Nonuniversal Critical Conductance Fluctuations of Chiral Surface States in the Bulk 
Integral Quantum Hall Effect – An Exact Calculation

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The chiral surface electrons in the bulk quantum Hall effect probably form the first extended 
system in which conductance fluctuations can be calculated non-perturbatively in the presence of disorder. By use of the Kubo formula with appropriate boundary conditions, we calculate exactly the variance of conductance with non-perturbative methods. We find that the conductance fluctuations of this system are nonuniversal and the variance of the conductance scales in a very peculiar way. This result can be checked with exact computation using the Landauer-Buttiker formula and both methods show the same scaling behavior. We have also calculated the diffusion constant fluctuations exactly. We find that the diffusion constant fluctuations vanish and thus play no role in the conductance fluctuations.

Electronic systems under the influence of disorder exhibit many intriguing phases most of which are still not well understood [1]. The understanding of the transition of electronic states from localized [2] to extended is still far from complete. In the standard scaling theory of localization [3], it is argued plausibly that the beta function \( \beta(g) \equiv d \ln g / d \ln L \) is a function of the dimensionless conductance \( g \) alone. This one-parameter scaling idea received further support in the metallic regime from the experimental appearance of universal conductance fluctuations (UCF) [4] in small metals as well as from an interesting numerical simulation [5] which revealed the UCF as a consequence of quantum interference. When the electronic states are localized, however, the system generally has a very broad conductance distribution [6]. The average conductance thus might not seem enough to fully characterize the system. Indeed, many-parameter scaling [6] has been proposed to take into account the broad conductance distribution near the delocalization transition. There is no method to determine from first principles whether one-parameter or many-parameter scaling is more appropriate except by calculating the higher cumulants of the conductance, such as conductance fluctuations (CF), directly. So far CF calculations have been done only perturbatively [7,8] and are valid only in the metallic regime. It is thus natural to ask will these perturbative results hold to all orders of perturbation and what happens in the regime that is inaccessible to perturbative methods. An exact conductance calculation in any physical system is thus important.

We now introduce an interesting 2D anisotropic system, the chiral surface states of the bulk (multi-layered) integral quantum Hall effect (IQHE). In the multi-layered quantum Hall sample, at the edge of each layer the electrons circulate in one direction only and can therefore be modeled as chiral Fermions [9]. When the tunneling of these edge electrons in between the layers is allowed, the edge states then smear over a two dimensional sheet where electrons move ballistically transverse to the field and diffusively otherwise. Bechgaard salts [10] and multi-layer heterostructures [11] are candidates for this system. The collection of these edge states turned out to be a novel metallic phase and thus can retain conductance much smaller than \( e^2 / h \) while still scaling ohmically. This should be contrasted with the ordinary metal for which ohmic scaling only happens at large conductance.

We shall focus on the limit in which the length along the chiral direction (circumference) \( L_x \) is much greater than the transverse dimension \( L_z \); this corresponds to the zero-dimensional regime defined by Balents et al. [4]. The finite \( L_x \) case will be discussed in a future publication [15]. Starting with linear response theory, we derive the Kubo formula suitable for this system and calculate the disorder-averaged conductance as well as perform the first exact conductance fluctuation calculation non-perturbatively. We find that the CF are non-universal because the variance of the conductance is a function of the hopping amplitude \( t \), which in turn is related to the system’s diffusion constant. The CF scale with the system size as \( L_x / L_z^2 \). We also find that the disorder-averaged conductance scales ohmically: \( \langle g \rangle \propto L_x / L_z^2 \) where \( \langle \rangle \) is used to denote the disorder average. Define \( \delta g \equiv g - \langle g \rangle \) and \( \Delta \equiv \langle \delta g^2 \rangle / \langle g \rangle^2 \). We have \( \Delta \sim 1 / L_x \ll 1 \) and thus \( g \) is narrowly distributed. Furthermore, \( \Delta \to 0 \) as \( L_x \to \infty \) means that \( g \) is self-averaging. We have also calculated the average conductance using the Landauer-Buttiker(LB) formula as well as the Einstein relation. These two results are identical. We have also calculated the diffusion constant fluctuations exactly and find surprisingly that there are no diffusion constant fluctuations at all. This should be contrasted with the conventional result [8] for ordinary metals with dimensionality two or lower for which the major contribution of CF is the diffusion...
fluctuation! This can be viewed as another salient feature of the novel chiral metallic phase. We can also calculate the CF by using LB formula, but the sums involved are difficult and are thus done numerically. The CF obtained from the LB formula agree qualitatively with the Kubo calculation and show the same scaling behavior. One interesting feature worth mentioning is that the CF vanish when $t = 0$ or $t = 2$ which correspond to the situations of insulating and maximum-conducting respectively. We find this happens both in LB type of calculation as well as in the Kubo type of calculation.

