Holographic entropy and brane FRW-dynamics from AdS black hole in $d5$ higher derivative gravity

SHIN’ICHI NOJIRI, SERGEI D. ODINTSOV, and SACHIKO OGUSHI

Department of Applied Physics
National Defence Academy, Hashirimizu Yokosuka 239-8686, JAPAN

♠ Instituto de Física de la Universidad de Guanajuato,
Lomas del Bosque 103, Apdo. Postal E-143, 37150 Leon, Gto., MEXICO

♦ Yukawa Institute for Theoretical Physics,
Kyoto University, Kyoto 606-8502, JAPAN

ABSTRACT

Higher derivative bulk gravity (without Riemann tensor square term) admits AdS-Schwarzschild black hole as exact solution. It is shown that induced brane geometry on such background is open, flat or closed FRW radiation dominated Universe. Higher derivative terms contributions appear in the Hawking temperature, entropy and Hubble parameter via the redefinition of 5-dimensional gravitational constant and AdS scale parameter. These higher derivative terms do not destroy the AdS-dual description of radiation represented by strongly-coupled CFT. Cardy-Verlinde formula which expresses cosmological entropy as square root from other parameters and entropies is derived in $R^2$ gravity. The corresponding cosmological entropy bounds are briefly discussed.

1 nojiri@cc.nda.ac.jp
2 On leave from Tomsk State Pedagogical University, 634041 Tomsk, RUSSIA.
odintsov@ifug5.ugto.mx, odintsov@mail.tomsknet.ru
3 ogushi@yukawa.kyoto-u.ac.jp
1 Introduction

Brane-world physics (especially in the form of Randall-Sundrum proposal [1]) may provide new revolutionary ideas on the structure of the early Universe and theory of fundamental interactions. Indeed, one can feel very uncomfortable with the picture of observable Universe as tiny boundary of higher-dimensional (fundamental?) object like black hole. Nevertheless, such situation is under current active discussion in high energy physics literature. Moreover, it was realized recently [2] (for related works, see [3]) that brane equations of motion are exactly Friedmann-Robertson-Walker (FRW) equations with radiation matter. This radiation matter plays the role of CFT in AdS/CFT correspondence [4]. This fact has direct interpretation as a consequence of holographic principle [5].

Indeed, it was demonstrated in ref. [5] that FRW equation can be related with three different cosmological entropies bounds, giving kind of constraint among them. Furthermore, FRW equation can be rewritten in the form of generalized Cardy-Verlinde formula [6, 5] relating cosmological entropy with the one of CFT filling the Universe. There was recently much activity on the studies of related questions [7, 3].

Our purpose in this work is further study of the CFT dominated Universe as the brane in the background of AdS Black Hole (BH). Our bulk theory is higher derivatives gravity which is known to possess AdS BH solution. The interest in the higher derivatives bulk gravity is caused by the following. First of all, any effective stringy gravity includes higher derivative terms of different order. Second, from the point of view of AdS/CFT correspondence the $R^2$-terms give next-to-leading terms in large $N$ expansion [8] as it was directly checked in the calculation of holographic conformal anomaly from bulk $R^2$ gravity [9]. Third, higher derivative gravity may serve as quite good candidate for the construction of realistic brane-world cosmologies [10, 11].

Our consideration is limited to specific model (c=0 model) of bulk $R^2$-gravity which does not contain the square of Riemann tensor in the action. The attractive feature of this model is that it admits Schwarzschild-Anti de Sitter black hole as exact solution in five dimensions. Moreover, as it was demonstrated in ref. [11] observable Universe could be the brane (boundary) of such BH. The extension to the case of c not equal zero (inclusion of Riemann tensor square) is evident [11], and qualitative conclusions are left the same.
The paper is organized as the following. In the next section the review of thermodynamics of AdS BH in d5 $R^2$ gravity is given. Hawking temperature, horizon radius and entropy are presented in the form suitable for later identification with the corresponding quantities in brane FRW Universe. Section 3 is devoted to the discussion of surface terms in higher derivative gravity, the surface counterterm on AdS BH background is given. Brane dynamical equations are constructed. In section 4 it is shown how induced geometry of brane takes the form of open, flat or closed radiation dominated FRW equations. The role of higher derivative terms is analyzed. The connection of higher derivative terms with AdS/CFT correspondence is mentioned. In the last section we construct Cardy-Verlinde formula in $R^2$ gravity. The cosmological entropy bounds are briefly discussed.

