SOME TOPOLOGICAL PROPERTIES OF SPACES BETWEEN
THE SORGENFREY AND USUAL TOPOLOGIES ON REAL
NUMBER

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Abstract. The H-space, denoted as $(\mathbb{R}, \tau_A)$, has $\mathbb{R}$ as its point set and a basis consisting of usual open interval neighborhood at points of $A$ while taking Sorgenfrey neighborhoods at points of $\mathbb{R} - A$. In this paper, we mainly discuss some topological properties of $H$-spaces. In particular, we prove that, for any subset $A \subset \mathbb{R}$,

1. Introduction

The usual, metric topology on $\mathbb{R}$ is a topological space which coarser than the Sorgenfrey line $\mathbb{S}$, which is a well known space and has been studied extensively. It is well known that Sorgenfrey line has a basis consisting of all half-open intervals of the form $[a, b)$, where $a < b$. The class of $H$-spaces mentioned in [12] is a space between usual topology of real number $\mathbb{R}$ and topology of Sorgenfrey line $\mathbb{S}$, which was described by Hattori in [8]. The $H$-space, denoted as $(\mathbb{R}, \tau_A)$, has $\mathbb{R}$ as its point set and a basis consisting of usual open interval neighborhood at points of $A$ while taking Sorgenfrey neighborhoods at points of $\mathbb{R} - A$, that is, the topology $\tau_A$ is defined as follows:

1. For each $x \in A$, $\{(x - \varepsilon, x + \varepsilon) : \varepsilon > 0\}$ is the neighborhood base at $x$, and
2. For each $x \in \mathbb{R} \setminus A$, $\{[x, x + \varepsilon) : \varepsilon > 0\}$ is the neighborhood base at $x$.

Chatyrko and Hattori first began to study the properties of such spaces, where many interesting results were obtained, see [4] and [5]. In particular, for any $A \subset \mathbb{R}$, $(\mathbb{R}, \tau_A)$ is regular, hereditarily Lindelöf, hereditarily separable and Baire space. Moreover, for any closed subset $A$ of $\mathbb{R}$, they proved that $(\mathbb{R}, \tau_A)$ is homeomorphic to the Sorgenfrey line $\mathbb{S}$ if and only if $A$ is countable. In 2017, Kulesza in [12] proved that $(\mathbb{R}, \tau_A)$ is perfectly subparacompact; quasi-metrizable.
made an improvement and a summary on the basis of Chatyrko and Hattori’s work, and the author called this kind of spaces as $H$-space, and demonstrated the properties of $H$-space with respect to homeomorphism, functions, completeness and reversibility. In particular, Kulesza proved that $(\mathbb{R}, \tau_A)$ is homeomorphic to $\mathbb{S}$ if and only if $A$ is scattered, and $(\mathbb{R}, \tau_A)$ is complete if and only if $\mathbb{R} \setminus A$ is countable, which implies that $(\mathbb{R}, \tau_{\mathbb{R}})$ is complete. Moreover, Bouziad and Sukhacheva in [1] gave some characterizations of some topological properties of $(\mathbb{R}, \tau_A)$, such as each compact subspace being countable, locally compactness and so on. In this paper, we continue the work of Chatyrko and Hattori by proving additional information about the spaces $(\mathbb{R}, \tau_A)$. The remaining of this paper is organized as follows.

Section 2 is dedicated to outline some concepts and terminologies. In Section 3, we mainly discuss some topological properties of $H$-spaces, such as zero-dimension, $\sigma$-compactness, $k_\omega$-property, perfect property, quasi-metrizability and so on. In particular, we give the characterizations of $A$ or $\mathbb{R} \setminus A$ such that $(\mathbb{R}, \tau_A)$ has topological properties of zero-dimension, $\sigma$-compactness, and $k_\omega$-property respectively. Moreover, we show that $(\mathbb{R}, \tau_A)^{\mathbb{N}}$ is perfectly subparacompact. Further, we discuss some generalized metric properties of $(\mathbb{R}, \tau_A)$, and prove that there exists a subset $A \subset \mathbb{R}$ such that $(\mathbb{R}, \tau_A)$ is not quasi-metrizable. In Section 4, we pose some interesting questions about $H$-spaces.

2. Preliminaries

In this section, we introduce the necessary notation and terminologies. First of all, let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of all positive integers, all integers and all real numbers, respectively. For undefined terminologies, the reader may refer to [6] and [7].

Definition 2.1. [6] Let $X$ be a topological space.

(1) $X$ is called zero-dimensional if it has a base of sets that are at the same time open and closed in it.

(2) $X$ is called a Baire space if every intersection of a countable collection of open dense sets in $X$ is also dense in $X$.

(3) $X$ is called locally compact, if every point $x$ of $X$ has a compact neighbourhood, i.e., there exists an open set $U$ and a compact set $K$, such that $x \in U \subseteq K$.

