated expansion ends because the vacuum energy density \( \rho_\Lambda \) is already too small to compete with the matter density. Clearly this is a very attractive scenario: one needs no ad hoc ingredients, such as an inflaton field or a special potential, either to trigger inflation or to stop it. It suffices to include the leading quantum effects in the gravity + matter system.

It is to be emphasized that the present investigations are not some sort of ‘model building’ of decaying-\( \Lambda \) cosmologies; they rather deal with consequences of the computable scale dependence of \( \Lambda \) and \( G \). Besides the validity of the mean field description and the above assumptions about the thermodynamical properties of matter, the only other assumption we make is that the renormalization effects of the matter fields, which are not taken into account explicitly, do not alter the RG flow of pure gravity as far as qualitative features and orders of magnitude are concerned. If so, it makes sense to confront the RG trajectories of pure QEG with observations in the real world.

Let us briefly review how the type IIIa trajectories of the Einstein–Hilbert truncation can be matched against the observational data [31]. This analysis is fairly robust and clear-cut; it does not involve the NGFP. All that is needed is the RG flow linearized about the GFP. It is [1]

\[
\Lambda(k) = \Lambda_0 + \nu \bar{G} k^4 + \cdots \\
G(k) = \bar{G} + \cdots.
\]  

(1.4)

Or, in terms of the dimensionless couplings \( g(k) \equiv k^2 G(k) \), and \( \lambda(k) \equiv \Lambda(k)/k^2 \):

\[
\lambda(k) = \Lambda_0/k^2 + \nu \bar{G} k^2 + \cdots \\
g(k) = \bar{G} k^2 + \cdots.
\]  

(1.5)
In the linear regime of the GFP, $\Lambda$ displays a running $\propto k^4$ which is seen in perturbation theory already, and $G$ is approximately constant. Here $\nu$ is a positive constant of order unity [1, 4],

$$\nu \equiv \frac{1}{4\pi} \Phi_2'(0) \equiv \frac{\varphi_2}{4\pi}. \tag{1.6}$$

Equations (1.5) are valid if $\lambda(k) \ll 1$ and $g(k) \ll 1$. They describe a two-parameter family of RG trajectories labelled by the pair $(\Lambda_0, \tilde{G})$. It will prove convenient to use an alternative labeling $(\lambda_T, k_T)$ with

$$\lambda_T \equiv (4\nu\Lambda_0\tilde{G})^{1/2}$$
$$k_T \equiv \left(\frac{\Lambda_0}{\nu\tilde{G}}\right)^{1/4}. \tag{1.7}$$

The old labels are expressed in terms of the new ones as

$$\Lambda_0 = \frac{1}{4}\lambda_T k_T^2$$
$$\tilde{G} = \frac{\lambda_T}{2\nu k_T^2}. \tag{1.8}$$

It us furthermore convenient to introduce the abbreviation (not an independent label)

$$g_T \equiv \frac{\lambda_T}{2\nu} \equiv \frac{\lambda_T}{(\varphi_2/2\pi)}. \tag{1.9}$$

When parametrized by the pair $(\lambda_T, k_T)$ the trajectories assume the form

$$\Lambda(k) = \frac{1}{4}\lambda_T k_T^2 \left[1 + \left(\frac{k}{k_T}\right)^4\right] \equiv \Lambda_0 \left[1 + \left(\frac{k}{k_T}\right)^4\right]$$
$$G(k) = \frac{\lambda_T}{2\nu k_T^2} \equiv \frac{g_T}{k_T^2}. \tag{1.10}$$

or, in dimensionless form,

$$\lambda(k) = \frac{1}{4}\lambda_T \left[\left(\frac{k_T}{k}\right)^2 + \left(\frac{k}{k_T}\right)^2\right]$$
$$g(k) = g_T \left(\frac{k}{k_T}\right)^2. \tag{1.11}$$

Note that $\lambda(k)$ is invariant under the ‘duality transformation’ [26] $k \mapsto k_T^2/k$. As for the interpretation of the new variables, it is clear that $\lambda_T \equiv \lambda(k \equiv k_T)$ and $g_T \equiv g(k = k_T)$, while $k_T$ is the scale at which $\beta_{\lambda}$ (but not $\beta_g$) vanishes according to the linearized running (1.11):

$$\beta_{\lambda}(k_T) \equiv \frac{d\lambda(k)}{dk} \bigg|_{k=k_T} = 0. \tag{1.12}$$
Thus we see that \((g_T, \lambda_T)\) are the coordinates of the turning point \(T\) of the type IIIa trajectory considered, and \(k_T\) is the scale at which it is passed. It is convenient to refer the ‘RG time’ \(\tau\) to this scale:

\[
\tau(k) \equiv \ln(k/k_T)
\]

(1.13)

so that \(\tau > 0\) \((\tau < 0)\) corresponds to the ‘UV regime’ (‘IR regime’) where \(k > k_T\) \((k < k_T)\).

In terms of the RG time,

\[
\lambda(\tau) = \lambda_T \cosh(2\tau)
\]

\[
g(\tau) = g_T \exp(2\tau).
\]

(1.14)

Let us now hypothesize that, within a certain range of \(k\) values, the RG trajectory realized in Nature can be approximated by (1.11). In order to determine its parameters \((\Lambda_0, \bar{G})\) or \((\lambda_T, k_T)\) we must perform a measurement of \(G\) and \(\Lambda\). If we interpret the observed values

\[
G_{\text{observed}} = m_{\text{Pl}}^{-2}, \quad m_{\text{Pl}} \approx 1.2 \times 10^{19} \text{ GeV}
\]

\[
\Lambda_{\text{observed}} = 3\Omega_0 H_0^2 \approx 10^{-120} m_{\text{Pl}}^2
\]

(1.15)

as the running \(G(k)\) and \(\Lambda(k)\) evaluated at a scale \(k \ll k_T\), then we get from (1.10) that \(\Lambda_0 = \Lambda_{\text{observed}}\) and \(\bar{G} = G_{\text{observed}}\). Using (1.7) and \(\nu = O(1)\) this leads to the order-of-magnitude estimates

\[
g_T \approx \lambda_T \approx 10^{-60}
\]

\[
k_T \approx 10^{-30} m_{\text{Pl}} \approx (10^{-3} \text{ cm})^{-1}.
\]

(1.16)

Because of the tiny values of \(g_T\) and \(\lambda_T\) the turning point lies in the linear regime of GFP.

Up to this point we discussed only that segment of the ‘trajectory realized in Nature’ which lies inside the linear regime of the GFP. The complete RG trajectory obtains by continuing this segment with the flow equation both into the IR and into the UV, where it ultimately spirals into the NGFP. While the UV continuation is possible within the Einstein–Hilbert truncation, this approximation breaks down in the IR when \(\lambda(k)\) approaches \(1/2\). A rough estimate for the ‘termination’ scale \(k_{\text{term}}\) at which this happens can be obtained from (1.5) for \(k \ll m_{\text{Pl}}\): \(\lambda(k) \approx \Lambda_0/k^2 = \Lambda_{\text{observed}}/k^2 = 3\Omega_0(H_0/k)^2\).

Since the observations show that \(\Omega_0 = O(1)\), we have \(\lambda(k) = O(1)\) exactly when \(k/H_0 = O(1)\). Stated differently, it is precisely for scales of the order of the present Hubble parameter that the Einstein–Hilbert truncation becomes insufficient. In [38,39] it was speculated that close to this regime strong IR renormalization effects could set in which perhaps might mimic the presence of dark matter. Whether these effects actually are there is an open problem; it will not affect the discussions in the present paper.

Let us try to interpret the trajectory found above in a cosmological context and let us ask during which cosmological epoch the Universe as a whole passed through the turning point. Using the natural cutoff identification \(k(t) \approx H(t)\) equation (1.16) tells us that this happened when the Hubble parameter was of the order of \(H_T \approx 10^{-30} m_{\text{Pl}}\). We can estimate the corresponding redshift \(z_T\) by exploiting that for \(k \lesssim k_T\) the impact of the cosmological constant is small so that we have a standard radiation-dominated FRW cosmology with \(a(t) \propto t^{1/2} \propto H(t)^{-1/2}\). Neglecting the comparatively short matter-dominated era we can then relate the scale factor at the turning point, \(a_T\), to its present
value $a_0$ by
\[ 1 + z_T = \frac{a_0}{a_T} = \left( \frac{H_T}{H_0} \right)^{1/2} \approx 10^{15}. \] (1.17)

Here we used that $H_T/H_0 \approx 10^{-30}(m_{Pl}/H_0) \approx 10^{30}$. Since the temperature behaves as $T \propto 1/a$ we can express its value at the turning point in terms of the present $T_0 \approx 2.7$ K:
\[ T_T = 10^{15}T_0 \approx 3 \times 10^{15} \text{ K} \approx 300 \text{ GeV}. \] (1.18)

Thus we see that the Universe passed through the turning point at about the time of the electroweak phase transition. This is a quite remarkable coincidence which might have a deeper meaning perhaps. It is, however, important to bear in mind how precisely this result is to be interpreted: at the cosmological time when the electroweak phase transition took place, $t_{\text{EWPT}}$, the Universe on scales of the Hubble radius $1/H(t_{\text{EWPT}})$ is effectively described by $G(k)$ and $\Lambda(k)$ evaluated at $k \approx k_T$. If, on the other hand, one wants to describe the microphysics of the phase transition where the pertinent scale is the transition temperature $T_{\text{EWPT}} = O(100 \text{ GeV})$ then one should set $k \approx T_{\text{EWPT}}$, and this scale is far higher than $k_T$.

The remaining sections of this paper are organized as follows. In section 2 we discuss the essential properties of the RG improved Einstein equations, and in section 3 we analyse the mechanism of entropy production they give rise to. In section 4 we obtain analytical solutions to those equations, valid during specific cosmological epochs, in particular in the very early Universe whose properties are crucially determined by the RG fixed point. Then, in section 5, we study the coupled system of RG and cosmological evolution equations with numerical methods; we obtain complete cosmological histories for a RG trajectory with realistic parameter values. The phenomenon of automatic inflation in the fixed point regime is discussed in section 6 and section 7 contains the conclusions.

2. The improved field equations

We assume that $G(k)$ and $\Lambda(k)$ have been converted to functions of the cosmological time, $G(t)$ and $\Lambda(t)$, by an appropriate cutoff identification $k = k(t)$ whose precise form is not important for the time being. We then ‘RG improve’ the Einstein equations by substituting these functions for their classical counterparts: $G_{\mu\nu} = -\Lambda(t)g_{\mu\nu} + 8\pi G(t)T_{\mu\nu}$. We specialize $g_{\mu\nu}$ to describe a spatially flat ($K = 0$) Robertson–Walker metric with scale factor $a(t)$, and we take $T_{\mu\nu} = \text{diag}[-\rho, p, p, p]$ to be the energy–momentum tensor of an ideal fluid with equation of state $p = w\rho$, where $w > -1$ is constant. Then the improved Einstein equation boils down to the modified Friedmann equation and a continuity equation:

\[ H^2 = \frac{8\pi}{3} \tilde{\rho}_{\text{eff}} \] (2.1a)

\[ \dot{\rho}_{\text{eff}} + 3H(\rho_{\text{eff}} + p_{\text{eff}}) = 0. \] (2.1b)
Here $\bar{G}$ is an arbitrary constant and

\begin{align}
\rho_{\text{eff}} & \equiv \frac{G(t)}{\bar{G}}(\rho + \rho_{\Lambda}) \\
p_{\text{eff}} & \equiv \frac{G(t)}{\bar{G}}(p + p_{\Lambda})
\end{align}

where

\begin{equation}
\rho_{\Lambda} = -p_{\Lambda} = \frac{\Lambda(t)}{8\pi G(t)}.
\end{equation}

Equations (2.1a) and (2.1b) have the same appearance as in classical FRW cosmology with $\Lambda = 0$ except that $\rho$ and $p$ are replaced by $\rho_{\text{eff}}$ and $p_{\text{eff}}$, respectively. Written more explicitly, this system of equations is

\begin{align}
H^2 & = \frac{8\pi}{3} G(t) \rho + \frac{1}{3} \Lambda(t) \\
\dot{\rho} + 3H(\rho + p) & = -\frac{\dot{\Lambda} + 8\pi \rho \dot{G}}{8\pi G}.
\end{align}

The modified continuity equation (2.4b) is the integrability condition for the improved Einstein equation implied by Bianchi’s identity, $D^{\mu}[\Lambda(t)g_{\mu\nu} + 8\pi G(t) T_{\mu\nu}] = 0$. It describes the energy exchange between the matter and gravitational degrees of freedom (geometry).

