Coulomb-Oscillator Duality  
and  
Scattering Problem in 5-Dimensional Coulomb Field  
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It is shown that the Hurwitz transformation connects the eight-dimensional repulsive oscillator problem with the five-dimensional Coulomb problem for continuous spectrum. The hyperspherical and parabolic bases for this system are calculated. The quantum mechanical scattering problem of charged particles in the 5-dimensional Coulomb field is solved.

1 Introduction

It is known [1, 2] that the eight-dimensional isotropic oscillator [3] with constraint is dual to the five-dimensional Coulomb problem in the discrete spectrum. This property is a particular case of a more general statement which says that the eight-dimensional isotropic oscillator (without additional constraint) is dual to the five-dimensional non-Abelian Yang monopole [4, 5, 6]. Similarly, it is possible to show that the eight-dimensional repulsive oscillator \(U = -\mu \omega^2 r^2 / 2\) with constraint is dual to the five-dimensional Coulomb problem in the continuous spectrum. This fact, just as for the bound states, later was named the Coulomb-oscillator duality. Due to the so \((5, 1)\) hidden symmetry the Coulomb problem can be factorized not only in hyperspherical but also in parabolic coordinates. The presence of the parabolic basis makes possible to construct the scattering theory for the charged particle in the five-dimensional Coulomb field. But the dyon-oscillator duality is inherent not only in the \(\mathbb{R}^8 \rightarrow \mathbb{R}^5\) mapping but also in the mappings \(\mathbb{R}^1 \rightarrow \mathbb{R}^1\), \(\mathbb{R}^2 \rightarrow \mathbb{R}^2\), and \(\mathbb{R}^4 \rightarrow \mathbb{R}^3\). As a result, we obtain one- and two-dimensional anions in the first two cases [7, 8] and the Dirac’s monopole in the third [9, 10, 11].

This article has the following structure. In section 2 it is shown that the eight-dimensional repulsive oscillator is dual to the five-dimensional Coulomb problem for the continuous spectrum. Section 3 presents the wave functions in five-dimensional hyperspherical and parabolic coordinates. In section 4 the quantum mechanical problem of the scattering of the charged particle in the five-dimensional Coulomb field is considered. The formulae for the amplitude scattering and cross section are calculated.

2 Coulomb-oscillator duality

Let us consider the Hurwitz transformation [11]:

\[
\begin{align*}
  x_0 &= u_0^2 + u_1^2 + u_2^2 + u_3^2 - u_4^2 - u_5^2 - u_6^2 - u_7^2, \\
  x_2 + ix_1 &= 2 \left[ (u_0 + iu_1)(u_5 + iu_4) + (u_2 - iu_3)(u_7 - iu_6) \right], \\
  x_4 + ix_3 &= 2 \left[ (u_0 + iu_1)(u_7 + iu_6) + (u_2 - iu_3)(u_5 - iu_4) \right].
\end{align*}
\]

(2.1)

Here \(u_\mu (\mu = 0, 1, ..., 7)\) are the coordinates of the space \(\mathbb{R}^8(\vec{u})\), and \(x_i (i = 0, 1, ..., 4)\), of the space \(\mathbb{R}^5(\vec{x})\). It is easily seen from (2.1) that the following equality holds:

\[
u^4 = (u_0^2 + u_1^2 + \cdots + u_7^2)^2 = x_0^2 + x_1^2 + \cdots + x_4^2 = r^2,
\]

(2.2)

which is called the Euler identity. According to [11], the connection of the Laplace operators in the spaces \(\mathbb{R}^8\) and \(\mathbb{R}^5\) has the form

\[
\Delta_8 = 4r\Delta_5 - \frac{4}{r}\hat{j}^2,
\]

