SICIAK’S HOMOGENEOUS EXTREMAL FUNCTIONS, 
HOLOMORPHIC EXTENSION AND A GENERALIZATION 
OF HELGASON’S SUPPORT THEOREM

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Dedicated to the memory of Professor Józef Siciak

Abstract. We prove that a function, which is defined on a union of 
lines $C_\mathcal{E}$ through the origin in $\mathbb{C}^n$ with direction vectors in $\mathcal{E} \subset \mathbb{C}^n$ 
and is holomorphic of fixed finite order and finite type along each line, 
extends to an entire holomorphic function on $\mathbb{C}^n$ of the same order and 
finite type, provided that $\mathcal{E}$ has positive homogeneous capacity in the 
sense of Siciak and all directional derivatives along the lines satisfy a 
necessary compatibility condition at the origin. We are able to estimate 
the indicator function of the extension in terms of Siciak’s weighted 
homogeneous extremal function, where the weight is a function of the 
type of the given function on each given line. As an application we 
prove a generalization of Helgason’s support theorem by showing how the 
support of a continuous function with rapid decrease at infinity can be 
located from partial information on the support of its Radon transform.

1. Introduction

This study grew out of the problem of locating the support of a continuous 
function $u$ on $\mathbb{R}^n$ with rapid decrease at infinity from partial information 
of the support of the Radon transform $(\omega, p) \mapsto R_u(\omega, p)$, which is defined 
for $(\omega, p) \in S^{n-1} \times \mathbb{R}$ as the integral of $u$ over the hyperplane given by the 
equation $\langle x, \omega \rangle = p$ with respect to the Lebesgue measure. More precisely, 
we assume that we have given a subset $\mathcal{E}$ of $S^{n-1}$ such that the convex hull 
of the support of the function $\mathbb{R} \ni p \mapsto R_u(\omega, p)$ is contained in a closed 
bounded interval $[a_\omega, b_\omega]$ for every $\omega$ in $\mathcal{E}$ and from this information only 
we want to locate the support of $u$.

Recall that Helgason’s support theorem states that if $u \in C(\mathbb{R}^n)$ is 
rapidly decreasing, i.e., $|x|^k u(x)$ is bounded for every $k = 1, 2, 3, \ldots$, and 
there exists a compact convex subset $K$ of $\mathbb{R}^n$ with the property that

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Theorem 6.1 states that if the set $E \subseteq \mathbb{S}^{n-1}$ is compact with positive homogeneous capacity in $\mathbb{C}^n$ in the sense of Siciak, e.g., if $E$ has non-empty interior in $\mathbb{S}^{n-1}$, and $\sigma: E \to \mathbb{R}_+$ defined by $\sigma(\omega) = \max\{-a_\omega, b_\omega\}$ for $\omega \in E$, is bounded above, then the support of $u$ is contained in the compact convex set

$$\{x \in \mathbb{R}^n ; \langle x, \omega \rangle \leq \Psi_{E, \sigma}^*(\omega), \forall \omega \in \mathbb{S}^{n-1}\}, \tag{1.1}$$

where $\Psi_{E, \sigma}^*$ is the upper semi-continuous regularization of Siciak’s homogeneous extremal function $\Psi_{E, \sigma}$ with weight $\sigma$, defined on $\mathbb{C}^n$ by

$$\Psi_{E, \sigma}(\zeta) = \sup\{|p(\zeta)|^{1/k} ; p \in \mathcal{P}^k(\mathbb{C}^n), k = \deg p \geq 1, |p|^{1/k} \leq \sigma \text{ on } E\},$$

and $\mathcal{P}^k(\mathbb{C}^n)$ is the set of homogeneous polynomials of $n$ complex variables. Furthermore, if $E$ is the closure of its relative interior in $\mathbb{S}^{n-1}$ and the functions $\omega \mapsto a_\omega$ and $\omega \mapsto b_\omega$ are lower and upper semicontinuous functions on $E$, respectively, then the support of $u$ is contained in

$$\{x \in \mathbb{R}^n ; \langle x, \omega \rangle \leq \Psi_{E, \sigma}^*(\omega), \forall \omega \in \mathbb{S}^{n-1}, a_\omega \leq \langle x, \omega \rangle \leq b_\omega, \forall \omega \in E\}. \tag{1.2}$$

If we take the Fourier transform of the function $\mathcal{R}u(\omega, \cdot)$, then we get the formula $\mathcal{F}_1(\mathcal{R}u(\omega, \cdot))(s) = \mathcal{F}_n u(s\omega) = \hat{u}(s\omega)$, where $\mathcal{F}_1$ and $\mathcal{F}_n$ are the Fourier transformations on $\mathbb{R}$ and $\mathbb{R}^n$, respectively. Since $u$ is rapidly decreasing, we have $\hat{u} \in C^\infty(\mathbb{R}^n)$ and the chain rule gives

$$\frac{1}{k!} \frac{d^k}{ds^k} \mathcal{F}_1(\mathcal{R}u(\omega, \cdot))(s) \bigg|_{s=0} = \sum_{|\alpha|=k} \frac{\partial^\alpha \hat{u}(0)}{\alpha!} \omega^\alpha, \quad \omega \in E, \tag{1.3}$$

with multi-index notation: $\alpha \in \mathbb{N}^n$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $s^\alpha = s_1^{\alpha_1} \cdots s_n^{\alpha_n}$, $\partial^\alpha \hat{u}(\xi) = (\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n})\hat{u}(\xi)$, and $\partial_j = \partial/\partial \xi_j$.

For every $\omega \in E$ the function $\mathcal{R}u(\omega, \cdot)$ has compact support, so the function $\mathcal{F}_1(\mathcal{R}u(\omega, \cdot))$ extends to an entire function of exponential type on $\mathbb{C}$. The family $(\mathcal{F}_1(\mathcal{R}u(\omega, \cdot)))_{\omega \in E}$ defines a function $f: \mathbb{C}E \to \mathbb{C}$, by $f(z\omega) = \mathcal{F}_1(\mathcal{R}u(\omega, \cdot))(z)$ for $z \in \mathbb{C}$ and $\omega \in E$, and it satisfies

$$|f(z\omega)| \leq \|\mathcal{R}u(\omega, \cdot)\|_{L^1(\mathbb{R})} e^{\sigma(\omega)|z|}, \quad z \in \mathbb{C}. \tag{1.4}$$

This leads us to a general problem of holomorphic extension. Assume that we have given a subset $E$ of $\mathbb{C}^n$ and a function $f: \mathbb{C}E \to \mathbb{C}$, which is holomorphic along each of the lines $\mathbb{C}\zeta$ and of type $\sigma(\zeta)$ with respect to the growth order $\sigma > 0$ with uniform estimates,

$$|f(z\zeta)| \leq C e^{\sigma(\zeta)|z|^\theta}, \quad z \in \mathbb{C}, \zeta \in E. \tag{1.5}$$
We would like to know under which conditions on $E$ it is possible to extend $f$ to an entire function on $\mathbb{C}^n$ with similar growth estimates. It is necessary to impose some condition at the origin where all the different lines $\mathbb{C} \ni \zeta \mapsto f(z\zeta)$ intersect, for if $F$ is some holomorphic extension of $f$ to a neighborhood of the origin, then $F$ links together the power series of the functions $\mathbb{C} \ni z \mapsto f(z\zeta)$ at the origin, because the chain rule implies

$$\frac{1}{k!} \frac{d^k}{dz^k} f(z\zeta) \bigg|_{z=0} = \sum_{|\alpha|=k} \frac{\partial^\alpha F(0)}{\alpha!} \zeta^\alpha, \quad \zeta \in E.$$ 

Our main result, Theorem 4.1, states that if we assume that $E$ is compact with positive homogeneous capacity in the sense of Siciak and $f$ satisfies (1.5) together with a certain compatibility condition at the origin, the $f$ extends to an entire function on $\mathbb{C}^n$ with the growth property that for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$|f(\zeta)| \leq C_\varepsilon e^{\Psi_{E,\gamma}(\zeta^\nu + \varepsilon |\zeta|^\nu), \quad \zeta \in \mathbb{C}^n,}$$

where $\gamma = \sigma^{1/\rho}$. As a consequence we get the inequality $i_f \leq \Psi_{E,\gamma}$, where $i_f$ is the indicator function of $f$.

