A geometric algebra approach to the hydrogen atom

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Monogenic functions in the algebra of 5-dimensional spacetime have been used previously by the author as first principle in different areas of fundamental physics; the paper recovers that principle applying it to the hydrogen atom. The equation that results from the monogenic condition is formally equivalent to Dirac’s and so its solutions resemble closely those found in the literature. The use of the monogenic condition as point of departure as not only the advantage of being a unified approach but also provides very strong links with geometry that are completely lost in the usual approach.

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I. INTRODUCTION

I have been advocating in recent papers that the majority of physics equations can be derived from an appropriately chosen geometry by exploration of the monogenic condition. Monogenic functions are not familiar to everybody but they are really the natural extension of analytic functions when one uses the formalism of geometric algebra [1, 2, 3]; those functions zero the vector derivative defined on the geometric algebra of the particular geometry under study.

In [4] I showed how special relativity and the Dirac equation could be derived from the monogenic condition applied in the geometric algebra of 5-dimensional spacetime $G_{4,1}$. An earlier paper [5] proved that the same condition in the same algebra was sufficient to produce a symmetry group isomorphic to the standard model gauge group; unfortunately this paper is incorrect in the formulation of particle dynamics but the flaw was recently corrected; [6] the same work introduces electrodynamics and electromagnetism in the monogenic formalism. Cosmological consequences were drawn from the addition of an hyperspherical symmetry hypothesis with the consequent choice of hyperspherical coordinates. [7] Summing up all those cited papers, I wrote a long book chapter. [8]

The present paper uses the 5D monogenic condition to study electron orbitals around a positively charged point sized nucleus, a problem known in quantum mechanics as the "Hydrogen atom problem." The equation that arises from the monogenic condition applied in $G_{4,1}$ algebra is formally equivalent to the Dirac equation and so one expects that its solutions are also formally equivalent to those obtained in relativistic quantum mechanics. It will be shown, however, that obtaining the solutions is greatly facilitated by the use of geometric algebra formalism and also that the resulting formulas are much more compact than standard ones and lend themselves to an easier geometrical interpretation. The derivations make use of some methods and strategies drawn from Refs. [1, 2] but depart from those works in many important aspects.

II. SOME GEOMETRIC ALGEBRA

Geometric algebra is not usually taught in university courses and its presence in the literature is scarce; good reference works are [1, 2, 3]. We will concentrate on the algebra of 5-dimensional spacetime because this will be our main working space; this algebra incorporates as subalgebras those of the usual 3-dimensional Euclidean space, Euclidean 4-space and Minkowski spacetime. We begin with the simpler 5-D flat space and progress to a 5-D spacetime of general curvature (see Appendix C for more details.)

The geometric algebra $G_{4,1}$ of the hyperbolic 5-dimensional space we consider is generated by the coordinate frame of orthonormal basis vectors $\sigma_\alpha$ such that

\begin{equation}
\begin{align*}
(\sigma_0)^2 &= -1, \\
(\sigma_i)^2 &= 1, \\
\sigma_\alpha \cdot \sigma_\beta &= 0, \quad \alpha \neq \beta.
\end{align*}
\end{equation}

Note that the English characters $i, j, k$ range from 1 to 4 while the Greek characters $\alpha, \beta, \gamma$ range from 0 to 4. See Appendix A for the complete notation convention used.

Any two basis vectors can be multiplied, producing the new entity called a bivector. This bivector is the geometric product or, quite simply, the product, and it is distributive. Similarly to the product of two basis vectors, the product of three different basis vectors produces a trivector and so forth up to the fivevector, because five is the dimension of space.

We will simplify the notation for basis vector products using multiple indices, i.e. $\sigma_\alpha \sigma_\beta \equiv \sigma_{\alpha\beta}$. The algebra is 32-dimensional and is spanned by the basis

- 1 scalar, $1$
- 5 vectors, $\sigma_\alpha$
- 10 bivectors (area), $\sigma_{\alpha\beta}$
- 10 trivectors (volume), $\sigma_{\alpha\beta\gamma}$

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• 5 tetravectors (4-volume), $i\sigma_\alpha$.
• 1 pseudoscalar (5-volume), $i \equiv \sigma_{01234}$.

Several elements of this basis square to unity:

$$ (\sigma_i)^2 = (\sigma_{0i})^2 = (\sigma_{0ij})^2 = (i\sigma_0)^2 = 1. \quad (2) $$

The remaining basis elements square to $-1$:

$$ (\sigma_0)^2 = (\sigma_{ij})^2 = (\sigma_{ijk})^2 = (i\sigma_i)^2 = i^2 = -1. \quad (3) $$

Note that the pseudoscalar $i$ commutes with all the other basis elements while being a square root of $-1$; this makes it a very special element which can play the role of the scalar imaginary in complex algebra.

