Classical and quantum geometric information flows and
entanglement of relativistic mechanical systems

Sergiu I. Vacaru1,2,3 · Laurenţiu Bubuianu4,5

Received: 20 May 2019 / Accepted: 24 October 2019 / Published online: 4 November 2019
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Abstract
This article elaborates on entanglement entropy and quantum information theory of
generic flows of (relativistic) Lagrange–Hamilton mechanical systems. A set of
basic geometric and quantum mechanics and probability concepts together with meth-
ods of computation are developed in general covariant form for curved phase spaces
modelled as cotangent Lorentz bundles. The constructions are based on ideas relating
the Grigori Perelman’s entropy for geometric flows and associated statistical thermo-
dynamic systems to the quantum von Neumann entropy, classical and quantum relative
and conditional entropy, mutual information, etc. We formulate the concept of the
entanglement entropy of quantum geometric information flows and study properties
and inequalities for quantum, thermodynamic and geometric entropies characterizing
such systems.

Keywords Perelman W-entropy · Quantum geometric information flows ·
Relativistic Lagrange–Hamilton mechanics · Entanglement entropy of quantum
generic information flows

The UAIC affiliation is for the director of a hosted Project IDEI2012-2-15. Address for post
correspondence in 2019–2020 as a visitor senior researcher at YF CNU Ukraine is: 37 Haharina Yu.
street, ap. 3, Chernivtsi, Ukraine, 58008.

Sergiu I. Vacaru
sergiu.vacaru@gmail.com; sergiuvacaru@mail.fresnostate.edu
Laurenţiu Bubuianu
laurentiu.bubuianu@tvr.ro

1 Physics Department, California State University at Fresno, Fresno, CA 93740, USA
2 Department of Theoretical Physics and Computer Modelling, Institute of Applied-Physics and
Computer Sciences, Yuri Fedkovych Chernivtsi National University, 101 Storozhynetska Street,
Chernivtsi 58029, Ukraine
3 Project IDEI - 2011, University “Al. I. Cuza”, Iaşi, Romania
4 SRTV - Studioul TVR Iaşi, 28 Alexandru Lapuşneanu Street, 700057 Iaşi, Romania
5 University Apollonia, 2 Muzicii Street, 700399 Iaşi, Romania
1 Introduction

A generic feature of quantum physics which is absent in classical physics is that of entanglement. There were introduced several entanglement measures of how much quantum a given system is. Because of computational accessibility, the entanglement entropy plays a particulary important role together with Rényi entropies, mutual information, etc. For recent reviews of most important ideas and results related to quantum information theory, we cite [1–9] and references therein. Here, we note that the concept of entanglement entropy originated from quantum information theory [10]. At present, it is connected to a wide range of applications in condensed matter physics, gravity theories and particle physics, etc. The progress in such directions included the holographic formula for entanglement entropy [11], a new type of order parameter for quantum-phase transitions [12–14], ideas of formulating quantum gravity from quantum entanglement and so-called ER = EPR [15,16].
There are many motivations to study quantum entanglement which depends on respective directions of research. For instance, we elaborated [17,18] on the idea that an intriguing connection exists between the Poincaré–Thurston conjecture (it became again a conjecture for relativistic Ricci flows even though a proof exists for Riemannian metrics [19]) and the emergent entropic gravity and/or other type modifications. G. Perelman introduced and applied in his famous preprints [19–21] the F- and W-functionals from which the R. Hamilton’s Ricci flow equations for Ricci flows [22–24] can be derived. A few years earlier, such equations were considered in physics by Friedan [25–27]. In a general context, such works and further developments provide strong motivations for elaborating a new direction (based on geometric flows and associated thermodynamical models) in classical and quantum information theory. For such models, the quantum entanglement can be exploited for computational tasks which are impossible if only classical methods are used but for performing on new type theories unifying quantum and geometric flow evolution scenarios.

This article is the fifth partner in a series of works [17,18,28,29] devoted to applications of G. Perelman’s entropic functionals [19] and nonholonomic geometric flow methods in classical and quantum information theory, geometric mechanics and thermodynamics, and modified (entropic and other types) gravity. For a review of mathematical results on Ricci flows of Riemannian and Kähler metrics, rigorous proofs and topological and geometric analysis methods, we cite [30–32]. In our approach, we consider nonholonomic deformations of the G. Perelman’s functionals and elaborated on new geometric methods and applications in (modified) gravity, geometric mechanics; locally anisotropic kinetics, diffusion and thermodynamics; and information theory. Here, we note that in this work we follow the notations on the so-called quantum geometric information flow, QGIF, theory (in brief, it is used GIF for classical models) introduced in [29]. Readers may study our previous works [33–37] and references therein, on nonholonomic (non) commutative/supersymmetric geometric flows and related kinetic and statistical thermodynamic models.

The aim of this paper is to specifically address the geometric flow evolution and dynamics of the entanglement in quantized Lagrange–Hamilton relativistic mechanical systems. We develop our approach on elaborating new principles and methods for formulating classical and quantum information theories encoding geometric flows and their analogous geometric thermodynamic models. The key ideas for developing such new directions in (quantum) information theory and applications are to extend the standard constructions involving the von Neumann, and related conditional and relative entropies. We introduce into consideration generalizations of the concepts of W-entropy and analogous thermodynamic entropy elaborated in original variants by G. Perelman for Ricci flows of Riemannian metrics.

We try to make this work self-contained and multi-disciplinary pedagogic enough but for advanced researchers working on geometry and physics, nonholonomic geometric mechanics and thermodynamics, quantum mechanics and quantum field theory and information theory. In our case, some typical Alice and Bob communicating using methods of quantum information theory should also have certain knowledge on geometric flows; systems of nonlinear partial differential equations, PDEs, and their applications in modern classical and quantum physics. It is assumed that readers have a background on modified gravity theories and modern astrophysics and cosmology.
because all such theories provide strong motivations and examples of applications of
the formalism elaborated in the cited monographs, reviews and series of works on
geometric flows and information theory. In this article, we study entanglement for
quantum geometric flows of mechanical systems and do not concern issues on grav-
ity, quantum field theory or condensed matter physics. On emergent gravity theories,
modified Ricci flow theories and gravity, exact solutions and related classical and
quantum mechanical entropic functionals from which generalized Einstein equations
can be derived, see our recent results [33–37] and references therein.

This article is organized as follows: in Sect. 2, we start with reviewing the fun-
damentals of the theory of geometric flows of relativistic Hamilton phase spaces. After defining the fundamental geometric objects such as the nonlinear connection,
N-connection, and distinguished metric, d-metric, structures, we show how the curva-
tures can be computed for general and preferred linear connections. Then, we introduce
the G. Perelman F- and W-functional (entropic type) for W. Hamilton mechanical sys-
tems and their formulation in general N-adapted variables.

Section 3 begins with a quick introduction into the statistical thermodynamic theory
of geometric information flows, GIFs, when the G. Perelman approach is generalized
for nonholonomic N-adapted variables. The approach is generalized for quantum geo-
metric information flows, QGIFs, using the statistical density matrix and its analogous
quantum density matrix. The von Neumann entropy for QGIFs and quantum general-
izations of the W- and thermodynamic entropy are considered.

In Sect. 4, we explore the entanglement and QGIFs as quantum mechanical systems.
There are defined QGIF analogs of two spin systems, thermofield double GIF states
and Bell-like geometric flow states. We outline the main properties and inequalities of
the entanglement entropy for such systems with mixed geometric and quantum flow
evolution. The entanglement and Rényi entropy and QGIFs at finite temperature are
studied. Conclusions are provided in Sect. 5.

2 Geometric flows of relativistic Hamilton phase spaces

We present a short review of the geometry of relativistic Hamilton phase spaces
modelled on cotangent bundle $T^*V$ of a nonholonomic Lorentz manifold $V$, see an
axiomatic approach and details in [38,39]. There are provided formulas for respective
generalizations of G. Perelman’s F- and W-entropy functionals for which we follow
the conventions from [17,18,29], see proofs and references therein.

2.1 The Hessian geometry of relativistic Hamilton spaces

2.1.1 Nonlinear connections and adapted metrics

We consider a cotangent Lorentz bundle $T^*V$, dim $V = 4$, enabled with local coordi-
nates $'u'^\alpha = (x^i, p_a)$, [in brief, $'u' = (x, p)$], where $x^i$ are base manifold coordinates
and $p_a$ are momentum like typical fiber coordinates. Such a model of relativistic phase
spacetime is enabled in any point with a total metric structure (phase space metric) of
signature (++++−; +++++), which for corresponding frames/coordinates transforms can be represented in the form

$\text{d} s^2 = \gamma_{\alpha\beta} \text{d} u^\alpha \text{d} u^\beta = \eta_{ij} x^i x^j + \eta^{ab} p_a p_b$, for $p_a \sim \text{d} x_a / \text{d} \tau$.  \hspace{1cm} (1)

In these formulas, $\eta_{ij} = \text{diag}[1, 1, 1, -1]$ and $\eta^{ab} = \text{diag}[1, 1, 1, -1]$.

In a more general context, we can elaborate on physical models on curved phase spaces when the metric structure (1) is determined by coefficients of type $\gamma_{\alpha\beta} = [g_{ij}(x), \gamma^{ab}(x, p)]$.

A relativistic Hamilton space $H^3.1 = (T^* V, H(x, p))$ is determined by a fundamental function $H(x, p)$ (it can be used a generating Hamilton function, Hamiltonian or Hamilton density). For classical models, it is considered that a map $T^* V \ni (x, p) \rightarrow H(x, p) \in \mathbb{R}$ defines a real valued function being differentiable on $T^* V := T^* V / \{0^*\}$, for $\{0^*\}$ being the null section of $T^* V$, and continuous on the null section of $\pi^*: T^* V \rightarrow V$. In a more general context, a $H(x, p)$ can be quantized following prescriptions for a respective quantum model (quantum mechanics, QM, or quantum field theory, QFT, with corresponding quasi-classical relativistic and nonrelativistic limits). In this work, we elaborate on relativistic mechanical models which are regular if the Hessian (cv-metric)

$\gamma^{ab}(x, p) := \frac{1}{2} \frac{\partial^2 H}{\partial p_a \partial p_b}$

for a $H = \tilde{H}$ is nondegenerate, i.e., $\det |\gamma^{ab}| \neq 0$, and of constant signature.

