Multiloop contributions to the $\overline{\text{MS}}$-on-shell mass relation for heavy quarks in QCD and charged leptons in QED and the asymptotic structure of the perturbative QCD series

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Abstract: The dependence on the numbers of flavors of the QCD corrections to the ratio between pole and $\overline{\text{MS}}$-scheme running masses of heavy quarks is fixed for the $\mathcal{O}(a_s^4)$-correction by the least squares method and estimated for the $\mathcal{O}(a_s^5)$ and $\mathcal{O}(a_s^6)$ contributions by means of the effective charges motivated approach and independently by the renormalon-based analysis. The application of the least squares method allows to determine the central values of the constant and linearly dependent on the number of flavors coefficients of the corresponding $a_s^4$ term and their theoretical uncertainties. The four-loop QED approximations for ratios of pole and $\overline{\text{MS}}$-scheme running masses of three charged leptons are determined and commented. The results of our researches indicate that the asymptotic nature of the perturbative QCD relation between pole and $\overline{\text{MS}}$-scheme running mass of $t$-quark starts to manifest itself from 7 order of PT. Therefore the concept of the pole mass of the top-quark can be safely used even at the $\mathcal{O}(a_s^6)$ level of PT. The impact of the typical to the Minkowskian region contributions, proportional to powers of $\pi^2$-terms, is discussed.

Keywords: Perturbative QCD, Heavy Quark Physics, Renormalization Group
1 Introduction

It is well known that the bare unrenormalized masses of quarks in QCD are related to the renormalized finite quarks masses, defined in a particular renormalization scheme. In this work we consider two renormalization schemes, namely the $\overline{\text{MS}}$-scheme and the on-shell (OS) scheme, which is used for defining pole masses of heavy quarks, namely for the charm, bottom and top-quarks. The relevant renormalization prescriptions for these quarks have the following form:

$$m_{0,q} = Z_\overline{\text{MS}} m_q,$$

$$m_{0,q} = Z_\text{OS} M_q,$$
where $m_{0,q}$, $m_q$, $M_q$ are the bare, running in $\overline{\text{MS}}$-scheme and pole masses of the heavy quarks respectively. The renormalization mass constants $Z_{m_{0}}^{\overline{\text{MS}}}$ and $Z_{m}^{\text{OS}}$ contain ultraviolet divergences and are represented by a perturbation theory (PT) series in the coupling constants of the strong interaction, defined in the corresponding renormalization schemes and depending on the renormalization scale parameter $\mu$.

Due to the fact that the renormalized masses $m_q(\mu^2)$ and $M_q(\mu^2)$ are the finite physical quantities, which can be extracted from the experimental data, their ratio must also be finite. It is convenient to introduce the following relation between $\overline{\text{MS}}$-scheme running and pole masses of heavy quarks, also called in the literature as $\overline{\text{MS}}$-on-shell mass relation, namely:

$$z_m(\mu^2) = \frac{m_q(\mu^2)}{M_q} = 1 + \sum_{i=1}^{\infty} z_m^{(i)} a_s(\mu^2)$$

with the strong coupling constant $a_s = \alpha_s/\pi$, defined in the $\overline{\text{MS}}$-scheme. The history of calculating the coefficients $z_m^{(i)}$ is rather long and does not stop so far. The one-loop term $z_m^{(1)}$ was calculated a long time ago in Ref.[1]. The two-loop correction $z_m^{(2)}$ was analytically computed in Ref.[2] and confirmed later in Refs.[3,4]. The $O(a_s^3)$ contribution was evaluated independently by analytical [5] and semi-analytical [6] methods.

Consider the four-loop term $z_m^{(4)}$ separately. Any-order term $z_m^{(i)}$ can be expanded in powers of the number of quarks which are lighter than the considered single heavy quark (in this approximation we assume that the one heavy quark is massive and the rest $n_l = n_f - 1$ are massless, where $n_f$ is the flavor number of the active quarks):

$$z_m^{(i)} = \sum_{j=0}^{i-1} z_m^{(i,j)} n_l^j.$$  \hspace{1cm} (1.4)

In particular, the coefficient of the $O(a_s^4)$-correction has the following form:

$$z_m^{(4)} = z_m^{(43)} n_f^3 + z_m^{(42)} n_f^2 + z_m^{(41)} n_f + z_m^{(40)}.$$ \hspace{1cm} (1.5)

The first two coefficients $z_m^{(43)}$ and $z_m^{(42)}$ in Eq.(1.5) were analytically calculated in Ref.[7]. The last two terms, namely the term with linear dependence on the number of massless quarks $z_m^{(41)}$ and the constant contribution $z_m^{(40)}$, are not yet computed in analytical form. However, work on the calculation of these two coefficients is underway and there may be progress in these computations. In Ref.[8] the initial semi-analytical evaluation of the overall value of $z_m^{(4)}$ coefficient at fixed number of flavors with $n_l = 3, 4, 5$ was performed. In the case of top-quark it contained a numerical uncertainty of about $\sim 2.5\%$. This inaccuracy was related to the impossibility of the direct analytical calculations of several concrete propagator master integrals. In order to get the unknown coefficients $z_m^{(40)}$ and $z_m^{(41)}$ from the numerical results of Ref.[8], which were obtained for three fixed values of $n_l$, in Ref.[9] the method of the least squares (LSM) was used. This approach is widely used for solving overdetermined systems of equations and allows to determine the uncertainties of the solutions of these systems. However, since the results, presented in Ref.[9], were obtained using only three numerical values for $z_m^{(4)}$-term, given in Ref.[8] for charm, bottom
and top-quarks, the inaccuracies of application of the least squares method turned out to be not small.

There are also independent estimations of the coefficients $z_{m}^{(4)}$ and $z_{m}^{(41)}$, which were obtained in Ref.[10] using the special fitting procedure and the renormalon-based analysis, first considered in the case of study the asymptotic structure of the relation between pole and $\overline{\text{MS}}$-scheme running mass in the works [11–14]. Unexpectedly these values turned out to be almost identical to the central values of the results, given in Ref.[9]. Note, however, that the uncertainties of the applied in Ref.[10] procedure were not clearly specified.

At the next step of the story the evaluation of the $z_{m}^{(4)}$-coefficient was done in work [15] with higher precision than in Ref.[8] for a much larger number of massless flavors in the interval $0 \leq n_l \leq 20$. As the result for the cases of $c$, $b$ and $t$-quarks the following values of $z_{m}^{(4)}$-coefficient were obtained $-1756.36 \pm 1.74$, $-1278.70 \pm 1.77$, $-871.73 \pm 1.80$. These results should be compared with the values of the same coefficients $-1744.8 \pm 21.5$, $-1267.0 \pm 21.5$, $-859.96 \pm 21.5$, obtained previously in Ref.[8]. Undoubtedly, an increase in statistics by a factor 7 and a reduction in errors of 12 times should lead to more accurate values of $z_{m}^{(40)}$ and $z_{m}^{(41)}$ terms in Eq.(1.5), than the ones obtained with the help of the LSM in Ref.[9]. Indeed, this fact was observed in Ref.[15] from the analysis of the results of direct calculations and in the “Note added” of Ref.[9], where combination of the LSM and the results, available from the preprint version of the work [15] was used. It should emphasized, that the central values of $z_{m}^{(40)}$ and $z_{m}^{(41)}$-terms, obtained in Ref.[9] by means of the LSM, depend weakly on the statistics with large number of $n_l$ and almost coincide with the numerical expressions, presented in Ref.[15] from the direct semi-analytical calculations. In Sec.2 of this paper the details of application of the LSM to the results of Ref.[15] for the determination of $z_{m}^{(40)}$ and $z_{m}^{(41)}$-terms with their theoretical uncertainties are described.

Despite the apparent smallness of the four-loop corrections to the relation between pole and running masses of heavy quarks, its knowledge is very important from both phenomenological and theoretical points of view. Indeed, in view of the asymptotic nature of the perturbative relation between pole and $\overline{\text{MS}}$-scheme running masses, which is governed by the dominant infrared renormalon contributions to this relation, discovered and discussed in the works of Refs.[11] and [12–14], it is necessary to fix the order of manifestation of this asymptotic behavior in the concrete cases when the charm, bottom and top-quark masses are considered. Moreover, it is clear that higher order perturbative QCD effects affect the available experimentally motivated results for the bottom and top-quark masses.

In the case of $b$-quark the values of $\overline{\text{MS}}$-scheme mass $m_{b}(m_{b}^2)$ were obtained at the N$^3$LO level as the final results of the QCD analysis of the properties of $\Upsilon$ system (see e.g. [16–19]) and of the production cross-section of the $b\bar{b}$-quarks in the $e^+e^-$ collisions [20]. Also worth mentioning the recent results of the four-flavor lattice QCD determination of the $m_{b}(m_{b}^2)$ mass [21]. These QCD lattice results are stimulating more careful study of the existing uncertainties of the four-loop QCD relation between pole and running $b$-quark masses.

In the case of $t$-quark mass the situation is even more intriguing. Indeed, the results of experimental analysis of different Tevatron and LHC data are expressed through so-called Monte-Carlo top-quark mass, which is closely related (though with investigated
process-dependent uncertainties) with to the concept of $t$-quark pole mass. In spite of the appearance of new LHC measurements and the updated Tevatron determinations of this important quantity, the presented in 2018 issue of PDG report [22] average value remained almost the same as in 2014 [23], namely $M_t = 173.34 \pm 0.27^{(\text{stat})} \pm 0.71^{(\text{syst})}$ GeV. The non-inclusion of the new available Tevatron and LHC data on the determination of the average value of $t$-quark mass may be related to the fact that the problem of the plausible relation of the Monte-Carlo top-quark mass, extracted from the experimental data, with its pole mass is still opened. No less important question is what are the real inaccuracies of different Monte-Carlo programs, used for determination of top-quark mass from the processes, which were studied at Tevatron and are investigated at LHC. These tasks are still in the process of careful investigations (see [24, 25]). Among the effects, which are under considerations, the issue of the theoretical uncertainties of the relations between pole and $\overline{\text{MS}}$-scheme mass $\overline{m}_t(\overline{m}_t^2)$ plays a decisive role at the $\mathcal{O}(a_s^4)$ level [26, 27] and beyond [25] and will be also considered in this work.

In Sec.3 we analyze the numerical relations between pole and $\overline{\text{MS}}$-scheme running heavy quark masses at the $\mathcal{O}(a_s^4)$ level using the obtained in Sec.2 LSM-results and taking into account the concrete theoretical uncertainties.

Sec.4, which is lying a bit far from the main stream of this work, is devoted to the consideration of the four-loop relations between pole and $\overline{\text{MS}}$-scheme running masses of charged leptons in QED and to the study of the structure of the corresponding perturbative series.

Sec.5 is dedicated to the estimations of the multiloop contributions to the $\overline{\text{MS}}$-on-shell heavy quark mass relation in the orders of perturbation theory greater than four. We evaluate the $\mathcal{O}(a_s^5)$ and $\mathcal{O}(a_s^6)$-corrections and restore their $n_l$-dependence with the help of the extended to the massive-dependent case [28] approach, proposed in Ref.[29]. It is based on the ideas of the effective-charges (ECH) method [30] and on the introduced in Ref.[31] concepts of scheme-invariants. This method allows to probe the asymptotic structure of the corresponding perturbative series and get theoretical information on the possible values of high-order corrections to the relation between pole and running heavy quark masses.

Further we consider results of the infrared renormalon studies of Refs.[12, 13], [32] and find the corresponding values of multiloop contributions to the mass conversion formula within the framework of this approach. Then we compare these results with the ones, obtained by means of the ECH-based analysis. It is demonstrated that the renormalon approach gives faster increase of the coefficients of the corresponding PT approximations for the studied relations between on-shell and running heavy quark masses. After taking into account results of the renormalon-based analysis of Ref.[32] for granted we estimate the values of the high-order perturbative corrections to the $t$-quark pole mass and state that the asymptotic nature of the relation between pole and running $t$-quark masses is manifesting itself starting from rather high level of PT, namely from the 7-th order. This means, that in the existing at present analysis of the Tevatron and LHC experimental data the definition of the $t$-quark pole mass may be safely used.
2 Application of the least squares method

Now we focus on getting the numerical values of two coefficients $z_m^{(41)}$ and $z_m^{(40)}$ in expression (1.5), which have not yet been computed analytically. In Ref.[9] we already obtained these two numbers applying the LSM to the results of semi-analytical calculations of the $O(a_s^4)$-correction to the relation between pole and running heavy quark masses, performed in Ref.[8] for the charm, bottom and top quarks only. Recently the more detailed and more precise numerical information for the values of $z_m^{(4)}$-term became available for a wider region of flavors including nonphysical ones, namely at fixed numbers of massless flavors $n_l$ in the interval $0 \leq n_l \leq 20$ [15]. In this work the coefficient $z_m^{(40)}$ was also determined as the numerical value of the whole $z_m^{(4)}$-contribution at $n_l=0$ [11]. However, such procedure for definition of the $z_m^{(40)}$-coefficient does not take into account the correlation effects, arising when the remaining equations with $1 \leq n_l \leq 20$ are also considered. In order to not to lose the influence of these correlation effects on the numerical expressions of $z_m^{(40)}$ and $z_m^{(41)}$-terms we use the mathematical least squares method (LSM). To link its application with the concrete theoretical problem, which appears within the perturbative QCD approach, we add extra physically-motivated input, namely restrict ourselves by the consideration of $n_l$ from the interval $3 \leq n_l \leq 15$, where the lower bound is fixed by us keeping in mind that we analyze the behavior of perturbative series for the relation between pole and running masses of heavy quarks, while the upper bound is following from the Banks-Zaks ansatz $n_l<31/2$ [33], which insures that in the considered interval of integer values of $n_l$ the QCD asymptotic freedom property is not violated.