2 AdS Black holes in bulk $R^2$-gravity and their thermodynamical properties

Let us consider thermodynamics of AdS BH in bulk $R^2$-gravity. The calculation of thermodynamical quantities like mass and entropy will be necessary to relate them with the corresponding ones in brane FRW Universe. The general action of $d + 1$-dimensional $R^2$ gravity is given by:

$$S = \int d^{d+1}x \sqrt{-\hat{G}} \left\{ a\hat{R}^2 + b\hat{R}_{\mu\nu}\hat{R}^{\mu\nu} + c\hat{R}_{\mu\nu\xi\sigma}\hat{R}^{\mu\nu\xi\sigma} + \frac{1}{\kappa^2} \hat{R} - \Lambda \right\}.$$  \hspace{1cm} (1)

The following conventions of curvatures are used

\begin{align*}
R &= g^{\mu\nu}R_{\mu\nu}, \\
R_{\mu\nu} &= -\Gamma_{\mu\lambda,\nu} + \Gamma_{\nu\mu,\lambda} - \Gamma_{\nu\lambda,\mu} + \Gamma_{\mu\nu,\lambda} + \Gamma_{\mu\lambda,\nu} - \Gamma_{\nu\lambda,\mu}, \\
\Gamma_{\mu\lambda} &= \frac{1}{2} g^{\nu\sigma} (g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu}). \hspace{1cm} (2)
\end{align*}

When $a = b = c = 0$, the action (1) becomes that of the Einstein gravity.

When $c = 0$ \footnote{For non-zero $c$ such S-AdS BH solution may be constructed perturbatively \cite{1}. It is useful to establish the higher-derivative AdS/CFT correspondence \cite{2} and find the strong coupling limit of super Yang-Mills theory with two supersymmetries in next-to-leading order.}, Schwarzschild-anti de Sitter space is an exact solution:

$$ds^2 = \hat{G}_{\mu\nu}dx^\mu dx^\nu$$
\[ e^{2\rho_0} = \frac{1}{r^{d-2}} \left( -\mu + \frac{k r^{d-2}}{d-2} + \frac{r^d}{l^2} \right). \] (3)

The curvatures have the following form:

\[ \hat{R} = -\frac{d(d+1)}{l^2}, \quad \hat{R}_{\mu\nu} = -\frac{d}{l^2} \hat{G}_{\mu\nu}. \] (4)

In (3), \( \mu \) is the parameter corresponding to mass and the scale parameter \( l \) is given by solving the following equation:

\[ 0 = \frac{d^2(d+1)(d-3)a}{l^4} + \frac{d^2(d-3)b}{l^4} - \frac{d(d-1)}{\kappa^2 l^2} - \Lambda. \] (5)

We also assume \( g_{ij} \) expresses the Einstein manifold, defined by \( r_{ij} = kg_{ij} \), where \( r_{ij} \) is the Ricci tensor defined by \( g_{ij} \) and \( k \) is the constant. For example, if \( k > 0 \) the boundary can be 4 dimensional de Sitter space (sphere when Wick-rotated), if \( k < 0 \), anti-de Sitter space or hyperboloid, or if \( k = 0 \), flat space. By properly normalizing the coordinates, one can choose \( k = 2, 0, \) or \(-2\).

The calculation of thermodynamical quantities like free energy \( F \), the entropy \( S \) and the energy \( E \) may be done with the help of the folllowing method [12]. After Wick-rotating the time variable by \( t \to i\tau \), the free energy \( F \) can be obtained from the action \( S \) where the classical solution is substituted:

\[ F = TS \] (6)

Substituting eqs. (3) into (1) in the case of \( d = 4 \) with \( c = 0 \), one gets

\[ S = -\int d^5x \sqrt{-G} \left( \frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} \right) \]

\[ = -\frac{V_3}{T} \int_{r_H}^{\infty} dr r^3 \left( \frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} \right). \] (7)

Here \( V_3 \) is the volume of 3d sphere and we assume \( \tau \) has a period of \( \frac{1}{T} \). The expression of \( S \) contains the divergence coming from large \( r \). In order to
subtract the divergence, we regularize $S$ in (4) by cutting off the integral at a large radius $r_{\text{max}}$ and subtracting the solution with $\mu = 0$ in a same way as in [13]:

$$S_{\text{reg}} = -\frac{V_3}{T} \left\{ \int_{r_H}^{r_{\text{max}}} drr^3 - e^{\rho(r=r_{\text{max}})-\rho(r=r_{\text{max}};\mu=0)} \int_{0}^{r_{\text{max}}} drr^3 \right\} \times \left( \frac{8}{\kappa^2 l^2} - \frac{320a}{l^4} - \frac{64b}{l^4} \right)$$  \hspace{1cm} (8)

The factor $e^{\rho(r=r_{\text{max}})-\rho(r=r_{\text{max}};\mu=0)}$ is chosen so that the proper length of the circle which corresponds to the period $\frac{1}{T}$ in the Euclidean time at $r = r_{\text{max}}$ concides with each other in the two solutions. Taking $r_{\text{max}} \to \infty$, one finds

$$F = V_3 \left( \frac{l^2 \mu}{8} - \frac{r_H^4}{4} \right) \left( \frac{8}{\kappa^2 l^2} - \frac{320a}{l^4} - \frac{64b}{l^4} \right)$$ \hspace{1cm} (9)

