THE EXPONENTIAL CONVERGENCE OF THE CR YAMABE FLOW

WEIMIN SHENG AND KUNBO WANG

Abstract. In this paper, we study the CR Yamabe flow with zero CR Yamabe invariant. We use the CR Poincaré inequality and a Gagliardo-Nirenberg type interpolation inequality to show that this flow has long time solution and the solution converges to a contact form with flat pseudo-Hermitian scalar curvature exponentially.

1. Introduction

Let \((M^n, g)\) be a smooth, compact Riemannian manifold without boundary, and its dimension \(n \geq 3\). The Yamabe problem [31] is to find a metric conformal to \(g\) such that it has constant scalar curvature. This problem was solved by Yamabe, Trudinger, Aubin and Schoen in [31, 29, 21, 25]. A different approach to the Yamabe problem is the Yamabe flow, which was proposed by Hamilton [18]. Denote \(R_g\) the scalar curvature of \(g\) and \(r_g\) the mean value of \(R_g\), i.e.

\[ r_g = \frac{\int_M R_g dV_g}{\int_M dV_g}. \]

Consider the following parabolic equation

\[ \frac{\partial g}{\partial t} = -(R_g - r_g)g. \]  

(1.1)

Hamilton showed the short time existence for (1.1) in [18]. Chow [9] proved that (1.1) approaches to a metric of constant scalar curvature provided that the initial metric is locally conformally flat and has positive Ricci curvature. In [32], Ye obtained uniform a priori \(C^1\) bounds for the solution of (1.1) on any conformally flat manifold, and showed that (1.1) smoothly converge to a metric of constant scalar curvature. Ye also proved that the Yamabe flow (1.1) exits for all time and converges smoothly to a unique limit of constant scalar curvature provided that the initial metric is scalar negative or scalar flat. By use of the general concentration-compactness result [27], Schwetlick and Struwe [26] proved the convergence of the Yamabe flow when \(3 \leq n \leq 5\) provided that the initial
metric has large energy. In [3], Brendle proved the convergence of the flow for arbitrary initial energy.

The CR geometry, which is the abstract model of real hypersurfaces in complex manifolds, has a lot of analogy with the geometry of Riemannian manifolds. Many mathematicians have made outstanding contributions in this field, such as Chern and Moser [8], Fefferman [10], Folland [11], Folland and Stein [12], Jerison and Lee [19, 20, 21], Tanaka [28], and Webster [30], etc. Jerison and Lee [19] studied a Yamabe type problem on CR manifolds. To distinguish it with the Riemannian Yamabe problem, it is called the CR Yamabe problem. Suppose that \((M, \theta)\) is a compact strongly pseudo-convex CR manifold of real dimension \(2n + 1\) with a given contact form \(\theta\). The CR Yamabe problem is to find a contact form \(\tilde{\theta}\) conformal to \(\theta\) such that its Webster scalar curvature is constant. If we define a new contact form \(\tilde{\theta} = u^{\frac{2}{2n}}\theta\), where \(u > 0\), and denote \(\tilde{R} (R \text{ resp.})\) the pseudo-Hermitian Webster scalar curvature with respect to the contact form \(\tilde{\theta} (\theta \text{ resp.}),\) then the CR Yamabe problem is reduced to solve the following CR Yamabe equation

\[
- (2 + \frac{2}{n})\triangle_b u + Ru = \tilde{R}u^{1+\frac{2}{n}},
\]

where \(\triangle_b\) is the sub-Lapacian of \(M\). The CR Yamabe invariant is defined as

\[
\lambda(M, \theta) = \inf \{ \int_M [(2 + \frac{2}{n})\|\nabla_\theta u\|^2 + Ru^2]dV_{\theta} \} : u > 0, u \in S^2_1(M) \}.
\]

Here \(dV_{\theta}\) is the volume form with respect to the contact form \(\theta\), \(S^2_1(M)\) is the Folland-Stein space, which is the completion of \(C^1(M)\) with respect to the norm

\[
||u||_{S^2_1(M)} = (\int_M (|\nabla_\theta u|^2 + |u|^2)dV_{\theta})^{\frac{1}{2}}.
\]

Jerison and Lee [19] solved the CR Yamabe problem when \(n \geq 2\) and \(M\) is not locally CR equivalent to the sphere. The remaining cases were solved by Gamara [13], and Gamara, Yacoub [14].

Since \(\lambda(M, \theta)\) is determined by the CR structure, which is independent of the choice of \(\theta\), we denote it by \(\lambda(M)\) from now on. It is natural to ask if we can solve the CR Yamabe problem by a parabolic argument. Namely, as an analogue to the Yamabe flow on a Riemannian manifold, one can construct the CR Yamabe flow as follows:

\[
\frac{\partial}{\partial t} \tilde{\theta}(t) = - (\tilde{R} - \bar{r})\tilde{\theta}(t).
\]

Here \(\bar{r}\) is the average value of the pseudohermitian scalar curvature \(\tilde{R}\), defined by

\[
\bar{r} = \frac{\int_M \tilde{R}dV_{\tilde{\theta}}}{\int_M dV_{\tilde{\theta}}},
\]
The CR Yamabe flow was firstly studied by Chang and Cheng [6]. They proved the short time existence in all dimensions and obtained a Harnack type inequality in dimension three. Zhang [33] proved the long time existence and convergence for the case $\lambda(M) < 0$. For the case $\lambda(M) > 0$, Ho [15] proved the long time existence for all dimensions, and the convergence when $M$ is the sphere. Ho and the authors [17] proved the convergence when $n = 1$ recently.

For a given contact form $\theta_0$ on $M$, we say $\tilde{\theta}$ is conformal to $\theta_0$ if there is a positive function $f$ such that

$$\tilde{\theta} = f\theta_0.$$  

Let $[\theta_0]$ be the conformal class of a given contact form $\theta_0$ on $M$. If we assume that $\lambda(M) = 0$, then we can find a contact form $\theta \in [\theta_0]$ with flat pseudohermitian scalar curvature. Without loss of generalization, we may assume it is $\theta_0$ itself. We consider the following CR Yamabe flow:

$$\begin{cases}
\frac{\partial}{\partial t}\tilde{\theta}(t) = -(\tilde{R} - \tilde{r})\tilde{\theta}(t), \\
\tilde{\theta}(t) = u^\frac{2}{n}(t)\theta_0, \\
\tilde{\theta}(t)|_{t=0} = \theta.
\end{cases}$$

(1.4)

Here $\theta$ may be $\theta_0$ or some other fixed contact form from the conformal class $[\theta_0]$, i.e.

$$\theta = u(\cdot, 0)^\frac{2}{n}\theta_0.$$  

In this paper, we follow the idea of Ye [32](Page 45-47) to prove the following main theorem:

**Theorem 1.1.** Let $(M, \theta_0)$ be a smooth, strictly pseudo-convex $2n+1$ dimensional compact CR manifold. Suppose $\lambda(M) = 0$, then the CR Yamabe flow (1.4) exists for all time, and converges to a contact form with flat pseudo-Hermitian scalar curvature exponentially.