Taking these circumstances into account, using analytical expressions for $z_m^{(43)}$ and $z_m^{(42)}$-terms, derived in Ref.[7], and keeping in mind the results of the four-loop numerical calculations of $z_m^{(4)}$-term at fixed number of massless quarks, performed in Ref.[15], we obtain the following overdetermined system of linear equations with two unknown quantities $z_m^{(40)}$ and $z_m^{(41)}$, normalized at the $\mu^2 = M_q^2$ renormalization point:

\[
\begin{pmatrix}
1 & 3 \\
1 & 4 \\
1 & 5 \\
1 & 6 \\
1 & 7 \\
1 & 8 \\
1 & 9 \\
1 & 10 \\
1 & 11 \\
1 & 12 \\
1 & 13 \\
1 & 14 \\
1 & 15
\end{pmatrix}
\begin{pmatrix}
z_m^{(40)} \\
z_m^{(41)}
\end{pmatrix} =
\begin{pmatrix}
-1383.33 \pm 1.74 \\
-626.38 \pm 1.77 \\
130.56 \pm 1.80 \\
887.50 \pm 1.84 \\
1644.45 \pm 1.87 \\
2401.39 \pm 1.91 \\
3158.33 \pm 1.94 \\
3915.27 \pm 1.98 \\
4672.22 \pm 2.01 \\
5429.15 \pm 2.05 \\
6186.09 \pm 2.08 \\
6943.03 \pm 2.12 \\
7699.98 \pm 2.16
\end{pmatrix}
\] (2.1)

Note that its central value is very close to the one, obtained in Ref.[9] with the help of the LSM, applied to less precise results of [8].
Note that the mean-square uncertainties on the right-hand side of the system (2.1), obtained in Ref. [15], arise when the corresponding master integrals, which contribute to the $O(a_s^4)$ correction of the ratio of Eq. (1.3), are computed using the Monte-Carlo methods. They are approximately 10 times smaller than the ones, presented previously in Ref. [8] for three values of $n_l = 3, 4, 5$ only. The reason for such sharp fall of the mean-square errors in the numerical calculations lies in a significant growth in the amount of sampling points, required for the evaluation by means of the Monte-Carlo methods. Moreover, these inaccuracies are not constants as it was given in the results of Ref. [8] for $n_l = 3, 4, 5$, but depend on the number of massless flavors. Therefore theoretical uncertainties, defined by the LSM, should also decrease in comparison with the case of Ref. [9] where equations for three flavors were used only. The rate of decline is proportional to $1/\sqrt{N}$, where $N$ is determined by the number of equations of system (2.1). It is important to find out to what changes the increase in statistics and the decrease of the mean-square errors of the four-loop numerical calculations, performed in Ref. [15], will lead for the determination of the $z_m^{(40)}$ and $z_m^{(41)}$-terms by means of the LSM method, previously used in Ref. [9].

To apply the LSM for solving the system of Eq. (2.1) one should first introduce the $\Phi$-function, which is equal to the sum of the squares of the deviations of all equations in this system:

$$\Phi(z_m^{(40)}, z_m^{(41)}) = \sum_{k=1}^{13} (z_m^{(40)} + z_m^{(41)} n_{l_k} - y_{l_k})^2, \quad (2.2)$$

where index $k$ runs through all values equal to the number of equations of the system (2.1), $y_{l_k}$ are the numbers presented on the r.h.s. of this system with their uncertainties $\Delta y_{l_k}$.

The solutions of the overdetermined system (2.1) are equal to values of the $z_m^{(40)}$ and $z_m^{(41)}$-terms, for which the function $\Phi(z_m^{(40)}, z_m^{(41)})$ has the minimum defined by the following requirements:

$$\frac{\partial \Phi}{\partial z_m^{(40)}} = 0, \quad \frac{\partial \Phi}{\partial z_m^{(41)}} = 0. \quad (2.3)$$

The LSM uncertainties of the solutions $z_m^{(40)}$ and $z_m^{(41)}$, obtained from conditions (2.3), are fixed as:

$$\Delta z_m^{(40)} = \sqrt{\sum_{k=1}^{13} \left( \frac{\partial z_m^{(40)}}{\partial y_{l_k}} \Delta y_{l_k} \right)^2} = \frac{1}{13 \sum_{k=1}^{13} n_{l_k}^2 - \left( \sum_{k=1}^{13} n_{l_k} \right)^2} \sqrt{\sum_{k=1}^{13} \Delta y_{l_k}^2 \left( \sum_{i=1}^{13} n_{l_i}^2 - \left( \sum_{i=1}^{13} n_{l_i} \right) \right)^2},$$

$$\Delta z_m^{(41)} = \sqrt{\sum_{k=1}^{13} \left( \frac{\partial z_m^{(41)}}{\partial y_{l_k}} \Delta y_{l_k} \right)^2} = \frac{1}{13 \sum_{k=1}^{13} n_{l_k}^2 - \left( \sum_{k=1}^{13} n_{l_k} \right)^2} \sqrt{\sum_{k=1}^{13} \Delta y_{l_k}^2 \left( 13 n_{l_k} - \sum_{i=1}^{13} n_{l_i} \right)^2}. \quad (2.4)$$

Applying formulas (2.3) and (2.4) we get the numerical values for the constant and linearly dependent on $n_l$ contributions to the four-loop contribution $z_m^{(4)}$ into the studied ratio of different definitions of heavy quark masses with their theoretical uncertainties, viz:

$$z_m^{(40)}(M^2_q) = -3654.14 \pm 1.34, \quad z_m^{(41)}(M^2_q) = 756.94 \pm 0.15. \quad (2.5a)$$
These values should be compared with the results, obtained in Ref.[15]:

\[
z_m^{(40)}(M_q^2) = -3654.15 \pm 1.64 \ , \quad z_m^{(41)}(M_q^2) = 756.942 \pm 0.040 . \tag{2.5b}
\]

Note that although the central values of the results (2.5a) and (2.5b) are derived by the different methods, they coincide. It should be stressed, that the result (2.5b) for \( z_m^{(40)} \)-term was obtained as the value of the four-loop contribution \( z_m^{(4)} \) at \( n_l = 0 \) and it did not take into account the correlation effects with the remaining equations for nonzero values of \( n_l \), whereas the result (2.5a), computed by the LSM, is obtained for \( 3 \leq n_l \leq 15 \) values and therefore takes into consideration these effects and does not rely on the value \( z_m^{(4)} \), obtained at \( n_l = 0 \). As the result the uncertainty of \( z_m^{(40)} \)-term in Eq.(2.5a) decreases, though only slightly.

Let us also remind that similar expressions were extracted previously in Ref.[10] from the related to \( n_l = 3,4,5 \) results of calculations of \( z_m^{(4)} \)-terms [8] with the help of the renormalon-based asymptotic formula, taken from Refs.[12, 13], and the special fitting procedure. These results read:\(^2\)

\[
z_m^{(40)}(M_q^2) = -3643.0 \pm 21.5 \ , \quad z_m^{(41)}(M_q^2) = 757.07 \ , \tag{2.5c}
\]

where the uncertainty of \( z_m^{(40)} \)-term is taken from the errors of the used in this work results of Ref.[8]. It is also worth reminding that the application of the LSM to the outcomes of computer calculations of Ref.[8] for three concrete values of \( n_l = 3,4,5 \) gives almost identical result for central values of \( z_m^{(40)} \) and \( z_m^{(41)} \)-terms and can be found in Ref.[9]

\[
z_m^{(40)}(M_q^2) = -3642.9 \pm 62.0 \ , \quad z_m^{(41)}(M_q^2) = 757.05 \pm 15.20 , \tag{2.5d}
\]

though with larger theoretical uncertainties. Thus we conclude that the application of the mathematical LSM gives the arguments in favor of the validity of the originally developed in Refs.[11–13] renormalon-based technique for the analysis of the asymptotic structure of the perturbative relation between \( \overline{\text{MS}} \)-scheme running and pole heavy quark masses.

In order to study whether the LSM solutions for the considered overdetermined system of linear equations are sensitive to the number of equations taken into account, we repeat the considerations of Ref.[9], changing the input of these studies from the numerical expressions \(-1744.8 \pm 21.5, -1267.0 \pm 21.5 \) and \(-859.96 \pm 21.5 \) for the overall values of the four-loop corrections to these relations, which were obtained for the cases of \( n_l = 3,4,5 \) numbers of flavors in Ref.[8], to their more precise analogs \(-1756.36 \pm 1.74, -1278.70 \pm 1.77, -871.73 \pm 1.80 \), obtained later on in Ref.[15] together with results for a wider region \( 0 \leq n_l \leq 20 \). The LSM solutions of the system of three correspondingly modified equations are:

\[
z_m^{(40)}(M_q^2) = -3654.16 \pm 5.08 \ , \quad z_m^{(41)}(M_q^2) = 756.95 \pm 1.25 . \tag{2.5e}
\]

Comparing them with the ones, given in Eq.(2.5a), one can conclude that the the central values of these two coefficients are practically indistinguishable, but the errors in Eq.(2.5e)
are much larger than their analogies in Eq. (2.5a). Thus we conclude that the number of equations of the system (2.1) strongly affects the final uncertainty of the solutions for $z_m^{(41)}$ and $z_m^{(40)}$-terms, obtained by the LSM, and does not change the central values of these results. Note also, that the decrease of the considerable uncertainties, presented in Eq. (2.5d) and obtained in our previous LSM results [9], are explained by the increase of the number of equations in system (2.1) and the drastic reduction of the errors in the results of Ref. [15] in comparison to the ones, given previously in Ref. [8].

3 Heavy quark pole masses: from the relation with $\overline{\text{MS}}$-scheme running masses to the numerical representation of their four-loop expressions

Now we turn to the numerical expressions for the four-loop relations between pole and running heavy quark masses. Substituting the results of Eq. (2.5a) into the expansion (1.3) one can get the following numerical perturbative representation for masses of heavy quarks at the renormalization point $\mu^2 = M_q^2$:

$$
\overline{m}_q(M_q^2) \approx M_q(1 - \alpha_s(M_q^2)/\pi) + (0.6781n_l^3 - 43.482n_l^2 + (756.94 \pm 0.15)n_l - 3654.14 \pm 1.34)\alpha_s^4.
$$

(3.1)

where $\alpha_s = \alpha_s(M_q^2)/\pi$. Using the solutions of the corresponding renormalization group (RG) equations (see Ref. [9] for the details), one can obtain the following four-loop expansion for the pole mass of heavy quarks when the scale parameter is fixed as $\mu^2 = \overline{m}_q^2$:

$$
M_q \approx \overline{m}_q(\overline{m}_q^2)(1 + 1.3333\pi_s + (-1.0441n_l + 13.443)\pi_s^2 + (0.6527n_l^2 - 26.655n_l + 190.60)\pi_s^3
+ (-0.6781n_l^3 + 43.396n_l^2 + (-745.72 \pm 0.15)n_l + 3567.60 \pm 1.34)\pi_s^4)
$$

(3.2)

where $\pi_s = \alpha_s(\overline{m}_q^2)/\pi$. This normalization condition has simple geometrical interpretation as a point of intersection of the logarithmically decreasing curve for the scale-dependence of the $\overline{\text{MS}}$-scheme running mass with the bisector of the angle in the coordinates $\overline{m}_q(\mu^2)$ on $\mu^2$. In particular, equation (3.2) for pole masses of $c$, $b$ and $t$-quarks leads to the following expansions:

$$
M_c \approx \overline{m}_c(\overline{m}_c^2)(1 + 1.3333 \pi_s + 10.318 \pi_s^2 + 116.49 \pi_s^3 + (1702.70 \pm 1.41) \pi_s^4)
$$

(3.3a)

$$
M_b \approx \overline{m}_b(\overline{m}_b^2)(1 + 1.3333 \pi_s + 9.277 \pi_s^2 + 94.41 \pi_s^3 + (1235.66 \pm 1.47) \pi_s^4)
$$

(3.3b)

$$
M_t \approx \overline{m}_t(\overline{m}_t^2)(1 + 1.3333 \pi_s + 8.236 \pi_s^2 + 73.63 \pi_s^3 + (839.14 \pm 1.54) \pi_s^4)
$$

(3.3c)

The results of expressions (3.3a-3.3c) demonstrate the property of the asymptotic structure of the perturbative QCD series. Indeed, one can see that all relations contain significantly growing and strictly sign-constant coefficients. Comparing these results with the ones, presented in Ref. [9], we conclude that the obtained in that work uncertainties of the four-loop corrections for the pole masses of heavy quarks are reduced 60 times approximately. With regard to the results of Ref. [8] this reduction factor is approximately equal to 15.

