Packing theory derived from phyllotaxis and products of linear forms

S. E. Graiff Zurita∗1, B. Kane†2, and R. Oishi-Tomiyasu‡3

1Graduate School of Mathematics, Kyushu University
2Department of Mathematics, University of Hong Kong
3Institute of Mathematics for Industry (IMI), Kyushu University

Abstract

Parastichies are spiral patterns observed in plants and numerical patterns generated using golden angle method. We generalize this method by using Markoff theory and the theory of product of linear forms, to obtain a theory for packing of Riemannian manifolds of general dimensions n with a locally diagonalizable metric, including the Euclidean spaces. For example, packings in a plane with logarithmic spirals and in a 3D ball (3D analogue of the Vogel spiral) are newly obtained. Using this method, we prove that it is possible to generate almost uniformly distributed point sets on any real analytic Riemannian surfaces in a local sense. We also discuss how to extend the packing to the whole manifold in some special cases including the Vogel spiral. The packing density is bounded below by approximately 0.7 for surfaces and 0.38 for 3-manifolds under the most general assumption.

Keywords: aperiodic packing; Markoff spectra; Lagrange spectra; products of linear forms; diagonalizable metric; inviscid Burgers equation; Vogel spiral; Doyle spiral

1 Introduction

In botany, spiral patterns observed in plants, such as sunflower heads, cacti, and pinecones, are called parastichies. Parastichies can also be observed in patterns that are numerically generated on a surface with circular symmetry using golden angle method. In the golden angle method, a new point is generated for each \(2\pi\varphi \approx 137.5^\circ\)-rotation, where \(\varphi = 1/(1 + \gamma)\) is the golden angle defined using

∗s-graiff@math.kyushu-u.ac.jp
†bkane@hku.hk
‡tomiyasu@imi.kyushu-u.ac.jp (corresponding author)
the golden ratio $\gamma_1 = (1 + \sqrt{5})/2$. With this method, packing patterns have been generated on a cylindrical surface [8], a disk known as the Vogel spiral [28], [41] (Figure 1), surfaces of revolution [33], sphere surface for mesh generation on globe [39], [18], and Poincaré disc [35].

Figure 1: Left: Vogel spiral $\sqrt{n e^{\frac{2 \pi n}{1 + \gamma_1}}}$ ($n > 0$: integer) and images of the lattice shortest vectors (arrows) that indicate the directions of parastichies, Right: Doyle spiral of type $(12, 24)$ dealt with in Example 2.

The generated patterns can be regarded as the image of the lattice with the basis:

$$\begin{pmatrix} 1 \\ 0 \\ \varphi \\ \varepsilon_0 \end{pmatrix}, \quad \varphi = \frac{1}{1 + \gamma_1}, \quad \varepsilon_0 : \text{constant.} \quad (1)$$

Variations of the Vogel spiral can be created by substituting different $\varphi$ [32]. While the golden angle method is applicable to surfaces with circular symmetry, it has not been known how to generalize it on general surfaces or manifolds in higher dimensions, maintaining the packing density in a certain range. The golden angle method for higher dimensions has been posed as an open problem [24], [2]. The main goal of this paper is to establish a general theory and a concrete method for that.

As a packing, we shall consider an arrangement of non-overlapping $n$-dimensional balls of the identical radius. Although the case of varying radii is not the scope of this article, the Doyle spiral (Figure 1) has also been addressed as a pattern of parastichies [4] and known as the case of conformal mapping. In fact, the Doyle spiral and the Vogel spiral can be constructed within the same framework (Example 2).

For the above goal, it is necessary to determine the optimal lattice of general rank that will be used in our packing method. As shown in (2) of Theorem 1, even the lattice of rank 2 given by (1) is not optimal for bounding the packing density by a larger lower limit.

For any full-rank lattice $L \subset \mathbb{R}^n$, let $\Delta(L)$ be the packing density [4] of the lattice packing given by $L$. For any $B_n \in GL_n(\mathbb{R})$, let $L(B_n)$ be the lattice generated by the column vectors of $B_n$. The following problem will be discussed in Section 2 for determining the optimal lattices, using known theorems for products of linear forms.
Problem: Determine the lattice basis $B \in \text{GL}_n(\mathbb{R})$ with $\Delta_n' = \Delta'(L(B))$.

\[ \Delta_n' := \sup_{B \in \text{GL}_n(\mathbb{R})} \Delta'(L(B)), \quad \Delta'(L(B)) := \inf_{D \in \text{GL}_n(\mathbb{R})} \Delta(L(DB)) \hspace{1cm} (2) \]

The answer for the above problem enables us to expand the scope of the golden angle method to higher dimensions. The packing densities of the obtained aperiodic packings are bounded below approximately by $\Delta_n'$ for each dimension $n$:

\[ \Delta_2' = \frac{\pi}{2\sqrt{5}} \approx 0.702, \quad \Delta_3' = \frac{\sqrt{3}\pi}{14} \approx 0.389, \]

\[ \Delta_4' \geq \frac{\pi^2}{10\sqrt{29}} \approx 0.183, \quad \Delta_5' \geq \frac{5\sqrt{5}\pi^2}{12 \cdot 11^2} \approx 0.076. \]

In addition to such a lattice, a map $f(x) = (f_1, \ldots, f_N)$ from an open subset $\mathcal{D} \subset \mathbb{R}^n$ to $\mathbb{R}^N$ ($n \leq N$) with the Jacobian matrix $Df(x) = (\partial f_i/\partial x_j)_{1 \leq i, j \leq n}$ satisfying $(\star)$ and $(\star\star)$ below, is used in our packing method.

$(\star)$ $\det(Df)Df$ is an invertible diagonal matrix for any $x \in \mathcal{D}$ (diagonalizable).

$(\star\star)$ $\det(Df)Df = c^2$ for some constant $c \neq 0$ (volume-preserving).

For a fixed pair of a map $f : \mathcal{D} \to \mathbb{R}^N$ and a lattice $L \subset \mathbb{R}^n$, a packing of $f(\mathcal{D})$ is provided as $f(L \cap \mathcal{D})$.

As a result of $(\star\star)$, the mesh generated as the Voronoi diagram of the point packing is equiareal, i.e., its Voronoi cells have approximately the same volume. $(\star)$ means that the metric on $f(\mathcal{D}) \subset \mathbb{R}^N$ induced from the Euclidean metric of $\mathbb{R}^N$ is diagonal. It is known that diagonalization of the metric is possible for any $C^\infty$ Riemannian manifolds of dimensions $n \leq 3$ in a local sense. The case of $n = 2$ follows from the existence of conformal metric $\lambda(x, y)(dx^2 + dy^2)$, and the case of $n = 3$ was proved in [14].

The main purpose of Section 3 is to present new aperiodic packings obtained by our generalization, discussing the system of partial differential equations (PDEs) that provide $f$ satisfying $(\star)$, $(\star\star)$ for $n = N$. The case of only $(\star)$ and $n = N$ is known as the problem on the $n$-orthogonal curvilinear coordinate systems and has been studied extensively [12], [43], but the case with additional constraint $(\star\star)$ has been much less studied with a few exceptions such as [22].

The results for the general case $n \leq N$ are summarized in Section 4. In Theorem 2, we prove that any real analytic Riemannian surface has an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ such that every $U_\alpha$ has the metric our packing method can be applied to. The same is clearly true for $n = 1$ in case of piecewise differentiable curves, because $(\star)$ and $(\star\star)$ means that $f$ is a parametrization of $f(\mathcal{D})$ by the arc-length.

Subsequently, a family of the PDE solutions for general dimensions is provided (Theorem 3) to explain the self-similarity observed among the aperiodic packings.
Another family obtained from solutions of the inviscid Burgers equation \(u_t + uu_x = 0\), also has the same self-similarity (Example 4).

As a motivation for Theorem 3, we remark that the Vogel spiral and the phyllotaxis model can be regarded as a growing disk and a growing cylindrical surface, respectively. This aspect of the golden angle method seems to have been not clear, while it is only applicable to surfaces with circular symmetry. Even after the generalization, the obtained packings still have the following commonly features and an appearance reminiscent of biological growth as seen in Figures 5, 6.

Firstly, the packings presented in this article are the images of lattice points contained in a rectangle \(\mathcal{D}\) (or a rectangular parallelepiped for the 3D case), which suggests that there is an inductive way to construct the PDE solutions. Secondly, \(f(\mathcal{D})\) has a codimension-1 foliation \(\{L_t\}\) such that any leaves \(L_t \subset f(\mathcal{D})\) are similar to each other in \(\mathbb{R}^N\). More specifically, if the last entry \(x_n\) is separated so that \(x = (x_{n-1}, x_n)\) and \(\mathcal{D} = \mathcal{D}_2 \times I (\mathcal{D}_2 \subset \mathbb{R}^{n-1}, I \subset \mathbb{R})\), every \(f\) can be represented by

\[
f(x) = e^{\alpha(h(x))}U(h(x))f_2(x_{n-1}) + v_0 \tag{3}
\]

for some functions \(h : \mathcal{D} \to \mathbb{R}\), \(\alpha : h(\mathcal{D}) \to \mathbb{R}\), \(U : h(\mathcal{D}) \to O(N)\), \(f_2 : \mathcal{D}_2 \to \mathbb{R}^N\) and \(v_0 \in \mathbb{R}^N\). Thus, if \(t = h(x)\) is considered as the time variable, the image of \(\{x \in \mathcal{D} : t = h(x)\}\) by \(f\) has the identical shape as \(f_2(\mathcal{D}_2)\) for any \(t \in I\).

Theorem 3 provides a method to construct \((n + 1)\)-dimensional Riemannian manifolds with a diagonal and constant-determinant metric from manifolds of dimension \(n\) with such a metric. In particular, for dimensions \(n = 2, 3\), Theorem 3 can provide a large family of lattice maps in the form of Eq. (3) that fulfill (⋆) and (⋆⋆).

As a result, the golden angle method has been generalized by using volume-preserving maps, based on algebraic properties of certain special lattices. Aperiodic packing has a number of applications. When quasicrystals were discovered [37], the Penrose tiling was immediately suggested as a mathematical model of aperiodic structures with a long-range order [25]. More recently, iterative algorithms for generating circle packings on Riemann surfaces have been studied [9], inspired by the Koebe-Andreev-Thurston theorem [20], [3], [40] and the Thurston conjecture proved in [34]. The generalized golden angle method will also broaden the range of applications.

### Methods to color point sets

All the presented packings are 2D or 3D scatter plots displayed by Wolfram Mathematica. The points are colored by either of two methods. The first one colors each point according to the local packing density in its neighborhood. In the considered situation in which the lattice basis \(B\) and the map \(f\) on \(\mathcal{D} \subset \mathbb{R}^n\) are specified, the local density around \(f(x)\) can be approximated from the packing density of the lattice \(L((Df(x))B)\) as justified in Theorem 3.
The second one colors each point, according to its birth time, by considering some \( x_i \) among \( x = (x_1, \ldots, x_n) \) as the time when the point \( f(x) \) is generated (e.g., Figure 5). This method allows easy observation of the self-similarity hidden in the image.

### Notation and symbols

The continued fractions are represented using squared brackets \([\ ]\) as follows.

\[
[a_0, a_1, a_2, \ldots, a_n, \ldots] := a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots + \frac{1}{a_n + \ldots}}}}.
\]

The \( n \)'th convergent \( p_n/q_n \) of \( \varphi = [a_0, a_1, a_2, \ldots] \) is the ratio of the coprime integers \( p_n, q_n \) that satisfy \( p_n/q_n = [a_0, a_1, a_2, \ldots, a_n] \). If there is an integer \( m \) > \( 0 \) such that \( a_{m+n} = a_n \) for any \( n \geq 0 \), \([a_0,a_1,\ldots]\) is called purely periodic. In such a case, \([a_0,a_1,\ldots]\) is abbreviated as \([a_0,a_1,\ldots,a_{m-1}]\).

For any lattice \( L \subset \mathbb{R}^n \) of full rank, we denote the packing density by

\[
\Delta(L) := \frac{\text{vol}(B(1))(\min L)^{n/2}}{2^n \text{vol}(\mathbb{R}^n/L)} = \frac{\pi^{n/2} (\min L)^{n/2}}{\Gamma(n/2 + 1) 2^n \text{vol}(\mathbb{R}^n/L)}, \tag{4}
\]

where \( \min L \) is defined by

\[
\min L := \{|l|^2 : 0 \neq l \in L\}, \tag{5}
\]

and \( \text{vol}(B(1)) \) and \( \text{vol}(\mathbb{R}^n/L) \) are the volumes of the \( n \)-ball with radius 1 and \( \mathbb{R}^n/L \).

\( B \in GL_n(\mathbb{R}) \) is called a basis matrix of \( L \) if the columns of \( B \) are a basis of \( L \).

Such a lattice \( L \) is also denoted as \( L(B) \) explicitly. For any diagonal \( D \in GL_n(\mathbb{R}) \), the lattice \( L(DB) \) is also denoted by \( D \cdot L \).

