ÉTALE AND CRYSTALLINE COMPANIONS, II

KIRAN S. KEDLAYA

Abstract. Let $X$ be a smooth scheme over a finite field of characteristic $p$. In answer to a conjecture of Deligne, we establish that for any prime $\ell \neq p$, an $\ell$-adic Weil sheaf on $X$ which is algebraic (or irreducible with finite determinant) admits a crystalline companion in the category of overconvergent $F$-isocrystals, for which the Frobenius characteristic polynomials agree at all closed points (with respect to some fixed identification of the algebraic closures of $\mathbb{Q}$ within fixed algebraic closures of $\mathbb{Q}_\ell$ and $\mathbb{Q}_p$). The argument depends heavily on the free passage between $\ell$-adic and $p$-adic coefficients for curves provided by the Langlands correspondence for $GL_n$ over global function fields (work of L. Lafforgue and T. Abe), and on the construction of Drinfeld (plus adaptations by Abe–Esnault and Kedlaya) giving rise to étale companions of overconvergent $F$-isocrystals. As corollaries, we transfer a number of statements from crystalline to étale coefficient objects, including properties of the Newton polygon stratification (results of Grothendieck–Katz and de Jong–Oort–Yang) and Wan’s theorem (previously Dwork’s conjecture) on $p$-adic meromorphicity of unit-root $L$-functions.

Introduction

0.1. Overview. Let $k$ be a finite field of characteristic $p$ and let $X$ be a smooth scheme over $k$. In a previous paper [40], we studied the relationship between coefficient objects (of locally constant rank) in Weil cohomology with $\ell$-adic coefficients for various primes $\ell$. For $\ell \neq p$, such objects are lisse Weil $\overline{\mathbb{Q}}_\ell$-sheaves, while for $\ell = p$ they are overconvergent $F$-isocrystals.

The purpose of this paper is to complete the proof of a conjecture of Deligne [17, Conjecture 1.2.10] which asserts that all coefficient objects “look motivic”, that is, they have various features that would hold if they were to arise in the cohomology of some family of smooth proper varieties over $X$. The deepest aspect of this conjecture is the fact that coefficient objects do not occur in isolation; in a certain sense, coefficient objects in one category have “companions” in the other categories.

To make this more precise, we say that an $\ell$-adic coefficient $E$ on $X$ is algebraic if for each closed point $x \in X$, the characteristic polynomial of (geometric) Frobenius $F_x$ acting on the fiber $E_x$ has coefficients which are algebraic over $\mathbb{Q}$.

To define the companion relation, consider two primes $\ell, \ell'$ and fix an identification of the algebraic closures of $\mathbb{Q}$ within $\overline{\mathbb{Q}}_\ell$ and $\overline{\mathbb{Q}}_{\ell'}$. We say that an algebraic $\ell$-adic coefficient $E$ on $X$
and an algebraic $\ell'$-adic coefficient $E'$ on $X$ are companions if for each closed point $x \in X$, the characteristic polynomials of Frobenius on $E'_x, E'_x$ coincide; note that $E$ then determines $E'$ up to semisimplification [40, Theorem 3.3.1].

With this definition, we can state a theorem answering [17, Conjecture 1.2.10] (and more precisely [40, Conjecture 0.1.1] or [9, Conjecture 1.1]), incorporating Crew’s proposal [13, Conjecture 4.13] to interpret (vi) by reading the phrase “petit camarade cristalline” to mean “companion in the category of overconvergent $F$-isocrystals.” (The definition of an overconvergent $F$-isocrystal was unavailable at the time of [17]; it was subsequently introduced by Berthelot [6].) See Corollary 8.1.6 for the proof.

**Theorem 0.1.1.** Let $E$ be an $\ell$-adic coefficient which is irreducible with determinant of finite order. (Recall that we allow $\ell = p$ here.) In the following statements, $x$ is always quantified over all closed points of $X$, and $\kappa(x)$ denotes the residue field of $x$.

(i) $E$ is pure of weight 0: for every algebraic embedding of $\overline{Q}_\ell$ into $\mathbb{C}$ and all $x$, the images of the eigenvalues of $F_x$ all have complex absolute value 1.

(ii) For some number field $E$, $E$ is $E$-algebraic: for all $x$, the characteristic polynomial of $F_x$ has coefficients in $E$. (Beware that the roots of this polynomial need not belong to a single number field as $x$ varies.)

(iii) $E$ is $p$-plain: for all $x$, the eigenvalues of $F_x$ have trivial $\lambda$-adic valuation at all finite places $\lambda$ of $E$ not lying above $p$.

(iv) For every place $\lambda$ of $E$ above $p$ and all $x$, every eigenvalue of $F_x$ has $\lambda$-adic valuation at most $\frac{1}{2} \text{rank}(E)$ times the valuation of $\#\kappa(x)$.

(v) For any prime $\ell' \neq p$ and any place $\lambda$ of $E$ above $\ell'$, there exists an $\ell'$-adic coefficient $E'$ which is irreducible with determinant of finite order and is a companion of $E$ with respect to $\lambda$.

(vi) As in (v), but with $\ell' = p$.

Of the various aspects of Theorem 0.1.1 all were previously known for $X$ of dimension 1, and all but (vi) for general $X$ (see below); consequently, the new content can also be expressed as follows (answering [40, Conjecture 0.5.1]). See Corollary 8.1.4 for the proof.

**Theorem 0.1.2.** Any algebraic $\ell$-adic coefficient on $X$ admits an $\ell'$-adic companion. (By [40, Theorem 3.5.2], this is only new for $\ell' = p$.)

In the remainder of this introduction, we summarize the preceding work in the direction of Theorem 0.1.1 including the results of [40]; we then describe the new ingredients in this paper that lead to a complete proof. See also the survey article [39] for background on $p$-adic coefficients.

0.2. Prior results: dimension 1. One conceivable approach to proving Theorem 0.1.2 would be to show that every $\ell$-adic coefficient arises as a realization of some motive, to which one could then apply the $\ell'$-adic realization functor to obtain the $\ell'$-adic companion. As part of the original formulation of [17, Conjecture 1.2.10], Deligne pointed out that for $X$ of dimension 1, one could hope to execute this strategy for $\ell, \ell' \neq p$ by establishing the Langlands correspondence for $GL_n$ over the function field of $X$ for all positive integers $n$; at the time, this was done only for $n = 1$ by class field theory, and for $n = 2$ by the work of Drinfeld [20]. An extension of Drinfeld’s work to general $n$ was subsequently achieved by L. Lafforgue [46], which yields parts (i)–(v) of Theorem 0.1.1 when $\dim(X) = 1$ except with a
slightly weaker inequality in part (iv); this was subsequently improved by V. Lafforgue [47] to obtain (iv) as written (and a bit more).

This work necessarily omitted cases where \( \ell = p \) or \( \ell' = p \) due to the limited development of \( p \)-adic cohomology (rigid cohomology) at the time. Building on recent advances in this direction, Abe [1] has replicated Lafforgue’s argument in \( p \)-adic cohomology; this completes parts (i)–(v) of Theorem 0.1.1 when \( \dim(X) = 1 \) by adding the cases where \( \ell = p \) or \( \ell' = p \).

0.3. Prior results: higher dimension. In higher dimensions, no general method for associating motives to coefficient objects seems to be known. The proofs of the various aspects of Theorem 0.1.1 for general \( X \) thus proceed by using the case of curves as a black box.

To begin with, suppose that \( \ell \neq p \). To make headway, one first shows that irreducibility is preserved by restriction to suitable curves; this was shown by Deligne [18, §1.7] by correcting an argument of L. Lafforgue [46, §VII]. Since parts (i), (iii), (iv) of Theorem 0.1.1 are statements about individual closed points, they follow almost immediately.

As for part (ii) of Theorem 0.1.1 by restricting to curves one sees that the coefficients in question are all algebraic, but one needs a uniformity argument over these curves to show that the extension of \( \mathbb{Q} \) generated by all of the coefficients is finite. Such an argument was provided by Deligne [18]. Building on this, Drinfeld [21] was then able to establish part (v) of Theorem 0.1.1 using an idea of Wiesend [69] to patch together tame Galois representations based on their restrictions to curves. (Esnault–Kerz [28] refer to this technique as the method of skeleton sheaves.)

It is not entirely automatic to extend these results to the case \( \ell = p \), as several key arguments (notably preservation of irreducibility) are made in terms of properties of \( \ell \)-adic sheaves with no direct \( p \)-adic analogues. Generally, arguments that refer to residual representations do not transfer (although there are some crucial exceptions), whereas arguments that refer only to monodromy groups or cohomology do transfer. With some effort, one can replace the offending arguments with alternates that can be ported to the \( p \)-adic setting, and thus recover the previously mentioned results with \( \ell = p \); this was carried out (in slightly different ways) by Abe–Esnault [2] and Kedlaya [40].

Crucially, this possibility of replacement only applies in cases where one starts with a \( p \)-adic coefficient; it therefore does not apply to part (vi) of Theorem 0.1.1 where the existence of a \( p \)-adic coefficient is itself at issue and cannot be established using a direct analogue of the \( \ell \)-adic construction. However, from the above discussion, one can at least deduce some reductions for the problem of constructing crystalline companions; notably, in any given case the existence of a crystalline companion can be checked after pullback along an open immersion with dense image, or an alteration in the sense of de Jong [16]. Also, we may ignore the case \( \ell = p \), as we may move from \( \ell = p \) to \( \ell' = p \) via an intermediate prime different from \( p \). This means that in most of this paper, we can focus on the case of an étale \( \ell \)-adic coefficient which is tame (that is, whose local monodromy representations are tamely ramified and quasi-unipotent) or even docile (replacing “quasi-unipotent” by “unipotent”). We can also focus most attention on the situation where \( X \) embeds into a smooth curve fibration, which provides some technical simplifications (for example, the existence of smooth lifts étale-locally on the base of the fibration).

0.4. Tame rigidity. We now arrive at the methods of the present paper. At a superficial level, the basic strategy is the same as in Drinfeld’s work. Given an étale coefficient on
X, the Langlands correspondence implies the existence of a crystalline companion for the restriction to any curve contained in X, and we wish to tie these objects together into a single crystalline coefficient.

We conceptualize the strategy in a framework we call the three principles of tame rigidity (see Definition 5.1.3). As above, assume that X is the total space of a smooth curve fibration. Consider two coefficient objects on X which are tame along the boundary of the relative compactification and constant along some other section. (This last condition arises naturally in Artin’s construction of elementary fibrations; see Corollary 1.3.3 and Remark 1.3.5.) The uniqueness principle states that if at least one of these objects is absolutely irreducible and their restrictions to some fiber are isomorphic, then the two objects are themselves isomorphic. The extension principle states that both objects extend uniquely along any extension of the fiberation (with even some control of behavior at the boundary).

This suggests an inadequate but instructive strategy for the construction of crystalline companions. Given an absolutely irreducible algebraic étale coefficient on X, we know its restrictions to fibers admit crystalline companions. If one of these fiberwise companions extends in some way to a coefficient object on X, then we can form an étale companion of the extension and compare it with the original object using the uniqueness principle.

This brings us to our third principle of tame rigidity, the obstruction principle. In the étale case, the existence of an extension of a coefficient object from a fiber to the whole fibration is obstructed by the homotopy exact sequence for a fibration, more precisely by the resulting outer the action of the fundamental group of the base on representations of the fundamental group of the fiber. From this description, it is not immediately clear how to transfer the obstruction principle to the crystalline case, let alone eliminate the obstruction. (The methods of Chin [10, 11] can be used to compare the Tannakian monodromy groups of companions on fibers, but this is not strong enough to transfer the vanishing of the obstruction.)

0.5. Moduli of truncated stacks. To address the aforementioned issue, we adopt a moduli-theoretic approach to constructing a crystalline companion. This proceeds by interpreting a crystalline coefficient as an object in the isogeny category associated to an inverse limit of categories of “truncated F-crystals”. We establish the algebraic nature of the resulting stacks by relating them to stacks of coherent sheaves. This then allows us to construct a crystalline version of the obstruction principle, where again the obstruction is a priori trivialized by some profinite covering of the base.

We then separate into cases depending on whether or not we start with an isoclinic coefficient object (one whose Newton polygon is constant). In the isoclinic case, the desired crystalline coefficient corresponds to an étale \( \overline{\mathbb{Q}}_p \)-sheaf and can be constructed using a direct adaptation of Drinfeld’s method (without worrying about the obstruction).

In the non-isoclinic case, we use the minimal slope theorem of Tsuzuki [65] (a special case of the paraboticity theorem of D’Addezio [15]) at one stage to replace the obstruction, an overconvergent F-isocrystal on a curve, with its (convergent) unit-root subobject. By taking a suitable exterior power, we can ensure that this subobject is rank 1 even though the original isocrystal is not; this makes the obstruction “abelian”. We can then eliminate the obstruction after a finite base change: otherwise, we would construct a violation of Deligne’s theorem that up to constant twist, there are only finitely many irreducible étale local systems on a prescribed curve over a finite field with prescribed rank and ramification [19, 28].
As a side effect of the proof, we show that over a finite field, the étale obstruction is always eliminated by an étale (not pro-étale) base change (Theorem 8.6.1). We do not know of any proof of this fact that does not go through crystalline companions.

0.6. Applications. We conclude the paper by giving some applications of the construction of crystalline companions. These include properties of Newton polygons (Theorem 8.2.1 Theorem 8.2.2) and Wan’s theorem on the $p$-adic meromorphy of unit-root $L$-functions (Theorem 8.3.1).

As remarked upon in [40, Remark 2.1.6], the existence of companions on curves suggests a new method for counting lisse Weil sheaves on curves, by directly relating these counts to the zeta functions of moduli spaces of vector bundles (see loc. cit. for some references on this question). Theorem 0.1.2 in turn provides an opportunity (albeit one not acted upon here) to make similar arguments on higher-dimensional varieties where techniques based on the Langlands correspondence do not apply, although one can at least use them to establish a finiteness result [28 Theorem 1.1] (see also [40 Corollary 3.7.6] and Remark 2.9.2).

Another application of crystalline companions is to improved “cut-by-curves” criteria for detecting whether a convergent $F$-isocrystal is overconvergent, building on the work of Shiho [63] (see also [39 Theorem 5.16]). See [32] for details.

We expect many additional applications to arise in due course, some of which do not explicitly refer to any $p$-adic behavior. For example, the existence of companions strengthens the work of Krishnamoorthy–Pál on the existence of abelian varieties associated to $\ell$-adic representations [14 [15]. It is an intriguing open question whether the existence of companions can be used to make even further progress on the existence of motives associated to étale or crystalline coefficients.

We note in passing that in place of coefficient objects of rank $n$, which are implicitly $\text{GL}_n$-torsors, one may consider similar objects with $\text{GL}_n$ replaced by a more general reductive group over $\mathbb{Q}$. We will not treat the resulting conjecture in any great detail, but see §8.5 for a limited discussion.

1. Background on algebraic stacks

We begin with some background material on algebraic stacks that will play a crucial role in our construction of crystalline companions. We follow the conventions of the Stacks Project [64, Tag 026M]; a more informal overview can be found in [64, Tag 0721].

Before proceeding, we fix some geometric conventions that run throughout the paper.

Notation 1.0.1. Throughout this paper, let $k$ be a perfect field of characteristic $p$; our main results require $k$ to be finite, but we will impose this hypothesis explicitly when needed. Let $K$ denote the fraction field of the ring $W(k)$ of $p$-typical Witt vectors with coefficients in $k$. Let $X$ denote a smooth (but not necessarily geometrically irreducible) separated scheme of finite type over $k$. Let $X^\circ$ denote the set of closed points of $X$ and let $|X|$ denote the entire underlying topological space (compare Definition 1.1.4).

Definition 1.0.2. By a curve over $k$, we will always mean a scheme which is smooth of dimension 1 and geometrically irreducible over $k$, but not necessarily proper over $k$ (this condition will be specified separately as needed). A curve in $X$ is a locally closed subscheme of $X$ which is a curve over $k$ in the above sense.
Definition 1.0.3. A smooth pair over a base scheme $S$ is a pair $(Y,Z)$ in which $Y$ is a smooth $S$-scheme and $Z$ is a relative strict normal crossings divisor on $Y$; we refer to $Z$ as the boundary of the pair. (Note that $Z = \emptyset$ is allowed.) A good compactification of $X$ is a smooth pair $(\overline{X},Z)$ over $k$ with $\overline{X}$ projective (not just proper) over $k$, together with an isomorphism $\overline{X} \cong X \setminus Z$; we will generally treat the latter as an identification.

1.1. Algebraic stacks. We start with very brief definitions, together with copious pointers to [64] which are crucial for making any sense of the definitions. As a critical link between schemes and stacks, we introduce the category of algebraic spaces as per [64, Tag 025X].

Definition 1.1.1. Let $\text{Sch}$ denote the category of schemes. For $S \in \text{Sch}$, let $\text{Sch}_S$ denote the category of schemes over $S$, equipped with the fppf topology. An algebraic space over $S$ is a sheaf $\mathcal{F}$ on $\text{Sch}_S$ valued in sets such that the diagonal $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is representable, and there exists a surjective étale morphism $h_U \to \mathcal{F}$ for some $U \in \text{Sch}$ (writing $h_U$ for the functor represented by $U$). These form a category in which morphisms are natural transformations of functors, containing $\text{Sch}$ as a full subcategory via the Yoneda embedding $U \mapsto h_U$.

With this definition in hand, we can define algebraic stacks.

Definition 1.1.2. As per [64, Tag 026N], by an algebraic stack over $S$, we will mean a stack $\mathcal{X}$ in groupoids over $\text{Sch}_S$ for the fppf topology whose diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces, and for which there exists a surjective smooth morphism $\text{Sch}_U \to \mathcal{X}$ for some scheme $U$. These form a 2-category as per [64, Tag 03YP], which contains $\text{Sch}_S$ as a full subcategory via the operation $U \mapsto \text{Sch}_U$.

A Deligne–Mumford (DM) stack over $S$ is an algebraic stack $\mathcal{X}$ for which the surjective smooth morphism $\text{Sch}_U \to \mathcal{X}$ can be taken to be étale.

The category of algebraic stacks admits fiber products, or more precisely 2-fiber products; see [64, Tag 04T2].

Remark 1.1.3. Any property of schemes which obeys sufficiently strong locality properties admits a natural generalization for algebraic stacks, which obeys corresponding locality properties and moreover specializes back to the original property when applied to the stacks corresponding to ordinary schemes. For example, such a generalization exists for the properties reduced [64, Tag 04YJ] and locally noetherian [64, Tag 04YE]; there is also a construction of the reduced closed substack of a given stack [64, Tag 0509].

Similarly, any property of morphisms of schemes which obeys sufficiently strong locality and descent properties admits a natural generalization for algebraic stacks, which obeys corresponding locality and descent properties and moreover specializes back to the original property when applied to the stacks corresponding to ordinary schemes. For example, such generalizations exist for the properties quasicompact [64, Tag 050S], quasiseparated and separated [64, Tag 04YV], finite type [64, Tag 06FR], smooth [64, Tag 075T], universally closed [64, Tag 0511], and proper [64, Tag 0CL4]. For the properties of being an open immersion, closed immersion, or immersion, we make a similar construction but also require that the morphism be representable (in schemes) [64, Tag 04YK].

Definition 1.1.4. Let $\mathcal{X}$ be an algebraic stack. A schematic point of $\mathcal{X}$ is a morphism of the form $\text{Spec} L \to \mathcal{X}$ where $L$ is an arbitrary field. A point of $\mathcal{X}$ is an equivalence class of
schematic points under the relation that two schematic points \( \text{Spec } L_1 \to \mathcal{X} \), \( \text{Spec } L_2 \to \mathcal{X} \) are equivalent if there exists a 2-commutative diagram

\[
\begin{array}{ccc}
\text{Spec } L_3 & \longrightarrow & \text{Spec } L_1 \\
\downarrow & & \downarrow \\
\text{Spec } L_2 & \longrightarrow & \mathcal{X}
\end{array}
\]

with \( L_3 \) being a third field. The set of equivalence classes of points is denoted \( |\mathcal{X}| \); this recovers the usual underlying set of a scheme. There is a natural quotient topology on \( |\mathcal{X}| \) which recovers the Zariski topology of a scheme \([64, \text{Tag } 04Y8]\).

There is no natural notion of “closed points” on a general algebraic stack \( \mathcal{X} \). Instead, it is more natural to speak of points of finite type of \( \mathcal{X} \), meaning schematic points \( \text{Spec } L \to \mathcal{X} \) for which the structure morphism is locally of finite type \([64, \text{Tag } 06FW]\); these form a dense subset of \( |\mathcal{X}| \) \([64, \text{Tag } 06G2]\).

**Remark 1.1.5.** Let \( \mathcal{X} \) be an algebraic stack of finite type over \( k \). If \( \text{Spec } L \to \mathcal{X} \) is a point of finite type, then by the Nullstellensatz \([64, \text{Tag } 00FY]\) the field \( L \) must be a finite extension of \( k \); in particular, if \( \mathcal{X} \) is a scheme of finite type over \( k \), then points of finite type are the same as closed points.

By contrast, suppose that \( \mathcal{X} = \text{Spec } R \) where \( R \) is a discrete valuation ring. Then the generic point of \( \mathcal{X} \) is not a closed point, but it is a point of finite type because \( \text{Frac } R \) is generated over \( R \) by the inverse of any single uniformizer.

### 1.2. Moduli stacks of stable curves

As a reminder of the practical meaning of some of the previous definitions, we recall the basic properties of moduli stacks of stable curves.

**Definition 1.2.1.** For \( g, n \geq 0 \), a stable \( n \)-pointed genus-\( g \) curve fibration (or for short a stable curve fibration) consists of a morphism \( f : Y \to S \) and \( n \) morphisms \( s_1, \ldots, s_n : S \to Y \) (all in \( \textbf{Sch} \)) satisfying the following conditions.

- The morphism \( f \) is flat, of finite presentation, and proper of relative dimension 1.
- Each geometric fiber of \( f \) is reduced and connected, has at worst nodal singularities, and has geometric genus \( g \).
- The morphisms \( s_1, \ldots, s_n \) are sections of \( f \) whose images are pairwise disjoint and do not meet the singular locus of any fiber.
- For each geometric fiber of \( f \), each irreducible component of genus 0 (resp. 1) contains at least 3 points (resp. at least 1 point) each of which either is a singularity of the fiber or lies in the image of some \( s_i \).

We refer to the union of the images of \( s_1, \ldots, s_n \) in \( Y \) as the pointed locus and the complement as the unpointed locus. We also occasionally consider the intersection of the unpointed locus with the smooth locus, called the smooth unpointed locus.

Note that for \( f \) as above, \( f \) is smooth if and only if each geometric fiber of \( f \) is smooth. In this case, we say that \( f \) is a smooth \( n \)-pointed genus-\( g \) curve fibration (or for short a smooth curve fibration).

**Remark 1.2.2.** It is customary to refer to the fibers of a stable \( n \)-pointed genus-\( g \) curve fibration as stable curves; however, this is not compatible with our running conventions.
**Definition 1.2.3.** Let $\overline{M}_{g,n}$ be the stack over $\text{Sch}$ whose fiber over $S \in \text{Sch}$ consists of stable $n$-pointed genus-$g$ curve fibrations over $S$. This is empty unless $2g + n \geq 3$.

Let $M_{g,n}$ be the substack of $\overline{M}_{g,n}$ whose fiber over $S \in \text{Sch}$ consists of smooth $n$-pointed genus-$g$ curve fibrations over $S$.

**Remark 1.2.4.** Note that the definition of the full moduli stack of curves in [64, Tag 0DMJ] requires consideration of families of curves in which the total space is an algebraic space rather than a scheme; this does not change anything over the spectrum of an artinian local ring or a noetherian complete local ring [64, Tag 0AE7], but does make a difference over a more general base [64, Tag 0D5D].

However, this discrepancy does not arise for stable curve fibrations: if $S$ is a scheme and $f : Y \to S$ is a stable $n$-pointed genus-$g$ fibration in the category of algebraic spaces, then $Y$ is a scheme. That is because the hypotheses ensure that the relative canonical bundle (for the logarithmic structure defined by $s_1, \ldots, s_n$) is ample, so we may realize $Y$ as a closed subscheme of a particular projective bundle over $S$ (compare [64, Tag 0E6F]).

In the language of stacks, the Deligne–Mumford stable reduction theorem takes the following form.

