The congruence subgroup property for the hyperelliptic modular group

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Abstract

Let \( \mathcal{M}_{g,n} \) and \( \mathcal{H}_{g,n} \), for \( 2g - 2 + n > 0 \), be, respectively, the moduli stack of \( n \)-pointed, genus \( g \) smooth curves and its closed substack consisting of hyperelliptic curves. Their topological fundamental groups can be identified, respectively, with \( \Gamma_{g,n} \) and \( H_{g,n} \), the so called Teichmüller modular group and hyperelliptic modular group. A choice of base point on \( \mathcal{H}_{g,n} \) defines a monomorphism \( H_{g,n} \hookrightarrow \Gamma_{g,n} \).

Let \( S_{g,n} \) be a compact Riemann surface of genus \( g \) with \( n \) points removed. The Teichmüller group \( \Gamma_{g,n} \) is the group of isotopy classes of diffeomorphisms of the surface \( S_{g,n} \) which preserve the orientation and a given order of the punctures. As a subgroup of \( \Gamma_{g,n} \), the hyperelliptic modular group then admits a natural faithful representation \( H_{g,n} \hookrightarrow \text{Out}(\pi_1(S_{g,n})) \).

The congruence subgroup problem for \( H_{g,n} \) asks whether, for any given finite index subgroup \( H^\lambda \) of \( H_{g,n} \), there exists a finite index characteristic subgroup \( K \) of \( \pi_1(S_{g,n}) \) such that the kernel of the induced representation \( H_{g,n} \rightarrow \text{Out}(\pi_1(S_{g,n})/K) \) is contained in \( H^\lambda \). The main result of the paper is an affirmative answer to this question.

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1 Introduction

Let \( S_{g,n} \), for \( 2g - 2 + n > 0 \), be the differentiable surface obtained from a compact Riemann surface \( S_g \) of genus \( g \) removing \( n \) distinct points \( P_i \in S_g \), for \( i = 1, \ldots, n \). The Teichmüller modular group of \( S_{g,n} \) is defined to be the group of isotopy classes of diffeomorphisms or, equivalently, of homeomorphisms of the surface \( S_{g,n} \) which preserve the orientation and the given order of the punctures:

\[
\Gamma_{g,n} := \text{Diff}^+(S_{g,n})/\text{Diff}_0(S_{g,n}) \cong \text{Hom}^+(S_{g,n})/\text{Hom}_0(S_{g,n}),
\]

where \( \text{Diff}_0(S_{g,n}) \) and \( \text{Hom}_0(S_{g,n}) \) denote the connected components of the identity in the topological groups of diffeomorphisms \( \text{Diff}^+(S_{g,n}) \) and of homeomorphisms \( \text{Hom}^+(S_{g,n}) \).
Let $\Pi_{g,n}$ denote the fundamental group of $S_{g,n}$ for some choice of base point. From the above definition and some elementary topology, it follows that there is a faithful representation:

$$\rho: \Gamma_{g,n} \hookrightarrow \text{Out}(\Pi_{g,n}).$$

A level of $\Gamma_{g,n}$ is just a finite index subgroup $H < \Gamma_{g,n}$. A characteristic finite index subgroup $\Pi^\lambda$ of $\Pi_{g,n}$ determines the geometric level $\Gamma^\lambda$, defined to be the kernel of the induced representation:

$$\rho^\lambda: \Gamma_{g,n} \twoheadrightarrow \text{Out}(\Pi_{g,n}/\Pi^\lambda).$$

The congruence subgroup problem asks whether geometric levels are cofinal in the set of all finite index subgroups of $\Gamma_{g,n}$, ordered by inclusion.

This problem is better formulated in the geometric context of moduli spaces of curves. Let $\mathcal{M}_{g,n}$, for $2g - 2 + n > 0$, the moduli stack of $n$-pointed, genus $g$, smooth algebraic complex curves. It is a smooth connected Deligne-Mumford stack (briefly $D-M$ stack) over $\mathbb{C}$ of dimension $3g - 3 + n$, whose associated underlying complex analytic and topological étale groupoids, we both denote by $\mathcal{M}_{g,n}$ as well.

In the category of analytic étale groupoids, there are natural and general definitions of topological homotopy groups (see [12]). However, for stacks of the kind of $\mathcal{M}_{g,n}$, such groups can be described in a simpler way. In fact, $\mathcal{M}_{g,n}$ has a universal cover $T_{g,n}$ in the category of analytic manifolds. The fundamental group $\pi_1(\mathcal{M}_{g,n}, [C])$ is then identified with the deck transformations’ group of the cover $T_{g,n} \rightarrow \mathcal{M}_{g,n}$ and the higher homotopy groups are naturally isomorphic to those of $T_{g,n}$.

From this perspective, Teichmüller theory is the study of the geometry of the universal cover $T_{g,n}$ of the moduli space $\mathcal{M}_{g,n}$, called Teichmüller space, and of its topological fundamental group $\pi_1(\mathcal{M}_{g,n}, [C])$. The basic facts of Teichmüller theory are that $T_{g,n}$ is contractible, thus making of $\mathcal{M}_{g,n}$ a classifying space for $\Gamma_{g,n}$, and that the choice of a lift of a point $[C] \in \mathcal{M}_{g,n}$ to $T_{g,n}$ and of a diffeomorphism $S_{g,n} \rightarrow \mathbb{C}\setminus\{\text{marked points}\}$ identifies the Teichmüller modular group $\Gamma_{g,n}$ with $\pi_1(\mathcal{M}_{g,n}, [C])$. The representation:

$$\rho: \pi_1(\mathcal{M}_{g,n}, [C]) \rightarrow \text{Out}(\Pi_{g,n}),$$

induced by the identification of $\Gamma_{g,n}$ with $\pi_1(\mathcal{M}_{g,n}, [C])$, is equivalent to the universal topological monodromy representation associated with the universal punctured curve $\mathcal{M}_{g,n+1} \rightarrow \mathcal{M}_{g,n}$. Algebraically, this may be recovered as the outer representation associated to the short exact sequence determined on topological fundamental groups by this curve:

$$1 \rightarrow \Pi_{g,n} \rightarrow \pi_1(\mathcal{M}_{g,n+1}) \rightarrow \pi_1(\mathcal{M}_{g,n}) \rightarrow 1.$$

The algebraic fundamental group of a $D-M$ stack $X$ over $\mathbb{C}$ is naturally isomorphic to the profinite completion $\hat{\pi}_1(X)$ of its topological fundamental group $\pi_1(X)$. It basically follows from the triviality of the center of the profinite completion $\hat{\Pi}_{g,n}$ of $\Pi_{g,n}$ that the above fibration induces on algebraic fundamental groups the short exact sequence:

$$1 \rightarrow \hat{\Pi}_{g,n} \rightarrow \hat{\pi}_1(\mathcal{M}_{g,n+1}) \rightarrow \hat{\pi}_1(\mathcal{M}_{g,n}) \rightarrow 1.$$
The universal algebraic monodromy representation is the outer representation:

\[ \hat{\rho} : \hat{\pi}_1(M_{g,n}) \to \text{Out}(\hat{\Pi}_{g,n}), \]

associated to the above short exact sequence. It is not hard to see that the congruence subgroup property holds for \( \Gamma_{g,n} \) if and only if the representation \( \hat{\rho} \) is faithful.

In [3], a positive answer to the above question was claimed but a gap emerged in an essential step of the proof (more precisely, in the proof of Theorem 5.4). As it is explained in detail below, this paper recovers some of the results of [3].

Indeed, the congruence subgroup problem can be formulated for any special subgroup of the Teichmüller group. The case we will deal with in this paper is that of the fundamental group of the closed sub-stack \( \mathcal{H}_{g,n} \) of \( \mathcal{M}_{g,n} \) parametrizing smooth hyperelliptic complex curves, for \( g \geq 1 \). Observe that, for \( g = 1, 2 \), all curves are hyperelliptic, i.e. admit a degree 2 morphism onto \( \mathbb{P}^1 \). We then define the hyperelliptic modular group to be the topological fundamental group of the stack \( \mathcal{H}_{g,n} \).

It is a classical fact of Teichmüller theory that the subspace of the Teichmüller space \( T_{g,n} \), parametrizing hyperelliptic curves, consists of a disjoint union of contractible analytic subspaces. The natural embedding \( \mathcal{H}_{g,n} \subset \mathcal{M}_{g,n} \) then induces, choosing for base points the isomorphism class \([C]\) of a hyperelliptic curve, a monomorphism of topological fundamental groups \( \pi_1(\mathcal{H}_{g,n},[C]) \hookrightarrow \pi_1(\mathcal{M}_{g,n},[C]) \). Let us remark that the image of the latter map, in general, is not a normal subgroup of \( \pi_1(\mathcal{M}_{g,n},[C]) \).

After the identification of \( \pi_1(\mathcal{M}_{g,n},[C]) \) with \( \Gamma_{g,n} \), we denote the subgroup corresponding to \( \pi_1(\mathcal{H}_{g,n},[C]) \) simply by \( H_{g,n} \). Let then \( \iota \) be the element of \( \Gamma_{g,n} \) corresponding to the hyperelliptic involution on \( C \). For \( g \geq 2 \) and \( n = 0 \) or \( g = 1 \) and \( n = 1 \), the subgroup \( H_{g,n} \) is the centralizer of \( \iota \) in \( \Gamma_{g,n} \).

For a given characteristic subgroup of finite index \( \Pi^\lambda \) of \( \Pi_{g,n} \), let us define \( H^\lambda := H_{g,n} \cap \Gamma^\lambda \) and call it the geometric level of \( H_{g,n} \) associated to \( \Pi^\lambda \). The congruence subgroup problem for the hyperelliptic modular group asks whether geometric levels of \( H_{g,n} \) are cofinal in the set of finite index subgroups of \( H_{g,n} \).

The natural morphism \( \mathcal{H}_{g,n+1} \to \mathcal{H}_{g,n} \) (forgetting the last marked point) is naturally isomorphic to the universal \( n \)-punctured, genus \( g \) curve over \( \mathcal{H}_{g,n} \) and the fiber over any closed point \([C]\) \( \in \mathcal{H}_{g,n} \) is diffeomorphic to \( S_{g,n} \). Identifying its fundamental group with \( \Pi_{g,n} \), we get, as above, a faithful topological monodromy representation:

\[ \rho_{g,n} : \pi_1(\mathcal{H}_{g,n},[C]) \to \text{Out}(\Pi_{g,n}). \]

Instead, the faithfulness of the corresponding algebraic monodromy representation:

\[ \hat{\rho}_{g,n} : \hat{\pi}_1(\mathcal{H}_{g,n},[C]) \to \text{Out}(\hat{\Pi}_{g,n}). \]

is a much deeper statement, equivalent to the congruence subgroup property for \( H_{g,n} \).

