On similarity of arithmetic sum of self-similar sets

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Abstract

Let $\beta > 1$. We define a class of similitudes

$$S := \left\{ f_i(x) = \frac{x}{\beta^{n_i}} + a_i : n_i \in \mathbb{N}^+, a_i \in \mathbb{R} \right\}.$$ 

Given any finite similitudes $\{f_i(x)\}_{i=1}^m \subset S$, it is well known that there is a unique self-similar set $K_1$ satisfying $K_1 = \bigcup_{i=1}^m f_i(K_1)$. Similarly, we can generate another self-similar set $K_2$. We call $K_1 + K_2 = \{x + y : x \in K_1, y \in K_2\}$ the arithmetic sum of two self-similar sets. In this paper, we prove that $K_1 + K_2$ is either a self-similar set or a unique attractor of some infinite iterated function system. Combining this result with Theorem 2 from [20] yields an interesting corollary, i.e., for any arbitrary two self-similar sets $F_1, F_2$ ($F_1, F_2$ may not be generated by the similitudes of $S$), we have $\dim_P(F_1 + F_2) = \dim_B(F_1 + F_2)$, provided that all the contractive ratios of the IFS’s of $F_1$ and $F_2$ are positive, where $\dim_P$ and $\dim_B$ denote Packing and upper Box dimensions respectively. We are primarily concerned with the case that $K_1 + K_2$ is the unique attractor of infinite iterated function system. Under some conditions we can explicitly compute the Hausdorff dimension of $K_1 + K_2$, which partially provides the dimensional result of $K_1 + K_2$ when the IFS’s of $K_1$ and $K_2$ fail the irrational assumption, see Peres and Shmerkin [20].

1 Introduction

Let $\{g_j\}_{j=1}^m$ be an iterated function system (IFS) of similitudes which are defined on $\mathbb{R}$ by

$$g_j(x) = r_j x + a_j,$$

where the similarity ratios satisfy $0 < r_j < 1$ and the translation parameter $a_j \in \mathbb{R}$. It is well known that there exists a unique non-empty compact set $K \subset \mathbb{R}$ such that

$$K = \bigcup_{j=1}^m g_j(K).$$

(1)
We call $K$ the self-similar set or attractor for the IFS $\{g_j\}_{j=1}^m$, see [10] for further details. The IFS $\{g_j\}_{j=1}^m$ is called homogeneous if all the similarity ratios are equal. We say that $\{g_j\}_{j=1}^m$ satisfies the open set condition (OSC) [10] if there exists an open set $V \subseteq \mathbb{R}$ such that $g_i(V) \cap g_j(V) = \emptyset$, $i \neq j$ and $g_j(V) \subseteq V$ for all $1 \leq j \leq m$. Under the OSC, Hausdorff dimension of $K$ is just the similarity dimension which is the unique solution $s$ of the equation $\sum_{j=1}^m r_j^s = 1$. Let $A, B \subseteq \mathbb{R}$, we say $A + B = \{x + y : x \in A, y \in B\}$ is the arithmetic sum of $A$ and $B$. In dynamical systems, arithmetic sum of two Cantor sets is essential [19]. Palis and Takens [19] restricted $A$ and $B$ to the Cantor sets, which came up in their studies of homoclinic bifurcations of dynamical systems. Palis [19] posed a celebrated conjecture on the structure of arithmetic sum of any two regular Cantor sets in line. More specifically, if $C_1 + C_2$ has positive Lebesgue measure, then generally it has an interior, where $C_1$ and $C_2$ are two Cantor sets of the real line. This conjecture led to much work, see [15], [14], [3], [4], [6], [2] and references therein. Moreira and Yoccoz [3] solved this conjecture eventually. Dekking and Károly [5] gave partial answer to the stochastic version of this conjecture. However, from fractal perspective relatively fewer papers considered the dimension and the fractal structure of the arithmetic sum of two self-similar sets. In this paper, we concentrate on the following two basic problems:

(1) What is the Hausdorff dimension of the arithmetic sum of two self-similar sets?

(2) What is the fractal structure of the arithmetic sum of two self-similar sets?

Peres and Solomyak [21] showed that if $C_1$ is a two-part homogenous Cantor set (the IFS of $C_1$ is $\{f_0(x) = ax, f_0(x) = ax + 1 - a\}$, where $0 < a < 1$), given a compact set $F \subseteq \mathbb{R}$, then for almost all $a \in (0, \frac{1}{2})$,

- if $\dim_H(C_1) + \dim_H(F) \leq 1$, then $\dim_H(C_1 + F) = \dim_H(C_1) + \dim_H(F)$.
- if $\dim_H(C_1) + \dim_H(F) > 1$, then $\mu(C_1 + F) > 0$, where $\mu$ denotes the Lebesgue measure.

The classical Marstrand projection theorem [11] states that

$$\dim_H(\pi(A \times B)) = \min\{1, \dim_H(A) + \dim_H(B)\}$$

holds for almost every projection $\pi$, where $\pi$ denotes the orthogonal projection from $\mathbb{R}^2$ to the one dimensional line and $A, B \subseteq \mathbb{R}$. Nevertheless, this result does not provide any information for a specific projection, hence it does not answer problem (1) explicitly. Let $F_1$ and $F_2$ be the
self-similar sets with IFS’s \( \{r_i x + a_i\}_{i=1}^n \) and \( \{r'_j x + b'_j\}_{j=1}^m \) respectively. Peres and Shmerkin showed in [20] that if there exist \( i, j \) such that \( \frac{\log r_i}{\log r'_j} \notin \mathbb{Q} \), then 

\[
\dim_H(F_1 + F_2) = \min\{1, \dim_H(F_1) + \dim_H(F_2)\}.
\]

The hypothesis of this theorem is called the irrational assumption. This theorem partially answers question (1).

The fractal structure of the arithmetic sum of two self-similar sets is complicated. Mendes and Oliveira classified in [15] that for the homogeneous Cantor sets \( C_\lambda \) and \( C_\gamma \), there are 5 types of topological structure. Anisca, Chlebovec and Ilie [1], making use of similar discussion, generalized Mendes and Oliveira’s result. They considered general Cantor sets which may not be homogeneous and their main result is similar to Mendes and Oliveira’s statement.

