DUALITY FOR $p$-ADIC ÉTALE TATE TWISTS WITH MODULUS

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Abstract. In this paper, we define $p$-adic étale Tate twists for a modulus pair $(X, D)$, where $X$ is a regular semi-stable family and $D$ is an effective Cartier divisor on $X$ which is flat over a base scheme. The main result of this paper is an arithmetic duality of $p$-adic étale Tate twists for proper modulus pairs $(X, D)$, which holds as a pro-system with respect to the multiplicities of the irreducible components of $D$.

Contents

1. Introduction 1
1.1. Logarithmic Hodge-Witt type sheaves ([Sat1], [Sat2], [Sat3]) 3
2. $p$-adic étale Tate twists with modulus 5
3. Structure of $\text{Ker}(\sigma^r_X|_{mD,n})$ 8
4. Logarithmic Hodge-Witt type sheaves with modulus 9
5. Explicit formula for $\mathcal{M}_m^r$ 21
5.1. Setting 22
5.2. Construction of $\Theta^r_D$ 22
5.3. Explicit formula for $\Theta^r_D$ 23
5.4. Proof of (⊛2) 28
6. Duality for $\mathcal{M}_m^r$ 28
6.1. Proof of Theorem 6.1 30
7. Proof of Theorem 1.1 33
8. Proof of non-degeneracy of the pairing in Theorem 1.1 37
8.1. Descending induction on $r$ 37
8.2. Proof of Proposition 39
9. Acknowledgements 42
References 42

1. Introduction

Let $p$ be a prime number. Let $K$ be a $p$-adic field and let $\mathcal{O}_K$ be its valuation ring with $k$ the residue filed. In recent years, theory of motives and the (higher) Chow group has been studied for a modulus pair $(X, D)$ ([KMSYI], [KMSYII], [KMSYIII], [BS], and [RS] etc.). One of the important questions with respect to the

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theory of motives with modulus is $p$-adic realization of it. In [Sat2] Conjecture 1.4.1 and [Sat3] Remark 7.2 imply that $p$-adic étale Tate twists have a deeply relation with motivic complexes.

In this paper, we introduce $p$-adic étale Tate twists $\mathcal{F}_n^{(r)}(X|D)$ for a modulus pair $(X, D)$, where $X$ is a regular semi-stable family over $\mathcal{O}_K$ and $D$ is an effective Cartier divisor on $X$ (we assume that $D$ is flat over $\mathcal{O}_K$). The object $\mathcal{F}_n(r|D)$ is a generalization of $p$-adic étale Tate twists (cf. [Sch], [Sat2]). We prove arithmetic duality of $\mathcal{F}_n^{(r)}(X|D)$ for proper modulus pairs $(X, D)$, where $X$ is a regular semi-stable family and proper over $\mathcal{O}_K$.

We proved the following result in this paper:

**Theorem 1.1.** Let $r \leq p - 2$ and $\zeta_p \in K$. We put $Y^0$ a set of generic points of $Y$. We assume that $X$ is a semi-stable family and proper over $\mathcal{O}_K$ and $|D| \cap Y^0 = \emptyset$.

The pairing

$$\lim_{m} H^i_\text{ét}(X, \mathcal{F}_n^{(r)}(X|mD)) \times H^{2d+1-i}(U, \mathcal{F}_n(d-r)U) \rightarrow \mathbb{Z}/p^n,$$

is a non-degenerate pairing for any $i \geq 0$ and $n \geq 0$.

When $D = \emptyset$ in Theorem 1.1 we have Theorem 1.2.2 in [Sat2]. Here $\mathcal{F}_n^{(d-r)}U$ is $p$-adic étale Tate twists, which is defined in [Sat2]. We prove Theorem 1.1 along the strategy of Theorem 1.2.2 in [Sat2]. The key idea of the construction of pairing in Theorem 1.1 is a Milne’s pairing (a pairing of two-term complexes) in [Mil2], p.175 or [Mil1], p.217. This construction idea is also applied to the proof of Theorem 4.1.4 in [JSZ]. We construct the pairing in Theorem 1.1 by extending Milne’s pairing to a pairing of multiple-term complexes. We introduce some logarithmic Hodge-Witt type sheaves for modulus pairs $(X, D)$ to show Theorem 1.1 and show its duality property.

An application of Theorem 1.1, we have the following Corollary (Take $i = 2d$ and $r = d$ in Theorem 1.1):

**Corollary 1.2.** (cf. [JSZ]) Under the same assumption of Theorem 1.1, we obtain a natural isomorphism

$$\lim_{m} H^{2d}(X, \mathcal{F}_n(d)X|mD) \cong \pi_1^{ab}(U)/p^n.$$

Here $\pi_1^{ab}(U)$ is the abelianized fundamental group of $U$.

**Notation and Conventions.**

(i) Let $K$ be a $p$-adic field, $\mathcal{O}_K$ denotes the integer ring of $K$, and $\pi_K$ denotes the prime element of $\mathcal{O}_K$. We denote $k$ the residue field of $\mathcal{O}_K$ which is a perfect field of characteristic $p$.

(ii) Let $p$ be a prime number, and let $\zeta_p$ be a primitive $p$-the roots of unity.

(iii) Unless indicated otherwise, all cohomology groups of schemes are taken over the étale topology. For a scheme $X$ and an integer $n \geq 2$, $D(X_\text{ét}, \mathbb{Z}/n)$ (resp. $D^b(X_\text{ét}, \mathbb{Z}/n)$) denotes the derived category of (resp. bounded) complexes of étale $\mathbb{Z}/n$-sheaves on $X_\text{ét}$. 
(iv) Throughout this paper, we assume that a scheme \( X \) is always separated over \( \mathcal{O}_K \).

(v) Let \( p \) be a prime number. For a scheme \( X \), we put \( X_n := X \otimes \mathbb{Z}/p^n\mathbb{Z} \).

(vi) Let \( k \) be a field, and let \( X \) be a pure-dimensional scheme which is of finite type over \( \text{Spec}(k) \). We call \( X \) a normal crossing scheme over \( \text{Spec}(k) \), if it is everywhere étale locally isomorphic to

\[
\text{Spec}(k[T_0, \ldots T_N]/(T_0 \cdots T_N)),
\]

for some integer \( a \) with \( 0 \leq a \leq N := \dim(X) \).

(vii) Let \( X \) be a pure-dimensional scheme which is flat of finite type over \( \text{Spec}(\mathcal{O}_K) \).

We call \( X \) be a regular semistable family over \( \text{Spec}(\mathcal{O}_K) \), if it is regular and everywhere étale locally isomorphic to

\[
\text{Spec}(\mathcal{O}_K[T_0, \ldots T_a]/(T_0 \cdots T_a - \pi_K)),
\]

for some \( a \) such that \( 0 \leq a \leq d := \dim(X/\mathcal{O}_K) \).

(viii) For a positive integer \( n \) invertible on a scheme \( X \), \( \mu_n \) denotes the étale sheaf of \( n \)-th roots of unity.

We assume the following situation (♣) :

- Let \( X \) be a regular semistable family over \( \text{Spec}(\mathcal{O}_K) \). We denote \( Y := X \otimes_{\mathcal{O}_K} k \) and \( X_K := X \otimes_{\mathcal{O}_K} K \). Let \( D \subset X \) be an effective Cartier divisor on \( X \) which is flat over \( \text{Spec}(\mathcal{O}_K) \) and \( Y \cup D_{\text{red}} \) has normal crossings on \( X \).
- \( M_X \) be a logarithmic structure on \( X \) associated with \( D_{\text{red}} \cup Y \). Let \( M_D \) be a logarithmic structure on \( D \) induced by the inverse image of \( M_X \). For \( n \geq 1 \), we write \( M_{X_n} \) for the inverse image of log structure of \( M_X \) into \( X_n \). Let \( (\bar{X}, M_X) \) be the reduction of mod \( \pi \) of \( (X, M_X) \).

Put \( U := X - D, V := Y - (Y \cap D), U_K := X - (Y \cup D) \), and \( V_X := X - Y \), and we consider a diagram of immersions.

\[
\begin{array}{ccc}
Y & \xrightarrow{i} & X \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
V & \xrightarrow{i'} & U & \xrightarrow{j'} & U_K \\
\downarrow{\psi} & & \downarrow{\psi} \\
V_X & \xrightarrow{j} & &
\end{array}
\]

1.1. Logarithmic Hodge-Witt type sheaves \( [\text{Sat1}], [\text{Sat2}], [\text{Sat3}] \).

Throughout this subsection, \( n \) denotes a non-negative integer and \( r \) denotes a positive integer. Let \( k \) be a perfect field of positive characteristic \( p \). Let \( X \) be a pure-dimensional scheme of finite type over \( \text{Spec}(k) \). We define the étale sheaves \( \nu_{X,r}^n \) and \( \lambda_{X,r}^n \) on \( X \) as follows:

\[
\nu_{X,r}^n := \text{Ker} \left( \bigoplus_{x \in X^0} i_{x*}W_r\Omega_{x,\log}^n \xrightarrow{j_{\text{red}}} \bigoplus_{x \in X^1} i_{x*}W_r\Omega_{x,\log}^{n-1} \right)
\]
\[ (1.2) \quad \lambda_{X,r}^n := \text{Im} \left( (\mathbb{G}_{m,X})^\otimes x \xrightarrow{d\log} \bigoplus_{x \in X^0} i_{x*} W_r \Omega_{X,\log}^n \right) \]

where \( i_x \) denote the canonical map \( x \hookrightarrow X \) for a point \( x \in X \) and \( d^{\text{val}} \) denote the sum of \( d^{\text{val}}_{y,x} \)'s with \( y \in X^0 \) and \( x \in X^1 \) (see \[ \text{Sat2}, \, \S 1.8 \]). Here \( W_r \Omega_{X,\log}^n \) denotes the étale subsheaf of \( W_r \Omega_X^n \), which defined in \[ B \] and \[ II \] for a smooth variety over \( \text{Spec}(K) \). It is obvious that \( \lambda_{X,r}^n \) is a sub sheaf of \( \nu_{X,r}^n \). If \( X \) is smooth then both \( \nu_{X,r}^n \) and \( \lambda_{X,r}^n \) agree with the sheaf \( W_r \Omega_{X,\log}^n \). We next review the duality result in \[ \text{Sat1} \]. For integers \( m, n \geq 0 \), there is a natural pairing of sheaves
\[ (1.3) \quad \nu_{X,r}^n \times \lambda_{X,r}^m \to \nu_{X,r}^n \]
(see loc. cit., 3.1.1).

**Theorem 1.3.** (Duality, loc. cit., 1.2.2). Let \( k \) be a finite field, and let \( X \) be a normal crossing scheme of dimension \( N \) which is proper over \( \text{Spec}(k) \). Then for integers \( q \) and \( n \) with 0 \( \leq n \leq N \), the natural mapping
\[ (1.4) \quad H^q(X, \nu_{X,r}^n) \times H^{N+1-q}(X, \lambda_{X,r}^{N-n}) \to H^{N+1}(X, \nu_{X,r}^n) \to \mathbb{Z}/p^r\mathbb{Z} \]
is non-degenerate pairing of finite \( \mathbb{Z}/p^r\mathbb{Z} \)-modules.

Here trace map \( \text{tr}_X \) is constructed in \[ \text{Sat1}, \, \text{Proposition 2.5.9 and Corollary 2.5.11} \]. In \[ \text{Sat3} \], Sato have introduced a following two new log arithmic sheaves under the condition "with Cartier divisor \( D \)":

**Definition 1.4.** (\[ \text{Sat3}, \, \S 2, \, \text{Definition 2.1} \])
\[ (1.5) \quad \nu_{(Y,Y \cap D),n}^q := \text{Ker} \left( \bigoplus_{y \in V^0} i_{y*} W_r \Omega_y^q \xrightarrow{d^{\text{val}}} \bigoplus_{y \in V^1} i_{y*} W_r \Omega_y^{q-1} \right), \]
\[ (1.6) \quad \lambda_{(Y,Y \cap D),n}^q := \text{Im} \left( d\log : (\alpha'_* \mathcal{O}_Y)^X \to \bigoplus_{y \in V^0} i_{y*} W_n \Omega_y^q \right). \]

Here \( V := Y - (Y \cap D) \) and \( i_y : y \to Y (y \in Y) \) denotes the composite map \( y \hookrightarrow V \hookrightarrow Y \).

We will introduce a new logarithmic sheaves as a subsheaf of these sheaves in section below.

**Definition 1.5.** (\( p \)-adic étale Tate twist \[ \text{Sat2}, \, \text{Definition 4.2.4} \])
Sato have defined the \( p \)-adic étale Tate twist \( \Sigma_n(r)_X \) by the following distinguished triangle:
\[ i_* \nu_{Y,n}^{r-1}[-r-1] \to \Sigma_n(r)_X \to \tau_{\leq r} Rj_* \mu_{p^n}^{\otimes r} \to i_* \nu_{Y,n}^{r-1}[-r] \]
in \( D^b(\mathbb{X}_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z}) \).

The \( p \)-adic étale Tate twist \( \Sigma_n(r)_X \) has some good properties (trivialization, acyclicity, purity, compatibility, product structure, etc.). See \[ \text{Sat2} \] for the details of properties of \( p \)-adic étale Tate twist.
2. \( p \)-adic étale Tate twists with modulus

We assume that \( 0 \leq r \leq p - 2 \) and \( X \) is in the setting \((\mathfrak{a})\). The aim of this section is to construct \( p \)-adic étale Tate twists with modulus \( \mathfrak{S}_n(r)_{X|D} \in D^b(Y_{\text{et}}, \mathbb{Z}/p^n \mathbb{Z}) \). We first review the definition of the syntomic complexes with modulus \( s_n(r)_{X|D} \in D^b(Y_{\text{et}}, \mathbb{Z}/p^n \mathbb{Z}) \), which is constructed in [Y]. To define the syntomic complex with modulus in local situation, we assume the existence of the following four datas:

**Assumption 2.1.**

- There exists exact closed immersions
  \[ \beta_n : (X_n, M_{X_n}) \hookrightarrow (Z_n, M_{Z_n}) \text{ and } \beta_{n,D} : (D_n, M_{D_n}) \hookrightarrow (\mathcal{D}_n, M_{\mathcal{D}_n}) \]
  of log schemes for \( n \geq 1 \) such that \( (Z_n, M_{Z_n}) \) and \( (\mathcal{D}_n, M_{\mathcal{D}_n}) \) are smooth over \( W := W(k) \), and such that the following diagram is cartesian:

\[
\begin{array}{ccc}
X_n & \hookrightarrow & Z_n \\
\downarrow & & \downarrow \\
D_n & \hookrightarrow & \mathcal{D}_n \\
\end{array}
\]

- There exist a compatible system of liftings of Frobenius endomorphisms \( \{F_{Z_n} : (Z_n, M_{Z_n}) \to (Z_n, M_{Z_n})\} \) and \( \{F_{\mathcal{D}_n} : (\mathcal{D}_n, M_{\mathcal{D}_n}) \to (\mathcal{D}_n, M_{\mathcal{D}_n})\} \) for each \( n \in \mathbb{N} \) ([Isu1, p.71, (2.1.1)–(2.1.3)]).

- The systems \( \{F_{Z_n} : (Z_n, M_{Z_n}) \to (Z_n, M_{Z_n})\} \) and \( \{F_{\mathcal{D}_n} : (\mathcal{D}_n, M_{\mathcal{D}_n}) \to (\mathcal{D}_n, M_{\mathcal{D}_n})\} \) fit into the following commutative diagram for each \( n \in \mathbb{N} \):

\[
\begin{array}{ccc}
(Z_n, M_{Z_n}) & \xrightarrow{F_{Z_n}} & (Z_n, M_{Z_n}) \\
\downarrow & & \downarrow \\
(\mathcal{D}_n, M_{\mathcal{D}_n}) & \xrightarrow{F_{\mathcal{D}_n}} & (\mathcal{D}_n, M_{\mathcal{D}_n})
\end{array}
\]

- \( \mathcal{D}_n \) is an effective Cartier divisor on \( Z_n \) such that \( \beta_{D,n}^* \mathcal{D}_n = D_n \) and \( F_{Z_n} \) which induces a morphism \( \mathcal{D}_n \to \mathcal{D}_n \).

In [Y], the following complex was defined for a modulus pair \((X, D)\):

**Definition 2.2.** (Definition 2.11, [Y] syntomic complex with modulus)

We assume \( r \leq p - 1 \). We define

\[ s_n(r)_{X|D}(Z_n, M_{Z_n}, \mathcal{D}_n, M_{\mathcal{D}_n}) := \text{Cone}(1 - \varphi_r : J_{\mathcal{D}_n}^{[r-1]} \otimes \mathcal{O}_{\mathcal{D}_n} \otimes \omega_{Z_n|\mathcal{D}_n} \to \mathcal{O}_{\mathcal{D}_n} \otimes \mathcal{O}_{\mathcal{D}_n} \otimes \omega_{Z_n|\mathcal{D}_n})[-1], \]

where \( \varphi_r = \varphi_{r-q} \otimes \wedge^{q+\text{deg}(\varphi)}p \) in degree \( q \). We denote this complex by \( s_n(r)_{X|D} \) for simplicity. We put \( \omega^q_{Z_n} := \Omega^q_{Z_n/W_n} (\log M_{Z_n}) \) and put \( \omega^q_{\mathcal{D}_n} := \omega^q_{Z_n} \otimes \mathcal{O}_{\mathcal{D}_n} \)
\( \mathcal{O}_{Z_n}(-\mathcal{D}_n) \) \( (q \geq 0) \) which are locally free \( \mathcal{O}_{Z_n} \)-modules.
When we consider the global situation of the syntomic complex with modulus, we can patch the another complex $\mathcal{L}_n(q)_{X,D}^{\text{loc}}$, which is quasi-isomorphic to $s_n(r)_{X,D}$ in local situation (see [1], Lemma 2.12). There is a product structure of syntomic complex $s_n(r)_{X,D}$:

**Definition 2.3.** (Product structure of $s_n(r)_{X,D}$, cf. §2.2 [1su2]) For $q, q' \geq 0$, we define a product morphism in $D^-(Y_{\text{et}}, \mathbb{Z}/p^n\mathbb{Z})$ as follows:

\[
s_n(q)_{X,D} \otimes L s_n(q')_{X,D} \to s_n(q + q')_{X,D}.
\]

\[(x, y) \otimes (x', y') \mapsto (x\cdot x', (-1)^j xy' + yq'(x')),
\]

where

\[(x, y) \in s_n^j(q)_{X,D} = \left( J^{[q-j]}_\mathcal{E}_n \otimes \mathcal{O}_{Z_{n}} \omega_{Z_{n}}^j \right) \oplus \left( \mathcal{O}_{\mathcal{E}_n} \otimes \mathcal{O}_{Z_{n}} \omega_{Z_{n}}^{j-1} \right)
\]

and

\[(x', y') \in s_n^j(q')_{X,D} = \left( J^{[q'-j']}_\mathcal{E}_n \otimes \mathcal{O}_{Z_{n}} \omega_{Z_{n}}^{j'} \right) \oplus \left( \mathcal{O}_{\mathcal{E}_n} \otimes \mathcal{O}_{Z_{n}} \omega_{Z_{n}}^{j'-1} \right).
\]

We define the following logarithmic Hodge-Witt type sheaf for a moduli pair $(X, D)$, which plays an important role in the proof of main result of this paper.

**Definition 2.4.** For $q \geq 1$, we define

\[
\nu_{Y,D,n}^{q-1} := \text{im} \left( H^q \left( s_n(q)_{X,D} \right) \otimes i^* R^q \psi_{1\mu} \sigma_{(X,D)} \nu_{Y,D,n}^{q-1} \right).
\]

Here morphism $\theta$ is induced by the map $s_n(q)_{X,D} \to S_n(q)_{(X,M_X)}$ and the map $\sigma_{(X,D)}$ is defined in [Sat3].

