THE X-CLASS AND ALMOST-INCREASING PERMUTATIONS

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Abstract. In this paper we give a bijection between the class of permutations that can be drawn on an X-shape and a certain set of permutations that appears in Knuth [4] in connection to sorting algorithms. A natural generalization of this set leads us to the definition of almost-increasing permutations, which is a one-parameter family of permutations that can be characterized in terms of forbidden patterns. We find generating functions for almost-increasing permutations by using their cycle structure to map them to colored Motzkin paths. We also give refined enumerations with respect to the number of cycles, fixed points, exceedances, and inversions.

1. Introduction and background

Permutations can sometimes describe the geometry of certain configurations of points in the plane. Picture classes, introduced by Waton [5], are sets of permutations that can be drawn on given shapes in the plane. These classes are closed under pattern containment, and sometimes can be characterized as the set of permutations that avoid a finite list of patterns. For example, it is shown in [5] that a permutation can be drawn on a circle if and only if it avoids a fixed list of 16 patterns. Another class that is studied in [5] is the X-class, which is the set of permutations that can be drawn on a pair of crossing lines forming an X-shape (see Section 3 for a precise definition). Waton characterized them in terms of forbidden patterns, and showed that their counting sequence has a rational generating function.

Interestingly, the same generating function also appears in Knuth [4], where it enumerates permutations $\pi$ such that for every $i$ there is at most one $j \leq i$ with $\pi(j) > i$. These permutations have a characterization in terms of forbidden patterns as well. The problem of finding a bijection between Knuth’s permutations and the X-class was proposed in the open problem session at the conference on Permutation Patterns held in 2006 in Reykjavik. The present paper originates from this open problem. The sought bijection between these two sets of permutations is given in Section 4 using an encoding of permutations as words.

Knuth’s permutations can be regarded intuitively as permutations whose entries are not too far from those of the identity permutation. The definition admits a natural generalization that depends on a parameter $k$, where Knuth’s permutations are simply the case $k = 1$. We call the permutations in this family almost-increasing permutations. Their precise definition is given in Section 2 where we also characterize them in terms of pattern avoidance.

The enumeration of almost-increasing permutations, for any value of the parameter $k$, is done in Section 5. Our method consists in using what we call the cycle diagrams of permutations to...
map them to certain colored Motzkin paths. In Section 6 we obtain multivariate generating functions that keep track of the number of cycles, fixed points, excedances, and inversions. When the parameter $k$ goes to infinity, our generating functions for almost-increasing permutations become continued fractions enumerating all permutations by these statistics. We also restrict our results to involutions.

Here is some notation about pattern avoidance that will be used in the paper. We denote by $S_n$ be the set of permutations of $\{1, 2, \ldots, n\}$. Using the standard notion of pattern avoidance, we say that $\pi \in S_n$ avoids $\sigma \in S_m$ if there is no subsequence of $\pi$ whose entries are in the same relative order as the entries of $\sigma$. $S_n(\sigma)$ denotes the set of permutations in $S_n$ that avoid the pattern $\sigma$. More generally, given $\sigma_i \in S_m_i$ for $1 \leq i \leq k$, $S_n(\sigma_1, \sigma_2, \ldots, \sigma_k)$ is the set of permutations that avoid all the patterns $\sigma_1, \sigma_2, \ldots, \sigma_k$.

We will use the terms fixed point, excedance, and deficiency to denote an entry of a permutation $\pi$ such that $\pi(i) = i$, $\pi(i) > i$, or $\pi(i) < i$, respectively. Recall also that $(i, j)$ is an inversion of $\pi$ if $i < j$ and $\pi(i) > \pi(j)$. The number of fixed points, excedances, inversions, and cycles of $\pi$ will be denoted by $fp(\pi)$, $exc(\pi)$, $inv(\pi)$, and $cyc(\pi)$, respectively.

2. Almost-increasing permutations

**Definition 2.1.** For any fixed $k \geq 0$, the set of $k$-almost-increasing permutations is defined as

$$A_n^{(k)} = \{ \pi \in S_n : |\{ j : j \leq i \text{ and } \pi(j) > i \}| \leq k \text{ for every } i \}.$$ 

Note that $A_n^{(0)}$ contains only the identity permutation $12\ldots n$, whereas $A_n^{([n/2])} = S_n$. Intuitively, the smaller $k$ is, the closer the permutations in $A_n^{(k)}$ are to the increasing permutation. This is the reason for the name almost-increasing. It is useful to observe that, for any permutation $\pi$,

$$|\{ j : j \leq i \text{ and } \pi(j) > i \}| = |\{ j : j > i \text{ and } \pi(j) \leq i \}|.$$ 

This identity, which is illustrated in Figure 1, follows by noticing that both sides of the equality are equal to $i - |\{ j : j \leq i \text{ and } \pi(j) \leq i \}|$.

The set $A_n^{(1)} = \{ \pi \in S_n : \text{for every } i \text{ there is at most one } j \leq i \text{ with } \pi(j) > i \}$ appears in Knuth’s book [3] Section 5.4.8, Exercise 8 in connection to sorting algorithms. It has an interesting characterization in terms of pattern avoidance as

$$A_n^{(1)} = S_n(3412, 3421, 4312, 4321).$$

This result can be generalized to arbitrary $k$ as follows.

