Dynamical Systems, Topology and Conductivity in Normal Metals.

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Abstract

New observable integer-valued numbers of the topological origin were revealed by the present authors studying the conductivity theory of single crystal 3D normal metals in the reasonably strong magnetic field ($B \leq 10^3 \text{Tl}$). Our investigation is based on the study of dynamical systems on Fermi surfaces for the motion of semi-classical electron in magnetic field. All possible asymptotic regimes are also found for $B \to \infty$ based on the topological classification of trajectories.

1 Introduction.

This is the survey article dedicated to the topological approach to the conductivity phenomena in metals in the presence of the rather strong magnetic field $B$. As can be shown, in the case of rather strong magnetic fields the topology of the Fermi surface plays the main role in the asymptotic behavior of conductivity $\sigma^k$ when $B \to \infty$. In this article we represent mainly the results in this area connected with the "Topological Quantum characteristics" introduced recently by the present authors and the classification
results for the regimes of conductivity behavior for $B \to \infty$ based on the topological results in the theory of special dynamical systems. The main consideration is made here for the case of the "generic" (or "Topologically regular") open electron trajectories according to classification recently introduced in the works of the present authors (see [20] [25] [31]). As was shown ([20]), this type of trajectories always leads to observable topological characteristics of the Fermi surface having the form of "stability zones" on the unit sphere (parameterizing directions of $B$) and the triples of the integer numbers observable in every "stability zone" in the conductivity measurements. Let us say that this type of trajectories corresponds to the "generic case" and it is the only type of non-closed trajectories stable with respect to the small perturbations. Besides that, the trajectories of this type appear with probability 1 in all cases when the non-closed quasiclassical electron can be observed on the fixed Fermi surface.

However, it can be shown that also other types of open orbits corresponding to more complicated "chaotic" behavior can be observed on the complicated Fermi surfaces. In this case, the direction of magnetic field should be chosen specially in the experiment and the behavior of conductivity reveals much more complicated features. Let us mention also, that the geometric forms of electron trajectories for such systems can be obtained as the intersections of the Fermi surface by the planes orthogonal to the magnetic field and in some sense these systems are all "analytically integrable" on the universal covering $\mathbb{R}^3$. However, after the identification of equivalent vectors modulo the reciprocal lattice these systems may become topologically very complicated since the two points in $\mathbb{R}^3$ different by the reciprocal lattice vector represent actually the same physical state of electron. For the generic irrational directions of $B$ the global geometry of these intersections is apriori unpredictable and can be rather complicated as we will see below.

Let us give here the brief historical survey concerning the "geometric strong magnetic field limit" in metals.

The investigation of the geometric effects in the magnetoresistance behavior in normal metals was started first in the school of I.M.Lifshitz (I.M.Lifshitz, M.Ya.Azbel, M.I.Kaganov, V.G.Peschansky) in 1950’s. Such, in the first paper [1] the crucial difference in the asymptotic behavior ($B \to \infty$) of conductivity in the cases of closed (compact) and open periodic electron trajectories on the Fermi surface was pointed out. The strong anisotropy of conductivity in the last case makes possible the observation of this phenomenon and the mean direction of the periodic open orbits in the experiment. The dif-
different examples of the complicated Fermi surfaces and the corresponding
non-closed electron orbits were considered in the works [2, 3]. In particular,
the open orbits, more general then just periodic ones were discovered in [2].
The geometry of such trajectories still was not very complicated and the
 corresponding electron motion was still ”in average” along the straight line
in the $p$-space. Again the strong anisotropy of the conductivity tensor was
expected in this case and the form of $\sigma^{ik}$ used in [2, 3] was actually the same
as for the periodic open orbits. In the paper [3] the important set of ana-
lytical dispersion relations was considered and the existence of open orbits
in the different parts of set parameters was discovered. Let us give here also
the references on papers [4, 5, 6, 7] (see also the survey articles [8, 9] and the
book [10]) where the different question connected with the geometry of open
orbits for concrete Fermi surfaces of real metals were considered.

The general problem of classification of different trajectories arising on
arbitrary periodic smooth surface in 3D space as the intersections with ar-
britary planes in $\mathbb{R}^3$ (S.P.Novikov problem) was set by S.P.Novikov in [12]
and then was considered in his topological school at the Moscow State Uni-
versity (A.V.Zorich, I.A.Dymnikov, S.P.Tsarev, 1980-2000). \footnote{The general
problem of S.P. Novikov has actually more general form and is connected
with the global geometry of level curves of quasiperiodic functions with $n$ quasiperiods
on the plane. Let us say, however, that the case $n > 3$ was started to investigate recently (see
[29]) and is still not so well studied as the case $n = 3$. Let us make also the reference on the
work [32] where the connection of general Novikov problem with the specially modulated
2D electron gas was considered.} Let us men-
tion here that the most important breakthroughs were made in the papers
[16, 19] where the important theorems about the non-compact trajectories
were proved. Using this results the concept of the ”Topological quantum
numbers” observable in the conductivity and characterizing the generic ”non-
trivial” conductivity behavior in metals was introduced by the present au-
thors in [20]. In particular, it was shown that these characteristics arise
always when the stable (with respect to the small variations of $B$) behav-
ior of conductivity different from the ”simple” behavior corresponding to
just compact electron orbits on the Fermi surface is observed. The corre-
sponding ”stability zones” on the unit sphere and the ”Triples of integer
numbers” (”Topological Quantum characteristics”) give the non-trivial geo-
metric characteristics of the complicated Fermi surfaces in metals. However,
as we already said above the conductivity can reveal also some ”degener-
ate” (unstable) behavior in the special cases of rather exotic electron tra-
jectories on the Fermi surface. The existence of such trajectories as well as the corresponding conductivity behavior was studied later in several works \([28, 22, 26, 24]\). Let say here that the full classification of all types of orbits for arbitrary smooth periodic surfaces in \(\mathbb{R}^3\) is finished in general now (see \([22, 27]\) and the total picture of different asymptotic behaviors of conductivity (as well as the conditions for each asymptotic regime) can be described in main order of \(1/B\) in general case \((25, 31)\). Let us say also that the general topological investigation required rather large set of methods of both 3-dimensional topology and the dynamical systems theory and we are going just to formulate here the corresponding statements needed for our purposes.

2 Some remarks on the Kinetic theory.

Let us describe briefly the standard approach to the conductivity in crystals based on one-particle quantum mechanical consideration. \(^2\) We should consider the Shrödinger equation

\[
-\frac{\hbar^2}{2m} \Delta \psi + V(x) \psi = \epsilon \psi
\]

in the periodic potential \(V(x)\) formed by the crystal lattice \(L\):

\[
V(x + m_1 l_1 + m_2 l_2 + m_3 l_3) \equiv V(x)
\]

for all \((m_1, m_2, m_3) \in \mathbb{Z}^3\).

The potential \(V(x)\) has also all the rotational symmetries of the crystal and in all physical cases we have the symmetry \(x \rightarrow -x\) after the appropriate choice of the origin in \(x\)-space.

The reciprocal lattice \(\Gamma^*\) is generated by the vectors:

\[
g_1 = \frac{\hbar l_2 \times l_3}{(l_1, l_2, l_3)}, \quad g_2 = \frac{\hbar l_3 \times l_1}{(l_1, l_2, l_3)}, \quad g_3 = \frac{\hbar l_1 \times l_2}{(l_1, l_2, l_3)}
\]

in the space of electron momenta.

The corresponding solutions of the stationary Shrödinger equation can be represented as Bloch functions

\(^2\)Let us note that we use here the Kinetic theory on the physical level of rigorousness. At the same time all use of differential topology is based on the mathematically rigorous topological results.
\[ \psi_{p,s}(x) = e^{\frac{i}{\hbar} \varphi_{p,s}(x)} \]

where \( \varphi_{p,s}(x) \) has the same periodicity as the potential \( V(x) \).

The electron states can be parameterized then by the discrete index \( s \) (energy band) and a continuous parameter \( p = (p_1, p_2, p_3) \) such that the states parameterized by \( p \) and \( p' \) (at the same \( s \)) different by any reciprocal lattice vector:

\[
    n_1 g_1 + n_2 g_2 + n_3 g_3, \quad (n_1, n_2, n_3) \in \mathbb{Z}^3
\]

are physically the same.

The electron energy \( \epsilon \) at fixed \( s \) becomes now the function of \( p \), \( \epsilon = \epsilon_s(p) \), periodic in \( p \)-space with periods equal to the reciprocal lattice vectors:

\[
    \epsilon_s(p + n_1 g_1 + n_2 g_2 + n_3 g_3) \equiv \epsilon_s(p), \quad (n_1, n_2, n_3) \in \mathbb{Z}^3
\]

The full set of functions \( \epsilon_s(p) \) inherits also the rotational symmetry of the potential \( V(x) \) and for any function \( \epsilon_s(p) \) we have a symmetry

\[
    \epsilon_s(p) = \epsilon_s(-p)
\]

after the appropriate choice of the initial point in \( p \)-space.

From topological point of view the correct phase space for any fixed energy band is the three-dimensional compact torus (Brillouin zone) \( \mathbb{T}^3 = \mathbb{R}^3/G \) rather then the open three-dimensional space \( \mathbb{R}^3 \). The functions \( \epsilon_s(p) \) become then the (smooth) functions on \( \mathbb{T}^3 \) with the same property \( \epsilon_s(p) = \epsilon_s(-p) \) for the appropriate initial point in \( \mathbb{T}^3 \).

The collective electron structure can be described by the one-particle distribution functions \( f_s(p) \) satisfying to Boltzmann equation.

According to the classical results in the absence of external fields the functions \( f_s(p) \) are given by the Fermi distribution

\[
    f_s(p) = \frac{1}{1 + e^{\frac{\epsilon_s(p) - \epsilon_F}{T}}}
\]

for the fixed temperature \( T \).

Let us mention also that every function \( \epsilon_s(p) \) is bounded by some minimal and maximal values \( \epsilon_s^{\text{min}}, \epsilon_s^{\text{max}} \) being the smooth function on the compact manifold.
For the case of normal metals we will always have a situation $T \ll \epsilon_F$ and the distribution function will change only in narrow energy interval ($\sim T$) near the Fermi energy $\epsilon_s(p) \sim \epsilon_F$, being equal to 1 for $\epsilon_F - \epsilon_s(p) \gg T$ and to 0 for $\epsilon_s(p) - \epsilon_F \gg T$. So, the energy bands with $\epsilon_{s}^{\text{max}} < \epsilon_F$ will be completely filled in this case ($f_s(p) \equiv 1$) and the energy bands with $\epsilon_{s}^{\text{min}} > \epsilon_F$ will be completely empty ($f_s(p) \equiv 0$).

Geometrically we have occupied all the states "inside" the surface $\epsilon_s(p) = \epsilon_F$ in 3D torus and empty all the states "outside". The union of surfaces $\epsilon_s(p) = \epsilon_F$ for all $s$ such that $\epsilon_{s}^{\text{min}} < \epsilon_F < \epsilon_{s}^{\text{max}}$ gives then the total Fermi surface $S_F$ of the given metal.

The Fermi surface will be called non-singular if $\nabla \epsilon_s(p) \neq 0$ for all $s$ on the Fermi level $\epsilon_F$, otherwise we will call it singular.

It will be convenient to consider also the natural covering $\hat{S}_F$ over the Fermi surface in $\mathbb{R}^3$ which is a smooth 3-periodic surface (for non-singular $S_F$) in the $p$-space with the periods equal to the reciprocal lattice vectors.