In this paper, we shall consider the IFS’s of \( F_1 \) and \( F_2 \) failing the irrational assumption. With a little effort, it may be shown that the IFS’s \( \{r_i x + a_i\}_{i=1}^n \) and \( \{r'_j x + b'_j\}_{j=1}^m \) do not satisfy the irrational assumption if and only if there exist \( \beta > 1, n_i \) and \( m_j \in \mathbb{N} \) such that \( r_i = \frac{1}{\beta^n_i}, 1 \leq i \leq n \) and \( r'_j = \frac{1}{\beta^m_j}, 1 \leq j \leq m \). Unless stated otherwise, in what follows we always assume that the similitudes of \( F_1 \) and \( F_2 \) are from 

\[
S := \left\{ f_i(x) = \frac{x}{\beta^{n_i}} + a_i : n_i \in \mathbb{N}^+, a_i \in \mathbb{R} \right\}.
\]

Without loss of generality, we let the IFS’s of \( K_1 \) and \( K_2 \) be \( \{f_i(x) = \frac{x}{\beta^{n_i}} + a_i\}_{i=1}^n \) and \( \{g_j(x) = \frac{x}{\beta^{m_j}} + b_j\}_{j=1}^m \) respectively. Here, we replace \( F_1, F_2 \) by \( K_1, K_2 \). In what follows we use \( F \) if the self-similar set is general, while \( K \) is utilised if the similitudes are taken form \( S \). For simplicity, we let the convex hull of \( K_i \) be \( [0, B_i] \), \( 1 \leq i \leq 2 \). This assumption yields that \( f_i([0, B_1]) \subset [0, B_1], 1 \leq s \leq n \) and \( g_t([0, B_2]) \subset [0, B_2], 1 \leq t \leq m \).

We shall prove that \( K_1 + K_2 \) is either a self-similar set or an attractor of some infinite iterated function system (IIFS) [13], [9]. This result implies that \( K_1 + K_2 \) has certain similarity. Hence, to calculate the Hausdorff dimension of \( K_1 + K_2 \) is reduced to consider the dimension of the attractor of some IFS (IIFS). It is well known that generally it is difficult to calculate the Hausdorff dimension of a self-similar set. There are more difficulties for the infinite iterated function system even if the infinite iterated function system satisfies some separation condition (Here the attractor of the IIFS is in the sense of Definition 2.1, we will introduce this definition in the next section). In fact, even Peres and Shmerkin’s dimensional formula above cannot find the exact Hausdorff dimension of \( F_1 + F_2 \) generally. Thus, we shall consider some cases which allow us to calculate the dimension. Due to this analysis, we observe that without the irrational assumption, the Hausdorff dimension of \( K_1 + K_2 \) is more complicated than Peres and Shmerkin’s
dimensional formula. In other words, it is not possible to offer a uniform dimensional formula for the case we consider.

The structure of the paper is as follows. In section 2, we introduce some basic results of infinite iterated function systems and define a crucial definition: Matching. At the end of this section, we state our main results. In section 3, we prove the main results. In section 4, we offer some examples for which we can explicitly calculate the Hausdorff dimension of \( K_1 + K_2 \). Finally in section 5, we give some further remarks.

## 2 Preliminaries and Main results

### 2.1 Infinite iterated function systems

Before stating the main results, we introduce some necessary definitions and results of infinite iterated function systems. Infinite iterated function systems behave differently from IFS \([12]\) \([8]\). There are two definitions of IIFS, see for example, \([8]\), \([13]\) and \([9]\). Here we adopt Fernau’s definition \([8]\).

**Definition 2.1.** Let \( A = \{ \phi_i(x) = r_i x + a_i : i \in \mathbb{N} \} \), where \( 0 < r_i < 1, a_i \in \mathbb{R} \). If there exists \( 0 < s < 1 \) such that for every \( \phi_i \in A \), \( |\phi_i(x) - \phi_i(y)| \leq s|x - y| \), then \( A \) is called an infinite iterated function system, abbreviated as IIFS. A unique non-empty compact \( J \) is called the attractor of \( A \) if

\[
J = \bigcup_{i \in \mathbb{N}} \phi_i(J).
\]

Here \( \overline{A} \) denotes the closure of \( A \).

**Remark 2.1.** If the cardinality of \( A \) is finite, then this definition coincides with Hutchinson’s definition of self-similar set \([10]\). The existence and uniqueness of \( J \) can be found in \([8]\) or in \([16]\). In \([13]\), Mauldin and Urbanski gave a similar definition of the attractor of IIFS, i.e. \( J_0 = \bigcup_{i \in \mathbb{N}} \phi_i(J_0) \). However, for their definition the attractor \( J_0 \) may not be unique or compact, see example 1.3 from \([8]\). It is easy to see that \( \overline{J_0} = J \).

An infinite iterated function system \( A = \{ \phi_i : i \in \mathbb{N} \} \) satisfies the open set condition if there exists an open set \( O \subseteq \mathbb{R} \) such that

\[
\phi_i(O) \cap \phi_j(O) = \emptyset, \ i \neq j
\]

and \( \phi_j(O) \subseteq O \) for all \( j \in \mathbb{N} \). Under this separation condition, we can find the Hausdorff dimension of \( J_0 \). The following result can be found in \([13]\), \([17]\) or \([9]\).
Theorem 2.1. For any IIFS satisfying the open set condition, we have
\[
\dim_H(J_0) = \inf \left\{ t : \sum_{i \in \mathbb{N}} r_i^t \leq 1 \right\}
\]

On the other hand, generally the Hausdorff dimension of \( J \) is more complicated even if the IIFS satisfies the open set condition, the main reason is the limit points of \( J_0 \), see \cite[Corollary 2]{9}. For most cases, we shall prove that \( J = K_1 + K_2 \) is an attractor of some IIFS in the sense of Definition 2.1. This makes the dimension of \( K_1 + K_2 \) complicated. If the IIFS does not satisfy the open set condition, we may still have techniques to calculate \( \dim_H(J_0) \), however when the limit points of \( J_0 \) is uncountable, calculating the Hausdorff dimension of \( K_1 + K_2 \) is difficult.

Therefore, whether the cardinality of the limit points of \( J_0 \) is countable or not is even more important than the open set condition. We mentioned above that \( J_0 = J \), if \( J_0 \) and \( J \) are the same except for a countable set, then by the countable stability of Hausdorff dimension we have \( \dim_H(J_0) = \dim_H(J) \). We will give a sufficient condition which is reasonable such that under this condition \( J_0 \) coincides with \( J \) apart from a countable set. This is the main idea we will utilise when \( K_1 + K_2 \) is the unique attractor of some IIFS.

2.2 Definition of Matching

We introduce some definitions of the symbolic space. Let \( \sum = \{ s_1, s_2, \ldots, s_n \}^\mathbb{N} \) be a symbolic space endowed with distance \( d((a_n), (b_n)) = \beta^{-\inf\{k: a_i = b_i, 1 \leq i < k \land a_k \neq b_k\}} \), where \( s_i \in \mathbb{R} \). We say \( c_1c_2 \cdots c_m \in \{ s_1, s_2, \ldots, s_n \}^m \) is a block with length \( m \).