(4) $X$ is called a $k_\omega$-space if there exists a family of countably many compact subsets $\{K_n : n \in \mathbb{N}\}$ of $X$ such that each subset $F$ of $X$ is closed in $X$ provided that $F \cap K_n$ is closed in $K_n$ for each $n \in \mathbb{N}$.

(5) $X$ is $\sigma$-compact if it is the union of countably many compact subsets of $X$.

(6) $X$ is Lindelöf if every open cover of $X$ has a countable subcover.

Clearly, each $k_\omega$-space is $\sigma$-compact and each $\sigma$-compact is Lindelöf.

Definition 2.2. [6, 7] (1) A space $X$ is subparacompact if each open cover of $X$ has a $\sigma$-locally finite closed refinement.

(2) A space $X$ is perfect if each closed subset of $X$ is a $G_\delta$ in $X$.

(3) A space $X$ is perfectly subparacompact if it is perfect and subparacompact.
(4) A space $X$ is weakly $\theta$-refinable if for each open cover $\mathcal{U}$ of $X$, there exists an open cover $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}(n)$ of $X$ which is refines $\mathcal{U}$ and which has the property that if $x \in X$, then there exists an $n \in \mathbb{N}$ such that $x$ belongs to exactly $k$ members of $\mathcal{V}(n)$ for some $k \in \mathbb{N}$.

(5) A family $\mathcal{U}$ of open sets in $X$ is called interior-preserving if for $\mathcal{F} \subset \mathcal{U}$ and $y \in \bigcap \mathcal{F}$, $\bigcap \mathcal{F}$ is an open neighborhood of $y$.

**Definition 2.3.** [6] A family $\mathcal{P}$ of subsets of a space $X$ is a network for $X$ if for each point $x \in X$ and any neighborhood $U$ of $x$ there is an $P \in \mathcal{P}$ such that $x \in P \subset U$. The network weight of a space $X$ is defined as the smallest cardinal number of the form $|\mathcal{P}|$, where $\mathcal{P}$ is a network for $X$, this cardinal number is denoted by $nw(X)$.

**Definition 2.4.** [7] Recall that $(X, \tau)$ is a $\beta$-space if there exists a function $g : \omega \times X \to \tau$ such that if $x \in g(n, x_n)$ for every $n \in \omega$ then the sequence $\{x_n\}$ has a cluster point in $X$.

**Definition 2.5.** [5] Let $A$ be a subset of $\mathbb{R}$ of the real number. Defined the topology $\tau_A$ on $\mathbb{R}$ as follows:

1. For each $x \in A$, $\{(x-\epsilon, x+\epsilon) : \epsilon > 0\}$ is the neighborhood base at $x$.
2. For each $x \in \mathbb{R} - A$, $\{[x, x+\epsilon) : \epsilon > 0\}$ is the neighborhood base at $x$.

Then $(\mathbb{R}, \tau_A)$ is called $H$-space. The point $x$ is called an $\mathbb{R}$-point, if $x \in A$, otherwise, $x$ is called an $\mathbb{S}$-point.

Let $\tau_E$ and $\tau_S$ denote the usual (Euclidean) topology of $\mathbb{R}$ and the topology of the Sorgenfrey line $\mathbb{S}$ respectively. It is clear that $\tau_A = \tau_E$ if $A = \mathbb{R}$ and $\tau_A = \tau_S$ if $A = \emptyset$. And it is also obvious that $\tau_E \subset \tau_A \subset \tau_S$.

Some topological properties of $(\mathbb{R}, \tau_E)$ and $(\mathbb{R}, \tau_S)$ is as the table below:

By the table, it is easy to see that, for every subset $A$ of real number, the $H$-space is always a hereditarily separable, paracompact, Lindelöf, normal and first-countable space. And we can also know that, the $H$-space is always not a compact, countably compact or sequentially compact space for any subset $A$ of $\mathbb{R}$. According to the [5, Proposition 2.3], $H$-space $(\mathbb{R}, \tau_A)$ is second-countable if and only if $\mathbb{R} - A$ is countable.

3. **Main Results**

In this section, we mainly discuss some topological properties of $H$-spaces, such as zero-dimension, $\sigma$-compactness, $k_\omega$-property, perfect property, quasi-metrizability and so on. We first, we give an obvious lemma.

**Lemma 3.1.** Let $D$ be a dense subset of $(\mathbb{R}, \tau_A)$. Then $D$ is dense in $(\mathbb{R}, \tau_E)$ and $(\mathbb{R}, \tau_S)$.

