QUANTUM TOROIDAL AND SHUFFLE ALGEBRAS,
$R$–MATRICES AND A CONJECTURE OF KUZNETSOV

ANDREI NEGUT

ABSTRACT. In this paper, we prove that the quantum toroidal algebra $U_q(\hat{\mathfrak{gl}}_n)$ is isomorphic to the double shuffle algebra of Feigin and Odesskii. The shuffle algebra viewpoint will allow us to prove a factorization formula for the universal $R$–matrix of the quantum toroidal algebra, and also prove a conjecture of Kuznetsov about the $K$–theory of affine Laumon spaces.

CONTENTS

1. Introduction
2. Quantum Affine and Quantum Toroidal Algebras
3. The Shuffle Algebra
4. The Slope Filtration and the Surjectivity of $\Upsilon$
5. The Universal $R$–matrix
6. $K$–theory of Affine Laumon Spaces
7. The fine correspondences
8. Back to $K$–theory
9. Appendix
References

1. INTRODUCTION

The quantum toroidal algebra $U_q(\hat{\mathfrak{gl}}_n)$ is defined, in the Drinfeld presentation, by certain generators and relations (see [7]) . It has a triangular decomposition:

$U_q(\hat{\mathfrak{gl}}_n) = U_q^-(\hat{\mathfrak{gl}}_n) \otimes U_q^0(\hat{\mathfrak{gl}}_n) \otimes U_q^+(\hat{\mathfrak{gl}}_n)$

The shuffle algebra was defined by Feigin and Odesskii ([8]) as the space of certain symmetric rational functions which satisfy the vanishing properties (3.6). We add certain Cartan generators and denote the resulting algebra by $\mathcal{A}$. A natural morphism between these two algebras was constructed in [4]:

$\Upsilon : U_q^{\geq}(\hat{\mathfrak{gl}}_n) \rightarrow \mathcal{A}$

Feigin conjectured that the above map is an isomorphism, and one of the main results of the present paper is to provide a proof of this. Essentially, proving this...
conjecture boils down to the statement that the shuffle algebra is generated by degree 1 elements. We will write down a bialgebra structure on the shuffle algebra \( A \), that corresponds to the Drinfeld coproduct on the quantum toroidal algebra. This will allow us to construct the double shuffle algebra \( A \). Summarizing, we prove:

**Theorem 1.1.** There exists an isomorphism \( \Upsilon : U_q(\hat{\mathfrak{g}l}_n) \cong A \).

We prove the above theorem by constructing a certain slope filtration of \( A \), which is not directly visible in the quantum toroidal picture. More concretely, we construct a decomposition:

\[
A = \prod_{\mu \in \mathbb{Q} \cup \{\infty\}} B^\geq_{\mu} \tag{1.1}
\]

by which we understand that elements of \( A \) can be written uniquely as finite sums of products of elements of the subalgebras \( B^\geq_{\mu} \), in increasing order of \( \mu \in \mathbb{Q} \cup \{\infty\} \). This decomposition respects the Hopf pairings on both sides, and we will use this in Section 5 to show that the universal \( R \)-matrix of \( A \) (with respect to the Drinfeld coproduct) is the product of the universal \( R \)-matrices in the factors of the RHS:

\[
R_A = \prod_{\mu \in \mathbb{Q} \cup \{\infty\}} R_{B_{\mu}} \in \hat{A} \hat{\otimes} \hat{A} \tag{1.2}
\]

The subalgebras \( B_\mu \) are defined by the limit property (4.1), and we will show in Lemma 4.4 that:

\[
U_q(\hat{\mathfrak{g}l}_d)^{\otimes g} \cong B^\geq_{\frac{a}{b}} \tag{1.3}
\]

where gcd\((a, b) = 1\) and gcd\((n, a) = g\). Taking into account Theorem 1.1, this constructs embeddings of quantum affine groups into the quantum toroidal algebra, associated to any rational slope \( \mu = \frac{b}{a} \). Moreover, the isomorphism \( \Upsilon \) respects the Hopf algebra structures of \( U_q(\hat{\mathfrak{g}l}_n) \) and \( A \), and thus (1.2) implies the following formula for the universal \( R \)-matrix of \( U_q(\hat{\mathfrak{g}l}_n) \):

\[
R_{U_q(\hat{\mathfrak{g}l}_n)} = \prod_{\frac{b}{a} \in \mathbb{Q} \cup \{\infty\}} R_{U_q(\hat{\mathfrak{g}l}_{\frac{a}{b} \otimes g})} \in U_q(\hat{\mathfrak{g}l}_d)^{\otimes U_q(\hat{\mathfrak{g}l}_n)} \tag{1.4}
\]

in terms of the universal \( R \)-matrices for the quantum affine groups \( U_q(\hat{\mathfrak{g}l}_d) \) for \( d \mid n \). The shuffle algebra viewpoint also has the advantage that one can write down elements explicitly as symmetric rational functions, and in particular we will show that the isomorphism (1.3) sends the root generators of the various \( U_q(\hat{\mathfrak{g}l}_{\frac{a}{b}}) \) to some particular shuffle elements of the form:

\[
X_m = \text{Sym} \left[ \frac{m(z_i, \ldots, z_j)}{1 - \frac{a_i}{z_i}} \ldots \frac{m(z_i, \ldots, z_j)}{1 - \frac{a_i}{z_i}} \prod_{i \leq a < b \leq j} \omega(z_b, z_a) \right] \in A \tag{1.5}
\]

\footnote{To be precise, \( A \) will be defined in Subsection 3.13 as an extension of the Drinfeld double \( A_* \), by adding certain Cartan elements}
in the notation of Section 4. The above shuffle elements $X_m$ can be defined for any $i \leq j$ and any Laurent polynomial $m(z_i, ..., z_j)$, and we will see that they play an important role in the formula for the $R-$matrix of $\mathcal{A} \cong U_q(\mathfrak{gl}_n)$.

We will show that shuffle elements of the form (1.5) also play a very important role in the geometric representation theory of affine Laumon spaces. These spaces appear naturally in mathematical physics, geometry and representation theory as semismall resolutions of singularities of Uhlenbeck spaces for the affine Lie algebra $\mathfrak{gl}_n$. In [13], we prove a conjecture of Braverman that relates the Nekrasov partition function of $N = 2$ rank $n$ SUSY gauge theory with adjoint matter in the presence of a complete surface operator to the elliptic Calogero-Moser system. The proof of this result makes use of $K$, the equivariant $K-$theory group of affine Laumon spaces. It was conjectured by Kuznetsov (see [5], [12]) that this group is acted on by $U_q(\mathfrak{gl}_n)$, and in the present paper we will prove this conjecture:

**Theorem 1.2.** There is a geometric action of the affine quantum group $U_q(\mathfrak{gl}_n)$ on $K$, and the latter is isomorphic to the universal Verma module.

In [1], Braverman and Finkelberg have constructed an action of the quantum group $U_q(\mathfrak{sl}_n)$ on $K$, which was extended to an action of the whole of $U_q(\mathfrak{sl}_n)$ by Tsymbaliuk in [15]. In Theorem 1.1, we will rewrite this action in terms of the shuffle algebra $\mathcal{A}$ via certain explicit formulas. The shuffle algebra viewpoint has the advantage that we can write down explicit operators on $K$, without having to resort to generators and relations. In particular, we will show that the elements $X_m$ of (1.5) act on $K$ via the fine correspondences $\mathcal{Z}_{[i;j]}$ of Section 7. We have:

$$\text{Pic}(\mathcal{Z}_{[i;j]}) \supset \mathbb{Z} \cdot l_i \oplus ... \oplus \mathbb{Z} \cdot l_j$$

where $l_k$ are the classes of the tautological line bundles (7.2). Then our main geometric result, which accounts for the word "geometric" in Theorem 1.2, is:

**Theorem 1.3.** The action of the operator $X_m \in \mathcal{A}$ on $K$ is via the class:

$$m(l_i, ..., l_j) \text{ on } \mathcal{Z}_{[i;j]}$$

which is interpreted as a correspondence on $K$ in (7.3).

A precise statement of the action in the above Theorem will be given in Section 7.26. The structure of the paper is the following:

- In Section 2 we recall the definitions of the quantum affine groups $U_q(\mathfrak{sl}_n)$, $U_q(\mathfrak{gl}_n)$ and the quantum toroidal algebras $U_q(\mathfrak{sl}_n)$, $U_q(\mathfrak{gl}_n)$

- In Section 3, we recall the definition of the shuffle algebra $\mathcal{A}^\geq$, and introduce its bialgebra structure

- In Section 4, we introduce the slope filtration on $\mathcal{A}^\geq$, and prove Theorem 1.1 and the decomposition results (1.1) and (1.3)
In Section 5, we use these decompositions to prove formulas (1.2) and (1.4) for the universal $R$–matrices.

In Section 6, we review affine Laumon spaces and their $K$–theory. We review the action of the quantum toroidal algebra ([15]) on $K$, and recast it in the language of the shuffle algebra.

In Section 7, we define the fine correspondences $3_{[i,j]}$, and use them to prove Theorem 1.3 by constructing the geometric operators (7.3).

In Section 8, we show how to derive Theorem 1.2 from Theorem 1.3. We also compute a group-like shuffle element which gives the action of the geometric Ext operator of Carlsson-Okounkov.

In Section 9, we provide proofs for some of the more computational results we need along the way.

I would like to thank Alexander Braverman, Michael Finkelberg and Andrei Okounkov for teaching me all about affine Laumon spaces, and for numerous very useful talks along the way. I am also grateful to Boris Feigin, Sachin Gautam, Aaron Silberstein, Valerio Toledano Laredo and Alexander Tsymbaliuk for their interest and help.

2. Quantum Affine and Quantum Toroidal Algebras

2.1. Consider the semigroup $\mathbb{N}^n$ of $n$–tuples of non-negative integers, which is partially ordered via $l \leq k \iff l_i \leq k_i$ for all $i \in \{1,...,n\}$. Such tuples $k \in \mathbb{N}^n$ will be called degree vectors. A particularly important example of degree vector is $k = [i;j]$ for natural numbers $i \leq j$, defined by:

$$k_a = \#\{\text{integers } \equiv a \text{ mod } n \in [i,...,j]\} \quad (2.1)$$

We introduce the following bilinear form:

$$\langle \cdot, \cdot \rangle : \mathbb{N}^n \otimes \mathbb{N}^n \to \mathbb{Z}, \quad \langle k, l \rangle = \sum_{i=1}^{n} (k_i l_i - k_i l_{i+1})$$

where $l_{n+1} := l_1$, as well as its symmetric version:

$$\langle k, l \rangle = \langle k, l \rangle + \langle l, k \rangle$$

Define:

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i = j \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

for any $i, j \in \mathbb{Z}$. Note that we do not take the indices modulo $n$ for the purpose of defining $a_{ij}$. 
2.2. In this section we will introduce the algebras \( U_q(\dot{\mathfrak{sl}}_n), U_q(\dot{\mathfrak{gl}}_n) \). The algebra \( U_q(\dot{\mathfrak{sl}}_n) \) is generated by \( e_1, \ldots, e_n \) together with invertible elements \( \varphi_1, \ldots, \varphi_n \) under the relation:

\[
[\varphi_i, \varphi_j] = 0, \quad e_i \varphi_j = q^{-a_{ij}} \varphi_j e_i \quad \forall i, j
\]

\[
[e_i, e_j] = 0 \quad \forall |j - i| > 1, \quad e_i^2 e_{i+1} - (q + q^{-1}) e_i e_{i+1} e_i + e_i e_{i+1}^2 = 0
\]

The indices \( i \) are taken modulo \( n \), so the above relations are periodic. Note that the element:

\[
\zeta = \varphi_1 \ldots \varphi_n
\]

(2.2)
is central. There is a Hopf algebra structure on \( U_q(\dot{\mathfrak{sl}}_n) \), given by the coproduct:

\[
\Delta(e_i) = \varphi_i \otimes e_i + e_i \otimes 1,
\]

\[
\Delta(\varphi_i) = \varphi_i \otimes \varphi_i,
\]

(2.3)

Finally, the assignment:

\[
(e_i, e_j) = \delta_j^i(q^2 - 1), \quad (\varphi_i, \varphi_j) = q^{a_{ij}}
\]

(2.4)
generates a Hopf pairing on \( U_q(\dot{\mathfrak{sl}}_n) \), namely a symmetric non-degenerate pairing such that:

\[
(a \cdot b, c) = (a \otimes b, \Delta(c))
\]

(2.5)

for all elements \( a, b, c \). A pairing which satisfies only the first of the relations above is called a bialgebra pairing.

2.3. To any datum as above, we may associate its Drinfeld double. Concretely, consider any bialgebra \( A \) with product \( \cdot \) and coproduct \( \Delta \). The latter will often be written in Sweedler notation:

\[
\Delta(a) = a_1 \otimes a_2
\]

by which we imply that there is a summation sign in front of the tensor in the RHS. To a bialgebra pairing on \( A \), we associate the Drinfeld double:

\[
DA = A^{\text{coop}} \otimes A
\]

It has the property that \( A^- = A^{\text{coop}} \otimes 1 \) and \( A^+ = 1 \otimes A \) are both sub-bialgebras of \( A \), and the extra condition that:

\[
a_i^+ \cdot b_2 (a_2, b_1) = b_3^+ \cdot a_2^+ (b_2, a_1) \quad \forall a, b \in A
\]

(2.5)

This latter condition governs the commutator of elements from the two factors, and it determines the bialgebra structure on the double from that of the two factors. if \( A \) is a Hopf algebra, this induces a Hopf algebra structure on \( DA \), although we will not need this. For example, the quantum algebra of affine type \( A \) is defined as:

\[
U_q(\dot{\mathfrak{sl}}_n) = DA \big|_{\varphi_i^+ \varphi_i^- = 1}
\]

(2.6)

4Since the antipode and counit can be derived quite easily from the defining properties of the Hopf algebra, and since they will not be of importance to us, we will henceforth not mention them explicitly. In other words, while all our objects are Hopf algebras, we will only need and write down their bialgebra structure and leave the rest as an exercise to the interested reader.
2.4. The larger quantum affine algebra $U_q(\mathfrak{gl}_n)$ admits a description as above, but we will prefer to use its RTT presentation ([3]). To present its generators and relations, consider the algebra $\text{Mat}_{\infty}^\geq$ of upper triangular infinite quasi-periodic matrices:

$$M = \{m_{ij}\}_{i,j \in \mathbb{Z}} \text{ such that } m_{ij} = \zeta \cdot m_{i+n,j+n} \quad \forall i,j \in \mathbb{Z}$$

where $\zeta$ is a central parameter, and multiplication of matrices is the map:

$$(MM')_{ij} = \sum_{i \leq k \leq j} \zeta^{-\lceil \frac{k}{n} \rceil} M_{ik} M'_{kj}$$

Similarly, one defines the algebra $\text{Mat}_{\infty}^\leq$ of lower triangular infinite periodic matrices, by replacing $\zeta$ with $\zeta^{-1}$. Letting $E_{ij}$ denote the elementary symmetric matrix corresponding to row $i$ and column $j$, we can define the following $R$–matrix:

$$R = \sum_{i,j=1}^{n} q^{ij} E_{ii} \boxtimes E_{jj} + (q - q^{-1}) \left[ \sum_{1 \leq i \leq n} \sum_{k \geq 1} q^{2k} E_{i,i+nk} \boxtimes E_{i,i+nk} + \sum_{1 \leq i \leq n, k \geq 0} \frac{q^{2k+1} + q^{-2k-1}}{q + q^{-1}} E_{ij} \boxtimes E_{ji} \right]$$

(2.6)

We denote the tensor product of matrices by $\boxtimes$ in order to distinguish it from the tensor product $\otimes$ in the definition of the coproduct $\Delta$. Note that the second tensor factor is lower triangular. It is straightforward to check that this matrix satisfies the Yang-Baxter equations:

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \quad (2.7)$$

among the $3$–tensors $R_{12} = R \boxtimes \text{Id}$ etc.

Remark 2.5. The above differs from the presentation of [3], but the difference is superficial. Indeed, loc. cit. uses $n \times n$ matrices valued in $\mathbb{C}[z]$ instead of periodic infinite matrices, and the passage from one language to the other is:

$$E_{ij} \cdot z^k \leftrightarrow E_{i,j+k}$$

In terms of the LHS, the $R$–matrix (2.6) takes the form:

$$R(z) = \sum_{1 \leq i \leq n} E_{ii} \boxtimes E_{ii} \frac{1 - z}{q^{-1} - qz} + \sum_{1 \leq i \neq j \leq n} E_{ii} \boxtimes E_{jj} \frac{(1 - z)^2}{(1 - qz^2)(1 - zq^{-2})} + \sum_{1 \leq i < j \leq n} E_{ij} \boxtimes E_{ij} \frac{(q - q^{-1})(1 - z)}{(1 - zq^2)(1 - zq^{-2})} + \sum_{1 \leq i < j \leq n} E_{ji} \boxtimes E_{ij} \frac{(q - q^{-1})z(1 - z)}{(1 - zq^2)(1 - zq^{-2})}$$

The above equals the $R$–matrix of (3.7) in [3], multiplied by the constant rational function:

$$\frac{1 - z}{q^{-1} - qz}$$

and hence satisfies the Yang-Baxter relation to the same extent as the construction in loc. cit. The two $R$–matrices give rise to the same algebra $U_q(\mathfrak{gl}_n)$ (via the construction that will be explained in the next section), but the systems of root
generators will be different.

2.6. Let us define the algebra $U_q^>(\mathfrak{gl}_n)$ to be generated by the central element $\zeta$, invertible elements $\psi_1, \ldots, \psi_n$, and the root generators:

$$e_{[i; j]}, \quad \forall i \in \{1, \ldots, n\}, \quad \forall i \leq j,$$

(2.8)

We write $e_{[i+n; j+n]} = e_{[i; j]}$ and $\psi_{i+n} = \zeta \cdot \psi_i$ for all $i, j$. Let us place all the generators of $U_q^>(\mathfrak{gl}_n)$ into an upper triangular infinite quasi-periodic matrix $T$:

$$T_{ij} = \psi_{j-1}^{-1} e_{[i; j-1]}$$

for all $i \leq j$, where we make the convention that $e_{[i; i-1]} = 1$. Then the relations between these generators are given by the following RTT relation:

$$RT_1 T_2 = T_2 T_1 R,$$

(2.9)

where $T_1 = T \boxtimes 1$ and $T_2 = 1 \boxtimes T$. Further, let us consider the coproduct:

$$\Delta(T) = T \otimes T$$

(2.10)

and the pairing:

$$(T_1, T_2^\dagger) = R$$

(2.11)

where $\dagger$ simply means transpose. The algebra $U_q^>(\mathfrak{gl}_n)$ is also graded by $\mathbb{N}^n$, with $\psi_i$ in degree 0 and $e_{[i; j]}$ in degree vector $[i; j] \in \mathbb{N}^n$ (see (2.1) for the definition).

Remark 2.7. While the graded components of $U_q^>(\mathfrak{gl}_n)$ are infinite-dimensional, those of the subalgebra:

$$U_q^+(\mathfrak{gl}_n) \subset U_q^>(\mathfrak{gl}_n)$$

generated by the $e_{[i; j]}$ (without the $\psi$’s) are finite-dimensional. The dimension of each graded component is equal to the number of partitions of that degree vector into the various $[i; j]$ (as in the PBW theorem).

2.8. Explicitly in the root generators, the coproduct is given by:

$$\Delta(e_{[i; j]}) = \sum_{k=1}^{j+1} \frac{\psi_{j+1}}{\psi_k} e_{[i; k-1]} \otimes e_{[k; j]},$$

(2.12)

for any $i \leq j$. The Hopf pairing is given by:

$$(e_{[i; i+nk-1]}, e_{[i; i+nk-1]}) = (q - q^{-1}) q^{2k-1}$$

(2.13)

and:

$$(e_{[i; i+nk-1]}, e_{[i'; i'+nk-1]}) = \frac{(q - q^{-1})(q^{2k} - q^{-2k})}{q + q^{-1}}$$

(2.14)

for $i' \neq i$ modulo $n$, while:

$$(e_{[i; ij]}, e_{[i; ij]}) = \frac{(q^2 - 1)(q^{2k+1} + q^{-2k-1})}{q + q^{-1}}$$

(2.15)
if \( j = i + nk + r \) for some \( r \in \{0, \ldots, n-2\} \). All other pairings between the \( e_{[i,j]} \) vanish for degree reasons.

### 2.9. We will write:

\[
U^{\geq}_{q,*}(\hat{\mathfrak{g}}(n)) \subset U^{\geq}_{q}(\hat{\mathfrak{g}}(n))
\]

for the slightly smaller Hopf subalgebra spanned by the root generators \( e_{[i,j]} \) together with the diagonal elements \( \varphi_i = \psi_i + / \psi_i \). Note that \( \varphi_i \) only depends on \( i \) modulo \( n \), and:

\[
\varphi_1 \cdots \varphi_n = \zeta
\]
is central. Observe that this matches (2.2).

#### Proposition 2.10. The above data makes \( U^{\geq}_{q,*}(\hat{\mathfrak{g}}(n)) \subset U^{\geq}_{q}(\hat{\mathfrak{g}}(n)) \) into Hopf algebras with Hopf pairings. Then we define:

\[
U_q(\hat{\mathfrak{g}}(n)) = DU^{\geq}_{q,*}(\hat{\mathfrak{g}}(n))|_{\psi_i^+ \psi_i^- = 1}
\]

\[
U_{q,*}(\hat{\mathfrak{g}}(n)) = DU^{\geq}_{q,*}(\hat{\mathfrak{g}}(n))|_{\varphi_i^+ \varphi_i^- = 1}
\]
The injective Hopf algebra morphism \( U_q^{\geq}(\mathfrak{sl}(n)) \rightarrow U_{q,*}(\hat{\mathfrak{g}}(n)) \subset U_q^{\geq}(\hat{\mathfrak{g}}(n)) \) given by the assignments \( e_i \rightarrow e_{[i,j]} \) and \( \varphi_i \rightarrow \varphi_i \) thus extends to an injective morphism:

\[
U_q(\mathfrak{sl}(n)) \rightarrow U_{q,*}(\hat{\mathfrak{g}}(n)) \subset U_q(\hat{\mathfrak{g}}(n))
\]
of the Drinfeld doubles.

