Optimal $C^{1+\frac{1}{2}}$-regularity of $H$-surfaces with a free boundary

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Abstract

We consider continuous stationary surfaces of prescribed mean curvature in $\mathbb{R}^3$ – shortly called $H$-surfaces – with part of their boundary varying on a smooth support manifold $S$ with non-empty boundary. We allow that the $H$-surface meets the support manifold non-perpendicularly and presume the $H$-surface to be continuous up to the boundary. Then we show: If $S$ belongs to $C^2$ resp. $C^{2,\mu}$, then the $H$-surface belongs to $C^{1,\alpha}$ for any $\alpha \in (0,\frac{1}{2})$ resp. $C^{1,1}$ up to the boundary. The latter conclusion is optimal by an example due to S. Hildebrandt and J.C.C. Nitsche. Our result extends a known theorem for the special case of minimal surfaces.

In addition, we present asymptotic expansions at boundary branch points.

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Let $S$ be a differentiable, two-dimensional manifold in $\mathbb{R}^3$ with boundary $\partial S$. Writing

$$B^+ := \{w = (u,v) = u + iv : |w| < 1, \ v > 0\}, \ I := (-1,1) \subset \partial B^+$$

for the upper unit half-disc in $\mathbb{R}^2 \simeq \mathbb{C}$ and the straight part of its boundary, we consider surfaces of prescribed mean curvature or shortly $H$-surfaces on $B^+$, i.e. solutions of the problem

$$\begin{align*}
\mathbf{x} &\in C^2(B^+, \mathbb{R}^3) \cap C^0(\overline{B^+}, \mathbb{R}^3) \cap H^1_2(B^+, \mathbb{R}^3), \\
\Delta \mathbf{x} &= 2\mathcal{H}(\mathbf{x})\mathbf{x}_u \wedge \mathbf{x}_v \quad \text{in} \ B^+, \\
|\mathbf{x}_u| &= |\mathbf{x}_v|, \quad \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0 \quad \text{in} \ B^+, \\
\end{align*}$$

which satisfy the free boundary condition

$$\mathbf{x}(I) \subset S \cup \partial S. \quad (2)$$

Here $H^1_2(B^+, \mathbb{R}^3)$ denotes the Sobolev-space of measurable mappings $\mathbf{x} : B^+ \to \mathbb{R}^3$, which are quadratically integrable together with their first derivatives. In addition, $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ stands for the Laplace operator in $\mathbb{R}^2$ and $y \wedge z, \langle y, z \rangle$ denote cross product and scalar product in $\mathbb{R}^3$, respectively; the latter notation
will be used for vectors in $\mathbb{R}^3$, too. Finally, $\mathcal{H} \in C^0(\mathbb{R}^3, \mathbb{R})$ is a prescribed function. In $\mathbb{R}^3$, the system in the second line is called Rellich’s system and the third line contains the conformality relations.

As is well-known, the restriction $x|_R$ of a solution of (1) to the set $\mathcal{R} := \{w \in B^+ : \nabla x(w) := (x_u(w), x_v(w)) \neq 0\}$ of regular points parametrizes a surface with mean curvature $H = \mathcal{H} \circ x$. We emphasize that singular points with $\nabla x(w) = 0$, so-called branch points, are specifically allowed. This is natural from the viewpoint of the calculus of variations: If $Q \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ is a vector field with $\text{div} Q = 2\mathcal{H}$, then solutions of (1) appear as stationary points of the functional

$$E_Q(y) := \int_{B^+} \left\{ \frac{1}{2} |\nabla y|^2 + \langle Q(y), y_u \wedge y_v \rangle \right\} \, du \, dv,$$

where so-called inner and outer variations $y$ of $x$ are allowed. Roughly speaking, inner variation means a perturbation in the parameters $(u,v)$ and outer variations are perturbations in the space that retain the boundary condition (2); see [DHT] Section 1.4 for the exact definitions in the minimal surface case $Q \equiv 0$. For our purposes, it suffices to give the exact definition of outer variations:

**Definition 1.** Let $x \in C^0(B^+, \mathbb{R}^3) \cap H^1_2(B^+, \mathbb{R}^3)$ fulfill the boundary condition (3). A perturbation $x^{(\varepsilon)}(w) := x(w) + \varepsilon \phi(w, \varepsilon), \ 0 \leq \varepsilon \ll 1$, is called outer variation of $x$, if $\phi(\cdot, \varepsilon)$ belongs to $A_x := \left\{ y \in H^1_2(B^+, \mathbb{R}^3) : y = x \text{ on } \partial B^+ \setminus I \right\}$ for any $\varepsilon$, if the family of Dirichlet’s integrals

$$D(\phi(\cdot, \varepsilon)) := \int_{B^+} \left( |\phi_u(w, \varepsilon)|^2 + |\phi_v(w, \varepsilon)|^2 \right) \, du \, dv, \quad 0 \leq \varepsilon \ll 1,$$

is uniformly bounded in $\varepsilon$, and if $\phi(\cdot, \varepsilon) \to \phi(\cdot, 0) \in H^1_2(B^+, \mathbb{R}^3)$ ($\varepsilon \to 0^+$) holds true a.e. on $B^+$. The function $\phi_0 := \phi(\cdot, 0)$ is to be termed direction of the variation.

**Definition 2.** Let $Q \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ be given, define $E_Q$ by formula (3) and set $\mathcal{H} := \frac{1}{2} \text{div} Q$. A solution $x : B^+ \to \mathbb{R}^3$ of (1)–(2) is called stationary free $\mathcal{H}$-surface (w.r.t. $E_Q$), if we have

$$\delta E_Q(x, \phi_0) := \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left[ E_Q(x^{(\varepsilon)}) - E_Q(x) \right] \geq 0$$

for any outer variation $x^{(\varepsilon)} = x + \varepsilon \phi(\cdot, \varepsilon), 0 \leq \varepsilon \ll 1$. The quantity $\delta E_Q(x, \phi_0)$ is called the first variation of $E_Q$ at $x$ in the direction $\phi_0$.

Now we are able to formulate our main result:
Theorem 1. Let $S \subset \mathbb{R}^3$ be a differentiable two-manifold and assume a vector-field $Q \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ to be given such that

$$|\langle Q, n \rangle| < 1 \quad \text{on } S \cup \partial S$$

is satisfied; here $n : S \cup \partial S \to \mathbb{R}^3$ denotes a unit normal field on $S$ which we locally extend continuously to $\partial S$. In addition, let $x \in C^2(B^+, \mathbb{R}^3) \cap C^0(B^+, \mathbb{R}^3) \cap H^1_2(B^+, \mathbb{R}^3)$ be a stationary free $H$-surface with $H := \frac{1}{2} \text{div} Q$.