(2.3)
where $I = I^2 + I^3 + I^5$, and
\[
\begin{align*}
I_1 &= \frac{i}{2} \left( u_1 \frac{\partial}{\partial u_0} - u_0 \frac{\partial}{\partial u_1} + u_3 \frac{\partial}{\partial u_2} - u_2 \frac{\partial}{\partial u_3} + u_5 \frac{\partial}{\partial u_4} - u_4 \frac{\partial}{\partial u_5} - u_7 \frac{\partial}{\partial u_6} - u_6 \frac{\partial}{\partial u_7} \right), \\
I_2 &= \frac{i}{2} \left( u_2 \frac{\partial}{\partial u_0} - u_0 \frac{\partial}{\partial u_2} + u_1 \frac{\partial}{\partial u_3} - u_3 \frac{\partial}{\partial u_1} + u_6 \frac{\partial}{\partial u_4} - u_4 \frac{\partial}{\partial u_6} - u_5 \frac{\partial}{\partial u_7} - u_7 \frac{\partial}{\partial u_5} \right), \\
I_3 &= \frac{i}{2} \left( u_3 \frac{\partial}{\partial u_0} + u_2 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_2} - u_0 \frac{\partial}{\partial u_3} - u_5 \frac{\partial}{\partial u_4} + u_4 \frac{\partial}{\partial u_5} + u_7 \frac{\partial}{\partial u_7} - u_7 \frac{\partial}{\partial u_7} \right).
\end{align*}
\] (2.4)

Using the explicit form of the operators one can prove by a direct calculation that the operators $I_1$, $I_2$ and $I_3$ satisfy the commutation relations
\[
[I_a, I_b] = i\varepsilon_{abc} I_c,
\]
where $a$, $b$ and $c$ are equal 1, 2 and 3, respectively.

Now let us connect the eight-dimensional problem of repulsive oscillator
\[
\left( -\frac{\hbar^2}{2\mu^2} \Delta_8 - \frac{\mu \omega^2 A^2}{2} \right) \psi(\vec{u}) = E \psi(\vec{u}) \tag{2.5}
\]
with the five-dimensional Coulomb problem. Since, the operators $I_a$ are independent of the coordinates $x_i$, we can represent the wave function $\psi(\vec{u})$ of the eight-dimensional repulsive oscillator in the following factorized form
\[
\psi(\vec{u}) = \psi(\vec{r}) \Phi(\Omega_a), \tag{2.6}
\]
where $\Omega_a$ denotes the angles, on which the operators $I_a$ depend, and $\Phi(\Omega_a)$ is the eigenfunction of the operator $I^2$, i.e.
\[
I^2 \Phi(\Omega_a) = J(J + 1) \Phi(\Omega_a). \tag{2.7}
\]
Here $J(J + 1)$ are the eigenvalues of the operator $I^2$. Now, substituting (2.8) into (2.9) and taking into account (2.7) we arrive at the equation
\[
\left[ -\frac{\hbar^2}{2\mu} \Delta_5 - \frac{e^2}{r} + \frac{\hbar^2}{2\mu^2} J(J + 1) \right] \psi(\vec{r}) = \varepsilon \psi(\vec{r}), \tag{2.8}
\]
where $\varepsilon = \mu \omega^2 / 8$ and $4e^2 = E$. Thus, we obtain that the eight-dimensional repulsive oscillator is dual to the infinite number of five-dimensional Coulomb systems with additional terms $1/r^2$ and with coupling constant $\hbar^2 J(J + 1)/2\mu$. At $J = 0$ we arrive at the equation for the ordinary Coulomb problem:
\[
\left( -\frac{\hbar^2}{2\mu} \Delta_5 - \frac{e^2}{r} \right) \psi(\vec{r}) = \varepsilon \psi(\vec{r}), \tag{2.9}
\]
The condition $J = 0$ is equivalent to requirement $I_a \psi(\vec{u}) = 0$. Moreover, it follows from $n_{12}$ that $\psi(\vec{x})$ is the even function of variables $u$
\[
\psi(\vec{x}(-\vec{u})) = \psi(\vec{x}(\vec{u})). \tag{2.10}
\]

