LINEAR COLORINGS OF SIMPLICIAL COMPLEXES AND COLLAPSING

YUSUF CIVAN AND ERGÜN YALÇIN

Abstract. A vertex coloring of a simplicial complex $\Delta$ is called a linear coloring if it satisfies the property that for every pair of facets $(F_1, F_2)$ of $\Delta$, there exists no pair of vertices $(v_1, v_2)$ with the same color such that $v_1 \in F_1 \setminus F_2$ and $v_2 \in F_2 \setminus F_1$. We show that every simplicial complex $\Delta$ which is linearly colored with $k$ colors includes a subcomplex $\Delta'$ with $k$ vertices such that $\Delta'$ is a strong deformation retract of $\Delta$. We also prove that this deformation is a nonevasive reduction, in particular, a collapsing.

1. INTRODUCTION

In this paper, we introduce a notion of linear coloring of a simplicial complex as a special type of vertex coloring. Recall that a vertex coloring of an abstract simplicial complex $\Delta$ with vertex set $V$ is a surjective map $\kappa : V \to [k]$ where $k$ is a positive integer and $[k] = \{1, \ldots, k\}$. We say a vertex coloring is linear if it satisfies the condition given in the abstract. Alternatively, a coloring is linear if for every two vertices $u, v$ of $\Delta$ having the same color, we have either $F(u) \subseteq F(v)$ or $F(v) \subseteq F(u)$ where $F(u)$ and $F(v)$ denote the set of facets including $u$ and $v$ respectively. This is actually equivalent to requiring that the set $F_i = \{F(u) \mid \kappa(u) = i\}$ is linearly ordered for every $i \in [k]$, which explains the rationale for our terminology.

The condition for linear coloring appears naturally when the multicomplex associated to a colored simplicial complex is studied closely. For example, in Theorem 3.6 we show that if a simplicial complex is linearly colored then we can recover it by using the multicomplex associated to it. The multicomplex associated to a simplicial complex $\Delta$ is the multicomplex whose simplices are the color combinations of the simplices on $\Delta$. We believe that this association between simplicial complexes and multicomplexes could be very useful to study the combinatorial properties of multicomplexes although we do not investigate this direction in the present work.

Another consequence of requiring a coloring to be a linear coloring is that it gives us a natural deformation of the colored complex to a subcomplex of itself where the subcomplex has as many vertices as the number of colors used. In fact, we can obtain such a deformation on any subcomplex which satisfies the following condition: Given a simplicial complex $\Delta$ and a linear coloring $\kappa$ of $\Delta$ with $k$ colors, we call a subcomplex
Δκ ⊆ Δ a representative subcomplex if for each i ∈ [k] there is one and only one vertex v in Δκ with κ(v) = i, and if it has the property that for every pair of vertices u, v with the same color, we have F(u) ⊆ F(v) whenever u ∈ Δ and v ∈ Δκ. The main result of the paper is the following:

**Theorem 1.1.** Let Δ be a simplicial complex on V, and let κ : V → [k] be a k-linear coloring map. If Δκ is a representative subcomplex of Δ, then Δκ is a strong deformation retract of Δ.

This allows us to gain information on the homotopy type of a simplicial complex by coloring it linearly. For example it is clear that if a simplicial complex can be linearly colored using k colors then its integral (simplicial) homology will be zero for dimensions greater than k.

We also introduce the notion of LC-reduction by saying that a simplicial complex Δ LC-reduces to its subcomplex Δ′, denoted by Δ \(_{\text{LC}}\) Δ′, if there exist a sequence of subcomplexes Δ = Δ₀ ⊇ Δ₁ ⊇ ... ⊇ Δₜ = Δ′ such that for all 0 ≤ r ≤ t - 1, the subcomplex Δₙ₊₁ is a representative subcomplex of Δₙ with respect to some linear coloring κₙ of Δₙ. We study various questions arising from this definition. For example, we show that if X₁ \(_{\text{LC}}\) X₂ and Y is any simplicial complex, then X₁ * Y \(_{\text{LC}}\) X₂ * Y.

The main result about LC-reduction is the following:

**Theorem 1.2.** Let Δ be a simplicial complex and Δ′ be a subcomplex in Δ. If Δ LC-reduces to Δ′, then Δ NE-reduces to Δ′ (also called strong collapsing), in particular Δ collapses to Δ′.

In fact, Theorem 1.2 implies Theorem 1.1 but we still give a separate proof for Theorem 1.1 using the basic techniques of poset homotopy due to Quillen [9]. The reason for this is that we believe that Theorem 1.1 is interesting in its own right for understanding the topology of simplicial complexes and should have an independent proof accessible to a topologist. We view Theorem 1.2 as a combinatorial version of Theorem 1.1.

It turns out that LC-reduction is stronger than the NE-reduction and hence also stronger than collapsing. In Example 6.3, we provide an example of a nonevasive simplicial complex which is not LC-reducible to a point.

In the rest of the paper, we give some applications of LC-reduction. The first application we give is closely related to an a theorem by Kozlov [4] about monotone maps and NE-reduction. We prove that if φ : P → P is a closure operator on a finite poset P, then Δ(P) \(_{\text{LC}}\) Δ(φ(P)), and we conclude that, in this case, Δ(P) collapses to Δ(φ(P)). Our second application is related to graph coloring. We show that a linear coloring of the neighborhood complex of a graph gives a (vertex) coloring for the graph. So, the linear chromatic number of the neighborhood complex of a simple graph gives an upper bound for the chromatic number of the graph.

We organize the paper as follows: In Section 2 we give the definition of a linear coloring and its equivalent formulations to ease the computations. Then, in Section 3 we describe an association between linearly colored simplicial complexes and multicomplexes. The following three sections contain the main results of our work, where we describe the strong deformation of a simplicial complex induced by a linear coloring,
introduce the notion of LC-reductions, and discuss its connections with known combinatorial reduction methods such as nonevasive reduction and collapsing. In particular, we prove Theorem 1.1 in Section 4 and Theorem 1.2 in Section 6.

The last two sections are devoted to applications of LC-reduction. In Section 7, we consider linear colorings of order complexes of posets and prove the reduction theorem for closure operators. Finally, in the last section, we consider the linear colorings of neighborhood complexes associated to graphs.

2. Linear coloring of a simplicial complex

We start with some basic definitions related to multisets.

**Definition 2.1.** A *multiset* $M$ on a set $S$ is a function $M : S \to \mathbb{N} := \{0, 1, 2, \ldots\}$, where $M(s)$ is regarded as the number of repetitions of $s \in S$. We say that $s \in S$ is an *element* of $M$, and write $s \in M$, if $M(s) > 0$. The *cardinality* (or *size*) of a multiset $M$ is defined by $\|M\| := \sum_{s \in S} M(s)$.

Note that every multiset $M$ on $S$ can be regarded as a monomial on the set $S$ where the degree of $s \in S$ is equal to $M(s)$. The elements of $M(s)$ will be the elements of $s$ with nonzero degree, and the cardinality will be equal to the total degree of the monomial. The usual division relation on monomials gives rise to the definition of submultisets and the union and the intersection of multisets can be defined with the following formulas:

$$(M_1 \cup M_2)(s) = M_1(s) + M_2(s);$$

$$(M_1 \cap M_2)(s) = \min(M_1(s), M_2(s)).$$

Now we recall the definition of vertex coloring of a simplicial complex.

**Definition 2.2.** Let $\Delta$ be a finite (abstract) simplicial complex on $V$. Let $[k]$ denote the set $\{1, \ldots, k\}$. A surjective map $\kappa : V \to [k]$ is called a (vertex) coloring of $\Delta$ using $k$ colors.

Given a coloring $\kappa$ of a simplicial complex $\Delta$, we can associate a multiset to each of its faces as follows: If $S$ is a face of $\Delta$, then we define the multiset $S\kappa$ on $[k]$ by setting $S\kappa(t)$ equal to the order of the set $\{v \in S : \kappa(v) = t\}$ for each $t \in [k]$. We define the linear coloring in its most technical form as follows:

**Definition 2.3.** Let $\Delta$ be a finite abstract simplicial complex on $V$ and let $\mathcal{F}$ denote the set of all facets of $\Delta$. A surjective map $\kappa : V \to [k]$ is called a $k$-linear coloring of $\Delta$ if and only if $\|F\kappa \cap F'\kappa\| = |F \cap F'|$ for any two facets $F, F' \in \mathcal{F}$.