(i) If $S \subset C^2$, then we have $x \in C^{1,\alpha}(B^+ \cup I, \mathbb{R}^3)$ for any $\alpha \in (0, \frac{1}{2})$.

(ii) If $S \subset C^{2,\beta}$ and $Q \in C^{1,\beta}(\mathbb{R}^3, \mathbb{R}^3)$ for some $\beta \in (0, 1)$, then we have $x \in C^{1,\beta}(B^+ \cup I, \mathbb{R}^3)$.

Remark 1. For minimal surfaces, i.e., the special case $Q \equiv 0$, the result of Theorem 1 is due to R. Ye [17]. Under higher regularity assumptions on $S$ - namely $S \subset C^3$ in case (i), $S \subset C^4$ in case (ii) - these results for minimal surfaces were already proved by S. Hildebrandt and J.C.C. Nitsche [19], [20]. In [21] the authors present an example showing the optimality of the regularity claimed in Theorem 1(ii).

Remark 2. In the minimal surface case, the assumption $x \in C^0(B^+, \mathbb{R}^3)$ in Theorem 1 becomes redundant provided $S$ satisfies an additional uniformity condition. This is the famous continuity result for stationary minimal surfaces up to the free boundary, which is due to M. Grütter, S. Hildebrandt, J.C.C. Nitsche [22]; see also C. Dziuk [23] regarding an analogue result for support surfaces without boundary. Concerning $H$-surfaces, it is an open question whether stationarity implies continuity up to the boundary. However, there is an affirmative answer in the special case of vector-fields $Q$ satisfying

$$\langle Q, n \rangle = 0 \quad \text{on } S \cup \partial S;$$

see [24] for support surfaces without boundary, in [25] the case of support surfaces with boundary is treated. In addition, minimality – instead of the weaker assumption of stationarity – implies continuity up to the boundary under very mild assumptions on $S$ and a smallness condition for $Q$: see [26] Section 2.5 or [27] Section 1.3.

Remark 3. In the general case $(Q, n) \neq 0$ on $S \cup \partial S$ the only results for stationary $H$-surfaces known to the author are addressed to the case of support surfaces with empty boundary $\partial S = \emptyset$, see [28], [29], [30], [31].

Our second theorem is concerned with boundary branch points:

Theorem 2. Let the assumptions of Theorem 1(i) be satisfied and let $w_0 \in I$ be a branch point of the stationary free $H$-surface $x$. If $x : B^+ \to \mathbb{R}^3$ is non-constant, then there exist an integer $m \geq 1$ and a vector $a \in \mathbb{C}^3 \setminus \{0\}$ with $\langle a, a \rangle = 0$, such that we have the representation

$$x_\omega(w) = a(w - w_0)^m + o(|w - w_0|^m) \quad \text{as } w \to w_0.$$

Remark 4. The proof of Theorem 2 can be found at the end of the paper; for branch points $w_0 \in I$ with $x(w_0) \in S$ the asymptotic expansion (5) has been already proved in [32] Theorem 1.13. The usual direct consequences as finiteness of boundary branch points in $B^+ \cap B_r(0)$ for any $r \in (0, 1)$ and continuity of the surface normal of $x$ up to the branch points follow; see e.g. [32] Remarks 5.1 and 5.2.
Preparing for the proof of Theorem 1, we first have to localize the setting: Obviously, it suffices to show that for any \( w_0 \in I \) there exists some \( \delta > 0 \) with \( x \in C^{1,\mu}(B^+_\delta(w_0), \mathbb{R}^3) \) for \( \mu \in (0, \frac{1}{2}) \) or \( \mu = \frac{1}{2} \), respectively. Here we abbreviated
\[
B^+_\delta(w_0) := \{ w = u + iv \in \mathbb{C} : |w - w_0| < \delta \},
\]
\[
B^+_\delta(0) := \{ w = u + iv \in B_\delta(0) : v > 0 \}.
\]
Since this result is included in Theorem 1.3 of [M4] for \( w_0 \in I \) with \( x_0 := x(w_0) \in S \), we may assume \( x_0 \in \partial S \). We localize around \( x_0 \) which is possible according to the assumption \( x \in C^0(\overline{B}^+, \mathbb{R}^3) \). After a suitable rotation and translation we can presume \( x_0 = 0 \) as well as the existence of some neighbourhood \( U = U(x_0) \subset \mathbb{R}^3 \) and functions \( \gamma \in C^2([-r, r]) \), \( \psi \in C^2(\overline{B}_r(0)) \), \( r > 0 \), with
\[
\gamma(0) = \frac{d}{ds} \gamma(0) = 0, \quad \psi(0) = \nabla \psi(0) = 0,
\]
such that we have the local representations
\[
\begin{align*}
S \cap U &= \{ p = (p^1, p^2, p^3) \in \Omega \times \mathbb{R} : p^3 = \psi(p^1, p^2) \}, \\
\partial S \cap U &= \{ p = (p^1, p^2, p^3) \in \Gamma \times \mathbb{R} : p^3 = \psi(p^1, p^2) \},
\end{align*}
\]
where we abbreviated
\[
\begin{align*}
\Omega &:= \{ (p^1, p^2) \in B_\varepsilon(0) : p^2 > \gamma(p^1) \}, \\
\Gamma &:= \{ (p^1, p^2) \in B_\varepsilon(0) : p^2 = \gamma(p^1) \}.
\end{align*}
\]
Now choose \( \delta > 0 \) with \( |x(w)| < r \) for all \( w \in B^+_\delta(w_0) \). Since the system (11) is conformally invariant, we may reparametrize \( x_{\mid B^+_\delta(w_0)} \) over \( B^+ \) without renaming and obtain
\[
x(B^+) \subset B_r := \{ p \in \mathbb{R}^3 : |p| < r \}, \quad x(0) = 0.
\]
In the following, we will repeatedly scale \( r > 0 \) down — sometimes without further command — always assuming (11) to be satisfied.

Next we define
\[
q = q(p) := Q^3(p) - \psi_{p^3}(p^1, p^2)Q^1(p) - \psi_{p^3}(p^1, p^2)Q^2(p),
\]
where \( Q^1, Q^2, Q^3 \) are the components of \( Q \). Note that the smallness condition (11) and the normalization (10) imply \( q \in C^1(\overline{B}_r) \) as well as
\[
|q(p)| \leq q_0 < 1 \quad \text{for all } p \in \overline{B}_r
\]
with sufficiently small \( r > 0 \); here \( q_0 \in (0, 1) \) denotes some suitable constant.

