Yangians and $\mathcal{W}$-algebras

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Abstract

We present a connection between $\mathcal{W}$-algebras and Yangians, in the case of $gl(N)$ algebras, as well as for twisted Yangians and/or super-Yangians. This connection allows to construct an $R$-matrix for the $\mathcal{W}$-algebras, and to classify their finite-dimensional irreducible representations. We illustrate it in the framework of nonlinear Schrödinger equation in 1+1 dimension.

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1 Introduction

$\mathcal{W}$-algebras have been introduced in the 2$d$-conformal models as a tool for the study of these theories. Then, these algebras and their finite-dimensional versions appeared to be relevant in several physical backgrounds. However, a full understanding of their algebraic structure (and of their geometrical interpretation) is lacking. The connection of some of these $\mathcal{W}$-algebras with Yangians appears to be a solution at least for the algebraic structure: it allows the construction of an $R$-matrix for $\mathcal{W}$-algebras, the classification of their irreducible finite-dimensional representations and the determination of their center.

The paper is structured as follows: in section 2 (section 3) we remind some basic definitions for Yangians (for $\mathcal{W}$-algebras). Then, the connection between these two objects is presented in section 4 for the case of $gl(N)$. Section 5 is devoted to a physical example where the connection explicitly appears, namely the Nonlinear Schrödinger equation in 1+1 dimension. The two following sections present various generalizations: the case of $so(M)$ and $sp(2M)$ algebras is studied in section 6 and the case of superalgebras in section 7. We conclude in section 8.

For the sake of concision, we have chosen to detail, as an illustrative case, the study of $Y(N) \equiv Y(gl(N))$ and $\mathcal{W}_p(N) \equiv \mathcal{W}[gl(Np), N, sl(p)]$, while being less precise on the generalizations, sending the interested reader back to the original papers.

2 Yangian $Y(\mathcal{G})$

Yangians $Y(\mathcal{G})$, associated to each simple Lie algebra $\mathcal{G}$, have been introduced by Drinfeld as deformation of (half) a loop algebra based on $\mathcal{G}[\mathbb{C}]$. They have generators:

$$Y(\mathcal{G}) = \mathcal{U} \left( Q_n^a, a = 1, \ldots \dim(\mathcal{G}); n = 0, 1, \ldots, \infty \right)$$

(2.1)

$n$ is the loop index and $a$ labels the $\mathcal{G}$-adjoint representation. In other words, we have an infinite set of adjoint representations (labeled by $n$), the first one being $\mathcal{G}$ itself. This is gathered in the relations:

$$[Q_n^a, Q_m^b] = f^{ab}_c Q_n^c$$

(2.2)

The deformation appears in the remaining relations:

$$[Q_m^a, Q_n^b] = f^{ab}_c Q_{m+n}^c + P_{nm}^a(Q)$$

(2.3)

where $P_{nm}^a$ is a polynomial in the $Q$’s.

Yangians are Hopf algebras, their coproduct being given by

$$\Delta(Q_n^a) = Q_n^a \otimes 1 + 1 \otimes Q_n^a$$

(2.4)

$$\Delta(Q_1^a) = Q_1^a \otimes 1 + 1 \otimes Q_1^a + \frac{1}{2} f^{ab}_c Q_0^b \otimes Q_0^c$$

(2.5)
which also shows the deformation with respect to the loop algebra coproduct.

There is a consistency relation, which takes the form of a Jacobi-like identity. Depending on the algebra, this Jacobi-like identity takes the form (2.6) when $G \neq sl(2)$, or the form (2.7) when $G = sl(2)$:

\[
f^{bc}_{\phantom{bc}d}[Q^a_1, Q^d_1] + f^{ca}_{\phantom{ca}d}[Q^b_1, Q^d_1] + f^{ab}_{\phantom{ab}d}[Q^c_1, Q^d_1] = f^a_{\phantom{a}pd}f^b_{\phantom{b}qx}f^c_{\phantom{c}ry}f^{xyd} s_3(Q^p_0, Q^q_0, Q^r_0) \tag{2.6}
\]

\[
f^{cd}_{\phantom{cd}e}[[Q^a_1, Q^b_1], Q^c_1] + f^{ab}_{\phantom{ab}e}[[Q^c_1, Q^d_1], Q^e_1] = 
\left(f^a_{\phantom{a}pe}f^b_{\phantom{b}qx}f^{cd}_{\phantom{cd}y}f^e_{\phantom{e}rz}f^{xy} + f^c_{\phantom{c}pe}f^d_{\phantom{d}qx}f^{ab}_{\phantom{ab}y}f^e_{\phantom{e}rz}f^{xz} \right) \eta^{eg} s_3(Q^p_0, Q^q_0, Q^r_1) \tag{2.7}
\]

where $s_3$ is the symmetrized product.

