Phase spaces related to standard classical $r$-matrices

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Abstract

Fundamental representations of real simple Poisson Lie groups are Poisson actions with a suitable choice of the Poisson structure on the underlying (real) vector space. We study these (mostly quadratic) Poisson structures and corresponding phase spaces (symplectic groupoids).

0 Introduction

The recent development of noncommutative geometry and, in particular, the theory of quantum groups, raises the question what happens with known models of physical systems when we pass from usual configurations to non-commutative ones. For the classical mechanical systems, this means that we allow the configuration space to be a Poisson manifold (positions need not commute). The phase space corresponding to usual configuration manifold (Poisson structure equal zero) is its cotangent bundle. For a general Poisson manifold, the role of the phase space plays the corresponding symplectic groupoid (if such exists, it is unique — if one restricts to connected and simply connected fibers).

It is natural to consider first mechanical systems with symmetry. In the Poisson case a symmetry is described by a Poisson action (of a Poisson group). This requirement imposes a reasonable limitation on a choice of the Poisson structure and actually leads to a construction of it.

In this paper we construct Poisson structures on real finite-dimensional vector spaces (the configuration spaces), such that the action of a chosen linear simple Poisson group becomes a Poisson action (the Poisson structure on the group is typically given by a standard classical $r$-matrix). We construct also the corresponding phase spaces.

1 Preliminaries and notation

For the theory of Poisson Lie groups we refer to [1, 2, 3, 4, 5]. Let us recall some basic notions and facts. We follow the notation used in our previous papers [6, 7, 8].
A Poisson Lie group is a Lie group $G$ equipped with a Poisson structure $\pi$ such that the multiplication map is Poisson. The latter property is equivalent to the following property (called multiplicativity of $\pi$)

$$\pi(gh) = \pi(g)h + g\pi(h) \quad \text{for } g, h \in G.$$ 

Here $\pi(g)h$ denotes the right translation of $\pi(g)$ by $h$ etc. This notation will be used throughout the paper.

A Poisson Lie group is said to be coboundary if

$$\pi(g) = rg - gr$$

for a certain element $r \in \mathfrak{g}$. Here $\mathfrak{g}$ denotes the Lie algebra of $G$. Any bivector field of the form (1) is multiplicative. It is Poisson if and only if

$[r, r] \in (\bigwedge \mathfrak{g})_{\text{inv}}$

(the Schouten bracket $[r, r]$ is $\mathfrak{g}$-invariant). In this case the element $r$ is said to be a classical $r$-matrix (on $\mathfrak{g}$).

If $G$ is semisimple, any Poisson Lie group structure on $G$ is coboundary. Standard classical $r$-matrix for a simple group — such that corresponds to ‘the standard (quantum) $q$-deformation’ — is given by (cf. [9], Prop. 2.1 in [10])

$$r = c \sum_{\alpha > 0} X_{\alpha} \wedge X_{-\alpha} / \langle X_{\alpha}, X_{-\alpha} \rangle,$$

where $X_{\pm \alpha}$ are (positive and negative) root vectors relative to a Cartan subalgebra in $\mathfrak{g}$, $\langle \cdot, \cdot \rangle$ is the Killing form and $c$ is a constant (if $G$ is compact, $X_{-\alpha} = X_{\alpha}$ and $c$ is imaginary).

Let $(G, \pi)$ be a Poisson Lie group. An action of $G$ on a Poisson manifold $(M, \pi_M)$ is said to be a Poisson action if the action map $G \times M \to M$ is Poisson. It holds if and only if the following $(G, \pi)$-multiplicativity of $\pi_M$ is satisfied

$$\pi_M(gx) = \pi(g)x + g\pi_M(x) \quad \text{for } g \in G, x \in M.$$ 

For any fixed action $G \times M \ni (g, x) \mapsto gx \in M$ and any $k$-vector $w \in \bigwedge^k \mathfrak{g}$ we denote by $w_M$ the associated $k$-vector field on $M$:

$$w_M(x) := wx.$$

## 2 Problem

The classical $r$-matrices for simple Lie groups like $SL(n, \mathbb{R})$, $SO(n, \mathbb{R})$, $SU(n)$ are relatively well investigated (in the sequel we shall consider mainly the standard $r$-matrices (2), which are indeed representing the non-trivial part of all classical $r$-matrices). In order to consider mechanical systems based on Poisson symmetry (typically being a ‘deformation’ of some ordinary symmetry), we have first to deal with the following problems:
1. Given an action $G \times M \to M$ (the ordinary symmetry) and a Poisson structure $\pi$ on $G$ making it a Poisson Lie group $(G, \pi)$ (a ‘deformation’ of the group), find all Poisson structures $\pi_M$ on $M$ such that the action becomes Poisson (the ‘deformed’ symmetry).

2. In cases when $M$ plays the role of the configurational manifold, construct the phase space $\text{Ph}(M, \pi_M)$ i.e. the symplectic groupoid of $(M, \pi_M)$.

For symplectic groupoids, phase spaces of Poisson manifolds etc., we refer to [11, 12, 13, 14, 15, 16].

For simplicity, in this paper we consider only the essential part of the structure of the symplectic groupoid (which is in most cases sufficient to formulate the classical model). Namely, for a given Poisson manifold $(M, \pi_M)$ of dimension $k$ we shall construct a symplectic manifold $S$ of dimension $2k$, a surjective Poisson map from $S$ to $M$ and a Lagrangian section of it. In this case, we shall simply call $S$ the phase space of $(M, \pi_M)$.

3 Fundamental bi-vector field

Let $G \times M \to M$ be an action. Let $r \in \bigwedge^2 g$ be a classical $r$-matrix and $\pi$ — the corresponding Poisson structure (1) (this notation is fixed throughout the section).

**Lemma 3.1**

1. $r_M$ is $(G, \pi)$-multiplicative

2. any $(G, \pi)$-multiplicative $\pi_M$ is given by $\pi_M = r_M + \pi_{\text{inv}}$, where $\pi_{\text{inv}}$ is a $G$-invariant bi-vector field

3. $[r_M, \pi_{\text{inv}}] = 0$

4. $[r_M, r_M] = [r, r]_M$.

Point 1 follows from $r(gx) = (rg - gr)x + g(rx)$. Point 3 follows from the fact that $r_M$ is built out of the fundamental vector fields of the action (and these vector fields preserve $\pi_{\text{inv}}$). From 3 follows that if both $r_M$ and $\pi_{\text{inv}}$ are Poisson then $\pi_M$ is also Poisson. Point 4 follows from the known property of fundamental fields of the action:

$$[X_M, Y_M] = [X, Y]_M$$

for $X, Y \in g$

(the Lie bracket on $g$ being defined by identifying elements of $g$ with the corresponding right-invariant vector fields on $G$).

In analogy with fundamental vector fields $X_M$, we call $r_M$ the fundamental bi-vector field. It is essential to know whether it is Poisson.