For numerical studies we use the following average PDG(18) values of the running masses of $c$ and $b$-quarks, namely $\overline{m}_c(\overline{m}_c^2) = 1.275^{+0.025}_{-0.030}$ GeV, $\overline{m}_b(\overline{m}_b^2) = 4.180^{+0.040}_{-0.030}$ GeV.
In accordance with the results of Ref.\cite{34} obtained from the LHC \( t\bar{t} \) experimental data and given in Ref.\cite{26}, for top quark we assume \( \overline{m}_t(m_t^2) = 164.3 \pm 0.6 \text{ GeV} \), that does not contradict the presented in PDG(18) values of the running \( t \)-quark mass. As the initial normalization point we take the average value of the strong coupling constant normalized at the mass of \( Z \)-boson \( \alpha_s(M_Z^2) = 0.1181(11) \) at \( M_Z = 91.1876(21) \text{ GeV} \) from PDG(18). Thence for the \( b \)-quark we obtain the following value of the scale parameter \( \Lambda^{(n_l=4)}_{\overline{MS}} = 210 \text{ MeV} \) in the N\(^3\)LO approximation for the inverse log representation for \( \alpha_s(M_Z^2) \). The numerical values of \( \Lambda^{(n_l=3)}_{\overline{MS}}, \Lambda^{(n_l=5)}_{\overline{MS}} \) are obtained using the corresponding N\(^3\)LO matching transformation conditions, derived in Ref.\cite{35, 36} where the matching scales are fixed by the values of the presented above numbers for the running heavy quark masses. Using the corresponding four-loop N\(^3\)LO approximation for \( \alpha_s \), normalized at the \( \overline{MS} \) masses, we find:

\[
\begin{align*}
\Lambda^{(n_l=3)}_{\overline{MS}} &= 292 \text{ MeV} , \quad \alpha_s(\overline{m}_c^2) = 0.3947 , \\
\Lambda^{(n_l=4)}_{\overline{MS}} &= 210 \text{ MeV} , \quad \alpha_s(\overline{m}_b^2) = 0.2256 , \\
\Lambda^{(n_l=5)}_{\overline{MS}} &= 89 \text{ MeV} , \quad \alpha_s(\overline{m}_t^2) = 0.1085 .
\end{align*}
\]

Note, that these numerical expressions are in agreement with the ones, given in \cite{22}. Using results (3.3a-3.3c) and (3.4a-3.4c), we obtain the following numerical expressions for pole masses of heavy quarks:

\[
\begin{align*}
\frac{M_c}{1 \text{ GeV}} &\approx (1.275 + 0.214 + 0.208 + 0.295 + 0.541)^{+0.033}_{-0.047} , \\
\frac{M_b}{1 \text{ GeV}} &\approx (4.180 + 0.400 + 0.200 + 0.146 + 0.137)^{+0.048}_{-0.036} , \\
\frac{M_t}{1 \text{ GeV}} &\approx (164.300 + 7.566 + 1.614 + 0.498 + 0.196) \pm 0.636 ,
\end{align*}
\]

where the estimates of the mean-square errors \( \sigma_{M_q} \) for pole masses of charm, bottom and top-quarks are made within the formula

\[
\sigma^2_{M_q} = \left( \frac{\partial M_q}{\partial \overline{m}_q} \right)^2 \sigma^2_{\overline{m}_q} + \left( \frac{\partial M_q}{\partial \alpha_s} \right)^2 \sigma^2_{\alpha_s} + \left( \frac{\partial M_q}{\partial z_m^{(4)}} \right)^2 \sigma^2_{z_m^{(4)}} .
\]

In expression (3.6a) the main contribution comes from the first term, which is determined by the measurement uncertainty of the running heavy quark masses. The second and third terms, responsible for inaccuracies in the coupling constant and four-loop \( z_m^{(4)} \)-expression, affect the change in the fourth decimal place only and therefore we can neglect them. Taking this into account we obtain the following approximate equality:

\[
\sigma_{M_q} \simeq \frac{M_q}{\overline{m}_q(m_q^2)} \sigma_{m_q} .
\]

From Eq.(3.5a) follows that the asymptotic nature of PT series for charm quark is manifesting itself from the second (or the third) order of PT. Therefore it is impossible to fix the

\footnote{The last work \cite{36} contains N\(^4\)LO matching transformation conditions.}
value of the pole mass of $c$-quark, beginning with the third order. For bottom quark the $O(a_s^3)$ and $O(a_s^4)$ corrections are comparable and give close contributions. These features demonstrate that the asymptotic behavior of the corresponding PT QCD series for $c$ and $b$ quark pole masses are manifesting themselves rather early. In connection with this it is interesting to find out from which order of the PT the whole numerical contributions to the bottom quark pole mass will start to grow up. This question will be studied in Sec.5.

In the $O(a_s^4)$ approximation the asymptotic nature of the PT series for the pole mass of $t$-quark does not yet manifest itself explicitly. Indeed, all subsequent corrections decrease, although rather slowly. The four-loop contribution in Eq.(3.5c) is not negligible and is comparable with modern uncertainties of the measured top-quark pole mass, which are of order $500 - 800$ MeV and are mainly determined by systematical inaccuracies of the concrete experimental results [22, 23, 37].

On the ground of the above-mentioned findings about the manifestation of asymptotic nature in the corresponding PT series for pole masses of heavy quarks we use the ratio $M_q/m_q(m_q^2)$ in Eq.(3.6b) up to two-loop level for case of charm quark and up to four-loop level for $b$ and $t$-quarks respectively.

4 The $\overline{\text{MS}}$-on-shell mass relation in QED at the four-loop level

Let us now consider various definitions of masses for charged leptons in QED. In the case of electron and muon the directly measurable parameters are their pole masses. In spite of the fact that the heavy $\tau$-lepton is decaying rather fast, one can also introduce as its main characteristic the pole mass as well and extract it from the corresponding experimental data for the threshold behavior of the $\tau^+\tau^-$ total cross-section production in $e^+e^-$ collisions [38]. However, like in the case of quarks, it is also possible to define the $\overline{\text{MS}}$-scheme running masses of charged leptons, which may be also used in the analysis of the experimental data [39]. In view of this it is useful to determine the relation between pole and running masses of charged leptons.

For this purpose we use the information about color structures of the four-loop $z_m^{(4)}$-term in QCD, presented in Ref.[15], where it is demonstrated that this term is decomposed into 23 color structures with $n_l$ massless and $n_h$ massive flavors (remind that in our work we consider only one heavy quark or lepton and in the notations of Ref.[15] we mean $n_h = 1$). The QCD results, proportional to the quadratic Casimir operator $C_A$ in adjoint representation of Lie algebra of the $SU(N_c)$ group and to the totally symmetric under permutation of indices tensor $\delta_{ab}^{bcd}$ in adjoint representation, do not contribute in the corresponding relations in QED. Therefore from the mentioned 23 terms in QCD of $z_m^{(4)}$-term, only 12 survive in QED. In its turn, from these 12 contributions 3 are known in analytical form [7] and 9, included in $n_l^0$ and $n_l$-dependent contributions to the four-loop $z_m^{(4)}$-correction, are derived numerically in Ref.[15]. One should note that from these 9 coefficients, yet unknown in the analytical form, only 3 are computed with a great enough uncertainty (the last 3 terms correspond to the QCD results for $z_m^{(4)}$-correction, proportional to $C_F^4$, $d_F^{abcd}d_F^{abcd}$ and $d_F^{abcd}d_F^{abcd}n_l$-structures with Casimir operator $C_F$ and symmetric tensor $\delta_{ab}^{bcd}$ in the fundamental representation). Under the final inaccuracy of the $n_l^0$ and $n_l$-dependent
contributions to the $z_{m}^{(4)}$-term in QED we imply the mean-square error of the corresponding coefficients from 9 unknown in the analytical form terms.

Thus, using the $U(1)$-limit of the results, presented in Refs.[1, 2, 4–7] and taking into account the values of 9 numerically derived in Ref.[15] terms, it is possible to get the $\overline{\text{MS}}$-on-shell mass relation for charged leptons in the $\mathcal{O}(a^4)$ approximation:

$$\overline{m}_l(M_l^2) \approx M_l(1 - a + (1.56205n_l - 0.6659)a^2 + (-1.95807n_l^2 + 0.7749n_l + 4.3602)a^3$$
$$+ (4.06885n_l^3 - 2.3576n_l^2 + (-4.097 \pm 0.178)n_l - 10.761 \pm 1.03)a^4) , \quad (4.1)$$

where $\overline{m}_l$ and $M_l$ are the running $\overline{\text{MS}}$-scheme and pole masses of charged leptons, $a = \alpha(M_l^2)/\pi$ is the QED coupling constant, defined in the $\overline{\text{MS}}$-scheme and normalized at the pole mass of $l$-th lepton.

Taking into account Eq.(4.1) and keeping in mind that when we consider the mass of electron we should put $n_l = 0$, and for $\mu$ and $\tau$-lepton masses $n_l = 1$ and $n_l = 2$ correspondingly, we obtain:

$$M_e \approx \overline{m}_e(M_e^2)(1 + a + 1.66591a^2 - 2.02839a^3 + (5.482 \pm 1.030)a^4) , \quad (4.2a)$$
$$M_\mu \approx \overline{m}_\mu(M_\mu^2)(1 + a + 0.10386a^2 - 3.96938a^3 + (5.907 \pm 1.045)a^4) , \quad (4.2b)$$
$$M_\tau \approx \overline{m}_\tau(M_\tau^2)(1 + a - 1.45819a^2 - 1.99421a^3 + (-0.653 \pm 1.099)a^4) . \quad (4.2c)$$

At the $\mathcal{O}(a^3)$ level these expressions agree with the ones, considered in the numerical form in [40] and independently presented later on in the analytical form in Ref.[41].

Even without detailed analysis of the perturbative structure of the presented expressions one can conclude that starting from the $\mathcal{O}(a^2)$ level the effects of the high-order corrections are really small and have minor impact even if one is interested in the rather precise determination of the numerical values of the running lepton masses in the $\overline{\text{MS}}$-scheme (note that the current CODATA values of the pole masses of charged leptons are $M_e = 510.9989461(31) \text{ KeV}$, $M_\mu = 105.6583745(24) \text{ MeV}$ and $M_\tau = 1776.86(12) \text{ MeV}$).

One should note the one interesting fact, namely that the mean-square inaccuracies of all four-loop contributions in Eqs.(4.2a-4.2c) are not negligibly small in comparison with the central values of these $\mathcal{O}(a^4)$ corrections. Indeed, for the case of electron and muon these current errors make up about 20% from the central values of the four-loop contributions and for $\tau$-lepton the uncertainty even exceeds the corresponding central value.

For the sake of completeness we also present the four-loop QED approximations for the relation between pole and running in $\overline{\text{MS}}$-scheme masses of charged leptons, defined at the scale $\mu^2 = \overline{m}_l^2$.

Applying the corresponding RG-transformations, we get

$$M_e \approx \overline{m}_e(\overline{m}_e^2)(1 + \bar{a} + 0.16591\bar{a}^2 - 2.13144\bar{a}^3 + (7.487 \pm 1.030)\bar{a}^4) , \quad (4.3a)$$
$$M_\mu \approx \overline{m}_\mu(\overline{m}_\mu^2)(1 + \bar{a} - 1.39614\bar{a}^2 - 0.64601\bar{a}^3 + (3.169 \pm 1.045)\bar{a}^4) , \quad (4.3b)$$
$$M_\tau \approx \overline{m}_\tau(\overline{m}_\tau^2)(1 + \bar{a} - 2.95819\bar{a}^2 + 4.75557\bar{a}^3 + (-21.238 \pm 1.090)\bar{a}^4) , \quad (4.3c)$$

where $\bar{a} = \alpha(\overline{m}_l^2)/\pi$. In contrast to the result, presented in Eq.(4.2c), at the renormalization point $\mu^2 = \overline{m}_l^2$ the inaccuracy of the four-loop contribution for $\tau$-lepton is much smaller than its central value and is about 5% of this value.
Expressions (4.3a-4.3c) demonstrate the absence of any regular sign-constant or sign-alternating structure of the related PT series. The same feature is observed in the case when the running QED parameters (masses of charged leptons and coupling constant) are normalized at the scale $\mu^2 = M^2_l$ (4.2a-4.2c). Therefore, the appearing from time-to-time in the literature point of view that the asymptotic perturbative series in QED should have sign-alternating structure, which is based in part on the theoretical studies, presented in Ref.[42], seems to be not the general rule. Perhaps the only place, where this property is vividly demonstrated nowadays, is the perturbative expression for the anomalous magnetic moment of electron, which is known at present up to five-loop term (for the most recent result of its evaluation see [43]). In Appendix A we present the explicit flavor and scale dependence of the $\overline{\text{MS}}$-on-shell relation in QED at the four-loop level.

5 Estimates of the perturbative QCD contributions to the relation between pole and $\overline{\text{MS}}$ running heavy quark masses beyond four-loop level

Now we turn to the main problem, which is studied in this work, namely to the analysis of the structure of the perturbative QCD relations between pole and defined in the $\overline{\text{MS}}$-scheme running heavy quark masses beyond the four-loop level. There are no doubts, that in general the corresponding perturbative QCD series are asymptotic ones. In Sec.3 it was discussed that in the case of the charm quark the asymptotic behavior of the related perturbative series begins to manifest itself starting from the two-loop approximation. For the $b$-quark mass there are indications that the asymptotic feature starts from the four-loop level, while for $t$-quark mass this feature should appear in higher orders.

In order to understand the structure of the perturbative relations between different definitions of $b$ and $t$-quark masses better it is interesting to estimate higher order perturbative QCD contributions to these relations. There are different approaches for probing high-order behavior of the perturbative QCD series by estimating their as yet unknown high-order contributions.

5.1 Application of the effective-charges motivated approach

Let us first consider the formulated on language of the effective-charges (ECH) method [30] for estimates of high-order perturbative QCD corrections to the physical quantities [29], or to be more precise its extension to the case of mass-dependent quantities, proposed in Ref.[28] and also used in Ref.[44] for estimation of the $O(a_s^4)$-contributions to the relation between pole and $\overline{\text{MS}}$ heavy quark masses.

