New moments criteria for convergence towards normal product/tetilla laws

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Abstract

We consider, in the classical probability, the distribution $F_\infty \sim N_1 \times N_2$ where $N_1, N_2$ are two independent standard normal random variable, and in the setting of free probability, $F_\infty \sim (S_1 S_2 + S_2 S_1) / \sqrt{2}$ known as tetilla law, where $S_1, S_2$ are freely independent normalized semicircular random variables. We provide new characterization of $F_\infty$ within the second Wiener (Wigner) chaos. Our characterizations can be seen as the classical moments matching problem. More precisely, we show that for any generic element $F$ in the second Wiener (Wigner) chaos with variance one the laws of $F$ and $F_\infty$ coincide if and only if $\mu_4(F) = 9$ (resp. $\varphi(F^4) = 2^5$) for some $r \geq 3$, where $\mu_r(F)$ stands for the $r$th moment of the random variable $F$, and $\varphi$ is the relevant tracial state. We use our moments criteria to study the non central limit theorems within the second Wiener (Wigner) chaos with target random variable $F_\infty$. Our results can be seen as a slight generalization of some findings in Nourdin & Poly [24], Azmoodeh, et. al [5] in the classical probability, and of Deya & Nourdin [8] in the free probability setting.

Keywords: Second Wiener/Wigner chaos, Normal product distribution, Tetilla law, Cumulants/Moments, Wasserstein distance, Weak convergence, Malliavin Calculus

MSC 2010: 60F05, 60G50, 46L54, 60H07

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1 Introduction

The present paper aims to provide the first study in the direction of the higher moments phenomenon of a sequence of random objects living in a fixed Wiener chaos converging in distribution towards a target random element in the second Wiener chaos. In the framework of Wiener space, higher moments phenomenon refers to the situation where convergence in law of a sequence of random variables living in a fixed chaos of a Gaussian random field is implied by convergence of finitely many higher (even) moments of the prelimit to the corresponding moments of the target random variable. We refer the reader to [23] below, for any unexplained notion evoked in the present section.

We consider the "natural" target random variable $F_\infty \sim N_1 \times N_2$ in the second Wiener chaos, the product of two independent standard normal random variables. The reason to focus on target random element $F_\infty$ is that in its canonical representation (2.3) in item 1, Proposition 2.1 there exist just only two non zero coefficients $\lambda_{f,\pm 1} = \pm 1/\sqrt{2}$, and so granting the symmetric flavor to the distribution. In the pioneering article [24], Nourdin & Poly (among many other results) showed that for a normalized sequence $\{F_n\}_{n \geq 1}$ in the second Wiener chaos, i.e $\mathbb{E}(F_n^2) = 1$ for all $n$, the convergence in law of the sequence $F_n$ towards random target $F_\infty$ is equivalent with $(\kappa_6(F_n)/5! - \kappa_4(F_n)/3 + 1) \to 0$, and $\kappa_3(F_n) \to 0$. It is worth here to point out that recently, the authors of [24] have shown that even for a target variable having only two non zero, rationally independent coefficients in the canonical representation (2.3), the requirement of convergence of the third cumulant can be removed. Using the relation between cumulants and moments, the later result reads $F_n \to F_\infty$ in distribution iff $\mu_6(F_n) := \mathbb{E}(F_n^6) \to (5!)^2$, and $\mu_4(F_n) = \mathbb{E}(F_n^4) \to 9$. Our main finding is the following.

**Theorem 1.1.** Let $\{F_n\}_{n \geq 1}$ be a sequence of the random elements living in a fixed Wiener chaos of order $p \geq 2$ such that $\mathbb{E}(F_n^2) = 1$ for all $n \geq 1$. Assume that $p = 2$. Then the following asymptotic assertions are equivalent.

(I) as $n \to \infty$, sequence $F_n \to F_\infty \sim N_1 \times N_2$ in distribution.

(II) as $n \to \infty$,

1. $\mu_4(F_n) \to 9$.
2. $\mu_{2r}(F_n) \to ((2r - 1)!!)^2$ for some $r \geq 3$.

In the free probability framework, the probability distribution $F_\infty \sim (S_1S_2 + S_2S_1)/\sqrt{2}$ plays the same role as the product of two independent normal random variables, where here $S_1$, and $S_2$ stand for two freely independent semicircular random variables. In [8], Deya & Nourdin named the probability distribution $F_\infty$ the tetilla law, and prove that for a sequence $\{F_n\}_{n \geq 1}$ of random elements in a fixed Wigner chaos of arbitrary order $p \geq 2$, the sequence $F_n \to F_\infty$ iff $\varphi(F_n^4) \to \varphi(F_\infty^4) = 2.5$, and $\varphi(F_n^6) \to \varphi(F_\infty^6) = 8.25$. Our next result slightly generalizes the main result in [8].

**Theorem 1.2.** Let $\{F_n\}_{n \geq 1}$ be a sequence of non-commutative random variables living in a fixed Wigner chaos of order $p \geq 2$ such that $\varphi(F_n^2) = 1$ for all $n \geq 1$. Assume that $F_\infty$ distributed as normalized tetilla distribution as explained in above. Let $p = 2$. Then, as $n \to \infty$, the following asymptotic assertions are equivalent.

(I) sequence $F_n \to F_\infty$ is distribution.

(II) as $n \to \infty$, 

\[ 5 \text{ Convergence in Wasserstein-2 distance in 2nd Wiener/Wigner chaos} \]

\[ 5.1 \text{ Quantitative estimates in Wasserstein-2 distance} \]

\[ 5.2 \text{ Asymptotic result for the coupled sequence} \]

\[ 6 \text{ Conjecture} \]
1. \( \varphi(F_n^4) \to 5/2 \).

2. \( \varphi(F_n^{2r}) \to \varphi(F_{\infty}^{2r}) \) for some \( r \geq 3 \).

A curious reader, for what happens when \( p \neq 2 \) is referred to Section 6.

1.1 Overview of relevant results

In the landmark article [19], Nualart & Peccati established an impressive result known nowadays as the fourth moment theorem providing a drastically simple criterion, in terms of the fourth moment, for normal approximation within the Wiener chaos. Immediately, their findings create a fertile line of research, and it is culminating in the popular article [20], introducing the so called Malliavin-Stein approach, an elegant combination of two probabilistic techniques in order to quantify the probability distance of a square integrable Wiener functional from a normal distribution. The following two results provide an exhaustive characterization of normal and Gamma approximations on Wiener chaos. As in [21], we denote by \( F(\nu) \) a centered random variable with the law of \( 2G(\nu/2) - \nu \), where \( G(\nu/2) \) has a Gamma distribution with parameter \( \nu/2 \). In particular, when \( \nu \geq 1 \) is an integer, then \( F(\nu) \) has a centered chi-squared distribution with \( \nu \) degrees of freedom, in other words, in the canonical representation (2.3), all the non zero coefficients \( \lambda_i = 1 \) for \( 1 \leq i \leq \nu \). For an alternative and general fourth moment-type statements based on the novel technique of Markov triplet, see [15, 8]. In addition, for a huge amount of applications and generalizations of the forthcoming theorem, see [1] for a constantly updated web resource, with links to all available papers.

Theorem 1.3. (A) (See [19, 18]) Let \( N \sim \mathcal{N}(0, 1) \), fix \( p \geq 2 \) and let \( F_n \) be a sequence of random variables living in the Wiener chaos of order \( p \) associated to \( W \), such that \( \mathbb{E}(F_n^2) = 1 \) for all \( n \geq 1 \). Then, the following are equivalent, as \( n \to \infty \):

(i) \( F_n \) converges in distribution to \( N \);

(ii) \( \mathbb{E}(F_n^4) \to \mathbb{E}(N^4) = 3 \).

(B) (See [21]) Fix \( \nu > 0 \), and let \( F(\nu) \) have the centered Gamma distribution described above. Let \( p \geq 2 \) be an even integer, and let \( F_n \) be a sequence of random variables living in the Wiener chaos of order \( p \) associated to \( W \) verifying \( \mathbb{E}(F_n^2) = 2\nu \). Then, the following are equivalent, as \( n \to \infty \):

(i) \( F_n \) converges in distribution to \( F(\nu) \);

(ii) \( \mathbb{E}(F_n^4) - 12\mathbb{E}(F_n^3) \to \mathbb{E}(F(\nu)^4) - 12\mathbb{E}(F(\nu)^3) = 12\nu^2 - 48\nu \).

The free counterpart of Theorem 1.3 is also investigated in [13], and [22], corresponding to parts (A) and (B) respectively. The most relevant work to the present paper, is the following result established in [4] using Markov triplet approach and spectral properties of the Ornstein-Uhlenbeck generator, where the higher moment phenomenon is studied within the framework of Wiener chaos and the normal distribution (in the first Wiener chaos) as the target random element.

Theorem 1.4. Let \( F_n \) be a sequence of random elements in a fixed Wiener chaos of arbitrary order \( p \geq 2 \) such that \( \mathbb{E}(F_n^2) = 1 \) for all \( n \geq 1 \). Then, as \( n \to \infty \), the following asymptotic assertions are equivalent.

(i) \( F_n \) converges in distribution to \( N \sim \mathcal{N}(0, 1) \);

(ii) \( \mathbb{E}(F_n^{2r}) \to (2r - 1)!! \) for some \( r \geq 2 \).

The following remarks are in order. Let \( F_n \) be a normalized sequence of random elements in a fixed Wiener chaos of order \( p \geq 2 \).
Let $L$ and $\Gamma$ stand for the Ornstein-Uhlenbeck, and the associated symmetric-positive carré-du-champ operators, see [15, 3, 4] for definitions. It is a well known fact that the total variation distance $d_{TV}(F_n, \mathcal{N}(0, 1))$ can be controlled from above by $\text{Var}(\Gamma(F_n))$. The crucial fact that the quantity $\Gamma(F_n) = P(L)(Q(F_n))$ for some polynomials $P$, and $Q$ allows one to estimate the later variance quantity by higher even moments relying on the knowledge of the spectrum $\text{sp}(L) = \mathbb{N}$.

In the classical probability setting, the target random variable $F_\infty \sim N_1 \times N_2$ belongs to the so-called Variance–Gamma class, and the Malliavin–Stein methodology is successfully implemented in [11]. There, it is shown that for some constant $C$, independent of $n$,

$$d_{W_1}(F_n, F_\infty) \leq C \left\{ \sqrt{\text{Var}(\Gamma^2(F_n) - F_n)} + |\kappa_3(F_n)| \right\},$$

where $d_{W_1}$ denotes the Wasserstein-1 metric. A similar result has been also obtained, in parallel, using a different path in [5].

The variance quantity, containing the double iterated Gamma operator, appearing in the previous item, only at the special case $p = 2$ can be reformulated as (or can be estimated from above by) a positive linear combination of finitely many cumulants/moments. The main reason is that, only in the case of the second Wiener chaos, the iterated Gamma operators are stable, in the sense that the resulting random objects remain in the second Wiener chaos. Furthermore, the successful Markov triplet technique [4] cannot be handy in our situation, because of, for a random variable $F \in \text{Ker}(L + p\text{Id})$, with $p \geq 2$, the quantity of interest $(\Gamma_2(F) - F)$, unlike the normal approximation case, cannot be written as a polynomial operator in $L$ acting on a polynomial in $F$ in order to produce moment criterion.

In order to turn around the above difficulties, we restrict ourself to the second Wiener/Wigner chaos, $p = 2$, and will take advantages of the spectral representations (2.3.3, 4.3), and use the combinatorial relation between moments and cumulants.

## 1.2 Plan

The paper is organized as follows. Section 2 contains some preliminary material including basic facts on second Wiener chaos and iterated Gamma operators. Sections 3 and 4 are devoted to characterizations of $N_1 \times N_2$ and the tetilla laws within the second Wiener/Wigner chaos respectively. Section 5 contains our main results on Wasserstein-2 convergence towards $N_1 \times N_2$ and the tetilla laws in terms of higher even moments criteria. Finally the paper ends with Section 6 including a conjecture arising from our study.

## 1.3 Cumulants

The notion of cumulant will be crucial throughout the paper. We refer the reader to the monograph [25] for an exhaustive discussion.

