Nowhere-zero flows in signed series-parallel graphs

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Abstract

Bouchet conjectured in 1983 that each signed graph that admits a nowhere-zero flow has a nowhere-zero 6-flow. We prove that the conjecture is true for all signed series-parallel graphs. Unlike the unsigned case, the restriction to series-parallel graphs is nontrivial; in fact, the result is tight for infinitely many graphs.

1 Introduction

A signed graph \((G,\sigma)\) is a graph \(G\) together with a signature, a mapping \(\sigma : E(G) \to \{+1, -1\}\), that assigns each edge with a sign. The graph \(G\) is called the underlying graph of \((G,\sigma)\). In this work we focus on signed graphs whose underlying graph is a series-parallel graph, that is, a graph that can be obtained from copies of \(K_2\) by iterated series and parallel connections.

A signed graph can be given an orientation as follows. Viewing each edge as composed of two half-edges, we orient each half-edge independently; it is required that of the two half-edges of an edge \(e\), exactly one points to its endvertex if \(e\) is positive, while none or both of them point to their endvertices if \(e\) is negative. A nowhere-zero \(k\)-flow \((D,\phi)\) on a signed graph \((G,\sigma)\) is an orientation \(D\) of edges of \((G,\sigma)\) and a valuation \(\phi\) of its arcs by non-zero integers whose absolute value is smaller than \(k\), such that for every vertex the sum of the incoming values (the inflow) is equal to the sum of the outgoing ones (the outflow). Graphs (signed or unsigned) admitting at least one nowhere-zero \(k\)-flow (for some \(k\)) are called flow-admissible.

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With respect to nowhere-zero flows, all-positive signed graphs (i.e., those with all edge signs positive) behave like ordinary unsigned graphs. Thus, problems about nowhere-zero flows in signed graphs include the celebrated 5-flow conjecture of Tutte [5]:

**Conjecture 1 (Tutte).** Every flow-admissible graph has a nowhere-zero 5-flow.

While there are examples showing that the analogue of Conjecture 1 is false for general signed graphs (as discussed below), Bouchet [1] conjectured that things do not get much worse:

**Conjecture 2 (Bouchet).** Every flow-admissible signed graph has a nowhere-zero 6-flow.

The best published partial result in the direction of Bouchet’s conjecture is a 30-flow theorem by Zýka [7] from 1987. Recently, a 12-flow theorem was announced by DeVos [2].

An infinite family of signed graphs reaching the bound stated in Conjecture 2 was found by Schubert and Steffen [4]. The smallest member of the family is shown in Figure 1. Interestingly, the underlying graphs of the members of this family are series-parallel. This is in sharp contrast with the situation in the unsigned case, where each flow-admissible series-parallel graph trivially admits a nowhere-zero 3-flow. In this paper, we concentrate on the family of signed series-parallel graphs and prove the corresponding restriction of Bouchet’s conjecture:

**Theorem 3.** Every flow-admissible signed series-parallel graph has a nowhere-zero 6-flow.

The proof is given in Section 5.

### 2 Preliminaries

In this section, we review the necessary terminology. Our graphs may contain parallel edges. The *switching at a vertex* \( v \) of a signed graph is the operation of inverting the signs on all the edges incident with \( v \). Two signed graphs are *switching equivalent* if one can be obtained from the other by a finite sequence of
switchings. Since switching does not affect the existence of a nowhere-zero $k$-flow (for any $k$), we may treat switching equivalent graphs as identical.

It is well known that an unsigned graph is flow-admissible if and only if it is bridgeless. Before we state a corresponding characterisation for signed graphs due to Bouchet [1], we recall several basic notions. A balanced cycle is a cycle with an even number of negative edges. An unbalanced cycle is a cycle with an odd number of negative edges. A signed graph is called unbalanced if it contains an unbalanced cycle. Otherwise it is balanced. A barbell in a signed graph $G$ is the union of two edge-disjoint unbalanced cycles $C_1$, $C_2$ and a path $P$ satisfying one of the following properties:

- $C_1$ and $C_2$ are vertex-disjoint, $P$ is internally vertex-disjoint from $C_1 \cup C_2$ and shares an endvertex with each $C_i$, or
- $V(C_1) \cap V(C_2)$ consists of a single vertex $w$, and $P$ is the trivial path consisting of $w$.