3 Hyperspherical and parabolic bases

Let us introduce the five dimensional hyperspherical coordinates in $\mathbb{R}^5$ as follows:
\[
\begin{align*}
x_0 &= r \cos \theta \\
x_2 + ix_1 &= r \sin \theta \sin \frac{\beta}{2} e^{i \frac{\pi}{4}} \\
x_4 + ix_3 &= r \sin \theta \cos \frac{\beta}{2} e^{i \frac{\pi}{4}},
\end{align*}
\] (3.1)
where \( r \in [0, \infty) \), \( \theta \in [0, \pi] \), \( \alpha \in [0, 2\pi) \), \( \beta \in [0, \pi] \), \( \gamma \in [0, 4\pi) \). In this coordinates the differential elements of length, volume and Laplace operator have the form

\[
\mathrm{d}l_{5}^{2} = dr^{2} + r^{2}d\theta^{2} + \frac{r^{2}}{4}\sin^{2}\theta \left( d\alpha^{2} + d\beta^{2} + d\gamma^{2} + 2\cos\beta d\alpha d\gamma \right),
\]

\[
dV_{5} = \frac{r^{4}}{8}\sin^{3}\theta \sin\beta drd\theta d\beta d\gamma,
\]

\[
\Delta_{5} = \frac{1}{r^{4}} \frac{\partial}{\partial r} \left( r^{4} \frac{\partial}{\partial r} \right) + \frac{1}{r^{2}\sin^{3}\theta} \frac{\partial}{\partial \theta} \left( \sin^{3}\theta \frac{\partial}{\partial \theta} \right) - \frac{4\hat{L}^{2}}{r^{2}\sin^{2}\theta},
\]

where

\[
\hat{L}^{2} = -\left[ \frac{\partial^{2}}{\partial \beta^{2}} + \cot\beta \frac{\partial}{\partial \beta} + \frac{1}{\sin^{2}\beta} \left( \frac{\partial^{2}}{\partial \alpha^{2}} - 2\cos\beta \frac{\partial^{2}}{\partial \alpha d\gamma} + \frac{\partial^{2}}{\partial \gamma^{2}} \right) \right].
\]

The components of the momentum operator \( \hat{L} \) have the form

\[
\hat{L}_{1} = i \left( \cos\alpha \cot\beta \frac{\partial}{\partial \alpha} + \sin\alpha \frac{\partial}{\partial \beta} - \frac{\cos\alpha}{\sin\beta} \frac{\partial}{\partial \gamma} \right),
\]

\[
\hat{L}_{2} = -i \left( \sin\alpha \cot\beta \frac{\partial}{\partial \alpha} - \cos\alpha \frac{\partial}{\partial \beta} - \frac{\sin\alpha}{\sin\beta} \frac{\partial}{\partial \gamma} \right),
\]

\[
\hat{L}_{3} = i \frac{\partial}{\partial \alpha}, \quad \hat{L}_{3}' = i \frac{\partial}{\partial \gamma}.
\]

The solution of the equation (2.9) in hyperspherical coordinates has the following form

\[
\psi_{k\lambda Lmm'}(r, \theta, \alpha, \beta, \gamma) = \sqrt{\frac{2L+1}{2\pi^{2}}} R_{k\lambda}(r) Z_{\lambda L}(\theta) D_{nm}^{L}(\alpha, \beta, \gamma),
\]

where Wigner D-function, normalized by the condition

\[
\frac{1}{8} \int \left| D_{mm'}^{L}(\alpha, \beta, \gamma) \right|^{2} \sin\beta drd\beta d\gamma = \frac{2\pi^{2}}{2L+1}
\]

is the eigenfunction of the set commuting operators \( \hat{L}^{2}, \hat{L}_{3} \) and \( \hat{L}_{3}' \). The function \( Z_{\lambda L}(\theta) \) is given by the formula

\[
Z_{\lambda L}(\theta) = 2^{L+1} \Gamma \left( \frac{2L+3}{2} \right) \sqrt{\frac{(2\lambda + 3)(\lambda - 2L)!}{2\pi(\lambda + 2L + 2)!}} \sin\theta^{2L} C_{\lambda-2L}^{2L+3/2}(\cos\theta),
\]

where \( C_{n}^{\lambda}(x) \) are Gegenbauer polynomials, and quantum number \( \lambda \) assumes the values \( \lambda = 2L, 2L+1, \ldots \).