**Example 3.2** Poisson Minkowski spaces (Poincaré group action). Any invariant element of $\bigwedge^3 g$, where $g = \mathbb{R}^4 \times o(1, 3)$ is the Poincaré Lie algebra, is proportional to

$$\Omega = g^{ik} g^{lm} e_j \wedge e_l \wedge \Omega_{km}$$

$$\Omega_{km} := e_k \otimes g(e_m) - e_m \otimes g(e_k) \in o(1, 3),$$

(3)
(summation convention), where \((e_j)_{j=0, \ldots, 3}\) is a basis in \(M = \mathbb{R}^4\), \(g\) is the Lorentz metric and \(g^{jk}\) are the components of the contravariant metric (cf. [8, 17]). Since
\[
\Omega_M(x) = g^{jk} g^{lm} e_j \wedge e_l \wedge (e_k g(e_m, x) - e_m g(e_k, x)) = 0,
\]
for each classical \(r\)-matrix on \(\mathfrak{g}\) the fundamental bi-vector field \(r_M\) on \(M\) is Poisson (because \([r_M, r_M] = [r, r]_M \sim \Omega_M = 0\). By point 2 of Lemma [7] this is the only \((G, \pi)\)-multiplicative bivector field on \(M\), since zero is the only \(G\)-invariant bivector field on \(M\). (Recall also that any Poisson structure on \(G\) comes from an \(r\)-matrix [8].) Concluding: for each Poisson Poincaré group there is exactly one Poisson Minkowski space (see also [7]). All this is true also for the case of arbitrary signature, \(\mathfrak{g} = \mathbb{R}^{p+q} \rtimes O(p, q)\), in dimension \(n = p + q > 3\). (Cf. [18] for the quantum case.)

**Example 3.3** Poisson Minkowski spaces (Lorentz group action). Classical \(r\)-matrices for the Lorentz Lie algebra \(o(1, 3)\) are classified in [6]. We know that \([r, r] = [r_-, r_-]\) and it is non-zero only in the case \(r_- = i\lambda X_+ \wedge X_-\) (in the classification of [6]) with \(\lambda \neq 0\):
\[
[r_-, r_-] = -\lambda^2 [X_+ \wedge X_-, X_+ \wedge X_-] = 2\lambda^2 X_+ \wedge [X_+, X_-] \wedge X_- = 4\lambda^2 X_+ \wedge H \wedge X_-,
\]
where \(X_+, X_-, H\) is the standard basis:
\[
H = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad X_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad X_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.
\]
Considering the usual action of the Lorentz Lie algebra on the Minkowski space \(M = \mathbb{R}^{1+3}\), we obtain
\[
(X_+ \wedge X_-)_{M}(x) = 2\Omega_{01}(x) \wedge \Omega_{13}(x),
\]
where (see [3])
\[
\Omega_{jk}(x) = e_j x_k - e_k x_j.
\]
Since \(\Omega_{jk}(x), \Omega_{kl}(x)\) and \(\Omega_{lj}(x)\) are linearly dependent for each fixed \(j, k, l\),
\[
(X_+ \wedge H \wedge X_-)_{M}(x) = -2\Omega_{30}(x) \wedge \Omega_{01}(x) \wedge \Omega_{13}(x) = 0
\]
\((H_M(x) = \Omega_{30}(x))\), but
\[
(X_+ \wedge JH \wedge X_-)_{M}(x) = -2\Omega_{21}(x) \wedge \Omega_{01}(x) \wedge \Omega_{13}(x)
\]
\((J\) is the complex structure in \(\mathfrak{g}\)) is not zero. It follows that \(r_M\) is Poisson if and only if \(\lambda^2\) is real, i.e. either \(\alpha\) or \(\beta\) in [6] has to be zero. Moreover, since the only Lorentz invariant bivector field on \(M\) is zero, \(r_M\) is the only \((G, \pi)\)-multiplicative field on \(M\). It follows that for \(\alpha \cdot \beta \neq 0\) there is no Poisson structure on \(M\) such that the action is Poisson. (Similar fact should hold for quantum Lorentz groups [19]: \(q\) should be real or of modulus one.)

Returning to a general technique, consider now two special cases of \(r\)-matrices.
3.1 Triangular case: \([r, r] = 0\)

Let \(\xi: T^*M \to M\) be the cotangent bundle projection and let \(\pi_0\) denote the canonical Poisson structure of \(T^*M\). In the triangular case

1. \(r_M\) is Poisson (by Lemma 3.1.4)

2. \(r_{T^*M}\) is Poisson (also Lemma 3.1.4); \(\xi^* r_{T^*M} = r_M\)

3. \(\pi_{T^*M} := r_{T^*M} + \pi_0\) is Poisson (by Lemma 3.1.3); \(\xi^* \pi_{T^*M} = r_M\).

This means that problems formulated in Sect. 2 are relatively easily solved. As the phase space one can take the open subset of points in \(T^*M\), in which the Poisson structure \(\pi_{T^*M}\) is non-degenerate (it is certainly non-degenerate in a neighbourhood of the zero section — that’s why we have added \(\pi_0\) in 3). (To construct the symplectic groupoid one should still find the foliation symplectically orthogonal to the fibers of the projection and choose points which have also the projection on \(M\) along this foliation.)

For another approach to this case, see [16].

3.2 Case of a simple \(g\)

In this case one can use the method of [20] to rewrite the condition \([r_M, r_M] = 0\). Denote by \(\Omega\) the canonical invariant element of \(\bigwedge^3 g\). Its Killing transported version to \(\bigwedge^3 g^*\) is defined by

\[
\Omega^\dagger(X, Y, Z) = \langle [X, Y], Z \rangle.
\]

It is known [21] that all invariant elements of \(\bigwedge^3 g\) are proportional to \(\Omega\), hence \([r, r] \sim \Omega\). Suppose \([r, r]\) is not zero. Then \(r_M\) is Poisson \(\Leftrightarrow \Omega_M = 0\) (in general, \(\Omega_M\) is just \(G\)-invariant). Now,

\[\Omega x = 0 \iff \text{the composition of linear maps} \]

\[\mathbb{R} \xrightarrow{\Omega} \bigwedge^3 g \xrightarrow{\bigwedge^3 (g/\mathfrak{g}_x)} \]

is zero \(\iff\) the composition of linear maps

\[\mathbb{R} \xrightarrow{\Omega^*} \bigwedge^3 g \xleftarrow{\bigwedge^3 (g_x)^0} \]

is zero \(\iff\) the composition of linear maps

\[\mathbb{R} \xrightarrow{\Omega^\dagger} \bigwedge^3 g \xleftarrow{\bigwedge^3 (g_x)^\perp} \]

is zero \(\iff\) \(\Omega^\dagger|_{(g_x)^\perp} = 0 \iff \langle [X, Y], Z \rangle = 0\) for \(X, Y, Z \in \mathfrak{g}_x \Rightarrow [\mathfrak{g}_x^+, \mathfrak{g}_x^+] \subset \mathfrak{g}_x\). Concluding,

\[[r_M, r_M] x = 0 \iff [\mathfrak{g}_x^+, \mathfrak{g}_x^+] \subset \mathfrak{g}_x. \]

The advantage of this method is that we do not have to use the explicit form of \(\Omega\).
Proposition 3.4  In the following three cases, for any classical $r$-matrix on $\mathfrak{g}$ the fundamental bi-vector field $r_M$ on $M$ is Poisson:

1. $\mathfrak{g} = \mathfrak{so}(n, \mathbb{R})$, $M = \mathbb{R}^n$
2. $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, $M = \mathbb{R}^n$
3. $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$, $M = \mathbb{R}^{2n}$

For $\mathfrak{g} = \mathfrak{su}(n)$, $M = \mathbb{C}^n = \mathbb{R}^{2n}$, the fundamental bi-vector field is Poisson if and only if $r$ is triangular.

Proof: Let $e_1, \ldots, e_n$ be the standard basis in $\mathbb{R}^n$. The dual basis in $(\mathbb{R}^n)^*$ is denoted by $e_1^*, \ldots, e_n^*$. We consider subsequent cases separately.