**Lemma 2.2.** Let $k \geq 0$, and let $\Sigma^{(k)} = \{ \sigma \in S_{2k+2} : \sigma_i > \sigma_j \text{ for every } i \leq k + 1 < j \}$. Then,

$$A_n^{(k)} = S_n(\Sigma^{(k)}).$$

**Proof.** If $\pi \in S_n \setminus A_n^{(k)}$, then there exist indices $i, j_1, j_2, \ldots, j_{k+1}$ such that $j_1 < j_2 < \ldots < j_{k+1} \leq i$ and $\pi(j_1), \pi(j_2), \ldots, \pi(j_{k+1}) > i$. By equation 1, this also implies that there are indices $\ell_1, \ell_2, \ldots, \ell_{k+1}$ such that $i < \ell_1 < \ell_2 < \ldots < \ell_{k+1}$ and $\pi(\ell_1), \pi(\ell_2), \ldots, \pi(\ell_{k+1}) \leq i$. But now $\pi(j_u) > \pi(\ell_v)$ for all $1 \leq u, v \leq k + 1$, so $\pi(j_1) \ldots \pi(j_{k+1})\pi(\ell_1)\ldots\pi(\ell_{k+1})$ forms an occurrence in $\pi$ of a pattern in $\Sigma^{(k)}$. 


To prove the reverse inclusion, assume that \( \pi \in S_n \) has an occurrence \( \pi(i_1)\pi(i_2)\ldots\pi(i_{2k+2}) \) (with \( i_1 < i_2 < \cdots < i_{2k+2} \)) of one of the patterns in \( \Sigma^{(k)} \). If \( \pi(i_1), \pi(i_2), \ldots, \pi(i_{k+1}) > i_{k+1} \), then \( \pi \not\in A_n^{(k)} \) and we are done. Otherwise, \( \pi(i_{k+2}), \pi(i_{k+3}), \ldots, \pi(i_{2k+2}) \leq i_{k+1} \), because of the shape of the patterns in \( \Sigma^{(k)} \). But then, again by (1), there must be \( k + 1 \) indices \( j_1, \ldots, j_{k+1} \leq i_{k+1} \) such that \( \pi(j_1), \ldots, \pi(j_{k+1}) > i_{k+1} \). Thus, \( \pi \not\in A_n^{(k)} \) in this case either. \( \square \)

3. The \( X \)-class

Consider two crossing lines in the plane with slopes 1 and \(-1\), forming an \( X \)-shape. Place \( n \) points anywhere on these lines, with no two of them having the same \( x \)- or \( y \)-coordinate, and label them \( 1, 2, \ldots, n \) by increasing \( y \)-coordinate. Reading the labels of the points by increasing \( x \)-coordinate determines a permutation. The set of permutations obtained in this way is called the \( X \)-class. This set was studied by Watson [5], who also showed that these are precisely those permutations that avoid the patterns 2143, 2413, 3142, and 3412. In other words, if we denote by \( X_n \) the set of permutations of length \( n \) in the \( X \)-class, we have that \( X_n = S_n(2143, 2413, 3142, 3412) \).

3.1. Enumeration of the \( X \)-class. We will show that \( X_n \) is in bijection with the set \( W_n \) of words of length \( n - 1 \) over the alphabet \( \{W, E, L, R\} \) not containing any occurrence (in consecutive positions) of \( LE \) or \( RW \), and always ending with a \( W \) or an \( E \).

It will be convenient to represent a permutation \( \pi \in S_n \) as an \( n \times n \) array that contains dots in positions \( (i, \pi(i)) \), for \( i = 1, 2, \ldots, n \) (see Figures 1 and 2). We use the convention that column numbers increase from left to right and row numbers increase from bottom to top. The following property of the permutations in the \( X \)-class will be useful to define the bijection.

**Lemma 3.1.** If \( \pi \in X_n \), then the array of \( \pi \) contains a dot in at least one of the four corners.

**Proof.** The statement of the lemma is equivalent to the fact that at least one of these four conditions holds: \( \pi(1) = 1, \pi(1) = n, \pi(n) = 1, \pi(n) = n \). Let us assume for contradiction
that none of them holds. Let \( i \) be the index such that \( \pi(i) = 1 \), and let \( j \) be such that \( \pi(j) = n \). We have that \( 1 < i, j < n \). If \( i < j \), the subsequence \( \pi(1)\pi(i)\pi(j)\pi(n) \) forms an occurrence of 2143 or 3142, depending on whether \( \pi(1) < \pi(n) \) or \( \pi(1) > \pi(n) \), respectively. Similarly, if \( j < i \), then \( \pi(1)\pi(j)\pi(i)\pi(n) \) forms an occurrence of 2413 or 3412. In any case, this contradicts the fact that \( \pi \in \mathcal{X}_n \).

\( \square \)

If the row and column containing a corner dot in the array of \( \pi \in \mathcal{X}_n \) are removed, then the resulting array determines a permutation in \( \mathcal{X}_{n-1} \). Furthermore, if we take the array of any permutation in \( \mathcal{X}_{n-1} \) and we add a row and a column intersecting at a corner, with a dot in that corner, then we obtain the array of a permutation in \( \mathcal{X}_n \). The reason is that the new corner dot cannot be part of an occurrence of any of the patterns 2143, 2413, 3142, 3412.

Next we describe the bijection \( f : \mathcal{W}_n \rightarrow \mathcal{X}_n \). Let a word \( w \in \mathcal{W}_n \) be given. Starting from an empty \( n \times n \) array, read \( w \) from left to right, and for each letter place a dot in an array according to the following rule. Every time an \( L \) (resp. \( R, W, E \)) is read, place a dot in the lower-left (resp. lower-right, upper-left, upper-right) corner of the unshaded region, and shade the row and column of the new dot. After reading the whole word \( w \), only one square remains unshaded. Place a dot in the unshaded square. We define \( f(w) \) to be the permutation whose array is constructed in this way. See the right side of Figure 4 for an example.