A signed circuit in $G$ is either a balanced cycle or a barbell in $G$. With respect to flows, signed circuits are analogous to cycles in unsigned graphs. The following characterisation theorem is due to Bouchet [1, Proposition 3.1]:

**Theorem 4.** A signed graph $G$ is flow-admissible if and only if each of its edges is contained in a signed circuit.

A two-terminal graph $(G, s, t)$ consists of a graph $G$ together with two distinguished vertices, the source terminal $s$ and the target terminal $t$, where $s \neq t$. We abbreviate $(G, s, t)$ to $G$.

Let $G_1, \ldots, G_n$ be two-terminal graphs and $H$ their disjoint union. The series connection $S(G_1, \ldots, G_n)$ of $G_1, \ldots, G_n$ is obtained from $H$ by identifying, for $i = 1, \ldots, n-1$, the target terminal of $G_i$ with the source terminal of $G_{i+1}$. By definition, the source and target terminal of $S(G_1, \ldots, G_n)$ is the source terminal of $G_1$ and the target terminal of $G_n$, respectively.

The parallel connection $P(G_1, \ldots, G_n)$ of $G_1, \ldots, G_n$ is obtained from $H$ by identifying their source terminals and identifying their target terminals. The source terminal of the resulting graph is the vertex obtained by the identification of the source terminals of the graphs $G_i$, and similarly for the target terminal.

A series-parallel graph is a two-terminal graph obtained by a sequence of series and parallel connections, starting with copies of $K_2$ (with some choice of the terminals).

If $G_1, \ldots, G_n$ ($n \geq 2$) are series-parallel graphs such that $G$ is a series or a parallel connection of $G_1, \ldots, G_n$ and $n$ is maximum with this property, then we refer to the $G_i$ as parts of $G$. In addition, in the case of a series connection, $G_1$ and $G_n$ are the endparts of $G$. We say that $G'$ is a piece of $G$ if there is a sequence $G' = H_0, H_1, \ldots, H_m = G$ such that for each $j = 0, \ldots, m-1$, $H_j$ is a part of $H_{j+1}$. In particular, $G$ itself is a piece of $G$. 

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The replacement of $G'$ by a series-parallel graph $H'$ in $G$ consists in removing all the edges and non-terminal vertices of $G'$ in $G$, adding $H'$ and identifying each of its terminals with the corresponding terminal of $G'$ in $G$.

We introduce the following notation for small signed series-parallel graphs: $K^+_2$ denotes the positive $K_2$, $K^-_2$ stands for the negative $K_2$, and $D$ is the unbalanced 2-cycle.

We define the depth $dp(G)$ of a signed series-parallel graph $G$ by letting $dp(K^+_2) = dp(K^-_2) = 0$, and

$$dp(G) = 1 + \max_H dp(H),$$

where $H$ ranges over all parts of $G$.

The following observation is immediate from the definition:

**Observation 5.** If $0 \leq k \leq dp(G)$, then $G$ contains a piece of depth $k$.

Another useful observation is the following one; it can be proved by straightforward induction:

**Observation 6.** Each non-terminal vertex of a series-parallel graph has at least two distinct neighbours.

We will need the following lemma, proved in [3] by induction:

**Lemma 7.** If $e = xy$ is an edge in a 2-connected series-parallel graph $G$, then $(G, x, y)$ is a series-parallel graph.

### 3 Reduced graphs

We begin by using easy reductions to prove the following observation:

**Lemma 8.** Let $G$ be a counterexample to Theorem 3 of minimum size. Then $G$ has the following properties:

(i) if $G$ is of series type, then its endparts are unbalanced,

(ii) the degree of each non-terminal vertex is at least three,

(iii) if a terminal vertex has degree two, then it is contained in a 2-cycle.

**Proof.** We prove (i). Suppose that $G$ has a balanced endpart $H$; let $u$ denote the terminal $u$ of $H$ that is a cutvertex of $G$, and let $H'$ be obtained from $G$ by removing $V(H) - \{u\}$. We show using Theorem 4 that $H'$ is flow-admissible. Suppose the contrary; then an edge $e$ of $H'$ is not contained in a signed circuit of $H'$. However, $G$ is flow-admissible, and it is not hard to see that any signed circuit of $G$ missing in $H'$ is a balanced circuit contained in $H$ (as $H$ is balanced...
and $u$ is a cut-vertex). This is a contradiction, so $H'$ is indeed flow-admissible. By the minimality of $G$, $H'$ admits a nowhere-zero 6-flow. Since any unsigned series-parallel graph admits a nowhere-zero 3-flow, so does the balanced signed graph $H$. Combining these two flows, we obtain a nowhere-zero 6-flow on $G$, a contradiction.

