NONEQUALITY OF DIMENSIONS FOR METRIC GROUPS

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Abstract. An embeddability criterion for zero-dimensional metrizable topological spaces in zero-dimensional metrizable topological groups is given. A space which can be embedded as a closed subspace in a zero-dimensional metrizable group but is not strongly zero-dimensional is constructed; thereby, an example of a metrizable group with noncoinciding dimensions ind and dim is obtained. It is proved that one of Kulesza's zero-dimensional metrizable spaces cannot be embedded in a metrizable zero-dimensional group.

The presence of a topological group structure on a topological space has a strong influence on many properties of the space; a classical illustration is the metrizability of any first countable topological group. The dimensional properties are no exception. Thus, ind\(G = \dim G = \text{Ind} G\) for any locally compact group \(G\) [14] and \(\text{ind} G = \text{Ind} G\) for any topological group \(G\) which is a Lindelöf \(\Sigma\)-space [16], while for a general topological space, these three dimensions can be pairwise different, even if the space is compact [2].

The purpose of this paper is to investigate the dimensional properties of metrizable topological groups. The celebrated theorem of Katetov [5] says that \(\dim X = \text{Ind} X\) for any metric space \(X\); however, there exist examples of metrizable spaces with noncoinciding dimensions ind and dim. The first (very involved) example of such a space was constructed by Roy in 1968 [15]. Since then, much simpler examples with various additional properties have been suggested (see, e.g., [6–8,11–13]), but the question about the coincidence of dimensions for metrizable topological groups has remained open (apparently, for the first time, it was stated by Mishchenko in 1964 [10]).

In the first section of this paper, we prove a criterion for the embeddability of zero-dimensional metrizable topological spaces in zero-dimensional metrizable topological groups. This criterion was formulated by Mishchenko in [10], but its proof has never been published; Mishchenko himself confessed to this author in a private communication that he had retained neither notes nor recollections of the proof. The spaces embeddable in zero-dimensional topological groups occupy an intermediate position between the zero-dimensional metrizable spaces and the strongly zero-dimensional metrizable spaces (a metrizable space \(X\) has dimension dim zero if and only if it is metrizable by a non-Archimedean metric, and this non-Archimedean metric can be assumed to take only rational values (see [1]). The Graev extension [4] of such a metric to the free group \(F(X)\) takes only rational values as well; therefore, the group \(F(X)\) with the Graev metric has dimension ind zero, and it contains \(X\) as a subspace). In the second section, we construct a space (this is a special case of Mrowka’s space \(\mu\nu_0\)) which can be embedded as a closed subspace in a zero-dimensional metrizable group but is not strongly zero-dimensional; thereby, an example of a metrizable group with noncoinciding dimensions ind and dim is obtained. The third section contains an example of a zero-dimensional metrizable space which cannot be embedded in a metrizable zero-dimensional group.

1. Spaces Embeddable in Zero-Dimensional Metrizable Groups

The purpose of this section is to prove the following theorem.

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Theorem 1.1. A topological space $X$ can be embedded in a metrizable topological group with dimension $\text{ind} \leq 1$ if and only if the topology of $X$ is generated by a uniformity which has a countable base consisting of open-and-closed sets.

The “only if” part is obvious: if $X$ is embedded in a group $G$ and clopen sets $U_n$, where $n \in \omega$, form a neighborhood base at the identity in $G$, then the required base of a uniformity on $X$ consists of the entourages $U_n = \{(x, y) : xy^{-1} \in U_n \cap U_n^{-1}\}$.

The rest of this section is devoted to the proof of the reverse implication. By $A(X)$ we denote the free Abelian group generated by $X$; the letters $a, b, u, v, w, x, y, z$ always denote elements of $X$, the letters $i, j, k, l, m, n, r, s, t$, and $N$ denote nonnegative integers, and $g$ and $h$ denote elements of the free Abelian group $A(X)$. We use the definition of uniformities and entourages given in [1]; in particular, all entourages are assumed to be symmetric. For $A, B \subseteq X \times X$, we write

$$A \circ B = \{(x, y) \in X \times X : \text{there exists a } z \in X \text{ for which } (x, z) \in A \text{ and } (z, y) \in B\}.$$ 

If $A$ or $B$ is a one-point set, we omit the braces in the notation of this set and write, e.g., $A \circ (x, y)$. In particular, $(x, y) \circ (y, z) = (x, z)$ and $(x, y) \circ (u, z) = \emptyset$ if $y \neq u$.

If $(x, y) = (x = x_1, y_1) \circ (y_1 = x_2, y_2) \circ \ldots \circ (y_{n-1} = x_n, y_n = y)$, then, obviously, $x - y = \sum_{i=1}^{n} (x_i - y_i)$ in $A(X)$. We write

$$x - y = \sum_{i=1}^{n} (x_i - y_i)$$

in this case.

Lemma 1.1. Let $V_0, V_1, \ldots$ be (symmetric) elements of a uniformity of a set $X$ such that $V_0 = X \times X$ and $V_{i+1} \circ V_{i+1} \circ V_{i+1} \subseteq V_i$ for $i = 1, 2, \ldots$, and let $U_i = V_i \rho$ for $i \leq \omega$. Suppose that $\{k_1, \ldots, k_n\}$ is a set of positive integers in which each number $i$ occurs at most $i$ times. Then $U_{k_1} \cap U_{k_2} \cap \ldots \cap U_{k_n} \subseteq U_{k-1}$, where $k_n = \min \{k_i\}$.

Proof. If $k_n = 1$, then the assertion holds trivially. Suppose that $k_n > 1$, i.e., all $k_i$ are larger than 1. Let $\rho$ be a pseudometric on $X$ such that $V_i \subseteq \{(x, y) : \rho(x, y) \leq \frac{1}{i}\} \subseteq V_{i-1}$ for any $i \geq 1$ (it exists by Theorem 8.1.10 from [1]). For $(x, y) \in U_{k_1} \cap U_{k_2} \cap \ldots \cap U_{k_n}$, we have

$$(x, y) = (z_1, z_2) \circ (z_2, z_3) \circ \ldots \circ (z_{n-1}, z_n) \circ (z_n, z_{n+1} = y),$$

where $(z_i, z_{i+1}) \in U_{k_i}$ for $i \leq n$. Hence

$$\rho(x, y) \leq \sum_{i=1}^{n} \frac{1}{2k_i^2} \leq \sum_{j=k_1}^{\infty} \frac{j}{2j^2} \leq \sum_{j=k_n}^{\infty} \frac{2^j - 1}{2j^2} = \frac{1}{2k_n^2} \leq \frac{1}{2k_n^2 - 1}.$$ 

Therefore, $(x, y) \in V_{(\min \{k_i\})^{-2}} = U_{(\min \{k_i\})^{-1}}$.

Let $X$ be a topological space whose topology is generated by a uniformity $\mathcal{U}$ having a countable base $\{V_n\}$ consisting of clopen sets. Take a sequence $\mathcal{U}, \mathcal{V}_1, \ldots$ of clopen entourages such that $\mathcal{V}_0 = X \times X$, $\mathcal{V}_1 = \mathcal{U}$, and $\mathcal{V}_i \circ \mathcal{V}_i \circ \mathcal{V}_i \subseteq \mathcal{V}_{i-1} \cap \mathcal{V}_{i+1}$ for $i = 2, 3, \ldots$. We set $\mathcal{U}_i = \mathcal{V}_i \rho$ for $i \in \omega$. The sequence $\mathcal{U} = \{\mathcal{U}_i\}$ is a base of the uniformity $\mathcal{U}$, and the sets

$$W_n(\mathcal{U}) = \bigcup_{k \in \omega} \{\sum_{i=1}^{k} (x_i - y_i) : (x_i, y_i) \in \mathcal{U}_{n-1}\}$$

form a neighborhood base at zero for some group topology $\mathcal{T}_\mathcal{U}$ on the free Abelian group $A(X)$ which induces the initial topology (generated by the uniformity $\mathcal{U}$) on $X$. Indeed, it is easy to show that $2W_{2n}(\mathcal{U}) \subseteq W_n(\mathcal{U})$ for $n \geq 1$ and that if $g = \sum_{i=1}^{k} (x_i - y_i) \in W_n(\mathcal{U})$, then $g + W_{n(k+1)}(\mathcal{U}) \subseteq W_n(\mathcal{U})$; in addition, all sets $W_n(\mathcal{U})$ are symmetric and contain the empty word (the zero of the group
On the other hand, clearly, $S$ (here $\omega$)

Sometimes, when it is clear what decomposition of $g$

$\sum_{i=1}^{k} \pi_{i}$.

**Definition 1.2.** Suppose that $\omega_{i} = 1$

$(x + W_{n}(U)) \cap X = \{ y \in X : (y, x) \in \bigcup \{ U_{n, \pi(1)} \circ \cdots \circ U_{n, \pi(k)} : k \geq 1, \pi \in S_{k} \} \}$

(here $S_{k}$ is the permutation group on $\{1, \ldots, k\}$). By Lemma 1.1,

$(x + W_{n}(U)) \cap X \subset \{ y \in X : (y, x) \in U_{n-1} \}$.

On the other hand, clearly,

$(x + W_{n}(U)) \cap X \supset \{ y \in X : (y, x) \in U_{n} \}$.

Our immediate goal is to construct a base of the topology $T_{n}$ on $A(X)$ consisting of open-and-closed (in this topology) sets.

**Definition 1.1.** For $x, y \in X$, we set

$$d(x, y) = \begin{cases} \frac{1}{\max \{ k : (x, y) \in U_{k} \} } & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Thus, for $x \neq y$, the number $d(x, y)$ is uniquely determined by the conditions $(x, y) \in U_{\frac{1}{d(x, y)}}$ and $(x, y) \notin U_{\frac{1}{d(x, y)}}+1$.

**Definition 1.2.** Suppose that $k \in \omega$, $x_{i}, y_{i} \in X$ for $i \leq k$, and

$$g = \sum_{i=1}^{k} (x_{i} - y_{i}) \in A(X).$$

We say that the sum (decomposition) $\sum_{i=1}^{k} (x_{i} - y_{i})$ satisfies condition $(\ast)$ if

$$d(x_{i}, y_{j}) \geq \min \{ d(x_{i}, y_{i}), d(x_{j}, y_{j}) \} \text{ for any } i, j \leq k. \tag{\ast}$$

Sometimes, when it is clear what decomposition of $g$ is meant, we say the word $g$ itself satisfies condition $(\ast)$ (meaning that condition $(\ast)$ holds for the decomposition).

**Remark 1.1.** Suppose that $d(x, y) \leq d(x, y_{i})$ and $d(x, y) \leq d(x, y)$ for all $i \leq k$. Then $\sum_{i=1}^{k} (x_{i} - y_{i})$ satisfies condition $(\ast)$ if and only if $\sum_{i=1}^{k} (x_{i} - y_{i}) + (x - y)$ satisfies condition $(\ast)$. Moreover, if $\sum_{i=1}^{k} (x_{i} - y_{i})$ satisfies condition $(\ast)$, then $\sum_{i \in I} (x_{i} - y_{i})$ satisfies condition $(\ast)$ for any $I \subset \{1, \ldots, k\}$.

**Lemma 1.2.** Suppose that

1. $g = \sum_{i=1}^{k} (x_{i} - y_{i})$;
2. $(x_{i}, y_{i}) = (x_{i} = x_{i}^{(1)}, y_{i}^{(1)}) \circ (y_{i}^{(1)} = x_{i}^{(2)}, y_{i}^{(2)}) \circ \cdots \circ (y_{i}^{(k_{i}-1)} = x_{i}^{(k_{i})}, y_{i}^{(k_{i})} = y_{i})$, i.e., $x_{i} - y_{i} = \sum_{j=1}^{k_{i}} (x_{i}^{(j)} - y_{i}^{(j)})$ for each $i \leq k$;
3. $(x_{i}^{(j)}, y_{i}^{(j)}) \in U_{N, n_{i}^{(j)}}$ for all $i \leq k$ and $j \leq k_{i}$;
4. if $m \leq k$, then $n_{i}^{(j)} = m$ for at most one pair $i, j$;
5. if $m > k$, then $n_{i}^{(j)} = m$ for at most $m - k + 1$ pairs $i, j$;
6. $k > 1$.

Then $g = \sum_{i=1}^{k} (x_{i}^{(j)} - y_{i}^{(j)}) + x'' - y''$, where each of the letters $x^{(j)}$, $y^{(j)}$, $x''$ and $y''$ is contained in one of the decompositions from (2) and $\sum_{i=1}^{k} (x_{i}^{(j)} - y_{i}^{(j)})$ satisfies conditions (2)–(5) with $x_{i}$ replaced by $x^{(j)}$, $x_{i}^{(j)}$ by $x_{i}^{(j)}$, $y_{i}$ by $y^{(j)}$, $y_{i}^{(j)}$ by $y^{(j)}$, $k$ by $k - 1$, $k_{i}$ by $k_{i}'$, and $n_{i}^{(j)}$ by $n'_{i}^{(j)}$; moreover,

7. $d(x'', y'') \leq d(x'', y^{(j)})$ and $d(x'', y'') \leq d(x^{(j)}, y'')$ for all $i \leq k - 1$ and $j \leq k_{i}';$
8. $(x'', y'') \in U_{N, k-1}$. 