In the special case $f(z\omega) = F_1(\mathcal{R}u(\omega, \cdot))(z)$ above we have $\rho = 1$. The indicator function of $\hat{u}$ satisfies $i_{\hat{u}}(\zeta) \leq i_{\hat{u}}(i \text{Im} \zeta)$ and the supporting function $H$ of $\text{ch supp} \ u$ is given on $\mathbb{R}^n$ by $H(\xi) = i_{\hat{u}}^*(i\xi)$. The inequality $H(\xi) \leq \Psi_{E,\sigma}(\xi)$ enables us to locate the support of $u$ in (1.1). We have $i_{\hat{u}}(-i\omega) \leq -a_\omega$ and $i_{\hat{u}}(i\omega) \leq b_\omega$ for every $\omega \in E$. In order to be able to conclude that $H(-\omega) \leq -a_\omega$ and $H(\omega) \leq b_\omega$ we need some regularity of $E$ and $\omega \mapsto (a_\omega, b_\omega)$. Then (1.2) holds.

The plan of the paper is as follows. In Section 2 we review a few facts on Siciak’s extremal functions and give some examples. In Section 3 we review a few results on growth properties of entire functions of one complex variable to be used later on. In Section 4 we state and prove our main result on the extension of a function on $\mathbb{C}^n$ with growth estimates to an entire function with similar growth estimates on $\mathbb{C}^n$. In Section 5 we review the variants of Paley-Wiener theorems which hold for $L^2$ functions with compact support, distributions with compact support, hyperfunctions with support in $\mathbb{R}^n$, and analytic functionals on $\mathbb{C}^n$, and show how the estimates proved in Section 4 can be used to locate supports and carriers. In Section 6 we finally prove the formulas (1.1) and (1.2) for the location of the support of $u$ from partial information of the support of $\mathcal{R} \ni p \mapsto \mathcal{R}u(\omega, p)$.

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2. Siciak’s extremal functions

In this section we have collected a few results on extremal plurisubharmonic functions and capacities, which are related to our results. First a few words on the notation. We use $\langle \cdot, \cdot \rangle$ both for the euclidean inner product on $\mathbb{R}^n$ and the natural bilinear form on $\mathbb{C}^n$,

$$\langle x, \xi \rangle = \sum_{j=1}^{n} x_j \xi_j, \quad x, \xi \in \mathbb{R}^n, \quad \langle z, \zeta \rangle = \sum_{j=1}^{n} z_j \zeta_j, \quad z, \zeta \in \mathbb{C}^n.$$ 

Then the hermitian form is $(z, \zeta) \mapsto \langle z, \bar{\zeta} \rangle$ and the euclidean norm is $\zeta \mapsto |\zeta| = \langle \zeta, \bar{\zeta} \rangle^{1/2}$.

We let $\mathcal{O}(X)$ denote space of all holomorphic functions on an open subset $X$ of $\mathbb{C}^n$ and $\mathcal{PSH}(X)$ the set of all plurisubharmonic functions on $X$ which are not identically $-\infty$ in any connected component of $X$.

We let $L = L(\mathbb{C}^n)$ denote the set of all $u \in \mathcal{PSH}(\mathbb{C}^n)$ satisfying

$$u(\zeta) \leq \log^+ |\zeta| + c_u, \quad \zeta \in \mathbb{C}^n,$$

for some constant depending on $u$, and we let $L^h = L^h(\mathbb{C}^n)$ denote the set of all $u$ in $L$ which are logarithmically homogeneous, i.e., $u(t\zeta) = \log |t| + u(\zeta)$ for every $t \in \mathbb{C}$ and $\zeta \in \mathbb{C}^n$.

We let $\mathcal{P}(\mathbb{C}^n)$ denote the space of polynomials in $n$ complex variables and $\mathcal{P}^h(\mathbb{C}^n)$ denotes the subset of homogeneous polynomials. For every subset $E$ of $\mathbb{C}^n$ and every function $\gamma: E \to \mathbb{R}_+$ we define Siciak’s extremal function with weight $\gamma$ on $\mathbb{C}^n$ by

$$\Phi_{E,\gamma}(\zeta) = \sup \{|p(\zeta)|^{1/k} : p \in \mathcal{P}(\mathbb{C}^n), \quad k = \deg p \geq 1, \quad |p|^{1/k} \leq \gamma \text{ on } E\},$$

and Siciak’s homogeneous extremal function with weight $\gamma$ by

$$\Psi_{E,\gamma}(\zeta) = \sup \{|p(\zeta)|^{1/k} : p \in \mathcal{P}^h(\mathbb{C}^n), \quad k = \deg p \geq 1, \quad |p|^{1/k} \leq \gamma \text{ on } E\}.$$ 

In the special case $\gamma = 1$ we denote these functions by $\Phi_E$ and $\Psi_E$ and call them Siciak’s extremal function of the set $E$ and Siciak’s homogeneous extremal function of the set $E$, respectively. We observe that the functions $\Psi_{E,\gamma}$ are absolutely homogeneous of degree 1, i.e., $\Psi_{E,\gamma}(t\zeta) = |t|\Psi_{E,\gamma}(\zeta)$ for every $t \in \mathbb{C}$ and $\zeta \in \mathbb{C}^n$. We also observe that if $\gamma$ is bounded above on $E$, $\gamma \leq \gamma_m$, for some constant $\gamma_m$, then $\Phi_{E,\gamma} \leq \gamma_m \Phi_E$ and $\Psi_{E,\gamma} \leq \gamma_m \Psi_E$.

Recall that a subset $E$ of $\mathbb{C}^n$ is said to be pluripolar if every $a \in E$ has a connected neighborhood $U_a$ and $v \in \mathcal{PSH}(U_a)$ not identically $-\infty$ such that $E \cap U_a \subseteq \{z \in U_a ; \quad v(z) = -\infty\}$. If $E$ is not pluripolar we say that $E$ is non-pluripolar. Josefson [13] proved that every pluripolar set $E$ is contained in a set $\{\zeta \in \mathbb{C}^n ; \quad u(\zeta) = -\infty\}$ for some $u \in \mathcal{PSH}(\mathbb{C}^n)$ and Siciak [28] proved that $u$ can even be chosen in $L$. 
For every function $\varphi : E \to \mathbb{R} = \mathbb{R} \cup \{\pm \infty\}$ we define the Siciak-Zakharyuta function with weight $\varphi$ by

$$V_{E,\varphi}(\zeta) = \sup\{u(\zeta) ; u \in \mathbb{L}, u|E| \leq \varphi\}, \quad \zeta \in \mathbb{C}^n,$$

and the homogeneous Siciak-Zakharyuta function with weight $\varphi$ by

$$V^h_{E,\varphi}(\zeta) = \sup\{u(\zeta) ; u \in \mathbb{L}^h, u|E| \leq \varphi\}, \quad \zeta \in \mathbb{C}^n.$$

If $\varphi$ is the constant function 0, then we call these functions the Siciak-Zakharyuta function of the set $E$ and the homogeneous Siciak-Zakharyuta function of the set $E$ and denote them by $V_E$ and $V^h_E$, respectively. In general, if the function $\varphi$ is bounded above by the constant $\varphi_m$, then we have $V_{E,\varphi} \leq \varphi_m + V_E$ and $V^h_{E,\varphi} \leq \varphi_m + V^h_E$. Siciak [28] and Zakharyuta [34] proved that $V_E = \log \Phi_E$ and $V^h_E = \log \Psi_E$ for every compact subset of $\mathbb{C}^n$.

