The reduction of Laplace equation in certain Riemannian spaces and the resulting Type II hidden symmetries

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Abstract

We prove a general theorem which allows the determination of Lie symmetries of Laplace equation in a general Riemannian space using the conformal group of the space. Algebraic computing is not necessary. We apply the theorem in the study of the reduction of Laplace equation in certain classes of Riemannian spaces which admit a gradient Killing vector, a gradient Homothetic vector and a special Conformal Killing vector. In each reduction we identify the source of Type II hidden symmetries. We find that in general the Type II hidden symmetries of Laplace equation are directly related to the transition of the CKVs from the space where the original equation is defined to the space where the reduced equation resides. In particular we consider the reduction of Laplace equation (i.e. the wave equation) in Minkowski space and obtain the results of all previous studies in a straightforward manner. We consider the reduction of Laplace equation in spaces which admit Lie point symmetries generated from a non gradient HV and a proper CKV and we show that the reduction with these vectors does not produce Type II hidden symmetries. We apply the results to General Relativity and consider the reduction of Laplace equation in locally rotational symmetric space times (LRS) and in algebraically special vacuum solutions of Einstein’s equations which admit a homothetic algebra acting simply transitively. In each case we determine the Type II hidden symmetries.

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1 Introduction

Lie point symmetries provide a systematic method to facilitate the solution of differential equations (DE) because they provide the first order invariants which can be used to reduce the DE. The reduction is different for ordinary differential equations (ODEs) and for partial differential equations (PDEs). In the case of ODEs the use of a Lie point symmetry reduces the order of the ODE by one while in the case of PDEs reduces by one the number of independent and dependent variables, but not the order of the PDE. A common characteristic in the reduction of both cases is that the Lie point symmetry which is used for the reduction is not admitted as such by the reduced DE, it is "lost". In this procedure it is possible that the reduced equation has more
Lie point symmetries than the original equation. These new point symmetries have been called Type II hidden symmetries and have been the subject of numerous papers during the recent years (e.g. [1, 2, 3, 4, 5, 6, 7, 8]).

The importance of Type II hidden symmetries is that they can be used to reduce further the reduced equation. Concerning the origin of Type II hidden symmetries, both for ODEs and PDEs, it has been shown that they can be viewed as having two sources. Either the point and the nonlocal / generalized symmetries of a given higher order equation, or the point symmetries of a variety of higher order DEs which reduce to this particular DE (see [5, 9, 10, 11] and references therein).

In the present paper we study the reduction and the consequent existence of Type II hidden symmetries of Laplace equation in certain classes of Riemannian spaces. In particular we prove a general theorem which allows us to study the reduction of Laplace equation in a general Riemannian space using the conformal group of the space. There is no need to employ algebraic computing and it is enough to work with pure geometric arguments. We note that the use of algebraic computing for higher dimensions and more complex metrics can be rather inapplicable, whereas the theorem applies irrespectively of the (finite) dimension and the complexity of the metric.

The Laplace equation in a general Riemannian space

\[
\frac{1}{\sqrt{|g|}} \partial_x \left( \sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j} \right) u(x^k) = 0
\]

has two Lie point symmetries, the \( u \partial_u \) and the \( b(x^k) \partial_u \) where \( b(x^k) \) is a solution of Laplace equation. These two symmetries are not useful for reduction. In order to find ‘sound’ reductions of Laplace equation we have to consider Riemannian spaces which admit some type of symmetry(ies) (these symmetries are not Lie point symmetries and are called collineations). Indeed it has been shown [12] that the Lie point symmetries of Laplace equation in a Riemannian space are generated from the conformal algebra of the space. Therefore in order to have more Lie point symmetries of DEs, hence the possibility of the existence of Type II hidden symmetries, we have to work in spaces which admit a conformal algebra.

The structure of the paper is as follows. In section 2 we give the basic definitions of Lie point symmetries and certain facts concerning the conformal algebra of a Riemannian space. In section 3 we prove a general theorem concerning the Lie point symmetries of the Poisson equation in a Riemannian space. We also give some results which relate the Lie point symmetries of the Poisson, the Laplace and the Klein Gordon equations with the conformal algebra of the space. In section 4 we consider the reduction of Laplace equation in various general classes of spaces which admit a nonzero conformal algebra. In particular we consider (a) decomposable spaces - that is Riemannian spaces which admit a gradient Killing (equivalently constant) vector field (KV) - (b) spaces which admit a gradient homothetic vector field (HV) and (c) spaces which admit a special conformal Killing vector field (sp.CKV). In section 5 we apply the results of section 4 and determine the Type II symmetries of Laplace equation in four dimensional Minkowski spacetime and we recover and complete well known results [6]. In section 6 we consider the reduction of Laplace equation in an LRS spacetime, which is an important class of spacetimes in General Relativity. In order to study the reduction of Laplace equation by a non-gradient HV and a proper CKV we consider two further examples. In section 7 we consider the algebraically special vacuum solutions of Einstein’s equations on which a homothetic vector field acts transitively [14] and make the reduction using the Lie point symmetry generated by the nongradient HV. In section 8 we do the same in a conformally flat FRW type space which admits a homothetic vector field. We reemphasize that all results are derived in a purely geometric manner without the use of a computer package. However they have been verified.
with the libraries PDEtools and SADE \cite{15, 16} of Maple\footnote{\url{www.maplesoft.com}}. Finally in section \ref{sec:conclusions} we draw our conclusions.

## 2 Lie point symmetries of a PDE and CKVs of a Riemannian space

In this section, we give the definition of Lie point symmetries of a DE and the definition of conformal Killing vector fields (CKVs) of a Riemannian space.

### 2.1 Lie point symmetries of differential equations

A partial differential equation (PDE) is a function \( H = H (x^i, u, u_i, u_{ij}, ...) \) in the jet space \( \mathcal{B\mathcal{M}} (x^i, u, u_i, u_{ij}, ...) \) where \( x^i \) are the independent variables and \( u^A \) are the dependent variables. An infinitesimal point transformation

\[
\begin{align*}
\bar{x}^i &= x^i + \varepsilon \xi^i (x^k, u) \\
\bar{u} &= \bar{u} + \varepsilon \eta (x^k, u)
\end{align*}
\]  

is a Lie point symmetry of the PDE \( H (x^i, u, u_i, u_{ij}, ...) \) with generator

\[
X = \xi^i (x^k, u^B) \partial_i + \eta^A (x^k, u^B) \partial_B
\]

where \( \varepsilon \) is an infinitesimal parameter, if there exists a function \( \lambda \) such as the following condition holds \cite{17, 18}

\[
X^{[n]} H (x^i, u, u_i, u_{ij}, ...) = \lambda H (x^i, u, u_i, u_{ij}, ...) \mod H = 0
\]  

\( X^{[n]} \) is the \( n \)th prolongation of (4) defined as follows,

\[
X^{[n]} = X + \eta_{[i]} \partial_{u_i} + ... + \eta_{[ij...i_n]} \partial_{u_{ij...i_n}}
\]

where

\[
\eta_{[i]} = D_i \eta - u_{,j} D_i \xi^j \\
\eta_{[ij...i_n]} = D_{i_n} \eta_{[ij...i_{n-1}]} - u_{i_{j...k}} D_{i_n} \xi^k.
\]

From condition (5) one defines the Lagrange system

\[
\frac{dx^i}{\xi^i} = \frac{du}{\eta} = \frac{du_i}{\eta_{[i]}} = ... = \frac{du_{ij...i_n}}{\eta_{[ij...i_n]}}
\]

whose solution provides the characteristic functions

\[
W^{[0]} (x^k, u), \ W^{[1]} (x^k, u, u_i), ... , W^{[n]} (x^k, u, u_i, ..., u_{ij...i_n}).
\]

The solution \( W^{[n]} \) is called the \( n \)th order invariant of the Lie symmetry vector (4). These invariants are used in order to reduce the order of the PDE (for details see e.g. \cite{18}).
2.2 Collineations of Riemannian spaces

In the following $L_\xi$ denotes Lie derivative with respect to the vector field $\xi^i$.