For Lagrange and Hamilton spaces, we can perform Legendre transforms $L \rightarrow H(x, p) := p_a y^a - L(x, y)$ and $y^a$ determining solutions of the equations $p_a = \partial L(x, y) / \partial y^a$. In a similar manner, the inverse Legendre transforms can be introduced, $H \rightarrow L$, when $L(x, y) := p_a y^a - H(x, p)$ for $p_a$ determining solutions of the equations $y^a = \partial H(x, p) / \partial p_a$. In this work, we consider Hamilton structures which allow canonical Hamilton formulations of some QM models and respective quasi-classical limits.

Any $\tilde{H}$ defines a canonical nonlinear connection (N-connection) structure

$\tilde{\mathcal{N}}: T T^* V = h T^* V \oplus v T^* V$ \hspace{1cm} (3)

and a N-adapted canonical distinguished metric (d-metric) structure parameterized with conventional horizontal, h, and covertical, cv, components,

$\tilde{g} = \tilde{\gamma}_{\alpha\beta}(x, p) \tilde{e}^\alpha \otimes \tilde{e}^\beta = \tilde{\gamma}_{ij}(x, p) e^i \otimes e^j + \gamma^{ab}(x, p) \tilde{e}_a \otimes \tilde{e}_b$. \hspace{1cm} (4)

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1 We follow such conventions: the “horizontal” indices, h-indices, run values $i, j, k, \ldots = 1, 2, 3, 4$; the vertical indices, v-vertical, run values $a, b, c, \ldots = 5, 6, 7, 8$; respectively, the v-indices can be identified/contracted with h-indices 1, 2, 3, 4 for lifts on total (co) bundles, when $\alpha = (i, a), \beta = (j, b), y = (k, c), \ldots = 1, 2, 3, \ldots 8$. There are used letters labeled by an abstract left up/down symbol “"” (for instance, $\gamma^{ab}$ and $\gamma_{\alpha\beta}$) in order to emphasize that certain geometric/physical objects are defined on $T^* V$. Similar formulas can be derived on $TV$ for geometric objects labeled without “".” Boldface symbols are used for geometric objects on spaces endowed with nonlinear connection structure [see below formula (3)].
where the canonical N-linear frames $\tilde{e}^a = (e^i, \tilde{e}_a)$ are canonically determined by data $(\tilde{H}, \tilde{g}^{ab})$.2

Considering general frame (vierbein) transforms, $e^a = e^a_\alpha (u) \partial / \partial u^\alpha$ and $e^\beta = e^\beta_\beta (u) du^\beta$, we write general N-connection and d-metric structures on a cotangent Lorentz bundle $T^*V$ in such general forms (without “tilde” on symbols):

$\mathcal{N} = \{ N_{ij}(x, p) \}$, with arbitrary coefficients;

$g = g_{\alpha\beta}(x, p) e^\alpha \otimes e^\beta = g_{ij}(x, p)e^i \otimes e^j + g^{ab}(x, p)e_a \otimes e_b$.

So, any classical regular Hamilton mechanics can be geometrized in general form on a phase spacetime $T^*V$ by some nonholonomic data $(\mathcal{N}, g)$. Inversely, using respective frame transforms on a nonholonomic cotangent bundle, we can always consider a relativistic Hamilton space model defined by some data $(\tilde{H}, \tilde{N}; \tilde{e}_a, \tilde{e}^a; \tilde{g}^{ab}, \tilde{g}_{ab})$.

2.1.2 Curvatures, torsions and nonmetricity of linear and distinguished connections

A physically realistic geometrization of physical models on $T^*V$ is possible if such a phase space is enabled with a linear (affine) connection structure. Using $g$, we can define in standard form the Levi–Civita connection $\nabla$ (as a unique one which is metric compatible and with zero torsion), but such a geometric object is not adapted to the N-connection structure. To elaborate N-adapted geometric models, we have to consider the concept of distinguished connection (d-connection) which is a linear connection $\mathcal{D}$ on $T^*V$ preserving under parallel transports a N-connection splitting $\mathcal{N}$. With respect to a N-adapted basis, the coefficients of a d-connection $\mathcal{D}$ are labeled $\Gamma^{\alpha}_{\beta\gamma} = \{ L^j_{ik}, \hat{L}_a^b, \hat{C}_j^i, \hat{C}^a_b \}$. This involves an explicit h- and cv-splitting, of covariant derivatives $\mathcal{D} = \{ d, \nabla \}$, where $d = \{ L^j_{ik}, \hat{L}_a^b \}$ and $\nabla = \{ \hat{C}_j^i, \hat{C}^a_b \}$.

Prescribing a d-connection structure $\mathcal{D}$, we can work alternatively with an arbitrary linear connection a linear connection $D$ (which is not obligatory a d-connection) $\mathcal{D}$ on $T^*V$. For such covector bundles, there are nonholonomic deformation relations with a respective distortion distinguished tensor, d-tensor, $Z := D - d$.

For any linear connection and/or d-connection structure, $\mathcal{D}$ and/or $\mathcal{D}$, we can define in standard form respective curvature, $\mathcal{R}$ and/or $\mathcal{R}$, torsion, $\mathcal{T}$ and/or $\mathcal{T}$, nonmetricity, $Q$ and/or $Q$, d-tensors,

The coefficients of the canonical N-connection are computed following formulas

$\tilde{N} = \{ \tilde{N}_{ij} := \frac{1}{2} \left[ \tilde{g}_{ij} - \frac{\partial^2 \tilde{H}}{\partial p_k \partial x^l} \tilde{g}_{jk} - \frac{\partial^2 \tilde{H}}{\partial p_k \partial x^l} \tilde{g}_{ik} \right] \}$, where $\tilde{g}_{ij}$ is inverse to $\tilde{g}^{ab}$ (2). The canonical N-adapted (co)frames are

$\tilde{e}_a = \left( \frac{\partial}{\partial x^i}, \tilde{N}_{ia}(x, p) \frac{\partial}{\partial p_a} \right), \tilde{e}^a = \left( e^i, x_a = dp_a + \tilde{N}_{ia}(x, p)dx^i \right)$,

being characterized by corresponding anholonomy relations $[ \tilde{e}_a, \tilde{e}_b ] = \tilde{e}_a \tilde{e}_b - \tilde{e}_b \tilde{e}_a = \tilde{W}^{\gamma}_{a\beta} \tilde{e}^\gamma$, with anholonomy coefficients $\tilde{W}^{\beta}_{a\gamma} = \partial_{a\beta} \tilde{N}^\gamma_{i\beta}, \tilde{W}^a_{ij} = \tilde{C}^{a}_{ij}$ and $\tilde{W}^{\gamma}_{i\beta} = \partial_i \tilde{N}^\gamma_{\beta\beta}$ and $\tilde{W}^a_{ji\alpha} = \tilde{N}^a_{ija}$.

Such a frame is holonomic (integrable) if the respective anholonomy coefficients are zero.
The N-adapted and/or coordinate formulas for coefficients of such geometric objects can be computed in explicit form, see appendices to [38,39] and references therein.

Using (5), we can define and compute respective Ricci tensors/d-tensors, scalar curvatures, etc. For instance, the Ricci d-tensor of a d-connection can be computed in explicit form, see appendices to [38,39] and references therein.

\[ \mathcal{R}(\{X, Y\}) := \mathcal{D}_X \mathcal{D}_Y \mathcal{Y} - \mathcal{D}_Y \mathcal{D}_X \mathcal{X} - \mathcal{D}_X \mathcal{D}_Y, \]

\[ \mathcal{T}(\{X, Y\}) := \mathcal{D}_X \mathcal{Y} - \mathcal{D}_Y \mathcal{X} - [\mathcal{X}, \mathcal{Y}], \]

\[ \mathcal{Q}(\{X\}) := \mathcal{D}_X \mathcal{g}, \]

and/or

\[ \mathcal{R}(\{X, Y\}) := \mathcal{D}_X \mathcal{D}_Y \mathcal{Y} - \mathcal{D}_Y \mathcal{D}_X \mathcal{X} - \mathcal{D}_X \mathcal{D}_Y \mathcal{X} - [\mathcal{X}, \mathcal{Y}], \]

\[ \mathcal{T}(\{X, Y\}) := \mathcal{D}_X \mathcal{Y} - \mathcal{D}_Y \mathcal{X} - [\mathcal{X}, \mathcal{Y}], \]

\[ \mathcal{Q}(\{X\}) := \mathcal{D}_X \mathcal{g}. \] 

(5)

Using (5), we can define and compute respective Ricci tensors/d-tensors, scalar curvatures, etc. For instance, the Ricci d-tensor of a d-connection can be defined and computed as

\[ \mathcal{R}_{\alpha\beta} = \begin{bmatrix} \mathcal{R}_{hh}^{i} & \mathcal{R}_{hi}^{i} \\ \mathcal{R}_{ij}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{ik}^{i} \\ \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} \\ \mathcal{R}_{ik}^{i} & \mathcal{R}_{ki}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} \\ \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} \\ \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} \\ \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} \\ \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} \\ \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} \\ \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} \\ \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} \\ \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} \\ \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} \\ \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} \\ \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} & \mathcal{R}_{jk}^{i} & \mathcal{R}_{kj}^{i} & \mathcal{R}_{ij}^{i} & \mathcal{R}_{ji}^{i} \end{bmatrix}. \]

(6)

Such formulas for \( \mathcal{D} \) can be written in a similar “underlined” form. Hereafter, for simplicity, we shall provide the formulas only for a general d-connection \( \mathcal{D} \) if that will not result in ambiguities.

If a phase space is enabled both with a d-connection, \( \mathcal{D} \), and a d-metric, \( \mathcal{g} \), structures, we can define and compute nonholonomic Ricci scalars,

\[ \mathcal{R} := \mathcal{g}^{ij} \mathcal{R}_{ij} + \mathcal{g}^{ab} \mathcal{R}_{ab} = \mathcal{R} + \mathcal{S}, \]

with respective h- and v-components, \( \mathcal{R} = \mathcal{g}^{ij} \mathcal{R}_{ij} \) and \( \mathcal{S} = \mathcal{g}^{ab} \mathcal{S}_{ab} \).

The geometric objects (5)–(7) can be defined for any special classes of linear connection structures. In next subsection, we consider three important classes of linear and/or d-connections determined by a d-metric structure \( \mathcal{g} \) or \( \mathcal{g} \).

