OPTIMAL REINSURANCE WITH DEFAULT RISK: A Reinsurer’s Perspective

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Abstract. In this paper, we study the optimal reinsurance design with default risk by minimizing the VaR (value at risk) of the reinsurer’s total risk exposure. The optimal reinsurance treaty is provided. When the reinsurance premium principle is specified to the expected value and exponential premium principles, the explicit expressions for the optimal reinsurance treaties are given, respectively.

1. Introduction. Reinsurance is an effective risk management tool for an insurer. By means of buying reinsurance contract, an insurer can transfer part of its liability to other insurers. Since the seminal papers of [4] and [6], many researchers have devoted themselves to the study about reinsurance design, and proposed many optimal reinsurance models. At the same time, reinsurance, as a risk management tool, is playing an increasingly important role in practice.

The aim of the optimal reinsurance design is to find a ceded loss function such that it satisfies an optimal objective. In many literature, to the perspective of the insurer, the optimal objective is usually assumed to maximize the expected utility of an insurer’s terminal wealth, or to minimize the risk measure of an insurer’s total retained risk. For example, see [3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 27, 28, 29, 30, 31, 32] and the references therein.

In most of the above-mentioned optimal reinsurance studies, it is assumed that the reinsurer will be able to compensate for the losses they commit. However in practice, the reinsurer may fail to pay the promised part of lose if this part of lose exceeds the reinsurer’s reserve. Therefore, it is reasonable to consider the default risk in the optimal reinsurance design. In recent years, to the perspective of the insurer, [1, 2, 5, 7] studied optimal reinsurance designs with default risk, among which [1, 2, 5] assumed that the reinsurer’s reserve is constant, while [7] assumed that the reinsurer’s reserve is determined by the losses it bears.

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Taking into account the default risk, [7] studied the optimal reinsurance design by minimizing the VaR of the insurer’s total retained risk. However, it is well known that the insurer and the reinsurer have conflict interests in a reinsurance contract. Therefore, a natural and interesting question is how about the optimal reinsurance design by minimizing the risk measure of the reinsurer’s total risk exposure with default risk. In the present paper, taking into account the default risk, we will study the optimal reinsurance treaty by minimizing the VaR of the reinsurer’s total risk exposure. The optimal reinsurance treaty is provided. Finally, examples are also given to illustrate the obtained results, where the explicit expressions for the optimal reinsurance treaties are provided. It turns out that the obtained optimal reinsurance treaty is quite different from the one of [7]. The present studies can be considered as the complements of those of [7].

The rest of this paper is organized as follows. In Section 2, we briefly introduce the preliminaries. In Section 3, the optimal reinsurance design is studied, and the optimal reinsurance treaty is provided in general. Finally, in Section 4, when the reinsurance premium is specified to be the expected value and exponential premium principles, respectively, the explicit expressions for the optimal reinsurance treaties are given.

2. Preliminaries. Let $X$ be a random loss initially faced by an insurer. We assume that $X \geq 0$ on some probability space $(\Omega, \mathcal{F}, P)$ with positive and finite expectation $E[X]$. Denote by $F_X(x) := P(X \leq x)$, $x \in \mathbb{R}$, the distribution function of $X$, and by $S_X(x) := 1 - F_X(x)$ the survival function of $X$. Denote by $X$ all the non-negative random variables with positive and finite expectations. Since it is possible that there is no claim for the insurer, to avoid any tedious discussion, through this paper, we assume that $S_X(0) > 0 \leq 1$ (or equivalently $0 \leq F_X(0) < 1$).

Next, we introduce the definition of the value at risk (VaR).

**Definition 2.1.** Let $\alpha \in (0, 1)$ and $X \in \mathcal{X}$. The VaR of $X$ at confidence level $(1 - \alpha)$ is defined as

$$\text{VaR}_\alpha(X) := \inf \{x \geq 0 : P(X > x) \leq \alpha\}.$$ (1)

**Proposition 1.** Let $X \in \mathcal{X}$. Then VaR has the following properties:

(i) For $0 < \alpha < 1$ and $x \geq 0$,

$$\text{VaR}_\alpha(X) \leq x \text{ if and only if } S_X(x) \leq \alpha.$$ (2)

(ii) Translation invariance: for $0 < \alpha < 1$ and any $c \in \mathbb{R}$,

$$\text{VaR}_\alpha(X + c) = \text{VaR}_\alpha(X) + c.$$ (3)

(iii) For any $0 < \alpha < 1$ and non-decreasing continuous function $\varphi(x)$, we have

$$\text{VaR}_\alpha(\varphi(X)) = \varphi(\text{VaR}_\alpha(X)).$$ (4)

Note that (iii) is from the equality (15) of [20]. Other properties are obvious.

We also denote $x_+ := \max \{x, 0\}$, $x_1 \wedge x_2 := \min \{x_1, x_2\}$, $x_1 \vee x_2 := \max \{x_1, x_2\}$ for $x, x_1, x_2 \in \mathbb{R}$.

In a classical reinsurance treaty, the insurer cedes part of the loss $X$, say $f(X)$, to a reinsurer, and retain the part of the loss $X$, say $R_f(X) := X - f(X)$. We call the function $f(x) : [0, +\infty) \to [0, +\infty)$ the ceded loss function, and $R_f(x) : [0, +\infty) \to [0, +\infty)$ the retained loss function. To preclude moral hazard, [21] suggested that
both the retained loss function and the ceded loss function are increasing. In the present paper, we adopt the class of ceded loss functions as in [21]. In other words, the class of admissible ceded loss functions considered in this paper is defined as 
\[ \mathcal{L} := \{ f : [0, +\infty) \to [0, +\infty); \ 0 \leq f(x) \leq x, \ R_f(x) \text{ and } f(x) \text{ are increasing in } x \}. \]
Note that any \( f \in \mathcal{L} \) is increasing and Lipschitz continuous, that is
\[ 0 \leq f(x_2) - f(x_1) \leq x_2 - x_1 \text{ for any } 0 \leq x_1 \leq x_2. \] (5)
A technical by-product is that every \( f \in \mathcal{L} \) is absolutely continuous with \( f(0) = 0 \).

When an insurer cedes part of the loss to a reinsurer, the insurer incurs an additional cost in the form of reinsurance premium which is payable to the reinsurer. Let \( \pi : \mathcal{R} \to [0, +\infty) \) be the reinsurance premium principle satisfying monotonicity, i.e. \( \pi(X) \leq \pi(Y) \) for any \( X, Y \in \mathcal{R} \) with \( X \leq Y \).