The main result of this paper is that \( \hat{\rho}_{g,n} \) is faithful for all \( 2g-2+n > 0 \). In particular, we prove that the congruence subgroup property holds for the genus 2 Teichmüller modular group (for genus 0 and 1, this has been proved by Asada in [1]).
2 The geometric profinite completion of $\Gamma_{g,n}$

Let us assume that the fundamental group $\Pi_{g,n}$ of $S_{g,n}$ has $P_{n+1}$ as base point. For $2g - 2 + n > 0$, the short exact sequence of topological fundamental groups, associated to the Serre fibration $M_{g,n+1} \to M_{g,n}$, is then identified with the classical short exact sequence of modular groups

$$1 \to \Pi_{g,n} \to \Gamma_{g,n+1} \to \Gamma_{g,n} \to 1,$$

while the corresponding short exact sequence of algebraic fundamental groups is identified with the short exact sequence

$$1 \to \hat{\Pi}_{g,n} \to \hat{\Gamma}_{g,n+1} \to \hat{\Gamma}_{g,n} \to 1.$$

The action by inner automorphisms of $\hat{\Gamma}_{g,n+1}$ on its normal subgroup $\hat{\Pi}_{g,n}$ induces the representations $\tilde{\rho}_{g,n} : \hat{\Gamma}_{g,n+1} \to \text{Aut}(\hat{\Pi}_{g,n})$ and $\hat{\rho}_{g,n} : \hat{\Gamma}_{g,n} \to \text{Out}(\hat{\Pi}_{g,n})$.

Let us mention here a fundamental result of Nikolov and Segal [11] which asserts that any finite index subgroup of any topologically finitely generated profinite group $G$ is open.

Since such a profinite group $G$ has also a basis of neighborhoods of the identity consisting of open characteristic subgroups, it follows that all automorphisms of $G$ are continuous and that $\text{Aut}(G)$ is a profinite group as well. Let us then give the following definitions:

**Definition 2.1.** Let us define the profinite groups $\tilde{\Gamma}_{g,n+1}$ and $\hat{\Gamma}_{g,n}$, for $2g - 2 + n > 0$, to be, respectively, the image of $\tilde{\rho}_{g,n}$ in $\text{Aut}(\hat{\Pi}_{g,n})$ and of $\hat{\rho}_{g,n}$ in $\text{Out}(\hat{\Pi}_{g,n})$.

By definition, there are natural maps with dense image $\Gamma_{g,n} \to \hat{\Gamma}_{g,n}$ and $\Gamma_{g,n+1} \to \hat{\Gamma}_{g,n+1}$, but it is a deep result by Grossman [6] that these maps are also injective.

By Definition 2.1 the representation $\hat{\Gamma}_{g,n+1} \to \text{Aut}(\hat{\Pi}_{g,n})$, induced by the action of inner automorphisms of $\hat{\Gamma}_{g,n+1}$ on its normal subgroup $\hat{\Pi}_{g,n}$, is injective. Therefore, it holds:

**Proposition 2.2.** The center of $\tilde{\Gamma}_{g,n+1}$ is trivial for $2g - 2 + n > 0$.

Another consequence of Definition 2.1 is the following:

**Proposition 2.3.** For $2g - 2 + n > 0$, there is a natural short exact sequence:

$$1 \to \hat{\Pi}_{g,n} \to \hat{\Gamma}_{g,n+1} \to \hat{\Gamma}_{g,n} \to 1.$$

In particular, $\tilde{\Gamma}_{g,n} \equiv \hat{\Gamma}_{g,n}$ if and only if $\tilde{\Gamma}_{g,n+1} \equiv \hat{\Gamma}_{g,n+1}$.

We then have the interesting corollary:

**Corollary 2.4.** If the congruence subgroup property holds for $\Gamma_{g,n}$, then $\hat{\Gamma}_{g,n+1}$ has trivial center.

A natural guess is that, for $2g - 2 + n > 0$, the two profinite completions $\tilde{\Gamma}_{g,n+1}$ and $\hat{\Gamma}_{g,n+1}$ of $\Gamma_{g,n+1}$ coincide. That this actually holds is a deep and subtle fact:
Theorem 2.5. For $2g - 2 + n > 0$, there is a natural isomorphism $\Phi: \tilde{\Gamma}_{g,n+1} \rightarrow \tilde{\Gamma}_{g,n+1}$.

Hence, a short exact sequence: $1 \rightarrow \hat{\Pi}_g \rightarrow \tilde{\Gamma}_{g,n+1} \rightarrow \hat{\Pi}_g \rightarrow 1$.

Theorem 2.5, for $n > 0$, is a direct consequence of Theorem 2.2 in [9]. However, the rest of this section will be devoted to giving an independent proof of the theorem for $n \geq 0$.

The existence of the epimorphism $\tilde{\Gamma}_{g,n+1} \rightarrow \tilde{\Gamma}_{g,n+1}$ was already remarked in the proof of Theorem 1 in [1]. Let us recall briefly the argument.

By the very definition of the two groups, there is a natural epimorphism $\tilde{\Gamma}_{g,n+2} \rightarrow \tilde{\Gamma}_{g,n+1}$. There is also an epimorphism $p: \tilde{\Gamma}_{g,n+2} \rightarrow \tilde{\Gamma}_{g,n+1}$, induced by the epimorphism $\pi: \hat{\Pi}_{g,n+1} \rightarrow \hat{\Pi}_g$ (filling up the $(n + 1)$-th puncture).

Let us give to the fundamental groups $\Pi_{g,n}$, for all $n \geq 0$, the standard presentations:

$$\Pi_{g,n} = \langle a_1, \ldots, a_g, b_1, \ldots, b_g, u_1, \ldots, u_n \mid \prod_{i=1}^{g}[a_i, b_i] \cdot u_n \cdots u_1 \rangle,$$

where $u_i$, for $i = 1, \ldots, n$, is a simple loop around the puncture $P_i$. The kernel of $\pi$ is then the closed subgroup normally generated by the element $u_{n+1}$ of $\Pi_{g,n+1}$.

Let $f \in \tilde{\Gamma}_{g,n+1}$ and let $\tilde{f}$ be a lift of $f$ in $\tilde{\Gamma}_{g,n+2}$ such that $\tilde{f}(u_{n+1}) = u_{n+1}$. Let us define the homomorphism $\Phi: \tilde{\Gamma}_{g,n+1} \rightarrow \tilde{\Gamma}_{g,n+1}$ by $\Phi(f) := p(\tilde{f})$. This is well defined, since, if $\tilde{f}'$ is another lift of $f$ in $\tilde{\Gamma}_{g,n+2}$, then $\tilde{f}'^{-1}\tilde{f}$ is an inner automorphism of $\hat{\Pi}_{g,n+1}$ which leaves $u_{n+1}$ fixed. So, $\Phi$ is well defined by the following well known fact:

Lemma 2.6. Let $x \in \hat{\Pi}_{g,n}$, for $n > 0$, be such that there is an automorphism $f$ of $\hat{\Pi}_{g,n}$ such that $f(x) \in \Pi_{g,n}$ is representable by a simple closed curve (briefly s.c.c.) on $S_{g,n}$. Then, the centralizer in $\hat{\Pi}_{g,n}$ of $x$ is the closed subgroup topologically generated by $x$.

Proof. The hypothesis on $x$ implies that the closed subgroup it spans inside $\hat{\Pi}_{g,n}$ is isomorphic to $\tilde{Z}$ and it is not contained properly in any other closed cyclic subgroup of $\hat{\Pi}_{g,n}$. Suppose then that there is an $y \notin \langle x \rangle$ which commutes with $x$. The torsion free abelian closed subgroup spanned by $x$ and $y$ inside $\hat{\Pi}_{g,n}$ has cohomological dimension 2. But this is in contrast with the fact that $\hat{\Pi}_{g,n}$, as a free profinite group, has cohomological dimension 1 and so, by Schapiro’s Lemma, the same holds for all of its closed subgroups.

So, there is, at least, a natural epimorphism $\Phi: \tilde{\Gamma}_{g,n+1} \rightarrow \tilde{\Gamma}_{g,n+1}$. An immediate consequence is the genus 0 case of the subgroup congruence property:

Corollary 2.7. For $n \geq 3$, it holds $\tilde{\Gamma}_{0,n} = \tilde{\Gamma}_{0,n} = \tilde{\Gamma}_{0,n}$.

Proof. The case $n = 3$ is trivial, since $\Gamma_{0,3} = \{1\}$. The general case follows by Proposition 2.3 the epimorphism $\Phi: \tilde{\Gamma}_{0,n} \rightarrow \tilde{\Gamma}_{0,n}$ and induction on $n$. 

\qed
Corollary 2.7 implies, a fortiori, that ker $\Phi = \{1\}$, for $g = 0$. For $g \geq 1$, the proof of the triviality of ker $\Phi$ is much trickier and a long preliminary digression is needed.

It is a classical fact of Teichmüller theory that each free isotopy class $\gamma$ of a simple, embedded, non-trivial simple closed curve (briefly s.c.c.) on $S_{g,n}$ uniquely determines a distinguished element $\tau_\gamma$ of $\Gamma_{g,n}$, called Dehn twist (or simply twist) along $\gamma$ and that two s.c.c. $\gamma$ and $\gamma'$ are freely isotopic on $S_{g,n}$ if and only if they are freely homotopic. The Dehn twists provide a natural set of generators for the Teichmüller group and, for $g \geq 1$, the Teichmüller group is generated just by Dehn twists along non-separating curves.

Let us denote by $D$ the set of isotopy classes of s.c.c. on $S_{g,n}$ and then by $\{\tau_\alpha\}_{\alpha \in D}$ the set of Dehn twists in $\Gamma_{g,n}$. In accordance with [3], we denote by $\{\tau_\alpha\}_{\alpha \in D}$ the closure of this set inside the profinite group $\hat{\Gamma}_{g,n}$. This is the profinite set of *profinite twists* of $\hat{\Gamma}_{g,n}$.