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Proof. Take any pair \((u, v)\) for which \(u \in \{x_1, \ldots, x_k\}, v \in \{y_1, \ldots, y_k\}\) and \(d(u, v)\) is minimal; if there exists a pair of the form \((x_i, y_i)\) with these properties, then let \((u, v)\) be such a pair. By condition (4), there exists an \(i \leq k\) for which \(\min_{j \leq k} n_i^{(j)} \geq k\). Conditions (2)–(5) and Lemma 1.1 imply that \((x_i, y_i) \in \mathcal{U}_{N, k-1}\) for this \(i\), i.e., \(d(x_i, y_i) \leq \frac{1}{N_{k-1}}\). Therefore, \(d(u, v) \leq \frac{1}{N_{k-1}}\) (by virtue of minimality), i.e., \((u, v) \in \mathcal{U}_{N, k-1}\). If \((u, v) = (x_i, y_i)\) for some \(i \leq k\), then the required decomposition consists of the term \(x_i - y_i\) and the sum of all other terms in the initial decomposition of the word \(g\); in other words, it suffices to set \(x'_j = x_j\) and \(y'_j = y_j\) for \(j < i\), \(x'_j = x_{j+1}\) and \(y'_j = y_{j+1}\) for \(j = i, \ldots, k-1\), \(x''_n = x_i\), and \(y''_n = y_i\). The decompositions from (2) remain the same for all \(x'_j - y'_j\).

If \(u = x_i, v = y_j\), and \(i \neq j\), i.e., the function \(d\) does not attain its minimum for pairs of the form \((x_r, y_r)\), then \(d(u, v) < \frac{1}{N_{k-1}}\), because, as mentioned above, \(d(x_s, y_s) \leq \frac{1}{N_{k-1}}\) for some \(s\). Therefore, \(d(u, v) \leq \frac{1}{N_{k-1}}\). Without loss of generality, we can assume that \(i < j\). We set \(x'_r = x_r\) and \(y'_r = y_r\) for \(r < j\) such that \(r \neq i\), \(x'_i = x_j\), \(y'_i = y_i\), \(x'_r = x_{r+1}\) and \(y'_r = y_{r+1}\) for \(r = j, \ldots, k-1\), \(x''_n = x_i\), and \(y''_n = y_j\); in other words, we replace the pairs \((x_i, y_i)\) and \((x_j, y_j)\) by \((x_i, y_j)\) and \((x_i, y_i)\). The fulfillment of condition (7) follows from the choice of the pair \((u, v)\), and (8) holds because \(d(u, v) < \frac{1}{N_{k-1}}\).

The decompositions from (2) and numbers of the form \(n^{(t)}_i\) remain the same for the pairs \((x'_r, y'_r) = (x_r, y_r)\) with \(r \neq i\), which coincide with \((x_r, y_r)\) or \((x_{r+1}, y_{r+1})\); for \((x'_i, y'_i) = (x_j, y_i)\), we set \(k'_i = k_i + k_j + 1\) and take the decomposition

\[
x'_i - y'_i = x_j - y_i = x_j - y_j + y_j - x_i + x_i - y_i = \sum_{r=1}^{k_i} (x_j^{(r)} - y_j^{(r)}) + (y_j - x_i) + \sum_{s=1}^{k_j} (x_i^{(s)} - y_i^{(s)});
\]

thus, we set \(x'^{(t)}_i = x^{(t)}_i \) and \(y'^{(t)}_i = y^{(t)}_i\) for \(t \leq k_j\), \(x^{(k_j)}_i = y_j\), \(y^{(k_j)}_i = x_i\), \(x'^{(t)}_i = x^{(t-k_j-1)}_i\), and \(y'^{(t)}_i = y^{(t-k_j-1)}_i\) for \(t = k_j + 2, \ldots, k_j + k_i + 1\). As mentioned above, \((u, v) = (y_j, x_i) \in \mathcal{U}_{N, k}\).

Therefore, setting \(n^{(t)}_i = n^{(t)}_j\) for \(t \leq k_j\) and \(n_i^{(k_j+1)} = k\) and \(n_i^{(t-k_j-1)} = n_i^{(t-k_j-1)}\) for \(t = k_j + 2, \ldots, k_j + k_i + 1\), we obtain \((x'^{(t)}_i, y'^{(t)}_i) \in \mathcal{U}_{N, n^{(t)}_i}\) for all \(t \leq k_i\). The term \(x^{(k_j+1)}_i - y^{(k_j+1)}_i = y_j - x_i\) is the only new element in the sum

\[
\sum_{r=1}^{k-1} \sum_{s=1}^{k_i} (x^{(r)}_r - y^{(s)}_r) = \sum_{r=1}^{k-1} (x'_r - y'_r)
\]

in comparison with the sum

\[
\sum_{r=1}^{k} \sum_{s=1}^{k_r} (x^{(s)}_r - y^{(s)}_r) = \sum_{r=1}^{k} (x_r - y_r),
\]

and we have \(n^{(k_j+1)}_i = k > k - 1\) for this element; the numbers of the form \(n^{(t)}_i\) corresponding to the other terms are equal to the numbers corresponding to them as terms of the sum \(\sum_{r=1}^{k} \sum_{s=1}^{k_r} (x^{(s)}_r - y^{(s)}_r)\). Therefore, \(\sum_{r=1}^{k-1} (x'_r - y'_r) = \sum_{r=1}^{k-1} \sum_{s=1}^{k'_r} (x^{(s)}_r - y^{(s)}_r)\) satisfies condition (4) with \(k\) replaced by \(k - 1\) and \(n_i^{(j)}\) by \(n_i^{(j)}\); it also satisfies the part of condition (5) (with the appropriate replacements) that relates to the number of \(n^{(t)}_r > k\). By condition (4), the sum \(\sum_{r=1}^{k} \sum_{s=1}^{k_r} (x^{(s)}_r - y^{(s)}_r)\) contains at most one term for which \(n^{(s)}_r = k\). Therefore, the sum \(\sum_{r=1}^{k-1} \sum_{s=1}^{k'_r} (x^{(s)}_r - y^{(s)}_r)\) contains at most two terms for which \(n^{(s)}_r = k\); thus, condition (5) with \(k\) replaced by \(k - 1\) and \(n_i^{(j)}\) by \(n_i^{(j)}\) is satisfied fully. Conditions (2) and (3) with the appropriate replacements hold by construction. \(\square\)
**Corollary 1.1.** If $k, n \in \omega$, $g = \sum_{i=1}^{k} (x_i - y_i)$, and $(x_i, y_i) \in U_{(n+1)i}$ for all $i \leq k$, then $g = \sum_{i=1}^{k} (\tilde{x}_i - \tilde{y}_i)$, where $(\tilde{x}_i, \tilde{y}_i) \in U_{ni}$ for all $i \leq k$ and the decomposition $\sum_{i=1}^{k} (\tilde{x}_i - \tilde{y}_i)$ satisfies condition $(\ast)$.

**Proof.** This assertion is proved by repeatedly applying Lemma 1.2 with $N = n + 1$ to the word $g$. If $k \leq 1$, then the assertion holds trivially. If $k > 1$, then we can apply Lemma 1.2 with $N = n + 1$ and obtain a decomposition

$$g = \sum_{i=1}^{k-1} (x'_i - y'_i) + x'' - y''$$

with the properties described in the lemma. We have $(x'', y'') \in U_{N-k-1} \subset U_{n-k}$ and, for each $i \leq k-1$, $x'_i - y'_i = \sum_{j=1}^{i} (x''^{(j)} - y''^{(j)})$, where $(x''^{(j)} , y''^{(j)}) \in U_{(n+1)j}$; moreover, if $m > k-1$, then $n''^{(j)} = m$ for at most $m - k + 2$ pairs $i, j$, and if $m < k-1$, then $n''^{(j)} = m$ for at most one pair $i, j$. We apply Lemma 1.2 first to the sum $\sum_{i=1}^{k-1} (x'_i - y'_i)$, then to the obtained decomposition, then to the new decomposition, and so on, while possible; in the end (after $k - 1$ steps), we obtain a decomposition

$$g = \sum_{j=1}^{\tilde{k}'_i} (\tilde{x}'^{(j)} - \tilde{y}'^{(j)}) + \sum (\tilde{x}'' - \tilde{y}''),$$

where $\sum_{j=1}^{\tilde{k}'_i} (\tilde{x}'^{(j)} - \tilde{y}'^{(j)}) = \tilde{x} - \tilde{y}$ for some $\tilde{x}, \tilde{y} \in X$ and $\sum (\tilde{x}'' - \tilde{y}'')$ denotes the sum of the residual terms of the form $x'' - y''$ obtained at each step. The pairs of letters in each residual term belongs to the entourage $U_{N-(k+s-1)} \subset U_{n-(k+s-1)}$, where $s < k$ is the number of the step at which this term has appeared (and $N = n + 1$). Moreover,

$$(\tilde{x}, \tilde{y}) = (\tilde{x}'^{(1)}, \tilde{y}'^{(1)}) \circ \cdots \circ (\tilde{x}'^{(\tilde{k}'_i)}, \tilde{y}'^{(\tilde{k}'_i)}) \in U_{N-n''^{(1)}} \circ \cdots \circ U_{N-n''^{(\tilde{k}'_i)}},$$

and, for $m = 1, 2, \ldots, n'_i(j) = m$ for at most $m$ indices $j$. Therefore, by Lemma 1.1

$$(\tilde{x}, \tilde{y}) \in U_{N-minj<\tilde{k}'_i{n''^{(j)}}-1} \subset U_{N-1} = U_n.$$

Condition (7) from lemma 1.2 and Remark 1.3 as well as the fact that no new letters appear in repeatedly applying Lemma 1.2 ensure the fulfillment of condition $(\ast)$. \hfill \Box

**Lemma 1.3.** Suppose that $I = \{k_1, \ldots, k_l\}$ is a finite set of different positive integers enumerated in increasing order, $g = \sum_{i=1}^{k} (x_i - y_i)$, $h = \sum_{j=1}^{l} (u_j - v_j)$, the decompositions $\sum_{i=1}^{k} (x_i - y_i)$ and $\sum_{j=1}^{l} (u_j - v_j)$ satisfy condition $(\ast)$, $(x_i, y_i) \in U_{n_i}$ for $i \leq k$, and $(u_j, v_j) \in U_{k_j}$ for $j \leq l$; suppose also that if $i, j \leq k$ and $F = (f_1, \ldots, f_r)$, $F' = (f'_1, \ldots, f'_{r'})$ are finite ordered sequences of elements of $I \cup \{k_1 + 1, k_1 + 2, \ldots\}$ in each of which every element of $I$ occurs at most once and every positive integer $s > k_1$ occurs at most $s$ times, then

1. $U_{f_1} \circ \cdots \circ U_{f_r} \circ (x_i, y_i) \circ U_{f'_1} \circ \cdots \circ U_{f'_{r'}} \subset U_{n_{2i} + 1}$ and
2. $U_{f_1} \circ \cdots \circ U_{f_r} \circ (x_i, y_i) \circ U_{f'_1} \circ \cdots \circ U_{f'_{r'}} \cap U_{n_{2i} + 1} = \emptyset$.

Then $g + h = \sum_{i=1}^{m} (z_i - w_i)$, where $m \leq k + l$, the decomposition $\sum_{i=1}^{m} (z_i - w_i)$ satisfies condition $(\ast)$, $z_i \in \{x_1, \ldots, x_k, u_1, \ldots, u_l\}$, $w_i \in \{y_1, \ldots, y_k, v_1, \ldots, v_l\}$, $(z_i, w_i) \in U_{n_i}$ for $i \leq k$, and $(z_{k+i}, w_{k+i}) \in U_{n_{k+i}}$ for $i \leq m - k$ (if $m > k$).

**Proof.** First, note that (2) implies $d(x_i, y_j) > \frac{1}{k_1} = \frac{1}{k_2}$ for any $i, j \leq k$ and $s \leq l$. Indeed, otherwise, $U_{k_1} \circ (x_i, y_j) \ni (y_j, y_j)$; clearly, $(y_j, y_j) \in U_{n_{2i} + 1}$, while by condition (2), $U_{k_1} \circ (x_i, y_j) \cap U_{n_{2i} + 1} = \emptyset$ (consider $F = \{k_1\}$ and $F' = \emptyset$). This implies, in particular, that $k_1 > n \cdot k$.

We shall prove the lemma by induction on $l$. If $l = 0$ (i.e., the word $h$ is empty), then the assertion holds trivially. Suppose that $l > 0$ and the assertion is true for smaller $l$. Choose
$h' \in \{x_1, \ldots, x_{l-1}, u_1, \ldots, u_t\}$ and $h'' \in \{y_1, \ldots, y_{k-1}, v_1, \ldots, v_t\}$ for which $d(h', h'')$ is minimal. Since $h$ is nonempty, we have

(i) $d(h', h'') \leq \frac{1}{k}$ (because $d(u_i, v_i) \leq \frac{1}{k}$) and $d(h', h'')$ is minimal and

(ii) either $h' \in \{u_1, \ldots, u_t\}$ or $h'' \in \{v_1, \ldots, v_t\}$ (this follows from (i) and because, by condition (2), $d(x_i, y_j) > \frac{1}{k}$ for all $i, j \leq k$); moreover, we can assume that if $h' = u_i$ and $h'' = v_j$, then $i = j$; otherwise, we replace the pair $h', h''$ by the pair $u_i, v_i$ or $u_j, v_j$ for which the value of $d$ does not exceed $d(h', h'')$ (such a pair exists because the decomposition $\sum_{i=1}^l (u_i - v_i)$ satisfies condition (\$)).

