PARTIAL DIFFERENTIAL INCLUSIONS OF TRANSPORT TYPE WITH STATE CONSTRAINTS

THOMAS LORENZ

Applied Mathematics
RheinMain University of Applied Sciences
Wiesbaden Rüsselsheim, Germany

Abstract. The focus is on the existence of weak solutions to the quasilinear first-order partial differential inclusion
\[ \partial_t f \in -\text{div}_x (G(t, f) \cdot f) + U(t, f) \cdot f + W(t, f) \]
with values in $L^p(\mathbb{R}^N)$ for $p \in (1, \infty)$. The solution is to satisfy state constraints in addition, i.e., all its values belong to a given set $V \subset L^p(\mathbb{R}^N)$ of constraints. We specify sufficient conditions such that every function in $V$ initializes at least one weak solution with all its values in $V$ (so-called weak invariance a.k.a. viability of $V$). Due to the regularity assumptions about the set-valued coefficient mappings, these solutions prove to be renormalized (in the sense of Di Perna and Lions).

1. Introduction. A large class of mathematical models is formulated in terms of a continuity equation or, more generally, a transport equation in divergence form
\[ \partial_t f + \text{div}_x (g f) = u \cdot f + w. \]
Whenever the models take a form of feedback into consideration, the coefficient functions $g$, $u$, $w$ do not depend just on time $t$ and space $x$, but also on the current state described as $f = f(t, x)$. Clearly, this leads to nonautonomous and nonlinear partial differential equations of first order. Although these analytical criteria make an inevitable impression, they open the door to various and extended theories of hyperbolic equations.

In this article, the motivation is based on hyperbolic models for cancer cell migration (like in [66]) and traffic flow (such as [58]). This context justifies the conceptual aspects that nonlocal dependence (in space) is taken into consideration. Indeed, drivers of vehicles on a highway can watch the vehicles in their respective neighborhoods (and strictly speaking, they are even obliged to do so). The resulting relationships between vehicles are usually described in terms of convolution operators instead of Nemytskii operators. Nonlocal models of hyperbolic type have already been investigated thoroughly by Colombo, Goatin, Quincampoix and collaborators, for example (see [1, 18, 28, 29, 30, 27, 31, 45, 67] and references therein).

From the analytical point of view, similar problems have occurred in models for migrating cancer cells. Indeed, each cell has a small, but positive

2010 Mathematics Subject Classification. Primary: 35F25, 35R70; Secondary: 35B30, 35D30, 47J35.
Key words and phrases. Partial differential inclusion of first order, nonlocal transport equation, renormalized solutions, weak invariance a.k.a. viability, Scorza-Dragoni theorem for closed graphs.
extension and the cells are communicating with each other by means of signaling substances. More details about chemotaxis and adhesion are described in, e.g., [32, 36, 37, 38, 39, 40, 44, 53, 56, 57, 66, 71]. These model classes motivate our interest in nonlinear hyperbolic problems with functional dependence, i.e.,

$$\partial_t f + \text{div}_x(G(t, f) f) = U(t, f) \cdot f + W(t, f) \quad \text{in} \ [0, T] \times \mathbb{R}^N$$

with given coefficient functions of appropriate regularity

$$
G : [0, T] \times L^p(\mathbb{R}^N) \to L^q_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N),
$$

$$
U : [0, T] \times L^p(\mathbb{R}^N) \to L^q_{\text{loc}}(\mathbb{R}^N),
$$

$$
W : [0, T] \times L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)
$$

for \( p \in (1, \infty) \) and \( q := \frac{p}{p-1} \in (1, \infty) \). Strictly speaking, we focus on renormalized solutions (in the sense of Di Perna and Lions [35, 60]) and so, the coefficients \( G, U \) will even be assumed to have their values in the Sobolev spaces \( W^{1, \infty}(\mathbb{R}^N, \mathbb{R}^N) \), \( W^{1, \infty}(\mathbb{R}^N) \) respectively. This class of partial differential equations has already been investigated in, e.g., [58] (see also [63, §§ 2.5, 3.8] and [65, 66]).

The new contribution of this article is motivated by uncertainty: The coefficient functions \( G, U \) and \( W \) are not single-, but set-valued. This mathematical change is required whenever the relationships between state function \( f(t) \in L^p(\mathbb{R}^N) \) and the coefficients are not known precisely or if the coefficients depend on an open-loop control in addition. We prefer set-valued maps to probabilistic approaches for preserving the deterministic character of the mathematical concept rather than neglecting a class of events from the very beginning. This leads to a so-called partial differential inclusion of first order

$$
\begin{align*}
\{ \partial_t f & \in -\text{div}_x(G(t, f) f) + U(t, f) \cdot f + W(t, f) \in [0, T] \times \mathbb{R}^N \\
 f(0) & = f_0 \}
\end{align*}
$$

for a function \( f : [0, T] \to L^p(\mathbb{R}^N) \) with the set-valued coefficient mappings

$$
G : [0, T] \times L^p(\mathbb{R}^N) \to W^{1, \infty}(\mathbb{R}^N, \mathbb{R}^N),
$$

$$
U : [0, T] \times L^p(\mathbb{R}^N) \to W^{1, \infty}(\mathbb{R}^N),
$$

$$
W : [0, T] \times L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)
$$

given. Our main result concerns sufficient conditions on the set-valued coefficients \( G, U, W \) and a set of state constraints \( V \subset L^p(\mathbb{R}^N) \) such that every function \( f_0 \in V \) initializes at least one renormalized solution \( f : [0, T] \to L^p(\mathbb{R}^N) \) with all its values in \( V \). Then the subset \( V \subset L^p(\mathbb{R}^N) \) is usually called weakly invariant or viable w.r.t. the corresponding differential inclusion (see, e.g., [8, 11, 13, 23, 26, 41, 52, 74, 75, 76] and references therein).

The criterion is based on a tangential condition – adapting Bouligand’s contingent cone to \( L^p(\mathbb{R}^N) \), but supplied with an appropriate metric \( d_{L^p} \) (closely related to its weak topology as we will specify in Appendix A.2). This aspect is not really surprising because many other types of differential inclusions have already been investigated in regard to viability since Nagumo’s first publication [70] about ordinary differential equations in 1942. Indeed, most viability results in the literature so far concern ordinary differential inclusions (usually for values in the Euclidean or a Banach space, see, e.g., [8, 23, 24, 26, 42, 41, 48, 49, 61, 77] and references therein) or, the authors consider semilinear evolution inclusions in Banach spaces (see, e.g., [19, 22, 23, 52, 74, 76] and references therein). There are, however, much less results available considering nonautonomous evolution inclusions with nonlinear semigroups, whose generators are usually hemicontinuous monotone w.r.t. state
and assumed to satisfy a coercivity-type condition (see, e.g., [52, § I.4] for details and [55, 68, 81, 82, 83] as well as references therein). To the best of our knowledge, initial value problem (1) does not belong to the classes of evolution problems covered in the literature about viability theory so far:

- We focus on nonautonomous and nonlinear transport inclusions of first order in $L^p(\mathbb{R}^N)$. They cannot be reformulated as a semilinear evolution inclusion by means of a semigroup in an obvious way and hence, this problem is not covered by the results concerning evolution inclusions (like in, e.g., [23, 69, 74, 76]).
- The influence of the respective state $f(t) \in L^p(\mathbb{R}^N)$ on the coefficient mappings $\mathcal{G}, \mathcal{U}, \mathcal{W}$ in inclusion (1) is of functional type and so, various forms of nonlocal dependence in space $\mathbb{R}^N$ can be taken into consideration.
- There are no assumptions about the set-valued coefficient mappings $\mathcal{G}, \mathcal{U}$ imitating the gist of monotonicity or parabolicity.
- The compositions of the coefficient mappings with $f(t)$ like $\mathcal{G}(t, f(t)) : [0, T] \ni t \rightarrow W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ have their set values in the bounded and Lipschitz continuous functions of space, but this regularity (in space) is not restricted to a pointwise and continuous dependence on $f(t) : \mathbb{R}^N \rightarrow \mathbb{R}$ (as suggested for quasilinear conservation laws in, e.g., [16, §§ 3.3, 5.5]).