**Definition 1.1 (Cumulants).** Let $F$ be a real-valued random variable such that $\mathbb{E}|F|^m < \infty$ for some integer $m \geq 1$. The $F$-cumulants $\kappa_\ell(F)$, $\ell = 1, \ldots, m$ are defined by the relations

$$\mathbb{E}(F^\ell) = \sum_{\pi \in \Pi_\ell} \prod_{\substack{A \in \pi \setminus \emptyset}} \kappa_{|A|}(F), \quad \ell = 1, \ldots, m,$$

where we sum over the partitions of $\{1, 2, \ldots, \ell\}$, and $|\pi|$ is the number of subsets of the partition $\pi$. Möbius inversion on the partitions lattice gives the explicit definition

$$\kappa_\ell(F) = \sum_{\pi \in \Pi_\ell} (|\pi| - 1)! (-1)^{|\pi| - 1} \prod_{A \in \pi} \mathbb{E}(F^{|A|}), \quad (1.1)$$
Remak 1.1. When $\mathbb{E}(F) = 0$, then the first six cumulants of $F$ are the following: $\kappa_1(F) = \mathbb{E}(F) = 0$, $\kappa_2(F) = \mathbb{E}(F^2) = \text{Var}(F)$, $\kappa_3(F) = \mathbb{E}(F^3)$, $\kappa_4(F) = \mathbb{E}(F^4) - 3\mathbb{E}(F^2)^2$, and

$$
\kappa_6(F) = \mathbb{E}(F^6) - 15\mathbb{E}(F^2)\mathbb{E}(F^4) - 10\mathbb{E}(F^3)^2 + 30\mathbb{E}(F^2)^3.
$$

Hence, $\mathbb{E}(F^6) = \kappa_6(F) + 15\kappa_2(F)\kappa_4(F) + 10\kappa_3^2(F) + 15\kappa_2^3(F)$.

2 Useful facts about the second Wiener chaos

We recall some relevant information about the elements in the second Wiener chaos. For a comprehensive treatment, we refer the reader to [23] Chapter 2. Consider an isonormal process $W = \{W(h); h \in \mathcal{H}\}$ over a separable Hilbert space $\mathcal{H}$. Recall that the second Wiener chaos $\mathcal{H}_2$ associated to the isonormal process $W$ consists of those random variables having the general form $F = I_2(f)$, with $f \in \mathcal{H} \otimes^2$. Notice that, if $f = h \otimes g$, where $h \in \mathcal{H}$ is such that $\|h\|_{\mathcal{H}} = 1$, then using the multiplication formula (see [23]), one has

$$
I_2(f) = W(h)^2 - 1 \overset{\text{law}}{=} N^2 - 1,
$$

where $N \sim \mathcal{N}(0, 1)$. To any kernel $f \in \mathcal{H} \otimes^2$, we associate the following Hilbert-Schmidt operator

$$
A_f: \mathcal{H} \mapsto \mathcal{H}; \quad g \mapsto f \otimes_1 g.
$$

It is also convenient to introduce the sequence of auxiliary kernels

$$
\left\{ f \otimes_1^{(p)} f : p \geq 1 \right\} \subset \mathcal{H} \otimes^2
$$

defined as follows: $f \otimes_1^{(1)} f = f$, and, for $p \geq 2$,

$$
f \otimes_1^{(p)} f = \left( f \otimes_1^{(p-1)} f \right) \otimes_1 f.
$$

In particular, $f \otimes_1^{(2)} f = f \otimes_1 f$. Finally, we write $\{\lambda_{f,j}\}_{j \geq 1}$ and $\{\epsilon_{f,j}\}_{j \geq 1}$, respectively, to indicate the (not necessarily distinct) eigenvalues of $A_f$ and the corresponding eigenvectors.

Proposition 2.1 (See e.g. Section 2.7.4 in [23]). Fix $F = I_2(f)$ with $f \in \mathcal{H} \otimes^2$.

1. The following equality holds:

$$
F = \sum_{z \in \mathbb{Z}} \lambda_{f,z} \frac{(N^2 - 1)}{\sqrt{2}},
$$

where $\{N_z\}_{z \in \mathbb{Z}}$ is a sequence of i.i.d. $\mathcal{N}(0, 1)$ random variables that are elements of the isonormal process $W$, and the series converges in $L^2$ and almost surely.

2. For any $i \geq 2$,

$$
\kappa_i(F) = 2^i - 1 \cdot (r - 1)! \sum_{z \in \mathbb{Z}} \lambda_{f,z}^i \cdot 2^r - 1 \cdot (r - 1)! \times \langle f \otimes_1^{(r-1)} f, f \rangle_{\mathcal{H} \otimes^2}.
$$

3. The law of the random variable $F$ is completely determined by its moments or equivalently by its cumulants.

Consider a generic element $F \in \mathcal{H}_2$. Since the reordering the coefficients in the representation $F$ at item 1 in Proposition 2.1 does not change the distribution of $F$, we will see that it is very useful with the convention $\lambda_{f,z} = \lambda_z$ to consider ordered coefficients with $\lambda_0 = 0$ and

$$
-\|F\|_{L^2} \leq \lambda_{-1} \leq \lambda_{-2} \leq \ldots \leq \lambda_{-n} \leq \ldots \leq 0 \leq \ldots \leq \lambda_n \leq \ldots \leq \lambda_2 \leq \lambda_1 \leq \|F\|_{L^2},
$$

(2.5)
and to separate positive and negative coefficients in the decomposition

\[ F = F_+ - F_- , \]

where \( F_\pm \) are the independent centered Generalized Gamma Convolutions (GCC)

\[ F_\pm = \sum_{\ell \in \mathbb{N}} |\lambda_{\pm \ell}| \left( \frac{N_{\pm \ell}^2 - 1}{\sqrt{2}} \right) , \]

We assume that \( F \) is normalized with \( \mathbb{E}(F^2) = \mathbb{E}(F_+^2) + \mathbb{E}(F_-^2) = 1 \).

Our aim is now to provide an explicit representation of cumulants in terms of Malliavin operators. To this end, it is convenient to introduce the following definition (see e.g. [23, Chapter 8] for a full multidimensional version).

**Definition 2.1.** Let \( F \in \mathbb{D}^\infty \). The sequence of random variables \( \{ \Gamma_i(F) \} \) is recursively defined as follows. Set \( \Gamma_0(F) = F \) and, for every \( i \geq 1 \),

\[ \Gamma_i(F) = (DF, -DL^{-1}\Gamma_{i-1}(F))_H. \]

The following statement explicitly connects the expectation of the random variables \( \Gamma_i(F) \) to the cumulants of \( F \).

**Proposition 2.2.** (See Chapter 8 in [23]) Let \( F \in \mathbb{D}^\infty \). Then \( F \) has finite moments of every order, and the following relation holds for every \( r \geq 0 \):

\[ \kappa_{r+1}(F) = r! \mathbb{E}[\Gamma_r(F)]. \quad (2.6) \]

**Proposition 2.3.** The law of

\[ N_1 \times N_2 \sim \frac{N_1^2 - N_2^2}{2} = \frac{(N_1^2 - 1) - (N_2^2 - 1)}{2} \]

is characterized in the second chaos by the coefficients \( \lambda_{\pm 1} = \pm 1/\sqrt{2} \), and \( \lambda_z = 0 \) for \( |z| \neq 1 \)

**Proof.** It is a direct consequence of the shape of the characteristic functions of elements in the second Wiener chaos, see [23, page 44]. \( \square \)

3 Classical probability and \( N_1 \times N_2 \) law

3.1 Characterization of \( N_1 \times N_2 \) within the second Wiener chaos

We start with the following general characterization of \( N_1 \times N_2 \) law inside the second Wiener chaos in terms of iterated Gamma operators.

**Proposition 3.1.** Assume \( F = I_2(f) \) be an element in the second Wiener chaos with \( \mathbb{E}(F^2) = 2||f||^2 = 1 \). Then the following assertions are equivalent.

(I) the laws of \( F \) and \( N_1 \times N_2 \) coincide, i.e \( F \sim N_1 \times N_2 \).

(II) for some \( m \neq n \in \mathbb{N} \) such that \( n + m \in 2\mathbb{N} \)

1. \( \Delta_{n,m}(F) := \text{Var}(\Gamma_{n-1}(F) - \Gamma_{m-1}(F)) = 0 \),
2. \( \kappa_r(F) = 0 \) for some odd \( r \geq 3 \).
Proof. Since \( F = I_2(f) = \sum_{i \in \mathbb{Z}} \lambda_i (N_{i}^2-1)/\sqrt{2} \) belongs to the second Wiener chaos, we have the nice representation (see [3] relation (3.7), Lemma 3.1))

\[
\Gamma_{n-1}(F) - \mathbb{E}(\Gamma_{n-1}(F)) - (\Gamma_{m-1}(F) - \mathbb{E}(\Gamma_{m-1}(F))) = I_2 \left( 2^{n/2-1} f \otimes_1^{(n)} f - 2^{m/2-1} f \otimes_1^{(m)} f \right).
\]

Therefore,

\[
\Delta_{n,m}(F) = \mathbb{E} \left( I_2(2^{n/2-1} f \otimes_1^{(n)} f - 2^{m/2-1} f \otimes_1^{(m)} f) \right)^2
= 2 \left\| 2^{n/2-1} f \otimes_1^{(n)} f - 2^{m/2-1} f \otimes_1^{(m)} f \right\|^2
= \frac{1}{2} \sum_{i \geq 1} \left( 2^{\frac{n}{2}} \lambda_i^n - 2^{\frac{m}{2}} \lambda_i^m \right)^2
= \frac{\kappa_{2n}(F)}{(2n-1)!} - 2 \frac{\kappa_{n+m}(F)}{(n+m-1)!} + \frac{\kappa_{2m}(F)}{(2m-1)!}.
\]  

(3.1)

Now assume that there are \( m < n \) with \( (n+m) \in 2\mathbb{N} \) and \( \Delta_{n,m}(F) = 0 \). Relation (3.1) implies that \( \lambda_i^2 = \{ 0, \frac{1}{2} \} \) for all \( i \geq 1 \). Combining this with the second moment assumption \( \mathbb{E}(F^2) = 1 \) we deduce that there are exactly two non zero coefficients with \( \lambda_i^2 = \frac{1}{2} \) and \( i \neq j \). If furthermore for some odd \( r \geq 3 \) we have \( \kappa_r(F) = 0 \) then necessarily \( \lambda_i = -\sqrt{\frac{1}{2}} \) and \( \lambda_j = \pm \sqrt{\frac{1}{2}} \) with opposite signs. Hence \( F \sim N_1 \times N_2 \). The other direction is obvious, because for \( F \sim N_1 \times N_2 \), we have \( \kappa_{2r}(F) = (2r-1)! \) for \( r \geq 1 \), and \( F \) is a symmetric distribution and as a result all the odd cumulants must be zero. \( \square \)

Remark 3.1. Let \( F = I_2(f) \) be a general element in the second Wiener chaos such that \( \mathbb{E}(F^2) = 2 \left\| f \right\|^2 = 1 \). Then proof of Proposition 3.1 reveals that condition \( \Delta_{n,m}(F) = 0 \) for some \( n \neq m \in \mathbb{N} \) with \( n+m \in 2\mathbb{N} \) implies that distribution of the random variable \( F \) belongs to the set of three possible probability distributions

\[
\left\{ \frac{(N_1^2 - N_2^2)}{2}, \pm \frac{(N_1^2 + N_2^2 - 2)}{2} \right\}.
\]

Hence, in order to distribution \( F \) lies exactly on the favorite target random variable \( F_\infty \sim N_1 \times N_2 \), one needs to fix \( \kappa_r(F) = 0 \) for at least one (and therefore for every) odd \( r \geq 3 \). Moreover, one has to note that outside of the second Wiener chaos those conditions stated in Proposition 3.1 do not characterize the law of random variable \( F_\infty \). A standard Gaussian random variable is a simple counterexample.

Corollary 3.1. Let \( F = I_2(f) \) be a general element in the second Wiener chaos such that \( \mathbb{E}(F^2) = 2 \left\| f \right\|^2 = 1 \). Let \( n, m \in \mathbb{N} \). Then the following chain of the estimates take place,

\[
\cdots \leq C \Delta_{n,m}(F) \leq C \Delta_{n-1,m-1}(F) \leq C \cdots \leq C \Delta_{3,1}(F) = \text{Var}(\Gamma_2(F) - F),
\]

where the quantity

\[
\Delta_{n,m}(F) = \frac{\kappa_{2n}(F)}{(2n-1)!} - 2 \frac{\kappa_{n+m}(F)}{(n+m-1)!} + \frac{\kappa_{2m}(F)}{(2m-1)!}
\]

is given at item 1 in Proposition 3.1. Moreover, if \( \Delta_{n,m}(F) = 0 \) for some \( n \neq m \in \mathbb{N} \) with \( (n+m) \in 2\mathbb{N} \), then \( \Delta_{3,1}(F) = 0 \), and therefore \( \Delta_{n,m}(F) = 0 \) for all \( n \neq m \in \mathbb{N} \) with \( (n+m) \in 2\mathbb{N} \). In particular, for \( F \sim N_1 \times N_2 \), we have \( \Delta_{n,m}(F) = 0 \) for all \( n \neq m \in \mathbb{N} \) with \( n+m \in 2\mathbb{N} \).