Note that if $\Delta$ is linearly colored with $\kappa$, then for distinct facets $F, F'$ of $\Delta$, the multisets $F\kappa$ and $F'\kappa$ must be also different. Otherwise, we would have $|F \cap F'| = |F| = |F'|$ which cannot happen since $F$ and $F'$ are distinct. We can rephrase this by saying that the color combinations (with multiplicities) used in different facets must be different.

Note that every complex with $n$ vertices can be linearly colored using $n$ colors by giving different color to each vertex. We call a linear coloring trivial if it is such a coloring.
**Definition 2.4.** The *linear chromatic number* of a simplicial complex $\Delta$, denoted by $\operatorname{lchr}(\Delta)$, is defined to be the minimum integer $k$ such that $\Delta$ has a $k$-linear coloring.

Since there is always the trivial linear coloring, the linear chromatic number of simplicial complex is well defined and it is less than or equal to the number of vertices of the complex.

**Definition 2.5.** Let $\Delta$ be a simplicial complex and let $\kappa$ be a $k$-linear coloring map. Define $V_i := \{v \in V \mid \kappa(v) = i\}$ and set $c^\kappa_i := \operatorname{card}(V_i)$ for each $i \in [k]$. Then, $\Delta$ is said to be a linear coloring of type $c^\kappa(\Delta) = (c^\kappa_1, \ldots, c^\kappa_k)$.

**Example 2.6.** In Figure 1(a), we illustrate a 2-dimensional simplicial complex admitting a 2-linear coloring of type $(3, 1)$, whereas Figure 1(b) shows linear coloring of type $(1, 1, 1, 1)$. Note that the complex in Figure 1(b) is a 1-dimensional complex with $\operatorname{lchr}(\Delta) = 4$. For the simplicial complex depicted in Figures 1(c) and 1(d), the map given at Figure 1(c) is a 4-linear coloring of type $(2, 1, 1, 2)$, while the coloring given in Figure 1(d) is not a linear coloring.

![Figure 1. Linear colorable complexes and a non-linear coloring](image)

To understand the definition of linear coloring better, we now give an equivalent condition for linear coloring. This is the same as the condition given in the abstract of the paper.

**Proposition 2.7.** Let $\Delta$ be a finite abstract simplicial complex on $V$. A coloring $\kappa: V \to [k]$ of its vertices is a $k$-linear coloring of $\Delta$ if and only if for every pair of facets $(F_1, F_2)$ of $\Delta$, there exists no pair of vertices $(v_1, v_2)$ with the same color such that $v_1 \in F_1 \setminus F_2$ and $v_2 \in F_2 \setminus F_1$.

**Proof.** In general $\| (F_1)_\kappa \cap (F_2)_\kappa \| \geq |F_1 \cap F_2|$ for every pair of facets $(F_1, F_2)$ of $\Delta$. So, the equality does not hold if and only if there is a pair of vertices $(v_1, v_2)$ with the same color such that $v_1 \in F_1 \setminus F_2$ and $v_2 \in F_2 \setminus F_1$. 

Note that the above condition for linear coloring can be rephrased as follows:
Proposition 2.8. Let $\Delta$ be a simplicial complex with vertex set $V$, and let $\kappa : V \to [k]$ be a coloring of $\Delta$. For every $v \in V$, let $F(v)$ denote the set of facets of $\Delta$ containing $v$. The coloring $\kappa$ is linear if and only if for every $i \in [k]$, the set $F_i = \{ F(v) : \kappa(v) = i \}$ is linearly ordered by inclusion.

Proof. Assume that $\kappa$ is a linear coloring. Let $v_1, v_2 \in V$ such that $\kappa(v_1) = \kappa(v_2)$. Suppose that there exist facets $F_1, F_2 \in F(v_1) \setminus F(v_2)$ and $F_2, F_1 \in F(v_2) \setminus F(v_1)$. Then, it is clear that $v_1 \in F_1 \setminus F_2$ and $v_2 \in F_2 \setminus F_1$. This contradicts with the fact that $\kappa$ is a linear coloring. So, either $F(v_1) \subseteq F(v_2)$ or $F(v_2) \subseteq F(v_1)$ holds. This shows that for each $i$, the set $F_i$ is linearly ordered by inclusion. It is clear that the converse also holds. □

We also have the following observation which will be used later in the paper.

Proposition 2.9. Let $\Delta$ be a simplicial complex with vertex set $V$, and let $\kappa : V \to [k]$ be a linear coloring of $\Delta$. Then, for each $i \in [k]$, there exists a facet $F$ which includes all the vertices $v \in V$ with $\kappa(v) = i$. On the other extreme, for each $i \in [k]$, there exists a vertex $v \in V$ such that $v$ lies on all the facets which include at least one vertex colored with the color $i$.

Proof. Take some $i \in [k]$. By Proposition 2.8, the set $F_i = \{ F(v) : \kappa(v) = i \}$ is linearly ordered by inclusion so there exists a vertex $v \in V$ such that $\kappa(v) = i$ and $F(v) \subseteq F(u)$ for every $u \in V$ with $\kappa(u) = i$. If we take $F \in F(v)$, then it is clear that $F$ will include all the vertices $u \in V$ with $\kappa(u) = i$. Note that on the other extreme, there is a vertex $v \in V$ such that $\kappa(v) = i$ and that $F(u) \subseteq F(v)$ for all vertices $u \in V$ with $\kappa(u) = i$. Then, $v$ is included in all the facets which include at least one vertex colored with the color $i$. □

3. Multicomplexes associated to linear colorings

In this section, we will discuss an association between multicomplexes and linearly colored simplicial complexes. We start with the definition of a multicomplex. More details on this material can be found in [2] and [11].

Definition 3.1. A multicomplex $\Gamma$ is a collection of multisets over a set $S$ such that if $M \in \Gamma$ and $M' \subseteq M$, then $M' \in \Gamma$. The elements of $\Gamma$ are usually called the faces of $\Gamma$.

Note that the faces of $\Gamma$ are ordered by inclusion, giving a lattice after adjoining a maximal element. We call the resulting lattice the face lattice of $\Gamma$ and denote it by $L(\Gamma)$. Every multiset $M$ includes a submultiset which is formed by all its elements with no repetitions. We denote this submultiset by $u(M)$ and call it the underlying set of $M$. If $M$ is a face of a multicomplex $\Gamma$, the underlying set $u(M)$ of $M$ is called the underlying face of $\Gamma$ with respect to $M$. We have the following simple observation:

Lemma 3.2. The collection of all underlying faces of a multicomplex $\Gamma$ is a simplicial complex. This simplicial complex is called the underlying simplicial complex of $\Gamma$ and denoted by $u(\Gamma)$.

Proof. Let $S = u(M)$ for some face $M$ of $\Gamma$ and $S' \subseteq S$. Then $S' \subseteq M$ as a multiset, so $S'$ must be a face of $\Gamma$. Since $S' = u(S')$, we have $S' \in u(\Gamma)$. □
Now, we consider complexes with a linear coloring.

**Proposition 3.3.** If $\Delta$ is a $k$-linear colored complex with coloring map $\kappa$, then the collection $\{S_\kappa : S \in \Delta\}$ of multisets is a multicomplex.

**Proof.** Let $M'$ be a submultiset of a $S_\kappa$ where $S$ is a simplex in $\Delta$. Then, it is clear that $S$ has a subset $S'$ such that $S'_\kappa$ is equal to $M'$.

**Definition 3.4.** Let $\Delta$ be a $k$-linear colored complex with coloring map $\kappa$. We call the multicomplex $\{S_\kappa : S \in \Delta\}$ the associated multicomplex of the couple $(\Delta, \kappa)$ and denote it by $\Gamma(\Delta, \kappa)$.

This gives us an assignment $(\Delta, \kappa) \to \Gamma(\Delta, \kappa)$ from the set of linearly colored simplicial complexes to multicomplexes. The following shows that this assignment is surjective.

**Proposition 3.5.** Given a multicomplex $\Gamma$ over $[k]$, there exists a simplicial complex $\Delta$ and a $k$-linear coloring map $\kappa : \Delta \to [k]$ such that $\Gamma = \Gamma(\Delta, \kappa)$.

