Universal Spectral Correlators and Massive Dirac Operators

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Abstract

We derive the large-$N$ spectral correlators of complex matrix ensembles with weights that in the context of Dirac spectra correspond to $N_f$ massive fermions, and prove that the results are universal in the appropriate scaling limits. The resulting microscopic spectral densities satisfy exact spectral sum rules of massive Dirac operators in QCD.
1 Introduction

There has recently been remarkable progress in the understanding of chiral symmetry breaking in QCD, based on universality conjectures from random matrix theory [1, 4, 5]. Central in this development is the idea that the spectra of massless Dirac operators in gauge theories that have non-vanishing chiral condensates (as is the case when there is spontaneous chiral symmetry breaking) display universal large-volume scaling laws near the origin. There is now mounting evidence that this scenario is correct. First, the appropriate microscopic spectral densities, derived from large-$N$ matrix ensembles with Gaussian weights [2, 3], have been shown to consistently reproduce the exact spectral sum rules of Leutwyler and Smilga [4]. Second, the microscopic spectral densities, and in fact all microscopic spectral correlators, have been proven to be universal within the given classes of matrix model ensembles [5]. Third, there is now direct numerical support from Monte Carlo simulations of lattice gauge theory (for the case of gauge group $SU(2)$ and quenched staggered fermions) that the microscopic limit of lattice Dirac spectra is as predicted from random matrix theory [6].

In quenched Monte Carlo simulations it is possible to compute, for finite volume, the lattice eigenvalues for massless quarks. In more realistic lattice computations one will keep finite quark masses, and at most consider mass rescalings towards smaller values as the volume is increased. So for the practical purpose of comparison with lattice gauge theory results, it is imperative that finite quark mass effects are under control and understood. From a purely theoretical point of view there is also interest in the problem of finite masses. Just as one considers a microscopic rescaling of eigenvalues $z$ it seems natural, from the Dirac eigenvalue equation, to consider also the corresponding microscopic scaling limit in $m$ [1, 7]. In view of the above, one might hope that such a double-microscopic limit of the Dirac spectrum is universal as well, and computable from random matrix theory.

The starting point of the following computations are the suggested matrix model ensembles that should be relevant for massive Dirac fermions [3, 8]. From this we shall derive the pertinent spectral correlators (and in particular the spectral densities themselves) for $N_f$ massive fermions in the double-microscopic limit where both eigenvalues $\lambda$ and masses $m$ are appropriately rescaled with $N$. As a straightforward by-product of the analysis in ref. [5], we shall prove that all of these double-microscopic spectral correlators are universal. We shall also show that the resulting microscopic spectral densities are consistent with exact spectral sum rules for massive Dirac operators in QCD and QCD-like theories. Such spectral sum rules are normally derived for the case of massless Dirac operators [8], but the generalization to massive Dirac operators is not complicated (see also ref. [4]).

Our paper is organized as follows. In the next section we introduce the chiral unitary matrix ensemble, and use the orthogonal polynomial method to iteratively derive the relevant orthogonal polynomials for increasing values of $N_f$. We prove that in the large-$N$ limit the $N$th orthogonal polynomials $P^{(N_f)}_N(\lambda; m_1^2, \ldots, m_{N_f}^2)$ for each $N_f$ have universal asymptotic limits near the origin in both $\lambda = x/N^2$ and masses $m^2_f = \mu_f^2/N^2$. From this it follows that also all microscopic correlators, and the microscopic spectral densities themselves, are universal. We also write down a simple set of consistency relations for the microscopic spectral densities, which follows from the decoupling of heavy fermions. These consistency relations are found to be satisfied by our microscopic spectral densities. In section 3 we briefly discuss some of the exact spectral sum rules for massive Dirac operators of QCD-like theories, and we verify that our double-microscopic spectral densities are consistent with these exact sum rules, which have been derived without the use of random matrix theory. As in the massless case [4, 5, 8], this provides strong support to the conjecture that the microscopic spectral densities are universal not only within the context of random matrix theory, but indeed are exact expressions also for the
full spectral densities of QCD in that limit. Section 4 contains our conclusions, and a proof of the general-$N_f$ expressions can be found in the appendix.

## 2 The Chiral Unitary Ensemble

Consider 4-dimensional $SU(N_c \geq 3)$ gauge theories coupled to $N_f$ fermions in the fundamental representation of the gauge group. Assume that a non-vanishing chiral condensate $\Sigma \equiv \langle \bar{\psi} \psi \rangle$ has been formed. According to the conjectures of ref. [2] the microscopic spectral density

$$\rho_S(\zeta) = \lim_{V_4 \to \infty} \frac{1}{V_4 \Sigma} \rho(\frac{\zeta}{V_4 \Sigma})$$  \hspace{1cm} (1)

of the Dirac operator can be computed exactly in an ensemble of complex block hermitian $(2N+|\nu|) \times (2N+|\nu|)$ matrices $M$:

$$M = \begin{pmatrix} 0 & W^\dagger \\ W & 0 \end{pmatrix}.$$  \hspace{1cm} (2)

The partition function is defined by

$$Z = \int dW \prod_{f=1}^{N_f} \det (M + i m_f) \exp \left[ -\frac{N}{2} \text{tr} V(M^2) \right].$$  \hspace{1cm} (3)

Here $W$ is a rectangular complex matrix of size $N \times (N+|\nu|)$. The integer $\nu$ is identified with topological charge, and the space-time volume $V_4$ is identified with $2N + |\nu|$, the size of the matrix $M$. The integration measure in eq. (3) is the Haar measure of $W$. From now on consider the sector of zero topological charge $\nu$, which is the case most relevant for comparison with Monte Carlo data of lattice gauge theory. However, the general-$\nu$ case can be extracted from the formulas we shall give below simply by setting $|\nu|$ fermion masses to zero.

Rewriting the matrix integral (3) in terms of the (positive definite) eigenvalues $\lambda_i$ of the hermitian matrix $W^\dagger W$, one gets, after discarding an irrelevant overall factor from the angular integrations [8]:

$$Z = \int_0^\infty \prod_{i=1}^N \left( d\lambda_i \prod_{f} (\lambda_i + m_f^2) e^{-N V(\lambda_i)} \right) \left| \det_{ij} \lambda_i^{j-1} \right|^2.$$  \hspace{1cm} (4)

In terms of the standard orthogonal polynomial method, we thus seek polynomials $P^{(N_f)}_n(\lambda; m_1^2, \ldots, m_{N_f}^2)$ orthogonal with respect to the weight functions

$$w(\lambda) = \prod_{f=1}^{N_f} (\lambda + m_f^2) e^{-N V(\lambda)}.$$  \hspace{1cm} (5)

So far we have not specified the potential $V(\lambda)$, which can be parametrized in a quite general way by

$$V(\lambda) = \sum_{k \geq 1} \frac{g_k}{k} \lambda^k.$$  \hspace{1cm} (6)

