The Rare Decay $B \to K^*\gamma$: A More Precise Calculation.

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Efforts to predict the rare exclusive decay $B \to K^*\gamma$ from the well known inclusive decay $b \to s\gamma$ are frustrated by the effect of the large recoil momentum. We show how to reduce the large uncertainty in calculating this decay by relating $B \to K^*\gamma$ to the semileptonic process $B \to \rho e\bar{\nu}$ using the heavy-quark symmetry in $B$ decays and $SU(3)$ flavor symmetry. A direct measurement of the $q^2$-spectrum for the semileptonic decay can provide accurate information for the exclusive rare decay.
The inclusive rare decay $B \to X_s\gamma$ is now well understood in the context of the standard model [1] and the experimental upper bound [2] of $8.4 \times 10^{-4}$ for the branching ratio is already playing an important role [3] in constraining the parameters of models other than the standard model. On the other hand, the most likely experimental observation to be made will be the exclusive decay $B \to K^*\gamma$. The recent limit from the CLEO collaboration [4] of the branching ratio for this mode is $0.92 \times 10^{-4}$. It is this exclusive rare decay $B \to K^*\gamma$, however, which is the least well known theoretically due to the large recoil momentum of the $K^*$ meson [5]. A recent paper [6] points out that heavy-quark symmetry together with $SU(3)$ flavor symmetry could relate the rare decay $B \to K^*\gamma$ to a measurement of the semileptonic decay $B \to \rho e\bar{\nu}$. However, the relation is only valid at a single point in the Dalitz plot, a point where the semileptonic decay vanishes, so that there would still be a large uncertainty in such a measurement.

In this paper we obtain a similar relation that relates the exclusive rare decay $B \to K^*\gamma$ to the spectrum in $q^2$ for the semileptonic decay $B \to \rho e\bar{\nu}$. The $q^2$-spectrum for $B \to \rho e\bar{\nu}$ does not vanish at $q^2 = 0$ and so a direct measurement of the spectrum at this point can provide accurate information for $B \to K^*\gamma$. Of necessity this new result requires an extension of the heavy-quark symmetries to a consideration of the $K^*$ and $\rho$ systems. We show that this is not the same as demanding the $K^*$ or $\rho$ to be a heavy-quark system in the conventional sense. Our result dramatically reduces the uncertainty from the earlier calculations.

First we discuss the application of the heavy-quark symmetry. Usually, it is the hadronic systems with a $b$ or a $c$ quark that have these symmetries. Here we derive the relations for matrix elements of either a $B$ or a $B^*$ with an unspecified vector meson $V$. We show how to extend the heavy-quark symmetry relations to the case when the meson $V$ is the $K^*$ or $\rho$, and we estimate the possible errors using a set of quark-model calculations, both for nonrelativistic and for relativistic cases. Then we use the results to give a reliable relation between the decay $B \to K^*\gamma$ and the $q^2$-spectrum for $B \to \rho e\bar{\nu}$.

I. Heavy-Quark Symmetry Relations

We first recapitulate the derivation of the heavy-quark symmetry relations for $B$ decay.
The hadronic matrix elements relevant to the decay \( B(b\bar{q}) \rightarrow V(Q\bar{q}) \) are given by

\[
\langle V(k,\epsilon)|\bar{Q}\gamma_\mu b|B(p_B)\rangle = 2T_1(q^2)i\varepsilon_{\mu\nu\lambda\sigma}\epsilon^{*\nu}p_B^\lambda k^\sigma ,
\]

\[
\langle V(k,\epsilon)|\bar{Q}\gamma_\mu\gamma_5 b|B(p_B)\rangle = -2(m_B^2 - m_V^2)T_2(q^2)\epsilon^{*\mu} - 2T_3(q^2)(\epsilon^* \cdot q)(p_B + k)_\mu
\]

\[
-2T_4(q^2)(\epsilon^* \cdot q)(p_B - k)_\mu ,
\]

