Point-sets in general position with many similar copies of a pattern

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Abstract

For every pattern $P$, consisting of a finite set of points in the plane, $S_P(n, m)$ is defined as the largest number of similar copies of $P$ among sets of $n$ points in the plane without $m$ points on a line. A general construction, based on iterated Minkovski sums, is used to obtain new lower bounds for $S_P(n, m)$ when $P$ is an arbitrary pattern. Improved bounds are obtained when $P$ is a triangle or a regular polygon with few sides. It is also shown that $S_P(n, m) \geq n^2 - \varepsilon$ whenever $m(n) \to \infty$ as $n \to \infty$. Finite sets with no collinear triples and not containing the 4 vertices of any parallelogram are called parallelogram-free. The more restricted function $S^0_P(n)$, defined as the maximum number of similar copies of $P$ among parallelogram-free sets of $n$ points, is also studied. It is proved that $\Omega(n \log n) \leq S^0_P(n) \leq O(n^{3/2})$.

Keywords: similar copy, pattern, general position, collinear points, parallelogram-free, Minkovski Sum.

1 Introduction

Sets $A$ and $B$ in the plane are similar, denoted by $A \sim B$, if there is an orientation-preserving isometry followed by a dilation that takes $A$ to $B$. Identifying the plane with $\mathbb{C}$, the set of complex numbers, $A \sim B$ if there are complex numbers $w$ and $z \neq 0$ such that $B = zA + w$. Here, $zA = \{za : a \in A\}$ and $A + w = \{a + w : a \in A\}$.

Consider a finite set of points $P$ in the plane, $|P| \geq 3$. We refer to $P$ as a pattern ($P$ is usually fixed). For any finite set of points $Q$, we define $S_P(Q)$ to be the number of similar copies of $P$ contained in $Q$. More precisely,

$$S_P(Q) = |\{P' \subseteq Q : P' \sim P\}|.$$

The main goal of this paper is to explicitly construct point sets $Q$ in general position with a large number of similar copies of the pattern $P$, that is, with large $S_P(Q)$. By general position we mean to forbid triples of collinear points, although we also consider the restriction of allowing at most $m$ points on a line, $m \geq 3$, and the stronger restriction of not allowing collinear points or parallelograms in $Q$.

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Some preliminary results appeared in [3]
To explain the motivation of this paper we first turn to the original problem. Erdős and Purdy [9]–[11] posed the problem of maximizing the number of similar copies of \( P \) contained in a set of \( n \) points in the plane. That is, to determine the function

\[
S_P(n) = \max_{|Q|=n} S_P(Q),
\]

where the maximum is taken over all point-sets \( Q \subseteq \mathbb{C} \) with \( n \) points. Elekes and Erdős [7] noted that \( S_P(n) \leq n(n-1) \) for any pattern \( P \). They gave a quadratic lower bound for \( S_P(n) \) when \( |P| = 3 \) or when all the coordinates of the points in \( P \) are algebraic numbers. They also proved a slightly subquadratic lower bound for all other patterns \( P \). (The precise statements can be found in Section 5.) Later, Laczkovich and Ruzsa [13] characterized precisely those patterns for which \( S_P(n) = \Theta(n^2) \). However, the coefficient of the quadratic term is not known for any non-trivial pattern, not even for the simplest case of \( P \) an equilateral triangle [2]. Elekes and the authors [4] investigated the structural properties of the \( n \)-sets \( Q \) that achieve a quadratic number of similar copies of a pattern \( P \). We proved that those sets contain large lattice-like structures, and therefore, many collinear points. In particular, the next result was obtained.

**Theorem A** (Ábrego et al. [4]). For every positive integer \( m \) and every real \( c > 0 \), there is a threshold function \( N_0 = N_0(c, m) \) with the following property: if \( n \geq N_0 \) and \( Q \) is an \( n \)-set with \( S_P(Q) \geq cn^2 \), then \( Q \) has \( m \) points on a line forming an arithmetic progression.

Thus having many points on a line is a required property to achieve \( \Theta(n^2) \) similar copies of a pattern \( P \). It is only natural to restrict the problem of maximizing \( S_P(Q) \) over sets \( Q \) with a limited number of possible collinear points. Given natural numbers \( n \) and \( m \geq 3 \), we restrict the maximum in \( S_P(n) \) to \( n \)-sets with at most \( m-1 \) collinear points. We denote this maximum by

\[
S_P(n, m) = \max \{ S_P(Q) : |Q| = n \text{ and no } m \text{ points of } Q \text{ are collinear} \}.
\]

Figure 1: Point-set in general position with many triples spanning equilateral triangles.
We mention here that Erdős’ unit distance problem [8] (arguably the most important problem in the area), has also been studied under general position assumptions, like no 3 points on a line [11] and also no parallelograms [4] (see also [6, Sections 5.1 and 5.5]).

When the maximum of $S_P(Q)$ is taken over $n$-sets in general position, that is, no three collinear points, we obtain the function $S_P(n, 3)$, which we simply denote by $S'_P(n)$. By definition, it is clear that $S'_P(n) \leq S_P(n, m_1) \leq S_P(n, m_2) \leq S_P(n)$ whenever $m_1 \leq m_2$. Moreover, if $m$ is constant, then Theorem 2 implies that $\lim_{n \to \infty} S_P(n, m)/n^2 = 0$, i.e., $S_P(n, m) = o\left(n^2\right)$. We believe that the true asymptotic value of $S_P(n, m)$ is close to quadratic; however, prior to this work there were no lower bounds other than the trivial $S'_P(n) = \Omega(n)$. The rest of the paper is devoted to the construction of point-sets giving non-trivial lower bounds for $S_P(n, m)$.

2 Statement of results

The symmetries of the pattern $P$ play an important role in the order of magnitude of our lower bound to the function $S_P(n, m)$. Let us denote by $\text{Iso}^+(P)$ the group of orientation-preserving isometries of the pattern $P$, also known as the proper symmetry group of $P$. Define the index of a point set $A$ with respect to the pattern $P$, denoted by $i_P(A)$, as

$$i_P(A) = \frac{\log (|\text{Iso}^+(P)| S_P(A) + |A|)}{\log |A|}.$$

Observe that $1 \leq i_P(A) \leq 2$, because $|\text{Iso}^+(P)| S_P(A) \leq |A|^2 - |A|$ as was noted by Elekes and Erdős, and moreover, $i_P(A) = 1$ if and only if $S_P(A) = 0$. Our main theorem gives a lower bound for $S_P(n, m)$ using the index as the corresponding exponent. Its proof, presented in Section 3, is based on iterated Minkovski Sums.

**Theorem 1.** For any $A$ and $P$ finite sets in the plane with at most $m - 1$ collinear points, there is a constant $c = c(P, A)$ such that, for $n$ large enough,

$$S_P(n, m) \geq cn^{i_P(A)}.$$

By using $A = P$, we conclude that the function $S_P(n, m)$ is superlinear, that is for any finite pattern $P$ with at most $m - 1$ collinear points, $S_P(n, m) \geq \Omega(n^{\log(1+|P|)/\log|P|})$. So the key to obtaining good lower bounds for these functions, is to begin with a set $A$ with large index. For a general pattern $P$, we can marginally improve the last inequality by constructing a better initial set $A$. The proof of next theorem is in Section 4.1

**Theorem 2.** For any finite pattern $P$ with at most $m - 1$ collinear points and $|P| = k \geq 3$, there is a constant $c = c(P)$ such that, for $n$ large enough,

$$S_P(n, m) \geq cn^{\log(k^2 - k + 1)/\log(k^2 - k + 1)}.$$

The following theorems summarize the lower bounds for $S'_P(n)$ obtained from the best known initial sets $A$ for some specific patterns $P$. We concentrate on triangles and regular polygons. We often refer to a finite pattern as a geometric figure. For instance, when we say “let $P$ be the equilateral triangle” we actually mean the set of vertices of an equilateral triangle.
Theorem 3. Let $P = T$ be a triangle.

If $T = \triangle$ is equilateral, then $S'_\triangle(n) \geq \Omega \left( n^{\log_{102}15} \right) \geq \Omega \left( n^{1.707} \right)$.

If $T$ is isosceles, then $S'_T(n) \geq \Omega \left( n^{\log_{17}8} \right) \geq \Omega \left( n^{1.362} \right)$.