In the case of $G = gl(N)$, Yangians admit an $R$-matrix presentation [2, 3]: gathering the generators into an $N \times N$ matrix, and using a spectral parameter $u$, one defines

\[
T(u) = \sum_{i,j=1}^{N} \sum_{n=0}^{\infty} u^{-n}T^{ij}_n E_{ij} = \sum_{i,j=1}^{N} T^{ij}(u) E_{ij} \quad \text{with} \quad T^0_{ij} = \delta^{ij} \tag{2.8}
\]

Then, the defining relations of $Y(gl(N)) = Y(N)$ become

\[
R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v) \tag{2.9}
\]

\[
\Delta T(u) = T(u) \otimes T(u) ; \quad S(T(u)) = T(u)^{-1} ; \quad \epsilon(T(u)) = 1 \tag{2.10}
\]

where

\[
R_{12}(x) = I_N \otimes I_N - \frac{1}{x} P_{12} \quad \text{with} \quad P_{12} = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji} \tag{2.11}
\]

\[
T_1(u) = T(u) \otimes I_N = \sum_{i,j=1}^{N} T^{ij}(u) E_{ij} \otimes I_N \tag{2.12}
\]

\[
T_2(u) = I_N \otimes T(u) = \sum_{i,j=1}^{N} T^{ij}(u) I_N \otimes E_{ij} \tag{2.13}
\]

$R$ is a rational solution to the Yang-Baxter equation, and $P_{12}$ is the permutation operator of the two auxiliary spaces (spanned by the $N \times N$ matrices).

### 2.1 Classical Yangians

In the following, we will be interested mainly by a classical version of the Yangians, where the commutators are replaced by Poisson brackets.

For the first presentation, the relations are the same, except for the Poisson bracket which now replaces the commutator. For instance, the relation (2.6) becomes:

\[
f^{bc}_{\phantom{bc}d}\{Q^a_1, Q^d_1\} + f^{ca}_{\phantom{ca}d}\{Q^b_1, Q^d_1\} + f^{ab}_{\phantom{ab}d}\{Q^c_1, Q^d_1\} = f^a_{\phantom{a}pd}f^b_{\phantom{b}qx}f^c_{\phantom{c}ry}f^{xyd} Q^p_0 Q^q_0 Q^r_0 \tag{2.14}
\]
For $Y(N)$, the Poisson bracket appears as a classical version of the commutator:

$$R(x) = \mathbb{I} - \hbar r(x) ; \; [\cdot, \cdot] = \hbar \{\cdot, \cdot \} ; \; T(u) = L(u)$$

(2.15)

$$\{L_1(u), L_2(v)\} = [r_{12}(u - v), L_1(u) L_2(v)] ; \; r(x) = \frac{1}{x} P_{12}$$

(2.16)

### 3 $\mathcal{W}(\mathcal{G}, \mathcal{H})$ algebras

$\mathcal{W}$-algebras have been first introduced in the context of $2d$-conformal theories by Zamolodchikov [4] as a tool for classifying the irreducible unitary representations of these theories. Since, they appeared to be symmetries of Toda field theories [5, 6]. In this context, $\mathcal{W}$-algebras are constructed as Hamiltonian reduction of affine (Kac-Moody) algebras. Later on, a simpler version of these algebras, called finite $\mathcal{W}$-algebras, was introduced by De Boer and Tjin [7]. They are constructed as Hamiltonian reduction of finite dimensional Lie algebras: the resulting algebra is a polynomial algebra with a finite number of generators.

More precisely, starting from a Poisson-Lie algebra $\mathcal{G}$, one constrains some of the generators of $\mathcal{G}$. The constraints are second class, and one considers the Dirac brackets deduced from these constraints: the $\mathcal{W}$-algebra is defined as the set of unconstrained generators provided with the Dirac brackets. The system of constraints is given by a subalgebra $\mathcal{H}$ of $\mathcal{G}$, whence the denomination $\mathcal{W}(\mathcal{G}, \mathcal{H})$, see [5, 6, 8] for more details.