The radial wave function for the continuous spectrum has the form

\[
R_{k\lambda}(r) = C_{k\lambda}(\frac{2ikr}{2\lambda + 3})^{\lambda} e^{-ikr} F \left( \lambda + 2 + \frac{i}{ak}; 2\lambda + 4; 2ikr \right),
\]

where \( k = \sqrt{2\mu \hbar} \), and \( a = \hbar^{2}/\mu e^{2} \) is the Bohr radius.

The representation for the confluent hypergeometric function is

\[
F(a; c; z) = \frac{\Gamma(c)}{\Gamma(c-a)} (-z)^{-a} G(a; a-c+1; -z) + \frac{\Gamma(c)}{\Gamma(a)} z^{a-c} G(c-a; 1-a; z),
\]

where

\[
G(a; c; z) = 1 + \frac{ac}{1!z} + \frac{a(a+1)c(c+1)}{2!z^{2}} + \cdots.
\]
makes it possible to obtain the following expression for \( R_{k\lambda}(r) \):

\[
R_{k\lambda}(r) = C_{k\lambda} \frac{(-i)^\lambda}{2\pi^2 r} e^{-\pi/2a k} \Re \left\{ e^{-i[kr - \lambda(\lambda+2)+\frac{i}{2}\ln 2kr]} \Gamma \left( \lambda + 2 - \frac{i}{ak}, \lambda - 1 \right) G \left( \lambda + 2 - \frac{i}{ak}, \lambda - 1; 2ikr \right) \right\}.
\] (3.13)

The normalization constant \( C_{k\lambda} \) is

\[
C_{k\lambda} = (-i)^\lambda 4k^2 e^{\pi/2ak} \left| \Gamma \left( \lambda + 2 - \frac{i}{ak} \right) \right|.
\] (3.14)

if the normalization condition for the radial wave function is

\[
\int_0^\infty r^4 R_{k\lambda}^*(r) R_{k\lambda}(r) dr = 2\pi \delta(k - k').
\] (3.15)

If we now let \( r \to \infty \) in formula (3.13) and restrict ourselves to the first term of the expansion, we obtain the asymptotic expression

\[
R_{k\lambda}(r) \approx \frac{2}{r^2} \sin \left[ kr + \frac{1}{ak} \ln 2kr - \frac{\pi}{2}(\lambda + 1) + \delta_\lambda \right]
\] (3.16)

for the radial wave function \( R_{k\lambda}(r) \), where

\[
\delta_\lambda = \arg \Gamma \left( \lambda + 2 - \frac{i}{ak} \right).
\] (3.17)

Now, we define the parabolic coordinates in \( \mathbb{R}^3 \) as

\[
x_0 = \frac{1}{2}(\xi - \eta), \quad x_2 + ix_1 = \sqrt{\xi \eta} \sin \frac{\beta}{2} e^{i\frac{\alpha \eta}{2}}, \quad x_4 + ix_3 = \sqrt{\xi \eta} \cos \frac{\beta}{2} e^{i\frac{\alpha \eta}{2}},
\] (3.18)

where \( \xi, \eta \in [0, \infty) \). The differential elements of length and volume and Laplace operator in the terms of these coordinates can be written as

\[
dl^2 = \frac{\xi + \eta}{4} \left( \frac{d\xi^2}{\xi} + \frac{d\eta^2}{\eta} \right) + \frac{\xi \eta}{4} \left( d\beta^2 + d\alpha^2 + 2\cos \beta d\alpha d\gamma + d\gamma^2 \right)
\] (3.19)

\[
dV_5 = \frac{\xi \eta}{32} (\xi + \eta) \sin \beta d\xi d\eta d\beta d\alpha d\gamma
\] (3.20)

\[
\Delta_5 = \frac{4}{\xi + \eta} \left[ \frac{1}{\xi} \frac{\partial}{\partial \xi} \left( \xi \frac{\partial}{\partial \xi} \right) + \frac{1}{\eta} \frac{\partial}{\partial \eta} \left( \eta \frac{\partial}{\partial \eta} \right) \right] - \frac{4}{\xi \eta} \hat{L}^2
\] (3.21)