A vector field $\xi^i$ is a CKV of a metric $g_{ij}$ if $L_\xi g_{ij} = 2\psi g_{ij}$. If $\psi = 0$ then $\xi^i$ is a KV, if $\psi_i = 0$ then $\xi^i$ is a HV and if $\psi_{ij} = 0$ then $\xi^i$ is a special CKV (sp.CKV) and also $\psi_i$ is a gradient KV or, equivalently, a constant vector field. A CKV which is neither of the above cases (i.e. $\psi_{ij} \neq 0$ ) is called a proper CKV.

Two metrics $g_{ij}, \bar{g}_{ij}$ are conformally related if $\bar{g}_{ij} = N^2 g_{ij}$ where the function $N^2$ is the conformal factor.

If $\xi^i$ is a CKV of the metric $\bar{g}_{ij}$ so that $L_\xi \bar{g}_{ij} = 2 \bar{\psi} \bar{g}_{ij}$ then $\xi^i$ is also a CKV of the metric $g_{ij}$, that is, $L_\xi g_{ij} = 2 \psi g_{ij}$ where the conformal factor $\psi = \bar{\psi} N^2 - \bar{N} N_i \xi^i$.

This means that two conformally related metrics have the same CKVs but different Killing/homothetic/Sp.CKVs. For example a KV of one is not in general a KV of the other and so on. This is an important observation which shall be useful in the following sections. In Appendix A we give the vector fields of the conformal algebra of the flat space in Cartesian coordinates. Details on the CKVs and their geometric properties can be found e.g. in [19]

3 Lie point symmetries of Laplace equation

In a general Riemannian space with metric $g_{ij}$ Poisson equation is

$$\Delta u - f(x^i, u) = 0 \quad (7)$$

where $\Delta$ is the Laplace operator $\Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right)$ and $q = q(t, x^k, u)$. Equation (7) can also be written

$$g^{ij} u_{ij} - \Gamma^i_{jk} u_j = f(x^i, u) \quad (8)$$

where $\Gamma^i = \Gamma^i_{jk} g^{jk}$ and $\Gamma^i_{jk}$ are the Christoffel Symbols of the metric $g_{ij}$. The Lie symmetries of the Poisson equation for $f = f(u)$ have been given in [12, 17]. Here we generalize the result of [12] by considering $f = f(x^i, u)$ and give a new concise proof.

**Theorem 1** The Lie point symmetries of the Poisson equation (7) are generated from the CKVs of the metric $g_{ij}$ of the $\Delta$-operator as follows

a) For $n > 2$, the Lie point symmetry vector is

$$X = \xi^i (x^k) \partial_i + \left( \frac{2-n}{2} \psi (x^k) u + a_0 u + b (x^k) \right) \partial_u \quad (9)$$

where $\xi^i (x^k)$ is a CKV with conformal factor $\psi (x^k)$ and the following condition holds

$$\frac{2-n}{2} \Delta \psi u + g^{ij} b_{ij} - \xi^k f_{,k} - \frac{2-n}{2} \psi u f_{,u} - \frac{n+2}{2} \psi f - b f_{,u} = 0 \quad (10)$$

b) For $n = 2$, the Lie point symmetry vector is

$$X = \xi^i (x^k) \partial_i + \left( a_0 u + b (x^k) \right) \partial_u \quad (11)$$

where $\xi^i$ is a CKV (i.e. $L_\xi g_{ij} = 2 \psi (x^k) g_{ij}$) and the following condition holds

$$g^{ij} b_{ij} - \xi^k f_{,k} - a_0 u f_{,u} + (a_0 - 2 \psi) f - b f_{,u} = 0 \quad (12)$$
where the function $b$ is solution of Laplace equation.

**Proof.** In [13] it has been shown that the Lie point symmetry conditions for the PDE of the form

$$A^{ij}u_{ij} - B^i(x,u)u_i - f(x,u) = 0$$  \hfill (13)

are as follows:

$$A^{ij}(a_{ij}u + b_{ij}) - (a_iu + b_i)B^i - \xi^k f_{,k} - au f_{,u} - bf_{,u} + \lambda f = 0$$  \hfill (14)

$$A^{ij} \xi_{ij}^k - 2A^{ik}a_{,i} + aB^k + auB^k_{,u} - \xi^k B^i - \xi^i B^k_{,i} - \lambda B^k + bB^k_{,u} = 0$$  \hfill (15)

where the generator of the Lie point symmetry is

$$X = \xi^i(x^i) \partial_i + (a(x^k)u + b(x^i)) \partial_u.$$  \hfill (17)

For the Poisson equation (7) we have $A^{ij} = g^{ij}$ and $B^i = \Gamma^i$. Replacing in conditions (14)-(16) we find

$$g^{ij}(a_{ij}u + b_{ij}) - (a_iu + b_i)\Gamma^i - \xi^k f_{,k} - au f_{,u} - bf_{,u} + \lambda f = 0$$  \hfill (18)

$$g^{ij} \xi_{ij}^k - 2g^{ik}a_{,i} + a\Gamma^k - \xi^k \Gamma^i + \xi^i \Gamma^k - \lambda \Gamma^k = 0$$  \hfill (19)

$$L\xi g_{ij} = (a - \lambda)g_{ij}.$$  \hfill (20)

Equation (19) becomes (see [12])

$$g^{jk}L\xi\Gamma^i_{,jk} = 2g^{ik}a_{,i}.$$  \hfill (21)

Equation (20) gives that $\xi^i$ is a CKV.

a) For $n > 2$, since $\xi^i$ is a CKV, equation (21) becomes

$$\frac{2 - n}{2} (a - \lambda)^i = 2a^i \rightarrow (a - \lambda)^i = \frac{4}{2 - n} a^i.$$  \hfill (22)

Therefore,

$$\psi = \frac{2}{2 - n} a + a_0$$

where $2\psi = (a - \lambda)$ is the conformal factor of $\xi^i$. Furthermore we have

$$\lambda^i = -\frac{(n + 2)}{(2 - n)} a^i.$$  \hfill (23)

Finally from (18) we have the constraint condition

$$g^{ij}a_{ij}u + g^{ij}b_{ij} - \xi^k f_{,k} - au f_{,u} + \lambda f - bf_{,u} = 0.$$  \hfill (24)

b) For $n = 2$ we have that $g^{jk}L\xi\Gamma^i_{,jk} = 0$, which implies $a_{,i} = 0 \rightarrow a = a_0$. Finally we have

$$\lambda = a_0 - 2\psi$$

and (18) becomes

$$g^{ij}b_{ij} - \xi^k f_{,k} - a_0 u f_{,u} + (a_0 - 2\psi) f - bf_{,u} = 0.$$  \hfill (25)

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2 The derivation of these conditions is a straightforward calculation using the Lie symmetry condition. The details of the calculation can be found in [13].
There are two important forms of the Poisson equation which are of special interest. The Laplace equation defined by
\[ f(x^i, u) = 0 \] and the Klein-Gordon equation defined by
\[ f(x^i, u) = V(x^i) u \]
(22)
where \( V(x^i) \) is the ‘potential’.

The Lie symmetries of (1) have been given in [12, 18]. For the convenience of the reader and because we shall make use of this result in the following, we state this result below.

**Theorem 2** The Lie point symmetries of Laplace equation are generated from the CKVs of the metric \( g_{ij} \) as follows:

a) for \( n > 2 \), the Lie point symmetry vector is
\[ X = \xi^i (x^k) \partial_i + \left( 2 - \frac{n}{2} \psi (x^k) + a_0 \right) u + b(x^k) \partial_u \]
(23)
where \( \xi^i (x^k) \) is a CKV with conformal factor \( \psi (x^k) \) and the conformal factor \( \psi (x^k) \) and the function \( b(x^k) \) satisfy Laplace equation.

b) for \( n = 2 \), the Lie point symmetry vector is
\[ X = \xi^i (x^k) \partial_i + (a_0 u + b(x^k)) \partial_u \]
(24)
where \( \xi^i (x^k) \) is a CKV with conformal factor \( \psi (x^k) \) and \( b(x^k) \) is a solution of Laplace equation.