In [I] and [II] the special case was considered where the coupled dynamics is such that there occurs no significant exchange between the two sectors. In this case equation (2.4b) is solved in the form $0 = 0$, i.e. both sides vanish separately:

\begin{align}
\dot{\rho} + 3H(\rho + p) & = 0 \\
\dot{\Lambda} + 8\pi \rho \dot{G} & = 0.
\end{align}

Equation (2.5a) was referred to as the ‘ordinary continuity equation’ and (2.5b) as the ‘consistency condition’. Clearly the set of equations (2.4a), (2.5a) and (2.5b) is stronger and more constraining than (2.4a) and (2.4b). In fact, it is quite nontrivial that the former has physically acceptable solutions at all. In [I] they were found analytically in the NGFP regime, and in [II] the complete cosmology was obtained using a special, dynamically adjusted cutoff identification (otherwise no solution exists).

The analyses in [I] and [II] dealt with a situation where, for an unspecified dynamical reason, the exchange of energy and momentum between matter and the running couplings is completely forbidden. In the following we consider the other limiting case, where this exchange is possible without any obstructions\(^4\). We shall analyse the coupled system (2.4a) and (2.4b) where we now accept any solution of (2.4b), not necessarily of the form ‘$0 = 0$’.

\(^4\) This is the case which has also been studied in most of the early phenomenological papers on cosmologies with time-dependent $\Lambda$ [52].
For later use let us note that upon defining the critical density
\[ \rho_{\text{crit}}(t) \equiv \frac{3H(t)^2}{8\pi G(t)} \] (2.6)
and the relative densities \( \Omega_M \equiv \rho/\rho_{\text{crit}} \) and \( \Omega_\Lambda = \rho_\Lambda/\rho_{\text{crit}} \) the modified Friedmann equation (2.4a) can be written as
\[ \Omega_M(t) + \Omega_\Lambda(t) = 1. \] (2.7)
We emphasize that this ‘sum rule’ is valid for arbitrary functions \( G(t) \) and \( \Lambda(t) \). (Recall that we consider flat time slices throughout.)

2.1. An algorithm for generating solutions
Let \( G(t) \) and \( \Lambda(t) \) be arbitrary prescribed functions. If \( H(t) \neq 0 \), the resulting solutions \((H(t), \rho(t))\) of the system (2.1a), (2.1b) or equivalently (2.4a), (2.4b) satisfy the equations
\[ \dot{H} = -4\pi(1+w)G(t)\rho \] (2.8a)
\[ \dot{H} = -\frac{1}{2}(3+3w)\left[H^2 - \frac{1}{3}\Lambda(t)\right] \] (2.8b)
\[ \rho = \frac{3}{8\pi G(t)}\left[H^2 - \frac{1}{3}\Lambda(t)\right]. \] (2.8c)
This statement is easily proven by differentiating the modified Friedmann equation (2.1a). What is less trivial, but more important from the point of view of finding solutions, is that its converse is also true. To be precise, we have the following algorithm for generating solutions to the improved field equations:

Let \( G(t) \) and \( \Lambda(t) \) be prescribed functions and \( H(t) \) a solution of
\[ \dot{H} = -\frac{1}{2}(3+3w)\left[H^2 - \frac{1}{3}\Lambda(t)\right]. \] (2.9)
Let furthermore \( \rho(t) \) be defined in terms of this solution according to
\[ \rho = \frac{3}{8\pi G(t)}\left[H^2 - \frac{1}{3}\Lambda(t)\right]. \] (2.10)
Then the pair \((H(t), \rho(t))\) is a solution to the system of differential equations (2.4a) and (2.4b) for the equation of state \( p = w\rho \), provided \( H(t) \neq 0 \).

From (2.10) it is obvious that the solution generated by the algorithm satisfies (2.4a). To show that it also satisfies (2.4b) one exploits that, for \( H \neq 0 \), (2.4b) together with (2.10) is equivalent to \( 2\dot{H} + (3+3w)(H^2 - \Lambda/3) = 0 \), which is nothing other than (2.9).

The existence of this simple algorithm is non-trivial since it amounts to decoupling the two differential equations for \((H(t), \rho(t))\). In fact, one has to solve only a single differential equation, (2.10), involving only one of the external functions, \( \Lambda(t) \). If this has been achieved, \( \rho(t) \) is given in terms of \( H(t) \) by an explicit formula, and only here \( G(t) \) enters.
For later use let us also note that by using the definition of the deceleration parameter
\[ q \equiv -\frac{a\ddot{a}}{a^2} = -\frac{\dot{H}}{H^2} - 1 \] (2.11)
together with the differential equation (2.10) one obtains a simple expression for \( q \) in terms of \( \Omega_\Lambda \):
\[ q = \frac{1}{2}(1 + 3w) - \frac{1}{2}(3 + 3w)\Omega_\Lambda. \] (2.12)
In particular,
\[ q = 1 - 2\Omega_\Lambda \quad \text{for} \quad w = 1/3 \] (2.13)
\[ q = (1 - 3\Omega_\Lambda)/2 \quad \text{for} \quad w = 0 \] (2.14)
for a radiation- and a matter-dominated Universe, respectively.

### 2.2. The cutoff identification

Up to now \( G \) and \( \Lambda \) were prescribed functions of time. Now we induce their \( t \) dependence by means of the cutoff identification
\[ k(t) = \xi H(t) \] (2.15)
from a given RG trajectory of the Einstein–Hilbert truncation, \((g(k), \lambda(k))\). Here \( \xi \) is a fixed positive constant of order unity. Equation (2.15) is a natural choice since in a Robertson–Walker geometry the Hubble parameter measures the curvature of spacetime; its inverse \( H^{-1} \) defines the size of the ‘Einstein elevator’. With \( G(k) = g(k)/k^2 \) and \( \Lambda(k) = \lambda(k)k^2 \) we have
\[
G(t) = \frac{g(\xi H(t))}{\xi^2 H(t)^2} \\
\Lambda(t) = \xi^2 H(t)^2 \lambda(\xi H(t)).
\] (2.16)
The algorithm for solving the cosmological evolution equation assumes the following form now:

\textit{Let} \((g(k), \lambda(k))\) \textit{be a prescribed RG trajectory and} \( H(t) \) \textit{a solution of}
\[
\dot{H}(t) = -\frac{1}{2}(3 + 3w)H(t)^2 \left[ 1 - \frac{1}{3} \xi^2 \lambda(\xi H(t)) \right].
\] (2.17)

\textit{Let} \( \rho(t) \) \textit{be defined in terms of this solution by}
\[
\rho(t) = \frac{3\xi^2}{8\pi g(\xi H(t))} \left[ 1 - \frac{1}{3} \xi^2 \lambda(\xi H(t)) \right] H(t)^4.
\] (2.18)

\textit{Then the pair} \((H(t), \rho(t))\) \textit{is a solution of the system (2.4a) and (2.4b) for the time dependence of} \( G \) \textit{and} \( \Lambda \) \textit{given by (2.16) and the equation of state} \( p = w\rho \), \textit{provided} \( H(t) \neq 0 \).

Later on we shall apply this algorithm to the various cosmological epochs of interest. Sometimes it is more convenient to regard \( H \) and \( \rho \) as functions of the scale factor rather
than time. Exploiting that \( a(dH/da) = (dH/dt)/H \) we see that equation (2.17) implies the following somewhat simpler differential equation for \( H = H(a) \):

\[
a \frac{dH(a)}{da} = -\frac{1}{2} (3 + 3w) H(a) \left[ 1 - \frac{1}{3} \xi^2 \lambda(\xi H(a)) \right].
\]  

(2.19)

Or, using logarithmic variables,

\[
\frac{d \ln H}{d \ln a} = -\frac{1}{2} (3 + 3w) \left[ 1 - \frac{1}{3} \xi^2 \lambda(\xi H) \right].
\]  

(2.20)

The latter equation is particularly convenient for the numerics.

### 2.3. Cosmology on theory space

The theory space of the Einstein–Hilbert truncation can be identified with a part of the \( g-\lambda \) plane. There are certain quantities of cosmological interest whose values at time \( t \) depend only on the point of theory space the Universe passes at this time, but not on \( t \) or on the dynamics directly. Such quantities are functions of \( g \) and \( \lambda \). Examples are \( \Omega_\Lambda \), \( \Omega_M \), and \( q \) which actually are functions of \( \lambda \) alone:

\[
\begin{align*}
\Omega_\Lambda(\lambda) &= \frac{\xi^2}{3} \lambda \\
\Omega_M(\lambda) &= 1 - \frac{\xi^2}{3} \lambda \\
q(\lambda) &= \frac{1}{2} \left[ (1 + 3w) - (1 + w) \xi^2 \lambda \right].
\end{align*}
\]  

(2.21)

Remarkably, whether or not the Universe decelerates at some time \( t \) depends only on the value of \( \lambda \) at this time. Defining the ‘zero acceleration’ value

\[
\lambda_{za}(w) = \frac{1 + 3w}{1 + w} \xi^2
\]  

(2.22)

we have deceleration \( (q > 0) \) if \( \lambda < \lambda_{za} \) and acceleration \( (q < 0) \) if \( \lambda > \lambda_{za} \). For \( \lambda = \lambda_{za} \) we get \( q = 0 \) and \( a \propto t \) therefore. The ‘zero acceleration line’ \( \{(g, \lambda_{za})| -\infty < g < \infty \} \) divides the \( g-\lambda \) plane in two parts, with deceleration on its left and acceleration on its right.

Another line on the \( g-\lambda \) plane which is of special significance is the so-called ‘\( \Omega \)-line’ [II] along which, by definition, \( \rho = 0 \), i.e. \( \Omega_M = 0 \). By (2.21), this line \( \{(g, \lambda_{\Omega})| -\infty < g < \infty \} \) is parallel to the zero acceleration line, with

\[
\lambda_{\Omega} = \frac{3}{\xi^2} \quad \text{for all } w.
\]  

(2.23)

Provided \( \lambda_{\Omega} < 1/2 \), cosmologies with an eternal expansion and a corresponding dilution of their matter contents terminate on the \( \Omega \)-line for \( t \to \infty \). If \( \lambda_{\Omega} > 1/2 \) this line is of no physical importance since it lies in a region where the Einstein–Hilbert truncation is not valid. (For the corresponding discussion when the consistency condition is imposed we refer to [II].)
3. Entropy generation

Let us return to the modified continuity equation (2.4b). After multiplication by $a^3$ it is
\[ [\dot{\rho} + 3H(\rho + p)]a^3 = \tilde{\mathcal{P}}(t) \]  
(3.1)
where we defined
\[ \tilde{\mathcal{P}} \equiv - \left( \frac{\dot{\Lambda} + 8\pi \rho \dot{G}}{8\pi G} \right) a^3. \]  
(3.2)
Without assuming any particular equation of state equation (3.1) can be rewritten as
\[ \frac{d}{dt}(\rho a^3) + p \frac{d}{dt}(a^3) = \tilde{\mathcal{P}}(t). \]  
(3.3)
The interpretation of this equation is as follows. Let us consider a unit coordinate, i.e. comoving volume in the Robertson–Walker spacetime. Its corresponding proper volume is $V = a^3$ and its energy contents is $U = \rho a^3$. The rate of change of these quantities is subject to (3.3):
\[ \frac{dU}{dt} + p \frac{dV}{dt} = \tilde{\mathcal{P}}(t). \]  
(3.4)
In classical cosmology where $\tilde{\mathcal{P}} \equiv 0$ this equation together with the standard thermodynamic relation $dU + p dV = T dS$ is used to conclude that the expansion of the Universe is adiabatic, i.e. the entropy inside a comoving volume does not change as the Universe expands, $dS/dt = 0$.

Here and in the following we write $S \equiv sa^3$ for the entropy carried by the matter inside a unit comoving volume and $s$ for the corresponding proper entropy density.

When $\Lambda$ and $G$ are time-dependent, $\tilde{\mathcal{P}}$ is nonzero and we interpret (3.4) as describing the process of energy (or ‘heat’) exchange between the scalar fields $\Lambda$ and $G$ and the ordinary matter. This interaction causes $S$ to change:
\[ T \frac{dS}{dt} = T \frac{d}{dt}(sa^3) = \tilde{\mathcal{P}}(t). \]  
(3.5)
The actual rate of change of the comoving entropy is
\[ \frac{dS}{dt} = \frac{d}{dt}(sa^3) = \mathcal{P}(t) \]  
(3.6)
where
\[ \mathcal{P} \equiv \frac{\tilde{\mathcal{P}}}{T}. \]  
(3.7)
If $T$ is known as a function of $t$ we can integrate (3.5) to obtain $S = S(t)$. In the RG improved cosmologies the entropy production rate per comoving volume
\[ \mathcal{P}(t) = - \left[ \frac{\dot{\Lambda} + 8\pi \rho \dot{G}}{8\pi G} \right] a^3 \]  
(3.8)
is nonzero because the gravitational ‘constants’ $\Lambda$ and $G$ have acquired a time dependence.