If $T$ is almost any scalene triangle, then $S'_T(n) \geq \Omega \left( n^{\log_{40}14} \right) \geq \Omega \left( n^{1.397} \right)$. For all others, $S'_T(n) \geq \Omega \left( n^{\log_{9}5} \right) \geq \Omega \left( n^{1.365} \right)$.

In general, if $P$ is a $k$-sided regular polygon, then $i_P = \log_{2k}(k)$ and thus $S_P(n) \geq \Omega \left( n^{\log_{2k}k} \right)$. If $k$ is even, $4 \leq k \leq 10$ or if $k = 5$ we have the following improvement.

Theorem 4. Let $P = R(k)$ be a regular $k$-gon. Then

$$S'_{R(4)}(n) \geq \Omega \left( n^{\log_{144}24} \right) \geq \Omega \left( n^{1.563} \right), \quad S'_{R(6)}(n) \geq \Omega \left( n^{\log_{528}84} \right) \geq \Omega \left( n^{1.414} \right),$$

$$S'_{R(8)}(n) \geq \Omega \left( n^{\log_{1312}208} \right) \geq \Omega \left( n^{1.345} \right), \quad S'_{R(10)}(n) \geq \Omega \left( n^{\log_{2640}420} \right) \geq \Omega \left( n^{1.304} \right), \quad S'_{R(5)}(n) \geq \Omega \left( n^{\log_{264}120} \right) \geq \Omega \left( n^{1.519} \right).$$

In Section 4.4, we briefly explore the behavior of the function $S_P(n, m)$ for larger values of $m$ and $P = \triangle$ the equilateral triangle. We then present general asymptotic results, for arbitrary patterns, when $m = m(n)$ is a function of $n$ such that $m(n) \to \infty$ when $n \to \infty$. In this case we prove that $S_P(n, m) \geq n^{2-\varepsilon}$ for every $n$ sufficiently large.

After having relative success constructing point sets with not many points on a line, we impose harder restrictions by prohibiting parallelograms (and collinear points) in the sets $Q$. This restriction immediately forbids the use of Minkovski Sums. We are able to construct $n$-sets $Q$ with $\Omega(n \log n)$ copies of a pattern $P$ without parallelograms (or collinear triples). We also show a non-trivial upper bound for these patterns, namely, we prove that at most $O(n^{3/2})$ copies of $P$ are possible. These results are presented in Section 6.

3 Proof of Theorem 1

Let $P$ and $A$ be sets with no $m$ collinear points. With $A$ as a base set, we recursively construct large sets with no $m$ collinear points and with large number of similar copies of $P$. Our main tool is the Minkovski Sum of two sets $A, B \subseteq C$, defined as the set $A + B = \{ a + b : a \in A, b \in B \}$. First we have the following observation.

Proposition 1. Let $P = \{ p_1, p_2, \ldots, p_k \}$ and $Q = \{ q_1, q_2, \ldots, q_k \}$ be sets with $k$ elements. $P$ is similar to $Q$, with $p_j$ corresponding to $q_j$ if and only if

\[ \frac{q_j - q_1}{q_2 - q_1} = \frac{p_j - p_1}{p_2 - p_1} \text{ for } j = 1, 2, \ldots, k. \]

Proof. If $P \sim Q$ with $q_j = zp_j + w$, where $z \neq 0$ and $w$ are fixed complex numbers, then

\[ \frac{q_j - q_1}{q_2 - q_1} = \frac{(zp_j + w) - (zp_1 + w)}{(zp_2 + w) - (zp_1 + w)} = \frac{p_j - p_1}{p_2 - p_1}. \]

Reciprocally, if $(q_j - q_1) / (q_2 - q_1) = (p_j - p_1) / (p_2 - p_1)$ for $1 \leq j \leq k$, then letting $z = (q_2 - q_1)/(p_2 - p_1)$ and $w = q_1 - zp_1$ we get that $q_j = zp_j + w$ and $z \neq 0$. \(\square\)

We now bound the number of copies of $P$ in the sum $A + B$. To be concise, let $I = |\text{Iso}^+(P)|$. 
**Lemma 1.** Let $P$ be any finite pattern, and $B$ and $C$ finite sets such that $B + C$ has exactly $|B||C|$ points. Then

$$I \cdot S_P(B + C) + |B||C| \geq (I \cdot S_P(B) + |B|)(I \cdot S_P(C) + |C|).$$

**Proof.** Suppose that $P = \{p_1, p_2, \ldots, p_k\}$ and let $\lambda_j$ denote the ratio $(p_j - p_1)/(p_2 - p_1)$. Let $P_B = \{b_1, b_2, \ldots, b_k\} \subseteq B$ and $P_C = \{c_1, c_2, \ldots, c_k\} \subseteq C$ be corresponding copies of $P$ with $P \sim P_B \sim P_C$. Then by the previous proposition,

$$\frac{b_j - b_1}{b_2 - b_1} = \frac{c_j - c_1}{c_2 - c_1} = \frac{p_j - p_1}{p_2 - p_1} = \lambda_j.$$

Since $B + C$ has exactly $|B||C|$ elements, then the relation $(b, c) \mapsto b + c$ for $b \in B$, $c \in C$ is bijective. For any orientation-preserving isometry $f$ of $P_C$ (which uniquely corresponds to an element of $\text{Iso}^+(P)$), consider the set $Q = \{q_j := b_j + f(c_j) : j = 1, 2, \ldots, k\} \subseteq B + C$. Since $(f(c_j) - f(c_1))/(f(c_2) - f(c_1)) = (c_j - c_1)/(c_2 - c_1) = \lambda_j$, then

$$\frac{q_j - q_1}{q_2 - q_1} = \frac{b_j - b_1 + f(c_j) - f(c_1)}{b_2 - b_1 + f(c_2) - f(c_1)} = \frac{\lambda_j(b_2 - b_1 + f(c_2) - f(c_1))}{\lambda_j(b_2 - b_1 + f(c_2) - f(c_1))} = \lambda_j.$$ 

That is, by the previous proposition, $Q \sim P$. Thus every similar copy $P_B$ of $P$ in $B$ together with a similar copy $P_C$ of $P$ in $C$ originate $I$ distinct similar copies of $P$ in $B + C$. We also have the ‘liftings’ of $P$ in $B$ and $C$. That is, similar copies of $P$ of the form $(b, c_1), (b, c_2), \ldots, (b, c_k)$ or $(b_1, c), (b_2, c), \ldots, (b_k, c)$, with $b \in B$ and $c \in C$. All these copies of $P$ in $B + C$ are different because $|B + C| = |B||C|$. Therefore the number of similar copies of $P$ in $B + C$ is at least

$$I \cdot S_P(B)S_P(C) + |B|S_P(C) + |C|S_P(B).$$ 

In other words

$$I \cdot S_P(B + C) + |B||C| \geq I^2 \cdot S_P(B)S_P(C) + I \cdot |B|S_P(C) + I \cdot |C|S_P(B) + |B||C| = (I \cdot S_P(B) + |B|)(I \cdot S_P(C) + |C|).$$

The next lemma allows us to preserve the maximum number of collinear points when we use the Minkovski Sum of two appropriate sets.

**Lemma 2.** Let $A$ and $B$ be two sets with no $m$ points on a line, $m \geq 3$. If $S$ is the set of points $v \in C$ for which $A + vB$ has less than $|A||B|$ points, or $m$ points on a line; then $S$ has zero Lebesgue measure.

For presentation purposes we defer its proof and instead proceed to the proof of the theorem.