Let us prove (ii). Let $u$ be a non-terminal vertex of degree 2, say incident with edges $e_1, e_2$. Switching at $u$ if necessary, we may assume that $e_1$ is positive. By Observation 6, the endvertices of $e_1, e_2$ different from $u$ are distinct. Contracting $e_1$, we therefore obtain a (loopless) series-parallel graph $G'$. Since the contraction of a positive edge preserves the existence of a nowhere-zero flow, $G'$ is flow-admissible and hence it has a nowhere-zero 6-flow by the minimality of $G$. This corresponds to a 6-flow $\psi$ on $G$, possibly with $\psi(e_1) = 0$. However, since $u$ has degree 2, we have $|\psi(e_1)| = |\psi(e_2)|$, so $\psi$ is nowhere-zero.

A similar argument works for terminal vertices of degree 2 with two distinct neighbours. This proves (iii). □

The following result will be useful in the proof of Lemmas 10 and 17 below.

**Lemma 9.** If a signed series-parallel graph $G'$ is of series type and has unbalanced endparts, then every edge of $G'$ is contained in a barbell.

*Proof.* Let the endparts of $G'$ be denoted by $R_1, R_2$. For $i = 1, 2$, let $r_i$ be the terminal of $R_i$ that is not a terminal of $G$. Let $C_i (i = 1, 2)$ be an unbalanced cycle in $R_i$.

Let $e$ be an edge of $G'$; we need to show that $e$ is contained in a barbell. If $e$ is not contained in an endpart, then $G'$ contains a path $P$ from $r_1$ to $r_2$ containing $e$. Extending $P$ to a path joining $C_1$ to $C_2$, we obtain a barbell in $G'$ containing $e$.

We may therefore assume that $e$ is contained in $R_1$. Since $R_1$ is unbalanced, $R_1$ is different from a signed $K_2$ and therefore 2-connected. Choose two vertex-disjoint paths in $R_1$ connecting the endvertices of $e$ to $C_1$. Taking the union of these paths with $e$ and a suitable subpath of $C_1$, we obtain an unbalanced cycle $C'_1$ in $R_1$ containing $e$. Then the union of $C'_1, C_2$ and an appropriate extension of $P$ is a barbell in $G'$ containing $e$. □

**Lemma 10.** If $G$ is a counterexample to Theorem 3 of minimum size, then $G$ contains no pair of parallel edges of the same sign.

*Proof.* Suppose that $G$ contains parallel edges $e, f$ of the same sign, say both positive. By Theorem 4, each edge of $G$ is contained in a signed circuit. Thus, each edge $e'$ of $G - e$ is clearly also contained in a signed circuit of $G - e$ unless $e' = f$.

Assume first that $f$ is also contained in a signed circuit of $G - e$. Then $G - e$ is flow-admissible. By the minimality of $G$, $G - e$ admits a nowhere-zero 6-flow
Adding to \( \varphi \) a suitable nowhere-zero 6-flow on the 2-cycle \( C \) comprised of \( e \) and \( f \), we obtain a nowhere-zero 6-flow in \( G \).

It follows that \( f \) is not contained in a signed circuit. Note that in this case, \( G - e - f \) is flow-admissible. If we can show that each component of \( G - e - f \) is series-parallel, then by the minimality of \( G \), \( G - e - f \) admits a nowhere-zero 6-flow, which is easily extended to \( G \) using a suitable flow on the above 2-cycle \( C \).

It remains to prove that \( G - e - f \) is comprised of series-parallel components. Let the terminals of \( G \) be \( u \) and \( v \). Suppose first that \( G - e \) is 2-connected. Then \( G - e - f \) is connected. By Lemma 7, \( G - e - f, x, y \) is a series-parallel graph, where \( x \) and \( y \) are the endvertices of \( f \).

We can therefore assume that \( G - e \) is not 2-connected. We claim that \( G \) is not 2-connected; assume the contrary. Since, clearly, \( G - e \) is different from \( K_2 \), there is a cutvertex of \( G - e \) separating the endvertices of \( e \). These are, however, connected by the edge \( f \), a contradiction.

Since \( G \) is not 2-connected, it has unbalanced endparts by Lemma 8(i), and the same holds for \( G - e \). By Lemma 9 \( f \) is contained in a barbell in \( G - e \), a contradiction with the hypothesis that \( f \) is not contained in a signed circuit of \( G - e \). This concludes the proof.

