Classification of gapped ground state phases in quantum spin systems

Yoshiko Ogata *

October 12, 2021

Abstract

Recently, classification problems of gapped ground state phases attract a lot of attention in quantum statistical mechanics. We explain about our operator algebraic approach to these problems.

1 Introduction

In quantum mechanics, physical models are determined in terms of some self-adjoint operators called Hamiltonians. Recently, Hamiltonians whose spectrum has a gap between the lowest eigenvalue (which coincides with the infimum of the spectrum) and the rest of the spectrum attract a lot of attention. Physically, these models are considered to be in normal phases, where no critical phenomena occur. Despite that, it has turned out that the structure of these normal gapped phases is actually mathematically interesting when we introduce some equivalence relation to them. Roughly speaking, we say two models are equivalent if we can connect them smoothly within those normal phases. In spacial dimensions higher than one, it is believed (and partially proven) there are multiple phases with respect to such classifications. If we further introduce some symmetry to the game, we obtain interesting mathematical structures, even in one dimension. In this talk, we explain the operator algebraic approach to those problems.

2 Finite-dimensional quantum mechanics

In order to motivate us for the operator algebraic framework of quantum statistical mechanics, we first recall finite-dimensional quantum mechanics in this section. In finite-dimensional quantum mechanics, physical observables are represented by elements of $M_n$, the algebra of $n \times n$-matrices. Each positive matrix $\rho$ with $\text{Tr} \rho = 1$ (called a density matrix) defines a physical state by

$$\omega_\rho : M_n \ni A \mapsto \text{Tr} (\rho A) \in \mathbb{C}. $$

We call this map $\omega_\rho$ a state. Clearly, it is positive i.e., $\omega_\rho (A^* A) \geq 0$ and normalized $\omega_\rho (\mathbb{1}) = 1$. This corresponds to the procedure of taking expectation values of each physical observables $A \in M_n$, in the physical state $\omega_\rho$. Note that the set of all states forms a convex compact set. Its extremal points are called pure states. A state $\omega_\rho$ is pure if and only if $\rho$ is a rank one projection.

Time evolution (Heisenberg dynamics) is given by a self-adjoint matrix $H$, called a Hamiltonian, via the formula

$$M_n \ni A \mapsto \tau_t (A) := e^{itH} A e^{-itH}, \quad t \in \mathbb{R}. \quad (2.1)$$

*Graduate School of Mathematical Sciences The University of Tokyo, Komaba, Tokyo, 153-8914, Japan Supported in part by the Grants-in-Aid for Scientific Research, JSPS.
Let $p$ be the spectral projection of $H$ corresponding to the lowest eigenvalue. A state $\omega_p(A) := \text{Tr} \rho A$ on $M_n$ is said to be a ground state of $H$ if the support of $\rho$ is under $p$. The ground state is unique if and only if $p$ is a rank one projection, i.e., if the lowest eigenvalue of $H$ is non-degenerated. In this case, the unique ground state is of the form $\omega_p(A) := \text{Tr} p A$, and it is pure because $p$ is rank one.

Sometimes, we consider time-dependent Hamiltonians $H(t)$. Then the time evolution of an observable $A \in M_n$ is given by a solution $\tau_t(A)$ of the differential equation

$$\frac{d}{dt} \tau_t(A) = i [H(t), \tau_t(A)], \quad \tau_0(A) = A, \quad A \in M_n.$$  

When the Hamiltonian is time-dependent $H(t) = H$, this reduces to the above Heisenberg dynamics $e^{itH} A e^{-itH}$.

Symmetry plays an important role in physics. Let $G$ be a finite group and suppose that there is a group action $\beta : G \to \text{Aut}(M_n)$ given by unitaries $V_g, g \in G$

$$\beta_g(A) := \text{Ad} (V_g)(A), \quad A \in M_n, \quad g \in G.$$  

Here and thereafter, $\text{Aut}(A)$ for a $*$-algebra $A$ denotes the automorphism group of $A$. If a Hamiltonian $H$ satisfies $\beta_g(H) = H$ for all $g \in G$, we say $H$ is $\beta$-invariant. If a $\beta$-invariant Hamiltonian $H$ has a unique ground state $\omega_p(A) := \text{Tr} p A$, then this unique ground state $\omega_p$ is $\beta$-invariant $\omega_p(\beta_g(A)) = \omega_p(A), A \in M_n$, because the spectral projection $p$ is $\beta$-invariant, i.e., $\beta_g(p) = p$.

3 Quantum spin systems

Operator algebraic framework of quantum statistical mechanics allows us to extend the framework of finite-dimensional quantum mechanical systems to infinite dimensions. Let $2 \leq d \in \mathbb{N}$ and $\nu \in \mathbb{N}$ be fixed. Physically, $\frac{d-1}{2}$ denotes the size of on-site spin (spin quantum number) and $\nu$ denotes the spacial dimension. We denote by $\mathcal{S}_{Z^\nu}$, the set of all finite subsets of $\mathbb{Z}^\nu$. For each finite subset $\Lambda \in \mathcal{S}_{Z^\nu}$, we associate a finite-dimensional $C^*$-algebra

$$A_\Lambda := \bigotimes_{\Lambda} M_d.$$  

Here, $M_d$ is the algebra of $d \times d$ -matrices. The $\nu$-dimensional quantum spin system $A_{Z^\nu}$ is the $C^*$-inductive limit of this inductive net, given by the natural inclusion. For each infinite subset $\Gamma$, we may define $A_{\Gamma}$ in exactly the same manner. The $C^*$-algebra $A_{\Gamma}$ can be naturally regarded as a $C^*$-subalgebra of $A_{Z^\nu}$. We say an element $A$ has support in $\Gamma$ if it belongs to $A_{\Gamma}$. If an automorphism $\alpha$ acts trivially on $A_{\Gamma}$ for some $\Gamma \subseteq \mathbb{Z}^\nu$, we say that $\alpha$ has support in $\Gamma$. The set of all elements in $A_{Z^\nu}$ with finite support are called local algebra and denoted by $A_{\text{loc}}$.

A state $\omega$ on $A_{\Gamma}$ is defined to be a linear functional on $A_{\Gamma}$ with $\omega(1) = 1$ which is positive in the sense that $\omega(A^* A) \geq 0$ for any $A \in A_{\Gamma}$. The map $A_{\Gamma} \ni A \mapsto \omega(A) \in \mathbb{C}$ corresponds to the procedure of taking the expectation value of a physical observable $A$ in our physical state $\omega$. The set of all states on $A_{\Gamma}$ forms a convex weak*-compact set. Its extremal points are called pure states. By the Krein-Milman theorem, the set of states is the weak*-closure of the convex envelope of pure states. See [BrK] for more details.

For each state, we can associate a representation of $A_{\Gamma}$ essentially uniquely.

**Theorem 3.1** (GNS representation). For each state $\omega$ on $A_{\Gamma}$, there exist a representation $\pi_\omega$ of $A_{\Gamma}$ on a Hilbert space $\mathcal{H}_\omega$ and a unit vector $\Omega_\omega \in \mathcal{H}_\omega$ such that

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A) \Omega_\omega \rangle, \quad A \in A_{\Gamma}, \quad \text{and} \quad \mathcal{H}_\omega = \overline{\pi_\omega(A_{\Gamma}) \Omega_\omega}.$$  

(3.1)

Here, $\overline{\cdot}$ denotes the norm closure. It is unique up to unitary equivalence.
The triple \((\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)\) is called the GNS triple of \(\omega\). We frequently consider the commutant or bicommutant of \(\pi_\omega(\mathcal{A}_\Gamma)\). For a \(*\)-algebra \(\mathcal{M}\) acting on a Hilbert space \(\mathcal{H}\), we denote by \(\mathcal{M}'\) the set of all elements in \(B(\mathcal{H})\) (the set of all bounded operators on \(\mathcal{H}\)) commuting with every element in \(\mathcal{M}\). The algebra \(\mathcal{M}'\) is called a commutant of \(\mathcal{M}\), and the commutant of \(\mathcal{M}'\) is called bicommutant and denoted by \(\mathcal{M}''\).

For a pure state \(\omega\), it is known that \(\pi_\omega\) is irreducible (i.e., there is no non-trivial closed subspace of \(\mathcal{H}_\omega\) invariant under \(\pi_\omega(\mathcal{A}_\Gamma)\)) and \(\pi_\omega(\mathcal{A}_\Gamma)\) is dense in \(B(\mathcal{H}_\omega)\) with respect to the strong operator topology. This property can be rephrased as \(\pi_\omega(\mathcal{A}_\Gamma)'' = B(\mathcal{H}_\omega)\).