In this paper, we propose a reinsurance model with default risk and assume that the initial capital or reserve of a reinsurer of a reinsurance contract \( f \) is determined through regulation by the VaR of its promised indemnity \( f(X) \), and denote the initial capital of the reinsurer by \( \upsilon_f := \text{VaR}_\alpha(f(X)) \). We assume that the reinsurer charges a reinsurance premium \( \pi_f := \pi[f(X)] \) based on the promised indemnity \( f(X) \). Both insurer and reinsurer are aware of the potential default by reinsurer but the worst case for insurer is that the reinsurer only pays \( \upsilon_f + \pi_f \) if \( f(X) > \upsilon_f + \pi_f \). Hence, under the proposed reinsurance model, the total risk exposure of the reinsurer is \( f(X) \land (\upsilon_f + \pi_f) - \pi_f \), which we denote by \( T_f(X) \), that is
\[ T_f(X) := f(X) \land (\upsilon_f + \pi_f) - \pi_f. \] (6)

Assume that the reinsurer adopt VaR at some confidence level \( (1 - \beta) \) to control its total risk exposure \( T_f(X) \) and seek the optimal ceded loss function \( f^*(x) \) that minimizes this VaR. In other words, the optimal reinsurance model which we concern in the present paper is described as
\[ \text{VaR}_\beta(T_f^*(X)) = \min_{f \in \mathcal{L}} \text{VaR}_\beta(T_f(X)), \] (7)
where \( f^* \) is the resulting optimal reinsurance treaty.

3. Optimal reinsurance treaty. In this section, we study the optimal reinsurance model (7). For any \( f \in \mathcal{L} \), we denote
\[ V(f) := \text{VaR}_\beta(T_f(X)) = \text{VaR}_\beta[f(X) \land (\upsilon_f + \pi_f) - \pi_f]. \]
Hence, the optimal model (7) can be reformulated as
\[ \text{VaR}_\beta(T_{f^*}(X)) = \min_{f \in \mathcal{L}} V(f). \] (8)

Denote \( a := \text{VaR}_\alpha(X) \) and \( b := \text{VaR}_\beta(X) \). For any \( f \in \mathcal{L} \), clearly, the function \( f(x) \land (\upsilon_f + \pi_f) \) is continuous and non-decreasing on \([0, \infty)\). Hence, by (4), \( V(f) \) can be rewritten as
\[ V(f) = f(b) \land (f(a) + \pi_f) - \pi_f. \] (9)

The optimal model (8) is an infinite-dimensional optimal one. Next, we will translate the optimal model (8) into a finite-dimensional optimization problem. For this purpose, we begin with a lemma, which will play an important role in later discussion.

**Lemma 3.1.** For any \( f_1, f_2 \in \mathcal{L} \), if \( f_1(a) = f_2(a) \), \( f_1(b) = f_2(b) \), and \( \pi_{f_1} \leq \pi_{f_2} \), then \( V(f_2) \leq V(f_1) \).
Proof. Denote \( \zeta_a := f_1(a) = f_2(a) \), \( \zeta_b := f_1(b) = f_2(b) \). From (9) it follows that
\[
V(f_2) - V(f_1) = f_2(b) \wedge (f_2(a) + \pi_f) - \pi_f - f_1(b) \wedge (f_1(a) + \pi_f) + \pi_f,
\]
\[
= \zeta_b \wedge (\zeta_a + \pi_f) - \pi_f - \zeta_b \wedge (\zeta_a + \pi_f) + \pi_f.
\]
If \( \zeta_b \leq \zeta_a \), then
\[
V(f_2) - V(f_1) = \pi_f - \pi_f \leq 0.
\]
If \( \zeta_a < \zeta_b \leq \zeta_a + \pi_f \), then
\[
V(f_2) - V(f_1) = \zeta_b \wedge (\zeta_a + \pi_f) - \zeta_b \wedge (\zeta_a + \pi_f) + \pi_f - \pi_f
\]
\[
= \begin{cases}
\pi_f - \pi_f, & \zeta_b \leq \zeta_a + \pi_f, \\
\zeta_b - (\zeta_a + \pi_f), & \zeta_b > \zeta_a + \pi_f,
\end{cases}
\]
\[
\leq 0.
\]
If \( \zeta_b > \zeta_a + \pi_f \), then
\[
V(f_2) - V(f_1) = (\zeta_a + \pi_f) - (\zeta_a + \pi_f) + \pi_f - \pi_f
\]
\[
= 0.
\]
In summary, \( V(f_2) - V(f_1) \leq 0 \). Lemma 3.1 is proved. \( \square \)

Let \( \mathcal{L}_{d_1,d_2} \) be the class of non-negative functions \( h(x; d_1, d_2) \) defined on \([0, \infty)\) with
\[
h(x; d_1, d_2) := x - (x - d_1)_+ + (x - a \wedge b)_+ - (x - d_2)_+ + (x - a \vee b)_+.
\]
where \( 0 \leq d_1 \leq a \wedge b \leq d_2 \leq a \vee b \). Clearly, \( \mathcal{L}_{d_1,d_2} \subset \mathcal{L} \).

Now, we are in a position to state the main result of this paper, which solves the optimization problem (8).

**Theorem 3.2.** For any \( f \in \mathcal{L} \), there exists an \( h_f \in \mathcal{L}_{d_1,d_2} \) such that
\[
V(h_f) \leq V(f).
\]
Particularly,
\[
\min_{f \in \mathcal{L}} V(f) = \min_{f \in \mathcal{L}_{d_1,d_2}} V(f)
\]
\[
= \min_{0 \leq d_1 \leq a \wedge b, \ a \wedge b \leq d_2 \leq a \vee b} v(d_1, d_2),
\]
where \( v(d_1, d_2) := V(f) = f(b) \wedge (f(a) + \pi_f) - \pi_f \) for \( f \in \mathcal{L}_{d_1,d_2} \).

**Remark 1.** Theorem 3.2 translates the infinite-dimensional optimization problem (8) into a two-dimensional optimization problem. Theorem 3.2 shows that from the reinsurer’s perspective, the optimal reinsurance treaty is different from the one to the insurer’s perspective, see [7] for the optimal reinsurance treaty from the insurer’s perspective. These different forms of the optimal treaties reflect the conflict interests of the insurer and the reinsurer in a reinsurance contract. How to balance...
Next, we show (11).

... both the insurer’s and reinsurer’s interests in a reinsurance contract is an interesting issue, and which will be the subject of further research.

Proof of Theorem 3.2. (12) follows from (11), since \( \mathcal{L}_{d_1,d_2} \subset \mathcal{L} \). (13) is obvious. Next, we show (11).