Let us call a graph **reduced** if the degree of each of its non-terminal vertices is at least 3, and there is no pair of parallel edges of the same sign. Observe that each part (and hence, each piece) of a reduced graph is reduced. It is easy to see that the only reduced graph of depth 1 is the unbalanced 2-cycle \( D \).

A **string** is a series connection of copies of \( K^+_2 \) and \( D \) where each non-terminal vertex is contained in a 2-cycle (see Figure 2). Thus, any string is a reduced graph. A string is **nontrivial** if it contains more than two vertices.

**Lemma 11.** Each reduced signed series-parallel graph of depth at most 2 is switching equivalent to a string.

**Proof.** Let \( G \) be a graph satisfying the assumption. If the depth of \( G \) is 0 or 1, then \( G \) is \( K^+_2 \), \( K^-_2 \) or \( D \) and the assertion holds. Assume that the depth of \( G \) is 2. Since the only reduced graph of depth 1 is \( D \), \( G \) is necessarily of series type; since each of its parts is reduced, \( G \) is a series connection of copies of \( K^+_2 \), \( K^-_2 \) and \( D \). Switching at the target terminal of each \( K^-_2 \), we obtain a string.

A **necklace** is a signed series-parallel graph obtained as the parallel connection of two strings, at least one of which is nontrivial (see Figure 2).

**Lemma 12.** Each reduced signed series-parallel signed graph of depth at least 3 contains a piece switching equivalent to a necklace.

**Proof.** Let \( G \) be a signed series-parallel graph of depth at least 3. By Observation 5 \( G \) contains a piece of depth 3 which is necessarily reduced. We may
therefore assume that the depth of $G$ is equal to 3. Since one of its parts has depth 2 and is reduced, it is of series type, so $G$ itself is of parallel type. Lemma 11 implies that $G$ is a parallel connection of graphs switching equivalent to a string. Let $H$ be the parallel connection of two of these graphs, say $H_1$ and $H_2$, where the depth of $H_1$ equals 2. Note that $H_1$ has more than two vertices. We show that $H$ is switching equivalent to a necklace. First, if $H_2$ is a $K_{2}^{-}$, then we perform a switch at one of the terminals to change its sign. Each of the remaining negative edges has an endvertex contained in an unbalanced 2-cycle; we switch at each such endvertex to obtain a necklace.

4 Pseudoflows

For the proof of Theorem 3, we utilise the concept of pseudoflow in a signed series-parallel graph $H$, defined just as a nowhere-zero 6-flow in $H$, except that at each terminal, the inflow is not required to equal the outflow. (In particular, pseudoflows are nowhere-zero by definition.) Let $I_5 = \{-5, -4, \ldots, 5\}$. A pseudoflow in $H$ is an $(a, b)$-pseudoflow (where $a, b \in I_5$) if the outflow at the source terminal equals $a$ and the inflow at the target terminal equals $b$. An example of a pseudoflow is shown in Figure 3. As another example, note that $K_2^+$ admits an $(a, b)$-pseudoflow if and only if $a = b \neq 0$ and $a \in I_5$.

We make a couple of observations related to pseudoflows. An $(a, b)$-pseudoflow can only exist if $a$ and $b$ have the same parity. A $(0, 0)$-pseudoflow coincides with a nowhere-zero 6-flow. Furthermore, if the source terminal of $H$ has degree 1, then $H$ admits no $(0, b)$-pseudoflow for any $b$. Based on the last observation, let us say that the pair $(a, b)$ is valid for $H = (H, s, t)$ if either $a \neq 0$ or $d(s) \geq 2$,
and at the same time $b \neq 0$ or $d(t) \geq 2$.

**Observation 13.** Let $a, b \in \mathbb{Z}$ such that $a \equiv b \pmod{2}$. If the unbalanced 2-cycle $D$ admits an $(a, b)$-pseudoflow, then $a \neq \pm b$. In the converse direction, if $a \neq \pm b$ and $a, b \in I_5$, then $D$ admits an $(a, b)$-pseudoflow.

The following lemma provides us with information on the types of pseudoflows that exist in strings. For a graph $G$, we define $\beta(G)$ as the number of distinct 2-cycles in $G$. Note that $\beta(G) \geq 1$ for any nontrivial string $G$.

