Dushnik-Miller dimension of $d$-dimensional tilings with boxes

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Abstract. Planar graphs are the graphs with Dushnik-Miller dimension at most three (W. Schnyder, Planar graphs and poset dimension, Order 5, 323-343, 1989). Consider the intersection graph of interior disjoint axis parallel rectangles in the plane. It is known that if at most three rectangles intersect on a point, then this intersection graph is planar, that is it has Dushnik-Miller dimension at most three. This paper aims at generalizing this from the plane to $\mathbb{R}^d$ by considering tilings of $\mathbb{R}^d$ with axis parallel boxes, where at most $d+1$ boxes intersect on a point. Such tilings induce simplicial complexes and we will show that those simplicial complexes have Dushnik-Miller dimension at most $d+1$.

1 Introduction

One can easily see that the intersection graph induced by a set of interior disjoint axis parallel rectangles, with at most three rectangles intersecting on a point, is a planar graph. C. Thomassen characterized those graphs \cite{thomassen} (See also \cite{thomassen2} for a combinatorial study of these representations). H. Zhang showed how such a representation (when it tiles a rectangle) also induces a Schnyder wood of the induced planar graph \cite{zhang}. Schnyder woods was the key structure that allowed W. Schnyder to prove that planar graphs are the graphs with Dushnik-Miller dimension at most three \cite{schnyder}. It is interesting to note that most planar graphs have Dushnik-Miller dimension equal to three. Indeed, a graph has Dushnik-Miller dimension at most two if and only if it is the subgraph of a path.

The main result of this paper is that the simplicial complexes induced by a wide family of tilings of $\mathbb{R}^d$ with axis parallel boxes, have Dushnik-Miller dimension at most $d+1$. As most of these simplicial complexes have a $d$-face, there Dushnik-Miller dimension is exactly $d+1$. Definitions are provided in the following. Note that both, the objects (graphs or simplicial complexes) with Dushnik-Miller dimension greater than three \cite{dushnik}, and the systems of interior disjoint axis parallel boxes in $\mathbb{R}^d$ \cite{goncalves}, are difficult to handle but are raising interest in the community. The proof of our result generalizes H. Zhang’s idea for constructing Schnyder woods in tilings of $\mathbb{R}^2$, and relies on some properties of tilings that are of independent interest. After providing a few basic definitions in Section 2 we present these properties in Section 3 and Section 4. We then prove our main result in Section 5. We then conclude with some open problems.
2 \(d\)-boxes

A \(d\)-box (or simply a box) is the Cartesian product of \(d\) closed intervals. The intervals of a \(d\)-box \(B\) are denoted \(B_1, \ldots, B_d\), that is \(B = B_1 \times \cdots \times B_d\). Similarly the coordinates of a point \(x \in \mathbb{R}^d\) are denoted \(x = (x_1, \ldots, x_d)\). The endpoints of an interval \(B_i\) are denoted \(B_i^-\) and \(B_i^+\) in such a way that \(B_i = [B_i^-, B_i^+]\). Such an interval is \textit{degenerate} if its endpoints coincide, that is if \(B_i^- = B_i^+\).

The \textit{dimension} \(\dim(B)\) of a \(d\)-box \(B\) is the number of non-degenerate intervals among \(\{B_1, \ldots, B_d\}\). For example, a 0-dimensional \(d\)-box is a point in \(\mathbb{R}^d\).

The \textit{interior} of a \((d')\)-dimensional box \(B\) is the open \((d')\)-dimensional box defined by the points \(p = (p_1, \ldots, p_d)\) such that \(B_i^- < p_i < B_i^+\) if \(B_i^- \neq B_i^+\) or such that \(p_i = B_i^- = B_i^+\) otherwise. The points of \(B\) that are not interior form the \textit{border} of \(B\). The border of \(B\) is the union of its sides. A \textit{side} of \(B\) is a \((d'-1)\)-dimensional box \(S(B, i, *) = [B_i^-, B_i^+] \times \cdots \times [B_d^-, B_d^+]\), for \(* \in \{-, +\}\), and some non-degenerate dimension \(i\) of \(B\) (i.e. such that \(B_i^- \neq B_i^+\)). Clearly, a box with dimension \(d'\) has \(2d'\) distinct sides. A \textit{corner} of \(B\) is a point \((x_1, x_2, \ldots, x_d)\) where each \(x_i\) is an endpoint of \(B_i\), that is either \(x_i = B_i^-\) or \(x_i = B_i^+\). Clearly, a box with dimension \(d'\) has \(2^{d'}\) corners.

The intersection \(A \cap B\) of two boxes is the box \((A_1 \cap B_1) \times \cdots \times (A_d \cap B_d)\). Two \(d\)-dimensional boxes are \textit{interior disjoint} if their interiors do not intersect, or equivalently if \(\dim(A \cap B) < d\).

\textbf{Definition 1.} A \(d\)-\textit{tiling} \(T\) is a collection of interior disjoint \(d\)-dimensional boxes contained in \([-1, +1]^d\) that tile \([-1, +1]^d\) (i.e. every point of \([-1, +1]^d\) belongs to at least one box of \(T\)).

Let us now define \(T_{ext} = \{T(i, *) : 1 \leq i \leq d, * \in \{-, +\}\}\), a set of \(2d\) \(d\)-dimensional boxes that tile \(\mathbb{R}^d \setminus [-1, +1]^d\).

\[
\begin{align*}
T(i, -)_j &= T(i, +)_j = [-1, +1] & \text{if } j < i \\
T(i, -)_j &= [-\infty, -1] & \text{if } j = i \\
T(i, +)_j &= [+1, +\infty] & \text{if } j = i \\
T(i, -)_j &= T(i, +)_j = [-\infty, +\infty] & \text{if } j > i
\end{align*}
\]

In particular,

\[
\begin{align*}
T(1, -) &= [-\infty, -1] \times [-\infty, +\infty] \times [-\infty, +\infty] \times \cdots \\
T(1, +) &= [+1, +\infty] \times [-\infty, +\infty] \times [-\infty, +\infty] \times \cdots \\
T(2, -) &= [-1, +1] \times [-\infty, -1] \times [-\infty, +\infty] \times \cdots \\
T(2, +) &= [-1, +1] \times [+1, +\infty] \times [-\infty, +\infty] \times \cdots \\
\vdots
\end{align*}
\]

Note that given a \(d\)-tiling \(T\), the set \(T \cup T_{ext}\) is a set of interior disjoint \(d\)-dimensional boxes that tile \(\mathbb{R}^d\). The set \(T_{ext}\) is needed to define \textit{proper} \(d\)-tilings in Section \ref{proper_tilings} and it is used in the following technical lemma.