We define the Lagrange number and Markoff spectrum for Theorem 1.

**Definition 1.** For any real number \( \alpha \), the supremum \( \sup M \) of \( M \) that fulfills (*) is called the Lagrange number of \( \alpha \), and denoted by \( \mathcal{L}(\alpha) \).

\((*)\) Infinitely many rationals \( p/q \) satisfy \( |\alpha - p/q| < \frac{1}{Mq^2} \).

The set \( \{\mathcal{L}(\alpha) : \alpha \in \mathbb{R} \setminus \mathbb{Q}\} \) is called the Lagrange spectrum.

For any irrational \( \alpha \), its Lagrange number can be calculated from the continued fraction expansion \( \alpha = [a_0, a_1, a_2, \ldots] \) by the following formula (cf. Proposition 1.22, [1]):

\[
\mathcal{L}(\alpha) = \limsup_{n \to \infty} ([a_{n+1}, a_{n+2}, \ldots] + [0, a_n, a_{n-1}, \ldots, a_1]).
\]

For any indefinite binary quadratic form \( f(x,y) = ax^2 + bxy + cy^2 \) with real coefficients, its discriminant \( d(f) \) and \( m(f) \) are defined by:

\[
d(f) = b^2 - 4ac, \\
m(f) = \inf \{|f(x,y)| : 0 \neq (x,y) \in \mathbb{Z}^2\}.
\]
**Definition 2.** The set of \( M(f) := \sqrt{d(f)}/m(f) \) of all the indefinite binary quadratic forms over \( \mathbb{R} \) is the Markoff spectrum.

The Markoff theorem states that the Lagrange spectrum and the Markoff spectrum coincide below 3 [27]. In fact, any indefinite binary quadratic forms \( f \) over \( \mathbb{R} \) with \( M(f) < 3 \) has a root \( \alpha \) of \( f(x, 1) = 0 \) that fulfills \( M(f) = L(\alpha) \). Such an \( \alpha \) with \( L(\alpha) < 3 \) is a quadratic irrational with the continued fraction expansion \( \alpha = [a_0, \ldots, a_n, \gamma] \), where \( \gamma \) is equal to one of the following \( \gamma_m \):

\[
\gamma_m = \frac{m + 2u + \sqrt{9m^2 - 4}}{2m},
\]

(6)

where \( m \) is one of the Markoff numbers \( m = 1, 2, 5, 13, \ldots \). The integer \( 0 \leq u \leq m/2 \) is the solution of \( u^2 \equiv -1 \mod m \).

## 2 Parastichies from a viewpoint of lattice-basis reduction and Markoff theory

Mathematically, parastichies are the images of lines that connect lattice points with the shortest vectors. In order to see this more precisely, it is useful to review Markov theory from a viewpoint of the reduction theory of lattices. Proposition 1 describes a result on indefinite binary quadratic forms in terms of the reduction of positive-definite quadratic forms.

Let \( \varphi_1 > 0 > \varphi_2 \) be real numbers with the continued fraction expansions:

\[
\varphi_1 = [a_0, a_1, a_2, \ldots, a_n, \cdot \cdot \cdot], \quad -\varphi_2^{-1} = [a_{-1}, a_{-2}, \cdot \cdot \cdot, a_{-n}, \cdot \cdot \cdot].
\]

The doubly infinite sequence \( \{a_n\}_{n=-\infty}^{\infty} \) associated with \( (\varphi_1, \varphi_2) \) is obtained from the expansions. For simplicity, \( \varphi_1, \varphi_2 \) are assumed to be irrational. Hence \( a_n > 0 \) for any \( n \neq 0, -1 \).

Let \( p_n^{(+)} / q_n^{(+)} = [a_0, a_1, a_2, \cdots, a_n] \), \( p_n^{(-)} / q_n^{(-)} = [a_{-1}, a_{-2}, a_{-3}, \cdots, a_{-n}] \) be the \( n \)'th convergent of \( \varphi_1 \) and \( -\varphi_2^{-1} \). For \( n = -1, -2 \), we put:

\[
\begin{pmatrix}
  p_{-1}^{(+)} & p_{-1}^{(-)} \\
  q_{-1}^{(+)} & q_{-1}^{(-)}
\end{pmatrix}
= \begin{pmatrix}
  p_{-2}^{(+)} & p_{-2}^{(-)} \\
  q_{-2}^{(+)} & q_{-2}^{(-)}
\end{pmatrix}
= \begin{pmatrix}
  1 & 0 \\
  0 & 1
\end{pmatrix}.
\]

If \( \varphi_1 = [a_0, a_1, \ldots, a_{m-1}] \), the following equality holds for the conjugate \( \overline{\varphi_1} \) of \( \varphi_1 \) (Lemma 1.28, [1]):

\[
-1/\varphi_1 = [a_{m-1}, a_{m-2}, \ldots, a_0].
\]
Therefore, if $\phi_2 = \overline{\phi_1}$, $\{a_n\}_{n=\infty}^\infty$ is also a periodic sequence.

For any $\epsilon > 0$, let $L_{\phi_1, \phi_2, \epsilon}$ be the lattice with the following basis:
\[
  b_0 := \begin{pmatrix} \epsilon^{-1/2} \\ \epsilon^{1/2} \end{pmatrix}, \quad b_{-1} := \begin{pmatrix} -\epsilon^{-1/2} \phi_1 \\ -\epsilon^{1/2} \phi_2 \end{pmatrix}.
\]

A new basis $b_n, b_{n-1}$ of $L_{\phi_1, \phi_2, \epsilon}$ can be provided by:
\[
b_n = \begin{cases} p_{n-1}^{(+)} b_0 + q_{n-1}^{(+)} b_{-1} & \text{if } n > 0, \\
(-1)^n(q_{-n-2}^{(-)} b_0 - p_{-n-2}^{(-)} b_{-1}) & \text{if } n < -1.
\end{cases}
\]

In fact, from the definition,
\[
(b_n, b_{n-1}) = \begin{cases} (b_0, b_{-1}) \begin{pmatrix} p_{n-1}^{(+)} & q_{n-1}^{(+)} \\ q_{n-1}^{(-)} & p_{n-1}^{(-)} \end{pmatrix} & \text{if } n \geq 0, \\
(-1)^n (b_0, b_{-1}) \begin{pmatrix} q_{-n-2}^{(-)} & -q_{-n-1}^{(-)} \\ -p_{-n-2}^{(-)} & p_{-n-1}^{(-)} \end{pmatrix} & \text{if } n \leq 0,
\end{cases}
\]
\[
= \begin{cases} (\epsilon^{-1/2} 0) \begin{pmatrix} 1 & -\phi_1 \\ 0 & \epsilon^{1/2} \end{pmatrix} \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} & \text{if } n \geq 0, \\
(\epsilon^{-1/2} 0) \begin{pmatrix} 1 & -\phi_2 \\ 0 & \epsilon^{1/2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -a_n \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & -a_n \end{pmatrix} & \text{if } n \leq 0.
\end{cases}
\]

Hence,
\[
(b_n, b_{n-1}) = \begin{cases} (b_{n-1}, b_{n-2}) \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} & \text{if } n \geq 1, \\
(b_{n+1}, b_n) \begin{pmatrix} 0 & 1 \\ 1 & -a_n \end{pmatrix} & \text{if } n \leq -1.
\end{cases}
\]

Thus, $b_n = a_{n+1} b_{n+1} + b_{n-2}$ holds for any $n \in \mathbb{Z}$. From $a_0, a_{-1} \geq 0$, $a_n > 0$ $(n \neq 0, -1)$ and the following, $\{b_n, b_{n-1}\}_{n=-\infty}^\infty$ increases monotonically:
\[
b_n \cdot b_{n-1} - b_{n+1} \cdot b_{n-2} = a_{n-1} b_{n-1} - b_{n-2}.
\]

In addition,
\[
b_n \cdot b_{n-1} = \begin{cases} \epsilon^{-1} q_{n-1}^{-1} q_{n-2}^{(-)} (p_{n-1}^{(+)} - \phi_1) (p_{n-2}^{(-)} - q_{n-2}^{(+)} \phi_1) + \epsilon \phi_2^{-1} q_{n-1}^{-1} q_{n-2}^{(-)} (p_{n-1}^{(+)} - \phi_2) (p_{n-2}^{(-)} - q_{n-2}^{(+)} \phi_2) & n \geq 0, \\
-\epsilon \phi_2^{-1} q_{n-1}^{-1} q_{n-2}^{(-)} (p_{n-1}^{(-)} - \phi_2) (p_{n-2}^{(-)} - q_{n-2}^{(+)} \phi_2) & n \leq 0.
\end{cases}
\]

In both cases, the first terms in the right-hand side converge to 0 as $n \to \pm \infty$.
The second terms diverge owing to $q_{n}^{(+)} \to \infty$ and $q_{n}^{(-)} \to -\infty$ $(n \to \infty)$. Hence, $\lim_{n \to \infty} b_n \cdot b_{n-1} = \infty$. Similarly, $\lim_{n \to -\infty} b_n \cdot b_{n-1} = -\infty$ is obtained.

The term superbase was first used in [10] to explain the Selling reduction [36].
Definition 3. For any lattice $L$ of rank 2, a basis $u_1, u_2 \in L$ and $u_3 = -u_1 - u_2$ are a Selling-reduced superbase if $u_1, u_2, u_3$ satisfy $u_1 \cdot u_2 \leq 0$, $u_1 \cdot u_3 \leq 0$, and $u_2 \cdot u_3 \leq 0$.

Proposition 1. For any $\varphi_1 > 0 > \varphi_2$ and $\varepsilon > 0$, suppose that a lattice $L_{\varphi_1, \varphi_2, \varepsilon}$ and its basis vectors $b_n \ (n \in \mathbb{Z})$ are defined as above. Let $N$ be the integer that fulfills

$$n \geq N \iff b_n \cdot b_{n-1} \geq 0.$$ 

Let $1 \leq d \leq a_{N-1}$ be the smallest integer that satisfies

$$(d b_{N-1} + b_{N-2}) \cdot b_{N-1} \geq 0.$$ 

In this case, the following $u_1, u_2, u_3$ are a Selling reduced superbase of $L_{\varphi_1, \varphi_2, \varepsilon}$.

$$u_1 := b_{N-1}, \quad u_2 := (d - 1)b_{N-1} + b_{N-2}, \quad u_3 := -u_1 - u_2 = -(d b_{N-1} + b_{N-2}).$$

In addition, one of the following holds:

(i) $|u_1| < |u_2|, |u_3|.$

(ii) $|u_2| \leq |u_1| < |u_3|$. In this case, $d = 1$, $u_2 = b_{N-2}$.

(iii) $|u_3| \leq |u_1| < |u_2|$. In this case, $d = a_{N-1}$, $u_3 = -b_N$.

(iv) $|u_2|, |u_3| \leq |u_1|$. In this case, $d = a_{N-1} = 1$ and $u_2 = b_{N-2}, u_3 = -b_N$.

In particular, one of $b_{N-2}, b_{N-1},$ or $b_N$ is the shortest vector of $L_{\varphi_1, \varphi_2, \varepsilon}$.

Proof. From the choice of $N$, the following hold:

$$b_N \cdot b_{N-1} = (a_N b_{N-1} + b_{N-2}) \cdot b_{N-1} \geq 0, \quad b_{N-1} \cdot b_{N-2} < 0.$$ 

Since $(x b_{N-1} + b_{N-2}) \cdot b_{N-1}$ monotonically increases as a function of $x$, there is an integer $1 \leq d \leq a_{N-1}$ as stated above.

We shall show that $u_1, u_2, u_3$ are a Selling-reduced superbase; clearly, $u_1, u_2$ are a basis of $L_{\varphi_1, \varphi_2, \varepsilon}$. From the definition, $u_1 \cdot u_2 < 0$ and $u_1 \cdot u_3 \leq 0$. Hence, we only need to show that $u_2 \cdot u_3 \leq 0$. For the proof, $N \geq 1$ may be assumed; in fact, for any integer $-M < N$, $(b_0, b_{-1})$ can be replaced with $(b_{-M}, b_{-M-1})$, by replacing $\varphi_1, \varphi_2$ with $[a_{-M}, a_{-M+1}, \ldots], [-[0, a_{-M-1}, a_{-M-2}, \ldots]]$, and changing $\varepsilon$ accordingly. As a result, $N$ can be set to a positive integer.

