Steps towards Lattice Virasoro Algebras: su(1,1)

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Abstract

An explicit construction is presented for the action of the su(1,1)-subalgebra of the Virasoro algebra on path spaces for the c(2,q) minimal models. In the case of the Lee-Yang model, we show how this action already fixes the central charge of the full Virasoro algebra. For this case, we additionally construct a representation in terms of generators of the corresponding Temperley-Lieb algebra.

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1. Introduction

Characters of the Virasoro minimal models [BPZ] are now for a decade known to correspond to spectra of CTM (corner transfer matrix) hamiltonians for certain 2d statistical models [ABF]. Recently, there have been some advances in the algebraic understanding of that relation [JM]. However, the geometric meaning of the Virasoro algebra in these lattice models is still unknown. To achieve a better understanding of the latter, an explicit construction of the Virasoro operation on the corresponding lattice or, as a first step, on the underlying path space seems to be promising. A natural starting point for such a construction are the path spaces directly obtained from the Virasoro irreducible representation module structure [KR,KRV], which in the cases we consider here are isomorphic to the 1d CTM-configuration spaces of the corresponding statistical models [Ri,KR]. Some recent results on the relation of statistical models and Virasoro character identities can be found e.g. in refs. [Be,BM,BLS,FQ,GG,WP]; different approaches to lattice Kac-Moody and Virasoro algebras have been considered by refs. [FG,FV] and [CE,IT,KS].

Some importance of path space structures also arises from the algebraic approach to conformal field theory: It provides a natural structure for the computation of fusion rules [Re1], the explicit construction of observable and field algebras [Sc], and finally of quantum symmetries [Re2]. Such constructions have already been given for the Ising model [MS,Bö] and some WZW-models [FGV].

Here, we consider the next simplest cases of Virasoro minimal models, which are not contained in the unitary series (which would be somewhat more natural from the point of view of algebraic quantum field theory), but the nonunitary minimal series of the Virasoro algebra

\[ [L_m, L_n] = (n - m) L_{m+n} + \frac{c}{12} (n^3 - n) \delta_{m+n,0}, \]

with central charges \( c(2, 2K + 3) \), where

\[ c(2, 2K + 3) = -\frac{2K(6K + 5)}{2K + 3}. \]

The lowest weight representations of these models have lowest weights

\[ h = h^{(2,2K+3)}_{1,m} = -\frac{(m - 1)(2K + 2 - m)}{2(2K + 3)}, \quad 1 \leq m \leq K + 1. \]

Actually, the relative simplicity of these cases in comparison to the unitary ones lies in the simpler path space structure of the superselection sectors [KR,KRV] or, in other words, in the simpler structure of the annihilating ideal [FNO,FF,FS].
More specifically, the path representation spaces for the $c(2,2K+3)$-Virasoro minimal models are well known [JM,FNO,KR] to be generated by the following set of sequences $\tilde{m} = (m_j)_{j \geq 0}$ taking values in $\{0, \ldots, K\}$, ending at zero, having initial value $m_0$ and obeying an additional constraint which is reminiscent of the null state structure of the model [FNO]:

$$S(K, m_0) = \{ \tilde{m} \in \{0, \ldots, K\}^\mathbb{N} | m_j = 0 \text{ for } j \gg 0; m_j + m_{j+1} \leq K \forall j \}.$$  \hfill (1.4)

$S$ is usually described as a set of paths running on graphs with nodes carrying labels $m_j$. The path space $\mathcal{P}(K, m)$ of formal linear combinations of elements of $S(K, K + 1 - m)$ after completion becomes a pseudo Hilbert space (not requiring positive definiteness of the — therefore pseudo — scalar product) for any choice of a non-degenerate pseudo scalar product in which all generating paths are orthogonal. In the following, we will make use of that bilinear form $(\cdot, \cdot) \equiv \langle \cdot, \cdot \rangle_\mathcal{P}$ in which they become orthonormal. Equipped with $L_0$ operating in a way motivated by the structure of corner transfer matrix hamiltonians [Ba], namely diagonally in the path basis [FNO,KR] by

$$L_0 \tilde{m} = \left( \sum_{k \geq 0} km_k \right) \tilde{m},$$  \hfill (1.5)

it reproduces the correct Virasoro characters

$$q^{c_{-1,2K+3}} \chi_{1,m}^{(2,2K+3)}(q) = ch_{1,m}^{(2,2K+3)}(q) = q^{-h_{1,m}^{(2,2K+3)}} \Tr_{\mathcal{P}(K, m)} qL_0 = \Tr_{V(c,h)} qL_0,$$

which directly follows from the sum expressions of their characters [FNO,NRT]

$$ch_{1,m}^{(2,2K+3)}(q) = \prod_{l \equiv 0, \pm m \mod (2K+3)} (1 - q^l)^{-1} = \sum_{n_1, \ldots, n_K \geq 0} \frac{q^{N_1^2 + \cdots + N_K^2 + N_m + \cdots + N_K}}{(q)_{n_1} \cdots (q)_{n_K}}$$  \hfill (1.6)

for the superselection sector with lowest weight $h_{1,m}^{(2,2K+3)}$, where $(q)_n = (1 - q) \cdots (1 - q^n)$ and $N_i = \sum_{j=i}^{K} n_j$.

