Representations of Lie Algebras
and Coding Theory

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Abstract

Linear codes with large minimal distances are important error correcting codes in information theory. Orthogonal codes have more applications in the other fields of mathematics. In this paper, we study the binary and ternary orthogonal codes generated by the weight matrices on finite-dimensional modules of simple Lie algebras. The Weyl groups of the Lie algebras act on these codes isometrically. It turns out that certain weight matrices of \( sl(n, \mathbb{C}) \) and \( o(2n, \mathbb{C}) \) generate doubly-even binary orthogonal codes and ternary orthogonal codes with large minimal distances. Moreover, we prove that the weight matrices of \( F_4, E_6, E_7 \) and \( E_8 \) on their minimal irreducible modules and adjoint modules all generate ternary orthogonal codes with large minimal distances. In determining the minimal distances, we have used the Weyl groups and branch rules of the irreducible representations of the related simple Lie algebras.

1 Introduction

Let \( m \) be a positive integer and denote \( \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z} \). A code \( \mathcal{C} \) of length \( n \) is a subset of \( (\mathbb{Z}_m)^n \) for some \( m \), where the ring structure of \( \mathbb{Z}_m \) may not be used. The elements of \( \mathcal{C} \) are called codewords. The (Hamming) distance between two codewords is the number of different coordinates. The minimal distance of a code is the minimal number among the distances of all its pairs of codewords in the code. A code with minimal distance \( d \) can be used to correct \( \lceil (d - 1)/2 \rceil \) errors in signal transmissions. We refer [6], [15], [23] for more details. Examples of the well-known infinite families of codes are cyclic codes, quadratic residue codes, Goppa codes, algebraic geometry codes, arithmetic codes, Hadamard codes and Pless double-circulant codes, etc. The names of these families also indicate the methods of constructing codes. In this paper, we introduce a new infinite family of codes arising from finite-dimensional representations of simple Lie algebras.

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which we may call *Lie theoretic codes*. One of the important features of these codes is that the corresponding Weyl group acts on them isometrically (may not be faithful).

A *linear code* $C$ over the ring $\mathbb{Z}_m$ is a $\mathbb{Z}_m$-submodule of $(\mathbb{Z}_m)^n$. The *(Hamming) weight* of a codeword in a linear code $C$ is the number of its nonzero coordinates. In this case, the minimal distance of $C$ is exactly the minimal weight of the nonzero codewords in $C$.

The inner product in $(\mathbb{Z}_m)^n$ is defined by

$$ (a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = \sum_{i=1}^{n} a_i b_i. \quad (1.1) $$

Moreover, $C$ is called *orthogonal* if

$$ C \subseteq \{ \vec{a} \in (\mathbb{Z}_m)^n \mid \vec{a} \cdot \vec{b} = 0 \text{ for } \vec{b} \in C \}. \quad (1.2) $$

When the equality holds, we call $C$ a *self-dual* code. Orthogonal linear codes (especially, self-dual codes) have important applications to the other mathematical fields such as sphere packings, integral linear lattices, finite group theory, etc. We refer References [2]-[6], [9]-[14], [17]-[21] and [24] for more details. A code is called *binary* if $m = 2$ and *ternary* when $m = 3$. A binary linear code is called *even* (doubly-even) if the weights of all its codewords are divisible by 2 (by 4).

Let $\mathcal{G}$ be a finite-dimensional simple Lie algebras over $\mathbb{C}$, the field of complex numbers. Take a Cartan subalgebra $H$ and simple positive roots $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Moreover, we denote by $\{h_1, h_2, \ldots, h_n\}$ the elements of $H$ such that the matrix

$$ (\alpha_i(h_j))_{n \times n} \text{ is the Cartan matrix of } \mathcal{G} \quad (1.3) $$

(e.g., cf. [7]). For a finite-dimensional $\mathcal{G}$-module $V$, it is well known that $V$ has a weight-subspace decomposition:

$$ V = \bigoplus_{\mu \in H^*} V_\mu, \quad V_\mu = \{ v \in V \mid h(v) = \mu(h)v \text{ for } h \in H \}. \quad (1.4) $$

Take a maximal linearly independent set $\{u_1, u_2, \ldots, u_k\}$ of weight vectors with nonzero weights in $V$ such that the order is compatible with the partial order of weights (e.g., cf. [H]). Write

$$ h_i(u_j) = c_{i,j} u_j, \quad C(V) = (c_{i,j})_{n \times k}. \quad (1.5) $$

By the representation theory of simple Lie algebras, all $c_{i,j}$ are integers. We call $C(V)$ the *weight matrix* of $\mathcal{G}$ on $V$. Identify integers with their images in $\mathbb{Z}_m$ when the context is clear. Denote by $C_m(V)$ the linear code over $\mathbb{Z}_m$ generated by $C(V)$.

In this paper, we prove that $C_2(V)$ and $C_3(V)$ for certain finite-dimensional irreducible modules of special linear Lie algebras are doubly-even binary orthogonal codes with large
minimal distances and ternary orthogonal codes with large minimal distances, respectively. Moreover, $C_3(V)$ for certain finite-dimensional modules of orthogonal Lie algebras are also ternary orthogonal codes with large minimal distances. Furthermore, we prove that the codes $C_3(V)$ of the exceptional simple Lie algebras $F_4$, $E_6$, $E_7$ and $E_8$ on their minimal irreducible modules and adjoint modules are all ternary orthogonal codes with large minimal distances. This coding theoretic phenomena was observed when we investigated the polynomial representations of these algebras in [26]-[28]. It is also well known that determining the minimal distance of a linear code is in general very difficult. We have used the Weyl groups and branch rules of irreducible representations of the related simple Lie algebras in determining the minimal distances of the concerned codes. Note also that our code $C_m(V)$ carries the important information of the simple root vectors acting on the weight vectors $u_i$ via the weight matrix $C(V)$ (e.g., c.f. [7]). Below we give more technical details.

Suppose that the weight of $u_i$ is $\mu_i$. Set

$$\mathcal{H}_m = \sum_{i=1}^{n} \mathbb{Z}_m h_i.$$  \hfill (1.6)

We define a map $\Im : \mathcal{H}_m \rightarrow (\mathbb{Z}_m)^k$ by

$$\Im \left( \sum_{i=1}^{n} l_i h_i \right) = \left( \sum_{i=1}^{n} l_i \mu_1(h_i), \sum_{i=1}^{n} l_i \mu_2(h_i), ..., \sum_{i=1}^{n} l_i \mu_k(h_i) \right).$$  \hfill (1.7)

Then

$$C_m(V) = \Im(\mathcal{H}_m).$$  \hfill (1.8)

Denote by $\mathcal{W}(G)$ the Weyl group of the simple Lie algebra $G$. For any $\sigma \in \mathcal{W}(G)$, there exists a linear automorphism $\hat{\sigma}$ of $V$ such that

$$\hat{\sigma}(V_\mu) = V_{\sigma(\mu)}, \quad \sigma(\mu)(\sigma(h)) = \mu(h) \quad \text{for} \ h \in H$$  \hfill (1.9)

(e.g., cf. [7]). Moreover, we define an action of $\mathcal{W}(G)$ on $\mathcal{H}_m$ by

$$\sigma \left( \sum_{i=1}^{n} l_i h_i \right) = \sum_{i=1}^{n} l_i \sigma(h_i) \quad \text{for} \ \sigma \in \mathcal{W}(G).$$  \hfill (1.10)

According to (1.9),

$$\text{wt} \ \Im(\sigma(h)) = \text{wt} \ \Im(h) \quad \text{for} \ \sigma \in \mathcal{W}(G), \ h \in \mathcal{H}_m.$$  \hfill (1.11)

So the number of the distinct weights of codewords in $C_m(V)$ is less than or equal to the number of $\mathcal{W}(G)$-orbits in $\mathcal{H}_m$. Expression (1.11) will be used later in determining minimal distances.
Let $\Lambda(V)$ be the set of nonzero weights of $V$. The module $V$ is called self-dual if $\Lambda(V) = -\Lambda(V)$. In this paper, we are only interested in the binary and ternary codes. We call $C_2(V)$ the binary weight code of $G$ on $V$. If $V$ is self-dual, then the weight matrix $C(V) = (-A, A)$ and $C_3(V)$ is orthogonal if and only if $A$ generates a ternary orthogonal code (e.g., cf. [15]). For this reason, we call the ternary code generated by $A$ the ternary weight code of $G$ on $V$ if $V$ is self-dual. When $V$ is not self-dual, then $C_3(V)$ is the ternary weight code of $G$ on $V$.

Denote by $V_X(\lambda)$ the finite-dimensional irreducible module of a simple Lie algebra of type $X$ with the highest weight $\lambda$. Let $p$ be a prime number. Then $\mathbb{Z}_p$ is a finite field, which is traditionally denoted as $\mathbb{F}_p$. A linear code $C$ of length $n$ over $\mathbb{F}_p$ is a linear subspace of $\mathbb{F}_p^n$ over $\mathbb{F}_p$. If $\dim C = k$, we say that $C$ is of type $[n, k]$. When $d$ is the minimal distance of $C$, we call $C$ an $[n, k, d]$-code. Take the labels of simple roots from [7].

Denote by $\lambda_i$ the $i$th fundamental weight of the related simple Lie algebra. We summarize the main results in this paper as the following three theorems.

The special linear Lie algebra $sl(n, \mathbb{C})$ consists of all $n \times n$ matrices with zero trace, which is a simple Lie algebra of type $A_{n-1}$.

**Theorem 1.**

1. The binary weight code $C_2(V_{A_{2m-1}}(\lambda_2))$ of $sl(2m, \mathbb{C})$ is a doubly-even orthogonal $[m(2m - 1), 2(m - 1), 4(m - 1)]$-code if $m \geq 2$.

2. The binary weight code $C_2(V_{A_{n-1}}(\lambda_3))$ of $sl(n, \mathbb{C})$ is a doubly-even orthogonal $[[n^3], n - 1, (n - 2)(n - 3)]$-code if $n > 9$ and $n \equiv 2, 3 \pmod{4}$.

3. The ternary weight code of $sl(3m + 2, \mathbb{C})$ on $V_{3m+1}(\lambda_2)$ is an orthogonal $[[3m+2], 3m + 1, 6m]$-code if $m > 0$.

4. The ternary weight code of $sl(3m, \mathbb{C})$ on $V_{3m-1}(\lambda_3)$ is an orthogonal $[[3m], 3m - 2, 3(m - 1)(3m - 2)]$-code. Moreover, the ternary weight code of $sl(3m+2, \mathbb{C})$ on $V_{3m+1}(\lambda_3)$ is an orthogonal $[[3m+2], 3m + 1, 3m + 1/2]$-code.

5. The ternary weight code of $sl(3m, \mathbb{C})$ on the adjoint module $sl(3m, \mathbb{C})$ is an orthogonal $[[3m], 3m - 2, 3(m - 1)]$-code if $m > 1$.

The Lie algebra $o(2n, \mathbb{C})$ consists of all $2n \times 2n$ skew-symmetric matrices, which is a simple Lie algebra of type $D_n$.

**Theorem 2.**

1. The ternary weight code of $o(6m + 2, \mathbb{C})$ on $V_{3m+1}(\lambda_2)$ is an orthogonal $[2m(3m + 1), 3m + 1, 6m]$-code if $m > 0$.

2. The ternary weight code of $o(2m, \mathbb{C})$ on $V_{3m}(\lambda_3)$ is an orthogonal $[m(m-1)(2m-1)/3, m, (m-1)(2m-3)]$-code if $m \not\equiv -1 \pmod{3}$ and $m > 3$.

3. The ternary code $C_3(V_{3m}(\lambda_m))$ of $o(2m, \mathbb{C})$ is of type $[2^{m-1}, m, 2m^2]$ if $6 \neq m > 3$. 


and of type $[32, 6, 12]$ when $m = 6$, where the representation of $o(2m, \mathbb{C})$ on $C_3(V_{D_m}(\lambda_m))$ is the spin representation.

(4) The ternary weight code of $o(12m + 4, \mathbb{C})$ on $o(12m + 4, \mathbb{C}) + V_{D_{6m+2}}(\lambda_{6m+2})$ is an orthogonal $[(6m + 2)(6m + 1) + 2^{6m}, 6m + 2, 24m + 1 + 2^{6m-1}]$-code for $m > 0$.

There are five exceptional finite-dimensional simple Lie algebras, labeled as $G_2$, $F_4$, $E_6$, $E_7$ and $E_8$. They have broad applications. We find the following common coding theoretic feature of the simple Lie algebras of types $F_4$, $E_6$, $E_7$ and $E_8$.

**Theorem 3.** (1) The ternary weight code of $F_4$ on its minimal module is an orthogonal $[12, 4, 6]$-code.

(2) The ternary weight code of $F_4$ on its adjoint module is an orthogonal $[24, 4, 15]$-code.

(3) The ternary weight code of $E_6$ on its minimal module is an orthogonal $[27, 6, 12]$-code.

(4) The ternary weight code of $E_6$ on its adjoint module is an orthogonal $[36, 5, 21]$-code.

(5) The ternary weight code of $E_7$ on its minimal module is an orthogonal $[28, 7, 12]$-code.

(6) The ternary weight code of $E_7$ on its adjoint module is an orthogonal $[63, 7, 27]$-code.

(7) The ternary weight code of $E_8$ on its minimal (adjoint) module is an orthogonal $[120, 8, 57]$-code.

Section 2 is devoted to the study of the binary and ternary weight codes of $sl(n, \mathbb{C})$. In Section 3, we prove Theorem 2. Section 4 is about the ternary weight codes of $F_4$ on its minimal module and adjoint module. In Section 5, we investigate the ternary weight codes of $E_6$ on its minimal module and adjoint module. We deal with the ternary weight codes of $E_7$ and $E_8$ on their minimal module and adjoint module in Section 6.

## 2 Codes Related to Representations of $sl(n, \mathbb{C})$

In this section, we study the binary and ternary codes related to representations of $sl(n, \mathbb{C})$, where $n > 1$ is an integer.