The convergence argument depends on a Poincaré inequality and a CR Gagliardo-Nirenberg type inequality. In section 2, we recall some basic concepts in CR geometry, derive a global version of Poincaré inequality on CR manifolds. In section 3, we prove the long time existence and exponential convergence of the CR Yamabe flow (1.4). In the appendix, we prove a Gagliardo-Nirenberg type interpolation inequality in CR geometry.

2. Preliminaries and Notations

Let $M$ be an orientable, real, $(2n + 1)$-dimensional manifold. A CR structure on $M$ is given by a complex $n$-dimensional subbundle $T_{1,0}$ of the complexified tangent bundle $\mathbb{C}T M$ of $M$, satisfying $T_{1,0} \cap T_{0,1} = \{0\}$, where $T_{0,1} = \bar{T}_{1,0}$. We assume the CR structure
is formally integrable, that is, \( T_{1,0} \) satisfies the Frobenius condition \([T_{1,0}, T_{1,0}] \subset T_{1,0}\). Set \( G = \text{Re}(T_{1,0} \oplus T_{0,1}) \). Then \( G \) is a real 2n-dimensional sub-bundle of \( TM \). Then \( G \) carries a natural complex structure map: \( J : G \to G \) given by \( J(V + \bar{V}) = \sqrt{-1}(V - \bar{V}) \) for \( V \in T_{1,0} \).

Let \( E \subset T^*M \) denote the real line bundle \( G \perp \). Because we assume \( M \) is orientable, and the complex structure \( J \) induces an orientation on \( G \), \( E \) has a global non-vanishing section. A choice of such a 1-form \( \theta \) is called a pseudo-Hermitian structure on \( M \). Associated with such \( \theta \), the real symmetric bilinear form \( L_\theta \) on \( G \):

\[
L_\theta(V, W) = d\theta(V, JW), \quad V, W \in G
\]

is called the Levi – form of \( \theta \). \( L_\theta \) extends by complex linearity to \( \mathbb{C}G \), and induces a Hermitian form on \( T_{1,0} \), which we write

\[
L_\theta(V, \bar{W}) = -\sqrt{-1}d\theta(V, \bar{W}), \quad V, W \in T_{1,0}
\]

If \( \theta \) is replaced by \( \tilde{\theta} = f\theta \), \( L_\theta \) changes conformally by \( L_{\tilde{\theta}} = fL_\theta \). We assume that \( M \) is strictly pseudo-convex, that is, \( L_\theta \) is positive definite for a suitable \( \theta \). In this case, \( \theta \) defines a contact structure on \( M \), and we call \( \theta \) a contact form. Then we define the volume form on \( M \) as

\[
dV_\theta = \theta \wedge (d\theta)^n.
\]

We can choose a unique \( T \) called the characteristic direction such that \( \theta(T) = 1 \), \( d\theta(T, \cdot) = 0 \), and \( TM = G \oplus \mathbb{R}T \). Then we can define a co-frame \( \{ \theta, \theta^1, \theta^2, \cdots, \theta^n \} \) satisfying \( \theta^a(T) = 0 \), which is called admissible coframe. Its dual frame \( \{ T, Z_1, Z_2, \cdots, Z_n \} \) is called admissible frame. In this co-frame, we have

\[
d\theta = \sqrt{-1}h_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}, \quad h_{\alpha\bar{\beta}} \text{ is a Hermitian matrix.}
\]

\( h_{\alpha\bar{\beta}} \) and \( h^{\alpha\bar{\beta}} \) are used to lower and raise the indices.

The sub-Laplacian operator \( \triangle_b \) is defined by

\[
\int_M (\triangle_b u)fdV_\theta = -\int_M \langle du, df \rangle_\theta dV_\theta,
\]

for all smooth function \( f \). Here \( \langle, \rangle_\theta \) is the inner product induced by \( L_\theta \). We denote \( |\nabla_\theta u|^2 = \langle du, du \rangle_\theta \). Tanaka \cite{28} and Webster \cite{30} showed there is a natural connection in the bundle \( T_{1,0} \) adapted to a pseudo-Hermitian structure, which is called the Tanaka-Webster connection. To define this connection, we choose an admissible co-frame \( \{ \theta^\alpha \} \) and dual frame \( \{ Z_\alpha \} \) for \( T_{1,0} \). Then there are uniquely determined 1-forms \( \omega_{\alpha\bar{\beta}}, \tau_\alpha \) on \( M \), satisfying

\[
\begin{align*}
d\theta^\alpha &= \omega^\alpha_{\beta} \wedge \theta^\beta + \theta \wedge \tau^\alpha, \\
dh_{\alpha\bar{\beta}} &= h_{\alpha\gamma} \omega^\gamma_{\beta} + \omega^\gamma_{\alpha} h_{\gamma\beta}, \\
\tau_\alpha \wedge \theta^\alpha &= 0.
\end{align*}
\]
Now let \( u \) be a relatively compact open subset of a normal coordinate neighborhood, with contact form \( \theta \) and pseudo-Hermitian frame \( \{W_1, \cdots, W_n\} \). Let \( X_j = \text{Re} W_j \) and \( \tilde{X}_j = \text{Im} W_j \) be the real and imaginary parts of \( W_j \), respectively. The real and imaginary parts of the pseudo-Hermitian connections satisfy

\[
\tilde{X}_j = \left( \begin{array}{c} 0 \\ X_j \end{array} \right), \quad X_j = \left( \begin{array}{c} X_j \\ 0 \end{array} \right).
\]

From this third equation, we can find \( A_{\alpha\gamma} \), such that

\[
\tau_\alpha = A_{\alpha\gamma} \theta^\gamma
\]

and \( A_{\alpha\gamma} = A_{\gamma\alpha} \). Here \( A_{\alpha\gamma} \) is called the pseudohermitian torsion. With this connection, the covariant differentiation is defined by

\[
\nabla Z_\alpha = \omega^\beta_\alpha \otimes Z_\beta, \quad \nabla Z_\bar{\alpha} = \omega^\bar{\beta}_{\bar{\alpha}} \otimes Z_{\bar{\beta}}, \quad \nabla T = 0.
\]

\( \{\omega^\beta_\alpha\} \) are called connection 1-forms. For a smooth function \( f \) on \( M \), we write \( f_\alpha = Z_\alpha f \), \( f_\bar{\alpha} = Z_{\bar{\alpha}} f \), \( f_0 = T f \), so that \( df = f_\alpha \theta_\alpha + f_{\bar{\alpha}} \theta_{\bar{\alpha}} + f_0 \theta \). The second covariant differential \( \nabla^2 f \) is the 2-tensor with components

\[
f_{\alpha\beta} = f_{\bar{\alpha}\bar{\beta}} = Z_{\bar{\beta}} Z_\alpha f - \omega^\gamma_\alpha (Z_\beta) Z_\gamma f, \quad f_{\alpha\bar{\beta}} = f_{\bar{\alpha}\beta} = Z_\beta Z_\alpha f - \omega^\gamma_{\bar{\alpha}} (Z_\beta) Z_\gamma f,
\]

\[
f_{0\alpha} = f_{\bar{0}\alpha} = Z_\alpha T f, \quad f_{0\bar{\alpha}} = f_{\bar{0}\bar{\alpha}} = T Z_\alpha f - \omega^\gamma_{\bar{\alpha}} (T) Z_\gamma f, \quad f_{00} = T^2 f.
\]

