HOLOMORPHIC INJECTIVE EXTENSIONS OF
FUNCTIONS IN \( P(K) \) AND ALGEBRA GENERATORS

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ABSTRACT. We present necessary and sufficient conditions on planar compacta \( K \) and continuous functions \( f \) on \( K \) in order that \( f \) generates the algebras \( P(K), R(K), A(K) \) or \( C(K) \). We also unveil quite surprisingly simple examples of non-polynomial convex compacta \( K \subseteq \mathbb{C} \) and \( f \in P(K) \) with the property that \( f \in P(K) \) is a homeomorphism, but for which \( f^{-1} \notin P(f(K)) \). As a consequence, such functions do not admit injective holomorphic extensions to the interior of the polynomial convex hull \( \hat{K} \). On the other hand, it will be shown that the restriction \( f^*|_G \) of the Gelfand-transform \( f^* \) of an injective function \( f \in P(K) \) is injective on every regular, bounded complementary component \( G \) of \( K \). A necessary and sufficient condition in terms of the behaviour of \( f \) on the outer boundary of \( K \) is given in order \( f \) admits a holomorphic injective extension to \( \hat{K} \). We also include some results on the existence of continuous logarithms on punctured compacta containing the origin in their boundary.

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INTRODUCTION

Let \( K \) be a compact set in the complex plane \( \mathbb{C} \). As usual, \( P(K) \) denotes the set of complex-valued continuous functions on \( K \) that can be uniformly approximated by polynomials. Endowed with the usual algebraic operations and the supremum norm, \( P(K) \) is a uniformly closed subalgebra of \( C(K) \). By definition, the monomial \( z \) is a generator for \( P(K) \). We recall the following definition:

**Definition 0.1.** If \( A \) is a commutative unital Banach algebra and \( S \) a subset of \( A \), then the smallest closed subalgebra of \( A \) containing \( S \) is denoted by \( [S]_{\text{alg}} \). We also say that \( [S]_{\text{alg}} \) is the algebra generated by \( S \).

Note that \( [S]_{\text{alg}} \) is the norm-closure of the set of all polynomials of the form \( \sum a_{ij} f_1^{n_1} \cdots f_j^{n_j} \), where \( f_k \in S, i = (n_1, \ldots, n_j) \in \mathbb{N}^j \) and \( j \in \mathbb{N}^* \).

We are interested in the following question: which functions are generators for \( P(K) \)? We also consider the associated algebras

\[
A(K) = \{ f \in C(K) : f \text{ holomorphic in the interior } K^o \text{ of } K \},
\]

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and $R(K)$, the uniform closure of the set $R_0(K)$ of rational functions without poles on $K$.

We present in Section 1, which represents the motivational part of this paper, the answer to this question. The description in the case of the algebra $P(K)$ leads to the following problem: if $f \in P(K)$ is a homeomorphism, is the unique holomorphic extension $f^*$ of $f$ to the polynomial convex hull $\hat{K}$ of $K$ injective?

In the case where $K$ is the unit circle $\mathbb{T}$, a classical result, known under the name of the Darboux-Picard theorem (see [3, p. 310]) tells us that $f^*$ actually is injective on the closed unit disk $D$. Generalizations in various directions had been established (see [3]). The general situation, however, does not seem to have been solved. We give a nice example showing that the answer to the preceding question is negative. Our main goal then will be achieved in Section 2, namely a proof of the following result: if $f \in P(K)$ then the Gelfand transform, $f^*$, of $f$ is injective on $\hat{K}$ if and only if $f$ maps the outer boundary of $K$ onto the outer boundary of $f(K)$. Our method involves Eilenberg’s representation theorem for zero-free functions on compacta as well as a homotopic variant of Rouché’s theorem. As a corollary we obtain that for every injective function $f \in P(K)$, the restriction $f^*|G$ of $f^*$ to a regular hole $G$ of $K$ is injective. Here a hole $G$ of $K$ is called regular if $G$ is the only hole of its boundary. In particular, if $K$ has a connected complement and a connected interior, then $f^*$ is injective on $K$ if and only if $f \in P(\partial K)$ is injective.

In Section 3 we deal with a feature not covered by Eilenberg’s theorem: under which conditions on $K$ with $0 \in \partial K$ does there exist a continuous branch of the logarithm on $K \setminus \{0\}$? (In Eilenberg’s theorem 0 belongs to the complement of $K$).

1. Algebra Generators

**Theorem 1.1.** Let $K \subseteq \mathbb{C}$ be compact and $\varphi \in C(K, \mathbb{C})$. The following assertions are equivalent:

1. $\varphi$ is a generator for $C(K, \mathbb{C})$; that is $C(K, \mathbb{C}) = [\varphi]_{\text{alg}}$;
2. $\varphi$ is a homeomorphism of $K$ onto $\varphi(K)$, $K^0 = \emptyset$ and $\mathbb{C} \setminus K$ is connected.

**Proof.** It is clear that every generator for $C(K, \mathbb{C})$ is point separating. Hence, $\varphi$ must be a homeomorphism of $K$ onto its image. Let $f \in C(K, \mathbb{C})$. We first show that $f \in [\varphi]_{\text{alg}}$ if and only if $f \circ \varphi^{-1} \in P(\varphi(K))$. In fact, $f \in [\varphi]_{\text{alg}}$ if and only if $p_n(\varphi) \to f$ uniformly on $K$ for some sequence of polynomials $p_n \in \mathbb{C}[z]$. But

$$\max_{z \in K} |p_n(\varphi(z)) - f(z)| \to 0 \iff \max_{w \in \varphi(K)} |p_n(w) - f(\varphi^{-1}(w))| \to 0.$$ 

This in turn is equivalent to $f \circ \varphi^{-1} \in P(\varphi(K))$. Next we observe that every $h \in C(\varphi(K), \mathbb{C})$ writes as $f \circ \varphi^{-1}$ for some $f \in C(K, \mathbb{C})$; just put
\[ f = h \circ \varphi. \] We conclude that the assumption \( C(K, \mathbb{C}) = [\varphi]_{\text{alg}} \) is equivalent to the assumption \( C(\varphi(K), \mathbb{C}) = P(\varphi(K)) \), whenever \( \varphi \) is an homeomorphism. By Lavrentiev’s theorem [2, p. 192], this happens if and only if \( \varphi(K)^\circ = \emptyset \) and \( \mathbb{C} \setminus \varphi(K) \) is connected. Now \( \varphi(K)^\circ = \emptyset \) if and only if \( K^\circ = \emptyset \). Moreover, the number of connected components of the complement of a compact set in \( \mathbb{C} \) is invariant under homeomorphisms (see [3, p. 99]). Hence condition (2) is necessary and sufficient for \( C(K, \mathbb{C}) \) to be singly generated by \( \varphi \). \( \square \)

**Remark 1.2.** Let \( K \subseteq \mathbb{R} \) be compact and \( \varphi \in C(K, \mathbb{R}) \). The following assertions are equivalent:

(1) \( \varphi \) is a generator for \( C(K, \mathbb{R}) \); that is \( C(K, \mathbb{R}) = [\varphi]_{\text{alg}} \);

(2) \( \varphi \) is a homeomorphism of \( K \) onto \( \varphi(K) \).