Let $\| \cdot \|$ be a complex norm on $\mathbb{C}^n$. For every compact subset $E$ of $\mathbb{C}^n$ we define the capacity of $E$ in the sense of Siciak by

$$\varrho(E) = \exp \left( - \sup_{\|\zeta\|=1} V^*_E(\zeta) \right) = \left( \sup_{\|\zeta\|=1} \Phi^*_E(\zeta) \right)^{-1},$$

and the homogeneous capacity of $E$ in the sense of Siciak by

$$\varrho^h(E) = \exp \left( - \sup_{\|\zeta\|=1} V^h*_E(\zeta) \right) = \left( \sup_{\|\zeta\|=1} \Psi^*_E(\zeta) \right)^{-1}.$$

The compact subset $E$ of $\mathbb{C}^n$ is pluripolar if and only if $V^*_E \equiv +\infty$ if and only if $\varrho(E) = 0$. (See Klimek [14].) Observe that the definitions of the set functions $\varrho$ and $\varrho^h$ are depending on choice of the norm. The following result of Siciak [28], Th. 1.10, gives a geometric description of the homogeneous capacity:

**Theorem 2.1.** Let $\| \cdot \|$ be a complex norm on $\mathbb{C}^n$ with unit ball $B$ and $\varrho^h$ be the corresponding homogeneous capacity. Let $E$ be a compact set in $\mathbb{C}^n$ and let $\hat{E} = \{z \in \mathbb{C}^n ; \Psi^*_E(z) < 1\}$ be the homogeneous hull of $E$. Then

$$\varrho^h(E) = \sup\{r \in \mathbb{R}_+ ; rB \subset \hat{E}\}.$$ 

Since all norms on $\mathbb{C}^n$ are equivalent the property of having zero or strictly positive homogeneous capacity is independent of the choice of norm. We have $\mathbb{S}^{n-1} \subset \{z \in \mathbb{C}^n ; \log|z^2 + \cdots + z^n| = -\infty\}$, so $\varrho(\mathbb{S}^{n-1}) = 0$. The following was proved by Korevaar [15].

**Theorem 2.2.** For every compact subset $E$ of $\mathbb{S}^{n-1}$ with non-empty relative interior $\varrho^h(E) > 0$.

Let $E \subset \mathbb{C}^n$ and let $T$ denote the unit circle in $\mathbb{C}$. We define then circular hull of $E$ by $E_c = TE = \{tz ; t \in \mathbb{T}, z \in E\}$, and we say that $E$ is circular if $E_c = E$. Since $\sup_E |p| = \sup_{E_c} |p|$ for every homogeneous polynomial $p$, it is clear that $\Psi_{E_c} = \Psi_E$. Furthermore, if $E_c$ is non-pluripolar then $\Psi^*_E \in PSH(\mathbb{C}^n)$ and $\Psi^*_E$ is absolutely homogeneous of degree 1, i.e.,

$$\Psi^*_E(z\zeta) = |z|\Psi^*_E(\zeta), \quad \zeta \in \mathbb{C}^n, z \in \mathbb{C}.$$
The following is a result of Siciak [27].

**Theorem 2.3.** If $E$ is circular compact set, then its polynomial hull is
\[ \hat{E} = \left\{ \zeta \in \mathbb{C}^n ; \Psi_E(\zeta) \leq 1 \right\}. \]

There are very few explicit formulas for $\Psi_E$. The most important is:

**Proposition 2.4.** If $E$ is the unit ball for a complex norm $\| \cdot \|$ on $\mathbb{C}^n$, then
\[ \Psi_E(\zeta) = \| \zeta \|, \quad \zeta \in \mathbb{C}^n. \]

If $\| \cdot \|$ is a norm on $\mathbb{R}^n$, then the largest complex norm on $\mathbb{C}^n$ which extends $\| \cdot \|$ is the *cross norm* $\| \cdot \|_c$. It is given by the formula
\[ \| \zeta \|_c = \inf \left\{ \sum_{j=1}^{N} |\alpha_j| \| \omega_j \| ; \ \zeta = \sum_{j=1}^{N} \alpha_j \omega_j, \alpha_j, \omega_j \in \mathbb{C}, \omega_j \in \mathbb{R}^n \right\}. \]

For a proof of the following result see Siciak [24] and Drużkowski [7].

**Proposition 2.5.** If $E \subset \mathbb{R}^n$ is the unit sphere in the norm $\| \cdot \|$, then
\[ \| \zeta \|_c \leq \Psi_E(\zeta), \quad \zeta \in \mathbb{C}^n \quad \text{and} \quad \| \zeta \|_c = \Psi_E(\zeta), \quad \zeta \in \mathbb{C} \mathbb{R}^n. \]

If the norm $\| \cdot \|$ in $\mathbb{R}^n$ is given by an inner product, then we have equality:

**Theorem 2.6.** If $x, \xi \mapsto \langle x, \xi \rangle$ is an inner product on $\mathbb{R}^n$, $\xi \mapsto |\xi| = |\langle \xi, \xi \rangle|^{\frac{1}{2}}$ is the corresponding norm and $E = \{ \xi \in \mathbb{R}^n ; |\xi| = 1 \}$ is the unit sphere, then $\Psi_E = | \cdot |$. If $\zeta = \xi + i \eta = e^{i\theta}(a + ib)$, $\theta \in \mathbb{R}, \xi, \eta, a, b \in \mathbb{R}^n$, $\langle a, b \rangle = 0$, and $|b| \leq |a|$, then
\[ |a| = \frac{1}{\sqrt{2}} (|\zeta|^2 + |\langle \zeta, \zeta \rangle|) \frac{1}{2} \quad \text{and} \quad |b| = \frac{1}{\sqrt{2}} (|\zeta|^2 - |\langle \zeta, \zeta \rangle|) \frac{1}{2}, \]
and
\[ |\zeta|_c = |a| + |b| = \left( |\zeta|^2 - d(\zeta, \mathbb{C} \mathbb{R}^n)^2 \right) \frac{1}{2} + d(\zeta, \mathbb{C} \mathbb{R}^n) = \left( |\zeta|^2 + (|\zeta|^4 - |\langle \zeta, \zeta \rangle|) \frac{1}{2} \right) \frac{1}{2} = \left( |\zeta|^2 + 2(|\zeta| \eta^2 - \langle \xi, \eta \rangle) \frac{1}{2} \right) \frac{1}{2}. \]

As a consequence we have $|\zeta| \leq |\zeta|_c \leq \sqrt{2}|\zeta|$ for every $\zeta \in \mathbb{C}^n$, we have $|\zeta| = |\zeta|_c$ if and only if $\zeta \in \mathbb{C} \mathbb{R}^n$, and we have $|\zeta|_c = \sqrt{2}|\zeta|$ if and only if $\langle \zeta, \zeta \rangle = \xi_1^2 + \cdots + \xi_n^2 = 0$.

See Drużkowski [7] and Sigurdsson and Snæbjarnarson [30] for a proof. As a consequence of Theorems 2.1 and 2.6 we have:

**Corollary 2.7.** If $| \cdot |$ is a norm with respect to an inner product on $\mathbb{R}^n$, then the corresponding homogeneous capacity of the unit sphere is $1/\sqrt{2}$. 
For the case when $E$ is the unit ball with respect to a norm in $\mathbb{R}^2$ we have a very interesting result of Baran [4]:

**Proposition 2.8.** Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$ with unit ball $E$, set $u(\xi) = \log \| (1, \xi) \|$ for $\xi$, and define

$$
P u(z) = \frac{|\text{Im} z|}{\pi} \int_{\mathbb{R}} \frac{u(\xi) d\xi}{|z - \xi|^2}, \quad z \in \mathbb{C} \setminus \mathbb{R}
$$

Then

$$
\Psi_E(z_1, z_2) = |z_1| \exp \left( P u(z_2/z_1) \right), \quad z \in \mathbb{C}^2, \ z_1 \neq 0,
$$

and it extends uniquely to a continuous function on $\mathbb{C}^2$.