The connections forms also satisfy

\[
d \omega^\beta_\alpha - \omega^\beta_{\bar{\alpha}} \wedge \omega^\alpha_{\bar{\gamma}} = \frac{1}{2} R^\alpha_{\beta\rho\theta} \theta^\rho \wedge \theta^\theta + \frac{1}{2} R^\alpha_{\beta\rho\theta} \theta^\rho \wedge \theta^\theta + R^\alpha_{\beta\rho\theta} \theta^\rho \wedge \theta^\theta + R^\alpha_{\beta\rho\theta} \theta^\rho \wedge \theta^\theta - R^\alpha_{\beta\rho a} \theta^\rho \wedge \theta^a.
\]

We call \( R_{\beta\alpha\rho\theta} \) the pseudohermitian curvature. Contractions of the pseudohermitian curvature yield the pseudohermitian Ricci curvature \( R_{\rho\theta} = R^\alpha_{\alpha\rho\theta} \), or \( R_{\rho\theta} = h^\beta_{\alpha\rho} R_{\alpha\beta\rho\theta} \), and the pseudohermitian scalar curvature \( R = h^\rho_{\alpha\rho} R_{\alpha\beta\rho\theta} \).

The sub-Laplacian operator in this connection can be expressed by

\[
(2.4) \quad \Delta_{\bar{\theta}} u = u^\alpha_{\alpha} + u^\bar{\alpha}_{\bar{\alpha}}
\]

If we define \( \tilde{\theta} = u^{\frac{2}{n}} \theta \), then we have

\[
\tilde{\Delta}_{\bar{\theta}} f = u^{-\left(1+\frac{2}{n}\right)} (u \Delta_{\bar{\theta}} f + 2 < du, df >_{\bar{\theta}}),
\]

where \( \tilde{\Delta}_{\bar{\theta}} \) is the sub-Laplacian operator with respect to the contact form \( \tilde{\theta} \) (see (2.4) in \cite{[15]} for example). If we set

\[
\tilde{u} = r^{-1} u,
\]

then we have the following CR transformation law

\[
(-2 + \frac{2}{n}) \tilde{\Delta}_{\bar{\theta}} \tilde{u} = r^{-1+\frac{2}{n}} (-2 + \frac{2}{n}) \Delta_{\bar{\theta}} u - R) u.
\]

If we substitute \( r = u \), then we get the CR Yamabe equation \cite{[12]}.

If \( \{W_1, \cdots, W_n\} \) is a frame for \( T^{1,0} \) over some open set \( U \subset M \) which is orthonormal with respect to the given pseudo-Hermitian structure on \( M \), we call \( \{W_1, \cdots, W_n\} \) a pseudo-Hermitian frame. \( \{W_1, \cdots, W_n, \overline{W}_1, \cdots, \overline{W}_n, T\} \) forms a local frame for \( \mathbb{C}T M \). Now let \( U \) be a relatively compact open subset of a normal coordinate neighborhood, with contact form \( \theta \) and pseudo-Hermitian frame \( \{W_1, \cdots, W_n\} \). Let \( X_j = \text{Re} W_j \) and
\[ X_{j+n} = \text{Im} W_j. \] 

Denote \( X^\alpha = X_{\alpha_1} \cdots X_{\alpha_k}, \) where \( \alpha = (\alpha_1, \ldots, \alpha_k). \) We also denote \( l(\alpha) = k. \) Define the norm

\[ \|f\|_{S_k^p(U)} = \sup_{l(\alpha) \leq k} \|X^\alpha f\|_{L^p(U)}. \]

The Folland-Stein space \( S_k^p(U) \) is the completion of \( C_0^\infty \) with respect to the norm \( \| \cdot \|_{S_k^p(U)} \) (See [12]). Now we use the notations in [12] as follows. Denote \( H^k \) the Hilbert space \( S_k^2. \) Define

\[ \Gamma_\beta(U) = \{ f \in C^0(\bar{U}) : |f(x) - f(y)| \leq C \rho(x, y)^\beta \}, \]

with norm

\[ \|f\|_{\Gamma_\beta(U)} = \sup_{x \in U} |f(x)| + \sup_{x, y \in U} \frac{|f(x) - f(y)|}{\rho(x, y)^\beta}. \]

For any integer \( k \geq 1 \) and \( k < \beta < k + 1, \) define

\[ \Gamma_\beta(U) = \{ f \in C^0(\bar{U}) : X^\alpha f \in \Gamma_{\beta-1}(U), l(\alpha) \leq k \}, \]

with norm

\[ \|f\|_{\Gamma_\beta(U)} = \sup_{x \in U} |f(x)| + \sup_{x, y \in U, l(\alpha) \leq k} \frac{|X^\alpha f(x) - X^\alpha f(y)|}{\rho(x, y)^{\beta-k}}. \]

If we fix local coordinates \((z, t) = \Theta_{\xi}\) for a fixed point \( \xi \in U, \) the standard Hölder space \( \Lambda_\beta(U) \) is defined for \( 0 < \beta < 1 \) by

\[ \Lambda_\beta(U) = \{ f \in C^0(\bar{U}) : |f(x) - f(y)| \leq C ||x - y||^\beta \}, \]

with norm

\[ \|f\|_{\Lambda_\beta(U)} = \sup_{x \in U} |f(x)| + \sup_{x, y \in U, l(\alpha) \leq k} \frac{|X^\alpha f(x) - X^\alpha f(y)|}{||x - y||^{\beta-k}}. \]

For any integer \( k \geq 1 \) and \( k < \beta < k + 1, \) define

\[ \Lambda_\beta(U) = \{ f \in C^0(\bar{U}) : (\partial/\partial x)^\alpha f \in \Lambda_{\beta-1}(U), l(\alpha) \leq k \}. \]

Now for a compact strictly pseudo-convex pseudo-Hermitian manifold \( M, \) choose a finite open covering \( U_1, \ldots, U_m, \) each \( U_j \) has the properties of \( U \) above. Choose a \( C^\infty \) partition of unity \( \varphi_i \) subordinate to this covering, and define

\[ S_k^p(M) = \{ f \in L^1(M) : \phi_j f \in S_k^p(U_j) \}; \]

\[ \Gamma_\beta(M) = \{ f \in C^0(M) : \phi_j f \in \Gamma_\beta(U_j) \}; \]

\[ \Lambda_\beta(M) = \{ f \in C^0(M) : \phi_j f \in \Lambda_\beta(U_j) \}. \]

Then we have the following Lemma, see [12], or Proposition 5.7 in [19]:
Lemma 2.1. For each positive non-integer $\beta$, each $r$, $1 < r < \infty$, and each integer $k \geq 1$, there exists a constant $C$ such that for every $f \in C_0^\infty(U)$,

1. $||f||_{r \beta(U)} \leq C||f||_{k(U)}$, where $\frac{1}{r} = \frac{k-\beta}{2n+2}$;
2. $||f||_{\Delta \beta(U)} \leq ||f||_{r \beta(U)}$;
3. $||f||_{S_r^2(U)} \leq C(||\Delta f||_{L^r(U)} + ||f||_{L^r(U)})$;
4. $||f||_{r \beta+2(U)} \leq C(||\Delta f||_{r \beta(U)} + ||f||_{r \beta(U)})$.