1. $\mathfrak{g} = \mathfrak{so}(n, \mathbb{R})$. Set $x = e_n$, then $\mathfrak{g}_x$ is spanned by $\Omega_{jk}$ for $j, k < n$ and $\mathfrak{g}_x^\perp$ is spanned by $\Omega_{jn}$ for $j < n$. It is easy to see that $[\Omega_{jn}, \Omega_{kn}]$ is proportional to $\Omega_{jk}$, hence it belongs to $\mathfrak{g}_x$. This shows (by (4)) that $[r_M, r_M] = 0$.

2. $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. For the same $x = e_n$, $\mathfrak{g}_x$ is spanned by $e_n^j$ (the matrix units) for $j < n$, but $[e_n^j, e_n^k] = 0$.

3. $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$. We use the basis $e_1, \ldots, e_n, e_1^*, \ldots, e_n^*$ in $M = \mathbb{R}^n \oplus (\mathbb{R}^n)^* \cong \mathbb{R}^{2n}$. We have four types of matrix units defined by

$$e_j^k := e_j \otimes e^k, \quad e_j^k := e^j \otimes e_k, \quad e_{jk} := e_j \otimes e_k, \quad e^{jk} := e^j \otimes e^k,$$

with the action on $(x, p) \in M = \mathbb{R}^n \oplus (\mathbb{R}^n)^*$ given explicitly by

$$e_j^k(x, p) = e_j x^k, \quad e_j^k(x, p) = e^j p_k, \quad e_{jk}(x, p) = e_j p_k, \quad e^{jk}(x, p) = e^j x^k.$$

We use the following basis in $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$:

$$a_{jk} := e_{jk} + e_{kj} \quad (j \leq k), \quad b^{jk} := e^{jk} + e^{kj} \quad (j \leq k), \quad d_j^k := e_j^k - e^k_j. \quad (5)$$

For $x := e_n$, $\mathfrak{g}_x$ is spanned by $a_{jk}$, $b^{jk}$ with $j, k < n$ and $d_j^k$ with $k < n$ and $\mathfrak{g}_x^\perp$ is spanned by $a_{jn}$, $d_n^j$ with $j = 1, \ldots, n$. Now, $[a_{jn}, a_{kn}] = 0$ and $[d_n^j, a_{kn}] = \delta_n^k a_{nn} + \delta_n^j a_{kn} \in \mathfrak{g}_x$.

We now pass to the case of $\mathfrak{g} = \mathfrak{su}(n)$, with the basis

$$F_j^k := e_j^k - e_k^j, \quad G_j^k := i(e_j^k + e_k^j), \quad H_j := e_{j+1}^j + e_j^j - e_{j+1}^j. \quad (6)$$

For $x := e_n$, $\mathfrak{g}_x$ is spanned by $F_j^k$, $G_j^k$ for $j, k < n$ and $H_j$ for $j < n - 1$ and $\mathfrak{g}_x^\perp$ contains $F_j^n$, $G_j^n$ for $j < n$ (and the component of $H_{n-1}$, orthogonal to $H_j$ for $j < n - 1$). We have

$$[F_1^n, G_1^n] = 2i(e_1^1 - e_n^n) \not\in \mathfrak{g}_x.$$

Actually one can see that

$$[\mathfrak{g}_x^\perp, \mathfrak{g}_x^\perp] = \mathfrak{g}_x + \langle i(e_1^1 - e_n^n) \rangle$$

(inclusion (P) is violated only by one dimension).

Q.E.D.
4 Standard \( r \)-matrices for \( \mathfrak{sl}(n, \mathbb{R}) \) and \( \mathfrak{sp}(n, \mathbb{R}) \)

According to (2), the standard \( r \)-matrix for \( \mathfrak{g} = \mathfrak{sl}(n, \mathbb{R}) \) is given by

\[
 r = \varepsilon \sum_{j<k} e_j^k \wedge e_k^j \quad (\varepsilon \in \mathbb{R})
\]  (7)

(the Cartan subalgebra consists of diagonal matrices and the ‘positive’ roots are contained in the upper-triangular matrices). Considering the natural action of \( \mathfrak{g} \) on \( M := \mathbb{R}^n \), we obtain

\[
 r_M(x) = \varepsilon \sum_{j<k} x^j x^k e_j \wedge e_k = \sum_{j<k} (x^j e_j) \wedge (x^k e_k),
\]  (8)

which defines the following Poisson brackets of coordinates:

\[
 \{x^j, x^k\} = \varepsilon x^j x^k \quad \text{for} \ j < k.
\]  (9)

For \( \tilde{\mathfrak{g}} := \mathfrak{sp}(n, \mathbb{R}) \) we choose \( d_j^k \ (j < k) \) and \( a_{jk} \ (j \leq k) \) as positive roots, which gives the following expression for the standard \( r \)-matrix (we denote it by \( \tilde{r} \)):

\[
 \tilde{r} = \varepsilon \left( \sum_{j<k} d_j^k \wedge d_k^j + \frac{1}{2} \sum_{j,k} a_{jk} \wedge b^{jk} \right)
\] (notation as in (5)). Considering the natural action of \( \tilde{\mathfrak{g}} \) on \( \tilde{M} = T^*M = \mathbb{R}^n \oplus (\mathbb{R}^n)^* \), we obtain

\[
 \tilde{r}_{\tilde{M}}(x, p) = \varepsilon \left[ \sum_{j<k} (e_j x^k - e^k p_j) \wedge (e_k x^j - e^j p_k) + \frac{1}{2} \sum_{j,k} (e_j p_k + e_k p_j) \wedge (e^j x^k + e^k x^j) \right]
\]  (10)

\[
 = \varepsilon \left[ x \wedge p + \sum_{j<k} (x^j x^k e_j \wedge e_k - p_j p_k e^j \wedge e^k) + \sum_j \left( \sum_k (1 - \text{sgn}(k-j)) x^k p_k \right) e_j \wedge e^j \right],
\]

which gives the following quadratic Poisson brackets of coordinates and momenta

\[
 \{x^j, x^k\} = \varepsilon x^j x^k, \quad \{p_j, p_k\} = -\varepsilon p_j p_k \quad \text{for} \ j < k
\]  (11)

\[
 \{x^j, p_k\} = \varepsilon \left[ x^j p_k + \delta^j_k \left( \sum_i (1 - \text{sgn}(i-j)) x^i p_i \right) \right].
\]  (12)

Now observe that we can add to (10) the canonical bi-vector \( \pi_0 \) on \( T^*M \) which modifies (12) in the following way

\[
 \{x^j, p_k\} = -\delta^j_k + \varepsilon \left[ x^j p_k + \delta^j_k \left( \sum_i (1 - \text{sgn}(i-j)) x^i p_i \right) \right].
\]  (13)

Poisson structure (11), (13) projects on (9) and is non-degenerate in a neighbourhood of \( (\mathbb{R}^n \oplus \{0\}) \cup (\{0\} \oplus (\mathbb{R}^n)^*) \). We have thus constructed the phase space for \( (M, r_M) \) \( (M \text{ is indeed embedded into this phase space as a lagrangian submanifold: } p_j = 0 \text{ are first class constraints}) \).

The quantum version of the above construction has been described in [22].
Remark 4.1 The natural embedding of $\mathfrak{g}$ into $\tilde{\mathfrak{g}}$ (the lift to $T^*M$), given by $e_j^k \mapsto d_j^k$, is a homomorphism of Lie bialgebras. It follows that the action of $SL(n, \mathbb{R})$ on $T^*M$ is Poisson.