\[
\langle V(k,\epsilon)|\bar{Q}i\sigma_{\mu\nu}q^\nu b|B(p_B)\rangle = f_1(q^2)i\varepsilon_{\mu\nu\lambda\sigma}\epsilon^{*\nu}p_B^\lambda k^\sigma
\]

\[
+ \left[ (m_B^2 - m_V^2)\epsilon^{\mu} - (\epsilon^* \cdot q)(p_B + k)_\mu \right] f_2(q^2)
\]

\[
+ (\epsilon^* \cdot q) \left[ (p_B - k)_\mu - \frac{q^2}{(m_B^2 - m_V^2)}(p_B + k)_\mu \right] f_3(q^2) ,
\]

where \( q = p_B - k \). We show below that the hadronic form factors \( f_{1,2,3}(q^2) \) and \( T_{1,2,3,4}(q^2) \) for the decay \( B \rightarrow V \) can all be related using just the spin symmetry and static limit of the heavy \( b \) quark.

In the heavy \( b \) limit, the spin of the \( b \) quark is decoupled from all other light fields in \( B \). We can therefore construct the spin operator \( S^Z_b \) for the \( b \) quark such that

\[
S^Z_b |B(b\bar{q})\rangle = \frac{1}{2} |B^*_1(b\bar{q})\rangle , \quad S^Z_b |B^*_1(b\bar{q})\rangle = \frac{1}{2} |B(b\bar{q})\rangle ,
\]

where \( B^*_1 \) stands for a longitudinal vector \( B^* \) meson. In \( |B\rangle \) and \( |B^*_1\rangle \), the spatial momentum of the \( b \) quark is in the \( z \)-direction for the \( b \) spinor to be an eigenstate of \( S^Z_b \). Using the relation \( \langle V|\bar{Q}\Gamma b|B\rangle = -2\langle V||S^Z_b,\bar{Q}\Gamma b||B^*_1\rangle \) for \( \Gamma \) any product of \( \gamma \) matrices, we have the following identities between the \( B \rightarrow V \) and \( B^*_1 \rightarrow V \) matrix elements:

\[
\langle V|A_0|B\rangle = -\langle V|V_3|B^*_1\rangle ,
\]

\[
\langle V|A_3|B\rangle = -\langle V|V_0|B^*_1\rangle ,
\]

\[
\langle V|V_\pm|B\rangle = \mp\langle V|V_\pm|B^*_1\rangle ,
\]

\[
\langle V|V_0|B\rangle = -\langle V|A_3|B^*_1\rangle ,
\]

\[
\langle V|V_3|B\rangle = -\langle V|A_0|B^*_1\rangle ,
\]

\[
\langle V|A_\pm|B\rangle = \mp\langle V|A_\pm|B^*_1\rangle ,
\]

where \( V_\mu = \bar{Q}\gamma_\mu b \) and \( A_\mu = \bar{Q}\gamma_\mu\gamma_5 b \). The covariant expansions of the vector and axial-vector matrix elements for the decay \( B^*(b\bar{q}) \rightarrow V(Q\bar{q}) \) are defined to be
(\langle V(k, \epsilon) | \bar{Q} \gamma_{\mu} b | B^* (p_B, \zeta) \rangle) = \left[ (\zeta \cdot \epsilon^*) A_1 (q^2) + (\zeta \cdot q) (\epsilon^* \cdot q) A_2 (q^2) \right] (p_B + k)_{\mu} \\
+ \left[ (\zeta \cdot \epsilon^*) B_1 (q^2) + (\zeta \cdot q) (\epsilon^* \cdot q) B_2 (q^2) \right] (p_B - k)_{\mu} \\
+ \mathcal{C} (q^2) (\epsilon^* \cdot q) \zeta_{\mu} + \mathcal{D} (q^2) (\zeta \cdot q) \epsilon^*_{\mu}, \tag{10} \\
(\langle V(k, \epsilon) | \bar{Q} \gamma_{\mu} \gamma_5 b | B^* (p_B, \zeta) \rangle) = \mathcal{E} (q^2) i \varepsilon_{\mu \nu \lambda \sigma} \epsilon^{\nu} \zeta^\lambda (p_B + k)^{\sigma} + \mathcal{F} (q^2) i \varepsilon_{\mu \nu \lambda \sigma} \epsilon^{\nu} \zeta^\lambda (p_B - k)^{\sigma}, \tag{11} \\