The horizon radius $r_h$ is given by solving the equation $e^{2\rho_0(r_H)} = 0$ in (3):

$$r_H^2 = -\frac{k l^2}{4} + \frac{1}{2} \sqrt{\frac{k^2}{4} l^4 + 4\mu l^2}. \hspace{1cm} (10)$$

The Hawking temperature $T_H$ is

$$T_H = \left( \frac{e^{2\rho}}{4\pi} \right)'_{r=r_H} = \frac{k}{4\pi r_H} + \frac{r_H}{\pi l^2}$$ \hspace{1cm} (11)

where $'$ denotes the derivative with respect to $r$. From the above equation (11), $r_H$ can be rewritten in terms of $T_H$ as

$$r_H = \frac{1}{2} \left( \pi l^2 T_H \pm \sqrt{(\pi l^2 T_H)^2 - kl^2} \right)$$ \hspace{1cm} (12)

In (12), the plus sign corresponds to $k = -2$ or $k = 0$ case and the minus sign to $k = 2$ case. One can also rewrite $\mu$ using $r_H$ or $T_H$ from (10) as

\footnote{When $k = 2$, as we can see from (10) and (11), $r_H$, and also $T_H$, are finite in the limit of $l \to \infty$, which corresponds to the flat background. Therefore we need to choose the minus sign in (12) for $k = 2$ case.}
follows:

\[
\mu = \frac{r_H^4}{l^2} + \frac{kr_H^2}{2} = r_H^2 \left( \frac{r_H^2}{l^2} + \frac{k}{2} \right)
\]

\[
= \frac{1}{4} \left( \pi l^2 T_H \pm \sqrt{(\pi l^2 T_H)^2 - kl^2} \right)^2 \times \left( \frac{1}{4l^2} \left( \pi l^2 T_H \pm \sqrt{(\pi l^2 T_H)^2 - kl^2} \right)^2 + \frac{k}{2} \right) . \tag{13}
\]

Then we can rewrite \( F \) using \( T_H \) or \( r_H \) as

\[
F = -\frac{V_3}{32l^2} \left( \frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} \right) \left( \pi l^2 T_H \pm \sqrt{(\pi l^2 T_H)^2 - kl^2} \right)^2 \times \left( -\left( \pi l^2 T_H + \sqrt{(\pi l^2 T_H)^2 - kl^2} \right)^2 - kl^2 \right),
\]

\[
= -\frac{V_3}{8} r_H^2 \left( \frac{r_H^2}{l^2} - \frac{k}{2} \right) \left( \frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} \right) . \tag{14}
\]

The entropy \( S \) and energy \( E \) are

\[
S = -\frac{dF}{dT_H} = -\frac{dF}{dr_H} \frac{dr_H}{dT_H}
\]

\[
= \frac{V_3}{16} \left( 4r_H^3 - kr_H \right) \left( \pi l^2 + \sqrt{\pi l^2 T_H^2} \right) \left( \frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} \right)
\]

\[
= \frac{V_3 \pi r_H^3}{2} \left( \frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} \right)
\]

\[
= \frac{V_3 \pi}{16} \left( \pi l^2 T_H \pm \sqrt{(\pi l^2 T_H)^2 - kl^2} \right)^3 \left( \frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} \right) . \tag{15}
\]

\[
E = F + TS
\]

\[
= \frac{3V_3 \mu}{8} \left( \frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} \right) . \tag{16}
\]

The above equations reproduce the standard Einstein theory results when \( a = b = 0 \). Note that one can consider the limit of \( l \to 0 \), where the background spacetime becomes flat Minkowski space. Since the scalar curvature and Ricci tensor vanishes in the flat Minkowski, we cannot derive the thermodynamical quantities by evaluating the action \( S \), which vanishes, if we start
with the flat Minkowski background from the beginning. Then finite \( l \) (or finite, non-vanishing cosmological constant) would give a kind of the regularization. Note also that above expressions will be used to get the cosmological entropy of brane FRW Universe.

3 Surface terms in \( R^2 \)-gravity

Before considering the dynamics of the brane, we review the problem of the variational principle in the Einstein gravity, whose action is given by

\[
S_E = \frac{1}{\kappa^2} \int d^{d+1}x \sqrt{-\hat{G}} \hat{R}.
\]