From the process of the continued fraction expansion, $a_N \ (N \geq 1)$ is the largest integer among all the $c$’s for which the following two have different signatures:

$$\frac{p_{N-1}^{(+)}}{q_{N-1}^{(+)}} - \varphi_1 \quad = \quad [a_0, a_1, \ldots, a_{N-1}] - \varphi_1,$$

$$\frac{c p_{N-1}^{(+)}}{c q_{N-1}^{(+)}} + p_{N-2}^{(+)}/q_{N-2}^{(+)}, - \varphi_1 \quad = \quad [a_0, a_1, \ldots, a_{N-1}, c] - \varphi_1.$$
In particular, the following two have the same signature for any $1 \leq r \leq a_N$:

\[
\frac{(r-1)p_{N-1}^{(+)}}{(r-1)q_{N-1}^{(+)}} - \varphi_1 = [a_0, a_1, \ldots, a_{N-1}, r - 1] - \varphi_1,
\]

\[
\frac{r p_{N-1}^{(+)}}{r q_{N-1}^{(+)}} - \varphi_1 = [a_0, a_1, \ldots, a_{N-1}, r] - \varphi_1.
\]

Thus, $u_2 \cdot u_3 \leq 0$ is obtained from the following:

\[
- u_2 \cdot u_3 = \left( (d-1)q_{N-1}^{(+)}/q_{N-2}^{(+)}) \right) \left( \left( \frac{d p_{N-1}^{(+)}/q_{N-2}^{(+)}}{d q_{N-1}^{(+)}/q_{N-2}^{(+)}} - \varphi_1 \right) \right) + e \left( \left( \frac{(d-1)p_{N-1}^{(+)}/q_{N-2}^{(+)}}{(d-1)q_{N-1}^{(+)}/q_{N-2}^{(+)}} - \varphi_2 \right) \right).
\]

Therefore, $u_1, u_2, u_3$ is Selling reduced, which implies that one of $u_1, u_2, u_3$ is the shortest vector of $L$.

As for the second statement, the following equation is obtained in a similar way as the above:

\[
|u_2| \leq |u_1| \iff - (u_2 - u_1) \cdot u_3 \leq 0
\]

\[
\Rightarrow \left( \frac{(d-2)p_{N-1}^{(+)}/q_{N-2}^{(+)}}{(d-2)q_{N-1}^{(+)}/q_{N-2}^{(+)}} - \varphi_1 \right) < 0,
\]

\[
|u_3| \leq |u_1| \iff - u_2 \cdot (u_3 - u_1) \leq 0
\]

\[
\Rightarrow \left( \frac{(d+1)p_{N-1}^{(+)}/q_{N-2}^{(+)}}{(d+1)q_{N-1}^{(+)}/q_{N-2}^{(+)}} - \varphi_1 \right) < 0.
\]

Because the values in the parentheses of Eq. (7) have different signatures owing to the inequality, $|u_2| \leq |u_1|$ implies $d = 1$. Similarly, $|u_3| \leq |u_1|$ implies $d = a_{N-1}$.

The determination of $\Delta_n$, $\Delta'(L(B_n))$ is equivalent to that of $\lambda_n'$, $\lambda'(B_n)$.

\[
\lambda_n' := \sup_{B_n \in GL_n(\mathbb{R}), \det B_n = \pm 1} \lambda'(B_n),
\]

\[
\lambda'(B_n) := \min_{D \in \text{GL}_n(\mathbb{R}) \text{ diagonal}} \det D = \pm 1 \min L(DB_n).
\]

In fact, for any $B_n \in GL_n(\mathbb{R})$ with the determinant $\pm 1$,

\[
\Delta'(L(B_n)) = \frac{(\pi \lambda'(B_n)/4)^{n/2}}{\Gamma(n/2 + 1)}.
\]
The determination of $\lambda_n'$ and $\lambda'(B_n)$ is boiled down to the following problem about the products of linear forms $\lambda_n$ and $\lambda(B_n)$, as proved in Lemma 1.

$$\lambda_n := \sup_{b_n=(b_{ij})\in GL_n(\mathbb{R}), \det B=\pm 1} \lambda(B_n),$$

$$\lambda(B_n) := \inf_{0 \neq (x_1, \ldots, x_n) \in \mathbb{Z}^n} \left| \prod_{i=1}^n (b_{ij}x_1 + \cdots + b_{in}x_n) \right|.$$ (10)

**Lemma 1.** (1) For any $B = (b_{ij}) \in GL_n(\mathbb{R})$ with $\lambda(B_n) \neq 0$, the following equality holds:

$$\lambda'(B_n) = n(\lambda(B_n))^{2/n}. \quad (11)$$

(2) If a totally real algebraic number field $K$ of degree $n$ has discriminant $d_K$, then $\lambda_n \geq d_K^{1/2}$. Furthermore, for each $n$, some $B_n$ attains the supremum $\Delta_n$, and

$$\Delta_n = \frac{(\pi \lambda_n/4)^{n/2}}{\Gamma(n/2+1)}.$$ (11)

**Proof.** (1) From the inequality of arithmetic and geometric means, the part $\geq$ is proved since we have the following:

$$\lambda'(B_n) = \inf_{d_1, \ldots, d_n \in \mathbb{R}, d_1, \ldots, d_n \neq 0} \sum_{i=1}^n d_i^2 (b_{1i}x_1 + \cdots + b_{in}x_n)^2,$$

$$\lambda(B_n) = \inf_{0 \neq (x_1, \ldots, x_n) \in \mathbb{Z}^n} \left| \prod_{i=1}^n (b_{ij}x_1 + \cdots + b_{in}x_n) \right|.$$

Furthermore, $>$ is impossible, because if $\left| \prod_{i=1}^n (b_{1i}x_1 + \cdots + b_{in}x_n) \right| = \nu$ for some $0 \neq (x_1, \ldots, x_n) \in \mathbb{Z}^n$, the left-hand side of Eq. (11) cannot be more than $\nu^{2/n}$, which is seen by putting $d_i = \sqrt[4]{\nu}/(b_{1i}x_1 + \cdots + b_{in}x_n)$.

(2) Let $\sigma_1, \ldots, \sigma_n$ be distinct embeddings of $K$ into $\mathbb{C}$ over $\mathbb{Q}$, $b_1, \ldots, b_n$ be a basis of the ring $\mathcal{O}_K$ of integers of $K_n$ over $\mathbb{Z}$, and $B_n \in GL_n(\mathbb{R})$ be the matrix with $b_{ij}^\sigma$ in the $(i, j)$-entry. From $|\det B_n| = \sqrt{d_K}$ and $\min_{\sigma \in \mathcal{O}_K} |\sigma_1(\alpha) \cdots \sigma_n(\alpha)| = 1$, we have

$$\lambda_n \geq \lambda(d_K^{-1/2}B_n) = d_K^{-1/2}.$$ (11)

This implies that $\lambda_n > 0$, and thus, the following star body is finite type.

$$\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_1 \cdots x_n| \leq 1 \}.$$ (10)

Hence, $\lambda(B_n) = \lambda_n$ holds for some $B_n \in GL_n(\mathbb{R})$ with $\det B_n = \pm 1$ (Theorem 9 of §17, chap.3 in [17]). For such a $B_n$, $\lambda'(B_n) = \lambda_n'$ and $\Delta'(L(B_n)) = \Delta_n'$ hold owing to Eqs. (9) and (11). The last inequality is also clear. □
The relation between Markoff theory and phyllotaxis was not clarified until very recently [6]. Theorem 1 of [6] handles a special case of our Eq. (12), as their growth capacity is a constant multiple of the packing density. Markoff theory can give us more insights. As seen in (2) of Theorem 1, the golden lattice given by \( B_2 \) is more optimal than the lattice by Eq. (1) that has been used for the golden angle method.

**Theorem 1.** (1) For \( n = 2, 3 \), let \( B_n \in GL_n(\mathbb{R}) \) be the matrix with \( b_{ij}^n \) in each \((i, j)\)-entry, where \( \sigma_1, \ldots, \sigma_n \) are all the embeddings of the following \( K_n \) into \( \mathbb{C} \) over \( \mathbb{Q} \), and \( b_1, \ldots, b_n \) are a basis of the ring of integers of the fields \( K_n \) as a \( \mathbb{Z} \)-module.

- \( K_2 = \mathbb{Q}(\zeta_5 + \zeta_5^{-1}) \), \( \zeta_5 = e^{2\pi \sqrt{-1}/5} \).
- \( K_3 = \mathbb{Q}(\zeta_7 + \zeta_7^{-1}) \), \( \zeta_7 = e^{2\pi \sqrt{-1}/7} \).
- \( K_4 = \mathbb{Q}(\sqrt{7} + 2\sqrt{5}) \).
- \( K_5 = \mathbb{Q}(\zeta_{11} + \zeta_{11}^{-1}) \), \( \zeta_{11} = e^{2\pi \sqrt{-1}/11} \).

For the above \( B_n \), the following hold:

\[
\Delta_2' = \Delta'(L(B_2)) = \frac{\pi}{2\sqrt{5}} \approx 0.702, \quad \Delta_3' = \Delta'(L(B_3)) = \frac{\sqrt{3}\pi}{14} \approx 0.389, \\
\Delta_4' \geq \Delta'(L(B_4)) = \frac{\pi^2}{10\sqrt{29}} \approx 0.183, \quad \Delta_5' \geq \Delta'(L(B_5)) = \frac{5\sqrt{5}\pi^2}{12 \cdot 112} \approx 0.076.
\]

(For comparison, the packing density of the densest lattice packings are \( \pi/2\sqrt{3} \approx 0.907 \) for \( n = 2 \), \( \pi/3\sqrt{2} \approx 0.740 \) [15] for \( n = 3 \), \( \pi^2/16 \approx 0.617 \) [27] for \( n = 4 \), and \( \pi^2/15\sqrt{2} \approx 0.465 \) [27] for \( n = 5 \).)

(2) For any distinct \( \varphi_1, \varphi_2 \in \mathbb{R} \setminus \mathbb{Q} \), let \( L_{\varphi_1, \varphi_2} \), \( L_{\varphi_1, \varphi_2, \varepsilon} \) be the lattices \( L(B_1) \), \( L(B_\varepsilon) \) generated by the column vectors of the following matrices:

\[
B_1 = \begin{pmatrix} 1 & -\varphi_1 \\ 1 & -\varphi_2 \end{pmatrix}, \quad B_\varepsilon = \begin{pmatrix} \varepsilon^{-1/2} & 0 \\ 0 & \varepsilon^{1/2} \end{pmatrix} \begin{pmatrix} 1 & -\varphi_1 \\ 1 & -\varphi_2 \end{pmatrix}.
\]

Then,

\[
\liminf_{\varepsilon \to +0} \Delta(L_{\varphi_1, \varphi_2, \varepsilon}) = \frac{\pi}{2d'(\varphi_1)}, \quad (12) \\
\Delta'(L_{\varphi_1, \varphi_2}) = \inf_{\varepsilon} \Delta(L_{\varphi_1, \varphi_2, \varepsilon}) = \frac{\pi}{2d(\varphi_1)}, \quad (13)
\]

where \( d'(\varphi_1) \) is the Lagrange number of \( \varphi_1 \), and \( d(\varphi) := \sqrt{d(f)}/m(f) \) is the element of the Markoff spectrum that corresponds to the quadratic form \( f(x, y) := (x - \varphi_1 y)(x - \varphi_2 y) \).
Remark 1. Eq. (12) does not hold for any rational $\varphi_1$, because $\mathcal{L}'(\varphi_1) = 0$ if $\varphi_1 \in \mathbb{Q}$ (Corollary 1.2, [1]).

Remark 2. When $B$ is as in (1) i.e., the rows are conjugate of each other, the product $\prod_{i=1}^{n} (b_1 x_1 + \cdots + b_n x_n)$ is called a norm form. It is conjectured that for $n \geq 3$, $\lambda(B_n) > 0$ implies that $B_n$ is a norm form [17].

Remark 3. The upper bounds $\lambda_4 \leq 3/(20\sqrt{5})$ [43] and $\lambda_5 \leq 1/57.02$ [16] are also known. Therefore, $\Delta_4 \leq 3\pi^2/(40\sqrt{5}) \approx 0.331$ and $\Delta_4 \leq 5\sqrt{5}\pi^2/(12 \cdot 57.02) \approx 0.161$.

Proof. (1) This part is an immediate consequence of Lemma [1] and the known results: $\lambda_2 = 1/\sqrt{5}$ [21], [26], $\lambda_3 = 1/7$ [13], $\lambda_4 \geq 1/5\sqrt{29}$ [29] and $\lambda_5 \geq 11^{-2}$ [19]. The lower bounds for $\lambda_4, \lambda_5$ are from the smallest discriminants among all the totally real quartic and quintic fields.

(2) As a result of Lemma [1] Eq. (13) is obtained as follows:

$$\Delta'(L_{\varphi_1, \varphi_2}) = \frac{\pi \lambda(B_1)}{2|\det B_1|} = \frac{\inf_{0 \neq (x,y) \in \mathbb{Z}^n} f(x,y)}{2|\varphi_1 - \varphi_2|} = \frac{\pi m(f)}{2\sqrt{d(f)}} = \frac{\pi}{2.\mathcal{M}(f)}.$$

Eq. (12) is proved as follows; $\varphi_1 > 0 > \varphi_2$ may be assumed, by replacing $B_\varepsilon$ with $DB_\varepsilon g$ for some diagonal $D \in GL_2(\mathbb{R})$ and $g \in GL_2(\mathbb{Z})$. This replaces each $\varphi_i$ by its linear fractional transformation. Hence, it does not change their Lagrange numbers.

