Linear instability leading to elastic turbulence in plane Couette flow

Miguel Beneitez,1 Jacob Page,2 and Rich R. Kerswell1

1DAMTP, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WA, UK
2School of Mathematics, University of Edinburgh, EH9 3FD, UK

(Dated: October 19, 2022)

Elastic turbulence (ET) is a chaotic flow state observed in dilute polymer solutions in the absence of inertia, which was originally discovered experimentally in curved geometries. It has long been thought that triggering ET in parallel flows requires a finite amplitude perturbation to generate streamline curvature. We demonstrate here that self-sustaining ET can be initiated via a linear instability in inertialess planar Couette flow, a flow configuration which has previously been assumed to be stable to all initial perturbations. The new linear instability is associated with the existence of finite polymer diffusivity and exists for a wide range of realistic parameter settings and different choices of boundary conditions on the polymer conformation tensor. Numerical simulations show that the instability leads to a three-dimensional self-sustaining chaotic state, which we believe is the first reported numerical evidence of ET in a parallel flow.

Dilute polymer solutions display a wealth of counterintuitive dynamics across a wide range of scales. Perhaps most strikingly, a chaotic flow state—so-called “elastic turbulence” (ET)—can occur in the absence of inertia in stark contrast to Newtonian turbulence (e.g. in water). This has important applications in, for example, small scale flows where enhanced mixing and/or heat transfer are highly desirable (e.g. microfluidics [1, 2]). The key ingredients for such a transition and for the existence of ET can be fluid elasticity provided by the polymers and streamline curvature which together give rise to a new linear elastic instability [3–6]. Experiments in curved geometries confirm the linear instability leads to sustained ET [7]. In contrast, a transition to ET in parallel flows is thought to require finite amplitude perturbations to generate sufficient streamline curvature to initiate this elastic instability [8, 9]. However, the requirements for both initiating such a transition and for the existence of a self-sustaining chaotic state in a planar geometry are unknown.

Recently a new linear elastic instability has been identified for the parallel flow in a pipe or channel at finite inertia [10, 11] and at vanishing inertia in a channel only [12, 13]. This instability is concentrated at centre of the channel and is strongly subcritical, giving rise to an arrowhead-shaped travelling wave solution [14, 15] over a large region of the parameter space. This solution has been seen in simulations at both finite [16] and vanishing inertia [15, 17], and may play a role in two-dimensional elasto-inertial turbulence where inertia is important [18], but no link has yet been found between this structure and ET in inertialess flows. This instability does not occur for the simplest parallel flow situation of constant shear between two differentially-moving, parallel plates, known as plane Couette flow, which is considered linearly stable for all inertia and elasticity parameters [10].

In this Letter we report that viscoelastic plane Couette flow is linearly unstable if polymer diffusion is included in the model. The linear instability mechanism differs significantly from the centre-mode present in channels, being concentrated instead at the walls. Significantly, there is no smallest diffusion threshold below which the instability vanishes: the wavelength of the instability decreases with the size of the diffusion so the instability could be misunderstood as a numerical instability. The new diffusive instability exists over a very wide area of the parameter space and is robust to the choice of boundary conditions on the polymer conformation. Direct numerical simulations show that the instability saturates onto a low-amplitude limit cycle in two-dimensions. Three-dimensional simulations show a transition to sustained ET, which we believe to be the first reported computation of such a state in a planar geometry.

We consider the inertialess flow of an incompressible, viscoelastic fluid between infinite plates at $y = \pm h$ moving with velocity $\pm U_0\hat{x}$. The governing equations are

$$\nabla p = \beta \Delta \mathbf{u} + (1 - \beta) \nabla \cdot \mathbf{T}(\mathbf{C}),$$

$$\nabla \cdot \mathbf{u} = 0,$$

$$\partial_t \mathbf{C} + (\mathbf{u} \cdot \nabla) \mathbf{C} + \mathbf{T}(\mathbf{C}) = \mathbf{C} \cdot \nabla \mathbf{u} + (\nabla \mathbf{u})^T \cdot \mathbf{C} + \varepsilon \Delta \mathbf{C},$$

where the polymeric stress $\mathbf{T}$ is related to the conformation tensor $\mathbf{C}$ using the FENE-P model,

$$\mathbf{T}(\mathbf{C}) := \frac{1}{W_i} \left[ \frac{\mathbf{C}}{(1 - (\text{tr} \mathbf{C} - 3)/L_{max}^2) - I} \right].$$