For a given solution to the coupled system of RG and cosmological equations it is sometimes more convenient to calculate $\mathcal{P}(t)$ from the LHS of the modified continuity
Entropy signature of the running cosmological constant

equation rather than its RHS (3.8):

\[ [\dot{\rho} + 3H(\rho + p)] \frac{a^3}{T} = \mathcal{P}(t). \]  

(3.9)

If \( S \) is to increase with time, by (3.8), we need that \( \dot{\Lambda} + 8\pi \dot{G} < 0 \). During most epochs of the RG improved cosmologies we have \( \dot{\Lambda} \leq 0 \) and \( \dot{G} \geq 0 \). The decreasing \( \Lambda \) and the increasing \( G \) have antagonistic effects therefore. We shall see that in the physically realistic cases \( \Lambda \) predominates so that there is indeed a transfer of energy from the vacuum to the matter sector rather than vice versa.

Clearly we can convert the heat exchanged, \( T dS \), to an entropy change only if the dependence of the temperature \( T \) on the other thermodynamical quantities, in particular \( \rho \) and \( p \), is known. For this reason we shall now make the following assumption about the matter system and its (non-equilibrium!) dynamics:

The matter system is assumed to consist of \( n_{\text{eff}} \) species of effectively massless degrees of freedom which all have the same temperature \( T \). The equation of state is \( p = \rho/3 \), i.e. \( w = 1/3 \), and \( \rho \) depends on \( T \) as

\[ \rho(T) = \kappa^4 T^4, \quad \kappa \equiv (\pi^2 n_{\text{eff}}/30)^{1/4}. \]  

(3.10)

No assumption is made about the relation \( s = s(T) \).

The first assumption, radiation dominance and equal temperature, is plausible since we shall find that there is no significant entropy production any more once \( H(t) \) has dropped substantially below \( m_{\text{Pl}} \), after the crossover from the NGFP to the GFP.

The second assumption, equation (3.10), amounts to the hypothesis formulated in the introduction. While entropy generation is a non-adiabatic process we assume, following Lima [44], that the non-adiabaticity is as small as possible. More precisely, the approximation is that the equilibrium relations among \( \rho \), \( p \) and \( T \) are still valid in the non-equilibrium situation of a cosmology with entropy production. In this sense, (3.10) is the extrapolation of the standard relation (1.3a) to a ‘slightly non-adiabatic’ process.

Note that while we used (1.3c) in relating \( \mathcal{P}(t) \) to the entropy production and also postulated equation (1.3a), we do not assume the validity of the formula for the entropy density, equation (1.3b), \( \text{a priori} \). We shall see that the latter is an automatic consequence of the cosmological equations.

To make the picture as clear as possible we shall neglect in the following all ordinary dissipative processes in the cosmological fluid.

Using \( p = \rho/3 \) and (3.10) in (3.9) the entropy production rate can be evaluated as follows:

\[ \mathcal{P}(t) = \kappa \left[ a^3 \rho^{-1/4} \dot{\rho} + 4a^3 H \rho^{3/4} \right] \]

\[ = \frac{4}{3} \kappa \left[ a^3 \frac{d}{dt}(\rho^{3/4}) + 3aa^2 \rho^{3/4} \right] \]

\[ = \frac{4}{3} \kappa \left[ a^3 \frac{d}{dt}(\rho^{3/4}) + \rho^{3/4} \frac{d}{dt}(a^3) \right]. \]  

(3.11)

Remarkably, \( \mathcal{P} \) turns out to be a total time derivative:

\[ \mathcal{P}(t) = \frac{d}{dt} \left[ \frac{4}{3} \kappa a^3 \rho^{3/4} \right]. \]  

(3.12)
Entropy signature of the running cosmological constant

Therefore we can immediately integrate (3.5) and obtain
\[ S(t) = \frac{4}{3} \kappa a^3 \rho^{3/4} + S_c \] (3.13)

or, in terms of the proper entropy density,
\[ s(t) = \frac{4}{3} \kappa \rho(t)^{3/4} + \frac{S_c}{a(t)^{3/4}}. \] (3.14)

Here \( S_c \) is a constant of integration. In terms of \( T \), using (3.10) again,
\[ s(t) = \frac{2\pi^2}{45} n_{\text{eff}} T(t)^3 + \frac{S_c}{a(t)^{3/4}}. \] (3.15)

The final result (3.15) is very remarkable for at least two reasons. First, for \( S_c = 0 \), equation (3.15) has exactly the form (1.3b) which is valid for radiation in equilibrium. Note that we did not postulate this relationship; only the \( \rho(T) \) law was assumed. The equilibrium formula \( s \propto T^3 \) was derived from the cosmological equations, i.e. the modified conservation law. This result makes the hypothesis ‘non-adiabatic, but as little as possible’ selfconsistent.

Second, if \( \lim_{t \to 0} a(t) \rho(t)^{1/4} = 0 \), which is actually the case for the most interesting class of cosmologies we shall find, then \( S(t \to 0) = S_c \) by equation (3.13). As we mentioned in the introduction, the most plausible initial value of \( S \) is \( S = 0 \) which means a vanishing constant of integration \( S_c \) here. But then, with \( S_c = 0 \), (3.13) tells us that the entire entropy carried by the massless degrees of freedom is due to the RG running. So it indeed seems to be true that the entropy of the CMBR photons we observe today is due to a coarse graining but, unexpectedly, not a coarse graining of the matter degrees of freedom but rather of the gravitational ones which determine the background spacetime the photons propagate on.

We close this section with various comments. As for the interpretation of the function \( \mathcal{P}(t) \), let us remark that it also measures the deviations from the classical laws \( a^4 \rho = \text{const} \) and \( aT = \text{const} \), respectively, since we have \( \mathcal{P} = \frac{4}{3} \kappa \frac{d(a^4 \rho)^{3/4}}{dt} = \frac{4}{3} \kappa a^{4} \frac{d(aT)^3}{dt} \).

Both in classical and in improved cosmology with the ‘consistency condition’ imposed the quantity \( \mathcal{M} = 8\pi a^4 \rho \) is conserved in time [32]. If energy transfer is permitted and the entropy of the ordinary matter grows, \( \mathcal{M} \) increases as well. This is obvious from
\[ \frac{d}{dt} \mathcal{M}(t)^{3/4} = \frac{3}{4\kappa} (8\pi)^{3/4} \mathcal{P}(t) \] (3.16)

or, in integrated form, \( \mathcal{M}(t) = \frac{4}{3} \kappa (8\pi)^{-3/4} \mathcal{M}(t)^{3/4} + S_c \).

In a spatially flat Robertson–Walker spacetime the overall scale of \( a(t) \) has no physical significance. If \( \mathcal{M} \) is time-independent, we can fix this gauge ambiguity by picking a specific value of \( \mathcal{M} \) and expressing \( a(t) \) correspondingly. For instance, parametrized in this way, the scale factor of the classical FRW cosmology with \( \Lambda = 0 \), \( w = 1/3 \) is [32]
\[ a(t) = \left[ 4G\mathcal{M}/3 \right]^{1/4} \sqrt{t}. \] (3.17)

If, during the expansion, \( \mathcal{M} \) increases slowly, equation (3.17) tells us that the expansion is actually faster than estimated classically. Of course, what we actually have to do in order to find the corrected \( a(t) \) is to solve the improved field equations and not insert \( \mathcal{M} = \mathcal{M}(t) \) into the classical solution, in particular when the change of \( \mathcal{M} \) is not ‘slow’. 

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Nevertheless, this simple argument makes it clear that entropy production implies an increase of \( \mathcal{M} \) which in turns implies an extra increase of the scale factor. This latter increase, or ‘inflation’, is a pure quantum effect. The explicit solutions to which we turn next will confirm this picture.

4. Explicit analytical solutions

In this and the next section we explicitly solve the cosmological evolution equations pertaining to type IIIa trajectories. In this section we discuss an analytical solution for the fixed point, the \( k^4 \) and the classical regime in turn, and then obtain complete cosmological histories by numerical methods in the next section.

4.1. The fixed point regime

For ‘\( k = \infty \)’ every trajectory stays very close to the NGFP with constant values \( g(k) \equiv \lambda_* \) and \( \lambda(k) \equiv \lambda_* \). In this regime the differential equation (2.17) is

\[
\dot{H}(t) = -\alpha^{-1} H^2(t)
\]

with the constant

\[
\alpha \equiv \frac{2}{(3 + 3w)[1 - \lambda_* \xi^2/3]}.
\]

Equation (4.1) describes a cosmology with an initial singularity. Fixing the constant of integration such that this singularity occurs at \( t = 0 \), the unique solution to (4.1) is

\[
H(t) = \frac{\alpha}{t}
\]

which integrates to \( a(t) \propto t^\alpha \). The exponent \( \alpha \) depends on the combination \( \lambda_* \xi^2 \); by equation (2.21) this is essentially the \( \Omega_\Lambda \) value in the NGFP regime:

\[
\Omega_\Lambda^* = \frac{\lambda_* \xi^2}{3}.
\]

Henceforth we shall always eliminate \( \xi \) in favour of the more physical parameter \( \Omega_\Lambda^* \). Then

\[
\alpha = \frac{2}{(3 + 3w)(1 - \Omega_\Lambda^*)}.
\]

Using \( \Omega_\Lambda^* \) as the free parameter which distinguishes different solutions, the fixed point cosmologies are characterized by the following power laws:

\[
a(t) = At^\alpha \quad A > 0
\]

\[
\rho(t) = \frac{\dot{\rho}}{\dot{a}^4}, \quad \dot{\rho} = \frac{2\Omega_\Lambda^*}{9\pi g_* \lambda_* (1 + w)^2 (1 - \Omega_\Lambda^*)^3}
\]

\[
G(t) = \frac{3g_* \lambda_* (1 + w)^2 (1 - \Omega_\Lambda^*)^2 t^2}{4\Omega_\Lambda^*}
\]

\[
\Lambda(t) = \frac{4\Omega_\Lambda^*}{3(1 + w)^2 (1 - \Omega_\Lambda^*)^2 t^2}.
\]
Figure 2. A trajectory of the type IIIa. The vertical lines are the ‘Ω line’ and
the zero-acceleration lines for \( w = 0 \) and \( 1/3 \). The Ω line is not necessarily in the
physical part of parameter space (\( \lambda < 1/2 \)).

Equation (4.6b) follows from (2.21) by inserting (4.3), while (4.6c) and (4.6d) are
equations (2.16) for \( H = \alpha/t \) and constant values of \( g \) and \( \lambda \). The parameter \( \xi \) has been
eliminated in favour of \( \Omega^* \) everywhere. Note that the RG data enter the solution (4.6)
only via the universal [3, 4] product \( g_\lambda^* \).

The solution (4.6) has time-independent values of \( \Omega_\Lambda = \Omega_\Lambda^* \), \( \Omega_M = 1 - \Omega_\Lambda^* \), and
\[
q = \frac{1}{\alpha} - 1 = \frac{1}{2} [1 + 3w - (3 + 3w)\Omega_\Lambda^*].
\] (4.7)

Eliminating \( \xi \) from (2.22) and (2.23) we can express \( \lambda_{za} \) and \( \lambda_\Omega \) in terms of \( \Omega_\Lambda^* \):
\[
\lambda_{za}(w) = \frac{1 + 3w}{3 + 3w} \frac{\lambda_*}{\Omega_\Lambda^*} \tag{4.8}
\]
\[
\lambda_\Omega = \frac{\lambda_*}{\Omega_\Lambda^*}. \tag{4.9}
\]

In the radiation-dominated case we have
\[
\lambda_{za}(1/3) = \frac{\lambda_*}{2\Omega_\Lambda^*} < \lambda_\Omega = \frac{\lambda_*}{\Omega_\Lambda^*}, \tag{4.10}
\]
so that for any value of \( \Omega_\Lambda^* \in (0, 1) \) the zero acceleration line is on the left of the Ω line.
The relative location of the NGFP depends on whether \( \Omega_\Lambda^* \) is bigger or smaller than 1/2:
\[
\begin{align*}
\lambda_{za}(1/3) &\leq \lambda_* < \lambda_\Omega & \text{if } \Omega_\Lambda^* \in [1/2, 1) \\
\lambda_* &< \lambda_{za}(1/3) < \lambda_\Omega & \text{if } \Omega_\Lambda^* \in (0, 1/2). \tag{4.11}
\end{align*}
\]

The Ω line is relevant only if it is in the physical part of parameter space (\( \lambda < 1/2 \)). If
indeed \( \lambda_\Omega < 1/2 \), the first case of (4.11) corresponds to the sketch in figure 2.

