Fredholm Determinants and the mKdV/Sinh–Gordon Hierarchies

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Abstract: For a particular class of integral operators $K$ we show that the quantity

$$\phi := \log \det (I + K) - \log \det (I - K)$$

satisfies both the integrated mKdV hierarchy and the Sinh–Gordon hierarchy. This proves a conjecture of Zamolodchikov.

I. Introduction

In recent years it has become apparent that there is a fundamental connection between certain Fredholm determinants and total systems of differential equations. This connection first appeared in work on the scaling limit of the 2-point correlation function in the 2D Ising model [7, 15] and the subsequent generalization to $n$-point correlations and holonomic quantum fields [12]. In applications the Fredholm determinants are either correlation functions or closely related to correlation functions in various statistical mechanical or quantum field-theoretic models. In the simplest of cases the differential equations are one of the Painlevé equations. Some, but by no means a complete set of, references to these further developments are [2–5, 13, 14, 16] The review paper [6] can be consulted for more examples of this connection.

In recent work by the present authors on random matrices, techniques were developed that gave simple proofs of the connection between a large class of Fredholm determinants and differential equations [13, 14]. In this paper we show how the philosophy of [3, 5, 13, 14] can be applied to study Fredholm determinants which are associated with operators $K$ having kernel of the form

$$K(x, y) = \frac{E(x) E(y)}{x + y},$$

where

$$E(x) = e(x) \exp \left( \sum \frac{1}{2} t_k x^k \right).$$
The (finite) sum is taken over odd positive and negative integers $k$. The domain of integration for the operator is $(0, \infty)$, and the function $e(x)$ can be very general. All that is required is that the operator be trace class for a range of values of the $t_k$ so the Fredholm determinants are defined. The quantity of interest is

$$\phi := \log \det (I + K) - \log \det (I - K).$$

We shall show that $\phi$ satisfies the equations of the integrated mKdV hierarchy if $t_1$ is the space variable and $t_3, t_5, \ldots$ the time variables, and that it satisfies the Sinh–Gordon hierarchy when $t_{-1}, t_{-3}, \ldots$ are the time variables.

To state the results precisely, the first assertion is that for $n \geq 1$,

$$\frac{\partial \phi}{\partial t_{2n+1}} = \left(D^2 - 4 \frac{\partial \phi}{\partial t_1} D^{-1} \frac{\partial \phi}{\partial t_1} D\right)^n \frac{\partial \phi}{\partial t_1},$$

where $D$ denotes $\partial/\partial t_1$ and $D^{-1}$ denotes the antiderivative which vanishes at $t_1 = -\infty$. (Observe that $\phi$ and all its derivatives vanish at $t_1 = -\infty$.) This is the integrated mKdV hierarchy of equations,

$$\frac{\partial \phi}{\partial t_3} = \frac{\partial^3 \phi}{\partial t_1^3} - 2 \left( \frac{\partial \phi}{\partial t_1} \right)^3,$$

$$\frac{\partial \phi}{\partial t_5} = \frac{\partial^5 \phi}{\partial t_1^5} - 10 \left( \frac{\partial^2 \phi}{\partial t_1^2} \right)^2 \frac{\partial \phi}{\partial t_1} - 10 \left( \frac{\partial \phi}{\partial t_1} \right)^2 \frac{\partial^3 \phi}{\partial t_1^3} + 6 \left( \frac{\partial \phi}{\partial t_1} \right)^5,$$

etc. (In general there are constant factors on the left sides which can be removed by changes of scale in the time variables; e.g. [1].)

To go in the other direction we introduce the inverse of the operator appearing in (2), which is given by

$$\left(D^2 - 4 \frac{\partial \phi}{\partial t_1} D^{-1} \frac{\partial \phi}{\partial t_1} D\right)^{-1} = \frac{1}{2} \left( D^{-1} e^{2\phi} D^{-1} e^{-2\phi} + D^{-1} e^{-2\phi} D^{-1} e^{2\phi} \right).$$

(Precisely, this is the inverse in a suitable space of functions. See Lemma 4 below.)