The equations are non-dimensionalised by the half the gap width, $h$, and the plate speed $U_0$, which defines the Weissenberg number $W_i := \lambda U_0/h$ (the ratio of the polymer relaxation time $\lambda$ to a flow timescale). The parameter $\beta := \mu_s/\mu_T$ is the ratio of the solvent-to-total viscosities while $\varepsilon$ is a dimensionless polymer diffusivity. In this configuration, the laminar basic state is simply $\mathbf{U} = y\hat{x}$ with only $T_{xx} = 2W_i$ and $T_{xy} = 1$ being non-zero stress components for an Oldroyd-B fluid ($L_{max} \to \infty$). The polymer equation (1c) changes character from hyperbolic at $\varepsilon = 0$ to parabolic for $\varepsilon \neq 0$ and extra boundary conditions are then needed. Three boundary conditions are considered: (i) application of the governing equations with $\varepsilon = 0$ at the walls [19], (ii) application of the governing equations with only the term $\varepsilon \partial_y \mathbf{C}_{ij}$ removed [13, 16] and (iii) Neumann so $\partial_y \mathbf{C}_{ij} = 0$. 
The linear stability of the basic state is examined by introducing small perturbations of the form \( \phi(x,t) = \tilde{\phi}(y) \exp(ik_x(x-ct)) + \text{c.c.} \), where \( k_x \in \mathbb{R} \) is the streamwise wavenumber and \( c = c_r + ic_i \) a complex wavespeed, with instability if \( c_i > 0 \). The linear eigenvalue problem is solved by expanding each flow variable using the first \( N \) Chebyshev polynomials (\( N = 300 \) is usually sufficient to ensure convergence). An example eigenvalue spectrum for an Oldroyd-B fluid with \( Wi = 100, \varepsilon = 10^{-3}, \beta = 0.95 \) and \( k_x = 3 \) is reported in figure 1 (the bottom right inset shows the equivalent spectrum with \( \varepsilon = 0 \)). The continuous spectra in the absence of polymeric diffusion are regularised with the introduction of \( \varepsilon \neq 0 \), with a pair of linear instabilities emerging with wavespeeds \( c_r \sim \pm 1 \).

We map out the unstable region for various parameters and boundary conditions in figure 2(a). In the top left panel of figure 2(a) we observe that the instability persists as \( \varepsilon \to 0 \) i.e. this is a singular limit for all three boundary conditions and occurs at a constant value of \( Wi \) for diffusivities \( \varepsilon \lesssim 10^{-2} \), requiring an increasingly large streamwise wavenumber \( k_x \propto \varepsilon^{-1/2} \). The unstable region appears unbounded as \( Wi \to \infty \) for boundary conditions (i) and (ii) but stability is restored in this limit for (iii). Henceforth boundary condition (i) is used.

In figure 2(b)-(c) we also examine the effect of the viscosity ratio and finite extensibility on the diffusive instability. The instability is realised for decreasing values of \( Wi \) at fixed \( \varepsilon \) as \( \beta \) is reduced (i.e. increasing polymer concentration), and the marginal stability curves collapse

---

**FIG. 1.** Spectrum at \( Re = 0, Wi = 100, \beta = 0.95, k_x = 3 \) for two different resolutions \( N_y = 300 \) (orange circles) and \( N_y = 400 \) (purple dots) with \( \varepsilon = 10^{-3} \) within the domain and \( \varepsilon = 0 \) at the boundaries. Top inset: zoom-in to the unstable eigenvalues \( c_i > 0 \). Bottom right inset: spectrum for the same parameters but \( \varepsilon = 0 \) everywhere in the domain and boundaries and two different resolutions \( N_y = 300 \) (red circles) and \( N_y = 400 \) (blue dots). The most unstable eigenvalues with finite \( \varepsilon \) here are \( c = \pm 1.0052607137 + 4.513293285i \times 10^{-5} \).

