Exponent of Cyclic Codes over $\mathbb{F}_q$

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Proposed running head: Exponent of a Cyclic Codes over $\mathbb{F}_q$
Abstract

In this article, we introduce and study the concept of the exponent of a cyclic code over a finite field $F_q$. We give a relation between the exponent of a cyclic code and its dual code. Finally, we introduce and determine the exponent distribution of the cyclic code.

Keywords: Order of the Polynomial, Cyclic Codes, Exponent of Cyclic Code, Dual Code.

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1 Introduction

Cyclic codes are a class of important linear codes and have generated great interest in coding theory. In general, they have natural encoding and decoding algorithm. Since cyclic codes can be described as ideals in polynomials residue rings, they have a rich algebraic structure. There is a lot of literature on cyclic codes over fields and more recently over rings. Cyclic codes have a unique property that is the codewords of cyclic code can be divided into a numbers of mutually disjoint equivalence classes according to an equivalence relation.

Let $F_q$ be a finite field with $q$ elements. A code $C$ of length $n$ over $F_q$ is a nonempty subset of $F_q^n$. If $C$ is a subspace of $F_q^n$, then $C$ is called a $q$-ary linear code of length $n$. If the dimension of $C$ is $k$, then the linear code $C$ is denoted by $[n,k]$ code. A linear code $C$ of length $n$ over $F_q$ is a cyclic code if for every $c = (c_0, c_1, \cdots, c_{n-1}) \in C$,

\[(c_{n-1}, c_0, \cdots, c_{n-2}) \in C.\]

Let $R_n = F_q[x]/(x^n-1)$ be a polynomial residue ring. Then the following theorem gives a relation between a cyclic code and an ideal of $R_n$.

**Theorem 1.1.** \[3\] The linear code $C$ is cyclic code if and only if $C$ is an ideal of $R_n = F_q[x]/(x^n-1)$.

**Theorem 1.2.** \[4\] Let $I$ be an ideal in $R_n = F_q[x]/(x^n-1)$. Then

1. there is a unique monic polynomial $g(x) \in I$ of minimal degree,
2. I is principal with generator \( g(x) \),

3. \( g(x) \) divides \((x^n - 1)\) in \( \mathbb{F}_q[x] \).

Note that \( C \) is an ideal of \( R_n \) and by Theorem 1.2 \( C \) is a principal ideal generated by a unique monic polynomial \( g(x) \in R_n \). The polynomial \( g(x) \) is called the generator polynomial of the cyclic code \( C \). Since the generator polynomial \( g(x) \) of \( C \) is a divisor \( x^n - 1, \frac{x^n-1}{g(x)} \in R_n \).

Let \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in \mathbb{F}_q[x] \) with \( a_n \neq 0 \). Then the reciprocal polynomial \( f^* \) of \( f \) is defined by

\[
f^*(x) = x^nf \left( \frac{1}{x} \right) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n.
\]

Let \( C \) be an \([n, k]_q\) code, then the dual code \( C^\perp \) of the linear code \( C \subseteq \mathbb{F}_q^n \) is defined by

\[
C^\perp = \{ x \in \mathbb{F}_q^n \mid \langle x, c \rangle = 0 \text{ for all } c \in C \}
\]

where \( \langle x, c \rangle = \sum_{i=1}^nx_ic_i \) is a scalar product. Clearly, \( C^\perp \) is an \([n, n-k]_q\) code.

**Theorem 1.3.** Let \( g(x) \) be the generator polynomial of the cyclic code \( C \) and let 
\[ h(x) = \frac{x^n-1}{g(x)} \in R_n. \]
Then the dual code \( C^\perp \) is cyclic with generator polynomial \( h^*(x) \).

Let \( f(x) \in \mathbb{F}_q[x] \) be a nonzero polynomial. If \( f(0) \neq 0 \), then the least positive integer \( e \) for which \( f(x) \) divides \( x^e - 1 \) is called the period of \( f(x) \) or the order of \( f(x) \) and denoted by \( \text{ord}(f) = \text{ord}(f(x)) \). If \( f(0) = 0 \), then \( f(x) = x^lg(x) \), where \( l \in \mathbb{N} \) and \( g(x) \in \mathbb{F}_q[x] \) with \( g(0) \neq 0 \) are uniquely determined and the \( \text{ord}(f) \) is then defined by \( \text{ord}(g) \).

We state the followings for our further discussion.

**Lemma 1.4.** Let \( r \) be a positive integer. Then the polynomial \( f(x) \in \mathbb{F}_q[x] \) with \( f(0) \neq 0 \) divides \( x^r - 1 \) if and only if \( \text{ord}(f) \) divides \( r \).

**Corollary 1.5.** If \( e_1 \) and \( e_2 \) are positive integers, then the greatest common divisor of \( x^{e_1} - 1 \) and \( x^{e_2} - 1 \) in \( \mathbb{F}_q[x] \) is \( x^d - 1 \), where \( d \) is the greatest common divisor of \( e_1 \) and \( e_2 \).