Given GNS representations, we can introduce some equivalence relation between states. We say two states \(\omega, \varphi\) on \(\mathcal{A}_\Gamma\) are equivalent (denoted \(\omega \simeq \varphi\)) if and only if the corresponding GNS representations are unitarily equivalent. For a state \(\omega\) and an automorphism \(\alpha\) on \(\mathcal{A}_\Gamma\), if \(\omega\) and \(\omega \circ \alpha\) are equivalent, then there is a unitary \(u\) on the GNS Hilbert space \(\mathcal{H}_\omega\) implementing \(\alpha\) in the sense

\[
\text{Ad}(u) \circ \pi_\omega = \pi_\omega \circ \alpha.
\] (3.2)

This is because \(\pi_\omega \circ \alpha\) is a GNS representation of \(\omega \circ \alpha\). In our context of quantum spin systems, we can see that two states \(\omega, \varphi\) are equivalent if they can be approximated by a local perturbation of each other. More precisely, \(\omega\) can be approximated arbitrarily well in the norm topology of \(\mathcal{A}_\Gamma^*\) by states of the form \(\varphi(A^* \cdot A)\), with \(A \in \mathcal{A}_{\text{loc}}\) and vice versa. Physically, it means \(\omega\) and \(\varphi\) are macroscopically the same.

There is yet another equivalence relation between states, which is called quasi-equivalence. Two states \(\omega, \varphi\) are said to be quasi-equivalent if there is a \(*\)-isomorphism \(\iota: \pi_\omega(\mathcal{A}_\Gamma)'' \to \pi_\varphi(\mathcal{A}_\Gamma)''\) such that \(\pi_\varphi(A) = \iota \circ \pi_\omega(A)\), for all \(A \in \mathcal{A}_\Gamma\). Note that if two states are equivalent, they are quasi-equivalent. The converse is not true in general, but if the states are pure, it is true.

In the operator algebraic framework of quantum spin systems, physical models are specified with a map called interactions. An interaction \(\Phi\) is a map \(\Phi: \mathfrak{S}_{\mathbb{Z}^d} \to \mathcal{A}_{\text{loc}}\) satisfying

\[
\Phi(X) = \Phi(X)^* \in \mathcal{A}_X
\]

for all \(X \in \mathfrak{S}_{\mathbb{Z}^d}\). Physically, this \(\Phi(X)\) indicates an interaction term between spins inside of \(X\). The easiest type of interaction is an on-site interaction, satisfying

\[
\Phi(X) = 0, \quad \text{if } |X| \neq 1.
\] (3.3)

It means the only possibly non-zero interaction terms are of the form \(\Phi(\{x\})\), with \(x \in \mathbb{Z}^d\). (Here and thereafter \(|X|\) indicates the number of elements in \(X\).) Note that all interaction terms commute with each other for such interactions.

Physically, we are more interested in interactions that have non-zero interaction terms between different sites of \(\mathbb{Z}^d\). For example, let \(\{S_j\}_{j=1,2,3}\) be generators of the irreducible representation of \(\mathfrak{su}(2)\) on \(\mathbb{C}^d\). Then an interaction of \(\mathcal{A}_{\mathbb{Z}}\) given by

\[
\Phi(\{x, x+1\}) = \sum_{j=1}^{3} S_j^{(x)} S_j^{(x+1)}, \quad x \in \mathbb{Z}
\] (3.4)

is called the antiferromagnetic Heisenberg chain and has been extensively studied.

Now, given an interaction, we would like to define a dynamics on \(\mathcal{A}_{\mathbb{Z}^d}\) out of it. In order for that, we need to assume that \(\Phi\) is "suitably local". The simplest condition among such is the condition of the uniformly bounded and finite range. An interaction is of finite range if there exists an \(m \in \mathbb{N}\) such that \(\Phi(X) = 0\) for \(X\) with a diameter larger than \(m\). It is uniformly bounded if it satisfies \(\sup_{X \in \mathfrak{S}_{\mathbb{Z}^d}} \|\Phi(X)\| < \infty\). We can relax this restriction extensively. More generally, we define norms on interactions and consider interactions with finite norms. See [NSY].
Given a suitably local interaction, we may define a $C^*$-dynamics, i.e., strongly continuous one-parameter group of automorphisms on $\mathcal{A}_{\mathbb{Z}^n}$. For an interaction $\Phi$ and a finite set $\Lambda \subset \mathbb{Z}^n$, we define the local Hamiltonian on $\Lambda$ by
\[
(H_{\Phi})_{\Lambda} := \sum_{X \subset \Lambda} \Phi(X). \tag{3.5}
\]
Then we consider the Heisenberg dynamics given by the local Hamiltonian $e^{i(H_{\Phi})_{\Lambda} A e^{-it(H_{\Phi})_{\Lambda}}}$ and take the thermodynamic limit. If our interaction $\Phi$ is suitably local, for example, if it is a uniformly bounded finite range interaction, the limit
\[
\tau_{\Phi}^t(A) = \lim_{\Lambda \to \mathbb{Z}^n} e^{i(H_{\Phi})_{\Lambda} A e^{-it(H_{\Phi})_{\Lambda}}, \quad t \in \mathbb{R}, \quad A \in \mathcal{A}_{\mathbb{Z}^n} \tag{3.6}
\]
exists and defines a dynamics $\tau_{\Phi}$ on $\mathcal{A}_{\mathbb{Z}^n}$. The reason why we consider the dynamics $\tau_{\Phi}$ instead of Hamiltonians is because there is no mathematically meaningful limit of local Hamiltonians $(H_{\Phi})_{\Lambda}$ as $\Lambda \to \mathbb{Z}^n$, while the limit (3.6) makes sense. For this reason, in the operator algebraic framework of quantum statistical mechanics, we talk about dynamics instead of Hamiltonians.

For the same reason, a ground state is defined in terms of the dynamics $\tau_{\Phi}$. Let $\delta_{\Phi}$ be the generator of $\tau_{\Phi}$. A state $\omega$ on $\mathcal{A}_{\mathbb{Z}^n}$ is called an $\tau_{\Phi}$-ground state if the inequality
\[
-i\omega(A^* \delta_{\Phi}(A)) \geq 0 \tag{3.7}
\]
holds for any element $A$ in the domain $\mathcal{D}(\delta_{\Phi})$ of $\delta_{\Phi}$. We occasionally say a ground state of $\Phi$ instead of a $\tau_{\Phi}$-ground state. We denote by $\mathcal{G}_\Phi$ the set of all ground states of $\Phi$. Clearly, $\mathcal{G}_\Phi$ is a weak-$*$-compact convex set, and it is known that its extremal points $\operatorname{ex} \mathcal{G}_\Phi$ consists of pure states. (See Theorem 5.3.37 [BR2].)

Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS triple of a $\tau_{\Phi}$-ground state $\omega$. Then there exists a unique positive operator $H_{\omega, \Phi}$ on $\mathcal{H}_\omega$ such that $e^{iH_{\omega, \Phi}}\pi_\omega(A)\Omega_\omega = \pi_\omega(\tau_{\Phi}^t(A))\Omega_\omega$, for all $A \in \mathcal{A}_{\mathbb{Z}^n}$ and $t \in \mathbb{R}$. We call this $H_{\omega, \Phi}$ the bulk Hamiltonian associated with $\omega$. Note that $\Omega_\omega$ is an eigenvector of $H_{\omega, \Phi}$ with eigenvalue $0$. (See Proposition 5.3.19 [BR2].)

Let us consider the corresponding condition for a finite quantum system $\mathcal{M}_n$ with dynamics given by a Hamiltonian $H$ (2.11). Let $p$ be the spectral projection of $H$ corresponding to the lowest eigenvalue $E_0$. Recall that a state $\omega$ on $\mathcal{M}_n$ is given by a density matrix $\rho$ with the formula $\omega(A) = \operatorname{Tr}\rho A$. Let $s(\rho)$ be the support projection of this $\rho$. Then one can check that $\omega$ is a $\tau$-ground state if and only if $s(\rho)$ satisfies $s(\rho) \leq p$. Recall that the last condition is the very definition of the ground state in finite-dimensional quantum mechanics. In fact, note that the generator $\delta$ of $\tau$ in (2.11) is $\delta(A) = i[H, A]$. If $s(\rho) \leq p$, then we have
\[
-i\omega(A^* \delta(A)) = \omega(A^*(H - E_0)A) \geq 0, \quad A \in \mathcal{M}_n,
\]
hence $\omega$ is a $\tau$-ground state. Conversely, suppose that $\omega$ is a $\tau$-ground state. For any unit eigenvectors $\xi, \eta$ of $H$ with $H\xi = E_0\xi, H\eta = E\eta$, for $E > E_0$, set $A \in \mathcal{M}_n$ to be a matrix satisfying $A\xi = \langle \eta, \zeta \rangle \xi$ for any $\zeta \in \mathbb{C}^n$. Substituting this $A$, we get
\[
0 \leq -i\omega(A^* \delta(A)) = (E_0 - E) \langle \eta, \rho \eta \rangle
\]
Because $E_0 - E < 0$, this means that $\langle \eta, \rho \eta \rangle = 0$ for any such $\eta$. Hence we conclude that $ppp = \rho$, namely, $s(\rho) \leq p$. It means that our definition in operator algebraic framework can be regarded as a natural generalization of the usual definition of a ground state to infinite systems.

Note, in general, that there can be many states satisfying the condition (3.7). Namely, the ground state does not need to be unique. If the ground state is unique, it is automatically an extremal point of $\mathcal{G}_\Phi$. As a result, it is pure.