### 2.1.3 Preferred linear and d-connection structures

Any relativistic phase space \( T^*V \) can be described as a Hamilton space using the canonical data \( (\tilde{\mathcal{N}}, \mathcal{g}) \) and/or in general nonholonomic (pseudo) Riemannian form for some \( (\mathcal{N}, \mathcal{g}) \). Respective canonical N-connections \( \tilde{\mathcal{N}} \) and/or \( \mathcal{N} \) define correspondingly certain canonical almost complex structures \( \tilde{\mathcal{J}} \) and/or \( \mathcal{J} \). For instance, we can consider a linear operator \( \tilde{\mathcal{J}} \) acting on \( \mathcal{e}_a = (\mathcal{e}_1, \mathcal{e}_2) \) using formulas \( \tilde{\mathcal{J}}(\mathcal{e}_i) = -\mathcal{e}^{a+i} \) and \( \tilde{\mathcal{J}}(\mathcal{e}^{a+i}) = \mathcal{e}_a \). Such a \( \tilde{\mathcal{J}} \) defines globally an almost complex structure \( \tilde{\mathcal{J}} \) (here \( \mathcal{I} \) is the unity matrix) on \( T^*V \). Using \( \tilde{\mathcal{J}} \) and \( \mathcal{J} \), we can define respective (canonical) almost symplectic structures, \( \mathcal{\tilde{\theta}} := \mathcal{g}(\tilde{\mathcal{J}}, \cdot) \) and \( \mathcal{\theta} := \mathcal{g}(\mathcal{J}, \cdot) \). In result, we can construct such preferred linear/distinguished connections:
The geometric objects in (8) are related via corresponding distortion relations

\[ \hat{D} = 'V + \hat{Z}, \quad \hat{D} = 'V + \hat{Z}, \quad \text{and} \quad \hat{D} = \hat{D} + 'Z, \text{ determined by ('g, 'N);} \]

with distortion d-tensors \( \hat{Z}, \quad \hat{Z}, \quad \text{and} \quad 'Z, \) on \( TT^*V. \) In principle, we can work with any such linear connection structure even though they have different geometric and physical meaning. The corresponding curvatures and Ricci d-tensors and scalar curvatures can be computed by introducing such distortion relations in respective formulas (5)–(7).

### 2.2 F- and W-functionals for mechanical systems in general N-adapted variables

The goal of this subsection is to generalize G. Perelman’s functionals and formulate and approach to the theory of nonholonomic geometric flows of relativistic mechanical systems. We shall consider canonical Hamilton variables and nonholonomic deformations to a general d-connection structure. This is important for further developments in classical and quantum information theories when the Hamilton variables are used in explicit form for analyzing certain analogous mechanical and thermodynamic models and, latter, the results are reformulated in general covariant forms.

We consider a family of nonholonomic cotangent Lorentz bundles \( T^*V(\tau) \) enabled with corresponding sets of canonical N-connections \( \tilde{N}(\tau) = \tilde{N}(\tau, 'u) \) and d-metrics \( \tilde{g}(\tau) = \tilde{g}(\tau, 'u) \) all parameterized by a positive parameter \( \tau, 0 \leq \tau \leq \tau_0. \) In general frame form, such sets of geometric objects are, respectively, denoted \( 'N(\tau) = 'N(\tau, 'u) \) and \( 'g(\tau) = 'g(\tau, 'u). \) Let us write correspondingly \( \tilde{\Xi} = (\tilde{\Xi}_t, \tilde{\Xi}_E) \) and \( '\tilde{\Xi} = (\tilde{\Xi}_t, '\tilde{\Xi}_E) \) for nonholonomic distributions of base and fiber hypersurfaces with conventional splitting 3 + 3 of signature \(+ + +; + + +\) on total phase space \( T^*V. \) On a typical cofiber of such a phase space, we can consider a 3-d cofiber hypersurface \( '\tilde{\Xi}_E, \) for instance, of signature \(+ + +\) with a label \( p_8 = E \) for an energy type parameter. Using N-adapted \( (3 + 1) + (3 + 1) \) frame and coordinate transforms of metrics with additional dependence on a flow parameter, we can parameterize the d-metric in the form

\[ \tilde{\Xi} = \text{Springer} \]
Introducing a respective thermodynamic generation function, all thermodynamic of type (11) defined for Riemannian metrics which has properties of “minus entropy”

**Geometric flows of Riemannian metrics** are characterized by a statistical thermodynamic model which can be elaborated in a self-consistent form using a W-functional for certain conditions when normalizing functions, and respective hypersurfaces. LC-configurations can be extracted via distorting relations, with correspondingly re-defined integration measures and normalizing functions, we can formulate [18] in explicit form using canonical data for a classical integration measure

\[ \int \mu |g_{\alpha \beta}| d^{8} u \left( \bar{T}_{R} + \left| \frac{i}{\hbar} D \left| f \right| \right|^{2} + 16 \right) \]

In these formulas, we use a brief notation for the integrals on phase space variables and the normalizing function \( |f(\tau, u)| \) is subjected to the conditions for a classical integration measure \( |\mu| = (4\pi)^{-8} e^{-\frac{1}{4}} \) and the Ricci scalar \( \frac{1}{4} \bar{R} \) is taken for the Ricci d-tensor \( |R_{\alpha \beta}| \) of a d-connection \( |D| \).

Similar F- and W-functionals can be postulated for nonholonomic geometric flows on \( T^{*}V \) using data (\( |g(\tau), D(\tau)| \), or (\( \bar{g}(\tau), \bar{D}(\tau)| \), and other type ones related via distorting relations, with correspondingly re-defined integration measures and normalizing functions, and respective hypersurfaces. LC-configurations can be extracted for certain conditions when \( |D| \big|_{T=0} = |\nabla| \).

### 3. G. Perelman & von Neumann entropies for geometric information flows

Geometric flows of Riemannian metrics are characterized by a statistical thermodynamic model which can be elaborated in a self-consistent form using a W-functional of type (11) defined for Riemannian metrics which has properties of “minus entropy” [19]. Introducing a respective thermodynamic generation function, all thermodynamic
values can be defined and computed by integrating with corresponding measures defined by the metric structure and a corresponding normalizing function. Similar constructions can be elaborated for various relativistic, supersymmetric, commutative and noncommutative generalizations if the geometric flow evolution is modelled for corresponding nonholonomic fibered structures preserving causality and basic postulates for self-consistent stochastic, kinetic and thermodynamics models [33–37], see also [17,18,29,38,39] and references therein. Originally, such nonholonomic transforms of geometric objects and deformations of the (non) linear connection structures were considered in [40,41] where the theory of geometric flows was generalized for Finsler–Lagrange geometries. Then, the approach was developed for flows of Hamilton classical and quantum mechanical systems with certain applications in information theory [29]. In this section, we re-define the constructions changing the “mechanical” variables into general N-adapted ones which is important for further developments in quantum information and gravity theories.

3.1 Analogous thermodynamic models for Hamiltonian GIFs

For relativistic geometric flows of mechanical systems described by Hamiltonians [18,29], the thermodynamic generating function can be written in the form ′Z[ ′g(τ)] = ′∫e−′f√′|g_{αβ}|d8′u(−′f + 16), on T*V, where the integral ′∫ is considered for canonical mechanical variables and the corresponding functional dependence is determined by ′g(τ). With respect to general frames (or with necessary (3 + 1) + (3 + 1) decomposition and a d-metric of type (9)), the integration measure can be re-defined in a form which allows us to consider

\[ \text{A variational N-adapted calculus for } \text{′Z} \text{ and geometric data (′N, ′g, ′D) allows us to compute such relativistic thermodynamic values:} \]

\[ \text{′E} = -τ^2 \int e^{-′f} \sqrt{|q_1 q_2 q_3 N q_5 q_6 q_7 N|} d8' u \left( τ R + |′D f|^2 - \frac{8}{τ} \right), \]

\[ \text{′S} = -\int e^{-′f} \sqrt{|q_1 q_2 q_3 N q_5 q_6 q_7 N|} d8' u \left[ 2 R + |′D f|^2 + f - 16 \right], \]

\[ \text{′η} = -\int e^{-′f} \sqrt{|q_1 q_2 q_3 N q_5 q_6 q_7 N|} d8' u \left[ 2 R_{αβ} + D_{α} D_{β} f - \frac{1}{2τ} |′g_{αβ}|^2 \right]. \]

Such values can be written in Hamilton mechanical variables with tilde as in (12) or re-defining the normalizing functions for the canonical d-connection ′D, see (8) and respective distorting relations.

4 Hereafter, we shall not write such dependencies in explicit form if that will not result in ambiguities.
Using the first two formulas in (13) for two d-metrics \( _1g \) and \( g \), we can define the respective free energy and relative entropy,

\[
^1F(1g) = E(1g) - \tau \cdot ^1S(1g) \quad \text{and} \quad ^1S(1g || g) = \beta [ ^1F(1g) - ^1F(g) ],
\]

where

\[
E(1g) = -\tau^2 \int e^{-f} \sqrt{|q_1 q_2 q_3 \tilde{N} q_5 q_6 q_7 \tilde{N}|} \delta^8 u \left[ s R(1g) + |D(1g) f(\tau, u)|^2 - \frac{8}{\tau} \right],
\]

\[
S(1g) = -\int e^{-f} \sqrt{|q_1 q_2 q_3 \tilde{N} q_5 q_6 q_7 \tilde{N}|} \delta^8 u \left[ \tau \left( s R(1g) + |D(1g) f(\tau, u)|^2 \right) + f(\tau, u) - 16 \right]
\]

are computed using the phase spacetime measures, the Ricci scalar and canonical d-connection are determined, respectively, by \( g \) and \( _1g \).

In this work, we study the geometric flow evolution of thermodynamics systems that preserves the thermal equilibrium at temperature \( \beta \) for maps \( _1g \rightarrow 2g \). A realistic physical interpretation for such systems exists if

\[
S(1g || g) \geq S(2g || g), \quad \text{i.e.,} \quad F(1g) \geq F(2g).
\]  \( \text{(14)} \)

These aspects connect general frame and mechanical variables flow models to the second law of thermodynamics. Values of type (13) are in relativistic thermodynamic relation if the second thermodynamic law (14) is satisfied. Such conditions impose additional constraints on the class of normalizing and generating functions.