When \( \lambda_{za} \) is smaller than \( \lambda_* \), the fixed point and the spiralling regime of the RG
trajectory close to the NGFP correspond to an accelerating epoch of the Universe. (In
the second case it would be decelerating.) If \( 1/2 < \Omega_\Lambda^* < 1 \), the NGFP regime is an epoch
of ‘power law inflation’ \( a \propto t^\alpha \), with \( \alpha > 1 \) and \( q < 0 \):

\[
\alpha(w = 1/3) = (2 - 2\Omega_\Lambda^*)^{-1}
\]

\[
q(w = 1/3) = 1 - 2\Omega_\Lambda^*.
\]

Note that for \( \Omega_\Lambda^* \not\to 1 \) the exponent \( \alpha(1/3) \) becomes very large and \( q(1/3) \) approaches \(-1\).

The equations (4.6) describe a one-parameter family of cosmologies labelled by \( \Omega_\Lambda^* \). The solution exists (and has \( \alpha > 0, \rho > 0 \)) for any value of \( \Omega_\Lambda^* \) in the interval \((0,1)\). The possibility of freely\(^5\) choosing \( \Omega_\Lambda^* \) is the main new feature as compared to [I] where the ‘consistency condition’ \((2.5b)\) had been imposed. The consequence of imposing \((2.5b)\) is exactly to eliminate all solutions of the family (4.6) except the one for \( \Omega_\Lambda^* = 1/2 \). In fact, it is easily checked that (4.6) for the special case \( \Omega_\Lambda^* = 1/2 \) is identical to the fixed point solution found in [I]. The solutions for \( \Omega_\Lambda^* \not= 1/2 \) are new.

The case \( \Omega_\Lambda^* = 1/2 \) is special for a variety of reasons. It has, for instance, equal relative matter and vacuum energy densities, \( \Omega_M^* = \Omega_\Lambda^* = 1/2 \), vanishing deceleration parameter \( (q = 0) \) corresponding to a linear expansion \( a \propto t, \alpha = 1 \), and the NGFP sits precisely on the zero acceleration line in this case [II]. If the ‘consistency condition’ is imposed, \( \mathcal{M} \equiv 8\pi\rho_\Lambda^{3/2} \) is time-independent, while in the present more general framework it is not. For the above fixed point solutions one finds \( \mathcal{M}(t) \propto t^{2/(1-\Omega_\Lambda^*)-4} \) which is constant in the exceptional case \( \Omega_\Lambda^* = 1/2 \) only. In [I] we reexpressed the constant \( A \) of (4.6) in terms of \( \mathcal{M} \) which is no longer possible here.

If the NGFP expansion \( a(t) \propto t^\alpha \) is realized for \( t \to 0 \), the Universe has no particle horizon if \( \alpha \geq 1 \), but does have a horizon of radius \( d_H = t/(1-\alpha) \) if \( \alpha < 1 \). In the case of \( w = 1/3 \) this means that there is a horizon for \( \Omega_\Lambda^* < 1/2 \), but none if \( \Omega_\Lambda^* \geq 1/2 \).

If \( w = 1/3 \), the discussion of section 3 on entropy generation applies to the NGFP regime of the improved cosmology. The corresponding rate of entropy production is

\[
\dot{\mathcal{P}}(t) = 4\kappa(\alpha - 1)A^{3}\tilde{\rho}^{3/4}\lambda^{3}(\alpha - 1)
\]

(4.14)

where \( \alpha \equiv \alpha(w = 1/3) \) is given by (4.12), and

\[
\tilde{\rho}(w = 1/3) = \frac{9\Omega_\Lambda^*}{128\pi g_*\lambda}(1 - \Omega_\Lambda^*)^{3/4}.
\]

(4.15)

As expected, \( \dot{\mathcal{P}} \) vanishes identically if \( \alpha = 1 \), i.e. \( \Omega_\Lambda^* = 1/2 \). In this case the solution obeys the ‘consistency condition’ and no energy is exchanged with the matter system. For the entropy per unit comoving volume we find, if \( \alpha \not= 1 \),

\[
S(t) = S_c + \frac{4\kappa A^{3}\tilde{\rho}^{3/4}\lambda^{3}(\alpha - 1)}{3t}
\]

(4.16)

and the corresponding proper entropy density is

\[
s(t) = \frac{S_c}{A^{3}\tilde{\rho}^{3}3t^{3}} + \frac{4\kappa A^{3}\tilde{\rho}^{3/4}\lambda^{3}}{3t^{3}}.
\]

(4.17)

Here \( S_c \) is an undetermined constant of integration. The temperature behaves as \( 1/t \) for any value of \( \alpha \):

\[
T(t) = \frac{\tilde{\rho}^{1/4}}{\kappa t}.
\]

(4.18)

\(^5\) However, physically plausible values of \( \Omega_\Lambda^* \) should be such that \( \xi^2 = 3\Omega_\Lambda^*\lambda_\star \) does not assume unnaturally small or large values.
Entropy signature of the running cosmological constant

For the discussion of the entropy we must distinguish three qualitatively different cases. They differ in particular with respect to the sign of $P(t)$, the behaviour of $S(t)$ close to the initial singularity, and with respect to the relative importance of the running cosmological and Newton constant. (Cf the remark after equation (3.9).)

(a) The case $\alpha > 1$, i.e. $1/2 < \Omega^*_\Lambda < 1$:
Here $P(t) > 0$ so that the entropy and energy content of the matter system increases with time. By equation (3.8), $P > 0$ implies $\Lambda + 8\pi \rho \dot{G} < 0$. Since $\Lambda < 0$ but $\dot{G} > 0$ in the NGFP regime, the energy exchange is predominantly due to the decrease of $\Lambda$ while the increase of $G$ is subdominant in this respect.

The comoving entropy (4.16) has a finite limit for $t \to 0$, $S(t \to 0) = S_c$, and $S(t)$ grows monotonically for $t > 0$. If $S_c = 0$, which would be the most natural value in view of the discussion in the introduction, all of the entropy carried by the matter fields is due to the energy injection from $\Lambda$. But even if $S_c \neq 0$, any such nonzero initial value will be irrelevant at a sufficiently late time (at least if this time is smaller than $\approx t_{Pl}$ above which (4.16) is invalid).

(b) The case $\alpha < 1$, i.e. $0 < \Omega^*_\Lambda < 1/2$:
Here $P(t) < 0$ so that the energy and entropy of matter decreases. Since $P < 0$ amounts to $\dot{\Lambda} + 8\pi \rho \dot{G} > 0$, the dominant physical effect is the increase of $G$ with time, the counteracting decrease of $\Lambda$ is less important. The comoving entropy starts out from an infinitely positive value at the initial singularity, $S(t \to 0) \to +\infty$, and decreases thereafter.

(c) The case $\alpha = 1$, i.e. $\Omega^*_\Lambda = 1/2$:
Here $P(t) \equiv 0$, $S(t) = \text{const}$, and $\dot{\Lambda} + 8\pi \rho \dot{G} = 0$. The effect of a decreasing $\Lambda$ and increasing $G$ cancel exactly so that there is no net energy exchange.

4.2. The $k^4$ regime

The ‘$k^4$ regime’ of a type IIIa trajectory is its part which can be described by the linearization about the GFP where $\Lambda$ has a quartic $k$ dependence. The corresponding trajectory $(\lambda(k), g(k))$ is given in equation (1.5). We shall set $\Lambda_0 = 0$ there which is a good approximation as long as $k^2 \gg \Lambda_0$. Since we are mostly interested in the entropy production in the early Universe (soon after the Planck regime) this approximation is sufficient for our purposes. (For the separatrix which has $\Lambda_0 = 0$ it represents no approximation at all.)

In this regime we have $G = \text{const}$ approximately and

$$\Lambda = \nu \xi^4 \bar{G} H^4.$$  \hfill (4.19)

As $\Lambda = 3\Omega_\Lambda H^2$, this implies directly

$$\Omega_\Lambda = L^2 H^2; \quad \Omega_M = 1 - L^2 H^2$$ \hfill (4.20)

$$q = \frac{1}{2} [1 + 3w - (3 + 3w) L^2 H^2]$$  \hfill (4.21)

where the scale is set by the quantity

$$L \equiv \sqrt{\nu/3\xi^2 \bar{G}}$$ \hfill (4.22)
which is a length of the order of the Planck length. The differential equation (2.17) for \( H(t) \) is, in the present case,

\[
\dot{H} = -\alpha_0^{-1} H^2 (1 - L^2 H^2) \tag{4.23}
\]

where

\[
\alpha_0 \equiv \frac{2}{3 + 3w}. \tag{4.24}
\]

The general solution can be found in the form \( t = t(H) \) by a simple integration, but the inversion is not elementary:

\[
t(H) = \alpha_0 [H^{-1} - L \artanh (LH)] + \text{const.} \tag{4.25}
\]

It is easier to solve equation (2.19) for \( H = H(a) \) which is in the case at hand

\[
a \frac{dH}{da} = -\alpha_0^{-1} H (1 - L^2 H^2). \tag{4.26}
\]

Its general solution is given by

\[
H(a) = \frac{1}{L} \left[ 1 + \left( \frac{a}{\tilde{a}} \right)^{3+3w} \right]^{-1/2} \tag{4.27}
\]

where \( \tilde{a} \) is a constant. As an aside we note that the cosmology described by (4.27), when taken seriously for any \( a \), does not have an initial singularity since the RHS of (4.27) is bounded above. This observation is of no relevance in the present context, however, because once \( H \) gets larger than the Planck mass (4.27) becomes invalid and the fixed point solution takes over, which does have a ‘big bang’.

Inserting (4.27) into (2.10) we obtain the energy density as a function of the scale factor:

\[
\rho(a) = \frac{3}{8\pi G L^2} \left( \frac{a}{\tilde{a}} \right)^{3+3w} \left[ 1 + \left( \frac{a}{\tilde{a}} \right)^{3+3w} \right]^{-2}. \tag{4.28}
\]

In a realistic cosmology the epoch during which (4.28) is valid is radiation-dominated so that we should set \( w = 1/3 \). Since \( L = O(\ell_{Pl}) \) we see that \( \tilde{a} \) is the scale factor at which \( k, H = O(m_{Pl}) \) and \( \rho = O(m_{Pl}^4) \). Equation (4.28) implies the following \( a \) dependence of the entropy:

\[
S(a) = S(\tilde{a}) + \frac{4}{3} \kappa \tilde{a}^3 \rho(\tilde{a})^{3/4} \left\{ 2\sqrt{2} \left( \frac{a}{\tilde{a}} \right)^6 \left[ 1 + \left( \frac{a}{\tilde{a}} \right)^4 \right]^{-3/2} - 1 \right\}. \tag{4.29}
\]

This formula predicts a significant entropy production only near \( a \approx \tilde{a}, k \approx m_{Pl} \), i.e. in the crossover regime. There the function inside the curly brackets of equation (4.29) grows from \(-1\) at \( a \ll \tilde{a} \) to \(+2\sqrt{2}\) at \( a \gg \tilde{a} \).
4.3. Classical regime and the value of $\tilde{a}$

For $k \ll k_T$, in the ‘IR regime’, Nature’s RG trajectory has a long classical regime where $G \approx \text{const}$ and $\Lambda \approx \text{const}$. In this regime the functions $H(t)$ and $\rho(t)$ are well known, of course\(^6\). Here we quote only the result for a negligible $\Lambda$:

$$a(t) = \left[\frac{3}{4}(1 + w)^2\mathcal{M}G\right]^{1/(3+3w)} t^{2/(3+3w)}. \quad (4.30)$$

There is no entropy production in the classical regime, $\mathcal{P} = 0$. The scale factor (4.30) corresponds to

$$H(a) = \sqrt{\frac{1}{2}\mathcal{M}Ga^{-3/2}} \quad (4.31)$$

with the invariant $\mathcal{M} \equiv 8\pi\rho a^{3+3w}$; its value can be determined by astrophysical observations in the late Universe.

We can use (4.31) in order to fix the constant $\tilde{a}$ of equation (4.27) valid in the preceding ‘$k^4$ regime’. For $a \gg \tilde{a}$ the latter equation yields

$$H(a) \approx \frac{1}{L} \left(\frac{a}{\tilde{a}}\right)^{-3/2} \quad (4.32)$$

Identifying (4.31) and (4.32) we read off that $\tilde{a}$ can be expressed in terms of $\mathcal{M}$ as

$$\tilde{a} = \left[\nu\xi^4\frac{G\mathcal{M}}{9}\right]^{1/(3+3w)}. \quad (4.33)$$

Since both the $k^4$ and the early classical regime are in the radiation-dominated epoch the above comparison requires $w = 1/3$.