**Remark 2.5.** When $D_s = \emptyset$, we have $\nu_{Y,D,n}^{q-1} = \nu_{Y,n}^{q-1}$, which is considered in Sato’s paper [Sat1]. The last morphism $\sigma_{(X,D)}$ is surjective by [Sat3], Theorem 3.4. When $Y \subset D_s$, we have $\nu_{Y,D,n}^{q-1} = 0$.

Let $\sigma_{X,D,n}^r$ be the morphism $s_n(r)_{X,D} \to \nu_{Y,D,n}^{q-1}[-r]$ in $D^b(Y_{\text{et}}, \mathbb{Z}/p^n\mathbb{Z})$, which naturally induced by the definition of $\nu_{Y,D,n}^{q-1}$. This is an analogue of the map $\sigma_{X,n}^r$ in (3.2.5), [Sat2].

**Lemma 2.6.** Suppose $r \geq 1$, and let

\[
\nu_{Y,D,n}^{q-1}[-r - 1] \to \mathcal{K} \to \varphi_{Y,D,n}^{r-1} \to \nu_{Y,D,n}^{q-1}[-r]
\]

be a distinguished triangle in $D^b(Y_{\text{et}}, \mathbb{Z}/p^n\mathbb{Z})$. Then $\mathcal{K}$ is concentrated in $[0, r]$, the triple $(\mathcal{K}, t, g)$ is unique up to a unique isomorphism and $g$ is determined by the pair $(\mathcal{K}, t)$.

**Proof.** The map $\sigma_{X,D,n}^r$ is surjective by the definition of $\nu_{Y,D,n}^{q-1}$. Since $\mathcal{K}$ is acyclic outside of $[0, r]$, by using Lemma 2.11 in [Sat2], we can compute $\text{Hom}_{D^b(Y_{\text{et}}, \mathbb{Z}/p^n\mathbb{Z})} \left( \mathcal{K}, \nu_{Y,D,n}^{q-1}[-r - 1] \right) = 0$. Then there is no non-zero morphism from $\mathcal{K}$ to $\nu_{Y,D,n}^{q-1}[-r - 1]$. The uniqueness assertion follows from this fact and Lemma 2.12 (3) in [Sat2].
**Definition 2.7.** For \( r \geq 1 \), we fix a pair \((\mathcal{K}, t)\) fitting into a distinguished triangle of the form (2.6), and define \( \mathcal{K} =: \mathcal{T}_n(r)^\text{syn}_{X|D} \).

**Lemma 2.8.** (Product Structure of \( \mathcal{T}_n(r)^\text{syn}_{X|D} \), cf. [Sat2]) For \( r, r' \geq 0 \), there is a unique morphism

\[
\{\mathcal{T}_n(r)^\text{syn}_{X|mD}\}_m \rightarrow \{\mathcal{T}_n(r+r')^\text{syn}_{X|mD}\}_m \quad \text{in} \quad D^- (\text{Y}^\text{et}, \mathbb{Z}/p^n\mathbb{Z})
\]

**Proof:** We show the assertion by the same argument in Prop 4.2.6 in [Sat2]. Put \( \mathcal{P} := \mathcal{T}_n(r)^\text{syn}_{X|mD} \otimes \mathcal{T}_n(r')^\text{syn}_{X|mD} \). By definition of \( \mathcal{T}_n(r+r')^\text{syn}_{X|mD} \) and Lemma 2.1.2 (i) in [Sat2], it suffices to show that the following composite morphism is zero in \( D^- (\text{Y}^\text{et}, \mathbb{Z}/p^n\mathbb{Z}) \):

\[
\mathcal{P} \rightarrow s_n(r)_{X|mD} \otimes s_n(r')_{X|mD} \rightarrow s_n(r+r')_{X|mD} \xrightarrow{\sigma_{X|D,n(r+r')}} \nu_{Y|D,n}^{r+r'-1} [-r-r'],
\]

where the second arrow is product structure of \( s_n(r)_{X|mD} \) (Lemma 2.3). Because \( \mathcal{P} \) is concentrated in degrees \([0, r + r']\), this composite morphism is determined by the composite map of the \((r + r')\)-th cohomology sheaves (Lemma 2.1.1 in [Sat2]). Then we consider the following composition maps:

\[
(2.5)
\mathcal{H}^{r+r'}(\mathcal{P}) \rightarrow \mathcal{H}^{r+r'}(s_n(r)_{X|mD}) \otimes \mathcal{H}^{r+r'}(s_n(r')_{X|mD}) \rightarrow \mathcal{H}^{r+r'}(s_n(r+r')_{X|mD}) \xrightarrow{\sigma_{X|D,n(r+r')}} \nu_{Y|D,n}^{r+r'-1}.
\]

The image of \( \mathcal{H}^{r+r'}(\mathcal{P}) \approx F_{A/m} \otimes F_{A/m} \) into \( \mathcal{H}^{r+r'}(s_n(r + r')_{X|mD}) \) is contained in \( F_{A/m}^{r+r'} \). Hence this composite map is zero by \((b)\) in Theorem 3.2 below.

We define a \( p \)-adic étale Tate twist for a modulus pair \((X, D)\):

**Definition 2.9.** (\( p \)-adic étale Tate twist with modulus)

We define \( \mathcal{T}_n(r)_{X|D} \) by the following homotopy square:

\[
\begin{array}{ccc}
\mathcal{T}_n(r)_{X|D} & \longrightarrow & \tau_{\leq r} R\psi_\ast \mu_{p^n}^{\otimes r} \\
\downarrow & & \downarrow \\
i_\ast \mathcal{T}_n(r)^\text{syn}_{X|D} & \longrightarrow & i_\ast \mathcal{S}_n(r)_{(X,M_X)},
\end{array}
\]

where the right vertical morphism is composite adjunction morphism and isomorphism in [Išu3] (here we use the assumption \( 0 \leq r \leq p - 2 \)):

\[
\tau_{\leq r} R\psi_\ast \mu_{p^n}^{\otimes r} \xrightarrow{\text{adj.}} \tau_{\leq r} (i_\ast i^\ast R\psi_\ast \mu_{p^n}^{\otimes r}) \cong i_\ast \mathcal{S}_n(r)_{(X,M_X)}.
\]

Note that by the definition of \( \mathcal{T}_n(r)_{X|D} \), there is a distinguished triangle:

\[
(2.6)
\mathcal{T}_n(r)_{X|D} \rightarrow i_\ast \mathcal{T}_n(r)^\text{syn}_{X|D} \oplus \tau_{\leq r} R\psi_\ast \mu_{p^n}^{\otimes r} \rightarrow i_\ast \mathcal{S}_n(r)_{(X,M_X)} \rightarrow \mathcal{T}_n(r)_{X|D}[1].
\]
When \( D = \emptyset \), we write \( \Xi_n(r)_X \) for \( \Xi_n(r)_X|_{\emptyset} \), which agrees with that in [Sat2] §4.

We call \( \Xi_n(r)_X|_D \) a \( p \)-adic étale Tate twist with modulus.

We have the cohomology sheaf of \( \Xi_n(r)_X|_D \):

**Lemma 2.10.**

(2.7)

\[
\mathcal{H}^q\left(\Xi_n(r)^{\text{syn}}_X|_D\right) \cong \begin{cases} 
\mathcal{H}^q\left(s_n(r)_X|_D\right) & (0 \leq q \leq r - 1), \\
\text{Ker}\left(\mathcal{H}^q\left(s_n(r)_X|_D\right) \xrightarrow{\mathcal{H}^r\left(\sigma_{X|D,n}\right)} \nu_{Y|D,n}^{r-1}\right) & (q = r).
\end{cases}
\]

**Proof:** This follows from the long exact sequence of cohomology sheaves associated with a distinguished triangle (2.6). The details are straightforward and left to reader. \( \square \)

## 3. Structure of Ker(\( \sigma^r_{X|MD,n} \))

In [Y], Theorem 3.5, we have the theorem on the structure of \( \mathcal{H}^r\left(s_n(r)_X|_D\right) \).

We put \( \mathcal{M}^r_{n,m} := \mathcal{H}^r\left(s_n(r)_X|_D\right) \). We define the étale subsheaf \( F.\mathcal{M}^r_{n,m} \) of \( \mathcal{M}^r_{n,m} \) as the part generated by the image of \( U^1.\mathcal{M}^r_{n,m} \) and \( i^* \left( 1 + \mathcal{O}_X(-mD) \right)^X \otimes i^*(\alpha_\mathcal{O}_U^X) \otimes \nu^{r-1} \). Here \( U^1.\mathcal{M}^r_{n,m} \) is defined in Definition 3.1, [Y] as for [Ts2].

**Theorem 3.1.** (cf. Theorem 3.4.2, [Sat2]) The map \( \sigma^r_{X|MD,n} \) induces an isomorphism

\[
\langle \psi \rangle \quad \{\mathcal{M}^r_{n,m}/F.\mathcal{M}^r_{n,m}\}_m \xrightarrow{\cong} \{\nu^{r-1}_{Y|MD,n}\}_m,
\]

that is, \( \{F.\mathcal{M}^r_{n,m}\}_m = \{\text{Ker}(\sigma^r_{X|MD,n})\}_m \).

Furthermore there is an exact sequence

\[
0 \rightarrow \{U^1.\mathcal{M}^r_{n,m}\}_m \rightarrow \{F.\mathcal{M}^r_{n,m}\}_m \rightarrow \{\lambda^r_{Y|MD}\}_m \rightarrow 0,
\]

i.e. \( \langle \psi \rangle \quad \{F.\mathcal{M}^r_{n,m}/U^1.\mathcal{M}^r_{n,m}\}_m \xrightarrow{\cong} \{\lambda^r_{Y|MD}\}_m\) .

**Proof:** By the Theorem 3.7 and 3.8 in [Y], we have \( U^0.\mathcal{M}^r_{n,m} = \mathcal{M}^r_{n,m} \) and a short exact sequence

\[
0 \rightarrow \{\text{gr}^0_U.\mathcal{M}^r_{n,m}\}_m \rightarrow \{\text{gr}^0.\mathcal{M}^r_{n,m}\}_m \rightarrow \{\text{gr}^0.\mathcal{M}^r_{n,m}\}_m \rightarrow 0
\]

as a pro-system. Then we have the natural adjunction map

\[
\{\mathcal{M}^r_{n,m}/U^1.\mathcal{M}^r_{n,m}\}_m \rightarrow \bigoplus_{y \in Y^0} i^{y*} \{\mathcal{M}^r_{n,m}/U^1.\mathcal{M}^r_{n,m}\}_m
\]

as a pro-system which is injective. We consider the following commutative diagram:

\[
0 \rightarrow \{\text{Ker}(\sigma_{X|D})/U^1.\mathcal{M}^r_{n,m}\}_m \rightarrow \{\mathcal{M}^r_{n,m}/U^1.\mathcal{M}^r_{n,m}\}_m \rightarrow \bigoplus_{y \in Y^0} i^{y*} \{\mathcal{M}^r_{n,m}/U^1.\mathcal{M}^r_{n,m}\}_m \rightarrow 0
\]
Here the bottom horizontal row induced by the direct decomposition \(\{\mathcal{M}_{n,m}/U^{1}.\mathcal{M}_{n,m}\}_{m} \cong \{\Omega_{Y|mD,\log}^{r} \oplus \Omega_{mD,\log}^{r-1}\}_{m}\) by Theorem 3.8 in \([Y]\) under the assumption \(Y\) is smooth over \(\text{Spec}(k)\). Since the central map is an injective, the map \(\tau_{D}\) is also. We next prove that the image of the map \(\tau_{D}\) is contained in \(\lambda_{Y|D}^{r}\). Here, we define the sheaf

\[
\lambda_{Y|D,n}^{r} := \text{Im} \left( (1 + \mathcal{O}_{Y}(-D_{s}))^{\times} \otimes (\alpha_{s}^{\ast} \mathcal{O}_{Y}^{\times})^{\otimes r-1} \cdot d \log_{\gamma} \bigoplus_{y \in V^{0}} W_{n} \Omega_{y,\log}^{r} \right).
\]

The exact sequence (3.12) in Proposition below, we have an isomorphism

\[
\lambda_{Y|mD_{s}}^{r} \cong \text{Ker} \left( a_{1} \cdot \Omega_{Y|D_{s}}^{r(1)} \rightarrow a_{2} \cdot \Omega_{Y|D_{s}}^{r(2)} \right).
\]

We consider the following commutative diagram as a pro-system:

\[
\begin{array}{ccc}
\text{Ker}(\sigma_{X|D}) & \longrightarrow & a_{1} \cdot \Omega_{Y|D_{s}}^{r(1)} \\
\downarrow & & \downarrow \\
\text{Ker}(\sigma_{X,D}) & \longrightarrow & a_{2} \cdot \Omega_{Y|D_{s}}^{r(2)}
\end{array}
\]

Here the bottom horizontal row induced by Theorem in \([Sat3]\), which is a zero map. Since the right vertical map is injective, the upper horizontal row is zero map. Therefore we have an inclusion \(\text{Im}(\tau_{D}) \subset \lambda_{Y|D}^{r}\). Finally, we show that the isomorphisms (b) and (\(\phi\)). We have the following injective maps:

\[
\left\{ \frac{F.\mathcal{M}_{n,m}}{U^{1}.\mathcal{M}_{n,m}} \right\}_{m} \hookrightarrow \left\{ \text{Ker}(\sigma_{X|D}) \right\}_{m} \\
\hookrightarrow \left\{ \lambda_{Y|mD_{s}}^{r} \right\}_{m}
\]

The sheaf \(F.\mathcal{M}_{n,m}/U^{1}.\mathcal{M}_{n,m}\) is generated by symbols which form \(i^{\ast} (1 + \mathcal{O}_{X}(-mD))^{\times} \otimes i^{\ast} (\alpha_{s}^{\ast} \mathcal{O}_{Y}^{\times})^{\otimes r-1}\). Since \(\lambda_{Y|D_{s}}^{r}\) is generated by a symbol form \((1 + \mathcal{O}_{Y}(-D_{s}))^{\times} \otimes (\alpha_{s}^{\ast} \mathcal{O}_{Y}^{\times})^{\otimes r-1}\), the two maps are bijective. Thus we have \(\{F.\mathcal{M}_{n,m}\}_{m} = \{\text{Ker}(\sigma_{X|mD})\}_{m}\). This completes the proof.

**Remark 3.2.** When \(D = \emptyset\) in Theorem , we have Theorem 3.4.2, \([Sat2]\).

### 4. Logarithmic Hodge-Witt type sheaves with modulus

In this section, we prove the structure and the duality theorem of \(\nu_{Y|D_{s},n}^{r-1}\) and introduce a sheaf \(\lambda_{Y,n}^{r}\). The duality of \(\nu_{Y|D_{s},n}^{r-1}\) and \(\lambda_{Y,n}^{r}\) plays an important role in our proof of the main result(see §below).

**Lemma 4.1.** If \(Y\) is smooth, we have \(\nu_{Y|D_{s},n}^{r-1} \cong W_{n} \Omega_{Y|D_{s,\log}}^{r-1}\) with respect to the multiplicity of the irreducible components of \(D\).
We define $\theta$ here and let $\tilde{D} := \tilde{D} \times_Y \tilde{Y}$. Then we have a short exact sequence

$$0 \to \nu_{Y|D_s,n}^r \to \nu_{Y,n}^r \to \beta_s^r \nu_{D_s,n}^r \to 0.$$  

Here the last map is a restriction.

**Proof.** We put $\mathcal{K}_{Y|D_s}^{r,-1} := \text{Ker}(\nu_{Y,n}^r \to \beta_s^r \nu_{D_s,n}^r)$. By the definition of $\nu_{Y|D_s,n}^r$ and the assumption of $|D| \cap Y^0 = \emptyset$, we have $\nu_{Y|D_s,n}^r = \nu_{Y,n}^r$. Let $Y^{\text{sing}}$ be the singular locus of $Y$ and let $j : \tilde{Y} := Y \setminus Y^{\text{sing}} \hookrightarrow Y$ be the open immersion. We define $D_s := D_s \times_Y \tilde{Y}$. There is a commutative diagram of exact rows:

Here the injectivity of the middle vertical (embedded) map and the right vertical map are due to [Sat1]. Then we have a relation (*) $\mathcal{K}_{Y|D_s}^{r,-1} = \nu_{Y,n}^r \cap j_2^* W_n \Omega_{D_s,\log}^{r,-1} \supset \nu_{Y|D_s,n}^r \cap j_2^* W_n \Omega_{D_s,\log}^{r,-1} = \nu_{Y|D_s,n}^r$ by using the embedding $(1)$ in Definition 4.3 below. There is a commutative diagram:

Here we define $\theta := g \circ f$. The upper horizontal exact row is from Theorem 3.4.2 in [Sat2]. By this diagram, we have a short exact sequence

$$0 \to \lambda_{Y,n}^r \to \text{Ker}(\theta) \to \nu_{Y|D_s,n}^r \to 0.$$
We consider the following commutative diagram of exact sequences:

\[
\begin{array}{c}
0 \\
\downarrow \\
\text{Ker}(\tau) \longrightarrow \lambda_{Y,n}^r \\
\downarrow \\
0 \\
\downarrow \\
\downarrow \\
0 \\
\downarrow \\
0
\end{array}
\]

\[
\begin{array}{c}
0 \\
\downarrow \\
\text{Ker}(\theta) \longrightarrow M_n^r/U^1M_n^r \longrightarrow \nu_{Y,n}^r/\nu_{Y|D_s,n}^r \\
\downarrow f \quad \quad \downarrow f' \\
\nu_{Y,n}^r \longrightarrow \beta_D^r \nu_{D_s,n}^r \\
\downarrow \\
0 \\
\downarrow \\
0
\end{array}
\]

The map \(f'\) is defined as the composition \(\nu_{Y,n}^r/\nu_{Y|D_s,n}^r \longrightarrow \nu_{Y,n}^r/\nu_{Y|D_s,n}^r \cong \beta_D^r \nu_{D_s,n}^r\). The well-definedness of this map is from (\(\star\)). By the fact that \(\text{Ker}(\tau) \cong \lambda_{Y,n}^r\), we have \(\mathcal{K}_{Y|D_s}^r = \nu_{Y|D_s,n}^r\).

We will investigate the sheaf \(\nu_{Y|D_s,n}^r\) by embedding into a differential module of a smooth variety as §2.5 in [Sat1].

**Definition 4.3.** We have a composite map:

\[
(\ast 1) \quad \nu_{Y|D_s,n}^r \hookrightarrow \tilde{j}_* j^{-1} \nu_{Y|D_s,n}^r = \tilde{j}_* \Omega_{Y|D_s,n}^r \hookrightarrow \tilde{j}_* \Omega_{Y|D_s,n}^r.
\]

By this embedding map (\(\ast 1\)), we define the étale sheaf \(\Xi_{Y|D_s}^r\) on \(Y\) as the \(\mathcal{O}_Y\)-submodule of \(\tilde{j}_* \Omega_{Y|D_s,n}^r\) generated by local sections of \(\nu_{Y|D_s,n}^r\). When \(D_s = \emptyset\), the étale sheaf \(\Xi_{Y|\emptyset}^r\) is the same as Sato’s differential module \(\Xi_Y^r\) of Definition 2.5.1 in [Sat1].