**Proposition 1.2.5.** The stack $\overline{M}_{g,n}$ is a smooth proper DM stack over $\mathbb{Z}$. The stack $M_{g,n}$ is a dense open substack, so it is a smooth separated DM stack over $\mathbb{Z}$.

**Proof.** In the case $n = 0$ this is [64, Tag 0E9C]; the general case is similar. □

**Remark 1.2.6.** At no point will we use the fact that the $n$ marked points in a stable curve fibration are distinguishable. That is, it would be sufficient for our purposes to replace the $n$ sections $S \to Y$ with a single closed immersion $Z \to Y$ such that $Z \to Y \to S$ is finite étale of constant degree $n$.

1.3. **Geometric corollaries of stable reduction.** We next recall some geometric corollaries of the stable reduction theorem. Chief among these is de Jong’s theorem on alterations, although the manner in which it is derived from stable reduction will not be relevant here except for one key detail (see Remark 1.3.6).

**Definition 1.3.1.** An alteration of a scheme $Y$ is a morphism $f : Y' \to Y$ which is proper, surjective, and generically finite étale. This corresponds to a separable alteration in the sense of de Jong [16].

**Proposition 1.3.2** (de Jong). For $X$ smooth of finite type over $k$ (as per our running convention), there exists an alteration $f : X' \to X$ such that $X'$ is smooth (but not necessarily geometrically irreducible over $k$) and admits a good compactification.

**Proof.** Keeping in mind that $k$ is perfect, see [16, Theorem 4.1]. □

**Corollary 1.3.3.** Suppose that $S$ is a smooth scheme of finite type over $k$ and $f : X \to S$ is a smooth $n$-pointed genus-$g$ curve fibration for some $n, g$ with $2g + n \geq 3$. Then there exist an alteration $S' \to S$, a good compactification $S' \to \overline{S}$, and a stable $n$-pointed genus-$g$ curve fibration over $\overline{S}$ whose pullback to $S'$ is $X \times_S S' \to S'$.

**Proof.** By Proposition 1.2.5 (see more precisely [64, Tag 0E98]), for any discrete valuation ring $R$ with fraction field $F$ and any morphism $\text{Spec} F \to M_{g,n}$, there exists a finite separable
extension \( F' \) of \( F \) such that, for \( R' \) the integral closure of \( R \) in \( F' \), the composition \( \text{Spec } F' \rightarrow \text{Spec } F \rightarrow M_{g,n} \rightarrow \overline{M}_{g,n} \) factors uniquely through the inclusion \( \text{Spec } F \rightarrow \text{Spec } R \).

Turning to the problem at hand, the original fibration corresponds to a morphism \( S \rightarrow M_{g,n} \). By applying the previous paragraph as in [16, 4.17], we obtain an alteration \( S' \rightarrow S \) such that the composition \( S' \rightarrow S \rightarrow M_{g,n} \rightarrow \overline{M}_{g,n} \) factors through some compactification \( \overline{S} \) of \( S' \). By applying Proposition 1.3.2, we may further ensure that \( \overline{S} \) is a good compactification of \( S' \). The induced morphism \( \overline{S} \rightarrow \overline{M}_{g,n} \) corresponds to the desired fibration over \( \overline{S} \). □

We will apply the previous result as follows (compare [40, Lemma 3.1.8]).

**Corollary 1.3.4.** For \( X \) smooth of finite type over \( k \) (as per our running convention), there exist a finite extension \( k' \) of \( k \), an alteration \( X' \rightarrow X_{k'} \), an open dense subscheme \( U \) of \( X' \), and a diagram

\[
\begin{array}{ccc}
U & \longrightarrow & X' \\
\uparrow & & \downarrow \\
S & \longrightarrow & \overline{S}
\end{array}
\]

in which \((\overline{S}, \overline{S} \setminus S)\) is a smooth pair over \( k' \), \( \overline{X} \rightarrow \overline{S} \) is a stable curve fibration, and \( \overline{X} \times_{\overline{S}} S \rightarrow S \) is a smooth curve fibration with unpointed locus equal to \( U \). For any given finite subset \( T \) of \( X^0 \), we may further ensure that every point of \( T(k') \) lifts to a point of \( X' \) contained in \( U \).

**Proof.** At any stage in the proof, we are free to replace \( X \) with either an alteration or an open dense subscheme, or to replace \( k \) with a finite extension. We may thus assume at once that \( X \) is quasiprojective; we then proceed as in [5, Proposition 3.3] (compare [40, Lemma 3.1.8]).

Choose a very ample line bundle \( L \) on \( X \), put \( n := \dim(X) \), and choose a point \( z \in X^0 \). After possibly enlarging \( k \), if we make a generic choice of \( n \) sections \( H_1, \ldots, H_n \) of \( L \) containing \( z \), then the intersection \( C \) will be zero-dimensional (but nonempty; see Remark 1.3.5 below). Let \( \tilde{X} \) be the blowup of \( X \) at \( C \) and let \( f : \tilde{X} \rightarrow \mathbb{P}^{n-1}_k \) be the morphism defined by \( H_1, \ldots, H_n \).

For \( S \) an open dense subscheme of \( \mathbb{P}^{n-1}_k \), write \( \tilde{X}_S := \tilde{X} \times_{\mathbb{P}^{n-1}_k} S \). For suitable \( S \), the map \( f \) is an **elementary fibration** in the sense of [5, Definition 3.1]; that is, it fits into a commutative diagram of the form

\[
\begin{array}{ccc}
\tilde{X}_S & \xrightarrow{j} & \tilde{X} \\
\downarrow f & & \downarrow i \\
S & \xrightarrow{\tilde{f}} & Z
\end{array}
\]

in which:

- \( j \) is an open immersion with dense image in each fiber, and \( i \) is a closed immersion such that \( \tilde{X}_S = \tilde{X} \setminus Z \);
- \( \tilde{f} \) is smooth projective with fibers which are geometrically irreducible of dimension 1;
- \( g \) is finite étale and surjective.

By shrinking \( X \), we may further ensure that \( \deg(g) \geq 3 \).
To make $\overline{f}$ into a smooth curve fibration, we must force $g$ to become a disjoint union of sections; this can be achieved by replacing $S$ with a finite étale cover. More precisely, take any component of $Z$ which does not map isomorphically to $S$; this component is itself a finite étale cover of $S$, and pulling back along it produces a fibration in which the inverse image of $Z$ splits off a component which maps isomorphically to $S$. We may repeat the construction to achieve the desired result.

Finally, by Corollary 1.3.3 after replacing $S$ with a suitable alteration, we get an extension of $\overline{f}$ to a stable curve fibration over some compactification $\overline{S}$ of $S$. By applying Proposition 1.3.2 we may ensure that $\overline{S}$ is in fact a good compactification of $S$; this completes the proof.

Remark 1.3.5. The construction in the proof of Corollary 1.3.4 has the following side effect which is germane to the principle of tame rigidity (Lemma 5.2.4, Lemma 5.2.7). For $\overline{f}$ the relative compactification of $f$, there exists a section $s$ of $\overline{f}$ whose image is the complement of $U$ in another open subscheme $U_1$ which is also an elementary fibration over $S$. The image of $s$ is the intersection of $U$ with one of the exceptional divisors of the blowup $\overline{X} \to X$; in particular, this image contracts to a point in $\overline{X}$.

Remark 1.3.6. In the notation of Corollary 1.3.4, the total space $\overline{X'}$ need not be smooth over $k$, but it has only toroidal singularities: every singularity is of the form

$$xy = e_1 \cdots e_m$$

for some $m \in \{1, \ldots, \dim(S)\}$.

Suppose now that $\dim(S) = 1$. If in all cases $e_1 = 1$, then $\overline{X'}$ is indeed smooth over $k$, but one cannot force this situation by a ramified base change (as this has the effect of increasing the base multiplicities). Instead, one can perform a resolution of singularities on the total space [64, Tag 0C2U] and then add sections to stabilize the resulting exceptional divisors.

1.4. Moduli of coherent sheaves. We now introduce a different moduli stack that will be more closely related to crystals.

Hypothesis 1.4.1. Throughout §1.4, let $f : Y \to B$ be a separated morphism of finite presentation of algebraic spaces over some base scheme $S$.

Definition 1.4.2. Let $\text{Coh}_{Y/B}$ denote the category in which:

- the objects are triples $(T, g, F)$ in which $T$ is a scheme over $S$, $g : T \to B$ is a morphism over $S$, and (writing $Y_T := Y \times_{B, g} T$) $F$ is a quasicoherent $\mathcal{O}_{Y_T}$-module of finite presentation which is flat over $T$ and has support which is proper over $T$;
- the morphisms $(T', g', F') \to (T, g, F)$ consist of pairs $(h, \psi)$ in which $h : T' \to T$ is a morphism of schemes over $B$ and (writing $h' : Y_{T'} \to Y_T$ for the base extension of $h$ along $f$) $\psi : (h')^* F \to F'$ is an isomorphism of $\mathcal{O}_{Y_{T'}}$-modules.

These form a stack over $B$ via the functor $(T, g, F) \mapsto (T, g)$.

Proposition 1.4.3. The category $\text{Coh}_{Y/B}$ is an algebraic stack over $S$.

Proof. See [64, Tag 09DS].

Proposition 1.4.4. The morphism $\text{Coh}_{Y/B} \to B$ is quasiseparated and locally of finite presentation.
Proof. See [64, Tag 0DLZ]. Additional references, which impose more restrictive hypotheses but would still suffice for our purposes, are [49, Théorème 4.6.2.1] and [50, Theorem 2.1]. □

Unless $f$ is finite, we cannot hope for $\text{Coh}_{Y/B} \to B$ to be quasicompact. However, when $f$ is projective, we can cover $\text{Coh}_{Y/B}$ with open substacks which are themselves quasicompact over $B$.

**Definition 1.4.5.** Assume that $f$ is projective and $B$ is quasicompact, and let $L$ be a line bundle on $Y$ which is very ample relative to $f$. For any object $(T, F) \to \text{Coh}_{Y/B}$, we may define the associated Hilbert function

$$P : T \mapsto \mathbb{Q}[t], \quad P(x)(t) = \chi(Y \times_T x, L \cdot L^\otimes t)$$

where $\iota_x : x \to T$ denotes the canonical inclusion. This function is locally constant [64, Tag 0D1Z].

For $P \in \mathbb{Q}[t]$, let $\text{Coh}^{P,L}_{Y/B}$ be the substack of $\text{Coh}_{Y/B}$ consisting of those triples $(T, g, F)$ for which the Hilbert function of $F$ (with respect to $L$) is identically equal to $P$. As per [64, Tag 0DNF], $\text{Coh}^{P,L}_{Y/B}$ is a closed-open substack of $\text{Coh}_{Y/B}$ and $\text{Coh}_{Y/B}$ is equal to the disjoint union of the $\text{Coh}^{P,L}_{Y/B}$ over all $P$.

For a positive integer, let $\text{Coh}^{P,L,m}_{Y/B}$ be the locally closed substack of $\text{Coh}^{P,L}_{Y/B}$ consisting of those triples $(T, g, F)$ for which $f^* f_* (F \otimes g^* \cdot L^\otimes m) \to F \otimes g^* \cdot L^\otimes m$ is surjective and $R^i f_* (F \otimes g^* \cdot L^\otimes m) = 0$ for all $i > 0$. Note that $\text{Coh}^{P,L,m}_{Y/B}$ is the union of the $\text{Coh}^{P,L,m}_{Y/B}$ over all $m$.

**Proposition 1.4.6.** Assume that $f$ is projective and $B$ is quasicompact. Then for any $P, L, m$ as in Definition 1.4.5, $\text{Coh}^{P,L,m}_{Y/B}$ is quasicompact.

**Proof.** It suffices to produce a quasicompact algebraic space $W$ which surjects onto $\text{Coh}^{P,L,m}_{Y/B}$. Let $Y \to P^n_B$ be the projective embedding defined by $L^\otimes m$. Let $P_m$ be the polynomial with $P_m(t) = P(m + t)$ and put $r = P_m(0)$; then each fiber of $F \otimes L^\otimes m$ is globally generated by its $r$-dimensional space of global sections. We may thus take $W$ to be the Quot space $\text{Quot}_{P^n_B/P^n_B}^{P_m/L^\otimes m}$, which is proper over $B$ by [64, Tag 0DPA]. □

**Remark 1.4.7.** If $f$ is projective, then by [64, Tag 0DM0] and [64, Tag 0CLW], the morphism $\text{Coh}^{P,L,m}_{Y/B} \to B$ is universally closed. However, in general it is not separated and hence not proper.

### 2. Coefficient objects

We review some relevant properties of coefficient objects and companions. Since we will be using terminology and notation from both [39] and [40], often with little comment, we recommend keeping those sources handy while reading.

**Hypothesis 2.0.1.** Throughout §2 assume that $k$ is finite.

#### 2.1. Coefficient objects and algebraicity.

**Definition 2.1.1.** By a coefficient object on $X$, we will mean an object of one of the categories $\text{Weil}(X) \otimes \overline{\mathbb{Q}}_\ell$, the category of lisse Weil $\overline{\mathbb{Q}}_\ell$-sheaves on $X$ for some prime $\ell \neq p$ (an étale coefficient); or $\text{F-Isoc}^!(X) \otimes \overline{\mathbb{Q}}_p$, the category of overconvergent $F$-isocrystals on $X$.  

[11]
with coefficients in \( \mathbb{Q}_p \), in the sense of [39, Definition 9.2] (a crystalline coefficient). By the full coefficient field of a coefficient object (or its ambient category), we mean the algebraic closure of the field \( \mathbb{Q}_\ell \) in the former case and \( \mathbb{Q}_p \) in the latter case (without completion).

A coefficient object on \( X \) is absolutely irreducible if for any finite extension \( k' \) of \( k \), the pullback of \( \mathcal{E} \) to \( X_{k'} \) is irreducible.

**Lemma 2.1.2.** Let \( U \) be an open dense subscheme of \( X \). For any category of coefficient objects, restriction from coefficient objects over \( X \) to coefficient objects over \( U \) is fully faithful.

**Proof.** See [40, Lemma 1.2.2].

**Definition 2.1.3.** We say that a coefficient object on \( X \) is constant if it arises by pullback from \( \text{Spec}(k) \). We refer to the result of tensoring with a constant coefficient object of rank 1 as a constant twist.

**Lemma 2.1.4.** Suppose that \( X \) is irreducible. For every coefficient object \( \mathcal{E} \) on \( X \), there exists a constant twist of \( \mathcal{E} \) whose determinant is of finite order.

**Proof.** See [40, Corollary 3.4.3].

**Corollary 2.1.7.** Suppose that \( X \) is irreducible. Let \( \mathcal{E} \) be an irreducible coefficient object on \( X \). Then some constant twist of \( \mathcal{E} \) is algebraic.

**Proof.** Combine Lemma 2.1.4 and Lemma 2.1.6.

**Remark 2.1.8.** Note that in a certain sense, the categories of coefficient objects on \( X \) do not depend on the base field \( k \); see [39, Definition 9.2] for discussion of this point in the crystalline case. For this reason, it will often be harmless for us to assume that \( X \) is geometrically irreducible.

**Lemma 2.1.9.** Fix a category \( \mathcal{C} \) of coefficient objects and an embedding of \( \overline{\mathbb{Q}}_p \) into the full coefficient field of \( \mathcal{C} \). Let \( E \) be a number field within \( \overline{\mathbb{Q}}_p \) and let \( L_0 \) be the completion of \( E \) in \( \overline{\mathbb{Q}}_p \). Let \( \mathcal{E} \) be an \( E \)-algebraic coefficient object on \( X \) in \( \mathcal{C} \) of rank \( r \). Then there exists a finite extension \( L_1 \) of \( L_0 \), depending only on \( L_0 \) and \( r \) (not on \( X \) or \( \mathcal{E} \) or \( E \)), for which \( \mathcal{E} \) can be realized as an object of \( \text{Weil}(X) \otimes L_1 \) (in the étale case) or \( \text{F-Isoc}^\dagger(X) \otimes L_1 \) (in the crystalline case).

**Proof.** See [40, Corollary 3.3.5].

**Remark 2.1.10.** It is also possible to define coefficient objects on nilpotent thickenings of \( X \). In both the étale and crystalline cases, the restriction functor from a nilpotent thickening back to \( X \) is an equivalence of categories.
2.2. Monodromy groups and Tannakian categories.

Definition 2.2.1. By a geometric coefficient object on $X$, we will mean one of the categories $\text{Weil}(X^\text{\acute{e}t}) \otimes \underline{\mathbb{Q}_\ell}$ for $\ell \neq p$ (an \acute{e}tale geometric coefficient); or $\text{Isoc}^\dagger(X) \otimes \underline{\mathbb{Q}_p}$, the category of overconvergent isocrystals on $X$ without Frobenius structure with coefficients in $\underline{\mathbb{Q}_p}$ (a crystalline geometric coefficient).

We will only consider such objects which appear in the Tannakian category generated by a coefficient object $E$. Any such object which is irreducible can be promoted to a coefficient object after a finite base extension on $k$ \cite[Remark 1.3.9]{example}. Consequently, $E$ is absolutely irreducible if and only if it is irreducible as a geometric coefficient object.

Definition 2.2.2. Let $E$ be a coefficient object on $X$. We write $G(E)$ and $\overline{G}(E)$ for the arithmetic and geometric monodromy groups of $E$, respectively, defined in the sense of Crew in the crystalline case \cite[Definition 1.3.4]{example}. In both cases these are Tannakian automorphism groups over the full coefficient field of $E$.

Lemma 2.2.3. Let $E$ be a coefficient object on $X$.

(a) If $E$ is semisimple, then so is $\overline{G}(E)^\circ$.
(b) There exists a finite \acute{e}tale cover $f : X' \rightarrow X$ such that $\overline{G}(f^*E)$ is connected.

Proof. See \cite[Proposition 1.3.11, Proposition 1.3.12]{example}.

Remark 2.2.4. The previous discussion has the following key consequence. Let $E$ be a coefficient object of rank $r$ such that $\overline{G}(E)$ is connected. Then for $i = 1, \ldots, r - 1$, the natural map $\overline{G}(\wedge^i E) \rightarrow \overline{G}(E)$ is an isomorphism; that is, $E$ and $\wedge^i E$ generate the same Tannakian subcategory in the category of geometric coefficient objects. Moreover, if $E$ is geometrically irreducible, then so is $\wedge^i E$.

2.3. The companion relation for coefficients. We next introduce the companion relation, together with a number of reduction steps for the construction of crystalline companions.

Definition 2.3.1. Fix coefficient objects $E$ and $F$ on $X$, as well as an isomorphism $\iota$ between the algebraic closures of $\mathbb{Q}$ in the full coefficient fields of $E$ and $F$. The map $\iota$ will often not be mentioned explicitly; in other case, it will be specified implicitly in terms of a place of the algebraic closure of $\mathbb{Q}$ in one of the full coefficient fields.

We say that $E$ and $F$ are companions (with respect to $\iota$) if for each $x \in X^\circ$, the coefficients of $P(E_x, T)$ and $P(F_x, T)$ are identified via $\iota$; in particular, this can only occur if both $E$ and $F$ are algebraic. Given $E$, the companion relation determines $F$ up to semisimplification (see Lemma 2.3.2 below). In case $F \in \text{F-Isoc}^\dagger(X) \otimes \underline{\mathbb{Q}_p}$, we also say that $F$ is a crystalline companion of $E$.

Lemma 2.3.2. Let $E$ and $F$ be coefficient objects on $X$ which are companions.

(a) If $E$ is irreducible, then so is $F$.
(b) If $E$ is absolutely irreducible, then so is $F$.
(c) If $\det(E)$ is of finite order, then so is $\det(F)$.
(d) If $F'$ is another coefficient object in the same category as $F$ which is also a companion of $E$, then $F$ and $F'$ have isomorphic semisimplifications. In particular, if one of $F$ and $F'$ is absolutely irreducible, then $F$ and $F'$ are isomorphic.
Proof. Part (a) is [40, Theorem 3.3.1(a)]. Part (b) is a formal consequence of (a). Part (c) is [40, Corollary 3.3.4]. Part (d) is [40, Theorem 3.3.2(b)] for the first assertion, plus (b) for the second assertion. □

Lemma 2.3.3. Let $U$ be an open dense subscheme of $X$. Let $\mathcal{E}$ be a semisimple coefficient object on $U$. Let $\mathcal{F}$ be a coefficient object on $X$ whose restriction to $U$ is a companion of $\mathcal{E}$. Then $\mathcal{E}$ extends to a coefficient object on $X$, and any such extension is a companion of $\mathcal{F}$.

Proof. By [40, Corollary 3.3.3], there exists an extension of $\mathcal{E}$ which is a companion of $\mathcal{F}$. By Lemma [2.1.2] this is the unique extension of $\mathcal{E}$ to $X$. □

Remark 2.3.4. Let $f : X' \to X$ be a radicial morphism. Then for any fixed category of coefficient objects, pullback via $f$ defines an equivalence of categories between the coefficients over $X$ and the coefficients over $X'$.

Lemma 2.3.5. Let $\mathcal{E}$ be a coefficient object on $X$. Let $f : X' \to X$ be a dominant morphism (this includes the case of a base extension on $k$). If $f^*\mathcal{E}$ admits a crystalline companion, then so does $\mathcal{E}$.

Proof. By restriction to a rational multisection of $f$, we may put ourselves in the position where $f$ is the composition of an open immersion with dense image and an alteration. We may handle the two cases separately; the former is treated by Lemma 2.3.3, while for the latter see [40, Corollary 3.6.3]. □

2.4. Tame and docile coefficients. Using the fact that the existence of companions can be checked after an alteration (Lemma 2.3.3), we will be able to limit our attention to $p$-adic coefficient objects of a relatively simple sort. We repeat here [40, Definition 1.4.1].

Definition 2.4.1. Let $X \hookrightarrow \overline{X}$ be an open immersion with dense image. Let $D$ be an irreducible divisor of $\overline{X}$ with generic point $\eta$.

- For $\ell$ a prime not equal to $p$, an object $\mathcal{E}$ of $\text{Weil}(X) \otimes \overline{\mathbb{Q}}_\ell$ is tame (resp. docile) along $D$ if the action of the inertia group at $\eta$ on $\mathcal{E}$ is tamely ramified (resp. tamely ramified and unipotent).
- An object $\mathcal{E}$ of $\text{F-Isoc}^1(X) \otimes \overline{\mathbb{Q}}_p$ is tame (resp. docile) along $D$ if $\mathcal{E}$ has $\mathbb{Q}$-unipotent monodromy in the sense of [62, Definition 1.3] (resp. unipotent monodromy in the sense of [36, Definition 4.4.2]) along $D$.

Lemma 2.4.2. For any prime $\ell \neq p$ and any object $\mathcal{E}$ of $\text{Weil}(X) \otimes \overline{\mathbb{Q}}_\ell$, there exists an alteration $f : X' \to X$ such that $X'$ admits a good compactification with respect to which $f^*\mathcal{E}$ is docile.

Proof. See [40, Proposition 1.4.6]. This also covers the case $\ell = p$, but we will not need that here. □

2.5. Companions on curves. We next recall the critical results on the existence of companions on curves, as extracted from the work of L. Lafforgue [46] and Abe [1] on the global Langlands correspondence in positive characteristic.

Theorem 2.5.1 (L. Lafforgue, Abe). For $X$ of dimension 1, every coefficient object on $X$ which is irreducible with determinant of finite order is uniformly algebraic and admits companions in all categories of coefficient objects, which are again irreducible with determinant of finite order.
Proof. See [40, Theorem 2.2.1] and references therein.

Although we frame the following statement as a corollary of Theorem 2.5.1, it also incorporates significant intermediate results of V. Lafforgue, Deligne, Drinfeld, Abe–Esnault, and Kedlaya; see [40] for more detailed attributions.

**Corollary 2.5.2.** Parts (i)–(v) of Theorem 0.1.1 hold in general. Under the additional hypothesis \( \dim(X) = 1 \), part (vi) of Theorem 0.1.1 also holds.

**Proof.** The first assertion is [40, Theorem 0.4.1]. The second assertion is [40, Theorem 0.2.1].

We record some refinements of this statement which we will also need.

**Corollary 2.5.3.** For \( X \) of dimension 1, any algebraic coefficient object admits companions in all categories of coefficient objects.

**Proof.** See [40, Theorem 2.2.1, Corollary 2.2.3].

**Corollary 2.5.4.** For any tame (resp. docile) coefficient object on \( X \), its companions are also tame (resp. docile).