As already clarified in the works of Refs.[28, 29, 44] it is more theoretically substantiated to apply the ECH-based procedures to the quantities, defined in the Euclidean region and then, if necessary, to add the arising kinematic corrections, which are proportional to powers of $\pi^2$-terms and contribute to the physical quantities, defined in the Minkowskian time-like region. They appear in the high-order perturbative QCD corrections as the result of analytical continuation from Euclidean to Minkowskian region. Note, that in the definite cases these $\pi^2$-dependent terms can be even summed up (see e.g. [45–49]).
Following the notations, introduced in Ref. [28], as the associated RG function, determined in the Euclidean region, we take $F(Q^2)$-function, related to the spectral function $T(s)$, defined in the Minkowskian region through the following Källen-Lehmann type representation:

$$F(Q^2) = Q^2 \int_0^\infty ds \frac{T(s)}{(s + Q^2)^2}.$$  \hspace{1cm} (5.1)

The perturbative expression for the spectral function $T(s)$ is expressed in the form, analogous to the one, used in Refs. [28, 44]:

$$T(s) = m_q(s) \sum_{n=0}^{\infty} t_n^M a_s^n(s),$$  \hspace{1cm} (5.2)

where $m_q(s)$ is the \overline{MS}-scheme running mass of heavy quark, normalized in the time-like region at the scale $\mu^2 = s$; $t_n^M$ are the numerical coefficients of the spectral function, defined in the Minkowskian region. Since $T(s)$-function is a perturbative series in powers of the coupling constant $a_s = \alpha_s(s)/\pi$, the associated function $F(Q^2)$ is a related perturbative series, where both running mass and the QCD coupling constant are defined in the Euclidean region at the scale $\mu^2 = Q^2$:

$$F(Q^2) = m_q(Q^2) \sum_{n=0}^{\infty} f_n^E a_s^n(Q^2).$$  \hspace{1cm} (5.3)

The evolution of these QCD parameters is determined by the system of the following RG equations:

$$\mu^2 \frac{\partial a_s}{\partial \mu^2} = \beta(a_s) = -\sum_{n=0}^{\infty} \beta_n a_s^{n+2}, \quad \mu^2 \frac{\partial \log(m_q)}{\partial \mu^2} = \gamma_m(a_s) = -\sum_{n=0}^{\infty} \gamma_n a_s^{n+1},$$  \hspace{1cm} (5.4)

where $\beta(a_s)$ and $\gamma_m(a_s)$ are the QCD $\beta$-function and the anomalous mass dimension function. In our further consideration we use their \overline{MS}-scheme expressions. The one and two-loop coefficients $\beta_0$ and $\beta_1$ of the QCD $\beta$-function was computed in Refs. [50, 51] and [52–54] respectively. The scheme-dependent three and four-loop coefficients $\beta_2$ and $\beta_3$ are known from the computations, performed in Refs. [55, 56] and [57, 58] correspondingly. The $\beta_4$-coefficient was obtained recently in $SU(3)$-group [59] and confirmed in Ref. [60] by computing this term in the general $SU(N_c)$ gauge group.

For our purposes it is convenient to present these coefficients $\beta_n$ in terms of the number of massless flavors $n_l = n_f - 1$. In the case of $SU(3)$ color gauge group their numerical expressions have the following form:

$$\beta_0 = 2.58333 - 0.166667 n_l, \quad \beta_1 = 5.58333 - 0.791667 n_l, \quad \beta_2 = 18.0454 - 4.18084 n_l + 0.994039 n_l^2, \quad \beta_3 = 88.684 - 23.951 n_l + 1.5999 n_l^2 + 0.005857 n_l^3, \quad \beta_4 = 359.687 - 148.1715 n_l + 16.46765 n_l^2 - 0.233054 n_l^3 - 0.0017993 n_l^4.$$  \hspace{1cm} (5.5a-d)
The first scheme-independent coefficient $\gamma_0$ was calculated in Refs.\cite{52, 53}. The two, three and four-loop expressions for anomalous mass dimension were computed in Refs.\cite{1, 61}, \cite{62, 63}, \cite{64, 65} correspondingly. The coefficient $\gamma_4$ of the fifth order was evaluated in Ref.\cite{66}. The numerical values of these coefficients read:

\[
\begin{align*}
\gamma_0 &= 1, & \gamma_1 &= 4.06944 - 0.138889n_l, \\
\gamma_2 &= 17.2045 - 2.33813n_l - 0.027006a_l^2, \\
\gamma_3 &= 80.117 - 18.5378n_l + 0.29354n_l^2 + 0.005793n_l^3, \\
\gamma_4 &= 423.611 - 128.3970n_l + 7.80682n_l^2 + 0.107977n_l^3 - 0.0000854n_l^4.
\end{align*}
\]

In the $O(a_s^6)$ approximation Eq.(5.4) can be re-written as:

\[
\log \frac{\mu^2}{s} = \int \frac{a_s(s)}{a_s(\mu^2)} dx = \log \left( \frac{\beta_0}{\beta_0 + t_1} \right) + \frac{\gamma_1 x^2}{\beta_0 x^2 + \beta_1 x^3 + \beta_2 x^4 + \beta_3 x^5 + \beta_4 x^6 + \beta_5 x^7}. \tag{5.7}
\]

Using Eq.(5.7) one can get the following relation between coupling constants, normalized at different points:

\[
a_s(s) = a_s(\mu^2) + \sum_{n=1}^{6} \theta_n a_s^{n+1}(\mu^2) + O(a_s^8), \tag{5.8}
\]

where the coefficients $\theta_i$ are presented in Appendix B.

The scale dependence of the running mass can be obtained from the system of two RG equations (5.4):

\[
\frac{\bar{m}_q(s)}{\bar{m}_q(\mu^2)} = \exp \left( \int \frac{a_s(s)}{a_s(\mu^2)} \frac{\gamma_0 x + \gamma_1 x^2 + \gamma_2 x^3 + \gamma_3 x^4 + \gamma_4 x^5 + \gamma_5 x^6}{\beta_0 x^2 + \beta_1 x^3 + \beta_2 x^4 + \beta_3 x^5 + \beta_4 x^6 + \beta_5 x^7} dx \right), \tag{5.9}
\]

The explicit expression for solution of this equation is contained in Appendix B.

Taking now into account the expressions (5.8-5.9) and substituting them into the spectral density $T(s)$ in the dispersion relation (5.1), \textit{we fix the relations} between Minkowskian coefficients $t_n^M$ and their Euclidean analogues $f_n^E$. The integrals, arising in this case, can be calculated in the complex plane using the Cauchy theorem on residues:

\[
Q^2 \int_0^\infty ds \frac{l_1 l_2 l_3 l_4 l_5}{(s + Q^2)^2} = \left\{ 1; \mathcal{L}; \mathcal{L}^2 + \frac{\pi^2}{3}; \mathcal{L}^3 + \pi^2 \mathcal{L}; \mathcal{L}^4 + 2\pi^2 \mathcal{L}^2 + \frac{7\pi^4}{15}; \right. \tag{5.10}
\]

\[
\mathcal{L}^5 + \frac{10}{3} \pi^2 \mathcal{L}^3 + \frac{7}{3} \pi^4 \mathcal{L}; \mathcal{L}^6 + 5\pi^2 \mathcal{L}^4 + 7\pi^4 \mathcal{L}^2 + \frac{31}{21} \pi^6 \left. \right\},
\]

with $l = \log(\mu^2/s)$ and $\mathcal{L} = \log(\mu^2/Q^2)$.

Taking $\mu^2 = Q^2$ we obtain the relations between the aforementioned coefficients $t_n^M$ and $f_n^E$: $f_n^E = t_n^M + \Delta_n$. Appearing due to the effects of analytic continuation $\Delta_n$-terms are presented in Appendix C for the cases when $0 \leq n \leq 6$. Starting from $n = 2$ the additional contributions $\Delta_n$ contain kinematic $\pi^2$-contributions, proportional to the coefficients of the
QCD $\beta$-function and the anomalous dimension function $\gamma_m$. Note, that the expression for $\Delta_6$ does not require knowledge of unknown six-loop RG-coefficients $\beta_5$ and $\gamma_5$. The reason for their absence is given in Appendix C.

Following theoretical studies, presented in Refs. [28, 29, 44], at the next stage of application of the ECH-motivated approach for estimating high-order QCD corrections to the relation between pole and running heavy quark masses it is necessary to determine the ECH $a_s^{eff}(Q^2)$ for the Euclidean quantity $F(Q^2)/\overline{m}_q(Q^2)$, introduced in Eq. (5.3):

$$
\frac{F(Q^2)}{\overline{m}_q(Q^2)} = f_k^E + f_k^E a_s^{eff}(Q^2) , \quad a_s^{eff}(Q^2) = a_s(Q^2) + \sum_{k=2}^{\infty} \phi_k a_s^k(Q^2) ,
$$

(5.11)

where $\phi_k = f_k^E / f_k^E$. Then we define the ECH $\beta$-function for coupling constant $a_s^{eff}(Q^2)$ through the scheme-independent combinations, first discovered in Ref. [31]. The general expressions for the coefficients of the four-loop ECH $\beta$-function are known (see e.g. Ref. [67], where this approach was applied for determination of the four-loop ECH $\beta$-function of the static potential in QCD). Here we present the explicit expressions for the coefficients of the six-loop ECH $\beta$-function, which is governing the $Q^2$-behavior of the ECH, defined in Eq. (5.11):

$$
\begin{align*}
\beta_0^{eff} &= \beta_0 , \quad \beta_1^{eff} = \beta_1 , \quad \beta_2^{eff} = \beta_2 - \phi_2 \beta_2 + (\phi_3 - \phi_2^2) \beta_0 , \\
\beta_3^{eff} &= \beta_3 - 2\phi_2 \beta_2 + \phi_2^2 \beta_1 + (2\phi_4 - 6\phi_2 \phi_3 + 4\phi_2^3) \beta_0 , \\
\beta_4^{eff} &= \beta_4 - 3\phi_2 \beta_3 + (4\phi_2^2 - \phi_3) \beta_2 + (\phi_4 - 2\phi_2 \phi_3) \beta_1 \\
&\quad + (3\phi_5 - 12\phi_2 \phi_4 - 5\phi_2^2 + 28\phi_2 \phi_3 - 14\phi_2^3) \beta_0 , \\
\beta_5^{eff} &= \beta_5 - 4\phi_2 \beta_4 + (8\phi_2^2 - 2\phi_3) \beta_3 + (4\phi_2 \phi_2 - 8\phi_2^3) \beta_2 \\
&\quad + (2\phi_5 - 8\phi_2 \phi_4 + 16\phi_2 \phi_3 - 3\phi_2^2 - 6\phi_2^4) \beta_1 \\
&\quad + (4\phi_6 - 20\phi_2 \phi_5 - 16\phi_3 \phi_4 + 48\phi_2 \phi_3 - 120\phi_2 \phi_3 + 56\phi_2 \phi_4 + 48\phi_2 \phi_2) \beta_0 ,
\end{align*}
$$

(5.12a-d)

where coefficients $\beta_k$ are defined in the $\overline{\text{MS}}$-scheme (5.5a-5.5d).

One of the basic ideas of the work [28] is that the coefficients $t_n^M$ in Eq. (5.2) are identical to the ones, which enter in the relation between pole and $\overline{m}_q(\overline{m}_q^2)$ running $\overline{\text{MS}}$-scheme heavy quark masses (see Eq. (3.2)) at $s = \overline{m}_q^2$:

$$
M_q = \overline{m}_q(\overline{m}_q^2) \sum_{n=0}^\infty t_n^M a_s^n(\overline{m}_q^2) ,
$$

(5.13)

This relation is supported by the following dispersion representation, postulated in Ref. [28]:

$$
M_q = \frac{1}{2\pi i} \int_{-\overline{m}_q(\overline{m}_q^2)-i\epsilon}^{\overline{m}_q(\overline{m}_q^2)+i\epsilon} ds' \int_0^\infty \frac{T(s)}{(s + s')^2} ds' .
$$

(5.14)

Note, that coefficients $t_n^M - t_n^M$ of the spectral function $T(s)$ (5.2) were presented in Sec. 3. Our further analysis is based on the theoretical studies, described in Refs. [28, 29, 44]. Their essence is as follows: if we would put $\beta_2^{eff} \approx \beta_2$, then from Eq. (5.12a) we would
get that \( f_E^E \approx (f_E^E)^2/f_1^E + f_2^E \beta_1/\beta_0 \) and using Appendix C we would restore the value of \( t_3^M \)-term. Similarly, supposing that \( \beta_3^{eff} \approx \beta_3 \) we could estimate the value of the four-loop contributions \( f_E^E \) and \( t_4^M \). Estimates of this kind were made in Ref.[44] to find the numerical value of the \( t_3^M \)-term to the \( \overline{\text{MS}} \)-on-shell mass relation for charm, bottom and top-quarks.

But nowadays terms \( t_3^M \) and \( t_4^M \), defined in the Minkowskian space, are known in analytical [5, 6] and numerical form with great enough accuracy [15] (3.2). Therefore we may compare the results of direct three and four-loop calculations for \( t_3^M \) and \( t_4^M \)-terms with the ones, obtained by the ECH-motivated method. Results of this study are presented in Table 1.

| \( n_l \) | \( t_3^M, \text{exact} \) | \( t_3^M, \text{ECH} \) | \( t_4^M, \text{exact} \) | \( t_4^M, \text{ECH} \) |
|-------|----------------|----------------|----------------|----------------|
| 3     | 116.494        | 124.097        | 1702.70 ± 1.41 | 1281.09        |
| 4     | 94.418         | 97.728         | 1235.66 ± 1.47 | 986.13         |
| 5     | 73.637         | 73.615         | 839.14 ± 1.54  | 719.38         |
| 6     | 54.161         | 51.775         | 509.07 ± 1.61  | 483.02         |
| 7     | 35.991         | 32.235         | 241.37 ± 1.70  | 279.37         |
| 8     | 19.126         | 15.034         | 31.99 ± 1.80   | 110.71         |

**Table 1:** The exact values and estimates of \( t_3^M \) and \( t_4^M \)-coefficients.

From the numbers, presented in Table 1, it is apparent that the ECH-motivated method gives quite good approximations for three and four-loop contributions in the \( \overline{\text{MS}} \)-on-shell relation\(^4\). This approach catches not only the correct signs of the \( \mathcal{O}(a_s^3) \) and \( \mathcal{O}(a_s^4) \)-terms, but also gives rather close values of \( t_3^M \) and \( t_4^M \)-terms to their exact expressions. This gives us a reason to believe that in the fifth and sixth order of PT the ECH-motivated approach will also give the quite satisfactory estimates with their correct signs. Thus, the above-mentioned reasoning allows us to estimate the numerical values of these corrections to the mass conversion formula. Our further studies contain the following steps:

\(^a\) At the first stage we find expressions for contributions \( f_5^E - f_4^E \), using the explicit form of the known \( t_0^M - t_4^M \)-terms and adding to them the \( \pi^2 \)-effects of analytical continuation, presented in Appendix C (C.1).