Writing \( \tilde{\gamma} := \frac{d}{dx} \gamma \), we set
\[
\begin{align*}
z^1 &:= -i \psi_{p^3} x^1_w - i \psi_{p^3} x^2_w + i x^3_w, \\
z^2 &:= (1 - iq\tilde{\gamma}) x^1_w + (\tilde{\gamma} + i q) x^2_w + (\psi_{p^1} + \psi_{p^2} \tilde{\gamma}) x^3_w \quad \text{on } B^+.
\end{align*}
\]
Here we abbreviated \( \psi_{p^j} = \psi_{p^j}(x^1, x^2) \), \( \gamma = \gamma(x^1) \), and \( q = q(x) \), and we used one of the Wirtinger derivatives \( x^j_w = \frac{\partial x^j}{\partial w} \) defined by the operators
\[
\begin{align*}
\frac{\partial}{\partial w} &:= \frac{1}{2} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right), \\
\frac{\partial}{\partial w} &:= \frac{1}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).
\end{align*}
\]
As a first important observation we infer the following
Proposition 1. The mapping $z := (z^1, z^2) : B^+ \to \mathbb{R}^3$ belongs to $C^1(B^+, C^2) \cap L_2(B^+, C^2)$ and satisfies the weak boundary condition

$$\lim_{\varepsilon \to 0} \int_{I_\varepsilon} \langle \lambda(w), \text{Im} z(w) \rangle \, du = 0 \quad \text{for all } \lambda \in C^1_0(B^+ \cup I, \mathbb{R}^2),$$

where we set $I_\varepsilon := \{ w = u + iv \in B^+ : v = \varepsilon \}$ for $\varepsilon > 0$.

Proof. The claimed regularity of $z$ is obvious by definition. In order to prove \[13\], we set $\eta(s) := \psi(s, \gamma(s))$ and $t(s) := (1, \tilde{\gamma}(s), \tilde{\eta}(s))$, $s \in (-r, r)$. Then $t(s)$ is tangential to $\partial S$ at the point $(s, \gamma(s), \eta(s))$. If we choose $\lambda \in C^1_0(B^+ \cup I)$ arbitrarily, the stationarity of $x$ yields

$$\lim_{\varepsilon \to 0^+} \int_{I_\varepsilon} \alpha(t(x^1), x_v + Q(x) \wedge x_u) \, du = 0;$$

this can be proved by combining the flow argument in \[DHT\] pp. 32–33 with \[M1\] Lemma 3. Now we set $\zeta := (t(x^1), x_v + Q(x) \wedge x_u)$ and claim

$$2 \text{Im } z^2 = -\zeta + (Q^2 - \gamma Q^1)(x_u^3 - \psi_{u^1} x_u^1 - \psi_{u^2} x_u^2) \quad \text{on } B^+,$$

where we again abbreviated $Q^j = Q^j(x)$, etc. Indeed, we compute

$$\begin{align*}
\zeta &= x^1 + Q^2 x_u^3 - Q^3 x_u^2 - \tilde{\gamma}(x_u^2 + Q^3 x_u^1 - Q^1 x_u^3) + \tilde{\eta}(x_u^3 + Q^1 x_u^2 - Q^2 x_u^1) \\
&= x^1 + \tilde{\gamma} x_u^2 - (Q^3 - \psi_{u^1} Q^1 - \psi_{u^2} Q^2)(x_u^1 - \tilde{\gamma} x_u^2) + (\psi_{u^1} + \psi_{u^2}) x_u^3 \\
&+ (Q^2 - \gamma Q^1)(x_u^3 - \psi_{u^1} x_u^1 - \psi_{u^2} x_u^2) \quad \text{on } B^+,
\end{align*}$$

having $\tilde{\eta} = \psi_{u^1} + \psi_{u^2} \gamma$ in mind. Hence, the definition \[12\] of $z^2$ yields \[16\]. Next we note the inequality

$$\int_{I_\varepsilon} [x^3 - \psi(x^1, x^2)]^2 \, du \leq c_0 \int_{B^+} |\nabla x|^2 \, du \, dv \leq c_0, \quad \delta \in (0, 1),$$

with some constant $c > 0$. This is an easy consequence of the boundary condition $x^3 = \psi(x^1, x^2)$ on $I$ and the boundedness of $|\nabla \psi|$.

Now let $\lambda = (\lambda_1, \lambda_2) \in C^1_0(B^+ \cup I, \mathbb{R}^2)$ be chosen arbitrarily. Then we find

$$\begin{align*}
\lim_{\varepsilon \to 0} &\int_{I_\varepsilon} \langle \lambda(w), \text{Im} z(w) \rangle \, du = \lim_{\varepsilon \to 0} \int_{I_\varepsilon} (\lambda_1 \text{Im } z^1 + \lambda_2 \text{Im } z^2) \, du \\
&= \lim_{\varepsilon \to 0} \frac{1}{4} \int_{I_\varepsilon} [\lambda_1 + \lambda_2 (Q^2 - \gamma Q^1)] [x_u^3 - \psi_{u^1} x_u^1 - \psi_{u^2} x_u^2] \, du^2 \\
&= \lim_{\varepsilon \to 0} \frac{1}{4} \int_{I_\varepsilon} (x^3 - \psi(x^1, x^2)) \frac{\partial}{\partial u} [\lambda_1 + \lambda_2 (Q^2 - \gamma Q^1)] \, du^2
\end{align*}$$
and are hence able to estimate
\[
\liminf_{\varepsilon \to 0} \left| \int_{I_\varepsilon} \langle \lambda(w), \text{Im} z(w) \rangle \, du \right|
\]
\[
\leq \liminf_{\varepsilon \to 0} \frac{1}{4} \int_{I_\varepsilon} [x^3 - \psi(x^1, x^2)]^2 \, du \cdot \int_{I_\varepsilon} \left\{ \frac{\partial}{\partial u} [\lambda_1 + \lambda_2 (Q^2 - \dot{\gamma} Q^1)] \right\}^2 \, du
\]
\[
\leq \liminf_{\varepsilon \to 0} c_\varepsilon \left( 1 + \int_{I_\varepsilon} |\nabla x|^2 \, du \right).
\]

with an adjusted constant \( c > 0 \). Using \( x \in H^1_h(B^+, \mathbb{R}^3) \), one can easily prove that the right hand side of this inequality vanishes (see e.g. [M4] Proposition 2.1).

In order to be able to relate the auxiliary function \( z \) with \( x \) we also need the following result:

**Proposition 2.** The mapping \( z = (z^1, z^2) \) defined in (12) fulfills the relations
\[
c^{-1} |\nabla x| \leq |z| \leq c |\nabla x| \quad \text{on } B^+
\]  
with some constant \( c > 0 \).