The scalar curvature contains the second order derivative of the metric tensor \( \hat{G}_{\mu\nu} \) with respect to the coordinates. Therefore if there is a boundary, which we denote by \( B \), under the variation \( \delta \hat{G}_{\mu\nu} \), \( \delta S_E \) contains, on the boundary, the derivative of \( \delta \hat{G}_{\mu\nu} \) with respect to the coordinate perpendicular to the boundary, which makes the variational principle ill-defined. Therefore we need to add a surface term to the action, which is called the Gibbons-Hawking surface term [14]. Note that by using the definition of the curvature in (2), the Einstein action can be rewritten as

\[
S_E = \frac{1}{\kappa^2} \int d^{d+1}x \left[ \sqrt{-\hat{G}} \left( -\hat{\Gamma}^\eta_{\mu\lambda} \hat{\Gamma}^\lambda_{\nu\eta} + \hat{\Gamma}^\eta_{\mu\nu} \hat{\Gamma}^\lambda_{\lambda\eta} \right) \hat{G}^{\mu\nu} \\
+ \hat{\Gamma}^\lambda_{\mu\lambda} \partial_\nu \left( \sqrt{-\hat{G}} \hat{G}^{\mu\nu} \right) - \hat{\Gamma}^\lambda_{\mu\nu} \partial_\lambda \left( \sqrt{-\hat{G}} \hat{G}^{\mu\nu} \right) \right] \\
+ \frac{1}{\kappa^2} \int_B d^d x \sqrt{-\hat{g}} \left[ -n_\nu \hat{\Gamma}^\lambda_{\mu\lambda} \hat{G}^{\mu\nu} + n_\lambda \hat{\Gamma}^\lambda_{\mu\nu} \hat{G}^{\mu\nu} \right].
\]

(19)

Here \( n_\mu \) is the unit vector perpendicular to the boundary and \( \hat{g}_{mn} \) is the boundary metric induced from \( \hat{G}_{\mu\nu} \). Since the bulk part of the action does not contain the second order derivative of \( \hat{G}_{\mu\nu} \) with respect to the coordinates, the variational principle becomes well-defined if we add the following boundary term to the Einstein action:

\[
\tilde{S}_b = -\frac{1}{\kappa^2} \int_B d^d x \sqrt{-\hat{g}} \left[ -n_\nu \hat{\Gamma}^\lambda_{\mu\lambda} \hat{G}^{\mu\nu} + n_\lambda \hat{\Gamma}^\lambda_{\mu\nu} \hat{G}^{\mu\nu} \right].
\]

(20)

The action (20), however, breaks the general covariance in general. We should note, however, that

\[
\nabla_\mu n_\nu = \partial_\mu n_\nu - \hat{\Gamma}^\lambda_{\mu\nu} n_\lambda, \quad \nabla_\mu n^\nu = \partial_\mu n^\nu + \hat{\Gamma}^\nu_{\mu\lambda} n^\lambda
\]

(21)
and therefore

\[ - n_\nu \hat{\Gamma}_\mu^\lambda \hat{G}^{\mu\nu} + n_\lambda \hat{\Gamma}_\mu^\nu \hat{G}^{\mu\nu} = \partial_\mu n^\nu + \hat{C}^{\mu\nu} \partial_\mu n_\nu - 2 \nabla_\mu n^\nu. \]  

(22)

Then one can replace the boundary action \( \tilde{S}_b \) with the Gibbons-Hawking one:

\[ S_{GH} = \frac{2}{\kappa^2} \int_B d^d x \sqrt{-\hat{g}} \nabla_\mu n^\mu, \]  

(23)

at least for the following metric

\[ ds^2 = \left(1 + \mathcal{O}(q^2)\right) dq^2 + \hat{g}_{mn}(q, x^m) dx^m dx^n. \]  

(24)

Here \( q \) is the radial coordinate and the brane exists at \( q = 0 \). The difference \( \partial_\mu n^\nu + \hat{C}^{\mu\nu} \partial_\mu n_\nu \) between \( \tilde{S}_b \) and \( S_{GH} \), which appears in (22), vanishes since both of \( n_\mu \) and \( n^\mu \) become constant vectors.

Motivated by the above argument, one assumes the surface term in the following form [13, 11]:

\[ S_b = S_b^{(1)} + S_b^{(2)}, \]  

(25)

\[ S_b^{(1)} = \frac{2}{\kappa^2} \int d^d x \sqrt{\hat{g}} \nabla_\mu n^\mu, \]  

(26)

\[ S_b^{(2)} = -\eta \int d^d x \sqrt{\hat{g}}. \]  

(27)

The parameter \( \eta \) (brane tension) which is usually free parameter in brane-world cosmology is not free any more and can be determined by the condition that the leading divergence of bulk AdS should vanish when one substitutes the classical solution (3) into the action (1) with \( c = 0 \) and into (25) or (26):

\[ S = \int d^4 x r_0^d \left\{ \frac{d^2(d+1)^2 a}{l^4} + \frac{d^2(d+1)b}{l^4} - \frac{d(d+1)}{\kappa^2 l^2} - \Lambda \right\} + \mathcal{O} \left( r_0^{d-1} \right) \]

\[ S_b = \int d^4 x r_0^d \left\{ + \frac{2d}{l^2 \kappa^2} - \frac{\eta}{l} \right\} + \mathcal{O} \left( r_0^{d-1} \right). \]  

(28)

Then one gets

\[ \frac{\eta}{l} = \frac{d(d+1)^2 a}{l^4} + \frac{d(d+1)b}{l^4} - \frac{d+1}{\kappa^2 l^2} - \frac{\Lambda}{d} + \frac{2d}{l^2 \kappa^2} \]  