The constants $C$ depend only on the frame constants.

We have the following corollary immediately.

Corollary 2.1. Let $(M, \theta)$ be a smooth, strictly pseudo-convex $2n+1$ dimensional compact CR manifold without boundary. Then there is an integer $k > 0$, such that $H^k(M)$ embeds into $C^0(M)$.

Proof. This is a direct consequence of Lemma 2.1 (1), and $\Gamma_\beta(M) \subset C^0(M)$.

Following CR version Sobolev Embedding Theorem was given by Jerison and Lee [19].

Proposition 2.1. (19) For $\frac{1}{s} = \frac{1}{r} - \frac{k}{2n+2}$, where $1 < r < s < \infty$. Then we have $S_k^s(M) \subset L^s(M)$.

Next we recall a CR version Poincaré inequality. In [19], Jerison and Lee proved a Poincaré type inequality for compact, strictly pseudo-convex CR manifolds.

Theorem 2.1. (See [19], Proposition 5.13) Let $(M, \theta_0)$ be a compact, strictly pseudo-convex CR manifold, $U$ is a relatively compact open subset of a normal coordinate neighborhood of $(M, \theta)$, $B_r$ is a ball of radius $r$, $B_r \subset U$. Then for any $f$ satisfying $|\nabla_{\theta_0} f| \in L^q(B_r)$, $1 < q < \infty$, there exits a constant $C$ independent of $f$ such that

$$\int_{B_r} |f(x) - f_{B_r}|^q dV_{\theta_0} \leq C r^q \int_{B_r} |\nabla_{\theta_0} f|^q dV_{\theta_0},$$

where $f_{B_r} = \frac{\int_{B_r} f(x) dV_{\theta_0}}{\int_{B_r} dV_{\theta_0}}$.

As a corollary of Theorem 2.1 we have

Lemma 2.2. Under the condition of Theorem 2.1 we have the following Poincaré type inequality:

$$\int_{B_r} |f(x)|^2 dV_{\theta_0} \leq C \int_{B_r} |\nabla_{\theta_0} f|^2 dV_{\theta_0},$$

where $C$ is a positive constant independent of $f$.\]
Proof. We choose \( v(x) \) satisfying \( f(x) = v(x) - v_{B_r} \). Since \( |\nabla_{\theta_0} f|^2 = |\nabla_{\theta_0} v|^2 \), this lemma follows from Theorem 2.1 by letting \( q = 2 \). \( \square \)

By the above Poincaré inequalities, we know for any \( x_0 \in M \), there exists a ball \( B_r(x_0) \) such that the above Poincaré inequalities are satisfied on \( B_r(x_0) \). Since \( (M, \theta_0) \) is compact, then we can obtain the following global Poincaré inequalities, which are the corollaries of Theorem 2.1 and Lemma 2.2.

**Corollary 2.2.** Under the condition of Theorem 2.1, for any \( f \in C^\infty(M) \), we have the following global Poincaré inequality:

\[
\int_M |f(x) - \bar{f}|^2 dV_{\theta_0} \leq C \int_M |\nabla_{\theta_0} f|^2 dV_{\theta_0},
\]

where \( C \) is a positive constant independent of \( f \), and \( \bar{f} = \frac{\int_M f(x) dV_{\theta_0}}{\int_M dV_{\theta_0}} \).

**Corollary 2.3.** Under the condition of Theorem 2.1, for any \( f \in C^\infty(M) \), we have the following global Poincaré inequality:

\[
\int_M |f(x)|^2 dV_{\theta_0} \leq C \int_M |\nabla_{\theta_0} f|^2 dV_{\theta_0},
\]

where \( C \) is a positive constant independent of \( f \).

Now we prove the following theorem, which is a Poincaré type inequality.

**Theorem 2.2.** Let \( (M, \theta_0) \) be a compact, strictly pseudoconvex CR manifold. For any \( f \in C^\infty(M) \), we have the following global Poincaré type inequality:

\[
\|\nabla_{\theta_0} f\|_{L^2(M, \theta_0)} \leq C \|\Delta_b f\|_{L^2(M, \theta_0)},
\]

for some \( C > 0 \) independent of \( f \).

**Proof.** From Proposition 5.7(c) in [19], we know there is a positive constant \( C \) independent of \( f \), such that

\[
\|f\|_{S^2(M, \theta_0)} \leq C (\|\Delta_b f\|_{L^2(M, \theta_0)} + \|f\|_{L^2(M, \theta_0)}).
\]

Therefore we obtain

\[
\|\nabla_{\theta_0} f\|_{L^2(M, \theta_0)} \leq C (\|\Delta_b f\|_{L^2(M, \theta_0)} + \|f\|_{L^2(M, \theta_0)}).
\]

We use the contradiction argument to prove the inequality. Suppose the inequality in the theorem is not true, then there exists a sequence \( \{f_j\} \) such that

\[
f_j \|\Delta_b f_j\|_{L^2(M, \theta_0)} \leq \|\nabla_{\theta_0} f_j\|_{L^2(M, \theta_0)}.
\]

Then by (2.8), we have

\[
\|\nabla_{\theta_0} f_j\|_{L^2(M, \theta_0)} \leq C (\|\Delta_b f_j\|_{L^2(M, \theta_0)} + \|f_j\|_{L^2(M, \theta_0)}).
\]
We may require that \( \|\nabla \theta_0 f_j\|_{L^2(M,\theta_0)} = 1 \), for any \( j \). Thus, as \( j \) tends to infinity, we have
\[
\|\Delta_b f_j\|_{L^2(M,\theta_0)} \to 0.
\]

Let \( u_j = f_j - \bar{f}_j \), here \( \bar{f}_j = \int_M f_j \, dV_{\theta_0} / \int_M dV_{\theta_0} \). By (2.6), we have
\[
\|f_j - \bar{f}_j\|_{L^2(M,\theta_0)} \leq \|\nabla \theta_0 f_j\|_{L^2(M,\theta_0)} \leq C.
\]
Then there is a subsequence of \( u_j \) converges weakly in \( S^2 \), we may assume it is \( u_j \) itself. Then we have \( u_j \to u \) in \( S^1 \) sense for some \( u \), and
\[
\int_M |\nabla \theta_0 u_j|^2 dV_{\theta_0} = - \int_M u_j \Delta_b u_j dV_{\theta_0} \leq \|u_j\|_{L^2(M,\theta_0)} \cdot \|\Delta_b u_j\|_{L^2(M,\theta_0)} \to 0
\]
as \( j \to \infty \), which means \( \|\nabla \theta_0 u\|_{L^2(M,\theta_0)} = \|\nabla \theta_0 f_j\|_{L^2(M,\theta_0)} = 1 \). This contradicts the fact that \( \|\nabla \theta_0 f_j\|_{L^2(M,\theta_0)} = 1 \).