In 5-dimensional spacetime of general curvature, spanned by 5 coordinate frame vectors $g_\alpha$, the indices follow the conventions set forth in Appendix \[A\]. We will also assume this spacetime to be a metric space whose metric tensor is given by

$$ g_{\alpha\beta} = g_\alpha \cdot g_\beta; \quad (4) $$

the double index is used with $g$ to denote the inner product of frame vectors and not their geometric product. The space signature is $(-+++)$, which amounts to saying that $g_{00} < 0$ and $g_{ii} > 0$. If the coordinate frame vectors can be expressed as a linear combination of the orthonormed ones, we have

$$ g_\alpha = n^\beta_\alpha \sigma_\beta, \quad (5) $$

where $n^\beta_\alpha$ is called the **refractive index tensor** or simply the **refractive index**; its 25 elements can vary from point to point as a function of the coordinates. In this work we will not consider spaces of general curvature but only those verifying condition \[4\]: in those spaces we define the vector and covariant derivatives (see appendix \[D\]).

### III. DIRAC’S EQUATION

There is a class of functions of great importance, called **monogenic functions**, \[1\] characterized by having null vector derivative; a function $\Psi$ is monogenic in flat space if and only if

$$ \nabla \Psi = 0. \quad (6) $$

A monogenic function is not usually a scalar and has by necessity null Laplacian, as can be seen by dotting Eq. \[9\] with $\nabla$ on the left; it has solutions of the type

$$ \Psi = \Psi_0 e^{i(p_0 x^0 + \delta)}. \quad (7) $$

The Dirac equation can be derived from the monogenic condition, as shown in \[4\] and briefly remembered here. For this effect it will be convenient to expand the monogenic condition \[10\] as $(i\nabla + \sigma^4 \partial_4)\Psi = 0$. If this is applied to the solution \[7\] and the derivative with respect to $x^4$ is evaluated we get

$$ (i\nabla + \sigma^4 p_4)\Psi = 0. \quad (8) $$

Let us now multiply both sides of the equation on the left by $\sigma^\alpha$ and note that bivector $\sigma^{0\alpha}$ squares to the identity while the 3 bivectors $\sigma^{4m}$ square to minus identity; we rename these bivectors as $\gamma$-bivectors in the form $\gamma^\mu = \sigma^{4\mu}$. Rewriting the equation in this form we get

$$ (\gamma^\mu \partial_\mu + ip_4)\Psi = 0. \quad (9) $$

Invoking the isomorphism between $G_{4,1}$ and the complex algebra of $4 \times 4$ matrices, the only thing this equation needs to be recognized as Dirac’s is the replacement of $p_4$ by the particle’s mass $m$; simultaneously we assign the energy $E$ to $p_0$ and 3D momentum $p$ to $\sigma^m p_m$. \[4, 6\] Alternatively we can multiply both sides of Eq. \[8\] by $\sigma^0$ on the left to obtain the $\alpha, \beta$ form of the Dirac equation, through the assignment $\sigma^m \equiv \sigma^m_0$, $\beta \equiv \sigma^4_0$.

Applying the monogenic condition to Eq. \[7\] we see that the following equation must be verified

$$ \Psi_0 (\sigma^\alpha p_\alpha) = 0. \quad (10) $$

It has been shown \[4\] that $p = \sigma^\alpha p_\alpha$ is a null vector and the wavefunction in Eq. \[7\] can be given a different form, taking in consideration the previous assignments

$$ \Psi = A(\pm \sigma_4 m + p + \sigma_0 E)e^{i(Et + p x \pm m x^4 + \delta)}, \quad (11) $$

where $A$ is the amplitude and $x = \sigma_m x^m$ is the 3-dimensional position. Although $\Psi$ does not look like a column Dirac spinor, it has the same number of components and can be written in that form, if desired.

In Eq. \[8\] we used $i$ as imaginary in the exponent but the monogenic condition would be equally verified if we had chosen any other algebra element whose square was minus unity; in \[5\] the various possible such elements are analysed and discussed; in the present work we will not explore such possibility. Applying the vector derivative to this solution we have

$$ \nabla \Psi = (E\sigma^0 \pm m\sigma^4)i\Psi + \nabla \Psi = 0. \quad (12) $$

### IV. ELECTROMAGNETIC POTENTIALS

When working in curved spaces the monogenic condition is naturally modified, replacing the vector derivative $\nabla$ with the covariant derivative $\nabla$. A generalized monogenic function is then a function that verifies the equation

$$ D\Psi = 0. \quad (13) $$

Remembering that an electron has minus unit charge in our units’ system, for an electron with rest mass $m$ in the presence of an electromagnetic potential $A$ we have to consider the reciprocal frame \[see \[F\] and \[1\]\]

$$ g^\mu = \sigma^\mu, \quad g^4 = \frac{-A_\mu}{m} \sigma^\mu + \sigma^4, \quad (14) $$
which corresponds to the refractive index frame

\[ g_0 = \sigma_\mu + \frac{A_\mu}{m} \sigma_4, \quad g_4 = \sigma_4; \]  

(15)

it is easily verified that \( g^\alpha g_\beta = \delta^\alpha_\beta \), as required by the definition of reciprocal frame. The covariant derivative is then

\[ D = \sigma^0 \partial_0 + \nabla + \left( -\frac{A_\mu}{m} \sigma^\mu + \sigma^4 \right) \partial_4. \]  

(16)

We expect solutions of Eq. (13) that are harmonic in \( t \) and \( \tau \), which we write as \( \Psi = \psi(x^m) \exp[i(Et - m\tau)] \), selecting the signs in the exponent for a forward propagating wave. The monogenic condition becomes

\[ [\sigma^0(E - A_0) - \nabla i - \sigma^m A_m - \sigma^4 m] \Psi = 0. \]  