**Lemma 14.** Let $G$ be a nontrivial string. Let $a, b \in I_5$ be integers such that $a \equiv b \pmod{2}$ and $(a, b)$ is valid for $G$. Then $G$ admits an $(a, b)$-pseudoflow if one of the following conditions holds:

(a) $\beta(G)$ is odd and $a \neq \pm b$,

(b) $\beta(G)$ is even and $a = \pm b$,

(c) $\beta(G) \geq 2$ and either $a$ is odd or $a = 0$ or $b = 0$.

**Proof.** Let $n = \beta(G)$ and let the 2-cycles of $G$ be $D_1, \ldots, D_n$. Orient the positive edges of $G$ so as to obtain a directed path from the source to the target terminal. We claim that it suffices to find a sequence of numbers $c_1, \ldots, c_{n+1} \in I_5$, all of the same parity, such that $a = c_1$, $b = c_{n+1}$, $c_i \neq \pm c_{i+1}$ for all $i = 1, \ldots, n$, and $c_i \neq 0$ unless $i \in \{1, n+1\}$. Given such a sequence, we construct an $(a, b)$-pseudoflow on $G$ as follows:

- using Observation 13 we find a $(c_i, c_{i+1})$-pseudoflow on each $D_i$, where $1 \leq i \leq n$,

- we assign flow value $c_{i+1}$ to a bridge joining $D_i$ to $D_{i+1}$ if there is one ($1 \leq i \leq n$), with respect to the fixed orientation,
• a bridge incident with a source (target) terminal, if such a bridge exists, will be assigned value \( a = c_1 \) \((b = c_{n+1}, \text{respectively})\) with respect to the fixed orientation.

It is not hard to see that this procedure defines an \((a,b)\)-pseudoflow on \(G\). Note that the values associated to any bridges incident with terminals are nonzero since the pair \((a,b)\) is valid for \(G\).

To find a sequence as above, we consider the possible cases one by one. In cases (a) and (b), we just take the alternating sequence \(a, b, a, b, \ldots\) of length \(n+1\). Consider case (c) and assume first that \(a\) is odd. Let \(d\) be an odd element of \(I_5\) such that \(\pm a \neq d \neq \pm b\). If \(n\) is odd, the alternating sequence \(a, b, \ldots\) of length \(n+1\) ends with \(b\) as required; otherwise, we insert \(d\) after its first element and delete the last element, again obtaining a sequence with the required property.

To finish the discussion of case (c), assume that \(a = 0\). Let \(d\) be an even element of \(I_5\) such that \(0 \neq d \neq \pm b\). Depending on the parity of \(n\), we take either the alternating sequence \(0, b, \ldots\) of length \(n+1\), or the sequence obtained by inserting \(d\) after the first 0 and dropping the last element. The case \(b = 0\) is symmetric. \(\square\)

We use Lemma 14 to obtain a similar result for necklaces:

**Lemma 15.** Let \(G\) be a necklace and \(a, b \in I_5\) integers such that \(a \equiv b \pmod{2}\). Then \(G\) admits an \((a,b)\)-pseudoflow if either \(a \neq \pm b\), or \(a = b = 0\) and \(\beta(G) \geq 2\).

**Proof.** Suppose that \(G\) is a parallel connection of strings \(G_1\) and \(G_2\). Label the strings in such a way that the following conditions hold if possible, in the order of precedence:

• \(\beta(G_2) = 0\),

• \(\beta(G_2)\) is odd.

We say that \(G\) is of type I if \(\beta(G_2)\) is odd. If \(\beta(G_2)\) is even, then \(G\) is of type II.

Suppose first that \(a = b = 0\) and \(\beta(G) \geq 2\). If both \(\beta(G_1)\) and \(\beta(G_2)\) are nonzero, then by Lemma 14 (a) and (c), there is a \((1,3)\)-pseudoflow in \(G_1\) and a \((-1,-3)\)-pseudoflow in \(G_2\); their sum is the required \((0,0)\)-pseudoflow in \(G\). Suppose then that \(\beta(G_2) = 0\), which means that \(\beta(G_1) \geq 2\). By Lemma 14 (c), \(G_1\) admits a \((1,1)\)-pseudoflow \(f_1\). Since \(G_2\) is necessarily the graph \(K_2^+\), it admits a \((-1,-1)\)-pseudoflow, which we again sum with \(f_1\) to obtain a \((0,0)\)-pseudoflow in \(G\).