Throughout this paper, we always take the following notion:

$$\overline{i, j} = \{i, i + 1, i + 2, \ldots, j\}$$

for any integers $i \leq j$. We denote

$$\varepsilon_i = (0, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^n.$$
So
\[ \mathbb{R}^n = \sum_{i=1}^{n} \mathbb{R} \varepsilon_i. \] (2.3)

Then inner product \( \langle \cdot, \cdot \rangle \) is Euclidian, that is,
\[ \sum_{i=1}^{n} k_i \varepsilon_i, \sum_{j=1}^{n} l_j \varepsilon_j = \sum_{i=1}^{n} k_i l_i. \] (2.4)

Denote by \( E_{i,j} \) the square matrix with 1 as its \((i, j)\)-entry and 0 as the others. The special linear Lie algebra
\[ sl(n, \mathbb{C}) = \sum_{1 \leq i<j \leq n} \mathbb{C} E_{i,j} + \mathbb{C} E_{j,i} + \sum_{r=1}^{n-1} \mathbb{C} h_r, \quad h_r = E_{r,r} - E_{r+1,r+1}. \] (2.5)

The subspace \( H_{A_{n-1}} = \sum_{i=1}^{n-1} \mathbb{C} h_i \) forms a Cartan subalgebra of \( sl(n, \mathbb{C}) \). The root system
\[ \Phi_{A_{n-1}} = \{ \varepsilon_i - \varepsilon_j \mid i, j \in \overline{1, n}, \ i \neq j \}. \] (2.6)

Take the simple positive roots
\[ \alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad \text{for} \ i \in \overline{1, n-1}. \] (2.7)

The corresponding Dynkin diagram is
\[ A_{n-1}: \quad \circ \quad \circ \cdots \circ \circ \]
\[ 1 \quad 2 \quad \cdots \quad n-2 \quad n-1 \]

The Weyl group \( W_{A_{n-1}} \) of \( sl(n, \mathbb{C}) \) is exactly the full permutation group \( S_n \) on \( \overline{1, n} \), which acts on \( H_{A_{n-1}} \) and \( \mathbb{R}^n \) by permuting sub-indices of \( E_{i,i} \) and \( \varepsilon_i \), respectively.

Let \( A \) be the associative algebra generated by \( \{ \theta_1, \theta_2, \ldots, \theta_n \} \) with the defining relations:
\[ \theta_i \theta_j = -\theta_j \theta_i \quad \text{for} \ i, j \in \overline{1, n}. \] (2.8)

The generators \( \theta_i \) are called spin variables. The representation of the Lie algebra \( sl(n, \mathbb{C}) \) on \( A \) is given by
\[ E_{i,j} = \theta_i \partial \theta_j \quad \text{for} \ i, j \in \overline{1, n}. \] (2.9)

Set
\[ A_r = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \mathbb{C} \theta_{i_1} \theta_{i_2} \cdots \theta_{i_r} \quad \text{for} \ r \in \overline{1, n}. \] (2.10)

Then \( A_r \) forms an irreducible \( sl(n, \mathbb{C}) \)-submodule of highest weight \( \lambda_r \) for \( r \in \overline{1, n-1} \), that is, \( A_r \cong V_{A_{n-1}}(\lambda_r) \). The Weyl group \( W_{A_{n-1}} \) acts on \( A \) by permuting sub-indices of \( \theta_i \).
Two $k_1 \times k_2$ matrices $A_1$ and $A_2$ with entries in $\mathbb{Z}_m$ are called *equivalent* in the sense of coding theory if there exist an invertible $k_1 \times k_1$ matrix $K_1$ and an invertible $k_2 \times k_2$ monomial matrix $K_2$ such that $A_1 = K_1 A_2 K_2$. Equivalent matrices generate isomorphic codes. Take any order of the basis 

$$\{ x_{r,1}, x_{r,2}, \ldots, x_{r,n} \} = \{ \theta_{i_1} \theta_{i_2} \cdots \theta_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n \}.$$  \hspace{1cm} (2.11)

Then we have 

$$h_i(x_{r,j}) = a_{i,j}(r) x_{r,j}, \quad a_{i,j}(r) \in \mathbb{Z}. \hspace{1cm} (2.12)$$

Modulo equivalence, the weight matrix 

$$C(A_r) = [a_{i,j}(r)]_{(n-1) \times n}.$$  \hspace{1cm} (2.13)

**Theorem 2.1.** When $n = 2m \geq 4$ is even, $C_2(A_2)$ is a doubly-even binary orthogonal $[m(2m-1), 2(m-1), 4(m-1)]$-code.

**Proof.** Denote by $\xi_i$ the $i$th row $C_2(A_2)$. Then 

$$\text{wt } \xi_i = 2(n-2) \quad \text{for } i \in \overline{1,n-1}. \hspace{1cm} (2.14)$$

Moreover, 

$$\sum_{i=0}^{m-1} \xi_{2i+1} = 0 \quad \text{in } C_2(A_2). \hspace{1cm} (2.15)$$

Furthermore, 

$$\xi_i \cdot \xi_j = 4 \equiv 0 \quad \text{if } i+1 < j \hspace{1cm} (2.16)$$

and 

$$\xi_i \cdot \xi_{i+1} = 2(m-1) \equiv 0. \hspace{1cm} (2.17)$$

Write 

$$E_{i,i}(x_{r,j}) = b_{i,j}(r) x_{r,j}, \quad B_r = [b_{i,j}(r)]_{n \times n}. \hspace{1cm} (2.18)$$

Denote by $\zeta_i$ the $i$th row of $B_2$. By symmetry (cf. (1.9)-(1.11)), any nonzero codeword in $C_2(A_2)$ has the same weight as the codeword 

$$u = \sum_{s=1}^{2t} \zeta_s \in \mathbb{F}_2^{n(n-1)/2} \quad \text{for some } t \in \overline{1,m-1}. \hspace{1cm} (2.19)$$

We calculate 

$$\text{wt } u = 4t(m-t) = -4t^2 + 4mt. \hspace{1cm} (2.20)$$

Since the function $-4t^2 + t(4m-1)$ attains maximal at $t = m/2$, $\text{wt } u$ is minimal at $t = 1$ or $m-1$. Note 

$$\text{wt } u = 4(m-1) \quad \text{if } t = 1 \text{ or } m-1. \hspace{1cm} (2.21)$$
Thus the code $C_2(A_2)$ has the minimal distance $4(m - 1)$. □

When $m = 2$, $C_2(A_2)$ is a doubly-even binary orthogonal $[6, 2, 4]$-code. If $m = 3$, $C_2(A_2)$ becomes a doubly-even binary orthogonal $[15, 4, 8]$-code. These two code are optimal linear codes (e.g., cf. [1]). In the case of $m = 4$, $C_2(A_2)$ is a doubly-even binary orthogonal $[28, 6, 12]$-code.

**Theorem 2.2.** The code $C_2(A_3)$ is a doubly-even binary orthogonal $[(n^3, n - 1, (n - 2)(n - 3))$-code if $n > 9$ and $n \equiv 2, 3 \pmod{4}$.

**Proof.** Denote by $\xi_i$ the $i$th row the weight matrix $C(A_3)$. Then
\[
\text{wt} \, \xi_i = (n - 2)(n - 3) \quad \text{for} \quad i \in [1, n - 1].
\]
Moreover,
\[
\xi_i \cdot \xi_j = 4(n - 4) \quad \text{if} \quad i + 1 < j
\]
and
\[
\xi_i \cdot \xi_{i+1} = n - 3 + \left( \frac{n - 3}{2} \right) = \frac{(n - 2)(n - 3)}{2}.
\]
So $C_2(A_3)$ is a doubly-even binary orthogonal code under the assumption.

Denote by $\zeta_i$ the $i$th row of $B_3$ (cf. (2.18)). By symmetry (cf. (1.9)-(1.11)), any nonzero codeword in $C_2(A_3)$ has the same weight as the codeword
\[
u(t) = \sum_{s=1}^{2t} \zeta_s \in \mathbb{F}_2^n \quad \text{for some} \quad t \in \left[1, \left\lfloor \frac{n}{2} \right\rfloor \right].
\]
We calculate
\[
f(t) = 3\text{wt} \, \nu(t) = 3t^2 + 6t \left( \frac{n - 2t}{2} \right) = t[16t^2 - 12nt + 3n(n - 1) + 2].
\]
Moreover,
\[
f'(t) = 48t^2 - 24nt + 3n(n - 1) + 2 = 48 \left( t - \frac{n}{4} \right)^2 - 3n + 2.
\]
Thus
\[
f'(t_0) = 0 \iff t_0 = \frac{n}{4} \pm \frac{1}{4} \sqrt{n - \frac{2}{3}}.
\]
Since $f'(0) = 3n(n - 1) + 2 > 0$, $f(t)$ attains local maximum at
\[
t = \frac{n}{4} - \frac{1}{4} \sqrt{n - \frac{2}{3}}
\]
and local minimum at
\[
t = \frac{n}{4} + \frac{1}{4} \sqrt{n - \frac{2}{3}}
\]
According to (2.22) and (2.26), \( f(1) = 3(n - 2)(n - 3) \). Furthermore,

\[
\begin{align*}
  f \left( \frac{n}{4} + \frac{1}{4} \sqrt{n - \frac{2}{3}} \right) & = \left( \frac{n}{4} + \frac{1}{4} \sqrt{n - \frac{2}{3}} \right) \left[ 16 \left( \frac{n}{4} + \frac{1}{4} \sqrt{n - \frac{2}{3}} \right)^2 - 12n \left( \frac{n}{4} + \frac{1}{4} \sqrt{n - \frac{2}{3}} \right) + 3n(n - 1) + 2 \right] \\
  & = \frac{1}{4} \left( n + \sqrt{n - \frac{2}{3}} \right) \left[ \left( n + \sqrt{n - \frac{2}{3}} \right)^2 - 3n \left( n + \sqrt{n - \frac{2}{3}} \right) + 3n(n - 1) + 2 \right] \\
  & = \frac{1}{4} \left[ n^3 - 3n^2 + 2n + \left( \frac{4}{3} - 2n \right) \sqrt{n - \frac{2}{3}} \right] \\
  & > \frac{1}{4}(n^3 - 5n^2 + 2n). \quad (2.31)
\end{align*}
\]

Thus

\[
\begin{align*}
  f \left( \frac{n}{4} + \frac{1}{4} \sqrt{n - \frac{2}{3}} \right) - f(1) & > \frac{1}{4}(n^3 - 5n^2 + 2n) - 3(n - 2)(n - 3) = \frac{1}{4}(n^3 - 17n^2 + 62n - 72) \\
  & > \frac{n^2(n - 17)}{4}. \quad (2.32)
\end{align*}
\]

If \( n \geq 17 \), we have

\[
\begin{align*}
  f \left( \frac{n}{4} + \frac{1}{4} \sqrt{n - \frac{2}{3}} \right) > f(1) \quad (2.33)
\end{align*}
\]

and

\[
\begin{align*}
  f(n/2) - f(1) & = \frac{n}{2} \left[ 4n^2 - 6n^2 + 3n(n - 1) + 2 \right] - 3(n - 2)(n - 3) \\
  & = \frac{n(n - 1)(n - 2)}{2} - 3(n - 2)(n - 3) \\
  & = \frac{(n - 2)(n^2 - 7n + 9)}{2} > 0 \text{ if } n \geq 6. \quad (2.34)
\end{align*}
\]

Thus the minimal weight is \( f(1)/3 = (n - 2)(n - 3) \) when \( n \geq 17 \).

When \( n = 10 \), we calculate

| Table 2.1 |
| --- | --- | --- | --- | --- | --- |
| \( t \) | 1 | 2 | 3 | 4 | 5 |
| wt \( u(t) \) | 56 | 64 | 56 | 64 | 120 |
If \( n = 11 \), we find

| \( t \) | 1 | 2 | 3 | 4 | 5 |
|--------|---|---|---|---|---|
| \( \text{wt } u(t) \) | 72 | 88 | 80 | 80 | 120 |

When \( n = 14 \), we obtain

| \( t \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|--------|---|---|---|---|---|---|---|
| \( \text{wt } u(t) \) | 132 | 184 | 188 | 176 | 180 | 232 | 364 |

If \( n = 15 \), we get

| \( t \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|--------|---|---|---|---|---|---|---|
| \( \text{wt } u(t) \) | 156 | 224 | 216 | 224 | 220 | 256 | 364 |

This prove the conclusion in the theorem. \( \square \)

Note that when \( n = 6 \), we find

| \( t \) | 1 | 2 | 3 |
|--------|---|---|---|
| \( \text{wt } u(t) \) | 12 | 8 | 20 |

So \( C_2(A_3) \) is a doubly-even binary orthogonal \([20, 5, 8]\)-code. Moreover, if \( n = 7 \), we find

| \( t \) | 1 | 2 | 3 |
|--------|---|---|---|
| \( \text{wt } u(t) \) | 20 | 16 | 20 |

Hence \( C_2(A_3) \) a doubly-even binary orthogonal \([35, 6, 16]\)-code. In both cases, the above theorem fails and both codes are the best even codes among the binary codes with the same length and dimension (e.g., cf. \([1]\)).

According to the above theorem, \( C_2(A_3) \) is a doubly-even binary orthogonal \([120, 9, 56]\)-code when \( n = 10 \), \([165, 10, 72]\)-code if \( n = 11 \), \([364, 13, 132]\)-code when \( n = 14 \) and \([455, 14, 156]\)-code if \( n = 15 \).

Next let us consider the ternary codes. Again by symmetry, any nonzero codeword in \( C_3(A_r) \) has the same weight as the codeword

\[
 u(s, t) = \sum_{r=1}^{s} \zeta_r - \sum_{i=1}^{t} \zeta_{s+i} \in \mathbb{F}_3^{(n)}
\]  
(2.35)
for some nonnegative integers \( s, t \), where \( \zeta_i \) is the \( i \)th row of the matrix \( B_r \) in (2.18).

Moreover,
\[
\text{wt } u(s, t) = \text{wt } u(t, s).
\]  
(2.36)

Furthermore, we have
\[
\text{wt } u(s, t) = (s + t)(n - s - t) + \left( \begin{array}{c} s \\ 2 \end{array} \right) + \left( \begin{array}{c} t \\ 2 \end{array} \right) \quad \text{in } C_3(A_2)
\]  
(2.37)

and
\[
\text{wt } u(s, t) = (s + t)\left( \frac{n - s - t}{2} \right) + (n - s)\left( \frac{s}{2} \right) + (n - t)\left( \frac{t}{2} \right) \quad \text{in } C_3(A_3).
\]  
(2.38)

For convenience, we denote
\[
\begin{align*}
f(s, t) &= 2\text{wt } u(s, t) = 2(s + t)(n - s - t) + s(s - 1) + t(t - 1) \\
&= (2n - 1)(s + t) - s^2 - t^2 - 4st
\end{align*}
\]  
(2.39)
in \( C_3(A_2) \) and
\[
\begin{align*}
g(s, t) &= 2\text{wt } u(s, t) = (s + t)(n - s - t)(n - s - t - 1) + (n - s)s(s - 1) + (n - t)t(t - 1) \\
&= (s + t)^3 - (2n - 1)(s + t)^2 + n(n - 1)(s + t) - s^3 - t^3 + (n + 1)(s^2 + t^2) - n(s + t) \\
&= 3st^2 + 3s^2t + (2 - n)(s^2 + t^2) - 2(2n - 1)st + n(n - 2)(s + t)
\end{align*}
\]  
(2.40)
in \( C_3(A_3) \).