**Proof.** For the proof when \( \dim(X) = 1 \) (which is the only case we will use here), see [40, Corollary 2.4.3]. The general case follows from this case using [40, Lemma 1.4.9].

### 2.6. Newton polygons.

The following argument corrects an error in the proof of [23, Theorem 1.3.3(i,ii)]: the proof of [23, Lemma 5.3.1] is incorrect, but is supplanted by part (e) of the following statement. We state these results because we need them as part of the construction of crystalline companions, but a posteriori we will get much more precise results (see §8.2).

**Lemma-Definition 2.6.1.** Fix a category \( C \) of coefficient objects and a normalized \( p \)-adic valuation \( v_p \) of the algebraic closure of \( \mathbb{Q} \) in the full coefficient field of \( C \). For \( \mathcal{E} \) an algebraic object of \( C \), there exists a unique function \( x \mapsto N_x(\mathcal{E}) \) from \( X \) to the set of polygons in \( \mathbb{R}^2 \) with the following properties.

(a) For \( x \in X^0 \), \( N_x(\mathcal{E}) \) equals the lower convex hull of the set of points

\[
\left\{ \left( i, \frac{1}{[\kappa(x) : k]} v_p(a_i) \right) : i = 0, \ldots, d \right\} \subset \mathbb{R}^2
\]

where \( P(\mathcal{E}_x, T) = \sum_{i=0}^d a_i T^i \in \mathbb{Q}[T] \) with \( a_0 = 1 \).

(b) For \( x \in X \) with Zariski closure \( Z \), for all \( z \in Z \), \( N_z(\mathcal{E}) \) lies on or above \( N_x(\mathcal{E}) \) with the same right endpoint. In particular, the right endpoint is constant on each connected component of \( X \).

(c) In (b), the set \( U \) of \( z \in Z^0 \) for which equality holds is nonempty.

(d) In (c), for each curve \( C \) in \( Z \), the inverse image of \( U \) in \( C \) is either empty or the complement of a finite set of closed points.

(e) For \( x \in X \), the vertices of \( N_x(\mathcal{E}) \) all belong to \( \mathbb{Z} \times \frac{1}{N} \mathbb{Z} \) for some positive integer \( N \) (which may depend on \( X \) and \( \mathcal{E} \), but not on \( x \)).
Proof. Using Lemma 2.1.4, we may reduce to the case where $E$ is irreducible with determinant of finite order. By Corollary 2.5.2, we may apply part (iv) of Theorem 0.1.1 to obtain uniform upper and lower bounds on $N_x(E)$ for all $x \in X^\circ$.

For each curve $C$ in $X$, let $F_C$ denote a crystalline companion of $E|_C$ given by Corollary 2.6.3. It has coefficients in some finite extension $L$ of $\mathbb{Q}_p$, which by Lemma 2.1.9 can be chosen independently of $C$. By comparison with the usual definition of Newton polygons for crystalline coefficients (see Definition 3.1.1 and Lemma 3.2.2), we deduce that all of the claims hold with $X$ replaced by $C$, taking $N = [L : \mathbb{Q}_p]$. This implies (e) for $x \in X^\circ$ by choosing $C$ to pass through $x$.

Now let $x \in X$ be arbitrary and let $Z$ be the Zariski closure of $x$. As $y$ varies over $Z^\circ$, the vertices of $N_y(E)$ are bounded above and below (by the first paragraph) and belong to $Z \times \frac{1}{p} \mathbb{Z}$ (by the second paragraph), and so $N_y(E)$ can assume only finitely many distinct values; we can thus find $y \in Z^\circ$ for which $N_y(E)$ is minimal (note that a priori there may be multiple minima). We then choose some such $y$ and set $N_x(E) := N_y(E)$; this definition clearly satisfies (a), (c), and (e).

To check (d), we compare it to the statement of (d) for $F_C$ which is already known (see above); the only possible discrepancy is the generic point $\eta$ of $C$. If $\eta \in U$, then by (c) and (d) for $F_C$ we deduce that $U$ is the complement of a finite set of closed points. Otherwise, by (b) for $F_C$ we deduce that $U$ is empty.

To check (b), let $Z'$ be the Zariski closure of $z$. By (c), there exists $z' \in Z^\circ$ for which $N_{z'}(E) = N_z(E)$. Choose a curve $C$ in $Z$ containing $z'$ and the point $y$ from the third paragraph. By (b) for $F_C$, we deduce the claim.

To establish uniqueness, we induct on $\dim(X)$. We may assume that $X$ is irreducible with generic point $\eta$. For $\dim(X) = 1$, the function $N_x(E)$ is determined by (a) for $x \in X^\circ$ and by (c) and (d) for $x = \eta$. For $\dim(X) > 1$, the induction hypothesis determines $N_x(E)$ for $x \neq \eta$. Meanwhile, (b) and (c) imply that $N_{\eta}(E)$ equals a minimal value of $N_x(E)$ for $x \in X^\circ$, so it suffices to check that there is a unique such value. Choose $x, y \in X^\circ$ for which $N_x(E)$ and $N_y(E)$ are minimal. Choose a curve $C$ in $X$ containing both $x$ and $y$; by (d), there is a nonempty set of $z \in C$ for which $N_z(E)$ is constant. By (b) and (c), this constant value lies below both $N_x(E)$ and $N_y(E)$, and so by minimality is equal to both of them.

Corollary 2.6.2. With notation as in Lemma-Definition 2.6.1 for $E$ a coefficient object on $X$, the function $x \mapsto N_x(E)$ on $X$ assumes only finitely many values. In particular, the slopes of $N_x(E)$ can be uniformly (in $x$) bounded above and below.

Proof. By Lemma-Definition 2.6.1(b), we obtain the second assertion by considering the Newton polygons at the generic points of the irreducible components of $X$. Adding part (e), we obtain the first assertion. 

Corollary 2.6.3. With notation as in Lemma-Definition 2.6.1, let $E_1, \ldots, E_n$ be algebraic objects of $C$. Then for $x \in X$ with Zariski closure $Z$, the set of $z \in Z^\circ$ for which $N_z(E_i) = N_x(E_i)$ for $i = 1, \ldots, n$ is nonempty.

Proof. By Lemma-Definition 2.6.1(c), there exists $y \in Z^\circ$ such that $N_y(E_1) = N_x(E_1)$. Choose a curve $C$ in $Z$ passing through $x$. By Lemma-Definition 2.6.1(d), for $i = 2, \ldots, n$, there exist only finitely many $z \in C^\circ$ for which $N_z(E_i) \neq N_x(E_i)$; consequently, for all but finitely many closed points $z \in C^\circ$, $N_z(E_i) = N_x(E_i)$ for $i = 1, \ldots, n$. This proves the claim.
Corollary 2.6.4. With notation as in Lemma-Definition 2.6.1, let \( f : Y \to X \) be a morphism of smooth \( k \)-schemes. Then for every \( y \in Y \), \( N_y(f^*E) = N_{f(y)}(E) \).

Proof. For \( y \in Y^\circ \) this is immediate from Lemma-Definition 2.6.1(a), as in this case \( f(y) \in X^\circ \). We may then deduce the general case by combining this with parts (b) and (c) of Lemma-Definition 2.6.1. \( \square \)

Remark 2.6.5. In the crystalline case of Lemma-Definition 2.6.1, if \( L \) is a finite extension of \( \mathbb{Q}_p \) for which \( E \in \text{F-Isoc}^+(X) \otimes L \), then the \( y \)-coordinates of the vertices of \( N_x(E) \) for all \( x \in X^\circ \) belong to \( [L : \mathbb{Q}_p]^{-1/2} \mathbb{Z} \) (see Lemma 3.2.2). In the étale case, one can derive a similar bound on denominators using Lemma 2.1.9, depending on the rank of \( E \) and the degree of the minimal number field \( E \) for which \( E \) is \( E \)-algebraic.

Definition 2.6.6. With notation as in Lemma-Definition 2.6.1, for \( X \) irreducible with generic point \( \eta \), we say that \( E \) is isoclinic if \( N_\eta(E) \) consists of a single slope. In this case, the function \( x \mapsto N_x(E) \) is constant.

2.7. Weights.

Hypothesis 2.7.1. Throughout §2.7, fix a category \( C \) of coefficient objects and an algebraic (but not topological!) embedding \( \iota \) of the full coefficient field of \( C \) into \( C \).

Definition 2.7.2. For \( E \in C \) and \( x \in X^\circ \), we define the \( \iota \)-weights of \( E \) at \( x \) as per [40, Definition 3.1.2]. We say that \( E \) is \( \iota \)-pure of weight \( w \) if for all \( x \in X^\circ \), the \( \iota \)-weights of \( E \) at \( x \) are all equal to \( w \).

Lemma 2.7.3. Suppose that \( E \in C \) is \( \iota \)-pure of weight \( w \). Then the eigenvalues of Frobenius on \( H^1(X,E) \) all have \( \iota \)-absolute value at least \( q^{(w+1)/2} \).

Proof. See [40, Lemma 3.1.3] and onward references. \( \square \)

Lemma 2.7.4. Let \( E \) be an irreducible coefficient object. Then \( E \) is \( \iota \)-pure of some weight.

Proof. See [40, Theorem 3.1.10]. \( \square \)

Corollary 2.7.5. Let \( f : Y \to X \) be a morphism of smooth schemes over \( k \). Let \( E \) be an irreducible coefficient object on \( X \).

(a) There exists an isotypical decomposition of \( E \) (i.e., each summand is a successive extension of a single coefficient object). Moreover, \( E \) is semisimple in the category of geometric coefficient objects.

(b) If \( f^*E \) admits a companion which is constant, then \( f^*E \) is itself constant.

Proof. By Lemma 2.7.4 \( E \) is \( \iota \)-pure of some weight, as then is \( f^*E \). We may thus deduce (a) from [39, Corollary 3.1.4]. To deduce (b), apply Lemma 2.3.2(d) to deduce that the semisimplification of \( f^*E \) is constant, then combine this with (a). \( \square \)

2.8. Cohomological rigidity. For \( E \) an absolutely irreducible coefficient object, \( H^0(X,E^\vee \otimes E) \) is equal to the full coefficient field; this means that \( E \) has no infinitesimal automorphisms. Using weights, we can also assert that \( E \) has no infinitesimal deformations.

Proposition 2.8.1. Let \( E \) be an absolutely irreducible coefficient object on \( X \). Let \( F \) be the trace-zero component of \( E^\vee \otimes E \). Let \( \varphi \) denote the action of (geometric) Frobenius on cohomology groups. Then

\[
H^0(X,F)_\varphi = H^1(X,F)_\varphi = H^1(X,E^\vee \otimes E)_\varphi = 0.
\]
Proof. Since $\mathcal{E}$ is absolutely irreducible, the multiplicity of 1 as an eigenvalue of $\varphi$ on $H^0(X, \mathcal{E}^\vee \otimes \mathcal{E})$ cannot exceed 1, as contributed by the trace component. Hence $H^0(X, \mathcal{F})_{\varphi} = 0$.

Let $\iota$ be an embedding of the full coefficient field of $E$ into $\mathbb{C}$. By Lemma 2.7.4, $E$ is $\iota$-pure of some weight $w$; then $\mathcal{E}^\vee$ is $\iota$-pure of weight $-w$ and $\mathcal{E}^\vee \otimes \mathcal{E}$ is $\iota$-pure of weight 0. By Lemma 2.7.3, the eigenvalues of $\varphi$ on $H^1(X, \mathcal{E}^\vee \otimes \mathcal{E})$ all have $\iota$-absolute value at least $q^{1/2}$; in particular, none of them is equal to 1. This proves the claim. □

Remark 2.8.2. The space $H^1(X, \mathcal{E}^\vee \otimes \mathcal{E})$ computes infinitesimal deformations of $\mathcal{E}$ as an object over $X_k$; compare [3] in the crystalline case. By the Hochschild–Serre spectral sequence, the infinitesimal deformations of $E$ as an object over $X$ form a space sandwiched between $H^0(X, \mathcal{F})_{\varphi}$ and $H^1(X, \mathcal{E}^\vee \otimes \mathcal{E})_{\varphi}$; Proposition 2.8.1 thus asserts the vanishing of this space.

2.9. A finiteness result for coefficients on curves. The following result of Deligne is the starting point for his conjectures on counting étale local systems [19]; it also plays a key role in our later arguments.

Proposition 2.9.1. Suppose that $\dim(X) = 1$. Fix a positive integer $r$. Fix a category of coefficient objects, and declare two objects in the category to be equivalent if they differ by a constant twist. Then the number of equivalence classes of tame irreducible objects of rank $r$ on $X$ is finite.

Proof. By Lemma 2.1.4 and Theorem 2.5.1, we may reduce to the étale case, for which see [28, Theorem 2.1]. Alternatively, see [10, Corollary 2.4.6]. □

Remark 2.9.2. Proposition 2.9.1 remains true, with the same proof, if we allow wild ramification at the boundary of $X$ with bounded Swan conductor. However, the tame case will suffice for our present purposes. See also [28, Theorem 1.1] or [10, Corollary 3.7.6] for a higher-dimensional analogue.

3. Convergent vs. overconvergent isocrystals

For the construction of crystalline companions, we will not be able to carry out the entire argument in the category of overconvergent $F$-isocrystals; instead, we make some steps in the closely related category of convergent $F$-isocrystals. We recall the relevant points here.

3.1. Convergent isocrystals.

Definition 3.1.1. For $L$ an algebraic extension of $\mathbb{Q}_p$, let $\textbf{F-Isoc}(X) \otimes L$ denote the category of convergent $F$-isocrystals with coefficients in $L$. There is a natural restriction functor from $\textbf{F-Isoc}^\dagger(X) \otimes L$ to $\textbf{F-Isoc}(X) \otimes L$, which in light of Lemma 3.1.2 we will view as a full embedding.

We may view both $\textbf{F-Isoc}(X) \otimes L$ and $\textbf{F-Isoc}^\dagger(X) \otimes L$ as special cases of the category $\textbf{F-Isoc}(X, Y) \otimes L$ of $F$-isocrystals which are overconvergent along an open immersion $X \to Y$; namely, one gets $\textbf{F-Isoc}(X) \otimes L$ when $X = Y$ and $\textbf{F-Isoc}^\dagger(X) \otimes L$ when $Y$ is proper over $k$ [39, Definition 2.4].

Lemma 3.1.2. For any open immersion $X \to Y$, the restriction functor $\textbf{F-Isoc}(X, Y) \otimes L \to \textbf{F-Isoc}(X) \otimes L$ is fully faithful. In particular, $\textbf{F-Isoc}^\dagger(X) \otimes L \to \textbf{F-Isoc}(X) \otimes L$ is fully faithful.
Proof. See [39, Theorem 5.3].

We use the following form of Zariski-Nagata purity.

**Lemma 3.1.3.** Let

\[
\begin{array}{c}
U \
\downarrow \\
\downarrow \\
W \\
\downarrow \\
X \\
\downarrow \\
Y
\end{array}
\]

be open immersions of smooth $k$-schemes such that:

- $Y \setminus X$ is a normal crossings divisor in $X$,
- $X \setminus U$ has codimension at least 2 in $X$, and
- $Y \setminus W$ has codimension at least 2 in $Y$.

Then the functor $F\text{-}\text{Isoc}(X,Y) \otimes L \to F\text{-}\text{Isoc}(U,W) \otimes L$ is an equivalence of categories.

*Proof. See [39, Theorem 5.1].*

**Definition 3.1.4.** We extend the definition of the companion relation (Definition 2.3.1) to the case where one or both objects belongs to $E \in F\text{-}\text{Isoc}(X) \otimes \mathbb{Q}_p$. Some care must be taken with this definition because Lemma 2.3.2 does not extend to this level of generality.

**Remark 3.1.5.** We will have occasion at several points to consider $F$-isocrystals on smooth schemes over an imperfect field $\ell$ with maximal perfect subfield $k$, or a valuation ring of such a field containing $k$. When realizing these objects as vector bundles with integrable connections, these connections will be $K$-linear; in particular, there will be an action of continuous $K$-linear differential operators on the base ring.

### 3.2. Newton polygons.

**Definition 3.2.1.** For $x \in X$ (not necessarily closed) and $E \in F\text{-}\text{Isoc}(X) \otimes L$, we may give an intrinsic definition of the Newton polygon $N_x(E)$ using the Dieudonné–Manin classification [39, Definition 3.3], [40, Definition 1.2.3].

In terms of this intrinsic definition of Newton polygons, Lemma-Definition 2.6.1 may be refined as follows.

**Lemma 3.2.2.** Let $L$ be a finite extension of $\mathbb{Q}_p$. For $E \in F\text{-}\text{Isoc}(X) \otimes L$, the function $x \mapsto N_x(E)$ has the following properties.

(a) For $k$ finite and $x = X = \text{Spec}(k)$, the construction agrees with Lemma-Definition 2.6.1 (a).

(b) For $x \in X$ with Zariski closure $Z$, for all $z \in Z$, $N_z(E)$ lies on or above $N_x(E)$ with the same right endpoint. In particular, the right endpoint is constant on each component of $X$.

(c) In (b), the set $U$ of $z \in Z$ for which equality holds is open and Zariski dense, and its complement is of pure codimension 1 in $Z$.

(d) In (c), for each curve $C$ in $Z$, the inverse image of $U$ in $C$ is either empty or the complement of a finite set of closed points.

(e) The vertices of $N_x(E)$ all belong to $Z \times \mathbb{Z}^N$ for $N = [L : \mathbb{Q}_p]$.
Proof. For (a), see [40, Lemma 1.2.4]. For (b) and (c), see [39, Theorem 3.12]. Part (d) is an immediate consequence of (c); we include it only for parallelism with Lemma-Definition 2.6.1. For part (e), we may assume that $x = X = \text{Spec}(k)$ and then deduce this directly from the Dieudonné–Manin classification (as in [40, Lemma 1.2.4] again). □

3.3. Slope filtrations. While the Dieudonné–Manin classification does not extend to the case where $X$ is not a point, a weaker version of the statement does generalize as follows. By Lemma 3.2.2(c), if $X$ is irreducible, then the hypothesis on the constancy of the Newton polygon can always be enforced after restricting from $X$ to a suitable open dense subscheme.

Proposition 3.3.1. Let $L$ be a finite extension of $\mathbb{Q}_p$. Suppose that $E \in \mathbf{F-Isoc}(X) \otimes L$ is such that the function $x \mapsto N_x(\mathcal{E})$ is constant. Let $\mu_1 < \cdots < \mu_l$ be the slopes of $N_x(\mathcal{E})$ for any $x \in X$. Then $E$ admits a filtration (the slope filtration)

$$0 = \mathcal{E}_0 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$$

such that for $i = 1, \ldots, l$, for all $x \in X$, $N_x(\mathcal{E}_i/\mathcal{E}_{i-1})$ consists of the single slope $\mu_i$.

Proof. Apply [39, Corollary 4.2]. □

The individual steps of the slope filtration can in turn be interpreted as representations of profinite fundamental groups.

Definition 3.3.2. An object $E \in \mathbf{F-Isoc}(X) \otimes L$ is unit-root if for all $x \in X$, $N_x(\mathcal{E})$ has all slopes equal to 0. By Lemma 3.2.2, it suffices to check this at the generic point of each irreducible component of $X$.

Proposition 3.3.3. Suppose that $X$ is irreducible and let $\overline{x} \to X$ be a geometric point. Let $L$ be a finite extension of $\mathbb{Q}_p$.

(a) There is a functorial (in $X$) equivalence of categories between unit-root objects of $\mathbf{F-Isoc}(X) \otimes L$ and continuous representations of the profinite étale fundamental group $\pi_1(X, \overline{x})$ on finite-dimensional $L$-vector spaces.

(b) Under this equivalence, unit-root objects of $\mathbf{F-Isoc}^\dagger(X) \otimes L$ correspond to representations of $\pi_1(X, \overline{x})$ which are potentially unramified (i.e., after pullback to some finite étale cover they become unramified).

Proof. See [39, Theorem 3.7, Theorem 3.9]. □

Lemma 3.3.4. Let $L$ be a finite extension of $\mathbb{Q}_p$. Let $U$ be an open dense subscheme of $X$. If $F \in \mathbf{F-Isoc}(U) \otimes L$ is unit-root and docile along $X \setminus U$, then $F \in \mathbf{F-Isoc}^\dagger(X) \otimes L$.

Proof. By Proposition 3.3.3(b), there exists a finite étale cover $\tilde{U}$ of $U$ such that $F|_{\tilde{U}}$ is everywhere unramified. Consequently, $F$ extends across the smooth locus of the normalization of $X$ in $\tilde{U}$; by faithfully flat descent, we deduce that $F$ extends across a subset of $X$ whose complement has codimension at least 2. By Lemma 3.1.3, this proves the claim. □

3.4. The minimal slope theorem. While by Lemma 3.1.2 the inclusion functor $\mathbf{F-Isoc}^\dagger(X) \otimes L \to \mathbf{F-Isoc}(X) \otimes L$ is fully faithful, it does not in general reflect subobjects; in particular, the slope filtration given by Proposition 3.3.1 does not in general lift back to $\mathbf{F-Isoc}^\dagger(X) \otimes L$. In some sense, one expects rather the opposite; a recent result of Tsuzuki [65] addresses a question raised in [39, Remark 5.14] to this effect. (We state only the one-dimensional case, but for $k$ finite there is also a higher-dimensional result in [65].)
Theorem 3.4.1 (Tsuzuki). Suppose that $X$ is irreducible of dimension 1. Let $\mathcal{E}, \mathcal{F}$ be irreducible objects in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$. Let $U$ be an open dense subset of $X$ on which the functions $x \mapsto N_x(\mathcal{E}), x \mapsto N_x(\mathcal{F})$ are constant (this set exists by Lemma 3.2.2). Let $\mathcal{E}_1, \mathcal{F}_1 \in \mathbf{F}\text{-}\mathbf{Isoc}(U) \otimes \overline{\mathbb{Q}}_p$ be the first steps of the slope filtrations of $\mathcal{E}, \mathcal{F}$, respectively, according to Proposition 3.3.1.

(a) Both $\mathcal{E}_1$ and $\mathcal{F}_1$ are irreducible in $\mathbf{F}\text{-}\mathbf{Isoc}(U) \otimes \overline{\mathbb{Q}}_p$.
(b) If $\mathcal{E}_1 \cong \mathcal{F}_1$ in $\mathbf{F}\text{-}\mathbf{Isoc}(U) \otimes \overline{\mathbb{Q}}_p$, then $\mathcal{E} \cong \mathcal{F}$ in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$.

Proof. For (a), see [65, Proposition 6.2]. For (b), see [65, Theorem 1.3]. □

Although we do not use it here, we mention an extension of Theorem 3.4.1 by D’Addezio. This resolves the parabolicity conjecture of Crew [13, p. 460] and also naturally includes Lemma 3.1.2.

Theorem 3.4.2 (D’Addezio). Suppose that $X$ is irreducible (of any dimension). Let $\mathcal{E}$ be an irreducible object in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$. Let $U$ be an open dense subset of $X$ on which the function $x \mapsto N_x(\mathcal{E})$ is constant (this set exists by Lemma 3.2.2). Let $\mathcal{E}$ be the image of $\mathcal{E}$ in $\mathbf{F}\text{-}\mathbf{Isoc}(U) \otimes \overline{\mathbb{Q}}_p$. Let $G(\mathcal{E}), G(\mathcal{F})$ be the algebraic monodromy groups of $\mathcal{E}, \mathcal{F}$ in the sense of Crew [13]; then $G(\mathcal{F})$ is the subgroup of $G(\mathcal{E})$ preserving the slope filtration (Proposition 3.3.1). Moreover, if $\mathcal{E}$ is semisimple, then $G(\mathcal{F})$ is a parabolic subgroup of $G(\mathcal{E})$.

Proof. Using Lemma 2.1.2 we may reduce to the case $X = U$. In this case, apply [15, Theorem 1.1.1]. □

4. RIGID AND DAGGER GEOMETRY

We recall some additional background in formal, rigid, and dagger geometry that will be useful in our later calculations.

4.1. LOCAL LIFTING BY FORMAL SCHEMES. We describe convergent isocrystals in terms of local lifts of varieties from characteristic $p$ to characteristic 0. (Not all terminology here is standard.)

Definition 4.1.1. By a smooth lift of $X$, we will mean a smooth formal scheme $P$ over $W(k)$ (for the $p$-adic topology) with $P_k \cong X$.

Lemma 4.1.2. Suppose that $X$ is affine. Then $X$ admits a smooth lift, which is unique up to noncanonical isomorphism.