3.2 Density matrix and entropies for quantum information flows

In this subsection, we develop the density matrix formalism for applications in the theory of classical and quantum geometric information flows (respectively, GIFs and QGIFs), see sections 4 and 5 in [29] for a formulation in Hamilton mechanical variables. Nonholonomic deformations of G. Perelman entropy like functionals will be used for relativistic formulations of the von Neumann entropy and QGIFs in arbitrary frames.

3.2.1 Statistical density matrix for relativistic classical GIFs

The thermodynamic generating function \( ^1Z[ ^1g(\tau) ] \) (12) with free energy \( ^1E \) can be used for defining the state density

\[
^1\sigma(\beta, ^1E, ^1g) = ^1Z^{-1} e^{-\beta ^1E}, \quad \text{(15)}
\]

with \( \beta = 1/T, \tau = T \). This value is the classical analog of the density matrix in QM. We shall use it for elaborating models of QGIFs.

We can consider that a density state \( ^1\sigma[ ^1g ] \) is associated with \( ^1g_{\alpha\beta} \), when but the geometric evolution may involve another density \( ^1\rho[ ^1\tilde{g} ] \), where the left label 1 is used for distinguishing two d-metrics \( ^1g \) and \( _1g \). In result, the concept of relative entropy
between any state density $\rho(\beta, E, g)$ and $\tilde{\sigma}(\beta, E, g)$ can be introduced. It can be computed for a prescribed measure $\omega(E)$ on a cotangent Lorentz bundle with $E$ considered as a thermodynamical energy parameter associated with $\tilde{\mathcal{E}}$.

The conditional entropy for GIFs is introduced

$$\mathcal{S}(\rho || \sigma) = \beta [\mathcal{F}(\rho) - \mathcal{F}(\sigma)],$$

where the free energy corresponding to $\rho$ is defined by formula

$$\mathcal{F}(\rho) := \mathcal{E}(\rho) - T \mathcal{S}(\rho)$$

with the average energy $\mathcal{E}(\rho) = \int \rho E d\omega(E)$. The thermodynamic entropy in (16) is computed following formula

$$\mathcal{S}(\rho) := \beta \mathcal{E}(\rho) + \log Z(\rho).$$

The condition $\mathcal{S}(\rho || \rho) = 0$ is satisfied if $\log Z$ is independent on $\rho$.

### 3.2.2 Entanglement and density matrix for QGIFs

Using canonical mechanical variables $(\tilde{\mathcal{H}}, \tilde{\mathcal{g}}_{ab})$, we can study special QM systems described by pure states. In a more general context, QM involves probabilities considered not for a quantum state but for densities matrices. In this subsection, we elaborate on how GIFs of classical mechanical systems can be generalized to QGIFs using basic concepts of quantum mechanics, QM, and information theory. We shall elaborate on quantum models of GIFs described in terms of density matrices defined as quantum analogs of state densities of type $\sigma$ (15).

For any point $u \in T^*V$ of a typical relativistic phase space used for modeling a classical GIF system $A = [\mathcal{E}, \mathcal{S}, \eta]$ (13), we associate a typical Hilbert space $\mathcal{H}_A$, which is denoted $\tilde{\mathcal{H}}_A$ for canonical Hamilton mechanical variables. A state vector $\tilde{\psi}_A \in \tilde{\mathcal{H}}_A$ can be defined as an infinite dimensional complex vector function. For applications in quantum information theory, there are considered approximations with finite dimensions. Such a $\tilde{\psi}_A$ is a solution of the Schrödinger equation with a Hamiltonian $\tilde{H}$ constructed as a well-defined quantum version of a canonical Hamiltonian $\mathcal{H}$. In a more general context, we can work with general covariant variables [or certain versions with $(3 + 1) + (3 + 1)$ splitting], when “nontilde” d-metrics $g$ [see (9)] are used for definition of certain quantum measures. Considering unitary transforms of type $\tilde{\psi}_A \rightarrow U \psi_A$, we can describe the system $A$ by an abstract Hilbert space $\mathcal{H}_A$, or to associate a complex vector space of dimension $N$ with Hermitian product, see details in [1,7].

The complex geometric arena for QGIFs models consists from complex bundles $\mathcal{H}_A(T^*V) = \bigcup_u \mathcal{H}_A^u$ associated with $T^*V$ and constructed as unities of Hilbert spaces $\mathcal{H}_A^u$ for $u \in T^*V$, or a points of a subspace of such a phase space. We consider that there are nonholonomic variables when $\psi_A \rightarrow \tilde{\psi}_A(\tilde{\psi}_A)$ and the integration measure is determined by a $\tilde{g}$, or its frame transforms to a $\tilde{g}$. It is assumed that such
constructions are possible at least for perturbations nearly a flat “double” Minkowski metric (1) nearly a point. This way a perturbative QGIF model with quasi-classical limits can be always elaborated. GIFs describe flow evolution of mechanical systems in causal relativistic classical forms.

The combined Hilbert space is defined as a tensor product, $\mathcal{H}_A \otimes \mathcal{H}_B$, with an associate Hilbert space $\mathcal{H}_A$ considered for a complementary system $A$. Here, we note that symbols $A, B, C, \ldots$ are used as labels for certain systems under geometric evolutions described by respective thermodynamical models of type (13). The state vectors for a combined QGIF system are written $\psi_{AB} = \psi_A \otimes \psi_B \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ for $\psi_A = 1_A$ taken as the unity state vector. Quantum systems subjected only to quantum evolution and not to geometric flows are denoted $A, B, C, \ldots$

**Entangled states** In QM and QGIF theories, a pure state $\psi_{AB} \in \mathcal{H}_{AB}$ may be not only a tensor product vector but also entangled and represented by a matrix of dimension $N \times M$ if $\dim \mathcal{H}_A = N$ and $\dim \mathcal{H}_B = M$. We underline such symbols in order to avoid ambiguities with the N-connection symbol $N$. A Schmidt decomposition can be considered for any pure state,

$$\psi_{AB} = \sum_i \sqrt{p_i} \psi^i_A \otimes \psi^i_B,$$

for any index $i = 1, 2, \ldots$ (up to a finite value). The state vectors $\psi^i_A$ can be taken to be orthonormal, $\langle \psi^i_A, \psi^j_A \rangle = \langle \psi^i_B, \psi^j_B \rangle = \delta^{ij}$, where $\delta^{ij}$ is the Kronecker symbol. If $p_i > 0$ and $\sum_i p_i = 1$, we can treat $p_i$ as probabilities. In general, such $\psi^i_A$ and/or $\psi^i_B$ do not define bases of $\mathcal{H}_A$ and/or $\mathcal{H}_B$ because we can take some vectors when, in principle, it is not enough for such bases. We can consider that such values split the GIFs into certain probable evolution scenarios.

The quantum density matrix for a QGIF-associated system $A$ is defined

$$\rho_A := \sum_a p_a |\psi^a_A\rangle \langle \psi^a_A|$$

as a Hermitian and positive semi-definite operator with trace $\text{Tr}_{\mathcal{H}_A} \rho_A = 1$. Using such a $\rho_A$, we can compute the expectation value of any operator $\hat{O}_A$ characterizing additionally such a system,

$$\langle \hat{O} \rangle_{AB} = \langle \psi_{AB} | \hat{O}_A \otimes 1_B | \psi_{AB} \rangle = \sum_i p_i \langle \psi^i_A | \hat{O}_A | \psi^i_A \rangle = \text{Tr}_{\mathcal{H}_A} \rho_A \hat{O}_A.$$

Such values encode both quantum information and geometric flow evolution of bipartite systems of type $A, B$, and $AB$ with both quantum and geometric entanglement defined by density matrices.
Joint probabilities for bipartite quantum systems and measurements

Bipartite QGIFs systems are described in general form by quantum density matrices of type $\rho_{AB}$ or (in canonical mechanical variables) $\tilde{\rho}_{\tilde{A}\tilde{B}}$. In the classical probability theory, we describe a bipartite system $XY$ by a joint probability distribution $P_{X,Y}(x_i, y_j)$, where $P_X(x_i) := \sum_j P_{X,Y}(x_i, y_j)$, see details in [1,7] and, for GIFs [29].

Considering $AB$ as a bipartite quantum system with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, we can define and parameterize a QGIF density matrix $\rho_{AB}$ in standard QM form:

$$\rho_{AB} = \sum_{a,a',b,b'} \rho_{a'a'bb'} |a>_A \otimes |b>_B <a'| \otimes <b'|.$$  

In this formula, $|a>_A, a = 1, 2, \ldots, n$ is an orthonormal basis of $\mathcal{H}_A$ and $|b>_B, b = 1, 2, \ldots, m$ as an orthonormal basis of $\mathcal{H}_B$.

A measurement of the system $A$ is characterized by a reduced density matrix obtained by respective contracting of indices,

$$\rho_A = \text{Tr}_{\mathcal{H}_B} \rho_{AB} = \sum_{a,a',b,b'} \rho_{a'a'bb'} |a>_A <a'| \otimes <b|.$$  

In a similar form, we can define and compute $\rho_B = \text{Tr}_{\mathcal{H}_A} \rho_{AB}$. For cotangent bundle constructions, we can distinguish the geometric and physical objects putting left labels “$\bar{\cdot}$”, $\rho_{\bar{A}} = \text{Tr}_{\mathcal{H}_B} \rho_{AB}$. Using such formulas, we can elaborate on QGIFs models and quantum information theory formulated in conventional mechanical variables or in a general covariant form.

### 3.2.3 Quantum density matrix for QGIFs

The quantum density matrix $\sigma_{AB}$ for a state density $\sigma$ (15) can be defined and computed using formulas (19),

$$\sigma_{AB} = <\sigma>_{AB} = <\psi_{AB}|\sigma \otimes 1_B|\psi_{AB}> = \sum_i p_i <\psi^i_A|\sigma|\psi^i_A> <\psi^i_B|1_B|\psi^i_B> =$$

$$\sigma_A = <\sigma>_{\bar{A}} = \sum_i p_i <\psi^i_A|\sigma|\psi^i_A> = \text{Tr}_{\mathcal{H}_A} \rho_{\bar{A}} \sigma,$$  

where the density matrix $\rho_{\bar{A}}$ is taken for computing the QGIF density matrix $\sigma_{\bar{A}}$. This matrix is determined by a state density of the thermodynamical model for GIFs of a classical system $\sigma$ which can be parameterized in nonholonomic variables of a mechanical Hamiltonian system $\sigma_{\bar{H}}$.