**Lemma 4.4.** (cf. Lemma 2.4.6, [Sat1]) We define

\[
\mathcal{Z}_{\mathcal{Y}|\mathcal{Z}}^{r+1}(\log Y) := \text{Ker} \left( d : \Omega_{\mathcal{Y}|\mathcal{Z}}^{r+1}(\log Y) \longrightarrow \Omega_{\mathcal{Y}|\mathcal{Z}}^{r+2}(\log Y) \right).
\]

Then there exists a short exact sequence on \(\mathcal{Y}_{et}\):

\[
0 \longrightarrow j_* \Omega_{\mathcal{Y}|(\mathcal{Y}\setminus D_s),\log}^{r+1} \longrightarrow \mathcal{Z}_{\mathcal{Y}|\mathcal{Z}}^{r+1}(\log Y) \xrightarrow{1-C} \Omega_{\mathcal{Y}|\mathcal{Z}}^{r+1}(\log Y) \longrightarrow 0.
\]

Here we put \(\mathcal{W} := \mathcal{Y}\setminus Y\).
**Proof.** We have a commutative diagram with exact sequences:

\[
\begin{array}{ccccccc}
0 & \rightarrow & \text{Ker}(1 - C) & \rightarrow & Z_{\mathcal{Y}|\mathcal{Y}}^{r+1}(\log Y) & \rightarrow & \Omega_{\mathcal{Y}|\mathcal{Y}}^{r+1}(\log Y) & \rightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & j_*\Omega_{\mathcal{Y}|\mathcal{Y}}^{r+1}(\log Y) & \rightarrow & Z_{\mathcal{Y}}^{r+1}(\log Y) & \rightarrow & \Omega_{\mathcal{Y}}^{r+1}(\log Y) & \rightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \beta_*j_*\Omega_{\mathcal{Y}|\mathcal{Y}}^{r+1}(\log Y) & \rightarrow & \beta_*Z_{\mathcal{Y}}^{r+1}(\log Y) & \rightarrow & \beta_*\Omega_{\mathcal{Y}}^{r+1}(\log Y) & \rightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

Here the two lower rows are from Lemma 2.4.6 in [Sat1]. Since \(\beta_*j_*\Omega_{\mathcal{Y}|\mathcal{Y}}^{r+1}(\log Y) \cong j_*\beta_*\Omega_{\mathcal{Y}|\mathcal{Y}}^{r+1}(\log Y)\), we have \(\text{Ker}(1 - C) \cong j_*\Omega_{\mathcal{Y}|\mathcal{Y}}^{r+1}(\log Y)\). Thus we have the assertion. 

**Proposition 4.5.** (cf. [Sat1], Lemma 2.5.3) Let \(Y\) be a simple and embedded into a smooth variety \(\mathcal{Y}\) over \(\text{Spec}(k)\) as a simple normal crossing divisor. Let \(\iota : Y \hookrightarrow \mathcal{Y}\) the closed immersion. Then we have an \(\iota^{-1}\mathcal{O}_{\mathcal{Y}}\)-linear isomorphism

\[
(\Phi) \quad \iota^{-1}\left(\Omega_{\mathcal{Y}|\mathcal{Y}}^{n+1}(\log Y)/\Omega_{\mathcal{Y}|\mathcal{Y}}^{r+1}\right) \cong \Xi_{Y|D_s}^{n}
\]

which induced by the Poincaré residue map

\[
\mathcal{P}_{Y|D_s} : \Omega_{\mathcal{Y}|\mathcal{Y}}^{n+1}(\log Y) \rightarrow \iota_*j_*\Omega_{Y|D_s}^{n}.
\]

**Proof.** We prove this Proposition by the same way as Lemma 2.5.3 in [Sat1]. We have a short exact sequences

\[
0 \rightarrow \Omega_{\mathcal{Y}|\mathcal{Y} \setminus D_s}^{n} \rightarrow \Omega_{r,Y \setminus D_s}^{n} \rightarrow \iota_*\nu_Y^{r-1} \rightarrow 0,
\]

\[
0 \rightarrow \beta_*\Omega_{\mathcal{Y}|\mathcal{Y} \setminus D_s}^{n} \rightarrow \Omega_{r,Y \setminus D_s}^{n} \rightarrow \iota_*\beta_*\nu^{r-1} \rightarrow 0.
\]

by [Sat3], p.200. Then we have a short exact sequence

\[
(\star) \quad 0 \rightarrow \Omega_{\mathcal{Y}|\mathcal{Y} \setminus D_s}^{n} \rightarrow j_*\Omega_{\mathcal{Y}|\mathcal{Y} \setminus D_s}^{n} \rightarrow \iota_*\nu_Y^{r-1} \rightarrow 0.
\]

Here \(j : \mathcal{U} := \mathcal{Y} \setminus Y \hookrightarrow \mathcal{Y}\). We prove that \(\text{Im}(\mathcal{P}_{Y|D_s}) = \iota_*\Xi_{Y|D_s}^{n}\). We consider the following anti-commutative diagram which considered in [Sat1] under no
We denote by $\pi$ the residue class of $\alpha \in \ell^{-1}O_{\mathcal{X}}$. The upper horizontal map $\mathfrak{R}$ is defined by sending a local section $\alpha \otimes \beta$ to $\overline{\alpha} \otimes \tau(\beta)$, which is surjective. Lemma 4.4 induces the left vertical map. The right vertical map is a product map. The image of this product map is $\mathfrak{M}^n_{Y|D_s}$. By the surjectivity of the top horizontal map and left vertical map, we have $\text{Im}(\mathfrak{M}_{Y|D_s}) \cong \ell_*\mathfrak{M}^n_{Y|D_s}$. One can show that the injectivity of the map (¶) by the same way as Lemma 2.5.3 in [Sat].

We next introduce a differential sheaf which is defined for a modulus pair $(X, D)$.

**Definition 4.6.** We define

$$\lambda^n_{Y|D_s,n} := \text{Im} \left( (1 + O_Y(-D_s))^x \otimes (\alpha_s^r O_{Y})^y \otimes 1 \to \bigoplus_{y \in Y^n} W_n \Omega^r_{Y,y, log} \right).$$

When $D = \emptyset$, we have $\lambda^n_Y|_{\emptyset,n} = \lambda_Y^{r|n}$, where $\lambda_Y^{r|n}$ is defined in [Sat]. We denote by $\mathcal{G}^{\otimes r}_{m,Y|D_s} := (1 + O_Y(-D_s))^x \otimes (\alpha_s^r O_{Y})^y \otimes 1$ for simplicity.

We first consider the structure of $\lambda^n_{Y|D_s}$. We apply Lemma 3.2.2 in [Sat] to the sheaf $\mathcal{F} = W_n \Omega^r_{Y \times (\bullet \times D_s), log}$, $Z = \mathcal{Y}$, we obtain an exact sequence by using the fact that $W_n \Omega^r_{Y,d, log} = 0$ for $q > d + 1 - r$:

$$(\mathfrak{M}1): \ell^{-1} W_n \Omega^r_{Y|\mathcal{Y}, log} \to a_{1*} W_n \Omega^r_{Y|D^{(1)}, log} \to \cdots \to a_{d+1-r*} W_n \Omega^r_{Y|D^{(d+1-r)}, log} \to 0.$$ 

Here $D^{(q)}_s$ is defined by $D^{(q)}_s := D_s \times Y^{(q)}$. See §8 of [Sat] about the notation $Y^{(q)} (q \geq 1)$. There is a commutative diagram:

$$\ell^{-1} \mathcal{G}^{\otimes r}_{m,\mathcal{Y}|\mathcal{G}} \to \ell^{-1} \mathcal{G}^{\otimes r}_{m,Y|D_s} \to \bigoplus_{y \in Y^n} W_n \Omega^r_{Y,y, log} \to a_{1*} W_n \Omega^r_{Y|D^{(1)}, log}.$$ 

Here the upper horizontal map is surjective by Proposition 2.10 in [RS]. The left vertical map are surjective by Theorem 1.1.5 in [SZ]; the left vertical arrow is a composition of maps

$$\ell^{-1} \mathcal{G}^{\otimes r}_{m,\mathcal{Y}|\mathcal{G}} \to \ell^{-1} \mathcal{K}^M_{r,\mathcal{Y}|\mathcal{G}} / (p^m \mathcal{K}^M_{r,\mathcal{Y}|\mathcal{G}} \cap \mathcal{K}^M_{r,\mathcal{Y}|\mathcal{G}}) \cong \ell^{-1} W_n \Omega^r_{Y|\mathcal{Y}, log}.$$
(the last isomorphism is due to Theorem 1.1.5 in [JSZ]). By the definition of $\lambda^{r}_{Y|D_{s},n}$, we have

$$\lambda^{r}_{Y|D_{s},n} = \text{Im} \left( \iota^{-1} W_{n, \Omega^{r}_{Y(D^{(1)}_{s})}} \log \rightarrow a_{1s} W_{n, \Omega^{r}_{Y^{(1)}|D^{(1)}_{s}} \log} \right).$$

Then we have the following exact sequence by (2):

$$(\Psi 2): 0 \rightarrow \lambda^{r}_{Y|D_{s},n} \rightarrow a_{1s} W_{n, \Omega^{r}_{Y^{(1)}|D^{(1)}_{s}} \log} \rightarrow \cdots$$

$$\rightarrow a_{d+1-rs} W_{n, \Omega^{r}_{Y^{(d+1-r)}|D^{(d+1-r)}_{s}} \log} \rightarrow 0.$$

Taking $D = \emptyset$ in (2), we have Proposition 3.2.1 in [Satl].

**Definition 4.7.** We have a composite map:

$$(1) \quad \lambda^{r}_{Y|D_{s},1} \hookrightarrow \tilde{j}_{*} \tilde{j}^{-1} \lambda^{r}_{Y|D_{s},1} = \tilde{j}_{*} \Omega^{r}_{Y|D_{s}} \log \hookrightarrow \tilde{j}_{*} \Omega^{r}_{Y|D_{s}}.$$

By this embedding map (1), we define the étale sheaf $\Lambda^{r}_{Y|D_{s}}$ on $Y$ as the $\Theta_{Y}$-submodule of $\tilde{j}_{*} \Omega^{r}_{Y|D_{s}}$ generated by local sections of $\lambda^{r}_{Y|D_{s},1}$. When $D_{s} = \emptyset$, the étale sheaf $\Lambda^{r}_{Y|\emptyset}$ is the same as Sato’s differential module $\Lambda^{r}_{Y}$ of Definiton 3.3.1 in [Satl].

**Lemma 4.8.** (cf. [SZ, Theorem 1.1.6, Satl, Corollary 2.2.5 (2)]) There are short exact sequences

$$(4.1) \quad 0 \rightarrow W_{n-1, \Omega^{r}_{Y|\emptyset/p}, \log} \rightarrow W_{n, \Omega^{r}_{Y|\emptyset/p}, \log} \rightarrow W_{1, \Omega^{r}_{Y|\emptyset/p}, \log} \rightarrow 0,$$

$$(4.2) \quad 0 \rightarrow W_{n-1, \Omega^{r}_{Y|\emptyset/p}, \log} \rightarrow W_{n, \Omega^{r}_{Y|\emptyset/p}, \log} \rightarrow W_{1, \Omega^{r}_{Y|\emptyset/p}, \log} \rightarrow 0.$$

Here $R$ is the natural projection operator and $p$ is the unique map such that $p \circ R = p$ (p denotes a multiplication by p) and we put $[D_{s}/p] := \sum_{\lambda \in \Lambda} [n_{\lambda}/p](D_{s})_{\lambda}$ with $[n_{\lambda}/p] := \min \{ n \in \mathbb{Z} \mid pm \geq n_{\lambda} \}$ (This notation is the same as [JSZ]).

**Proof.** We first show that the short exact sequence (4.1). There is a commutative diagram

$$\begin{array}{c}
0 & \rightarrow & W_{n-1, \Omega^{r}_{Y|\emptyset/p}, \log} & \rightarrow & W_{n, \Omega^{r}_{Y|\emptyset/p}, \log} & \rightarrow & W_{1, \Omega^{r}_{Y|\emptyset/p}, \log} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & W_{n-1, \Omega^{r}_{Y|\emptyset/p}, \log} & \rightarrow & W_{n, \Omega^{r}_{Y|\emptyset/p}, \log} & \rightarrow & W_{1, \Omega^{r}_{Y|\emptyset/p}, \log} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & W_{n-1, \Omega^{r}_{Y|\emptyset/p}, \log} & \rightarrow & W_{n, \Omega^{r}_{Y|\emptyset/p}, \log} & \rightarrow & W_{1, \Omega^{r}_{Y|\emptyset/p}, \log} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0. & & 0.
\end{array}$$
Here the upper horizontal row is given by [JSZ], Theorem 1.1.6, and the middle horizontal row is given by [CTSS], p.779, Lemma 3. Then the lower horizontal row is exact. By the definition of $\nu_{D_s,m}^r$, we obtain a short exact sequence:

$$0 \to \nu_{[D_s/p],n-1}^r \overset{p}{\to} \nu_{D_s,n}^r \overset{\mathcal{R}^{r-1}}{\to} \nu_{D_s,1}^r \to 0.$$ 

We consider the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \to & \nu_{[D_s/p],n-1}^r & \overset{p}{\to} & \nu_{Y|D_s}^r & \overset{\mathcal{R}^{r-1}}{\to} & \nu_{Y|D_s,1}^r & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \nu_{Y,n-1}^r & \overset{p}{\to} & \nu_{Y,n}^r & \overset{\mathcal{R}^{r-1}}{\to} & \nu_{Y,1}^r & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \nu_{[D_s/p],n-1}^r & \overset{p}{\to} & \nu_{D_s,n}^r & \overset{\mathcal{R}^{r-1}}{\to} & \nu_{D_s,1}^r & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0,
\end{array}
\]

where the exactness of the middle horizontal row is given by [Sat1], Corollary 2.2.5 (2), and the vertical rows are exact by Proposition 4.2. Thus the upper horizontal row is exact. The exactness of (4.2) is given by [JSZ], Theorem 1.1.6 and the exact sequence (V2) of $\lambda_r^Y|_{D_s,n}$. This completes the proof of the assertion. □

We have the following duality which is a modulus version of Proposition 3.3.5 in [Sat1].

**Proposition 4.9.** (cf. Proposition 3.3.5, [Sat1]) Assume that $0 \leq m \leq d$. Let

$$\Xi_{Y|D_s}^m \times \Lambda_{Y|D_s}^{d-m} \to \Xi_Y^d$$

and

$$\Lambda_{Y|D_s}^m \times \Xi_{Y|D_s}^{d-m} \to \Xi_Y^d$$

are the $\mathcal{O}_Y$-bilinear pairing obtained by the pairing $\nu_{Y|mD_s,1}^r \otimes \alpha_r^Y \lambda_{D_s}^r \to \nu_{Y,1}^d$ which is defined below. Here $\Xi_{Y|D_s}^r := \Xi_Y^r \otimes \mathcal{O}_Y(D_s)$ and $\Lambda_{Y|D_s}^r := \Lambda_Y^r \otimes \mathcal{O}_Y(D_s)$. Then the induced morphisms

$$(a1) : \Lambda_{Y|D_s}^{d-m} \to \mathcal{R}\text{Hom}_{\mathcal{O}_Y}(\Xi_{Y|D_s}^m, \Xi_Y^d)$$

and

$$(a2) : \Xi_{Y|D_s}^{d-m} \to \mathcal{R}\text{Hom}_{\mathcal{O}_Y}(\Lambda_{Y|D_s}^m, \Xi_Y^d)$$

are isomorphisms.

**Proof.** We use the same argument of Proposition 3.3.5 in [Sat1]. This problem is étale local on $Y$, we may suppose that $Y$ is embedded into a smooth variety $\mathcal{Y}$.
over $s$ as a simple normal crossing divisor. We put $\iota : Y \hookrightarrow \mathcal{Y}$, $f : Y \to s$ and $h : \mathcal{Y} \to s$. Let $\mathcal{I}_Y$ be the definition ideal of $Y$. We have a short exact sequence

$$0 \to \Omega^m_{\mathcal{Y}/\mathcal{Y}}(-\log Y) \to \Omega^m_{\mathcal{Y}/\mathcal{Y}} \to \iota_* \Lambda^m_{\mathcal{Y}/\mathcal{D}_s} \to 0$$

by Theorem in [Sat1]. Here we put $\Omega^m_{\mathcal{Y}/\mathcal{Y}}(-\log Y) := \Omega^m_{\mathcal{Y}/\mathcal{Y}}(\log Y) \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{I}_Y$. There are the $\mathcal{O}_{\mathcal{Y}}$-perfect pairings of locally free $\mathcal{O}_{\mathcal{Y}}$-modules

$$\Omega^{m+1}_{\mathcal{Y}/\mathcal{Y}} \times \Omega^{d-m}_{\mathcal{Y}/\mathcal{Y}} \to \Omega^{d+1}_{\mathcal{Y}/\mathcal{Y}},$$

$$\Omega^{m+1}_{\mathcal{Y}/\mathcal{Y}}(\log Y) \times \Omega^{d-m}_{\mathcal{Y}/\mathcal{Y}}(-\log Y) \to \Omega^{d+1}_{\mathcal{Y}/\mathcal{Y}}.$$

We have an isomorphism $\tilde{\Xi}^m$ in the proof of Proposition 3.3.5 in [Sat1] and projection formula (see Exercise 5.1 (d) in [Har1]) of $\mathcal{I}_Y$. Then we can consider Then we obtain an isomorphism

$$\tilde{\Xi}^{m+1} \times \tilde{\Xi}^{d-m} \to \tilde{\Xi}^{d+1},$$

$$\tilde{\Xi}^{m+1}(\log Y) \times \tilde{\Xi}^{d-m}(-\log Y) \to \tilde{\Xi}^{d+1}.$$

We have an isomorphism $\tilde{\Xi}^{m+1}$ in the proof of Proposition 3.3.5 in [Sat1] and projection formula (see Exercise 5.1 (d) in [Har1]) of $\iota_*$. Then we obtain an isomorphism $(a1)$. We have an isomorphism $(a2)$ by the same argument. This completes the proof.

The differential operator $d : \Omega^r_Y \to \Omega^{r+1}_Y$ induces a differential operators $d^r : \Xi^r_{Y|\mathcal{D}_s,n} \to \Xi^{r+1}_{Y|\mathcal{D}_s,n}$ and $d^r : \Lambda^r_{Y|\mathcal{D}_s,n} \to \Lambda^{r+1}_{Y|\mathcal{D}_s,n}$. Then we can consider $Z\Xi^r_{Y|\mathcal{D}_s} := \text{Ker}(d^r)$, $Z\Lambda^r_{Y|\mathcal{D}_s} := \text{Ker}(d^r)$, $B\Xi^r_{Y|\mathcal{D}_s} := \text{Im}(d^{r-1})$, and $B\Lambda^r_{Y|\mathcal{D}_s} := \text{Im}(d^{r-1})$.

**Lemma 4.10.** (cf. [Sat1], Lemma 2.5.7 and Lemma 3.3.4)

We have a Cartier isomorphisms

1. $C : Z\Xi^r_{Y|\mathcal{D}_s}/B\Xi^r_{Y|\mathcal{D}_s} \cong \Xi^r_{Y|([D_s/p])}$

2. $C : Z\Lambda^r_{Y|\mathcal{D}_s}/B\Lambda^r_{Y|\mathcal{D}_s} \cong \Lambda^r_{Y|([D_s/p])}$

which are induced by the Cartier isomorphism $Z^\mathcal{U}_Y/B^\mathcal{U}_Y \cong \Omega^\mathcal{U}_Y$.