The systems we are interested in, in this talk, are the ones with gapped ground states.

**Definition 3.2.** We say $\Phi$ has gapped ground states in the bulk if the followings hold.

---

(i) The bulk Hamiltonian $H_{\omega,\Phi}$ of any pure $\tau_{\Phi}$-ground state $\omega$ has 0 as its non-degenerate eigenvalue.

(ii) There exists a constant $\gamma > 0$ such that

$$\sigma(H_{\omega,\Phi}) \setminus \{0\} \subset [\gamma, \infty),$$

for any pure $\tau_{\Phi}$-ground state $\omega$. Here $\sigma(H_{\omega,\Phi})$ denotes the spectrum of $H_{\omega,\Phi}$.

We denote by $P$, the set of all uniformly bounded finite range interactions with gapped ground states in the bulk. An interaction $\Phi$ is said to have a unique gapped ground state if its ground state is unique and gapped in the sense of Definition 3.2. See [AKLT], [FNW], [FNW2], [O1], [O2], [O3] for examples of such models. If we consider the corresponding condition for a finite system $M_n$ with dynamics (2.1). This condition corresponds to the situation that “the lowest eigenvalue of $H$ is non-degenerated and the difference between the lowest eigenvalue and the second-lowest eigenvalue is at least $\gamma$”. One remarkable property of the unique gapped ground state is the exponential decay of correlation functions.

Theorem 3.3 ([HK] [NS06] [NS09]). Let $\Phi$ be a uniformly bounded finite range interaction with a unique gapped ground state $\omega_{\Phi}$. Then the correlation functions of $\omega_{\Phi}$ decay exponentially fast: there exist constants $\mu > 0$ and $C > 0$ such that for all $A \in A_X$, $B \in A_Y$, with finite $X, Y \subset \mathbb{Z}^\nu$,

$$|\omega_{\Phi}(AB) - \omega_{\Phi}(A)\omega_{\Phi}(B)| \leq C \|A\| \|B\| \|X\| e^{-\mu d(X,Y)},$$

hold. Here $d(X,Y)$ denotes the distance between $X$ and $Y$.

This means $\omega_{\Phi}$ is “almost like a product state”.

4 Paths of automorphisms generated by time-dependent interactions

In the previous section, we considered time-independent interactions, and derived a $C^*$-dynamics out of them. The same procedure can be carried out for time-dependent interactions to derive a strongly continuous paths of automorphisms. (Recall that in finite dimensional quantum mechanics, we also considered time-dependent Hamiltonians.) Let $\Phi : [0, 1] \ni t \rightarrow \Phi_t = (\Phi(X; t))$ be a piecewise continuous path of interactions. Namely, for each finite $X$, the matrix-valued function $[0, 1] \ni t \rightarrow \Phi(X; t) \in A_X$ is piecewise continuous. We then define the path of local Hamiltonians $(H_{\Phi_t})_\Lambda := \sum_{X \subset \Lambda} \Phi(X; t)$ for each finite subset $\Lambda$ of $\mathbb{Z}^\nu$ and consider the solution $\alpha_{\Phi,t,\Lambda}(A)$ of the differential equation

$$\frac{d}{dt} \alpha_{\Phi,t,\Lambda}(A) = i [(H_{\Phi_t})_\Lambda, \alpha_{\Phi,t,\Lambda}(A)], \quad \alpha_{\Phi,0,\Lambda}(A) = A.$$

If the interactions along this path are suitably local, analogous to the ones considered in the previous section, then the thermodynamic limit

$$\alpha_{\Phi,t}(A) = \lim_{\Lambda \rightarrow \mathbb{Z}^\nu} \alpha_{\Phi,t,\Lambda}(A), \quad A \in A_{\mathbb{Z}^\nu},$$

exists and defines a strongly continuous path of automorphisms $\alpha_{\Phi,t}$. We denote by $Q\text{Aut}(A_{\mathbb{Z}^\nu})$ the set of all automorphisms $\alpha = \alpha_{\Phi,t}$ generated by some time-dependent interactions $\Phi$ in this manner. It forms a subgroup of the automorphism group $\text{Aut}(A_{\mathbb{Z}^\nu})$ on $A_{\mathbb{Z}^\nu}$.

Due to the fact that $\alpha \in Q\text{Aut}(A_{\mathbb{Z}^\nu})$ is given out of local interactions, it shows some nice locality properties. The most famous one is the Lieb-Robinson bound, which has been extensively studied...
and used [HK] [NS06] [NS09] BMNS NSY. It gives an estimate on \( \| \alpha(A), B \| \) for \( A \in \mathcal{A}_X, B \in \mathcal{A}_Y \), which decays as the distance between finite subsets \( X \) and \( Y \) goes to infinity.

The other property that is satisfied by \( \alpha \in \text{QAut}(\mathcal{A}_{\mathbb{Z}^r}) \) is the factorization property. It basically says that we can split \( \alpha \) into two along any cut of the system modulo some error terms localized around the boundary. For example, in one-dimensional systems, if we cut the system into two parts at the origin, we have

\[
\alpha = \text{Ad}(v) \circ (\alpha_L \otimes \alpha_R),
\]

(4.1)

where \( \alpha_L \) is an automorphism on the left infinite chain \( \mathcal{A}_L := \mathcal{A}_{[−∞,−1[\cap \mathbb{Z}}, \alpha_R \) an automorphism on the right infinite chain \( \mathcal{A}_R := \mathcal{A}_{[0,∞[\cap \mathbb{Z}} \). The term \( \text{Ad}(v) \) is an inner automorphism given by some unitary \( v \) in \( \mathcal{A}_{\mathbb{Z}} \), which corresponds to the “error around the boundary”. In a two-dimensional system, for example, we have the following when we cut the system into two by the \( y \)-axis. For \( 0 < \theta < \frac{\pi}{2} \), we define a double cone \( C_\theta \) by

\[
C_\theta := \{(x, y) \in \mathbb{Z}^2 \mid |y| \leq \tan \theta \cdot |x|\}.
\]

(4.2)

Furthermore, \( H_L, H_R, H_U, H_D \) denotes half left/right and upper/lower planes, and \( C_{\theta,L} := C_\theta \cap H_L, C_{\theta,R} := C_\theta \cap H_R \). For any \( 0 < \theta < \frac{\pi}{2} \), there is \( \alpha_L \in \text{Aut} \mathcal{A}_{H_L}, \alpha_R \in \text{Aut} \mathcal{A}_{H_R}, \) and \( \Theta \in \text{Aut} \mathcal{A}_{(C_\theta)\cap} \) such that

\[
\alpha = \text{Ad}(v) (\alpha_L \otimes \alpha_R) \circ \Theta.
\]

(4.3)

Actually, \( \alpha \) can be cut in many directions simultaneously. Factorization property is simple but strong analytical property, which turns out to be useful in the analysis of gapped ground state phases [Q4] [Q6] [Q7] [Q8] [Q9].

Another property we note about \( \alpha \in \text{QAut}(\mathcal{A}_{\mathbb{Z}^r}) \) is that it does not create a long-range entanglement. For example, it satisfies the following property. If \( A \) and \( B \) are observables localized in finite regions far away from each other, then \( \alpha \) almost preserves the tensor product form of \( A \otimes B \), namely, there are operators \( \hat{A}, \hat{B} \) strictly localized in some finite disjoint areas such that \( A \otimes B \) approximates \( \alpha(A \otimes B) \) in the norm topology. In fact, our \( \alpha \) can be regarded as a version of a quantum circuit with finite depth, which is regarded as a quantum circuit which does not create long-range entanglement [BL]. From this point of view, we say a state has a short-range entanglement if it is of the form

\[
\bigotimes_{x \in \mathbb{Z}^r} \rho_x \circ \alpha,
\]

(4.4)

with infinite tensor product state \( \bigotimes_{x \in \mathbb{Z}^r} \rho_x \) and an automorphism \( \alpha \in \text{QAut}(\mathcal{A}_{\mathbb{Z}^r}) \). Otherwise, we say it has a long-range entanglement.

In physics literature, the classification of states with respect to local unitaries is considered [CGW1]. Two states are equivalent if there is a local unitary connecting them. In our framework, these local unitaries can be understood as automorphisms in \( \text{QAut}(\mathcal{A}_{\mathbb{Z}^r}) \), and the classification in [CGW1] can be reformulated as follows. For two states \( \omega_1, \omega_0 \) on \( \mathcal{A}_{\mathbb{Z}^r} \), we write \( \omega_1 \sim_{l.u.} \omega_0 \) if there is an automorphism \( \alpha \in \text{QAut}(\mathcal{A}_{\mathbb{Z}^r}) \) such that \( \omega_1 = \omega_0 \circ \alpha \). This gives some equivalence relation. From the fact that automorphisms in \( \text{QAut}(\mathcal{A}_{\mathbb{Z}^r}) \) do not create long-range entanglement, this is one physically natural criterion of classification of states.