Given two intersecting boxes \(A\) and \(B\), if \(A_i \cap B_i\) is degenerate, then these two boxes are said to \textit{touch} in dimension \(i\).
Lemma 1. In a $d$-tiling $\mathcal{T}$, for any $A \in \mathcal{T}$ and any point $p \in A$. If $p_i = A_i^{-}$ (resp. $p_i = A_i^{+}$), there is a box $B \in \mathcal{T} \cup \mathcal{T}_{\text{ext}}$ such that $p \in A \cap B$, and such that $A$ and $B$ touch only in dimension $i$ (i.e. such that $A_j \cap B_j$ is degenerate only for $j = i$). In particular, $\dim(A \cap B) = d - 1$.

Proof. Define $Z \subseteq \mathbb{R}$ to be the set of all distinct numbers that appear as a coordinate of some corner of some box in $\mathcal{T} \cup \mathcal{T}_{\text{ext}}$, i.e.,

$$Z = \bigcup_{B \in \mathcal{T} \cup \mathcal{T}_{\text{ext}}} \bigcup_{j \in \{1, \ldots, d\}} \{B_j^{-}, B_j^{+}\}$$

Choose a real number $\epsilon > 0$ such that $\epsilon < \min\{|a - b| : a, b \in Z, a \neq b\}$. We now choose a point $q \in \mathbb{R}^d$ such that for each $j \in \{1, \ldots, d\} \setminus \{i\}$, $q_j \in A_j$ and $q_i \notin A_i$. We also make sure that $q$ is so close to $p$ that any box $B$ that contains $q$ also contains $p$. This can be achieved by choosing $q$ as follows.

$$q_j = \begin{cases} p_j - \epsilon & \text{if } j = i \text{ and } p_j = A_j^{-} \\ p_j + \epsilon & \text{if } j = i \text{ and } p_j = A_j^{+} \\ p_j & \text{if } A_j^{-} < p_j < A_j^{+} \\ p_j + \epsilon & \text{if } j \neq i \text{ and } p_j = A_j^{-} \\ p_j - \epsilon & \text{if } j \neq i \text{ and } p_j = A_j^{+} \end{cases}$$

Clearly, there is some box $B \in \mathcal{T} \cup \mathcal{T}_{\text{ext}}$ such that $q \in B$. Suppose that $p \notin B$, i.e., there exists some $j \in \{1, \ldots, d\}$ such that $p_j \notin B_j$. Then since $q_j \in B_j$, we have either $p_j < B_j^{-} \leq q_j$ or $q_j \leq B_j^{+} < p_j$. From the definition of $q$, we have that $p_j \in \{A_j^{+}, A_j^{-}\}$ (as otherwise $q_j = p_j$) and that $|p_j - q_j| = \epsilon$. This means that there exist distinct $a, b \in \{B_j^{+}, B_j^{-}, A_j^{+}, A_j^{-}\}$ such that $|a - b| \leq \epsilon$, which contradicts our choice of $\epsilon$. Therefore, we conclude that $p \in B$, and so $A$ and $B$ intersect.

Actually for each $j \in \{1, \ldots, d\} \setminus \{i\}$, by construction $q_j \in A_j$ and $q_j \notin \{A_j^{+}, A_j^{-}\}$. As $\{p_j, q_j\} \subset A_j \cap B_j$, we thus have that $A_j \cap B_j$ is degenerate. As $A$ and $B$ intersect on a box of dimension $d' < d$, it must be the case that $A_i \cap B_j$ is degenerate. This completes the proof. \hfill \square

Let $\mathcal{H}^{(i)}_x$ denote the hyperplane defined by $\{p \in \mathbb{R}^d : p_i = x\}$. Given a $d'$-dimensional box $B$, let us denote $B^{(i)}_x$ the intersection between $B$ and $\mathcal{H}^{(i)}_x$. Note that this intersection is either empty, or the box $B$ itself (if $B_i = [x, x]$), or it is a $(d' - 1)$-dimensional box. Depending on the context, $B^{(i)}_x$ is considered as a box of $\mathbb{R}^d$, or of $\mathbb{R}^{d-1}$ if we omit the $i$-th dimension. The hyperplane $\mathcal{H}^{(i)}_x$ is said to be generic w.r.t. a $d$-tiling $\mathcal{T}$, if for every $B \in \mathcal{T}$, $x \notin B^{-}_i$ and $x \neq B^{+}_i$. Given a $d$-tiling $\mathcal{T}$, let the intersection of $\mathcal{T}$ and $\mathcal{H}^{(i)}_x$ be the set $\mathcal{T}^{(i)}_x = \{B^{(i)}_x : B \in \mathcal{T} \text{ with } B^{-}_i \leq x \leq B^{+}_i\}$. The following lemma indicates when $\mathcal{T}^{(i)}_x$ defines a $(d-1)$-tiling.

Lemma 2. For every $d$-tiling $\mathcal{T}$, and every hyperplane $\mathcal{H}^{(i)}_x$ that is generic w.r.t. $\mathcal{T}$, $\mathcal{T}^{(i)}_x$ is a $(d-1)$-tiling.
Proof. As the boxes in $\mathcal{T}$ are $d$-dimensional, it is clear that $\mathcal{T}_{x}^{(i)}$ is a set of $(d-1)$-dimensional $(d-1)$-boxes (by omitting the $i$-th dimension). It is also clear that these boxes span $[-1, +1]^{d-1}$. What remains to prove is that these boxes are interior disjoint.