At the end of this section, we recall some basic properties of the CR Yamabe flow (1.3). Under this flow, we have the following evolution equations [15].

**Lemma 2.3.** Under the CR-Yamabe flow (1.3), we have

1. \( \frac{\partial}{\partial t} dV_{\bar{\theta}} = -(n + 1)(\bar{R} - \bar{r}) dV_{\bar{\theta}}; \)
2. \( \frac{\partial}{\partial t} \bar{u} = -\frac{n}{2}(\bar{R} - \bar{r}) \bar{u}; \)
3. \( \frac{\partial}{\partial t} \bar{r} = -n \int_M (\bar{R} - \bar{r})^2 dV_{\bar{\theta}}; \)
4. \( \frac{\partial}{\partial t} \bar{R} = (n + 1) \bar{\Delta}_b \bar{R} + (\bar{R} - \bar{r}) \bar{R}; \)

We also need the following lemmata, which were proved in [15] (Propositions 3.1, 3.3 and 3.4).

**Lemma 2.4.** The volume of \( M \) does not change under the CR Yamabe flow.

**Lemma 2.5.** The function \( t \mapsto \bar{r}(t) \) is bounded from below and non-increasing under (1.3).

### 3. Scalar flat case of the CR Yamabe flow

By the CR Yamabe equation (1.2), we can reduce the CR Yamabe flow (1.3) to the following evolution equation of the conformal factor:

\[
\begin{align*}
\frac{\partial}{\partial t} u^{\frac{n+2}{n}} &= \frac{(n + 2)(n + 1)}{n} (\Delta_b u + \frac{n}{2n + 2} \bar{r} u^{\frac{n+2}{n}}) \\
\text{with } u(\cdot, 0)^{\frac{n}{n+2}} \theta_0 &= \theta. \end{align*}
\]

We have the following lemma.

**Lemma 3.1.** Under the condition of Theorem 1.1, \( \bar{r} \geq 0 \) for all the time.
Proof. By the definition of $\lambda(M)$, we obtain

$$
\lambda(M) = \inf \left\{ \frac{\tilde{r}}{(\int_M u^{2+\frac{2}{n}} dV_{\theta_0})^{\frac{n}{n+2}}} : u > 0, u \in S^2_1(M) \right\}.
$$

Since $\lambda(M) = 0$, we therefore have $\tilde{r} \geq 0$. \qed

Then we have the following corollary:

**Corollary 3.1.** Under the condition of Theorem 1.1, if $\theta = \theta_0$, then the Yamabe flow (1.4) exists for all time, and $\tilde{r} \equiv 0$, $u \equiv 1$.

**Proof.** This is a direct consequence of Lemmata 3.1 and 2.5. \qed

Now we prove the following theorem.

**Theorem 3.1.** Under the condition of Theorem 1.1, for any $T > 0$, there exists a constant $C(T)$, such that $u_{\min}(0) \leq u(x,t) \leq C(T)$ for $t \in [0,T]$.

**Proof.** Since $M$ is compact, we denote $x(t)$ to be the set of points in $M$ where $u_{\min}(t)$ is obtained. Then we have

$$
\frac{du_{\min}^{\frac{n+2}{n}}}{dt}(t) \geq \inf \left\{ \frac{\partial}{\partial t} \left( u^{\frac{n+2}{n}}(x,t) \right) : x \in x(t) \right\}
= \inf \left\{ \frac{(n+2)(n+1)}{n} (\Delta_b u + \frac{n}{2n+2} \tilde{r} u^{\frac{n+2}{n}}(t)) : x \in x(t) \right\}
\geq \frac{n+2}{2} \tilde{r} u_{\min}^{\frac{n+2}{n}}(t)
\geq 0,
$$

which means

$$
u_{\min}(t) \geq u_{\min}(0).
$$

Similarly we get

$$
\frac{du_{\max}^{\frac{n+2}{n}}}{dt}(t) \leq \frac{n+2}{2} \tilde{r} u_{\max}^{\frac{n+2}{n}}(t) \leq \frac{n+2}{2} \tilde{r}(0) u_{\max}^{\frac{n+2}{n}}(t).
$$

Therefore, we can obtain

$$
u_{\min}(0) \leq u(x,t) \leq u_{\max}(0) e^\frac{n}{2} \tilde{r}(0)t.
$$

\qed

**Theorem 3.2.** Under the condition of Theorem 1.1, for any $T > 0$, there exists a constant $C > 0$ independent of $T$ such that

$$
\frac{1}{C} \leq u(x,t) \leq C,
$$

where
for any $t \in [0, T]$.

**Proof.** First we show that the function $f(t) := \left(\frac{u_{\max}(t)}{u_{\min}(t)}\right)^{\frac{n+2}{n}}$ is non-increasing. In fact, for any $h > 0$, we have

$$
\frac{f(t+h) - f(t)}{h} = \frac{1}{h} \left(\frac{u_{\max}^{n+2}(t+h)}{u_{\min}^{n+2}(t+h)} - \frac{u_{\max}^{n+2}(t)}{u_{\min}^{n+2}(t)}\right)
$$

$$
= \frac{1}{u_{\min}^{n+2}(t+h)u_{\min}^{n+2}(t)} \left(\frac{u_{\max}^{n+2}(t+h) - u_{\max}^{n+2}(t)}{h}\right)
$$

$$
- \frac{1}{u_{\min}^{n+2}(t+h)} \left(\frac{u_{\max}^{n+2}(t) - u_{\min}^{n+2}(t)}{h}\right).
$$

Thus we have

$$
\limsup_{h \to 0} \frac{f(t+h) - f(t)}{h} \leq \limsup_{h \to 0} \frac{1}{u_{\min}^{n+2}(t+h)u_{\min}^{n+2}(t)} \left(\frac{u_{\max}^{n+2}(t+h) - u_{\max}^{n+2}(t)}{h}\right)
$$

$$
- \liminf_{h \to 0} \frac{1}{u_{\min}^{n+2}(t+h)} \left(\frac{u_{\max}^{n+2}(t) - u_{\min}^{n+2}(t)}{h}\right)
$$

$$
\leq \frac{1}{u_{\min}^{n+2}(t+h)} \left(\frac{\frac{n+2}{2} du_{\max}^{n+2}(t)}{dt} - \frac{\frac{n+2}{2} du_{\min}^{n+2}(t)}{dt}\right)
$$

$$
\leq \frac{1}{u_{\min}^{n+2}(t+h)} \left(\frac{n+2}{2} \frac{\frac{n+2}{2} du_{\max}^{n+2}(t)}{dt} - \frac{n+2}{2} \frac{\frac{n+2}{2} du_{\min}^{n+2}(t)}{dt}\right)
$$

$$
= 0.
$$

Then we get

$$
(3.2) \quad \frac{u_{\max}(t)}{u_{\min}(t)} \leq \frac{u_{\max}(0)}{u_{\min}(0)}.
$$
It has been shown in Lemma 2.4 that the volume is invariant under the CR Yamabe flow. We therefore have
\[ \text{Vol}(M, \theta) = \int_M u^{2+\frac{4}{n}} dV_{\theta_0} \geq u_{\text{min}}^{2+\frac{4}{n}} \text{Vol}(M, \theta_0), \]
thus
\[ u_{\text{min}}(t) \leq \left( \frac{\text{Vol}(M, \theta)}{\text{Vol}(M, \theta_0)} \right)^{\frac{n}{2n+2}}. \]
Putting these together, we obtain
\[ u_{\text{max}}(t) \leq \frac{u_{\text{max}}(0)}{u_{\text{min}}(0)} \left( \frac{\text{Vol}(M, \theta)}{\text{Vol}(M, \theta_0)} \right)^{\frac{n}{2n+2}}. \]