#### Proof. The coassociativity of \( \Delta \) is immediate from (2.10). To show that it extends multiplicatively from the generators to the whole algebra, we need to prove that it respects relation (2.9):

\[
\Delta(RT_1 T_2) = R\Delta(T_1)\Delta(T_2) = R(T_1 \otimes T_1)(T_2 \otimes T_2) =
\]

\[
= RT_1 T_2 \otimes T_1 T_2 = T_2 T_1 R \otimes T_1 T_2 = T_2 T_1 \otimes RT_1 T_2 = T_2 T_1 \otimes T_2 T_1 R =
\]

\[
= (T_2 \otimes T_2)(T_1 \otimes T_1) R = \Delta(T_2)\Delta(T_1) R = \Delta(T_2 T_1 R)
\]

where the equalities on the first and third lines are simply formal, and those on the second line (except for the middle ones) are applications of (2.9). In order to prove that (2.11) extends to a bialgebra pairing on the whole algebra, we need to check that it preserves the defining relations (2.9). In other words, we need to check that:

\[
(R_{12} T_1 T_2, T_3) = (T_2 T_1 R_{12}, T_3)
\]

where now \( R_{12} = R \otimes 1 \), and \( T_1 = T \otimes 1 \otimes 1 \), \( T_2 = 1 \otimes T \otimes 1 \), \( T_3 = 1 \otimes 1 \otimes T \). Using the bialgebra property, we see that:

\[
(R_{12} T_1 T_2, T_3) = R_{12} (T_1 \otimes T_2, \Delta(T_3)) = R_{12} (T_1, T_1^\dagger)(T_2, T_3^\dagger) = R_{12} R_{13} R_{23}
\]

\[
(T_2 T_1 R_{12}, T_3) = (T_2 \otimes T_1, \Delta(T_3)) R_{12} = (T_2, T_3^\dagger)(T_1, T_1^\dagger) R_{12} = R_{23} R_{13} R_{12}
\]
The equality between the RHS of the two above relations is precisely the Yang-Baxter equation (2.7), so we conclude that the two LHS are also equal. \( \Box \)
2.11. Consider the algebra $U_q(\tilde{\mathfrak{sl}}_n)$ generated by a central element $\kappa$ and the coefficients of the series:

$$e_i(z) = \sum_{k \in \mathbb{Z}} e_{i,k} z^{-k}, \quad \varphi_i(z) = \varphi_i + \sum_{k \geq 1} \varphi_{i,k} z^{-k}, \quad \forall \ i \in \{1, \ldots, n\}$$

such that the leading coefficients $\varphi_i$ are invertible. We will write:

$$e_i(\kappa z) = e_i(z)$$

and:

$$\varphi_i(\kappa z) = \varphi_i(z)$$

(2.16)

for all $i$, and hence we may think of the indices as being arbitrary integers. Then we require that the coefficients of the series $\varphi_i(z)$ commute with each other, and also that:

$$e_i(z) \cdot \varphi_j(w) = \frac{zq^{a_{ij}} - w}{z - wq^{a_{ij}}} \varphi_j(w) \cdot e_i(z)$$

(2.17)

$$e_i(z) \cdot e_j(w) = \frac{zq^{a_{ij}} - w}{z - wq^{a_{ij}}} e_j(w) \cdot e_i(z)$$

(2.18)

and:

$$e_{i\pm 1}(w)e_i(z)e_{i'}(z') - (q + q^{-1})e_i(z)e_{i\pm 1}(w)e_i(z') + e_i(z)e_{i'}(z')e_{i\pm 1}(w) +$$

+ same expression with $z$ and $z'$ switched = 0

(2.19)

for all $|i - j| \leq n/2$. This algebra is bigraded by $\mathbb{N}^n \times \mathbb{Z}$:

$$\deg e_{i,k} = (\varsigma_i, k), \quad \deg \varphi_{i,k} = (0, k)$$

where $\varsigma_i = (0, ..., 0, 1, 0, ..., 0)$. We will write $U_q(\tilde{\mathfrak{sl}}_n)_{k,d}$ for the graded components.

2.12. $U_q(\tilde{\mathfrak{sl}}_n)$ is a Hopf algebra with coproduct $\Delta(\varphi_i(z)) = \varphi_i(z) \otimes \varphi_i(z)$ and:

$$\Delta(e_i(z)) = \varphi_i(z) \otimes e_i(z) + e_i(z) \otimes 1$$

(2.20)

We claim that the following assignments generate a Hopf pairing via (2.4):

$$(e_i(z), e_j(w^{-1})) = \delta_i^j(q^2 - 1)\delta \left( \frac{z}{w} \right)$$

$$(\varphi_i(z), \varphi_j(w^{-1})) = \frac{zq^{a_{ij}} - w}{z - wq^{a_{ij}}}$$

(2.21)

for all $|i - j| \leq n/2$. The quantum toroidal algebra $U_q(\tilde{\mathfrak{sl}}_n)$ is defined as the double of this Hopf algebra:

$$U_q(\tilde{\mathfrak{sl}}_n) = DU_q(\tilde{\mathfrak{sl}}_n)|_{\varphi_i^+\varphi_i^- = 1}$$

---

6There are such commutation relations for any pair $i, j \in \mathbb{Z}$, and they follow from the ones above and the quasi-periodicity relation (2.16)
2.13. We will enrich the algebra $U_q(\hat{\mathfrak{sl}}_n)$ by adding one more series of Cartan generators $\psi_i^\pm(z) = \psi_i^\pm + \sum_{k \geq 1} \psi_{i,k}^\pm z^{-k}$ such that:

$$\varphi_i^\pm(z) = \frac{\psi_{i+1}(z)}{\psi_i^\pm(zq^{-1})}$$  

(2.22)

We require these new generators to satisfy the quasi-periodicity conditions:

$$\psi_{i+n}(z) = \zeta_0 \psi_i(\zeta z)$$

for any $i \in \mathbb{Z}$, where the central element $\zeta_0$ is given by (2.2). We impose the following relations among these and the $e_i^\pm(z)$:

$$e_i^+(z) \cdot \psi_j^\pm(w) = \frac{z - wq^{j-i} - wq^{j-i+1}}{wq^{j-i} - z} \cdot \psi_j^\pm(w) \cdot e_i^+(z)$$  

(2.23)

$$e_i^-(z) \cdot \psi_j^\pm(w) = \frac{wq^{j-i+1} - zq^{j-i+1}}{wq^{j-i} - z} \cdot \psi_j^\pm(w) \cdot e_i^-(z)$$  

(2.24)

for any $|i - j| \leq n/2$, and note that they are compatible with (2.17). We further require that $\psi_i^\pm \psi_i^\pm = 1$, and denote the resulting algebra by $U_q(\hat{\mathfrak{sl}}_n)$. It contains the algebra $U_q(\mathfrak{sl}_n)$ by embedding $\varphi$’s into $\psi$’s via (2.22).

2.14. We have embeddings:

$$U_q(\mathfrak{sl}_n) \hookrightarrow U_q(\hat{\mathfrak{sl}}_n), \quad e_i^\pm \mapsto e_i^\pm, \quad \varphi_i \mapsto \varphi_i$$

which can be upgraded to:

$$U_q(\mathfrak{gl}_n) \hookrightarrow U_q(\hat{\mathfrak{sl}}_n), \quad U_q(\hat{\mathfrak{gl}}_n) \hookrightarrow U_q(\hat{\mathfrak{sl}}_n),$$

(2.25)

A priori, the latter two maps are not very explicit in terms of the root generators $e_{[i:j]}$. The particular case of Lemma 4.4 when $a = 1$ and $b = 0$ shows how $U_q(\mathfrak{gl}_n)$ embeds into the double shuffle algebra $A$ (to be defined in the next section), which by Theorem 1.1 is isomorphic to $U_q(\hat{\mathfrak{gl}}_n)$.

2.15. An important step which has so far been overlooked is the non-degeneracy of the Hopf pairings on our algebras. Since this aspect will be crucial for us, we will give a proof:

Proposition 2.16. The Hopf pairings (2.11) and (2.21) on $U_q(\mathfrak{gl}_n)$ and $U_q(\hat{\mathfrak{sl}}_n)$ are non-degenerate.

**Proof** We will prove the result for $U_q^\pm(\mathfrak{sl}_n)$ and leave the simpler case of $U_q^\pm(\mathfrak{gl}_n)$ as an exercise. We will use a modification of the argument in [10] to show that the Hopf pairing on $U_q^\pm(\mathfrak{sl}_n)$ is non-degenerate. Assume that the exists $x^+ \in U_q(\mathfrak{sl}_n)_{k,d}$ such that $(x, \cdot) = 0$, and suppose $k \in \mathbb{N}$ is minimal with respect to this property.

---

7In fact, as we will see in Subsection 5.6, $U_q^\pm(\mathfrak{gl}_n) = U_q^\pm(\mathfrak{sl}_n) \otimes U_q^\pm(\mathfrak{gl}_1)$, and the non-degeneracy of the pairing on $U_q^\pm(\mathfrak{sl}_n)$ can be proved word by word as in the present proof.
It is easy to see that \( k \notin \{ \varsigma_1, \ldots, \varsigma_n \} \). Together with property (2.4), the minimality implies that all intermediate terms in the coproduct \( \Delta(x^+) \) vanish:

\[
\Delta(x^+) \in \varphi \otimes x^+ + U_q^> (\mathfrak{sl}_n)_{0,>0} \otimes U_q^> (\mathfrak{sl}_n)_{k,<d} + x^+ \otimes 1
\]

where \( \varphi = \prod_{i=1}^{n} \varphi_i^j \). Looking back at the definition of the Drinfeld double \( U_q (\mathfrak{sl}_n) \) in (2.5), the above relation implies that:

\[
[x^+, f_{i,e}] = 0 \quad \forall i \in \{1, \ldots, n\} \quad \forall e \in \mathbb{Z}
\]

(2.26)

Consider now any simple \( U_q (\mathfrak{sl}_n) \) module \( L_\lambda \) with lowest weight vector. By (2.26), we have:

\[
U_q (\mathfrak{sl}_n) x \cdot v_\lambda = U_q^> (\mathfrak{sl}_n) \otimes U_q^- (\mathfrak{sl}_n) x \cdot v_\lambda = U_q^> (\mathfrak{sl}_n) x \cdot v_\lambda \subset L_\lambda
\]

Because \( x \) has degree \( k > 0 \), the submodule \( U_q^> (\mathfrak{sl}_n) x \cdot v_\lambda \) will not contain the lowest weight vector \( v_\lambda \). It will therefore be a proper submodule of the simple module \( L_\lambda \), and therefore zero. We conclude that \( x \cdot v_\lambda = 0 \), which is impossible for a generic lowest weight \( \lambda \).

\[
\square
\]

### 3. The Shuffle Algebra

#### 3.1. We will now present the shuffle algebra, which is a close variant of the construction of [8]. For each index \( i \in \{1, \ldots, n\} \), we consider an infinite set of variables \( z_{i1}, z_{i2}, \ldots \). We endow the variable \( z_{ij} \) with weight \( i \), and we will often use the shorthand notation:

\[
\text{wt } z_{ij} = i
\]

for this. We may (and sometimes will) consider the weight to be quasi-periodic modulo \( n \), in the sense that a variable \( z \) of weight \( i + n \) may be replaced by the variable \( \kappa z \) of weight \( i \) \(^9\). Let us define \( \mathbb{K} = \mathbb{C}(q, \kappa) \), and consider the \( \mathbb{K} \)-vector space:

\[
\mathcal{V} = \bigoplus_{k \in \mathbb{N}^n} \text{Sym}_k (z_{11}, \ldots, z_{1k_1}, \ldots, z_{n1}, \ldots, z_{nk_n})
\]

(3.1)

where \( \text{Sym}_k \) will always mean rational functions that are symmetric in \( z_{i1}, z_{i2}, \ldots \) for each \( i \in \{1, \ldots, n\} \) separately.

#### 3.2. We endow the vector space \( \mathcal{V} \) with a \( \mathbb{K} \)-algebra structure via the shuffle product:

\[
P(..., z_{i1}, \ldots, z_{ik_i}, \ldots) \ast P'(..., z_{i1}, \ldots, z_{ik'_i}, \ldots) =
\]

\[
= \text{Sym } \left[ P(..., z_{i1}, \ldots, z_{ik_i}, \ldots) P'(..., z_{i,k_i+1}, \ldots, z_{i,k_i+k'_i}, \ldots) \prod_{i,i'=1 \ldots d} \prod_{j,j'>k'_i} \omega(z_{ij}, z_{i'j'}) \right]
\]

---

\(^8\)See [9] for more details on category \( \mathcal{O} \) for quantum toroidal algebras

\(^9\)This convention is meant to match (2.16)
where for $|\text{wt } x - \text{wt } y| \leq n/2$ we write:

$$
\omega(x, y) = \begin{cases} 
\frac{x-y}{xq^{i-yq}} & \text{if } \text{wt } x = \text{wt } y \\
\frac{x-y}{xq^{i-yq}} & \text{if } \text{wt } x = \text{wt } y + 1 \\
1 & \text{otherwise}
\end{cases} \quad (3.3)
$$

The function $\omega$ may be defined for variables with arbitrary weights, via the quasi-periodicity by $\kappa$ modulo $n$. In the above, $\text{Sym}$ denotes symmetrization with respect to all:

$$
k! := \prod_{i=1}^{n} k_i! \quad (3.4)
$$

permutations that preserve the weights of the variables modulo $n$. The algebra $\mathcal{V}$ is graded by $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ (the number of variables $z_{ij}$) and also by the total homogenous degree $d \in \mathbb{Z}$ of the rational functions.

3.3. Define the **shuffle algebra** $A^+$ as the subspace of (3.1) consisting of rational functions of the form:

$$
P(z_{11}, \ldots, z_{nk_n}) = \frac{1}{\prod_{1 \leq i < j \leq k_i} z_{ij} - z_{ij'}} \cdot \frac{p(z_{11}, \ldots, z_{nk_n})}{\prod_{i=1}^{n} \prod_{1 \leq j \leq k_i} \prod_{1 \leq j' \leq k_i+1} \omega(z_{i+1,j}, z_{ij'})} \quad (3.5)
$$

where $p$ is a symmetric Laurent polynomial that satisfies the **wheel conditions**. By this we mean that for all $i \in \{1, \ldots, n\}$, we require:

$$
p(..., zq^{-1}, ..., z, ..., zq, ...) = 0 \quad (3.6)
$$

where the three variables singled out have weights $i$, $i \pm 1$, and all other variables of $p$ are arbitrary. The following is easy to prove, and we leave it to the reader:

**Proposition 3.4.** $A^+$ is closed under the product (3.2), and is thus an algebra.

3.5. The shuffle algebra is bigraded by the number of variables $k \in \mathbb{N}^n$ and also by the total homogenous degree $d \in \mathbb{Z}$ of our rational functions. We denote by:

$$
A^+ = \bigoplus_{k \in \mathbb{N}^n} A_k, \quad A_k = \bigoplus_{d \in \mathbb{Z}} A_{k,d}
$$

the bigraded pieces of the shuffle algebra. We also have the map $\tau : A^+ \to A^+$:

$$
A_{k,d} \ni P(..., z_{ij}, ...) \overset{\tau}{\mapsto} P(..., z_{ij}^{-1}, ...) \prod_{i=1}^{n} \prod_{j=1}^{k_i} \prod_{j'=1}^{k_i+1} \omega(z_{i+1,j}, z_{ij'}) \in A_{k,-d} \quad (3.7)
$$

It is easy to see that this map is an involution and an anti-automorphism:

$$
\tau \circ \tau = 1, \quad \tau(f \ast g) = \tau(g) \ast \tau(f)
$$
3.6. Define the algebra $A^\geq$ to be generated by $A^+$ and the coefficients of the series:

$$\varphi_i(z) = \varphi_i + \sum_{k \geq 1} \varphi_{i,k} z^{-k}$$

for all $i \in \{1, ..., n\}$. We ask that the $\varphi_i$ are invertible and that all the $\varphi_{i,k}$ commute. We impose the following relation between the $\varphi_{i,k}$ and shuffle elements $P \in A_k$:

$$P \ast \varphi_j(w) = \varphi_j(w) \ast \left[ P(..., z_{ih}, ...) \prod_{i=1}^n \prod_{h=1}^{k_i} \frac{z_{ih} q^{a_{ij}} - w}{z_{ih} - w q^{a_{ij}}} \right]$$

(3.8)

To ensure that the factor in the RHS is correct, the weights of the $z$ variables must be changed so that $|i - j| \leq n/2$. We will see that the larger algebra $A^\geq$ is actually a Hopf algebra, and we will prove this in stages.

**Proposition 3.7.** There is a bialgebra structure on $A^\geq$, with coproduct given by:

$$\Delta(\varphi_i(z)) = \varphi_i(z) \otimes \varphi_i(z),$$

$$\Delta(P) = \sum_{l \leq k} \left[ \prod_{i \leq l, j \leq \nu} \varphi_i(z_{ij}) \right] P(z_{ij} \leq l, z_{ij} > l) \prod_{i \leq j' \leq n} \prod_{j' > l} \omega(z_{ij'}, z_{ij})$$

(3.9)

for all $P \in A_k$.

**Remark 3.8.** To think of (3.9) as a well-defined tensor, we expand the rational function in the RHS in non-negative powers of $z_{ij}/z_{ij'}$ for $j \leq l_i, j' > l_i$, thus obtaining an infinite sum of monomials. In each of these monomials, we put the $\varphi_{i,k}$ to the very left of the expression, then all powers of $z_{ij}$ with $j \leq l_i$, then the $\otimes$ sign, and finally all powers of $z_{ij}$ with $j > l_i$. The resulting expression will be a power series, and therefore lies in a completion of $A^\geq \otimes A^\geq$. The above Proposition is proved exactly like Proposition 4.1 of [14], so we will leave the proof as an exercise.

3.9. Let us consider the rational function:

$$\eta(x, y) = \frac{(xq^{-1} - yq)^2}{(x - y)(xr^{-1} - yr)}$$

(3.10)

When we specialize $r = q$, we have $\eta(x, y) = \omega^{-1}(x, y)$ with $wt x = wt y$. However, in the integral formulas to come, we will often think of $|q| < 1$ and $|r| > 1$ and then specialize $r = q$. The meaning of this apparently paradoxical situation is that the size of $r$ and $q$ determine which residues are picked up by our contour integrals. The values of these residues are analytic functions of $r$ and $q$, and they make sense irrespective of the sizes of the parameters.

**Proposition 3.10.** Assume $|q| > 1$ and $|r| < 1$. There exists a bialgebra pairing on $A^\geq$, given by:

$$\langle \varphi_i(z), \varphi_j(w^{-1}) \rangle = \frac{zq^{a_{ij}} - w}{z - w q^{a_{ij}}}$$
for all $|i - j| \leq n/2$, and:

$$(P, P') = \left(\frac{q^2 - 1}{k!}\right) \int_{|z_{ij}| = 1} P(..., z_{ij}, ...) P'(..., z_{ij}^{-1}, ...) \prod_{1 \leq i \leq n} \prod_{1 \leq j \neq j' \leq k_i} \eta(z_{ij}, z_{ij'}) \prod_{1 \leq j \leq k_i} Dz_{ij} \bigg|_{r=q}$$

for any $P, P' \in A_k$, where we always write $Dz = \frac{dz}{2\pi i}$.

The above Proposition will be proved in the Appendix. Note that it gives a structure of bialgebra on $A^\geq$, which turns out to be a Hopf algebra. We will not explicitly spell out the antipode and counit map, since they are forced upon us by the rest of the data. We will not need them in this paper.

3.11. We are now ready to give a full statement of Theorem 1.1, when $\mathfrak{gl}_n$ is replaced by $\mathfrak{sl}_n$. The claim is that there exists a Hopf algebra isomorphism:

$$\Upsilon : U_q^\geq (\mathfrak{sl}_n) \to A^\geq$$

given by:

$$U_q^\geq (\mathfrak{sl}_n) \ni e_{i,k} \to z_i^k \in A^\geq$$

$$U_q^\geq (\mathfrak{sl}_n) \ni \varphi_{i,k} \to \varphi_{i,k} \in A^\geq$$

It has already been proved in [4] that the assignment (3.12) extends to an algebra morphism, and the fact that the assignment (3.13) also extends is trivial from our choice of relations. It is easy to check that the above assignments match the coproduct and the pairing on the two sides (and the antipode and counit maps are forced by the data). Since all the Hopf data is preserved, we conclude that $\Upsilon$ determines a morphism between the Drinfeld doubles:

$$\Upsilon : U_q (\mathfrak{sl}_n) \to A_\ast := DA^\geq |_{\varphi_1^\ast = \varphi_1^{-1}}$$

which is the $\mathfrak{sl}_n$ case of Theorem 1.1. The main work we need to do is to establish the above is to prove the injectivity and surjectivity of the map $\Upsilon$ in (3.11).

3.12. To prove the injectivity of the map (3.11), assume that $\Upsilon(a) = 0$ for some $a \in U_q^\geq (\mathfrak{sl}_n)$. Because $\Upsilon$ preserves the pairing, we have:

$$(\Upsilon(a), \Upsilon(b)) = 0 \implies (a, b) = 0 \quad \forall b \in U_q^\geq (\mathfrak{sl}_n)$$

The fact that $a = 0$ is equivalent to the non-degeneracy of the pairing on $U_q^\geq (\mathfrak{sl}_n)$, which we established in Proposition 2.16. We conclude that the map of (3.11) is injective, and therefore all we still need to show is its surjectivity. This will be the content of Section 4, but first we will give the full statement of Theorem 1.1.
3.13. By analogy with Subsection 2.13, we define the algebra $A$ to be generated by the double shuffle algebra $A^*$ and commuting Cartan generators $\psi_\pm(z) = \psi_\pm(z) + \sum_{k \geq 1} \psi_{\pm, k} z^{-k}$ such that:

$$\varphi_\pm(z) = \frac{\psi_{\pm, 1}(z)}{\psi_\pm(zq^{-1})}$$

We impose the following relations among these and shuffle elements:

$$P_\pm \ast \psi_\pm^j(w) = \psi_\pm^j(w) \ast \left[ P_\pm(..., z_{ih}, ...) \prod_{i=1}^{n} \prod_{h=1}^{k_i} \omega(wq, z_{ih})^{\pm 1} \right]$$

(3.15)

as well as the condition $\psi_+ \psi_- = 1$. Note that the algebras $A$ and $U_q(\hat{\mathfrak{sl}}_n)$ are obtained from $A^*$ and $U_q(\hat{\mathfrak{gl}}_n)$ (respectively) by adding these extra Cartan generators, which satisfy the same commutation relations. Hence the induced map:

$$\Upsilon : U_q(\hat{\mathfrak{gl}}_n) \rightarrow A$$

is an isomorphism if and only if the map (3.14) is. The latter is a map between Drinfeld doubles, which we have shown to be an isomorphism, as long as we can prove that the map (3.11) is surjective. Hence this concludes the proof of Theorem 1.1, modulo the surjectivity of the map (3.11). Proving surjectivity was our initial drive for introducing many of the concepts in the next section.