*Proof.* As above, if \( \varphi \) is a homeomorphism of \( K \) onto its image, the assumption \( C(K, \mathbb{R}) = [\varphi]_{\text{alg}} \) is equivalent to the assumption \( C(\varphi(K), \mathbb{R}) = P_{\mathbb{R}}(\varphi(K)) \). \( ^1 \) This is always true, though, by Weierstrass’ approximation theorem. \( \square \)

**Theorem 1.3.** Let \( K \subseteq \mathbb{C} \) be compact and \( \varphi \in A(K) \). The following assertions are equivalent:

(1) \( \varphi \) is a generator for \( A(K) \); that is \( A(K) = [\varphi]_{\text{alg}} \);

(2) \( \varphi \) is a homeomorphism of \( K \) onto \( \varphi(K) \) and \( \mathbb{C} \setminus K \) is connected.

*Proof.* As in the previous theorem, we obtain that the assumption \( A(K) = [\varphi]_{\text{alg}} \) is equivalent to the assumption \( A(\varphi(K)) = P(\varphi(K)) \) whenever \( \varphi \in A(K) \) is an homeomorphism. Note that \( \varphi^{-1} \in A(\varphi(K)) \). By Mergelyan’s theorem [9], this happens if and only if \( \mathbb{C} \setminus K \) is connected. \( \square \)

The proof of the corresponding result for \( R(K) \) and \( P(K) \) needs an additional argument:

**Lemma 1.4.** Let \( K \subseteq \mathbb{C} \) be compact and \( \varphi \in C(K) \). The following assertions hold:

(1) If \( \varphi \in R(K) \), then \( h \in R(\varphi(K)) \) implies that \( f := h \circ \varphi \in R(K) \).

(2) If \( \varphi \in P(K) \), then \( h \in P(\varphi(K)) \) implies that \( f := h \circ \varphi \in P(K) \).

*Proof.* (1) Let \( (r_n(w)) \) denote a sequence of rational functions without poles on \( \varphi(K) \) converging uniformly on \( \varphi(K) \) to \( h(w) \). Then

\[ \max_{z \in K} |r_n(\varphi(z)) - h(\varphi(z))| \to 0. \]

Next, let \( (\varphi_n(z)) \) be a sequence of rational functions without poles on \( K \) converging uniformly on \( K \) to \( \varphi(z) \). We claim that the following assertions hold:

i) For every \( n \) there exists \( j_n > n \) such that \( r_n \circ \varphi_{j_n} \) is a rational function without poles on \( K \).

ii) \( (r_n \circ \varphi_{j_n}) \) converges uniformly on \( K \) to \( h \circ \varphi \).

\( ^1 \) This is, per definition, the uniform closure of the set of real polynomials on \( \varphi(K) \).
In fact, since it is obvious that $r_n \circ \varphi_j$ is a rational function again, it remains to prove for i) that $j \geq n$ can be chosen so that $r_n \circ \varphi_j$ has no poles on $K$. To see this, we observe that $r_n$ has no poles in the closure of an open neighborhood $U_n$ of $\varphi(K)$. Let $\varepsilon_n = \text{dist}(\varphi(K), \mathbb{C} \setminus U_n)$. The compactness of $\varphi(K)$ implies that $\varepsilon_n > 0$. Since $||\varphi_j - \varphi||_K \to 0$, $\text{dist}(\varphi_j(z), \varphi(K)) < \varepsilon_n/2$ for every $z \in K$ and $j \geq j_n^* > n$. Thus, for all $z \in K$ and $j \geq j_n^*$, $\varphi_j^*(z) \in U_n$. Hence $r_n \circ \varphi_j$ has no poles on $K$ when $j \geq j_n^*$. This gives i).

ii) Fix $n$. Since $r_n$ is uniformly continuous on $U_n$, we may choose $j_n \geq j_n^*$ so big that

$$||r_n \circ \varphi_{j_n} - r_n \circ \varphi||_K < 1/n.$$  

Then ii) is a consequence of the following estimations:

$$|r_n \circ \varphi_{j_n} - h \circ \varphi| \leq |r_n \circ \varphi_{j_n} - r_n \circ \varphi| + |r_n \circ \varphi - h \circ \varphi|$$

$$\leq 1/n + \varepsilon/2 < \varepsilon$$

for $n \geq n_0$. We conclude that $h \circ \varphi \in R(K)$.

(2) This works as in part ii) above, where rational functions are replaced by polynomials. Note that i) is irrelevant here.

\textbf{Theorem 1.5.} Let $K \subseteq \mathbb{C}$ be compact and $\varphi \in R(K)$. The following assertions are equivalent:

1. $\varphi$ is a generator for $R(K)$; that is $R(K) = [\varphi]_{\text{alg}}$; 
2. $\varphi$ is a homeomorphism of $K$ onto $\varphi(K)$ and $\mathbb{C} \setminus K$ is connected.

\textbf{Proof.} As usual, we see that for homeomorphic maps $\varphi$ and $f \in R(K)$ one has $f \in [\varphi]_{\text{alg}}$ if and only if $f \circ \varphi^{-1} \in P(\varphi(K))$.

(1) $\implies$ (2) Let $h \in R(\varphi(K))$. Since, by assumption, $\varphi \in R(K)$, we deduce from Lemma 1.4 that $f := h \circ \varphi \in R(K)$. Hence $h = f \circ \varphi^{-1} \in P(\varphi(K))$ if $\varphi$ is a generator for $R(K)$. Thus $P(\varphi(K)) = R(\varphi(K))$. By Runge’s theorem, $\varphi(K)$ has connected complement, and so the same is true for $K$.

(2) $\implies$ (1) If $K$ (and so $\varphi(K)$), has connected complement, then by Mergelyan’s Theorem, see [9], $P(\varphi(K)) = R(\varphi(K)) = A(\varphi(K))$. Consider any $f \in R(K)$ and let $h := f \circ \varphi^{-1}$. Then $h \in A(\varphi(K))$. Hence $f \circ \varphi^{-1} = h \in P(\varphi(K))$. Thus $f \in [\varphi]_{\text{alg}}$. Consequently, $R(K) = [\varphi]_{\text{alg}}$. 

\textbf{Corollary 1.6.} If $A = C(K)$, $A(K)$ or $R(K)$ is singly generated, then $K$ is polynomially convex and $A = P(K)$.

\textbf{Proof.} This follows from the previous Theorems which imply that under the given assumption, $K$ is polynomially convex. Hence, by Mergelyan’s Theorem, $P(K) = R(K) = A(K)$, and in the remaining case, the additional condition $K^c = \emptyset$ implies that $C(K) = P(K)$. 