As was shown in ref. [3], when all $m_f = 0$ the orthogonal polynomials have, for fixed $x = N^2 \lambda$ and $t = n/N$, a universal limiting behavior. When the polynomials can be normalized according to
\( P_n^{(N_f)}(0) = 1 \) the limit is
\[
\lim_{N \to \infty} P_n^{(N_f)} \left( \frac{x}{N^2} \right) \bigg|_{n=Nt} = N_f! \frac{J_{N_f}(u(t) \sqrt{x})}{(u(t) \sqrt{x}/2)^{N_f}} ,
\]
with
\[
u(t) = \int_0^t \frac{dt'}{\sqrt{r(t')}} \quad \text{and} \quad t = \sum \frac{g_k}{2 \choose k} r(t)^k .
\]
The restriction to normalizability according to \( P_n^{(N_f)}(0) = 1 \) for \( n = Nt \) can be rephrased as \( \rho(0) \neq 0 \), so the condition is precisely as expected: the macroscopic spectral density must be non-vanishing at the origin. One can then identify
\[
a = 2 \sqrt{r(1)} , \quad \rho(0) = \frac{u(1)}{2\pi} ,
\]
where \( a \) gives the upper limit of the support of \( \rho(\lambda) \). Once universality of the orthogonal polynomials has been established, the general proof of universality of all microscopic spectral correlators follows as a simple corollary.

2.1 One massive flavor

To generalize these results to the case of massive fermions, we first note that for quenched fermions \( N_f = 0 \) there is no distinction between the massive and massless cases. Consider next the first non-trivial case of \( N_f = 1 \). We here need polynomials orthogonalized according to
\[
\int_0^\infty d\lambda (\lambda + m^2) e^{-NV(\lambda)} P_i^{(1)}(\lambda; m^2) P_j^{(1)}(\lambda; m^2) = h_i^{(1)} \delta_{ij} .
\]
Our basic observation is that these polynomials are readily found by expanding in terms of the polynomials of the quenched case, \( P_j^{(0)}(\lambda) \) by use of Christoffel’s theorem:
\[
P_i^{(1)}(\lambda; m^2) = \sum_{j=0}^i \frac{P_j^{(0)}(-m^2)}{h_j^{(0)}} P_j^{(0)}(\lambda) ,
\]
up to an arbitrary normalization condition. By the Christoffel-Darboux formula we can thus choose
\[
P_i^{(1)}(\lambda; m) = \frac{P_i^{(0)}(-m^2) P_{i+1}^{(0)}(\lambda) - P_{i+1}^{(0)}(-m^2) P_i^{(0)}(\lambda)}{\lambda + m^2} ,
\]
and fix the normalization subsequently. Note that the r.h.s., despite appearances, indeed is a polynomial in \( \lambda \).

Because the new factor \((\lambda + m^2)\) in the weight function \( w(\lambda) \) can be viewed as a contribution to the generic potential \( V(\lambda) \) which is subdominant in \( 1/N \), the macroscopic spectral density is unchanged. Since \( \rho(0) \neq 0 \) this means that \( P_N^{(1)}(0; m^2) \neq 0 \) as well in the large-\( N \) limit, and by redefining, for sufficiently large index \( i \),
\[
P_i^{(1)}(\lambda; m^2) \to P_i^{(1)}(\lambda; m^2)/P_i^{(1)}(0; m^2) ,
\]
the orthogonal polynomials of eq. (12) become normalized according to \( P_i^{(1)}(0; m^2) = 1 \). The normalization constants \( h_i^{(1)} \) of course become modified accordingly.
In the double-microscopic limit where \( x = N^2 \lambda \) and \( \mu = N \lambda \) are kept fixed as \( N \to \infty \) we thus get from eq. (12) and the previous universal result (7):

\[
\lim_{N \to \infty} P_n^{(1)} \left( \frac{x}{N^2}; \frac{\mu^2}{N^2} \right) \bigg|_{n=Nt} = \frac{\mu^2}{x + \mu^2} \left( J_0(u(t) \sqrt{x}) + \frac{\sqrt{x} J_1(u(t) \sqrt{x})}{\mu} I_0(u(t) \mu) \right)
\]

(14)

where \( J_n(x) \) and \( I_n(x) \) are ordinary and modified Bessel functions, respectively. This depends on the given potential \( V(\lambda) \) only implicitly through the two functions \( u(t) \) and \( r(t) \), which are inherited from the massless case. Because of relation (9), we have therefore proved that the double-microscopic limit of the polynomial \( P_n^{(1)}(x/N^2; \mu^2/N^2) \) is universal, depending only on the end-point of the cut \( a \) (which one can choose at some fixed value), and the value of the macroscopic spectral density at the origin, \( \rho(0) \). In applications to Dirac spectra, this value is, by the Banks-Casher relation [10], fixed by the chiral condensate: \( \rho(0) = \langle \bar{\psi} \psi \rangle / \pi \). As a first simple check we note that \( P_n^{(1)}(x/N^2; \mu^2/N^2) \to J_1(u(t) \sqrt{x})/(u(t) \sqrt{x}) \) as \( \mu \to 0 \), and we hence recover the result of ref. [9]. In the opposite limit, \( \mu \to \infty \), we see that the above universal polynomial approaches \( J_0(u(t) \sqrt{x}) \), the result of the quenched case. This is the first example of the decoupling of a massive fermion, which we shall return to in greater generality below.

It now follows as a simple corollary that also all microscopic spectral correlators are universal in the above scaling limit. Consider first the kernel

\[
K_n^{(1)}(z_1, z_2; m) = \sqrt{|z_1 z_2|} \left| e^{-N(V(z_1^2) + V(z_2^2))} (z_1^2 + m^2)(z_2^2 + m^2) \right|
\]

\[
\times \frac{P_{n-1}^{(1)}(z_1^2; m^2)P_n^{(1)}(z_2^2; m^2) - P_n^{(1)}(z_1^2; m^2)P_{n-1}^{(1)}(z_2^2; m^2)}{z_1^2 - z_2^2}
\]

(15)

governing the correlation of \( M \). In the scaling limit it becomes of the universal form

\[
K_S^{(1)}(\zeta, \zeta'; \mu) = \lim_{N \to \infty} \frac{1}{N} K_n^{(1)}(\zeta \zeta'/N, \mu/N)
\]

\[
\det \begin{pmatrix}
J_0(2\pi \rho(0)\zeta) & \zeta J_1(2\pi \rho(0)\zeta) & \zeta^2 J_2(2\pi \rho(0)\zeta)
J_0(2\pi \rho(0)\zeta') & \zeta' J_1(2\pi \rho(0)\zeta') & \zeta'^2 J_2(2\pi \rho(0)\zeta')
I_0(2\pi \rho(0)\mu) & -\mu I_1(2\pi \rho(0)\mu) & \mu^2 I_2(2\pi \rho(0)\mu)
\end{pmatrix}
\]

\[
C(\mu) \sqrt{|\zeta \zeta'| (\zeta^2 - \zeta'^2) \sqrt{(\zeta^2 + \mu^2)(\zeta'^2 + \mu^2)}}
\]

(16)

\( C(\mu) \) is a function of \( \mu \) only, to be determined below.