Proof. Let assume the nontrivial case \( n \neq m \). Then relation (3.1) yields that

\[
\Delta_{n,m}(F) = 2 \left\| 2^{n/2-1} f \otimes_1^{(n)} f - 2^{m/2-1} f \otimes_1^{(m)} f \right\|^2
= 8 \left\| f \otimes_1 g \right\|^2,
\quad g = 2^{n/2-2} f \otimes_1^{(n-1)} f - 2^{m/2-2} f \otimes_1^{(m-1)} f
\]

(3.2)

\[
\leq 8 \left\| f \right\|^2 \left\| g \right\|^2
= 2 \Delta_{n-1,m-1}(F).
\]
Now assume that we have $\Delta_{n,m}(F) = 0$ for some $n \neq m \in \mathbb{N}$ with $(n + m) \in 2\mathbb{N}$. Then proof of Proposition 3.1 tells us that all the nonzero coefficients must satisfy in $\lambda_i^2 = \frac{1}{2}$, and therefore $\Delta_{3,1}(F) = 0$ which implies that $\Delta_{n,m}(F) = 0$ for all $n \neq m \in \mathbb{N}$ with $(n + m) \in 2\mathbb{N}$.

The following result aims to provide some variance calculus of the iterated Gamma random variables.

**Proposition 3.2.** Let $F = I_2(f)$ be a random variable in the second Wiener chaos with $\mathbb{E}(F^2) = 1$. Then for $r \in \mathbb{N}$ there exists a constant $C = C_r$ such that the following variance estimates take place.

$$\mathbb{Var}(\Gamma_r(F) - \Gamma_{r-1}(F)) \leq C \mathbb{Var}(\Gamma_r(F) - \Gamma_{r-2}(F)) \times \mathbb{Var}(\Gamma_{r+2}(F) - \Gamma_r(F)), \quad r \geq 2$$

$$\mathbb{Var}^{2r}(\Gamma_3(F) - \Gamma_1(F)) \leq C \mathbb{Var}^{2r-1}(\Gamma_2(F) - F) \times \mathbb{Var}(\Gamma_{2r+2}(F) - \Gamma_{2r}(F)), \quad r \geq 1.$$

In particular case, we obtain

$$\mathbb{Var}^{2}(\Gamma_3(F) - \Gamma_1(F)) \leq C \mathbb{Var}(\Gamma_2(F) - F) \times \mathbb{Var}(\Gamma_4(F) - \Gamma_2(F)).$$

**Proof.** Denote $A_f: \mathcal{H} \to \mathcal{H}$ defined as $g \mapsto (f,g)_{\mathcal{H}}$ the associated Hilbert-Schmidt operator to the kernel $f$. It is well known that for $r \geq 2$ (see for example [23])

$$\kappa_r(F) = 2^{2r-1}(r-1)! \mathbb{Tr}(A_f^r)$$

where $\mathbb{Tr}(A_f^r)$ stands for the trace of the $r$th power of $A_f$. Using relation (3.1) together with some direct computations one can get that for $r \geq 2$,

$$\mathbb{Var}(\Gamma_r(F) - \Gamma_{r-2}(F)) = 2^{2r-3} \mathbb{Tr}((2^2A_f^{r+1} - A_f^{r-1})^2).$$

Now, the first variance estimate is an application of [9] Corollary 1 with $P = (A_f^{r+2} - A_f^2)\mathbb{I}_N, C = A_f^2, \mathbb{I}_N$ and the second variance estimate can be deduced from [10] Corollary 1 with $P = (2^2A_f^3 - A_f)^2$ and the convex function $f(x) = x^{2r}$.

**3.2 Case $\mathbb{E}(F^4) \geq 9$**

**Proposition 3.3.** Let $F$ be a general element in the second Wiener chaos, and $F_\infty \sim N_1 \times N_2$. Then, for every $r \geq 2$,

$$\frac{\kappa_{2r}(F)}{(2r-1)! \kappa_2(F)} - 1 \geq (r-1)\left\{ \frac{\kappa_4(F)}{3! \kappa_2(F)} - 1 \right\}.$$ (3.3)

When $\kappa_2(F) = 1$, we have for $r \geq 2$

1. $\kappa_4(F) - \kappa_4(F_\infty) \leq \frac{3!}{(r-1)(2r-1)!} \left\{ \kappa_{2r}(F) - \kappa_{2r}(F_\infty) \right\}$.

2. $\frac{r}{2r-1} \left\{ \frac{\kappa_{2r}(F)}{(2r-1)!} - 1 \right\} \leq \left\{ \frac{\kappa_{2r+2}(F)}{(2r+1)!} - 1 \right\}$.

Furthermore, assume that $\kappa_2(F) = 1$, and $\kappa_4(F) \geq 6$, then for all $r \geq 1$

$$\kappa_{2r}(F) \geq \kappa_{2r}(F_\infty) = (2r-1)!,$$ (3.4)

and, if (3.4) becomes equality for some $r \geq 3$, then it becomes equality for all $r \geq 3$.

**Proof.** First note that using relation (3.1),

$$0 \leq \mathbb{Var}(\Gamma_2(F) - F) \leq \sum_{2 \leq s \leq r-1} \mathbb{Var}(\Gamma_s(F) - \Gamma_{s-2}(F))$$

$$= \left\{ \frac{\kappa_{2r}(F)}{(2r-1)!} - \frac{\kappa_{2r-2}(F)}{(2r-2)!} \right\} - \left\{ \frac{\kappa_4(F)}{3!} - \kappa_2(F) \right\}. $$
Hence
\[
\frac{\kappa_{2r}(F)}{(2r-1)!} - \frac{\kappa_{2r-2}(F)}{(2r-2)!} \geq \frac{\kappa_4(F)}{3!} - \kappa_2(F), \quad r \geq 2.
\]

Using a telescopic argument yields that
\[
\frac{\kappa_{2r}(F)}{(2r-1)!} - \kappa_2(F) = \sum_{2 \leq s \leq r} \left\{ \frac{\kappa_{2s}(F)}{(2s-1)!} - \frac{\kappa_{2s-2}(F)}{(2s-2)!} \right\} \geq (r-1)\left\{ \frac{\kappa_4(F)}{3!} - \kappa_2(F) \right\}.
\]

Next, we prove item 2. We proceed with induction on \( r \geq 2 \). Let \( r = 2 \), and we assume that \( \kappa_2(F) = 1 \), then
\[
0 \leq \Delta_{3,1}(F) = \frac{\kappa_6(F)}{5!} - 2\frac{\kappa_4(F)}{3!} + 1 = \left( \frac{\kappa_6(F)}{5!} - 1 \right) - 2 \left( \frac{\kappa_4(F)}{3!} - 1 \right).
\]

Similarly, using induction hypothesis,
\[
0 \leq \Delta_{r+1,r-1}(F) = \frac{\kappa_{2r+2}(F)}{(2r+1)!} - 2\frac{\kappa_{2r}(F)}{(2r-1)!} + \frac{\kappa_{2r-2}(F)}{(2r-3)!} = \left( \frac{\kappa_{2r+2}(F)}{(2r+1)!} - 1 \right) - 2 \left( \frac{\kappa_{2r}(F)}{(2r-1)!} - 1 \right) + \frac{r-2}{r-1} \left( \frac{\kappa_{2r}(F)}{(2r-1)!} - 1 \right),
\]
which implies the claim. Item 1 can be also shown in similar way. Moreover (3.3) is a direct application of (3.3).

**Proposition 3.4.** Let \( F \) be a generic element in the second Wiener chaos such that \( \mathbb{E}(F^2) = \kappa_2(F) = 1 \), and \( \mathbb{E}(F^4) \geq 9 \) (or equivalently \( \kappa_4(F) \geq 6 \)). Then
\[
\mu_{2r}(F) := \mathbb{E}(F^{2r}) \geq \mu_{2r}(N_1 \times N_2) = (2r-1)!! = \left( \frac{2r!}{r!2^r} \right)^2. \tag{3.5}
\]

If (3.5) is an equality for some \( r \geq 3 \), it holds as equality for all \( r \geq 1 \). In such case we have also \( \mathbb{E}(F^{2r+1}) = \kappa_{2r+1}(F) = 0 \) for all \( r \geq 1 \), and necessarily \( F \stackrel{\text{law}}{=} N_1 \times N_2 \).

**Proof.** First we recall that moments and cumulants are related by
\[
\mu_n(F) = \mathbb{E}(F^n) = \sum_{\pi \in \Pi_n} \prod_{A \in \pi} \kappa_{|A|}(F) \tag{3.6}
\]
where the sum is over the set of partitions of \( \Pi_n \) of the set \([n] := \{1, \ldots, n\}\), and the product is over the partition components. Note also that \( \kappa_n(\alpha F) = \alpha^n \kappa_n(F) \) for any scalar \( \alpha \) and \( \kappa_n(F) = \kappa_n(F_+) + \kappa_n(-F_-) \), since \( F = (F_+ - F_-) \) with independent \( F_\pm \). Next we compare the even moments of \( F \) with the even moments of \( N_1 N_2 \) by using (3.4). Since by assumption \( F \) and \( N_1 \times N_2 \) have the same 2nd moment and \( \mathbb{E}(F^4) \geq \mathbb{E}(N_1^4)^2 = 9 \), necessarily also \( \kappa_4(F) \geq \kappa_4(N_1 \times N_2) \) and from the cumulant inequalities (3.3) it follows that
\[
\kappa_{2n}(F) \geq \kappa_{2n}(N_1 \times N_2), \quad \forall n \geq 3 \tag{3.7}
\]
and if this inequality is an equality for some \( n \geq 3 \), it holds as equality \( \forall n \geq 3 \) as well. Note that
\[
\mu_{2n} = \mathbb{E}(F^{2n}) = \sum_{\pi \in \Pi_{2n}} \prod_{A \in \pi} \kappa_{|A|}(F) + \sum_{\rho \in \Pi_{2n}^\rho} \prod_{B \in \rho} \kappa_{|B|}(F) \tag{3.8}
\]
where $\Pi^\prime_{2n}$ are the partitions of $2n$ containing only components of even size, and $\Pi^\prime_{2n} = \Pi_{2n} \setminus \Pi^\prime_{2n}$ is its complement, whose partition elements contain a non-zero even number of components with odd size. By \[\eqref{3.7}\], it is clear that
\[
\sum_{\pi \in \Pi^\prime_{2n}} \prod_{A \in \pi} \kappa_{|A|}(F) \geq \sum_{\pi \in \Pi^\prime_{2n}} \prod_{A \in \pi} \kappa_{|A|}(N_1N_2)
\]
when all sets $A$ have even size. In order to show that $\mu_{2n}(F) \geq \mu_{2n}(N_1N_2)$, it is enough to show that under the assumptions all the odd cumulants of $F$ have the same sign or vanish, namely
\[
\kappa_{2n+1}(F)\kappa_{2m+1}(F) = (\kappa_{2n+1}(F_+) - \kappa_{2n+1}(F_-))(\kappa_{2m+1}(F_+) - \kappa_{2m+1}(F_-)) \geq 0 \quad \forall n > m,
\]
implies that the second sum in \[\eqref{3.8}\] is always non-negative.

