The Noncommutative Standard Model and Forbidden Decays

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1 Introduction

In this contribution we discuss the Noncommutative Standard Model and the associated Standard Model-forbidden decays that can possibly serve as an experimental signature of space-time noncommutativity.

The idea of quantized space-time and noncommutative field theory has a long history that can be traced back to Heisenberg [1] and Snyder [2]. A noncommutative structure of spacetime can be introduced by promoting the usual spacetime coordinates $x$ to noncommutative (NC) coordinates $\hat{x}$ with

$$[\hat{x}^\mu, \hat{x}^\nu] = i \theta^{\mu\nu}, \quad (1)$$

were $\theta^{\mu\nu}$ is a real antisymmetric matrix. A noncommutativity scale $\Lambda_{NC}$ is fixed by choosing dimensionless matrix elements $\theta^{\mu\nu} = \Lambda_{NC}^2 \theta^{\mu\nu}$ of order one. The original motivation to study such a scenario was the hope that the introduction of a fundamental scale could deal with the infinities of quantum field theory in a natural way [3]. The mathematical theory that replaces ordinary differential geometry in the description of quantized spacetime is noncommutative geometry [4]. A realization of the electroweak sector of the Standard Model in the framework of noncommutative geometry can be found [5], where the Higgs field plays the role of a gauge boson in the non-commutative (discrete) direction. This model is noncommutative in an extra internal direction but not in spacetime itself. It is therefore not the focus of the present work, although it can in principle be combined with it.

Noncommutativity of spacetime is very natural in string theory and can be understood as an effect of the interplay of closed and open strings. The commutation relation (1) enters in string theory through the Moyal-Weyl star product

$$f \star g = \sum_{n=0}^{\infty} \frac{\theta^{\mu_1 \nu_1} \cdots \theta^{\mu_n \nu_n}}{(-2i)^n n!} \partial_{\mu_1} \cdots \partial_{\nu_1} f \cdot \partial_{\nu_1} \cdots \partial_{\nu_n} g. \quad (2)$$
For coordinate functions: $x^\mu \star x^\nu - x^\nu \star x^\mu = i \theta^{\mu \nu}$. The tensor $\theta^{\mu \nu}$ is determined by a NS $B^{\mu \nu}$-field and the open string metric $G^{\mu \nu}$ [6], which both depend on a given closed string background. The effective physics on D-branes is most naturally captured by noncommutative gauge theory, but it can also be described by ordinary gauge theory. Both descriptions are related by the Seiberg-Witten (SW) map [7], which expresses noncommutative gauge fields in terms of fields with ordinary “commutative” gauge transformation properties.

The star product formalism in conjunction with the Seiberg-Witten map of fields naturally leads to a perturbative approach to field theory on noncommutative spaces. It is particularly well-suited to study Standard Model-forbidden processes induced by spacetime noncommutativity. This formalism can also be used to study non-perturbative noncommutative effects. In particular cases an algebraic approach may be more convenient for actual computations but the structure of the star product results can still be a useful guideline.

A method for implementing non-Abelian $SU(N)$ Yang-Mills theories on non-commutative spacetime has been proposed in [8, 9, 10, 11]. In [12] this method has been applied to the full Standard Model of particle physics [13] resulting in a minimal non-commutative extension of the Standard Model with structure group $SU(3)_C \times SU(2)_L \times U(1)_Y$ and with the same fields and the same number of coupling parameters as in the original Standard Model. It is the only known approach that allows to build models of the electroweak sector directly based on the structure group $SU(2)_L \times U(1)_Y$ in a noncommutative background. Previously only $U(N)$ gauge theories were under control, and it was thus only possible to consider extensions of the Standard Model. Furthermore there were problems with the allowed charges and with the gauge invariance of the Yukawa terms in the action.

In an alternative approach to the construction of a noncommutative generalization of the Standard Model the usual problems of noncommutative model buildings, i.e., charge quantization and the restriction of the noncommutative gauge group are circumvented by enlarging the gauge group to $U(3) \times U(2) \times U(1)$ [14]. The hypercharges and the electric charges are quantized to the correct values of the usual quarks and leptons, however, there are some open issues with the NC gauge invariance of the Yukawa terms. In principle the two approaches can be combined.

2 The Noncommutative Standard Model

2.1 Noncommutative Yang-Mills

Consider an ordinary Yang-Mills action with gauge group $G$, where $G$ is a compact simple Lie group, and a fermion multiplet $\Psi$

$$S = \int d^4x \frac{-1}{2g^2} \text{Tr}(F_{\mu \nu} F^{\mu \nu}) + \overline{\Psi} \slashed{D} \Psi$$