The Teichmüller group $\Gamma_{g,n}$ acts by conjugation on the set of all Dehn twists $\{\tau_\alpha\}_{\alpha \in D}$, with a finite number of orbits, which corresponds to the possible topological types of the surface $S_{g,n} \setminus \gamma$, for $\gamma$ a s.c.c. on $S_{g,n}$. The profinite group $\hat{\Gamma}_{g,n}$ acts continuously by conjugation on $\{\tau_\alpha\}_{\alpha \in D}$ and each orbit contains all the discrete twists corresponding to a fixed topological type and no other discrete twist. This separation property is a consequence of the local monodromy description of geometric levels (see, for instance, Proposition 2.8. in [4] or Theorem 3.3.3 in [13]). For $f \in \hat{\Gamma}_{g,n}$, let us then define:

$$\tau_{f(\alpha)} := f \tau_\alpha f^{-1}.$$  

For $2g - 2 + n > 0$, there is a natural monomorphism $i: \Pi_{g,n} \hookrightarrow \Gamma_{g,n+1}$ which, more intrinsically, is described as follows (see Birman [2], for a proof and more details). The base point of $\Pi_{g,n}$ is assumed, as usual, to be the marked point $P_{n+1}$. Let $D$ be a closed disc on $S_{g,n}$ centered in $P_{n+1}$, let $\partial D$ be its boundary and $\hat{D} := D \setminus \partial D$. Then, any element of $\Pi_{g,n}$, which is representable by an oriented s.c.c. $\gamma$, determines the free isotopy classes of a pair $\{\gamma_0, \gamma_1\}$ of s.c.c. on $S_{g,n+1}$ in the following way. Let us first orient the boundary $\partial D$ clockwise. The s.c.c. $\gamma_0$ is then obtained gluing to the closed oriented arc $\gamma \setminus \hat{D}$ the connected component of $\partial D \setminus \gamma$ which is oriented compatibly. Similarly, the s.c.c. $\gamma_1$ is obtained by applying the same procedure with $\partial D$ oriented counterclockwise.

The monomorphism $i$ then assigns to the class $[\gamma] \in \Pi_{g,n}$ of the oriented s.c.c. $\gamma$ the product of Dehn twists $\tau_{\gamma_0} \cdot \tau_{\gamma_1}^{-1}$. We have seen that the monodromy representation $\Gamma_{g,n+1} \hookrightarrow \text{Aut}(\Pi_{g,n})$ is given by the action by inner automorphisms of $\Gamma_{g,n+1}$ on its normal subgroup $\Pi_{g,n}$. Coherently, for all $f \in \Gamma_{g,n+1}$ and a s.c.c. $\gamma \in \Pi_{g,n}$, it holds:

$$i(f(\gamma)) = \tau_{f(\gamma_0)} \tau_{f(\gamma_1)}^{-1} = \tau_{f(\gamma_0)} \tau_{f(\gamma_1)}^{-1} = (f \tau_{\gamma_0} f^{-1})(f \tau_{\gamma_1}^{-1} f^{-1}) = f i(\gamma) f^{-1}.$$  

The map $i$ extends to a continuous monomorphism $i: \hat{\Pi}_{g,n} \hookrightarrow \hat{\Gamma}_{g,n+1}$ and, as above, the profinite monodromy representation $\hat{\Gamma}_{g,n+1} \rightarrow \text{Aut}(\hat{\Pi}_{g,n})$ is given by the action by inner automorphisms of $\hat{\Gamma}_{g,n+1}$ on its normal subgroup $\hat{\Pi}_{g,n}$. Let $[\gamma]$ be an element of $\hat{\Pi}_{g,n}$ such that there is a sequence of s.c.c. $\{\gamma_k\}_{k \in \mathbb{N}} \subset S_{g,n}$, such that the sequence $\{[\gamma_k]\}_{k \in \mathbb{N}} \subset \hat{\Pi}_{g,n}$ converges to $[\gamma]$ in the profinite topology. Let then $\{\tau_{\gamma_{k_0}}\}_{k \in \mathbb{N}}$ and $\{\tau_{\gamma_{k_1}}\}_{k \in \mathbb{N}}$ be the two sequences of elements of $\Gamma_{g,n+1}$ obtained from the given sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ by means of the
procedure described above. It holds:

$$i([\gamma]) = \lim_{k \to \infty} \tau_{\eta_k} \tau_{\eta_{k-1}}^{-1},$$

where the limit is taken in the profinite topology of $\tilde{\Gamma}_{g,n+1}$.

Let $E$ be a cylinder which compactifies the punctured disc $D \setminus P_{n+1}$ in such a way that $U := E \setminus (D \setminus P_{n+1})$ is a circle. Let $S^U_{g,n}$ be the surface with boundary $U$ obtained gluing $E$ to $S_{g,n+1}$ along the punctured disc $D \setminus P_{n+1}$. The natural inclusion $S_{g,n+1} \hookrightarrow S^U_{g,n}$ induces an isomorphism on fundamental groups. The choice of a tangential base point in $P_{n+1}$ for the fundamental group of the surface $S_{g,n+1}$ consists in taking a point $O \in U$ as the base point for the fundamental group of the extended surface $S^U_{g,n}$. Let us fix such a tangential base point and let us maintain for the corresponding fundamental group the usual notation $\Pi_{g,n+1}$.

Let $u$ be a simple loop based in $O$ which has $U$ for support. Let then $\langle \text{inn} \rangle$ be the closed subgroup topologically generated by $\text{inn} u$ inside $\text{Aut}(\tilde{\Pi}_{g,n+1})$ and let $\text{Aut}^*(\tilde{\Pi}_{g,n+1})$ be its normalizer there. With the above choice of tangential base point, the associated profinite outer representation $\tilde{\Gamma}_{g,n+1} \to \text{Out}(\tilde{\Pi}_{g,n+1})$ takes the form:

$$\tilde{\Gamma}_{g,n+1} \to \text{Aut}^*(\tilde{\Pi}_{g,n+1}) / \langle \text{inn} u \rangle$$

and its image is still isomorphic to the geometric profinite completion $\tilde{\Gamma}_{g,n+1}$. For the rest of the proof, the profinite group $\tilde{\Gamma}_{g,n+1}$ will be identified with such image.

Let $\Pi_{g,n}$ be the fundamental group of $S_{g,n}$ based at the last marked point $P_{n+1}$, as above. Then, there is also a natural surjective $\Gamma_{g,n+1}$-equivariant map, induced by the surjective map $S^U_{g,n} \twoheadrightarrow S_{g,n}$ which contracts the circle $U$ onto the point $P_{n+1} \in S_{g,n}$:

$$\pi' : \Pi_{g,n+1} / \langle \text{inn} u \rangle \twoheadrightarrow \Pi_{g,n};$$

and then a surjective $\tilde{\Gamma}_{g,n+1}$-equivariant map:

$$\tilde{\pi}' : \tilde{\Pi}_{g,n+1} / \langle \text{inn} u \rangle \twoheadrightarrow \tilde{\Pi}_{g,n}.$$  

Above, we described a procedure which assigned to an element of $\Pi_{g,n}$, representable by an oriented s.c.c. $\gamma$, the free isotopy classes of two oriented s.c.c. $\gamma_i$ on $S_{g,n+1}$, for $i = 0, 1$, i.e. elements of the orbit set $\Pi_{g,n+1} / \langle \text{inn}(\Pi_{g,n+1}) \rangle$. This procedure can be refined in order to get sections $s_i$ and $\tilde{s}_i$, for $i = 0, 1$, of suitable restrictions of the maps $\pi'$ and $\tilde{\pi}'$.

**Definition 2.8.** Let $\mathcal{L}$ be the set of relative isotopy classes of $P_{n+1}$-pointed oriented non-separating s.c.c. on $S_{g,n}$. It is a classical fact in the theory of Riemann surfaces (see Theorem 3.4.15 in [5]) that this set embeds in the fundamental group $\Pi_{g,n}$ of $S_{g,n}$. Let then $\mathcal{L}$ be the closure of $\mathcal{L}$ inside the profinite completion $\tilde{\Pi}_{g,n}$ of $\Pi_{g,n}$.

We are going to define two natural sections for the maps $\pi'^{-1}(\mathcal{L}) \to \mathcal{L}$ and $\tilde{\pi}'^{-1}(\hat{\mathcal{L}}) \to \hat{\mathcal{L}}$ induced, by restriction, respectively, by $\pi'$ and $\tilde{\pi}'$. 
Let $\gamma$ be a $P_{n+1}$-pointed oriented s.c.c. on $S_{g,n}$ and let $\{\gamma_0, \gamma_1\}$ the couple of oriented s.c.c. on $S_{g,n+1}$ which we associated to $\gamma$ in order to describe the natural monomorphism $i:\Pi_{g,n} \twoheadrightarrow \Gamma_{g,n+1}$. We can associate to each of the $\gamma_i$, for $i = 0, 1$, an element of $\Pi_{g,n+1}$, connecting $\gamma_i$ to the tangential base point by means of a simple path $\theta$ on the cylinder $E$ from $O \subset U$ to a point in the arc $\gamma_i \cap \partial D$, for $i = 0, 1$. The classes $[\theta \gamma_i \theta^{-1}] \in \Pi_{g,n+1}$, for $i = 0, 1$, then are determined by the oriented s.c.c. $\gamma$ modulo conjugation by a power of $u$.

Now, two $P_{n+1}$-pointed oriented s.c.c. $\gamma$ and $\gamma'$ which are linked by an isotopy which preserves $P_{n+1}$ determine the same classes in $\Pi_{g,n+1}/\langle \text{inn} \, u \rangle$.

Indeed, an isotopy from $\gamma$ to $\gamma'$ is given, on the disc $D$, by the curve $\gamma$ approaching $\gamma'$ rotating around the fixed point $P_{n+1}$. Such isotopy induces an isotopy from $\gamma_i$ to $\gamma_i'$, for $i = 0, 1$, which, on $D$, is given by the arc $\gamma_i \cap \partial D$ approaching the arc $\gamma_i' \cap \partial D$, moving around the circle $\partial D$. Deforming the path $\theta$ along with this isotopy, we get a simple path $\theta'$ which connect the tangential base point $O$ with the arc $\gamma_i' \cap \partial D$, for $i = 0, 1$.