All double-microscopic spectral correlators

\[
\rho_n^{(1)}(z_1, \ldots, z_s; m) = \left\langle \prod_{a=1}^s \frac{1}{2N} \text{tr} \delta(z_a - M) \right\rangle = \det_{a,b} K_n^{(1)}(z_a, z_b; m)
\]

(17)

therefore reach universal limits as well:

\[
\rho_S^{(1)}(\zeta_1, \ldots, \zeta_s; \mu) = \lim_{N \to \infty} \frac{1}{N^s} \rho_n^{(1)}\left( \frac{\zeta_1}{N}, \ldots, \frac{\zeta_s}{N}; \frac{\mu}{N} \right) = \det_{a,b} K_S^{(1)}(\zeta_a, \zeta_a; \mu).
\]

(18)

In particular, the spectral density itself,

\[
\rho_n^{(1)}(z; m) = K_n^{(1)}(z, z; m)
\]

(19)
takes a universal form in the double-microscopic limit, which, after choosing the conventional normalization \[\bar{\rho}_S^{(1)}(\zeta; \mu) = \frac{1}{\pi^2(0)^2} \mathcal{K}_S^{(1)}(\frac{\zeta}{2\pi(0)^2}, \frac{\zeta}{2\pi(0)^2}; \frac{\mu}{2\pi(0)^2})\], reads

\[
\rho_S^{(1)}(\zeta; \mu) = \frac{|\zeta|}{2} \left( J_0(\zeta)^2 + J_1(\zeta)^2 \right) - |\zeta| \frac{J_0(\zeta)[\mu I_1(\mu) J_0(\zeta) + \mu I_0(\mu) \zeta J_1(\zeta)]}{(\zeta^2 + \mu^2) \mu I_0(\mu)}.
\]

The microscopic spectral correlators, and the density itself, have had their overall normalization fixed by the matching condition (the compensating factor of \(\pi(0)^2\) on the left hand side of the relation below is due to the normalization convention \([2]\)) between microscopic and macroscopic densities,

\[
\lim_{\zeta \to \infty} \left[ \pi(0)^2 \rho_S^{(1)}(\zeta; \mu) \right] = \rho(0),
\]

the latter of which is independent of \(\mu\). This is just a convenient short-cut. We could have avoided it by explicitly computing the normalization constants \(n_0^{(1)}\) of eq. \([\ref{match}]\), but which we see no need to do that here. In addition we of course have that \(\rho_S^{(1)}(\zeta; 0)\) agrees with the result obtained directly from the massless case \([2]\). A more interesting condition comes from the decoupling of a very massive fermion. This implies

\[
\lim_{\mu \to \infty} \rho_S^{(1)}(\zeta; \mu) = \rho_S^{(0)}(\zeta).
\]

One verifies that indeed this relation is satisfied by the density \([\ref{result}]\).

### 2.2 More Massive Flavors

Next, consider the case of more quark flavors. The decoupling of heavy fermions, of which we already saw one example in eq. \([\ref{result}]\), leads to a hierarchy of consistency relations which must be satisfied by the microscopic spectral densities:

\[
\rho_S^{(N_f)}(\zeta; \mu_1, \ldots, \mu_{N_f}) \rightarrow \rho_S^{(N_f-1)}(\zeta; \mu_1, \ldots, \mu_{N_f-1}) \quad \text{as} \quad \mu_{N_f} \to \infty
\]

\[
\rho_S^{(N_f)}(\zeta; \mu_1, \ldots, \mu_{N_f}) \rightarrow \rho_S^{(N_f-2)}(\zeta; \mu_1, \ldots, \mu_{N_f-2}) \quad \text{as} \quad \mu_{N_f-1}, \mu_{N_f} \to \infty
\]

\[
\vdots
\]

\[
\rho_S^{(N_f)}(\zeta; \mu_1, \ldots, \mu_{N_f}) \rightarrow \rho_S^{(0)}(\zeta) \quad \text{as} \quad \mu_1, \ldots, \mu_{N_f} \to \infty,
\]

and all other relations obtained by permutations of the \(\mu_i\). There are similar consistency relations for all microscopic spectral correlators. In addition, we of course also have that in the limit where all masses are set to zero, the result must agree with that of ref. \([3]\).

We construct higher-\(N_f\) microscopic spectral densities by straightforward iteration of the procedure described above. For \(N_f = 2\) the new orthogonal polynomials become

\[
P_i^{(2)}(\lambda; m_1^2, m_2^2) = \frac{P_i^{(1)}(-m_2^2; m_1^2)P_{i+1}^{(1)}(\lambda; m_1^2) - P_{i+1}^{(1)}(-m_2^2; m_1^2)P_i^{(1)}(\lambda; m_1^2)}{\lambda + m_2^2},
\]

*Our expression for the double-microscopic spectral density does not agree with that of ref. \([3]\), where a derivation was given in terms of a saddle-point evaluation of a matrix model with Gaussian action. Because of this disagreement, we have taken special pains to check that the result presented here is correct. While both the result presented in ref. \([3]\) and the expression shown here (eq. \([\ref{result}]\)) reduce to the previously known result in the massless limit, only the expression \([\ref{result}]\) reduces correctly to the quenched result in the limit \(\mu \to \infty\) (see below). One further check on the result \([\ref{result}]\) comes from the fact that it correctly reproduces an exact spectral sum rule of QCD; see next section.*
which we again can give our conventional normalization by letting

\[ P_i^{(2)}(\lambda; m_1^2, m_2^2) \rightarrow P_i^{(2)}(\lambda; m_1^2, m_2^2)/P_i^{(2)}(0; m_1^2, m_2^2). \]  

(26)

Taking the large-\(N\) limit with \(x = N^2 \lambda\), \(\mu_1 = Nm_1\) and \(\mu_2 = Nm_2\) fixed, leads to

\[
\lim_{N \to \infty} P_n^{(2)}(x N^2 \mu_1^2 N^2, \mu_2^2 N^2, \mu_3^2 N^2)_{n=Nt} = \frac{\mu_1^2}{x + \mu_1^2 x + \mu_2^2} \det \begin{pmatrix} J_0(u(t)\sqrt{x}) & \sqrt{x} J_1(u(t)\sqrt{x}) & x J_2(u(t)\sqrt{x}) \\ I_0(u(t)\mu_1) & -\mu_1 I_1(u(t)\mu_1) & \mu_1^2 I_2(u(t)\mu_1) \\ I_0(u(t)\mu_2) & -\mu_2 I_1(u(t)\mu_2) & \mu_2^2 I_2(u(t)\mu_2) \end{pmatrix} \det \begin{pmatrix} -\mu_1 I_1(u(t)\mu_1) & \mu_1^2 I_2(u(t)\mu_1) \\ -\mu_2 I_1(u(t)\mu_2) & \mu_2^2 I_2(u(t)\mu_2) \end{pmatrix}.
\]

(27)

Also here, and for the appropriate generalization to all higher values of \(N_f\), the universality proof of ref. [5] immediately implies that \(P_n^{(2)}(x/N^2; \mu_1^2/N^2, \mu_2^2/N^2)\) is universal. All double-microscopic spectral correlators (13) therefore are universal as well. The results (generalized Bessel kernels) can immediately be written down, following the definitions (18). Most compact expressions seem to be obtained by writing everything exclusively in terms of the zeroth and first (modified) Bessel functions.