If $h' = u_i$ and $h'' = v_i$ for some $i \leq l$, then we set $\tilde{u}_j = u_j$ and $\tilde{v}_j = v_j$ for $j < i$, $\tilde{u}_j = u_{j+1}$ and $\tilde{v}_j = v_{j+1}$ for $j = i, \ldots, l-1$, $\tilde{h} = \sum_{i=1}^{l-1} (\tilde{u}_i - \tilde{v}_i)$, and $\tilde{I} = \{k_1, \ldots, k_{l-1}\}$. Note that the conditions of the lemma hold for $\tilde{I}$, $g = \sum_{i=1}^k (x_i - y_i)$, and $\tilde{h}$. By the induction assumption, $g + \tilde{h} = \sum_{i=1}^{m_i} (\tilde{z}_i - \tilde{w}_i)$, where $m_i \leq k + l - 1$, the decomposition $\sum_{i=1}^{m_i} (\tilde{z}_i - \tilde{w}_i)$ satisfies condition (\$), $\tilde{z}_i \in \{x_1, \ldots, x_k, u_1, \ldots, u_t\}$, $\tilde{w}_i \in \{y_1, \ldots, y_k, v_1, \ldots, v_t\}$ for all $i \leq k$, and $\tilde{z}_i, \tilde{w}_i \in U_{k_i}$ for $i \leq k$. By the definition of the pair $h' = u_i$, $h'' = v_i$ and Remark [\[i\] 1], the decomposition $g + \tilde{h} = \sum_{i=1}^{m_i} (\tilde{z}_i - \tilde{w}_i) + (h' - h'')$ has the required properties (recall that $d(h', h'') \leq \frac{1}{k} < \frac{1}{n^{-1}}$).

Suppose that $h'$ and $h''$ cannot be chosen among the letters of the form $u_i$ and $v_j$, i.e., either $h' = x_i$ and $h'' = v_j$ for some $i \leq k$ and $j \leq l$ and $d(x_i, v_j) < d(u_s, v_s)$ for all $r, s \leq l$ (i.e., $d(x_i, v_j) < \frac{1}{k}$) or $h' = u_i$ and $h'' = y_j$ for some $i \leq k$ and $j \leq l$ and $d(u_i, y_j) < d(u_r, v_s)$ for all $r, s \leq l$ (i.e., $d(u_i, y_j) < \frac{1}{k}$). For definiteness, suppose that $h' = x_i$ and $h'' = v_j$. We have $(u_j, x_i) \circ (v_j, x_i) \in U_{k_j} \circ U_{k_i}$, and conditions (1) and (2) imply $d(u_j, y_j) = d(x_i, y_j)$ for all $r \leq k$. We set $\tilde{x}_i = u_j$, $\tilde{s} = s$, and $\tilde{y}_s = v_j$ for all $s \leq k$; thus, the word $\sum_{s=1}^l (\tilde{s}_s - \tilde{y}_s)$ differs from $\sum_{s=1}^l (x_s - y_s)$ in one letter $\tilde{x_i}$, and $d(\tilde{s}_s, \tilde{y}_s) = d(x_s, y_s)$ for all $s, t \leq k$ (this means that $\sum_{s=1}^l (\tilde{s}_s, \tilde{y}_s)$ satisfies condition (\$) and $(\tilde{s}_s, \tilde{y}_s) \in U_{k_j}$ for $s \leq k$). We also set $\tilde{u}_s = \tilde{u}_s$ and $\tilde{v}_s = \tilde{v}_s$ for all $s \leq l$, $\tilde{u}_s = \tilde{u}_{s+1}$ and $\tilde{v}_s = \tilde{v}_{s+1}$ for all $s \leq l$, $\tilde{u}_l = x_i$, and $\tilde{v}_l = v_j$; thus, the word $\sum_{s=1}^l (\tilde{u}_s - \tilde{v}_s)$ is obtained from $\sum_{s=1}^l (u_s - v_s)$ by deleting the term $u_j - v_j$ and inserting $\tilde{u}_j - \tilde{v}_j = x_i - v_j$. Since $\tilde{u}_i = \tilde{v}_i$, $\tilde{v}_i = \tilde{v}_j$, and $d(x_i, v_j)$ is minimal, it follows that $d(\tilde{u}_i, \tilde{v}_j) \leq d(\tilde{u}_i, \tilde{v}_j)$ and $d(\tilde{u}_j, \tilde{v}_j) \leq d(\tilde{u}_j, \tilde{v}_j)$ for all $r \leq k$. Therefore, the word $\sum_{s=1}^l (\tilde{u}_s - \tilde{v}_s)$ satisfies condition (\$). Indeed, the word $h'$ satisfies condition (\$); according to Remark [\[i\] 1], deleting the term $u_j - v_j$ does not violate condition (\$); applying Remark [\[i\] 1] again with taking into account the minimality of $d(\tilde{u}_i, \tilde{v}_j)$, we conclude that $\sum_{s=1}^l (\tilde{u}_s - \tilde{v}_s)$ satisfies condition (\$). We set $\tilde{k}_s = k_s$ for $s = 1, \ldots, j-1$, $\tilde{k}_j = k_j + 1$, and $\tilde{k}_j = k_j + 1$, and

$$\tilde{I} = \{k_1, \ldots, k_{l-1}\} \cup \{k_{l-1} + 1\}.$$ 

We have $(\tilde{u}_s, \tilde{v}_s) \in U_{k_s}$ for all $s \leq l$. Finally, $\tilde{k}_l = k_l + 1$ and $\tilde{I}$ does not contain $k_j$; therefore, if $F = (f_1, \ldots, f_r)$ is a finite ordered sequence of elements of the set $\tilde{I} \cup \{k_l + 1, k_l + 2, \ldots\}$ with the properties (a) each element from $\tilde{I}$ occurs in $F$ at most once and (b) each element $s$ larger than all elements of $\tilde{I}$ occurs at most $s$ times, then the sequences $F$ and $(f_1, \ldots, f_r, k_j, k_{j+1})$ have the same properties with respect to the set $I$. This observation, conditions (1) and (2) of the lemma being proved, and the relations

$$(\tilde{x}_i, \tilde{y}_t) = (u_j, v_j) = (u_j, v_j) \circ (v_j, x_i) \circ (x_i, y_t) \in U_{k_j} \circ U_{k_l}$$

and $(\tilde{s}_s, \tilde{y}_t) = (x_s, y_t)$ for $s \neq i$ and any $t$ imply that, for any $s, t \leq k$ and any two finite ordered sequences $(f_1, \ldots, f_k)$ and $(f'_1, \ldots, f'_r)$ of elements of the set $\tilde{I} \cup \{k_j + 1, k_j + 2, \ldots\}$ in each of which every element of $\tilde{I}$ occurs at most once and every element $s > \tilde{k}_l$ occurs at most $s$ times, we have

$$(1) \quad U_{f_1} \circ \cdots \circ U_{f_r} \circ (\tilde{x}_s, \tilde{y}_t) \circ U_{f'_1} \circ \cdots \circ U_{f'_r} \subset U_{\frac{1}{d(\tilde{x}_s, \tilde{y}_t)}}$$

and

$$(2) \quad U_{f_1} \circ \cdots \circ U_{f_r} \circ (\tilde{x}_s, \tilde{y}_t) \circ U_{f'_1} \circ \cdots \circ U_{f'_r} \cap U_{\frac{1}{d(\tilde{x}_s, \tilde{y}_t)}} + 1 = \emptyset.$$
Thus, the set $I$ and the words $\sum_{s=1}^{k}(\tilde{x_s} - \tilde{y_s})$ and $\sum_{s=1}^{l}(\tilde{u_s} - \tilde{v_s})$ satisfy the conditions of the lemma. Moreover, the set of letters (with signs) of which these words consist coincides with the set of letters in the words $\sum_{s=1}^{k}(x_s - y_s)$ and $\sum_{s=1}^{l}(u_s - v_s)$; therefore, the function $d$ takes minimal value at the same pair of letters $(h', h'') = (x_j, v_j) = (\tilde{u}_j, \tilde{v}_j)$. However, these letters form a summand in the decomposition $\sum_{s=1}^{l}(\tilde{u}_s - \tilde{v}_s)$; this situation was considered at the beginning of the proof. As there, we delete this summand, apply the induction assumption, and insert the deleted summand back; as a result, we obtain a representation $g + h = \sum_{s=1}^{m}(\tilde{z}_s - \tilde{w}_s) + (h' - h'')$, where $m > k + l - 1$, $(\tilde{z}_s, \tilde{w}_s) \in U_{i}$ for $i \leq k$, $(\tilde{z}_{k+i}, \tilde{w}_{k+i}) \in U_{k+i}$ for $i \leq m - k$ (if $m > k$), and $(h', h'') \in U_{ki}$. Since $k_i = k_l + 1 \geq k_l$ and $k_i \geq k_i$ for all $i \leq l - 1$, this representation is as required. 

**Corollary 1.2.** Suppose that $g = \sum_{i=1}^{k}(x_i - y_i)$, $h = \sum_{i=1}^{l}(u_i - v_i)$, the decompositions $\sum_{i=1}^{k}(x_i - y_i)$ and $\sum_{i=1}^{l}(u_i - v_i)$ satisfy condition (*), $(x_i, y_i) \in U_{n_i}$ for $i \leq k$, $(u_i, v_i) \in U_{(N+1) - i}$ for $i \leq l$, $N \geq 2n \cdot k$, and, for any $i, j \leq k$, 

1. $U_{n} \circ (x_i, y_j) \cap U_{\frac{1}{X}} = \emptyset$
2. $U_{N} \cap \mathbb{U}_{(\frac{1}{X} - 1)} = \emptyset$

Then $g + h = \sum_{i=1}^{m}(z_i - w_i)$, where the decomposition $\sum_{i=1}^{m}(z_i - w_i)$ satisfies condition (*), $z_i \in \{x_1, \ldots, x_k, l_1, \ldots, u_1\}$, $w_i \in \{y_1, \ldots, y_k, l_1, \ldots, u_1\}$, and $(z_i, w_i) \in U_{n_i}$ for $i \leq m$.

**Proof.** This assertion follows immediately from Lemmas 1.3 and 1.4. 

**Lemma 1.4.** Suppose that $n \geq 2$ and 

1. $\sum_{i=1}^{n}(z_i - w_i)$ satisfies condition (*);
2. $\sum_{i=1}^{n-1}(z_i - w_i)$ satisfies condition (*);
3. $(z_{i+1}, w_i) \in U_{k_i}$ for all $i \leq n - 1$, and the $k_i$ are different positive integers larger than 1;
4. $d(z_1, w_1) \geq d(z_i, w_i)$ and $d(z_n, w_n) \geq d(z_i, w_i)$ for $i = 2, \ldots, n - 2$;
5. $k_n = \min\{k_1, \ldots, k_{n-1}\}$;
6. either $d(z_1, w_1) \geq d(z_n, w_n)$ or $d(z_n, w_n) \geq d(z_1, w_1)$ (the last condition is included for convenience). Then there exists a one-to-one map 

$$f : \{2, \ldots, n - 1\} \rightarrow \{k_1, \ldots, k_{n-1}\} \setminus \{k_m\}$$

such that $d(z_i, w_i) \leq \frac{1}{m-1}$ for $i = 2, \ldots, n - 1$ and $d(z_n, w_n) \leq \frac{1}{m-1}$ (if (6left) holds) or $d(z_1, w_1) \leq \frac{1}{m-1}$ (if (6right) holds).

**Proof.** We prove the lemma by induction. For $n = 2$, the map $f$ is trivial, and $d(w_1, z_2) \geq \min\{d(w_1, z_1), d(w_2, z_2)\}$ by condition (*). This implies the required assertion, because it follows from (3) and (5) that $d(w_1, z_2) \geq \frac{1}{m}$. Suppose that $n > 2$ and the assertion is true for smaller $n$. Let $m_1, \ldots, m_r$ be the indices (or index) from $\{2, \ldots, n - 1\}$ for which the numbers $d(z_{m_j}, w_{m_j})$ are maximal (and equal to each other). These indices divide the set of all indices into intervals. Suppose that $m$ belongs to the $s$th interval, i.e., $k_s$ is minimal for $i \in \{m_s, m_s + 1, \ldots, m_{s+1} - 1\}$, where $s = 0, \ldots, r$ (we assume that $m_0 = 1$ and $m_r + 1 = n$). Suppose that $s > 0$; for $s = 0$, the argument is the same except that we must replace the conditions $j < s$ and $j > s$ by $j \leq s$ and $j > s$ (that is, by $j = 0$ and $j > 0$), respectively, every time they are encountered. Consider the words $z_{m_j} - w_{m_j} + z_{m_{j+1}} - w_{m_{j+1}} + \cdots + z_{m_{j+1} - 1} - w_{m_{j+1} - 1} + z_{m_{j+1}} - w_{m_{j+1}}$. They satisfy condition (*), being subsums of a sum satisfying condition (*), and to these words the induction hypothesis applies. Using the left version of the lemma for $j < s$ and the right version for $j > s$ (recall that, for $s = 0$, the condition $j < s$ should be replaced by $j = 0$ and the condition $j > s$ by $j > 0$; in the situation under consideration, this means that if $s = 0$, then the left version should be applied to $j = 0$ and the right version to $j > 0$), we obtain one-to-one maps 

$$f_j : \{m_j + 1, \ldots, m_{j+1} - 1\} \rightarrow \{k_{m_j}, \ldots, k_{m_{j+1} - 1}\} \setminus \{k_{m_j}\}$$
such that \( d(z_{m_j+i}, w_{m_j+i}) \leq \frac{1}{f_j(m_j+i)-1} \) for \( i = 1, \ldots, m_j+1 - m_j - 1 \) and \( d(z_{m_j+1}, w_{m_j+1}) \leq \frac{1}{k_{m_j+1} - 1} \) (if \( j < s \)) or \( d(z_{m_j}, w_{m_j}) \leq \frac{1}{k_{m_j} - 1} \) (if \( j \geq s \)); here \( k_{m_j} \) is the least number among \( k_{m_1}, \ldots, k_{m_j+1} - 1 \).