5 The classification of gapped ground state phases

The automorphisms in \( \text{QAut}(\mathcal{A}_{\mathbb{Z}^r}) \) are of fundamental importance in the classification problem of gapped ground state phases. In a word, ground state spaces of two interactions \( \Phi_0, \Phi_1 \in \mathcal{P} \) (Definition [22]) are connected to each other via such automorphisms if they are equivalent in the
classification of gapped ground state phases. In this section, we introduce such theorem, called
the automorphic equivalence. The automorphic equivalence started as Hasting’s adiabatic Lemma
[HW] in finite-dimensional quantum mechanical system. There have been seminal mathematical
polishment and generalization after that [BMNS], [NSY] in the context of the thermodynamic limit
of quantum spin systems. Here we introduce a version in [MO], where we require the spectral gap
only in the infinite systems (i.e., the setting in section 3).

The classification problem of gapped ground states in infinite systems can be roughly described
as follows.

We say two interactions $\Phi_0, \Phi_1 \in \mathcal{P}$ are equivalent if there is a path of interactions $\Phi : [0, 1] \to \mathcal{P}$
satisfying the following conditions

1. $\Phi(0) = \Phi_0$ and $\Phi(1) = \Phi_1$
2. $\forall s \in [0, 1], \quad \Phi_X : \mathcal{A}_X \to \mathcal{A}_0$ is continuous and piecewise $C^1$. The interaction $\Phi_X(s)$ and its
derivative are of finite range, bounded with respect to some norm uniformly in $s \in [0, 1]$. (See (ii)-(iv) of Assumption 1.2 of [MO].)
3. for each pure $\tau_{\Phi_0}$-ground state $\varphi_0$, there is a unique smooth path of states $\varphi_s$ where each
$\varphi_s$ is a pure $\tau_{\Phi(s)}$-ground state. (Here, smooth means the expectation value of some class
of elements in $\mathcal{A}_Z$ with respect to $\varphi_s$ is differentiable, and its derivative is not too large
compared to some norm. See [MO] Assumption 1.2 (vii).) For each $s \in [0, 1]$, the map
$\varphi_0 \mapsto \varphi_s$ gives a bijection.
4. The gap is uniformly bounded from below by some $\gamma > 0$ along the path, i.e., $\sigma(H_{\varphi_s}) \subset [\gamma, \infty)$ for all $s \in [0, 1]$ and a pure $\tau_{\Phi(s)}$-ground state $\psi_s$.

We write $\Phi_0 \sim \Phi_1$ if $\Phi_0, \Phi_1 \in \mathcal{P}$ are equivalent in this sense.

The automorphic equivalence in this setting is given as follows.

**Theorem 5.1.** [MO] If $\Phi_0 \sim \Phi_1$, then there is an $\alpha \in \text{QAut}(\mathcal{A}_Z)$ such that

$$\mathcal{G}_{\Phi_1} = \mathcal{G}_{\Phi_0} \circ \alpha.$$  

**Proof.** We use the notation above for $\Phi_0 \sim \Phi_1$. From Remark 1.4. of [MO], there is a path of
automorphisms $\alpha_s \in \text{QAut}(\mathcal{A}_Z)$ satisfying $\varphi_s = \varphi_0 \circ \alpha_s$ for each state $\varphi_0, \varphi_s$ in (3).
This $\alpha_s$ is independent of the choice of $\varphi_0$. Because $\mathcal{G}_{\Phi(s)}$ is a convex weak$^*$-compact set, it coincides with the weak$^*$-closure of the convex hull of extremal points of $\mathcal{G}_{\Phi(s)}$. Hence we see that this $\alpha_s$ maps $\mathcal{G}_{\Phi_0}$ to $\mathcal{G}_{\Phi_1}$ bijectively.

Hence automorphisms in $\text{QAut}(\mathcal{A}_Z)$ connect ground state spaces of $\Phi_0$ and $\Phi_1$. For this reason,
this class of automorphisms is of fundamental importance. The point here is that it is not only
that there is some automorphism connecting the ground state spaces, but also, we know the details
of the automorphisms.

Note that for interactions $\Phi_1, \Phi_0 \in \mathcal{P}$ with unique ground states $\omega_{\Phi_1}, \omega_{\Phi_0}, \Phi_1 \sim \Phi_0$ implies
$\omega_{\Phi_1} \sim_{1,u} \omega_{\Phi_0}$ from Theorem 5.1. For the moment of writing, it is not clear for us if the converse
is true.

We call an on-site interaction (defined in (3.3)) with a unique gapped ground state a trivial
interaction. The unique ground state $\omega_{\Phi_0}$ of a trivial interaction $\Phi_0$ is of infinite tensor product
form. One can easily see that any two trivial interactions are equivalent. The equivalence class $\mathcal{P}_0$
of interactions including these trivial interactions is called a trivial phase. Any interaction $\Phi$ in the
trivial phase has a unique ground state, and from Theorem 5.1 it has a short-range entanglement
(4.4).
6 Symmetry protected topological (SPT) phases

The trivial phase $P_0$ consists of interactions that are connected to trivial interactions, and as a result, its ground state has a short-range entanglement and is basically the same as product states. From this point of view, the trivial phase itself may not be that interesting. However, if we introduce some symmetry to the game, we can extract some interesting mathematical structure out of it. This is so-called symmetry protected topological (SPT) phases, which were introduced by Gu and Wen [GW] [CGLW] [CGW2]. Throughout this section $\omega_\Phi$ for $\Phi \in P_0$ indicates the unique ground state of $\Phi$.

In this talk, as a symmetry, we consider an on-site finite group symmetry, which is defined as follows. (A study on the global reflection symmetry in one-dimensional systems can be found in [O6].) We fix a finite group $G$ and a (projective) unitary representation $U$ of $G$ on $\mathbb{C}^d$. Then there is a unique automorphism $\beta_g$ satisfying

$$\beta_g(A) = \left( \bigotimes_{x \in \Lambda} U(g) \right) A \left( \bigotimes_{x \in \Lambda} (U(g))^* \right), \quad g \in G, \quad A \in \mathcal{A}_\Lambda, \quad \Lambda \in \mathbb{S}_{2^\nu}.$$ 

Clearly, this gives an action of $G$ on $\mathcal{A}_{2^\nu}$, i.e., $\beta_g \beta_h = \beta_{gh}$ for $g, h \in G$. We call this action of $G$, an on-site symmetry given by $G$ and $U$. We say an interaction $\Phi$ is $\beta$-invariant if $\beta_g(\Phi(X)) = \Phi(X)$ for all $X \in \mathbb{S}_{2^\nu}$ and $g \in G$. For a ground state $\varphi$ of a $\beta$-invariant interaction $\Phi$, one can check that $\varphi \circ \beta_g$ is also a ground state of $\Phi$. Therefore, if a $\beta$-invariant interaction $\Phi$ has a unique ground state $\omega_\Phi$, the ground state is $\beta$-invariant, $\omega_\Phi \circ \beta_g = \omega_\Phi$.

What we are interested in, in this section is the set of all $\beta$-invariant interactions in the trivial phase $P_0$. We denote the set of all such interactions by $P_{0,\beta}$. We would like to classify them with respect to the following criterion. Two interactions $\Phi_0$, $\Phi_1$ are $\beta$-equivalent if there is a smooth path of interactions in $P_{0,\beta}$ satisfying the conditions (1)-(4) we saw in section 5. We write $\Phi_0 \sim_\beta \Phi_1$ in this case. The difference between $\sim$ and $\sim_\beta$ is that we require the symmetry to be preserved along the path. Because of this additional condition, there can be interactions $\Phi_0, \Phi_1 \in P_{0,\beta}$ which satisfy $\Phi_0 \sim \Phi_1$ (by definition) but not $\Phi_0 \sim_\beta \Phi_1$. In other words, $P_{0,\beta}$ may split into possibly multiple equivalence classes. The resulting equivalence classes are the symmetry protected topological (SPT) phases.

For this SPT classification problem, physicists and algebraic topologists have a conjecture [KTTW] [Yon]. They say that SPT-phases should be classified by the Pontryagin dual of bordism group on the classifying space $BG$ of $G$. In one and two-dimensions, these Pontryagin duals are $H^2(G, U(1))$, $H^3(G, U(1))$. In fact, we can derive these group cohomology valued invariants out of our general microscopic models of in those dimensions.

**Theorem 6.1.** [O4] [O7] There is a $H^2(G, U(1))$-valued invariant for one-dimensional SPT-phases. There is a $H^3(G, U(1))$-valued invariant for two-dimensional SPT-phases.

For the rest of this section, we explain how to find such invariants out of general models. In the analysis of gapped ground state phases, there is a general guiding principle to find an invariant. That is, cut the system into two and look at the edge. This principle is sometimes called the bulk-edge correspondence. In order to derive the invariant in the Theorem 6.1 we follow this principle and restrict our group action $\beta$ to the half of the system. Namely, we consider the group actions

$$\beta^R_g := \text{id}_{\mathcal{A}_L} \otimes \bigotimes_{x \geq 0} \text{Ad} \left( U(g) \right), \quad \beta^U_g := \text{id}_{\mathcal{A}_{H^U}} \otimes \bigotimes_{(x,y) \in H^U} \text{Ad} \left( U(g) \right), \quad (6.1)$$

in one and two dimensions, respectively. We investigate the effect of these actions on our unique ground state $\omega_\Phi$ for $\Phi \in P_{0,\beta}$.