For any \( f \in \mathcal{L} \), denote \( d_1^* := f(a \wedge b), \ d_2^* := a \wedge b + f(a \vee b) - f(a \wedge b) \), then \( 0 \leq d_1^* \leq a \wedge b \leq d_2^* \leq a \vee b \). Substituting \( d_1^* \) and \( d_2^* \) into (10), we have

\[
h(a \wedge b; d_1^*, d_2^*) = d_1^* = f(a \wedge b),
\]

and

\[
h(a \vee b; d_1^*, d_2^*) = d_2^* - a \wedge b + d_1^*,
\]

\[
= a \wedge b + f(a \vee b) - f(a \wedge b) - a \wedge b + f(a \wedge b)
\]

\[
= f(a \vee b).
\]

Define a function \( h_f(x) \) on \([0, \infty)\) by

\[
h_f(x) := h(x; d_1^*, d_2^*).
\]

Clearly, \( h_f \in \mathcal{L}_{d_1,d_2} \). Then (14) and (15) are equivalent to

\[
h_f(a \wedge b) = f(a \wedge b),
\]

and

\[
h_f(a \vee b) = f(a \vee b).
\]

Next we show that for any \( f \in \mathcal{L} \), the following inequality

\[
h_f(x) \geq f(x), \text{ for any } x \geq 0,
\]

holds, where \( h_f(x) \) is defined in (16). To see this, let us recall that the ceded loss function \( f \in \mathcal{L} \) is non-negative and Lipschitz-continuous and hence (5) implies that

(i) when \( x \in [0, d_1^*] \),

\[
f(x) - f(0) \leq x \Rightarrow f(x) \leq x = h_f(x),
\]

since \( f(0) = 0 \);

(ii) when \( x \in [a \wedge b, d_2^*] \),

\[
f(x) - f(a \wedge b) \leq x - a \wedge b \Rightarrow f(x) \leq x - a \wedge b + f(a \wedge b)
\]

\[
= x - a \wedge b + d_1^*
\]

\[
= h_f(x);
\]

(iii) when \( x \in [a \vee b, \infty] \),

\[
f(x) - f(a \vee b) \leq x - a \vee b
\]

\[
\Rightarrow f(x) \leq x - a \vee b + f(a \vee b)
\]

\[
= x - a \vee b - a \wedge b + a \vee b + f(a \vee b) - f(a \wedge b) + f(a \wedge b)
\]

\[
= x - a - b + d_1^* + d_2^*
\]

\[
= h_f(x).
\]

On the other hand, the increasing property of \( f(x) \) leads to

\[
h_f(x) = d_1^* = f(a \wedge b) \geq f(x), \ x \in [d_1^*, a \wedge b],
\]

and

\[
h_f(x) = d_2^* - a \wedge b + d_1 = f(a \vee b) \geq f(x), \ x \in [d_2^*, a \vee b].
\]
(20), (21) and (22) together with (23) and (24), lead to (19). Then,
\[ \pi_{h_f} \geq \pi_f, \]
due to the monotonicity of \( \pi \), where \( \pi_{h_f} := \pi[h_f(X)] \).

Consequently, combining (17), (18) and (25), (11) follows from Lemma 3.1. Theorem 3.2 is then proved.

4. Examples. In this section, we will derive the explicit expressions for the optimal reinsurance treaties when the reinsurance premium principle is specified to the expected value and exponential premium principle, respectively.

**Definition 4.1.** The expected value premium principle is defined as
\[ \pi(X) := (1 + \theta)E(X), \quad X \in \mathcal{X}, \]
where \( \theta > 0 \) is the safety loading factor.

**Definition 4.2.** The exponential premium principle is defined as
\[ \pi(X) := \frac{1}{\lambda} \ln \left[ E(e^{\lambda X}) \right], \quad X \in \mathcal{X}, \]
where \( \lambda > 0 \) is called the risk aversion coefficient.

**Remark 2.** (i) When the premium principle \( \pi \) is calculated by the expected value premium principle, then for any \( f \in \mathcal{L} \),
\[ \pi_f = (1 + \theta)\mathbb{E}(f(X)) = (1 + \theta) \int_0^\infty S_X(x)f'(x)dx, \]
which can be found in Lemma 2.1 of [13].

(ii) The exponential premium increases as \( \lambda \) increases. When \( \lambda \to 0 \), the exponential premium is pure premium. When \( \lambda \to \infty \), the exponential premium tends to be the maximum of \( X \). When the premium principle \( \pi \) is calculated by the exponential premium principle, then for any \( f \in \mathcal{L} \),
\[ \pi_f = \frac{1}{\lambda} \ln \left\{ \mathbb{E} \left[ e^{\lambda f(X)} \right] \right\} = \frac{1}{\lambda} \ln \left\{ \mathbb{E} \left[ \int_0^X dt e^{\lambda f(t)} + 1 \right] \right\} = \frac{1}{\lambda} \ln \left\{ \mathbb{E} \left[ \int_0^\infty 1_{(X>t)} dt e^{\lambda f(t)} + 1 \right] \right\} = \frac{1}{\lambda} \ln \left\{ \int_0^\infty \mathbb{E} \left[ 1_{(X>t)} \lambda e^{\lambda f(t)} dt + 1 \right] \right\} \]
\[ = \frac{1}{\lambda} \ln \left\{ \lambda \int_0^\infty S_X(t)e^{\lambda f(t)} dt + 1 \right\} = \frac{1}{\lambda} \ln \left\{ \lambda \int_0^\infty S_X(t)e^{\lambda f(t)} f'(t) dt + 1 \right\}. \]
f'(t)dt because of the absolute continuity of $f \in \mathcal{L}$, whose derivative exists almost everywhere (see Remark (v)(a) on page 285 of [26]).

4.1. Expected value premium principle. In this subsection, we will derive the explicit solution to the optimization problem (8), when the premium is calculated by the expected value premium principle (26).

**Theorem 4.3.** Denote by $f^*$ the optimal solution to the optimization problem (8).

(1) If $\alpha \leq \beta$, then

$$f^*(x) = x - (x - \text{VaR}_{\theta^*}(X))_+ + (x - b)_+$$

$$= \begin{cases} x, & x \leq \text{VaR}_{\theta^*}(X) \land b, \\ \text{VaR}_{\theta^*}(X) \land b, & \text{VaR}_{\theta^*}(X) \land b < x \leq b, \\ x + \text{VaR}_{\theta^*}(X) \land b - b, & x > b. \end{cases}$$

(2) If $\alpha > \beta \geq \theta^*$, then

$$f^*(x) = x \quad \text{for all} \quad x \geq 0.$$

(3) If $\theta^* \geq \alpha > \beta$, then

$$f^*(x) = x - (x - \text{VaR}_{\theta^*}(X))_+ + (x - b)_+$$

$$= \begin{cases} x, & x \leq \text{VaR}_{\theta^*}(X), \\ \text{VaR}_{\theta^*}(X), & \text{VaR}_{\theta^*}(X) \land b < x \leq b, \\ x + \text{VaR}_{\theta^*}(X) \land b - b, & x > b, \end{cases}$$

where $\theta^* := \frac{1}{1+\theta}$.