**Proof.** Let $\Gamma$ be an arbitrary multicomplex over $[k]$. For each $i \in [k]$, let $n_i := \max\{M(i) : M \in \Gamma\}$ and let $V_i := \{a_{ir}^i : 1 \leq r \leq n_i\}$. We next define a simplicial complex $\Delta(\Gamma)$ on $V := \cup_{i=1}^k V_i$ as follows: We first associate a subset $S_M$ of $V$ to every multiset $M \in \Gamma$ by taking $a_{ir}^i$, $a_{js}^j$, $\ldots$, $a_{lt}^t \in S_M$ whenever $M(i) = j$ for any $i \in [k]$. Now, $\Delta(\Gamma)$ is the $k$-linear colorable simplicial complex generated by the subsets $F_M \subseteq V$ for which $M$ is a facet of $\Gamma$, and the linear coloring map $\kappa : V \to [k]$ of $\Delta(\Gamma)$ is given by $\kappa(a_{ir}^i) = i$ for all $i \in [k]$.

The construction given above gives us a unique simplicial complex associated to a multicomplex $\Gamma$. Let us denote this simplicial complex $\Delta(\Gamma)$. The following shows that the assignment $\Gamma \to \Delta(\Gamma)$ is, in fact, inverse to the assignment $(\Delta, \kappa) \to \Gamma(\Delta, \kappa)$.

**Theorem 3.6.** Let $\Delta$ be a simplicial complex on $V$, and let $\kappa : V \to [k]$ be a $k$-linear coloring. Suppose $\Gamma = \Gamma(\Delta, \kappa)$ is the multicomplex associated to the linear coloring $\kappa$ and let $\Delta(\Gamma)$ be the simplicial complex as in Proposition 3.5. Then, $\Delta(\Gamma)$ is isomorphic to $\Delta$.

**Proof.** One can show this using a delicate labeling technique. Note that the coloring $\kappa : V \to [k]$ gives a partitioning of $V = \cup_{i=1}^k V_i$ such that $V_i$ is the set of vertices colored by $i$. Let $n_i$ denote the number of elements in $V_i$ for each $i \in [k]$. As before let $F(v)$ denote the set of facets in $\Delta$ including $v$ as a vertex. Recall that by Proposition 2.8, for each $i \in [k]$, the set $F_i := \{F(v) : v \in V_i\}$ is linearly ordered by inclusion. We can label the vertices of $\Delta$ in the following way: Let $V = \{v_{ir}^i : i \in [k], r \in [n_i]\}$ where for all $i$, the vertex $v_{ir}^i$ belongs to $V_i$ and $F(v_{ir}^i) \subseteq F(v_{is}^i)$ whenever $1 \leq r \leq t \leq n_i$.

Recall that the simplicial complex $\Delta(\Gamma)$ on $V := \cup_{i=1}^k V_i$ is defined as follows. The subset $S_M$ of $V$ to every multiset $M \in \Gamma$ is defined by taking $a_{ir}^i$, $a_{js}^j$, $\ldots$, $a_{lt}^t \in S_M$ whenever $M(i) = j$ for any $i \in [k]$. Now, $\Delta(\Gamma)$ is the simplicial complex generated by the subsets $F_M \subseteq V$ for which $M$ is a facet of $\Gamma$.

We claim that the assignment $f : \Delta \to \Delta(\Gamma)$ defined by $f(v_{ir}^i) = a_{ir}^i$ for every $i \in [k]$ and $r \in [n_i]$ is an isomorphism of simplicial complexes. To prove this claim, it is enough
to show that $S$ is a simplex in $\Delta$ if and only if $f(S)$ is a simplex in $\Delta(\Gamma)$. Note that we can prove each direction starting with a facet. Let $F$ be a facet in $\Delta$. To show that $f(F)$ is a simplex in $\Delta(\Gamma)$, we need to show that $F$ satisfies the property that if $v_i^r \in F$, then $v_i^t$ is in $F$ for every $1 \leq r \leq t$. This follows from the fact that $F(v_i^r) \subseteq F(v_i^t)$ for every $1 \leq r \leq t \leq n_i$. So, $f(F) \in \Delta(\Gamma)$ as desired. For the other direction, let $F$ be a facet in $\Delta(\Gamma)$, and let $M$ be the corresponding face in $\Gamma$. Then, there is a facet $F'$ in $\Delta$ such that for each $i \in [k]$, a vertex from $V_i$ appears exactly $M(i)$ times. Recall that the facets of $\Delta$ satisfy the property that if $v_i^t$ is in a facet, then $v_i^r$ is also in that facet for every $1 \leq r \leq t$. So, we can conclude that $F' = f^{-1}(F)$, and hence $f^{-1}(F)$ is in $\Delta$. This completes the proof. □

This shows, in particular, that we can recover a linearly colored simplicial complex from its associated multicomplex. Another way to state the above result is the following:

Corollary 3.7. Let $\Delta$ be a simplicial complex on $V$, and let $\kappa : V \to [k]$ be a $k$-linear coloring. For each $i \in [k]$, let $V_i := \{ v \in V : \kappa(v) = i \}$ and let $n_i = |V_i|$. Then, we can label the vertices of $\Delta$ in such a way that $V = \{ v_i^r : i \in [k], r \in [n_i] \}$ and that whenever $v_i^r$ is in a facet of $\Delta$, then $v_i^t$ is also in that facet for every $1 \leq r \leq t$.

The labeling technique given in the above corollary can also be used to produce some poset maps between the face posets of the simplex $\Delta$, the associated multicomplex $\Gamma(\Delta, \kappa)$, and the underlying simplicial complex $u(\Gamma(\Delta, \kappa))$. We now explain these.

Lemma 3.8. Let $\Delta$ be a simplicial complex linearly colored with a coloring map $\kappa : V \to [k]$, and let $\Gamma$ be the associated multicomplex. Then the map
\[
c : \Delta \to \Gamma
\]
defined by $S \to S_\kappa$ for every $S \in \Delta$ is a poset map (between corresponding face posets).

Proof. This is clear since for every $S' \subseteq S$, the number of times a color used in $S'$ is less than or equal to the number of times it is used in $S$. □

We also have the following:

Lemma 3.9. Let $\Gamma$ be a multicomplex, and let $u(\Gamma)$ denote its underlying simplicial complex. The canonical map
\[
u : \Gamma \to u(\Gamma)
\]
defined by $M \to u(M)$ for every $M \in \Gamma$ is a poset map.

Proof. If $M' \leq M$, then $M'(t) \leq M(t)$ for all $t$. In particular, if $M'(t) > 0$, then $M(t) > 0$. □

Given a linear coloring $\kappa : V(\Delta) \to [k]$ for $\Delta$, let
\[
\varphi_\kappa : \Delta \to u(\Gamma(\Delta, \kappa))
\]
denote the composite map $u \circ c$. It is clear by the above two lemmas that $\varphi_\kappa$ is a poset map between face posets of two simplicial complexes. So, considered as a map between two simplicial complexes, it is a simplicial map.
Proposition 3.10. Let $\Delta$ be a simplicial complex and $\kappa : V \to [k]$ be a $k$-linear coloring. Then, there exists a simplicial map

$$i_\kappa : u(\Gamma(\Delta, \kappa)) \to \Delta$$

such that $\varphi_\kappa \circ i_\kappa = id$.

Proof. Suppose that the vertices of $\Delta$ are labeled as in Corollary 3.7. We first show that for each $S \in u(\Gamma(\Delta, \kappa))$, the set $V_S = \{v_1^s : s \in S\}$ is a simplex in $\Delta$. Note that if $F$ is a facet of $\Delta$ such that $v_1^s \in F$ for some $s \in S$ and some $i \in [n]$, then the vertex $v_1^s$ belongs to $F$. Since $S$ is equal to $u(M)$ for some multiset $M$ in $\Gamma = \Gamma(\Delta, \kappa)$, the set $S$ considered as a multiset belongs to multicompex $\Gamma$. We also observed earlier that there is a one-to-one correspondence between facets of $\Delta$ and $\Gamma$, so we can choose a facet $F$ of $\Delta$ such that $S \leq F$. This facet has to include $v_1^s$ for all $s \in S$ by the above argument. So, $V_S$ is a simplex of $\Delta$.