We set
\[
\begin{align*}
  f|\{z_1, \ldots, z_{m-1}\} &= f_1, & f(m_1) &= k_{m_0}, \\
  f|\{z_{m_1}, \ldots, z_{m_2} - 1\} &= f_2, & f(m_2) &= k_{m_1}, \\
  \vdots & & \\
  f|\{z_{m_{s-1} + 1}, \ldots, z_{m_s - 1}\} &= f_s, & f(m_s) &= k_{m_{s-1}}, \\
  f|\{z_{m_s + 1}, \ldots, z_{m_{s+1} - 1}\} &= f_s, & f(m_{s+1}) &= k_{m_{s+1}}, \\
  f|\{z_{m_{s+1} + 1}, \ldots, z_{m_{s+2} - 1}\} &= f_{s+1}, & f(m_{s+2}) &= k_{m_{s+2}}, \\
  \vdots & & \\
  f|\{z_{m_{r-1} + 1}, \ldots, z_{m_r - 1}\} &= f_r, & f(m_r) &= k_{m_r}, \\
  f|\{z_{m_r + 1}, \ldots, z_{m_{r+1} - 1}\} &= f_{r+1}.
\end{align*}
\]

For \( i \leq n - 1 \), we have \((z_i, w_i) \in U_{k_i}\) (by assumption), \((z_i, w_i) \in U_{f(i)-1}\) (by construction), the \( k_i \) are different numbers larger than 1, the \( f(i) \) are different numbers of the form \( k_j \), \( f(i) > k_m \) for all \( i \), and \( k_i > k_m \) for \( i \neq m \). By Lemma 1.4

\[
(w_1, w_m) = (w_1, z_2) \circ (z_2, w_2) \circ (w_2, z_3) \circ \cdots \circ (w_{m-1}, z_m) \circ (z_m, w_m)
\]

\[
\in U_{k_1} \circ U_{f(2)} \circ U_{k_2} \circ \cdots \circ U_{k_{m-1}} \circ U_{f(m)} \subset U_{k_m}
\]

and, similarly,

\[
(z_{m+1}, z_n) \in U_{k_m};
\]

moreover, by assumption, we have

\[
(z_{m+1}, w_m) \in U_{k_m}.
\]

Therefore,

\[
(w_1, z_n) \in 3U_{k_m} = 3V_{k_m} \subset V_{k_m-1} \subset V_{k_m-1}^2 = U_{k_m-1}.
\]

The word \((z_1 - w_1) + (z_n - w_n)\) satisfies condition (*), because the word \(\sum_{i=1}^{n}(z_i - w_i)\) satisfies this condition by assumption (see Remark 1.4); hence \(d(z_m, w_n) \leq \frac{1}{k_{m-1}}\) (if (6left) holds) or \(d(z_1, w_1) \leq \frac{1}{k_{m-1}}\) (if (6right) holds).

\[\square\]

**Corollary 1.3.** Suppose that, for \( j \leq k \), \(\sum_{i=1}^{n_j}(z_i^{(j)} - w_i^{(j)})\) and \(\sum_{i=1}^{n_j-1}(z_i^{(j)} - w_i^{(j)})\) are words satisfying condition (*), \(x_j = z_1^{(j)}\), and \(y_j = w_1^{(j)}\). Suppose also that \(U_N \circ (x_r, y_s) \circ U_N \subset U_{1/d(x_r, y_s)}\)
and \(U_N \circ (x_r, y_s) \circ U_N \cap U_{1/d(x_r, y_s)} = \emptyset\) for all \(r, s \leq k\). Finally, suppose that \((z_i^{(j)}, w_i^{(j)}) \in U_{k_i^{(j)}}\) for any \(j \leq k\) and \(i \leq n_i - 1\), where the \(k_i^{(j)}\) are different positive integers larger than \(N + 2\) for each \(j\). Then, for any \(j \leq k\), there exists an \(n_j^{(j)} \leq n_j\) such that \(d(x_r, y_s) = d(z_{n_j^{(r)}}, w_{n_j^{(s)}})\) for all \(r, s \leq k\).

**Proof.** Take \(j \leq k\) and consider the word \(\sum_{i=1}^{n_j}(z_i^{(j)} - w_i^{(j)})\); for convenience, we omit the index \(j\).

Take some \(n_0 \leq n\) for which \(d(z_{n_0}, w_{n_0})\) is maximal among all \(d(z_i, w_i)\) with \(1 \leq i \leq n\). Suppose for definiteness that \(n_0 < n\); if \(n_0 > n\), then the left-to-right argument described below should be replaced by a similar right-to-left argument. Let \(n_1 > n_0\) be the minimum number for which \(d(z_{n_1}, w_{n_1})\) is largest among all \(d(z_i, w_i)\) with \(i = n_0 + 1, \ldots, n\), and let \(m_1\) be such that \(k_{m_1}\) is minimal among all \(k_i\) with \(i = n_0, \ldots, n_1 - 1\). Applying Lemma 1.4 to the word

\[
z_{n_0} - w_{n_0} + z_{n_0+1} - w_{n_0+1} + \cdots + z_{n_1} - w_{n_1},
\]

we obtain a one-to-one map

\[
f_1: \{n_0 + 1, \ldots, n_1 - 1\} \to \{k_{n_0}, \ldots, k_{n_1-1}\} \setminus \{k_{m_1}\}
\]
and the inequalities
\[ d(z_{n_1}, w_{n_1}) \leq \frac{1}{k_{m_1} - 1} \quad \text{and} \quad d(z_{n_i}, w_{n_i}) \leq \frac{1}{f_1(i) - 1} \]
for \( i = n_0 + 1, \ldots, n_1 - 1 \). Moreover, by assumption, \( d(z_{i+1}, w_i) \leq \frac{1}{k_i} \) for \( i = n_0, \ldots, n_1 - 1 \).

Therefore, by Lemma 1.4, the word \( (w_{n_0}, w_{n_1}) = (w_{n_0}, z_{n_0}+1) \circ (z_{n_0}+1, w_{n_0}+1) \circ \cdots \circ (z_{n_1}-1, w_{n_1}-1) \circ (w_{n_1}-1, z_{n_1}) \circ (z_{n_1}, w_{n_1}) \in U_{k_{n_0}} \circ U_{f_1(n_0+1)-1} \circ \cdots \circ U_{f_1(n_1-1)-1} \circ U_{k_{n_1}-1} \circ U_{k_{n_1}-1} \subset U_{k_{n_2}-2} \).

Consider the word
\[ z_{n_1} - w_{n_1} + z_{n_1+1} - w_{n_1+1} + \cdots + z_{n_2} - w_{n_2}, \]
where \( n_2 > n_1 \) is the least number for which \( d(z_{n_2}, w_{n_2}) \) is maximal among all \( d(z_i, w_i) \) with \( i = n_1 + 1, \ldots, n \), and let \( m_2 \) be such that \( k_{m_2} \) is minimal among all \( k_i \) with \( i = n_1, \ldots, n_2 - 1 \). Arguing as above, we obtain
\[ (w_{n_1}, w_{n_2}) \in U_{k_{n_2}-2}. \]
In the end, we join the letters \( w_{n_0} \) and \( w_{n_1} \) by a chain
\[ (w_{n_0}, w_{n_1}) = (w_{n_0}, w_{n_1}) \circ (w_{n_1}, w_{n_2}) \circ \cdots \circ (w_{n_{t-1}}, w_{n_t}) \circ (w_{n_t}, w_{n_1}), \]
where \( (w_{n_{i-1}}, w_{n_i}) \in U_{k_{m_i}-2} \) for \( i = 1, \ldots, t \) and all numbers \( m_i \) (and, therefore, \( k_{m_i} \)) are different.

By assumption, \( k_{m_i} > N + 2 \) and \( w_n = y \); hence Lemma 1.4 implies
\[ (w_{n_0}, y) \in U_N. \]

Similarly,
\[ (x, z_{n_0}) \in U_N. \]

Thus, we have shown that, for each \( j \leq k \), there exists an \( n^{(j)}_0 \leq n_j \) such that \((x_j, z_{n^{(j)}_0}) \in U_N \) and \((w_{n^{(j)}_0}, y_j) \in U_N \). This means that
\[ (z^{(r)}_{n^{(s)}}, w^{(r)}_{n^{(s)}}) \in U_N \circ (x_r, y_s) \circ U_N \]
for any \( r, s \leq k \), which immediately implies the required assertion. \( \square \)

**Remark 1.2.** In Corollary 1.3, if \( z^{(j)}_1 = w^{(j)}_1 \), then \( n^{(j)}_0 \neq 1 \), and if \( z^{(j)}_0 = w^{(j)}_0 \), then \( n^{(j)}_0 = n^j \).

**Lemma 1.5.** Suppose that \( g = \sum_{i=1}^{k} (a_i - b_i) \) is an irreducible word; \( h = \sum_{i=1}^{l} (u_i - v_i) \) is an irreducible word satisfying condition (\( \ast \)); \( U_N \circ (a_i, b_j) \circ U_N \subset U_{\frac{1}{d(a_i, b_j) + 1}} \) and \( U_N \circ (a_i, b_j) \circ U_N \cap U_{\frac{1}{d(a_i, b_j) + 1}} = \emptyset \) for \( i, j \leq k; (u_i, v_i) \in U_{(N+3)i} \) for \( i \leq l \); a decomposition \( g + h = \sum_{i=1}^{m} (z_i - w_i) \) is irreducible and satisfies condition (\( \ast \)); and \( (z_i, w_i) \in U_{n_{0i}} \) for \( i \leq m \). Then there exists a decomposition \( g = \sum_{i=1}^{k} (x_i - y_i) \) satisfying (\( \ast \)) in which \( (x_i, y_i) \in U_{n_{0i}} \) for \( i \leq k \).

**Proof.** The decomposition \( g + h = \sum_{i=1}^{m} (z_i - w_i) \) is obtained from \( \sum_{i=1}^{k} (a_i - b_i) + \sum_{i=1}^{l} (u_i - v_i) = g + h \) by canceling pairs of equal letters with opposite signs. We assume that the cancellations are fixed and each letter in this decomposition remembers to which word \( g \) or \( h \) it belonged before cancellation and which position in this word it occupied. In other words, when we say, e.g., that \( z_i \) is a letter from \( g \), this does not merely means that \( z_i \) equals some letter \( a_j \); this means also that some letter \( a_j \) from the word \( g \) has not been canceled in \( \sum_{i=1}^{k} (a_i - b_i) + \sum_{i=1}^{l} (u_i - v_i) \) (while some other letter equal to \( a_j \) might have been canceled) and has become the letter \( z_i \). Possibly, some other letter \( z_r \) also equals \( a_j \), but \( z_r \) is not \( a_j \), because \( a_j \) is \( z_i \); this letter \( z_r \) is some other letter \( a_s \), or even a letter from \( h \). To emphasize that, considering letters of \( g + h = \sum_{i=1}^{m} (z_i - w_i) \), we mean letters together with their origins, we use the sign \( \equiv \) instead of \( = \); thus, in the above example, \( z_i \equiv a_j \) but \( z_r \not\equiv a_j \) (although \( z_r \equiv a_j \)).
Take any letter $x_1$ included in the word $g$ with coefficient 1 (e.g., $x_1 \equiv a_1$). Our immediate goal is to define a letter $y_1$. For this purpose, we shall construct a chain of letters of the forms $z_i \equiv v_j$ and $w_i \equiv u_j$, until we reach a letter from $g$; this letter will be $y_1$.

**Link 1.** If $x_1$ is not canceled in the word $g+h$, then $x_1 \equiv z_i$ for some $i_1 \leq m$. If the corresponding letter $-w_{i_1}$ is a letter from $g$, then we set $y_1 \equiv w_{i_1}$; otherwise (i.e., if this is a letter from $h$), we have $w_{j_1} \equiv v_{j_1}$ for some $j_1 \leq l$. If the letter $x_1$ is canceled in the word $g+h$, then it is canceled by a letter from $h$ (because $g$ is irreducible), i.e., $-v_{j_1}$ for some $j_1 \leq l$. We have either found $y_1$ or defined $v_{i_1}$ and (possibly) $w_{i_1} \equiv v_{j_1}$ and $z_{i_1}$.