For quantum systems, we can work with quantum density matrices $\sigma_{AB}$ and $\sigma_{\bar{A}}$ and respective partial traces $\sigma_{\bar{A}} = \text{Tr}_{\mathcal{H}_B} \sigma_{AB}$ and $\sigma_B = \text{Tr}_{\mathcal{H}_A} \sigma_{AB}$. Such formulas can be written in coefficient forms.
\[
\sum_{a, a', b, b'} \sigma_{aa'bb'} |a > A \otimes |b > B, A < a' | \otimes B < b' | \text{ and } \\
\sum_{a, a', b, b} \sigma_{aa'bb} |a > A, A < a' |.
\]

Using a density matrix encoding the data for QGIFs of Hamilton mechanical system described in general covariant variables, we can compute respective thermodynamical values.

### 3.2.4 The von Neumann entropy and QGIFs

QGIFs can be described in standard QM form for the von Neumann entropy determined by (20) as a probability distribution,

\[ q_S(\sigma_A) := \text{Tr} \sigma_A \log \sigma_A. \]  

Hereafter, we shall write the trace in a simplified form without a label for the corresponding Hilbert space if that will not result in ambiguities. We use also a left label \( q \) to state the quantum character of such values. It should be also emphasized that such an entropy is a quantum analog of a \( \tilde{S} \) used in the thermodynamic model for geometric flow evolution of Hamilton mechanical systems. Tilde can be omitted for general frame transforms when \( \sigma \) encode a different frame structure. Such a QGIF entropy satisfies two conditions: \( q_S(\sigma_A) \geq 0 \) and it is manifestly invariant under a unitary transformation \( \sigma_A \rightarrow U \sigma_A U^{-1} \).

The von Neumann entropy for QGIFs, \( q_S(\sigma_A) \), has a purifying property which does not have a classical analog. Considering a bipartite system \( \psi_{AB} = \sum_i \sqrt{p_i} \psi_i^A \otimes \psi_i^B \) and \( \rho_A := \sum_i p_i |\psi_i^A > \otimes < \psi_i^A | \), we compute

\[
\begin{align*}
\sigma_A &:= \sum_{a, a', b, b} \sum_{k} \sigma_{aa'bb} p_k |a' > A, A < a' | \otimes \psi_k^A |a > A, \\
\sigma_B &:= \sum_{a, a', b, b} \sum_{k} \sigma_{aa'bb} p_k < a' | \psi_k^B < b > B.
\end{align*}
\]

In these formulas, we have the same probabilities \( p_k \) for two formulas with different matrices and bases. This proves that \( q_S(\sigma_A) = q_S(\sigma_B) \) when a system \( A \) and a purifying system \( B \) have the same von Neumann entropy.

### 3.2.5 Quantum generalizations of the W- and thermodynamic entropy

QGIFs can be characterized not only by a von Neumann entropy of type (21) but also by quantum analogs of entropy values used for classical geometric flows. We can consider both an associated thermodynamics entropy and a W-entropy in classical...
variants and then quantize such systems using a respective Hamiltonian which allows a self-consistent QM formulation. Such values can be introduced and computed in explicit form using respective formulas (20), (22) for classical conditional (16) and mutual entropy considered for GIFs and in information theory [1,7,29]. We define, respectively,

\[ \bar{q}_W^{AB} = \text{Tr} H_{AB} \left( (\bar{\sigma}^{AB}) (\bar{W}^{AB}) \right) \]

\[ \bar{q}_W^A = \text{Tr} H_A \left( (\bar{\sigma}^A) (\bar{W}^A) \right) \]

\[ \bar{q}_W^B = \text{Tr} H_B \left( (\bar{\sigma}^B) (\bar{W}^B) \right) \]

\[ \bar{q}_S^{AB} = \text{Tr} H_{AB} \left( (\bar{\sigma}^{AB}) (\bar{S}^{AB}) \right) \]

\[ \bar{q}_S^A = \text{Tr} H_A \left( (\bar{\sigma}^A) (\bar{S}^A) \right) \]

\[ \bar{q}_S^B = \text{Tr} H_B \left( (\bar{\sigma}^B) (\bar{S}^B) \right) \]

Such values describe corresponding entropic properties of quantum systems with rich geometric structure under geometric flow evolution.

The quantum probabilistic characteristics are described by the von Neumann entropy \( \bar{q}_S(\bar{\sigma}^A) \) (21) and corresponding generalizations for \( AB \) and \( B \) systems

\[ \bar{q}_S(\bar{\sigma}^{AB}) := \text{Tr} \bar{\sigma}^{AB} \log \bar{\sigma}^{AB} \]

\[ \bar{q}_S(\bar{\sigma}^A) := \text{Tr} \bar{\sigma}^A \log \bar{\sigma}^A \]

Such values also encode thermodynamic, geometric flow and probabilistic properties of QGIFs and can be used for elaborating a standard approach to quantum information theory for systems with geometric mechanical Hamilton flows and their covariant frame transforms.

**4 Entanglement and QGIFs of quantum mechanical systems**

Originally, the notion of bipartite entanglement was introduced for pure states and density matrix generalizations in description of finite-dimensional QM systems, see review of results in [1,3,6,7]. In this section, we analyze how the concept of entanglement can be generalized for QGIFs when, for instance, there are considered two relativistic mechanical systems under geometric flow evolution. Such systems and their thermodynamic and QM analogs are characterized by a set of entropies like G. Perelman’s W-entropy and geometric thermodynamic entropy and the nontrivial entanglement entropy in the von Neumann sense. Each of such entropic values characterize classical and quantum correlations determined by geometric flow evolution and quantifies the amount of quantum entanglement. A set of inequalities involving Perelman and entanglement entropies play a crucial role in definition and description of such systems. We provide such formulas without rigorous proofs following two reasons: the W-entropy \( \bar{\mathcal{W}} \) (11), thermodynamic entropy \( \bar{\mathcal{S}} \) (13) and related von Neumann \( \bar{q}_S \) (21) realizations are well-defined classical and quantum entropic type values. For physicists, such formulas have a natural and intuitive motivation and interpretation in terms of thermodynamical generation functions and density matrices for GIFs. Rigorous mathematical proofs on hundreds of papers use methods of geometric
On main ideas and key steps for checking such results and selecting causal and realistic physical scenarios, we discuss in footnote 10 of our partner work [29].

### 4.1 Geometric flows with entanglement

The goal of this subsection is to study how the concept of quantum entanglement can be developed for QGIF systems characterized by an associated statistical thermodynamic model with respective generating function which transforms into a respective density matrix in a related quantum theory.

#### 4.1.1 Bipartite entanglement for QGIFs

For any (relativistic) mechanic model, continuous or a lattice model of quantum field theory, thermofield theory, QGIF model, etc., we can associate a QM mechanical model with a pure ground state $|\Psi\rangle$ for a total Hilbert space $i\mathcal{H}$ when the density matrix is

$$i\rho = |\Psi\rangle \langle \Psi|$$

(23)
can be normalized following the conditions $\langle \Psi | \Psi \rangle = 1$ so that the total trace $i\text{tr}(i\rho) = 1$. Such a conventional total quantum system is divided into a two subsystems $A$ and $B$. In this section, we consider that $A = [iE, iS, i\eta]$ (13) is a typical GIF system (in mechanical, or general covariant variables) for with a QGIF model is elaborated. A similar model (in principle, for a different associated relativistic Hamiltonian and d-metric $i\mathbf{g}$) is considered for $B = [iE, iS, i\eta]$. Such subsystems $A$ and $B = \overline{A}$ are complimentary to each other if in a $2n$-dimensional cotangent bundle space there is a common boundary $\partial A = \partial B$ of codimension 2, where the nonsingular geometric flow evolution $A$ transforms into a necessary analytic class of flows on $\overline{A}$. In principle, we can consider two completely different and classically separated GIF systems $A$ and $B$ which are correlated as quantum systems. We can consider that for bipartite QGIFs $i\mathcal{H} = \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ as we considered in Sect. 3.2. Such an approximation is less suitable, for instance, if there are considered theories with gauge symmetries, see discussion and references in footnote 3 of [6]. (We omit such constructions in this work.)

The measure of entanglement of a QGIF subsystem $A$ is just the von Neumann entropy $iS (21)$ but defined for the reduced density matrix $i\rho_A = \text{Tr}_{\mathcal{H}_B}(i\rho)$, when the entanglement entropy of $A$ is

$$iS(i\rho_A) := \text{Tr}(i\rho_A \log i\rho_A).$$

(24)

Such a $i\rho_A$ is associated with a state density $i\rho(\beta, i\mathcal{E}, i\mathbf{g})$ of type (15). We note that the total entropy $iS = 0$ for a pure grand state (23).
4.1.2 Separable and entangled QGIFs

Considering \( |a> \in \mathcal{H}_A \) and \( |b> \in \mathcal{H}_B \) as orthonormal bases, we can parameterize a pure total ground state in the form

\[
|\Psi> = \sum_{ab} c_{ab} |a> \otimes |b>,
\]

(25)

where \( c_{ab} \) is a complex matrix of dimension \( \dim \mathcal{H}_A \times \dim \mathcal{H}_B \). When such coefficients factorize, \( c_{ab} = c_a c_b \), we obtain a separable ground state (equivalently, pure product state), when

\[
|\Psi> = |\Psi_A> \otimes |\Psi_B>, \quad \text{for} \quad |\Psi_A> = \sum_a c_a |a> \quad \text{and} \quad |\Psi_B> = \sum_b c_b |b>.
\]

The entanglement entropy \( \overline{q}S(\overline{\rho}_A) = 0 \) if and only if the pure ground state is separable. For QGIFs, such definitions are motivated because corresponding subsystems are described by corresponding effective relativistic Hamilton functions, \( \tilde{H}_A \) and \( \tilde{H}_B \), and/or effective thermodynamics energies, \( \overline{A}E \) and \( \overline{B}E \).