In the following, first steps towards an implementation of an explicit operation of the Virasoro algebra on these path spaces are given, starting from the simplest case: The
su(1, 1) subalgebra in the c(2,5)-model. For convenience, we restrict our considerations to the vacuum sector. For general sectors, the operation remains basically the same; they involve, however, some additional technicalities at the beginning of the sequence \([R \delta]\).

2. su(1,1) for the Lee Yang edge singularity c(2,5)

The paths \((m_j)\) \(\in \mathcal{P}(1, m)\) generating the c(2,5)-Hilbert space just consist of zeros and ones. Hence, the states of lowest energy \(E (L_0\)-eigenvalue) in both sectors of that theory are given by the sequences in table 1 – in agreement with the beginning of the two characters:

\[
\text{ch}_{1,1}(q) = 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + \ldots \quad \text{and} \quad \text{ch}_{1,2}(q) = 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + \ldots
\]

| E   | \(h_{1,1} = 0\)         | \(h_{1,2} = -1/5\)         |
|-----|-------------------------|---------------------------|
| 0   | (1,0,0,0,0,0,0,0,0,0,\ldots) | (0,0,0,0,0,0,0,0,0,0,\ldots) |
| 1   | (1,0,1,0,0,0,0,0,0,0,\ldots) | (0,1,0,0,0,0,0,0,0,0,\ldots) |
| 2   | (1,0,0,1,0,0,0,0,0,0,\ldots) | (0,0,1,0,0,0,0,0,0,0,\ldots) |
| 3   | (1,0,0,0,1,0,0,0,0,0,\ldots) | (0,0,0,1,0,0,0,0,0,0,\ldots) |
| 4   | (1,0,0,0,0,1,0,0,0,0,\ldots) | (0,0,0,0,1,0,0,0,0,0,\ldots) |
| 5   | (1,0,0,0,0,0,1,0,0,0,\ldots) | (0,0,0,0,0,1,0,0,0,0,\ldots) |
| 6   | (1,0,0,0,0,0,0,1,0,0,\ldots) | (0,0,0,0,0,0,1,0,0,0,\ldots) |
| ... | ...                     | ...                       |

Table 1: lowest states for \(c = -22/5\)

Therefore, the definition of \(L_0\) suggests that \(L_1\) – which we now want to construct – should basically shift a single 1 in the sequence to the right by one step, for example

\[
L_1(1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, \ldots) = \alpha_1(1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, \ldots) + \alpha_2(1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, \ldots) + \alpha_3(1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, \ldots)
\]

(the coefficient \(\alpha_2\) for the shift of the third 1 being zero due to the obstruction given by the fourth 1 — see eq. (1.4)) and \(L_{-1}\) should do the same job in the reverse direction. This will already ensure the two commutators

\[
[L_0, L_{\pm 1}] = \pm L_{\pm 1},
\]
to be fulfilled such that it only remains to adjust coefficients $\alpha$ in order to obtain the third commutator, $[L_{-1}, L_1] = 2L_0$, which will be our aim for the remaining part of this section.

In order to get a somewhat natural operation, we make use of the following assumption:

- $L_{\pm 1}$ should act only “locally” on the path, which means that it should be a linear combination of shifts of any single 1 in the sequence, weighted by coefficients $\alpha$ which only depend on the position of the shifted 1 and global quantities (as the number of 1’s to the left or the right of the shifted 1 in the sequence, and the total number of 1’s), but not on where exactly all the other 1’s are located in the sequence. In particular, it should not make any difference whether by a successive application of $L_n$-operators first some specific 1 and than another 1 is shifted or vice versa. Therefore, this requirement already ensures $[L_{-1}, L_1]$ to act diagonally on $\mathcal{P}$ in the path basis. Finally, the locality assumption ensures well-definedness of the Temperley-Lieb expressions in section 4, and is motivated by the relation of “path-locality” to locality in the underlying model configuration space.

Now, we necessarily have to fulfill a physical requirement:

- $L_{\pm 1}$ of course are not allowed to change the superselection sector; therefore, they have to keep the beginning of a path fixed.