The condition $\mathbb{E}(F^4) \geq 9$, which together with $\mathbb{E}(F^2) = 1$, $\mathbb{E}(F) = 0$ is equivalent to $\kappa_4(F) \geq 6$, implies that
\[
\lambda_1^2 \leq \frac{1}{2} \leq \lambda_1^2 \quad \text{or} \quad \lambda_1^2 \leq \frac{1}{2} \leq \lambda_{-1}^2.
\]

By contradiction, assume that $\lambda_1^2 < 1/2$ strictly $\forall z$, which implies
\[
\frac{1}{2} = \frac{\kappa_4(N_1N_2)}{12} \leq \sum_{z \in \mathbb{Z}} \lambda_z^4 < \frac{1}{2} \sum_{z \in \mathbb{Z}} \lambda_z^2 = \frac{\mathbb{E}(F^2)}{2} = \frac{1}{2}
\]
with strict inequality, which is a contradiction. Therefore, $\exists z \in \mathbb{Z}$ with $\lambda_z^2 \geq 1/2$. Assume without loss of generality that
\[
\frac{1}{2} \leq \lambda_1^2 \leq \sum_{\ell \in \mathbb{N}} \lambda_\ell^2 = \mathbb{E}(F^2) = 1 - \mathbb{E}(F^2).
\]

Then, $\forall \ell \geq 1$
\[
\lambda_\ell^2 \leq \sum_{\ell \in \mathbb{N}} \lambda_\ell^2 = \mathbb{E}(F^2) \leq \frac{1}{2}.
\]

Now it follows $\kappa_n(F_+) \geq \kappa_n(F_-) \forall n \geq 2$, since
\[
\sum_{\ell \in \mathbb{N}} \lambda_\ell^2 \geq \lambda_1^n \geq 2^{-n/2} \geq \left(\sum_{\ell \in \mathbb{N}} \lambda_\ell^2\right)^{n/2} \geq \sum_{\ell \in \mathbb{N}} |\lambda_{-\ell}|^n
\]
where the last inequality is referred as Jensen inequality for sequences, which is strict unless the series has at most one nonzero term $\Omega 2$.

\[\Box\]

**Corollary 3.2.** For a random element $F$ in the second Wiener chaos with $\mathbb{E}(F^2) = 1$, $\mathbb{E}(F^4) \geq 9$, and $\mathbb{E}(F^{2r}) \leq ((2r - 1)!!)^2$ for some $r \geq 3$ necessarily we have $F \overset{\text{law}}{=} N_1 \times N_2$.

**Remark 3.2.** It worth to separately point it out that the random variable $F_\infty \sim N_1 \times N_2$ minimizes all the even moments/cumulants among the class of random elements in the second Wiener chaos having the moment properties $\mathbb{E}(F^2) = 1$, and $\mathbb{E}(F^4) \geq 9$, see also Proposition \[3.6\].

**Remark 3.3.** The assumption $\mathbb{E}(F^4) \geq 9$ in Corollary \[3.2\] is essential and cannot be dropped. For example, consider a random element $F$ in the second Wiener chaos with three non zero coefficients $\lambda_1 = 0.7624, \lambda_2 = 0.5370, \lambda_{-1} = 0.3610$, i.e
\[
F = \frac{1}{\sqrt{2}}(\lambda_1(N_1^2 - 1) + \lambda_2(N_2^2 - 1) - \lambda_{-1}(N_{-1}^2 - 1))
\]
where \( N_1, N_2, N_\perp \sim \mathcal{N}(0,1) \) are independent. We found these \( \lambda_i \) values by minimizing numerically the 4-th moment with 2nd and 6th moment constraints. For such random variable \( F \) (up to numerical precision) we get we get \( \mathbb{E}(F^2) = 1, \mathbb{E}(F^6) = (5!)^2 = 225 \), and obviously \( F \) is not distributed as \( N_1 \times N_2 \). This is because of \( \mathbb{E}(F^4) = 8.2567 < 9 \).

**Proposition 3.5.** Under the assumptions of Proposition 3.4, for \( 2 \leq m \leq n \in \mathbb{N} \), we have

\[
\mu_{2n}(F) - \mu_{2n}(N_1 \times N_2) \geq \left( \frac{2n - 2m}{2} \right) \left( \mu_{2m}(F) - \mu_{2m}(N_1 \times N_2) \right).
\]

**Proof.** By the cumulants-to-moments formula

\[
\mu_{2n}(F) - \mu_{2n}(N_1N_2) = \sum_{\pi \in \Pi_{2n}} \left\{ \prod_{A \in \pi} \kappa_{|A|}(F) - \prod_{A \in \pi} \kappa_{|A|}(N_1N_2) \right\}.
\]

Now for each partition \( \pi \in \Pi_{2n} \),

\[
\prod_{A \in \pi} \kappa_{|A|}(F) \geq \prod_{A \in \pi} \kappa_{|A|}(N_1N_2).
\]

Indeed if the partition \( \pi \) contains any part \( A \) with odd size, then the right side is zero, and the left side is non-negative since even cumulants are non-negative, there must be an even number of odd parts in the partition and under the assumptions all odd cumulants have the same sign. Otherwise the partition \( \pi \) contains only parts of even size, but then we have shown that under the assumptions

\[
\kappa_{2\ell}(F) \geq \kappa_{2\ell}(N_1N_2) \geq 0, \quad \forall \ell \in \mathbb{N},
\]

and the inequality is preserved when we take product over the partition. If \( \pi \) is partition of \( 2m \), let’s say \( \pi = \{A_1, A_2, \ldots, A_r\} \) with \( A_i \cap A_j = \emptyset \) for \( i \neq j \) and \( A_1 \cup A_2 \cup \ldots \cup A_r = \{1, 2, \ldots, 2m\} \), then we can add \((n - m)\) pairs to obtain

\[
\pi' = \{A_1, A_2, \ldots, A_r, \{2m + 1, 2(m + 1)\}, \ldots, \{2n - 1, 2n\}\}
\]

which is a partition of \( \{1, 2, \ldots, 2n\} \), and

\[
\prod_{A' \in \pi'} \kappa_{|A'|}(F) = \kappa_2(F)^{n-m} \prod_{A \in \pi} \kappa_{|A|}(F) = \prod_{A \in \pi} \kappa_{|A|}(F).
\]

Since we could choose those pairs differently, for every partition of \( 2m \) there are at least \( \left( \frac{2n - 2m}{2} \right) \) partitions of \( 2n \) which contribute equally to the sum, and we get

\[
\sum_{\pi' \in \Pi_{2n}} \left( \prod_{A' \in \pi'} \kappa_{|A'|}(F) - \prod_{A' \in \pi'} \kappa_{|A'|}(N_1N_2) \right) \geq \left( \frac{2n - 2m}{2} \right) \sum_{\pi \in \Pi_{2m}} \left( \prod_{A \in \pi} \kappa_{|A|}(F) - \prod_{A \in \pi} \kappa_{|A|}(N_1N_2) \right) \geq 0
\]

\( \blacksquare \)

In what follows the notation \( \mathcal{H}_2^{\text{symm}} \) stands for the collection of random variables in the second Wiener chaos with symmetric distributions.

**Proposition 3.6.** Let \( F \in \mathcal{H}_2^{\text{symm}} \) such that \( \mathbb{E}(F^2) \leq 1 \). Then

1. for \( r \in \mathbb{N} \), we have \( \kappa_{2r}(F) \leq \kappa_{2r}(N_1 \times N_2) \).
2. for \( r \in \mathbb{N} \), we have \( \mu_{2r}(F) \leq \mu_{2r}(N_1 \times N_2) \).
3. if one of these cumulant or moment inequalities at items 1 or 2 is an equality for some \( r \geq 2 \), then \( F \) law \( N_1 \times N_2 \).
Proof. Since \( \lambda_\ell = -\lambda_{-\ell} \) for \( F \in \mathcal{H}^{\text{symm}}_2, \ell \in \mathbb{N}, \) and \( \mathbb{E}(F^2) \leq 1, \) for \( r \in \mathbb{N} \) by using Jensen inequality
\[
\frac{2^{1-r}}{(2r-1)!} \kappa_{2r}(F) = \sum_{\ell \in \mathbb{Z}} \lambda_{2\ell}^r = 2 \sum_{\ell \in \mathbb{N}} \lambda_{2\ell}^r \leq 2 \left( \sum_{\ell \in \mathbb{N}} \lambda_{2\ell}^2 \right)^r = \frac{2^{1-r}}{(2r-1)!} \kappa_{2r}(N_1 \times N_2)
\]
with equality if and only if \( \lambda_\ell = 0 \forall \ell \notin \{-1, 1\} \). This proves item 1, with equality if and only if \( F \overset{\text{law}}{=} N_1 \times N_2 \) and hence the half of the item 3. Since \( F \) is symmetric, the odd cumulants are zero, and by the cumulant to moments formula we obtain
\[
\mu_{2r}(F) = \sum_{\pi \in \Pi_{2r}} \prod_{A \in \pi} \kappa_{|A|}(F) \leq \sum_{\pi \in \Pi_{2r}} \prod_{A \in \pi} \kappa_{|A|}(N_1 \times N_2) = \mu_{2r}(N_1 \times N_2), \tag{3.9}
\]
which is an equality if and only if \( \kappa_{2s}(F) = \kappa_{2s}(N_1 \times N_2) \forall s \leq r \), meaning that \( F \overset{\text{law}}{=} N_1 \times N_2 \). Hence item 2 is shown together with the remaining half part of item 3. \( \square \)

3.3 Case \( \mathbb{E}(F^4) < 9 \)

In this section, we aim to cover the case when \( F_n \) is a sequence of random elements in the second Wiener chaos such that \( \lim \inf_n \mathbb{E}(F_n^4) \leq \mathbb{E}((N_1 \times N_2)^4) = 9. \) For example, imagine the case when \( \mu_4(F_n) \to 9 \) from below as \( n \to \infty. \) We start with the following useful observation on the geometry of \( \ell^p \) spaces. Let \( p > 0. \) For a sequence \( x \) we denote \( \|x\|_p := \left( \sum_{i \geq 1} |x_i|^p \right)^{1/p}, \|x\|_\infty := \sup_{i \in \mathbb{N}} |x_i|. \)

Lemma 3.1. Let \( \epsilon < \frac{1}{6}. \) Assume \( x = (x_1, x_2, \cdots) \in \mathbb{R}^{\mathbb{N}} \) such that \( \|x\|_1 = 1, \|x\|_\infty < \frac{1}{2}, \) and \( \|x\|_2^2 > \frac{1}{2} - \epsilon. \) Then, there are exactly two indices \( k \neq l \) such that \( \frac{1}{2} - |x_k| < \epsilon, \frac{1}{2} - |x_l| < \epsilon, \) and \( \sum_{i \neq k,l} |x_i| < 2\epsilon. \)

Proof. Assume \( \epsilon > 0 \) is sufficiently small. We will make it clear at the end. Without loss of generality assume that \( x_i \geq 0 \forall i, \) (otherwise consider the sequence \( |x_i| \)). Denote
\[
A := \sum_{i \geq 1} \left( \frac{1}{2} - x_i \right) x_i.
\]
Then \( 0 < A = \frac{1}{2} \|x\|_1 - \|x\|_2^2 = \frac{1}{2} - \|x\|_2^2 < \epsilon. \) Set \( I = \{i \geq 1 : x_i > \frac{1}{2} - \epsilon\}. \) Then \( I \neq \emptyset, \) otherwise \( x_i \leq \frac{1}{2} - \epsilon \) for all \( i \geq 1, \) and therefore \( \|x\|_2^2 \leq (\frac{1}{2} - \epsilon) \sum_{i \geq 1} x_i = \frac{1}{2} - \epsilon \) which is a contradiction. Next, we show that \( \#I \geq 2. \) By contrary assume that \( \#I = 1, \) and \( j \in I. \) Then
\[
\|x\|_2^2 = \sum_{i \geq 1} x_i^2 = x_j^2 + \sum_{i \neq j} x_i^2 < x_j^2 + \epsilon (1 - x_j) < \frac{1}{4} + \epsilon (\epsilon + \frac{1}{2}) < \frac{1}{2} - \epsilon
\]
for every \( \epsilon < \frac{1}{4}, \) which is again a contradiction. Obviously \( \#I < \infty, \) and now we are going to show that in fact \( \#I = 2. \) To this end, note that
\[
\epsilon > A \geq \sum_{i \in I} (\frac{1}{2} - x_i) x_i > (\frac{1}{2} - \epsilon) \sum_{i \in I} (\frac{1}{2} - x_i)
= (\frac{1}{2} - \epsilon) \left( \frac{\#I}{2} - \sum_{i \in I} x_i \right)
\geq (\frac{1}{2} - \epsilon) \left( \frac{\#I}{2} - 1 \right).
\]
Hence \( \#I = 2, \) otherwise for \( \epsilon < \frac{1}{6} \) the above chain of inequalities do not takes place. \( \square \)
Take an element $F$ in the second Wiener chaos. As it indicates in the proof of Proposition 3.4 the key point to control the signs of the products of the odd cumulants of $F$ was to realize at least one coefficient $\lambda_i$ in the representation of $F$ such that $\lambda_i^2 \geq \frac{1}{2}$.

The next corollary studies the situation that all $\lambda_i^2 \leq \frac{1}{2}$.