**Proof.** Obviously, $D$ is a dense subset of $(\mathbb{R}, \tau_E)$ since $\tau_A$ is finer than $\tau_E$. In order to prove $D$ being dense in $(\mathbb{R}, \tau_S)$, take an arbitrary non-empty open subset $U$ in $\tau_S$, then there exists a non-empty open subset $V$ in $\tau_E$, hence $\emptyset \neq V \cap D \subset U \cap D$. Therefore, $D$ is also dense in $(\mathbb{R}, \tau_S)$. \[\square\]

By Lemma 3.1, we have the following corollary.
Some topological properties of \((\mathbb{R}, \tau_E)\) and \((\mathbb{R}, \tau_S)\)

| Number | Property                | \((\mathbb{R}, \tau_E)\) | \((\mathbb{R}, \tau_S)\) |
|--------|-------------------------|-----------------------------|-----------------------------|
| 1      | metrizable              | Yes                         | No                          |
| 2      | Hereditarily Separable  | Yes                         | Yes                         |
| 3      | Normality               | Yes                         | Yes                         |
| 4      | Lindelöf                | Yes                         | Yes                         |
| 5      | Measurable              | Yes                         | No                          |
| 6      | Baire Space             | Yes                         | Yes                         |
| 7      | Zero-dimension          | No                          | Yes                         |
| 8      | Compactness             | No                          | No                          |
| 9      | Countably Compact       | No                          | No                          |
| 10     | Local Compactness       | Yes                         | No                          |
| 11     | Sequential Compactness  | No                          | No                          |
| 12     | Paracompactness         | Yes                         | Yes                         |
| 13     | \(\sigma\)-Compactness | Yes                         | No                          |
| 14     | Connectedness           | Yes                         | No                          |
| 15     | Path Connectedness      | Yes                         | No                          |
| 16     | Local Connectedness     | Yes                         | No                          |
| 17     | Every compact subset is countable | No | Yes |

**Corollary 3.2.** For an arbitrary subset \(A\) of \(\mathbb{R}\), we have \(d(\mathbb{R}, \tau_A) = d(\mathbb{R}, \tau_E) = d(\mathbb{R}, \tau_S) = \omega\).

Since \((\mathbb{R}, \tau_E)\) and \((\mathbb{R}, \tau_S)\) are all Baire, we have the following corollary.

**Corollary 3.3.** For an arbitrary subset \(A\) of \(\mathbb{R}\), the \(H\)-space \((\mathbb{R}, \tau_A)\) is a Baire space.

**Proposition 3.4.** For an arbitrary subset \(A\) of \(\mathbb{R}\), the \(H\)-space \((\mathbb{R}, \tau_A)\) is homeomorphic to \((\mathbb{R}, \tau_E)\) if and only if \(A = \mathbb{R}\).

**Proof.** Assume that \((\mathbb{R}, \tau_A)\) is homeomorphic to \((\mathbb{R}, \tau_E)\) and \(A \neq \mathbb{R}\). Hence \(\mathbb{R} \setminus A \neq \emptyset\). Take an arbitrary point \(a \in \mathbb{R} \setminus A\). Then \((-\infty, a)\) is an open and closed subset in \((\mathbb{R}, \tau_A)\). Hence \((\mathbb{R}, \tau_A)\) is not connected. However, \((\mathbb{R}, \tau_E)\) is connected. Hence \(A = \mathbb{R}\). \(\square\)

Now, we can prove one of main results of this paper, which gives a characterization of subset \(\mathbb{R} \setminus A\) such that \((\mathbb{R}, \tau_A)\) is zero-dimensional.

**Theorem 3.5.** For an arbitrary subset \(A\) of \(\mathbb{R}\), the \(H\)-space \((\mathbb{R}, \tau_A)\) is zero-dimensional if and only if \(\mathbb{R} \setminus A\) is dense in \((\mathbb{R}, \tau_E)\).

**Proof.** Necessity. Let \((\mathbb{R}, \tau_A)\) be zero-dimensional. Assume that \(\mathbb{R} \setminus A\) is not dense in \((\mathbb{R}, \tau_E)\), then it follows from Lemma 3.1 that \(\mathbb{R} \setminus A\) is also not dense in \((\mathbb{R}, \tau_A)\). Then there exists an open subset \(U\) in \((\mathbb{R}, \tau_A)\) such that \(U \cap (\mathbb{R} \setminus A) = \emptyset\). Hence \(U \subset A\), which implies that there exists an open interval \((c,d)\) such that \((c,d) \subset U \subset A\). Since \((\mathbb{R}, \tau_A)\) is zero-dimensional and \((c,d) \subset A\), it follows that \((c,d)\) contains an open and closed subset \(V\). Since each neighborhood of each
point of \( A \) belongs to \( \tau_E \), it is easy to see that \( V \) is an open and closed subset in \((\mathbb{R}, \tau_E)\). However, \((\mathbb{R}, \tau_E)\) is not zero-dimensional, which is a contradiction.

Sufficiency. Let \( \mathbb{R} \setminus A \) be dense in \((\mathbb{R}, \tau_E)\). Take an arbitrary \( x_0 \in \mathbb{R} \). We divide into the proof into the following two cases.