As we have already mentioned there are only a few explicit examples of homogeneous extremal functions. *Disc envelope formulas* are an alternative way of expressing extremal functions, see Lárusson and Sigurdsson [18, 19, 20], Magnússon and Sigurdsson [22], and Drinovec Drnovšek and Sigurdsson [6].

We have only mentioned results on extremal functions that are needed for proving our results. For the general theory see the works of Siciak [24, 25, 26, 27, 28], Baran [3, 4, 5], and Klimek [14]. Some interesting applications are given in Korevaar [15, 16]. For a related capacity on projective spaces see Alexander [1, 2].

### 3. Growth properties entire functions

We say that a function $f \in \mathcal{O}(\mathbb{C}^n)$ is of *finite order* if there exist positive constants $C, \sigma$ and $\varrho$ such that

$$
|f(\zeta)| \leq Ce^{\sigma|\zeta|^\varrho}, \quad \zeta \in \mathbb{C}^n,
$$

and we define the *order* $\varrho_f$ of $f$ as the infimum over all $\varrho$ for which such an estimate exists. If $f$ is not of finite order, then we say that $f$ is of *infinite order* and define $\varrho_f = +\infty$. If we set $M_f(r) = \sup_{|\zeta| \leq r} |f(\zeta)|$, then

$$
\varrho_f = \lim_{r \to +\infty} \frac{\log \log M_f(r)}{\log r}.
$$

If $f$ is of finite order, then we say that $f$ is of *finite type with respect to the growth order* $\varrho$ if there exist positive constants $C$ and $\sigma$ such that (3.1) holds. Then we define the type $\sigma_f$ of $f$ (with respect to the order $\varrho$) as the infimum over all $\sigma$ for which (3.1) holds for some $C$. We have

$$
\sigma_f = \lim_{r \to +\infty} \frac{\log M_f(r)}{r^\varrho}.
$$

The function $f$ is said to be of *exponential type* if it is of finite type with respect to the growth order 1.
For every $f \in \mathcal{O}(\mathbb{C}^n)$ of finite type with respect to the order $\varrho$ we define the indicator function $i_f$ by

$$i_f(\zeta) = \lim_{t \to \infty} \frac{1}{t^\varrho} \log |f(t\zeta)|,$$

then its upper semi-continuous regularization,

$$i_f^*(\zeta) = \lim_{\varphi \to \zeta} i_f(\varphi), \quad \zeta \in \mathbb{C}^n,$$

is plurisubharmonic and we have

$$\sigma_f = \sup_{|\zeta|=1} i_f^*(\zeta).$$

Let $\varphi \in \mathcal{O}(\mathbb{C})$ be given by $\varphi(z) = \sum_{k=0}^{\infty} c_k z^k$ and assume that $|\varphi(z)| \leq C e^{e \varrho |z|^\varrho}$ for all $z \in \mathbb{C}$, where $C$, $\varrho$, and $\sigma$ are positive constants. Then Cauchy’s inequalities give that for every $r > 0$ we have

$$|c_k| = \frac{|\varphi^{(k)}(0)|}{k!} \leq C e^{k \varrho \sigma} r^k, \quad k = 0, 1, 2, \ldots.$$

The minimal value of the right hand side is taken for $r = (k/\varrho)^{1/k}$, so we conclude that

$$|c_k| \leq C \left( \frac{e \varrho \sigma}{k} \right)^{k/\varrho}, \quad k \geq 0, \quad (3.2)$$

(with the abuse of notation $(e\varrho/0)^{0/0} = 1$), and

$$|\varphi(z)| \leq C \sum_{k=0}^{\infty} \left( \frac{e \varrho \sigma}{k} \right)^{k/\varrho} |z|^k, \quad z \in \mathbb{C}. \quad (3.3)$$

By Levin [21], Ch. 1., it is possible to express the order and type of any $\varphi \in \mathcal{O}(\mathbb{C})$ in terms of the coefficients $c_k$ of its power series at 0. Its order $\varrho_\varphi \in [0, +\infty]$ is given by the formula

$$\varrho_\varphi = \lim_{k \to \infty} \frac{k \log k}{-\log |c_k|}.$$

If $\varrho_\varphi \in [0, +\infty]$, then the type $\sigma_\varphi \in [0, +\infty]$ with respect to $\varrho_\varphi$ is given by the equation

$$\left( e\varrho_\varphi \varrho_\varphi \right)^{1/\varrho_\varphi} = \lim_{k \to \infty} \frac{1}{k \varrho_\varphi} \log |c_k|.$$

These formulas tell us that for any given positive numbers $\varrho$ and $\sigma$ the entire function

$$z \mapsto \sum_{k=0}^{\infty} \left( \frac{e \varrho \sigma}{k} \right)^{k/\varrho} z^k, \quad z \in \mathbb{C}, \quad (3.4)$$

is of order $\varrho$ and type $\sigma$ with respect to $\varrho$. 
Our main result is

**Theorem 4.1.** Let $E$ be a compact subset of $\mathbb{C}^n$ and assume that $E$ has positive homogeneous capacity in the sense of Siciak. Let $f : \mathbb{C}E \to \mathbb{C}$ be a function, such that for every $\zeta \in E$ the function $\mathbb{C} \ni z \mapsto f(z\zeta)$ is holomorphic and satisfies

$$|f(z\zeta)| \leq Ce^{\sigma(\zeta)|z|^\varrho}, \quad z \in \mathbb{C}, \, \zeta \in E,$$

(4.1)

with positive constants $C$ and $\varrho$ and a function $\sigma : E \to \mathbb{R}_+ = \{x \geq 0\}$, which is bounded above. Assume that for every $k = 0, 1, 2, \ldots$ there exists a $k$-homogeneous complex polynomial $P_k$ on $\mathbb{C}^n$ such that

$$\left.\frac{1}{k!} \frac{d^k}{dz^k} f(z\zeta)\right|_{z=0} = P_k(\zeta), \quad k = 0, 1, 2, \ldots, \, \zeta \in E.$$  

(4.2)

Then the series $\sum_{k=0}^{\infty} P_k$ converges locally uniformly in $\mathbb{C}^n$ and gives a unique holomorphic extension of $f$ to $\mathbb{C}^n$ by $f(\zeta) = \sum_{k=0}^{\infty} P_k(\zeta)$ for $\zeta \in \mathbb{C}^n$. We have

$$|f(\zeta)| \leq C \sum_{k=0}^{\infty} \left(\frac{e\varrho}{k}\right)^{k/\varrho} \Psi_{E,\gamma}(\zeta)^k, \quad \zeta \in \mathbb{C}^n,$$

(4.3)

where $\Psi_{E,\gamma}$ is Siciak’s weighted homogeneous extremal function with weight $\gamma = \sigma^{1/\varrho}$. This implies that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|f(\zeta)| \leq C_\epsilon e^{\Psi_{E,\gamma}(\zeta)+\epsilon||\zeta||^\varrho}, \quad \zeta \in \mathbb{C}^n.$$  

(4.4)

Hence $f$ is of order $\leq \varrho$ and of type $\leq \alpha^\varrho$ with respect to $\varrho$, where $\alpha = \sup_{|\zeta|=1} \Psi_{E,\gamma}(\zeta)$, and $i_f \leq \Psi_{E,\gamma}^\ast$.