Let us start with one-dimensional systems. Recall that $\omega_\Phi$ has a short-range entanglement, and is $\beta$-invariant. From these facts, we expect that the effect of $\beta^R$ is not much recognizable on
the right infinite chain, far away from the origin. On the other hand, on the right infinite chain, far away from the origin, the difference between \( \beta \) and \( \beta^R \) are not much recognizable. Combining this and the fact that \( \omega_\Phi \) is \( \beta \)-invariant, we conclude that the effect of \( \beta^R \) is not much recognizable on the right infinite chain, far away from the origin. As a result, we expect that the effect of \( \beta^R \) on \( \omega_\Phi \) should be localized around the origin. In other words, \( \omega_\Phi \) and \( \omega_\Phi \circ \beta^R_g \) are macroscopically the same. It turns out to be true, mathematically, in the following sense.

**Proposition 6.2.** The state \( \omega_\Phi \) and \( \omega_\Phi \circ \beta^R_g \) are equivalent.

This can be seen very easily. Recall from the definition that \( \Phi \in \mathcal{P}_0 \) means \( \Phi \sim \Phi_0 \) with some trivial interaction \( \Phi_0 \). From Theorem 5.1 we have \( \omega_\Phi = \omega_{\Phi_0} \circ \alpha \) with some \( \alpha \in \text{QAut}(\mathcal{A}_Z) \). Recall that as a trivial interaction, \( \Phi_0 \) has a unique ground state of infinite tensor product form. In particular, we can write \( \omega_{\Phi_0} \), as \( \omega_{\Phi_0} = \omega_L \otimes \omega_R \) with pure states \( \omega_L, \omega_R \) on the left and right infinite chains \( \mathcal{A}_L, \mathcal{A}_R \), respectively. Recall also that our \( \alpha \) satisfies the factorization property \([6.1]\). Combining these, we conclude that

\[
\omega_\Phi \simeq (\omega_L \otimes \omega_R) \circ (\alpha_L \otimes \alpha_R),
\]

with some automorphisms \( \alpha_L, \alpha_R \) on \( \mathcal{A}_L, \mathcal{A}_R \). From this and the invariance of \( \omega_\Phi \) under \( \beta_g \), we see that \( \omega_L \alpha_L \beta^R_g \otimes \omega_R \alpha_R \beta^R_g \simeq \omega_L \alpha_L \otimes \omega_R \alpha_R \), where \( \beta^L_g, \beta^R_g \) are the restrictions of \( \beta \) to the left, right infinite chains. This implies \( \omega_R \alpha_R \beta^R_g \simeq \omega_R \alpha_R \), hence we get

\[
\omega_\Phi \circ \beta^R_g \simeq \omega_L \alpha_L \otimes \omega_R \alpha_R \beta^R_g \simeq \omega_L \alpha_L \otimes \omega_R \alpha_R \simeq \omega_\Phi,
\]

proving the claim.

Note from section 3 that Proposition 6.2 means \( \beta^R_g \) is implementable by a unitary \( u_g \) in the GNS representation \( (\mathcal{H}_{\omega_\Phi}, \pi_{\omega_\Phi}) \) of \( \omega_\Phi \), i.e.,

\[
\text{Ad} \,( u_g ) \circ \pi_{\omega_\Phi} = \pi_{\omega_\Phi} \circ \beta^R_g .
\]

Because \( \beta^R \) is a group action, we have

\[
\text{Ad} \,( u_g u_h ) \circ \pi_{\omega_\Phi} = \pi_{\omega_\Phi} \circ \beta^R_g \beta^R_h = \pi_{\omega_\Phi} \circ \beta^R_{gh} = \text{Ad} \,( u_{gh} ) \circ \pi_{\omega_\Phi}, \quad g, h \in G.
\]

Recall that \( \omega_\Phi \) is a unique ground state of \( \Phi \), hence it is pure. As a result, \( \pi_{\omega_\Phi}(\mathcal{A}_Z) \) is dense in \( \mathcal{B}(\mathcal{H}_{\omega_\Phi}) \) with respect to the strong operator topology. From this, \(6.5\) implies there is some \( \sigma(g,h) \in U(1) \) such that

\[
u_g u_h = \sigma(g,h) u_{gh}, \quad g, h \in G.
\]

In other words, \((u_g)\) forms a projective representation. As a result, we obtain \( H^2(G, U(1)) \)-valued index out of it.

Using the automorphic equivalence Theorem 5.1 and the factorization property of the automorphism therein, one can show that it is in fact an invariant of our classification \( \sim_\beta \) \([\Omega]\). The point of the proof is, when \( \Phi_0 \sim_\beta \Phi_1 \), the time-dependent interactions giving \( \alpha \in \text{QAut}(\mathcal{A}_Z) \) in Theorem 5.1 can be taken to be \( \beta \)-invariant. Proposition 6.2 itself holds for general \( \beta \)-invariant unique gapped ground state. This is thanks to the theorem by Matsui \([M2]\) showing the split property for unique gapped ground states. Projective representations associated to split states have been known from the time around 00 \([M1]\) among operator algebraists. What is new here is that the associated cohomology class is an invariant of our classification. In fact, this \( H^2(G, U(1)) \)-valued index is a complete invariant of pure \( \beta \)-invariant split states with respect to some classification \([\Omega]\). This index can be used to show Lieb-Schultz-Mattis type theorems \([LSM]\) \([AL]\) \([M1]\) \([NS07]\) (no-go theorems for the existence of unique gapped ground state under some symmetry), for finite groups symmetries \(\text{OT} \) \([OTT]\).

For two dimension, \( \omega_\Phi \circ \beta^R_g \) is not equivalent to \( \omega_\Phi \) in general. However, an analogous argument as the one-dimensional case lets us expect that the effect of \( \beta^L_g \) should be localized around the \( x \)-axis. In fact, it turns out to be true mathematically.
Proposition 6.3. For any $0 < \theta < \frac{\pi}{2}$, there are $\eta_{g,L} \in \text{Aut} \left( \mathcal{A}_{C_0,L} \right)$ and $\eta_{g,R} \in \text{Aut} \left( \mathcal{A}_{C_0,R} \right)$ such that

$$\omega_\Phi \circ \beta^U_g \simeq \omega_\Phi (\eta_{g,L} \otimes \eta_{g,R}).$$

It means macroscopically, the effect of $\beta^U_g$ on $\omega_\Phi$ is localized around $C_{0,L}$ and $C_{0,R}$ for any $0 < \theta < \frac{\pi}{2}$. This $\eta_{g,R}$ is our source of $H^3(G, U(1))$-valued index.

Now we fix some $0 < \theta < \frac{\pi}{2}$, and set $\gamma^R_g := \beta^U_g \circ \eta_{g,R}^{-1}, \gamma^L_g := \beta^U_g \circ \eta_{g,L}^{-1}$ with $\eta_{g,R}, \eta_{g,L}$ for this $\theta$. Here, $\beta^U_g, \beta^L_g$ are group actions of $G$ given by

$$\beta^U_g := \text{id}_{(H \cap H_R)^c} \otimes \bigotimes_{(x,y) \in (H \cap H_R)} \text{Ad} \left( U(g) \right), \quad \beta^L_g := \text{id}_{(H \cap H_L)^c} \otimes \bigotimes_{(x,y) \in (H \cap H_L)} \text{Ad} \left( U(g) \right).$$

From Proposition 6.3, we have

$$\omega_\Phi \circ \left( \gamma^L_g \otimes \gamma^R_g \right) \simeq \omega_\Phi, \quad g \in G. \quad (6.7)$$

On the other hand, recall from the definition that $\Phi \in \mathcal{P}_0$ means $\Phi \sim \Phi_0$ with some trivial interaction $\Phi_0$. From Theorem 5.1 we have $\omega_\Phi = \omega_{\Phi_0} \circ \alpha$, with $\alpha \in Q\text{Aut}(A_{Z^r})$ satisfying the factorization property, i.e.,

$$\alpha = \text{Ad} (v) \circ (\alpha_L \otimes \alpha_R) \circ \Theta, \quad \alpha_L \in \text{Aut} \mathcal{A}_{H_L}, \quad \alpha_R \in \text{Aut} \mathcal{A}_{H_R}, \quad \Theta \in \text{Aut} \mathcal{A}_{Z^r}, \quad (6.8)$$

for our fixed $\theta$. Recall that as a trivial interaction, $\Phi_0$ has a unique ground state $\omega_{\Phi_0}$ of infinite tensor product form. In particular, we can write $\omega_{\Phi_0}$ as $\omega_{\Phi_0} = \omega_L \otimes \omega_R$ with pure states $\omega_L, \omega_R$ on $\mathcal{A}_{H_L}, \mathcal{A}_{H_R}$, respectively. Combining these, we conclude that

$$\omega_\Phi \simeq (\omega_L \otimes \omega_R) \circ (\alpha_L \otimes \alpha_R) \circ \Theta. \quad (6.9)$$