The gradient of function $\epsilon_s(p)$ is the group velocity of a particle in the corresponding quantum state and describes the mean velocity of the localized wave packet formed by the wave functions $\psi_{s,p}(x)$ with values $p'$ close to $p$. Easy to see that the group velocity is the odd function of $p$ (after the appropriate choice of the initial point)

$$v_{s \text{ gr}}(-p) = -v_{s \text{ gr}}(p)$$

since all the functions $\epsilon_s(p)$ are even. The total flux density

$$i = 2 \sum_s \int \ldots \int v_{s \text{ gr}}(p) f_s(p) \frac{d^3p}{(2\pi\hbar)^3} = 2 \sum_s \int \ldots \int \nabla \epsilon_s(p) f_s(p) \frac{d^3p}{(2\pi\hbar)^3}$$

as well as the electric currents density $j = ei$ are zero for the equilibrium functions given by (II). In our further consideration we will deal with the small deviations from the Fermi distributions (II) caused by small external forces (electric field) and the correspondent small response in the total electric current density.

Let us put now $v_{gr}(p) = v_{s \text{ gr}}(p)$ on the corresponding components of the Fermi surface and consider just one function $v_{gr}(p)$ on $S_F$.

We will use now the fact that all the components of $S_F$ give the independent contributions to conductivity tensor $\sigma^{ik}$. This means that we
can actually consider only one conductivity zone with the dispersion relation \( \epsilon(p) = \epsilon_s(p) \) and the corresponding distribution function \( f(p) = f_s(p) \) defined everywhere in \( T^3 \). The corresponding contributions to the electric conductivity (in the presence of magnetic field) should be then just added in the 3-dimensional tensor \( \sigma^{ik} \).

So we will consider now only one dispersion relation \( \epsilon(p) \) and the function \( f(p) \) which satisfies to the Boltzmann equation for the spatially homogeneous case (we omit here the spin dependence of the distribution function for simplicity):

\[
\frac{\partial f}{\partial t} + F_{\text{ext}} \frac{\partial f}{\partial p} = I[f](p)
\]

(2)

Here \( F_{\text{ext}} \) is the homogeneous external force and \( I[f](p) \) is the collision integral. For our situation of low temperatures (\( T \sim 1K \)) only the scattering on the impurities will play the main role so we will write here the functional \( I[f](p) \) in the general form:

\[
I[f](p) = \int \ldots \int [f(p')(1 - f(p)) - f(p)(1 - f(p'))] \sigma(p, p') \frac{d^3 p'}{(2\pi \hbar)^3} = \\
= \int \ldots \int [f(p') - f(p)] \sigma(p, p') \frac{d^3 p'}{(2\pi \hbar)^3}
\]

Using of the Boltzmann equation implies the applicability of the semiclassical approach for the electron motion and the evolution of the quantum state within the Brillouin zone can be described just by the "classical" dynamical system:

\[
\dot{p} = F_{\text{ext}}
\]

with the only difference that \( p \) now is a quasimomentum and belongs to 3-dimensional torus \( T^3 \) instead of the Euclidean space.

In our case we will have

\[
F_{\text{ext}} = \frac{e}{c} [v_{gr} \times B] + eE = [\nabla \epsilon(p) \times B] + eE
\]

where \( B \) is the homogeneous magnetic field and \( E \) is the small electric field responsible for the electric current.
The applicability of semiclassical approach requires then that the fields \( B \) and \( E \) are small with respect to the internal fields in the crystal. We will require also the condition \( \hbar \omega_B \ll \epsilon_F \) where \( \omega_B = eB/m^*c \) is the cyclotron frequency which will permit to consider the quantization of levels as a small effect with respect to the classical motion picture. The electric field is going to be infinitesimally small so we don’t put any conditions on it. Let us just mention here that these semiclassical conditions are satisfied very well for normal metals in all experimentally available fields \( B \) (the theoretical limit is \( B \sim 10^3 - 10^4 Tl \)). Thus we use the Boltzmann equation (2) to describe the main part of the conductivity tensor \( \sigma^{ik} \) while the quantum phenomena will be considered as the small corrections in our situation.

Now the only reminiscent of the quantization will be the dispersion relation \( \epsilon(p) \) and the corresponding dynamical system on \( T^3 \) given by

\[
p = \frac{e}{c} [\nabla \epsilon(p) \times B] + eE
\]

For the stationary distribution function \( f(p) \) we can write now the Boltzmann equation as

\[
\frac{e}{c} [\nabla \epsilon(p) \times B] \nabla f(p) + eE \nabla f(p) = I[f](p)
\]

Any Fermi distribution function (1) satisfies the condition \( I[f_0](p) \equiv 0 \) and gives an equilibrium distribution for given temperature \( T \) in the absence of external fields.

Let us now consider in details the main dynamical system:

\[
p = \frac{e}{c} [\nabla \epsilon(p) \times B]
\]

on \( T^3 \) for general periodic \( \epsilon(p) \).

System (4) is Hamiltonian with respect to the (non-standard) Poisson bracket

\[
\{p_1, p_2\} = \frac{e}{c} B_3 \ , \ \{p_2, p_3\} = \frac{e}{c} B_1 \ , \ \{p_3, p_1\} = \frac{e}{c} B_2
\]

with the Hamiltonian function \( \epsilon(p) \). The bracket (5) is degenerate with the Casimir

\[
C = \sum_{i=1}^{3} p_i B_i = (p \cdot B)
\]
However, the Casimir function $C$ is a multi-valued function on the three-dimensional torus $\mathbb{T}^3$ and should be actually considered as a 1-form in it. Geometrically, the trajectories of (4) in $\mathbb{T}^3$ will be given on every energy level $\epsilon(p) = \text{const}$ by the level curves of this 1-form restricted on these two-dimensional surfaces. So we have locally the analytic integrability of the system (4). However, the global geometry of the trajectories of (4) can be highly nontrivial because of the non-uniqueness of values of Casimir function $C$.

The picture becomes more visible if we consider the corresponding covering $\mathbb{R}^3 \to \mathbb{T}^3$ defined by the reciprocal lattice in $p$-space. The corresponding function $\epsilon(p)$ is then the three-periodic function in $\mathbb{R}^3$. The Fermi surface (the covering over the Fermi surface in $\mathbb{T}^3$) also becomes the three-periodic (smooth) surface in the $p$-space. The Casimir function $C$ is now well defined function in $\mathbb{R}^3$ (height function) and the trajectories of (4) on the Fermi level will be given by the intersections of the three-periodic surface $\epsilon(p) = \epsilon_F$ with all different planes orthogonal to $B$. The most important thing for us will be the global geometry of the trajectories in the planes $\Pi(B)$ orthogonal to the magnetic field.

The dynamical system (4) conserves also the volume element $d^3p$ in $\mathbb{T}^3$ and does not change at all the Fermi distribution (4). So, in the absence of the electric field $E$ we will have the electron distribution unchanged (up to the quantum corrections). Nevertheless, the response of this system to small perturbations will be completely different from the case $B = 0$ and depend strongly on the geometry of trajectories of the dynamical system (4).

Let us now come back to the Boltzmann equation (3) and analyze the effects of infinitesimally small electric field $E$.

We will use for simplicity the $\tau$-approximation for the collision integral throughout this paper and just put for small perturbations $\delta f$:

$$ I [f_T + \delta f] (p) - I [f_T] (p) = \hat{J}_T (\delta f) (p) = -\frac{1}{\tau} \delta f(p) $$

Let us make now also some (standard) remarks about $\tau$ for our case of normal metals at low temperatures ($\sim 1K$). As it is well-known for the low temperatures both the electron-phonon and electron-electron scattering decrease (as $T^5$ and $T^2$ respectively) and the only reminiscent at $T \to 0$ is the scattering by the impurities which is constant for $T \to 0$. For the elastic scattering by impurities we can usually write the collision integral in the form
\[ I[f](\mathbf{p}) = \frac{2\pi}{\hbar} n_i \int \ldots \int |v_{pp'}|^2 [f(\mathbf{p'}) - f(\mathbf{p})] \delta(\epsilon(\mathbf{p}) - \epsilon(\mathbf{p}')) \frac{d^3 p}{(2\pi \hbar)^3} \]

where \( n_i \) is the concentration of impurities and \( v_{pp'} \) is the amplitude of scattering \( \mathbf{p} \rightarrow \mathbf{p}' \) on the impurity. Such we can put

\[ I[f](\mathbf{p}) = \frac{2\pi}{\hbar} n_i \int \int_{S_F} |v_{pp'}|^2 [f(\mathbf{p'}) - f(\mathbf{p})] \frac{dS'}{v_{gr}(\mathbf{p'})} \]
on the Fermi surface.

Easy to see then that we have the property \( I[f] \equiv 0 \) for any function \( f \) depending only on energy: \( f(\mathbf{p}) = f(\epsilon(\mathbf{p})) \). So for this form of \( I[f] \) the corresponding operator \( \hat{J}_T \) will have a lot of zero modes corresponding to the perturbations \( \delta f(\mathbf{p}) = \delta f(\epsilon(\mathbf{p})) \). Practically this means that the processes of electron-electron and electron-phonon scattering responsible for the mixing of different energy levels become very small at low temperatures. The corresponding "energy relaxation time" \( \tau_{en} \rightarrow \infty \) as \( T \rightarrow 0 \) and the characteristic times of energy relaxation become very big at low \( T \).

However the energy relaxation time \( \tau_{en} \) will not be important for us in the consideration of the conductivity phenomena. Indeed, the mean electric flux density given by

\[ \mathbf{j} = e \int \ldots \int v_{gr}(\mathbf{p}) \delta f(\mathbf{p}) \frac{d^3 p}{(2\pi \hbar)^3} \]
remains zero for any distribution perturbation of form \( \delta f(\mathbf{p}) = \delta f(\epsilon(\mathbf{p})) \) because of the property \( v_{gr}(-\mathbf{p}) = -v_{gr}(\mathbf{p}) \). We have so that all the zero modes of \( \hat{J}_T \) do not give any contribution to the conductivity and we can omit them in our consideration.

We will consider now only the perturbations \( \delta f(\mathbf{p}) \) such that the total number of particles on each energy level remains unchanged w.r.t. the initial Fermi distribution. So we put an additional restriction

\[ \int \int_{\epsilon(\mathbf{p})=\text{const}} \delta f(\mathbf{p}) \frac{dS}{v_{gr}(\mathbf{p})} \equiv 0 \quad (6) \]
for all the energy levels. Let us say also that the only factor which can disturb this condition according to the full Boltzmann equation
\[
\frac{\partial f}{\partial t} + \frac{e}{c} [\nabla \epsilon(p) \times B] \frac{\partial f}{\partial p} + eE \frac{\partial f}{\partial p} = I[f]
\]
is the electric field \(E\). However, because of the equality
\[
\int \ldots \int E \frac{\partial f_T}{\partial p} \frac{d^3p}{(2\pi \hbar)^3} = \int \ldots \int E \nabla \epsilon(p) \frac{\partial f_T}{\partial \epsilon} \frac{d^3p}{(2\pi \hbar)^3} \equiv 0
\]
this effect has the order \(E^2\) and should be omitted in the computing a linear approximation for the infinitesimally small \(E\). We can now forget about the zero modes of the functional \(\hat{J}_T\) using the functional space defined by (6) and put \(\tau\) to be the characteristic relaxation time on each energy level due to the scattering on the impurities. This time \(\tau = \tau_{mom}\) is then the momentum relaxation time and it is this quantity which is responsible for conductivity at \(T \to 0\). We just use then the approximation
\[
\tilde{J}_T[\delta f](p) = -\frac{1}{\tau} \delta f(p)
\]
on the space of function satisfying (6) where \(\tau = \tau_{mom}\) has the meaning of the free electron motion time before the scattering by the impurity and remains constant at \(T \to 0\).

The Boltzmann equation (3) can be rewritten now in linear order of \(E\) as
\[
\frac{e}{c} [\nabla \epsilon(p) \times B] \nabla f_1(p) + eE v_{gr}(p) \frac{\partial f_T}{\partial \epsilon} = -\frac{1}{\tau} f_1(p)
\]
(7)
where \(f_1(p)\) is a linear in \(E\) perturbation of \(f_T\).