Let $i_\kappa : u(\Gamma(\Delta, \kappa)) \to \Delta$ be the map defined by $i_\kappa(S) = V_S$ for every $S \in u(\Gamma(\Delta, \kappa))$. It is easy to see that $i_\kappa$ satisfies the desired properties. □

Note that the simplicial map $i_\kappa$ is not uniquely defined in general. This is because the set of faces $F_i = \{F(v) : \kappa(v) = i\}$ can be linearly ordered in many different ways, and as a result of these different orderings there could be more than one vertex that can be chosen as the vertex with label $v_i^1$. On the other hand the subcomplexes which can be the image of $i_\kappa$ have something in common. Their vertices are colored with distinct colors and have the property that for every pair of vertices $x, y$ with $x \in \Delta$ and $y \in i_\kappa(u(\Gamma(\Delta, \kappa)))$ such that $\kappa(x) = \kappa(y)$, we have $F(x) \subseteq F(y)$. Conversely any subcomplex having these properties is the image of $i_\kappa$ for some choice of ordering. We will study such subcomplexes further in the next section.

4. Deformation to a representative subcomplex

In this section we prove Theorem 1.1 stated in the introduction. We start with the definition of representative subcomplex.

Definition 4.1. Let $\Delta$ be a simplicial complex with linear coloring map $\kappa : V(\Delta) \to [k]$ where $V(\Delta)$ denotes the vertex set of $\Delta$. A subcomplex $\Delta_\kappa$ of $\Delta$ is said to be a representative subcomplex with respect to $\kappa$ if for each $i \in [k]$ there is one and only one vertex in $x \in \Delta_\kappa$ with $\kappa(x) = i$ and if it has the property that for every pair of vertices $x, y$ with $x \in \Delta$, $y \in \Delta_\kappa$ and $\kappa(x) = \kappa(y)$, we have $F(x) \subseteq F(y)$, where $F(x)$ and $F(y)$ denote the set of faces including $x$ and $y$ respectively.

Although a linearly colored complex may have many different representing subcomplexes, the following result shows that as simplicial complexes they are all same.

Proposition 4.2. Let $\Delta$ be a simplicial complex with linear coloring $\kappa$. Suppose that $\Delta_\kappa$ and $\Delta'_\kappa$ are two subcomplexes of $\Delta$ which are representative with respect to $\kappa$. Then, $\Delta_\kappa$ and $\Delta'_\kappa$ are isomorphic as simplicial complexes.

Proof. Let $x, y$ be two vertices in a simplicial complex with $F(x) = F(y)$. Consider the map $f : V(\Delta) \to V(\Delta)$ such that $f(x) = y, f(y) = x$ and $f(z) = z$ for all the other vertices. We claim that $f$ extends to an isomorphism of simplicial complexes. For this
it is enough to show that if $S \in \Delta$, then $f(S) \in \Delta$. This is clear if $x, y$ are both in $S$ or if neither of them are in $S$. Suppose $S$ is such that $x \in S$ and $y \notin S$. Let $F$ be a facet that includes $S$. Since $x \in F$, we must have $y \in F$ by the assumption that $F(x) = F(y)$. This gives that $f(F) = F$. From this we can conclude that $f(S) \subseteq F$ and hence $f(S)$ is a simplex in $\Delta$. Similarly, if $S$ is a simplex with $y \in S$ and $x \notin S$, we can prove again $f(S)$ is in $\Delta$ using the equality $F(x) = F(y)$.

Let $\Delta_\kappa$ and $\Delta_\kappa'$ be two different choices of representative subcomplexes. Composing isomorphisms of the above type, we can find an isomorphism $f : \Delta \to \Delta$ such that $f$ takes the image of $\Delta_\kappa$ to the image of $\Delta_\kappa'$.

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** We need to show that the composition

$$f : \Delta \xrightarrow{i_{\kappa} \circ \phi_{\kappa}} \Delta_\kappa \xrightarrow{inc} \Delta$$

is homotopic to identity with a homotopy relative to $\Delta$. Note that there exists a unique inclusion $i_{\kappa}$ once $\Delta_\kappa$ is chosen. Also, it is clear that $f$ is a poset map between corresponding face posets. If there exists another poset map $g : \Delta \to \Delta$ such that $S \leq g(S) \geq f(S)$ for all $S \in \Delta$, then by Quillen’s criteria for homotopy equivalence of poset maps (see, for example, [9]), we can conclude that $id \simeq g \simeq f$. Below we show that for every $S \in \Delta$, the set $S \cup f(S)$ is a simplex of $\Delta$. This allows us to define $g : \Delta \to \Delta$ as the map $g(S) = S \cup f(S)$ and conclude that $f$ is homotopic to identity. Since both $f$ and $g$ are equal to identity on $\Delta_\kappa$, the required relativeness condition for the homotopy also holds.

To show that $S \cup f(S)$ belongs to $\Delta$ for all $S \in \Delta$, we use the labeling given in Corollary 3.7. Suppose that the vertices of $\Delta$ are labeled as in Corollary 3.7. Note that

$$f(S) = \{v^i_r : i \in u(S_\kappa)\}$$

for every simplex $S \in \Delta$. Let $S$ be a simplex in $\Delta$ and $F$ be a facet including $S$. If the color $i$ is used to color a vertex in $S$, then $S$ must include a vertex of the form $v^i_r$ for some $r \in [n]$. The way we have chosen the labeling implies that $v^i_r \in F$. Since this is true for all $i \in u(S_\kappa)$, we can conclude that $f(S) \subseteq F$. Since $F$ includes both $S$ and $f(S)$, it includes $S \cup f(S)$. This shows that $S \cup f(S)$ is a simplex of $\Delta$. This completes the proof.

The following is an immediate corollary of Theorem 1.1.

**Corollary 4.3.** Let $\Delta$ be a $k$-linear colorable simplicial complex. Then, $H_i(\Delta, \mathbb{Z}) = 0$ for all $i \geq k$.

Another important consequence of Theorem 1.1 is that it provides a lower bound for the linear chromatic number of a simplicial complex by the topology of the complex. To see this, we first introduce some terminology about connectedness. Let $\tilde{H}_r(\Delta)$ denote the reduced simplicial homology groups (over $\mathbb{Z}$) of a simplicial complex $\Delta$ (see [7] for details). A simplicial complex $\Delta$ is said to be $k$-acyclic if $\tilde{H}_r(\Delta) = 0$ for all $r \leq k$, and it is called acyclic if it is $k$-acyclic for all $k \in \mathbb{Z}$. Further, $\Delta$ is called $k$-connected if it is $k$-acyclic and simply connected, $k \geq 1$. The following is the linear coloring analogue of a well-known result of Lovász on graph colorability (see [8]).
Corollary 4.4. If $\Delta$ is non-acyclic and $k$-connected ($k \geq 1$), then $lchr(\Delta) \geq k + 3$.

Proof. Assume that $\Delta$ admits a $(k+2)$-linear coloring $\kappa$ and let $\Delta_\kappa$ be a representative subcomplex of $\Delta$ with respect to $\kappa$. Then, $\Delta$ is homotopy equivalent to $\Delta_\kappa$ by Theorem 1.1 where $\Delta_\kappa$ is a simplicial complex with $k+2$ vertices. Such a complex is at most $(k+1)$-dimensional. Since $\Delta$ is non-acyclic, the dimension of $\Delta_\kappa$ cannot be less than $k+1$ by $k$-connectivity. On the other hand, if $\dim(\Delta_\kappa) = k + 1$, then it is a $(k+1)$-simplex which is contractible; hence, it is acyclic, a contradiction. □

5. LC-reduction of a simplicial complex

In this section we introduce the concept of LC-reduction and study its basic properties. We start with the definition of LC-reduction.

Definition 5.1. Let $\Delta$ be a simplicial complex and $\Delta'$ be a subcomplex of $\Delta$. If there exist a sequence of subcomplexes $\Delta = \Delta_0 \supseteq \Delta_1 \supseteq \ldots \supseteq \Delta_t = \Delta'$ such that $\Delta_{r+1}$ is a representative subcomplex in $\Delta_r$ with respect to some linear coloring $\kappa_r$ of $\Delta_r$ for all $0 \leq r \leq t-1$, then we say $\Delta$ LC-reduces to $\Delta'$, and write $\Delta \xrightarrow{\text{LC}} \Delta'$.

By Theorem 1.1 it is easy to see that if $\Delta$ LC-reduces to a subcomplex $\Delta'$, then $\Delta'$ is a strong deformation retract of $\Delta$.