Once we get the \( C^0 \) estimate of \( u(x, t) \), we may use the same argument in [17] (page 12) to show all higher order derivatives of \( u(x, t) \) are uniformly bounded on \([0, \infty)\). Then \( u(t) \) converges to a smooth function \( u_\infty \) as \( t \to \infty \). Next we show that \( u(t) \) converges to a smooth function \( u_\infty \) at an exponential rate. Actually, we will show that \( u_\infty \) is a constant. We first prove the following lemma.

**Lemma 3.2.** Under the condition of Theorem 1.1, \( \tilde{r} \to 0 \) as \( t \to \infty \).

**Proof.** If \( \tilde{r} \geq C > 0 \), for some positive constant \( C \), then from the proof of Theorem 3.1, we get
\[ \frac{d u_{\text{min}}^{\frac{n+2}{n}}}{dt}(t) \geq \frac{n+2}{2} \tilde{r} u_{\text{min}}^{\frac{n+2}{n}}(t) \geq C \cdot \frac{n+2}{2} \cdot u_{\text{min}}^{\frac{n+2}{n}}(t). \]
Thus
\[ u_{\text{min}}^{\frac{n+2}{n}}(t) \geq e^{\frac{n+2}{2} C t} u_{\text{min}}^{\frac{n+2}{n}}(0). \]
But this contradicts with Theorem 3.2. Therefore we have \( \tilde{r} \to 0 \) as \( t \to \infty \). \( \square \)

Next we show that the convergence is exponential.

**Lemma 3.3.** Under the condition of Theorem 1.1, the pseudo-Hermitian scalar curvature \( \tilde{r}(t) \to 0 \) exponentially as \( t \to \infty \).

**Proof.** Since
\[ \frac{\partial}{\partial t} u = (n+1) \triangle_b u \cdot u^{-\frac{2}{n}} + \frac{n}{2} \tilde{r} u, \]
we have
\[ \frac{1}{n+1} \frac{\partial}{\partial t} u = \triangle_b u \cdot u^{-\frac{2}{n}} + \frac{n}{2n+2} \tilde{r} u, \]
and
\[
\frac{1}{n+1} \frac{\partial}{\partial t} u \cdot \Delta_b u = (\Delta_b u)^2 \cdot u^{-2} + \frac{n}{2n+2} \tilde{r} u \cdot \Delta_b u.
\]
Integrating both sides of the above equality over \(M\), we have
\[
\frac{1}{n+1} \int_M \frac{\partial}{\partial t} u \cdot \Delta_b u dV_{\theta_0} = \int_M (\Delta_b u)^2 \cdot u^{-2} dV_{\theta_0} + \frac{n}{2n+2} \tilde{r} \int_M u \cdot \Delta_b u dV_{\theta_0}.
\]
Since
\[
\frac{1}{n+1} \int_M \frac{\partial}{\partial t} u \cdot \Delta_b u dV_{\theta_0} = -\frac{1}{n+1} \int_M \nabla_{\theta_0} u \cdot \nabla_{\theta_0} (\frac{\partial}{\partial t} u) dV_{\theta_0}
\]
\[
= -\frac{1}{2n+2} \int_M \frac{\partial}{\partial t} |\nabla_{\theta_0} u|^2 dV_{\theta_0}
\]
\[
= -\frac{1}{2n+2} \frac{d}{dt} \int_M |\nabla_{\theta_0} u|^2 dV_{\theta_0},
\]
then we get
\[
(3.3) \quad \frac{1}{n+1} \int_M |\nabla_{\theta_0} u|^2 dV_{\theta_0} = -2 \int_M (\Delta_b u)^2 \cdot u^{-2} dV_{\theta_0} + \frac{n}{n+1} \tilde{r} \int_M |\nabla_{\theta_0} u|^2 dV_{\theta_0}.
\]
By Theorem 2.2, we have
\[
(3.4) \quad \| \nabla_{\theta_0} u \|_{L^2(M, \theta_0)} \leq C \| \Delta_b u \|_{L^2(M, \theta_0)}.
\]
Here \(C\) is some positive constant independent of \(u\). By (3.4), we have
\[
\int_M (\Delta_b u)^2 \cdot u^{-2} dV_{\theta_0} \geq \frac{1}{u_{\max}^2} \int_M (\Delta_b u)^2 dV_{\theta_0}
\]
\[
\geq C \int_M |\nabla_{\theta_0} u|^2 dV_{\theta_0},
\]
for some positive constant \(C\). Substituting this inequality into (3.3), we get
\[
\frac{1}{n+1} \frac{d}{dt} \int_M |\nabla_{\theta_0} u|^2 dV_{\theta_0} \leq \left( \frac{n}{n+1} \tilde{r} - 2C \right) \int_M |\nabla_{\theta_0} u|^2 dV_{\theta_0}.
\]
Then for sufficiently large \(t\), there exists a positive constant \(A\), such that
\[
\frac{d}{dt} \log \int_M |\nabla_{\theta_0} u|^2 dV_{\theta_0} \leq (n+1) \left( \frac{n}{n+1} \tilde{r} - 2C \right) \leq -A,
\]
from which we get
\[
(3.5) \quad \tilde{r}(t) = \frac{\int_M (2 + \frac{2}{n}) |\nabla_{\theta_0} u|^2 dV_{\theta_0}}{\text{Vol}(M, \theta)} \leq C \cdot e^{-At},
\]
for \(t\) sufficiently large.
From the proof of Lemma 3.3, we also get
\[ \| \nabla_{\theta_0} u \|^2_{L^2(M, \theta_0)} \leq C \cdot e^{-At} , \]
which will be used later.

Now we prove the following theorem:

**Theorem 3.3.** Under the condition of Theorem 1.1, the solution \( u(t) \) of the CR Yamabe flow (1.3) converges to a constant at an exponential rate.