Remark 4.2 Set $\nabla_j := x^j e_j$ (no summation). According to (8), $r_M = \sum_{j<k} \nabla_j \wedge \nabla_k$. We see that $r_M$ is built of commuting vector fields and, actually, $r_M = \rho_M$, where $\rho$ is a $r$-matrix on the abelian Lie algebra spanned by those fields. Since $\rho$ is triangular and even abelian [23], one can easily construct the phase space of $(M, \rho_M)$ using the method described in Sect. 3.1, or, another method, described in [23]. In the present paper we shall exploit only the original $r$-matrix, because such an approach can be generalized and applied to more difficult situations (see next sections).

5 Crossed product phase spaces and quasitriangularity

Let $\mathfrak{g}$ be a Lie subalgebra of $\text{End} V$, where $V = \mathbb{R}^n$. Any classical $r$-matrix on $\mathfrak{g}$ may be identified as an element of $\bigwedge^2 \text{End} V$, which can be expressed in terms of matrix units:

$$r = \sum_{jklm} r^{jk}_{lm} e_j^l \otimes e_k^m$$

$$r^{jk}_{lm} = -r^{kj}_{ml}.$$ 

If $r$ is triangular, the phase space of $(V, r_V)$ can be realized on $T^*V = V \times V^*$ with the Poisson structure $\pi_{T^*V}$ being the sum of the canonical Poisson structure $\pi_0$ and $r_{T^*V}$ (cf. Sect. 3.1). We have then

$$\pi_{T^*V}(x, p) = \pi_0 + \sum_{jklm} r^{jk}_{lm}(e_j^l x^m - p_j^l e^m) \otimes (e_k^m x^m - p_k^m e^m),$$

which leads to the following Poisson brackets

$$\{x^j, x^k\} = \sum_{lm} r^{jk}_{lm} x^l x^m,$$

$$\{p_l, p_m\} = \sum_{jk} p_j^k p_k^m r^{jk}_{lm},$$

$$\{x^k, p_l\} = -\delta^k_l + \sum_{jm} p_j^r r^{jk}_{lm} x^m.$$ (14)

(15)

We shall use also the following abbreviated notation:

$$\{x_1, x_2\} = r x_1 x_2, \quad \{p_1, p_2\} = p_1 p_2 r, \quad \{x_1, p_2\} = -I + p_1 r x_2.$$ (16)

Definition 5.1 Let $(M, \pi_M)$ and $(N, \pi_N)$ be two Poisson manifolds and $P := M \times N$. If $\pi_P$ is a Poisson structure on $P$ such that the cartesian projections $P \to M, N$ are Poisson, then $(P, \pi_P)$ is said to be a crossed product of $(M, \pi_M)$ and $(N, \pi_N)$.
We see that in the triangular case, the phase space is a crossed product of \((V, r_V)\) and \((V^*, r_{V^*})\). Note that the Poisson brackets between \(x^k\) and \(p_t\) (the cross-relations) are expressed in terms of the same \(r\)-matrix as \(r_V\) and \(r_{V^*}\).

In the case of the non-triangular \(r\)-matrix [7] for \(sl(n, \mathbb{R})\), the phase space turns out to be also a crossed product of \((V, r_V)\) and \((V^*, r_{V^*})\). Poisson structure [14], [13] is realized on \(T^*V\) and has the following form

\[
\pi_{T^*V} = \pi_0 + r_{T^*V} + \Delta,
\]

where \(\Delta\) is some additional quadratic term (in the cross-relations), which was not present in the triangular case.

We shall now explain the nature of the additional term \(\Delta\) in a general situation. We assume that our \(r\)-matrix is such that \(r_V\) and \(r_{V^*}\) are Poisson (with explicit form of Poisson brackets given by [14]) and we ask what conditions should satisfy a bi-vector field \(\Delta\) of the form

\[
\Delta(x, p) = \sum_{jklm} p_j \Delta^{jk}_{lm} x^m e_k \wedge e^l
\]

on \(T^*V\), in order to make \([17]\) a Poisson bi-vector field. Since \(r_{T^*V}(x, p) = r_V(x) + r_{V^*}(p) + \sum_{jklm} p_j \Delta^{jk}_{lm} x^m e_k \wedge e^l\), we have

\[
\pi_{T^*V}(x, p) = \pi_0 + r_V + r_{V^*} + \sum_{jklm} p_j \Delta^{jk}_{lm} x^m e_k \wedge e^l,
\]

where \(\Delta^{jk}_{lm} = r^{jk}_{lm} + \Delta^{jk}_{lm}\). We look therefore for conditions on \(\Delta^{jk}_{lm}\) under which the brackets

\[
\{x_1, x_2\} = r_{x_1, x_2}, \quad \{p_1, p_2\} = p_1 p_2 r, \quad \{p_1, x_2\} = I - p_1 w x_2
\]

(abbreviated notation) satisfy the Jacobi identity. We first consider only the quadratic part

\[
\{x_1, x_2\} = r_{x_1, x_2}, \quad \{p_1, p_2\} = p_1 p_2 r, \quad \{p_1, x_2\} = -p_1 w x_2
\]

(it is easy to show that the quadratic part itself must also define a Poisson bracket). Since

\[
\{p_1, \{x_2, x_3\}\} = \{p_1, r_{23} x_2 x_3\} = -p_1 r_{23} (w_{12} + w_{13}) x_1 x_3
\]

and

\[
\{\{p_1, x_2\}, x_3\} - \{\{p_1, x_3\}, x_2\} = -p_1 w_{12} x_2 x_3 - \{p_1 w_{13} x_3, x_2\}
\]

\[
= p_1 (w_{13} w_{12} - w_{12} r_{23} x_2 x_3 - (p_1 (w_{12} w_{13} - w_{13} r_{23} x_2 x_3)) x_2 x_3,
\]

the part of the Jacobi identity corresponding to the equality \(\{p_1, \{x_2, x_3\}\} = \{\{p_1, x_2\}, x_3\} - \{\{p_1, x_3\}, x_2\}\) is equivalent to

\[
p_1 ([w_{12}, w_{13}] + [w_{12}, r_{23}] + [w_{13}, r_{23}]) x_2 x_3 = 0.
\]

Similarly, the part of the Jacobi identity corresponding to \(\{\{p_1, p_2\}, x_3\} = \{p_1, \{p_2, x_3\}\} - \{p_2, \{p_1, x_3\}\}\) is equivalent to

\[
p_1 p_2 ([r_{12}, w_{13}] + [r_{12}, w_{23}] + [w_{13}, w_{23}]) x_3 = 0.
\]
**Theorem 5.2** Let \((g, r)\) be quasitriangular, i.e. there exists an invariant symmetric element \(s\) of \(g \otimes g\) such that \(w := r + s\) satisfies the classical Yang-Baxter equation:

\[
[[w, w]] := [w_{12}, w_{13}] + [w_{12}, w_{23}] + [w_{13}, w_{23}] = 0.
\]

Then the brackets (21) satisfy the Jacobi identity (we assume that (14) satisfy already the Jacobi identity).

**Proof:** Since \(s\) is invariant,

\[
[w_{12}, s_{23}] + [w_{13}, s_{23}] = 0
\]

(for any \(w\)), hence \([[w, w]] = 0\) if and only if

\[
[w_{12}, w_{13}] + [w_{12}, r_{23}] + [w_{13}, r_{23}] = 0.
\]

This obviously implies (21). Similarly, since \([s_{12}, w_{13}] + [s_{12}, w_{23}] = 0\), \([[w, w]]\) is zero if and only if

\[
[r_{12}, w_{13}] + [r_{12}, w_{23}] + [w_{13}, w_{23}] = 0,
\]

which implies (22).