The Lie point symmetries of the Klein Gordon equation follow from Theorem 1.

**Theorem 3** The Lie point symmetries of the Klein Gordon equation (22) are generated from the CKVs of the metric \( g_{ij} \) as follows:

a) for \( n > 2 \), the Lie symmetry vector is
\[ X = \xi^i (x^k) \partial_i + \left( 2 - \frac{n}{2} \psi (x^k) + a_0 \right) u + b(x^k) \partial_u \]
(25)
where \( \xi^i (x^k) \) is a CKV with conformal factor \( \psi (x^k) \), \( b(x^k) \) is a solution of the Klein Gordon equation (22) and the following condition is satisfied
\[ \xi^k V_{,k} + 2 \psi V - \frac{2 - n}{2} \Delta \psi = 0 \]
(26)

b) for \( n = 2 \), the Lie symmetry vector is
\[ X = \xi^i (x^k) \partial_i + (a_0 u + b(x^k)) \partial_u \]
(27)
where \( \xi^i \) is a CKV with conformal factor \( \psi (x^k) \), \( b(x^k) \) is a solution of the Klein Gordon equation (22) and the following condition is satisfied
\[ \xi^k V_{,k} + 2 \psi V = 0. \]
(28)

We recall that the conformal algebra of a two-dimensional space has infinite dimension, a result which we shall need below. Because all two dimensional spaces are conformally flat they all have the same conformal algebra. In the subsequent sections we apply theorem 1 to study the reduction of Laplace equation in certain classes of Riemannian spaces which admit a conformal algebra.
3.1 The conformal algebra of certain classes of Riemannian spaces

In a general Riemannian space Laplace equation (1) admits the Lie point symmetries

\[ X_u = u\partial_u, \quad X_b = b(t, x)\partial_u \]

where \( b(t, x) \) is a solution of Laplace equation. These symmetries are too general to provide useful reductions and lead to reduced PDEs which possess Type II hidden symmetries. However the above theorems indicate that if we restrict our considerations to spaces which admit a conformal algebra (proper or not) then we will have new Lie point symmetries hence new reductions of Laplace equation, which might lead to Type II hidden symmetries.

In the following we consider the following classes of Riemannian spaces which admit a conformal algebra.

a. Spaces which admit a gradient KV (decomposable spaces)

b. Spaces which admit a gradient HV and
c. Spaces which admit a sp.CKV (this case is a subcase of a.).

The generic form of the metric for each type of space is as follows (\( A, B, \ldots = 1, 2, \ldots, n \)):

a. If an \( 1 + n \) dimensional Riemannian space admits a non null gradient KV, the \( S^i = \partial_z \) (\( S = z \)) say, then the space is decomposable along \( \partial_z \) and the metric takes the form (see e.g. [20])

\[ ds^2 = dz^2 + h_{AB}y^A y^B, \quad h_{AB} = h_{AB}(y^C) \]  

(29)

b. If an \( n \) dimensional Riemannian space admits a gradient HV, the \( H^i = r\partial_r \) (\( H = H_r^2 \)), \( \psi_H = 1 \) say, then the metric can be written in the form [21]

\[ ds^2 = dr^2 + r^2 h_{AB}dy^A dy^B, \quad h_{AB} = h_{AB}(y^C) \]  

(30)

c. If an \( n \) dimensional Riemannian space admits a non null KV and a gradient HV, then the space admits a sp.CKV and the metric can be written in the form

\[ ds^2 = -dz^2 + dR^2 + R^2 f_{AB}(y^C) dy^A dy^B \]  

(31)

where the sp.CKV is \( C_S = \frac{z^2 + R^2}{2} \partial_z + zR \partial_R \) with conformal factor \( \psi_{CS} = z \) [21] [22].

The Riemannian spaces which admit a non-gradient proper HV do not have a generic form for their metric. However the spaces for which the HV acts simply transitively are a few and are given together with their homothetic algebra in [14]. A special class of these spaces are the algebraically special vacuum solutions of Einstein equations [14]. In the following we shall consider the reduction of Laplace equation in the Petrov type III solution only with line element

\[ ds^2 = 2d\rho dv + \frac{3}{2}xd\rho^2 + \frac{v^2}{x} (dx^2 + dy^2) \]  

(32)

in which the symmetry generated by the non gradient HV \( H = v\partial_v + \rho\partial_\rho \), \( \psi_H = 1 \). The reduction of Laplace equation in the rest of the Petrov types in this class of solutions is similar both in the working method and the results and there is no need to consider them explicitly.

3This is not the most general Petrov type III space-time
Finally we shall consider the conformally flat space
\[ ds^2 = e^{2t} \left[ dt^2 - \delta_{AB} y^A y^B \right] \]
which admits a non-gradient HV.

In what follows all spaces are assumed to be of dimension \( n > 2 \).

4 Reduction of Laplace equation in certain Riemannian spaces

As we have seen in Theorem 2 the Lie point symmetries of Laplace equation (1) in a Riemannian space are the CKVs (not necessarily proper) whose conformal factor satisfies Laplace equation. This condition is satisfied trivially by the KVs (\( \psi = 0 \)), the HV (\( \psi_i = 0 \)) and the sp.CKVs (\( \psi_{ij} = 0 \)). Therefore these vector fields (which span a subalgebra of the conformal group) are among the Lie symmetries of Laplace equation. Concerning the proper CKVs it is not necessary that their conformal factor satisfies Laplace equation (1) therefore they may not produce Lie point symmetries of Laplace equation.

4.1 Riemannian spaces admitting a gradient KV

Without loss of generality we assume the gradient KV to be the \( \partial_z \) so that the metric has the generic form (29) where \( h_{AB}, A, B, C = 1, \ldots, n \) is the metric of the \( n \)-dimensional space. For the metric (29) Laplace equation (1) takes the form
\[ u_{zz} + h^{AB} u_{AB} - \Gamma^A u_B = 0 \]
and admits as extra Lie point symmetry the gradient KV \( \partial_z \).

We reduce (34) using the zeroth order invariants \( y^A, w = u \) of the extra Lie point symmetry \( \partial_z \). Taking these invariants as new coordinates eqn (34) reduces to
\[ h \Delta w = 0 \]
which is Laplace equation in the \( n \)-dimensional space with metric \( h_{AB} \). Now we recall the (easy to show) result that the conformal algebra of the \( n \) metric \( h_{AB} \) and the \( 1 + n \) metric (29) are related as follows [20]:

a. The KVs of the \( n \)-metric are identical with those of the \( n + 1 \) metric (apart from the vector \( \partial_z \)).

b. The \( 1 + n \) metric admits a HV if and only if the \( n \) metric admits one and if \( n H^A \) is the HV of the \( n \) metric then the HV of the \( 1 + n \) metric is given by the expression
\[ 1+nH^\mu = z \delta^\mu_z + n H^A \delta^\mu_A \quad \mu = x, 1, \ldots, n. \]

c. The \( 1 + n \) metric admits CKVs if and only if the \( n \) metric \( h_{AB} \) admits a gradient CKV (for details see proposition 2.1 of [20]).