Towards a contradiction, suppose that $A'$ and $B' \in T_{x}^{(i)}$ intersect on a $(d-1)$-dimensional $(d-1)$-box $C'$. Thus if $A$ and $B$ are the corresponding original boxes in $\mathcal{T}$, i.e. those such that $A' = A|_{x}^{(i)}$ and $B' = B|_{x}^{(i)}$, then $C' = A \cap B = \emptyset$ is non-empty and non-degenerate for each dimension $j$ other than $i$. Moreover, $A \cap B$ is also non-empty as $x \in A \cap B$. As $\mathcal{T}$ is a $d$-tiling, this means that $A \cap B$ is degenerate, or in other words, $A$ and $B$ touch in dimension $i$. This implies that $x = A_i^-$ or $x = A_i^+$, a contradiction to the fact that $\mathcal{H}_x^{(i)}$ is generic w.r.t. $\mathcal{T}$. $\square$

For any $d$-tiling $\mathcal{T}$ and any hyperplane $\mathcal{H}_x^{(i)}$ that is generic w.r.t. $\mathcal{T}$, let us define $\mathcal{T}_x^{(i)-}$ and $\mathcal{T}_x^{(i)+}$ as the two parts obtained by cutting $\mathcal{T}$ through $\mathcal{H}_x^{(i)}$. In order to obtain $d$-tilings, we prolong the sides on $\mathcal{H}_x^{(i)}$ towards $\mathcal{H}_{x+1}^{(i)}$, or towards $\mathcal{H}_{x-1}^{(i)}$. Formally, for any $B \in \mathcal{T}$, let $B|_{x}^{(i)-}$ and $B|_{x}^{(i)+}$ be the boxes $B_1 \times \ldots \times \alpha^- (B_i) \times \ldots \times B_d$ and $B_1 \times \ldots \times \alpha^+ (B_i) \times \ldots \times B_d$, where,

\[
\alpha^-(B_i) = \begin{cases} 
\emptyset & \text{if } x \leq B_i^- \\
B_i & \text{if } B_i^+ < x \\
[B_i^-, +1] & \text{otherwise}
\end{cases}
\]

and

\[
\alpha^+(B_i) = \begin{cases} 
\emptyset & \text{if } B_i^+ \leq x \\
B_i & \text{if } x < B_i^- \\
[-1, B_i^+] & \text{otherwise}
\end{cases}
\]

Now, let $\mathcal{T}_x^{(i)-}$ (resp. $\mathcal{T}_x^{(i)+}$) be the set of non-empty boxes of the form $B|_{x}^{(i)-}$ (resp. $B|_{x}^{(i)+}$) for each $B \in \mathcal{T}$. The following lemma is trivial.

**Lemma 3.** For every $d$-tiling $\mathcal{T}$, and every hyperplane $\mathcal{H}_x^{(i)}$ that is generic w.r.t. $\mathcal{T}$, both $\mathcal{T}_x^{(i)-}$ and $\mathcal{T}_x^{(i)+}$ are $d$-tilings.

### 3 Proper $d$-tilings

The boxes satisfy the Helly property. Indeed, given a set $\mathcal{B}$ of pairwise intersecting boxes, the set $\bigcap_{B \in \mathcal{B}} B$ is a non-empty box. Graham-Pollak’s Theorem [8,12] asserts that to partition the edges of $K_n$ into complete bipartite graphs, one needs at least $n-1$ such graphs. Using this theorem Zaks proved the following.

**Lemma 4 (13 Zaks 1985).** Consider a set $\mathcal{B}$ of $d$-dimensional boxes that intersect, that is $\bigcap_{B \in \mathcal{B}} B \neq \emptyset$. If for every pair $A, B \in \mathcal{B}$, $\dim(A \cap B) = d-1$, then $|\mathcal{B}| \leq d + 1$.

We include a proof of this result for completeness.

**Proof.** Consider a point $x \in \bigcap_{B \in \mathcal{B}} B$. For any two boxes $A, B \in \mathcal{B}$, since $\dim(A \cap B) = d-1$, there exists exactly one dimension in which they touch. If this dimension is $t$, then $A_t \cap B_t = \{x_t\}$. As $A_t$ and $B_t$ are non-degenerate, we either have $A_t^+ = x_t = B_t^-$ or $B_t^+ = x_t = A_t^-$. 


Let $K$ be the complete graph with vertex set $B$. Now label each edge $AB$ of $K$ with $t$ if $A$ and $B$ touch in dimension $t$. As every pair of boxes touch in exactly one dimension, this labeling defines an edge partition of $K$ into $d$ subgraphs $G_1, \ldots, G_d$. Let us now prove that every such graph $G_i$ is a complete bipartite graph. The vertices $A$ with an incident edge in $G_t$ divide into two categories, those such that $x_t = A_t^-$ and those such that $x_t = A_t^+$. Any two boxes in the same category do not touch in dimension $t$, so these categories induce two independent sets in $G_t$. On the other hand, any two boxes in different categories do touch in dimension $t$, so they are adjacent in $G_t$.

So, by Graham-Pollak’s Theorem, $k \leq d + 1$. $\Box$

A $d$-tiling $T$ is proper if every point $p \in \mathbb{R}^d$ is contained in at most $d + 1$ boxes of $T \cup T_{\text{ext}}$. Figure 1 provides two configurations that are forbidden in proper $d$-tilings.

![Fig. 1. (left) A configuration that can appear in a tiling of $[-1, +1]^3$. Here every point of $[-1, +1]^3$ belongs to at most 4 boxes of $T$, but it is not proper when considering $T_{\text{ext}}$ as two points would belong to 5 boxes. (right) A configuration of 3-boxes pairwise intersecting on a 2-dimensional box. If this configuration was part of a 3-tiling there would be 5 boxes intersecting on a point, and thus it would not be proper.](image)

**Lemma 5.** In a proper $d$-tiling $T$, every pair of intersecting boxes intersect on a $(d-1)$-dimensional box.

**Proof.** We shall prove the following stronger statement:

(*) In a $d$-tiling $T$, if two boxes $A, B \in T \cup T_{\text{ext}}$ touch in more than one dimension, then there exists a point in $A \cap B$ that is contained in at least $d + 2$ boxes of $T \cup T_{\text{ext}}$.