The main result is formulated in subsequent Theorem 2.4. Briefly speaking, the tangent condition is related to the so-called contingent cone by Bouligand as introduced for subsets of finite-dimensional vector spaces in, e.g., [8, 11, 12, 14], formulated for Banach spaces in, e.g., [19, 23, 26, 33] and finally extended to metric spaces in, e.g., [9, 10, 63, 64]. In more detail, for Lebesgue-almost every $(\mathcal{L}^1$-a.e.) $t \in [0, T)$ and every element $\phi \in \mathcal{V} \subset L^p(\mathbb{R}^N)$, we assume (at least) one tuple $(g, u, w) \in \mathcal{G}(t, \phi) \times \mathcal{U}(t, \phi) \times \mathcal{W}(t, \phi)$ such that the renormalized solution $\partial_{g, u, w}(\cdot, \phi) := f : [0, 1] \rightarrow L^p(\mathbb{R}^N)$ of the autonomous linear initial value problem

$$
\begin{align*}
\partial_t f + \text{div}_x(g \cdot f) &= u \cdot f + w \\
\quad \quad \quad \; f(0) &= \phi \in L^p(\mathbb{R}^N)
\end{align*}
$$

satisfies

$$
\lim_{h \downarrow 0} \frac{1}{h} \cdot \text{dist}(\partial_{g, u, w}(h, \phi), \mathcal{V}) = 0
$$

(3)

where the distance from $\mathcal{V}$ is understood w.r.t. $d_{L^p} : \text{dist}(\phi, \mathcal{V}) := \inf_{\psi \in \mathcal{V}} d_{L^p}(\phi, \psi)$. Here the choice of the metric $d_{L^p}$ on $L^p(\mathbb{R}^N)$ proves to play a key role. On the one hand, it should be sufficiently “weak” so that the asymptotic condition (3) is not too restrictive. On the other hand, it specifies the topology on the domain of the set-valued coefficient mappings

$$
\begin{align*}
\mathcal{G} : [0, T] \times (L^p(\mathbb{R}^N), d_{L^p}) &\ni W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N), \\
\mathcal{U} : [0, T] \times (L^p(\mathbb{R}^N), d_{L^p}) &\ni W^{1,\infty}(\mathbb{R}^N), \\
\mathcal{W} : [0, T] \times (L^p(\mathbb{R}^N), d_{L^p}) &\ni L^p(\mathbb{R}^N).
\end{align*}
$$

Based on earlier results about transport equations in [58, 63, 66], the state space $L^p(\mathbb{R}^N)$ is now supplied with the metric $d_{L^p}(f, g) := \sup \left\{ \int_{\mathbb{R}^N} \varphi \cdot (f - g) \, dx \quad \middle| \quad \varphi \in C_c^1(\mathbb{R}^N), \; \|\varphi\|_{L^\infty(\mathbb{R}^N)} \leq 1, \right.$

$$
\|\varphi\|_{L^\infty(\mathbb{R}^N)} \leq 1, \; \|\nabla \varphi\|_{L^\infty} \leq 1 \right\}.
$$

It metrizes the weak topology on (norm-) bounded and tight subsets of $L^p(\mathbb{R}^N)$ as formulated in more detail in Appendix A.2 below.
In other words, the main theorem, Theorem 2.4, specifies an existence result for solutions under state constraints. The proof follows essentially the same steps of constructing approximate solutions and extracting appropriately converging subsequences as they are known for ordinary differential inclusions (see, e.g., [8, 12]) and as they have also been applied to set-valued states in, e.g., [62, 63, 64]. This approximating approach is reported to go back to Haddad [48, 49].

Here we consider nonautonomous differential inclusions though and, we would like to keep the regularity assumptions w.r.t. time rather weak so that the typical situation of open-loop control problems is still covered. In this context, the standard choice of regularity is based on Lebesgue measurability. The contingent hypothesis (3) is formulated in terms of the flow \( \vartheta_{g,u,w} : [0,1] \times L^p(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N) \) induced by the autonomous linear initial value problem (2). Hence, it suggests itself to construct the approximate solutions by means of autonomous linear auxiliary problems in a way piecewise in time (see Subsection 3.2 below). In particular, we can benefit from earlier results about transport equations proved in [58] and summarized in Appendix B. After selecting appropriately converging subsequences of both the approximate solutions and the coefficient mappings, we need a form of closed graph property w.r.t. time and state for verifying that the limits lead to a solution of the original partial differential inclusion (1).

This step is based on a extension of the Scorza-Dragoni theorem. Originally established for real-valued functions of two real variables in [73], the statement about “almost continuity” has been extended to set-valued maps in various regards. Our focus of interest is on restrictions with closed graphs – and in the general situation of metric spaces like \((L^p(\mathbb{R}^N), d_{L^p})\). There are several generalizations in these directions available in the literature (such as, e.g., [54, 72]), but they usually provide just a further set-valued map whose closed graph is contained in the given graph and whose set values might be empty ([72, Theorem 1], for example, is cited in subsequent Lemma A.1). At first glance, it is not clear at all how to draw any conclusions about this additional set-valued mapping in regard to contingent hypothesis (3).

Motivated by the general interest in Scorza-Dragoni-type results, we investigate in more detail which conclusions can be drawn from the established assumptions in [54, 72]. In Corollary A.3 below, the (joint) measurability of the set-valued coefficient mappings is identified as an appropriate assumption for guaranteeing the “almost” closed graph (as required for the proof of main Theorem 2.4). It might be worth mentioning that this result is a special case of [15, Theorem 2], but we give a different proof (by means of other auxiliary functions) and formulate it here for the sake of a self-contained and clarifying presentation.

The article has the following structure. In Section 2, we give the definition of a weak solution to the first-order partial differential inclusion (1) and specify the metric \( d_{L^p} \) on the basic set \( L^p(\mathbb{R}^N) \). This lays the foundations for formulating our main result, i.e., Theorem 2.4. The detailed proofs concerning Theorem 2.4 can be found in Section 3. Several analytical tools are collected in Appendices A and B. Indeed, Section A.1 provides extensions of the Scorza-Dragoni theorem to set-valued maps between metric spaces – with special focus on the closed graph property (not on continuity or lower semicontinuity as frequently formulated in the literature). In Appendix A.2, we collect various results about the metrics \( d_{L^p}, \tilde{e}_{L^p} \) used on tight subsets of the state space \( L^p(\mathbb{R}^N) \). Then, Section A.3 provides the metric
\[ \delta_{L^q, \text{loc}} \text{ of local } L^q \text{ norm convergence in } \mathbb{R}^N \text{ which proves to be suitable for the values of coefficient mappings } G, U. \text{ In Appendix B, we collect several properties of the renormalized solutions to nonautonomous linear transport equations. Their proofs can be found in [58], for example.} \]

2. **Main results.** On our way to the main result, the first step consists in the analytical characterization which type of solution we are interested in. In particular, we choose the state space \( L^p(\mathbb{R}^N) \) deliberately because some forms of singularities (w.r.t. space) are still mathematically admissible. This supplementary advantage has already been pointed out in the article [66] about cancer cell migration.

**General Hypothesis.** Fix \( p \in (1, \infty) \) arbitrarily and \( q := \frac{p}{p-1} \in (1, \infty) \). Furthermore, \( L^q_{\text{loc}}(\mathbb{R}^N) \) and \( L^q_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N) \) are supplied with the topology of \( L^q \) norm convergence on each compact subset of \( \mathbb{R}^N \). \( L^1 \) and \( \mathcal{L}^N \) abbreviate the Lebesgue measure on \( \mathbb{R} \) and \( \mathbb{R}^N \) respectively.

**Definition 2.1.** Let the set-valued maps
\[
G : [0, T] \times L^p(\mathbb{R}^N) \Rightarrow L^q_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N), \\
U : [0, T] \times L^p(\mathbb{R}^N) \Rightarrow L^q_{\text{loc}}(\mathbb{R}^N), \\
\mathcal{W} : [0, T] \times L^p(\mathbb{R}^N) \Rightarrow L^p(\mathbb{R}^N)
\]
be given. \( f : [0, T] \longrightarrow L^p(\mathbb{R}^N) \) is called a weak solution of the first-order partial differential inclusion
\[
\begin{cases}
\partial_t f \in -\text{div}_x (G(t,f) f) + U(t,f) \cdot f + \mathcal{W}(t,f) & \text{in } [0, T] \times \mathbb{R}^N \\
f(0) = f_0
\end{cases}
\]
if \( f : [0, T] \longrightarrow (L^p(\mathbb{R}^N), \text{weak}) \) is continuous with \( f(0) = f_0 \) and if there exist Lebesgue integrable selections \( g : [0, T] \longrightarrow L^1_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N) \), \( u : [0, T] \longrightarrow L^q_{\text{loc}}(\mathbb{R}^N) \) and \( w : [0, T] \longrightarrow L^p(\mathbb{R}^N) \) satisfying
\[
\begin{cases}
g(t) \in G(t, f(t)) \quad \text{for } L^1\text{-a.e. } t \in [0, T], \\
u(t) \in U(t, f(t)) \quad \text{for } L^1\text{-a.e. } t \in [0, T], \\
w(t) \in \mathcal{W}(t, f(t)) \quad \text{for } L^1\text{-a.e. } t \in [0, T], \\
\int_{\mathbb{R}^N} \varphi \left( f(t_2) - f(t_1) \right) \, dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^N} f(s, x) \, g(s, x) \cdot \nabla_x \varphi(x) \, ds \, dx + \\
\quad \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (f(s, x) \, u(s, x) + w(s, x)) \, \varphi(x) \, ds \, dx
\end{cases}
\]
for any \( 0 \leq t_1 < t_2 \leq T \) and \( \varphi \in C^1_c(\mathbb{R}^N) \).