We shall show that for $n \geq 1$ we have the Sinh–Gordon hierarchy of equations

$$\frac{\partial \phi}{\partial t_{-2n+1}} = 2^{-n} \left( D^{-1} e^{2\phi} D^{-1} e^{-2\phi} + D^{-1} e^{-2\phi} D^{-1} e^{2\phi} \right) \frac{\partial \phi}{\partial t_1}.$$

The case $n = 1$ of this is equivalent to the Sinh–Gordon equation

$$\frac{\partial^2 \phi}{\partial t_{-1} \partial t_1} = \frac{1}{2} \sinh 2\phi.$$

Observe that (2) and (4) can be combined into the single statement that either of them holds for all values of the integer $n$. Further observe that these results hold independently of the function $e(x)$ appearing in the kernel $K(x, y)$. The function $e(x)$ affects the boundary conditions for (2) and (4) at $t_k = -\infty$.

That $\phi$ satisfies the integrated mKdV hierarchy was conjectured in [16], and that it satisfies the Sinh–Gordon equation (5) was conjectured in [16] and proved in [2].
A related identity,
\[- \frac{\partial^2}{\partial t_{\text{-}1} \partial t_1} \log \det (I - K) = \frac{e^{2\phi} - 1}{4}, \tag{6}\]
was also conjectured in [16] and proved in [2], and will be rederived here.

We prove our results by expressing all relevant quantities in terms of inner products
\[u_{i,j} := ((I - K^2)^{-1} E_i, E_j), \quad v_{i,j} := ((I - K^2)^{-1} K E_i, E_j), \tag{7}\]
where \(E_i(x) := x^i E(x)\), and showing that these quantities satisfy nice differentiation and recursion formulas. Observe that both \(u_{i,j}\) and \(v_{i,j}\) are symmetric in the indices, since the operator \(K\) is symmetric. That these inner products are basic is expected from earlier investigations; e.g. [3, 5, 13, 14].

II. Recursion and Differentiation Formulas

If we denote by \(M\) multiplication by the independent variable, then the form of the kernel of \(K\) shows that
\[MK + KM = E \otimes E, \tag{8}\]
where in general we denote by \(X \otimes Y\) the operator with kernel \(X(x)Y(y)\). Applying this twice we see that, with brackets denoting the commutator as usual,
\[[M, K^2] = E \otimes KE - KE \otimes E.\]
It follows immediately that if \(Q_i := (I - K^2)^{-1} E_i\) and \(P_i := (I - K^2)^{-1} K E_i\), then
\[[M, (I - K^2)^{-1}] = Q_0 \otimes P_0 - P_0 \otimes Q_0.\]
Applying these operators to the function \(E_j\) gives the recursion formula
\[x Q_j(x) - Q_{j+1}(x) = Q_0(x) v_j - P_0(x) u_j, \tag{9}\]
where we write \(u_j\) for \(u_{j,0}\) and \(v_j\) for \(v_{j,0}\). Taking inner products with \(E_i\) gives
\[u_{i+1,j} - u_{i,j+1} = u_i v_j - v_i u_j. \tag{10}\]

To obtain the analogous relations for the \(v_{i,j}\) we temporarily define
\[w_i := ((I - K^2)^{-1} K E_i, KE_i),\]
and take inner products with \(KE_i\) in (9), obtaining
\[(MKE_i, Q_j) - v_{i,j+1} = v_i v_j - w_i u_j.\]
The identity \((I - K^2)^{-1} K^2 = (I - K^2)^{-1} - I\) gives
\[w_i = u_i - (E, E_i),\]
and by (8)
\[(MKE_i, Q_j) = -(KE_{i+1}, Q_j) + (E, E_i) (E, Q_j) = -v_{i+1,j} + (E, E_i) u_j.\]
Thus we obtain

\[ v_{i+1,j} + v_{i,j+1} = u_i u_j - v_i v_j . \]  