For simplicity, we consider the situation where only the first Landau level is filled, i.e. only one edge state per quantum Hall layer. The system thus consists of a collection of 1+1 dimensional ballistic states moving in the positive $x$ (chiral) direction, each described by a chiral Fermion, which are coupled together by a nearest-neighbor hopping amplitude $t(n, x)$:

$$\mathcal{H}_0 = \sum_n \int dx \left[ -\psi_n^\dagger i \partial_x \psi_n - \left( t(n, x)\psi_n^\dagger \psi_{n+1} + \text{h.c.} \right) \right]. \quad (1)$$

There are two versions of the chiral model distinguished by the form of the hopping amplitude $t(n, x)$. The first one we shall call continuous hopping has $t(n, x) = t$ being a constant. The second one which we shall call discrete hopping has $t(n, x)$ as a sharply peaked function at even(odd) integer $x$ for $n$ being even(odd) and $t(n, x)$ being zero otherwise. We shall focus only on the discrete hopping case. The chiral fermion velocity $v$ is 1 in our choice of units. The full Hamiltonian including random scattering is $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$, where

$$\mathcal{H}_1 = \sum_n \int dx V_n(x) \psi_n^\dagger \psi_n. \quad (2)$$

An illustration of this system is given in Fig. 1. The random potential $V_n(x)$ is assumed to have zero mean. To make our discrete hopping model identical to the one used in ref. [12], we require that the integral of the random potential over each link $\int_{\text{link}} V_n(x)dx \mod (2\pi)$ is a random variable distributed uniformly over $(0, 2\pi]$. In the large $L_x$ limit, due to the ballistic nature of the electron motion along the $x$ (chiral) direction the Green’s function which is retarded in time must also be retarded in $x$ [16]. The retarded Green’s function with constant frequency $E$, $R_E$, satisfies the following equation

$$\begin{pmatrix} -i \partial_x + V_{2n}(x) - E \end{pmatrix} R_E(2n, x, m, x') + \tilde{\Lambda}_o R_E(2n - 1, x, m, x') + \tilde{\Lambda}_r R_E(2n + 1, x, m, x') = -\delta_{2n, m} \delta(x - x')$$

$$\begin{pmatrix} -i \partial_x + V_{2n-1}(x) - E \end{pmatrix} R_E(2n - 1, x, m, x') + \tilde{\Lambda}_o R_E(2n, x, m, x') + \tilde{\Lambda}_r R_E(2n - 2, x, m, x') = -\delta_{2n-1, m} \delta(x - x') \quad (3)$$

where $\tilde{\Lambda}_o(e)(x) = t \sum_{\text{odd(even)}} f(x - t)$ and $f(x)$ is a symmetric function sharply peaked around zero with the condition that $\int f(x) dx = 1$ [14]. Note that the nonzero fixed frequency $E$ can be gauged away by a simple local gauge transformation $G \rightarrow e^{iEz} G$.

The retarded Green’s function $R(n, x; n', x')$ obeying Eq. $R$ can be expressed as a sum over paths of electron going from $(n', x')$ to $(n, x)$

$$R(n, x; n', x') = \sum_p \prod_{l \in p} \left[ \chi_l(t) \exp(i\phi_l) \right]. \quad (4)$$

As depicted in Fig. 1, the electron only moves upwards along the links and can hop horizontally to its neighboring layer across the nodes. Each path contributes an amplitude which is a product of the phases $\phi = \int_{\text{link}} V_n(x')dx'$ of the constituent links together with a prefactor $\chi(t)$ for each node which depends on whether the fermion hops to a neighboring layer or remains in the same layer. The prefactors can be obtained by integrating Eq. $R$ across a node [18]. An analogous expression can be written for the advanced Green’s functions.

Let us first consider the disorder-averaged conductance which will illustrate the methods used here to calculate the CF. The disorder-averaged conductance has previously been calculated for this model by Chalker and Dohmen [12]. Using the standard linear response theory a Kubo formula can then be written that expresses the conductivity in terms of the Green’s function as

$$\sigma_{zz}(n, x; n', x') = -\Lambda_{n-1}(x)\Lambda_{n'-1}(x') \left\{ \left[ R(n - 1, x; n' - 1, x') A(n', x'; n, x) + R(n, x; n', x') A(n' - 1, x; n - 1, x) \right. \right.$$

$$\left. - R(n, x; n' - 1, x') A(n', x'; n - 1, x) - R(n - 1, x; n', x') A(n' - 1, x; n, x) \right] + \left[ R \leftrightarrow A \right] e^{2\pi^2/4}\pi \quad (5)$$
where the $E = 0$ subscript is suppressed from the Green’s functions. In deriving the above expression it is important to keep in mind that the current operator along the non-chiral direction is $J_n(x) = i e t \Lambda_{n-1}(x) [\psi_n^\dagger \psi_{n-1} - \psi_{n-1}^\dagger \psi_n]$ where $\Lambda_n(x) = \tilde{\Lambda}_n(x)/t$ if $n$ is even and $\Lambda_n(x) = \tilde{\Lambda}_o(x)/t$ if $n$ is odd \cite{12}.