Note
\[
\begin{align*}
f(3, 0) &= 3(2n - 1) - 9 = 6(n - 2), \quad f(n, 0) = n(2n - 1) - n^2 = n(n - 1), \\
f(1, 1) &= 2(2n - 1) - 6 = 4(n - 2), \quad f(1, n - 1) = (n - 1)(n - 2).
\end{align*}
\]  
(2.41)

Since geometrically \( f(s, t) \) has only local minimum, it attains the absolute minimum at boundary points. Thus
\[
\min\{f(s, t) \mid s \equiv t \pmod{3}\} = 4(n - 2) \quad \text{if } n \geq 5.
\]  
(2.43)

Now
\[
\begin{align*}
g_s(s, t) &= 3t^2 + 6st + 2(2 - n)s - 2(2n - 1)t + n(n - 2), \\
g_t(s, t) &= 3s^2 + 6st + 2(2 - n)t - 2(2n - 1)s + n(n - 2).
\end{align*}
\]  
(2.44)

(2.45)

Suppose that \( g_s(s_0, t_0) = g_t(s_0, t_0) = 0 \) for \( s_0, t_0 \geq 0 \), that is,
\[
3t_0^2 + 6s_0t_0 + 2(2 - n)s_0 - 2(2n - 1)t_0 + n(n - 2) = 0,
\]  
(2.46)
In summary, we have:

\[ 3s_0^2 + 6s_0t_0 + 2(2 - n)t_0 - 2(2n - 1)s_0 + n(n - 2) = 0. \quad (2.47) \]

By (2.46) – (2.47), we get

\[ (t_0 - s_0)(3t_0 + 3s_0 - 2(n + 1)) = 0 \implies t_0 = s_0 \ \text{or} \ 3t_0 + 3s_0 = 2(n + 1). \quad (2.48) \]

If \( s_0 = t_0 \), then we find

\[ 9s_0^2 - 2(n - 1)s_0 + n(n - 2) = 0 \sim 8s_0^2 + (s_0 - n + 1)^2 - 1 = 0, \quad (2.49) \]

which leads to a contradiction because \( n > 1 \). Thus \( 3t_0 + 3s_0 = 2(n - 1) \). Denote \( s_1 = 3t_0 \) and \( t_1 = 3t_0 \). Then \( s_1 + t_1 = 2(n + 1) \) and (2.46) becomes

\[ t_1^2 + 2(2(n + 1) - t_1)t_1 + 2(2 - n)(2(n + 1) - t_1) - 2(2n - 1)t_1 + 3n(n - 2) = 0, \quad (2.50) \]

equivalently,

\[ t_1^2 - 2(n + 1)t_1 + (n - 2)(n + 4) = 0 \sim (t_1 - n - 1)^2 - 9 = 0 \implies t_1 = n + 4, \ n - 2. \quad (2.51) \]

Therefore,

\[ s_0 = \frac{n + 4}{3}, \ t_0 = \frac{n - 2}{3} \quad \text{or} \quad t_0 = \frac{n + 4}{3}, \ s_0 = \frac{n - 2}{3}. \quad (2.52) \]

We calculate

\[ g(s_0, t_0) = \frac{2(n - 2)(n^2 - n - 3)}{9}, \quad (2.53) \]

\[ g(1, 0) = g(n - 1, 0) = (n - 1)(n - 2), \ g(3, 0) = 3(n - 2)(n - 3), \ g(n, 0) = 0. \quad (2.54) \]

\[ g(1, 1) = g(n - 2, 1) = 2(n - 2)(n - 3), \ g(n - 2, 0) = 2(n - 2)^2. \quad (2.55) \]

Moreover,

\[ g(s_0, t_0) \geq g(1, 0), \ g(1, 1) \quad \text{if} \ n \geq 6. \quad (2.56) \]

When \( n = 5 \), we calculate

\[ g(1, 0) = g(1, 1) = g(2, 1) = g(2, 2) = g(3, 1) = g(4, 0) = g(4, 1) = 12, \quad (2.57) \]

\[ g(2, 0) = g(3, 0) = g(3, 2) = 18. \quad (2.58) \]

In summary, we have:

**Theorem 2.3.** Let \( n \geq 5 \). The matrix \( B_3 \) (cf. (2.18)) generates a ternary \( \left[ \binom{n}{3}, n - 1, \binom{n - 1}{2} \right] \)-code, which is equal to \( C_3(A_3) \) if \( n \not\equiv 0 \pmod{3} \). If \( n = 3m + 2 \) for some positive integer \( m \), \( C_3(A_2) \) is a ternary orthogonal \( \left[ \binom{3m+2}{2}, 3m + 1, 6m \right] \)-code and \( C_3(A_3) \) is a ternary orthogonal \( \left[ \binom{3m+2}{3}, 3m + 1, 3m(3m + 1)/2 \right] \)-code. The code \( C_3(A_3) \) is a ternary orthogonal \( \left[ \binom{n}{3}, n - 2, (n - 2)(n - 3) \right] \)-code when \( n \equiv 0 \pmod{3} \).
Proof. The part of minimal distances has been proved by the above arguments. We only need to prove orthogonality.

Suppose $n = 3m + 2$. In $C_3(A_2)$, $\xi_r$ stands for the $r$th row of the weight matrix $C(A_2)$ and

$$\xi_i \cdot \xi_j = 2 - 2 = 0 \quad \text{for } 1 \leq i < j - 1 \leq n - 2,$$

$$\xi_r \cdot \xi_{r+1} = -(n - 2) = -3m, \quad \xi_s \cdot \xi_s = 2(n - 2) = 6m$$

for $r \in \overline{1, n-2}$ and $s \in \overline{1, n-1}$. So $C_3(A_2)$ is orthogonal. Now $\zeta_r$ stands for the $r$th row of $B_3$ (cf. (2.18)). Observe

$$\sum_{i=1}^{n} \zeta_i = 0 \in \mathbb{F}_3^n$$

by (2.9) and (2.10). Moreover,

$$\zeta_i \cdot \zeta_j = n - 2 = 3m, \quad \zeta_i \cdot \zeta_i = \left(\frac{n-1}{2}\right) = \frac{3m(3m+1)}{2}, \quad i \neq j.$$

Thus $B_3$ generate a ternary orthogonal code.

Assume that $n = 3m$ for some nonnegative integer $m$. In $C_3(A_3)$, we also use $\xi_r$ for the $r$th row of the weight code $C(A_3)$ and

$$\xi_i \cdot \xi_j = 2(n - 4) - 2(n - 4) = 0 \quad \text{for } 1 \leq i < j - 1 \leq n - 2,$$

$$\xi_s \cdot \xi_s = 2\xi_r \cdot \xi_{r+1} = (n - 2)(n - 3) = 3(3m - 2)(m - 1) \equiv 0$$

for $r \in \overline{1, n-2}$ and $s \in \overline{1, n-1}$. So $C_3(A_3)$ is orthogonal. □

According to the above theorem, $C_3(A_2)$ is a ternary orthogonal $[10, 4, 6]$-code when $n = 5$ (which is optimal (e.g., cf. [1])), $[28, 7, 12]$-code when $n = 8$, and $[55, 10, 18]$-code when $n = 11$. Moreover, $C_3(A_3)$ is a ternary orthogonal $[10, 4, 6]$-code when $n = 5$, $[15, 4, 12]$-code if $n = 6$, $[56, 7, 21]$-code when $n = 8$, $[84, 7, 42]$-code if $n = 9$, $[165, 10, 45]$-code when $n = 11$ and $[220, 10, 90]$-code when $n = 12$.

Finally, we consider the adjoint representation of $sl(n, \mathbb{C})$. Note that $\{E_{i,j} \mid 1 \leq i < j \leq n\}$ are positive root vectors. Given an order

$$\{y_1, \ldots, y_{\frac{n(n+1)}{2}}\} = \{E_{i,j} \mid 1 \leq i < j \leq n\},$$

we write

$$[h_i, y_j] = k_{i,j} y_j, \quad [E_{r,r}, y_j] = l_{r,j} y_j.$$ 

Denote

$$K = (k_{i,j})_{(n-1) \times \binom{n}{2}}, \quad L = (l_{i,j})_{n \times \binom{n}{2}}.$$ (2.67)
Let $K$ be the ternary code generated by $K$ and let $L$ be the ternary code generated by $L$. Moreover, $\vec{k}_i$ stands for the $i$th row of $K$ and $\vec{l}_r$ stands for the $r$th row of $L$. Set

$$u(s, t) = \sum_{i=1}^{s} \vec{l}_i - \sum_{j=1}^{t} \vec{l}_{s+j}. \quad (2.68)$$

For any nonzero codeword $v \in L$, using negative root vectors, we can prove

$$\text{wt } (v, -v) = \text{wt } (u(s, t), -u(s, t)) \quad (2.69)$$

for some $s$ and $t$ by symmetry (cf. (1.9)-(1.11)). Thus

$$\text{wt } v = \text{wt } u(s, t) = (s + t)(n - s - t) + st = \phi(s, t). \quad (2.70)$$

Note

$$\phi(s, t) = n^2 - \frac{1}{2} [(s - n)^2 + (t - n)^2 + (s - t)^2]. \quad (2.71)$$

So $\phi(s, t)$ has only local maximum. Thus it attains the absolute minimum at the boundary points. We calculate

$$\phi(1, 0) = \phi(n - 1, 0) = n - 1, \quad \phi(n - 3, 0) = 3(n - 3),$$

$$\phi(1, 1) = 2n - 3, \quad \phi(n - 2, 1) = 2(n - 1). \quad (2.72)$$

Since

$$\sum_{i=1}^{n} \vec{l}_i = 0, \quad (2.74)$$

$$K = L \quad \text{if } n \neq 0 \pmod{3}. \quad (2.75)$$

$$\vec{k}_i \cdot \vec{k}_j = 2 - 2 = 0 \quad 1 \leq i < j - 1 \leq n, \quad (2.76)$$

$$\vec{k}_r \cdot \vec{k}_{r+1} = 6 - n, \quad \vec{k}_s \cdot \vec{k}_s = 2n - 3. \quad (2.77)$$

In summary, we have:

**Theorem 2.4.** The code $L$ is a ternary $[(\binom{n}{2}, n - 1, n - 1)]$-code if $n \geq 4$, which is also the ternary weight code on the adjoint module $\text{sl}(n, \mathbb{C})$ when $n \neq 0 \pmod{3}$. If $n = 3m$ for some integer $m > 1$, then the ternary weight code $K$ on $\text{sl}(3m, \mathbb{C})$ is an orthogonal $[(\binom{3m}{2}, 3m - 2, 3(m - 1))]$-code.

### 3 Codes Related to Representations of $o(2m, \mathbb{C})$

In this section, we only study ternary codes related to certain representations of $so(2m, \mathbb{C})$, some of which will be used to investigate the codes related to exceptional simple Lie algebras.
Let \( n = 2m \) be a positive even integer. Take the settings in (2.1)-(2.4) (with \( n \to m \)).

The orthogonal Lie algebra

\[
\mathfrak{o}(2m, \mathbb{C}) = \sum_{1 \leq i < j \leq m} \left[ \mathbb{C}(E_{i,j} - E_{m+j,m+i}) + \mathbb{C}(E_{j,i} - E_{m+i,m+j}) + \mathbb{C}(E_{i,m+j} - E_{j,m+i}) \right] + \sum_{r=1}^{m} \mathbb{C}h_r,
\]

(3.1)

where

\[
h_s = E_{s,s} - E_{s+1,s+1} - E_{m+s,m+s} - E_{m+s+1,m+s+1} \quad \text{for} \quad s \in \overline{1, m-1}
\]

(3.2)

and

\[
h_m = E_{m-1,m-1} + E_{m,m} - E_{2m-1,2m-1} - E_{2m,2m}.
\]

(3.3)

Indeed, we take the Cartan subalgebra

\[
H_{D_m} = \sum_{i=1}^{m} \mathbb{C}h_i
\]

(3.4)

of \( \mathfrak{o}(2m, \mathbb{C}) \). The root system

\[
\Phi_{D_m} = \{ \pm \varepsilon_i \pm \varepsilon_j \mid i, j \in \overline{1, m}, i \neq j \}
\]

(3.5)

and simple positive roots are:

\[
\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad \alpha_m = \varepsilon_{m-1} + \varepsilon_m, \quad i \in \overline{1, m-1}.
\]

(3.6)

The corresponding Dynkin diagram is

\[
D_m
\]

The Weyl group is \( S_m \times \mathbb{Z}_2^{m-1} \), which acts \( H_{D_m} \) and \( \mathbb{R}^m \) by permuting sub-indices of \( \varepsilon_i \) and \( E_{i,i} - E_{m+i,m+i} \), and changing sign on even number of their coefficients.