In the case of more regularity w.r.t. space, solutions to the corresponding transport equations prove to be even renormalized and, this leads to the following class (investigated in [58] in more detail):

**Definition 2.2.** For arbitrary functions \( g \in W^{1, \infty}(\mathbb{R}^N, \mathbb{R}^N) \), \( u \in W^{1, \infty}(\mathbb{R}^N) \) and \( w \in L^p(\mathbb{R}^N) \), let
\[
\partial_{g, u, w} : [0, 1] \times L^p(\mathbb{R}^N) \longrightarrow L^p(\mathbb{R}^N), \quad (t, f_0) \longmapsto f(t)
\]
be defined by means of the renormalized (and thus, unique weak) solution to the autonomous linear transport equation
\[
\partial_t f + \text{div}_x (f \, g) = u \cdot f + w, \quad f(0) = f_0
\]
Let $V$ be a nonempty subset of $L^p(\mathbb{R}^N)$ with the following properties:

(i) $V \subset L^p(\mathbb{R}^N)$ is norm-bounded and weakly closed.

(ii) $\{ \| \phi \| \ : \phi \in V \}$ is tight in $\mathbb{R}^N$, i.e., $\lim_{\rho \to \infty} \sup_{\phi \in V} \| \phi \|_{L^p(\mathbb{R}^N \setminus B_\rho(0))} = 0$.

(iii) For $L^1$-a.e. $t \in [0,T]$ and every element $\phi \in V \subset L^p(\mathbb{R}^N)$, there exists a tuple $(g,u,w) \in \mathcal{G}(t,\phi) \times \mathcal{U}(t,\phi) \times \mathcal{W}(t,\phi)$ with

$$\liminf_{h \downarrow 0} \frac{1}{h} \cdot \text{dist}(\partial_{g,u,w}(h,\phi), V) = 0$$

(6)
where the distance from $V$ is understood w.r.t. $d_{L^p}$, i.e.,
\[
\text{dist}(\phi, V) := \inf_{\psi \in V} d_{L^p}(\phi, \psi).
\]
Then, every function $f_0 \in V$ initializes a solution $f : [0,T] \to L^p(\mathbb{R}^N)$ of the following partial differential inclusion with state constraint
\[
\begin{cases}
\partial_t f \in -\text{div}_x (G(t, f)) + \mathcal{U}(t, f) \cdot f + \mathcal{W}(t, f) & \text{in } [0,T] \times \mathbb{R}^N \\
f(t) \in V & \text{for every } t \in [0,T].
\end{cases}
\]

3. Proof of main theorem 2.4. In a word, we follow a concept of approximate solutions well-established in the literature of viability theory (see, e.g., [8, 19, 23, 47]) and then extract appropriately converging subsequences (see §3.2 – 3.4 below).

The limit curve in $L^p(\mathbb{R}^N)$ proves to be a renormalized solution of a transport equation (§3.5). Finally, the SCORZA-DRAGONI-type result in Corollary A.3 lays the foundations for concluding indirectly that the limits of the coefficient curves are related to selections of the set-valued coefficient mappings $L^1$-a.e. in $[0, T]$ (§3.6).

Hence, the limit curve in $L^p(\mathbb{R}^N)$ is a viable solution of the partial differential inclusion (in the sense of Definition 2.1).

3.1. A priori bounds for solutions to the PDI with state constraints.

**Lemma 3.1.** Under the assumptions of Theorem 2.4, consider any Lebesgue measurable functions
\[
g : [0,T] \to L^q_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N),
\]
\[
u : [0,T] \to L^q_{\text{loc}}(\mathbb{R}^N),
\]
\[
w : [0,T] \to L^p(\mathbb{R}^N)
\]
whose values belong to the image sets of $G$, $\mathcal{U}$ and $\mathcal{W}$ respectively.

Every solution $f : [0,T] \to L^p(\mathbb{R}^N)$ to the nonautonomous linear transport equation (8) satisfies
\[
\|f(t)\|_{L^p(\mathbb{R}^N)} \leq (\|f_0\|_{L^p(\mathbb{R}^N)} + \gamma t) \cdot e^{\text{const}(\gamma,p,N) t}
\]
for each $t \in [0,T]$.

This uniform a priori bound results directly from Proposition B.1 (1.) due to hypothesis 2.4 (i). In particular, it provides the radius $r_{\text{max}} := (\|f_0\|_{L^p(\mathbb{R}^N)} + \gamma T) \cdot e^{\text{const}(\gamma,p,N) T}$. By assumption 2.4 (vi), the set $V \subset L^p(\mathbb{R}^N)$ is norm-bounded, i.e., $r_V := \sup \{\|\phi\|_{L^p} \mid \phi \in V\}$ is finite. For all the considerations from now on, hypothesis 2.4 (ii) provides a priori bounds $\gamma_a, \gamma_b$ for the time-dependent coefficients as required in Proposition B.1 (i), i.e., for all $t \in [0,T]$ and every $\phi \in L^p(\mathbb{R}^N)$ with $\|\phi\|_{L^p(\mathbb{R}^N)} \leq r_{\text{max}} + r_V$, all the functions $g \in G(t, \phi)$, $u \in \mathcal{U}(t, \phi)$, $w \in \mathcal{W}(t, \phi)$ satisfy
\[
\begin{cases}
\|\text{div}_x g\|_{L^\infty(\mathbb{R}^N)} + \|u\|_{L^\infty(\mathbb{R}^N)} + \|w\|_{L^p(\mathbb{R}^N)} \leq \gamma_a, \\
\|g\|_{W^{1,\infty}} + ||\nabla_x u||_{L^\infty(\mathbb{R}^N, \mathbb{R}^N)} \leq \gamma_b.
\end{cases}
\]

According to hypotheses 2.4 (vii), the subset $\{\|\phi\|_{L^p(\mathbb{R}^N \setminus B_\rho(0))} \to 0 \quad (\rho \to \infty)$. 

\[
\omega_V(\rho) := \sup_{\phi \in V} \|\phi\|_{L^p(\mathbb{R}^N \setminus B_\rho(0))} \to 0
\]
Using the modulus of tightness \( \omega(\cdot) \) specified on the basis of the a priori bounds \( r_{\max} + r_{\nu}, \gamma_a, \gamma_b \) in Corollary B.2, the subset
\[
\mathcal{L} := \{ \phi \in L^p(\mathbb{R}^N) \mid \|\phi\|_{L^p(\mathbb{R}^N)} \leq r_{\nu} + r_{\max}, \|\phi\|^p_{L^p(\mathbb{R}^N \setminus B_\rho)} \leq e^\kappa T \cdot (\omega(\rho) + \omega_{\nu}(\rho)) \text{ for all } \rho > 0 \}
\]
is tight, convex and norm-bounded with \( V \subset \mathcal{L} \). Hence, the metric space \((\mathcal{L}, d_{L^p})\) is compact due to Corollary A.12 and the lower semicontinuity of \( L^p \) norms w.r.t. weak convergence. It is worth mentioning that all subsequent approximate solutions of the PDI will have their values in \( \mathcal{L} \).

### 3.2. Constructing approximate solutions of PDI with state constraints.

We adapt the construction of approximate solutions which was developed by Hadad [47, 48] for functional differential inclusions and is presented for the case of ordinary differential inclusions in [8], for example. There is a modification though: We do not use the projections on the constraint set \( V \subset L^p(\mathbb{R}^N) \) (i.e., minimizers of the \( d_{L^p} \) distance to \( V \)) because their existence results from local compactness of the underlying basic set, which is usually regarded as a rather restrictive requirement.

Instead, we prefer their approximations as a separate component of the tuple – similarly to what Bothe suggested in [20].

Contingent hypothesis 2.4 (viii) is made at \( \mathcal{L}^1 \)-a.e. time instant \( t \in [0, T] \). For the rest of this Subsection 3.2, let \( J_\varepsilon \subset [0, T] \) denote a closed subset of \( \mathbb{R} \) with \( \mathcal{L}^1([0, T] \setminus J_\varepsilon) < \varepsilon \) and such that this contingent condition (i.e., equation (6) in Theorem 2.4) holds for every \( t \in J_\varepsilon \).

Now we formulate the approximate solutions and their relevant features for each \( \varepsilon \in (0, 1) \).