**FIG. 2.** (a) Neutral curves for Oldroyd-B in the \( \varepsilon-Wi \) space for \( \beta = 0.9 \) with different boundary conditions: (i) \( \varepsilon = 0 \) at the walls (blue), (ii) \( \varepsilon \to 0 \) only for \( \partial_y \phi \) (red) and (iii) \( \partial_y \phi = 0 \) (purple). Dashed lines show the neutral curves for individual wavenumbers \( \{1, 2, 3\} \) with b.c. (ii). Inset shows the wavenumber \( k_x \) along the neutral curves and its scaling with \( \varepsilon^{-1/2} \) for all boundary conditions. (b) Neutral curves in the \( \varepsilon-Wi \) for different \( \beta \) in Oldroyd-B. Inset: collapse of the curves in the ultra-dilute limit \( \beta \to 1, Wi \to \infty \). (c) Effect of the adding finite extensibility to the flow in the FENE-P model. Curves show different values of \( L_{\text{max}} \) for \( \varepsilon = 10^{-3} \). Coloured area indicates instability.
when plotted against $Wi(1−\beta)$, which is the magnitude of the perturbation stresses $\tau_{xx}'$ and $\tau_{xy}'$ in the streamwise momentum equation. Furthermore, the instability survives for realistic values of the polymer extensibility of $L_{max} = O(100)$, in the FENE-P model, though it is pushed to increasingly low values of $\beta$ and suppressed beyond a critical $Wi$.

A neutral eigenfunction is shown in figure 3(a), where we visualise contours of the perturbation trace of the polymer conformation and streamlines for an example set of parameters $Wi = 47.84$, $\beta = 0.9$, $\varepsilon = 10^{-3}$. The spanwise vorticity is shown below in figure 3(b) for various values of $\varepsilon$, $\beta$ and $Wi$.

The mechanism of instability is reminiscent of the destabilizing effect of viscosity in Newtonian channel flow which subtly adjusts the relative phases of key dynamical processes. The polymer is highly stretched in a boundary layer $\delta \sim \varepsilon^{1/2}$ at one of the walls where the vorticity is concentrated. Without diffusion, the ensuing polymer stress would be exactly out of phase with the velocity field preventing any feedback. A diffusive phase lag in the polymer stretch response, however, allows the polymer stress to reinforce the perturbing flow.

The instability described here is realised for experimentally relevant parameters (e.g. $Wi \sim 50$ at $\beta = 0.9$) over a wide range of realistic diffusivities. For example, in a microfluidic context, $\varepsilon$ can range from $O(10^{-6})$ for long polymer molecules such as haemoglobin in aqueous solution to $O(10^{-3})$ for short polymers. We now show that this instability forms a pathway to ET in planar Couette flow.

The nonlinear evolution of the diffusive instability is examined by first conducting two-dimensional DNS of the full governing equations (1) using the open-source codebase Dedalus [20]. We perform calculations at parameter settings $Wi = 100$, $\beta = 0.9$, $L_{max} = 600$ and $\varepsilon = 10^{-3}$ in a box of length of length $L_x = 2\pi$ with resolution $[N_x, N_y] = [128, 256]$. The simulations are initialised with low amplitude white noise, which excites the most unstable mode $k_x = 3$. The unstable mode amplifies exponentially before saturating onto a limit cycle with period $T \sim h/U_0$. We report a snapshot of the instantaneous polymer stretch field in figure 4(a) alongside the volume-averaged – denoted $(\cdot)$ – vertical velocity squared in figure 4(b). While the polymer is significantly stretched relative to the laminar value, the velocity fluctuations are relatively low amplitude, $v = O(10^{-2}U_0)$.