**Corollary 3.3.** Let $\varepsilon < \frac{1}{\sqrt{2}}$, and $F = \sum_{i \in \mathbb{Z}} \lambda_i (N_i^2 - 1)/\sqrt{2}$ be a random variable in the second Wiener chaos such that $\mathbb{E}(F^2) = 1$, and $9 \geq \mathbb{E}(F^4) > 9 - \varepsilon$ (or equivalently $6 \geq \kappa_4(F) > 6 - \varepsilon$). If $\lambda_i^2 \leq 1/2$ for all $i \geq 1$, there exist exactly two indices $i \neq l$ such that

(i) $0 \leq 1/2 - \lambda_i^2 < \varepsilon$, and also $0 \leq 1/2 - \lambda_l^2 < \varepsilon$.

(ii) $\sum_{i \neq k, l} \lambda_i^2 < 2\varepsilon$.

**Proof.** This is a direct application of Lemma 3.1 with $x_i = \lambda_i^2$.

**Remark 3.4.** One has to note that under the assumptions of Corollary 3.3 even for very tiny $\varepsilon > 0$ the laws of $F$ and $N_1 \times N_2$ might be very different. For example, consider the simple random variable $F = \frac{1}{2}(N_1^2 - 1) + \frac{1}{2}(N_2^2 - 1)$ where $N_1, N_2 \sim \mathcal{N}(0, 1)$ are independent. We get $\mathbb{E}(F^2) = 1$, and $\mathbb{E}(F^4) = 9$ matching the fourth moment of $N_1 \times N_2$, however $F$ has centered chi squared distribution with two degrees of freedom. Indeed this observation highlights the role of a higher even moment matching.

The next lemma is a well known fact in the Wiener analysis for all chaoses, see for example [23, Corollary 2.8.14]. However, to be self-contained, we provide a simple proof of the fact in the case of the second Wiener chaos. We will use it in Section 5.

**Lemma 3.2.** (hypercontractivity) The cumulants and moments of a r.v. $F \in \mathcal{H}_2$ satisfy

$$|\kappa_n(F)| \leq 2^{n/2-1}(n-1)! \kappa_2(F)^{n/2}, \quad |\mathbb{E}(F^n)| \leq C_n \mathbb{E}(F^2)^{n/2}$$

with constants

$$C_n = 2^{n/2} \sum_{\pi \in \Pi_n} 2^{-|\pi|} \prod_{A \in \pi} (|A|-1)! = 2^{-n/2} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{(2k)!}{k! 2^k}$$

with equalities if and only if $F \overset{law}{=} \pm (N^2 - 1)/\sqrt{2}$, where $N \sim \mathcal{N}(0, 1)$ is standard Gaussian.

**Proof.** For $F \in \mathcal{H}_2$ with representation (2.3), by applying Jensen inequality to (2.4) we obtain

$$|\kappa_n(F)| = 2^{n/2-1}(n-1)! \left| \sum_{i \in \mathbb{Z}} \lambda_i^2 \right| \leq 2^{n/2-1}(n-1)! \left( \sum_{i \in \mathbb{Z}} \lambda_i^2 \right)^{n/2} = 2^{n/2-1}(n-1)! \kappa_2(F)^{n/2}$$

with equality if and only if the series have at most one nonzero term. Then the claim follows by the cumulant to moment relation.

### 4 Free probability and the "tetilla law"

We introduce some basic notions of non-commutative probability theory, following very closely [17, Ch.8], [24]. A free probability space is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a Von-Neumann algebra (that is, an algebra of bounded operators on a complex separable Hilbert space, closed under the adjoint and convergence in weak operator topology) and a trace $\varphi : \mathcal{A} \rightarrow \mathbb{R}$, that is a weakly continuous linear operator satisfying $\varphi(1) = 1$, which is tracial (meaning that $\varphi(XY) = \varphi(YX) \forall X, Y \in \mathcal{A}$), positive and faithful (meaning that $\varphi(XX^*) \geq 0 \forall X \in \mathcal{A}$, with equality if and only if $X = 0$). Elements of the algebra $\mathcal{A}$ are called non-commutative random variables.
We say that the unital subalgebras \(A_1, \ldots, A_n\) of \(A\) are freely independent when the following property holds: \(\forall m, \forall X_1, \ldots, X_m\) such that \(\varphi(X_i) = 0\) and \(X_i \in A_j\) for some \(1 \leq j \leq m\), and, \(\forall i = 1, \ldots, m - 1\), consecutive \(X_i, X_{i+1}\) do not belong to the same \(A_j\) subalgebra, then \(\varphi(X_1X_2\ldots X_n) = 0\). We say that the non-commutative random variables \(X_1, \ldots, X_n\) are freely independent if the unital subalgebras they generate are freely independent. If \(X, Y\) are free, we have \(\varphi(X^nY^n) = \varphi(X^n)\varphi(Y^n)\) as in the classical case, however \(\varphi((XY)^2) = \varphi(Y)^2\varphi(X^2) + \varphi(Y^2)\varphi(X)^2 - \varphi(Y)^2\varphi(X)^2\). We remark that classical probability is included in free probability theory as a special case, when we consider

\[
\mathcal{A} = \bigcap_{p<\infty} L^p(\Omega, \mathcal{F}, \mathbb{P}), \quad \varphi(X) = \mathbb{E}(X).
\]

A partition \(\rho\) of \(\{1, \ldots, n\}\) is said to be non-crossing if there are integers \(1 \leq p_1 < q_1 < p_2 < q_2 \leq n\) such that \(p_1, p_2\) are in the \(\rho\)-partition block \(B\), and \(q_1, q_2\) are in the \(\rho\)-partition block \(B'\), then necessarily \(B = B'\).

Moments \(\mu_n(F)\) and free cumulants \(\hat{\kappa}_\ell(F)\) of a non-commutative random variable \(F\) are defined by the relations

\[
\mu_n(F) := \varphi(F^n) = \sum_{\rho \in NC_n} \prod_{A \in \rho} \hat{\kappa}_{|A|}(F), \quad \text{ (4.1)}
\]

where the sum is over the non-crossing partitions of \(\{1, \ldots, n\}\). Möbius inversion formula is given by

\[
\hat{\kappa}_n(F) = \sum_{\rho \in NC_n} (-1)^{|\rho|-1} C_{|\rho|-1} \prod_{A \in \rho} \mu_{|A|}(F).
\]

where \(C_n = (2n)!(n+1) = \#\{\text{non-crossing partitions of } n\}\) are the Catalan numbers [13].

### 4.1 Semi-circular law

Following [16], we say that a probability distribution \(Q\) on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) is the law of the non-commutative random variable \(F\) if

\[
\varphi(F^n) = \int_{\mathbb{R}} x^n Q(dx), \quad \forall n \in \mathbb{N}.
\]

The semicircle law of parameter \(t > 0\) has density with respect to the Lebesgue measure given by

\[
q_t(x) = \frac{1}{2\pi t} \sqrt{4t - x^2} 1(-2\sqrt{t} < x < 2\sqrt{t}).
\]

Since the semicircle law is symmetric, the odd moments vanish, and for the even moments we have

\[
\int_{-2\sqrt{t}}^{2\sqrt{t}} x^{2n} q_t(x) dx = C_n t^n.
\]

In particular a classical or non-commutative \(t\)-semicircular random variable \(S(t)\) has \(\mu_2(S(t)) = t\) and \(\mu_4(S(t)) = 2t^2\). The free cumulants to moment relation (4.1) implies that \(\hat{\kappa}_2(S(t)) = t\) and \(\hat{\kappa}_n(S(t)) = 0\), for all \(n \neq 2\). In non-commutative probability the semi-circular law plays the same role as the Gaussian law in classical probability.
4.2 Tetilla law

The tetilla law is the distribution of the non-commutative random variable

\[ F_\infty := \frac{S_1^2 - S_2^2}{\sqrt{2}} = \frac{S_1S_2 + S_2S_1}{\sqrt{2}}, \]  

where \( S_1 \) and \( S_2 \) are freely independent semicircular random variables with unit variance. It takes its name from the resemblance of the density function with the anatomical profile [8]. It can be shown that \( \hat{\kappa}_n(S(t)^2) = t^n \) [16, Proposition 12.13]. By symmetry \( \varphi(F_{2n+1}) = \hat{\rho}_{2n+1}(F_\infty) = 0 \), while the free cumulants and moments of the tetilla law are

\[ \hat{\kappa}_{2n}(F_\infty) = 2^{1-n}, \quad \varphi(F_{2n}^2) = \frac{1}{2^{2n}} \sum_{k=1}^{n} 2^k \binom{n}{k} \binom{2n}{k-1}, \quad \text{see} \ [8]. \]

4.3 Free Brownian motion

A free Brownian motion on the non-commutative \((\mathcal{A}, \varphi)\) consists in a filtration \((\mathcal{A}_t)_{t \geq 0}\), which is a sequence of unital sub-algebra of \( \mathcal{A} \) with \( \mathcal{A}_u \subset \mathcal{A}_t \) for \( 0 \leq u < t \), and a collection of self-adjoint operators \((S(t) : t \geq 0)\) such that

1. \( S(t) \in \mathcal{A}_t \) for all \( t \),
2. each \( S(t) \) has the semicircular law with parameter \( t \),
3. for every \( 0 \leq u \leq t \) the increment \((S(t) - S(u))\) is freely independent from \( \mathcal{A}_u \) and it has the semicircular law with variance parameter \((t-u)\).

The free Brownian motion can thought as a matrix-valued Brownian motion in infinite dimension.

4.4 Characterization of the tetilla law: case \( \varphi(F^4) \geq 5/2 \)

Throughout this section, the random element \( F_\infty \) distributed as a normalized tetilla law given as [12]. In analogy with the Wiener chaos with respect to classical Brownian motion, the \( q \)-th Wigner chaos with respect to free Brownian motion is constructed in [6] as follows: for a simple function of the form

\[ f(t_1, \ldots, t_q) = 1(a_1 < t_1 < b_1) \times \ldots \times 1(a_q < t_q < b_q) \]

with \( 0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_q < b_q \) define

\[ I_q^S(f) = (S_{b_1} - S_{a_1})(S_{b_2} - S_{a_2}) \ldots (S_{b_q} - S_{a_q}). \]

Let \( f \in L^2(\mathbb{R}_+^q) \), and define the adjoint function \( f^*(t_1, t_2, \ldots, t_q) := f(t_q, \ldots, t_2, t_1) \). In general, object \( I_q^S(f) \) for \( f \in L^2(\mathbb{R}_+^q) \) can be defined by a density argument, using linearity and the isometry

\[ \langle I_q^S(f), I_q^S(g) \rangle_{L^2(\mathcal{A}, \varphi)} = \varphi(I_q^S(f^*)I_q^S(g)) = \varphi(I_q^S(f^*)I_q^S(g)) = \int_{\mathbb{R}_+^q} f(x)g(x)dx = \langle f, g \rangle_{L^2(\mathbb{R}_+^q)}, \]

which follows immediately for simple functions \( f, g \) vanishing on diagonals. Let \( S = (S(t) : t \geq 0) \) be a free Brownian motion defined on a non-commutative probability space \((\mathcal{A}, \varphi)\). Similarly, every element \( F = I_q^S(f) \) with \( f \in L^2_{\text{symm}}(\mathbb{R}_+^q) \) in the second Wigner chaos \( \mathcal{H}^S_2 \) allows the following representation

\[ I_q^S(f) = \sum_{z \in \mathbb{Z}} \lambda_z (S_2^z - 1) \]  

(4.3)
where $S_z$ are freely independent centered semicircular non-commutative random variables with unit variance and the series converges in $L^2(\mathcal{A}, \varphi)$ [24, Proposition 2.3]. The free cumulants of $F = I_2^2(f)$ are given by $\hat{\kappa}_1(F) = 0$ and

$$\hat{\kappa}_n(F) = \sum_{\pi \in \mathcal{Z}} \lambda_\pi^n, \quad n \geq 2. \tag{4.4}$$

For the free cumulants, we have if $F \in \mathcal{H}_2^S$, with $\varphi(F^2) = 1$ and $(nm) \in 2\mathbb{N}$

$$2^n \hat{\kappa}_{2m}(F) + 2^n \hat{\kappa}_{2n}(F) - 2^{(n+m+2)/2} \hat{\kappa}_{n+m}(F) \geq 0$$

with equality if $F = F_\infty$. As before, by telescoping

$$2^{n-1} \hat{\kappa}_{2n}(F) - \hat{\kappa}_2(F) = \sum_{\ell=1}^{n-1} (2^\ell \hat{\kappa}_{2\ell+2}(F) - 2^{\ell-1} \hat{\kappa}_{2\ell}(F)) \geq (n-1)(2\hat{\kappa}_4(F) - \hat{\kappa}_2(F))$$

since

$$2^n \hat{\kappa}_{2n}(F) - 2^{n-1} \hat{\kappa}_{2n-2}(F) \geq 2^{n-1} \hat{\kappa}_{2n-2}(F) - 2^{n-2} \hat{\kappa}_{2n-4}(F) \geq \ldots \geq 4\hat{\kappa}_4(F) - 2\hat{\kappa}_2(F),$$

and if one of these inequalities is an equality, and $\hat{\kappa}_{2m+1}(F) = 0$ for some $m \geq 1$, then $F = F_\infty$ in distribution. We summarize these facts in the following lemma.