**Proof of Theorem.** Let $A_1 = A_1^* = A$ and suppose $A_j$ and $A_j^*$ have been defined. By Lemma 2 there is a set $A_{j+1}$, similar to $A$, such that $A_{j+1}^* := A_j^* + A_{j+1}$ does not have $m$ points on a line and $|A_{j+1}^*| = |A_j^*||A_{j+1}|$. For every $j \geq 1$, $|A_j^*| = |A_j^*||A| = |A_{j-2}^*||A|^2 = \cdots = |A|^j$. Moreover, by Lemma 1 it follows that

$$I \cdot S_P(A_j^*) + |A_j^*| = I \cdot S_P(A_{j-1}^* + A_j) + |A_{j-1}^*||A_j| \geq (I \cdot S_P(A_{j-1}^*) + |A_{j-1}^*|)(I \cdot S_P(A) + |A|) \geq \cdots \geq (I \cdot S_P(A) + |A|)^j.$$
If \( S_P(A) = 0 \), then \( i_P(A) = 1 \) and the result is trivial. Assume \( i_P(A) > 1 \) and suppose \( |A|^j \leq n < |A|^{j+1} \). The previous inequality yields

\[
S_P(n, m) \geq S_P(A, m) \geq \frac{1}{I}(I \cdot S_P(A) + |A|^j - |A|^j) \geq \frac{1}{I} \left( |A|^j \cdot i_P(A) - n \right) \geq \frac{n}{|A|^j} \cdot i_P(A) - n \geq cn^{i_P(A)},
\]

for some constant \( c = c(A, P) \). Therefore, every \( n \) large enough. For instance, \( c = (2I/|A|^{i_P(A)})^{-1} \) works whenever \( n^{i_P(A) - 1} \geq 2 |A|^{i_P(A)} \).

Figure 1 shows a set \( A_{\lambda}^3 \) obtained from this procedure when \( P = \Delta \) the equilateral triangle and \( A \) is the starting set with 15 points and 29 equilateral triangles constructed in Section 4.2.3. Finally, we present the proof of Lemma 2.

Proof of Lemma 2 We show that \( \mathcal{S} \) is the union of a finite number of algebraic sets, all of them of real dimension at most one. This immediately implies that the Lebesgue measure of such a set is zero. For every \( a \in A \) and \( b \in B \), let \( q(a, b) = a + vb \). Suppose that \( q(a_1, b_1) = q(a_2, b_2) \) with \( (a_1, b_1) \neq (a_2, b_2) \). Then \( v(b_2 - b_1) + (a_2 - a_1) = 0 \) and \( b_1 \neq b_2 \). Thus \( v = -(a_2 - a_1)/(b_2 - b_1) \). Therefore there are at most \( \left( \binom{|A|}{2} \right) \) values of \( v \) for which \( A + vB \) has less than \( |A| \cdot |B| \) points.

Now, suppose that the set \( \{q(a_j, b_j) : 1 \leq j \leq m\} \) consists of \( m \) points on a line, where \( \{a_j\} \subseteq A \) and \( \{b_j\} \subseteq B \). Then for \( 3 \leq j \leq m \), we have \( q(a_j, b_j) - q(a_1, b_1) = \lambda_j (q(a_2, b_2) - q(a_1, b_1)) \) where \( \lambda_j \neq 0, 1 \) is a real number. Thus for \( 3 \leq j \leq m \),

\[
v(b_j - b_1 - \lambda_j (b_2 - b_1)) = \lambda_j (a_2 - a_1) - (a_j - a_1) .
\]

First assume all \( b_j \) are equal. Then all \( a_j \) are pairwise different, otherwise we would have less than \( m \) points initially. Moreover, Equation (3) implies that \( (a_j - a_1)/(a_2 - a_1) = \lambda_j \in \mathbb{R} \) for all \( 3 \leq j \leq m \). But this contradicts the fact that there are no \( m \) points on a line in \( A \). By possibly relabeling the points, we can now assume that \( b_1 \neq b_2 \). If \( b_j - b_1 - \lambda_j (b_2 - b_1) \neq 0 \) for some \( j \), then

\[
v = -\frac{\lambda_j (a_2 - a_1) - (a_j - a_1)}{\lambda_j (b_2 - b_1) - (b_j - b_1)}.
\]

Möbius Transformations send circles (or lines) to circles (or lines); thus the last equation, seen as a parametric equation on the real variable \( \lambda_j \), represents a circle (or a line) in the plane. The remaining case is when

\[
b_j - b_1 - \lambda_j (b_2 - b_1) = 0 \text{ for } 3 \leq j \leq m.
\]

Since \( \lambda_j \neq 0, 1 \), then \( b_j \neq b_1, b_2 \). Suppose \( b_j = b_k \) for \( 3 \leq j < k \leq m \), then \( \lambda_j = \lambda_k \) and by (3) and (4) we deduce that \( a_j = a_k \). This contradicts the fact that \( q(a_j, b_j) \) and \( q(a_k, b_k) \) are two different points. Therefore all the \( b_j \) are pairwise different, and then by (3) all the \( b_j \) are on a line. This is a contradiction since there are no \( m \) points on a line in \( B \).

Therefore \( \mathcal{S} \) is the union of a finite number of points and at most \( \binom{|A|}{m} \) circles (or lines), and consequently it has zero Lebesgue measure. \( \square \)
4 Constructions of the initial sets

All the constructions of the initial sets $A$ we provide have explicit coordinates so it is possible to calculate $S_P(A)$ via the following algorithm \cite{5}: Fix two points $p_1, p_2 \in P$, for every ordered pair $(a_1, a_2) \in A \times A$ of distinct points consider the unique orientation-preserving similarity transformation $f$ that maps $p_1 \mapsto a_1$ and $p_2 \mapsto a_2$. An explicit expression is $f(z) = \frac{a_1 - a_2}{p_1 - p_2}z + \frac{a_2 p_1 - a_1 p_2}{p_1 - p_2}$.

Then verify whether $f(P) \subseteq A$. If $N$ is the number of pairs $(a_1, a_2)$ for which $f(P) \subseteq A$, then $S_P(A) = N/|\text{Iso}^+(P)|$. The running time of this algorithm is $O(|P|^2 \log |A|)$. Likewise, it is possible to verify that no 3 points are on a line by simply checking the pairwise slopes of every triple of distinct points. We first present our construction for arbitrary patterns $P$.

4.1 Arbitrary pattern $P$

Proof of Theorem 2. Let $z_0 \in \mathbb{C}\setminus P$ be an arbitrary point and let $p_1, p_2 \in P$ be two fixed points in $P$. For every $p \in P$, there is exactly one orientation-preserving similarity function $f_p$ such that $p_1 \mapsto z_0$ and $p_2 \mapsto p$; indeed an explicit expression of such function is $f_p(z) = \frac{z - p}{p_1 - p_2}z + \frac{p_1 (z - p_2)}{p_1 - p_2}$. Let $A$ be the point set obtained by taking the image of $P$ under every one of the functions $f_p$, that is $A = \bigcup_{p \in P} f_p(P)$. For almost all $z_0$, except for a subset of real dimension 1, the set $A$ does not have $m$ points on a line and all the sets $f_p(P)\{z_0\}$ are pairwise disjoint, that is $|A| = 1 + k(k - 1) = k^2 - k + 1$. This fact can be proved along the same lines as Lemma 2; we omit the details. By construction, each of the $k$ sets $f_p(P)$ is similar to $P$. For every $q \in P\{p_1\},$

\begin{align*}
\{f_p(q) : p \in P\} = \frac{p_1 - q}{p_1 - p_2} P + \frac{z_0 (q - p_2)}{p_1 - p_2}
\end{align*}

is also similar to $P$ and different from the previous copies of $P$ we counted before. Thus $S_P(A) \geq 2k - 1$ and

$$i_P(A) \geq \frac{\log (S_P(A) + |A|)}{\log |A|} = \frac{\log (k^2 + k)}{\log (k^2 - k + 1)}.$$ 

The conclusion follows from Theorem 1.

Figure 2: $A$ has no $m$ points on a line, $|A| = |P|^2 - |P| + 1$ points and $S_P(A) = 2|P| - 1$. 

\begin{equation}
(5)
\end{equation}
4.2 Triangles

The following table gives the currently best available initial set $A$ for each pattern $P$ in Theorem 3. That is, the set $A$ with the largest index $i_P(A)$ known to date. The lower bound stated in Theorem 3 is then given by Theorem 1 applied to $A$.

| Triangular pattern $P = T$ | $|\text{Iso}^+(T)|$ | $|A|$ | $S_T(A)$ | $i_T(A)$ |
|----------------------------|-------------------|------|----------|----------|
| most scalene triangles     | 1                 | 14   | 26       | $\log 40 / \log 14 > 1.397$ |
| all scalene triangles      | 1                 | 5    | 4        | $\log 9 / \log 5 > 1.365$ |
| isosceles triangle with    | 1                 | 8    | 9        | $\log 17 / \log 8 > 1.362$ |
| largest angle $\neq 2\pi/3, \pi/2, \pi/3$ | 1     | 84   | 444      | $\log 528 / \log 84 > 1.414$ |
| $(2\pi/3, \pi/6, \pi/6)$-isosceles triangle | 1     | 24   | 120      | $\log 144 / \log 24 > 1.563$ |
| $(\pi/2, \pi/4, \pi/4)$-isosceles triangle | 1     | 15   | 29       | $\log 102 / \log 15 = 1.707$ |
| equilateral triangle       | 3                 |      |          |          |

Table 1: Indices for the best initial sets when $P = T$ is a triangle.