(17)

For the following derivations we will follow closely the procedure explained in detail in Doran and Lasenby [1], making the necessary changes to conform to our monogenic formalism. In the case of a central field we have to make \( A_m = 0 \) and \( A_0 = \phi(r) \), with \( r \) the radial coordinate and the previous equation becomes

\[ [\sigma^0(E - \phi) - \nabla i - \sigma^4 m] \Psi = 0. \]  

(18)

Under those conditions we write the previous equation as

\[ \nabla \Psi = D \Psi, \]

\[ D = -(E\sigma^0 - \phi\sigma^0 - m\sigma^4)i. \]  

(19)

The wavefunction \( \Psi \) is multiplied by a vector on both sides of Eq. (19) but the vector on the lhs has only 3D components while the vector on the rhs has only \( \sigma^0 \) and \( \sigma^4 \) components; this arrangement is particularly useful to study commutativity of operators, as we shall see. Suppose we have an operator \( \mathcal{O} \) that commutes with both \( \nabla \) and \( D \) and suppose also that \( \Psi \) is an eigenstate of \( \mathcal{O} \), that is \( \mathcal{O}\Psi = \lambda \Psi \); Eq. (19) is then automatically verified. Our task is then to find the operators that verify the commutation conditions and solve for their eigenstates; this procedure is very similar to what is done in quantum mechanics for the hydrogen atom.

In Appendix E we define the total angular momentum \( J_t \) whose eigenstates define the angular solutions; another associated operator is

\[ \mathcal{K} = (1 - x \wedge \nabla)\sigma^{40}. \]  

(20)

\( \mathcal{K} \) anti-commutes with \( D \) because the bivectors present in \( x \wedge \nabla \) belong to 3D space and \( \sigma^{40} \) anti-commutes with both \( \sigma^0 \) and \( \sigma^4 \). The fact that it does not commute with \( D \) is not a problem as long as \( \mathcal{K} \) has symmetric eigenvalues; it commutes with \( \nabla \) as shown in Appendix E. So, we can assume \( \Psi \) to be in an eigenstate of \( \mathcal{K} \), that is

\[ \mathcal{K}\Psi = k\Psi. \]  

(21)

With central potentials we will use spherical coordinates; this implies that the frame vectors undergo rotations and their derivatives must be considered as follows

\[ \partial_0 \sigma_\tau = \sigma_\phi, \quad \partial_\tau \sigma_\phi = \sin \theta \sigma_\phi, \]

\[ \partial_0 \sigma_\phi = -\sigma_\tau, \quad \partial_\tau \sigma_\phi = \cos \theta \sigma_\phi, \]

\[ \partial_0 \sigma_\phi = 0, \quad \partial_\tau \sigma_\phi = -\sin \theta \sigma_\tau - \cos \theta \sigma_\phi. \]  

(22)

The 3D part of the vector derivative becomes

\[ \nabla = \sigma^r \partial_r + \frac{1}{r} \left( \sigma^\theta \partial_\theta + \csc \theta \sigma^\phi \partial_\phi \right). \]  

(23)

V. ANGULAR SOLUTIONS

In order to solve Eq. (18) we make the ansatz \( \Psi = R Y_t^s \), where \( Y_t^s \) contains all the angular dependence and \( R \) is a function of \( r \), \( t \) and \( \tau \) only. We note also that \( x \wedge \nabla R = 0 \); in terms of this operator we can write

\[ x \wedge \nabla \Psi = (1 - \sigma^{40} k) \Psi = R x \wedge \nabla Y_t^s. \]  

(24)

The analysis starts with 3D monogenic functions, or spherical harmonics, defined by \( \nabla \psi_t^s = 0 \); these functions are of the type \( \psi_t^s = r^t Y_t^s(\theta, \phi) \) and the vector derivative is

\[ \nabla \psi_t^s = t \sigma^r r^{t-1} Y_t^s + r^t \nabla Y_t^s. \]  

(25)

Since the first member must be null

\[ \nabla Y_t^s = -\frac{t}{r} \sigma^s Y_t^s. \]  

(26)

In terms of operator \( x \wedge \nabla \) the equation is

\[ -x \wedge \nabla Y_t^s = t Y_t^s; \]  

(27)

comparing with Eq. (24) we see that \( k = t + 1 \). Equation (27) is an eigenvalue equation satisfied by spherical harmonics with the general formula [1]

\[ Y_t^s = [(s + t + 1) P_t^s(\cos \theta)]^{-t-1} e^{2i\sigma^3 t}, \]  

(28)

where \( P_t^s(x) \) are Legendre polynomials, \( t \geq 0 \) and \(-1 \leq s \leq t \); solutions for \( t \leq -2 \) can be found through the relation

\[ -x \wedge \nabla(\sigma^s \psi^3) = -(t + 2) \sigma^r \psi^3. \]  

(29)

Because \( k = t + 1 \), the allowed values for \( k \) are given by \( |k| > 0 \); this ensures that the eigenvalues of \( \mathcal{K} \) operator are indeed symmetric, as we required in the previous section.

