A DETAILED ACCOUNT OF ALAIN CONNES’ VERSION OF THE STANDARD MODEL IV

Daniel KASTLER and Thomas SCHÜCKER

Abstract

We give a detailed account of the computation of the Yang-Mills action for the Connes-Lott model with general coupling constant in the commutant of the $K$-cycle. This leads to tree-approximation results amazingly compatible with experiment, yielding a first indication on the Higgs mass.

PACS-92: 11.15 Gauge field theories
MSC-91: 81E13 Yang-Mills and other gauge theories

January 1995
CPT-94/P.3092
hep-th/9501077

anonymous ftp or gopher: cpt.univ-mrs.fr

* Unité Propre de Recherche 7061
1 and Université d’Aix-Marseille II
2 and Université de Provence
The preceding papers I, II and III of this series presented computations based on the following Ansatz for the non-commutative Yang-Mills action:

\[ Y_M = \alpha_l (\theta_l, \theta_l)_I + \alpha_q (\theta_q, \theta_q)_q \quad (\alpha_l + \alpha_q = 1), \]

with the scalar products \((\cdot, \cdot)_l\) and \((\cdot, \cdot)_q\) stemming as follows from the traces \(\tau_{D_l}\) and \(\tau_{D_q}\):

\[
\begin{align*}
(\omega, \omega')_l &= \text{Re} \tau_{D_l} (\omega^* \omega') = \text{Re} T r_{\omega} \left\{ D_l^{-4} \pi_l (\omega^* \omega') \right\}, \\
(\omega, \omega')_q &= \text{Re} \tau_{D_q} (\omega^* \omega') = \text{Re} T r_{\omega} \left\{ D_q^{-4} \pi_q (\omega^* \omega') \right\},
\end{align*}
\]

\(\omega, \omega' \in \Omega A^n (\in \Omega B^n)\).

These scalar products are however not the most general and natural ones \[3\]: with \(\sum\), resp. \(\sum_q\) (\(\sum'_l\), resp. \(\sum'_q\)), positive elements of the respective commutants \(\{\pi_l(B), D_l\}', \{\pi_q(B), D_q\}'\) \((\{\pi_l(B), D_l\}', \{\pi_q(B), D_q\}')\), the alternative Ansatz:

\[
\begin{align*}
(\omega, \omega')_l &= \text{Re} T r_{\omega} \left\{ D_l^{-4} \pi_l (\omega^* \omega') \sum_l \right\}, \\
(\omega, \omega')_q &= \text{Re} T r_{\omega} \left\{ D_q^{-4} \pi_q (\omega^* \omega') \sum_q \right\},
\end{align*}
\]

yield indeed, in contrast to the previous Ansatz \[2\], a non-committed choice of (as required) gauge-invariant scalar products. According to the Poincaré-duality philosophy the new Yang-Mills action will be the sum of its electroweak and chromodynamics parts respectively stemming from the scalar products \(3\) and \(3\). \[4\] Due to the product structure of the traces \(\tau_{D_l}\) and \(\tau_{D_q}\) \[4\]

\[
\begin{align*}
\tau_{D_l} &= \tau_D \otimes T r_2 \otimes T r_N, \\
\tau_{D_q} &= \tau_D \otimes T r_2 \otimes T r_N \otimes T r_3,
\end{align*}
\]

together with the “fiberwise” nature of the combined space-time-inner space theory, the problem of adapting our previous computations to the generalized Ansatz \(3\) will de facto reduce to computations within the representations \(\pi_l\) and \(\pi_q\) pertaining to the inner space. The present treatment differs also from that of III in that the modular coalescence of the three \(U(1)\) groups – and their Lie algebras is performed after squaring the curvature and not before.

The results reached through the new computations constitute a relatively mild modification of our previous results based on the former Ansatz \[2\], however leading to much more satisfying tree-approximation results, suppressing the former inconsistencies\[3\] and amazingly compatible

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\[3\] The scalar products \(3\) incorporate a generalized coupling constant “in the commutant of the \(K\)-cycle”. The leptonic and quark sectors appear independently since their intertwiners are trivial cf. \[16\], \[17\] below.

\[4\] \(T r_2\) denotes the \(R \times L = 2 \times 2\) matrix trace with entries matrices given by the weak isotopic spin (overall \(4 \times 4\) matrices for quarks, cf. \[13\], \[17\], \[18\] below; and \(3 \times 3\) matrices for leptons, cf. \[13\], \[20\], \[21\] below; whilst \(T r_N, T r_3\), and \(T r_5\) respectively stand for the fermion family \(N \times N\) matrix trace, the colour \(3 \times 3\) matrix trace, and the Dirac-operator trace \(T r_D\).

\[5\] e.g. the fact that, for \(x = 0\), the values of \(g_3/g_2 = 1\) and \(\sin^2 \theta = 3/8\) were of the “grand unification type” whilst the mass-ratio \(m_\ell/m_W = 2\) was near the experimental value \[31\].
with the experimental evidence. For a specific choice of the “coupling constant” within the commutant, one computes the tree-approximation values of the ratio between strong and electroweak coupling constant, the weak angle, and the ratios between the top and the $W$, and the Higgs and the top masses. One can fit the three first items with the known experimental values: in fact $g_3/g_2$ and $\sin^2 \theta_W$ turn out to be mutually uncorrelated, and uncorrelated with the ratios $m_t/m_W$ and $m_H/m_W$ which determine each other, thus yielding, since the top has been found, a “prediction” of the Higgs mass: the latter equals $1.5698 m_t$ for $m_t/m_W = 2$, the value (near experiment !) fixed by the (canonical ?) choice of the “coupling constant” in the center of the $K$-cycle, a choice without incidence on $g_3/g_2$ and $\sin^2 \theta_W$.

Of course these results are “classical”. Reliable results await a renormalized field quantization which should be effected with due consideration of the (still to be found) esoteric symmetry brought about by the Higgs boson as a fifth gauge boson.

This paper is the companion paper to [5a] with whom it shares its subject matter with a different style and emphasis. Whilst [5a], destined to a physical audience, adopts a notational setting congenial to the habits of elementary particle physicists, and insists on the global strategy [6] without giving all computational details, we here address mathematical physicists in the notation of our former reports [4b], giving a line-by-line account of computations. These computations have been performed independently in the two papers in different notation, thus affording a mutual check.

For the convenience of our reader, we begin by recalling the definitions of the inner space structure with its Poincaré-dual $A_{ew} \otimes B_{chrom-K}$-cycles $(H_l, D_l, \chi_l)$ and $(H_q, D_q, \chi_q)$. We begin with the uncolored leptonic and quarkonic $A_{ew-K}$-cycles.

[0] Reminder (the inner space).

The algebra $A_{ew}$. With $H = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} ; a, b \in C \right\}$, we have:

$$A_{ew} = C \oplus H = \{(p, q); p \in C, q \in H\} = \left\{ \begin{pmatrix} \bar{p} & 0 \\ 0 & p \end{pmatrix}, q \right\}; p \in C, q \in H \right\} \quad (5)$$

$$G_{ew} = \{u = (u, v) \in A_{ew}; u^* u = u u^* = 1, v^* v = v v^* = 1\} = U(1) \times SU(2) \quad (6)$$

The leptonic and quarkonic $A_{ew-K}$-cycles $(H_l, D_l, \chi_l)$ and $(H_q, D_q, \chi_q)$. Leptonic $K$-cycle: Hilbert space:

$$H_l = (C_R^l \oplus C_L^l) \otimes C^N, \quad e_R \nu_L e_L \quad (8)$$
Operators (endomorphisms of $H_l$ as $3 \times 3$ matrices with entries in $M_N(C)$):

\[
\chi_l = \begin{pmatrix} e_R & \nu_L & e_L \\ 1_N & 0 & 0 \\ 0 & -1_N & 0 \end{pmatrix} e_R \nu_L.
\]

(9)

\[
\pi_l((p, q)) = \begin{pmatrix} e_R & \nu_L & e_L \\ p1_N & 0 & b1_N \\ 0 & -b1_N & \alpha1_N \end{pmatrix} e_R \nu_L, \quad (p = \begin{pmatrix} \overline{p} & 0 \end{pmatrix}, q = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}) \in A_{ew},
\]

(10)

\[
D_l = \begin{pmatrix} e_R & \nu_L & e_L \\ 0 & 0 & M_e^* \\ M_e & 0 & 0 \end{pmatrix} e_R \nu_L.
\]

(11)

**Quarkonic K-cycle:** Hilbert space:

\[
H_q = (C_R^q \oplus C_L^q) \otimes C^N,
\]

Operators (endomorphisms of $H_q$ as $4 \times 4$ matrices with entries in $M_N(C)$):

\[
\chi_q = \begin{pmatrix} u_R & d_R & u_L & d_L \\ 1_N & 0 & 0 & 0 \\ 0 & 1_N & 0 & 0 \\ 0 & 0 & -1_N & 0 \end{pmatrix} u_R d_R \nu_L.
\]

(13)

\[
\pi_q((p, q)) = \begin{pmatrix} u_R & d_R & u_L & d_L \\ \overline{p}1_N & 0 & 0 & 0 \\ 0 & p1_N & 0 & 0 \\ 0 & 0 & a1_N & b1_N \end{pmatrix} u_R d_R \nu_L, \quad (p = \begin{pmatrix} \overline{p} & 0 \end{pmatrix}, q = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}) \in A_{ew},
\]

(14)

\[
D_q = \begin{pmatrix} u_R & d_R & u_L & d_L \\ 0 & 0 & M_u^* & 0 \\ 0 & 0 & 0 & M_d^* \\ M_u & 0 & 0 & 0 \end{pmatrix} u_R d_R \nu_L.
\]

(15)

Remark: One passes from the matrices (5), (6), (7) to the matrices (8), (9), (10) through the changes $M_u \to 0$, $M_d \to M_e$ followed by restriction to the right-lower corner $3 \times 3$ matrix. This procedure applied to a $3 \times 3$ depending upon $M_u$ and $M_d$ is called **leptonic reduction**. We will in fact also use the following
**Two-by-two matrix versions. Quark sector:** version with $2 \times 2$ matrices with entries in $M_2(C) \otimes M_N(C)$, corresponding to the decomposition:

\begin{equation}
\pi_q((p, q)) = \begin{pmatrix} R & L \\ p \otimes 1_N & q \otimes 1_N \end{pmatrix} R L,
\end{equation}

\begin{equation}
D_q = \begin{pmatrix} R & L \\ 0 & M^* \end{pmatrix} R L,
\end{equation}

\begin{equation}
\chi_q = \begin{pmatrix} 1 \otimes 1_N & 0 \\ 0 & -1 \otimes 1_N \end{pmatrix} R L.
\end{equation}

