Obstructions for two-vertex alternating embeddings of graphs in surfaces

Bojan Mohar *†  Petr Škoda

Department of Mathematics,
Simon Fraser University,
8888 University Drive,
Burnaby, BC, Canada.

Abstract

A class of graphs that lies strictly between the classes of graphs of genus (at most) $k - 1$ and $k$ is studied. For a fixed orientable surface $S_k$ of genus $k$, let $A_{xy}^k$ be the minor-closed class of graphs with terminals $x$ and $y$ that either embed into $S_{k-1}$ or admit an embedding $\Pi$ into $S_k$ such that there is a $\Pi$-face where $x$ and $y$ appear twice in the alternating order. In this paper, the obstructions for the classes $A_{xy}^k$ are studied. In particular, the complete list of obstructions for $A_{xy}^1$ is presented.

1 Introduction

For a simple graph $G$, let $g(G)$ be the genus of $G$, that is, the minimum $k$ such that $G$ embeds into the orientable surface $S_k$. Similarly $\hat{g}(G)$ stands for the Euler genus of $G$. A combinatorial embedding $\Pi$ of $G$ is a pair $(\pi, \lambda)$ where $\pi$ assigns each vertex $v \in V(G)$ a cyclic permutation of edges adjacent

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†On leave from: IMFM & FMF, Department of Mathematics, University of Ljubljana, Ljubljana, Slovenia.
to \( v \) called the local rotation around \( v \) and the function \( \lambda : E(G) \to \{-1, 1\} \) describes the signature of edges when \( \Pi \) is non-orientable. A \( \Pi \)-face is a walk in \( G \) around a face of \( \Pi \) (for a formal definition see for example \([5]\)). Vertices \( v_1, \ldots, v_k \) are \( \Pi \)-cofacial if there is a \( \Pi \)-face where the vertices \( v_1, \ldots, v_k \) appear in some order.

For an edge \( e \) of \( G \), the two standard graph operations, deletion of \( e \), \( G - e \), and contraction of \( e \), \( G/e \), are called minor operations and are denoted by \( G \ast e \) when no distinction is necessary. A graph \( H \) is a minor of \( G \) if \( H \) is obtained from a subgraph of \( G \) by a sequence of minor operations. A family of graphs \( \mathcal{C} \) is minor-closed if, for each graph \( G \in \mathcal{C} \), all minors of \( G \) belong to \( \mathcal{C} \). A graph \( G \) is a (minimal) obstruction for a family \( \mathcal{C} \) if \( G \) does not belong to \( \mathcal{C} \) but for every edge \( e \) of \( G \), both \( G - e \) and \( G/e \) belong to \( \mathcal{C} \).

The well-known result of Robertson and Seymour \([12]\) asserts that the list of obstructions is finite for every minor-closed family of graphs.

For a fixed surface \( S \), the graphs that embed into \( S \) form a minor-closed family and it is of general interest to understand the obstructions \( \text{Forb}(S) \) for these families. Unfortunately, \( \text{Forb}(S_1) \) already contains thousands of graphs and is not yet determined \([6]\). We approach the problem by studying graphs in \( \text{Forb}(S_k) \) of small connectivity (see \([10]\)).

In this paper we study a phenomenon that arises when joining two graphs by two vertices. Given graphs \( G_1 \) and \( G_2 \) such that \( V(G_1) \cap V(G_2) = \{x, y\} \), the union of \( G_1 \) and \( G_2 \), that is the graph \((V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))\), is an \( xy \)-sum of \( G_1 \) and \( G_2 \) (or a \( 2 \)-sum if the vertices are not important). Sometimes, we also call \( G \) to be an \( xy \)-sum of \( G_1 \) and \( G_2 \) even if the edge \( xy \) is an edge of \( G_1 \) or \( G_2 \) but is not present in \( G \). To determine the genus of the \( xy \)-sum of \( G_1 \) and \( G_2 \), it is necessary to know if \( G_1 \) (and \( G_2 \)) has a minimum genus embedding \( \Pi \) such that there is a \( \Pi \)-face in which \( x \) and \( y \) appear twice in the alternating order (see \([4, 5]\)). For vertices \( x, y \in V(G) \), we say that \( G \) is \( xy \)-alternating on \( S_k \) if \( g(G) = k \) and \( G \) has an embedding \( \Pi \) of genus \( k \) with a \( \Pi \)-face \( W = v_1 \ldots v_l \) and indices \( i_1, \ldots, i_4 \) such that \( 1 \leq i_1 < i_2 < i_3 < i_4 \leq l \), \( v_{i_1} = v_{i_3} = x \), and \( v_{i_2} = v_{i_4} = y \).

A graph \( G \) is \( k \)-connected if \( G \) has at least \( k + 1 \) vertices and \( G \) remains connected after deletion of any \( k - 1 \) vertices. A graph has connectivity \( k \) if it is \( k \)-connected but not \( (k + 1) \)-connected.

To determine minimal obstructions of connectivity 2, we need to know which graphs are minimal not \( xy \)-alternating (see \([10]\)). For \( k \geq 1 \), let \( A^k_{xy} \) be the class of graphs with terminals \( x \) and \( y \) that are either embeddable in \( S_{k-1} \) or are \( xy \)-alternating on \( S_k \). When performing minor operations on graphs...
with terminals, we do not allow a contraction identifying two terminals to a single vertex. Also, when contracting an edge joining a terminal and a non-terminal vertex, the new vertex is a terminal. Thus the number of terminals of a minor is the same as of the original graph. A homomorphism of two graphs with terminals is an isomorphism if it is a graph isomorphism and (non-)terminals are mapped onto (non-)terminals. In particular, automorphisms that switch the terminals are considered. Under these restrictions, $\mathcal{A}^k_{xy}$ is a minor-closed family of graphs with two terminals. Let $\mathcal{F}^k_{xy}$ be the set of minimal obstructions for $\mathcal{A}^k_{xy}$, that is, a graph $G$ belongs to $\mathcal{F}^k_{xy}$ if $G \notin \mathcal{A}^k_{xy}$ and, for each edge $e \in E(G)$ and each allowed minor operation $*$, $G*e \in \mathcal{A}^k_{xy}$. It is shown in Sect. 2 that $\mathcal{F}^k_{xy}$ is finite for each $k \geq 1$. Note that each vertex of a graph in $\mathcal{F}^k_{xy}$ has degree at least 3 unless it is a terminal.

A Kuratowski graph is a graph isomorphic to $K_5$, the complete graph on five vertices, or to $K_{3,3}$, the complete bipartite graph on three and three vertices. For a fixed Kuratowski graph $K$, a Kuratowski subgraph in $G$ is a minimal subgraph of $G$ that contains $K$ as a minor. A $K$-graph $L$ in $G$ is a subdivision of $K_4$ or $K_{2,3}$ that can be extended to a Kuratowski subgraph in $G$. We are using extensively the following well-known theorem.

**Theorem 1** (Kuratowski [8]). A graph is planar if and only if it does not contain a Kuratowski subgraph.

We also use the following classical theorem (see [9, Theorem 6.3.1]).

**Theorem 2.** Let $G$ be a connected graph and $C$ a cycle in $G$. Let $G'$ be a graph obtained from $G$ by adding a new vertex joined to all vertices of $C$. Then $G$ can be embedded in plane with $C$ as an outer cycle unless $G$ contains an obstruction of the following type:

(a) a pair of disjoint crossing paths,

(b) a tripod, or

(c) a Kuratowski subgraph contained in a 3-connected block of $G'$ distinct from the 3-connected block of $G'$ containing $C$.

Let $G$ be a 2-connected graph. Each vertex of degree different from 2 is a branch vertex. A branch of $G$ is a path in $G$ whose endvertices are branch vertices and such that each intermediate vertex has degree 2.

Let $H$ be a subgraph of $G$. An $H$-bridge in $G$ is a subgraph of $G$ which is either an edge not in $H$ but with both ends in $H$, or a connected component
of $G - V(H)$ together with all edges which have one end in this component and the other end in $H$. For a $H$-bridge $B$, the interior of $B$, $B^\circ$, is the set $E(B) \cup (V(B) \setminus V(H))$ containing the edges of $B$ and the vertices inside $B$. Thus, $G - B^\circ$ is the graph obtained from $G$ by deleting $B$.

Let $B$ be an $H$-bridge in $G$. The vertices in $V(B) \cap V(H)$ are called attachments of $B$. The bridge $B$ is a local bridge if all attachments of $B$ lie on a single branch of $H$.