From Proposition [1] the shortest vector of $L_{\varphi_1, \varphi_2, \varepsilon}$ is equal to $p_N^{(+)}b_1 - q_N^{(+)}b_2$ for sufficiently small $\varepsilon > 0$, where $p_N^{(+)}, q_N^{(+)}$ is the $N$’th convergent of $\varphi_1$. Furthermore, the following is proved as in Lemma [1]

$$\liminf_{\varepsilon \to 0} \min_{L_{\varphi_1, \varphi_2, \varepsilon}} \Delta'(L_{\varphi_1, \varphi_2}) = \liminf_{\varepsilon \to 0} \inf_{N > 0} \left\{ \varepsilon^{-1}(p_N^{(+)} - q_N^{(+)} \varphi_1)^2 + \varepsilon(p_N^{(+)} - q_N^{(+)} \varphi_2)^2 \right\}$$

$$= 2 \liminf_{N \to \infty} \left| (p_N^{(+)} - q_N^{(+)} \varphi_1)(p_N^{(+)} - q_N^{(+)} \varphi_2) \right|$$

$$= 2|\varphi_1 - \varphi_2| \liminf_{N \to \infty} \left| q_N^{(+)}(p_N^{(+)} - q_N^{(+)} \varphi_1) \right|$$

$$= \frac{2|\varphi_1 - \varphi_2|}{\limsup_{N \to \infty} \ell_N(\varphi_1)}.$$ (cf. proof of Proposition 1.22 of [1])

$$\ell_N(\varphi_1) := [a_{n+1}, a_{n+2}, \ldots] + [0, a_n, a_{n-1}, \ldots, a_1].$$

Thus, Eq. (12) is proved as follows:

$$\liminf_{\varepsilon \to 0} \Delta'(L_{\varphi_1, \varphi_2}) = \frac{\pi}{2\limsup_{N \to \infty} \ell_N(\varphi_1)} = \frac{\pi}{2.\mathcal{L}(\varphi)}.$$
As an immediate consequence of Theorem 1, it is possible to improve the packing around the origin of the Vogel spiral by using the optimal lattice basis $B_2$ in Theorem 1 (Figure 2). More obvious examples will be provided in Figure 4 of Section 3.2.

![Figure 2: Original Vogel spiral (left) and a packing obtained from the lattice basis $B_2$ in Theorem 1 (right). The lattice map $f$ to use is explained in Example 5.](image)

### Example 1 (Vogel spirals for quadratic irrationals $\phi$ with $L(\phi) < 3$).

The $\gamma_m$ defined by Eq. (6) has the Lagrange number $L(\gamma_m) = \sqrt{9 - 4/m^2}$. Hence, the basis matrix $B_m = \begin{pmatrix} 1 & -\gamma_m \\ 1 & -\gamma_m \end{pmatrix}$ fulfills the following for any Markoff number $m$.

$$\Delta'(L(B_m)) = \frac{\pi}{2L(\gamma_m)} = \frac{\pi}{2\sqrt{9 - 4/m^2}} > \frac{\pi}{6} \approx 0.5236.$$

The quadratic irrationals with the smallest Lagrange numbers are $\gamma_1 = (1 + \sqrt{5})/2 = [1]$, $\gamma_2 = 1 + \sqrt{2} = [2]$, $\gamma_5 = (9 + \sqrt{221})/10 = [2,2,1,1]$. For each $m = 1,2,5$, the packing density of $L(DB_m)$ is not less than the following value, regardless of the diagonal $D \in \text{GL}_2(\mathbb{R})$:

$$\Delta'(L(B_1)) = \frac{\pi}{2\sqrt{5}} \approx 0.7025,$$

$$\Delta'(L(B_2)) = \frac{\pi}{4\sqrt{2}} \approx 0.5554,$$

$$\Delta'(L(B_5)) = \frac{5\pi}{2\sqrt{221}} \approx 0.5283.$$

### 3 Lattice maps for packing of the Euclidean spaces

In this section, it is assumed that $n = N$, and $f(x) = (f_1(x), \ldots, f_n(x)) \in C^3(D, \mathbb{R}^n)$ defined on open subset $D \subset \mathbb{R}^n$, satisfies the properties ($\ast$), ($\ast\ast$):
(∗) For any \( x \in \mathcal{D} \), there are an orthogonal matrix \( U(x) \) of degree \( n \) and a diagonal matrix \( \Phi(x) \) with the diagonal entries \( \phi_1(x), \ldots, \phi_n(x) \neq 0 \) that satisfy the following:

\[
Df(x) := \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{pmatrix} = U(x)\Phi(x).
\]

(∗∗) \( \prod_{i=1}^{n} \phi_i(x) = c \) for some constant \( c \neq 0 \).

After deriving all the PDEs for \( \Phi(x) \) and \( U(x) \) in Proposition 2, a family of the PDE solutions are provided by using solutions of inviscid Burgers equation (Section 3.1) and separation of variables (Section 3.2).

### 3.1 System of PDEs and solutions provided by inviscid Burgers equation

First we assume only (∗), and the constraint (∗∗) will not be used until Example 3.

Since \( f(x) \in C^3(\mathcal{D}, \mathbb{R}^n) \) and the Jacobian matrix \( Df(x) \) fulfills \( (Df)Df = \Phi(x) \), \( \phi_1(x), \ldots, \phi_n(x) \) are functions of class \( C^2 \). Let \( A_k(x) \) be the matrix defined by \( A_k(x) := U(x)^{-1}(U(x))_{ik} \). From \( (UU)_{ik} = O \) and \( (U_{x_k})_{ij} = (U_{x_j})_{ik} \), the following are obtained:

(a) \( A_k A_k = O \).

(b) \( A_j A_k - A_k A_j = (A_j)_{x_k} - (A_k)_{x_j} \).

In particular, Eqs. (14) and (15), which are derived only from (∗) and \( n = N \), are the PDEs called Lamé equations [43].

**Proposition 2.** Let \( \mathcal{D} \subset \mathbb{R}^n \) be a simply-connected open subset. The \( \phi_1(x), \ldots, \phi_n(x) \) that fulfill (∗) for some \( f \in C^3(\mathcal{D}, \mathbb{R}^n) \), are provided by solving the following PDEs:

\[
\phi_j^{-1}(\phi_i)_{x_i} + \phi_i^{-1}(\phi_k)_{x_j}(\phi_j)_{x_i} = (\phi_j)_{x_i x_i} \quad (1 \leq i, j, k \leq n : \text{distinct}),
\]

(14)

\[
(\phi_k^{-1}(\phi_j)_{x_j})_{x_k} + (\phi_j^{-1}(\phi_k)_{x_j})_{x_j} = - \sum_{i \neq j, k} \phi_i^{-2}(\phi_k)_{x_i}(\phi_j)_{x_i} \quad (1 \leq j, k \leq n : \text{distinct}).
\]

(15)

For fixed \( \Phi(x) \), the matrix \( A_k = \left( a_{ij}^{(k)} \right) (k = 1, \ldots, n) \) that satisfy the above (a) and (b) are provided by:

\[
A_k = \mathbf{e}_k \mathbf{e}_k - \mathbf{e}_k \mathbf{e}_k,
\]

(16)

where \( \mathbf{c}_k := (\phi_1^{-1}(\phi_k)_{x_1}, \ldots, \phi_n^{-1}(\phi_k)_{x_n}) \) and \( \mathbf{e}_k \) is the unit vector with 1 in the \( k \)’th entry. \( U(x) \in O(n) \) is obtained by solving \( U_{x_k} = UA_k \) \( (k = 1, \ldots, n) \).
Proof. If such an \( f \) exists, from \((f_{ij})_{x_k} = ((f_{ij})_{x_k}, \phi_1(x), \ldots, \phi_n(x))\) and \(U(x) = (u_1(x), \ldots, u_n(x))\) satisfy:

\[
(\phi_j)_{x_i} u_j + \phi_j \sum_{i \neq j} a_{ij}^{(k)} u_i = (\phi_k)_{x_i} u_k + \phi_k \sum_{i \neq k} a_{ik}^{(i)} u_i \quad (1 \leq j, k \leq n).
\]

Since \(u_1, \ldots, u_n\) are linearly independent over \(\mathbb{R}\), the following are obtained:

\[
\begin{align*}
\phi_j a_{ij}^{(k)} &= \phi_k a_{ik}^{(j)} \quad (1 \leq i, j, k \leq n : \text{distinct}), \\
\phi_k a_{ij}^{(k)} &= \phi_j a_{kj}^{(i)} \quad (1 \leq j, k \leq n : \text{distinct}).
\end{align*}
\]

It is concluded that \(a_{ij}^{(k)} = 0\) for any distinct \(1 \leq i, j, k \leq n\), owing to \(a_{ij}^{(k)} = -a_{ji}^{(k)}\) and

\[
a_{ij}^{(k)} = \frac{\phi_k}{\phi_j} a_{ij}^{(j)} = -\frac{\phi_k}{\phi_j} a_{ij}^{(j)} = -\frac{\phi_k}{\phi_j} a_{ij}^{(j)} = \frac{\phi_k}{\phi_j} a_{ij}^{(j)}.
\]

Thus, \(A_k = \mathbf{e}_k c_k - \mathbf{c}_k e_k\) is obtained. Because \(A_k\) also fulfills the above (b),

\[
\begin{align*}
\langle \mathbf{e}_j c_j - \mathbf{c}_j e_j, \mathbf{e}_k c_k - \mathbf{c}_k e_k \rangle - \langle \mathbf{e}_k c_k - \mathbf{c}_k e_k, \mathbf{e}_j c_j - \mathbf{c}_j e_j \rangle &= \phi_k^{-1}(\phi_j)_{x_k} \langle \mathbf{e}_j c_k - \mathbf{c}_j e_k, \mathbf{e}_k c_k - \mathbf{c}_k e_k \rangle - \phi_j^{-1}(\phi_k)_{x_j} \langle \mathbf{e}_j c_j - \mathbf{c}_j e_j, \mathbf{e}_k c_k - \mathbf{c}_k e_k \rangle \\
&= \langle \mathbf{e}_j c_j - \mathbf{c}_j e_j, \mathbf{e}_k c_k - \mathbf{c}_k e_k \rangle.
\end{align*}
\]

By comparing the \((j, i)\)- and \((j, k)\)-components of Eq.(17), we can obtain:

\[
\begin{align*}
(\phi_j \phi_k)^{-1}(\phi_j)_{x_k} & (\phi_k)_{x_k} = (\phi_j^{-1}(\phi_j)_{x_k})_{x_k} \quad (1 \leq i, j, k \leq n : \text{distinct}), \\
\phi_j^{-2}(\phi_j)_{x_k} & (\phi_k)_{x_k} + \phi_k^{-2}(\phi_j)_{x_k} (\phi_k)_{x_k} - \mathbf{c}_j \cdot \mathbf{c}_k \\
&= (\phi_k^{-1}(\phi_j)_{x_k})_{x_k} + (\phi_j^{-1}(\phi_j)_{x_k})_{x_k} \quad (1 \leq j, k \leq n : \text{distinct}).
\end{align*}
\]

Each equation leads to Eqs.((14), (15), respectively.}

For any constants \(t_1, \ldots, t_n \neq 0, f(x_1, \ldots, x_n)\) fulfills (18) if and only if \(f(t_1 x_1, \ldots, t_n x_n)\) does. Accordingly, \(\phi_j(x_1, \ldots, x_n) (j = 1, \ldots, n)\) fulfill Eqs.((14) and (15), if and only if \(t_j \phi_j(t_1 x_1, \ldots, t_n x_n) \quad (j = 1, \ldots, n)\) do. Thus, the self-similar solutions of the PDEs are provided by those that fulfill \(\phi_j(x_1, \ldots, x_n) = t_j \phi_j(t_1 x_1, \ldots, t_n x_n)\) for any \(0 \neq t_1, \ldots, t_n \in \mathbb{R}\), which leads to the self-similar solution \(\phi_j = c_j / x_j\ (c_j \in \mathbb{R})\). In this case, (18) does not hold clearly. The map \(f\) is given by

\[
f(x_1, \ldots, x_n) = U_0 \begin{pmatrix} c_1 \log x_1 \\ \vdots \\ c_n \log x_n \end{pmatrix} + \mathbf{v}_0 \quad (U_0 \in O(n), \mathbf{v}_0 \in \mathbb{R}^N).
\]

Next, suppose that \(\phi_1(x) = \cdots = \phi_n(x)\ i.e., \ f\ is a conformal map. In this case, the condition (18) only provides trivial lattice packings, because \(\phi_1 = \cdots = \phi_n\) and \(\phi_1 \cdots \phi_n = c\) imply that \(\Phi(x)\) and \(U(x)\) are constant.
Example 2 (Case of conformal mapping without the condition (++)). We can put $\phi(x):=\phi_1(x)=\cdots=\phi_n(x)$. For $n\geq 3$, any conformal maps are Möbius transformations of the following form as a result of Liouville’s theorem.

$$f(x) = \frac{\alpha U(x-a)}{|x-a|^\epsilon} + b. \quad (a,b \in \mathbb{R}^n, \alpha \in \mathbb{R}, U \in O(n), \epsilon = 0, 2)$$

From Proposition 2, the PDE for $n=2$ is

$$(\log \phi)_{xx} + (\log \phi)_{yy} = 0.$$  

Thus, $u(x,y) := \log \phi(x,y)$ is a harmonic function. If $v(x,y)$ is harmonic conjugate to $u(x,y) \ (i.e., \ u_x = v_y \ \text{and} \ \ u_y = -v_x)$, the following is obtained from Eq. (16).

$$A_1 = \begin{pmatrix} 0 & \phi^{-1}_y \\ -\phi^{-1}_y & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -\phi^{-1}_x \\ \phi^{-1}_x & 0 \end{pmatrix}, \quad U(x,y) = U_0 \exp \begin{pmatrix} 0 & -v \\ v & 0 \end{pmatrix} = U_0 \begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix}, \quad U_0 \in O(2).$$

The Doyle spiral is included in this case, in which each circle is tangent to six other circles, and the seven circles have the radii with the ratio $1, a, b, 1/a, 1/b, a/b, b/a \ (a, b > 0)$ as in Figure 3.