**Proof.** For each $d_1 \in [0, a \land b]$, we define $v_{d_1}(d_2) := v(d_1, d_2)$ as a function of $d_2 \in [a \land b, \infty)$. In order to obtain the optimal ceded function, we need to consider the following two exclusive cases.

(1) Assume that $\alpha \leq \beta$ (or equivalently $b \leq a$). In this case, for any fixed $d_1 \in [0, b]$ and for any $d_2 \in [b, a]$ and $f \in \mathcal{L}_{d_1, d_2}$, we have

$$v_{d_1}(d_2) = v(d_1, d_2) = V(f) = d_1 - (1 + \theta) \left( \int_0^{d_1} + \int_b^{d_2} + \int_a^{\infty} \right) S_X(x)dx.$$

The first-order derivative of $v_{d_1}(d_2)$ with respect to $d_2$, denoted by $v'_{d_1}(d_2)$, is

$$v'_{d_1}(d_2) := \frac{\partial}{\partial d_2} v(d_1, d_2) = -(1 + \theta)S_X(d_2) < 0.$$ 

Hence, for any $d_1 \in [0, b]$, the minimum of $v_{d_1}(d_2)$ over $[b, a]$ can be attained at $d_2 = a$. Thus,

$$\min_{0 \leq d_1 \leq b, b \leq d_2 \leq a} v(d_1, d_2) = \min_{0 \leq d_1 \leq b} v(d_1, a).$$

Next, we consider the function $d_1 \rightarrow v(d_1, a)$ on $[0, b]$. Note that

$$v(d_1, a) = d_1 - (1 + \theta) \left( \int_0^{d_1} + \int_b^{\infty} \right) S_X(x)dx.$$
Obviously, the function \( v(d_1, a) \) is continuous in \( d_1 \), and its first-order derivative is
\[
\frac{d}{dd_1} v(d_1, a) = 1 - (1 + \theta)S_X(d_1).
\]

Since
\[
1 - (1 + \theta)S_X(d_1) \leq 0 \text{ if and only if } S_X(d_1) \geq \theta^* \text{ if and only if } \text{VaR}_{\theta^*}(X) \geq d_1,
\]
we know that the minimum of \( v(d_1, a) \) over \([0, b]\) can be attained at \( d_1 = \text{VaR}_{\theta^*}(X) \land b \). Hence, by Theorem 3.2, we know that the optimal ceded loss function \( f^* \) to model (8) is \( h(x; d_1, d_2) \) with \( d_1 = \text{VaR}_{\theta^*}(X) \land b \) and \( d_2 = a \), that is
\[
f^*(x) = x - (x - \text{VaR}_{\theta^*}(X) \land b) + (x - b)
\]
\[
= \begin{cases} 
  x, & x \leq \text{VaR}_{\theta^*}(X) \land b, \\
  \text{VaR}_{\theta^*}(X) \land b, & \text{VaR}_{\theta^*}(X) \land b < x \leq b, \\
  x + \text{VaR}_{\theta^*}(X) \land b - b, & x > b.
\end{cases}
\]

(2) Assume that \( \alpha > \beta \) (or equivalently \( b > a \)). In this case, for any fixed \( d_1 \in [0, a] \) and \( f \in \mathcal{L}_{d_1, d_2} \), we first define \( v_{d_1}(d_2) \) and \( G_{d_1}(d_2) \) as function of \( d_2 \) by
\[
v_{d_1}(d_2) := v(d_1, d_2)
= V(f)
= (d_2 - a + d_1) \land (d_1 + \pi_f) - \pi_f
= \begin{cases} 
  d_2 - a + d_1 - \pi_f, & \pi_f \geq d_2 - a, \\
  d_1, & \pi_f < d_2 - a,
\end{cases}
\]
where
\[
\pi_f = (1 + \theta) \left( \int_0^{d_1} + \int_a^{d_2} + \int_\infty^{\infty} \right) S_X(x)dx,
\]
and
\[
G_{d_1}(d_2) := (d_1 + \pi_f) - (d_2 - a + d_1)
= \pi_f - d_2 + a
= (1 + \theta) \left( \int_0^{d_1} + \int_a^{d_2} + \int_\infty^{\infty} \right) S_X(x)dx - d_2 + a.
\]

Obviously, the function \( G_{d_1}(d_2) \) is continuous in \( d_2 \), and its first-order derivative is
\[
\frac{d}{dd_2} G_{d_1}(d_2) = (1 + \theta)S_X(d_2) - 1.
\]

To determine the optimal optimal reinsurance treaty \( f^* \), we need to consider the following two cases.

Case 1. \( \alpha > \beta \geq \theta^* \). In this case,
\[
\frac{d}{dd_2} G_{d_1}(d_2) = (1 + \theta)S_X(d_2) - 1 \geq 0.
\]

Hence, for any \( d_1 \in [0, a] \), the minimum of \( G_{d_1}(d_2) \) over \([a, b]\) can be attained at \( d_2 = a \). Thus,
\[
\min_{a \leq d_2 \leq b} G_{d_1}(d_2) = G_{d_1}(a) = (1 + \theta) \left( \int_0^{d_1} + \int_\infty^{\infty} \right) S_X(x)dx \geq 0,
\]
which implies that \( G_{d_1}(d_2) \geq 0 \) on \([a,b]\). Therefore, we have

\[
v_{d_1}(d_2) = d_2 - a + d_1 - \pi_f.
\]

Obviously, the function \( v_{d_1}(d_2) \) is continuous in \( d_2 \), and its first-order derivative is

\[
v'_{d_1}(d_2) = 1 - (1 + \theta)S_X(d_2) \leq 0.
\]

Hence, for any \( d_1 \in [0,a] \), the minimum of \( v_{d_1}(d_2) \) over \([a,b]\) can be attained at \( d_2 = b \). Thus,

\[
\min_{0 \leq d_1 \leq a, a \leq d_2 \leq b} v(d_1, d_2) = \min_{0 \leq d_1 \leq a} v(d_1, b).
\]

Next, we consider the function \( d_1 \rightarrow v(d_1, b) \) on \([0,a]\). Note that

\[
v(d_1, b) = b - a + d_1 - (1 + \theta) \left( \int_0^{d_1} + \int_a^{\infty} \right) S_X(x)dx.
\]