4. The Slope Filtration and the Surjectivity of $\Upsilon$

4.1. One of the main actors of this section will be the slope filtration of the shuffle algebra $A$, which will allow us to prove the surjectivity of the map $\Upsilon$ of (3.11), and also to factor the universal $R$–matrix. Note that the graded components $A_{k,d}$ of the shuffle algebra are, in general, infinite-dimensional $\mathbb{K}$–vector spaces. For $P \in A_{k,d}$ and $\mu \in \mathbb{R}$, if the limits:

$$\lim_{\xi \rightarrow \infty} P(..., z_{i1}, ..., z_{il}, \xi z_{i1}, ..., \xi z_{il}, ..., \xi z_{ik}, ...)$$

exist and are finite for all degree vectors $0 \leq l \leq k$, then we say that $P$ has slope $\leq \mu$. We denote by:

$$A_{k,d}^\mu \subset A_{k,d}$$

the subspace of shuffle elements of slope $\leq \mu$. The following proposition is a simple exercise, based on the fact that $\omega(x, y)$ remains finite when either of the variables goes to $\infty$. We leave its proof to the interested reader, and note that it is analogous to Proposition 2.3 of [14].

**Proposition 4.2.** The vector space:

$$A^\mu = \bigoplus_{k \in \mathbb{N}^n} A^\mu_{k,d} \subset A^+$$

is a subalgebra. In other words, the property of having slope $\leq \mu$ is preserved under the shuffle product.
4.3. The notion of slope is related to the coproduct of (3.9). Namely, suppose we take a shuffle element $P \in A^\mu_{k,d}$ and we look at the tensor $\Delta(P)$. By definition of the coproduct in (3.9), we have:

$$\Delta(P) = \sum_{l \leq k} \varphi_{k-l} \cdot \lim_{\xi \to -\infty} \frac{P(z_{i,j} \leq l_i \otimes \xi \cdot z_{i,j} > l_i)}{\xi^{k-l} q^{l(k-l)}} + \text{(slope > } \mu) \otimes \text{(slope < } \mu)$$

(4.2)

where $\varphi_l = \prod_{n=1}^n \varphi_l^i$ for all $l \in \mathbb{N}^n$. Recall that the tensor product inside the rational function $P$ means that all powers of $z_{i,j} \leq l_i$ go to the left of the $\otimes$ sign, while all powers of $z_{i,j} > l_i$ go to the right. In particular, it is easy to see that the leading term:

$$\Delta_\mu(P) = \sum_{l \leq k} \varphi_{k-l} \cdot \lim_{\xi \to -\infty} \frac{P(z_{i,j} \leq l_i \otimes \xi \cdot z_{i,j} > l_i)}{\xi^{k-l} q^{l(k-l)}}$$

(4.3)

is a coproduct on the subalgebra:

$$\mathcal{B}^\geq_\mu = \langle \varphi^\pm_1, ..., \varphi^\pm_n \rangle \ast \bigsqcup_{k \in \mathbb{N}^n} A^\mu_{k,d} \subset A^\geq$$

(4.4)

The coproduct $\Delta_\mu$ should be interpreted as the leading term of $\Delta$ to order $\mu$. It is straightforward to show that the pairing $(\cdot, \cdot)$ on $\mathcal{B}^\geq_\mu$ respects the bialgebra property with the above coproduct. One of the main results of this section is:

**Lemma 4.4.** For any coprime integers $a$ and $b$, we have an isomorphism of Hopf algebras:

$$\left[ U_q^\geq_* (\mathfrak{gl}_{\gcd(n,a)}) \right] \otimes \mathbb{Z}_{\gcd(n,a)} \cong \mathcal{B}^\geq_\mu$$

(4.5)

We will prove this by constructing the elements in the RHS that correspond to the root generators of $U_q^\geq_* (\mathfrak{gl}_{\gcd(n,a)})$ of Subsection 2.6.

**Remark 4.5.** Note that this implies the isomorphism of the Drinfeld doubles of the algebras in question:

$$\left[ U_q_* (\mathfrak{gl}_{\gcd(n,a)}) \right] \otimes \mathbb{Z}_{\gcd(n,a)} \cong \mathcal{B}^\geq_\mu \cong \mathcal{D}B^\geq_\mu_{\mu} |_{\varphi^+_i \varphi^-_i = 1}$$

If we enrich both these algebras by Cartan generators $\psi^\pm_i$ such that $\varphi^\pm_i = \psi^\pm_{i+1}/\psi^\pm_i$ and $\psi^+_i \psi^-_i = 1$, modulo relations as in Subsections 2.13 or 3.13, then this implies the following isomorphism:

$$\left[ U_q (\mathfrak{gl}_{\gcd(n,a)}) \right] \otimes \mathbb{Z}_{\gcd(n,a)} \cong \mathcal{B}^\geq_\mu$$

precisely what is being claimed in (1.3).
As our first step toward proving the above lemma, let us prove an upper bound on the dimension of the graded pieces of the subalgebra $B^+_{\mu} = B_{\mu} \cap A^+$. We will actually prove the following more general result.

**Lemma 4.7.** For any $\mu, d, k$, the dimension of $A_{k,a}^\mu$ is at most equal to the number of unordered collections:

$$C = \{(i_a; j_a, d_a)\}_{a \in \{1, \ldots, t\}},$$

such that the following conditions hold:

$$\sum_{a=1}^t [i_a; j_a] = k \quad \sum_{a=1}^t d_a = d$$

$$d_a \leq \mu(j_a - i_a + 1) \quad \forall a \in \{1, \ldots, t\}$$

Recall that $[i; j]$ denotes the particular degree vector in (2.1).

**Remark 4.8.** When $\mu = \frac{k}{a} = \frac{d}{t}$, note that the number of such collections is precisely the dimension of:

$$U_q^+ (\mathfrak{gl}_{n, \gcd(n, a)}) \otimes \gcd(n, a)$$

(see the LHS of (4.5)). This is a consequence of Remark 2.7.

**Proof** This lemma and its proof are a generalization of Lemma 2.14 from [6] (see also Proposition 2.4 of [14]). Any shuffle element $P$ of bidegree $(k, d)$ is determined by the Laurent polynomial $p$ of (3.5), which has total degree $d + \sum_{i=1}^n k_i k_{i+1}$. The fact that the limits (4.1) are finite implies that for all $0 \leq l \leq k$,

$$\deg \ldots, z_{i_1}, \ldots, z_{i_l}, \ldots, p(\ldots, z_{i_1}, \ldots, z_{i_l}, \ldots) \leq \mu \sum_{i=1}^n l_i + \sum_{i=1}^n (k_i l_{i+1} + l_i k_{i+1} - l_i l_{i+1})$$

(4.9)

Let $A_{k,d}$ denote the set of symmetric Laurent polynomials $p$ which satisfy the wheel conditions (3.6) and the above degree constraints. Then we need to prove the desired bound for the dimension of $A_{k,d}^\mu$. Consider any unordered set:

$$C = \{[i_1; j_1], \ldots, [i_t; j_t]\}$$

of intervals modulo $n$. If $k = \sum_{a=1}^t [i_a; j_a]$, then we will call such a set a partition of $k$ and write $C \vdash k$. We order the constituent intervals from 1 to $t$ in descending order of length, where those of the same length can be placed in any order. Given $k$ variables (which means $k_i$ variables of weight $i$ for each $i \in \{1, \ldots, n\}$), we can play the following game. Split up the variables into groups, each associated to one of the intervals $[i_a; j_a]$, and set the variables in each interval equal to $y_a q^{-i_a}, \ldots, y_a q^{-j_a}$. This evaluation gives rise to a linear operator:

$$\phi_C : A_{k,d}^\mu \longrightarrow \mathbb{K}[y_1, \ldots, y_t], \quad p(\ldots, z_{i,j}, \ldots) \longrightarrow p|_{y_a q^{-i_a}, \ldots, y_a q^{-j_a}, \ldots}$$

(4.10)

Because $p$ is symmetric, the way we split up the $k$ variables among the intervals $[i_a; j_a]$ does not matter. The image of $\phi_C$ is a partially symmetric polynomial, i.e. is symmetric in $y_a$ and $y_b$ whenever $(i_a, j_a) - (i_b, j_b) \in Z(1, 1)$. Let us denote
by $S$ the space of such partially symmetric Laurent polynomials which satisfy the above degree condition and also the wheel conditions (3.6). We can filter the vector space $A_{k,d}^{\mu}$ by using these evaluation maps:

$$A_{k,d}^{\mu} = \bigcap_{C' > C} \ker \phi_{C'}$$

where $C' > C$ is the partial ordering on partitions, lexicographic in the set of lengths of these intervals (if the partition $C$ is maximal, then we set $A_{k,d}^{\mu} = A_{k,d}^{0}$).

It is easy to see that:

$$\dim A_{k,d}^{\mu} \leq \sum_{C} \dim \phi_{C}(A_{k,d}^{\mu})$$

Then our lemma is implied by the following assertion:

$$\dim \phi_{C}(A_{k,d}^{\mu,C}) \leq \#(d_1, ..., d_t) \quad (4.11)$$

such that conditions (4.7) and (4.8) hold. The above collections $(d_1, ..., d_t)$ are partially ordered: we ignore the order of $d_a$ and $d_b$ if $(i_a, j_a) - (i_b, j_b) \in \mathbb{Z}(1,1)$. Given $p \in A_{k,d}^{\mu,C}$, the Laurent polynomial $r = \phi_{C}(p)$ has total degree $d + \sum_{i=1}^{n} k_i k_i + 1$, while the degree in each variable is controlled by (4.9):

$$\deg_{y_a}(r) \leq \mu |l_a^n| + \sum_{i=1}^{n} (k_i l_{i+1}^a + l_i^a k_{i+1} - l_i^a l_{i+1}^a), \quad \forall a \in \{1, ..., t\}$$

where $l_a := [i_a : j_a]$. We now claim that for all $1 \leq a < b \leq t$, the Laurent polynomial $r$ vanishes for:

- $y_b q^{-x'} = y_a q^{-x+1}$ for all $x \in [i_a, j_a]$ and $x' \in [i_b, j_b]$ of weight $\equiv x + 1$
- $y_b q^{-x'} = y_a q^{-x-1}$ for all $x \in (i_a, j_a)$ and $x' \in [i_b, j_b]$ of weight $\equiv x - 1$
- $y_b q^{-x'+1} = y_a q^{-i_a}$ for all $x' \in [i_b, j_b]$ of the same weight as $i_a - 1$
- $y_b q^{-x'-1} = y_a q^{-j_a}$ for all $x' \in [i_b, j_b]$ of the same weight as $j_a + 1$

The above are counted with the correct multiplicities. The first and second cases take place because of the wheel conditions (3.6). As for the last two cases, if we could set the variables equal to each other as prescribed there, we could splice together the longer interval $[i_a : j_a]$ to part of $[i_b : j_b]$, thus obtaining a collection $C'$ which is larger than $C$ in lexicographic order. The assumption $p \in \cap_{C' > C} \ker \phi_{C'}$ tells us that this assignment must yield 0, hence the vanishing of $r$. It follows that $r$ is divisible by the polynomial $r_0$ which is the product of the linear factors in the four bullets above. One sees that:

$$\deg_{y_a}(r_0) = \sum_{i=1}^{n} \sum_{b \neq a} (l_i^a l_{i+1}^b + l_i^a l_{i+1}^b) = \sum_{i=1}^{n} l_i^a k_{i+1} + l_i^a k_i - 2 l_i^a l_{i+1}^a$$

while the total degree of $r_0$ equals:

$$\deg(r_0) = \sum_{i=1}^{n} \left( k_i k_{i+1} - \sum_{a=1}^{t} l_i^a l_{i+1}^a \right)$$
We conclude that the Laurent polynomial $r/r_0$ has total degree equal to $d + \sum_{i=1}^{n} \sum_{a=1}^{t} l_{i}^{a} t_{i+1}$, while (4.9) and (4.12) imply that:

$$\deg_{y_a} \left( \frac{r}{r_0} \right) \leq l_0 |l_a| + \sum_{i=1}^{n} l_i^{a} t_{i+1}$$

Therefore, the Laurent polynomial $r/r_0$ is a sum of monomials:

$$\prod_{a} y_{d_a + \sum_{i=1}^{n} l_i^{a} t_{i+1}}$$

such that $d_a \leq \mu |l_a| = \mu (j_a - i_a + 1)$. Since $r/r_0$ is symmetric in $y_a$ and $y_b$ if $(i_a, j_a) - (i_b, j_b) \in \mathbb{Z}(1, 1)$, this proves that $r/r_0$ belongs to a vector space of dimension equal to the RHS of (4.11). This proves (4.11), and with it, the lemma.

4.9. Now set $\mu = \frac{d}{l_0}$, and consider any $P \in A_{
u}^{\mu}$. Note that the numbers in the RHS of (4.11) are all at most equal to 1, for any partition:

$$C = \{ [i_1; j_1], ..., [i_t; j_t] \}$$

of the degree vector $k$. The reason for this is that the $d_s$ are constrained by:

$$\frac{d_s}{j_s - i_s + 1} = \mu, \quad \forall \ s \in \{1, ..., t\}$$

Such integers $d_s$ may exist only when $\mu (j_s - i_s + 1) \in \mathbb{Z}$ for all intervals $[i_s; j_s]$ in the collection $C$. We call such a partition viable, and it follows that the dimension of $A_{
u}^{\mu}$ is at most equal to the number of viable partitions. However, we can say more. Because the vector space in (4.11) is at most one-dimensional, tracing through the proof of Lemma 4.7 shows that it is actually spanned by a single monomial (4.13).

For such a monomial to vanish, it is enough for it to vanish in the limit when $y_1 \gg ... \gg y_t$, that is to say:

$$\lim_{y_1 \gg ... \gg y_t} \frac{P(y_1 q^{-i_1}, ..., y_1 q^{-j_1}, ..., y_t q^{-i_t}, ..., y_t q^{-j_t})}{y_1^{d_1} ... y_t^{d_t}} = 0$$

(4.14)

The above limit is obtained, up to a non-zero constant, by applying the coproduct $t - 1$ times to $P$ (and ignoring the Cartan elements $\phi$ in front of the coproduct). Therefore, (4.14) is equivalent to:

$$\phi^{\otimes t} \left( \text{component of } \Delta^{(t-1)}(P) \text{ in } A_{[i_1; j_1]}^{\mu} \otimes ... \otimes A_{[i_t; j_t]}^{\mu} \right) = 0$$

(4.15)

where $\phi$ is the scaled evaluation map:

$$\phi : A_{[i;j]}^{\mu} \to \mathbb{K}, \quad \phi(P) = P(q^{-i_1}, ..., q^{-j}) / \prod_{1 \leq a < b \leq j} \omega(a, q^{-b}, q^{-a})$$

(4.16)

11 If the expressions (4.15) vanish for any viable partition $C \vdash k$, then $P$ lies in the kernel of all the linear maps $\phi_C$, and the proof of Lemma 4.7 implies that $P = 0$. To summarize:

The denominator has to be introduced in order to cancel the poles of $P(z_i, ..., z_j)$ of the form $z_{a+1}q - z_a$, for all $a \in \{i, ..., j-1\}$
Proposition 4.10. If $P \in \mathcal{A}_{k,d}^\mu$ is such that \eqref{eq:4.15} holds for all viable partitions $C \vdash k$, then $P = 0$.

4.11. To prove that the upper bounds in Lemma 4.7 are attained, we need to construct certain explicit shuffle elements, which will be interpreted geometrically in Section 7. They are:

$$X_m(z_i, \ldots, z_j) := \text{Sym} \left[ \frac{m(z_i, \ldots, z_j)}{(1 - \frac{z_{i+1}}{z_i}) \cdots (1 - \frac{z_{j+1}}{z_j})} \prod_{i \leq a < b \leq j} \omega(z_b, z_a) \right]$$

for any $i \leq j$ and Laurent polynomial $m(z_i, \ldots, z_j)$. In virtue of quasi-periodicity, $X_m$ should be viewed as an element of the space \eqref{eq:3.1} by replacing the variables:

$$z_a, z_{a+n}, z_{a+2n} \cdots \rightarrow z_a, \kappa z_a, \kappa^2 z_a, \ldots$$

for each $a \in \{1, \ldots, n\}$. The following result will be proved in the appendix:

Proposition 4.12. For any $i \leq j$ and Laurent polynomial $m(z_i, \ldots, z_j)$, we have:

$$X_m \in \text{Im } \Upsilon$$

where $\Upsilon : U^\geq q(\mathfrak{sl}_n) \longrightarrow \mathcal{A}^\geq$ is the map of \eqref{eq:3.11}.

4.13. For any degree vector $[i; j]$ and homogenous degree $d$, we will now construct particular cases of the above shuffle elements $X_m$. Let $\mu = \frac{d}{j-i+1}$.

Proposition 4.14. For $d > 0$, consider the multiset:

$$S_{i,j,d} = \left\{ i + \left\lfloor \frac{a}{\mu} \right\rfloor - \delta_a^d, \quad 0 \leq a \leq d \right\}$$

Then the shuffle element:

$$P^d_{[i,j]} := \text{Sym} \left[ \prod_{k \in S_{i,j,d}} \frac{z_k}{(1 - \frac{z_{i+1}}{z_i}) \cdots (1 - \frac{z_{j+1}}{z_j})} \prod_{i \leq a < b \leq j} \omega(z_b, z_a) \right]$$

has bidegrees $([i; j], d)$ and slope $\leq \mu$. In other words, $P^d_{[i,j]} \in \mathcal{B}_\mu^+$. 

Proof. Replacing $d$ by $d+(j-i+1)$ has the effect of multiplying the shuffle element by $z_i \cdots z_j$, so it is enough to prove the proposition for $d \in \{0, \ldots, j-i\}$ (moreover, this periodicity allows us to extend the definition of $P^d_{[i,j]}$ to $d \leq 0$). The advantage of this assumption is that $S_{i,j,d}$ becomes a set instead of a multiset. The statement about the bidegrees is immediate, so we will prove the statement about the slope. The slope condition asks that for any subset $T \subset \{i, \ldots, j\}$, multiplying the variables $\{z_k, k \in T\}$ by $\xi$ leaves the rational function \eqref{eq:4.19} of order:

$$O\left(\xi^{\frac{d}{j-i+1}}\right)$$

From the form of $\omega(x, y)$ in \eqref{eq:3.3}, we see that it has a finite limit when either of its variables is sent to $\infty$. Therefore, it doesn’t contribute to the limit as $\xi \rightarrow \infty$. Meanwhile, the power of $\xi$ that we pick up from the numerator of \eqref{eq:4.19} is
#(S_{i,j,d} \cap T), while the power of \( \xi \) that we pick up from the denominator is the number of clusters of consecutive elements of \( T \). So we need to show that for any subset \( T \subset \{i, \ldots, j\} \), we have:

\[
\#(S_{i,j,d} \cap T) - \# \text{ clusters in } T \leq \frac{d \# T}{j - i + 1} \tag{4.20}
\]

To maximize the LHS of (4.20) while keeping \( \# T \) fixed, each cluster in \( T \) must start and end with an element of \( S \), so it must be of the form:

\[
i + \left\lfloor \frac{a(j - i + 1)}{d} \right\rfloor - \delta_a, \ldots, i + \left\lfloor \frac{b(j - i + 1)}{d} \right\rfloor - \delta_b
\]

for some \( a \leq b \). The contribution of this cluster to (4.20) is:

\[
b - a \leq \frac{d}{j - i + 1} \left( \left\lfloor \frac{b(j - i + 1)}{d} \right\rfloor - \delta_b - \left\lfloor \frac{a(j - i + 1)}{d} \right\rfloor + \delta_a + 1 \right) \Leftrightarrow \left\lfloor \frac{b(j - i + 1)}{d} \right\rfloor + \delta_b - \left\lfloor \frac{a(j - i + 1)}{d} \right\rfloor - \delta_a \leq 1
\]

Since the fractional part always lies in the interval \([0, 1)\), the above inequality holds for all \( a \) and \( b \). Moreover, we have equality if and only if \( b = d \), \( a < d \) and \( d \mid a(j - i + 1) \) (we allow \( a = d \) only if \( d = 0 \)). This means that we have equality in (4.20) if and only if the variables we multiply by \( \xi \) are:

\[
z_k, \ldots, z_j
\]

for some \( k \) such that \( j - i + 1 \mid d(k - i) \).

\[\Box\]

4.15. Keep \( \mu = \frac{d}{j - i + 1} \). By (4.2), we have on general grounds:

\[
\Delta(P_{[i,j,d]}^d) = \Delta \mu(P_{[i,j]}^d) + (\text{slope} > \mu) \otimes (\text{slope} < \mu) \tag{4.21}
\]

where \( \Delta \mu \) is determined by letting some of the variables go off to \( \infty \). A consequence of the above Proposition is that the only terms which survive as we do so are:

\[
\Delta \mu(P_{[i,j]}^d) = \sum_k \varphi(k)[j] P_{[i,k-1]}^\mu(k-i) \otimes P_{[k,j]}^\mu(j-k+1) \tag{4.22}
\]

where the sum goes over all \( k \in \{i, \ldots, j + 1\} \) such that the superscripts of \( P \) are integers. We will need to compute these elements under the scaled evaluation map (4.16).

**Proposition 4.16.** For any \( i \leq j \) and \( d \in \mathbb{Z} \), we have:

\[
\phi(P_{[i,j]}^d) = q^{-\sum_{x=0}^{d-1}(i+\left\lfloor \frac{x}{d} \right\rfloor)} \frac{1-q^2}{1-q^2}^{j-i}
\]

**Proof** We need to compute \( P_{[i,j]}^d \) when its variables are specialized to \( z_a = q^{-a} \).

There are as many summands to \( P_{[i,j]}^d \) as there are weighted permutations of the variables \( z_i, \ldots, z_j \), and it is easy to see that the only summand which does not vanish in our specialization is the one corresponding to the identity permutation. This summand produces exactly the contribution in the RHS.

\[\Box\]
Finally, to completely understand the elements $P^d_{[i,j]}$, we will need to compute the pairings between them and other elements. It turns out that when $d = 0$, they coincide with the pairings between the root generators of Subsection 2.6. More precisely, the following result will be proved in the Appendix.

**Proposition 4.18.** The pairing $(\cdot, \cdot)$ is non-degenerate on $B^\bar{\times}_{\bar{n}}$. We have:

$$
\left( P^d_{[i; i+kn-1]}, P^d_{[i'; i'+kn-1]} \right) = (q - q^{-1})q^{2gcd(k,d)-1} \quad (4.23)
$$

For $i' \neq i$ modulo $n$, we have:

$$
\left( P^d_{[i; i+kn-1]}, P^d_{[i'; i'+kn-1]} \right) = \frac{(q - q^{-1})^2 gcd(k,d) - q^{-2} gcd(k,d)}{q - q^{-1}} \quad (4.24)
$$

or 0, depending on whether $n gcd(k,d)$ divides $d(i' - i + 1)$ or not. Finally, for $j \neq i - 1$ modulo $n$, we have:

$$
\left( P^d_{[i; j]}, P^d_{[i; j]} \right) = \frac{(q^2 - 1)(q^{2g+1} + q^{-2g-1})}{q + q^{-1}} \quad (4.25)
$$

where $g = \left\lfloor \frac{gcd(j-i+1, m)}{n} \right\rfloor$. All other pairings among the $P^d_{[i,j]}$ vanish for degree reasons.