Definition 2.2. Let \( \hat{P}_1 = d_1d_2 \cdots d_m, \hat{P}_2 = c_1c_2 \cdots c_m \) be two blocks in \( \{ s_1, s_2, \ldots, s_n \}^m \). We define the concatenation of \( \hat{P}_1 \) and \( \hat{P}_2 \) by \( \hat{P}_1 \ast \hat{P}_2 = d_1d_2 \cdots d_mc_1c_2 \cdots c_m \). The sum of \( \hat{P}_1 \) and \( \hat{P}_2 \) is defined by \( \hat{P}_1 + \hat{P}_2 = (d_1 + c_1)(d_2 + c_2) \cdots (d_m + c_m) \). Concatenating \( A_1 \in \mathbb{N} \) blocks is denoted by
\[
A_1\hat{P}_1 = \underbrace{\hat{P}_1 \ast \hat{P}_1 \ast \cdots \ast \hat{P}_1}_{A_1 \text{ times}}.
\]
The value of the block \( \hat{P}_1 = d_1d_2 \cdots d_m \) with respect to \( \beta > 1 \) is
\[
(d_1d_2 \cdots d_m)_\beta = \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \cdots + \frac{d_m}{\beta^m}.
\]
Similarly, we can define the value of an infinite sequence \( (d_n) \in \sum \) by \( (d_n)_\beta = \sum_{n=1}^{\infty} \frac{d_n}{\beta^n} \).
we can translate over the problem, i.e. in order to study the sum of two numbers form lemma. Not affect any results of our paper. Using this new representation, we have following simple Remark 2.2. Although the lemma above is very simple, the significance of this lemma is that $K_1$ and $K_2$ are $\{f_i(x) = \frac{x}{\beta^{m_i}} + a_i\}_{i=1}^n$ and $\{g_j(x) = \frac{x}{\beta^n} + b_j\}_{j=1}^m$ respectively. Note that

$$f_i(x) = \frac{x}{\beta^{m_i}} + a_i = \frac{x + \beta^{m_i}a_i}{\beta^{m_i}} = \frac{x}{\beta^{m_i}} + \frac{0}{\beta^{m_i-1}} + \frac{0}{\beta^{m_i-2}} + \cdots + \frac{0}{\beta^{m_i-1}} + \frac{\beta^{m_i}a_i}{\beta^{m_i}},$$

therefore $f_i(x)$ corresponds to a block $(000 \cdots a'_i)_{\beta}$, $a'_i = \beta^{m_i}a_i$, we denote it by $\hat{P}_i = (000 \cdots a'_i)$ if there is no fear of ambiguity. We identify $f_i(x)$ with $f_{\hat{P}_i}$. The only difference between $f_i(x)$ and $f_{\hat{P}_i}$ is the symbol, both of them represent the map $\hat{f}_i(x) = \frac{x}{\beta^{m_i}} + a_i$. Define $D_1 = \{\hat{P}_1, \hat{P}_2, \cdots, \hat{P}_n\}$ and $D_2 = \{\hat{Q}_1, \hat{Q}_2, \cdots, \hat{Q}_m\}$, where $\hat{P}_i = (000 \cdots a'_i)$, $a'_i = \beta^{m_i}a_i$, $\hat{Q}_j = (000 \cdots b'_j)$ and $b'_j = \beta^{m_j}b_j$. We call $D_1, D_2$ digital sets or block sets. However, different blocks may stand for the same similitude, for example $\hat{R} = (08) = (22)$ with respect to base 3, both of their corresponding similitudes are $\varphi_{\hat{R}}(x) = \frac{x}{3^2} + 0 + \frac{8}{3^2} = \frac{2}{3} + \frac{2}{3}$. This replacement does not affect any results of our paper. Using this new representation, we have following simple lemma.

**Lemma 2.1.**

$$K_1 = \{x = \lim_{n \to \infty} f_{\hat{P}_{i_1}} \circ f_{\hat{P}_{i_2}} \circ \cdots \circ f_{\hat{P}_{i_n}}(0) : \hat{P}_{i_j} \in D_1\}. $$

$$K_2 = \{y = \lim_{n \to \infty} g_{\hat{Q}_{j_1}} \circ g_{\hat{Q}_{j_2}} \circ \cdots \circ g_{\hat{Q}_{j_m}}(0) : \hat{Q}_{j_k} \in D_2\}. $$

**Proof.** For any $x \in K_1$, we know that there exists $(i_n)_{n=1}^{\infty}$ such that

$$x = \lim_{n \to \infty} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(0).$$

The lemma is a restatement of this fact. \hfill $\Box$

**Remark 2.2.** Although the lemma above is very simple, the significance of this lemma is that we can translate over the problem, i.e. in order to study the sum of two numbers form $K_1$ and $K_2$ respectively, we only need to consider the sum of blocks form $D_1$ and $D_2$.

Motivated by this lemma, we can define a crucial definition of this paper.

**Definition 2.3.** Take $\nu_1$ blocks

$$\hat{P}_{i_1}, \hat{P}_{i_2}, \hat{P}_{i_3}, \cdots, \hat{P}_{i_{\nu_1}}$$

from $D_1$ with the length $P_1, P_2, P_3, \cdots, P_{\nu_1}$, $\nu_2$ blocks

$$\hat{Q}_{j_1}, \hat{Q}_{j_2}, \hat{Q}_{j_3}, \cdots, \hat{Q}_{j_{\nu_2}}$$

from $D_2$ with the length $Q_1, Q_2, Q_3, \cdots, Q_{\nu_2}$. If there exist integers $A_1, A_2, A_3 \cdots A_{\nu_1}, B_1, B_2, B_3 \cdots B_{\nu_2}$ such that

$$\sum_{i=1}^{\nu_1} A_i P_i = \sum_{i=1}^{\nu_2} B_i Q_i$$

and there exist integers $M_1, M_2, M_3 \cdots M_{\nu_1} \in [0, \beta^{m_i}-1]$ such that

$$\sum_{i=1}^{\nu_1} M_i P_i = \sum_{i=1}^{\nu_2} N_i Q_i$$

and there exist integers $M_1, M_2, M_3 \cdots M_{\nu_1} \in [0, \beta^{m_i}-1]$ such that

$$\sum_{i=1}^{\nu_1} M_i P_i = \sum_{i=1}^{\nu_2} N_i Q_i$$

and there exist integers $M_1, M_2, M_3 \cdots M_{\nu_1} \in [0, \beta^{m_i}-1]$ such that

$$\sum_{i=1}^{\nu_1} M_i P_i = \sum_{i=1}^{\nu_2} N_i Q_i$$

and there exist integers $M_1, M_2, M_3 \cdots M_{\nu_1} \in [0, \beta^{m_i}-1]$ such that

$$\sum_{i=1}^{\nu_1} M_i P_i = \sum_{i=1}^{\nu_2} N_i Q_i$$

and there exist integers $M_1, M_2, M_3 \cdots M_{\nu_1} \in [0, \beta^{m_i}-1]$ such that

$$\sum_{i=1}^{\nu_1} M_i P_i = \sum_{i=1}^{\nu_2} N_i Q_i$$

and there exist integers $M_1, M_2, M_3 \cdots M_{\nu_1} \in [0, \beta^{m_i}-1]$ such that

$$\sum_{i=1}^{\nu_1} M_i P_i = \sum_{i=1}^{\nu_2} N_i Q_i.$$
then the block \((A_1\hat{P}_{i_1} \ast A_2\hat{P}_{i_2} \ast \cdots \ast A_t\hat{P}_{i_t}) + (B_1\hat{Q}_{j_1} \ast B_2\hat{Q}_{j_2} \ast \cdots \ast B_t\hat{Q}_{j_t})\) is called a Matching with respect to \(\beta\).

