DIFFEROMORPHISM GROUPS OF NON-COMPACT MANIFOLDS
ENDOWED WITH THE WHITNEY $C^\infty$-TOPOLOGY

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ABSTRACT. Suppose $M$ is a non-compact connected $n$-manifold without boundary. $\mathcal{D}(M)$ is the group of $C^\infty$-diffeomorphisms of $M$ endowed with the Whitney $C^\infty$-topology and $\mathcal{D}_0(M)$ is the identity connected component of $\mathcal{D}(M)$, which is an open subgroup in the group $\mathcal{D}_c(M) \subset \mathcal{D}(M)$ of compactly supported diffeomorphisms of $M$. It is shown that $\mathcal{D}_c(M)$ is homeomorphic to $N \times \mathbb{R}^\infty$ for an $l_2$-manifold $N$ whose topological type is uniquely determined by the homotopy type of $\mathcal{D}_0(M)$. For instance, $\mathcal{D}_0(M)$ is homeomorphic to $l_2 \times \mathbb{R}^\infty$ if $n = 1, 2$ or $n = 3$ and $M$ is orientable and irreducible.

1. Introduction

This paper is a continuation of the study of topological types of diffeomorphism groups of non-compact smooth manifolds endowed with the Whitney $C^\infty$-topology. Suppose $M$ is a $\sigma$-compact smooth $n$-manifold without boundary. Let $\mathcal{D}(M)$ denote the group of diffeomorphisms of $M$ endowed with the Whitney $C^\infty$-topology (= the very-strong $C^\infty$-topology in [15]) and $\mathcal{D}_0(M)$ the identity connected component of $\mathcal{D}(M)$. The group $\mathcal{D}(M)$ includes the normal subgroup $\mathcal{D}_c(M)$ consisting of diffeomorphisms with compact support.

In [2] Theorem 4, Theorem 6.8 we have shown that $\mathcal{D}_c(M)$ is an $(l_2 \times \mathbb{R}^\infty)$-manifold and $\mathcal{D}_0(M)$ is an open subgroup of $\mathcal{D}_c(M)$. Here $l_2$ is the separable Hilbert space and $\mathbb{R}^\infty$ is the direct limit of the sequence $(\mathbb{R}^n)_{n \in \omega}$, where $\mathbb{R}^n$ is identified with the hyperspace $\mathbb{R}^n \times \{0\}$ in $\mathbb{R}^{n+1}$.

In a series of papers [4, 5, 6] T. Banakh and D. Repovš studied topological properties of direct limits in the categories of uniform spaces. These results were applied in [1] to yield a simple criterion for recognizing topological groups homeomorphic to open subspaces of $l_2 \times \mathbb{R}^\infty$ (see Theorem 2 in Section 2).

In this paper we apply the above criterion to obtain the following important conclusion on the group $\mathcal{D}_0(M)$.

**Theorem 1.** For any non-compact $\sigma$-compact smooth $n$-manifold without boundary the group $\mathcal{D}_c(M)$ is homeomorphic to an open subspace of $l_2 \times \mathbb{R}^\infty$.

In [20] K. Mine and K. Sakai obtained the triangulation theorem of open subsets of $l_2 \times \mathbb{R}^\infty$ (see Theorem 3 in Section 2). This means that any open subset $U$ of $l_2 \times \mathbb{R}^\infty$ is homeomorphic to $N \times \mathbb{R}^\infty$ for some $l_2$-manifold $N$ whose topological type is uniquely determined by the homotopy type of $U$. Thus we obtain the following conclusion of the group $\mathcal{D}_0(M)$.

**Corollary 1.** The group $\mathcal{D}_0(M)$ is homeomorphic to $N \times \mathbb{R}^\infty$ for some $l_2$-manifold $N$ whose topological type is uniquely determined by the homotopy type of $\mathcal{D}_0(M)$.

In some specific cases we can detect the homotopy type of $\mathcal{D}_0(M)$.

**Corollary 2.** Let $M$ be a non-compact connected smooth $n$-manifold without boundary.

1. If $1 \leq n \leq 2$, then $\mathcal{D}_0(M) \approx l_2 \times \mathbb{R}^\infty$.
2. If $n = 3$ and the manifold $M$ is orientable and irreducible (i.e., any smooth 2-sphere in $M$ bounds a 3-ball in $M$), then $\mathcal{D}_0(M) \approx l_2 \times \mathbb{R}^\infty$.
3. If $M$ is the interior $\text{Int}X = X \setminus \partial X$ of a compact connected smooth $n$-manifold $X$ with boundary, then

$$\mathcal{D}_0(M) \approx \mathcal{D}_0(X, \partial X) \times \mathbb{R}^\infty.$$
In particular, if $\mathcal{D}_0(X, \partial X)$ is contractible, then $\mathcal{D}_0(M) \approx l_2 \times \mathbb{R}^\infty$.