Using the path integral expressions for the retarded and advanced Green’s function, Eq. \((3)\) can be expressed as a double sum over paths of retarded Green’s function multiplied by the paths of advanced Green’s function. Upon disorder-averaging, due to the random link phases, only the diagonal terms, in which the retarded and advanced paths pair up, will survive in this double sum. Hence, the disorder-averaged conductivity is nonvanishing only when both $x$ and $x'$ are equal to the same integer and $n = n' \equiv n'$. \(\sigma_{zz}(n, x; n', x') = e^2 t^2 / 2 \pi \delta_{n,n'} \delta_{x,x'} \).

This leads to $\langle g \rangle = (L_x/2L_z)e^2 t^2 / 2\pi$; the factor $L_x/2$ comes from the fact that $\Lambda_{e(o)}$ only sum over even(odd) integers along the chiral direction.

The disorder-averaged conductance can also be evaluated using the LB formula. This approach differs from the Kubo method in that scattering boundary conditions at the probes are implemented exactly. To proceed we have to calculate the probability $|t_{ij}|^2$ of finding outgoing current at $z = L_z$, $x = j$ while constant current is injected at $z = 1$, $x = i$. The conductance is given by $g = \sum_{i,j=1}^{L_z} |t_{ij}|^2$. Note that in terms of diffusion $D(n, x; n', x') \equiv R(n, x; n', x') A(n', x'; n, x)$, $|t_{ij}|^2$ is given by $D(L_z, j; 1, i)$. Calculating $\langle g \rangle$ is thus equivalent to counting paths with appropriate weights. A direct calculation \cite{18} thus shows that

$$\langle g \rangle = e^2 \frac{L_z}{2\pi t^2} \frac{(t^2/4)^2}{L_z}$$

where the factor $L_x/2$ originates from the fact that along the chiral direction there is only one lead out for every two lattice spacings. The above result holds for all $t$ and agrees with the result from the Kubo formula for small $t$. At $t = 2$ where the system becomes ballistic also in the non-chiral direction, the disorder-averaged conductance saturates as one might expect from a Landauer conductance calculation. The result from Kubo formula, however, does not show this behavior as $t \to 2$. It will be interesting to understand the discrepancies between Kubo and Landauer results. For $t \neq 2$ and large $L_z$, $\langle g \rangle \sim e^2 t^2 L_z/[4\pi(1 - t^2/4)^2 L_z]$ which reduces to the result obtained in ref \cite{12} with $t/(1 + t^2/4)$ identified as the $t$ used in ref \cite{12}.

To calculate the average conductance in the third way, let us first consider the diffusion constant $D$ which is defined as the large $L$ limit of the quantity $\sum_n n^2 D(n, L)/2L$ where $D(n, x) \equiv D(n, x; n', 0, x' = 0)$. We find that \cite{18} the disorder-averaged diffusion constant is given by $\langle D \rangle = t^2/2(1 - t^2/4)^2 + O(1/L)$. As $t \to 2$, it seems that $\langle D \rangle$ diverges. A careful analysis by having $L$ large but fixed shows that as $t \to 2$ one gets $\langle D \rangle \sim L/2$. This signifies that when $t \to 2$ the electron motion becomes ballistic in the non-chiral direction also. We also calculate exactly \cite{18} the diffusion constant fluctuations and find them vanish for arbitrary $L$: $\langle (D - \langle D \rangle)^2 \rangle \equiv \sum_{n,m} n^2 m^2 \langle [D(n, L)D(m, L)] - \langle D(n, L)\rangle \langle D(m, L)\rangle \rangle / 4L^2 = 0$.

This indicates that the distribution of the diffusion constant is $\delta(D - D_0)$ independent of the disorder. If we wish to use the Einstein relation to obtain conductivity in this case, we only need to calculate the disorder-averaged density of states which is $1/2\pi$. Assuming Ohmic scaling, we then obtain the disorder-averaged conductance $\langle g \rangle = e^2 t^2 L_x/[4\pi L_z(1 - t^2/4)^2]$ for $t \neq 2$. This result is exactly the same as the result from using the LB formula at large $L_z$ and agrees with the result from the Kubo formula for small $t$.