In elasto-inertial flows, a sustained chaotic state can be realised in purely two-dimensional configurations [21], which is not something we observe in our inertialess calculations. The stability of the limit cycle is examined in a three-dimensional configuration with spanwise width $L_z = 2\pi$ by adding small-amplitude white noise in the velocity component $w$. Three-dimensional simulations were performed with resolutions ranging from $[N_x, N_y, N_z] = [64, 32, 128]$ to $[N_x, N_y, N_z] = [128, 64, 256]$ to check robustness. A time series of the volume-averaged polymer trace from this computation is reported in figure 4(d) which shows a departure from the simple time-periodic solution to a chaotic trajectory which is maintained for over a $10^4h/U_0$ time period. We believe this to be a numerical realisation of elastic turbulence in parallel flow.

An instantaneous snapshot of the elastic turbulence can be seen in figure 4(c). The horizontal pseudocolor plane located at $y = -0.99h$ (in the boundary layer at the lower wall for $\varepsilon = 10^{-3}$) shows how $\text{tr}(C)$ is modulated in the both the streamwise and spanwise direction forming high and low high stretch regions. The vertical back plane contours the vertical velocity $v$, which is amplified in small scale near-wall patches reminiscent of suction and ejection events in high Reynolds number wall bounded turbulence. The three-dimensional contours (in the box) represent $w$, illustrating the spanwise motion of the flow.

The statistics of both the limit cycle and ET are examined further in panels (e)-(h) of figure 4, which indicates (i) that the limit cycle is a significant departure from the laminar base state, (ii) EIT departs from the limit cycle while retaining certain features (iii) the polymer is substantially stretched in the centre of the domain for both the limit cycle and ET. There are also sharp variations of $u_{rms}$ in a boundary layer $\delta \sim \varepsilon^{1/2}$ at the walls (verified in computations at various $\varepsilon$ but not shown). The values of
\(v_{\text{rms}}\) in ET are significantly different from the limit cycle and reach their largest magnitude at the centreline.

**Summary.** The presence of polymeric diffusivity is commonly disregarded in the linear stability analyses of viscoelastic flows \([8, 10, 11, 22–24]\). However, we have found that its presence, even at vanishingly small values, fundamentally changes the stability of viscoelastic plane Couette flow in the absence of inertia (i.e. vanishing polymer diffusion is a singular limit). The diffusive instability is a ‘wall mode’, travelling with the roughly the wall speed, and has a streamwise wavelength comparable to the boundary layer thickness. This last feature could easily have led to the instability being dismissed in the past as a numerical instability. The onset of instability is found at \(Wi \approx 8\) independently of \(L_{\text{max}}\) and \(\varepsilon\) for a broad range of \(\beta\). DNS of the instability leads to a stable periodic orbit in 2D and to ET in 3D. The later represents the first numerical demonstration of sustained ET in a parallel flow configuration. That it is generated by a new diffusive instability in a simple constant shear
configuration suggests that this may represent a generic pathway to ET in other viscoelastic flows (e.g. the pathway is also present in channel flow [25]).

The authors gratefully acknowledge the support of EP-SRC through grant EP/V027247/1.