Now, the elements $[\theta \gamma_i \theta^{-1}] \in \Pi_{g,n+1}$, for $i = 0, 1$, project to the classes in $\Pi_{g,n+1}/\langle \text{inn} \, u \rangle$ associated to the $P_{n+1}$-pointed oriented s.c.c. $\gamma'$ by means of the above procedure and each $[\theta \gamma_i \theta^{-1}]$, for $i = 0, 1$, is conjugated in $\Pi_{g,n+1}$ by a power of $u$ to the element $[\theta \gamma_i \theta^{-1}]$.

So, the assignment $\gamma \mapsto [\theta \gamma_i \theta^{-1}]$, for $i = 0, 1$:

$$s_i: \mathcal{L} \rightarrow \Pi_{g,n+1}/\langle \text{inn} \, u \rangle.$$

An element of $\Gamma_{g,n+1}$ can be represented by a homeomorphism which fixes point-wise a closed neighborhood of the disc $D$ in $S_{g,n}$. It is then clear that the maps $s_i$, for $i = 0, 1$, commute with the action of $\Gamma_{g,n+1}$ on $\mathcal{L}$ and $\Pi_{g,n+1}/\langle \text{inn} \, u \rangle$, respectively, i.e. they are $\Gamma_{g,n+1}$-equivariant.

From the conjugacy separability of $\Pi_{g,n+1}$, it follows that the set $\Pi_{g,n+1}/\langle \text{inn} \, u \rangle$ embeds in the profinite set $\hat{\Pi}_{g,n+1}/\langle \text{inn} \, u \rangle$. The key to the proof of Theorem 2.5 is the following lemma:

**Lemma 2.9.** Let $2g - 2 + n > 0$. The natural $\hat{\Gamma}_{g,n+1}$-equivariant map $\tilde{\pi}|_{\pi^{-1}(\mathcal{L})}$ has two continuous $\hat{\Gamma}_{g,n+1}$-equivariant sections:

$$\hat{s}_i: \hat{\mathcal{L}} \rightarrow \hat{\Pi}_{g,n+1}/\langle \text{inn} \, u \rangle, \quad \text{for } i = 0, 1,$$

which extend the maps $s_i$ defined above.

**Proof.** There is a more algebraic way to recover the maps $s_i$ which may serve as a model to define the profinite maps $\hat{s}_i$, for $i = 0, 1$. As we saw above, the natural monomorphism $i: \Pi_{g,n} \twoheadrightarrow \Gamma_{g,n+1}$ associates to a $\gamma \in \mathcal{L}$ the product of Dehn twists $\tau_{\gamma_0} \tau_{\gamma_1}^{-1}$. Let us see how the classes of $s_i(\gamma)$, for $i = 0, 1$, can be recovered from this product.

Let $\Gamma^U_{g,n}$ be the group of relative homotopy classes of homeomorphism of $S^U_{g,n}$ which fix pointwise the boundary $U$ and respect the given order of punctures on $S^U_{g,n}$. There is a natural faithful representation $\Gamma^U_{g,n} \hookrightarrow \text{Aut}(\Pi_{g,n+1})$ and a natural epimorphism $\Gamma^U_{g,n} \rightarrow \Gamma_{g,n+1}$ which determines the central extension:

$$1 \rightarrow \mathbb{Z} \cdot \tau_u \rightarrow \Gamma^U_{g,n} \rightarrow \Gamma_{g,n+1} \rightarrow 1,$$
where $\tau_u$ is the Dehn twist along a s.c.c. isotopic to the boundary circle $U$.

A non-trivial isotopy class of a s.c.c. $\alpha \subset S_{g,n+1} \subset S_{g,n}$, not bounding a single puncture, determines both the Dehn twist $\tau_\alpha \in \Gamma_{g,n+1}$ and a lift of this to $\Gamma_{g,n}^U$, which we denote as well by $\tau_\alpha$. In particular, the given $\gamma \in \mathcal{L}$ determines the product of Dehn twists $\tau_{\gamma_0} \tau^{-1}_{\gamma_1} \in \Gamma_{g,n}$.

Now, the class $s_i(\gamma)$ in $\Pi_{g,n+1}/\langle\text{inn} \, u\rangle$, for $i = 0, 1$, can be recovered from the action of $\tau_{\gamma_0} \tau^{-1}_{\gamma_1}$ on $\Pi_{g,n+1}$.

Fix a point $Q \in S_{g,n+1} \setminus D$ and let $\alpha$ be a non-trivial loop on $S_{g,n+1}$ with base point $Q$ such that its support is disjoint from $D \cup \gamma$. Fix simple paths $\theta_i$, for $i = 0, 1$, from the tangential base point $O \in S_{g,n}^U$ to $Q$, which cross the circle $\partial D$ in just one point contained in the arc $\gamma_i \cap \partial D$. Let then $[\theta_i, \alpha \theta^{-1}_i]$, for $i = 0, 1$, be the elements determined in the fundamental group $\Pi_{g,n+1}$.

The product of Dehn twists $\tau_{\gamma_0} \tau^{-1}_{\gamma_1} \in \Gamma_{g,n}$ acts on all the elements of the fundamental group $\Pi_{g,n+1}$, which have the form above, by the inner automorphism $\text{inn}(x_i)$, where $x_i := [\theta_i, \gamma \theta^{-1}_i]$, for $i = 0, 1$. Then, $x_{i'}$ for $i = 0, 1$, is determined by the restriction of $\text{inn}(x_i)$ to the subgroup of $\Pi_{g,n+1}$ generated by elements of the form above.

Different choices $\theta'_i$ of the paths $\theta_i$ determine elements $x'_{i'}$ which are conjugated to the elements $x_i$ above by powers of $u$, for $i = 1, 2$. Therefore, at any rate, the equivalence class $s_i(\gamma) \equiv x_i$ in $\Pi_{g,n+1}/\langle\text{inn} \, u\rangle$, for $i = 0, 1$, is uniquely determined.

Let us now try to see how this construction can be translated to the profinite case. Let us begin with the following definition:

**Definition 2.10.** There is a natural faithful representation $\Gamma_{g,n}^U \hookrightarrow \text{Aut}(\hat{\Pi}_{g,n+1})$. Let us then define the profinite group $\hat{\Gamma}_{g,n}^U$ to be the closure of the image of the discrete group $\Gamma_{g,n}^U$ in the profinite group $\text{Aut}(\hat{\Pi}_{g,n+1})$.

The group $\hat{\Gamma}_{g,n}^U$ is a central extension of $\Gamma_{g,n+1}$ by the cyclic group generated by $\tau_u$, the Dehn twist along the s.c.c. $u$. Therefore, the profinite group $\hat{\Gamma}_{g,n}^U$ is a central extension of $\hat{\Gamma}_{g,n+1}$ by the closed subgroup generated by $\tau_u$:

$$1 \to \hat{\mathbb{Z}} \cdot \tau_u \to \hat{\Gamma}_{g,n}^U \to \hat{\Gamma}_{g,n+1} \to 1.$$  

The first difficulty arises when we want to define a canonical lift in $\hat{\Gamma}_{g,n}^U$ of a profinite twist in $\hat{\Gamma}_{g,n+1}$. However, it is easy to see that this problem reduces to show that, if, for a discrete Dehn twist $\tau_{\alpha} \in \Gamma_{g,n+1}$ and some element $f \in \hat{\Gamma}_{g,n+1}$, it holds $f \tau_{\alpha} f^{-1} = \tau_{\alpha}$, then, for any lift $\tilde{f}$ of $f$ in the profinite group $\Gamma_{g,n}^U$, it holds also $\tilde{f} \tau_{\alpha} \tilde{f}^{-1} = \tau_{\alpha}$, where, as above, we denote as well by $\tau_{\alpha}$ the canonical lift of $\tau_{\alpha}$ to $\Gamma_{g,n}^U$. It is not difficult to settle this problem proceeding like in the proof of Lemma 2.11.

If the above property holds, the canonical lifts of the discrete twists determine canonical lifts for all the profinite twists which are in their conjugacy classes inside $\hat{\Gamma}_{g,n+1}$ and so we are done because each profinite twist is conjugated to a discrete one.

Indeed, if $\tau_{\alpha'} = f \tau_{\alpha} f^{-1}$, for $f \in \hat{\Gamma}_{g,n+1}$, and $\bar{\tau}_{\alpha}$ is a lift of $\tau_{\alpha}$ in $\hat{\Gamma}_{g,n}^U$, then, for any $\tilde{f} \in \hat{\Gamma}_{g,n}^U$ which lifts $f$, the element $\tilde{f} \bar{\tau}_{\alpha} \tilde{f}^{-1}$ is a lift of $\tau_{\alpha'}$ in $\hat{\Gamma}_{g,n}^U$ which does not depend on the particular $\tilde{f}$ chosen.
It is true as well, by Lemma \ref{lem:2.6} that, for a s.c.c. \( x \in \hat{\Pi}_{g,n+1} \), the restriction of the corresponding inner automorphism \( \text{inn}(x) \) to a rank two subgroup determines \( x \).

The real difficulty, in trying to define the maps \( \hat{s}_i \), for \( i = 0,1 \), directly from the monomorphism \( \hat{i} \), arise when we try to make sense of the notion of support on \( S_{g,n+1} \) of a profinite twist (or simple product of them).

A profinite twist \( \tau_\alpha \in \hat{\Gamma}_{g,n+1} \) can be realized as a sequence \( \{ \tau_{\alpha_k} \}_{k \in \mathbb{N}} \) of discrete twists which converges in the profinite group \( \hat{\Gamma}_{g,n+1} \). However, even when elements of this sequence get very close in the profinite topology of \( \hat{\Gamma}_{g,n+1} \), the isotopy classes of the s.c.c. \( \alpha_k \) are not commensurable with respect to the ordinary topology of \( S_{g,n+1} \). This makes it impossible to find elements of \( \hat{\Pi}_{g,n+1} \) on which the action of the product of profinite Dehn twists \( \hat{i}(\gamma) \) takes the simple form we need in order to carry out the argument above.

In order to overcome this difficulty, we need to use the fact that we have already defined the discrete sections \( s_i : \mathcal{L} \to \Pi_{g,n+1}/\langle \text{inn} u \rangle \), for \( i = 0,1 \).