Let $B$ be a cycle of a fixed orientation and $u$ and $v$ two vertices in $C$. The segment $C[u,v]$ is the path $P$ in $C$ from $u$ to $v$ (in the given orientation of $C$). Similarly, $C(u,v)$ denotes $P$ without the endvertices and any combination of brackets can be used to indicate which endvertices are included in the path. Let $P$ be a segment of $C$ and $B$ a $C$-bridge whose attachments are contained in $P$. The support of $B$ in $P$ is the smallest subsegment of $P$ that contains all attachments of $B$.

For a cycle $C$, two $C$-bridges avoid each other if there are vertices $u$ and $v$ such that all attachments of one bridge lie on $C[u,v]$ and all attachments of the other bridge lie on $C[v,u]$. Otherwise, they overlap. A $C$-bridge $B$ is planar if $C \cup B$ is planar. Let $B$ be a set of $C$-bridges. The bridge-overlap graph of $B$ has vertex set $B$ and two bridges are adjacent if they overlap. We use the following well-known theorem.

**Theorem 3.** Let $G$ be a graph that consists of a cycle $C$ and a set $B$ of planar $C$-bridges. Then $G$ is planar if and only if the bridge-overlap graph of $B$ is bipartite.

The paper is organized as follows. In Sec. 2 we study the classes $F^k_{xy}$ in general. The rest of the paper is focused on the class $F^1_{xy}$. A basic classification of $F^1_{xy}$ is shown in Sec. 3 and the complete list of $F^1_{xy}$ is provided in the subsequent chapters. The paper is concluded in Sec. 7 where the main theorem is proven.

## 2 General properties

In this section we present some general results about graphs in $F^k_{xy}$, where $k \geq 1$. In the following, $G/xy$ is the underlying simple graph of the multigraph obtained by identifying vertices $x$ and $y$. Note that the edge $xy$ does not have to be present and, if $xy \in E(G)$, we delete $xy$ before identifying $x$ and $y$. Let $v_{xy}$ be the vertex obtained after the identification. For a graph $G$ with
terminals \( x \) and \( y \), let \( G^+ \) denote the graph \( G + xy \) if \( xy \notin E(G) \) and the graph \( G \) otherwise. We will use the following lemma (see [9, Prop. 6.1.2.]).

**Lemma 4.** Let \( G \) be an \( xy \)-sum of graphs \( G_1 \) and \( G_2 \). If \( G^+_2 \) is planar, then each embedding of \( G^+_1 \) into a surface can be extended to an embedding of \( G \) into the same surface.

**Proof.** Since \( G^+_2 \) is planar, there is a planar embedding of \( G_2 \) such that \( x \) and \( y \) are on the infinite face. A given embedding \( \Pi \) of \( G^+_1 \) can be extended into the embedding of \( G \) by embedding \( G_2 \) into a \( \Pi \)-face incident with the edge \( xy \).

In the sequel, we shall use another graph \( G^* \) obtained from a given graph \( G \) with given terminals \( x \) and \( y \). The graph \( G^* \) is obtained as an \( xy \)-sum of \( G \) and \( K_5 - xy \) (the graph obtained from \( K_5 \) with two terminals \( x \) and \( y \) by deleting the edge \( xy \)). We will use a characterization of \( xy \)-alternating graphs by Decker et al. [5], that a graph \( G \) with terminals \( x \) and \( y \) is \( xy \)-alternating if and only if \( g(G^*) = g(G) \). They also proved the following theorem:

**Theorem 5** (Decker, Glover, and Huneke [5]). If \( G \) is an \( xy \)-sum of graphs \( G_1 \) and \( G_2 \), then

\[
g(G) = \min\{g(G^+_1) + g(G^+_2) - \epsilon(G_1)\epsilon(G_2), g(G_1) + g(G_2) + 1\}
\]

where \( \epsilon(G) = 1 \) if \( G^+ \) is \( xy \)-alternating and \( \epsilon(G) = 0 \) otherwise.

Note that both \( K_5 \) and \( K_{3,3} \) are \( xy \)-alternating on the torus for any pair of vertices \( x \) and \( y \) (see Fig. 1).

For a graph \( G \) and a vertex \( x \) of \( G \), the graph \( G' \) is obtained by splitting \( G \) at \( x \) if \( x \) is replaced by two adjacent vertices \( x_1 \) and \( x_2 \) and edges incident with \( x \) in \( G \) are distributed arbitrarily to \( x_1 \) and \( x_2 \) in \( G' \). By doing the same except that \( x_1 \) and \( x_2 \) are non-adjacent, a resulting graph \( G' \) is said to be obtained by cutting of \( G \) at \( x \).

Suppose that a graph \( G \) is embedded in some surface \( S \). Let \( \gamma \) be a simple closed curve in \( S \) that intersects the embedded graph \( G \) only at vertices of \( G \). The number of vertices in \( \gamma \cap V(G) \) is called the width of \( \gamma \) (with respect to the embedded graph). If \( \gamma \) intersects \( G \) at a vertex \( z \), then it separates the edges incident with \( z \) into two parts, \( \gamma \)-sides at \( z \), according to their appearance in the local rotation around \( z \). The graph obtained by cutting \( G \) at each vertex \( v \) in \( \gamma \cap V(G) \) using the \( \gamma \)-sides to partition the edges is said
Figure 1: Kuratowski graphs and their two-vertex alternating embeddings in the torus.

to be obtained by cutting $G$ along $\gamma$. The curve $\gamma$ also induces the cutting of the surface $\mathbb{S}$ along $\gamma$, and the cut graph is embedded in the cut surface. A curve is orientizing for a $\Pi$-embedded graph $G$ if cutting $G$ along $\gamma$ yields an orientable embedding of the resulting graph using the embedding induced by $\Pi$. The orientizing face-width of $G$ is the minimum width of an orientizing curve.

The next lemma outlines three characterizations of $A_{xy}^k$.

**Lemma 6.** Let $G$ be a graph with terminals $x$ and $y$. If $G$ does not embed into $S_{k-1}$, then the following statements are equivalent:

(i) $G$ is in $A_{xy}^k$.

(ii) $G$ has an embedding $\Pi$ into $N_{2k-1}$ with an orientizing 1-sided simple closed curve $\gamma$ of width 2 going through $x$ and $y$.

(iii) $G$ can be cut at $x$ and $y$ so that the resulting graph embeds into $S_{k-1}$ with $x_1, y_1, x_2$ and $y_2$ appearing on a common face (in the stated order).

(iv) $G^*$ embeds into $S_k$.

The proof of Lemma 6 uses the following result by Archdeacon and Huneke [2].

**Lemma 7.** Let $G$ be a $\Pi$-embedded graph and $W$ a $\Pi$-facial walk. If two vertices $x$ and $y$ appear twice in $W$ in the alternating order $x, y, x, y$, then
there exists an embedding $\Pi'$ of $G$ of Euler genus $\hat{g}(\Pi) - 1$ such that every $\Pi$-facial walk is $\Pi'$-facial except for $W$ which turns into two $\Pi'$-facial walks $W_1$ and $W_2$, each of which contains both $x$ and $y$. Moreover, the curve $\gamma$ passing through $x$ and $y$ and the faces $W_1$ and $W_2$ is 1-sided in $\Pi'$ and the signatures of edges in $\Pi'$ differ from $\Pi$ only by switching the signatures of a $\gamma$-side at $x$ and a $\gamma$-side at $y$.

Proof of Lemma 6. The equivalence of (i) and (iv) was proven by Decker et al. [5].

(i) $\Rightarrow$ (ii): Since $G$ does not embed into $S_{k-1}$, it $\Pi$-embeds into $S_k$ with $x$ and $y$ alternating in a $\Pi$-face $W$. By Lemma 7, there is an embedding $\Pi'$ of Euler genus $2k - 1$ with two $\Pi'$-faces $W_1$ and $W_2$, both containing $x$ and $y$. The curve $\gamma$ obtained by connecting vertices $x$ and $y$ in both faces $W_1$ and $W_2$ is the sought 1-sided curve of width 2. Since the signatures of edges in $\Pi$ are positive, the edges of negative signature in $\Pi'$ form two $\gamma$-sides of $x$ and $y$ (respectively). Thus cutting $G$ along $\gamma$ yields an orientable embedding and $\gamma$ is orientizing.

(ii) $\Rightarrow$ (iii): Cutting along the 1-sided orientizing curve $\gamma$ yields an orientable embedding $\Pi$ of genus $k - 1$. Since $\gamma$ is 1-sided, the vertices obtained by cutting $G$ along $\gamma$ lie on a common face in the interlaced order.