Obviously, the function \( v(d_1, b) \) is continuous in \( d_1 \), and its first-order derivative is

\[
\frac{d}{dd_1} v(d_1, b) = 1 - (1 + \theta)S_X(d_1) < 0.
\]

which implies that the minimum of \( v(d_1, b) \) over \([0,a]\) can be attained at \( d_1 = a \). Hence, by Theorem 3.2, we know that the optimal ceded function \( f^* \) to model (8) is \( h(x;d_1, d_2) \) with \( d_1 = a \) and \( d_2 = b \), that is

\[
f^*(x) = x \quad \text{for all} \quad x \geq 0.
\]

Case 2. \( \theta^* \geq \alpha > \beta \). In this case,

\[
\frac{d}{dd_2} G_{d_1}(d_2) = (1 + \theta)S_X(d_2) - 1 \leq 0.
\]

Hence, for any \( d_1 \in [0,a] \), the minimum of \( G_{d_1}(d_2) \) over \([a,b]\) can be attained at \( d_2 = b \). Thus,

\[
\min_{a \leq d_2 \leq b} G_{d_1}(d_2) = G_{d_1}(b) = (1 + \theta) \left( \int_0^{d_1} + \int_a^{\infty} \right) S_X(x)dx - b + a.
\]

Note that if \( G_{d_1}(b) \geq 0 \), then \( G_{d_1}(d_2) \geq 0 \) on the interval \([a,b]\). If \( G_{d_1}(b) \leq 0 \), then there exists a \( c(d_1) \in [a,b] \) such that \( G_{d_1}(d_2) \geq 0 \) for any \( d_2 \in [a,c(d_1)] \) and \( G_{d_1}(d_2) \leq 0 \) for any \( d_2 \in [c(d_1),b] \). Thus, we need to consider the following three subcases in order to obtain the optimal ceded function \( f^* \).

(i) Suppose that \( (1 + \theta) \int_0^\infty S_X(x)dx - b + a \leq 0 \). In this subcase, \( G_{d_1}(b) \leq 0 \) for any \( d_1 \in [0,a] \). Hence,

\[
v_{d_1}(d_2) = \begin{cases} 
  d_2 - a + d_1 - \pi_f, & a \leq d_2 \leq c(d_1), \\
  d_1, & c(d_1) \leq d_2 \leq b.
\end{cases}
\]  

Note that

\[
v_{d_1}(c(d_1)) = c(d_1) - a + d_1 - (1 + \theta) \left( \int_0^{d_1} + \int_a^{c(d_1)} + \int_b^{\infty} \right) S_X(x)dx
\]

\[
= d_1 - G_{d_1}(c(d_1))
\]

\[
= d_1.
\]
It implies that $v_{d_1}(d_2)$ is continuous in $[a, b]$. Then its first-order derivation is

$$v'_{d_1}(d_2) = \begin{cases} 1 - (1 + \theta)S_X(d_2), & a \leq d_2 \leq c(d_1), \\ 0, & c(d_1) \leq d_2 \leq b, \end{cases}$$

$\geq 0$.

Hence, for any $d_1 \in [0, a]$, the minimum of $v_{d_1}(d_2)$ over $[a, b]$ can be attained at $d_2 = a$. Thus,

$$\min_{0 \leq d_1 \leq a, \ a \leq d_2 \leq b} v(d_1, d_2) = \min_{0 \leq d_1 \leq a} v(d_1, a).$$

Next, we consider the function $d_1 \to v(d_1, a)$ on $[0, a]$. Note that

$$v(d_1, a) = d_1 - (1 + \theta) \left( \int_0^{d_1} + \int_b^\infty \right) S_X(x)dx.$$  \hspace{1cm} (35)

Obviously, the function $v(d_1, a)$ is continuous in $d_1$, and its first-order derivative is

$$\frac{d}{dd_1} v(d_1, a) = 1 - (1 + \theta)S_X(d_1).$$

Since

$$1 - (1 + \theta)S_X(d_1) \leq 0$$

if and only if $S_X(d_1) \geq \theta^*$ if and only if $\text{VaR}_{\theta^*}(X) \geq d_1$, and $\text{VaR}_{\theta^*}(X) \leq a$ by assumption, we know that the minimum of $v(d_1, a)$ over $[0, a]$ can be attained at $d_1 = \text{VaR}_{\theta^*}(X)$. Hence, by Theorem 3.2, we know that the optimal ceded loss function $f^*$ to model (8) is $h(x; d_1, d_2)$ with $d_1 = \text{VaR}_{\theta^*}(X)$ and $d_2 = a$, that is

$$f^*(x) = x - (x - \text{VaR}_{\theta^*}(X)) + (x - b)^+,$$

$$= \begin{cases} x, & x \leq \text{VaR}_{\theta^*}(X), \\ \text{VaR}_{\theta^*}(X), & \text{VaR}_{\theta^*}(X) < x \leq b, \\ x + \text{VaR}_{\theta^*}(X) - b, & x > b. \end{cases}$$  \hspace{1cm} (36)

(ii) Suppose that $(1 + \theta) \int_a^\infty S_X(x)dx - b + a \geq 0$. In this subcase, $G_{d_1}(b) \geq 0$ for any $d_1 \in [0, a]$. Hence,

$$v_{d_1}(d_2) = v(d_1, d_2) = d_2 - a + d_1 - \pi_f.$$  

The first-order derivation of $v_{d_1}(d_2)$ is

$$v'_{d_1}(d_2) = 1 - (1 + \theta)S_X(d_2) \geq 0.$$  

Hence, for any $d_1 \in [0, a]$, the minimum of $v_{d_1}(d_2)$ over $[a, b]$ can be attained at $d_2 = a$. Thus,

$$\min_{0 \leq d_1 \leq a, \ a \leq d_2 \leq b} v(d_1, d_2) = \min_{0 \leq d_1 \leq a} v(d_1, a).$$

Next, we consider the function $d_1 \to v(d_1, a)$ on $[0, a]$. Note that

$$v(d_1, a) = d_1 - (1 + \theta) \left( \int_0^{d_1} + \int_b^\infty \right) S_X(x)dx$$

is equal to (35). Hence, the optimal reinsurance treaty $f^*$ is the same as (36).