**Remark 2.3.** Obviously, every Matching corresponds to a similitude. For instance, let \((abc)\) be a Matching with respect to \(\beta\), then the corresponding similitude is \(\varphi(x) = \frac{x}{\beta^3} + \frac{a}{\beta^2} + \frac{b}{\beta} + \frac{c}{\beta}\).

We show that \(D_1\) and \(D_2\) generate countably many Matchings.

**Lemma 2.2.** The cardinality of Matchings which are generated by \(D_1\) and \(D_2\) is at most countable.

**Proof.** We give a constructive proof. Firstly, we find out all the possible Matchings which have length 1. The cardinality of Matchings with length 1 is finite due to the finite cardinalities of \(D_1\) and \(D_2\). If there are no such Matchings, we then consider the Matchings with length 2. Similarly, we can find finite Matchings which are of length 2. If there do not exist such Matchings, then we may consider the Matchings with length 3. We continue this procedure and prove the lemma.

**Remark 2.4.** There are some cases which need special handling, i.e. if the new born Matchings can be concatenated by the old Matchings, then we do not choose these new Matchings. In what follows we always abide by this rule. In some cases, after \(n\) steps, all the new Matchings can be concatenated by the former old Matchings, then we stop the procedure. For these cases, the cardinality of Matchings is finite.

**Remark 2.5.** We shall prove that if the cardinality of Matchings is finite, then \(K_1 + K_2\) is a self-similar set, while \(K_1 + K_2\) is the unique attractor of some IIFS, provided that the cardinality of Matchings is infinitely countable.

**Example 2.1.** Let \(K_1 = K_2\) be the attractor of the IFS \(\{g_1(x) = \frac{x}{3}, g_2(x) = \frac{x + 8}{9}\}\). All the possible Matchings are

\[
\{(0), (22), (44), (242), (2442), (24442), (244442), (2444442), \cdots \},
\]

where \(D_1 = D_2 = \{(0), (08) = (22)\}\). Here, for simplicity we assume that \(\hat{R} = (08) = (22)\) as their corresponding similitudes are the same \(\varphi_{\hat{R}}(x) = \frac{x}{3^2} + \frac{0}{3} + \frac{8}{3^2} = \frac{x}{3^2} + \frac{2}{3} + \frac{2}{3^2}\).

**Example 2.2.** Let \(\{f_1(x) = \frac{x}{3}, f_2(x) = \frac{x + 2}{3}\}\) be the IFS of \(K_1, K_2\) is generated by \(\{g_1(x) = \frac{x}{3}, g_2(x) = \frac{x + 8}{9}\}\). Then the Matchings generated by \(D_1\) and \(D_2\) are \(\{(0), (2), (24), (42), (44)\}\), where \(D_1 = \{(0), (2)\} \) and \(D_2 = \{(0), (22)\}\).

After we find all the possible Matchings, we denote this set by

\[
D = \{\hat{R}_1, \hat{R}_2, \cdots, \hat{R}_{n-1}, \hat{R}_n \cdots \},
\]
the lengths of these blocks are increasing. By Remark 2.3, $D$ uniquely determines a set of similarities $\Phi^\infty \triangleq \{\phi_1, \phi_2, \phi_3, \phi_4 \cdots\}$, where $\phi_i$ corresponds to $\hat{R}_i$. We define $E \triangleq \bigcup_{\phi_n \in \Phi^\infty} \bigcap_{n=1}^\infty \phi_1 \circ \phi_2 \cdots \phi_n([0, B_1 + B_2])$ and have $E = \bigcup_{i \in \mathbb{N}} \phi_i(E)$, see section 2 from [13].

Now we state the main results of this paper.

### 2.3 Main results

**Theorem 2.2.** $K_1 + K_2$ is either a self-similar set or an attractor of some infinite iterated function system. More precisely, if the cardinality of Matchings is finite, we then have that $K_1 + K_2$ is a self-similar set. Otherwise

\[ K_1 + K_2 = \bigcup_{i \in \Phi^\infty} \phi_i(K_1 + K_2). \]

We have an interesting corollary of Theorem 2.2.

**Corollary 2.1.** Let $F_1$ and $F_2$ be the self-similar sets with IFS’s \{\(r_i x + a_i\)\}_{i=1}^n and \{\(r'_j x + b'_j\)\}_{j=1}^m, if $0 < r_i, r'_j < 1$ for any $1 \leq i \leq n$ and $1 \leq j \leq m$, then

\[ \dim_P(F_1 + F_2) = \dim_B(F_1 + F_2). \]

**Theorem 2.3.** For any $k \in \mathbb{N}^+$, let the IFS’s of $K_1$ and $K_2$ be

\[
\begin{align*}
\{ f_i(x) = \frac{x}{\beta^k} + \sum_{t=1}^{k} \frac{a^{(i)}_t}{\beta^t}, 1 \leq i \leq n - 1, f_n(x) = \frac{x}{\beta^{2k}} + \sum_{t=1}^{2k} \frac{a^{(n)}_t}{\beta^t} \} \\
\{ g_j(x) = \frac{x}{\beta^k} + \sum_{s=1}^{k} \frac{b^{(j)}_s}{\beta^s}, 1 \leq j \leq m - 1, f_m(x) = \frac{x}{\beta^{2k}} + \sum_{s=1}^{2k} \frac{b^{(m)}_s}{\beta^s} \}
\end{align*}
\]

respectively, where $a^{(i)}_t, b^{(j)}_s \in \mathbb{R}^+$. We have that $E = K_1 + K_2$ except for a countable set. Moreover, if

\[ \max_{i,t} a^{(i)}_t + \max_{j,s} b^{(j)}_s + B_1 + B_2 < c(\beta - 1) \]

where $c$ is a positive constant. More precisely,

\[ c = \min\{|c_i - c_j| : c_i \text{ and } c_j \text{ are any numbers which are from two different Matchings}\} \]

and $c > 0$. Then $\Phi^\infty$ satisfies the open set condition and $\dim_H(K_1 + K_2)$ is computable.
Remark 2.6. It is well known that for any self-similar set $F$, $\dim_P(F) = \dim_B(F)$ (in fact we have $\dim_H(F) = \dim_P(F) = \dim_B(F)$). Corollary 2.4 states that for arithmetic sum of self-similar sets this is still true. A minor modification enables us to prove following stronger results: for any $n \in \mathbb{N}^+$ and $\{K_i\}_{i=1}^n$, $K_1 + K_2 + \cdots + K_n = \{\sum_{i=1}^n x_i : x_i \in K_i\}$ is either a self-similar set or a unique attractor of some IIFS, where $\{K_i\}_{i=1}^n$ are generated by the similitudes of $S$. Moreover,

$$\dim_P(K_1 + K_2 + \cdots + K_n) = \dim_B(K_1 + K_2 + \cdots + K_n).$$

Remark 2.7. The assumptions in Theorem 2.3 seems to be strong. However this class is the case which can guarantee that $E = K_1 + K_2$ except for a countable set.