Let us introduce the vector field \(\xi(p)\) on \(T_3\) according to our system (4):
\[
\xi(p) = \frac{e}{c} [\nabla \epsilon(p) \times B]
\]
We can then rewrite the equation (7) on \(f_1\) in the form:
\[
\nabla_\xi f_1(p) + \frac{1}{\tau} f_1(p) = -eE v_{gr}(p) \frac{\partial f_T}{\partial \epsilon}
\]
(8)
The equation (8) can be now solved separately on every trajectory of the system (4). Indeed, let us introduce the coordinate \(t\) (time) along the trajectories of (4). We have then
\[
e^{-t/\tau} \frac{d}{dt} \left[ e^{t/\tau} f_1(t) \right] = e \left( -\frac{\partial f_T}{\partial \epsilon} \right) E v_{gr}(t)
\]
on every trajectory. We put here $f_1 = f_1(t)$, $\mathbf{v}_{gr} = \mathbf{v}_{gr}(t)$ and the value $\partial f_T / \partial \epsilon$ is constant on the trajectories of (II).

The solution of (II) can be written now as (||):

$$f_1(t) = e \left( -\frac{\partial f_T}{\partial \epsilon} \right) \int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} \mathbf{E} \mathbf{v}_{gr}(t') dt' =$$

$$= e \left( -\frac{\partial f_T}{\partial \epsilon} \right) \int_{0}^{+\infty} e^{-\frac{t'}{\tau}} \mathbf{E} \mathbf{v}_{gr}(t-t') dt'$$

(9)

Let us introduce the "geometric" time $s = teB/c$ where $t$ is the time along the trajectories according to the system (II). For the derivative of $\mathbf{p}$ with respect to $s$ we then will have the system:

$$\frac{d\mathbf{p}}{ds} = [\nabla \epsilon(\mathbf{p}) \times \mathbf{n}]$$

(10)

where $\mathbf{n} = \mathbf{B}/\mathbf{B}$ is a unit vector along $\mathbf{B}$. Let us introduce the notation $\mathbf{v}_{gr}(\mathbf{p}, t) = \mathbf{v}_{gr}(\mathbf{p}(t))$ where $\mathbf{p}(t)$ is a solution of the system (II) with the initial data $\mathbf{p}(0) = \mathbf{p}$. The formula (9) can be then written as

$$f_1(\mathbf{p}) = e \left( -\frac{\partial f_T}{\partial \epsilon} \right) \int_{0}^{+\infty} \mathbf{E} \mathbf{v}_{gr}(\mathbf{p}, -t) e^{-t/\tau} dt$$

(11)

The parameter $s$ plays the role of geometric parameter along the trajectories, such that $dl/ds = v^\perp_{gr}(s)$ where $dl$ is the length element in the $\mathbf{p}$-space and $v^\perp_{gr}(s)$ is the length of component of $\mathbf{v}_{gr}(s)$ orthogonal to $\mathbf{B}$. The size of the Brillouin zone corresponds then to $s \sim p_F/v_F \sim m^*$ where $m^*$ is some parameter with mass dimension playing the role of "typical" electron mass on the Fermi surface for a given dispersion relation.

We will be interested in the "strong geometric limit" when the parameter $m^*c/eB\tau$ is much less then $1$. This condition can be also written as $\omega_B \tau \gg 1$ where $\omega_B = eB/m^*c$ is the cyclotron frequency corresponding to the mass $m^*$. It is easy to see from (II) that this case corresponds to "long integrations" of $\mathbf{v}_{gr}(\mathbf{p})$ along the trajectories of (II) (much longer then one Brillouin zone for open trajectories) such that the global geometry of the trajectories becomes the most important factor for the linear response $f_1(\mathbf{p})$.

The correspondent current density $\mathbf{j}$ has the form

$$\mathbf{j} = e \int \cdots \int \mathbf{v}_{gr}(\mathbf{p}) f_1(\mathbf{p}) \frac{d^3p}{(2\pi \hbar)^3} =$$

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\[ e^2 \int \ldots \int \mathbf{v}_{gr}(\mathbf{p}) \left( -\frac{\partial f_T}{\partial \epsilon} \right) \left[ \int_0^{+\infty} \mathbf{E} \mathbf{v}_{gr}(\mathbf{p}, -t) e^{-t/\tau} dt \right] \frac{d^3 p}{(2\pi \hbar)^3} \]

and we have for the conductivity tensor \( \sigma^{ik} \) ([10], §27):

\[ \sigma^{ik}(\mathbf{B}) = e^2 \int \ldots \int \left( -\frac{\partial f_T}{\partial \epsilon} \right) v^{i}_{gr}(\mathbf{p}) \left[ \int_0^{+\infty} v^k_{gr}(\mathbf{p}, -t) e^{-t/\tau} dt \right] \frac{d^3 p}{(2\pi \hbar)^3} \quad (12) \]

Easy to prove also the Onsager relations for tensor \( \sigma^{ik}(\mathbf{B}) \):

\[ \sigma^{ik}(\mathbf{B}) = \sigma^{ki}(-\mathbf{B}) \]

It is easy to see that the expression

\[ \int \ldots \int \left( -\frac{\partial f_T}{\partial \epsilon} \right) v^{i}_{gr}(\mathbf{p}) v^k_{gr}(\mathbf{p}, -t) \frac{d^3 p}{(2\pi \hbar)^3} \]

is invariant under the action of the dynamical system (4). We have so:

\[ \sigma^{ik}(\mathbf{B}) = e^2 \int_0^{+\infty} \left[ \int \ldots \int \left( -\frac{\partial f_T}{\partial \epsilon} \right) v^{i}_{gr}(\mathbf{p}) v^k_{gr}(\mathbf{p}, -t) \frac{d^3 p}{(2\pi \hbar)^3} \right] e^{-t/\tau} dt = \]

\[ = e^2 \int_0^{+\infty} \left[ \int \ldots \int \left( -\frac{\partial f_T}{\partial \epsilon} \right) v^{i}_{gr}(\mathbf{p}, t) v^k_{gr}(\mathbf{p}) \frac{d^3 p}{(2\pi \hbar)^3} \right] e^{-t/\tau} dt \quad (13) \]

Using now the fact that \( v^{i}_{gr}(\mathbf{p}, t) \) coincides with the value \( v^{i}_{gr}(\mathbf{p}, -t) \) defined for the system (4) for \(-\mathbf{B}\) we get the Onsager identity.

3 Topological considerations and theorems.

Before starting the investigation of the dependence of \( \sigma^{ik} \) on the geometry of orbits of (4) let us make here some basic definitions and formulate the topological results which we are going to use in our considerations.

First of all, we consider the intersections of the smooth 3-periodic Fermi surface in \( \mathbf{p} \)-space, with the set of parallel planes orthogonal to magnetic field.
B. We can consider the corresponding trajectories also as the level curves of the restrictions $\epsilon(p)|_\Pi$ of the dispersion relation to the planes orthogonal to $B$ which will then be the quasiperiodic functions in this planes with 3 quasiperiods according to the standard definition. Let us make now the generic assumption that the restrictions $\epsilon(p)|_\Pi$ are the Morse functions in any plane $\Pi$ orthogonal to $B$, i.e. all the critical points of this functions can be the non-degenerate local minima, non-degenerate saddle points or the non-degenerate local maxima in the plane. On the Fig. 1 we show the local behavior of the trajectories close to these critical points and we don’t admit here any other degeneration of trajectories in the plane.

Let us give here the following definitions:

**Definition 1.** We call the trajectory non-singular if it is not adjacent to the critical (saddle) point of the function $V(r)$. The trajectories adjacent to the critical points as well as the critical points themselves we call singular trajectories (see Fig. 1).

**Definition 2.** We call the non-singular trajectory compact if it is closed on the plane. We call the non-singular trajectory open if it is unbounded in...
Figure 2: The singular, compact and open non-singular quasiclassical trajectories. The signs ”+” and ”−” show the regions of larger and smaller values of $\epsilon(p)|_{\Pi}$ respectively.

The examples of singular, compact and open non-singular trajectories are shown on the Fig. 2 a-c.

It is easy to see also that the singular trajectories have the measure zero among all the trajectories on the plane.

In our consideration we will not make any difference between the compact trajectories of different forms and consider the global geometry of the non-compact trajectories only. According to the topological results on the classification of non-compact orbits it will be convenient to use the following definition concerning the global behavior of open trajectories in the planes orthogonal to $B$:

**Definition 3.** We call the open trajectory topologically regular (corresponding to "topologically integrable" case) if it lies within the straight line of finite width in $\mathbb{R}^2$ and passes through it from $-\infty$ to $\infty$ (see Fig. 3 a).
Figure 3: "Topologically regular" (a) and "chaotic" (b) open trajectories in the plane II orthogonal to B.

All other open trajectories we will call chaotic (Fig. 3 b).

We will see below that this approach to the classification of electron trajectories is closely connected with the form of the Fermi surface itself and the forms of the "carriers of non-compact trajectories" belonging to the Fermi surface. Thus the "topological integrability" is actually the property characterizing the topology of the carriers of open trajectories in $T^3$ (and $R^3$) and has the direct analog in the theory of integrable systems although the reasons of "integrability" are completely different in this situation and have completely topological origin.

Let us give now the basic definitions concerning the topology of the Fermi surface and later the topology of "carriers of open trajectories" on the Fermi surface for the fixed direction of B.

**Definition 4.**
1) Genus.
Let us now come back to the original phase space $T^3 = R^3/\Gamma^*$. Every com-
ponent of the Fermi surface becomes then the smooth orientable 2-dimensional surface embedded in $\mathbb{T}^3$. We can then introduce the standard genus of every component of the Fermi surface $g = 0, 1, 2, \ldots$ according to standard topological classification depending on if this component is topological sphere, torus, sphere with two holes, etc ... (see Fig. 4).

II) Topological Rank.

Let us introduce the Topological Rank $r$ as the characteristic of the embedding of the Fermi surface in $\mathbb{T}^3$. It’s much more convenient in this case to come back to the total $\mathbf{p}$-space and consider the connected components of the three-periodic surface in $\mathbb{R}^3$.

1) The Fermi surface has Rank 0 if every its connected component can be bounded by a sphere of finite radius.

2) The Fermi surface has Rank 1 if every its connected component can be bounded by the periodic cylinder of finite radius and there are components which can not be bounded by the sphere.

3) The Fermi surface has Rank 2 if every its connected component can be bounded by two parallel (integral) planes in $\mathbb{R}^3$ and there are components which can not be bounded by cylinder.

4) The Fermi surface has Rank 3 if it contains components which can not be bounded by two parallel planes in $\mathbb{R}^3$.

On the Fig. 4 we show the topological form of the surfaces of genuses 0, 1 and 2 (without the embedding in $\mathbb{T}^3$). It is easy to generalize this picture to the arbitrary genus $g$.

The pictures on Fig. 5 a-d represent the pieces of the Fermi surfaces in $\mathbb{R}^3$ with the Topological Ranks 0, 1, 2 and 3 respectively.
Figure 5: The Fermi surfaces with Topological Ranks 0, 1, 2 and 3 respectively.
It is easy to see also that the topological Rank coincides with the maximal Rank of the image of mapping \( \pi_1(S^i) \rightarrow \pi_1(T^3) \) for all the connected components of the Fermi surface.

As can be seen the genuses of the surfaces represented on the Fig. a-d are also equal to 0, 1, 2 and 3 respectively. However, the genus and the Topological Rank are not necessary equal to each other in the general situation.

Let us discuss briefly the connection between the genus and the Topological Rank since this will play the crucial role in further consideration. It is easy to see that the Topological Rank of the sphere can be only zero and the Fermi surface consists in this case of the infinite set of the periodically repeated spheres \( S^2 \) in \( \mathbb{R}^3 \).