**Case 1:** \( x_0 \in A \).

By Lemma 3.1 there exist a strictly increasing sequence \( \{y_n\} \) and a strictly decreasing sequence \( \{z_n\} \) such that \( \{y_n\} \subset \mathbb{R} \setminus A \), \( \{z_n\} \subset \mathbb{R} \setminus A \) and two sequences all converge to \( x_0 \) in \((\mathbb{R}, \tau_E)\). For any \( n \in \mathbb{N} \), put \( U_n = [y_n, z_n) \). Clearly, each \( U_n \) is an open and closed subset of \((\mathbb{R}, \tau_A)\). However, it easily check that the family \( \{U_n : n \in \mathbb{N}\} \) is a base at \( x_0 \) in \((\mathbb{R}, \tau_A)\). Hence the point \( x_0 \) has a base consisting of open and closed subsets in \((\mathbb{R}, \tau_A)\).

**Case 2:** \( x_0 \not\in A \).

Obviously, there exists a strictly decreasing sequence \( \{x_n\} \) such that \( \{x_n\} \subset \mathbb{R} \setminus A \) and \( \{x_n\} \) converges to \( x_0 \) in \((\mathbb{R}, \tau_E)\). Then the family \( \{[x_0, x_n) : n \in \mathbb{N}\} \) is base consisting of open and closed subsets in \((\mathbb{R}, \tau_A)\).

In a word, \((\mathbb{R}, \tau_A)\) is zero-dimensional. 

Next we prove the second main result of this paper, which shows that local compactness is equivalent to \( k_\omega \) property in \((\mathbb{R}, \tau_A)\). Indeed, A. Bouziad and E. Sukhacheva in [1] has proved that, for an arbitrary subset \( A \) of \( \mathbb{R} \), we have \((\mathbb{R}, \tau_A)\) is locally compact if and only if \( \mathbb{R} \setminus A \) is closed in \( \mathbb{R} \) and discrete in \( \mathbb{S} \).

**Theorem 3.6.** For an arbitrary subset \( A \) of \( \mathbb{R} \), then the following statements are equivalent:

1. \((\mathbb{R}, \tau_A)\) is locally compact;
2. \((\mathbb{R}, \tau_A)\) is a \( k_\omega \)-space;
3. \( \mathbb{R} \setminus A \) is discrete and closed in \((\mathbb{R}, \tau_A)\).

**Proof.** From [1], we have \((1) \iff (3)\). The implication \((1) \Rightarrow (2)\) is easily to be checked. It suffices to prove that \((2) \Rightarrow (1)\).

\((2) \Rightarrow (1)\). Assume that \((\mathbb{R}, \tau_A)\) is a \( k_\omega \)-space, and let \( \{K_n\} \) be an increasing sequence of compact subsets such that the family \( \{K_n\} \) determines the topology of \((\mathbb{R}, \tau_A)\). Take an arbitrary \( x_0 \in \mathbb{R} \). Since \((\mathbb{R}, \tau_A)\) is first-countable, choose an open neighborhood base \( \{U_n : n \in \mathbb{N}\} \) of point \( x_0 \) in \((\mathbb{R}, \tau_A)\) such that \( U_{n+1} \subset U_n \) for each \( n \in \mathbb{N} \). We claim that there exist \( m, p \in \mathbb{N} \) such that \( U_m \subset K_p \). Suppose not, then for each \( n \in \mathbb{N} \) it follows that \( U_n \setminus K_n \neq \emptyset \), hence choose a point \( x_n \in U_n \setminus K_n \). Put \( K = \{x_n : n \in \mathbb{N}\} \cup \{x_0\} \). Then \( K \) is a compact subset in \((\mathbb{R}, \tau_A)\) and \( K \setminus K_n \neq \emptyset \) for each \( n \in \mathbb{N} \). However, since \((\mathbb{R}, \tau_A)\) is a \( k_\omega \)-space, it easily check that there exists \( n_0 \in \mathbb{N} \) such that \( K \subset K_{n_0} \), which is a contradiction.

From Theorem 3.6 it natural to pose the following question.

**Question 3.7.** What subsets \( A \) of \( \mathbb{R} \) are \((\mathbb{R}, \tau_A)\) being \( \sigma \)-compact?

Next we give some a partial answer to this question. First, we give some lemmas.

**Proposition 3.8.** For arbitrary \( B \) of \((\mathbb{R}, \tau_S)\), we have \( nw(B) \geq |B| \). In particular, \( w(B) \geq |B| \).
Proof. Let $\mathcal{P}$ be an arbitrary network of the subspace $B$ with $|\mathcal{P}| = nw(B)$. For each $x \in B$, let

$$
\mathcal{B}_x = \{x \in P : P \in \mathcal{P}, P \subset [0, \frac{1}{n}) \cap B \text{ for some } n \in \mathbb{N}\}.
$$

Then $\bigcup_{x \in B} \mathcal{B}_x \subset \mathcal{P}$ and is also a network of the subspace $B$. However, for any $x, y \in B$ with $x \neq y$, we have $\mathcal{B}_x \cap \mathcal{B}_y = \emptyset$, hence $nw(B) \geq |B|$.