**Proof of Lemma 4.4:** We need to prove that:

$$
\left[ U_{\bar{d}}^{\bar{\times}}, (\hat{g}_n^{\bar{\times}}) \right] \overset{\otimes \Xi}{\cong} B^\bar{\times}_{\bar{n}} \quad (4.26)
$$

where $g = gcd(n, a)$. Denote the root generators in the factors of the LHS by $e^{(1)}_{[i,j]}, \ldots, e^{(g)}_{[i,j]}$ for indices $i$ and $j$ modulo $\bar{n}$, and the Cartan generators by $\varphi^{(1)}_i, \ldots, \varphi^{(g)}_i$ for $i \in \{1, \ldots, \frac{n}{g}\}$. Construct the map $\Xi$ of (4.26) by:

$$
\varphi^{(x)}_i \overset{\Xi}{\mapsto} \varphi_{ia+x}, \quad e^{(x)}_{[i,j]} \overset{\Xi}{\mapsto} P_{[a+x; ja+x-1]}^{b(j-i)}
$$

for all $x \in \{1, \ldots, g\}$, $i \in \{1, \ldots, n/g\}$ and $j > i$. Comparing (4.22)-(4.25) with (2.12)-(2.15), we see that the coproduct $\Delta$ and the pairing are preserved under the above correspondence. We will use this fact to show that the map $\Xi$ preserves relations, and thus gives rise to an algebra homomorphism. Any relation in the domain of $\Xi$ is of the form:

$$
\sum \text{const} \prod \varphi^{(x)}_i \prod e^{(x)}_{[i,j-1]} = 0 \quad (4.27)
$$

and we want to show that the corresponding relation:

$$
\sum \text{const} \prod \varphi_{ia+x} \prod P_{[a+x; ja+x-1]}^{b(j-i)} = 0
$$

holds in $B^\bar{\times}_{\bar{n}}$. Since the pairing on $B^\bar{\times}_{\bar{n}}$ is non-degenerate (as proved in Proposition 4.18), it is enough to show that the above relation pairs trivially with all possible products of $P_{[a+x; ja+x-1]}^{b(j-i)}$’s and $\varphi_{ia+x}$’s. By properties (2.4) and (4.22), the coproduct reduces such pairings to expressions of the form:

$$
\sum \prod (\varphi_{ia+x}, \varphi_{i'a+x'}) \prod \left( P_{[a+x; ja+x-1]}^{b(j-i)}, P_{[a+x'; j'a+x'-1]}^{b(j'-i')} \right) = 0
$$
Since these pairings are preserved under $\Xi$, it is enough to check the above relation between the $e_{[i.j-1]}^{(x)}$ and $\varphi_{i}^{(x)}$, where we already know that it holds by backtracking to (4.27). We have thus showed that $\Xi$ is an algebra homomorphism. By construction, $\Xi$ preserves the coproduct and the pairing. To prove that it is injective, we use the fact that the pairing on the LHS of (4.26) is non-degenerate, and use the argument of Section 3.12. To prove that it is surjective, we use the dimension bound of Remark 4.8.

\[\square\]

4.19. We will now complete the proof of Theorem 1.1.

**Proof of Theorem 1.1** First of all, note that it is enough to show that the map of (3.14) is an isomorphism. Indeed, the two algebras in the theorem are obtained from the two algebras in (3.14) by adding similar Cartan elements $\psi_{i}$ (see the definition in Subsection 3.13).

To prove that (3.14) is an isomorphism, it is enough to show that its restriction to the positive halves, namely the map $\Upsilon$ of (3.11) is an isomorphism. As explained in Subsection 3.11, all we still need to prove is surjectivity:

\[A^{\geq} = \text{Im } \Upsilon\]

As was shown in Proposition 4.12, the particular shuffle elements $X_{m}$ belong to $\text{Im } \Upsilon$. Hence so do the shuffle elements $P_{i}^{d}$, and Lemma 4.4 implies that:

\[B_{\mu}^{\geq} \subset \text{Im } \Upsilon\]

So now all that remains to prove is that the subalgebras $B_{\mu}^{\geq}$ (as $\mu$ ranges over all rational numbers) generate the algebra $A^{\geq}$, and we will do so by proving the decomposition result (1.1). Since $A^{\geq}$ is infinite-dimensional in most degrees, we will actually prove a stronger and finer result:

\[A^{\mu} = \bigoplus_{\mu' \leq \mu} B_{\mu'}^{\geq}\]

for all $\mu$. To do so, let us pick an orthonormal basis $\{v_{i}^{\mu}\}$ of the subalgebra $B_{\mu}^{\geq}$. Then it is enough to show that the elements:

\[v_{i_{1}}^{\mu_{1}} ... v_{i_{t}}^{\mu_{t}}, \quad \text{as } \mu_{1} < \mu_{2} < ... < \mu_{t} \leq \mu\]

\[(4.28)\]

are an orthonormal basis of $A^{\mu}$. Note that the above are all elements of $A^{\mu}$, and moreover their number is precisely equal to the upper bound on $A^{\mu}$ discussed in Lemma 4.7. All we still need do is to prove that:

\[\left( v_{i_{1}}^{\mu_{1}} ... v_{i_{t}}^{\mu_{t}}, v_{j_{1}}^{\mu'_{1}} ... v_{j_{s}}^{\mu'_{s}} \right) = \text{Kronecker delta}\]

Since the pairing is symmetric, we may assume $\mu_{1} \leq \mu'_{1}$. Then applying the bialgebra property, we see that the above equals:

\[\left( v_{i_{1}}^{\mu_{1}} \otimes v_{i_{2}}^{\mu_{2}} ... v_{i_{t}}^{\mu_{t}}, \Delta \left( v_{j_{1}}^{\mu'_{1}} ... v_{j_{s}}^{\mu'_{s}} \right) \right)\]

\[\text{If there are several consecutive } \mu_{a}' \text{'s equal to each other, then we order the corresponding } v_{i_{a}}^{\mu_{a}} \text{ in non-increasing order of their degrees}\]
As discussed in Subsection 4.3, the very definition of the subalgebra $B_{\mu'}$ entails the fact that the coproduct in the above expression only has first tensor factors of slope $\geq \mu_1'$. So if $\mu_1' > \mu_1$, the above pairing is trivial. If $\mu_1' = \mu_1$, we may assume that the degree of $v_{i_1}^{\mu_1}$ is no less than the degree of $v_{j_1}^{\mu_1}$. Then the only non-trivial term in the above pairing is:

\[ \left( v_{i_1}^{\mu_1} \otimes v_{i_2}^{\mu_2} \ldots v_{i_t}^{\mu_t}, v_{j_1} \otimes v_{j_2}^{\mu_2} \ldots v_{j_s}^{\mu_s} \right) \]

Since $\{v_{i_1}^{\mu_1}\}$ is an orthonormal basis, the above pairing is non-trivial only if $i_1 = j_1$. We may repeat the argument to prove that non-triviality of the pairing forces $\mu_2' = \mu_2$ and $i_2 = j_2$ etc. This proves that (4.28) form an orthonormal basis of $A_{\mu}$ for any $\mu$, thus proving that $A_{\mu}$ is generated by the $B_{\mu'}$, and hence is contained in the image of $\Upsilon$.

\[ \blacksquare \]

5. The Universal $R$–matrix

5.1. Given a Hopf algebra $A$, the universal $R$–matrix is an element $R \in A\hat{\otimes}A$ such that:

\[ R \cdot \Delta(a) = \Delta^{op}(a) \cdot R, \quad \forall a \in A \]

and:

\[ (\Delta \otimes 1)R = R_{13}R_{23}, \quad (1 \otimes \Delta)R = R_{13}R_{12} \]

The first property implies that for any $V,W \in \text{Rep}(A)$, the operator $R_{VW}$ given by:

\[ A\hat{\otimes}A \rightarrow \text{End}(V \otimes W), \quad R \rightarrow R_{VW} \]

intertwines the tensor product representations $V \otimes W$ and $W \otimes V$, which explains the terminology ”universal” and ”matrix”.

5.2. When $A$ is the Drinfeld double of a certain subalgebra $A^+$, then a universal $R$–matrix always exists, and we recall the construction below:

\[ R = \sum_i P_i^+ \otimes Q_i^- \in A\hat{\otimes}A \]

is a universal $R$–matrix. Note that the definition does not depend on the choice of dual bases.

**Proposition 5.3.** Let $\{P_i\}$ and $\{Q_i\}$ be dual bases of $A^+$ with respect to the Hopf pairing, the tensor:

\[ R = \sum_i P_i^+ \otimes Q_i^- \in A\hat{\otimes}A \]

is a universal $R$–matrix. Note that the definition does not depend on the choice of dual bases.

**Proof** It is enough to check property (5.1) for $a \in A^+$, because the relation is multiplicative in $a$ and the case $a \in A^-$ simply follows by changing all the signs from $+$ to $-$. In other words, we need to check that for any basis element $P_x \in A^+$ we have:

\[ R \cdot \Delta(P_x) = \Delta^{op}(P_x) \cdot R \]

(5.3)
Because the $P_i$ and $Q_i$ are dual bases of $A_\infty^\geq$, relation (2.4) implies that the structure constants for their multiplication and comultiplication are the same, in the sense that:

$$\Delta(P_x) = \sum_{y,z} Q_y \otimes P_x e_{xy}^{yz}, \quad \text{where} \quad P_y Q_z = \sum_x Q_x e_{xy}^{yz}$$

$$\Delta(P_x) = \sum_{y,z} Q_y \otimes Q_z d_{xy}^{yz}, \quad \text{where} \quad P_y P_z = \sum_x Q_x d_{xy}^{yz}$$

Therefore, the desired relation (5.3) becomes equivalent to:

$$\sum_{i,y,z} P_i^+ Q_y^+ \otimes Q_i^- Q_z^- d_{xy}^{yz} = \sum_{i,y,z} P_i^+ P_i^- \otimes Q_y^+ Q_i^- e_{xy}^{yz} \iff$$

$$\iff \sum_{i,j,y,z} Q_j^+ \otimes Q_i^- Q_z^- c_{ij}^{yz} d_{xy}^{yz} = \sum_{i,j,y,z} Q_j^+ \otimes Q_y^+ Q_i^- e_{xy}^{yz} d_{ij}^{yz}$$

For any fixed $j$ and $x$, the above equality follows from:

$$\sum_{i,y,z} Q_i^+ Q_j^- c_{ij}^{yz} d_{xy}^{yz} = \sum_{i,y,z} Q_j^+ Q_i^- e_{xy}^{yz} d_{ij}^{yz}$$

which is simply (2.5) for $a = P_x$ and $b = P_j$. As for (5.2), we have:

$$(\Delta \otimes 1) R = \sum_{i,x,y} Q_x^+ P_y^+ \otimes Q_i^- e_{xy}^{yz} = \sum_{x,y} Q_x^+ P_y^+ \otimes P_x^- Q_y^- = R_{13} R_{23}$$

$$(1 \otimes \Delta) R = \sum_{i,x,y} Q_i^+ P_y^- \otimes Q_x^- e_{xy}^{yz} = \sum_{x,y} P_x^+ Q_y^+ \otimes P_y^- Q_x^- = R_{13} R_{12}$$

Here we used the fact that the $R$ matrix can be defined with respect to any dual bases, i.e. it does not change if we swap the roles of $P$ and $Q$.

5.4. We will use the above Proposition to present the universal $R$–matrix on the algebra $U_q(\hat{\mathfrak{gl}}_n) \cong A$. As was proved in subsection 4.19, the factorization (1.1) respects the pairing. In other words, to pair two elements of $A_\infty^\geq$, it is enough to take the product of the pairings of their factors in each $B_{ij}^\geq$. Together with the previous Proposition on how to construct the $R$–matrix from dual bases, this implies the factorization formula:

$$R_A = \prod_{\mu \in Q \cup \{\infty\}} R_{\mathfrak{g}_\mu} \in A \hat{\otimes} A$$

Using Lemma 4.4, we know exactly what each of the factors is equal to:

$$R_{\mathfrak{g}_\mu} = \left( R_{U_q(\hat{\mathfrak{g}}_{\frac{1}{2}})} \right)^{\otimes g}$$

where $g = \gcd(n, a)$. For quantum groups associated to Kac-Moody algebras (note that $U_q(\mathfrak{gl}_n)$ and $U_q(\hat{\mathfrak{gl}}_n)$ are not of this type), Khoroshkin and Tolstoy prove that the universal $R$–matrix factorizes as a product over the positive roots (see [11]). In the case of $U_q(\hat{\mathfrak{gl}}_n)$, which can roughly be thought of as an affinization of a Kac-Moody type quantum group, we see that the $R$–matrix factorizes as a product over rational slopes. The slope is a substitute for positive roots in the
To make (1.2) more useful, one would need to better understand the universal $R$–matrix of $U_q(\mathfrak{gl}_n)$, and to use this to understand the universal $R$–matrix of each subalgebra $\mathcal{B}_\mu \subset \mathcal{A}$. We see two approaches of doing so, although we do not know how to carry either of them out completely:

- Express the universal $R$–matrix of $U_q(\mathfrak{gl}_n)$ in terms of the root generators $e_{[i,j]}^\pm$, which would immediately express the universal $R$–matrix of $\mathcal{B}_\mu$ in terms of the shuffle elements $P_{[i,j]}^\ell$ of (4.19).

- Write the universal $R$–matrix of $U_q(\mathfrak{gl}_n)$ in terms of the $R$–matrices of $U_q(\mathfrak{sl}_n)$ and $U_q(\mathfrak{gl}_1)$, as in (5.6) below, by a procedure which we will explain in the next Subsection. These $R$–matrices are understood, the former by [11] and the latter by direct computation (5.7). The problem with this approach is that we do not know which shuffle elements in $\mathcal{B}_\mu$ correspond to the loop generators of $U_q(\mathfrak{gl}_1)$.

Let us now expand on the second approach above. There exists an isomorphism:

$$U_{q,*}(\mathfrak{gl}_n) \cong U_q(\mathfrak{sl}_n) \otimes U_q(\mathfrak{gl}_1) \quad (5.5)$$

We could obtain a similar result for $U_q(\mathfrak{gl}_n)$ if we enriched the RHS with Cartan elements $\psi_i$ (since this is a trivial procedure, we will not spell it out). In the above, the subalgebra $U_q(\mathfrak{sl}_n) \subset U_{q,*}(\mathfrak{gl}_n)$ is generated by the $e_i^\pm$ and the $\varphi_i$, while:

$$U_q(\mathfrak{gl}_1) = \mathbb{C}(q, \zeta)(a_1^+, a_2^+, ...)$$

modulo the relations $[a_i^+, a_j^-] = 0$ and:

$$[a_i^+, a_j^-] = \delta_i^j (q^{nk} - q^{-nk})(\zeta^k - \zeta^{-k})$$

Since the algebras $U_q(\mathfrak{sl}_n)$ and $U_q(\mathfrak{gl}_1)$ commute, the universal $R$–matrix also has a similar decomposition:

$$R_{U_{q,*}(\mathfrak{gl}_n)} = R_{U_q(\mathfrak{sl}_n)} \otimes R_{U_q(\mathfrak{gl}_1)} \quad (5.6)$$

The universal $R$–matrix of $U_q(\mathfrak{sl}_n)$ was discussed in [11], and expressed as a product of exponentials of certain root generators. Meanwhile, it is elementary to compute:

$$R_{U_q(\mathfrak{gl}_1)} = \exp \left( \sum_{k=1}^{\infty} \frac{a_k^+ \otimes a_k^-}{q^{nk} - q^{-nk}} \right) \quad (5.7)$$

6. $K$–theory of Affine Laumon Spaces

6.1. In this section, we will define affine Laumon spaces. Consider the surface $\mathbb{P}^1 \times \mathbb{P}^1$ and the divisors:

$$D = \mathbb{P}^1 \times \{0\}, \quad \infty = \mathbb{P}^1 \times \{\infty\} \cup \{\infty\} \times \mathbb{P}^1.$$
A parabolic sheaf \( \mathcal{F} \) is a flag of rank \( n \) torsion free sheaves:

\[
\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \subset \mathcal{F}_1(D)
\]  

(6.1)
on \( \mathbb{P}^1 \times \mathbb{P}^1 \), together with an isomorphism:

\[
\begin{array}{cccc}
\mathcal{F}_1|_\infty & \cdots & \mathcal{F}_n|_\infty & \mathcal{F}_1(D)|_\infty \\
\cong & \cdots & \cong & \cong \\
O^\otimes_n(-D) & \cdots & O^\otimes_{n-1} \oplus O_\infty(-D) & O^\otimes_n
\end{array}
\]  

(6.2)

The above is called framing at \( \infty \), and it forces \( c_1(\mathcal{F}_i) = -(n + 1 - i)D \). On the other hand, \( -c_2(\mathcal{F}_i) := d_i \) can vary over all non-negative integers, and we therefore call the vector \( d = (d_1, \ldots, d_n) \in \mathbb{N}^n \) the degree of the parabolic sheaf. The moduli space \( \mathcal{M}_d \) of rank \( n \) degree \( d \) parabolic sheaves is a smooth quasiprojective variety of dimension \( 2|d| := 2(d_1 + \ldots + d_n) \) called the affine Laumon space.

6.2. The maximal torus \( T_n \subset GL_n \) and the product \( \mathbb{C}^* \times \mathbb{C}^* \) act on \( \mathcal{M}_d \) by changing the trivialization at \( \infty \), respectively by multiplying the base \( \mathbb{P}^1 \times \mathbb{P}^1 \). We will write \( T \) for the \( 2n^2 + 2 \)-fold cover of \( T_n \times \mathbb{C}^* \times \mathbb{C}^* \), and in this paper we will be mostly concerned with the \( T \)-equivariant \( K \)-theory groups of Laumon spaces:

\[
K_T^*(\mathcal{M}_d),
\]

which are all modules over \( K_T^*(\text{pt}) = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}, q^{\pm 1}, (q')^{\pm 1}] \). Here, \( t_1, \ldots, t_n, q, q' \) are the square roots of the usual coordinates on \( T_n \times \mathbb{C}^* \times \mathbb{C}^* \). Let us consider:

\[
K = \bigoplus_{d \in \mathbb{N}^n} K_T^*(\mathcal{M}_d) \otimes_{\mathbb{C}[t_1, \ldots, t_n, q, q']} \mathbb{C}(t_1, \ldots, t_n, q, q'),
\]

(6.3)

which is a graded vector space. It is convenient to extend our definitions by:

\[
t_{k+n} = t_k q', \quad \mathcal{F}_{k+n} = \mathcal{F}_k(D)
\]

(6.4)

for all integers \( k \). This will give rise to a quasi-periodicity of the form (2.16).

6.3. Affine Laumon spaces come with universal sheaves \( \mathcal{S}_i \) on \( \mathcal{M}_d \times \mathbb{P}^1 \times \mathbb{P}^1 \). Pushing these forward along the standard projection \( p \) to \( \mathcal{M}_d \) gives rise to the tautological vector bundles \( \mathcal{T}_k \):

\[
\mathcal{T}_i = R^1 p_* \mathcal{S}_i(-\infty), \quad \mathcal{T}_i|_x = H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{F}_i(-\infty))
\]

(6.5)

which have rank \( d_k \) over the connected component \( \mathcal{M}_d \). We will also be concerned with the universal sheaves:

\[
[\mathcal{W}_i] = t_i^{-2} q^{-1} + (q - q^{-1}) ([\mathcal{T}_i] - [\mathcal{T}_{i-1}])
\]

(6.6)

The above defines \([\mathcal{W}_i]\) as a class in \( K \)-theory, but one can define it as the restriction of the sheaf \( \mathcal{S}_i/\mathcal{S}_{i-1} \) to \((0,0) \in \mathbb{P}^1 \times \mathbb{P}^1 \). We will not need this construction, and will therefore contend ourselves with (6.6) as the definition.
6.4. On the $K$–theory of any variety, an important role is played by exterior classes. These are given by:

$$ \Lambda(\mathcal{V}, u) = \sum_{i=0}^{\text{rank } \mathcal{V}} (-u)^i [\Lambda^i \mathcal{V}] $$

for a vector bundle $\mathcal{V}$, and they extend multiplicatively to all $K$–theory classes. For example, if we start off with a class:

$$ c = \sum l_i - \sum l_j $$

where $l_i, l_j \in K_1^j$, then its $\Lambda$ class equals:

$$ \Lambda(c, u) = \prod_i \left( 1 - \frac{u}{l_i} \right) \prod_j \left( 1 - \frac{u}{l_j} \right) $$

(6.7)

We can apply this construction to the sheaves (6.5) and (6.6), and obtain:

$$ \Lambda(\mathcal{T}, u) \in K[u], \quad \Lambda(\pm W, u) \in K(u) $$

We call the coefficients of $\Lambda(\mathcal{T}, u)$ **tautological classes**. They are important because of the following result, to be proved in subsection 7.15.

**Proposition 6.5.** For any $d \in \mathbb{N}^n$, the vector space:

$$ K_d := K^* \mathcal{T}(M_d) \otimes_{\mathbb{C}[t_1, \ldots, t_n, q, q']} \mathbb{C}(t_1, \ldots, t_n, q, q') $$

is spanned by products of tautological classes.