(iii) $\Rightarrow$ (i): Take an embedding $\Pi$ of the resulting graph $G'$ into $S_{k-1}$ with $x_1, y_1, x_2, y_2$ on a common face $W$. Let $G'' = G' + x_1x_2 + y_1y_2$. We extend $\Pi$ to an embedding $\Pi'$ of $G''$ into $S_k$ by embedding the new edges into $W$ (and adding a handle). The number of faces of $\Pi'$ stays the same but the number of edges is increased by two. Thus $g(\Pi') = g(\Pi) + 1$. By contracting the edges $x_1x_2$ and $y_1y_2$, we obtain $G$ and its $xy$-alternating embedding in $S_k$. □

The classical result of Robertson and Seymour [11] asserts that the set of obstructions for each minor-closed family of graphs is finite. In particular, this implies that $\text{Forb}(S_k)$ is finite for each $k \geq 0$. A topological obstruction $G$ for $S_k$ is a graph with no vertices of degree 2 that does not embed in $S_k$ but each proper subgraph of $G$ does. Since minors and topological minors ($H$ is a topological minor of $G$ if $G$ contains a subdivision of $H$ as a subgraph) are closely related, the set $\text{Forb}'(S_k)$ of topological obstructions is also finite for each $k \geq 0$ (see [9, Prop. 6.1.1.]). Unfortunately, since the graphs in the classes $\mathcal{A}_{xy}^k$ have terminals, the result of Robertson and Seymour does not directly apply and thus it is not clear a priori whether the sets $\mathcal{F}_{xy}^k$ are finite.
The next lemma shows that the graphs in $\mathcal{F}_{xy}^k$ are derived from graphs in $\text{Forb}^*(S_k)$ and thus the finiteness of $\text{Forb}^*(S_k)$ implies the finiteness of $\mathcal{F}_{xy}^k$.

**Lemma 8.** Let $G \in \mathcal{F}_{xy}^k$. Then precisely one of the graphs $G$, $G^+$, or $G^*$ belongs to $\text{Forb}^*(S_k)$.

*Proof.* For each $e \in E(G)$ (possibly $e = xy$), we have $G - e \in \mathcal{A}_{xy}^k$. Therefore, we have that $g(G - e) \leq k$ and, if $g(G - e) = k$, then $G - e$ is $xy$-alternating on $S_k$. If $g(G) > k$, then $G \in \text{Forb}^*(S_k)$, since $g(G - e) \leq k$ for each $e \in E(G)$.

Thus we may assume that $g(G) = k$ and $G$ is not $xy$-alternating on $S_k$.

Suppose that $g(G^+) > g(G)$. For each $e \in E(G)$, we have $G - e \in \mathcal{A}_{xy}^k$, hence either $g(G - e) = k - 1$ and thus $g(G^+ - e) = k$, or $G - e$ is $xy$-alternating on $S_k$ and then $g(G^+ - e) = k$ since the edge $xy$ can be embedded into the $xy$-alternating face. Therefore, $G^+ \in \text{Forb}^*(S_k)$.

Suppose now that $g(G^+) = g(G)$. We shall show that $G^* \in \text{Forb}^*(S_k)$. Since $G$ is not $xy$-alternating on $S_k$, $g(G^*) > g(G)$ by Lemma 6. For $e \in E(G)$, either $g(G - e) = k - 1$ and thus $g(G^* - e) \leq k$ by (1), or $G - e$ is $xy$-alternating on $S_k$ and so $g(G^* - e) = k$ also by (1). Let $H$ be the $xy$-bridge of $G^*$ induced by the edges not in $G$, which is isomorphic to $K_5$ minus an edge. For $e \in E(H)$, since $H^+ - e$ is planar, Lemma 1 gives that $g(G^* - e) = g(G^*) = g(G) = k$. This shows that $G^* \in \text{Forb}^*(S_k)$.

In conclusion, $G$, $G^+$, or $G^*$ belongs to $\text{Forb}^*(S_k)$, and it is clear that only one of these graphs is in $\text{Forb}^*(S_k)$ since they are topological minors of each other. \qed

Lemma 8 has the following immediate corollary.

**Corollary 9.** For $k \geq 1$, the class of graphs $\mathcal{F}_{xy}^k$ is finite.

*Proof.* Let $\mathcal{F}$ be the family of all graphs with two terminals obtained from graphs $H \in \text{Forb}^*(S_k)$ by declaring two vertices of $H$ to be terminals (in all possible ways), by declaring two adjacent vertices to be terminals and deleting the edge joining them or by removing a bridge isomorphic to $K_5$ minus an edge and declaring its two vertices of attachments to be terminals. By Lemma 6, $\mathcal{F}_{xy}^k \subseteq \mathcal{F}$. This completes the proof since $\text{Forb}^*(S_k)$ is finite. \qed

**Lemma 10.** For $k \geq 1$, let $G \in \mathcal{F}_{xy}^k$. If $G$ is not embeddable into $S_k$, then $xy \notin E(G)$, $G$ is $xy$-alternating on $S_{k+1}$ and an obstruction for $S_k$. 

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Proof. If \( xy \in E(G) \), then \( G - xy \in \mathcal{A}_{xy}^k \). Since \( g(G) > k \), we have \( g(G - xy) = k \) and thus \( G - xy \) has an \( xy \)-alternating embedding in \( S_k \). But then \( G \) also embeds in \( S_k \). This contradiction shows that \( xy \notin E(G) \).

Since \( G \ast e \in \mathcal{A}_{xy}^k \) for every edge \( e \in E(G) \) and each minor operation \( \ast \), \( G \ast e \) embeds into \( S_k \). Hence \( G \) is an obstruction for \( S_k \).

Let us construct an \( xy \)-alternating embedding in \( S_{k+1} \). Let \( e = uv \) be an arbitrary edge in \( G \) and consider the graph \( G - e \). Clearly, the genus of \( G - e \) cannot drop by more than one and since \( G \in \mathcal{F}_{xy}^k \), there has to be an \( xy \)-alternating embedding of \( G - e \) in \( S_k \). Let \( \Pi \) be this \( xy \)-alternating embedding of \( G - e \) in \( S_k \) and let \( W \) be an \( xy \)-alternating \( \Pi \)-face. Pick two arbitrary \( \Pi \)-faces \( W_u \) and \( W_v \) incident with \( u \) and \( v \), respectively. Since \( u \) and \( v \) are not \( \Pi \)-cofacial, \( W_u \) and \( W_v \) are distinct. Extend \( \Pi \) to an embedding \( \Pi' \) of \( G \) in \( S_{k+1} \) by adding a handle into faces \( W_u \) and \( W_v \). Since at most one of \( W_u \) or \( W_v \) is \( \Pi \)-labeled, the \( xy \)-alternating \( \Pi' \)-face \( W \) is extended to an \( xy \)-alternating \( \Pi' \)-face.

The following is an immediate corollary of Lemma 10.

**Corollary 11.** For \( k \geq 1 \), we have \( \mathcal{F}_{xy}^k \subseteq \mathcal{A}_{xy}^{k+1} \).

We think that the scenario forced by Lemma 10 when \( G \in \mathcal{F}_{xy}^k \) is not embeddable in \( S_k \), is quite unlikely, and we would like to pose the following conjecture.

**Conjecture 12.** Let \( G \) be in \( \mathcal{F}_{xy}^k \). Then \( G \) embeds in \( S_k \).

In this paper we confirm the conjecture for \( k = 1 \).

### 3 Basic classification

To classify all minimal obstructions for the torus of connectivity 2, we aim to understand the class \( \mathcal{A}_{xy}^1 \) of \( xy \)-alternating graphs on the torus and the set \( \mathcal{F}_{xy}^1 \) of its obstructions.

Lemma 6 gives the following characterizations of \( \mathcal{A}_{xy}^1 \).

**Corollary 13.** Let \( G \) be a non-planar graph with terminals \( x \) and \( y \). The following statements are equivalent:

(i) \( G \) is in \( \mathcal{A}_{xy}^1 \).
(ii) \( G \) has an embedding \( \Pi \) into the projective plane of face-width 2 with a non-contractible curve of width 2 going through \( x \) and \( y \).

(iii) \( G \) can be cut at \( x \) and \( y \) so that the resulting graph is planar with \( x_1, x_2, y_1 \) and \( y_2 \) on a common face.

(iv) \( G^* \) embeds into the torus.

By Corollary 13, a non-planar graph \( G \) belongs to \( A_{xy}^1 \) if and only if the vertices \( x \) and \( y \) can be split so that the resulting graph is planar with the new vertices on a common face. This implies that \( G/xy \) is planar.

**Corollary 14.** If \( G \) is a non-planar graph in \( A_{xy}^1 \), then \( G/xy \) is planar.

We will show below that, if \( G/xy \) is non-planar, then there is a Kuratowski subgraph in \( G \) with a K-graph disjoint from \( x \) and \( y \). The following lemma by Juvan et al. \(^7\) allows us to choose a subgraph without local bridges provided that we have an almost 3-connected graph. Let \( K \) be a subgraph of \( G \). The graph \( G \) is 3-connected modulo \( K \) if for every vertex set \( U \subseteq V(G) \) with at most 2 elements, every connected component of \( G - U \) contains a branch vertex of \( K \).