(iii) Suppose that

$$(1 + \theta) \int_a^\infty S_X(x)dx - b + a \leq 0 \leq (1 + \theta) \int_0^\infty S_X(x)dx - b + a.$$
In this subcase, there exists a $d_0 \in [0, a]$ such that

$$G_{d_1}(b) = (1 + \theta) \left( \int_0^{d_1} + \int_a^\infty \right) S_X(x)dx - b + a \leq 0 \text{ for } 0 \leq d_1 \leq d_0,$$

and $G_{d_1}(b) \geq 0$ for $d_0 \leq d_1 \leq a$. For $d_1 \in [0, d_0]$ or $G_{d_1}(b) \leq 0$, one has that $v(d_1, a)$ is equal to (35). For $d_1 \in [d_0, a]$ or $G_{d_1}(b) \geq 0$, $v(d_1, a)$ is equal to (35), too. Hence, on the whole interval $[0, a]$, $v(d_1, a)$ is equal to (35), which implies that

$$\min_{0 \leq d_1 \leq a, a \leq d_2 \leq b} v(d_1, d_2) = \min_{0 \leq d_1 \leq a} v(d_1, a) = v(\text{VaR}_\theta(X), a).$$

Consequently, the optimal reinsurance treaty $f^*$ in (iii) is the same with (i) and (ii) and equal to (36).

Combining the above three subcases, the optimal solution to optimization problem (8) is

$$f^*(x) = x - (x - \text{VaR}_\theta(X))_+ + (x - b)_+$$

$$= \begin{cases} x, & x \leq \text{VaR}_\theta(X), \\ \text{VaR}_\theta(X), & \text{VaR}_\theta(X) < x \leq b, \\ x + \text{VaR}_\theta(X) - b, & x > b. \end{cases}$$

Theorem 4.3 is proved. \hfill \Box

**Remark 3.** The optimal solution $f^*$ in Theorem 4.3 can be uniformly expressed as $f^*(x) = x - (x - d^*)_+ + (x - b)_+$, where $d^* = b \wedge \text{VaR}_\theta(X)$ if $\alpha \leq \beta$, $d^* = \text{VaR}_\theta(X)$ if $\theta^* \geq \alpha > \beta$ and $d^* = b$ if $\alpha > \beta \geq \theta^*$. In addition, for the case where $\alpha \geq \theta^* > \beta$ and $\alpha > \theta^* \geq \beta$, the optimal solution $f^*$ has no closed form.

### 4.2. Exponential premium principle

In this subsection, we will derive the explicit solution to the optimization problem (8), when the premium is calculated by the exponential premium principle (27).

Through this subsection, we assume that the $X \in \mathcal{X}$ is continuous.

**Theorem 4.4.** Denote by $f^*$ the optimal solution to the optimization problem (8),

$$f^*(x) = (x - b)_+$$

$$= \begin{cases} 0, & 0 \leq x < b, \\ x - b, & x \geq b. \end{cases}$$

**Proof.** In order to obtain the optimal ceded function, we need to consider the following two exclusive cases.

(1) Assume that $\alpha \leq \beta$ (or equivalently $b \leq a$). In this case, for any fixed $d_1 \in [0, b]$ and for any $d_2 \in [b, a]$ and $f \in \mathcal{L}_{d_1, d_2}$, we have that

$$v(d_1, d_2) = v(d_1, d_2)$$

$$= V(f)$$

$$= d_1 - \frac{1}{\lambda} \ln \left\{ \lambda \left[ \int_0^{d_1} e^\lambda x + \int_b^{d_2} e^\lambda (x - b + d_1) \\ + \int_a^\infty e^\lambda (x - b + d_2 - a + d_1) \right] S_X(x)dx + 1 \right\}.$$
The first-order derivation of $v_{d_1}(d_2)$ with respect to $d_2$ is
\[
v'_{d_1}(d_2) = \frac{\partial}{\partial d_2} v(d_1, d_2) = -\frac{1}{\lambda} e^{\lambda d_1} S_X(d_1) + \frac{\lambda^2}{\lambda} \int_a^\infty e^{\lambda(x-b-a+d_1)} S_X(x) dx + 1.
\]

Hence, for any $d_1 \in [0, b]$, the minimum of $v_{d_1}(d_2)$ over $[b, a]$ can be attained at $d_2 = a$. Thus,
\[
\min_{0 \leq d_1 \leq b, b \leq d_2 \leq a} v(d_1, d_2) = \min_{0 \leq d_1 \leq b} v(d_1, a).
\]

Next, we consider the function $d_1 \rightarrow v(d_1, a)$ on $[0, b]$. Note that
\[
v(d_1, a) = d_1 - \frac{1}{\lambda} \ln \left\{ \lambda \left[ \int_0^{d_1} e^{\lambda x} dx + \int_a^\infty e^{\lambda(x-b+d_1)} S_X(x) dx + 1 \right] S_X(x) dx + 1 \right\}.
\]

Obviously, the function $v(d_1, a)$ is continuous in $d_1$, and its first-order derivation is
\[
\frac{d}{dd_1} v(d_1, a) = 1 - \frac{1}{\lambda} e^{\lambda d_1} S_X(d_1) + \frac{\lambda^2}{\lambda} \int_a^\infty e^{\lambda(x-b-a+d_1)} S_X(x) dx + 1
\]
\[
= 1 - \frac{\lambda \int_0^{d_1} e^{\lambda x} S_X(x) dx + \lambda \int_b^\infty e^{\lambda(x-b+d_1)} S_X(x) dx + 1}{\lambda \int_0^{d_1} e^{\lambda x} S_X(x) dx + 1}
\]
\[
= \frac{\lambda \int_0^{d_1} e^{\lambda x} S_X(x) dx + 1}{\lambda \int_0^{d_1} e^{\lambda x} S_X(x) dx + 1 - S_X(0) + \int_0^{d_1} e^{\lambda x} dS_X(x)}
\]
\[
= \frac{1 - S_X(0) + \int_0^{d_1} e^{\lambda x} dS_X(x)}{\lambda \int_0^{d_1} e^{\lambda x} S_X(x) dx + \lambda \int_0^\infty e^{\lambda(x-b+d_1)} S_X(x) dx + 1}
\]
\[
\geq 0,
\]

where $p_X(x)$ is the density function of $X$ and the inequality is implied by $0 < S_X(0) \leq 1$. Hence, we know that the minimum of $v(d_1, a)$ over $[0, b]$ can be attained at $d_1 = 0$. Thus, by Theorem 3.2, we know that the optimal ceded loss function $f^*$ to model (8) is $h(x; d_1, d_2)$ with $d_1 = 0$ and $d_2 = a$, that is
\[
f^*(x) = (x - b)_+
\]
\[
= \begin{cases} 
0, & 0 \leq x < b, \\
x - b, & x \geq b.
\end{cases}
\]

(2) Assume that $\alpha \geq \beta$ (or equivalently $b \geq a$). In this case, for any fixed $d_1 \in [0, a]$ and $f \in \mathcal{L}_{d_1, d_2}$, we first introduce the function $v_{d_1}(d_2)$ from (32) and
$G_{d_1}(d_2)$ from (33).