A particularly simple formula for 3D monogenic functions is obtained when the two quantum numbers are identical, in which case we obtain

\[ \psi_t^s = (r \sin \theta)^t e^{-i \phi \sigma^{12}}. \]  

(30)

Since Eq. (20) does not depend on \( s \), we can always use this particular case for the discussion of radial solutions.
VI. RADIAL SOLUTIONS

For an hydrogen-like atom we make \( \phi = Z\alpha/r \), with \( Z \) the positive charge of the nucleus and \( \alpha \) the fine structure constant. We will multiply Eq. (19) on the left by \( x \) noting that \( x\nabla = r\partial_r + \mathbf{x} \cdot \nabla \); considering Eq. (20)

\[
 r\partial_r\Psi + (1 - k\sigma r^4)\Psi = r \left( E\sigma r - \frac{Z\alpha}{r}\sigma r - m\sigma r^4 \right) i\Psi. 
\]

Introducing the function \( \Upsilon = \psi e^{-y} \), this can be rearranged isolating \( \partial_r \Upsilon \)

\[
 \partial_r\Upsilon = (E\sigma r + m\sigma r^4)i\Upsilon + \frac{1}{r} (k\sigma r^4 + Z\alpha\sigma r^4)i\Upsilon. 
\]

It is useful to define the two multivectors

\[
 F = -(E\sigma r - m\sigma r^4)i, \\
 G = -(k\sigma r^4 + Z\alpha\sigma r^4); 
\]

so that the monogenic condition becomes

\[
 \partial_r\Upsilon + \left( F + \frac{G}{r} \right) \Upsilon. 
\]

The \( F \) and \( G \) operators satisfy

\[
 F^2 = m^2 - E^2 = f^2, \\
 G^2 = k^2 - (Z\alpha)^2 = \nu^2. 
\]

They also observe the anticommutation relation

\[
 FG + GF = -2Z\alpha E. 
\]

We will now make a change of variable to allow the separation of large and small \( r \) behaviour; this is \( y = fr \) with which we write

\[
 \Upsilon = \Phi e^{-y}. 
\]

The new function \( \Phi \) satisfies

\[
 \partial_r\Phi + \frac{G}{y} \Phi + \left( \frac{F}{f} - 1 \right) \Phi = 0. 
\]

We can certainly express \( \Phi \) as a power series; moreover this power series must not be infinite, otherwise it would not fall to zero at large \( r \). Calling \( C_l \) to the series coefficients

\[
 \Phi = y^s \sum_{l=0}^{n} C_l y^l. 
\]

The coefficients verify the recursion relation

\[
 (l + s + G)C_l = - \left( \frac{F}{f} - 1 \right) C_{l-1}. 
\]

For \( l = 0 \) it is

\[
 (s + G)C_0 = 0. 
\]

Multiplying on the left by \( (s - G) \) we can see that \( s^2 = G^2 = \nu^2 \) and we set \( s = \nu \) to avoid a central singularity.

Since the series terminates at \( l = n \), the coefficient \( C_{n+1} \) must be null but it must still verify the recursion relation

\[
 \left( \frac{F}{f} - 1 \right) C_n = 0, 
\]

and so \( FC_n = fC_n \). Multiplying both sides of the recursion relation on the left by \( (F/f + 1) \) and replacing \( s \) by \( \nu \)

\[
 \left( \frac{F}{f} + 1 \right) (n + \nu + G)C_n = - \left( \frac{F}{f} + 1 \right) \left( \frac{F}{f} - 1 \right) C_{n-1} = 0, 
\]

Combining the two equations we get

\[
 \left[ 2(n + \nu) + G \left( \frac{F}{f} \right) \right] C_n = 0. 
\]

This can in turn be manipulated to give

\[
 \left[ 2(n + \nu) + \frac{1}{f} (GF + FG) \right] = 0; 
\]

and finally

\[
 n + \nu = \frac{Z\alpha E}{f} = 0. 
\]

This is the energy quantization equation, which can be arranged into the usual form by first manipulating to

\[
 \frac{E}{\sqrt{m^2 - E^2}} = n + \nu \frac{1}{Z\alpha}, 
\]

and then rearranging to

\[
 E^2 = m^2 \left[ 1 - \frac{Z\alpha^2}{n^2 + 2n\nu + k^2} \right]. 
\]

VII. GROUND STATE OF THE HYDROGEN ATOM

It is useful to analyse the ground state in order to find the character of the series coefficients. Since this is a spherically symmetric solution we ignore the \( \theta \) and \( \varphi \) dependence. At large \( r \) we can neglect the angular dependence of \( \sigma \) and a second order equation can be written for \( \psi(x^m) \) as

\[
 \partial_{rr}\psi = (E\sigma^0 - m\sigma^4) \sigma^r \partial_r\psi = (m^2 - E^2)\psi. 
\]