Repeated use of (6.7) gives

$$\omega_\Phi \circ \left( \gamma^L_g \gamma^R_g \left( \gamma^L_{gh} \right)^{-1} \otimes \gamma^R_g \gamma^R_{gh} \right) \simeq \omega_\Phi. \quad (6.10)$$

Applying (6.9) to this, we obtain

$$(\omega_L \otimes \omega_R) \circ (\alpha_L \otimes \alpha_R) \circ \Theta \circ \left( \gamma^L_g \gamma^R_g \left( \gamma^L_{gh} \right)^{-1} \otimes \gamma^R_g \gamma^R_{gh} \right) \simeq (\omega_L \otimes \omega_R) \circ (\alpha_L \otimes \alpha_R) \circ \Theta. \quad (6.11)$$

Note that

$$\gamma^R_{gh} \gamma^R_g \left( \gamma^L_{gh} \right)^{-1} = \left( \gamma^L_{gh} \gamma^R_{gh} \right)^{-1} \left( \gamma^L_{gh} \gamma^R_{gh} \right)^{-1} \left( \gamma^L_{gh} \gamma^R_{gh} \right)^{-1} \in \text{Aut} \left( \mathcal{A}_{C_0,R} \right). \quad (6.12)$$

Similarly, we have $\gamma^L_g \gamma^L_{gh} \left( \gamma^L_{gh} \right)^{-1} \in \text{Aut} \left( \mathcal{A}_{C_0,L} \right)$. Therefore, they commute with $\Theta \in \text{Aut} \left( \mathcal{A}_{C_0} \right)$. From this and (6.11), we obtain

$$(\omega_L \otimes \omega_R) \circ (\alpha_L \otimes \alpha_R) \circ \left( \gamma^L_g \gamma^R_g \left( \gamma^L_{gh} \right)^{-1} \otimes \gamma^R_g \gamma^R_{gh} \right) \simeq (\omega_L \otimes \omega_R) \circ (\alpha_L \otimes \alpha_R),$$

which implies

$$\omega_R \alpha_R \gamma^R_g \gamma^R_{gh} \left( \gamma^R_{gh} \right)^{-1} \simeq \omega_R \alpha_R. \quad (6.13)$$

Recall from section 3, this means that the automorphism $\gamma^R_g \gamma^R_{gh} \left( \gamma^R_{gh} \right)^{-1}$ is implementable by a unitary $u(g, h)$ in the GNS representation $(\mathcal{H}_R, \pi_R)$ of $\omega_R \alpha_R$ i.e.,

$$\text{Ad} \left( u(g, h) \right) \pi_R = \pi_R \gamma^R_g \gamma^R_{gh} \left( \gamma^R_{gh} \right)^{-1}. \quad (6.14)$$
Therefore, with \( \pi \) implementing \( \Theta \), i.e.,

\[
\omega_L \otimes \omega_R) \circ (\alpha_L \otimes \alpha_R) \circ \Theta \circ (\gamma_g^L \otimes \gamma_g^R) \cong (\omega_L \otimes \omega_R) \circ (\alpha_L \otimes \alpha_R) \circ \Theta.
\] (6.15)

Note also that (6.9) and (6.7) implies

\[
(\omega_L \otimes \omega_R) = (\omega_L \otimes \omega_R) \circ \Theta \circ (\gamma_g^L \otimes \gamma_g^R) \circ \Theta^{-1}.
\] (6.16)

For these \( u(g, h) \) and \( W_g \), we claim there are \( c(g, h, k) \in U(1) \) such that

\[
\text{Ad} (W_g) (\pi_L \otimes \pi_R) = (\pi_L \otimes \pi_R) \circ \Theta \circ (\gamma_g^L \otimes \gamma_g^R) \circ \Theta^{-1}.
\] (6.17)

To see this, consider \( \pi_L \otimes \pi_R^R \gamma_h^R \gamma_k^R \). On the one hand, with the repeated use of (6.14), we have

\[
\pi_L \otimes \pi_R^R \gamma_h^R \gamma_k^R = \text{Ad} (\pi_L \otimes u(g, h)) (\pi_L \otimes \pi_R^R \gamma_h^R \gamma_k^R) = \text{Ad} (\pi_L \otimes u(g, h)u(g(h), k)) (\pi_L \otimes \pi_R \circ \gamma^R_{ghk}).
\] (6.18)

On the other hand, note that both of \( \gamma_h^R \gamma_k^R \) and \( \gamma^R_{ghk} \) commute with \( \Theta \) as before. Hence, we have

\[
\text{id}_L \otimes \gamma^R_g (\gamma_h^R \gamma_k^R (\gamma^R_{ghk})^{-1}) (\gamma^R_{g})^{-1} = \Theta \circ (\gamma^R_g \otimes \gamma^R_g) \Theta^{-1} \left( \text{id}_L \otimes \gamma^R_h \gamma^R_k (\gamma^R_{ghk})^{-1} \right) \Theta \circ (\gamma^R_g \otimes \gamma^R_g)^{-1} \Theta^{-1}.
\] (6.19)

From this and repeated use of (6.14), (6.16), we have

\[
\pi_L \otimes \pi_R^R \gamma_h^R \gamma_k^R = (\pi_L \otimes \pi_R) \Theta \circ (\gamma^R_g \otimes \gamma^R_g) \Theta^{-1} \left( \text{id}_L \otimes \gamma^R_h \gamma^R_k (\gamma^R_{ghk})^{-1} \right) \Theta \circ (\gamma^R_g \otimes \gamma^R_g)^{-1} \Theta^{-1} \left( \text{id}_L \otimes \gamma^R_h \gamma^R_k (\gamma^R_{ghk})^{-1} \right) \Theta \circ (\gamma^R_g \otimes \gamma^R_g)^{-1} \Theta^{-1}.
\] (6.20)

Comparing this and (6.18), we have

\[
\text{Ad} (\pi_L \otimes \pi_R) = \text{Ad} (W_g (\pi_L \otimes u(g, h)) W^*_g (\pi_L \otimes u(g, h))) (\pi_L \otimes \pi_R).
\] (6.21)

Note, because \( \pi_L \otimes \pi_R \) is a GNS representation of a pure state \( \omega \otimes \omega \otimes \omega \), \( \pi_L \otimes \pi_R \) is dense in the \( \text{B}(\mathcal{H}_L \otimes \mathcal{H}_R) \), with the strong operator topology. As a result, (6.21) implies our claim (6.17).

The situation in (6.14), (6.17) is pretty much similar to that of cocycle actions [C, J]. In fact, following the argument in [J], we can show that \( c(g, h, k) \) satisfies the 3-cocycle relation. Hence, out of it, we obtain a \( H^3(G, U(1)) \)-valued index. Using the automorphic equivalence Theorem 5.1 and the factorization property of the automorphism therein, one can show that it is in fact an invariant of our classification \( \sim_\beta \).

A derivation of indices for SPT-phases was initially carried out in tensor network models, matrix product states MPS [PTBO1], [PTBO2], [PWSVC] in one dimension, and projected entangled pair states [MGSC]. Our indices coincide with theirs in those models. In other words, thanks to those works, there are many examples. Our approach introduced in this section is operator algebraic. Recently, some quantum information based approach are reported [KSY].
7 Anyons in topological phases

In this section, we consider the classification \( \sim_{\text{u.u.}} \) in two-dimension. Recall that states which are equivalent to an infinite tensor product state with respect to \( \sim_{\text{u.u.}} \) are said to have a short-range entanglement, and otherwise it is said to have a long-range entanglement. It is frequently said that in the two dimensional systems, the existence of "anyon" means the long-range entanglement of the state. In this section, we formulate this statement in our operator algebraic setting.

An anyon is a string-like excitation with a braiding structure. How to formulate an anyon mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motivated by AQFT and studies of Kitaev models [N1] [N2] [FN] [CNN1] is that it is a superselection sector. It is mathematically is a non-trivial question of mathematical physics. Our answer, motiva...
In other words, the existence of non-trivial superselection sectors implies the long-range entanglement. If we regard superselection sectors as anyons, it is a mathematical realization of the folklore saying that the existence of anyons implies long-range entanglement of the state.

The reason why we expect superselection sectors to be related to anyons comes from AQFT. Using the tools from AQFT, Cha-Naaikens-Nachtergaele derived a braiding structure in a general setting of semi-group of almost localized endomorphisms in quantum spin systems. It is well known that anyons show up in AQFT surprisingly naturally [BDMRS] [BF]. More precisely, under some condition called Haag duality, a braided $C^*$-tensor category can be associated to the irreducible representation with non-trivial sector theory. The Haag duality is the property $\pi_0(A_{\Lambda^\epsilon}^\epsilon) = \pi_0(A_{\Lambda})^\epsilon$, for all cones $\Lambda$ in $\mathbb{Z}^2$.