Take the settings in (2.8)-(2.13) and (2.18). Moreover, the representation of \( \mathfrak{o}(2m, \mathbb{C}) \) on \( \mathcal{A} \) determined by (2.9). For any \( \vec{\iota} = (\iota_1, \ldots, \iota_m) \) with \( \iota_i \in \{0, 1\} \) and \( \tau \in S_m \), we have an associative algebra automorphism \( \sigma_{\tau, \vec{\iota}} \) of \( \mathcal{A} \) determined by

\[
\sigma_{\tau, \vec{\iota}}(\theta_i) = \theta_{m\delta_{\iota_i,1}+\tau(i)}, \quad \sigma_{\tau, \vec{\iota}}(\theta_{m+i}) = \theta_{m\delta_{\iota_i,0}+\tau(i)} \quad \text{for} \quad i \in \overline{1, m}.
\]

(3.6)

Moreover, we define a linear map \( \sigma_{\tau, \vec{\iota}} \) on \( \mathcal{H} \) by

\[
\sigma_{\tau, \vec{\iota}}(E_{i,i} - E_{m+i,m+i}) = (-1)^{\iota_i}(E_{\tau(i),\tau(i)} - E_{m+\tau(i),m+\tau(i)}) \quad \text{for} \quad i \in \overline{1, m}.
\]

(3.7)
Then
\[ \sigma_{\tau}(h(w)) = \sigma_{\tau}(h)[\sigma_{\tau}(w)] \quad \text{for } h \in \mathcal{H}, \ w \in \mathcal{A}. \] (3.8)

Note that all \( \mathcal{A}_r \cong V_{D_m}(\lambda_r) \) are self-dual \( o(2m, \mathbb{C}) \)-submodules for \( r \in \mathbb{Z}, m - 2 \). In particular, the ternary weight code \( \mathcal{C}_2 \) of \( o(2m, \mathbb{C}) \) on \( \mathcal{A}_2 \) is given by the weight matrix on its subspace
\[ \mathcal{A}_{2,1} = \sum_{1 \leq i < j \leq m} (\mathbb{C}\theta_i \theta_j + \mathbb{C}\theta_i \theta_{m+j}). \] (3.9)

We take any order
\[ \{x_1, x_2, \ldots, x_{m(m-1)}\} = \{\theta_i \theta_j, \theta_i \theta_{m+j} \mid 1 \leq i < j \leq m\} \] (3.10)
and write
\[ (E_{i,i} - E_{m+i,m+i})(x_j) = c_{i,j}(2)x_j, \quad C_2 = (c_{i,j}(2))_{m \times m(m-1)}. \] (3.11)

Moreover,
the weight matrix on \( \mathcal{A}_2 \) is equivalent to \( (C_2, -C_2) \). (3.12)

Since
\[ \sum_{i=1}^{m} F_3 h_i = \sum_{i=1}^{m} F_3 (E_{i,i} - E_{m+i,m+i}), \] (3.13)

\( C_2 \) is a generator matrix of the ternary code \( \mathcal{C}_2 \). Denote by \( \zeta_i \) the \( i \)th row of \( C_2 \). By (3.8) and (3.12), any nonzero codeword in \( C_2 \) has the same weight as the codeword
\[ u(t) = \sum_{i=1}^{t} \zeta_t \quad \text{for some } t \in \overline{1, m}. \] (3.14)

Moreover,
\[ f(t) = \text{wt } u(t) = \left(\frac{t}{2}\right) + 2t(m-t) = \frac{(4m-1)t - 3t^2}{2} \] (3.15)
So \( f(t) \) has only local maximum and it attains the absolute minimum at the boundary points. Note that
\[ f(1) = 2(m - 1), \quad f(m) = \frac{m(m-1)}{2}. \] (3.16)
Hence
the minimal distance of \( C_2 \) is \( 2(m - 1) \) if \( m \geq 4 \). (3.17)

**Theorem 3.1.** When \( m = 3m_1 + 1 \) for some positive integer \( m_1 \), the ternary weight code \( C_2 \) of \( o(2m, \mathbb{C}) \) on \( \mathcal{A}_2 \) is an orthogonal \([m(m-1), m, 2(m-1)]\)-code.

**Proof.** Note that for \( i, j \in \overline{1, m} \) with \( i \neq j \),
\[ \zeta_i \cdot \zeta_j = f(1) = 6m_1, \quad (\zeta_i + \zeta_j) \cdot (\zeta_i + \zeta_j) = f(2) = 1 + 4(m-2) = 4m - 7 = 12(m_1-1). \] (3.18)
Thus
\[ \zeta_i \cdot \zeta_j = \frac{f(2) - 2f(1)}{2} = -6. \] (3.19)

Hence \( C_2 \) is an orthogonal ternary code. \( \square \)

In particular, \( C_2 \) is an orthogonal ternary \([12, 4, 6]\)-code when \( m_1 = 1 \), \([42, 7, 12]\)-code when \( m_1 = 2 \) and \([90, 10, 18]\)-code when \( m_1 = 3 \). It can be proved that \( C_2 \) is also the weight code on the adjoint module of \( o(2m, \mathbb{C}) \).

The ternary weight code \( C_3 \) of \( o(2m, \mathbb{C}) \) on \( A_3 \) is given by the weight matrix on its subspace
\[ A_{3,1} = \sum_{1 \leq i < j < l \leq m} \mathbb{C} \theta_i \theta_j \theta_l + \sum_{1 \leq i < j \leq m} \sum_{l=1}^{m} \mathbb{C} \theta_i \theta_j \theta_{m+l}. \] (3.20)

We take any order
\[ \{y_1, y_2, \cdots, y_{\binom{m}{3} + \binom{m}{2}}\} = \{\theta_i \theta_j \theta_l, \theta_i \theta_s \theta_{m+q} | 1 \leq i < j < l \leq m; 1 \leq r < s \leq m; q \in \overline{1,m}\} \] (3.21)

and write
\[ (E_{i,i} - E_{m+i,m+i})(y_j) = c_{i,j}(3)y_j, \quad C_3 = (c_{i,j}(3))_{m \times \binom{m}{3} + \binom{m}{2}}. \] (3.22)

Moreover,
the weight matrix on \( A_3 \) is equivalent to \((C_3, -C_3)\). (3.23)

Denote by \( \eta_i \) the \( i \)th row of \( C_3 \). By (3.8) and (3.23), any nonzero codeword in \( C_3 \) has the same weight as the codeword
\[ u(t) = \sum_{i=1}^{t} \eta_i \quad \text{for some } t \in \overline{1,m}. \] (3.24)

Moreover,
\[ g(t) = \text{wt } u(t) = (2m-t)\Big(\frac{t}{2}\Big) + 2t\Big(\frac{m-t}{2}\Big) + t(m-t)^2 \]
\[ = \frac{t(t-1)(2m-t) + 2t(m-t)(2m-2t-1)}{2} \]
\[ = \frac{t}{2}[3t^2 + 3(1 - 2m)t + 4(m^2 - m)]. \] (3.25)

Observe that
\[ g'(t) = \frac{1}{2}[9t^2 + 6(1 - 2m)t + 4(m^2 - m)] = \frac{1}{2}[(3t + 1 - 2m)^2 - 1]. \] (3.26)

Thus
\[ g'(t_0) = 0 \implies t_0 = \frac{2(m-1)}{3}, \quad \frac{2m}{3}. \] (3.27)
Since \( g'(0) = (m^2 - m)/2 \geq 0 \), \( t = 2(m - 1)/3 \) is a point of local maximum and \( t = 2m/3 \) is a point of local minimum. We calculate

\[
g(1) = (m - 1)(2m - 3), \quad g(m) = \frac{m^2(m - 1)}{2}, \quad g(2m/3) = \frac{2}{9}m^2(2m - 3). \quad (3.28)
\]

Note that \( g(m) \geq g(1) \) and \( g(2m/3) \geq g(1) \) if \( m \geq 3 \).

**Theorem 3.2.** Let \( m \geq 3 \). The ternary weight code \( C_3 \) of \( o(2m, \mathbb{C}) \) on \( A_3 \) is of type \([m(m - 1)(2m - 1)/3, m, (m - 1)(2m - 3)]\). Moreover, it is orthogonal if \( m \not\equiv -1 \pmod{3} \).

**Proof.** Note

\[
\eta_i \cdot \eta_i = g(1) = (m - 1)(2m - 3) \quad (3.29)
\]

and

\[
(\eta_i + \eta_j) \cdot (\eta_i + \eta_j) = g(2) = 2(2m - 2)^2 + 1 \quad (3.30)
\]

for \( i, j \in \overline{1, m} \) such that \( i \neq j \). Thus

\[
\eta_i \cdot \eta_j = \frac{g(2) - 2g(1)}{2} = 3(2 - m). \quad (3.31)
\]

So \( C_3 \) is orthogonal if \( m \not\equiv -1 \pmod{3} \). \( \square \)

Remark that \( C_3 \) is an orthogonal \([10, 3, 6]\)-code when \( m = 3 \), \([28, 4, 15]\)-code when \( m = 4 \), \([110, 6, 45]\)-code when \( m = 6 \) and \([182, 7, 66]\)-code when \( m = 7 \).

Let \( B \) be the subalgebra of \( A \) generated by \( \{1_A, \theta_i \mid i \in \overline{1, m}\} \) and

\[
B_r = \mathcal{A}_r \cap B \quad \text{for} \quad r \in \overline{0, m}. \quad (3.32)
\]

The spin representation of \( so(2m, \mathbb{C}) \) is given by

\[
E_{i,j} - E_{m+j,m+i} = \theta_i \partial_{\theta_j} - \frac{\delta_{i,j}}{2} \quad \text{for} \quad i, j \in \overline{1, m}, \quad (3.33)
\]

\[
E_{m+s,r} - E_{m+r,s} = \partial_{\theta_s} \partial_{\theta_r}, \quad E_{r,m+s} - E_{s,m+r} = \theta_r \theta_s \quad (3.34)
\]

for \( 1 \leq r < s \leq m \). Then the subspace

\[
\mathcal{V} = \sum_{i=1}^{[m/2]} B_{m-i} \quad (3.35)
\]

is the irreducible module with highest weight \( \lambda_m \), that is, \( \mathcal{V} \cong V_{D_m}(\lambda_m) \).

If \( m = 2m_1 + 1 \) is odd, then

\[
\{\theta_{i_1} \cdots \theta_{i_{m-2r}} \mid r \in \overline{0, m_1}; \ 1 \leq i_1 < \cdots < i_{m-2r} \leq m\} \quad (3.36)
\]

forms a weight-vector basis of \( \mathcal{V} \). When \( m = 2m_1 \) is even,

\[
\{1, \theta_{i_1} \cdots \theta_{i_{m-2r}} \mid r \in \overline{0, m_1 - 1}; \ 1 \leq i_1 < \cdots < i_{m-2r} \leq m\} \quad (3.37)
\]
is a weight-vector basis of \( \mathcal{V} \). Take any order \( \{ z_1, z_2, ..., z_{2^m - 1} \} \) of the above base vectors.

Denote
\[
(E_{r,r} - E_{m+r,m+r})(z_i) = q_{r,i}z_i, \quad C(\mathcal{V}) = (q_{r,i})_{m \times 2^m - 1}.
\]

(3.38)

Note that
\[
\frac{1}{2} \equiv -1 \quad \text{in } \mathbb{F}_3.
\]

(3.39)

Denote by \( \xi_r \) the \( r \)-th row of the weight matrix \( C(\mathcal{V}) \). Set
\[
\bar{u} = \sum_{r=1}^{m-1} \xi_r - \xi_m, \quad u(t) = \sum_{i=1}^{t} \xi_i \quad \text{for } t \in \overline{1,m}.
\]

(3.40)

Then any nonzero codeword in \( C_3(\mathcal{V}) \) is conjugated to some \( u(t) \) or \( \bar{u} \) under the action of the Weyl group of \( o(2m, \mathbb{C}) \) (cf. (1.10) and (1.11)). It has the same weight as \( u(t) \) or \( \bar{u} \).

We calculate
\[
\text{wt } u(1) = 2^{m-1}, \quad \text{wt } u(2) = 2^{m-2}.
\]

(3.41)

Moreover, we have the following more general estimates. For any positive integer \( k > 2 \), we always have
\[
\binom{k}{l - 1} + \binom{k}{l + 1} > \binom{k}{l} \quad \text{for } l \in \overline{0,k},
\]

(3.42)

where we treat \( \binom{k}{l} = 0 \). If \( t = 3t_1 \) for some positive integer \( t_1 \), we have

\[
\text{wt } u(t) = 2^{m-3t_1-1} \sum_{i=0}^{t_1} \left[ \binom{3t_1}{6i + 1} + \binom{3t_1}{6i + 2} + \binom{3t_1}{6i + 4} + \binom{3t_1}{6i + 5} \right] > 2^{m-3t_1-1} \sum_{i=0}^{t_1} \left[ \binom{3t_1}{6i + 1} + \binom{3t_1}{6i + 3} + \binom{3t_1}{6i + 5} \right] = 2^{m-2}.
\]

(3.43)

When \( t = 3t_1 + 1 \) for some positive integer \( t_1 \), we obtain

\[
\text{wt } u(t) = 2^{m-3t_1-2} \sum_{i=0}^{t_1} \left[ \binom{3t_1 + 1}{6i} + \binom{3t_1 + 1}{6i + 1} + \binom{3t_1 + 1}{6i + 3} + \binom{3t_1 + 1}{6i + 4} \right] > 2^{m-3t_1-2} \sum_{i=0}^{t_1} \left[ \binom{3t_1 + 1}{6i} + \binom{3t_1 + 1}{6i + 2} + \binom{3t_1 + 1}{6i + 4} \right] = 2^{m-2}.
\]

(3.44)

If \( t = 3t_1 + 2 \) for some positive integer \( t_1 \), we get

\[
\text{wt } u(t) = 2^{m-3t_1-3} \sum_{i=0}^{t_1} \left[ \binom{3t_1 + 2}{6i} + \binom{3t_1 + 2}{6i + 2} + \binom{3t_1 + 2}{6i + 3} + \binom{3t_1 + 2}{6i + 4} \right] > 2^{m-3t_1-3} \sum_{i=0}^{t_1} \left[ \binom{3t_1 + 2}{6i} + \binom{3t_1 + 2}{6i + 2} + \binom{3t_1 + 2}{6i + 4} \right] = 2^{m-2}.
\]

(3.45)

Let \( k \) be a positive integer. We have
\[
\binom{2k}{i} > \binom{2k}{i + 1}
\]

(3.46)
if \( i \leq k - 3 \) or \( i \geq k \). Moreover,

\[
\binom{2k}{k-2} + \binom{2k}{k+2} - \binom{2k}{k-1} = \frac{k - 4}{k - 1} \binom{2k}{k-2}.
\] (3.47)

\[
\binom{2k}{k-1} + \binom{2k}{k+3} - \binom{2k}{k} = \frac{k^3 - 4k^2 - 3k - 6}{k(k-1)(k-2)} \binom{2k}{k-3}.
\] (3.48)

Thus (3.46) always holds if \( k \geq 5 \). Furthermore,

\[
\binom{2k+1}{i} + \binom{2k+1}{i+4} > \binom{2k+1}{i+1}
\] (3.49)

if \( i \neq k - 1 \). Observe that

\[
\binom{2k+1}{k - 1} + \binom{2k+1}{i + 3} - \binom{2k+1}{k} = \frac{k^2 - 3k - 6}{k(k-1)} \binom{2k+1}{k - 2}.
\] (3.50)

So (3.49) holds whenever \( k \geq 5 \). Therefore,

\[
\binom{k}{i} + \binom{k}{i+4} > \binom{k}{i+1}
\] if \( k \geq 10 \). (3.51)

If \( m = 3m_1 \) for some positive integer \( m_1 \),

\[
\text{wt } \bar{u} = \sum_{i=0}^{m} \left[ \binom{m}{6i} + \binom{m-1}{6i+1} + \binom{m-1}{6i+4} \right]
\]

\[
= \sum_{i=0}^{m} \left[ \binom{m-1}{6i} + \binom{m-1}{6i+5} + \binom{m-1}{6i+1} + \binom{m-1}{6i+4} \right],
\] (3.52)

which is \( > 2^{m-2} \) if \( m_1 \geq 4 \) by (3.51). When \( m = 3m_1 + 1 \) for some positive integer \( m_1 \),

\[
\text{wt } \bar{u} = \sum_{i=0}^{m} \left[ \binom{m-1}{6i} + \binom{m}{6i+2} + \binom{m-1}{6i+3} \right]
\]

\[
= \sum_{i=0}^{m} \left[ \binom{m-1}{6i} + \binom{m-1}{6i+5} + \binom{m-1}{6i+2} + \binom{m-1}{6i+3} \right]
\]

\[
= 1 + \sum_{i=0}^{m} \left[ \binom{m-1}{6i+1} + \binom{m-1}{6i+3} + \binom{m-1}{6i+2} + \binom{m-1}{6i+6} \right],
\] (3.53)

which is again \( > 2^{m-2} \) if \( m_1 \geq 4 \) by (3.51). Assuming \( m = 3m_1 + 2 \) for some positive integer \( m_1 \), we have

\[
\text{wt } \bar{u} = \sum_{i=0}^{m} \left[ \binom{m-1}{6i+2} + \binom{m}{6i+4} + \binom{m-1}{6i+5} \right]
\]

\[
= \binom{m-1}{3} + \sum_{i=0}^{m} \left[ \binom{m-1}{6i+2} + \binom{m-1}{6i+4} + \binom{m-1}{6i+5} + \binom{m-1}{6i+9} \right],
\] (3.54)

which is \( > 2^{m-2} \) if \( m_1 \geq 3 \) by (3.51). Moreover, we have the following table:
In summary, we have:

**Theorem 3.3.** Let $m > 3$ be an integer. The ternary code $C_3(V)$ is of type $[2^{m-1}, m, 2^{m-2}]$ if $m \neq 6$ and of type $[32, 6, 12]$ when $n = 6$.