(3)
This action is gauge invariant under
\[ \delta \Psi = i \rho_\Psi (A) \Psi \]
where \( \rho_\Psi \) is the representation of \( G \) determined by the multiplet \( \Psi \). The noncommutative generalization of (3) is given by
\[ \hat{S} = \int d^4 x \frac{-1}{2g^2} \text{Tr}(\hat{F}_{\mu\nu} \hat{F}^{\mu\nu}) + \overline{\hat{\Psi}} \hat{i} \hat{D} \hat{\Psi} \]
where the noncommutative field strength \( \hat{F} \) is defined by
\[ \hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i [\hat{A}_\mu, \hat{A}_\nu]_\ast. \]
The covariant derivative is given by
\[ \hat{D}_\mu \hat{\Psi} = \partial_\mu \hat{\Psi} - i \rho_\Psi (\hat{A}_\mu) * \hat{\Psi}. \]
The action (5) is invariant under the noncommutative gauge transformations
\[ \hat{\delta} \hat{\Psi} = i \rho_\Psi (\hat{A}) * \hat{\Psi}, \quad \hat{\delta} \hat{A}_\mu = \partial_\mu \hat{A} + i [\hat{A}, \hat{A}_\mu]_\ast, \quad \hat{\delta} \hat{F}_{\mu\nu} = i [\hat{A}, \hat{F}_{\mu\nu}]_\ast. \]
If the gauge fields are assumed to be Lie-algebra valued, it appears that only \( U(N) \) in the fundamental representation is consistent with noncommutative gauge transformations: Only in this case the commutator
\[ [\hat{A}, \hat{A}]_\ast = \frac{1}{2} \{ A_a (x) ; A'_a (x) \} [T^a, T^b] + \frac{1}{2} \{ A_a (x) ; A'_a (x) \} \{ T^a, T^b \} \]
of two Lie algebra-valued non-commutative gauge parameters \( \hat{A} = A_a (x) T^a \) and \( \hat{A}' = A'_a (x) T^a \) again closes in the Lie algebra [8, 9]. The fact that a \( U(1) \) factor cannot easily be decoupled from NC \( U(N) \), can also be seen by noting the interactions of \( SU(N) \) gluons and \( U(1) \) (hyper) photons in NC Yang-Mills theory [15]. For a sensible phenomenology of particle physics on noncommutative spacetime we need to be able to use other gauge groups. Furthermore, in the special case of \( U(1) \) a similar argument show that charges are quantized to values \( \pm e \) and zero. These restrictions can be avoided if we allow gauge fields and gauge transformation parameters that are valued in the enveloping algebra of the gauge group.
\[ \hat{A} = A_a^0 (x) T^a + A_{ab}^1 (x) T^a T^b + A_{abc}^2 (x) T^a T^b T^c + \ldots \]
A priori we now face the problem that we have an infinite number of parameters \( A_a^0 (x), A_{ab}^1 (x), A_{abc}^2 (x), \ldots \), but these are not independent. They can in fact all be expressed in terms of the right number of classical parameters and fields via the Seiberg-Witten maps. The non-commutative fields \( \hat{A}, \hat{\Psi} \) and non-commutative gauge parameter \( \hat{A} \) can be expressed as “towers” built upon the corresponding ordinary fields \( A, \Psi \) and ordinary gauge parameter \( A \). The
Seiberg-Witten maps [16] express non-commutative fields and parameters as local functions of the ordinary fields and parameters,

\[
\hat{A}_\xi[A] = A_\xi + \frac{1}{4} \theta^\mu{}^\nu \{ A_\nu, \partial_\mu A_\xi \} + \frac{1}{4} \theta^\mu{}^\nu \{ F_\mu{}^\xi, A_\nu \} + \mathcal{O}(\theta^2) \tag{11}
\]

\[
\hat{\Psi}[\Psi, A] = \Psi + \frac{1}{2} \theta^\mu{}^\nu \rho_\phi(A_\nu) \partial_\mu \Psi + \frac{i}{8} \theta^\mu{}^\nu \{ \rho_\phi(A_\mu), \rho_\phi(A_\nu) \} \Psi + \mathcal{O}(\theta^2) \tag{12}
\]

\[
\hat{\Lambda}[\Lambda, A] = \Lambda + \frac{1}{4} \theta^\mu{}^\nu \{ A_\nu, \partial_\mu \Lambda \} + \mathcal{O}(\theta^2) \tag{13}
\]

where \( F_\mu{}^\nu = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu] \) is the ordinary field strength. The Seiberg-Witten maps have the remarkable property that ordinary gauge transformations \( \delta A_\mu = \partial_\mu \Lambda + i [A_\mu, A_\nu] \) and \( \delta \Psi = i \Lambda \cdot \Psi \) induce non-commutative gauge transformations (8) of the fields \( \hat{A}, \hat{\Psi}, \hat{\Lambda} \).

2.2 Standard model fields

The Standard Model gauge group is \( G_{SM} = SU(3)_C \times SU(2)_L \times U(1)_Y \). The gauge potential \( A_\mu \) and gauge parameter \( \Lambda \) are valued in \( \text{Lie}(G_{SM}) \):

\[
A_\nu = g' A_\nu(x) Y + g \sum_{a=1}^3 B_{\nu a}(x) T_L^a + g_s \sum_{b=1}^8 G_{\nu b}(x) T_S^b \tag{14}
\]

\[
\Lambda = g' \alpha(x) Y + g \sum_{a=1}^3 \alpha^L_a(x) T_L^a + g_s \sum_{b=1}^8 \alpha^S_b(x) T_S^b \tag{15}
\]

where \( Y, T_L^a, T_S^b \) are the generators of \( u(1)_Y, su(2)_L \) and \( su(3)_C \) respectively. In addition to the gauge bosons we have three families of left- and right-handed fermions and a Higgs doublet

\[
\Psi^{(i)}_L = \begin{pmatrix} L^{(i)}_L \\ Q^{(i)}_L \end{pmatrix}, \quad \Psi^{(i)}_R = \begin{pmatrix} e^{(i)}_R \\ u^{(i)}_R \\ d^{(i)}_R \end{pmatrix}, \quad \Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{16}
\]

where \( i = 1,2,3 \) is the generation index and \( \phi^+, \phi^0 \) are complex scalar fields. We shall now apply the appropriate SW maps to the fields \( A_\mu, \Psi^{(i)}_L, \Phi \), expand to first order in \( \theta \) and write the corresponding NC Yang-Mills action [12].