For the sake of contradiction, consider two boxes $A$ and $B$ such that $\dim(A \cap B) = d - s$, with $s \geq 2$. Without loss of generality, we assume that these boxes touch in dimension $i$ for $1 \leq i \leq s$. Furthermore, we assume that $A_t^+ = B_t^-$ and let us call $p_i$ this value, for $1 \leq i \leq s$. 

Clearly, \( A \) and \( B \) do not belong both to \( T_{\text{ext}} \). Furthermore, as any box of \( T \) is contained in \([-1, +1]^d\), it can touch a box of \( T_{\text{ext}} \) in at most one dimension. Therefore, we have that \( A \) and \( B \in T \).

We shall prove \((*)\) by induction on \( d-s \). As the base case, we shall show that it is true when \( d-s = 0 \). We claim that the point \( p = (p_1, \ldots, p_d) \) is contained in at least \( d+2 \) boxes. By Lemma 1 there is a box \( H^{(i)} \neq A \) in \( T \cup T_{\text{ext}} \) such that \( H^{(i)} \) touches \( A \) only in dimension \( i \) and contains the point \( p \).

As \( H^{(i)} \) touches \( A \) only in dimension \( i \), all these boxes are distinct. Furthermore, as \( \dim(A \cap H^{(i)}) = d-1 \), each \( H^{(i)} \) is different from \( B \). So, together with \( A \) and \( B \) they form a collection of \( d+2 \) boxes that contain the point \( p \).

We consider now the case \( d-s > 0 \). As \( A \cap B \) is proper, we assume without loss of generality that \( B^-_d < A^+_d \leq B^+_d \). For an arbitrarily small \( \epsilon > 0 \), we have that no box of \( T \) has \( x = A^+_d - \epsilon \) as an endpoint of its \( d\text{st} \) interval.

By Lemma 2, \( T^{(d)}_x \) is a \((d-1)\)-tiling. Note that this tiling contains \( A \) and \( B \), and that those still touch in \( s \geq 2 \) dimensions. Therefore, by the induction hypothesis, there exists a point \( p' = (p'_1, \ldots, p'_{d-1}) \) that is contained in at least \( d+1 \) of these \((d-1)\)-boxes and also such that \( p' \in A \cap B \). Let these \((d-1)\)-boxes be \( A, B, H^{(1)}, \ldots, H^{(d-1)} \). Coming back to \( \mathbb{R}^d \), the point \( p = (p'_1, \ldots, p'_{d-1}, A^+_d) \) is contained in \( A \cap B \cap H^{(1)} \cap \cdots \cap H^{(d-1)} \). By Lemma 1 there exists a box \( F \) that contains the point \( p \) and touches \( A \) in dimension \( d \). This means that \( F^-_d = A^+_d > x \), therefore \( F \notin \{A, B, H^{(1)}, \ldots, H^{(d-1)}\} \). The point \( p \) is thus contained in \( d+2 \) distinct boxes. This concludes the proof of the lemma.

Lemma 4 and Lemma 5 imply the following.

**Theorem 1.** A \( d \text{-tiling} \) is proper if and only if for any two intersecting boxes \( A,B \) we have \( \dim(A \cap B) = d-1 \).

Lemma 6 also allows us to prove the following improvement of Lemma 2.

**Lemma 6.** For every proper \( d \)-tiling \( T \), and every hyperplane \( H_x^{(i)} \) that is generic w.r.t. \( T \), \( T^{(i)}_x \) is a proper \((d-1)\)-tiling.

**Proof.** By Lemma 2, \( T^{(i)}_x \) is a \((d-1)\)-tiling. It remains to prove that it is proper. For any pair of intersecting boxes \( A^{(i)}_x \) and \( B^{(i)}_x \), Theorem 1 implies that the original boxes \( A, B \in T \) touch in exactly one dimension. As \( x \) is not an endpoint of \( A_i \) or \( B_i \), this dimension cannot be \( i \). So the boxes \( A^{(i)}_x \) and \( B^{(i)}_x \) touch in exactly one dimension. By Theorem 1 this implies that \( T^{(i)}_x \) is proper.

Theorem 1 provides us a simple proof for the following strengthening of Lemma 3.

**Lemma 7.** For every proper \( d \)-tiling \( T \), and every hyperplane \( H_x^{(i)} \) that is generic w.r.t. \( T \), both \( T^{(i)}_x^- \) and \( T^{(i)}_x^+ \) are proper \( d \)-tilings.
Proof. By Theorem 1, we can suppose towards a contradiction that there are two boxes $A$ and $B$ of $\mathcal{T}|_{x}^{(i)-}\mathcal{T}_{ext}$ intersecting on a $d'$-dimensional box for some $d' < d - 1$. Clearly, $A$ and $B$ cannot belong to $\mathcal{T}_{ext}$ both. If $A \in \mathcal{T}|_{x}^{(i)-}$ and $B \in \mathcal{T}_{ext}$, say that $B = T(i, \ast)$ for some $i \in \{1, \ldots, d\}$ and $\ast \in \{-, +\}$, then as $A_j \subseteq [-1, +1] \subseteq B_j$ for all $j \neq i$ we have that $d' = d - 1$, a contradiction. Finally if both $A$ and $B$ belong to $\mathcal{T}|_{x}^{(i)-}$, and if they respectively come from $A'$ and $B'$ of $\mathcal{T}$, then we have that $A_j \cap B_j = A'_j \cap B'_j$ for all $j \neq i$. Furthermore, by construction $A_i \cap B_i$ is a non-degenerate interval if and only if $A'_i \cap B'_i$ is non-degenerate. Thus $A' \cap B'$ is also $d'$-dimensional, a contradiction. \qed

Lemma 5 now allows us to strengthen Theorem 1 as follows.

Theorem 2. A $d$-tiling is proper, if and only if for any set $\mathcal{B}$ of pairwise intersecting boxes we have $\dim(\cap_{B \in \mathcal{B}} B) = d + 1 - |\mathcal{B}|$.