**Proof.** The right-hand inequality in (17) is obvious by definition. In order to prove the left-hand inequality we write (12) as
\[
z = A(x) \cdot \begin{pmatrix} x^1_w \\ x^3_w \end{pmatrix} + b(x) x^2_w \quad \text{on } B^+
\]  
with
\[
A := \begin{pmatrix} -i \dot{\psi}_p & i \\ 1 - i \dot{\gamma} \psi_{p^1} + \psi_{p^2} \dot{\gamma} \end{pmatrix}, \quad b := \begin{pmatrix} -i \dot{\psi}_{p^2} \psi_{p^1} + (1 + \dot{\gamma}) \dot{\gamma} \\ \dot{\gamma} + i \dot{\gamma} \end{pmatrix}.
\]

Pick \( 0 < \varepsilon < 1 - q_0 \) arbitrarily. According to the normalization (6) we may choose \( r = r(\varepsilon) > 0 \) sufficiently small to ensure
\[
|\det A(p)| \geq 1 - \varepsilon > 0 \quad \text{for } p \in \overline{B}_r.
\]  
In particular, the inverse \( A^{-1}(p) \) exists on \( \overline{B}_r \), and we conclude
\[
\begin{pmatrix} x^1_w \\ x^3_w \end{pmatrix} = A^{-1}(x) \cdot z - A^{-1}(x) \cdot b(x) x^2_w \quad \text{on } B^+.
\]  
Computing
\[
A^{-1} \cdot b = \frac{1}{\det A} \left( q - i[\psi_{p^1} \psi_{p^2} + (1 + \dot{\gamma}) \dot{\gamma}] \right)
\]
the smallness (14) of \( q \), inequality (20), and the normalization (6) imply
\[
|A^{-1}(p) \cdot b(p)| \leq q_0 + \varepsilon \quad \text{for } p \in \overline{B}_r.
\]
with sufficiently small \( r = r(\varepsilon) > 0 \). Finally, we write the conformality relations in \( (1) \) as \( \langle x_w, x_w \rangle = 0 \) in \( B^+ \), which yields
\[
|x_w|^2 \leq |x_1|^2 + |x_3|^2 \quad \text{on} \quad B^+.
\]
With these estimates we conclude
\[
\sqrt{|x_1^2| + |x_3^2|} \leq c|x| + (q_0 + \varepsilon)\sqrt{|x_1^2| + |x_3^2|} \quad \text{on} \quad B^+
\]
from \( (21) \), where \( c > 0 \) denotes a constant. Choosing e.g. \( \varepsilon = \frac{1-q_0}{2} \), we hence obtain the claimed estimate \( (13) \) with an aligned \( c > 0 \).

Combining Propositions \( (1) \) and \( (2) \), we arrive at the following

Lemma 1. Let \( z = (z_1, z_2) \) be defined by \( (12) \). Set \( B := B_1(0) \), \( B^- := B \setminus (B^+ \cup I) \), and consider the reflected function
\[
\hat{z}(w) := \begin{cases} 
\frac{z(w)}{\overline{z(w)}}, & w \in B^- \\
|z(w)|^{-2} \hat{z} w, & w \in B \setminus I, \quad \text{with} \quad |z(w)| \neq 0 \\
0, & \text{otherwise}
\end{cases} \quad \in L_\infty(B, \mathbb{C}^2).
\]
Then there exists \( h \in L_\infty(B, \mathbb{C}^2) \) such that \( \hat{z} \) solves the equation
\[
\int_B \left( (\hat{z}, \varphi) + |\hat{z}|^2 (h, \varphi) \right) dudv = 0 \quad \text{for all} \quad \varphi \in C^0(B, \mathbb{C}^2) \cap H^1(B, \mathbb{C}^2). \quad (23)
\]

Proof. The assertion follows from the estimate
\[
|\hat{z}| \leq c|z|^2 \quad \text{on} \quad B \setminus I, \quad (24)
\]
which we will prove below. Indeed, defining
\[
h(w) := \begin{cases} 
|z(w)|^{-2} \hat{z} w, & \text{for} \quad w \in B \setminus I \quad \text{with} \quad |z(w)| \neq 0 \\
0, & \text{otherwise}
\end{cases} \quad \in L_\infty(B, \mathbb{C}^2),
\]
we infer \( \hat{z} w = |z(w)|^2 h(w) \) away from isolated points in \( B \setminus I \), because points \( w \in B^+ \) with \( |z(w)| = 0 \) are exactly the isolated branch points of \( x \).
If we multiply this relation with an arbitrary \( \varphi \in C_c(B, \mathbb{C}^2) \), integrate over \( B^+_q := \{ w \in B^+ : -q < \varphi < 0 \} \), and apply Gauss’ integral theorem as well as the boundary condition, Proposition \( (1) \) we arrive at \( (24) \) for such \( \varphi \). By a standard approximation argument we can also allow \( \varphi \in C_c^0(B, \mathbb{C}^2) \cap H^1(B, \mathbb{C}^2) \) in \( (24) \).

In showing \( (24) \), the proof will be completed. To this end, we reflect \( x \) trivially across \( I \),
\[
\hat{x}(w) := \begin{cases} 
x(w), & w \in B^+ \cup I \\
x(\overline{w}), & w \in B^-
\end{cases} \quad (25)
\]
Defining \( A, b \in C^1(B) \) by \( (19) \) and having \( (18) \) in mind, we now may write \( \hat{z} \) as
\[
\hat{z} = A(\hat{x}) \cdot \left( \frac{\hat{x}_1}{\hat{x}_3} \right) + b(\hat{x}) \hat{x}_w^2 \quad \text{on} \quad B^+ \quad (26)
\]
and as
\[
\hat{z} = A(\hat{x}) \cdot \left( \frac{\hat{x}_1}{\hat{x}_3} \right) + b(\hat{x}) \hat{x}_w^2 \quad \text{on} \quad B^-. \quad (27)
\]
On the other hand, Rellich’s system in (11) can be written as
\[ \hat{x}_w = \pm i \mathcal{H}(\hat{x}) \hat{x}_w \wedge \hat{x}_w \quad \text{on } B^\pm. \] (28)

Differentiating (26), (27) and applying (28), we obtain
\[ |\hat{z}| \leq c |\nabla \hat{x}|^2 \quad \text{on } B \setminus I \]
with some constant \( c > 0 \). Hence, Proposition 2 yields the asserted relation (24).

Now the crucial step in the proof of Theorem 1 is the following

**Lemma 2.** For any \( \mu \in (0, 1) \), the mapping \( \hat{z} \) defined in Lemma 1 can be extended to a mapping of class \( C^\mu(B, \mathbb{C}^2) \) with the property \( \text{Im} \hat{z} = 0 \) on \( I \).

**Proof.** We attempt to recover the steps in Section 3 of [M4], which were used there to prove an analogue result, namely Lemma 3.4.