There is only one \( \Gamma_{g,n+1} \)-orbit in \( \mathcal{L} \) and \( \hat{\mathcal{L}} \) is its closure in \( \hat{\Pi}_{g,n} \). So, it is enough to show that, for a given \( f \in \hat{\Gamma}_{g,n+1} \) and a \( [\gamma] \in \mathcal{L} \) such that \( f([\gamma]) = [\gamma] \), in the profinite set \( \hat{\Pi}_{g,n+1}/\langle \text{inn} u \rangle \), it holds as well \( f(s_i([\gamma])) = s_i([\gamma]) \), for \( i = 0,1 \).

The natural monomorphism \( \hat{i} : \hat{\Pi}_{g,n} \hookrightarrow \hat{\Gamma}_{g,n+1} \) translates the condition \( f([\gamma]) = [\gamma] \) into the identitiy:

\[
\tau_{\gamma_0}^{-1} \tau_{\gamma_1}^{-1} = f(\tau_{\gamma_0}^{-1})f^{-1}.
\]

In order to make more explicit these identities, it is useful to work with an actual automorphism of \( \hat{\Pi}_{g,n+1} \) rather than an outer automorphism like \( f \). Let \( f \in \hat{\Gamma}_{g,n+1} \) and \( [\gamma] \in \mathcal{L} \) be such that, in \( \hat{\mathcal{L}} \), it holds \( f([\gamma]) = [\gamma] \) and let \( \hat{f} \) be a lift of \( f \) in \( \hat{\Gamma}_{g,n} \). Then, for some \( k \in \hat{\mathbb{Z}} \), which does not depend on the lift chosen, it holds:

\[
\hat{f}(\tau_{\gamma_0}^{-1})\hat{f}^{-1} = \tau_{\gamma_0}^{-1}\tau_{\gamma_1}^{-1}\tau_{\gamma_0}^{-1}\tau_{\gamma_1}^{-1}u^k.
\]

We claim that the above identity makes sense only for \( k = 0 \):

**Lemma 2.11.** Let be given an \( f \in \hat{\Gamma}_{g,n+1} \) which acts trivially on \( [\gamma] \in \mathcal{L} \subset \hat{\mathcal{L}} \). Then, for any lift \( \hat{f} \) of \( f \) in \( \hat{\Gamma}_{g,n} \), it holds:

\[
\hat{f}(\tau_{\gamma_0}^{-1})\hat{f}^{-1} = \tau_{\gamma_0}^{-1}\tau_{\gamma_1}^{-1}.
\]

**Proof.** Let us embed \( S^U_{g,n} \) in the Riemann surface \( S_{g+1,n} \) so that \( S_{g+1,n} \setminus U \cong S_{g,n+1} \bigcup S_{1,1} \). By definition, the elements of \( \Gamma_{g,n} \) are homeomorphisms which fix \( U \). Let us then assume that the \( (n+1) \)-th marked point \( P_{g+1} \) on our surface of reference \( S_{g+1,n} \) coincides with the tangential base point \( O \) fixed on \( S^U_{g,n} \) and let this be also the base point of the fundamental group \( \Pi_{g+1,n} \) of \( S_{g+1,n} \). The inclusion of surfaces \( S^U_{g,n} \hookrightarrow S_{g+1,n} \) induces the monomorphisms \( \hat{\Pi}_{g,n+1} \hookrightarrow \hat{\Pi}_{g+1,n} \) and \( \Gamma^U_{g,n} \hookrightarrow \Gamma_{g+1,n+1} \), and then also \( \hat{\Gamma}^U_{g,n} \hookrightarrow \hat{\Gamma}_{g+1,n+1} \).

A further reduction to the case without punctures can be operated observing that the mirror embedding of the reference surface \( S_{g+1,n} \) in the un-punctured surface \( S_{2g+n+1} \), induces a homomorphism of profinite groups \( \hat{\Gamma}_{g+1,n+1} \to \hat{\Gamma}_{2g+n+1} \) and then \( \hat{\Gamma}^U_{g,n} \to \hat{\Gamma}_{2g+n+1} \).
Letting $f'$ be the image of $\tilde{f}$ by the latter homomorphism, the above identity projects in $\tilde{\Gamma}_{2g+n+1}$ to the identity:

$$\tau_{f'(\gamma_0)}^{-1} f'(\gamma_1) = f'(\tau_{\gamma_0} \tau_{\gamma_1}) f'^{-1} = \tau_{\gamma_0} \tau_{\gamma_1}^{-1} f'^{-1}.$$

To show that this identity is consistent only for $k = 0$, we make use of the Prym cover trick (see Looijenga [8]). Let $\hat{\Pi}^{(2)}$ be the closed subgroup spanned by squares in $\hat{\Pi}_{2g+n+1}$. It is a characteristic subgroup of $\hat{\Pi}_{2g+n+1}$ with cokernel $G := H_1(S_{2g+n+1}, \mathbb{Z}/2)$. Let then $\rho_G: \hat{S} \rightarrow S_{2g+n+1}$ be the corresponding étale cover with Galois group $G$.

Let $\Gamma(\hat{S})$ the orientation preserving mapping class group of the compact Riemann surface $\hat{S}$. There is a natural faithfull representation $G \hookrightarrow \Gamma(\hat{S})$. Let us identify $G$ with the image of this representation. A lift of an element of $\Gamma_{2g+n+1}$ to $\Gamma(\hat{S})$ is then uniquely determined modulo an element of $G$. The elements of the abelian level $\Gamma(2)$ lift to elements of $\Gamma(\hat{S})$ which commute with the covering transformations $G$.

A Dehn twist along a separating curve $\alpha \subset S_{2g+n+1}$ lifts canonically to the product of $2^{2g+n+1}$ non-separating Dehn twists $\prod_{\alpha \in \rho_G^{-1}(\alpha)} \tau_\alpha$ while the $2^{nd}$ order power $\tau_\gamma^2$ of a Dehn twist along a non-separating curve $\gamma \subset S_{2g+n+1}$ lifts canonically to the product of $2^{2g+n}$ non-separating Dehn twists $\prod_{\gamma \in \rho_G^{-1}(\alpha)} \tau_\gamma$ (see [8]).

The above construction extends to the profinite case. In particular, also profinite Dehn twists along separating curves and $2^{nd}$ order powers of profinite twists along non-separating curves can be lifted canonically to products of profinite Dehn twists in $\hat{\Gamma}(\hat{S})$. Indeed, they are conjugated in $\hat{\Gamma}(2)$ to discrete twists and to $2^{nd}$ order powers of discrete twists, respectively. Since lifts of elements of $\hat{\Gamma}(2)$ in $\hat{\Gamma}(\hat{S})$ commute with the covering transformations $G$, the canonical lift of a power of a twist in $\Gamma(2)$ to $\Gamma(\hat{S})$ determines uniquely the lifts of all elements in its conjugacy class inside the profinite group $\hat{\Gamma}(2)$.

Now, the identity above implies the identity $\tau_{f'(\gamma_0)}^2 \tau_{f'(\gamma_1)}^{-2} = \tau_{\gamma_0}^2 \tau_{\gamma_1}^{-2} s_{u,2}$, where $\gamma_i$, for $i = 0, 1$, are non-separating s.c.c. and $u$ is a separating s.c.c. on $S_{2g+n+1}$. Let $\tilde{f}'$ be a lift to $\hat{\Gamma}(\hat{S})$ of the given $f' \in \hat{\Gamma}_{2g+n+1}$, then this identity, lifts to:

$$\prod_{\gamma_0 \in \rho_G^{-1}(\gamma_0)} \tau_{\gamma_0} \prod_{\gamma_1 \in \rho_G^{-1}(\gamma_1)} \tau_{\gamma_1}^{-1} = \prod_{\alpha \in \rho_G^{-1}(\alpha)} \tau_\alpha \prod_{\gamma \in \rho_G^{-1}(\gamma_1)} \tau_{\gamma_1}^{-1} \prod_{\alpha \in \rho_G^{-1}(\alpha)} \tau_\alpha^{2k}.$$

Projecting to the group $\text{Sp}(H_1(\hat{S}, \hat{\mathbb{Z}}))$, from the unicity of the Jordan normal form, it follows that $k = 0$.

Thus, by Lemma 2.11, given $f \in \hat{\Gamma}_{g,n+1}$ and an element of $L$, represented by the the s.c.c. $\gamma$, such that $f([\gamma]) = [\gamma]$ in $L$, we know that for any lift $\tilde{f}$ of $f$ in $\hat{\Gamma}_{g,n}^U$, it holds:

$$\tilde{f}(\tau_{\gamma_0} \tau_{\gamma_1}^{-1}) \tilde{f}^{-1} = \tau_{\gamma_0} \tau_{\gamma_1}^{-1}.$$

Let us show how the above identity implies that, in the profinite set $\hat{\Gamma}_{g,n+1}/\overline{\langle \text{im} \ u \rangle}$, it holds $\tilde{f}(s_i([\gamma])) = s_i([\gamma])$, for $i = 0, 1$. This will complete the proof of Lemma 2.9.
Let \( \{f_k\}_{k \in \mathbb{N}} \) be a sequence of elements of \( \Gamma_{g,n}^U \) converging to \( f \) inside \( \hat{\Gamma}_{g,n}^U \) and let us choose, as representatives for the \( f_k \), homeomorphism of \( S_{g,n}^U \) which restrict to the identity on a fixed closed neighborhood \( V \) of \( E \) and which we denote also by \( f_k \). For all \( k \in \mathbb{N} \), then \( f_k(\gamma) \) is a s.c.c. on \( S_{g,n} \), whose support inside \( V \) coincides with that of \( \gamma \).

The group \( \Gamma_{g,n}^U \) acts on the sets \( \mathcal{L} \) and \( \Pi_{g,n+1}/(\text{inn } u) \) via the natural epimorphism \( \Gamma_{g,n}^U \to \Gamma_{g,n+1} \). Therefore, the maps \( s_i \) are \( \Gamma_{g,n}^U \)-equivariant as well and it holds \( s_i([f_k(\gamma)]) = f_k(s_i([\gamma])) \), for \( i = 0, 1 \) and all \( k \in \mathbb{N} \).