\textbf{Theorem 1.7.} Let $K \subseteq \mathbb{C}$ be compact and $\varphi \in P(K)$. The following assertions are equivalent:

1. $\varphi$ is a generator for $P(K)$; that is $P(K) = [\varphi]_{\text{alg}}$;
(2) \( \varphi \) is a homeomorphism of \( K \) onto \( \varphi(K) \) and \( \varphi^{-1} \in P(\varphi(K)) \).

Proof. (1) \( \Rightarrow \) (2) As usual, if \( \varphi \) is a generator, then \( \varphi \) is point separating, hence a homeomorphism of \( K \) onto \( \varphi(K) \). Note also that for \( f \in P(K) \), \( f \in [\varphi]_{\text{alg}} \) if and only if \( f \circ \varphi^{-1} \in P(\varphi(K)) \). In particular, if \( f(z) = z \) then \( \varphi^{-1} \in P(\varphi(K)) \).

(2) \( \Rightarrow \) (1) Let \( f \in P(K) \). By Lemma 1.4 (2) applied to the inverse function, the assumption \( \varphi^{-1} \in P(\varphi(K)) \) implies that \( f \circ \varphi^{-1} \in P(\varphi(K)) \). Hence \( f \in [\varphi]_{\text{alg}} \) and so \( P(K) = [\varphi]_{\text{alg}} \). \( \square \)

It is now a natural question to ask whether the condition \( \varphi^{-1} \in P(\varphi(K)) \) is redundant or not? The following example shows that it is not.

**Example 1.8.** Let \( K = \{ z \in \mathbb{C} : |z + 1| = 1 \} \cup \{ z \in \mathbb{C} : |z - 2| = 2 \} \) (see figure 1).

\[ \begin{align*}
\text{Figure 1. No injective extension} \\
\text{Then the function } f(z) &= -z \text{ for } |z + 1| = 1 \text{ and } f(z) = z \text{ for } |z - 2| = 2 \text{ is injective on } K \text{ and belongs to } P(K), \text{ because } f \text{ has a holomorphic extension to the polynomial convex hull } \\
\hat{K} &= \{ z \in \mathbb{C} : |z + 1| \leq 1 \} \cup \{ z \in \mathbb{C} : |z - 2| \leq 2 \} \\
of K \text{ and so, by Mergelyan’s theorem, } f \text{ can be uniformly approximated on } \hat{K} \text{ by polynomials.}
\end{align*} \]

The image \( f(K) \) of \( K \) under \( F \) coincides with the set 
\[ \{ w \in \mathbb{C} : |w - 1| = 1 \} \cup \{ w \in \mathbb{C} : |w - 2| = 2 \}. \]
Moreover, \( f^{-1}(w) = -w \) on \( D_1 := \{ w \in \mathbb{C} : |w - 1| = 1 \} \) and \( f^{-1}(w) = w \) on \( D_2 := \{ w \in \mathbb{C} : |w - 2| = 2 \} \). It is clear that this function does not belong to \( P(f(K)) \), because otherwise, \( f^{-1}|_{D_2} \) would have a holomorphic extension to the polynomial convex hull \( \hat{D}_2 \) of \( D_2 \). Since this extension can
only be \( w \) itself, it does not coincide with \( f^{-1}|_{D_1}(w) = -w \) on \( D_1 \subseteq \hat{D}_2 \). Note also, that \( f \) does not admit a holomorphic injective extension to \( \hat{K} \).

**Proposition 1.9.** Let \( f \in P(K) \) be a homeomorphism and suppose that \( f \) has an injective, holomorphic extension to the interior of the polynomial convex hull, \( \hat{K} \), of \( K \). \(^2\) Then \( f^{-1} \in P(f(K)) \).

**Proof.** If \( f^* \) denotes this extension, then \( f^* \) coincides with the Gelfand transform \( \hat{f} \) of \( f \) (in fact, \( f^* \) and \( \hat{f} \) belong to \( A(\hat{K}) \) and \( f^* = \hat{f} = f \) on the Shilov boundary of \( A(\hat{K}) \), which coincides with \( \partial K \)). Now \( (f^*)^{-1} \in A(f^*(\hat{K})) \). Since \( \hat{K} \) has connected complement, the invariance theorem 2.5(4) implies that \( S := f^*(\hat{K}) \) has connected complement, too. Hence, by Mergelyan’s Theorem, \( (f^*)^{-1} \in P(S) \). Restricting to \( f(K) \subseteq S \) yields that \( f^{-1} = (f^*)^{-1}|_{f(K)} \in P(f(K)) \), because any sequence of polynomials converging uniformly on \( S \) to \( (f^*)^{-1} \) converges a fortiori uniformly on \( f(K) \). \( \square \)

### 2. Injective Extensions

Example 1.8 shows that \( P(K) \)-functions which are injective on \( K \) do not necessarily have an injective holomorphic extension to the polynomial convex hull of \( K \). A positive result in this direction is known, though:

**Theorem 2.1** (Darboux-Picard). [3, p. 310], [8] Let \( f \in A(\mathbb{D}) \) and suppose that \( f \) is injective on \( \partial \mathbb{D} \). Then \( f \) is injective on \( \mathbb{D} \).

In the following we shall deal with the general case of arbitrary compacta. Recall that a **hole** of a compact set \( K \) is a bounded component of \( \mathbb{C} \setminus K \) and that the **outer boundary**, \( S_\infty \), of \( K \) is the boundary of the polynomial convex hull \( \hat{K} \) of \( K \). We need Eilenberg’s theorem (see below) and the following homotopic variant of Rouché’s theorem, the proof of which is based on an areal analogue of the argument principle (see [7, p. 105]). Here, as usual, the maps \( f, g \in C(X,Y) \), defined on Hausdorff spaces \( X \) and \( Y \), are said to be **homotopic** in \( C(X,Y) \) if there exists a continuous map \( H : X \times [0,1] \to Y \) such that \( H(x,0) = f(x) \) and \( H(x,1) = g(x) \) for every \( x \in X \).

**Definition 2.2.** For a compact set \( K \subseteq \mathbb{C} \), let \( M(K) \) denote the set of continuous functions on \( K \) that are meromorphic in \( K^\circ \).

Thus, a function in \( M(K) \) has only a finite number of poles in \( K^\circ \) and none on the boundary. Of course, \( A(K) \subseteq M(K) \). Finally, for a function \( f \in M(K) \), \( n_K(f) \) denotes the number of zeros (possibly infinite) of \( f \) in \( K^\circ \) and \( p_K(f) \) the number of poles of \( f \) in \( K^\circ \) (including multiplicities).

**Theorem 2.3** (Rouché for homotopic maps). Let \( K \subseteq \mathbb{C} \) be compact and let \( f, g \in M(K) \) be zero-free on \( \partial K \). Suppose that \( f \) and \( g \) are homotopic in \( C(\partial K, \mathbb{C}^\ast) \). Then \( n_K(f) - p_K(f) = n_K(g) - p_K(g) \).