Here, we will be concerned with a class of finite $\mathcal{W}$-algebras: $\mathcal{W}[gl(Np), N.sl(p)]$ algebras. We will denote these algebras $\mathcal{W}_p(N)$. The generators of $\mathcal{W}_p(N)$ are in finite number:

$$\mathcal{W}_p(N) = \mathcal{U} \left( W_m^a; \; a = 1, 2, \ldots, N^2; \; n = 1, 2, \ldots, p \right)$$

(3.1)

They obey to

$$\{W_0^a, W_n^b\} = f^{ab}_{\; \; c} W_n^c \quad \text{and} \quad \{W_m^a, W_n^b\} = f^{ab}_{\; \; c} W_m^c + P_{nm}^{ab}(W)$$

(3.2)

where $P_{nm}^{ab}(W)$ are polynomials in the $W$ generators.

Its similarity with the Yangian presentation is quite appealing and has motivated the studies in this direction.

### 4 $Y(N)$ and $\mathcal{W}_p(N)$

From the previous presentations, it is natural to seek for a relation between $\mathcal{W}_p(N)$ algebras and Yangians $Y(N)$. Indeed, such a relation exists, and it has been proven in [9]:

**Theorem 1** There is an algebra homomorphism between $\mathcal{W}_p(N)$ algebras and Yangians $Y(N)$. More precisely, there is a one-to-one connection between the first $p N^2$ generators
of $Y(N)$ and the generators of the $W_p(N)$ algebra:

$$Q_n^a \rightarrow \beta_n^a W_n^a + R_n^a(W) \text{ with } \beta_n^a \in \mathbb{R} \setminus \{0\}$$

(4.1)

$R_n^a(W)$ are polynomials in the $W_m^b$ with $m < n$. The remaining generators of $Y(N)$ are polynomials in the $W$-generators.

It has been proven that the generators of the $W$-algebra obey to the Jacobi-like relations that define the Yangian.

The $R$-matrix approach is an easier way to tackle this relation \[10\]:

**Theorem 2** $W_p(N)$ is isomorphic to the truncated Yangian $Y_p(N)$, defined by

$$Y_p(N) = Y(N) / \mathcal{J}_p \text{ with } \mathcal{J}_p \text{ ideal generated by } T_{ij}^n, \ i,j = 1, \ldots, N; \ n > p$$

Thanks to this theorem, one gets an $R$-matrix formulation of the $W_p(N)$ algebras:

$$\{W_1(u), W_2(v)\} = [r_{12}(u - v), W_1(u)W_2(v)]$$

with $W(u) = \sum_{i,j=1}^{N} \sum_{n=0}^{p} W_{ij}^n u^{-n} E_{ij}$ and $r(x) = \frac{1}{x} P_{12}$

**Remark:** The Hopf structure of $Y(N)$ does not survive the coset, so that the algebra isomorphism of theorem 2 is in this sense a no-go theorem about the existence of a natural Hopf structure for $W$-algebras.

One can also determine the center of the $W_p(N)$ algebra:

**Theorem 3** The center of the $W_p(N)$ has dimension $Np$ and is canonically associated to the center of the underlying $gl(Np)$ algebra.

Moreover, since the irreducible finite-dimensional representations of the Yangian have been classified, one can prove:

**Theorem 4** All the finite dimensional irreducible representations of $W_p(N)$ are highest weight. They are in one-to-one correspondence with the families $\{P_1(u), \ldots, P_{N-1}(u), \rho(u)\}$, where $P_i(u)$ are polynomials of the form

$$P_i(u) = \prod_{k=1}^{d_i} (u - \gamma_k^i) \text{ with } \sum_i d_i \leq p \text{ and } \gamma_k^i \in \mathbb{C}$$

(4.2)

$$\rho(u) = 1 + \sum_{n=1}^{Np} c_n u^{-n} \text{ codes the values } c_n \text{ of the Casimir operators in the representation.}$$
All these representations are highest weight, the highest weight being reconstructed from the polynomials \( P_i \) through:
\[
\frac{\mu^i(u)}{\mu^{i+1}(u)} = \frac{P_i(u + 1)}{P_i(u)}, \quad i = 1, \ldots, N
\] (4.3)
with the highest weight vector \( \xi \) defined by:
\[
W_{ii}(u)\xi = \mu^i(u)\xi \quad 1 \leq i \leq N \quad \text{and} \quad W_{ij}(u)\xi = 0, \quad 1 \leq i < j \leq N
\] (4.4)

Finally, let us remark that a detailed analysis of the decomposition of \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) algebras with respect to their Lie subalgebras (using the technics developed in [8]) shows that such a connection cannot exist with Yangians \( Y(\mathcal{G}) \) when \( \mathcal{G} = so(N) \) or \( sp(2N) \). Indeed, when \( \mathcal{G} \) is not a \( gl(N) \) algebra, there is no \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) algebra such that all its generators are in adjoint representations of the Lie subalgebra of \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \). We will see below that the connection apply to objects different from the \( Y(\mathcal{G}) \) Yangians.