We restrict ourselves to displaying only the most important of them, the double-microscopic spectral density itself, which becomes

\[
\rho_S^{(2)}(\zeta; \mu_1, \mu_2) = \frac{|\zeta|}{2} \left( J_0(\zeta)^2 + J_1(\zeta)^2 \right) - \frac{|\zeta|((\mu_1^2 - \mu_2^2))}{(\zeta^2 + \mu_1^2)(\zeta^2 + \mu_2^2)} \left[ \mu_1 I_1(\mu_1) J_0(\zeta) + \zeta I_0(\mu_1) J_1(\zeta) \right] \left[ \mu_2 I_1(\mu_2) J_0(\zeta) + \zeta I_0(\mu_2) J_1(\zeta) \right].
\]

(28)

We note that it satisfies all the required consistency conditions (24).

The iterative procedure described above gives the relevant orthogonal polynomials, and hence all the microscopic spectral correlators and densities for arbitrary \(N_f\). For example, the orthogonal polynomial for \(N_f=3\) reads

\[
\lim_{N \to \infty} P_n^{(3)}(x N^2 \mu_1^2 N^2, \mu_2^2 N^2, \mu_3^2 N^2)_{n=Nt} = \frac{\mu_1^2}{x + \mu_1^2 x + \mu_2^2 x + \mu_3^2} \det \begin{pmatrix} J_0(u(t)\sqrt{x}) & \sqrt{x} J_1(u(t)\sqrt{x}) & x J_2(u(t)\sqrt{x}) & x^{3/2} J_3(u(t)\sqrt{x}) \\ I_0(u(t)\mu_1) & -\mu_1 I_1(u(t)\mu_1) & \mu_1^2 I_2(u(t)\mu_1) & -\mu_1^3 I_3(u(t)\mu_1) \\ I_0(u(t)\mu_2) & -\mu_2 I_1(u(t)\mu_2) & \mu_2^2 I_2(u(t)\mu_2) & -\mu_2^3 I_3(u(t)\mu_2) \\ I_0(u(t)\mu_3) & -\mu_3 I_1(u(t)\mu_3) & \mu_3^2 I_2(u(t)\mu_3) & -\mu_3^3 I_3(u(t)\mu_3) \end{pmatrix} \det \begin{pmatrix} -\mu_1 I_1(u(t)\mu_1) & \mu_1^2 I_2(u(t)\mu_1) & -\mu_1^3 I_3(u(t)\mu_1) \\ -\mu_2 I_1(u(t)\mu_2) & \mu_2^2 I_2(u(t)\mu_2) & -\mu_2^3 I_3(u(t)\mu_2) \\ -\mu_3 I_1(u(t)\mu_3) & \mu_3^2 I_2(u(t)\mu_3) & -\mu_3^3 I_3(u(t)\mu_3) \end{pmatrix}.
\]

(29)

We shall give the general expression for spectral correlators for an arbitrary number of flavors \(N_f\) below. In fig. 1 we show the microscopic spectral density \(\rho_S^{(3)}(\zeta; \mu, \mu, \mu)\) with, for simplicity, degenerate masses. The decoupling of heavy fermions that is expressed by eq. (24) is easily seen in the convergence towards the quenched result \(\rho_S^{(0)}(\zeta)\), while in the opposite limit of zero masses the density approaches the result of ref. [5].
Before proceeding to the general formula for arbitrary $N_f$, we note that there are still a few particular cases that can be expressed in a very simple analytical form. As an example, consider the case of one massive fermion (of rescaled mass $\mu$), and $N_f-1$ massless ones. For $N_f=1$ this case of course coincides with what we have already described above. For general $N_f$ the formula can worked out completely analogous to the case of one massive flavor, by starting from the asymptotic form of the polynomial (7). The resulting double-microscopic spectral density reads as follows:

$$\rho_S^{(N_f)}(\zeta; \mu, 0, \cdots, 0) = \frac{|\zeta|^2}{2} \left\{ J_{N_f}(\zeta)^2 - J_{N_f-1}(\zeta) J_{N_f+1}(\zeta) + \frac{\mu^2 J_{N_f-1}(\zeta)}{N_f(\zeta^2 + \mu^2)} \left( I_{N_f+1}(\mu) J_{N_f-1}(\zeta) + J_{N_f+1}(\zeta) \right) \right\}.$$  \hspace{1cm} (30)

One verifies that also this microscopic spectral densities satisfies the consistency condition (24).

2.3 AN ARBITRARY NUMBER OF MASSIVE FLAVORS

For practical purposes, only the massive microscopic spectral densities for a few flavors may be of relevance. Nevertheless, it is worthwhile to note that the iterative procedure described above can be carried through for an arbitrary number of flavors $N_f$, which we here for notational convenience denote by $\alpha$.

We shall first state the main results. The orthogonal polynomials, normalized by

$$P_n^{(\alpha)}(0; \frac{\mu_1^2}{N^2}, \cdots, \frac{\mu_\alpha^2}{N^2}) = 1,$$

read

$$\lim_{N \rightarrow \infty} P_n^{(\alpha)}(\frac{x}{N}; \frac{\mu_1^2}{N^2}, \cdots, \frac{\mu_\alpha^2}{N^2}) = \left. \prod_{f=1}^{\alpha} \frac{\mu_f^2}{x + \mu_f^2} \right|_{n=Nt} \det \begin{pmatrix} J_0(u(t)\sqrt{x}) & \sqrt{x} J_1(u(t)\sqrt{x}) & \cdots & x^{(\alpha+1)/2} J_{\alpha+1}(u(t)\sqrt{x}) \\ I_0(u(t)\mu_1) & -\mu_1 I_1(u(t)\mu_1) & \cdots & (-\mu_1)^{\alpha+1} I_{\alpha+1}(u(t)\mu_1) \\ \vdots & \vdots & \ddots & \vdots \\ I_0(u(t)\mu_\alpha) & -\mu_\alpha I_1(u(t)\mu_\alpha) & \cdots & (-\mu_\alpha)^{\alpha+1} I_{\alpha+1}(u(t)\mu_\alpha) \end{pmatrix} \det_{i,j}(-\mu_i^2 J_j(u(t)\mu_i)) \right). \hspace{1cm} (31)
The double-microscopic spectral correlators