By Proposition 3.8 and the separability of $(\mathbb{R}, \tau_S)$, we have the following corollary.

**Corollary 3.9.** [5 Proposition 2.3] For any subset $X$ of $(\mathbb{R}, \tau_S)$, $X$ is metrizable if and only if $X$ is countable.

**Lemma 3.10.** For an arbitrary subset $A$ of $\mathbb{R}$, $(\mathbb{R}, \tau_A)$ is submetrizable.

Proof. Since $(\mathbb{R}, \tau_A)$ and $\tau_E \subset \tau_A$, it follows that $(\mathbb{R}, \tau_A)$ is submetrizable.

**Lemma 3.11.** If $K$ is a compact subset of $(\mathbb{R}, \tau_A)$, then $K \cap (\mathbb{R} \setminus A)$ is countable.

Proof. By Lemma 3.10, $K$ is metrizable. Put $X = K \cap (\mathbb{R} \setminus A)$. Then $X$ is metrizable. Moreover, $X$ is subspace of $(\mathbb{R}, \tau_S)$. By Corollary 3.9, $X$ is countable. Therefore, $K \cap (\mathbb{R} \setminus A)$ is countable.

**Lemma 3.12.** For an arbitrary subset $A$ of $\mathbb{R}$, if $\mathbb{R} \setminus A$ is countable then $(\mathbb{R}, \tau_A)$ is $\sigma$-compact.

Proof. Put $U = \mathbb{R} \setminus \overline{\mathbb{R} \setminus A}$. Then $U$ is open in $(\mathbb{R}, \tau_A)$ and $U \subset A$, hence $U$ is open in $(\mathbb{R}, \tau_E)$, which implies that $U$ is $\sigma$-compact. From $U \subset A$, it follows that $U$ is $\sigma$-compact $(\mathbb{R}, \tau_A)$. By the countability of $\mathbb{R} \setminus A$, $(\mathbb{R}, \tau_A)$ is $\sigma$-compact.

Now we have the following two results.

**Theorem 3.13.** For an arbitrary subset $A$ of $\mathbb{R}$, $(\mathbb{R}, \tau_A)$ is $\sigma$-compact, then $\mathbb{R} \setminus A$ is countable and nowhere dense in $(\mathbb{R}, \tau_A)$.

Proof. By Lemma 3.11, it is easy to see that $\mathbb{R} \setminus A$ is countable. It suffices to prove that $\mathbb{R} \setminus A$ is nowhere dense.

Assume that $\mathbb{R} \setminus A$ is not nowhere dense. Then there exists an open subset $V$ being contained in the closure of $\mathbb{R} \setminus A$. Hence there exist $a, b \in \mathbb{R} \setminus A$ such that $[a, b] \subset V$. Then $[a, b]$ is $\sigma$-compact since $[a, b]$ is open and closed in $(\mathbb{R}, \tau_A)$, hence there exists a sequence of compact subsets $\{K_n\}$ of $(\mathbb{R}, \tau_A)$ such that $[a, b] = \bigcup_{n \in \mathbb{N}} K_n$. By Lemma 3.11, it easily check that $[a, b]$ is a Baire space, then there exists $n \in \mathbb{N}$ such that $K_n$ contains an empty open subset $W$ in $(\mathbb{R}, \tau_A)$. Since $W \subset [a, b] \subset V$, there exist $c, d \in \mathbb{R} \setminus A$ such that $[c, d] \subset [a, b]$. Since $[c, d]$ is closed and $[c, d] \subset K_n$, it follows that $[c, d]$ is compact, which is a contradiction. Therefore, $\mathbb{R} \setminus A$ is nowhere dense.

**Theorem 3.14.** For an arbitrary subset $A$ of $\mathbb{R}$, if $\mathbb{R} \setminus A$ is countable and scattered in $(\mathbb{R}, \tau_A)$, then $(\mathbb{R}, \tau_A)$ is $\sigma$-compact.
Proof. Assume that $\mathbb{R} \setminus A$ is countable and scattered. Then it follows from [10, Corollary 3] that $\mathbb{R} \setminus A$ is homeomorphic to a subspace of $[0, \omega_1)$. Hence it easily see that the closure of $\mathbb{R} \setminus A$ in $(\mathbb{R}, \tau_A)$ is countable scattered and non-discrete. By Lemma 3.12, $(\mathbb{R}, \tau_A)$ is $\sigma$-compact.

The following example shows that the property of $\sigma$-compact in $(\mathbb{R}, \tau_A)$ does not imply local compactness.

Example 3.15. There exists a subset $A$ such that $(\mathbb{R}, \tau_A)$ is $\sigma$-compact but not locally compact.