For our purposes it is desirable to be able to express an LC-reduction as a composition of LC-reductions which are primitive in some sense. In this context, the appropriate definition of primitiveness can be given as follows:

Definition 5.2. A linear coloring of a simplicial complex $\Delta$ with $n$ vertices is called a primitive linear coloring if it uses exactly $n-1$ colors. An LC-reduction is called primitive if it involves only one linear coloring and that coloring is primitive.

Note that if $\kappa$ is a primitive linear coloring then there is a pair of vertices $u, v$ in $\Delta$ such that $\kappa(u) = \kappa(v)$ and the remaining vertices of $\Delta$ are colored using distinct colors. By the condition of a linear coloring, we have either $F(u) \subseteq F(v)$ or $F(v) \subseteq F(u)$. In the first case, the subcomplex $\text{del}_\Delta(u) = \{S \in \Delta \mid u \notin S\}$ will be a representative subcomplex, and in the second case $\text{del}_\Delta(v) = \{S \in \Delta \mid v \notin S\}$ will be representative. In the case of equality either of these sets can be taken as a representative subcomplex. Note that an LC-reduction $\Delta \xrightarrow{\text{LC}} \Delta'$ is primitive if and only if the number of vertices in $\Delta'$ is exactly one less than the number of vertices in $\Delta$.

Proposition 5.3. Any LC-reduction $\Delta \xrightarrow{\text{LC}} \Delta'$ can be expressed as a sequence of primitive LC-reductions.

Proof. It is enough to prove the proposition for a LC-reduction involving only one coloring. So, we can assume $\Delta' = \Delta_\kappa$ for some coloring $\kappa$ of $\Delta$. Suppose that the vertices $\Delta$ are labeled as in Corollary 3.7. So, if $V$ is the set of vertices of $\Delta$, then we can write $V = \{v^i_r : i \in [k], r \in [n_i]\}$ where $F(v^i_r) \subseteq F(v^i_s)$ whenever $1 \leq r \leq t \leq n_i$. We can assume that $\Delta_\kappa$ is the subcomplex generated by the vertices $\{v^i_r : i = 1, \ldots, k\}$.

Let $\kappa(i, j)$ denote the primitive linear coloring involving vertices $v^i_j$ and $v^i_{j+1}$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n_i - 1$. It is easy to see that if we apply LC-reductions associated to primitive linear colorings $\kappa(i, n_i - 1), \kappa(i, n_i - 2), \ldots, \kappa(i, 1)$ in this order for each $i = 1, \ldots, k$, then we obtain an LC-reduction to $\Delta_\kappa$. □
Some complexes cannot be LC-reduced further to any proper subcomplex.

**Definition 5.4.** A simplicial complex $\Delta$ on a set $V$ is called LC-irreducible if it admits only a trivial linear coloring.

The following is clear from the definition.

**Proposition 5.5.** A simplicial complex $\Delta$ is LC-irreducible if and only if for every pair of vertices $u, v$, the facet sets $\mathcal{F}(u)$ and $\mathcal{F}(v)$ are not comparable by inclusion.

A typical example of an LC-irreducible complex is the boundary of a simplex. Another example would be a complex whose realization is an $n$-gon.

It is easy to see that every simplicial complex $\Delta$ LC-reduces to an LC-irreducible subcomplex, although the resulting LC-irreducible subcomplex can be quite different depending on the choices we make. Let us call a subcomplex $\Delta'$ of $\Delta$ an LC-core of $\Delta$ if it is irreducible and if $\Delta$ LC-reduces to it. The homotopy type of an LC-core is uniquely determined by the homotopy type of $\Delta$, but it is not easy to see what other properties of LC-cores of $\Delta$ are invariants of $\Delta$. It is reasonable to ask:

**Question 5.6.** Let $\Delta$ be a simplicial complex and $\Delta_1$ and $\Delta_2$ are two different LC-cores for $\Delta$. Is it true that $\Delta_1$ and $\Delta_2$ are isomorphic as simplicial complexes?

At this point we do not know the answer to this question. One would expect that at least the number of vertices of a core is an invariant of the simplicial complex. Until finding an answer to this question we can define such an invariant as follows:

**Definition 5.7.** Let $\Delta$ be a simplicial complex. The linear dimension of $\Delta$, denoted by $\text{lindim}(\Delta)$, is defined to be the smallest integer $n$ such that $\Delta$ has a core with $n$ vertices.

Note that $\text{lindim}(\Delta)$ is also the smallest integer $n$ such that $\Delta$ LC-reduces to a simplicial complex with $n$ vertices. It is easy to see that linear dimension is related to the homological dimension of the complex. Recall that the homology dimension $\text{homdim}(\Delta)$ of a finite simplicial complex $\Delta$ is defined to be the integer

$$\text{homdim}(\Delta) := \min\{i \mid \tilde{H}_j(\Delta; \mathbb{Z}) = 0 \text{ for all } j > i\}$$

with the convention that $\tilde{H}_{-1}(\Delta; \mathbb{Z}) = \mathbb{Z}$. We can easily adopt the proof of Corollary 4.4 to obtain the following.

**Proposition 5.8.** For any finite simplicial complex $\Delta$, we have

$$\text{lchr}(\Delta) \geq \text{lindim}(\Delta) \geq \text{homdim}(\Delta) + 2.$$

An interesting family of simplicial complexes are the ones with linear dimension equal to one. These are the complexes which can be LC-reduced to a point. We say a simplicial complex $\Delta$ is LC-contractible if $\Delta \backslash_{\text{LC}} \{x\}$ for some vertex $x$ of $\Delta$. We use this terminology later in the paper.

Now, we investigate the behavior of LC-reduction under the join operator. Recall that the join of two simplicial complexes $X$ and $Y$, denoted by $X \ast Y$, is defined as the simplicial complex which includes both $X$ and $Y$ as subcomplexes and includes also the sets of the form $S \cup T$ where $S \in X$ and $T \in Y$. 
Proposition 5.9. Let $X_1 \setminus_{LC} X_2$ and let $Y$ be an arbitrary simplicial complex. Then, $X_1 \ast Y \setminus_{LC} X_2 \ast Y$.

Proof. It is enough to prove the result for a primitive LC-reduction. Let $X_1 \setminus_{LC} X_2$ be a primitive reduction involving vertices $u, v \in X_1$. Without loss of generality we can assume $v \in X_2$. Recall that in this case $X_2$ is the subcomplex $\text{del}_{X_1}(u) = \{ S \in X_1 \mid u \notin S \}$. Since $\text{del}_{X_1 \ast Y}(u) = \text{del}_{X_1}(u) \ast Y$, we just need to show that primitive coloring involving $u$ and $v$ is still a linear coloring in $X_1 \ast Y$. We know that $\mathcal{F}(u) \subseteq \mathcal{F}(v)$ in $X_1$. Let $F$ be a facet of $X_1 \ast Y$ including the vertex $u$. Then either $F$ is a facet of $X$ or $F$ is of the form $S \cup T$ where $S$ and $T$ are facets of $X$ and $Y$ respectively. In the first case, $F \in \mathcal{F}(u)$, so $v \in F$ can be seen easily. In the second case, the facet $S$ belongs to the set $\mathcal{F}(u)$, and again we can conclude $v \in S$. This gives $v \in F$ since $F = S \cup T$. This shows that the inclusion $\mathcal{F}(u) \subseteq \mathcal{F}(v)$ still holds for facet sets in $X_1 \ast Y$. This completes the proof. \hfill \Box

6. LC-reduction, nonevasive reduction and collapsing

The aim of this section is to prove Theorem 1.2 stated in the introduction. We first recall the definition of collapsing.

Definition 6.1. A face $S$ of a simplicial complex $\Delta$ is called free if $S$ is not maximal and there is a unique maximal face in $\Delta$ that contains $S$. If $S$ is a free face of $\Delta$ then the simplicial complex $\Delta[S] := \Delta \setminus \{ T \in \Delta \mid S \subseteq T \}$ is called an elementary collapse of $\Delta$. If $\Delta$ can be reduced to a subcomplex $\Delta'$ by a sequence of elementary collapses, then we say $\Delta$ collapses to $\Delta'$. In this case, we write $\Delta \setminus \Delta'$. If a complex collapses to a point then we say it is collapsible.