For a convenient handling of special configurations in the path, we additionally require:

- If an obstruction against the shift of a 1 in the path occurs — which is a pattern of type $(\ldots, 1, 0, 1, \ldots)$ such that neither the left 1 is allowed to be shifted to the right nor the right 1 to the left — the two missing contributions to the commutator $[L_{-1}, L_1]$ should just cancel each other, such that its total value is still fixed to the correct value of $2L_0$.

This requirement can easily be fulfilled by the ansatz

$$
\langle (\ldots, 1_i, \ldots), L_{-1}(\ldots, 1_{i+1}, \ldots) \rangle \mathcal{P} \langle (\ldots, 1_{i+1}, \ldots), L_1(\ldots, 1_i, \ldots) \rangle \mathcal{P} = f(i - K_i, K_\infty), \tag{2.1}
$$

where $(\ldots, 1_{i(\pm 1)}, \ldots)$ is a sequence with a 1 at position $i(\pm 1)$, $K_i$ 1’s to the left of position $i$ and containing totally $K_\infty$ 1’s (note that due to the locality of the shift operations, they leave invariant all $K_i$’s evaluated on any path), and where $f$ is a polynomial function. Therefore, the effect of the commutator $[L_{-1}, L_1]$ on the 1 at position $i$ gives a diagonal contribution of

$$
\langle (\ldots, 1_i, \ldots), L_{-1}(\ldots, 1_{i+1}, \ldots) \rangle \mathcal{P} \langle (\ldots, 1_{i+1}, \ldots), L_1(\ldots, 1_i, \ldots) \rangle \mathcal{P}
- \langle (\ldots, 1_i, \ldots), L_1(\ldots, 1_{i-1}, \ldots) \rangle \mathcal{P} \langle (\ldots, 1_{i-1}, \ldots), L_{-1}(\ldots, 1_i, \ldots) \rangle \mathcal{P} \tag{2.2}
= f(i - K_i, K_\infty) - f(i - 1 - K_i, K_\infty).
$$
Now, we have to put everything together to get the family of $su(1,1)$-operations which is fixed by these requirements. In the following explicit formulas, we restrict our considerations to the vacuum sector ($m = 1$). First of all, note that $f$ can only be of degree two as a polynomial in $i$, since higher degree $d$ polynomials will leave the commutator $[L_{-1}, L_1]$ with terms $i^{d-1}$ as contribution from the shift of the 1 at position $i$ up and down (and vice versa), which due to the locality of the operation cannot be compensated generically. Thus, $d = 2$, since what we like to obtain from the commutator is $2L_0$ which is linear in the position $i$.

Summing (2.2) over all positions $i$ with a 1, and requiring this to equal $2L_0 \tilde{a} = (2 \sum_i ia_i) \tilde{a}$, together with the initial condition $f(0, K_\infty) = 0$ fixes $f$ to be

$$f(j - K_j, K_\infty) = (j + K_\infty - K_j + 2)(j - K_j), \quad (2.3)$$

which essentially consists of the position ± the number of 1’s in the sequence to the left or the right. We can now arbitrarily factorize $f$ into a part for $L_1$ and one for $L_{-1}$; any choice will just fix another pseudo scalar product on the path Hilbert space. The nicest choices are either the symmetric one or the one just making use of the above factorization into polynomials of degree 1; the first one will be used for the remaining part of this section, the other one in the next section.

We therefore have derived a family of $su(1,1)$ representations on $\mathcal{P}(1,1)$ in terms of counting operators $K_1, K_\infty$ and shift operators: Using shift operators defined by

$$S_+^j(..., 1_j, 0, 0, ...) = (... , 0_j, 1, 0, ...) \quad S_-^j(..., 0, 0_j, 1, ...) = (... , 0, 1_j, 0, ...)$$

as long as the image path is allowed by (1.4), and operating identically zero otherwise, the operators

$$L_{\pm 1} := \sum_{j \geq 1} L_{\pm 1}^{[1]:j} \quad \text{with} \quad L_{\pm 1}^{[1]:j} := \sqrt{(j + K_\infty - K_j + 2)(j - K_j)} S_{\pm}^j \quad (2.4)$$

are just the most symmetric representative of this family of $su(1,1)$ representations, fulfilling

$$[L_0, L_{\pm 1}] = \pm L_{\pm 1} \quad [L_{-1}, L_1] = 2L_0.$$  

Another version will be given along with the formulas for the more general case $\mathcal{P}(K,1)$ treated in the next section. The $h = -1/5$ sector can essentially be treated in an analogous
way; however, the commutator \([L_{-1}, L_1] = 2L_0\) introduces \(h\) on the right hand side of the equations for the coefficients, such that the latter explicitly contain \(h [R\partial]\); furthermore, the number of 1’s in an irreducible \(su(1,1)\) lowest weight representation is no longer constant in generic sectors due to an additional contribution on the beginning of the path \([R\partial, K\alpha]\).