2.3 Noncommutative Yukawa terms

Special care must be taken in the definition of the trace in the gauge kinetic terms and in the construction of covariant Yukawa terms. The classical Higgs field \( \Phi(x) \) commutes with the generators of the \( U(1) \) and \( SU(3) \) gauge transformations. It also commutes with the corresponding gauge parameters. The latter is no longer true in the noncommutative setting: The coefficients
\( \alpha(x) \) and \( \alpha_S(x) \) of the \( U(1) \) and \( SU(3) \) generators in the gauge parameter are functions and therefore do not \(*\)-commute with the Higgs field. This makes it hard to write down covariant Yukawa terms. The solution to the problem is the hybrid SW map [17]

\[
\hat{\Phi}[\Phi, A, A'] = \Phi + \frac{1}{2} \theta^{\mu \nu} A_\nu \left( \partial_\mu \Phi - \frac{i}{2} (A_\mu \Phi - \Phi A'_\mu) \right) + \frac{1}{2} \theta^{\mu \nu} \left( \partial_\mu \Phi - \frac{i}{2} (A_\mu \Phi - \Phi A'_\mu) \right) A'_\nu + O(\theta^2).
\] (17)

By choosing appropriate representations it allows us to assign separate left and right charges to the noncommutative Higgs field \( \hat{\Phi} \) that add up to its usual charge [12]. Here are two examples:

\[
Y = \begin{array}{cccc}
1/2 & -1/2 + 1 & -1 & -1/6 + 1/3 & -1/3 \\
1/2 & & & & \\
\end{array}
\]

(18)

We see here two instances of a general rule: The gauge fields in the SW maps and in the covariant derivatives inherit their representation (charge for \( Y \), trivial or fundamental representation for \( T_a^L, T_b^S \)) from the fermion fields \( \Psi^{(i)} \) to their left and to their right.

In GUTs it is more natural to first combine the left-handed and right-handed fermion fields and then contract the resulting expression with Higgs fields to obtain a gauge invariant Yukawa term. Consequently in NC GUTs we need to use the hybrid SW map for the left-handed fermion fields and then sandwich them between the NC Higgs on the left and the right-handed fermion fields on the right [18].

2.4 The minimal NCSM

The trace in the kinetic terms for the gauge bosons is not unique, it depends on the choice of representation. This would not matter if the gauge fields were Lie algebra valued, but in the noncommutative case they live in the enveloping algebra. The simplest choice is a sum of three traces over the \( U(1), SU(2), \) \( SU(3) \) sectors with \( Y = \frac{1}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \) in the definition of \( \text{tr}_1 \) and the fundamental representation for \( \text{tr}_2 \) and \( \text{tr}_3 \). This leads to the following gauge kinetic terms

\[
S_{\text{gauge}} = -\frac{1}{4} \int d^4x f_{\mu \nu \rho \sigma} f^{\mu \nu} - \frac{1}{2} \text{Tr} \int d^4x F_{\mu \nu}^L F_{\mu \nu}^L + \frac{1}{2} \text{Tr} \int d^4x F_{\mu \nu}^S F_{\mu \nu}^S + \frac{1}{4} g_S \theta^{\mu \nu} \text{Tr} \int d^4x F_{\mu \nu}^S F_{\mu \nu}^S F_{\rho \sigma} + g_S \theta^{\mu \nu} \text{Tr} \int d^4x F_{\mu \nu}^S F_{\mu \nu}^S F_{\rho \sigma} + O(\theta^2).
\] (19)
Note, that there are no new triple $f$ or triple $F$-terms. The full action of the Minimal Noncommutative Standard Model is [12]:

$$S_{\text{NCSM}} = \int d^4x \sum_{i=1}^3 \bar{\Psi}_L(i) \gamma^\mu \gamma^5 \gamma^\nu \Psi_L(i) + \int d^4x \sum_{i=1}^3 \bar{\Psi}_R(i) \gamma^\mu \gamma^5 \gamma^\nu \Psi_R(i)$$

$$- \int d^4x \frac{1}{2g} \text{tr}_1 F_{\mu\nu} \gamma^5 F^{\mu\nu} - \int d^4x \frac{1}{2g} \text{tr}_2 F_{\mu\nu} \gamma^5 F^{\mu\nu}$$

$$- \int d^4x \frac{1}{2g_S} \text{tr}_3 F_{\mu\nu} \gamma^5 F^{\mu\nu} + \int d^4x \left( \rho_0(\hat{D}_\mu \hat{\Phi})^\dagger \rho_0(\hat{D}_\nu \hat{\Phi}) - i \mu^2 \rho_0(\hat{\Phi})^\dagger \rho_0(\hat{\Phi}) - \lambda \rho_0(\hat{\Phi})^\dagger \rho_0(\hat{\Phi})^\dagger \rho_0(\hat{\Phi}) \right)$$