We start with the following result:

Proposition 6.2. Let $\Delta$ be a simplicial complex and $\Delta'$ be a subcomplex in $\Delta$. If $\Delta \setminus_{LC} \Delta'$, then $\Delta \setminus \Delta'$.

Proof. It is enough to prove the proposition for a primitive linear coloring. So, assume that $\Delta' = \Delta_\kappa$ for some primitive linear coloring $\kappa$ which involves vertices $u$ and $v$. Without loss of generality, we may assume that $u$ lies on $\Delta_\kappa$. Note that this implies in particular that $\mathcal{F}(v) \subseteq \mathcal{F}(u)$. Let $F_1 \in \mathcal{F}(v)$ be given. Then, we claim that the face $S_1 := F_1 \setminus \{ u \}$ is contained in $F_1$, i.e., it is free in $\Delta$. Indeed, if $F'$ is any facet containing $S_1$, then $v \in F'$. This gives $u \in F'$ because $\mathcal{F}(v) \subseteq \mathcal{F}(u)$. But then $F_1 \subseteq F'$, and we can conclude that $F_1 = F'$. Let $\Delta_1$ denote the elementary collapse of $\Delta$ through the face $S_1$, that is, $\Delta_1 = \Delta[S_1]$. For the simplicial complex $\Delta_1$, we note that any facet containing the vertex $v$ must also contain $u$. Therefore, we may similarly collapse $\Delta_1$ by choosing a facet $F_2$ of $\Delta_1$ containing $v$. We iterate the same process until we obtain a simplicial complex $\Delta_m$ in which $\mathcal{F}(v)$ is empty. It is easy to see that $\Delta_m = \text{del}_\Delta(v)$, and hence it is equal to $\Delta_\kappa$.

The converse of Proposition 6.2 does not hold in general.

Example 6.3. Let $\Delta$ be the 2-dimensional simplicial complex on $V = \{ a, b, c, d, e, f \}$ with the set of facets

$$\mathcal{F}(\Delta) = \{ \{ a, b, c \}, \{ a, b, e \}, \{ a, d, e \}, \{ b, e, f \}, \{ d, e, f \}, \{ b, c, f \}, \{ c, d, f \} \}.$$
The realization of $\Delta$ is given in Figure 2, where the picture is intended to be three dimensional like a pyramid. Note also that the interior of the shaded simplex is not part of the complex. It is clear that $\Delta$ is collapsible and NE-reduces to a point (i.e. nonevasive), but it does not LC-reduce to a point (in fact it is $LC$-irreducible).

Another type of reduction of simplicial complexes is nonevasive reduction (see Kozlov [4], Welker [12]) which is also known as strong collapsing (see Kurzweil [6]). Recall that for a vertex $v$ in a simplicial complex $\Delta$, the deletion of $v$ is defined as the subcomplex $\text{del}_\Delta(v) = \{ S \in \Delta \mid v \notin S \}$ and the link of $v$ in $\Delta$ is defined as the subcomplex $\text{lk}_\Delta(v) = \{ S \in \Delta \mid v \notin S, S \cup \{ v \} \in \Delta \}$. Nonevasiveness of a simplicial complex is defined inductively by declaring that a point is nonevasive and a simplicial complex $\Delta$ is nonevasive if it has a vertex $v$ such that both its deletion $\text{del}_\Delta(v)$ and its link $\text{lk}_\Delta(v)$ are nonevasive.

**Definition 6.4.** Let $\Delta$ be a simplicial complex and $\Delta'$ be a subcomplex of $\Delta$. We say that $\Delta$ NE-reduces to $\Delta'$, denoted by $\Delta \searrow_{\text{NE}} \Delta'$, if there exist a sequence $\Delta = \Delta_1, \Delta_2, \ldots, \Delta_{t+1} = \Delta'$ of subcomplexes and a sequence of vertices $v_1, \ldots, v_t$ such that $V(\Delta_r) = V(\Delta_{r+1}) \cup \{ v_r \}$ and $\text{lk}_{\Delta_r}(v_r)$ is nonevasive for any $1 \leq r \leq t$.

We have the following:

**Proposition 6.5.** Let $\Delta$ be a simplicial complex and $\Delta'$ be a subcomplex of $\Delta$. If $\Delta \searrow_{\text{LC}} \Delta'$, then $\Delta \searrow_{\text{NE}} \Delta'$.

**Proof.** As before it is enough to prove the proposition for a primitive linear coloring. Let $\Delta' = \Delta_\kappa$ where $\kappa$ is a primitive coloring involving vertices $u$ and $v$. Without loss of generality we can assume $u \in \Delta'$. We have $F(v) \subseteq F(u)$. We claim that $\text{lk}_\Delta(v)$ is nonevasive. This will imply that $\Delta \searrow_{\text{NE}} \Delta'$ as desired.

It is well known that if a simplicial complex is a cone then it is nonevasive. So, it is enough to show that $\text{lk}_\Delta(v)$ is a cone. Let $S$ be a simplex in $\text{lk}_\Delta(v)$. Then, $S \cup \{ v \}$ is a simplex in $\Delta$. Let $F$ be a facet of $\Delta$ which includes $S \cup \{ v \}$. Since $F \in F(v)$, we have $F \in F(u)$ by our assumption that $F(v) \subseteq F(u)$. This implies that $S \cup \{ u \}$ is a simplex in $\text{lk}_\Delta(v)$. We have shown that for every simplex $S$ in $\text{lk}_\Delta(v)$, $S \cup \{ u \}$ is also a simplex in $\text{lk}_\Delta(v)$. This means $\text{lk}_\Delta(v)$ is a cone with cone point $u$. \qed

Note that the proof of Theorem 1.2 is now complete.
Remark 6.6. It is well known that nonevasive reduction is a collapsing by a result of Kahn, Saks, and Sturtevant (see Proposition 1 in [3]). So, Proposition 6.5 can also be obtained as a corollary of Proposition 6.6.

7. Linear coloring of posets

Let $P$ be a finite partially ordered set. We denote by $\Delta(P)$ its order complex, i.e., the set of all chains in $P$. When $P$ has maximal and minimal elements, we denote them by $\hat{0}$ and $\hat{1}$, respectively. The elements of $P$ that cover $\hat{0}$ are called atoms, and the elements that are covered by $\hat{1}$ are called coatoms. We denote the set of atoms and coatoms of a bounded poset $P$ by $at(P)$ and $co(P)$ respectively. We write $P$ for the poset $P\setminus\{\hat{0},\hat{1}\}$, and call it the proper part of $P$. The set of maximal chains of $P$ is denoted by $\mathcal{M}$, and in particular $\mathcal{M}_x$ denotes the maximal chains containing the element $x \in P$. For a given subset $S \subseteq P$, we denote by $\bigwedge S$ and $\bigvee S$, the greatest lower bound and the least upper bound (when exist) of $S$ respectively.

Throughout, by a linear coloring of $P$, we mean a linear coloring of $\Delta(P)$. We may rephrase the definition of a linear coloring for posets as follows.

Lemma 7.1. A surjective mapping $\kappa: P \rightarrow [k]$ is a $k$-linear coloring of $P$ if and only if $\kappa(x) = \kappa(y)$ implies either $\mathcal{M}_x \subseteq \mathcal{M}_y$ or $\mathcal{M}_y \subseteq \mathcal{M}_x$ for any two elements $x, y \in P$.

This implies, in particular, that in a linearly colored poset $P$ any two elements $x, y \in P$ having the same color must be comparable. In fact, more is true. Let $P$ be a poset linearly colored with $\kappa$, and let $x, y \in P$ be such that $\kappa(x) = \kappa(y)$. Suppose $\mathcal{M}_x \subseteq \mathcal{M}_y$. Let $z$ be an element in $P$ such that $x$ is comparable with $z$, i.e., either $x < z$ or $z < x$. Then, there is a maximal chain $M$ including $x$ and $z$. Since $\mathcal{M}_x$ is included in $\mathcal{M}_y$, the chain $M$ must also include $y$. Thus, $z$ and $y$ are also comparable. Similarly, we can show that if $\mathcal{M}_x \subseteq \mathcal{M}_y$, then every element of $P$ which is comparable with $y$ is also comparable with $x$. We define the following:

Definition 7.2. Let $P$ be a poset and $x, y \in P$. We say $y$ dominates $x$, denoted by $x \prec y$, if every element $z$ which is comparable with $x$ is also comparable with $y$.