The Topological Rank of the torus \( T^2 \) can take three values \( r = 0, 1, 2 \). Indeed, it is easy to see that all the three cases of periodically repeated tori \( T^2 \) in \( \mathbb{R}^3 \), periodically repeated ”warped” integral cylinders and the periodically repeated ”warped” integral planes give the topological 2-dimensional tori \( T^2 \) in \( \mathbb{T}^3 \) after the factorization (see Fig. 6).

Let us note here that we call the cylinder in \( \mathbb{R}^3 \) integral if it’s axis is parallel to some vector of the reciprocal lattice, while the plane in \( \mathbb{R}^3 \) is called integral if it is generated by some two reciprocal lattice vectors. The case \( r = 2 \), however, has an important difference from the cases \( r = 0 \) and \( r = 1 \). The matter is that the plane in \( \mathbb{R}^3 \) is not homological to zero in \( \mathbb{T}^3 \) (i.e. does not restrict any domain of ”lower energies”) after the factorization. We can conclude so that if these planes appear as the connected components of the physical Fermi surface they should always come in pairs, \( \Pi_+ \) and \( \Pi_- \), which are parallel to each other in \( \mathbb{R}^3 \). The factorization of \( \Pi_+ \) and \( \Pi_- \) gives then the two tori \( T^2_+ \), \( T^2_- \) with the opposite homological classes in \( \mathbb{T}^3 \) after the factorization. The space between the \( \Pi_+ \) and \( \Pi_- \) in \( \mathbb{R}^3 \) can now be taken as the domain of lower (or higher) energies and the disjoint union \( \Pi_+ \cup \Pi_- \) will correspond to the union \( T^2_+ \cup T^2_- \) homological to zero in \( \mathbb{T}^3 \).

It can be shown that the Topological Rank of any component of genus 2 can not exceed 2 also. The example of the corresponding immersion of such component with maximal Rank is shown at Fig. 6 c and represents the two parallel planes connected by cylinders.

At last we say that the Topological Rank of the components with genus \( g \geq 3 \) can take any value \( r = 0, 1, 2, 3 \).

Let us give also the definitions of ”rationality” and ”irrationality” of the
Figure 6: The periodically repeated tori $\mathbb{T}^2$, periodically repeated "warped" integral cylinders and the periodically repeated "warped" integral planes in $\mathbb{R}^3$. 
Definition 5. Let \( \{g_1, g_2, g_3\} \) be the basis of the reciprocal lattice \( \Gamma^* \). Then:

1) The direction of \( B \) is rational (or has irrationality 1) if the numbers \((B, g_1), (B, g_2), (B, g_3)\) are proportional to each other with rational coefficients.

2) The direction of \( B \) has irrationality 2 if the numbers \((B, g_1), (B, g_2), (B, g_3)\) generate the linear space of dimension 2 over \( \mathbb{Q} \).

3) The direction of \( B \) has irrationality 3 if the numbers \((B, g_1), (B, g_2), (B, g_3)\) are linearly independent over \( \mathbb{Q} \).

The conditions (1)-(3) can be formulated also as if the plane \( \Pi(B) \) orthogonal to \( B \) contains two linearly independent reciprocal lattice vectors, just one linearly independent reciprocal lattice vector or no reciprocal lattice vectors at all respectively.

It can be seen also that if \( \{l_1, l_2, l_3\} \) is the basis of the original lattice in \( x \)-space then the irrationality of the direction of \( B \) will be given by the dimension of the vector space generated by numbers

\[
(B, l_2, l_3) , \ (B, l_3, l_1) , \ (B, l_1, l_2)
\]

over \( \mathbb{Q} \). Easy to see that these numbers have the meanings of the magnetic fluxes through the faces of elementary lattice cell.

Let us discuss now the connection between the geometry of the non-singular electron orbits and the topological properties of the Fermi surface. We will briefly consider here the simple cases of Fermi surfaces of Rank 0, 1 and 2 and come then to our basic case of general Fermi surfaces having the maximal rank \( r = 3 \). We have then the following situations:

1) The Fermi surface has Topological Rank 0.

Easy to see that in this simplest case all the components of the Fermi surface are compact (Fig. 7 a) in \( \mathbb{R}^3 \) and there is no open trajectories at all.

2) The Fermi surface has Topological Rank 1.

In this case we can have both open and compact electron trajectories. However the open trajectories (if they exist) should be quite simple in this case. They can arise only if the magnetic field is orthogonal to the mean
direction of one of the components of Rank 1 (periodic cylinder) and are periodic with the same integer mean direction (Fig. 7 b). There is only the finite number of possible mean directions of open orbits in this case and a finite "net" of one-dimensional curves on the unit sphere giving the directions of \( \mathbf{B} \) corresponding to the open orbits. In some special points we can have the trajectories with different mean directions lying in different parallel planes orthogonal to \( \mathbf{B} \). Easy to see that in this case the direction of \( \mathbf{B} \) should be purely rational such that the orthogonal plane \( \Pi(\mathbf{B}) \) contains two different reciprocal lattice vectors. It is evident also that there is only the finite number of such directions of \( \mathbf{B} \) clearly determined by the mean directions of the components of Rank 1. Let us mention also that the existence of open orbits is not necessary here even for \( \mathbf{B} \) orthogonal to the mean direction of some component of Rank 1 as can be seen from the example of the "helix" represented on Fig. 7 a. Easy to see also the this type of trajectories corresponds to topologically integrable case according to the Definition 3 and gives the simplest example of topologically regular open orbit in the plane \( \Pi(\mathbf{B}) \).

Let us point out here that both cases of compact and periodic open tra-
Figure 8: (a) Connected component of Rank 1 having the form of "helix". Open orbits are absent for any direction of $\mathbf{B}$. (b) The example of the Fermi surface of Rank 2 containing two components with different integral directions.
jectories were considered in details in the work [1] where the form of the conductivity tensor in the limit $B \to \infty$ was obtained for these situations. Let us say also, that these two situations are actually the only two cases where the "dynamical" coordinates $(\epsilon, p_z, t)$ considered in [1] (corresponding to the dynamical system (4)) can be introduced globally on the Fermi surface and the full analytic expansion of $\sigma^{ik}(B)$ in the powers of $1/B$ can be obtained for $\omega B \tau \gg 1$. As we will see later this situation does not take place even for more general cases of quasiperiodic topologically regular trajectories on the Fermi surface (as well as for more complicated "chaotic" trajectories). However, the expressions for $\sigma^{ik}(B)$ for the case of the open periodic trajectories give also the correct main approximation to $\sigma^{ik}(B)$ in the main order of $1/B$ as we will see below. The corresponding dependence was also assumed in the papers [2, 3] where the more complicated cases of open trajectories appeared. They were also used in [20, 25, 31] where the "Topological quantum numbers" corresponding to general topologically regular trajectories were introduced. As we will see, however, the "ergodic dynamics" of (4) in the more general cases can be observed in the next orders in $1/B$ in precise conductivity measurements. Let us give here the corresponding expressions for conductivity for cases of compact and open periodic trajectories obtained in [1]. We assume here that the $z$-axis is always directed along the magnetic field $B$ and the $x$-axis in the plane $\Pi(B)$ (orthogonal to $B$) is directed along the mean direction of the periodic trajectory considered in $p$-space. The asymptotic values of $\sigma^{ik}$ can then be written in the following for in the main order on $1/B$ (14):

Case 1 (compact orbits):

$$\sigma^{ik} \simeq \frac{ne^2}{m^*} \left( \begin{array}{ccc} (\omega_B \tau)^{-2} & (\omega_B \tau)^{-1} & (\omega_B \tau)^{-1} \\ (\omega_B \tau)^{-1} & (\omega_B \tau)^{-2} & (\omega_B \tau)^{-1} \\ (\omega_B \tau)^{-1} & (\omega_B \tau)^{-1} & * \end{array} \right)$$  \hspace{1cm} (14)$$

Case 2 (open periodic orbits):

$$\sigma^{ik} \simeq \frac{ne^2}{m^*} \left( \begin{array}{ccc} (\omega_B \tau)^{-2} & (\omega_B \tau)^{-1} & (\omega_B \tau)^{-1} \\ (\omega_B \tau)^{-1} & * & * \\ (\omega_B \tau)^{-1} & * & * \end{array} \right)$$  \hspace{1cm} (15)$$

Here $\simeq$ means "of the same order in $\omega_B \tau$ and $*$ are some constants $\sim 1$. 

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Let us mention also that the relations (14)-(15) give only the absolute values of $\sigma^{ik}$.

Let us mention also that the electron dynamics in $\mathbf{x}$-space can be described by additional system

$$\dot{\mathbf{x}} = v_{gr}(\mathbf{p})$$

and can be easily reconstructed for any known trajectory of (4) in $\mathbf{p}$-space. Such the $xy$-projection of any trajectory in $\mathbf{x}$-space can be obtained just by rotation of corresponding trajectory of (4) by $\pi/2$. Easy to see then that the mean direction of open periodic electron trajectory in $\mathbf{x}$-space coincides with the $y$-direction in our coordinate system. The last fact can be clearly seen in the form (15) of the tensor $\sigma^{ik}$ as the strong anisotropy of longitudinal conductivity in the plane $\Pi(\mathbf{B})$ for $\omega_B \tau \gg 1$.

Let us now consider the cases of Topological rank 2 and 3 of the Fermi surface.

3) The Fermi surface has Topological Rank 2.

It can be easily seen that this case gives much more possibilities for the existence of open orbits for different directions of the magnetic field. In particular, this is the first case where the open orbits can exist for the generic direction of $\mathbf{B}$ with irrationality 3. So, in this case we can have the whole regions on the unit sphere such that the open orbits present for any direction of $\mathbf{B}$ belonging to the corresponding region. It is easy to see, however, that the open orbits have also a quite simple description in this case. Namely, any open orbit (if they exist) lies in this case in the straight strip of the finite width for any direction of $\mathbf{B}$ not orthogonal to the integral planes given by the components of Rank 2. The boundaries of the corresponding strips in the planes $\Pi(\mathbf{B})$ orthogonal to $\mathbf{B}$ will be given by the intersection of $\Pi(\mathbf{B})$ with the pairs of integral planes bounding the corresponding components of Rank 2. It can be also shown ([17], [18]) that every open orbit passes through the strip from $-\infty$ to $+\infty$ and can not turn back.

According to the remarks above the contribution to the conductivity given by every family of orbits with the same mean direction reveals the strong anisotropy when $\omega_B \tau \rightarrow \infty$ and coincides in the main order with the formula (15) for the open periodic trajectories.

For purely rational directions of $\mathbf{B}$ we can have the situation when the
open trajectories with different mean directions present on different components of the Fermi surface. For example, for the "exotic" surface shown at Fig. 8, we will have the periodic trajectories along both the $x$ and $y$ directions in different planes orthogonal to $B$ if $B\parallel z$. However, for any direction of $B$ which is not purely rational this situation is impossible. We have so that for any direction of $B$ with irrationality 2 or 3 all the open orbits will have the same mean direction and can exist only on the components of Rank 2 with the same (parallel) integral orientation. This statement is a corollary of more general topological theorem which we will discuss below.

At last we note that the directions of $B\parallel z$ orthogonal to one of the components of Rank 2 are purely rational and all the non-singular open orbits (if they exist) are rational periodic in this case. For any family of such orbits with the same mean direction the corresponding contribution to the conductivity can then be written in the form (15) in the appropriate coordinate system. However, the direction of open orbits can not be predicted apriori in this case.

Let us say that the trajectories of this type have already all the features of the general topologically integrable situation and the topologically regular trajectories appearing in the more complicated case of topological rank 3 have in fact very similar structure to the described above. However, we will need the special procedure of topological reconstruction preserving all the open trajectories on the Fermi surface which permits to see this fact in more complicated general situation. We will consider here in details also the "ergodic" properties of such trajectories on the corresponding pieces of the Fermi surface using the topological structure of this kind.