**Lemma 15** (Juvan, Marinček and Mohar \(^7\)). Let \( K \) be a subgraph of a graph \( G \). If \( G \) is 3-connected modulo \( K \), then \( G \) contains a subgraph \( K' \) such that

(a) \( K' \) is homeomorphic to \( K \) and has the same branch vertices as \( K \).

(b) For each branch \( e \) of \( K \), the corresponding branch \( e' \) of \( K' \) joins the same pair of branch vertices as \( e \) and is contained in the union of \( e \) and all \( K \)-bridges that are local on \( e \).

(c) \( K' \) has no local bridges.

Now, we are ready to prove that if \( G/xy \) is non-planar, then there is a Kuratowski subgraph in \( G \) with a K-graph disjoint from \( x \) and \( y \).

**Lemma 16.** Let \( G \) be a non-planar graph and \( x, y \in V(G) \). If \( G/xy \) is non-planar, then \( G \) contains a K-graph disjoint from \( x \) and \( y \).
Proof. Suppose that the conclusion of the lemma is false. Let $G$ be a counterexample with $|V(G)| + |E(G)|$ minimum. It is easy to see that $G$ is connected. If $G-x$ is non-planar, then by Theorem 1, $G-x$ contains a Kuratowski graph $K$ and thus $K-y$ contains a K-graph in $G$ that is disjoint from $x$ and $y$. Hence $G-x$ is planar. Similarly, $G-y$ is planar.

Let $K$ be a Kuratowski subgraph in $G/xy$ and $L$ a K-graph contained in $K-v_{xy}$. Let $B_x$ and $B_y$ be the L-bridges of $G$ containing $x$ and $y$, respectively. Necessarily, $B_x$ and $B_y$ are different $L$-bridges of $G$ since otherwise $L$ is a K-graph in $G$ disjoint from $x$ and $y$. We aim to get rid of the local $L$-bridges by applying Lemma 15 but also preserve the property that the graph is a K-graph in $G/xy$ that is disjoint from $x$ and $y$. In order to achieve that, we consider the graph $\hat{G} = G - B_x^o - B_y^o - w_1w_2$ in the case when $L$ is isomorphic to $K_{2,3}$, $w_1, w_2$ are the vertices of degree 3 in $L$, and $w_1w_2 \in E(G)$. Otherwise, let $\hat{G} = G - B_x - B_y$.

If $\hat{G}$ is not 3-connected modulo $L$, then there is a (minimal) vertex set $U$ with $|U| \leq 2$ such that a $U$-bridge $C$ does not contain any branch vertex (in $C^o$). If $|U| \leq 1$, then $C$ is a block of $\hat{G}$. Since genus is additive over blocks (see 3), the block $C$ is planar and its removal from $G$ yields a subgraph of $G$ that satisfies the assumptions of the lemma. This is a contradiction with the choice of $G$ being minimal. Thus $U$ contains exactly two vertices, $u$ and $v$, and there is a path in $C$ that connects $u$ and $v$. Let $G'$ be the graph obtained from $G$ by contracting $C$ into a single edge $uv$. Since $C$ does not contain $x$ and $y$, if $C + uv$ is non-planar, then $C$ contains a K-graph disjoint from $x$ and $y$ in $G$. Hence $C + uv$ is planar and Lemma 4 gives that $G'$ is non-planar. It is not difficult to see that $G'/xy$ is also non-planar. By the choice of $G$, there is a K-graph $L'$ in $G'$ disjoint from $x$ and $y$. Since the edge $uv$ in $G'$ can be replaced in $G$ by a path in $C$, $L'$ induces in a straightforward way a K-graph in $G$ disjoint from $x$ and $y$.

Therefore, we may assume that $\hat{G}$ is 3-connected modulo $L$. By Lemma 15, there exists a subgraph $L'$ of $\hat{G}$ homeomorphic to $L$ that has no local bridges, and has the same branch vertices as $K'$ and also satisfies property (b) of Lemma 15. Note that, since $K_{2,3}$ and $K_4$ are uniquely embeddable in the plane, $L'$ has a unique planar embedding. Let $B'_x$ and $B'_y$ be the $L'$-bridges in $G$ containing $x$ and $y$, respectively. By using (b) of Lemma 15, it is not difficult to check that $L'$ is still a K-graph in $G/xy$. It follows that $B'_x$ and $B'_y$ are different $L'$-bridges in $G$.

Case 1: $L'$ is a subdivision of $K_4$ or $w_1w_2 \notin E(G)$.
Since \( G - B_x^t \) and \( G - B_y^t \) are planar, each \( L' \)-bridge can be embedded into some \( \Pi \)-face. Since only \( B_x' \) and \( B_y' \) can be local \( L' \)-bridges in \( G \), each other \( L' \)-bridge in \( G \) embeds into a unique \( \Pi \)-face. Since the vertices of the union of the attachments of \( B_x' \) and \( B_y' \) do not lie on a single \( \Pi \)-face, the bridges \( B_x' \) and \( B_y' \) embed into different \( \Pi \)-faces. We conclude that each \( L' \)-bridge in \( G \) can be assigned a \( \Pi \)-face such that all bridges assigned to a single \( \Pi \)-face can be embedded there simultaneously. Hence \( G \) is planar — a contradiction.

**Case 2:** \( L \) is a subdivision of \( K_{2,3} \) and \( w_1w_2 \in E(G) \).

Consider the graph \( G' = G - w_1w_2 \). Since \( G' \) is a subgraph of \( G \) and \( G' / xy \) is non-planar, \( G'' \) is planar by the choice of \( G \). Since the planar embedding of \( G' \) cannot be extended into a planar embedding of \( G \) by adding the edge \( w_1w_2 \) into one of the three \( \Pi \)-faces, there are three paths \( P_1, P_2, P_3 \) that connect the three pairs of open branches of \( L' \), respectively (see Fig. 2). Let \( L'' \) be the subgraph of \( G \) that consists of \( w_1w_2 \), the path \( P_i \) that is embedded in the \( \Pi \)-face containing neither \( x \) nor \( y \) and the two branches of \( L' \) that \( P_i \) connects to. It is easy to see that \( L'' \) forms a K-graph in \( G \) that is disjoint from \( x \) and \( y \), a contradiction.

Lemma 16 leads to the following dichotomy of graphs is \( F_{xy}^1 \).

**Lemma 17.** Let \( G \) be a graph in \( F_{xy}^1 \). Then one of the following is true.

(i) \( G \) is a split of a Kuratowski graph with \( x \) and \( y \) being the two vertices resulting after the split (see Fig. 3) or \( G \) is a Kuratowski graph plus one or two isolated vertices that are terminals.

(ii) \( G/xy \) is planar.
Proof. Suppose that $G$ does not satisfy (ii). By Lemma [16], there is a Kuratowski subgraph $K$ in $G$ with a K-graph $L$ disjoint from $x$ and $y$. If there is an edge $e$ and a minor operation $*$ such that $G * e$ still contains a K-graph disjoint from $x$ and $y$, then $(G * e)/xy$ is non-planar and thus $G * e \notin A^1_{xy}$ by Corollary [14]. Hence $E(G) = E(K)$. If $e$ is a subdivided edge of $K$, then $G/e$ still contains a K-graph disjoint from $x$ and $y$ unless a terminal and a branch vertex of $K$ are the endvertices of $e$. Now it is easy to see that $G$ satisfies (i).

4 XY-labelled graphs

Let $G$ be a graph with terminals $x$ and $y$. To investigate graphs in $G \in F^1_{xy}$ where $G/xy$ is planar, we study the graph $H = G - x - y$. Let us label each vertex of $H$ by the label $X$ ($Y$) if it is adjacent to $x$ ($y$) in $G$. Thus each vertex of $H$ is given up to two labels. Let $\lambda(v)$ denote the set of labels given to the vertex $v$ of $H$. A vertex $v$ is labelled if $\lambda(v)$ is non-empty. The graph $H$ together with the labels carries all information about $G$. Let us call $H$ an XY-labelled graph. The notion of a minor of a graph is extended to XY-labelled graphs naturally: an XY-labelled graph $H_1$ is a minor of an XY-labelled graph $H_2$ if the graph with terminals corresponding to $H_1$ is a minor of the graph with terminals corresponding to $H_2$. For example, the deletion of a label is a minor operation that corresponds to an edge deletion and, when contracting an edge $uv$ in an XY-labelled graph, the resulting vertex is labelled by $\lambda(u) \cup \lambda(v)$.