[1] A. Groisman and V. Steinberg, Efficient mixing at low Reynolds numbers using polymer additives, Nature 410, 905 (2001).
[2] T. M. Squires and S. R. Quake, Microfluidics: Fluid physics at the nanoscale rate, Reviews of Modern Physics 77, 977 (2005).
[3] E. S. G. Shaqfeh, Purely elastic instabilities in viscometric flows, Annual Review of Fluid Mechanics 28, 129 (1996).
[4] A. Groisman and V. Steinberg, Elastic turbulence in a polymer solution flow, Nature 405, 53 (2000).
[5] R. G. Larson, Turbulence without inertia, Nature 405, 27 (2000).
[6] A. Groisman and V. Steinberg, Elastic turbulence in curvilinear flows of polymer solutions, New Journal of Physics 6, 29 (2004).
[7] V. Steinberg, Elastic turbulence: an experimental view on inertialess random flow, Annual Review of Fluid Mechanics 53, 27 (2021).
[8] R. van Buel and H. Stark, Characterizing elastic turbulence in the three-dimensional von Kármán swirling flow using the Oldroyd-B model, Physics of Fluids 34, 043112 (2022).
[9] S. S. Datta, A. M. Ardekani, P. E. Arratia, A. N. Beris, I. Bischofberger, G. H. McKinley, J. G. Eggers, J. E. Lopez-Aguilar, S. M. Fielding, A. Frishman, et al., Perspectives on viscoelastic flow instabilities and elastic turbulence, Physical Review Fluids 7, 080701 (2022).
[10] P. Garg, I. Chaudhary, M. Khalid, V. Shankar, and G. Subramanian, Viscoelastic pipe flow is linearly unstable, Physical Review Letters 121, 024502 (2018).
[11] M. Khalid, I. Chaudhary, P. Garg, V. Shankar, and G. Subramanian, The centre-mode instability of viscoelastic plane Poiseuille flow, Journal of Fluid Mechanics 915 (2021).
[12] M. Khalid, V. Shankar, and G. Subramanian, Continuous pathway between the elasto-inertial and elastic turbulent states in viscoelastic channel flow, Physical Review Letters 127, 134502 (2021).
[13] G. Buza, J. Page, and R. R. Kerswell, Weakly nonlinear analysis of the viscoelastic instability in channel flow for finite and vanishing Reynolds numbers, Journal of Fluid Mechanics 940 (2022).
[14] J. Page, Y. Dubief, and R. R. Kerswell, Exact traveling wave solutions in viscoelastic channel flow, Physical Review Letters 125, 154501 (2020).
[15] G. Buza, M. Benitez, J. Page, and R. R. Kerswell, Finite-amplitude elastic waves in viscoelastic channel flow from large to zero Reynolds number, arXiv preprint arXiv:2202.08047 (2022).
[16] Y. Dubief, J. Page, R. R. Kerswell, V. E. Terrapon, and V. Steinberg, First coherent structure in elasto-inertial turbulence, Physical Review Fluids 7, 073301 (2022).
[17] A. Morozov, Coherent structures in plane channel flow of dilute polymer solutions with vanishing inertia, Physical Review Letters 129, 017801 (2022).
[18] G. H. Choueiri, J. M. Lopez, A. Varshney, S. Sankar, and B. Hof, Experimental observation of the origin and structure of elasto-inertial turbulence, Proceedings of the National Academy of Sciences 118, e2102350118 (2021).
[19] R. Sureshkumar, A. N. Beris, and R. A. Handler, Direct numerical simulation of the turbulent channel flow of a polymer solution, Physics of Fluids 9, 743 (1997).
[20] K. J. Burns, G. M. Vasil, J. S. Oishi, D. Lecoanet, and B. P. Brown, Dedalus: A flexible framework for numerical simulations with spectral methods, Physical Review Research 2, 023068 (2020).
[21] S. Sid, V. Terrapon, and Y. Dubief, Two-dimensional dynamics of elasto-inertial turbulence and its role in polymer drag reduction, Physical Review Fluids 3, 011301 (2018).
[22] R. G. Larson, E. S. Shaqfeh, and S. J. Muller, A purely elastic instability in Taylor-Couette flow, Journal of Fluid Mechanics 218, 573 (1990).
[23] L. Pan, A. Morozov, C. Wagner, and P. Arratia, Nonlinear elastic instability in channel flows at low Reynolds numbers, Physical Review Letters 110, 174502 (2013).
[24] H. A. C. Sanchez, M. R. Jovanovic, S. Kumar, A. Morozov, V. Shankar, G. Subramanian, and H. J. Wilson, Understanding viscoelastic flow instabilities: Oldroyd-B and beyond, Journal of Non-Newtonian Fluid Mechanics 302, 104742 (2022).
[25] Manuscript in preparation.