For example, if $M$ is the 3-dimensional Euclidean space $\mathbb{R}^3$ or the Whitehead contractible 3-manifold \([14]\), then $\mathcal{D}_0(M) \approx l_2 \times \mathbb{R}^\infty$.

2. OPEN SUBSPACES OF LF-SPACES

In this preliminary section we recall the criterion for recognizing topological groups homeomorphic to open subsets of $l_2 \times \mathbb{R}^\infty$. First we recall some necessary definitions. Below a Polish space means a separable completely metrizable space; a Polish group is a topological group whose underlying topological space is Polish.

A subgroup $H$ of a topological group $G$ is called locally topologically complemented (LTC) in $G$ if $H$ is closed in $G$ and the quotient map $q : G \to G/H = \{xH : x \in G\}$ is a locally trivial bundle. Here, a local section of a map $q : X \to Y$ at a point $y \in Y$ means a continuous map $s : U \to X$ defined on a neighborhood $U$ of $y$ in $Y$ such that $q \circ s = \text{id}_U$.

Following [1], we say that a topological group $G$ carries the strong topology with respect to a tower of subgroups

$$G_0 \subset G_1 \subset G_2 \subset \cdots$$

if $G = \bigcup_{n \in \omega} G_n$ and for any neighborhood $U_n$ of the neutral element $e$ in $G_n$, $n \in \omega$, the group product

$$\prod_{n \in \omega} U_n = \bigcup_{n \in \omega} U_0 U_1 \cdots U_n$$

is a neighborhood of $e$ in $G$. In this case the topology of $G$ coincides with the topology of the direct limit $\text{g-lim}_{n \in \omega} G_n$ of the tower $(G_n)_{n \in \omega}$ in the category of topological groups, which means that $G$ carries the strongest group topology such that the identity maps $G_n \to G$, $n \in \omega$, are continuous.

The following criterion is obtained in [1, Theorem 11].

**Theorem 2** (Banakh-Mine-Repovš-Sakai-Yagasaki). A non-metrizable topological group $G$ is homeomorphic to an open subset of $\mathbb{R}^\infty$ or $l_2 \times \mathbb{R}^\infty$ if $G$ carries the strong topology with respect to an LTC-tower of Polish ANR-groups $(G_n)_{n \in \omega}$.

Open subspaces of $l_2 \times \mathbb{R}^\infty$ were studied in [20, 21] and the following Triangulation Theorem was obtained.

**Theorem 3** (Mine-Sakai). 1. Each open subspace $X$ of $l_2 \times \mathbb{R}^\infty$ is homeomorphic to the product $K \times l_2 \times \mathbb{R}^\infty$ for a locally finite simplicial complex $K$.

2. Two open subspaces of $l_2 \times \mathbb{R}^\infty$ are homeomorphic if and only if they are homotopically equivalent.

Note that the product $N = K \times l_2$ is an $l_2$-manifold and its topological type is determined by its homotopy type.

3. DIFFEOMORPHISM GROUPS OF NON-COMPACT MANIFOLDS

Suppose $M$ is a non-compact $\sigma$-compact smooth $n$-manifold without boundary. We can represent $M$ as the countable union $M = \bigcup_{i \in \omega} M_i$ of compact $n$-submanifolds $M_i \subset M$, $i \in \omega$, such that $M_i \subset \text{Int}M_{i+1}$. Let $M_{-1} = \emptyset$ and consider the $n$-submanifolds $K_i = M \setminus \text{Int}M_i$, $i \in \omega$, of $M$ and closed subgroups $\mathcal{D}(M; K_i) = \{h \in \mathcal{D}(M) : h|_{K_i} = \text{id}_{K_i}\}$ of the diffeomorphism group $\mathcal{D}(M)$ endowed with the Whitney $C^\infty$-topology, see [13]. Thus we obtain the group $G = \mathcal{D}_c(M)$ and the tower $(G_i)_{i \in \omega}$ of closed subgroups $G_i = \mathcal{D}(M; K_i)$ of $G$.

