On the well posedness of
Robinson–Trautman–Maxwell solutions

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Abstract
We show that the so-called Robinson–Trautman–Maxwell equations do not constitute a well-posed initial-value problem. That is, the dependence of the solution on the initial data is not continuous in any norm built out from the initial data and a finite number of its derivatives. Thus, they cannot be used to solve for solutions outside the analytic domain.

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1. Introduction
One of the most challenging problems in GR is to construct non-stationary spacetimes for which a null boundary $\mathcal{S}^{\text{cri}}$ can be attached. Those spacetimes are particularly useful since the radiation fields at $\mathcal{S}^{\text{cri}}$ represent the gravitational and/or electromagnetic radiation emitted from some source [1–4]. Finding explicit solutions of the Einstein equations with this property, however, is a formidable task unless some extra conditions are assumed for the spacetime under consideration. In some cases a set of extra conditions simplifies the problem in such a way that either an explicit solution can be found or the resulting equations are sufficiently simple that they can be integrated numerically.

In particular, if we assume that the spacetime admits a shear-free and twist-free null congruence, the field equations adopt an extremely simple form. In 1960, Robinson and Trautman (RT from now on) obtained a class of Ricci flat spacetimes where all the field equations but one could be explicitly integrated [5]. This last equation gives the dynamical evolution of the radiating gravitational field at $\mathcal{S}^{\text{cri}}$, and the solution is regarded as a radiating spacetime since it has a non-trivial Bondi mass that obeys the Bondi mass loss equation and decays exponentially to a stationary spacetime. Although one could argue that gravitational waves should not decay exponentially but rather with a power-law formula, the RT solutions have been used as a working model and several stability theorems have been proved to show that it might be regarded as a physical solution [6–9].
It is an interesting problem to generalize the RT equations to include electromagnetic radiation and call Robinson–Trautman–Maxwell (RTM from now on) spacetimes to the new class of spacetimes. In this paper we address this problem, write the RTM equations and then show that if the total charge is different from zero the field equations are unstable against linear perturbations, but worst than that, the growth rate increases without bound as the wave number of the perturbation tends to infinity. Thus, by a simple argument we conclude that the RTM equations are not well posed and so they cannot be used as an ansatz to construct physical solutions.

In section 2 we write down the relevant field equations, whereas in section 3 we analyze the stability of those equations. The use of the stability theorems and its implications on the existence of those spacetimes is given in the conclusions.

2. The equations

The type II algebraically special spacetimes have two principal null directions. Thus, it is possible to find a null tetrad such that two of the scalars built out of the Weyl tensor by contracting it with the frame, usually denoted by $\psi_0$ and $\psi_1$, vanish. It can be shown that all of the radial equation of the Einstein–Maxwell field equations can be integrated leaving just the angular and time evolution to be solved.

If in addition we impose the extra condition that the tetrad is shear and twist free the resulting equations adopt a very simple form when written in a suitable coordinate system instead of the standard Bondi coordinates $(u, \varsigma, \overline{\varsigma})$ [10]. This particular coordinate system $(\tau, \varsigma, \overline{\varsigma})$ is called Newman–Unti and the relationship between the NU and Bondi time $\tau = T(u, \varsigma, \overline{\varsigma})$ is one of the basic variables for the problem. It can be shown that the Einstein–Maxwell field equations constitute a set of four equations for the variables $(\phi_1, \phi_2, V, \psi_2)$, where $\phi_1$ and $\phi_2$ represent the non-vanishing Maxwell scalars, $\psi_2$ is the Weyl scalar that represents the mass aspect and $V = X'$ with $X(\tau, \varsigma, \overline{\varsigma})$ the inverse function of $T$ and prime its derivative with respect to $\tau$. With this in mind we write the RTM equations and suitable definitions as follows.

Maxwell 1. Defining $q := \phi_1 V^2$, we write the first Maxwell equation as

$$\delta q = 0 \Rightarrow q = q(\tau).$$

Restricting the freedom left in $\tau$ we can set $q = \text{const.}$

Maxwell 2. We define $\Phi := \phi_2 V$ and the second Maxwell equation reads

$$\delta \Phi = -[q V^{-2}]',$$

from which we obtain an evolution equation for $V$:

$$2q V' = V^3 \delta \Phi.$$

GR 1. Starting with

$$V^{-3} \delta (\psi_2 V^3) = 2\kappa \phi_1 \overline{\phi}_2,$$

using $\phi_1 = \frac{\psi_2}{V}$ and defining $\chi := -\psi_2 V^3$, we can rewrite the above equation as

$$\delta \chi = -2\kappa q \overline{\Phi}.$$

An equivalent equation is given by

$$4\kappa q^2 V' + V^3 (\nabla^2 \chi) = 0,$$

with $\nabla^2 \chi = \delta \delta \chi$ the Laplacian operator on the unit sphere.
\begin{equation}
(V^{-3} \chi)' - [(\delta \delta^2)V + 2 \delta \delta V - V^{-1} \delta^2 V \cdot \delta^2 V] + \kappa V \phi_2 \phi_2 = 0,
\end{equation}

which can be rewritten as
\begin{equation}
4 \kappa q^2 \chi' + 3 \chi V^2(\nabla^2 \chi) - 4 \kappa q^2(V^3(\nabla^2 V + 2V^2) - V^2(\nabla^2 V)^2) + (\nabla \chi)^2 = 0.
\end{equation}