Q.E.D.

**Remark 5.3** Let \(r, s\) and \(w\) satisfy the assumptions of Theorem 5.2. For any \(\lambda \in \mathbb{R}\), the element

\[
w_\lambda := r + s + \lambda I \otimes I \in \text{End} V \otimes \text{End} V,
\]

satisfies the Yang-Baxter equation and the proof of Theorem 5.2 works also for \(w_\lambda\) (once \(w\) satisfies (23), \(w_\lambda\) also satisfies (23)). It means that one can replace \(w\) by \(w_\lambda\) in (21).

**Theorem 5.4** Under assumptions of Theorem 5.2, brackets (19) satisfy the Jacobi identity if and only if

\[
s^{jk}_{lm} = s^{kj}_{lm}.
\]

Brackets (19) with \(w\) replaced by \(w_\lambda\) satisfy the Jacobi identity if and only if

\[(s_\lambda)^{jk}_{lm} = (s_\lambda)^{kj}_{lm},\]

where \(s_\lambda = s + \lambda I \otimes I\).

**Proof:** In the part of the Jacobi identity corresponding to \(\{p_1, \{x_2, x_3\}\} = \{\{p_1, x_2\}, x_3\} - \{\{p_1, x_3\}, x_2\}\), we must take care of the linear terms (the cubic terms were taken into account in the previous theorem). This gives

\[
r x - (r x)^t = w x - (w x)^t,
\]

where \((r x)^{jk}_{lm} := \sum_m r^{jk}_{in} x^m, ((r x)^t)^{jk}_{lm} := (r x)^{kj}_{lm}\), etc. Of course, (27) means that \(sx = (sx)^t\), i.e. (25). The modification \(w \mapsto w + \lambda I \otimes I, s \mapsto s + \lambda I \otimes I\) leading to condition (29) is straightforward. The remaining part of the Jacobi identity leads to the same condition.

Q.E.D.
Example 5.5 It is convenient to consider (7) as a \( r \)-matrix on \( \mathfrak{gl}(n, \mathbb{R}) \), because it is easy to write down a natural invariant symmetric element (trace form) of \( \mathfrak{g} \otimes \mathfrak{g} \). Taking this element with the appropriate coefficient:

\[
    s = \varepsilon \sum_{j,k} e_j^k \otimes e_k^j,
\]

we obtain

\[
    w = r + s = \varepsilon \left( \sum_j e_j^j \otimes e_j^j + 2 \sum_{j<k} e_j^k \otimes e_k^j \right),
\]

which satisfies the classical Yang-Baxter equation. There is a unique modification \( s_\lambda = s + \lambda I \otimes I \) of \( s \) satisfying the symmetry (26), namely for \( \lambda = \varepsilon \):

\[
    s_\lambda = s_\varepsilon = s + \varepsilon I \otimes I, \quad (s_\varepsilon)^{jk}_{lm} = \varepsilon (\delta^j_m \delta^k_l + \delta^j_l \delta^k_m).
\]

Poisson brackets (19) with \( w_\lambda = w_\varepsilon \) coincide with (11), (13).

Concluding, if \((\mathfrak{g}, r)\) is quasitriangular and if there exists a modification \( s_\lambda = s + \lambda I \otimes I \) of \( s \) satisfying (26), then one can realize the phase space of \((V, r_V)\) on \( T^*V \) with the Poisson structure (17), where \( \Delta \) is given by (18) with

\[
    \Delta^{jk}_{lm} = (s_\lambda)^{jk}_{lm}.
\]

It is convenient to introduce the following notation. For each

\[
    \rho = \sum_{jklm} \rho^{jk}_{lm} e_j^l \otimes e_k^m \in \text{End} \, V \otimes \text{End} \, V,
\]

we denote by \( \rho_{V^*} \) the bi-vector field on \( T^*V = V \oplus V^* \) defined by

\[
    \rho_{V^*}(x, p) = \sum_{jklm} \rho^{jk}_{lm} x^m \circ e_k \wedge e^l.
\]

Using this notation we can write (17) as follows

\[
    \pi_{T^*V} = \pi_0 + r_{T^*V} + (s_\lambda)_{V^*V^*} = \pi_0 + r_V + r_{V^*} + (w_\lambda)_{V^*V^*}.
\]

6 \( \text{SO}(n, \mathbb{R}) \), imaginary quasitriangularity and reality condition

In this section we construct the phase space of \((V, r_V)\), where \( V := \mathbb{R}^n \), for a standard \( r \)-matrix on \( \mathfrak{so}(n, \mathbb{R}) \), using methods which are completely analogous to those used in [24] for the investigation of a real differential calculus on quantum Euclidean spaces.

In terms of the ‘angular momentum’ generators \( M_j^k := e_j^k - e_k^j \), the standard \( r \)-matrix on \( \mathfrak{g} = \mathfrak{so}(n, \mathbb{R}) \) is given by

\[
    r = \frac{\varepsilon}{4} \sum_{j<k} (M_j^k + M_j^{k'}) \wedge (M_j^{k'} + M_k^{j'}) = \varepsilon \sum_{j<k<j'} M_j^k \wedge M_k^{j'},
\]
where \( j' := n + 1 - j \) (the underlying Cartan subalgebra consists of anti-diagonal matrices in this case).

The above \( r \)-matrix is not quasi-triangular in the real sense (this is a characteristic feature of compact simple groups). Instead, one can find an invariant symmetric element \( s \) of \( \mathfrak{g} \otimes \mathfrak{g} \) such that \( w := r \pm is \) satisfies the classical Yang-Baxter equation. We say that \((\mathfrak{g}, r)\) is imaginary quasitriangular in this case.

In our case, \( s = \varepsilon \sum_{j,k} M^k_j \otimes M^j_k = \varepsilon \sum_{j,k} (e^k_j \otimes e^j_k - e^j_k \otimes e^k_j) \)

(the simplest way to obtain \( w \) is to extract the first order term from the known \( R \)-matrix for the quantum \( SO(n) \) group). In order to satisfy (26), we add \( \lambda I \otimes I \) to \( s \) with \( \lambda = \varepsilon \):

\[
s_\lambda = \varepsilon \sum_{j,k} (e^k_j \otimes e^j_k - e^j_k \otimes e^k_j + e^j_j \otimes e^k_k).
\]

It is clear that (19) with \( w \) replaced by \( w_\lambda := r - is_\lambda \) defines a complex-valued Poisson structure on \( T^*V \cong \mathbb{R}^{2n} \). Equally well we can treat (19) as defining a holomorphic Poisson structure on the complexification \((T^*V)^C \cong \mathbb{C}^{2n}\) of \( T^*V \). In the sequel we shall construct a real form of this holomorphic Poisson manifold, playing the role of the phase space of \((V, r_V)\).