The inverse of \( f \) can be easily described. We start with the array of a permutation \( \pi \in \mathcal{X}_n \), with no shaded squares. We will successively shade some squares as we write a word \( w \in \mathcal{W}_n \). At each step, the permutation given by the dots in the unshaded area belongs to the \( X \)-class, so by Lemma 3.1, one of its corners must contain a dot. If two of the corners (which are necessarily opposite) contain a dot, we choose the dot in the upper corner. If this dot is in the lower-left (resp. lower-right, upper-left, upper-right) corner of the unshaded region, we append an \( L \) (resp. \( R, W, E \)) to the word and we shade the row and column containing the dot. We repeat this process until the unshaded area contains only one square.

It is clear that the procedures described in the above two paragraphs are the inverse of each other, so \( f \) is a bijection between \( \mathcal{W}_n \) and \( \mathcal{X}_n \). Enumerating \( \mathcal{W}_n \) is straightforward, as the following proposition shows.
Proposition 3.2. The generating function for the sequence $b_n = |W_n| = |X_n|$ is
\[
1 + \sum_{n \geq 1} b_n x^n = \frac{1 - 3x}{1 - 4x + 2x^2}.
\]

Proof. We show that the sequence satisfies the recurrence
\[
b_n = 4b_{n-1} - 2b_{n-2}
\]
for $n \geq 3$. To see this, notice that to construct a word in $W_n$ we have 4 choices for the first letter, which can be followed by any word in $W_{n-1}$, except for the case where the first two letters create an occurrence of either $LE$ or $RW$, followed by a word in $W_{n-2}$. Using that the initial terms are $b_1 = 1$ and $b_2 = 1$, we get the generating function above. \qed

3.2. A bijection with lattice paths. The $X$-class is also equinumerous to the following family of paths on the integer lattice $\mathbb{Z}^2$. Let $P_n$ be the set of paths from $(0,0)$ to $(2n-2,0)$ with steps $U = (1,1)$ and $D = (1,-1)$ whose second coordinate satisfies $|y| \leq 3$. We next describe a simple bijection between $W_n$ and $P_n$.

Given a word $w \in W_n$, we divide it into blocks by inserting a divider after every $E$ and after every $W$. From each block we construct a piece of the path returning to the $x$-axis according to the following rules.

- If the block contains only one letter, the corresponding piece of path is $UD$ if the letter is an $E$, and $DU$ if the letter is a $W$.
- Otherwise, the block ends in $RE$ or $LW$, preceded by a sequence of $R$s and $L$s.
  - If the block ends in $RE$, start the corresponding piece of path with $UU$, followed by $UD$ (resp. $DU$) for each $R$ (resp. $L$) in the sequence, from left to right, not including the $R$ immediately before the $E$, and end with $DD$.
  - If the block ends in $LW$, start the piece of path with $DD$, followed by $DU$ (resp. $UD$) for each $R$ (resp. $L$) in the sequence, from left to right, not including the $L$ immediately before the $W$, and end with $UU$.

For example, if $w = WRLWERLRE$, the path is $DUDDUUUUDUUUDDUD$, which is drawn in Figure 3. This bijection has additional properties. Recall that return of a path is a step whose right endpoint is on the $x$-axis (for example, the path in Figure 3 has four returns). In our construction, $Es$ in the word are mapped to returns from above in the path, and $Ws$ are mapped to returns from below. Also, $Rs$ not followed by an $E$ correspond to points in the path with $|y| = 3$.

\[\text{Figure 3. An element of } P_{10}.\]
It is an easy exercise to check directly that the generating function for these paths is given by equation \((2)\) as well.

4. From the \(X\)-class to almost-increasing permutations

In this section we show that there is a bijection between the \(X\)-class and 1-almost-increasing permutations. In terms of pattern avoidance, we will prove that \(|\mathcal{S}_n(2143, 2413, 3142, 3412)| = |\mathcal{S}_n(3412, 3421, 4312, 4321)|\) for all \(n\). This equality follows from \([4, 5]\), but no bijective proof of it was known.

In Section 3.1 we described a bijection \(f : \mathcal{W}_n \rightarrow \mathcal{X}_n\). Next we present a bijection \(\varphi\) between \(\mathcal{A}_n^{(1)}\) and \(\mathcal{W}_n\) which, composed with \(f\), will give the desired bijection between \(\mathcal{A}_n^{(1)}\) and \(\mathcal{X}_n\).

To construct \(\varphi(\pi)\) for a given \(\pi \in \mathcal{A}_n^{(1)}\), we read its entries \(\pi(1), \pi(2), \ldots, \pi(n - 1)\) from left to right, and write a sequence of letters in the alphabet \(\{W, E, L, R\}\) as follows.

Assume that \(\pi(i)\) is the current entry. Let \(m = |\{j > i : \pi(j) < \pi(i)\}|\). We consider three cases:

- If \(m = 0\), write a \(W\).
- If \(m = 1\), write an \(E\).
- If \(m \geq 2\), read the next \(m - 1\) entries, namely \(\pi(j)\) for \(j = i + 1, \ldots, i + m - 1\). For each entry \(\pi(j)\), write an \(R\) if \(\pi(j) = j\) and write an \(L\) otherwise. Next, if the last letter (the one corresponding to \(\pi(i + m - 1)\)) was an \(R\) (resp. an \(L\)), write an \(E\) (resp. a \(W\)) right after it.

The entry to the right of the last one that was read becomes the new current entry. We repeat this process until entry \(\pi(n - 1)\) has been read.