Therefore Type II hidden symmetries for \( 1 \) exist if the \( n \)-metric \( h_{AB} \) admits more symmetries. In particular the sp.CKVs of the \( h_{AB} \) metric as well as the proper CKVs whose conformal function is a solution of Laplace equation (33) generate Type II hidden symmetries.
4.2 Riemannian spaces admitting a gradient HV

In Riemannian spaces which admit a gradient HV, $H$ say, there exists a coordinate system in which the metric is written in the form (30) and the gradient HV is $H = r \partial_r$. In these coordinates the Laplacian takes the form

$$u_{rr} + \frac{1}{r} h^{AB} u_{AB} + \frac{(n - 1)}{r} u_r - \frac{1}{r^2} \Gamma^A u_A = 0$$

and admits the extra Lie point symmetry $H$ (see Theorem 2). We reduce (37) using $H$.

The zero order invariants of $H$ are $y^A$, $w(y^A)$ and using them it follows easily that the reduced equation is

$$h \Delta w = 0$$

that is, the Laplacian defined by the metric $h_{AB}$.

It is easy to establish the following results concerning the conformal algebras of the metrics (30) and $h_{AB}$.
1. The KVs of $h_{AB}$ are also KVs of (30).
2. The HV of (30), if it exist, is independent from that of $h_{AB}$.
3. The metric (30) admits proper CKVs if and only if the $n$ metric $h_{AB}$ admits gradient CKVs [20]. This is because (30) is conformally related with the decomposable metric

$$ds^2 = d\bar{r}^2 + h_{AB}(y^C) dy^A dy^B.$$  

(39)

The above imply that Type II hidden symmetries we shall have from the HV of the metric $h_{AB}$, the sp.CKVs and finally from the proper CKVs of $h_{AB}$ whose conformal factor is a solution of Laplace equation (38).

4.3 Riemannian spaces admitting a sp.CKV

It is known [22] that if a decomposable $n = m + 1$ dimensional ($n > 2$) Riemannian space which admits the non null gradient KV $K_G = \partial_z$, also admits sp.CKVs (there is a 1:1 correspondence between the non null gradient KVs and the sp.CKVs in a Riemannian space see [22]) then also admits a gradient HV. In these spaces there exists always a coordinate system in which the metric is written in the form (31) where $\partial_z$ is the gradient KV and $z \partial_z + R \partial_R$ is the gradient HV. $f_{AB}(y^C)$, $A, B, C, .. = 1, 2, ... m - 1$ is an $m - 1$ dimensional metric. For a general $m - 1$ dimensional metric $f_{AB}$ the $n$–dimensional metric (31) admits the following special conformal group

$$K_G = \partial_z, \quad H = z \partial_z + R \partial_R$$

$$C_S = \frac{z^2 + R^2}{2} \partial_z + z R \partial_R$$

where $K_G$ is a gradient KV, $H$ is a gradient HV and $C_S$ is a sp.CKV with conformal factor $\psi_{C_S} = z$. In these coordinates Laplace equation takes the form

$$- u_{zz} + u_{RR} + \frac{1}{R^2} h^{AB} u_{AB} + \frac{(m - 1)}{R} u_R - \frac{1}{R^2} \Gamma^A u_A = 0.$$  

(40)

From Theorem 2 we have that the extra Lie point symmetries of (40) are the vector fields

$$X^1 = K_G, \quad X^2 = H$$

$$X^3 = C_S + 2pz \partial_u$$

(41)

(42)
where \( 2p = \frac{1-m}{2} \). The non-zero commutators are

\[
[X^1, X^2] = X^1, \quad [X^2, X^3] = X^3, \quad [X^1, X^3] = X^2 + 2pu\partial_u.
\]

We consider the reduction of (40) with each of the extra Lie symmetries.

Reduction with the gradient KV \( X^1 \) reduces the Laplacian (40) to (37) which admits the Lie point symmetry \( X^2 \) generated by the HV. This result is expected [6] because \([X^1, X^2] = X^1\) hence the Lie point symmetry \( X^2 \) is inherited. Therefore in this reduction the Type II symmetries are generated from the CKVs of the metric (30). It is possible to continue the reduction by the gradient HV and then we find the results of section 4.2.

The reduction with a gradient HV has been studied in section 4.2. To apply the results of section 4.2 in the present case we have to bring the metric (31) to the form (30). For this we consider the transformation

\[
z = r \sinh \theta, \quad R = r \cosh \theta
\]

which brings (31) to

\[
ds^2 = dr^2 + r^2 \left( -d\theta^2 + \cosh^2 \theta f_{AB} \gamma^A \gamma^B \right)
\]

so that the metric \( h_{AB} \) of (30) is:

\[
ds^2_h = \left( -d\theta^2 + \cosh^2 \theta f_{AB} \gamma^A \gamma^B \right).
\]

The reduced equation of (40) under the Lie point symmetry generated by the gradient HV is Laplace equation in the space (44). For this reduction we do not have inherited symmetries and there exist Type II hidden symmetries as stated in section 4.2.

Before we reduce (40) with the symmetry generated from the sp.CKV \( X^3 \), it is best to write the metric (31) in new coordinates. We introduce the new variable \( x \) via the relation

\[
z = \sqrt{\frac{R (xR - 1)}{x}}.
\]

In the new variables the Laplacian (40) becomes

\[
0 = \frac{x^2}{R^2} u_{xx} - 2 \frac{x}{R} (2xR - 1) u_{xR} + u_{RR} + \frac{1}{R^2} f_{AB} u_{AB} + \frac{m-1}{R} u_R - \frac{x}{R^2} (m-1) (2xR - 1) u_x - \frac{1}{R^2} \Gamma^A u_A
\]

whereas the Lie symmetry \( X^3 \) becomes

\[
X^3 = \sqrt{\frac{R (xR - 1)}{x}} R\partial_R + 2p \sqrt{\frac{R (xR - 1)}{x}} u\partial_u.
\]

The zero order invariants of (47) are \( x, y^A, \quad w = uR^{-2p} \). We choose \( x, y^A \) to be the independent variables and \( w = w \left( x, y^A \right) \) the dependent one. Replacing in (46) we find the reduced equation:

\[
x^2 w_{xx} + f_{AB} w_{AB} - \Gamma^A w_A - 2p (2p + 1) w = 0
\]

We consider cases.

**The case** \( m \geq 4 \).

If \( 2p + 1 \neq 0, \quad m \geq 4 \) then (48) becomes

\[
(m \geq 4) \tilde{\Delta} w - 2p (2p + 1) V (x) w = 0
\]
where \( V(x) = x^{\frac{2}{2-m}} \) and \( (m \geq 4) \) is the Laplace operator for the metric

\[
ds^2_{(m \geq 4)} = \frac{1}{V(x)} \left( \frac{1}{x^2} dx^2 + f_{AB} dy^A dy^B \right).
\]

(50)

Considering the new transformation \( \phi = \int \sqrt{\frac{1}{V}} dx \) or \( x = (m - 2)^{2-m} \phi^{m-2} \) the metric (50) is written

\[
ds^2_{(m \geq 4)} = d\phi^2 + \phi^2 \tilde{f}_{AB} dy^A dy^B.
\]

(51)

where \( \tilde{f}_{AB} = (m - 2)^{-2} f_{AB} \) whereas the potential \( V(\phi) = \left( \frac{2}{m} \right)^{2} \phi^2 \) which is the well known Ermakov potential [24].

This means that the gradient HV \( \phi \partial \phi \), \( \psi_\phi = 1 \) of the metric (51) satisfies condition (26) of Theorem 3 hence it is a Lie point symmetry of (49), which is the Lie symmetry \( X^2 \) of (41).

Therefore if the metric (51) admits proper CKVs which satisfy the conditions of Theorem 3 then these vectors generate Type II hidden symmetries of (40).

**The case** \( m = 3 \).

If \( 2p + 1 = 0 \) then \( m = 3 \) and \( f_{AB} \) is a two dimensional metric. In this case (48) becomes

\[
x^2 w_{xx} + f^{AB} w_{AB} - \Gamma^A w_A = 0
\]

(52)

or, by multiplying with \( x^2 \)

\[
(m=3) \Delta w = 0
\]

(53)

which is the Laplacian in the three dimensional space with metric

\[
ds^2_{(m=3)} = \frac{1}{x^4} dx^2 + \frac{1}{x^2} f_{AB} dy^A dy^B.
\]

(54)

By making the new transformation \( x = \frac{1}{\phi} \), (53) is of the form (30) and admits the gradient HV \( \phi \partial \phi \) which gives an inherited symmetry.