We first derive Poisson brackets of coordinates with basic \( \mathfrak{g} \)-invariant functions:

\[
x^2 := g_{jk}x^j x^k, \quad p^2 := p_j p_k g^{jk}, \quad E := \langle p, x \rangle = p_j x^j
\]

(summation convention assumed), where \( g_{jk} \) is the metric tensor (equal to the Kronecker delta in our orthonormal basis \( e_j \)). Since \( s \) is universal (the Killing form) — independent of \( r \) (only the coefficient at \( s \) depends on the proportionality constant between \([r, r]\) and the canonical element of \( \bigwedge^3 \mathfrak{g} \)), the part

\[
\{p_j, x^k\}_{\text{univ}} = \delta^k_j + i\varepsilon (E \delta^k_j + p_j x^k - x_j p^k)
\]

of Poisson brackets (19) corresponding to \( \pi_0 - i(s_\lambda)_{V^*} \) is universal. It follows that the brackets with \( \mathfrak{g} \)-invariant functions are also universal (these functions are Casimirs of \( r_{T^*V} \) and it is sufficient to use only \( \pi_0 - i(s_\lambda)_{V^*} \)). From (33), it is now easy to obtain the following brackets:

\[
\{x^2, x^j\} = 0, \quad \{p^2, x^j\} = p^j + i\varepsilon p^2 x^j, \quad \{p^2, p^j\} = 0, \quad \{p_j, \frac{1}{2} x^2\} = x_j + i\varepsilon x^2 p_j
\]

(34)

\[
\{E, x^j\} = x^j + 2i\varepsilon Ex^j - i\varepsilon x^2 p^j, \quad \{p_j, E\} = p_j + 2i\varepsilon Ep_j - i\varepsilon p^2 x_j.
\]

Let us now introduce one more invariant function:

\[
\Lambda := 1 + 2i\varepsilon E - \varepsilon^2 x^2 p^2
\]

(35)

(following the method of [24]). From (34) it follows that

\[
\{\Lambda, x^j\} = 2i\varepsilon \Lambda x^j, \quad \{p_j, \Lambda\} = 2i\varepsilon p_j \Lambda
\]

(36)
Denote by $\text{Hol}(Y)$ the algebra of holomorphic functions on a complex manifold $Y$. Recall that any holomorphic map $\phi: Y \to Z$ (of complex manifolds) defines a linear multiplicative map $\Phi: \text{Hol}(Z) \to \text{Hol}(Y)$ by the pullback: $\Phi(f) = f \circ \phi$. Similarly, any anti-holomorphic map $\psi: Y \to Z$ defines an anti-linear multiplicative map $\Psi: \text{Hol}(Z) \to \text{Hol}(Y)$ by pullback followed by the complex conjugation: $\Psi(f) = \overline{f \circ \phi}$. Using $\Lambda$, we define an anti-holomorphic map $\psi$ from $P_C := \{(x, p) \in (T^*V)^C : \Lambda \neq 0\} \subset (T^*V)^C$

into $(T^*V)^C$ by

$$\Psi(x^j) = x^j, \quad \Psi(p_j) = \frac{p_j + i\varepsilon p^2 x_j}{\Lambda}.$$  \hspace{1cm} (37)

Since

$$\Psi(p^2) = \frac{p^2}{\Lambda}, \quad \Psi(E) = \frac{E + i\varepsilon p^2 x^2}{\Lambda}, \quad \Psi(\Lambda) = \frac{1}{\Lambda},$$  \hspace{1cm} (38)

the underlying map $\psi$ maps $P_C$ into $P_C$. Moreover, since $\Psi(\Psi(x^j)) = x^j$ and

$$\Psi(\Psi(p_j)) = \Psi \left( \frac{p_j + i\varepsilon p^2 x_j}{\Lambda} \right) = \Lambda \left( \frac{p_j + i\varepsilon p^2 x_j}{\Lambda} - i\varepsilon \frac{p^2}{\Lambda} x_j \right) = p_j,$$

the anti-holomorphic map $\psi: P_C \to P_C$ is an involution. Therefore we can define the corresponding real form $P$ of $P_C$ as the set of fixed points of $\psi$:

$$P := \{z \in P_C : \psi(z) = z\}. \hspace{1cm} (39)$$

The antilinear multiplicative involution $\Psi$ corresponding to the map $\psi$ will be henceforth denoted by a star:

$$(x^j)^* = x^j, \quad (p_j)^* = \frac{p_j + i\varepsilon p^2 x_j}{\Lambda}. \hspace{1cm} (40)$$

Let us collect once more the basic formulas (38):

$$\left(\frac{p^2}{\Lambda}\right)^*, \quad \left(\frac{E + i\varepsilon p^2 x^2}{\Lambda}\right)^*, \quad \left(\frac{1}{\Lambda}\right)^*, \quad P := \{(x, p) : (x^j)^* = \overline{x^j}, (p_j)^* = \overline{p_j}\}.$$  \hspace{1cm} (41)

The fundamental theorem of this section says that the star operation is compatible with the Poisson brackets (19).

**Theorem 6.1** The Poisson structure (13) is real with respect to the star operation (40):

$$\{f, g\}^* = \{f^*, g^*\}. \hspace{1cm} (41)$$

**Proof:** We have to prove (11) in two cases: 1) $f = p_j$, $g = x^k$ and 2) $f = p_j$, $g = p_k$. The case $f = x^j$, $g = x^k$ is trivial.

1) We set $T_j := p_j + i\varepsilon p^2 x_j$. Since

$$\{p_j, x^k\} = \delta_j^k - (p_l)^*(\overline{\mu}_n)^{lk} x^m$$

and

$$\{(p_j)^*, x^k\} = \left\{ \frac{T_j}{\Lambda}, x^k \right\} = \frac{1}{\Lambda} \left( \left\{ T_j, x^k \right\} - 2i\varepsilon T_j x^k \right),$$
we have to show that
\[ \{T_j, x^k\} - 2i\varepsilon T_j x^k = \Lambda \delta^k_j - T_l (\overline{w_\lambda})_{jm}^l x^m. \] (42)

Since
\[ \{T_j, x^k\} = \delta^k_j - p_l (w_\lambda)_{jm}^l x^m + i\varepsilon (2x_j p^k + 2i\varepsilon x_j x^k + p^2 g_{jl} x^m n^m x^n), \]
the left hand side of (42) equals
\[ \delta^k_j - p_l (w_\lambda)_{jm}^l x^m + i\varepsilon (2(x_j p^k - p_j x^k) + p^2 g_{jl} x^m n^m x^n), \]
whereas the right hand side of (42) equals
\[ (1 + 2i\varepsilon E - \varepsilon^2 x^2 p^2)\delta^k_j - (p_l + i\varepsilon x_l) (\overline{w_\lambda})_{jm}^l x^m. \]

Since
\[ p_l ((w_\lambda)_{jm}^l - (\overline{w_\lambda})_{jm}^l) x^m = 2p_l (-is_\lambda)_{jm}^l x^m = -2i\varepsilon (E \delta^k_j + p_j x^k - x_j p^k) \]
(cf. (33)), it follows that (42) is equivalent to
\[ i\varepsilon p^2 g_{jl} x^m n^m x^n = -\varepsilon^2 x^2 p^2 \delta^k_j - i\varepsilon p^2 x_l (\overline{w_\lambda})_{jm}^l x^m, \]
or,
\[ ig_{jl} x^m n^m x^n = -\varepsilon^2 \delta^k_j - i x_l (\overline{w_\lambda})_{jm}^l x^m. \]

Taking into account
\[ g_{jl} r_{mn}^l = -g_{lm} r_{jn}^l \]
(\(r_{mn}^l \) belongs to \( \mathfrak{g} \) with respect to indices \( l, m \)), it means that (42) is equivalent to
\[ ig_{lm} (is_\lambda)_{jm}^l x^m x^n = \varepsilon x^2 \delta^k_j, \]
which can be easily verified.

