Superstring, Unifications and Dualities*

Joseph Kouneiher,
U.M.R. 7596 (C.N.R.S, Université Diderot Paris7)
2, place Jussieu 75005 Paris, France
kouneiher@paris7.jussieu.fr

Abstract

I describe our understanding of physics near the planck length, in particular the great progress of the last four years in string theory. Superstring theory, and a recent extension called M theory, are leading candidates for a quantum theory that unifies gravity with the other forces. As such, they are certainly not ordinary quantum field theories. However, recent duality conjectures suggest that a more complete definition of these theories can be provided by the large $N$ limits of suitably chosen $U(N)$ gauge theories associated to the asymptotic boundary of spacetime.

1 Introduction

Philosophically, one of the main achievement of quantum fields theory is to have constructed (in the transcendental sense) the third dynamical category of interaction. Weyl’s gauge principle converting gauge invariances into dynamical principles. According to quantum field theories the constitutive principles (relativity, symmetry) provides lagrangians, which in turn provide Feynman’s integrals, which provide themselves the models. Gauges theories have shown that if we localise the internal symmetries and if we impose the invariance of the Lagrangians for these supplementary symmetries, we can reconstruct in a purely mathematical manner the interactions Lagrangians. The mathematical constraints are so strong (renormalizability, elimination of anomalies, 

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*IASSNS-I.H.P-Mars,99

1 This conference is the first part of two lectures treating the geometric principle lying behind superstring theory. It is an introductory one.

2 General relativity constructed the second dynamical category of causality i.e. the force. Indeed, in general relativity, the metric is no longer an a priori component but on the contrary a physical phenomenon which has to be determined. It is for this reason that metric can absorb the forces. So we have a conversion of the kinematical moment concerning metric into the dynamical moment concerning forces, a shifts from the metrical and global level to the local and differentiable one.
Higgs mechanism of symmetry breaking for conferring a mass to the gauge bosons, etc.) that it is possible to infer from very few empirical data to the choice of a symmetry group. In superstring theory this fact is even more evident.

In addition, as we all know, quantum field theory has been extremely successful in providing a description of elementary particles and their interactions. However, it does not work so well for gravity. If we naively try to quantize general relativity (which is a classical field theory) using the methods of quantum field theory, we run into divergences which cannot be removed by using the conventional renormalization techniques of quantum field theory. String theory is an attempt to solve this problem.

2 A Review of Perturbative String Theory

The basic idea in string theory is quite simple. According to string theory, different elementary particles, instead of being point like objects, are different vibrational modes of a string. Fig. 1 shows some of the oscillation modes of closed strings and open strings.

![Figure 1: Some vibrational modes of closed and open strings](image)

The energy per unit length of the string, known as string tension, is parametrized as \((2\pi a')^{-1}\), where \(a'\) has the dimension of \((\text{length})^2\). This theory automatically contains gravitational interaction between elementary particles, but in order to correctly reproduce the strength of this interaction, we need to choose \(\sqrt{a'}\) to be of the order of \(10^{-33}\text{cm}\). Since \(\sqrt{a'}\) is the only length parameter in the theory, the typical size of a string is of the order of \(\sqrt{a'} \sim 10^{-33}\text{cm}\) – a distance that cannot be resolved by present day experiments. Thus there is no direct way of testing string theory, and its appeal lies in its theoretical consistency.
The basic principle behind constructing a quantum theory of relativistic string is quite simple. Consider propagation of a string from a space-time configuration A to a space-time configuration B. During this motion the string sweeps out a two dimensional surface in space-time, known as the string world-sheet (see Fig.2). The amplitude for the propagation of the string from the space-time position A to space-time position B is given by the weighted sum over all world-sheet bounded by the initial and the final locations of the string. The weight factor is given by $e^{-S}$ where $S$ is the product of the string tension and the area of the world-sheet. Let $\sigma$ be a parametrization of the string. If $\tau$ is its proper time, the parametrization of its world leaf is $X_\mu(\sigma, \tau)$ endowed with the metric $g_{ab} = g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu (a, b = \sigma, \tau)$. This leads to the introduction of new Lagrangians, for instance the Polyakov Lagrangian:

$$L = \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\nu$$  \hspace{1cm} (2.1)

with $g = \left| \text{det}(g_{ab}) \right|$. One has to compute functional integrals of the following type:

$$Z = \sum_{\text{topologies}} \int_{\text{metrics}} Dg_{ab} \int_{\text{leaves}} DX^\mu \exp(\frac{iS}{\hbar})$$  \hspace{1cm} (2.2)

It turns out that this procedure by itself does not give rise to a fully consistent string theory. In order to get a fully consistent string theory we need to add some internal fermionic degrees of freedom to the string and generalize the notion of area by adding new terms involving these fermionic degrees of freedom. This leads to five (apparently) different consistent string theories in (9+1) dimensional space-time.

In the first quantized formalism, the dynamics of a point particle is described by quantum mechanics. Generalizing this we see that the first quantized description of a string will involve a (1+1) dimensional quantum field theory. However unlike a conventional quantum field theory where the spatial directions have infinite extent, here the spatial direction, which labels the coordinate on the string, has finite extent. It represents a compact circle if the string is closed (Fig.3(a)) and a finite line interval if the string is open (Fig.3(b)).
This (1+1) dimensional field theory is known as the world-sheet theory. The fields in this (1+1) dimensional quantum field theory and the boundary conditions on these fields vary in different string theories. Since the spatial direction of the world-sheet theory has finite extent, each world-sheet field can be regarded as a collection of infinite number of harmonic oscillators labelled by the quantized momentum along this spatial direction. Different states of the string are obtained by acting on the Fock vacuum by these oscillators. This gives an infinite tower of states. Typically each string theory contains a set of massless states and an infinite tower of massive states. The massive string states typically have mass of the order of $10^{-33} \text{cm}^{-1} \sim 10^{19} \text{GeV}$ and are far beyond the reach of the present day accelerators. Thus the interesting part of the theory is the one involving the massless states. We shall now briefly describe the interaction in various string theories and their compactifications.

2.1 Interactions

To describe the theory we must also describe the interaction between various particles in the spectrum (in string theories). In particular, we would like to know how to compute a scattering amplitude involving various string states. It turns out that there is a unique way of introducing interaction in string theory. Consider for example a scattering involving four external strings, situated along some specific curves in space-time. The prescription for computing the scattering amplitude is to compute the weighted sum over all possible string world-sheet bounded by the four strings with weight factor $e^{-S}$, $S$ being the string tension multiplied by the generalized area of this surface (taking into account the fermionic degrees of freedom of the world-sheet). One such surface is shown in Fig.3.