\(^2\) in the sense that there is \( g \in C(\hat{K}) \) such that \( g \) is holomorphic in \( \hat{K}^\circ \) and injective on \( \hat{K} \).
Proof. For a proof where $f$ and $g$ have no poles, that is in the case where $f, g \in A(K)$, we refer to [6]. Now suppose that $f, g \in M(K)$. Since $f$ and $g$ have only a finite number of poles and zeros in $K$, we may write them as

$$f(z) = \prod_{j=1}^{p}(z - a_j)^{n_j} \bar{f}(z), \quad g(z) = \prod_{j=1}^{q}(z - b_j)^{m_j} \bar{g}(z),$$

where $\bar{f}, \bar{g} \in A(K)$ are zero-free and $m_j, n_j, p_j, q_j \in \mathbb{N}^*$. Note that a zero of $g$ may be a pole or zero of $f$ and vice versa. Put

$$h(z) := \prod_{j=1}^{p}(z - z_j)^{p_j} \prod_{j=1}^{q}(z - w_j)^{q_j}$$

and consider the functions $F := hf$ and $G := hg$.

Then $F, G \in A(K)$ and $F$ and $G$ are homotopic in $K(\partial K, \mathbb{C}^*)$ (note that if $H(z, t)$ is a homotopy between $f$ and $g$, then

$$\tilde{H}(z, t) := h(z) H(z, t)$$

is a homotopy in $K(\partial K, \mathbb{C}^*)$ between $F$ and $G$). Hence, by the homotopic version of Rouché’s theorem for holomorphic functions [6], $n_K(F) = n_K(G)$; that is

$$\sum_{j=1}^{n} n_j + \sum_{j=1}^{q} q_j = \sum_{j=1}^{m} m_j + \sum_{j=1}^{p} p_j.$$

In other words, $n_K(f) - p_K(f) = n_K(g) - p_K(g)$. \hfill \square

Here is a variant of the preceding result. For a bounded open set $G$ in $\mathbb{C}$, let $MC(G)$ denote the set of functions continuous on $\overline{G}$ and meromorphic in $G^*$. Note that, in general, $MC(G)$ cannot be represented as $M(K)$ for some compact space $K$. For example, if $E \subseteq \mathbb{D}$ is a compact, nowhere dense set having positive Lebesgue measure, then the planar integral

$$f(z) = \int \int_{E} \frac{1}{w - z} d\sigma_2(w)$$

belongs to $MC(\mathbb{D} \setminus E)$, but not to $M(\mathbb{D})$.

**Corollary 2.4.** For a bounded open set $G \subseteq \mathbb{C}$, suppose that $f, g \in MC(G)$ are homotopic in $C(\partial G, \mathbb{C}^*)$. Then $n_G(f) - p_G(f) = n_G(g) - p_G(g)$.

**Proof.** By assumption, $f$ and $g$ have no zeros and poles on $\partial G$. Hence, there are open neighborhoods $U$ and $V$ of $\partial G$ with $\partial G \subseteq U \subseteq \overline{U} \subseteq V$ such that $f, g \in M(\overline{G} \setminus U)$ and $f$ and $g$ are homotopic in $C(V \cap \overline{G}, \mathbb{C}^*)$ (for this latter point see [6]). The assertion now follows from Theorem 2.3 if we set $K := \overline{G} \setminus U$. \hfill \square

A proof of the next Theorem is in [3, p. 97-101].

**Theorem 2.5 (Eilenberg).** Let $K \subseteq \mathbb{C}$ be compact and for each bounded component $C$ of $\mathbb{C} \setminus K$, let $a_C \in C$. 

(1) Suppose that \( f : K \to \mathbb{C} \setminus \{0\} \) is continuous. Then there exist finitely many bounded components \( C_j \) of \( \mathbb{C} \setminus K \), integers \( s_j \in \mathbb{Z} \) \((j = 1, \ldots, n)\), and \( L \in \mathbb{C}(K) \) such that for all \( z \in K \)
\[
f(z) = \prod_{j=1}^{n} (z - a_{C_j})^{s_j} e^{L(z)}.
\]
(2) If for some \( f \in \mathbb{C}(K) \), \( 0 \) belongs to the unbounded component of \( \mathbb{C} \setminus f(K) \), then \( f \) has a continuous logarithm on \( K \).
(3) Suppose that \( C_1, \ldots, C_n \) are distinct holes for \( K \) and that for some \( s_j \in \mathbb{Z} \) \((j = 1, \ldots, n)\), the function
\[
f(z) = \prod_{j=1}^{n} (z - a_{C_j})^{s_j}, \quad (z \in K)
\]
has a continuous logarithm on \( K \). Then \( s_1 = \cdots = s_n = 0 \).
(4) If \( f : K \to \mathbb{C} \) is a homeomorphism, then the number of holes of \( K \) and \( f(K) \) coincide.

**Proposition 2.6.** Let \( K \subseteq \mathbb{C} \) be a compact set for which \( \mathbb{C} \setminus K \) is connected and let \( G \) be a bounded component of \( \mathbb{C} \setminus \partial K \). The following assertions hold:

1. \( G \) is simply connected.
2. \( \partial G = \partial \hat{G} \).
3. \( \overline{G} = G \).

Item (1) and the equivalence of (2) with (3) for non-void open sets in general topological spaces are well known. We include a proof of (1) and (2) for the reader’s convenience.

**Proof.** (1) Let \( \mathcal{H} := \{G_n : n \in I\} \) be the set of holes of \( \partial K \) and let \( C := (\mathbb{C} \setminus K) \cup \partial K \). Let \( n_0 \in I \) be chosen so that \( G = G_{n_0} \). Note that \( G_{n_0} \) is an open set and that for every \( n \), \( \partial G_n \subseteq \partial K \subseteq C \). Hence
\[
\mathbb{C} \setminus G_{n_0} = C \cup \bigcup_{n \in I \setminus n_0} G_n = C \cup \bigcup_{n \in I \setminus n_0} \overline{G_n}.
\]
Since \( C = \overline{C \setminus K} \), the assumption of the connectedness of \( \mathbb{C} \setminus K \) implies that \( C \) is connected. Moreover, \( \overline{G_n} \) is connected for every \( n \) and \( \overline{G_n} \cap C \neq \emptyset \). Hence the union of all of these connected sets is connected; that is \( C \setminus G_{n_0} \) is connected. Thus \( G_{n_0} \) is a simply connected domain.