$$- \frac{1}{g^2} \sum_{i,j=1}^3 W^{ij} \left( \bar{L}_L(i) \star \rho_L(\hat{\Phi}) \gamma^\mu \gamma^5 \gamma^\nu \Psi_L(j) + \bar{L}_L(i) \star \rho_L(\hat{\Phi}) \gamma^\mu \gamma^5 \gamma^\nu \Psi_L(j) \right)$$

$$+ \sum_{i,j=1}^3 G^{ij}_{q}(\bar{Q}_L(i) \star \rho_Q(\hat{\Phi}) \gamma^\mu \gamma^5 \gamma^\nu \Psi_L(j) + \bar{Q}_L(i) \star \rho_Q(\hat{\Phi}) \gamma^\mu \gamma^5 \gamma^\nu \Psi_L(j) \right)$$

$$+ \sum_{i,j=1}^3 G^{ij}_{d}(\bar{u}_R(i) \star \rho_Q(\hat{\Phi}) \gamma^\mu \gamma^5 \gamma^\nu \Psi_L(j) + \bar{u}_R(i) \star \rho_Q(\hat{\Phi}) \gamma^\mu \gamma^5 \gamma^\nu \Psi_L(j) \right)$$

(20)

where $W^{ij}, G^{ij}_{q}, G^{ij}_{d}$ are Yukawa couplings and $\hat{\Phi} = i\tau_2 \Phi^*$.  

### 2.5 Non-minimal versions of the NCSM

We can use the freedom in the choice of traces in kinetic terms for the gauge fields to construct non-minimal versions of the NCSM. The general form of the gauge kinetic terms is [12, 18]

$$S_{\text{gauge}} = -\frac{1}{2} \int d^4x \sum_{\rho} \kappa_\rho \text{Tr} \left( \rho(\tilde{F}_{\mu\nu}) \gamma^5 \gamma^\mu \gamma^5 \gamma^\nu \rho(\tilde{F}_{\mu\nu}) \right),$$

(21)

where the sum is over all unitary irreducible inequivalent representations $\rho$ of the gauge group $G$. The freedom in the kinetic terms is parametrized by real coefficients $\kappa_\rho$ that are subject to the constraints

$$\frac{1}{g^2} = \sum_{\rho} \kappa_\rho \text{Tr} \left( \rho(T^a_{\rho}) \rho(T^a_{\rho}) \right),$$

(22)

where $g_1$ and $T^a_1$ are the usual “commutative” coupling constants and generators of $U(1)_Y, SU(2)_L, SU(3)_C$, respectively. Both formulas can also be written more compactly as
\[ S_{\text{gauge}} = -\frac{1}{2} \int d^4x \, \text{Tr} \frac{1}{G^2} \hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu}, \quad \frac{1}{g_I^2} = \text{Tr} \frac{1}{G^2} T_I^a T_I^a, \quad (23) \]

where the trace \( \text{Tr} \) is again over all representations and \( G \) is an operator that commutes with all generators \( T_I^a \) and encodes the coupling constants. The possibility of new parameters in gauge theories on noncommutative spacetime is a consequence of the fact that the gauge fields are in general valued in the enveloping algebra of the gauge group.

The expansion in \( \theta \) is at the same time an expansion in the momenta. The \( \theta \)-expanded action can thus be interpreted as a low energy effective action. In such an effective low energy description it is natural to expect that all representations that appear in the commutative theory (matter multiplets and adjoint representation) are important. We should therefore consider the non-minimal version of the NCSM with non-zero coefficients \( \kappa_\rho \) at least for these representations. The new parameters in the non-minimal NCSM can be restricted by considering GUTs on noncommutative spacetime [18].

2.6 Properties of the NCSM

The key properties of the Noncommutative Standard Model (NCSM) are:

- The known elementary particles can be accommodated with their correct charges as in the original “commutative” Standard Model. There is no need to introduce new fields.
- The noncommutative Higgs field in the minimal NCSM has distinct left and right hyper (and colour) charges, whose sum are the regular SM charges. This is necessary to obtain gauge invariant Yukawa terms.
- In versions of the NCSM that arise from NC GUTs it is more natural to equip the neutrino (and other left-handed fermion fields) with left and right charges. The neutrino can in principle couple to photons in the presence of spacetime noncommutativity, even though its total charge is zero.
- Noncommutative gauge invariance implies the existence of many new couplings of gauge fields: Abelian gauge bosons self-interact via a star-commutator term that resembles the self-interaction of non-abelian gauge bosons and we find many new interaction terms that involve gauge fields as a consequence of the Seiberg-Witten maps.
- The perturbation theory is based on the free commutative action. Asymptotic states are the plane-wave eigenstates of the free commutative Hamiltonian. Both ordinary interaction terms and interactions due to noncommutative effects are treated on equal footing. This makes it particularly simple to derive Feynman rules and compute the invariant matrix elements of fundamental processes. While there is no need to reinvent perturbation theory, care has to be taken nevertheless for a time-like \( \theta \)-tensor to avoid problems with unitarity.
- Violation of spacetime symmetries and in particular of angular momentum conservation and discrete symmetries like P, CP, and possibly even CPT
can be induced by spacetime noncommutativity. This symmetry breaking is spontaneous in the sense that it is with respect to a fixed $\theta$-"vacuum". (As long as $\theta$ is also transformed as a tensor, everything is fully covariant.)