Figure 3: Left: hexagonal lattice packing in $\mathbb{R}^2$, Right: radius ratios of adjacent circles in the Doyle spiral

The Doyle spirals are the image of a hexagonal lattice in the complex plane $\mathbb{C}$ by exponential maps [5]. In [7], conformally symmetric circle packings were defined as a generalization of the Doyle spirals. From the aspect of the golden angle method, packings with the Voronoi cells of varying volumes have been investigated in [38], [42]. In order to generate such circle packings, the determination of the radius of each circle is also necessary, which is not discussed in this article.
Example 3 (PDE for $n = 2$). From $(\phi^{-1}_2(\phi_1)_y)_y + (\phi^{-1}_1(\phi_2)_y)_x = 0$, if $(\ast\ast)$ is also assumed, $\varepsilon(x,y) := e^{-1}(\phi^{-1}_2)_x, \phi_1 \phi_2 = c$ satisfies the following equations:

$$
\begin{align*}
\varepsilon_{xx} + (\varepsilon^{-1})_{yy} &= -e^{-1}(\phi^{-1}_2)_{xx} + (\phi^{-1}_1)_{yy} = 0, \\
A_1 = U^{-1} U_x &= \begin{pmatrix} 0 & \phi^{-1}_2(\phi_1)_y \\ -\phi^{-1}_2(\phi_1)_y & 0 \end{pmatrix} = \begin{pmatrix} 0 & -e^{-1}_y/2 \\ -e^{-1}_y/2 & 0 \end{pmatrix}, \\
A_2 = U^{-1} U_y &= \begin{pmatrix} \phi^{-1}_1(\phi_2)_x & 0 \\ 0 & \phi^{-1}_1(\phi_2)_x \end{pmatrix} = \begin{pmatrix} 0 & -\varepsilon_x/2 \\ \varepsilon_x/2 & 0 \end{pmatrix}.
\end{align*}
$$

$U(x,y)$ is explicitly represented by using $\theta(x,y)$ with $\varepsilon_x = 2\theta_y$ and $(\varepsilon^{-1})_y = -2\theta_x$ as follows:

$$
U(x,y) = U_0 \exp \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = U_0 \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (U_0 \in O(2)).
$$

All the solutions of inviscid Burgers equation $\varepsilon \varepsilon_x + \varepsilon_y = 0$ fulfill $\varepsilon_{xx} + (\varepsilon^{-1})_{yy} = 0$, which is also seen from the following decomposition:

$$
\varepsilon_{xx} + (\varepsilon^{-1})_{yy} = \left( \frac{\partial}{\partial x} - \frac{\partial e^{-1}}{\partial y} \right) \left( \frac{\partial}{\partial x} + e^{-1} \frac{\partial}{\partial y} \right) \varepsilon.
$$

As a result, non-trivial solutions of Eqs.(14), (15) for dimensions $n > 2$ are obtained by setting $\phi^{-1}_{2n-1}(x) = c_n/\varepsilon_x(x_{2n-1},x_{2n}), \phi^{-1}_{2n}(x) = c_n \varepsilon_x(x_{2n-1},x_{2n}), (and \phi_n(x) = c_{(n+1)/2} if n is odd) for some constants $c_n$ and solutions of $\varepsilon_x(\varepsilon_x)_x + (\varepsilon_y)_y = 0$.

Example 4 (Case of inviscid Burgers equation $\varepsilon_{xx} + (\varepsilon^{-1})_{yy} = 0$). Let $\varepsilon(x,y)$ be the solution of the following inviscid Burgers equation:

$$
\begin{align*}
\left\{ \begin{array}{l}
\varepsilon \varepsilon_x + \varepsilon_y = 0, \\
\varepsilon(t,0) = h(t) \text{ for any } t \in I,
\end{array} \right.
\end{align*}
$$

where $h(x) : I \to \mathbb{R}$ is the initial condition given on an interval $I \subset \mathbb{R}$.

The map $f(x,y)$ with the following Jacobian matrix, can be determined as follows:

$$
\begin{pmatrix}
(f_1)_x & (f_1)_y \\
(f_2)_x & (f_2)_y
\end{pmatrix} = \begin{pmatrix} \cos \theta(x,y) & -\sin \theta(x,y) \\ \sin \theta(x,y) & \cos \theta(x,y) \end{pmatrix} \begin{pmatrix} e^{-1/2} & 0 \\ 0 & e^{1/2} \end{pmatrix}.
$$

From $\varepsilon \varepsilon_x + \varepsilon_y = 0,$

$$
\begin{align*}
2\theta_x &= -(\varepsilon^{-1})_y = -\varepsilon^{-1} \varepsilon_x, \\
2\theta_y &= \varepsilon_x = -\varepsilon^{-1} \varepsilon_y,
\end{align*}
$$

Thus, $\theta = -(\log \varepsilon + d)/2$ for some constant $d$. The inviscid Burgers equation can be solved by using the characteristic equation:

$$
\begin{align*}
\left\{ \begin{array}{l}
q'(s) = \varepsilon(q(s),s), \\
q(0) = t.
\end{array} \right.
\end{align*}
$$
From Eq. (20), \( q'(0) = \varepsilon(t, 0) = h(t) \). On the characteristic curve, \( q'(s) = \varepsilon(q(s), s) \) is constant because

\[
\frac{d}{ds} \varepsilon(q(s), s) = q'(s)\varepsilon_1(q(s), s) + \varepsilon_2(q(s), s) = \varepsilon(q(s), s)\varepsilon_1(q(s), s) + \varepsilon_2(q(s), s) = 0.
\]

Hence, \( q(s) = t + h(t)s \) (s \in \mathbb{R}) is the characteristic line, on which \( f = (f_1, f_2) \) satisfies:

\[
\begin{align*}
\frac{d}{ds} f_1(q(s), s) &= q'(s)(f_1)_x + (f_1)_s = \varepsilon^{1/2} \left( \cos \frac{\log d + \varepsilon}{2} + \sin \frac{\log d + \varepsilon}{2} \right), \\
\frac{d}{dt} f_2(q(s), s) &= q'(s)(f_2)_x + (f_2)_s = \varepsilon^{1/2} \left( \cos \frac{\log d + \varepsilon}{2} - \sin \frac{\log d + \varepsilon}{2} \right).
\end{align*}
\]

Without loss of generality, \( d = -\pi/2 \) may be assumed. In this case, for any \( (x, y) \in \mathbb{R}^2 \) and \( t \in I \) that satisfies \( t + h(t)y = x \), \( f = (f_1, f_2) \) is given by

\[
\begin{align*}
f(x, y) &= f(t, 0) + y\sqrt{2h(t)} \begin{pmatrix} \sin \frac{\log h(t)}{2} \\ \cos \frac{\log h(t)}{2} \end{pmatrix}, \\
f(t, 0) &= \begin{pmatrix} f_1(t_1, t_0) \cos \frac{h(t)}{2} \\ f_2(t_1, t_0) \sin \frac{h(t)}{2} \end{pmatrix} = \begin{pmatrix} \int \frac{1}{\sqrt{h(t)}} \cos \frac{\log h(t) - \pi/2}{2} dt \\ -\int \frac{1}{\sqrt{h(t)}} \sin \frac{\log h(t) - \pi/2}{2} dt \end{pmatrix}.
\end{align*}
\]

In particular, if \( f_1(t_0) \cos \frac{\log h(t)}{2} \neq f_2(t_0) \sin \frac{\log h(t)}{2} \),

\[
f(x, y) = \left( f_1(t_0) \cos \frac{\log h(t)}{2} - f_2(t_0) \sin \frac{\log h(t)}{2} \right) \begin{pmatrix} Y & 1 \\ -1 & Y \end{pmatrix} \begin{pmatrix} \sin \frac{\log h(t)}{2} \\ \cos \frac{\log h(t)}{2} \end{pmatrix},
\]

\[
Y = \frac{y\sqrt{2h(t)} + f_1(t_0) \sin \frac{\log h(t)}{2} + f_2(t_0) \cos \frac{\log h(t)}{2}}{f_1(t_0) \cos \frac{\log h(t)}{2} - f_2(t_0) \sin \frac{\log h(t)}{2}}.
\]

Otherwise, by using \( g(t) := f_1(t_0)/\sin \frac{\log h(t)}{2} = f_2(t_0)/\cos \frac{\log h(t)}{2} \),

\[
f(x, y) = Y \begin{pmatrix} \sin \frac{\log h(t)}{2} \\ \cos \frac{\log h(t)}{2} \end{pmatrix}, \quad Y = y\sqrt{2h(t)} + g(t).
\]

Both correspond to the case of Eq. (3), if \( f \) is regarded as the function of \( (t, Y) \).

### 3.2 A family of solutions obtained by separation of variables

Another family of solutions of \( \varepsilon_{xx} + (\varepsilon^{-1})_{yy} = 0 \) can be obtained by separation of variables. If we put \( \varepsilon(x, y) = F(x)/G(y) \), \( F''(x)/G(y) = -G''(y)/F(x) \) is obtained from \( \varepsilon_{xx} = -(\varepsilon^{-1})_{yy} \). Hence, for some constant \( \alpha \),

\[
F(x)F''(x) = \alpha, \quad G(y)G''(y) = -\alpha.
\]
If $\alpha \neq 0$, $(F'(x))^2 = 2\alpha \log F(x) + d_1$, $(G'(y))^2 = -2\alpha \log G(y) + d_2$. The solutions $F(x)$ and $G(y)$ are functions represented by incomplete gamma functions in general. Only the special case of $\alpha = 0$ is discussed below to obtain 2D and 3D analogues of the Vogel spiral.

If $\alpha = 0$, we have $F(x) = c_1 x + d_1$ and $G(y) = c_2 y + d_2$. By translating the $x$ and $y$-coordinates, it may be assumed that $\varepsilon(x, y) = F(x)/G(y)$ is equal to either of (a) $\varepsilon = 1/\beta y$, (b) $\varepsilon = \beta x$, (c) $\varepsilon = \beta x/y$. In case of (a)–(c), $\varepsilon$ satisfies $\varepsilon \varepsilon_x + \varepsilon_y = 0$ if and only if (c) and $\beta = 1$.

The following examples explains each case. The case (b) may be omitted, because it can be obtained from (a) by exchanging $x$ and $y$, and $f_1$ and $f_2$. The case (a) contains the Vogel spiral as a special case.

**Example 5** ((a) $\varepsilon(x, y) = 1/\beta y$: packing of a disk). In this example, $\varphi$ is always set to $1/(1 + \gamma_1) = (3 - \sqrt{5})/2$. Without loss of generality, $\beta > 0$ may be assumed.

From $\varepsilon(x, y) = 1/\beta y$, the map $f$ is as follows, up to similarity transformations.

$$f(x, y) = \sqrt{\frac{\sin(\beta x/2)}{\cos(\beta x/2)}},$$

which is injective on the following $\mathcal{D}$:

$$\mathcal{D} := \{(x, y) \in \mathbb{R}^2 : 0 \leq x < 4\pi/\beta, 0 < y < M\}.$$

The following are considered as the basis matrix $B$ of $L$.

\begin{align*}
(i) & \begin{pmatrix} 1 & -\varphi \\ 0 & -1 \end{pmatrix}, & (ii) & \begin{pmatrix} 1 & -\varphi \\ 1 & -\varphi \end{pmatrix}, & (iii) & \begin{pmatrix} 0 & -1 \\ 1 & -\varphi \end{pmatrix},
\end{align*}

Although it was proved in Theorem[7] that the lattice basis in (ii) is optimal with respect to the stability of the packing density, the mapped lattice $L$ also needs to contain a vector very close to $\left(4\pi/\beta, 0\right)$ in order to smoothly connect the spiral patterns near the half-lines $x = 0$ and $x = 4\pi/\beta$ (see Figure[4]).