Thus, the non-trivial equations to solve are
\begin{align}
4 \kappa q^2 \chi' + 3 \chi V^2(\nabla^2 \chi) - 4 \kappa q^2(V^3(\nabla^2 V + 2V^2) - V^2(\nabla^2 V)^2) + (\nabla \chi)^2 &= 0,
\end{align}

3. The stability of the equations

To show that the equations are not well posed we study their linearized version, first of a constant solution, which allows a full treatment and then around any solution, using a theorem by Strang [11].

3.1. Lack of well posedness around constant solutions

To analyze the stability of the above set we first write down their linearized versions of the Reissner–Nordström solution (given by \( q = \text{const}, \chi = \chi_0 = \text{const}, V = 1 \)),
\begin{equation}
4 \kappa q^2 \chi' + 3 \chi_0 (\nabla^2 \chi) - 4 \kappa q^2(V^3(\nabla^2 V + 2V^2) - V^2(\nabla^2 V)^2) = 0,
\end{equation}
\begin{equation}
4 \kappa q^2 V' + (\nabla^2 \chi) = 0.
\end{equation}

Taking a time derivative on the top equation gives
\begin{equation}
4 \kappa q^2 \chi'' + 3 \chi_0 (\nabla^2 \chi') + (\nabla^6 \chi + 2 \nabla^4 \chi) = 0.
\end{equation}

Looking for a solution proportional to an eigenfunction of \( \nabla^2 \) with eigenvalue \( k^2 \), we get
\begin{equation}
4 \kappa q^2 \chi'' - 3 \chi_0 k^2 \chi' - (k^6 - 2k^4) \chi = 0,
\end{equation}
and so the solutions have a time dependence of the form \( e^{\alpha t} \),
\begin{equation}
4 \kappa q^2 \alpha^2 - 3 \chi_0 k^2 \alpha - (k^6 - 2k^4) = 0,
\end{equation}
with \( \alpha \) given by
\begin{equation}
\alpha = \frac{3 \chi_0 k^2 \pm \sqrt{(3 \chi_0 k^2)^2 + 16 \kappa q^2(k^6 - 2k^4)}}{8 \kappa q^2}.
\end{equation}

For fixed values of \( k \) and arbitrary values of \( t \), either positive or negative, one branch always blows up and the linear system obtained by studying perturbations around a constant solution is unstable. Worse than that, for any fixed value of \( t \), as \( k \to \infty \), both roots of \( \alpha \) blow up, i.e., the growth rate of the perturbation increases without bound as the wave number increases. This implies that the system is not well posed in the sense that the solution is not a continuous function of the initial data when the topology is given by a norm controlling a finite number of derivatives.

To see this, take a sequence of initial data of the form
\begin{equation}
\chi_k = \frac{1}{|k|^p} f_k(x), \quad \chi'_k = 0,
\end{equation}
where \( f_k(x) \) is an eigenfunction of \( \nabla^2 \) with eigenvalue \( k^2 \), and \( p \) is the maximum number of derivatives controlled by the norm. Thus, as \( k \to \infty \) the norm of the initial data remains
bounded, on the other hand, since the solution corresponding to this data is of the form $f_k(x)(e^{\alpha t} - e^{-\alpha t})$ the norm at time $T > 0$ would have grown by a factor $e^{\alpha T}$ (or by a factor $e^{-\alpha T}$ if $T < 0$). In both cases, the norm of the solution at $T$ would grow with no bound as $k \to \infty$ showing that there cannot be a bound of the solution in terms of the bound on the initial data for any finite time and for all initial data.

This result is not only valid for linear perturbations but, as we shall see below, extends around arbitrary smooth solutions. The reason is that by being essentially a high wave number phenomena we can consider an arbitrary small neighborhood of an arbitrary point in spacetime and localize the perturbation there. In such a small neighborhood, the background solution can be considered of constant coefficients and so the previous analysis is valid. The perturbation need not be of the order of the background solution to show ill posedness, for it is enough to see that the difference of solutions growth as in the linear case and that difference can be made arbitrarily small on the initial data.

Even if we find a real analytic solution of this equation, an arbitrarily small $C^\infty$ perturbation of initial data will produce an arbitrary large solution when we evolve that data with the field equations.

We now turn to the more general analysis of the stability around arbitrary smooth solutions.