**Lepton sector:** version with $2 \times 2$ matrices with entries

\begin{equation}
\begin{pmatrix} M_1 \otimes M_N(C) & M(C^2, C) \otimes M_N(C) \\ M(C, C^2) \otimes M_N(C) & M_2(C) \otimes M_N(C) \end{pmatrix} ;
\end{equation}

\begin{equation}
p_l((p, q)) = \begin{pmatrix} R & L \\ p \otimes 1_N & q \otimes 1_N \end{pmatrix} R L, \quad \left( p \in C, \ q = \begin{pmatrix} a \\ -b \end{pmatrix} \right) \in A_{ew},
\end{equation}

\begin{equation}
D_l = \begin{pmatrix} R & L \\ 0 & (0 M_e^*) \end{pmatrix} R L.
\end{equation}

\begin{equation}
\chi_l = \begin{pmatrix} 1 \otimes 1 & 0 \\ 0 & -1 \otimes 1_N \end{pmatrix} R L.
\end{equation}

**Coloured Poincaré-dual** $A_{ew} \otimes B_{chrom}$-**K-cycles** $(H_l, D_l, \chi_l)$ and $(H_q, D_q, \chi_q)$.

**The algebra $B_{chrom}$**.

\begin{equation}
B_{chrom} = C \oplus M_3(C) = \{(p', M); p \in C, \ m \in M_3(C)\}
\end{equation}

\begin{equation}
(p', m)^* = (\overline{p}', m^*), \quad p' \in C, \ m \in M_3(C),
\end{equation}

\begin{equation}
G_{chrom} = \{u' = (u', v) \in A_{ew}; u'u = uu^* = 1, v^*v = vv^* = 1\} = U(1) \times U(3).
\end{equation}

**The leptonic and quarkonic** $A_{ew} \otimes B_{chrom}$-**K-cycles** $(H_l, D_l, \chi_l)$ and $(H_q, D_q, \chi_q)$.

**Leptonic K-cycle** $(H_l, D_l, \chi_l)$:

\begin{equation}
\begin{cases}
H_l = H_l \otimes C_{chrom}, & \chi_l = \chi_l \otimes 1_{chrom} \\
D_l = D_l \otimes 1_{chrom} \\
\pi_l(p, q) = \pi_l(p, q) \otimes 1_{chrom} \\
\pi_l(p', m) = 1_l \otimes p' = p'
\end{cases}
\end{equation}
Quarkonic K-cycle $(H_q, D_q, \chi_q)$:

\[
\begin{align*}
H_q &= H_q \otimes C_{\text{chrom}}^3, \\
D_q &= D_q \otimes 1_{\text{chrom}}, \\
\pi_q(p, q) &= \pi_q(p, q) \otimes 1_{\text{chrom}}, \\
\pi_q(p', m) &= 1_q \otimes m
\end{align*}
\]  

(note the relation $[D_l, \pi_l(a)] = [D_q, \pi_q(a)] = 0$ implying the algebraic Poincaré duality condition).

We recall the formulae (concerning the quark sector):

\[
M = \begin{pmatrix} M_u & 0 \\ 0 & M_d \end{pmatrix} = E \otimes M_u + F \otimes M_d \\
\text{with } E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } F = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]  

COMMUTANTS OF THE INNER SPACE ELECTROWEAK, RESP. CHROMODYNAMICS K-CYCLE.

We now analyze the commutants of the inner space electroweak, resp. chromodynamics K-cycle, calling so the respective subalgebras of $\text{End} (H \oplus H_q)$ consisting of the elements commuting with $D_l \oplus D_q$ and with all $\pi_l(a) \oplus \pi_q(a)$, $a \in A_{\text{ew}}$, resp. all $\pi_l(b) \oplus \pi_q(b)$, $b \in B_{\text{chrom}}$.

We begin with a remark relative to a notation which we shall use in order to spare writing:

[1] Remark.

With $a$ and $b$ linear operators of the respective complex vector spaces $H$ and $K$, we write $\text{Int} (a, b)$ for the set of linear maps: $H \to K$ intertwining $a$ and $b$:

\[
\text{Int} (a, b) = \{ S \in \text{End} (H, K); Sa = bS \}.
\]  

We then have that:

(i): With $a, H$ and $b, K$ as above, and using a $2 \times 2$ matrix notation for the endomorphisms of $H \oplus K$, we have that:

\[
\text{Int} (a \oplus b, a \oplus b) = \begin{pmatrix} \text{Int} (a, a) & \text{Int} (a, b) \\ \text{Int} (b, a) & \text{Int} (b, b) \end{pmatrix}.
\]  

(ii): With $a, H$ and $b, K$ as above, $H$ and $K$ finite-dimensional, and $a$ and $b$ self-adjoint with non-intersecting respective sets of eigenvalues, we have $\text{Int} (a, b) = \{0\}$.  

5
Proof:

(i): follows from:

\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} \cdot 
\begin{pmatrix}
  S & 0 \\
  0 & T
\end{pmatrix} = 
\begin{pmatrix}
  aS - Sa & bT - Sb \\
  cS - Tc & dT -Td
\end{pmatrix},
\]

(ii): With \( S = (S_i^k) \), \( a = (\lambda_i^k) \), \( b = (\mu_i^k) \), we have

\[
(Sa - bS)^i_k = \Sigma_h \left( S_h^i \lambda_h^k \delta_h^i - \mu_h^k \delta_h^i S_h^k \right) = (\lambda_k - \mu_i) S_i^k = 0.
\]

In what follows we shall comply to the common usage of choosing our fermion mass-matrices such that \( M_e \) and \( M_u \) are diagonal, positive matrices, whilst \( M_d = C|M_d| \), with \( C \) (the \textbf{Kobayashi-Maskawa matrix}) unitary and \( |M_d| \) strictly positive. Furthermore we assume that all fermion masses are different (the eigenvalues of \( M_e \), \( M_u \) and \( |M_d| \) consists of positive numbers (the masses of leptons and quarks) all different from one another – experiment!). We further assume that no eigenstate of \( |M_d| \) is an eigenstate of \( C \) (experiment!). We use the shorthands:

\[
\begin{align*}
\mu &= MM^*, & \bar{\mu} &= M^*M, \\
\mu_e &= M_e^2, & \mu_u &= M_u^2, \\
\mu_d &= M_dM_d^*, & \bar{\mu}_d &= M_d^*M_d,
\end{align*}
\]

we then have:

\[
\bar{\mu}_d = |M_d|^2 = C\mu_d C^*,
\]

\[(33a) \] \( \bar{\mu} = |M|^2 = C\mu C^*, \)

where \( C = id \oplus C \) in the second line.

[2]  \textbf{Lemma.}

We have that:

(i): The most general self-adjoint element \( \Gamma'_l \) of \( \text{Int} (D_l, D_l) \) is as follows: one has in 3 x 3 matrix notation:

\[
\Gamma'_l = \begin{pmatrix}
  h(\mu_e) & 0 & k(\mu_e) \\
  0 & \delta & 0 \\
  k(\mu_e) & 0 & h(\mu_e)
\end{pmatrix},
\]

where \( h \) and \( k \) are arbitrary real functions, and \( \delta \) is any self-adjoint element of \( M_N(C) \).
(ii): The most general self-adjoint element $\Gamma_q'$ of $\text{Int} (D_q, D_q)$ is as follows: one has in $4 \times 4$ matrix notation:

$$
\Gamma_q' = \begin{pmatrix}
    f(\mu_u) & 0 & l(\mu_u) & 0 \\
    0 & g(\mu_d) & 0 & m(\mu_d)C^* \\
    l(\mu_u) & 0 & f(\mu_u) & 0 \\
    0 & C\mu(\mu_d) & 0 & Cg(\mu_d)C^*
\end{pmatrix},
$$

where $f, g, l$ and $m$ are arbitrary real functions. Thus the most general self-adjoint element $\Gamma_q'$ of $\text{Int} (D_q, D_q)$ is of the form $\Gamma_q' \otimes S$ with $\Gamma_q'$ as in (35) and $S \in M_N(C)$ self-adjoint.

(iii): The self-adjoint elements of $\text{Int} (D_q, D_q)$ or of $\text{Int} (D_q, D_i)$ vanish. The same holds for self-adjoint elements of $\text{Int} (D_q, D_q)$ or of $\text{Int} (D_q, D_i)$.

Proof:

(i): With $\Gamma_q' = \begin{pmatrix}
    \alpha & \beta & \mu \\
    \beta & \delta & \nu \\
    \nu & \sigma & \sigma
\end{pmatrix}$, in $3 \times 3$ matrix notation, equating:

$$
D_q\Gamma_q' = \begin{pmatrix}
    0 & 0 & M_e \\
    0 & 0 & 0 \\
    M_e & 0 & 0
\end{pmatrix}\begin{pmatrix}
    \alpha & \beta & \mu \\
    \beta & \delta & \nu \\
    \nu & \sigma & \sigma
\end{pmatrix} = \begin{pmatrix}
    M_e\mu & M_e\nu & M_e\sigma \\
    0 & 0 & 0 \\
    M_e\alpha & M_e\beta & M_e\mu
\end{pmatrix},
$$

and

$$
\Gamma_q'D_q = \begin{pmatrix}
    \alpha & \beta & \mu \\
    \beta & \delta & \nu \\
    \nu & \sigma & \sigma
\end{pmatrix}\begin{pmatrix}
    0 & 0 & M_e \\
    0 & 0 & 0 \\
    M_e & 0 & 0
\end{pmatrix} = \begin{pmatrix}
    \mu M_e & 0 & \alpha M_e \\
    \nu M_e & 0 & \nu M_e \\
    \sigma M_e & 0 & \sigma M_e
\end{pmatrix},
$$

yields $\beta = \nu = 0$; further $M_e\mu = \mu M_e$ and $M_e\mu = \mu M_e$, whence $M_e^2 = M_e \mu M_e = \mu M_e^2$, whence $\mu = k(M_e)$, $\nu = k'(M_e)$ for some functions $k, k'$, with in addition $k = k'$ since the relation $M_e\mu = \mu M_e$ now reads $M_e(k(M_e) - k'(M_e)) = 0$; finally $M_e\alpha = \sigma M_e$ and $M_e\sigma = \alpha M_e$, whence $M_e^2 = M_e \sigma M_e = \alpha M_e^2$, whence $\alpha = h(M_e^2)$ which also equals $\sigma$ because the relation $M_e\alpha = \sigma M_e$ now reads $\alpha M_e = \sigma M_e$.