**Link 2.** If the letter $u_{j_1}$ corresponding to the $v_{j_1}$ found at the preceding step is not canceled in the word $g+h$, then $u_{j_1} \equiv z_{i_2}$ for some $i_2 \leq m$. If the corresponding $-w_{i_2}$ is a letter from $g$, then we set $y_1 \equiv w_{i_2}$; otherwise, we have $w_{j_2} \equiv v_{j_2}$ for some $j_2 \leq l$. If the letter $u_{j_1}$ is canceled in the word $g+h$, then it is necessarily canceled by a letter $b_{\alpha}$ from $g$, and we take this letter for $y_1$; then $y_1 \equiv b_{\alpha} = u_{j_1}$. We have either found $y_1$ or defined $z_{i_2} \equiv u_{j_1}$ and $w_{j_2} \equiv v_{j_2}$.

Continuing, we obtain $y_1$ in the end.

Applying this procedure to all letters of $g$ with positive coefficients in turn, we obtain a partitioning of the letters of $g$ into pairs $x_s, y_s$ together with chains of letters

$$z'_{i_1(s)} \equiv x_s, \quad w'_{i_1(s)} \equiv v_{j_1(s)}, \quad z'_{i_2(s)} \equiv u_{j_2(s)}, \quad w'_{i_2(s)} \equiv v_{j_2(s)}, \quad \ldots, \quad z'_{i_{s_2(s)}(s)} \equiv u_{j_{s_2(s)}-1(s)}, \quad w'_{i_{s_2(s)}(s)} \equiv y_s,$$

where $z'_{i_1(s)} \equiv z_{i_1(s)}$ (if $x_s$ is not canceled in $g+h$; in this case, $w'_{i_1(s)} \equiv w_{i_1(s)}$) or $z'_{i_1(s)} \equiv v_{j_1(s)}$ (if $x_s$ is canceled by $-v_{j_1(s)}$), $z'_{i_2(s)} \equiv u_{j_2(s)}$, and $w'_{i_2(s)} \equiv v_{j_2(s)}$ for $t = 2, \ldots, r_s - 1$, $z'_{i_{s_2(s)}(s)} \equiv u_{j_{s_2(s)}-1(s)}$, and $w'_{i_{s_2(s)}(s)} \equiv w_{i_{s_2(s)}(s)}$ or $w'_{i_{s_2(s)}(s)} \equiv b_{\alpha} = u_{j_{s_2(s)}-1(s)}$ for some $\alpha_s$. The sets $\{i_0(s)\}$ are disjoint for different $s$. The sums $\sum_{t=1}^{r_s} (z'_{i_t(s)} - w'_{i_t(s)})$ satisfy the conditions of Corollary 1.3.

Indeed, these sums satisfy condition (*), because their terms are divided into the pairs $z'_{i_t(s)} - w'_{i_t(s)}$, which belong to a decomposition of $g+h$ satisfying condition (*). The first and last pairs may differ from the corresponding terms of the decomposition of $g+h$, but they equal zero (the empty word) in this case; i.e., either $z'_{i_1(s)} = z_{i_1(s)}$ and $w'_{i_1(s)} = w_{i_1(s)}$ or $z'_{i_1(s)} = v_{j_1(s)}$, and either $z'_{i_{s_2(s)}(s)} = z_{i_{s_2(s)}(s)}$ and $w'_{i_{s_2(s)}(s)} = w_{i_{s_2(s)}(s)}$ or $w'_{i_{s_2(s)}(s)} = z_{i_{s_2(s)}(s)}$; so, condition (*) is not violated. The sums $\sum_{t=1}^{r_s} (z'_{i_t(s)} - w'_{i_t(s)})$ also satisfy condition (*), because each pair $z'_{i_t(s)} - w'_{i_t(s)} = u_{j_t(s)} - v_{j_t(s)}$ is contained in a decomposition of $h$ satisfying condition (*). Moreover, by assumption, we have $(z'_{i_{t+1}(s)} - w'_{i_{t+1}(s)}) = (u_{j_{t+1}(s)} - v_{j_{t+1}(s)}) \in U_{N+3 \cdot |j_{t}(s)|}$, and the $(N+3) \cdot |j_{t}(s)|$ are different numbers larger than $N+2$. Finally, since all $x_r$ and $y_s$ are letters of the word $g = \sum_{i=1}^{k} (a_i - b_i)$, it follows from the conditions of the lemma being proved that the remaining condition of Corollary 1.3 holds too; namely, $U\cap (x_r, y_s) \cap U N \subset U_{\frac{1}{a_{r,t} - b_{s,t}}}$ and $U\cap (x_r, y_s) \cap U N \cap U_{\frac{1}{a_{r,t} - b_{s,t}}+1} = \emptyset$ for all $r, s \leq k$.

Therefore, for all $s$, there exist $n_0^{(s)} \in \{i_1(s) : t = 1, \ldots, r_s\}$ such that $d(x_r, y_t) = d(z'_{n_0^{(s)}}, w'_{n_0^{(s)}})$ for any $r, t \leq k$, and the numbers $n_0^{(s)}$ are different for different $s$ (because the sets $\{i_t^{(s)} : t = 1, \ldots, r_s\}$ are disjoint). By Remark 1.2, $n_0^{(s)} \neq i_1(s)$ if $z'_{i_1(s)} \neq z_{i_1(s)}$ or $w'_{i_1(s)} \neq w_{i_1(s)}$ (i.e., if $z'_{i_1(s)} = w_{i_1(s)}$ or $z_{i_1(s)} \neq w_{i_1(s)}$), and $n_0^{(s)} \neq i_r(s)$ if $z'_{i_r(s)} \neq z_{i_r(s)}$ or $w'_{i_r(s)} \neq w_{i_r(s)}$ (i.e., if $z'_{i_r(s)} = w_{i_r(s)}$). Thus, we have $d(x_r, y_t) = d(z_{n_0^{(s)}}, w_{n_0^{(s)}})$ for $r, t \leq k$. Since the sum $\sum_{t=1}^{k} (z_{n_0^{(s)}} - w_{n_0^{(s)}})$ also satisfies condition (*), it follows that the sum $\sum_{t=1}^{k} (z_{n_0^{(s)}} - w_{n_0^{(s)}})$ also satisfies condition (*). Hence, $g = \sum_{i=1}^{k} (x_i - y_i)$ satisfies condition (*). Finally, it follows from $d(x_i, y_i) = d(z_{n_0^{(i)}}, w_{n_0^{(i)}})$ that $(x_i, y_i) \in U_{n_0^{(i)}}$ for $i \leq k$. Since all $n_0^{(i)}$ are different, we can assume that each $(x_i, y_i)$ belongs to $U_i$ (otherwise, we renumber the terms $x_i - y_i$ and recall that $U_i \supset U_s$ for $r \leq s$).
Recall that, at the beginning of the paper, we defined the sets \( W_n(\mathcal{U}) \), which form a neighborhood base at zero for a (metrizable) group topology \( \mathcal{T}_\mathcal{U} \) on \( A(X) \). We set

\[
W^*_n(\mathcal{U}) = \bigcup_{k \in \omega} \left\{ \sum_{i=1}^{k} (x_i - y_i) : (x_i, y_i) \in \mathcal{U}_{n-i}, \sum_{i=1}^{k} (x_i - y_i) \right\}.
\]

the decomposition \( \sum_{i=1}^{k} (x_i - y_i) \) satisfies condition \((*)\).

**Claim 1.1.** 
(i) \( W^*_n(\mathcal{U}) \subset W_n(\mathcal{U}) \) for all \( n \);  
(ii) \( W_{n+1}(\mathcal{U}) \subset W^*_n(\mathcal{U}) \) for all \( n \);  
(iii) for any \( n \in \omega \) and any \( g \in W^*_n(\mathcal{U}) \), there exists an \( n_0 \in \omega \) for which \( g + W_{n_0}(\mathcal{U}) \subset W^*_n(\mathcal{U}) \);  
(iv) for any \( k, n \in \omega \) and any word \( g = \sum_{i=1}^{k}(a_i - b_i) \), there exists an \( n_0 \in \omega \) such that the condition \( g + W^*_n(\mathcal{U}) \cap W^*_n(\mathcal{U}) \neq \emptyset \) implies \( g \in W^*_n(\mathcal{U}) \).

**Proof.** Assertion (i) is obvious; (ii) is Corollary 1.1. 
Assertion (iii) follows from Corollary 1.2. Indeed, suppose that \( g = \sum_{i=1}^{k}(x_i - y_i) \), the decomposition \( \sum_{i=1}^{k}(x_i - y_i) \) satisfies condition \((*)\), and \( (x_i, y_i) \in \mathcal{U}_{n-i} \) for \( i \leq k \). We can assume that \( x_i \neq y_i \) for \( i \leq k \), because if \( x_j = y_j \) for some \( j \), then we can delete the term \( x_j - y_j \) from the sum \( \sum_{i=1}^{k}(x_i - y_i) \); i.e., we can set \( x_i' = x_i \) and \( y_i' = y_i \) for \( i < j \) and \( x_j' = x_{i+1} \) and \( y_j' = y_{i+1} \) for \( i = j, \ldots, k-1 \); we have \( g = \sum_{i=1}^{k-1}(x_i' - y_i') \), the decomposition \( \sum_{i=1}^{k-1}(x_i' - y_i') \) satisfies condition \((*)\) (see Remark 1.1), and \( (x_i', y_i') \in \mathcal{U}_{n-i} \) for \( i \leq k-1 \) (for \( i \geq j \), we have \( (x_i', y_i') \in \mathcal{U}_{n-i+1} \subset \mathcal{U}_{n-i} \)). Thus, suppose that \( x_i \neq y_i \); in this case, the decomposition \( \sum_{i=1}^{k}(x_i - y_i) \) is irreducible (i.e., \( x_i \neq y_i \) for any \( i, j \leq k \)), because it satisfies \((*)\). Since all \( \mathcal{U}_n \) are clopen and form a base for a uniformity generating the initial (completely regular) topology on \( X \), we can find \( N \) for which the conditions of Corollary 1.2 hold; after that, it remains to set \( n_0 = N + 2 \); if \( h \in W_{N+2}(\mathcal{U}) \), then \( h \in W_{N+1}(\mathcal{U}) \) (see (ii)) and, by Corollary 1.2, \( g + h \in W^*_n(\mathcal{U}) \). Assertion (iv) is derived from Lemma 1.3 in a similar way \((n_0 = N + 4)\). \( \square \)

It follows from (i)–(iii) that the sets \( W^*_n(\mathcal{U}) \) are open in the topology \( \mathcal{T}_\mathcal{U} \) and form a neighborhood base at zero for this topology; (iv) says that each \( W^*_n(\mathcal{U}) \) is closed in \( \mathcal{T}_\mathcal{U} \).

**Remark 1.3.** Let \( \rho \) be a metric on \( X \) such that \( \mathcal{U}_i \subset \{(x, y) : \rho(x, y) \leq \frac{1}{i} \} \subset \mathcal{U}_{i-1} \) for any \( i \geq 1 \) (it exists by Theorem 8.1.10 from [1]). Then the topology on \( A(X) \) generated by the Graev extension of \( \rho \) is no stronger than \( \mathcal{T}_\mathcal{U} \). Indeed, if \( g \in W_n(\mathcal{U}) \), then \( g = \sum_{i=1}^{k}(x_i - y_i) \), where \( (x_i, y_i) \in \mathcal{U}_{n-i} \) for \( i \leq k \), and \( \sum_{i=1}^{k}\rho(x_i - y_i) \leq \sum_{i=1}^{k}\frac{1}{2^{m-i}} < \frac{1}{2^{m-1}} \). Since the Graev norm \( \|g\|_\rho \) of the element \( g \) is defined as \( \min\left\{ \sum_{i=1}^{m}\rho(u_i, v_i) : m \geq 1, g = \sum_{i=1}^{m}(u_i - v_i) \right\} \), we have \( \|g\|_\rho < \frac{1}{2^m} \). Thus, each Graev ball of radius \( \frac{1}{2^m} \) centered at zero contains some base neighborhood \( W_n(\mathcal{U}) \) of zero in the topology \( \mathcal{T}_\mathcal{U} \). Since the space \( X \) is closed in the free group with the Graev topology, it is also closed in the free group with the topology \( \mathcal{T}_\mathcal{U} \).

### 2. A Metrizable Group with Noncoinciding Dimensions

We denote the Cantor set \( 2^\omega \) by \( C \). The elements of \( C \) are infinite sequences of zeros and ones. The topology of \( C \) has a standard base, which is a tree under inclusion; the \( n \)-th-level elements of this tree are sets of sequences whose first \( n \) members coincide; different elements of the same level do not intersect. Clearly, all base neighborhoods of the same point of \( C \) are comparable, and larger neighborhoods belong to levels with smaller numbers. We denote the elements of the Cantor set \( C \) itself by the letters \( x, y, z, \ldots \) and the infinite sequences of such elements (i.e., the elements of the set \( C^\omega \)) by the same letters in boldface: \( \mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots \); we denote the value of a sequence \( \mathbf{x} \) at \( n \) by \( \mathbf{x}(n) \). The restriction of a sequence \( \mathbf{x} \in C \) to \( \{0, \ldots, n-1\} \) (i.e., the ordered set of the first \( n \) elements of this sequence) is denoted by \( \mathbf{x}|_n \). Thus, the \( n \)-th-level elements of the base-tree have the form \( \{y \in C : y|_{n+1} = \mathbf{x}|_{n+1} \} \) for \( x \in C \).
By I we denote the usual interval $[0, 1]$. Let $t \in I$. If $t = (2k + 1)/2^n$ for some positive integers $k$ and $n$, then we define the order $t$ as $\text{ord } t = n$. We assume that $\text{ord } 0 = \text{ord } 1 = 0$. For all other numbers $t \in [0, 1]$, we set $\text{ord } t = \infty$.