Fortunately, at least for the \(c(2,5)\) model \(\mathcal{P}\) is even “slim enough” to demonstrate that also \(L_2\) is already determined, fixing the central charge \(c\) to \(-22/5\). This can be shown by inspection of the commutators on the states of the lowest four energy levels: Taking \(\alpha, \beta\) to be the coefficients of \(L_2\), which can be chosen symmetrically as those of \(L_1\) are, and which are to be determined by the Virasoro commutators, we have

\[
(1,0,0,0,0,0,0,0,0,\ldots) \xrightarrow{L_2} (1,0,1,0,0,0,0,0,0,0,0,\ldots) \xrightarrow{L_2} (1,0,0,0,0,0,0,0,0,0,0,0,\ldots)
\]

\[
(1,0,1,0,0,0,0,0,0,0,0,0,\ldots) \xrightarrow{L_2} (1,0,0,0,1,0,0,0,0,0,0,0,\ldots) \xrightarrow{L_2} (1,0,1,0,0,0,0,0,0,0,0,0,\ldots)
\]

\[
(1,0,0,0,0,1,0,0,0,0,0,0,0,\ldots) \xrightarrow{L_2} (1,0,0,1,0,0,0,0,0,0,0,0,\ldots)
\]

\[
(1,0,1,0,0,0,0,0,0,0,0,0,\ldots) \xrightarrow{L_2} (1,0,0,1,0,0,0,0,0,0,0,0,\ldots)
\]

\[
(1,0,0,0,1,0,0,0,0,0,0,0,\ldots) \xrightarrow{L_2} (1,0,0,1,0,0,0,0,0,0,0,0,\ldots)
\]

and the commutators

\[
[L_0, L_2] = 2L_2 \quad [L_{-1}, L_2] = 3L_1 \quad [L_{-2}, L_2] = 4L_0 + c/2
\]

fix \(\alpha, \beta\) and \(c\) to be

\[
\beta = 3\sqrt{3/5} \quad c = 2\alpha^2 = -22/5.
\]

Therefore, our construction of the \(su(1,1)\)-subalgebra passed the natural consistency check on extensibility to the full Virasoro algebra. However, an explicit construction of \(L_{\pm 2}\) — and therefore, by computing commutators, also for the full Virasoro algebra — is much more involved than the one of \(L_{\pm 1}\). In particular, \(L_{\pm 2}\) has a part which does not commute with \(K\infty\), and it is actually just this part of \(L_{\pm 2}\) which fixes the central charge \(c\). Therefore, the situation here is in total analogy to that of the Ising model \([CE, KS]\), which however does not involve additional counting operators \(K_i [R\partial]\) and in that sense is less complicated.

Another approach to the construction of both \(L_{\pm 1}\) and \(L_{\pm 2}\) for the \(c(2,5)\) model was given in reference \([Ka]\). However, the consistency of that construction has only been shown for subspaces of sufficiently low energy. Yet another possibility of defining a Virasoro operation on paths is given implicitly by the basis of ref. \([FNO]\); however, the resulting operation is not local on the paths.
3. su(1,1) for the c(2,2K+3)-series

The sum form of characters (1.6) can be interpreted as the partition function for \( K \) different types of equivalence classes of local patterns in paths which for convenience will be called “quasiparticles” in the following (see e.g. [KM,Me,KRV,Be,BM]), and which can be excited independently: The summation indices \( n_i \) correspond to the number of particles of type (or weight) \( i \) in a given configuration, and the \((q)_n\) in the denominator generate the combinatorics for independent excitations in steps of 1. The \( q \)-exponent in the numerator is just the minimal energy for a configuration of \( n_1, \ldots, n_K \) quasiparticles of weights 1, \ldots, \( K \), which corresponds to the ground state of the fixed particle number sector. That ground state is for the \( h_{1,m+1} \)-sector \((0 \leq m \leq K)\) of the \( c(2,2K+3) \) model given by \((K-m)\&(m, K-m)^{n_K} \& (m, K-1-m)^{n_{K-1}} \& \cdots \& (m, 1)^{n_1} \& (m, 0)^{n_m} \& (m-1, 0)^{n_{m-1}} \& \cdots \& (1, 0)^{n_1} \& z_\infty\). In that expression and for later convenience, a path \( \tilde{m} \in P \) is considered as the concatenation — defined by \((a_1, \ldots, a_r) \& (b_1, \ldots, b_s) = (a_1, \ldots, a_r, b_1, \ldots, b_s)\) — of a finite set of patterns \( p_j^k \) which are sequences of length two, \( p_j^k = (j-k, k) \) being called “quasiparticle of weight \( j \)”, each of which comes in \( j \) versions \( k = 0, \ldots, j-1 \), and of (finite or infinite) zero sequences \( z_j = (0, 0, 0, 0, \ldots, 0) \) (\( j \) zeros): \( \tilde{m} = p_{j_1}^{m_1} \& z_{r_1} \& p_{j_2}^{m_2} \& z_{r_2} \& \cdots \). The expression \((a, b)^n := (a, b, a, b, a, b, \ldots, a, b)\) denotes the \( n \)-fold concatenation of the finite sequence \((a, b)\). If one or more of the \( r_i \) are zero, it may as well happen in some special configurations that a quasiparticle is split into two parts \( p_{-j}^k = (j-k) \) and \( p_{+j}^k = (k)\).