\(^b\) Secondly, fixing \( \beta_4^{eff} \approx \beta_4 \) (this is our main guess-conjecture), we obtain the approximate form of the \( f_5^E \)-term in the Euclidean region for concrete values of \( n_l \).

\(^c\) We subtract from \( f_5^E \)-term, obtained numerically at the previous step, the contribution of analytical continuation \( \Delta_5 \), given in the Appendix C, and get \( \mathcal{O}(a_s^5) \) coefficient \( t_5^M \) in Eq.(5.13).

\(^4\) Note that the values of the \( t_3^M, \text{ECH} \)-contributions, presented in Table 1, are slightly different from the analogous coefficients, obtained by means of the same ECH-motivated method in Ref.[44]. The difference between these coefficients lies in the slip made in Ref.[44]. However this fact did not affect the final results of the four-loop \( t_4^M, \text{ECH} \)-term.
d) Repeating this procedure at the next order of PT (assuming that \( \beta_5^{eff} \approx \beta_5 \) and using the numerical expression for \( f_5^E \)-term, obtained at the previous stage\(^5\)), we estimate the value of \( f_5^E \)-contribution initially and then using Appendix C we evaluate the value of the \( t_6^M \)-correction in the \( \mathcal{O}(\alpha_s^6) \) approximation.

Relying on the foregoing, we obtain the following estimates for five and six-loop contributions \( f_5^E \) and \( f_6^E \) in Eq. (5.3):

\[
f_5^E \approx \frac{1}{3\beta_0} \left[ 3 f_2^E \beta_3 + \left( f_3^E - 4 \left( \frac{f_2^E}{f_1^E} \right)^2 \right) \beta_2 + \left( 2 f_2^E f_3^E - f_1^E \right) \beta_1 \right] \quad (5.15a)
\]
\[
+ 4 \frac{f_2^E f_4^E}{f_1^E} + 5 \left( \frac{f_3^E}{f_1^E} \right)^2 - \frac{28}{3} \frac{f_3^E}{f_1^E} \left( \frac{f_2^E}{f_1^E} \right)^2 + \frac{14}{3} \left( \frac{f_2^E}{f_1^E} \right)^4 \]
\]
\[
f_6^E \approx \frac{1}{4\beta_0} \left[ 4 f_2^E \beta_4 + \left( 2 f_3^E - 8 \left( \frac{f_2^E}{f_1^E} \right)^2 \right) \beta_3 + \left( 8 \left( \frac{f_2^E}{f_1^E} \right)^3 - 4 \frac{f_2^E f_4^E}{f_1^E} \right) \beta_2 \right]
\]
\[
+ \left( 6 \left( \frac{f_2^E}{f_1^E} \right)^4 + 3 \left( \frac{f_3^E}{f_1^E} \right)^2 + 8 \frac{f_3^E f_4^E}{f_1^E} - 16 f_3^E \left( \frac{f_2^E}{f_1^E} \right)^2 - 2 f_5^E \right) \beta_1 \]
\]
\[
+ 5 \frac{f_2^E f_5^E}{f_1^E} + 4 \frac{f_3^E f_4^E}{f_1^E} + 30 f_3^E \left( \frac{f_2^E}{f_1^E} \right)^3 - 12 f_2^E \left( \frac{f_3^E}{f_1^E} \right)^2 - 12 \left( \frac{f_2^E}{f_1^E} \right)^3 \left( \frac{f_2^E}{f_1^E} \right) - 14 f_4^E \left( \frac{f_2^E}{f_1^E} \right)^2 \right].
\]

Taking into account these expressions and relations \( t_5^M = f_5^E - \Delta_5, t_6^M = f_6^E - \Delta_6 \), presented in Appendix C, we estimate the five and six-loop corrections \( t_5^M \) and \( t_6^M \) in Eq. (5.13). The results of the corresponding numerical calculations are presented in Table 2.

| \( n_t \) | \( t_5^{M, ECH} \) | \( t_6^{M, ECH} \) |
|---|---|---|
| 3 | 28435 | 476522 |
| 4 | 17255 | 238025 |
| 5 | 9122 | 90739 |
| 6 | 3490 | 8412 |
| 7 | -127 | -29701 |
| 8 | -2153 | -39432 |

**Table 2:** The estimates of \( t_5^M \) and \( t_6^M \)-coefficients by the ECH method.

As indicated in Eq. (1.4) the five-loop contribution \( t_5^M \) can be expanded in powers of number of massless flavors \( t_5^M = t_{54}^M n_t^4 + t_{53}^M n_t^3 + t_{52}^M n_t^2 + t_{51}^M n_t + t_{50}^M \) with five unknown variables \( t_{54}^M - t_{50}^M \). In order to define them within the ECH-motivated method we should use at least 5 equations for \( 3 \leq n_t \leq 7 \). The similar situation takes place for six-loop \( t_6^M \) term also, but in view of the fact that it contains one unknown more, we should consider

\(^5\) Thereby the uncertainty in the definition of the six-loop corrections to the mass conversion formula will certainly be greater than the five-loop ones.
cases with $3 \leq n_l \leq 8$. For this reason we present the results of the ECH-based calculations for 6 values of massless flavors, namely $3 \leq n_l \leq 8$.

Using the figures, presented in Table 2 for $t_{5}^{M, ECH}$-correction, we obtain the following matrix equation with $3 \leq n_{l} \leq 7$:

$$
\begin{pmatrix}
1 & 3 & 9 & 27 & 81 \\
1 & 4 & 16 & 64 & 256 \\
1 & 5 & 25 & 125 & 625 \\
1 & 6 & 36 & 216 & 1296 \\
1 & 7 & 49 & 343 & 2401
\end{pmatrix}
\begin{pmatrix}
t_{50}^{M, ECH} \\
t_{51}^{M, ECH} \\
t_{52}^{M, ECH} \\
t_{53}^{M, ECH} \\
t_{54}^{M, ECH}
\end{pmatrix}
= \begin{pmatrix}
28435 \\
17255 \\
9122 \\
3490 \\
-127
\end{pmatrix}
$$

(5.16)

The numerical solution of this system is

$$
t_{5}^{M, ECH} = 2.5n_{l}^{4} - 136n_{l}^{3} + 2912n_{l}^{2} - 26976n_{l} + 86620 .
$$

(5.17)

One should make one informative remark on the square matrix in Eq.(5.16). This matrix is the Vandermonde matrix possessing interesting mathematical properties. Indeed, elements of each row of this matrix are the terms of a geometric progression and the Vandermonde determinant is equal to $\det V = \Delta = \prod_{0 \leq i < j < k} ((n_{l} + j) - (n_{l} + i)) = \prod_{0 \leq i < j < k} (j - i)$, provided that we vary the number of massless quarks from $n_{l}$ to $(n_{l} + k)$, $k \in \mathbb{N}$. In a number of theoretical issues the number $n_{l}$ is also assumed to be a negative quantity, therefore, strictly speaking the index $k$ does not have to belong to the set of natural numbers only. However, in our paper we use the positive integer values of $k$ only. The Cramer’s rule allows to find the unknown coefficients $t_{5}^{M, ECH} = \Delta_{s}/\Delta$ with $0 \leq s \leq 4$, where $\Delta_{s}$ is the determinant of the matrix, obtained from the Vandermonde matrix by replacing the $s$-th column by the column with $t_{5}^{M, ECH}$-terms at fixed $n_{l}$.

Repeating the similar reasoning for $t_{6}^{M}$-contribution with $3 \leq n_{l} \leq 8$, we obtain

$$
t_{6}^{M, ECH} = -4.9n_{l}^{5} + 352n_{l}^{4} - 9708n_{l}^{3} + 131176n_{l}^{2} - 855342n_{l} + 2096737 .
$$

(5.18)

Thus the application of the ECH-motivated method to finding the coefficients $t_{50}^{M, ECH}$, $t_{51}^{M, ECH}$ and $t_{52}^{M, ECH}$ predicts the sign-alternating structure of the $t_{5}^{M}$-contributions in expansion in powers of $n_{l}$. In Appendix D we present the explicit scale and flavor dependence of the conversion mass formula for heavy quarks at the $\mathcal{O}(a_{s}^{6})$ approximation for special case of the $SU(3)$ color gauge group with five and six-loop contributions, obtained by the ECH-motivated method.

Note that the leading on $n_{l}$ contributions into the presented in Eqs.(5.17-5.18) estimated $n_{l}$-dependent expressions of the five-loop ans six-loop coefficients for the relation between pole and running heavy quark masses are essentially larger than their two, three and four-loop analogs, directly evaluated in the analytical form and presented in Eq.(3.2). Indeed, the values of these leading on $n_{l}$ coefficients are $t_{21}^{M} = -1.0414$, $t_{32}^{M} = 0.6527$, $t_{43}^{M} = -0.6781$, while the ECH-inspired estimates of the leading on $n_{l}$ contributions to the five and six-loop corrections are $t_{50}^{M, ECH} = 2.5$ and $t_{54}^{M, ECH} = -4.9$. This is a hint that the ECH-inspired estimates respect the general tendency to sign-alternating factorial growth of these terms,
which, after application of the renormalon-based naive non-abelianization procedure, define the behavior of the rest sub-leading on $n_l$ high-order corrections to the $\overline{\text{MS}}$-on-shell mass relation for heavy quarks \cite{14, 25, 32}. It would be interesting to check the found by us feature of ECH-inspired estimates by direct analytical or numerical calculation of these leading on $n_l$ corrections, which follow from the subset of the concrete renormalon-chain Feynman diagrams.

Let us now consider the numerical impact of the five and six-loop QCD estimations, computed by the ECH-motivated method, on the magnitudes of the pole masses of heavy quarks in more detail. Using the numerical expressions for $t_{5}^{M, ECH}$ and $t_{6}^{M, ECH}$-contributions, given in Table 2, the $\text{N}^3\text{LO}$-inverse log representation for the QCD coupling constant and our corresponding numerical results, presented in Eqs.(3.4a-3.4c), we arrive to the following expansions:

$$\frac{M_c}{1 \text{ GeV}} \approx 1.275 + 0.214 + 0.208 + 0.295 + 0.541 + \left[ 1.135 + 2.389 \right], \quad (5.19a)$$

$$\frac{M_b}{1 \text{ GeV}} \approx 4.180 + 0.400 + 0.200 + 0.146 + 0.137 + \left[ 0.137 + 0.137 \right], \quad (5.19b)$$

$$\frac{M_t}{1 \text{ GeV}} \approx 164.300 + 7.566 + 1.614 + 0.498 + 0.196 + \left[ 0.074 + 0.025 \right]. \quad (5.19c)$$

The boxed terms are the numerical contributions, obtained by means of the ECH-motivated approach. Despite the fact that these values are approximate, they reflect the specific behavior of the high-order PT corrections to the $\overline{\text{MS}}$-on-shell mass relation of heavy quarks, namely

- In the case of $c$-quark the numerical PT QCD corrections to the value of its pole mass, which are starting to increase from the $\mathcal{O}(a_3^2)$ level, are continuing their asymptotic growth at the $\mathcal{O}(a_5^2)$ and $\mathcal{O}(a_6^2)$-orders. Indeed, as demonstrated within the used and studied by us ECH-inspired method, the $\mathcal{O}(a_5^2)$-contribution is almost 2 times larger than the four-loop expression, and the $\mathcal{O}(a_6^2)$-correction is 2 times greater than the estimated five-loop term and is even larger than the first term of this PT series.

- For $b$-quark pole mass the ECH-motivated approach demonstrates the peculiar stabilization feature. Actually, the four, five and six-loop corrections to the $\overline{\text{MS}}$-on-shell mass relation coincide for bottom quark\textsuperscript{6}. This property signals that the asymptotic nature of Eq.(5.19b) is starting to manifest itself from the $\mathcal{O}(a_4^2)$-contribution.

- The relation (5.19c) shows a decrease of the $\mathcal{O}(a_5^2)$ and $\mathcal{O}(a_6^2)$-corrections. This means that the asymptotic behavior of the PT series for the relation between pole and running top quark masses is not yet observed at six-loop level. Thus, it is most likely that the concept of pole mass of $t$-quark can be used safely even at the six-loop level. Therefore we can sum all these corrections and obtain $M_t^{ECH} \approx 174.273$ GeV. However, it is interesting to determine from which order of PT this behavior starts to come out. This problem is studied by us in Sec.5.3.

\textsuperscript{6}The inclusion in the numerical analysis of the five-loop threshold effects, derived in Ref.\cite{36, 68, 69} and of the five-loop contribution to the QCD $\beta$-function \cite{59, 60} does not affects this feature.
5.2 Comment on application of the ECH-motivated procedure directly in the Minkowskian region

As stated in the previous subsection the application of the ECH-motivated estimating procedure in the Euclidean region allows one to control the magnitude and structure of the proportional to $\pi^2$-effects of the analytic continuation to the Minkowski space getting predictions for the unknown coefficients of the PT series for the concrete measurable physical quantities, defined in the time-like region of energies. This approach was first considered in Ref.[29] in the analysis of the effects of high order QCD corrections to the characteristics of the $e^+e^-$ annihilation to hadrons process. Moreover, it turned out later on that in the \( \overline{\text{MS}} \)-scheme these explicitly determined kinematic $\pi^2$-terms are dominating in the whole four-loop QCD contribution to the $e^+e^-$-annihilation $R(s)$-ratio, analytically evaluated in Ref.[70]. However, this feature is not the general rule. For instance, in the total decay width of the Higgs boson to $b\bar{b}$-pair, computed in the \( \overline{\text{MS}} \)-scheme up to $O(a_s^4)$ level [71]$^7$, the total contribution of negative analytical continuation effects is not dominant over the positive corrections of the direct calculations of the corresponding Feynman integrals in the Euclidean region. In view of this it is also of interest to study what will be the results of applications of various procedures of estimating high-order PT QCD corrections to different physical quantities directly in the Minkowski space. The further comparison of the outcomes, predicted by the different estimating approaches in the Euclidean and Minkowskian regions, may clarify whether and when the separate treatment of the effects of the analytical continuation is important and to what extent. Consideration of these effects and the subsequent possible procedures of their resummation may affect the theoretical uncertainties of the concrete experimentally measurable physical quantities (for more detailed studies of this problem see [45–49] and the review talk of Ref.[73]). Following this trend, we supplement the presented above ECH-based method for the PT QCD relation between pole and running heavy quark masses by the analysis of its results, obtained in the Minkowskian region directly (see [28, 44]).