\[ \rho_S^{(\alpha)}(\zeta_1, \cdots, \zeta_s; \mu_1, \cdots, \mu_\alpha) \equiv \lim_{N \to \infty} \frac{1}{N^2} \rho_N^{(\alpha)}(\frac{\zeta_1}{N}, \cdots, \frac{\zeta_s}{N}, \frac{\mu_1}{N}, \cdots, \frac{\mu_\alpha}{N}) = \det K^{(\alpha)}(\zeta, \zeta'; \{\mu_f\}), \tag{32} \]

are expressed through the kernel which is given by

\[ K^{(\alpha)}(\zeta, \zeta'; \{\mu_f\}) = \det \begin{pmatrix} J_0(2\pi \rho(0)\zeta) & J_0(2\pi \rho(0)\zeta') & \cdots & \zeta^{\alpha+1} J_{\alpha+1}(2\pi \rho(0)\zeta) \\ J_0(2\pi \rho(0)\zeta) & J_1(2\pi \rho(0)\zeta) & \cdots & \zeta^{\alpha+1} J_{\alpha+1}(2\pi \rho(0)\zeta') \\ \vdots & \vdots & \ddots & \vdots \\ I_0(2\pi \rho(0)\mu_1) & -\mu_1 I_1(2\pi \rho(0)\mu_1) & \cdots & (-\mu_1)^{\alpha+1} I_{\alpha+1}(2\pi \rho(0)\mu_1) \end{pmatrix}. \tag{33} \]

The double-microscopic spectral density thus explicitly reads

\[ \rho_S^{(\alpha)}(\zeta; \{\mu_f\}) = \frac{-|\zeta|}{2} \frac{\sqrt{|\zeta|}}{\prod_{j=1}^s (\zeta^2 + \mu_j^2) \det M} \det \begin{pmatrix} \zeta^{-1} J_{-1}(\zeta) & J_0(\zeta) & \cdots & \zeta^{\alpha} J_\alpha(\zeta) \\ J_0(\zeta) & \zeta J_1(\zeta) & \cdots & \zeta^{\alpha+1} J_{\alpha+1}(\zeta) \\ I_0(\mu_1) & -\mu_1 I_1(\mu_1) & \cdots & (-\mu_1)^{\alpha+1} I_{\alpha+1}(\mu_1) \\ \vdots & \vdots & \ddots & \vdots \\ I_0(\mu_\alpha) & -\mu_\alpha I_1(\mu_\alpha) & \cdots & (-\mu_\alpha)^{\alpha+1} I_{\alpha+1}(\mu_\alpha) \end{pmatrix}. \tag{34} \]

Here \( M \) is an \( \alpha \times \alpha \) matrix defined by \( M_{ij} = (-\mu_i)^{j-1} I_{j-1}(\mu_i) \). The proof of the above formulae is summarized in Appendix A.

We can now easily check that the microscopic kernel \([33]\) and density \([34]\) for arbitrary \( \alpha \) satisfies all the decoupling relations \([24]\). Suppose \( \mu_\alpha \) is taken to be infinity in eq. \([34]\). Due to the asymptotic behavior of the modified Bessel functions

\[ I_i(\mu) \sim \frac{e^\mu}{\sqrt{2\pi \mu}} \quad \text{as} \quad \mu \to \infty, \tag{35} \]

the determinant in the numerator of \([34]\) is dominated by its minor for the lower right corner,

\[ (-\mu_\alpha)^{\alpha+1} \frac{e^{\mu_\alpha}}{\sqrt{2\pi \mu_\alpha}} \det \begin{pmatrix} \zeta^{-1} J_{-1}(\zeta) & J_0(\zeta) & \cdots & \zeta^{\alpha-1} J_{\alpha-1}(\zeta) \\ J_0(\zeta) & \zeta J_1(\zeta) & \cdots & \zeta^{\alpha} J_\alpha(\zeta) \\ I_0(\mu_1) & -\mu_1 I_1(\mu_1) & \cdots & (-\mu_1)^{\alpha} I_\alpha(\mu_1) \\ \vdots & \vdots & \ddots & \vdots \\ I_0(\mu_{\alpha-1}) & -\mu_{\alpha-1} I_1(\mu_{\alpha-1}) & \cdots & (-\mu_{\alpha-1})^{\alpha} I_\alpha(\mu_{\alpha-1}) \end{pmatrix}. \tag{36} \]

Similarly, its denominator is approximated by

\[ \mu_\alpha^2 \prod_{j=1}^{\alpha-1} (\zeta^2 + \mu_j^2) \cdot (-\mu_\alpha)^{\alpha-1} \frac{e^{\mu_\alpha}}{\sqrt{2\pi \mu_\alpha}} \det_{1 \leq i,j \leq \alpha-1} (-\mu_i)^{j-1} I_{j-1}(\mu_i). \tag{37} \]

Thus we recover the same expression \([34]\) for \( \alpha \to \alpha - 1 \). By iteration, we confirm that the tower of consistency relations \([24]\) is satisfied.
On the other hand, we have another consistency relation that when all masses vanishes, $\rho_s^{(\alpha)}(\zeta; 0, \ldots, 0)$ should agree with the known formula for massless quarks [2]. In order to check this, let us take the limit where the masses are taken to be zero one by one. Suppose $\mu_\alpha$ is taken to be zero first. Since
\[ I_0(0) = 1, \quad I_n(0) = 0 \quad (n \geq 1), \] (38)
the determinants in the numerator and denominator of (34) are replaced by their minors for the lower left corner. Then we obtain
\[ \rho_s^{(\alpha)}(\zeta; \mu_1, \ldots, \mu_{\alpha-1}, 0) = \frac{-|\zeta|}{2} \frac{\zeta^2 \prod_{j=1}^{\alpha-1} (\zeta^2 + \mu_j^2) \det_{1 \leq i,j \leq \alpha-1} ( -\mu_i J_i(\mu_1) )}{\zeta^{2(\alpha-1)} (\zeta^2 + \mu_\alpha^2) ( -\mu_\alpha^{\alpha-1} I_{\alpha-1}(\mu_1) )}. \] (39)

We may iterate this procedure as many times as required to obtain,
\[ \rho_s^{(\alpha)}(\zeta; \mu, 0, \ldots, 0) = \frac{-|\zeta|}{2} \frac{\zeta^2 \prod_{j=1}^{\alpha-1} (\zeta^2 + \mu_j^2) \det_{1 \leq i,j \leq \alpha-1} ( -\mu_i J_i(\mu_1) )}{\zeta^{2(\alpha-1)} (\zeta^2 + \mu_\alpha^2) ( -\mu_\alpha^{\alpha-1} I_{\alpha-1}(\mu_1) )}, \] (40)
\[ \rho_s^{(\alpha)}(\zeta; 0, \ldots, 0) = \frac{-|\zeta|}{2} \frac{\zeta^2 \prod_{j=1}^{\alpha-1} (\zeta^2 + \mu_j^2) \det_{1 \leq i,j \leq \alpha-1} ( -\mu_i J_i(\mu_1) )}{\zeta^{2\alpha}} = \frac{|\zeta|}{2} ( J_\alpha(\zeta) - J_{\alpha-1}(\zeta) J_{\alpha+1}(\zeta) ). \] (41)

We note that the expression (41) coincides with that of eq. (30). The last expression, (41) indeed also agrees with the known formula for massless quarks [3].