3 Proofs of the main results

3.1 Proofs of Theorem 2.2 and Corollary 2.1

To begin with we assume that the cardinality of all Matchings is infinitely countable. Before we prove the main results, we need some preliminaries. For any $x + y \in K_1 + K_2$ with coding $(x+y)_{n=1}^\infty$, where $(x_n)$ and $(y_n)$ are the codings of $x$ and $y$ respectively. We know that $(x_n)$ ($(y_n)$) can be decomposed into infinite blocks from $D_1(D_2)$, see the following figure

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $\cdots$ |
|-------|-------|-------|-------|---------|
| $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ | $\cdots$ |

There are two floors in this figure. We call the top floor (bottom floor) the first floor (second floor). In the first floor the concatenation of each block $X_i$ is the coding of $(x_n)_{n=1}^\infty$, this is also true for the second floor. Let $(a_n)_{n=1}^\infty$ be a coding of some point $x + y \in K_1 + K_2$, i.e., $(a_n) = (x_n + y_n)$, where $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are the coding of $x \in K_1$ and $y \in K_2$ respectively. We define

$$C = \{(a_n) : \text{there exists } N \in \mathbb{N}^+ \text{ such that every finite block of }(a_{N+i})_{i=1}^\infty \text{ is not a Matching}\}$$

Lemma 3.1. Let $(a_n) \in C$, for any $\epsilon > 0$ we can find a coding $(b_n)_{n=1}^\infty$ which is the concatenation of Matchings such that

$$|(a_n)_\beta - (b_n)_\beta| < \epsilon$$

Proof. Let $(a_n) \in C$ and $\epsilon > 0$, then there exists $n_0 \in \mathbb{N}$ such that $\beta^{-n_0} < \epsilon$. Now, we choose $(b_n)_{n=1}^\infty$ which is a coding of some point of $E$. We let $b_1b_2b_3\cdots b_{n_0} = a_1a_2a_3\cdots a_{n_0}$. 


If $a_1a_2a_3\cdots a_n$ is a Matching or the concatenation of some Matchings, then we can choose arbitrary tail $(b_{n_0+i})_{i=1}^{\infty}$ which is the concatenation of Matchings. Subsequently we have that

$$|(a_n)_{\beta} - (b_n)_{\beta}| = |(a_{n_0+1}a_{n_0+2}a_{n_0+3}\cdots)_{\beta} - (b_{n_0+1}b_{n_0+2}b_{n_0+3}\cdots)_{\beta}| \leq M \sum_{i=n_0+1}^{\infty} \beta^{-i} < M(\beta^{-1})^{-1} \epsilon,$$

where $M$ is a positive constant and is independent on $\beta$. Hence we prove that there exists a point $b \in E$, i.e. $b = (b_n)_{\beta}$, such that

$$|(a_n)_{\beta} - (b_n)_{\beta}| < \epsilon.$$

If $a_1a_2a_3\cdots a_n$ is not the concatenation of some Matchings, we assume $a_1a_2a_3\cdots a_n$ is the prefix of $X + Y$, here $X$ and $Y$ may not have the same length, but we can still add their prefixes. We assume that $\sum_{i=1}^{p} |X_i| < \sum_{i=1}^{q} |Y_i|$, where $|X_i|$ denotes the length of the block $X_i$, this condition means that $X = X_1 * X_2 * \cdots * X_p$ and $Y = Y_1 * Y_2 * \cdots * Y_q$ are not matched.

Let $k_1 = \sum_{i=1}^{p} |X_i|$ and $k_2 = \sum_{i=1}^{q} |Y_i|$. Then $k_2X + k_1Y$ is a Matching. The initial $k_1$ digits of $k_2X + k_1Y$ is $a_1a_2a_3\cdots a_n$. Now the remaining proof is the same as the first case. \hfill \Box

**Lemma 3.2.** $E = K_1 + K_2$.

**Proof.** For every $\epsilon > 0$ and $x + y \in K_1 + K_2$, we can find a coding $(a_n)$ such that $x + y = \sum_{n=1}^{\infty} a_n \beta^{-n}$. If there exists a subsequence of integer $n_k \to \infty$ such that $(a_1, a_2, a_3, \cdots, a_{n_k})$ is a block generated by the concatenation of some Matchings, then by the definition of $E \triangleq \bigcup_{\{\phi_n\} \in \Phi_{\infty}} \bigcap_{n=1}^{\infty} \phi_1 \circ \phi_2 \cdots \phi_n([0, B_1 + B_2])$ we have $x + y \in E$. If $(a_n) \in C$, by Lemma 3.1 there exists $b \in E$ such that $|b - x - y| < \epsilon$. \hfill \Box

**Lemma 3.3.** $\bigcup_{i \in \mathbb{N}} \phi_i(K_1 + K_2) = K_1 + K_2$.

**Proof.** On the one hand, $E = \bigcup_{i \in \mathbb{N}} \phi_i(E)$, this equality implies that

$$E = \bigcup_{i \in \mathbb{N}} \phi_i(E) = \bigcup_{i \in \mathbb{N}} \phi_i(E) \supseteq \bigcup_{i \in \mathbb{N}} \phi_i(E) = \bigcup_{i \in \mathbb{N}} \phi_i(K_1 + K_2),$$

i.e. we have

$$\bigcup_{i \in \mathbb{N}} \phi_i(K_1 + K_2) \subseteq K_1 + K_2.$$

On the other hand, $E = \bigcup_{i \in \mathbb{N}} \phi_i(E) \subseteq \bigcup_{i \in \mathbb{N}} \phi_i(K_1 + K_2)$, therefore we prove the converse inclusion. \hfill \Box

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Proof of Theorem 2.2. By Lemma 2.2, we know that there are at most countably many Matchings generated by \( D_1 \) and \( D_2 \). If the cardinality of Matchings is infinite (countable), then by Lemma 3.3 \( K_1 + K_2 \) is an attractor of \( \Phi^\infty \). If the cardinality is finite, then \( K_1 + K_2 \) is a self-similar set since for every point \( x + y \in K_1 + K_2 \), we can find a coding which is the concatenation of Matchings such that the value of this infinite coding is \( x + y \).

Now, we can prove Corollary 2.1. When the IFS’s of \( F_1 \) and \( F_2 \) satisfy the irrational assumption, it is easy to prove Corollary 2.1 due to Peres and Shmerkin [20]. In fact, we can prove a stronger result. We recall their main result.

Theorem 3.1. Let \( F_1 \) and \( F_2 \) be the attractors of \( \{r_i x + a_i\}^n_{i=1}, \{r'_j x + b_j\}^m_{j=1} \) respectively. If there exist \( i, j \) such that \( \frac{\log r_i}{\log r_j} \notin \mathbb{Q} \), then \( \dim_H(F_1 + F_2) = \min\{\dim_H F_1 + \dim_H F_2, 1\} \).