Then the equation \( \Box_{\mu\nu} \) in the parabolic coordinates \( \xi, \eta \) has the form

\[
\left\{ \frac{4}{\xi + \eta} \left[ \frac{1}{\xi} \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial}{\partial \xi} \right) + \frac{1}{\eta} \frac{\partial}{\partial \eta} \left( \eta^2 \frac{\partial}{\partial \eta} \right) - 4\hat{L}^2 \right] \right\} \psi + \frac{2\mu}{\hbar^2} \left( \varepsilon + \frac{2\varepsilon^2}{\xi + \eta} \right) \psi = 0.
\] (3.22)

After the substitution

\[
\psi (\xi, \eta, \alpha, \beta, \gamma) = \Phi_1 (\xi) \Phi_2 (\eta) D_{\alpha\beta\gamma}(\xi, \eta),
\] (3.23)

the variables in Eq. (3.22) are separated, and we arrive at the set of differential equations

\[
\frac{1}{\xi} \frac{d}{d\xi} \left( \xi^2 \frac{d\Phi_1}{d\xi} \right) + \left[ \frac{k^2}{4} \xi - \frac{L(L+1)}{\xi} + \frac{\sqrt{\Omega}}{2\hbar} + \frac{1}{2a} \right] \Phi_1 = 0
\] (3.24)

\[
\frac{1}{\eta} \frac{d}{d\eta} \left( \eta^2 \frac{d\Phi_2}{d\eta} \right) + \left[ \frac{k^2}{4} \eta - \frac{L(L+1)}{\eta} - \frac{\sqrt{\Omega}}{2\hbar} + \frac{1}{2a} \right] \Phi_2 = 0.
\] (3.24)
where $\Omega$ is the parabolic separation constant. The function $\psi(\xi, \eta, \alpha, \beta, \gamma)$ normalized by the condition

$$\frac{1}{4} \int \psi^*_{\lambda \lambda L_m m'} \psi_{\lambda \lambda L_m m'} \xi \eta d\xi d\eta = 2\pi \delta(k - k') \delta(\Omega - \Omega') \delta_{L L'} \delta_{m m'} \delta_{m' m'},$$

leads to the parabolic basis

$$\psi_{\lambda \lambda L_m m'}(\xi, \eta, \alpha, \beta, \gamma) = \sqrt{\frac{2L + 1}{2\pi^2}} \kappa_{\lambda \lambda L} \Phi_{\lambda \lambda L}(\xi) \Phi_{\lambda - \Omega L}(\eta) D_{m m'}(\alpha, \beta, \gamma),$$

where

$$\Phi_{\lambda \lambda L}(x) = \frac{(ikx)^L}{(2L + 1)!} e^{-i k x/2} \Gamma(L + 1 + i \frac{\sqrt{\mu}}{2\hbar k} \Omega; 2L + 2; ikx),$$

$$C_{\lambda \lambda L} = (-1)^L \sqrt{\frac{\hbar^2 k^3}{2\pi \mu}} e^{\frac{\pi \mu}{4\hbar^2}} \frac{\Gamma(L + 1 - i \frac{\sqrt{\mu}}{2\hbar k} \Omega; \Omega + 1)}{\hbar^2} \frac{\Gamma(L + 1 + i \frac{\sqrt{\mu}}{2\hbar k} \Omega)}{\hbar^2}.$$ We calculate the normalization constant $C_{\lambda \lambda L}$ using the representation (3.11).