We remark that the spin module $V$ is self-dual if and only if $m$ is even.

**Corollary 3.4.** When $m = 6m_1 + 2$ for some positive integer $m_1$, the ternary weight code of $o(2m, \mathbb{C})$ on $o(2m, \mathbb{C}) + V$ is an orthogonal ternary $[m(m - 1) + 2^{m-2}, m, 4m - 7 + 2^{m-3}]$-code. If $m = 6m_1 + 3$ for some positive integer $m_1$, the ternary weight code of $o(2m, \mathbb{C})$ on $o(2m, \mathbb{C}) + V$ is an orthogonal ternary $[2m(m - 1) + 2^{m-1}, m, 8m - 14 + 2^{m-2}]$-code. In the case $m = 6m_1 + 5$ and $m = 6m_1 + 12$ for some nonnegative integer $m_1$, the code $C_2 \oplus C_3(V)$ is an orthogonal ternary $[m(m - 1) + 2^{m-1}, m, 4m - 7 + 2^{m-2}]$-code. When $m = 6$, the code $C_2 \oplus C_3(V)$ is an orthogonal ternary $[62, 6, 27]$-code.

**Proof.** Suppose $m = 6m_1 + 2$ for some positive integer $m_1$. Then the weight matrix of $o(2m, \mathbb{C})$ on $o(2m, \mathbb{C}) + V$ is equivalent to $(A, -A)$, where $A$ generates the weight code $C$ of $o(2m, \mathbb{C}) + V$. Moreover, $C$ is orthogonal if and only if the matrix $(A, -A)$ generates an orthogonal code. But

$$(A, -A) \sim (C_2, C_2, C(V)). \quad (3.55)$$

Note that

$$\text{wt} (\zeta_i, \zeta_i, \xi_i) = 2f(1) + 2^{m-1} = 4(m - 1) + 2^{m-1} \equiv 1 + (-1)^{6m_1+1} \equiv 0 \pmod{3}, \quad (3.56)$$

$$\text{wt} (\zeta_i + \zeta_j, \zeta_i + \xi_i, \xi_j + \xi_j)$$

$$= 2f(2) + 2^{m-2} = 8m - 14 + 2^{m-2} \equiv 2 + (-1)^{6m_1} \equiv 0 \pmod{3} \quad (3.57)$$

for $i, j \in \mathbb{Z}/m$ with $i \neq j$ by (3.16) and (3.41). Thus

$$(\zeta_i, \zeta_i, \xi_i) \cdot (\zeta_i, \zeta_i, \xi_i) \equiv \text{wt} (\zeta_i, \zeta_i, \xi_i) \equiv 0, \quad (3.58)$$

$$(\zeta_i, \zeta_i, \xi_i) \cdot (\zeta_j, \zeta_j, \xi_j)$$

$$\equiv -[\text{wt} (\zeta_i + \zeta_j, \zeta_i + \xi_j, \xi_i + \xi_j) - \text{wt} (\zeta_i, \zeta_i, \xi_i) - \text{wt} (\zeta_j, \zeta_j, \xi_j)] \equiv 0 \quad (3.59)$$

by (3.39). Thus $C$ is orthogonal. Note

$$f(2) = 4m - 7 \leq \frac{m(m - 1)}{2} = f(m) \quad \text{if} \ m \geq 7. \quad (3.60)$$
Thus
\[ f(2) \leq f(t) \quad \text{for } t \in \mathbb{Z}, m. \] (3.61)

By (3.8),
\[ \text{wt} \left( \sum_{i=1}^{m-1} \zeta_i - \zeta_m \right) = f(m) \geq f(2). \] (3.62)

Thus the minimum distance of \( C \) is
\[ \min \{ f(1) + 2^{m-2}, f(2) + 2^{m-3} \} = 4m - 7 + 2^{m-3} \quad \text{if } m \geq 6. \] (3.63)

This proves the first conclusion. The other conclusions for \( m \geq 7 \) can be proved similarly.

In the case \( m = 5 \), we have

| Table 3.2 |
|---|---|---|---|---|
| t | 1 | 2 | 3 | 4 |
| f(t) | 8 | 13 | 15 | 14 |

and on the \( \mathcal{V} \),

| Table 3.3 |
|---|---|---|---|---|---|
| t | 1 | 2 | 3 | 4 | 5 |
| \text{wt } u(t) | 16 | 8 | 12 | 10 | 11 |

By Tables 3.1-3.3 and the fact \( \text{wt} \left( \sum_{i=1}^{4} \zeta_i - \zeta_5 \right) = f(5) \) in \( C_3(\mathcal{A}_2) \), the third conclusion holds for \( m = 5 \).

If \( m = 6 \),

| Table 3.4 |
|---|---|---|---|---|---|---|
| t | 1 | 2 | 3 | 4 | 5 | 6 |
| f(t) | 10 | 17 | 21 | 22 | 20 | 15 |

and on the \( \mathcal{V} \),

| Table 3.5 |
|---|---|---|---|---|---|---|
| t | 1 | 2 | 3 | 4 | 5 | 6 |
| \text{wt } u(t) | 32 | 16 | 24 | 20 | 22 | 21 |

By Tables 3.1, 3.4, and 3.5, and the fact \( \text{wt} \left( \sum_{i=1}^{5} \zeta_i - \zeta_6 \right) = f(6) \) in \( C_3(\mathcal{A}_2) \), the last conclusion holds.

When \( m = 8 \), the ternary weight code of \( o(16, \mathbb{C}) \) on \( o(16, \mathbb{C}) + \mathcal{V} \) is a ternary orthogonal \([120, 8, 57] \)-code, which will later be proved also to be the ternary weight code of \( E_8 \) on its adjoint module. If \( m = 9 \), the ternary weight code of \( o(18, \mathbb{C}) \) on \( o(18, \mathbb{C}) + \mathcal{V} \) is a ternary orthogonal \([400, 8, 186] \)-code. When \( m = 5 \), the code \( C_2 \oplus C_3(\mathcal{V}) \) is a ternary orthogonal \([36, 5, 21] \)-code, which will later be proved also to be the ternary weight code of \( E_6 \) on its adjoint module. In the case \( m = 11 \), the code \( C_2 \oplus C_3(\mathcal{V}) \) is a ternary orthogonal \([1134, 8, 549] \)-code.
4 Representations of $F_4$ and Ternary Codes

In this section, we study the ternary weight codes of $F_4$ on its minimal irreducible module and adjoint module.

We go back to the settings in (2.2)-(2.4) with $n = 4$. The root system of $F_4$ is

$$\Phi_{F_4} = \left\{ \pm \varepsilon_1, \pm \varepsilon_i \pm \varepsilon_j, \frac{1}{2}(\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \mid i \neq j \right\}$$  \hspace{1cm} (4.1)

and the positive simple roots are

$$\alpha_1 = \varepsilon_2 - \varepsilon_3, \alpha_2 = \varepsilon_3 - \varepsilon_4, \alpha_3 = \varepsilon_4, \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4).$$ \hspace{1cm} (4.2)

The corresponding Dynkin diagram is

$$F_4: \hspace{1cm} \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}$$

The Weyl group $W_{F_4}$ of $F_4$ contains the permutation group $S_4$ on the sub-indices of $\varepsilon_i$ and all reflections with respect to the coordinate hyperplanes. Moreover, there is an identification:

$$h_1 \leftrightarrow \alpha_1, \ h_2 \leftrightarrow \alpha_2, \ h_3 \leftrightarrow 2\alpha_3, \ h_4 \leftrightarrow 2\alpha_4$$ \hspace{1cm} (4.3)

(e.g., cf. [7]). Thus

$$H_2 = \sum_{i=1}^{4} \mathbb{F}_2 h_i = \sum_{i=1}^{4} \mathbb{F}_2 \varepsilon_i.$$ \hspace{1cm} (4.4)

Moreover,

$$H_2 = \{W_{F_4}(h_1), W_{F_4}(h_1 + h_3), W_{F_4}(h_3), W_{F_4}(h_4)\}.$$ \hspace{1cm} (4.5)

The basic (minimal) irreducible module $V_{F_4}$ of the 52-dimensional Lie algebra $G^{F_4}$ has a basis $\{x_i \mid 1, 26\}$ and with the representation determined by the following formulas in terms of differential operators:

$$E_{\alpha_1}|_V = x_4 \partial_{x_6} + x_5 \partial_{x_8} + x_7 \partial_{x_9} - x_{18} \partial_{x_{20}} - x_{19} \partial_{x_{22}} - x_{21} \partial_{x_{23}},$$ \hspace{1cm} (4.6)

$$E_{\alpha_2}|_V = x_3 \partial_{x_4} + x_8 \partial_{x_{10}} + x_9 \partial_{x_{11}} - x_{16} \partial_{x_{18}} - x_{17} \partial_{x_{19}} - x_{23} \partial_{x_{24}},$$ \hspace{1cm} (4.7)

$$E_{\alpha_3}|_V = -x_2 \partial_{x_3} - x_4 \partial_{x_5} - x_6 \partial_{x_8} + x_{10} \partial_{x_{12}} + x_{11}(\partial_{x_{13}} - 2\partial_{x_{14}})$$
$$-x_{14} \partial_{x_{16}} - x_{15} \partial_{x_{17}} + x_{19} \partial_{x_{21}} + x_{22} \partial_{x_{23}} + x_{24} \partial_{x_{25}},$$ \hspace{1cm} (4.8)

$$E_{\alpha_4}|_V = -x_1 \partial_{x_2} - x_5 \partial_{x_7} - x_8 \partial_{x_9} - x_{10} \partial_{x_{11}} + x_{12}(\partial_{x_{14}} - 2\partial_{x_{13}})$$
$$-x_{13} \partial_{x_{15}} + x_{16} \partial_{x_{17}} + x_{18} \partial_{x_{19}} + x_{20} \partial_{x_{22}} + x_{25} \partial_{x_{26}},$$ \hspace{1cm} (4.9)
\[
E_{-\alpha_1}|_V = -x_6 \partial_{x_4} - x_8 \partial_{x_5} - x_9 \partial_{x_7} + x_{20} \partial_{x_{18}} + x_{22} \partial_{x_{19}} + x_{23} \partial_{x_{21}},
\]
(4.10)
\[
E_{-\alpha_2}|_V = -x_4 \partial_{x_3} - x_{10} \partial_{x_8} - x_{11} \partial_{x_9} + x_{18} \partial_{x_{16}} + x_{19} \partial_{x_{17}} + x_{24} \partial_{x_{23}},
\]
(4.11)
\[
E_{-\alpha_3}|_V = x_3 \partial_{x_2} + x_5 \partial_{x_4} + x_8 \partial_{x_6} - x_{12} \partial_{x_{10}} + x_{16} (2 \partial_{x_{14}} - \partial_{x_{13}})
+ x_{14} \partial_{x_{11}} + x_{17} \partial_{x_{15}} - x_{21} \partial_{x_{19}} - x_{23} \partial_{x_{22}} - x_{25} \partial_{x_{24}},
\]
(4.12)
\[
E_{-\alpha_4}|_V = x_2 \partial_{x_1} + x_7 \partial_{x_5} + x_9 \partial_{x_8} + x_{11} \partial_{x_{10}} + x_{15} (2 \partial_{x_{13}} - \partial_{x_{14}})
+ x_{13} \partial_{x_{12}} - x_{17} \partial_{x_{16}} - x_{19} \partial_{x_{18}} - x_{22} \partial_{x_{20}} - x_{26} \partial_{x_{25}},
\]
(4.13)
\[
h_1|_V = x_4 \partial_{x_1} + x_5 \partial_{x_2} - x_6 \partial_{x_4} + x_7 \partial_{x_5} - x_8 \partial_{x_7} - x_9 \partial_{x_8} + x_{18} \partial_{x_{21}}
+ x_{19} \partial_{x_{18}} - x_{20} \partial_{x_{20}} + x_{21} \partial_{x_{21}} - x_{22} \partial_{x_{22}} - x_{23} \partial_{x_{23}},
\]
(4.14)
\[
h_2|_V = x_3 \partial_{x_3} - x_4 \partial_{x_4} + x_8 \partial_{x_6} + x_9 \partial_{x_9} - x_{10} \partial_{x_{10}} - x_{11} \partial_{x_{11}} + x_{16} \partial_{x_{16}}
+ x_{17} \partial_{x_{17}} - x_{18} \partial_{x_{18}} + x_{19} \partial_{x_{19}} - x_{23} \partial_{x_{23}} - x_{24} \partial_{x_{24}},
\]
(4.15)
\[
h_3|_V = x_2 \partial_{x_2} - x_3 \partial_{x_3} + x_4 \partial_{x_4} - x_5 \partial_{x_5} + x_6 \partial_{x_6} - x_8 \partial_{x_8} + x_{10} \partial_{x_{10}}
+ 2x_{11} \partial_{x_{11}} - x_{12} \partial_{x_{12}} + x_{15} \partial_{x_{15}} - 2x_{16} \partial_{x_{16}} - x_{17} \partial_{x_{17}} + x_{19} \partial_{x_{19}}
- x_{21} \partial_{x_{21}} + x_{22} \partial_{x_{22}} - x_{23} \partial_{x_{23}} + x_{24} \partial_{x_{24}} - x_{25} \partial_{x_{25}},
\]
(4.16)
\[
h_4|_V = x_1 \partial_{x_1} - x_2 \partial_{x_2} + x_5 \partial_{x_5} + x_7 \partial_{x_7} + x_8 \partial_{x_8} - x_9 \partial_{x_9} + x_{16} \partial_{x_{16}}
- x_{11} \partial_{x_{11}} + 2x_{12} \partial_{x_{12}} - 2x_{15} \partial_{x_{15}} - x_{16} \partial_{x_{16}} - x_{17} \partial_{x_{17}} + x_{18} \partial_{x_{18}}
- x_{19} \partial_{x_{19}} + x_{20} \partial_{x_{20}} - x_{22} \partial_{x_{22}} + x_{25} \partial_{x_{25}} - x_{26} \partial_{x_{26}}.
\]
(4.17)

(e.g., cf. [26])

The module \(V_{F_4}\) is self-dual. The weight matrix of \(V_{F_4}\) is \((A_{F_4}, -A_{F_4})\) with
\[
A_{F_4} = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 & -1 & 1 & 0 & -1 & 0 & 1 & 2 & -1 & 0 & 0 \\
1 & -1 & 0 & 0 & 1 & 0 & 1 & -1 & 1 & -1 & -1 & 2 & -1 & 0
\end{bmatrix}.
\]
(4.18)

**Theorem 4.1.** The ternary weight code \(C_{F_4,1}\) (generated by \(A_{F_4}\)) of \(F_4\) on \(V_{F_4}\) is an orthogonal \([12,4,6]\)-code.