We conclude that Type II hidden symmetries of (54) will be generated from the proper CKVs of the metric (54) which satisfy the conditions of Theorem 2.

**The case** \( m = 2 \).

For \( m = 2 \), \( f_{AB} \) is a one dimensional metric and (31) is

\[
ds^2 = -dz^2 + dR^2 + R^2 d\theta^2
\]

(55)

which is a flat metric [4]. In this space Laplace equation (41) admits ten Lie point symmetries, as many as the elements of the conformal algebra of the flat 3d-space. Six of these vectors are KVs, one vector is a gradient HV and three are sp.CKVs (see Appendix). We reduce Laplace equation with the symmetry \( X^3 \) and the reduced equation is (48) which for \( f_{AB} = \delta_{\theta \theta} \) becomes

\[
x^2 w_{xx} + w_{\theta \theta} + \frac{1}{4} w = 0.
\]

(56)

Equation (56) is in the form of (13) with \( A^{ij} = \text{diag} (x^2, 1) \) and \( B^i = 0 \). Replacing in the symmetry conditions (14)-(16) we find the Lie symmetries

\[
X = \xi^i \partial_i + (a_0 w + b) \partial_w
\]

where \( \xi^i \) are the CKVs of the two dimensional space with metric \( A^{ij} \). In this case all proper CKVs of the two dimensional space \( A^{ij} \) generate Type II Lie symmetries. We recall that the conformal algebra of a two dimensional space is infinite dimensional.

\[\text{The only three dimensional space which admits sp.CKV is the flat space, because in that case we also have a gradient HV and a gradient KV.}\]
5 The Wave equation

In this section we apply the general results of the previous section to the 3+1 wave equation in Minkowski spacetime $M^4$. As will be shown we recover the results of previous studies [6] easily and in a straightforward manner. We also amend some of them.

Laplace equation in $M^4$

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

is the wave equation in $E^3$

$$u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0.$$  

The conformal algebra of the metric (57) is generated by 15 vector fields (see Appendix). From Theorem 2 we have that the extra Lie point symmetries of (58) are the following vector fields

$$K^1_G, \ K^A_G, \ X^1_A, \ X^{AB}, \ H$$

(59)

$$X^1_L = X^1_C - tu \partial_u, \ X^L_A = X^A_C - y^A u \partial_u$$

(60)

where $y^A = (x, y, z)$ with nonzero commutators

$$[K^I_G, X^J_R] = -K^J_G, \ [K^I_G, H] = K^I_G$$

$$[K^I_G, X^A] = H - u \partial_u, \ [K^I_G, X^L] = X^{IJ}_R$$

$$[H, X^L] = X^L, \ [X^{IJ}_R, X^{KL}_R] = X^{IJ}_L$$

and the commutators of the rotations $X^{1A}_R, X^{AB}_R$.

5.1 Reduction with a gradient KV

We consider reduction of (58) with the gradient KV $K^z_G = \partial_z$. The reduced equation is

$$w_{tt} - w_{xx} - w_{yy} = 0$$

(61)

which is Laplace equation in the space $M^3$. The extra Lie point symmetries of (61) are

$$K^1_C, \ K^x_G, \ K^y_G, \ X^{1a}_R, \ X^{ab}_R, \ H$$

(62)

and are inherited symmetries (see also the relevant commutators). The Type II symmetries are the vector fields

$$\bar{X}^1_C = -\frac{1}{2} tu \partial_u, \ \bar{X}^x_C = -\frac{1}{2} xu \partial_u, \ \bar{X}^y_C = -\frac{1}{2} yu \partial_u$$

(63)

that is, the Type II hidden symmetries are generated from the sp.CKVs of $M^3$.

5.2 Reduction with the gradient HV

In this case it is better to switch to hyperspherical coordinates $(r, \theta, \phi, \zeta)$. In these coordinates the metric (57) is

$$ds^2 = dr^2 - r^2 \left[ d\theta^2 + \cosh^2 \theta \left( d\phi^2 + \cosh^2 \phi d\zeta^2 \right) \right]$$

(64)

and the wave equation (58) becomes

$$u_{rr} - \frac{1}{r^2} \left( u_{\theta \theta} + \frac{u_{\phi \phi}}{\cosh^2 \theta} \right) + \frac{u_{\zeta \zeta}}{\cosh^2 \theta \cosh^2 \phi} + \frac{3}{r} u_r - 2 \frac{\tanh \theta}{r^2} u_\theta - \frac{\tanh \phi}{r^2 \cosh^2 \theta} u_\phi = 0.$$  

(65)
According to the analysis of section 4.2, the reduced equation is (38), which is the Laplacian in the three-dimensional space of the variables \((\theta, \phi, \zeta)\):

\[
\frac{w_{\theta\theta}}{\cosh^2 \theta} + \frac{w_{\phi\phi}}{\cosh^2 \theta \cosh^2 \phi^2} + 2 \frac{\tanh \theta}{r^2} w_\theta + \frac{\tanh \phi}{r^2 \cosh^2 \theta} w_\phi = 0.
\]

(66)

This space is a space of constant curvature. The conformal algebra of a 3d space of constant non-vanishing curvature consists of 6 non-gradient KVs and 4 proper CKVs \([19, 20]\) which in Cartesian coordinates are given in Appendix. (The rotations and the sp.CKVs of the flat space are KVs for the space of constant curvature, the rest are proper gradient CKVs). The conformal factors of the CKVs do not satisfy the condition \(h \Delta \psi = 0\), hence they do not generate Lie point symmetries for the reduced equation (66) (see theorem 2) whereas for the same reason the KVs are Lie point symmetries of (66). Therefore all point Lie symmetries are inherited and we do not have Type II hidden symmetries.

We note that the proper CKVs of a space of constant non-vanishing curvature are gradient and their conformal factor satisfies the relation \([19]\)

\[
\psi_{;ab} = \frac{\psi R}{n-1} g_{ab} \rightarrow g^{ab} \psi_{;ab} = \frac{n}{n-1} R \psi \rightarrow h \Delta \psi = \frac{n}{n-1} R \psi
\]

where \(R\) is the Ricci scalar of the space of constant curvature. This implies that they are Lie symmetries of the conformally invariant Laplace-Beltrami operator \([23]\) but not of the Laplace equation (66).

### 5.3 Reduction with a sp.CKV

Following the steps of section 4.3, we consider the transformation to axi-symmetric coordinates \((t, R, \theta, \phi)\) in which (58) takes the form

\[
u_{tt} - \frac{1}{R^2} \left( u_{\theta\theta} + \frac{u_{\phi\phi}}{\cosh^2 \theta} \right) - \frac{2}{R} u_r - \frac{\tanh \theta}{R^2} u_\theta = 0.
\]

(67)

Applying the transformation (45) \(t = \sqrt{\frac{R(t - R)}{\tau}}\) we find (note that this is the case \(m = 3\)) that (67) is written as (46) and the reduced equation is the Laplacian \((m=3) \Delta w\) for the 3d metric

\[
ds^2 = \frac{1}{\tau^4} d\tau^2 - \frac{1}{\tau^2} (d\theta^2 + \cosh^2 \theta d\phi^2).
\]

(68)

However the metric (68) under the coordinate transformation \(\tau = \frac{1}{t}\) is written

\[
ds^2 = dT^2 - T^2 (d\theta^2 + \cosh^2 \theta d\phi^2)
\]

(69)

which is the flat 3d Lorentzian metric, which does not admit proper CKVs. This implies that the Lie point symmetries of the reduced equation are generated from the KVs/HV/sp.CKVs of the flat \(M^3\) metric and all are inherited. Therefore we do not have Type II hidden symmetries for the reduction with a sp.CKV.