The fact that \( f \), and then its lift \( f \), fixes \([\gamma]\) simply means that the sequence \( \{[f_k(\gamma)]\}_{k \in \mathbb{N}} \) converges to \([\gamma]\) in the profinite group \( \hat{\Pi}_{g,n} \).

In order to simplify the notations, let \( \gamma_{k,i} \) denote the s.c.c. \( f_k(\gamma)_i \), for \( i = 0, 1 \), obtained from the s.c.c. \( f_k(\gamma) \) by means of the procedure explained above, for all \( k \in \mathbb{N} \). So, in particular, we have that \( i([f_k(\gamma)]) = \tau_{\gamma_{0,k}}^{-1}. \)

By the hypothesis above, then, the sequence \( \{\tau_{\gamma_{0,k}}^{-1}\}_{k \in \mathbb{N}} \) of elements of \( \Gamma_{g,n}^U \) converges to \( i([\gamma]) = \tau_{\gamma_0}^{-1} \) in the profinite topology of \( \Gamma_{g,n}^U \). We claim that this implies that the element \( f \) acts trivially on \( s_i([\gamma]) \), for \( i = 0, 1 \). In the argument given below, in order to simplify the notations, we just consider the case \( i = 0 \).

Let \( Q \) be a point in \( V \setminus E \subset S_{g,n}^U \) and let \( \theta \) be a simple path from the tangential base point \( O \) of \( \Pi_{g,n+1} \) to \( Q \) contained in the neighborhood \( V \) of \( E \) and crossing transversally \( \gamma_0 \) only once in a point \( R \) contained in the circle \( \partial D \). As remarked above, inside \( V \), the s.c.c. \( \gamma_{k,0} \) and \( \gamma_0 \) have the same support. Therefore, it holds \( \gamma_{k,0} \cap \theta = \gamma_0 \cap \theta = R \), for all \( k \in \mathbb{N} \).

Let \( \beta' \) be a loop on \( S_{g,n}^U \), with base point \( Q \), disjoint from \( E \cup \gamma_0 \) and meeting the path \( \theta \) only in \( Q \). Thus, the action of \( \tau_{\gamma_0}^{-1} \) on \( \beta' \) is trivial. Since the sequence \( \{\tau_{\gamma_{0,k}}^{-1}\}_{k \in \mathbb{N}} \) converges to \( \tau_{\gamma_0}^{-1} \) in \( \hat{\Gamma}_{g,n}^U \), it holds, in \( \hat{\Pi}_{g,n+1} \):

\[
\lim_{k \to \infty} \tau_{\gamma_{0,k}}^{-1}([\beta']) = [\beta'].
\]

Let then \( \beta := \theta \beta' \theta^{-1} \) and \( \beta_k := \theta(\tau_{\gamma_{0,k}}^{-1}(\beta'))\theta^{-1} \), for \( k \in \mathbb{N} \), be the loops on \( S_{g,n+1} \) with base point \( O \), obtained changing the base point by means of the path \( \theta \). By the above identity, the sequence \( \{[\beta_k]\}_{k \in \mathbb{N}} \) converges to \([\beta]\) in the profinite topology of \( \hat{\Pi}_{g,n+1} \). Moreover, it holds:

\[
\tau_{\gamma_{0,k}}^{-1}(\beta) = x_k \theta(\tau_{\gamma_{0,k}}^{-1}(\beta'))\theta^{-1} x_k^{-1} = x_k \beta_k x_k^{-1},
\]

where \( x_k := e_{\gamma_{0,k}}e^{-1} \) with \( e \) the sub-path of \( \theta \) going from \( O \) to \( R \). Indeed, the loop \( \beta \) crosses the union of the s.c.c. \( \gamma_{k,0} \) and \( \gamma_1 \) in the same points at which \( \beta' \) does plus the point \( R \in \gamma_{k,0} \). Thus, in \( \Pi_{g,n+1} \), it holds \( \tau_{\gamma_{0,k}}^{-1}([\beta]) = \text{inn } [x_k]([\beta_k]) \).

So, passing to the limit in the profinite group \( \hat{\Pi}_{g,n+1} \), we get:

\[
\lim_{k \to \infty} [x_k]([\beta]) = \lim_{k \to \infty} [x_k]([\beta_k]) = \lim_{k \to \infty} \tau_{\gamma_{0,k}}^{-1}([\beta]) = \tau_{\gamma_0}^{-1}([\beta]) = \text{inn } [x]([\beta]),
\]

where \( x := e_{\gamma_0}e^{-1} \). Indeed, the loop \( \beta \) crosses the union of the s.c.c. \( \gamma_0 \) and \( \gamma_1 \) only in the point \( R \in \gamma_0 \).
So, we have proved that the sequence \{\text{inn}[x_k](|[\beta]|)\} converges to \text{inn}[x](|[\beta]|), for all \beta of the type above. As we remarked, by Lemma 2.6, this implies that the sequence \{[x_k]\} converges to \[ \text{inn} u \] in the profinite group \( \hat{\Pi}_{g,n+1} \).

Since \[ [x_k] \equiv \bar{f}_k(s_0([\gamma])) \] and \[ [x] \equiv s_0([\gamma]) \], in the orbit set \( \hat{\Pi}_{g,n+1} / \langle \text{inn} u \rangle \), it follows that there is a series of identities:

\[
\bar{f}(s_0([\gamma])) = \lim_{k \to \infty} \bar{f}_k(s_0([\gamma])) = s_0([\gamma]),
\]

thus completing the proof of Lemma 2.9.

By Lemma 2.9, a given \( f \in \hat{\Gamma}_{g,n+1} \), such that \( \Phi(f) \) acts trivially on \( \hat{\Pi}_{g,n} \), lifts to an automorphism of \( \hat{\Pi}_{g,n+1} \) which acts by conjugation by powers of \( u \) on all non-separating s.c.c.. So, the following lemma completes the proof of Theorem 2.5.

**Lemma 2.12.** Let \( f \in \text{Aut}(\hat{\Pi}_{g,n+1}) \), for \( g \geq 1 \), be such that its restriction to any non-separating s.c.c. \( \gamma \in \Pi_{g,n+1} \) is conjugation by \( u^\gamma \), for some \( k, \gamma \in \mathbb{Z} \). Then, \( f \in \langle \text{inn} u \rangle \).

**Proof.** For \( g \geq 1 \), the fundamental group \( \Pi_{g,n+1} \) has a generating set \( B \) consisting of non-separating s.c.c. such that any two elements of \( B \) either have trivial geometric intersection on \( S_{g,n+1} \) or intersect transversally once (in the base point of the fundamental group). The lemma follows if we prove that, for all \( x, y \in B \), it holds \( k_x = k_y \).

If \( x, y \in B \) are such that \( |x \cap y| = 1 \), then either \( xy \) or \( x^{-1}y \) is representable by a non-separating s.c.c.. Otherwise, if \( |x \cap y| = 0 \), there is a \( z \in \mathcal{L} \) such that, in each of the subsets \( \{xz, x^{-1}z\} \) and \( \{yz, y^{-1}z\} \) of \( \Pi_{g,n+1} \), there is at least one element representable by a non-separating s.c.c.. In any case, given a \( z \in \mathcal{L} \) such that \( xz \) is representable by a non-separating s.c.c., it holds \( f(xz) = \text{inn}(u^{k_{xz}})(xz) \) and then:

\[
\text{inn} (u^{k_{xz}})(xz) = u^{k_{xz}xu^{k_x}zu^{-k_z}} = u^{k_{xz}xu^{k_x}zu^{-k_z}}.
\]

Let \( S' \) be the topological surface obtained from \( S_{g,n+1} \) contracting the s.c.c. \( z \) to a point and denote by \( \Pi' \) its fundamental group. The natural map \( S_{g,n+1} \to S' \) induces an epimorphism \( p: \Pi_{g,n+1} \to \Pi' \), where \( \Pi' \) is the profinite completion of the free group \( \Pi' \). The push-forward of the above identity by \( p \) is the identity \( u^{k_xxu^{-k_x}} = u^{k_{xz}xu^{-k_z}} \).

By the same argument of the proof of Lemma 2.6, the centralizer of \( x \) in \( \Pi' \) is the closed subgroup generated by \( x \). Since the latter group has trivial intersection with the closed subgroup spanned by \( u \) in \( \Pi' \), it follows that \( k_x = k_{xz} \). Inverting the roles of \( x \) and \( z \), we see that \( k_z = k_{xz} \), as well, and so that \( k_x = k_z \).

Similarly, it is possible to prove that \( k_y = k_z \) and so that \( k_x = k_y \) for all \( x, y \in B \).

By Theorem 2.5, we can now define unambiguously the geometric profinite completion of \( \Pi_{g,n} \) to be the group \( \hat{\Gamma}_{g,n} \), for \( 2g - 2 + n > 0 \). An important corollary of the theorem is also the following:
Corollary 2.13. For $2g - 2 + n > 0$, the geometric profinite completion $\hat{\Gamma}_{g,n+1}$ has trivial center.

Asada in [1] has proved the genus 1 case of the subgroup congruence conjecture:

Theorem 2.14 (Asada). It holds $\hat{\Gamma}_{1,n} \equiv \hat{\Gamma}_{1,n}$, for $n \geq 1$.

Let us remark, however, that the natural epimorphism $\hat{\Gamma}_{1,1} \to \text{SL}_2(\hat{\mathbb{Z}})$ is not injective (see §8.8 in [14] for details). This is not a surprise, since $S_1$ is not a hyperbolic surface.

In the next section, we will also provide an alternative proof of Asada’s Theorem.