(29)

\footnote{In [13, 11], more general counterterm corresponding to \( S_b^{(1)} \) has been considered. From the requirements of the finiteness of bulk AdS and well-defined variational principle of the action, the counterterm has finally the form (26).}
or deleting $\Lambda$ by using (5)

\[ \frac{\eta}{l} = \frac{4d(d+1)a}{l^4} + \frac{4db}{l^4} - \frac{2}{\kappa^2l^2} + \frac{2d}{l^2\kappa^2}. \tag{30} \]

Motivated with (24), we choose the metric in the following form:

\[ ds^2 = dq^2 - \varepsilon(q, \tau) d\tau^2 + \varepsilon(q, \tau) g_{ij} dx^i dx^j. \tag{31} \]

Here $g_{ij}$ is the metric of the Einstein manifold as in (3). Then the curvatures etc. can be expressed as follows:

\[
\begin{align*}
\hat{R}_{qq} &= -\frac{\zeta_{qq}}{2} - \frac{\zeta_{q}^2}{4} + (d-1) \left( -\frac{\xi_{qq}}{2} - \frac{\xi_{q}^2}{4} \right), \\
\hat{R}_{\tau\tau} &= \varepsilon \left( \frac{\zeta_{qq}}{2} + \frac{\zeta_{q}^2}{4} + \frac{(d-1)\zeta_{q}\zeta_{q}}{4} \right) \\
&\quad + (d-1) \left( -\frac{\xi_{\tau\tau}}{2} - \frac{\xi_{\tau}^2}{4} + \frac{\zeta_{\tau}\zeta_{\tau}}{4} \right), \\
\hat{R}_{q\tau} &= R_{q\tau} = (d-1) \left\{ -\frac{\xi_{q\tau}}{2} - \frac{\xi_{\tau}(\xi_{q} - \zeta_{q})}{4} \right\}, \\
\hat{R}_{ij} &= g_{ij} \left[ k + e^{-\zeta + \xi} \left( \frac{\xi_{\tau\tau}}{2} + \frac{(d-1)\xi_{\tau}^2}{4} - \frac{\zeta_{\tau}\zeta_{\tau}}{4} \right) \right. \\
&\quad + e^{-\xi} \left( -\frac{\xi_{qq}}{2} - \frac{(d-1)\xi_{q}^2}{4} - \frac{\zeta_{q}\zeta_{q}}{4} \right) \right], \\
\text{other components of Ricci tensor} &= 0, \tag{32}
\end{align*}
\]

\[
\begin{align*}
R &= (d-1)ke^{-\zeta} + (d-1)e^{-\xi} \left( \xi_{\tau\tau} + d\xi_{\tau}^2 - \frac{\zeta_{\tau}\xi_{\tau}}{2} \right) \\
&\quad + (d-1) \left( -\xi_{qq} - \frac{d\xi_{q}^2}{4} - \frac{\zeta_{q}\zeta_{q}}{2} \right) \xi_{qq} - \frac{\zeta_{q}^2}{2}, \\
\nabla_{\mu}n^{\mu} &= \frac{\zeta_{q}}{2} + \frac{(d-1)\zeta_{q}}{2}. \tag{33}
\end{align*}
\]

Assume the bulk solution has the form of (4). Then the variation of the action at the boundary has the following form:

\[ \delta S + \delta S_b \]
\[
\frac{1}{\bar{\kappa}^2} = \frac{1}{\kappa^2} - \frac{2d(d+1)a}{l^2} - \frac{2db}{l^2},
\]

(35)

Then taking \(\bar{\kappa}\) as in the previous work [11],

\[
\frac{1}{\bar{\kappa}^2} \left( \zeta_q + (d-1)\xi_q \right) - \frac{\eta}{2} \delta\xi
\]

\[
\left( \zeta_q + \frac{(d-1)\xi_q}{2} \right)
\]

\[
\left( \zeta_q + \frac{(d-1)\xi_q}{2} \right)
\]

\[
\zeta_q + \frac{(d-1)\xi_q}{2}
\]

\[
\zeta_q + \frac{(d-1)\xi_q}{2}
\]

(36)

The dynamical brane equations look as

\[
(\zeta_q + (d-1)\xi_q)_{q=0} = (d-1)\xi_q|_{q=0} = \bar{\kappa}^2 \eta .
\]

(37)

In Eqs.(36) and (37), it is supposed \(\bar{\kappa}\) is given by (35). Combining (30) and (35), we find

\[
\eta = \frac{2(d-1)}{l\bar{\kappa}^2} .
\]

(38)

Then (37) can be rewritten as

\[
(\zeta_q + (d-1)\xi_q)_{q=0} = (d-1)\xi_q|_{q=0} = \frac{2(d-1)}{l} .
\]

(39)
Especially when $e^\xi = e^\zeta = l^2 e^{2A}$, where the metric is given by

$$ds^2 = dq^2 + l^2 e^{2A} \left( -d\tau^2 + \sum_{i,j=1}^{d-1} g_{ij} dx^i dx^j \right),$$  \hspace{1cm} (40)$$

one obtains

$$A_{q|q=0} = \frac{1}{l}. \hspace{1cm} (41)$$

The contribution of the higher derivative terms come through $l \left( \frac{l}{\tau} \right)$. We should note, however, that $\frac{1}{l}$ is finite even if bulk cosmological constant $\Lambda$ vanishes.