Another useful representation of $G$ is as follows. Consider the multigraph $\tilde{H}$ and the vertex $v_{xy}$ obtained by identification of $x$ and $y$ in $G$ (in contrast to the simple graph $G/xy$ used in the previous sections). Label each edge $e$ of $\tilde{H}$ incident to $v_{xy}$ by the label $X$ ($Y$) if the edge was incident to $x$ ($y$) in $G$. Let $\Pi$ be a planar embedding of $\tilde{H}$. The local rotation around $v_{xy}$ gives a cyclic sequence $S$ of labels that appear on the edges incident with

Figure 3: Splits of Kuratowski graphs.
Call $S$ a label sequence of $\hat{H}$. A label transition in a label sequence is a pair of (cyclically) consecutive labels that are different. The number of transitions $\tau(Q)$ of $S$ is the number of label transitions in $S$. In the case when $S$ contains only two different labels, $\tau(Q)$ is a multiple of 2. Thus we say that a label sequence $S$ is $k$-alternating if $\tau(Q) = 2k$. A planar embedding of $\hat{H}$ is $k$-alternating if the induced label sequence is $k$-alternating and $H$ is called $k$-alternating if $\hat{H}$ admits a $k$-alternating embedding in the plane. Note that Lemma 13 implies that, if $H$ is 2-alternating, then the corresponding graph $G$ is in $A_{xy}^1$.

When $H$ is connected, a planar embedding of $\hat{H}$ induces a planar embedding of $H$ with a special face $W$ in which $v_{xy}$ is embedded. Call the cyclic sequence of vertices of $W$ (with some possibly appearing more than once) a boundary of $H$. If $H$ is 2-connected, then $W$ is a cycle of $H$ (see [9, Thm. 2.2.3]). To understand when a planar embedding of $\hat{H}$ induces a 2-alternating label sequence, we study the possible boundaries of $H$. If $M$ is a block of $H$ that is not an edge, then a boundary of $H$ induces a boundary cycle in $M$.

A sequence $R = v_1, \ldots, v_k$ of consecutive vertices on a boundary $Q$ is called an $X$-block in $Q$ if no vertices in $R$ except possibly the endvertices $v_1$ and $v_k$ are labelled with $Y$. Define a $Y$-block similarly.

The following lemma states the observation that, if two $X$-blocks contain all vertices that are labelled $X$, then it is easy to construct a 2-alternating embedding of $\hat{H}$. In this case, we say that the labels $X$ are covered by the two $X$-blocks.

**Lemma 18.** Let $H$ be an XY-labelled graph, $Q$ a boundary of $H$, and $A \in \{X, Y\}$. If the $A$-labelled vertices of $H$ are covered by two $A$-blocks in $Q$, then $H$ is 2-alternating.

For $A \in \{X, Y\}$, an induced subgraph $H'$ of $H$ contains the label $A$ if there is a vertex in $H'$ labelled $A$. Let $S = A_1 \ldots A_k$ be a label sequence. Here we consider $S$ as a linear label sequence as opposed to cyclic. Let $R$ be a subsequence of a boundary of $H$. We say that $R$ contains the label sequence $S$ if there are distinct vertices $v_1, \ldots, v_k$ that appear in $R$ in this order (or the reverse order) and $v_i$ is labelled $A_i$ for $i = 1, \ldots, k$. We say that $H'$ contains the label sequence $S$ if for every boundary $Q$ of $H$, the subsequence of $Q$ induced by $V(H')$ contains the label sequence $S$. Let $B$ be a block of $H$ and $v$ a vertex of $B$. We say that label $A$ is attached to $B$ at $v$ if either $v$ is labelled $A$ or there is a $v$-bridge in $H$ not containing $B$ that contains $A$. 

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Lemma 19. Let $H$ be an $XY$-labelled graph such that at most four vertices of $H$ have both labels $X$ and $Y$. If $H$ is not 2-alternating, then $H$ contains the label sequence $XYXYXY$.

Proof. Suppose that $H$ is not 2-alternating and let $Q$ be a boundary of $H$. Let $R$ be a subsequence of $Q$ with no unlabelled vertices such that each labelled vertex appears in $R$ exactly once. A stronger claim is proved instead. If $R$ does not contain the label sequence $XYXYXY$, then the labels of vertices in $R$ can be arranged in the order given by $R$ to obtain a 2-alternating sequence of labels. Suppose that this is not true and choose a counter-example $R$ with minimum total number of labels.

Suppose there are cyclically consecutive vertices $u$ and $v$ in $R$ such that both $u$ and $v$ have label $A$ and $v$ has only one label. By deleting $A$ from $u$ we obtain a sequence $R'$ with smaller total number of labels. By the construction of $R'$, $R'$ does not contain the label sequence $XYXYXY$. Thus there is a 2-alternating label sequence $S'$ of labels in $R'$. By inserting the label $A$ before the occurrence of $A$ at $v$, we obtain a valid 2-alternating label sequence for $R$. Therefore, every two consecutive vertices in $R$ have either distinct labels or both labels.

Two cases remain: Either $R$ contains at most four labelled vertices, all with both labels, or there are at most four vertices that have alternating labels (six vertices give the label sequence $XYXYXY$ and five vertices are not possible because of parity). In both cases, we see immediately that the labels in $R$ can be arranged into a 2-alternating label sequence.

For graphs in $\mathcal{F}_{xy}^1$, Lemma 4 gives the following result.

Corollary 20. Let $G \in \mathcal{F}_{xy}^1$ and let $\{u, v\}$ be a 2-vertex-cut in $G$. If $C$ is a non-trivial $uv$-bridge such that $C + uv$ is planar, then $C - u - v$ contains a terminal.

The following lemma describes the structure of a graphs in $\mathcal{F}_{xy}^1$ when the $XY$-labelled graph is disconnected.

Lemma 21. Let $G$ be a graph in $\mathcal{F}_{xy}^1$ such that $G/xy$ is planar and let $H$ be the $XY$-labelled graph corresponding to $G$. If $H$ is disconnected, then $G$ is an $xy$-sum of two Kuratowski graphs and $xy \notin E(G)$ (this yields precisely six non-isomorphic graphs; see Fig. 4).
Figure 4: The two-sums of Kuratowski graphs.

Figure 5: Kuratowski graphs as an alternating extension to outerplanar graphs.

Proof. Each $xy$-sum of two Kuratowski graphs (without the edge $xy$ even if it is present in a summand) is a projective planar obstruction (see [1]) and it is straightforward to check that if belongs to $\mathcal{F}_{xy}^1$. Fig. 5 shows the three possible $XY$-labelled blocks that arise.

Since $H$ is disconnected, $G$ has at least two non-trivial $xy$-bridges $C_1$ and $C_2$. Since neither $C_1 - x - y$ nor $C_2 - x - y$ contains a terminal, Corollary 20 gives that both $C_1 + xy$ and $C_2 + xy$ are non-planar. Hence $G$ contains an $xy$-sum of two Kuratowski graphs as a minor.

□

5 Connectivity 2

This section is devoted to the proof of the following lemma characterizing graphs in $\mathcal{F}_{xy}^1$ that correspond to a 2-connected $XY$-labelled graph.

Lemma 22. Let $G$ be a graph in $\mathcal{F}_{xy}^1$ such that $G/xy$ is planar and such that the $XY$-labelled graph $H$ corresponding to $G$ is 2-connected. If $xy \in E(G)$, then $H$ is one of the graphs in Fig. 6. Otherwise, $H$ is one of the graphs in Fig. 7.
First, we derive two lemmas that will be used in the proof of Lemma 22.

**Lemma 23.** Let $G$ be a graph that consists of a cycle $C$ and $C$-bridges $B_1, B_2$ such that all other $C$-bridges avoid each other. If $G$ is non-planar, then there is a $C$-bridge $B$ (different from $B_1$ and $B_2$) such that $B$, $B_1$, and $B_2$ all pairwise overlap.

**Proof.** Let $B$ be the set of $C$-bridges in $H$ different from $B_1$ and $B_2$. Since the bridges in $B$ avoid each other, $B$ forms an independent set in the bridge-overlap graph $H$ of $B \cup \{B_1, B_2\}$. Since $G$ is non-planar, Theorem 3 asserts that $H$ is non-bipartite and thus contains an odd cycle. Since every edge in $H$ is incident with $B_1$ or $B_2$, this odd cycle is a triangle that consists of $B_1$, $B_2$ and a bridge $B \in B$. \hfill \Box

**Lemma 24.** Let $H$ be an $XY$-labelled planar graph that consists of an $XY$-labelled cycle $C$ and a $C$-bridge $B$. Let $C[w_1, w_2]$ be a segment of $C$ that contains all attachments of $B$. If $C$ contains all labels of $H$ and the graph with terminals corresponding to $H$ is non-planar, then $C(w_1, w_2)$ contains both labels.
Proof. Let $G$ be the graph with terminals $x, y$ corresponding to $H$. Let $B_x$ and $B_y$ be the $C$-bridges that contain $x$ and $y$, respectively. Since $C$ contains all labels of $H$, $B_x$ and $B_y$ are stars attached only to $C$. By Lemma 23, the bridges $B$, $B_x$ and $B_y$ pairwise overlap. Theorem 2 implies that, for each $z \in \{x, y\}$, either

(i) there are disjoint crossing paths $P_1$ in $B$ and $P_2$ in $B_z$, or

(ii) the bridges $B$ and $B_z$ have three vertices of attachment in common.