Proof. This may be obtained by the method of Elkik [27], or more precisely by a result of Arabia [4, Théorème 3.3.2]. □

Lemma 4.1.3. Let $\bar{f} : X' \to X$ be an étale morphism and let $P$ be a smooth lift of $X$. Then $\bar{f}$ lifts functorially to an étale morphism $f : P' \to P$ of formal schemes over $W(k)$, where $P'$ is a certain smooth lift of $X'$ (determined by $P$ and $\bar{f}$).

Proof. This is a consequence of the henselian property of the pair $(W(k), pW(k))$. See for example [29, Theorem 5.5.7], which is written in the more general context of almost commutative algebra, but is nonetheless a good reference for this point. □
Definition 4.1.4. Let \( P \) be a smooth lift of \( X \). A Frobenius lift on \( P \) is a morphism \( \sigma : P \to P \) which acts on \( W(k) \) via the Witt vector Frobenius and lifts the absolute Frobenius morphism on \( X \).

The following construction gives a particular class of Frobenius lifts.

Definition 4.1.5. Let \((X, Z)\) be a smooth pair over \( k \). A smooth chart for \((X, Z)\) is a sequence \( t_1, \ldots, t_n \) of elements of \( O_X(X) \) such that the induced morphism \( f : X \to A^n_k \) is étale and there exists \( m \in \{0, \ldots, n\} \) for which the zero loci of \( t_1, \ldots, t_m \) on \( X \) are the irreducible components of \( Z \) (and in particular are reduced).

Lemma 4.1.6. Let \((X, Z)\) be a smooth pair over \( k \). Then for each \( x \in X \), there exist an open subscheme \( U \) of \( X \) containing \( x \) and a smooth chart for \((U, Z \cap U)\).

Proof. By replacing \( x \) with a specialization, we may assume that \( x \in X^o \). Since \( X \) is smooth, it satisfies the Jacobian criterion; we can thus find elements \( t_1, \ldots, t_n \in O_{X,x} \) such that \( dt_1, \ldots, dt_n \) form a basis of \( \Omega_{X/k,x} \) over \( O_{X,x} \). By adjusting the choice of coordinates, we may further ensure that \( Z \) is cut out locally at \( x \) by \( t_1 \cdots t_m \). Choose an open affine neighborhood \( U \) of \( x \) in \( X \) omitting every irreducible component of \( Z \) not passing through \( x \). Then \( t_1, \ldots, t_n \) form a smooth chart for \((U, Z \cap U)\). \( \square \)

Definition 4.1.7. Let \((X, Z)\) be a smooth pair and let \( \bar{t}_1, \ldots, \bar{t}_n \) be a smooth chart for \((X, Z)\). Let \( P_0 \) be the formal completion of \( \text{Spec} W(k)[t_1, \ldots, t_n] \) along the zero locus of \( p \). By Lemma 4.1.3, there exists a unique smooth affine formal scheme \( P \) over \( W(k) \) equipped with an étale morphism \( f : P \to P_0 \) lifting \( \bar{f} \); we refer to \((P, t_1, \ldots, t_n)\) as the lifted smooth chart associated to the original smooth chart.

Let \( \sigma_0 : P_0 \to P_0 \) be the Frobenius lift for which \( \sigma_0^* (t_i) = t_i^p \) for \( i = 1, \ldots, n \). By the functoriality aspect of Lemma 4.1.3, there exists a unique Frobenius lift \( \sigma \) on \( P \) making the diagram

\[
\begin{array}{ccc}
\text{P} & \to & \text{P} \\
\downarrow^f & & \downarrow^f \\
P_0 & \to & P_0 \\
\end{array}
\]

commute. We call \( \sigma \) the associated Frobenius lift of the lifted smooth chart.

Definition 4.1.8. For \( P \) a smooth lift of \( X \), denote by \( P_K \) the Raynaud generic fiber of \( P \); this is a rigid analytic space whose points correspond to formal subschemes of \( P \) which are integral and finite flat over \( W(k) \) (see [7, §7.4]). In particular, there is a specialization map taking any such point to the intersection of \( X \) with the corresponding formal subscheme. For \( S \subseteq P \), let \( |S|_P \) denote the inverse image of \( S \) under the specialization map; this set is called the tube of \( S \) within \( P_K \).

4.2. Weak formal schemes and dagger spaces. We then upgrade the previous discussion to handle overconvergent isocrystals, by introducing certain analogues of formal schemes and rigid analytic spaces which directly incorporate the notion of overconvergence. These can also be handled in terms of partially overconvergent isocrystals (compare Definition 3.1.1), but as this description will not appear explicitly we will not spell it out.
Definition 4.2.1. A ring $R$ is weakly complete with respect to an ideal $I$ if it is $I$-adically separated and, for any positive real numbers $a, b$ and any elements $x_1, \ldots, x_n \in R$, any infinite sum of the form

$$\sum_{i_1, \ldots, i_n=0}^{\infty} a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n} \quad (a_{i_1 \cdots i_n} \in I^{f(i_1, \ldots, i_n) - n})$$

converges in $R$. (By contrast, if $R$ is complete with respect to $I$, then the sum also converges under the weaker condition that $a_{i_1 \cdots i_n} \in I^{f(i_1, \ldots, i_n)}$ where $f$ is any function for which $f(i_1, \ldots, i_n) \to \infty$ as $i_1 + \cdots + i_n \to \infty$.)

By replacing complete rings with weakly complete rings, we obtain Meredith’s concept of a weak formal scheme\cite{52}; there is an obvious forgetful functor from weak formal schemes to formal schemes.

By a dagger lift of $X$, we will mean a smooth weak formal scheme $P^\dagger$ over $\text{Spf} W(k)$ with $P_k^\dagger \cong X$. We write $P$ for the underlying formal scheme of $P^\dagger$.

Lemma 4.2.2. Let $P^\dagger$ be a smooth weak formal scheme over $\text{Spf} W(k)$. Then $P^\dagger[p^{-1}]$ is noetherian and excellent. (The same is true of $P^\dagger$, but we will not need this fact.)

Proof. This reduces at once to the case where $P^\dagger$ is the weak completion of an affine space. Since $K$ is of characteristic 0, this case is an easy consequence of the Nullstellensatz for dagger algebras (see\cite{30} §1.4) plus the weak Jacobian criterion in the form of \cite{51} Theorem 102. □

Lemma 4.2.3. For $P^\dagger$ a smooth weak formal scheme over $\text{Spf} W(k)$, the morphism $P \to P^\dagger$ is faithfully flat.

Proof. Since we have surjectivity on points, it suffices to check flatness of the morphism on coordinate rings. This morphism is the direct limit of a family of morphisms, each of which corresponds to an open immersion of affinoid spaces over $K$ and so is flat\cite{5} Corollary 7.3.2/6]. □

Lemma 4.2.4. Let $\overline{f} : X' \to X$ be an étale morphism and let $P^\dagger$ be a dagger lift of $X$. Then $\overline{f}$ lifts functorially to an étale morphism $f^\dagger : P^\dagger \to P^\dagger$ of weak formal schemes, where $P^\dagger$ is a certain dagger lift of $X'$ (determined by $P^\dagger$ and $\overline{f}$).

Proof. Note that if $R$ is weakly complete with respect to $I$, then the pair $(R, I)$ is henselian. We may thus argue as in the proof of Lemma \cite{41,3}.

Definition 4.2.5. With notation as in Definition \cite{41,7} let $P^\dagger_0$ be the weak formal completion of $\text{Spec} W(k)[t_1, \ldots, t_n]$ along the zero locus of $p$. By Lemma 4.2.4 $f$ descends uniquely to an étale morphism $f^\dagger : P^\dagger \to P^\dagger_0$ of weak formal schemes, and $\sigma$ descends to a morphism $\sigma^\dagger : P^\dagger \to P^\dagger$. We refer to $(P^\dagger, t_1, \ldots, t_n)$ as the lifted dagger chart associated to the original smooth chart.

Definition 4.2.6. For $P^\dagger$ a dagger lift of $X$, we may again define the generic fiber $P^\dagger_K$ as a locally G-ringed space with the same underlying G-topological space as $P_K$, but with a modified structure sheaf. The space $P^\dagger_K$ lives in the category of dagger spaces of Grosse-Klönne\cite{30}; to summarize, these are built in a fashion analogous to rigid analytic spaces, but with the role of standard Tate algebras (i.e., the coordinate rings of generic fibers of formal completions of affine spaces over $W(k)$) being played by their overconvergent analogues (in which the formal completions become weak formal completions).
4.3. Relative GAGA for rigid and dagger spaces. It is well known that Serre’s GAGA theorem for varieties over \( \mathbb{C} \) [58] has an analogue over a nonarchimedean field, in which complex analytic spaces are replaced by rigid analytic spaces. For completeness, we give a somewhat more general statement than the one we actually need. (Here we only use the fact that \( K \) is a nonarchimedean field; the discreteness of the valuation plays no role.)

**Proposition 4.3.1.** Let \( \mathcal{C} \) denote either the category of rigid analytic spaces over \( K \) or the category of dagger spaces over \( K \). Suppose that \( S \) is an affinoid space in \( \mathcal{C} \), put \( S_0 := \text{Spec} \mathcal{O}(S) \), and let \( \pi_S : S \to S_0 \) be the adjunction morphism. Let \( f_0 : Y_0 \to S_0 \) be a proper morphism, let \( f : Y \to S \) be the analytiﬁcation of \( f_0 \) in \( \mathcal{C} \), and let \( \pi_Y : Y \to Y_0 \) be the adjunction morphism.

(a) The morphisms \( \pi_S, \pi_Y \) are flat and every closed point has a unique preimage.
(b) Pullback along the induced morphism \( Y \to Y_0 \) deﬁnes an equivalence of categories between coherent sheaves on \( Y_0 \) and on \( Y \).
(c) Let \( E_0 \) be a coherent sheaf on \( Y_0 \) and let \( E \) be the pullback of \( E_0 \) to \( Y \). Then the natural morphisms \( \pi^*(R^i f_0^* E_0) \to R^i f_* E \) of sheaves on \( S \) are isomorphisms for all \( i \geq 0 \).

**Proof.** To prove (a), we need only treat the case \( S \to S_0 \). For this, see [30, Theorem 1.7] for the dagger case, and references therein for the rigid-analytic case.

We next skip to (c). Using Chow’s lemma as in the complex-analytic case, we may reduce to the case where \( f \) is projective, and then further to the case where \( Y_0 = \mathbb{P}^n_{S_0} \). By (a), pullback of coherent sheaves along \( Y \to Y_0 \) is an exact functor; we are thus free to make homological reductions. Using the relative ampleness of \( \mathcal{O}(1) \) with respect to \( \mathbb{P}^n_{S_0} \to S_0 \), we may reduce to the cases where \( E = \mathcal{O}(m) \) for \( m \in \mathbb{Z} \). These cases amount to the fact that the morphism

\[
\mathcal{O}(S_0)[T_1^\pm, \ldots, T_n^\pm] \to \mathcal{O}(S_0)[T_1^\pm, \ldots, T_n^\pm]
\]

in the rigid case, and the morphism

\[
\mathcal{O}(S_0)[T_1^\pm, \ldots, T_n^\pm] \to \mathcal{O}(S_0)[T_1^\pm, \ldots, T_n^\pm]^{\dagger}
\]

in the dagger case, induces isomorphisms of associated graded rings for the grading by homogeneous degree. (This is essentially the method of Serre; see the ﬁrst paragraph of [12, Example 3.2.6] for another approach that directly handles the case where \( f_0 \) is proper, without having to add Chow’s lemma.)

To prove (b), using (a) and (c) we know that the pullback functor is fully faithful. To establish essential surjectivity, we may again assume that \( Y_0 = \mathbb{P}^n_{S_0} \); again using (c), it sufﬁces to check that for every coherent sheaf \( \mathcal{F} \) on \( \mathbb{P}^n_S \), there exists an integer \( m \) such that \( \mathcal{F}(m) \) is generated by global sections. In the rigid-analytic case, we may appeal to [12, Theorem 3.2.4] to deduce this immediately. In the dagger case, we may make the corresponding argument after we note that as in Kiehl’s theorem, one knows that the higher direct images of a coherent sheaf along a proper morphism of dagger spaces are again coherent [30, Theorem 3.5].

**Remark 4.3.2.** Proposition 4.3.1 specialized to the case where \( S \) is a point, includes the usual GAGA theorem for rigid analytic spaces over \( K \), which has been known for some time (see the discussion in [12, Example 3.2.6]). It also includes GAGA for dagger spaces over \( K \), but this is not new either because the category of proper dagger spaces over \( K \) is equivalent to the category of proper rigid analytic spaces over \( K \) [30, Theorem 2.27].
As an application, we record a description of docile overconvergent isocrystals in the case where \( X \) has a liftable smooth compactification.

**Definition 4.3.3.** By a smooth lift of a smooth pair \((\overline{X}, \mathfrak{Z})\) over \( k \), we will mean a smooth pair \((\overline{X}, \mathfrak{Z})\) of schemes (not formal schemes) over \( W(k) \) equipped with compatible identifications \( \overline{X}_k \cong \overline{X}, \mathfrak{Z}_k \cong \mathfrak{Z} \).

**Proposition 4.3.4.** Let \((\overline{X}, \mathfrak{Z})\) be a smooth lift of a smooth pair \((X, \mathfrak{Z})\) with \( X \cong \overline{X} \setminus \mathfrak{Z} \). Let \( L \) be a finite extension of \( \mathbb{Q}_p \).

(a) There is a fully faithful functor from the category of objects of \( \text{Isoc}(X, \overline{X}) \otimes L \) (isocrystals without Frobenius structure) which are docile along \( \mathfrak{Z} \) to the category of vector bundles on the Raynaud generic fiber \( \overline{X}_K \times_{\mathbb{Q}_p} L \) equipped with an integrable \( K \)-linear logarithmic (with respect to \( \mathfrak{Z}_K \times_{\mathbb{Q}_p} L \)) connection with nilpotent residues.

(b) Suppose that \( \overline{X} \) is proper over \( k \). Then the functor in (a) factors through the category of vector bundles on the scheme \( \overline{X}_K \times_{\mathbb{Q}_p} L \) equipped with an integrable \( K \)-linear logarithmic (with respect to \( \mathfrak{Z}_K \times_{\mathbb{Q}_p} L \)) connection with nilpotent residues.

**Proof.** We may reduce at once to the case \( L = \mathbb{Q}_p \). By Proposition 4.3.1 it suffices to prove the corresponding assertion with \((\overline{X}, \mathfrak{Z})\) replaced by its \( p \)-adic formal completion. In that case, apply [36, Theorem 6.4.1] to obtain a fully faithful functor from the latter category to vector bundles on \( \overline{X}_K \) equipped with logarithmic (with respect to \( \mathfrak{Z}_K \)) integrable connections with nilpotent residues. \( \Box \)

**Remark 4.3.5.** In Proposition 4.3.4 one can get an equivalence of categories if \((\overline{X}, \mathfrak{Z})\) admits a Frobenius lift, in which case the vector bundles also carry an action of this lift compatible with the connection). In practice such a lift almost never exists, so the Frobenius structure has to be described locally using smooth lifts of affine subschemes of \( \overline{X} \).

5. Tame rigidity

We now focus on the setting of stable curve fibrations and study the problem of extending a coefficient object from a single fiber across the whole fibration. By examining the analogous complex-analytic situation carefully, we identify results which can be transferred to étale and crystalline coefficients, as well as an obstruction which seems unsurmountable in the étale setting but which we will eventually overcome in the crystalline setting.

5.1. The complex-analytic situation. We begin by summarizing some key observations from complex analytic geometry.

**Hypothesis 5.1.1.** Throughout §5.1 let \( f : Y \to S \) be a smooth curve fibration of connected complex analytic spaces with unpointed locus \( U \) and pointed locus \( Z \). Let \( s : S \to U \) be an additional section.

**Definition 5.1.2.** Pick any point \( x \in S \). Under Hypothesis 5.1.1, there is a short exact sequence

\[
1 \to \pi_1(U \times_S x) \to \pi_1(U) \to \pi_1(S) \to 1
\]

which is split by \( s \). (The splitting ensures that the map \( \pi_2(S) \to \pi_1(U \times_S x) \) is trivial.) For \( \rho \) a finite-dimensional \( \mathbb{C} \)-representation of \( \pi_1(U \times_S x) \), a spreading of \( \rho \) over \( S \) is an extension of
ρ to a representation of π₁(U); we say such a spreading is pinned if its restriction to s(π₁(S)) is trivial.

Via Riemann–Hilbert, ρ corresponds to a vector bundle Eₓ on Y ×ₛ x with a ℂ-linear logarithmic (with respect to Z ×ₛ x) connection. A spreading of ρ over S corresponds to an extension of Eₓ to a vector bundle E on Y with a ℂ-linear logarithmic (with respect to Z) integrable connection. A pinned spreading is one for which s∗E is trivial.

Definition 5.1.3. With definitions as in Definition 5.1.2, we formulate the following three principles of tame rigidity.

• Uniqueness principle: a pinned spreading of ρ over S is unique if it exists.
• Extension principle: let T be an open subspace of S such that π₁(T) → π₁(S) is surjective. Then any pinned spreading of ρ over T extends over S.
• Obstruction principle: for a spreading of ρ over S to exist, the outer action of π₁(S) on π₁(U ×ₛ x) must stabilize the isomorphism class of ρ. For a pinned spreading, it must also be true that [π₁(U ×ₛ x), π₁(S)] must be contained in ker(ρ).

Remark 5.1.4. Although this is not germane here, we point out for completeness that the primary obstruction in the obstruction principle is not a perfect obstruction; it must be supplemented by the Clifford obstruction. However, the latter vanishes in many cases of interest, e.g., when π₁(S) is a free group (in particular, when S is a hyperbolic curve).

Remark 5.1.5. When ρ is one-dimensional, we can make the obstruction principle more explicit: the obstruction to spreading is a homomorphism from π₁(S) to the character group of π₁(U ×ₛ x), or equivalently a bilinear homomorphism

\[ \pi₁^{\text{ab}}(S) \times \pi₁^{\text{ab}}(U ×ₛ x) \to \mathbb{C}×. \]

Note that in this case, there is no analogue of either the Clifford obstruction (Remark 5.1.4) or the further obstruction to pinned spreading.

5.2. Uniqueness principle for coefficient objects. We next establish the uniqueness principle of tame rigidity for étale and crystalline coefficient objects on curve fibrations.

Remark 5.2.1. We first reformulate the uniqueness principle. With notation as in Definition 5.1.3 let E₁, E₂ be vector bundles on Y equipped with ℂ-linear logarithmic (with respect to Z) integrable connections. Since f is a fiber bundle, the sheaves R²f∗E_j are again local systems.

Now suppose in addition that s : S → U is a section such that s∗E_j is constant for j = 1, 2. Then the following statements hold.

(a) For every x ∈ S, the natural map End(E₁) → End(E₁|U ×ₛ x) is an isomorphism. In particular, the restriction of E₁ to U ×ₙ x is irreducible if and only if E₁ is.
(b) If there exists some x ∈ S such that the restrictions of E₁ and E₂ to U ×ₛ x are irreducible and isomorphic, then E₁ ≅ E₂.

We next make a translation into the étale setting.

Hypothesis 5.2.2. For the remainder of §5.2, suppose that k is finite, let S be a smooth, geometrically irreducible k-scheme, let f : X → S be a smooth curve fibration with unpointed locus U and pointed locus Z, and let s : S → U be an additional section.

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Lemma 5.2.3. Let $\mathcal{E}$ be an étale coefficient object on $U$ which is tame along $Z$. Then the sheaves $R^i\text{f}_{\text{et},*}\mathcal{E}$ are étale coefficient objects on $S$ for $i \geq 0$, and vanish for $i > 2$; moreover, the formation of these commutes with arbitrary base change on $S$.

Proof. These statements reduce to the corresponding statements for a torsion étale sheaf on $X_K$ of order prime to $p$. In this context, the sheaves $R^i\text{f}_{\text{et},*}\mathcal{E}$ are constructible and vanish for $i > 2$. To check that they are lisse, using the proper base change theorem it suffices to check that the cohomology groups of $\mathcal{E}$ on geometric fibers have locally constant dimension. For $i = 0$, this follows from the tame specialization theorem [31 Théorème 3.8]; this implies the case $i = 2$ by Poincaré duality. Given these cases, the case $i = 1$ follows from the local constancy of the Euler characteristic, which is implied by the Grothendieck–Ogg–Shafarevich formula. 

In the following, part (a) is a variant of [40 Lemma 3.2.1].

Lemma 5.2.4 (Étale uniqueness principle). Let $\mathcal{E}_1, \mathcal{E}_2$ be two étale coefficient objects on $U$ which are tame along $Z$, such that $s^*\mathcal{E}_1$ and $s^*\mathcal{E}_2$ are constant.

(a) For every $x \in S$, the natural map $\text{End}(\mathcal{E}_1) \to \text{End}(\mathcal{E}_1|_{U \times_S x})$ is an isomorphism. In particular, the restriction of $\mathcal{E}_1$ to $U \times_S x$ is absolutely irreducible if and only if $\mathcal{E}_1$ is.

(b) If there exists some $x \in S$ such that the restrictions of $\mathcal{E}_1$ and $\mathcal{E}_2$ to $U \times_S x$ are absolutely irreducible and isomorphic, then $\mathcal{E}_1 \cong \mathcal{E}_2$.

Proof. By Lemma 5.2.3, $\text{f}_{\text{et},*}(\mathcal{E}_1^\vee \otimes \mathcal{E}_1)$ and $\text{f}_{\text{et},*}(\mathcal{E}_2^\vee \otimes \mathcal{E}_2)$ are étale coefficient objects. Since they are also subobjects of the respective constant coefficient objects $s^*_\text{et}(\mathcal{E}_1^\vee \otimes \mathcal{E}_1)$ and $s^*_\text{et}(\mathcal{E}_2^\vee \otimes \mathcal{E}_2)$, they must themselves be constant.

This immediately yields the first assertion of (a). For the second assertion, note that by Schur’s lemma, the rank of $\text{f}_{\text{et},*}(\mathcal{E}_1^\vee \otimes \mathcal{E}_1)$ equals 1 if and only if $\mathcal{E}_1$ is absolutely irreducible.

To establish (b), combine the hypothesis on $x$ with the previous analysis to deduce that $\text{f}_{\text{et},*}(\mathcal{E}_1^\vee \otimes \mathcal{E}_2)$ is trivial of rank 1. 

We next make a translation into the crystalline case, which for later use we formulate slightly more generally. In particular, we allow isocrystals without Frobenius structure.

Lemma 5.2.5. Let $P$ be a smooth affine formal scheme over $W(k)$. Let $M$ be a finitely generated $\mathcal{O}(P)$-module equipped with a $W(k)$-linear integrable connection. Then $M[p^{-1}]$ is a locally free $\mathcal{O}(P)[p^{-1}]$-module.

Proof. See [38 Lemma 3.3.3].

We now obtain an analogue of Lemma 5.2.3 which we need in two different forms.

Lemma 5.2.6. Let $L$ be a finite extension of $\mathbb{Q}_p$. Let $\mathcal{E}$ be an object of $\text{F-Isoc}(U, X) \otimes L$ (resp. $\text{F-Isoc}^!(U) \otimes L$) which is tame along $Z$. Then the higher direct images $R^i\text{f}_{\text{crys},*}\mathcal{E}$ exist in $\text{F-Isoc}(S) \otimes L$ (resp. $\text{F-Isoc}^!(S) \otimes L$) for $i \geq 0$, and vanish for $i > 2$; moreover, the formation of these commutes with arbitrary base change on $S$.

Proof. We may check the claim étale-locally on $S$. By Proposition 1.2.5, we may reduce to the case where $S$ admits a smooth dagger lift $Q^!$ and $X$ admits a smooth dagger lift $P^!$ mapping to $Q^!$. Using Proposition 4.3.3 we may then realize $\mathcal{E}$ as a vector bundle on $P_K \times_{\mathbb{Q}_p} L$ (resp. $P_K^! \times_{\mathbb{Q}_p} L$) equipped with a $K$-linear logarithmic integrable connection.
and a compatible Frobenius structure (in the sense of Remark 4.3.5). Using the algebraic description of the Gauss–Manin connection [35], pushing forward yields coherent sheaves on \(Q_K \times_{Q_p} L\) (resp. \(Q_K^\dagger \times_{Q_p} L\)) equipped with a \(K\)-linear integrable connection and a compatible Frobenius structure, which vanish for \(i > 2\).