2) Since
\[ \{T_j, \Lambda\} = 2i\varepsilon T_j \Lambda, \]
we have
\[ \left\{ \frac{T_j}{\Lambda}, \frac{T_k}{\Lambda} \right\} = \frac{\{T_j, T_k\}}{\Lambda^2}. \]

Therefore it is sufficient to show that
\[ \{T_j, T_k\} = T_l T_m r_{jk}^l m. \]

We have
\[ \{T_j, T_k\} = \{p_j, p_k\} + i\varepsilon p^2 \{p_j, x_k\} + (i\varepsilon)^2 \{p^2 x_j, p^2 x_k\} = p_l p_m r_{jk}^l m + i\varepsilon p^2 [-p_l r_{jm}^l m g_k + p_m r_{km}^l m g_n + 2i\varepsilon (p_j x_k - x_j p_k)] = -\varepsilon^2 p^2 p^2 \{x_j, x_k\} + 2(p_k x_j - x_k p_j). \]

Since
\[ -p_l r_{jm}^l m g_k = p_l x_m r_{jk}^l m \]
(by the argument similar to (43)), we have finally \[
\{T_j, T_k\} = p_ip_m r_{jk}^{lm} + \varepsilon p^2 p_i x_m (r_{jk}^{lm} - r_{lj}^{mk}) + (i \varepsilon p^2)^2 x_i x_m r_{jk}^{lm} = (p_i + \varepsilon p^2 x_i)(p_m + \varepsilon p^2 x_m) r_{jk}^{lm}.
\]
Q.E.D.

Corollary: \(P\) is endowed with a structure of a real analytic Poisson manifold. If \(f_0, g_0\) are two real analytic functions on \(P\), then their Poisson bracket is defined by
\[
\{f_0, g_0\} := \{f, g\}|_P,
\]
where \(f, g\) are the (local) holomorphic extensions of \(f_0, g_0\) to \(P\) (by (41), the restriction of \(\{f, g\}\) to \(P\) is real).

\(P\) is the required phase space of \((V, r_V)\).

7 Poisson action of \(SU(n)\) on \(\mathbb{C}^n\)

We treat here \(V = \mathbb{C}^n\) as a real manifold \((V \cong \mathbb{R}^{2n})\). Specifying (2) to the case of \(SU(n)\) we get the following standard \(r\)-matrix (see (6) for the basis of \(su(n)\))
\[
r = \varepsilon \frac{1}{2} \sum_{j<k} (e_j^k - e_k^j) \wedge J(e_j^k + e_k^j),
\]
(44)
where \(J: V \to V\) is the complex structure of \(V\) (multiplication by the imaginary unit). From now on we set \(\varepsilon = 1\) (arbitrary \(\varepsilon\) will be restored in the final formulas). It is convenient to work with the complexification \(V^c \cong V \oplus iV\) and the complex-linear embedding
\[
V \ni z \mapsto z^c := \frac{1}{2}(z - iJz) \in V^c.
\]
We have
\[
z = z^c + \bar{z}^c, \quad Jz = i(z^c - \bar{z}^c)
\]
and the typical notation:
\[
(e_k)^c = \left(\frac{\partial}{\partial x_k}\right)^c = \frac{\partial}{\partial z_k} = \partial_k.
\]
Note that
\[
e_j^k z = (e_j^k z)^c + (e_j^k \bar{z})^c = (e_j^k z)^c + (e_j \bar{z}^k)^c = z^k \partial_j + \bar{z}^k \bar{\partial}_j.
\]
In this notation, the fundamental bi-vector field \(r_V\) looks as follows
\[
r_V(z) = i \sum_{j<k} \text{sgn}(k - j) \left(\frac{1}{2} \nabla_j \wedge \nabla_k - \frac{1}{2} \bar{\nabla}_j \wedge \nabla_k + |z|^2 \partial_k \wedge \partial_k\right),
\]
(45)
where \(\nabla_k := z^k \partial_k, \bar{\nabla}_k := \bar{z}^k \bar{\partial}_k\).

Lemma 7.1 \([r_V, r_V](z) = -||z||^2 Jz \wedge \pi_0\), where
\[
\pi_0 = 2i \sum_k \partial_k \wedge \bar{\partial}_k
\]
is the canonical constant bi-vector on \(V = \mathbb{C}^n = \mathbb{R}^{2n}\).
\textbf{Proof:} Taking into account

\begin{align*}
\left[|z|^2 \partial_k \wedge \tilde{\partial}_k, \nabla_a \wedge \nabla_b\right] &= z^j(-\tilde{\partial}_k) \wedge [z^j \nabla_k, \nabla_a \wedge \nabla_b] \\
&= -z^j \tilde{\partial}_k \wedge [(z^j \delta_k^a \partial_a - z^a \delta_j^a \partial_k) \wedge \nabla_b + \nabla_a \wedge (z^j \delta_k^b \partial_b - z^b \delta_j^b \partial_k)] \\
&= -|z|^2 \tilde{\partial}_k \wedge \partial_k \wedge [(\delta_k^a - \delta_j^a)\nabla_b - (\delta_k^b - \delta_j^b)\nabla_a]
\end{align*}

and

\begin{align*}
\left[|z|^2 \partial_k \wedge \tilde{\partial}_k, |z|^2 \partial_b \wedge \tilde{\partial}_b\right] &= [z^j \partial_k \wedge z^j \tilde{\partial}_k, z^a \partial_b \wedge z^a \tilde{\partial}_b] = \\
&= [z^j \partial_k, z^a \partial_b] \wedge [z^j \tilde{\partial}_k, z^a \tilde{\partial}_b] + z^j \partial_k \wedge z^a \partial_b \wedge [z^j \tilde{\partial}_k, z^a \tilde{\partial}_b] = z^j \partial_k \wedge z^a \partial_b \wedge (z^j \delta_k^a \tilde{\partial}_b - z^a \delta_j^a \tilde{\partial}_k) + c.c. \\
&\quad (+c.c. \text{ means 'plus complex conjugated terms'}, \text{we see that } [\tilde{r}_V, r_V](z) \text{ equals})
\end{align*}

\begin{align*}
- \sum_{jkab} \text{sgn}(k-j)\text{sgn}(b-a)\{|z|^2((\delta_k^a - \delta_j^a)\nabla_b - (\delta_k^b - \delta_j^b)\nabla_a)| + 2|z|^2 \delta_k^b \nabla_b\} \wedge \partial_k \wedge \tilde{\partial}_k + c.c. &= \\
&= -2 \sum_{jkab} \text{sgn}(k-j)\text{sgn}(b-a)|z|^2((\delta_k^a - \delta_j^a)\nabla_b| + |z|^2 \delta_k^b \nabla_b\wedge \partial_k \wedge \tilde{\partial}_k + c.c.
\end{align*}

Note that

\begin{align*}
\sum_{ja} \text{sgn}(k-j)\text{sgn}(b-a)|z|^2(2|z|^2 \delta_k^a - |z|^2 \delta_j^a) \nabla_b \wedge \partial_k \wedge \tilde{\partial}_k &= \\
&= \sum_{j} |z|^2(\text{sgn}(k-j)\text{sgn}(b-k) + \text{sgn}(b-k)\text{sgn}(j-b) + \text{sgn}(j-b)\text{sgn}(k-j))
\end{align*}

and

\begin{align*}
\text{sgn}(k-j)\text{sgn}(b-k) + \text{sgn}(b-k)\text{sgn}(j-b) + \text{sgn}(j-b)\text{sgn}(k-j) &= -1
\end{align*}

for $b \neq k$. It follows that

\begin{align*}
[r_V, r_V](z) &= 2||z||^2 \sum_b (\nabla_b - \tilde{\nabla}_b) \wedge \sum_k \partial_k \wedge \tilde{\partial}_k = -||z||^2 \sum_b i(\nabla_b - \tilde{\nabla}_b) \wedge \sum_k 2i\partial_k \wedge \tilde{\partial}_k.
\end{align*}

Q.E.D.