**Proof.** Denote by \(\xi_i\) the \(i\)th row of the matrix \(A_{F_4}\). Then
\[
\text{wt } \xi_1 = 6, \quad \text{wt } (\xi_1 + \xi_3) = \text{wt } \xi_3 = \text{wt } \xi_4 = 9.
\]
(4.19)
According to (4.5), any nonzero codeword in $C_{F_4,1}$ has weight 6 or 9. By an argument as (3.29)-(3.31), $C_{F_4,1}$ is orthogonal. □

Next we consider the adjoint representation of $F_4$. Its weight code $C_{F_4,2}$ is determined by the set $\Phi_{F_4}^+$ of positive roots. The followings are positive roots of $F_4$:

$$\begin{align*}
&\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \quad (4.20) \\
&\alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \quad (4.21) \\
&\alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \quad (4.22) \\
&\alpha_1 + \alpha_2 + 3\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4, \quad (2.22) \\
&\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4. \quad (4.23)
\end{align*}$$

Let $E_\alpha$ be a root vector associated with the root $\alpha$. The weight matrix $B_{F_4}$ on $\sum_{\alpha \in \Phi_{F_4}^+} \mathbb{F} E_\alpha$ is given by

$$\begin{bmatrix}
2 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\
-1 & 2 & -1 & 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & -2 & 2 & -1 & -2 & 0 & 1 & 0 & -1 & 2 & 2 & 1 & -1 & 0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & -1 & 2 & 0 & -1 & 1 & -1 & 1 & -2 & -2 & 0 & 1 & -2 & 0 & 2 & 2 & -1 & 1 & 0 & 0 & 0
\end{bmatrix}. \quad (4.24)$$

**Theorem 4.2.** The ternary weight code $C_{F_4,2}$ (generated by $B_{F_4}$) of $F_4$ on its adjoint module is an orthogonal [24, 4, 15]-code.

**Proof.** Denote by $\eta_i$ the $i$th row of the above matrix. Then

$$\text{wt} \, \eta_i = 15, \quad \text{wt} \, (\eta_1 + \eta_3) = 18. \quad (4.25)$$

According to (4.5), any nonzero codeword in $C_{F_4,2}$ has weight 15 or 18. By an argument as (3.29)-(3.31), $C_{F_4,2}$ is orthogonal. □

## 5 Representations of $E_6$ and Ternary Codes

In this section, we investigate the ternary weight codes of $E_6$ on its minimal irreducible module and adjoint module.

First we give a lattice construction of the exceptional simple Lie algebras of type $E$. Let $\{\alpha_i \mid i \in \overline{1,m}\}$ be the simple positive roots of type $E_m$. Set

$$Q_{E_m} = \sum_{i=1}^{m} \mathbb{Z} \alpha_i, \quad (5.1)$$
the root lattice of type $E_m$. Denote by $(\cdot, \cdot)$ the symmetric $\mathbb{Z}$-bilinear form on $Q_{E_m}$ such that the root system

$$\Phi_{E_m} = \{ \alpha \in Q_{E_m} \mid (\alpha, \alpha) = 2 \}. \quad (5.2)$$

Define $F(\cdot, \cdot) : Q_{E_m} \times Q_{E_m} \to \{ \pm 1 \}$ by

$$F(\sum_{i=1}^m k_i \alpha_i, \sum_{j=1}^m l_j \alpha_j) = (-1)^{\sum_{i=1}^m k_i l_i + \sum_{i>j}^m k_i l_j (\alpha_i, \alpha_j)}, \quad k_i, l_j \in \mathbb{Z}. \quad (5.3)$$

Denote

$$H_{E_m} = \sum_{i=1}^m C \alpha_i. \quad (5.4)$$

The simple Lie algebra of type $E_m$ is

$$\mathcal{G}_{E_m} = H_{E_m} \oplus \bigoplus_{\alpha \in \Phi_{E_m}} \mathbb{C}E_{\alpha} \quad (5.5)$$

with the Lie bracket $[\cdot, \cdot]$ determined by:

$$[H_{E_m}, H_{E_m}] = 0, \quad [h, E_{\alpha}] = (h, \alpha)E_{\alpha}, \quad [E_{\alpha}, E_{-\alpha}] = -\alpha, \quad (5.6)$$

$$[E_{\alpha}, E_{\beta}] = \begin{cases} 0 & \text{if } \alpha + \beta \notin \Phi_{E_m}, \\ F(\alpha, \beta)E_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi_{E_m}. \end{cases} \quad (5.7)$$

for $\alpha, \beta \in \Phi_{E_m}$ and $h \in H_{E_m}$ (e.g., cf. [8], [25]). Moreover,

$$h_i = \alpha_i \quad \text{for } i \in 1, m. \quad (5.8)$$

Recall the settings in (2.2)-(2.4). Taking $n = 7$, we have the following root system of $E_6$:

$$\Phi_{E_6} = \left\{ \varepsilon_i - \varepsilon_j, \frac{1}{2} \left( \sum_{s=1}^6 \iota_s \varepsilon_s \pm \sqrt{2} \varepsilon_7 \right), \pm \sqrt{2} \varepsilon_7 \mid i, j \in \overline{1, 6}, i \neq j; \iota_s = \pm 1; \sum_{i=1}^6 \iota_i = 0 \right\}. \quad (5.9)$$

and the simple positive roots are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \frac{1}{2} \left( \sum_{j=1}^3 (\varepsilon_{3+j} - \varepsilon_j) + \sqrt{2} \varepsilon_7 \right), \quad \alpha_i = \varepsilon_{i-1} - \varepsilon_i, \quad i \in \overline{3, 6}. \quad (5.10)$$

The Dynkin diagram is:

```
α₁  α₂  α₃  α₄  α₅  α₆
```

Note

$$\mathcal{H}_{E_6,3} = \sum_{i=1}^6 \mathbb{F}_3 h_i = \left\{ \sum_{i=1}^6 \iota_i \varepsilon_i + \iota_7 \sqrt{2} \varepsilon_7 \mid \iota_r \in \mathbb{F}_3, \sum_{i=1}^6 \iota_i = 0 \right\}. \quad (5.11)$$
Moreover, the Weyl group $\mathcal{W}_{E_6}$ contains the permutation group $S_6$ on the first six sub-indices of $\varepsilon_i$ and the reflection

$$\sum_{i=1}^{6} t_i \varepsilon_i + t_7 \sqrt{2} \varepsilon_7 \mapsto \sum_{i=1}^{6} t_i \varepsilon_i - t_7 \sqrt{2} \varepsilon_7.$$ (5.12)

So

$$\mathcal{H}_{E_6,3} = \mathcal{W}_{E_6}(\{\sum_{i=1}^{s} \varepsilon_i - \sum_{j=1}^{t} \varepsilon_{s+j} + t \sqrt{2} \varepsilon_7, \sqrt{2} \varepsilon_7 \mid t = 0, 1; s - t \equiv 0 \text{ (mod 3)}\}).$$ (5.13)

The 27-dimensional basic irreducible module $V_{E_6}$ of weight $\lambda_1$ for $E_6$ has a basis $\{x_i \mid i \in \{1, 27\}\}$ with the representation formulas determined by

$$E_{\alpha_1}|_V = -x_1 \partial_{x_2} + x_{11} \partial_{x_{14}} + x_{15} \partial_{x_{17}} + x_{16} \partial_{x_{19}} + x_{18} \partial_{x_{21}} + x_{20} \partial_{x_{23}},$$ (5.14)

$$E_{\alpha_2}|_V = -x_4 \partial_{x_6} - x_5 \partial_{x_7} - x_8 \partial_{x_{10}} + x_{18} \partial_{x_{20}} + x_{21} \partial_{x_{23}} + x_{22} \partial_{x_{24}},$$ (5.15)

$$E_{\alpha_3}|_V = -x_2 \partial_{x_3} + x_9 \partial_{x_{11}} + x_{12} \partial_{x_{15}} + x_{13} \partial_{x_{16}} + x_{21} \partial_{x_{22}} + x_{23} \partial_{x_{24}},$$ (5.16)

$$E_{\alpha_4}|_V = -x_3 \partial_{x_4} - x_7 \partial_{x_9} - x_{10} \partial_{x_{12}} - x_{16} \partial_{x_{18}} - x_{19} \partial_{x_{21}} + x_{24} \partial_{x_{25}},$$ (5.17)

$$E_{\alpha_5}|_V = -x_4 \partial_{x_5} - x_6 \partial_{x_7} - x_{12} \partial_{x_{13}} - x_{15} \partial_{x_{16}} - x_{17} \partial_{x_{19}} + x_{25} \partial_{x_{26}},$$ (5.18)

$$E_{\alpha_6}|_V = -x_5 \partial_{x_8} - x_7 \partial_{x_{10}} - x_9 \partial_{x_{12}} - x_{11} \partial_{x_{15}} - x_{14} \partial_{x_{17}} + x_{26} \partial_{x_{27}},$$ (5.19)

$$h_r|_{V_{E_6}} = \sum_{i=1}^{27} a_{r,i} x_i \partial_{x_i}$$ (5.20)

with $a_{r,i}$ given by the following table

| $i$ | $a_{1,i}$ | $a_{2,i}$ | $a_{3,i}$ | $a_{4,i}$ | $a_{5,i}$ | $a_{6,i}$ | $i$ | $a_{1,i}$ | $a_{2,i}$ | $a_{3,i}$ | $a_{4,i}$ | $a_{5,i}$ | $a_{6,i}$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----|-----------|-----------|-----------|-----------|-----------|-----------|
| 1   | 1         | 0         | 0         | 0         | 0         | 0         | 2   | -1        | 0         | 1         | 0         | 0         | 0         |
| 3   | 0         | 0         | -1        | 0         | 0         | 0         | 4   | 0         | 1         | 0         | -1        | 1         | 0         |
| 5   | 0         | 1         | 0         | 0         | -1        | 1         | 6   | 0         | -1        | 0         | 0         | 1         | 0         |
| 7   | 0         | -1        | 0         | 1         | -1        | 1         | 8   | 0         | 1         | 0         | 0         | 0         | -1        |
| 9   | 0         | 0         | 1         | -1        | 0         | 1         | 10  | 0         | -1        | 0         | 1         | 0         | -1        |
| 11  | 1         | 0         | -1        | 0         | 0         | 1         | 12  | 0         | 0         | 1         | -1        | 1         | -1        |
| 13  | 0         | 0         | 1         | 0         | -1        | 0         | 14  | -1        | 0         | 0         | 0         | 0         | 1         |
| 15  | 1         | 0         | -1        | 0         | 1         | -1        | 16  | 1         | 0         | -1        | 1         | -1        | 0         |
| 17  | -1        | 0         | 0         | 0         | 1         | -1        | 18  | 1         | 1         | 0         | -1        | 0         | 0         |
| 19  | -1        | 0         | 0         | 1         | -1        | 0         | 20  | 1         | -1        | 0         | 0         | 0         | 0         |
| 21  | -1        | 1         | 1         | -1        | 0         | 0         | 22  | 0         | 1         | -1        | 0         | 0         | 0         |
| 23  | -1        | -1        | 1         | 0         | 0         | 0         | 24  | 0         | -1        | -1        | 1         | 0         | 0         |
| 25  | 0         | 0         | 0         | -1        | 1         | 0         | 26  | 0         | 0         | 0         | 0         | -1        | 1         |
| 27  | 0         | 0         | 0         | 0         | 0         | -1        | 0   | 0         | 0         | 0         | 0         | 0         | 0         |
Denote by \( G \) (e.g., cf. [27]). Moreover, the algebra \( G \) are annihilated by its positive root vectors. By Table 5.1 and (5.27), the

\[
E_{-\alpha_1} V = x_2 \partial_{x_1} - x_{14} \partial_{x_{11}} - x_{17} \partial_{x_{15}} - x_{19} \partial_{x_{16}} - x_{21} \partial_{x_{18}} - x_{23} \partial_{x_{20}},
\]

(5.21)

\[
E_{-\alpha_2} V = x_6 \partial_{x_4} + x_7 \partial_{x_5} + x_{10} \partial_{x_8} - x_{20} \partial_{x_{18}} - x_{23} \partial_{x_{21}} - x_{24} \partial_{x_{22}},
\]

(5.22)

\[
E_{-\alpha_3} V = x_3 \partial_{x_2} - x_{11} \partial_{x_9} - x_{15} \partial_{x_{12}} - x_{16} \partial_{x_{13}} - x_{22} \partial_{x_{21}} - x_{24} \partial_{x_{23}},
\]

(5.23)

\[
E_{-\alpha_4} V = x_4 \partial_{x_3} + x_9 \partial_{x_7} + x_{12} \partial_{x_{10}} + x_{18} \partial_{x_{16}} + x_{21} \partial_{x_{19}} - x_{25} \partial_{x_{24}},
\]

(5.24)

\[
E_{-\alpha_5} V = x_5 \partial_{x_4} + x_7 \partial_{x_6} + x_{13} \partial_{x_{12}} + x_{16} \partial_{x_{15}} + x_{19} \partial_{x_{17}} - x_{26} \partial_{x_{25}},
\]

(5.25)

\[
E_{-\alpha_6} V = x_3 \partial_{x_5} + x_{10} \partial_{x_7} + x_{12} \partial_{x_9} + x_{15} \partial_{x_{11}} + x_{17} \partial_{x_{14}} - x_{27} \partial_{x_{26}},
\]

(5.26)

(e.g., cf. [27]). Moreover,

\[
E_{\alpha_i}(x_i) \neq 0 \iff a_{r,i} < 0, \quad E_{-\alpha_i}(x_i) \neq 0 \iff a_{r,i} > 0.
\]

(5.27)

**Theorem 5.1.** The ternary weight code \( C_{E_6,1} \) of \( E_6 \) on \( V_{E_6} \) is an orthogonal \([27, 6, 12]\)-code.