For example, let \(\pi = (5, 2, 1, 4, 3, 7, 6, 10, 8, 13, 11, 9, 12) \in \mathcal{A}_n^{(1)}\). We start with \(\pi(1) = 5\), so \(m = 4\). The next \(m - 1\) entries are \(\pi(2) = 2\), \(\pi(3) = 1\), and \(\pi(4) = 4\), so the first letters of \(\varphi(\pi)\) are \(RLR\), followed by an \(E\). The next entry is \(3\), and now \(m = 0\), so we write a \(W\). Next we look at the entry \(7\), so \(m = 1\) and we write an \(E\). For the entry \(6\), \(m = 0\) again so we write a \(W\). The next entry is \(10\), which means that \(m = 2\), so we have to look at \(\pi(9) = 8\) and write an \(L\), followed by a \(W\) (see Figure 4). We now read \(13\), so \(m = 3\), and we have to read the entries \(\pi(11) = 11\), \(\pi(12) = 9\), and write \(RL\), followed by a \(W\). At this point, all the entries up to \(\pi(n - 1)\) have been read, so the word obtained is \(\varphi(\pi) = RLREWELWRLW\), and \(f(\varphi(\pi)) = (2, 12, 10, 4, 9, 6, 8, 7, 5, 11, 13, 3, 1)\). As another example, if \(\pi = (2, 6, 1, 4, 5, 8, 7, 3, 10, 9, 11)\), then \(\varphi(\pi) = ELRREREWWE\), and \(f(\varphi(\pi)) = (1, 8, 6, 5, 7, 9, 4, 10, 3, 2, 11)\).

**Proposition 4.1.** The map \(\varphi\) described above is a bijection between \(\mathcal{A}_n^{(1)}\) and \(\mathcal{W}_n\).

*Proof.* Consider the following recursive construction of permutations in \(\mathcal{A}_n^{(1)}\). Let \(\pi \in \mathcal{A}_n^{(1)}\), represented as an \(n \times n\) array of dots as described above.

If \(\pi(1) = 1\) (which corresponds to \(m = 0\) in the bijection), deleting the first row and the first column of the array we obtain a permutation \(\pi' \in \mathcal{A}_{n-1}^{(1)}\), and every permutation in \(\mathcal{A}_{n-1}^{(1)}\) can be obtained in this way. In this case, \(\varphi\) maps \(\pi\) to the word \(W\varphi(\pi')\).
If \( \pi(1) = 2 \) (which corresponds to \( m = 1 \) in the bijection), again deleting row 1 and column 2 in the array, we obtain the array of a permutation \( \pi' \in A_{n-1}^{(1)} \), and every permutation in \( A_{n-1}^{(1)} \) can be obtained in this way. In this case, \( \varphi \) maps \( \pi \) to the word \( E\varphi(\pi') \).

If \( \pi(1) \geq 3 \), we let \( m = \pi(1) - 1 \). By definition of \( A_n^{(1)} \), this forces \( \pi(2) \leq 2, \pi(3) \leq 3, \ldots, \pi(m) \leq m \). Besides, deleting the rows and columns of the dots in the array corresponding to the first \( m \) entries of \( \pi \), we obtain the array of a permutation \( \pi' \in A_{n-m}^{(1)} \), and every permutation in \( A_{n-m}^{(1)} \) can be obtained in this way. Note that there are two choices for \( \pi(2) \), namely \( \{1,2\} \), two choices for \( \pi(3) \), namely \( \{1,2,3\} \setminus \{\pi(2)\} \), and in general two choices for each \( \pi(j) \) for \( k = 2, \ldots, m \), one being \( m \) and the other being the value in \( \{1, \ldots, m-1\} \) that is not attained by any of \( \pi(2), \ldots, \pi(m-1) \). These choices determine whether each one of the first \( m-1 \) entries of \( \varphi(\pi) \) is an \( R \) or an \( L \). The \( m \)th entry, an \( E \) or a \( W \), is then determined by the condition that the word must avoid occurrences of \( LE \) or \( RW \). The remaining \( n-m \) letters of \( \varphi(\pi) \) are just \( \varphi(\pi') \). \( \square \)

The above recursive description of \( A_n^{(1)} \) shows that the numbers \( a_n^{(1)} = |A_n^{(1)}| \) satisfy the recurrence

\[
a_n^{(1)} = 2a_{n-1}^{(1)} + \sum_{m=2}^{n-1} 2^{m-1}a_{n-m}^{(1)}
\]

for \( n \geq 2 \), with initial condition \( a_1 = 1 \). This recurrence is equivalent to (3). The above proof implies that the sets \( W_n \) admit a parallel recursive construction: any \( w \in W_n \) can be written as either a \( W \) or an \( E \) followed by a word in \( W_{n-1} \), or a sequence of \( m-1 \) (with \( m \geq 2 \)) \( R \)'s and \( L \)'s followed by the letter in \( \{W,E\} \) that does not create an occurrence of \( LE \) or \( RW \), followed by a word in \( W_{n-m} \).

**5. Enumeration of almost-increasing permutations**

For \( k \geq 0, n \geq 1 \), let \( a_n^{(k)} = |A_n^{(k)}| \), and let \( a_0^{(k)} = 1 \) by convention. For each \( k \), define the generating function

\[
A^{(k)}(x) = \sum_{n \geq 0} a_n^{(k)} x^n.
\]
An expression for $A^{(1)}(x)$ follows from [4] or, alternatively, from Proposition 3.2 and the bijection in Section 4.

**Corollary 5.1.**

$$A^{(1)}(x) = \frac{1 - 3x}{1 - 4x + 2x^2}.$$  

In this section we generalize this result by finding simple expressions for $A^{(k)}(x)$ for any $k$. As we will see, all these generating functions are rational. Similar expressions for $A^{(k)}(x)$ have been found by Atkinson [1] using inclusion-exclusion to obtain recurrence relations for the numbers $a^{(k)}_n$.