1. At first, we prove \( \hat{x} \in C^{\beta}(B, \mathbb{R}^3) \) for some \( \beta \in (0, 1) \). To this end, we consider the function
\[ \chi := \begin{cases} \hat{x}^3 - \psi(\hat{x}_1, \hat{x}_2) & \text{on } B^+ \cup I, \\ -\hat{x}^3 + \psi(\hat{x}_1, \hat{x}_2) & \text{on } B^-. \end{cases} \] (29)

Note that \( \chi \in C^0(B) \cap H_1^1(B) \) is satisfied according to the boundary condition (2). Choose any disc \( B_\rho (w_0) \subset B \) and define \( y = (y^1, y^2) \in C^\infty(B_\rho (w_0), \mathbb{R}^2) \cap C^0(\overline{B_\rho (w_0)}, \mathbb{R}^2) \) as harmonic vector with boundary values
\[ y^1 = \hat{x}_1, \quad y^2 = \chi \quad \text{on } \partial B_\rho (w_0). \]

Setting
\[ \varphi := \begin{cases} -i(\chi - y^2) & \text{on } B_\rho (w_0), \\ \hat{x}_1 - y^1 & \text{on } B \setminus B_\rho (w_0), \end{cases} \]

we obtain an admissible test function \( \varphi \in C^0(B) \cap H_1^1(B) \) for (29). We now insert \( \varphi \) and the relations (26), (27) for \( \hat{z} \) into (23) and use the special form (19) of \( A \) and \( b \). Writing \( \xi := (\hat{x}_1, \hat{x}_3) \), we then find
\[
(1 - d(r)) \int_{B_\rho (w_0)} |\xi_w|^2 \, du \, dv \\
\leq (q_0 + d(r)) \int_{B_\rho (w_0)} |\xi_w| |\hat{x}_w^2| \, du \, dv \\
+ c \int_{B_\rho (w_0)} |y_w| |\hat{x}_w| \, du \, dv + \int_{B_\rho (w_0)} |\hat{z}| |\mathbf{h}| |\varphi| \, du \, dv,
\]
where \( c > 0 \) is a constant and \( d(r) \), \( 0 < r \ll 1 \), denotes some (possibly varying) positive function satisfying \( d(r) \to 0 \) \( (r \to 0+) \). By our general assumption (3), the maximum principle, and the normalization \( \psi(0, 0) = 0 \).
we further get $|\phi| \leq d(r)$. Using the conformality relations as well as Proposition 2, we hence conclude

$$(1 - q_0 - d(r)) \int_{B_{\varepsilon}(w_0)} |\xi_w|^2 \, du \, dv \leq c \int_{B_{\varepsilon}(w_0)} |\chi_w| \, du \, dv.$$  

Applying the inequality of Cauchy-Schwarz and assuming $d(r) \leq \frac{1}{2}(1 - q_0)$, we finally arrive at

$$\int_{B_{\varepsilon}(w_0)} |\nabla \hat{x}|^2 \, du \, dv \leq c \int_{B_{\varepsilon}(w_0)} |\nabla \hat{y}|^2 \, du \, dv \quad \text{for all } B_{\varepsilon}(w_0) \subset B.$$  

(30)

Note that there is a constant $c > 0$ with

$$c^{-1} |\nabla \hat{x}| \leq |\nabla (\hat{x}^1, \chi)| \leq c |\nabla \hat{x}|$$

on $B$, due to the conformality relations and the condition $\nabla \psi(0, 0) = 0$. Employing C. B. Morrey's Dirichlet growth theorem, we hence infer $\hat{x} \in C^3(B, \mathbb{R}^3)$ for some $\beta \in (0, 1)$ from (30).

2. Next we show: For any $\alpha \in [0, 2\beta]$ and any compact subset $K \subset B$ we have

$$\int_B |w - w_0|^{-\alpha} |\hat{z}(w)|^2 \, du \, dv \leq c \quad \text{for all } w_0 \in K,$$ 

(31)

where $c > 0$ denotes a constant depending on $\alpha$ and $K$.

We fix some $w_0 \in K$ and define $\chi$ as in (29). We consider

$$\psi(w) := \left( -i(\chi(w) - \chi(w_0)) \right), \quad w \in B.$$  

According to part 1 of the proof we have $\chi, \hat{x}^1 \in C^3(B)$ and conclude

$$|\psi(w)| \leq c |w - w_0|^\beta, \quad w \in K.$$  

(32)

Moreover, we can estimate (remember $\xi = (\hat{x}^1, \hat{x}^3)$)

$$|\hat{z}, \psi| \geq |\xi_w|^2 - d(r)|\xi_w|^2 - (q_0 + d(r))|\xi_w||\xi_w|^2| 
\geq (1 - q_0 - d(r))|\xi_w|^2 \geq c(1 - q_0 - d(r))|\hat{z}|^2 \quad \text{in } B,$$ 

(33)

where we retained the notation of part 1 and used Proposition 2.

Now we choose some $\delta \in (0, \delta_0)$, $\delta_0 := \frac{1}{4}\text{dist}(K, \partial B)$, and set

$$\gamma(w) := \begin{cases} 
\delta^{-\alpha} - \delta_0^{-\alpha}, & 0 \leq |w - w_0| < \delta \\
|w - w_0|^{-\alpha} - \delta_0^{-\alpha}, & \delta \leq |w - w_0| < \delta_0 \\
0, & \delta_0 \leq |w - w_0| 
\end{cases}.$$  

(34)
We assume \( d(r) \leq \frac{1}{2}(1 - q_0) \) and apply the inequalities
\[
\int_B |\hat{z}|^2 \, du \, dv \geq \int_{\delta < |w-w_0| < \delta_0} |w-w_0|^{-\alpha}|\hat{z}|^2 \, du \, dv - \delta_0^{-\alpha} \int_B |\hat{z}|^2 \, du \, dv
\]
and
\[
\int_{\delta < |w-w_0| < \delta_0} |w-w_0|^{-\alpha-1+\beta} |\hat{z}| \, du \, dv \leq \frac{\varepsilon}{2} \int_{\delta < |w-w_0| < \delta_0} |w-w_0|^{-\alpha}|\hat{z}|^2 \, du \, dv + \frac{1}{2\varepsilon} \int_{\delta < |w-w_0| < \delta_0} |w-w_0|^{-\alpha-2+2\beta} \, du \, dv
\]
with sufficiently small \( \varepsilon > 0 \) to (34). Having \( \int_B |\hat{z}|^2 \, du \, dv < +\infty \) as well as \( 2\beta > \alpha \) in mind, we arrive at
\[
\int_{\delta < |w-w_0| < \delta_0} |w-w_0|^{-\alpha}|\hat{z}|^2 \, du \, dv \leq c
\]
with some constant \( c > 0 \) which is independent of \( w_0 \in K \) and \( \delta \in (0, \delta_0) \). For \( \delta \to 0^+ \) we obtain the asserted estimate (31).