Since we are looking for bound states it must be \( E < m \) and \( \psi \) goes with \( \exp(-fr) \), with \( f = \sqrt{m^2 - E^2} \); the large \( r \) solution is

\[
 \psi = \psi_1 e^{-fr}. 
\]
Inserting into Eq. (13)
\[ f \sigma^r \psi_i = (E \sigma^0 - m \sigma^4) i \psi_i. \] (51)
The equation is solved if \( \psi_i \) contains a factor \((E \sigma^0 - m \sigma^4) + f \sigma^i_1 \). For small \( r \) Eq. (13) becomes
\[ \nabla \psi \approx -\frac{Z\alpha}{r} \sigma^0 \psi_i. \] (52)
Because we are assuming \( \psi \) to be a radial function we try a solution of the type
\[ \psi = a \sigma^r + b \sigma^0, \] (53)
where \( a \) and \( b \) are scalar functions of \( r \). We note that \( \nabla \sigma^r = 2/r \) and insert in Eq. (52) to get
\[ a' + \frac{2a}{r} + b' \sigma^0 = -\frac{Z\alpha}{r} \sigma^0 (a \sigma^r + b \sigma^0)i = (a \sigma^0 + b) \frac{Z\alpha}{i}. \] (54)
This implies the simultaneous equations
\[ \begin{cases} a' + \frac{2a}{r} = \frac{Z\alpha b}{r}i \\ b' = \frac{Z\alpha a}{r}i \end{cases} \] (55)
From the first equation in this set we take the derivative of \( b \) as
\[ b' = -(3a' + ra'') \frac{i}{Z\alpha}. \] (56)
We can now combine the two equations for \( b' \) into the single differential equation
\[ \frac{Z\alpha a}{r} = -\frac{3a' + ra''}{Z\alpha}, \] (57)
which we solve as
\[ a = r^\eta, \] (58)
with
\[ \eta = -1 \pm \sqrt{1 - (Z\alpha)^2}; \] (59)
we will later argue that only the plus sign is physically meaningful. Inserting \( a \) into the second Eq. (55) and solving we get for \( b \)
\[ b = \frac{Z\alpha}{\eta} r^\eta. \] (60)
Actually it is convenient to multiply \( a \) and \( b \) by \( i \). Summarizing the results for small and large \( r \), respectively,
\[ \psi = \begin{cases} \left( \frac{Z\alpha}{-\eta} \sigma^0 + \sigma^i_1 \right) r^\eta, & r \text{ small,} \\ \left( \frac{E \sigma^0 - m \sigma^4}{f} + \sigma^i_1 \right)e^{-fr}, & r \text{ large.} \end{cases} \] (61)
In order to make the two factors in brackets compatible we must include a factor \((1 - \sigma^4)\) to get
\[ \psi = \begin{cases} \left( \frac{Z\alpha}{-\eta} \sigma^0 + \sigma^i_1 \right)(1 - \sigma^4)\eta^\eta, \\ \left( \frac{E \sigma^0 - m \sigma^4}{f} + \sigma^i_1 \right)(1 - \sigma^4)e^{-fr}. \end{cases} \] (62)
\[ \psi = \begin{cases} \left[ \frac{Z\alpha}{-\eta} \sigma^0 (\sigma^0 - \sigma^4) + (1 - \sigma^4)\sigma^i_1 \right] r^\eta, \\ \left[ \frac{E + m}{f} \sigma^0 (\sigma^0 - \sigma^4) + (1 - \sigma^4)\sigma^i_1 \right] e^{-fr}, \end{cases} \] (63)
The complete radial solution for the ground state is
\[ \psi = \begin{cases} \left[ \frac{Z\alpha}{-\eta} (\sigma^0 - \sigma^4) + (1 - \sigma^4)\sigma^i_1 \right] r^\eta e^{-fr}, \end{cases} \] (64)
under the condition
\[ \frac{E + m}{f} = -\frac{Z\alpha}{\eta} = \frac{Z\alpha}{1 + \sqrt{1 - (Z\alpha)^2}}. \] (65)
The left hand side can be written as
\[ \frac{E + m}{f} = \frac{f/m}{1 - E/m} = \frac{f/m}{1 + \sqrt{1 - (f/m)^2}}. \] (66)
In order to have a positive energy it must be \( f = mZ\alpha \) and
\[ E = m\sqrt{1 - (Z\alpha)^2}. \] (67)
This is a particular case of the general energy quantization formula derived above, for \( n = 0 \) and \( k = 1 \).

**VIII. CONCLUSION**

The relativistic solutions for the hydrogen atom can be derived from the monogenic condition applied to functions in the algebra of 5D spacetime. When the space is bent by the consideration of a central potential, the equation that follow from the monogenic condition is formally equivalent to Dirac’s and so one could expect equivalent solutions from the onset. The paper revisits the solutions of that equation applying the formalism consistent with the monogenic condition.

There is nothing fundamentally new in the paper, in the sense that the energy levels that one obtains with the monogenic formalism are the same that can be found in the literature. However, the monogenic condition had previously been applied in other areas of physics showing high unifying potential; this work is just one further step in the path of unification. Besides that, the author believes the monogenic formalism to be easier to apprehend than the more usual matrix formalism; this is a question of taste, though.
APPENDIX A: INDEXING CONVENTIONS

In this section we establish the indexing conventions used in the paper. We deal with 5-dimensional space but we are also interested in two of its 4-dimensional subspaces and one 3-dimensional subspace; ideally our choice of indices should clearly identify their ranges in order to avoid the need to specify the latter in every equation. The diagram in Fig. 1 shows the index naming choice of indices should clearly identify their ranges in subspaces and one 3-dimensional subspace; ideally our but we are also interested in two of its 4-dimensional letters α, β, γ. Indices in the range for partial derivatives will be adopted as well as the compact notation for partial derivatives ∂ _α = ∂/∂x_α.