(ii): We have, using the $2 \times 2$ matrix notation, with $\Gamma_q' = \begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix}$, $a, b, c, d \in M_2(C) \otimes M_N(C)$:

$$
[\Gamma_q', D_q] = \begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix} \begin{pmatrix}
    0 & M^* \\
    M & 0
\end{pmatrix} = \begin{pmatrix}
    bM - M^*c & aM^* - M^*d \\
    dM - Ma & cM^* - Mb
\end{pmatrix},
$$

we must thus have:

$$
\begin{align*}
\text{(39a)} & \quad dM - Ma = 0, \\
\text{(39b)} & \quad aM^* - M^*d = 0, \\
\text{(39c)} & \quad bM - M^*c = 0, \\
\text{(39d)} & \quad cM^* - Mb = 0,
\end{align*}
$$
We compute $a$ and $d$: \textbf{(39a)} and \textbf{(39b)} entail:

\[(40)\quad aM^*M = M^*Ma, \quad dMM^* = MM^*d.\]

whence the existence of a real function $F$ with $a = F(\bar{\mu})$.\textbf{(39a)} then reads $dC|M| = C[M]F(\bar{\mu}) = CF(\bar{\mu})|M|$ whence $dC = CF(\bar{\mu})$, $d = CF(\bar{\mu})C^\ast$.

We now compute $b$ and $c$: \textbf{(39c)} and \textbf{(39d)} entail $M^*Mb = M^*cM^* = bMM^*$, $\bar{\mu}b = b\mu = bC\bar{\mu}C^\ast$, $\bar{\mu}bC = bC\bar{\mu}$, whence the existence of a real function $G$ with $b = G(\bar{\mu})C^\ast$.

For $c = b^\ast$, $G$ is real: indeed \textbf{(39d)} then reads $Mb = C|M|b = b^\ast M^* = b^\ast|M|C^\ast$, i.e. $|M|bC = C^*b^\ast|M|$, whence, since $|M|bC = b|C|M$, $bC = C^*b^\ast$.

We showed the existence of real functions $F$ and $G$ for which:

\[(41)\quad \Gamma'_q = \begin{pmatrix} F(\bar{\mu}) & G(\bar{\mu})C^\ast \\ CG(\bar{\mu}) & CF(\bar{\mu})C^\ast \end{pmatrix},\]

which comes to \textbf{(B3)} in $4 \times 4$ matrix notation.

\[\text{(iii): Let } S = \begin{pmatrix} \alpha' & \beta & \gamma & \delta \\ \alpha'' & \beta' & \gamma' & \delta' \\ \alpha''' & \beta'' & \gamma'' & \delta'' \end{pmatrix} \in \text{Int}(D_q, D_1): \text{equating} \]

\[(42) \quad \begin{pmatrix} 0 & 0 & M_u & 0 \\ 0 & 0 & 0 & M_e \\ M_e & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \\ \alpha'' & \beta'' & \gamma'' & \delta'' \end{pmatrix} = \begin{pmatrix} M_e\alpha'' & M_e\beta'' & M_e\gamma'' & M_e\delta'' \\ 0 & 0 & 0 & 0 \end{pmatrix},\]

and

\[(43) \quad \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \\ \alpha'' & \beta'' & \gamma'' & \delta'' \end{pmatrix} \begin{pmatrix} 0 & 0 & M_u & 0 \\ 0 & 0 & 0 & M^*_d \\ M_u & 0 & 0 & 0 \\ 0 & M_d & 0 & 0 \end{pmatrix} = \begin{pmatrix} \gamma M_u & \delta M_d & \alpha M_u & \beta M^*_d \\ \gamma' M_u & \delta' M_d & \alpha' M_u & \beta' M^*_d \\ \gamma'' M_u & \delta'' M_d & \alpha'' M_u & \beta'' M^*_d \end{pmatrix},\]

yields $\alpha' = \beta' = \gamma' = \delta' = 0$; further $M_d\alpha'' = \gamma M_u$ and $M_e\gamma = \alpha''M_u$ whence $M_d\gamma = M_e\alpha''M_u = \gamma M^*_u$ implying $\gamma = 0$ by \textbf{[I]}(ii), and thus $\alpha'' = 0$ – the changes $\alpha'' \rightarrow \gamma''$ and $\gamma \rightarrow \alpha$ then yielding $\alpha = \gamma'' = 0$; analogously $M_d\beta'' = \delta M_d$ and $M_e\delta = \beta''M^*_d$ whence $M_d\beta'' = M_e\delta M_d = \beta''M^*_dM_d$ implying $\beta'' = 0 = \delta$; the changes $\beta'' \rightarrow \delta''$ and $\delta \rightarrow \beta$ then yielding $\beta = \delta'' = 0$. We proved that $S = 0$. For $S \in \text{Int}(D_q, D_2)$, writing $H_q \cong C^3$ as a direct sum, the restrictions of $S$ to all summands vanish.

\[\textbf{[3]} \quad \text{Lemma.}\]

\begin{itemize}
  \item[(i):] The commutant of the $K$-cycle $(H_q, D_q, \chi_q)$ of $A_{ew}$ coincides with $\mathbf{1}_2 \otimes \mathbf{1}_N \otimes M_3(C)$ \[\textbf{[F]}\]
\end{itemize}

\[\text{We recall that we call commutant of the $K$-cycle $(H, D, \chi)$ of an algebra } A \text{ the set of operators of } H \text{ commuting with } \pi(A) \text{ and } D.\]

\[\text{\textbf{[F]} } 1_2 \text{ denotes here the unit of the } 2 \times 2 \text{ matrix algebra with entries in } M_2(C). \text{ Note that in fact the colorless } K\text{-cycle } (H_q, D_q, \chi_q) \text{ is irreducible in the sense that the only operators of } H_q \text{ commuting with } \pi_q(A_{ew}) \text{ and } D_q \text{ are the scalars. Equivalently } \pi_q(A_{ew}) \text{ and } D_q \text{ generate } B(H_q).\]

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(ii): A self-adjoint $\Gamma_l$ acting on $H_l$ belongs to the commutant of the $K$-cycle $(H_l, D_l, \chi_l)$ of $A_{ew}$ iff it is of the type $1_2 \otimes \Gamma_N$, with $\Gamma_N$ a real function of $M_e$.

(iii): The commutant of the $K$-cycle $(H_q, D_q, \chi_q)$ of $B_{chrom}$ coincides with $\text{Int}(D_q, D_q) \otimes 1_3$.

(iv): The commutant of the $K$-cycle $(H_l, D_l, \chi_l)$ of $B_{chrom}$ coincides with $\text{Int}(D_l, D_l)$.

Proof:

(i): If $S$ commutes with $D_q$ and $\pi_q(A_{ew})$, it is of the form $S \otimes T, T \in M_N(C)$, $S$ commuting with $D_q$ and $\pi_q(A_{ew})$. $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in M_2(C) \otimes M_N(C)$ is then both of the form (35) and fulfills for all $p \in H_{\text{diag}}, q \in H$:

\begin{align*}
(a, p \otimes 1_N) = 0, \\
(d, q \otimes 1_N) = 0, \\
b(q \otimes 1_N) - (p \otimes 1_N)b = 0, \\
c(p \otimes 1_N) - (q \otimes 1_N)c = 0,
\end{align*}

(44a) \hspace{1cm} (44b) \hspace{1cm} (44c) \hspace{1cm} (44d)

Now (44c), resp. (44d), with $p = 0$ yield $b(q \otimes 1_N) = 0$, resp. $(q \otimes 1_N)c = 0$ whence $b = c = 0$ since $q \otimes 1_N$ is invertible. Since $H$ generates $M_2(C)$ linearly, (44d) further implies that $d = 1 \otimes 1$, with $1 \in M_N(C)$. Together these imply that $l = m = 0, f(\mu_e) = C_q(\tilde{\mu}_d)C^*$ being a multiple of the identity: this then also holds for $S$.

(ii): If $S$ commutes with $D_l$ and $\pi_l(A_{ew})$, it is of the form (34) and is contained in

\begin{align*}
\pi_l(A_{ew})' = \left\{ \begin{pmatrix} 1_1 \otimes X & 0 \\ 0 & 1_2 \otimes Y \end{pmatrix}; X, Y \in M_N(C) \right\},
\end{align*}

(45)

(cf. (1)(i)). Both facts together imply $k(\mu_e) = 0$, and $h(\mu_e) = \delta = X = Y$, whence the claim.

[4] Proposition.

We have that:

(i): The positive elements of the commutant of the $K$-cycle $(H_q \oplus H_l, D_q \oplus D_l, \chi_q \oplus \chi_l)$ of $A_{ew}$ are the operators of the form

\begin{align*}
\Gamma = \begin{pmatrix} \Gamma_q & 0 \\ 0 & \Gamma_l \end{pmatrix},
\end{align*}

(46)

$81_2$ denotes here the unit $2 \times 2$ matrix of the type $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a \in M_{1} \otimes M_N(C), b \in M(C^2, C) \otimes M_N(C), c \in M(C, C^2) \otimes M_N(C), d \in M_2(C) \otimes M_N(C)$. Note that $A$ belongs to the commutant of $(H_l, D_l, \chi_l)$ iff one has $b = c = 0$ and $a = 1 \otimes \Gamma_N, d = 1 \otimes \Gamma_N$ with $\Gamma_N$ an arbitrary function of $M_e$. 

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with \( \Gamma_q \in 1_2 \otimes 1_N \otimes M_3(C)^+ \) and \( \Gamma_q \) of the type \( 1_2 \otimes \Gamma_N \), with \( \Gamma_N \) a positive function of \( M_e \).

(ii): Consequently \( \Gamma \) in (47) belongs to the center of \((\pi \oplus \pi_q)(A_{ew})\) iff \( \Gamma_l = \lambda 1_{H_l} \) and \( \Gamma_q = \lambda 1_{H_q} \) for some real \( \lambda \).

(iii): The positive elements of the commutant of the \( K \)-cycle \((H_q \oplus H_l, D_q \oplus D_l, \chi_q \oplus \chi_l)\) of \( B_{chrom} \) are the operators of the form

\[
\Gamma' = \begin{pmatrix} \Gamma'_q & 0 \\ 0 & \Gamma'_l \end{pmatrix},
\]

where \( \Gamma'_q = \Gamma_q \otimes 1_3 \) and \( \Gamma'_l = \Gamma_l \), with \( \Gamma'_q \) positive as in (34) and \( \Gamma'_l \) positive as in (32).