A ground state \( |\Psi> \) is entangled (inseparable) if \( c_{ab} \neq c_a c_b \). For such a state, the entanglement entropy is positive, \( \overline{q}S(\overline{\rho}_A) > 0 \). Using quantum Schmidt decompositions (17) and (18), we prove that

\[
\overline{q}S = - \sum_a p_a \log p_a \quad \text{and} \quad \overline{q}S_{\text{max}} = \log \min(a, b) \quad \text{for} \quad \sum_a p_a = 1 \quad \text{and} \quad p_a > 0.
\]

(26)

In summary, an entangled state of QGIFs is a superposition of several quantum states associated with GIFs. An observer having access only to a subsystem \( A \) will find him/herself in a mixed state when the total ground state \( |\Psi> \) is entangled following such conditions:

\[
|\Psi>: \text{ separable} \quad \leftrightarrow \quad |\rho_A>: \text{ pure state},
\]

\[
|\Psi>: \text{ entangled} \quad \leftrightarrow \quad |\rho_A>: \text{ mixed state}.
\]

The von Neumann entanglement entropy \( \overline{q}S \) encodes two types of information: (1) how geometric evolution is quantum flow correlated and (2) how much a given QGIF state differs from a separable QM state. A maximum value of quantum correlations is reached when a given QGIF state is a superposition of all possible quantum states with an equal weight. Additional GIF properties are characterized by W-entropy \( \overline{W} \) and thermodynamic entropy \( \overline{S} \) which can be computed in certain quasi-classical QM limits, for a \( 3+1 \) splitting, for instance, along a time-like curve.

4.1.3 Two QGIFs systems as analogs of two spin and/or bipartite systems

The most simple example of an entangled system \([1,3,6,7]\) is that of two particles \( A \) and \( B \) with spin \( 1/2 \). In the information theory, such quantum spin systems can be used
to encode binary information as bits and, with further generalizations, to elaborate on quantum bits, qubits. Respective theoretical descriptions use density matrices and the von Neumann entropy.

To study similar entanglement properties of geometric flows in classical and quantum information theory, we can consider two thermodynamical models of general covariant mechanical systems $A = [g, \mathcal{E}, S, \eta]$ and $B = [g, \mathcal{E}, S, \eta]$, see formulas (13). A respective QGIF model with entanglement is elaborated for different associated relativistic Hamiltonians and respective d-metrics $g$ and $\mathring{g}$. For simplicity, we consider that the conventional Hilbert spaces are spanned by two orthonormal basic states in the form $\{|a\rangle_A; a = 1, 2\} \in \mathcal{H}_A$ and $\{|b\rangle_B; b = 1, 2\} \in \mathcal{H}_B$, when $\langle a|b\rangle_A,B = \delta_{ab}$. The total Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ has a 4-dim orthonormal basis $\mathcal{H}_{AB} = \{|11\rangle, |12\rangle, |21\rangle, |22\rangle\}$, where $|ab\rangle = |a\rangle_A \otimes |b\rangle_B$ are tensor product states.

As a general state, we can consider $|\Psi\rangle = \cos \theta |12\rangle - \sin \theta |21\rangle$, (27) where $0 \leq \theta \leq \pi/2$. The corresponding entanglement entropy (24) is computed

$$\langle S(\rho_A) = -\cos^2 \theta \log(\cos^2 \theta) - \sin^2 \theta \log(\sin^2 \theta).$$

The above formulas show that for $\theta = 0, \pi/2$ we obtain pure product states with zero entanglement entropy. For a system $|\Psi\rangle = \frac{1}{\sqrt{2}} (|12\rangle - |21\rangle)$, when the density matrix

$$\rho_A = \frac{1}{2} (|1\rangle_A A <1| + |2\rangle_A A <2|) = \frac{1}{2} diag(1, 1)$$

results in

$$\langle S(\rho_A) = -\text{tr}_{A}(\rho_A \log \rho_A) = \log 2.$$ 

So, the maximal entanglement is for $\theta = \pi/4$. If the GIF structure is “ignored” for such a quantum system [or (27)], we can treat it as conventional QM system, for instance, with up-spin $|1\rangle$ and down-spin $|2\rangle$. In a general context, QGIFs with nonholonomic structure determined by Hamilton mechanical systems are characterized additionally by respective values of W-entropy $\mathcal{W}$ (11) and thermodynamic entropy $\langle S \rangle$ (13). In orthonormal quantum bases, the entanglement entropy is the measure of “pure” quantum entanglement. The information flows with rich nonholonomic geometric structure are characterized additionally by geometric type entropies.

### 4.1.4 Thermofield double QGIF states and entanglement and W-entropy

If the evolution parameter $\beta = T^{-1}$ is treated as a temperature one like in the standard G. Perelma’s approach, we can consider respective geometric flow theories as certain classical and/or quantum thermofield models. Such a nontrivial example with entanglement and a thermofield double GIFs state is defined by a ground state (25) parameterized in the form
where the normalization of the states is taken for the partition function $Z = \sum_k e^{-\beta E_k/2}$. Such values are associated with the thermodynamic generating function $\bar{Z} = \sum_k e^{-\beta E_k/2}$, see discussions related to formulas (22). Coping the state vectors $\{|k\rangle_B\}$ from $\mathcal{H}_A$ to $\mathcal{H}_B$, we can purify the QGIF thermal system $A$ in the extended Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. In result, every expectation value of local operators in $A$ can be represented using the thermofield double state $|\Psi\rangle$ of the total system $A \cup B$. For such models, the entanglement entropy is a measure of the thermal entropy of the subsystem $A$ when

$$\bar{S}(\bar{\rho}_A) = -\text{tr}_A [\bar{\rho}_A (-\beta \bar{\mathcal{E}}_A - \log Z)] = \beta (<\bar{\mathcal{E}}_A> - \bar{\mathcal{F}}_A),$$

where the thermal free energy is computed $\bar{\mathcal{F}}_A = -\log Z$. Here, we note that for the thermofield values it omits the label “q” considered, for instance, for $\bar{\mathcal{S}}$ (24), see also formulas (16).

Thermofield GIF configurations are also characterized by the respective W-entropy $\bar{\mathcal{W}}$ (11) which can be defined even though thermodynamic models are not elaborated. For nonholonomic kinetic, diffusion and thermodynamic structures including relativistic Ricci flows, such models were studied in detail in [33,36,44], see references therein. We also cite some important works on geometric thermodynamics and thermofield theories, see [42,43,45] and references. The thermofield double states were considered in black hole thermodynamics and QFT, see reviews of results in [1–3,6,7].

4.1.5 Bell-like QGIF states

In a two-QGIF system, a state (27) is maximally entangled for $\theta = \pi/4$. Analogs of Bell state (or Einstein–Podolsky–Rosen pairs) in quantum geometric flow theory are defined

$$|\Psi^1_B\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle), \quad |\Psi^2_B\rangle = \frac{1}{\sqrt{2}}(|11\rangle - |22\rangle),$$

$$|\Psi^3_B\rangle = \frac{1}{\sqrt{2}}(|12\rangle + |21\rangle), \quad |\Psi^4_B\rangle = \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle).$$

(29)
In QM models, these states violate the Bell’s inequalities. Such inequalities hold in a hidden variable theory for the probabilistic features of QM with a hidden variable and a probability density. In this work, the states (29) encode also information of geometric flows characterized by W-entropy.

**EPR pairs and multi-qubits for QGIFs** The constructions can be extended for systems of \( k \) qubits. The first example generalizes the concept of Greenberger–Horne–Zeilinger, GHZ, states [6,46,47],

\[
|\Psi_{GHZ}^{B}> = \frac{1}{\sqrt{2}}(|1> \otimes k + |2> \otimes k).
\]

In quantum information theory, there are used another type of entangled states (called W states; do not confuse with W-entropy) [48],

\[
|\Psi_{W}^{B}> = \frac{1}{\sqrt{2}}(|21 \ldots 11> + |12 \ldots 1> + \ldots + |11 \ldots 12>).
\]

We emphasize that \(|\Psi_{GHZ}^{B}>\) is fully separable but not \(|\Psi_{W}^{B}>\) which we shall prove in the example below.

**Tripartite QGIFs** For \( k = 3 \) with subsystems \( A, B \) and \( C \), we write

\[
|\Psi_{GHZ}^{B}> = \frac{1}{\sqrt{2}}(|111> + |222>) \text{ and } |\Psi_{W}^{B}> = \frac{1}{\sqrt{2}}(|112> + |121> + |211>).
\]

Considering \( \text{Tr}_C \), we define the reduced density matrices for the system \( A \cup B \),

\[
\rho_{A\cup B}^{GHZ} = \frac{1}{2}(|11><11| + |22><22>) \text{ and } \rho_{A\cup B}^{W} = \frac{2}{3} |\Psi_{B}^{3}> <\Psi_{B}^{3}| + \frac{1}{3} |11><11|.
\]

This describes two different QGIF states. The first one is fully separable and can be represented in the form \( \rho_{A\cup B}^{GHZ} = \sum_{k=1}^{2} p_k \rho_{A}^{k} \otimes \rho_{B}^{k} \), where \( p_k = 1/2 \) and \( \rho_{A\cup B}^{W} = |11><11| + |22><22| \). Because of the Bell state \(|\Psi_{B}^{3}> (29)\), the \( \rho_{A\cup B}^{W} \) cannot be written in a separable form. So, the state \(|\Psi_{B}^{W}>\) is still entangled even though we have taken \( \text{Tr}_C \). This establishes a quantum correlation between QGIFs. Additionally, such values are characterized by W-entropies of type \( W(11) \) computed for \( A, B, C \) and \( A \cup B \).

### 4.2 Important properties and entanglement inequalities for QGIFs entropies

We summarize several useful properties of the entanglement entropy (24) for QGIFs formulated in terms of the density matrix of type \( \rho_{A} = \text{Tr}_{H_{B}}(\rho) \). We omit explicit cumbersome and technical proofs because they are similar to derivations in [10]. For any \( \rho_{A} \) associated with a state density \( \rho(\beta, \mathcal{E}, g) \) of type (15), we can compute the respective W-entropy and geometric thermodynamic entropy taking measures determined by \( g \) and/or respective Hamilton mechanical variables. Rigorous mathematical proofs involve a geometric analysis technique summarized in [19,30–32].
applications in modern gravity and particle physics theories, we can elaborate on alternative approaches using the anholonomic frame method of constructing off-diagonal solutions in relativistic geometric flow theories and generalizations [28,36,37]. Using explicit classes of solutions and re-defining normalizing functions, we can always compute Perelman’s like entropy functionals at least in the quasi-classical limit with respective measures and related to $\hat{q}S (24)$ for a QGIF or a thermofield GIF model.