Keeping in mind the outlined above aim, we should redefine Eqs.(5.12a-5.12d) for Minkowskian quantity \( T(s)/m_q(s) \), determined in Eq.(5.2), changing coefficients $\phi_k = f_k^E/f_k^E$ to $\phi_k^M = t_k^M/t_k^M$. As the result, starting from the third order of PT the scheme-invariant coefficients of the effective $\beta$-function governing the energy dependence of the ECH coupling constant $a_s^{\text{eff}}(s)$, which is the Minkowskian analog of Eq.(5.11), start to differ from their Euclidean analogs by proportional to $\pi^2$-terms extra corrections $\Delta_n$, presented in Appendix C.

Repeating in part the reasoning, described in Sec.5.1, viz assuming that the five and six-loop coefficients of the effective Minkowskian $\beta$-function are approximately equal to its analogs, determined in the $\overline{\text{MS}}$-scheme, we obtain the following estimating expressions for

$\footnote{The analytical results of Refs.[70, 71] were recently confirmed in Ref.[72].}$
the $\mathcal{O}(a_s^5)$ and $\mathcal{O}(a_s^6)$-terms entering in Eq.\,(5.13):

\[
t^5_M, \text{ECH direct} \approx \frac{1}{3\beta_0(t_1^M)^3} \left[ 3t^M_1 (t_1^M)^3 \beta_3 + t^M_3 (t_1^M)^3 \beta_2 - 4(t^M_2 t_1^M)^2 \beta_2 \\
+ 2t^M_3 t_2^M (t_1^M)^2 \beta_1 - t^M_3 (t_1^M)^3 \beta_1 + 12t^M_4 t_2^M (t_1^M)^2 \beta_0 + 5(t^M_3 t_1^M)^2 \beta_0 \\
+ 14(t^M_2)^4 \beta_0 - 28t^M_3 (t_2^M t_1^M)^2 \beta_0 \right],
\]

\[
t^6_M, \text{ECH direct} \approx \frac{1}{12\beta_0(t_1^M)^4} \left[ 48t^M_4 t_3^M (t_1^M)^3 \beta_0^3 + 72t^M_4 (t_1^M t_2^M)^2 \beta_0^2 + 12t^M_2 (t_1^M)^4 \beta_0 \beta_4 \right. \\
\left. + 136(t^M_2)^5 \beta_0^5 - 200t^M_3 t_1^M (t_2^M)^3 \beta_0^2 - 20t^M_4 t_2^M (t_1^M)^3 \beta_0 \beta_1 - (t^M_1)^3 (t_3^M)^2 \beta_0 \beta_1 \\
+ 48t^M_3 (t_1^M t_2^M)^2 \beta_0 \beta_1 - 10t^M_4 (t_2^M)^4 \beta_0 \beta_1 - 44t^M_4 (t_1^M t_2^M)^2 \beta_0^2 + 6t^M_3 (t_1^M)^4 \beta_0 \beta_3 \\
+ 36(t_1^M)^3 (t_2^M)^2 \beta_0 \beta_3 - 56(t_1^M t_2^M)^3 \beta_0 \beta_2 + 2t^M_4 (t_1^M)^4 \beta_2^2 - 4t^M_3 t_2^M (t_1^M)^3 \beta_1^2 \\
+ 8t^M_3 t_2^M (t_1^M)^3 \beta_1 \beta_2 - 6t^M_3 (t_1^M)^4 \beta_1 \beta_3 - 2t^M_3 (t_1^M)^4 \beta_1 \beta_2 + 8(t^M_1)^3 (t_2^M)^2 \beta_1 \beta_2 \right] .
\]

It turns out that at fixed $n_t$ the numerical expressions for $t^5_M, \text{ECH direct}$ and $t^6_M, \text{ECH direct}$ are in rather good agreement with the results, obtained in previous subsection by means of the ECH-motivated method, applied directly in the Euclidean region with addition of the proportional to $\pi^2$ kinematic contributions. Therefore we do not present here the concrete results of the discussed above numerical analysis, made directly in the Minkowskian region. The above mentioned feature is similar to the one, observed in the studies of high-order PT QCD corrections to the total decay width of Higgs boson to $b\bar{b}$-pair, namely the proportional to $\pi^2$ analytical continuation contributions, which can be reproduced from Eqs.\,(5.20a) and Eq.\,(5.20b) by the substitution $t^M_n = f_B^n - \Delta_n$ and the subsequent grouping of all terms proportional to $\Delta_n$-contributions, are rather closed to values of the corrections, obtained from the analytical continuation from the Euclidean to the Minkowski region and presented in the Appendix C.

One can see from the computations, demonstrated in Appendix C, that effects of analytical continuation from the Euclidean to the Minkowski region, where the pole and running heavy quark masses are defined, are producing not negligible and increasing with order of PT negative contributions. Moreover, they are not explicitly identified from the whole contributions, calculated in Refs.\,[2–6, 15], apart of possible consideration, based on renormalon-oriented large $\beta_0$ analysis, which was used in Ref.\,[48] to fix these effects partly in the PT QCD expression for the total decay width of Higgs boson to $b\bar{b}$ pair, proportional to square of $b$-quark running mass. In view of this and in order to probe theoretical ambiguities of different approaches for estimating high-order QCD PT contributions to the $\overline{\mathbf{MS}}$-on-shell heavy quark mass relation we also compare the results of the ECH-motivated method and of the formulated in the Minkowskian region directly renormalon-based approach, used in Ref.\,[32] and recently summarized in Ref.\,[25] (for its related consideration see [74]).
5.3 Renormalon-based estimating procedure: results and discussions

As is known the relation between pole and \( \overline{\text{MS}} \) running heavy quark masses contains the infrared renormalon (IRR) contributions, which lead to the sign-constant factorial increase of the coefficients in this asymptotic PT series \([11–14]\). Therefore it is important to find out from which order of PT the corresponding loop corrections to the \( \overline{\text{MS}} \)-on-shell QCD mass relation begin their growth. Since the IRR approach is asymptotic, and therefore approximate certainly, the answer to this important from phenomenological point of view question should be found either from the results of the direct calculations or from the analysis of expressions for the coefficients, obtained by different estimate methods.

We have already discussed in Sec. 3 that for the charm quark mass the asymptotic structure of the relation we are interested in is starting to demonstrate itself at rather low order of PT, namely from the third term. In the case of the bottom quark mass the situation is not so transparent. Indeed, four-loop results indicate that possible asymptotic growth manifests itself from the analyzed in Sec. 3 \( \mathcal{O}(a_s^4) \)-contribution. The extra support for this statement comes from the ECH-motivated estimates of the five and six-loop corrections to this PT QCD series, described in Sec. 5.1. However, in the case of the top quark the question of fixing the order of PT, for which its \( \overline{\text{MS}} \)-on-shell mass relation starts to diverge, remain unclear even after the estimates of these additional terms.

To analyze this problem in more detail we apply the IRR-based technique for estimating of high-order corrections to the relation between pole and \( \overline{\text{MS}} \)-scheme running heavy quark masses. As was shown in Refs. [12–14, 75, 76] the application of the Borel transformation to the relation (5.13) allows to study the asymptotic behavior of the perturbative coefficients \( t_{n,r}^M \) at large orders of PT. The renormalon dominance hypothesis leads to the derived in Ref. [13] following factorially growing expression of \( t_{n,r}^M \)-coefficients:

\[
\lim_{n \to \infty} t_{n,r}^M \sim \pi N_m(2\beta_0)^{n-1} \Gamma(n+b) \Gamma(1+b) \left( 1 + \frac{s_1}{n+b-1} \right) \frac{s_2}{(n+b-1)(n+b-2)} + \frac{s_3}{(n+b-1)(n+b-2)(n+b-3)} + \mathcal{O}\left( \frac{1}{n^4} \right),
\]

where \( \Gamma(x) \) is the Euler Gamma-function, \( b = \beta_1/(2\beta_0^2) \) and the values of the sub-leading coefficients \( s_k \) are presented below. The normalization factor \( N_m \) depends on \( n_l \) and on the order of PT. Note that our notations and normalizations differ from those introduced in Refs. [12, 13, 32, 76]. To clarify the differences in our notations and the ones, used in these works, one should note that in our consideration the first scheme-independent coefficient \( \beta_0 \) of the QCD \( \beta \)-function, which enters Eq. (5.21), depends on \( (n_l - 1) \) number of massless flavors, viz \( \beta_0 = \beta_0(n_l - 1) \). The rest high-order coefficients \( \beta_k \) also depend on the number \( (n_l - 1) \). These values are presented in Eqs. (5.5a-5.5d).

The expressions for coefficients \( s_k \) can be found in Ref.[32] and in our notations read:

\[
s_1 = \frac{1}{4\beta_0^2} (\beta_1^2 - \beta_0 \beta_2), \quad (5.22a)
\]

\[
s_2 = \frac{1}{32\beta_0^5} (\beta_1^4 - 2\beta_1^3 \beta_0^2 - 2\beta_1^2 \beta_2 \beta_0 + 4\beta_1 \beta_2 \beta_0^2 + \beta_2^2 \beta_0^2 - 2\beta_3 \beta_0^3), \quad (5.22b)
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\[ s_3 = \frac{1}{384\beta_0^2} (\beta_1^6 - 6\beta_1^5\beta_0^2 + 8\beta_1^4\beta_0^4 - 3\beta_1^3\beta_0^5 + 18\beta_1^2\beta_2\beta_0^3 - 24\beta_1^2\beta_2^2\beta_0^1 + 3\beta_1\beta_2^2\beta_0^2 - 6\beta_1\beta_2^3\beta_0^1 + 16\beta_1\beta_3\beta_0^4 - 3\beta_2^2\beta_3^1 + 8\beta_2^2\beta_3^2 + 6\beta_2\beta_3\beta_0^5 - 8\beta_4\beta_0^6) \]  

(5.22c)

Among theoretical uncertainties of the IRR-based formula (5.21) the absence of the information about explicit dependence of the factor \(N_m\) on the number of massless flavors \(n_l\) and on the order \(n\) of PT approximation is one of the most significant ambiguity. The value of \(N_m\) was extracted from rather careful studies, performed in the number of works, devoted to the consideration of the IRR contributions to the relation between pole and running masses of charm, bottom and top quarks (see e.g. [17, 32, 76] and the review of Ref.[25]). The most reasonable extracted results of \(N_m\), which depend on \(n_l\), vary in the interval \(N_m = 0.45 - 0.55\) for charm, bottom and top-quarks [32, 76]. Since we are interested in the comparison of the obtained in Sec.5.1 approximate ECH-motivated estimates with the results of the IRR-based estimates, which may have yet unfixed additional theoretical ambiguities [25], we use in our further studies the rather rough approximation putting \(N_m = 0.5\) for \(3 \leq n_l \leq 5\) at five and six-loop levels, which corresponds to the central point in the above mentioned interval.

Taking this into account and using Eq.(5.21), we obtain the corresponding values of \(t_{5}^{M,r-n}\) and \(t_{6}^{M,r-n}\)-coefficients, obtained within the IRR-based approach and presented in Table 3:

| \(n_l\) | \(t_{5}^{M,r-n}\) | \(t_{6}^{M,r-n}\) |
|-------|----------------|----------------|
| 3     | 31527          | 768520         |
| 4     | 22335          | 501230         |
| 5     | 15089          | 308590         |

**Table 3:** The estimates of \(t_{5}^{M}\) and \(t_{6}^{M}\)-contributions, obtained within the IRR technique with \(N_m = 0.5\).

In order to understand as far as the results, obtained by means of the ECH-motivated method and the IRR-based technique with approximation \(N_m = 0.5\), differ from each other for cases of charm, bottom and top-quarks, we consider the numerical values of \(t_{n}^{M,ECH}/t_{n}^{M,r-n}\)-terms. The results of these studies are presented in Table 4:

| \(n_l\) | \(t_{3}^{M,ECH}/t_{3}^{M,r-n}\) | \(t_{4}^{M,ECH}/t_{4}^{M,r-n}\) | \(t_{5}^{M,ECH}/t_{5}^{M,r-n}\) | \(t_{6}^{M,ECH}/t_{6}^{M,r-n}\) |
|-------|----------------|----------------|----------------|----------------|
| 3     | 1.21           | 0.81           | 0.90           | 0.62           |
| 4     | 1.14           | 0.80           | 0.77           | 0.47           |
| 5     | 1.05           | 0.79           | 0.60           | 0.29           |

**Table 4:** The numerical values of the \(t_{n}^{M,ECH}/t_{n}^{M,r-n}\)-ratio at \(N_m = 0.5\).
The numbers in the Table 4 indicate that all ratios decline monotonically with increasing of order of PT from 3 to 6 (the exception is the $t_5^{M,ECH}/t_5^{M,r-n}$-ratio for charm-quark only). From the figures one can see that up to $O(a_4^s)$ level the approximation $N_m = 0.5$ leads to values of considered ratios close to each other. At five-loop level the ratio $t_5^{M,ECH}/t_5^{M,r-n}$ is the most distant from unity for case of $t$-quark, but nevertheless these terms for $3 \leq n_l \leq 5$ are still quite close to unity. At six-loop level the numerical values of the ratios $t_6^{M,ECH}/t_6^{M,r-n}$ do not give satisfactory results. Indeed, in the $O(a_5^s)$ order of PT the numerical values for coefficients of the $\overline{\text{MS}}$-on-shell mass relation, obtained by means of the ECH-motivated method and the IRR-based approach with $N_m = 0.5$, differ by factor 2 and 3 for the cases of $c$, $b$ and $t$-quarks correspondingly. This may indicate to the existence of 1 theoretical uncertainties in the estimates, made by both approaches. In the case of the IRR-based estimates these uncertainties may be related say to missed effects of proportional to $\pi^2$ analytical continuation high order terms or extra inaccuracies, which may manifest themselves beyond consideration of the leading IRR contribution only [25], whereas the ambiguity of the ECH-motivated estimates of $t_6^{M,ECH}$-term is related to the quantitatively unfixed uncertainty of the assumption $\beta_5^{eff} \approx \beta_5$ and additional indeterminacy, outlined in the footnote at page 17 of this manuscript.