Proof of Corollary 2.1. Firstly, we prove under the irrational assumption that \( \dim_H(F_1 + F_2) = \dim_P(F_1 + F_2) = \dim_B(F_1 + F_2) = \min\{\dim_H F_1 + \dim_H F_2, 1\} \). Using the theorem above, if \( \dim_H(F_1 + F_2) = 1 \), then

\[
1 = \dim_H(F_1 + F_2) \leq \dim_P(F_1 + F_2) \leq \dim_B(F_1 + F_2) \leq 1.
\]

If \( \dim_H(F_1 + F_2) = \dim_H(F_1) + \dim_H(F_2) \). We note that for any \( A, B \subseteq \mathbb{R} \), we have \( B - A = P_y(A \times B) \), where \( P_y(A \times B) \) denotes the projection of \( A \times B \) on the \( y \) axis along lines having 45° angle with the \( x \) axis. Therefore,

\[
\dim_H(F_1 + F_2) \leq \dim_B(F_1 + F_2) \\
\leq \dim_B((-F_2) \times F_1) \\
\leq \dim_B(F_1) + \dim_B(F_2) \\
= \dim_H(F_1) + \dim_H(F_2)
\]

The second inequality holds since the projection is a Lipschitz map, the third inequality is due to the property of product of fractal sets, see the product formula 7.5, page 102, [7]. For the last equality, we use the fact that for any self-similar set, its Hausdorff dimension and the Box dimension coincide.

If \( K_1 \) and \( K_2 \) are generated by the similitudes of \( S \) and the cardinality of Matchings is infinitely countable, then we have \( \dim_P(K_1 + K_2) = \dim_B(K_1 + K_2) = \dim_P(E) = \dim_B(E) \) due to Lemma 3.2 and Theorem 3.1 form [13]. By Theorem 2.2 we know that \( K_1 + K_2 \) is a self-similar set if the cardinality of Matchings is finite. Hence, whether the irrational assumption holds or not we always have \( \dim_P(K_1 + K_2) = \dim_B(K_1 + K_2) \).

3.2 Dimension of \( K_1 + K_2 \)

Before proving Theorem 2.3 we discuss the cardinality of Matchings. Let \( \#D \) be the cardinality of the Matchings generated by \( D_1 \) and \( D_2 \). We say that \( D_i \) is homogeneous if the length of
every block in $D_i$ coincides. For simplicity we identify the blocks of $D_i$ with the lengths of these blocks. There is one point we should keep in mind, i.e. different blocks of $D_i$ may have the same length. Hence we should count the multiplicity when some blocks have the same length.

We know that the digits of $D_i$ stand for the length of the blocks of $D_i$ and the similarity ratios, for example $D_2 = \{6, 10\}$, 6 refers to the similarity ratio $\beta^{-6}$ and 10 means $\beta^{-10}$. Iterating the IFS of $K_2$ (see the formula (1)) the similarity ratios change. The original similarity ratios are $\beta^{-6}$ and $\beta^{-10}$, if we iterate the IFS one time, there are four similarity ratios $\beta^{-16}, \beta^{-16}, \beta^{-12}, \beta^{-20}$, we denote the new digit set by $D_2^i = \{16, 16, 12, 20\}$. Similarly, we can iterate $K_2$ several times and obtain new digital set $D_2^i$.

**Lemma 3.4.** $\sharp D$ is finite if and only if $D_1$ (or $D_2$) is homogeneous and if we iterate the IFS of $K_2$ ($K_1$) for finite times the lengths of all the blocks of $D_2^i$ (or $D_1^i$) are the multipliers of the length of blocks of $D_1$ (or $D_2$).

**Proof.** $\iff$, for simplicity, we let $D_1 = \{k, k\}, D_2 = \{2k, 3k\}$. For other cases the discussion is similar. It is easy to find that $\sharp D$ is finite.

$\Rightarrow$, if $\sharp D$ is finite, we know that $K_1 + K_2$ is a self-similar set. Let $(a_n)_{n=1}^{\infty}$ be the coding of some point $x + y \in K_1 + K_2$. Therefore, $(a_n)_{n=1}^{\infty}$ is the concatenation of Matchings. i.e., $(a_n)_{n=1}^{\infty}$ can be decomposed into infinitely many blocks, these blocks are Matchings. We assume that $D_1 = \{a, b, \cdots\}$ and $D_2 = \{c, d, \cdots\}$, where $a \neq b, c \neq d \in \mathbb{N}^+$. If we can construct an arbitrary long block (Here we should explain the meaning of block. Usually the concatenation of the blocks from $D_1$ and from $D_2$ are not matched, if they are matched, then we find a Matching. Here we cut the tail of the sum of the concatenation of the blocks from $D_1$ and $D_2$. We find the block in this way.) such that there are no Matchings in this block, then our assumption, i.e., $a \neq b, c \neq d \in \mathbb{N}^+$, is not correct. On the other hand, it is easy to construct such block under our assumption. If $a \neq b, c \neq d$, we do not form the Matchings deliberately at each step when we construct an arbitrary long block. For instance, when $a + b = c + d$, if we choose the the blocks $a, b$ from $D_1$ and $c, d$ from $D_2$, then we have a new Matching with length $a + b$. Since $a \neq b$ ($c \neq d$), we have $a + a \neq c + d$ or $a + b \neq d + d$, therefore we choose these blocks $(a, a$ from $D_1$ and $c, d$ from $D_2$ or $a, b$ from $D_1$ and $d, d$ from $D_2$) and the new Matchings cannot appear in the first step. If $a + b \neq c + d$, we can choose blocks $a, b$ and $c, d$. The Matching cannot appear either. For the second step, we choose the blocks in a similar way. The rule of choosing blocks is that Matchings cannot appear in each step.

It follows from this procedure that we can find an arbitrary long block (the block is constructed via the method mentioned above) which does not have Matchings. This contradicts with the fact that $\sharp D$ is finite. Thus, we assume that the length of all the blocks in $D_1$ is $a$. If for any iteration there always exist some blocks whose lengths are not the multipliers of $a$, then at every step we can construct a block (here the block is constructed via the methodology mentioned above) such that there are no Matchings appearing in this block, moreover, the block can be
arbitrary long, which contradicts with the hypothesis again.

The following two examples can illustrate this idea of the proof.

**Example 3.1.** $D_1 = \{4, 4\}$, $D_2 = \{6, 10\}$. We iterate $K_2$ and have a new digits set $D'_2 = \{12, 16, 16, 20\}$, the length of all the new blocks in $D'_2$ are the multiples of 4. Hence, the Matchings generated by $D_1$ and $D_2$ are finite.

**Example 3.2.** $D_1 = \{4, 4\}$, $D_2 = \{6, 8\}$. We iterate $K_2$, $K_2 = f_6(K_2) \cup f_8(K_2) = f_{14}(K_2) \cup f_{12}(K_2) \cup f_8(K_2) = f_{20}(K_2) \cup f_{22}(K_2) \cup f_{12}(K_2) \cup f_8(K_2) = \cdots$, here we use the subscripts to represent the similarity ratios, e.g., 8 stands for the similarity ratios $\beta^{-8}$. Obviously we can find an increasing sequence $(a_n) = (6, 14, 22, 30, 38, 46, \cdots)$ such that $4 \nmid a_n$, for any $n \geq 1$, which yields that $E_D$ is not finite.