If we imagine the time axis running from left to right, then this diagram represents two strings joining into one string and then splitting into two strings, — the analog of a tree diagram in field theory. A more complicated surface is shown in
Figure 4: A string world-sheet bounded by four external strings.

Figure 5: A more complicated string world-sheet.

two and joins again, and finally splits into two strings. This is the analog of a one
loop diagram in field theory. The relative normalization between the contributions
from these two diagrams is not determined by any consistency requirement. This
introduces an arbitrary parameter in string theory, known as the string coupling
constant. However, once the relative normalization between these two diagrams is
fixed, the relative normalization between all other diagrams is fixed due to various consistency requirement. Thus besides the dimensionful parameter $\alpha'$, string theory has a single dimensionless coupling constant.

So Feynman’s interaction graphs are substituted by Riemann surfaces (which are topological configurations of interactions). For doing that, we need Riemann surfaces theory. For example we need Teichmüller theory of moduli spaces for knowing exactly what are the automorphisms of a Riemann surface (what are its diffeomorphisms which are not isotopic to the identity, what are the complex structures compatible with a given differentiable structure, etc.). We need also the solution of the Schottky problem. Let $S$ be a Riemann surface of genus $g$. It is well known that it is possible to find a basis $(a_i, b_i), i = 1, ..., g$ of the homology of $S$ and a basis $(\omega_i), j = 1, ..., g$ of the space of differentiable 1–forms which are the simplest possible, that is to say which satisfy:

$$\int_{a_i} \omega_j = \delta_{ij} \quad (2.3)$$

and

$$\int_{b_i} \omega_j = \Omega_{ij} \quad (2.4)$$

the matrix $\Omega = (\Omega_{ij})$ of periods being symmetric and of imaginary part positive definite: $\text{Im} \Omega > 0$. But if $g > 3$, the space of the matrices $\Omega$ which are symmetric and of imaginary part $> 0$ has a dimension $\frac{1}{2}g(g + 1)$ which is greater than the dimension $3g - 3$ of the moduli space of $S$. Therefore we must characterize the $\Omega$ which can be the period matrices of Riemann surfaces. This is the Schottky problem. It has been solved only in 1984.

### 2.2 Compactification

The five different string theories mentioned above, all live in ten space-time dimensions. Since our world is (3+1) dimensional, these are not realistic string theories. However one can construct string theories in lower dimensions using the idea of compactification. The idea is to take the (9+1) dimensional space-time as the product of a $(9 - d)$ dimensional compact manifold $\mathcal{M}$ with euclidean signature and a $(d + 1)$ dimensional Minkowski space $R^{d,1}$. Then, in the limit when the size of the compact manifold is sufficiently small so that the present day experiments cannot resolve this distance, the world will effectively appear to be $(d + 1)$ dimensional. Choosing $d = 3$ will give us a (3+1) dimensional theory. Of course we cannot choose any arbitrary manifold $\mathcal{M}$ for this purpose; it must satisfy the equations of motion of the effective field theory that comes out of string theory. One also normally considers only those manifolds which preserve part of the space-time supersymmetry of the original ten dimensional theory, since this guarantees vanishing of the cosmological constant, and hence consistency of the corresponding string theory order by order in perturbation theory. There are many known examples of manifolds satisfying these
restrictions \textit{e.g.} tori of different dimensions, K3, Calabi-Yau manifolds etc. For instance -using a Kaluza-Klein device - we can compactify 16 dimensions (starting from $D = 26$) using the lattice of the roots of the Lie gauge group $E_8 \otimes E_8$ and then compactify again 6 dimensions :$M^{10} \rightarrow M^4 \times K^6$. Physical constraints of preservation of supersymmetry impose for example that there exists on $K$ a spinor field $\xi$ which is constant for the covariant derivation (i.e. $D_i \xi = 0$). This fact imposes drastic constraints upon the geometry of $K^6$: the Ricci curvature must be $= 0$, the holonomy group must be $= SU(3)$, the first Chern class $c_1(K)$ of $K$ must be $= 0$, there must be exist a kähler metric on $K$, etc. (In fact, according to a celebrated theorem of Calabi and Yau, a Kähler manifold $K^{2n}$ with $c_1(K) = 0$ admits necessarily a kähler metric with holonomy group $SU(n)$ (and not $O(2n)$)).

Instead of going via the effective action, one can also directly describe these compactified theories as string theories. For this one needs to modify the string world-sheet action in such a way that it describes string propagation in the new manifold $M \times R^{d,1}$, instead of in flat ten dimensional space-time. This modifies the world-sheet theory to an interacting non-linear $\sigma$-model instead of a free field theory. Consistency of string theory puts restriction on the kind of manifold on which the string can propagate. At the end both approaches yield identical results.

The effect of this compactification is to periodically identify some of the bosonic fields in the string world-sheet field theory -- the fields which represent coordinates tangential to the compact circles. One effect of this is that the momentum carried by any string state along any of these circles is quantized in units of $1/R$ where $R$ is the radius of the circle. But that is another novel effect: we now have new states that correspond to strings wrapped around a compact circle. For such a states, as we go once around the string, we also go once around the compact circle. These states are known as winding states and play a crucial role in the analysis of duality symmetries.

![Figure 6: Example of Compactification. The two dimensional cylinder appears to be one dimensional if the radius is very small.](image-url)

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$10^{10}$ cm

$10^{10}$ light years
It turns out that there are many different choices for this six dimensional compact manifold (case $d = 3$). Thus each of the five string theories in $(9+1)$ dimensions gives rise to many possible string theories in $(3+1)$ dimensions after compactification. Some of these theories come tantalizingly close to the observed universe. In particular one can construct models with:

i) Gauge group containing the standard model gauge group $SU(3) \times SU(2) \times U(1)$,

ii) Chiral fermions representing three generations of quarks and leptons,

iii) N=1 supersymmetry,

iv) Gravity.

Furthermore unlike conventional quantum field theories which are ultraviolet divergent but renormalizable, and quantum general relativity which is ultraviolet divergent and not renormalizable, string theories have no ultraviolet divergence at all!

### 3 Unification

The motivation for supersymmetry comes from the idea of gauge unification. Recent experiments have yielded precise determinations of the strengths of the $SU(3) \times SU(2) \times U(1)$ gauges interactions - the analogs of the structure constant for these interactions. In quantum field theory these values depend on the energy at which they are measured in a way that depends on the particle content of the theory. Using the measured values of the coupling constants and the particle content of the standard model, one can extrapolate to higher energies and determine the coupling constants there. The result is that the three coupling constants do not meet at the same point. However, repeating this extrapolation with the particles belonging to the minimal supersymmetric extension of the standard model, the three gauge coupling constants meet at a point as sketched in fig.7.