5 Nonlinear Schrödinger equation in 2 dimensions

The Nonlinear Schrödinger equation in two dimensions (NLS) is a nice framework where the connection between \( Y(N) \) and \( \mathcal{W}_p(N) \) can be visualized.

We start with
\[
i \partial_t \Phi = \partial_x^2 \Phi + g |\Phi|^2 \Phi \quad \text{with} \quad \Phi = (\varphi_1, \ldots, \varphi_N) \text{ and } g < 0
\] (5.1)
The (quantum) solutions to this equation are known for a long time [11]. They take the form
\[
\Phi = \sum_{n=0}^{\infty} g^n \Phi(n) \quad ; \quad \Omega_n = q_0 x - q_0^2 t + \sum_{i=1}^{n} (q_i - p_i) x - (q_i^2 - p_i^2) t
\]
\[
\Phi(n) = \int d^{n+1}q d^n p \ a_1^\dagger(p_1) \ldots a_n^\dagger(p_n) a_n(q_n) \ldots a_0(q_0) \prod_{i=1}^{n} \exp(i\Omega_i) \prod_{i=1}^{n} (p_i - q_i - 1)(p_i - q_i)
\]
where the \( a \)'s and \( a^\dagger \)'s obey a Zamolodchikov-Faddeev (ZZF) algebra [12]:
\[
a_1(k_1) a_2(k_2) = R_{12}(k_2 - k_1) a_2(k_2) a_1(k_1)
\] (5.2)
\[
a_1^\dagger(k_1) a_2^\dagger(k_2) = a_2^\dagger(k_2) a_1^\dagger(k_1) R_{12}(k_2 - k_1)
\] (5.3)
\[
a_1(k_1) a_2^\dagger(k_2) = a_2^\dagger(k_2) R_{12}(k_1 - k_2) a_1(k_1) + \delta_{12}(k_1 - k_2)
\] (5.4)
where \( R \) is the matrix of the Yangian \( Y(N) \). We use the notation
\[
R_{12}(x) = R_{ik}^{jl}(x) E_{ij} \otimes E_{kl}
\]
\[
a_1(k) = a_1(k) v_i \otimes I \quad a_2(k) = a_2(k) I \otimes v_i
\]
\[
a_1^\dagger(k) = a_1^\dagger(k) v_i \otimes I \quad a_2^\dagger(k) = a_2^\dagger(k) I \otimes v_i
\]
\[
E_{ij} v_k = \delta_{jk} v_i \quad ; \quad E_{ij} v_k^\dagger = \delta_{ik} v_j^\dagger \quad ; \quad v_i \cdot v_j = \delta_{ij} \quad ; \quad v_i^\dagger \cdot v_j = E_{ij}
\]
The apparition of the Yangian’s $R$-matrix is not surprising in this context, since the Yangian is a symmetry of NLS. Indeed, in [13], the generators of this algebra have been expressed in term of the ZZF algebra. They take the form

$$Q^a_s = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} Q^a_{s,(n)} \quad \text{with } s = 0, 1$$  \hspace{1cm} (5.5)

$$Q^a_{s,(n)} = \int d^n k a_1^\dagger(k_1) \ldots a_n^\dagger(k_n) J^a_{s,(n)} a_n(k_n) \ldots a_1(k_1)$$  \hspace{1cm} (5.6)

where $J^a_{s,(n)}$ belongs to $M(N, \mathbb{C})^\otimes n(k_1, \ldots, k_n)$, $M(N, \mathbb{C})$ being the space of $N \times N$ matrices (see [13] for the exact expression). The Yangian is a symmetry of the whole hierarchy associated to NLS, as it can be seen from the expression of the Hamiltonians $H_m$ in terms of the ZZF algebra:

$$H_m = \int dk k^m a_1^\dagger(k) a(k) \Rightarrow [H_m, Q^a_s] = 0$$  \hspace{1cm} (5.7)

In fact, the generators $a_1^\dagger(k)$ correspond to the asymptotic states of the NLS hierarchy, and it is natural to look at the Fock space $\mathcal{F}$ spanned by the $a_1^\dagger$’s. This Fock space naturally decomposes into eigenspaces of the particle number $H_0$: $\mathcal{F} = \bigoplus_p \mathcal{F}_p$.

Now, on each subspace $\mathcal{F}_p$, the sums (5.5) truncate at level $n = p$, in the same way one defines $\mathcal{W}_p(N)$ from $Y(N)$.

Thus, on each subspace $\mathcal{F}_p$, the action of $Y(N)$ reduces to the $\mathcal{W}_p(N)$ algebra.