Proof. Let $A = \mathbb{R} \setminus \{(0) \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $\mathbb{R} \setminus A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is closed, countable scattered and non-discrete. By Theorem 3.14, $(\mathbb{R}, \tau_A)$ is $\sigma$-compact. However, it follows from Theorem 3.6 that $(\mathbb{R}, \tau_A)$ is not locally compact and not a $k_\omega$-space.

Next we prove that $(\mathbb{R}, \tau_A)^{\omega_0}$ is perfectly subparacompact for arbitrary subset $A \subset \mathbb{R}$. The proof of the following theorem is similar to the proof of [9, Lemma 2.3]. However, for the convenience to the reader, we give out the proof.

Theorem 3.16. For an arbitrary subset $A$ of $\mathbb{R}$, $(\mathbb{R}, \tau_A)^n$ is perfect for every $n \in \mathbb{N}$.

Proof. By induction. The theorem is clear for $n = 1$ since $(\mathbb{R}, \tau_A)$ is a Lindelöf space. Therefore let us suppose the theorem for $n$ and let us prove it for $n + 1$.

Let $Z = \prod_{i=1}^{n+1} Z_i$ with $Z_i = (\mathbb{R}, \tau_A)$ for all $i \leq n + 1$. For every $m \leq n + 1$, put $Z(m) = \prod_{i=1}^{n+1} Z_i(m)$, where $Z_m(m) = (\mathbb{R}, \tau_E)$ and $Z_i(m) = (\mathbb{R}, \tau_A)$ if $i \neq m$.

Now it suffices to prove that arbitrary open subset $U$ of $Z$ is an $F_\sigma$ in $Z$. For every $m \leq n + 1$, let $U(m)$ be the interior of $U$ as a subset of $Z(m)$, and let $U^* = \bigcup_{m=1}^{n+1} U(m)$. It follows from [9, Lemma 2.2] that each $Z(m)$ is perfect, hence $U(m)$ is an $F_\sigma$ in $Z(m)$ and thus also in $Z$. Therefore, $U^*$ is also an $F_\sigma$ in $Z$. Put $U' = U \setminus U^*$. Thus it only remains to prove that $U'$ is an $F_\sigma$ in $Z$. Clearly, for each $x = (x_1, \ldots, x_{n+1}) \in U'$, it has $x_i \in \mathbb{R} \setminus A$ for each $i \leq n + 1$.

For each $z \in U'$, let $\{W_j(z) : j \in \mathbb{N}\}$ denote the base of neighborhoods of $z$ in $Z$ defined by

$$W_j(z) = \{y \in Z : y_i \in [z_i, z_i + \frac{1}{j}) \text{ for each } i \leq n + 1\}.$$

For every $j \in \mathbb{N}$, let

$$U'_j = \{z \in U' : W_j(z) \subset U\}.$$

It easily see that $U' = \bigcup_{j=1}^{\infty} U'_j$. Next we shall prove that each $U'_j$ is closed in $Z$.

Take an arbitrary $j \in \mathbb{N}$, assume $z \notin U'_j$, it suffices to prove that $z$ is not in the closure of $U'_j$ in $Z$.

For each $F \subset \{1, \cdots, n + 1\}$, let

$$U'_{j,F}(z) = \{y \in U'_j : z_i = y_i \text{ iff } i \in F\}.$$

Clearly, $U'_j = \bigcup\{U'_{j,F}(z) : F \subset \{1, \cdots, n + 1\}\}$. Then it suffices to prove that for each $F \subset \{1, \cdots, n + 1\}$ there exists a neighborhood of $z$ in $Z$ disjoint from
Indeed, suppose that \( W_j(z) \cap U'_{j,F}(z) \neq \emptyset \), then it can choose a point \( x \in W_j(z) \cap U'_{j,F}(z) \). Then the set

\[
V = W_j(z) \cap \{y \in Z : y_i < x_i \text{ if } i \notin F\}
\]

is a neighborhood of \( z \) in \( Z \), and it will suffice to prove that \( V \cap U'_{j,F} = \emptyset \).

Suppose not, then there exists some \( y \in V \cap U'_{j,F} \). Clearly, \( y \in W_j(z) \) and \( y \neq z \), thus there is an \( m \leq n + 1 \) such that \( y_m > z_m \). Then \( m \notin F \). Put

\[
W = W_j(y) \cap \{u \in Z : y_m < u_m\}.
\]

Clearly, \( W \) is open in \( Z(m) \) and \( W \subset W_j(y) \subset U \). It follows from the definition of \( U(m) \) that \( W \subset U(m) \subset U^* \). Moreover, it easily check that \( x \in W \). Therefore, \( x \in U^* \), which is a contradiction. That completes the proof.

**Theorem 3.17.** For arbitrary \( n \in \mathbb{N} \), \((\mathbb{R}, \tau_A)^n\) is perfectly subparacompact.

**Proof.** By Theorem 3.16, \((\mathbb{R}, \tau_A)^n\) is perfect. It suffices to prove that \((\mathbb{R}, \tau_A)^n\) is subparacompact.