**Lemma 4.1.** Let $F \in \mathcal{H}_2^S$, with $\varphi(F^2) = 1$, $\hat{\kappa}_4(F) \geq \hat{\kappa}_4(F_\infty) = 1/2$, $\hat{\kappa}_{2m+1}(F) = 0$, and $\hat{\kappa}_{2n}(F) \leq \hat{\kappa}_{2n}(F_\infty)$ for some $m \geq 1$, $n \geq 3$. Then, these inequalities are equalities and $F \overset{\text{law}}{=} F_\infty$.

Next, we shall derive the corresponding characterizations by using moments instead of free cumulants. The following result extends [8, Theorem 1.1].

**Theorem 4.1.** Let $F \in \mathcal{H}_2^S$ a non-commutative random variable in the second Wigner chaos such that $\varphi(F^2) = \varphi(F_\infty^2) = 1$ and $\varphi(F^4) \geq \varphi(F_\infty^4) = 5/2$ where $F_\infty = (S_2^1 - S_2^2)/\sqrt{2}$ has the tetilla law. Then

$$\varphi(F^{2n}) \geq \varphi(F_\infty^{2n}), \quad n \in \mathbb{N},$$

and if this inequality is an equality for some $n \geq 3$, then $F = F_\infty$ in distribution.

**Proof.** We follow the steps of the proof for commutative random variables, just note that by (4.1)

$$\mu_{2n}(F) = \varphi(F^{2n}) = \sum_{\pi \in NC^\prime_{2n}} \prod_{A \in \pi} \hat{\kappa}_{|A|}(F) + \sum_{\rho \in NC^\prime_{2n}} \prod_{B \in \rho} \hat{\kappa}_{|B|}(F)$$

where $NC^\prime_{2n}$ are the non-crossing partitions of $2n$ containing only components of even size, and $NC^\prime_{2n} = NC_{2n} \setminus NC^\prime_{2n}$ is its complement, where the non-crossing partition elements contain an even number of components with odd size. As in the classical case, the problem is to deal with the free cumulants of odd order. Note that $F = F_+ - F_-$ with free $F_\pm$, and $\hat{\kappa}_n(F) = \hat{\kappa}_n(F_+) - \hat{\kappa}_n(F_-)$, where $\hat{\kappa}_n(\alpha F) = \alpha^n \hat{\kappa}_n(F)$. As in the classical case, the assumptions $\varphi(F^2) = \varphi(F_\infty^2) = 1$ and $\varphi(F^4) \geq \varphi(F_\infty^4)$ imply that all odd free cumulants have the same sign, i.e.

$$\hat{\kappa}_{2m+1}(F) \hat{\kappa}_{2n+1}(F) \geq 0, \quad \forall n, m.$$

Therefore, all the terms in the sum are non-negative and minorized by the corresponding products of $F_\infty$-free cumulants, and when one of these even moment inequalities is an equality Lemma 4.1 applies.
Proposition 4.1. Under the conditions of Theorem 4.1 for \(2 \leq m \leq n \in \mathbb{N}\), we have
\[
\varphi(F^{2m}) - \varphi(F^{2n}) \geq C_{n-m} \left( \varphi(F^{2m}) - \varphi(F^{2n}) \right),
\]
where \(C_k\) denotes the \(k\)-th Catalan number.

Proof. By using (4.1)
\[
\varphi(F^{2n}) - \varphi(F^{2n}) = \sum_{\rho \in NC(2n)} \left\{ \prod_{A \in \rho} \hat{\kappa}_{|A|}(F) - \prod_{A \in \rho} \hat{\kappa}_{|A|}(F_\infty) \right\}.
\]

Now for each non-crossing partition \(\rho \in NC_{2n}\),
\[
\prod_{A \in \rho} \hat{\kappa}_{|A|}(F) \geq \prod_{A \in \rho} \hat{\kappa}_{|A|}(F_\infty).
\]

Indeed if the non-crossing partition \(\rho\) contains any part \(A\) with odd size, then the right side is zero, and the left side is non-negative since even free cumulants are non-negative, there must be an even number of odd parts in the partition and under the assumptions all odd free cumulants have the same sign. Otherwise the non-crossing partition \(\rho\) contains only parts of even size, but then we have shown that under the assumptions
\[
\hat{\kappa}_{2\ell}(F) \geq \hat{\kappa}_{2\ell}(N_1N_2) \geq 0, \quad \forall \ell \in \mathbb{N},
\]
and the inequality is preserved when we take product over the parts. If \(\rho\) is a non-crossing partition of \(2m\), let’s say \(\rho = \{A_1, A_2, \ldots, A_r\}\) with \(A_i \cap A_j = \emptyset\) for \(i \neq j\) and \(A_1 \cup A_2 \cup \ldots \cup A_r = \{1, 2, \ldots, 2m\}\) then we can add \((n-m)\) pairs of consecutive elements to obtain
\[
\rho' = \{A_1, A_2, \ldots, A_r, \{2m+1, 2(m+1)\}, \ldots, \{2n-1, 2n\}\}
\]
which is a non-crossing partition of \(\{1, 2, \ldots, 2n\}\), and
\[
\prod_{A \in \rho} \hat{\kappa}_{|A|}(F) = \hat{\kappa}_{2\ell}(F)^{n-m} \left( \prod_{A \in \rho} \hat{\kappa}_{|A|}(F) \right) = \prod_{A \in \rho} \hat{\kappa}_{|A|}(F).
\]

Since \(C_{n-m}\) is also the number of non-crossing pairings of \(\{1, 2, \ldots, 2(n-m)\}\), for every non-crossing partition of \(2m\) there are at least \(C_{n-m}\) non-crossing partitions of \(2n\) which contribute equally to the sum and we get
\[
\sum_{\rho \in NC_{2n}} \left( \prod_{A' \in \rho'} \hat{\kappa}_{|A'|}(F) - \prod_{A' \in \rho'} \hat{\kappa}_{|A'|}(F_\infty) \right) \geq 0 \quad \sum_{\rho \in NC_{2n}} \left( \prod_{A \in \rho} \hat{\kappa}_{|A|}(F) - \prod_{A \in \rho} \hat{\kappa}_{|A|}(F_\infty) \right) \geq 0
\]

In what follows the notation \(\mathcal{H}_{2}^{S,\text{symm}}\) stands for the collection of non-commutative random variables in the second Wigner chaos with symmetric distributions.

Proposition 4.2. Let \(F \in \mathcal{H}_{2}^{S,\text{symm}}\) such that \(\varphi(F^2) \leq 1\). Then
\begin{enumerate}
\item for \(r \in \mathbb{N}\), we have \(\hat{\kappa}_{2r}(F) \leq \hat{\kappa}_{2r}(F_\infty)\).
\item for \(r \in \mathbb{N}\), we have \(\varphi(F^{2r}) \leq \varphi(F_{\infty}^{2r})\).
\item if one of these free cumulant or moment inequalities at items 1 or 2 is an equality for some \(r \geq 2\), then \(F \overset{\text{law}}{=} F_\infty\).
\end{enumerate}
Proof. Since $\lambda_z = -\lambda_{-z}$ for $F \in \mathcal{H}_2^{\text{symm}}$, and $\varphi(F^2) \leq 1$, for $r \in \mathbb{N}$ by using Jensen inequality

$$\hat{\kappa}_{2r}(F) = \sum_{z \in \mathbb{Z}} \lambda_z^{2r} = 2 \sum_{n \in \mathbb{N}} \lambda_n^{2r} \leq 2 \left( \sum_{\ell \in \mathbb{N}} \lambda_\ell^2 \right)^r = 2^{1-r} \hat{\kappa}_2(F)^r \leq \hat{\kappa}_{2r}(F_\infty)$$

with equality if and only if $\lambda_z = 0$ $\forall z \notin \{-1,1\}$. This proves item 1, with equality if and only if $F \overset{\text{law}}{=} F_\infty$ and hence the half of the item 3. Since $F$ is symmetric, the odd free cumulants are zero, and by the free cumulant to moments formula we obtain

$$\varphi(F^{2r}) = \sum_{\rho \in NC_{2r}} \prod_{A \in \rho} \hat{\kappa}_{|A|}(F) \leq \sum_{\rho \in NC_{2r}} \prod_{A \in \rho} \hat{\kappa}_{|A|}(F_\infty) = \varphi(F^{2r}_\infty), \quad (4.5)$$

which is an equality if and only if $\hat{\kappa}_{2s}(F) = \hat{\kappa}_{2s}(F_\infty)$ $\forall s \leq r$, meaning that $F \overset{\text{law}}{=} F_\infty$. Hence item 2 is shown together with the remaining half part of item 3. $\square$

4.5 Case $\varphi(F^4) < 5/2$

In this section, we aim to analysis the situation $\liminf_n \varphi(F_n^4) \leq \varphi(F_\infty^4) = 5/2$ for a sequence $\{F_n\}_{n \geq 1}$ of random elements in the second Wiener chaos. For example, imagine the case when $\varphi(F_n^4) \to 5/2$ from below as $n \to \infty$. Take an element $F$ in the second Wigner chaos. As it indicates in the proof of Proposition 4.2, the key point to control the signs of the products of the odd free cumulants of $F$ was to realize at least one coefficient $\lambda_i$ in the representation of $F$ such that $\lambda_i^2 \geq \frac{1}{2}$.

Proposition 4.3. Let $\epsilon < 1/72$, and $F = \sum_{i \geq 1} \lambda_i(S_i^2 - 1)$ be a random variable in the second Wigner chaos such that $\varphi(F^2) = 1$, $|\lambda_i| < \frac{1}{\sqrt{7}}$ for all $i \geq 1$, and $\varphi(F^4) > 5/2 - \epsilon$ (or equivalently $\hat{\kappa}_4(F) > 1/2 - \epsilon$). Then there exist exactly two indices $k \neq l$ such that

(i) $|\lambda_k^2 - \frac{1}{2}| < \epsilon$, and also $|\lambda_l^2 - \frac{1}{2}| < \epsilon$.
(ii) $\sum_{i \neq k, l} \lambda_i^2 < 2\epsilon$ for all the other indices.

Proof. As in Corollary 3.3 $\square$

Lemma 4.2. (hypercontractivity) The free cumulants and moments of a non-commutative random variable $F \in \mathcal{H}_2^S$

$$|\hat{\kappa}_n(F)| \leq C_n \varphi(F^n)^{n/2}, \quad |\varphi(F^n)| \leq C_n \varphi(F^2)^{n/2}$$

where $C_n$ denotes the $n$-th Catalan number, with equalities if and only if $F \overset{\text{law}}{=} \pm(S_1^2 - 1)$, where $S_1$ has the circular law with unit variance or $F = 0$.

Proof. As in Lemma 3.2 $\square$

5 Convergence in Wasserstein-2 distance in 2nd Wiener/Wigner chaos

The Wasserstein–2 distance between two probability distributions $Q_1, Q_2$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is given by

$$d_{W_2}(Q_1, Q_2) := \inf_{(X_1, X_2)} \left\{ E \left( (X_1 - X_2)^2 \right)^{1/2} \right\}$$

where the supremum is taken over the random pairs $(X_1, X_2)$ defined on the same classical probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ with marginal distributions $Q_1$ and $Q_2$. Relevant information
about Wasserstein distances can be found, e.g. in [27, Section 6]. It is shown in [7, Thm 5.1] that

\[ d_{W_2}(Q_1, Q_2) = \inf_{(X_1, X_2)} \left\{ \varphi((X_1 - X_2)^2)^{1/2} \right\} \]

where the infimum is over the larger class of non-commutative r.v’s \((X_1, X_2)\) defined on a common non-commutative probability space \((\mathcal{A}, \varphi)\), with marginal laws \(Q_1, Q_2\).