(iv): Consequently \( \Gamma' \) in (47) belongs to the center of \((\pi_l \oplus \pi_q)(B_{chrom})\) iff one has \( \Gamma'_l = \lambda' 1_{H_l} \) and \( \Gamma'_q = \lambda' 1_{H_q} \) for some real \( \lambda' \).

Proof:

(i): (resp. (iii)): [1](i) together with [2](i)(ii) (resp. [3](ii)(iv) and [2](i)(iii)), leads to the the diagonal elements of the matrix (10) (resp. (16)); and together with [2](ii), it leads to the vanishing of its off-diagonal elements.

(ii): Asking simultaneously for \( \pi_q(p, q) = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \otimes 1_N \otimes 1_3 \in 1_2 \otimes 1_N \otimes M_3(C)^+ \) and \( \pi_l(p, q) = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \otimes 1_N = 1_2 \otimes \Gamma_N \) requires \( \Gamma_N = \lambda 1_N \) and \( p = q = \lambda \) for some real \( \lambda \).

(iii): Asking simultaneously for \( \pi_l(p', m) = p' 1_2 \otimes 1_N = \Gamma'_l = \begin{pmatrix} h(\mu_e) & 0 & k(\mu_e) \\ 0 & \delta & 0 \\ k(\mu_e) & 0 & h(\mu_e) \end{pmatrix} \) and

\[
\pi_q(p', m) = 1_2 \otimes 1_N \otimes m = \Gamma'_q = \begin{pmatrix} f(\mu_u) & 0 & l(\mu_u) \\ 0 & g(\bar{\mu}_d) & 0 & m(\bar{\mu}_d) C^* \\ l(\mu_u) & 0 & f(\mu_u) \\ 0 & C m(\bar{\mu}_d) & 0 & C g(\bar{\mu}_d) C^* \end{pmatrix} \otimes 1_3
\]

implies vanishing of \( k(\mu_e), l(\mu_u) \) and \( m(\bar{\mu}_d) \), and equality of \( h(\mu_e), \delta \) to \( \lambda' 1_N \) and of \( f(\mu_u), g(\bar{\mu}_d) \) to \( \lambda'' 1_N \).

Knowing the commutants of the \( K \)-cycle \((H_q \oplus H_l, D_q \oplus D_l, \chi_q \oplus \chi_l)\) of \( A_{ew} \) and of \( B_{chrom} \) we know the general form of the new scalar products (3a), (3b). We can thus proceed to our computation of the Yang-Mills action. We begin with the electroweak sector. We first calculate the projection \( P_2 \) key to the computation of quantum two-forms, this will enable us to compute the electroweak curvature corresponding to the new scalar product (3a), and the

\[9\] The \( 2 \times 2 \) matrix corresponds to the decomposition \( H_q \oplus H_l \). Note that, as a consequence, we should in Ansatz (3a), (3b) plug in \( \Gamma_q = 1_2 \otimes 1_N \otimes \Gamma_3 \) with \( \Gamma_3 \in M^+(C^3) \), and \( \Gamma_l = 1_2 \otimes \Gamma_N \) with \( \Gamma_N \) a positive function of \( M_e \).
new form of the electroweak Yang-Mills action (differing slightly from that we had with the scalar product \( \mathcal{A}^2 \), the deviation being negligible in the valid approximation where all fermion masses are neglected against that of the top quark). We then compute the same objects for the chromodynamics sector for which the projection \( P_2 \) reduces to its spatial tensorial part due to the fact that the gluons are of vectorial type, the Yang-Mills action needs however to be computed anew. We then combine the electroweak and chromodynamics sectors making use of the modular condition: this yields the total Yang-Mills action, comprising terms identical to those of the (bosonic) action of the traditional full standard model, however with interesting constraints which we discuss at the end.

**COMPUTATION OF THE ELECTROWEAK PROJECTION \( P_2 \).**

**[5] Proposition.**

(canonical representant of quantum two-forms).

Consider the class \( \pi_D(\Omega^2) \) modulo \( \pi_D(\delta K^1) \), indexed by \( (\lambda, \mu, q, q', Q, Q') \), consisting of the direct sum of the elements

\[
\begin{align*}
\eta_{[2,0]} &= \left( \begin{array}{c|c}
\gamma(\lambda_k) - (X_k^i) & 0 \\
0 & \gamma(\mu_k) - (Y_k^i)
\end{array} \right) 1_N \\
\eta_{[1,1]} &= \left( \begin{array}{cc}
0 & M^* i\gamma(q') \gamma^5 \\
M i\gamma(q) \gamma^5 & 0
\end{array} \right) 1_N \\
\eta_{[0,2]} &= \left( \begin{array}{cc}
M^* (Q \otimes 1_N) M & 0 \\
0 & Q' \otimes \Sigma + iQ'' \otimes \Delta
\end{array} \right)
\end{align*}
\]

(48)

with

\[
\begin{align*}
(\lambda_k^i) & \in \Omega (M, H_{\text{diag}})^2 \\
(X_k^i) & \in C^\infty (M, H_{\text{diag}}) \\
(\mu_k^i) & \in \Omega (M, H)^2 \\
(Y_k^i) & \in C^\infty (M, H) \\
q, q', Q, Q' & \in \Omega (M, H) \\
Q, Q', Q'' & \in C^\infty (M, H)
\end{align*}
\]

and their leptonic reduction, where one fixes \( (\lambda_k^i), (\mu_k^i), q, q', Q, Q' \), and lets \( (X_k^i), (Y_k^i), Q'' \) range through all possible values.

And take as scalar product the convex combination \( \alpha_q(\omega, \omega') + \alpha_l(\omega, \omega')_l \) of the scalar products (34) with the choices \( \Sigma = 1_2 \otimes 1_N \otimes \Gamma_3, \Gamma_3 \in M_3(C) \) positive with \( Tr_3 \Gamma_3 = 3 \), and \( \Sigma = 1_2 \otimes \Gamma_N, \Gamma_N a \text{ positive function of } M_e \text{ with } Tr_N \Gamma_N = N \).\(^{10}\)

\(^{10}\) As already noted these choices encompass the special cases \( \Gamma_3 = 1_3 \) and \( \Gamma_N = 1_N \) corresponding to the previous Ansatz (3).
Now the canonical representant of this class \((\mathfrak{R})\) (obtained by projecting orthogonally parallel to \((\pi_q \oplus \pi_\ell)(\delta K^1)\) and indexed by \((\lambda, \mu, q, q', Q, Q')\) is given as follows: make in \((\mathfrak{R})\):

\[
\begin{cases}
X = (\alpha_l + 6\alpha_q)^{-1} N^{-1} L Q_{\text{diag}} \\
Y_k^i = (\alpha_l + 3\alpha_q)^{-1} (2N)^{-1} L Q_k^{i'} \\
Q'' = 0
\end{cases}
\]

(49)

where:

\[L = Tr [3\alpha_q(\mu_u + \mu_d) + \alpha_l \Gamma_n \mu_e].\]

Note that these results are obtained from those of the previous work \([10]\) by effecting the change \(\mu_e \rightarrow \Gamma_n \mu_e\). This will allow to exploit them easily.

**Proof:** One finds the representative \((\lambda, \mu, q, q', Q, Q')\) by asking the direct sum of the element \((\mathfrak{R})\) and its leptonic reduction to be orthogonal to the direct sum of all elements

\[
\begin{pmatrix}
(S_k^i) \otimes 1_N \\
0 \otimes 1_N
\end{pmatrix} + \begin{pmatrix} 0 & 0 \\
0 & iR \otimes \Delta
\end{pmatrix}
\]

(51)

\[
(S_k^i) \otimes 1_N \in C^\infty(M, H_{\text{diag}})
\]

and their leptonic reductions: this amounts to \([1]\)

\[
0 = \alpha_q \text{Re} \left( (Tr_2 \otimes Tr_3) \left[ \left[ (X_k^i) \otimes 1_N - M^*(Q \otimes 1_N) M \right] (S_k^i) \otimes 1_N \right] + \left[ (Y_k^i) \otimes 1_N - Q' \otimes \Sigma - iQ'' \otimes \Delta \right] (T_k^i) \otimes 1_N + iR \otimes \Delta \right) \right) (1_2 \otimes 1_N \otimes \Gamma_3)
\]

(52)

\[
+ \alpha_l \text{Re} \left( (Tr_2 \otimes Tr_3) \left[ \{\text{leptonic reduction}\} \ (1_2 \otimes \Gamma_N) \right],
\]

(we omitted the purely imaginary term \((Tr_2 \otimes Tr_3) \ i(Y_k^i)R \otimes \Delta - iQ'R \otimes \Sigma \Delta - iQ''(T_k^i) \otimes \Delta\) vanishing under \(\text{Re}\)). This yields (with independent vanishing of the three terms in \((S_k^i), \ (T_k^i),\) and \(R\)):

\[
0 = 3\alpha_q \cdot \text{Re} (Tr_2 \otimes Tr_3) \left[ \left[ (X_k^i) (S_k^i) \otimes 1_N - M^*(S_k^i) Q \otimes 1_N M \right] + \left[ (Y_k^i) (T_k^i) \otimes 1_N - Q' (T_k^i) \otimes \Sigma \right] + \left[ Q'' R \otimes \Delta^2 \right] \right] + \alpha_l \text{Re} (Tr_2 \otimes Tr_3) \left[ \{\text{leptonic reduction}\} \ (1_2 \otimes \Gamma_N) \right]
\]

(53)