For the rest of the proof suppose that \(a, b \in I_5\) are integers of the same parity such that \(a \neq \pm b\). The desired \((a,b)\)-pseudoflow on \(G\) will be constructed as a sum of an \((a_1,b_1)\)-pseudoflow on \(G_1\) and an \((a_2,b_2)\)-pseudoflow on \(G_2\) for suitable \(a_1, b_1, a_2\) and \(b_2\).
Let us consider possible pseudoflows in $G_1$. Let $a_1, b_1 \in I_5$ such that $a_1 \equiv b_1 \pmod{2}$. By the choice of $G_2$ and the fact that at least one string of a necklace is nontrivial, $\beta(G_1) \geq 1$. If $\beta(G_1) = 1$, by Lemma 14 (a), $G_1$ admits an $(a_1, b_1)$-pseudoflow if $a_1 \neq b_1$. If $\beta(G_1) \geq 2$, then, by Lemma 14 (c), $G_1$ admits an $(a_1, b_1)$-pseudoflow if $a_1$ and $b_1$ are odd. In summary, regardless of the type of $G_2$, $G_1$ admits an $(a_1, b_1)$-pseudoflow if $a_1$ and $b_1$ are odd and $a_1 \neq \pm b_1$. (1)

Next, we consider pseudoflows in $G_2$. Let $a_2, b_2 \in I_5$ such that $a_2 \equiv b_2 \pmod{2}$. By Lemma 14 (a) and (b), $G_2$ admits an $(a_2, b_2)$-pseudoflow if either $G$ is of type I and $a_2 \neq \pm b_2$, or $G$ is of type II and $a_2 = \pm b_2$. (2)

For each possible $(a, b)$ we now exhibit a choice of $(a_1, b_1)$ and $(a_2, b_2)$ satisfying the conditions (1) and (2), respectively, and such that $a_1 + a_2 = a$, $b_1 + b_2 = b$. The $(a, b)$-pseudoflow in $G$ will be the sum of an $(a_1, b_1)$-pseudoflow in $G_1$ and an $(a_2, b_2)$-pseudoflow in $G_2$.

The choices of $(a_1, b_1)$ and $(a_2, b_2)$ are given in Table 1. We assume, without loss of generality, that $|a| \leq |b|$. By inverting all signs if necessary, we may further assume that $a \geq 0$, and if $a = 0$, then $b \geq 0$.

For example, if $(a, b) = (0, 2)$ and $G$ is of type II, then the table suggests taking $(a_1, b_1) = (1, 3)$ and $(a_2, b_2) = (−1, −1)$, in accordance with conditions (1) and (2). The rest of the proof is a routine inspection of the table.

**Corollary 16.** Every flow-admissible string or necklace admits a $(0, 0)$-pseudoflow (that is, a nowhere-zero 6-flow).

### 5 Proof of Theorem 3

Let $G$ be a counterexample to Theorem 3 with minimum number of edges and, subject to this condition, maximum number of vertices. By Lemmas 8 and 10, $G$ is reduced, and by Corollary 16, its depth is at least 3. Using Lemma 12, we may assume that $G$ contains a piece $H$ that is a necklace. Furthermore, by Corollary 16, $G \neq H$.

We choose $H$ in such a way that $\beta(H)$ is minimized. Recall that $\beta(H)$ is the number of 2-cycles in $H$, and that $\beta(H) \geq 1$.

**Case A:** $H$ is not an endpart of $G$.

We replace $H$ with $D' = S(K_2^+, D, K_2^+)$ in $G$, where $D$ is the unbalanced 2-cycle. Let us call the resulting graph $G'$. The following lemma provides the last missing piece in our argument.

**Lemma 17.** If $G$ contains an unbalanced cycle edge-disjoint from $H$, then $G'$ is flow-admissible.
| $a$ | $b$ | $a_1$ | $b_1$ | $a_2$ | $b_2$ | $a_1$ | $b_1$ | $a_2$ | $b_2$ |
|-----|-----|-------|-------|-------|-------|-------|-------|-------|-------|
| 0   | 1   | -1    | 1     | -1    |
| 2   | 5   | -3    | 3     | -1    |
| 0   | 3   | -3    | 1     | -1    |
| 4   | 5   | -1    | 5     | -1    |
| 2   | -1  | 3     | 1     | 1     |
| 4   | 5   | -1    | 3     | 1     |
| 2   | -1  | 3     | 1     | 1     |
| -4  | -5  | 1     | -5    | 1     |
| 1   | 3   | -2    | 3     | -2    |
| 3   | -1  | 4     | 5     | -2    |
| -3  | 1   | -4    | -1    | -2    |
| 1   | 3   | -2    | -1    | 2     |
| 5   | 1   | 4     | 3     | 2     |
| 1   | 3   | -2    | 5     | -4    |
| -5  | -1  | -4    | -1    | -4    |
| 3   | 5   | -2    | 1     | 2     |
| 5   | 1   | 4     | 3     | 2     |
| 3   | 5   | -2    | 5     | -2    |
| -5  | -1  | -4    | -3    | -2    |