APPENDIX B: NON-DIMENSIONAL UNITS

The interpretation of t and τ as time coordinates implies the use of a scale parameter which is naturally chosen as the vacuum speed of light c. We don’t need to include this constant in our equations because we can always recover time intervals, if needed, introducing the speed of light at a later stage. We can even go a step further and eliminate all units from our equations so that they become pure number equations; in this way we will avoid cumbersome constants whenever coordinates have to appear as arguments of exponentials or trigonometric functions. We note that, at least for the macroscopic world, physical units can all be reduced to four fundamental ones; we can, for instance, choose length, time, mass and electric charge as fundamental, as we could just as well have chosen others. Measurements are then made by comparison with standards; of course we need four standards, one for each fundamental unit. But now note that there are four fundamental constants: Planck constant (ℏ), gravitational constant (G), speed of light in vacuum (c) and proton electric charge (e), with which we can build four standards for the fundamental units. Table I lists the standards of this units’ system, frequently called Planck units, which the authors prefer to designate by non-dimensional units. In this system all the fundamental constants, ℏ, G, c, e, become unity, a particle’s Compton frequency, defined by υ = mc²/ℏ, becomes equal to the particle’s mass and the frequent term GM/(c²r) is simplified to M/r. We can, in fact, take all measures to be non-dimensional, since the standards are defined with recourse to universal constants; this will be our posture. Geometry and physics become relations between pure numbers, vectors, bivectors, etc. and the geometric concept of distance is needed only for graphical representation.

APPENDIX C: SOME COMPLEMENTS OF GEOMETRIC ALGEBRA

In this section we expand the concepts given in Sec. II introducing some useful relations and definitions. Starting with the basis elements that square to unity Eq. (2), repeated here,

(σ_0)_² = (σ_0i)_² = (σ_0ij)_² = (iσ_0)_² = 1,  \hspace{1cm} (C1)

it is easy to verify any of the above equations; suppose we want to check that (σ_0ij)_² = 1. Start by expanding the square and remove the compact notation (σ_0ij)_² = σ_0σ_0σ_iσ_jσ_0, then swap the last σ_j twice to bring it next to its homonymous; each swap changes the sign, so an even number of swaps preserves the sign: (σ_0ij)_² = σ_0σ_i(σ_j)_²σ_0σ_i. From the third equation (1) we know that the squared vector is unity and we get successively (σ_0ij)_² = σ_0σ_i(σ_j)_²σ_0σ_i. Using the first equation (1) we get finally (σ_0ij)_² = 1 as desired.

The remaining basis elements square to −1 as can be verified in a similar manner, Eq. (3):

(σ_0)_² = (σ_0i)_² = (σ_0ij)_² = (0σ_i)_² = 0² = −1.  \hspace{1cm} (C2)

Note that the pseudoscalar i commutes with all the other basis elements while being a square root of −1; this makes it a very special element which can play the role of the scalar imaginary in complex algebra.

We can now address the geometric product of any two vectors a = a_ασ_α and b = b_βσ_β making use of the distributive property

\( ab = \left( -a_0b_0 + \sum_i a_ib_i^* \right) + \sum_{\alpha \neq \beta} a_\alpha b_\beta \sigma_{\alpha\beta}; \)  \hspace{1cm} (C3)

and we notice it can be decomposed into a symmetric part, a scalar called the inner or interior product, and an anti-symmetric part, a bivector called the outer or exterior product.

\( ab = a \cdot b + a \wedge b, \quad ba = a \cdot b - a \wedge b. \)  \hspace{1cm} (C4)
Reversing the definition one can write inner and outer products as
\[ a \cdot b = \frac{1}{2} (ab + ba), \quad a \wedge b = \frac{1}{2} (ab - ba). \] (C5)

The inner product is the same as the usual "dot product," the only difference being in the negative sign of the \( ab \) term; this is to be expected and is similar to what one finds in special relativity. The outer product represents an oriented area; in Euclidean 3-space it can be linked to the "cross product" by the relation \( \text{cross}(a, b) = -\sigma_{123} a \wedge b; \) here we introduced bold characters for 3-dimensional vectors and avoided defining a symbol for the cross product because we will not use it again. We also used the convention that interior and exterior products take precedence over geometric product in an expression.

When a vector is operated with a multivector the inner product reduces the grade of each element by one unit and the outer product increases the grade by one. We will generalize the definition of inner and outer products below; under this generalized definition the inner product between a vector and a scalar produces a vector. Given a multivector \( a \) we refer to its grade-\( r \) part by writing \( <a>_{r}; \) the scalar or grade zero part is simply designated as \( <a>. \) By operating a vector with itself we obtain a scalar equal to the square of the vector’s length
\[ a^2 = aa = a \cdot a + a \wedge a = a \cdot a. \] (C6)

The definitions of inner and outer products can be extended to general multivectors
\[ a \cdot b = \sum_{\alpha, \beta} <a>_{\alpha} <b>_{\beta} \epsilon_{\alpha-\beta}, \] (C7)
\[ a \wedge b = \sum_{\alpha, \beta} <a>_{\alpha} <b>_{\beta} \epsilon_{\alpha+\beta}. \] (C8)

Two other useful products are the scalar product, denoted as \( <ab> \) and commutator product, defined by
\[ a \times b = \frac{1}{2} (ab - ba). \] (C9)

In mixed product expressions we will use the convention that inner and outer products take precedence over geometric products as said above.