**Lemma 3.2.** In connection with the assumptions of Theorem 2.4, consider \( \omega : [0, \infty) \rightarrow [0, \infty) \), \( \kappa = \kappa(p, N, \gamma_a) \geq 1 \) and \( \tilde{C} = \tilde{C}(\gamma, p, N, T, \|f_0\|_{L^p(\mathbb{R}^N)}) > 0 \) introduced in Corollary B.2 and Lemma B.3 respectively. For every \( \varepsilon \in (0, 1) \), there exist functions
\[
\begin{align*}
f_\varepsilon : [0, T] &\longrightarrow L^p(\mathbb{R}^N), \\
g_\varepsilon : [0, T] &\longrightarrow W^{1, \infty}(\mathbb{R}^N, \mathbb{R}^N), \\
P_\varepsilon : [0, T] &\longrightarrow L^p(\mathbb{R}^N), \\
u_\varepsilon : [0, T] &\longrightarrow W^{1, \infty}(\mathbb{R}^N), \\
\tau_\varepsilon : [0, T] &\longrightarrow [0, T], \\
w_\varepsilon : [0, T] &\longrightarrow L^p(\mathbb{R}^N)
\end{align*}
\]
satisfying with the auxiliary radius \( R_\varepsilon(\cdot) := e^{\tilde{C} \cdot T} \cdot \varepsilon \cdot \varepsilon \)

(a) \( f_\varepsilon(0) = f_0 \),

(b) \( d_{L^p}(f_\varepsilon(t), P_\varepsilon(t)) \leq R_\varepsilon(T), P_\varepsilon(t) \in V \) and \( \max \{0, t - \varepsilon\} \leq \tau_\varepsilon(t) \leq t \) hold for all \( t \in [0, T] \),

(c) for all \( t \in J_\varepsilon \cap [0, T] \), it holds
\[
\begin{cases}
g_\varepsilon(t) &\in \mathcal{G}(\tau_\varepsilon(t), P_\varepsilon(\tau_\varepsilon(t))), \\
u_\varepsilon(t) &\in \mathcal{U}(\tau_\varepsilon(t), P_\varepsilon(\tau_\varepsilon(t))), \\
w_\varepsilon(t) &\in \mathcal{W}(\tau_\varepsilon(t), P_\varepsilon(\tau_\varepsilon(t)))
\end{cases}
\]

(d) \( f_\varepsilon : [0, T] \longrightarrow L^p(\mathbb{R}^N) \) is a renormalized solution of the nonautonomous linear transport equation
\[
\partial_t f_\varepsilon + \text{div}_x(f_\varepsilon g_\varepsilon(t)) = u_\varepsilon(t) \cdot f_\varepsilon + w_\varepsilon(t),
\]

(e) \( f_\varepsilon \) is \( \Lambda \)-LIPSCHITZ continuous w.r.t. \( \varepsilon_{L^p} \) with a constant \( \Lambda = \Lambda(\gamma, T, \|f_0\|_{L^p}) \),

(f) \( \|f_\varepsilon(t)\|^p_{L^p(\mathbb{R}^N \setminus B_\rho)} \leq e^{- \kappa T} \cdot \omega(\rho) \) holds for all \( t \in [0, T] \) and \( \rho > 0 \),
(g) \( P_\varepsilon \) is piecewise constant “to the left” in \((0, T)\), i.e., in the sense that for each \( t \in (0, T) \), there exists some \( \delta > 0 \) such that \( P_\varepsilon([t - \delta, t]) \) is constant,

(h) \( \dot{\varepsilon}_L P_\varepsilon(\tau(t), P(t)) \leq (\Lambda + 2 \varepsilon) \cdot \varepsilon \) for all \( t \in [0, T] \),

(i) \( g_\varepsilon, u_\varepsilon, w_\varepsilon \) are piecewise constant “to the right”, i.e., in the sense that for each \( t \in [0, T] \), there exists some \( \delta > 0 \) such that each of their restrictions to \([t, t + \delta)\) is constant,

(j) \( \tau_\varepsilon \) is nondecreasing.

Its proof is based on Zorn’s lemma applied to the set \( A_\varepsilon(f_0) \) of all tuples \((\hat{\tau}, f(\cdot), P(\cdot), \tau(\cdot), g(\cdot), u(\cdot), w(\cdot))\) consisting of a scalar \( \hat{\tau} \in [0, T] \) and functions

\[
\begin{align*}
  f & : [0, \hat{\tau}] \rightarrow L^p(\mathbb{R}^N), & g & : [0, \hat{\tau}] \rightarrow W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N), \\
  P & : [0, \hat{\tau}] \rightarrow L^p(\mathbb{R}^N), & u & : [0, \hat{\tau}] \rightarrow W^{1,\infty}(\mathbb{R}^N), \\
  \tau & : [0, \hat{\tau}] \rightarrow [0, \hat{\tau}], & w & : [0, \hat{\tau}] \rightarrow L^p(\mathbb{R}^N)
\end{align*}
\]

which satisfy the preceding conditions (a), (d) – (g), (i), (j) in the respective domain \( \subset [0, \hat{\tau}] \) and

(b') \( d_{L^p}(f(t), P(t)) \leq R_\varepsilon(T), \ P(t) \in V, \ \max \{0, t - \varepsilon\} \leq \tau(t) \leq t \)

for all \( t \in [0, \hat{\tau}] \) and \( d_{L^p}(f(\tau), P(\tau)) \leq R_\varepsilon(\tau) \),

(c') for all \( t \in J_\varepsilon \cap [0, \hat{\tau}] \), it holds

\[
\begin{align*}
  \{ & g_\varepsilon(t) \in G(\tau(t), P_\varepsilon(\tau(t))), \\
  & u_\varepsilon(t) \in U(\tau(t), P_\varepsilon(\tau(t))), \\
  & w_\varepsilon(t) \in W(\tau(t), P_\varepsilon(\tau(t))) \}
\end{align*}
\]

(h') there exists an (at most) countable family \((\{a_j, b_j\})_{j \in J_P}\) of pairwise disjoint intervals whose union is \([0, \hat{\tau}]\) and which satisfies for all \( j \in J_P \) and \( t \in [a_j, b_j] \)

\[
\begin{align*}
  \tau(t) &= a_j, \\
  0 &< b_j - a_j < \varepsilon, \\
  \dot{\varepsilon}_L P(a_j, P(t)) &\leq (\Lambda + \varepsilon) \cdot (b_j - a_j), \\
  \dot{\varepsilon}_L P(t, P(b_j)) &\leq (\Lambda + 2 \varepsilon) \cdot (b_j - a_j), \\
  P &\text{ has at most finitely many points of discontinuity in } [\frac{a_j + b_j}{2}, b_j].
\end{align*}
\]

\( A_\varepsilon(f_0) \) is supplied with the order relation \( \preceq \) motivated by extension, i.e.,

\[
\begin{align*}
  (\hat{\tau}_1, f_1, P_1, \tau_1, g_1, u_1, w_1) &\preceq (\hat{\tau}_2, f_2, P_2, \tau_2, g_2, u_2, w_2) \iff \\
  \{ & \hat{\tau}_1 \leq \hat{\tau}_2, \\
  & f_2 |_{[0, \hat{\tau}_1]} = f_1, \\
  & P_2 |_{[0, \hat{\tau}_1]} = P_1, \tau_2 |_{[0, \hat{\tau}_1]} = \tau_1 \text{ and } g_2 |_{[0, \hat{\tau}_1]} = g_1, \ u_2 |_{[0, \hat{\tau}_1]} = u_1, \ w_2 |_{[0, \hat{\tau}_1]} = w_1 \text{ and } \\
  & \{ [a_{1,j}, b_{1,j}] \} \subset \{ [a_{2,j}, b_{2,j}] \cap [0, \hat{\tau}_1] \} \text{ for all } j \in J_{P_1} \}
\end{align*}
\]

Obviously, \( A_\varepsilon(f_0) \) is nonempty because it contains the tuple \((0, f(\cdot) \equiv f_0, P(\cdot) \equiv f_0, \tau(\cdot) \equiv 0)\) in combination with arbitrary selections of \( G(0, f_0), U(0, f_0), W(0, f_0) \neq \emptyset \) respectively. We present the further steps of the proof in the following lemmata. Finally, an indirect conclusion from Lemma 3.3 reveals \( \hat{\tau} = T \) for the maximal element of \( (A_\varepsilon(f_0), \preceq) \).