### 3.1 Supersymmetry

Supersymmetry is a symmetry that relates bosons to fermions, though every fermion has a bosonic superpartner and vice versa. For example, fermionic quarks are partners of bosonic squarks. By this we mean that quarks and squarks belong to the same irreducible representation of the supersymmetry, if supersymmetry were an broken symmetry, particles and their superpartners would have exactly the same masse. So it is inherently quantum mechanical symmetry, since the very concept of fermions is a quantum mechanical.

In quantum field theory boson fields have dimension one and fermion fields have $\frac{3}{2}$ in order that the action be dimensionless (in units $\hbar = c = 1$). The reason is that boson fields have two derivatives in their action while fermion fields have
only one. It is not difficult to see that two supersymmetry transformations will
certainly lead to a gap of one unit of dimension. The only dimensional object
different from fields themselves available to fill this gap is the derivative. Thus in
any global supersymmetry model we can always find a derivative appearing in a
double transformation relation, purely on dimensional grounds.
Mathematically therefore the global supersymmetry resemble taking the square root
of the transformation operator. So actually it is not an internal symmetry but an
enlargement of the Poincaré groupe. This amounts to an extension of space-time
to *superspace* that includes extra spinorial anticommuting coordinates as well as
ordinary coordinates. We do not change the structure of space-time but we add
structure to it. We start with the usual coordinates \( X^\mu = t, x, y, z \) and add an odd
dimensions \( \theta^\alpha (\alpha = 1, ..., n) \). These dimensions are fermionic and anticommute
\[
\theta^\alpha \theta^\beta = -\theta^\beta \theta^\alpha
\] (3.5)
They are quantum dimension that have no classical analog, which makes it difficult
to visualize or to understand them intuitively.

More formally, a general Grassmann algebra with \( n \) generators \( G_n \) is defined as
follows:

i) \( G_n \) is a vector space over the complex numbers

ii) a product is defined over \( G_n \), which is associative and bilinear with respect to
addition and multiplication by scalars

iii) \( G_n \) contains the unit element for this product

iv) \( G_n \) is generated by \( n \) elements \( \xi^A, A = 1, ..., n \) which obey the relation
\[
\xi^A \xi^B + \xi^B \xi^A = 0
\] (3.6)
Furthermore, there is no other independent relations among the generators. It follows from the anticommutation of the $\xi$'s that $G_n$ is $2^n$-dimensional as a vector space. A basis of $G_n$ is given by the monomials $1, \xi^A, \xi^A \xi^B (A < B), \ldots \xi^1 \xi^2 \ldots \xi^n$. A general element $g$ of $G_n$ reads

$$g = g_0 + g_A \xi^A + g_{AB} \xi^A \xi^B + \ldots + g_{A_1 A_2 \ldots A_n} \xi^{A_1} \xi^{A_2} \ldots \xi^{A_n} \quad (3.7)$$

where the coefficient $g_{AB}, \ldots, g_{A_1 A_2 \ldots A_n}$ can be assumed to be completely antisymmetric. The coefficient $g_0$ is the component of $g$ along unity.

So we can distinguish between even dynamical $X^\mu$ variables (commuting c-numbers) and odd dynamical variables $\theta^\alpha$ (anticommuting c-numbers). where

$$X^\mu(t) = X_0^\mu(t) + X^\mu_{AB}(t) \xi^B \xi^A + \ldots \quad (3.8)$$

$$\theta^\alpha(t) = \theta^\alpha_A(t) \xi^A + \theta^\alpha_{ABC}(t) \xi^A \xi^B \xi^C + \ldots \quad (3.9)$$

A Grassmann-valued function of the dynamical variables $(X^\mu, \theta^\alpha)$ is an element of the Grassmann algebra, to which $X^\mu$ and $\theta^\alpha$ belong, which depends on $X^\mu$ and $\theta^\alpha$. In terms of components, a function $F(X^\mu, \theta^\alpha)$ is equivalent to a set of function $F_0, F_A, F_{AB}, \ldots$ of the components $X_0^\mu, X^\mu_{AB}, \ldots, \theta^\alpha_A, \ldots$ of $X^\mu$ and $\theta^\alpha$ such that

$$F = F_0(X_0^\mu, X_{AB}^\mu, \ldots, \theta^\alpha_A, \ldots) + F_A(X_0^\mu, X_{AB}^\mu, \ldots, \theta^\alpha_A, \ldots) \xi^A + \ldots \quad (3.10)$$

of particular importance are the so-called superfunctions. These depends on the individual components $X_0^\mu, X_{AB}^\mu, \ldots, \theta^\alpha_A, \ldots$ only through $X^\mu$ and $\theta^\alpha$ and have no explicit dependence on $\xi^A$.

The fact that the odd directions are anticommuting has important consequences. Consider a function of superspace (with $\alpha = 4$)

$$F(X, \theta) = \phi(X) + \theta_\alpha \psi_\alpha(X) + \ldots + \theta^4 F(X) \quad (3.11)$$

Since the square of any $\theta$ is zero and there are only four different $\theta$'s the expansion in powers of $\theta$ terminates at the fourth order. Therefore, a function of superspace includes only a finite numbers of functions of $X$ (16 in this case). Hence, we can replace any function of superspace $F(X, \theta)$ with the component functions $\phi(X), \psi(X), \ldots$. This include bosons $\phi(X)$ and fermions $\psi(X), \ldots$. This is one way of understanding the pairing between bosons and fermions.

A supersymmetric theory looks like an ordinary theory with degrees of freedom and interactions that satisfy certain symmetry requirement. In this sense, supersymmetry by itself is not a very radical proposal. However, the fact that bosons and fermions come in pairs in supersymmetric theories had important consequences. In some loop diagrams the bosons and fermions cancel each other. This boson-fermion cancellation is at the heart of most of the applications of supersymmetry.
Just as for usual symmetries, one can distinguish between two kinds of supersymmetries: global ones (rigid supersymmetry) and local ones (gauge supersymmetry). In local theory the translation operator differs from point to point. This is precisely the notion of a general coordinate transformation and leads us to expect that gravity must be present. Indeed, guided by the requirement of local supersymmetry invariance and using 'Noether’s method', we can actually get massless spin \( \frac{3}{2} \) field gauging supersymmetry, i.e. gravitino and massless spin-2 field gauging space-time symmetry, i.e. the graviton. So the local gauge theory of supersymmetry implies a local gauge theory of gravity. This is the reason for such local supersymmetry being called supergravity. Still that in supergravity and extended supergravity the relation between supersymmetry and extra internal symmetry is still unclear. So the relation between external and internal geometries are still vague within the context of supergravity.