As before, we require \( su(1, 1) \) to leave invariant the quasiparticle structure (i.e. the particle numbers \( n_j \)); thus, the fixed particle number sector ground state becomes an \( su(1, 1) \) lowest weight vector.

Therefore, it is reasonable to decompose \( P(K, 1) \) — for explicit formulas, we again restrict our considerations to the vacuum sector — into a tensor product of path spaces \( Q(k) \) of single quasiparticle types with energy weight \( k \),

\[
Q(k) := \{ \tilde{a} = (a_n)_{n \in \mathbb{N}} \in \{0, 1\}^\mathbb{N} \mid a_0 = 1, a_n = 0 \text{ almost everywhere} , \\
(a_i, a_j \neq 0, i \neq j \Rightarrow |i-j| \geq 2k) \}. \tag{3.1}
\]

The minimal distance \( 2k \) between two particles in \( Q(k) \) corresponds to the distance of the particles in the ground state of a sector with fixed particle numbers according to the above interpretation of the character formula. On each of the tensor factors, \( su(1, 1) \) operates in a fashion similar to the case \( P(1, 1) \) in the previous section, and the operation on the
Our strategy is to define an operation on an appropriate subspace $Q_K \subset \bigotimes_{1 \leq k \leq K} Q(k)$ of the tensor product and an isomorphism $\mathcal{I}: \mathcal{P}(K, 1) \xrightarrow{\cong} Q_K$ which pulls that operation as well as the quasiparticle interpretation back to $\mathcal{P}(K, 1)$. For that purpose, we define in analogy to the previous section some endomorphisms acting on sequences $\tilde{\alpha} \in Q(n)$:

$$\tilde{K}_j^{[n]} \tilde{\alpha} = \left(\sum_{1 \leq k \leq j+1} a_k\right) \tilde{\alpha}$$

$$\tilde{S}_+^{[n]}: (\ldots, 1j, 0, \ldots) \mapsto (\ldots, 0j, 1, \ldots)$$

$$\tilde{S}_-^{[n]}: (\ldots, 0j, 1, \ldots) \mapsto (\ldots, 1j, 0, \ldots).$$

The shift operators $\tilde{S}_+^{[n]}, \tilde{S}_-^{[n]}$ are zero, if the resulting image sequence is forbidden in $Q(n)$. For later convenience, we put these into operators acting on the tensor product $\bigotimes_{1 \leq n \leq K} Q(n)$, namely for $\mathcal{O}^{[n]} = K_j^{[n]}, K_\infty^{[n]}, S_+^{[n]}, S_-^{[n]}$

$$\mathcal{O}^{[n]} = \text{id}_{Q(1)} \otimes \cdots \otimes \text{id}_{Q(n-1)} \otimes \tilde{O}^{[n]} \otimes \text{id}_{Q(n+1)} \otimes \cdots \otimes \text{id}_{Q(K)}$$

and

$$\Xi^{[n]} = 2n \left(1 + \sum_{n+1 \leq k \leq K} K_\infty^{[k]} \right).$$

The position restrictions on “light” particles by the “heavier” particles, which in an $su(1, 1)$ lowest weight path are located closer to the beginning of the path, can now be implemented into the definition of $Q_K$ by means of those endomorphisms:

$$Q_K := \{ \tilde{\alpha} = \tilde{a}^{(1)} \otimes \ldots \otimes \tilde{a}^{(K)} | a_j^{(n)} = 0 \text{ for } 0 < j < \xi^{(n)} \}$$

and $1 \leq n \leq K - 1$, where $\Xi^{(n)} \tilde{\alpha} = \xi^{(n)} \tilde{\alpha}$.

Alternatively, these restrictions could also be completely implemented into the coefficients of the $su(1, 1)$-operation on the original tensor product $\bigotimes Q$ as an additional offset in the position evaluation, but for technical reasons, we prefer to use $Q_K$ here.

