Inversion formula for determining parameters of an astrometric binary

Hideki Asada, Toshio Akasaka, Masumi Kasai
Faculty of Science and Technology, Hirosaki University, Hirosaki 036-8561, Japan
asada@phys.hirosaki-u.ac.jp
kasai@phys.hirosaki-u.ac.jp

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Abstract

It is believed that some numerical technique must be employed for the determination of the system parameters of a visual binary or a star with a planet because the relevant equations are not only highly nonlinear but also transcendental owing to the Kepler’s equation. Such a common sense, however, is not true; we discover an analytic inversion formula, in which the original orbital parameters are expressed as elementary functions of the observable quantities such as the location of four observed points and the time interval between these points. The key thing is that we use the time interval but not the time of each observation in order to avoid treating the Kepler’s equation. The present formula can be applied even in cases where the observations cover a short arc of the orbit during less than one period. Thus the formula will be useful in the future astrometric missions such as SIM, GAIA and JASMINE.

Key words: astrometry — celestial mechanics — stars: binaries: general — stars: planetary systems

1. Introduction

The astrometric observation of binaries gives us a lot of invaluable informations on the mass of the component stars and the orbital parameters such as the ellipticity (Danby 1988, Roy 1988). In the near future, space missions such as SIM ¹, GAIA ² and JASMINE ³ find a number of binaries nearly within 10kpc. A problem in the astrometry arises from a fact that we make a measurement of the angular position of the celestial object, which is projected on a

¹ Space Interferometry Mission (SIM), http://sim.jpl.nasa.gov/
² Global Astrometric Interferometer for Astrophysics (GAIA), http://astro.estec.esa.nl/GAIA/
³ Japan Astrometry Satellite Mission for INfrared Exploration (JASMINE), http://www.jasmine-galaxy.org/
plane perpendicular to the line of sight. For simplicity we call this plane the observed plane. In a case of a binary system, the plane of the Keplerian orbit is inclined with respect to the line of sight. As a result, the observed ellipse can be considerably different from the original Keplerian orbit. To determine the original orbital parameter, we thus need make a kind of data fitting (Olevic and Cvetkovic 2004). It rather consumes time and computer resources, especially for a huge amount of data which will be available by the near future missions. Obviously it would be preferable to use an inversion formula, which enables us to directly determine from the observed quantities the original orbital parameters and the inclination. However, the conditional equations which connect the observable quantities with the orbital parameters are not only highly nonlinear but also transcendental because of the Kepler’s equation. It is thus believed that an explicit solution is impossible so that some numerical technique must be employed to solve the equations for the orbital parameters (Eichhorn and Xu 1990, Catovic and Olevic 1992). Such a common sense in this field, however, is not true as shown below.

The purpose of this letter is to derive the inversion formula: The key point is that we use the time interval between the observations but not the time of each observation. First, we determine the observed ellipse from the positional data. Next, together with the time interval between observations, all of the ellipticity, the orbital period, the major and minor axes of the original Keplerian orbit and the inclination are expressed in the measured quantities.

2. Inversion Formula

First, we determine the observed ellipse. Next, we discuss how to determine the original orbital parameters and the inclination angle. In this letter, we consider only the Keplerian motion of the binary by neglecting the motion of the observer and the galactic motion.

2.1. Observed Ellipse

We observe an ellipse on the plane perpendicular to the line of sight. The Cartesian coordinates on the plane are denoted by \((\bar{x}, \bar{y})\). A general form of the ellipse is

\[
\alpha \bar{x}^2 + \beta \bar{y}^2 + 2\gamma \bar{x} \bar{y} + 2\delta \bar{x} + 2\varepsilon \bar{y} = 1,
\]

which is specified by five parameters since the center, the major/minor axes and the orientation of the ellipse are arbitrary. We need make at least five observations to determine the parameters. The location of each observed point is \((\bar{x}_i, \bar{y}_i)\) for \(i = 1, \ldots, 5\). Then, we find