(i) If $s := 4\pi/\beta$ is an integer, then $\left(s, 0\right)$ is precisely contained in $L(B)$ in this case. If $s = 1$, $f\left(L(B) \cap \mathcal{D}\right)$ is the same as the Vogel spiral.

$$f\left(L(B) \cap \mathcal{D}\right) := \left\{ \sqrt{n}(\cos(2\pi n\varphi/s), \sin(2\pi n\varphi/s)) : n \in \mathbb{Z}, 0 < n < M \right\}.$$

However, as shown in (i) of Figure[4], the points $f\left(L(B) \cap \mathcal{D}\right)$ are not well distributed around the origin for large $s$, because $s = 4\pi/\beta$ is not sufficiently small (cf. (2) of Theorem[7]). The non-uniformity can be avoided by using the basis (ii) instead.

(ii) In this case, $L(B)$ does not contain any vectors of the form $\left(s, 0\right)$. However, the $n$'th convergent $p_{-n}^{-}/q_{-n}^{-}$ of $-1/\varphi$ fulfills:

\begin{align*}
\begin{pmatrix} 1 & -\varphi \\ 1 & -\varphi \end{pmatrix} \begin{pmatrix} q_{-n}^{-} \\ -p_{-n}^{-} \end{pmatrix} & = \begin{pmatrix} q_{-n}^{-} + p_{-n}^{-} \varphi \\ q_{-n}^{-} + p_{-n}^{-} \varphi \end{pmatrix} \\
& = \begin{pmatrix} 2q_{-n}^{-} + p_{-n}^{-} (\varphi + \varphi) \\ 0 \end{pmatrix} + q_{-n}^{-} \varphi \begin{pmatrix} \varphi^{-1} + p_{-n}^{-} \varphi \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\end{align*}
Therefore, by setting $s$ to one of $\left| 2q_n^{(-)} + p_n^{(-)}(\varphi + \overline{\varphi}) \right|$ ($n \geq 0$), it is possible to glue the spiral patterns near the boundary of $\mathcal{D}$ apparently smoothly, as seen in (ii) of Figure 4. This technique of setting $s$ to a special value will be also used in the other examples. For the given parameters, the discontinuity in spirals could be completely removed by violating the assumptions (⋆) and (⋆⋆) very slightly.

(iii) As in case (ii), although $L(B)$ does not contain any vectors $t(s, 0)$, the boundary problem can be avoided, by putting $s = \left| p_n^{(-)} \right|$, because

\[
\begin{pmatrix}
0 & -1 \\
1 & -\varphi
\end{pmatrix}
\begin{pmatrix}
q_n^{(-)} \\
-p_n^{(-)}
\end{pmatrix}
= 
\begin{pmatrix}
p_n^{(-)} \\
q_n^{(-)} + p_n^{(-)}\varphi
\end{pmatrix}
\approx 
\begin{pmatrix}
p_n^{(-)} \\
0
\end{pmatrix}.
\]

The packing becomes sparse at the coordinates farther from the origin, as seen in (iii) of Figure 4. The number of spines is equal to the parameter $s = \left| p_n^{(-)} \right|$.

Figure 4: Packings in a disk obtained from the lattice bases (i)–(iii) of Example 5. The parameter $s$ is set to (i), (ii) $s = 2q_9^{(-)} + (\varphi + \overline{\varphi})p_9^{(-)} = 47$, where $p_9^{(-)} = -5$, $q_9^{(-)} = 55$ are the ninth convergent of $-1/\overline{\varphi} = (-3 + \sqrt{5})/2$, and (iii) $s = -p_{11}^{(-)} = 55$. As for (ii), the case of $s = 45 \neq 2q_n^{(-)} + (\varphi + \overline{\varphi})p_n^{(-)}$ ($n \in \mathbb{Z}_{>0}$) is also presented. The arrow indicates that the spirals are not smoothly connected around the $x > 0$ part of the $x$-axis.
It is known that the parastichies in the Vogel spiral are the Fermat spiral. In fact, the image of \((x, y)\) by the following \(f\) has the polar coordinate \((r, \theta) = (\sqrt{r}, \beta x/2)\).

\[
 f(x, y) = \sqrt{\frac{\cos(\beta x/2)}{\sin(\beta x/2)}}.
\]

If \(x, y\) is colinear, \((r, \theta) = (\sqrt{r}, \beta x/2)\) satisfies a linear equation \(r^2 = a + b\theta\) for some constants \(a\) and \(b\), which is an equation of the Fermat spiral.

In the following case of (c), the image of any line \(y = ax\) passing through the origin by the map \(f\) in Eq. (21), is a logarithmic spiral, because \(\log r\) and \(\theta\) fulfill a linear equation.

**Example 6** (c) \(\varepsilon(x, y) = \beta x/y\), case of logarithmic spirals. The map \(f\) is as follows, up to similarity transformations:

\[
f(x, y) = \sqrt{xy} \left(\begin{array}{c}
\cos \theta(x, y) \\
\sin \theta(x, y)
\end{array}\right), \quad \theta(x, y) = -\frac{\beta^{-1}}{2} \log |x| + \frac{\beta}{2} \log |y|.
\]

The map \(f\) is injective on the \(\mathcal{D}\):

\[
\mathcal{D} := \left\{(x, y) \in \mathbb{R}^2 : 0 < \log x \leq \frac{4\pi}{\beta + \beta^{-1}}, 0 < y \leq M\right\}.
\]

As in the previous example, it is necessary to set \(s := \exp(4\pi/(\beta + \beta^{-1}))\) to a positive integer \(s = 2q_n^{(-)} + (\varphi + \overline{\varphi})p_n^{(-)}\). Since \(X^2 - (4\pi/\log s)X + 1 = 0\) has a real root \(\beta\), \(1 \leq s \leq e^{2\pi} \approx 535.5\) is also required.

Proposition [3] is a case of dimension \(n = 3\).

**Proposition 3.** If \(n = 3\) and \(A_3 = 0\), the solutions \(\phi_1(x), \phi_2(x), \phi_3(x)\) for the system of PDEs in Eqs. (14), (15) and (**) are given by

\[
\begin{align*}
\phi_1(x) &= e^{1/3}(d^3 + 3d_2x_3)^{1/3} \varepsilon^{-1/2}(x_1, x_2), \\
\phi_2(x) &= e^{1/3}(d^3 + 3d_2x_3)^{1/3} \varepsilon^{1/2}(x_1, x_2), \\
\phi_3(x) &= e^{1/3}(d^3 + 3d_2x_3)^{-2/3}
\end{align*}
\]

for some constants \(d, d_2\) and \(\varepsilon(x_1, x_2) = \varepsilon_{x_1 x_2} + (\varepsilon^{-1})_{x_1 x_2} = -2d_2^2\).

**Proof.** From \(A_3 = U^{-1}U_{x_3} = O\), \(U\) is independent of \(x_3\). From [16], \(\phi_3(x)_{x_3} = (\phi_3)_{x_3} = 0\) also holds. Eq. (15) implies that \((\phi_3^{-1}(\phi_1)_{x_3})_{x_3} = (\phi_3^{-1}(\phi_2)_{x_3})_{x_3} = 0\). Thus, if we put \(G(x_3) := \int_0^x \phi_3(x_1, x_2, x) dx\), the equation

\[
\phi_i(x) = F_i(x_1, x_2)G(x_3) + H_i(x_1, x_2) \quad (i = 1, 2).
\]

holds for some \(F_i(x_1, x_2)\) and \(H_i(x_1, x_2)\).
Figure 5: Packing of planes with logarithmic spirals. Each point is colored according to the \( y \)-value (birth time) of its preimage (cf. Eq.(22)). The points with the same \( y \)-value form the identical shape, regardless of \( 0 < y \leq M \). This self-similarity explains their biological shapes. The last \( s = e^{2\pi} \approx 535.5 \) is also the case of an inviscid Burgers solution.
From $\phi_1\phi_2\phi_3 = c$, $(\phi_1\phi_2)_{x_j} = 0$ holds for $j = 1, 2$. Thus, for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ and $\tilde{F}(x_1, x_2)$,

$$\phi_1(x) = (\alpha_1G(x_3) + \beta_1)\tilde{F}(x_1, x_2),$$
$$\phi_2(x) = (\alpha_2G(x_3) + \beta_2)/\tilde{F}(x_1, x_2).$$

The following equations are obtained from [14] and [15].

$$\phi_2^{-1}(\phi_1)_{x_3} = 1/2 \frac{\alpha_1G(x_3) + \beta_1}{\alpha_2G(x_3) + \beta_2} (F^{-2})_{x_3},$$
$$\phi_1^{-1}(\phi_2)_{x_1} = 1/2 \frac{\alpha_2G(x_3) + \beta_2}{\alpha_1G(x_3) + \beta_1} (F^{-2})_{x_1}. \tag{23}$$

Hence, it is possible to choose $r_1, r_2 \neq 0, d, d_2 \in \mathbb{R}$ and $F(x, y)$ so that $\phi_1 = r_1(d + d_2G(x_3))\tilde{F}(x_1, x_2)$ and $\phi_2 = r_2(d + d_2G(x_3))/\tilde{F}(x_1, x_2)$. The following equations are obtained from $\phi_1\phi_2\phi_3 = r_1r_2(d + d_2G(x_3))^2\phi_3(x_3) = c$, $\phi_3(x_3) = G'(x_3)$, $G(0) = 0$ and Eq. (25):

$$r_1r_2(d + d_2G(x_3))^3 = r_1r_2d^3 + 3cd_2x_3,$$
$$r_2(\tilde{F}^{-2})_{x_1} + \frac{r_2}{2r_1}(\tilde{F}^{-2}_{x_2})_{x_2} = -r_1r_2d_2^2. \tag{26}$$

By putting $\epsilon(x_1, x_2) := (r_2/r_1)^2\tilde{F}(x_1, x_2)^{2}, \epsilon_{x_1x_1} + (\epsilon^{-1})_{x_2x_2} = -2(r_1r_2)d_2^2$ is obtained in addition to the following.

$$\phi_1(x) = (r_1r_2)^{1/2}(d^3 + 3(c_2/d_2r_1r_2)x_3)^{1/3}\epsilon^{-1/2}(x_1, x_2),$$
$$\phi_2(x) = (r_1r_2)^{1/2}(d^3 + 3(c_2/d_2r_1r_2)x_3)^{1/3}\epsilon^{1/2}(x_1, x_2),$$
$$\phi_3(x) = c(r_1r_2)^{-2}(d + d_2G(x_3))^{-2} = c(r_1r_2)^{-2}(d^3 + 3(c_2/d_2r_1r_2)x_3)^{-2/3}. \tag{27}$$

The statement is proved if $d, d_2$ are replaced by $c^{1/3}(r_1r_2)^{-1/2}d, (r_1r_2)^{-1/2}d_2$. \hfill \Box

Separation of variables can also be used to obtain a family of solutions of $\epsilon_{x_1x_1} + (\epsilon^{-1})_{x_2x_2} = -2d_2^2$ for $d_2 \neq 0$. A packing of a ball (i.e., 3D analogue of the Vogel spiral), is obtained as a result; if we put $\epsilon(x_1, x_2) = F(x_1)/G(x_2)$, $\epsilon_{x_1x_1} + (\epsilon^{-1})_{x_2x_2} + 2d_2^2 = 0$ implies:

$$F(x_1)F''(x_1) + G(x_2)G''(x_2) + 2d_2^2F(x_1)G(x_2) = 0.$$ 

Hence $F(x_1)$ or $G(x_2)$ must be a constant function, and either of (a) $\epsilon = d_0 + d_1x_1 - (d_2x_1)^2$ or (b) $\epsilon = 1/\{d_0 + d_1x_2 - (d_2x_2)^2\}$ holds. As the part (b) can be obtained by swapping the roles of $x_1$ and $x_2$, and $f_1$ and $f_2$ in (a), we will discuss only the case (a) below.

\hfill 23
Example 7 (Packing of a ball, 3D Vogel spiral). From $\varepsilon = d_0 + d_1 x_1 - (d_2 x_1)^2$, $d_2 \neq 0$ and $A_k = \mathcal{e}_k \mathcal{e}_k - \mathcal{c}_k \mathcal{c}_k$.

$A_1 = \begin{pmatrix} 0 & 0 & d_2 \varepsilon^{-1/2} \\ 0 & 0 & 0 \\ -d_2 \varepsilon^{-1/2} & 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & -\varepsilon_{x_1}/2 & 0 \\ \varepsilon_{x_1}/2 & 0 & d_2 \varepsilon^{1/2} \\ 0 & -d_2 \varepsilon^{1/2} & 0 \end{pmatrix}$, $A_3 = O$.