Based on the results, presented in Table 3, we obtain that within the IRR-based approach with $N_m = 0.5$ at the six-loop level the expansion for pole masses of heavy quarks has the following form:

\[
\frac{M_c}{1 \ \text{GeV}} \approx 1.275 + 0.214 + 0.208 + 0.295 + 0.541 + [1.258 + 3.854], \quad (5.23a)
\]

\[
\frac{M_b}{1 \ \text{GeV}} \approx 4.180 + 0.400 + 0.200 + 0.146 + 0.137 + [0.178 + 0.287], \quad (5.23b)
\]

\[
\frac{M_t}{1 \ \text{GeV}} \approx 164.300 + 7.566 + 1.614 + 0.498 + 0.196 + [0.122 + 0.086], \quad (5.23c)
\]

where all four-loop approximations are taken from Sec.3. For the case of charm-quark the application of the renormalon-dominated hypothesis for estimation of the five-loop correction gives very close value of $t_5^{M,ECH}$-coefficient to the ECH estimation, presented in Eq.(5.19a). At the six-loop approximation the difference between corresponding contributions, obtained by the both considered estimate procedures, is more substantial. However, they predict the same in order of magnitude corrections, which exceed the value of the first term in expansion of the pole mass of $c$-quark. Concerning the $b$-quark we have a completely different situation. Indeed, the ECH-motivated approach demonstrates output to some kind of plateau, whereas the renormalon-based estimations testify the growth of these corrections. One should emphasize that the both ways indicate the five-loop contribution to the mass conversion formula is either equal to or greater than the four-loop expression, but not less than it. Therefore we conclude that the strict asymptoticity of the series for pole mass of bottom quark begins with the $O(a_5^s)$ level of PT. For $t$-quark the both estimate methods outline the decrease of the five and six-loop corrections. However, the ECH-inspired method predicts slightly smaller values of these contributions. The IRR analysis provides us the following $O(a_6^s)$ value of the pole mass of top-quark $M_t^{r-n} \approx 174.382 \ \text{GeV}$ against $M_t^{ECH} \approx 174.273 \ \text{GeV}$, obtained by the ECH-motivated approach.
Thus, both estimate procedures, namely the ECH-inspired method and the IRR-based approach, indicate that the conception of pole mass of the top-quark can be safely used even at the $O(a_s^6)$ level. But we are also interested in finding the number of order of PT, starting from which the asymptotic behavior of the corresponding QCD PT series for $t$-quark begins to manifest itself. This goal is not yet achieved by us. Therefore we apply the IRR-based technique to estimate it. On the average, neglecting the dependence of the $N_m$-term on the order of PT and putting $N_m = 0.5$ we estimate roughly all multiloop corrections to the top-quark pole mass up to the order when the coefficients of its series begin to grow. Using Eq.(5.21) we obtain the following expansion:

$$\frac{M_t}{1\text{ GeV}} \approx 164.300 + 7.566 + 1.614 + 0.196 + 0.122 + 0.086 + 0.072 + 0.070 + 0.077 + 0.095 + \ldots$$

This estimate IRR procedure allows us to understand approximately, from what level of PT the asymptotic behavior of the QCD series for pole mass of $t$-quark begins to manifest itself. The first traces of this effect can already be observed in the seven order of PT. The eighth and ninth contributions are either comparable or exceed the value of the seventh correction.

6 Conclusion

In this work we evaluate the two unknown in analytical form four-loop coefficients and their uncertainties in the $\overline{\text{MS}}$-on-shell mass relation for heavy quarks in QCD. Herewith we use the results of numerical calculations [15] and the least squares method that allows one to take into account the correlation in overdetermined systems of linear equations. In the case of $t$-quark correction of the fourth order may be important for theoretical and experimental studies. Indeed, it is comparable with the current statistical uncertainty of the experimental top-quark pole mass value. Also the $\overline{\text{MS}}$-on-shell mass relation for charged leptons in QED are presented up to the four-loop level. Applying the ECH-motivated approach with the $\pi^2$-effects of the analytic continuation from the Euclidean to Minkowskian region we obtain the five and six-loop contributions to the QCD $\overline{\text{MS}}$-on-shell relation. The plausibility of the obtained values is demonstrated. Using the numerical results for these corrections we find the explicit flavor dependence in the $O(a_s^5)$ and $O(a_s^6)$ orders of PT. We indicate that ECH-motivated method for pole mass of $b$-quark leads to the effect of a plateau, whereas for top-quark the five and six-loop corrections are decreased. The infrared renormalon-based analysis is also considered. In the framework of the renormalon-dominated hypothesis we estimate $O(a_s^5)$ and $O(a_s^6)$-contributions to the pole masses of $c$, $b$ and $t$-quarks. The results of this approach show qualitatively different behavior of the five and six-loop corrections for $b$-quark and similar for $c$ and $t$-quark. The renormalon-based analysis is applied up to 10 order of PT and we conclude that the asymptotic behavior for expansion of the pole mass of top-quark through its running mass begins to manifest itself somewhere at the 7 or 8 level of PT. Therefore the concept of the pole mass of $t$-quark is applicable up to 6 order of PT for sure.
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A The explicit flavor and scale dependence of the \( \overline{\text{MS}} \)-on-shell relation in QED at the four-loop level

Applying the procedure of solving RG-equations, described in Ref. [9], we obtain the explicit dependence on scale parameter \( \mu^2 \) and number of massless charged leptons \( n_l \) for the expansion of the pole masses through its running analogs in QED in the \( \mathcal{O}(a^4) \) approximation:

\[
M_l = \overline{m}_l(\mu^2)^2 \left(1 + \sum_{i=1}^{4} \delta_i(\mu^2)a_i(\mu^2) + \mathcal{O}(a^5)\right),
\]

(A.1)

\[\delta_1 = 1 + 0.75L,\]
\[\delta_2 = 1.66591 - 1.56205n_l + (0.30208 - 0.541667n_l)L + (0.15625 - 0.125n_l)L^2,\]
\[\delta_3 = -2.02840 - 3.899056n_l + 1.9580722n_l^2 + (0.88383 - 1.678357n_l + 0.9603484n_l^2)L\]
\[+ (-0.12413 - 0.295139n_l + 0.1805556n_l^2)L^2 + (0.00434 - 0.038194n_l + 0.0277778n_l^2)L^3,\]
\[\delta_4 = 5.482 \pm 1.030 + (-4.219 \pm 0.178)n_l + 8.7138n_l^2 - 4.06885n_l^3\]
\[+ (-3.480 + 0.7686n_l + 4.31421n_l^2 - 1.923313n_l^3)L\]
\[+ (0.253 - 0.0959n_l + 0.90924n_l^2 - 0.48017n_l^3)L^2\]
\[+ (-0.022 + 0.0547n_l + 0.112847n_l^2 - 0.0601852n_l^3)L^3\]
\[+ (-0.00027 + 0.001302n_l + 0.0078125n_l^2 - 0.00694444n_l^3)L^4,\]

where \( L = \log(\mu^2/M_l^2), \) \( M_l \) is the pole mass of charged lepton. For electron case we presume \( n_l = 0, \) for muon \( n_l = 1 \) and taon \( n_l = 2 \) correspondingly.

B The solutions of the RG equations at six-loop level

The solution of the RG-equation (5.4) can be expressed through the log-dependent terms \( \theta_1 - \theta_6, \) through known coefficients \( \beta_0 - \beta_4 \) of the \( \beta \)-function, calculated in the \( \overline{\text{MS}} \)-scheme, and yet unknown six-loop coefficient \( \beta_5 \) in the following form ( \( l = \log(\mu^2/s) \)):

\[
a_5(s) = a_5(\mu^2) + \sum_{n=1}^{6} \theta_n a_n^{n+1}(\mu^2) + \mathcal{O}(a_6^5),
\]

(B.1)

\[
\theta_1 = \beta_0 l, \quad \theta_2 = \beta_0^3 l^2 + \beta_1 l, \quad \theta_3 = \beta_0^3 l^3 + \frac{5}{2} \beta_0 \beta_1 l^2 + \beta_2 l,
\]
\[
\theta_4 = \beta_0^5 l^4 + \frac{13}{3} \beta_0^3 \beta_1 l^3 + \frac{3}{2} \beta_0 \beta_2 + \frac{3}{2} \beta_1^2 l^2 + \beta_3 l,
\]
\[
\theta_5 = \frac{77}{12} \beta_0^5 \beta_1 l^4 + \frac{35}{6} \beta_0 \beta_2 + \frac{35}{6} \beta_0^3 \beta_1^2 l^3 + \frac{7}{2} \beta_0 \beta_3 + \beta_1 \beta_2 l^2 + \beta_4 l,
\]

\[
\theta_6 = \frac{77}{12} \beta_0^5 \beta_1 l^4 + \frac{35}{6} \beta_0 \beta_2 + \frac{35}{6} \beta_0^3 \beta_1^2 l^3 + \frac{7}{2} \beta_0 \beta_3 + \beta_1 \beta_2 l^2 + \beta_4 l,
\]
Immediate integration of Eq. (5.9) and using of the expansion (B.1) leads us to the following cumbersome formulas:

\[
\bar{m}(s) = \bar{m}(\mu^2) \left( 1 + \sum_{n=1}^{6} b_n a_n^\omega(\mu^2) \right),
\]

\[
b_1 = \gamma_0 l, \quad b_2 = \frac{\gamma_0}{2} (\beta_0 + \gamma_0) t^2 + \gamma_1 l,
\]

\[
b_3 = \frac{\gamma_0}{3} (\beta_0 + \gamma_0) \left( \beta_0 + \frac{\gamma_0}{2} \right) \left( \beta_1 \frac{\gamma_0}{2} + \gamma_1 \beta_0 + \gamma_1 \gamma_0 \right) l^2 + \gamma_2 l,
\]

\[
b_4 = \frac{\gamma_0}{4} (\beta_0 + \gamma_0) \left( \beta_0 + \frac{\gamma_0}{2} \right) \left( \beta_0 + \frac{\gamma_0}{3} \right) \left( \beta_0 + \frac{\gamma_0}{4} \right) l^3
\]

\[
+ \left( \beta_2 \frac{\gamma_0}{2} + \gamma_1 \beta_1 + \frac{\gamma_1}{2} \right) l^2 + \gamma_3 l,
\]

\[
b_5 = \frac{\gamma_0}{5} \left( \beta_0 + \gamma_0 \right) \left( \beta_0 + \frac{\gamma_0}{2} \right) \left( \beta_0 + \frac{\gamma_0}{3} \right) \left( \beta_0 + \frac{\gamma_0}{4} \right) l^3
\]

\[
+ \left( \gamma_1^{\beta_0} + \frac{13}{12} \gamma_0 \beta_1^{\beta_0} + \frac{13}{12} \gamma_0 \beta_1^{\beta_0} + \frac{11}{6} \gamma_0 \gamma_1 \beta_0 + \gamma_0 \gamma_1 \beta_0 + \frac{1}{4} \beta_1^{\gamma_0} + \frac{1}{6} \gamma_1 \gamma_0 \right) l^4
\]

\[
+ \left( \gamma_0 \beta_2 \beta_0 + 2 \gamma_0 \beta_0 \gamma_2 + \gamma_0 \gamma_0 \gamma_0 \gamma_0 + \frac{3}{\gamma_0} \gamma_0 \gamma_0 \gamma_0 + \frac{1}{\gamma_0} \gamma_0 \gamma_0 \gamma_0 + \frac{1}{2} \gamma_0 \gamma_0 \gamma_0 + \frac{1}{2} \gamma_0 \gamma_0 \gamma_0 \right) l^2 + \gamma_4 l,
\]

\[
b_6 = \frac{\gamma_0}{6} \left( \beta_0 + \gamma_0 \right) \left( \beta_0 + \frac{\gamma_0}{2} \right) \left( \beta_0 + \frac{\gamma_0}{3} \right) \left( \beta_0 + \frac{\gamma_0}{4} \right) l^4
\]

\[
+ \left( \frac{5}{12} \beta_0 \gamma_0 + \gamma_0 \beta_0 + \frac{1}{24} \gamma_1 \gamma_0 + \gamma_0 \beta_0 \gamma_0 + \frac{5}{2} \beta_0 \beta_1 \gamma_0 + \frac{35}{24} \beta_0 \gamma_0 \gamma_1 + \frac{2}{3} \beta_0 \beta_1 \gamma_0 + \frac{77}{60} \beta_0 \gamma_0 \gamma_0 \right) l^5
\]

\[
+ \left( \frac{1}{12} \beta_0 \gamma_0 + \gamma_0 \beta_0 + \frac{1}{24} \gamma_1 \gamma_0 + \gamma_0 \beta_0 \gamma_0 + \frac{5}{2} \beta_0 \beta_1 \gamma_0 + \frac{35}{24} \beta_0 \gamma_0 \gamma_1 + \frac{2}{3} \beta_0 \beta_1 \gamma_0 + \frac{77}{60} \beta_0 \gamma_0 \gamma_0 \right) l^5
\]

\[
+ \gamma_5 l,
\]

where coefficients \( \gamma_0 - \gamma_4 \) are known contributions to the anomalous dimension of mass in the \( \overline{\text{MS}} \)-scheme and six-loop term \( \gamma_5 \) is not yet evaluated.
C The relation between Euclidean and Minkowskian coefficients