Proof. Theorem 1 clearly implies that this condition is sufficient. Let us then prove that this condition is necessary. We have to prove that in a proper $d$-tiling, for any set $\mathcal{B}$ of pairwise intersecting boxes we have $\dim(\cap_{B \in \mathcal{B}} B) = d + 1 - |\mathcal{B}|$.

Let $k = |\mathcal{B}|$. By Lemma 4 we know that $1 \leq k \leq d + 1$. We already know that the implication holds for $k = 1$ or $k = 2$. For the remaining cases we proceed by induction on the pair $(d, k)$. That is, we assume the theorem holds for $(d', k')$ ($k'$ pairwise intersecting boxes in a proper $d'$-tiling) with $d' < d$ or with $d' = d$ and $k' < k$.

Consider any set $\mathcal{B}$ of $k$ pairwise intersecting boxes, and let $I = \cap_{B \in \mathcal{B}} B$.

We consider first that $\dim(I) > 0$, and we assume without loss of generality that $I$ is non-degenerate in dimension $d$. Let $x$ be a value in the interior of $I_d$, and such that $x$ is not an endpoint of $A_d$, for any box $A \in \mathcal{T}$. By Lemma 6 $\mathcal{T}|_{x}^{(d)}$ is a proper $(d - 1)$-tiling. In this tiling, $\mathcal{B}$ is also a set of $k$ pairwise intersecting boxes, which intersect on $I|_{x}^{(d)}$. By induction on $(d - 1, k)$ we have that $\dim(I|_{x}^{(d)}) = d - k$, and thus that $\dim(I) = d + 1 - k$.

We now consider that $\dim(I) = 0$ and that $k \leq d$ (if $k = d + 1$ we are fine). Consider any $B \in \mathcal{B}$, and let $J = \cap_{B \in \mathcal{B} \setminus B} H$. By induction hypothesis $\dim(J) = d + 2 - k \geq 2$, and without loss of generality we consider that $J$ is non-degenerate in dimension $i$ and only if $1 \leq i \leq \dim(J)$. As $I = B \cap J$ we have that $\dim(B \cap J) = 0$ and we can assume without loss of generality that $B_i^+ = J_i^-$ for every $i \in \{1, \ldots, \dim(J)\}$. Let us also denote by $p$ the point where $B$ and $J$ intersect.

Claim. For every $i \in \{1, \ldots, \dim(J)\}$ there is a box $F$ of $(\mathcal{T} \cup \mathcal{T}_{ext}) \setminus \mathcal{B}$ such that $p \in F$, such that $\dim(F \cap J) = \dim(J) - 1 = d + 1 - k$, and such that $F_i \cap J_i$ is degenerate.

For each $i \in \{1, \ldots, \dim(J)\}$, if $J_i^- = p_i$ it is because some box $A \in \mathcal{B} \setminus B$ is such that $A_i^- = p_i$. Let $q$ be an interior point of $J|_{p_i}^{(i)}$ that is arbitrarily close to $p$ (thus every box containing $q$ also contains $p$). By Lemma 1, there is a box $F \in \mathcal{T} \cup \mathcal{T}_{ext}$ such that $q \in A \cap F$, and such that $A$ and $F$ touch only in
dimension $i$. As $F_i^+ = p_i$, we have that $F_i \cap J_i = [p_i, p_i]$ is degenerate while for every $j \in \{1, \ldots, \dim(J)\} \setminus \{i\}$ as $q_j \in F_i \cap J_i = F_i \cap H \in B \setminus B$ the interval $F_i \cap J_i$ is non-degenerate. We thus have that $\dim(F \cap J) = \dim(J) - 1 \geq 1$. As $\dim(B \cap J) = 0$ we have that $F \neq B$, and as $J \not\subset F$, $F \not\in B \setminus B$. So this box $F$ does not belong to $B$.

Claim. There are $\dim(J)$ such boxes $F$.

If a box $F$ is such that $F_i \cap J_i$ is degenerate for two distinct values in $\{1, \ldots, \dim(J)\}$ then $\dim(F \cap J) \leq \dim(J) - 2$, and so $F$ does not verify the previous claim. So for each value $i \in \{1, \ldots, \dim(J)\}$ there is a distinct box $F$ fulfilling the previous claim.

The theorem now follows from the fact that all the $k$ boxes of $B$ and all the $\dim(J) = d + 2 - k$ boxes $F$ intersect at $p$, contradicting the fact that $T$ is proper. □

4 Separations

Let us now define an equivalence relation $\sim$ on the set of sides of $T \cup T_{ext}$. The relation $\sim$ is the transitive closure of the relation linking two sides if and only if they intersect on a $(d - 1)$-dimensional box. If the boxes $A$ and $B$ touch in dimension $i$, then $S(A, i, *)$ and $S(B, i, *^{-1})$ intersect on a $(d - 1)$-dimensional box, for some $* \in \{-1, +1\}$. A separation is then defined as the union of all the boxes of some equivalence class of $\sim$. Note that a separation is a finite union of $(d - 1)$-dimensional boxes that are degenerate in the same dimension. If this dimension is $i$, by extension we say that this separation is degenerated in dimension $i$.

Lemma 8. Any separation $S$ of a proper $d$-tiling $T$ is a $(d - 1)$-dimensional box.

Proof. This clearly holds for $d \leq 2$, we thus assume that $d \geq 3$. Consider a separation $S$ that is degenerated in dimension $i$. By induction on $d$ we obtain that for any $j \in \{1, \ldots, d\} \setminus \{i\}$ and any $x \in [-1, +1]$ $S_{[x]}^{(j)}$ is either empty, or it is a box. Indeed, for every generic $H_{[x]}^{(j)}$, $S_{[x]}^{(j)}$ is either empty, or it is a separation of $T_{[x]}^{(i)}$.

Let us first prove the lemma for $d = 3$. If $S$ is not a 2-dimensional box it has a $3\pi/2$ angle at some point $p$. It is clear that two boxes are necessary below (resp. above) $p$ with respect to the dimension $i$ to form the separation $S$, while the remaining $\pi/2$ angle at $p$ has to be covered by another (at least) fifth box, contradicting the fact that $T$ is proper.

For $d \geq 4$, the lemma follows from the following claim (considering $S \subseteq H_{[x]}^{(i)} \simeq \mathbb{R}^{d-1}$).