\[ v_{d_1}(d_2) := v(d_1, d_2) = V(f) = (d_2 - a + d_1) \wedge (d_1 + \pi_f) - \pi_f \]

\[ = \begin{cases} d_2 - a + d_1 - \pi_f, & \pi_f \geq d_2 - a, \\ d_1, & \pi_f < d_2 - a, \end{cases} \quad (40) \]

where

\[ \pi_f := \frac{1}{\lambda} \ln \left\{ \lambda \left[ \int_0^{d_1} e^{\lambda x} + \int_a^{d_2} e^{\lambda(x-a+d_1)} + \int_b^{\infty} e^{\lambda(x-b+d_2-a+d_1)} \right] S_X(x)dx + 1 \right\}, \]

and

\[ G_{d_1}(d_2) := (d_1 + \pi_f) - (d_2 - a + d_1) = \pi_f - d_2 + a \]

\[ = \frac{1}{\lambda} \ln \left\{ \lambda \left[ \int_0^{d_1} e^{\lambda x} + \int_a^{d_2} e^{\lambda(x-a+d_1)} + \int_b^{\infty} e^{\lambda(x-b+d_2-a+d_1)} \right] S_X(x)dx + 1 \right\} - d_2 + a. \quad (41) \]

Obviously, the function $G_{d_1}(d_2)$ is continuous in $d_2$, and its first-order derivative is

\[ \frac{d}{dd_2} G_{d_1}(d_2) = \frac{e^{\lambda(x-d_2-a+d_1)} S_X(d_2) + \lambda \int_b^{\infty} e^{\lambda(x-b+d_2-a+d_1)} S_X(x)dx}{\lambda \left[ \int_0^{d_1} e^{\lambda x} + \int_a^{d_2} e^{\lambda(x-a+d_1)} + \int_b^{\infty} e^{\lambda(x-b+d_2-a+d_1)} \right] S_X(x)dx + 1} - 1 \]

\[ = \frac{P(d_1, d_2)}{Q(d_1, d_2)} - 1, \]

where

\[ P(d_1, d_2) := e^{\lambda(d_2-a+d_1)} S_X(d_2) + \lambda \int_b^{\infty} e^{\lambda(x-b+d_2-a+d_1)} S_X(x)dx, \]

and

\[ Q(d_1, d_2) := \lambda \left[ \int_0^{d_1} e^{\lambda x} + \int_a^{d_2} e^{\lambda(x-a+d_1)} + \int_b^{\infty} e^{\lambda(x-b+d_2-a+d_1)} \right] S_X(x)dx + 1. \]

Obviously, $P(d_1, d_2) > 0, Q(d_1, d_2) > 0$ for any $(d_1, d_2) \in [0, a] \times [a, b]$.

Next, we prove that $\frac{d}{dd_2} G_{d_1}(d_2) < 0$ for any $d_2 \in [a, b]$. For the sake of illustration, we denote $R(d_1, d_2) := P(d_1, d_2) - Q(d_1, d_2)$ as a function of $d_1 \in [0, a]$ and $d_2 \in [a, b]$. For any fixed $d_1 \in [0, a]$, define $R_{d_1}(d_2) := R(d_1, d_2)$ as a function of
\[ d_2 \in [a, b]. \]

\[ R(d_1, d_2) = P(d_1, d_2) - Q(d_1, d_2) \]

\[ = e^{\lambda(d_2 - a + d_1)}S_X(d_2) + \lambda \int_b^\infty e^{\lambda(x - b + d_2 - a + d_1)}S_X(x)dx - \lambda \int_0^{d_1} e^{\lambda x}S_X(x)dx \]

\[ - \lambda \int_a^{d_2} e^{\lambda(x - a + d_1)}S_X(x)dx - \lambda \int_b^\infty e^{\lambda(x - b + d_2 - a + d_1)}S_X(x)dx - 1 \]

\[ = e^{\lambda(d_2 - a + d_1)}S_X(d_2) - \lambda \int_0^{d_1} e^{\lambda x}S_X(x)dx - \lambda \int_a^{d_2} e^{\lambda(x - a + d_1)}S_X(x)dx - 1. \]

Obviously, for any fixed \( d_1 \in [0, a] \), the function \( R_{d_1}(d_2) \) is continuous in \( d_2 \), and its first-order derivation is

\[ \frac{d}{dd_2}R_{d_1}(d_2) = \lambda e^{\lambda(d_2 - a + d_1)}S_X(d_2) - e^{\lambda(d_2 - a + d_1)}p_X(d_2) - \lambda e^{\lambda(d_2 - a + d_1)}S_X(d_2) \]

\[ = -e^{\lambda(d_2 - a + d_1)}p_X(d_2) \leq 0. \]

Hence, for any \( d_1 \in [0, a] \), the maximum of \( R_{d_1}(d_2) \) over \([a, b]\) can be attained at \( d_2 = a \). Thus,

\[ \max_{0 \leq d_1 \leq a, a \leq d_2 \leq b} R(d_1, d_2) = \max_{0 \leq d_1 \leq a} R(d_1, a). \]

Next, we consider the function \( d_1 \to R(d_1, a) \) on \([0, a]\). Note that

\[ R(d_1, a) = e^{\lambda d_1}S_X(a) - \lambda \int_0^{d_1} e^{\lambda x}S_X(x)dx - 1. \]

Obviously, the function \( R(d_1, a) \) is continuous in \( d_1 \), and its first-order derivation is

\[ \frac{d}{dd_1}R(d_1, a) = \lambda e^{\lambda d_1}S_X(a) - \lambda e^{\lambda d_1}S_X(d_1) \]

\[ = \lambda e^{\lambda d_1}(S_X(a) - S_X(d_1)) \leq 0, \]

which implies that the maximum of \( R(d_1, a) \) over \([0, a]\) can be attained at \( d_1 = 0 \). Hence,

\[ \max_{0 \leq d_1 \leq a, a \leq d_2 \leq b} R(d_1, d_2) = R(0, a) = S_X(a) - 1 \leq 0. \]