5.1 Quantitative estimates in Wasserstein-2 distance

Proposition 5.1. Let \(F\) be an element in the second Wiener (Wigner) chaos such that \(\mathbb{E}(F^2) = 1\) \((\varphi(F^2) = 1\) respectively). Assume that \(F_\infty \sim N_1 \times N_2\) (normalized tetilla law respectively) where \(N_1, N_2 \sim \mathcal{N}(0, 1)\) are independent. Then there exists a constant \(C\) such that

\[ d_{W_2}(F, F_\infty) \leq C \left\{ \begin{array}{ll} \sqrt{(\mu_6(F) - 225) - 55 (\mu_4(F) - 9)}, & \text{Wiener case,} \\ \sqrt{(\varphi(F^6) - 8.25) - 7(\varphi(F^4) - 2.5)}, & \text{Wigner case.} \end{array} \right. \]  

(5.1)

If moreover assume that \(\mu_4(F) \geq 9\) \((\varphi(F^4) \geq 2.5\) respectively), then for every \(r \geq 3\), we obtain \(\mu_{2r}(F) \geq (2r - 1)!^2\), \((\varphi(F^{2r}) \geq \varphi(F^{2r}_\infty)\), where \(F_\infty\) stands for normalized tetilla law), and in addition

\[ d_{W_2}(F, F_\infty) \leq C \left\{ \begin{array}{ll} \sqrt{\mu_{2r}(F) - (2r - 1)!^2}, & \text{Wiener case,} \\ \sqrt{\varphi(F^{2r}) - \varphi(F^{2r}_\infty)}, & \text{Wigner case.} \end{array} \right. \]

(5.2)

Proof. Consider the polynomial \(P(x) = x^6 - 55x^4 + 331x^2 - 61\). A straightforward computation yields that \(\mathbb{E}(P(F)) = 5! \Delta_{3,1}(F) + 10 \kappa_2(F) \geq 0\) in the light of \(\Delta_{3,1}(F) = \text{Var}(\Gamma_2(F) - F) \geq 0\). Now, relying on [2, Theorem 2.4] for some constant \(C\) we obtain that

\[ d_{W_2}(F, N_1 \times N_2) \leq C \sqrt{\mathbb{E}(P(F))} \leq C \sqrt{(\mu_6(F) - 225) - 55 (\mu_4(F) - 9)}. \]

In particular, for \(F\) in the second Wiener chaos with variance 1,

\[ (\mu_6(F) - 225) - 55 (\mu_4(F) - 9) \geq 0. \]

When \(\mathbb{E}(F^4) \geq 9\), then Proposition 3.4 tells us that the even moment \(\mu_{2r}(F) \geq (2r - 1)!^2\) holds, and also estimate (5.2) is an application of Proposition 3.5. For the Wigner case the proof follows the same lines, by using \(P(x) = x^6 - 7x^4 + \frac{3}{2} x^2\), satisfying

\[ \varphi(P(F)) = \tilde{\kappa}_6(F) - \tilde{\kappa}_4(F) + \frac{1}{4} \tilde{\kappa}_2(F) + 3 \tilde{\kappa}_3^2(F) \geq 0. \]

\[ \square \]

Now, we are ready to present the main result of the section.

Theorem 5.1. Let \(\{F_n\}_{n \geq 1}\) be a sequence in the second Wiener chaos such that \(\mathbb{E}(F_n^2) = 1\) for all \(n \geq 1\). Then the following asymptotic assertions are equivalent.

\((\text{I})\) as \(n \to \infty\), \(d_{W_2}(F_n, N_1 \times N_2) \to 0\).

\((\text{II})\) as \(n \to \infty\), sequence \(F_n \to F_\infty \sim N_1 \times N_2\) in distribution.

\((\text{III})\) as \(n \to \infty\),

1. \(\mu_4(F_n) \to 9\).
2. \(\mu_{2r}(F_n) \to (2r - 1)!^2\) for some \(r \geq 3\).
If moreover $\mu_4(F_n) \geq 9$ for all $n \geq 1$, then $\mu_{2r}(F_n) \geq ((2r-1)!!)^2$ for $r \geq 3$, and in addition for some constant $C$ (independent of $n$) we obtain

$$d_{W_2}(F_n, N_1 \times N_2) \leq C \sqrt{\mu_{2r}(F_n) - (2r-1)!!^2}.$$  \hspace{1cm} (5.3)

**Proof.** (I) $\rightarrow$ (II): It is well-known that convergence with respect to probability metric $W_p$, $(p \geq 1)$ is equivalent to the usual weak convergence of measures plus convergence of the first $p$th moments, see [27]. (II) $\rightarrow$ (III): Let’s assume that $F_n$ converges in distribution towards $F_\infty \sim N_1 \times N_2$. Then because of hypercontractivity of Wiener chaoses (see [24, Lemma 2.4], or Lemma [3.2]),

$$\sup_{n \geq 1} |E[F_n^r]| < +\infty, \quad \forall r \geq 1.$$ 

Hence, an application of continuous mapping Theorem yields that $\mu_{2r}(F_n) \rightarrow (2r-1)!!^2$ for any $r \geq 2$. (III) $\rightarrow$ (I): note that, since $\sup_{n \geq 1} E[F_n^2] < +\infty$, so the sequence $\{F_n\}_{n \geq 1}$ is tight, and therefore any subsequence $\{F_{n_k}\}_{k \geq 1}$ contains a further subsequence $\{F_{n_{k_l}}\}_{l \geq 1}$, and a random variable $F$ such that $F_{n_{k_l}}$ converges in distribution towards $F$ as $l \rightarrow \infty$.

We need to show that $F \sim F_\infty$. To simplify our argument, we consider two separate cases and also we assume that $\{F_{n_{k_l}}\}_{l \geq 1} = \{F_n\}_{n \geq 1}$.

*Case (i):* assume that $\mu_4(F_n) \geq 9$ for all $n \geq 1$. Then, item 2 in (III) together with Proposition [5.1] indicates that $F_n$ converges to $F_\infty$ in Wasserstein-2 distance.

*Case (ii):* assume that $\mu_4(F_n) < 9$ for all $n \geq 1$. Suppose that there is a subsequence of indexes, such that for each $F_n$ in the subsequence $E[F_n^2] = 1$, $9 > \kappa_4(F_n) \rightarrow \kappa_4(N_1 N_2)$, $\mu_{2r}(F_n) \rightarrow \mu_{2r}(N_1 N_2)$ for some $r \geq 3$, and we assume that for each $n$ either $\lambda_{n,1}^2 > 1/2$ or $\lambda_{n,2}^2 > 1/2$. The last condition implies that all odd cumulants have the same sign i.e. $\kappa_{2\ell-1}(F_n)\kappa_{2\ell-1}(F_n) \geq 0$, $\forall \ell, r \in \mathbb{N}$.

By assumption for such subsequence $\mu_4(F_n) \rightarrow \mu_4(N_1 N_2)$, and $\mu_{2r}(F_n) \rightarrow \mu_{2r}(N_1 N_2)$ for some $r \geq 3$. We write the cumulant to moment formula into two parts,

$$\mu_{2\ell}(F_n) = \sum_{\pi \in \Pi_{2\ell}} \prod_{A \in \pi} \kappa_{|A|}(F_n) + \sum_{\rho \in \Pi_{2\ell}} \prod_{B \in \rho} \kappa_{|B|}(F_n)$$  \hspace{1cm} (5.4)

where the first sum is over partitions containing only parts of even size, and the second sum is over partitions containing an even number of odd parts. Since all odd cumulants have the same sign, the second sum is non-negative, and it vanishes when $F_n$ is a symmetric random variable. On the other hand, in the first sum, for every even $|A| = 2\ell$ with $3 \leq \ell \leq r$, we have

$$\kappa_{2\ell}(F_n) \geq \frac{(2\ell-1)!}{6} \kappa_4(F_n) + \left( \kappa_{2\ell}(F_n) - \frac{(2\ell-1)!}{6} \kappa_4(F_n) \right) \geq \frac{(2\ell-1)!}{6} \kappa_4(F_n) + (2\ell-1)! (\ell - 2) \left( \frac{\kappa_4(F_n)}{6} - \kappa_{2\ell}(F_n) \right)$$

and the last inequality is an equality if and only if $F_n = N_1 N_2$. Hence, by the assumption $\mu_4(F_n) \rightarrow \mu_4(N_1 N_2)$, for every $|A| = 2\ell$, with $3 \leq \ell \leq r$, we obtain

$$\limsup_{n \rightarrow \infty} \kappa_{2\ell}(F_n) \geq \liminf_{n \rightarrow \infty} \kappa_{2\ell}(F_n) \geq (2\ell-1)! = \kappa_{2\ell}(N_1 N_2).$$  \hspace{1cm} (5.5)

We need to show that these are equalities $\forall 3 \leq \ell \leq r$. Otherwise there would be some $3 \leq \ell \leq r$ and $\varepsilon > 0$ such that

$$\limsup_{n \rightarrow \infty} \kappa_{2\ell}(F_n) = \kappa_{2\ell}(N_1 N_2) + \varepsilon.$$
which would lead to
\[
\sum_{\pi \in \Pi_{2r}} \prod_{A \in \pi} \kappa_{|A|}(N_1 N_2) = \mu_{2r}(N_1 N_2) = \lim_{n \to \infty} \mu_{2r}(F_n) \geq \limsup_{n \to \infty} \sum_{\pi \in \Pi_{2r}} \prod_{A \in \pi} \kappa_{|A|}(F_n) \geq \sum_{\pi \in \Pi_{2r}} \prod_{A \in \pi} \left( \kappa_{|A|}(N_1 N_2) + \varepsilon 1(|A| = 2\ell) \right) > \mu_{2r}(N_1 N_2),
\]
with strict inequality.

Hence, the above observation together with relation (5.5) imply that \( \kappa_{2\ell}(F_n) \to \kappa_{2\ell}(N_1 N_2) \) for every \( 3 \leq \ell \leq r \), and therefore, \( \mu_{2\ell}(F_n) \to \mu_{2\ell}(N_1 N_2) \) \( \forall \ell \leq r \). In particular also \( \mu_6(F_n) \to \mu_6(N_1 N_2) \), and the conclusion for this subsequence follows by Proposition 5.1. It follows also that
\[
\lim_{n \to \infty} \sum_{\rho \in \Pi_{2r}} \prod_{B \in \rho} \kappa_{|B|}(F_n) \to 0 \quad \text{where the sum is over partitions containing an even number of odd parts, and since all summands are non-negative, this implies that } \kappa_3(F_n) \to 0.
\]

**Case (iii):** Otherwise \( \lambda_{1,n}^2 < 1/2 \ \forall \ i \in \mathbb{Z} \), and by item 1 together with Proposition 3.3 yield that in the representations of \( F_n \)'s there are exactly two indices \( k, l \) such that \( \lambda_{n,k}^2, \lambda_{n,l}^2 \to 1/2 \) from below, and also all the rest of coefficients tend to \( 0 \) as \( n \to \infty \). (Note that in principle the indices \( k, l \) may depend on \( n \). However, this does not affect our argument in below). Since reordering the coefficients does not change the law of \( F_n \), we can assume that
\[
F_n \sim G_n + H_n := \left( \lambda_{n,1} \frac{(N_1^2 - 1)}{\sqrt{2}} + \lambda_{n,2} \frac{(N_2^2 - 1)}{\sqrt{2}} \right) + H_n, \quad (5.6)
\]
where \( \lambda_{n,k}^2 \to 1/2 \) for \( k = 1, 2 \), \( G_n \) and \( H_n \) are independent, and also \( H_n \) belongs to the second Wiener chaos. First note that \( 1 = \mathbb{E}(F_n^2) = \lambda_{n,1}^2 + \lambda_{n,2}^2 + \mathbb{E}(H_n^2) \), and so one can infer that \( \mathbb{E}(H_n^p) \to 0 \) as \( n \to \infty \), imply by hypercontractivity argument that as \( n \to \infty \),
\[
\mathbb{E}(|H_n|^p) \to 0, \quad \forall p \geq 2. \quad (5.7)
\]
We claim that \( (\lambda_{n,1} \lambda_{n,2}) \to -1/2 \) and \( G_n \overset{L^2}{\to} N_1 \times N_2 \). By contradiction assume that this is not the case and without loss of generality there is subsequence with both \( \lambda_{n,1}, \lambda_{n,2} \to 1/\sqrt{2} \) (for a subsequence with both \( \lambda_{n,1}, \lambda_{n,2} \to -1/\sqrt{2} \) we can flip the sign of all the random variables). Then using item 2, relation (5.7), and exploring the independence between \( G_n \), and \( H_n \), we get that
\[
(2r - 1)!^2 \varepsilon - \mu_{2r}(F_n) \approx \mu_{2r}(G_n) \to \sum_{s=0}^{2r} (-1)^s \frac{(2r)!}{(2r-s)!} \geq \left( \frac{2r!}{r! r^r} \right)^2,
\]
which is a contradiction. The strict inequality for \( r \geq 3 \) follows by the cumulant to moment formula, since for \( n \) large enough all odd cumulants of \( F_n \) in the subsequence have the same sign.