**Proof.** The proofs are essentially same of Lemma 2.5.7 and Lemma 3.3.4 in [Sat1]. We first show (I). We assume that $Y$ is simple (if any irreducible component is smooth over $k$; this is the terminology of [Sat1]) and embedded into a smooth variety $\mathcal{Y}/k$ as a simple normal crossing divisor since the problem is étale local. We put $\iota$ the closed immersion $\iota : Y \hookrightarrow \mathcal{Y}$. We have an exact sequence of complexes

$$0 \to \Omega^\bullet_{\mathcal{Y}/\mathcal{Y}} \to \Omega^\bullet_{\mathcal{Y}/\mathcal{Y}}(\log Y) \xrightarrow{\mathcal{P}_{Y|\mathcal{D}_s}} \iota_* \Xi^\bullet_{Y|\mathcal{D}_s} \to 0$$
by Proposition 4.5. Then we taking the cohomology of this sequence, we get the following short exact sequence:

$$0 \to \mathcal{H}^{q+1}\left(\Omega^\bullet_{\mathcal{Y}|\mathcal{D}}\right) \to \mathcal{H}^{q+1}\left(\Omega^\bullet_{\mathcal{Y}|\mathcal{D}}(\log Y)\right) \to \iota_*\mathcal{H}^q\left(\Xi^\bullet_{Y|D_s}\right) \to 0.$$  

There are Cartier isomorphisms (see [RS], Theorem 2.16):

$$C : \mathcal{H}^{q+1}\left(\Omega^\bullet_{\mathcal{Y}|\mathcal{D}}\right) \xrightarrow{\cong} \mathcal{H}^{q+1}\left(\Omega^\bullet_{\mathcal{Y}|\mathcal{D}}(\log Y)\right), \quad C : \mathcal{H}^{q+1}\left(\Omega^\bullet_{\mathcal{Y}|\mathcal{D}}(\log Y)\right) \xrightarrow{\cong} \mathcal{H}^{q+1}\left(\mathcal{D}/p\right)(\log Y).$$

Hence we obtain an isomorphism in (1);\(\mathcal{H}^{q+1}\left(\Xi^\bullet_{Y|D_s}\right) \xrightarrow{\cong} \mathcal{H}^{q+1}\left(\mathcal{D}/p\right)(\log Y)\) because the Poincaré residue map \(\mathcal{Q}_{Y|D_s}\) is compatible with Cartier operator. The Cartier isomorphism in (2) is reduced to the smooth case by using an exact sequence \(\mathcal{Q}^2\) and Lemma 3.2.2 in [Sat2].

**Lemma 4.11.** (cf. [Sat], p.731) We consider the following cartesian diagram:

$$\begin{array}{ccc}
Y' & \xrightarrow{p^{r2}} & Y \\
\downarrow f_1 & & \downarrow f_2 \\
\mathcal{S} & \xrightarrow{F_{\mathcal{S}}} & \mathcal{S}.
\end{array}$$

Here \(F_{\mathcal{S}}\) denotes the absolute Frobenius automorphism of \(\mathcal{S}\). For \(q \geq 0\), we have two isomorphisms:

1. \((C1)\) \(C^{\text{lin}}_{\mathcal{S}^{r2}} : F_{Y/\mathcal{S}^{r2}}\left(Z\Xi^q_{Y|D_s}/B\Xi^q_{Y|D_s}\right) \xrightarrow{\cong} \Xi^q_{Y'/D_s/p}\)
2. \((C2)\) \(C^{\text{lin}}_{\mathcal{S}^{r2}} : F_{Y/\mathcal{S}^{r2}}\left(Z\Lambda^q_{Y|D_s}/B\Lambda^q_{Y|D_s}\right) \xrightarrow{\cong} \Lambda^q_{Y'/D_s/p}\)

Here we define \(D'_s := D_s \times_Y Y'\) and let \(F_{Y/\mathcal{S}^{r2}} : Y \to Y'\) be the unique finite morphism that the absolute Frobenius map \(F_Y : Y \to Y\) factors through (see [Sat]). If \(D_s = \emptyset\), we have an isomorphisms (3.4.6) and (3.4.7) in [Sat], p.731.

**Proof.** Let \(F_{\mathcal{Y}^{r2}} : \mathcal{Y} \to \mathcal{Y}'\) be a relative Frobenius morphism. We have a short exact sequence by Proposition 4.5

$$0 \to F_{\mathcal{Y}^{r2}}\Omega^\bullet_{\mathcal{Y}|\mathcal{D}} \to F_{\mathcal{Y}^{r2}}\Omega^\bullet_{\mathcal{Y}|\mathcal{D}}(\log Y) \xrightarrow{F_{\mathcal{Y}^{r2}}\mathcal{Q}_{Y|D_s}} F_{\mathcal{Y}^{r2}}\mathcal{Q}_{Y|D_s} \to 0.$$  

We have \(F_{\mathcal{Y}^{r2}}\mathcal{Q}_{\mathcal{Y}^{r2}}\iota_* = t'_*F_{Y/\mathcal{S}^{r2}},\) where \(t'\) is an immersion \(t' : Y' \to \mathcal{Y}'\).

We have isomorphisms

$$C : \mathcal{Q}_{\mathcal{Y}'|\mathcal{D}/p}(\log Y') \xrightarrow{\cong} \mathcal{H}^q\left(F_{\mathcal{Y}^{r2}}(\Omega^\bullet_{\mathcal{Y}|\mathcal{D}}(\log Y))\right)$$

and

$$C : \mathcal{Q}_{\mathcal{Y}'|\mathcal{D}/p} \xrightarrow{\cong} \mathcal{H}^q\left(F_{\mathcal{Y}^{r2}}(\Omega^\bullet_{\mathcal{Y}|\mathcal{D}})\right)$$

by the similar argument in the proof of Theorem 2.16, [RS]. This isomorphisms are induced by the linear Cartier isomorphism given by Katz ([Katz], 7.2). Taking
cohomology sheaves, we obtain a short exact sequence:

\[ 0 \rightarrow \mathcal{H}^{q+1} \left( F_{Y/s} \Omega^q_{Y/G} \right) \rightarrow \mathcal{H}^{q+1} \left( F_{Y/s} \Omega^q_{Y,\log}(\log Y) \right) \]

\[ \xrightarrow{F_{Y/s},\mathcal{P}_{Y/D_s}} t'_s \mathcal{H}^q \left( F_{Y/s} \mathcal{E}^\bullet_{Y/D_s} \right) \rightarrow 0. \]

The assertion follows from the fact that the map \( \mathcal{P}_{Y/D} \) is compatible with Cartier operators. We obtain the isomorphism \((C2)\) by the similar argument as Lemma 3.3.4 in [Sat1]. □

The following is the main result of this chapter.

**Theorem 4.12.** (cf. [Sat1], Theorem 1.2.2 (2))

(1) For any integers \( i \) and \( n \) with \( 0 \leq i \leq d \), there exist natural pairings

\[(4.3) \quad \lim_m H^i_{\text{et}}(Y, \nu^r_{Y,\lfloor d+s \cdot n \rfloor}) \times H^{d+1-i}_{\text{et}}(Y, \alpha'_s \lambda^{d-r}_{V,1}) \rightarrow \mathbb{Z}/p^n\mathbb{Z},\]

\[(4.4) \quad \lim_m H^i_{\text{et}}(Y, \lambda^r_{Y,\lfloor d+s \cdot n \rfloor}) \times H^{d+1-i}_{\text{et}}(Y, \alpha'_s \nu^{d-r}_{V,1}) \rightarrow \mathbb{Z}/p^n\mathbb{Z}.\]

(2) The pairings \((4.3), (4.4)\) are non-degenerate pairings.

**Proof.** We first prove (1). It is enough to show that the case \( n = 1 \) by using Lemma 4.8. Consider a diagram

\[
\begin{array}{ccc}
\nu^r_{Y,\lfloor d+s \cdot 1 \rfloor} \otimes \alpha'_s \lambda^{d-r}_{V,1} & \xrightarrow{f} & \nu^r_{Y,1} \otimes \alpha'_s \lambda^{d-r}_{V,1} \\
\downarrow g & & \downarrow \delta \\
\nu^d_{Y,1} & \xrightarrow{\delta} & \beta'_s \nu^{d-1}_{D,1} \\
\end{array}
\]

where \( g \) is a composition of a map \( \nu^r_{Y,1} \otimes \alpha'_s \lambda^{d-r}_{V,1} \rightarrow \alpha'_s \nu^r_{Y,1} \otimes \alpha'_s \lambda^{d-r}_{V,1} \rightarrow \alpha'_s \nu^d_{Y,1} \). Here last morphism is product morphism defined in [Sat1]. Since \( \delta \circ g \circ f = 0 \) and \( \text{Hom} \left( \nu^r_{Y,\lfloor d+s \cdot 1 \rfloor} \otimes \alpha'_s \lambda^{d-r}_{V,1}, \beta'_s \nu^{d-1}_{D,1}[-1] \right) = 0 \), we have a morphism \( \nu^r_{Y,\lfloor d+s \cdot 1 \rfloor} \otimes \alpha'_s \lambda^{d-r}_{V,1} \rightarrow \nu^d_{Y,1} \) by Lemma 2.1.2 (1) in [Sat2]. By taking the cohomology of this morphism, we have a pairing \((4.3)\): \( \lim_m H^i_{\text{et}}(Y, \nu^r_{Y,\lfloor d+s \cdot 1 \rfloor}) \times H^{d+1-i}_{\text{et}}(Y, \nu^d_{Y,1}) \rightarrow \mathbb{Z}/p^n\mathbb{Z} \). We next construct the pairing \((4.4)\) by the similar argument of §3 in [Sat1]. Since \( \alpha'_s \nu^{d-r}_{V,1} \cong \lim_m \nu^{d-r}_{Y,1} \otimes \mathcal{O}_Y(m'D_s) \), we define the map \( \lambda^r_{Y,\lfloor d+s \cdot n \rfloor} \otimes \{ \nu^{d-r}_{Y,1} \otimes \mathcal{O}_Y(m'D_s) \}_m \rightarrow \nu^d_{Y,1} \) of sheaves on \( Y_{\text{et}} \). We consider the map

\[ p : \lambda^r_{Y,\lfloor d+s \cdot n \rfloor} \otimes \{ \nu^{d-r}_{Y,1} \otimes \mathcal{O}_Y(m'D_s) \}_m \rightarrow \bigoplus_{y \in Y_0} i_{y^*} \Omega^d_{Y,\log} \]

defined by \( d \log(f_m) \otimes (y_{m'} \otimes \omega) \mapsto \omega \cdot d \log((1 + y_{m'} z) \cdot f'_m) \), where \( y_{m'} \in \mathcal{O}_Y(m'D_s), \omega \in i_{y^*} \Omega^d_{Y,\log}, f_m = (1 + z) \cdot f'_m \in (1 + \mathcal{O}_Y(-mD_s)) \otimes (\alpha'_s \mathcal{O}_Y) \otimes (r-1) \).
By the equation \( \partial_{y,x}^{val}(\omega \cdot d log(\gamma_{m'} \cdot f_m)) = \partial_{y,x}^{val}(\omega) \cdot d log((1 + \gamma_{m'} z) \cdot f_m) \) in Definition 3.1 (2) of \([Sat1]\), the image of the map \( p \) lies in \( \nu_{Y,1}^d \). Thus we obtain the map \( \lambda_{Y|D_s,n} \otimes \nu_{Y,1}^{d-r} \otimes \vartheta_Y(m'D_s) \rightarrow \nu_{Y,1}^d \). We next prove (2) assuming \( n = 1 \). We show the non-degeneracy of the pairing (4.3)

\[
\lim_{m} H^i_{et}(Y, \nu_{Y|mD_s,1}^r) \times H^{d+1-i}(Y, \alpha_{d-r}' \lambda_{Y,1}^{d-r} \nu_{Y,1}^r) \rightarrow H^{d+1}(Y, \nu_{Y,1}^d) \xrightarrow{tr} \mathbb{Z}/p^n\mathbb{Z}.
\]

It suffices to show that the natural two pairings

(4.5) \[
\lim_{m} H^i_{et}(Y, \Xi^{r}_{Y|D_s}) \times H^{d+1-i}(Y, \alpha_{d-r}' \Lambda_{V}^{d-r} \nu_{Y,1}^r) \rightarrow H^{d}(Y, \Xi^{d}_{Y}) \xrightarrow{tr} \mathbb{Z}/p^n\mathbb{Z}
\]

(4.6) \[
\lim_{m} H^i_{et}(Y, Z\Xi^{r}_{Y|D_s}) \times H^{d+1-i}(Y, \alpha_{d-r}' (\Lambda_{V}^{d-r} / B\Lambda_{V}^{d-r})) \rightarrow H^{d}(Y, Z\Xi^{d}_{Y}) \xrightarrow{tr} \mathbb{Z}/p^n\mathbb{Z}
\]

are non-degenerate pairings of \( \mathbb{Z}/p^n\mathbb{Z} \)-modules for any \( i \in \mathbb{Z} \) by the following Lemma 4.13 below. Here \( tr_1 \) and \( tr_2 \) are defined as follows (see [Sat1] p.731):

\[
tr_1 : H^d_{et}(Y, \Xi^{d}_{Y}) \xrightarrow{tr} \mathbb{Z}/p^n\mathbb{Z}, \quad tr_2 : H^d_{et}(Y, Z\Xi^{d}_{Y} / B\Xi^{d}_{Y}) \xrightarrow{C} H^d_{et}(Y, \Xi^{d}_{Y}) \xrightarrow{tr} \mathbb{Z}/p^n\mathbb{Z}.
\]

**Lemma 4.13.** (cf. [Sat1], Lemma 2.5.7 (2), Lemma 3.3.4 (2))

(1) There exists a short exact sequence on \( Y_{et} \)

\[
0 \rightarrow \nu_{Y|D_s,1}^q \rightarrow Z\Xi^{q}_{Y|D_s} \xrightarrow{1-C} \Xi^{q}_{Y|D_s} \rightarrow 0
\]

(2) There exists a short exact sequence on \( Y_{et} \)

\[
0 \rightarrow \Lambda_{Y|D_s,1}^q \rightarrow \Lambda_{Y|D_s}^{q-1} \xrightarrow{1-C} \Xi^{q}_{Y|D_s}/B\Xi^{q}_{Y|D_s} \rightarrow 0
\]
Proof. We first show that the assertion (1). There is a commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \to & 0 & \to & Z^{q+1} & \to & 0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & j_* \Omega^{q+1} & \to & Z^{q+1}(\log Y) & \to & 0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & t_* Z^Y & \to & t_* Z^Y & \to & 0 & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0
\end{array}
\]

Here the left vertical row is due to (1) in the proof of Proposition [4.3], the middle and right vertical rows follows from an isomorphism (1) of Proposition [4.3]. The top and middle horizontal rows are due to Lemma 4.4. Thus we obtain the lower horizontal row, which is an exact. We next prove the assertion (2). Since the problem are étale local on \( Y \), we may assume that \( Y \) is simple. By the definition of \( \lambda^r \) and \( \Lambda^r_{Y|D_s} \) we have \( \Lambda^r_{Y|D_s} = \text{Im} (\iota^{-1} \Omega^{Y|D_s} \to a_1 \Omega^{Y|D_s}) \).

Here the map \( \iota^* \) is the pull-back map of differential forms. Then we get exact sequences by similar argument as the proof of Lemma 3.3.4 in [Sat1]:

\[
\begin{align*}
0 & \to \Lambda^r_{Y|D_s} \xrightarrow{a_1^r} a_1 \Omega^n_{Y|D_s} \to \cdots \to a_{d+1-r_\ast} \Omega^n_{Y|D_s} \to 0 \\
0 & \to Z \Lambda^r_{Y|D_s} \xrightarrow{a_1^r} a_1 \Omega^n_{Y|D_s} \to \cdots \to a_{d+1-r_\ast} \Omega^n_{Y|D_s} \to 0 \\
0 & \to B \Lambda^r_{Y|D_s} \xrightarrow{a_1^r} a_1 \Omega^n_{Y|D_s} \to \cdots \to a_{d+1-r_\ast} \Omega^n_{Y|D_s} \to 0
\end{align*}
\]

By this exact sequences, we can reduce the assertion to the smooth case and exact sequence (1). The smooth case of the assertion due to Theorem 1.2.1 of [JSZ]. This completes the proof.

We return to the proof of the non-degeneracy of the pairings (4.5) and (4.6). The non-degeneracy of the pairing (4.5) follows from Proposition 4.9 VII. 3, [Har2], and an isomorphism \( \alpha^r \Lambda_Y^{d-r} \cong \lim_{m \to \infty} \Lambda_{Y,1}^{d-r} \otimes \mathcal{O}_Y(m^r D_s) \) because \( \Lambda_{Y,1}^{d-r} \) is a coherent sheaf (see [Sat1], Lemma 3.3.2 [1]). We consider the non-degeneracy
of \([4.6]\). We can decompose the isomorphism \(C\) into isomorphisms

\[
Z \Xi^r_{Y/Y_D} / B \Xi^r_{Y/Y_D} \xrightarrow{F_{Y/s}^{-1}(C_{mD})} F_{Y/s}^{-1}(\Xi^r_{Y/Y_D}) \xrightarrow{(pr_2^* )^{-1}} F_{Y/s}^{-1}pr_2^{-1}(\Xi^r_{Y/Y_D}) = F_{Y/s}^{-1}\Xi^r_{Y/Y_D} = \Xi^r_{Y/Y_D}
\]

as \((3.4.8)\) in \([Sat]\). The pairing of pro-system with respect to \(m\):

\[
F_{Y/s}(Z \Xi^r_{Y/Y_D}) \times F_{Y/s}(\Lambda^{d-r}_{ZY/Ds}) \to F_{Y/s}(\Xi^r_{Y/Y_D}) \xrightarrow{C_{mD}} \Xi^r_{Y/Y_D}
\]

induces an isomorphism

\[
\{ F_{Y/s}(\Lambda^{d-r}_{ZY/Ds}) \} \to \{ R \text{Hom}_{\mathcal{O}_Y}(F_{Y/s}(Z \Xi^r_{Y/Y_D}), \Xi^r_{Y/Y_D}) \}.\]

We have an isomorphism as a pro-system with respect to \(m\):

\[
\{ F_{Y/s}(\Lambda^{d-r}_{ZY/Ds}) \} \cong \{ \Lambda^{d-r}_{ZY/Ds} \} \cong \{ R \text{Hom}_{\mathcal{O}_Y}(\Xi^r_{Y/Y_D}, \Xi^r_{Y/Y_D}) \} \cong \{ R \text{Hom}_{\mathcal{O}_Y}(F_{Y/s}(Z \Xi^r_{Y/Y_D}), \Xi^r_{Y/Y_D}) \}.\]

Here the second isomorphism is due to Proposition \(4.9\). For the third isomorphism, we use Lemma \(4.11\). Thus, by using this fact and an isomorphism

\[
\{ F_{Y/s}(\Lambda^{d-r}_{ZY/Ds}) \} \approx \{ F_{Y/s}(\Lambda^{d-r}_{ZY/Ds}) \} m \approx \{ R \text{Hom}_{\mathcal{O}_Y}(\Xi^r_{Y/Y_D}, \Xi^r_{Y/Y_D}) \} m.
\]

We have a perfect pairing

\[
\lim_{m} H^i_{\text{ét}}(Y, Z \Xi^r_{Y/Y_D}) \times H^{d+1-i}_{\text{ét}}(Y, \alpha'_s \Lambda^{d-r}_{V} / B \Lambda^{d-r}_{V}) \to H^{d+1}(Y', \Xi^r_{Y/Y_D}) \xrightarrow{\text{tr}_y} k.
\]

The non-degeneracy of the pairing \((4.6)\) is due to the commutative diagram in the proof of Theorem 1.2.2 (2) in \([Sat]\), p.732. The proof of non-degeneracy of the pairing \((4.4)\) is similar argument as \((4.3)\)'s one. This completes the proof. \(\square\)

**Remark 4.14.** When \(Y\) is smooth in Theorem \(4.12\) we obtain the Theorem 4.1.4 in \(JZ\).

### 5. Explicit formula for \(\mathcal{M}^r_m\)

In this section we construct a canonical pairing \((5)\) below and prove an explicit formula for that pairing, which will be used in the proof of \(11\). In this section, we use a Sato’s arguments of §8 in \([Sat2]\).
5.1. Setting. We put \( \nu_Y^r := \nu_{\eta,1}^r \), \( \mu^r := i^* \psi_* \mu_p \) and \( \mu := \mu_p(K) \) for simplicity.