(2) First we note that for any set \( M \) in any topological space, \( \partial \overline{M} \subseteq \partial M \). The reverse inclusion now is a specific property of the set \( G \). So let \( x \in \partial G \) and \( U \) a neighborhood of \( x \). Since the connectivity of \( \mathbb{C} \setminus K \) implies that \( \partial K = \partial \hat{K} \) we deduce from \( \partial G \subseteq \partial K \) that \( U \) meets the unbounded component of \( \mathbb{C} \setminus \hat{K} \). Since \( \overline{G} = G \cup \partial G \subseteq \hat{K} = K \), \( U \) cannot be entirely contained in \( \overline{G} \). Hence \( U \) meets the complement of \( \overline{G} \) as well as \( \overline{G} \). That is \( x \in \partial \overline{G} \). We conclude that \( \partial \overline{G} = \partial G \). \( \square \)
Here is now the main result of this paper. Recall that if \( f \in P(K) \), then the Gelfand transform \( f^* \) of \( f \) is the unique continuous extension of \( f \) to \( \hat{K} \) that is holomorphic in \( \hat{K}^\circ \). In particular, if \( K \neq \hat{K} \), then every function \( f \in P(K) \) is holomorphic in a neighborhood of each “inner-boundary” point \( z_0 \in \partial K \cap \hat{K}^\circ \) (whenever they exist).

**Theorem 2.7.** Let \( K \subseteq \mathbb{C} \) be compact. Suppose that \( f \in P(K) \) is injective. Then \( f^* \) is injective on \( \hat{K} \) if and only if the outer boundary \( S_\infty \) of \( K \) is mapped under \( f \) onto the outer boundary of \( f(K) \). Moreover, in that case, \( f^*(\hat{K}) = \hat{f(K)} \) and each hole of \( f(S_\infty) \) is the image under \( f^* \) of a unique hole of \( S_\infty \).

Let us mention that Example 1.8 provides an injective function \( f \in P(K) \) that does not map the outer boundary to the outer boundary.

**Proof.** (1) Let \( f^* \) be injective on \( \hat{K} \). Note that \( S_\infty = \partial \hat{K} \subseteq \partial K \) and that the outer boundary of \( f(K) \) coincides with \( \partial \hat{f(K)} \). It remains to show that
\[
\partial \hat{f(K)} = \partial f^*(\hat{K}) = f^*(\partial \hat{K}).
\]
Here the second equality is satisfied due to the assumption that \( f^* \) is a homeomorphism between \( \hat{K} \) and \( f^*(\hat{K}) \). Now \( \hat{K} \) is polynomially convex. Hence, by Theorem 2.5 (4), \( f^*(\hat{K}) \) has no holes. Consequently, \( \partial f^*(\hat{K}) \) is the outer boundary of \( f^*(\hat{K}) \) and the polynomial convexity of \( f^*(\hat{K}) \) implies that
\[
\hat{f(K)} \subseteq f^*(\hat{K}).
\]
But we also have the reverse inclusion. In fact, let \( \hat{w} = f^*(\hat{z}) \in f^*(\hat{K}) \), where \( \hat{z} \in \hat{K} \). Since \( p \circ f \in P(K) \) for every polynomial \( p \in \mathbb{C}[z] \), we conclude from
\[
\max_{\hat{K}} |h| = \max_\hat{K} |h^*|
\]
for every \( h \in P(K) \), that
\[
|(p \circ f)^*(\hat{z})| \leq \max_{\hat{z} \in \hat{K}} |(p \circ f)(\hat{z})|.
\]
Hence
\[
|p(\hat{w})| \leq \max\{|p(y)| : y \in f(K)\}.
\]
In other words, \( \hat{w} \in \hat{f(K)} \). This implies that
\[
f^*(\hat{K}) \subseteq \hat{f(K)}. \tag{2.2}
\]
(Note that (2.2) holds independently of \( f^* \) being injective or not.) Thus
\[
f^*(\hat{K}) = f(\hat{K}), \tag{2.3}
\]
and therefore \( \partial \hat{f(K)} = \partial f^*(\hat{K}) \), which establishes (2.1).

(2) Next we prove the converse. We may assume that \( K \) is not polynomially convex, otherwise there is nothing to show. In particular, \( \hat{K}^\circ \neq \emptyset \). So suppose that \( \partial \hat{f(K)} = f(\partial \hat{K}) \).
Step 1 We show that $f^*|G$ is injective for every hole $G$ of $\partial \hat{K}$.

Let $M := f(\partial G)$ and $S := \hat{f}(\hat{K})$. Then $\partial S$ is the outer boundary of $f(K)$, and

$$M = f(\partial G) \subseteq f(\partial \hat{K}) = \partial \hat{f}(\hat{K}) = \partial S.$$

Let $a$ belong to the unbounded component, $\Omega_\infty$, of $\mathbb{C} \setminus M$. Then 0 belongs to the unbounded component of $\mathbb{C} \setminus (f - a)(\partial G)$. By Theorem 2.5(2), $f(z) - a = e^{L(z)}$ for some $L \in C(\partial G, \mathbb{C})$. Hence $f - a$ is homotopic in $C(\partial G, \mathbb{C}^*)$ to 1. Since $\partial G = \partial \hat{G}$, (Proposition 2.6) we conclude from Theorem 2.3 that $f^* - a$ has no zeros in $\hat{G}^o$. Hence

$$f^*(G) \subseteq \hat{M}.$$  \hfill (2.4)

Next, we claim that $f^*(G) \cap \partial S = \emptyset$. To see this, let us suppose that there exists $z \in G$ with $f^*(z) \in \partial S$. Since $f^*$ is holomorphic in $G$ (and due to the injectivity on the boundary, not constant on $G$), we conclude that $f^*$ is an open map on $G$. Hence a whole disk $D(f^*(z), \varepsilon)$ belongs to $f^*(G)$. Thus $f^*(G)$ meets the unbounded component $C_\infty$, of $\mathbb{C} \setminus S$ (note that $S$ is polynomially convex). This is a contradiction because $C_\infty \subseteq \Omega_\infty$ and no point in $\Omega_\infty$ belongs to $f^*(G)$, as was shown above. Consequently, $f^*(G) \cap \partial S = \emptyset$.

Because $\hat{M} = f(\partial G) \subseteq f(\hat{K}) = S$, we then conclude from (2.4) that $f^*(G) \subseteq \hat{M} \setminus \partial S \subseteq S \setminus \partial S$. But $S^o \neq \emptyset$, since the open set $f^*(G)$ is contained in $f^*(\hat{K})$ (2.2), $\hat{f}(\hat{K}) = S$. Hence $S \setminus \partial S$ is a non-void open set.

Because $\mathbb{C} \setminus S$ is connected, $S \setminus \partial S$ consists of the union of all holes of $\partial S$. Thus the connected set $f^*(G)$ is contained in a unique hole, $H$, of $\partial S$.

Next we show that every point in $H$ is taken once by $f^*$ on $G$. For technical reasons, we suppose that $0 \in G$ (otherwise we use an appropriate translation).