\text{where } \zeta \text{ and } \epsilon^* \text{ are the polarization vectors of } B^* \text{ and } V, \text{ respectively. Using the matrix identities in Eqs. (4)-(9), we can relate } T_{1, 2, 3, 4} \text{ to the } B^* \rightarrow V \text{ form factors:-} \\

2 m_B T_1 = (A_1 - B_1), \\
2 (m_B^2 - m_V^2) T_2 = m_B (A_1 + B_1) + E_V (A_1 - B_1), \\
2 m_B (T_3 - T_4) = -(A_1 - B_1), \\
2 m_B (T_3 + T_4) = -(A_1 + B_1) - \mathcal{C}, \\
\mathcal{D} = (A_1 - B_1), \\
\mathcal{E} = A_1, \\
\mathcal{F} = B_1, \\
A_2 = B_2 = 0. \tag{12} \\

\text{Since the spatial momentum of the } b \text{ quark is defined in the } z\text{-direction, the above relations are worked out in the } B \text{ rest frame. We choose the longitudinal polarization vector for } B_i^* \text{ to be } \zeta_i^\mu = (0, 0, 0, 1) \text{ and define the momentum of } V \text{ to be } k^\mu = (E_V; k^1, k^2, k^3), \text{ where } E_V = (m_B^2 + m_V^2 - q^2) / (2 m_B). \text{ The resulting form-factor relations in Eq. (12) are consistent with those in Ref. [4] using the spin symmetry of a heavy } Q, \text{ except for the relation } T_3 + T_4 = 0, \text{ which is missing here.} \\

\text{We can relate the form factors } f_{1, 2, 3} \text{ to the form factors } T_{1, 2, 3, 4} \text{ using the static limit of the } b \text{ quark. In the } B \text{ rest frame, the static } b\text{-quark spinor satisfies the equation of motion } \gamma_0 b = b. \text{ We then have the relations between the } \gamma_\mu \text{ and } \sigma_{\mu \nu} \text{ matrix elements [8]:-} \\

\langle V | \bar{Q} \gamma_i b | B \rangle = \langle V | \bar{Q} i \sigma_{0i} b | B \rangle, \tag{13} \\
\langle V | \bar{Q} \gamma_i \gamma_5 b | B \rangle = -\langle V | \bar{Q} i \sigma_{0i} \gamma_5 b | B \rangle. \tag{14} \\

\text{This gives the form-factor relations}
\begin{align*}
  f_1 &= -(m_B - E_V)T_1 - \frac{(m_B^2 - m_V^2)}{m_B}T_2 , \\
  f_2 &= \frac{1}{2} \left[ (m_B - E_V) - (m_B + E_V) \frac{q^2}{m_B^2 - m_V^2} \right] T_1 - \frac{1}{2m_B} \left( m_B^2 - m_V^2 + q^2 \right) T_2 , \\
  f_3 &= \frac{1}{2}(m_B + E_V)T_1 + \frac{1}{2m_B}(m_B^2 - m_V^2)(T_1 + T_2 + T_3 - T_4) .
\end{align*}

Using the spin-symmetry relations in Eq. (12), we can also write \( f_{1,2,3} \) in terms of the \( B^* \to V \) form factors as,

\begin{align*}
  f_1 &= -A_1 , \\
  f_2 &= -\frac{1}{2} A_1 - \frac{1}{2} \left( \frac{q^2}{m_B^2 - m_V^2} \right) B_1 , \\
  f_3 &= \frac{1}{2} B_1 .
\end{align*}