The reduction of the 3+1 and 2+1 wave equation has been done previously by Abraham-Shrauner et. al \([6]\) and our results coincide with theirs. We note that in \([6]\) they use algebraic computing to find the Lie symmetry generators whereas our approach is geometric and general and makes no use of algebraic computing programs. Furthermore our analysis can be generalized to higher dimensions - where algebraic computing is rather cumbersome - in a straightforward manner.

In the following sections we apply theorem 2 to determine the Lie point symmetries and in addition determine the Type II hidden symmetries of some spacetimes which are of interest in General Relativity.

\[\text{All spaces of constant curvature are conformally flat hence they admit the same conformal algebra with the flat space but in general with different KVs and CKVs.}\]
6 LRS Spacetime

A spatially Locally Rotational Symmetric (LRS) spacetime has a metric which admits a group of motions $G_4$ acting transitively on spacelike hypersurfaces $S_3$. In coordinates \( \{t, x, y, z\} \) the classes of metrics describing the LRS spacetimes are the following [25]:

\[
\begin{align*}
    ds^2 &= \varepsilon[-dt^2 + A^2(t)dx^2] + B^2(t) \left[ dy^2 + \Sigma^2(y, k)dz^2 \right] \\
    ds^2 &= \varepsilon \left[-dt^2 + A^2(t) \left[ dx + \Lambda(k, y)dz \right] \right] + B^2(t) \left[ dy^2 + \Sigma^2(y, k)dz^2 \right] \\
    ds^2 &= \varepsilon[-dt^2 + A^2(t)dx^2] + B^2(t)e^{2x}(dy^2 + dz^2)
\end{align*}
\]

where $\varepsilon = \pm 1, \Sigma(y, k) = \sin y, \sinh y, y$ and $\Lambda(k, y) = \cos y, \cosh y, y$ for $k = 1, -1, 0$ respectively. (The factor $\varepsilon \pm 1$ essentially distinguishes between the static and the non-static cases as it can be seen by interchanging the coordinates $t, x$). These metrics are quite general and contain many well known and important families of spacetimes. They contain the diagonal Bianchi Type I, III metrics (with two of the three spacelike metric components equal), the static spherically/hyperbolically/plane symmetric metrics (interchange $t, x$ in (70) and the signs of $dt, dx$), some of the \{2+2\} and \{1+3\} decomposable metrics etc. The LRS metrics have been classified by Ellis [26]. The metrics (71) with $\varepsilon = 1$ in Ellis classification are class I LRS metrics, the metrics (70) and (72) are class II LRS and the metrics (71) with $\varepsilon = -1$ are the class III LRS metrics.

In this section we consider the particular LRS space-time with line element

\[
    ds^2 = -dt^2 + dR^2 + R^s \left( dz^2 + dy^2 \right)
\]

When $s = 0$ the space-time (73) reduces to Minkowski space $M^4$ considered in the last section.

For $s \neq 0, 2$ spacetime (73) admits four KVs [27]

\[
    K^1 = \partial_t, \; K^2, \; K^3, \; K^4
\]

where $K^1$ is a gradient KV, $K^{2-4}$ are the elements of $E^2$ Lie algebra, that is,

\[
    K^2 = \sin y \partial_z + \frac{\cos y}{z} \partial_z, \; K^3 = \cos y \partial_z - \frac{\sin y}{z} \partial_y, \; K^4 = \partial_y
\]

and one non-gradient HV

\[
    H = t\partial_t + R\partial_R + \frac{(2 - n)}{2} (z\partial_z + y\partial_y), \; \psi_H = 1.
\]

Moreover, in the special case $s = 2$ the metric (73) admits the special CKV

\[
    C_{sp} = \frac{t^2 + R^2}{2} \partial_t + tR\partial_R, \; \psi_C = t
\]

and in that case, $H$ becomes a gradient HV. In all cases the metric (73) does not admit proper non special CKVs.

Laplace equation (1) for the line element (73) becomes

\[
    -u_{tt} + u_{RR} + \frac{1}{Rs} (u_{zz} + u_{yy}) + \frac{s}{R} u_R = 0.
\]
From theorem 2 we have that the extra Lie point symmetries of (74) are the vector fields

\[ K^1, K^{2-4}, H \text{ for } s \neq 0,2 \]
\[ K^1, K^{2-4}, H, C_{sp} - tu\partial_u \text{ for } s = 2 \]

with non-zero commutators:

For \( s \neq 0,2 \)

\[ [K^1, H] = K^1, \quad [K^2, K^{4}] = -K^3, \quad [K^3, K^{4}] = K^2 \]
\[ [K^2, H] = \left(1 - \frac{s}{2}\right)K^2, \quad [K^3, H] = \left(1 - \frac{s}{2}\right)K^3 \]

and for \( s = 2 \)

\[ [K^1, H] = K^1, \quad [K^1, C_{sp} - tu\partial_u] = H - u\partial_u \]
\[ [H, C_{sp} - tu\partial_u] = C_{sp} - tu\partial_u, \quad [K^2, K^{4}] = -K^3, \quad [K^3, K^{4}] = K^2. \]

Below we study the reduction of (74) using the zero order invariant \( s \) generated by the gradient \( K^1 \), the \( HV \) and the \( sp.CKV \).

6.1 Reduction with a gradient \( KV \)

The gradient \( KV \) \( K^1 \) is a non-null vector field and the results of section 4.1 apply. Therefore the reduced equation of (74) is Laplace equation of the 3d space

\[ ds^2_{(3)} = dR^2 + R^s (dz^2 + dy^2) \]

that is,

\[ w_{RR} + \frac{1}{R^s} (w_{zz} + w_{yy}) + \frac{s}{R} w_R = 0. \]

The 3d space (75) is a conformally flat space hence admits a ten-dimensional conformal algebra. Applying theorem 2 we find that the extra Lie symmetries of (76) are:

For \( s \neq 0,2 \)

\[ w\partial_w, \ b(\bar{x}, z, y)\partial_w \]
\[ H_{(3)} = R\partial_R + \frac{(2 - n)}{2} (z\partial_z + y\partial_y), \ K^2, \ K^3, \ K^4 \]
\[ C^1 = Rz\partial_R + \left[ \frac{2 - s}{4} (z^2 - y^2) - \frac{R^{2-s}}{2 - s} \right] \partial_z + \frac{2 - s}{2} zy\partial_y - \frac{1}{2} z w\partial_w \]
\[ C^2 = R y\partial_R + \frac{2 - s}{2} zy\partial_z + \left[ \frac{2 - s}{4} (y^2 - z^2) - \frac{R^{2-s}}{2 - s} \right] \partial_y - \frac{1}{2} y w\partial_w \]

and for \( s = 2 \)

\[ w\partial_w, \ b(\bar{x}, z, y)\partial_w \]
\[ H_{(3)} = R\partial_R, \ K^2, \ K^3, \ K^4 \]
\[ C^1 = Rz\partial_R - \ln R\partial_z - \frac{1}{2} z w\partial_w, \quad C^2 = R y\partial_R - \ln R\partial_y - \frac{1}{2} y w\partial_w. \]

We observe that the Lie symmetries \( H_{(3)}, K^{2-4} \) are inherited symmetries. Lie point symmetries \( C^{1-2} \) are generated from the proper CKVs of (75); therefore symmetries \( C^{1-2} \) are Type II hidden symmetries.
6.2 Reduction with a HV

In this subsection we reduce Laplace equation \((74)\) using the Lie point symmetry generated by the HV \(H\).

Recall that \(H\) is a gradient HV only for \(s = 2\).