## 4 The 5-dimensional generalization of the Rutherford’s formula

We now consider the scattering problem of a charged particle in the five-dimensional Coulomb field. Since the motion in Coulomb field of arbitrary dimensions $d \geq 3$ is two-dimensional problem, the wave function is independent of the angles $\alpha$, $\beta$ and $\gamma$, i.e. independent of the quantum numbers $L$, $m$ and $m'$. Substituting $L = 0$ in Eqs. (3.26), we obtain

$$\frac{1}{\xi} \frac{d}{d\xi} \left( \xi^2 \frac{d\Phi_1}{d\xi} \right) + \left[ \frac{k^2}{4} \xi + \sqrt{\mu} + \frac{1}{2\alpha} \right] \Phi_1 = 0$$

$$\frac{1}{\eta} \frac{d}{d\eta} \left( \eta^2 \frac{d\Phi_2}{d\eta} \right) + \left[ \frac{k^2}{4} \eta - \frac{\sqrt{\mu}}{2\hbar k} \Omega \right] \Phi_2 = 0,$$

We seek such solutions of Eqs. (4.2) that the solution of the Schrödinger equation for negative $x_0 \in (-\infty; 0)$ and large $r \to \infty$ has the form of a flat wave:

$$\psi_{\lambda \lambda L}(\xi, \eta) \sim e^{ikx_0} e^{\frac{\pi}{4\hbar} \Omega}.$$ This condition can be satisfied if we set the parabolic separation constant equal to

$$\Omega = -\frac{\hbar}{a\sqrt{\mu}} - i\frac{2\hbar k}{\sqrt{\mu}}.$$ Substituting this relation in Eqs. (4.2), we obtain a solution of the Schrödinger equation that describes the scattering of a charged particle in the five-dimensional Coulomb field:

$$\psi_k(\xi, \eta) = k e^{ik(\xi - \eta)} F\left(\frac{iak}{2}; ik\eta\right),$$

where $C_k$ is the normalization constant. To separate the incident and scattered waves in function (4.4), we must investigate the behavior of this function at large distances from the scattering center. Using the first two terms in representation (3.11) for a confluent hypergeometric function, we obtain

$$F\left(\frac{iak}{2}; ik\eta\right) \approx e^{-\frac{2\hbar k}{\sqrt{\mu}}} \frac{\Gamma(2 - \frac{1}{ak})}{\Gamma(2 + \frac{1}{ak})} a^2 k^2 e^{\frac{2\hbar k}{\sqrt{\mu}}}.$$
for the large $\eta$. We now substitute this relation in wave function 4.4) and select the normalization constant $C_k$ in the form

$$C_k = e^{\pi/2ak} \Gamma \left( 2 - \frac{i}{ak} \right)$$

(4.6)

for the incident wave to have a unit amplitude. Using the formulae $r = (\xi + \eta)/2$ and $\eta = r - x_0 = r(1 - \cos \theta)$ to change to spherical coordinates, we obtain

$$\psi_k(\xi, \eta) = \left[ 1 + \frac{ak - i}{2a^2 k^3 r \sin^2 \theta/2} \right] \exp \left[ i k x_0 - \frac{i}{ak} \ln \left( 2 k r \frac{\sin \theta}{2} \right) \right] +$$

$$+ \frac{f(\theta)}{r^2} \exp \left[ i k r + \frac{i}{ak} \ln (2 k r) \right],$$

(4.7)

(4.8)

where $f(\theta)$ is the scattering amplitude,

$$f(\theta) = \frac{(1 - i ak)}{4a^2 k^4 \sin^2 \theta/2} \frac{\Gamma(2 - i/ak)}{\Gamma(2 + i/ak)} \exp \left( \frac{2i}{ak} \ln \sin \frac{\theta}{2} \right).$$

(4.9)

Therefore, for the scattering cross section $d\sigma = |f(\theta)|^2 d\Omega$ ($d\Omega$ is the element of the solid angle), we obtain the formula

$$d\sigma = \frac{1 + a^2 k^2}{16a^2 k^8 \sin^8 \theta/2} d\Omega,$$

(4.10)

which generalizes the Rutherford’s formula in five-dimensional case.

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