Let $Z$ be the label corresponding to the vertex $z$. When (i) holds, $C(w_1, w_2)$ contains one of the endvertices of $P_2$ and thus contains the label $Z$. When (ii) holds, each attachment of $B$ is labelled $Z$. Since $C(w_1, w_2)$ contains at least one of the attachments of $B$, $C(w_1, w_2)$ contains the label $Z$. Therefore, $C(w_1, w_2)$ contains both labels $X$ and $Y$ as claimed. \hfill $\Box$

Proof of Lemma 22. Let $C$ be a boundary cycle of $H$ and $\Pi$ the corresponding planar embedding of $H$.

Suppose that the edge $xy$ is present in $G$. By Lemma 19, either $C$ contains the label sequence $XYXYXY$, and then $H$ has HEXAGON as a minor, or there are five vertices in $C$ with both labels, and then $H$ has PENTAGON as a minor.

Therefore, we may assume that the edge $xy$ is not present in $G$. Let us consider the $C$-bridges $B_x$ and $B_y$ in $G$ that are the stars with centers $x$ and $y$, respectively. We may assume that, in $\Pi$, $C$ is the boundary of the infinite face. By Lemma 23 there is a $C$-bridge $B$ such that $B$, $B_x$, and $B_y$ pairwise overlap.

Let us first consider the case when $C$ does not contain the label sequence $XYXYXY$. By Lemma 19 $C$ contains five vertices with both labels. Let $v_1, \ldots, v_5$ be the vertices with both labels. We may assume by symmetry that an attachment of $B$ lies in $C(v_1, v_3)$. If there is an attachment of $B$ in the segment $C(v_3, v_1)$, then $H$ has ROCKET as a minor. Otherwise, all attachments of $B$ are in the segment $C[v_1, v_3]$. Let $S$ be the support of $B$ in $C[v_1, v_3]$. By Lemma 24, the segment $S$ (excluding the endvertices of $S$) contains both labels. Thus $H$ has ROCKET as a minor.

Now, assume that $C$ contains the label sequence $XYXYXY$ and let $v_1, \ldots, v_6$ be the vertices manifesting that (so $X \in \lambda(v_1)$, $Y \in \lambda(v_2)$, etc.). Let $w_1, \ldots, w_k$ be the attachments of $B$. Note that $k \geq 2$. By symmetry, we may assume that $w_1$ lies in the segment $C(v_1, v_3)$. 

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If all attachments of $B$ lie in $C[v_1, v_3]$, then the support $S$ of $B$ in $C[v_1, v_3]$ (excluding the endvertices of $S$) contains both labels by Lemma 24. Thus $H$ has BULLET as a minor. Hence we may assume that not all attachments of $B$ are in $C[v_1, v_3]$ and similarly in $C[v_2, v_4]$ and so on. If there is an attachment of $B$ in the segment $C(v_4, v_6)$, then $H$ has FROG as a minor. Hence we may assume that all attachments of $B$ lie in the segment $C[v_6, v_4]$.

By using reflection symmetry exchanging $v_1, v_3$ and $v_4, v_6$, since not all attachments of $B$ are in $C[v_1, v_3]$, there is an attachment $w_2$ of $B$ in the segment $C(v_3, v_4)$. By the same argument as above, there is no attachment of $B$ in $C(v_6, v_2)$. Since not all attachments of $B$ are in $C[v_2, v_4]$, the vertex $v_6$ is an attachment of $B$. We conclude that $H$ has HIVE as a minor.

\section{Connectivity 1}

In this section, we describe all obstructions in $\mathcal{F}^1_{xy}$ that correspond to an XY-labelled graph of connectivity 1.

\textbf{Lemma 25.} Let $G$ be a graph in $\mathcal{F}^1_{xy}$ such that $G/xy$ is planar and let $H$ be the XY-labelled graph corresponding to $G$. If $H$ has connectivity 1, then $H$ is one of the graphs in Fig. 8. Furthermore, $xy \notin E(G)$.

The following observation is useful.

\textbf{Lemma 26.} Let $G$ be a graph and $uvw$ be a triangle in $G$. If $u$ has degree 3 in $G$, then every embedding of $G − vw$ into a surface can be extended into an embedding of $G$ into the same surface.

\textit{Proof.} Let $H$ be the graph obtained from $G − vw$ by subdividing the edge incident to $u$ that is not in the triangle $uvw$. Then $G$ is the graph obtained from $H$ by applying a $\Delta$-operation on $u$. The result follows.

For graphs in $\mathcal{F}^1_{xy}$, Lemma 26 has the following consequence.

\textbf{Corollary 27.} Let $G \in \mathcal{F}^1_{xy}$ and $uvw$ be a triangle in $G$. If $u$ has degree at most 3 in $G$, then $u$ is a terminal.

\textit{Proof.} Since $G − vw \in \mathcal{A}^1_{xy}$, either $G − vw$ is planar or $G − vw$ is $xy$-alternating on $S_1$. By Lemma 26, the first outcome is not possible since then $G$ would be planar. In the second case, Lemma 26 shows that the $xy$-alternating embedding of $G − vw$ can be extended into an embedding of $G$ in $S_1$ by
embedding \( vw \) along the path \( vuw \). This extension would be \( xy \)-alternating if \( u \notin \{ x, y \} \). Thus, \( u \) is one of the terminals.

The next lemma will be used throughout the rest of the paper.

**Lemma 28.** Let \( H \) be an \( XY \)-labelled graph that has distinct blocks \( B_1 \) and \( B_2 \). Suppose that each of \( B_1 \) and \( B_2 \) contains both labels \( X \) and \( Y \) on vertices that do not belong to another block. Let \( G \) be the graph with terminals corresponding to \( H \). If \( H \) is not 1-alternating, then \( G \) is non-planar.

**Proof.** Suppose for contradiction that \( G \) is planar and take a planar embedding \( \Pi \) of \( G \). If \( x \) and \( y \) are cofacial in \( \Pi \), then \( \Pi \) gives a 1-alternating embedding of \( \hat{H} \). If \( x \) and \( y \) are not cofacial in \( \Pi \), then there is a cycle \( C \) in \( H \) that separates \( x \) and \( y \) (since \( x \) and \( y \) lie inside different faces of the induced embedding of \( H \)). Since \( C \) is a cycle of \( H \), it intersects either \( B_1 \) or \( B_2 \) in at most one vertex. Say, \( B_1 \) shares at most one vertex with \( C \) and is embedded on the other side of \( C \) than \( x \) is. By assumption, there is a vertex \( v \in V(B_1) \setminus V(C) \) that is labelled \( X \). Clearly, \( v \) and \( x \) are not cofacial in \( \Pi \) since they are separated by \( C \). But \( v \) and \( x \) are adjacent and thus cofacial in \( \Pi \), a contradiction. \( \square \)

Let \( C \) be a block in a graph \( G \). The \( C \)-bridge set \( B_v \) at a vertex \( v \) of \( C \) is the union of all \( C \)-bridges in \( G \) that are attached to \( v \). The following lemma asserts several properties of \( H \) and its labels and it is used to classify the graphs of connectivity 1 in \( F_{xy}^1 \).

**Lemma 29.** Let \( G \) be a graph in \( F_{xy}^1 \) such that \( G/xy \) is planar and the corresponding \( XY \)-labelled graph \( H \) has connectivity 1. Then the following statements hold.

(S1) Vertices of degree at most 2 in \( H \) are labelled. Leaves in \( H \) have both labels.

(S2) If \( B \) is an endblock of \( H \), and \( v \) is a cutvertex that separates \( B \) from the rest of \( H \), then the graph \( B - v \) contains both labels.

(S3) Let \( M \) be a block of \( H \) that is not an edge and \( C \) a boundary cycle of \( M \). Let \( B \) be the subgraph of \( M \) that consists of \( C \)-bridges in \( M \). If \( B \) is non-empty, then \( H - B^o \) is not 2-alternating.

(S4) Each block of \( H \) is either an edge or a cycle.
Let $u$ be a vertex of degree 2 in $H$. If $u$ has only one label, then the neighbors of $u$ are not labelled by $\lambda(u)$. In particular, if $P$ is a path in $H$ such that each vertex of $P$ has degree 2 in $H$, then either each vertex of $P$ has both labels or each vertex of $P$ has precisely one label that is different from the labels of its neighbors.