When $d = 4$, Eq. (33) has the following form:

$$1 = \frac{1}{\tilde{\kappa}^2} - \frac{40a}{l^2} - \frac{8b}{l^2}. \hspace{1cm} (42)$$

It is non-zero even if bulk Einstein action is absent. The entropy (16) and the energy (17) have the following form:

$$S = \frac{4V_3 \pi r^3 H}{\tilde{\kappa}^2}, \hspace{1cm} (43)$$

$$E = \frac{3V_3 \mu}{\tilde{\kappa}^2}. \hspace{1cm} (44)$$

Therefore the corrections from the higher derivative terms appear through the redefinition of gravitational coupling $\kappa$ to $\tilde{\kappa}$ through (33) or (42) and the length scale $l$ given by (5). As one sees in a moment the above entropy is cosmological entropy of FRW Universe.

### 4 The FRW equation of the brane cosmology from $R^2$-gravity

Let us rewrite the metric (3) of Schwarzschild-anti de Sitter space in a form of (31) or (40). If one chooses coordinates $(q, \tau)$ as

$$l^2 e^{2A-2\rho_0} A_{q,q}^2 - e^{2\rho_0} t_{q}^2 = 1, \quad l^2 e^{2A-2\rho_0} A_{q,\tau} A_{q,\tau} - e^{2\rho_0} t_{q} t_{\tau} = 0, \quad l^2 e^{2A-2\rho_0} A_{\tau,\tau} - e^{2\rho_0} t_{\tau}^2 = -l^2 e^{2A}. \hspace{1cm} (45)$$
the metric takes the form (40). Here \( r = le^A \). Furthermore choosing a coordinate \( \tilde{t} \) by \( d\tilde{t} = le^A d\tau \), the metric on the brane takes FRW form:

\[
ds^2_{\text{brane}} = -d\tilde{t}^2 + l^2 e^{2A} \sum_{i,j=1}^{d-1} g_{ij} dx^i dx^j .
\]

(46)

By solving Eqs.(45), we have

\[
H^2 = A^2 - \frac{e^{2\rho_0} e^{-2A}}{l^2} .
\]

(47)

Here the Hubble constant \( H \) is defined by \( H = \frac{dA}{d\tilde{t}} \). Then using (3) and (41), one obtains the following equation:

\[
H^2 = \frac{1}{l^2} - \frac{1}{r^d} \left( -\mu + \frac{k r^{d-2}}{d-2} + \frac{r^d}{l^2} \right) = - \frac{k}{(d-2)r^2} + \frac{\mu}{r^d} .
\]

(48)

Especially, when \( k = d - 2 > 0 \), the spacial part of the brane has the shape of the \((d-1)\)-dimensional sphere and \( r \) can be regarded as the radius of the spacial part of the brane universe.

Eq.(48) can be rewritten in the form of the FRW equation (compare with [2]):

\[
H^2 = - \frac{k}{(d-2)r^2} + \frac{\kappa^2_d}{(d-1)(d-2)V} \tilde{E} ,
\]

\[
\tilde{E} = \frac{(d-1)(d-2)\mu V_{d-1}}{\kappa^2_d r} ,
\]

\[
V = r^{d-1} V_{d-1} .
\]

(49)

Here \( V_{d-1} \) is the volume of the \((d-1)\)-dimensional sphere with a unit radius and \( \kappa_d \) is the \( d \)-dimensional gravitational coupling, which is given by

\[
\kappa^2_d = \frac{2\kappa^2}{l} .
\]

(50)

By differentiating Eq.(49) with respect to \( \tilde{t} \), since \( H = \frac{1}{r} \frac{dx}{dt} \), we obtain the second FRW equation

\[
\dot{H} = - \frac{\kappa^2_d}{2(d-2)} \left( \frac{\tilde{E}}{V} + p \right) + \frac{k}{(d-2)r^2} ,
\]

\[
p = \frac{(d-2)\mu}{r^d \kappa^2_d} .
\]

(51)
Here $p$ can be regarded as the pressure of the matter on the brane.

In case of the Einstein gravity ($a = b = c = 0$ in (1)), the equation corresponding to (50) has the form

$$\kappa^2_{(\text{Ein})d} = \frac{2\kappa^2}{l} .$$

(52)

Then the effects of the higher derivative terms appear through the redefinition of $\kappa^2$ to $\tilde{\kappa}^2$. Note, however, that one can obtain (50) even without the Einstein term ($\frac{1}{\kappa^2} = 0$).