**Lemma 3.3.** Under the assumptions of Lemma 3.2, consider \((A_\varepsilon(f_0), \preceq)\) just defined.

For every tuple \((\hat{\tau}, f, P, \tau, g, u, w) \in A_\varepsilon(f_0)\) with \( \hat{\tau} < T \), there exist some \( \rho \in (0, T - \hat{\tau}) \) and a related tuple \((\hat{\tau} + \rho, \hat{\tilde{f}}, \tilde{P}, \tilde{\tau}, \tilde{g}, \tilde{u}, \tilde{w}) \in A_\varepsilon(f_0)\) with

\[
(\hat{\tau}, f, P, \tau, g, u, w) \preceq (\hat{\tau} + \rho, \hat{\tilde{f}}, \tilde{P}, \tilde{\tau}, \tilde{g}, \tilde{u}, \tilde{w}).
\]
Proof. There are two cases excluding each other:

**Case 1.** $\bar{\tau} \in J_r$ In a word, we follow the proof strategy of [64, Lemma 4.15], but for states in $L^p(\mathbb{R}^N)$ (rather than compact subsets of a HILBERT space). Due to the construction of $J_r$, hypothesis 2.4 (viii) guarantees $\tilde{g} \in G(\bar{\tau}, P(\bar{\tau})) \subset W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$, $\tilde{u} \in U(\bar{\tau}, P(\bar{\tau})) \subset W^1(\mathbb{R}^N, \mathbb{R}^N)$ and $\hat{w} \in W(\bar{\tau}, P(\bar{\tau})) \subset L^p(\mathbb{R}^N)$ satisfying

$$\liminf_{h \downarrow 0} \frac{1}{h} \cdot \text{dist}(\vartheta_{g, u, \hat{w}}(h, P(\bar{\tau})), \mathcal{V}) = 0.$$ 

In particular, there is a sequence $h_m \downarrow 0$ in $(0, \min\{\varepsilon, T - \bar{\tau}\})$ with

$$\text{dist}(\vartheta_{g, u, \hat{w}}(h_m, P(\bar{\tau})), \mathcal{V}) < \frac{\varepsilon}{2} h_m$$

for each $m \in \mathbb{N}$. Now we define the respective extensions

$$\hat{f}(t) = \begin{cases} f(t) & \text{for } t \in [0, \bar{\tau}] \\ \vartheta_{g, u, \hat{w}}(t - \bar{\tau}, f(\bar{\tau})) & \text{for } t \in (\bar{\tau}, \bar{\tau} + h_1], \end{cases}$$

$$\hat{g}(t) = \begin{cases} g(t) & \text{for } t \in [0, \bar{\tau}] \\ \tilde{g} \in G(\bar{\tau}, P(\bar{\tau})) & \text{for } t \in [\bar{\tau}, \bar{\tau} + h_1). \end{cases}$$

$$\hat{u}(t) = \begin{cases} u(t) & \text{for } t \in [0, \bar{\tau}] \\ \tilde{u} \in U(\bar{\tau}, P(\bar{\tau})) & \text{for } t \in [\bar{\tau}, \bar{\tau} + h_1), \end{cases}$$

$$\hat{w}(t) = \begin{cases} w(t) & \text{for } t \in [0, \bar{\tau}] \\ \tilde{w} \in W(\bar{\tau}, P(\bar{\tau})) & \text{for } t \in [\bar{\tau}, \bar{\tau} + h_1), \end{cases}$$

$$\hat{\tau}(t) = \begin{cases} \tau(t) & \text{for } t \in [0, \bar{\tau}] \\ \bar{\tau} & \text{for } t \in [\bar{\tau}, \bar{\tau} + h_1], \end{cases}$$

$$\hat{P}(t) = \begin{cases} P(t) & \text{for } t \in [0, \bar{\tau}]. \end{cases}$$

Just $\hat{P}$ remains to be defined on $(\bar{\tau}, \bar{\tau} + h_1]$. Furthermore we are going to specify some sufficiently small $\rho \in (0, h_1)$ such that the respective restrictions to $[0, \bar{\tau} + \rho]$ fulfill the conditions of $\mathcal{A}_r(f_0)$. This will be in form of $\rho := h_{m_0}$ with an index $m_0 \in \mathbb{N}$. Then the index set $J_{\rho}$ of $(\bar{\tau}, f, \tau, g, u, w) \in \mathcal{A}_r(f_0)$ mentioned in condition (h’ is extended by a new index $\bar{j}$ and we set $a_{\bar{j}} := \bar{\tau}$, $b_{\bar{j}} := \bar{\tau} + \rho$.

For each index $m \in \mathbb{N}$, the function $\hat{P}(\tau + h_m) \in L^p(\mathbb{R}^N)$ is selected as an arbitrary element of $\mathcal{V}$ with

$$d_{L^p}\left(\vartheta_{g, u, \hat{w}}(h_m, P(\bar{\tau})), \hat{P}(\tau + h_m)\right) \leq \text{dist}(\vartheta_{g, u, \hat{w}}(h_m, P(\bar{\tau})), \mathcal{V}) + \frac{\varepsilon}{2} h_m < \varepsilon h_m.$$ 

Then $\hat{P}$ is extended constantly to the interval $(\bar{\tau} + h_m + 1, \bar{\tau} + h_m]$ implying that $\hat{P}$ is piecewise constant “to the left” on $[0, \bar{\tau} + h_1]$. Hence, properties (a), (c’), (d) – (g), (i), (j) are rather obvious consequences of the construction, Corollaries B.2 and B.4.

For every index $m \in \mathbb{N}$, we conclude from Lemma B.3

$$d_{L^p}(f(\tau + h_m), \hat{P}(\tau + h_m))$$

$$\leq d_{L^p}(\vartheta_{g, u, \hat{w}}(h_m, f(\bar{\tau})), \vartheta_{g, u, \hat{w}}(h_m, P(\bar{\tau}))) + d_{L^p}(\vartheta_{g, u, \hat{w}}(h_m, P(\bar{\tau})), \hat{P}(\tau + h_m))$$

$$< d_{L^p}(f(\bar{\tau}), P(\bar{\tau})) \cdot e^{\hat{C}_m h_m} + \varepsilon \cdot h_m$$

$$\leq R_\varepsilon(\bar{\tau} + h_m),$$
i.e., the last inequality of condition (b') is also satisfied at time \( t = \tau + h_m \) for every index \( m \in \mathbb{N} \).

Next, the first inequality of condition (b') is verified for each \( m \in \mathbb{N} \) sufficiently large. The functions

\[
[0, T - \hat{\tau}) \rightarrow L^p(\mathbb{R}^N), \quad h \mapsto \tilde{f}(\hat{\tau} + h) = \partial_{\tilde{g}, \tilde{u}, \tilde{w}}(h, f(\hat{\tau})),
\]

\[
[0, T - \hat{\tau}) \rightarrow L^p(\mathbb{R}^N), \quad h \mapsto \partial_{\tilde{g}, \tilde{u}, \tilde{w}}(h, P(\hat{\tau}))
\]

are continuous w.r.t. \( \tilde{c}_{L^p} \) according to Proposition B.1 (2.). Due to the tightness property in Proposition B.1 (6.), Corollary A.11 guarantees their continuity w.r.t. \( d_{L^p} \) as well. Hence, there is \( \delta \in (0, T - \hat{\tau}) \) such that all \( h \in [0, \delta] \) satisfy

\[
\begin{align*}
\delta_{L^p}(t, f(\hat{\tau}), \partial_{\tilde{g}, \tilde{u}, \tilde{w}}(h, f(\hat{\tau}))) &< \frac{1}{3} \cdot (R_\epsilon(T) - R_\epsilon(\hat{\tau} + h_1)), \\
\delta_{L^p}(P(\hat{\tau}), \partial_{\tilde{g}, \tilde{u}, \tilde{w}}(h, P(\hat{\tau}))) &< \frac{1}{3} \cdot (R_\epsilon(T) - R_\epsilon(\hat{\tau} + h_1)).
\end{align*}
\]

Due to \( h_m \downarrow 0 \), there is always an index \( m_0 \in \mathbb{N} \) with \( 0 < h_m < \delta \) for all \( m \geq m_0 \). Then for each \( t \in [\hat{\tau}, \hat{\tau} + h_m] \), we select the unique index \( m \geq m_0 \) with \( h_{m+1} < t - \hat{\tau} \leq h_m \) and conclude from the triangle inequality of \( d_{L^p} \)

\[
d_{L^p}(\tilde{f}(t), P(t)) \leq d_{L^p}(\tilde{f}(t), \tilde{f}(\hat{\tau})) + d_{L^p}(\tilde{f}(\hat{\tau}), \tilde{f}(\hat{\tau} + h_m)) + d_{L^p}(\tilde{f}(\hat{\tau} + h_m), \tilde{f}(t))
\]

\[
\leq d_{L^p}(\partial_{\tilde{g}, \tilde{u}, \tilde{w}}(t - \hat{\tau}, \tilde{f}(\hat{\tau})), \tilde{f}(\hat{\tau})) + R_\epsilon(\hat{\tau}) +
\]

\[
d_{L^p}(\partial_{\tilde{g}, \tilde{u}, \tilde{w}}(h_m, P(\hat{\tau}))), P(\hat{\tau}))) + 0
\]

\[
< \frac{R_\epsilon(T) - R_\epsilon(\hat{\tau} + h_1)}{3} + R_\epsilon(\hat{\tau}) + \frac{R_\epsilon(T) - R_\epsilon(\hat{\tau} + h_1)}{3} < R_\epsilon(T).
\]

The remaining parts of condition (b') result from the construction of \( \tilde{P} \) and \( \hat{\tau} \).