The physically interpretation of these violations of conservation laws is that angular momentum (and even energy-momentum) can be transferred to the noncommutative spacetime structure in much the same way as energy can be carried away from binary stars by gravitational waves.

3 Standard Model forbidden processes

A general feature of gauge theories on noncommutative spacetime is the appearance of many new interactions including Standard Model-forbidden processes. The origin of these new interactions is two-fold: One source are the star products that let abelian gauge theory on NC spacetime resemble Yang-Mills theory with the possibility of triple and quadruple gauge boson vertices. The other source are the gauge fields in the Seiberg-Witten maps for the gauge and matter fields. These can be pictured as a cloud of gauge bosons that dress the original ‘commutative’ fields and that have their origin in the interaction between gauge fields and the NC structure of spacetime. One of the perhaps most striking effects and a possible signature of spacetime noncommutativity is the spontaneous breaking of continuous and discrete spacetime symmetries.

3.1 Triple gauge boson couplings

New anomalous triple gauge boson interactions that are usually forbidden by Lorentz invariance, angular moment conservation and Bose statistics (Yang theorem) can arise within the framework of the non-minimal noncommutative standard model [19, 20], and also in the alternative approach to the NCSM given in [14].

The new triple gauge boson (TGB) terms in the action have the following form [19, 20]:

$$S_{\text{gauge}} = -\frac{1}{4} \int d^4x f_{\mu\nu} f^{\mu\nu}$$

$$- \frac{1}{2} \int d^4x \text{Tr} \left( F_{\mu\nu} F^{\mu\nu} \right) - \frac{1}{2} \int d^4x \text{Tr} \left( G_{\mu\nu} G^{\mu\nu} \right)$$

$$+ g_s \theta^{\rho\sigma} \int d^4x \text{Tr} \left( \frac{1}{4} G_{\rho\nu} G_{\mu\sigma} - G_{\mu\rho} G_{\nu\sigma} \right) G^{\mu\nu}$$

$$+ g^3 \kappa_1 \theta^{\rho\sigma} \int d^4x \left( \frac{1}{4} f_{\rho\nu} f_{\sigma\mu} - f_{\mu\rho} f_{\sigma\nu} \right) f^{\mu\nu}$$

$$+ g^2 g^2 \kappa_2 \theta^{\rho\sigma} \int d^4x \sum_{a=1}^{3} \left( \frac{1}{4} f_{\rho\nu} F^{\alpha}_{\mu\nu} - f_{\mu\rho} F^{\alpha}_{\nu\sigma} \right) F^{\mu\nu,\alpha} + c.p. \right]$$
\[ + g' g^2 \kappa_3 \theta^{\alpha \beta} \int d^4x \sum_{b=1}^8 \left[ \frac{1}{4} f_{\rho \nu} G_{\mu \nu}^b - f_{\mu \rho} G_{\nu \tau}^b \right] G^{\mu \nu, b} + c.p. \],
\]
where \( c.p. \) means cyclic permutations. Here \( f_{\mu \nu}, F_{\mu \nu}^a \) and \( G_{\mu \nu}^b \) are the physical field strengths corresponding to the groups \( U(1)_Y, \text{SU}(2)_L \) and \( \text{SU}(3)_C \), respectively. The constants \( \kappa_1, \kappa_2 \) and \( \kappa_3 \) are functions of \( 1/g_i^2 \) (\( i = 1, \ldots, 6 \)):
\[
\kappa_1 = - \frac{1}{g_1^2} - \frac{1}{4g_2^2} + \frac{8}{9g_3^2} - \frac{1}{9g_4^2} + \frac{1}{36g_5^2} + \frac{1}{4g_6^2},
\]
\[
\kappa_2 = - \frac{1}{4g_2^2} + \frac{1}{4g_5^2} + \frac{1}{4g_6^2},
\]
\[
\kappa_3 = + \frac{1}{3g_3^2} - \frac{1}{6g_4^2} + \frac{1}{6g_5^2}.
\]

The \( g_i \) are the coupling constants of the non-commutative electroweak sector up to first order in \( \theta \). The appearance of new coupling constants beyond those of the standard model reflect a freedom in the strength of the new TGB couplings. Matching the SM action at zeroth order in \( \theta \), three consistency conditions are imposed on (24):
\[
\frac{1}{g'^2} = \frac{2}{g_1^2} + \frac{2}{g_2^2} + \frac{2}{g_3^2} + \frac{1}{g_4^2} + \frac{1}{g_5^2} + \frac{1}{g_6^2},
\]
\[
\frac{1}{g'^2} = \frac{1}{g_2^2} + \frac{3}{g_5^2} + \frac{1}{g_6^2},
\]
\[
\frac{1}{g'^2} = \frac{1}{g_2^2} + \frac{1}{g_4^2} + \frac{2}{g_5^2}.
\]