On the other hand in the revitalized Kaluza-Klein theory, where the number of extra space dimensions become seven (taking into account the number of symmetry operations embodied in grand unified theories and extended N=8 supergravity), the internal symmetries are the manifestations of the geometrical symmetries associated with the extra compactified space dimensions and that all the kinds of geometries associated with internal symmetries are genuine space geometries, i.e. geometries associated with extra space dimensions. So the question concerning the relation between internal and external geometries, in a deeper sense, remains profound. Nevertheless, the modern Kaluza-Klein theory does open a door for establishing the correlation between non-gravitational gauge potentials and the geometrical structures in four dimensional space-time via the geometrical structures in extra dimensions. Within this theoretical context the unifications of gravity and other gauge interactions, is in principle testable, and cannot be accused of being irrelevant to the future development of fundamental physics.

In superstring the introduction of extra compactified space dimensions is due to different considerations from just reproducing the gauge symmetry. Therefore, the properties and structures of the compactified dimensions are totally different from those in the Kaluza-Klein version. For example there is no symmetry in the compact dimensions from which the gauge symmetries emerge. The gauge interactions are correlated with the geometrical structure of ten-dimensional space-time as a whole and not with the extra dimensions. More, in ten-dimensional quantum superstring theories there are gravitational and Yang-Mills anomalies, that is the violation of the conservation of the Yang-Mills charges and the energy-momentum. Requiring the absence of all anomalies leads to requiring a very intimate relationship between gravitational and Yang-Mills interactions.
3.2 Supersymmetry and strong coupling

Supersymmetry gives new information about strong coupling. To see this (we follow Polchinski [30]), let us consider in quantum theory the Hamiltonian operator $H$, the simplest example is the Hamiltonian:

$$H = w_a a^+ a + w_b b^+ b$$  \hspace{1cm} (3.12)

where we have bosonic and fermionic harmonic operators that obey:

$$[a, a^+] = [b, b^+] = 1$$  \hspace{1cm} (3.13)

The supersymmetric operator $Q$ is defined as:

$$Q \equiv b^+ a + a^+ b$$  \hspace{1cm} (3.14)

if $a^+ | 0 \rangle$ is a one boson state, then $Qa^+ | 0 \rangle$ becomes a one fermion state, and vice versa $Q$ obeys the following identity

$$[Q, H] = (w_a - w_b)Q$$  \hspace{1cm} (3.15)

If $w_a = w_b = w$, then the supersymmetric operator $Q$ commutes with the Hamiltonian and:

$$\{Q, Q^+\} = \frac{2}{w} H$$  \hspace{1cm} (3.16)

These identities show that $Q$ and $Q^+$ form a closed algebra with the Hamiltonian if the fermions and bosons have equal energy.

Take in addition the charge operator $G$ associated with an ordinary symmetry like electric charge or baryon number. The fact that $G$ is a symmetry means that it commutes with the Hamiltonian,

$$[H, G] = 0 .$$  \hspace{1cm} (3.17)

as we have said for supersymmetry one has the same,

$$[H, Q] = 0 ,$$  \hspace{1cm} (3.18)

but there is an additional relation

$$Q^2 = H + G ,$$  \hspace{1cm} (3.19)

in which the Hamiltonian and ordinary symmetries appear on the right. It is this equation that gives the extra information. Consider now a state $| \psi \rangle$ which is neutral under supersymmetry:

$$Q | \psi \rangle = 0 .$$  \hspace{1cm} (3.20)

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3There are usually several $G$s and several $Q$s, so that there should be additional indices and constants in these equations.
We are interested in states that are neutral under at least one $Q$ but usually not all of them. These are known as BPS (Bogomolnyi–Prasad–Sommerfield) states. The expectation value of the relation (3.19) in this state gives us:

$$\langle \psi | Q^2 | \psi \rangle = \langle \psi | H | \psi \rangle + \langle \psi | G | \psi \rangle .$$  

(3.21)

The left side vanishes by the BPS property, while the two terms on the right are the energy $E$ of the state $|\psi\rangle$ and its charge $q$ under the operator $G$. Thus

$$E = -q ,$$  

(3.22)

and so the energy of the state is determined in terms of its charge. So a dynamical quantity is determined entirely by symmetry information.

Since the calculation of $E$ uses only symmetry information, it does not depend on any coupling being weak. Thus we know something about the spectrum at strong coupling. The BPS states are only a small part of the spectrum, but by using this and similar types of information from supersymmetry, together with general properties of quantum systems, one can usually recognize a distinctive pattern in the strongly coupled theory and so deduce the dual theory.

### 3.3 Vacuum Selection

Another property of many supersymmetric theories that make them tractable is that they have a family of inequivalent vacua. To understand this fact we should contrast it with the situation in ferromagnet, which has a continuum of vacua, labeled by the common orientation of the spins. These vacua are all equivalent, i.e. the physical observables in one of these vacua are exactly the same as in any other. The reason is that these vacua are related by a symmetry. The system must choose one of them, which leads to spontaneous symmetry breaking. However, in a supersymmetric theory the zero-point energy of the fermions exactly cancels that of the bosons this is the source of the presence of some degeneracy. So we see that a supersymmetric system has a continuous family of vacua. This family, or manifold, is referred to as a moduli space of vacua.

The analysis of supersymmetric theories is usually simplified by the presence of these manifolds of vacua, by using the asymptotic behavior along several directions, where the analysis of the system is simple and various approximation techniques are applicable, as well as the constraints from holomorphy. A unique solution is

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4The main point is that the supersymmetric quantum field theories are very constrained. The dependance of some observables on the parameters of the problem is so constrained that there is only one solution that satisfies all the consistency conditions. More technically, because of supersymmetry some observables vary holomorphically with the coupling constants, which are complex numbers in these theories. due to Cauchy’s theorem, such analytic functions are determined in terms of very little data: the singularities and the asymptotic behavior. Therefore, if supersymmetry requires an observables to depend holomorphically on the parameters and we know the singularities and the asymptotic behavior, we can determine the exact answer.
obtained. So an approximate calculations, which are valid only in some regime, gives us the exact answer.

4 Dualities

Few words have been used with more different meanings than the word duality. Even within the restricted framework of string theories, duality originally meant a symmetry between the s and the t-channels in strong interactions (coming from the demands in the S-matrix approach of the sixties of Regge behavior without fixed poles and analiticity, which were shown to imply the existence of an infinite number of resonances) \cite{73}. Somewhat related ideas, also termed duality, appear in the context of Conformal Field Theory (CFT) as simple consequences of locality and associativity of the operator product expansion (OPE) \cite{74}.