3 Exact Massive Spectral Sum Rules in QCD

So far our discussion has been entirely within the framework of random matrix theory. A highly non-trivial question is whether the resulting universal spectral correlators are exact statements about the spectral correlators in QCD as well. With the increasing evidence that this is the case for massless Dirac operators, it is natural to expect that it extends to massive Dirac operators as well, once considered in the appropriate double-microscopic limit which we defined in the preceding section. We shall here present some evidence that this is the case. We do this by temporarily leaving the framework of random matrix models, and turning to the QCD partition function, as it can be represented in the range $1/\Lambda_{QCD} \ll N^{1/d} \ll 1/m_\pi$. Here $N$ gives the size of the volume, and $\Lambda_{QCD}$ is a typical hadronic scale in QCD. Following the conventional notation, we denote (up to a sign, see the discussion in ref. [4]) the chiral condensate by $\Sigma \equiv \langle \bar{\psi} \psi \rangle$.

Leutwyler and Smilga [4] have shown how to derive exact spectral sum rules for massless Dirac operators in the above range. One generalization to the case of massive spectral sum rules has

\footnote{See also the work of ref. [11] for generalizations to gauge group $SU(2)$, and to more exotic flavor symmetry breaking patterns.}
briefly been considered by Shuryak and Verbaarschot [4]. Since the masses and the eigenvalues are rescaled at the same rate in the $N \to \infty$ double-microscopic limit, it is clear that we should consider spectral sum rules for which the masses and the eigenvalues enter on equal footing. Consider first the case of one flavor, $N_f = 1$, in the sector of vanishing topological charge. The natural generalization of the spectral sum rules in terms of inverse powers of $z^2$ is to consider inverse powers of $z^2 + m^2$. From the expansion [4] of the QCD partition function in terms of Bessel functions in the above range, one finds that

$$\frac{1}{\Sigma^2 N^2} \left( \sum_n \frac{1}{z_n^2 + m^2} \right) = \sum_n \frac{1}{J_{0,n} + \mu^2} \frac{I_1(\mu)}{2\mu I_0(\mu)} . \quad (42)$$

where we have defined $\mu \equiv mN\Sigma$. The sum on the right hand side runs over the real zeros $j_{0,n}$ of the Bessel function $J_0(x)$. Defining the double-microscopic spectral density

$$\rho_S(\zeta; \mu) = \lim_{N \to \infty} \frac{1}{\Sigma N} \rho \left( \frac{\zeta}{\Sigma N} \right) , \quad \mu = mN\Sigma \text{ fixed} \quad (43)$$

in terms of the ordinary macroscopic spectral density

$$\rho(z) = \langle \sum_n '\delta(z - z_n) \rangle , \quad (44)$$

one sees that the above spectral sum rule can be written

$$\int_0^\infty d\zeta \frac{\rho_S(\zeta; \mu)}{\zeta^2 + \mu^2} = \frac{I_1(\mu)}{2\mu I_0(\mu)} . \quad (45)$$

This massive spectral sum rule, derived entirely within the framework of QCD (a very simple chiral Lagrangian, in the above range), can now be tested on the double-microscopic spectral density $\rho_S^{(1)}(\zeta; \mu)$ of eq. (21). Performing the required integrals, one finds that it indeed is satisfied. As in the massless case, the microscopic spectral density of the chiral unitary ensemble is thus consistent with the spectral sum rule of QCD.

Another example of an exact spectral sum rule which can be derived directly from QCD, and for which we can compare with our present results, is given by the case of one massive fermion of mass $m$, and $N_f - 1$ massless fermions. Again expanding the Bessel functions of the relevant partition function of QCD with one massive flavor and $N_f - 1$ massless flavors [4], one finds the following exact spectral sum rule ($j_{m,n}$ denotes the $n$th zero of the Bessel function $J_m(x)$):

$$\int_0^\infty d\zeta \frac{\rho_S(\zeta; \mu)}{\zeta^2 + \mu^2} = \sum_n \frac{1}{J_{N_f-1,n} + \mu^2} \frac{I_{N_f}(\mu)}{2\mu I_{N_f-1}(\mu)} . \quad (46)$$

Inserting the microscopic spectral density $\rho_S^{(N_f)}(\zeta; \mu, 0, \ldots, 0)$ of eq. (30) we verify that also this identity is satisfied. Other spectral sum rules can be worked out analogously.
4 Conclusions

Motivated by the recent progress in numerically computing the microscopic spectral densities of the Dirac operator in realistic four-dimensional lattice gauge theories [1], we have set out to compute the microscopic spectral densities of what should correspond to massive Dirac operators. Such an extension will presumably be required before a detailed comparison with lattice gauge theory data can be performed beyond the quenched approximation. To reach the universal limit, masses must be scaled at a very precise rate as the volume is increased, but this is entirely feasible in the context of lattice gauge theory. We thus expect that the results presented here may be of practical value when it comes to detailed comparisons between lattice Monte Carlo data for QCD and the universality predictions from random matrix theory.

We have succeeded in deriving the relevant microscopic (we have called these “double-microscopic” because both masses and eigenvalues are rescaled as $N \to \infty$) spectral correlators, and in particular the microscopic spectral densities themselves, within the framework of random matrix models. In doing so, we have simultaneously extended the universality proofs of ref. [5] to this more general situation. The case of the unitary matrix ensemble, conjectured to be possibly relevant for $SU(N_c)$ gauge theories in (2+1) dimensions [3], can be worked out analogously and will be presented elsewhere [14]. As in ref. [5], the proven universality is strictly limited to the framework of random matrix theory.

Based on the decoupling of heavy fermions, we have derived a set of consistency conditions for the microscopic spectral densities. These are non-trivial relations that show how the various microscopic spectral densities $\rho^{(N_f)}_S(\zeta; \mu_1, \ldots, \mu_{N_f})$ for different values of $N_f$ must be related to each other as one or more of the masses are sent to infinity. We have verified that our spectral densities satisfy all of these general consistency conditions. In the other extreme limit where all masses are taken to zero, we of course recover the known results [2, 3].