— Vanishing of the \((S_k^i)\)-term: we have, with \((X_k^i) = \begin{pmatrix} X & 0 \\
0 & X \end{pmatrix}, \ (S_k^i) = \begin{pmatrix} S & 0 \\
0 & S \end{pmatrix}, \ Q = \begin{pmatrix} Q'_2 & Q_1 \\
-Q_2 & Q_1 \end{pmatrix}\):

\[\text{We here use the fact that the traces} \ \tau_D, \ \tau_{D'}, \ \text{are proportional to the spatial integrals of} \ Tr_2 \otimes Tr_3, \ \text{resp.} \ Tr_2 \otimes Tr_N \ \text{(note that the latter vanishes on} \chi_q\text{-odd elements).}\]
\[ 0 = Re(Tr_2 \otimes Tr_N) \left\{ 3\alpha_q \left( \begin{pmatrix} XS1_N & 0 \\ 0 & XS1_N \end{pmatrix} - \begin{pmatrix} SQ_qM^*_u & SQ_qM^*_d \\ -SQ_qM^*_d & SQ_qM^*_u \end{pmatrix} \right) \right\} + \alpha_q \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} SQ_qM^*_e & 0 \\ 0 & SQ_qM_q^*_e \end{pmatrix} \right) \right\}(\Gamma_N 0 \quad 0 \quad \Gamma_N) \]
\[ = Re(Tr_2 \otimes Tr_N) \left\{ 3\alpha_q \left( \begin{pmatrix} XS1_N & 0 \\ 0 & XS1_N \end{pmatrix} - \begin{pmatrix} SQ_qM^*_u & SQ_qM^*_d \\ -SQ_qM^*_d & SQ_qM^*_u \end{pmatrix} \right) \right\} + \alpha_q \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} SQ_qM^*_e & 0 \\ 0 & SQ_qM_q^*_e \end{pmatrix} \right) \right\}(\Gamma_N 0 \quad 0 \quad \Gamma_N) \]

\[ = Re \left\{ 3\alpha_q \left[ NXS + XS - Tr\mu_u \cdot SQ_q - Tr\mu_d \cdot SQ_q \right] \right\} + \alpha_q \left[ NXS - SQ_q Tr(\Gamma_N \mu_e) \right] \]
\[ = 3\alpha_q \left[ 2NXS - (Tr\mu_u + Tr\mu_d)SQ_q \right] + \alpha_q \left[ NXS - SQ_q Tr(\Gamma_N \mu_e) \right] \]
\[ = S \left\{ 3\alpha_q \left[ 2NX - (Tr\mu_u + Tr\mu_d)Q_2 \right] + \alpha_q \left[ N \mu - SQ_q Tr(\Gamma_N \mu_e) \right] \right\} \]
\[ = S \left\{ 6\alpha_q + \alpha_q \right\} N \mu - Tr \left\{ 3\alpha_q (\mu_u + \mu_d) + \alpha_q \Gamma_N \mu_e \right\} \]
\[ = S \left\{ 6\alpha_q + \alpha_q \right\} N \mu - LQ_2, \]

whence the relation first line in (49).

— Vanishing of the \( T^i_k \)-term: we have:

\[ 0 = Re(Tr_2 \otimes Tr_N) \left\{ 3\alpha_q \left[ (Y^i_k T^i_k) \otimes 1_N - Q^i_k T^i_k \otimes \Sigma \right] \right\} + \alpha_q \left[ (Y^i_k T^i_k) \otimes 1_N - Q^i_k T^i_k \otimes \frac{1}{2} \mu_e \right] (1_2 \otimes \Gamma_N) \]
\[ = Re \left\{ 3\alpha_q \left[ NY^i_k T^i_k - \frac{1}{2} Tr(\mu_u + \mu_d)Q^i_k T^i_k \right] \right\} + \alpha_q \left[ NY^i_k T^i_k - \frac{1}{2} Tr(\Gamma_N \mu_e)Q^i_k T^i_k \right] \]
\[ = Re \left\{ T^i_k \left\{ 3\alpha_q + \alpha_q \right\} NY^i_k - \frac{1}{2} Tr \left\{ 3\alpha_q (\mu_u + \mu_d) + \Gamma_N \mu_e \right\} Q^i_k \right\} \]
\[ = Re \left\{ T^i_k \left\{ 3\alpha_q + \alpha_q \right\} NY^i_k - \frac{1}{2} LQ^i_k \right\} \]

whence the relation second line in (49).

— Vanishing of the \( R \)-term: \( Re(Tr_2 \otimes Tr_N) \left[ Q''R \otimes (\alpha_q \Delta^2 + \alpha_q \mu_u^2) \right] \) vanishes for all \( R \) iff \( Q'' = 0 \), the relation third line in (49).
[6] Proposition.

Consider the class $\pi_D(\Omega B^2)$ modulo $\pi_D(\delta K^1)$, indexed by $(g^i_k, f')$, consisting of the direct sum of the elements

\begin{align}
(56) & \quad 1_{H_q} \otimes 1_N \otimes \left[ \gamma(g^i_k) + X'^i \right], \quad \begin{cases} (g^i_k) \in \Omega(\mathbb{M}, M_3(C))^2 \\ (X'^i) \in C^\infty(\mathbb{M}, M_3(C)) \end{cases}, \\
(57) & \quad 1_{H_i} \otimes 1_N \otimes \left[ \gamma(f') + X' \right], \quad \begin{cases} f' \in \Omega(\mathbb{M}, C)^2 \\ X' \in C^\infty(\mathbb{M}, C) \end{cases},
\end{align}

where one fixes $g^i_k$, $f'$, and lets $X'^i$, $X'$ range through all possible values. And take as scalar product the sum of the scalar products \((\Omega)\) with the choices $\Gamma_q' = \Gamma_q \otimes 1_{\text{chrom}}$, $\Gamma_i'$ positive in the commutant of $D_q$, and $\Gamma_i' = \Gamma_i$, $\Gamma_i'$ positive in the commutant of $D_l$.

The canonical representant of this class (obtained by projecting orthogonally parallel to $(\pi_q \oplus \pi_l)(\delta K^1)$ and indexed by $(g^i_k, f')$), is given as follows: make $(X'^i) = 0$ in \((57)\) and $X' = 0$ in \((57)\). In other terms $P_2$ acts only on the space-time tensorial factor.

Proof: Commutation of the generalized Dirac operator with the action of the chromodynamics algebra (cf. \((\mathbb{F})\)) makes the situation easy to handle. Since $B_{\text{chrom}}$ acts irreducibly on $C^3_{\text{chrom}}$, the quarkonic commutant is of the form $\Gamma_q' = \Gamma_q \otimes 1_{\text{chrom}}$, $\Gamma_q'$ in the commutant of $D_q$. And since $B_{\text{chrom}}$ acts on $C_{\text{chrom}}$ by scalars, the leptonic commutant is of the form $\Gamma_i' = \Gamma_i$, $\Gamma_i'$ in the commutant of $D_l$. We recall that, by the commutation of $(\pi_q \oplus \pi_l)(B_{\text{chrom}})$ with $D_q \otimes D_l$, $(\pi_q \oplus \pi_l)((\Omega B^2)^n)$ vanishes for $n \geq 1$, implying the simple situation $\pi_q((\Omega B^2)) = \pi_D((\Omega A)^2) \otimes \pi_{\text{chrom}}(B_{\text{chrom}})$ and $\pi_l((\Omega B^2)) = \pi_D((\Omega A)^2) \otimes \pi_{\text{chrom}}(B_{\text{chrom}})$, i.e. the form \((\mathbb{F})\), \((\mathbb{G})\) for the Hilbert space representant of formal two-forms. Together with the product structure \((\mathbb{I})\) of the traces, this implies that, for the computation of $P_2$ and of the chromodynamics part of the action, $\Gamma_q$ and $\Gamma_l$ enter only through their traces\((\mathbb{I})\) $Tr_N(\Gamma_q)$ and $Tr_N(\Gamma_l)$. Hence $P_2$ restricts to its space-time tensorial factor. Here is the explicit calculation: $P_2$ is found by asking the direct sum of \((\mathbb{F})\) and \((\mathbb{G})\) with fixed $g^i_k$, $f'$ to be fiberwise orthogonal to $\pi_D(\delta K^1)$, i.e. to all direct sums of

\begin{align}
(58) & \quad 1_2 \otimes 1_N \otimes G^i_k, \quad G^i_k \in C^\infty(\mathbb{M}, M_3(C)), \\
(59) & \quad 1_2 \otimes 1_N \otimes F', \quad F' \in C^\infty(\mathbb{M}, C).
\end{align}

This amounts to vanishing under the Clifford trace, for all $G^i_k$ and $F'$, of the following expressions:

\begin{align}
(60) & \quad Tr(\Gamma_q') \otimes Tr_3 \left\{ 1_{H_q} \otimes 1_N \otimes \left[ \gamma(g^i_k) + X'^i \right] \right\} = Tr(\Gamma_q)G^i_k X'^i, \\
(61) & \quad Tr_7 \otimes Tr(\Gamma_q') \otimes Tr_3 \left\{ 1_{H_i} \otimes 1_N \otimes \left[ \gamma(f') + X' \right] \right\} = Tr(\Gamma_l)F' X',
\end{align}

Thus we do not need to compute their precise form.
this leading indeed to \( X^u_k = X' = 0 \).

Our task is to calculate the Yang-Mills action for the compound electroweak-chromodynamics system, for which the curvature is obtained via modular condition. We first ignore this subtlety and compute the (unphysical) electroweak action. Once this is done; it will be easy to adapt the computation to the full problem.

**COMPUTATION OF THE ELECTROWEAK YANG-MILLS ACTION.**

For the convenience of the reader we reproduce the expression of the curvature \[ HI \]: here is the quark component: with \( L = Tr [3\alpha_q(\mu_u + \mu_d) + \alpha_l \Gamma_N \mu_e] \), \( f \) the \( U(1) \)-curvature, \( h_i^k \) the \( SU(2) \)-curvature, \( \Phi \) the Higgs doublet, and \( V_\Phi = v_\Phi^2 \) the Higgs potential:

\[
\begin{align*}
-\theta_{q[2,0]} &= \left( \begin{array}{cc}
\frac{i}{2} \gamma(f) 1 - (\alpha_l + 6\alpha_q)^{-1} N^{-1} L v_\Phi 1 \\
0 & 0
\end{array} \right) \\
-\theta_{q[1,1]} &= \left( \begin{array}{cc}
0 & M^* [\gamma^5 \gamma(iD\Phi^*) \otimes 1_N] \\
\gamma^5 (iD\Phi) \otimes 1_N & 0
\end{array} \right)
\end{align*}
\]

(62)

from which the leptonic component is obtained by suppression of the first line and the first column of \( 4 \times 4 \) matrices, after the changes:

\[
\begin{align*}
M &= \left( \begin{array}{cc}
\mu_u & 0 \\
0 & \mu_d
\end{array} \right) \rightarrow M = \left( \begin{array}{cc}
0 & 0 \\
0 & \mu_e
\end{array} \right),
\end{align*}
\]

\[
\Sigma = \frac{1}{2} (\mu_u + \mu_d) \rightarrow \Sigma = \frac{1}{2} \mu_e
\]