**Proof.** Since
\[ \frac{d}{dt} \int_M u^{n+2} \, dV_{\theta_0} = \int_M \frac{d}{dt} (u^{n+2}) \, dV_{\theta_0} = \frac{(n+2)(n+1)}{n} \int_M \triangle_b u \, dV_{\theta_0} + \frac{n+2}{2} \int_M u^{n+2} \, dV_{\theta_0} \]
\[ = \frac{n+2}{2} \int_M u^{n+2} \, dV_{\theta_0} \leq C \cdot e^{-At} \cdot \int_M u^{n+2} \, dV_{\theta_0} , \]
therefore \( \int_M u^{n+2} \, dV_{\theta_0} \) is bounded from above and non-decreasing, which means
\[ \lim_{t \to \infty} \int_M u^{n+2} (x, t) \, dV_{\theta_0} = L , \]
for some positive constant \( L \). Hence, there exists a constant \( C \) such that
\[ \frac{d}{dt} \int_M u^{n+2} \, dV_{\theta_0} \leq C \cdot e^{-At} . \]

Then for \( t_2 > t_1 \), and \( t_1 \) sufficiently large, we have
\[ | \int_M u^{n+2} (x, t_2) \, dV_{\theta_0} - \int_M u^{n+2} (x, t_1) \, dV_{\theta_0} | = \int_M u^{n+2} (x, t_2) \, dV_{\theta_0} - \int_M u^{n+2} (x, t_1) \, dV_{\theta_0} \leq C (e^{-At_1} - e^{-At_2}) . \]

Let \( t_2 \to \infty \), we get
\[ | \int_M u^{n+2} \, dV_{\theta_0} - L | \leq C \cdot e^{-At} . \]

for \( t \) sufficiently large. By Corollary 2.2 and Hölder inequality, we have
\[ \| u^{n+2} - \frac{1}{V} \int_M u^{n+2} \, dV_{\theta_0} \|^2_{L^2(M, \theta_0)} \leq C \| \nabla_{\theta_0} u \|^2_{L^2(M, \theta_0)} \leq C \cdot e^{-At} . \]

Let \( f = u^{n+2} - \frac{1}{V} \int_M u^{n+2} \, dV_{\theta_0} \), then \( \int_M f \, dV_{\theta_0} = 0 \). We apply Theorem 4.1 in the Appendix below by choosing \( a = \frac{1}{2} \), \( p = q = r = 2 \), \( j = k \) and \( m = 2k \), and use the fact that the higher order derivatives of \( u \) are uniformly bounded for all \( t \geq 0 \), we get
\[ \| u^{n+2} - \frac{1}{V} \int_M u^{n+2} \, dV_{\theta_0} \|_{H^k(M, \theta_0)} \leq C \cdot e^{-At} . \]
Then by Corollary 2.1 we obtain
\[ \left| u^{\frac{n+2}{n}} - \frac{1}{V} \int_M u^{\frac{n+2}{n}} dV_{\theta_0} \right| \leq C \cdot e^{-\lambda t}. \]

Let \( t \to \infty \), we get \( u^{\frac{n+2}{n}} \to L^p \) exponentially. \( \square \)

4. Appendix

The Gagliardo-Nirenberg interpolation inequality is a result in the theory of Sobolev spaces that estimates the weak derivatives of a function. The estimates are in terms of \( L^p \) norms of the function and its derivatives, and the inequality "interpolates" into various values of \( p \) and orders of differentiation. The result is of particular importance in the theory of elliptic partial differential equations. It was proposed by Nirenberg and Gagliardo, see [24]. For Riemannian case, the Gagliardo-Nirenberg type interpolation inequality was proved by Aubin (see [2], Theorem 3.70). Due to the lack of relevant references, we did not find the similar inequalities in CR geometry. In this section, we try to establish a Gagliardo-Nirenberg type inequality in CR geometry.

Let \((M, \theta)\) be a smooth, strictly pseudoconvex \(2n+1\) dimensional compact CR manifold without boundary. We choose an admissible coframe \(\{\theta^\alpha\}\) and dual frame \(\{Z_\alpha\}\) for \(T_{1,0}\). We adopt the same notations as in [21]. Let \(\alpha, \beta, \gamma, \cdots \in \{1, 2, \cdots, n\}\), and \(a, b, c, \cdots \in \{1, 2, \cdots, 2n\}\), and \(\bar{\alpha} = \alpha + n\). We denote \(\nabla_j^{|f}|\) the \(j\)-th covariant derivative of \(f\) in the Tanaka-Webster connection in the sense
\[ \| \nabla_j^{|f} \|^2 = \nabla^{a_1} \nabla^{a_2} \cdots \nabla^{a_j} f \nabla_{a_1} \nabla_{a_2} \cdots \nabla_{a_j} f, \]
here \(a_i \in \{1, 2, \cdots, 2n\}\) and \(\nabla_{a_i}\) means \(\nabla_{Z_{a_i}}\). From now on we denote \(\| f \|_p\) be the \(L^p\) norm of \(f\).

By the existence of the Possion type equation \(\triangle_b f = C\) (see [22]). We denote \(G_P(x)\) is the Green’s function of the sub-Laplacian operator \(\triangle_b\) which satisfies
\[ \triangle_b G_P(X) = \delta_P(x) - \frac{1}{V}, \]
where \(V\) is the volume of \((M, \theta)\), and \(\delta_P(x)\) is the Dirac function at \(P\). For the general case of the Green’s function see [7]. By the definition of Dirac function, we have
\[ (4.1) \quad \varphi(P) = \frac{1}{V} \int_M \varphi dV_\theta + \int_M G_P(x) \triangle_b \varphi(x) dV_\theta. \]

We now prove the following theorem:

**Theorem 4.1.** Let \((M, \theta)\) be a smooth, strictly pseudoconvex \(2n+1\) dimensional compact CR manifold without boundary. Let \(q, r\) be real numbers \(1 \leq q, r < \infty\) and \(j, m\) integers
$0 \leq j < m$. Then there exists a constant $K$ depending only on $n$, $m$, $j$, $q$, $r$ and $(M, \theta_0)$, such that for all $f \in C^\infty$ with $\int f \, dV_\theta = 0$, we have:

$$\| \nabla^j f \|_p \leq K \| \nabla^m f \|_p \cdot \| f \|_q^{1-a}.$$  

Here $\frac{1}{p} = \frac{2}{2n+2} + a \left( \frac{1}{r} - \frac{m}{2n+2} \right) + (1-a) \frac{1}{q}$, for all $a$ in the interval $\frac{1}{m} \leq a < 1$, for which $p$ is non-negative.