Let us start now with the most general and complicated case of arbitrary Fermi surface of Topological rank 3.

We describe first the convenient procedure (22) of reconstruction of the constant energy surface when the direction of $B$ is fixed.

As we said already, we admit only the non-degenerate singularities of the dynamical system having the form of the non-degenerate poles or non-degenerate saddle points. The singular trajectories passing through the critical points (and the critical points themselves) divide the set of trajectories into the different parts corresponding to different types of trajectories on the Fermi surface. As we also already said, we will not be interested here in the geometry of compact electron orbit in the "geometric limit" $\omega B/T \to \infty$. It's not difficult to show that the pieces of the Fermi surface carrying the
compact orbits can be either infinite or finite cylinders in $\mathbb{R}^3$ bounded by the singular trajectories (some of them maybe just points of minimum or maximum) at the bottom and at the top (see Fig. 9).

Easy to see that the first case corresponds then to the whole component of rank 1 carrying just the compact trajectories while the second case gives the part of more complicated Fermi surface filled by the compact trajectories.

Let us remove now all the parts containing the non-singular compact trajectories from the Fermi surface. The remaining part

$$S_F/(\text{Compact Nonsingular Trajectories}) = \bigcup_j S_j$$

is a union of the 2-manifolds $S_j$ with boundaries $\partial S_j$ who are the compact singular trajectories. The generic type in this case is a separatrix orbit with just one critical point like on the Fig. 9

Easy to see that the open orbit will not be affected at all by the construction described above and the rest of the Fermi surface gives the same open orbits as all possible intersections with different planes orthogonal to $\mathbf{B}$. 
Definition 6.

We call every piece $S_j$ the ”Carrier of open trajectories”. The trajectory is ”chaotic” if the genus $g(S_j)$ is greater than 1. The case $g(S_j) = 1$ we call ”Topologically Completely Integrable”. ³

Let us fill in the holes by topological 2D discs lying in the planes orthogonal to $B$ and get the closed surfaces

$$\bar{S}_j = S_j \cup (2\text{ - discs})$$

(see Fig. 10).

This procedure gives again the periodic surface $\bar{S}_i$ after the reconstruction and we can define the ”compactified carriers of open trajectories” both in $\mathbb{R}^3$ and $\mathbb{T}^3$.

Let us formulate now the main topological theorems concerning the geometry of open trajectories which made a breakthrough in the theory of such dynamical systems on the Fermi surfaces ([16], [19]).

Theorem 1. [16]

Let us fix the energy level $\epsilon_0$ and any rational direction $B_0$ such that no two saddle points on $S_\epsilon$ are connected in $\mathbb{R}^3$ by the singular electron trajectory. Then for all the directions of $B$ close enough to $B_0$ every open trajectory lies in the strip of the finite width between two parallel lines in the plane orthogonal to $B$.

In fact, the proof of the Theorem 1 was based on the statement that genus of every compactified carrier of open orbits $\bar{S}_j$ is equal to 1 in this case.

Theorem 2. [19]

Let a generic dispersion relation $\epsilon(p) : \mathbb{T}^3 \to \mathbb{R}$ be given such that for level $\epsilon(p) = \epsilon_0$ the genus $g$ of some carrier of open trajectories $\bar{S}_i$ is greater than 1. Then there exists an open interval $(\epsilon_1, \epsilon_2)$

³Such systems on $\mathbb{T}^2$ were discussed for example in [33]: the generic open orbits are topologically equivalent to the straight lines. Ergodic properties of such systems indeed can be nontrivial as it was found by Ya.Sinai and K.Khanin in [34].
Figure 10: The reconstructed constant energy surface with removed compact orbits the two-dimensional discs attached to the singular orbits in the generic case of just one critical point on every singular orbit.
containing $\epsilon_0$ such that for all $\epsilon \neq \epsilon_0$ in this interval the genus of carrier of open trajectories is less than $g$. Let us note now that the genus 1 corresponds precisely to the two-dimensional tori $\mathbb{T}^2$ embedded to the three-dimensional torus $\mathbb{T}^3$ as the compactified carriers of open trajectories on the Fermi surface. The corresponding components in the covering space can thus be either the periodically deformed cylinders or the periodically deformed integral plane in $\mathbb{R}^3$ filled by the open orbits. Easy to see then that the first case can arise only for the directions of $\mathbf{B}$ orthogonal to some integer vector in $\mathbb{R}^3$ (i.e. to the mean direction of the corresponding component carrying open orbits) and corresponds to the periodic open orbits in $\Pi(\mathbf{B})$. The second case corresponds to the more general situation and can arise for generic directions of $\mathbf{B}$ being stable with respect to all small rotations of the magnetic field. Easy to see the direct analogy of this situation with the cases of Fermi surfaces of rank 1 and 2 respectively. Actually, we show in this construction that for the fixed direction of $\mathbf{B}$ we can replace the Fermi surface by some other surface having the simpler structure which gives us the same open trajectories in all planes $\Pi(\mathbf{B})$ orthogonal to the magnetic field. As can be easily seen also, all the trajectories can correspond only to topologically integrable case if the carriers of open trajectories all have the genus 1. The mean direction of generic open trajectories can be obtained then as the intersections of the planes orthogonal to $\mathbf{B}$ with the integral planes in reciprocal lattice represented by the periodically deformed integral planes carrying open orbits.

The Theorem 2 claims then that only the "Topologically Integrable case" can be stable with respect to the small variations of energy level and has a generic properties in this situation. It follows also from Theorem 1 that this case is the only stable case with respect to the small rotations of $\mathbf{B}$ and then should be considered as the generic situation among all the situations when the non-compact electron trajectories arise on the Fermi surface.

The important property of the compactified components of genus 1 arising for the generic directions of $\mathbf{B}$ is following: they are all parallel in average in $\mathbb{R}^3$ and do not intersect each other. This property, playing the crucial role for conductivity phenomena, was first pointed out in [20] and called later the "Topological resonance" for the system (1). According to this property, all the stable topologically regular open orbits in all planes orthogonal to $\mathbf{B}$ have the same mean direction and then give the same form (15) of contribution to conductivity in the appropriate coordinate system common for all of them.
This fact gives the experimental possibility to measure the mean direction of non-compact topologically regular orbits both in $x$ and $p$ spaces from the anisotropy of conductivity tensor $\sigma^{ik}$. From the physical point of view, all the regions on the unit sphere (giving the directions of magnetic field) where the stable open orbits exist can be represented as the "stability zones" $\Omega_\alpha$ such that each zone corresponds to some integral plane $\Gamma_\alpha$ common to all the points of stability zone $\Omega_\alpha$. The plane $\Gamma_\alpha$ is then the integral plane in reciprocal lattice given by the integral mean direction of the components of genus 1 carrying open orbits and defining the mean directions of open orbits in $p$-space for any direction of $B$ belonging to $\Omega_\alpha$ just as the intersection with the plane orthogonal to $B$. As can be easily seen from the form of (15) this direction always coincides with the unique direction in $\mathbb{R}^3$ corresponding to the decreasing of longitudinal conductivity as $\omega_B \tau \to \infty$.

The corresponding integral planes $\Gamma_\alpha$ can then be given by three integer numbers $(n_1^\alpha, n_2^\alpha, n_3^\alpha)$ (up to the common multiplier) from the equation

$$n_1^\alpha [x]_1 + n_2^\alpha [x]_2 + n_3^\alpha [x]_3 = 0$$

where $[x]_i$ are the coordinates in the basis $\{g_1, g_2, g_3\}$ of the reciprocal lattice, or equivalently

$$n_1^\alpha(x, l_1) + n_2^\alpha(x, l_2) + n_3^\alpha(x, l_3) = 0$$

where $\{l_1, l_2, l_3\}$ is the basis of the initial lattice in the coordinate space.

We see then that the direction of conductivity decreasing $\hat{\eta} = (\eta_1, \eta_2, \eta_3)$ satisfies to relation

$$n_1^\alpha(\hat{\eta}, l_1) + n_2^\alpha(\hat{\eta}, l_2) + n_3^\alpha(\hat{\eta}, l_3) = 0$$

for all the points of stability zone $\Omega_\alpha$ which makes possible the experimental observation of numbers $(n_1^\alpha, n_2^\alpha, n_3^\alpha)$.

The numbers $(n_1^\alpha, n_2^\alpha, n_3^\alpha)$ were called in [20] the "Topological Quantum numbers" of a dispersion relation in metal.

Let us add also that the number of tori $T_i^2$ being even can still be different for the different points of stability zone $\Omega_\alpha$. We can then introduce in the general situation the "sub-boundaries" of the stability zone which are the piecewise smooth curves inside $\Omega_\alpha$ where the number of tori generically changes by 2. The asymptotic behavior of conductivity will still be described by the formula (15) in this case but the dimensionless coefficients will then
"jump" on the sub-boundaries of stability zone. Let us however mention here that this situation can be observed only for rather complicated Fermi surfaces.

Let us say now some words about the special situations when the carriers of open trajectories have genus more than 1 which can also happen for rather complicated Fermi surfaces.

It was first shown by S.P.Tsarev (28) the more complicated chaotic open orbits can still exist on rather complicated Fermi surfaces $S_F$. Such, the example of open trajectory which does not lie in any finite strip of finite width was constructed. The corresponding direction of $B$ had the irrationality 2 in this example and the closure of the open orbit was a "half" of the surface of genus 3 separated by the singular closed trajectory non-homotopic to zero in $T^3$. However, the trajectory had in this case the asymptotic direction even not being restricted by any straight strip of finite width in the plane orthogonal to $B$.

As was shown later in [22] this situation always takes place for any chaotic trajectory for the directions of $B$ with irrationality 2. We have so, that for non-generic "partly rational" directions of $B$ the chaotic behavior is still not "very complicated" and resembles some features of stable open electron trajectories.

The corresponding asymptotic behavior of conductivity should reveal also the strong anisotropy properties in the plane orthogonal to $B$ although the exact form of $\sigma^{ik}$ will be slightly different from (15) for this type of trajectories. By the same reason, the asymptotic direction of orbit can be measured experimentally in this case as the direction of lowest longitudinal conductivity in $\mathbb{R}^3$ according to kinetic theory. The measure of the corresponding set on the unit sphere is obviously zero for such type of trajectories being restricted by the measure of directions of irrationality 2. We will consider here the Tsarev example in more details below and discuss the corresponding conductivity behavior for $B \to \infty$.

The more complicated examples of chaotic open orbits were constructed in [22] for the Fermi surface having genus 3. The direction of the magnetic field has the irrationality 3 in this case and the closure of the chaotic trajectory covers the whole Fermi surface in $T^3$. These types of the open orbits do not have any asymptotic direction in the planes orthogonal to $B$ and have rather complicated form "walking everywhere" in these planes. Let us also discuss later this case in more details.
Let us formulate now the recent topological results concerning the general behavior of quasiclassical electron orbits on the energy levels of the generic dispersion relation $\epsilon = \epsilon(p)$ on $T^3$.

The systematic investigation of the open orbits was completed after the works [16, 19, 20] by I.A. Dynnikov (see [22, 27]). In particular the total picture of different types of the open orbits for generic dispersion relations was presented in [27]. Let us just formulate here the main results of [22, 27] in the form of Theorem.

**Theorem 3 ([22], [27]).**

Let us fix the dispersion relation $\epsilon = \epsilon(p)$ and the direction of $B$ of irrationality 3 and consider all the energy levels for $\epsilon_{\min} \leq \epsilon \leq \epsilon_{\max}$. Then:

1) The open electron trajectories exist for all the energy values $\epsilon$ belonging to the closed connected energy interval $\epsilon_1(B) \leq \epsilon \leq \epsilon_2(B)$ which can degenerate to just one energy level $\epsilon_1(B) = \epsilon_2(B) = \epsilon_0(B)$.