Duality symmetry plays an important role in Statistical Mechanics, in particular in the analysis of the phase diagram of spin systems. It can also be understood as a way to show the equivalence between two apparently different theories. On a lattice system described by a Hamiltonian \( H(g_i) \) with coupling constants \( g_i \) the duality transformation produces a new Hamiltonian \( H^*(g_i^*) \) with coupling constants \( g_i^* \) on the dual lattice. In this way one can often relate the strong coupling regime of \( H(g) \) with the weak coupling regime of \( H^*(g^*) \). An important application was the determination of the exact temperature at which the phase transition of the two-dimensional Ising model takes place \cite{76}.

More recently, the word duality (space-time duality) has been introduced in yet another sense.

4.1 String Duality, M-theory

Existence of duality symmetries in string theory started out as a conjecture and still remains a conjecture. However so many non-trivial tests of these conjectures have been performed by now that most people in the field are convinced of the validity of these conjectures.

A duality conjecture is a statement of equivalence between two or more apparently different string theories. Two of the most important features of duality are as follows:

i) Often under the duality map, an elementary particle in one theory gets mapped to a composite particle in a dual theory and vice versa. Thus classification of particles into elementary and composite loses significance as it depends on which particular theory we use to describe the system.

ii) Often duality relates a weakly coupled string theory to a strongly coupled string theory and vice versa. In many simple cases the coupling constants \( g \)
and $\tilde{g}$ in the two theories are related via the simple relation:

$$g = \tilde{g}^{-1}.$$  

(4.23)

Thus a perturbation expansion in $g$ contains information about non-perturbative effects in the dual theory. In particular the tree level (classical) results in one theory can contain contribution from perturbative and non-perturbative terms in the dual theory. This also clearly shows that duality is a property of the full quantum string theory, and not of its classical limit.

Thus there are two aspects of duality

- elementary ↔ composite
- classical ↔ quantum

Let me now give some examples of dual pairs of string theories.

i) $(9+1)$ dimensional SO(32) heterotic and type I string theories are conjectured to be dual to each other.[3, 4]

ii) SO(32) heterotic string theory compactified on a four dimensional torus (denoted as $T^4$) is conjectured to be dual to type IIA string theory compactified on a different four dimensional manifold, denoted by $K3$.[5]

iii) Type IIB string theory is conjectured to be self-dual, in the sense that the type IIB string theories at two different couplings $g$ and $\tilde{g}$ related by eq.(4.23) are conjectured to describe the same physical theory.[6]

iv) Heterotic string theory compactified on a six dimensional torus, denoted by $T^6$, is conjectured to be self-dual in the same sense as above.[5, 7]

So due to the fact that a duality conjecture relates two apparently different theories, we see that it gives a unified picture of all string theories. The situation is summarized in Fig.8.

According to this picture the apparently different string theories and their compactifications are just different limits of the same theory, with a large parameter

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5 One should keep in mind that this is only a schematic representation.
There is no universally accepted name for this central theory. We call it $U$-theory. $U$ can be taken to stand for Unknown or Unified. Some small regions of the parameter space of $U$-theory, which can be represented by some weakly coupled string theory, are reasonably well understood and correspond to the weak coupling regime of the five different string theories and their compactifications. But for most of the parameter space $U$-theory does not have a description in terms of weakly coupled string theory. Note that in one corner of the parameter space of $U$-theory, there is a theory called $M$-theory [9, 3, 10, 11, 12] which has not been introduced before. At present not much is known about $M$-theory except that its low energy limit is the eleven dimensional supergravity theory, and that various string theories and their compactifications approach $M$-theory in certain limits. However, unlike string theory, $M$-theory does not have any coupling constant, and no systematic procedure for doing computations in $M$-theory beyond the low energy supergravity limit is known.

The $M$-theory point in the figure is in fact a point of $SO(10, 1)$ symmetry: the spacetime symmetry of string theory is larger than had been suspected. The extra piece is badly spontaneously broken, at weak coupling, and not visible in the perturbation theory, but it is a property of the exact theory. It is interesting that $SO(10, 1)$ is known to be the largest spacetime symmetry compatible with supersymmetry.

Another way to describe this is that in the $M$-theory limit the theory lives in eleven spacetime dimensions: a new dimension has appeared. This is one of the surprising discoveries of the past few years.

4.2 The canonical approach to T-duality

In String Theory and Two-Dimensional Conformal Field Theory duality is an important tool to show the equivalence of different geometries and/or topologies and in determining some of the genuinely stringy implications on the structure of the low energy Quantum Field Theory limit. T-Duality symmetry was first described on the context of toroidal compactifications [5]. For the simplest case of a single compactified dimension of radius $R$, the entire physics of the interacting theory is left unchanged under the replacement $R \to \alpha'/R$ provided one also transforms the dilaton field $\phi \to \phi - \log (R/\sqrt{\alpha'})$ [2]. This simple case can be generalized to arbitrary toroidal compactifications described by constant metric $g_{ij}$ and antisymmetric tensor $b_{ij}$ [3]. The generalization of duality to this case becomes $(g + b) \to (g + b)^{-1}$ and $\phi \to \phi - \frac{1}{2} \log \det (g + b)$. In fact this transformation is an element of an infinite order discrete symmetry group $O(d, d; Z)$ for $d$-dimensional toroidal compactifications [5, 6]. The symmetry was later extended to the case of non-flat conformal backgrounds in [8]. In Buscher’s construction one starts with a manifold $M$ with metric $g_{ij}, i, j = 0, \ldots, d - 1$, antisymmetric tensor $b_{ij}$ and dilaton field $\phi(x_{i})$. One requires the metric to admit at least one continuous abelian isometry leaving invar-

\footnote{In string theory parameters themselves are related to vacuum expectation values of different fields and are expected to be determined dynamically.}
The $\sigma$-model action constructed out of $(g, b, \phi)$. Choosing an adapted coordinate system $(x^0, x^\alpha) = (\theta, x^\alpha)$, $\alpha = 1, \ldots, d - 1$ where the isometry acts by translations of $\theta$, the change of $g, b, \phi$ is given by

\[
\begin{align*}
\tilde{g}_{00} &= 1/g_{00}, \\
\tilde{g}_{0\alpha} &= b_{0\alpha}/g_{00}, \\
\tilde{g}_{\alpha\beta} &= g_{\alpha\beta} - (g_{00}g_{\beta\beta} - b_{00}b_{0\beta})/g_{00} \\
\tilde{b}_{0\alpha} &= g_{0\alpha}/g_{00}, \\
\tilde{b}_{\alpha\beta} &= b_{\alpha\beta} - (g_{0\alpha}b_{0\beta} - g_{0\beta}b_{0\alpha})/g_{00}, \\
\tilde{\phi} &= \phi - \frac{1}{2} \log g_{00}. \quad (4.24)
\end{align*}
\]

The final outcome is that for any continuous isometry of the metric which is a symmetry of the action one obtains the equivalence of two apparently very different non-linear $\sigma$-models. The transformation (4.24) is referred to in the literature as abelian T-duality due to the abelian character of the isometry of the original $\sigma$-model. If $n$ is the maximal number of commuting isometries, one gets a duality group of the form $O(n, n; \mathbb{Z})$ [90]. T-Duality symmetries are useful in determining important properties of the low-energy effective action, in particular in questions related to supersymmetry breaking and to the lifting of flat directions from the potential [87]. Although the transformation (4.24) was originally obtained using a method apparently not compatible with general covariance, it is not difficult to modify the construction to eliminate this drawback [91].