(16)

Thus, using only the spin symmetry and static limit of the heavy \( b \) quark, we can relate the \( B \to V \) hadronic form factors as,

\begin{align*}
  2(m_B^2 - m_V^2)T_2 &= \left[ (m_B + m_V)^2 - q^2 \right] T_1 + \alpha , \\
  2T_3 &= -T_1 + \frac{\beta}{2m_B} , \\
  2T_4 &= T_1 + \frac{\beta}{2m_B} , \\
  f_1 &= -(m_B + m_V)T_1 - \frac{\alpha}{2m_B} , \\
  2f_2 &= -\left[ (m_B + m_V)^2 - q^2 \right] \frac{T_1}{m_B + m_V} - \frac{\alpha}{2m_B} \left( 1 + \frac{q^2}{m_B^2 - m_V^2} \right) , \\
  2f_3 &= -(m_B - m_V)T_1 + \frac{\alpha}{2m_B} ,
\end{align*}

(17)

where

\begin{align*}
  \alpha &= m_B(A_1 + B_1) - m_V(A_1 - B_1) , \\
  \beta &= -C - (A_1 + B_1) .
\end{align*}

If we ignore the \( \alpha \) and \( \beta \) terms in Eq. (17), the resulting form-factor relations are exactly the heavy-quark symmetry relations obtained using the large-mass limits of both \( b \) and \( Q \) quarks [5,7,8]. The function \(-\sqrt{4m_Bm_V}T_1\) in this limit resembles the role of the Isgur-Wise function, with absolute normalization at \( q^2 = t_m \) given in the quark model as [3].
\[-\sqrt{4m_B m_V}T_1(t_m) = \left( \frac{2\beta_B \beta_V}{\beta_B^2 + \beta_V^2} \right)^{3/2} \approx 1,\]

where \(\beta_B\) and \(\beta_V\) are variational parameters of the momentum wave functions for \(B\) and \(V\) respectively. We shall show below that the \(\alpha\) and \(\beta\) terms in Eq. (17) can be regarded as small corrections to the heavy-quark relations coming from the weak binding, or \(\Lambda_{QCD}\) effects. Thus, the symmetry relations are dominated by the \(T_1\) terms.

To show that the RHS of Eq. (17) is dominated by the \(T_1\) terms, we use the equation of motion for the \(b\) quark to get the matrix relation

\[\langle V|\bar{Q}\gamma_0 b|B^*\rangle = \frac{\sqrt{4m_B m_V}}{m_B} \langle V|\bar{Q}b|B^*\rangle\]

and work in the \(V\) rest frame. If we assume the static limit of \(Q\) so that the equation of motion \(\bar{Q} = \bar{Q}\gamma_0\) is satisfied we can relate the matrix elements for \(\bar{Q}b\) and \(\bar{Q}\gamma_0 b\) as,

\[\langle V|\bar{Q}\gamma_0 b|B^*\rangle \approx \frac{\sqrt{4m_B m_V}}{m_B} \langle V|\bar{Q}b|B^*\rangle.\]  

(18)

The additional form-factor relations that follow from Eq. (18) are given by

\[-C = (A_1 + B_1) = \frac{m_V}{m_B} (A_1 - B_1),\]

(19)

Corresponding to \(\alpha = \beta = 0\) in Eq. (17). The correction to the static \(Q\) assumption is proportional to \(p/m_Q\), where \(p\) is the spatial momentum of \(Q\) in the \(V\) rest frame. Since

\[m_V = \sqrt{p^2 + m_Q^2} + \sqrt{p^2 + m_q^2} + \text{Binding Energy},\]

and \(m_V \approx m_Q + m_q\) for a heavy enough vector meson, in the weak binding limit of \(V\) it is easy to show that \(p/m_Q \ll 1\). The corrections to Eq. (19) arising from the static \(Q\) assumption are therefore small, and consequently we have the suppression of \(\alpha, \beta \sim O(p/m_Q)\) in this limit.