The following $D_j, V_j$ ($j = 1, 2$) provide diagonalizations $A_1 = V_1 D_1 V_1^*$, $A_2 = V_2 D_2 V_2^*$ of $A_1$ and $A_2$.

$D_1 = d_2 \varepsilon^{-1/2} \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $V_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$.

$D_2 = \sqrt{d_1^2/4 + d_0 d_2^2} \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$V_2 = \frac{1}{\sqrt{2(d_1^2/4 + d_0 d_2^2)}} \begin{pmatrix} \varepsilon_{x_1}/2 & -i \sqrt{d_1^2/4 + d_0 d_2^2} & \varepsilon_{x_1}/2 \\ -i \sqrt{d_1^2/4 + d_0 d_2^2} & 0 & \varepsilon_{x_1}/2 \\ d_2 \varepsilon^{1/2} & d_2 \varepsilon^{1/2} & \varepsilon_{x_1}/\sqrt{2} \end{pmatrix}.$

From $(V_j)_{x_j} = O$ and $U_{x_j} = UA_j$ ($j = 1, 2$), $U(x)$ satisfies $(UV_j)_{x_j} = UV_j D_j$. Hence, for some $U_0 \in O(3)$,

$U(x) = \frac{1}{\sqrt{d_1^2/4 + d_0 d_2^2}} U_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \sqrt{d_1^2/4 + d_0 d_2^2} & \sin \sqrt{d_1^2/4 + d_0 d_2^2} \\ 0 & -\sin \sqrt{d_1^2/4 + d_0 d_2^2} & \cos \sqrt{d_1^2/4 + d_0 d_2^2} \end{pmatrix} \begin{pmatrix} d_2 \varepsilon^{1/2} & 0 & -\varepsilon_{x_1}/2 \\ 0 & \sqrt{d_1^2/4 + d_0 d_2^2} & 0 \\ \varepsilon_{x_1}/2 & 0 & d_2 \varepsilon^{1/2} \end{pmatrix}.$

The Jacobian matrix of the map $f$ is provided as $U(x) \Phi(x)$. Hence, for some $v_0 \in \mathbb{R}^3$,

$f(x) = \frac{c^{1/3} \sqrt{d_1^2/4 + d_0 d_2^2}}{d_2} \begin{pmatrix} (d^3 + 3d_2 x_3)^{1/3} & 0 & \frac{-\varepsilon_{x_1}/2}{d_2 \varepsilon^{1/2} \sin \sqrt{d_1^2/4 + d_0 d_2^2}} \\ d_2 \varepsilon^{1/2} \cos \sqrt{d_1^2/4 + d_0 d_2^2} & d_2 \varepsilon^{1/2} \sin \sqrt{d_1^2/4 + d_0 d_2^2} & \v_0 \end{pmatrix}

\propto (d^3 + 3d_2 x_3)^{1/3} U_0 \begin{pmatrix} \sqrt{d_1^2/4 + d_0 d_2^2} - (d_2^2 x_1 - d_1/2)^2 \sin \sqrt{d_1^2/4 + d_0 d_2^2} & 0 & -\v_0 \\ \sqrt{d_1^2/4 + d_0 d_2^2} - (d_2^2 x_1 - d_1/2)^2 \cos \sqrt{d_1^2/4 + d_0 d_2^2} & 0 & 0 \end{pmatrix}$.

For constant $r > 0$, by putting $s := \frac{1}{\sqrt{d_1^2/4 + d_0 d_2^2}} U_0 I$ and $v_0 = 0$, and replacing $x_1$, $x_2$, $x_3$ by $(x_2/ri + d_1/2)^2$, $2\pi x_1$, $(x_3 - d^3)/3d_2$ respectively,

$f(x) \propto x_3^{1/3} \begin{pmatrix} \sqrt{r^2 - x_3^2} \sin(2\pi x_1/s) \\ \sqrt{r^2 - x_3^2} \cos(2\pi x_1/s) \end{pmatrix}.$
The above $f$ is injective on the $\mathcal{D}$, and maps $\mathcal{D}$ onto a ball of radius $rR$:

$$\mathcal{D} := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leq x_1 < s, -r < x_2 < r, 0 < x_3 < R\}.$$
4 General case: packing of Riemannian manifolds

The determination of the Riemannian manifolds that can be packed by the proposed method is a problem raised for the first time in this study. Theorem 2 deals with this problem in a local sense in the case of real analytic surfaces (class $C^\omega$). The general treatment of the same problem in a global sense is left as a challenging problem for future research.

Let $N \geq n > 0$ be integers, and $f(x) = (f_1(x), \ldots, f_n(x)) : \mathcal{D} \rightarrow \mathbb{R}^N$ be a function defined on an open subset $\mathcal{D} \subset \mathbb{R}^n$ that fulfills ($*$), ($**$) in the Introduction section. Thus,

$$Df(x) = U(x) \begin{pmatrix} \Phi(x) \\ O \end{pmatrix}, \quad O : (N-n) \times n \text{ zero matrix.} \tag{26}$$

As in the previous section, if $n$ and $N$ are fixed, it is possible to derive the PDE systems for the diagonal entries of $\Phi(x)$ and the orthogonal matrix $U(x)$.

The Nash embedding theorem states that there is a lower bound $N_0$ only depending on $k$ such that any $C^k$ Riemannian $n$-manifolds $(M, g)$ can be isometrically embedded into the Euclidean space $\mathbb{R}^N$ if $N \geq n_0$, by an injective map of class $C^k$ ($3 \leq k \leq \infty$ or $k = \omega$) [30], [31].

Therefore we can fix an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ of $M$ and isometries $\iota_\alpha : U_\alpha \rightarrow \iota_\alpha(U_\alpha) \subset \mathbb{R}^N$. Let $Df_\alpha(x)$ be the Jacobian matrix of $f_\alpha := \iota_\alpha \circ \varphi_\alpha^{-1}$. For another diffeomorphism $\psi_\alpha : U_\alpha \rightarrow \psi_\alpha(U_\alpha) \subset \mathbb{R}^n$ and any open subset $V \subset U_\alpha$, it is straightforward to see that the Jacobian matrix of $\iota_\alpha \circ \psi_\alpha^{-1}$ fulfills ($*$) and ($**$) on $V$ if and only if the following ($\beta$) holds:

$(\beta)$ The diffeomorphism $x = \sigma(y) := \varphi_\alpha \circ \psi_\alpha^{-1} : \psi_\alpha(V) \overset{\simeq}{\rightarrow} \varphi_\alpha(V)$ has the Jacobian matrix $D\sigma$ such that $\left(D\sigma(y)\right)^t(Df_\alpha(\sigma(y)))Df_\alpha(\sigma(y))D\sigma(y)$ is diagonal, and has a constant determinant.

In particular, if $\varphi_\alpha$ is an isothermal coordinate system, i.e., $(Df_\alpha(x))Df_\alpha(x) = \lambda(x)I$ for some positive-valued function $\lambda(x)$, then ($\beta$) occurs if and only if $(D\sigma(y))D\sigma(y)$ is diagonal, and

$$\det((D\sigma(y))D\sigma(y)) = e^2 \lambda^{-n}(\sigma(y)).$$

In Theorem 2, we assume that $n = 2$, and the Riemannian surface $(M, g)$ is real analytic in order to use the Cauchy-Kovalevskaya theorem.

**Theorem 2.** For any constant $c \neq 0$ and real analytic Riemannian surface $(M, g)$, there is an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ of $M$ such that for any $\alpha \in I$, we have a real analytic function $\varepsilon(x_1, x_2)$ on $\varphi_\alpha(U_\alpha)$ with

$$g|_{U_\alpha} = \frac{c^2}{\varepsilon(x_1, x_2)} dx_1^2 + \varepsilon(x_1, x_2) dx_2^2.$$

Thus, for any isometry $\iota_\alpha : U_\alpha \rightarrow \iota_\alpha(U_\alpha) \subset \mathbb{R}^N$, $\iota_\alpha \circ \varphi_\alpha^{-1}$ fulfills ($*$) and ($**$).
Proof. For any \( p \in M \), fix a neighborhood \( p \in U_\alpha \subset M \), a diffeomorphism \( \varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{R}^2 \) and an isometry \( \iota_\alpha : U_\alpha \to \iota_\alpha(U_\alpha) \subset \mathbb{R}^N \). We may assume that \( \varphi_\alpha \) is an isothermal coordinate system of \( U_\alpha \). We shall prove that some neighborhood \( V \subset U_\alpha \) of \( p \) and a diffeomorphism \( \sigma : \sigma^{-1}(\varphi_\alpha(V)) \to \varphi_\alpha(V) \) satisfy (b) with respect to \( f_\alpha := \iota_\alpha \circ \varphi_\alpha^{-1} \) and \( \psi_\alpha := \sigma^{-1} \circ \varphi_\alpha | V \). Hence the chart \((V, \psi_\alpha)\) has the desired property from the above discussion.

\[
D\sigma^{-1}(x)^t(D\sigma^{-1}(x)) = (\sigma^{-1}(\varphi_\alpha(V)))^{-1} \text{ is diagonal and has a determinant } c^{-2}\lambda^2(x) \text{ for some positive function } \lambda(x) \text{ on } \varphi_\alpha(V) \text{ if and only if } \]

\[
D\sigma^{-1} \text{ is represented as follows:}
\]

\[
D\sigma^{-1}(x_1, x_2) = c^{-1}\lambda(x_1, x_2) \begin{pmatrix} e^{1/2}(x_1, x_2) & 0 \\ 0 & \pm e^{-1/2}(x_1, x_2) \end{pmatrix} U(x_1, x_2),
\]

\[
U(x_1, x_2) = \begin{pmatrix} \cos \theta(x_1, x_2) & \sin \theta(x_1, x_2) \\ -\sin \theta(x_1, x_2) & \cos \theta(x_1, x_2) \end{pmatrix}.
\]

If we put \( B_i := (D\sigma^{-1})^{-1}(D\sigma^{-1})_{x_i} \) and \( A_i := U^{-1}U_i \), then

\[
B_i = \frac{\lambda_{x_i}}{\lambda} I + \frac{e_{x_i}}{2\epsilon} U \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U + A_i
\]

\[
= \frac{\lambda_{x_i}}{\lambda} I + \frac{e_{x_i}}{2\epsilon} \begin{pmatrix} \cos 2\theta(x_1, x_2) & \sin 2\theta(x_1, x_2) \\ \sin 2\theta(x_1, x_2) & -\cos 2\theta(x_1, x_2) \end{pmatrix} + \theta_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

From \((f_{x_1})_{x_1} = (f_{x_2})_{x_2}\), the second column of \((D\sigma^{-1})_{x_1} = (D\sigma^{-1})_{B_1} \) and the first column of \((D\sigma^{-1})_{x_2} = (D\sigma^{-1})_{B_2} \) must be equal. Hence, the second column of \( B_1 \) and the first column of \( B_2 \) are also equal, which implies

\[
\frac{\lambda_{x_1}}{\lambda} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{e_{x_1}}{2\epsilon} \begin{pmatrix} \sin 2\theta(x_1, x_2) \\ -\cos 2\theta(x_1, x_2) \end{pmatrix} + \theta_{x_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\lambda_{x_2}}{\lambda} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{e_{x_2}}{2\epsilon} \begin{pmatrix} \cos 2\theta(x_1, x_2) \\ \sin 2\theta(x_1, x_2) \end{pmatrix} + \theta_{x_2} \begin{pmatrix} 0 \\ -1 \end{pmatrix}.
\]

Thus,

\[
\begin{pmatrix} \cos 2\theta(x_1, x_2) & -\sin 2\theta(x_1, x_2) \\ \sin 2\theta(x_1, x_2) & \cos 2\theta(x_1, x_2) \end{pmatrix} \begin{pmatrix} (\log \epsilon)_{x_2} \\ (\log \epsilon)_{x_1} \end{pmatrix} = 2 \begin{pmatrix} -\theta_{x_2} - \theta_{x_1} \\ \theta_{x_1} + \theta_{x_2} \end{pmatrix}.
\]

From \(((\log \epsilon)_{x_1})_{x_1} = ((\log \epsilon)_{x_1})_{x_2} = (\log \epsilon)_{x_1} \),

\[
\frac{\partial}{\partial x_1} \begin{pmatrix} \cos 2\theta(x_1, x_2) & \sin 2\theta(x_1, x_2) \end{pmatrix} \begin{pmatrix} (\log \lambda)_{x_2} + \theta_{x_1} \\ (\log \lambda)_{x_1} + \theta_{x_2} \end{pmatrix} = \frac{\partial}{\partial x_2} \begin{pmatrix} -\sin 2\theta(x_1, x_2) & \cos 2\theta(x_1, x_2) \end{pmatrix} \begin{pmatrix} (\log \lambda)_{x_2} + \theta_{x_1} \\ (\log \lambda)_{x_1} + \theta_{x_2} \end{pmatrix}.
\]

which implies

\[
\begin{pmatrix} 1 & \tan 2\theta \\ -\tan 2\theta & 1 \end{pmatrix} \begin{pmatrix} (\log \lambda)_{x_1,x_2} + \theta_{x_1 x_2} \\ (\log \lambda)_{x_2} + \theta_{x_2} \end{pmatrix} + 2\theta_{x_2} \begin{pmatrix} (\log \lambda)_{x_1} + \theta_{x_1} \\ (\log \lambda)_{x_1} + \theta_{x_2} \end{pmatrix} = \begin{pmatrix} 1 & \tan 2\theta \\ -\tan 2\theta & 1 \end{pmatrix} \begin{pmatrix} (\log \lambda)_{x_1} + \theta_{x_1} \\ (\log \lambda)_{x_2} + \theta_{x_2} \end{pmatrix}.
\]

\[
(1) \quad (\log \lambda)_{x_1} + \theta_{x_2} + 2\theta_{x_2} (\log \lambda)_{x_1} + \theta_{x_2} = (\log \lambda)_{x_1} + \theta_{x_1}.
\]

\[
(2) \quad (\log \lambda)_{x_2} + \theta_{x_2} - 2\theta_{x_2} (\log \lambda)_{x_2} + \theta_{x_2} = (\log \lambda)_{x_2} + \theta_{x_2}.
\]
From the Cauchy-Kovalevskaya theorem, Eq. [27] has a real analytic solution \( \theta \) defined on \( \varphi_\alpha(V) \) for a simply-connected neighborhood \( p \in V \subset U \). From this \( \theta \), \( \log \varepsilon \) and \( \sigma^{-1}(x_1,x_2) \) can be constructed by solving the corresponding PDEs above.