Matching the observed data against the classical FRW cosmology in the usual way one finds that the quantity $\mathcal{M}$ for the radiation-dominated era of the Universe we live in is approximately given by

$$\mathcal{M} \approx (10^{-30}a_0/\ell_{\text{Pl}})^4 \quad (4.34)$$

with $\ell_{\text{Pl}} \equiv m_{\text{Pl}}^{-1} \equiv \sqrt{G}$, and $a_0$ the scale factor today. Since $\nu$, $\xi = \text{O}(1)$, the order of magnitude estimate (4.34) implies that

$$\tilde{a} \approx 10^{-30}a_0. \quad (4.35)$$

Hence in the $k^4$ regime, approximately,

$$H(a) \approx m_{\text{Pl}}\left[1 + c(10^{30}a_0)^4\right]^{-1/2} \quad (4.36)$$

where $c$ is a constant of order unity.

Note that using (4.34) in (4.30) for $w = 1/3$ we get that, classically, and up to factors of the order of unity,

$$a(t) \approx 10^{-30}a_0\sqrt{t/\ell_{\text{Pl}}}. \quad (4.37)$$

Therefore $\tilde{a}$ has the interpretation of the scale factor $a(t_{\text{Pl}})$ predicted by classical cosmology for the time $t = t_{\text{Pl}}$. It is just at this scale, however, where according to (4.36) deviations from the classical behaviour start to occur.

\(^6\) See, for instance, appendix B of [I] where the present notation is employed.
5. Complete cosmological histories

In this section we construct complete cosmologies by numerically integrating the coupled system of RG and cosmological evolution equations from the ‘big bang’ up to asymptotically late times. As for the RG equations we use the same cutoff scheme, the sharp cutoff, as in [II] in order to facilitate the comparison. In four dimensions the flow equations for the Einstein–Hilbert truncation with a sharp cutoff are [4]

\begin{align}
  k \partial_k g(k) &= \beta_g(g,\lambda) \equiv [2 + \eta_N(g,\lambda)]g \\
  k \partial_k \lambda(k) &= \beta_\lambda(g,\lambda)
\end{align}

with \( \beta_\lambda \) and \( \eta_N \) given by

\begin{align}
  \beta_\lambda(g,\lambda) &= -(2 - \eta_N)\lambda - \frac{g}{\pi} \left[ 5 \ln(1 - 2\lambda) - \varphi_2 + \frac{5}{4} \eta_N \right] \\
  \eta_N(g,\lambda) &= -\frac{2g}{6\pi} + \frac{5g}{5} \left[ \frac{18}{1 - 2\lambda} + 5 \ln(1 - 2\lambda) - \varphi_1 + 6 \right].
\end{align}

As in [II] we use the variant of the sharp cutoff with ‘shape parameter’ \( s = 1 \) for which \( \varphi_1 = \zeta(2) \), \( \varphi_2 = 2\zeta(3) \), see [4]. The NGFP is at \( g_* = 0.403 \), \( \lambda_* = 0.330 \) then.

In particular in numerical computations employing Nature’s RG trajectory which comprises very many orders of magnitude it is advantageous to use logarithmic variables. We normalize them with respect to their value at the turning point and express the RG time, the scale factor, the cosmological time and the Hubble parameter by, respectively,

\begin{align}
  \tau &\equiv \ln(k/k_T) \\
  x &\equiv \ln(a/a_T) \\
  y &\equiv \ln(t/t_T) \\
  U &\equiv \ln(H/H_T)
\end{align}

By definition, \( x \) and \( y \) are negative in what we call the ‘UV regime’, the upper branch of the trajectory where \( k > k_T \), and positive in the ‘IR regime’, the lower branch with \( k < k_T \). For \( \tau \) and \( U \) it is the other way around. The variable \( x \) is the number of ‘e folds’ relative to the size of the Universe when it passes the turning point of the underlying RG trajectory. In these variables, the cutoff identification \( k = \xi H \) implies \( U = \tau \).

In a regime with power law expansion \( a \propto t^\alpha \) we have

\begin{align}
  x &= \alpha y, \quad U = \tau = -y.
\end{align}

For instance, in the NGFP regime with \( w = 1/3 \),

\begin{align}
  x &= (2 - 2\Omega_\Lambda^*)^{-1} y, \quad U = \tau = -y = -(2 - 2\Omega_\Lambda^*)x
\end{align}

while for a classical FRW cosmology with \( \Lambda = 0 \),

\begin{align}
  x &= \frac{1}{2} y, \quad U = \tau = -y = -2x \quad \text{if } w = 1/3 \\
  x &= \frac{2}{3} y, \quad U = \tau = -y = -\frac{3}{2} x \quad \text{if } w = 0.
\end{align}

We shall need these simple relations repeatedly.
For reasons of numerical stability it is advantageous to integrate the coupled system of the RG and cosmological differential equations not with respect to \( t \) but rather \( x \). Let us write

\[ g_s(x) \equiv g(k(x)), \quad \lambda_s(x) \equiv \lambda(k(x)) \] (5.9)

for the gravitational couplings regarded functions of the logarithmic scale factor (whence the subscript ‘s’). Then equation (2.20) for the relationship \( H = H(a) \) translates to

\[ \frac{d}{dx} U(x) = -\frac{1}{2} \left[ 1 - \frac{\xi^2}{3} \lambda_s(x) \right]. \] (5.10)

Furthermore, upon differentiating (5.9) and using \( k(x) = \xi H(x) \), along with (5.10) and (5.1) we obtain the following system for \( g_s \) and \( \lambda_s \):

\[ \frac{d}{dx} g_s(x) = -\frac{(3 + 3w)}{2} \left[ 1 - \frac{\xi^2}{3} \lambda_s(x) \right] \beta_g \left( g_s(x), \lambda_s(x) \right) \] (5.11a)

\[ \frac{d}{dx} \lambda_s(x) = -\frac{(3 + 3w)}{2} \left[ 1 - \frac{\xi^2}{3} \lambda_s(x) \right] \beta_\lambda \left( g_s(x), \lambda_s(x) \right). \] (5.11b)

The system (5.11a) and (5.11b) is closed: it can be integrated in \( x \) without involving \( U \) or any other of the cosmological quantities. Solving this system directly is numerically more stable than integrating with respect to \( k \) and substituting

\[ k(x) = \xi H(x) = \xi H_T e^{U(x)} \] (5.12)

into the result, in particular as \( k(x) \) becomes functionally dependent on \( U \) then.

Sometimes it is helpful to analyse the system of RG and cosmological equations in still another way, namely by treating the dimensionless cosmological constant \( \lambda \) as the independent parameter. Considered functions of \( \lambda \), the quantities \( g, U \) and \( x \) are easily seen to obey the evolution equations

\[ \frac{d}{d\lambda} g(\lambda) = \frac{\beta_g(g(\lambda), \lambda)}{\beta_\lambda(g(\lambda), \lambda)} \]

\[ \frac{d}{d\lambda} U(\lambda) = \frac{1}{\beta_\lambda(g(\lambda), \lambda)} \] (5.13)

\[ \frac{d}{d\lambda} x(\lambda) = -\frac{1}{2} \left[ \left( 1 - \frac{\xi^2}{3} \lambda \right) \beta_\lambda(g(\lambda), \lambda) \right]^{-1}. \]

Below we shall apply these equations to the IR branch of the cosmology. Here the relationship between \( g, U, x \) and \( \lambda \) is single valued, which of course is necessary for the method to work.

### 5.1. A trajectory with realistic parameter values

Since through every point in the \( g-\lambda \) plane there passes exactly one RG trajectory, we can specify a trajectory by specifying one of its points. Dealing with type IIIa trajectories,
we shall use the turning point for this purpose. The line of all turning points \((g_T, \lambda_T)\) is given by \(\beta_A(g_T, \lambda_T) = 0\), which yields the condition (1.9) in the linear regime of the GFP. Therefore a trajectory is uniquely specified by the \(g\) coordinate of its turning point, \(g_T\):

\[
(g_T, \lambda_T) = \left(g_T, \frac{\varphi_2}{2\pi g_T}\right). \quad (5.14)
\]

Having fixed \(g_T\), we then integrate both upward and downward from the turning point, obtaining the UV and IR branch of the trajectory, respectively.

In the numerical calculations we shall use the value

\[
g_T = 10^{-60} \quad (5.15)
\]

which is motivated by the order of magnitude estimates in the introduction where we matched the linearized IIIa trajectories against the experimental data. Furthermore, in order to fix the zero-point of the RG time \(\tau \equiv \ln(k/k_T)\) we use the estimate (1.16) for the turning point scale: \(k_T = 10^{-30} m_{\text{Pl}}\). A result,

\[
\tau(k) = \ln(k/m_{\text{Pl}}) + 30 \ln(10). \quad (5.16)
\]

In particular, recalling \(H_0 \approx 10^{-60} m_{\text{Pl}}\),

\[
\tau(k = m_{\text{Pl}}) = +30 \ln(10) \approx +69
\]

\[
\tau(k = k_T) = 0
\]

\[
\tau(k = H_0) = -30 \ln(10) \approx -69. \quad (5.17)
\]

Integrating the equations for \(g(\tau), \lambda(\tau)\) numerically towards positive values of \(\tau\) with the initial condition (5.14) imposed at \(\tau = 0\) we obtain the UV branch of the trajectory approximately ‘realized in Nature’. It is displayed in figure 3 and compared to the approximation (1.14) which we had obtained by linearizing about the GFP.

The plots of \(g\) and \(\lambda\) versus \(\tau\) show that for \(\tau \lesssim 69\), i.e. \(k \lesssim m_{\text{Pl}}\), the linearization provides a reliable approximation while \(g\) and \(\lambda\) assume their constant fixed point values for \(\tau \gtrsim 69\). The very sharp bend in the \(g(\tau)\) and \(\lambda(\tau)\) curves indicates that the crossover from the GFP to the NGFP is very rapid for the extreme initial condition (5.15). Because of the extreme smallness of the beta functions, the trajectory spends a very long RG time near the GFP and the NGFP, respectively. The transition from the linear scaling regime of the GFP to that of the NGFP happens in a short \(\tau\) interval of the order of \(\Delta \tau \approx 6\). This is more clearly seen in figure 3(d) which shows the anomalous dimension plotted versus \(\tau\). It crosses over from \(\eta_*(\text{GFP}) = 0\) to \(\eta_*(\text{NGFP}) = -2\) between \(\tau = 66\) and 72, approximately.

Likewise one can obtain the IR branch of the trajectory by numerically integrating the equations for \(g(\tau), \lambda(\tau)\) from \(\tau = 0\) towards negative values of \(\tau\). We shall not display the plots here since the GFP linearization (1.14) is a very precise approximation for most values of \(\tau < 0\). The equations can be integrated downward only to a finite termination scale \(\tau_{\text{term}} \equiv \ln(k_{\text{term}}/k_T) > -\infty\), i.e. \(k_{\text{term}} > 0\). At this point \(\eta_N\) diverges to \(-\infty\) and the \(\beta\) functions are undefined. For the realistic initial conditions (5.14) and (5.15) we find

\[
\tau_{\text{term}} \approx -69 \implies k_{\text{term}} \approx H_0 \quad (5.18)
\]

where (5.17) was used. Quite remarkably, and in accordance with the discussion in [39], the termination scale of the realistic trajectory is about the present Hubble parameter. At this scale \(\lambda\) has reached the value 1/2, while \(g\) is of the order of \(10^{-120}\) there.
Figure 3. The realistic RG trajectory discussed in the text: (a) shows its shape on the $g$–$\lambda$ plane and compares it to the approximation (1.14) obtained by linearizing about the GFP (dashed line). (b) and (c) show the scale dependence of $g$ and $\lambda$; the dashed lines are the approximations from the GFP linearization. (d) Displays the scale dependence of the anomalous dimension.

As a consequence of this tiny $g$ value, the breakdown of the Einstein–Hilbert truncation near $\tau_{\text{term}}$ happens in a very abrupt way. In this regime the anomalous dimension (5.3) is well approximated by $\eta_N \approx -(6/\pi)g(1 - 2\lambda)^{-1} \approx -10^{-120}(1 - 2\lambda)^{-1}$, and this function jumps almost step-function-like from $\eta_N \approx 0$ at $\lambda < 1/2$ to $\eta_N = -\infty$ at $\lambda = 1/2$. Thus, immediately before the termination, the trajectory is still essentially classical: $\Lambda = \text{const}$, $G = \text{const}$. In fact, the numerics confirm the discussion in [39]: in
the IR branch the quantum effects die off already one or two orders of magnitude below \( k_T \) where then a very long classical regime starts (between \( \tau \approx -3 \) and \( \tau \approx -69 \), say). In this regime the GFP linearization (1.14) provides an excellent approximation. In order to determine the fate of the trajectory below \( \tau_{\text{term}} \) a better truncation would be needed.

5.2. The UV branch of the cosmology

In this subsection we describe the results obtained by numerically integrating equations (5.10) and (5.11) from \( x = 0 \) towards large negative values. This amounts to going back in time, towards the UV, starting at the turning point. For the equation of state parameter we choose \( w = 1/3 \) which corresponds to radiation dominance.