— Square of upper left corner: the leptonic part:

\[
\left\{ \left[ \frac{i}{2} \gamma(f) - (\alpha_l + 6\alpha_q)^{-1} N^{-1} L v_\Phi \right] \otimes 1_N + v_\Phi M^* M \right\}^2
\]

\[
= \left[ -\frac{1}{4} \gamma(f)^2 + (\alpha_l + 6\alpha_q)^{-2} N^{-2} L^2 V_\Phi \right] \otimes 1_N + V_\Phi M^* M M^* M
\]

\[
-2(\alpha_l + 6\alpha_q)^{-1} N^{-1} L V_\Phi M^* M + \text{terms linear in } \gamma(f)
\]

yielding after the substitutions \[ HI \] under \( \alpha_l \) \( tr_{\text{Clifford}} \otimes Tr_N(\Gamma_N \cdot) \[ 13 \]

\[
\begin{align*}
\alpha_l \left\{ -\frac{1}{2} N f_{\mu\nu} f^{\mu\nu} + (\alpha_l + 6\alpha_q)^{-2} N^{-1} L^2 V_\Phi + Tr \left( \Gamma_N \mu_e^2 \right) V_\Phi \\
-2(\alpha_l + 6\alpha_q)^{-1} N^{-1} L Tr_N(\Gamma_N \mu_e) V_\Phi \right\},
\end{align*}
\]

\[ 13 \] We use the fact that the normalized Clifford trace of \( \gamma(f)^2 \) equals \( 2f_{\mu\nu} f^{\mu\nu} \).
and the quark part:
\[
\left\{ \frac{i}{2} \gamma(f) 1 - (\alpha_l + 6\alpha_q)^{-1} N^{-1} L_{\ell} \right\} \otimes 1_N + v_{\Phi} M^* M \right\}^2 \\
= \left[ -\frac{1}{4} \gamma(f)^2 1 + (\alpha_l + 6\alpha_q)^{-2} N^{-2} L^2 V_{\Phi} 1 \right] \otimes 1_N + V_{\Phi} M^* M M^* M \\
-2(\alpha_l + 6\alpha_q)^{-1} N^{-1} L_{\ell} V_{\Phi} M^* M + \text{terms linear in } \gamma(f)
\]
yielding under \( \alpha_q tr_{\text{Clifford}} \otimes Tr_2 \otimes Tr_N \otimes Tr_3(\Gamma_3) \):
\[
3\alpha_q \left\{ -N f_{\mu\nu} f^{\mu\nu} + 2(\alpha_l + 6\alpha_q)^{-2} N^{-1} L^2 V_{\Phi} + Tr(\mu_u^2 + \mu_d^2) V_{\Phi} \\
-2(\alpha_l + 6\alpha_q)^{-1} N^{-1} LTr(\mu_u + \mu_d) V_{\Phi} \right\},
\]
combine to give:
\[
-\frac{1}{2} N(\alpha_l + 6\alpha_q) f_{\mu\nu} f^{\mu\nu} + (\alpha_l + 6\alpha_q)^{-1} N^{-1} L^2 V_{\Phi} + Tr[\alpha_l \Gamma_N \mu_e^2 + 3\alpha_q(\mu_u^2 + \mu_d^2)] V_{\Phi} \\
-2(\alpha_l + 6\alpha_q)^{-1} N^{-1} LTr[\alpha_l \Gamma_N \mu_e + 3\alpha_q(\mu_u + \mu_d)] V_{\Phi},
\]
i.e. (cf. (50)):
\[
(*) \quad -\frac{1}{2} N(\alpha_l + 6\alpha_q) f_{\mu\nu} f^{\mu\nu} - (\alpha_l + 6\alpha_q)^{-1} N^{-1} L^2 V_{\Phi} + Tr[\alpha_l \Gamma_N \mu_e^2 + 3\alpha_q(\mu_u^2 + \mu_d^2)] V_{\Phi}
\]
— Contribution stemming from \( \theta_{q[1,1]} \): we have:
\[
(\theta_{q[1,1]})^2 = \left( M^* \left[ \frac{1}{2} \gamma(\alpha_{\Phi}^+ \alpha_{\Phi}^-) \otimes 1_N \right] + \left[ \gamma^5 \gamma(\alpha_{\Phi}^+ \alpha_{\Phi}^-) \otimes 1_N \right] M \quad 0 \\
0 \quad \left[ \gamma^5 \gamma(\alpha_{\Phi}^+ \alpha_{\Phi}^-) \otimes 1_N \right] \right) M M^* \left[ \gamma^5 \gamma(\alpha_{\Phi}^+ \alpha_{\Phi}^-) \otimes 1_N \right] M
\]
where the upper left corner can be replaced by its Hermitean part
\[
\frac{1}{2} M^* \left\{ \left[ \gamma^5 \gamma(\alpha_{\Phi}^+ \alpha_{\Phi}^-) \right]_+ \otimes 1_N \right\} M = (D_{\Phi t})(D_{\Phi t}^i) M^* M,
\]
also equal to the Hermitean part of an order-permuted lower right corner.
We thus get under \( \alpha_l tr_{\text{Clifford}} \otimes Tr_2 \otimes Tr_N(\Gamma_N) \cdot \) the leptonic part:
\[
2\alpha_l Tr(\Gamma_N \mu_e) \cdot (D_{\Phi t})(D_{\Phi t}^i),
\]
and under \( \alpha_q tr_{\text{Clifford}} \otimes Tr_2 \otimes Tr_N \otimes Tr_3(\Gamma_3) \). The quark part:
\[
6\alpha_q Tr(\mu_u + \mu_u) \cdot (D_{\Phi t})(D_{\Phi t}^i),
\]
combining to give (cf. (50)):
\[
(**) \quad 2L(D_{\Phi t})(D_{\Phi t}^i),
\]
\[^{14}\text{Tr}_2 \text{ now denotes a trace in } M_2(C).\]
We use the fact that the normalized Clifford trace of

\[ \frac{i}{2} \gamma(h_j) - (\alpha_l + 3\alpha_q)^{-1}(2N)^{-1}Lv_\Phi 1 \otimes 1_N + v_\Phi 1 \otimes \Sigma \]^2 

\[ = -\frac{1}{4} (\gamma(h^i_k) \gamma(h^i_j)) \otimes 1_N + (\alpha_l + 3\alpha_q)^{-2}(2N)^{-2}L^2V_\Phi 1 \otimes 1_N + V_\Phi 1 \otimes \frac{1}{4}\mu_e^2 

- 2(\alpha_l + 3\alpha_q)^{-1}(2N)^{-1}LV_\Phi 1 \otimes \frac{1}{2}\mu_e + \text{terms linear in } \gamma(f) \]

yielding under $\alpha_l$ $tr_{\text{Clifford}} \otimes T_{r2} \otimes T_{rN}(\Gamma_N)$:

\[ \alpha_l \left\{ -\frac{1}{4}Nh_{\mu}^s h_{\mu}^{\mu} + (\alpha_l + 3\alpha_q)^{-2}(2N)^{-1}L^2V_\Phi + \frac{1}{2}Tr(\Gamma_N\mu_\mu^2)V_\Phi 

- 2(\alpha_l + 3\alpha_q)^{-1}(2N)^{-1}LTr(\Gamma_N\mu_\mu) \cdot V_\Phi \right\} \]

and the quark part:

\[ \left\{ \frac{i}{2} \gamma(h_j) - (\alpha_l + 3\alpha_q)^{-1}(2N)^{-1}Lv_\Phi 1 \otimes 1_N + v_\Phi 1 \otimes \Sigma \right\}^2 

\[ = -\frac{1}{4} (\gamma(h^i_k) \gamma(h^i_j)) \otimes 1_N + (\alpha_l + 3\alpha_q)^{-2}(2N)^{-2}L^2V_\Phi 1 \otimes 1_N + V_\Phi 1 \otimes \frac{1}{4}(\mu_u + \mu_d)^2 

- 2(\alpha_l + 3\alpha_q)^{-1}(2N)^{-1}LV_\Phi 1 \otimes \frac{1}{2}(\mu_u + \mu_d) + \text{terms linear in } \gamma(f), \]

yielding under $\alpha_q$ $tr_{\text{Clifford}} \otimes T_{r2} \otimes T_{rN} \otimes T_{r3}(\Gamma_3)$:

\[ 3\alpha_q \left\{ -\frac{1}{4}Nh_{\mu}^s h_{\mu}^{\mu} + (\alpha_l + 3\alpha_q)^{-2}(2N)^{-1}L^2V_\Phi + \frac{1}{2}Tr[(\mu_u + \mu_d)^2] V_\Phi 

- 2(\alpha_l + 3\alpha_q)^{-1}(2N)^{-1}LTr(\mu_u + \mu_d) \cdot V_\Phi \right\} \]

combine to give:

\[ -\frac{1}{4}(\alpha_l + 3\alpha_q)Nh_{\mu}^s h_{\mu}^{\mu} - (\alpha_l + 3\alpha_q)^{-1}(2N)^{-1}L^2V_\Phi + \frac{1}{2}Tr[\alpha_l \Gamma_N\mu_\mu^2 + 3\alpha_q(\mu_u + \mu_d)^2] V_\Phi 

- 2(\alpha_l + 3\alpha_q)^{-1}(2N)^{-1}LTr[\alpha_l \Gamma_N\mu_e + 3\alpha_q(\mu_u + \mu_d)] V_\Phi \]

i.e. (cf. \( [50] \)):

\( (***) \)

\[ -\frac{1}{4}(\alpha_l + 3\alpha_q)Nh_{\mu}^s h_{\mu}^{\mu} 

- (\alpha_l + 3\alpha_q)^{-1}(2N)^{-1}L^2V_\Phi + \frac{1}{2}Tr[\alpha_l \Gamma_N\mu_\mu^2 + 3\alpha_q(\mu_u + \mu_d)^2] V_\Phi \]

Collecting the terms (*), (**), and (***) yields the electroweak contribution.

\( ^{15} \) We use the fact that the normalized Clifford trace of \( (\gamma(h^i_k) \gamma(h^i_j)) \) equals \( h_{\mu}^s h_{\mu}^{\mu} \).
[7] Proposition.

The electroweak Yang-Mills action equals\[16\]

\[Y_{\text{ew}} = -\left(\alpha_l + 6\alpha_q\right)NF_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\left(\alpha_l + 3\alpha_q\right)NH_{\mu\nu}h_{\nu}^{\mu}\]

\[+ 2L(D\Phi_j)(D\Phi^j) + K(\Phi_i\Phi^i - 1)^2,\]

(64)

Where \(\alpha_l\) and \(\alpha_q\) are positive constants, and where

\[
\begin{align*}
L &= Tr \left[ \alpha_l \Gamma_N \mu_e + 3\alpha_q(\mu_u + \mu_d) \right] \\
K &= \frac{3}{2} Tr \left[ \alpha_l \Gamma_N \mu_e^2 + 3\alpha_q(\mu_u^2 + \mu_d^2) \right] \\
&\quad + 3\alpha_q Tr(\mu_u + \mu_d) - 2^{-1}(\alpha_l + 3\alpha_q)^{-1} + (\alpha_l + 6\alpha_q)^{-1} \right] N^{-1}L^2
\end{align*}
\]

COMPUTATION OF THE CHROMODYNAMICS YANG-MILLS ACTION.