Fix $b \in H$. Let $g : \partial S \to S_\infty \subseteq K$ be the restriction to $\partial S$ of the inverse of $f$ (here we have used the hypothesis that $f$ maps the outer boundary $S_\infty$ of $K$ onto the outer boundary $S$ of $f(K)$). Note that $g$ does not take the value 0 because, by assumption, $0 \in \mathbb{C} \setminus \partial \hat{K}$. By Theorem 2.5(4), $\partial S$ and $S_\infty$ have the same number of holes. Let $\mathcal{H} := \{H_j : j \in I\}$ be the set of holes of $\partial S$. We may assume that $H_1 = H$. Fix in each hole $H_j$ of $\partial S$ a point $b_j$, $(j \in I \subseteq \mathbb{N}^*)$, where we take $b_1 = b$. By Eilenberg’s Theorem 2.5, there exists $n \in \mathbb{N}$, $L \in C(\partial S, \mathbb{C})$ and $s_j \in \mathbb{Z}$ such that

$$g(w) = \prod_{j=1}^n (w - b_j)^{s_j} e^{L(w)} \text{ for every } w \in \partial S.$$

If $z := g(w)$ (or equivalently $w = f(z)$), then $z \in \partial \hat{K} = S_\infty \subseteq \partial K$ and

$$z = \prod_{j=1}^n (f(z) - b_j)^{s_j} e^{L(f(z))} \text{ for these } z.$$  \hfill (2.5)
In particular

\[ H(z,t) := \prod_{j=1}^{n} (f(z) - b_j)^{s_j} e^{tL(f(z))} \]

is a homotopy in \( C(\partial G, \mathbb{C}^*) \) between the function \( \prod_{j=1}^{n} (f(z) - b_j)^{s_j} \) and the identity function \( z \). Now, for \( z \in \hat{K} \),

\[ \psi(z) := \prod_{j=1}^{n} (f^*(z) - b_j)^{s_j} \]

is a meromorphic function in \( M(\hat{K}) \). Also, \( \partial G = \partial \hat{G} \) and \( \overline{C}^\circ = G \) (Proposition 2.6). Hence, by Theorem 2.3, \( n_G(\psi) - p_G(\psi) = 1 \). Since \( f^*(G) \subseteq H_1 \), \( \psi(G)(z) = (f^*(z) - b_1)^{s_1} R(z) \), where \( R \) is zero-free and holomorphic on \( G \).

We conclude that \( s_1 = 1 \) and \( f^*(z_1) = b_1 \) for a unique \( z_1 \in G \). Hence \( f^* \) is a bijection of \( G \) onto \( H_1 \). Since \( f(\partial G) \subseteq \partial S \), \( f^* \) actually is a bijection from \( \overline{G} \) onto \( \overline{H_1} \).

**Step 2** We claim that \( f^* \) is injective on \( \hat{K} \). It only remains to show that \( f^*(G) \cap f^*(C) = \emptyset \) whenever \( G \) and \( C \) are two different holes of \( S_\infty = \partial \hat{K} \).

To see this, suppose that \( f^*(G) \cap f^*(C) \neq \emptyset \). Since the images of \( G \) and \( C \) under \( f^* \) are holes of \( \partial S \), we conclude that \( f^*(C) = f^*(G) = H_1 \). Moreover,

\[ f^*(\partial G) = \partial f^*(G) = \partial f^*(C) = f^*(\partial C) \]

The injectivity of \( f \) on \( \partial K \) and the fact that \( \partial C \cup \partial G \subseteq \partial K \) now imply that \( \partial G = \partial C \). Moreover, \( \partial \overline{C} = \partial C \). Since \( 0 \in G \neq C \), we conclude from (2.5) and Theorem 2.3 that \( n_C(\psi) - p_C(\psi) = 0 \). On the other hand, since \( f^*(C) \subseteq H_1 \), \( \psi|_C(z) = (f^*(z) - b_1)^{s_1} R(z) \), where \( R \) is zero-free and holomorphic on \( C \). Now \( s_1 = 1 \) implies that \( p_C(\psi) = 0 \). Hence \( n_C(\psi) = 0 \), too. This is contradiction, though, because \( f^*(C) = H_1 \) and \( b_1 \in H_1 \). Thus we have shown that \( f^* \) is a bijection of \( \hat{K} \) onto \( f^*(\hat{K}) \).

(3) If \( f^* \) is a homeomorphism of \( \hat{K} \) onto its image \( f^*(\hat{K}) \), then we have already shown that \( f^*(\hat{K}) = \hat{f}(\hat{K}) \) (see 2.3). Hence, we conclude from the preceding paragraphs (applied to \( (f^*)^{-1} \)) that each hole \( H \) of \( \partial \hat{f}(\hat{K}) = f(S_\infty) \) writes as \( H = f^*(G) \) for some uniquely determined hole \( G \) of \( S_\infty = \partial \hat{K} \).

A natural question is whether a compactum \( K \) with a single hole has the so-called extension property, that is if \( f \in P(K) \) is injective, then \( f^* \) is injective on \( \hat{K} \). A slight modification of Example 1.8 shows that this is not true, either:

**Example 2.8.** Let

\[ K_1 = \{ z \in \mathbb{C} : |z + 1| \leq 1 \} \cup \{ z \in \mathbb{C} : |z - 2| = 2 \} \]

(see figure 2).
Then the function \( f(z) = -z \) for \( |z + 1| \leq 1 \) and \( f(z) = z \) for \( |z - 2| = 2 \) belongs to \( P(K_1) \), but of course, by the same reasoning as in Example 1.8 \( f^* \) is not injective on \( \bar{K}_1 \).

So let us modify the question a little bit: let \( G \) be a hole of \( K \) and suppose that \( f \in P(K) \) is injective. Is \( f^*|_G \) injective? See figure 2 for several examples. In the following, a positive answer will be given for a special class of holes.

**Definition 2.9.** Let \( K \subseteq \mathbb{C} \) be compact and \( G \) a hole of \( K \). Then \( G \) is called a regular hole if \( G \) is the only hole of its boundary \( \partial G \); that is if \( \partial G = \overline{G} = G \cup \partial G = \bar{G} \).

In figure 2, the holes of \( K_1 \) and \( K_2 \) are regular as well as the hole \( G_2 \) of \( K_3 \), but \( G_1 \) is not regular. A more interesting class of non-regular holes is provided by Example 2.10. It has the additional property that \( G_1 \) is a component of the interior of a polynomially convex set \( K \).

**Example 2.10.** There is a compact set \( K \subseteq \mathbb{C} \) with connected complement such that some hole \( G_1 \) of \( \partial K \) has the property that \( G_1 \) is not the unique hole of \( \partial G_1 \).