We can use the quark model to show explicitly the suppression of \(\alpha\) and \(\beta\) in the weak binding limit. In the quark-model calculations of \(\alpha, \beta,\) and \(T_1\), we have in the \(B\) rest frame

\[m_B \beta = -\alpha = \sqrt{4m_B E_V} \left( h_1 - \frac{h_2}{E_V - m_V} \right),\]

(20)

\[T_1 = -\sqrt{\frac{E_V}{m_B (E_V + m_V)(E_V - m_V)}} h_2,\]

(21)

where
The term \( k \) is the recoil momentum of \( V \) in the \( B \) rest frame. The energies of the \( b \) and \( Q \) quarks in Eq. (24) are given by \( E_b = \sqrt{p^2 + m_b^2} \) and \( E_Q = \sqrt{(k+p)^2 + m_Q^2} \). The terms \( \phi_B(p) \) and \( \phi_V(p + r k) \), where \( r = m_q/(m_Q + m_q) \), are the momentum wave functions of \( B \) and \( V \), respectively. In the weak binding limit of \( V \) and with \( m_V \approx m_Q + m_q \), we have \([5]\)

\[
\frac{k \cdot p}{(E_b + m_b)} + \frac{k \cdot (k + p)}{(E_Q + m_Q)} \approx (E_V - m_V) \left( 1 + \frac{(k + p) \cdot p}{(E_Q + m_Q)(E_b + m_b)} \right).
\]

This gives in Eq. (22) the relation

\[
(E_V - m_V)h_1 \approx h_2,
\]

which is insensitive to the problem of how the overlap occurs between \( \phi_B \) and \( \phi_V \). The suppression of \( \alpha \) and \( \beta \) in this limit are then obvious from Eq. (24).

In the quark model, the relation between \(-\alpha \) (and \( m_B \beta \)) and \( T_1 \) can also be written

\[
m_B \beta = -\alpha = [ (m_B + m_V)^2 - q^2 ] T_1 \varepsilon,
\]

where \( \varepsilon = 1 - (E_V - m_V)h_1/h_2 \). The correction to Eq. (24) is given by

\[
\delta = [ (m_b - m_Q) - (E_b - E_Q) - (E_V - m_V) ] \left( 1 + \frac{(k + p) \cdot p}{(E_Q + m_Q)(E_b + m_b)} \right)
\]

\[
-2(m_b - m_Q) \frac{(k + p) \cdot p}{(E_Q + m_Q)(E_b + m_b)},
\]

which corresponds to the binding effects in \( V \). From Eq. (26), it is then easy to show that \( \varepsilon \approx 0 \) in the weak binding limit of \( V \) and with \( m_V \approx m_Q + m_q \).

Using the Gaussian wave functions for \( \phi_B \) and \( \phi_V \) \([5]\), we obtain throughout the whole kinematic region a rather stable ratio for \( (E_V - m_V)h_1/h_2 \), which is between 0.95 and 1.04 for \( B \to K^* \), and between 0.89 and 0.98 for \( B \to \rho \). These two ranges include the large uncertainty in the quark model associated with different recoil dependencies in the wave function overlap \([5]\). In all cases the value for \( \varepsilon \) is very small with \( |\varepsilon| < 0.05 \) for \( B \to K^* \) and \( |\varepsilon| < 0.11 \) for \( B \to \rho \) throughout the full kinematic range.
It is then clear from Eq. (25) that the symmetry relations in Eq. (17) are dominated by the \( T_1 \) terms as the \( \alpha \) and \( \beta \) terms are suppressed by \( \varepsilon \). (Near \( q^2 = t_m \) some relations are further suppressed by \( \varepsilon (m_V/m_B) \)). Thus, the \( B \to V \) hadronic form factors satisfy the heavy-quark symmetry relations even if the quark \( Q \) is much lighter than the \( b \) quark. The breakdown of the relations is a measure of the weak binding approximation and is a small correction.