6 Orthogonal and symplectic cases

6.1 Folding $\mathcal{W}_p(N)$

It is known for a long time that $so(M)$ and $sp(2M)$ algebras can be obtained from $gl(N)$ ones using their outer automorphism. If $\tau$ is such an automorphism, the $so(M)$ and $sp(2M)$ algebras are obtained as $\text{Ker}(\mathbb{I} - \tau)$, i.e. the algebra of $\tau$-invariant generators of $gl(N)$.

It is the same technics that is used for $\mathcal{W}$-algebras. Indeed, it has already been showed that $\mathcal{W}$-algebras based on $so(M)$ and $sp(2M)$ can be constructed from the ones based on $gl(N)$ [14]. The Hamiltonian reduction (i.e. the constraints) must be compatible with the folding (i.e. the automorphism) so that not all the $\mathcal{W}(gl(N), \mathcal{H})$ algebras can be folded. However, it is enough to produce all the $\mathcal{W}$-algebras based on $so(M)$ and $sp(2M)$.

Here, we will consider only the folding of $\mathcal{W}_p(N)$, which can indeed be folded. The automorphism we consider has been defined in [15]. It takes the form:

$$\tau_{\pm}(W_{ij}^{N}) = (-1)^{n+1} \theta^i \theta^j W_{n}^{N+1-j, N+1-i} \begin{cases} \theta^i = 1 \text{ for } \tau_+ \\ \theta^i = \text{sg}(\frac{N+1}{2} - i) \text{ for } \tau_- \text{ and } N = 2n \end{cases}$$  \hspace{1cm} (6.1)

The folded $\mathcal{W}$-algebra is then defined by:
\textbf{Definition 1} The folded $W_p(N)^\pm$ algebra is defined by the coset $W_p(N)/J$ where $J$ is the ideal generated by $W_{n}^{ij} - \tau_{\pm}(W_{n}^{ij})$.

Note that $J$ is an ideal for the product law, and one can show that the coset can be provided with the bracket of the $W_p(N)$ algebra (see \cite{15} for more detail).

Now, one can prove \cite{14,15}:

\textbf{Theorem 5} $W_p(2n)^+$ (resp. $W_p(2n+1)^+$ and $p = 2k+1$, resp. $W_p(2n)^-$) is $W[so(2np), n.sl(p)]$ (resp. $W[so((2n+1)p), n.sl(p) \oplus so(k)]$, resp. $W[sp(2np), n.sl(p)]$).

6.2 Twisted Yangians

In the same way $U[gl(Np)]$ and $W_p(N)$ have been folded into $U(G)$ and $W(G,H)$ with $G = so(M)$ and $sp(2M)$, one naturally considers the case of $Y(N)$. However, although Yangians based on $so(M)$ and $sp(2M)$ exist, it is not these Hopf algebras that are obtained through this procedure, but another type of algebras, named twisted Yangians \cite{16}. More precisely, the automorphism \eqref{6.1} takes here the form

$$\tau(T(u)) = T^t(-u) \quad \text{with} \quad T^t(u) = \sum_{i,j} T^{ij}(u) E_{ij}^t \quad \text{and} \quad E_{ij}^t = \theta^i \theta^j E_{N+1-j,N+1-i}$$ \eqref{6.2}

It can be shown that $\tau$ is an automorphism of $Y(N)$. From this automorphism, one defines

$$S(u) = T(u) \tau(T(u))$$ \eqref{6.3}

Essentially, two classes of automorphisms appear, labeled by a parameter $\theta_0 = \pm 1$:

\begin{align*}
\text{For } Y^+(N) : & \quad \theta^i = 1, \forall i \quad (\theta_0 = 1) \\
\text{For } Y^-(2n) : & \quad \theta^i = \text{sg}(\frac{N+1}{2} - i), \forall i \quad (\theta_0 = -1) \quad \text{(6.4)}
\end{align*}

This defines a subalgebra $Y^\pm(N)$ of $Y(N)$, whose commutation relations are coded in

$$R_{12}(u-v) S_1(u) R'_{12}(u+v) S_2(v) = S_2(v) R'_{12}(u+v) S_1(u) R_{12}(u-v)$$ \eqref{6.5}

where $R(x)$ is given by \eqref{2.11}, and

$$R'(x) = (\tau \otimes \mathbb{1})(R(x)) = (\mathbb{1} \otimes \tau)(R(x)) = \mathbb{1} - \frac{1}{x} Q_{12}$$

with $Q_{12} = \sum_{i,j=1}^{N} \theta^i \theta^j E_{ij} \otimes E_{N+1-i,N+1-j}$ \eqref{6.6}

The finite dimensional irreducible representations and the center of $Y^\pm(N)$ have been determined in \cite{17}. 