2) For the case of the nontrivial energy interval the set of compactified carriers of open trajectories $\bar{S}_\epsilon$ is always a disjoint union of two-dimensional tori $T^2$ in $T^3$ for all $\epsilon_1(B) \leq \epsilon \leq \epsilon_2(B)$. All the tori $T^2$ for all the energy levels do not intersect each other and have the same (up to the sign) indivisible homology class $c \in H_2(T^3, \mathbb{Z})$, $c \neq 0$. The number of tori $T^2$ is even for every fixed energy level and the corresponding covering $\bar{S}_\epsilon$ in $\mathbb{R}^3$ is a locally stable family of parallel ("warped") integral planes $\Pi^2_i \subset \mathbb{R}^3$ with common direction given by $c$. The form of $\bar{S}_\epsilon$ described above is locally stable with the same homology class $c \in H_2(T^3)$ under small rotations of $B$. All the open electron trajectories at all the energy levels lie in the strips of finite width with the same direction and pass through them. The mean direction of the trajectories is given by the intersections of planes $\Pi(B)$ with the integral family $\Pi^2_i$ for the corresponding "stability zone" on the unit sphere.

3) The functions $\epsilon_1(B)$, $\epsilon_2(B)$ defined for the directions of $B$ of irrationality 3 can be continued on the unit sphere $S^2$ as the piecewise smooth functions such that $\epsilon_1(B) \geq \epsilon_2(B)$ everywhere on the unit sphere.

4) For the case of trivial energy interval $\epsilon_1 = \epsilon_2 = \epsilon_0$ the corresponding open trajectories may be chaotic. Carrier of the chaotic open trajectory is homologous to zero in $H_2(T^3, \mathbb{Z})$ and has genus $\geq 3$. For the generic energy level $\epsilon = \epsilon_0$ the corresponding directions of magnetic fields belong to the countable union of the codimension 1 subsets. Therefore a measure of this set is equal to zero on $S^2$. 

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The whole manifold $\bar{S}_H$ is always homologous to zero in $T^3$ and all the two dimensional tori $T^2$ can be always divided into two equal groups $\{T^2_{i+}\}, \{T^2_{i-}\}$ according to the direction of the electron motion. As can be proved using Theorem 1 ([27]) the ”stability zones” form the everywhere dense set on the unit sphere for the generic dispersion relations. All the non-compact trajectories stable under the small rotations of $B$ should have thus the form described above.

Let us however remind here that for the conductivity phenomena in metals only the Fermi level $\epsilon(p) = \epsilon_F$ is important and we can not get any information about stability zones of dispersion relation such that $\epsilon_F \notin [\epsilon_2(B), \epsilon_1(B)]$.

Let us say also that Theorem 3 describes the generic situation of the directions of $B$ of irrationality 3 and some additional features can arise for the purely (irrationality 1) or ”partly” (irrationality 2) rational directions of $B$. We will not discuss here all these features in details and just make a reference on the articles [25, 31] where the general situation was considered.

4 The behavior of conductivity in the case of topologically regular open orbits.

Let us now briefly discuss the conductivity behavior in ”geometric strong magnetic field limit”. We prove below the analog of Lifshitz-Azbel-Kaganov-Peschanskii formulae in this case ([1, 2, 3]). We will start with generic ”topologically regular” open orbits having the ”strong asymptotic directions in $R^3$.

Let us come back to the formula (12) and consider the case of the open orbits having the form shown at Fig. 3 a. Let us take the $x$-axis in $R^3$ with the direction corresponding to the mean directions of open orbits in $p$-space and the $z$-axis directed along the magnetic field $B$. The projection of trajectory on the plane orthogonal to $B$ will have the same form in $x$-space just rotated by $\pi/2$ such that the mean direction of open orbits will be directed along the $y$-axis in our coordinate system.

We can write then that generally

$$\langle v^x_{\text{gr}} \rangle = 0, \quad \langle v^y_{\text{gr}} \rangle \neq 0, \quad \langle v^z_{\text{gr}} \rangle \neq 0$$

on these trajectories, where $\langle \ldots \rangle$ means the averaging along the trajectory according to system (4).

Moreover we can claim that the function
\[ p^y(p, -t) - p^y(p, 0) = \frac{eB}{c} \int_0^t v^x_{gr}(p, -t') dt' \]

is a bounded function for any trajectory shown at Fig. 3, as follows from its geometric form.

Let us consider now the components \( \sigma^{11} \) (or \( \sigma^{11'} \)) of the conductivity tensor \( \sigma^{ik}(B) \). Using the integration by parts we can write:

\[
\sigma^{11}(B) = e^2 \frac{c^2}{eB\tau} \int \ldots \int \left( -\frac{\partial f_T}{\partial \epsilon} \right) v^i_{gr}(p) \times \nabla_x [\int_0^{\infty} (p^y(p, -t) - p^y(p, 0)) e^{-t/\tau} dt] \frac{d^3p}{(2\pi\hbar)^3}
\]

(16)

For the component \( \sigma^{11} \) we can use then the formula

\[
\sigma^{11}(B) = e^2 \frac{c^2}{eB\tau} \int \ldots \int \left( -\frac{\partial f_T}{\partial \epsilon} \right) \times \nabla_x \left[ \int_0^{\infty} \frac{1}{2} (p^y(p, -t) - p^y(p, 0))^2 e^{-t/\tau} dt \right] \frac{d^3p}{(2\pi\hbar)^3} =
\]

\[
e^2 \left( \frac{c}{eB\tau} \right)^2 \int \ldots \int \left( -\frac{\partial f_T}{\partial \epsilon} \right) \left[ \int_0^{\infty} \frac{1}{2} (p^y(p, -t) - p^y(p, 0))^2 e^{-t/\tau} dt \right] \frac{d^3p}{(2\pi\hbar)^3}
\]

The integral

\[
I = \int_0^{\infty} \frac{1}{2} (p^y(p, -t) - p^y(p, 0))^2 e^{-t/\tau} dt
\]

can be evaluated as \( I \leq 1/2\tau P_0^2 \) where \( P_0 \) is the common constant bounding the values \( |p^y(p, -t) - p^y(p, 0)| \) for all the regular trajectories (the width of straight strip). The value \( \sigma^{11}(B) \) can then be written as

\[
\sigma^{11}(B) \sim \frac{ne^2\tau}{m^*} \frac{\alpha}{(\omega_B\tau)^2} + o \left( \frac{1}{(\omega_B\tau)^2} \right)
\]

(17)

when \( B \rightarrow \infty \).
The value of constant $\alpha$ can then be defined in terms of the integral

$$\frac{1}{2} \int \int \langle (p^y(p, -t) - p^y(p, 0))^2 \rangle \frac{d^2 S}{(2\pi\hbar)^3 v_{gr}(p)}$$

over the Fermi surface and is proportional to the square of effective width of straight lines bounding the topologically regular trajectories.

Let us say now that the total expansion in $(1/B)$ can not be actually made for the generic regular trajectories because of their ergodic behavior. However, the difference between the quasiperiodic and periodic trajectories does not appear in the main terms of the formula (15) and can be detected only in the next smaller approximations for $B \to \infty$. This fact permits actually to use the formula (15) also for regular quasiperiodic trajectories when speaking just about the geometric properties of $\sigma^{ik}$. Let us add also that the next approximations to (15) depend on whether the saddle critical points really present on the carriers of open orbits or not.

For the components $\sigma^{i1}$ or $\sigma^{1i}$, $i = 2, 3$ we can not use the second integration by parts and we should put then

$$\sigma^{i1}(B) \sim \sigma^{1i}(B) \sim \frac{ne^2 \tau}{m^* \omega_B \tau} + o \left( \frac{1}{\omega_B \tau} \right)$$

according to the formula (16).

Again the same remarks about next approximations in $\sigma^{i1}$ and $\sigma^{1i}$ can also be made in this case.

Let us consider now components $\sigma^{ik}$ where both $i,j \neq 1$. We can write the the formula (12) using the averaged values of $v_{gr}^y$ and $v_{gr}^z$ in the following way:

$$\sigma^{ik}(B) = e^2 \tau \int \ldots \int \left( -\frac{\partial f_T}{\partial \epsilon} \right) \langle v_{gr}^i \rangle \langle v_{gr}^k \rangle \frac{d^3 p}{(2\pi\hbar)^3} +$$

$$+ e^2 \int \ldots \int \left( -\frac{\partial f_T}{\partial \epsilon} \right) (v_{gr}^i(p) - \langle v_{gr}^i \rangle) \times$$

$$\times \left[ \int_0^{+\infty} (v_{gr}^k(p, -t) - \langle v_{gr}^k \rangle) e^{-t/\tau} dt \right] \frac{d^3 p}{(2\pi\hbar)^3}$$

(19)
(It’s not difficult to see the the ”cross terms” will disappear after the integration over $T^3$.)

The first part of this formula gives the finite values for $\sigma^{ik}(B)$ when $B \to \infty$ while the second becomes zero in the same limit. Easy to see that the last statement is just a simple corollary of the fact that the integral with decreasing exponent approaches the mean value on the trajectory in the limit $B \to \infty$ (up to the factor $1/\tau$) according to the system (4). Let us say again, however, that the $B$-dependence is more complicated in general here than in the case of purely periodic trajectories and can not be generically expanded in the powers of $1/B$.

Thus we can see that the formula (15) gives the main part of $\sigma^{ik}(B)$ also for general topologically regular trajectories with irrational mean directions. The cases of compact and purely periodic trajectories, however, are characterized by the existence of full analytic expansions in $1/B$ which do exist in general for more complicated quasiperiodic orbits.

Let us make also some additional remark on the periodic and quasiperiodic open trajectories. Namely, the periodic open orbits arise actually in every ”stability zone” when the intersection of the carrier of open orbits with the plane $\Pi(B)$ has the rational mean direction. Easy to see that this situation can appear if the direction of $B$ has irrationality 2 and always arise for the purely rational directions of $B$. This actually means that the corresponding orbits in $T^3$ are not everywhere dense on the tori $T^2_i$ (defined for a given stability zone) but become closed (in $T^3$) unlike the situation with irrational mean direction of open orbits. The mean values of $v^{i}_{gr}$, then actually depend on the open periodic trajectory and do not coincide with the value of $v^{i}_{gr}$ averaged over the carrier of open trajectory. We know just that averaging of these mean values over all the periodic trajectories will give us the values of $\langle v^{i}_{gr} \rangle$ close to the same values for close irrational directions of $B$. However, the limit $B \to \infty$ for tensor $\sigma^{ik}(B)$ ($i, k = 2, 3$) will be given by formula

$$\sigma^{ik} = e^{2\tau} \int \cdots \int \left( -\frac{\partial f_T}{\partial \epsilon} \right) \langle v^{i}_{gr} \rangle(p) \langle v^{k}_{gr} \rangle(p) \frac{d^3p}{(2\pi \hbar)^3}$$

containing the integration of products $\langle v^{i}_{gr} \rangle(p) \langle v^{k}_{gr} \rangle(p)$ which can not be replaced by the ”global” mean values $\langle v^{i}_{gr} \rangle$, $\langle v^{k}_{gr} \rangle$. As a corollary the corresponding values of $\sigma^{ik}$ will be actually different in this limit from the case of purely irrational directions of $B$. Such we can claim that the longitudinal
conductivities $\sigma_{22}$ and $\sigma_{33}$ should be actually bigger for $B \to \infty$ than the same values for close irrational directions of $B$.

We can see then that except the smooth dependence on the direction of $B$ (connected with the dependence on $B$ of total phase volume of open trajectories) we will have also the "sharp" peaks in the conductivity for the rational mean directions of open orbits. The corresponding "peaks" however will be quite small for the rational directions with big denominators. Easy to see also that this behavior does not affect the "Topological characteristics" of conductivity introduced above.