From the action (24) we extract neutral triple-gauge boson terms which are not present in the SM Lagrangian. The allowed range of values for the coupling constants
\[
K_{\gamma \gamma \gamma} = \frac{1}{2} g' (\kappa_1 + 3\kappa_2),
\]
\[
K_{Z \gamma \gamma} = \frac{1}{2} \left[ g'^2 \kappa_1 + \left( g'^2 - 2g^2 \right) \kappa_2 \right],
\]
\[
K_{Z gg} = \frac{g^2}{2} \left[ 1 + \left( \frac{g'}{g} \right)^2 \right] \kappa_3,
\]
compatible with conditions (26) and the requirement that \( 1/g_i^2 > 0 \) are plotted in figure 1. The remaining three coupling constants \( K_{ZZ \gamma} \), \( K_{Z ZZ} \) and \( K_{\gamma gg} \), are uniquely fixed by the equations
\[
K_{ZZ \gamma} = \frac{1}{2} \left( \frac{g}{g'} - 3 \frac{g'}{g} \right) K_{Z \gamma \gamma} - \frac{1}{2} \left( 1 - \frac{g'^2}{g^2} \right) K_{\gamma \gamma \gamma},
\]
\[
K_{Z ZZ} = \frac{3}{2} \left( 1 - \frac{g'^2}{g^2} \right) K_{Z \gamma \gamma} - \frac{1}{2} \frac{g^2}{g'^2} \left( 3 - \frac{g'^2}{g^2} \right) K_{\gamma \gamma \gamma},
\]
\[
K_{\gamma gg} = - \frac{2}{g} K_{Z gg}.
\]
Fig. 1. The three-dimensional pentahedron that bounds possible values for the coupling constants $K_{\gamma\gamma\gamma}$, $K_{Z\gamma\gamma}$ and $K_{Z\gamma\gamma}$ at the $M_Z$ scale.

We see that any combination of two TGB coupling constants does not vanish simultaneously due to the constraints set by the values of the SM coupling constants at the $M_Z$ scale [20].

We conclude that the gauge sector is a possible place for an experimental search for noncommutative effects. The experimental discovery of the kinematically allowed $Z \rightarrow \gamma\gamma$ decay would indicate a violation of the Yang theorem and would be a possible signal of spacetime non-commutativity.

3.2 Electromagnetic properties of neutrinos

In the presence of spacetime noncommutativity, neutral particles can couple to gauge bosons via a $\star$-commutator

$$D_{\mu}^{\text{NC}}\hat{\psi} = \partial_{\mu}\hat{\psi} - ie\hat{A}_{\mu} \star \hat{\psi} + ie\hat{\psi} \star \hat{A}_{\mu}. \quad (28)$$

Expanding the $\star$-product in (28) to first order in the antisymmetric (Poisson) tensor $\theta^{\mu\nu}$, we find the following covariant derivative on neutral spinor fields:

$$D_{\mu}^{\text{NC}}\hat{\psi} = \partial_{\mu}\hat{\psi} + e\theta^{\rho\nu} \partial_{\nu}\hat{A}_{\mu} \partial_{\rho}\hat{\psi}. \quad (29)$$

We treat $\theta^{\mu\nu}$ as a constant background field of strength $|\theta^{\mu\nu}| = 1/A_{\text{NC}}^2$ that models the non-commutative structure of spacetime in the neighborhood of the interaction region. As $\theta$ is not invariant under Lorentz transformations, the neutrino field can pick up angular momentum in the interaction. The gauge-invariant action for a neutral fermion that couples to an Abelian gauge boson via (29) is
\[ S = \int d^4 x \bar{\psi} \left[ \left( i\gamma^\mu \partial_\mu - m \right) - \frac{\theta}{2} F_{\mu\nu} (i\theta^{\mu\nu\rho} \partial_\rho - \theta^{\mu\nu} m) \right] \psi, \quad (30) \]

\[ \theta^{\mu\nu\rho} = \theta^{\mu\nu} \gamma^\rho + \theta^{\rho\nu} \gamma^\mu + \theta^{\rho\mu} \gamma^\nu, \]

up to first order in \( \theta [21, 22, 23] \). The noncommutative part of (30) induces a
force, proportional to the gradient of the field strengths, which represents an
interaction of Stern-Gerlach type [24]. This interaction is non-zero even for
\( m_\nu = 0 \) and in this case reduces to the coupling between the stress–energy
tensor of the neutrino \( T^{\mu\nu} \) and the symmetric tensor composed from \( \theta \) and \( F \)[21]. The following is based on [22].

**Neutrino dipole moments in the mass-extended Standard Model**

Following the general arguments of [25, 26, 27, 28] only the Dirac neutrino can
have a magnetic moment. However, the transition matrix elements relevant for
\( \nu_i \rightarrow \nu_j \) may exist for both Dirac and Majorana neutrinos. In the neutrino-
mass extended standard model [28], the photon–neutrino effective vertex is
determined from the \( \nu_i \rightarrow \nu_j + \gamma \) transition, which is generated through
1-loop electroweak process that arise from the so-called “neutrino–penguin”
diagrams via the exchange of \( \ell = e, \mu, \tau \) leptons and weak bosons, and is given
by [25, 23]

\[ J_{\mu}^{\text{eff}}(q) = \{ F_1(q^2) \bar{\nu}_i(p') (\gamma_\mu q^2 - q_\mu e q) \nu_i(p)L \\
- iF_2(q^2) \left[ m_{\nu_i} \bar{\nu}_j(p') \sigma_\mu q^\nu \nu_i(p)L \right. \\
+ m_{\nu_i} \bar{\nu}_j(p') \sigma_\mu q^\nu \nu_i(p)R \left. \} \right] \}
\]

(31)

The above effective interaction is invariant under the electromagnetic gauge
transformation. The first term in (31) vanishes identically for real photon due
to the electromagnetic gauge condition.