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At the classical level, $S(u)$ generates a Poisson subalgebra $Y(N)$, the Poisson brackets being defined by
\[ \{S_1(u), S_2(v)\} = [r_{12}(u - v), S_1(u)S_2(v)] + S_2(v)r'_{12}(u + v)S_1(u) - S_1(u)r'_{12}(u + v)S_2(v) \]
where \( r'_{12}(x) = (I \otimes \tau) r_{12}(x) = (\tau \otimes I) r_{12}(x) \)

The level one generators of this subalgebra form the Lie algebra \( so(M) \) or \( sp(2M) \), but the total subalgebra is not the Yangian based on \( so(M) \) or \( sp(2M) \) (see [18] for more details).

However, it is these algebras that are involved in the comparison with $\mathcal{W}$-algebras:

**Theorem 6** The truncated classical Yangians $Y_p^\pm(N)$ are $\mathcal{W}$-algebras. More precisely:

\[
Y_p(2n)^- \equiv \mathcal{W}[sp(2np), n.sl(p)] ; \quad Y_p(2n)^+ \equiv \mathcal{W}[so(2np), n.sl(p)] \\
Y_p(2n+1)^+ \equiv \mathcal{W}[so((2n+1)p), n.sl(p) \oplus so(k)] \text{ with } p = 2k + 1
\]

where \( \equiv \) denotes algebra isomorphisms. The truncation is defined as in theorem [2]

Let us remark that, as in the case of $Y_p(N)$, the isomorphism cannot be extended to a Hopf algebra isomorphism, the (untruncated) twisted Yangians $Y(N)$ being even not Hopf algebras (only left coideals in $Y(N)$).

As for the Yangian $Y(N)$, this isomorphism provides a simple way of quantizing the $\mathcal{W}$-algebras. One can also use it to determine the center and the finite-dimensional irreducible representations of these $\mathcal{W}$-algebras, see [15] for more details.

## 7 Generalization to superalgebras

Once again, one can apply the same technics to the case of super-Yangians and $\mathcal{W}$-superalgebras. As for $Y(N)$ and $gl(N)$, the case of $gl(M|N)$ singles out.

### 7.1 Super-Yangian $Y(M|N)$

They are based on the superalgebra $gl(M|N)$ in the same way $Y(N)$ is based on $gl(N)$. They have been defined in [19] and their representations are studied in [20]. One defines a $\mathbb{Z}_2$-grading

\[
[T_{(n)}^{ij}] = [i] + [j] \quad \text{with} \quad \left\{ \begin{array}{ll} [i] = 0 & \text{for } 1 \leq i \leq M \\ [i] = 1 & \text{for } M + 1 \leq i \leq M + N \end{array} \right.
\]

and introduces as usual:

\[
T(u) = \sum_{i,j=1}^{M+N} \sum_{n \geq 0} u^{-n} T_{(n)}^{ij} E_{ij} = \sum_{i,j=1}^{M+N} T^{ij}(u) E_{ij} \quad \text{and} \quad P_{12} = \sum_{i,j} (-1)^{[i][j]} E_{ij} \otimes E_{ji}
\]
The super-Yangian is then defined by
\[ R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v) \quad \text{with} \quad R_{12}(u) = I - \frac{1}{u}P_{12} \quad (7.3) \]
where we have introduced graded tensor products:
\[ T_1(u) = \sum_{i,j,k,l} (-1)^{([i]+[j])[k]} T_{ij}(u) \delta_{kl} \otimes E_{ij} \quad \text{and} \quad T_2(u) = \sum_{i,j} T_{ij}(u) \mathbb{1} \otimes E_{ij} \quad (7.4) \]
It is a graded Hopf algebra, and its \( R \)-matrix obeys a graded Yang-Baxter algebra. Their classical version is defined as in section 2.1. We refer to [19, 20, 21] for more details.

7.2 \( \mathcal{W}(M|N) \)-superalgebras
Starting from the superalgebra \( gl(M|N) \) and using \( sl(2) \) embeddings, one can construct \( \mathcal{W} \)-superalgebras. The \( sl(2) \) generators being bosonic, they belong to the \( gl(M) \oplus gl(N) \) subalgebra, and the procedure is the same as in section 3. The only difference comes with the fermionic generators which have to be constrained to Grassmann constant for consistency: see [8] for details.