**Proof.** There is no condition ensuring convergence of the series $\sum_{k=0}^{\infty} P_k$, so we show that it is locally uniformly convergent. For every $\zeta \in E$ we define $\varphi_\zeta \in \mathcal{O}(\mathbb{C})$ by $\varphi_\zeta(z) = f(z\zeta)$ for $z \in \mathbb{C}$. By (4.1) $\varphi_\zeta$ is of type $\leq \sigma(\zeta)$ with respect to the growth order $\varrho$, so (3.2) gives

$$|P_k(\zeta)| = \frac{||\varphi_\zeta^{(k)}(0)||}{k!} \leq C \left(\frac{e\sigma(\zeta)\varrho}{k}\right)^{k/\varrho} = C \left(\frac{e\varrho}{k}\right)^{k/\varrho} \gamma(\zeta)^k, \quad \zeta \in E.$$

and by the definition of $\Psi_{E,\gamma}$ we have

$$|P_k(\zeta)| \leq C \left(\frac{e\varrho}{k}\right)^{k/\varrho} \Psi_{E,\gamma}(\zeta)^k, \quad \zeta \in \mathbb{C}^n.$$  

(4.5)

Since $\sigma$ is bounded above, $\gamma \leq \gamma_m$ for some constant $\gamma_m$, and we have $\Psi_{E,\gamma} \leq \gamma_m \Psi_E$. Since $E$ has positive homogeneous capacity, it now follows from Siciak’s theorem that both $\Psi_E^\ast$ and $\Psi_{E,\gamma}^\ast$ are plurisubharmonic and therefore locally bounded above in $\mathbb{C}^n$. Hence we conclude that $\sum_{k=0}^{\infty} P_k$ converges locally uniformly in $\mathbb{C}^n$ and from (4.5) we see that the limit defines
a function \( F \in \mathcal{O}(\mathbb{C}^n) \) satisfying (4.3). The function \( F \) is an extension of \( f \), for if \( \zeta \in E \), then for every \( z \in \mathbb{C} \)

\[
F(z\zeta) = \sum_{k=0}^{\infty} P_k(z\zeta) = \sum_{k=0}^{\infty} P_k(\zeta)z^k = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} z^k = \varphi(\zeta) = f(z\zeta).
\]

Since the function (3.4) is of order \( \rho \) and of type \( \sigma \) with respect to \( \rho \), we even have (4.4) and the inequality \( i_f \leq \Psi^\rho_{E,\gamma} \) follows from homogeneity. \( \square \)

It is interesting to state a special case of Theorem 4.1 for functions of exponential type and combine it with Theorem 2.6 and Corollary 2.7:

**Corollary 4.2.** If the assumptions of Theorem 4.1 are satisfied with \( \rho = 1 \) and \( E \subset \mathbb{S}^{n-1} \), then the extension \( f \) satisfies

\[
|f(\zeta)| \leq C_e^{\sigma_m(\zeta)} e^{\sqrt{2} \sigma_m(\zeta)} e^{\varepsilon |\zeta|}, \quad \zeta \in \mathbb{C}^n,
\]

for every \( \varepsilon > 0 \), where \( \sigma_m = \sup_{E} \sigma \). As a consequence we note that if the restriction of \( f \) to the union of lines \( CE \subset \mathbb{C}P^n \) has type \( \leq \sigma_m \), then the extension is of type \( \leq \sqrt{2} \sigma_m \).

Extremal plurisubharmonic functions have been applied for extension of holomorphic functions in the same spirit as Theorem 4.1. For example the following result of Siciak [28], Th. 13.4:

**Theorem 4.3.** Let \( E \) be a subset of \( \mathbb{C}^n \) with \( \rho^h(E) > 0 \) and \( P_k \) be a homogeneous polynomial on \( \mathbb{C}^n \) of degree \( k \) for \( k = 0, 1, 2, \ldots \). Assume that the series \( \sum_{k=0}^{\infty} P_k \) converges at every point of \( E \) outside a subset of homogeneous capacity zero. Then the series converges locally uniformly in \( \{ z \in \mathbb{C}^n ; \Psi^*_E(z) < 1 \} \).

For similar results see Forelli [3] and Wiegerinck and Korevaar [33]. Estimates of the growth of a plurisubharmonic function in \( \mathbb{C}^n \) in terms of the growth along a complex cone \( CE \) were proved in Sibony and Wong [23] with optimized constants in Siciak [28], Cor. 11.2. See also Korevaar [15, 16].

5. **Fourier-Laplace transforms and Paley-Wiener theorems**

The main motivation for studying entire functions is the fact that Fourier-Laplace transforms of analytic functionals, hyperfunctions with compact support, distributions with compact support, and \( L^2 \)-functions with compact support, are entire functions of exponential type. For each of these classes of functionals and functions there is a variant of the Paley-Wiener theorem which describes how estimates of Fourier-Laplace transforms are used to locate hulls of carriers or supports. This is based on the fact that every convex function \( H \) on \( \mathbb{R}^n \), which is positively homogeneous of degree 1,
implies that

\[ H(\xi) = \sup_{x \in K} \langle x, \xi \rangle, \quad \xi \in \mathbb{R}^n, \]

of a unique compact convex set \( K \) which is related to \( H \) by

\[ K = \{ x \in \mathbb{R}^n ; \langle x, \xi \rangle \leq H(\xi), \forall \xi \in \mathbb{R}^n \}. \]

(For a proof see [10], Th. 4.3.2.) We define the Fourier transform of \( u \in L^1(\mathbb{R}^n) \) by

\[ \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} u(x) \, dx, \quad \xi \in \mathbb{R}^n. \]

If \( u \) has a compact support, i.e., \( u \) vanishes almost everywhere outside a compact set, then its Fourier transform extends to an entire function on \( \mathbb{C}^n \), which is called the Fourier-Laplace transform of \( u \) and is given by the formula

\[ \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} u(x) \, dx, \quad \xi \in \mathbb{C}^n, \]

and if \( K = \text{ch supp} \, u \) with supporting function \( H_K \), we have the estimate

\[ |\hat{u}(\xi)| \leq \int_K e^{(x, \text{Im} \xi)} |u(x)| \, dx \leq \|u\|_{L^1(\mathbb{R}^n)} e^{H_K(\text{Im} \xi)}, \]

for \( u \in L^1 \cap L^2(\mathbb{R}^n) \), not necessarily with compact support, then \( \hat{u} \in L^2(\mathbb{R}^n) \) and the Plancherel formula

\[ \|\hat{u}\|_{L^2(\mathbb{R}^n)} = (2\pi)^{\frac{n}{2}} \|u\|_{L^2(\mathbb{R}^n)} \]

implies that \( u \mapsto \hat{u}/(2\pi)^{\frac{n}{2}} \) extends from \( L^1 \cap L^2(\mathbb{R}^n) \) to an isometry on \( L^2(\mathbb{R}^n) \). If \( u \in L^2(\mathbb{R}^n) \) has compact support and \( K = \text{ch supp} \, u \) has supporting function \( H_K \), then we have the growth estimate

\[ |\hat{u}(\xi)| \leq \int_K e^{(x, \text{Im} \xi)} |u(x)| \, dx \leq \lambda(K)^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^n)} e^{H_K(\text{Im} \xi)}, \]

where \( \lambda \) is the Lebesgue measure on \( \mathbb{R}^n \). The following is the original Paley-Wiener theorem on \( \mathbb{R}^n \):

**Theorem 5.1.** The function \( f \in \mathcal{O}(\mathbb{C}^n) \) is the Fourier-Laplace transform of an \( L^2 \)-function with support contained in the compact convex subset \( K \) of \( \mathbb{R}^n \) if and only if \( f \|_{\mathbb{R}^n} \in L^2(\mathbb{R}^n) \) and there exists \( C > 0 \) such that

\[ |f(\zeta)| \leq C e^{H_K(\text{Im} \zeta)}, \quad \zeta \in \mathbb{C}^n. \]

If \( u \in \mathcal{E}'(\mathbb{R}^n) \) is a distribution with compact support, then its Fourier transform \( \hat{u} \) is in \( C^\infty(\mathbb{R}^n) \) and it extends to an entire holomorphic function \( \hat{u} \) which is given by \( \hat{u}(\zeta) = u(e^{-i(\cdot, \zeta)}) \) for \( \zeta \in \mathbb{C}^n \), i.e., we get \( \hat{u}(\zeta) \) by letting \( u \) act on the \( C^\infty \) function \( x \mapsto e^{-i(\cdot, \zeta)} \). The following is called the Paley-Wiener-Schwartz-theorem. (For a proof see [10], Th. 7.3.1.)