\[
\begin{pmatrix}
\bar{x}_1^2 & \bar{y}_1^2 & 2\bar{x}_1\bar{y}_1 & 2\bar{x}_1 & 2\bar{y}_1 \\
\bar{x}_2^2 & \bar{y}_2^2 & 2\bar{x}_2\bar{y}_2 & 2\bar{x}_2 & 2\bar{y}_2 \\
\bar{x}_3^2 & \bar{y}_3^2 & 2\bar{x}_3\bar{y}_3 & 2\bar{x}_3 & 2\bar{y}_3 \\
\bar{x}_4^2 & \bar{y}_4^2 & 2\bar{x}_4\bar{y}_4 & 2\bar{x}_4 & 2\bar{y}_4 \\
\bar{x}_5^2 & \bar{y}_5^2 & 2\bar{x}_5\bar{y}_5 & 2\bar{x}_5 & 2\bar{y}_5 \\
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta \\
\varepsilon \\
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
\end{pmatrix}.
\]

By using the inverse matrix, we can determine \((\alpha, \beta, \gamma, \delta, \varepsilon)\) in terms of \((\bar{x}_i, \bar{y}_i)\). Henceforth, we
choose the Cartesian coordinates \((x, y)\) so that the observed ellipse can be reexpressed in the standard form as
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,
\]
where \(a \geq b\). The ellipticity \(e\) is \(\sqrt{1 - b^2/a^2}\).

2.2. Time interval

Before considering a general case, we study a special case to illustrate a problem due to the inclination; we assume that an ellipse is inclined around its major axis by the angle \(i\). The length of the major and minor axes is denoted by \(a\) and \(b'\), respectively. The length of the major and minor axes of the observed ellipse becomes \(a\) and \(b'\cos i\), respectively. Only by using the shape of the ellipse, we cannot determine \(b'\) and \(i\) separately. This degeneracy can be broken if we use not only the location of each point but also its time. More rigorously speaking, we use the time interval between points as shown below. We adopt four observational data at the time of \(t_4 > t_3 > t_2 > t_1\). The star is observed at the location of \(P_i = (x_i, y_i) = (a\cos u_i, b\sin u_i)\) at each time \(t_i\) for \(i = 1, \ldots, 4\), where \(a\) and \(b\) have been determined by fixing the observed ellipse in the preceding subsection, and \(u_i\) denotes the eccentric anomaly. We assume the anti-clockwise motion such that \(u_i > u_j\) for \(i > j\). All we have to do in the case of the clockwise motion is to change the signature of Eq. (5) for the area in the following. We define the time interval as \(t_{ij} = t_i - t_j\). The number of these additional quantities \(t_{21}, t_{32}\) and \(t_{43}\) is three, which equals the number of the unknown parameters of the period, the orientation and angle of the inclination.

The original Keplerian orbit is assumed to be parameterized by the length of the major and minor axes, \(a_K\) and \(b_K\), and the period \(T\). The ellipticity of the orbit is denoted by \(e_K\). The focus of the original ellipse is projected onto the observed plane at \(P_e = (x_e, y_e)\). Figure 1 shows the geometrical configuration.

One of the important things on the Keplerian orbit is that the area velocity is constant, where the area is swept by the line interval between the focus and the object in the Keplerian motion (For instance, Goldstein 1980). Henceforth, the object in the Keplerian motion is called the star for simplicity. We should note that the projected focus is not necessarily the focus of the observed ellipse. Even after the projection, however, the law of the constant area velocity still holds, where the area is swept by the line interval between the projected focus and the star. The area swept during the time interval \(t_{ij}\) is denoted by \(S_{ij}\). The total area of the observed ellipse \(S\) is \(\pi ab\). The law of the constant area velocity on the observed plane becomes
\[
\frac{S}{T} = \frac{S_{ij}}{t_{ij}}.
\]

2.3. Determining the ellipticity and the orbital period

By elementary computations, the area \(S_{ij}\) is obtained as
Fig. 1. A relation between the Keplerian ellipse and the observed one. The line of sight is along the $z$-axis.

$$S_{ij} = \frac{1}{2} ab \left( u_i - u_j - \frac{x_e}{a} (\sin u_i - \sin u_j) + \frac{y_e}{b} (\cos u_i - \cos u_j) \right).$$

It is noteworthy that $S_{ij}$ is linear in $x_e$ and $y_e$. By using Eq. (4), we obtain

$$\frac{S_{21}}{t_{21}} = \frac{S_{32}}{t_{32}},$$
$$\frac{S_{32}}{t_{32}} = \frac{S_{43}}{t_{43}}.$$  