We recall the expression of the leptonic, resp. quark components of the gluonic curvature one has, with \(f'\) and \(g_0\) \(U(1)\)-curvature-two-forms, and \(g^a, a = 1, \ldots, 8\, a SU(3)\)-curvature-two-form (the \(\lambda_a\) are the eight Gell-Man matrices):\(^4\,^5\)

\[
\theta_l = \frac{i}{2}\gamma(f')1_2 \otimes 1_N,
\]

(66)

resp.

\[
\theta_q = \frac{i}{2}\gamma(g_0)1_2 \otimes 1_N \otimes 1_3 + \frac{i}{2}\gamma(g^a)1_2 \otimes 1_N \otimes \frac{\lambda_a}{2}.
\]

(67)

The gluonic new scalar product (3b) is given as \(Tr(\Gamma'\cdot),\Gamma' = \begin{pmatrix} \Gamma'_l & 0 \\ 0 & \Gamma'_q \end{pmatrix}\), (cf. (46)), with (cf. (34), (35)) here reproduced (in shorthand) for the convenience of the reader:

\[
\Gamma'_l = \begin{pmatrix}
 h(\mu_e) & 0 & k(\mu_e) \\
 0 & \delta & 0 \\
 k(\mu_e) & 0 & h(\mu_e)
\end{pmatrix},
\]

(34)

\[
\Gamma'_q = \begin{pmatrix}
 f(\mu_u) & 0 & l(\mu_u) & 0 & m(\bar{\mu}_d)C^* \\
 0 & g(\bar{\mu}_d) & 0 & m(\bar{\mu}_d)C^* \\
 l(\mu_u) & 0 & f(\mu_u) & 0 \\
 0 & Cm(\bar{\mu}_d) & 0 & Cg(\bar{\mu}_d)C^*
\end{pmatrix} \otimes 1_3.
\]

(35)

Recalling the formula:

\[Tr_3 \left\{ \left[ M \otimes 1_3 + N^a \otimes \frac{\lambda_a}{2} \right]^2 \right\} = 3M^2 + \frac{1}{2}N^aN_a,\]

(68)

\(^{16}\) We recall that \(\Gamma' = 1_2 \otimes \Gamma_N, \Gamma_N M_c Tr_N \Gamma_N = N.\)
\[ \begin{align*}
M \text{ and } N^a, a = 1, \ldots, 8, p \times p \text{ matrices, and since} \\
(69) \\
\left\{ \begin{array}{l}
\theta_l^2 = -\frac{1}{4} \gamma(f')^2 1_2 \otimes 1_N, \\
\theta_q^2 = -\left[ \frac{3}{4} \gamma(g_0)^2 + \frac{1}{8} \gamma(g^a) \gamma(g_a) \right] 1_2 \otimes 1_N \otimes 1_3,
\end{array} \right.
\end{align*} \]

we have:
\[
(Tr_2 \otimes Tr_N) [\Gamma' l \theta^2_l] + (Tr_2' \otimes Tr_N \otimes Tr_3) [\Gamma' q \theta^2_q]
\]
\[
= -\frac{1}{4} \gamma(f')^2 \cdot (Tr_2 \otimes Tr_N) \left[ h(\mu_e) + \frac{\delta}{2} \right] - \frac{1}{4} \left[ 3 \gamma(g_0)^2 + \frac{1}{2} \gamma(g^a) \gamma(g_a) \right] \cdot (Tr_2 \otimes Tr_N \otimes Tr_3) \left[ 2 h(\mu_e) + 2 g(\bar{\mu}_d) \right],
\]

whose Clifford trace is then the sought gluonic Yang-Mills action:

[8] Proposition.

The chromodynamics Yang-Mills action equals
\[
YM_{\text{chrom}} = -Tr_N [h(\mu_e) + \frac{\delta}{2}] f_{\mu\nu} f^{\mu\nu}
\]
\[
- Tr_N \left[ f(\mu_u) + g(\bar{\mu}_d) \right] \left[ 3 g_0 \delta_{\mu\nu} g_0^{\mu\nu} + \frac{1}{2} g^a_{\mu\nu} g^a_{\mu\nu} \right]
\]

where \( h, f, \) and \( g \) are positive real functions.

MODULAR COMBINATION OF THE ELECTROWEAK AND CHROMODYNAMICS SECTORS. THE FULL YANG-MILLS ACTION.

We now combine our results [7] and [8] using the modular condition coalescing the unwanted three \( U(1) \)-gauge groups into a single one. At the level of connexion the modular condition amounts to the identifications:

\[
(72) \quad \begin{align*}
a' &= a \\
c^0 &= -\frac{1}{3} a
\end{align*}
\]

\[
(73) \quad \begin{align*}
f' &= f \\
g^0 &= -\frac{1}{3} f
\end{align*}
\]

Adding the electroweak action (64) to the gluonic action (71) with these identifications then gives the total Yang-Mills action:

\[
YM = -\frac{1}{2} \left( N(\alpha_l + 6 \alpha_q) + Tr_N \left[ 2 h(\mu_e) + \frac{\delta}{2} \left[ f(\mu_u) + g(\bar{\mu}_d) \right] \right] \right) f_{\mu\nu} f^{\mu\nu}
\]
\[
- \frac{1}{4} N(\alpha_l + 3 \alpha_q) h^s_{\mu\nu} h^s_{\mu\nu} - \frac{1}{2} Tr_N \left[ f(\mu_u) + g(\bar{\mu}_d) \right] g^a_{\mu\nu} g^a_{\mu\nu}
\]
\[
+ 2 L(D\Phi)(D\Phi) + K(\Phi_1 \Phi^1 - 1)^2.
\]
[9] Proposition.

The bosonic action of the Connes-Lott version of the standard model is of the form

\[
YM = -A g_{\mu\nu} \delta^\mu_a - B f_{\mu\nu} f^{\mu\nu} - \frac{1}{4} C h_{\mu\nu} h^{\mu\nu} + 2 L D_{\mu} \Phi D^\mu \Phi^i + K (\Phi, \Phi^i - 1)^2,
\]

where

\[
\begin{align*}
A &= \frac{1}{2} \text{Tr}_N [f(\mu_a) + g(\bar{\mu}_d)] \\
B &= \frac{1}{2} \left\{ N(\alpha_l + 6\alpha_q) + \text{Tr}_N \left[ 2h(\mu_e) + \delta + \frac{2}{3} [f(\mu_a) + g(\bar{\mu}_d)] \right] \right\} \\
C &= N(3\alpha_q + \alpha_l) \\
L &= \text{Tr} \left[ \alpha_l \Gamma_N \mu_e + 3\alpha_q (\mu_u + \mu_d) \right] \\
K &= \frac{3}{2} \text{Tr} \left[ \alpha_l \Gamma_N \mu_e^2 + 3\alpha_q (\mu_u^2 + \mu_d^2) \right] \\
&\quad + 3\alpha_q \text{Tr}(\mu_u + \mu_d) - [2^{-1}(\alpha_l + 3\alpha_q)^{-1} + (\alpha_l + 6\alpha_q)^{-1}] N^{-1} L^2
\end{align*}
\]

with \( \alpha_l, \alpha_q \geq 0, \Gamma_N a \text{ positive function of } \mu_e = M_e^2 \text{ such that } \text{Tr}_N \Gamma_N = N, \text{ and } f \text{ and } g \text{ positive functions. The coupling constant is in the center iff (a): } \alpha_l = \alpha_q \text{ ; (b): } h(\mu_e) = \delta = \lambda N \text{ for some } \lambda \geq 0 \text{ ; (c): } f(\mu_a) = g(\bar{\mu}_d) = \lambda\nu N \text{ for some } \lambda'' \geq 0. \)

The last claim follows from [3](ii) (requiring \( \alpha_q 1_2 \otimes 1_N \otimes 1_3 = \alpha_l 1_2 \otimes 1_N \) and [3](iv)).

Setting \( \alpha_l = \rho \frac{1-x}{2}, \alpha_q = \frac{x}{2}, \rho > 0, -1 \leq x \leq 1, \) \( \frac{1}{2\rho} \text{Tr}_N [f(\mu_a) + g(\bar{\mu}_d)] = y, \) \( \frac{1}{2\rho} \text{Tr}_N [2h(\mu_e) + \delta] = z, \) this reads:

\[
\begin{align*}
A/\rho &= y \\
B/\rho &= N \frac{7 - 5x}{4} + z + \frac{2}{3} y \\
C/\rho &= N(2 - x) \\
L/\rho &= \frac{1}{2} \text{Tr} \left[ (1 + x) \Gamma_N \mu_e + 3(1 - x)(\mu_u + \mu_d) \right] \\
K/\rho &= \frac{3}{4} \text{Tr} \left[ (1 + x) \Gamma_N \mu_e^2 + (1 - x) \left[ 3(\mu_u^2 + \mu_d^2) + 2\mu_u \mu_d \right] \right] \\
&\quad - \frac{3(5 - 3x)(1 - x)}{2(7 - 5x)(2 - x)} N^{-1} (L/\rho)^2
\end{align*}
\]

with the coupling constants in the center iff \( x = 0. \)

The expressions for \( L \) and \( K \) differ only from those obtained with the former scalar product (3) (besides the overall factor \( \rho \)) by the factors \( \Gamma_N \) figuring in their first terms [4] – the latter however without inference on the (very good) approximation which consists in

\[17\] The corresponding error on the tree-level computation of masses – see below – is of the order of the present error on the measurement of the \( W \) mass. The overall factor \( \rho \) drops out from the tree-level computation of masses.
neglecting all fermion masses against the top mass. This approximation leads to the values:

\[
\begin{align*}
L/\rho &\approx \frac{3}{2}(1-x)m_t^2 \\
K/\rho &\approx \frac{9}{4}(1-x) \left[ 1 - \frac{(5-3x)(1-x)}{2(7-5x)(2-x)} \right] m_t^4 .
\end{align*}
\]