It is worth noting that our bound for scalene triangles is better than the one for (most) isosceles triangles. The intuitive reason for this is that it is harder to obtain better initial sets when the pattern has any type of symmetries. This difficulty is overtaken by the factor $|\text{Iso}^+(P)| = 3$ when $P$ is an equilateral triangle. We extend our comments in the concluding remarks.

4.2.1 Scalene triangles

We first give a construction for all scalene triangles. If the pattern $P = T$ consists of the points (complex numbers) $0, 1,$ and $z \notin \mathbb{R}$, then the initial set $A_1 = \{0, 1, z, w = z - 1 + 1/z, wz\}$ is in general position for every scalene triangle $T$, see Figure 3(a). $A_1$ has 4 triangles similar to $T$: $(0, 1, z), (z, w, 1), (1, z, wz)$ and $(0, w, wz)$. For all $z$, but a 1-dimensional subset of $\mathbb{C}$, the construction in Figure 3(b) is in general position and gives a better lower bound for $S_T'(n)$. The
corresponding initial set is \( A_2 = A \cup A' \) where \( A = A_1 \cup zA_1 \cup \{ wz^2 / (z - 1) \} \) and \( A' \) is the \( \pi \)-rotation of \( A \) about \( wz/2 \), that is \( A' = -A + wz \). Because some points overlap, \( A_2 \) has only 14 points,

\[
A_2 = \{ 0, 1, z, w, wz, z^2, wz^2, wz^2 / (z - 1), wz - 1, wz - z, wz - w, 1 - z, wz (1 - z), wz / (1 - z) \}.
\]

Among the points in \( A_2 \), there are 6 similar copies of \( A_1 \) with no triangles similar to \( T \) in common: \( A_1, zA_1, wz (A_1 - w), \) and their \( \pi \)-rotations about \( wz/2 \). In addition, the triangles \( (w, wz(1 - z), wz^2 / (z - 1)) \) and \( (wz^2, wz - w, wz / (1 - z)) \) are similar to \( T \) and are not contained in the 6 sets similar to \( A_1 \) mentioned before, so \( S_T(A) \geq 26 \).

![Figure 4](image_url)

**Figure 4:** Sets \( A_i \) with \( |A_i| = 8 \) and \( S_{T(\beta)}(A_i) = 9 \). \( A_1 \) is in general position for \( \alpha \neq \pi/12, \pi/6, \pi/4, \pi/3, \) or \( 5\pi/12 \), \( A_2 \) is in general position for \( \alpha \neq \arccos \sqrt{(3 + \sqrt{17})/8}, \pi/6, \pi/4, \) or \( \pi/3 \).

### 4.2.2 Isosceles triangles

Let the pattern \( P = T(\alpha) \) be an isosceles triangle with angles \( \alpha, \alpha, \) and \( \pi - 2\alpha \), where \( 0 < \alpha < \pi/2 \). We use as initial set one of the following two constructions, each with 8 points and 9 copies of \( T(\alpha) \), i.e., \( S_{T(\alpha)}(A) = 9 \). There are three exceptions that are analyzed later in Section 4.3.3 \( T(\pi/6), T(\pi/4), \) and the equilateral triangle \( T(\pi/3) \). Let \( u = e^{2\alpha i} \) so that \( T(\alpha) = \{ 0, 1, -u \} \). The first initial set is \( A_1 = B_1 \cup B_1^{\prime} \) where \( B_1 = \{ 0, 1, u, 1 + u, 2u + 1 / (u + 1) \} \) and \( B_1^{\prime} \) is the conjugate of \( B_1 \), i.e., \( B_1 = \{ b : b \in B_1 \} \). (See Figure 4(a).) This configuration is in general position as long as \( \alpha \neq k\pi/12, k \in \mathbb{Z} \). It has 9 copies of \( T(\alpha) \): \( (0, 1, 1 + u), (1 + u, u, 0), (1, 2u + 1 / (u + 1), 1 + u), (1 + 1/u, 2u + 1 / (u + 1), u) \), their reflections about the real axis, and \( (1/u, 1, u) \). The actual points in \( A_1 \) are

\[
A_1 = \left\{ 0, 1, u, \frac{1}{u}, 1 + u, 1 + \frac{1}{u}, \frac{u + 2}{u + 1}, \frac{2u + 1}{u + 1} \right\}.
\]

The second initial set is \( A_2 = B_2 \cup B_2^{\prime} \) where \( B_2 = \{ 0, 1, u, 2u + 1 / (u + 1), 1 - \frac{1}{(u + 1)^2} \} \). (See Figure 4(b).) This set is in general position as long as \( \alpha \neq \arccos \sqrt{(3 + \sqrt{17})/8}, \pi/6, \pi/4, \) or \( \pi/3 \). It has 9 copies
of $T(\alpha)$: $\left(1, \frac{u}{u+1}, 0\right)$, $\left(0, \frac{u}{u+1}, u\right)$, $\left(1, 1 - \frac{1}{(u+1)^2}, \frac{u}{u+1}\right)$, $\left(1+u, 1 - \frac{1}{(u+1)^2}, u\right)$, their reflections about the real axis, and $\left(1/u, 1, u\right)$. The actual points in $A_2$ are $A_2 = \left\{0, 1, u, \frac{1}{u}, 1 - \frac{1}{(u+1)^2}, \frac{1}{u}, 1 - \frac{u^2}{(u+1)^2}\right\}$.

### 4.2.3 Equilateral triangle

Let $z \in \mathbb{C}$ and $\omega = e^{2\pi i/3}$ so that $\omega^2 + \omega + 1 = 0$. When the pattern $P = \triangle = \{1, \omega, \omega^2\}$ is the equilateral triangle, we use as initial set $A = B \cup \omega B \cup \omega^2 B$ where $B = \{1, -z\} \cup (-1 + z P)$. For all $z$ but a 1-dimensional subset of $\mathbb{C}$, the set $A$ is in general position and $|A| = 15$. The set $B_1 = \bigcup_{k=0}^2 \omega^k(-1 + z P)$ is the Minkovski Sum $-P + z P$, thus by Lemma 1 there are at least 9 equilateral triangles in $B_1$. In addition $(1, \omega, \omega^2)$ and $(-z, -z \omega, -z \omega^2)$ are equilateral, and each of the points $1, \omega,$ and $\omega^2$ is incident to 6 more equilateral triangles: $(1, -\omega^2 + z, -\omega^2 z)$, $(1, -\omega^2 + z \omega, -z)$, $(1, -\omega^2 + z \omega^2, -z \omega)$, $(1, -\omega^2 z, -\omega + z \omega)$, $(1, -z, -\omega + z \omega^2)$, and $(1, -\omega z, -\omega^2 + z)$, together with the $\pi/3$- and $2\pi/3$-rotations of these triangles about the origin. Thus $S_{\triangle}(A) \geq 29$. It can be checked that there are only 29 equilateral triangles in $A$.

![Figure 5: Initial set $A$ with $|A| = 15$ and $S_{\triangle}(A) = 29.$](image)

### 4.3 Regular polygons

As in the previous section, we construct initial sets for each $k$-regular polygon with $k \in \{4, 5, 6, 8, 10\}$. The following table gives the currently best available initial set $A$ for each of these regular polygons.

| Regular polygon $R(k)$ | $|\text{Iso}^+(R(k))|$ | $|A|$ | $S_{R(k)}(A)$ | $i_{R(k)}(A)$ |
|------------------------|-----------------|------|--------------|--------------|
| Square = $R(4)$        | 4               | 24   | 30           | $\log 144/\log 24 > 1.563$ |
| Hexagon = $R(6)$       | 6               | 74   | 84           | $\log 528/\log 84 > 1.414$ |
| Octagon = $R(8)$       | 8               | 208  | 138          | $\log 1312/\log 208 > 1.345$ |
| Decagon = $R(10)$      | 10              | 420  | 222          | $\log 2640/\log 420 > 1.304$ |
| Pentagon = $R(5)$      | 5               | 120  | 264          | $\log 1440/\log 120 > 1.519$ |

Table 2: Indices for the best initial sets for regular polygons.
We first present the construction for the even-sided polygons and then the construction for the regular pentagon. Finally, we explain how to use these initial sets for any pattern that is a subset of a regular polygon.