3 The hyperelliptic modular group

In this section, we are going to prove the results announced in the introduction. The main feature of the moduli stack of $n$-pointed, genus $g$ smooth hyperelliptic complex curves $\mathcal{H}_{g,n}$ is that it can be described in terms of moduli of pointed genus 0 curves. More precisely, there is a natural $\mathbb{Z}/2$-gerbe $\mathcal{H}_g \to \mathcal{M}_{0,[2g+2]}$, for $g \geq 2$, defined assigning, to a genus $g$ hyperelliptic curve $C$, the genus zero curve $C/\iota$, where $\iota$ is the hyperelliptic involution of $C$, labeled by the branch points of the cover $C \to C/\iota$. In the genus 1 case, there is a $\mathbb{Z}/2$-gerbe $\mathcal{M}_{1,1} \to \mathcal{M}_{0,1[3]}$, where, by the notation “1[3]”, we mean that one label is distinguished while the others are unordered. For $2g - 2 + n > 0$, there is also a natural representable morphism $\mathcal{H}_{g,n+1} \to \mathcal{H}_{g,n}$, forgetting the $(n + 1)$-th labeled point, which is isomorphic to the universal $n$-punctured curve over $\mathcal{H}_{g,n}$. So, the fiber above an arbitrary closed point $x \in \mathcal{H}_{g,n}$ is diffeomorphic to $S_{g,n}$ and its fundamental group is isomorphic to $\Pi_{g,n}$. These morphisms induce, on topological fundamental groups, the short exact sequences, for $g \geq 2$:

$$1 \to \mathbb{Z}/2 \to H_g \to \Gamma_{0,[2g+2]} \to 1 \quad \text{and} \quad 1 \to \Pi_{g,n} \to H_{g,n+1} \to H_{g,n} \to 1.$$  

Similarly, for the algebraic fundamental groups, there are short exact sequences:

$$1 \to \mathbb{Z}/2 \to \hat{H}_g \to \hat{\Gamma}_{0,[2g+2]} \to 1 \quad \text{and} \quad 1 \to \hat{\Pi}_{g,n} \to \hat{H}_{g,n+1} \to \hat{H}_{g,n} \to 1.$$  

The outer representation $\hat{\rho}_{g,n} : \hat{H}_{g,n} \to \text{Out}(\hat{\Pi}_{g,n})$, induced by the last of the above short exact sequences, is the algebraic monodromy representation of the punctured universal curve over $\mathcal{H}_{g,n}$. As already remarked, the congruence subgroup property for $H_{g,n}$ is equivalent to the faithfullness of $\hat{\rho}_{g,n}$.

Let us prove some general properties of the groups $H_{g,n}$. For definitions and elementary properties of good groups, we refer to exercise 1 in Section 2.6 of [16]. From the above exact sequences, it then follows immediately:

**Proposition 3.1.** For $2g - 2 + n > 0$ and $g \geq 1$, the group $H_{g,n}$ is good.

It is well known that the centralizer of a finite index subgroup $U$ of $H_{g,n}$, for $g \geq 2$ and $n = 0$ or $g = 1$ and $n = 1$, is spanned by the hyperelliptic involution $\iota$ while it is trivial for $g \geq 2$ and $n \geq 1$ or $g = 1$ and $n \geq 2$. An analogue statement holds for the profinite completion $\hat{H}_{g,n}$.
Proposition 3.2. Let $U$ be an open subgroup of $\hat{H}_{g,n}$, for $2g - 2 + n > 0$. Then, for $g \geq 2$ and $n = 0$ or $g = 1$ and $n = 1$, the centralizer of $U$ in $\hat{H}_{g,n}$ is spanned by the hyperelliptic involution. In all the other cases, the centralizer of $U$ in $\hat{H}_{g,n}$ is trivial.

Proof. Let us consider first the cases $g \geq 2$ and $n = 0$ or $g = 1$ and $n = 1$. It is clearly enough to prove that for any open subgroup $U$ of $\hat{H}_{g,n}$, which contains the hyperelliptic involution $\iota$, the center $Z(U)$ is equal to the subgroup spanned by $\iota$.

The center of any open subgroup of $\hat{\Gamma}_{0,[2g+2]}$ is trivial. From the exact sequences:

$$1 \to \mathbb{Z}/2 \cdot \iota \to \hat{H}_g \to \hat{\Gamma}_{0,[2g+2]} \to 1 \quad \text{and} \quad 1 \to \mathbb{Z}/2 \cdot \iota \to \hat{\Gamma}_{1,1} \to \hat{\Gamma}_{0,[4]},$$

it then follows that $Z(U) = \langle \iota \rangle$.

For the cases $g \geq 2$ and $n \geq 1$ or $g = 1$ and $n \geq 2$, we have to prove that the center is trivial for any open subgroup $U$ of $\hat{H}_{g,n}$. By induction on $n$, thanks to the short exact sequences:

$$1 \to \hat{\Pi}_{g,n-1} \to \hat{H}_{g,n} \to \hat{H}_{g,n-1} \to 1,$$

it is enough to prove the proposition for the cases $g \geq 2$, $n = 1$ and $g = 1$, $n = 2$.

From the above short exact sequence, we then see that the center $Z(U)$, if non-trivial, projects to the subgroup of $\hat{H}_{g,n-1}$ spanned by the hyperelliptic involution.

In this case, the subgroup $Z(U) \cdot \hat{\Pi}_{g,n-1}$ of $\hat{H}_{g,n}$ would be generated by a hyperelliptic involution $\mu$ in $\hat{H}_{g,n}$ and $\hat{\Pi}_{g,n-1}$. So, $Z(U)$ would be generated by a conjugate $f \mu f^{-1}$ for some $f \in \hat{\Pi}_{g,n-1}$. Let $U' := fUf^{-1}$, then it is clear that $Z(U') = \langle \mu \rangle$.

Hence, such $\mu$ would commute with the elements of the finite index subgroup $U' \cap \Pi_{g,n-1}$ of $\Pi_{g,n-1}$. By the simple topological description of a hyperelliptic involution, there is a simple loop $\gamma \in \Pi_{g,n-1}$ such that $\mu(\gamma) = \gamma^{-1}$.

As already remarked in §2 if we identify $\hat{\Pi}_{g,n-1}$ with its image in $\hat{H}_{g,n}$, then $\mu(\gamma) = \mu \gamma \mu^{-1}$. For some $k > 0$, it holds $\gamma^k \in U' \cap \Pi_{g,n-1}$ and then:

$$\gamma^{-k} = \mu(\gamma^k) = \mu \gamma^k \mu^{-1},$$

which contradicts the fact that $Z(U') = \langle \mu \rangle$. Therefore, it holds $Z(U) = \{1\}$.

We call a finite index subgroup $H^\lambda$ of $H_{g,n}$ a level of $H_{g,n}$ and the corresponding étale cover $H^\lambda \to H_{g,n}$ a level structure over $H_{g,n}$. Geometric levels of $H_{g,n}$ are defined by means of the monodromy representation $\rho: H_{g,n} \to \text{Out}(\Pi_{g,n})$. For a characteristic subgroup $\Pi^\lambda$ of $\Pi_{g,n}$, the geometric level $H^\lambda$ is defined to be the kernel of the induced representation $\rho^\lambda: H_{g,n} \to \text{Out}(\Pi_{g,n}/\Pi^\lambda)$. The abelian level $H(m)$ of order $m \geq 2$ is then defined to be the kernel of the representation $\rho_{(m)}: H_{g,n} \to \text{Sp}_{2g}(\mathbb{Z}/m)$ and we let $H^{(m)}$ be the corresponding abelian level structure.

There is a standard procedure to simplify the structure of an algebraic stack $X$ by erasing a generic group of automorphisms $G$ (see, for instance, [15]). The algebraic stack thus obtained is usually denoted by $X/\!/G$. So, the natural map $H_g \to \mathcal{M}_{0,[2g+2]}$ yields an isomorphism $H_g/\!/\langle \iota \rangle \cong \mathcal{M}_{0,[2g+2]}$. A natural question is then which level structure over $H_g$ corresponds to the Galois étale cover $\mathcal{M}_{0,2g+2} \to \mathcal{M}_{0,2g+2}$. 
Proposition 3.3. For \( g \geq 2 \), there is a natural isomorphism \( \mathcal{H}(2)/\langle \iota \rangle \cong \mathcal{M}_{0,2g+2} \).

Proof. The groups \( \mathcal{H}_g/\langle \iota \rangle \) and \( \Gamma_{0,[2g+2]} \) are naturally isomorphic. By means of this isomorphism, the normal subgroup \( \Gamma_{0,[2g+2]} \triangleleft \Gamma_{0,[2g+2]} \) identifies with the subgroup of \( \mathcal{H}_g/\langle \iota \rangle \) spanned by squares of Dehn twists along non-separating s.c. c. on \( S_{g,n} \). Squares of Dehn twists all act trivially on homology with \( \mathbb{Z}/2 \)-coefficients. Therefore, \( \Gamma_{0,2g+2} \) identifies with a normal finite index subgroup of \( \mathcal{H}(2)/\langle \iota \rangle \). So, there is a commutative diagram with exact rows:

\[
1 \to \Gamma_{0,2g+2} \to \Gamma_{0,[2g+2]} \to S_{2g+2} \to 1
\]

At this point, observe that the representation \( \rho: S_{2g+2} \to \text{PGL}_{2g}(\mathbb{Z}/2) \) is induced by the permutation of \( 2g + 2 \) points in general position in the projective space \( \mathbb{P}^{2g-1}_{\mathbb{Z}/2} \) and so is faithful. Thus, the injection \( \Gamma_{0,2g+2} \hookrightarrow \mathcal{H}(2)/\langle \iota \rangle \) is actually an isomorphism and then \( \Phi \) is an isomorphism as well.

Remark 3.4. Likewise, it is not hard to prove that, for the abelian level structure \( \mathcal{M}(2) \) over \( \mathcal{M}_{1,1} \), there is a natural isomorphism \( \mathcal{M}(2)/\langle \iota \rangle \cong \mathcal{M}_{0,4} \), where \( \iota \) here denotes the generic elliptic involution.

From now on, we will mostly stick to moduli spaces of hyperelliptic curves \( \mathcal{H}_{g,n} \), with \( g \geq 2 \), and leave to the reader the formulation and the proof of the analogous statements for \( g = 1, n \geq 1 \).

Let \( C_g \to \mathcal{H}_g \), for \( g \geq 2 \), be the universal curve. Removing Weierstrass points from its fibers, we obtain a \((2g + 2)\)-punctured, genus \( g \) curve \( C_0 \to \mathcal{H}_g \). A weak version of the congruence subgroup property for \( \mathcal{H}_g \) is then the assertion that the algebraic monodromy representation, associated to \( C_0 \to \mathcal{H}_g \), is faithful:

\[
\hat{\rho}_0: \hat{\pi}_1(\mathcal{H}_g, x) \to \text{Out}(\hat{\pi}_1(C_0, \overline{x})),
\]

where \( C_0 \) is the fiber of \( C_0 \to \mathcal{H}_g \) over the closed point \( x \). Let us show how this assertion reduces to Corollary 2.7.