6.6. According to the above proposition, a basis for $K$ is given by the coefficients of the classes:

$$ \gamma_d^S = \prod_{1 \leq i \leq n} \Lambda(\mathcal{T}^d_i, s) $$

as $S = (S_1, \ldots, S_n)$ runs over all collections of sets of variables $S_i$. For any $k \in \mathbb{N}^n$, we will write $k \pm 1$ for the vector whose $i$–th spot equals $k_i \mp 1$, and also set:

$$ t^k = \prod_{i=1}^n t_i^{k_i} $$

We will now restate Theorem 4.13 of [15] in terms of the shuffle algebra:

**Theorem 6.7.** There is an action of the extended double shuffle algebra $\mathcal{A}$ on $K$, given by $\kappa = q^n q'^2$,

$$ \psi_i^\pm(z) \cdot c = (t_i q^{d_i - 1 - d_{i+1}}) \pm 1 \cdot \Lambda \left( W_i^\pm, (z q^i)^{\mp 1} \right) \cdot c $$

(6.8)
for all $c \in K_d$, \[ P^+ \cdot \gamma_d^S = \frac{\gamma d + k}{k!} : \int P(\ldots, u_{ij}, \ldots) \prod_{1 \leq i \leq n, 1 \leq j \leq k} \frac{\Lambda(W_{i+1}^\vee, u_{ij} q^{-i-1})}{\prod_{s \in S_i} \left(1 - \frac{s}{u_{ij} q^s}\right)} Du_{ij} \] (6.9) and:
\[ P^- \cdot \gamma_d^S = \frac{t^k t^{k+1} q^{k(k+1)/(k,d)} - (d,k) \gamma_d^S}{k!} \]
\[ : \int \tau(P)(\ldots, u_{ij}, \ldots) \prod_{1 \leq i \leq n, 1 \leq j \leq k} \frac{u_{ij} q^i \Lambda(-W_i, u_{ij} q^{-i-1})}{\prod_{s \in S_i} \left(1 - \frac{s}{u_{ij} q^s}\right)} Du_{ij} \] (6.10)
for any $P \in A_k$ and any set of variables $S$. The normal-ordered integral $\int : \tau$ is defined by a modified residue count, as in Subsection 3.9. Namely, we assume $|q| > 1$ and $|r| < 1$ and define:
\[ : \int P(\ldots, u_{ij}, \ldots) Du_{ij} = \int_{|u_{ij}| = \rho} P(\ldots, u_{ij}, \ldots) \prod_{i=1}^n \prod_{j \neq j'} u_{ij} q^{-i-1} u_{i'j'} q \prod_{i,j} Du_{ij} \bigg|_{r=q} \]
The contours $\rho$ are chosen so as to separate the set $S \cup \{0, \infty\}$ from the poles of the various $\Lambda(\pm W_i, \cdot)$.

**Proof** As follows from Theorem 1.1, $A \cong U_q(G_{1,r})$ is generated by the Cartan elements $e_{i,k}$, together with the degree 1 currents:
\[ e_i^+ (z) = \delta(z/z_{i1}) \in A^+ \quad \text{and} \quad e_i^- (z) = \delta(z^{-1}/z_{i1}) \in A^- \]
For the positive generators, formula (6.9) reads:
\[ (z_i^c)^+ \cdot \gamma_d^S = \frac{\gamma d + c}{d} : \int u^c \frac{\Lambda(W^\vee_{i+1}, u^{-1} q^{-i-1})}{\prod_{s \in S_i} \left(1 - \frac{s}{u q^s}\right)} Du \] (6.11)
We will first show that (6.11) implies formulas (6.9). Indeed, iterating (6.11) tells us that $(z_i^c \ast \ldots \ast z_i^c)^+ \in A^+$ sends $\gamma_d^S$ to:
\[ \int P^c \prod_{a=1}^k u_a^c \prod_{1 \leq a < b \leq k} \omega(u_a, u_b) \prod_{a=1}^k \frac{\Lambda(W^\vee_{i_a+1}, u^{-1} q^{-i_a-1})}{\prod_{s \in S_{i_a}} \left(1 - \frac{s}{u_a q^s}\right)} Du_a \] (6.12)
where the contours of integration separate the poles of $\Lambda(-W^\vee, \cdot)$ from $\cup_{i=1}^n S_i \cup \{0, \infty\}$, and are arranged from the former set to the latter in decreasing order of the indices $a$. They are far away from each other so as to not pick up any poles from the $\omega$'s. Hence, we may rewrite the above as:
\[ \int P^c \prod_{a=1}^k u_a^c \prod_{a < b} \omega(u_a, u_b) \prod_{a \equiv b} u_a q^{-1} - u_b q \prod_{a=1}^k \frac{\Lambda(W^\vee_{i_a+1}, u^{-1} q^{-i_a-1})}{\prod_{s \in S_{i_a}} \left(1 - \frac{s}{u_a q^s}\right)} Du_a \bigg|_{r=q} \]
Because of the new factor we introduced, we may move the contours on top of each other without picking up any new poles, so the above becomes:
\[ \int_{|u_a| = \rho} P^c \prod_{a=1}^k u_a^c \prod_{a < b} \omega(u_a, u_b) \prod_{a \equiv b} u_a q^{-1} - u_b q \prod_{a=1}^k \frac{\Lambda(W^\vee_{i_a+1}, u^{-1} q^{-i_a-1})}{\prod_{s \in S_{i_a}} \left(1 - \frac{s}{u_a q^s}\right)} Du_a \bigg|_{r=q} \]
Since \( \rho \) is now one and the same for all the variables, the above integrand may be symmetrized without changing the value of the integral. This yields precisely formula (6.9). Formula (6.10) is proved analogously.

We are therefore left to show that (6.8) and (6.11) give a well-defined action of the algebra \( U_q(\mathfrak{gl}_n) \) on \( K \). Tsymbaliuk ([15]) has already constructed such action, so all one needs to do is to show that it is the same as ours. In loc.cit., the action of \( z \) was given by the \( c \)-th power of the tautological bundle on a certain correspondence. The computation of Subsection 7.26 (for \( i = j \)) shows that this correspondence acts on \( K \) via (6.11). This proves that our action coincides with the one in [15], up to adjusting our normalization and conventions with theirs. Hence the action of (6.11) is well-defined.

Alternatively, we can prove that \( U_q(\mathfrak{gl}_n) \) acts on \( K \) via the formulas in the theorem by showing directly that they satisfy the defining relations. Explicitly, they are:

\[
e^+ (z) \psi^\pm (w) = \frac{z - wq^{\delta_j}}{zq^{j+1} - wq^{j+1}} \cdot \psi^\pm (w) e^+ (z),
\]

\[
e^- (z) \psi^\pm (w) = \frac{wq^{\delta_j+1} - zq^{j+1}}{wq^{j+1} - z} \cdot \psi^\pm (w) e^- (z),
\]

\[
e^+ (z) e^+_j (w) = \left( \frac{zq^{a_{ij}} - w}{z - wq^{a_{ij}}} \right)^{\pm 1} \cdot e^+_j (w) e^+ (z),
\]

\[
[e^+_i (z), e^-_j (w)] = \delta^i_j (q - q^{-1}) \delta \left( \frac{z}{w} \right) \left[ \frac{\psi^+_{i+1} (w)}{\psi^+_i (wq^{-1})} - \frac{\psi^+_i (z)}{\psi^+_i (zq^{-1})} \right]
\]

for any \( |i - j| \leq n/2 \), together with the Serre relations (2.19) for both \( e^+ \)'s and \( e^- \)'s. This is a straightforward residue computation, and we leave it as an exercise to the interested reader.

\[\Box\]

6.8. Formulas (6.9) and (6.10) have the advantage that they take the whole shuffle element as an integrand, and hence it’s easy to see that the action is well-defined. The disadvantage is that the shuffle elements we mostly care about are:

\[
X_m = \text{Sym} \left[ \frac{m(z_1, \ldots, z_j)}{(1 - \frac{qz_i}{z_{i+1}}) \cdots (1 - \frac{qz_j}{z_{j+1}})} \prod_{i \leq a < b \leq j} \omega(z_b, z_a) \right] \in \mathcal{A}
\]

of (1.5). These shuffle elements are very difficult to write down explicitly, though the above formula as a symmetrization is rather concise. The following Proposition is proved quite similarly with Theorem 6.7 above, so we will leave it as an exercise to the interested reader. Let us specify that the only thing that sets it apart from the computation of \( (z_{i_1}^c \ast \ldots \ast z_{i_k}^c) \) in formula (6.12) above is the presence of the denominators:

\[
\left( 1 - \frac{qz_a}{z_{a+1}} \right)
\]

in the definition of \( X_m \). However, since these denominators are canceled by certain numerators of \( \omega \), they do not affect the residues of the integrals involved.
Proposition 6.9. For any $i < j$ and any Laurent polynomial $m(z_i, ..., z_j)$, we have:

$$X_m^+ \cdot \gamma_d^S = \gamma_d^{S+i} \int \frac{m(u_i, ..., u_j) \prod_{a,b} \omega(u_b, u_a)}{(1 - \frac{u_a}{u_i+1}) ... (1 - \frac{u_a}{u_j})} \prod_{a=1}^j \Lambda(W_{a+1}, u_a q^{-a-1}) D u_a$$

and:

$$\tau(X_m^+ \psi_{i,j}^1) \psi_{i,j+1}^1 \cdot \gamma_d^S = \gamma_d^{S+i-j} \int \frac{m(u_i, ..., u_j) \prod_{a,b} \omega(u_b, u_a)}{(1 - \frac{u_a}{u_i+1}) ... (1 - \frac{u_a}{u_j})} \prod_{a=1}^j u_a \Lambda(-W_{a+1}, u_a q^{-a-1}) D u_a$$

where the contours of integration separate the sets $\text{Poles}(\Lambda(\pm W))$ from $\cup_{k=1}^n S_k \cup \{0, \infty\}$, and are arranged from the former set to the latter in increasing / decreasing order of the indices $a$, depending on whether the sign is $+$ / $-$. 

7. The fine correspondences

7.1. We will now give a geometric interpretation to the formulas in Proposition 6.9, by proving Theorem 1.3. Since the root generators of $U_q(\mathfrak{sl}_n)$ are particular cases of the shuffle elements $X_m$, this will also give a geometric interpretation to the action of:

$$U_q(\mathfrak{gl}_n) \subset U_q(\mathfrak{sl}_n) \cong \mathcal{A}$$

on $K$, as per Theorem 1.2. Historically, the conjecture of Kuznetsov that motivated Theorem 1.2 (see [5]) suggests that the action of $\mathfrak{gl}_n$ on the cohomology of affine Laumon spaces is given by the correspondences:

$$\mathcal{C}_{[i,j]} = \{ F^- \supset_x^{i,j} F^+ \mid x \in D \setminus \infty \} \subset \mathcal{M}_{d^-} \times \mathcal{M}_{d^+},$$

where the notation $\supset_x^{i,j}$ means that the individual component sheaves of $F^-$ and $F^+$ are contained inside each other, with the quotient giving a type $[i,j]$ indecomposable representation of the cyclic quiver supported at $x \in D$. The main problem with these correspondences is that they are not lci, and thus their structure sheaves do not give good operators in $K$–theory.

7.2. Instead, we will use the following fine correspondences:

$$\mathcal{F}_{[i,j]} = \{ F^+ \supset_x^i F^{i+1} \supset_x^{i+1} \ldots \supset_x^{j-1} F^j \supset_x^j F^{j+1}, \quad x \in D \setminus \infty \}, \quad (7.1)$$

where the notation $F^+ \supset_x^k F'$ means that $F_l \supset F_l'$ for all $l \in \{1, ..., n\}$, and the quotients have length 1 if $k \equiv l$ modulo $n$, and 0 otherwise. We expect the map $\mathcal{F}_{[i,j]} \rightarrow \mathcal{C}_{[i,j]}$ to be birational and proper, and thus the two correspondences should give rise to the same operator in cohomology. The advantage of the correspondence $\mathcal{F}_{[i,j]}$ is that it behaves like a linear combination of lci varieties (as will be explained in Subsection 7.12), so it has a well-behaved structure sheaf in $K$–theory.
7.3. For each \( k \in \{i, ..., j\} \), there is a **tautological** line bundle \( \mathcal{L}_k \) on \( \mathfrak{f}_{[i:j]} \):

\[
\mathcal{L}_k|_{\mathcal{F}_1 \supset \cdots \supset \mathcal{F}_{j+1}} = \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{F}_k / \mathcal{F}_k^{k+1}) \tag{7.2}
\]

and let us write \( l_k = [\mathcal{L}_k] \) for its class in \( K \)-theory. Moreover, we consider the diagram:

\[
\mathfrak{f}_{[i:j]} \xrightarrow{p^-} M_{d-} \times \cdots \times M_{d+} \xrightarrow{p^+} M_{d+}
\]

which forget all but the first / last sheaf in the flag. For any Laurent polynomial \( m(z_i, ..., z_j) \), we define the operator:

\[
x^\pm_m : K \rightarrow K, \quad \alpha \mapsto p^\pm_n (m(l_iq^{-i}, ..., l_jq^{-j}) \cdot [\mathfrak{f}_{[i:j]}] \cdot p^{\mp}(\alpha)) \tag{7.3}
\]

where \( [\mathfrak{f}_{[i:j]}] \) denotes the **modified** structure sheaf of the fine correspondence. We do not claim that it equals the ordinary structure sheaf of the singular variety \( \mathfrak{f}_{[i:j]} \), although we expect it to be true. Instead, we will build up the modified structure sheaf out of ieci varieties, via a procedure which we now describe. The defining formula will be given in (7.10).

7.4. To have a better grasp on the operator \( x^\pm_m \), consider the larger variety:

\[
\mathfrak{f}^\lambda_{[i:j]} = \{\mathcal{F}^i \supset \mathcal{F}^{i+1} \supset \cdots \supset \mathcal{F}^j \supset \mathcal{F}^{j+1}\}, \tag{7.4}
\]

where now each inclusion is free to be supported at any point \( \in D \). It is equidimensional of dimension \( |d| + |d'| \), and it breaks up into components:

\[
\mathfrak{f}^\lambda_{[i:j]} = \{\mathcal{F}^i \supset x_1 \supset \cdots \supset x_{\lambda_1} \mathcal{F}^{i+\lambda_1} \supset \cdots \supset x_{\lambda_2} \mathcal{F}^{i+\lambda_1+\lambda_2} \supset \cdots \}
\]

as \( \lambda = (\lambda_1, \lambda_2, ...) \) goes over all **ordered** partitions of \( j - i + 1 \) and \( x_1, x_2, ... \) over all sets of distinct points of \( D \). We expect that the \( \mathfrak{f}^\lambda_{[i:j]} \) are all irreducible, although we will not need this.

7.5. In (7.4), points of the variety \( \mathfrak{f}^\lambda_{[i:j]} \) are described as flags of flags of sheaves. They can be alternatively written as commutative square diagrams of sheaves, as follows. Without loss of generality, let us assume \( i = 1 \) and \( j = n(k - 1) + r \) for \( 0 \leq r < n \), in order to ensure clear notation:

\[
\begin{array}{cccccccc}
\mathcal{F}_1 & \xrightarrow{\mathcal{F}_2} & \cdots & \xrightarrow{\mathcal{F}_n} & \mathcal{F}_1(D) \\
\mathcal{F}_{n+1} & \xrightarrow{\mathcal{F}_{n+2}} & \cdots & \xrightarrow{\mathcal{F}_{n+r}} & \mathcal{F}_{n+1}(D) \\
\mathcal{F}_{nk+1} & \xrightarrow{\mathcal{F}_{nk+2}} & \cdots & \xrightarrow{\mathcal{F}_{nk+r}} & \mathcal{F}_{nk+1}(D)
\end{array}
\tag{7.6}
\]

All the vertical arrows are colength 1 inclusions, with the support point located on \( D \). There are exactly \( nk + r = j - i + 1 \) of them, once the rightmost column is
identified with the twist of the leftmost column. To pass from this description to the one of (7.4), we simply set:

$$\mathcal{F}_1 := \{ \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_n \subset \mathcal{F}_1(D) \}$$

and then the parabolic sheaves $\mathcal{F}_2, \mathcal{F}_3, \ldots$ are obtained from each other by successively replacing the sheaves $\mathcal{F}_1, \mathcal{F}_2, \ldots$ by the one directly underneath it in the square diagram.

7.6. For $\lambda = (\lambda_1, \lambda_2, \ldots)$, the component $Z^\lambda_{[i;j]}$ can be described as the locus of diagrams (7.6) which satisfy the following additional conditions:

- if we read the vertical co-length 1 inclusions from left-right and then top-down (like the words on a page), the first $\lambda_1$ support points must coincide, the next $\lambda_2$ support points must coincide etc.
- there are inclusions $\mathcal{F}_{\lambda_1+\ldots+\lambda_t} \subset \mathcal{F}_{\lambda_1+\ldots+\lambda_t+n_s+1}$ for all $t \geq 1$, $s \geq 1$, in a way compatible with the rest of the diagram.

In particular, for $Z^0_{[i;j]} = Z^{(j-i+1)}_{[i;j]}$ the second condition is vacuous and the first requires that all support points coincide, thus giving the fine correspondence (7.1).

7.7. On the other hand, for $U_{[i;j]} := Z^{(1,\ldots,1)}_{[i;j]}$ the first bullet is vacuous, but the second bullet requires that we have diagonal arrows from the upper left to the lower right corners of each square in the diagram (7.6). Therefore, a point of $U_{[i;j]}$ can be written as a linear flag of torsion free sheaves (again take $i = 1$ and $j = n(k-1)+r$):

$$\mathcal{F}_{nk+1} \subset \ldots \subset \mathcal{F}_1 \subset \mathcal{F}_{nk+2} \subset \ldots \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_{nk} \subset \ldots \subset \mathcal{F}_n \subset \mathcal{F}_{nk+1}(D)$$

This is a flag just as in (6.1), with the sole exception that the framing at the divisor $\infty \subset \mathbb{P}^1 \times \mathbb{P}^1$ is required to be:

$$\mathcal{O}_{\infty}^n(-D) \subset \ldots \subset \mathcal{O}_{\infty}^n(-D) \subset \mathcal{O}_{\infty}^{n-1}(-D) \oplus \mathcal{O}_{\infty} \subset \ldots \subset \mathcal{O}_{\infty}(-D) \oplus \mathcal{O}_{\infty}^{n-1} \subset \mathcal{O}_{\infty}^n$$

Spaces of flags of torsion free sheaves with such more general framing conditions will be called **generalized Laumon spaces**, and they are all smooth by the same reason as affine Laumon spaces.

7.8. In [5], the authors have introduced the following rank $|d| + |d'|$ vector bundle on $\mathcal{M}_d \times \mathcal{M}_{d'}$:

$$E|_{\mathcal{F},\mathcal{F}'} = \text{Ext}^1(\mathcal{F}, \mathcal{F}'(-\infty))$$  \hspace{1cm} (7.7)

The notion of Ext space of flags of sheaves is defined in [5], where the following result is proved:

---

13This condition models the fact that the support points from the previous bullet are distinct, for a generic point of $Z^\lambda_{[i;j]}$. 

---
Proposition 7.9. The vector bundle $E$ has a section $s$ which vanishes set-theoretically on the locus:

$$C = \{(\mathcal{F} \subset \mathcal{F}') \} \subset \mathcal{M}_d \times \mathcal{M}_{d'}$$

(7.8)

Remark 7.10. When $d = d'$, the locus is simply the diagonal and thus has the expected dimension. We conclude that the vanishing locus of $s$ is scheme-theoretically equal to a multiple of the diagonal $[\Delta]$. We conjecture that this multiple is equal to 1, although we will not need this.

Remark 7.11. If $d - d' \in \mathbb{N}$, the locus $C$ is empty, and hence the vector bundle $E$ has a nowhere vanishing section. This implies that its top exterior power is zero.

7.12. Now we are able to explain why the subvarieties $Z_{\lambda'}^\lambda_{[i;j]} \subset \mathcal{M}_d \times \cdots \times \mathcal{M}_{d+[i;j]}$ are linear combinations of lci subvarieties. Every ordered partition $\lambda$ of $j - i + 1$ can be perceived as a $(j - i + 1) \times 1$ rectangle where we draw some black vertical bars to separate the parts of the partition $\lambda$. The remaining vertical bars (which we will call white) determine another ordered partition $\mu$ of $k$, simply by inverting black and white:

```

Diagram of a rectangle with black and white bars.
```

In the above example, $j - i + 1 = 11$ and:

$$\lambda = (3, 1, 1, 2, 4) \rightarrow \mu = (1, 1, 4, 2, 1, 1, 1)$$

With this in mind, we consider the smooth variety:

$$U_{[i;i+\mu_1-1]} \times \cdots \times U_{[i+\mu_1+\cdots+\mu_{s-1};j]}$$

(7.9)

and the following vector bundle on it:

$$E_\mu = \bigoplus_{s=1}^{t-1} \text{Ext}^1(F_s^+, F_{s+1}^-)$$

where $F_s^-$, $F_s^+$ denote the first / last sheaf in $U_{[i+\mu_1+\cdots+\mu_{s-1};i+\mu_1+\cdots+\mu_s-1]}$. We can consider the direct sum of the sections $s$ from Proposition 7.9. By Remark 7.10, this direct sum vanishes on the locus:

$$\{(\mathcal{F}^i \supset^i \cdots \supset^j \mathcal{F}^{j+1})\} \subset U_{[i;i+\mu_1-1]} \times \cdots \times U_{[i+\mu_1+\cdots+\mu_{s-1};j]}$$

This locus is simply the union of $Z_{\lambda'}^\lambda_{[i;j]}$ for all partitions $\lambda' \leq \lambda$. Here, the notation $\lambda' \leq \lambda$ means that the partition $\lambda'$ is obtained from $\lambda$ by breaking up some of its parts into smaller ones. We conclude that we have the following equality in the Chow group:

$$Z_{[i;j]}^\lambda + \sum_{\lambda' < \lambda} Z_{[i;j]}^{\lambda'} = \text{euler}(E_\mu)$$
If we multiply these relations by $(-1)^{\# \text{ parts of } \lambda}$ and add them up over all $\lambda$, then all terms in the LHS drop out, except for $Z_{[i;j]}$:

$$Z_{[i;j]} = (-1)^{j-i+1} \sum_{\mu} (-1)^{\# \text{ parts of } \mu} \cdot \text{euler}(E_\mu)$$

We can translate the above equality of Chow group classes to the definition of the modified structure sheaf that goes into (7.3):

$$[Z_{[i;j]}] = (-1)^{j-i+1} \sum_{\mu} (-1)^{\# \text{ parts of } \mu} \cdot \Lambda^{\text{top}} E_\mu$$  \hspace{1cm} (7.10)

7.13. In [5], it was shown that the $T-$fixed points of $M_d$ are indexed by collections of non-negative integers:

$$\tilde{d} = \{d_{ii} \geq d_{i+1,i} \geq \ldots \geq 1 \leq i \leq n\}$$

such that $\sum_{i \leq j} d_{ij} = d_j$. All our indices will be periodic modulo $n$, in the sense that $d_{i+n,j+n} = d_{ij}$. This latter condition implies that the above infinite descending sequences eventually stabilize to 0. Fixed points are important because equivariant $K-$theory is “concentrated” around these fixed points, as in the following localization formula:

$$c = \sum_{\tilde{d}} [\tilde{d}] \cdot \frac{c[\tilde{d}]}{A(T_{\tilde{d}} \mathcal{M}, 1)} \forall c \in K$$  \hspace{1cm} (7.11)

where $[\tilde{d}]$ is the class of the skyscraper sheaf at the fixed point corresponding to $\tilde{d}$. Therefore, if we know that two $K-$theory classes are equal when restricted to all $T-$fixed points, we conclude that they are equal. We will call this delocalization.