Put \( \mathcal{M}^q := \mathcal{H}^q(s_1(q)_{X[Y]}), M^q_Y = i^* R^q j_* \mu_p^{\otimes q}, M^q_Y = i^* R^q j'_* \mu_p^{\otimes q} \), and let \( U^* \) be the filtration on \( \mathcal{M}^q \) defined in [Sat2]. The purpose of this section is to construct a morphism in \( D^b(Y_{et}, \mathbb{Z}/p\mathbb{Z}) \):

\[
(5) \quad \Theta_{\mu, D} : \{ U^1 \mathcal{M}^r \}_{m} \otimes \alpha'_r U^1 M^r_{d-r+1} [-d - 2] \longrightarrow \alpha'_r \nu^{d-r} [-d - 1]
\]

and to prove an explicit formula for this morphism (Theorem 5.4 below).

5.2. Construction of \( \Theta'_{\mu} \). The sheaf \( \mu' \) is non-canonically isomorphic to the constant sheaf \( \mathbb{Z}/p\mathbb{Z} \), then we will write \( \mu' \otimes \mathcal{K} (\mathcal{K} \in D^b(Y_{et}, \mathbb{Z}/p\mathbb{Z})) \) for \( \mu' \otimes L \mathcal{K} \) in \( D^b(Y_{et}, \mathbb{Z}/p\mathbb{Z}) \). We consider the following distinguished triangle of pro-system with respect to \( m \):

\[
\{ (\mathcal{M}^r_{m}/U^1 \mathcal{M}^r)[r-1] \}_{m} \xrightarrow{g} \{ \mathbb{K}(r) \}_{m} \xrightarrow{t'} \{ \tau_{\leq s_1(r)_{X} mD} \}_{m} \xrightarrow{\mu} \{ (\mathcal{M}^r_{m}/U^1 \mathcal{M}^r)[r] \}_{m},
\]

where the last morphism is defined as the composite \( \tau_{\leq s_1(r)_{X} mD} \xrightarrow{} \mathcal{M}^r_{m}[r] \). It is easy to see that \( \mathbb{K}(r) \) is concentrated in \([0, r]\), and the triple \( (\mathbb{K}(r), g', t') \) is unique up to a unique isomorphism by Lemma. We also consider the following distinguished triangle which is induced by [Sat2]:

\[
\alpha'_s(M^d_{V}^{d-r+1}/U^1 M^d_{V}^{d-r+1})[-d + r - 2] \xrightarrow{h^V} \mathbb{K}^V(d - r + 1) \xrightarrow{u^V} \tau_{d-r+1} \alpha'_s i^* R^d j'_* \mu_p^{\otimes d-r+1} \longrightarrow \alpha'_s(M^d_{V}^{d-r+1}/U^1 M^d_{V}^{d-r+1})[-d + r - 1].
\]

Here the last morphism is defined as the composite

\[
(5.1) \quad \tau_{d-r+1} \alpha'_s i^* R^d j'_* \mu_p^{\otimes d-r+1} \longrightarrow \alpha'_s M^d_{V}^{d-r+1}[-d + r - 1] \longrightarrow \alpha'_s(M^d_{V}^{d-r+1}/U^1 M^d_{V}^{d-r+1})[-d + r - 1].
\]

It is easy to see that \( \mathbb{K}^V(d - r + 1) \) is concentrated in \([0, d - r + 1]\), and the triple \( (\mathbb{K}^V_n(d - r + 1), h^V, u^V) \) is unique up to a unique isomorphism by Lemma (3).

We construct \( \Theta_{\mu} \) by decomposing the morphism

\[
\{ \mathbb{K}(r) \}_{m} \otimes \mathbb{K}^V(d - r + 1) \longrightarrow \{ \tau_{\leq s_1(r)_{X} mD} \}_{m} \otimes \{ \tau_{\leq d-r+1} \alpha'_s i^* R^d j'_* \mu_p^{\otimes d-r+1} \}_{m} \xrightarrow{\cong} \{ \tau_{\leq s_1(r)_{X} mD} \}_{m} \otimes \{ \tau_{\leq d-r+1} \mathbb{K}^V_{mD} \}_{m} \xrightarrow{\cong} \tau_{d+1} \mathcal{S}(d+1)(X,M) \xrightarrow{\cong} i^* R^d j'_* \mu_p^{\otimes d+1}.
\]

Here the second morphism is a quasi-isomorphism and the complex \( s_{mD}^{\nu_{\bullet}} \) is defined as follows:

\[
\left[ s_n(d-r)_{X|mD}^{\nu_{\bullet}} \longrightarrow \cdots \longrightarrow s_n(d-r)_{X|mD}^{d-r+1} \longrightarrow \cdots \right],
\]

where \( s_n(q)_{X|mD}^{\nu_0} \) is a complex which degree \( i \)-part defined by

\[
s_n(q)_{X|mD}^{\nu_0} := \left( \omega_{Z_n} \otimes \mathcal{O}_{Z_n}((m'+i)\mathcal{I}) \otimes J_{q-i}^{q-p} \right) \oplus \left( \omega_{Z_n}^{q-1} \otimes \mathcal{O}_{Z_n}((m'+i)\mathcal{I}) \otimes \mathcal{O}_{\mathcal{E}_n} \right).
\]
The third morphism are induced by the product morphism $s_1(r)_{X|mD} \otimes \mathfrak{S}_{mD}^\vee \to S(d + 1)(X,M_X)$, which is defined as for product structure (see [Tsu2]).

By Lemma 7.3.2 [Sat2] and the assumption that $\zeta_p \in K$, there is a morphism

$$(\sigma) \ i^* R\psi_* \mu_p^\otimes [d + 1] \cong \mu' \otimes (\tau \leq d i^* R\psi_* \mu_p^\otimes d) \overset{\text{id} \otimes \sigma_n^d [-d]}{\longrightarrow} \mu' \otimes \nu_Y^{d - 1} [-d],$$

where the morphism $\sigma_n^d [-d]$ is defined in Proposition 3.6 of [Sat3]. Thus we have a morphism

$$(\ast 1) \ \{\mathbb{B}_m(r)\}_m \otimes^L \mathbb{B}^\vee (d - r + 1) \longrightarrow \mu' \otimes \nu_Y^{d - 1} [-d].$$

Noting that $\{\mathbb{B}_m(r)\}_m$ is concentrated in $[0, r]$ with $\mathcal{H}^r(\mathbb{B}_m(r)) \cong U^1 \mathcal{M}_m$, and $\mathbb{B}^\vee (d - r + 1)$ is concentrated in $[0, d - r + 1]$ with $\mathcal{H}^{d - r + 1}(\mathbb{B}^\vee (d - r + 1)) \cong \alpha'_U U^1 M^d_{d + 1}$, we show the following:

**Lemma 5.1.** There is a unique morphism

$$(\ast 2) \ \{\mathbb{B}_m(r)\}_m \otimes^L (\alpha'_U U^1 M^d_{d + 1} [-d + r - 1]) \longrightarrow \mu' \otimes \nu_Y^{d - 1} [-d] \ \text{in} \ D^- (Y_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z})$$

for each $m$ that the morphism $\ast 1$ factors through.

**Proof.** One can check this by a similar argument as for Lemma 8.2.4 in [Sat2].

Applying a similar argument as for this lemma to the morphism $\ast 2$, we obtain a morphism

$$(\ast 3) \ \{U^1 \mathcal{M}_m\}_m [-r] \otimes^L \alpha'_U U^1 M^d_{d + 1} [-d + r - 1] \longrightarrow \mu' \otimes \nu_Y^{d - 1} [-d].$$

Because $\mathbb{Z}/p\mathbb{Z}$-sheaves are flat over $\mathbb{Z}/p\mathbb{Z}$, there is a natural isomorphism in $D^- (Y_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z})$:

$$\{U^1 \mathcal{M}_m\}_m [-r] \otimes^L \alpha'_U U^1 M^d_{d + 1} [-d + r - 1] \cong \{U^1 \mathcal{M}_m\}_m \otimes \alpha'_U U^1 M^d_{d + 1} [-d + r - 1]$$

induced by the identity map on the $(r + (d - r + 1))$-th cohomology sheaves.

We thus define the morphism $\Theta|_D$ by composing the inverse of this isomorphism and the morphism $\ast 3$ with shifting $[-1]$.

### 5.3. Explicit formula for $\Theta|_D$

We calculate the morphism $\Theta|_D$ explicitly below. We use the idea of the construction of maps of [Sat2]. Let $n$ and $q$ satisfy $1 \leq n \leq d$ and $1 \leq q \leq e' - 1$. We put $n' := d + 1 - n$. We define

$$\text{Symb}_{n', D} := \{r \otimes (i^* \psi \otimes \mathcal{O}_{U, K})^\otimes [r - 1] \otimes \mathcal{O}_Y (m' D)\}.$$

Here we remark that there is an isomorphism

$$\alpha'_U U^1 M^d_{d + 1} \cong \lim_{m'} U^1 M^{d - r + 1} \otimes \mathcal{O}_Y (m' D),$$

because $\text{gr}_1^1 M^{d - r + 1}$ is a quasi-coherent sheaf. The sheaf $U^q \mathcal{M}_m \otimes (U^{e - q} M^{d - r + 1} \otimes \mathcal{O}_Y (m' D))$ is a quotient of $\text{Symb}_{n', D}$.

We define the homomorphism of étale sheaves

$$F^{\text{Gr},\text{r}}_{n', D} : \text{Symb}^{\text{Gr},\text{r}}_{n', D} \longrightarrow \omega^{d - 1}/B^{d - 1}$$
by sending a local section \((1 + \pi^q \alpha_{1,m}) \otimes (\bigotimes_{i=1}^{r-1} \beta_{i,m}) \otimes (\bigotimes_{i=r}^{d-r} \beta'_{i,m'}) \otimes \gamma_{m'}\) with \(\alpha_{1,m} \in i^* \Theta_X(-mD), \beta_{i,m} \in i^* \psi_i \Theta_X^\times, \alpha_{2,m} \in i^* \Theta_X, \beta_{i,m} \in i^* j_* \Theta_X^\times, \gamma_{m'} \in \Theta_Y(m'D),\) to the following:

\[
(5.2) \quad q \cdot \alpha_{1,m} \cdot \alpha_{2,m} \cdot \gamma_{m'} \cdot \left( \bigwedge_{i=1}^{r-1} d \log \beta_{i,m} \wedge \bigwedge_{i=r}^{d-r} d \log \beta'_{i,m'} \right) + g^{-1} \left( \bigwedge_{i=1}^{r-1} d(\alpha_{2,m} \cdot \gamma_{m'}) \wedge \bigwedge_{i=r}^{d-r} d \log \beta'_{i,m'} \right) \pmod{\mathcal{B}_Y^{l-1}}.
\]

Here we put \(\mathfrak{r}\) its residue class in \(\Theta_Y\) for \(x \in i^* \Theta_X\) and put \(g\) the following \(\Theta_Y\)-linear isomorphism which is defined in \textbf{[Sat2]}, :\]

\[
g : \omega_{Y}^N = \omega_{(Y, Y')}/(s, N) \rightarrow d \log(\mathfrak{r}) \wedge \omega.
\]

In what follows, we put \(U^e-q M_{n,Y,m',D}^{d-r+1} := U^{e-a} M_{Y}^{d-r+1} \otimes \Theta_Y(m'D)\) for simplicity.

**Lemma 5.2.** (cf. \textbf{[Sat2]}, Lemma 8.3.4) Let \(r\) and \(q\) be as above. Then \(F_{q,r}^{m,D}\) factors through \(U^{q, \mathcal{M}^{-}}_m \otimes U^{e-a} M_{n,Y,m',D}^{d-r+1}\).

**Proof:** We prove this Lemma by the method of Lemma 8.3.4, \textbf{[Sat2]}. We denote \(Y^{\text{sing}}\) the singular locus of \(Y\), and let \(j_Y\) be the open immersion \(Y - Y^{\text{sing}} \rightarrow Y\). Since \(\omega_{Y}^N / \mathcal{B}_Y^{l}\) is a locally free \((\Theta_Y)^p\)-module, we may assume that \(Y\) is smooth over \(s = \text{Spec}(k)\). We prove that \(F_{q,r}^{m,D}\) factors \(\text{gr}_U^{q, \mathcal{M}^{-}} \otimes \text{gr}_{U}^{e-a} M_{n,Y,m',D}^{d-r+1}\) under the assumption that \(Y\) is smooth. We put \(\Omega_u_{Y,-m'D} := \Omega_{Y}^u \otimes \Theta_Y(m'D)\) below. Let

\[
\begin{align*}
(1) \quad \rho_{m,D}^{l,u} : & \omega_{Y,m,D}^{u-2} \otimes \omega_{Y,m,D}^{u-1} \rightarrow \{ \text{gr}_U^{l} \mathcal{M}^{-}_m \}^{m}, \\
(2) \quad \rho_{m,D}^{l,u} : & \omega_{Y,-m'D}^{u-2} \otimes \omega_{Y,-m'D}^{u-1} \rightarrow \text{gr}_U^{l} M_{n,Y,m',D}^{u},
\end{align*}
\]

be the map (cf. \textbf{[BK]}, (4.3)) defined as follows:

\[
\begin{align*}
(1') \quad & \begin{cases} (\alpha_m \cdot d \log \beta_1 \wedge \cdots \wedge d \log \beta_{u-2}, 0) & \rightarrow \quad (1 + \pi^{u} \tilde{\alpha}_m, \tilde{\beta}_1, \ldots, \tilde{\beta}_{u-2}, \pi) \mod U^{l+1} \mathcal{M}^{-}_m, \\
(0, \alpha_m \cdot d \log \beta_1 \wedge \cdots \wedge d \log \beta_{u-1}) & \rightarrow \quad (1 + \pi^{u} \tilde{\alpha}_m, \tilde{\beta}_1, \ldots, \tilde{\beta}_{u-1}, \pi) \mod U^{l+1} \mathcal{M}^{-}_m,
\end{cases} \\
(2') \quad & \begin{cases} (\eta_m \cdot d \log \beta_1 \wedge \cdots \wedge d \log \beta_{u-2}, 0) & \rightarrow \quad (1 + \pi^{u} \tilde{\eta}_m, \tilde{\beta}_1, \ldots, \tilde{\beta}_{u-2}, \pi) \mod U^{l+1} M_{n,Y,m',D}^{u}, \\
(0, \eta_m \cdot d \log \beta_1 \wedge \cdots \wedge d \log \beta_{u-1}) & \rightarrow \quad (1 + \pi^{u} \tilde{\eta}_m, \tilde{\beta}_1, \ldots, \tilde{\beta}_{u-1}, \pi) \mod U^{l+1} M_{n,Y,m',D}^{u},
\end{cases}
\end{align*}
\]

for \(\alpha_m \in \Theta_Y(-mD), \eta_m \in \Theta_Y(m'D)\) and each \(\beta_i \in \Theta_X^\times\), where \(\tilde{\alpha}_m \in \Theta_X^\times (-mD)\) (resp. \(\tilde{\beta}_i \in \Theta_X^\times\)) denotes a lift of \(\alpha_m\) (resp. \(\beta_i\)). We have the following short exact sequences which is considered in \textbf{[BK]} Lemma 4.5:

\[
\begin{align*}
(a) \quad & 0 \rightarrow \omega_{Y,m,D}^{u-2} \otimes \omega_{Y,m,D}^{u-1} \otimes \omega_{Y,m,D}^{u} \rightarrow \text{gr}_U^{l} \mathcal{M}^{-}_m \rightarrow 0 \quad (p \not\mid l), \\
(b) \quad & 0 \rightarrow \omega_{Y,-m'D}^{u-2} \otimes \omega_{Y,-m'D}^{u-1} \otimes \omega_{Y,-m'D}^{u} \rightarrow \text{gr}_U^{l} M_{n,Y,m',D}^{u} \rightarrow 0 \quad (p \not\mid l), \\
(c) \quad & 0 \rightarrow \omega_{Y,-m'D}^{u-2} \otimes \omega_{Y,-m'D}^{u-1} \otimes \omega_{Y,-m'D}^{u} \rightarrow \text{gr}_U^{l} M_{n,Y,m',D}^{u} \rightarrow 0 \quad (p \not\mid l),
\end{align*}
\]
Here \( h^\ell_{l,n} \) and \( \theta_{l,n}^\ell \) are given by \( \omega \mapsto \((-1)^l \cdot l \cdot \omega, \, d\omega) \). We define the following maps:

\[
h^\ell_{l,D} : i^* U^0 \left( 1 + I_{mD} \right)^{\times} \otimes \left( i^* \psi_s \Theta_{U_K}^\ell \right) \otimes \left( (-1)^{u-1} \right) \longrightarrow \Omega^u \otimes \Omega^u_{mD};
\]

\[
(1+\pi \alpha_m) \otimes \left( \otimes_{i=1}^{u-1} \beta_i \right) \mapsto \begin{cases} 
(0, \pi \cdot \Lambda_{1\leq i\leq u-1} d \log \beta_i) & \text{if } \beta_i \in \pi \Theta_X^\ell \text{ for all } i, \\
(-1)^{u-1} \cdot \pi \cdot \Lambda_{1\leq i\leq u-1, i \neq i'} d \log \beta_i, 0) & \text{if } \beta_i = \pi \text{ for exactly one } i \neq i', \\
(0, 0) & \text{otherwise},
\end{cases}
\]

\[
h^\ell_{l,m} : U_{X_K} \otimes \left( i^* \psi_s \Theta_{U_K}^\ell \right) \otimes \Theta_Y(m'D_s) \longrightarrow \Omega^u \otimes \Omega^u_{mD};
\]

\[
(1+\pi \alpha) \otimes \left( \otimes_{i=1}^{u-1} \beta_i \right) \mapsto \begin{cases} 
(0, \gamma \cdot \pi \cdot \Lambda_{1\leq i\leq u-1} d \log \beta_i) & \text{if } \beta_i \in i^* \Theta_X^\ell \text{ for all } i, \\
(-1)^{u-1} \cdot \gamma \cdot \Lambda_{1\leq i\leq u-1, i \neq i'} d \log \beta_i, 0) & \text{if } \beta_i = \pi \text{ for exactly one } i \neq i', \\
(0, 0) & \text{otherwise},
\end{cases}
\]

with \( \alpha \in i^* \Theta_X \), \( \alpha_m \in i^* \Theta_X(-mD) \), \( \gamma_m \in \Theta_Y(m'D_s) \) and \( \beta_i \in i^* \Theta_X^\ell \cup \{ \pi \} \). Here for \( x \in i^* \Theta_X \) (resp. \( x \in i^* \Theta_X^\ell \)), \( \pi \) denotes its residue class in \( \Theta_Y \) (resp. \( i^* \Theta_X^\ell \)). We consider a commutative diagram (cf. 
§8.3, [Sat2]):

\[
\text{symbol map} \quad \begin{array}{c}
\text{Symb}^q_{m,n} \left( \otimes_{i=1}^{u-1} \pi^l \cdot \Lambda_{1\leq i\leq u-1} d \log \beta_i \right) \\
\otimes \left( (-1)^{u-1} \cdot \pi \cdot \Lambda_{1\leq i\leq u-1, i \neq i'} d \log \beta_i, 0) \right) & \otimes \quad \text{mod } \mathcal{B}_N
\end{array} \]

where \( \varphi^{q,r} \) is defined as for §8.3, [Sat2]:

\[
(\omega_1 \otimes \gamma_{1,m}, \, \omega_2 \otimes \gamma_{2,m}) \ominus (\omega_3 \otimes \delta_{1,m}, \, \omega_4 \otimes \delta_{2,m})
\]

\[
\mapsto \begin{pmatrix} q \cdot \gamma_{2,m} \cdot \delta_{2,m} (\omega_2 \wedge \omega_4) + (-1)^{r-1} \gamma_{1,m} \cdot \delta_{2,m} d(\omega_4) \wedge \omega_4 + (-1)^{d-r} \gamma_{2,m} \cdot \delta_{1,m} (\omega_2 \wedge d(\omega_3)) \end{pmatrix}
\]

mod \( \mathcal{B}_N \).