Hence, as \( n \to \infty \),
\[
d_{W_2}(F_n, N_1 \times N_2)^2 \leq \mathbb{E}|F_n - N_1 N_2|^2 \leq \mathbb{E}|F_n - G_n|^2 + \mathbb{E}(H_n^2) \to 0.
\]

\[\square\]

**Remark 5.1.** The proof of Theorem 5.1 reveals that, under the knowledge of \( \mu_4(F_n) \geq 9 \) for all \( n \), the assumption in item 1 at (III) is immaterial, and it automatically takes place.
Remark 5.2. The quantitative estimate (5.3) is in the similar spirit as the main result in [4] in the case of normal approximation. There it is shown that for a sequence \( \{F_n\}_{n \geq 1} \) in the fixed Wiener chaos of arbitrary order \( p \geq 2 \) such that \( \mathbb{E}(F_n^2) = 1 \) for all \( n \geq 1 \), always \( \mu_{2r}(F_n) \geq (2r-1)! \) for \( r \geq 2 \), and in addition

\[
d_{TV}(F_n, \mathcal{N}(0, 1)) \leq C \sqrt{\mu_{2r}(F) - (2r-1)!}.
\]

Hence to capture the distance in the total variation metric with only one higher even moment, in the case of normal approximation, it is enough to fix the first even moment (i.e. the second moment). However in the case of \( N_1 \times N_2 \) approximation in Wasserstein-2 distance, one needs to perfectly match the first two even moments, i.e. the second and the fourth moments. One has to note that the restriction \( p = 2 \) in our result, and in the time being it is hard to generalize our results to the higher Wiener chaoses.

Corollary 5.1. Let \( \{F_n\}_{n \geq 1} \) be a sequence of non-commutative random variables in the second Wiener chaos such that \( \varphi(F_n^2) = 1 \) for all \( n \geq 1 \). Assume that \( F_\infty \) distributed as normalized tetilla distribution. Then, as \( n \to \infty \), the following asymptotic assertions are equivalent.

(I) \( d_{W_2}(F_n, F_\infty) \to 0. \)

(II) sequence \( F_n \to F_\infty \) is distribution.

(III) as \( n \to \infty \),

1. \( \varphi(F_n^1) \to 5/2. \)
2. \( \varphi(F_n^{2r}) \to \varphi(F_\infty^{2r}) \) for some \( r \geq 3. \)

If moreover \( \varphi(F_n^1) \geq 5/2 \) for all \( n \geq 1 \), then \( \varphi(F_n^{2r}) \geq \varphi(F_\infty^{2r}) \) for all \( r \geq 3 \), and in addition for some constant \( C \) (independent of \( n \)) we obtain

\[
d_{W_2}(F_n, F_\infty) \leq C \sqrt{\varphi(F_n^{2r}) - \varphi(F_\infty^{2r})}. \quad (5.8)
\]

Proof. As in Theorem 5.1. \( \square \)

5.2 Asymptotic result for the coupled sequence

The materials of this section are inspired from the proof of Theorem 5.1. Our new setup is the following. We consider now the convergence in Wasserstein-2 distances for the laws of a sequence of non commutative random variables \( \{F_n : n \in \mathbb{N}\} \in L^2 \) with representation

\[
F_n = \sum_{z \in \mathbb{Z}} \lambda_{n,z} X_z, \quad (5.9)
\]

where \( \{X_z : z \in \mathbb{Z}\} \subseteq L^2(\mathcal{A}, \varphi) \) is a sequence of identically distributed free random variables with \( \varphi(X_1) = 0, \varphi(X_1^2) = 1 \), and for each \( n \geq 0 \) the sequence \( \lambda_n = (\lambda_{n,z} : z \in \mathbb{Z}) \in \ell^2(\mathbb{Z}) \) is ordered as in (5.5), i.e. we consider the situation that \( \lambda_{n,0} = 0 \), and moreover

\[
-\|F\|_{L^2} \leq \lambda_{n,-1} \leq \lambda_{n,-2} \leq \ldots \leq \lambda_{n,z} \leq \ldots \leq 0 \leq \ldots \leq \lambda_{n,z} \leq \ldots \leq \lambda_{n,2} \leq \lambda_{n,1} \leq \|F\|_{L^2}. \quad (5.10)
\]

In the commutative case \( \{X_z : z \in \mathbb{Z}\} \subseteq L^2(\omega, \mathcal{F}, P) \) is a sequence of classically independent and identically distributed random variables with \( \mathbb{E}(X_1) = 0, \mathbb{E}(X_1^2) = 1 \). In the special case where \( X_1 \overset{law}{=} -X_1 \), we also assume without loss of generality that \( \lambda_{n,z} = 0, \forall z \leq n \). First, we need the following enlightening lemma telling us that the best permutation in the definition of \( d_{\text{tort}} \) distance in [2] relation (2.1), page 4] is given by ordering (5.10).

Lemma 5.1. Under the above setting, for every two (non) commutative square integrable random variables \( F_1 \) and \( F_2 \) following representation (5.9), and (5.10), we have

\[
d_{W_2}(F_1, F_2)^2 \leq \|\lambda_1 - \lambda_2\|_{\ell^2(\mathbb{Z})}^2 \leq \|\lambda_1 \circ \pi - \lambda_2\|_{\ell^2(\mathbb{Z})}^2 \quad (5.11)
\]

for any bijection \( \pi : \mathbb{Z} \to \mathbb{Z} \).
Proof. The representation \[ (5.9) \] with the same free sequence \((X_n : n \in \mathbb{N})\) gives a coupling of the non-commutative random variables \((F_n : n \in \mathbb{N})\), and by using freeness we obtain

\[
\varphi((F_1 - F_2)^2) = \sum_{z \in \mathbb{Z}} (\lambda_{1,z} - \lambda_{2,z})(\lambda_{1,z'} - \lambda_{2,z'})\varphi(X_zX_{z'}) = \sum_{z \in \mathbb{Z}} (\lambda_{1,z} - \lambda_{2,z})^2 \varphi(X_z^2) = \|\lambda_1 - \lambda_2\|_{\ell^2(\mathbb{Z})}^2.
\]

The inequality in \[ (5.11) \] is equivalent to

\[
\sum_{z \in \mathbb{Z}} \lambda_{1,z}\lambda_{2,\pi(z)} \leq \sum_{z \in \mathbb{Z}} \lambda_{1,z}\lambda_{2,z}, \quad \forall \text{ bijection } \pi : \mathbb{Z} \to \mathbb{Z}.
\]

(5.12)

For finite vectors this is known as rearrangement inequality [12 Thm. 368]. For infinite sequences, it is clear that non-negative (non-positive) \(\lambda_1\) coordinates should be matched with non-negative (non-positive) \(\lambda_2\) coordinates, and \[ (5.12) \] follows from the Hardy-Littlewood inequality for function rearrangements [12 Thm. 378], applied separately to the functions piecewise constant on unit intervals corresponding to the non-negative (non-positive) subsequences.

We next study two possible situations:

(C): \((F_n : n \in \mathbb{N}) \subset \mathcal{H}_2\), the second Wiener chaos. Equivalently, \(F_n\) have representation \[ (5.9) \] and there is a sequence \((N_z : z \in \mathbb{Z})\) of independent standard Gaussian random variables on a classical probability space \((\Omega, \mathcal{F}, P)\) such that \(X_z = (N_z^2 - 1)/\sqrt{2}, \forall z \in \mathbb{Z}\).

(N-C): \((F_n : n \in \mathbb{N}) \subset \mathcal{H}_2^S\), the second Wigner chaos. Equivalently, \(F_n\) have representation \[ (5.9) \] and there is a sequence \((S_z : z \in \mathbb{Z})\) of freely independent normalized semicircular random variables defined on a non-commutative probability space \((\mathcal{A}, \varphi)\) such that \(X_z = (S_z^2 - 1), \forall z \in \mathbb{Z}\).

The main result in this section is the following.

**Theorem 5.2.** In both commutative and non-commutative settings, under the above settings, and assumptions (C) and (N-C), and the second moment condition \(\mathbb{E}(F_n^2) = 1(\varphi(F_n^2) = 1)\) for all \(n \geq 1\), with the target random variable

\[
F_\infty := \frac{X_1 - X_{-1}}{\sqrt{2}} = \begin{cases} (S_1S_2 + S_2S_1)/\sqrt{2}, & \text{(N-C)} \\ N_1 \times N_2, & \text{(C)} \end{cases}
\]

as \(n \to \infty\), the following asymptotic assertions are equivalent.

1. \(\forall p \geq 1, F_n \overset{L^p}{\to} F_\infty\).
2. \(F_n \overset{L^2}{\to} F_\infty\).
3. \(\|\lambda_n - \lambda_\infty\|_{\ell^2(\mathbb{Z})} = (2 + \sqrt{2}\lambda_{n,-1} - \sqrt{2}\lambda_{n,1}) \to 0\), with limiting sequence \(\lambda_\infty,z = 0\) for \(|z| \neq 1\), and \(\lambda_\infty,\pm 1 = \pm 1/\sqrt{2}\).
4. \(d_{F_2}(F_n, F_\infty) \to 0\).
5. \(F_n \overset{law}{\to} F_\infty\).
6. For some \(r \geq 3\);

\[
\mu_4(F_n) \to \mu_4(F_\infty), \quad \mu_{2r}(F_n) \to \mu_{2r}(F_\infty)
\]

\[\varphi(F_n^4) \to 2.5, \quad \varphi(F_n^{2r}) \to \varphi(F_\infty^{2r}).\]
Proof. We consider the commutative case. The chain of implications (I) → (II) → (III) → (IV) → (V) → (VI) is straightforward. Note that under ([5.9]), ([5.10], and (C), using Lemma 5.1 and [2] Theorem 2.3) together with Proposition 5.1 one can infer that
\[
d_{W_2}(F_n, F_\infty) \leq \mathbb{E}((F_n - F_\infty)^2) = ||\lambda_n - \lambda_\infty||_{\ell_2}^2 \\
\leq C \sqrt{(\mu_4(F_n) - \mu_4(F_\infty))} + 55 (\mu_4(F_n) - \mu_4(F_\infty)).
\]
This implies implication (VI) → (II). Lastly, (II) → (I) is just the hypercontractivity of the second Wiener chaos, see Lemma 3.2.

6 Conjecture

The ultimate message of our study in a concise writing is the following. For a normalized sequence \( \{F_n\}_{n \geq 1} \), i.e. \( \mu_2(F_n) = 1 \) for \( n \geq 1 \) of classical random variables in the second Wiener chaos (non-commutative random variables in the second Wigner chaos, respectively) the convergences of the fourth and another higher even moments to the corresponding even moments of \( N_1 \times N_2 \) (of the normalized tetilla law, respectively) are necessary and sufficient conditions for convergence in distribution of the sequence \( F_n \) towards these target distributions in classical and non-commutative framework, respectively.

It turns out that in our analysis knowledge on the fourth moment plays a substantial role, and therefore, at present, becomes unclear how one can replace the convergence of the fourth moment in Theorems 5.1 and 4.1 by convergence of another higher even moments. However, in general, we believe that such moment replacement would be possible, and we leave it in the shape of the conjecture in below.

Conjecture 6.1. Let \( \{F_n\}_{n \geq 1} \) be a sequence of random variables in the second Wiener (Wigner) chaos such that \( \mu_2(F_n) = 1 (\varphi(F_n^2) = 1) \), for all \( n \geq 1 \). Assume that the target distribution \( F_\infty \sim N_1 \times N_2 \) (normalized tetilla law), where \( N_1, N_2 \) are independent standard Gaussian random variables. Then the following asymptotic assertions are equivalent;

(I) \( F_n \rightarrow F_\infty \) in distribution as \( n \rightarrow \infty \).

(II) as \( n \rightarrow \infty \), for some \( 3 \leq r \neq s \),

1. \( \mu_{2s}(F_n) \rightarrow \mu_{2s}(F_\infty) \), \( (\varphi(F_n^{2s}) \rightarrow \varphi(F_\infty^{2s})) \).

2. \( \mu_{2r}(F_n) \rightarrow \mu_{2r}(F_\infty) \), \( (\varphi(F_n^{2r}) \rightarrow \varphi(F_\infty^{2r})) \).

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