We have proved that if $E_D$ is finite, then $K_1 + K_2$ is a self-similar set. Comparing with the IIFS, there are more techniques which can calculate the dimensions of self-similar sets. For example, when the IFS of $K_1 + K_2$ does not satisfy the open set condition, if $a_i, b_j \in \frac{1}{\beta} \mathbb{Z}[\frac{1}{\beta}]$ and $\beta$ is a Pisot number, we can explicitly calculate $\dim_H(K_1 + K_2)$, the details can be found in [18].

Our main consideration is the case that $K_1 + K_2$ is the attractor of some IIFS.

By Lemma 3.2, we know that when $E_D$ is infinitely countable, $E = K_1 + K_2$. If the limit points of $E$ is uncountable, we cannot calculate the dimension of $K_1 + K_2$ in terms of the dimensional theory of IIFS. Hence, we need to find some classes that have $\dim_H(E) = \dim_H(K_1 + K_2)$.

Let $(a_n)_{n=1}^{\infty}$ be the coding of some point $x + y \in K_1 + K_2$, i.e., $(a_n) = (x_n + y_n)$ where $(x_n)$ and $(y_n)$ are the codings of $x$ and $y$ respectively. Recall the definition of $C$,

$$C = \{(a_n) : \text{there exists } N \in \mathbb{N}^+ \text{ such that every finite block of } (a_{N+i})_{i=1}^{\infty} \text{ is not a Matching}\}.$$

We have

**Lemma 3.5.** If $C$ is countable, then we have that $E = K_1 + K_2$ apart from a countable set.

*Proof.* Firstly we have $E = K_1 + K_2$, it remains to prove that there are only countable limit points of $E$ which are not in $E$. For any $x + y \in K_1 + K_2 = E$, there is a coding $(a_n)$ such that the value of this coding is $x + y$. If there exists $n_k \to \infty$ such that $(a_1 a_2 \cdots a_{n_k})$ is a Matching (or the concatenation of some Matchings), by the definition of $E \triangleq \bigcup_{\{\phi_n\} \in \Phi^n} \bigcap_{n=1}^{\infty} \phi_1 \circ \phi_2 \cdots \phi_n([0, B_1 + B_2])$, we know that $x + y \in E$. If $(a_n) \in C$, then $E \setminus E$ is countable as $C$ and the cardinality of all the Matchings are countable. 

\]
However, the converse statement of this lemma may not be correct. The following lemma gives the sufficient condition such that $C$ is countable.

**Lemma 3.6.** $C$ is countable if there exists $k$ such that $D_1 = \{k, k, \cdots, k, 2k\}$ and $D_2 = \{k, k, \cdots, k, 2k\}$, i.e. both $D_1$ and $D_2$ have the blocks with length $k$ apart from the last block with length $2k$.

**Proof.** If $D_1 = \{k, k, \cdots, k, 2k\}$ and $D_2 = \{k, k, \cdots, k, 2k\}$, we need to find all the possible sequences of $C$. Firstly, we choose one block form $D_1$ and $D_2$ and stack on the first and second floor respectively. We can pick only $k$ from $D_1$ and $2k$ from $D_2$ (or $2k$ from $D_1$ and $k$ from $D_2$). Otherwise, a Matching will appear, see the following figure:

\[
\begin{array}{ccc}
  & & \\
  & k & \cdots \\
 2k & & \\
\end{array}
\]

Then at the second step for the first floor (top floor) we cannot take any block of $D_1$ with length $k$ as $k + k = 2k$ and new Matching can appear. Hence for the first floor we can pick only the block with length $2k$. Similarly, for the second floor (bottom floor) we cannot take a block of $D_2$ with length $k$ as $k + 2k = 2k + k$, which can generate a new Matching. Therefore, we must take block with length $2k$ for the second floor if we do not want new Matching to appear.

The figure now is:

\[
\begin{array}{ccc}
  & & \\
  & k & 2k \cdots \\
 2k & & 2k \cdots \\
\end{array}
\]

It is easy to find that if we do not want the new Matchings to appear in the summed blocks of two floors we cannot choose blocks freely from the second step on. The figure below illustrates this idea:

\[
\begin{array}{ccccccc}
  & & & & & & \\
  & k & 2k & 2k & 2k \cdots \\
 2k & & 2k & 2k & 2k \cdots \\
\end{array}
\]

From the analysis above, we see that the sequences in $C$ are eventually periodical. Thus, we prove that $C$ is countable.

**Remark 3.1.** The condition of the lemma is not necessary, for instance, let $D_1 = \{k, 2k\}$ and $D_2 = \{k, k, \cdots, k, 3k\}$, i.e., $D_1$ only has two blocks and the blocks in $D_2$ have length $k$ except the last block with length $3k$. We can similarly prove that in this case $C$ is countable.

**Proof of Theorem 2.3:** By Lemmas 3.5 and 3.6, we prove the first statement. For the second statement, given any two Matchings $(s_1s_2 \cdots s_p), (t_1t_2 \cdots t_q)$, their corresponding similitudes
are \(\phi_{s_1s_2\ldots s_p}(x) = \beta^{-p}x + \sum_{i=1}^{p} s_i\beta^{-i}\) and \(\phi_{t_1t_2\ldots t_q}(x) = \beta^{-q}x + \sum_{i=1}^{q} t_i\beta^{-i}\) respectively. Let \(V = (0, B_1 + B_2)\), simple calculation implies that

\[
\phi_{s_1s_2\ldots s_p}(V) = \left(\sum_{i=1}^{p} s_i\beta^{-i}, \sum_{i=1}^{p} s_i\beta^{-i} + (B_1 + B_2)\beta^{-p}\right)
\]

\[
\phi_{t_1t_2\ldots t_q}(V) = \left(\sum_{i=1}^{q} t_i\beta^{-i}, \sum_{i=1}^{q} t_i\beta^{-i} + (B_1 + B_2)\beta^{-q}\right)
\]

We assume that \((s_1s_2\ldots s_p) < (t_1t_2\ldots t_q)\), i.e., there exists \(1 \leq i_0 < p\) such that \(s_k = t_k\) for any \(1 \leq k \leq i_0 - 1\) and \(s_{i_0} < t_{i_0}\). By the definition of \(c\), \(\phi_{s_1s_2\ldots s_p}(V) \cap \phi_{t_1t_2\ldots t_q}(V) = \emptyset\). It remains to prove that \(\phi(V) \subset V\) for any \(V \in \Phi^\infty\). Let \(\phi\) be generated by the Matching \(\hat{R}_1 \ast \hat{R}_2 + \hat{T}_1 \ast \hat{T}_2\), where \(\hat{R}_i\) and \(\hat{T}_i\) correspond to the similitudes \(H_i(x)\) and \(I_i(x)\) respectively. Let the length of \(\hat{R}_1 \ast \hat{R}_2 + \hat{T}_1 \ast \hat{T}_2\) be \(k_0\). It is easy to find that

\[
\phi(x) = H_1 \circ H_2(x) + I_1 \circ I_2(0)
\]