The neighbor of a leaf in $H$ is unlabelled.

Let $C$ be a cycle of $H$ and $T$ a $C$-bridge set that is a tree. If $H$ consists of at least three blocks, then $T$ contains at least two leaves of $H$.

Let $B$ be a block of $H$ that is a triangle and $v$ a vertex of $B$. If $v$ is not a cutvertex, then it has both labels. Otherwise, both labels are attached to $B$ at $v$.

Proof. Each property is proved separately.

(S1): Vertex of degree 2 in $H$ with no label would be a vertex of degree 2 in $G$. Similarly, a vertex of degree 1 with at most one label would be a vertex of degree at most 2 in $G$.

(S2): Let $B$ be an endblock of $H$ and $v \in V(B)$ the cutvertex that separates $B$ from the rest of the graph. If $B$ is an edge, then the result follows from (S1). Suppose for contradiction that $B - v$ does not contain the label $Y$. Since $G/xy$ is planar, $B$ is either a planar block of $G$ or $B$ is in an $xv$-bridge $C$ of $G$ such that $C + xv$ is planar. Corollary 20 asserts that this cannot happen in $G$.

(S3): Suppose $B$ is non-empty and $\Pi$ is a 2-alternating embedding of $\hat{H} - B^o$ in the plane. Suppose that there is an edge $e$ of $H - B^o$ with one end $v$ in $C$. By construction of $\hat{H} - B^o$, $e$ lies in a different $v$-block $B$ of $H$ than $C$. By (S2), there is a vertex $u$ in $B$ labelled $X$. Thus there is a path $P$ in $\hat{H} - B^o$ that connects $v_{xy}$ and $v$ and is internally disjoint from $C$. It follows that $e$ is embedded on the same side of $C$ in $\Pi$ as $x$ and $y$. We conclude that $C$ is a $\Pi$-face. By construction of $C$, $\Pi$ can be extended to a 2-alternating embedding of $\hat{H}$ by embedding $B$ inside $C$ — a contradiction.

(S4): Let $M$ be a block of $H$ that is neither a cycle nor an edge. Let $C$ be a boundary cycle of $M$ and $B$ the subgraph that consists of $C$-bridges in $M$. By (S3), $G - B^o$ is not $xy$-alternating on the torus. By (S2), $H - B^o$ contains two endblocks that contain both labels. By Lemma 28, $G - B^o$ is non-planar, a contradiction with $G - B^o \in A^1_{xy}$. 

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(S5): By (S1), u is labelled, say by X. If v is a neighbor of u with label X, then u is a vertex of degree 3 in the triangle uvx which is not possible by Corollary 27 unless u is also labelled Y.

(S6): Let v be a leaf and u its neighbor. If u is labelled, say with label X, then v is a vertex of degree 3 in the triangle vxu which is not possible by Corollary 27.

(S7): Let C be a cycle and T be a C-bridge set that is a tree. Assume that H has at least 3 blocks and that T contains only one leaf. We see that T is a path and, by (S6) and (S1), it is a path of length 1. Contract T to C to get H'. Let G' be the graph corresponding to H'. By the choice of G, G' is either xy-alternating on the torus or planar. Since H either contains 3 endblocks or two disjoint endblocks, if G' is not xy-alternating on the torus, then Lemma 28 gives that G' is non-planar. Hence G' is xy-alternating on the torus. Let Π be a 2-alternating embedding of H' in the plane. Uncontract T to get a 2-alternating embedding of H — a contradiction.

(S8): Let v be a vertex in a triangle C with at most one label. If v is not a cutvertex, then v is has degree at most 3 in G. By Corollary 27, this is a contradiction. If v is a cutvertex, then there is a v-bridge B' that does not contain C. Since B' contains an endblock of H, (S2) implies that B' contains both labels. These labels are attached to B at v. □

We use the structural properties from Lemma 29 to prove Lemma 25.

Proof of Lemma 25. Let G and H be as in the statement of the lemma. Our goal is to show that H has one of the graphs from Fig. 8 as a minor.

If H has at least five leaves, then all leaves are labelled X and Y, by (S1). Since H is connected, H has STAR as a minor. We assume henceforth that H has at most four leaves.

By (S4), every block of H that is not an edge is a cycle. We split the discussion according to the number of cycles in H.

Case 1: H is acyclic.

Suppose H has k leaves w1, . . . , wk, where k ≤ 4. Let u1, . . . , uk be their neighbors (possibly not distinct). By (S6) and (S1), vertices ui (where i = 1, . . . , k) have no labels and are of degree at least 3. By a counting argument, there are at most two such vertices in H. If there is only one vertex u of degree at least 3, H is a star with center u and thus H is a proper minor of STAR and hence G is in A1xy. Thus, there are two of them, say u1
Figure 8: The $XY$-labelled graphs of connectivity 1 that correspond to graphs in $\mathcal{F}^1_{xy}$. 
and $u_2$, and they are connected by a path $P$. If $P$ contains both labels $X$ and $Y$, then $H$ has SADDLE as a minor. If $P$ contains at most one of the labels, say $X$, then the two pairs of leaves are covered by two $Y$-blocks and thus $G$ is in $A_{xy}^1$ by Lemma 18 — a contradiction.

**Case 2:** $H$ has precisely one cycle $C$.

Since $C$ is the only cycle in $H$, every $C$-bridge is a tree attached to a vertex of $C$. The proof is split according to the number of leaves of $H$. Note that $H$ has at least one leaf since $H$ is not 2-connected.

**Subcase i:** $H$ has precisely four leaves.

If $C$ is an endblock, then a single $C$-bridge set $B_v$ contains all four leaves $w_1, \ldots, w_4$. By (S2), $C - v$ contains both labels. Therefore, $H$ has STAR as a minor.

Otherwise, by (S7), there are precisely two non-trivial $C$-bridge sets $B_{v_1}$ and $B_{v_2}$, and each contains two leaves. Hence each of $B_{v_1} - v_1$ and $B_{v_2} - v_2$ contains at most one vertex of degree 3 in $H$. When $B_{v_1} - v_1$ contains a vertex of degree 3, let $u_1$ be this vertex. Otherwise, let $u_1 = v_1$. Define $u_2$ similarly. Note that $u_1$ and $u_2$ are unlabelled by (S6) and (S1). If there is a path $P$ in $H$ connecting $u_1$ and $u_2$ and both labels $X$ and $Y$ appear on $P$, then $H$ has SADDLE as a minor. Let $P_1$ and $P_2$ be the two paths in $C$ connecting $v_1$ and $v_2$. If $P_1$ contains $X$ and $P_2$ contains $Y$ (or vice versa), then $H$ has RIBBON (or SADDLE) as a minor. Otherwise, there is a label missing from $H - \{w_i : i = 1, \ldots, 4\}$, say $X$, so the leaves are covered by two $X$-blocks. Lemma 18 implies that $G \in A_{xy}^1$, a contradiction.

**Subcase ii:** $H$ has precisely three leaves.

By (S7), there is a single $C$-bridge set $B_v$ that contains all three leaves. Suppose $C$ is a triangle. By (S8), both vertices of $C$ different from $v$ have both labels and $H$ contains STAR as a minor.

Suppose $C$ has length at least 4. By (S5), $C - v$ contains the label sequence $XYX$ or $YXY$. Thus $H$ has TRIPOD as a minor.

**Subcase iii:** $H$ has precisely two leaves.

By (S7), there is a single $C$-bridge set $B_v$ that contains both leaves. Let $u$ be a vertex of degree 3 in $B_v - v$ if there is one and let $u = v$ otherwise. Let $P$ be the path from $u$ to $v$, possibly of zero length.

Suppose $C$ is a triangle. Again by (S8), both vertices of $C$ different from $v$ have both labels. If $P$ contains both labels, then $H$ has ALIEN as a minor (by (S6)). Thus $P$ contains at most one label, say $X$, and then labels $Y$
are covered by two $Y$-blocks, one at the leaves and one on the triangle. By Lemma 18, $G \in \mathcal{A}_{xy}^1$, a contradiction.

Suppose $C$ has length at least 4. If all vertices in $C - v$ have both labels, then $H$ has FOUR as a minor. If $C - v$ contains the label sequence $XYXY$, then $H$ has FIVE as a minor. Otherwise, (S5) implies that $C$ has length 4 and $C - v$ form the label sequence $YXY$ or $XYX$, say the former. If $P$ contains $X$, then $H$ has HUMAN as a minor. Otherwise, the labels $X$ are covered by two $X$-blocks, one at the leaves and one covering the label $X$ at $C$ — a contradiction by Lemma 18.