We will encounter exponentials with multivector exponents; two particular cases of exponentiation are specially important. If \( u \) is such that \( u^2 = -1 \) and \( \theta \) is a scalar
\[ e^{u\theta} = 1 + u\theta - \frac{u^2}{2!} \theta^2 + \frac{u^4}{4!} \theta^4 + \ldots = \cos \theta \]
\[ +u\theta - \frac{u^2}{2!} \theta^2 + \frac{u^4}{4!} \theta^4 + \ldots \{= u \sin \theta \} \] (C10)
Conversely if \( h \) is such that \( h^2 = 1 \)
\[ e^{h\theta} = 1 + h\theta + \frac{h^2}{2!} \theta^2 + \frac{h^3}{4!} \theta^4 + \ldots \]
\[ = 1 + \frac{h^2}{2!} \theta^2 + \frac{h^4}{3!} \theta^4 + \ldots \{= \cosh \theta \}
\[ +h\theta + h\frac{h^3}{3!} \theta^3 + \ldots \{= h \sinh \theta \} \] (C11)

The exponential of bivectors is useful for defining rotations; a rotation of vector \( a \) by angle \( \theta \) on the \( \sigma_{12} \) plane is performed by
\[ a' = e^{\sigma_{12} \theta/2} a e^{\sigma_{12} \theta/2} = \hat{R} a R; \] (C12)
the tilde denotes reversion and reverses the order of all products. As a check we make \( a = \sigma_1 \)
\[ e^{-\sigma_{12} \theta/2} \sigma_1 e^{\sigma_{12} \theta/2} = \left( \cos \frac{\theta}{2} - \sigma_{12} \sin \frac{\theta}{2} \right) \sigma_1 \]
\[ \times \left( \cos \frac{\theta}{2} + \sigma_{12} \sin \frac{\theta}{2} \right) \] (C13)
\[ = \cos \theta \sigma_1 + \sin \theta \sigma_2. \]

Similarly, if we had made \( a = \sigma_2 \), the result would have been \(- \sin \theta \sigma_1 + \cos \theta \sigma_2 \).

If we use \( B \) to represent a bivector whose plane is normal to \( \sigma_0 \) and define its norm by \( |B| = |BB|^{1/2}, \) a general rotation in 4-space is represented by the rotor
\[ R = e^{-B/2} = \cos \left( \frac{|B|}{2} \right) - \frac{B}{|B|} \sin \left( \frac{|B|}{2} \right). \] (C14)
The rotation angle is \( |B| \) and the rotation plane is defined by \( B \). A rotor is defined as a unitary even multivector (a multivector with even grade components only) which squares to unity; we are particularly interested in rotors with bivector components. It is more general to define a rotation by a plane (bivector) then by an axis (vector) because the latter only works in 3D while the former is applicable in any dimension. When the plane of bivector \( B \) contains \( \sigma_0 \), a similar operation does not produce a rotation but produces a boost instead. Take for instance \( B = \sigma_0 \theta/2 \) and define the transformation operator \( T = \exp(B); \) a transformation of the basis vector \( \sigma_0 \) produces
\[ a' = T \sigma_0 T = e^{-\sigma_0 \theta/2} \sigma_0 e^{\sigma_0 \theta/2} \]
\[ = \left( \cosh \frac{\theta}{2} - \sigma_0 \sinh \frac{\theta}{2} \right) \sigma_0 \]
\[ \times \left( \cosh \frac{\theta}{2} + \sigma_0 \sinh \frac{\theta}{2} \right) \] (C15)
\[ = \cosh \theta \sigma_0 + \sinh \theta \sigma_1. \]

APPENDIX D: RECIPROCAL FRAME AND DERIVATIVE OPERATORS

A reciprocal frame is defined by the condition
\[ g^{\alpha} \cdot g_{\beta} = \delta^\alpha_\beta. \] (D1)
Defining $g^{\alpha\beta}$ as the inverse of $g_{\alpha\beta}$, the matrix product of the two must be the identity matrix, which we can state as

$$g^{\alpha\gamma}g_{\beta\gamma} = \delta^{\alpha}_{\beta}. \quad (D2)$$

Using the definition we have

$$(g^{\alpha\gamma}g_{\beta\gamma})_\beta = \delta^\alpha_{\beta}; \quad (D3)$$

comparing with Eq. $\text{(D1)}$ we determine $g^\alpha$ and $g_\alpha = g_{\alpha\gamma}g^\gamma$.

It would be easy to verify that it is also $g^{\alpha\beta} = g^\alpha \cdot g^\beta$ and $g_\alpha = g_{\alpha\gamma}g^\gamma$.