Thus, \( R(d_1, d_2) \leq 0 \) for any \( d_1 \in [0, a], d_2 \in [a, b] \). Therefore, \( 0 < P(d_1, d_2) \leq Q(d_1, d_2) \Rightarrow \frac{P(d_1, d_2)}{Q(d_1, d_2)} \leq 1 \Rightarrow \frac{d}{dd_2}G_{d_1}(d_2) = \frac{P(d_1, d_2)}{Q(d_1, d_2)} - 1 \leq 0 \). Hence, for any \( d_1 \in [0, a] \), the minimum of \( G_{d_1}(d_2) \) over \([a, b]\) can be attained at \( d_2 = b \). Thus,

\[ \min_{a \leq d_2 \leq b} G_{d_1}(d_2) = G_{d_1}(b) \]

\[ = \frac{1}{\lambda} \ln \left\{ \lambda \left[ \int_0^{d_1} e^{\lambda x} + \int_a^{\infty} e^{\lambda(x - a + d_1)} \right] S_X(x)dx + 1 \right\} - b + a. \]

Note that if \( G_{d_1}(b) \geq 0 \), then \( G_{d_1}(d_2) \geq 0 \) on the interval \([a, b]\). If \( G_{d_1}(b) \leq 0 \), then there exists \( c(d_1) \in [a, b] \), such that \( G_{d_1}(d_2) \geq 0 \) for any \( d_2 \in [a, c(d_1)] \) and \( G_{d_1}(d_2) \leq 0 \) for any \( d_2 \in [c(d_1), b] \). Thus, we need to consider the following three cases in order to obtain the optimal ceded function \( f^* \).
Case 1. Suppose \( \frac{1}{\lambda} \ln(\lambda \int_0^\infty e^{\lambda x} S_X(x) dx + 1) - b + a \leq 0 \). In this case, \( G_{d_i}(b) \leq 0 \) for any \( d_i \in [0, a] \). Thus,

\[
v_{d_i}(d_2) = \begin{cases} 
  d_2 - a + d_1 - \pi_f, & a \leq d_2 \leq c(d_1), \\
  d_1, & c(d_1) \leq d_2 \leq b.
\end{cases}
\]

Note that

\[
v_{d_1}(c(d_1)) = c(d_1) - a + d_1 - \frac{1}{\lambda} \ln \left\{ \lambda \left[ \int_0^{d_1} e^{\lambda x} + \int_a^{c(d_1)} e^{\lambda(x-a+d_1)} + \int_b^{\infty} e^{\lambda(x-b+c(d_1)-a+d_1)} \right] S_X(x) dx + 1 \right\} 
= d_1 - G_{d_i}(c(d_1)) 
= d_1.
\]

It implies that \( v_{d_1}(d_2) \) is continuous in \([a, b]\). Then its first-order derivative is

\[
v'_{d_1}(d_2) = \begin{cases} 
  1 - \frac{P(d_1, d_2)}{Q(d_1, d_2)}, & a \leq d_2 \leq c(d_1), \\
  0, & c(d_1) \leq d_2 \leq b,
\end{cases}
\]

\[ \geq 0. \]

Hence, for any \( d_1 \in [0, a] \), the minimum of \( v_{d_1}(d_2) \) over \([a, b]\) can be attained at \( d_2 = a \). Thus,

\[
\min_{0 \leq d_1 \leq a, a \leq d_2 \leq b} v(d_1, d_2) = \min_{0 \leq d_1 \leq a} v(d_1, a).
\]

Next, we consider the function \( d_1 \to v(d_1, a) \) on \([0, a]\). Note that

\[
v(d_1, a) = d_1 - \frac{1}{\lambda} \ln \left\{ \lambda \left[ \int_0^{d_1} e^{\lambda x} + \int_b^{\infty} e^{\lambda(x-b+c(d_1)-a+d_1)} \right] S_X(x) dx + 1 \right\}.
\]

Obviously, the function \( v(d_1, a) \) is continuous in \( d_1 \), and its first-order derivative is equal to \( (38) \). Hence, \( \frac{d}{d_1} v(d_1, a) \geq 0 \). It implies that the minimum of \( v(d_1, a) \) over \([0, a]\) can be attained at \( d_1 = 0 \). Hence, by Theorem 3.2, we know that the optimal ceded loss function \( f^* \) to model (8) is \( h(x; d_1, d_2) \) with \( d_1 = 0 \) and \( d_2 = a \), that is

\[
f^*(x) = (x - b)_+
= \begin{cases} 
  0, & 0 \leq x < b, \\
  x - b, & x \geq b.
\end{cases}
\]

Case 2. Suppose \( \frac{1}{\lambda} \ln(\lambda \int_a^\infty e^{\lambda x} S_X(x) dx + 1) - b + a \leq 0 \). In this case, \( G_{d_i}(b) \geq 0 \) for any \( d_i \in [0, a] \). Hence,

\[
v_{d_i}(d_2) = v(d_1, d_2) = d_2 - a + d_1 - \pi_f.
\]

The first-order derivation of \( v_{d_1}(d_2) \) is

\[
v'_{d_1}(d_2) = 1 - \frac{P(d_1, d_2)}{Q(d_1, d_2)} \geq 0.
\]

Hence, for any \( d_1 \in [0, a] \), the minimum of \( v_{d_1}(d_2) \) over \([a, b]\) can be attained at \( d_2 = a \). Thus,

\[
\min_{0 \leq d_1 \leq a, a \leq d_2 \leq b} v(d_1, d_2) = \min_{0 \leq d_1 \leq a} v(d_1, a).
\]
Next, we consider the function \( d_1 \rightarrow v(d_1, a) \) on \([0, a]\). Note that
\[
v(d_1, a) = d_1 - \frac{1}{\lambda} \ln \left\{ \lambda \left[ \int_0^{d_1} e^{\lambda x} + \int_b^\infty e^{\lambda(x-b-d_1)} \right] S_X(x) dx + 1 \right\}
\]
is equal to (37). Hence, the optimal reinsurance treaty \( f^* \) is the same as (39).

Case 3. We can see that \( f^* \) in Case 1 is the same as \( f^* \) in Case 2. Following the arguments in Case 2 (iii) of the proof of Theorem 4.3, it is not hard to get that when
\[
\frac{1}{\lambda} \ln(\lambda \int_a^\infty e^{\lambda x} S_X(x) dx + 1) - b + a \leq 0 \leq \frac{1}{\lambda} \ln(\lambda \int_0^\infty e^{\lambda x} S_X(x) dx + 1) - b + a,
\]
f* in Case 3 is the same as f* in Case 1 and Case 2 and is equal to (39). Theorem 4.4 is then proved.

5. **Concluding remarks.** In this paper, in the presence of default risk, we have studied an optimal reinsurance model from the reinsurer’s perspective. The optimal reinsurance treaties are obtained in general. When the reinsurance principle is calculated by the expected value or exponential premium principles, the explicit expressions for the optimal reinsurance treaties are given, respectively. It turns out that the forms of the optimal reinsurance treaties are quite different from those in the absence of default risk. It also turns out that the forms of the optimal reinsurance treaties are quite different from those from the insurer’s perspective.

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