Since the effect is small and stable across the full kinematic range we can use the model to investigate, with some confidence, the \( 1/m_Q \) behaviour of \( \varepsilon \). This behaviour can most easily be checked near the zero recoil part, in which the recoil effect becomes insignificant. In Fig. [1], we show the \( m_Q \) dependence of \( \varepsilon (t_m) \) with \( m_V - m_Q \) fixed at \( m_\rho - m_u \). For \( m_Q \) greater than about 0.9 GeV, \( \varepsilon \) falls off like power law of \( 1/m_Q \). As shown in the figure, the curve can be approximated, above 1.25 GeV, by a Taylor expansion of \( \varepsilon (t_m) \) with respect to \( \langle p^2 \rangle / m_Q \) using Eq. (26). The leading terms in the expansion are given by

\[
\varepsilon (t_m) \approx \left( \frac{m_V - m_Q}{m_Q} \right) - \frac{1}{3} \frac{m_q}{m_Q + m_q} \left( \frac{\langle p^2 \rangle}{m_V} \right) \frac{m_V}{m_b} \left( 1 + \frac{m_Q}{m_b} \right) - \frac{3}{8} \frac{\langle p^2 \rangle}{m_Q} \frac{1 + \frac{7}{9} m_Q}{m_Q} \beta^2 V m_Q (1 + \frac{7 m_Q}{9 m_b}) ,
\]

where

\[
\langle p^2 \rangle = \frac{\int d\mathbf{p} \phi_V(\mathbf{p}) \phi_B(\mathbf{p}) p^2}{\int d\mathbf{p} \phi_V(\mathbf{p}) \phi_B(\mathbf{p})} = \frac{3 \beta_B^2 \beta_V^2}{\beta_B^2 + \beta_V^2} .
\]

The expansion parameter \( \langle p^2 \rangle \) is a stable function of \( m_Q \) with \( \sqrt{\langle p^2 \rangle} = 428 \text{ MeV} \) for \( m_u \) and \( \sqrt{\langle p^2 \rangle} = 502 \text{ MeV} \) for \( m_b \). At about \( m_Q = 0.9 \text{ GeV} \), \( \varepsilon \) has a maximum and ceases to follow the \( 1/m_Q \) power law. It is this turn-over that stops the correction \( \varepsilon \) from becoming very large for smaller \( m_Q \) and keeps the correction to the symmetry relation in Eq. (17) small for \( s \) and \( u \) quarks.

II. The Decays \( B \to K^*\gamma \) and \( B \to \rho e\bar{\nu} \)

The branching ratio for the exclusive \( B \to K^*\gamma \) to the inclusive \( b \to s\gamma \) processes can be written in terms of \( f_1 \) and \( f_2 \) at \( q^2 = 0 \), as [10,11]

\[
R(B \to K^*\gamma) = \frac{\Gamma (B \to K^*\gamma)}{\Gamma (b \to s\gamma)} \cong \frac{m_b^3 (m_B^2 - m_{K^*}^2)^3}{m^3_B (m_b^2 - m_s^2)^3} \frac{1}{2} \left[ |f_1(0)|^2 + 4|f_2(0)|^2 \right] .
\]

(28)
Using Eq. (17), we can write \( f_2(0) = (1/2)f_1(0) \) at \( q^2 = 0 \). Although there is now only one form factor to calculate in Eq. (28), this is still a controversial model-dependent calculation \([5,10–13]\). There is an uncertainty of about a factor of about ten depending on the way the large recoil of the \( K^* \) is handled.