Finally, we focus on property (h'). For every \( t \in [\hat{\tau}, \hat{\tau} + h_m] \), there exists a unique index \( m \geq m_0 \) with \( t - \hat{\tau} \in (h_{m+1}, h_m) \) and, we conclude from Remark A.7 and Proposition B.1 (2.)

\[
\tilde{c}_{L^p}(\tilde{P}(a_j), \tilde{P}(t)) = \tilde{c}_{L^p}(\tilde{P}(\hat{\tau}), \tilde{P}(\hat{\tau} + h_m))
\]

\[
\leq \tilde{c}_{L^p}(\tilde{P}(\hat{\tau}), \partial_{\tilde{g}, \tilde{u}, \tilde{w}}(h_m, P(\hat{\tau}))), P(\hat{\tau} + h_m)) + \partial_{\tilde{g}, \tilde{u}, \tilde{w}}(h_m, P(\hat{\tau})), P(\hat{\tau} + h_m))
\]

\[
< \Lambda \cdot h_m + \varepsilon \cdot h_m
\]

\[
\leq (\Lambda + \varepsilon) \cdot h_m = (\Lambda + \varepsilon) \cdot (b_j - a_j).
\]

Moreover, we obtain

\[
\tilde{c}_{L^p}(P(t), \tilde{P}(b_j)) = \tilde{c}_{L^p}(\tilde{P}(\hat{\tau} + h_m), \tilde{P}(\hat{\tau} + h_m))
\]

\[
\leq \tilde{c}_{L^p}(\tilde{P}(\hat{\tau} + h_m), \partial_{\tilde{g}, \tilde{u}, \tilde{w}}(h_m, P(\hat{\tau}))), P(\hat{\tau} + h_m))
\]

\[
\leq \tilde{c}_{L^p}(\partial_{\tilde{g}, \tilde{u}, \tilde{w}}(h_m, P(\hat{\tau}))), \partial_{\tilde{g}, \tilde{u}, \tilde{w}}(h_m, P(\hat{\tau}))), P(\hat{\tau} + h_m))
\]

\[
< \Lambda \cdot |h_m - h_m| + d_{L^p}(\partial_{\tilde{g}, \tilde{u}, \tilde{w}}(h_m, P(\hat{\tau}))), P(\hat{\tau} + h_m))
\]
Lemma 3.4 implies the following statement: Every totally ordered subset of 

\[ A \]

Case 2. \( \hat{\tau} \in [0, T) \setminus J, \) We simply select the functions constant 0, i.e., \( \hat{g} := 0 \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N), \) \( \hat{u} := 0 \in W^{1,\infty}(\mathbb{R}^N) \) and \( \hat{w} := 0 \in L^p(\mathbb{R}^N). \) For \( \rho := \inf \{ s - \hat{\tau}, T - \hat{\tau} \mid s \in J, \cap [\hat{\tau}, T] \} > 0, \) define the extensions \( \hat{f} : [0, \hat{\tau} + \rho] \rightarrow L^p(\mathbb{R}^N), \)

\[ \hat{g} : [0, \hat{\tau} + \rho] \rightarrow W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N), \hat{u} : [0, \hat{\tau} + \rho] \rightarrow W^{1,\infty}(\mathbb{R}^N), \hat{w} : [0, \hat{\tau} + \rho] \rightarrow L^p(\mathbb{R}^N) \)

and \( \hat{\tau} : [0, \hat{\tau} + \rho] \rightarrow [0, T] \) as in the first case and, set

\[ \hat{P}(t) = \begin{cases} P(t) & \text{for } t \in [0, \hat{\tau}) \\ P(\hat{\tau}) \in \mathcal{V} & \text{for } t \in [\hat{\tau}, \hat{\tau} + \rho]. \end{cases} \]

The index set \( \mathcal{J}_P \) of \((\hat{\tau}, f, P, \tau, g, u, w) \in A_c(f_0)\) mentioned in condition \((h')\) is extended by a new index \( j \) again and, we set \( a_j := \hat{\tau}, b_j := \hat{\tau} + \rho. \) Then properties \((a), (b'), (d) - (j)\) are obvious since \( \hat{f}(t) = f(\hat{\tau}) \) holds for all \( t \in (\hat{\tau}, \hat{\tau} + \rho). \)

In regard to the assumptions of ZORN’s lemma (see, e.g., [80, § 0.1]), subsequent Lemma 3.4 implies the following statement: Every totally ordered subset of \( A_c(f_0) \) has an upper bound w.r.t. \( \preceq. \)

**Lemma 3.4.** Under the assumptions of Lemma 3.2, consider \( (A_c(f_0), \preceq) \) defined previously.

For every sequence \((\tilde{\tau}_k, f_k, P_k, \tau_k, g_k, u_k, w_k) \) \( (k \in \mathbb{N}) \) in \( A_c(f_0) \) satisfying

\[ (\tilde{\tau}_k, f_k, P_k, \tau_k, g_k, u_k, w_k) \preceq (\tilde{\tau}_{k+1}, f_{k+1}, P_{k+1}, \tau_{k+1}, g_{k+1}, u_{k+1}, w_{k+1}) \]

for each \( k \in \mathbb{N}, \) there exists a tuple \((\hat{\tau}, f, P, \tau, g, u, w) \in A_c(f_0)\) with

\[ (\tilde{\tau}_k, f_k, P_k, \tau_k, g_k, u_k, w_k) \preceq (\hat{\tau}, f, P, \tau, g, u, w) \]

for every \( k \in \mathbb{N}. \)

**Proof.** We adapt the notions proving [64, Lemma 4.16] to renormalized solutions in \( L^p(\mathbb{R}^N) \) (again). Assuming that \( ((\tilde{\tau}_k, f_k, P_k, \tau_k, g_k, u_k, w_k))_{k \in \mathbb{N}} \) is monotone w.r.t. \( \preceq, \) the sequence \((\tilde{\tau}_k)_{k \in \mathbb{N}} \) is non-decreasing in \([0, T)\) and so, it tends to some \( \hat{\tau} \in [0, T]. \)

Next, we specify candidates for the wanted functions \( f, P, \tau, g, u, w \in [0, \hat{\tau} \setminus J] \) by choosing the index \( k \in \mathbb{N} \) sufficiently large: For each \( t \in [0, \tilde{\tau}_k], \) there exists some \( k_0 \in \mathbb{N} \) with \( t < \tau_k \leq \tilde{\tau} \) for every \( k \geq k_0 \) and, set \( f(t) := f_k(t), P(t) := P_k(t), \tau(t) := \tau_k(t), g(t) := g_k(t), u(t) := u_k(t), w(t) := w_k(t) \) with an arbitrary index \( k \geq k_0. \) The functions \( f(\tilde{\tau}) \in L^p(\mathbb{R}^N), P(\tilde{\tau}) \in \mathcal{V} \subset L^p(\mathbb{R}^N) \) remain to be specified appropriately such that \( (\tilde{\tau}, f, P, \tau, g, u, w) \in A_c(f_0). \)

Due the \( \Lambda\)-LIPSCHITZ continuity of each \( f_k \) \( (k \in \mathbb{N}) \) in condition 3.2 (e), \( (f(\tilde{\tau}_k))_{k \in \mathbb{N}} \) is a CAUCHY sequence w.r.t. \( \hat{e}_{L^p}. \) The uniform tightness property formulated in Corollary B.2 implies in addition that \( f(\tilde{\tau}_k) = f_k(\tilde{\tau}_k) \) \( (k \in \mathbb{N}) \) induce a tight sequence and so, they converge w.r.t. both \( d_{L^p} \) and \( \hat{e}_{L^p} \) according to Corollary A.10 and A.11. Its limit is called \( f(\tilde{\tau}) \in L^p(\mathbb{R}^N) \) and coincides with the corresponding weak limit. Hence, the lower semicontoimity of the norm w.r.t. weak convergence (see, e.g., [21, Prop. 3.5 (iii)]) leads to \( \|f(\tilde{\tau})\|_{L^p(\mathbb{R}^N)}^p \leq c^* \frac{\tau}{\rho} \) for every \( \rho > 0. \)