**Theorem 1.6.** Let \( f \) be a nonzero polynomial in \( \mathbb{F}_q[x] \) and \( f^* \) its reciprocal polynomial. Then \( \text{ord}(f) = \text{ord}(f^*) \).

**Theorem 1.7.** Let \( g_1, g_2, \ldots, g_k \) be pairwise prime nonzero polynomials over \( \mathbb{F}_q \) and let \( f = g_1g_2\cdots g_k \). Then \( \text{ord}(f) = \text{lcm}(\text{ord}(g_1), \text{ord}(g_2), \ldots, \text{ord}(g_k)) \).

In this article, we introduced the concept of the exponent of a cyclic code over \( \mathbb{F}_q \) and studied the exponent of the dual cyclic code in section 3. We introduced and determined the exponent distribution of the cyclic code over \( \mathbb{F}_q \) in section 4.
2 Exponent of Cyclic Codes over $\mathbb{F}_q$

In this section, we define the exponent of a cyclic code over $\mathbb{F}_q$ and discussed its properties.

**Definition 2.1.** Let $C$ be a cyclic code of length $n$ over $\mathbb{F}_q$ with generator polynomial $g(x)$. Then the least positive integer $e$ for which $g(x)$ divides $x^e - 1$ is called the exponent of the cyclic code $C$ and denoted by $e = \exp(C)$.

Clearly, the exponent of $C$ is the same as the order of its generator polynomial.

**Example 2.2.** Let $C = \{000, 101, 110, 011\}$ be a cyclic code of length 3 over $\mathbb{F}_2$. Then the generator polynomial of $C$ is $1 + x$ and $1 + x | x^1 - 1$. Hence $\exp(C) = 1$.

**Theorem 2.3.** Let $C$ be an $[n, k]_q$ cyclic code over $\mathbb{F}_q$ and let $e = \exp(C)$. Then $e | n$.

**Proof.** Let $C$ be an $[n, k]_q$ cyclic code over $\mathbb{F}_q$ and let $g(x)$ be its generator polynomial. Then $g(x) | x^n - 1$ and hence by Lemma 1.4, $\ord(g) | n$. Since $e = \exp(C)$ and $\ord(g) = \exp(C)$, hence $e | n$. \qed

Note that by the above theorem, the exponent of a cyclic code is always less than or equal to its length. The bound is reached by the following example.

**Example 2.4.** Let $C = \{000, 111, 222\}$ be a cyclic code of length 3 over $\mathbb{F}_3$. Then the generator polynomial $1 + x + x^2$ of $C$ divides $x^3 - 1$ and hence $\exp(C) = 3$.

**Theorem 2.5.** Let $C$ be an $[n, k]_q$ cyclic code over $\mathbb{F}_q$. If $e = \exp(C)$, then $n - k \leq e$.

**Proof.** Let $C$ be an $[n, k]_q$ cyclic code over $\mathbb{F}_q$ and let $g(x)$ be its generator polynomial. Then $\deg(g(x)) = n - k$. If $e = \exp(C)$, then $g(x) | x^e - 1$. This implies that, $\deg(g(x)) \leq \deg(x^e - 1)$. That is, $n - k \leq e$. \qed

By Theorems 2.3 and 2.5, we have

**Corollary 2.6.** Let $C$ be an $[n, k]_q$ cyclic code over $\mathbb{F}_q$. If $e = \exp(C)$, then $n - k \leq e \leq n$.

The lower bound of the above corollary is reached by the Example 2.2.

**Theorem 2.7.** Let $C$ be an $[n, k]_q$ cyclic code over $\mathbb{F}_q$ and let $g(x), h(x)$ be the generator and parity-check polynomials of $C$, respectively. If $(g(x), h(x)) = 1$, $\exp(C) = e$ and $r = \exp(C^\perp)$, then $n = \lcm(e, r)$.

**Proof.** Let $g(x)$ be a generator polynomial and let $h(x)$ be a parity check polynomial of $C$. Since $x^n - 1 = g(x)h(x)$ and $(g(x), h(x)) = 1$, by Theorem 1.7 we have $n = \lcm(e, r)$. \qed
Theorem 2.8. Let $C_i$ be an $[n, k_i]_q$ cyclic code over $\mathbb{F}_q$ for $i = 1, 2$. If $e_i$ is the exponent of $C_i$, then $\text{exp}(C_1 + C_2) = \gcd(e_1, e_2)$.

Proof. Let $C_1, C_2$ be two cyclic codes of length $n$ over $\mathbb{F}_q$. Then

$$C_1 + C_2 = \{c_1 + c_2 \mid c_i \in C_i, i = 1, 2\}$$

is a cyclic code of length $n$ over $\mathbb{F}_q$. Let $g_1(x), g_2(x), g(x)$ be the generator polynomial of $C_1, C_2$ and $C_1 + C_2$, respectively and let $\text{exp}(C_1 + C_2) = e$. Then $g(x) \mid x^e - 1$. Since $C_i = \langle g_i(x) \rangle$, $C_1 + C_2 = \langle \gcd(g_1(x), g_2(x)) \rangle$. This implies that, $g(x) \mid g_i(x)$ for $i = 1, 2$. Then $e$ divides $e_i$ for $i = 1, 2$. Hence $e$ divides $\gcd(e_1, e_2) = d$. That is, $e \leq d$.