They are reexpressed as

$$A_3 - \frac{x_e}{a} A_1 + \frac{y_e}{b} A_2 = 0,$$
$$B_3 - \frac{x_e}{a} B_1 + \frac{y_e}{b} B_2 = 0,$$

where

$$A_1 = t_{21} \sin u_3 + t_{32} \sin u_1 - t_{31} \sin u_2,$$
$$A_2 = t_{21} \cos u_3 + t_{32} \cos u_1 - t_{31} \cos u_2,$$
$$A_3 = t_{21} u_3 + t_{32} u_1 - t_{31} u_2,$$
$$B_1 = t_{32} \sin u_4 + t_{43} \sin u_2 - t_{42} \sin u_3,$$
$$B_2 = t_{32} \cos u_4 + t_{43} \cos u_2 - t_{42} \cos u_3,$$
$$B_3 = t_{32} u_4 + t_{43} u_2 - t_{42} u_3.$$
\[ x_e = -\frac{A_2B_3 - A_3B_2}{A_1B_2 - A_2B_1}, \quad (16) \]
\[ y_e = b\frac{A_3B_1 - A_1B_3}{A_1B_2 - A_2B_1}, \quad (17) \]

which determine the location of the projected focus.

The original major axis is projected onto the observed ellipse at \( P_L \equiv (x_L, y_L) = (a \cos u_L, b \sin u_L). \) The ratio of the major axis to the distance between the origin and the focus remains same even after the projection. Hence we find

\[ P_L = \frac{1}{e_K} P_e. \quad (18) \]

The positional vector \( P_L \) is located on the observed ellipse given by Eq. (3). Thus we obtain the ellipticity as

\[ e_K = \sqrt{\frac{x_e^2}{a^2} + \frac{y_e^2}{b^2}}. \quad (19) \]

The parameter \( u_L \in [0, 2\pi) \) is thus given by

\[ \cos u_L = \frac{x_e}{ae_K}, \quad (20) \]
\[ \sin u_L = \frac{y_e}{be_K}. \quad (21) \]

The location of the projected focus is given by Eqs. (16) and (17) so that we can determine the area \( S_{ij} \) from Eq. (5). By using \( S_{21} \) for instance in Eq. (4), we obtain the orbital period as

\[ T = \frac{S^{t_{21}}}{S_{21}}. \quad (22) \]

2.4. Determining the major axis and the inclination angle

First, from the above result, we determine the location of the intersection of the observed ellipse and the projected minor axis, denoted by \( P_S \equiv (x_S, y_S) = (a \cos u_S, b \sin u_S). \) It is not necessary to make an observation of the intersection. The major and minor axes divide the area of the ellipse in quarter. Even after projecting the ellipse, the projected major and minor axes, which are not those of the observed ellipse, divide the observed ellipse in quarter areas. This fact implies that \( u_S = u_L + \pi/2, \) which does not mean that \( P_L \) is perpendicular to \( P_S. \)

Up to this point, we have not specified the angle and orientation of the inclination. Let the Keplerian ellipse inclined with the angle \( i \) in a way that the angle at the origin between the periastron and the ascending node is \( \omega \) called the angular distance of the periastron. We can assume that the inclination angle is in \([0, \pi/2)\), though it takes a value in \([0, \pi)\) in the standard context of the celestial mechanics. This is because in the present paper we do not specify whether the motion of the star in the Keplerian orbit is prograde or retrograde. Hence, rigorously speaking, the ascending node, which we have called above for convenience, may be the descending node. In short, our inclination angle is between the line of sight and a unit
normal vector to the orbital plane, where there exist two unit normal vectors and we choose one such as \( \cos i \geq 0 \). Only in this paragraph, we adopt another Cartesian coordinates \((x', y')\) so that the ascending node is located on the \(x'\)-axis. The periastron of the original ellipse is projected at \( P_L \equiv (x'_L, y'_L) = (a_K \cos \omega, a_K \sin \omega \cos i) \). Similarly an intersection of the ellipse and the minor axis is projected at \( P_S \equiv (x'_S, y'_S) = (-b_K \sin \omega, b_K \cos \omega \cos i) \). The component of the vector depends on the adopted coordinates. Therefore, it is useful to consider the invariants such as \( |P_L|, |P_S| \) and \( |P_L \times P_S| \). We find

\[
|P_L| = a_K \sqrt{\cos^2 \omega + \sin^2 \omega \cos^2 i}, \quad (23)
\]

\[
|P_S| = b_K \sqrt{\sin^2 \omega + \cos^2 \omega \cos^2 i}, \quad (24)
\]

\[
|P_L \times P_S| = a_K b_K \cos i. \quad (25)
\]