**Proof.** Write

\[
A_{E_6} = (a_{r,i})_{6 \times 27}.
\]

(5.28)

Denote by \( \xi_r \) the \( r \)th row of the matrix \( A_{E_6} \). Then

\[
\text{wt } \xi_r = 12 \quad \text{for } r \in [1, 6].
\]

(5.29)

Moreover,

\[
\text{wt } (\xi_1 + \xi_3) = \text{wt } (\xi_2 + \xi_4) = 12, \quad \text{wt } (\xi_1 + \xi_4) = 18,
\]

(5.30)

\[
\text{wt } (\xi_1 + \xi_2) = \text{wt } (\xi_2 + \xi_3) = \text{wt } (\xi_2 + \xi_5) = \text{wt } (\xi_2 + \xi_6) = 18.
\]

(5.31)

By an argument as (3.29)-(3.31) and symmetry, we have

\[
\xi_i \cdot \xi_j \equiv 0 \pmod{3} \quad \text{for } i, j \in [1, 6],
\]

(5.32)

that is \( C_{E_6,1} \) is orthogonal.

Note that the Lie subalgebra \( G_{A_{1,1}}^{E_6} \) generated by \( \{E_{\pm \alpha_i} \mid 2 \neq i \in [1, 6]\} \) is isomorphic to \( sl(6, \mathbb{C}) \). Recall that a singular vector in a module of simple Lie algebra is a weight vector annihilated by its positive root vectors. By Table 5.1 and (5.27), the \( G_{A_{1,1}}^{E_6} \)-singular vectors are \( x_1 \) of weight \( \lambda_1 \), \( x_6 \) of weight \( \lambda_4 \) and \( x_{20} \) of weight \( \lambda_1 \). So the \( (G_{A_{1,1}}^{E_6}, G_{A_{1,1}}^{E_6}) \)-branch rule on \( V_{E_6} \) is

\[
V_{E_6} \cong V_{A_5}(\lambda_1) \oplus V_{A_5}(\lambda_4) \oplus V_{A_5}(\lambda_1).
\]

(5.33)

Denote by \( G_{A_{1,2}}^{E_6} \) the Lie subalgebra of \( G^{E_6} \) generated by \( \{E_{\pm \alpha_r}, \ E_{\pm(\alpha_2 + \alpha_4)} \mid 2, 4 \neq r \in [1, 6]\} \).

The algebra \( G_{A_{1,2}}^{E_6} \) is also isomorphic to \( sl(6, \mathbb{C}) \). According to Table 5.1 and (5.27), the
$G_{A,2}$-singular vectors are $x_1$ of weight $\lambda_1$, $x_4$ of weight $\lambda_4$ and $x_{18}$ of weight $\lambda_1$. Hence (5.33) is also the $(G^E_6, G_{A,2}^E)$-branch rule. Since the module $V_{A_5}(\lambda_2)$ is contragredient to $V_{A_5}(\lambda_4)$, they have the same ternary weight code. By (2.39) and (2.43) with $n = 6$, the minimal distances of the subcodes $\sum_{2 \neq i \in 1,6} \mathbb{F}_3 \xi_i$ and $\mathbb{F}_3(\xi_2 + \xi_4) + \sum_{2,4 \neq i \in 1,6} \mathbb{F}_3 \xi_i$ are $\text{wt} \xi_1 = 12$.

Recall $\frac{1}{2} = -1$ in $\mathbb{F}_3$. Moreover,

$$-(\alpha_2 + \alpha_4) = -\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \sqrt{2}\varepsilon_7 \text{ in } \mathcal{H}_{E_6;3}. \quad (5.34)$$

Thus in $\mathcal{H}_{E_6;3}$,

$$\alpha_1 - (\alpha_2 + \alpha_4) = \varepsilon_2 + \varepsilon_3 - \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \sqrt{2}\varepsilon_7, \quad (5.35)$$

$$\alpha_1 - \alpha_2 - (\alpha_2 + \alpha_4) = -\varepsilon_3 - \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \sqrt{2}\varepsilon_7, \quad (5.36)$$

$$\alpha_1 - \alpha_2 - (\alpha_2 + \alpha_4) + \alpha_6 = -\varepsilon_3 - \varepsilon_4 - \varepsilon_5 + \sqrt{2}\varepsilon_7, \quad (5.37)$$

$$\alpha_1 - \alpha_2 - (\alpha_2 + \alpha_4) - \alpha_5 + \alpha_6 = -\varepsilon_3 + \varepsilon_4 + \sqrt{2}\varepsilon_7. \quad (5.38)$$

Note that

$$\text{wt} (\xi_1 - (\xi_2 + \xi_4)), \text{ wt} (\xi_1 - \xi_2 - (\xi_2 + \xi_4)) \geq 12, \quad (5.39)$$

$$\text{wt} (\xi_1 - \xi_2 - (\xi_2 + \xi_4) + \xi_6), \text{ wt} (\xi_1 - \xi_2 - (\xi_2 + \xi_4) - \xi_5 + \xi_6) \geq 12 \quad (5.40)$$

because the minimal distance of $\mathbb{F}_3(\xi_2 + \xi_4) + \sum_{2,4 \neq i \in 1,6} \mathbb{F}_3 \xi_i$ is 12. Furthermore,

$$-\sum_{i=1}^{6} \varepsilon_i + \sqrt{2}\varepsilon_7 = \alpha_1 - \alpha_2 - \alpha_3 \text{ in } \mathcal{H}_{E_6;3}. \quad (5.41)$$

We calculate

$$\text{wt} (\xi_1 - \xi_2 - \xi_3) = 21. \quad (5.42)$$

By (5.13), the minimal distance of the ternary code $C_{E_6;1}$ is 12. \hfill $\Box$

Next we consider the ternary weight code $C_{E_6;2}$ of $E_6$ on its adjoint module. Take any order

$$\{y_1, \ldots, y_{36}\} = \{E_\alpha \mid \alpha \in \Phi^+_E\}. \quad (5.43)$$

Write

$$[\alpha_i, y_j] = b_{i,j}, \quad B_{E_6} = (b_{i,j})_{6 \times 36}. \quad (5.44)$$

**Theorem 5.2.** The ternary weight code $C_{E_6;2}$ (generated $B_{E_6}$) of $E_6$ on its adjoint module is an orthogonal $[36, 5, 21]$-code.
Proof. Denote by $\zeta_i$ the $i$th row of $B_{E_6}$. Note that
\[
\zeta_1 - \zeta_3 + \zeta_5 - \zeta_6 \equiv 0 \quad \text{in } \mathbb{F}_3.
\] (5.45)
Thus
\[
C_{E_6,2} = \sum_{i=2}^{6} F_3 \zeta_i.
\] (5.46)
Denote by $G_{E_6}^D$ the Lie subalgebra of $G_{E_6}^D$ generated by $\{E_{\pm \alpha} \mid r \in \overline{2, 6}\}$. According to the Dynkin diagram of $E_6$,
\[
G_{E_6}^D \cong o(10, \mathbb{C}).
\] (5.47)
Let $G_{E_6}^D = \sum_{i=1}^{36} C y_i$ and denote by $G_{E_6}^{D+}$ the subspace spanned by the root vectors $E_\alpha \in G_{E_6}^D$ with $\alpha \in \Phi_{E_6}^\pm$. Then $[G_{E_6}^{D+}, G_{E_6}^{D+}] \subset G_{E_6}^{D+}$. Moreover, the space $G_{E_6}^{D+}$ contains $G_{E_6}^{E_6}$-singular vectors $E_{\alpha_4 + \alpha_5 + \sum_{i=1}^{6} \alpha_i}$ of weight $\lambda_2$ (the highest root) and $E_{8 + \sum_{i=3}^{5} \alpha_i + \sum_{i=1}^{6} \alpha_i}$ of weight $\lambda_5$. Hence, we have the partial $(C_{E_6}, G_{E_6}^D)$-branch rule on $G_{E_6}$:
\[
G_{E_6}^{D+} \cong G_{E_6}^{E_6} \oplus V_{D_5}(\lambda_5).
\] (5.48)
Thus the ternary weight code $C_{E_6,2}$ of $E_6$ on its adjoint module is exactly the code $C_2 \oplus C_3(\mathcal{V})$ with $n = 5$ in Corollary 3.4, which is a ternary orthogonal $[36, 5, 21]$-code. \qed

6 \hspace{1em} \textbf{Representations of } E_7, E_8 \text{ and Ternary Codes}

In this section, we study the ternary weight codes of $E_7$ on its minimal irreducible module and adjoint module, and the ternary weight code of $E_8$ on its minimal irreducible module (adjoint module).

Recall the settings in (2.2)-(2.4) and (5.1)-(5.8). Taking $n = 8$, we have the root system of $E_7$:
\[
\Phi_{E_7} = \left\{ \varepsilon_i - \varepsilon_j, \frac{1}{2} \sum_{s=1}^{8} t_s \varepsilon_s \mid i, j \in \overline{1, 8}, i \neq j; \ t_s = \pm 1, \ \sum_{s=1}^{8} t_s = 0 \right\}
\] (6.1)
and the simple positive roots are:
\[
\alpha_1 = \varepsilon_1 - \varepsilon_3, \ \alpha_2 = \frac{1}{2} \sum_{j=1}^{4} (\varepsilon_{4+j} - \varepsilon_j), \ \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad i \in \overline{3, 7}.
\] (6.2)

The Dynkin diagram of $E_7$ is as follows:

\[ E_7: \]

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (1,0) -- (2,0);
\draw (2,0) -- (3,0);
\draw (3,0) -- (4,0);
\draw (4,0) -- (5,0);
\draw (5,0) -- (6,0);
\draw (6,0) -- (7,0);
\draw (1,0) -- (2,0);
\draw (3,0) -- (4,0);
\draw (5,0) -- (6,0);
\draw (7,0) -- (8,0);
\draw (2,0) -- (3,0);
\draw (4,0) -- (5,0);
\draw (6,0) -- (7,0);
\draw (3,0) -- (4,0);
\draw (5,0) -- (6,0);
\draw (7,0) -- (8,0);
\node at (0,0) {$1$};
\node at (1,0) {$3$};
\node at (2,0) {$4$};
\node at (3,0) {$5$};
\node at (4,0) {$6$};
\node at (5,0) {$7$};
\node at (6,0) {$8$};
\end{tikzpicture}
\end{center}
The minimal module $V_{E_7}$ of $E_7$ is of 56-dimensional and has a basis $\{x_i \mid i \in \{1, 56\}\}$ with the representation formulas determined by

\begin{align}
E_{\alpha_1}\mid V & = -x_6 \partial_{x_8} - x_9 \partial_{x_{11}} - x_{10} \partial_{x_{13}} - x_{12} \partial_{x_{16}} - x_{14} \partial_{x_{19}} - x_{17} \partial_{x_{22}} \\
& + x_{35} \partial_{x_{40}} + x_{38} \partial_{x_{43}} + x_{41} \partial_{x_{45}} + x_{44} \partial_{x_{47}} + x_{46} \partial_{x_{48}} + x_{49} \partial_{x_{51}}, \quad (6.3) \\
E_{\alpha_2}\mid V & = x_5 \partial_{x_7} + x_6 \partial_{x_9} + x_8 \partial_{x_{11}} - x_{20} \partial_{x_{23}} - x_{24} \partial_{x_{26}} - x_{27} \partial_{x_{29}} \\
& - x_{28} \partial_{x_{30}} - x_{31} \partial_{x_{33}} - x_{34} \partial_{x_{37}} + x_{46} \partial_{x_{49}} + x_{48} \partial_{x_{51}} + x_{50} \partial_{x_{52}}, \quad (6.4) \\
E_{\alpha_3}\mid V & = -x_5 \partial_{x_6} - x_7 \partial_{x_9} - x_{13} \partial_{x_{15}} - x_{16} \partial_{x_{18}} - x_{19} \partial_{x_{21}} - x_{22} \partial_{x_{25}} \\
& + x_{32} \partial_{x_{35}} + x_{36} \partial_{x_{38}} + x_{39} \partial_{x_{41}} + x_{42} \partial_{x_{44}} + x_{48} \partial_{x_{50}} + x_{51} \partial_{x_{52}}, \quad (6.5) \\
E_{\alpha_4}\mid V & = x_4 \partial_{x_5} - x_9 \partial_{x_{10}} - x_{11} \partial_{x_{13}} - x_{18} \partial_{x_{20}} - x_{21} \partial_{x_{24}} - x_{25} \partial_{x_{28}} \\
& - x_{29} \partial_{x_{32}} - x_{33} \partial_{x_{36}} - x_{37} \partial_{x_{39}} - x_{41} \partial_{x_{44}} - x_{45} \partial_{x_{48}} + x_{52} \partial_{x_{53}}, \quad (6.6) \\
E_{\alpha_5}\mid V & = x_3 \partial_{x_4} - x_{10} \partial_{x_{12}} - x_{13} \partial_{x_{16}} - x_{15} \partial_{x_{18}} - x_{24} \partial_{x_{27}} - x_{26} \partial_{x_{29}} \\
& - x_{28} \partial_{x_{31}} - x_{30} \partial_{x_{33}} - x_{39} \partial_{x_{42}} - x_{41} \partial_{x_{44}} - x_{45} \partial_{x_{47}} + x_{53} \partial_{x_{54}}, \quad (6.7) \\
E_{\alpha_6}\mid V & = x_2 \partial_{x_3} - x_{12} \partial_{x_{14}} - x_{16} \partial_{x_{19}} - x_{18} \partial_{x_{21}} - x_{20} \partial_{x_{24}} - x_{23} \partial_{x_{26}} \\
& - x_{31} \partial_{x_{34}} - x_{33} \partial_{x_{37}} - x_{36} \partial_{x_{39}} - x_{38} \partial_{x_{41}} - x_{43} \partial_{x_{45}} + x_{54} \partial_{x_{55}}, \quad (6.8) \\
E_{\alpha_7}\mid V & = x_1 \partial_{x_2} - x_{14} \partial_{x_{17}} - x_{19} \partial_{x_{22}} - x_{21} \partial_{x_{25}} - x_{24} \partial_{x_{28}} - x_{26} \partial_{x_{30}} \\
& - x_{27} \partial_{x_{31}} - x_{29} \partial_{x_{33}} - x_{32} \partial_{x_{36}} - x_{35} \partial_{x_{38}} - x_{40} \partial_{x_{43}} + x_{55} \partial_{x_{56}}, \quad (6.9) \\
E_{-\alpha_1}\mid V & = x_8 \partial_{x_6} + x_{11} \partial_{x_9} + x_{13} \partial_{x_{10}} + x_{16} \partial_{x_{12}} + x_{19} \partial_{x_{14}} + x_{22} \partial_{x_{17}} \\
& - x_{40} \partial_{x_{35}} - x_{43} \partial_{x_{38}} - x_{45} \partial_{x_{41}} - x_{47} \partial_{x_{44}} - x_{48} \partial_{x_{46}} - x_{51} \partial_{x_{49}}, \quad (6.10) \\
E_{-\alpha_2}\mid V & = -x_7 \partial_{x_5} - x_9 \partial_{x_{6}} - x_{11} \partial_{x_8} + x_{23} \partial_{x_{20}} + x_{26} \partial_{x_{24}} + x_{29} \partial_{x_{27}} \\
& + x_{30} \partial_{x_{28}} + x_{33} \partial_{x_{31}} + x_{37} \partial_{x_{34}} - x_{49} \partial_{x_{46}} - x_{51} \partial_{x_{48}} - x_{52} \partial_{x_{50}}, \quad (6.11) \\
E_{-\alpha_3}\mid V & = x_6 \partial_{x_5} + x_9 \partial_{x_7} + x_{15} \partial_{x_{13}} + x_{18} \partial_{x_{16}} + x_{21} \partial_{x_{19}} + x_{25} \partial_{x_{22}} \\
& - x_{35} \partial_{x_{32}} - x_{38} \partial_{x_{36}} - x_{41} \partial_{x_{39}} - x_{44} \partial_{x_{42}} - x_{50} \partial_{x_{48}} - x_{52} \partial_{x_{51}}, \quad (6.12) \\
E_{-\alpha_4}\mid V & = -x_5 \partial_{x_4} + x_{10} \partial_{x_9} + x_{13} \partial_{x_{11}} + x_{20} \partial_{x_{18}} + x_{24} \partial_{x_{21}} + x_{28} \partial_{x_{25}} \\
& + x_{32} \partial_{x_{29}} + x_{36} \partial_{x_{33}} + x_{39} \partial_{x_{37}} + x_{46} \partial_{x_{44}} + x_{48} \partial_{x_{47}} - x_{53} \partial_{x_{52}}, \quad (6.13)
\end{align}
\[ E_{-\alpha_i}|_V = -x_4 \partial_{x_3} + x_{12} \partial_{x_{10}} + x_{16} \partial_{x_{13}} + x_{18} \partial_{x_{15}} + x_{27} \partial_{x_{24}} + x_{29} \partial_{x_{26}} + x_{31} \partial_{x_{28}} + x_{33} \partial_{x_{30}} + x_{42} \partial_{x_{39}} + x_{44} \partial_{x_{41}} + x_{47} \partial_{x_{45}} - x_{54} \partial_{x_{53}}, \quad (6.14) \]