After fixing the initial conditions (5.14) and (5.15) only one parameter remains to be fixed, namely \( \xi \), or equivalently the \( \Omega_\Lambda \) value in the NGFP regime:

\[
\xi^2 = \left( \frac{3}{\lambda_*} \right) \Omega_\Lambda^*.
\]

(5.19)

Since \( k \) is supposed to be of the order of \( H \) we require \( \xi = O(1) \). As \( \lambda_* = O(1) \) this is indeed the case if \( \Omega_\Lambda^* = O(1) \). In principle \( \Omega_\Lambda^* \) can vary over the full interval \((0,1)\). For \( \Omega_\Lambda^* \) ‘anomalously’ close to zero, the condition \( \Omega_\Lambda^* = O(1) \) is violated, however, and we exclude such choices. On the other hand, \( \Omega_\Lambda^* \) values very close to 1 are perfectly allowed.

We shall study the UV cosmology in dependence on \( \Omega_\Lambda^* \) and are particularly interested in the limit \( \Omega_\Lambda^* \to 1 \), i.e. \( \Omega_M \to 0 \). (A possible dominance of the vacuum over the matter energy density would be nicely consistent with the physical picture that part or all of the matter is generated by ‘cosmological particle production’ during the NGFP regime.)

It will be helpful to compare the exact numerical results to the predictions of the classical FRW cosmology with \( \Lambda = 0 \). This yields, for instance, \( U(x) = \tau(x) = -2x \). Since, by (5.17), \( k \) and hence \( H \) are of the order of \( m_{\text{Pl}} \) for \( \tau \approx -69 \), the classical prediction for the logarithmic scale factor at which \( k \approx H = O(m_{\text{Pl}}) \) is

\[
x_{\text{FRW}}^\text{PI} \approx -34.5.
\]

(5.20)

Figure 4 displays the result of the numerical solution for \( \Omega_\Lambda^* = 1/2 \). The plots (a), (b) and (c) show \( U, \Omega_\Lambda, \lambda \) and \( g \) as a function of the logarithmic scale factor \( x \). We observe a crossover quite precisely at \( x_{\text{FRW}}^\text{PI} \approx -34.5 \). For smaller scale factors (earlier times) the exact numerical solution is well approximated by the analytic fixed point solution (2.16) with \( w = 1/3 \) and \( \Omega_\Lambda^* = 1/2 \). It has \( \alpha = 1, U(x) = -x, \Omega_\Lambda = \text{const} = 0.5, q = 0, (g, \lambda) = (g_*, \lambda_*) = \text{const} \), in accord with the plots. For scale factors larger than \( x_{\text{FRW}}^\text{PI} \) the behaviour is essentially that of a classical FRW cosmology without a substantial cosmological constant: \( \alpha = 1/2, U(x) = -2x, \Omega_\Lambda = \text{const} \approx 0, q = 1 \). This explains why the crossover occurs almost exactly at the value (5.20) predicted by the classical \( \Lambda = 0 \) theory. In figure 4(d) we redisplay the RG trajectory on the \( g-\lambda \) plane and indicate by a diamond the point on the trajectory corresponding to \( x = -34.5 \). In the plots 4(b) and (c) we see that the ‘width’ of the crossover is about 2 units of \( x \) (‘e folds’). In particular, the anomalous dimension changes from the canonical value \( \eta_N = 0 \) to the NGFP value \( \eta_N = -2 \) between \( x \approx -34 \) and \( x \approx -36 \). From figure 4(f) we learn that only in this interval the entropy production rate \( \mathcal{P} \) is significantly different from zero. Note that \( \mathcal{P} \) is negative during short time intervals; its time integral is positive though.
Figure 4. The UV cosmology for $\Omega_\Lambda^* = 0.5$. The plots (a), (b) and (c) display the logarithmic Hubble parameter $U$, as well as $q$, $\Omega_\Lambda$, $g$ and $\lambda$ as a function of the logarithmic scale factor $x$. A crossover is observed near $x \approx -34.5$. The diamond in plot (d) indicates the point on the RG trajectory corresponding to this $x$ value. The plots (e) and (f) show the $x$ dependence of the anomalous dimension and entropy production rate, respectively.

Figure 5 shows the analogous plots for $\Omega_\Lambda^* = 0.98$. The crossover is found at the same scale $x_{\text{Pl}}^{\text{FRW}} \approx -34.5$. The cosmology for $x \gtrsim x_{\text{Pl}}^{\text{FRW}}$, for any value of $\Omega_\Lambda^*$ in fact, is essentially the classical $\Lambda = 0$ cosmology again. The numerical results for $x < x_{\text{Pl}}^{\text{FRW}}$ approach the analytic fixed point solution with an exponent $\alpha = (2 - 2\Omega_\Lambda^*)^{-1} > 1$ for $\Omega_\Lambda^* > 0.5$ corresponding to a `power law inflation' $a(t) \propto t^\alpha$. Consistent with that we see that when $\Omega_\Lambda^* \sim 1$ the slope of $U(x) = -2(1 - \Omega_\Lambda^*)x$ decreases and finally vanishes at $\Omega_\Lambda^* = 1$. This limiting case corresponds to a constant Hubble parameter, i.e. to de Sitter space. For values of $\Omega_\Lambda^*$ smaller than, but close to 1 this de Sitter limit is approximated by an expansion $a \sim t^\alpha$ with a very large exponent $\alpha$. We can see this trend when we compare the plots (a) of figures 4 and 5, respectively. In figure 5, the logarithmic Hubble parameter has almost no visible $x$ dependence in the NGFP regime. We shall come back to this power law inflation in more detail later on.

Another feature which distinguishes the $\Omega_\Lambda^* > 1/2$ cosmologies from the case $\Omega_\Lambda^* = 1/2$ is that entropy is produced in the NGFP regime, see section 4.1. The entropy production
Figure 5. The same set of plots as in figure 4, but for a UV cosmology with $\Omega^*_\Lambda = 0.98$. Note the almost vanishing slope of $\mathcal{U}$ in the NGFP regime.

rate $\mathcal{P}$ is plotted in figures 4(f) and 5(f), respectively. The contribution from the NGFP regime is not visible in figure 5(f), however, since there $\mathcal{P}$ is much smaller than at the peak in the crossover region.

Summarizing the numerical results we can say that for any value of $\Omega^*_\Lambda$ the UV cosmologies consist of two scaling regimes with a relatively sharp crossover region near $k, H \approx m_{Pl}$ which separates them. At higher $k$ scales the fixed point approximation (4.6) is valid; at lower scales one has a classical FRW cosmology in which $\Lambda$ can be neglected. The $k^4$ cosmology discussed analytically in section 4.2 would be valid near the crossover, but it seems not to be realized for a significant number of $e$-folds.

We have not yet related the (logarithmic) cosmological time $y$ to the scale factor $x$. In principle the function $y = y(x)$ could be obtained by integrating $dt(a)/da = [aH(a)]^{-1}$. We shall not need the exact relationship here. Since FRW cosmology is valid for $t \gtrsim t_{Pl}$ the classical relation $t \propto a^2$ or $y = 2x$ is an excellent approximation for all $t \gtrsim t_{Pl}$.

5.3. The IR branch of the cosmology

By integrating the improved cosmological equations from $x = 0$ towards positive values of $x$ we obtain a 1-parameter family of cosmologies which are valid after the turning point of the RG trajectory has been passed. The free parameter is $\xi = k/H$. In the UV cosmology
we used (5.19) in order to express $\xi$ in terms of the more physical parameter $\Omega^*_\Lambda$. If $\xi$ was strictly constant all the way from the Planck regime to asymptotically late times then we could keep using (5.19) in the IR, of course. However, the cutoff identification $k = \xi H$ is only an approximation. Hence it would be unrealistic to assume that, in the case $k$ is always approximately proportional to $H$, the constant of proportionality is strictly time-independent. For this reason we allow $\xi$ in the late Universe to be different from its value in the very early Universe. So the parameter $\xi$ labelling the different IR cosmologies does not necessarily satisfy (5.19).

In section 2.3 we saw that the ‘$\Omega$ line’ along which $\rho = 0$ is a straight line on the $g$–$\lambda$ theory space, parallel to the $g$ axis at $\lambda_\Omega = 3/\xi^2$. Depending on $\xi$, the $\Omega$ line can be within the domain of validity of the Einstein–Hilbert truncation ($\lambda_\Omega \lesssim 1/2$) or outside ($\lambda \geq 1/2$). Only in the first case the RG improved field equations possess solutions which realize what was called the ‘$\Omega$ mechanism’ in [II].

Consider a solution with $\text{d}a/\text{d}t > 0$ at late times describing a Universe which keeps expanding for ever, i.e. there is no recontraction. Hence its matter contents gets continuously diluted (at least in the absence of particle creation) and at asymptotically late times one has $\rho(t \to \infty) = 0$. In the first case above this entails that, for $t \to \infty$, the Universe is described by a pair $(g, \lambda)$ on the $\Omega$ line. Remarkably, if $\lambda_\Omega < 1/2$, any RG trajectory of type IIIa hits the $\Omega$ line at a non-zero value of $k$, see figure 2. As a result, the asymptotically late Universe is characterized by a constant and non-zero scale $k_{\text{asym}} = k(t \to \infty) > 0$. (5.21)

During its entire history the Universe does not probe the complete RG trajectory, but only the portion with $k > k_{\text{asym}}$. If $\lambda(k_{\text{asym}}) = \lambda_\Omega$ is still sufficiently far below 1/2, the Einstein–Hilbert truncation can describe the latest stages of the cosmological evolution even. In fact, since its breakdown happens very abruptly near $\lambda = 1/2$ and before that $\eta_N$ is almost zero, we see that, if the $\Omega$ mechanism takes place, the late cosmology is essentially classical; no significant renormalization effects occur.

With the cutoff identification $k = \xi H$ adopted in this paper the interpretation of the asymptotic regime with $k = k_{\text{asym}} = \text{const}$ is clear$: it amounts to an asymptotic de Sitter phase with a constant Hubble parameter $H_{\text{asym}} = k_{\text{asym}}/\xi$. Or, using (2.4a) for $\rho = 0$,

$$H_{\text{asym}} = \sqrt{\Lambda(k_{\text{asym}})/3}. \tag{5.22}$$

Since $k_{\text{asym}} \ll k_T$ we may use (1.10) to rewrite (5.22) as

$$H_{\text{asym}} = \sqrt{\Lambda_0/3} = k_T \sqrt{\lambda_T/6}. \tag{5.23}$$

Neglecting factors of order unity, this relation yields the asymptotic $U$ value

$$U_{\text{asym}} \approx \frac{1}{2} \ln(\lambda_T). \tag{5.24}$$

For the realistic trajectory with the initial conditions (5.14) and (5.15) we have $U_{\text{asym}} \approx -30 \ln(10) \approx -69$.

For a numerical investigation of the $\Omega$ mechanism it is most convenient to integrate the evolution equations with respect to $\lambda$ as the independent parameter, see equations (5.13).

$^7$ In [II] the situation was slightly more complicated since a dynamical cutoff identification had been used.
In figure 6 we show the results for the example with $\xi = 2.86$ which has $\lambda_\Omega = 0.367$. (If we insist on a strictly constant $\xi$ this would correspond to $\Omega_\Lambda^* = 0.90$.)

Figure 6(b) shows that $\mathcal{U}$ indeed approaches the constant value $-69$ asymptotically. The plot in figure 6(c) displays the logarithmic scale factor as a function of $\lambda$. We see that $x$ diverges for $\lambda \nearrow \lambda_\Omega$, which is precisely the signature of the $\Omega$ mechanism. Figures 6(d) and (e) confirm that the cosmology directly at $\lambda = \lambda_\Omega$ is an almost ‘empty’ de Sitter Universe with $q = -1$ and $\Omega_\Lambda = 1$. The energy density of ordinary matter has dropped to zero there. One can check that this entire cosmology is essentially classical. The quantum corrections to the beta functions are negligible on the entire interval $\lambda_T \leq \lambda \leq \lambda_\Omega$, which implies in particular that there is no entropy production.

This calculation has been performed for $w = 1/3$. To be more realistic one should compute the last few $e$ folds with $w = 0$ corresponding to matter dominance. This will not change the overall conclusion, however, that there are no significant quantum effects in the IR cosmologies with $\Omega$ mechanism. As long as $\lambda_\Omega < 1/2$, every value of $\xi$ yields essentially the same cosmology.

As a result, we may use the classical FRW formulae to estimate the time when the Universe starts accelerating due to the transition from matter or radiation dominance to
A dominance. For a first orientation we can use (5.7) to estimate \( x \) and \( y \) at the onset of the acceleration:

\[
x_{\text{acc}} \approx -U_{\text{asym}}/2 \approx 34.5
\]

(5.25)

\[
y_{\text{acc}} \approx -U_{\text{asym}} \approx 69.
\]

(5.26)

Interestingly enough, these numbers correspond roughly to the scale factor and age of the Universe we live in. In particular its age is \( t_0 \approx t_{\text{acc}} \approx 10^{30} \), \( t_T \approx 10^{60} \) \( t_{\text{Pl}} \).