(11)

For the differentiation formulas we use the fact

\[ \frac{\partial}{\partial t_k} E(x) E(y) = \frac{1}{2} (x^k + y^k) E(x) E(y) \]

and elementary algebra to deduce that for \( k > 0, \)

\[ \frac{\partial K}{\partial t_k} = \frac{1}{2} \sum_{i+j=k-1} (-1)^i E_i \otimes E_j, \quad \frac{\partial K}{\partial t_{-k}} = \frac{1}{2} \sum_{i+j=-k-1} (-1)^{i+1} E_i \otimes E_j. \]  

(12)

In the first sum we take \( i, j \geq 0 \) and in the second \( i, j \leq -1. \) This will be our convention throughout. (Here we use the fact that \( k \) is odd; the reader will find other such places later.) Since, with \( t = t_k \) or \( t_{-k}, \)

\[ \frac{\partial \phi}{\partial t} = \text{tr} (I + K)^{-1} \frac{\partial K}{\partial t} + \text{tr} (I - K)^{-1} \frac{\partial K}{\partial t} = 2 \text{tr} (I - K^2)^{-1} \frac{\partial K}{\partial t}, \]

we find that

\[ \frac{\partial \phi}{\partial t_k} = \sum_{i+j=k-1} (-1)^i u_{i,j}, \quad \frac{\partial \phi}{\partial t_{-k}} = \sum_{i+j=-k-1} (-1)^{i+1} u_{i,j}. \]  

(13)

Notice especially the important fact

\[ \frac{\partial \phi}{\partial t_1} = u_0 . \]  

(14)

To obtain differentiation formulas for the \( u_{i,j} \) and \( v_{i,j} \) themselves we use

\[ \frac{\partial}{\partial t_k} (I - K^2)^{-1} = (I - K^2)^{-1} \frac{\partial K^2}{\partial t_k} (I - K^2)^{-1} \]

and, by (12),

\[ \frac{\partial K^2}{\partial t_k} = K \frac{\partial K}{\partial t_k} + \frac{\partial K}{\partial t_k} K = \frac{1}{2} \sum_{i+j=k-1} (-1)^i (K E_i \otimes E_j + E_i \otimes K E_j) \]

to deduce

\[ \frac{\partial}{\partial t_k} (I - K^2)^{-1} = \frac{1}{2} \sum_{i+j=k-1} (-1)^i (P_i \otimes Q_j + Q_i \otimes P_j). \]

From this and the fact \( \partial E_i / \partial t_k = \frac{1}{2} E_{i+k} \) we deduce from the definition (7) that

\[ \frac{\partial u_{p,q}}{\partial t_k} = \frac{1}{2} \sum_{i+j=k-1} (-1)^i (u_{p,j} v_{q,i} + v_{p,j} u_{q,i}) + \frac{1}{2} (u_{p+k,q} + u_{p,q+k}). \]  

(15)

If we introduce \( R_i := (I - K^2)^{-1} K^2 E_i = Q_i - E_i, \) then we find similarly first

\[ \frac{\partial}{\partial t_k} (I - K^2)^{-1} K = \frac{1}{2} \sum_{i+j=k-1} (-1)^i (Q_i \otimes R_j + P_i \otimes P_j) + \frac{1}{2} \sum_{i+j=k-1} (-1)^j Q_i \otimes E_j \]

\[ = \frac{1}{2} \sum_{i+j=k-1} (-1)^i (Q_i \otimes Q_j + P_i \otimes P_j), \]
and then
\[
\frac{\partial v_{p,q}}{\partial t_k} = \frac{1}{2} \sum_{i+j=k-1} (-1)^i (u_{p,j} u_{q,i} + v_{p,j} v_{q,i}) + \frac{1}{2} (v_{p+k,q} + v_{p,q+k}). \tag{16}
\]