3.2. Lack of well posedness around an arbitrary solution

To study the general case we invoke a theorem of Strang [11] asserting that if a linear, smooth coefficient system is well posed in the $L^2$ norm of its components, then the principal part of it with its coefficients frozen at any point must also be a (constant coefficient) well-posed system. This theorem illustrates the fact that the issue of well posedness is a microlocal problem or, equivalently, a high-frequency phenomena. We shall use Strang’s theorem in the double false way, that is, showing that if the principal part is not well posed, then neither is the variable coefficient system.

In order to apply Strang’s theorem we first take the gradient of the first equation of the full system (1), and then consider the principal part of that new system, which, for quasilinear systems, is the same as the one corresponding to their linearization, at an arbitrary point $p$:

$$
\tilde{V}' + \frac{V^3(p)}{4\sqrt{q}} \nabla^2 \nabla \chi = 0,
\chi' - V^3(p) \nabla^2 \nabla \cdot \tilde{V} = 0,
$$

where we have defined the new variable $\tilde{V} = \nabla V$, and so the vector of variables is now $(V, \tilde{V}, \chi)$.

The eigenvalues of this system are

$$
\alpha = \pm \frac{k^3}{2\sqrt{\lambda q} V^3(p)},
$$

that is, the limit for large $k$ of the ones previously found for linearizations of constant solutions with $V = 1$. We clearly see that the frozen coefficient system is ill posed and so, by Strang’s theorem, is the linearized equation of the full system around any smooth solution.

3.3. The RT branch

It is interesting to see why the RT equations can be recovered in the limit when the charge $q$ tends to zero, and so why are they well posed in the negative time direction. To see that we now assume that the solution can be written as

$$
\chi = \chi_0 + q^2 \chi_0.
$$
Thus, the linearized equations for $\tilde{\chi}$ and $V$ read
\[
4\kappa q^2 \tilde{\chi}'' + 3 \chi_0 (\nabla^2 \tilde{\chi}) - 4\kappa (\nabla^4 V + 2\nabla^3 V) = 0,
\]
\[
4\kappa V' + (\nabla^2 \chi) = 0.
\]
Taking a time derivative on the top equation gives
\[
4\kappa q^2 \tilde{\chi}''' + 3 \chi_0 (\nabla^2 \tilde{\chi}') + (\nabla^6 \tilde{\chi} + 2\nabla^4 \tilde{\chi}) = 0.
\]
The Fourier-transformed equation is
\[
4\kappa q^2 \tilde{\chi}'' - 3 \chi_0 k^2 \tilde{\chi}' - (k^6 - 2k^4) \tilde{\chi} = 0,
\]
the Laplace transform gives
\[
4\kappa q^2 \alpha^2 - 3 \chi_0 k^2 \alpha - (k^6 - 2k^4) = 0,
\]
so
\[
\alpha = \frac{3 \chi_0 k^2 \pm \sqrt{(3 \chi_0 k^2)^2 + 16\kappa q^2 (k^6 - 2k^4)}}{8\kappa q^2},
\]
i.e., we obtain the same roots as before. Taking $\chi_0 < 0$, and the positive root and then the limit $q \to 0$ yields the standard RT linearized solutions. Thus, in the limit, we see that there is a branch whose growth rate is bounded, in this case for positive times. Those are the ones which in that limit go to the usual RT linearized solutions. Even the structure of the RT solutions is puzzling, for one can only evolve them to the future but not to the past, while Einstein’s equations in their symmetric hyperbolic formulations allow for the evolution both into the past and future4.

4. Conclusions

We have shown that the RTM equations are not well posed and so they cannot be used as an ansatz to construct physical solutions beyond the real analytic domain. Any $C^\infty$ perturbation of an exact solution will produce an arbitrary large solution when we evolve the data with the RTM equations.

This is rather unexpected, since it is well known that the full set of Einstein’s equations without any special conditions is a well-posed problem under suitable gauge conditions. The problem seems to arise from the special condition assumed for the spacetime which forces some Weyl scalars to vanish throughout the evolution. This condition is not a typical local gauge condition on the initial data set. Thus, even if we can find a real analytic data for which a solution exists, an arbitrary $C^\infty$ perturbation would not preserve the algebraic condition and that would be reflected on the appearance of an unbound growth.

This lack of well posedness makes the system totally unsuited for finding numerical solutions. Any numerical scheme is prone to errors (particularly truncation errors) and even if we start with analytic data, the errors (considered as perturbations) would completely overcome the exact solution. There will not be any way of having a stable converging algorithm.

Moreover, the above analysis shows that if we perturb a Robinson–Trautman spacetime adding a small amount of electric charge, the whole construction blows up. Since we cannot prevent the presence of tiny amounts of excess charge in any compact source it appears that RT spacetimes do not represent physical sources.

4 In some cases, like the one where the initial surface extends to future null infinity, only future evolution is possible for the maximal domain of dependence, but even in this case local solutions on the causal domain of the initial data surface is possible.
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