**Theorem 5.2.** The function \( f \in \mathcal{O}(\mathbb{C}^n) \) is the Fourier-Laplace transform of a distribution with support contained in the compact convex subset \( K \) of \( \mathbb{R}^n \) if and only if there exist \( C > 0 \) and \( N \geq 0 \) such that

\[ |f(\zeta)| \leq C(1 + |\zeta|)^N e^{H_K(\text{Im} \zeta)}, \quad \zeta \in \mathbb{C}^n. \]
Recall that an analytic functional is an element \( u \) in \( \mathcal{O}(\mathbb{C}^n) \), i.e., a continuous linear functional acting on the space of entire functions. By continuity there exists a compact subset \( M \) of \( \mathbb{C}^n \) and a positive constant \( C \) such that
\[
|u(\varphi)| \leq C \sup_{\zeta \in M} |\varphi(\zeta)|, \quad \varphi \in \mathcal{O}(\mathbb{C}^n).
\]
Observe that the Hahn-Banach theorem implies that \( u \) extends to a linear functional on \( C(M) \) satisfying (5.1) for \( \varphi \in C(M) \). By the Riesz representation theorem there exists a complex measure \( \mu \) on \( M \) such that
\[
\varphi \mapsto \int_M \varphi \, d\mu, \quad \varphi \in C(M).
\]
We say that \( u \) is carried by the compact subset \( K \) of \( \mathbb{C}^n \) if for every neighbourhood \( U \) of \( K \) in \( \mathbb{C}^n \) there exists a constant \( C_U \) such that
\[
|u(\varphi)| \leq C_U \sup_{\zeta \in U} |\varphi(\zeta)|, \quad \varphi \in \mathcal{O}(\mathbb{C}^n).
\]
In this case we also say that \( K \) is a carrier for \( u \). The Fourier-Laplace transform of \( u \in \mathcal{O}(\mathbb{C}^n) \) is defined by
\[
\widehat{u}(\zeta) = u(e^{-i\langle \cdot, \zeta \rangle}), \quad \zeta \in \mathbb{C}^n.
\]
We can differentiate with respect to \( \zeta_j \) and \( \bar{\zeta}_j \) under the \( u \)-sign and conclude that \( \widehat{u} \in \mathcal{O}(\mathbb{C}^n) \). Furthermore, if \( u \) is carried by the compact set \( K \) then we take \( \varepsilon > 0 \) and \( K_\varepsilon = K + B(0, \varepsilon) \) as \( U \) in the definition of a carrier and conclude that
\[
|\widehat{u}(\zeta)| \leq C_\varepsilon e^{H_K(-i\zeta)+\varepsilon|\zeta|}, \quad \zeta \in \mathbb{C}^n,
\]
where \( H_K \) is the supporting function of the set \( K \), now defined as
\[
H_K(\zeta) = \sup_{z \in K} \Re \langle z, \zeta \rangle, \quad \zeta \in \mathbb{C}^n.
\]
Observe that here we use the real bilinear form
\[
(z, \zeta) \mapsto \Re \langle z, \zeta \rangle = \langle x, \xi \rangle - \langle y, \eta \rangle, \quad z = x + iy, \zeta = \xi + i\eta \in \mathbb{C}^n
\]
instead of the euclidean inner product
\[
(z, \zeta) \mapsto \Re \langle z, \zeta \rangle = \langle x, \xi \rangle + \langle y, \eta \rangle, \quad z = x + iy, \zeta = \xi + i\eta \in \mathbb{C}^n,
\]
which we use to identify \( \mathbb{C}^n \) with the real euclidean space \( \mathbb{R}^{2n} \). This means that if \( H : \mathbb{C}^n \to \mathbb{R} \) is convex and positively homogeneous of degree 1, \( L = \{ z \in \mathbb{C}^n ; \Re \langle \bar{z}, \zeta \rangle \leq H(\zeta) \} \) and \( K = \{ \bar{z} ; z \in L \} \), then
\[
H_K(\zeta) = \sup_{z \in K} \Re \langle z, \zeta \rangle = H(\zeta), \quad \zeta \in \mathbb{C}^n.
\]
The following variant of the Paley-Wiener theorem is usually called the Pólya-Ehrenpreis-Martineau theorem. (For a proof see [I], Th. 4.5.3.)

**Theorem 5.3.** The function \( f \in \mathcal{O}(\mathbb{C}^n) \) is the Fourier-Laplace transform of an analytic functional carried by the compact convex subset \( K \) of \( \mathbb{C}^n \) if and only if for every \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon \) such that
\[
|f(\zeta)| \leq C_\varepsilon e^{H_K(-i\zeta)+\varepsilon|\zeta|}, \quad \zeta \in \mathbb{C}^n.
\]
For every subset $A$ of $\mathbb{C}^n$ we let $\mathcal{O}'(A)$ denote the set of all analytic functionals carried by a compact subset of $A$. Even if $u$ is carried by two compact subsets $K_1$ and $K_2$, it does not mean that $u$ is carried by $K_1 \cap K_2$, i.e., in general analytic functionals do not have a unique minimal carrier.

If, on the other hand, $u \in \mathcal{O}'(\mathbb{R}^n)$ then by a theorem of Martineau, $u$ has a minimal carrier in $\mathbb{R}^n$ which is called the support of $u$ and is denoted by $\text{supp} \ u$. (For a proof see [10], Th. 9.1.6.) The space $\mathcal{O}'(\mathbb{R}^n)$ is identified with the space of hyperfunctions with compact support. The following theorem is a variant of the Paley-Wiener theorem for hyperfunctions. It is sometimes called the Paley-Wiener-Martineau theorem. (For a proof see [10], Th. 15.1.5.)

**Theorem 5.4.** The function $f \in \mathcal{O}(\mathbb{C}^n)$ is the Fourier-Laplace transform of a hyperfunction with support contained in the compact convex subset $K$ of $\mathbb{R}^n$ if and only if for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(\zeta)| \leq C_\varepsilon e^{H_K(\text{Im}\zeta) + \varepsilon |\zeta|}, \quad \zeta \in \mathbb{C}^n.$$ 

If $u$ is a hyperfunction, distribution or an $L^2$ function with compact support and $i_u^*$ is the regularized indicator function of its Fourier-Laplace transform then the function $\mathbb{R}^n \ni \xi \mapsto i_u^*(i\xi)$, is the supporting function $H_K$ of $\text{ch supp} \ u$. (For a proof see [29], Th. 2.1.1.) This implies that if we have a growth estimate of the form

$$|\hat{u}(\zeta)| \leq C_\varepsilon e^{\Psi(\zeta) + \varepsilon |\zeta|}, \quad \zeta \in \mathbb{C}^n,$$

where $\Psi$ is a function on $\mathbb{C}^n$, which is absolutely homogeneous of degree 1, i.e., $\Psi(t\zeta) = |t|\Psi(\zeta)$ for every $\zeta \in \mathbb{C}^n$ and $t \in \mathbb{C}$, and $C_\varepsilon > 0$ is a constant for every $\varepsilon > 0$, then

$$\text{ch supp} \ u \subseteq \{x \in \mathbb{R}^n : \langle x, \xi \rangle \leq \Psi(\xi), \forall \xi \in \mathbb{R}^n\}.$$ 

Now we have reviewed all the Paley-Wiener theorems which are relevant to our application of Theorem 4.1.