Claim. Consider a connected set $S \subset \mathbb{R}^d$ with $d \geq 3$, that is a finite union of $d$-dimensional boxes. If for any $i \in \{1, \ldots, d\}$ and any $x \in [-1, +1]$ we have that $S_{[x]}^{(i)}$ is either empty or a box, then $S$ is a box.
Towards a contradiction suppose that $S$ is not a box. In such case for some dimension $i \in \{1, \ldots, d\}$, and for some values $x, x' \in (-1, 1)$ the sets $S_{i}^{(i)}$ and $S_{i}^{(i)}$ are two boxes that differ (at least) on their $j$th interval for some $j \neq i$. Thus following a curve going from $S_{j}^{(i)}$ to $S_{j}^{(i)}$ one goes through a point $p \in \mathbb{R}^d$ with the following property: For any sufficiently small $\epsilon$, the sets $S_{i}^{(i)}$ and $S_{i}^{(i)}$ are two boxes that differ (at least) on their $j$th interval for some $j \neq i$ and, as $S$ is a finite union of boxes, these boxes respectively contain the points $p$ and $p' = (p_1, \ldots, p_{i-1}, p_i + \epsilon, p_{i+1}, \ldots, p_d)$. As their $j$th interval differ, we can assume that for some $y \in (-1, 1)$ we have that $q = (p_1, \ldots, p_{i-1}, p_j - 1, y, p_{j+1}, \ldots, p_d) \in S_{i}^{(i)}$ and that $q' = (p_1, \ldots, p_{i-1}, p_j + 1, y, p_{j+1}, \ldots, p_d) \notin S_{i}^{(i)}$. This implies that for any $k \neq i, j$, the set $S_{i}^{(k)}$ contains exactly three of the aforementioned four points, contradicting the fact that it is a box.

A $d$-tiling $\mathcal{T}$ is said in general position if it does not contain two coplanar separations, that is two separations belonging to the same hyperplane $\mathcal{H}_x^{(i)}$.

**Lemma 9.** Any proper $d$-tiling $\mathcal{T}$ can be slightly perturbed to obtain an equivalent one $\mathcal{T}'$ that is in general position. Here equivalent means that the elements of $\mathcal{T}$ and $\mathcal{T}'$ are in bijection and that two boxes of $\mathcal{T} \cup \mathcal{T}_{ext}$ touch in dimension $i$ if and only if the corresponding two boxes of $\mathcal{T}' \cup \mathcal{T}_{ext}$ touch in dimension $i$.

**Proof.** First note that two coplanar separations do not intersect. Otherwise, if two such separations $S$ and $S'$ would intersect at some point $p$, then there would be a box $A$ below $S$ (with respect to $i$) that would contain $p$ and a box $B$ above $S'$ (with respect to $i$) that would also contain $p$. As $A$ and $B$ intersect they should intersect on a $(d-1)$-dimensional box, but as this intersection is degenerated in dimension $i$ then $S(A, i, +) \sim S(B, i, -)$, contradicting the fact that they belong to distinct equivalence classes.

Let us now proceed by induction on the number of separations that are coplanar with another separation. Consider such a separation $S$ that is degenerated in dimension $i$. For any box $A$ below $S$ (resp. $B$ above $S$) with respect to dimension $i$, replace $A_i = [A_i^-, A_i^+]$ with $[A_i^-, A_i^+ + \epsilon]$ (resp. replace $B_i = [B_i^-, B_i^+]$ with $[B_i^- + \epsilon, B_i^+]$). It is clear that for a sufficiently small $\epsilon$ any two boxes intersect and touch on a dimension $j \neq i$ in $\mathcal{T} \cup \mathcal{T}_{ext}$ if and only if they did in $\mathcal{T} \cup \mathcal{T}_{ext}$. The intersections degenerated in dimension $i$ either remained the same, either were simply translated (for those belonging to $S$). As we have now one separation less that is coplanar with another separation, we are done.

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### 5 Simplicial complexes and Dushnik-Miller dimension

Given a $d$-tiling $\mathcal{T}$, the simplicial complex $\mathcal{S}(\mathcal{T})$ **induced** by $\mathcal{T}$ is defined as follows. Let $\mathcal{T}$ be the vertex set of $\mathcal{S}(\mathcal{T})$ and let a set $F \subseteq \mathcal{T}$ be a face of $\mathcal{S}(\mathcal{T})$ if and only if the elements of $F$ intersect, that is if $\bigcap_{B \in F} B \neq \emptyset$. From this definition, it is clear that if $F \subseteq \mathcal{T}$ is a face of $\mathcal{S}(\mathcal{T})$, any subset of $F$ is also a face of $\mathcal{S}(\mathcal{T})$. So $\mathcal{S}(\mathcal{T})$ is indeed a simplicial complex.
The Dushnik-Miller dimension of a simplicial complex $S$, denoted by $\dim_{DM}(S)$, is the minimum integer $k$ such that there exist $k$ linear orders $<_{1}, \ldots, <_{k}$ on $V$, where $V$ is the vertex set of $S$, such that for every face $F$ of $S$ and for every vertex $u \in V$, there exists some $i$ such that $\forall v \in F, v \leq_{i} u$. Such set of orders is said to be a realizer of $S$. Note that if $T$ has $p$ pairwise intersecting boxes, $S(T)$ has a face $F$ of size $p$ (usually such face is said to have dimension $p - 1$), and this implies that $\dim_{DM}(S(T)) \geq p$. Indeed, every vertex $v \in F$ has to be greater than the other vertices of $F$ in some order. This shows why Theorem 3 is tight in most cases.

**Definition 2.** Given a proper $d$-tiling $T$, let $\overrightarrow{G}(T)$ be the digraph with vertex set $T$ and with an arc $AB$ if and only if there exist points $a \in A$ and $b \in B$ such that $b_{i} < a_{i}$ for all $i \in \{1, \ldots, d\}$. Note that this is equivalent as saying that $\overrightarrow{G}(T)$ has an arc $AB$ if and only if $B_{i}^{-} < A_{i}^{+}$ for all $i \in \{1, \ldots, d\}$.