By induction. The result is certainly true for \( n = 1 \). Let us assume that \((\mathbb{R}, \tau_A)^n\) is subparacompact. Next it will prove that \((\mathbb{R}, \tau_A)^{n+1}\) is subparacompact. By [13, Proposition 2.9], it suffices to prove that \((\mathbb{R}, \tau_A)^{n+1}\) is weakly \( \theta \)-refinable. Put \( Z = (\mathbb{R}, \tau_A)^{n+1} \). Let \( \mathscr{W} = \{W(\alpha) : \alpha \in A\} \) be a basic open cover of the space \( Z \), where \( W(\alpha) = U(1, \alpha) \times \cdots \times U(n+1, \alpha) \) such that \( U(k, \alpha) = (a(k, \alpha), b(k, \alpha)) \) if \( a(k, \alpha) \in A \) and \( U(k, \alpha) = [a(k, \alpha), b(k, \alpha)) \) if \( a(k, \alpha) \notin A \). By the same notations in the proof of Theorem 3.16, let \( W(\alpha, m) \) be the interior of the set \( W(\alpha) \) in \( Z(m) \) for each \( m \leq n + 1 \), thus \( W(\alpha, m) \) is open in \( Z(m) \), hence also open in \( Z \). For each \( m \leq n + 1 \), put \( \mathscr{G}(m) = \{W(\alpha, m) : \alpha \in A\} \). By [13, Corollary 2.6], each \( Z(m) \) is perfect subparacompact. Then it follows from [13, Proposition 2.9] that \( \mathscr{G}(m) \) has a weakly \( \theta \)-refinement \( \mathscr{H}(m) \) which covers \( \bigcup \mathscr{H}(m) \) and which consists of open subsets of \( Z(m) \). Clearly, \( \mathscr{H}(m) \) is also a collection of open subsets of \( Z \). Let

\[
Y = Z \setminus \bigcup \{\mathscr{H}(m) : 1 \leq m \leq n + 1\}.
\]

Then for each \( y \in Y \) it has \( y_i \in \mathbb{R} \setminus A \) for each \( i \leq n + 1 \), hence there exists \( \alpha \in A \) such that \( y_i = a(i, \alpha) \) for each \( i \leq n + 1 \). For each \( y \in Y \), pick \( \alpha(y) \in A \) such that \( y \in W(\alpha(y)) \). Put

\[
\mathscr{H}(0) = \{W(\alpha(y)) : y \in Y\}.
\]

It easily see that if \( x \) and \( y \) are distinct elements of \( Y \), then \( y \notin W(\alpha(x)) \). Therefore, \( \mathscr{H}(0) \) is a collection of open subsets of \( Z \) which covers \( Y \) in such a way that each point of \( Y \) belongs to exactly one member of \( \mathscr{H}(0) \). Hence \( \mathscr{H} = \{\mathscr{H}(m) : m \in \omega\} \) is a weak \( \theta \)-refinement of \( \mathscr{W} \). Therefore, \( Z \) is weakly \( \theta \)-refinable.

By [9, Proposition 2.1] and [13, Proposition 2.7], we have the following theorem.

**Theorem 3.18.** For an arbitrary subset \( A \) of \( \mathbb{R} \), \((\mathbb{R}, \tau_A)^{\aleph_0}\) is perfectly subparacompact.

**Corollary 3.19.** [9, 13] The space \((\mathbb{R}, \tau_S)^{\aleph_0}\) is perfectly subparacompact.
Finally, we consider the quasi-metrizability of $H$-spaces. It is well-known that $(\mathbb{R}, \tau_E)$ and $(\mathbb{R}, \tau_S)$ are all quasi-metrizable, it natural to pose the following question.

**Question 3.20.** For an arbitrary $A \subset \mathbb{R}$, is $(\mathbb{R}, \tau_A)$ quasi-metrizable?

We give some a negative answer to Question 3.20 in Example 3.23. Indeed, from the definition of generalized ordered space, we have the following proposition.

**Proposition 3.21.** For arbitrary $A \subset \mathbb{R}$, the $H$-space $(\mathbb{R}, \tau_A)$ is a generalized ordered space.

By [11, Theorem 10], we can easily give a characterization of subset $A$ of $\mathbb{R}$ such that $(\mathbb{R}, \tau_A)$ is quasi-metrizable, see Theorem 3.22.

**Theorem 3.22.** For any subset $A \subset \mathbb{R}$, the $H$-space $(\mathbb{R}, \tau_A)$ is quasi-metrizable if and only if $\mathbb{R} \setminus A$ is a $F_\sigma$-set in $(\mathbb{R}, \tau_{S^-})$, where $(\mathbb{R}, \tau_{S^-})$ is the set of real numbers with the topology generated by the base $\{(a, b] : a, b \in \mathbb{R}, a < b\}$.