\textbf{Corollary:} For any classical $r$-matrix $\tilde{r}$ on $\mathfrak{g} = \text{su}(n)$ there is a constant $c$ such that $[\tilde{r}_V, \tilde{r}_V] = -c||z||^2 Jz \wedge \pi_0$. Poisson structures $\pi_V$ on $V$ for which the action of $SU(n)$ on $V$ is Poisson are exactly bi-vector fields

\begin{equation}
\pi_V = \tilde{r}_V + \Delta, \tag{46}
\end{equation}

such that the bi-vector field $\Delta$ on $V$ is $SU(n)$-invariant and satisfies

\begin{align*}
[\Delta, \Delta](z) &= c||z||^2 Jz \wedge \pi_0. \tag{47}
\end{align*}

It is easy to show that all $SU(n)$-invariant bi-vector fields $\Delta$ on $V$ are of the following form:

\begin{equation}
\Delta = \frac{1}{2}a\pi_0 + \frac{1}{2}bz \wedge Jz, \tag{48}
\end{equation}

where $a = a(||z||^2)$, $b = b(||z||^2)$ are arbitrary functions of $||z||^2$. We shall write condition (17) in terms of these functions.
Lemma 7.2 \([\Delta, \Delta](z) = ||z||^2 Jz \wedge \pi_0\) if and only if
\[\quad aa' + b(a - a't) = t.\]  \hspace{1cm} (49)

Here \(t \equiv ||z||^2\) and \(\text{prim}\) means the differentiating with respect to the variable \(t\).

Proof: If \(K, L\) are bi-vector fields and \(f, g\) are functions, then
\[\quad [fK, gL] = fg[K, L] - fK(dg) \wedge -gK \wedge L(dg),\] \hspace{1cm} (50)

where by \(K(dg)\) we denote the contraction of \(K\) with \(dg\) on the first place. In particular,
\[\quad [fK, fK] = f^2 - 2fK \wedge K(df).\]

Using
\[\quad \pi_0 \left(\frac{1}{2}d||z||^2\right) = -Jz, \quad (z \wedge Jz) \left(\frac{1}{2}d||z||^2\right) = ||z||^2 Jz,\]
and
\[\quad [\pi_0, z \wedge Jz] = 2Jz \wedge \pi_0,\]
we obtain
\[\begin{align*}
\frac{1}{2}a\pi_0, \frac{1}{2}a\pi_0 & = aa'Jz \wedge \pi_0 \\
[bz \wedge Jz, bz \wedge Jz] & = 0 \\
\frac{1}{2}a\pi_0, \frac{1}{2}bz \wedge Jz & = \frac{1}{2}b(a - a' ||z||^2)Jz \wedge \pi_0.
\end{align*}\]

From this, (49) follows immediately.

Of course, the easy way to solve (49) is to write
\[\quad b = \frac{t - aa'}{a - a't},\] \hspace{1cm} (51)

but in such a way we have no control of regularity of those functions and we do not see the simplest cases. To pick up the simplest cases, let us consider \(\Delta\) at most quadratic, i.e. \(a = a_0 + a_1t, b = b_0,\) where \(a_0, a_1, b_0\) are some constants. Inserting this form of \(a\) and \(b\) in (49) gives the following two cases:
1. \(a_0 = 0, a_1 = \pm 1, b_0\) arbitrary,
2. \(a_0\) arbitrary, \(a_1 = \pm 1, b_0 = \mp 1.\)

One of the simplest non-quadratic solution for \(\Delta\) is the following solution of degree 4:
3. \(a = h = \text{const} \neq 0, b = \frac{1}{h}t\)

An example of a non-singular rational solution is given by \(a = 1 - t^2\) (in this case the denominator of (51) is positive: \(a - a't = 1 + t^2\)).

Another way to pick up a simple case is to assume that \(\Delta\) is (as \(r_V\)) tangent to spheres \(||z|| = \text{const}\). It is easy to show that this conditions holds if and only if \(a = bt\). In this case (49) reduces to
\[\quad ab = t.\]
It means that \( a = \pm t, \ b = \pm 1 \). This is a special case of type 1 above.

We end by listing the explicit form of the Poisson brackets corresponding to the mentioned cases. From the general form (46), (48) with the standard \( r \)-matrix (44), we obtain

\[
\begin{align*}
\{ z^j, z^k \} &= i\varepsilon z^j z^k \quad \text{for } j < k \\
\{ z^j, \bar{z}^k \} &= -i\varepsilon b z^j \bar{z}^k \quad \text{for } j \neq k \\
\{ z^j, \bar{z}^j \} &= i\varepsilon \sum_k \text{sgn}(j - k) \cdot |z^k| + i\varepsilon a - i\varepsilon b |z^j|
\end{align*}
\] (52) (53) (54)

(we have restored the parameter \( \varepsilon \)). Now we list the cases which seem to be most interesting ones.

1. **Poisson \( SU(n) \)-spheres.** According to the discussion above, there are only two Poisson structures on \( \mathbb{C}^n \) solving our problem and tangent to spheres, namely

\[
\begin{align*}
\{ z^j, z^k \} &= i\varepsilon z^j z^k \quad \text{for } j < k \\
\{ z^j, \bar{z}^k \} &= -i\varepsilon b z^j \bar{z}^k \quad \text{for } j \neq k \\
\{ z^j, \bar{z}^j \} &= i\varepsilon (\sigma |z|)^2 - \sigma |z|^j + \sum_k \text{sgn}(j - k) \cdot |z^k|) = 2\varepsilon \sigma \sum_{\sigma k < \sigma j} |z^k|^2,
\end{align*}
\]

where \( \sigma = \pm 1 \). The function \( z \mapsto ||z||^2 \) is a Casimir function of this Poisson structure (and can be fixed, which leads to a sphere \( S^{2n-1} \)).

2. **Twisted annihilation and creation ‘operators’.** Setting \( h = \varepsilon a_0 \) in the case 2 above, we obtain

\[
\begin{align*}
\{ z^j, z^k \} &= i\varepsilon z^j z^k \quad \text{for } j < k \\
\{ z^j, \bar{z}^k \} &= i\varepsilon b z^j \bar{z}^k \quad \text{for } j \neq k \\
\{ z^j, \bar{z}^j \} &= ih + i\varepsilon (\sigma |z|)^2 + \sigma |z|^j + \sum_k \text{sgn}(j - k) \cdot |z^k|) = ih + 2\varepsilon \sigma \sum_{\sigma k \leq \sigma j} |z^k|^2,
\end{align*}
\]

where \( \sigma = \pm 1 \). This is the Poisson version of the ‘twisted canonical commutation relations’ of (23) (see also (20)). It may describe the phase space of a Poisson deformed harmonic oscillator.

3. The non-quadratic brackets corresponding to the case 3 above are given by

\[
\begin{align*}
\{ z^j, z^k \} &= i\varepsilon z^j z^k \quad \text{for } j < k \\
\{ z^j, \bar{z}^k \} &= -i\varepsilon \frac{1}{h} |z|^2 z^j \bar{z}^k \quad \text{for } j \neq k \\
\{ z^j, \bar{z}^j \} &= i\varepsilon h + i\varepsilon (-\frac{1}{h} |z|^2 |z|^j + \sum_k \text{sgn}(j - k) \cdot |z^k|).
\end{align*}
\]

(55) (56) (57)

**Problem:** What is the quantum counterpart of condition (49)? The quantum counterpart of relations (56), (57)?
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