From Eqs. (23) and (24), we obtain

\[
C^2 + D^2 = a_K^2 (1 + \cos^2 i), \quad (26)
\]

\[
C^2 - D^2 = a_K^2 \cos 2\omega \sin^2 i. \quad (27)
\]

Here we used \( b_K = a_K \sqrt{1 - e_K^2} \) and defined

\[
C = |P_L|, \quad (28)
\]

\[
D = |Q_S|, \quad (29)
\]

by introducing

\[
Q_S = \frac{P_S}{\sqrt{1 - e_K^2}}, \quad (30)
\]

where \( Q_S \) denotes a positional vector on a circle which is made by stretching the observed ellipse \( a_K/b_K \) times along its minor axis. Equation (25) is rewritten as

\[
|P_L \times Q_S| = a_K^2 \cos i. \quad (31)
\]

By using another expression of \( P_L = (a \cos u_L, b \sin u_L) \) and \( P_S = (-a \sin u_L, b \cos u_L) \), \( C, D \) and \( |P_L \times Q_S| \) are rewritten as

\[
C = \frac{1}{e_K} \sqrt{x_e^2 + y_e^2}, \quad (32)
\]

\[
D = \frac{1}{abe_K} \sqrt{a^4 y_e^2 + b^4 x_e^2}, \quad (33)
\]

\[
|P_L \times Q_S| = \frac{ab}{\sqrt{1 - e_K^2}}, \quad (34)
\]

From Eqs. (26) and (31), we obtain

\[
\cos^2 i - \xi \cos i + 1 = 0, \quad (35)
\]

where we defined
\[ \xi = \frac{C^2 + D^2}{|P_L \times Q_S|}. \]  

(36)

By using \( C^2 + D^2 \geq 2CD \geq 2|P_L \times Q_S| \), we can show that

\[ \xi \geq 2. \]  

(37)

Equation (35) is solved as

\[ \cos i = \frac{1}{2}(\xi \pm \sqrt{\xi^2 - 4}). \]  

(38)

We can show that \( \xi + \sqrt{\xi^2 - 4} \geq 2 \). Since \( \cos i \) must be no more than unity, we find the only solution as

\[ \cos i = \frac{1}{2}(\xi - \sqrt{\xi^2 - 4}), \]  

(39)

which satisfies \( \cos i \in [0, 1] \). Hence we can uniquely determine the inclination angle \( i \) for \( i \in [0, \pi/2) \), which has been discussed in the subsection 2.4. Here, by substituting Eqs. (32), (33) and (34) into Eq. (36), \( \xi \) is expressed in terms of \( a, b, e_K, x_e \) and \( y_e \).

Next, Eq. (26) determines the length of the major axis as

\[ a_K = \sqrt{\frac{C^2 + D^2}{1 + \cos^2 i}}. \]  

(40)

Finally, Eq. (27) determines the orientation of the inclination as

\[ \cos 2\omega = \frac{C^2 - D^2}{a_K^2 \sin^2 i}. \]  

(41)

3. Conclusion

We derive the inversion formula for astrometric observations of binaries. It is summarized as follows. First, we fix the observed ellipse by using the location of five points. Second, we choose four points and use their locations and the time intervals between these points. The projected focus is determined by Eqs. (16) and (17). The ellipticity is given by Eq. (19). The orbital period is determined by Eq. (22). The inclination angle is given by Eq. (39). The length of the major axis is computed from Eq. (40). Finally, the orientation of the inclination is given by Eq. (41).

Moreover, our result proves that the mapping between a point in the Keplerian motion and the observed point (in time and space) is one-to-one if the number of the observation is more than four. For instance, a sixth observed point must satisfy our equations with the determined values of the parameters. In practice, the observation inevitably associates errors so that we must make a kind of fittings for instance by the least square method. Even in such a case, our formula would give likely values of parameters so quickly that we could save CPU time for fittings.

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