Next, we prove Theorem 3. For any diffeomorphism \( \sigma : U \to V \) between open subsets of two manifolds and metric \( g \) on \( V \), for any \( p \in U \) and \( u,v \in T_pU \), the pullback metric \( \sigma^*g \) is defined by

\[
(\sigma^*g)_p(u,v) = g_{\sigma(p)}(d\sigma_p(u),d\sigma_p(v)).
\]

**Theorem 3.** For any integers \( 0 < n < N \), and \( 2 \leq k \leq \infty \) or \( k = \omega \), let \( f(x) \in C^k(\mathcal{D},\mathbb{R}^N) \) be a function on a simply-connected open subset \( \mathcal{D} \subset \mathbb{R}^n \) with the Jacobian matrix \( J_f(x) \). Suppose that \( f(x) \) satisfies (i) and (ii) for some constant \( \alpha \in \mathbb{R} \) and antisymmetric matrix \( A \) of degree \( N \):

(i) \( J_f(x)Q(x)J_f(x) = 0 \) for any \( x \in \mathcal{D} \), if we put:

\[
\begin{align*}
J^*(x) &:= (\alpha I + A)f(x), \\
Q(x) &:= f^*(x)^tJ^*(x)A + Af^*(x)^tJ^*(x) - (f^*(x) \cdot f^*(x))A.
\end{align*}
\]

(ii) \( \det(J_f(x)) \neq 0 \), and \( f^*(x) \) is linearly independent of \( f_{x_1}(x),\ldots,f_{x_n}(x) \) over \( \mathbb{R} \) for any \( x \in \mathcal{D} \).

Let \( H_f(x) \) be the function on \( \mathcal{D} \) determined by the following equations, except for the constant term:

\[
(H_f)_{s_j}(x) = \frac{f^*(x) \cdot f_{s_j}(x)}{f^*(x) \cdot f^*(x)} \quad (j = 1,\ldots,n).
\]

Let \( q_f = (q_{ij})_{1 \leq i,j \leq n} \) be the positive-definite symmetric matrix.

\[
q_f(x) := e^{-\frac{2(n+1)}{n} \alpha H_f(x)} (\frac{f^*(x) \cdot f^*(x)}{f^*(x) \cdot f^*(x)})^\frac{1}{2} J_f(x) \left( I - \frac{f^*(x) \cdot f^*(x)}{f^*(x) \cdot f^*(x)} \right) J_f(x).
\]

Hence, \( g_f = \sum_{i,j=1}^{n} q_{ij}dx_idx_j \) is a Riemannian metric on \( \mathcal{D} \). For any function \( h(x,x_{n+1}) \) on \( \mathcal{D} \times \mathbb{R}_{>0}, F_{f,h}(x,x_{n+1}) : \mathcal{D} \times \mathbb{R}_{>0} \to \mathbb{R}^N \) is defined by

\[
F_{f,h}(x,x_{n+1}) := e^{\alpha h(x,x_{n+1})} \exp(Ah(x,x_{n+1}))f(x).
\]

The following (a) and (b) are equivalent:

(a) \( F_{f,h}(x,x_{n+1}) \) satisfies (\( * \), (\( ** \)) for some \( h(x,x_{n+1}) \in C^k(\mathcal{D} \times \mathbb{R}_{>0},\mathbb{R}) \).

(b) The metric \( g_f \) is diagonal and has a constant determinant on \( \mathcal{D} \).

For any \( C^k \) diffeomorphism \( x = \sigma(y) \) on \( \mathcal{D} \), (\( a' \), (\( b' \)) are also equivalent:
(a') \( F_{f,\sigma,h}(y, x_{n+1}) \) fulfills (\( \ast \)), (\( \ast \ast \)) for some \( h(y, x_{n+1}) \in C^k(\mathcal{O} \times \mathbb{R}_{>0}, \mathbb{R}) \).

(b') The pull-back \( \sigma^* g_f = g_{f,\sigma} \) is diagonal and has a constant determinant on \( \mathcal{O} \).

**Remark 4.** In (b'), such a \( \sigma \) exists whenever \( n = 1 \), or \( n = 2 \) and \( k = \omega \) as a result of Theorem 2. In (i), \( Q(x) = 0 \) holds whenever \( A = 0 \), or \( n = 1 \), or \( N = 2, 3 \) and \( \alpha = 0 \).

**Proof.** From (i),

\[
\left( \frac{f^* \cdot f_{x_j}}{f^* \cdot f_i} \right)_{x_i} = 2J_f (f^* A + Af^* f^*) \frac{J_f}{(f^* f^*)^2} = O.
\]

The existence of \( H_f \) is obtained from this and the generalized Stokes theorem for 1-forms of class \( C^1 \) (Theorem 6.1, XXIII, [23]). Since \( I - J_f (J_f J_f)^{-1} J_f \) is the projection onto the linear space generated by \( f_{x_1}, \ldots, f_{x_n} \), the assumption (ii) implies:

\[
\begin{aligned}
\det q_f(x) &= e^{-2(n+1)\alpha H_f(x)}(f^*(x) (I - J_f (J_f J_f)^{-1} J_f) f^*(x)) \det(J_f(x) J_f(x)) \neq 0.
\end{aligned}
\]

Thus, \( q_f(x) \) is positive-definite, and \( g_f \) is a Riemannian metric on \( \mathcal{O} \).

For any diffeomorphism \( \sigma \) on \( \mathcal{O} \), (i) and (ii) hold for \( f \) if and only if they do for \( f \circ \sigma \). Thus, (a') \( \Leftrightarrow \) (b') is immediately obtained from (a) \( \Leftrightarrow \) (b). Hence, only (a) \( \Leftrightarrow \) (b) is proved in the following.

To show (a) \( \Rightarrow \) (b), we assume that \( F_{f,h}(x, x_{n+1}) \) satisfies (\( \ast \)) and (\( \ast \ast \)).

\[
\begin{align*}
(F_{f,h})_{x_j}(x, x_{n+1}) &= e^{\alpha h} \exp(Ah)(h_{x_j} f^*(x) + f_{x_j}(x)), \quad (j = 1, \ldots, n) \\
(F_{f,h})_{x_{n+1}}(x, x_{n+1}) &= e^{\alpha h} \exp(Ah)h_{x_{n+1}} f^*(x).
\end{align*}
\]

Because of (\( \ast \ast \)), \( h_{x_{n+1}}(x, x_{n+1}) \neq 0 \) must hold for any \( (x, x_{n+1}) \in \mathcal{O} \times \mathbb{R}_{>0}. \) In addition, \( (F_{f,h})_{x_j}(x, x_{n+1}) = 0 \) \( (j = 1, \ldots, n) \) from (\( \ast \)), and \( f^*(x) \cdot f^*(x) \neq 0 \) from (ii). These imply:

\[
\begin{align*}
h_{x_j}(x, x_{n+1}) &= \frac{f^*(x) \cdot f_{x_j}(x)}{f^*(x) \cdot f^*(x)}. 
\end{align*}
\]

Therefore, \( h(x, x_{n+1}) \) fulfills for some function \( h_0(x_{n+1}) \):

\[
h(x, x_{n+1}) = -H_f(x) + h_0(x_{n+1}). \tag{29}
\]

We also have:

\[
(F_{f,h})_{x_j}(x, x_{n+1}) = e^{\alpha h} \exp(Ah) \left( I - \frac{f^*(x) f^*(x)}{f^*(x) f^*(x)} \right) f_{x_j}(x) \quad (j = 1, \ldots, n).
\]
Thus, using the Jacobian matrix $J_f(x)$ of $f(x)$,

$$\left. (F_{j,h})_{x_i} \cdot (F_{j,h})_{x_i} \right|_{i,j \leq n} = e^{2\alpha h t} J_f(x) \left( I - \frac{f^*(x)^t f^*(x)}{f^*(x) f^*(x)} \right) J_f(x) \quad (30)$$

$$= e^{2\alpha (h + \frac{\alpha - 1}{2} H_f)} \left( f^*(x) \cdot f^*(x) \right)^{-\frac{1}{2}} q_f(x).$$

$F(x, x_{n+1})$ fulfills $(\ast \ast)$ if and only if the following holds for some constant $c \neq 0$:

$$\prod_{j=1}^{n+1} ((F_{j,h})_{x_j} \cdot (F_{j,h})_{x_j}) = h_{n+1}^2 e^{2(n+1)\alpha (h + H_f)} \det q_f(x)$$

$$= (h_{n+1}'(x_{n+1}))^2 e^{2(n+1)\alpha h_0(x_{n+1})} \det q_f(x) = c^2,$$

which implies: $\det q_f(x) = \gamma^2$ for some $\gamma \neq 0$. Thus, (a) $\implies$ (b) is proved.

Conversely, the above discussion shows that (b) $\implies$ (a) is obtained if there is a constant $d_1 \neq 0$ such that $h_{n+1}'(x_{n+1}) e^{(n+1)\alpha h_0(x_{n+1})} = d_1$. Namely,

$$h(x, x_{n+1}) = \begin{cases} -H_f(x) + \frac{1}{(n+1)\alpha} \log(d_1 x_{n+1} + d_2) & \text{if } \alpha \neq 0, \\ -H_f(x) + d_1 x_{n+1} + d_2 & \text{if } \alpha = 0. \end{cases} \quad (31)$$

In particular, if $d_1 = 1$ and $d_2 = 0$, the above $h(x, x_{n+1})$ is defined on $\mathbb{D} \times \mathbb{R}_{>0}$, which proves (b) $\implies$ (a). \hfill \Box

As seen from Eq. (30), if $N = n + 1$ and a diffeomorphism $\sigma$ on $\mathbb{D}$ satisfies (b') of Theorem 3 for some $3 \leq k \leq \infty$ or $k = \infty$, the following $\phi_j^2(x, x_{n+1})$ ($j = 1, \ldots, n$) and $\phi_{n+1}^2(x, x_{n+1}) = c^2 / \prod_{j=1}^n \phi_j^2(x, x_{n+1})$ are a solution of the PDEs of Theorem 2.

$$\phi_j^2(x, x_{n+1}) = e^{2\alpha h(x, x_{n+1})} \left( (f(\sigma(x)))_{x_j} \cdot (f(\sigma(x)))_{x_j} \right)^{\frac{1}{2}} \left. \frac{((f(\sigma(x)))_{x_j} \cdot f^*(\sigma(x)))^2}{f^*(\sigma(x)) \cdot f^*(\sigma(x))} \right|_{x_j} = \left( f(\sigma(x))_{x_j} \cdot f^*(\sigma(x))_{x_j} \right)^{\frac{1}{2}},$$

where $h(x, x_{n+1})$ is the function obtained by replacing $H_f$ in Eq. (31) with $H_{f \circ \sigma}$.

The 3D Vogel spiral can be obtained from Theorem 4 and the following parameters as the case of $n = 2, N = 3$:

$$\alpha = 0, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad f(s, t) = s^{1/3} \left( \begin{array}{c} t \\ 0 \\ \sqrt{t^2 - t^2} \end{array} \right).$$

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Author contributions

The last author designed the project, and performed the majority of the mathematical research. The first author derived the PDEs and coded programs with the last author. The second author supervised the project as the mentor of the SENTAN-Q program.

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