At this point, we obtain the desired conclusion for \(i > 2\). We may then deduce the claim for all \(i\) by descending induction, as follows. The fact that \(R^{i+1}f_{crys,*}\mathcal{E}\) is a locally free sheaf implies compatibility with base change for the \(i\)-th higher direct image. Lemma 5.2.5 now implies that the \(i\)-th higher direct image is locally free, yielding the desired conclusion for \(R^if_{crys,*}\mathcal{E}\).

This in turn yields an analogue of Lemma 5.2.4.

**Lemma 5.2.7** (Crystalline uniqueness principle). Let \(L\) be a finite extension of \(Q_p\). Let \(\mathcal{E}_1, \mathcal{E}_2\) be two objects of \(\mathbf{F}_{\text{Isoc}}(U, X) \otimes L\) (resp. \(\mathbf{F}_{\text{Isoc}}^\dagger(U) \otimes L\)) which are tame along \(Z\), such that \(s^*\mathcal{E}_1\) and \(s^*\mathcal{E}_2\) are constant.

(a) For every \(x \in S\), the natural map \(\text{End}(\mathcal{E}_1) \to \text{End}(\mathcal{E}_1|_{U \times_S x})\) is an isomorphism. In particular, the restriction of \(\mathcal{E}_1\) to \(U \times_S x\) is absolutely irreducible if and only if \(\mathcal{E}_1\) is.

(b) If there exists some \(x \in S\) such that the restrictions of \(\mathcal{E}_1\) and \(\mathcal{E}_2\) to \(U \times_S x\) are absolutely irreducible and isomorphic, then \(\mathcal{E}_1 \cong \mathcal{E}_2\).

**Proof.** The proof of Lemma 5.2.4 applies with the use of Lemma 5.2.3 replaced by Lemma 5.2.6.

5.3. **Extension principle for coefficient objects.** We next turn to the extension principle; this is the only point at which we are forced to consider stable curve fibrations rather than smooth curve fibrations. We first refine the discussion in the complex case.

**Remark 5.3.1.** With notation as in Hypothesis 5.1.1, let \(S \to \overline{S}\) be an open immersion whose complement is Zariski closed and nowhere dense, and let \(\overline{f} : \overline{Y} \to \overline{S}\) be a stable curve fibration extending \(f\), with smooth unpointed locus \(\overline{U}\) and pointed locus \(\overline{Z}\). Suppose also that \(s\) extends to a section \(\overline{s} : \overline{S} \to \overline{U}\) meeting all fibers at smooth points. Let \(Z'\) be the union of the components not meeting \(\overline{s}\) of the singular fibers of \(\overline{f}\).

Let \(\mathcal{E}\) be a vector bundle on \(Y\) equipped with a \(\mathbb{C}\)-linear logarithmic (with respect to \(Z\)) integrable connection whose pullback along \(s\) is trivial. By the extension principle, \(\mathcal{E}\) extends (but not uniquely) across \(Y\) with the connection also having logarithmic singularities across the singular fibers. However, the trivialization along \(s\) means we can eliminate the logarithmic singularities along components of singular fibers meeting \(f\); we thus obtain a unique extension of \(\mathcal{E}\) with logarithmic singularities along \(\overline{Z} \cup Z'\).

We now translate into the étale and crystalline settings.

**Hypothesis 5.3.2.** Throughout \(\S 5.3\), when specified, assume that \(k\) is finite. Let \(S \to \overline{S}\) be an open immersion of smooth, geometrically connected \(k\)-schemes with dense image. Let \(\overline{f} : \overline{X} \to \overline{S}\) be a stable curve fibration with smooth unpointed locus \(\overline{U}\) and pointed locus \(\overline{Z}\), such that the pullback \(f : X \to S\) of \(\overline{f}\) along \(S \to \overline{S}\) is a smooth curve fibration with unpointed locus \(U\) and pointed locus \(Z\). Let \(\overline{s} : \overline{S} \to \overline{U}\) be a section. Let \(Z'\) be the union of the components not meeting \(\overline{s}\) of the singular fibers of \(\overline{f}\). Let \(T\) be an open dense subscheme of \(S\).
Proposition 5.3.3 (Étale extension principle). Under Hypothesis 5.3.2, suppose that $\mathcal{E}$ is an absolutely irreducible étale coefficient on $U \times_S T$ which is tame along $Z$ and whose pullback along $s$ is constant. Then $\mathcal{F}$ extends uniquely to a coefficient object on $\overline{X} \setminus (Z \cup Z')$.

Proof. We may argue directly as in §5.1 and Remark 5.3.1 using étale fundamental groups. □

We now turn to the crystalline case.

Lemma 5.3.4. Let $R$ be a noetherian ring and let $t \in R$ be a non-zero divisor. Put $S = \text{Spec } R$, $T = \text{Spec } R[t^{-1}]$, and let $j : T \to S$ be the canonical inclusion. Let $f : Y \to S$ be a stable curve fibration which is smooth over $T$, with smooth unpointed locus $U$ and pointed locus $Z$, and let $s : S \to U$ be another section. For each positive integer $n$, let $S_n$ be the $n$-th nilpotent thickening of $s(S)$ within $Y$ and put $T_n := S_n \times_S T$. Let $\mathcal{F}$ be a vector bundle on $S$ and let $\mathcal{F}_n$ be the pullback of $\mathcal{F}$ along the projection $S_n \to S$. Let $\mathcal{E}$ be a coherent sheaf on $Y \times_S T$ equipped with a coherent sequence of isomorphisms $\mathcal{E}|_{T_n} \cong \mathcal{F}_n|_{T_n}$. Then there exists a unique (up to unique isomorphism) extension of $\mathcal{E}$ to a coherent sheaf $\mathcal{E}$ on $Y$ for which the isomorphisms $\mathcal{E}|_{T_n} \cong \mathcal{F}_n|_{T_n}$ extend to isomorphisms $\mathcal{E}|_{S_n} \cong \mathcal{F}_n$.

Proof. Fix an ample line bundle $\mathcal{O}(1)$ on $Y$. Let $M_{n,m}$ denote the image of the map

$$H^0(Y \times_S T, \mathcal{E}(m)) \to H^0(T_n, \mathcal{E}(m)) \cong H^0(T_n, \mathcal{F}_n(m));$$

note that this map is injective for $n \gg 0$, so $M_{n,m}$ is isomorphic to the inverse limit $\lim_{\leftarrow n} M_{n,m}$ where the transition maps are pullbacks along $T_n \to T_{n+1}$. In particular, $\mathcal{E}$ is the coherent sheaf associated to the graded module

$$\bigoplus_{m=0}^{\infty} \lim_{\leftarrow n} M_{n,m}$$

via the Proj construction. We now identify $H^0(T_n, j^* \mathcal{F}_n(m))$ with $H^0(S_n, \mathcal{F}_n(m)) \otimes_R R[t^{-1}]$, which contains $H^0(S_n, \mathcal{F}_n(m))$ because $\mathcal{F}$ is a vector bundle and $t$ is not a zero divisor. With this in mind, we take $\mathcal{E}$ to be the coherent sheaf associated to the graded module

$$\bigoplus_{m=0}^{\infty} \lim_{\leftarrow n}(M_{n,m} \cap H^0(S_n, \mathcal{F}_n(m)))$$

(where the intersection is taken in $H^0(T_n, j^* \mathcal{F}_n)$); again the inverse system stabilizes at some $n$ (depending on $m$). This sheaf has the desired property. □

Translating into the language of isocrystals yields the following.

Proposition 5.3.5 (Crystalline extension principle). Under Hypothesis 5.3.2, let $L$ be a finite extension of $\mathbb{Q}_p$. Suppose that $\mathcal{E} \in \mathbf{F} \text{-Isoc}(U \times_S T, X \times_S T) \otimes L$ is absolutely irreducible and tame along $Z \times_S T$ and that the pullback of $\mathcal{E}$ along $s$ is constant. Then $\mathcal{E}$ extends uniquely to an object of $\mathbf{F} \text{-Isoc}^!(\overline{X} \setminus (Z \cup Z')) \otimes L$.

Proof. By crystalline tame rigidity (Lemma 5.2.7), we may work étale-locally on $\overline{S}$; by Proposition 1.2.5 we may assume that $\overline{S}$ admits a smooth lift $\overline{S}$ and $\overline{f}$ lifts to a stable curve fibration $\overline{f} : \overline{X} \to \overline{S}$. We may then realize $\mathcal{E}$ as a vector bundle with integrable logarithmic connection and Frobenius structure on $\overline{X}_K \times_{\overline{S}} T$ (Proposition 4.3.4), then choose a coherent sheaf $\mathcal{E}$ on $\overline{X} \times_{\overline{S}} T$ to which the connection extends; we may also ensure that the pullback
along $s$ is constant. The connection then defines trivializations along the nilpotent thickenings of $s$, so we may apply Lemma 5.3.4 (modulo each power of $p$) to extend $\mathcal{E}$ across $\overline{X}$. This extension, being unique, automatically carries an extended connection with the specified logarithmic singularities, as well as a Frobenius structure. This yields the desired object of $F$-Isoc$^\dagger(\overline{X} \setminus (Z \cup Z')) \otimes L$. □

Remark 5.3.6. For $\eta \in S$ the generic point, Proposition 5.3.5 carries over to the case where the initial object $\mathcal{E}$ belongs to $F$-Isoc$(U \times_S \eta, X \times_S \eta)$ in the sense of Remark 3.1.5.

5.4. Obstruction principle for coefficient objects. At this point, it is natural to turn to the obstruction principle. For étale coefficient objects, one can again directly follow the complex-analytic setup to obtain an obstruction to spreading a tame coefficient object on $U \times_S x$ across a fibration. For $k$ algebraically closed, one can use the structure of tame fundamental groups [31, Corollaire 3.9] to see that the primary obstruction can indeed occur.

For crystalline coefficient objects, it is less clear how to even construct a meaningful obstruction. We will answer this question in §7.3 using moduli stacks of truncated $F$-crystals and Tsuzuki’s minimal slope theorem; the latter allows us to pin down overconvergent $F$-isocrystals using their (convergent) unit-root parts, thus putting us back into the setting of representations of fundamental groups.

The use of Tsuzuki’s theorem will have an important side effect: in most cases, we can use exterior powers to arrange for the unit-root to be of rank 1 without changing the monodromy group of the whole isocrystal; this will allow us to “abelianize” the obstruction. This in turn will show that while the obstruction is (presumably) genuine for general $k$, we can eliminate it for $k$ finite using Deligne’s finiteness theorem for local systems with prescribed ramification (Proposition 2.9.1).

The vanishing of the obstruction will then allow us to construct crystalline companions from a single fiber in the fibration, except in the cases excluded by the previous paragraph. These are the isoclinic cases, where the whole isocrystal corresponds to a representation of the étale fundamental group. Fortunately, in these cases we can directly simulate Drinfeld’s original construction of crystalline companions. We can then turn around and show that for $k$ finite, the obstruction to spreading also vanishes in the étale case (and the isoclinic crystalline case); see Theorem 8.6.1.

6. Moduli of truncated $F$-crystals

In this section, we introduce a somewhat ad hoc notion of a “truncated $F$-crystal” on a smooth scheme (with logarithmic structure) over a perfect field. We then study some moduli stacks associated to this definition.

Remark 6.0.1. We warn the reader that the letter $P$ is used three different ways within §6 alone: as a finitely generated monoid (Definition 6.1.2), a smooth lift of $X$ (Hypothesis 6.3.1), and a polynomial over $\mathbb{Q}$ (Definition 6.5.5). However, these usages are in such different contexts that there should be no risk of confusion.

6.1. Logarithmic schemes. In order to deal with boundaries of compactifications, we introduce log schemes. See [54] for a comprehensive treatment.

Definition 6.1.1. By a logarithmic (log) scheme, we will always mean a scheme equipped with a fine log structure in the sense of [33]. For example, this can (and generally will) be
the log structure associated to a smooth pair as in [33 (1.5)(1)], which is indeed fine [33 Example (2.5)].

In (slightly) more detail, for $X$ a scheme, a pre-log structure on $X$ consists of a sheaf of (commutative) monoids $M$ on $X_{et}$ together with a homomorphism $\alpha : M \to \mathcal{O}_X$ with respect to multiplication on $\mathcal{O}_X$; a log structure is a pre-log structure for which $\alpha$ induces an isomorphism $\alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$; and a fine log structure is a log-structure for which $M$ is cancellative and $M/\alpha^{-1}(\mathcal{O}_X^\times)$ is locally finitely generated.

**Definition 6.1.2.** Let $(X, M)$ be a log scheme. As per [33 Definition 2.9], a chart for $X$ is a homomorphism $\beta : P_X \to M$ where $P$ is a finitely generated monoid, $P_X$ is the locally constant sheaf on $X$ associated to $P$, and

$$P_X/((\alpha \circ \beta)^{-1}(\mathcal{O}_X^\times)) \xrightarrow{\beta} M/\alpha^{-1}(\mathcal{O}_X^\times)$$

is an isomorphism of sheaves of monoids on $X_{et}$. (This last condition asserts that the adjunction from pre-log structures to log structures [33 (1.3)] promotes $P$ to a fine log structure which is isomorphic to $M$ via $\beta$.) Such a chart always exists étale-locally on $X$ [33 Lemma 2.10].

**Definition 6.1.3.** For $(Y, N) \to (X, M)$ a morphism of log schemes, let $\Omega_{Y/X}^1$ denote the module of relative logarithmic differentials [33 (1.7)].

**Remark 6.1.4.** It is also possible to define logarithmic structures on algebraic stacks; see [55]. This will not be necessary for our purposes: the stacks we need to work with are moduli stacks of objects associated to logarithmic schemes, but no logarithmic structure on these stacks will be relevant here.

### 6.2. Interlude on Witt vectors.

In order to get a better handle on crystals, we recall some facts about rings of finite $p$-typical Witt vectors over not necessarily perfect rings of characteristic $p$.

**Hypothesis 6.2.1.** Throughout §6.2 fix a positive integer $n$ and let $R$ be a ring of characteristic $p$.

**Definition 6.2.2.** Let $W_n$ denote the endofunctor on rings given by taking $p$-typical Witt vectors of length $n$. The projection $W_n(R) \to W_1(R) = R$ admits a natural section [•] : $R \to W_n(R)$ at the level of multiplicative monoids. For $\varphi : R \to R$ the Frobenius map (and the induced map on $W_n(R)$), there is a functorial (in $R$) additive homomorphism $V : W_n(R) \to W_n(R)$, the Verschiebung map, with the property that $\varphi \circ V = V \circ \varphi$ is multiplication by $p$.

**Remark 6.2.3.** For $R$ perfect, $W_n(R)$ is the mod-$p^n$ truncation of the usual Witt ring $W(R)$, which is $p$-adically separated and complete with $W(R)/(p) \cong R$ via the first projection. For general $R$, the structure of $W_n(R)$ is somewhat more complicated; for example, it is no longer generated over $\mathbb{Z}$ by the image of [•]. However, it is true that $W_n(R^{p^n})$ is the subring of $W_n(R)$ generated by $V^i([r^{p^n}]) = p^i[r^{p^{n-i}}]$ for $i = 0, \ldots, n - 1$ and $r \in R$.

**Lemma 6.2.4.** Let $S$ be a ring and let $I$ be an ideal in $S$. Suppose that $I$ is nilpotent (that is, $I^n = 0$ for some positive integer $n$), $S/I$ is a noetherian ring, and $I/I^2$ is a finitely generated $S/I$-module. Then $S$ is noetherian.
Proof. For \( h = 0, \ldots, n-1 \), \( I^h/I^{h+1} \) is a quotient of the \( h \)-th tensor power of \( I/I^2 \), and hence is a finitely generated \( S/I \)-module. Consequently, for any ideal \( J \) of \( S \), \((J \cap I^h)/(J \cap I^{h+1})\) is also a finitely generated \( S/I \)-module, and hence a finitely generated \( S \)-module. Since \( J \) is a successive extension of these modules, it is also finitely generated.

Corollary 6.2.5. The ring \( W_n(R) \) is noetherian if and only if \( R \) is noetherian and \( F \)-finite (i.e., finite as an \( R^p \)-module). For example, this happens if \( R \) is a localization of a finitely generated \( k \)-algebra.

Proof. We prove the “if” assertion, as the “only if” assertion will become clear during the argument. We apply Lemma 6.2.4 to the ring \( S := W_n(R) \) and the ideal \( I = \ker(W_n(R) \to R) \). We are given that \( S/I \cong R \) is noetherian.

For \( i = 1, \ldots, n \), the image \( V^i(W_n(R)) \) of \( V^i \) equals the kernel of \( W_n(R) \to W_i(R) \). Since

\[
V^i([x])V^i([y]) = p^iV^i([xy]),
\]

the ideal \( V^i(W_n(R)) \) squares into \( V^{i+1}(W_n(R)) \), so all of these ideals are nilpotent. Since \( I = V(W_n(R)) \), it follows that \( I \) is nilpotent.

To check that \( I/I^2 \) is finitely generated, by the previous paragraph we may assume \( n = 2 \). In this case, every element of \( I \) can be written as \( V([x]) \) for some \( x \in R \), and we have \( [x]V([y]) = V([xp^i y]) \). Consequently, \( I = I/I^2 \) is isomorphic as an \( R \)-module to \( R \) via Frobenius. Since \( R \) is \( F \)-finite, it follows that \( I \) is finitely generated.

Definition 6.2.6. Let \( S \) be a ring equipped with a homomorphism \( \overline{f} : R \to S/pS \). Then there exists a natural homomorphism \( f_n : W_n(R^{p^{n-1}}) \to S \) which makes the diagram

\[
\begin{array}{ccc}
W_n(R^{p^{n-1}}) & \xrightarrow{f_n} & S \\
\downarrow & & \downarrow \\
R & \xrightarrow{\overline{f}} & S/pS
\end{array}
\]

commute; namely, for \( i = 0, \ldots, n-1 \), \( f_n \) carries \( V^i([x^{p^{n-1}}]) \) to \( p^i x^{p^{n-1-i}} \) where \( x \in S \) is any lift of \( \overline{f}(r) \). In particular, any homomorphism \( f : W_n(R) \to S \) lifting \( \overline{f} \) must restrict to \( f_n \).

6.3. Truncated crystals: local definition. We now give the promised \textit{ad hoc} definition of truncated crystals which we will use to build genuine isocrystals via inverse limits (see §6.7). We first give a local description in coordinates.

Hypothesis 6.3.1. Throughout §6.3, fix a smooth affine scheme \( S \) over \( k \). Let \((X, Z)\) be a smooth pair over \( k \) with \( X \) affine over \( S \), and let \((P, t_1, \ldots, t_m)\) be a lifted smooth chart for \((X, Z)\) where the reductions of \( t_1, \ldots, t_d \) cut out \( Z \) within \( X \). (We will only use this setup when \( X \to S \) is smooth, but we do not yet need this restriction.) Fix a positive integer \( n \) and let \( P_n \) denote the reduction of \( P \) mod \( p^n \).

Definition 6.3.2. For a scheme \( T \) over \( k \), let \( T^{(n)} \) be a copy of \( T \) and let \( \varphi_n : T \to T^{(n)} \) denote the \( p^{n-1} \)-st power Frobenius. By Definition 6.2.6 the identification of \( P_n \times_{W_n(k)} k \)
with $X$ induces a morphism $P_n \to W_n(X^{(n)})$ such that

\[
\begin{array}{c}
X \\
\downarrow \varphi_n \\
X^{(n)} \rightarrow W_n(X^{(n)}) \\
\downarrow \\
S \\
\downarrow \varphi_n \\
S^{(n)} \rightarrow W_n(S^{(n)})
\end{array}
\]

commutes.

Let $P_{n,2}$ be the closure of the graph of the rational map $P_n \times_{W_n(X^{(n)})} P_n \to \mathbb{G}^{d}_{m,k}$ given by $\pi_1^*(t_1)/\pi_2^*(t_1), \ldots, \pi_1^*(t_d)/\pi_2^*(t_d)$. Let $P_{n,3}$ be the closure of the graph of the rational map $P_n \times_{W_n(X^{(n)})} P_n \times_{W_n(X^{(n)})} P_n \to \mathbb{G}^{2d}_{m,k}$ given by

$$\pi_1^*(t_1)/\pi_2^*(t_1), \ldots, \pi_1^*(t_d)/\pi_2^*(t_d), \pi_2^*(t_1)/\pi_3^*(t_1), \ldots, \pi_2^*(t_d)/\pi_3^*(t_d)$$

We then have projection maps

$$\pi_1, \pi_2 : P_{n,2} \to P_n, \quad \pi_{12}, \pi_{13}, \pi_{23} : P_{n,3} \to P_{n,2}.$$

Let $\text{Crys}_{X^{\log}/S,n}$ be the category consisting of pairs $(T, F)$ in which $T \in \text{Sch}_S$ and $F$ is a finitely generated $O$-module on $P_n \times_{W_n(S^{(n)})} W_n(T^{(n)})$, flat over $W_n(T^{(n)})$, equipped with an isomorphism $\iota : \pi_1^*F \cong \pi_2^*F$ on $P_{n,2} \times_{W_n(S^{(n)})} W_n(T^{(n)})$ satisfying the cocycle condition

$$\pi_{13}^*(\iota) = \pi_{23}^*(\iota) \circ \pi_{12}^*(\iota)$$
on $P_{n,3} \times_{W_n(S^{(n)})} W_n(T^{(n)})$. A morphism $(T', F') \to (T, F)$ consists of a morphism $f : T' \to T$ and a morphism $F' \to f^* F$ compatible with $\iota$.

**Remark 6.3.3.** We may think of an object of $\text{Crys}_{X^{\log}/S,n}$ as a tuple $(T, F, F_1, F_2, t_1, t_2)$ in which $T \in \text{Sch}_S$; $F$ is a finitely generated $O$-module on $P_n \times_{W_n(S^{(n)})} W_n(T^{(n)})$, flat over $W_n(T^{(n)})$; $F_1, F_2$ are finitely generated $O$-modules on $P_{n,2} \times_{W_n(S^{(n)})} W_n(T^{(n)})$; and for $j = 1, 2$, $t_j : \pi_j^*F \to F_1$ is a morphism of $O$-modules on $P_{n,2} \times_{W_n(S^{(n)})} W_n(T^{(n)})$. This tuple also has the property that $t_1, t_2$ are isomorphisms and the cocycle condition is satisfied.

**Lemma 6.3.4.** Let $(P', t'_1, \ldots, t'_m)$ be a second lifted smooth chart for $(X, Z)$. Let $\text{Crys}'_{X^{\log}/S,n}$ be the category defined by analogy with $\text{Crys}_{X^{\log}/S,n}$ using the chart $(P', t'_1, \ldots, t'_m)$. Then there is a canonical equivalence of categories $\text{Crys}_{X^{\log}/S,n} \cong \text{Crys}'_{X^{\log}/S,n}$; in particular, for a third choice of lifted smooth chart, the cocycle condition holds.

**Proof.** By Lemma 4.1.2 we can find an isomorphism $P \cong P'$ reducing to the identity modulo $p$ which we may use to define a functor $\text{Crys}_{X^{\log}/S,n} \cong \text{Crys}'_{X^{\log}/S,n}$. The claim is that for two isomorphisms $f_1, f_2 : P \to P'$, we can find a natural isomorphism between the pullback functors $f_1^*$ and $f_2^*$ in a manner compatible with composition on either side. To see this, note that by Definition 6.2.6 we have a map $P_{n,2} \to P_{n,2}$ induced by $f_1$ and $f_2$; pulling $\iota$ back along this isomorphism gives rise to a natural isomorphism of the desired form, with compatibility with composition being guaranteed by the cocycle condition. 

□
6.4. **Truncated crystals: global definition.** We now globalize the previous definition to obtain truncated crystals without the requirement of local coordinates.

**Hypothesis 6.4.1.** For the remainder of §6, let $S$ be a smooth scheme of finite type over $k$, let $(X, Z)$ be a smooth pair over $k$, let $f : X \to S$ be a morphism, and let $n$ be a positive integer.

**Definition 6.4.2.** By Lemma 6.3.4, the category $\mathcal{Crys}_{X, S, n}$ is canonically independent of the choice of a smooth lifted chart. We may thus consider these categories to form a stack for the étale topology on $S$; since smooth lifted charts always exist locally (Lemma 4.1.6), we may extend the definition of $\mathcal{Crys}_{X, S, n}$ to the case where neither $X$ nor $S$ is required to be affine.