**Lemma 10.** For any proper $d$-tiling $T$, the digraph $\overrightarrow{G}(T)$ is acyclic.

**Proof.** We proceed by induction on $d$ and on the size of $T$. The lemma clearly hold for $d = 1$ or for $|T| = 1$, and we thus focus on the induction step. We thus consider any proper $d$-tiling $T$ with $d > 1$ and $|T| > 1$, and we assume that the lemma holds for any $d'$-tiling with $d' < d$, and for any $d$-tiling with less boxes than $T$. By Lemma 9 we also assume that $T$ is in general position.

Let $X$ be the unique box of $T$ containing the point $(-1, \ldots, -1)$. It is clear that $X$ is a sink in $\overrightarrow{G}(T)$, so it suffices to show that $\overrightarrow{G}(T) \setminus X$ is acyclic. In the following we show this by constructing a proper $d$-tiling $T'$ with one box less than $T$ and such that $\overrightarrow{G}(T) \setminus X$ is a subgraph of $\overrightarrow{G}(T')$. The digraph $\overrightarrow{G}(T')$ being acyclic by induction hypothesis, so is its subgraph $\overrightarrow{G}(T) \setminus X$.

**Claim.** There exists an $i \in \{1, \ldots, d\}$ and a box $Y \in T$ such that $X_{i}^{+} = Y_{i}^{-}$, and such that $X_{j}^{+} = Y_{j}^{+}$ for all $j \neq i$.

For any point $p \in \mathbb{R}^{d}$, any $\epsilon > 0$, and any box $B$, the set $\mathcal{P}(p, \epsilon)$ of $2^{d}$ points $(p_{1} + \epsilon_{1}, \ldots, p_{d} + \epsilon_{d})$, for $\epsilon_{i} \in \{-\epsilon, +\epsilon\}^{d}$, intersect $B$ on a number of points that is a power of two. Furthermore, when $\epsilon$ is sufficiently small, all the boxes of $T$ intersecting $\mathcal{P}(p, \epsilon)$ contain the point $p$ (and thus intersect each other), and any point of $\mathcal{P}(p, \epsilon)$ belongs to exactly one box. Thus for the point $p$ defined by $p_{i} = X_{i}^{+}$, and for a sufficiently small $\epsilon > 0$, the box $X$ contains exactly one point of $\mathcal{P}(p, \epsilon)$. But as $|\mathcal{P}(p, \epsilon)|$ is even, there is (at least) one other box in $T$ that contains exactly one point of $\mathcal{P}(p, \epsilon)$, let us denote $Y$ this box. Since $X$ and $Y$ intersect on a $(d - 1)$-dimensional box, let us denote $i$ the dimension where they touch, and note that $X_{i}^{-} = Y_{i}^{-}$, and that $X_{j}^{+} = Y_{j}^{+}$ for all $j \neq i$, otherwise $Y$ would intersect $\mathcal{P}(p, \epsilon)$ on more points.

**Claim.** For any box $B \in T$ touching $X$ in dimension $i$ we have that its side $S(B, i, -)$ is contained in $X$’s side $S(X, i, +)$. 
The previous claim implies that for each $j \neq i$ there exists a separation containing $S(X, j, +)$ and $S(Y, j, +)$. By Lemma 9 such separation contains the box $[-1, X_i^+] \times \cdots \times [-1, X_{i+1}^+] \times [-1, X_i^+ + \epsilon] \times [-1, X_{i+1}^+ + \epsilon] \times \cdots \times [-1, X_j^+]$. If some box $B \in T$ touching $X$ in dimension $i$ has its side $S(B, i, -)$ not contained in $X$’s side, for example because $S(B, i, -) \subseteq S(X, i, +)$, then some interior point of $S(B, i, -)$ is also in the interior of $S(X, j, +)$, a contradiction.

We thus define $T'$ from $T \setminus X$ in the following way:

- For any box $B \in T \setminus X$ touching $X$ in dimension $i$ we define a box $B'$ in $T'$ by setting $B'_i = [-1, B_i^+]$ and $B'_j = B_j$ for $j \neq i$.
- Any other box $B \in T \setminus X$ is contained in $T'$. In this context this box is denoted $B'$.

**Claim.** $T'$ is a proper $d$-tiling.

Every box $B' \in T'$ contains the corresponding box $B \in T \setminus X$ so if there is a point $p' \in [-1, +1]^d$ not covered by $T'$, it is a point of $X$. But by construction this would imply that the point $p$, defined by $p_i = X_i^+ + \epsilon$ and $p_j = p_j'$ for any $j \neq i$, is not covered by $T$, a contradiction. One can similarly prove that the boxes of $T'$ are interior disjoint. $T'$ is thus a $d$-tiling. It remains to prove that it is a proper one. Towards a contradiction, assume that there exist two intersecting boxes $A, B' \in T'$ that touch in at least two dimensions. As $A \subseteq A'$ and $B \subseteq B'$ the boxes $A$ and $B$ do not intersect. This implies that one of these boxes, say $A$, touches $X$ in dimension $i$ while the other, $B$, touches $X$ in a dimension $j \neq i$. This implies that $A'$ and $B'$ touch in dimension $j$ and in another dimension $k \in \{1, \ldots, d\} \setminus \{i, j\}$. We thus either have that $A_k^+ = A_k'^+ = B_k^- = B_k'$ or that $B_k^+ = B_k'^+ = A_k^- = A_k'$. Whatever the case we denote $x$ this value, and as both $A_k$ and $B_k$ intersect $X_k$ in more than one point, we have that $-1 < x < A_k^+$. As $T$ is in general position, it admits a separation $S$ such that $S_k = [x, x]$ containing a point in the interior of $S(X, i, +)$ (as it is bordered by $A$) and a point in the interior of $S(X, j, +)$ (as it is bordered by $B$). As $S$ is a box it thus contains a point in $X$’s interior, a contradiction.

As any box $B \in T \setminus X$ is contained in the corresponding box $B' \in T'$, for any arc $AB \in \overline{G}(T) \setminus X$ we have that $B_i^- \leq B_i^+ < A_i^+ = A_i'^+$ for all $i \in \{1, \ldots, d\}$, and thus there is an arc $A'B' \in \overline{G}(T')$. This concludes the proof of the lemma.