Under the coordinate transformation

\[
t = e^{Sr} \sinh \theta, \quad R = e^r \cosh \theta,
\]

\[
z = \zeta e^{Sr}, \quad y = ve^{Sr},
\]

where \(S = \frac{2-n}{2}\) the line element \((73)\) becomes

\[
ds^2 = -\left(1 - \frac{(2-s)^2}{4} \cosh^s \theta\right) e^{2r} dr^2
+ e^{2r} d\theta + (2-s) e^{2r} \cosh^s \theta dr d\zeta + e^{2r} \cosh^s \theta (d\zeta^2 + \zeta^2 dv^2)
\]

and Laplace equation is written as follows

\[
0 = -u_{rr} + u_{\theta\theta} + (2-s) e^{2r} u_{r\zeta} + \frac{1}{\cosh^s \theta} \left[ \left(1 - \frac{(2-s)^2}{4} \cosh^s \theta\right) u_{\zeta\zeta} + \frac{1}{\zeta^2} u_{vv}\right]
+ s \tanh \theta u_{\theta} - su_R + \frac{\left(\frac{1}{4} \cosh^s (3s-2) (2-s) - 1\right)}{\zeta \cosh^s \theta} u_\zeta
\]

In these coordinates the HV becomes \(H = \partial_r\). Hence the zero order invariants are \(\alpha, \beta, \gamma\) and \(w\). We choose \(\alpha, \beta, \gamma\) as the independent variables and \(w = w(\alpha, \beta, \gamma)\) as the dependent variable and find the reduced equation

\[
0 = w_{\alpha\alpha} + \frac{1}{\cosh^s \alpha} \left[ \left(1 - \frac{(2-s)^2}{4} \beta^2 \cosh^s \alpha\right) w_{\beta\beta} + \frac{1}{\beta^2} w_{\gamma\gamma}\right]
+ s \tanh \alpha w_{\alpha} + \frac{\left(\frac{1}{4} \beta^2 \cosh^s (3s-2) (2-s) - 1\right)}{\beta \cosh^s \alpha} w_{\beta}
\]

Equation \((78)\) admits the Lie point symmetry \(K^4\) for \(s \neq 0, 2\) plus the symmetries \(w \partial_w, b(\alpha, \beta, \gamma) \partial_w\) because \((78)\) is linear in \(w\). \(K^4\) is inherited therefore we do not have Type II hidden symmetries.

In the case where \(s = 2\), \(H\) is a gradient HV and the results of section 4.2 apply. In that case \((78)\) becomes

\[
h\tilde{\Delta} w = 0
\]

where \(h\tilde{\Delta}\) is Laplace operator for the 3d metric

\[
ds^2_{(3)} = da^2 + \cosh^2 \alpha \left(d\beta^2 + \beta^2 d\gamma^2\right).
\]

This metric is conformally flat hence admits a ten dimensional conformal algebra with a three dimensional Killing algebra. From theorem 2 we have that \((79)\) admits as Lie point symmetries the \(E^2\) Lie algebra plus the vectors \(w \partial_w, b(\alpha, \beta, \gamma) \partial_w\). In that case all symmetries are inherited and we do not have Type II hidden symmetries.

6.3 Reduction with a sp.CKV

In the case \(s = 2\) Laplace equation admits a Lie point symmetry generated by the sp.CKV \(C_{sp}\) and the results of section 4.3 apply. That is, from subsection 4.3 and for \(m = 3\) the reduced equation is

\[
(s_p)\Delta w = 0
\]
where \((sp)\Delta\) is the Laplace operator for the metric
\[
d\bar{s}^2_{(m=3)} = dx^2 + \bar{x}^2 \left( dz^2 + dy^2 \right),
\]
and we have defined \(x = \frac{1}{\bar{x}}\). From theorem 2 we have that \((81)\) admits the Lie point symmetries
\[
\bar{H}_{(3)} = \partial_{\bar{x}} , \ w\partial_{w} , \ b(\bar{x},z,y) \partial_{w}
\]
\[
\bar{C}^1 = \bar{x}z\partial_{\bar{x}} - \ln \bar{x}\partial_z - \frac{1}{2}z w\partial_{w}, \ \bar{C}^2 = \bar{x}y\partial_{\bar{x}} - \ln \bar{x}\partial_y - \frac{1}{2}yw\partial_{w}.
\]
\(H_{(3)}\) is a gradient HV for space \((82)\) and the Lie point symmetries \(\bar{C}^{1-2}\) are generated by proper CKVs of \((82)\). From subsection 4.3 we conclude have that \(\bar{H}_{(3)}\) is an inherited symmetry whereas \(\bar{C}^{1-2}\) are Type II hidden symmetries.

7 The algebraically special empty space Petrov Type III solution

In this section we consider the reduction of Laplace equation in spaces which do not admit gradient KVs or a gradient HV. As it has been mentioned in section 3.1 we shall consider the algebraically special solutions of Einstein equations which admit a homothetic algebra acting simply transitively. These spacetimes have been determined in [14] and are of Petrov type D,N,II and III. In the following we restrict our discussion to Petrov Type III only with metric \((32)\), because both the method of work and the results are the same for the remaining Petrov types in this class of spacetimes. Spacetime \((32)\) admits the four dimensional conformal algebra generated by the vector fields
\[
K^1 = \partial_{\rho} , \ K^2 = \partial_{y} , \ K^3 = v\partial_{v} - \rho\partial_{\rho} + 2x\partial_{x} + 2y\partial_{y} \\
H = v\partial_{v} + \rho\partial_{\rho} , \ \psi = 1
\]
where \(K^{1-3}\) are KVs and \(H\) is a non-gradient HV. (The space does not admit proper CKVs).

In this spacetime the Laplacian takes the form:
\[
-\frac{3}{2}x^2 u_{vv} + 2x u_{v\rho} + \frac{x^3}{v^2} (u_{xx} + u_{yy}) - 3xu_{xv} + \frac{2}{v} u_{\rho} = 0.
\]
From theorem 2 we have that the extra Lie point symmetries are the vector fields
\[
X_{1-3} = K_{1-3} , \ X_4 = H
\]
with nonzero commutators:
\[
[X_2,X_3] = 2X_2 \\
[X_3,X_1] = X_1 , \ [X_1,X_4] = X_1.
\]
We use \(X_4\) to reduce the PDE because this is the Lie symmetry generated by the non-gradient HV.

The zero order invariants of \(X_4\) are \(\sigma = x, y, w\). We choose \(\sigma, x, y\) as the independent variables and \(w = w(\sigma, x, y)\) as the dependent variable and we find the reduced equation
\[
- \sigma \left( \frac{3}{2} x \sigma + 2 \right) w_{\sigma \sigma} + x^3 (w_{xx} + w_{yy}) = 0.
\]
Equation (84) can be written

\[ \Delta^* w = \left( \frac{3x\sigma}{2} + 1 \right) w - \frac{3x^3}{2(3x\sigma + 4)} w_x = 0 \] (85)

where \( \Delta^* \) is the Laplacian for the metric

\[ ds^2 = -\frac{1}{\sigma \left( \frac{3x\sigma}{2} + 2 \right)} d\sigma^2 + \frac{1}{x^3} (dx^2 + dy^2) . \] (86)

The Lie point symmetries of (85) are generated from the conformal algebra of (86) with some extra conditions (see eqs. (14)-(16)). We find that equation (85) admits as Lie point symmetries the vectors \( \partial_y, x\partial_x + y\partial_x - \sigma\partial_\sigma \) which are inherited symmetries. Hence we do not have Type II hidden symmetries with this reduction.