7.14. Take a fixed point $\tilde{d} = \{d_{kl}\}$ and we claim that the character of $T$ in the fiber of the tautological bundle $T_k$ is:

$$\text{char}_T(T_k|_{\tilde{d}}) = \sum_{l \leq k} q^{d_{kl} - 1} t_l^{-2}$$  \hspace{1cm} (7.12)

The dependence on $q'$ can be found inside $t_l := t_{l \mod n} \cdot q^{[\frac{l}{n}]}$. By (6.6), the character in the fiber of the quotient sheaf $\mathcal{W}_k$ is:

$$\text{char}_T(\mathcal{W}_k|_{\tilde{d}}) = \sum_{l \leq k} q^{2d_{kl} - 1} t_l^{-2} - \sum_{l \leq k-1} q^{2d_{k-1,l} - 1} t_l^{-2}$$  \hspace{1cm} (7.13)

7.15. We can use our description of the fixed points to prove Proposition 6.5. In general, suppose we are given a vector space $V = \text{span}(v_1, \ldots, v_p)$. Take a vector $v = \sum \alpha_i v_i$ with all $\alpha_i \neq 0$, and a collection of commuting endomorphisms $A_1, \ldots, A_k$ that are diagonal in the basis $\{v_1, v_2, \ldots\}$. It is easy to prove the assertion that the products:

$$A_1^{\beta_1} \ldots A_k^{\beta_k} \cdot v, \quad \beta_1, \ldots, \beta_k \in \mathbb{N}_0$$

span the vector space $V$ if and only if for any two different basis vectors $v_i$ and $v_j$, there exists an $l \in \{1, \ldots, k\}$ such that $A_l$ has different eigenvalues on $v_i$ and $v_j$.

We will use this assertion in the case when $V = K^*_T(M_d)$, $v = 1$ is the class of the
structure sheaf, and \(\{v_1, ..., v_p\}\) is the basis of torus fixed points. We will choose \(A_k\) to be the operator of multiplication by the \(K\)-theory class:

\[
\Lambda^1(\mathcal{T}_k) = \text{coefficient of } u \text{ in } \Lambda(\mathcal{T}_k, u)
\]

This class has restriction equal to minus the dual of \((7.12)\) at each of the fixed points, and therefore, at least one of these restrictions will be different between any two different fixed points. This proves Proposition 6.5.

**7.16.** In [5], it was shown that the character of \(T\) in the tangent space to \(\mathcal{M}_d\) at a fixed point \(\tilde{d} = \{d_{ij}\}\) equals:

\[
\text{char}_T(T_{\tilde{d}}) = \sum_{k=1}^{n} \sum_{l \leq k} t_l^2 \frac{q^{-2d_{kl}} - 1}{q^2 - 1} + \sum_{k=1}^{n} \sum_{l' \leq k-1} t_{l'}^2 \frac{q^{2d_{(k-1)l'}} + 2 - q^2}{q^2 - 1} + \\
+ \sum_{k=1}^{n} \sum_{l \leq k} t_l^2 \frac{(q^{2d_{kl}} - 1)}{q^2 - 1} \sum_{k=1}^{n} \sum_{l' \leq k-1} t_{l'}^2 \frac{(q^{2d_{kl'}} + 1 - q^2)}{q^2 - 1}
\]

We will now delocalize this equality, meaning we will write a equality of \(K\)-theory classes which restricts to the above at all torus fixed points:

\[
[T, \mathcal{M}] = \sum_{k=1}^{n} t_k^{-2} \cdot [\mathcal{T}_k]^\vee + q^2 t_k^2 \cdot \mathcal{T}_{k-1} + (q^2 - 1)[\mathcal{T}_k]^\vee \otimes (\mathcal{T}_k - \mathcal{T}_{k-1}) = \\
= \sum_{k=1}^{n} q^2 t_k^2 \cdot [\mathcal{T}_{k-1}] + q[\mathcal{T}_k]^\vee \otimes [\mathcal{W}_k] = \sum_{k=1}^{n} t_k^{-2} \cdot [\mathcal{T}_k]^\vee + q[\mathcal{T}_k] \otimes [\mathcal{W}_{k+1}]^\vee
\]

**7.17.** Another formula which was worked out in [5] is the character in the tangent space to \(\mathfrak{Z}_i := \mathfrak{Z}_{[i,j]}\) at the torus fixed points. Since \(\mathfrak{Z}_i\) parametrizes pairs of parabolic sheaves \(\mathcal{F} \supseteq \mathcal{F}'\) such that the quotient has length 1 and is supported on the \(i-th\) piece of the flag, fixed points of \(\mathfrak{Z}_i\) are parametrized by pairs of partitions:

\[
(d, \tilde{d}) \quad \text{such that} \quad \tilde{d}^- \leq_i \tilde{d}^+
\]

The notation \(\tilde{d}^- \leq_i \tilde{d}^+\) means that \(d_{i,j}^- = d_{i,j}^+ + \delta_i^j\delta^l_k\) for a unique \(l \leq i\). The following is the delocalized version of Proposition 4.21 of [5]:

\[
[T, \mathfrak{Z}_i] = p^{\pm}([T, \mathcal{M}]) \mp q \cdot p^{\pm}([\mathcal{W}_{i+1}]^\vee) \cdot l_i^{k+1} + q^2 - 1
\]

where we write \(\mathcal{W}^+ = \mathcal{W}, \mathcal{W}^- = \mathcal{W}^\vee, \) and \(\varepsilon = 1\) or 0 depending on whether the sign is + or -. In the above, \(p^\pm : \mathfrak{Z}_i \rightarrow \mathcal{M}_{d^\pm}\) are the standard projections, and \(l_i = [L_i]\) is the tautological line bundle on \(\mathfrak{Z}_i\), whose restrictions are given by:

\[
l_i\big|_{(\tilde{d}^- \leq_i \tilde{d}^+)} = q^{2d_{ij}^+} t_j^{-2}
\]

where \(j\) is the unique index such that \(d_{ij}^- = d_{ij}^+ - 1\).
7.18. A more general formula is the one for the character in the torus fixed fibers of the vector bundle $E$ of (7.7). Given any two fixed points $d^1, d^2 \in \mathcal{M}^T$, we have:

$$\chi_T(E_{d^1, d^2}) = \sum_{k=1}^{n} \frac{t_k^2}{q^{2d^1_k} - 1} - \frac{1}{q^2 - 1} + \sum_{k=1}^{n} \frac{t_k^2}{q^{2d^1_k} - 1} - \frac{1}{q^2 - 1} + \sum_{k=1}^{n} \frac{t_k^2}{q^{2d^2_k} - 1} - \frac{1}{q^2 - 1} + \sum_{k=1}^{n} \frac{t_k^2}{q^{2d^2_k} - 1} - \frac{1}{q^2 - 1} + \sum_{k=1}^{n} \frac{t_k^2}{q^{2d^2_k} - 1} - \frac{1}{q^2 - 1}$$

After simplification, the above will be a sum of monomials in $q, q', t_1, ..., t_n$, and it contains the trivial character 1 as a summand unless $d^1 \geq d^2$. Delocalizing the above gives us:

$$|E| = \sum_{k=1}^{n} t_k^2 \cdot [T^1_k] \cdot (q^2 - 1) [T^2_k] \cdot (q^2 - 1) \cdot (|T_k^1| - |T_k^2|) = \sum_{k=1}^{n} t_k^2 \cdot [T^1_k] \cdot (q^2 - 1) [T^2_k] \cdot (q^2 - 1) \cdot (|T_k^1| - |T_k^2|) = \sum_{k=1}^{n} t_k^2 \cdot [T^1_k] \cdot (q^2 - 1) [T^2_k] \cdot (q^2 - 1) \cdot (|T_k^1| - |T_k^2|)$$

(7.17)

where $T^1_k$ and $T^2_k$ (respectively $W^1_k$ and $W^2_k$) denote the tautological vector bundles (respectively universal sheaves) on the two factors of $\mathcal{M} \times \mathcal{M}$.

7.19. We will now use the above fixed point computations to compute some push-forwards. But first, we need to consider some generalities. Take a proper map of algebraic varieties $\pi : X \rightarrow X_0$ which is equivariant under a torus action, and a class $l \in K^1_T(X)$. For any rational function $P(u) \in K^1_T(X_0)(u)$, we may consider the push-forward:

$$\pi_*(P(l)) \in K^1_T(X_0).$$

(7.18)

Now suppose that $\pi$ factors as:

$$\pi : X_k \xrightarrow{\pi_k} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0$$

(7.19)

where each $\pi_i$ is proper. We call the above a tower, and fix a class $l_i \in K^1_T(X_i)$ for each $i \in \{1, ..., k\}$. For any rational function $P(u_1, ..., u_k) \in K^1_T(X_0)(u_1, ..., u_k)$, we are interested in computing push-forwards of the form:

$$\pi_*(P(l_1, ..., l_k)) \in K^1_T(X_0).$$

(7.20)

Though each $l_i$ is a class on $X_i$, we view it as a class on $X_k$ and suppress the obvious pull-back maps.

7.20. For a single proper map $\pi : X \rightarrow X_0$, let us assume that there exists a rational function $f(u) \in K^1_T(X_0)(u)$ such that:

$$\pi_*(P(l)) = \int P(u)f(u)Du, \quad \forall P(u) \in K^1_T(X_0)(u)$$

(7.21)

where $Du = \frac{du}{2\pi i}$ and the integral is taken over a contour that separates the poles of $P$ from those of $f$. Specifying the contour is necessary, because we want it to only pick up the poles of the rational function $P$. Relation (7.21) might seem like...
a formal artifice, but it comes up naturally in important situations:

**Example 7.21.** Suppose $X = \mathbb{P}V$ for some vector bundle $V$ on $X_0$, and $l = [\mathcal{O}(1)]$ is the tautological line bundle. Then $f(u) = \Lambda(-V, u)$ of subsection 6.4. In this case, the push-forwards (7.18) are simply $K$-theoretic analogues of Segre classes.

**7.22.** Take a tower as in (7.19) and suppose that for each proper map therein there exists a rational function $f_i(u)$ fulfilling (7.21). Then by writing $\pi = \pi_1, \ldots, \pi_k$, we can inductively conclude that:

$$\pi_*(P(l_1, \ldots, l_k)) = \int P(u_1, \ldots, u_k) f_1(u_1) \cdots f_k(u_k) Du_1 \cdots D u_k$$

(7.22)

where the contours of integration all separate the poles of the functions $f_i$ from those of the rational function $P$, in such a way that we only pick up the residues at the former. We will now apply the above setup to our situation.

**Proposition 7.23.** Consider the projections $p^\pm : \mathcal{Z}_i \rightarrow \mathcal{M}_{d\pm}$, and the tautological line bundle $l_i$ on $\mathcal{Z}_i$. Then for any rational function $m(u)$ with coefficients in $K^*_T(\mathcal{M}_{d\pm})$, we have:

$$p^\pm_*(m(l_i q^{-i})) = \frac{\pm 1}{1 - q^{-2}} \int m(u) \cdot \Lambda(\pm W_i^{\pm}, u^{\pm 1} q^{\mp i - 1}) Du$$

(7.23)

where $\varepsilon = 1$ or $0$, depending on whether the sign is $+$ or $-$. The integral is over a contour which separates the poles of $\Lambda(\pm W_i^{\pm}, \varepsilon q^{\mp i - 1})$ from $\text{Poles}(m) \cup \{0, \infty\}$.

**Proof** We can apply the equivariant localization formula (7.11) on $\mathcal{Z}_i$ and obtain:

$$m(l_i q^{-i}) = \sum_{d^- \leq s, d^+} \sum_{[\tilde{d}^-], [\tilde{d}^+]} m(l_i_{[\tilde{d}^-]_{d^- \leq s, d^+}} q^{-i}) \frac{m(l_i_{[\tilde{d}^-]_{d^- \leq s, d^+}} : q^{-i})}{\Lambda(T_{d^- \leq s, d^+}, \mathcal{M}_i, 1)}$$

$$\Rightarrow p^\pm_*(m(l_i)) = \sum_{d^\pm} \left( \frac{[\tilde{d}^\pm]}{\Lambda(T_{d^\pm}, \mathcal{M}_i, 1)} \sum_{d^- \leq s, d^+} m(l_i_{[\tilde{d}^-]_{d^- \leq s, d^+}} : q^{-i}) \frac{1 - 1}{1 - q^{-2}} \cdot \Lambda(\pm q W_i^{\pm}, l_i_{d^\pm}, 1) \right)$$

(7.24)

where $Tp^\pm = \text{Cone}(T \mathcal{Z}_i \rightarrow p^{\pm*} T \mathcal{M})$ is the vertical tangent sheaf. Using (7.15), we see that:

$$\sum_{d^\pm} \left( \frac{[\tilde{d}^\pm]}{\Lambda(T_{d^\pm}, \mathcal{M}_d, 1)} \sum_{d^- \leq s, d^+} \sum_{d^- \leq i, d^+} \lambda l_i_{d^- \leq s, d^+} : q^{-i}) \frac{1 - 1}{1 - q^{-2}} \cdot \Lambda(\pm q W_i^{\pm}, l_i_{d^\pm}, 1) \right)$$

The term $1 - 1 = 0$ which appears in the numerator of the above will cancel out with a single zero which appears in the rightmost $\Lambda$. By looking at (7.13), we can rewrite the above as:

$$\sum_{d^\pm} \left( \frac{[\tilde{d}^\pm]}{\Lambda(T_{d^\pm}, \mathcal{M}_d)} \sum_{d^- \leq s, d^+} m(l_1 q^{-i}) \frac{1 - 1}{1 - q^{-2}} \prod_{j \leq i, j \leq 1} (1 - q^{\pm 2d_j^{\pm 1} l_j^{\mp 1} l_j^{\pm 1}}) \prod_{j \leq i} (1 - q^{\pm 2d_j^{\pm 1} l_j^{\mp 1} l_j^{\pm 1}}) \right)$$
As a function of $l$, the poles of the above occur precisely at $l = q^{2d^2_{i,j} + 1} t_j^{-2}$, and by (7.16), these are precisely the various $l_{i|d- \frac{d^+}{d^+}}$. Then we can apply the residue formula to write the above as:

$$
\pm \frac{1}{1 - q^{-2}} \sum_{d \pm} \left( \frac{[d^\pm]}{\Lambda(T_{d \pm} \mathcal{M}_{d \pm})} \int m(uq^{-i}) \prod_{j \leq i \pm 1} (1 - q^{2d^2_{i,j} + 1} t_j^{-2} u^{1}) \frac{Du}{u} \right)
$$

$$
= \pm \frac{1}{1 - q^{-2}} \sum_{d \pm} \left( \frac{[d^\pm]}{\Lambda(T_{d \pm} \mathcal{M}_{d \pm})} \int m(uq^{-i}) \Lambda(\pm W_{i+1}^{\frac{1}{2}}, u^{1} q^{-1}) Du \right)
$$

Delocalizing and substituting $u \rightarrow uq^i$, this gives precisely the RHS of (7.23).

\[\square\]

7.24. We now need to generalize the above proposition to allow for any fine correspondences. We will do so by putting together Proposition 7.23 with the general formula (7.22).

**Proposition 7.25.** For a general projection $p^\pm : \mathfrak{J}_{[i:j]} \rightarrow \mathcal{M}_{d \pm}$ and any rational function $m(z_i, ..., z_j)$ with coefficients in $\mathcal{K}_\mathfrak{J}(\mathcal{M}_{d \pm})$, we have:

$$
p^\pm_m = - \left(1 \pm 1\right)^{j-i+1} q^{j-i+|\frac{i-j}{m}|} \frac{1}{1 - q^{-2}}
$$

$$
\int \frac{m(z_i, ..., z_j)}{(1 - \frac{u \omega(z_i, u_i)}{u_i+1}) ... (1 - \frac{u_j - u \omega(z_j, u_j)}{u_j})} \prod_{k=i}^j \Lambda(\pm W_{k+e_i}^\pm, u_k^1 q^k q^{-1}) Du_k
$$

(7.25)

where the contours separate $\text{Poles}(\Lambda(\pm W))$ from the set $\text{Poles}(m) \cup \{0, \infty\}$, and are arranged in order of the variables $u_i, ..., u_j$ with $u_i$ closest to / farthest from the former set, depending on whether the sign is $+$ or $-$.

**Proof** The maps $p^\pm : \mathfrak{J}_{[i:j]} \rightarrow \mathcal{M}_{d \pm}$ can be written as projective towers (7.19):

$$
p^+ : \mathfrak{J}_{[i:j]} \xrightarrow{p^+_{i+1}} \mathfrak{J}_{[i+1:j]} \rightarrow ... \rightarrow \mathfrak{J}_{[j:j]} \xrightarrow{p^+_{j+1}} \mathcal{M}_{d^+}
$$

$$
p^- : \mathfrak{J}_{[i:j]} \xrightarrow{p^-_{j-1}} \mathfrak{J}_{[i:j-1]} \rightarrow ... \rightarrow \mathfrak{J}_{[i:i]} \xrightarrow{p^-_{i-1}} \mathcal{M}_{d^-}
$$

We would like to compute the push-forwards under the individual maps $p^k \pm$. When the sign is $+$, consider the vector bundle $E$ similar to that of Section 7.8, whose fibers are given by:

$$
\text{Ext}^1(F^k+1, F^{k}(\infty)) \xrightarrow{\mathcal{G}} E
$$

$$
\mathcal{F}^k \times (\mathcal{F}^{k+1} \supset \mathcal{G}_{k+1}^x \supset ... \supset \mathcal{G}_j^x \mathcal{F}^{j+1} \supset \mathcal{G}_j^x \mathcal{F}^{j+1}) \rightarrow \mathcal{M}_{d^+} \times \mathfrak{J}_{[k+1:j]}
$$

It was shown in [5] that this bundle has a section which vanishes on the locus:

$$\{ \mathcal{F}^k \supset \mathcal{F}^{k+1} \supset \mathcal{G}_{k+1}^x \supset ... \supset \mathcal{G}_j^x \mathcal{F}^{j+1} \supset \mathcal{G}_j^x \mathcal{F}^{j+1} \}$$
Note that we have removed the condition on support for the first quotient in the above. This locus has two components: the first corresponds to the case where the first support point is \( x \), and the second corresponds to the case where the second support point is different from \( x \) (in this latter case we need to take the closure). Both these components have the expected codimension, so we have:

\[
[E] = [3_{[k,j]}] + [3_{[k+1,j]} \times \mathcal{M}_{d,k+2} 3_k]
\]

Letting:

\[
\pi : \mathcal{M}_{d,k} \times 3_{[k+1,j]} \to 3_{[k+1,j]}, \quad \rho : 3_{[k+1,j]} \times \mathcal{M}_{d,k+2} 3_k \to 3_{[k+1,j]}
\]
denote the standard projections, we can use the above equality to compute our push-forwards from \( 3_{[k,j]} \) to \( 3_{[k+1,j]} \), for any rational function \( r \):

\[
p^+_s (r(l_k q^{-k})) = \pi_* \left( [E] \cdot r(l_k q^{-k}) \right) - \rho_* \left( r(l_k q^{-k}) \right),
\]

The map \( \rho_* \) reduces to pushing forward along the projection \( 3_k \to \mathcal{M}_{d,k+2} \). As for computing \( \pi_* \left( [E] \cdot r(l_k q^{-k}) \right) \), note that \( E \) only takes into account the first two sheaves of the flag. Therefore, it reduces to pushing forward along the projection \( 3_k \to \mathcal{M}_{d,k+1} \), so both cases are covered by Proposition 7.23. We conclude that:

\[
p^+_s (r(l_k q^{-k})) = \int \frac{r(u) \left[ \Lambda(W^{k+1}_{k+1}, u^{-1} q^{-k-1}) - \Lambda(W^{k+2}_{k+1}, u^{-1} q^{-k-1}) \right] Du}{1 - q^{-2}}
\]

and:

\[
p^-_s (r(l_k q^{-k})) = -\int \frac{r(u) \left[ \Lambda(-W^k_{k+1}, uq^{-k-1}) - \Lambda(-W^k_{k+1}, uq^{-k+1}) \right] Du}{1 - q^{-2}}
\]

The fixed point formula tells us that, on a simple correspondence \( F \supset \mathcal{F}' \), we have the relation:

\[
[W'_m] = [W_m] + (q - q^{-1}) \cdot l, \quad [W'_{m+1}] = [W_{m+1}] + (q^{-1} - q) \cdot l,
\]

and \( [W'_m] = [W_m] \) for all \( m' \neq m, m+1 \). The above has the following effect on the \( \Lambda \) construction:

\[
\Lambda(W^m_{m'}, u^{-1} q^{-m'}) = \Lambda(W^m_{m'}, u^{-1} q^{-m'}) \cdot q^{-\langle m, m' \rangle} \omega(lq^{-m}, u)
\]

\[
\Lambda(-W^m_{m'}, uq^{m'-1}) = \Lambda(-W^m_{m'}, uq^{m'-1}) \cdot q^{\langle m, m' \rangle} \omega(u, lq^{-m})
\]

for all weights \( m \) and \( m' \), where \( u \) is thought to have weight \( m' - \varepsilon \) and \( l \) is thought to have weight \( m \) (this is relevant for the definition of \( \omega \) from (3.3)). Therefore, (7.26) and (7.27) give rise to:

\[
p^+_s (r(l_k)) = -q^2 \int \frac{r(u) \Lambda(W^{k+1}_{k+1}, u^{-1} q^{-k-1}) Du}{1 - \frac{q^{k+2}}{l_{k+1}}} =
\]

\[
= -q^2 \int \frac{r(u) \Lambda(W^{k+1}_{k+1}, u^{-1} q^{-k-1})}{1 - \frac{q^{k+2}}{l_{k+1}}} \prod_{m \geq k+1} q^{-\langle m, \varepsilon_{k+1} \rangle} \omega(l_m q^{-m}, u) Du
\]

and:

\[
p^-_s (r(l_k)) = q^2 \int \frac{r(u) \Lambda(-W^k_{k+1}, uq^{-k-1})}{1 - \frac{q^{k+1}}{uq^{-k}}} Du =
\]
\[ q = q^2 \int \frac{r(u) \Lambda(-W_{k}^{m}, u q^{k-1})}{1 - \frac{1}{u q^{k-1}}} \prod_{m<k} \omega(u, l_{m} q^{-m}) Du \]

We can use (7.22) to iterate the above push-forwards, obtaining precisely (7.23).

Note that we are using the elementary observations that \(-\langle \zeta_{m}, \zeta_{k+1} \rangle = \langle \zeta_{k}, \zeta_{m} \rangle\) and that:

\[ \sum_{i \leq a < b \leq j} \langle \zeta_{a}, \zeta_{b} \rangle = i - j + \left\lfloor \frac{j - i}{n} \right\rfloor \]

\[ \Box \]

7.26. We can now use the above computation to connect the geometric operators \(x_{m}^{\pm}\) of (7.3) to the shuffle elements \(X_{m}^{\pm}\), for every Laurent polynomial \(m(z_{i}, ..., z_{j})\).