In an attempt to overcome this uncertainty, Burdman and Donoghue \([6]\) have discussed a method of relating \( B \to K^*\gamma \) to the semileptonic process \( B \to \rho e\bar{\nu} \) using the static \( b \)-quark limit and \( SU(3) \) flavor symmetry. Their main result is that the ratio

\[
\Gamma(B \to K^*\gamma) \left( \lim_{q^2 \to 0, \text{curve}} \frac{1}{q^2} \frac{d\Gamma(B \to \rho e\bar{\nu})}{dE_\rho dE_e} \right)^{-1} = \frac{4\pi^2}{G_F^2 |V_{ub}|^2} \frac{|\eta|^2 (m_B^2 - m_{K^*}^2)^3}{m_B^4},
\]

is independent of hadronic form factors. Here, \( \eta \) represents the QCD corrections \([1]\) to the decay \( b \to s\gamma \), and the word “curve” denotes the region in the Dalitz plot for which \( q^2 = 4E_e(m_B - E_\rho - E_e) \). The only uncertainty on the right hand side is that of \( |V_{ub}| \) for which \([14]\) \( |V_{ub}|/|V_{cb}| = 0.10 \pm 0.03 \).

Their method proposes to overcome the uncertainty in the calculation at large recoil \((q^2 = 0)\) of the \( B \to K^* \) form factors by making a direct measurement of the semileptonic decay \( B \to \rho e\bar{\nu} \). Notice that we use only the \( q^2 = 0 \) point on the “curve” to compare with the photonic decay in Eq. (29). The problem with this is that the semileptonic decay vanishes at the \( q^2 = 0 \) point on the “curve,” which is why this kinematic factor is divided out in Eq. (29). This means that experimentally there should be no events at that point and very few in the neighborhood, making it a very difficult measurement.

We shall overcome this by considering instead the \( q^2 \)-spectrum for the semileptonic decay \( B \to \rho e\bar{\nu} \). The advantage here is that the \( q^2 \)-spectrum does not vanish at \( q^2 = 0 \) since we integrate over the events from different electron energies across the Dalitz plot. The disadvantage is that in taking the ratio we do not have the simple cancellation of form factors, which made the previous relationship so appealing. However, we can relate the ratio to the knowledge of \( \varepsilon \), which we have demonstrated to be a small number anywhere in the Dalitz plot.

The differential width for \( B \to \rho e\bar{\nu} \) is given by

\[
\frac{d\Gamma(B \to \rho e\bar{\nu})}{dq^2} = \frac{G_F^2 |V_{ub}|^2 |k|^3}{12\pi^3} \Lambda_T,
\]

where
\[ \Lambda_T = T_1^2 q^2 + T_2^2 \frac{(m_B^2 - m_\rho^2)^2}{2m_\rho^2} \left( 1 + \frac{3q^2 m_\rho^2}{m_B^2 |k|^2} \right) + T_3^2 \frac{2m_B^2 |k|^2}{m_\rho^2} + T_2 T_3 \frac{(m_B^2 - m_\rho^2)}{m_\rho^2} (m_B^2 - m_\rho^2 - q^2) . \]

At \( q^2 = 0 \), the differential width for \( B \to \rho \bar{\nu} \) reduces to

\[ \frac{d\Gamma(B \to \rho \bar{\nu})}{dq^2} \bigg|_{q^2=0} = \frac{G_F}{192\pi^3} |V_{ub}|^2 \frac{(m_B^2 - m_\rho^2)^5}{m_B^3 m_\rho^2} |T_2(0) + T_3(0)|^2 . \] (31)