5.1. **A map to Motzkin paths.** Recall that a Motzkin path of length $n$ is a lattice path from $(0,0)$ to $(n,0)$ with steps $U = (1,1)$, $D = (1,-1)$, $L = (1,0)$ with the condition that it never goes below the $x$-axis. The height of a step is the $y$-coordinate of its right endpoint. The height of a path is the maximum height of any of its steps. A key ingredient in the enumeration of almost-increasing permutations will be a map from permutations to Motzkin paths.

Given the representation of a permutation $\pi$ as an $n \times n$ array, we will depict its cycles in the following way. Take any index $i_1 \in \{1,2,\ldots,n\}$. If $i_1$ is a fixed point, then it forms a cycle of length 1. Otherwise, let $i_2 = \pi(i_1)$, and draw a vertical line in the array from the center of the square $(i_1,i_1)$ to the center of the square $(i_1,i_2)$, followed by a horizontal line from $(i_1,i_2)$ to $(i_2,i_2)$. Now we let $i_3 = \pi(i_2)$ and repeat the process, until we eventually return to $(i_1,i_1)$ after going through all the elements of the cycle containing $i_1$. We do this for each cycle of $\pi$, obtaining a picture like the example in Figure 5. We call this the **cycle diagram** of $\pi$.

![Figure 5](image.png)

**Figure 5.** The cycle diagram of $\pi = (5,7,2,4,3,8,1,6,9,12,10,11)$.  

The squares in the diagonal of the cycle diagram, namely, those of the form $(i,i)$, can be classified into five types:

- a **fixed point**, 
- a **fixed point**, 
- a **fixed point**, 
- a **fixed point**, 
- a **fixed point**.
The sequence of types of the squares in the diagonal of the cycle diagram of \( \pi \), read from bottom-left to top-right, will be called the **diagonal sequence** of \( \pi \), and denoted \( D(\pi) \). Note that \( D(\pi) \in \{s, L, \uparrow, \downarrow \}^n \). We map the permutation to a Motzkin path of length \( n \) by considering its diagonal sequence and drawing an up step \( U \) for each \( \uparrow \), a down step \( D \) for each \( \downarrow \), and a level step \( L \) for each \( s \), \( \uparrow \) or \( \downarrow \). Let us denote \( \theta(\pi) \) the path defined in this way. For example, if \( \pi \) is the permutation in Figure 5, \( \theta(\pi) \) is the path in Figure 6.

**Figure 6.** The Motzkin path \( \theta(\pi) \), where \( \pi = (5, 7, 2, 4, 3, 8, 1, 6, 9, 12, 10, 11) \).

**Proposition 5.2.** Let \( \pi \in S_n \). Then \( \pi \in A_n^{(k)} \) if and only if the height of \( \theta(\pi) \) is at most \( k \).

**Proof.** For \( 1 \leq i \leq n \), let \( h_i = |\{ j : j \leq i \text{ and } \pi(j) > i \}| \). The key observation is that the height of the \( i \)-th step of \( \theta(\pi) \) is precisely \( h_i \). This can be checked by induction on \( i \), since \( h_i - h_{i-1} \) is 1, -1, or 0 depending on whether there is an opening bracket, a closing bracket, or any other symbol in position \((i, i)\), respectively. \( \square \)

Despite the above useful property, one disadvantage of the map \( \theta \) is that it is not injective. To fix this problem, we will modify the map using colored steps in the Motzkin paths.

### 5.2. Bijections to colored Motzkin paths

**Proposition 5.3.** There is a bijection \( \psi_1 \) between \( A_n^{(1)} \) and colored Motzkin paths of length \( n \) and height at most 1 where the \( L \) steps at height 1 can receive three colors.

**Proof.** For \( \pi \in A_n^{(1)} \), Proposition 5.2 implies that in \( D(\pi) \), the positions of opening and closing brackets alternate, starting with an opening bracket and ending with a closing bracket. From the construction of the cycle diagram, one sees that in the intervals between an opening and a closing bracket, any combination of the other three types of squares (fixed point, upper bounce, and lower bounce) is possible, while in the intervals not enclosed by brackets, only fixed points can occur. Besides, any sequence in \( \{s, \uparrow, \downarrow, \cdot, \cdot \} \) satisfying these conditions uniquely determines the permutation \( \pi \in A_n^{(1)} \) that it came from.

Define \( \psi_1 \) to be a variation of the map \( \theta \) where each level step at height 1 of the Motzkin path receives one of three different colors, depending on whether the corresponding element in the diagonal sequence is a fixed point, an upper bounce, or a lower bounce. Then \( \psi_1 \) is the desired bijection. \( \square \)
Enumerating these colored Motzkin paths (see [2]) we recover the expression for $A^{(1)}(x)$:

$$A^{(1)}(x) = \frac{1}{1 - x - x^2} = \frac{1 - 3x}{1 - 4x + 2x^2}. $$

The above idea can be generalized to enumerate $A^{(k)}_n$ for any $k$. By Proposition 5.2 we know that if $\pi \in A^{(k)}_n$, then the Motzkin path $\theta(\pi)$ has height at most $k$. A variation of the map $\theta$ will produce a bijection between $A^{(k)}_n$ and certain colored Motzkin paths.

**Proposition 5.4.** There is a bijection $\psi_k$ between $A^{(k)}_n$ and colored Motzkin paths of length $n$ and height at most $k$, where for each $h$,

- each $L$ step at height $h$ receives one of $2h + 1$ possible colors,
- each $U$ step at height $h$ receives one of $h$ possible colors, and
- each $D$ step at height $h - 1$ receives one of $h$ possible colors.