Similarly, we specify \( P(\tilde{\tau}) \in \mathcal{V} \subset L^p(\mathbb{R}^N) \) : Each function \( P_k : [0, \tau_k] \rightarrow L^p(\mathbb{R}^N) \) is related to an (at most) countable family of pairwise disjoint subintervals \([a_{k,j}, b_{k,j}], j \in \mathcal{J}_{P_k}\), as specified in condition \((h')\). The monotonicity w.r.t. \( \preceq \) implies for every \( k \in \mathbb{N} \)

\[ \{ [a_{k,j}, b_{k,j}] \mid j \in \mathcal{J}_{P_k} \} \subset \{ [a_{k+1,j}, b_{k+1,j}] \cap [0, \tau_k) \mid j \in \mathcal{J}_{P_{k+1}} \}. \]
Then the family \([a_{k,j}, b_{k,j}]\) for any \(k \in \mathbb{N}, j \in \mathcal{J}_{P_k}\) is countable and has the property that any two subintervals are either disjoint or their left boundary points coincide. This induces a countable family of pairwise disjoint subintervals \([a_{j}, b_{j}]\), \(j \in \mathcal{J}_{P}\), whose union is \([0, \hat{T}]\) and which satisfy for any \(j \in \mathcal{J}_{P}\) and \(t \in [a_{j}, b_{j}] \subset [0, \hat{T}]\)

\[
\begin{align*}
\tau(t) &= a_{j}, \\
0 &< b_{j} - a_{j} < \varepsilon, \\
\hat{e}_{L^p}(P(a_{j}), P(t)) &\leq (\Lambda + \varepsilon) \cdot (b_{j} - a_{j}), \\
\hat{e}_{L^p}(P(t), P(b_{j})) &\leq (\Lambda + 2 \varepsilon) \cdot (b_{j} - a_{j}), \\
P \text{ has at most finitely many points of discontinuity in } [\frac{a_{j}+b_{j}}{2}, b_{j}].
\end{align*}
\]

These features result from property \((h')\) of \(P_{k}\) with any index \(k = k(j) \in \mathbb{N}\) sufficiently large such that either \(\hat{T}_{k} = \hat{T}\) (if \(b_{j} = \hat{T}\)) or \(b_{j} < \hat{T}_{k} \leq \hat{T}\) (if \(b_{j} < \hat{T}\)).

There are two cases excluding each other:

In the first case, there is no sequence \((\hat{t}_{\ell})_{\ell \in \mathbb{N}}\) in \([0, \hat{T}]\) which converges to \(\hat{T}\) and consists of points of discontinuity of \(P\). Then, there is a sequence \((\hat{t}_{\ell})_{\ell \in \mathbb{N}}\) of indices in \(\mathcal{J}_{P}\) such that \(\hat{t}_{\ell} \in [a_{(j_{\ell})}, b_{(j_{\ell})}]\) holds for every \(\ell \in \mathbb{N}\) and \(a_{(j_{\ell})}\) converges to \(\hat{T}\) for \(\ell \to \infty\) (as we conclude from the last property in condition \((h')\) indirectly). Hence, \((P(\hat{t}_{\ell}))_{\ell \in \mathbb{N}}\) proves to be a CAUCHY sequence w.r.t. the metric \(\hat{e}_{L^p}\) since

\[
\hat{e}_{L^p}(P(\hat{t}_{\ell}), P(\hat{t}_{m})) \leq (\Lambda + 2 \varepsilon) \cdot (b_{(j_{m})} - a_{(j_{m})}) \leq (\Lambda + 2 \varepsilon) \cdot (\hat{T} - a_{(j_{m})})
\]

holds for all \(\ell, m \in \mathbb{N}\) \((\ell \leq m)\). Furthermore, \((|P(\hat{t}_{\ell})|)_{\ell \in \mathbb{N}}\) is tight in \(\mathbb{R}^{N}\) due to hypothesis \(2.4\) \((vii)\). As a consequence of Corollary A.10 and A.11, we obtain the joint limit \(P(\hat{T}) \in L^p(\mathbb{R}^{N})\) w.r.t. both \(\hat{e}_{L^p}\) and \(d_{L^p}\). (In particular, this limit does not depend on the sequence \((\hat{t}_{\ell})_{\ell \in \mathbb{N}}\) as an indirect standard conclusion reveals.)

This construction of \(f(\hat{T}), P(\hat{T}) \in L^p(\mathbb{R}^{N})\) preserves conditions \((b')\), \((h')\) and \((d) - (g)\) because \(\mathcal{V}\) is assumed to be closed in \((L^p(\mathbb{R}^{N}), d_{L^p})\). In particular, \(P : [0, \hat{T}] \to L^p(\mathbb{R}^{N})\) is continuous w.r.t. \(\hat{e}_{L^p}\) at time instant \(t = \hat{T}\). \qed

### 3.3. Aspects of compactness for coefficient functions of time

Suppose the hypotheses of Theorem 2.4 and for \(\varepsilon \in (0, 1)\), consider any sequence tending to 0. Then, Lemma 3.2 provides sequences

\[
\begin{align*}
f_{k} : [0, T] &\to L^p(\mathbb{R}^{N}), & g_{k} : [0, T] &\to W^{1, \infty}(\mathbb{R}^{N}, \mathbb{R}^{N}), \\
P_{k} : [0, T] &\to L^p(\mathbb{R}^{N}), & u_{k} : [0, T] &\to W^{1, \infty}(\mathbb{R}^{N}), \\
\tau_{k} : [0, T] &\to [0, T], & w_{k} : [0, T] &\to L^p(\mathbb{R}^{N})
\end{align*}
(k \in \mathbb{N})
\]

with the following properties:

(a) \(f_{k}(0) = f_{0}\),
(b) \(d_{L^p}(f_{k}(t), P_{k}(t)) \leq \frac{1}{k}, P_{k}(t) \in \mathcal{V}, \text{ max } \{0, t - \frac{1}{k}\} \leq \tau_{k}(t) \leq t \) hold for all \(t \in [0, T]\)
Lemma 3.5 \[ \text{There exist both a sequence} \]

\[ g_k(t) \subset \mathcal{G}(\tau_k(t), P_k(\tau_k(t))), \]

\[ u_k(t) \subset \mathcal{U}(\tau_k(t), P_k(\tau_k(t))), \]

\[ w_k(t) \subset W(\tau_k(t), P_k(\tau_k(t))) \]

and \( g_k(t) = 0, u_k(t) = 0, w_k(t) = 0 \) for all \( t \in [0, T] \setminus J_k \).

Proof. As mentioned in §3.1, assumptions 2.4 (i), (ii) imply global a priori bounds \( \gamma_a, \gamma_b > 0 \) (depending on \( \rho, N, T \) and \( \|f_0\|_{L^p(\mathbb{R}^N)} \)) such that

\[
\begin{cases}
\|\text{div}_x g_k(t)\|_{L^\infty(\mathbb{R}^N)} + \|u_k(t)\|_{L^\infty(\mathbb{R}^N)} + \|w_k(t)\|_{L^p(\mathbb{R}^N)} \leq \gamma_a, \\
\|g_k(t)\|_{W^{2,\infty}(\mathbb{R}^N, \mathbb{R}^N)} + \|\nabla_x u_k(t)\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^N)} \leq \gamma_b
\end{cases}
\]

hold for every \( t \in [0, T] \) and each index \( k \in \mathbb{N} \). In particular, \( \{u_k(t) \mid k \in \mathbb{N}, t \in [0, T]\} \) is contained in a closed ball in \( L^p(\mathbb{R}^N) \) which is weakly compact due to reflexivity \((1 < p < \infty) \) and the theorem of KAKUTANI (see, e.g., [21, Theorems 3.17, 4.10]). Preceding Lemma 3.5 guarantees a subsequence of \( \{u_k \}_{k \in \mathbb{N}} \) converging weakly to some \( u \in L^1(0, T; L^p(\mathbb{R}^N)) \).

Next, let us fix the radius \( \rho \in \mathbb{N} \) arbitrarily and, consider all the restrictions to the open ball \( B_\rho \subset \mathbb{R}^N \) (in space). The \( L^q \) norms are uniformly bounded, i.e., we obtain for all \( t \in [0, T], k \in \mathbb{N} \)

\[
\|g_k(t)\|_{L^q(\mathbb{B}_\rho, \mathbb{R}^N)} + \|u_k(t)\|_{L^q(\mathbb{B}_\rho)} \leq \gamma_a + \gamma_b \cdot \mathcal{L}^N(B_\rho)^{1/q}.
\]