4.3.1 Even sided regular polygons.

The following construction of the initial set $A$ is in general position for all even $k$, however the index $i_{R(k)}(A)$ is only better than $i_{R(k)}(R(k))$ when $k \leq 10$. Let $\omega = e^{2\pi i/k}$, $k$ even, and $z \in \mathbb{C}$ an arbitrary nonzero complex number. Suppose that the regular $k$-gon $R(k)$ is given by $R(k) = P = 1 + \omega + z\{\omega^j : 0 \leq j \leq k - 1\}$. The reason why we translated the canonical regular polygon by $1 + \omega$ and rotated and magnified it by $z$ will become apparent soon. To construct our initial set $A$, we first follow the construction of Theorem 2 applied to $P$ with $p_1 = 1 + \omega + z$, $p_2 = 1 + \omega + z\omega$, and $z_0 = 2$. We obtain a set $A_1$ consisting of $z_0$ and $k - 1$ disjoint similar copies of $P$ given by (5). That is, $A_1 = \{2\} \cup \bigcup_{j=1}^{k-1} B_j$ where

$$\omega^j B_{k-j} = \frac{1 - \omega^j}{1 - \omega} (1 + \omega) + \frac{2(1 - \omega^{j+1})}{1 - \omega} = \frac{1 - \omega^j}{1 - \omega} P + \frac{2(\omega^j - \omega)}{1 - \omega} = B_j.$$
For almost all $z \in \mathbb{C}$, except for a subset of real dimension 1, there are no 3 collinear points in $A$ and also for every $j \neq k/2$ the sets $B_j$ and $\omega^j B_i$ are disjoint except for the pairs $(i, l) = (j, 0)$ and $(i, l) = (k - j, j)$. It follows that every set of the form $\omega^j B_j$ with $j \neq k/2$ is a subset of exactly two terms in the union from Equation (6) and it is disjoint from the rest. Thus

$$
|A| = |B_{k/2}| + \left| \bigcup_{0 \leq l \leq k-1, j \neq k/2} \omega^j B_j \right| + \left| \left\{ \omega^j z_0 : 0 \leq l \leq k-1 \right\} \right| = k + \frac{1}{2} k (k - 2) + k = \frac{k}{2} (k^2 - 2k + 4) .
$$

To bound the number of $k$-regular polygons in $A$, first note that for each $0 \leq l \leq k-1$, there are at least $k$ regular polygons with a vertex in $\omega^j z_0$ contained in $\omega^j A_1$ and all of these $k^2$ copies of $P$ are different. For every $1 \leq j \leq k/2 - 1$ the set $\bigcup_{l=0}^{k-1} \omega^j B_j$ is the Minkowski Sum of two copies of $P$, namely

$$P_1 = (1 + \omega^j) \left\{ \omega^l : 0 \leq l \leq k-1 \right\} \text{ and } P_2 = \frac{1 - \omega^j}{1 - \omega} \left\{ \omega^l : 0 \leq l \leq k-1 \right\} ,$$

with exactly $k^2$ points. Thus, by Lemma 1 we have that $S_P(\bigcup_{l=0}^{k-1} \omega^j B_j) = S_P(P_1 + P_2) \geq 3k$. Furthermore, all these $3k(k/2 - 1)$ regular polygons are distinct and also different from those previously counted. Finally, there are two extra polygons not yet counted, namely $B_{k/2}$ and $\left\{ \omega^j z_0 : 0 \leq l \leq k-1 \right\}$. Thus $S_P(A) \geq k^2 + 3k(k/2 - 2) + 2 = \frac{k}{2} (5k^2 - 6k + 4)$ and then

$$i_P(A) \geq \frac{\log \left( \frac{k}{2} (5k^2 - 6k + 4) + k \left( k^2 - 2k + 4 \right) \right)}{\log \left( \frac{k}{2} (k^2 - 2k + 4) \right)} = \frac{\log \left( 3k^3 - 4k^2 + 4k \right)}{\log \left( \frac{k}{2} (k^2 - 2k + 4) \right)} .$$

The conclusion follows by setting $k = 4, 6, 8, \text{ or } 10$.

4.3.2 The regular pentagon

Let $\omega = e^{2\pi i/5}$, for every $z \in \mathbb{C}$ set $R(5) = P = z\{1, \omega, \omega^2, \omega^3, \omega^4\}$. Define

$$A_1 = P + \frac{\sqrt{5} + 3}{2}, \quad A_2 = \frac{\sqrt{5} + 1}{2} (-P + 1), \quad \text{and } A_3 = (\omega^2 - 1) \{z, \omega + 1\} .$$

![Figure 7: Best known initial sets for $P = R(6)$ and $P = R(8)$.](image-url)
Now we consider all the $2\pi k/5$ rotations of these points, $0 \leq k \leq 4$, as well as their symmetrical points with respect to the origin. That is, we define $B = A_1 \cup (-A_1) \cup A_2 \cup (-A_2) \cup A_3 \cup (-A_3)$ and $A = \bigcup_{k=0}^{4} \omega^k B$. For almost all $z \in \mathbb{C}$, except for a subset of real dimension one, $A$ has exactly 120 points and has no three points on a line. There are at least 264 regular pentagons with vertices in $A$:

for each $j = 1, 2$, the set $\bigcup_{k=0}^{4} \omega^k (\pm A_j)$ is the Minkovski Sum of two regular pentagons, and thus by Lemma 1 each of these 4 sets has 15 regular pentagons, the point set $\bigcup_{k=0}^{4} \omega^k (\pm A_3)$ consists of two regular decagons so it contains 4 pentagons, finally each of the 20 points in $\bigcup_{k=0}^{4} \omega^k (\pm A_3)$ is incident to 10 more regular pentagons different from the ones previously counted (see Figure 8).

In fact every point in $A$ is incident to exactly 11 regular pentagons and it turns out that the set $A$ has an interesting set of automorphisms that preserve the regular pentagons.

![Figure 8: The best initial set $A$ for the regular pentagon](image)

### 4.3.3 Subsets of regular polygons

If $P$ is a subset of a regular polygon $R$, then the constructions we have previously obtained for $R$ would be also suitable for $P$. More precisely we have the following result.

**Theorem 5.** Let $R$ be a regular polygon and $P \subseteq R$ with $|P| \geq 3$. For every nonempty finite $A \subseteq \mathbb{C}$, we have that

$$S_P(n) \geq \Omega \left( n^{i_R(A)} \right).$$

**Proof.** Let $I = |\text{Iso}^+(P)|$. Because $R$ is a regular polygon and $|P| \geq 3$, each similar copy of $P$ in $A$ is contained in at most one copy of $R$ in $A$. Thus $S_P(A) \geq S_R(A) \cdot S_P(R)$. On the other hand, $S_P(R) = |R| / I$ and thus

$$I \cdot S_P(A) \geq I \cdot S_R(A) \cdot S_P(R) = |R| S_R(A).$$

Consequently

$$i_P(A) = \frac{\log (I \cdot S_P(A) + |A|)}{\log |A|} \geq \frac{\log (|R| S_R(A) + |A|)}{\log |A|} = i_R(A).$$

Finally, by Theorem 1, $S_P(n) \geq \Omega \left( n^{i_P(A)} \right) \geq \Omega \left( n^{i_R(A)} \right)$. □

As a direct consequence of this theorem, we take care of the isosceles triangles for which the construction in Section 4.2.2 yielded collinear triples. For the isosceles triangles $T(\alpha)$ with $\alpha = \pi/6$
or π/4 we have that $S_{T(\pi/6)}(n) \geq \Omega \left( n^{\log 25/\log 24} \right)$ and $S_{T(\pi/4)}(n) \geq \Omega \left( n^{\log 14/\log 12} \right)$, both of which exceed the bound in Theorem 3. Other point sets treated before can be improved this way as well. For instance $S_{T(\pi/5)}(n), S_{T(2\pi/5)}(n) \geq \Omega \left( n^{\log 14/\log 12} \right) \geq \Omega \left( n^{1.519} \right)$.