In a completely analogous fashion, using the second part of (12), we obtain formulas for differentiation with respect to the \(t_{-k}\):
\[
\frac{\partial u_{p,q}}{\partial t_{-k}} = \frac{1}{2} \sum_{i+j=-(k-1)} (-1)^{i+1} (u_{p,j} v_{q,i} + v_{p,j} u_{q,i}) + \frac{1}{2} (u_{p-k,q} + u_{p,q-k}), \tag{17}
\]
\[
\frac{\partial v_{p,q}}{\partial t_{-k}} = \frac{1}{2} \sum_{i+j=-(k-1)} (-1)^{i+1} (u_{p,j} u_{q,i} + v_{p,j} v_{q,i}) + \frac{1}{2} (v_{p-k,q} + v_{p,q-k}). \tag{18}
\]

### III. The mKdV Hierarchy

We begin by showing how to derive the first of the integrated mKdV equations,
\[
\frac{\partial \phi}{\partial t_3} = \frac{\partial^3 \phi}{\partial t_3^3} - 2 \left( \frac{\partial \phi}{\partial t_1} \right)^3.
\]
This will illustrate the procedure. By (14) \(\partial \phi/\partial t_1 = u_0\), and we differentiate twice more with respect to \(t_1\), using (15) and (16). We find that the quantities \(u_0, u_1, u_{1,1}, v_0, v_1\) arise. But the recursion formulas (10) and (11) allow us to express two of these in terms of the others:
\[
v_1 = (u_0^2 - v_0^2)/2, \quad u_{1,1} = u_2 + u_0 v_1 - u_1 v_0 = u_2 + \frac{1}{2} u_0 (u_0^2 - v_0^2) - u_1 v_0.
\]
In the end the formula becomes
\[
\frac{\partial^3 \phi}{\partial t_3^3} = \frac{3}{2} u_0^3 + \frac{1}{2} u_0 v_0^2 + u_1 v_0 + u_2.
\]
Now from (13), \(\partial \phi/\partial t_5 = 2 u_2 - u_{1,1}\) and by the above representation of \(u_{1,1}\) this may be written
\[
\frac{\partial \phi}{\partial t_3} = -\frac{1}{2} u_0^3 + \frac{1}{2} u_0 v_0^2 + u_1 v_0 + u_2.
\]
This gives
\[
\frac{\partial^3 \phi}{\partial t_3^3} - \frac{\partial \phi}{\partial t_3} = 2 u_0^3 = 2 \left( \frac{\partial \phi}{\partial t_1} \right)^3,
\]
which is the desired equation.

The proof of the general formula (2) follows from a series of three lemmas.

**Lemma 1.** We have
\[
2 u_0 \frac{\partial u_0}{\partial t_k} = \frac{\partial}{\partial t_1} \sum_{i+j=k-1} (-1)^i u_i u_j. \tag{19}
\]
Proof. We begin by noting that from (15)
\[ \frac{\partial u_0}{\partial t_k} = \sum_{i+j=k-1} (-1)^j u_i v_j + u_k \]
and, from (15), (16), (10) and (11),
\[ \frac{\partial u_p}{\partial t_1} = u_0 v_p + u_{p+1}, \quad \frac{\partial v_p}{\partial t_1} = u_0 u_p. \] (20)

We find that the right side of (19) equals
\[ \sum_{i+j=k-1} (-1)^j [u_i (u_0 v_j + u_{j+1}) + u_j (u_0 v_i + u_{i+1})] \]
\[ = 2u_0 \sum_{i+j=k-1} (-1)^j u_i v_j + 2 \sum_{i+j=k-1} (-1)^j u_i u_{j+1}. \]

The last sum equals
\[ u_0 u_k - u_1 u_{k-1} + u_2 u_{k-2} - \cdots - u_{k-2} u_2 + u_{k-1} u_1 = u_0 u_k. \]

It follows that the right side of (19) equals the left side of (19).