In many situations of great interest the frame vectors $g_\alpha$ can be expressed in terms of an orthonormed frame given by Eqs. $\text{(D1)}$. If the frame vectors can be expressed as linear combination of the orthonormed ones we have

$$g_\alpha = n^\beta_{\alpha}g_\beta, \quad (D5)$$

where $n^\beta_{\alpha}$ is called the refractive index tensor or simply the refractive index as said in the main text. When the refractive index is the identity we have $g_\alpha = \sigma_\alpha$ for the main or direct frame and $g^\beta = -\sigma_0$, $g_\beta = \sigma_i$ for the reciprocal frame, so that Eq. $\text{(D1)}$ is verified.

The first use we will make of the reciprocal frame is for the definition of two derivative operators. In flat space we define the vector derivative

$$\nabla = \sigma_\alpha \partial_\alpha. \quad (D6)$$

It will be convenient, sometimes, to use vector derivatives in subspaces of 5D space; these will be denoted by an upper index before the vector derivative applies; For instance $m \nabla = \sigma^m \partial_m = \sigma^1 \partial_1 + \sigma^2 \partial_2 + \sigma^3 \partial_3$. In 5-dimensional space it will be useful to split the vector derivative into its time and 4-dimensional parts

$$\nabla = -\sigma_0 \partial_t + \sigma^\alpha \partial_\alpha = -\sigma_0 \partial_t + \nabla. \quad (D7)$$

Consistently with the boldface notation for 3-dimensional vectors $m \nabla$ will be denoted by $\nabla$. We will use over arrows, when necessary, to imply that the vector derivative is applied to a function which is not immediately on its right; for instance in $\nabla AB$ and in $B A \nabla$ the derivative operator is applied to function $B$.

The second derivative operator is called covariant derivative, sometimes also designated by Dirac operator, and it is defined with recourse to the reciprocal frame $g^\alpha$

$$D = g^\alpha \partial_\alpha. \quad (D8)$$

Taking into account the definition of the reciprocal frame $\text{(D1)}$ we see that the covariant derivative is also a vector. In cases where there is a refractive index, it will be possible to define both derivatives in the same space.

Vector derivatives can also be left or right multiplied with other vectors or multivectors. For instance, when $\nabla$ is multiplied by vector $a$ on the right the result comprises scalar and bivector terms $\nabla a = \nabla \cdot a + \nabla \wedge a$. The scalar part can be immediately associated with the divergence and the bivector part is called the exterior derivative; in the particular case of Euclidean 3-dimensional space it is possible to define the curl of a vector by $\text{curl}(a) = -\sigma_{123} \nabla \wedge a$.

We define also second order differential operators, designated Laplacian and covariant Laplacian respectively, resulting from the inner product of one derivative operator by itself. The square of a vector is always a scalar and the vector derivative is no exception, so the Laplacian is a scalar operator, which consequently acts separately in each component of a multivector. For 4 + 1 flat space it is

$$\nabla^2 = -\frac{\partial^2}{\partial t^2} + \nabla^2. \quad (D9)$$

One sees immediately that a 4-dimensional wave equation is obtained zeroing the Laplacian of some function

$$\nabla^2 \Psi = \left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) \Psi = 0. \quad (D10)$$

This procedure was used in Ref. [3] for the derivation of special relativity and extended in Ref. [6] to general curved spaces.

**APPENDIX E: COMMUTATION RELATIONS**

We examine here the commutation of operators with $\nabla$. First of all we let us expand $x \wedge \nabla$

$$x \wedge \nabla = (x^1 \partial_2 - x^2 \partial_1)\sigma^{12} + (x^2 \partial_3 - x^3 \partial_2)\sigma^{23} + (x^3 \partial_1 - x^1 \partial_3)\sigma^{31}. \quad (E1)$$

The angular momentum operator associated with the plane of 3D bivector $B$ is defined as

$$L_B = \text{i}B \cdot (x \wedge \nabla). \quad (E2)$$

For instance, if we are interested in the angular momentum relative to the $\sigma^3$ direction

$$L_{\sigma^3} = \text{i} \sigma^{12} \cdot (x \wedge \nabla) = \text{i} (x^2 \partial_1 - x^1 \partial_2). \quad (E3)$$

The angular momentum operator does not commute with $\nabla$; following Ref. [1] we have

$$[B \cdot (x \wedge \nabla), \nabla] = -\nabla B \cdot (x \wedge \nabla) = B \times \nabla. \quad (E4)$$

Since $B \times \nabla = (B \nabla - \nabla B)/2$ we can define an operator $B \cdot (x \wedge \nabla) - B/2$ which commutes with both $\nabla$ and $D$. The conserved total angular momentum operator is then

$$J_B = L_B - \frac{1}{2} iB. \quad (E5)$$
We simplify the notation for the case of bivectors normal to frame vectors by writing $J_m \equiv J_{\sigma^{m\sigma}}$, with $\sigma^m = \sigma^{123}\sigma^{\sigma}$.

If we ignore second derivatives $(x \wedge \nabla) \nabla$ is zero but

$$\nabla (x \wedge \nabla) = 2 \nabla. \quad (E6)$$

For the $\mathcal{K}$ operator we have then

$$[\sigma^{40}(1 - x \wedge \nabla), \nabla] = 2\sigma^{40} \nabla - \sigma^{40} \nabla (x \wedge \nabla) = 0. \quad (E7)$$

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