The integration over $s$ in Eq. (5.1) and the setting $\mu^2 = Q^2$ lead to the following relations for $0 \leq n \leq 6$:

$$f_n^E = t_n^M + \Delta_n,$$

(C.1)

$$\Delta_0 = 0, \quad \Delta_1 = 0, \quad \Delta_2 = \frac{\pi^2}{6} \gamma_0 (\beta_0 + \gamma_0) t_0^M,$$

$$\Delta_3 = \frac{\pi^2}{3} \left[ t_1^M (\beta_0 + \gamma_0) \left( \beta_0 + \frac{1}{2} \gamma_0 \right) + t_0^M \left( \frac{1}{2} \beta_1 \gamma_0 + \gamma_1 \beta_0 + \gamma_1 \gamma_0 \right) \right],$$

$$\Delta_4 = \frac{\pi^2}{3} \left[ t_2^M \left( 3 \beta_0^2 + \frac{5}{2} \beta_0 \gamma_0 + \frac{1}{2} \gamma_0^2 \right) + t_1^M \left( \frac{3}{2} \beta_1 \gamma_0 + \frac{5}{2} \beta_1 \beta_0 + 2 \gamma_1 \beta_0 + \gamma_1 \gamma_0 \right) \right. \right.$$

$$+ t_0^M \left( \frac{1}{2} \beta_2 \gamma_0 + \gamma_1 \beta_1 + \frac{1}{2} \gamma_1^2 + \frac{3}{2} \gamma_2 \beta_0 + \gamma_2 \gamma_0 \right) \left. \right]$$

$$+ \frac{7 \pi^4}{60} t_0^M \gamma_0 (\beta_0 + \gamma_0) \left( \beta_0 + \frac{1}{2} \gamma_0 \right) \left( \beta_0 + \frac{1}{3} \gamma_0 \right),$$

$$\Delta_5 = \frac{\pi^2}{3} \left[ t_3^M \left( 6 \beta_0^3 + \frac{7}{2} \beta_0 \gamma_0 + \frac{1}{2} \gamma_0^2 \right) + t_2^M \left( 7 \beta_1 \beta_0 + 3 \gamma_1 \beta_0 + \frac{5}{2} \beta_1 \beta_0 + \gamma_1 \gamma_0 \right) \right.$$

$$+ t_1^M \left( \frac{3}{2} \beta_2 \gamma_0 + \gamma_2 \beta_1 + \frac{3}{2} \gamma_2 \beta_1 + 2 \gamma_3 \beta_0 + \gamma_1 \gamma_2 + \gamma_0 \gamma_3 \right) \left. \right]$$

$$+ \frac{7 \pi^4}{15} t_1^M \left( \beta_0^4 + \frac{25}{12} \beta_0^3 \gamma_0 + \frac{35}{24} \beta_0^2 \gamma_0^2 + \frac{5}{12} \beta_0 \gamma_0^3 + \frac{1}{24} \gamma_0^4 \right)$$

$$+ t_0^M \left( \gamma_1 \beta_0^3 + \frac{13}{12} \gamma_0 \beta_1 \beta_0^2 + \frac{13}{12} \gamma_0^2 \beta_0 \beta_1 + \frac{11}{6} \gamma_0 \gamma_1 \beta_0^2 + \gamma_0^2 \beta_0 \gamma_1 + \frac{1}{4} \beta_1 \gamma_0^3 + \frac{1}{6} \gamma_1 \gamma_0^3 \right),$$

$$\Delta_6 = \frac{\pi^2}{3} \left[ t_4^M \left( 10 \beta_0^2 + \frac{9}{2} \beta_0 \gamma_0 + \frac{1}{2} \gamma_0^2 \right) + t_3^M \left( \frac{27}{2} \beta_0 \beta_1 + 4 \beta_0 \gamma_1 + \frac{7}{2} \beta_1 \gamma_0 + \gamma_0 \gamma_1 \right) \right.$$

$$+ t_2^M \left( 8 \beta_0 \beta_2 + \frac{7}{2} \beta_0 \gamma_2 + 3 \beta_2 \beta_0 + \frac{5}{2} \beta_2 \gamma_0 + 4 \beta_1^2 + \frac{1}{2} \gamma_1^2 + \gamma_0 \gamma_2 \right) \left. \right]$$

$$+ t_1^M \left( \frac{7}{2} \beta_0 \beta_3 + \frac{7}{2} \beta_1 \beta_2 + 3 \beta_0 \gamma_3 + \frac{5}{2} \beta_0 \gamma_2 + 2 \gamma_2 \beta_1 + \frac{3}{2} \beta_1 \gamma_0 + \gamma_0 \gamma_3 + \gamma_1 \gamma_2 \right)$$

$$+ t_0^M \left( \frac{1}{2} \gamma_2^2 + \frac{3}{2} \beta_2 \gamma_2 + \frac{5}{2} \beta_0 \gamma_4 + 2 \beta_1 \gamma_3 + \beta_3 \gamma_1 + \frac{1}{2} \beta_4 \gamma_0 + \gamma_0 \gamma_4 + \gamma_1 \gamma_3 \right) \left. \right]$$

$$+ \frac{7 \pi^4}{15} t_2^M \left( \frac{25}{12} \beta_0^4 + \frac{277}{24} \beta_0^3 \gamma_0 + \frac{71}{12} \beta_0^2 \gamma_0^2 + \frac{7}{12} \beta_0 \gamma_0^3 + \frac{1}{24} \gamma_0^4 \right)$$

$$+ t_1^M \left( \frac{77}{12} \beta_0^3 \beta_1 + \frac{5}{2} \beta_1 \gamma_0 + 4 \beta_0 \gamma_1 + \frac{10}{3} \beta_0 \beta_1 \gamma_0^2 + \frac{25}{3} \beta_0^2 \beta_1 \gamma_0 + \frac{3}{2} \beta_0 \gamma_2 \gamma_1 \right. \left. \right]$$

$$+ \frac{13}{3} \beta_0 \gamma_0 \gamma_1 \right) + t_0^M \left( \frac{1}{4} \beta_2 \gamma_0^3 + \frac{5}{2} \beta_0 \beta_2 \gamma_2 + \frac{1}{6} \gamma_0 \gamma_2^3 + \frac{3}{2} \beta_0^2 \gamma_1 + \frac{5}{8} \beta_1 \gamma_0^3 + \frac{1}{4} \gamma_0 \gamma_1^2 \right.$$

$$+ \frac{35}{24} \beta_0 \beta_2^2 \gamma_0 + \frac{5}{4} \beta_0 \beta_2 \gamma_2^2 + \frac{47}{12} \beta_0 \beta_1 \gamma_1 + \frac{3}{2} \beta_0^2 \beta_2 \gamma_0 + \frac{5}{4} \beta_0 \gamma_0 \gamma_1^2 + \frac{5}{4} \beta_0 \gamma_0^2 \gamma_2 + \beta_1 \gamma_0 \gamma_1$$
\[
\Delta_6 = 5.89434 - 0.274156 n_l, \quad (C.2)
\]
\[
\Delta_3 = 105.6221 - 10.04477 n_l + 0.198002 n_l^2, \\
\Delta_4 = 2272.002 - 403.9489 n_l + 20.67673 n_l^2 - 0.315898 n_l^3, \\
\Delta_5 = 56304.639 - 13767.2725 n_l + 1137.17794 n_l^2 - 37.745285 n_l^3 + 0.427523 n_l^4, \\
\Delta_6 = 163315.62 \pm 347.27 + (-518511.694 \pm 56.723) n_l + (61128.1666 \pm 4.7791) n_l^2 \\
+ (-3345.0818 \pm 0.1371) n_l^3 + 85.37937 n_l^4 - 0.818446 n_l^5. \\
\]

Note that analytical expressions for \(\Delta_0 - \Delta_4\)-terms are presented in Refs.\[28, 44\]. The mean-square uncertainties in \(\Delta_6\)-term are defined by the errors, obtained by the least squares method in Eq.(2.5a).

One should emphasize that Eqs.(C.2) demonstrates the significant growth of the \(\Delta_n\)-terms with increasing of the order \(n\) of PT. Moreover, their values are comparable with coefficients \(t^M_n\), determined in the Minkowski region. Therefore, we conclude that the kinematic \(\pi^2\) effects of analytical continuation are not negligible. Their numerical values are presented in Table 5:

| \(n_l\) | \(\Delta_2\) | \(\Delta_3\) | \(\Delta_4\) | \(\Delta_5\) | \(\Delta_6\) |
|--------|--------|--------|--------|--------|--------|
| 3      | 5.072  | 77.270 | 1237.717 | 24252.930 | 544133.68±389.46 |
| 4      | 4.798  | 68.611 | 966.817  | 17124.144 | 344053.30±422.21 |
| 5      | 4.524  | 60.348 | 729.689  | 11446.766 | 201430.55±464.61 |

Table 5: The numerical values of the \(\Delta_n\)-contributions.

D The explicit flavor and scale dependence of the \(\overline{\text{MS}}\)-on-shell relation in QCD at the six-loop level

The solution of the RG-equations for evolution of the QCD coupling constant and of the running masses of the heavy quarks at six-loop level allows us to restore all characteristic RG
logarithms. Taking into account the expressions for five and six-loop contributions $t_5^{M,ECH}$ and $t_6^{M,ECH}$, expanded in powers of the number of massless flavors $n_l$ in Eq.(5.17-5.18), we obtain the following relation between pole and running mass of heavy quarks:

\[
M_q = \bar{m}_q(\mu^2) \left( 1 + \sum_{n=1}^{6} \rho_n a^n_s(\mu^2) + \mathcal{O}(a_s^7) \right), \quad (D.1)
\]

\[
\rho_1 = \frac{4}{3} + L,
\]

\[
\rho_2 = 16.1100 - 1.04136 n_l + (8.8472 - 0.3611 n_l) L + (1.7917 - 0.08333 n_l) L^2,
\]

\[
\rho_3 = 239.296 - 29.7008 n_l + 0.65269 n_l^2 + (129.420 - 15.3706 n_l + 0.32011 n_l^2)
\]

\[
+ (32.105 - 3.0533 n_l + 0.06019 n_l^2) L^2 + (3.683 - 0.3704 n_l + 0.00926 n_l^2) L^3,
\]

\[
\rho_4 = 4469.04 \pm 1.34 + (-864.14 \pm 0.15) n_l + 46.3073 n_l^2 - 0.67814 n_l^3
\]

\[
+ (2466.41 - 450.371 n_l + 22.7378 n_l^2 - 0.32055 n_l^3) L + (651.23 - 113.230 n_l
\]

\[
+ 5.5876 n_l^2 - 0.08003 n_l^3) L^2 + (102.74 - 15.708 n_l + 0.7323 n_l^2 - 0.01003 n_l^3) L^3
\]

\[
+ (8.06 - 1.271 n_l + 0.0666 n_l^2 - 0.00116 n_l^3) L^4,
\]

\[
\rho_5 \approx 105540 - 30748 n_l + 3125 n_l^2 - 139 n_l^3 + 2.5 n_l^4 + (57138 - 14406 n_l
\]

\[
+ 1213 n_l^2 - 40 n_l^3 + 0.45 n_l^4) L + (15644 - 3775 n_l + 305 n_l^2 - 9.8 n_l^3 + 0.11 n_l^4) L^2
\]

\[
+ (2715 - 628 n_l + 50 n_l^2 - 1.6 n_l^3 + 0.018 n_l^4) L^3 + (310 - 66 n_l + 5 n_l^2 - 0.16 n_l^3
\]

\[
+ 0.0017 n_l^4) L^4 + (18 - 4 n_l + 0.3 n_l^2 - 0.011 n_l^3 + 0.00015 n_l^4) L^5,
\]

\[
\rho_6 \approx 2559767 - 985344 n_l + 143731 n_l^2 - 10216 n_l^3 + 2 n_l^4 - 4.9 n_l^5
\]

\[
+ (\gamma_5(n_l) + 1609024 - 560836 n_l + 73900 n_l^2 - 4735 n_l^3 + 153 n_l^4 - 2.1 n_l^5) L
\]

\[
+ (436294 - 136060 n_l + 15671 n_l^2 - 832 n_l^3 + 20.5 n_l^4 - 0.19 n_l^5) L^2
\]

\[
+ (79218 - 23850 n_l + 2662 n_l^2 - 138 n_l^3 + 3.3 n_l^4 - 0.03 n_l^5) L^3
\]

\[
+ (10189 - 2961 n_l + 323 n_l^2 - 16.6 n_l^3 + 0.4 n_l^4 - 0.004 n_l^5) L^4
\]

\[
+ (906 - 247 n_l + 26 n_l^2 - 1.3 n_l^3 + 0.03 n_l^4 - 0.0003 n_l^5) L^5
\]

\[
+ (42 - 11.7 n_l + 1.3 n_l^2 - 0.07 n_l^3 + 0.002 n_l^4 - 0.00002 n_l^5) L^6,
\]

where $L = \log(\mu^2/M_q^2)$, the five and six-loop contributions $\rho_5$ and $\rho_6$ are derived by the method of the effective charges and are rather rough estimates. Nevertheless this approach allows to reproduce the sign-alternating structure of these corrections in front of each logarithmic-dependent term. All these corrections possesses one common property: they all have a large positive term with $n_l^0$-dependence and contain decreasing in absolute values of the remaining $n_l$-dependent terms. The $\mathcal{O}(a_s^6)$-contribution $\rho_6$ contains also in the linear logarithmic-dependent term with still unknown six-loop coefficient $\gamma_5$ of the anomalous mass dimension. However, note that this unknown term does not affect the sign-alternating structure of the L-dependent correction.

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