As for the \( gl(N) \) case, one selects a special class of \( \mathcal{W} \)-superalgebras: the finite \( \mathcal{W} \)-superalgebras \( \mathcal{W}_p(M|N) = \mathcal{W}(gl(pM|pN), (M+N).sl(p)) \), where \( (M+N).sl(p) \) denotes the direct sum of \( (M+N) \) algebras \( sl(p) \), \( M \) of them being included in the \( gl(M) \) subalgebra of \( gl(M|N) \) and the \( N \) remaining in its \( gl(N) \) subalgebra. Then, one can prove

**Theorem 7** The \( \mathcal{W}_p(M|N) \) superalgebras are isomorphic to the truncation at level \( p \) of the classical super-Yangian \( Y(M|N) \).

This isomorphism allows us to classify the irreducible finite-dimensional representations of the \( \mathcal{W}_p(M|N) \) superalgebras [21].

7.3 Twisted super-Yangians
Similarly to the twisted Yangians, one can define the twisting of super-Yangians [22]. This leads to subalgebras of \( Y(M|N) \) which contains the orthosymplectic superalgebras. Mimicking the case of twisted Yangian, one introduces an automorphism of \( Y(M|N) \):
\[ \tau(T_{ij}(u)) = (-1)^{[i][j]+1} \theta_i \theta_j T_{ij}(-u) \quad \text{with} \quad \theta_i = \pm 1 \quad ; \quad (-1)^{[i]} \theta_i \theta_0 = \theta_0 = \pm 1 \\
\bar{i} = M + 1 - i \quad \text{for} \quad 1 \leq i \leq M \\
\bar{i} = 2M + N + 1 - i \quad \text{for} \quad M + 1 \leq i \leq M + N \quad (7.5) \]
However, the presence of fermionic generators forces to have\(^*\) \( N = 2n \) and \( \theta_0 = +1 \). Thus, one is led to the definition:

\(^*\)Up to the Hopf algebra isomorphism \( Y(M|N) \leftrightarrow Y(N|M) \) which identifies \( M = 2m \) and \( \theta_0 = -1 \) with \( N = 2n \) and \( \theta_0 = 1 \), see [22].
Definition 2  The twisted Yangian $Y(M|2n)^+ \text{ is the subalgebra of } Y(M|2n)$ generated by $S(u) = T(u)\tau(T(u))$, where $\tau$ is defined in (7.5) with
\[ \begin{align*}
\theta_i &= 1 \quad \text{for } 1 \leq i \leq M \\
\theta_i &= \text{sg}(\frac{2M+N+1}{2} - i) \quad \text{for } M + 1 \leq i \leq M + 2n
\end{align*} \] (7.6)

From this definition, one proves that $S(u)$ obeys the rules:
\[ R_{12}(u-v) S_1(u) R'_{12}(u+v) S_2(v) = S_2(v) R'_{12}(u+v) S_1(u) R_{12}(u-v) \] (7.7)
\[ \tau(S(u)) = S(-u) + \frac{1}{2u} (S(u) - S(-u)) \] (7.8)

The irreducible finite-dimensional representations of the twisted super-Yangian are studied in [22].

As far as $\mathcal{W}$-superalgebras are concerned, their folding have been introduced in [14] and shown to lead to $\mathcal{W}$-superalgebras based on $osp(M|N)$ superalgebras. Considering a special class of folded $\mathcal{W}$-superalgebras, one gets again:

Theorem 8  Let $\mathcal{W}_p(M|2n)^+$ be the $\mathcal{W}[osp(Mp|2np), ([\frac{M}{2}] + n)sl(p) \oplus \epsilon_M so(k)]$ superalgebra, where $\epsilon_M \equiv M \pmod{2}$ and $p$ is chosen odd ($p = 2k+1$) when $M$ is odd. $\mathcal{W}_p(M|2n)^+$ is isomorphic to the truncation at level $p$ of the twisted super-Yangian $Y(M|2n)^+$.

It allows to classify the finite dimensional representations of the $\mathcal{W}_p(M|2n)^+$ algebras [22].

8  Conclusion

A wide class of $\mathcal{W}$-(super)algebras are shown to be isomorphic to the truncation of (super)(twisted) Yangians. This isomorphism allows to classify all the irreducible finite-dimensional representations of these $\mathcal{W}$-algebras.

Moreover, since there are many more $\mathcal{W}$-algebras, the connection let us hope that a generalization of Yangians (as Hopf algebras) is available. The same is valid for affine $\mathcal{W}$-algebras, which should lead to two-parameters generalization of Yangians.

Finally, the application to physical models, such as Nonlinear Schrödinger equation has to be studied.