3. Finally, it turns out that (31) is valid for \( \alpha = 1 \). This can be proved exactly as in [M4] Proposition 3.3 via an induction argument using the representation formula of Pompeiu and Vekua, namely
\[
\hat{z}(w) = y(w) - \frac{1}{\pi} \int_B \frac{|\hat{z}(\zeta)|^2 h(\zeta)}{\zeta - w} \, d\xi \, d\eta, \quad w \in B; \quad \zeta = \xi + i\eta, \quad (35)
\]
with some holomorphic vector \( y : B \to \mathbb{C}^2 \). Hence \( \hat{z} \) is locally bounded in \( B \). By applying E. Schmidt’s inequality (see e.g. [DHT] pp. 219–221) to a local version of (35), we conclude \( \hat{z} \in C^\mu(B, \mathbb{C}^2) \) for any \( \mu \in (0, 1) \), as asserted. The property \( \text{Im}(\hat{z}) = 0 \) on \( I \) is now an immediate consequence of Proposition 1.

As the last preliminaries towards the proof of Theorem we need two further lemmata; the first one is due to E. Heinz, S. Hildebrandt, and J.C.C. Nitsche and we present it in a special appropriate form:

**Lemma 3. (Heinz–Hildebrandt–Nitsche)**

(a) Let \( f \in C^0(B^+, \mathbb{C}) \) be given such that its square \( f^2 \) has a continuous extension to \( B^+ \cup I \). Then \( f \) can be extended to a continuous function \( f \in C^0(B^+ \cup I, \mathbb{C}) \).

(b) Let \( f \in C^0([-\varrho_0, \varrho_0], \mathbb{C}) \) be given with some \( \varrho_0 \in (0, 1) \). Suppose that \( \text{Re}(f) \cdot \text{Im}(f) = 0 \) on \( [-\varrho_0, \varrho_0] \) is satisfied and that there exist numbers \( c > 0, \alpha \in (0, 1] \) with
\[
|f^2(u_1) - f^2(u_2)| \leq c|u_1 - u_2|^{2\alpha} \quad \text{for all } u_1, u_2 \in [-\varrho_0, \varrho_0]. \quad (36)
\]
Then we have \( f \in C^\alpha([-\varrho_0, \varrho_0], \mathbb{C}) \).
Proof. We refer to the Lemmata 3 and 4 in [DHT] Section 2.7.

The second of the announced lemmata contains a regularity result for generalized analytic functions, which we may attribute to I.N. Vekua; for the sake of completeness, we give the proof of a local version needed here:

**Lemma 4. (Vekua)** Let \( z \in C^1(B^+, \mathbb{C}) \cap C^0(B^+ \cup I, \mathbb{C}) \) be a solution of

\[
zw = g \quad \text{in } B^+, \quad \text{Im } z = h \quad \text{on } [-\varrho_0, \varrho_0]
\]

for some \( \varrho_0 \in (0, 1) \). Then there hold:

(a) If \( g \in C^0(B^+ \cup I, \mathbb{C}) \) and \( h \in C^\alpha([-\varrho_0, \varrho_0]) \) for some \( \alpha \in (0, 1) \), then we have \( z \in C^\alpha(B^+ - \varrho_0(0), \mathbb{C}) \) for any \( \varrho \in (0, \varrho_0) \).

(b) If \( g \in C^\alpha(B^+ \cup I, \mathbb{C}) \) and \( h \in C^{1,\alpha}([-\varrho, \varrho]) \) for some \( \alpha \in (0, 1) \), then we have \( z \in C^{1,\alpha}(B^+ - \varrho_0(0), \mathbb{C}) \) for any \( \varrho \in (0, \varrho_0) \).

Proof. 1. We first prove assertion (a). Fix some \( \varrho \in (0, \varrho_0) \) and choose a test function \( \phi \in C^\infty_c(B) \) with \( \phi = 1 \) in \( B_{\varrho} \) and \( \phi = 0 \) in \( B \setminus B_{\varrho}(0) \) as well as a simply connected domain \( B_{\varrho}(0) \subset G \subset B_{\varrho_0}(0) \) with \( C^2 \)-boundary. Let \( \sigma : B \to G \) be a conformal mapping. Then the function \( \tilde{z} := (\phi z) \circ \sigma \in C^1(B, \mathbb{C}) \cap C^0(B, \mathbb{C}) \) solves a boundary value problem

\[
\tilde{z}w = \tilde{g} \quad \text{on } B, \quad \text{Im } \tilde{z} = \tilde{h} \quad \text{on } \partial B,
\]

where \( \tilde{g} \in C^0(\overline{B}, \mathbb{C}) \), \( \tilde{h} \in C^\alpha(\partial B) \) is satisfied; here one has to use (37) as well as the well-known Kellogg-Warschawski theorem on the boundary behaviour of conformal mappings, see e.g. [P]. By subtracting a holomorphic function in \( B \) with boundary values \( \tilde{h} \) we may assume \( \tilde{h} \equiv 0 \); note that this holomorphic function belongs to \( C^\alpha(\overline{B}, \mathbb{C}) \) by a well-known result of I.I. Privalov. Now, any solution of (38) with \( \tilde{h} \equiv 0 \) has the form

\[
\tilde{z}(w) = -\frac{1}{\pi} \int_B \frac{\tilde{g}(\zeta)}{\zeta - w} d\xi d\eta - \frac{1}{\pi} \int_B \frac{g(\zeta)}{1 - w\zeta} d\xi d\eta + z_0, \quad w \in \overline{B},
\]

with some constant \( z_0 \in \mathbb{R} \); see Theorem 2 in [S] Chap. IX, § 4. Defining the Vekua-Operator

\[
T[\tilde{g}](w) := -\frac{1}{\pi} \int_B \frac{\tilde{g}(\zeta)}{\zeta - w} d\xi d\eta, \quad w \in \mathbb{C},
\]

we may rewrite (39) as

\[
\tilde{z}(w) = T[\tilde{g}](w) + T[g]\left(\frac{1}{w}\right) + z_0, \quad w \in \overline{B}.
\]

Well-known estimates for the Vekua-operator (see [V] Chap. I, § 6) now show \( \tilde{z} \in C^\alpha(\overline{B}) \) and hence \( z \in C^\alpha(B^+ - \varrho_0(0), \mathbb{C}) \). This proves (a).
2. For the proof of claim (b) we repeat the construction above and note that, by (a), the right hand sides in (35) satisfy \( \tilde{g} \in C^{\alpha}(\overline{B}, \mathbb{C}) \), \( \tilde{h} \in C^{1,\alpha}(\partial B) \).