The factors $\Gamma_N$ are harmless: indeed we have

\[
\text{Sup}\left\{ Tr(\Gamma_N^2); Tr\Gamma_N = N \right\} = \text{Sup}\left\{ \lambda_1^2 + \lambda_2^2 + \lambda_3^2; \lambda_1 + \lambda_2 + \lambda_3 = N \right\} = 3,
\]

thus, by the Schwartz inequality, one can neglect the terms:

\[
\begin{align*}
| Tr(\Gamma_N \mu e) | &\leq | Tr(\Gamma_N^2) |^{1/2} | Tr(\mu^2) |^{1/2} \leq \sqrt{3} Tr(\mu^2) |^{1/2} \\
| Tr(\Gamma_N \mu^2) | &\leq | Tr(\Gamma_N^2) |^{1/2} | Tr(\mu^4) |^{1/2} \leq \sqrt{3} Tr(\mu^4) |^{1/2} ,
\end{align*}
\]

**TREE-LEVEL COMPUTATIONS.**

We now compare the Connes-Lott Lagrangian (75) assorted with the covariant derivatives [4b]:

\[
\begin{align*}
D_\mu & = \nabla_\mu + i \left( a_\mu - b_\mu \frac{\tau_3}{2} \right) \\
D_{R\mu} & = \nabla_\mu - 2i b_\mu \\
D_{L\mu} & = \nabla_\mu - ia_\mu - ib_\mu \frac{\tau_3}{2} \\
D_{R\mu} & = \nabla_\mu + 4i a_\mu - i c_\mu \frac{\lambda_a}{2} \\
D_{R\mu} & = \nabla_\mu - 2i c_\mu \frac{\lambda_a}{2} \\
D_{L\mu} & = \nabla_\mu + \frac{1}{3} i a_\mu - ib_\mu \frac{\tau_3}{2} - ic_\mu \frac{\lambda_a}{2}
\end{align*}
\]

with the bosonic part of the traditional full standard model [5]:

\[
\begin{align*}
L_{\text{gauge}} + L_{\text{Higgs}} & = -\frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W_{\mu\nu}^s W_s^{\mu\nu} \\
&\quad + (D_\mu \phi)^* (D_\mu \phi) + \frac{\mu^2}{v^2} \left( \phi^* \phi - \frac{v^2}{2} \right)^2
\end{align*}
\]

\[\text{18} \text{ We recall the following relationships of constants in (81) with masses and the weak angle: one has } m_W = \frac{1}{2} mg_2, m_H = \sqrt{2} \mu, \text{ and } \tan\theta_W = g_1/g_2.\]
assorted with the covariant derivatives:

\[
\begin{align*}
D_\mu &= \partial_\mu - ig_1 B_\mu - ig_2 W^s_\mu \tau_s / 2 \\
D^R_\mu &= \partial_\mu + ig_1 B_\mu \\
D^L_\mu &= \partial_\mu + ig_1 B_\mu - ig_2 W^s_\mu \tau_s / 2 \\
D^R_\mu &= \partial_\mu - 2ig_1 B_\mu - ig_3 G^a_\mu \lambda_a / 2 \\
D^L_\mu &= \partial_\mu + ig_1 B_\mu - ig_3 G^a_\mu \lambda_a / 2 \\
D^L_R &= \partial_\mu - ig_1 B_\mu - ig_2 W^s_\mu \tau_s / 2 - ig_3 G^a_\mu \lambda_a / 2
\end{align*}
\]

Identification of the covariant derivatives (80) and (82) is synonymous with the identifications:

\[
\begin{align*}
c &= g_3 G \\
\text{in components } c_\mu^a &= g_3 G^a_\mu, \quad a = 1, \ldots, 8
\end{align*}
\]

\[
\begin{align*}
a &= -\frac{1}{2} g_1 B \\
\text{in components } a'_\mu &= -\frac{1}{2} g_1 B_\mu
\end{align*}
\]

\[
\begin{align*}
b &= g_2 W \\
\text{in components } b^s_\mu &= g_2 W^s_\mu, \quad s = 1, 2, 3
\end{align*}
\]

implying:

\[
\begin{align*}
f_{\mu \nu} &= -\frac{1}{2} g_1 B_{\mu \nu} = -\frac{1}{2} g_2 \cos \theta W_{\mu \nu} \\
h^s_{\mu \nu} &= g_2 W^s_{\mu \nu}, \quad s = 1, 2, 3, \\
g^a_{\mu \nu} &= g G^a_{\mu \nu}, \quad a = 1, \ldots, 8,
\end{align*}
\]

Assuming that \( \phi \) and \( \Phi \) differ by a constant (insensitive to multiplication of \( \phi \), resp. \( \Phi \), by constants), the latter follows from comparison of the fourth terms of (75) and (81), yielding:

\[
\frac{v}{\sqrt{2}} \Phi = \phi.
\]

Inserting (83), (84) and (85) into (81) yields:

\[
YM = -g_3^2 A \cdot G^a_{\mu \nu} G^a_{\mu \nu} - g_1^2 B_{\mu \nu} B_{\mu \nu} - \frac{1}{4} g_2^2 C W^s_{\mu \nu} W^s_{\mu \nu}
\]

\[
+ \frac{4L}{v^2} (D_\mu \phi)^*(D^\mu \phi) + \frac{4K}{v^4} \left( \phi^* \phi - \frac{v^2}{2} \right)^2.
\]

Comparison with (75) yields

\[
4g_3^2 A = g_1^2 B = g_2^2 C = \frac{4L}{v^2} = \frac{4K}{\mu^2 v^2},
\]
\( g_3 = \frac{1}{2}(C/A)^{1/2} g_2, \) \hfill (90)

\( \frac{g_2^2}{g_1} = \tan^{-2} \theta_W = \frac{B}{C} \) \hfill (91)

whence \( \sin^2 \theta_W = \frac{C}{B+C} \)

\[ v^2 g_2^2 = 4 m_W^2 = \frac{4L}{C} \] \hfill (92)

whence \( m_W = (L/C)^{1/2} \)

\( \mu^2 = \frac{K}{L} \) \hfill (93)

whence \( m_H = (2K/L)^{1/2} \).

Plugging into those relations \( A, B, C \) as in (76a) and \( K, L \) as in (77) gives:

[10] Proposition.

We have the following tree-level evaluations:

\( g_3 = \frac{1}{2}(N(2-x)/y)^{1/2} g_2 \) \hfill (94)

\( \sin^2 \theta_W = \frac{N(2-x)}{N^3(5-3x) + z + \frac{2}{3} y} \) \hfill (95)

\( m_W \cong \left[ \frac{3(1-x)}{2N(2-x)} \right]^{1/2} m_t \) \hfill (96)

and

\( m_H \cong \sqrt{3} \left[ 1 - \frac{(5-3x)(1-x)}{2(7-5x)(2-x)} \right]^{1/2} m_t, \) \hfill (97)

where \( \cong \) denotes approximation neglecting all fermion masses against the Higgs mass. The real numbers \( x, y, \) and \( z \) range independently in the respective intervals \([-1, +1], [0, +\infty] \) and \([0, -\infty] \).

The choice of the coupling constant in the center gives

\[ \begin{cases} m_t = 2m_W \\ m_H = \sqrt{3} \cdot \sqrt{(23/28)} m_t = 1.5698 m_t \end{cases} \] \hfill (98)

We now study the constraints affecting the above quantities. Since \( y \) is an arbitrary positive number, (94) contains no information.

— Constraint on \( \sin^2 \theta_W \): since \( y, z \in [0, +\infty] \) one has \( \sin^2 \theta_W < \frac{4-3x}{3} \leq \frac{2}{3} m_t^2/ m_W \).

— Constraint on \( m_W/m_t \): continuous decreasing function of \( x \) varying from 0 to \( N^{-1/2} \).

— Constraint on \( m_H/m_t \): continuous increasing function of \( x \) varying from \( \sqrt{(7/3)} \) to \( \sqrt{3} \).
These are individual constraints. But we have also a correlation due to the additional information obtained by eliminating $x$ between (96) and (97): if we set:

\[ S = \frac{2N}{3} \left[ \frac{m_W}{m_t} \right]^{1/2} = \frac{1 - x}{2 - x} \quad \text{whence} \quad x = \frac{2S - 1}{S - 1} \]

we have

\[ T = \frac{1}{3} \left[ \frac{M_H}{m_t} \right]^{1/2} = 1 - \frac{(5 - 3x)(1 - x)}{2(7 - 5x)(2 - x)} = 1 - \frac{(5 - 3x)}{2(7 - 5x)} S = 1 - \frac{1}{2} \frac{S + 2}{3S + 2}. \]

**[11] Proposition.**

The Connes-Lott version of the full standard model differs in tree-approximation from the traditional standard model by the following constraints relative to the weak angle and the ratios of the top quark, resp. the Higgs boson mass to the $W$-boson mass. Irrespective of the choice of coupling constant in the $K$-cycle commutant we have the inequalities:

\[ \sin^2 \theta_W \leq \frac{2}{3} \frac{m_t^2}{m_t^2 + m_W^2}, \]

\[ 0 \leq \frac{m_W}{m_t} \leq N^{-1/2}, \]

\[ \sqrt{(7/3)} \leq \frac{m_W}{m_t} \leq \sqrt{3}. \]

Furthermore the ratios $m_W/m_t$ and $m_H/m_t$ determine each other as follows: one has:

\[ T = 1 - \frac{1}{2} \frac{S + 2}{3S + 2} \]

where

\[ S = \frac{2N}{3} \left[ \frac{m_W}{m_t} \right]^{1/2}, \quad T = \frac{1}{3} \left[ \frac{m_H}{m_t} \right]^{1/2}, \]

where the values (98) corresponding to the coupling constant in the $K$-cycle center are obtained for $S = \frac{1}{2}$.

We conclude with the remark that the correlation (104) together with the fact that the top has approximately twice the $W$-mass leads to propose that the Higgs mass might lie around $1.5698 m_t$. 

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Appendix.

Present notation versus notation in the companion paper [5a]

| [5a] | this paper |
|------|------------|
| $x$  | $\alpha_q \Gamma_3$ | $(tr \Gamma_3 = 3)$ $trx = 3\alpha_q$ |
| $y$  | $\alpha_l \Gamma_N$ | $(tr \Gamma_N = N)$ $try = N\alpha_l$ |
| $r$  | $f(\mu_u)$ |
| $s$  | $g(\bar{\mu}_d)$ |
| $u$  | $\delta$ |
| $v$  | $h(\mu_e)$ |
| $k$  | $l(\mu_u)$ |
| $p$  | $m(\bar{\mu}_d)$ |
| $w$  | $k(\mu_e)$ |

$C_{KM}$ $C$
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