We check the composition map \((G :=) \varphi^{q,r}_D \circ (h^{q,r} \otimes h^{q',d-r+1}) = F^{q,r}_{m,n}) \). We take any local section \( \tau := (1 + \pi^q \alpha_{1,m}) \otimes \left( \otimes_{i=1}^{r-1} \beta_i \right) \otimes (1 + \pi^{e-q} \alpha_2) \otimes \left( \otimes \beta_{i,m} \right) \otimes \gamma_{m'} \) in Symb^{q,r}_{m,n}. We can consider (3×3)-cases, but we calculate
the following cases (other cases left to the reader):

- If $\beta_{i,m} \in i^* G^X$ for all $i$ and $\beta_{i,m'} = \pi$ for exactly one $i'$, we denote by $k$.

$$G(\tau) = \varphi_{D,n}^{\tau'} \left( \left( 0, \pi_{1,\tau} \cdot \bigwedge_{1 \leq r \leq r-1} d \log \beta_{i,m} \right), \left( (-1)^{d-r-i'} \cdot \gamma_{m'} \cdot \pi_{2,\tau} \cdot \bigwedge_{r \leq s \leq r-i'} d \log \beta_{i,m'} \cdot 0 \right) \right)$$

$$= (-1)^{d-r-i'} \left( \pi_{1,\tau} \cdot \bigwedge_{1 \leq r \leq r-1} d \log \beta_{i,m} \right) \land d \left( (-1)^{d-r-i'} \cdot \gamma_{m'} \cdot \pi_{2,\tau} \cdot \bigwedge_{r \leq s \leq r-i'} d \log \beta_{i,m'} \right)$$

$$= \pi_{1,\tau} d(\gamma_{m'} \cdot \pi_{2,\tau}) \land \bigwedge_{1 \leq r \leq r-1} d \log \beta_{i,m} \land \bigwedge_{r \leq s \leq r-i'} d \log \beta_{i,m'}$$

$$= F_{m,D}(\tau) \mod B^{d-1}_{V}.$$  

Hence it suffices to prove that the subsheaf $\ker(P^{q,r}_{m,D} \otimes P_{m'}^{e'-q,d-r+1})$ of $\left( O^u_{Y[m,D] \oplus O^{u-1}_{Y[m,D]}} \right) \oplus (O^u_{Y[m,m'-D] \oplus O^{u-1}_{Y[m,m'-D]}})$ has trivial image under $\varphi_{D,n}^{q,r}$. We only consider the case (i) $p \mid l$ for simplicity. For any element $x = (x_1, x_2) \in \ker(P^{q,r}_{m,D} \otimes P_{m'}^{e'-q,d-r+1})$, there exists an element $\eta \otimes \gamma_{1,m'} \in O^u_{Y[m,D]}$ such that $\theta^{u}_{m,D}(\eta \otimes \gamma_{1,m'}) = x_1$ and there exists an element $\eta_2 \otimes \gamma_{2,m'} \in O^u_{Y[m,m'-D]}$ such that $\theta^{u}_{m,D}(\eta_2 \otimes \gamma_{2,m'}) = x_2$ by the exact sequences (a) and (c). Then we can compute $\varphi_{D,n}^{q,r}(x)$ as follows:

$$\varphi_{D,n}^{q,r}(x) = \varphi_{D,n}^{\tau'} \left( ((-1)^r \eta_1 \otimes \gamma_{1,m'}, \eta_2 \otimes \gamma_{2,m'}), \left( (-1)^{d-r}(e' - q) \cdot \eta_2 \otimes \gamma_{2,m'}, d \eta_2 \otimes \gamma_{2,m'} \right) \right)$$

$$= ((-1)^r \eta_1 \cdot \gamma_{1,m'} d \eta_1 \wedge d \eta_2 + (-1)^{d-r} \eta_2 d \eta_2) \mod B^{d}_{V}$$

Since $d(\eta_1 \wedge d \eta_2) = d \eta_1 \wedge d \eta_2$, we obtain the assertion in the case (i). This completes the proof.

\[\square\]

**Definition 5.3.** *cf. [Sat2], Definition 8.3.6* For $\zeta(\neq 1) \in \mu_{p}(K)$, let $v(\zeta) \in k^X$ be the residue class of $(1 - \zeta)/\pi^{e(p-1)} \in G^X$. We put $w := \zeta \otimes v(\zeta)^{-p} \in \mu \otimes k$ (see Definition 8.3.6 in [Sat2]). We define the homomorphism as for $f_{m,n}^{q,r}$ in Definition 8.3.6 in [Sat2]

$$f_{m,D}^{q,r} : U^q \mathcal{M}_{m} \otimes U^{e'-q} M_{n,Y,m'}^{d-r+1} \longrightarrow (\mu \otimes k) \otimes \omega_N^Y/B_Y$$

as $w \otimes (-1)^{N+n} F^{q,r}_{m,D}$. Here $k$ is the constant sheaf on $Y_{et}$ associated with $k$. We regard $w \in \mu \otimes k$ as a global section of $\mu \otimes k$ and $F^{q,r}_{m,D}$ denotes the map induced by $f_{m,n}^{q,r}$. The map $f_{m,D}^{q,r}$ is independent of the choice of $\pi$ by the definition of $w$ and $F^{q,r}_{m,D}$.

We state the main result of this section.
**Theorem 5.4.** (Explicit reciprocity law, cf. Theorem 8.3.8, [Sat2]) We have the following commutative square of pro-systems in $D^b(Y_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$

$$
\begin{array}{ccc}
\{U^q M_m^\tau\}_m \otimes \{U^{e'-q} M_{n,Y,m'D}^{d-r+1}\}_m' & \xrightarrow{\text{canonical}} & \mu' \otimes \omega^N_Y/B_Y^N \\
\{U^1 M_m^\tau\}_m \otimes \{U^1 M_{n,Y,m'D}^{d-r+1}\}_m' & \xrightarrow{\Theta_{m,D}^{r,d+2}} & \mu' \otimes \nu^N_{Y,m}[1],
\end{array}
$$

for $(q, r)$ with $1 \leq q \leq e' - 1$ and $1 \leq r \leq d$. Here $\chi$ is defined in §8.3, [Sat2].

We will prove Theorem 5.4 in two steps below by the same arguments as for the proof of Theorem 8.3.8, [Sat2].

**I. Reduction to cohomology groups (cf. §8.4, [Sat2]):**

Without loss of generality, we may assume that $X$ is connected. Thus by Proposition 8.4.1, [Sat2] for $i = N$, we have

$$
\text{Hom}_{D^b(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})} \left( \{U^q M_m^\tau\}_m \otimes \{U^{e'-q} M_{n,Y,m'D}^{d-r+1}\}_m', \mu' \otimes \nu^N_{Y,m}[1] \right)
\cong \text{Hom} \left( H^N(Y, \{U^q M_m^\tau\}_m \otimes \{U^{e'-q} M_{n,Y,m'D}^{d-r+1}\}_m'), \mu \otimes H^{N+1}(Y, \nu^r_Y) \right).
$$

Therefore we are reduced to the equality of maps on cohomology groups

$$
(\otimes 1) \quad H^N(Y, \Theta^{q,r}_{m,D}) = H^N(Y, \chi \circ f_{m,D}^{q,r}),
$$

where we denote $\Theta^{q,r}_{m,D}$ the composite morphism

$$
\Theta^{q,r}_{m,D} : \{U^q M_m^\tau\}_m \otimes \{U^{e'-q} M_{n,Y,m'D}^{d-r+1}\}_m' \xrightarrow{\text{canonical}} \{U^1 M_m^\tau\}_m \otimes \{U^1 M_{n,Y,m'D}^{d-r+1}\}_m' \xrightarrow{\Theta_{m,D}^{r,d+2}} \mu' \otimes \nu^N_{Y,m}[1].
$$

**II. Reduction to higher local fields (cf. §8.5, [Sat2]):**

In this subsection, we reduce $(\otimes 1)$ to $(\otimes 2)$ below by Sato’s trick used in §8.5, [Sat2]. For $K \in D^b(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$, we can consider the map

$$
\delta_Y(K) : \bigoplus_{(y_0, y_1, \ldots, y_N) \in \text{Ch}(Y)} H^0(y_N, K) \to H^N(Y, K).
$$

Here $(y_0, y_1, \ldots, y_N) \in \text{Ch}(Y)$ is a chain on $Y$ (see §8.5, [Sat2] for details). We obtain the following Lemma by Sublemma 8.5.2, [Sat2]:

**Lemma 5.5.** The map $\delta_Y \left( \{U^q M_m^\tau\}_m \otimes \{U^{e'-q} M_{n,Y,m'D}^{d-r+1}\}_m' \right)$ is surjective.

By the above lemma, $(\otimes 1)$ is reduced to the formula

$$
(\otimes 2) \quad H^0(y_N, \Theta^{q,r}_{m,D}) = H^0(y_N, \chi \circ f_{m,D}^{q,r}),
$$
for all chains \((y_0, y_1, \ldots, y_N) \in \text{Ch}(Y)\), which will be proved in next subsection below.

5.4. **Proof of (2).** We begin the proof of the (2) by the same manner of the proof of (8.5.3) in §8.7, [Sat2]. We fix an arbitrary chain \((y_0, y_1, \ldots, y_N) \in \text{Ch}(Y)\). Put \(F_N := \kappa(y_N)\) and \(L_{N+1} := \text{Frac}\left[\cdots((\Theta_{X,0}^{h_0}, y_1, \ldots, y_N)^h_{Y_N})\right]\) which is a henselian discrete valuation field of characteristic 0 with residue field \(F_N\). Let \(F/F_N\) be a finite separable field extension. We put \(y := \text{Spec}(F)\) and

\[
\rho^{q,r}_m(F) := H^0(y, U^q \mathcal{M}^r_m) \otimes H^0(y, U^{d-r-q} M_{n,Y}^{d-r+1}) \subset H^0(y, U^q \mathcal{M}^r_m \otimes U^{d-r-q} M_{n,Y}^{d-r+1}).
\]

We use Lemma 8.6.1, [Sat2] for the subfield \((F_N)^p \subset F_N\), the formula (2) is reduced to the formula

\[
(\oplus 3) \quad H^0(y, \Theta_{Y,D}^{q,r}_m|_{\rho^{q,r}_m(F)}) = H^0(y, \alpha \circ f^{q,r}_{m,D}|_{\rho^{q,r}_m(F)}).
\]

Here we have \(H^0(y, U^q \mathcal{M}^r_m) \cong H^0(y, U^q M^r)\) and \(H^0(y, U^{d-r-q} M_{n,Y}^{d-r+1}) \cong H^0(y, U^{d-r-q} M_{n,Y}^{d-r+1})\). Thus we obtain the (2) by the same argument as for §8.5 in [Sat2]. This completes the proof of Theorem 5.4.

6. **Duality for \(\mathcal{M}^r_m\)**

In this section, we prove the following Theorem 6.1 which will be used in §8.

**Theorem 6.1.** (cf. Theorem 9.1.1, [Sat2]) Let \(r \geq 1\), the following pairing induced by \(\Theta_{r,m,D}\) and \(\text{tr}_Y\):

\[
\lim_{m} H^i(Y, U^1 \mathcal{M}_m^r) \times H^{2d+1-i}(Y, \alpha U^1 M_{n,Y}^{d-r+1}) \xrightarrow{\Theta_{r,m,D}} \mu \otimes H^d(Y, U^{d-1}) \xrightarrow{\text{tr}_Y} \mu
\]

is a non-degenerate pairing.

We will calculate the map \(f^{q,r}_{m,D}\) constructed in §5. We recall and define some filtrations on \(M_{n,Y}^{d-r+1}\) and \(M^r_m\).

**Definition 6.2.** (§9 in [Sat2] and §3 in [Y])

\(1\) We define the subsheaf \(T^q M_Y^r \subset U^q M_Y^r\) \((q \geq 1)\) as the part generated by \(V^q M_Y^r\) and symbols of the form

\[
\{1 + \pi^q \alpha^p; \beta_1, \ldots, \beta_{r-1}\}
\]

with \(\alpha \in i^* \mathcal{O}_X\) and each \(\beta_i \in i^* \mathcal{O}_X^\times\).

\(2\) We similarly define the subsheaf \(T^q \mathcal{M}_m^r \subset U^q \mathcal{M}_m^r\) \((q \geq 1)\) as the part generated by \(V^q \mathcal{M}_m^r\) and symbols of the form

\[
\{1 + \pi^q \alpha^p; \beta_1, \ldots, \beta_{r-1}\}
\]

with \(\alpha \in i^* \mathcal{O}_X(-mD)\) and each \(\beta_i \in i^* \mathcal{O}_U^\times\) for each \(m\).
Lemma 6.3. (cf. Lemma 9.1.4, [Sat2]) Assume that $1 \leq q \leq e' - 1$. Then:
(1) $f_{m,D}^q$ annihilates the subsheaf of $\{U^q, M^r_m\}_m \otimes \{U^q M^d_{r+1,m'}\}_m$ generated by $\{U^{q+1}, M^r_m\}_m \otimes \{U^{q+1} M^d_{r+1,m'}\}_m$, $\{V^q, M^r_m\}_m \otimes \{V^q M^d_{r+1,m'}\}_m$, and $\{T^q, M^r_m\}_m \otimes \{T^q M^d_{r+1,m'}\}_m$.

(2) The composite map

$$\frac{(\omega^{r-1}_{Y|m'} / Z^{r-1}_{Y|m'})_m \otimes \{\omega^{d-r}_{Y|m'} / Z^{d-r}_{Y|m'}\}_m}{\langle f_{3,m} \otimes f_{3,m} \rangle} \rightarrow \{\omega^{d-r}_{Y|m'} / Z^{d-r}_{Y|m'}\}_m \otimes \{\omega^{d-r}_{Y|m'} / Z^{d-r}_{Y|m'}\}_m$$

sends a local section $\{x_m \otimes \gamma_1, m\}_m \otimes \{y_m \otimes \gamma_2, m\}_m$ to $w \otimes k (-1)^r$. Similarly, the composite map

$$\frac{(\omega^{r-2}_{Y|m'} / Z^{r-2}_{Y|m'})_m \otimes \{\omega^{d-r+1}_{Y|m'} / Z^{d-r+1}_{Y|m'}\}_m}{\langle f_{2,m} \otimes f_{2,m} \rangle} \rightarrow \{\omega^{d-r+1}_{Y|m'} / Z^{d-r+1}_{Y|m'}\}_m \otimes \{\omega^{d-r+1}_{Y|m'} / Z^{d-r+1}_{Y|m'}\}_m$$

sends a local section $\{x_m \otimes \gamma_1, m\}_m \otimes \{y_m \otimes \gamma_2, m\}_m$ to $w \otimes k (-1)^N$.

(3) If $q$ is prime to $p$, then the composite map

$$\frac{(\omega^{r-1}_{Y|m'} / Z^{r-1}_{Y|m'})_m \otimes \{\omega^{d-r+1}_{Y|m'} / Z^{d-r+1}_{Y|m'}\}_m}{\langle f_{1,m} \otimes f_{1,m} \rangle} \rightarrow \{\omega^{d-r+1}_{Y|m'} / Z^{d-r+1}_{Y|m'}\}_m \otimes \{\omega^{d-r+1}_{Y|m'} / Z^{d-r+1}_{Y|m'}\}_m$$

sends a local section $\{x_m \otimes \gamma_1, m\}_m \otimes \{y_m \otimes \gamma_2, m\}_m$ to $w \otimes k (-1)^{N+r} q$.

Proof: The proof of these assertions are straightforward.

6.1. Proof of Theorem 6.1

In this subsection, we prove Theorem 6.1 by a similar arguments as for §9.2, [Sat2]. Since the sheaves $U^1, M^r_m$ and $U^1 M^d_{r+1,m'}$ are finitely successive extension of coherent $(\mathcal{O}_Y)^p$-modules, then we have the finiteness of the groups in the pairing in Theorem 6.1 for each $m$. We introduce a descending filtration $Z^s M^q$ on $U^1, M^r_m$ and $Z^s M^q$ on $U^1 M^d_{r+1,m'}$ defined as follows:

$$Z^s M^q = \begin{cases} U^m M^q & \text{if } s \equiv 1 \mod 3, n = (s + 2)/3, \\ T^m M^q & \text{if } s \equiv 2 \mod 3, n = (s + 1)/3, \\ V^m M^q & \text{if } s \equiv 0 \mod 3, n = s/3, \end{cases}$$

$$Z^s M^q_{Y,m'} = \begin{cases} U^m M^q_{Y,m'} & \text{if } s \equiv 1 \mod 3, n = (s + 2)/3, \\ T^m M^q_{Y,m'} & \text{if } s \equiv 2 \mod 3, n = (s + 1)/3, \\ V^m M^q_{Y,m'} & \text{if } s \equiv 0 \mod 3, n = s/3. \end{cases}$$
We have that the following Proposition.

**Proposition 6.4.** ([Sat2], §9.2) Let \( l \) be an integer \( 1 \leq l \leq 3e' - 3 \). We have the following:

(i) There is a map

\[
H^N(Y, (U^1 \mathcal{M}_m/Z^{l+1} \mathcal{M}_m) \otimes Z^{3e' - 2 - l} M_{Y,m'}^l) \rightarrow \mu.
\]

Here this map is induced by a map

\[
\alpha : H^N(Y, U^1 \mathcal{M}_m^r \otimes Z^{3e' - 2 - r} M_{Y,m'}^r) \rightarrow H^N(Y, U^1 \mathcal{M}_m^r \otimes U^1 M_{Y,m'}^r) \rightarrow \mu \otimes H^{N+1}(Y, \nu_N^Y) \cong \mu,
\]

(ii) There is a map

\[
H^N(Y, \text{gr} Z M_l^m \otimes Z^{3e' - 2 - l} M_{Y,m'}^l) \rightarrow \mu.
\]

Here this map is induced by a map

\[
H^N(Y, \text{gr} Z M_l^m \otimes Z^{3e' - 2 - l} M_{Y,m'}^l) \rightarrow H^N(Y, (U^1 \mathcal{M}_m^r/Z^{r+1} \mathcal{M}_m) \otimes Z^{3e' - 2 - r} M_{Y,m'}^r) \rightarrow \mu.
\]

(iii) There is a map

\[
H^N\left(Y, \frac{(U^1 \mathcal{M}_m^r/Z^{l+1} \mathcal{M}_m) \otimes Z^{3e' - 2 - l} M_{Y,m'}^l}{\text{gr} Z \mathcal{M}_m^r \otimes Z^{3e' - 2 - l} M_{Y,m'}^l}\right) \rightarrow \mu.
\]

We put \( I := \frac{(U^1 \mathcal{M}_m^r/Z^{l+1} \mathcal{M}_m) \otimes Z^{3e' - 2 - l} M_{Y,m'}^l}{\text{gr} Z \mathcal{M}_m^r \otimes Z^{3e' - 2 - l} M_{Y,m'}^l} \) for simplicity. If \( r \geq 2 \), there is a commutative diagram:

\[
\begin{array}{ccc}
H^N(Y, \text{gr} Z \mathcal{M}_m^r \otimes Z^{3e' - 2 - l} M_{Y,m'}^l) & \oplus & H^N(Y, \frac{U^1 \mathcal{M}_m^r/Z^{l+1} \mathcal{M}_m}{\text{gr} Z \mathcal{M}_m^r \otimes Z^{3e' - 2 - l} M_{Y,m'}^l}) \\
\mu + \mu & \mapsto & \mu
\end{array}
\]

The map (iii) is induced by the map (i).