Subtracting a holomorphic function with boundary values \( \tilde{h} \), which belongs to \( C^{1,\alpha}(\overline{B}, \mathbb{C}) \) by Privalov's theorem, we may again assume \( \tilde{h} \equiv 0 \).

According to Theorem 2 in [S] Chap. IX, § 4 (see also [V] Chap. I, §8) the solution (39) of this problem belongs to \( C^{1,\alpha}(\overline{B}, \mathbb{C}) \) and we conclude \( z \in C^{1,\alpha}(\overline{B}_{\delta}(0), \mathbb{C}) \), as asserted. \( \square \)

We are now prepared to give the proof of our main result, Theorem 1. To this end, we define a further auxiliary function, namely

\[
z^3 := -(\dot{\gamma} + iq)x_w^1 + (1-iq\dot{\gamma})x_w^2 + (\psi_{p\dot{\gamma}} - \psi_{\dot{\gamma}p})x_w^3 \in C^1(B^+, \mathbb{C}) \cap H_1^1(B^+, \mathbb{C})
\]

with \( q = q(x), \dot{\gamma} = \dot{\gamma}(x^1), \psi_{p\dot{\gamma}} = \psi_{\dot{\gamma}p}(x^1, x^2) \); remember the definitions of \( \psi, \gamma \), and \( q \) in (7), (8), and (10). If we set \( \zeta := (z, z^3) = (z^1, z^2, z^3) : B^+ \to \mathbb{C}^3 \), we have the identity

\[
\zeta(w) = B(x(w)) \cdot x_w(w), \quad w \in B^+,
\]

where we abbreviated

\[
B := \begin{pmatrix} -i\psi_{p\dot{\gamma}} & -i\psi_{\dot{\gamma}p} & i \\ 1 - iq\dot{\gamma} & \dot{\gamma} + iq & \psi_{p\dot{\gamma}} + \psi_{\dot{\gamma}p} \\ -(\dot{\gamma} + iq) & 1 - iq\dot{\gamma} & \psi_{p\dot{\gamma}} - \psi_{\dot{\gamma}p} \end{pmatrix} \in C^1(\overline{B_r}, \mathbb{C}^{3 \times 3})
\]

see (12) for the definition of \( z = (z^1, z^2) \). Note that

\[
det B = i(1 + \dot{\gamma}^2)(1 - q^2 + |\nabla \psi|^2) \neq 0 \quad \text{on } \overline{B_r}
\]

is true according to the smallness condition (11). Hence, the inverse \( B^{-1}(p) \) exists for any \( p \in \overline{B_r} \) and we have \( B^{-1} \in C^1(\overline{B_r}, \mathbb{C}^{3 \times 3}) \).

We intend to employ the conformality relations, which now can be written as

\[
0 = \langle x_w, x_w \rangle = \langle B^{-1}(x) \zeta, B^{-1}(x) \zeta \rangle = \langle \zeta, C(x) \zeta \rangle \quad \text{on } B^+
\]

with the matrix \( C = (c_{ij})_{i,j=1,2,3} = B^{-T}B^{-1} \in C^1(\overline{B_r}, \mathbb{C}^{3 \times 3}) \). A lengthy but straightforward computation yields

\[
c_{11} = \frac{1 - q^2}{1 - q^2 + |\nabla \psi|^2};
\]

\[
c_{12} = \frac{q(\psi_{p\dot{\gamma}} - \psi_{\dot{\gamma}p})}{(1 + \dot{\gamma}^2)(1 - q^2 + |\nabla \psi|^2)} = c_{21},
\]

\[
c_{13} = \frac{q(\psi_{p\dot{\gamma}} + \psi_{\dot{\gamma}p})}{(1 + \dot{\gamma}^2)(1 - q^2 + |\nabla \psi|^2)} = c_{31},
\]

\[
c_{22} = \frac{1 + \dot{\gamma}^2 + (\psi_{p\dot{\gamma}} - \psi_{\dot{\gamma}p})^2}{(1 + \dot{\gamma}^2)(1 - q^2 + |\nabla \psi|^2)};
\]

\[
c_{23} = \frac{(\psi_{p\dot{\gamma}} + \psi_{\dot{\gamma}p})^2(\psi_{p\dot{\gamma}} - \psi_{\dot{\gamma}p})}{(1 + \dot{\gamma}^2)(1 - q^2 + |\nabla \psi|^2)} = c_{32},
\]

\[
c_{33} = \frac{1 + \dot{\gamma}^2 + (\psi_{p\dot{\gamma}} + \psi_{\dot{\gamma}p})^2}{(1 + \dot{\gamma}^2)(1 - q^2 + |\nabla \psi|^2)}.
\]

In particular, we have \( C : \overline{B_r} \to \mathbb{R}^{3 \times 3} \). We are now ready to give the
Proof of Theorem 1. 1. We write (43) in the form
\[ 0 = \sum_{j,k=1}^{3} c_{jk} z^j z^k = c_{33}(z^3)^2 + 2(c_{13}z^1 + c_{23}z^2)z^3 + \sum_{j,k=1}^{2} c_{jk} z^j z^k \quad \text{on } B^+, \]

where we abbreviated \( c_{jk} = c_{jk} \circ \chi \). Since \( c_{33} > 0 \) holds on \( \overline{B^+} \) due to (11), we may rewrite this identity as
\[ \left( z^3 + \sum_{j=1}^{2} \frac{c_{jk}}{c_{33}} z^j \right)^2 = \left( \sum_{j=1}^{2} \frac{c_{jk}}{c_{33}} z^j \right)^2 - \sum_{j,k=1}^{2} \frac{c_{jk}}{c_{33}} z^j z^k \quad \text{on } B^+. \quad (45) \]

With Lemma 2, we extend the right hand side of (45) to a continuous function on \( B^+ \cup I \). Lemma 3(a) thus yields that also \( z^3 + \sum_{j=1}^{2} \frac{c_{jk}}{c_{33}} z^j \) and, again due to Lemma 2, \( \zeta = (z^1, z^2, z^3) \) can be extended continuously to \( B^+ \cup I \). The definition (11) of \( \zeta \) as well as det \( B \neq 0 \) then imply \( x \in C^1(B^+ \cup I, \mathbb{R}^3) \).