References

[1] V.G. Drinfel’d, Hopf algebras and the quantum Yang-Baxter equation, Sov. Math. Dokl. 32 (1985) 254.
[2] A.A. Kirillov and N. Yu. Reshetikhin, *The Yangians, Bethe ansatz and combinatorics*, Lett. Math. Phys. 12 (1986) 199.

[3] P.P. Kulish, N. Yu. Reshetikhin, E.K. Sklyanin, *Yang-Baxter equation and representation theory*, Lett. Math. Phys. 5 (1981) 393.
V.O. Tarasov, *Irred. monodromy matrices for the R-matrix of the XXZ model and local lattice quantum Hamiltonians*, Theor. Math. Phys. 63 (1985) 440.
V. Chari, A. Pressley, *Yangians and R-matrices*, Enseign. Math. 36 (1990) 267.

[4] A.B. Zamolodchikov, *Infinite additional symmetries in two-dimensional conformal quantum field theory*, Theor. Math. Phys. 63 (1985) 347.

[5] L. Feher, L. O’Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, *On the general structure of Hamiltonian reductions of the WZNW theories*, Phys. Rep. 222 (1992) 1, and references therein.

[6] P. Bouwknegt and K. Schoutens, *W-symmetry in conformal field theory*, Phys. Rep. 223 (1993) 183, and references therein.

[7] J. De Boer, F. Harmsze and T. Tjin, *Nonlinear finite W-symmetries and applications in elementary systems*, Phys. Rep. 272 (1996) 139, and ref. therein.

[8] L. Frappat, E. Ragoucy, P. Sorba, *W-algebras and superalgebras from constrained WZW models: a group theoretical classification*, Comm. Math. Phys. 157 (1993) 499.

[9] E. Ragoucy and P. Sorba, *Yangian realization from finite W-algebras*, Comm. Math. Phys. 203 (1999) 551.

[10] C. Briot and E. Ragoucy, *RTT presentation of finite W-algebras*, Preprint LAPTH-792/00, [math.QA/0005111](http://arxiv.org/abs/math.QA/0005111).

[11] R. Rosales, Stud. Appl. Math. 59 (1978) 117; E. Sklyanin, L. D. Faddeev, Sov. Phys. Dokl. 23 (1978) 902; E. Sklyanin, Sov. Phys. Dokl. 24 (1979) 107; H.B. Tacker, D. Wilkinson, Phys. Rev. D19 (1979) 3660; D.B. Creamer, H.B. Tacker, D. Wilkinson, Phys. Rev. D21 (1980) 1523; J. Honerkamp, P. Weber, A. Wiesler, Nucl. Phys. B152 (1979) 266; B. Davies, J. Phys. A14 (1981) 2631.

[12] A.B. Zamolodchikov, A.B. Zamolodchikov, Ann. Phys. 120 (1979) 253; L.D. Faddeev, Sov. Scient. Rev. C1 (1980) 107.

[13] M. Mintchev, E. Ragoucy, P. Sorba and Ph. Zaugg, *Yangian symmetry in the Nonlinear Schrödinger hierarchy*, J. Phys. A32 (1999) 5885, [hep-th/9905105](http://arxiv.org/abs/hep-th/9905105)
[14] L. Frappat, E. Ragoucy and P. Sorba, *Folding the \(W\)-algebras*, Nucl. Phys. B404 (1993) 805.

[15] E. Ragoucy, *Folded \(W\)-algebras as truncations of twisted Yangians*, preprint LAPTH-824/00, math.QA/0012182, Int. J. Mod. Phys. A, to appear.

[16] G. Olshanski, *Representations of infinite dimensional classical Lie algebras*, Lect. Notes Math. bf 1510 (1992) 103, Springer, Berlin-Heidelberg.

[17] A. Molev, *Finite-dimensional irreducible representations of twisted Yangians*, J. Math. Phys. 39 (1998) 5559, q-alg/9711022.

[18] A. Molev, M. Nazarov and G. Olshanski, *Yangians and classical Lie algebras*, Russian Math. Survey 51 (1996) 205, hep-th/9409025.

[19] M. Nazarov, *Quantum Berezinian and the classical Capelli identity*, Lett. Math. Phys. 21 (1991) 123.

[20] R.B. Zhang, *Representations of super Yangian*, hep-th/9411243.

R.B. Zhang, *The \(gl(M|N)\) super Yangian and its finite dimensional representations*, Lett. Math. Phys. 37 (1996) 419, hep-th/9507029.

[21] C. Briot and E. Ragoucy, *\(W\)-superalgebras and super-Yangians*, in preparation.

[22] C. Briot and E. Ragoucy, *Twisted super-Yangians*, in preparation.