We follow the idea of Aubin in [2], we first prove the following lemma:

**Lemma 4.1.** Let $(M, \theta)$ be a smooth, strictly pseudoconvex $2n+1$ dimensional compact CR manifold without boundary, and $p$, $q$ real numbers satisfying $\frac{1}{p} = \frac{1}{q} - \frac{1}{2n+2}$, $1 \leq q < 2n+2$. Then there exists a constant $K$ depending only on $p$, $q$, $n$ and $(M, \theta)$, for any function $\varphi \in C^1(M)$ with $\int_M \varphi \, dV_\theta = 0$, we have

$$\| \varphi \|_p \leq \| \nabla \varphi \|_q .$$

**Proof.** Since $\int_M \varphi \, dV_\theta = 0$, by (4.1), we have

$$\varphi(P) = \int_M G_P(x) \Delta_k \varphi(x) \, dV_\theta ,$$

from which we get

$$|\varphi(P)| \leq \int_M \| \nabla G_P \| \cdot \| \nabla \varphi \| \, dV_\theta = \int_M (\| \nabla G_P \| \cdot \| \nabla \varphi \|)^{\frac{q}{q'}} \cdot \| \nabla G_P \|^{1-\frac{q}{q'}} \, dV_\theta \leq (\int_M \| \nabla G_P \| \cdot \| \nabla \varphi \|^{q} \, dV_\theta)^{\frac{1}{q'}} \cdot (\int_M \| \nabla G_P \| \, dV_\theta)^{1-\frac{1}{q'}} .$$

Here we have used the H"older inequality. Then we obtain

$$\| \varphi \|_q \leq \| \nabla \varphi \|_q \sup_{P \in M} \int_M \| \nabla G_P \| \, dV_\theta .$$

Then by Folland-Stein imbedding theorem, we obtain

$$\| \varphi \|_p \leq C(\| \nabla \varphi \|_q + \| \varphi \|_q) \leq K \| \nabla \varphi \|_q .$$

Here $K = C + C \cdot \sup_{P \in M} \int_M \| \nabla G_P \| \, dV_\theta$.

Next, we prove the following Lemma, which is a generalized Poincaré type inequality.

**Lemma 4.2.** Let $(M, \theta)$ be a smooth, strictly pseudoconvex $2n+1$ dimensional compact CR manifold without boundary, and $p$, $q$, $r$ real numbers satisfying $1 \leq q, r < \infty$, $p \geq 2$. Set $\frac{2}{p} = \frac{1}{q} + \frac{1}{r}$. Then for any functions $f \in C^\infty(M)$, we have:

$$\| \nabla f \|_p^2 \leq (\sqrt{2n} + |p-2|) \| f \|_q \cdot \| \nabla |^2 f \|_r .$$
Proof. By a direct computation, we have
\[
\nabla^a (f \parallel \nabla f \parallel^{p-2} \nabla_a f) = \parallel \nabla f \parallel^p + f \parallel \nabla f \parallel^{p-2} \nabla^a \nabla_a f + (p-2) \parallel \nabla f \parallel^{p-4} f \nabla_{ab} f \nabla^a f \nabla^b f.
\]
Especially, if \( p = 2 \), we have \( \parallel \nabla f \parallel_2^2 = -\int_M f \triangle_b f \, dV_\theta \). Then Lemma 4.2 is just the Poincaré type inequality we proved above. If \( p > 2 \), we have
\[
\parallel \nabla f \parallel_p^p = \int_M f \triangle_b f \parallel \nabla f \parallel^{p-2} + (2-p) \int_M \parallel \nabla f \parallel^{p-4} f \nabla_{ab} f \nabla^a f \nabla^b f \, dV_\theta.
\]
Since \( |\triangle_b f|^2 \leq 2n \parallel \nabla^2 f \parallel^2 \) and \( |\nabla_{ab} f \nabla^a f \nabla^b f| \leq \parallel \nabla^2 f \parallel \parallel \nabla f \parallel^2 \), we choose \( r \) such that \( \frac{1}{q} + \frac{1}{r} + \frac{p-2}{p} = 1 \). By Hölder inequality, we have
\[
\parallel \nabla f \parallel_p^p \leq (\sqrt{2n} + |p-2|) \parallel f \parallel_q \parallel \nabla^2 f \parallel_r \parallel \nabla f \parallel^{p-2},
\]
and the desired result follows. □

Now we prove Theorem 4.1. First we note if the two cases \( j = 0, m = 1 \) and \( j = 1, m = 2 \) are proved, the general case will be followed by induction by applying the inequality
\[
\parallel \nabla \parallel \nabla^{[l]} f \parallel \leq \parallel \nabla^{[l+1]} f \parallel,
\]
which follows from the fact that the Tanaka-Webster connection is compatible with the inner product \( \langle \cdot, \cdot \rangle_\theta \) and Cauchy-Schwarz inequality. From Lemma 4.1, we have
\[
\parallel f \parallel_s \leq C \parallel \nabla f \parallel_t,
\]
where \( \frac{1}{s} = \frac{1}{t} - \frac{1}{2n+2} > 0 \).

For the case \( j = 0, m = 1 \). By Hölder inequality, we have
\[
\parallel f \parallel_p \leq \parallel f \parallel_s^{\frac{a}{q}} \parallel f \parallel^{1-a}_q.
\]
Here \( \frac{1}{p} = \frac{a}{s} + \frac{1-a}{q} \), i.e. \( \frac{1}{p} - \frac{1}{q} = a(\frac{1}{s} - \frac{1}{q}) \). Then we choose \( t = r < 2n+2 \), from which we get
\[
\parallel f \parallel_p \leq C \parallel \nabla f \parallel_a \parallel f \parallel^{1-a}_q,
\]
which means \( \frac{1}{p} = a(\frac{1}{r} - \frac{1}{2n+2}) + (1-a)\frac{1}{q} \).

If \( r \geq 2n+2 \), we choose \( \mu \) such that \( \frac{1}{ap} = \frac{1}{\mu} - \frac{1}{2n+2} \). Let \( h = |f|^\frac{1}{a} \), we have
\[
\parallel h \parallel_{ap} \leq C \parallel \nabla h \parallel_\mu,
\]
again by Hölder inequality, we have
\[
\parallel f \parallel_p^{\frac{1}{a}} \leq C_a \parallel \nabla f \parallel \cdot |f|^{\frac{a}{a}-1} \parallel f \parallel_\mu \leq C \frac{a}{a} \parallel \nabla f \parallel_r \parallel f \parallel^{\frac{1}{a}-1},
\]
the desired consequence follows.
For the case $j = 1$, $m = 2$. If $a = \frac{j}{m} = \frac{1}{2}$, Theorem 4.1 is just Lemma 4.2. Then for $r \geq 2n + 2$, and $\frac{1}{2} < a < 1$, the interpolation inequality follows from Hölder inequality. If $r \geq 2n + 2$, by induction, we apply the first case to $\| \nabla f \|$ and get

$$\| \nabla f \|_p \leq C \| \nabla|^{2|f} \|_b \| \nabla f \|_s^{-b},$$

where $\frac{1}{p} = \frac{1}{s} + b(\frac{1}{r} - \frac{1}{2n+2} - \frac{1}{s}) > 0$, $\frac{2}{s} = \frac{1}{r} + \frac{1}{q}$, and $a = \frac{1+b}{2}$. i.e.

$$\frac{1}{p} = \frac{1}{2n+2} + a(\frac{1}{r} - \frac{2}{2n+2}) + (1-a)\frac{1}{q},$$

and the proof is completed.

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SCHOOL OF MATHEMATICAL SCIENCES, ZHEJIANG UNIVERSITY, HANGZHOU 310027, CHINA.
E-mail address: weimins@zju.edu.cn

SCHOOL OF MATHEMATICAL SCIENCES, ZHEJIANG UNIVERSITY, HANGZHOU 310027, CHINA.,
CURRENT ADDRESS: COLLEGE OF SCIENCES, CHINA JILIANG UNIVERSITY, HANGZHOU 310018,
CHINA.
E-mail address: 21235005@zju.edu.cn