The isomorphism $\mathcal{I}$ is now defined step by step, starting from the $su(1, 1)$ lowest weights:

The path $(K, 0, K, 0, \ldots, 0, K - 1, 0, K - 1, 0, \ldots, 2, 0, 1, 0, 1, 0, 0, 0, \ldots) \in \mathcal{P}(K, 1)$ containing $n_k$ particles of weight $k = 1, \ldots, K$ is mapped to the path $\tilde{a}^{(1)} \otimes \ldots \otimes \tilde{a}^{(K)} \in Q_K$ with $a_j^{(k)} = 1$ if either $j = 0$ or $2k(n_{k+1} + \ldots + n_K) < j \leq 2k(n_k + \ldots + n_K)$ with $j \in 2k\mathbb{N}$, all other $a_j^{(k)}$ being zero.
Making use of shift operators $\hat{S}_{\pm}^{[n]:j}$ on $\mathcal{P}(K,1)$ which are defined below, $\mathcal{I}$ is completely defined by the relation $\mathcal{I}\hat{S}_{\pm}^{[n]:j} = S_{\pm}^{[n]:j}\mathcal{I}$. By definition, the $S_{\pm}^{[n]:j}$ of different particle index $n$ commute, and so the $\hat{S}_{\pm}^{[n]:j}$ do. Therefore without loss of generality, any path configuration can now be obtained from the corresponding lowest weight by first shifting the “light” quasiparticles (which are most right in the ground state) to their actual position to the right, and then to do the same for more and more “heavy” ones.

In order to complete the definition of $\mathcal{I}$ we have to define the operation of the $\hat{S}_{\pm}^{[n]}\bullet$ operators on $\mathcal{P}(K,1)$:

The shift operator $\hat{S}_{\pm}^{[j]:E}$ operates on a quasiparticle $p_j^m$ — which has already been shifted to energy $E$ — by $\hat{S}_{\pm}^{[j]:E}(...&p_j^m&\&z_{n_k+1}&\&...)=...&z_{n_k}&\&p_j^{m+1}&\&z_{n_{k+1}}&\&...$ as long as $m < j$ and $n_k,n_{k+1} > 0$. Furthermore, there are obvious rules $p_j^m&z_n = z_1&p_j^0&z_{n-1}$ and $z_n&z_{n_2} = z_{n_1+n_2}$. Note that here the actual energy $E$ is the sum of the original $L_0$ contribution $E_0$ of $p_j^m$ in the lowest weight path and the number of applications of appropriate $\hat{S}_{\pm}^{[j]}\bullet$ on it for shifting it to the right, namely by $\hat{S}_{\pm}^{[j]:E-1}\hat{S}_{\pm}^{[j]:E-2}\ldots\hat{S}_{\pm}^{[j]:E_0}$.

For a definition of $\mathcal{I}$ it only remains to define the operation of shifts $\hat{S}_{\pm}^{[j]}\bullet$ on successive pairs of quasiparticles: Without loss of generality (see above), let $j_1 > j_2$. Then $\hat{S}_{\pm}^{[j_1]}\bullet$ (where $\bullet$ is appropriately chosen to act on $p_{j_1}$) operates by

$$
\begin{align*}
&\ldots&p_{j_1}^{m_1}&\&p_{j_2}^{m_2}&\&\ldots \quad \rightarrow \quad \ldots&p_{j_1}^{m_1+1}&\&p_{j_2}^{m_2}&\&\ldots \\
&\ldots&p_{j_1}^{j_1-j_2+m_2-1}&\&p_{j_2}^{m_2}&\&\ldots \quad \rightarrow \quad \ldots&p_{j_2}^{m_2}&\&p_{j_1}^{j_2-m_2+1}&\&p_{j_2}^{m_2}&\&\ldots \\
&\ldots&p_{j_2}^{m_2}&\&p_{j_1}^{m_1+1}+1&\&p_{j_2}^{m_2}&\&\ldots \quad \rightarrow \quad \ldots&p_{j_2}^{m_2}&\&p_{j_1}^{m_1+1}&\&p_{j_2}^{m_2}&\&\ldots \\
&\ldots&p_{j_2}^{m_2}&\&p_{j_1}^{j_2-m_2-1}&\&p_{j_2}^{m_2}&\&\ldots \quad \rightarrow \quad \ldots&p_{j_2}^{m_2}&\&p_{j_1}^{j_2}&\&p_{j_2}^{m_2}&\&\ldots \\
&\ldots&p_{j_2}^{m_2}&\&p_{j_1}^{m_1}&\&\ldots \quad \rightarrow \quad \ldots&p_{j_2}^{m_2}&\&p_{j_1}^{m_1+1}&\&\ldots
\end{align*}

(m_1 < j_1 - j_2 + m_2 - 1)

(j_2 - m_2 \leq m_1 < j_1 - m_2 - 1)

(m_2 \leq m_1 < j_1).