**Theorem 3.** Given a proper $d$-tiling $T$, we have that $\dim_{\text{DM}}(S(T)) \leq d + 1$.

**Proof.** Consider the orders $(\leq_1, \ldots, \leq_{d+1})$ defined as follows. If two distinct boxes $A, B \in T$ are such that $B_i^- < A_i^+$ for all $1 \leq i \leq d$, then $A <_{d+1} B$. By Lemma 10 the transitive closure of this relation is antisymmetric. In the following let $<_{d+1}$ be any of its linear extensions. For $<_i$ with $1 \leq i \leq d$, given two boxes $A, B \in T$, $A <_i B$ if and only if $A_i^- < B_i^-$, or if $A_i^- = B_i^-$ and $A <_{d+1} B$. Those relations are clearly total orders.
Let us now prove that \( \{<1,\ldots,<_{d+1}\} \) is a realizer of \( S(T) \). To do so, we prove that for any point \( p \) and any box \( B \in T \), that the set \( A(p) \) of boxes containing \( p \) is dominated by \( B \) in some order \( <_i \). By extension of notation, in such case we say that \( A(p) <_i B \). Note that \( B \) verifies one of the following cases:

1. \( p_i < B^-_i \) for some \( 1 \leq i \leq d \),
2. \( B^-_i < p_i \) for all \( 1 \leq i \leq d \), or
3. \( B^-_i = p_i \) for all \( i \in I \), for some non-empty set \( I \subseteq \{1,\ldots,d\} \), and \( B^-_i < p_i \), otherwise.

In case (1), as \( A^-_i \leq p_i < B^-_i \) for all \( A \in A(p) \), then \( A <_i B \) for all \( A \in A(p) \), that is \( A(p) <_i B \).

In case (2), as \( B^-_i < p_i \leq A^+_j \) for all \( A \in A(p) \) and all \( 1 \leq i \leq d \), then \( A(p) <_{i+1} B \).

Case (3) is more intricate. Towards a contradiction we suppose that \( A(p) \not<_i B \) for all \( 1 \leq i \leq d + 1 \). Let \( I \subseteq \{1,\ldots,d\} \) be the non-empty set such that \( B^-_i = p_i \) if and only if \( i \in I \). Consider the directed graph \( D \) with vertex set \( I \) and which contains an arc \((i,j)\) if and only if there exists a box \( A \in A(p) \) such that \( A^-_i = p_i \) and \( A^+_j = p_j \). Note that as every box is \( d \)-dimensional \( D \) has no loop \((i,i)\).

**Claim.** Every vertex in \( D \) has at least one outgoing arc.

For any \( i \in I \), as \( A(p) \not<_i B \), there exists at least one box \( A \in A(p) \) such that \( B <_i A \). By definition of \( <_i \) this box is such that \( A^-_i = p_i \) (as \( p_i = B^-_i \leq A^-_i \leq p_i \)) and such that \( B <_{i+1} A \). As \( A \not<_{i+1} B \) there exists a \( j \in \{1,\ldots,d\} \) such that \( A^+_j \leq B^-_j \), but as \( B^-_j \leq p_j \leq A^+_j \), we have \( A^+_j = p_j = B^-_j \). Thus \( j \in I \) and \( D \) has an arc \((i,j)\).

Thus \( D \) is not acyclic and we can consider a circuit of minimum length \( C = (i_0,\ldots,i_k) \) in \( D \), with \( k \geq 1 \). For every \( j \in \{0,\ldots,k\} \) let \( A(j) \) be a box of \( A(p) \) such that \( A(j)^{i_j} = p_{i_j} \) and \( A(j)^{i_{j+1}} = p_{i_{j+1}} \) (where \( j + 1 \) is understood modulo \( k + 1 \)).

**Claim.** All the \( k + 1 \) boxes \( A(j) \) are distinct.

Towards a contradiction, consider there exists two distinct values \( j \) and \( j' \) such that \( A(j) \) and \( A(j') \) are identical. Let us call \( A \) this box. By definition of \( A(j) \) and \( A(j') \), this box \( A \) is such that \( A^-_{i_j} = p_{i_j} \), \( A^+_{i_{j+1}} = p_{i_{j+1}} \). Thus there is an arc from \( i_j \) to \( i_{j+1} \), contradicting the minimality of \( C \).

Note that the intersection of these \( k + 1 \) boxes is degenerate in dimensions \( i_j \) for \( 0 \leq j \leq k \). Thus these \( k + 1 \) boxes intersect in a box of dimension at most \( d - k - 1 \). This contradicts Theorem 2 and concludes the proof of the theorem.

\[ \square \]

6 Conclusion

It would be appreciable to deal with contact system of boxes instead of \( d \)-tilings, that is to deal with sets of interior disjoint \( d \)-dimensional boxes not necessarily spanning \([-1,+1]^d\) or \( \mathbb{R}^d \). For this purpose, we conjecture the following.
Conjecture 1. A set $C$ of $d$-dimensional boxes in $[-1,+1]^d$ is a subset of a proper $d$-tiling $T$ if and only if every set $B \subset C$ of pairwise intersecting boxes is such that $\dim(\bigcap_{B \in B} B) = d + 1 - |B|$.

Thomassen [11] (see also [6]) characterized the intersection graphs of proper 2-tilings, exactly as the strict subgraphs of the 4-connected planar triangulations. The 4-connected planar triangulations are those where every triangle bounds a face. A simplicial complex has the Helly property if every clique in its skeleton is a face of the simplicial complex. As the simplicial complexes defined by $d$-tilings have the Helly property, we conjecture the following:

Conjecture 2. A simplicial complex $S$ is such that $S = S(T \cup T_{ext})$ for some proper $d$-tiling $T$, if and only if $S$ is a triangulation of the $d$-dimensional octahedron with the Helly property and with Dushnik-Miller dimension $d + 1$.

Similarly, is it possible to generalize the fact that bipartite planar graphs are the intersection graphs of non-intersecting and axis parallel segments in the plane [5]?

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