To investigate the question as to whether QCD with massive flavors indeed fall into the universality classes derived here, we have confirmed that exact massive spectral sum rules (generalizations of the Leutwyler-Smilga sum rules [1, 11]) derived directly from QCD are satisfied if we identify the double-microscopic spectral density of QCD with that of the chiral unitary ensemble. This, together with the mounting evidence from the massless case, makes it highly plausible that all the universal double-microscopic spectral correlators we have derived here are exact statements about QCD in the above limit. The crucial test will be a comparison with results from lattice gauge theory, where these microscopic spectral correlators and densities are very convenient finite-size scaling functions, almost tailored for Monte Carlo simulations. Hopefully such data will soon be available.

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Note added: Our main results (33) and (34) of this paper have been derived independently for [12].
the case of the Laguerre ensemble by Wilke, Guhr and Wettig ([hep-th/9711057]), in a paper which appeared on the e-print archives just a few days after ours (these authors did not consider the question of universality). The sum rule (45) has been derived earlier by Gasser and Leutwyler in Phys. Lett. B188 (1987) 477, and has been compared to lattice QCD data by Verbaarschot in Phys. Lett. B368 (1996) 137. The authors of ref. [7] have found an error in their previous calculation of the $N_f = 1$ case in the Gaussian ensemble, and they now agree with the expression given in eq. (23) (M. Nowak, private communication).

A  Proof of the Formulae in section 2.3

Since the procedure of constructing the kernel out of the polynomials ([13]) is essentially the same as that of adding another massive flavor ([12]) up to the sign of $m^2$, it suffices to prove ([38]). We first demonstrate the following lemma:

**Lemma:**
Let $P^{[\alpha]}(t; \lambda_0, \lambda_1, \cdots, \lambda_\alpha)$, $\alpha = 0, 1, 2, \cdots$, be a set of functions generated by the iteration

$$P^{[\alpha+1]}(t; \lambda_0, \lambda_1, \cdots, \lambda_{\alpha+1}) = \frac{P^{[\alpha]}(t; \lambda_0, \lambda_1, \cdots, \lambda_{\alpha}) P^{[\alpha]}(t; \lambda_{\alpha+1}, \lambda_1, \cdots, \lambda_{\alpha}) - P^{[\alpha]}(t; \lambda_0, \lambda_1, \cdots, \lambda_{\alpha}) P^{[\alpha]}(t; \lambda_{\alpha+1}, \lambda_1, \cdots, \lambda_{\alpha})}{\lambda_0 - \lambda_{\alpha+1}}. \quad (A.1)$$

Then they are given by

$$P^{[\alpha]}(t; \lambda_0, \lambda_1, \cdots, \lambda_{\alpha-1}, \lambda_\alpha) = c(t; \lambda_1, \cdots, \lambda_{\alpha-1}) \frac{\det_{i,j} P^{[i]}(t; \lambda_j)}{\prod_{i<j}(\lambda_i - \lambda_j)} \quad (A.2)$$

where $P^{[i]}(t; \lambda) = \frac{\partial^t}{\partial \lambda^i} P^{[0]}(t; \lambda)$, and $c(t; \lambda_1, \cdots, \lambda_{\alpha-1})$ is a function in $t$ and in $\{\lambda_1, \cdots, \lambda_{\alpha-1}\}$.

**Proof:** We prove it by induction. Suppose ([A.2]) holds for an $\alpha$. Let $P^{[i]}(\lambda) \equiv \frac{\partial^t}{\partial \lambda^i} P^{[0]}(t; \lambda)$ (the argument $t$ is for convenience suppressed below). Consider the numerator in (A.1),

$$c(\lambda_1, \cdots, \lambda_{\alpha-1}) \begin{vmatrix} P & P' & \cdots & P^{(a-1)} & P^{(a)}(\lambda_0) \\ P & P' & \cdots & P^{(a-1)} & P^{(a)}(\lambda_1) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ P & P' & \cdots & P^{(a-1)} & P^{(a)}(\lambda_\alpha) \end{vmatrix} - (\lambda_0 \leftrightarrow \lambda_{\alpha+1})$$

$$= c(\lambda_1, \cdots, \lambda_{\alpha-1})^2 \begin{vmatrix} P & P' & \cdots & P^{(a-1)} & P^{(a)}(\lambda_0) \\ P & P' & \cdots & P^{(a-1)} & P^{(a)}(\lambda_1) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ P & P' & \cdots & P^{(a-1)} & P^{(a)}(\lambda_\alpha) \end{vmatrix} - (\lambda_0 \leftrightarrow \lambda_{\alpha+1})$$

$$\equiv c(\lambda_1, \cdots, \lambda_{\alpha-1})^2 D(\lambda_0, \lambda_1, \cdots, \lambda_{\alpha-1}, \lambda_\alpha). \quad (A.3)$$

The above identity holds because the $t$–derivatives acting on $c(\lambda_1, \cdots, \lambda_{\alpha-1})$ produce terms independent of $\lambda_0$ and $\lambda_{\alpha+1}$ and are thus cancelled by the subtraction interchanging $\lambda_0 \leftrightarrow \lambda_{\alpha+1}$, and those
acting on the columns
\[
\begin{pmatrix}
P \\
\vdots \\
P
\end{pmatrix}, \begin{pmatrix}
P' \\
\vdots \\
P'
\end{pmatrix}, \ldots, \begin{pmatrix}
P^{(a-1)} \\
\vdots \\
P^{(a-1)}
\end{pmatrix}
\]
vanish by themselves (the same columns are already present in the determinant).

By definition, \(D(\lambda_0, \lambda_1, \cdots, \lambda_{\alpha-1}, \lambda_{\alpha})\) enjoys the following properties:

- It is a sum of monomials of the form
  \[
  \pm P^{(#)}(\lambda_0) P^{(#)}(\lambda_1) \cdots P^{(#)}(\lambda_{\alpha}) P^{(#)}(\lambda_{\alpha}) P^{(#)}(\lambda_{\alpha+1})
  \]
  with (total # of derivatives) = \(2(0 + 1 + \cdots + \alpha) + 1 = \alpha^2 + \alpha + 1\).

- It vanishes if any two \(\lambda\)'s are coincident.

- It is antisymmetric under \(\lambda_0 \leftrightarrow \lambda_{\alpha+1}\).

- It is completely symmetric under \(\lambda_i \leftrightarrow \lambda_j, 1 \leq i, j \leq \alpha\).