The problem for us about introducing this condition in quantum spin systems is that it does not look to be plausible that this condition is stable under automorphisms in $Q\text{Aut}(A_{\mathbb{Z}^2})$. More precisely, under some condition called Haag duality, a braided $C^*$-tensor category can be associated to the irreducible representation with non-trivial sector theory. The Haag duality is the property $\pi_0(A_{\Lambda^\epsilon}^\epsilon) = \pi_0(A_{\Lambda})^\epsilon$, for all cones $\Lambda$ in $\mathbb{Z}^2$.

The problem for us about introducing this condition in quantum spin systems is that it does not look to be plausible that this condition is stable under automorphisms in $Q\text{Aut}(A_{\mathbb{Z}^2})$. More precisely, under some condition called Haag duality, a braided $C^*$-tensor category can be associated to the irreducible representation with non-trivial sector theory. The Haag duality is the property $\pi_0(A_{\Lambda^\epsilon}^\epsilon) = \pi_0(A_{\Lambda})^\epsilon$, for all cones $\Lambda$ in $\mathbb{Z}^2$.

**Definition 7.4.** [Approximate Haag duality] [OS] Let $(\mathcal{H}, \pi_0)$ be an irreducible representation of $A_{\mathbb{Z}^2}$. We say that $(\mathcal{H}, \pi_0)$ satisfies the approximate Haag duality if the following conditions hold:

- For any $\varphi \in (0, 2\pi)$ and $\epsilon > 0$ with $\varphi + 4\epsilon < 2\pi$, there is some $R_{\varphi, \epsilon} > 0$ and decreasing functions $f_{\varphi, \epsilon, \delta}(t), \delta > 0$ on $\mathbb{R}_{\geq 0}$ with $\lim_{t \rightarrow \infty} f_{\varphi, \epsilon, \delta}(t) = 0$ such that

(i) for any cone $\Lambda$ with $|\arg \Lambda| = \varphi$, there is a unitary $U_{\Lambda, \epsilon} \in \mathcal{U}(\mathcal{H})$ satisfying

$$\pi_0(A_{\Lambda^\epsilon}^\epsilon) \subset \text{Ad}(U_{\Lambda, \epsilon})(\pi_0(A_{\Lambda^\epsilon}^\epsilon))''.$$  

(7.3)

and

(ii) for any $\delta > 0$ and $t \geq 0$, there is a unitary $\tilde{U}_{\Lambda, \epsilon, \delta, t} \in \pi_0(A_{\Lambda^\epsilon+\delta-t\epsilon_\Lambda})''$ satisfying

$$\|U_{\Lambda, \epsilon} - \tilde{U}_{\Lambda, \epsilon, \delta, t}\| \leq f_{\varphi, \epsilon, \delta}(t).$$  

(7.4)

The good point about this weaker version is that we know it is stable under automorphisms in $Q\text{Aut}(A_{\mathbb{Z}^2})$.

**Proposition 7.5.** Let $(\mathcal{H}, \pi_0)$ be an irreducible representation of $A_{\mathbb{Z}^2}$ satisfying the approximate Haag duality. Then for any automorphism $\alpha \in Q\text{Aut}(A_{\mathbb{Z}^2})$, $(\mathcal{H}, \pi_0 \circ \alpha)$ also satisfies the approximate Haag duality.

It turns out that even with this weaker version of Haag duality and the setting of gapped ground state phases (which is different from that of AQFT), we can still derive a braided $C^*$-tensor category (see [NT] for the definition) out of superselection sectors where, unlike endomorphisms, the multiplication rule is not a priori given [OS]. The proof is a modification of the argument in AQFT and some additional argument using the gap condition Definition 3.2. More precisely, let $\Phi$ be a uniformly bounded finite range interaction on $A_{\mathbb{Z}^2}$ with gapped ground states. Let $\omega$ be a pure $\tau_\varphi$-ground state with a GNS representation $(\mathcal{H}, \pi_0, \Omega)$. We assume that $\pi_0$ has a non-trivial sector theory, and $\pi_0$ satisfies the approximate Haag duality. Fix some $\theta \in \mathbb{R}$ and $\varphi \in (0, \pi)$ and denote by $\mathcal{C}((\theta, \varphi))$ the set of all cones whose angle does not intersects with $[\theta - \varphi, \theta + \varphi]$. We set

$$\mathcal{B}(\theta, \varphi) := \bigcup_{\Lambda \in \mathcal{C}((\theta, \varphi))} \pi_0(A_{\Lambda^\epsilon}^\epsilon).$$  

Here $\cdot$ denotes the norm closure. Using the approximate Haag duality, using the argument in [BF], each superselection sector $\rho : A_{\mathbb{Z}^2} \rightarrow \mathcal{B}(\mathcal{H}_\omega)$ for $\pi_0$ extends to an endomorphism on $\mathcal{B}(\theta, \varphi)$. We denote the extension by the same symbol $\rho$. Via these extensions, we can introduce compositions.
between superselection sectors. With this composition as a tensor, the superselection sectors of $\pi_0$ are the objects of our braided $C^*$-tensor category. Our morphisms are given by the intertwiners. Namely, for objects $\rho, \sigma$, the morphisms from $\rho$ to $\sigma$ are bounded operators $R$ on $H$ such that $R\rho(A) = \sigma(A)R$, for any $A \in \mathcal{A}_{\mathbb{Z}^2}$. The set of all morphisms from $\rho$ to $\sigma$ is denoted by $(\rho, \sigma)$. Note that $(\rho, \sigma)$ is a Banach space and $(\rho, \rho)$ is a $C^*$-algebra. Following AQFT, the tensor of morphisms $R_1 \in (\rho_1, \sigma_1)$, $R_2 \in (\rho_2, \sigma_2)$ are defined by

$$R_1 \otimes R_2 := R_1\rho_1(R_2) \in (\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2).$$

(7.6)

In fact, each intertwiner belongs to $\mathcal{B}_{(\theta, \varphi)}$ that $\rho_1(R_2)$ is well-defined. Using the gap inequality and the non-triviality of the sector theory, we can show for any cone $\Lambda$ that $\pi_0(\mathcal{A}_{\Lambda})''$ is either type $II_\infty$ or type $III$ factor. It means that there are isometries $u_{\Lambda}, v_{\Lambda} \in \pi_0(\mathcal{A}_{\Lambda})''$ such that $u_{\Lambda}u_{\Lambda}^* + v_{\Lambda}v_{\Lambda}^* = I$. Using this, for any superselection sectors $\rho, \sigma$, we can define their direct sum $\rho \bigoplus \sigma : \mathcal{A}_{\mathbb{Z}^2} \to \mathcal{B}(H_0)$ by

$$\left(\rho \bigoplus \sigma\right)(A) := u_{\Lambda}\rho(A)u_{\Lambda}^* + v_{\Lambda}\sigma(A)v_{\Lambda}^*, \quad A \in \mathcal{A}_{\mathbb{Z}^2}. $$

(7.7)

From the same fact, we can also define subobjects. Namely, if $p \in (\rho, \rho)$ is a non-zero projection, we can find some super selection sector $\sigma$ and an isometry $v$ such that $vv^* = p$ and $\rho(A)v = v\sigma(A)$ for all $A \in \mathcal{A}_{\mathbb{Z}^2}$. Hence we obtain the following theorem.

**Theorem 7.6.** [O8] In the above setting, superselection sectors of $\pi_0$ form a braided $C^*$-tensor category. If two of such states $\omega_{\Phi_1}, \omega_{\Phi_2}$ satisfy $\omega_{\Phi_1} \sim_{l.u.} \omega_{\Phi_2}$, then corresponding braided $C^*$-tensor categories are monoidally equivalent.

**References**

[AKLT] I. Affleck, T. Kennedy, E.H. Lieb, and H. Tasaki, Valence bond ground states in isotropic quantum antiferromagnets. *Commun. Math. Phys.*, **115** (1988), 477–528.

[AL] I. Affleck and E. H. Lieb, A proof of part of Haldane’s conjecture on spin chains. *Lett. Math. Phys.* **12** (1986), 57–69 .

[BL] S. Bachmann and M. Lange, Trotter product formulae for $*$-automorphisms of quantum lattice systems. 2021, arXiv:2105.14168.

[BMNS] S. Bachmann, S. Michalakis, B. Nachtergaele, and R. Sims, Automorphic Equivalence within Gapped Phases of Quantum Lattice Systems. *Commun. Math. Phys.* **309** (2012), 835–871.

[BKLR] M. Bischoff, Y. Kawahigashi, R. Longo, K. H. Rehren, *Tensor categories and endomorphisms of von Neumann algebras (with applications to Quantum Field Theory)*. SpringerBriefs in Mathematical Physics 3, 2015.

[BR1] O. Bratteli, D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics 1*. Springer-Verlag, 1986.

[BR2] O. Bratteli, D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics 2*. Springer-Verlag, 1996.

[BDMRS] D. Buchholz, S. Doplicher, G. Morchio, J.E. Roberts, F. Strocchi, Asymptotic Abelian-ness and Braided Tensor $C^*$-Categories. In *Rigorous Quantum Field Theory. Progress in Mathematics* edited by A. B. de Monvel, D. Buchholz and U. Moschella. vol 251. Birkhauser, Basel. 2007.
[BF] D. Buchholz, K. Fredenhagen, Locality and the structure of particle states. Commun.Math.Phys. 84 (1982), 1–54.