Let us denote by \( C^\lambda \to \mathcal{H}^\lambda \) the pull-back of the universal curve \( C_g \to \mathcal{H}_g \) to the level structure \( \mathcal{H}^\lambda \to \mathcal{H}_g \) and by \( C^\lambda_0 \to \mathcal{H}^\lambda \) the pull-back of the punctured curve \( C_0 \to \mathcal{H}_g \).

By Proposition 3.3, there is a natural étale Galois morphism \( \mathcal{H}(4) \to \mathcal{M}_{0,2g+2} \) which is also representable, since \( \iota \notin H(4) \). Let \( R \to \mathcal{H}(4) \) be the pull-back of the universal \((2g + 2)\)-punctured, genus 0 curve \( \mathcal{M}_{2g+3} \to \mathcal{M}_{2g+2} \). There is then a commutative diagram:

\[
C^0_0 \xrightarrow{\psi} \mathcal{R} \xrightarrow{\rho} \mathcal{H}(4),
\]

where \( \psi \) is the étale, degree 2 map which, fiberwise, is the quotient by the hyperelliptic involution. The algebraic monodromy representation \( \tilde{\pi}_1(\mathcal{H}(4), a) \to \text{Out}(\tilde{\pi}_1(\mathcal{R}_a, \overline{a})) \), associated to the rational curve \( \mathcal{R} \to \mathcal{H}(4) \), is faithful by Corollary 2.7. Then, by Lemma 8 in
the algebraic monodromy representation \( \hat{\pi}_1(\mathcal{H}^{(4)}, a) \to \text{Out}(\hat{\pi}_1(C_0^{(4)}, \bar{a})) \), associated to the curve \( C_0^{(4)} \to \mathcal{H}^{(4)} \), is faithful as well, where \( C_0 \) denotes the fiber over the closed point \( a \). This immediately implies the faithfulness of the representation \( \hat{\rho}_0 \).

We can now state and prove the main result of the paper:

**Theorem 3.5.** For \( 2g - 2 + n > 0 \) and \( g \geq 1 \), let \( \mathcal{H}_{g,n} \) be the moduli stack of \( n \)-pointed, genus \( g \) hyperelliptic complex curves. The universal algebraic monodromy representation \( \hat{\rho}_{g,n}: \hat{\pi}_1(\mathcal{H}_{g,n}) \to \text{Out}(\hat{\Pi}_{g,n}) \), associated to the universal \( n \)-punctured, genus \( g \) hyperelliptic curve \( \mathcal{H}_{g,n+1} \to \mathcal{H}_{g,n} \), is faithful.

**Proof.** The proof of Theorem 3.5 consists of two steps. In the first, we show that the faithfulness of \( \hat{\rho}_{g,n} \), for a given \( g \geq 2 \) and all \( n \geq 0 \) or for a given \( g = 1 \) and all \( n \geq 1 \), can be deduced from that of \( \hat{\rho}_{g,n'} \), for any given \( n' \). In the second, we prove that \( \hat{\rho}_{g,2g+2} \) is faithful for all \( g \geq 1 \). The first step is accomplished, by induction, in the following lemma:

**Lemma 3.6.** Let \( g \geq 2 \) and \( n \geq 0 \) or \( g = 1 \) and \( n \geq 1 \). Then, the monodromy representation \( \hat{\rho}_{g,n} \) is faithful if and only if \( \hat{\rho}_{g,n+1} \) is.

**Proof.** By Theorem 2.5, there is a commutative diagram with exact rows:

\[
\begin{array}{cccccc}
1 & \to & \hat{\Pi}_{g,n} & \to & \hat{H}_{g,n+1} & \to \hat{H}_{g,n} & \to 1 \\
& \parallel & \downarrow \hat{\rho}_{g,n+1} & & \downarrow \hat{\rho}_{g,n} & & \\
1 & \to & \hat{\Pi}_{g,n} & \to & \hat{\Gamma}_{g,n+1} & \to \hat{\Gamma}_{g,n} & \to 1
\end{array}
\]

and the lemma follows immediately.

**Lemma 3.7.** For \( g \geq 1 \), the algebraic monodromy representation \( \hat{\rho}_{g,2g+2} \) is faithful.

**Proof.** Here, as usual, for notational reason, we assume \( g \geq 2 \) and leave to the reader the transposition of the argument to the genus 1 case.

The universal curve \( C^{(2)} \to \mathcal{H}^{(2)}_g \) is endowed with \( 2g + 2 \) ordered sections, corresponding to the Weierstrass points on the fibers. So, by the universal property of \( \mathcal{H}^{(2)}_{g,n} \), there is a morphism \( s: \mathcal{H}^{(2)}_g \to \mathcal{H}^{(2)}_{g,2g+2} \) which is a section of the natural projection \( p: \mathcal{H}^{(2)}_{g,2g+2} \to \mathcal{H}^{(2)}_g \) (forgetting the labels). The morphism \( p \) is smooth and its fiber above a closed point \( [C] \in \mathcal{H}_g \) is the configuration space of \( 2g + 2 \) points on the curve \( C \). Let us denote by \( S_g(n) \) the configuration space of \( n \) points on the compact Riemann surface \( S_g \) and by \( \Pi_g(n) \) its fundamental group. Then, all fibers of \( p \) above closed points of \( \mathcal{H}_g \) are diffeomorphic to \( S_g(n) \). Therefore, the fundamental group \( H_{g,2g+2}(2) \) of \( \mathcal{H}^{(2)}_{g,2g+2} \) fits in the short exact sequence:

\[
1 \to \Pi_g(2g + 2) \to H_{g,2g+2}(2) \to H_g(2) \to 1,
\]

which is splitted by \( s_*: H_g(2) \to H_{g,2g+2}(2) \). Moreover, since the space \( S_g(n + 1) \) is fibered in \( n \)-punctured, genus \( g \) curves over \( S_g(n) \), for all \( n \geq 0 \), there is a short exact sequence:

\[
1 \to \Pi_g(n) \to \Pi_g(n + 1) \to \Pi_g(n) \to 1.
\]
From Theorem 2.5 and a simple induction on $n$, it follows that the profinite completion $\hat{\Pi}_g(n)$ embeds in $\hat{\Gamma}_{g,n}$ (this is essentially the same argument of Asada in Theorem 1, [1], where this was first proved). Therefore, passing to profinite completions, we get the short exact sequences:

$$1 \to \hat{\Pi}_g(2g + 2) \to \hat{H}_{g,2g+2}(2) \to \hat{H}_g(2) \to 1,$$

$$1 \to \hat{\Pi}_g(n) \to \hat{\Pi}_g(n + 1) \to \hat{\Pi}_g(n) \to 1.$$ 

The former is split by $\hat{s}_*: \hat{H}_g(2) \to \hat{H}_{g,2g+2}(2)$. So there is an isomorphism:

$$\hat{H}_{g,2g+2}(2) \cong \hat{\Pi}_g(2g + 2) \rtimes \hat{H}_g(2).$$

In order to prove that the algebraic monodromy representation $\hat{\rho}_{g,2g+2}$ is faithful, it is enough to show that this holds for its restriction to $\hat{H}_{g,2g+2}(2)$, which we denote also by $\hat{\rho}_{g,2g+2}$. But we have already seen that $\hat{\rho}_{g,2g+2} \circ \hat{s}_* = \hat{\rho}_0: \hat{H}_g(2) \to \text{Out}(\hat{\Pi}_{g,2g+2})$ is faithful and, as remarked above, the restriction of $\hat{\rho}_{g,2g+2}$ to the normal subgroup $\hat{\Pi}_g(2g + 2)$ of $\hat{H}_{g,2g+2}(2)$ is faithful as well. So, Lemma 3.7 follows, if we prove that:

$$\hat{\rho}_{g,2g+2}(\hat{\Pi}_g(2g + 2)) \cap \hat{\rho}_{g,2g+2}(\hat{s}_*(\hat{H}_g(2))) = \{1\} \quad (\ast).$$

The subgroup $s_*(\hat{H}_g(2))$ of $\hat{H}_{g,2g+2}(2)$ centralizes the hyperelliptic involution $s_*(\iota) \in \hat{H}_{g,2g+2}(2)$. Passing to profinite completions, the subgroup $s_*(\hat{H}_g(2))$ of $\hat{H}_{g,2g+2}(2)$ then centralizes the hyperelliptic involution $s_*(\iota) \in \hat{H}_{g,2g+2}(2)$. It is clear that $\hat{\rho}_{g,2g+2}(\hat{s}_*(\iota)) \neq 1$. Hence, since $\hat{\Pi}_g(2g + 2)$ is torsion free:

$$\hat{\rho}_{g,2g+2}(\hat{\Pi}_g(2g + 2) \cdot s_*(\iota)) \cong \hat{\Pi}_g(2g + 2) \cdot s_*(\iota) \cong \hat{\Pi}_g(2g + 2) \rtimes \mathbb{Z}/2.$$

All elements of $\hat{\rho}_{g,2g+2}(\hat{s}_*(\hat{H}_g(2)))$ commute with $\hat{\rho}_{g,2g+2}(s_*(\iota))$. So, in order to prove the identity $(\ast)$, it is enough to show that no element of $\hat{\rho}_{g,2g+2}(\hat{\Pi}_g(2g + 2))$ does.

From item (ii) of Lemma 2.1 in [10], it follows that a primitive finite subgroup of the algebraic fundamental group of a hyperbolic orbi-curve is self-normalizing.

For all $0 \leq n \leq 2g + 2$, a given hyperelliptic involution $\iota' \in H_{g,n+1}$ and $\hat{\Pi}_{g,n}$ span inside of $\hat{H}_{g,n+1}$ a group isomorphic to the algebraic fundamental group of an $n$-punctured, genus $g$ hyperelliptic orbi-curve $[C/\iota']$. In particular, by Lemma 2.1 in [10], there is no element of $\hat{\Pi}_{g,n}$ with which $\iota'$ commutes.

The short exact sequences $1 \to \hat{\Pi}_{g,n} \to \hat{\Pi}_g(n + 1) \to \hat{\Pi}_g(n) \to 1$ and a simple induction on $n \geq 0$ then imply that $s_*(\iota)$ does not commute with any given element of $\hat{\Pi}_g(2g + 2)$, as claimed above. This completes the proof of Lemma 3.7 and then that of Theorem 3.5.

\[\square\]
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