The only function which satisfies these properties is, up to a constant,

\[
\begin{pmatrix}
P & P' & \cdots & P^{(a-1)}(\lambda_1) \\
\vdots & \vdots & & \vdots \\
P & P' & \cdots & P^{(a-1)}(\lambda_\alpha)
\end{pmatrix} = \begin{pmatrix}
P & P' & \cdots & P^{(a-1)} & P^{(a)} & P^{(a+1)}(\lambda_0) \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
P & P' & \cdots & P^{(a-1)} & P^{(a)} & P^{(a+1)}(\lambda_1) \\
P & P' & \cdots & P^{(a-1)} & P^{(a)} & P^{(a+1)}(\lambda_\alpha)
\end{pmatrix}
\]

(A.5)

On the other hand, the denominator in (A.1) is given by

\[
(\lambda_0 - \lambda_{\alpha+1}) \prod_{0 \leq i,j \leq \alpha} (\lambda_i - \lambda_j) \prod_{1 \leq i,j \leq \alpha+1} (\lambda_i - \lambda_j) = \prod_{1 \leq i,j \leq \alpha} (\lambda_i - \lambda_j) \prod_{0 \leq i,j \leq \alpha+1} (\lambda_i - \lambda_j)
\]

(A.6)

Since the first factors in (A.5) and (A.6) as well as \(c(\lambda_1, \cdots, \lambda_{\alpha-1})^2\) are independent of \(\lambda_0\) and \(\lambda_{\alpha+1}\), it can be absorbed into the redefinition of \(c(\lambda_1, \cdots, \lambda_{\alpha})\). Then we recover (A.2) for \(\alpha + 1\). QED.

Now we replace \(\lambda_i \to \frac{\zeta_i^2}{N^2}\) and \(P^{[0]}(t, \lambda)\) by its microscopic limit, \(P^{[0]}(t; \zeta^2/N^2) \to J_0(u(t)\zeta)\). Then we can inductively prove that its \(t\)-derivatives are expressed as

\[
P^{(i)}(t; \frac{\zeta^2}{N^2}) \to \frac{d^i}{dt^i} J_0(u(t)\zeta) = \sum_{k=0}^i d_{i,k}(t) \zeta^k J_k(u(t)\zeta)
\]

(A.7)

Once it is substituted inside the determinant \(\det P^{(i)}(\lambda_j)\), only the top term proportional to \(\zeta^i J_i(u(t)\zeta)\) contributes. Thus the determinant in (A.2) is replaced by

\[
\left( \prod_{i=0}^{\alpha+1} d_{i,i}(t) \right) \det_{0 \leq i,j \leq \alpha+1} \zeta^i J_i(u(t)\zeta_j)
\]

(A.8)
In order to construct the kernel, we make an analytical continuation of \((\zeta_1, \cdots, \zeta_\alpha)\) to imaginary 
\((i \mu_1, \cdots, i \mu_\alpha)\), denote the rest as \(\zeta_0 = \zeta, \ \zeta_{\alpha+1} = \zeta'\), and multiply the whole expression \((A.2)\) at \(t = 1\) 
by the factors from the measure and the Jacobian, \(\sqrt{\zeta' - \zeta} \prod_{f=1}^{\alpha} \left(\zeta^2 + \mu_f^2\right)\). Then we obtain

\[
K^{(\alpha)}(\zeta, \zeta'; \{\mu_f\}) = C(\{\mu_f\}) \sqrt{|\zeta \zeta'|} \begin{vmatrix}
J_0(2\pi \rho(0) \zeta) & \zeta J_1(2\pi \rho(0) \zeta) & \cdots & \zeta^{a+1} J_{\alpha+1}(2\pi \rho(0) \zeta) \\
J_0(2\pi \rho(0) \zeta') & \zeta J_1(2\pi \rho(0) \zeta') & \cdots & \zeta^{a+1} J_{\alpha+1}(2\pi \rho(0) \zeta') \\
I_0(2\pi \rho(0) \mu_1) & -\mu_1 I_1(2\pi \rho(0) \mu_1) & \cdots & (-\mu_1)^{a+1} I_{\alpha+1}(2\pi \rho(0) \mu_1) \\
\vdots & \vdots & \ddots & \vdots \\
I_0(2\pi \rho(0) \mu_\alpha) & -\mu_\alpha I_1(2\pi \rho(0) \mu_\alpha) & \cdots & (-\mu_\alpha)^{a+1} I_{\alpha+1}(2\pi \rho(0) \mu_\alpha) \\
\end{vmatrix} \prod_{f=1}^{\alpha} \left(\zeta^2 + \mu_f^2\right) \left(\zeta'^2 + \mu_f^2\right).
\]

(A.9)

\[
\rho^{(\alpha)}_S(\zeta; \{\mu_f\}) = C(\{\mu_f\}) |\zeta| \begin{vmatrix}
\zeta^{-1} J_{-1}(\zeta) & J_0(\zeta) & \cdots & \zeta^{a-1} J_{a-1}(\zeta) & \zeta^a J_0(\zeta) \\
J_0(\zeta) & \zeta J_1(\zeta) & \cdots & \zeta^a J_{a}(\zeta) & \zeta^{a+1} J_{a+1}(\zeta) \\
I_0(\mu_1) & -\mu_1 I_1(\mu_1) & \cdots & (-\mu_1)^a I_a(\mu_1) & (-\mu_1)^{a+1} I_{a+1}(\mu_1) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
I_0(\mu_\alpha) & -\mu_\alpha I_1(\mu_\alpha) & \cdots & (-\mu_\alpha)^a I_a(\mu_\alpha) & (-\mu_\alpha)^{a+1} I_{a+1}(\mu_\alpha) \\
\end{vmatrix} \prod_{f=1}^{\alpha} \left(\zeta^2 + \mu_f^2\right).
\]

(A.10)

The constant \(C(\{\mu_f\}) = (2\pi \rho(0))(1-\frac{\pi}{\alpha})(1+\alpha) C(\{\mu_f\})\) is determined to be \(-(2 \det \mathcal{M})^{-1}\) by requiring 
the matching between the \(\zeta \to \infty\) limit of the microscopic density \((A.10)\) and the macroscopic density:

\[
\rho^{(\alpha)}_S(\zeta \to \infty; \{\mu_f\}) \to C(\{\mu_f\}) |\zeta| \begin{vmatrix}
\det \mathcal{M} & \zeta^{-1} J_{a-1}(\zeta) & \cdots & \zeta^{a-1} J_{a-1}(\zeta) \\
\zeta J_{a}(\zeta) & \zeta^a J_{a}(\zeta) & \cdots & \zeta^{a+1} J_{a+1}(\zeta) \\
(-\mu_1)^a I_a(\mu_1) & (-\mu_1)^{a+1} I_{a+1}(\mu_1) & \cdots & (-\mu_\alpha)^a I_a(\mu_\alpha) \\
\end{vmatrix} \zeta^{2a}.
\]

(A.11)

By our normalization \([2]\), this should equal \(1/\pi\). Substituting the resulting \(C\) back to \((A.9)\) and 
\((A.10)\), we establish the result announced in section 2.3.

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