Suppose that $e < d$. Then by Division Algorithm, there exist $q, r \in \mathbb{Z}$ such that $d = eq + r$ where $0 \leq r < e$. Consider

$$x^d - 1 = x^{eq+r} - 1 = x^{eq}x^r - x^r + x^r - 1 = x^r(x^{eq} - 1) + x^r - 1$$

$$(x^d - 1) - x^r(x^{eq} - 1) = x^r - 1.$$ 

Since $g(x)$ divides both $x^e - 1$ and $x^d - 1$, $g(x)$ divides $x^r - 1$. By Lemma 1.3, we have $e \leq r$, a contradiction to $r < e$. Therefore, $e < d$ is impossible and hence $e = d = \gcd(e_1, e_2)$. \hfill \Box

3 Exponent Distribution of a Cyclic Codes over $\mathbb{F}_q$

In this section, we introduce the exponent distribution of a cyclic code over $\mathbb{F}_q$ and discuss its properties.

We know that, if $c = (c_0, c_1, \cdots, c_{n-1})$ is a codeword in the cyclic code $C$, then its corresponding polynomial representation is $c(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$. Since $c(x) \in C = \langle g(x) \rangle$, there exists a polynomial $a(x) \in \mathbb{F}_n$ such that $c(x) = a(x)g(x)$. We call $a(x)$ the information polynomial of $c$.

For $1 \leq t \leq n$, we define $A_t := \{c(x) \in C \mid \text{ord}(c(x)) = t\}$. Let $B_t = |A_t|$, then the sequence $\{B_t\}$ is called the exponent distribution of the cyclic code $C$ over $\mathbb{F}_q$.

Example 3.1. Let $C = \{000, 011, 101, 110, 111\}$ be a cyclic code over $\mathbb{F}_2$. Then the corresponding codeword polynomials are $0, 1+x^2, 1+x, x+x^2$ and $\text{ord}(1+x^2) = 2, \text{ord}(x+x^2) = 1, \text{ord}(1+x) = 1$. This implies that, $A_1 = \{1 + x, x + x^2\}, A_2 = \{1 + x^2\}$ and $A_3 = \emptyset$. Hence $(2, 1, 0)$ is the exponent distribution of the cyclic code $C$. 


Example 3.2. Let $C = \{000, 111, 222\}$ be a cyclic code over $\mathbb{F}_3$. Then the corresponding codeword polynomials are $0, 1 + x + x^2, 2 + 2x + 2x^2$ and $\text{ord}(1 + x + x^2) = 3, \text{ord}(2 + 2x + 2x^2) = 3$. This implies that, $A_1 = \emptyset, A_2 = \emptyset, A_3 = \{1 + x + x^2, 2 + 2x + 2x^2\}$. Hence $(0, 0, 2)$ is the exponent distribution of the cyclic code $C$.

Example 3.3. Let $C = \{000, 010, 001, 110, 011, 101, 111\}$ be a cyclic code over $\mathbb{F}_2$. Then the corresponding codeword polynomials are $0, 1, x, x^2, 1 + x, 1 + x^2, x + x^2, 1 + x + x^2$. This implies that, $A_1 = \{1, x, x^2\}, A_2 = \{1 + x^2\}, A_3 = \{1 + x + x^2\}$. Hence $(4, 1, 1)$ is the exponent distribution of the cyclic code $C$.

Theorem 3.4. Let $C$ be an $[n, k]_q$ cyclic code over $\mathbb{F}_q$ with generator polynomial $g(x)$. If $G$ is a generator matrix of $C$, then the order of the polynomials corresponding to a basis of $C$ are same.

Proof. Let $g(x)$ be a generator polynomial of $C$. Then $\{g(x), xg(x), x^2g(x), \ldots, x^{k-1}g(x)\}$ is a basis of $C$. Since $\text{ord}(g(x)) = \text{ord}(x^i g(x))$ for all $i \geq 0$ and hence the order of the polynomials corresponding to a basis of $C$ are same.

Corollary 3.5. Let $C$ be an $[n, k]_q$ cyclic code over $\mathbb{F}_q$ with generator polynomial $g(x)$. If $\text{ord}(g(x)) = e$, then $B_e \geq k$.

Proof. Given that $\text{ord}(g(x)) = e$. By the above theorem, there are $k$ elements in $A_e$ and hence $B_e \geq k$.

4 Conclusion

In this paper, we have introduced and studied the concept of the exponent of a cyclic code over a finite field $\mathbb{F}_q$. We have given a relation between the exponent of a cyclic code and its dual code. Finally, we introduced determined the exponent distribution of the cyclic code. A future work is to find bounds for the number of cyclic codes of given length and exponent over $\mathbb{F}_q$. Finding the exponents of the BCH, Reed-Soloman and Goppa codes are other direction to work further.

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