The small box product $\boxdot_{i \in \omega} G_i$ is defined by

$$\boxdot_{i \in \omega} G_i = \{(x_i)_{i \in \omega} \in \boxdot_{i \in \omega} G_i : \exists k \in \omega \forall i \geq k, x_i = e\}.$$ 

This space is endowed with the box topology generated by the base consisting of boxes $\boxdot_{i \in \omega} U_i$, where $U_i$ is an open set of $G_i$. The left multiplication map

$$\pi : \boxdot_{i \in \omega} G_i \to G, \pi(x_0, \ldots, x_k, e, e, \ldots) = x_0 \cdot x_1 \cdots x_k$$

is continuous ([2, Lemma 2.10]).
We shall show that the tower \((G_i)_{i \in \omega}\) has the properties listed in Lemma 1 below. Hence, Theorem 1 now follows from Theorem 2.

**Lemma 1.** (1) \(G = \bigcup_{i \in \omega} G_i\) and the group \(G\) is not metrizable.

(2) \(G_i\) is a separable \(l_2\)-manifold for each \(i \in \omega\).

(3) \(G_i\) is TLC in \(G_{i+1}\) for each \(i \in \omega\).

(4) The left multiplication map \(\pi : \coprod_{i \in \omega} G_i \to G\) admits a local section at any points. Hence, the map \(\pi\) is an open map and the group \(G\) carries the strong topology with respect to the sequence \(G_i\) (\(i \in \omega\)).

(5) \(G = g \lim_l G_i\).

(6) Let \(H\) and \(H_i\) (\(i \in \omega\)) denote the identity connected components of \(G\) and \(G_i\) (\(i \in \omega\)) respectively. Then \(H\) is an open subgroup of \(G\) and the sequence \(H_i\) (\(i \in \omega\)) of closed subgroups of \(H\) also have the above properties (1) \(\sim\) (5).

**Proof.** The statement (1) easily follows from the definitions of \(G\) and \(G_i\) themselves. The statement (4) follows directly from [2] Proposition 5.5 (2). (The proof of Theorem 6.8 in [2] assures that Proposition 5.5 can be applied to this setting.) The statement (5) now follows from (4) and [1] Proposition 1.

(2) Since \(M - \text{Int } K_i = M_i\) is compact, the group \(G_i\) is an infinite-dimensional separable Fréchet manifold (cf. [10], [19]).

(3) The assertion follows directly from the bundle theorem, Theorem 4, explained below.

(6) Since \(H_i\) is an open subgroup of \(G_i\) for each \(i \in \omega\), it suffices to show that \(H = \bigcup_{i \in \omega} H_i\). The group \(H\) is path-connected, since it is the identity connected component of \(G\) and the latter is locally path-connected by (2) and (4). Hence any \(h \in H\) can be joined to \(\text{id}_M\) by an arc \(A\) in \(H\). By [2] Proposition 3.3] the compact subset \(A\) lies in \(G_n\) for some \(n \in \omega\). This means that \(h \in H_n\). \(\square\)

For an \(n\)-submanifold \(L\) of \(M\) and a subset \(K \subset L\) let \(E_K(L, M)\) denote the space of \(C^\infty\)-embeddings \(f : L \to M\) with \(f|_K = \text{id}_K\) endowed with the compact-open \(C^\infty\)-topology. There is a natural restriction map

\[ r : \mathcal{D}(M, K) \to E_K(L, M), \quad r(h) = h|_L. \]

The following is the classical bundle theorem in codimension 0 (cf. [7], [10], [22], [24]).

**Theorem 4.** Suppose \(K, L\) are \(n\)-submanifolds of \(M\) which are closed subsets of \(M\), \(K \subset \text{Int } L\) and \(\text{cl}_M(L \setminus K)\) is compact. Then, for any closed subset \(C\) of \(M\) with \(C \cap L = \emptyset\), the restriction map

\[ r : \mathcal{D}(M, K) \to E_K(L, M) \]

has a local section

\[ s : U \to \mathcal{D}_0(M, K \cup C) \subset \mathcal{D}(M, K) \]

at the inclusion \(i_L : L \subset M\) such that \(s(i_L) = \text{id}_M\).

For the proof of Corollary 2 we need a preliminary. For any pairs of spaces \((X, A)\) and \((Y, B)\) let \([X, A; Y, B]\) denote the set of homotopy classes of maps of pairs. Any map of pairs \(f : (Y, B) \to (Z, C)\) induces a function

\[ f_\# : [X, A; Y, B] \to [X, A; Z, C], \quad f_\# : [g] \to [fg]. \]

Suppose \(L\) is a compact space and \(K\) is a closed subset of \(L\). The inclusion maps \(H_i \subset H_{i+1}\) and \(H_i \subset H\) (\(i \in \omega\)) induce the associated functions between pointed sets:

\[ [L, K; H_i, \text{id}_M] \rightarrow [L, K; H_{i+1}, \text{id}_M] \]

\[ [L, K; H, \text{id}_M] \]

Taking the direct limit, we obtain a function between pointed sets

\[ \iota : \varinjlim [L, K; H_i, \text{id}_M] \to [L, K; H, \text{id}_M]. \]

Since any compact subset of \(H\) is included in some \(H_i\), we have the following conclusion.
Lemma 2. For any pair of compact spaces \((L, K)\) the inclusion induced function
\[
i : \lim_{i \to \infty} [L, K; H_i, \text{id}_M] \to [L, K; H, \text{id}_M]
\]
is a bijection.