5 The chaotic cases.

Let us say now some words about chaotic trajectories which can arise in the special cases for rather complicated Fermi surfaces. We will first describe the Tsarev’s example of chaotic trajectory having an asymptotic direction in $\mathbb{R}^3$ (28). Let us consider the Fermi surface $S_F$ consisting of two sets of horizontal parallel layers $\{Q_+\}$ and $\{Q_-\}$ connected by two sets of inclined cylinders $\{C_1\}$, $\{C_2\}$ in $\mathbb{R}^3$ (see Fig. 11).

We will assume that both types of the cylinders have the identical forms and two cylinders shown at Fig. 11 have the same vertical symmetry plane $\Pi$ which intersects the horizontal layers in some irrational direction $\hat{\alpha}$ (Fig. 12).

We can assume now that the dark region on Fig. 12 represents actually the "jumping place" where the intersection curves $\Pi' \cap S_F$ "jump" from one horizontal layer to another (Fig. 11) for any plane $\Pi'$ parallel to $\Pi$. Let us put now $\Pi = \Pi(B)$ and consider the intersections $\Pi \cap S_F$ as the electron trajectories in $p$-space for given $B$.

We will put for simplicity that the horizontal periods of Fermi surface are orthogonal to each other and we can divide the horizontal layers into identical periodic strips parallel to one of the periods (say vertical strips formed by the repeated domains shown at Fig. 12). Easy to see that for our specific direction $\hat{\alpha}$ any trajectory will either pass through the strip being at the same layer in $\mathbb{R}^3$ or meet the pair of cylinders (of identical form) and jump by two layers preserving the layer type ("positive" or "negative"). The horizontal projection of corresponding orbits parts are always the same in this situation while the vertical displacements can be either zero or the vertical period of $S_F$ if the vertical jump takes place in the corresponding strip. The probability
Figure 11: The Fermi surface for Tsarev’s example of chaotic trajectory.
Figure 12: The intersection of the horizontal planes by the vertical plane with irrational direction passing through the axes of inclined cylinders in Tsarev example.
of jump is proportional to the ratio $\delta/a$ (Fig. 12) and all the non-singular trajectories have then the same asymptotic directions.

It can be also easily seen that the trajectories belonging to different types of layers go and "jump" in the opposite directions being divided by the singular trajectories passing through the saddle-points on the sides of the cylinders $C_1$ and $C_2$.

It can be proved, however, that there exist the special values of $\hat{\alpha}$ such that the trajectories will not belong here to the straight strips of any finite width as follows from the numbers theory. In this situation we can not use anymore the same arguments which we used for calculating of $\sigma^{\text{ii}}$ and $\sigma^{\text{ii}}$ in the case of topologically regular trajectories and so we don’t have formulas (17) and (18) in this situation. However, we can still use the fact

\[
\langle v_x^{gr} \rangle = 0, \quad \langle v_y^{gr} \rangle \neq 0, \quad \langle v_z^{gr} \rangle \neq 0
\]

(in the appropriate coordinate system) also in this case. Using the same arguments for the formula (19) we can prove then the following formulae for the behavior of $\sigma^{ik}(B)$:

\[
\sigma^{ik}(B) \simeq \frac{n e^2 \tau}{m^*} \begin{pmatrix}
    o(1) & o(1) & o(1) \\
    o(1) & * & * \\
    o(1) & * & *
\end{pmatrix}
\]

which replaces the formula (15) for the case of Tsarev’s ergodic trajectories. Let us omit here all the details of Tsarev’s chaotic trajectories and just point out that the asymptotic direction of ergodic trajectory of Tsarev type can be also observed experimentally due to the same reasons as in the case of topologically regular trajectories. However, unlike the topologically regular case, the ergodic trajectories of Tsarev type are unstable with respect to generic small rotations of $B$ and do not correspond to "stability zones" on the unit sphere.

Let us say now some words about more general ergodic trajectories of Dynnikov type which do not have any asymptotic direction in $\mathbb{R}^3$. We will not describe here the corresponding construction (see [22]) and just give the main features of such trajectories.

First of all, these trajectories can arise only in the case of magnetic field of irrationality 3 and the corresponding carriers have then the genus $\geq 3$. This kind of trajectories are completely unstable with respect to the small rotations of $B$ and can be observed just for special fixed directions of $B$
in the case of rather complicated Fermi surfaces. The approximate form of trajectories of this kind is shown at Fig. 3 b. The carrier of such trajectory can then resemble the Fermi surface shown at the Fig. 5 d and is homologous to zero in the 3-dimensional torus $T^3$. Moreover, if the genus of the Fermi surface is not very high ($< 6$) it can always be stated that the corresponding carrier of open ergodic trajectories is invariant under the involution $p \to -p$ (after the appropriate choice of the initial point in $T^3$). The ergodicity of the open trajectories on the carrier gives then immediately the relations:

$$
\langle v_{gr}^x \rangle = 0 \, , \, \langle v_{gr}^y \rangle = 0 \, , \, \langle v_{gr}^z \rangle = 0
$$

for all three components of the group velocity on any of such trajectories. This important fact leads to the rather non-trivial behavior of corresponding contribution to the conductivity for $B \to \infty$. Namely, using the same formula \cite{12} in this case we can show that all the components of the corresponding contribution to $\sigma_{ik}(B)$ become actually zero in the limit $B \to \infty$ \cite{21}. We can write then for this contribution:

$$
\sigma_{ik}(B) \simeq \frac{ne^2 \tau}{m^*} \begin{pmatrix}
o(1) & o(1) & o(1) 
o(1) & o(1) & o(1) 
o(1) & o(1) & o(1)
\end{pmatrix}
$$

\text{(21)}

for $B \to \infty$.\footnote{Actually the component $\sigma^{zz}(B)$ contains the non-vanishing term of order of $T^2/\epsilon_F^2$ for $B \to \infty$ for non-zero temperatures \cite{21}. However, this parameter is very small for the normal metals and we don’t take it here in the account.}

We see then that the chaotic trajectories of Dynnikov type do not give any contribution even for conductivity along the magnetic field $B$ for rather big values of $B$. In the work \cite{24} also the special ”scaling” asymptotic behavior of $\sigma_{ik}(B)$ were suggested. Let us note, however, that the full conductivity tensor include also the contribution of compact (closed) electron trajectories having the form \cite{14} which presents in general as the additional contribution in all the cases described above. We can so claim that the ergodic behavior of Dynnikov type does not actually completely removes the conductivity along the magnetic field $B$ because of the contribution of compact trajectories. However, the sharp local minimum in this conductivity can still be observed in this case since a part of the Fermi surface will be effectively excluded from the conductivity in this situation.
It can be proved (see [27]) that for generic Fermi surfaces the measure of directions of magnetic field \(B\) where the chaotic behavior of Dynnikov type can be found on the Fermi surface is zero. Let us say that the restriction on just one energy level is connected closely with the situation in metals where only the energy levels close to \(\epsilon_F\) are important. Let us formulate also the more general conjecture of S.P.Novikov about the total set of ”chaotic directions” on the unit sphere, where we do not restrict the consideration to just one energy level:

**Conjecture.** (S.P.Novikov).

The total set of the directions of \(B\) corresponding to the chaotic behavior has the measure 0 for the whole generic dispersion relation and the Hausdorff dimension strictly less than 2.

Let us mention also that in the paper [23] the possibilities of the investigation of total topological characteristics of whole dispersion relation \(\epsilon(p)\) were discussed. In particular the behavior of electrons injected in the empty band of semiconductor in the presence of magnetic field was considered. However, the corresponding magnetic fields should be extremely high (\(\sim 10^2 Tl\)) in this case which make such experiments very difficult. Most probably this situation should be considered now just as theoretical possibility.

### 6 The different regimes of conductivity behavior. Classification.

We are going to give now the full classification of possible conductivity regimes corresponding to different topological types of trajectories given by system (4). First we will need to add here the non-generic cases of irrationality 1 and 2. The Theorem 3 should be slightly modified in this case but has the same main features as in the case of fully irrational magnetic field ([27]). Namely, the set of carriers of open trajectories \(\tilde{S}_\epsilon\) can contain now the two-dimensional tori \(T^2_s\) having the zero homology class in \(H_2(T^3)\) in addition to the family of parallel tori with non-zero homology classes described above. The corresponding covering of these components in \(R^3\) are ”warped” periodic cylinders and all the open trajectories belonging to these components are purely periodic. As it is easy to see these components of \(\tilde{S}_\epsilon\) are stable with respect to the small rotations of \(B\) in the plane orthogonal to the axis
of cylinder and disappear after any other small rotation. The part (1) of the Theorem 2 will be true also for rational or ”partly rational” directions of \( \mathbf{B} \) with some connected energy interval \( \epsilon'_1(\mathbf{B}) \leq \epsilon \leq \epsilon'_2(\mathbf{B}) \). However, the boundary values \( \epsilon'_1(\mathbf{B}), \epsilon'_2(\mathbf{B}) \) do not necessarily coincide in this case with the values of piecewise smooth functions \( \epsilon_1(\mathbf{B}), \epsilon_2(\mathbf{B}) \) defined everywhere on \( S^2 \) according to Theorem 3 ([27]). Namely, we will have instead the relations \( \epsilon'_1(\mathbf{B}) \leq \epsilon_1(\mathbf{B}) \leq \epsilon_2(\mathbf{B}) \leq \epsilon'_2(\mathbf{B}) \) for all such directions of \( \mathbf{B} \) where all the components of \( \tilde{S}_i \) belonging to intervals \( [\epsilon'_1(\mathbf{B}), \epsilon_1(\mathbf{B})) \), \( (\epsilon_2(\mathbf{B}), \epsilon'_2(\mathbf{B})] \) consist of the tori homologous to zero in \( T^3 \). As we will see this cases can be observed experimentally for those \( \mathbf{B} \) where the Fermi level lies in the one of such intervals and only the ”partly stable” non-compact trajectories exist on the Fermi surface.

The ”partly-stable” cylinders described above do not intersect the ”absolutely stable” components of \( \tilde{S}_i \) and all the open trajectories will still have the same mean direction if \( \mathbf{B} \) is not orthogonal to corresponding integral plane \( \Gamma_\alpha \) (let us mention that all the trajectories lying on ”regular” parallel integral planes in \( \tilde{S}_i \subset R^3 \) will be also periodic with the same period in this case). The form of the conductivity tensor will still be described by the formula (15) but the numerical values of dimensionless coefficients will jump for those non-generic directions of \( \mathbf{B} \) where the situation described appears. In particular, the asymptotic behavior (15) will arise on the one-dimensional curves for those directions of \( \mathbf{B} \) where \( \epsilon_F \in [\epsilon'_1(\mathbf{B}), \epsilon_1(\mathbf{B})) \cup (\epsilon_2(\mathbf{B}), \epsilon'_2(\mathbf{B})] \) being stable only for rotations of \( \mathbf{B} \) in the corresponding direction. As follows from the statements above the corresponding directions of \( \mathbf{B} \) can have at most the irrationality 2 and the corresponding one-dimensional curves are always the parts of the circles orthogonal to some integer vector in the reciprocal lattice \( \Gamma^* \).

Let us make now a special remark about the ”Special directions” of \( \mathbf{B} \) orthogonal to the integral planes \( \Gamma_\alpha \) if this direction belongs to the corresponding stability zone \( \Omega_\alpha \). The direction of \( \mathbf{B} \) is then purely rational and all the corresponding open orbits (if they exist) should be periodic in \( R^3 \). However, the mean directions of these open orbits can be different in this case for the different planes orthogonal to the magnetic field and the corresponding contributions to conductivity can not be then written in the form (15) in the same coordinate system. The conductivity tensor \( \sigma^{ik} \) can have then the full rank in the limit \( B\tau \to \infty \) and the conductivity remains constant for all the directions in \( R^3 \) in the strong magnetic field limit. This situation, however, is completely unstable and disappear after any small rotation of \( \mathbf{B} \).
The second possibility in the case of such directions is that all the open trajectories become singular and form the "singular periodic nets" in the planes orthogonal to $B$ (see Fig. 13). The asymptotic behavior of conductivity is described then by formula (14) but is also completely unstable and changes to (15) after any small rotation of $B$.