**Proof of Theorem 1.3:** The precise claim is that:

\[ X_{m}^{\pm} = (-1)^{j-i} q^{-\lfloor \frac{j-i}{n} \rfloor - \sum a_i = 1 (q - q^{-1})} \cdot x_{m(z_{i}, ..., z_{j})}^{\pm} \quad (7.30) \]

\[ \tau(X_{m})^{-1} q^{-1} = -q^{-\lfloor \frac{b-a}{n} \rfloor - \sum a_i = 1 (q - q^{-1})} \cdot x_{m(z_{i}, ..., z_{j})}^{-\pm} \quad (7.31) \]

as operators on \(K\). To prove this, note that it suffices by Proposition 6.5 to show that the two sides have the same action on tautological classes. By (7.3), we have:

\[ x_{m}^{\pm} \left( \prod_{1 \leq k \leq n} \Lambda(T_{k}, s) \right) = p_{s}^{\pm} \left( m(l_{i} q^{-i}, ..., l_{j} q^{-j}) \prod_{1 \leq a \leq n} \Lambda(p_{a}^{\mp} T_{a}, s) \right) \]

Comparing the various tautological sheaves on the variety \(3_{[i,j]}\), we can write:

\[ \Lambda(p_{a}^{\mp} T_{a}, s) = \Lambda(p_{b}^{\mp} T_{a}, s) \prod_{i \leq b \leq j} \left( 1 - \frac{s}{t_{a}} \right)^{\mp 1} \]

and therefore:

\[ x_{m}^{\pm} \left( \prod_{1 \leq a \leq n} \Lambda(T_{a}, s) \right) = \prod_{s \in S_{n}} \Lambda(T_{a}, s) \cdot p_{s}^{\pm} \left( m(l_{i} q^{-i}, ..., l_{j} q^{-j}) \prod_{i \leq a \leq j} \left( 1 - \frac{s}{t_{a}} \right)^{\mp 1} \right) \]

which according to (7.25) equals:

\[ -\frac{(\mp 1)^{j-i+1} q^{j-i+\lfloor \frac{j-i}{n} \rfloor}}{1 - q^{-2}} \prod_{1 \leq a \leq n} \Lambda(T_{a}, s). \]

\[ \int \frac{m(u_{i}, ..., u_{j})}{(1 - \frac{u_{m}}{m_{i+j}})} \prod_{a \leq k} \omega(u_{h}, u_{a}) \prod_{a=1}^{j} \Lambda(\pm W_{k}^{a}, u_{a}^{\mp 1} q^{-1}) Du_{a} \]

Comparing this with Proposition 6.9 for the (positive and negative) actions of the shuffle algebra elements \(X_{m}\) of (4.17) on \(K\) gives us the desired result. \[ \Box \]
8. BACK TO $K$–THEORY

8.1. With the algebraic results of the previous section, we may revisit the $K$–theory group of affine Laumon spaces, and complete the proof of Theorem 1.2. The embedding of Hopf algebras of Lemma 4.4 implies the following embedding of their Drinfeld doubles:

$$U_q(\mathfrak{gl}_n) \cong B_0 \subset A \cong U_q(\mathfrak{gl}_n), \quad e^\pm_{[i:j]} \rightarrow P^\pm_{[i:j]},$$

where:

$$P_{[i:j]} := P^0_{[i:j]} = \text{Sym} \left[ \frac{\prod_{i\leq a \leq b \leq j} \omega(z_b, z_a)}{(1 - z_a q_{z+1}) \cdots (1 - z_{i-1} q_{z_j})} \right] \in A^+$$

By (7.30), the above copy of $U_q(\mathfrak{gl}_n)$ acts on $K$ by sending the positive root generators $e^+_i$ to the geometric correspondence:

$$\alpha \rightarrow (-1)^{j-i} q^{-\frac{1}{2}(j-i)} - \sum_{a=1}^j (q - q^{-1}) \cdot p^+_a \left( [j_{[i:j]}] \cdot p^{-a}(\alpha) \right) \quad (8.1)$$

As for the negative root generators, (7.31) claims that $\tau(e_{[i:j]})^{-1} \psi^{-1}_{[i:j]} \psi^{-1}_{[i:j] + 1}$ acts via the transposed correspondence:

$$\alpha \rightarrow -q^{-\frac{1}{2}(i+j)} - \sum_{a=1}^j (q - q^{-1}) \cdot p^-_a \left( 1 \cdot [j_{[i:j]}] \cdot p^{+a}(\alpha) \right) \quad (8.2)$$

where $l = \prod_{a=1}^j l_a q^{-a}$ and $\tau$ is the involution of (3.7). We will now give a representation theoretic understanding of this involution, as restricted to $U_q(\mathfrak{gl}_n)$.

8.2. More concretely, we will seek to express $\tau(e_{[i:j]})$ in terms of the root generators themselves. To do that, we need to study the shuffle element:

$$R_{[i:j]} := \tau(P_{[i:j]}) = \text{Sym} \left[ \frac{\prod_{i\leq a \leq b \leq j} \omega(z_b, z_a)}{(1 - z_a q_{z+1}) \cdots (1 - z_{i-1} q_{z_j})} \right] \prod_{b=a+1} \omega(z_b, z_a) \in A^+$$

and compute its coproduct and various pairings. By applying (3.7), one sees that:

$$R_{[i:j]} = \text{Sym} \left[ \prod_{i\leq a \leq b \leq j} \omega(z_a, z_b) \right] \left[ \frac{1}{(1 - z_a q_{z+1}) \cdots (1 - z_{i-1} q_{z_j})} \right]$$

This shuffle element lies in $B_0^+ \cong U_q^+(\mathfrak{gl}_n)$, since it remains finite when we send any number of variables to $\infty$. By analogy with (4.22), we see that:

$$\Delta_0(R_{[i:j]}) = \sum \varphi_{[i:k-1]} R_{[k:j]} \otimes R_{[i:k-1]} \quad (8.3)$$

To completely characterize the element $R_{[i:j]} \in B_0^+ \cong U_q^+(\mathfrak{gl}_n)$, we also need to calculate its pairings. By analogy with Proposition 4.18, we obtain:

$$(R_{[i:i+k-1]}, R_{[i:i+k-n-1]}) = (q^2 - 1) q^{2k-2} \quad (8.4)$$

For $i' \neq i$ modulo $n$, we have:

$$(R_{[i:i+k-1]}, R_{[i':i'+k-n-1]}) = \frac{(q^2 - 1)(q^{2k} - q^{-2k})}{q^2 + 1} \quad (8.5)$$
Finally, for \( j \neq i - 1 \) modulo \( n \), we have:

\[
(R_{[i;j]}, R_{[i;j]}) = \frac{(q^2 - 1)(q^{2g+1} - q^{2g-1})}{q + q^{-1}}
\]

(8.6)

if \( j \neq i - 1 \) modulo \( n \), and \( g = \lfloor \frac{j-i+1}{n} \rfloor \). All other pairings among the \( R_{[i;j]} \) vanish for degree reasons.

8.3. Since the Hopf pairing on \( U_q(\hat{\mathfrak{g}}_n) \) is non-degenerate, properties (8.3)-(8.6) determine the elements \( R_{[i;j]} \) uniquely. Let us therefore look at the antipode antiautomorphism:

\[
S : U_q(\hat{\mathfrak{g}}_n) \rightarrow U_q(\hat{\mathfrak{g}}_n), \quad S(T) = T^{-1}
\]

for which we have the opposite coproduct rule from (2.10):

\[
\Delta(S(T)) = (12) \cdot S(T) \otimes S(T),
\]

(8.7)

In terms of generators, let us look at:

\[
f_{[i;j]} := (-1)^{j-i+1} \frac{\psi_{j+1}}{\psi_i} S(e_{[i;j]}) \quad \in U_q^\geq(\hat{\mathfrak{g}}_n)
\]

Properties (8.7) imply:

\[
\Delta(f_{[i;j]}) = \sum_{k=1}^{j+1} \frac{\psi_k}{\psi_i} f_{[k;j]} \otimes f_{[i;k-1]}
\]

as well as formulas (2.13)-(2.15) for \( f \) instead of \( e \). The fact that the coproduct and pairings determine elements of \( U_q^\geq(\hat{\mathfrak{g}}_n) \) completely, we can compare the above relations with (8.3)-(8.6) to obtain:

**Proposition 8.4.** The isomorphism \( \Upsilon : U_q^\geq(\hat{\mathfrak{g}}_n) \rightarrow B_0^\geq \) defined by sending \( e_{[i;j]} \rightarrow P_{[i;j]} \) (see Lemma 4.4) also sends:

\[
f_{[i;j]} = (-1)^{j-i+1} \frac{\psi_{j+1}}{\psi_i} S(e_{[i;j]}) \quad \rightarrow \quad R_{[i;j]} = \tau(P_{[i;j]})
\]

8.5. With this in mind, we note that the antipode of the root generators gives the action of the transposed correspondences from (8.2). We are now in position to complete the proof of the last Theorem stated in the introduction:

**Proof of Theorem 1.2:** Formulas (8.1) and (8.2) define the action of the positive and negative root generators of \( U_q(\hat{\mathfrak{g}}_n) \) on \( K \):

\[
e_{[i;j]}^+(\alpha) = (-1)^{j-i} q^{-\lfloor \frac{j-i}{n} \rfloor - \sum_{a=1}^{i} (q - q^{-1}) \cdot p_+^a ([3_{[i;j]}] \cdot p^{-}^{a})(\alpha))
\]

\[
\psi_{[i;j]}^{-2} S(e_{[i;j]})^{-} (\alpha) = (-1)^{j-i} q^{-\lfloor \frac{j-i}{n} \rfloor - \sum_{a=1}^{i} (q - q^{-1}) \cdot p_+^a ([3_{[i;j]}] \cdot p^{+}^{a})(\alpha))
\]
where \( l = \prod_{a} l_a q^{-a} \). It follows from Theorem 1.1, Lemma 4.4 and 6.7 that this action is well-defined. The last thing we need to prove is that this action makes \( K \) isomorphic to the Verma module:

\[ V = U_q(\mathfrak{g}_n) \otimes U_{\infty}(\mathfrak{g}_n) \mathbb{C} \]

where \( U_q(\mathfrak{g}_n) \to \mathbb{C} \) annihilates \( \mathbb{C} \) and the diagonal generators \( \psi_i \) act on \( \mathbb{C} \) by multiplication with \( t_i q^i \). Sending \( 1 \in V \) to the unit \( K \) theory class \( \in K^0 = K^*_{T}(pt) \) gives us a map of \( U_q(\mathfrak{g}_n) \) modules:

\[ V \to K \]

For generic values of the \( t_i \)'s, the Verma module is irreducible, so the above map is injective. To prove that it is surjective, it is enough to show that any class in \( K \) can be brought to a non-zero multiple of \( 1 \) by acting on it with sufficiently many annihilators. For generic values of \( t_i \), this is a standard argument.

\[ \Box \]

8.6. In this section, we will study the vector bundle \( E \) of (7.7) in the context of the previous section. The results will be necessary in [13]. More precisely, we will consider the operator:

\[ y: K_{d'} \to K_d, \quad \alpha \to p_1^i(\Lambda(E,1) \cdot p_2^i(\alpha)) \]

where:

\[ \begin{array}{ccc}
\mathcal{M}_d & \xrightarrow{p_1} & \mathcal{M}_{d'} \\
& \xleftarrow{p_2} & \\
\mathcal{M}_d & \to & \mathcal{M}_{d'}
\end{array} \] (8.8)

are the standard projections. Because of Remark 7.11, the top exterior power \( \Lambda(E,1) \) vanishes unless \( d - d' = k \in \mathbb{N}^n \). Hence we will write \( y_k \) for the restriction of the above operator to \( \text{Hom}(K_*,K_{*+k}) \), and this is the geometric operator we will now study.

**Proposition 8.7.** The operator \( y_k \) acts on \( K \) via the element:

\[ Y_k := q^{(k,k)} (ij) \neq (i'j') \prod_{1 \leq i \leq n, j \leq k_i} \omega(z_{ij}, z_{i'j'}) \in \mathcal{A}^+ \] (8.9)

**Proof** To emphasize the fact that \( d - d' = k \in \mathbb{N}^n \), let us relabel \( d = d^+, \ d' = d^- \), \( p^1 = p^+, \ p^2 = p^- \) in diagram (8.8). By the equivariant localization formula, we have:

\[ y_k \left( \prod_{1 \leq i \leq n} \Lambda(T, s) \right) = \sum_{d^-, d^+ \in \mathcal{M}} [d^+] \frac{\Lambda(E_{d^-, d^+, 1})}{\Lambda(T_{d^+, M}, 1) \Lambda(T_{d^-, M}, 1)} \prod_{1 \leq i \leq n} \Lambda(T_{d^-}, s) \]

As was mentioned in Remark 7.11, \( E_{d^-, d^+} \) contains a trivial character (and so the corresponding term in the numerator of the RHS above vanishes) unless we have
\[ \tilde{d}^{-} \leq \tilde{d}^{+}. \] From the fixed point formulas of (7.17), the above equals:

\[ \sum_{\tilde{d} \in \mathcal{M}^{T}} \frac{[\tilde{d}^{+}] \cdot \prod_{1 \leq i \leq n} \Lambda(T_{i} | \tilde{d}^{+}, s)}{\Lambda(T_{\tilde{d}^{+}}, M, 1)} \sum_{\tilde{d}^{-} \leq \tilde{d}^{+}} \frac{\Lambda(E_{\tilde{d}^{-}, \tilde{d}^{+}, 1})}{\Lambda(T_{\tilde{d}^{-}}, M, 1)} \prod_{1 \leq i \leq n} \frac{\Lambda(T_{i} | \tilde{d}^{-}, s)}{\Lambda(T_{i} | \tilde{d}^{+}, s)} \]

Given any \( \tilde{d}^{-} \leq \tilde{d}^{+} \), let us write the difference between the corresponding characters as \( \tilde{d}^{+} = \tilde{d}^{-} + \sum l_{a} \), so the above equals:

\[ = \sum_{\tilde{d} \in \mathcal{M}^{T}} \frac{[\tilde{d}^{+}] \cdot \prod_{1 \leq i \leq n} \Lambda(T_{i} | \tilde{d}^{+}, s)}{\Lambda(T_{\tilde{d}^{+}}, M, 1)} \sum_{\tilde{d}^{+} = \tilde{d}^{-} + \sum l_{a}} \frac{\Lambda(E_{\tilde{d}^{-}, \tilde{d}^{+}, 1})}{\Lambda(T_{\tilde{d}^{-}}, M, 1)} \prod_{i \leq n} \Lambda(T_{i} | \tilde{d}^{-}, s) \prod_{a} \left( 1 - \frac{s}{l_{a}} \right)^{-1} \]

The last equality used (7.17). We can use (7.28) and (7.29) to express the \( \Lambda \) classes of \( \mathcal{W} | \tilde{d}^{-} \) in terms of the classes of \( \mathcal{W} | \tilde{d}^{+} \), and so the above becomes:

\[ = q^{(k,k)} \sum_{\tilde{d} \in \mathcal{M}^{T}} \frac{[\tilde{d}^{+}] \cdot \prod_{1 \leq i \leq n} \Lambda(T_{i} | \tilde{d}^{+}, s)}{\Lambda(T_{\tilde{d}^{+}}, M, 1)} \sum_{\tilde{d}^{+} = \tilde{d}^{-} + \sum l_{a}} \prod_{i \leq n} \frac{\Lambda(W_{i+1} | l_{a}^{-1} q^{-1})}{\prod_{a,b} \left( 1 - \frac{s}{l_{a}} \right)} \prod_{i \leq n} \omega(l_{a} q^{-a}, l_{b} q^{-b}) \]

When \( a = b \) in the last product above, the factors \( \omega \) contribute \( |k| \) factors \( \frac{1}{q^{-1} - q} \), which precisely cancel out \( |k| \) zeroes from the denominators of \( \Lambda(W_{i}^{|k|} \cdot \cdot \cdot) \). The above sum is therefore a sum of residues for a certain integral, so we have:

\[ y_{k} \left( \prod_{1 \leq i \leq n} \Lambda(T_{i}, s) \right) = q^{(k,k)} \left( \frac{[\tilde{d}^{+}] \cdot \prod_{1 \leq i \leq n} \Lambda(T_{i} | \tilde{d}^{+}, s)}{\Lambda(T_{\tilde{d}^{+}}, M, 1)} \sum_{\tilde{d} \in \mathcal{M}^{T}} \sum_{\tilde{d}^{+} = \tilde{d}^{-} + \sum l_{a}} \prod_{i \leq n} \frac{\Lambda(W_{i+1} | l_{a}^{-1} q^{-1})}{\prod_{a,b} \left( 1 - \frac{s}{l_{a}} \right)} \prod_{i \leq n} \omega(l_{a} q^{-a}, l_{b} q^{-b}) \right) \]

Changing the variables to \( u_{ij} = z_{ij} q^{-i} \) and delocalizing makes the above equal to:

\[ = q^{(k,k)} \left( \prod_{1 \leq i \leq n} \Lambda(T_{i}, s) \right) \left( \int \prod_{1 \leq j \leq n, i \leq k} \omega(u_{ij}, u_{ij} q^{-i}) \frac{\Lambda(W_{i+1} | u_{ij}^{-1} q^{-i})}{\prod_{a \in S_{i}} \left( 1 - \frac{s}{u_{ij}} \right)} D u_{ij} \right) \]

Comparing this with Theorem 6.7 produces the desired result.
8.8. In virtue of Proposition 8.7, we need to understand the shuffle element $Y_k$ of (8.9). It is easy to see that:

$$Y_k \in B_0^+ \cong U_q^+(\mathfrak{gl}_n),$$

because all $\omega$’s have finite limit when we let one of the variables go to $\infty$:

$$\lim_{\xi \to \infty} \omega(x, y) = q^{(\omega x, y - \omega x, x)}, \quad \lim_{\xi \to \infty} \omega(x, \xi y) = q^{- (\omega x, \xi - \omega x, x)}$$

(8.10)

Therefore, we may take:

$$g_k = \Upsilon^{-1}(Y_k) \in U_q^+(\mathfrak{gl}_n)$$

and would like to express this element in terms of the root generators $e_{ij}$. We do not know a formula for this expression, but we expect it to be some kind of vertex operator for $U_q(\mathfrak{gl}_n)$. We will describe it by computing the coproduct and pairing rules of the $g_k$, as in the following Proposition.

**Proposition 8.9.** We have $g_0 = 1$, as well as:

$$\Delta(g_k) = \sum_{0 \leq l \leq k} q^{(l,k-l)} \frac{l!}{k!(1 - q^{-2})^{|k|}} g_{k-l} \otimes g_k$$

(8.11)

$$\langle g_k, g_k \rangle = \frac{q^{(k,k)}}{k!(1 - q^{-2})^{|k|}}$$

(8.12)

for any $k \in \mathbb{N}^n$.

**Proof** Since the isomorphism $\Upsilon$ preserves the coproduct and pairing, we can prove the above Proposition by replacing $g_k$ with $Y_k$ and $e_{[ij]}$ with $P_{[ij]}$. By definition, the coproduct of the shuffle elements $Y_k$ is computed by letting some of the variables go to $\infty$, as in (4.3):

$$\Delta_0(Y_k) = \sum_{0 \leq l \leq k} q^{(l,k-l)} \varphi_{k-l} Y_l \otimes Y_{k-l} q^{(l,k-l)-(k-l,l)}$$

The powers of $q$ in the numerator come from the limits (8.10). Since:

$$\varphi_{k-l} = \frac{\psi_{k-l+1}}{\psi_{k-l}}$$

and

$$\varphi_{k-l+1} Y_l = Y_l \psi_{k-l+1} q^{-(l,k-l+1)} = Y_l \psi_{k-l+1} q^{(k-l,l)}$$

the above implies (8.11). Meanwhile, Proposition 3.10 gives us:

$$\left( \frac{q^{(k,k)}}{(q^2 - 1)^{|k|}} \prod_{(ij) \neq (i'j')} \omega(z_{ij}, z_{i'j'}) \frac{q^{(k,k)}}{(q^2 - 1)^{|k|}} \prod_{(ij) \neq (i'j')} \omega(z_{ij}, z_{i'j'}) \right) =$$

$$= \frac{q^{(k,k)}}{(q^2 - 1)^{|k|}} \cdot \frac{(q^2 - 1)^{|k|}}{k!} \int_{|z_{ij}| = 1} \prod_{1 \leq i \leq n} \prod_{1 \leq j \neq k_i} \prod_{1 \leq j \leq k_i} \frac{z_{ij}q^{-1} - z_{ij}q}{z_{ij}r^{-1} - z_{ij}r} D z_{ij} \Big|_{r = q}$$

The only residue which contributes to the above is the one where all the variables are 0, so we conclude that the integral above is 1. This implies (8.12). 

\qed
9. Appendix

Proof of Proposition 3.10: The pairing is clearly symmetric, because replacing the variables $z$ by their inverses switches the role of $P$ and $P'$, and:

$$\eta(x^{-1}, y^{-1}) = \eta(y, x)$$

The non-degeneracy of the pairing is a straightforward exercise which we leave to the interested reader. In Subsection 4.19, we exhibited an orthonormal basis of $A^\mathbb{Z}$, which provides a constructive argument for the non-degeneracy of the pairing.