It is important to understand what determines the time \( t_{\text{acc}} \) at which the Universe switches from deceleration to acceleration as the vacuum energy starts dominating the matter energy density. According to the observations, this has happened only ‘very recently’ in our cosmological history, so the natural and frequently posed question is ‘why just now?’ [46,47].

In the present setting the answer to this question is clear: \( t_{\text{acc}} \) is determined by the asymptotic value of the cosmological constant which, in turn, is dictated by the RG trajectory. Hence, \( t_{\text{acc}} \) is what it is because Nature’s RG trajectory is what it is.

This might appear to be a rather tautological statement at first sight, in particular since we actually used the observed cosmological constant to fix the parameters of the trajectory. However, QEG is believed to be a predictive theory [6,18] in the sense that at the exact level only \textit{finitely many} parameters need to be taken from the experiment in order to completely determine the trajectory, and that then \textit{infinitely many} predictions are possible in terms of those\(^8\). One of the input parameters is \( \Lambda \) at some scale, and there are \textit{a few} more such input parameters. One of the predictions is \( t_{\text{acc}} \), but there are \textit{infinitely many} more such predictions.

It is only because of the observational situation we are in that the above statement about \( t_{\text{acc}} \) appears tautological. Since the only determination of \( \Lambda \) which is available to date is on cosmological scales, we are forced to use this cosmological \( \Lambda \) as an input parameter and therefore cannot predict \( \Lambda(k_{\text{asym}}) \) or \( t_{\text{acc}} \) in terms of anything independent. Instead, we are able to make ‘predictions’ about the early Universe in terms of the parameters fixed in the late Universe. Conceptually the situation is the same as in (perturbative) QED for instance. The pertinent RG trajectories have two free parameters. It is convenient, but by no means compulsory, to choose them as the mass \( m \) and charge \( e \) of the electron. If one does so, \( e \) and \( m \) are no predictions of course, but if instead we parametrize the trajectory by two different couplings \( g_1 \equiv g_1(e,m) \) and \( g_2 \equiv g_2(e,m) \) and measure \( g_1 \) and \( g_2 \) then \( e \) and \( m \) are predicted by the theory in terms of \( g_1 \) and \( g_2 \). Likewise we can imagine a (considerably) improved experimental situation in which all the parameters of the gravitational RG trajectory can be determined from laboratory measurements. The time \( t_{\text{acc}} \) and similar cosmological quantities are true predictions then.

If \( \lambda_0 > 1/2 \) there is no \( \Omega \) line which would prevent the RG trajectory underlying the cosmological evolution to run into the singularity at \( \lambda = 1/2 \). In this case the coupled RG and cosmological equations cannot be integrated beyond a certain point where the Einstein–Hilbert truncation breaks down. As we explained in the introduction already, this breakdown, if it occurs, is expected to happen when \( k \approx H_0 \), i.e. ‘just now’. For smaller \( k \) scales one would have to use a more general truncation of theory space whose

\(^8\) See [18] for a more precise discussion of this point.
implications for the cosmology cannot be guessed offhand. A theoretically attractive (and phenomenologically viable) possibility could be the IR fixed point model of refs. [34,35].

6. Inflation in the fixed point regime

In this section we discuss in some detail the epoch of power law inflation which is realized in the NGFP regime if $\Omega_\Lambda^* > 1/2$. Since, as we saw in the previous section, the transition from the fixed point to the classical FRW regime is rather sharp it will be sufficient to approximate the RG improved UV cosmologies by the following caricature. For $0 < t < t_{tr}$, where $t_{tr}$ is a transition time, the scale factor behaves as $a(t) \propto t^\alpha$, $\alpha > 1$. Here $\alpha = (2 - 2\Omega_\Lambda^*)^{-1}$ since $w = 1/3$ will be assumed. Thereafter, for $t > t_{tr}$, we have a classical, entirely matter-driven expansion $a(t) \propto t^{1/2}$.

6.1. Transition time and apparent initial singularity

The transition time $t_{tr}$ is dictated by the RG trajectory. The latter leaves the asymptotic scaling regime near $k \approx m_{Pl}$. Hence $t_{tr}$ is the time at which $k(t_{tr}) = \xi H(t_{tr}) \approx m_{Pl}$. In the following we only consider values of $\Omega_\Lambda^*$ in the interval $(1/2, 1)$ because there is no inflation otherwise. For such values of $\Omega_\Lambda^*$, and since $\lambda^* = O(1)$, equation (4.4) tells us that

$$\xi = \sqrt{3\Omega_\Lambda^*/\lambda^*}$$

is of order unity so that we can determine $t_{tr}$ from $H(t_{tr}) \approx m_{Pl}$. Using (4.3) at the matching point we find

$$t_{tr} = \alpha t_{Pl}.$$  \hspace{1cm} (6.2)

This is an important relation and several comments are in order here. Let us recall that, as always in this paper, the Planck mass, time and length are defined in terms of the value of Newton’s constant in the classical regime, cf. the discussion following equation (1.15):

$$t_{Pl} = \ell_{Pl} = m_{Pl}^{-1} = \sqrt{\frac{G}{G_{obs}}} = \sqrt{\frac{G}{G_{obs}}}.$$  \hspace{1cm} (6.3)

For the sake of the argument, let us now assume that $\Omega_\Lambda^*$ is very close to 1 so that $\alpha$ is large: $\alpha \gg 1$. Then (6.2) implies that the transition takes place at a cosmological time which is much later than the Planck time. At the transition the Hubble parameter is of order $m_{Pl}$, but the cosmological time is in general not of the order of $t_{Pl}$. Stated differently, the ‘Planck time’ is not the time at which $H$ and the related physical quantities assume Planckian values. Turning (6.2) around we conclude that the Planck time as defined above is well within the NGFP regime: $t_{Pl} = t_{tr}/\alpha \ll t_{tr}$.

At $t = t_{tr}$ the NGFP solution (4.6) is to be matched continuously with a FRW cosmology (with vanishing cosmological constant). We may use the familiar classical formulae such as (3.17) for the scale factor, but we must shift the time axis on the classical side such that $a, H$, and then as a result of (2.4a) also $\rho$ are continuous at $t_{tr}$. Therefore $a(t) \propto (t - t_{as})^{1/2}$ and

$$H(t) = \frac{1}{2}(t - t_{as})^{-1} \quad \text{for } t > t_{tr}.$$  \hspace{1cm} (6.4)
Figure 7. Shown is the proper length $L$ and the Hubble radius as a function of time. The NGFP and FRW cosmologies are valid for $t < t_{tr}$ and $t > t_{tr}$, respectively. The classical cosmology has an apparent initial singularity at $t_{as}$ outside its domain of validity. Structures of size $e^N l_P$ at $t_{tr}$ cross the Hubble radius at $t_N$, a time which can be larger than the Planck time.

Equating the Hubble parameter (6.4) at $t = t_{tr}$ to $H(t) = \alpha/t$, valid in the NGFP regime, we find that the shift $t_{as}$ must be chosen as

$$t_{as} = \left(\alpha - \frac{1}{2}\right) t_P = \left(1 - \frac{1}{2\alpha}\right) t_{tr} < t_{tr}. \quad (6.5)$$

Here the subscript ‘as’ stands for ‘apparent singularity’. This is to indicate that if one continues the classical cosmology to times $t < t_{tr}$, it has an initial singularity (‘big bang’) at $t = t_{as}$. Since, however, the FRW solution is not valid there nothing special happens at $t_{as}$; the true initial singularity is located at $t = 0$ in the NGFP regime. (See figure 7.)

We emphasize that for any choice of $\Omega_\Lambda^*$, and hence $\alpha$, one always has

$$H(t_{tr}) = m_{Pl}. \quad (6.6)$$

At the moment when the classical cosmology starts becoming valid, whatever was its ‘prehistory’, it starts with $H \approx m_{Pl}$ and $\rho \approx m_{Pl}^4$.

6.2. Crossing the Hubble radius

In the NGFP regime $0 < t < t_{tr}$ the Hubble radius $\ell_H(t) \equiv 1/H(t)$, i.e.

$$\ell_H(t) = \frac{1}{\alpha} t, \quad (6.7)$$

increases linearly with time but, for $\alpha \gg 1$, with a very small slope. At the transition, the slope jumps from $1/\alpha$ to the value 2 since $H = 1/(2t)$ and $l_P = 2t$ in the FRW regime. This behaviour is sketched in figure 7. The length scale $\ell_H$ measures the radius of curvature of spacetime. It has no interpretation as the distance to a horizon: Robertson–Walker spacetimes with $a(t \to 0) \propto t^\alpha$, $\alpha > 1$, have no particle horizon. At the transition time $\ell_H(t_{tr}) = l_P$. 
Entropy signature of the running cosmological constant

Let us consider some structure of comoving length \( \Delta x \), a single wavelength of a density perturbation, for instance. The corresponding physical, i.e. proper, length is \( L(t) = a(t)\Delta x \) then. In the NGFP regime it has the time dependence

\[
L(t) = \left( \frac{t}{t_{\text{tr}}} \right)^\alpha L(t_{\text{tr}}).
\]  

(6.8)

The ratio of \( L(t) \) and the Hubble radius evolves according to

\[
\frac{L(t)}{\ell_H(t)} = \left( \frac{t}{t_{\text{tr}}} \right)^{\alpha-1} \frac{L(t_{\text{tr}})}{\ell_H(t_{\text{tr}})}.
\]  

(6.9)

For \( \alpha > 1 \), i.e. \( \Omega^*_\Lambda > 1/2 \), the proper length of any object grows faster than the Hubble radius. So objects which are of ‘sub-Hubble’ size at early times can cross the Hubble radius and become ‘super-Hubble’ at later times, see figure 7.

Let us focus on a structure which, at \( t = t_{\text{tr}} \), is \( e^N \) times larger than the Hubble radius. Before the transition we have

\[
L(t)/\ell_H(t) = e^N (t/t_{\text{tr}})^{\alpha-1}.
\]  

(6.10)

Assuming \( e^N > 1 \), there exists a time \( t_N < t_{\text{tr}} \) at which \( L(t_N) = \ell_H(t_N) \) so that the structure considered ‘crosses’ the Hubble radius at the time \( t_N \). Using (4.12) it is given by

\[
t_N = t_{\text{tr}} \exp \left( -\frac{N}{\alpha - 1} \right) = t_{\text{tr}} \exp \left[ -\frac{(1 - \Omega^*_\Lambda)N}{(\Omega^*_\Lambda - 1/2)} \right].
\]  

(6.11)

What is remarkable about this result is that, even with rather moderate values of \( \alpha \), one can easily ‘inflate’ structures to a size which is by many factors larger than the Hubble radius during a very short time interval at the end of the NGFP epoch.

Let us illustrate this phenomenon by means of an example, namely the choice \( \Omega^*_\Lambda = 0.98 \) used in figure 5. Corresponding to 98% vacuum and 2% matter energy density in the NGFP regime, this value is still ‘generic’ in the sense that \( \Omega^*_\Lambda \) is not fine tuned to equal unity with a precision of many decimal places. It leads to the exponent \( \alpha = 25 \), the transition time \( t_{\text{tr}} = 25t_{\text{Pl}} \) and \( t_{\text{tr}} = 24.5t_{\text{Pl}} \).

The largest structures in the present Universe, evolved backward in time by the classical equations to the point where \( H = m_{\text{Pl}} \), have a size of about \( e^{60}t_{\text{Pl}} \) there. We can use (6.11) with \( N = 60 \) to find the time \( t_{60} \) at which those structures crossed the Hubble radius. With \( \alpha = 25 \) the result is \( t_{60} = 2.05t_{\text{Pl}} = t_{\text{tr}}/12.2 \). Remarkably, \( t_{60} \) is smaller than \( t_{\text{tr}} \) by one order of magnitude only. As a consequence, the physical conditions prevailing at the time of the crossing are not overly ‘exotic’ yet. The Hubble parameter, for instance, is only one order of magnitude larger than at the transition: \( H(t_{60}) \approx 12m_{\text{Pl}} \). The same is true for the temperature; equation (4.18) implies \( T(t_{60}) \approx 12T(t_{\text{tr}}) \) where \( T(t_{\text{tr}}) \) is of the order of \( m_{\text{Pl}} \). Note also that \( t_{60} \) is larger than \( t_{\text{Pl}} \).