[CNN1] M. Cha, P. Naaijkens, B. Nachtergaele, The complete set of infinite volume ground states for Kitaev’s abelian quantum double models. Commun. Math. Phys. 357 (2018), 125–157.

[CNN2] M. Cha, P. Naaijkens, B. Nachtergaele, On the Stability of Charges in Infinite Quantum Spin Systems. Commun. Math. Phys. 373 (2020), 219–264.

[CGLW] X. Chen, Z. C. Gu, Z. X. Liu, and X. G. Wen, Symmetry protected topological orders and the group cohomology of their symmetry group. Phys. Rev. B 87 (2013), 155114.

[CGW1] X. Chen, Z. C. Gu, and X.G. Wen, Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order. Phys. Rev. B 82 (2010), 155138.

[CGW2] X. Chen, Z.-C. Gu, and X.-G. Wen, Classification of gapped symmetric phases in one-dimensional spin systems. Phys. Rev. B 83 (2011), 035107.

[C] A. Connes, Periodic automorphisms of the hyperfinite factor of type II$_1$. Acta Sci. Math. 39 (1977), 39–66.

[DHRI] S. Doplicher, R. Haag, J.E. Roberts, Local observables and particle statistics I. Commun.Math. Phys. 23 (1971), 199–23.

[FNW] M. Fannes, B. Nachtergaele, and R.F. Werner, Finitely correlated states on quantum spin chains. Commun. Math. Phys. 144 (1992) 443–490.

[FNW2] M. Fannes, B. Nachtergaele, and R.F. Werner, Finitely correlated pure states. Journal of Functional Analysis., 120 (1994), 511–534.

[FN] L. Fiedler, P. Naaijkens, Haag duality for Kitaev’s quantum double model for abelian groups Rev. Math. Phys. 27 (2015)1550021:1–43 (2015).

[FRS] K. Fredenhagen, K.H. Rehren, and B. Schroer, Superselection sectors with braid group statistics and exchange algebras. Commun. Math. Phys.125, (1989) 201–226.

[GW] Z.-C. Gu, X.-G. Wen, Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order, Phys. Rev. B 80 (2009) 155131.

[HK] M. B. Hastings, T. Koma, Spectral Gap and Exponential Decay of Correlations. Commun. Math. Phys. 265 (2006) 781–804.

[HW] M. B. Hastings, X. G. Wen, Quasi-adiabatic continuation of quantum states: The stability of topological ground-state degeneracy and emergent gauge invariance, Phys. Rev. B 72 (2005) 045141.

[J] V. Jones, Actions of finite groups on the hyperfinite type II$_1$ factor. Mem. Amer. Math. Soc. 28 (1980), no. 237.

[KTTW] A. Kapustin, R. Thorngren, A. Turzillo, Z. Wang., Fermionic symmetry protected topological phases and cobordisms. J. High Energ. Phys. (2015), 1–21.

[KSY] A. Kapustin, N. Sopenko, B. Yang A classification of phases of bosonic quantum lattice systems in one dimension arXiv:2012.15491

[K] Y. Kawahigashi, Conformal field theory, Vertex operator algebras and Operator algebras. In Proceedings of the International Congress of Mathematicians (ICM 2018), pp. 2597–2616 2019.
[Ki] A. Kitaev, Fault-tolerant quantum computation by anyons. *Ann. Phys.* **303** (2003) 2–30.

[LSM] E. Lieb, T. Schultz, and D. Mattis, Two soluble models of an antiferromagnetic chain. *Ann. Phys.* **16** (1961) 407–466.

[M1] T. Matsui, The split property and the symmetry breaking of the quantum spin chain. *Commun. Math. Phys.*, **218** (2001) 393–416.

[M2] T. Matsui, Boundedness of entanglement entropy and split property of quantum spin chains, *Rev. Math. Phys.* (2013) 1350017.

[MGSC] A. Molnar, Y. Ge, N. Schuch, J. I. Cirac, A generalization of the injectivity condition for projected entangled pair states *J. Math. Phys.* **59** (2018) 021902.

[MO] A. Moon and Y. Ogata, Automorphic equivalence within gapped phases in the bulk *Journal of Functional Analysis* **278** (2020), 108422.

[N1] P. Naaijkens, Haag duality and the distal split property for cones in the toric code. *Lett. Math. Phys.* **101** (2012), 341–354.

[N2] P. Naaijkens Localized endomorphisms in Kitaev’s toric code on the plane *Rev. Math. Phys.* **23**(2011) 347–373.

[NO] P. Naaijkens, Y. Ogata, The split and approximate split property in 2D systems: stability and absence of superselection sectors, 2021, arXiv:2102.07707

[NS06] B. Nachtergaele and R. Sims, Lieb-Robinson Bounds and the Exponential Clustering Theorem. *Commun. Math. Phys.* **265** (2006) 119–130.

[NS07] B. Nachtergaele, R. Sims, A multi-dimensional Lieb-Schultz-Mattis theorem *Commun. Math. Phys.* **276** (2007), 437–472.

[NSY] B. Nachtergaele, R. Sims, and A. Young, Quasi-locality bounds for quantum lattice systems. I. Lieb-Robinson bounds, quasi-local maps, and spectral flow automorphisms. *J. Math. Phys.* **60** (2019) 061101.

[NS09] B. Nachtergaele, R. Sims, Locality Estimates for Quantum Spin Systems. In: *New Trends in Mathematical Physics*. edited by V. Sidoravicus, Springer 2009.

[NT] S. Neshveyev, L. Tuset, *Compact Quantum Groups and Their Representation Categories* AMS, 2014.

[O1] Y. Ogata, A class of asymmetric gapped Hamiltonians on quantum spin chains and its classification I. *Commun. Math. Phys.*, **348** (2016) 847–895.

[O2] Y. Ogata, A class of asymmetric gapped Hamiltonians on quantum spin chains and its classification II. *Commun. Math. Phys.*, **348** (2016) 897–957.

[O3] Y. Ogata, A class of asymmetric gapped Hamiltonians on quantum spin chains and its classification III. *Commun. Math. Phys.*, **352** (2017) 1205–1263.

[O4] Y. Ogata, A $\mathbb{Z}_2$-Index of Symmetry Protected Topological Phases with Time Reversal Symmetry for Quantum Spin Chains. *Commun Math. Phys.*, **374** (2020) 705–734.
[O5] Y. Ogata, A classification of pure states on quantum spin chains satisfying the split property with on-site finite group symmetries. Transactions of the American Mathematical Society, Series B 8 (2021), 39 – 65.

[O6] Y. Ogata, A $\mathbb{Z}_2$-index of symmetry protected topological phases with reflection symmetry for quantum spin chains Commun. Math. Phys. 385 (2021) 1247–1272.

[O7] Y. Ogata, A $H^3(G, T)$-valued index of symmetry protected topological phases with on-site finite group symmetry for two-dimensional quantum spin systems, 2021 arXiv:2101.00426

[O8] Y. Ogata, A derivation of braided $C^*$-tensor categories from gapped ground states satisfying the approximate Haag duality, 2021,

[OTT] Y. Ogata, Y. Tachikawa, H. Tasaki, General Lieb-Schultz-Mattis type theorems for quantum spin chains, Commun. Math. Phys. 385 (2021), 79–99.

[OT] Y. Ogata, H. Tasaki, Lieb-Schultz-Mattis type theorems for quantum spin chains without continuous symmetry. Commun. Math. Phys 372 (2019), 951–962.

[Os] M. Oshikawa, Commensurability, excitation gap, and topology in quantum many-particle systems on a periodic lattice, Phys. Rev. Lett. 84 (2000) 1535.

[PTBO1] F. Pollmann, A. Turner, E. Berg, and M. Oshikawa, Entanglement spectrum of a topological phase in one dimension. Phys. Rev. B 81 (2010) 064439.

[PTBO2] F. Pollmann, A. Turner, E. Berg, and M. Oshikawa, Symmetry protection of topological phases in one-dimensional quantum spin systems. Phys. Rev. B 81 (2012) 075125.

[PWSVC] D. Perez-Garcia, M. M. Wolf, M. Sanz, F. Verstraete, and J. I. Cirac, String order and symmetries in quantum spin lattices, Phys. Rev. Lett. 100, (2008)167202.

[S] Sopenko An index for two-dimensional SPT states 2021 arxiv 2101.00801.

[Yon] K. Yonekura, On the Cobordism Classification of Symmetry Protected Topological Phases. Commun. Math. Phys. 368 (2019), 1121–1173.

[Yos] B. Yoshida, Topological phases with generalized global symmetries, Phys. Rev. B 93 (2016) 155131.

[WPVZ] H. Watanabe, H. C. Po, A. Vishwanath, and M.P. Zaletel, Filling constraints for spin-orbit coupled insulators in symmorphic and nonsymmorphic crystals, Proc. Natl. Acad. Sci. U.S.A. 112 (2015) 14551–14556.