When $d = 4$, by using (44) we find

$$\tilde{E} = \frac{l}{r} E .$$

(53)

Note that when $r$ is large, the metric (3) has the following form:

$$ds^2_{\text{AdS-S}} \rightarrow \frac{r^2}{l^2} \left( -dt^2 + l^2 \sum_{i,j} g_{ij} dx^i dx^j \right) ,$$

(54)

which tells that the CFT time $\tilde{t}$ is equal to the AdS time $t$ times the factor $\frac{r}{l}$:

$$t_{\text{CFT}} = \frac{a}{l} \tilde{t} .$$

(55)

Therefore Eq.(53) expresses that the energy in CFT is related with the energy $E$ in AdS by a factor $\frac{l}{r}$.

Eq.(19) or (53) tells that $\tilde{E}$ scales as $\tilde{E} \rightarrow \lambda^{-1} \tilde{E}$ when one scales the radius of the brane universe as $r \rightarrow \lambda r$. From Eqs.(19) and (51), we find

$$0 = -\frac{\tilde{E}}{V} + (d - 1)p ,$$

(56)

which tells that the trace of the energy-stress tensor coming from the matter on the brane vanishes:

$$T^\text{matter}{}_{\mu}{}^{\mu} = 0 .$$

(57)

Therefore the matter on the brane can be regarded as the radiation, i.e., the massless fields. In other words, field theory on the brane should be conformal one as in case of Einstein brane [2]. This supports the claim that even for.
$c = 0$ case, the higher derivative terms in bulk gravity correspond to next-to-leading corrections in AdS/CFT set-up.

Using (35) and (50), we can rewrite $\tilde{E}$ as
\[
\tilde{E} = \frac{2(d - 1)(d - 2)\kappa^2 \mu V_{d-1}}{r \left(1 - \frac{2d(d+1)\kappa^2 a}{l^2} - \frac{2db\kappa^2}{l^2}\right)}.
\] (58)

Assuming AdS/CFT correspondence, the higher derivative terms in (1) correspond to the $1/N$ corrections in the large $N$ limit of some gauge theory, which could be a CFT on the brane. For $\kappa^2 a, \kappa^2 b \ll 1$, one can rewrite (58) as
\[
\tilde{E} \sim \frac{2(d - 1)(d - 2)\kappa^2 \mu V_{d-1}}{r} \left(1 + \frac{2d(d + 1)\kappa^2 a}{l^2} + \frac{2db\kappa^2}{l^2}\right).
\] (59)

Then the parameters $a$ and $b$ in (59) could express the $1/N$ correction of the next-to-leading order of $1/N$ expansion.

We now restrict to $d = 4$ case and let the entropy $S$ of CFT on the brane is given by the entropy (16) of the AdS$_5$ black hole. If the total entropy $S$ is constant during the cosmological evolution, the entropy density $s$ is given by (see [2])
\[
s = \frac{S}{r^3 V_3} = \frac{4V_3 \pi r_H^3}{\kappa^2 l} \frac{8\pi r_H^3}{l\kappa^4 r^3}.
\] (60)

Here Eq.(16) is used. The temperature $T$ on the brane is different from Hawking temperature $T_H$ by the factor $\frac{l}{r}$:
\[
T = \frac{l}{r} T_H = \frac{r_H}{\pi rl} + \frac{kl}{4\pi rr_H}.
\] (61)

Then especially when $r = r_H$
\[
T = \frac{1}{\pi l} + \frac{k}{4\pi r_H^2}.
\] (62)

If the energy and entropy are purely extensive, the quantity $\tilde{E} + pV - TS$ vanishes. But in general, this quantity does not vanish and one can define the Casimir energy $E_C$ by
\[
E_C = 3 \left(\tilde{E} + pV - TS\right).
\] (63)
(The factor 3 is replaced by \(d - 1\) in the general dimensions). Then by using, Eqs.(49), (51) for \(d = 4\) and Eqs.(60), (61), we find

\[
E_C = \frac{6kr_H^2 V}{\kappa_4^2 r^4} = \frac{6kr_H^2 V_3}{\kappa_4^2 r} = \frac{3lk r_H^2 V_3}{\kappa_4^2 r} \left( 1 - \frac{40a\kappa^2}{l^2} - \frac{8b\kappa^2}{l^2} \right). \tag{64}
\]

When \(k = 0\), the Casimir energy vanishes. When \(a\) and \(b\) are small, \(E_C\) is positive (negative) when \(k = 2\) \((k = -2)\) but if \(a\) or \(b\) is large and positive, \(E_C\) can be negative (positive) even if \(k = 2\) \((k = -2)\). In case of absence of higher derivative terms corrections the above Casimir energy coincides with the one calculated in ref.\[14\].