**Subcase iv:** $H$ has precisely 1 leaf.

Let $w$ be this leaf and $u$ its neighbor. By (S6) and (S1), $u$ is unlabelled vertex of degree at least 3 and thus lies on $C$. If $C$ has length at most 5, then $H$ contains five vertices with both labels, by Lemma 19. Thus $H$ is isomorphic to Lollipop. If $C$ has length at least 6 (and, then $H$ has MIRROR as a minor, by (S5)).

**Case 3:** $H$ has (at least) two cycles, $C_1$ and $C_2$.

Pick $C_1$ and $C_2$ such that, first, the distance between them is maximal and, second, their size is maximal. By (S4), $C_1$ and $C_2$ are blocks of $H$ that share at most one vertex. Let $P$ be a shortest path (possibly of zero length) joining vertices $v_1 \in V(C_1)$ and $v_2 \in V(C_2)$. Note that by the choice of $C_1$ and $C_2$, all $C_1$-bridges attached to $C_1 - v_1$ and all $C_2$-bridges attached to $C_2 - v_2$ are trees.

**Subcase i:** $C_1$ and $C_2$ are triangles.

Suppose there is more than one $C_1$-bridge at $v_1$ and let $B$ be a $C_1$-bridge at $v_1$ not containing $P$. By (S2), $B$ contains both labels. By (S8), all vertices of $C_1 - v_1$ and $C_2 - v_2$ have both labels attached. Thus $H$ has STAR as a minor. So we may assume that there is only one $C_1$-bridge attached at $v_1$. Similarly, there is only one $C_2$-bridge attached at $v_2$.

If there is a $C_1$-bridge attached to a vertex $v$ of $C_1 - v_1$, then the $C_1$-bridge set at $v$ is a tree containing at least two leaves by (S7). This implies that $H$ has STAR as a minor. Thus there are no $C_1$-bridges attached to $C_1 - v_1$. The same holds for $C_2$ by symmetry.

If the component $M$ of $H - E(C_1) - E(C_2)$ containing $P$ has both labels, then $H$ has BOWTIE as a minor. Suppose to the contrary that $M$ has at most one label, say $X$. Since there are no other bridges attached to $C_1$ and $C_2$, the $Y$-labelled vertices of $H$ are covered by two $Y$-blocks, a contradiction by Lemma 18.
Subcase ii: $C_1$ is a triangle and $C_2$ has length at least 4.

If $H$ contains four leaves, then it is not difficult to check that $H$ has Star as a minor. Hence there is at most one non-trivial bridge set attached to $C_1 - v_1$ or $C_2 - v_2$ (by (S7)). Suppose that there is a $C_2$-bridge set $B$ attached to a vertex $v$ in $C_2 - v_2$. By (S7), $B$ contains at least two leaves. If $B$ contains three leaves, then $H$ has Star as a minor. Therefore, $B$ has precisely two leaves $w_1, w_2$. Let $M$ be the component of $H - E(C_1) - w_1 - w_2$ containing $P$. By using (S6), it is easy to see that, if $M$ contains both leaves, then $H$ has ALIEN as a minor. Otherwise, $M$ has at most one label, say $X$. Thus labels $Y$ are covered by two $Y$-blocks, one at $C_1 - v_1$ and the other at $w_1, w_2$. A contradiction by Lemma 18.

Therefore, there is no $C_2$-bridge attached to $C_2 - v_2$. By (S5), $C_2 - v_2$ either contains the sequence $YXY$ or $XYX$, say the former. Suppose there is a $C_1$-bridge $B$ attached at $C_1 - v_1$. By (S7), $B$ has at least two leaves. Hence $H$ has Tripod as a minor. Therefore, there is no $C_1$-bridge attached at $C_1 - v_1$ and both vertices in $C_1 - v_1$ have both labels.

Let $M$ be the component of $H - E(C_1) - E(C_2)$ containing $P$. If $M$ contains $X$, then $H$ has Doll as a minor. Hence $M$ contains at most one label, say $X$. Hence all $Y$ labels are at the leaves and can be covered by two $Y$-blocks. A contradiction by Lemma 18. If there are two non-trivial bridge sets $B_1, B_2$ attached to one of the cycles, say to $C_1$, then $B_1$ and $B_2$ contain together four leaves. By (S2), there are both labels attached to a vertex of $C_2 - v_2$. Hence $H$ has Star as a minor.

Therefore, there is at most one non-trivial bridge set attached to $C_1 - v_1$.
and $C_2 - v_2$. Suppose there is a $C_1$-bridge set $B$ attached to a vertex $v$ in $C_1 - v_1$. By (S5), $C_2 - v_2$ contains the label sequence $YXY$ or $XYX$, say the former. If $B$ contains at least three leaves, then $H$ has Tripod as a minor. By (S7), $B$ has precisely two leaves $w_1, w_2$. If $C_2$ has length at least 5, then $C_2$ contains the sequence $XYXY$, by (S5). Hence $H$ has Five as a minor. If $C_2$ contains three vertices with both labels, then $H$ has Four as a minor. If $H - w_1 - w_2 - (C_2 - v_2)$ contains label $X$, then $H$ has Human as a minor. Otherwise, the $X$ labels at $C_2 - v_2$ can be covered by a single $X$ block and all other $X$ labels are at $w_1, w_2$ which are covered by a second $X$ block. By Lemma 18, $H$ is 2-alternating, a contradiction.

By symmetry of $C_1$ and $C_2$, we conclude that there are no non-trivial bridge sets attached to $C_1 - v_1$ and $C_2 - v_2$. By (S5), $C_2 - v_2$ contains the label sequence $YXY$ or $XYX$, say the former. By (S5), $C_1 - v_1$ contains the label sequence $YXY$ or $XYX$. If $C_1 - v_1$ contains the sequence $XYX$, then $H$ has Pinch as a minor. Thus $C_1 - v_1$ contains the sequence $YXY$. If $C_2 - v_2$ contains the sequence $XYX$, then $H$ has Pinch as a minor. Let $M$ be the component of $H - E(C_1) - E(C_2)$ that contains $P$. If $M$ contains label $X$, then $H$ has Extra as a minor. Otherwise, the labels $X$ can be covered by two $X$-blocks, one at $C_1 - v_1$ and one at $C_2 - v_2$. A contradiction by Lemma 18.

7 The main theorem

The previous lemmas give rise to the following theorem.

**Theorem 30.** Let $G$ be a graph in $F_{xy}^1$. Then one of the following holds:

(i) $G$ is a split of a Kuratowski graph with $x$ and $y$ being the two vertices resulting after the split (see Fig. 3) or $G$ is a Kuratowski graph plus one or two isolated vertices that are terminals.

(ii) $G$ is an $xy$-sum of two Kuratowski graphs (see Fig. 4).

(iii) $G$ corresponds to one of the $XY$-labelled graphs in Fig. 6, 7, or 8.

**Proof.** By Lemma 17, either (i) holds or $G/xy$ is planar. In the latter case, let $H$ be the $XY$-labelled graph that corresponds to $G$. We will now show that $H$ contains one of these graphs as a minor. If $H$ is disconnected, then (ii) holds by Lemma 21. If $H$ is 2-connected, then $H$ is one of the graphs
in Fig. 6 or 7 by Lemma 22. Otherwise, $H$ is one of the graphs in Fig. 8 by Lemma 25.

It is easy to see that none of the graphs in (i)–(iii) contains another one as a minor. Thus, in order to prove that each of them is in $F_{xy}^1$, it suffices to see that they are not in $A_{xy}^1$. This is clear for (i) since the graphs in (i) are non-planar after identifying $x$ and $y$. Similarly, graphs in (ii) cannot be in $A_{xy}^1$ since they do not have an embedding in the projective plane. Finally, graphs in (iii) are not in $A_{xy}^1$ since their corresponding $XY$-labelled graphs are not 2-alternating.

Note that the edge $xy$ is present in a graph $G \in F_{xy}^1$ if and only if $G - xy$ is planar. There are only five graphs in $F_{xy}^1$ with the edge $xy$, the three splits of Kuratowski graphs (see Fig. 3) and the two graphs in Fig. 6.

**Corollary 31.** All graphs in $F_{xy}^1$ embed into the torus.

**Proof.** By Theorem 30, graphs in $F_{xy}^1$ are of three types, (i)–(iii). For graphs in (i) and (ii), embeddings in the torus are easily constructed. The graphs in (iii) are 3-alternating and thus have a planar embedding with three $X$-blocks covering the $X$-labels. This embedding can be extended to an embedding in the torus by adding a single handle; see Fig. 9 where the $X$-blocks are shown as thick intervals on the boundary of the planar part (and $Y$-blocks are shown by thick broken line).

\[ \square \]
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