4.2.1 (Strong) subadditivity

We present four important properties of QGIFs which result in the strong subadditivity property of entanglement and Perelman’s entropies.

**Entanglement entropy for complementary subsystems** If $B = \overline{A}$, the entanglement entropies are the same

$$\hat{q}S_A = \hat{q}S_{\overline{A}}$$

which follows from formulas (26) for a pure ground state wave function. Similar equalities for the W-entropy $\hat{q}W (11)$ and/or thermodynamic entropy $\hat{q}S (13)$ can be proven only for the same d-metrics $\hat{g}$ and respective normalizations on $A$ and $\overline{A}$. Here, we note that $\hat{q}S_A \neq \hat{q}S_B$ if $A \cup B$ is a mixed state, for instance, at a finite temperature. So, in general,

$$\hat{q}S_A \neq \hat{q}S_B \text{ and } \hat{q}W_A \neq \hat{q}W_B.$$  \hspace{1cm} (30)

We have to consider a subclass of nonholonomic deformations when conditions transform into equalities for respective relativistic flow evolution scenarios and associated thermodynamic and QM systems.

**Subadditivity** For disjoint subsystems $A$ and $B$, the conditions of subadditivity are satisfied

$$\hat{q}S_{A \cup B} \leq \hat{q}S_A + \hat{q}S_B \text{ and } |\hat{q}S_A - \hat{q}S_B| \leq \hat{q}S_{A \cup B}. \hspace{1cm} (30)$$

The second equation transforms into the triangle inequality [49]. In the quasi-classical limit, we obtain similar inequalities for the thermodynamic entropy $\hat{q}S (13)$. We claim that similar conditions hold for the W-entropy $\hat{q}W (11)$. They can be computed as quantum perturbations in a QFT associated with a bipartite QGIF model

$$\hat{q}W_{A \cup B} \leq \hat{q}W_A + \hat{q}W_B \text{ and } |\hat{q}W_A - \hat{q}W_B| \leq \hat{q}W_{A \cup B}.$$  \hspace{1cm} (30)

Such flow evolution and QM scenarios are elaborated for mixed geometric and quantum probabilistic information flows.

**Strong subadditivity** Considering three disjointed QGIF subsystems $A$, $B$ and $C$ and certain conditions of convexity of a function built from respective density matrix and unitarity of systems [6,7,50,51], one holds the following inequalities of *strong subadditivity*:

$$\hat{q}S_{A \cup B \cup C} + \hat{q}S_B \leq \hat{q}S_{A \cup B} + \hat{q}S_{B \cup C} \text{ and } \hat{q}S_A + \hat{q}S_C \leq \hat{q}S_{A \cup B} + \hat{q}S_{B \cup C}.$$
From these conditions, the conditions of subadditivity (30) can be derived as particular cases. Along causal curves on respective cotangent Lorentz manifolds, we can prove similar formulas for the W-entropy and small quantum perturbations

$$\frac{1}{q} W_{A\cup B\cup C} + \frac{1}{q} W_B \leq \frac{1}{q} W_{A\cup B} + \frac{1}{q} W_{B\cup C}$$

and

$$\frac{1}{q} W_A + \frac{1}{q} W_C \leq \frac{1}{q} W_{A\cup B} + \frac{1}{q} W_{B\cup C}.$$  

We claim such properties for respective QGIFs. They play vital roles in the entropic proofs of the so-called c- F-theorems for renormalization group flows in QFT, see review of results in section VIII of [6]. In our approach, we elaborate on a different geometric formalism with nonholonomic flow evolution and respective applications in quantum information theory.

### 4.2.2 Relative entropy and QGIF entanglement

There are several measures of quantum entanglement which are determined by geometric and thermodynamic values for QGIFs. We begin with the concept of relative entropy in geometric information theories.

$$\mathcal{S}\left(\rho_A \parallel \sigma_A\right) = \text{Tr}_{\mathcal{B}} \left[ \rho_A \left( \log \rho_A - \log \sigma_A \right) \right], \quad (31)$$

where \( \mathcal{S}(\rho_A \parallel \sigma_A) = 0 \). This value is a measure of “distance” between two QGIFs with a norm \( ||\rho_A|| = tr(\sqrt{\rho_A}) \). For thermodynamical GIF systems, it transforms into the conditional entropy (16). It was introduced and studied for standard density matrices in QM and information theory, respectively, in [52–54], see reviews [1,6,7]. In straightforward form, we can check that there are satisfied certain important properties and inequalities.

**Two-QGIF systems** are characterized by formulas and conditions:

1. for tensor products of density matrices,

$$\mathcal{S}(\rho_A \otimes \sigma_A) = \mathcal{S}(\rho_A \parallel \sigma_A) + \mathcal{S}(\sigma_A \parallel \sigma_A);$$

2. positivity:

$$\mathcal{S}(\rho_A \parallel \sigma_A) \geq \frac{1}{2} ||\rho_A - \sigma_A||^2,$$

i.e., \( \mathcal{S}(\rho_A \parallel \sigma_A) \geq 0 \);

3. monotonicity:

$$\mathcal{S}(\rho_A \parallel \sigma_A) \geq \mathcal{S}(\text{tr}_s \rho_A |\text{tr}_s \sigma_A),$$

where \( \text{tr}_s \) is the trace for a subsystem of \( \mathcal{A} \).

Using the above positivity formula and the Schwarz inequality \( ||X|| \geq tr(XY)/||X|| \), we obtain that

$$\mathcal{S}(\rho_A \parallel \sigma_A) \geq \frac{1}{2} \frac{(\langle \mathcal{O}_\rho \rangle - \langle \mathcal{O}_\sigma \rangle)^2}{||\mathcal{O}||^2}$$
for any expectation value $<O>_{\rho}$ of an operator $O$ computed with the density matrix $^{1}\rho_{A}$, see formulas (19).

The relative entropy $^{1}S(^{1}\rho_{A} \parallel ^{1}\sigma_{A})$ (31) can be related to the entanglement entropy $^{q}S(^{1}\rho_{A})$ (24) using formula

$$^{1}S(^{1}\rho_{A} \parallel 1_{A}/k_{A}) = \log k_{A} - ^{q}S(^{1}\rho_{A}), \quad (32)$$

where $1_{A}$ is the $k_{A} \times k_{A}$ unit matrix for a $k_{A}$-dimensional Hilbert space associated with the region $A$. The above properties can be re-defined by the entanglement entropy $^{q}S$, see similar formulas for QGIFs in Hamilton mechanical variables in [29].

Three-QGIF systems Let us denote by $^{1}\rho_{AUB\cup C}$ the density matrix of three QGIFs subsystems $A \cup B \cup C$ and, for instance, $^{1}\rho_{AUB}$ for its restriction on $A \cup B$ and $^{1}\rho_{B}$ for its restriction on $B$. Using the formula for computing traces of reduced density matrices,

$$\text{tr}_{AUB\cup C}[^{1}\rho_{AUB\cup C}(O_{AUB} \otimes 1C/kC)] = \text{tr}_{AUB}(^{1}\rho_{AUB}O_{AUB})$$

we prove such identities

$$^{1}S(^{1}\rho_{AUB\cup C} \parallel 1_{AUB\cup C}/k_{AUB\cup C}) = ^{1}S(^{1}\rho_{AUB} \parallel 1_{AUB}/k_{AUB})$$

$$+ ^{1}S(^{1}\rho_{AUB\cup C} \parallel ^{1}\rho_{AUB} \otimes 1C/kC),$$

$$^{1}S(^{1}\rho_{B\cup C} \parallel 1_{B\cup C}/k_{B\cup C}) = ^{1}S(^{1}\rho_{B} \parallel 1_{B}/k_{B}) + ^{1}S(^{1}\rho_{B\cup C} \parallel ^{1}\rho_{B} \otimes 1C/kC);$$

and inequalities

$$^{1}S(^{1}\rho_{AUB\cup C} \parallel ^{1}\rho_{AUB} \otimes 1C/kC) \geq ^{1}S(^{1}\rho_{B\cup C} \parallel ^{1}\rho_{B} \otimes 1C/kC),$$

$$^{1}S(^{1}\rho_{AUB\cup C} \parallel 1_{AUB\cup C}/k_{AUB\cup C}) + ^{1}S(^{1}\rho_{B} \parallel 1_{B}/k_{B})$$

$$\geq ^{1}S(^{1}\rho_{AUB} \parallel 1_{AUB}/k_{AUB}) + ^{1}S(^{1}\rho_{B\cup C} \parallel 1_{B\cup C}/k_{B\cup C}).$$

These formulas can be re-written [after corresponding applications of the rule (32)] for the entanglement entropies $^{q}S$ and Hamilton mechanical variables with “tilde” [29].

4.2.3 Mutual information for QGIFs

The correlation between two-QGIF systems $A$ and $B$ (it can be involved also a third system $C$) is characterized by the mutual information $^{1}J(A, B)$ and respective inequalities which follow from the above formulas for relative entropy,

$$^{1}J(A, B) := ^{1}S_{A} + ^{1}S_{B} - ^{1}S_{AUB} \geq 0 \text{ and } ^{1}J(A, B \cup C) \leq ^{1}J(A, B).$$

The mutual information is related to the relative entropy following formula

$$^{1}J(A, B) = ^{1}S(^{1}\rho_{AUB} \parallel ^{1}\rho_{A} \otimes ^{1}\rho_{B}), \quad (33)$$
which allows to consider similar concepts and inequalities for the entanglement of QGIF systems:

\[ \bar{q} J(A, B) := \bar{q} S_A + \bar{q} S_B - \bar{q} S_{A \cup B} \geq 0, \]
\[ \bar{q} J(A, B \cup C) \leq \bar{q} J(A, B), \quad \text{for} \quad \bar{q} J(A, B) = \bar{q} S(\rho_{A \cup B} \parallel \rho_A \otimes \rho_B). \]

In the classical variant of GIFs, one holds similar formulas for GIFs and associated thermodynamic models with statistical density \( \bar{\rho}(\beta, \bar{\mathcal{E}}, \bar{\mathbf{g}}) \) (15). For relativistic geometric flows, we claim that similar properties hold for the constructions using the W-entropy. In particular, this can be proven for causal configurations in nonholonomic Hamilton variables [29].