**Proof.** Given a permutation $\pi$, the underlying uncolored Motzkin path that $\psi_k$ maps it to is just $\theta(\pi)$. It remains to show how the steps are colored. For each entry in the diagonal sequence of $\pi$, define its height to be the height of the corresponding step in $\theta(\pi)$. Let $h_i$ be the height of the $i$-th entry. Let us first consider entries that are not opening or closing brackets, that is, those that correspond to $L$ steps. It is obvious that such entries at height 0 can only be fixed points. Such entries at height one or more can be either fixed points, upper bounces, or lower bounces. However, the map $\pi \mapsto D(\pi)$ is not injective outside of $A^{(1)}_n$.

For any given diagonal sequence $\Delta = D(\pi)$, let us analyze the set $\{\pi' \in S_n \mid D(\pi') = \Delta\}$. Any permutation $\pi'$ with $D(\pi') = \Delta$ can be obtained in the following way. Place the entries of $\Delta$ on the diagonal of an $n \times n$ array. Think of each symbol $\cdot$, $\cdot$, $\cdot$, $\cdot$ as a gadget with a vertical and a horizontal ray that can be extended until they intersect another ray. Read these symbols from bottom-left to top-right and proceed as follows.

- Every time a $\cdot$ is read, just place a dot (a fixed point) there.
- Every time a $\cdot$ is read, it creates a new open vertical ray in its column and a new open horizontal ray in its row.
- Every time a $\cdot$ is read, take any of the open vertical rays coming from the symbols read so far, extend it upward until it intersects the leftward extension of the horizontal ray of the $\cdot$, and place a dot in the intersection. The intersected vertical ray becomes closed after this, but the $\cdot$ creates a new open vertical ray in its column (see Figure 7).
- Every time a $\cdot$ is read, proceed similarly with any of the horizontal rays that are open at that time.
- Every time a $\cdot$ is read, take any of the open vertical rays and any of the open horizontal rays, extend them until they intersect the two extended rays of the $\cdot$, and place a dot in each of the two intersections.

The placed dots determine a permutation $\pi'$ with diagonal sequence $\Delta$.

To see how many permutations $\pi'$ satisfy $D(\pi') = \Delta$, note that when a $\cdot$ or $\cdot$ is seen in position $(i, i)$, the number of vertical (equivalently, horizontal) rays that are open at that
time equals $h_i$. Thus, the number of possibilities for which ray to close is the height of the corresponding $L$ step in the path. Similarly, when a $\Uparrow$ is seen in position $(i,i)$, the number of vertical (equivalently, horizontal) rays that are open at that time is equal to $h_i + 1$. So, the number of possibilities for which rays to close is $(h_i + 1)^2$, where $h_i$ is the height of the corresponding $D$ step in the path.

This argument determines how many colors we need for each step of the Motzkin path in order to obtain a bijection. For each $h \geq 0$, level steps at height $h$ of $\theta(\pi)$ can receive one of $2h + 1$ colors, corresponding to the $h$ possibilities of which rays to close when the symbol is a $\Uparrow$, plus the $h$ possibilities when the symbol is a $\Downarrow$, plus the case where the symbol is a $\Rightarrow$. For each $h \geq 1$, down steps at height $h - 1$ can receive one of $h^2$ colors, corresponding to the possibilities of which rays to close when a $\Uparrow$ is read. Instead of using $h^2$ colors for down steps at height $h - 1$, another equivalent (and more symmetric) way to obtain the $h^2$ factor is by coloring up steps at height $h$ with one of $h$ possible colors and down steps at height $h - 1$ with one of $h$ possible colors. This is the coloring that we use to define the bijection $\psi_k$.

The description of the inverse map $\psi_k^{-1}$ is clear. Given a colored Motzkin path, read its steps from left to right while building the array of a permutation from the lower-left to the upper-right corner. For each step of the path, place a symbol in the diagonal of the array accordingly, that is,

- if the step is a $U$, put a $\Uparrow$, creating an open vertical ray and an open horizontal ray;
- if the step is a $D$, put a $\Downarrow$, and use the colors of that $D$ and its matching $U$ to determine which horizontal and which vertical ray to close, placing dots where the closed rays intersect the extended rays of the $\Uparrow$;
- if the step is a $L$, use its color to determine whether to put a $\Uparrow$, a $\Downarrow$, or a $\Rightarrow$, and in the last two cases, also to determine which rays to close and where to place a dot.

This bijection reduces the enumeration $\mathcal{A}^{(k)}_n$ to finding a generating function for colored Motzkin paths with bounded height.
Theorem 5.5. Let $k \geq 0$. The generating function for the numbers $a_n^{(k)}$ is

$$A^{(k)}(x) = \frac{1}{1 - x - \frac{x^2}{1 - 3x - \frac{4x^2}{1 - 5x - \frac{9x^2}{1 - 11x + \frac{k^2 x^2}{1 - (2k - 1)x}}}}}. $$

Proof. By Proposition 5.4, $A^{(k)}(x)$ is the generating function for weighed Motzkin paths where $L$ steps at height $h$ have weight $2h + 1$, and $U$ steps at height $h$ and $D$ steps at height $h - 1$ have weight $h$. Obtaining the generating function is now a straightforward application of the tools from [2].

For small values of $k$, the expressions of $A^{(k)}(x)$ as a quotient of polynomials are

$$A^{(2)}(x) = \frac{1 - 8x + 11x^2}{1 - 9x + 18x^2 - 6x^3},$$

$$A^{(3)}(x) = \frac{1 - 15x + 58x^2 - 50x^3}{1 - 16x + 72x^2 - 96x^3 + 24x^4},$$

$$A^{(4)}(x) = \frac{1 - 24x + 177x^2 - 444x^3 + 274x^4}{1 - 25x + 200x^2 - 600x^3 + 600x^4 - 120x^5}.$$

These results agree with [1], where Atkinson determines the coefficients of these polynomials from the recurrence relation satisfied by the $a_n^{(k)}$.