Table 1: The pairs $(a_1, b_1)$ and $(a_2, b_2)$ (the first and second column in each field, respectively) for each possible choice of $(a, b)$ and type of the necklace in the proof of Lemma 15. The case $(a, b) = (0, 0)$ is discussed separately.
Proof. Let \( u' \) and \( v' \) be the source and target terminal, respectively, of the necklace \( H \) (as well as of the graph \( D' \)). Let \( D_0 \) denote the 2-cycle in \( D' \). Consider an arbitrary edge \( e \) of \( G' \). We need to show that \( e \) is contained in a signed circuit of \( G' \). Suppose the contrary.

Case 1: \( e \notin E(D') \).

We first observe that \( G \) contains no \( u'v' \)-path that contains \( e \) and is vertex-disjoint from \( H \) except for its endvertices (let us call such a path an \( e \)-detour). Indeed, combining such a path with one of the two \( u'v' \)-paths in \( D' \) would provide us with a balanced cycle of \( G' \) containing \( e \).

In particular, each cycle of \( G \) containing \( e \) is edge-disjoint from \( H \). Since \( e \) is not contained in any signed circuit of \( G' \), any such cycle must be unbalanced. On the other hand, since \( G \) is flow-admissible, \( e \) is contained in a signed circuit \( B \) of \( G \), which must therefore be a barbell. Clearly, \( B \) is not edge-disjoint from \( H \), for otherwise \( B \) is a signed circuit containing \( e \) in \( G' \). Let \( A_1 \) and \( A_2 \) be the unbalanced cycles in \( B \) and let \( Q \) be the path connecting them.

If \( e \) belongs to an unbalanced cycle of \( B \), say \( A_1 \), then \( A_1 \) is edge-disjoint from \( H \) in \( G \), and thus also from \( D' \) in \( G' \). Since \( G' \) is connected, there exists a path connecting \( A_1 \) and \( D_0 \) and therefore also a barbell of \( G' \) containing \( e \), a contradiction. Hence we can assume that \( e \in Q \).

Note first that \( A_1 \) and \( A_2 \) are not both edge-disjoint from \( H \), for otherwise \( Q \) contains both \( u' \) and \( v' \), and replacing the part of \( Q \) inside \( H \) with a \( u'v' \)-path in \( D' \) yields a barbell in \( G' \) containing \( e \).

Suppose then that \( A_1 \) contains an edge of \( H \). We claim that \( A_2 \) is edge-disjoint from \( H \). For the sake of a contradiction, assume that \( A_2 \) contains an edge of \( H \). If both \( A_1 \) and \( A_2 \) were contained in \( H \), then a subpath of \( Q \) would be an \( e \)-detour. On the other hand, at least one \( A_i \) has to be contained in \( H \), for otherwise they both contain \( u' \) and \( v' \), violating the definition of a barbell.

We may thus assume that \( A_1 \) is contained in \( H \) and \( A_2 \) is not. Since \( A_2 \) contains both terminals of \( H \), we have \( Q \subseteq H \), which is a contradiction, since \( e \notin E(D') \). Therefore \( A_2 \) is edge-disjoint from \( H \) as claimed. Let us choose a shortest path \( P \) in \( Q \cup A_1 \) connecting \( A_2 \) to a terminal of \( H \); the union of \( P \), \( A_2 \), \( D_0 \) and an edge of \( D' \) connecting \( D_0 \) to an endvertex of \( P \) is then a barbell in \( G' \) containing \( e \). This finishes the discussion of Case 1.

Case 2: \( e \in E(D') \).

We show that \( G \) is of parallel type. Otherwise, it would be of series type, its endparts would be unbalanced (Lemma 8(i)) and by Lemma 9, \( e \) would be contained in a barbell, which is a contradiction.

The graph \( G' \) is also of parallel type. This is clear if \( H \) is a (proper) subgraph of one of the parts of \( G \). Otherwise, the two strings forming \( H \) are parts of \( G \), and by the assumption that \( G \) contains an unbalanced cycle edge-disjoint from \( H \), there are some more parts. Then \( G' \) is the parallel connection of these parts with \( D' \).