**Remark 6.4.3.** We view $\mathcal{Crys}_{X, S, n}$ as a stack over $S$ via the functor $(T, \mathcal{F}) \mapsto T$. This requires some care: we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{Crys}_{X, S, n+1} & \rightarrow & \mathcal{Crys}_{X, S, n} \\
\downarrow & & \downarrow \\
S & \xrightarrow{\phi_S} & S
\end{array}
\]

in which the bottom horizontal arrow is Frobenius mapping right-to-left. In particular, there is no natural $S$-linear structure on the inverse limit over $n$ (see Lemma 6.7.1).

**Definition 6.4.4.** Given a commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{h} & X \\
\downarrow & & \downarrow \\
S' & \xrightarrow{\phi} & S
\end{array}
\]

by lifting $h$ locally to charts and arguing as in Lemma 6.3.4 to eliminate dependence on choices, we obtain the pullback functor $h^*$ appearing in the diagram

\[
\begin{array}{ccc}
\mathcal{Crys}_{X, S, n} & \rightarrow & \mathcal{Crys}_{X', S', n} \\
\downarrow & & \downarrow \\
S & \xrightarrow{\phi} & S'
\end{array}
\]

**Remark 6.4.5.** For a general morphism $S' \to S$, we do not have natural pullback functors $\mathcal{Crys}_{X, S, n} \to \mathcal{Crys}_{(X \times_S S'), S', n}$, roughly speaking because $\mathcal{Crys}_{X, S, n}$ cannot detect parallel transport along directions in $S'$ that project trivially to $S$. However, we do have such pullback functors when $S' \to S$ is unramified (e.g., a locally closed immersion).

6.5. **Moduli stacks of truncated crystals.** We now discuss representability for the stacks $\mathcal{Crys}_{X, S, n}$. Our strategy is to reduce to the study of moduli of coherent sheaves as described in §1.4, however, in order to comply with the condition on properness of supports, we must assume at a minimum that $f$ is proper. For the sake of expediency, we further restrict to the case where $f$ is a smooth curve filtration, so that we can lift $f$ étale-locally on $S$; see Remark 6.5.9. (It would be possible to handle stable curve fibrations also, but we will not need them in this context.)

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Hypothesis 6.5.1. For the remainder of §6, let \( f : X \to S \) be a smooth curve fibration with pointed locus \( Z \).

Definition 6.5.2. By applying Definition 6.4.4 to the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi_X} & X \\
\downarrow & & \downarrow \\
S & \xrightarrow{\varphi_S} & S
\end{array}
\]

we obtain a pullback functor which we denote simply by \( \varphi^* \). Crucially, this makes sense even though we cannot hope to form a smooth lift of \( f \) carrying a Frobenius lift, even \( \acute{e} \text{tale-locally} \) on \( S \) (or even when \( S \) is a point).

Remark 6.5.3. Suppose that there exist a smooth lift \( \mathcal{G} \) of \( S \) and a smooth curve fibration \( \tilde{f} : \mathfrak{X} \to \mathcal{G} \) lifting \( f \). In this case, if we take \( P = \mathfrak{X} \), we may reinterpret Remark 6.3.3 as the description of a locally closed immersion

\[
(6.5.3.1) \quad \text{Crys}_{X_{\log/S,n}} \to \text{Coh}_{P_{n,2}/W_n(S^{(n)})} \times_{\text{Coh}_{P_{n,2}/W_n(S^{(n)})}} \text{Coh}_{P_{n,2}/W_n(S^{(n)})},
\]

of algebraic stacks over \( W_n(S^{(n)}) \). (Note that when constructing the stacks on the right, the condition on proper supports drops out because \( \tilde{f} \) is itself proper.)

Proposition 6.5.4. The category \( \text{Crys}^{\log/S,n}_{X} \) is an algebraic stack which is quasiseparated and locally of finite presentation over \( S \).

Proof. It suffices to check this \( \acute{e} \text{tale-locally} \) on \( S \). By Proposition 1.2.5, this means that we can assume that there exist a smooth lift \( \mathcal{G} \) of \( S \) and a smooth curve fibration \( \tilde{f} : \mathfrak{X} \to \mathcal{G} \) lifting \( f \). Let us view \( \text{Crys}^{\log/S,n}_{X} \) as a stack over \( W_n(S^{(n)}) \) via the functor \( (T, F) \mapsto W_n(T^{(n)}) \). In light of Remark 6.5.3, we may apply Proposition 1.4.3 and Proposition 1.4.4 to deduce that \( \text{Crys}^{\log/S,n}_{X} \) is an algebraic stack over \( W_n(S^{(n)}) \) which is quasiseparated and locally of finite presentation. Since the functor \( T \mapsto W_n(T^{(n)}) \) is itself represented by a scheme of finite type over \( S \), this yields the desired result. \( \square \)

Definition 6.5.5. Fix a line bundle \( \mathcal{L} \) on \( X \) which is very ample relative to \( f \). For \( P \in \mathbb{Q}[t] \), using the identification \( \text{Crys}^{\log/S,1}_{X} = \text{Coh}_{X/S} \) (independently of the choice of \( Z \)), we may follow Definition 1.4.5 and define

\[
\text{Crys}^{P,\mathcal{L}}_{X_{\log/S,n}} := \text{Crys}^{\log/S,n}_{X} \times_{\text{Coh}_{X/S}} \text{Coh}^{P,\mathcal{L}}_{X/S},
\]

view \( \text{Crys}^{P,\mathcal{L}}_{X_{\log/S,n}} \) as a closed-open substack of \( \text{Crys}^{\log/S,n}_{X} \), and observe that \( \text{Crys}^{P,\mathcal{L}}_{X_{\log/S,n}} \) is the disjoint union of the \( \text{Crys}^{P,\mathcal{L}}_{X_{\log/S,n}} \) over all \( P \).

Similarly, for \( m \) a positive integer, we define

\[
\text{Crys}^{P,\mathcal{L},m}_{X_{\log/S,n}} := \text{Crys}^{\log/S,n}_{X} \times_{\text{Coh}_{X/S}} \text{Coh}^{P,\mathcal{L},m}_{X/S},
\]

view \( \text{Crys}^{P,\mathcal{L},m}_{X_{\log/S,n}} \) as a locally closed substack of \( \text{Crys}^{P,\mathcal{L}}_{X_{\log/S,n}} \), and observe that \( \text{Crys}^{P,\mathcal{L},m}_{X_{\log/S,n}} \) is the union of the \( \text{Crys}^{P,\mathcal{L},m}_{X_{\log/S,n}} \) over all \( m \).

Proposition 6.5.6. For any \( P, \mathcal{L}, m \) as in Definition 6.5.5, the morphism \( \text{Crys}^{P,\mathcal{L},m}_{X_{\log/S,n+1}} \to \text{Crys}^{P,\mathcal{L},m}_{X_{\log/S,n}} \) is of finite type.
Proof. As in the proof of Proposition 6.5.4, we may reduce to the case where \( f \) lifts over a smooth formal lift of \( S \). In this case, we may deduce the finite type property directly from Proposition 1.4.4 and Proposition 1.4.6.

Proposition 6.5.7. For any \( P, \mathcal{L}, m \) as in Definition 6.5.5, \( \text{Crys}_{X^\log/S, n}^{P, \mathcal{L}, m} \) is of finite type over \( S \).

Proof. We proceed by induction on \( n \). For \( n = 1 \), we identify \( \text{Crys}_{X^\log/S, n}^{P, \mathcal{L}, m} \) with \( \text{Coh}_{X/S}^{P, \mathcal{L}, m} \) and invoke Proposition 1.4.4 and Proposition 1.4.6. Given the claim for some \( n \), we factor the structure morphism \( \text{Crys}_{X^\log/S, n+1}^{P, \mathcal{L}, m} \to S \) as per Remark 6.4.3 and note that the other three arrows in the diagram are morphisms of finite type: the top horizontal arrow by Proposition 6.5.6, the right vertical arrow by the induction hypothesis, and the bottom horizontal arrow because \( S \) is of finite type over the perfect field \( k \).

Remark 6.5.8. Notwithstanding Remark 1.4.7, the projection maps \( \text{Crys}_{X^\log/S, n+1}^{P, \mathcal{L}, m} \to \text{Crys}_{X^\log/S, n}^{P, \mathcal{L}, m} \) are not universally closed (as was erroneously asserted in a prior version of this article). The problem is that these stacks parametrize coherent sheaves plus some extra data which is “affine” in nature. Another way to say it is that on the right-hand side of (6.5.3.1), each factor individually satisfies the existence part of the valuative criterion, but not necessarily in a compatible way.

Remark 6.5.9. It is likely that all of the previous results continue to hold assuming only that \( f \) is proper, not necessarily a smooth curve fibration. Since in this case \( f \) need not lift to characteristic 0, the arguments for this may require de Jong’s theorem on semistable reduction via alterations [16, Theorem 8.2].

6.6. Frobenius structures on truncated crystals. We now add Frobenius structures to the previous discussion. Keep in mind that \( f \) is still required to be a smooth curve fibration as per Hypothesis 6.5.1.

Definition 6.6.1. By applying Definition 6.4.4 to the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi_X} & X \\
\downarrow & & \downarrow \\
S & \xrightarrow{\varphi_S} & S
\end{array}
\]

we obtain a pullback functor which we denote simply by \( \varphi^* \). Crucially, this makes sense even though we cannot hope to form a smooth lift of \( f \) carrying a Frobenius lift, even étale-locally on \( S \) (or even when \( S \) is a point).

Remark 6.6.2. The pullback functor \( \varphi^* : \text{Crys}_{X^\log/S, n} \to \text{Crys}_{X^\log/S, n} \) carries \( \text{Crys}_{X^\log/S, n}^{P, \mathcal{L}, m} \) into \( \text{Crys}_{X^\log/S, n}^{qP, \varphi^* \mathcal{L}, m} \).

Definition 6.6.3. Let \( \mathbf{F} \text{-Crys}_{X^\log/S, n} \) be the category in which an object is an object \( (T, \mathcal{F}) \) of \( \text{Crys}_{X^\log/S, n} \), together with a morphism \( \Phi : \varphi^* \mathcal{F} \to \varphi^* \mathcal{F} \) in \( (\text{Crys}_{X^\log/S, n})_{\mathcal{F} \times S, \varphi S} \), where \( \varphi^* \mathcal{F} \) is the pullback of \( \mathcal{F} \) along \( \varphi : T \times_{S, \varphi S} S \to T \).

In addition to the natural first projection \( \pi_1 : \mathbf{F} \text{-Crys}_{X^\log/S, n} \to \text{Crys}_{X^\log/S, n} \) which forgets the extra data, there is a second projection \( \pi_2 : \mathbf{F} \text{-Crys}_{X^\log/S, n} \to \text{Crys}_{X^\log/S, n} \).
carrying \((T, \mathcal{F})\) to \((T \times_{S, \varphi} S, \varphi^* \mathcal{F})\). Since \(F\text{-Crys}_{X, \log/S, n}\) admits a natural projection to \(\text{Crys}_{X, \log/S, n} \times \text{Crys}_{X, \log/S, n}\) via these two maps, we may view it as a stack over \(S \times S\).

As in Remark \([6.4.3]\) we have natural projection maps \(F\text{-Crys}_{X, \log/S, n+1} \to F\text{-Crys}_{X, \log/S, n}\).

**Proposition 6.6.4.** The category \(F\text{-Crys}_{X, \log/S, n}\) is an algebraic stack which is quasiseparated and locally of finite presentation over \(S \times S\).

**Proof.** This is immediate from Proposition \([6.5.4]\). \(\square\)

**Definition 6.6.5.** For \(P, L, m\) as in Definition \([6.5.5]\), define
\[
F\text{-Crys}_{X, \log/S, n}^{P, L, m} := F\text{-Crys}_{X, \log/S, n} \times_{\pi_1, \text{crys}_{X, \log/S, n}} \text{Crys}_{X, \log/S, n}^{P, L, m}.
\]

**Proposition 6.6.6.** For any \(P, L, m\) as in Definition \([6.5.5]\), \(F\text{-Crys}_{X, \log/S, n}^{P, L, m}\) is of finite type over \(S \times S\).

**Proof.** This follows from Proposition \([6.5.7]\) plus Remark \([6.6.2]\). \(\square\)

### 6.7. Comparison with isocrystals

We finally reconcile our previous work with convergent isocrystals.

**Lemma 6.7.1.** There is a restriction functor from the category of convergent log-isocrystals on \((X, Z)\) to the isogeny category of \(\text{lim}_{\leftarrow n} (\text{Crys}_{X, \log/S, n})_S\) (inverting \(p\) in Hom sets).

**Proof.** We use the interpretation of the category of convergent log-isocrystals as crystals on the log-crystalline topos of Shiho (see \([61, Proposition 2.2.7]\) for the comparison with the definition we have been using up to now). Each smooth lifted chart \((P, t_1, \ldots, t_m)\) for an open subscheme of \(X\) corresponds to an object of the log-crystalline site, on which we can evaluate a convergent log-isocrystal to get a vector bundle on \(P \times_{\mathbb{Q}_p} \mathbb{Q}_p\). By choosing a coherent (but not necessarily projective) lattice, we may descend to a coherent sheaf on \(P\) which we may then restrict to \(P_n\); we obtain the isomorphism \(\iota\) (and its cocycle condition) from the rigidity property of crystals. \(\square\)

**Remark 6.7.2.** We cannot directly invert the construction of Lemma \([6.7.1]\) because the base spaces for our truncated crystals are too restrictive: we are modeling not convergent isocrystals, but rather \(p\text{-adically convergent isocrystals}\) in the sense of \([53]\). The difference between these and true convergent isocrystals is that the formal Taylor isomorphism defined by the connection is required to be convergent on a smaller region, corresponding to the closed disc \(|T| \leq p^{-1}\) inside the open disc \(|T| < 1\). Crucially, in the presence of Frobenius structures this difference goes away \([53, Proposition 2.18]\); this corresponds to Dwork’s observation (a/k/a “Dwork’s trick”) that while a \(p\)-adic differential equation without singularities on an open unit disc in general only admits solutions on some smaller disc (by the \(p\)-adic Cauchy theorem), in the presence of a Frobenius structure it admits solutions on the whole disc \([37, Corollary 17.2.2]\).

In light of the previous remark, we restrict our comparison statement from truncated crystals to isocrystals to the case where Frobenius structures are present.

**Definition 6.7.3.** Let \(F\text{-Isoc}^\log(X)\) be the category of convergent log-isocrystals on \((X, Z)\) equipped with a Frobenius structure, which we insist is an isomorphism even over \(Z\). Note that this enforces that the underlying log-isocrystal has nilpotent residues along \(Z\): its residues form a finite multiset of a field of characteristic 0 stable under multiplication by \(p\).
Let $\mathcal{C}_{X^{\log}/S}$ be the full subcategory of the isogeny category of $\lim_{n}(\text{F-Crys}_{X^{\log}/S,n})_S$ (inverting $p$ in Hom sets) consisting of objects for which the cokernel of Frobenius is killed by some power of $p$. Then Lemma 6.7.1 formally promotes to give a restriction functor $\text{F-Isoc}^{\log}(X) \to \mathcal{C}_{X^{\log}/S}$.

**Proposition 6.7.4.** For the category $\mathcal{C}_{X^{\log}/S}$ of Definition 6.7.3, there is a functor $\mathcal{C}_{X^{\log}/S} \to \text{F-Isoc}(X \setminus Z, X)$ (where the target consists of partially overconvergent isocrystals; see Definition 3.1.1) whose composition with the functor $\text{F-Isoc}^{\log}(X) \to \mathcal{C}_{X^{\log}/S}$ from Lemma 6.7.1 is the usual restriction functor $\text{F-Isoc}^{\log}(X) \to \text{F-Isoc}(X \setminus Z, X)$.

**Proof.** Lemma 6.3.4 gives rise to a functor from $\mathcal{C}_{X^{\log}/S}$ to the category of “$p$-adically convergent $F$-isocrystals on $(X, Z)$” in the sense of [53, Proposition 2.18] (extrapolated from ordinary schemes to log schemes). As in [53, Proposition 2.18], these may then be promoted to true $F$-isocrystals. \qed

## 7. Moduli stacks and crystalline obstructions

Using moduli stacks of truncated crystals, we study the crystalline analogue of the obstruction principle.

### 7.1. Setup

We begin by describing the geometric setup in which we will work, including the relevant moduli stacks of truncated $F$-crystals.

**Hypothesis 7.1.1.** Throughout §7 let $S$ be a smooth, geometrically irreducible scheme of finite type over $k$. Let $f : X \to S$ be a smooth curve fibration with pointed locus $Z$ and unpointed locus $U$, admitting an additional section $s : S \to X$.

**Definition 7.1.2.** For $n$ a positive integer and $L$ a finite extension of $\mathbb{Q}_p$, for $P, \mathcal{L}, m$ as in Definition 6.5.5 let $\text{F-Crys}^{P,\mathcal{L},m,0}_{X^{\log}/S,n} \otimes \mathfrak{o}_L$ be the category of objects of $\text{F-Crys}^{P,\mathcal{L},m}_{X^{\log}/S,n}$ equipped with a $\mathbb{Z}_p$-linear action of $\mathfrak{o}_L$ and a trivialization of the pullback along $s_0$ without Frobenius structures.

**Proposition 7.1.3.** For any $n, L, P, \mathcal{L}, m$ as in Definition 7.1.2, the category $\text{F-Crys}^{P,\mathcal{L},m,0}_{X^{\log}/S,n} \otimes \mathfrak{o}_L$ is a quasiseparated finite type algebraic stack over $S$ (via either projection).

**Proof.** The quasiseparated and finite type assertions follow from Proposition 6.6.4 and Proposition 6.6.6, respectively. \qed

### 7.2. Stable local geometry

We next examine the local geometry of these moduli stacks around a coherent sequence of points coming from an $F$-isocrystal.

**Hypothesis 7.2.1.** For the remainder of §7 choose $x \in S^o$ and let $\mathcal{F}_x$ be an absolutely irreducible object of $\text{F-Isoc}^\dagger(U \times_S x) \otimes \overline{\mathbb{Q}}_p$ which is docile along $Z \times_S x$. By Lemma 6.7.1 for some choice of $L, P, \mathcal{L}, m$, $\mathcal{F}_x$ arises from some sequence

\[(x_n \cong x \to \text{F-Crys}^{P,\mathcal{L},m,0}_{X^{\log}/S,n} \otimes \mathfrak{o}_L) \in \lim_{n}(\text{F-Crys}^{P,\mathcal{L},m,0}_{X^{\log}/S,n} \otimes \mathfrak{o}_L)_x.\]

Fix such a choice and such a sequence for the remainder of §7 (The effect of the trivialization along $s$ here is to rigidify; see the proof of Lemma 7.2.1)

**Lemma 7.2.2.** For every nilpotent thickening $\tilde{x}$ of $x$ in $S$, the sequence (7.2.1.1) extends uniquely to a coherent sequence $\{\tilde{x} \to \text{F-Crys}^{P,\mathcal{L},m,0}_{X^{\log}/S,n} \otimes \mathfrak{o}_L\}$. 


Proof. This is immediate from Remark 2.1.10. □

Definition 7.2.3. Let $M_n$ be the smallest closed substack of $\mathbf{F-Crys}_{X^{\text{log}}_{S,n}}^{P,L,m,0} \otimes \mathcal{O}_L$ through which all of the $\tilde{x}$-valued points from Lemma 7.2.2 factor. By Proposition 7.1.3, it is again quasiseparated of finite type over $S$.

As per Remark 6.4.3, the stacks $M_n$ fit into an inverse system “over $S$” in the sense that there are commutative diagrams

\[
\begin{array}{ccc}
M_{n+1} & \longrightarrow & M_n \\
\downarrow & & \downarrow \\
S & \varphi_S & S \\
\end{array}
\]

where $\varphi_S : S \to S$ is Frobenius. To keep track of the Frobenius twists, we again write $S^{(n)}$ for a copy of $S$ with the “natural” map $S^{(n)} \to S^{(n+1)}$ being $\varphi_S$; we may then rewrite the previous diagram as

\[
\begin{array}{ccc}
M_{n+1} & \longrightarrow & M_n \\
\downarrow & & \downarrow \\
S^{(n+1)} & \longrightarrow & S^{(n)}.
\end{array}
\]

Lemma 7.2.4. For each $n$, the map $M_n \to S^{(n)}$ is representable and étale in some neighborhood of $x_n \to M_n$. In particular, there exist a tower of étale morphisms $\cdots \to Y_2 \to Y_1$ over $\mathcal{O}_{S,x}$ with $Y_i$ connected, a coherent sequence of sections $\{x \to Y_n\}_n$, and a coherent sequence of sections $\{Y_n^{(n)} \to M_n\}_n$.

Proof. By Proposition 2.8.1 to rule out infinitesimal deformations within the fiber of $M_n$ over $x$ and Lemma 7.2.2 in some neighborhood of $x$, $M_n \to S^{(n)}$ is formally smooth and formally unramified. Since it also of finite presentation by Proposition 7.1.3, it is smooth [64 Tag 0DP0] (hence flat) and unramified (by a similar argument), hence étale [64 Tag 0CJ1]. In particular, in some neighborhood of $x$ the morphism is Deligne–Mumford [64 Tag 04YW], and so $M_n$ is a DM stack.

Now note that $\mathcal{F}_x$ admits no nontrivial automorphisms that preserve the trivialization along $s$. Consequently, $M_n$ has trivial stabilizer at $x$, and so $M_n \to S^{(n)}$ is representable in some neighborhood of $x$. □

Remark 7.2.5. The surjectivity of the transition maps $M_{n+1} \to M_n$ allows us to control the local geometry of these stacks in terms of properties of isocrystals. By contrast, even at the points $x_n$, the individual stacks $\mathbf{F-Crys}_{X^{\text{log}}_{S,n}}^{P,L,m,0} \otimes \mathcal{O}_L$, in particular in some neighborhood of $x$ the morphism is Deligne–Mumford [64 Tag 04YW], and so $M_n$ is a DM stack.

Now note that $\mathcal{F}_x$ admits no nontrivial automorphisms that preserve the trivialization along $s$. Consequently, $M_n$ has trivial stabilizer at $x$, and so $M_n \to S^{(n)}$ is representable in some neighborhood of $x$. □

7.3. A crystalline obstruction. We next pick up the theme of §5.4 and describe an adaptation of the obstruction principle of tame rigidity (Definition 5.1.3) to crystalline coefficients, built using the moduli stacks of truncated $F$-crystals.

Proposition 7.3.1. Let $\mathcal{O}_{S,x}^h$ denote the henselization of $\mathcal{O}_{S,x}$. Then there exists a docile object $\mathcal{F} \in \mathbf{F-Isoc}(U \times_S \mathcal{O}_{S,x}^h, X \times_S \mathcal{O}_{S,x}^h)$ extending $\mathcal{F}_x$ (interpreting the category as in Remark 3.1.3).
Proposition 7.4.2. Suppose that □ (Theorem 3.4.1), this implies the desired descent statement.

Proof. With notation as in Lemma 7.2.4, the embeddings \( Y_n \to M_n \) induce a coherent sequence of maps \( \{ \text{Spec } \mathcal{O}^{h}_{S,x} \to M_n \}_n \). Apply Proposition 6.7.4 to obtain a docile object \( \mathcal{F} \in \mathbf{F-Isoc}(U \times_S \mathcal{O}^{h}_{S,x}, X \times_S \mathcal{O}^{h}_{S,x}) \).

Remark 7.3.2. Let \( \eta \) denote the generic point of \( \mathcal{O}^{h}_{S,x} \). We will later need to apply Lemma 3.2.2(b) to see that the generic Newton polygon of \( \mathcal{F}_\eta \) lies on or above the generic Newton polygon of \( \mathcal{F}_\eta \). The references cited in the proof of Lemma 3.2.2 do not cover the situation we are considering, of an \( F \)-isocrystal on a smooth scheme over a valuation ring; however, by taking sections we may reduce to the analogue of Lemma 3.2.2(b) for an \( F \)-isocrystal over a perfect valuation ring. In this case, the \( F \)-isocrystal does always arise from a locally free \( F \)-crystal, so for example [34, Theorem 2.3.1] does apply.

As an aside, we note that this argument does not give any \emph{a priori} bound on the slopes of \( \mathcal{F}_\eta \), but since \( \mathcal{F} \) is irreducible we do have such a bound from [23]. That said, we will not use such a bound in the sequel.