Of more recent history is the notion of non-abelian T-duality [94, 95, 96, 97], which has no analogue in Statistical Mechanics. The basic idea of [94], inspired in the treatment of abelian T-duality presented in [89], is to consider a conformal field theory with a non-abelian symmetry group $G$.

In the abelian case it is also possible to work out the mapping between some operators in the original and dual theories, as well as the global topology of the dual manifold [91]. Thus for $G$ abelian we have a rather thorough understanding of the detailed local and global properties of T-duality. In the non-abelian case global information can only be extracted for $\sigma$-models with chiral currents [97].

5 The Canonical Approach

Some suggestions have been made in the literature pointing (at least in the simplified situation where all backgrounds are constant or dependent only on time) towards an understanding of T-duality as particular instances of canonical transformations [86, 116].

Following [117] we can show that this idea works well when the background admits an abelian isometry, laying T-duality on a simpler setting, namely as a (privileged) subgroup of the whole group of (non-anomalous, that is implementable in Quantum Field Theory [118]) canonical transformations on the phase space of the theory.
So Buscher’s transformation formulae can be derived by performing a given canonical transformation on the Hamiltonian of the initial theory. This is a minimal approach in the sense that no extraneous structure has to be introduced, and all standard results in the abelian case are easily recovered using it. In particular it is possible to perform the T-duality transformation in arbitrary coordinates not only in the original manifold (which was also possible in Roček and Verlinde’s formulation) but also in the dual one. Even more, all the generators of the full T-duality group \( O(d,d; \mathbb{Z}) \) can be described in terms of canonical transformations. This gives the impression that the T-duality group should be understood in terms of global symplectic diffeomorphisms. It would be useful to formulate it in the context of some analogue of the group of disconnected diffeomorphisms, but for the time being such a construction is lacking.

Concerning non-abelian duality, it seems to fall beyond the scope of the Hamiltonian point of view. There is one example [119] in which the non-abelian dual has been constructed out of a canonical transformation but it is still early to say whether the general case can be treated similarly.

### 5.1 The Abelian Case

We start with a bosonic sigma model written in arbitrary coordinates on a manifold \( M \) with Lagrangian

\[
L = \frac{1}{2} (g_{ab} + b_{ab}) (\dot{\phi}) \partial_+ \phi^a \partial_- \phi^b \tag{5.1}
\]

where \( x^\pm = (\tau \pm \sigma)/2 \), \( a, b = 1, \ldots, d = \dim M \). The corresponding Hamiltonian is

\[
H = \frac{1}{2} (g_{ab} (p_a - b_{ac} \phi^c) (p_b - b_{bd} \phi^d) + g_{ab} \dot{\phi}_a \phi^b) \tag{5.2}
\]

where \( \phi^a \equiv d\phi^a/d\sigma \). We assume moreover that there is a Killing vector field \( k^a \), \( \mathcal{L}_k g_{ab} = 0 \) and \( i_k H = -dv \) for some one-form \( v \), where \( (i_k H)_{ab} \equiv k^c H_{cab} \) and \( H = db \) locally. This guarantees the existence of a particular system of coordinates, “adapted coordinates”, which we denote by \( x^i \equiv (\theta, x^\alpha) \), such that \( \vec{k} = \partial/\partial \theta \). We denote the jacobian matrix by \( e_{ib}^i \equiv \partial x^i / \partial \phi^a \).

This defines a point transformation in the original Lagrangian \( [2] \) which acts on the Hamiltonian as a canonical transformation with generating function \( \Phi = x^i(\phi)p_i \), and yields:

\[
\begin{align*}
p_a &= e_a^i p_i \\
x^i &= x^i(\phi).
\end{align*}
\tag{5.3}
\]

Once in adapted coordinates we can write the sigma model Lagrangian as

\[
L = \frac{1}{2} G(\dot{\theta}^2 - \dot{\phi}^2) + (\dot{\theta} + \theta^i J^- + (\dot{\theta} - \theta^i) J^+ + V \tag{5.4}
\]
where
\[ G = g_{00} = k^2 \quad V = \frac{1}{2} (g_{\alpha\beta} + b_{\alpha\beta}) \partial_+ x^\alpha \partial_- x^\beta \]
\[ J_- = \frac{1}{2} (g_{0\alpha} + b_{0\alpha}) \partial_- x^\alpha \quad J_+ = \frac{1}{2} (g_{0\alpha} - b_{0\alpha}) \partial_+ x^\alpha. \] (5.5)

In finding the dual with a canonical transformation we can use the Routh function with respect to \( \theta \), i.e. we only apply the Legendre transformation to \((\theta, \dot{\theta})\). The canonical momentum is given by
\[ p_\theta = G\dot{\theta} + (J_+ + J_-) \] (5.6)
and the Hamiltonian
\[ H = p_\theta \dot{\theta} - L = \frac{1}{2} G^{-1} p_\theta^2 - G^{-1} (J_+ + J_-) p_\theta + \frac{1}{2} G\theta' \quad + \]
\[ + \frac{1}{2} G^{-1} (J_+ + J_-)^2 + \theta' (J_+ - J_-) - V. \] (5.7)

The Hamilton equations are:
\[ \dot{\theta} = \frac{\delta H}{\delta p_\theta} = G^{-1} (p_\theta - J_+ - J_-) \]
\[ \dot{p}_\theta = -\frac{\delta H}{\delta \theta} = (G\theta' + J_+ - J_-)' \] (5.8)

The generator of the canonical transformation we choose is:
\[ F = \frac{1}{2} \int_{D, \partial D = S^1} d\tilde{\theta} \wedge d\theta = \frac{1}{2} \int_{S^1} (\theta' \tilde{\theta}' - \theta \tilde{\theta}') d\sigma \] (5.9)
that is,
\[ p_\theta = \frac{\delta F}{\delta \theta} = -\tilde{\theta}' \]
\[ p_{\tilde{\theta}} = -\frac{\delta F}{\delta \tilde{\theta}} = -\theta'. \] (5.10)