Now, we can give a negative answer to Question 3.20.

**Example 3.23.** There exists a subset $A$ of $\mathbb{R}$ such that $(\mathbb{R}, \tau_A)$ is not quasi-metrizable.

**Proof.** Indeed, let $A = \mathbb{Q}$ be the rational number. By Theorem 3.22 assume $\mathbb{R} \setminus A$ is a $F_\sigma$-set in $(\mathbb{R}, \tau_{S^-})$, then $\mathbb{Q}$ is a $G_\delta$-set in $(\mathbb{R}, \tau_{S^-})$. However, it follows from [3, Theorem 3.4] that $(\mathbb{R}, \tau_{S^-})$ does not have a dense metrizable $G_\delta$-space, which is a contradiction.

 Obviously, if $\mathbb{R} \setminus A$ is a $F_\sigma$-set in $(\mathbb{R}, \tau_A)$, then $\mathbb{R} \setminus A$ is a $F_\sigma$-set in $(\mathbb{R}, \tau_{S^-})$, hence we have the following corollary.

**Corollary 3.24.** If $\mathbb{R} \setminus A$ is a $F_\sigma$-set in $(\mathbb{R}, \tau_A)$, then $(\mathbb{R}, \tau_A)$ is quasi-metrizable.

We now close this section with a result about generalized metric property of $H$-space.

**Theorem 3.25.** For an arbitrary $A \subset \mathbb{R}$, then the following statements are equivalent:

1. $(\mathbb{R}, \tau_A)$ is metrizable;
2. $(\mathbb{R}, \tau_A)$ is a $\beta$-space;
3. $\mathbb{R} \setminus A$ is countable.

**Proof.** Obviously, it suffices to prove (2) $\Rightarrow$ (3). Assume that $(\mathbb{R}, \tau_A)$ is a $\beta$-space. Since $(\mathbb{R}, \tau_A)$ is a paracompact submetrizable space, it follows from [7, Theorem 7.8 (ii)] that $(\mathbb{R}, \tau_A)$ is semi-stratifiable. By Proposition 3.21 and [7, Theorems 5.16 and 5.21], $(\mathbb{R}, \tau_A)$ is a stratifiable space, hence $(\mathbb{R}, \tau_A)$ is a $\sigma$-space by [7, Theorem 5.9]. Then $(\mathbb{R}, \tau_A)$ has a countable network since $(\mathbb{R}, \tau_A)$ is separable, hence $\mathbb{R} \setminus A$ has a countable network. Therefore, it follows from Proposition 3.8 that $\mathbb{R} \setminus A$ must be countable. □
4. Open questions

It is well known that $(\mathbb{R}, \tau_E) \times (\mathbb{R}, \tau_E)$ is Lindelöf, and $(\mathbb{R}, \tau_S) \times (\mathbb{R}, \tau_S)$ is not Lindelöf, hence it is natural to pose the following question.

**Question 4.1.** For an arbitrary subset $A$ of $\mathbb{R}$, are the following statements equivalent?

1. $(\mathbb{R}, \tau_A) \times (\mathbb{R}, \tau_A)$ is Lindelöf;
2. $(\mathbb{R}, \tau_A) \times (\mathbb{R}, \tau_A)$ is normal;
3. $(\mathbb{R}, \tau_A)$ is metizable.

The following example gives a negative answer to Question 4.1 under the assumption of CH.

**Example 4.2.** Under the assumption of CH, there exists a subspace $A \subset \mathbb{R}$ such that $\mathbb{R} \setminus A$ is uncountable and $(\mathbb{R}, \tau_A) \times (\mathbb{R}, \tau_A)$ is Lindelöf.

**Proof.** By [2, Theorem 3.4], there exists an uncountable subset $Y \subset S$ such that $Y^2$ is Lindelöf. Put $A = \mathbb{R} \setminus Y$. Then $(\mathbb{R}, \tau_A) \times (\mathbb{R}, \tau_A)$ is Lindelöf. Indeed, it is obvious that

$$(\mathbb{R}, \tau_A) \times (\mathbb{R}, \tau_A) = (A \times A) \cup (A \times Y) \cup (Y \times A) \cup (Y \times Y).$$

Since $A$ is a separable metrizable space, the subspace $A \times A$, $A \times Y$ and $Y \times A$ are Lindelöf. Therefore, $(\mathbb{R}, \tau_A) \times (\mathbb{R}, \tau_A)$ is Lindelöf. □

By Theorem 3.14, we have the following question.

**Question 4.3.** If $(\mathbb{R}, \tau_A)$ is $\sigma$-compact, is $A$ a scattered subspace?

The following question was posed by Boaz Tsaban.

**Question 4.4.** When is the space $(\mathbb{R}, \tau_A)$ Menger (Hurewicz) for any $A \subset \mathbb{R}$?

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