2. Now we prove part (i) of the theorem. For fixed \( \varrho_0 \in (0, 1) \) and any \( \mu \in (0, 1) \) the right hand side of (15) belongs to \( C^{\varrho}([-\varrho_0, \varrho_0], \mathcal{C}) \) according to Lemma 2 and \( x \in C^1(B^+ \cup I, \mathbb{R}^3) \). In addition, the imaginary part of the right hand side vanishes on \([-\varrho_0, \varrho_0]\) due to \( \Im(z^1) = \Im(z^2) = 0 \) on \( I \) (see again Lemma 2) and to \( \mathcal{C} : \overline{B^+} \to \mathbb{R}^{3 \times 3} \) as shown above. Hence, the function \( f = z^3 + \sum_{j=1}^{2} \frac{c_{jk}}{c_{33}} z^j \in C^{\alpha}(I, \mathcal{C}) \) satisfies the assumptions of Lemma 3(b) for any \( \alpha \in (0, \frac{\mu}{2}) \). We conclude \( f \in C^{\alpha}([-\varrho_0, \varrho_0], \mathcal{C}) \) and, by Lemma 2 also \( \zeta \in C^{\alpha}([-\varrho_0, \varrho_0], \mathbb{C}^3) \) for any \( \alpha \in (0, \frac{\mu}{2}) \). If we differentiate (11) w.r.t. \( \overline{\varpi} \) and apply Rellich’s system (23) we obtain
\[ \zeta = g_0 \quad \text{on } B^+ \quad \text{with some } g \in C^0(B^+ \cup I, \mathbb{C}^3). \]

Consequently, we may apply Lemma 4(a) to \( \zeta \) and find \( \zeta \in C^1(B^+(0), \mathbb{C}^3) \) as well as \( x \in C^{1,\gamma}(\overline{B^+(0)}, \mathbb{R}^3) \) for any \( \varrho \in (0, \varrho_0) \) and any \( \alpha \in (0, \frac{\mu}{2}) \). Since we localized around an arbitrary point \( \varrho_0 \in I \), the proof of Theorem 4(a) is completed.

3. For the proof of Theorem 1(ii) we assume \( S \in C^{2,\beta}, Q \in C^{1,\beta}(\mathbb{R}^3, \mathbb{R}^3) \) with some \( \beta \in (0, 1) \). Then we also have \( B \in C^{1,\beta}(\overline{B^+}, \mathbb{R}^{3 \times 3}) \) and by part (i) we know, for instance, \( x \in C^{1,\gamma}(B^+ \cup I) \). Set \( \gamma := \min\{1, \frac{\mu}{2}\} \), define \( z = (z^1, z^2) \) by (12) and differentiate these equations w.r.t. \( \overline{\varpi} \). Then we obtain
\[ z = g_0 \quad \text{on } B^+ \quad \text{and } \Im z = 0 \quad \text{on } I \]

with some \( g_0 \in C^1(B^+ \cup I, \mathbb{C}^2) \). From Lemma 4(b) we thus conclude \( z \in C^{1,\gamma}([-\varrho, \varrho], \mathbb{C}^2) \) for any \( \varrho \in (0, 1) \). In particular, the right hand side of equation (15) belongs to \( C^1([-\varrho, \varrho], \mathcal{C}) \) and Lemma 4(b) shows \( \zeta \in C^{1,\gamma}([-\varrho, \varrho], \mathbb{C}^3) \) for any \( \varrho \in (0, 1) \). Now Lemma 4(a) can be applied to get \( \zeta \in C^{1,\gamma}(B^+(0), \mathbb{C}^3) \) and we finally arrive at \( x \in C^{1,\gamma}(B^+ \cup I, \mathbb{R}^3) \), as asserted.

We conclude the paper with the
Proof of Theorem \[4\]. We choose a branch point \( w_0 \in I \) and assume \( x(w_0) \in \partial S \); compare Remark \[4\] above. We localize as above – note especially \( w_0 \mapsto 0 \) – and define \( z = (z^1, z^2) \) by \[12\]. Reflecting \( z \) as in \[22\], the resulting function \( \hat{z} : B \to \mathbb{C}^2 \) satisfies \( \hat{z} \in C^1(B \setminus I, \mathbb{C}^2) \cap C^0(B, \mathbb{C}^2) \) and \( \text{Im} \hat{z} = 0 \) on \( I \) according to Lemma \[2\].

Now choose an arbitrary domain \( D \subset B \) with piecewise smooth boundary. Then the arguments leading to formula \[23\] in Lemma \[1\] yield
\[
\frac{1}{2i} \oint_{\partial D} \langle \hat{z}, \varphi \rangle dw = \frac{1}{2} \int_D \left( |\hat{z}|^2 (h, \varphi) + |\hat{z}|^2 (\hat{h}, \varphi) \right) du \, dv \quad \text{for all } \varphi \in C^1(B, \mathbb{C}^2);
\]
here \( h : B \to \mathbb{C}^2 \) denotes some bounded function. According to the boundedness of \( \hat{z} \) on \( D \) we find a constant \( c > 0 \) such that
\[
\left| \frac{1}{2i} \oint_{\partial D} \langle \hat{z}, \varphi \rangle dw \right| \leq 2 \int_D (|\varphi| + c|\varphi|)|\hat{z}| du \, dv \quad \text{for all } \varphi \in C^1(B, \mathbb{C}^2).
\]

The Hartman-Wintner technique – see e.g. Theorem 1 in \[DHT\] Section 3.1 – now implies the existence of some \( m \in \mathbb{N} \) and some vector \( \hat{b} \in \mathbb{C}^2 \) \( \setminus \{0\} \) such that
\[
\hat{z}(w) = \hat{b}w^m + o(|w|^m) \quad \text{as } w \to 0.
\]
(46)

Note here that \( \hat{z} \) cannot vanish identically in \( B \) since, otherwise, we would have \( \nabla x \equiv 0 \) near \( w_0 \) due to Proposition \[2\].

This is impossible by our assumption \( x \neq \text{const} \) as can easily be seen by employing the well-known asymptotic expansions at interior branch points.

Next, we define \( z^3 \) by \[40\] and consider \( \zeta = (z^1, z^2, z^3) = (z, z^3) \), which can be extended to a continuous function on \( B_C^1(0) \) for any \( \rho \in (0, 1) \), according to part 2 in the proof of Theorem \[1\]. In addition, we recall the relation \[45\], where the quantities \( c_{jk} = c_{jk} \circ x \) are continuous functions on \( B_C^1 \).

Now we multiply \[44\] by \( w^{-2m} \) and let \( w \in B_{\rho}(0) \) tend to 0. Due to \[45\], the right hand side and hence also the left hand side converge. Applying \[45\] again as well as a variant of Lemma \[3\](a), we find \( w^{-m}z^3(w) \to b^3 \) as \( w \to 0 \) with some limit \( b^3 \in \mathbb{C} \). Setting \( b := (b, b^3) \in \mathbb{C}^4 \), we conclude
\[
\zeta(w) = bw^m + o(|w|^m) \quad \text{as } w \to 0.
\]
(47)

This relation finally yields the announced expansion \[53\] according to \( x_w = (B^{-1} \circ x) \zeta \); see \[11\] and recall \( B \neq 0 \). The relation \( \langle a, a \rangle = 0 \) is now a direct consequence of the conformality relations and \[5\].

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