### 4.4 $S_P(n, m)$ for an equilateral triangle $P$

When $m \geq 4$, the initial sets $A_m$ with the largest indices we know are clusters of points of the equilateral triangle lattice in the shape of a circular disk.

![Figure 9: Best known constructions of initial sets $A_m$ with many equilateral triangles and at most $m - 1$ points on a line.](image)

**Theorem 6.** For $4 \leq m \leq 9$ and $P = \triangle$ the equilateral triangle, we have

\[
\begin{align*}
S_{\triangle}(n, 4) &\geq \Omega \left( n^{\log 31/\log 7} \right) \geq \Omega \left( n^{1.764} \right) & S_{\triangle}(n, 5) &\geq \Omega \left( n^{\log 116/\log 14} \right) \geq \Omega \left( n^{1.801} \right) \\
S_{\triangle}(n, 6) &\geq \Omega \left( n^{\log 217/\log 19} \right) \geq \Omega \left( n^{1.827} \right) & S_{\triangle}(n, 7) &\geq \Omega \left( n^{\log 528/\log 30} \right) \geq \Omega \left( n^{1.843} \right) \\
S_{\triangle}(n, 8) &\geq \Omega \left( n^{\log 811/\log 37} \right) \geq \Omega \left( n^{1.855} \right) & S_{\triangle}(n, 9) &\geq \Omega \left( n^{\log 1600/\log 52} \right) \geq \Omega \left( n^{1.867} \right)
\end{align*}
\]

and in general if $m$ is even then

\[
S_{\triangle}(n, m) \geq \Omega \left( n^{i_{\triangle}(A_m)} \right)
\]

where

\[
i_{\triangle}(A_m) = \frac{\log(21m^4 - 84m^3 + 156m^2 - 144m + 64) - \log 64}{\log(3m^2 - 6m + 4) - \log 4}
\]

**Proof.** For the first part refer to Figure 9 where the sets $A_m$ and their corresponding indices are shown. For the second part we consider as our set $A_m$ the lattice points inside a regular hexagon of side $m/2 - 1$ with sides parallel to the lattice. Clearly $A_m$ contains at most $m - 1$ collinear points. Also $|A_m| = (3m^2 - 6m + 4)/4$ and $S_{\triangle}(A_m) = (7m^4 - 28m^3 + 36m^2 - 16m)/64$ (see [1]), therefore the result follows from Theorem [1].
After some elementary estimations, we get that the index on the last theorem satisfies that

\[ i_\Delta(A_m) \geq 2 - \frac{\log(12/7)}{2 \log m} + \Theta(\log m)^{-2} > 2 - \frac{0.269}{\log m} + \Theta(\log m)^{-2}. \]

This suggests that \( S_\Delta(n, m) \geq \Omega(n^{2-0.269/\log m}) \), however the constant term hidden in the \( \Omega \) may depend on \( A \), and thus on \( m \). We see this with more detail on the next section.

5 When \( m \) grows together with \( n \)

We investigate the function \( S_P(n, m) \) when the pattern \( P \) is fixed and \( m = m(n) \to \infty \) when \( n \to \infty \). For instance, the maximum number of squares in a \( n \)-point set without \( \log n \) points on a line, is at least \( \Omega(n^{2-c(\log \log n)^{-1}}) \). For the proof of our result, we use the following two theorems mentioned in the introduction as the best bounds for the function \( S_P(n) \) without restrictions.

**Theorem B** (Elekes and Erdős [7]). For any pattern \( P \) there are constants \( a, b, c > 0 \) such that

\[ S_P(n) \geq cn^2 - a(\log n)^{-b}, \]

moreover, if the coordinates of \( P \) are algebraic or if \( |P| = 3 \), then \( S_P(n) \geq cn^2 \).

If \( u, v, w, z \in \mathbb{C} \) then the cross-ratio of the 4-tuple \( (u, v, w, z) \) is defined as

\[ \frac{(w-u)(z-v)}{(z-u)(w-v)}. \]

**Theorem C** (Laczkovich and Ruzsa [13]). \( S_P(n) = \Theta(n^2) \) if and only if the cross-ratio of every 4-tuple in \( P \) is algebraic.

For the sake of clarity, let us call a pattern \( P \) cross-algebraic if the cross-ratio of every 4-tuple is algebraic, and cross-transcendental otherwise.

**Theorem 7.** Let \( P \) be an arbitrary pattern and suppose \( m = m(n) \to \infty \), then for every \( \varepsilon > 0 \) there is a threshold function \( N_0 = N_0(\varepsilon, P) \) such that

\[ S_P(n, m) \geq n^{2-\varepsilon} \text{ for every } n \geq N_0. \]

**Proof.** Suppose \( m = m(n) \to \infty \). We actually prove the following stronger result.

(i) If \( \log (m) \leq \sqrt{\log n} \) and \( P \) is cross-algebraic, then there is a constant \( c_1 > 0 \) depending only on \( P \) such that

\[ S_P(n, m) \geq \Omega\left(n^{2-c_1/\log m}\right). \]

(ii) If \( \log (m) \leq \sqrt{\log n} \) and \( P \) is cross-transcendental, then there are constants \( c_1, c_2, c_3 > 0 \) depending only on \( P \) such that

\[ S_P(n, m) \geq \Omega\left(n^{2-c_2/(\log m)^c_3-c_1/\log m}\right). \]

(iii) If \( \log (m) > \sqrt{\log n} \), then \( S_P(n, m) \geq S_P(n, e^{\sqrt{\log n}}) \) and thus either (i) or (ii) holds with \( \log m = \sqrt{\log n} \).
We first prove (i). Suppose that $P$ is cross-algebraic. By Theorem \textup{[1]} there is a constant $c$, depending only on $P$, and a $([m] - 1)$-set $A$ such that $|A| + S_P(A) \geq cm^2$. Clearly $A$ does not have $m$ points on a line. Then, by \textup{[2]} in the proof of Theorem \textup{[1]} we have that

$$S_P(n, m) \geq \frac{1}{I} \left( \left( \frac{n}{|A|} \right)^{i_P(A)} - n \right) \geq \frac{1}{I} \left( \left( \frac{n}{m} \right)^{\log \left( \frac{cm^2}{\log m} \right)} - n \right) = \frac{1}{Ic} \left( n^{2 + \frac{\log c}{\log m} - \frac{2\log m}{\log m} - cn \right).$$

By assumption, $(2\log m)/\log n \leq 2/\log m$. Since $c < 1$ it follows that $\log c < 0$. Let $c_1 = 2 - \log c > 0$, then

$$S_P(n, m) \geq \frac{1}{Ic} \left( n^{2 + \frac{\log c}{\log m} - \frac{2\log m}{\log m} - n \right) \geq \frac{1}{Ic} \left( n^{2-c_1/\log m} - n \right).$$

That is, $S_P(n, m) \geq \Omega \left( n^{2-c_1/\log m} \right)$, where the constant in the $\Omega$ term does not depend on $n$ or $m$.

Similarly, to prove (ii), assume $P$ is cross-transcendental, then by Theorem \textup{[3]} there are constants $c_i, c_2, c_3 > 0$, depending only on $P$, such that $S_P(n) \geq cn^{2-c_2/\log(m)^c}$. Then there is a $([m] - 1)$-set $A$ such that $|A| + S_P(A) \geq cm^{2-c_2/\log(m)^c}$. Again $A$ does not have $m$ points on a line and setting $c_1 = 2 - \log c$ we get

$$S_P(n, m) \geq \frac{1}{Ic} \left( \left( \frac{n}{|A|} \right)^{i_P(A)} - n \right) \geq \frac{1}{I} \left( \left( \frac{n}{m} \right)^{2 + \frac{\log c}{\log m} - \frac{3\log m}{(\log m)^2}} - n \right) \geq \frac{1}{Ic} \left( n^{2-c_2/\log m} - c_1/\log m - cn \right)$$

for $n$ and $m$ large enough depending only on $P$. That is, $S_P(n, m) \geq \Omega(n^{2-c_2/\log(m)^c-c_1/\log m})$, where the constant in the $\Omega$ term does not depend on $n$ or $m$.

If $m$ grows like a fixed power of $n$ and $P$ is cross-algebraic (i.e., $S_P(n) = \Theta(n^2)$), then we can improve our bound.