7.4. Eliminating the obstruction. In general, we expect that the crystalline obstruction principle provides a genuine obstruction against spreading \( \mathcal{F}_x \) over \( S \) without a pro-\( \acute{e} \text{tale} \) base change on \( S \). However, for \( k \) finite, Deligne’s finiteness theorem for \( \acute{e} \text{tale} \) local systems with prescribed ramification (Proposition 2.9.1) will enable us in many cases of interest to eliminate the obstruction after an \( \acute{e} \text{tale} \) base change. To see this, we first “abelianize” the obstruction.

Lemma 7.4.1. Let \( \eta \) denote the generic point of \( \mathcal{O}^{h}_{S,x} \). For \( \mathcal{F} \) as in Proposition 7.3.1, suppose that the least Newton slope of \( \mathcal{F}_\eta \) occurs with multiplicity 1. Then the object \( \mathcal{F} \) of Proposition 7.3.1 descends from \( \mathcal{O}^{h}_{S,x} \) to the abelian henselization \( \mathcal{O}^{h,\text{ab}}_{S,x} := \mathcal{O}^{h}_{S,x} \cap (\text{Frac } \mathcal{O}_{S,x})^{\text{ab}} \).

Proof. Let \( \eta \) denote the generic point of \( \mathcal{O}^{h}_{S,x} \). By hypothesis, the first step \( \mathcal{G} \) in the slope filtration of \( \mathcal{F}_\eta \) (Proposition 3.3.1) has rank 1. Using Proposition 3.3.3(a) to convert \( \mathcal{G} \) into an \( L^\times \)-valued character, we may now emulate Remark 5.2.3 to obtain a pairing
\[
\pi^\text{ab}_1(\mathcal{O}_{S,x}) \times \pi^\text{ab}_1(\eta) \to L^\times.
\]
This pairing becomes trivial upon pullback from \( \mathcal{O}_{S,x} \) to \( \mathcal{O}^{h,\text{ab}}_{S,x} \); by the minimal slope theorem (Theorem 3.4.1), this implies the desired descent statement. □

This yields the following key result.

Proposition 7.4.2. Suppose that \( k \) is finite, \( \mathcal{F}_x \) is not isoclinic, and \( \overline{\mathcal{G}}(\mathcal{F}_x) \) is connected. Then there exists an (abelian) \( \acute{e} \text{tale} \) neighborhood \( T \to S \) of \( x \) admitting a section \( x \to T \) such that the object \( \mathcal{F} \) of Proposition 7.3.1 descends from \( \mathcal{O}^{h}_{S,x} \) to \( \mathcal{O}^{h}_{T,x} \).

Proof. Since \( \overline{\mathcal{G}}(\mathcal{F}_x) \) is connected, we may check the claim after replacing \( \mathcal{F} \) with \( \wedge^i \mathcal{F} \) for any \( i \in \{1, \ldots, \text{rank}(\mathcal{F}_x) - 1\} \) (compare Remark 2.2.4). Since \( \mathcal{F}_x \) is not isoclinic, by Lemma 3.2.2 and Remark 7.3.2, \( \mathcal{F}_\eta \) is not isoclinic either. We may thus reduce to the case where the least Newton slope of \( \mathcal{F}_\eta \) occurs with multiplicity 1.

By Lemma 7.4.1, there exists a finite or infinite tower \( \cdots \to Y_2 \to Y_1 \) of connected finite \( \acute{e} \text{tale} \) abelian extensions, admitting a coherent sequence of sections \( \{x \to Y_n\}_n \), such that \( \mathcal{F} \) descends to \( \varprojlim_n \mathcal{O}_{Y_n,x} \). We may further make this tower minimal; it then suffices to check that this tower stabilizes. Suppose the contrary; the Galois group of the tower is then an \emph{uncountable} profinite abelian group. Since the tower admits a coherent sequence of
sections over $x$, it in fact splits completely over $x$; this yields a collection of docile objects in $\text{F-Isoc}^\dagger(U \times_S x) \otimes L$ indexed by a profinite topological space $W$. Let $w \in W$ be the point corresponding to the sequence $\{x_n \to M_n\}$; for some neighborhood $Y$ of $w$ in $W$, all of the objects of $\text{F-Isoc}^\dagger(U \times_S x) \otimes L$ indexed by $Y$ are absolutely irreducible. After grouping together points of $W$ corresponding to different lattices in the same isocrystal (which results in equivalence classes which are at most countable), we remain with uncountably many distinct absolutely irreducible docile objects in $\text{F-Isoc}^\dagger(U \times_S x) \otimes L$, no two of which differ by a constant twist (thanks to the constancy of the pullbacks along $s$). However, this contradicts Proposition 2.9.1.

7.5. An extension result. We now use the elimination of the geometric obstruction to deduce a result about extension of $F$-isocrystals from a single fiber across a fibration. We will upgrade this result later by eliminating the isoclinic hypothesis (Theorem 8.6.1).

Proposition 7.5.1. Under Hypothesis 7.1.1 and Hypothesis 7.2.1, suppose that $k$ is finite and $F_x$ is not isoclinic. Then there exist an étale neighborhood $T \to S$ of $x$, an $x$-valued point $x \to T$, and an object $F \in \text{F-Isoc}^\dagger(U \times_S T) \otimes L$ which is docile along $Z \times_S T$, whose pullback along $s$ is constant, and whose pullback along $x \to T \to S$ is isomorphic to $F_x$.

Proof. By the crystalline uniqueness principle (Lemma 5.2.7), we may check the claim after shrinking $S$ and replacing $X$ with a finite étale cover; more precisely, if $g : X' \to X$ is such a cover, then the decomposition of $g^*g^*F_x$ into a direct sum decomposition including $F_x$ (adjunction) spreads over the pushforward of a spreading of $g^*F_x$. Consequently, we may ensure that $G(F_x)$ is connected. Now apply Proposition 7.4.2 to choose $T$ so that the crystalline obstruction descends to a docile object in $\text{F-Isoc}^\dagger(U \times_S O_{T,x} \times_S O_{T,x}) \otimes L$. Then apply Proposition 5.3.5 in the sense of Remark 5.3.6 to produce the desired object. □

8. Companions and corollaries

With the key construction in hand, we complete the construction of crystalline companions, and record some corollaries.

Hypothesis 8.0.1. Throughout §8 assume that $k$ is finite.

8.1. Proof of the main theorems. We complete the proofs of Theorem 0.1.1 and Theorem 0.1.2. Note in particular the use of étale companions in the construction of crystalline companions, to ensure that spreading out a crystalline companion on a fiber indeed yields a crystalline companion on a smooth curve fibration. Since this argument excludes the isoclinic case, we handle that separately by an adaptation of Drinfeld’s original method.

Theorem 8.1.1. Any algebraic étale coefficient object on $X$ admits all crystalline companions.

Proof. Let $E$ be an algebraic étale coefficient object on $X$, and fix a $p$-adic valuation $v$ on $E$. By Lemma 2.3.5 we may check the claim after replacing $X$ with an alteration or an open dense subspace. we may thus assume that $E$ is absolutely irreducible and docile. By Lemma 2.1.4 we may make a constant twist to reduce to the case where $\det(E)$ is of finite order. By Corollary 1.3.4 we may assume that $X$ is the unpointed locus of some smooth curve fibration $f : \overline{X} \to S$. In light of Remark 1.3.5 we may also assume that $X \to S$ admits...
a section \( s \) on which \( \mathcal{E} \) is constant. By the étale uniqueness principle (Lemma 5.2.4(a)), the fibers of \( \mathcal{E} \) over \( S \) are also absolutely irreducible.

We now pick a point \( x \in S^o \) in the image of \( X \) and set \( \mathcal{E}_x := \mathcal{E}|_{X \times S^x} \). If \( \mathcal{E}_x \) is isoclinic with respect to \( v \), then we apply a form of Drinfeld’s argument which we present below (Lemma 8.1.3). Otherwise, apply Corollary 8.1.2 to produce a crystalline companion \( \mathcal{F}_x \) of \( \mathcal{E}_x \) which is absolutely irreducible (by Lemma 2.3.2(b)), docile (by Corollary 2.3.4), \( E \)-algebraic, and not isoclinic. After an étale base change on \( S \), Proposition 7.5.1 guarantees the existence of some absolutely irreducible crystalline coefficient object \( \mathcal{F} \) on \( X \) extending \( \mathcal{F}_x \). By Lemma 2.1.4 and Lemma 2.1.6, \( \mathcal{F} \) is algebraic. By the previously known part of Theorem 0.1.2, \( \mathcal{F} \) itself admits an étale companion \( \mathcal{E}' \). Since \( \mathcal{E}'|_{X \times S^x} \) is a companion of \( \mathcal{E}_x \), by Lemma 2.3.2(d) the two are isomorphic. By the étale uniqueness principle (Lemma 5.2.4(b)), \( \mathcal{E} \) and \( \mathcal{E}' \) are isomorphic, so \( \mathcal{F} \) is indeed a crystalline companion of \( \mathcal{E} \).

**Lemma 8.1.2.** Let \( G \to H \) be a surjective homomorphism of topologically finitely presented pro-\( p \) groups. Then \( G \to H \) is an isomorphism if and only if the maps \( H^i(H, \mathbb{F}_p) \to H^i(G, \mathbb{F}_p) \) are isomorphisms for \( i = 1, 2 \).

**Proof.** The rank of \( H^1(G, \mathbb{F}_p) \) equals the minimum number of generators of \( G \), while the rank of \( H^2(G, \mathbb{F}_p) \) equals the minimum number of relations among a minimal system of generators of \( G \) (see [59, §4.2, 4.3]). Consequently, the isomorphism for \( i = 1 \) ensures that a minimal system of generators of \( G \) remains minimal as a system of generators of \( H \), while the isomorphism for \( i = 2 \) ensures that no extra relations are needed.

**Lemma 8.1.3.** With notation as in the proof of Theorem 8.1.1, if \( \mathcal{E}_x \) is isoclinic with respect to \( v \), then \( \mathcal{E} \) admits a crystalline companion.

**Proof.** Since \( \mathcal{E} \) is algebraic, it is \( E \)-algebraic for some number field \( E \). Let \( L_0 \) be the completion of \( E \) with respect to \( v \). By Corollary 2.5.3, for every curve \( C \) in \( \overline{X} \), \( \mathcal{E}|_{C \times X} \) admits a crystalline companion which is again \( E \)-algebraic. Since \( \mathcal{E} \) is docile, so is this companion (by Corollary 2.3.4), and Lemma 3.3.4 implies that the companion extends over \( C \); consequently, we may reduce to the case \( Z = \emptyset \). Applying Proposition 3.3.3(a) produces a skeleton sheaf (see Definition 8.4.5) of \( \mathbb{Q}_p \)-local systems on \( X \), with no ramification at all; by Lemma 2.1.9, the skeleton sheaf consists entirely of \( L \)-local systems for some finite extension \( L \) of \( \mathbb{Q}_p \).

For \( \overline{\pi} \to S \) a geometric point, let \( \pi^0_1(X \times_S \overline{\pi}) \) denote the maximal pro-\( p \) quotient of the étale fundamental group of \( X \times_S \overline{\pi} \) (which is equal to the tame fundamental group because \( f \) is proper). We have natural identifications

\[
H^i(\pi^0_1(X \times_S \overline{\pi}), \mathbb{F}_p) \cong H^i_{\text{ét}}(X \times_S \overline{\pi}, \mathbb{F}_p).
\]

Since in context we are free to shrink \( S \), by Lemma 3.2.2, we may ensure that \( R^1f_{\text{crys}} \mathcal{O} \) has constant Newton polygon on \( S \) (or even just that the multiplicity of the slope is constant). By Lemma 8.1.2, this ensures that for \( \overline{\pi} \to S \) a geometric point lying over the generic point of \( S \), the surjective maps

\[
\pi^0_1(X \times_S \overline{\pi}) \to \pi^0_1(X \times_S \overline{\pi})
\]

provided by the tame specialization theorem [31, Corollaire 2.4] are always isomorphisms.

We now apply Drinfeld’s method as exposed in [9, §7] but with \( \ell = p \). Let \( \varpi \) be a uniformizer of \( L \) and put \( r := \text{rank}(\mathcal{E}) \). We first argue as in [9, Lemme 7.4] that after shrinking \( S \), we can find a connected finite étale cover \( X_1 \) of \( X \) which trivializes all of the
Frobenius characteristic polynomials modulo \(\varpi\). Let \(\eta\) be the generic point of \(S\) and let \(\eta\) be a geometric point above \(\eta\). Since \(\pi_1(X \times_S \eta)\) is topologically finitely generated [31, Théorème 2.9], all continuous homomorphisms \(\pi_1(X \times_S \eta) \to \text{GL}_r(\mathcal{O}_L/\varpi)\) factor through some characteristic open subgroup \(N\). We may thus choose a cover of \(X\) whose pullback along \(\eta \to S\) corresponds to a subgroup of \(N\) and which trivializes the pullback of \(E\) along \(s\) modulo \(\varpi\) (note that in our setup the latter only requires a constant field extension).

At this point we may assume that all of the Frobenius characteristic polynomials of \(E\) vanish modulo \(\varpi\). We then argue as in [31, Lemme 7.4] that with no further shrinking of \(S\), for each \(n\) we can find a connected finite étale cover \(X_n\) of \(X\) which trivializes all of the Frobenius characteristic polynomials modulo \(\varpi^n\). Namely, the map (8.1.3.1) is an isomorphism for all \(s \in S\), so the joint kernel of all continuous homomorphisms \(\pi_1(X \times_S \eta) \to \text{GL}_r(\mathcal{O}_L/\varpi^n)\) corresponds to a finite étale cover that spreads out over all of \(S\).

We finally argue as in [31, Lemme 7.3] to recover a \(\mathbb{Q}_p\)-local system on \(X\). Let \(\Pi\) be the intersection of the subgroups \(\pi_1(X_n)\) of \(\pi_1(X)\); the group \(\pi_1(X)/\Pi\) is topologically finitely generated and virtually pro-\(p\). In particular, the Frattini subgroup \(F\) of \(\pi_1(X)/\Pi\) is open and the set

\[
H := \text{Hom}(\pi_1(X)/\Pi, \text{GL}_r(\mathcal{O}_L)) = \lim_{\downarrow n} \text{Hom}(\pi_1(X)/\Pi, \text{GL}_r(\mathcal{O}_L/\varpi^n))
\]
equipped with the inverse limit of the discrete topologies is compact. For each \(x \in X^\circ\), let \(H_x \subseteq H\) be the closed subset of representations whose Frobenius characteristic polynomial at \(x\) matches that of \(E_x\); by compactness, it will suffice to check that \(\bigcap_{x \in W} H_x \neq \emptyset\) for every finite subset \(W\) of \(X^\circ\). By [20, Theorem 2.15], there exists a curve \(C\) in \(X\) containing \(W\) such that \(\pi_1(C) \to \pi_1(X) \to (\pi_1(X)/\Pi)/F\) is surjective, as then is \(\pi_1(C) \to \pi_1(X)/\Pi\); let \(K_{\Pi}\) be the kernel of the latter map. A semisimple crystalline companion of \(E|_C\) corresponds to a semisimple representation \(\rho : \pi_1(C) \to \text{GL}_r(L)\); the restriction of \(\rho\) to \(K_{\Pi}\) is both semisimple (because \(K_{\Pi}\) is normal in \(\pi_1(C)\)) and unipotent (by Chebotarev and Brauer–Nesbitt), hence trivial. Hence \(\rho \in \bigcap_{x \in W} H_x\) as desired.

From the resulting \(\mathbb{Q}_p\)-local system on \(X\), by Proposition [3.3.3] we recover an object of \(\mathcal{F}\)-Isoc\((X) \otimes \mathbb{Q}_p\) which is a convergent companion of \(E\) in the sense of Definition 8.1.4. By Proposition 5.3.5, this object is in fact overconvergent and hence is the desired crystalline companion.

\[\square\]

**Corollary 8.1.4.** Theorem 0.1.2 holds in all cases.

**Proof.** As noted previously, for \(\ell' \neq p\) this is included in [40, Theorem 3.5.2]. For \(\ell \neq p, \ell' = p\), this is included in Theorem 8.1.1. For \(\ell = \ell' = p\), we may first apply Corollary 2.5.2 to change to the case \(\ell \neq p\), then proceed as before. \[\square\]

We mention explicitly the following special case of Theorem 0.1.2.

**Corollary 8.1.5.** Let \(E\) be an \(E\)-algebraic coefficient object on \(X\). Then for any automorphism \(\tau\) of \(E\), there exists a coefficient object \(E_x\) such that for each \(x \in X^\circ\), we have the equality \(P((E_x)_x, T) = \tau(P(E_x, T))\) in \(E[T]\) (where \(\tau\) acts coefficientwise).

**Corollary 8.1.6.** Theorem 0.1.1 holds in all cases.

**Proof.** By Corollary 2.5.2, parts (i)–(v) hold. To prove (vi), note that (ii) implies that \(E\) is algebraic, so Corollary 8.1.4 implies the existence of a crystalline companion \(F\). By Lemma 2.3.2(a) and (c), \(F\) is irreducible and \(\det(F)\) is of finite order. \[\square\]
8.2. **Newton polygons revisited.** With Theorem 0.1.2 in hand, we can now assert much stronger properties of Newton polygons of Weil sheaves than were asserted in Lemma-Definition 2.6.1. These extend the Grothendieck–Katz semicontinuity theorem and the de Jong–Oort–Yang purity theorem to étale coefficients (see [39, §3]). The first of these extensions had been informally conjectured by Drinfeld [22, D.2.4].

**Theorem 8.2.1** (after Grothendieck–Katz). Let $E$ be an $E$-algebraic ℓ-adic coefficient object for some number field $E$, and fix an embedding $E \hookrightarrow \mathbb{Q}_p$. Then the function $x \mapsto N_x(E)$ from Lemma-Definition 2.6.1 on $X$ is upper semicontinuous, with the endpoints being locally constant; in particular, this function defines a locally closed stratification of $X$.

**Proof.** By Theorem 0.1.2, this follows from Lemma 3.2.2. □

**Theorem 8.2.2** (after de Jong–Oort–Yang). With notation as in Theorem 8.2.1, the Newton polygon stratification jumps purely in codimension 1. More precisely, for $X$ irreducible, each breakpoint of the generic Newton polygon disappears purely in codimension 1.

**Proof.** By Theorem 0.1.2, this follows from Lemma 3.2.2. □

**Remark 8.2.3.** Suppose that $X$ admits a good compactification $\overline{X}$ and that $E$ is a docile coefficient object on $X$. Apply Theorem 0.1.2 to construct a crystalline companion $F$ of $E$; by Corollary 2.5.4, $F$ is again docile. We then define the Newton polygon function $N_x(E) := N_x(F)$; by retracing through the arguments cited in [39, §3], it can be shown that it satisfies the analogues of the theorems of Grothendieck–Katz and de Jong–Oort–Yang. In particular, the conclusions of Theorem 8.2.1 and Theorem 8.2.2 can be seen to carry over to this definition; we leave a detailed development of this statement to a later occasion.

**Example 8.2.4.** Take $X = \mathbb{P}_k^1 \setminus \{0, 1, \infty\}$ with coordinate $\lambda$ and let $E$ be the middle cohomology of the Legendre elliptic curve $y^2 = x(x-1)(x-\lambda)$. (This is similar to [39, Example 4.6] except with $N = 2$.) Using the Tate uniformization of elliptic curves with split multiplicative reduction, one can show that $E$ is docile and for $\lambda \in \{0, 1, \infty\}$, $N_\lambda(E)$ equals the generic value (i.e., its slopes are 0 and 1).

We next extend some results of Koshikawa [41] from crystalline to étale coefficients. (This is only meant to be a representative sample to illustrate the technique.)

**Definition 8.2.5.** Let $E$ be an $E$-algebraic coefficient object for some number field $E$, and fix an embedding $E \hookrightarrow \mathbb{Q}_p$. As in Definition 2.6.6, we say that $E$ is *isoclinic* if for all $x \in X$, $N_x(E)$ consists of a single slope with some multiplicity; if this slope is always 0, we say moreover that $E$ is *unit-root*. It suffices to check this condition at generic points; it implies local constancy of $N_x(E)$ on $X$.

We say that $E$ is *absolutely isoclinic/unit-root* if it is isoclinic/unit-root with respect to every choice of the embedding $E \hookrightarrow \mathbb{Q}_p$. If $E$ is absolutely isoclinic, then we can make $E$ absolutely unit-root using a constant twist (at the expense of possibly enlarging the number field generated by Frobenius traces). See [41, Example 2.2] for an example of a coefficient object which is unit-root but not absolutely isoclinic.

**Definition 8.2.6.** A coefficient object $E$ on $X$ is *isotrivial* if there exists a finite étale cover $f : Y \to X$ such that $f^*E$ is trivial. This implies that $E$ is absolutely unit-root, but not
conversely [41, Example 3.5]. By Lemma [2.3.2d], if two semisimple coefficient objects are companions, then one is isotrivial if and only if the other is.

**Theorem 8.2.7** (after Koshikawa). *Let $E$ be a coefficient object on $X$ which is absolutely unit-root. Assume in addition either that $E$ is semisimple, or that each irreducible constituent of $E$ has determinant of finite order. Then $E$ is isotrivial.*

*Proof.* In both cases, we may reduce to the case where $E$ is irreducible with determinant of finite order (using Lemma [2.1.4] in the first case). Using Corollary [8.1.6], we may reduce to the crystalline case. We may then apply [41, Theorem 1.4] to conclude. □

**Theorem 8.2.8** (after Koshikawa). *Suppose that $X$ is proper. Let $E$ be a semisimple, $\mathbb{Q}$-algebraic coefficient object on $X$ with constant Newton polygon. Then $E$ is isotrivial.*

*Proof.* Using Theorem [8.1.1], we may reduce to the crystalline case. We may then apply [41, Theorem 1.6] to conclude. □

**Remark 8.2.9.** In Theorem 8.2.8 [41, Example 2.2] shows that the hypothesis of $\mathbb{Q}$-algebraicity cannot be relaxed to mere algebraicity.

8.3. **Wan’s theorem on fixed-slope $L$-functions.** We give a statement on a certain factorization of the $L$-function of $E$.

**Theorem 8.3.1** (after Wan). *Let $E$ be an algebraic coefficient object on $X$. For $s \in \mathbb{Q}$, let $P_s(E_x, T)$ be the factor of $P(E_x, T)$ with constant term 1 corresponding to the slope-$s$ segment of $N_x(E)$. Then the associated $L$-function

$$L_s(X, E, T) = \prod_{x \in X^\circ} P_s(E_x, T^{[\kappa(x): k]})^{-1}$$

is $p$-adic meromorphic (i.e., a ratio of two $p$-adic entire series).

*Proof.* For any locally closed stratification of $X$, $L_s(X, E, T)$ equals the product of $L_s(Y, E, T)$ as $Y$ varies over the strata; we may thus assume that $X$ is affine. Apply Theorem 0.1.2 to construct a crystalline companion $F$ of $E$; we then have $L_s(X, E, T) = L_s(X, F, T)$. The $p$-adic meromorphicity of $L_s(X, F, T)$ is a theorem of Wan [67, Theorem 1.1] (see also [66, 68]); this proves the claim. □

**Remark 8.3.2.** In the crystalline case, Theorem 8.3.1 had been conjectured by Dwork [26]; this conjecture, originally resolved by the work of Wan cited above, was motivated by his original study of zeta functions of algebraic varieties via $p$-adic analytic methods. Indeed, Dwork’s original proof of rationality of zeta functions of varieties over finite fields [24, 25] involved combining archimedean and $p$-adic analytic information about zeta functions as power series.

8.4. **Skeleton sheaves.** We reformulate [21, Theorem 2.5], restricted to smooth $k$-schemes, to include $p$-adic coefficients.

**Definition 8.4.1.** Let $E$ be a coefficient object on $X$. For $x \in X^\circ$, let $c(E, x)$ be the number of eigenvalues of Frobenius on $E_x$ which belong to $\overline{\mathbb{Q}}$ (counted with multiplicity); then $E$ is algebraic if and only if $c(E, x) = \text{rank}(E_x)$ for all $x \in X^\circ$.

**Lemma 8.4.2.** For any coefficient object $E$ on $X$, the function $c(E, \bullet) : X^\circ \to \mathbb{Z}$ is locally constant. In particular, if $X$ is irreducible then $c$ is constant.
Proof. Since we can pass a smooth curve through any two given closed points of $X$ in the same component, we may reduce to the case where $X$ is a curve. We may then further reduce to the case where $E$ is irreducible; in this case, we deduce the claim directly from Corollary 2.1.7.