**Theorem 5.5.** Let $E$ be a compact subset of $\mathbb{C}^n$ with positive homogeneous capacity and $f : \mathcal{C}E \to \mathbb{C}$ be a function satisfying the conditions in Theorem 4.1 with $\varrho = 1$.

(i) Then $f$ extends to an entire function on $\mathbb{C}^n$ which is the Fourier-Laplace transform of an analytic functional with a carrier contained in the ball with center at the origin and radius $\alpha = \sup_{|\zeta|=1} \Psi(\zeta)$.

(ii) If, in addition, there exists a constant $A > 0$ such that $i_f^*(\zeta) \leq A|\text{Im}\zeta|$ for every $\zeta \in \mathbb{C}^n$, then $f$ is the Fourier-Laplace transform of a hyperfunction with support contained in the compact convex set

$$K_{E,\varrho} = \{x \in \mathbb{R}^n : \langle x, \xi \rangle \leq \Psi(\xi), \forall \xi \in \mathbb{R}^n\}.$$
(iii) If we have the estimate $|f(\xi)| \leq C(1 + |\xi|)^N$ for all $\xi \in \mathbb{R}^n$, for some constants $C > 0$ and $N \geq 0$, then $f$ is the Fourier-Laplace transform of a distribution with compact support in $K_{E,\sigma}$.

(iv) Finally, if we can conclude that $f$ is in $L^2(\mathbb{R}^n)$, then $f$ is the Fourier-Laplace transform of a function in $L^2(\mathbb{R}^n)$ with support in $K_{E,\sigma}$.

Proof.  (i) By Theorem 4.1, $f$ has an extension to an entire function satisfying (4.4). The supporting function of the closed ball with center at the origin and radius $\alpha$ is $\zeta \mapsto \alpha|\zeta|$, so (i) follows from Theorem 5.3.

(ii) By [12], Th. 3.9, it follows that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $f(\xi) \leq C_\varepsilon e^{A|\Im \xi|+|\xi|}$ for $\xi \in \mathbb{C}^n$, and by Theorem 5.4 the function $f$ is the Fourier-Laplace transform of a hyperfunction $u$ with support contained in the ball $\{x \in \mathbb{R}^n; |x| \leq A\}$. Furthermore, the supporting function of $K = \text{chsupp} u$ is given by $H_K(\xi) = i\zeta_j(i\xi)$ for $\xi \in \mathbb{R}^n$. Since $i\zeta_j \leq \Psi_{E,\sigma}$, we have $H_K(\xi) \leq \Psi_{E,\sigma}(\xi)$ for every $\xi \in \mathbb{R}^n$ and it follows that $K \subseteq K_{E,\sigma}$.

(iii) Recall [12], Lemma 2.1, which is an application of the Phragmén-Lindelöf principle: Let $v$ be a subharmonic function in the upper half plane $\mathbb{C}_+ = \{z \in \mathbb{C}; \Im z > 0\}$ such that for some real constants $c$ and $a$ we have $v(z) \leq c + a|z|$ for every $z \in \mathbb{C}_+$ and $\lim_{z \to x} v(z) \leq 0$ for every $x \in \mathbb{R}$. Then $v(z) \leq a|\Im z|$ for every $z \in \mathbb{C}_+$.

We are now going to apply this lemma to the function

$$v(z) = \log |f(\xi + z\eta)| - \log C - \frac{1}{2}N \log 2 - N \log |1 + |\xi| - iz|\eta||,$$

$z \in \mathbb{C}_+$.

It is subharmonic for $\log |f|$ is plurisubharmonic on $\mathbb{C}^n$ and the logarithmic term is a real part of a holomorphic function and thus harmonic. Since $f$ is of exponential type, there exist constants $A$ and $B$ such that we have for every $\xi, \eta \in \mathbb{R}^n$ and every $z \in \mathbb{C}$ that $|f(\xi + z\eta)| \leq Be^{A|\xi + z\eta|} \leq e^{\log B + A|\xi| + A|\eta|}|z|$. By estimating the logarithmic term by $|\xi|\eta||z|$ we see that for every $\varepsilon > 0$ there exists a constant $c_\varepsilon$ such that $v(z) \leq c_\varepsilon + (A + \varepsilon)|\eta||z|$ for every $z \in \mathbb{C}_+$. The estimate $|\xi + x\eta| \leq \sqrt{2}||\xi| - ix|\eta||$ implies that $\lim_{z \to x} v(z) \leq 0$. By the lemma, $v(z) \leq A|\eta||\Im z$. The inequality $v(i) \leq A|\eta|$ implies $|f(\xi)| \leq 2^NC(1 + |\xi|)^N e^{A|\Im \xi|}$, $\xi \in \mathbb{C}^n$, and Theorem 5.2 implies that $f$ is the Fourier-Laplace transform of a distribution $u$ with support in the ball $\{x \in \mathbb{R}^n; |x| \leq A\}$. As in (ii) it follows that $\text{chsupp} u \subseteq K_{E,\sigma}$.

(iv) Let $u \in L^2(\mathbb{R})$ be the inverse Fourier-transform of the restriction of $f$ to $\mathbb{R}^n$. Take $0 \leq \varphi \in C_0^\infty(\mathbb{R}^n)$ with support in the closed unit ball, $\|\varphi\|_{L^1(\mathbb{R}^n)} = 1$, and for every $\delta > 0$ define $\varphi_\delta$ by $\varphi_\delta(x) = \varphi(x/\delta)/\delta^n$. Then $\|u * \varphi_\delta\|_{L^2(\mathbb{R}^n)} \leq \delta^{-\frac{n}{2}}\|u\|_{L^2(\mathbb{R}^n)}\|\varphi_\delta\|_{L^1(\mathbb{R}^n)}$, so $u * \varphi_\delta$ is in $L^1(\mathbb{R}^n)$ and consequently its Fourier transform $f_\delta = f\varphi_\delta$ is bounded on the real axis. We
have $|\hat{\varphi}_t(\zeta)| \leq e^{\delta|\text{Im}(\zeta)|}$ for every $\zeta \in \mathbb{C}^n$, so $f_\delta$ is of exponential type $\leq A + \delta$, if $f$ is of type $\leq A$. From (iii) we conclude that $u * \varphi_\delta$ has support in the ball $\{ x \in \mathbb{R}^n \mid |x| \leq A + \delta \}$ and since $u * \varphi_\delta \to u$ as $\delta \to 0$ in the sense of distributions we conclude that $u$ has support in $\{ x \in \mathbb{R}^n \mid |x| \leq A \}$. Again, with the same argument as in (ii) it follows that $\text{ch supp } u \subseteq K_{E,\sigma}$. □

For a detailed study of growth properties of plurisubharmonic functions related to Fourier-Laplace transforms see Hörmander and Sigurdsson [12].