Hence,

\[
\phi(V) = \left(H_1 \circ H_2(0) + I_1 \circ I_2(0), H_1 \circ H_2(0) + I_1 \circ I_2(0) + \frac{B_1 + B_2}{\beta^{k_0}}\right)
\]

Recall the assumption of \(K_1, K_2\), the convex hull of \(K_i\) is \([0, B_i], 1 \leq i \leq 2\), i.e., \(H_i([0, B_i]) \subset [0, B_i], 1 \leq s \leq n\) and \(I_i([0, B_i]) \subset [0, B_i], 1 \leq t \leq m\). Therefore \(0 < \phi(x) < B_1 + B_2\). Similarly, we can prove that \(\phi(V) \subset V\) for any \(V \in \Phi^\infty\). As such \(\Phi^\infty\) satisfies the open set condition. The calculation of \(\text{dim}_H(K_1 + K_2)\) now is a straightforward application of Theorem 2.1.

## 4 Examples

In this section, we give some examples for which Theorem 3.1 cannot calculate the dimension.

**Example 4.1.** Let \(K_1 = K_2\) be the self-similar sets with IFS \(\{g_1(x) = \frac{x}{7}, g_2(x) = \frac{x+8}{9}\}\), then \(\text{dim}_H(K_1 + K_2) = \text{dim}_B(K_1 + K_2) = \frac{\ln t_0}{\ln 3}\), where \(t_0 = \sup\{t : t^3 - t^2 - 2t + 1 \geq 0, 0 < t < 1\}\).

We know that \(D_1 = D_2 = \{(0), (22)\}\), all the Matchings which are generated by \(D_1\) and \(D_2\) are

\[
D = \{(0), (22), (44), (2422), (2442), (24442), (244442)\}
\]

The corresponding IIFS of \(D\) is

\[
\Phi^\infty = \{\varphi_1 = f_0, \varphi_2 = f_2 \circ f_2, \varphi_3 = f_4 \circ f_4, \varphi_4 = f_2 \circ f_4 \circ f_2, \ldots\}
\]
where \( f_0(x) = \frac{x}{3}, f_2(x) = \frac{x+2}{3}, f_4(x) = \frac{x+4}{3} \).

Therefore, \( E = \bigcup_{i \in \mathbb{N}} \varphi_i(E) \). By Theorem 2.3, \( \dim_h(K_1 + K_2) = \dim_h(E) \). Obviously this IIFS satisfies the OSC, i.e.

\[
\varphi_i((0, 2)) \cap \varphi_j((0, 2)) = \emptyset
\]

for any \( i \neq j \) and \( \varphi_i((0, 2)) \subseteq (0, 2) \) for any \( i \in \mathbb{N} \), now we can use Theorem 2.1 to calculate the dimension.

**Example 4.2.** Let \( \{f_1(x) = \frac{x}{3}, f_2(x) = \frac{x+2}{3}\} \) and \( \{g_1(x) = \frac{x}{3}, g_2(x) = \frac{x+8}{9}\} \) be the IFS’s of \( K_1 \) and \( K_2 \) respectively. Then \( K_1 + K_2 \) is a self-similar set, the IFS is \( \{\varphi_1(x) = \frac{x}{3}, \varphi_2(x) = \frac{x+2}{3}, \varphi_3(x) = \frac{x}{3} + \frac{2}{3}, \varphi_4(x) = \frac{x}{3} + \frac{2}{3} + \frac{2}{3}, \varphi_5(x) = \frac{x}{3} + \frac{4}{3} + \frac{2}{3}\} \). This IFS does not satisfy the OSC, in fact it is of finite type, see [18, Theorem 2.5]. Hence, we can calculate the Hausdorff dimension of \( K_1 + K_2 \) in terms of the main result of [18]. We omit the details.

**Example 4.3.** Let \( K_1 \) and \( K_2 \) be the attractor of the IFS’s \( \{f_1(x) = \frac{x}{\beta}, f_2(x) = \frac{x+2}{\beta} + \frac{2}{\beta^2}\} \) and \( \{g_1(x) = \frac{x}{\beta}, g_2(x) = \frac{x}{\beta} + \frac{2}{\beta} + \frac{2}{\beta^2}\} \), where \( \beta > 1 \) satisfies \( \beta^2 - 4\beta + 1 > 0 \).

We know that \( D_1 = \{(0), (22)\} \), \( D_2 = \{(0), (222)\} \), it is easy to find that all the Matchings are

\[
\{(0), (22), (244), (2242), 2244(244)^n22, 2244(244)^n44, 2244(244)^nn242, (224), (442), 4444(244)^n22, 4444(244)^n44, 4444(244)^nn242),
(22), 24(244)^nn22, 24(244)^n44, 24(244)^nn242)\}
\]

where \( n_i, n_j, n_k \geq 0, (244)^n \) means the concatenation of \( (244) \) for \( n_i \) times. Hence, we can find all the similitudes \( \Phi^{\infty} \), since \( \beta > 1 \) satisfies \( \beta^2 - 4\beta + 1 > 0 \), by Theorem 2.3 the IIFS satisfies the open set condition. Hence

\[
\dim_h(K_1 + K_2) = \frac{\log t}{\log \beta},
\]

where \( t \) is the largest positive root of the equation \( x^7 - x^6 - x^5 - 4x^4 - 4x^3 + x^2 - 3x + 2 = 0 \).

## 5 Final remark and an open problem

The main result of this paper is that \( K_1 + K_2 \) is either a self-similar set or a unique attractor of some IIFS. Moreover, we can find all the possible similitudes. However, to calculate the dimension of \( K_1 + K_2 \) is difficult, especially the IIFS case. As in this case, we should consider the limit points of \( E \) as well as the open set condition. Ignoring either of them may hinder the calculation of the dimension of \( K_1 + K_2 \). We may implement the Vitali process if the IIFS has overlaps, see [17, Theorem 3.1].
Finally, we give an interesting open problem, in the Corollary 2.1, we have proved that \( \dim_P(F_1 + F_2) = \dim_B(F_1 + F_2) \), we know that if the IFS’s of \( K_1 \) and \( K_2 \) fall in the setting of satisfying the irrational assumption, \( \dim_H(F_1 + F_2) = \dim_P(F_1 + F_2) = \dim_B(F_1 + F_2) \). Otherwise, this beautiful equality still holds if the cardinality of all the Matchings is finite (\( K_1 + K_2 \) is a self-similar set). Hence, according to this analysis, we pose following open problem:

**Open problem 5.1.** If the cardinality of all the Matchings is infinite, when can we have \( \dim_H(K_1 + K_2) = \dim_P(K_1 + K_2) = \dim_B(K_1 + K_2) \).

If the equality above fails, we have that \( K_1 + K_2 \) is not a self-similar set and that \( K_1 + K_2 \) is only a unique attractor of some IIFS.

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