**Proof.** Let \( K \) be the union of the closed unit disk with a “thick” spiral \( S \) surrounding the unit circle infinitely often and clustering exactly at every point of \( \mathbb{T} \) (see figure 3). Then \( \mathbb{C} \setminus K \) is connected, and the holes of \( \partial K \) are the components of \( K^\circ \); these are the interior \( G_1 \) of the spiral \( S \) and the open unit disk, denoted here by \( G_2 \). Then \( \partial G_1 = \partial K \); hence \( G_1 \) and \( G_2 \) are the holes of the boundary of the hole \( G_1 \) of \( \partial K \). \( \square \)

This example also shows that the closure \( \overline{G_1} \) of the component \( G_1 \) of the polynomial convex set \( K \), may have a disconnected complement, although \( G_1 \) itself is simply connected.

It actually can happen that two, or even infinitely many, holes of a compactum may have the same boundary. These sets are known under the name “lakes of Wada”, first discovered by L.E.J. Brouwer [1], see also [5, p. 138].
Figure 3. A p.c. compactum with a boundary hole whose boundary induces two holes.

Lemma 2.11. Let $G \subseteq \mathbb{C}$ be a bounded domain with $\overline{G} = G$ and
$$
\hat{\partial}G = G \cup \partial G. 
$$
If $f : \partial G \to \mathbb{C}$ is a continuous injective map, then $f(\partial G)$ is the boundary of a bounded domain $H$ with $\overline{H} = H$ and
$$
\hat{\partial}H = H \cup \partial H.
$$

Proof. By Theorem 2.5(4), $E := f(\partial G)$ has a single hole, too. Let us denote this hole by $H$. Since $\partial H \subseteq \partial E$, we have
$$
(2.7) \quad \hat{E} = E \cup H = E \cup \overline{\Pi}.
$$

Note that $\partial \overline{\Pi} \subseteq \partial H \subseteq E$. We claim that $\partial \overline{\Pi} = E$. Suppose, to the contrary, that $S := \partial \overline{\Pi} \subset E$, the inclusion being strict. Let $F := f^{-1}(S)$. Then $F$ is a proper, closed subset of $\partial G$. Since $\partial G \setminus F$ is relatively open in the closed set $\partial G$, there is $\xi \in \partial G$ and a disk $D = D(\xi, \varepsilon)$ such that $D \cap F = \emptyset$. Let
$$
U := G \cup (\mathbb{C} \setminus \overline{G}) \cup D.
$$
By hypothesis, $\hat{\partial}G = \overline{G}$. Hence $\mathbb{C} \setminus \overline{G}$ is connected (because it coincides with the unbounded complementary component of the polynomially convex set $\hat{\partial}G$).

Because the hypothesis $\overline{G} = G$ implies that $\partial G = \partial \overline{G}$, we conclude that $D$ meets $G$ as well as $\mathbb{C} \setminus \overline{G}$. Hence, $U$ is an unbounded open connected set contained in the open set $\mathbb{C} \setminus F$. Thus $U$ is contained in the unbounded component of $\mathbb{C} \setminus F$. Since the remaining part $(\mathbb{C} \setminus F) \setminus U \subseteq \partial G \setminus F$ of $\mathbb{C} \setminus F$ is small in the sense that it does not contain interior points, $\mathbb{C} \setminus F$ does not have a bounded component. In other words, $F$ has no holes. This

---

3 In other words, $G$ is the only hole of $\partial G$. 
is a contradiction, because $F$ has the same number of holes as $S$; that is at least one hole. Thus we have shown that $\partial \overline{P} = \partial H = E$. The identity $\hat{E} = E \cup H$ (see (2.7)) now implies that $\hat{\partial H} = \partial H \cup H$.

**Theorem 2.12.** Let $K \subseteq \mathbb{C}$ be compact and suppose that $f \in P(K)$ is injective. If $G$ is a hole of the outer boundary $S_\infty$ of $K$, then the restriction $f^*|_G$ of the Gelfand transform $f^*$ of $f$ to $G$ is injective whenever $G$ is the only hole of $\partial G$.

Example 2.10 shows that the strange condition “whenever $G$ is the only hole of $\partial G$” is not always satisfied.

**Proof.** Because $G$ is the only hole of $\partial G$, we have $\hat{\partial G} = G \cup \partial G = \overline{G}$. Thus $M := \overline{G}$ is polynomially convex. Hence, the outer boundary of $M$ coincides with $\partial M = \partial \overline{G}$. Moreover, since $G$ is a hole of the boundary $S_\infty$ of the polynomially convex set $\hat{S}_\infty$, we obtain from Proposition 2.6 that $\partial G = \partial \overline{G}$ and that $\hat{\partial G} = \partial \overline{G}$.

Since $\partial M$ has a single hole, namely, $G = \overline{G}^o$, and since $f$ is injective on $\partial M$, $E := f(\partial M)$ has a single hole, too. Let $H$ be that hole. By Lemma 2.11, $\hat{\partial H} = H \cup \partial H$ and $\partial H = \partial \overline{H} = E$. We conclude that $f$ maps the outer boundary $\partial M$ of $M$ onto the outer boundary $E$ of $\hat{f}(\partial M)$. By Theorem 2.7, $f^*$ is injective on $M = \overline{G}$. \qed

Example 1.8 shows that, in general, $f^*$ is not injective on the union of two bounded components $G_j$ of $\mathbb{C} \setminus S_\infty$. However, we don’t know whether $f^*|_G$ is injective in case $G$ is not a regular hole of $S_\infty$.

**Corollary 2.13.** Let $X \subseteq \mathbb{C}$ be compact and $H$ a hole of $X$. Suppose that $f \in P(X)$ is injective. Under each of the following conditions $f^*$ is injective on $\overline{P}$:

1. $(\partial \overline{P}, f)$ satisfies the condition of Theorem 2.7 with $K = \partial \overline{P}$.
2. $H$ is contained in a hole $G$ of the outer boundary of $X$ which has the property that $G$ is the only hole of $\partial G$.
3. $H$ is a regular hole of $X$.

**Proof.** (1) and (2) are clear.

(3) Let $M = \overline{P}$. By hypothesis, $\hat{\partial H} = H \cup \partial H$. Thus $M$ is polynomially convex. Since $H \subseteq \overline{P}^o \subseteq \overline{P}$, we conclude from the connectedness of $H$ that $G := \overline{P}^o$ is connected. Hence $G$ is the only hole of $\partial \overline{P}$. Since $\partial \overline{P}$ is the outer boundary of $\overline{H}$, it follows that $\partial \overline{P} = \partial G$ and $\overline{G} = \overline{H}$. In particular, $\hat{\partial G} = \partial G \cup G$. By Theorem 2.12, $f^*|_\overline{G}$ is injective. \qed

**Corollary 2.14.** Let $K \subseteq \mathbb{C}$ be compact. Suppose that $\mathbb{C} \setminus K$ and $K^o$ are connected. Then $\partial K$ has the extension property.