6.3. Primordial density fluctuations

QEG offers a natural mechanism for generating primordial fluctuations during the NGFP epoch which have a scale-free spectrum with a spectral index close to \( n = 1 \). This
mechanism is at the very heart of the ‘asymptotic safety’ underlying the nonperturbative renormalizability of QEG. It might open an observational window which allows us a view of the gravitational physics in a regime where we expect qualitatively important quantum effects. Hence this issue could be of interest for the program of asymptotic safety per se and not only for cosmology.

The cosmology of the very early Universe reflects properties of the RG trajectory close to the fixed point. In this regime the anomalous dimension of the graviton is very close to \( \eta_N^* = -2 \), its value directly at the NGFP (in \( d = 4 \)).

Using the effective field theory properties of \( \Gamma_k \) it was shown in [3] that the graviton propagator implied by the standard effective action \( \Gamma_{k=0} \), on a flat background, has a large momentum behaviour \( \tilde{G}(p) \propto 1/p^4 \) which amounts to \( G(x; y) \propto \ln(x - y)^2 \) in position space. This form of the propagator is valid for \( p^2 \gg m^2_{\text{Pl}} \) and \( (x - y)^2 \ll \ell_{\text{Pl}}^2 \), respectively. It is a direct consequence of \( \eta_N^* = -2 \). In fact, the logarithmic position dependence can be understood as a limiting case of the standard critical two-point function \( G(x; y) \propto 1/(x - y)^{d-2+\eta} \) for \( d = 4 \) and \( \eta \to -2 \).

Following [I], let us now consider curvature fluctuations \( \delta R \) caused by metric fluctuations \( h_{\mu\nu}(x) \). In a symbolic notation we have \( \delta R \propto \partial \delta h \) where \( R \) stands for any component of the Riemann tensor. As \( \langle h_{\mu\nu}(x)h_{\rho\sigma}(y) \rangle \propto \ln(x - y)^2 \) the two-point function of \( \delta R \) is found to be \( \langle \delta R(x)\delta R(y) \rangle \propto 1/(x - y)^4 \). (In the classical regime we would have a decay \( \propto 1/(x - y)^6 \) instead.) Up to now the background was assumed flat. Allowing for a curved background spacetime, the above formulae will give the leading short distance behaviour:

\[
G(x; y) \propto \ln d(x, y)^2, \quad \langle \delta R(x)\delta R(y) \rangle \propto \frac{1}{d(x, y)^4}.
\] (6.12)

Here \( d(x, y) \) is the geodesic distance of \( x \) and \( y \). Equations (6.12) are valid provided \( d(x, y) \) is smaller than the radius of curvature of the background spacetime (and \( \ell_{\text{Pl}} \), of course). In a Robertson–Walker geometry this condition amounts to \( d(x, y) < \ell_{\text{H}}(t) \), i.e. (6.12) is valid on ‘sub-Hubble’ scales.

Next assume \( x \) and \( y \) are two points on the same time slice of a Robertson–Walker spacetime. Their distance is \( d(x, y) = a(t)|x - y| \), where \( x \) and \( y \) are the comoving Cartesian coordinates of \( x \) and \( y \). Ignoring the time dependence, (6.12) yields

\[
\langle \delta R(x, t)\delta R(y, t) \rangle \propto \frac{1}{|x - y|^4}.
\] (6.13)

The above general arguments imply that these relations should be valid if \( a(t)|x - y| \ll \ell_{\text{H}}(t) \ll \ell_{\text{Pl}} \). (At larger distances the two-point function can be determined by a detailed computation only which has not been performed yet.) In the improved cosmologies with inflation we found that for any value of \( \Omega^*_\Lambda \) the inequality \( \ell_{\text{H}}(t) < \ell_{\text{Pl}} \) is satisfied for all \( t < t_{\text{tr}} \). Hence (6.13) is applicable, on sub-Hubble distances, during the entire NGFP era.

Let us now come back to the problem of primordial density perturbations which could act as seeds for structure formation in the Universe. Here we adopt the same hypothesis as in the standard inflationary scenarios [50, 51], namely that they stem from quantum
fluctuations which have effectively become classical. In models of scalar-driven inflation it is usually the fluctuations of the ‘inflaton’ itself which serves this purpose. In our case inflation happens automatically as a consequence of the RG running and no inflaton is needed. Instead, it is the fluctuations of the geometry itself, i.e. of the metric and its curvature, which are the natural candidates for the seeds of structure formation.

A quantity we have observational access to is the classical correlator of density perturbations,

$$\xi(x) = \langle \delta(x + y) \delta(y) \rangle,$$

(6.14)

where $\delta(x) \equiv \delta\rho(x)/\rho$. If its power spectrum at a fixed instant of time,

$$|\delta_k|^2 \equiv V \int dx^3 \xi(x) \exp(-i k \cdot x),$$

(6.15)

behaves as $|\delta_k|^2 \propto |k|^n$ the spectrum is said to have the spectral index $n$. From the observation of the CMBR we know that the perturbations $\delta\rho$ which got imprinted in the microwave background at decoupling had an almost scale-free (Harrison–Zeldovich) spectrum with $n \approx 1$.

Remarkably, this is exactly the spectrum one obtains if the seeds of the density perturbations are sub-Hubble fluctuations in the NGFP era. The reasoning in [I] was as follows. Already at the level of the classical Einstein equations, density fluctuations $\delta\rho$ are proportional to fluctuations $\delta G_{00}$ of the Einstein tensor $G_{\mu\nu}$, i.e. a special combination of $\delta R$ components. Therefore, if fluctuations of the geometry are the source of the density fluctuations, the correlators of $\delta\rho$ should at least approximatively be proportional to that of $\delta R$ as given in equation (6.13): $\xi(x) \propto 1/|x|^4$. Taking the Fourier transform one finds $|\delta_k|^2 \propto |k|$, i.e. the spectral index $n = 1$. This argument is similar in spirit to the discussion in [53]. It suggests that, near $t_{tr}$, when the Universe has become classical, density perturbations have been created from quantum fluctuations with a nearly scale-free spectrum, $n \approx 1$.

Since the evolution of the perturbations after $t_{tr}$ is essentially classical we know that, to be of phenomenological relevance, the $n = 1$ spectrum thus generated should apply to fluctuation modes with wavelengths as large as $e^{30} \ell_{Pl}$, say, at $t = t_{tr}$. If the wavelength is larger than $\ell_{Pl}$ the scale-free correlator (6.13) is not valid, most probably. However, in any of the cosmologies with $\Omega_\Lambda^* > 1/2$ there is an inflationary NGFP era. As a consequence, there exists always a time $t_{60}$ before which the modes are completely within the Hubble radius and the above argument does apply. (See figure 7.) If they keep their spectrum during the expansion to super-Hubble scales we end up with a $n \approx 1$ spectrum at $t = t_{tr}$.

If $\alpha$ is large the crossing times $t_N$ of all modes relevant to structure formation are close to $t_{tr}$. Hence they all become ‘super-Hubble’ at about the same value of $H$ and $\ell_H$.

At this point there is a clear difference between the RG improved cosmology found in [I] by imposing the ‘consistency condition’ and the new ones found in the present paper. The cosmology of [I] has $a(t) \propto t$, i.e. no inflation. Therefore the above argument applies only to the modes which were sub-Hubble at $t_H = t_{Pl}$ and are of millimetre size today.
6.4. No reheating is necessary

By combining (4.18) with (4.15) and (6.2) we obtain the following expression for the temperature at the end of the NGFP regime:

\[ T(t_{tr}) = \left[ \frac{135}{8\pi^3 n_{\text{eff}} g_* \lambda_*} \right]^{1/4} \left( 1 - \frac{1}{2\alpha} \right)^{1/4} \alpha^{-1/4} m_{\text{Pl}}. \]  

This temperature is the initial value for the classical cosmology after \( t_{tr} \). If we evolve the present \( T_0 = 2.7 \) K backward by means of the classical equations to the time when \( H = m_{\text{Pl}}, \) i.e. to \( t = t_{tr} \), we obtain a temperature of the order of \( m_{\text{Pl}} \) and this value should coincide with (6.16). Because of the very weak \( \alpha \) dependence of \( T(t_{tr}) \), and assuming \( n_{\text{eff}} g_* \lambda_* \) is of order unity, this is indeed the case for a wide range of \( \alpha \) values, namely those for which \( \Omega^*_{\Lambda} \) is not ‘anomalously close’ to 1.

Hence, contrary to many of the conventional models of inflation [50], the RG cosmology does not require a phase of ‘reheating’ before the classical FRW evolution can (re-)start. The reason is clear: because of the energy transfer from the cosmological constant to the radiation there was a continuous ‘heating’ during the NGFP epoch, as a consequence of which the temperature decreased only very slowly as \( T \propto 1/t \) even though \( a \propto t^{\alpha} \) inflated rapidly with a large exponent \( \alpha \) possibly.

We shall give a more quantitative description of the generation of density fluctuations elsewhere [54]. Once their ‘birth’ as quantum fluctuations of geometry and their subsequent decoherence is understood in more detail the next task will consist in evolving them from about \( t_{tr} \) towards later times. Since we found no significant quantum gravity effects in the very early Universe much later than \( t_{tr} \), this evolution will be essentially the standard one, at least as long as \( \lambda \lesssim 1/2 \). The UV quantum effects we discussed in this paper would not interfere with the usual refinements of the minimal (homogeneous, isotropic, adiabatic, etc.) standard model of cosmology which concerns later epochs only. In particular one might include standard dissipative processes in the classical regime. Or one could study inhomogeneous and/or anisotropic cosmologies and explore also the classical RG flow related to an averaging over spatial inhomogeneities [55]. In classical cosmology one assumes, and this assumption is to some extent justified by its success, that these modifications do not completely invalidate the minimal FRW cosmology as a kind of ‘zeroth approximation’. The same assumption underlies the present investigations. We discussed one specific source of modification, the running of the gravitational parameters, which exists over and above all standard refinements and generalizations which mostly concerns later stages of the cosmological evolution only.

Since \( t_{tr} \) is not very much later than the Planck time, the UV effects are certainly irrelevant during the era of structure formation after decoupling. The situation is different for the as-yet poorly understood IR effects which might or might not show up near \( \lambda = 1/2 \). As conjectured in [38, 39] they could lead to observable effects at the scale of galaxies, but it was not yet possible to test their conjecture by explicit flow equation studies. If it should turn out to be correct, then also the dynamics of structure formation is likely to be affected by QEG effects. Structure formation in cosmologies with a running Newton constant was first discussed in [56]. The qualitative phenomenological analysis of this work might perhaps also apply to the QEG case; it is not clear, however, whether in this context it is an admissible approximation to consider only \( G \) scale-dependent and ignore
all other induced couplings, in particular as any reliable approximation to the effective average action in this regime is bound to be nonanalytic in the curvature and/or nonlocal. While these issues clearly have to be studied further they are unrelated to the main topic of this paper, the cosmological implications of the UV effects encapsulated in the explicitly known local truncations.

7. Discussion and conclusion

In this paper we advocated the point of view that the scale dependence of the gravitational parameters has an impact on the physics of the Universe we live in and we tried to identify known features of the Universe which could possibly be due to this scale dependence. We discussed two possible candidates for such features: the entropy carried by the radiation which fills the Universe today, and a period of Λ-driven inflation directly after the big bang.

As for the first point, we argued that within QEG the most likely RG trajectory is of type IIIa, predicting a positive cosmological constant whose magnitude decreases with scale. We saw that this leads to a continuous transfer of energy from the vacuum to the matter sector. Already this process alone could generate the entropy carried by the CMBR photons today. In this picture the cosmological evolution started from a pure state; the entropy of the matter system is caused by the ‘coarse graining’ of the quantum gravitational dynamics which is forced upon us because the optimal effective field theory \( \Gamma_k[g_{\mu\nu}] \) changes as the Universe expands. The time dependence of \( k \) leads in particular to a time-dependent cosmological constant. It acts like a ‘quintessence’ field \([48, 49]\) in that it explains the present value of \( \Lambda \) dynamically, its smallness being due to the Universe’s old age. This ‘quintessence’ field is a natural consequence of quantum field theory and does not have to be introduced by hand.

As for inflation, there is clearly no direct observational evidence for an inflationary epoch in the early Universe which theory necessarily would have to explain. However, such an epoch would help in understanding certain properties of the observed Universe, in particular as it can stretch primordial density perturbations from sub-to super-Hubble scales. Allowing for an unrestricted energy exchange between the vacuum and the matter sector we found solutions of the RG improved cosmological evolution equations with a phase of power law inflation immediately after the initial singularity. In this phase \( \Lambda \) dominates the matter energy density. The inflationary expansion gets ‘switched off’ automatically due to the RG running of \( \Lambda(k) \). For \( k \lesssim m_{\text{Pl}} \) the cosmology approaches that of a classical FRW model. In this context the scale-, and hence time-, dependent cosmological constant plays the role of an inflaton which, again, does not need to be introduced by hand but rather arises as a quantum effect.

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