We have seen above that in a linearly colored poset $P$ any two elements $x, y \in P$ having the same color must be comparable by domination. The converse of this statement also holds:

Proposition 7.3. Let $P$ be a poset and $\kappa: P \rightarrow [k]$ be a coloring of $P$. Then, $\kappa$ is a linear coloring if and only if for every pair $x, y \in P$ with $\kappa(x) = \kappa(y)$, either $x \prec y$ or $y \prec x$.

Proof. We only need to prove one direction. Let $x, y \in P$ be such that $\kappa(x) = \kappa(y)$ and $x \prec y$. Then every element $z \in P$ which is comparable with $x$ is also comparable with $y$. We claim that in this case the inclusion $\mathcal{M}_x \subseteq \mathcal{M}_y$ holds. Let $M$ be a maximal chain in $\mathcal{M}_x$. Note that all the elements in $M$ are comparable with $x$, so they must be also comparable with $y$. If $y$ is not in $M$, then by adding $y$ to $M$ we would get a longer chain which will contradict with the maximality of $M$. So, $y$ must lie already in $M$. Thus, $M \in \mathcal{M}_y$. \qed
We have the following:

**Proposition 7.4.** Let $P$ be a poset and let $x, y \in P$ such that $x < y$. Then, $\Delta(P) \setminus_{LC} \Delta(P\setminus\{x\})$.

**Proof.** Consider the primitive linear coloring $\kappa$ that involves only $x$ and $y$. The proposition follows from the fact that $\Delta(P)_k = \text{del}_{\Delta(P)}(x) = \Delta(P\setminus\{x\})$. \hfill $\square$

It is easy to see that if an element is minimal or maximal, then it dominates all other elements. So, if a poset has a minimal or maximal element, then it is LC-contractible.

Now, we consider monotone poset maps and prove a reduction theorem for them.

**Definition 7.5.** Let $P$ be a poset. An order-preserving map $\varphi : P \to P$ is called a monotone map if either $x \leq \varphi(x)$ or $x \geq \varphi(x)$ for any $x \in P$. If $\varphi$ is a monotone map which also satisfies $\varphi^2 = \varphi$, then it is called a closure operator on $P$.

Note that when $\varphi : P \to P$ is a closure operator then $\text{Fix}(\varphi) = \varphi(P)$, and the equality $P = \varphi(P)$ holds only when $\varphi$ is the identity map.

**Lemma 7.6.** Let $P$ be a finite poset, and let $\psi : P \to P$ be a monotone map on $P$ which is different than the identity map. Then there exists a $x \in P\setminus\text{Fix}(\psi)$ such that $x < \psi(x)$.

**Proof.** Assume to the contrary that for all $x \in P\setminus\text{Fix}(\psi)$, we have $x \not< \psi(x)$. Start with $y_0 \in P\setminus\text{Fix}(\psi)$ such that $y_0 \not< \psi(y_0)$. This means that there exists an element $y_1 \in P$ such that $y_1$ is comparable with $y_0$ but not with $\psi(y_0)$.

Note that since $\psi$ is a monotone map either $y_0 < \psi(y_0)$ or $\psi(y_0) < y_0$ holds. We look at each case separately.

Case 1: Assume $y_0 < \psi(y_0)$ holds. Then, we must have $y_0 < y_1$, because otherwise we have $y_1 < y_0 < \psi(y_0)$ which contradicts the assumption that $y_1$ and $\psi(y_0)$ are not comparable. Also note that $y_1$ cannot be an element of $\text{Fix}(\psi)$, because otherwise $y_1 = \psi(y_1) < \psi(y_0)$ implies that $y_1$ and $\psi(y_0)$ are comparable, which is again a contradiction. So, we have $y_1 \in P\setminus\text{Fix}(\psi)$.

Now, let’s apply the same arguments for $y_1$. First we have $y_1 \not< \psi(y_1)$ by our starting assumption, so there exists a $y_2$ such that $y_2$ comparable with $y_1$ but not with $\psi(y_1)$. Since $\psi$ is a monotone map, we again have either $y_1 < \psi(y_1)$ or $\psi(y_1) < y_1$. Now we claim that actually the second inequality cannot hold. Suppose it holds, i.e., $\psi(y_1) < y_1$. Then we get $\psi(y_0) < \psi(y_1) < y_1$ which gives $\psi(y_0)$ and $y_1$ are comparable and hence a contradiction. So, we have $y_1 < \psi(y_1)$. This allows us to continue in the same way and obtain an infinite ascending sequence $y_0 < y_1 < y_2 < \cdots$ of distinct elements in $P$. But, this is in contradiction with the fact that $P$ is a finite poset.

Case 2: Assume $y_0 > \psi(y_0)$ holds. Then, arguing as above we find a descending infinite sequence $y_0 > y_1 > y_2 > \cdots$ of distinct elements in $P$ and again reach a contradiction. \hfill $\square$

The main result of this section is the following:

**Theorem 7.7.** Let $\varphi : P \to P$ be a closure operator on a finite poset $P$. Then, $\Delta(P) \setminus_{LC} \Delta(\varphi(P))$. 

Proof. We will prove the result by induction on $n = |P \setminus \varphi(P)|$. If $n = 0$, then there is nothing to prove. So assume $n \geq 1$, i.e., $\varphi$ is not identity. Then, by Lemma 7.6 there exists a $x \in P \setminus \varphi(P)$ such that $x < \varphi(x)$. By Proposition 7.4, we have $\Delta(P) \setminus_{\text{LC}} \Delta(P \setminus \{x\})$. Since $x \not\leq \varphi(P)$, the restriction of $\varphi$ to $P \setminus \{x\}$ induces a closure operator $\overline{\varphi} : P \setminus \{x\} \to P \setminus \{x\}$. Applying the induction assumption, we obtain $\Delta(P \setminus \{x\}) \setminus_{\text{LC}} \Delta(\overline{\varphi}(P \setminus \{x\}))$ which gives $\Delta(P \setminus \{x\}) \setminus_{\text{LC}} \Delta(\varphi(P))$ since $\overline{\varphi}(P \setminus \{x\}) = \varphi(P)$. Combining this with the above reduction, we conclude that $\Delta(P) \setminus_{\text{LC}} \Delta(\varphi(P))$. □

Corollary 7.8. For a finite poset $P$, if $\bar{x} = \bigwedge\{c \in \text{co}(P) : x \leq c\}$ exists for all $x \in P$ then $P \setminus_{\text{LC}} R$, where $R = \{\bar{x} \mid x \in P\}$. If in addition, $\bigwedge \text{co}(P)$ exists then $\Delta(P)$ is LC-contractible.

Proof. The map $\varphi : P \to P$ defined by $\varphi(x) = \bar{x}$ is a closure operator. Hence, by Theorem 7.7, $\Delta(P) \setminus_{\text{LC}} \Delta(R)$, since $\text{Fix}(\varphi) = \varphi(P) = R$. On the other hand, when it exists, $\bigwedge \text{co}(L)$ is the minimal element of $R$, therefore $\Delta(R)$ is LC-contractible so is $\Delta(P)$. □

In particular, the above corollary says that the proper part of a lattice is LC-reducible to the proper part of the sublattice of elements that are the meet of coatoms. This result is well-known when the LC-reduction is replaced by homotopy equivalence (see Theorem 10.8 in [1]).

Another interesting invariant in poset theory is the order dimension of a poset which is defined as follows:

Definition 7.9. The order dimension of a finite poset $P$, denoted by $\text{ordim}(P)$, is defined to be the smallest integer $n$ such that $P$ can be embedded in $\mathbb{N}^n$ as an induced subposet (an induced subposet is a subposet which inherits all the relations of the poset.)

There is a very nice paper by Reiner and Welker [10] which proves that the order dimension of a lattice $L$ is greater than $\text{homdim}(L) + 2$ where $L$ denotes the proper part of the lattice $L$. Recall that there is a similar inequality for the linear dimension of a poset (see Proposition 5.8). The obvious question is whether there is any connection between the order dimension of a lattice and the linear dimension of its proper part. Unfortunately these invariants are not comparable by inclusion as the following examples show.