The mutual information between two QGIFs shows how much for a union \( A \cup B \) the density matrix \( \rho_{A \cup B} \) differs from a separable state \( \rho_A \otimes \rho_B \). Quantum correlations entangle even spacetime disconnected regions of the phase spacetime under geometric flow evolution. For bounded operators \( O_A \) and \( O_B \) under geometric evolution in respective regions, one holds true (the proof is similar to that in [55]) the inequality

\[ \bar{J}(A, B) \geq \frac{1}{2} \left( \langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle \right)^2 \left| |O_A|^2 |O_B|^2 \right|. \]

Such formulas can be proven for associated thermodynamic systems to classical GIFs using the statistical density if, for instance, \( A \) and \( B \) are certain subsystems of phase spaces and respective geometric flows.

4.2.4 The Rényi entropy for QGIFs

We can introduce another type of parametric entropy which provides us more information about the eigenvalues of reduced entropy matrices than the entanglement entropy. This is the Rényi entropy [56] which is important for computing the entanglement entropy of QFTs using the replica method, see section IV of [6]. Such constructions are possible in QGIF theory because the thermodynamic generating function \( \bar{Z}[\bar{\mathbf{g}}(\tau)] \) (12) and related statistical density \( \bar{\rho}(\beta, \bar{\mathcal{E}}, \bar{\mathbf{g}}) \) (15) can be used for defining \( \bar{\sigma}_A \) (20) as a probability distribution.

**Replica method and G. Perelman’s thermodynamics model** Let us consider an integer \( r \) called as the replica parameter and introduce the Rényi entropy

\[ \bar{r}S(A) := \frac{1}{1 - r} \log[\text{tr}_A(\bar{\rho}_A)^r] \quad (34) \]

for a QGIF system determined by a density matrix \( \rho_A \).\(^5\) We use the symbol \( r \) for the replica parameter (and not \( n \) as in the typical works in information theory) because the symbol \( n \) is used in our works for the dimension of base manifolds. To elaborate a computational formalism, one considers an analytic continuation of \( r \) to a real number

\(^5\) We use the symbol \( r \) for the replica parameter (and not \( n \) as in the typical works in information theory) because the symbol \( n \) is used in our works for the dimension of base manifolds.
which allows us to define the limit $\bar{q}(\bar{\rho}_A) = \lim_{r \to 1} \bar{r}(\bar{r}S(A))$, with the normalization $\text{tr}_A(\bar{\rho}_A)$ for $r \to 1$, when the Rényi entropy (34) reduces to the entanglement entropy (24).

There are satisfied certain important inequalities for derivatives on replica parameter, $\partial_r$, of the Rényi entropy $\bar{r}(\bar{r}S)$ (proofs are similar to [57]):

$$\partial_r(\bar{r}S) \leq 0,$$

$$\partial_r \left( \frac{r - 1}{r} \bar{r}S \right) \geq 0, \quad \partial_r [(r - 1)\bar{r}S] \geq 0, \quad \partial^2_{rr} [(r - 1)\bar{r}S] \leq 0. \quad (35)$$

These formulas have usual thermodynamical interpretations for a system with a modular Hamiltonian $H_A$ and effective statistical density $\bar{\rho}_A := e^{-2\pi H_A}$. Considering $\beta_r = 2\pi r$ as the inverse temperature, we introduce the effective “thermal” statistical generation (partition) function,

$$\bar{r}(\beta_r) := \text{tr}_A(\bar{r}\rho_A)^r = \text{tr}_A(e^{-\beta_r H_A}).$$

similarly to $\bar{Z}[\bar{g}(\tau)]$ (12). In analogy to the thermodynamical model for geometric flows (13), we compute by canonical relations such statistical mechanics values

$$\bar{r}E(\beta_r) := -\partial_{\beta_r} \log[\bar{r}(\beta_r)] \geq 0, \quad \text{for. the modular energy;}$$

$$\bar{r}S(\beta_r) := (1 - \beta_r \partial_{\beta_r}) \log[\bar{r}(\beta_r)] \geq 0, \quad \text{for. the modular entropy;}$$

$$\bar{r}C(\beta_r) := \beta^2_r \partial^2_{\beta_r} \log[\bar{r}(\beta_r)] \geq 0, \quad \text{for. the modular capacity.}$$

These inequalities are equivalent to the second line in (35) and characterize the stability if GIFs as a thermal system with replica parameter regarded as the inverse temperature for a respective modular Hamiltonian. Such replica criteria of stability were not considered in the original works on Ricci flows [19,30–32]. They define a new direction for the theory of geometric flows and applications in modern physics with respective generalizations for nonholonomic structures [17,18,29,33–41].

We note that the constructions with the modular entropy can be transformed into models derived with the Rényi entropy and inversely. Such transforms can be performed using formulas

$$\bar{r}S := r^2 \partial_r \left( \frac{r - 1}{r} \bar{r}S \right) \quad \text{and, inversely,} \quad \bar{r}S = \frac{r}{r - 1} \int_1^r d\bar{r} \frac{1}{(r')^2} \bar{r}S.$$

The implications of the inequalities for the Rényi entropy were analyzed for the gravitational systems with holographic description, see reviews [1,2,6]. In this subsection, the approach is generalized for nonholonomic geometric structures and covariant mechanical systems with applications in information theory.

**Relative Rényi entropy for QGIFs** The concept of relative entropy $\bar{r}(\bar{\rho}_A \parallel \bar{\sigma}_A)$ (31) can be extended to that of relative Rényi entropy [58,59] (for a review, see section II.E.3b in [6]). For a QGIF system with two density matrices $\bar{\rho}_A$ and $\bar{\sigma}_A$, we introduce

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\[ r S(\rho_A \parallel \sigma_A) = \frac{1}{r - 1} \log \left[ \text{tr} \left( \left( \rho_A \right)^{(1-r)/2r} \left( \rho_A \right)^{(1-r)/2r} \right) \right], \]

for \( r \in (0, 1) \cup (1, \infty) \);

or \( r S(\rho_A \parallel \sigma_A) = S(\rho_A \parallel \sigma_A) \) and \( S(\rho_A \parallel \sigma_A) \)

\[ \geq \log \left\| \left( \frac{1}{r} \rho_A \right)^{-1/2} \rho_A \left( \frac{1}{r} \rho_A \right)^{-1/2} \right\|_\infty \].

(36)

Such definitions allow us to prove certain monotonic properties,

\[ r S(\rho_A \parallel \sigma_A) \geq S(\text{tr} \rho_A | \text{tr} \sigma_A) \] and \( \partial_r [r S(\rho_A \parallel \sigma_A)] \geq 0 \),

and to reduce the relative Rényi entropy to the Rényi entropy using a formula similar to (32),

\[ r S(\rho_A \parallel 1_A \! \! / k_A) = \log k_A - r S(A). \]

Nevertheless, the values (36) do not allow a naive generalization of the concept of mutual information and interpretation as an entanglement measure of quantum information because of possible negative values of relative Rényi entropy for \( r \neq 1 \). This problem is solved by the \( r \)-Rényi mutual information [61],

\[ r \mathcal{J}(A, B) := \min_{\sigma_B} r S(\rho_{A \cup B} \parallel \rho_A \otimes \sigma_B) \geq 0, \]

when the minimum is taken over all \( \sigma_B \). This formula reduced to the mutual information (33) for \( r = 1 \). In result, we can elaborate a self-consistent geometric information thermodynamic theory for QGIFs. This is possible if the statistical density \( \rho(\beta, \mathcal{E}, \mathcal{g}) \) (15) is used for defining \( \rho_A \) (20) as a probability distribution and respective von Neumann density matrix formulation of the quantum models. It is not clear at present whether a version of relative Rényi entropy can be elaborated for the W-entropy.

5 Conclusions

The geometric flows of Riemannian metrics can be characterized by G. Perelman’s W-entropy and associated statistical thermodynamic model with respective mean energy, mean entropy and fluctuation parameter [19]. Such constructions can be generalized for nonholonomic geometric flows (subjected to certain nonintegrable, i.e., anholonomic, equivalently, nonholonomic conditions) with generalized entropy-type functionals and related locally anisotropic diffusion, kinetic and thermodynamic theories [33, 36, 44]. In result, we can elaborate on advanced geometric methods for modeling relativistic geometric flows of classical and quantum mechanical systems and modified commutative and noncommutative/supersymmetric gravity theories, etc. [35–37].

A series of our recent works, see [17, 29] and references therein, is devoted to formulation and applications on the theory of geometric information flows, GIFs and quantum information flows, QGIFs. In such approaches, the geometric thermodynamic models involve G. Perelman like entropic constructions [18] which are more general
than those elaborated using the Bekenstein–Hawking surface-area entropy and respective holographic, dual CFT-gauge theory generalizations, etc. [62–68]. New classes of generic off-diagonal solutions (various locally anisotropic cosmological ones, generalized black hole metrics) with the coefficients of metrics and generalized connections depending, in principle, on all spacetime and possible phase space coordinates can be constructed [28,39] in general relativity and modified gravity theories. Such new classes of exact and parametric solutions, and related quantized models, are characterized by G. Perelman entropies and do not have Bekenstein–Hawking analogs.

In this article, we have focused on developing the notion of entanglement for quantum mechanical, QM, and geometric thermodynamic models derived for QGIFs. This specific problem is of utmost importance within vast domains of studies of properties of entanglement entropy of general relativistic quantum systems and, for instance, new types of QGIF teleportation, geometric flow testing, and encoding classical mechanical flow information in quantum states. In addition to the results of [40,41] formulated for nonholonomic Lagrange and Hamilton variables, we elaborated such constructions for covariant classical and quantum mechanical systems and explicit applications in quantum information theory.

Finally, we note that important questions connected to entanglement of QGIF and modified gravity theories still remain as open challenges and promising research directions in modern geometric classical and quantum mechanics, thermodynamics and modified gravity, see [17,18,28,29].

Acknowledgements This research develops the former programs partially supported by IDEI, PN-II-ID-PCE-2011-3-0256, CERN 2012–2014, DAAD-2015 and QGR 2016–2017 and contains certain results for new grant proposals. The UAIC affiliation for S. V. refers to a Project IDEI hosted by that University during 2012–2015, when the bulk of geometric ideas and methods of this and partner works were elaborated (to put a relevant co-/affiliation for further related results was the condition of that grant). Performing rigorous mathematical proofs and respective manuscripts request many years of technical work and further collaborations. S. V. is grateful to D. Singleton, S. Rajpoot and P. Stavrinos for collaboration and supporting his research on geometric methods in physics.

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