By symmetry, we may assume that \( e \) is not incident with the terminal \( v' \) of
Let \( A \) be an unbalanced cycle in \( G \), edge-disjoint from \( D' \). Since \( G' \) is 2-connected, it contains a path \( R \) joining \( u' \) to \( A \) and avoiding \( v' \). The union of \( D_0, R, A \) and the edge connecting \( D_0 \) to \( u' \) is a barbell containing \( e \). This concludes the proof.

We claim that there is indeed an unbalanced cycle of \( G \) that is edge-disjoint from \( H \), as required in Lemma 17. This is clear if \( G \) is of series type, because every endpart of \( G \) is unbalanced (Lemma 8(i)).

Let \( G \) be a parallel connection of its parts \( H_1, \ldots, H_k \). The necklace \( H \) is either a union of two strings \( H_i \cup H_j \) for some \( i \) and \( j \), or it is a proper subgraph of some \( H_i \). Suppose first the former — say, \( H = H_1 \cup H_2 \). Then \( k \geq 3 \), because \( G \neq H \). If there exists \( i \in \{3, \ldots, k\} \) such that \( H_i \) contains an unbalanced cycle, we are done. Since \( G \) is reduced, each \( H_i \) is reduced as well, and thus every \( H_i \) is a signed \( K_2 \). If there exist \( H_i \) and \( H_j \) \((i, j \in \{3, \ldots, k\})\) such that they have opposite signs, then \( H_i \cup H_j \) is the sought unbalanced cycle edge-disjoint from \( H \).

We conclude that \( k = 3 \), because \( G \) is reduced and it does not contain parallel edges of the same sign. If neither \( H_1 \) nor \( H_2 \) is a \( K_2^+ \), then \( H_1 \cup H_2 \) is a necklace of \( G \) with \( \beta(H_1 \cup H_2) < \beta(H) \), which is a contradiction with the choice of \( H \).

On the other hand, if \( H \) contains a string that is a \( K_2^+ \), then the string forms an unbalanced 2-cycle with \( H_3 \) (as \( G \) is reduced), and \( G \) is a necklace, which is a contradiction with Corollary 16.

Suppose now that \( H \) is a proper subgraph of one of the parts of \( G \), say \( H_1 \).

By a similar argument as above we conclude that \( k = 2 \) and \( H_2 \) is a signed \( K_2^+ \).

Moreover \( H_1 \) is a series connection of \( H_1 \), say, \( \bigcup_{i=1}^{q} H_{1i} \), for some \( q \).

If any of the graphs \( H_{1i} \) that does not contain \( H \) contains an unbalanced cycle, we are done. Therefore each \( H_{1i} \) that does not contain \( H \) is a signed \( K_2^+ \).

Since \( H \) is a proper subgraph of \( H_{1} \), we conclude that at least one of the terminals of \( G \) must be of degree 2. But the terminal is not contained in an unbalanced 2-cycle, which is a contradiction with Lemma 8(c). This proves the claim that \( G \) satisfies the hypothesis of Lemma 17 and the graph \( G' \) is flow-admissible.

The graph \( D' \), used to obtain \( G' \), has four edges. All necklaces have at least four edges, with \( P(K_2^+, S(K_2^+, D)) \) and its mirror image being the only ones with exactly four. These necklaces, however, have one vertex fewer than \( D' \), so the choice of \( G \) implies that \( G' \) admits a nowhere-zero 6-flow \( \varphi \).

The restriction of \( \varphi \) to \( D_0 \) (the 2-cycle in \( D' \)) is an \((a, b)\)-pseudoflow for some \( a \) and \( b \). By Observation 13 \( a \equiv b \pmod{2} \) and \( a \neq \pm b \).

Furthermore, since the values of \( \varphi \) on the edges of \( D' - E(D_0) \) are \( \pm a \) and \( \pm b \), we find that \( a, b \) are nonzero elements of \( I_5 \). By Lemma 15, \( H \) admits an \((a, b)\)-pseudoflow. Combining it with a restriction of \( \varphi \), we obtain a nowhere-zero 6-flow in \( G \), contradicting the hypothesis.

**Case B:** \( H \) is an endpart of \( G \).

The argument is similar to the argument of Case A, except that we use the graph \( S(K_2^+, D) \) or \( S(D, K_2^+) \) in place of \( D' \) (so as to obtain a flow-admissible
graph) and invoke an analogue of Lemma 17. We arrive at a similar contradiction, which concludes the proof.

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