\textbf{Theorem 8.} If $P$ is cross-algebraic and $m = m(n) \geq n^\alpha$ for some fixed $0 < \alpha < 1$, then there is $c_1 = c_1(P, \alpha) > 0$ such that

$$S_P(n, m) \geq c_1 n^2 \text{ for every } n \geq |P|.$$

\textit{Proof.} Choose an integer $j \geq 2$ such that $\alpha > 1/j$. Consider an optimal set $A$ for the function $S_P([n^{1/j}])$. Then $|A| = [n^{1/j}]$ and by Theorem \textup{[1]} there is a constant $c = c(P)$ such that $S_P(A) = S_P(|A|) \geq c|A|^2$. Since $|A| \leq n^{1/j} < n^\alpha \leq m$, then $A$ has no $m$ collinear points. By Identity \textup{[1]} in Theorem \textup{[1]} we have

$$S_P(n, m) \geq S_P(|A|^j, m) \geq \frac{1}{I} \left( (I \cdot S_P(A) + |A|^j) - |A|^j \right) \geq I^{j-1}S_P(A)^j \geq c^j I^{j-1} |A|^{2j}.$$

Now, if $n \geq |P|$ then $|A| \geq n^{1/j} - 1 \geq \left( 1 - |P|^{-1/j} \right) n^{1/j}$. By letting $c_1 = c^j I^{j-1} \left( 1 - |P|^{-1/j} \right)^{2j}$ we get

$$S_P(n, m) \geq c_1 n^2.$$
6 Parallelogram-free sets

We consider the restriction of the function $S_P(n)$ to sets of points $A$ in general position (no 3 points on a line) and without parallelograms. We say that such a set $A$ is parallelogram-free. This immediately prohibits the use of Minkovski Sums to obtain good constructions. More precisely, for a parallelogram-free pattern $P$, define

$$S^\parallel_P(n) = \max \{ S_P(A) : |A| = n \text{ and } A \text{ is parallelogram-free} \}.$$

We obtain the following upper bound on $S^\parallel_P(n)$.

Figure 10: A parallelogram-free point set with $n$ points and $cn \log n$ similar copies of $P$.

**Theorem 9.** Let $P$ be a parallelogram-free pattern with $|P| \geq 3$. Then for all $n$,

$$S^\parallel_P(n) \leq n^{3/2} + n.$$

**Proof.** Suppose $A$ is an $n$-set in the plane in general position and with no parallelograms. Let $p_1, p_2, p_3$ be three points in $P$. Consider the following bipartite graph $B$. The vertex bipartition is $(A, A)$; the edges are the pairs $(a_1, a_2) \in A \times A$ such that there is a point $a_3 \in A$ with $\triangle a_1a_2a_3 \sim \triangle p_1p_2p_3$. Every similar copy of $P$ in $A$ has at least one edge $(a_1, a_2)$ associated to it. Thus the number of edges $E$ in our graph satisfies that $E \geq S_P(A)$. By a theorem of Kővari et al. [12] (also referred in the literature as Zarankiewicz problem [15]), it is known that a bipartite graph with $n$ vertices on each class and without subgraphs isomorphic to $K_{2,2}$ contains at most $(n - 1)n^{1/2} + n$ edges. To finish our proof we now show that $B$ has no subgraphs isomorphic to $K_{2,2}$. Suppose by contradiction that $(a_1, a_3), (a_1, a_4), (a_2, a_3), (a_2, a_4)$ are edges in $B$. Let $\lambda = (p_3 - p_1)/(p_2 - p_1)$. By definition, there are points $a_{13}, a_{14}, a_{23}, a_{24} \in A$ such that $\triangle a_1p_2p_3 \sim \triangle a_1a_3a_{13} \sim \triangle a_1a_4a_{14} \sim \triangle a_2a_3a_{23} \sim \triangle a_2a_4a_{24}$. Thus $a_{13} = a_1 + \lambda(a_3 - a_1), a_{14} = a_1 + \lambda(a_4 - a_1), a_{23} = a_2 + \lambda(a_3 - a_2)$, and $a_{24} = a_2 + \lambda(a_4 - a_2)$. Then $a_{13} - a_{14} = a_{23} - a_{24} = \lambda(a_3 - a_4)$, which means that $a_{13}a_{14}a_{24}a_{23}$ is a parallelogram. This contradicts the parallelogram-free assumption on $A$. 

\[ \square \]
We make no attempt to optimize the coefficient of the $n^{3/2}$ term, since we do not believe that $n^{3/2}$ is the right order of magnitude.

**Theorem 10.** Let $P$ be a parallelogram-free pattern with $|P| \geq 3$. Then there is a constant $c = c(P)$ such that for $n \geq |P|$, $$S_P^\|_1(n) \geq cn \log n.$$ For every pattern $P$, we recursively construct a parallelogram-free point set with many occurrences of $P$. For any $u, v \in \mathbb{C}$, we define $$Q(P, A, u, v) = \bigcup_{p \in P} (up + (vp - p + 1) A) = \bigcup_{p \in P \ A} (up + (vp - p + 1) a). \quad (7)$$ Almost all selections of $u$ and $v$ yield a set $Q = Q(P, A, u, v)$ that is parallelogram-free and such that all the terms in the double union are pairwise different. The proof of this technical fact is given by next lemma.

**Lemma 3.** Let $A$ and $P$ be parallelogram-free sets. If $S$ is the set of points $(u, v) \in \mathbb{C}^2$ for which $Q = Q(P, A, u, v)$ satisfies that $|Q| < |A| |P|$, $Q$ has three collinear points, or $Q$ has a parallelogram; then $S$ has zero Lebesgue measure.

We defer the proof of this lemma and instead proceed to bound the number of similar copies of $P$ in $Q$.

**Lemma 4.** If $A$ and $P$ are finite parallelogram-free sets, and $Q = Q(P, A, u, v)$ defined in (7) satisfies that $|Q| = |A| |P|$, then

$$S_P(Q) \geq |P| S_P(A) + |A|. \quad \text{(8)}$$

**Proof.** Because $|Q| = |A||P|$ it follows that each term in the first union contributes exactly $S_P(A)$ similar copies of $P$, all of them pairwise different. In addition note that

$$Q = \bigcup_{a \in A} (a + (u + va - a) P).$$

So each term in the new union is a similar copy of $P$, all of them different and also different from the ones we had counted before. Therefore $S_P(Q) \geq |P| S_P(A) + |A|$. \hfill $\square$

We now prove the theorem.

**Proof of Theorem 10.** Let $A_1 = P$ and for $m \geq 1$ let $A_{m+1}$ be the parallelogram-free set $Q$ obtained from Lemma 3 with $A = A_m$. Because $|A_{m+1}| = |A_1||A_m|$, it follows that $|A_m| = |P|^m$ for all $m$. Further, by Lemma 4 for every $0 \leq k \leq m - 2$, $S_P(A_{m-k}) \geq |P| S_P(A_{m-k-1}) + |A_{m-k-1}| = |P| S_P(A_{m-k-1}) + |P|^{m-k-1}$. Thus

$$S_P(A_m) \geq |P| S_P(A_{m-1}) + |P|^{m-1} \geq |P|^2 S_P(A_{m-2}) + 2 |P|^{m-1}$$

$$\geq \cdots \geq |P|^{m-1} S_P(A_1) + (m - 1) |P|^{m-1} = m |P|^{m-1}. \quad \text{(9)}$$

Now, suppose $|P|^m \leq n < |P|^{m+1}$ with $m \geq 2$. Let $c = 1/(2 |P|^2 \log |P|)$, then

$$S_P^\|_1(n) \geq S_P(A_m) \geq m |P|^{m-1} > \left( \frac{\log n}{\log |P|} - 1 \right) \frac{n}{|P|^2} \geq cn \log n.$$ 

If $|P| \leq n < |P|^2$ then $S_P^\|_1(n) \geq S_P(P) = 1 > cn \log n$. \hfill $\square$
Finally, we present the proof of Lemma 3.