This generating functional does not receive any quantum corrections (as explained in [118]) since it is linear in \( \theta \) and \( \tilde{\theta} \). If \( \theta \) was not an adapted coordinate to a continuous isometry, the canonical transformation would generically lead to a non-local form of the dual Hamiltonian. Since the Lagrangian and Hamiltonian in our case only depend on the time- and space-derivatives of \( \theta \), there are no problems with non-locality. The transformation (5.10) in (5.7) gives:
\[ \tilde{H} = \frac{1}{2} G^{-1} \tilde{\theta}'^2 + G^{-1} (J_+ + J_-) \tilde{\theta}' + \]
\[ + \frac{1}{2} G p_\theta^2 - (J_+ - J_-) p_\theta + \frac{1}{2} G^{-1} (J_+ + J_-)^2 - V. \] (5.11)
Since:
\[ \dot{\tilde{\theta}} = \frac{\delta \tilde{H}}{\delta \tilde{p}_\tilde{\theta}} = G p_{\tilde{\theta}} - (J_+ - J_-), \]  
we can perform the inverse Legendre transform:
\[ \tilde{L} = \frac{1}{2} G^{-1} (\dot{\theta}^2 - \dot{\tilde{\theta}}^2) + G^{-1} J_+ (\dot{\tilde{\theta}} - \dot{\theta}) \]
\[ - G^{-1} J_- (\dot{\tilde{\theta}} + \dot{\theta}) + V - 2 G^{-1} J_+ J_- \]  
(5.12)

From this expression we can read the dual metric and torsion and check that they are given by Buscher’s formulae:
\[ \tilde{g}_{00} = \frac{1}{g_{00}}, \quad \tilde{g}_{0\alpha} = - \frac{b_{0\alpha}}{g_{00}}, \quad \tilde{g}_{\alpha\beta} = \frac{g_{0\alpha} g_{0\beta} - b_{0\alpha} b_{0\beta}}{g_{00}} \]
\[ \tilde{b}_{0\alpha} = - \frac{g_{0\alpha}}{g_{00}}, \quad \tilde{b}_{\alpha\beta} = \frac{g_{0\alpha} b_{0\beta} - g_{0\beta} b_{0\alpha}}{g_{00}} \]  
(5.13)

For the dual theory to be conformal invariant the dilaton must transform as \( \Phi' = \Phi - \frac{1}{2} \log g_{00} \) \[8\] \[82\].

The dual manifold \( \tilde{M} \) is automatically expressed in coordinates adapted to the dual Killing vector \( \tilde{k} = \partial/\partial \tilde{\theta} \). We can now perform another point transformation, with the same jacobian as (5.3) to express the dual manifold in coordinates which are as close as possible to the original ones.

The transformations we perform are then: First a point transformation \( \phi^a \rightarrow \{ \theta, x^\alpha \} \), to go to adapted coordinates in the original manifold. Then a canonical transformation \( \{ \theta, x^\alpha \} \rightarrow \{ \tilde{\theta}, x^\alpha \} \), which is the true duality transformation. And finally another point transformation \( \{ \tilde{\theta}, x^\alpha \} \rightarrow \tilde{\phi}^a \), with the same jacobian as the first point transformation, to express the dual manifold in general coordinates.

It turns out that the composition of these three transformations can be expressed in geometrical terms using only the Killing vector \( k^a, \omega_a \equiv e^0_a \) and the corresponding dual quantities:\[8\]

The canonical approach has been very useful in order to obtain the dual manifold in an arbitrary coordinate system. With the usual approaches it is expressed in adapted coordinates to the dual isometry. This happens because the dual variables appear as Lagrange multipliers and after an integration by parts only the derivatives of them emerge, being then adapted coordinates automatically.

---

7 The minus signs in \( \tilde{g}_{0\alpha} \) and \( \tilde{b}_{0\alpha} \) can be absorbed in a redefinition \( \tilde{\theta} \rightarrow -\tilde{\theta} \).

8 Note that we must raise and lower indices with the dual metric, i.e. \( \tilde{e}^a_i = \tilde{g}_{ij} \tilde{e}^j_a, \tilde{e}^i_a = \tilde{g}^{ab} \tilde{e}_b^j \), which implies \( \tilde{\omega}_a = \omega_a \), but \( \tilde{\omega}^a = k^a (k^2 + v^2) + \tilde{e}^a \cdot v \) (where \( \tilde{e}^a \equiv e^0_a \)), \( \tilde{k}^a = k^a \) but \( \tilde{k}_a = (\omega_a - (\tilde{e}^a \cdot v))/k^2 \). We have moreover \( \tilde{\omega}^2 = k^2 + v^2 + g^{ab} v_b \omega_a \) and \( \tilde{k}^2 = 1/k^2 \).


## 5.2 Gauge Theories from Branes

The study of branes has been useful in deriving gauge theory results from string theory\cite{54, 55, 56, 57}.

A \( p \)-brane denotes a static configuration which extends along \( p \) spatial direction (the tangential directions) and is localized in all other spatial directions (the transverse directions). Thus the solution is invariant under translation along the \( p \) directions tangential to the brane, as well as the time direction, and approaches the vacuum configuration as we go away from the brane in any one of the transverse direction. Thus in this language,

\[
0 - \text{brane} \equiv \text{particle} \\
1 - \text{brane} \equiv \text{string} \\
2 - \text{brane} \equiv \text{membrane}
\]

eetc. Typically the quantum dynamics of a configuration of \( p \)-branes is described by a \((p + 1)\) dimensional gauge field theory,\cite{58, 59} and the coupling constant of this quantum field theory is related to the coupling constant of the string theory of which the brane configuration is a solution. In this case duality symmetries relating strong and weak coupling limits of the original string theory can be used to derive duality relations involving the quantum field theories describing the dynamics of the brane. This approach has been used to derive many different results in supersymmetric gauge theories. Some example are:

i) Derivation of Montonen-Olive duality\cite{60} in \( \mathcal{N} = 4 \) supersymmetric gauge theories.\cite{61}

ii) Derivation of Seiberg-Witten like results\cite{62} in \( \mathcal{N} = 2 \) supersymmetric gauge theories.\cite{54, 55, 57}

iii) Derivation of a special kind of symmetry, known as the mirror symmetry,\cite{63} in \((2+1)\) dimensional gauge theories.\cite{56}

iv) Derivation of Seiberg dualities\cite{64} involving \( \mathcal{N} = 1 \) supersymmetric gauge theories in \((3+1)\) dimensions.\cite{65, 66}

A special class of \( p \)-branes are called Dirichlet \( p \)-branes (D-branes).\cite{67} The name derives from the boundary conditions assigned to the ends of open strings. More general, in type II theories, one can consider an open string boundary conditions at the end given by \( \sigma = 0 \)

\[
\frac{\partial X_\mu}{\partial \sigma} = 0 \quad \mu = 0, 1, ..., p \\
X_\mu = X_0^\mu \quad \mu = p + 1, ..., 9
\]

and similar boundary conditions at the other end\cite{67}.