It is worth mentioning that $\psi_k$ can naturally be extended to a bijection between $S_n$ and colored Motzkin paths of length $n$ with no height restriction, where the possible colors of the steps at each height are given by the same rules as in Proposition 5.4.

6. Statistics on almost-increasing permutations

Having a bijection $\psi_k$ between almost-increasing permutations and colored Motzkin paths enables us to study the distribution of some statistics on almost-increasing permutations. The following is a refinement of Theorem 5.5 by considering the number of cycles, the number of fixed points, and the number of excedances.

Theorem 6.1. Let $k \geq 0$, and let

$$F^{(k)}(t, u, v, x) = \sum_{c, i, j, n \geq 0} |\{\pi \in A_n^{(k)} : \text{cyc}(\pi) = c, \text{fp}(\pi) = i, \text{exc}(\pi) = j\}| \, t^c u^i v^j x^n.$$
Then \( F^{(k)}(t, u, v, x) = \)

\[
\frac{1}{1 - tux - (1 + v + tu)x - \frac{2(1 + v)tx^2}{1 - (2(1 + v) + tu)x - \frac{3(2 + t)vx^2}{1 - (2(1 + v) + tu)x - \frac{k(k - 1 + t)vx^2}{1 - (k(1 + v) + tu)x}}}.
\]

**Proof.** To calculate the contribution of each step of the colored Motzkin path to the number of cycles, fixed points, and excedances of the permutation, we think of the array of the permutation as being built as the steps of the path are read from left to right, using the description of \( \psi_{k}^{-1} \). Every time a cycle, fixed point, or excedance is created, this will be reflected in the generating function. We begin by justifying that the contribution of a level step at height \( h \) in the generating function is \( h(1 + v) + tu \). Recall that level steps of the Motzkin path correspond to \( \bullet \), \( \bigcirc \), and \( \bigcirc \bullet \) in the diagonal sequence of the permutation. A level step at height \( h \) can be receive \( 2h + 1 \) colors. One of these colors indicates a fixed point in the permutation, which contributes \( tu \) to the generating function. Of the remaining \( 2h \) colors, half of them come from a \( \bigcirc \), which creates an excedance in the permutation, while the other half come from a \( \bigcirc \bullet \), which produces a deficiency. This explains the contribution \( h(1 + v) \).

![Figure 8](image)

**Figure 8.** Two different ways of closing the rays with a closing bracket: completing a cycle (center) or leaving it incomplete (right).

Next we show that the joint contribution of a \( U \) at height \( h \) and the matching \( D \) at height \( h - 1 \) is \( h(h - 1 + t)vx^2 = (h^2 - h + ht)v \). Recall from the construction of \( \psi_k \) that the \( h^2 \) possible ways of coloring this pair of steps correspond to the choices among the \( h \) vertical and the \( h \) horizontal rays that can be closed when the symbol \( \bigcirc \) appears in the diagonal sequence. When building the array of the permutation from the colored Motzkin path, these open rays can be thought of as “incomplete” cycles. For each open vertical ray, there is a unique open horizontal ray that belongs to the same cycle (see Figure 8). Closing these two rays simultaneously completes the cycle. Closing two rays that belong to different incomplete cycles merges them into one cycle; this decreases by one the number of incomplete cycles, but does not complete any cycle. Therefore, of the \( h^2 \) possible ways of choosing the pair of
rays to close, exactly \( h \) of them complete a cycle. This justifies the factor \( h^2 - h + ht \). The factor \( v \) is explained by the fact that the dot placed at the intersection of the closed vertical ray with the horizontal ray of the \( \Phi \) produces an excedance.

Another statistic whose distribution we can obtain is the number of inversions. While we have not been able to use our method to keep track of inversions and cycles simultaneously, the following result gives the joint distribution of the number of inversions, the number of fixed points, and the number of excedances. We use the notation \([k]_q = 1 + q + q^2 + \cdots + q^{k-1}\).

**Theorem 6.2.** Let \( k \geq 0 \), and let

\[
G^{(k)}(q, u, v, x) = \sum_{r,i,j,n \geq 0} |\{\pi \in A^{(k)}_n : \text{inv}(\pi) = r, \text{fp}(\pi) = i, \text{exc}(\pi) = j\}| q^r u^i v^j x^n.
\]

Then \( G^{(k)}(q, u, v, x) = \frac{1 - ux - vqx^2}{1 - ((1 + v)q + uq^2)x - vq^3(1 + q)^2x^2} \).

**Proof.** The idea of this proof is similar to that of Theorem 6.1. We think of the array of the permutation \( \pi = \psi_k^{-1}(M) \) as being built as we go through the steps of the colored Motzkin path \( M \). Each step determines what symbol to place in the diagonal of the array, which rays to open and close, and where to place dots. We will consider that a step of the path “creates” an inversion in the permutation when the changes in the array produced by that step force the inversion to occur.

First we show that the contribution of a level step at height \( h \) to the generating function is \( (1 + v)q^h[k]_q + uq^{2h} \). If this step corresponds to a fixed point \( i \) in the permutation, then \( i \) will form an inversion with the \( h \) dots that will be placed (further along the construction of \( \pi \)) in the currently open vertical rays, above and to the left of \((i,i)\), in the array, and also with the \( h \) dots that will be placed in the currently open horizontal rays, below and to the right of \((i,i)\). This situation is depicted in Figure 9(a). The contribution in this case is \( uq^{2h} \).