The calculations of CF as well as the diffusion constant fluctuations involve disorder-averaging the product of four Green’s functions. But as already mentioned, since the Green’s functions must pair up, we can turn this average into a two-diffusion problem with contact interaction which comes from the possibility of exchanging paired-partners when two diffusion cross spatially. The quantity of central importance is the probability of finding one diffusion at position $r_1$ and the other at position $r_2$ for two diffusions starting at their specified positions. In the large $L_z$ limit where the loops circled the circumference once or more can be neglected, we can write down an evolution equation along the chiral direction for the interacting two-diffusion problem. We then perform a discrete Laplace transform and introduce the center of mass momentum as well as the relative momentum of the two diffusions. The results are obtained in Fourier space and are then converted back to real space \cite{18}. A simpler version of this calculation, although it appears in a different context, has also been done \cite{13}.

To calculate the CF exactly via the Kubo formula, we first observe that $g - \langle g \rangle$ is obtained by throwing away terms with $x = x'$ in Eq. \((3)\). Note that the integral of $\Lambda_{n-1}(x)$ over $x$ gives rise to $\sum_i^{\text{even(odd)}}$ for $n - 1$ even(odd).
Furthermore, since the contribution to the CF from $x > x'$ and $x < x'$ will be the same due to the symmetry in Eq. (4), we can work out the CF contributed from $x > x'$ and then multiply it by two. Denote $\hat{D}(n, n') \equiv D(l_-, n; l'_+, n')$ with both $l$ and $l'$ being integers. After some calculations, in which the zero wave-vector mode along the non-chiral direction of diffusion was taken out, we obtain

$$
\langle \delta g^2 \rangle = \langle e^{4t^4/4\pi L_x^4} \sum_{l=1}^{L_x} \sum_{l'=0 \text{ or } n(l), n' (l')}^\infty \langle \hat{D}(n-1; n' -1) \hat{D}(n; n') + \hat{D}(n-1; n') \hat{D}(n; n' -1) + (1 - t^2/4)^2 t^2/(1 + t^2/4)^4 [\hat{D}(n-1; n' -1) \hat{D}(n; n' -1) + \hat{D}(n; n') \hat{D}(n-1; n')]\rangle
$$

$$
= \langle e^{4t^2(1 - t^2/4)^2(1 + \Gamma)L_x}/[4\pi^3(1 + 2\Gamma)L_x^3] \rangle \quad \text{when } L_x \gg L_x^2 \gg 1
$$

(9)

where $\Gamma = t^2(1 - t^2/4)^2/(1 + t^2/4)^4$. For $t \neq 2$ and large $L_z$, we find that $\langle \delta g^2 \rangle$ scales with the system size as $L_x/L_z^2$. It is interesting to note that only in the $t \to 0$ limit where perturbative calculation being plausible, our exact calculation agrees with the result of a perturbative calculation. For $t = 2$ where electronic motion becomes ballistic also in the non-chiral direction, we find the CF vanish as physically expected. For general $t$, CF exhibit complicated $t$ dependence which is not accessible to perturbative calculations.

Starting from LB formula $\langle g^2 \rangle$ contains disorder-averaged terms of products of two $|t_{i,j}|^2$. To show how this is done, let us consider the case $i' > i$ and $j > j'$ for concreteness. The quantity $\langle |t_{i,j}|^2|t_{i',j'}|^2 \rangle$ now can be written as

$$
\langle |t_{i,j}|^2|t_{i',j'}|^2 \rangle = \sum_{z_1, z_2 = 1}^{L_x} \langle D(z_1, i; 1, i) \rangle \langle D(L_z, j; z_2; j') \rangle \langle D(z_2, j'; z_1; i') \rangle \langle D(L_z; j', 1, i') \rangle.
$$

(10)

Following the methods outlined above, one can calculate the disorder-averaged one-diffuson or two-diffuson explicitly. The sums over $z_1, z_2, i, i', j, j'$ turn out to be quite complicated. We thus perform the sums numerically. We have performed the calculations for $L_x$ up to 10 and $L_z$ up to 6000. To extract the scaling behavior of the CF, we plot $\langle L_x/\langle \delta g^2 \rangle \rangle^{1/2}$ against $L_z$, and find that it is very much a straight line for most values of $t$ in good agreement with the results found in using the Kubo formula. The case for $t = 0.31$ is shown in Fig. 2. We have also check the specific cases where $t = 0$ and $t = 2$; and we see that the CF vanish in both cases in agreement with the Kubo calculation.

To conclude, we have performed the CF non-perturbatively by using the path integral expression of the Green’s function. We find that the CF are non-universal and scale in a peculiar way. It has been speculated that UCF always appear for mesoscopic system with extended quantum states. Our exact calculation shows that at least this system is an exception.

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For an arbitrary function $H(x)$ we define $\int f(x)H(x)dx = [H(0_--) + H(0_+)]/2$ to regularize the situation where $H(x)$ is discontinuous at point 0; for $H(x)$ continuous at 0 this definition is the same as taking $f(x) \to \delta(x)$ at the end of calculation.

The detailed calculation will be described elsewhere.