For $n \in \omega$, we define the neighborhood $I_n(t)$ of a number $t \in (0, 1)$ to be the interval $I_n(t) = (a_n(t), b_n(t))$, where $a_n(t)$ and $b_n(t)$ are the dyadic rationals of minimal order for which $b_n(t) - a_n(t) = 1/2^n$ and $t \in (a_n(t), b_n(t))$: we set $I_n(0) = [0, 1/2^n + 1]$ and $I_n(1) = (1 - 1/2^{n+1}, 1]$. Thus, if $0 < \text{ord } t \leq n$, i.e., $t = k/2^n$ for some (possibly, even) $k$, then $a_n(t) = (2k - 1)/2^{n+1}$ and $b_n(t) = (2k + 1)/2^{n+1}$ (and hence $\text{ord } a_n(t) = \text{ord } b_n(t) = n + 1$), and if $\text{ord } t > n$, then $a_n(t) = k/2^n$ and $b_n(t) = (k + 1)/2^n$ for some $k$ (and hence the order of one of the numbers $a_n(t)$ and $b_n(t)$ equals $n$ and the order of the other is strictly less than $n$).

Let $A \subset C$. We set

$$\nu_0(A) = \{(x, t) \in C^\omega \times I : x(n) \in A \text{ for } n \neq \text{ord } t, x(n) \in C \setminus A \text{ for } n = \text{ord } t\}$$

and endow $\nu_0(A)$ with the topology generated by the sets of the form

$$U_n(x, t) = \begin{cases} \{(y, s) \in \nu_0(A) : s \in I_n(t), y(i) = x(i) \text{ for } i \leq n\} & \text{if } n < \text{ord } t, \\
\{(y, s) \in \nu_0(A) : s \in I_n(t); y(i) = x(i) \text{ for } i \leq n + 1, i \neq \text{ord } t; \\
y(i)|_{n+1} = x(i)|_{n+1} \text{ for } i = \text{ord } t\} & \text{if } \text{ord } t \leq n.\end{cases}$$

According to Mrowka [12], the space $\nu_0(A)$ is metrizable and $\text{ind } \nu_0(A) = 0$; moreover, if $A$ is everywhere dense in $C$ and the set $C \setminus A$ is of second category, then $\dim \nu_0(A) > 0$.

The projection $\pi(\nu_0(A))$ of the set $\nu_0(A) \subset C^\omega \times I$ on the first factor consists of all sequences $x \in C^\omega$ each of which takes at most one value not in $A$.

For $A$ we take the set $\sigma 2^\omega$ of binary sequences with only finitely many elements different from 0. For each nonzero $x \in A$, we define its length $\text{len } x$ to be the number of the last nonzero term of the sequence $x$: we set $\text{len } 0 = 0$ (thus, $\text{len } 0100 \cdots = 2$).

For $x \in \pi(\nu_0(A))$ and $n, i \in \omega$, we fix a maximal base neighborhood $J_n^i(x)$ of $x(i)$ of level $\geq n$ such that

1. if $x(j) \in A$ for all $j \leq n$, then the lengths of all elements of the intersection $J_n^i(x) \cap A$ (except, possibly, the point $x(i)$ itself) are larger than all lengths $\text{len } x(j)$ for $j \leq n$;
2. if $x(j) \notin A$ for some $j \leq n$, then the lengths of all elements of the intersection $J_n^i(x) \cap A$ (except, possibly, the point $x(i)$ itself) are larger than the lengths $\text{len } x(j)$ for all $j \leq n + 1$ such that $x(j) \in A$.

Since all sets of the form $J_n^i(x)$ are elements of the base-tree, it follows that, for any $x, y \in \pi(\nu_0(A))$ and any $n, k, i, j \in \omega$, either the sets $J_n^i(x)$ and $J_k^j(y)$ are disjoint or one of them is contained in the other.

For $(x, t) \in \nu_0(A)$, we set

$$V_n(x, t) = \begin{cases} \{(y, s) \in \nu_0(A) : s \in I_n(t), y(i) = x(i) \text{ for } i \leq n, y(n + 1) \in J_n^{n+1}(x)\} & \text{if } n < \text{ord } t, \\
\{(y, s) \in \nu_0(A) : s \in I_n(t); y(i) = x(i) \text{ for } i \leq n + 1, i \neq \text{ord } t; \\
y(i) \in J_n^i(x) \text{ for } i = \text{ord } t\} & \text{if } \text{ord } t \leq n.\end{cases}$$

Clearly, the sets of the form $V_n(x, t)$ constitute a base for the topology of $\nu_0(A)$.

**Remark 2.1.** Suppose that $(x, t), (y, s) \in \nu_0(A)$, $n, n' \in \omega$, and $I_n'(s) \cap \{r \in [0, 1] : \text{ord } r \leq n + 1, r \neq s\} = \emptyset$ (this implies, in particular, that $n' \geq n$). Then one of following four cases occurs:

1. $(y, s) \in V_n(x, t)$;
2. $V_n'(y, s) \cap V_n(x, t) = \emptyset$;
3. $s \in I_n(t), \text{ord } s, \text{ord } t > n, y(i) = x(i) \text{ for } i \leq n$, and $x(n + 1) \in J_n^{n+1}(x)$; moreover, in this case, $V_n(x, t) \subset V_n(y, s)$;
4. $s \in I_n(t) \setminus I_n(t) = \{a_n(t), b_n(t)\}$ and $y(i) = x(i) \text{ for } i \leq n$ such that $i \neq \text{ord } t, \text{ord } s$. 
Indeed, if $s \notin I_n(t)$, then $I_n'(s) \cap I_n(t) = \emptyset$ and $V_n'(y, s) \cap V_n(x, t) = \emptyset$, i.e., condition (ii) holds.

If $s \in I_n(t)$, ord $s \leq n$ (this can happen only if $s = t$), and $y(i) \neq x(i)$ for some $i \leq n + 1$ such that $i \neq \text{ord} t$, then (ii) holds.

If $s \in I_n(t)$, ord $s = k \leq n$ (then $s = t$), and $y(i) = x(i)$ for all $i \leq n + 1$ such that $i \neq k$, then either (a) $y(k) \in J_n^k(x)$ (and then (i) holds), (b) $J_n^k(y) \cap J_n^k(x) = \emptyset$ (and then (ii) holds), or (c) $J_n^k(y) \supset J_n^k(x)$ and $y(k) \notin J_n^k(x)$. In case (c), $J_n^k(y)$ is a base neighborhood of the point $x(k)$, its level is at least $n' \geq n$, and the lengths of all elements of the intersection $J_n^k(y) \cap A$ are larger than the length $\text{len} x(j) = \text{len} y(j)$ for all $j \leq n + 1$ such that $x(j) = y(j) \in A$ (the points $x(k)$ and $y(k)$ themselves do not belong to $A$). This contradicts the maximality of the neighborhood $J_n^k(x)$.

Suppose that $s \in I_n(t)$, ord $s > n$, and ord $t = k \leq n$. Then $y(k) \in A$, $x(k) \notin A$, and $x(n + 1) \in A$. If (ii) does not hold, then there exists a $(z, h) \in V_n'(y, s) \cap V_n(x, t)$.

Take $z(i) = y(i) = x(i)$ for all $i \leq n$ different from $k$, $z(k) = y(k) \in J_n^k(x)$, and $z(n + 1) = x(n + 1) = J_n^{n + 1}(y)$. We have $h \notin \{x(k) \neq A, y(k) \neq x(k) \}$. Thus, $y(k) \in J_n^k(A)$, and $x(n + 1) \in A$; thus, it follows from $y(k) \in J_n^k(x)$ that $\text{len} y(k) > \text{len} x(i)$ for all $i \leq n + 1$ different from $k$ (in particular, $\text{len} y(k) > \text{len} x(n + 1)$). The inclusion $x(n + 1) = x(n + 1) \in J_n^{n + 1}(y)$ implies that either $\text{len} x(n + 1) > \text{len} y(i)$ for all $i \leq n$ or $x(n + 1) = y(n + 1)$. The former inequality cannot hold, because $\text{len} x(n + 1) < \text{len} y(k)$; hence $x(n + 1) = y(n + 1)$. Thus, $y(i) = x(i)$ for $i \leq n + 1$, $i \neq k$, and $y(k) \in J_n^k(x)$. This means that (i) holds.

If $s \in I_n(t)$, ord $s > n$, ord $t > n$, and $y(i) \neq x(i)$ for some $i \leq n$, then (ii) holds.

If $s \in I_n(t)$, ord $s > n$, ord $t > n$, and $y(i) = x(i)$ for all $i \leq n$, then either (a) $y(n + 1) \in J_n^{n + 1}(x)$ (and hence (i) holds), (b) $J_n^{n + 1}(y) \cap J_n^{n + 1}(x) = \emptyset$ (and then (ii) holds), or (c) $J_n^{n + 1}(y) \supset J_n^{n + 1}(x)$ (i.e., (iii) holds). The inclusion $V_n(x, t) \subset V_n(y, s)$ follows from the obvious inclusion $J_n^{n + 1}(y) \subset J_n^{n + 1}(y)$ (which is an immediate consequence of $n' \geq n$).

If $s \in I_n(t) \setminus I_n(t)$ and $y(i) \neq x(i)$ for some $i \leq n$ such that $i \neq \text{ord} t$, ord $s$, then (ii) holds.

Claim 2.1. For any $n \in \omega$, the set

$$U_n = \bigcup \{V_n(x, t) \times V_n(x, t) : (x, t) \in \nu \mu_0(A)\}$$

has empty boundary.

Proof. Suppose that $(y, s), (z, r) \in \nu \mu_0(A)$, and $n \in \omega$. Take $n' \in \omega$ such that

$$I_n'(s) \cap \{t \in [0, 1) : \text{ord} t \leq n + 1, t \neq s\} = \emptyset,$$

$$I_n'(r) \cap \{t \in [0, 1) : \text{ord} t \leq n + 1, t \neq r\} = \emptyset,$$

and if $y(i) \neq z(i)$ for $i \leq n + 1$, then

$$J_{n'}^{i}(y) \cap J_{n'}^{j}(z) = \emptyset.$$

Suppose that $V_{n'}(y, s) \times V_{n'}(z, r) \cap U_n \neq \emptyset$ but $((y, s), (z, r)) \notin U_n$. This means that there exist $(x, t) \in \nu \mu_0(A)$, $(y', s') \in V_{n'}(y, s)$, and $(z', r') \in V_{n'}(z, r)$ such that $(y', s') \in V_n(x, t)$, $(z', r') \in V_n(x, t)$, and either $(y, s) \notin V_n(x, t)$ or $(z, r) \notin V_n(x, t)$. For definiteness, suppose that $(y, s) \notin V_n(x, t)$. Then (iii) or (iv) from Remark 2.1 holds. Suppose that (iv) holds. There are the following possibilities:

1. ord $t = k \leq n$. In this case, ord $s = n + 1$ and $y(i) = y'(i) \in A$ for $i \leq n$. Moreover, $y'(n + 1) = x(n + 1) \in A$ and $y'(k) \in J_n^k(x)$ (because $(y', s') \in V_n(x, t)$). Therefore, $\text{len} y'(k) > \text{len} x(j)$ for all $j \leq n + 1$ different from $k$ (in particular, $\text{len} y'(k) > \text{len} x(n + 1)$).

2. ord $t > n$. In this case, ord $s = k \leq n$. Suppose that $r \neq s$. If $r \notin I_n(t)$, then $I_{n'}(r) \cap I_n(t) = \emptyset$ and $V_{n'}(z, r) \cap V_n(x, t) = \emptyset$, which contradicts the assumption. Therefore, $r \in I_n(t)$, and ord $r \neq \text{ord} s = k$ (the endpoints of the interval $I_n(t)$ are of different orders, and all interior
points of this interval have orders larger than \( n \). Thus, \( y(k) \in C \setminus A \), whereas \( z(k) \in A \).

The number \( n' \) was chosen so that \( J^k_n(y) \cap J^k_n(z) = \emptyset \); in particular, \( z(k) \notin J^k_n(y) \).

Since \((y',s') \in V_n(y,s), (z',r') \in V_n(z,r)\), and, moreover, \( \text{ord } s = k \), \( \text{ord } r \neq k \), and \( k \leq n \), it follows that \( y'(k) \in J^k_n(y) \) and \( z'(k) = z(k) \notin J^k_n(y) \). Therefore, \( y'(k) \neq z'(k) \), and at least one of these numbers is not equal to \( x(k) \), i.e., at least one of the pairs \((y',s')\) and \((z',r')\) does not belong to the set \( V_n(x,t) \), which contradicts the definition of these pairs. Hence \( r = s \). The same argument shows that \( y(i) = z(i) \) for all \( i \leq n \): if \( y(i) \neq z(i) \), then at least one of the numbers \( y'(i) \) and \( z'(i) \) is not equal to \( x(i) \), and the corresponding pair does not belong to \( V_n(x,t) \). Since \((y',s') \in V_n(x,t)\) and \( \text{ord } t > n \), we have \( y(k) \notin A \) and \( y'(k) = x(k) \in J^k_n(y,k) \). Therefore, \( x(k) \neq y(i) \) for all \( i \leq n \) different from \( k \) (in particular, \( \text{len } y(k) > n + 1 \)).