From the general decomposition of the second term of the transition matrix
element \( T (31) \),

\[ T = -ie^\mu(q)\bar{\nu}(p') \left[ A(q^2) - B(q^2)\gamma_5 \right] \sigma_\mu q^\nu \nu(p), \quad (32) \]

we found the following expression for the electric and magnetic dipole moments

\[ d_{ji}^{el} \equiv B(0) = \frac{-e}{M^2} \left( m_{\nu_i} - m_{\nu_j} \right) \sum_{\ell=e,\mu,\tau} U^\dagger_{jk} U_{ki} F\left( \frac{m_{\ell}^2}{m_W^2} \right), \quad (33) \]

\[ \mu_{ji} \equiv A(0) = \frac{-e}{M^2} \left( m_{\nu_i} + m_{\nu_j} \right) \sum_{\ell=e,\mu,\tau} U^\dagger_{jk} U_{ki} F\left( \frac{m_{\ell}^2}{m_W^2} \right), \quad (34) \]

where \( i, j, k = 1, 2, 3 \) denotes neutrino species, and

\[ F\left( \frac{m_{\ell}^2}{m_W^2} \right) \approx -\frac{3}{2} + \frac{3}{4} \frac{m_{\ell}^2}{m_W^2}, \quad \frac{m_{\ell}^2}{m_W^2} \ll 1, \quad (35) \]
was obtained after the loop integration. In Eqs. (33) and (34) \( M^* = 4\pi v = 3.1 \) TeV, where \( v = (\sqrt{2} G_F)^{-1/2} = 246 \) GeV represents the vacuum expectation value of the scalar Higgs field [29].

The neutrino mixing matrix \( U [30] \) is governing the decomposition of a coherently produced left-handed neutrino \( \tilde{\nu}_L,\ell \) associated with charged-lepton-flavour \( \ell = e, \mu, \tau \) into the mass eigenstates \( \nu_L,i \):

\[
| \tilde{\nu}_L,\ell; \mathbf{p} \rangle = \sum_i U_{\ell i} | \nu_L,i; \mathbf{p}, m_i \rangle, \quad (36)
\]

For a Dirac neutrino \( i = j \) [26, 31], and using \( m_\nu = 0.05 \) eV [32], from (34), in units of \( [\text{e cm}] \) and Bohr magneton, we obtain

\[
\mu_{\nu_i} = \frac{3e}{2M^*^2} m_{\nu_i} \left[ 1 - \frac{1}{2} \sum_{\ell=e,\mu,\tau} \frac{m_\ell^2}{m_W^2} |U_{\ell i}|^2 \right],
\]

\[
= 3.0 \times 10^{-31} \,[\text{e cm}] = 1.6 \times 10^{-20} \mu_B. \quad (37)
\]

From formula (37) it is clear that the chirality flip, which is necessary to induce the magnetic moment, arises only from the neutrino masses: Dirac neutrino magnetic moment (37) is still much smaller than the bounds obtained from astrophysics [33, 34]. More details about Dirac neutrinos can be found in [35, 36].

In the case of the off-diagonal transition moments, the first term in (35) vanishes in the summation over \( \ell \) due to the orthogonality condition of \( U \) (GIM cancellation)

\[
d_{\tilde{\nu}_j,\nu_i}^1 = \frac{3e}{2M^*^2} (m_{\nu_i} - m_{\nu_j}) \sum_{\ell=e,\mu,\tau} \frac{m_\ell^2}{m_W^2} U_{jk}^\dagger U_{ki}, \quad (38)
\]

\[
\mu_{\tilde{\nu}_j,\nu_i} = \frac{3e}{2M^*^2} (m_{\nu_i} + m_{\nu_j}) \sum_{\ell=e,\mu,\tau} \frac{m_\ell^2}{m_W^2} U_{jk}^\dagger U_{ki}. \quad (39)
\]

In Majorana 4-component notation the Hermitian, neutrino–flavor antisymmetric, electric and magnetic dipole operators are

\[
\begin{pmatrix} D_5 \\ D \end{pmatrix}^{\mu\nu}_{ij} = e \psi_i^\dagger \begin{pmatrix} C \sigma^{\mu\nu} \gamma_5 \\ 1 \end{pmatrix} \psi_j. \quad (40)
\]

Majorana fields have the property that the particle is not distinguished from antiparticle. This forces us to use both charged lepton and antilepton propagators in the loop calculation of “neutrino-penguin” diagrams. This results in a complex antisymmetric transition matrix element \( T \) in lepton-flavour space:

\[
T_{ji} = -ie^{\mu}\tilde{\nu}_j ([A_{ji} - A_{ij}] - (B_{ji} - B_{ij})\gamma_5) \sigma_{\mu\nu} q^\nu \nu_i
\]

\[
= -ie^{\mu}\tilde{\nu}_j [2i\text{Im}A_{ji} - 2\text{Re}B_{ji}\gamma_5] \sigma_{\mu\nu} q^\nu \nu_i. \quad (41)
\]
From this equation it is explicitly clear that for \( i = j \), \( d_{\nu_i}^{el} = \mu_{\nu_i} = 0 \). Also, considering transition moment, only one of two terms in (41) is non-vanishing if the interaction respects the CP invariance, i.e. the first term vanishes if the relative CP of \( \nu_i \) and \( \nu_j \) is even, and the second term vanishes if odd [27].