Let us add that the same situations can arise also on the stable two-dimensional tori $T^2_i$ in this case where the directions of open orbits cannot be defined anymore as the intersection of $\Gamma_\alpha$ with the plane $\Pi(B)$. Such, we can have either the "singular nets" or the regular periodic open orbits on these tori for this special direction. Also we can have the open orbits with different mean directions on different tori in this case but the number of such tori should be $\geq 4$ for any physical type of dispersion relations. This situation can thus also be observed only for rather complicated Fermi surfaces.
Figure 14: The wide finite strip in the plane $\Pi(B)$ containing the open orbits for $B$ close to "Special rational direction" within the stability zone.

Let us mention also that for the directions of $B$ close to this special rational directions the widths of the straight strips containing the regular open orbit can become very big in the planes $\Pi(B)$ (see Fig. 14).

So from physical point of view the conductivity phenomena will not "feel" the mean directions of open orbits for the directions of $B$ close enough to these special points even for rather big (but finite) values of magnetic field. Instead, the oscillations of the trajectory within the strip (Fig. 14) will be essential for conductivity up to the rather big values of $B$ such that $\omega_B \tau \sim L/p_0$ (where $L$ is the width of the strip and $p_0$ is the size of the Brillouin zone). These situation exists, however, if the open orbits with corresponding direction exist also for $B = B_0$ where $B_0$ is the special rational direction in the stability zone. For $B$ close to $B_0$ we will observe then exactly this direction up to the values of $B$ such that $\omega_B \tau \sim L/p_0$ and then the regime will change to
the common situation corresponding to given stability zone. Experimentally we will observe then the "small spots" around these directions on the unit sphere where the anisotropy of $\sigma^{ik}$ corresponds to the direction of orbits for $B = B_0$. In the most complicated case when we have the open orbits with different mean directions for $B = B_0$ we will have then the finite conductivity for all the directions in $R^3$ in the corresponding spot for rather big values of $B$.

For the "special rational directions" corresponding to the "singular net" on the stable tori $T_i^2$ the behavior of $\sigma^{ik}$ will correspond to the common form for a given stability zone even for $B$ very close to $B_0$. However, the measure of open orbits will tend to zero as $B \to B_0$ in the stability zone. The dimensionless coefficients ($*$) in the formula (15) will vanish then for $B \to B_0$ although the integral plane $\Gamma_\alpha$ will be observable up to $B = B_0$.

We mention here at last that both cases when $B_0^\alpha$ belongs or does not belong to the corresponding stability zone $\Omega_\alpha$ are possible in the examples.

We can describe now the total picture for the angle diagram of conductivity in normal metal in the case of geometric strong magnetic field limit. Namely, we can observe the following objects on the unit sphere parameterizing the directions of $B$:

1) The "stability zones" $\Omega_\alpha$ corresponding to some integral planes $\Gamma_\alpha$ in the reciprocal lattice ("Topological Quantum numbers"). This "Topological Type" of open trajectories is stable with respect to small rotations of $B$ and this is the only open orbits regime which can have a non-zero measure on the unit sphere. All the "stability zones" have the piecewise smooth boundaries on $S^2$ and are given by the condition

$$\epsilon_1(B) \leq \epsilon_F \leq \epsilon_2(B)$$

where $\epsilon_1(B)$, $\epsilon_2(B)$ are the piecewise smooth functions defined in the Theorem 3. Generally speaking, this set is not everywhere dense anymore being just a subset of the corresponding set for the whole dispersion relation and have some rather complicated geometry on the unit sphere.

The corresponding behavior of conductivity is described by the formula (15) and reveals the strong anisotropy in the planes orthogonal to the magnetic field. For rather complicated Fermi surfaces we can observe also the "sub-boundaries" of the stability zones where the coefficients in (15) have the sharp "jump".
2) The net of the one-dimensional curves containing directions of irrationality $\leq 2$ where the additional two-dimensional tori (homologous to zero in $T^3$) can appear. The corresponding parts of the net are always the parts of the big (passing through the center of $S^2$) circles orthogonal to some reciprocal lattice vector where the condition

$$\epsilon_1'(B) \leq \epsilon_F \leq \epsilon_2'(B)$$

is satisfied. The asymptotic behavior of conductivity is given again by the formula (15).

Let note also that these special curves on $S^2$ can be considered actually as the reminiscent of the bigger stability zones if we don’t restrict ourselves just by one Fermi surface. The structure of such sets thus can be used to get more information about the corresponding total structure for the whole dispersion relation in metal. In particular, the mean direction of such open orbits coincides with the mean direction of the generic open orbits in the intersections of the net with the stability zones (except the "Special rational directions"). The corresponding conductivity tensor is given then by the same formula (15) where the dimensionless coefficients ”jump” on the curves of the net.

3) The "Special rational directions".

Let us remind that we call the special rational direction the direction of $B$ orthogonal to some plane $\Gamma_\alpha$ in case when this direction belongs to the same stability zone on the unit sphere. We can have here all the possibilities described earlier for this situation (i.e. regular behavior with vanishing coefficients in (15), spots with isotropic or anisotropic behavior of conductivity different from the given by corresponding ”Topological quantum numbers”, ”partly stable” isotropic or anisotropic addition to the (15), etc.)

4) The chaotic open orbits of Tsarev type ($B$ of irrationality 2).

We can have points on the unit sphere where the open orbits are chaotic in Tsarev sense. All open trajectories still have the asymptotic direction in this case and the conductivity reveals the strong anisotropy in the plane orthogonal to $B$ as $B \to \infty$. The $B$ dependence, however is slightly different from the formula (15) in this case.

5) The chaotic open orbits of Dynnikov type ($B$ of irrationality 3).
For some points on $S^2$ we can have the chaotic open orbits of Dynnikov type on the Fermi surface. At these points the local minimum of conductivity along the magnetic field is expected. The conductivity along $B$ however remains finite as $B \to \infty$ in general situation because of the contribution of compact trajectories.

6) At last we can have the open regions on the unit sphere where only the compact trajectories on the Fermi level are present. The asymptotic behavior of conductivity tensor is given then by the formula (12).

Let now point out some new features connected with the ”magnetic breakdown” (self-intersecting Fermi surfaces) which can be observed for rather strong magnetic fields. Up to this point it has been assumed throughout that different parts of the Fermi surface do not intersect with each other. However, it is possible for some special lattices that the different components of the Fermi surface (parts corresponding to different conductivity bands) come very close to each other and may have an effective ”reconstruction” as a result of the ”magnetic breakdown” in strong magnetic field limit. In this case we can have the situation of the electron motion on the self-intersecting Fermi surface such that the intersections with other pieces do not affect at all the motion on one component. (The physical conditions for the corresponding values of $B$ can be found in [10]). In this case the picture described above should be considered independently for all the non-selfintersecting pieces of Fermi surface and we can have simultaneously several independent angle diagrams of this form on the unit sphere. Such we can have here the overlapping stability zones where the open orbits can have different mean directions. The correspondent conductivity tensor will then be given just as a sum of all conductivity tensors corresponding to different non-selfintersecting components. (The problem of the magnetic breakdown was brought to the authors’ attention by M.I.Kaganov.)

References

[1] I.M.Lifshitz, M.Ya.Azbel, M.I.Kaganov. Sov. Phys. JETP 4, 41 (1957).
[2] I.M.Lifshitz, V.G.Peschansky. Sov. Phys. JETP 8, 875 (1959).
[3] I.M.Lifshitz, V.G.Peschansky. Sov. Phys. JETP 11, 137 (1960).
[4] N.E.Alexeevsky, Yu.P.Gaidukov. *Sov. Phys. JETP* **8**, 383 (1959).
[5] N.E.Alexeevsky, Yu.P.Gaidukov. *Sov. Phys. JETP* **9**, 311 (1959).
[6] N.E.Alexeevsky, Yu.P.Gaidukov. *Sov. Phys. JETP* **10**, 481 (1960).
[7] Yu.P.Gaidukov. *Sov. Phys. JETP* **10**, 913 (1960).
[8] I.M.Lifshitz, M.I.Kaganov. *Sov. Phys. Usp.* **2**, 831 (1960).
[9] I.M.Lifshitz, M.I.Kaganov. *Sov. Phys. Usp.* **5**, 411 (1962).
[10] I.M.Lifshitz, M.Ya.Azbel, M.I.Kaganov. Electron Theory of Metals. Moscow, Nauka, 1971. Translated: New York: Consultants Bureau, 1973.
[11] A.A.Abrikosov. Fundamentals of the Theory of Metals. ”Nauka”, Moscow (1987). Translated: Amsterdam: North-Holland, 1998.
[12] S.P.Novikov. *Russian Math. Surveys* **37**, 1 (1982).
[13] S.P.Novikov. Proc. Steklov Inst. Math. 1 (1986).
[14] S.P.Novikov. ”Quasiperiodic structures in topology”. Proc. Conference ”Topological Methods in Mathematics”, dedicated to the 60th birthday of J.Milnor, June 15-22, S.U.N.Y. Stony Brook, 1991. Publish of Perish, Houston, TX, pp. 223-233 (1993).
[15] S.P.Novikov. Proc. Conf. of Geometry, December 15-26, 1993, Tel Aviv University (1995).
[16] A.V.Zorich. *Russian Math. Surveys* **39**, 287 (1984).
[17] I.A.Dynnikov. *Russian Math. Surveys* **57**, 172 (1992).
[18] I.A.Dynnikov. *Russian Math. Surveys* **58** (1993).
[19] I.A.Dynnikov. ”A proof of Novikov’s conjecture on semiclassical motion of electron.” *Math. Notes* **53**:5, 495 (1993).
[20] S.P.Novikov, A.Ya.Maltsev. *ZhETP Lett.* **63**, 855 (1996).
[21] I.A.Dynnikov. ”Surfaces in 3-Torus: Geometry of plane sections.” Proc.of ECM2, BuDA, 1996.
[22] I.A.Dynnikov. "Semiclassical motion of the electron. A proof of the Novikov conjecture in general position and counterexamples." American Mathematical Society Translations, Series 2, Vol. 179, Advances in the Mathematical Sciences. Solitons, Geometry, and Topology: On the Crossroad. Editors: V.M.Buchstaber, S.P.Novikov. (1997)

[23] I.A.Dynnikov, A.Ya.Maltsev. ZhETP 85, 205 (1997).

[24] A.Ya. Maltsev. ZhETP 85, 934 (1997).

[25] S.P.Novikov, A.Ya.Maltsev. Physics-Uspekhi 41(3), 231 (1998).

[26] A.V.Zorich. Proc. ”Geometric Study of Foliations” (Tokyo, November 1993)/ ed. T.Mizutani et al. Singapore: World Scientific, 479-498 (1994).

[27] I.A.Dynnikov. Russian Math. Surveys 54, 21 (1999).

[28] S.P.Tsarev. Private communication. (1992-93).

[29] S.P.Novikov. Russian Math. Surveys 54:3, 1031 (1999).

[30] R.D.Leo. PhD Theses. University of Maryland. Department of Math., College Park, MD 20742, USA.

[31] A.Ya.Maltsev, S.P.Novikov, ArXiv: math-ph/0301033, to appear in Special Volume of Bulletin of Braz. Math. Society.

[32] A.Ya. Maltsev, Arxiv: cond-mat/0302014

[33] V.I.Arnold. Functional analysis and its applications bf 25:2 (1991).

[34] Ya.G.Sinai, K.M.Khanin. Functional analysis and its applications bf 26:3 (1992).

[35] A.B.Pippard. Phil. Trans. Roy. Soc., A250, 325 (1957).

[36] M.H.Cohen, L.M.Falicov. Phys. Rev. Lett. 7, 231 (1961).