If the level step corresponds to a \( \mathcal{J} \) in the array, then we have \( h \) choices for which vertical ray to close and place a dot on. If the \( i \)-th from the left of these vertical rays is chosen, the new placed dot will create inversions with the dots that will later be placed on the \( i - 1 \) open vertical rays to the left of it, but not with the dots placed on the \( h - i \) open vertical rays to its right. Additionally, the new open vertical ray produced by the \( \mathcal{J} \) will create (once a dot is placed on it) inversions with the future dots on each one of the \( h \) currently open horizontal rays. See Figure 9(b) for an example with \( i = 2 \). Since \( 1 \leq i \leq h \), the contribution of a \( \mathcal{J} \) to the generating function is \( \sum_{i=1}^{h} vq^{i-1} = vq^h(1 + q + \cdots + q^{h-1}) \), where the \( v \) indicates that the new placed dot is an excedance. The case of a \( \mathcal{F} \) in the array is analogous, only that the new dot in this case does not produce an excedance, so the contribution is just \( q^h(1 + q + \cdots + q^{h-1}) \).

Next we show that the joint contribution of a \( \mathcal{U} \) at height \( h \) and the matching \( \mathcal{D} \) at height \( h - 1 \) is \( vq^{2h-1}[k]_q^2 \). A \( \mathcal{U} \) in the path corresponds to a \( \mathcal{J} \) in the diagonal of the array of \( \pi \).
Figure 9. Inversions created by a fixed point (a), an upper bounce (b), an opening bracket (c), and a closing bracket (d), all corresponding to steps at height $h = 3$ in the Motzkin path. The white dots represent entries that will be placed further along the construction of the permutation.

The new vertical ray emanating from this symbol forces an inversion with each of the $h - 1$ currently open horizontal rays (once dots are placed on them), and the new horizontal ray forces an inversion with each of the $h - 1$ currently open vertical rays. Additionally, the two future dots on the two new rays will also form an inversion pair. Figure 9(c) represents this situation. The contribution of a $\nearrow$ is therefore $q^{(h-1)+(h-1)+1} = q^{2h-1}$. Finally, a $D$ in the path corresponds to a $\nearrow$ in the array of $\pi$, which can close any one of the $h$ currently open vertical rays and any one of the $h$ currently open horizontal rays. If the $i$-th vertical ray from the left is closed, the placed dot will create inversions with the future dots on the $i - 1$ open vertical rays to its left. Similarly, if the $j$-th horizontal ray from the bottom is closed, the placed dot will create inversions with the future dots on the $j - 1$ open horizontal rays below it. Figure 9(d) shows an example with $i = 2$ and $j = 3$. Since $1 \leq i, j \leq h$, the contribution of a $\nearrow$ to the generating function is $\sum_{i,j=1}^{h} v q^{(i-1)+(j-1)} = v(1 + q + \ldots + q^{h-1})^2$, where again the $v$ indicates that one of the two placed dots is an excedance. □
Finally, it is not difficult to restrict our results to involutions. Let $I_n^{(k)} = \{ \pi \in A_n^{(k)} : \pi = \pi^{-1} \}$ be the set of $k$-almost-increasing involutions of length $n$.

**Corollary 6.3.** Let $k \geq 0$, and let

$$H^{(k)}(q,u,v,x) = \sum_{r,i,j,n \geq 0} |\{ \pi \in I_n^{(k)} : \text{inv}(\pi) = r, \text{fp}(\pi) = i, \text{exc}(\pi) = j \}| q^r u^i v^j x^n.$$ 

Then $H^{(k)}(q,u,v,x) = \frac{1}{1 - u x - \frac{v q x^2}{1 - u q^2 x - \frac{v q^3 (1 + q^2) x^2}{1 - u q^{2k-2} x - \frac{v q^{2k-1} (1 + q^2 + q^4 + \ldots + q^{2k-2}) x^2}{1 - u q^{2k} x}}}}.$

**Proof.** A permutation $\pi$ is an involution if and only if its array is symmetric with respect to the diagonal from the bottom-left to the top-right corner. This implies that its diagonal sequence $D(\pi)$ does not contain any $\uparrow$ or $\downarrow$ symbols, and also that every $\uparrow$ in the array closes a vertical and a horizontal ray that are symmetric with respect to the diagonal. This allows us to restrict $\psi_k$ to a bijection between $I_n^{(k)}$ and colored Motzkin paths of length $n$ and height at most $k$, where $L$ and $U$ steps can only receive one color, and $D$ steps at height $h - 1$ receive one of $h$ possible colors, corresponding to the $h$ choices of which rays to close when a $\uparrow$ is placed in the diagonal, since the choice of horizontal ray determines the choice of vertical ray.

To find the generating function for involutions with respect to the number of inversions, fixed points, and excedances, we argue as in the proof of Theorem 6.2, with the following two changes. First, we have to exclude the contributions of colored $L$ steps corresponding to $\uparrow$ or $\downarrow$ symbols. This kills the $(1 + v)q^h [h]_q$ terms, so an $L$ step at height $h$ only contributes $u q^{2h}$. Second, we have to take into account that when a $\uparrow$ closes the $i$-th open vertical ray from the left, it must also close the $i$-th open horizontal ray from the bottom, therefore creating $2(i - 1)$ inversions. So the contribution of a $\uparrow$ is now $\sum_{i=1}^{h} v q^{2(i-1)}$. \(\square\)

We conclude by mentioning that if we let $k$ go to infinity, Theorem 6.1 gives a continued fraction expression for the generating function of all permutations with respect to the number of cycles, fixed points, and excedances. By Foata’s correspondence [3], this also provides the enumeration of permutations by the number of left-to-right minima and descents. Similarly, taking $k \to \infty$ in Theorem 6.2 (resp. Corollary 6.3), we get a continued fraction that enumerates all permutations (resp. all involutions) by the inversion number and the number of fixed points and excedances.

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