**Proof.** The proof is the same way as for Lemma 9.2.1 in [Sat2]. We show only the case (i). One can show that the cases (ii) and (iii) are similar way. There is a map

\[
F : H^N\left(Y, Z^l \mathcal{M}_m^r \otimes Z^{3e' - 2 - l} M_{Y,m'}^r\right) \rightarrow H^N\left(Y, U^1 \mathcal{M}_m^r \otimes Z^{3e' - 2 - l} M_{Y,m'}^r\right) \rightarrow \mu.
\]
Theorem 3.4 implies that $F = \chi \circ f_D^r$. We consider the following short exact sequence

$$0 \to Z^{l+1} \cdot \mathcal{M}_m^r \otimes Z^{3e'-2-l}M_{Y,m'}^r \to U^1 \cdot \mathcal{M}_m^r \otimes Z^{3e'-2-l}M_{Y,m'}^r \to \left( U^1 \cdot \mathcal{M}_m^r / Z^{l+1} \cdot \mathcal{M}_m^r \right) \otimes Z^{3e'-2-l}M_{Y,m'}^r \to 0.$$ 

By this exact sequence, we have the following diagram:

$$H^N \left( Z^{l+1} \cdot \mathcal{M}_m^r \otimes Z^{3e'-2-l}M_{Y,m'}^r \right) \xrightarrow{f_1} H^N \left( U^1 \cdot \mathcal{M}_m^r \otimes Z^{3e'-2-l}M_{Y,m'}^r \right) \xrightarrow{f_2} H^N \left( U^1 \cdot \mathcal{M}_m^r \otimes Z^{3e'-2-l}M_{Y,m'}^r \right).$$

Here the map $f_1$ is surjective by SubLemma 8.5.2 (2) in [Sat2] and $g_1 \circ f_1$ is a zero map by $f_D^r$ annihilates the subsheaf $Z^{l+1} \cdot \mathcal{M}_m^r \otimes Z^{3e'-2-l}M_{Y,m'}^r$ by Lemma 6.3 (1). Then we get the map $g_2$. 

By the same arguments as in [Sat2], §9.2, we obtain the following pairings and Lemma below:

$$i_D^{i,l} : H^i(Y, U^1 \cdot \mathcal{M}_m^r / Z^{l+1} \cdot \mathcal{M}_m^r) \times H^N-i(Y, Z^{3e'-2-l}M_{Y,m'}^{d-r+1}) \to \mu,$$

$$c_D^{i,l} : H^i(Y, \text{gr}_Z^l \cdot \mathcal{M}_m^r) \times H^N-i(Y, \text{gr}_Z^{3e'-2-l}M_{Y,m'}^{d-r+1}) \to \mu,$$

for $i$ and $l$ with $1 \leq l \leq 3e' - 3$.

**Lemma 6.5.** (cf. Lemma 9.2.7, [Sat2]) The pairing $c_D^{i,l}$ is non-degenerate for any $i$ and $l$ with $1 \leq l \leq 3e' - 3$.

**Proof.** See the proof of Lemma 9.2.7 in [Sat2] for the brief explanation of a linear Cartier isomorphism $C_{\text{lin}}$. We consider the following maps:

$$F_{Y/s}^r(\omega_{Y|mD}^{r-1} / Z_{Y|mD}^{r-1}) \times F_{Y/s}^r(\omega_Y^{d-r} / Z_Y^{d-r} \otimes \mathcal{O}_Y(mD)) \to \omega_Y^d,$$

$$(x \otimes \gamma_1, y \otimes \gamma_2) \mapsto C_{\text{lin}}(\gamma_1 \cdot \gamma_2 \cdot dx \wedge y),$$

$$F_{Y/s}^r(\omega_{Y|mD}^{r-1} / Z_{Y|mD}^{r-1}) \times F_{Y/s}^r(\omega_Y^{d-r+1} / Z_Y^{d-r+1} \otimes \mathcal{O}_Y(mD)) \to \omega_Y^d,$$

$$(x \otimes \gamma_1, y \otimes \gamma_2) \mapsto C_{\text{lin}}(\gamma_1 \cdot \gamma_2 \cdot x \wedge y),$$

$$F_{Y/s}^r(\omega_{Y|mD}^{r-2} / Z_{Y|mD}^{r-2}) \times F_{Y/s}^r(\omega_Y^{d-r+1} / Z_Y^{d-r+1} \otimes \mathcal{O}_Y(mD)) \to \omega_Y^d,$$

$$(x \otimes \gamma_1, y \otimes \gamma_2) \mapsto C_{\text{lin}}(\gamma_1 \cdot \gamma_2 \cdot x \wedge dy).$$

By 3.1, [Hy] and the Serre-Hartshorne duality, $\omega_Y^d$ is a dualizing sheaf on $\tilde{Y}$ in the sense of Definition, p.241, [Har]. We put $q$ the maximal integer such that $3(q - 1) < l$. Then the pairing

$$H^N(Y, \text{gr}_Z^l \cdot \mathcal{M}_m^r) \times H^N-i(Y, \text{gr}_Z^{3e'-2-l}M_{Y,m'}^{d-r+1}) \xrightarrow{f_m^l} \mu \otimes H^N(Y, \omega_Y^{N} / B_Y^{N}) \xrightarrow{id \otimes i_Y^l} \mu \otimes k$$

is non-degenerate.
We show the non-degeneracy of Lemma and Theorem 6.1. This completes the proof of the theorem: is a non-degenerate pairing. Here \( τ_1 \) is the \( k \)-linear map (see §9.3, [Sat2]). We show the non-degeneracy of \( \tau_1 \) by the commutative diagrams:

\[
\begin{array}{ccc}
H^N(Y, \mathcal{O}_Y) & \xrightarrow{\mu} & H^N(Y, \mathcal{O}_Y) \\
\downarrow{\text{(*)}} & & \downarrow{\text{(*)}} \\
\mu & \xrightarrow{id} & \mu \\
\end{array}
\]

where the left square commutes by Theorem 5.4 and the right square commutes by Remark 2.2.6(4), [Sat2] and 3.4.1, [Sat1]. The morphism (\( \tau_1 \)) is given by the same arguments as for Lemma 9.2.1 (2), [Sat2]. This completes the proof of Lemma and Theorem 6.1

\[ \square \]

7. Proof of Theorem 1.1

First, we define the following pairing, which is a generalization of Milne's pairing of two-term complexes:

**Definition 7.1.** (cf. [Mil2], p.175, [Mil1], p.277)

Let

\[ \mathcal{F}^\bullet := \left[ \mathcal{F}^0 \xrightarrow{d_0} \mathcal{F}^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \mathcal{F}^n \rightarrow \cdots \right], \]

\[ \mathcal{G}^\bullet := \left[ \mathcal{G}^0 \xrightarrow{d_0} \mathcal{G}^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \mathcal{G}^n \rightarrow \cdots \right], \]

and

\[ \mathcal{H}^\bullet := \left[ \mathcal{H}^0 \xrightarrow{d_0} \mathcal{H}^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \mathcal{H}^n \rightarrow \cdots \right] \]

be complexes of sheaves. A pairing of complexes \( \langle \cdot, \cdot \rangle : \mathcal{F}^\bullet \times \mathcal{G}^\bullet \rightarrow \mathcal{H}^\bullet \) is a system of pairings

\[ \langle \langle \cdot, \cdot \rangle \rangle_{i,j,k} : \mathcal{F}^i \times \mathcal{G}^j \rightarrow \mathcal{H}^k \]

such that

\[ \sum_{i+j=n} d_{ij}^\mathcal{F} \left( \langle x_i, y_j \rangle \right)^{n+1} = \sum_{i+j=n} \left\{ \langle d_\mathcal{F}^i (x_i), y_j \rangle \right\}^{n+1} + (-1)^i \left\{ \langle x_i, d_\mathcal{G}^j (y_j) \rangle \right\}^{n+1} \]

for all \( n \geq 0, x_i \in \mathcal{F}^i \), and \( y_j \in \mathcal{G}^j \).

**Remark 7.2.** Milne defines a pairing of two term complexes in [Mil2], p.175 or [Mil1], p.217. In [JSZ], they have used Milne's two-term pairing to prove their results.

The aim of this section is to construct the following pairing

\[ \lim_{\mathcal{M}} H^i(Y, \mathcal{T}_n(r)_{X_{\mathcal{M}}}^{\text{sym}}(X_{\mathcal{M}D})) \times H^{2d+1-i}(Y_{\mathcal{M}D}(d-r)_{U}) \rightarrow \mathbb{Z}/p^n, \]

which is equivalent to giving the following pairing by the proper base change theorem:

\[ \lim_{\mathcal{M}} H^i(Y, \mathcal{T}_n(r)_{X_{\mathcal{M}}}^{\text{sym}}(X_{\mathcal{M}D})) \times H^{2d+1-i}(Y, R^iR_{\mathcal{M}}\mathcal{O}r, \mathcal{T}_n(d-r)_{U}) \rightarrow \mathbb{Z}/p^n. \]
For this purpose, we construct a morphism
\[
(\Upsilon) \quad \{\Sigma_n(r)_{X|\text{mD}}\}_{m} \otimes^\mathbb{L} R^1\alpha_+\Sigma_n(d - r)_U \to \nu^{d-1}_Y [-d - 1].
\]

We first consider a presentation as complex of \(R^1\alpha_+\Sigma_n(d - r)_U\). We have \(R^1\alpha_+\Sigma_n(d - r)_U = R\alpha'_+ R^d\Sigma_n(d - r)_U\) since the diagram in Setting (\(\Phi\)) is commutative. There is a spectral sequence \(E^2_{p,q} = R^p\alpha'_+ \mathcal{H}^q(R^d\Sigma_n(d - r)_U) \Rightarrow H^{p+q}R\alpha'_+ R^d\Sigma_n(d - r)_U\) and we have \(E^2_{p,q} = 0\) for \(p \geq 2\). We introduce an augmented complex \(s_n(q)_{X'|\text{m}'D}\) to construct a pairing \((\Upsilon)\):

**Definition 7.3.** (cf. [1su1], [1su2], [1su3])

We define a complex \(s_n(q)_{X'|\text{m}'D}\), which degree \(i\)-part defined by

\[
s_n(q)_{X'|\text{m}'D} := \left(\begin{array}{c}
\text{Coker}

(n + i') D_n \to s_n(d - r)_{X|\text{m}'D} \\
\Omega
\end{array}\right).
\]

Its differential is given by \(d^i(x,y) := (dx, (1 - \varphi_q)x - dy)\). Here we take \(m'\) satisfies \(m' + i - m \leq 0\).

We consider a complex (as a pro-system)

\[
\mathcal{G}_{m'} := \left[\begin{array}{c}
\text{Coker}

(n + d - r)_{X|\text{m}'D} \to s_n(d - r)_{X|\text{m}'D} \\
\Omega
\end{array}\right],
\]

where \(\Omega = \text{Ker} \left(\mathcal{H}^{d-r}(s_n(d - r)_{X'|\text{m}'D}) \to \nu^{d-r-1}_Y \otimes \mathcal{O}_Y (m'D_n)\right)\). Here the first term of \(\mathcal{G}_{m'}\) is a degree \(d - r + 1\). The transition map of \(\mathcal{G}_{m'}\) is a differential \(d^i\) of \(s_n(d - r)_{X'|\text{m}'D}\). We check the well-definedness of the map \(\mathcal{G}_{m'}\). It is easy to see that the map \(d^{d-r}\) induces \(\text{Coker} \left(s_n(d - r)_{X|\text{m}'D} \to s_n(d - r)_{X|\text{m}'D}\right) = \nu^{d-r-1}_Y \otimes \mathcal{O}_Y (m'D_n)\). Then we obtain the following representation (quasi-isomorphism) of \(R\alpha'_+ R^d\Sigma_n(d - r)_U\) by using the calculation of the cohomology sheaf of \(R^d\Sigma_n(d - r)_U\) in [Sat2]:

\[
R\alpha'_+ R^d\Sigma_n(d - r)_U \cong \lim_{m'} \mathcal{G}_{m'}.
\]

Taking the cohomology of the right hand side, one can check this quasi-isomorphism.

Next we consider \(\Sigma_n(r)_{X|\text{mD}}\). By the definition and Lemma . We have the following presentation of \(\Sigma_n(r)_{X|\text{mD}}\):

\[
\Sigma_n(r)_{X|\text{mD}} = \left[\begin{array}{c}
s_n(r)_{X|\text{mD}} \to s_n(r)_{X|\text{mD}} \to \cdots \to s_n(r)_{X|\text{mD}} \\
\to \text{Ker} \left(s_n(r)_{X|\text{mD}} \to s_n(r)_{X|\text{mD}} \to \nu^{r-1}_Y (m'|m,D_n)\right)\end{array}\right].
\]

We define the following maps by a similar way as for product structure of syntomic complex (cf. [1su2], §2.2):

\[
(7.3) \quad \begin{align*}
& s_n(r)_{X|\text{mD}} \oplus^i s_n(d - r)_{X|\text{mD}} \\
& \to \left(\begin{array}{c}
\psi_{Z_n} \otimes \mathcal{O}_Z (m' + i' - m) D_n \end{array}\right) \otimes \mathcal{J}_{Z_n}^{d-2}(m' + i' - m) \otimes \mathcal{O}_Z (m' + i' - m) D_n \otimes \mathcal{O}_Z (m' + i' - m). 
\end{align*}
\]

Since \(m' + i' - m \leq 0\) by assumption of \(m'\), the target of this map is contained in \(\mathcal{S}_n(d)_{X,M_X}\).
Lemma 7.4. We have the morphism
\[ \{ \Xi_n(r)_{X|mD} \} \otimes L R^1 R\alpha_* \Xi_n(d - r)_U \to \mathcal{H}^*. \]

Proof. We put
\[ \mathcal{F}^*_m := [s_n(r)_{X|mD} \to s_n(r)_{X|mD} \to \cdots \to s_n(r)_{X|mD} \to \ker \left( \ker(s_n(r)_{X|mD} \to s_n(r)_{X|mD}^{r+1}) \to \nu_{m,n}^{-1} \right)], \]
and
\[ \mathcal{H}^* := \left[ \text{Coker} \left( S_n(d)_{(X,M_X)}^{d-1} \to S_n(d)_{(X,M_X)}^d \right) \text{FM}^d_{X} \right] \to S_n(d)_{(X,M_X)}^{d+1} \to \cdots. \]

Here \( S_n(d)_{(X,M_X)} \) is syntomic complex defined by Tsuji (see [Tsu1], [Tsu2] and [Tsu3]). The pairing \( \langle \cdot, \cdot \rangle : \mathcal{F}_m \times \mathcal{G}_m' \to \mathcal{H} \) is a product map of syntomic complexes defined in Definition. We have to check the pairing \( \langle \cdot, \cdot \rangle : \mathcal{F}_m \times \mathcal{G}_m' \to \mathcal{H} \) satisfies the relation \( \circ \) for each \( m \). It is enough to show that the equation
\[ (\ast) \quad d_{\mathcal{H}}^m(\langle x_i, y_j \rangle_{i,j}^n) = (d_{\mathcal{F}_m}(x_i), y_j)_{i+1,j}^{n+1} + (-1)^i \left( x_i, d_{\mathcal{G}_m'}(y_j) \right)_{i,j+1}^{n+1}. \]

We will check the relation \((\ast)\) for the following two cases (I), (II).

(I) The case \( s_n(r)_{X|mD} \otimes s_n(d - r)_{X|m'D} \to S_n(d)_{(X,M_X)}^{i+j} \). The calculation of this case is similar as for product structure of syntomic complexes ([Tsu2]). We denote by \( x_i = (x_{i1}, x_{i2}) \in s_n(r)_{X|mD}, y_j = (y_{j1}, y_{j2}) \in s_n(d - r)_{X|m'D}. \)

On the other hand, we can calculate the RHS of \((\ast)\) as follows:
\[ \left( d_{\mathcal{F}_m}(x_i), y_j \right)_{i+1,j}^{n+1} + (-1)^i \left( x_i, d_{\mathcal{G}_m'}(y_j) \right)_{i,j+1}^{n+1} \]
\[ = ((dx_{i1}, (1 - \varphi_d)(x_{i1}y_{j1}) - dy_{j1}), (1 - \varphi_d)(x_{ij2}y_{j2}) - d((1 - \varphi_d)(x_{ij2}y_{j2}))) \]
\[ = (-1)^i x_{i1} dy_{j1} + y_{j1} dx_{i1}, (1 - \varphi_d)(x_{ij1}y_{j1}) - (1 - \varphi_d)(x_{ij2}y_{j2}) - d((1 - \varphi_d)(x_{ij2}y_{j2}))). \]

Then this case, the map \( s_n(r)_{X|mD} \otimes s_n(d - r)_{X|m'D} \to S_n(d)_{(X,M_X)}^{i+j} \) satisfies the relation \((\ast)\).
(II) The case

\[
\text{Ker} \left( \text{Ker}(s_n(r)_{X|m,D} \to s_n(r)_{X|m,D}^{d+1}) \to \nu_{r-1}^{d+1} \right) \otimes \text{Coker} \left( s_n(d-r)_{X|m,D}^{d-r-1} \to s_n(d-r)_{X|m,D}^{d-r} \right) \\
\to \text{Coker} \left( S_n(d)_{(X,M_X)}^{d-1} \to S_n(d)_{(X,M_X)}^d \right) \frac{FM_n^d}{FM_n^d,}
\]

First we check the well-definedness of the map (we denote by this map (\(\star\)))

\[
\text{Ker} \left( \text{Ker}(s_n(r)_{X|m,D} \to s_n(r)_{X|m,D}^{d+1}) \to \nu_{r-1}^{d+1} \right) \otimes \text{Coker} \left( s_n(d-r)_{X|m,D}^{d-r-1} \to s_n(d-r)_{X|m,D}^{d-r} \right) \\
\to \text{Coker} \left( S_n(d)_{(X,M_X)}^{d-1} \to S_n(d)_{(X,M_X)}^d \right) .
\]

The element \((z_1, z_2)\) is \(d(z_1), \ (1 - \varphi_{d-r})(z')_1 - d(z')_2\) for some \((z'_1, z'_2) \in s_n(d-r)_{X|m,D}^{d-r-1}\). We calculate the image of \((x_1, x_2) \otimes (z_1, z_2)\) under the product map:

\[
(x_1, x_2) \otimes (z_1, z_2) = (x_1, x_2) \otimes \left( d(z'_1), \ (1 - \varphi_{d-r})(z')_1 - d(z')_2 \right) \\
\to (x_1 d(z'_1), \ (-1)^{r} x_1 \cdot \left\{ (1 - \varphi_{d-r})z'_1 - d(z'_2) \right\} + x_2 \varphi_{d-r}(d z'_1)).
\]

It is enough to find a solution \((\eta_1, \eta_2)\) satisfies the following two equations:

1. \(d \eta_1 = x_1 d(z'_1),\)
2. \((1 - \varphi_{d-r})\eta_1 - d \eta_2 = (-1)^{r} x_1 \cdot \left\{ (1 - \varphi_{d-r})z'_1 - d(z'_2) \right\} + x_2 \varphi_{d-r}(d z'_1).\)

We take \(\eta_1 := x_1 z'_1,\) this element satisfies the condition (1) because \(d(x_1) = 0.\)

Next we put \(\eta_2 := (-1)^{r} \{ x_1 z'_2 - x_2 \varphi_{d-r}(z'_1) \},\) this satisfies the condition (2) by the direct calculation. Then the pair \((\eta_1, \eta_2)\) is the solution of the equation (1) and (2). Lastly, we have the following map, which induced by the map (\(\star\)):

\[
\text{Ker} \left( \text{Ker}(s_n(r)_{X|m,D} \to s_n(r)_{X|m,D}^{d+1}) \to \nu_{r-1}^{d+1} \right) \otimes \text{Coker} \left( s_n(d-r)_{X|m,D}^{d-r-1} \to s_n(d-r)_{X|m,D}^{d-r} \right) \\
\to \text{Coker} \left( S_n(d)_{(X,M_X)}^{d-1} \to S_n(d)_{(X,M_X)}^d \right) \frac{FM_n^d}{FM_n^d,}
\]

Here we used the fact that \(FM_n^d \cong \text{Ker} \left( H^d \left( S_n(d)_{(X,M_X)} \right) \to \nu_{d-n}^{d-1} \right)\) (see [Sat3], Theorem 3.4). We check that this map (\(\star\)) satisfies the relation (*). It suffices to show that the relation \(d^m_n \left( \left(x_i, y_j \right)_{i,j} \right) = (-1)^j \left\langle x_i, d^m_{n-1} \left( y_j \right) \right\rangle \)

This relation follows from straightforward computations. Then we have a morphism \(\{T_n(r)^{sym}_{X|m,D}\} \otimes \text{R}^{\text{op}} \text{R}_* \xi_n(d-r) \to H^\bullet \). For the complex \(H^\bullet,\) we have \(H^{d+1}(H^\bullet)[-d-1] \cong \nu_{d-n}^{d-1}[-d-1].\) Thus we obtain the map (\(Y\)) by the following composition:

\[
\{T_n(r)^{sym}_{X|m,D}\} \otimes \text{R}^{\text{op}} \text{R}_* \xi_n(d-r) \to H^\bullet \to H^{d+1}(H^\bullet)[-d-1] \cong \nu_{d-n}^{d-1}[-d-1]
\]

\(\square\)
We obtain a pairing

\[(7.4) \lim\limits_{\overset{\longleftarrow}{m}} H^r(Y, \Xi_n(r)_{\text{syn}}^\text{sym}_{X|mD}) \times H^{2d+1-i}(U, \Xi_n(d-r)_U) \longrightarrow \mathbb{Z}/p^n.\]

This completes the proof of the main result [1](1). In the next section, we will prove that the main result [1](2).