\[ E_{-\alpha_6}|_V = -x_3 \partial_{x_2} + x_{14} \partial_{x_{12}} + x_{19} \partial_{x_{16}} + x_{21} \partial_{x_{18}} + x_{24} \partial_{x_{20}} + x_{26} \partial_{x_{23}} + x_{34} \partial_{x_{31}} + x_{37} \partial_{x_{33}} + x_{39} \partial_{x_{36}} + x_{41} \partial_{x_{38}} + x_{45} \partial_{x_{43}} - x_{55} \partial_{x_{54}}, \quad (6.15) \]

\[ E_{-\alpha_7}|_V = -x_2 \partial_{x_1} + x_{17} \partial_{x_{14}} + x_{22} \partial_{x_{19}} + x_{25} \partial_{x_{21}} + x_{28} \partial_{x_{24}} + x_{30} \partial_{x_{26}} + x_{31} \partial_{x_{27}} + x_{33} \partial_{x_{29}} + x_{36} \partial_{x_{32}} + x_{38} \partial_{x_{35}} + x_{43} \partial_{x_{40}} - x_{56} \partial_{x_{55}}, \quad (6.16) \]

\[ h_r|_V = \sum_{i=1}^{28} a_{r,i} (x_i \partial x_i - x_{57-i} \partial x_{57-i}) \quad \text{for} \ r \in \overline{1,7}, \quad (6.17) \]

where \( a_{r,i} \) are constants given by the following table:

| \( i \) | \( a_{1,i} \) | \( a_{2,i} \) | \( a_{3,i} \) | \( a_{4,i} \) | \( a_{5,i} \) | \( a_{6,i} \) | \( a_{7,i} \) | \( i \) | \( a_{1,i} \) | \( a_{2,i} \) | \( a_{3,i} \) | \( a_{4,i} \) | \( a_{5,i} \) | \( a_{6,i} \) | \( a_{7,i} \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 3   | 0   | 0   | 0   | 0   | 1   | -1  | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 5   | 0   | 1   | 1   | -1  | 0   | 0   | 0   | 6   | 1   | 1   | -1  | 0   | 0   | 0   | 0   |
| 7   | 0   | -1  | 1   | 0   | 0   | 0   | 0   | 8   | -1  | 1   | 0   | 0   | 0   | 0   | 0   |
| 9   | 1   | -1  | -1  | 1   | 0   | 0   | 0   | 12  | 1   | 0   | 0   | 0   | 0   | -1  | 1   |
| 11  | -1  | -1  | 0   | 1   | 0   | 0   | 0   | 10  | 1   | 0   | 0   | 0   | 0   | 0   | 1   |
| 13  | -1  | 0   | 1   | -1  | 1   | 0   | 0   | 14  | 1   | 0   | 0   | 0   | 0   | 0   | -1  |
| 15  | 0   | 0   | -1  | 0   | 1   | 0   | 0   | 16  | -1  | 0   | 1   | 0   | -1  | 1   | 0   |
| 17  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 18  | 0   | 0   | -1  | 1   | -1  | 2   | 1   |
| 19  | -1  | 0   | 0   | 0   | -1  | 1   | 20  | 0   | 1   | 0   | -1  | 1   | 0   | 0   | 1   |
| 21  | 0   | 0   | -1  | 1   | 0   | -1  | 1   | 22  | -1  | 0   | 1   | 0   | 0   | 0   | -1  |
| 23  | 0   | -1  | 0   | 0   | 0   | 1   | 0   | 24  | 0   | 1   | 0   | -1  | 1   | -1  | 1   |
| 25  | 0   | 0   | -1  | 1   | 0   | 0   | -1  | 26  | 0   | -1  | 0   | 0   | 1   | -1  | 1   |
| 27  | 0   | 1   | 0   | 0   | -1  | 0   | 1   | 28  | 0   | 1   | 0   | -1  | 1   | 0   | -1  |

(e.g., cf. [28]). Again we have

\[ E_\alpha(x_i) \neq 0 \iff a_{r,i} < 0, \quad E_{-\alpha_r}(x_i) \neq 0 \iff a_{r,i} > 0. \quad (6.18) \]

Denote

\[ A_{E_7} = (a_{r,i})_{7 \times 28}. \quad (6.19) \]

**Theorem 6.1.** The ternary weight code \( C_{E_7,1} \) of \( E_7 \) on \( V_{E_7} \) is an orthogonal \([28,7,12]\)-code.

**Proof.** Note that the root system of \( A_7 \):

\[ \Phi_{A_7} = \{ \epsilon_i - \epsilon_j \mid i, j \in \overline{1,8}, \ i \neq j \} \subset \Phi_{E_7}. \quad (6.20) \]
Thus we have the Lie subalgebra of $G^E_7$ (cf. (5.1)-(5.7) with $m = 7$):

$$G^E_7 = \sum_{i=1}^{7} \mathbb{C} \alpha_i + \sum_{\alpha \in \Phi_{A_7}} \mathbb{C} E_\alpha \cong sl(8, \mathbb{C}).$$  

(6.21)

Moreover,

$$\alpha'_1 = \varepsilon_1 - \varepsilon_2 = -2\alpha_2 - 2\alpha_1 - 3\alpha_3 - 4\alpha_4 - 3\alpha_5 - 2\alpha_6 - \alpha_7. \quad (6.22)$$

Note that $x_{23}$ is a $G^E_7$-singular vector of weight $\lambda_6$ and $x_{49}$ is a $G^E_7$-singular vector of weight $\lambda_2$ by (6.17), (6.18) and Table 6.1. Thus the $(G^E_7, G^E_7)$-branch rule on $V_{E_7}$ is

$$V_{E_7} \cong V_{A_7}(\lambda_2) \oplus V_{A_7}(\lambda_6). \quad (6.23)$$

Since $V_{A_7}(\lambda_6)$ is contragredient to $V_{A_7}(\lambda_2)$, they have the same ternary weight code of $G^E_7$, which is the $C_3(A_2)$ with $m = 2$ in Theorem 2.3. Hence the weight matrix of $G^E_7$ on $V_{E_7}$ generates a ternary orthogonal $[56, 7, 24]$-code.

On the other hand,

$$\sum_{i=1}^{7} F_3 \alpha_i = F_3 \alpha'_1 + \sum_{2 \neq i \in 17} F_3 \alpha_i$$

by (6.1) and the fact $1/2 \equiv -1$ in $\mathbb{F}_3$. Thus the weight matrix $(A_{E_7}, -A_{E_7})$ of $E_7$ on $V_{E_7}$ generates the same ternary code as the weight matrix of $G^E_7$ on $V_{E_7}$. So $(A_{E_7}, -A_{E_7})$ generates a ternary orthogonal $[56, 7, 24]$-code. Hence the ternary code $C_{E_7,1}$ generated by $A_{E_7}$ is an orthogonal $[28, 7, 12]$-code. \( \square \)

Next we consider the ternary weight code of $E_7$ on its adjoint module. Recall the construction of $G^E_7$ in (5.1)-(5.7) with $m = 7$. The $(G^E_7, G^E_7)$-branch rule on $G^E_7$ is

$$G^E_7 \cong G^E_7 \oplus V_{A_7}(\lambda_4). \quad (6.25)$$

The module $V_{A_7}(\lambda_4)$ of $sl(8, \mathbb{C})$ ($\cong G^E_7$) is exactly $A_4$ in (2.10) with $n = 8$, which is self-dual. For convenience, we study the ternary code generated by the weight matrix of $sl(8, \mathbb{C})$ on $A_4$. Taking any order of its basis

$$\{z_1, ..., z_{70}\} = \{\theta_{i_1} \theta_{i_2} \theta_{i_3} \theta_{i_4} \mid 1 \leq i_1 < i_2 < i_3 < i_4 \leq 8\}, \quad (6.26)$$

we write

$$[E_{r,r}, z_i] = b_{r,i} z_i, \quad B_{E_7} = (b_{r,i})_{7 \times 70}. \quad (6.27)$$

Denote by $\eta_r$ the $r$th row of $B_{E_7}$ and by $C'$ the ternary code generated by $B_{E_7}$. Set

$$v(s, t) = \sum_{i=1}^{s} \eta_i - \sum_{j=1}^{t} \eta_{s+j} \in C'. \quad (6.28)$$

Moreover, we only calculate the related weights:
Table 6.2

| (s,t) | (1,1) | (2,2) | (3,3) | (4,4) | (3,0) | (6,0) | (4,1) | (5,2) |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| wt v(s,t) | 40    | 44    | 48    | 34    | 60    | 30    | 46    | 50    |

Recall (2.65)-(2.70). We have

Table 6.3

| (s,t) | (1,1) | (2,2) | (3,3) | (4,4) | (3,0) | (6,0) | (4,1) | (5,2) |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 2wt u(s,t) | 26    | 40    | 42    | 32    | 30    | 24    | 38    | 34    |

According to (6.1), the Weyl group $\mathcal{W}_{E_7}$ contains the permutation group $S_8$ on the sub-indices of $\varepsilon_i$. By (1.9), (1.11) and the values of $\text{wt } v(s,t) + 2\text{wt } u(s,t)$ from the above tables, 54, 66, 84 and 90 are the only weights of the nonzero codewords in $C_3(G^{E_7})$, the ternary code generated by the weight matrix of $G^{E_7}_A$ on $G^{E_7}$. By (6.24) and an argument as (3.29)-(3.31), we have:

**Theorem 6.2.** The ternary weight code of $E_7$ on its adjoint module is an orthogonal $[63, 7, 27]$-code.

The minimal representation of $E_8$ is its adjoint module. Recall the settings in (2.2)-(2.4) and construction of the simple Lie algebra $G^{E_8}$ given in (5.1)-(5.8) with $m = 8$. we have the $E_8$ root system

$$\Phi_{E_8} = \left\{ \pm \varepsilon_i \pm \varepsilon_j, \frac{1}{2} \sum_{i=1}^{8} t_i \varepsilon_i \mid i, j \in \overline{1, 8}, i \neq j; t_i = \pm 1, \sum_{i=1}^{8} t_i \in 2\mathbb{Z} \right\} \quad (6.29)$$

and positive simple roots:

$$\alpha_1 = \frac{1}{2} \left( \sum_{j=2}^{7} \varepsilon_j - \varepsilon_1 - \varepsilon_8 \right), \quad \alpha_2 = -\varepsilon_1 - \varepsilon_2, \quad \alpha_r = \varepsilon_{r-2} - \varepsilon_{r-1}, \quad r \in \overline{3, 8}. \quad (6.30)$$

The Dynkin diagram of $E_8$ is as follows:

![Dynkin diagram of E8]

Observe that the root system of $o(16, \mathbb{C})$:

$$\Phi_{D_8} = \left\{ \pm \varepsilon_i \pm \varepsilon_j \mid i, j \in \overline{1, 8}, i \neq j \right\} \subset \Phi_{E_8}. \quad (6.31)$$

So the Lie subalgebra

$$G^{E_8}_D = H_{E_8} + \sum_{\alpha \in \Phi_{D_8}} CE_\alpha \quad (6.32)$$
of $G^{E_8}$ is exactly isomorphic to $o(16, \mathbb{C})$. Moreover, the $(G^{E_8}, G_D^{E_8})$-branch rule on $G^{E_8}$ is

$$G^{E_8} \cong G_D^{E_8} \oplus V_{D_8}(\lambda_8). \quad (6.33)$$

In fact, $V_{D_8}(\lambda_8)$ is exactly the spin module $\mathcal{V}$ in (3.35). Since

$$\sum_{i=1}^{8} F_3 \alpha_i = \sum_{\alpha \in \Phi_{D_8}} F_3 \alpha, \quad (6.34)$$

the ternary weight code of $E_8$ on $G^{E_8}$ is the same as that of $G_D^{E_8}$ on $G^{E_8}$. By Corollary 3.4 with $m = 8$, we have:

**Theorem 6.3.** The ternary weight code of $E_8$ on its adjoint module is an orthogonal $[120, 8, 57]$-code.

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