If we use the symmetry relations in Eq. (17) and \( SU(3) \) flavor symmetry in which \( T_1^{B \to K^*} = T_1^{B \to \rho} \), we can express the ratio between \( R(B \to K^*\gamma) \) and \( d\Gamma(B \to \rho \bar{\nu})/dq^2 \) at \( q^2 = 0 \) as

\[ R(B \to K^*\gamma) \left( \frac{d\Gamma(B \to \rho \bar{\nu})}{dq^2} \bigg|_{q^2=0} \right) ^{-1} \]

\[ = \frac{192\pi^3}{G_F} \frac{1}{|V_{ub}|^2} \frac{(m_B^2 - m_\rho^2)^5}{m_B^3 m_\rho^2} \frac{(m_B^2 - m_{K^*}^2)^2}{(m_B - m_{K^*})^2} \frac{m_\rho^3}{(m_B^2 - m_{K^*}^2)^3} |I|^2 \]

\[ = 1.9 \times 10^{16} \text{GeV} \cdot \left( \frac{0.1}{|V_{ub}/V_{cb}|} \right)^2 |I|^2 . \] (32)

In the limit \( \varepsilon \approx 0 \), that is \( \alpha \approx 0 \) and \( \beta \approx 0 \) in Eq. (17), we have \( I = 1 \) in Eq. (32). We estimate the correction to \( I \) using the quark-model results for \( \alpha \) and \( \beta \) in Eq. (23) as

\[ I = \left( 1 - \frac{(m_B + m_{K^*})}{2m_B} \varepsilon^{B \to K^*}(0) \right) / \left( 1 - \frac{(m_B + m_\rho)(m_B^2 + m_\rho^2)}{4m_B^2 m_\rho} \varepsilon^{B \to \rho}(0) \right) . \] (33)

Although \( \varepsilon \) is a small number, the value of \( \varepsilon(0) \) is sensitive to the details of the wavefunction overlap in the quark model because of the large recoil momentum. In an extreme case with a relativistic recoil dependence in the overlap, we obtain \( \varepsilon^{B \to K^*}(0) = 0.010 \) and \( \varepsilon^{B \to \rho}(0) = 0.043 \) which gives \( I = 1.09 \). On the other hand, using a milder nonrelativistic recoil dependence, we obtain \( \varepsilon^{B \to K^*}(0) = 0.046 \) and \( \varepsilon^{B \to \rho}(0) = 0.087 \) which gives \( I = 1.18 \). The uncertainty in calculating the branching ratio \( R(B \to K^*\gamma) \) due to the recoil problem has now been reduced by an order of magnitude.

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We have derived a relation between the branching ratio \( R(B \to K^*\gamma) \) and the \( q^2 \)-spectrum for \( B \to \rho \bar{\nu} \). Since the \( q^2 \)-spectrum for \( B \to \rho \bar{\nu} \) does not vanish at \( q^2 = 0 \), this reduces the uncertainty in the measurement of the semileptonic decay in contrast to the case in Eq. (29). Now ARGUS has given the result [15] of \( BR(B^- \to \rho^0 l\bar{\nu}) = (11.3 \pm 3.6 \pm 2.7) \times 10^{-4} \),
and by isospin symmetry, \( \Gamma(B^0 \rightarrow \rho^+ l \bar{\nu}) = 2 \Gamma(B^- \rightarrow \rho^0 l \bar{\nu}) \). This allows us to estimate \( d\Gamma(B \rightarrow \rho e \bar{\nu})/dq^2 \) at \( q^2 = 0 \) to be about \( 10^{-17} \) GeV\(^{-1}\). Eq. (32) then gives \( R(B \rightarrow K^*\gamma) \) about \( 10^{-1} \), which is quantitatively correct. A direct measurement of \( d\Gamma(B \rightarrow \rho e \bar{\nu})/dq^2 \) at \( q^2 = 0 \) can therefore provide reliable information for \( R(B \rightarrow K^*\gamma) \).

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FIGURES

FIG. 1. The $m_Q$ dependence of $\varepsilon(t_m)$ with $m_V - m_Q$ fixed at $m_\rho - m_u$. The solid line is the exact numerical result of $\varepsilon(t_m)$ using Eq. (26). The dotted line is a Taylor expansion of $\varepsilon(t_m)$ with respect to $\langle p^2 \rangle/m_Q^2$. 

![Graph showing the dependence of $\varepsilon(t_m)$ on $m_Q$]