Lemma 2. We have
\[ 2v_k = \sum_{i+j=k-1} (-1)^j (u_i u_j - v_i v_j). \] (21)

Proof. By the recursion formulas (11),
\[ v_{k,0} + v_{k-1,1} = u_0 u_{k-1} - v_0 v_{k-1} \]
\[-(v_{k-1,1} + v_{k-2,2}) = -(u_1 u_{k-2} - v_1 v_{k-2}) \]
\[ \vdots \]
\[-(v_{2,k-2} + v_{1,k-1}) = -(u_{k-2} u_1 - v_{k-2} v_1) \]
\[ v_{1,k-1} + v_{0,k} = u_{k-1} u_0 - v_{k-1} v_0. \]

Adding gives (21).

Lemma 3. We have for \( k \geq 1 \),
\[ \frac{\partial \phi}{\partial t_{k+2}} = D \frac{\partial u_0}{\partial t_k} - 4 u_0 D^{-1} \left( u_0 \frac{\partial u_0}{\partial t_k} \right). \] (22)

Proof. By Lemma 1 and the differentiation formula (15) the right side of (22) equals
\[ \frac{\partial}{\partial t_1} \left( \sum_{i+j=k-1} (-1)^j u_i v_j + u_k \right) - 2u_0 \sum_{i+j=k-1} (-1)^j u_i u_j, \]
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and by (20) this equals
\[ \sum_{i+j=k-1} (-1)^i (u_i u_j + u_i v_j + u_{i+1} v_j) + u_0 v_k + u_{k+1} - 2 u_0 \sum_{i+j=k-1} (-1)^i u_i u_j \]

This is the right side of (22). By (13) the left side equals
\[ u_{k+1} - (u_{1, k} - u_{2, k-1}) - (u_{3, k-2} - u_{4, k-3}) - \cdots - (u_{k, 1} - u_{k+1, 0}) , \]

and by (10) this equals
\[ u_{k+1} + (u_1 v_{k-1} - u_{k-1} v_1) + (u_3 v_{k-3} - u_{k-3} v_3) + \cdots + (u_k v_0 - u_0 v_k) \]

Thus the difference between the right and left sides of (22) equals
\[ u_0 \sum_{i+j=k-1} (-1)^i (v_i v_j - u_i u_j) + 2 u_0 v_k , \]

and by Lemma 2 this equals 0.

The proof of (2) is now immediate. In fact (22) may be rewritten
\[ (23) \]
and this together with (14) gives (2).

IV. The Sinh–Gordon Hierarchy

We begin by deriving (3).

Lemma 4. The operator \( D^2 - 4 u_0 D^{-1} u_0 D \) is invertible in the space of smooth functions all of whose derivatives are rapidly decreasing as \( t_1 \to -\infty \), and its inverse is given by (3).

Remark. The function \( \phi \) and all the \( u_{i,j} \) and \( v_{i,j} \) belong to the space of functions in the statement of the lemma.

Proof. We have
\[ D^2 - 4 u_0 D^{-1} u_0 D = (I - 4 u_0 D^{-1} u_0 D^{-1}) D^2 . \]

Both factors on the right are invertible (the Neumann series inverts the first factor) so the operator on the left is also, and its inverse is equal to
\[ D^{-2} (I - 4 u_0 D^{-1} u_0 D^{-1})^{-1} = \frac{1}{2} D^{-2}[(I - 2 u_0 D^{-1})^{-1} + (I + 2 u_0 D^{-1})^{-1}] \]
\[ = \frac{1}{2} D^{-1}[(D - 2 u_0)^{-1} + (D + 2 u_0)^{-1}] . \]

Since \( (D + p)^{-1} = e^{-D^{-1}p} D^{-1} e^{D^{-1}p} \) and \( D^{-1} u_0 = \phi \), the above is equal to
\[ \frac{1}{2} (D^{-1} e^{2\phi} D^{-1} e^{-2\phi} + D^{-1} e^{-2\phi} D^{-1} e^{2\phi}) . \]
Lemma 5. Relation (23) holds for $k \leq -1$.