Of course, as horizon radius has higher derivative terms corrections, the quantities found in this section are different from the ones in Einstein theory. This finishes our discussion of open, flat or closed radiation-dominated FRW Universe equations as it follows from induced geometry of brane living in d5 AdS BH (solution of bulk \(R^2\) gravity).

### 5 Cardy-Verlinde formula in \(R^2\)-gravity

In \[5\], it was shown that the FRW equation in \(d\)-dimensions can be regarded as a \(d\)-dimensional analogue of the Cardy formula of 2d conformal field theory (CFT) \[6\]:

\[
\tilde{S} = 2\pi \sqrt{\frac{c}{6} \left( L_0 - \frac{k}{d - 2} \frac{c}{24} \right)}. \tag{65}
\]

In the present case, identifying

\[
\begin{align*}
\frac{2\pi \tilde{E} r}{d - 1} & \Rightarrow 2\pi L_0, \\
\frac{(d - 2)V}{\kappa_4^2 r} & \Rightarrow \frac{c}{24}, \\
\frac{4\pi(d - 2)HV}{\kappa_d^2} & \Rightarrow \tilde{S},
\end{align*}
\tag{66}
\]

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the FRW-like equation (49) has the form (65).

The total entropy of the universe could be conserved in the expansion. Then one can evaluate holographic (Hubble) entropy \( \tilde{S} \) in (66) when the brane crosses the horizon \( r = r_H \). When \( r = r_H \), Eq. (49) tells that

\[
H = \pm \frac{1}{l} .
\] (67)

Here the plus sign corresponds to the expanding brane universe and the minus one to the contracting universe. Taking the expanding case and using (66), we find

\[
\tilde{S} = \frac{4\pi(d-2)V}{l\kappa_d^2} = \frac{2\pi(d-2)r_H^{d-1}V_{d-1}}{\tilde{\kappa}^2} .
\] (68)

Especially for \( d = 4 \), the entropy \( \tilde{S} \) is identical with \( S \) in (10), which is nothing but the black hole entropy

\[
\tilde{S} = S .
\] (69)

Generally, one gets

\[
\tilde{S} > S \quad \text{if } Hl > 1
\]
\[
\tilde{S} < S \quad \text{if } Hl < 1 .
\] (70)

As in [5], if defining the Bekenstein entropy \( S_B \) and the Bekenstein-Hawking entropy \( S_{BH} \) by

\[
S_B = \frac{2\pi}{d-1}E_T = 2\pi L_0 , \quad S_{BH} = \frac{4\pi(d-2)V}{\kappa_d^2 r} = \frac{\pi}{6} c ,
\] (71)

we have

\[
\tilde{S}^2 = 2S_BS_{BH} - \frac{k}{d-2}S_{BH}^2 ,
\] (72)

It is interesting that this equation becomes \( k \)-dependent as in [13]. \( R^2 \) gravity corrections are hidden in the entropies. One can also define the Casimir entropy \( S_C \) by

\[
S_C = \frac{4\pi r}{(d-1)k} E_C .
\] (73)
When \( d = 4 \), using (64) one gets
\[
S_C = \frac{8\pi r_H^2 V}{\kappa^2 r^3}.
\] (74)

By comparing the above expression \( S_{BH} \) in (71), we find
\[
S_C > S_{BH} \text{ if } r < r_H \quad \text{and} \quad S_C < S_{BH} \text{ if } r > r_H.
\] (75)

We now stress again that compared with the Einstein gravity case in [1], the corrections from the higher derivative terms always appear through the redefinition of gravitational coupling \( \kappa \) to \( \tilde{\kappa} \) via (35) or (42) and when the length scale \( l \) is given by (5). From the viewpoint of AdS/CFT correspondence, the higher derivative terms in (1) correspond to the \( 1/N \) corrections in the large \( N \) limit of some gauge theory, which could be a CFT on the brane. Then Eqs. (53) and (69) would tell that AdS/CFT correspondence could be valid in the next-to-leading order of the \( 1/N \).

Of course, it is interesting to investigate the \( c \neq 0 \) case. It may include also the situation when bulk gravity is Gauss-Bonnet one. When \( c \neq 0 \) case, however, the Schwarzschild-AdS space in (3) is not an exact solution. Then one should treat \( c \) as a perturbation. Although the calculation becomes tedious, it is straightforward and does not change the qualitative conclusions. It will be investigated elsewhere.

In summary, the emergence of brane FRW Universe dynamics as well as appearance of holographic cosmological entropy from AdS BH in d5 \( R^2 \) gravity is demonstrated. Despite the presence of parameters from higher derivative bulk terms, the radiation is represented by a strongly coupled CFT as it happens in purely Einstein theory. Cardy-Verlinde formula for cosmological entropy in \( R^2 \) gravity is derived. The corresponding cosmological entropy bounds are briefly discussed. Our study indicates to affirmative answer to the question of [11]: Can we live on the brane in SAdS black hole?

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