**Lemma 8.4.3.** Suppose that $X$ is an affine curve. Let

$$0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$$

be a short exact sequence of coefficient objects on $X$ such that $c(E, x) = c(E_2, x) = \text{rank}(E_{2,x})$ for all $x \in X^\circ$. Then this exact sequence splits.

**Proof.** We may use internal Homs to reduce to the case where $E_2 = \mathcal{O}$, in which case we must show that $\text{Ext} (\mathcal{O}, E_1) = 0$. For this, we may reduce to the case where $E_1$ is irreducible. In this case, by Corollary 2.1.7 for some transcendental element $λ$ of the full coefficient field, the constant twist $\mathcal{F}$ of $E_1$ by $λ$ has the property that the Frobenius eigenvalues of $\mathcal{F}_x$ are algebraic for each $x \in X^\circ$.

The eigenvalues of Frobenius on $H^0(X, \mathcal{F})$ are all roots of unity and hence algebraic; since $X$ is an affine curve, we can combine the algebraicity of the eigenvalues of $\mathcal{F}$ with the Lefschetz trace formula [40, (1.1.7.1)] to deduce that the eigenvalues of Frobenius on $H^1(X, \mathcal{F})$ are algebraic. Since $λ$ is not algebraic, we deduce that none of the eigenvalues of Frobenius on $H^0(X, E_1)$ or $H^1(X, E_1)$ is equal to 1; that is, in the Hochschild–Serre exact sequence

$$H^0(X, E_1) \xrightarrow{φ} \text{Ext} (\mathcal{O}, E_1) \rightarrow H^1(X, E_1)$$

the two terms on the ends are both zero. We thus deduce that $\text{Ext} (\mathcal{O}, E_1) = 0$ as desired. □

**Corollary 8.4.4.** Suppose that $X$ is an affine curve. For any coefficient object $E$ on $X$, there exists a direct sum decomposition $E = E_1 \oplus E_2$ such that $c(E, x) = c_1(E_2, x) = \text{rank}(E_{2,x})$ for all $x \in X^\circ$.

**Proof.** We proceed by induction on $\text{rank}(E)$. If $E$ is nonzero, let $E_1$ be an irreducible subobject of $E$. By first applying Corollary 2.1.7 to $E_1$, then applying Lemma 8.4.3 and the induction hypothesis to $E/E_1$, we deduce the claim. □

**Definition 8.4.5.** Fix a category $\mathcal{C}$ of coefficient objects with full coefficient field $F$. A **skeleton sheaf** on $X$ valued in $\mathcal{C}$ (of rank $n$) is a function $χ : X^\circ \rightarrow F[T]$ such that for each morphism $f : C \rightarrow X$ of $k$-schemes with $C$ a curve over $k$, there exists a coefficient object $E_C$ on $C$ in $\mathcal{C}$ (of rank $n$) such that $P(E_{C,x}, T) = χ(f(x))$ for each $x \in C^\circ$.

We say that $χ$ is **tame** (resp. **docile** if the coefficient objects $E_C$ can all be taken to be tame (resp. docile).

We say that $χ$ is **representable** if there exists a coefficient object $E$ on $X$ in the specified category such that $P(E_{x}, T) = χ(x)$ for each $x \in X^\circ$.

**Theorem 8.4.6.** Let $χ$ be a skeleton sheaf on $X$ such that for some alteration $g : X' \rightarrow X$, the pullback of $χ$ along $g$ is tame. Then $χ$ is representable.

**Proof.** We may assume that $X$ is irreducible. The function $x \mapsto \text{deg}(χ(x))$ on $X^\circ$ is constant on smooth irreducible curves in $X$, and hence is constant; we induct on this constant value. For $x \in X^\circ$, let $c(x, χ)$ be the number of algebraic eigenvalues of $χ(x)$ (counted with multiplicity); by Lemma 8.4.2 this function is also constant. Since it is harmless to make a single constant twist on all of the $E_C$, we may assume that $c(x, χ) \neq 0$. 


For each $x \in X^\circ$, factor $\chi(x)$ as $\chi_1(x)\chi_2(x)$ where $\chi_1$ has all roots transcendental and $\chi_2$ has all roots algebraic (and $\deg \chi_2 > 0$). By Lemma 8.4.3 we may split $E$ as a direct sum $E_{C,1} \oplus E_{C,2}$ such that $P(E_{C,i,x}, T) = \chi_i(x)$ for $i \in \{1, 2\}$ and $x \in C^\circ$. By the induction hypothesis, there exists a coefficient object $E_1$ on $X$ in the specified category such that $P(E_{1,x}, T) = \chi_1(x)$ for each $x \in X^\circ$. Meanwhile, after applying Corollary 2.5.3 if needed to convert $E_{C,2}$ into an étale coefficient, we may apply [18 Théorème 3.1] (compare [9 Théorème 5.1]) to deduce that the polynomials $\chi(x)$ for $x \in X^\circ$ are all defined over a single number field. We may then apply [21 Theorem 2.5] to construct an étale coefficient $E_2$ such that $P(E_{2,x}, T) = \chi_2(x)$ for each $x \in X^\circ$. By applying Theorem 8.1.1 if needed to convert $E_2$ into a crystalline coefficient, we may take $E = E_1 \oplus E_2$ to conclude. □

8.5. Reduction of the structure group. As mentioned in the introduction, one can interpret the problem of constructing companions, for coefficient objects of rank $r$, as a problem implicitly associated to the group $GL_r$; it is then natural to consider the corresponding problem for other groups. A closely related question is the extent to which monodromy groups are preserved by the companion relation. In the étale-to-étale case, this question was studied in some detail by Chin [10]. The adaptations of Chin’s work to incorporate the crystalline case were originally made by Pál [56]; we follow more closely the treatment by D’Addezio [14].

Hypothesis 8.5.1. Throughout §8.5 we impose [40 Hypothesis 1.3.1]: assume that $X$ is connected, fix a closed point $x \in X^\circ$ and a geometric point $\overline{x} = \text{Spec}(\overline{k})$ of $X$ lying over $x$, and for each category of crystalline coefficient objects fix an embedding of $W(\overline{k})$ into the full coefficient field.

The following result illustrates a certain limitation of Corollary 2.5.2; see Remark 8.5.3.

Theorem 8.5.2. Let $E$ be an algebraic coefficient object on $X$. Then there exists a number field $E$ with the following property: for every category of coefficient fields and every embedding of $E$ into the full coefficient field, for $L$ the corresponding completion of $E$, $E$ admits an companion in $\text{Weil}(X) \otimes L$ or $\text{F-Isoc}^1(X) \otimes L$.

Proof. In the étale-to-étale case, this is due to Chin [10, Main Theorem]; we may add the crystalline case using Corollary 8.1.6. See also [14, Theorem 3.7.2]. □

Remark 8.5.3. The point of Theorem 8.5.2 is that given that $E$ is $E$-algebraic, any companion is also $E$-algebraic (this being a condition solely involving Frobenius traces) but may not be realizable over the completion of $E$; see [14, Remark 3.7.3] for an illustrative example. The content of Chin’s result is that this phenomenon can be eliminated everywhere at once by a single finite extension of $E$ itself; note that this does not follow from Lemma 2.1.9 because that result does not give enough control on the local extensions.

We next address the question of independence of $\ell$ in the formation of monodromy groups. At the level of component groups, one has the following statement.

Theorem 8.5.4. Let $E$ be a pure algebraic coefficient object on $X$ and let $F$ be a companion of $E$. Then there exists an isomorphism $\pi_0(G(E)) \cong \pi_0(G(F))$ which is compatible with the surjections $\psi_E, \psi_F$ from $\pi_0^1(X, \overline{x})$ (see [40 Proposition 1.3.11]), and which induces an isomorphism $\pi_0(G(E)) \cong \pi_0(G(F))$. 47
Proof. In the étale case, this is due to Serre [60] and Larsen–Pink [48, Proposition 2.2]; the key step is to show that $G(\mathcal{E})$ (resp. $\overline{G}(\mathcal{E})$) is connected if and only if $G(F)$ (resp. $\overline{G}(F)$) is. For the adaptation of this result, see [56, Proposition 8.22] for the case of $\overline{G}$, or [14, Theorem 4.1.1] for both cases.

At the level of connected components, one has the following result.

**Theorem 8.5.5.** Let $\mathcal{E}$ be a semisimple algebraic coefficient object on $X$. Then for some number field $E$ as in Theorem 8.5.2, there exists a connected split reductive group $G_0$ over $E$ such that for every place $\lambda$ of $E$ at which $\mathcal{E}$ admits a companion $F$ (possibly including $\mathcal{E}$ itself), the group $G_0 \otimes_E E_\lambda$ is isomorphic to $G(F)^o$. Moreover, these isomorphisms can be chosen so that for some faithful $E$-linear representation $\rho_0$ of $G_0$ (independent of $\lambda$), the representation $\rho_0 \otimes_E E_\lambda$ corresponds to the restriction to $G(F)^o$ of the canonical representation of $G(F)$.

**Proof.** Since we assume $\mathcal{E}$ is semisimple, we may reduce to the case where it is irreducible; then $\mathcal{E}$ is pure by Lemma 2.7.4 and by Corollary 8.1.6 it becomes $p$-plain after a constant twist. We may then apply [11, Theorem 1.4] in the étale case and [56, Theorem 8.23] or [14, Theorem 4.3.2] in the crystalline case.

**Remark 8.5.6.** It seems likely that one can integrate Theorem 8.5.4 and Theorem 8.5.5 into a single independence statement for arithmetic monodromy groups; see [10, Conjecture 1.1]. Such a result would give an alternate proof of [40, Corollary 3.7.5].

In connection with the discussion of Lefschetz slicing in [40, §3.7], we also mention the following result.

**Theorem 8.5.7.** Let $f : Y \to X$ be a morphism of smooth connected $k$-schemes, and fix a consistent choice of base points. Let $\mathcal{E}, F$ be semisimple coefficient objects on $X$ which are companions. Then the inclusion $G(f^*\mathcal{E}) \to G(\mathcal{E})$ (resp. $\overline{G}(f^*\mathcal{E}) \to \overline{G}(\mathcal{E})$) is an isomorphism if and only if $G(f^*F) \to G(F)$ (resp. $\overline{G}(f^*F) \to \overline{G}(F)$) is an isomorphism.

**Proof.** By Lemma 2.7.4, we may reduce to the case where $\mathcal{E}$ and $F$ are pure. We may then apply [14, Theorem 4.4.2] to conclude.

**Remark 8.5.8.** On the crystalline side, there is a rich theory of isocrystals with additional structure encoding the replacement of $GL_n$ by a more general group $G$; in particular, this theory provides the analogue of a Newton polygon for a $G$-isocrystal at a point, which carries more information than just the Newton polygon of the underlying isocrystals. The basic references for this are the papers of Kottwitz [42, 43] and Rapoport–Richartz [57].

### 8.6. Extensions from fibers

We simultaneously upgrade Proposition 7.5.1 (in the case where $k$ is finite) and Proposition 7.5.1 to the following result. Notably, this says that over a finite field, the étale obstruction principle only imposes a finite obstruction to spreading a local system from a single fiber across a fibration.

**Theorem 8.6.1.** Let $f : X \to S$ be a smooth curve fibration with pointed locus $Z$ and unpointed locus $U$, and let $s : S \to U$ be a section. Choose $x \in S^0$ and let $\mathcal{E}_x$ be an absolutely irreducible coefficient object on $U \times_S x$ which is docile along $Z \times_S x$. Then there exist a finite étale morphism $T \to S$, a section $x \to T$, and a coefficient object $\mathcal{E}$ on $U \times_S T$ whose pullback along $s$ is constant and whose pullback along $x \to T$ is isomorphic to $\mathcal{E}_x$. Moreover, if $\overline{G}(\mathcal{E}_x)$ is connected, then $T \to S$ can be taken to be an abelian cover.
To simplify the proof of Theorem 8.6.1, we introduce a lemma.

**Lemma 8.6.2.** With notation as in the conclusion of Theorem 8.6.1, \(E\) is absolutely irreducible and admits a constant twist which is algebraic with finite determinant. Moreover, if \(E_x\) is \(E\)-algebraic for some number field \(E\), then so is \(E\).

**Proof.** Absolute irreducibility follows from the uniqueness principle (Lemma 5.2.4 or Lemma 5.2.7). By Lemma 2.1.4 and Lemma 2.1.6, \(E\) admits a constant twist which is algebraic with finite determinant. If \(E_x\) is \(E\)-algebraic for some number field \(E\), we may apply the uniqueness principle and Corollary 8.1.5 to deduce that \(E\) is also \(E\)-algebraic. □

**Proof of Theorem 8.6.1.** By Lemma 2.1.4 and Lemma 2.1.6, we may make a constant twist to reduce to the case where \(E\) is algebraic. By the uniqueness principle (Lemma 5.2.4 or Lemma 5.2.7), it suffices to prove the claim with \(T \to S\) being merely étale, not necessarily finite.

Suppose first that \(E_x\) is an étale coefficient. If \(E_x\) is isotrivial, then the claim holds trivially after making a finite étale cover on \(X\); we may recover the original statement by the étale uniqueness principle (Lemma 5.2.4). Otherwise, by Theorem 8.2.7, \(E_x\) is not isoclinic at some \(p\)-adic place. By Corollary 2.5.3, \(E_x\) admits a crystalline companion \(F_x\) which is absolutely irreducible (by Lemma 2.3.2(b)), docile (by Corollary 2.5.4), algebraic, and not isoclinic. By Proposition 7.5.1, the desired conclusion holds with \(E_x\) replaced by \(F_x\); if \(\overline{G}(E_x)\) is connected, then so is \(\overline{G}(F_x)\) (by Theorem 8.5.4) and by Proposition 7.4.2, we may take \(T \to S\) to be abelian. By Lemma 8.6.2, the resulting crystalline coefficient \(F\) is algebraic. By Theorem 0.1.2, \(F\) admits a crystalline companion \(E\). By Corollary 2.7.5, the pullback of \(E\) along \(s\) is constant. By Lemma 2.3.2(d), \(E|_{X \times S}^{x}\) is isomorphic to \(E_x\).

Suppose next that \(E_x\) is a (possibly isoclinic) crystalline coefficient. We may then reverse the previous logic as follows. By Corollary 2.5.3, \(E_x\) admits an étale companion \(F_x\), which is absolutely irreducible (by Lemma 2.3.2(b)), docile (by Corollary 2.5.4), and algebraic. By the previous paragraph, the desired conclusion holds with \(E_x\) replaced by \(F_x\). If \(\overline{G}(E_x)\) is connected, then so is \(\overline{G}(F_x)\) (by Theorem 8.5.4) and we may take \(T \to S\) to be abelian. By Lemma 8.6.2, the resulting étale coefficient \(F\) is algebraic. By Theorem 0.1.2, \(F\) admits a crystalline companion \(E\). By Corollary 2.7.5, the pullback of \(E\) along \(s\) is constant. By Lemma 2.3.2(d), \(E|_{X \times S}^{x}\) is isomorphic to \(E_x\).

**Remark 8.6.3.** Theorem 8.6.1 suggests an approach to Deligne’s broader conjecture that coefficient objects always admit a “geometric origin”: for \(f : X \to S\) a smooth curve fibration and \(x \in S^0\), show that any tame automorphic representation on \(X \times_S x\) spreads out after an étale base change on \(S\).

**References**

[1] T. Abe, Langlands correspondence for isocrystals and existence of crystalline companion for curves, *J. Amer. Math. Soc.* 31 (2018), 921–1057.

[2] T. Abe and H. Esnault, A Lefschetz theorem for overconvergent isocrystals with Frobenius structure, *Ann. Sci. Éc. Norm. Supér.* 52 (2019), 1243–1264.

[3] S. Agrawal, Deformations of overconvergent isocrystals on the projective line, *J. Number Theory* 237 (2022), 167–241.

[4] A. Arabia, Relèvements des algèbres lisses et de leurs morphismes, *Comment. Math. Helv.* 76 (2001), 607–639.
[5] M. Artin, Comparaison avec la cohomologie classique: cas d’un schema lisse, Expose XI in Theorie des Topos et Cohomologie Etale des Schemas (SGA 4 III), Lecture Notes in Math. 305, Springer-Verlag, Berlin, 1973.

[6] P. Berthelot, Geometrie rigide et cohomologie des varietes algebriques de caracteristique p, in Introductions aux cohomologies p-adiques (Luminy, 1982), Mém. Soc. Math. France (N.S.) 23 (1986), 7–32.

[7] S. Bosch, Lectures on Formal and Rigid Geometry, Lecture Notes in Math. 2105, Springer, Cham, 2014.

[8] S. Bosch, U. Güntzer, and R. Remmert, Non-Archimedean Analysis, Grundlehren der Math. Wiss. 261, Springer-Verlag, Berlin, 1984.

[9] A. Cadoret, La conjecture des compagnons d’après Deligne, Drinfeld, Lafforgue, T. Abe., ..., Exposé Bourbaki 1155, Astérisque 422 (2020), 173–223.

[10] C. Chin, Independence of ℓ in Lafforgue’s theorem, Adv. Math. 180 (2003), 64–86.

[11] C. Chin, Independence of ℓ of monodromy groups, J. Amer. Math. Soc. 17 (2004), 723–747.

[12] B. Conrad, Relative ampleness in rigid geometry, Ann. Inst. Fourier 56 (2006), 1049–1126.

[13] R. Crew, F-isocrystals and their monodromy groups, Ann. Scient. Éc. Norm. Sup. 25 (1992), 429–464.

[14] M. D’Addezio, The monodromy groups of lisse sheaves and overconvergent F-isocrystals, Sel. Math. 26 (2020).

[15] M. D’Addezio, Parabolicity conjecture of F-isocrystals, arXiv:2012.12879v3 (2022).

[16] A.J. de Jong, Smoothness, semi-stability, and alterations, Publ. Math. IHÉS 83 (1996), 51–93.

[17] R. Elkik, Solutions d’équations à coefficients dans un anneau hensélien, Ann. Scient. Éc. Norm. Sup. 6 (1973), 553–604.

[18] H. Esnault and M. Kerz, A finiteness theorem for Galois representations of function fields over finite fields (after Deligne), Acta Math. Vietnam. 37 (2012), 531–562.

[19] O. Gabber and L. Ramero, Anost Ring Theory, Lecture Notes in Math. 1800, Springer-Verlag, Berlin, 2003.

[20] E. Grosse-Klönne, Rigid analytic spaces with overconvergent structure sheaf, J. reine angew. Math. 519 (2000), 73–95.

[21] A. Grothendieck, Theorie de la specialization du groupe fondamental, Exposé X in Revêtements Étales et Groupe Fondamental (SGA 1), Lecture Notes in Math. 224, Springer-Verlag, Berlin, 1971.

[22] T. Grubb, K.S. Kedlaya, and J. Upton, A cut-by-curves criterion for overconvergence of F-isocrystals, arXiv:2202.03604v1 (2022).

[23] K. Kato, Logarithmic structures of Fontaine–Illusie, in Algebraic Analysis, Geometry, and Number Theory, Johns Hopkins University, Baltimore, 1988, 191–224.

[24] N. Katz, Slope filtrations of F-crystals, Journées de Géométrie Algébriques (Rennes, 1978), Astérisque 63 (1979), 113–164.

[25] N. Katz and T. Oda, On the differentiation of De Rham cohomology classes with respect to parameters, J. Math. Kyoto Univ. 8 (1968), 199–213.
[36] K.S. Kedlaya, Semistable reduction for overconvergent $F$-isocrystals, I: Unipotence and logarithmic extensions, *Compos. Math.* 143 (2007), 1164–1212; errata, [62].

[37] K.S. Kedlaya, *p*-adic Differential Equations, Cambridge Univ. Press, Cambridge, 2010.

[38] K.S. Kedlaya, Good formal structures for flat meromorphic connections, II: excellent schemes, *J. Amer. Math. Soc.* 24 (2011), 183–229.

[39] K.S. Kedlaya, Notes on isocrystals, *J. Num. Theory* 237 (2022), 353–394.

[40] K.S. Kedlaya, ´Etale and crystalline companions, I, arXiv:1811.00204v5 (2022); to appear in *Épijournal Géom. Alg.*

[41] T. Koshikawa, Overconvergent unit-root $F$-isocrystals and isotriviality, *Math. Res. Lett.* 24 (2017), 1707–1727.

[42] R.E. Kottwitz, Isocrystals with additional structure, *Compos. Math.* 56 (1985), 201–220.

[43] R.E. Kottwitz, Isocrystals with additional structure, II, *Compos. Math.* 109 (1997), 255–339.

[44] R. Krishnamoorthy and A. Pál, Rank 2 local systems and abelian varieties, *Sel. Math.* 27 (2021), paper no. 51.

[45] R. Krishnamoorthy and A. Pál, Rank 2 local systems and abelian varieties II, *Compos. Math.* 158 (2022), 868–892.

[46] L. Lafforgue, Chtoucas de Drinfeld et correspondance de Langlands, *Invent. Math.* 147 (2002), 1–241.

[47] V. Lafforgue, Estimations pour les valuations $p$-adiques des valeurs propres des opérateurs de Hecke, *Bull. Soc. Math. France* 139 (2011), 455–477.

[48] M. Larsen and R. Pink, Abelian varieties, $\ell$-adic representations, and $\ell$-independence, *Math. Ann.* 302 (1995), 561–580.

[49] G. Laumon and L. Moret-Bailly, *Champs Algébriques*, Springer, Berlin, 2000.

[50] M. Lieblich, Remarks on the stack of coherent algebras, *Int. Math. Res. Notices* (2006), article ID 75273.

[51] H. Matsumura, *Commutative Algebra*, second edition, Benjamin/Cummings, Reading, Mass., 1980.

[52] D. Meredith, Weak formal schemes, *Nagoya Math. J.* 45 (1971), 1–38.

[53] A. Ogus, $F$-isocrystals and de Rham cohomology II—convergent isocrystals, *Duke Math. J.* 51 (1984), 765–850.

[54] A. Ogus, *Lectures on Logarithmic Algebraic Geometry*, Cambridge Univ. Press, Cambridge, 2018.

[55] M. Olsson, Logarithmic geometry and algebraic stacks, *Ann. Sci. Éc. Norm. Sup.* 36 (2003), 747–791.

[56] A. Pál, The $p$-adic monodromy group of abelian varieties over global function fields of characteristic $p$, arXiv:1512.03587v1 (2015).

[57] M. Rapoport and M. Richartz, On the classification and specialization of $F$-isocrystals with additional structure, *Compos. Math.* 103 (1996), 153–181.

[58] J.-P. Serre, Géométrie algébrique et géométrie analytique, *Ann. Inst. Fourier* 6 (1956), 1–42.

[59] J.-P. Serre, *Galois Cohomology*, corrected second printing, Springer-Verlag, Berlin, 1997.

[60] J.-P. Serre, Lettres à Ken Ribet du 1/1/1981 et du 29/1/1981, in *Œuvres – Collected Papers IV*, Springer–Verlag, Heidelberg, 2000, 1–12.

[61] A. Shiho, Crystalline fundamental groups II—log convergent cohomology and rigid cohomology, *J. Math. Sci. Univ. Tokyo* 9 (2002), 1–163.

[62] A. Shiho, On logarithmic extension of overconvergent isocrystals, *Math. Ann.* 348 (2010), 467–512.

[63] A. Shiho, Cut-by-curves criterion for the overconvergence of $p$-adic differential equations, *Manuscripta Math.* 132 (2010), 517–537.

[64] The Stacks Project Authors, *Stacks Project*, [http://stacks.math.columbia.edu](http://stacks.math.columbia.edu) retrieved March 2019.

[65] N. Tsuzuki, Minimal slope conjecture of $F$-isocrystals, *Invent. Math.* 205 (2022).

[66] D. Wan, Dwork’s conjecture on unit root zeta functions, *Ann. Math.* 150 (1999), 867–927.

[67] D. Wan, Higher rank case of Dwork’s conjecture, *J. Amer. Math. Soc.* 13 (2000), 807–852.

[68] D. Wan, Rank one case of Dwork’s conjecture, *J. Amer. Math. Soc.* 13 (2000), 853–908.

[69] G. Wiesend, A construction of covers of arithmetic schemes, *J. Number Theory* 121 (2006), 118–131.