6. Applications to Radon transforms

Recall that the Radon transform $\mathcal{R}u$ of rapidly decreasing continuous function $u$ on $\mathbb{R}^n$ is defined by

$$\mathcal{R}u(\omega, p) = \int_{\langle x, \omega \rangle = p} u(x) \, dm(x), \quad \omega \in S^{n-1}, \ p \in \mathbb{R},$$

where $dm$ is the Lebesgue measure in the hyperplane given by $\langle x, \omega \rangle = p$. If we fix $\omega \in S^{n-1}$, take the Fourier transform of $p \mapsto \mathcal{R}u(\omega, p)$ and apply Fubini’s theorem, then we get for $s \in \mathbb{R}$

$$\mathcal{F}_1(\mathcal{R}u(\omega, \cdot))(s) = \int_{-\infty}^{+\infty} e^{-isp} \int_{\langle x, \omega \rangle = p} u(x) \, dm(x) \, dp$$

$$= \int_{-\infty}^{+\infty} \int_{\langle x, \omega \rangle = p} e^{-is\langle x, \omega \rangle} u(x) \, dm(x) \, dp = \mathcal{F}_n u(s\omega) = \hat{u}(s\omega). \quad (6.1)$$

Since $u$ is rapidly decreasing, we have $\hat{u} \in C^\infty(\mathbb{R}^n)$. For $k = 0, 1, 2, \ldots$ we define the $k$-homogeneous complex polynomial $P_k$ by

$$P_k(\zeta) = \sum_{|\alpha| = k} \frac{\partial^\alpha \hat{u}(0)}{\alpha!} \zeta^\alpha, \quad \zeta \in \mathbb{C}^n.$$ 

Observe that a priori nothing is known about the convergence of $\sum_{k=0}^\infty P_k$.

If $E \subseteq S^{n-1}$ and for every $\omega \in E$ the function $\mathbb{R} \ni p \mapsto \mathcal{R}u(\omega, p)$ has compact support, contained in the interval $[a_\omega, b_\omega]$, then $\mathcal{F}_1(\mathcal{R}u(\omega, \cdot))$ extends to an entire function on $\mathbb{C}$ satisfying the estimate

$$|\mathcal{F}_1(\mathcal{R}u(\omega, \cdot))(z)| \leq \|\mathcal{R}u(\omega, \cdot)\|_{L^1(\mathbb{R})} e^{H_\omega(\text{Im} z)}, \quad z \in \mathbb{C}, \quad (6.2)$$

where $H_\omega$ is the supporting function of the interval $[a_\omega, b_\omega]$, i.e., $H_\omega(t) = a_\omega t$ for $t \leq 0$ and $H_\omega(t) = b_\omega t$ for $t \geq 0$.

If we assume that $E$ has positive homogeneous capacity in the sense of Siciak, and the function $\sigma$ defined by $\sigma(\omega) = \max\{-a_\omega, b_\omega\}$ is bounded above on $E$, then the assumptions of Theorem 4.1 are satisfied for the function $f : \mathbb{C}E \to \mathbb{C}$ defined by $f(z\omega) = \mathcal{F}_1(\mathcal{R}u(\omega, \cdot))(z)$. Hence $f$ extends to an entire function on $\mathbb{C}^n$ and it is given by the Taylor series of $\hat{u}$ at
the origin. Furthermore, Theorems 5.1 and 5.5 imply that \( u \) has compact support contained in \( K_{E,\sigma} = \{ x \in \mathbb{R}^n ; \langle x, \xi \rangle \leq \Psi^*_E(\xi), \forall \xi \in \mathbb{R}^n \} \).

By (6.1) and (6.2) we have
\[
i_\omega(-i\omega) = \lim_{t \to +\infty} \frac{1}{t} \log |F_t R_u(\omega, \cdot)(-it)| \leq -a_\omega, \quad \omega \in E,
\]
and
\[
i_\omega(i\omega) = \lim_{t \to +\infty} \frac{1}{t} \log |F_t R_u(\omega, \cdot)(it)| \leq b_\omega, \quad \omega \in E.
\]
By Sigurdsson [29], Th. 2.1.1, and by Wiegerinck [31], Th. 2, the supporting function \( H \) of \( \text{ch supp} \ u \) is given by \( H(\omega) = i_\omega^*(i\omega) \) for every \( \omega \in S^{n-1} \) and we have
\[
H(\omega) = \lim_{\omega \to \omega} i_\omega(i\omega), \quad \omega \in S^{n-1}.
\]
If we assume that \( \omega \mapsto a_\omega \) and \( \omega \mapsto b_\omega \) are lower and upper semi-continuous, respectively, then this implies that for every interior point \( \omega \) of \( E \)
\[
H(-\omega) \leq -a_\omega \quad \text{and} \quad H(\omega) \leq b_\omega.
\]
If we assume that \( E \) is the closure of its interior in \( S^{n-1} \), then these inequalities hold at every point in \( E \), and we conclude that for every \( x \in \text{ch supp} \ u \) and every \( \omega \in E \) we have
\[
a_\omega \leq -H(-\omega) \leq \langle x, \omega \rangle \leq H(\omega) \leq b_\omega.
\]
We summarize our argument in

**Theorem 6.1.** Let \( u \in C(\mathbb{R}^n) \) be rapidly decreasing and \( E \) be a non-empty compact subset of \( S^{n-1} \), which has positive homogeneous capacity. Assume that the function \( \mathbb{R} \ni p \mapsto R_u(\omega, p) \) has support contained in the closed bounded interval \([a_\omega, b_\omega]\) for every \( \omega \in E \), where the functions \( \omega \mapsto a_\omega \) and \( \omega \mapsto b_\omega \) are bounded from below and above, respectively, and set \( \sigma(\omega) = \max\{-a_\omega, b_\omega\} \) for \( \omega \in E \). Then the support of \( u \) is contained in the compact convex set
\[
\{ x \in \mathbb{R}^n ; \langle x, \omega \rangle \leq \Psi^*_E(\omega), \forall \omega \in S^{n-1} \}.
\]
If \( E \) is the closure of its relative interior in \( S^{n-1} \) and the functions \( \omega \mapsto a_\omega \) and \( \omega \mapsto b_\omega \) are lower and upper semi-continuous, respectively, then the support of \( u \) is contained in the compact convex set
\[
\{ x \in \mathbb{R}^n ; \langle x, \omega \rangle \leq \Psi^*_E(\omega), \forall \omega \in S^{n-1}, a_\omega \leq \langle x, \omega \rangle \leq b_\omega, \forall \omega \in E \}.
\]
If \( K \) satisfies the assumptions in Helgason’s theorem, i.e., \( K \) is compact, convex, and \( R(\omega, p) = 0 \) for every \( (\omega, p) \) such that the hyperplane defined by the equation \( \langle x, \omega \rangle = p \) does not intersect \( K \), and we define for \( \omega \in S^{n-1} \)
\[
a_\omega = \inf_{x \in K} \langle x, \omega \rangle = -H_K(-\omega) \quad \text{and} \quad b_\omega = \sup_{x \in K} \langle x, \omega \rangle = H_K(\omega),
\]
then \( \omega \mapsto a_\omega \) and \( \omega \mapsto b_\omega \) are continuous on \( S^{n-1} \), \( a_{-\omega} = -b_\omega, b_{-\omega} = -a_\omega \), the function \( p \mapsto R_u(\omega, p) \) has support in \([a_\omega, b_\omega]\), and
\[
K = \{ x \in \mathbb{R}^n ; a_\omega \leq \langle x, \omega \rangle \leq b_\omega, \forall \omega \in S^{n-1} \}.
\]
Hence Theorem 6.1 is a generalization of Helgason’s support theorem. It is also a generalization of a theorem of Wiegerinck [32] which states that a rapidly decreasing function \( u \) has compact support under the assumption that \( p \mapsto R_u(\omega, p) \) decreases exponentially as \( |p| \to +\infty \) for every \( \omega \in S^{n-1} \) and has compact support for every \( \omega \) in an open subset \( E \) of \( S^{n-1} \). His proof is based on a very interesting result of Korevaar and Wiegerinck [17, 33] on representation of mixed derivatives of functions in terms of higher order directional derivatives.

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