Note that particles of equal type $j_1 = j_2$ are not allowed to change their mutual order. Finally, $\hat{S}_-$ just reverses the arrows. Successive application of arbitrary shift operators can be transformed to the abovementioned normal form by making use of the commutativity of shifts on different particle indices.

Having collected all ingredients for the $su(1,1)$ representation under construction including the quasiparticle decomposition map $\mathcal{I}$ for the vacuum sector $\mathcal{P}(K,1)$ of the model $c(2,2K+3)$, we therefore get the following explicit $su(1,1)$ representation on this path.
space:

\[ L_{-1}^{[n]:j} = I^{-1} \circ \left( j - (2n - 1)K_j^{[n]} - \Xi^{[n]} + 2n \right) S_{-1}^{[n]:j} \circ I \]

\[ L_1^{[n]:j} = I^{-1} \circ \left( j + (2n - 1) \left( K_\infty^{[n]} - K_j^{[n]} \right) + \Xi^{[n]} \right) S_1^{[n]:j} \circ I \]

\[ L_{-1} = \sum_{1 \leq n \leq K} \sum_{j \geq 1} L_{-1}^{[n]:j} \quad \text{and} \quad L_1 = \sum_{1 \leq n \leq K} \sum_{j \geq 1} L_1^{[n]:j} \quad (3.4) \]

\[ L_0(l_k)_k = \left( \sum_{j \geq 1} jl_j \right)(l_k)_k \]

The details of the proof are essentially identical to the arguments given in the previous section, and for that reason are omitted here.

Finally we have to construct a pseudo inner product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{P} \), such that \( L_{\pm 1}^* = L_{\mp 1} \) and \( \mathcal{P} \) becomes pseudo Hilbert. As long as we do not care about the operation of \( L_{\pm 2} \), this inner product is not fixed on the ground state \( v_0 \) of any sector with fixed quasiparticle numbers, that is, on the \( su(1,1) \) lowest weight vectors; it is, however, fixed by \( L_{\pm 1}^* = L_{\mp 1} \) as soon as the pseudo norms of just these ground states are chosen, and can explicitly be constructed in terms of the inner product \( \langle \cdot, \cdot \rangle \equiv \langle \cdot, \cdot \rangle_\mathcal{P} \) on \( \mathcal{P} \) which is defined by orthonormality of all paths generating the space, namely by successive evaluation of the condition \( L_{\pm 1}^* = L_{\mp 1} \) starting from the \( su(1,1) \) lowest weight vector.

It follows by construction that the sign of \( \langle v, v \rangle \) is constant for any path \( v \) taken from the same quasiparticle number sector. Therefore, our result is in compatibility with Nahm’s conjecture on the signature characters \([Na,Ke]\) for \( c(2,2K+3) \) defined by

\[ \text{sign} - \text{ch}_{1,m}^{(2,2K+3)}(q) \equiv q^{-h_{1,m}^{(2,2K+3)}} \text{Tr}_{\mathcal{P}(K,m)} q^{L_0} \text{sign}(\langle \cdot, \cdot \rangle), \]

namely that they are given by

\[ \text{sign} - \text{ch}_{1,m}^{(2,2K+3)}(q) = \mathcal{K}_{1,m}^{(2,2K+3)}(q,-1) \]

where

\[ \mathcal{K}_{1,m}^{(2,2K+3)}(q,z) = \sum_{n_1,\ldots,n_K \geq 0} q^{N_1^2 + \cdots + N_K^2 + N_m + \cdots + N_K} \frac{(q)_{n_1} \cdots (q)_{n_K}}{(q)_{n_1+\cdots+n_K}} z^{N_1+\cdots+N_K} \quad (3.5) \]

in obvious similarity to eq. (1.6). However, for a completion of the proof it remains to show that \( L_2 \) can be consistently defined with this choice of \( L_{\pm 1} \).
4. Temperley-Lieb algebra for c(2,5)

For $r$ the golden ratio $\frac{\sqrt{5}-1}{2}$ — satisfying $r^2 = 1 - r$ — we express the Virasoro generators for the c(2,5)-model vacuum sector in terms of the corresponding Temperley-Lieb projections $e_i$, which obey [Ba,Jo]

\begin{align*}
  e_i e_j &= e_j e_i \quad |i - j| > 1 \\
  e_i e_{i \pm 1} e_i &= \tau e_i \quad (\tau = r^2) \\
  e_i^2 &= e_i = e_i^* \quad (4.1)
\end{align*}