For \(m = 0, 1, \ldots, \infty\), a map \(f : X \to Y\) between path-connected spaces is called an \(m\)-equivalence if for some base point \(x \in X\), the induced homotopy group on the \(k\)-th homotopy group
\[
f\# : \pi_k(X, x) \to \pi_k(Y, f(x))
\]
is an isomorphism for \(k = 0, 1, \ldots, m - 1\) and an epimorphism for \(k = m\). An \(\infty\)-equivalence is called a weak equivalence. If both \(X\) and \(Y\) have the homotopy type of CW-complexes, then every weak equivalence is a homotopy equivalence. Note that the groups \(H\) and \(H_i (i \in \omega)\) are path-connected and have the homotopy type of CW-complexes.

Corollary 3. For \(m = 0, 1, \ldots, \infty\), if each inclusion \(H_i \subset H_{i+1}\) is an \(m\)-equivalence, then so is the inclusion \(H_1 \subset H\). For example, each \(H_i\) is contractible, then so is \(H\) and hence \(H \approx l_2 \times \mathbb{R}^\infty\).

Proof of Corollary 2. We keep the notations \(M_i, K_i, H_i (i \in \omega)\) and \(H\).

(1), (2) Since \(M\) is connected, we may assume that for each \(i \in \omega\) (a) \(M_i\) is connected and (b) each connected component of \(K_i = M \setminus \text{Int} \, M_i\) is non-compact. By Corollary 3 it suffices to show that each \(H_i\) is contractible. Note that the inclusion \(H_i \subset D_0(M_i, \partial M_i)\) is a homotopy equivalence.

For \(n = 1, 2\) the assertion follows from [9], [18], Section 2.7], [23], [25], etc. In the case \(n = 3\), if \(M_i\) is a 3-ball, then \(D_0(M_i, \partial M_i)\) is contractible by the Smale conjecture [12, Appendix (1)]. If \(M_i\) is not a 3-ball, then by the assumption, \(M_i\) is an orientable Haken 3-manifold with boundary [4, 26] and \(D_0(M_i, \partial M_i)\) is contractible by [11], [16], [17].

(3) Take a collar \(\partial X \times [0, 1]\) of \(\partial X = \partial X \times \{0\}\) in \(X\) and let \(M_i = X \setminus (\partial X \times [0, 1/i])\) and \(K_i = M - \text{Int} \, M_i = \partial X \times (0, 1/i)\) \((i \in \mathbb{N})\). First we shall show that the inclusion \(H_i \subset H_{i+1}\) is a homotopy equivalence. Consider the restriction map \(\pi : H_{i+1} \to \mathcal{E}_{K_{i+1}}(K_i, M)\). Since \(\mathcal{E}_{K_{i+1}}(K_i, M)\) is the space of embeddings of the collar \(K_i\) relative to \(K_{i+1}\), it is seen that \(\mathcal{E}_{K_{i+1}}(K_i, M)\) is contractible. Since \(\mathcal{E}_{K_{i+1}}(K_i, M)\) is path-connected, from Theorem 1 it follows that the map \(\pi\) is onto and is a principal bundle with the structure group \(H_{i+1} \cap G_i\). Since \(\mathcal{E}_{K_{i+1}}(K_i, M)\) is contractible and paracompact, this bundle is trivial and \(H_{i+1} \approx \mathcal{E}_{K_{i+1}}(K_i, M) \times (H_{i+1} \cap G_i)\). Since \(H_{i+1}\) is path-connected, it follows that \(H_i = H_{i+1} \cap G_i\) and the inclusion \(H_i \subset H_{i+1}\) is a homotopy equivalence.

From Corollary 3 it follows that \(H \approx H_1 \approx D_0(M_1, \partial M_1) \approx D_0(X, \partial X)\). Since the last one is an \(l_2\)-manifold, we have \(H \approx D_0(X, \partial X) \times \mathbb{R}^\infty\). By Corollary 1.

\[\square\]

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