The proof of this is almost exactly the same as for $k \geq 1$ and so is omitted. Lemma 5 is equivalent to the statement that for $k = 1, 3, 5, \ldots$, 

$$\frac{\partial \phi}{\partial t_{-k+2}} = (D^2 - 4 u_0 D^{-1} u_0 D) \frac{\partial \phi}{\partial t_{-k}},$$

or by (3),

$$\frac{\partial \phi}{\partial t_{-k}} = \frac{1}{2} (D^{-1} e^{2\phi} D^{-1} e^{-2\phi} + D^{-1} e^{-2\phi} D^{-1} e^{2\phi}) \frac{\partial \phi}{\partial t_{-k+2}}.$$

This establishes (4) by induction.

The case $n = 1$ of (4) is

$$\frac{\partial \phi}{\partial t_{-1}} = \frac{1}{2} (D^{-1} e^{2\phi} D^{-1} e^{-2\phi} + D^{-1} e^{-2\phi} D^{-1} e^{2\phi}) \frac{\partial \phi}{\partial t_1},$$

which gives (keep in mind that $D^{-1}$ is the antiderivative which vanishes at $-\infty$)

$$4 \frac{\partial^2 \phi}{\partial t_{-1} \partial t_1} = 4 D \frac{\partial \phi}{\partial t_{-1}} = 2 (e^{2\phi} D^{-1} e^{-2\phi} + e^{-2\phi} D^{-1} e^{2\phi}) \frac{\partial \phi}{\partial t_1}$$

$$= e^{2\phi} (1 - e^{-2\phi}) + e^{-2\phi} (e^{2\phi} - 1) = 2 \sinh 2\phi.$$

This is (5).

Finally we derive (6). By (17) we have

$$\frac{\partial^2 \phi}{\partial t_{-1} \partial t_1} = \frac{\partial u_0}{\partial t_{-1}} = u_{-1} (1 + v_{-1}),$$

and so we know that

$$u_{-1} (1 + v_{-1}) = \frac{1}{2} \sinh 2\phi.$$

Now we use a special case of (11), $2 v_{-1} = u_{-1}^2 - v_{-1}^2$, which has the more useful form

$$(1 + v_{-1})^2 = 1 + u_{-1}^2.$$

These equations can be solved for $u_{-1}$ and $v_{-1}$, giving

$$u_{-1} = \sinh \phi, \quad v_{-1} = \cosh \phi - 1. \quad (24)$$

Now we use the fact $(I - K)^{-1} = (I - K^2)^{-1} + (I - K^2)^{-1} K$ and (12) to obtain

$$-2 \frac{\partial}{\partial t_1} \log \det (I - K) = ((I - K)^{-1} E, E) = u_0 + v_0.$$

Therefore by (17) and (18),

$$-2 \frac{\partial^2}{\partial t_{-1} \partial t_1} \log \det (I - K) = u_{-1} (v_{-1} + 1 + u_{-1}).$$

Using (24) we find that the right side equals $(e^{2\phi} - 1)/2$, which gives (6).
Note added in proof. After this work was completed, the authors became aware of the work [8–11]) which also considers integral equations, similar to the ones considered here, which yield solutions of a broad class of nonlinear evolution equations. In these papers one finds methods for deriving differentiation formulas for quantities similar to our $u_{i,j}$ and $v_{i,j}$.

Using the Miura transformation,

$$u_0 \to u_0^2 + \frac{\partial u_0}{\partial t_1},$$

we can show that

$$2 \frac{\partial^2}{\partial t_1^2} \log \det(I - K) = \frac{\partial}{\partial t_1}(u_0 + v_0)$$

satisfies the KdV hierarchy.

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