On the path space $P(1,1)$, these generators are represented by [Oc,Pa]

\begin{align*}
  e_{j+1}(\ldots, l_j, l_{j+1}, l_{j+2}, \ldots) &= \\
  &\delta_{l_j,l_{j+2}} \left( [\delta_{l_j,1} + r^{1+l_{j+1}} \delta_{l_j,0}] (\ldots, l_j, l_{j+1}, l_{j+2}, \ldots) + \delta_{l_j,0} r^{3/2} (\ldots, l_j, 1 - l_{j+1}, l_{j+2}, \ldots) \right),
\end{align*}

which obviously fulfill the Temperley-Lieb relations. Explicitly, we have

\begin{equation}
  L_0 = \sum_{j \geq 1} j L_0^{(j)},
\end{equation}

where the $L_0^{(j)}$ are defined recursively by

\begin{align*}
  L_0^{(1)} &= 0 \\
  L_0^{(2)} &= e_1 \\
  L_0^{(n)} &= \left( 1 - L_0^{(n-2)} - \frac{1}{r} e_{n-1} \right) \left( 1 - L_0^{(n-1)} \right) \left( 1 - \frac{1}{r} e_{n-1} \right) \left( 1 - L_0^{(n-1)} \right), \quad (4.4)
\end{align*}

which is hermitean with respect to the adjoint in the Temperley-Lieb path representation, since in that representation the first bracket commutes with the product of the three following ones. More explicitly, we can make use of the fact that the Temperley Lieb algebra for $\tau = r^2$ contains an ideal [Jo] generated by

\begin{equation}
  ( -e_2 e_3 e_1 e_2 + e_2 e_3 e_1 + e_1 e_3 e_2 - e_1 e_3 ) + r ( e_3 e_2 e_1 + e_1 e_2 e_3 - e_1 e_2 - e_2 e_1 - e_2 e_3 - e_3 e_2 ) \\
  + (1 - r) (e_1 + e_3) + r e_2 + (3r - 2)
\end{equation}

and its translates (by index shifts). Since that ideal is mapped to zero in the path representation, the first terms turn out to be

\begin{align*}
  L_0^{(3)} &= \frac{1}{r} (e_1 e_2 + e_2 e_1 - e_1 - e_2) + 1 \\
  L_0^{(4)} &= \frac{1}{r^3} e_2 e_3 e_1 e_2 + \frac{1}{r} (e_1 e_3 - e_2 e_3 e_1 - e_1 e_3 e_2) .
\end{align*}

(4.5)
Similar expressions also hold for $L_{\pm 1}$: After rewriting the counting operators implicitly in terms of the Temperley-Lieb generators,

$$K_{\infty} = 1 + \sum_{n \geq 2} L_0^{(n)} \quad \quad K^{(n)} = 1 + \sum_{j=2}^{n} L_0^{(j)},$$

the building blocks for $L_1$ are given by

$$L_1^{(n)} = \frac{1}{(1-r^{1/2})(1-r^{-1/2})} \left( 1 - \frac{1}{r} e_{n+1} \right) \left( 1 - \frac{1}{r^2} e_n \right) \left( 1 - e_{n+1} \right) L_0^{(n)},$$

and the expressions for $L_{-1}^{(n)}$ can be obtained by hermitean conjugation. Therefore, we finally get

$$L_{\pm 1} = \sum_{j \geq 2} \sqrt{(j + K_{\infty} - K_j + 2)(j - K_j)} L_{\pm 1}^{(j)} \quad \quad (4.6)$$

or analogously instead of the square root some coefficients of any other appropriate choice of the function $f$ of coefficient products.

5. Discussion and Outlook

We gave an explicit construction of the operation of some Virasoro generators on the path spaces of $c(2,2K+3)$-Virasoro minimal models; although the full algebra seems already to be fixed by the choice of the $su(1,1)$-subalgebra, an explicit formula for $L_2$ is not yet known. To go beyond the vacuum sector in our description, the $L_1$-operation at the beginning of the paths and the coefficients have to be modified slightly: The quasi particle number $n_1$ is no longer conserved in this case [Ro,Ka].

For the $c(2,5)$-model we additionally showed how to express the $su(1,1)$ subalgebra in terms of the Temperley Lieb Jones algebra on the corresponding graph. However, in this case, explicit expressions for the CTM from statistical mechanics are structurally simpler, and the basis change in the algebra transforming one CTM expression to the other still has to be worked out.

For other models apart from the considered series of minimal models, there still exist similar path spaces in some cases [KRV], but the explicit construction of the Virasoro operation — and, more generally, of larger W-algebras — seems to be more involved. At least in the cases of ref. [KRV] with characters of type (1.6), the $su(1,1)$ operation on the corresponding paths should work in the same way.
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