**Proof.** If $K^o = \emptyset$, then the polynomial convexity of $K$ implies that $\hat{K} = K = \partial K$. Hence the assertion is trivial. So let us assume that $K^o \neq \emptyset$. Let
$M = \overline{K^\circ}$. We claim that $M$ is polynomially convex. In fact,

$$\overline{K^\circ} \subseteq \hat{\overline{K^\circ}} \subseteq \hat{K} = K.$$ 

If $\overline{K^\circ}$ would be a strict subset of $\hat{\overline{K^\circ}}$, then $\overline{K^\circ}$ would have a hole $H$. Hence

$$\overline{K^\circ} \cup H \subseteq \hat{\overline{K^\circ}} \subseteq K.$$

Consequently, $K^\circ \cup H \subseteq K^\circ$; this is an obvious contradiction. We conclude that

$$\partial K^\circ = \hat{\overline{K^\circ}} = \overline{K^\circ} = K^\circ \cup \partial K^\circ.$$

Thus $K^\circ$ is a regular hole for $\partial M$. The conclusion now follows from Corollary 2.13. □

Examples 2.8 and 1.8 (this latter for the full disks) show that neither of the conditions $C \setminus K$ connected or $K^\circ$ connected implies that $\partial K$ has the extension property.

Now let $K \subseteq \mathbb{C}$ be a compact set for which $\partial K$ has the extension property (for $P(K)$-functions). If $f \in R(K)$ is injective on $\partial K$, does this imply that $f$ is injective on $K$? The following example shows that this is not necessarily the case:

**Example 2.15.** Let $K = \{z \in \mathbb{C} : r \leq |z| \leq R \}$ where $0 < r < 1 < R$ and $rR \neq 1$. Then the function $f$, given by $f(z) = z + \frac{1}{z}$ belongs to $R(K)$, is injective on $\partial K$, but not on $K$. In fact, $f(z) = f(w)$ implies that $z - w = (w - z)/zw$. Since on $\partial K$, $zw \neq 1$, we have $z = w$. On the other hand, $f(i) = f(-i) = 0$.

Finally, we want to present the following problem: suppose that $f \in C(\partial K, \mathbb{C})$ is injective. Under which conditions $f$ admits a continuous injective extension to $\hat{K}$ or even $\mathbb{C}$? Note that if $K$ is the closure of a Jordan domain, then the Schoenflies theorem guarantees the existence of a homeomorphism of $\mathbb{C}$ extending $f$.

### 3. Continuous logarithms on compact sets containing the origin on their boundary

Eilenberg’s Theorem 2.5(2) shows that if 0 belongs to the unbounded complementary component of a compact set $K$ in $\mathbb{C}$, then there exists a continuous branch of the logarithm of $z$ on $K$. On the other hand, by 2.5(3), if 0 belongs to a bounded complementary component of $K$, then there does not exist a continuous function $h$ on $K$ such that $e^{h(z)} = z$ for every $z \in K$. We will investigate now the case when 0 belongs to the boundary of $K$. Does there exist a continuous branch of $\log z$ on $K \setminus \{0\}$? The answer is “not necessarily”\(^4\).

\(^4\)This refutes statements and invalidates the associated “proofs” in [10, p. 62] and its verbatim copy in [4, p. 348]
Proposition 3.1. There exists a compact set $K$ in $\mathbb{C}$ with $0 \in \partial K$ and connected complement such that no continuous branch of $\log z$ can be defined on $K \setminus \{0\}$.

Proof. Let $E$ be the disk $\{z \in \mathbb{C} : |z + 1| \leq 1\}$ and $S$ a spiral starting at 1 and surrounding $E$ infinitely often and clustering at every point on the boundary of $E$; for example one may describe $S$ as the half-open curve

$$z(t) = -1 + \left(1 + \frac{1}{1+t}\right) e^{it}, \quad 0 \leq t < \infty.$$ 

Let $K = E \cup S$. Then $K$ is compact and polynomially convex. Note also that $S \cap E = \partial E$. Moreover, 0 is a boundary point of $K$. We show that there does not exist a continuous branch of $\log z$ on $K \setminus \{0\}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{spiral.png}
\caption{A spiral clustering at a circle}
\end{figure}

In fact, since $S$ is a connected set surrounding 0 infinitely often, any continuous determination of the argument of $z$ when $z$ runs through the spiral $S$ has to be unbounded. This can be seen by geometric intuition or by the following analytic argument:

If we look at $w(t) := \exp(-it)z(t) = 1 + 1/(1+t) - \exp(-it)$, $0 \leq t < \infty$, then $\Re w(t) \geq 1/(1+t) > 0$. Hence $w(t)$ belongs to the right half-plane. Let $L(z) = \log z$ be the principal branch of the logarithm on the right half-plane and set $h(t) := L(w(t))$. Then

$$\exp(-it)z(t) = \exp(h(t)).$$

Therefore, $z(t) = \exp(it + h(t))$. Because $|\Im h(t)| \leq \pi/2$,

$$\arg z(t) = \Im (it + h(t))$$

behaves as $t$ for large $t$. Thus the imaginary part of $\log z$ is unbounded, for $z \in S$.

Since the spiral $S$ clusters at every point of the circle $C := \{|z + 1| = 1\}$ and $C \subseteq S \subseteq K$, $\log z$ cannot be continuous on $K \setminus \{0\}$. \qed
Next we give a sufficient condition for the existence of such logarithms.

**Definition 3.2.** A boundary point $z_0$ of a compact set $K$ is said to be accessible, if there is a Jordan arc $\gamma : ]0,1[ \to \mathbb{C} \setminus K$ coming from infinity and ending at $z_0$ (that is $\lim_{t \to 0} \gamma(t) = \infty$ and $\lim_{t \to 1} \gamma(t) = z_0$).

We note that it is well known that the set of accessible boundary points for $K$ is dense in the boundary $\partial K$ of $K$.

**Theorem 3.3.** Let $K$ be a compact set in $\mathbb{C}$ and suppose that $0 \in \partial K$. If $0$ is an accessible boundary point, then there is a continuous branch of $\log z$ on $K \setminus \{0\}$.

**Proof.** Let $J = \gamma([0,1])$ be a Jordan arc in the complement of $K$, joining $\infty$ with $0$; in particular, $\lim_{t \to 0} \gamma(t) = \infty$ and $\lim_{t \to 1} \gamma(t) = 0$. Note that $J = J \cup \{0\}$. Then $\Omega := \mathbb{C} \setminus J$ is a simply connected domain in $\mathbb{C}$ with $0 \notin \Omega$. Hence there is a holomorphic branch of $\log z$ in $\Omega$. Because $K \setminus \{0\} \subseteq \Omega$, we have obtained the desired logarithm.

For example if $K$ is the union of $\{0\}$ with the spiral parametrized by

$$z(t) = \left\{ \frac{1}{1 + t}e^{it} : 0 \leq t < \infty \right\},$$

then $0$ is an accessible boundary point of $K = \partial K$ and $\log z(t) = it - \log(1+t)$ is a continuous branch of the logarithm on $K \setminus \{0\}$.

**Figure 5.** A spiral ending at the origin

It is not known at present, whether accessibility characterizes the compact sets under discussion.

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