Finally, dipole moments describing the transition from Majorana neutrino mass eigenstate-flavour \( \nu_j \) to \( \nu_k \) in the mass extended standard model reads:

\[
del_{\nu_i,\nu_j}^{el} = \frac{3e}{2M^2} (m_{\nu_i} - m_{\nu_j}) \sum_{\ell=e,\mu,\tau} \frac{m_{\ell}}{m_W} \Re U_{jk}^\dag U_{ki},
\]

(42)

\[
\mu_{\nu_i,\nu_j} = \frac{3e}{2M^2} (m_{\nu_i} + m_{\nu_j}) \sum_{\ell=e,\mu,\tau} \frac{m_{\ell}}{m_W} \Im U_{jk}^\dag U_{ki},
\]

(43)

For the Majorana case the neutrino-flavour mixing matrix \( U \) is approximately unitary, i.e. it is necessarily of the following form [29]

\[
\sum_{i=1}^{3} U_{jk}^\dag U_{ki} = \delta_{ji} - \varepsilon_{ji},
\]

(44)

where \( \varepsilon \) is a hermitian nonnegative matrix (i.e. with all eigenvalues nonnegative) and

\[
|\varepsilon| = \sqrt{\text{Tr} \varepsilon^2} = O \left( \frac{m_{\nu_{\text{light}}}}{m_{\nu_{\text{heavy}}}} \right),
\]

\[
\sim 10^{-22} \text{ to } 10^{-21}.
\]

(45)

For the sum and difference of neutrino masses we assume hierarchical structure and take \( |m_3 + m_2| \simeq |m_3 - m_2| \simeq |\Delta m^2_{32}|^{1/2} = 0.05 \text{ eV} \) [32]. For the MNS matrix elements we set \( |\Re U_{2\tau}^* U_{\tau 3}| \simeq |\Im U_{2\tau}^* U_{\tau 3}| \leq 0.5 \). The electric and magnetic transition dipole moments of neutrinos \( \nu_{2\nu_3} \) and \( \mu_{\nu_2,\nu_3} \) are then denoted as \( (d_{\text{mag}}^{el})_{23} \) and given by

\[
\left| (d_{\text{mag}}^{el})_{23} \right| = \frac{3e}{2M^2} \frac{m_2^2}{m_W} |\Delta m^2_{32}|^{1/2} \frac{|\Re U_{2\tau}^* U_{\tau 3}|}{|\Im U_{2\tau}^* U_{\tau 3}|},
\]

\[
\lesssim 1.95 \times 10^{-30} [e/\text{eV}] = 3.8 \times 10^{-35} [e \text{ cm}],
\]

\[
= 2.0 \times 10^{-24} \mu_B.
\]

(46)

The electric transition dipole moments of light neutrinos are smaller than the one of the d-quark. This is the order of magnitude of light neutrino transition dipole moments underlying the see–saw mechanism. It is by orders of magnitude smaller than in unprotected SUSY models.

**Limits on the noncommutativity scale**

Now we extract an upper limit on the \( \ast \)-gradient interaction. The strength of the interaction (30) becomes \( |m_\nu e \theta F| \). We compare it with the dipole interaction \( |F \mu_{\nu_i}| \) for Dirac neutrino (37), and with the dipole transition interactions \( |F d_{\text{mag}}^{el}| \) for Majorana case (42,43). Assuming that contributions from
the neutrino-mass extended standard model are at least as large as those from noncommutativity, we derive the following two bounds on noncommutativity arising from the Dirac and Majorana nature of neutrinos, respectively:

\[
\Lambda_{\text{Dirac}}^{NC} \gtrsim \left| \frac{e m_{\nu}}{\mu_{\nu}} \right|^{1/2} \approx 1.7 \text{ TeV.} \tag{47}
\]

\[
\Lambda_{\text{Majorana}}^{NC} \gtrsim \left| \frac{e m_{\nu}}{(\delta_{\text{mag}}^2)_{23}} \right|^{1/2} \approx 150 \text{ TeV.} \tag{48}
\]

The fact that the neutrino mass extended standard model, as a consequence of (35), produces very different dipole moments for Dirac neutrinos (37) and Majorana neutrinos (46) respectively, manifests in two different scales of noncommutativity (47) and (48). The \( (m_{\nu}^2/m_W^2) \) suppression of Majorana dipole moments (46) relative to the Dirac ones (37), is the main source for the different scales of noncommutativity. The bounds on noncommutativity thus obtained fix the scale \( \Lambda_{NC} \) at which the expected values of the neutrino electromagnetic dipole moments due to noncommutativity matches the standard model contributions.

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