Now, suppose that condition (iii) from Remark 2.1 holds. If \((z,r) \in V_n(x,t)\), then \((z,r) \in V_n(y,s) \) and \((y,s),(z,r)) \in U_n \). Suppose that \((z,r) \notin V_n(x,t) \). Then \( V_n(z,r) \cap V_n(x,t) = \emptyset \), it follows that one of conditions (iii) and (iv) with \( z \) instead of \( y \) and \( r \) instead of \( s \) holds.

The case in which (iv) holds has just been considered. Suppose that (iii) holds. We have \( s,r \in I_n(t) \); \( \text{ord } s, \text{ord } r, \text{ord } t > n \); \( y(i) = z(i) = x(i) \) for all \( i \leq n \) (because \( x(t) \in V_n(x,t), V_n(x,t) \subset V_n(y,s) \) by condition (iii) for \( x(t) \) and \( y(s) \), and \( V_n(x,t) \subset V_n(z,r) \) by condition (iii) for \( x(t) \) and \( z(r) \); \( x(n+1) \in J^k_n(y) \); and \( x(n+1) \in J^k_n(z) \). Therefore, \( J^k_n(y) \subset J^k_n(z) \) or \( J^k_n(z) \subset J^k_n(y) \). For definiteness, suppose that \( J^k_n(y) \subset J^k_n(z) \). Then \( y(n+1) \in J^k_n(z) \). It remains to note that \( I_n(r) = I_n(t) \) (because \( \text{ord } t > n \) and \( r \in I_n(t) \)). This immediately implies \( s \in I_n(r) \) and \( (y,s) \in V_n(z,r) \), i.e., \( (y,s),(z,r) \in U_n \). This contradiction completes the proof.

It follows immediately from Claim 2.1 and Theorem 3.1 that the space \( \nu \mu_0(A) \) can be embedded in a metrizable topological group \( G \) with \( \text{ind } G = 0 \); moreover, \( \nu \mu_0(A) \) is closed in \( G \) (see Remark 3.3). Since \( \dim \nu \mu_0(A) > 0 \) and the group \( G \) is metrizable, we have \( \dim G > 0 \). Thus, we have obtained an example of a metrizable group with noncoinciding dimensions ind and dim.

3. A Zero-Dimensional Metrizable Space Which is not Embedded in a Zero-Dimensional Metrizable Group

In this section, by a sequence we mean a map from an at most countable ordinal to some set and consider only sequences with values in \( \omega_1 \). We identify all sequences with ordered sets of their values and write them in the form of (finite or infinite) words. As in the preceding section, we denote sequences by boldface Latin letters, but their elements we denote by the same letters with subscript-numbers. Thus, the symbol \( a_n \) always denotes the element number \( n \) in the sequence \( a: a_n = a(n) \). The word whose letters are sequences (all but the last must be finite) denotes the concatenation of these sequences. For example, if \( a = a_0a_1 \ldots a_n \) and \( b = b_0b_1 \ldots \), then \( ab = a_0a_1 \ldots a_nb_0b_1 \ldots \).

If \( a \) is a sequence of length \( \geq n \), then

\[ a|_n = a_0a_1 \ldots a_{n-1} \]

(recall that we assume that \( a = a_0a_1 \ldots \)); we set \( a|_0 = \emptyset \). For \( m < n \),

\[ a|m = a_ma_{m+1} \ldots \quad \text{and} \quad a|m = a_m \ldots a_{n-1} \]

For a set \( A \) of sequences of length \( \geq n \), we put

\[ A|_n = \{ a|_n : a \in A \}, \quad A|m = \{ a|m : a \in A \}, \]
and
\[ A^m_n = \{a^m_n : a \in A\}. \]

If \( A \) is a set of finite sequences, \( c \) is a finite sequence, \( B \) is a set of sequences, and \( d \) is a sequence, then
\[ cB = \{cb : b \in B\}, \quad AB = \{ab : a \in A, b \in B\}, \]
and
\[ Ad = \{ad : a \in A\}. \]

Let \( L \) be the set of all limit ordinals smaller than \( \omega_1 \), and let \( S = \omega_1 \setminus L \). We have \( \omega_1 = \omega \cup \bigcup_{k \in \omega}(L + k) \), where \( L + k = \{\alpha + k : \alpha \in L\} \).

Kulesza’s space \( Z \subset \omega_1^\omega \) is defined as
\[ Z = \{a = a_0a_1 \cdots \in \omega_1^\omega : a_0 \in \omega_1 \setminus L, a_k \in L \text{ for at most one } k \in \omega, \]
and if \( a_k \in L \), then \( a_{k+1} = a_k + k \) and \( a_{k+i} \in L + k \) for all \( i \geq 2 \).

Kulesza proved that the space \( Z \) with the topology induced by the topological product \( \omega_1^\omega \) of countably many copies of the space \( \omega_1 \) with the usual order topology is metrizable and \( \text{Ind } Z = \dim Z = 1 \) (while, obviously, \( \text{ind } Z = 0 \)) [6].

Kulesza did not give an explicit formula for a metric on \( Z \), but he described base neighborhoods of the points of \( Z \). They look as follows.

For each limit ordinal \( \alpha \in \omega_1 \), we fix an increasing sequence \( \tilde{\alpha}_0\tilde{\alpha}_1 \cdots \in \omega_1 \) with limit \( \alpha \) and put \( M_n(\alpha) = (\tilde{\alpha}_n, \alpha) \).

Let \( m \in \omega \). If a sequence \( a \in Z \) is such that \( a|m \in S^m \), then we set
\[ N_m(a) = \{b \in Z : b|m = a|m\}. \]

If \( 1 \leq k < m \) and \( a_k \in L \), then
\[ N_m(a) = \{b \in Z : b|k = a|k, b|k+1 = a|k+1, b_k \in M_m(a_k)\}. \]

The sets \( N_m(a) \) form a neighborhood base at the point \( a \) in the space \( Z \).

To prove the inequality \( \dim Z > 0 \), Kulesza used the notion of full sets introduced by Fleissner in [3].

**Definition 3.1** ([3]). A set \( T \subset \omega_1^n \) is said to be full if \( \{b_j : b \in T, b|j = a|j\} \) is uncountable for any \( a \in T \) and \( j < n \) (in particular, \( T|1 \) is uncountable).

We say that a set \( T \subset \omega_1^n \) is full if \( T|n \) is full for all \( n \in \omega \).

We need the following two combinatorial properties of full sets.

**Lemma 3.1** ([3, Lemma 6.4(b)]). If a set \( T \subset \omega_1^n \) is full and \( h : T \to \omega \), then \( T \) contains a full subset on which \( h \) is constant.

**Lemma 3.2.** If a set \( T \subset \omega_1^n \) is full and \( \{C_m : m \in \omega\} \) is a family of sets such that \( C_m \subset T|m \) for \( m \in \omega \) and, for any \( a \in T \), there exists an \( n \in \omega \) for which \( a|n \subset C_n \), then \( C_t \) contains a full set (a subset of \( T|t \subset \omega_1^n \)) for some \( t \in \omega \).

**Proof.** This lemma is similar to Lemma 6.4(a) from [3]. In [3], the role of \( T \) is played by \( \omega_1^\omega \). There exists a natural bijection
\[ \psi : [\omega_1]^\omega \to \bigcup_{n \leq \omega} T|n. \]

It is constructed as follows. For all \( n \in \omega \) and \( x \in T|n \), we fix bijections \( \varphi_{x_1} : \omega_1 \to \{y : xy \in T|n+1\} \) and put
[\[ \psi(a_0a_1 \cdots) = \varphi(\alpha_0)\varphi(\alpha_0)(\alpha_1)\varphi(\alpha_0)\varphi(\alpha_0)(\alpha_2) \cdots \]
for any (finite or infinite) sequence $\alpha_0\alpha_1\cdots \in [\omega_1]^{<\omega}$. The map $\psi$ respects restrictions in the sense that if $\alpha, \beta \in [\omega_1]^{<\omega}$ and $\alpha|_n = \beta|_n$, then $\psi(\alpha)|_n = \psi(\beta)|_n$; moreover, $\psi(\omega^n) = T|_n$. The family \( \{\psi^{-1}(C_m) : m \in \omega \} \) has the properties
\[
\psi^{-1}(C_m) \subseteq \omega^1_m \quad \text{for all } m \in \omega
\]
and
\[
\text{for any } \alpha \in \omega^\omega, \text{ there exists an } n \in \omega \text{ such that } \alpha|_n \in \psi^{-1}(C_n).
\]
According to [3, Lemma 6.4(a)], there exists a $t \in \omega$ for which $\psi^{-1}(C_t)$ contains a full set. For this $t$, $C_t$ contains a full set. □

Levin [9] suggested a simple short proof of the inequality $\dim Z > 0$ based on the notion of regular sets. We need the following modification of this notion.

Definition 3.2. Let $U \subseteq Z \times Z$ be any set containing the diagonal. We say that a pair of sequences $(x, y) \in S^n \times S^n$ is $U$-regular (or simply regular; when it is clear what set $U$ is meant) if there exists a map (regulator) $f: ([S]^{<\omega})^2 \to \omega_1$ such that $(xa, yb) \in U$ whenever the sequences $a, b \in S^\omega$ satisfy the condition $a_i, b_i > f(a_i, b_i)$ for all $i \in \omega$ (in particular, $a_0, b_0 > f(\varnothing)$).

Let $U$ be an arbitrary subset of $Z \times Z$ containing the diagonal. For $a \in Z$, we put
\[
U(a) = \{b \in Z : (a, b) \in U\}.
\]
The set $U^2$ is defined standardly as
\[
U^2 = \{(a, b) : \text{there exists } c \in Z \text{ such that } (a, c) \in U \text{ and } (c, b) \in U\}.
\]
Thus,
\[
U^2(a) = \{b \in Z : \text{there exists } c \in Z \text{ such that } (a, c) \in U \text{ and } (c, b) \in U\}.
\]

Suppose that $\{U_n : n \in \omega\}$ is a countable base for a uniformity on $Z$ generating the topology of the space $Z$. For each $a \in S^\omega \subseteq Z$, fix $m_a \geq 2$ for which $U^2_{m_a}(a) \subset N_2(a)$. For $k \in \omega$, we set
\[
C_k = \{a|_k : a \in S^\omega, \ m_a \leq k, \ N_2(a) \supseteq U^2_{m_a}(a) \supseteq U_{m_a}(a) \supseteq N_k(a)\}.
\]
Clearly, for any sequence $a \in Z$, there exists a $k \geq m_a$ for which $N_k(a) \subset U_{m_a}(a)$ (because the sets $U_{m_a}(a)$ are open and the $N_k(a)$ form a base for the topology of $Z$ at the point $a$). Hence, for any sequence $a \in S^\omega$, there exists a $k$ for which $a|_k \in C_k$. By Lemma 3.2, there exists a $t$ such that $C_t$ contains a full set (clearly, $t \geq 2$, because the sets $C_t$ are empty for $k < 2$). Using Lemma 3.1, we choose a number $m \in \omega$ and a full set $T \subset C_t$ such that $\min\{m_a : a|_t = a_0 \ldots a_{t-1}\} = m$ for any $a_0 \ldots a_{t-1} \in T$; note that $m \leq t$ by the definition of $C_t$. We put $U = U_m$. Our purpose is to show that $U \neq \overline{U}$. Suppose that $U = \overline{U}$.

Remark 3.1. For any $x \in Z$ such that $x|_t \in T$, we have $U(x) \subset N_2(x)$. Indeed, by the definition of $T$, there exists an $a \in S^\omega$ for which $a|_t = x|_t$, $N_2(a) \supseteq U^2_{m_a}(a) \supseteq U_{m_a}(a) \supseteq N_t(a)$, and $m_a = m \leq t$ (i.e., $U_{m_a} = U$). Since $x|_t = a|_t \in S^t$, we have $x \in N_t(a)$. Therefore, $x \in U(a)$, and $U(x) \subset U^2(a) \subset N_2(a)$. Since $t \geq m \geq 2$ and $x|_t = a|_t \in S^t$, it follows that $N_2(a) = N_2(x)$; thus, $U(x) \subset N_2(x)$.