\footnote{quantum versions of objects that were first found as solitonic solutions of supergravity}

\footnote{The interpretation of these equations is that strings end on a \( p \)-dimensional object in space- a \( D-\)}
Figure 9: a) A D0-brane with two attached strings. b) A D1-brane (bold) with attached string. c) A D2-brane with attached string.
Existence of branes in string theory has also given rise to the possibility that the standard model gauge fields arise from branes rather than in the bulk of space-time. This corresponds to novel compactifications in which gravity lives in the bulk of the ten dimensional space-time, but the other observed fields (quarks, leptons, gauge particles etc.) live on a brane of lower dimension\textsuperscript{[67]}.

### 5.3 Maldacena Conjecture

A $p$-brane, or collection of $p$-branes, gives rise to a certain space-time geometry and gauge field configuration, which can be analyzed using the appropriate supergravity field equations. In a number of cases one finds that the geometry has an event horizon, giving a higher-dimensional analog of black holes. In some of these cases the geometry near the horizon is approximated by $AdS_{p+2} \times S^{D- p-2}$. This means that the AdS space has $p+2$ dimensions and the remainder of the $D$ dimensions form a sphere of $D - p - 2$ dimensions. There are three basic examples that have maximal supersymmetry (32 conserved supercharges). A stack of D3 branes in type IIB superstring theory has near horizon geometry $AdS_5 \times S^5$, a stack of M2-branes in M theory gives $AdS_4 \times S^7$, and a stack of M5-branes in M theory gives $AdS_7 \times S^4$.

Let me briefly describe some features of anti de Sitter space. $AdS_{n+1}$ is a maximally symmetric spacetime with a negative cosmological constant. It can be described as a hypersurface in flat space by the equation

\[
u_1^2 + \nu_2^2 - v_1^2 - v_2^2 - \ldots - v_n^2 = R^2,
\]

where $R$ is called the AdS radius. This spacetime has Lorentzian signature and reduces to Minkowski spacetime in $n+1$ dimensions in the limit $R \to \infty$. Just as an $n+1$-dimensional sphere ($S^{n+1}$) has $SO(n+2)$ symmetry, the symmetry of this spacetime is $SO(2,n)$, a noncompact version of the rotation group in $n+2$ dimensions. This contracts to the Poincaré group (consisting of the Lorentz group $SO(1,n)$ and translations) in the $R \to \infty$ limit. An intrinsic description of $AdS_{n+1}$ is given by the metric

\[
 ds^2 = \frac{R^2}{z^2}(dz^2 + dx^\mu dx_\mu), \quad z > 0,
\]

where

\[
 dx^\mu dx_\mu = dx_1^2 + \ldots + dx_{n-1}^2 - dt^2.
\]

Note that the $z = 0$ boundary of $AdS_{n+1}$ is an $n$-dimensional Minkowski spacetime, aside from a divergent factor. What matters is the \textit{conformal} structure, which is not sensitive to this divergent factor.
The $SO(2, n)$ isometries of the $(n + 1)$-dimensional anti de Sitter space induce the group of conformal transformations on its $n$-dimensional Minkowski boundary. (Strictly speaking, the boundary should be compactified by adding a point at infinity.) The conformal group is therefore also $SO(2, n)$. Let me illustrate how this works with a couple of examples. The $SO(1, n - 1)$ subgroup of $SO(2, n)$ given by Lorentz transformations of the $x^\mu$ corresponds to the Lorentz group of the boundary. The important point is that these transformations map $z = 0$ to $z = 0$, so that they are well-defined on the boundary. Another example is the isometry $x^\mu \rightarrow \lambda x^\mu, z \rightarrow \lambda z$ where $\lambda$ is a positive scale factor. This clearly leaves the AdS metric in eq. (5.18) invariant and preserves the boundary. Thus the corresponding conformal transformations of the boundary are scale transformations $x^\mu \rightarrow \lambda x^\mu$.

The basic idea of AdS/CFT duality is to identify a conformally invariant field theory (CFT) on the $n$-dimensional boundary with a suitable quantum gravity theory in the $(n + 1)$-dimensional AdS bulk. The IIB theory contains a four-index field $A_{\mu\nu\rho\lambda}$ for which the D3-brane is a source. It has a field strength $F_{\mu\nu\rho\lambda\sigma}$, which is self-dual (in ten dimensions). In the $AdS_5 \times S^5$ solution of the theory, the field $F$ has a quantized flux on the sphere. Schematically,

$$\int_{S^5} F = N,$$

(5.20)

where $N$ is a positive integer. This integer determines the radius $R$ of the $AdS_5$ and of the $S^5$, which are the same. Aside from a constant numerical factor, one finds that

$$R = (g_s N)^{1/4} \ell_s.$$  

(5.21)

Thus the curvatures (which are proportional to $R^{-2}$) are small compared to the string scale for $g_s N \gg 1$ and small compared to the Planck scale for $N \gg 1$. The first limit suppresses stringy corrections to supergravity, whereas the latter suppresses quantum corrections to classical string theory.

Maldacena’s duality conjecture is that type IIB superstring theory on $AdS_5 \times S^5$ with $N$ units of $F$ flux is equivalent to $\mathcal{N} = 4, D = 3 + 1 \ U(N)$ Yang–Mills theory with $g_Y^2 = g_s$. For this conjecture to be plausible, it is a crucial fact the $\mathcal{N} = 4$ super Yang–Mills theory [128] is a CFT with vanishing beta function, a fact that was proved in the early 1980s [129]. As should be clear from our presentation, this conjecture arose from considering the near-horizon geometry of a stack of $N$ D3-branes, in the limit $N \to \infty$.

6 conclusion

The underlying conception of space-time advocated by string theories is very interesting. Yet the conservative side of its conception is also striking: it takes the enlarged space-time coordinates themselves directly as physical degrees of freedom that have to be treated dynamically and quantum mechanically. Nevertheless, superstring theory gives an existence proof that gravity can be treated quantum
mechanically in a consistent way. More, it has predicted many new physical degrees of freedom: strings, classical objects such as smooth solitons and singular black hole, and new types of topological defects such as D-branes. It sheds much light on field theory duality and on the gauge-theoretical structure of the standard model. In field theory, we now understand that not just confinement but the whole range of surprise of strongly coupled field theory should be derived from duality, at least in the supersymmetric case. For string theory the change in viewpoint is perhaps even wider and includes the discovery that there is only one theory. To understand the dualities, or the relations between the different string theories, we have had to learn about this new degrees of freedom in string theory, such as D-branes.

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