Example 7.10. Consider the poset $P$ which is an antichain with three elements. Let $L$ be the lattice obtained form $P$ by adding minimal and maximal elements. It is clear that $L = P$ has linear dimension exactly 3. But, the order dimension of $L$ is equal to 2 since we can embed $L$ in $\mathbb{N}^2$ by taking the minimal element to $(0,0)$, the maximal element to $(2,2)$ and the 3 middle points to the points $(0,2), (1,1), (2,0)$. This shows that there is a lattice $L$ where $\text{ordim}(L) < \text{lindim}(L)$.

For the other direction, consider the poset $P = \{a, b, c\}$ where $a \leq b$, $a \leq c$, and $b$ and $c$ are not comparable. It is easy to see that $P$ is LC-reducible to a point so $\text{lindim}(P) = 1$. Let $L$ be the lattice obtained from $P$ by adding $\hat{0}$ and $\hat{1}$. It is clear that $L$ is not linear, so $\text{ordim}(L) > 1 = \text{lindim}(L)$. 
We end the section with an application of Corollary 7.11 to subgroup lattices.

**Corollary 7.11.** Let $G$ be a finite $p$-group ($p$ a prime). Then, $\mathcal{L}(G)$ is LC-contractible if and only if $G$ is not elementary abelian, where $\mathcal{L}(G)$ is the subgroup lattice of $G$.

**Proof.** It is known that if $G$ is elementary abelian, then the Euler characteristic of $\mathcal{L}(G)$ is bigger than 1. Thus, $\mathcal{L}(G)$ cannot be LC-contractible. Conversely, if $G$ is not elementary abelian, then the intersection of the maximal subgroups of $G$ is non-trivial. Therefore, by Lemma 7.8, $\mathcal{L}(G)$ is LC-contractible. □

8. Linear Graph Colorings

In this final section, we consider linear colorings of neighborhood complexes associated to simple graphs.

Let $G = (V,E)$ be a simple graph. We recall that a (vertex) coloring of $G$ is a surjective mapping $\nu: V \to [n]$ such that $\nu(x) \neq \nu(y)$ whenever $(x,y) \in E$. The neighborhood complex of $G$, denoted by $\mathcal{N}(G)$, is defined as the simplicial complex whose simplices are those subsets of $V$ which have a common neighbor. We start with the following easy observation.

**Proposition 8.1.** Let $G = (V,E)$ be a simple graph and let $\mathcal{N}(G)$ denote its neighborhood complex. If $\kappa: V \to [k]$ is a $k$-linear coloring of $\mathcal{N}(G)$, then $\kappa$ is a coloring of the underlying graph $G$.

**Proof.** Assume that $\kappa$ is not a coloring of the underlying graph $G$. Therefore, there exist $x, y \in V$ such that $(x,y) \in E$ and $\kappa(x) = \kappa(y)$. By the definition of a linear coloring, either $F(x) \subseteq F(y)$ or $F(y) \subseteq F(x)$. So, without loss of generality, assume $F(x) \subseteq F(y)$. Let $\mathcal{N}(z)$ be a facet of $\mathcal{N}(G)$ such that $\mathcal{N}(y) \subseteq \mathcal{N}(z)$. Since there is an edge between $x$ and $y$, we have $x \in \mathcal{N}(y)$, and hence $x \in \mathcal{N}(z)$. This implies that $\mathcal{N}(z) \in F(x)$, and gives $\mathcal{N}(z) \in F(y)$. Therefore, $y \in \mathcal{N}(z)$ and hence $z \in \mathcal{N}(y)$. However, together with $\mathcal{N}(y) \subseteq \mathcal{N}(z)$, this implies $z \in \mathcal{N}(z)$ which is a contradiction since $G$ is a simple graph and has no loops. □

The following is immediate:

**Corollary 8.2.** For any graph $G$, we have $\text{lchr}(\mathcal{N}(G)) \geq \chi(G)$, where $\chi(G)$ denotes the (vertex) chromatic number of $G$.

It is easy to see that a coloring of $G$ may not give rise to a linear coloring of its neighborhood complex $\mathcal{N}(G)$. So, in general the equality does not hold.

**Example 8.3.** Consider the graph which is an hexagon, i.e., $G = (V,E)$ with $V = \{v_1, \ldots, v_6\}$ and $E = \{(v_i, v_{i+1}) \mid 1 \leq i \leq 5\} \cup \{(6,1)\}$. Note that $\chi(G) = 2$, but $\text{lchr}(\mathcal{N}(G)) = 6$ since $\mathcal{N}(G)$ is a disjoint union of two triangles.

We now give a sufficient condition for a coloring of a graph to be a linear coloring of its neighborhood complex.

**Proposition 8.4.** A coloring $\nu: V \to [k]$ of $G = (V,E)$ is a $k$-linear coloring of $\mathcal{N}(G)$ if either $\mathcal{N}(v) \subseteq \mathcal{N}(u)$ or $\mathcal{N}(u) \subseteq \mathcal{N}(v)$ holds for every $x, y \in V$ with $\nu(x) = \nu(y)$. 
Proof. Assume that whenever \( \nu(u) = \nu(v) \) for any two vertices \( u, v \in V(G) \), then one of the inclusions \( N(v) \subseteq N(u) \) or \( N(u) \subseteq N(v) \) holds. Let \( u, v \in V(G) \) be two such vertices and let \( N(u) \subseteq N(v) \). To verify that \( F(u) \subseteq F(v) \), let \( N(y) \) be a facet of \( N(G) \) containing \( u \). Then we must have \( y \in N(v) \), since \( y \in N(u) \subseteq N(v) \). Hence, \( v \in N(y) \).

The converse of Proposition 8.4 does not hold in general as illustrated in Figure 3. It is easy to see that the given vertex coloring of \( G \) is indeed a linear coloring of \( N(G) \) with \( \nu(u) = \nu(v) = 1 \); however, there is no inclusion relation between the neighborhoods of \( u \) and \( v \).

\[
\begin{array}{c}
1 & 2 & 1 & 3 \\
1 & 2 & 1 & \\
x & y & z & u & w \\
1 & 2 & 1 & 3 \\
1 & 2 & 3 & \\
\end{array}
\]

\[ G \quad N(G) \]

\textbf{Figure 3.}

References

[1] A. Björner, \textit{Topological Methods}, Handbook of Combinatorics, R. Graham, M. Grötschel, and L. Lovász (eds), North-Holland/Elsevier, Amsterdam, 1995, 1819-1872.
[2] A. Björner and S. Vrečica, \textit{On f-vectors and Betti numbers of multicomplexes}, Combinatorica, 17 (1997), 53-65.
[3] J. Kahn, M. Saks, and D. Sturtevant, \textit{A topological approach to evasiveness}, Combinatorica, 4 (1984), 297–306.
[4] D.N. Kozlov, \textit{Collapsing along monotone poset maps}, to appear in the International Journal of Mathematics and Mathematical Sciences, arXiv:math.CO/0503416.
[5] D.N. Kozlov, \textit{Simple homotopy types of Hom-complexes, neighborhood complexes, Lovász complexes, and atom crosscut complexes}, to appear in Topology Appl., arXiv:math.AT/0503613.
[6] H. Kurzweil, \textit{A combinatorial technique for simplicial complexes and some applications to finite groups}, Discrete Math., 82 (1990), 263-278.
[7] J.R. Munkres, \textit{Elements of Algebraic Topology}, Addison-Wesley Pub., New York, 1993.
[8] L. Lovász, \textit{Kneser’s conjecture, chromatic number, and homotopy}, J. Combinatorial Theory, Series A, 25 (1978), 319-324.
[9] D. Quillen, \textit{Homotopy properties of the poset of nontrivial p-subgroups of a group}, Adv. in Math., 28(2) (1978), 101-128.
[10] V. Reiner and V. Welker, \textit{A homological lower bound for order dimension of lattices}, Order, 16 (1999), 165-170.
[11] R.P. Stanley, \textit{Combinatorics and Commutative Algebra}, Progress in Mathematics, 41, Birkhäuser, Boston, 1997.
[12] V. Welker, \textit{Constructions preserving evasiveness and collapsibility}, Discrete Math., 207 (1999), 243-255.

Department of Mathematics, Suleyman Demirel University, Isparta, 32260, Turkey.

Department of Mathematics, Bilkent University, Ankara, 06800, Turkey.

E-mail address: ycivan@fef.sdu.edu.tr, yalcine@fen.bilkent.edu.tr