8. Proof of non-degeneracy of the pairing in Theorem [1](1)

We start the proof of Theorem [1](2) by the same strategy of [Sat2] in §10. We can reduce the problem to the case \(n = 1\) by the distinguished triangle in Proposition.

8.1. Descending induction on \(r\). We assume that \(\zeta_\rho \in K\). We obtain the following Lemma by the same arguments as in [Sat2], Proposition 4.3.1 (3), and by Lemma 4.8.

**Lemma 8.1.** (Bockstein triangle, cf. [Sat2], Proposition 4.3.1 (3)) There is a distinguished triangle as a projective system with respect to the multiplicities of an irreducible components of \(D\).

\[
\left\{ \Xi_n(r)_{\text{syn}}^\text{sym}_{X|D} \right\}_D \xrightarrow{R} \left\{ \Xi_{n-1}(r)_{\text{syn}}^\text{sym}_{X|D} \right\}_D \xrightarrow{\delta_n} \left\{ \Xi_1(r)_{\text{syn}}^\text{sym}_{X|D}[1] \right\}_D \xrightarrow{\varphi^n[1]} \left\{ \Xi_n(r)_{\text{syn}}^\text{sym}_{X|D}[1] \right\}_D
\]

Therefore it suffices to show that the case \(n = 1\). We show this case of (2) in Theorem [1](1) by induction on \(r\). If \(r = 0\), the pairing in [1](2) is isomorphic to the pairing

\[\lim\limits_{\overset{\longleftarrow}{m}} H^r(Y, \Xi_1(0)_{\text{syn}}^\text{sym}_{X|mD}) \times H^{2d+1-i}(Y, \alpha'_*\nu_{V,1}^{d+1}) \longrightarrow \mathbb{Z}/p\mathbb{Z}\]

by the proper-base change theorem and Lemma 7.3.3, [Sat2]. We put \(\nu'_Y := \nu_Y\), \(\Xi(r)_{\text{syn}}^\text{sym}_{X|mD} := \Xi_1(r)_{\text{syn}}^\text{sym}_{X|mD}, \mu := \mu_p(K), \mu' := \nu_{*}\psi_*\mu_p\) and \(\mu'' := \nu^{*}\psi;'\mu_p\) for simplicity. Note that \(\mu'\) is the constant étale sheaf on \(Y\) associated with the abstract group \(\mu(\cong \mathbb{Z}/p\mathbb{Z})\). We define the morphism

\[(8.1) \quad \text{ind}_n : \mu' \otimes^{\mathbb{L}} \Xi(r-1)_{\text{syn}}^\text{sym}_{X|mD} \longrightarrow \Xi(r)_{\text{syn}}^\text{sym}_{X|mD}\]

by restricting the product structure (cf. Proposition) \(\Xi(1)_{\text{syn}}^\text{sym}_{X|mD} \otimes^{\mathbb{L}} \Xi(r-1)_{\text{syn}}^\text{sym}_{X|mD} \longrightarrow \Xi(r)_{\text{syn}}^\text{sym}_{X|mD}\) to the 0-th cohomology sheaf \(\mu'\) of \(\Xi(1)_{\text{syn}}^\text{sym}_{X|mD}\). We have the following Lemma (cf. [Sat2], Lemma 10.4.1).

**Lemma 8.2.** Let

\[(8.2) \quad \mathbb{K}(r)_m \xrightarrow{b_r} \mu' \otimes^{\mathbb{L}} \Xi(r-1)_{\text{syn}}^\text{sym}_{X|mD} \longrightarrow \Xi(r)_{\text{syn}}^\text{sym}_{X|mD} \xrightarrow{a_{r}} \mathbb{K}(r)_m[1]\]

be a distinguished triangle in \(D^b(Y_\text{ét}, \mathbb{Z}/p\mathbb{Z})\). Then:

1. The triple \((\mathbb{K}(r), a_{r}, b_r)\) is unique up to a unique isomorphism in \(D^b(Y_\text{ét}, \mathbb{Z}/p\mathbb{Z}))\),
and \( b_r \) is determined by the pair \((\mathbb{K}(r), a_r)\).

(2) \( \mathbb{K}(r)_m \) is concentrated in \([r, r + 1]\) and \( a_r \) induces isomorphism

\[
\mathcal{H}^q(\mathbb{K}(r)_m) \cong \begin{cases} 
\mu' \otimes \nu_{Y|mD}^{r-2} & (q = r), \\
\text{Ker} \left( \mu' \left( s_1(r)_{X|mD} \right) \rightarrow \nu_{Y|mD}^{r-1} \right) & (q = r + 1).
\end{cases}
\]

**Proof.** The assertion (2) follows from the long exact sequence of cohomology sheaves associated with \((8.2)\). The details are straightforward and left to reader. We have \( \text{Hom}(\mu' \otimes L \mathcal{I}(r - 1)^{\text{sgn}}_{X|mD}, \mathbb{K}(r)_m[-1]) = 0 \) by (2) and Lemma 2.1.1 in \[\text{Sat2}\]. Then we obtain the assertion (1) by Lemma 2.1.2 (3) in \[\text{Sat2}\]. \( \square \)

In what follows, we fix a pair \((\mathbb{K}(r), a_r)\) fitting into () for each \( r \) with \( 0 \leq r \leq p - 2 \).

**Lemma 8.3.** \((\text{Sat2}, \text{Lemma } 10.4.1)\)

There is a distinguished triangle in \( D^b(V_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) \)

\[
\mathcal{M}(d - r)[-1] \xrightarrow{c_r} \mu'' \otimes L \text{R} \xi_1(d - r) \xrightarrow{d_r} \mathcal{M}(d - r).
\]

(1) The triple \((\mathcal{M}(d - r), c_r, d_r)\) is unique up to a unique isomorphism in \( D^b(V_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) \), and \( b_r \), is determined by the pair \((\mathbb{K}(r), c_r)\).

(2) \( \mathcal{M}(d - r)_m \) is concentrated in \([d - r, d - r + 1]\) and \( c_r \) induces isomorphism

\[
\mathcal{H}^q(\mathcal{M}(d - r)) \cong \begin{cases} 
\mu'' \otimes \nu_{V,1}^{d-r-1} & (q = d - r), \\
FM_{1}^{d-r+1} & (q = d - r + 1).
\end{cases}
\]

Let us note that for objects \( K_1, K_2 \in D^-(V_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z}) \), and \( K_3 \in D^+(V_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z}) \), we have

\[
\text{Hom}_{D(V_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})}(K_1 \otimes L K_2, K_3) \cong \text{Hom}_{D(V_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})}(K_1, R\text{Hom}_{V, \mathbb{Z}/p\mathbb{Z}}(K_2, K_3)).
\]

We now introduce some notations. For \( K \in D^-(V_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z}) \), we define

\[
\mathbb{D}(K) := R\text{Hom}_{V, \mathbb{Z}/p\mathbb{Z}}(K, \mu'' \otimes \nu_{V}^{d-1}[-d - 1]) \in D^+(V_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z}).
\]

This notation is due to \[\text{Sat2}.\]

**Lemma 8.4.** Then there is a unique morphism

\[
\left( \lim_m \mathbb{K}(r)_m \right) \otimes L \text{Ra}_r(\mathcal{M}(d - r)) \rightarrow \mu' \otimes \nu_{Y}^{d-1}[-d - 1] \quad \text{in } D^-(V_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})
\]
whose adjoint morphism \( \varprojlim_{m} \mathbb{K}(r)_m \to R\alpha'_* \mathbb{D}(\mathbb{M}(d-r)) \) fits into a commutative diagram with distinguished rows:

\[
\begin{array}{cccc}
\varprojlim_{m} \mathbb{K}(r)_m & \longrightarrow & \varprojlim_{m} \mu' \otimes^L T(r-1) & \longrightarrow & \varprojlim_{m} T(r) & \longrightarrow & \varprojlim_{m} \mathbb{K}(r)_m[1] \\
\end{array}
\]

Here lower row arises from a distinguished triangle obtained by truncation and taking \( R\alpha'_* \), and the vertical arrows \((\triangledown)\) comes from the pairing \((\Upsilon)\).

\textbf{Proof.} The assertion follows from Lemma 2.1.2 (1) and the fact that

\[
\text{Hom}_{\mathcal{D}^+((\mathbb{Y},\mathbb{Z}/p\mathbb{Z}))} \left( \mathbb{K}(r)_m, \mathbb{D}(\mu' \otimes^L \mathbb{1}(d-r)\nu) \right) \cong \text{Hom}_{\mathcal{D}^-((\mathbb{Y},\mathbb{Z}/p\mathbb{Z}))} \left( \mathbb{K}(r)_m \otimes^L (\mu' \otimes^L \mathbb{1}(d-r)\nu), \mu' \otimes \nu_{\mathcal{Y}}^{d-1}[-d-1] \right)
\]

\[
= 0,
\]

where the last equality follows from Lemma 8.2 (2) and Lemma 2.1.1 in [Sat2].

We turn to the proof of the Theorem [L1] (2) and claim the following:

\textbf{Proposition 8.5.} Let \( 0 \leq r \leq p - 2 \). Then for \( i \in \mathbb{Z} \), the pairing

\[
(8.9) \quad \varprojlim_{m} H^i(Y, \mathbb{K}(r)_m) \times H^{2d+1-i}(Y, \alpha'_* \mathbb{M}(d-r)) \to \mu \otimes H^{d}(Y, \nu_{\mathcal{Y}}^{d-1}) \xrightarrow{id \otimes \text{tr}} \mu
\]

is a non-degenerate pairing of compact group and discrete group.

We will prove this proposition [8.5] in the next subsection. We first complete the proof of Theorem [L1] (2) by descending induction on \( n \leq d \), admitting Proposition. We get the Theorem 1.3 (2) by applying [Sat2], Lemma 10.4.10 to the diagram in Lemma [8.4].

\textbf{8.2. Proof of Proposition.}

Let \( V(d-r) \) be an object of \( \mathcal{D}^b(\mathcal{V}_{\text{et}}, \mathbb{Z}/p^n\mathbb{Z}) \) fitting into a distinguished triangle

\[
(8.10) \quad \lambda_{V,1}^{d-r+1}[-d+r-2] \to V(d-r) \to \mathbb{M}(d-r) \to \lambda_{V,1}^{d-r+1}[-d+r-1],
\]

where the last morphism is defined as the composite

\[
(8.11) \quad \mathbb{M}(d-r) \to H^{d-r+1}(\mathbb{M}(d-r))[-d+r-1] \cong FM_{V,1} \to \lambda_{V,1}^{d-r+1}[-d+r-1].
\]

Here the last map is due to [Sat2]. By Lemma 8.2 (2) and Lemma 2.1.2(3) in [Sat2], \( V(d-r) \) is concentrated in \([d-r, d-r+1]\) and unique up to a unique
isomorphism. We have

\begin{equation}
\mathcal{H}^q(\mathbb{V}(d - r)) \cong \begin{cases} 
\mu^l \otimes \nu^{d-r-1}_{\mathbb{V}} & (q = d - r) \\
U^1 \mathcal{M}_{\mathbb{V}}^{d-r+1} & (q = d - r + 1).
\end{cases}
\end{equation}

**Lemma 8.6.** (cf. [Sat2], Lemma 10.5.3) There is a unique morphism

\begin{equation}
R\alpha'_* \mathbb{V}(d - r)[-1] \longrightarrow D(\lim_{m} \mathcal{M}^{r}_{\mathcal{M}}[-r]) \quad \text{in } D^+(Y_{et}, \mathbb{Z}/p\mathbb{Z})
\end{equation}

fitting into a commutative diagram with distinguished rows

\begin{equation}
R\alpha'_* \mathbb{V}(d - r)[-1] \quad R\alpha'_* \mathbb{M}(d - r)[-1] \quad R\alpha'_* \mathcal{M}^{d-r+1}_{\mathbb{V}}[-d + r - 2] \quad R\alpha'_* \mathbb{V}(d - r)
\end{equation}

\begin{equation}
\begin{array}{c}
\Downarrow \text{(1)} \\
\Downarrow \text{(2)} \\
\Downarrow \text{(3)} \\
\Downarrow \text{(4)}
\end{array}
\end{equation}

\begin{equation}
\begin{array}{c}
D(\lim_{m} \mathcal{M}^{r}_{\mathcal{M}}[-r]) \\
D(\lim_{m} \mathcal{M}(r)_m) \\
D(\lim_{m} \mu^l \otimes \nu^{r-2}_{\mathbb{V}} m[-r + 1]) \\
D(\lim_{m} \mathcal{M}^{r}_{\mathcal{M}}[-r])[-1]
\end{array}
\end{equation}

Here the morphism (2) obtained in Lemma 8.4 and the morphism (3) obtained in section §4.

**Proof.** We have $\text{Hom}\left(R\alpha'_* \mathbb{V}(d - r), D(\lim_{m} \mu^l \otimes \nu^{r-2}_{\mathbb{V}} m m[-r + 1])\right) = 0$. It suffices to show that the commutativity of the central square of (8.11) by Lemma 2.1.2 (1) in [Sat2]. We consider the composition of the morphisms

\begin{equation}
v : \left(R\alpha'_* \mathbb{M}(d - r)[-1]\right) \otimes^L \left(\lim_{m} \mu^l \otimes \nu^{r-2}_{\mathbb{V}} m[-r + 1]\right) \\
\longrightarrow \left(R\alpha'_* \mathbb{M}(d - r)[-1]\right) \otimes^L \left(\lim_{m} \mathbb{K}(r)_m\right) \longrightarrow \mu^l \otimes \nu^{d-1}_{\mathbb{V}}[-d - 1]
\end{equation}

, where the last morphism follows from the morphism (2). It remains to show that the commutativity of the central square (2) to prove the commutativity of the square (2), we first prove the following Lemma:

**Lemma 8.7.** We have the following commutative diagram (we denote by (CSQ)) in $D(Y_{et}, \mathbb{Z}/p\mathbb{Z})$:

\begin{equation}
\begin{array}{c}
R\alpha'_* i^* \mathfrak{X}(d - r)_{U}[-1] \otimes^L \left\{\mathfrak{X}^{\text{syn}}(r)_{X|mD}\right\}_m \quad \text{product} \quad i^* R\psi_* \mu^l \otimes^{d+1} [-1] \\
\downarrow a_{d-r}[-1] \otimes^L a_r \\
R\alpha'_* \mathbb{M}(d - r)[-1] \otimes^L \left\{\mathbb{K}(r)_m\right\}_m \quad \mu^l \otimes \nu^{d-1}_{\mathbb{V}}[-d - 1].
\end{array}
\end{equation}
Proof. We have a commutative diagram in $\mathcal{D}^b(Y_{\text{et}}, \mathbb{Z}/p\mathbb{Z})$ by using an anticommutative diagram (10.4.4.1) in [Sat2]:

We have a commutative diagram in Proof.

By Lemma 10.5.6 in [Sat2], we obtain the following commutative diagram:

Thus we combine the diagram (I) with (II), we obtain the following commutative diagram:

The commutativity of (III) follows from the construction of the pairing. This completes the proof.

We return to the proof of the commutativity of the square (□). There is a commutative diagram (Diagram):

(CSQ)
Here the morphism $(\ast)$ correspondings to the pairing in section §4, which follows from the fact that

$$
\text{Hom} \left( R\alpha'_r \lambda^{d-r+1}_{V,1} [-d+r-2] \otimes \mathcal{L} \{ \mathbb{K}(r) \}_m, \mu'' \otimes \nu^{d-1}_{V} [-d-1] \right) \cong 
\text{Hom} \left( R\alpha'_r \lambda^{d-r+1}_{V,1} \otimes \mathcal{L} \left( \mu' \otimes \nu^{d-2}_{Y(m_D)} \right), \mu' \otimes \nu^{d-1}_{V} \right)
$$

by Lemma 2.1.1 in [Sat2]. In the right hand side Hom, there is a product morphism constructed in §4. Then we get the morphism $R\alpha'_r \lambda^{d-r+1}_{V,1} [-d+r-2] \otimes \mathcal{L} \{ \mathbb{K}(r) \}_m \to \mu'' \otimes \nu^{d-1}_{V} [-d-1]$. The above diagram (Diag) commutes by using the commutativity of (CSQ) (the upside of the right square). Then we obtain the middle commutative square of (8.14).

**Lemma 8.8.**

\[ \mu' \otimes \alpha'_r \nu^{d-r-1}_{V} [-d+r] \quad \xrightarrow{t_1} \quad R\alpha'_r \mathcal{V}(d-r) \quad \xrightarrow{t_2} \quad R\alpha'_r U^1 M_{V}^{d-r+1} [-d+r+1] \quad \xrightarrow{t_3} \quad \mu' \otimes \alpha'_r \nu^{d-r-1}_{V} [-d+r+1]. \]

\[ \mathbb{D}(\text{lim}_{m} \lambda^{V}_{m \otimes D} [-r]) \quad \xrightarrow{t_4} \quad \mathbb{D}(\text{lim}_{m} F, \mathcal{M}_{m}^{r} [-r]) \quad \xrightarrow{t_5} \quad \mathbb{D}(\text{lim}_{m} U^1 \mathcal{M}_{m}^{r} [-r]) \quad \xrightarrow{t_6} \quad \mathbb{D}(\text{lim}_{m} \lambda^{V}_{m \otimes D} [-r][1]). \]

**Proof:** We have $\text{Hom} \left( \mu' \otimes \alpha'_r \nu^{d-r-1}_{V} [-d+r+1], \mathbb{D}(\text{lim}_{m} U^1 \mathcal{M}_{m}^{r} [-r]) \right) = 0$, and the left square commutes by a similar argument in (1). Then there is a unique morphism $f'_3 : R\alpha'_r U^1 M_{V}^{d-r+1} [-d+r] \to \mathbb{D}(\text{lim}_{m} U^1 \mathcal{M}_{m}^{r} [-r] - 1)$ fitting into (8.15) by Lemma 2.1.2 (2) in [Sat2]. We have $f'_3 = f_3$ by the commutativity of (CSQ) and the construction of maps. □

**proof of the main result Theorem 1.1(2):**

In Lemma 8.8 since morphisms $f_1$ and $f_3$ are isomorphisms, the morphism $f_2$ is an isomorphism by Theorem and . Thus in Lemma 8.6, the morphism (1) is an isomorphism. This completes the proof of the main result Theorem 1.1(2). □

Take $i = 2d, r = d$ in Theorem 1.1 we obtain the following:

**Corollary 8.9.** *(cf. [SZ])* We obtain a natural isomorphism

\[ \text{lim}_{m} H^{2d}(X, \mathfrak{S}(d)_{X|mD}) \xrightarrow{\cong} \pi_{1}^{ab}(U)/p^{n}. \]

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