8 The spatially flat \( n \)-dimensional FRW like metric

As a final example we consider the \( n \) dimensional FRW like space \( (n > 2) \) with metric \( \delta_{AB} \) where \( \delta_{AB} \) is the \( n-1 \) dimensional Euclidian metric. The reduction of Laplace equation in this space (for \( n = 4 \)) has been studied previously in [28]. The metric \( \delta_{AB} \) is conformally flat hence admits the same CKVs with the flat space but with different conformal factors. More precisely the space admits

a. \( (n - 1) + \frac{(n-2)(n-3)}{2} \) KVs the \( K^A_G, X^{AB}_R \)

b. 1 gradient HV the \( K^1_G = \partial_t \)

the rest vectors being proper CKVs [29]. In this space, Laplace equation (1) becomes

\[ e^{-2t} \left[ u_{tt} - \left( \delta^{AB} u_{AB} \right) + (n - 2) u_t \right] = 0 \] (87)

and the extra Lie point symmetries are

\[ K^A_G, X^{AB}_R, K^1_G \]
\[ C^A = X^{1A}_R - 2py^Au_\alpha \]

where \( 2p = \frac{2-n}{2} \) and with nonzero commutators

\[ [K^1_G, C^A] = K^A_G \]
\[ [C^A, C^B] = -X^{AB}_R, \quad [X^{AB}_R, C^A] = C^B \]

and the commutators of the rotations \( X^{AB}_R \).

8.1 Reduction with the gradient HV

The gradient HV \( K^1_G = \partial_t \) is a Lie point symmetry of the Laplacian (87) hence we consider the reduction by this vector. The zero order invariants are \( y^A, w \) and lead to the reduced equation

\[ \delta^{AB} u_{AB} = 0 \] (88)

which is Laplace equation in the flat space \( E^{n-1} \). We consider again cases.

Case \( n > 3 \)

This is a special form of the FRW metric which admits a HV.
In this case the Lie point symmetries of \( \mathbf{SS} \) are given by the vectors (see Appendix A)

\[
K_G^A, \; X_{R}^{AB}, \; n-1 \; H, \; X_{C}^{A} - y^A u \partial_u.
\] (89)

From these the \( K_G^A, X_{R}^{AB} \) are inherited symmetries and the rest - which are produced by the HV and the sp.CKVs of the space \( E^{n-1} \) - are Type II hidden symmetries.

**Case \( n = 3 \)**

In this case, the reduced equation \( \mathbf{SS} \) is the Laplacian in \( E^2 \), hence admits an infinite dimensional Lie algebra \([18]\). Type II hidden symmetries are generated from the HV and the CKVs of \( E^2 \).

**8.2 Reduction with a proper CKV**

We consider next the reduction with a proper CKV. We may take any of the vectors \( X_{1}^{R} \) (because, as one can see in the Appendix, there is a symmetry between the coordinates \( y^A \)). We choose the vector field

\[
X_{1}^{R} = x \partial_t + t \partial_x + 2pxu \partial_u.
\]

whose zero order invariants are

\[
R = t^2 - x^2, \quad y^C, \quad w = e^{-2pt}u.
\]

We take the dependent variable to be the \( w = w(R, y^C) \) and find the reduced equation

\[
4Rw_{RR} - \delta^{ab}w_{ab} + 4w_R - 4p^2w = 0 \quad (90)
\]

where \( a = 1, \ldots, n - 2 \). We consider cases.

**Case \( n > 3 \).**

For \( n > 3 \) equation (90) is

\[
c \Delta w - 4p^2 f(R) w = 0 \quad (91)
\]

where \( c \Delta \) is the Laplace operator for the \((n-1)\) dimensional metric

\[
dx_c^2 = \frac{1}{f(R)} \left( \frac{1}{4R} dR^2 - \delta_{ab} dy^a dy^b \right) \quad (92)
\]

and \( f(R) = R^{-\frac{1}{2(n-2)}} \). The metric (92) is conformally flat hence we know its conformal algebra. Application of theorem [3] gives that the Lie point symmetries of (91) are the vector fields

\[
X_u = u \partial_u, \quad X_b = b \partial_u
\]

\[
X_{K}^{a} = \partial_{y^a}, \quad X_{R}^{ab} = y^b \partial_a - y^a \partial_b.
\]

These are inherited symmetries (this result agrees with the commutators). We conclude that for this reduction we do not have Type II hidden symmetries.

**Case \( n = 3 \).**

For \( n = 3 \) the reduced equation is a two dimensional equation (that is \( \delta_{AB} = \delta_{yy} \))

\[
4Rw_{RR} - w_{yy} + 4w_R - \frac{1}{4} w = 0 \quad (93)
\]

and admits as Lie point symmetry the KV \( \partial_y \) which is an inherited symmetry. Hence we do not have Type II hidden symmetries.

We conclude that the reduction of Laplace equation in a \( n \) dimensional FRW like space with the proper CKV does not produce Type II hidden symmetries and in fact the inherited symmetries of the reduced equation are the KVs of the flat metric.
9 Conclusion

Up to now the study of Type II hidden symmetries has been done by counter examples or by considering very special PDEs and in low dimensional flat spaces. In this paper we improve this scenario and study the problem of Type II hidden symmetries of second order PDEs from a geometric point of view in $n$-dimensional Riemannian spaces. We have considered the reduction of Laplace equation and the consequent possibility of existence of Type II hidden symmetries in some general classes of spaces which admit some kind of symmetry hence admit nontrivial Lie symmetries. The general conclusion of this study is that the Type II hidden symmetries of Laplace equation are directly related to the transition of the CKVs from the space where the original equation is defined to the space where the reduced equation resides.

We may summarize the general conclusions of this study as follows:

- If we reduce Laplace equation with a non null gradient KV the reduced equation is again Laplace equation in the non-decomposable space. In this case the Type II hidden symmetries are generated from the special and the proper CKVs of the non-decomposable space.

- If we reduce Laplace equation with a gradient HV the reduced equation is again Laplace equation for an appropriate metric. In this case the Type II hidden symmetries are generated from the HV and the special/proper CKVs.

- If we reduce Laplace equation with the symmetry generated by a sp.CKV in a space which admits a non null KV, the reduced equation is the Klein Gordon equation (22) for an appropriate metric which inherits the Lie symmetry generated by the gradient HV. In this case the Type II symmetries are generated from the proper CKVs.

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A Appendix: Conformal algebra of a flat space of Lorentzian / Euclidian character

We consider a flat space of dimension $n > 2$ with metric

$$ds^2 = \varepsilon dt^2 + \delta_{AB}dy^Ady^B, \varepsilon = \pm 1.$$ 

The conformal algebra of the space consists of the following vectors

$n$ gradient KVs

$$K^1_G = \partial_t, K^A_G = \partial_A$$

$n(n-1)$ non gradient KVs (rotations)

$$X^A_K = y^A\partial_t - \varepsilon t\partial_A, X^{AB}_R = y^B\partial_A - y^A\partial_B$$

1 gradient HV

$$H = t\partial_t + \sum_A y^A\partial_A$$
$n$ sp.CKVs

$$X_{C}^{1} = \frac{1}{2} \left( t^2 - \varepsilon \sum_{A} (y^{A})^2 \right) \partial_{t} + t \sum_{A} y^{A} \partial_{A}$$

$$X_{C}^{A} = ty^{A} \partial_{t} + \frac{1}{2} \left( \varepsilon t^2 + (y^{A})^2 - \sum_{B \neq A} (y^{B})^2 \right) \partial_{A} + y^{A} \sum_{B \neq A} y^{B} \partial_{B}$$

where $y^{A} = 1...n - 1$ with conformal factor $\psi_{C}^{1} = t$ and $\psi_{C}^{A} = y^{A}$. For $n > 2$ the flat space does not admit proper CKVs [20].

For $n = 2$ the vector field

$$X = \left[ f \left( t + i\sqrt{\varepsilon}x \right) - g \left( t - i\sqrt{\varepsilon}x \right) + c_{0} \right] \partial_{t} + i\sqrt{\varepsilon} \left[ f \left( t + i\sqrt{\varepsilon}x \right) + g \left( t - i\sqrt{\varepsilon}x \right) \right] \partial_{x}$$

is the generic CKV, that is, includes the KVs, the HV, and the sp.CKVs.

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