Proof of Lemma 3. We show that $S$ is made of a finite number of algebraic sets, all of them of real dimension at most three. This immediately implies that the Lebesgue measure of such a set is zero. For every $p \in P$ and $a \in A$, let $q(a, p) = (up + (vp - p + 1)a)$. Suppose that $q(a_1, p_1) = q(a_2, p_2)$ with $(a_1, p_1) \neq (a_2, p_2)$. Then

$$(p_1 - p_2)u + (p_1a_1 - p_2a_2) v + a_1(1 - p_1) - a_2(1 - p_2) = 0.$$  

This is the equation of a complex-line in $C^2$ (with real-dimension two) unless the coefficients of $u$ and $v$, as well as the independent term are equal to zero. That is, $p_1 - p_2 = 0$, $(p_1a_1 - p_2a_2) = 0$, and $a_1(1 - p_1) - a_2(1 - p_2) = 0$. These equations imply that $(a_1, p_1) = (a_2, p_2)$ which contradicts our assumption. Thus the set of pairs $(u, v)$ for which $|Q| < |A| |P|$ is the union of $\frac{|A| |P|}{2}$ sets of real-dimension two.

Assume that $q(a_1, p_1), q(a_2, p_2), \text{ and } q(a_3, p_3)$ are three collinear points. Thus there is a real $\lambda \neq 0, 1$ such that $q(a_2, p_2) - q(a_1, p_1) = \lambda (q(a_3, p_3) - q(a_1, p_1))$. Then

$$(p_2 - p_1 - \lambda(p_3 - p_1))u + (p_2a_2 - p_1a_1 - \lambda(p_3a_3 - p_1a_1)) v + a_2(1 - p_2) - a_1(1 - p_1) - \lambda (a_3(1 - p_3) - a_1(1 - p_1)) = 0.$$  

For every $\lambda$, the last equation represents a complex-line in $C^2$. Considering $\lambda$ as a real variable, this equation represents an algebraic set of real-dimension three. This happens unless the coefficients of $u$ and $v$, as well as the independent term are equal to zero. That is, $p_2 - p_1 - \lambda(p_3 - p_1) = 0$, $p_2a_2 - p_1a_1 - \lambda(p_3a_3 - p_1a_1) = 0$, and $a_2 - a_1 - \lambda(a_3 - a_1) = 0$. If $p_1, p_2, p_3$ are three different points then, since no three points in $P$ are collinear, $p_2 - p_1 - \lambda(p_3 - p_1) \neq 0$. If any two of $p_1, p_2, p_3$ are equal and the remaining is different, then we still have $p_2 - p_1 - \lambda(p_3 - p_1) \neq 0$. Thus $p_1 = p_2 = p_3$, and by symmetry $a_1 = a_2 = a_3$; which contradicts the fact that we started with three distinct points $q(a_3, p_3)$. Thus the set of pairs $(u, v)$ for which $|Q|$ has collinear points is the union of $\frac{|A| |P|}{3}$ sets of real-dimension three.

Finally, assume that $q(a_1, p_1)q(a_2, p_2)q(a_4, p_4)q(a_3, p_3)$ is a parallelogram. That is $q(a_2, p_2) - q(a_1, p_1) = q(a_4, p_4) - q(a_3, p_3)$, and thus

$$(p_2 - p_1 - (p_4 - p_3))u + (p_2a_2 - p_1a_1 - (p_4a_4 - p_3a_3)) v + a_2(1 - p_2) - a_1(1 - p_1) - (a_4(1 - p_4) - a_3(1 - p_3)) = 0.$$  

Again this equation represents a complex-line in $C^2$, unless the coefficients of $u$ and $v$, as well as the independent term are equal to zero. That is, $p_2 - p_1 - (p_4 - p_3) = 0$, $p_2a_2 - p_1a_1 - (p_4a_4 - p_3a_3) = 0$, and $a_2 - a_1 - (a_4 - a_3) = 0$. If $p_1, p_2, p_3, p_4$ are four different points then, since $P$ has no parallelograms, $p_2 - p_1 - (p_4 - p_3) \neq 0$. Since no three points of $P$ are collinear there are two extra possibilities: $(p_1, p_2) = (p_3, p_4)$ or $(p_1, p_3) = (p_2, p_4)$. By symmetry we also have $(a_1, a_2) = (a_3, a_4)$ or $(a_1, a_3) = (a_2, a_4)$. If $(p_1, p_2) = (p_3, p_4)$ and $(a_1, a_2) = (a_3, a_4)$, then $q(a_1, p_1) = q(a_3, p_3)$; which contradicts our assumption. Similarly, if $(p_1, p_3) = (p_2, p_4)$ and $(a_1, a_3) = (a_2, a_4)$, then $q(a_1, p_1) = q(a_2, p_2)$. Assume $(p_1, p_2) = (p_3, p_4)$ and $(a_1, a_3) = (a_2, a_4)$. Then the equation $p_2a_2 - p_1a_1 - (p_4a_4 - p_3a_3) = 0$ becomes $(p_2 - p_1)(a_1 - a_3) = 0$. But if $p_1 = p_2$ then $q(p_1, a_1) = q(p_2, a_2)$, and if $a_1 = a_3$ then $q(p_1, a_1) = q(p_3, a_3)$; a contradiction in both cases. The remaining case when $(p_1, p_3) = (p_2, p_4)$ and $(a_1, a_2) = (a_3, a_4)$ follows by symmetry. Thus the set of pairs $(u, v)$ for which $|Q|$ has parallelograms is the union of $\frac{|A| |P|}{4}$ sets of real-dimension two. 

Remark 1. If the sets $A$ and $P$ have no two parallel segments then it can be proved, along the lines of last lemma, that almost all the sets $Q(A, P, u, v)$ are free of pairs of parallel segments as well.
7 Concluding Remarks

The main relevance of Theorem 1 is that it provides an effective tool to obtain better lower bounds for $S_P(n, m)$. Indeed, any of the results for specific patterns in this paper, can be improved by finding initial sets with larger indices. For a general pattern $P$, Theorem 2 is only slightly better than the bound $\Omega(n^{\log(1+|P|)/\log|P|})$ which is obtained using $A = P$ as the initial set. There must be a way to construct a better initial set.

When the pattern $P = T$ is a triangle we obtained a considerably larger bound when $T$ is equilateral. The reason behind this is the fact that there is a multiplying factor of $|\text{Iso}^+(T)| = 3$ in the index of $i_T(A)$. We could not construct initial sets for arbitrary isosceles triangles that would improve the bound for scalene triangles. The mirror symmetry of the isosceles triangles became an obstacle when trying to construct sets with large indices. For instance, the set $A_1$ in Section 4.2.1 always yields collinear points when $T$ is an isosceles triangle.

**Problem 1.** For every isosceles triangle $T$, find a set $A$ such that $i_T(A) \geq \log 9/\log 5$.

We also had some limitations to construct initial sets when $P = R(k)$ is a regular polygon. In this case, the index obtained using $P$ itself as initial set is $\log(2k)/\log k$. We do not have a better initial set for odd $k \geq 7$ and in fact our construction of the initial set $A$ for even-sided regular $k$-gons in Section 4.3.1 only gives an index $i_{R(k)}(A)$ better than $\log(2k)/\log k$ when $k < 12$.

**Problem 2.** Let $R(k)$ be a regular $k$-gon. For every even $k \geq 12$ and odd $k \geq 7$ find a set $A$ such that $i_{R(k)}(A) \geq \log(2k)/\log k$.

We are confident that there are some yet undiscovered methods for getting initial sets with larger indices. We would like to find such sets for some other classes of interesting geometric patterns. For instance, right triangles, trapezoids, parallelograms, and sets already having some points on a line, like subsets of a lattice.

According to Theorem 7, if we let the number of allowed collinear points to increase with $n$, then we can achieve $n^{2-\varepsilon}$ similar copies of a pattern $P$. We actually believe this is true even when $m$ is constant.

**Conjecture 1.** Let $m \geq 3$ be a positive integer and $P$ a finite pattern with no $m$ collinear points. For every real $\varepsilon > 0$, there is $N(\varepsilon) > 0$ such that for all $n \geq N(\varepsilon)$,

$$S_P(n, m) \geq n^{2-\varepsilon}.$$  

A proof of this conjecture cannot follow from Theorem 1, so such a proof will require a different way of constructing sets in general position and with many similar copies of the pattern $P$. On the other hand, we believe that the construction in Theorem 10 for the function $S^2_P(n)$ is close to optimal. Here we believe that a stronger upper bound is needed.

**Conjecture 2.** Let $P$ be a parallelogram-free pattern. For every real $\varepsilon > 0$, there is $N(\varepsilon) > 0$ such that for all $n \geq N(\varepsilon)$,

$$S^2_P(n) \leq n^{1+\varepsilon}.$$  

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