The most important thing is to prove that the above pairing respects relations (2.4):

$$(P_1 * P_2, P') = (P_1 \otimes P_2, \Delta(P'))$$  \hspace{1cm} (9.1)

for all $P_1 \in A_k^+$, $P_2 \in A_k^-$ and $P' \in A_k$, where $k = k^1 + k^2$. This property essentially follows from the fact that:

$$\frac{zq_{x^{ij}}w - w}{z - wq_{x^{ij}}} = \frac{\omega(z, w)}{\omega(w, z)} \quad \text{whenever } \text{wt } z = i, \text{ wt } w = j$$

Plugging in the formula for $\Delta(P')$ from (3.9) tells us that the RHS of (9.1) equals:

$$\left( P_1(z_{ij}) \otimes P_2(w_{ij'}'), \frac{\prod_{ij'} \varphi_{ij'}(w_{ij'}') \cdot P'(z_{ij} \otimes w_{ij'}')} {\prod_{ij'} \omega(w_{ij'}', z_{ij})} \right) =$$

$$= \left( \prod_{ij} \varphi_{ij}(z_{ij}) \otimes P_1(z_{ij}) \otimes P_2(w_{ij'}'), \frac{\prod_{ij'} \varphi_{ij'}(w_{ij'}') \cdot P'(z_{ij} \otimes w_{ij'}')} {\prod_{ij'} \omega(w_{ij'}', z_{ij})} \right) =$$

$$= \frac{k_k!}{k^1! k^2!} \int_{|z_{ij}| = |w_{ij'}'|} \prod_{ij} \left( \varphi_{ij}(z_{ij}), \varphi_{ij'}(w_{ij'}') \right) \cdot P_1(z_{ij}) P_2(w_{ij'}') \cdot \prod_{ij'} \omega(w_{ij'}', z_{ij})^P$$

$$= \frac{k_k!}{k^1! k^2!} \int_{|z_{ij}| = |w_{ij'}'|} \prod_{ij} \left( \frac{zq_{x^{ij}}w - w}{z - wq_{x^{ij}}} \right) \cdot P_1(z_{ij}) P_2(w_{ij'}') \cdot \prod_{ij'} \omega(w_{ij'}', z_{ij})^P$$
\[ k! \int_{\text{z} \neq \text{w} \neq \text{w}'} \left[ P_1(z_{ij}) P_2(w_{ij'}') \prod^{ij}_{i'j'} \omega(z_{ij}, w_{ij'}') \right] \prod_{1 \leq i' \leq n} Dz_{ij} \prod_{1 \leq j' \leq k^2} Dw_{ij'} \bigg|_{r=q} \]

Note that the denominator of the above expression vanishes whenever the variables \( z \) and \( w \) have different weights, so it becomes equal to:

\[ \frac{k!}{k^2!} \int_{|z_{ij}|=|z_{ij}|=1} \left[ P_1(z_{ij}) P_2(w_{ij'}) \prod^{ij}_{i'j'} \omega(z_{ij}, w_{ij'}) \right] \prod_{1 \leq i' \leq n} Dz_{ij} \prod_{1 \leq j' \leq k^2} Dw_{ij'} \bigg|_{r=q} \]

Since \( z \) is away from \( w \) in the above integral, we can replace it by:

\[ = \frac{k!}{k^2!} \int_{|z_{ij}|=|z_{ij}|=1} \left[ P_1(z_{ij}) P_2(w_{ij'}) \prod^{ij}_{i'j'} \omega(z_{ij}, w_{ij'}) \right] \prod_{1 \leq i' \leq n} Dz_{ij} \prod_{1 \leq j' \leq k^2} Dw_{ij'} \bigg|_{r=q} \]

As we move \( z \) toward \( w \), the only poles picked up by the above expression are \( w_{ij'}'r^{-1} = z_{ij} \). However, the residue at this pole becomes zero as we set \( r = q \), hence we may assume \( z = w = 1 \) in the integral. Once we have done this, we can symmetrize all the variables concerned without worrying about changing the integral:

\[ = \frac{k!}{k^2!} \int_{|z_{ij}|=|w_{ij'}|=1} (P_1 * P_2)(z_{ij}, w_{ij'}) P'(z_{ij}, w_{ij'}) \]

\[ \prod_{1 \neq j'} \eta(z_{ij}, w_{ij'}) \eta(z_{ij}, w_{ij'}') \eta(z_{ij}, w_{ij'}) \eta(z_{ij}, w_{ij'}') \prod_{1 \leq j' \leq k^2} Dz_{ij} \prod_{1 \leq j' \leq k^2} Dw_{ij'} \bigg|_{r=q} \]

The above is precisely the LHS of (9.1).

**Proof of Proposition 4.12:** First let us show that \( X_m \) is indeed a shuffle element, i.e. that it belongs to \( A^+ \). We can write \( X_m \) in the form (3.5), with \( p \) equal to the symmetrization of:

\[ m(z_1, \ldots, z_j) \prod_{b < c} (z_b q^{-1} - z_c q) \prod_{b < c} (z_b - z_c) \prod_{b < c} (z_b - z_a) \prod_{b < c} \frac{1 - \frac{z_b}{z_{i+1}}}{z_j} \prod_{b < c} \left( 1 - \frac{z_b}{z_{i+1}} \right) \prod_{b < c} (z_b - z_a) \]

The above is indeed a Laurent polynomial, since the factors \( z_{k+1} - qz_k \) in the denominator are canceled by the same factors in the numerator, and the simple pole at \( z_a = z_b \) (for \( a \equiv b \)) disappears after we take the symmetrization (since the order of the pole in a symmetric rational function must be even). Therefore, all we need
to do is to show that the above Laurent polynomial vanishes at the specializations (3.6). We will show that by setting any three variables of weights $i, i \pm 1, i$ equal to $zq^{-1}, z, zq$. Suppose these variables are indexed by $a, b, c$, respectively. Depending on the order of these indices, we fall into one or more of the following situations:

- If $c < a$, then the first product in the numerator vanishes.
- If $a < b - 1$ (respectively $b < c - 1$) and $\pm = +$ (respectively $\pm = -$), then the second product in the numerator vanishes.
- If $b < c$ (respectively $a < b$) and $\pm = +$ (respectively $\pm = -$), then the third product in the numerator vanishes.

Regardless of the relative order of $a \neq b \neq c$, we will be in at least one of the above cases, so the Laurent polynomial vanishes at the wheel condition (3.6). Note that if we added any more factors to the denominator, the second bullet might fail to be true and our proof would break down. Therefore, the set of denominators in our elements $X_m$ is maximal such that they still satisfy the wheel conditions that are defining of shuffle elements.

We will now prove that $X_m \in \text{Im } \Upsilon$, and to do so it is enough to assume $m$ homogenous. We will prove the statement by induction on $j - i$, where the case $j = i$ is trivial. Note that for any $k \in \{i, \ldots, j - 1\}$ and any homogenous Laurent polynomials $m_1(z_1, \ldots, z_k)$ and $m_2(z_{k+1}, \ldots, z_j)$,

\[ X_{m_2} \ast X_{m_1} = \text{Sym} \left[ \frac{m_1(z_i, \ldots, z_k)m_2(z_{k+1}, \ldots, z_j)}{(1 - z_{i+1}q) \ldots (1 - z_jq)} \prod_{i \leq a < b \leq j} \omega(z_b, z_a) \right] \]

By the induction hypothesis, the above shuffle element lies in $\text{Im } \Upsilon$ for all $m_1$ and $m_2$. Therefore, so does $X_m$ for all Laurent polynomials $m$ in the homogenous ideal:

\[ (z_j - qz_{j-1}, \ldots, z_{i+1} - qz_i) \in \mathbb{K}[z_i^{\pm 1}, \ldots, z_j^{\pm 1}] \]

Clearly, this graded ideal consists of all homogenous Laurent polynomials such that $m(q^i, \ldots, q^j) = 0$. So in order to prove that $X_m \in \text{Im } \Upsilon$ for all homogenous Laurent polynomials $m$, it is enough to do so for a single homogenous polynomial $m$ of any given degree such that $m(q^i, \ldots, q^j) \neq 0$. To this end, consider the shuffle element:

\[ z_i^{c_i} \ast \ldots \ast z_j^{c_j} = \text{Sym} \left[ z_i^{c_i} \ldots z_j^{c_j} \prod_{i \leq a < b \leq j} \omega(z_a, z_b) \right] \]

It lies in $\text{Im } \Upsilon$ for all $c_i, \ldots, c_j \in \mathbb{Z}$. A suitable linear combination of these elements allows us to cancel the denominators, and we conclude that:

\[ \text{Sym} \left[ z_i^{c_i} \ldots z_j^{c_j} \prod_{i \leq a < b \leq j} (z_a - z_b) \prod_{i \leq a < b \leq j} (z_a - qz_b) \right] \in \text{Im } \Upsilon \]
for all $c_1, \ldots, c_j \in \mathbb{Z}$. Another linear combination of these elements allows us to add some more factors, and we conclude that:

$$\text{Im } \mathcal{Y} \ni \text{Sym} \left[ z_i^{c_i} \ldots z_j^{c_j} \prod_{i \leq a < b \leq j} (z_a - z_b) \prod_{i \leq a < b \leq j} (z_a - qz_b) \prod_{i < a+1 < b \leq j} (z_b - qz_a) \right] =$$

$$= \text{Sym} \left[ z_i^{c_i} \ldots z_j^{c_j} \prod_{a \leq b} (z_a q - z_b q^{-1}) \omega(z_a, z_b) \prod_{a \leq b} (z_a - qz_b) \prod_{b \leq a+1} (z_b - qz_a) \right] =$$

$$= \text{Sym} \left[ z_i^{c_i} \ldots z_j^{c_j} \prod_{i \leq a < b \leq j} (z_a q - z_b q^{-1}) \prod_{i \leq a < b \leq j} (z_a - qz_b) \prod_{i < a+1 < b \leq j} (z_b - qz_a) \right] \frac{(q - \frac{z_{i+1}}{z_i}) \ldots (q - \frac{z_{j+1}}{z_j})}{(1 - \frac{z_{i+1}}{z_i}) \ldots (1 - \frac{z_{j+1}}{z_j})} \prod_{i \leq a < b \leq j} \omega(z_b, z_a)$$

For any choice of $c_i, \ldots, c_j \in \mathbb{Z}$, the numerator of the fraction does not vanish when we set $z_k = q^k$ for all $k \in \{i, \ldots, j\}$. This concludes our proof.

\[\square\]

**Proof of Proposition 4.18:** As far as the general elements $X_m$ of (4.17) are concerned, note that Proposition 3.10 implies the following formula:

$$(X_m, Q) = (q^2 - 1)^{-i+1} \int_{|u_a| = 1} \frac{Q(u_i, \ldots, u_j) m(u^{-1}_i, \ldots, u^{-1}_j) \prod_{a \leq b} \omega(u^{-1}_a, u^{-1}_b) \prod_{a \equiv b} \eta(u_a, u_b) }{1 - \frac{q u_{i+1}}{u_i} \ldots \frac{q u_{j-1}}{u_{j-1}}} \bigg|_{r=q}$$

for any $Q \in \mathcal{A}^+$, where we still assume $|q| > 1$. Therefore, we have:

$$(P^d_{[i;j]}, Q) = (q^2 - 1)^{-i+1} \int_{|u_a| = 1} \frac{Q(u_i, \ldots, u_j) \prod_{a \in S_{i,j,d}} u^{-1}_a \prod_{a \leq b} \omega(u^{-1}_a, u^{-1}_b) \prod_{a \equiv b} \eta(u_a, u_b) }{1 - \frac{q u_{i+1}}{u_i} \ldots \frac{q u_{j-1}}{u_{j-1}}} \bigg|_{r=q}$$

As we move the contours past each other so as to insure $u_a \gg u_{a+1}$, the only poles that hinder us are $u_a = q u_{a+1}$. As we do so, we will pick residues whenever:

$$\{u_i, \ldots, u_j\} = \{y_1 q^{-i}, \ldots, y_1 q^{-j}, \ldots, y_i q^{-i}, \ldots, y_j q^{-j}\}$$

over partitions:

$$(i, \ldots, j) = (i_1, \ldots, j_1, i_2, \ldots, j_2, \ldots, i_t, \ldots, j_t) \quad (9.2)$$

and then letting $y_1 \gg \ldots \gg y_t$. If we assume $Q \in \mathcal{B}^+_\mu$, then the integrand has degree:

$$\leq d_s - \#(S_{i, j, d} \cap [i_t; j_t]) + \delta_i \quad (9.3)$$

in the variable $y_s$, where $d_s = \mu(j_s - i_s + 1)$. By (4.20), the above quantity is $\leq 0$ for all $s$, so the integrand produces a non-zero residue if and only:

- the RHS of (9.3) equals 0 for all $s$, and

- we have equality in (9.3) for all $s$. 


The first bullet happens if only if $d_s$ is an integer for all $s$, and implies the following formula for all $Q \in B^+_\mu$:

$$
(P_{[i;j]}^d, Q) = (q^2 - 1)^{j-i+1} \sum_{t \geq 1} \sum_{[i;j]=\delta_{[i]}} (\sum_{d_s = \mu(j_s - i_s + 1) \in \mathbb{Z}} q^{\sum_{s=1}^t d_s + [\mathbb{Z}]} (i_s + \frac{\mathbb{Z}}{\mathbb{Z}})).
$$

leading term $y_{1} \cdots y_{n} Q(y_{1} q^{-i_{1}}, \ldots, y_{n} q^{-j_{n}})$

By the definition of the map $\phi$ in Subsection 4.9, the above equals:

$$
(P_{[i;j]}^d, Q) = (q^2 - 1)^{j-i+1} \sum_{t \geq 1} \sum_{[i;j]=\delta_{[i]}} (\sum_{d_s = \mu(j_s - i_s + 1) \in \mathbb{Z}} q^{\sum_{s=1}^t d_s + [\mathbb{Z}]} (i_s + \frac{\mathbb{Z}}{\mathbb{Z}})).
$$

\cdot \phi^{\otimes t} \left( \text{component of } \Delta^{(t-1)}(Q) \text{ in } A^\mu_{\{i;\bar{j}\}} \otimes \ldots \otimes A^\mu_{\{i_1;\bar{j}_1\}} \right) \prod_{1 \leq s \neq s' \leq t} q^{\langle i_s; j_s; \{i_s'; j_s'\} \rangle} \quad (9.4)

We can use this formula to prove the non-degeneracy of $(\cdot, \cdot)$ on $B^+_\mu$. Suppose there exists $Q \in B^+_\mu$ in the kernel of the pairing. By taking its pairing with various products of $P_{[i;j]}^d$, Proposition 4.18 inductively shows that all the terms in the RHS of (9.4) vanish. Hence, we have:

$$
\phi^{\otimes t} \left( \text{component of } \Delta^{(t-1)}(Q) \text{ in } A^\mu_{\{i;\bar{j}\}} \otimes \ldots \otimes A^\mu_{\{i_1;\bar{j}_1\}} \right) = 0
$$

for all viable partitions of $[i; j]$, where $\phi$ is the scaled evaluation map of (4.16). Then Proposition 4.10 implies that $Q = 0$.

Let us use (9.4) to compute the various pairings when $Q = P_{[i';\bar{j}']}$ given by:

$$
\Delta^{(t-1)}(P_{[i';\bar{j}']}^d) = \sum_{[i;j] \in \mathbb{Z}} \phi_{[i;j]} \otimes \ldots \otimes \phi_{[i_1;j_1]}
$$

only has non-zero terms when the interval $[i_{s+1}; j_{s+1}]$ comes directly before the interval $[i_s; j_s]$. So (9.4) produces non-zero terms only when:

$$
(i', \ldots, j') = (i_t, \ldots, j_t, i_{t-1}, \ldots, j_{t-1}, \ldots, i_1, \ldots, j_1) \quad (9.5)
$$
on top of (9.2). We call a partition that satisfies both (9.2) and (9.5) an acceptable composition. Then we have established that:

$$
(P_{[i;j]}^d, P_{[i';\bar{j}']}^d) = (q^2 - 1)^{j-i+1} \sum_{t \geq 1} \sum_{[i;j]=\delta_{[i]}} (\sum_{d_s = \mu(j_s - i_s + 1) \in \mathbb{Z}} q^{\sum_{s=1}^t d_s + [\mathbb{Z}]} (i_s + \frac{\mathbb{Z}}{\mathbb{Z}})).
$$

Applying Proposition 4.16, the above gives us:

$$
(P_{[i;j]}^d, P_{[i';\bar{j}']}^d) = \sum_{t \geq 1} (q^2 - 1)^t \sum_{\text{acceptable composition}} \prod_{1 \leq s \neq s' \leq t} q^{\langle i_s; j_s; \{i_s'; j_s'\} \rangle} \quad (9.6)
$$
Now we must figure out which are the acceptable compositions in each of the cases (4.23), (4.24) and (4.25). In the first case, an acceptable composition is of the form:

\[(i, \ldots, i + nk - 1) = (i, \ldots, i + nk_1 - 1, i + nk_1, \ldots, i + nk_1 + nk_2 - 1, \ldots)\]

where \(k = k_1 + k_2 + \ldots + k_t\) is a composition. To ensure that each \(d_s = \frac{dk_s}{k}\) is an integer, we must have \(k_s = \frac{gk}{gcd(k, d)}\) for a composition \(gcd(k, d) = g_1 + g_2 + \ldots + g_t\). Therefore, (9.6) gives us:

\[
\left( P_{d_{i+znk-1}}^d, P_{d_{i+znk-1}}^d \right) = \sum_{t \geq 1} (q^2 - 1)^t \# \{gcd(k, d) = g_1 + \ldots + g_t\} = \sum_{t \geq 1} (q^2 - 1)^t \left( \frac{gcd(k, d) - 1}{t - 1} \right) = (q^2 - 1)^t q^{gcd(k, d) - 2}
\]

which proves (4.23). As for (4.24), an acceptable composition is of the form:

\[(i, \ldots, i + nk - 1) = (i, \ldots, i' + nl_1 - 1, i' + nl_1, \ldots, i + nk_1 - 1, \ldots)\]

where \(t = 2\tau\) is even, \(k = k_1 + k_2 + \ldots + k_t\) is a composition, and \(l_s \in \{0, \ldots, k_s - 1\}\) is a natural number. To ensure that each \(d_s\) is an integer, we must have \(k_s = \frac{gk}{gcd(k, d)}\) for a composition \(gcd(k, d) = g_1 + g_2 + \ldots + g_t\). Moreover, congruence forces the numbers \(l_s\) to only take \(g_s\) values, hence (9.6) gives us:

\[
\left( P_{d_{i+znk-1}}^d, P_{d_{i+znk-1}}^d \right) = \sum_{\tau \geq 1} (q - q^{-1})^{2\tau} \sum_{gcd(k, d) = g_1 + \ldots + g_{\tau}} g_1 \ldots g_{\tau}
\]

We leave it as an elementary exercise to the interested reader (e.g. via generating functions) to show that the above number equals the RHS of (4.24). Finally, in the case of (4.25), an acceptable composition is one such that:

\[(i, \ldots, j) = (i, \ldots, j + nl_1, j + nl_1 + 1, \ldots, i + nk' - 1, i + nk', \ldots, j)\]

where \(t = 2\tau + 1\) is odd,

\[k' \leq k := \left\lfloor \frac{j - i}{n} \right\rfloor \quad \text{and} \quad k' = k_1 + k_2 + \ldots + k_\tau\]

is a composition, and \(l_s \in \{0, \ldots, k_s - 1\}\) is a natural number. Note that the part before the last interval \((i + nk', \ldots, j)\) is precisely the same as in the previous case. To ensure that each \(d_s\) is an integer, we must have \(k_s = \frac{g}{gcd(j - i + 1, dn)}\) \(\in \mathbb{Z}\) for a composition:

\[g' := k' \frac{gcd(j - i + 1, dn)}{j - i + 1} = g_1 + g_2 + \ldots + g_t\]

Moreover, congruence forces \(j - i + 1\) to divide \(nl_s\), so this leaves \(g_s\) ways to choose \(l_s\). We conclude that (9.6) gives us:

\[
\left( P_{d_{i+znk-1}}^d, P_{d_{i+znk-1}}^d \right) = (q^2 - 1) + (q^2 - 1) \sum_{g' = 0}^{\left\lfloor \frac{gcd(j - i + 1, dn)}{d} \right\rfloor} \sum_{\tau \geq 1} (q - q^{-1})^{2\tau} \sum_{g' = g_1 + \ldots + g_{\tau}} g_1 \ldots g_{\tau}
\]

By the computation we did in the case (4.24), the above equals:

\[
= (q^2 - 1) + (q^2 - 1)^2 \sum_{g' = 0}^{g} \frac{q^{2g'} - q^{-2g'}}{q^2 + 1} = (q^2 - 1) \frac{q^{2g+1} + q^{-2g-1}}{q + q^{-1}}
\]

This completes the proof of (4.25).
References

[1] Braverman A., Finkelberg M. Finite difference quantum Toda lattice via equivariant
K−theory, Transform. Groups 10 (2005), no. 3-4, 363-386
[2] Carlsson E., Okounkov A. Euts and Vertex Operators, Duke Math. J. 161 (2012), no. 9,
1797-1815
[3] Ding J., Frenkel I. Isomorphism of two realizations of quantum affine algebra \( U_q(\hat{g}_n) \),
Comm. Math. Phys. 156 (1993), no. 2, 277-300
[4] Enriquez B., On correlation functions of Drinfeld currents and shuﬄe algebras, Transform.
Groups 5 (2000), no. 2, 111 - 120
[5] Feigin B., Finkelberg M., Negut A., Rybnikov L. Yangians and cohomology rings of Lau-
mon spaces Selecta Math. (N.S.) 17 (2011), no. 3, 573-607
[6] Feigin B., Hashizume K., Hoshnio A., Shiraishi J., Yanagida S. A Commutative Algebra
on Degenerate \( C^1 \) and MacDonald Polynomials, J. Math. Phys. 50 (2009), no. 9
[7] Feigin B., Jimbo M., Miwa T., Mukhin E. Representations of quantum toroidal \( gl_n \),
preprint arXiv:1204.5378
[8] Feigin B., Odesskii A. Quantized moduli spaces of the bundles on the elliptic curve and
their applications, Integrable structures of exactly solvable two-dimensional models of
quantum ﬁeld theory (Kiev, 2000), 123-137, NATO Sci. Ser. II Math. Phys. Chem., 35,
Kluwer Acad. Publ., Dordrecht, 2001
[9] Hernandez D. Quantum toroidal and representations, Selecta Math. (N.S.) 14 (2009), no.
3-4, 701-725
[10] Jantzen, Lectures on Quantum Groups, Graduate Studies in Mathematics, 6. American
Mathematical Society, Providence, RI, 1996. viii+266 pp. ISBN: 0-8218-0478-2
[11] Khoroshkin S., Tolstoy V. The universal \( R \)-matrix for quantum untwisted aﬃne Lie
algebras, Functional Analysis and Its Applications, January-March, 1992, Vol 26, Issue 1,
pp 69-71
[12] Finkelberg M., Gaitsgory D., Kuznetsov A. Uhlenbeck spaces for \( k^2 \) and aﬃne Lie algebra
\( \hat{g}_n \), Publ. Res. Inst. Math. Sci. Vol 39 (2003), no. 4, pp 721-766.
[13] Negut A. Aﬃne Laumon Spaces and the Calogero-Moser Integrable System, in preparation
[14] Negut A. The shuﬄe algebra revisited, Int. Math. Res. Not. (2013), doi: 10.1093/imrn/rnt156
[15] Tsymbaliuk A. Quantum aﬃne Gelfand-Tsetlin bases and quantum toroidal algebra via
K−theory of aﬃne Laumon spaces, Selecta Math. (N.S.) 16 (2010), no. 2, 173-200

Columbia University, Department of Mathematics, New York, USA

Simion Stoilow Institute of Mathematics, Bucharest, Romania

E-mail address: andrei.negut@gmail.com