Remark 3.2. The pair $(x, x)$ is not $U$-regular for any $x \in C_t|_1$. Indeed, suppose that $x \in C_t|_1$, the pair $(x, x)$ is regular, and $f: ([S]^{<\omega})^2 \to \omega_1$ is the corresponding regulator. Since the set $C_t$ is full, we can find $a_1, a_2, \ldots, b_1, b_2, \ldots \in S$ such that
\[
a_1 \neq b_1, \quad xa_1a_2 \ldots a_{t-1}, xb_1b_2 \ldots b_{t-1} \in C_t,
\]
\[
a_1, b_1 > f(\varnothing), \quad \text{and} \quad a_{i+1}, b_{i+1} > f(a_1 \ldots a_i, b_1 \ldots b_i) \quad \text{for all } i \geq 1.
\]
Let $a_0 = b_0 = x$. We have $a|_t, b|_t \in C_t$. According to Remark 3.1, $U(a) \subset N_2(a)$. However, by the definition of a regular pair, we also have $(a, b) \in U$, i.e., $b \in U(a)$. Therefore, $b \in N_2(a)$, which is false, because $b_1 \neq a_1$. 
Therefore, if a sequence $a \sim x$, where $x \in S^ω$, there exists an $n \in ω$ such that the pair $(x|_n, y|_n)$ is regular. Indeed, since $U$ is open and the sets $N_k(x)$ and $N_k(y)$ form bases of neighborhoods of the points $x$ and $y$, it follows that there exists an $n \in ω$ for which

$$N_n(x) \times N_n(y) \subset U;$$

this means that $(x_0x_1...x_{n-1}a, y_0y_1...y_{n-1}b) \subset U$ for any $a$ and $b$ from $Z$, not only for those satisfying the condition from the definition of regular pairs.

**Lemma 3.3.** Suppose that $k > 0$; $x = x_0...x_{k-1}, y = y_0...y_{k-1} \in S^k$; the pairs $(x|_n, y|_n)$ with $n \leq k$ are not regular; and there exists an uncountable set $S' \subset S$ such that the pair $(xz, yz)$ is regular for any $z \in S'$. Then there exists a number $l > 0$, points $x_k, ..., x_{k+l-1}, y_k, ..., y_{k+l-1} \in S$, and an uncountable set $S'' \subset S$ such that the pairs $(x_0...x_n, y_0...y_n)$ with $n < k+l$ are not regular and the pair $(x_0...x_{k+l-1}z, y_0...y_{k+l-1}z)$ is regular for any $z \in S''$.

**Proof.** Let $C \subset L$ be an arbitrary closed unbounded set of limit ordinals. Take $c_0 \in C$ and $z_0 \in S'$ for which $z_0 > c_0$. By assumption, the pair $(xz_0, yz_0)$ is regular; let $f_0$ be the corresponding regulator. Take $c_1 \in C$ such that $c_1 > \max \{f_0(\emptyset), z_0\}$ and $z_1 \in S^1$ such that $z_1 > c_1$. By assumption, the pair $(xz_1, yz_1)$ is regular; let $f_1$ be the corresponding regulator. Suppose that we made $n$ steps, i.e., chose ordinals $c_{n-1} \in C$ and $z_{n-1} \in S'$ and a regulator $f_{n-1}$. At the $(n + 1)$th step, we take $c_n \in C$ and $z_n \in S'$ such that

$$c_n > \max \{f_{n-1}(\emptyset), z_{n-1}\} \quad \text{and} \quad z_n > c_n,$$

and choose a map $f_n$ witnessing the regularity of the pair $(xz_n, yz_n)$.

As a result, we obtain an increasing sequence of elements of $C$. Let $c = \sup \{c_n : n \in ω\}$. We have $c \in C$, because $C$ is closed. Moreover, for any $n \in ω$, the pair $(xz_n, yz_n)$ is regular, $f_n$ is the corresponding regulator, and $c + k > c > f_n(\emptyset)$. Therefore, if $a \in S^ω$ is a sequence such that

$$a_i > \sup \{f_n((c+k)a|_i, (c+k)a|_i) : n \in ω\}$$

for all $i \in ω$, then

$$(xz_n(c+k)a, yz_n(c+k)a) \subset U.$$

Recall that $c = \sup \{c_n : n \in ω\} = \sup \{z_n : n \in ω\}$; thus, any neighborhood in $Z \times X$ of any point of the form $(xc(c+k)a, yc(c+k)a)$ contains the point $(xz_n(c+k)a, yz_n(c+k)a)$ for some $n$. Therefore, if a sequence $a$ satisfies condition (1), then

$$(xc(c+k)a, yc(c+k)a) \in U = U.$$

Clearly, the set of sequences $a \in S^ω$ satisfying (1) is full.

Thus, any closed unbounded set of limit ordinals contains a point $c \in L$ for which there exists a full set $Y_c \subset S^ω$ such that

$$(xc(c+k)z, yc(c+k)z) \subset U \quad \text{for any } z \in Y_c.$$

Therefore, the set $L'$ of such points $c$ is stationary.

Since $U$ open, it follows that, for any $c \in L'$ and $z \in Y_c$, there exists an $n = n(z, c) > k + 2$ such that

$$N_n(xc(c+k)z) \times N_n(yc(c+k)z) \subset U.$$

For $m \in ω$ and $c \in L'$, we set

$$C_m(c) = \{z \in Y_c : n(z, c) = m\}.\]{L^\prime}_m.$$

For any $c \in L'$, using Lemma 3.3.2 and the definition of the neighborhoods of the form $N_n(a)$, we can find an $m_c > 0$ and a full set $Y'_c \subset Y_{l_m+c-2}$ such that

$$\{x \mu(c+k)za, y \nu(c+k)zb \} \subset U \quad \text{for any } \mu, \nu \in M_{mc}, \ z \in Y'_c, \text{ and } a, b \in Z^{mc}.\]{L^\prime}$$

Using the pressing down lemma, we choose a stationary subset $L''$ of the stationary set $L'$ such that

$$\tilde{c}_m = \beta \quad \text{for all } c \in L''.$$
where \( \beta \) is a countable ordinal (here the \( c_n \) are the ordinals converging to \( c \) that are used in the definition of the sets \( M_n(c) \) involved in the definition of the neighborhoods \( N_n(xc(c+k)z) \) and \( N_n(yc(c+k)z) \)).

Suppose that the pairs \((xx, yy)\) are regular for any \( x, y > \beta \) from \( S \) and \( f_{xy} \) are the corresponding regulators. Then the pair \((x, y)\) itself is regular; the corresponding regulator is defined by

\[
f(\varnothing) = \beta, \quad f(xa, yb) = \begin{cases} 
  f_{xy}(a, b) & \text{if } x, y > \beta, \\
  0 & \text{if } x \leq \beta \text{ or } y \leq \beta.
\end{cases}
\]

The pair \((x, y)\) is not regular by assumption; hence there exist \( x, y \) such that for which the pair \((xx, yy)\) is not regular.

If the pairs \((xx_k(c+k), yy_k(c+k))\) are regular for all \( c \in L'' \) such that \( c > x, y \), then we can set \( l = 1 \) and \( S'' = L'' + k \). Otherwise, we take \( c \in L'' \) for which \( c > x, y \), and the pair \((xx(c+k), yy(c+k))\) is not regular. Condition \((2)\) implies

\[
(xx_k(c+k)za, yy_k(c+k)zb) \in U \quad \text{for any } z \in Y_c' \text{ and } a, b \in Z^{me}.
\]

The set \( Y_c' \) is full; hence \( Y_c'|_1 \) is uncountable. If the pairs \((xx_k(c+k)z, yy_k(c+k)z)\) are regular for all \( z \in Y_c'|_1 \), then we have obtained what is required. Otherwise, we take \( z_0 \in Y_c'|_1 \) for which the pair \((xx_k(c+k)z_0, yy_k(c+k)z_0)\) is not regular. Relation \((2)\) implies

\[
(xx_k(c+k)z_0a, yy_k(c+k)z_0b) \in U \quad \text{for any } z \text{ such that } z_0z \in Y_c' \text{ and } a, b \in Z^{me}.
\]

The set \( Y_c' \) is full; hence \( Y_c'|_1 \) is uncountable. If the pairs \((xx_k(c+k)z_0, yy_k(c+k)z_0)\) are regular for all \( z \in S \) such that \( z_0z \in Y_c'|_2 \), then we have obtained what is required. Otherwise, we continue the construction. Sooner or later, the procedure will terminate: we shall find either an \( n < mc - k - 4 \) such that the pairs \((xx_k(c+k)z_0 \ldots z_{n-1}z, yy_k(c+k)z_0 \ldots z_{n-1}z)\) are regular for all \( z \in S \) with \( z_0 \ldots z_{n-1}z \in Y_c'|_{n+1} \) or \( z_0 \ldots z_{mc-k-4} \in Y_c'|_{mc-k-3} \) such that all pairs \((xx_k(c+k)z_0 \ldots z_{mc-k-4}n, yy_k(c+k)z_0 \ldots z_{mc-k-4}n)\), where \( n \leq mc \), are not regular. In the latter case, the pair

\[
(xx_k(c+k)z_0 \ldots z_{mc-k-4}z, yy_k(c+k)z_0 \ldots z_{mc-k-4}z)
\]

is regular for any \( z \) such that \( z_0 \ldots z_{mc-k-4}z \in Y_c' \) (and there are uncountably many such \( z \), because \( Y_c' \) is full) by virtue of \((3)\).

Take any point \( x \) such that \( z_0 \ldots z_{mc-k-4}z \in Y_c' \) (and there are uncountably many such \( z \), because \( Y_c' \) is full) by virtue of \((3)\).
N_n(x) \times N_n(y) \cap U \neq \emptyset \text{ for any } n \in \omega, \text{ and hence } (x,y) \in \overline{U} = U. \text{ Remark 3.3 implies that the pair } (x|_n, y|_n) \text{ must be regular for some } n. \text{ This contradiction shows that } \overline{U} \neq U.

4. Concluding Remarks

We have considered two metrizable spaces with noncoinciding dimensions, Mrowka’s and Kulesza’s, and shown that one of them can be embedded in a zero-dimensional metrizable group and the other cannot. The natural question arises: What properties of Kulesza’s space obstruct its embedding into a zero-dimensional metrizable group? The most manifest difference between Mrowka’s and Kulesza’s spaces is that Kulesza’s space is metrizable by a complete metric. This suggests the conjecture that a space metrizable by a complete metric can be embedded in a zero-dimensional metrizable group only if it is strongly zero-dimensional. This conjecture is based not only on purely formal grounds but also on some intuitive reasons; in this author’s opinion, it is fairly likely. Even more likely is the following auxiliary conjecture: If \((X, \rho)\) is a metric space with complete metric \(\rho\), \(A_\rho(X)\) is the free group of \(X\) metrized by the Graev extension of \(\rho\), and \(\text{ind } A_\rho(X) = 0\), then \(\dim X = 0\).

It is also unclear how the dimension of metrizable groups behaves under completion\(^1\). It is only clear that the free and free Abelian groups with Graev metrics (as well as the metrizable groups of the form \((A(X), \mathcal{T}_U)\) described in the first section, into which we can embed zero-dimensional metrizable spaces) are never complete; we can always construct a fundamental sequence consisting of words with unboundedly increasing lengths, which converges to no word of finite length.\(^2\)

We conclude this paper with several questions.

**Problem 1.** Is it true that if the uniformity generated by a metric \(\rho\) on a set \(X\) has a countable base consisting of open-and-closed sets, then the free (Abelian) group of \(X\) metrized with the Graev extension of \(\rho\) is zero-dimensional?

**Problem 2.** Does there exist a complete metric group with noncoinciding dimensions \(\text{ind}\) and \(\dim\)?

**Problem 3.** Is it true that any complete metric space which can be embedded into a zero-dimensional metrizable group is strongly zero-dimensional?

**Problem 4.** Is it true that if \((X, \rho)\) is a complete metric space with metric \(\rho\), \(G_\rho(X)\) is the free (Abelian) group of \(X\) metrized by the Graev extension of \(\rho\), and \(\text{ind } G_\rho(X) = 0\), then \(\dim X = 0\)?

**Problem 5.** Is it true that if \((X, \rho)\) is a metric space with metric \(\rho\), \(G_\rho(X)\) is the free (Abelian) group of \(X\) metrized by the Graev extension of \(\rho\), and the completion of \(G_\rho(X) = 0\) is zero-dimensional, then \(\dim X = 0\)? What if the metric \(\rho\) is complete?

**Problem 6.** How large can the gap between the dimensions \(\text{ind}\) and \(\dim\) of a metrizable group be? What values can the dimension \(\dim\) of a metrizable topological group \(G\) with \(\text{ind } G = 0\) take?

**Problem 7.** Let \((\nu_\mu_0(A), \rho)\) be Mrowka’s space described in the second section with a metric \(\rho\) generating the uniformity with a clopen base described in the same section, and let be \(G\) the metrizable group with \(\text{ind } G = 0\) into which \(\nu_\mu_0(A)\) is embedded by Theorem 1.1.

(a) Find \(\dim G\);
(b) Find \(\text{ind } G_\rho(\nu_\mu_0(A))\) and \(\dim G_\rho(\nu_\mu_0(A))\), where \(G_\rho(\nu_\mu_0(A))\) is the free (Abelian) group of \(\nu_\mu_0(A)\) metrized by the Graev extension of the metric \(\rho\).

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\(^1\)This question is difficult even for general topological spaces. Thus, Mrowka’s space \(\nu_\mu_0\) has a zero-dimensional completion under the continuum hypothesis \([11]\); however, Mrowka also proved that the assertion that the small inductive dimension of all metric completions of \(\nu_\mu_0\) is larger than zero is possibly consistent \([11]\), i.e., it holds under certain set-theoretic assumption whose consistency with ZFC is very likely.

\(^2\)More details on topologies on free groups (including the Graev metric topology) can be found in \([17]\).
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