Hence, the values of \( g_k(t) \) and \( u_k(t) \) for all \( t \in [0, T], k \in \mathbb{N} \) are contained in sufficiently large closed balls in \( L^q \) which are also weakly compact. As a consequence of Lemma 3.5, the sets \( \{g_k|_{B_\rho} \mid k \in \mathbb{N}\} \subset L^1(0, T; L^q(\mathbb{B}_\rho, \mathbb{R}^N)) \) and \( \{u_k|_{B_\rho} \mid k \in \mathbb{N}\} \subset L^1(0, T; L^q(\mathbb{B}_\rho)) \) are relatively weakly compact. Their weakly converging subsequences depend on the radius \( \rho \in \mathbb{N} \) though.
CANTOR’s diagonal method w.r.t. \( \rho \in \mathbb{N} \) leads to the claimed sequence \( k_\ell \nearrow \infty \) of indices such that in addition to the weak convergence of \( \{w_{k_\ell}\}_{\ell \in \mathbb{N}} \), the sequences \( \{g_{(k_\ell)}\}_{\ell \in \mathbb{N}}, \{u_{(k_\ell)}\}_{\ell \in \mathbb{N}} \) are converging weakly in \( L^1(0,T; L^q(\mathbb{B}_\rho)) \) for each radius \( \rho \in \mathbb{N} \). Their limits induce two further functions
\[
g : [0,T) \to L^q_{\text{loc}}(\mathbb{R}^N,\mathbb{R}^N),
\quad u : [0,T) \to L^q_{\text{loc}}(\mathbb{R}^N).
\]
It remains to verify both \( g(t) \in W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N) \) and \( u(t) \in W^{1,\infty}(\mathbb{R}^N) \) for \( L^1\)-a.e. \( t \in [0,T) \). We focus on the scalar function \( u(t) \). For the radius \( \rho > 0 \) fixed arbitrarily, \( u_{(k_\ell)} \mid_{\mathbb{B}_\rho} : [0,T) \to L^q(\mathbb{B}_\rho) \) is the strong \( L^1 \) limit of a sequence of convex combinations of \( \{u_{(k_\ell)}\}_{\ell \in \mathbb{N}} \) as a consequence of MAZUR’s lemma (see, e.g., [21, Corollary 3.8], [80, \S V.1 Theorem 2]). Considering an appropriate subsequence (on the basis of [21, Theorems 3.18, 4.9]), we conclude for \( L^1\)-a.e. \( t \in [0,T) \) that there exists a sequence \( (v_{p,t,j})_{j \in \mathbb{N}} \) in \( W^{1,\infty}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \) and some \( v_\rho \in L^q(\mathbb{B}_\rho, \mathbb{R}^N) \) satisfying
\[
\begin{cases}
\|v_{p,t,j}\|_{L^\infty(\mathbb{B}_\rho)} \leq \gamma_a & \text{for every } j \in \mathbb{N},
\|\nabla_x v_{p,t,j}\|_{L^\infty(\mathbb{B}_\rho,\mathbb{R}^N)} \leq \gamma_b & \text{for every } j \in \mathbb{N},

v_{p,t,j} \rightharpoonup u(t)_{\mid_{\mathbb{B}_\rho}} \text{ w.r.t. the } L^q(\mathbb{B}_\rho) \text{ norm for } j \to \infty
\end{cases}
\]}

In combination with \( j \to \infty \), the standard condition on weak derivatives implies that \( v_{p,t,j} \) is the weak gradient of \( u(t)_{\mid_{\mathbb{B}_\rho}} \) and so, \( u(t)_{\mid_{\mathbb{B}_\rho}} \) belongs to \( W^{1,q}(\mathbb{B}_\rho) \) for every \( \rho > 0 \) and \( L^1\)-a.e. \( t \in [0,T) \). Due to the uniform \( L^\infty \) bound of \( (v_{p,t,j})_{j \in \mathbb{N}} \), the convergence \( L^\infty\)-a.e. in \( \mathbb{B}_\rho \subset \mathbb{R}^N \) guarantees \( u(t)_{\mid_{\mathbb{B}_\rho}} \in L^\infty(\mathbb{B}_\rho) \) and \( \|u(t)_{\mid_{\mathbb{B}_\rho}}\|_{L^\infty(\mathbb{B}_\rho)} \leq \gamma_a \) for every \( \rho > 0 \), i.e., \( u(t) \in L^\infty(\mathbb{R}^N) \) and \( \|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq \gamma_a \) hold for \( L^1\)-a.e. \( t \in [0,T) \).

Next, we apply the corresponding arguments to the weak gradient \( \nabla_x u(t)\mid_{\mathbb{B}_\rho} \) as the weak \( L^q \) limit of \( (\nabla_x v_{p,t,j})_{j \in \mathbb{N}} \). This leads to
\[
\nabla_x u(t)\mid_{\mathbb{B}_\rho} \in L^\infty(\mathbb{B}_\rho, \mathbb{R}^N), \quad \|\nabla_x u(t)\|_{L^\infty(\mathbb{B}_\rho,\mathbb{R}^N)} \leq \gamma_b
\]
for every \( \rho > 0 \). Hence, \( u(t) \in W^{1,\infty}(\mathbb{R}^N) \) and \( \|\nabla_x u(t)\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^N)} \leq \gamma_b \) hold for \( L^1\)-a.e. \( t \in [0,T) \). Essentially the same arguments used componentwise reveal \( g(t) \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N) \) and \( \|g(t)\|_{W^{1,\infty}} \leq \gamma_a + \gamma_b \) for \( L^1\)-a.e. \( t \in [0,T) \).

3.4. A uniformly converging subsequence of continuous approximate solutions. In regard to the sequence of approximate solutions \( (f_k)_{k \in \mathbb{N}} \), the well-known theorem of ARZELÀ-ASCOLI proves to be useful. Continuous functions between metric spaces can be handled by the generalizations in [6, Theorem 4.4.3], [46], for example:

**Lemma 3.7** (ARZELÀ-ASCOLI in metric spaces, [6, 46]). Let \( (X, d_X), (Y, d_Y) \) be two metric space such that \( X \) is compact. Whenever a sequence \( \{\varphi_k\}_{k \in \mathbb{N}} \) of continuous functions \( X \to Y \) fulfills
- \( \{\varphi_k\}_{k \in \mathbb{N}} \) is equicontinuous, i.e., for every \( \varepsilon > 0 \), there exists \( \delta = \delta(\varepsilon) > 0 \) such that all \( x_1, x_2 \in X \) with \( d_X(x_1,x_2) < \delta \) and every \( k \in \mathbb{N} \) satisfy \( d_Y(\varphi_k(x_1), \varphi_k(x_2)) < \varepsilon \) and
• there exists a compact set $C \subset Y$ such that for every $\varepsilon$-neighborhood $B_\varepsilon(C) \subset Y$ of $C$, the inclusion $\varphi_k(X) \subset B_\varepsilon(C)$ holds for all $k \in \mathbb{N}$ sufficiently large, then it has a uniformly converging subsequence.

**Corollary 3.8.** Under the assumptions of Theorem 2.4, consider the sequences specified at the beginning of \S 3.3. Then, there exist a sequence of indices (again denoted by) $k_\ell \nearrow \infty$ and a function $f : [0, T] \rightarrow L^p(\mathbb{R}^N)$ with

$$
\begin{cases}
\sup_{t \in [0, T]} \left| \varepsilon_{L^p}(f(k_\ell)(t), f(t)) \right| \rightarrow 0 \\
\forall t \in [0, T]: \quad d_{L^p}(f(k_\ell)(t), f(t)) \rightarrow 0 \quad (\ell \rightarrow \infty).
\end{cases}
$$

**Proof.** Consider the set $\mathcal{L} \subset L^p(\mathbb{R}^N)$ specified in Subsection 3.1, i.e.,

$$
\mathcal{L} := \left\{ \phi \in L^p(\mathbb{R}^N) \left| \begin{array}{l}
\|\phi\|_{L^p(\mathbb{R}^N)} \leq r_N + r_{\text{max}}, \\
\|\phi\|_{L^p(\mathbb{R}^N \setminus B_\rho)} \leq e^{r T} \cdot (\omega(\rho) + \omega_N(\rho)) \text{ for all } \rho > 0
\end{array} \right. \right\}.
$$

It is relatively compact w.r.t. $\varepsilon_{L^p}$ due to Corollary A.10. Moreover, it contains all the values $f_k(t)$ for $k \in \mathbb{N}$, $t \in [0, T]$. As each $f_k$ ($k \in \mathbb{N}$) is $\text{L-LIPSCHITZ}$ continuous w.r.t. $\varepsilon_{L^p}$, we conclude the existence of a (w.r.t. $\varepsilon_{L^p}$) uniformly converging subsequence from preceding Lemma 3.7. Finally, the pointwise convergence w.r.t. $d_{L^p}$ results from Corollary A.11.

Hypotheses 2.4 (vi),(vii) ensure that all $|P_k(t)|^p \in L^1(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $t \in [0, T]$ are norm-bounded and tight. Hence, we conclude from the sequence properties (b), (h) and Corollary A.11 successively:

**Corollary 3.9.** At every time instant $t \in [0, T]$, the following convergences for $\ell \rightarrow \infty$ hold

$$
\begin{cases}
d_{L^p}(P(k_\ell)(t), f(t)) \rightarrow 0 \\
d_{L^p}(P(k_\ell)(\tau(k_\ell)(t)), f(t)) \rightarrow 0.
\end{cases}
$$

**3.5. Limits induce a renormalized solution of a transport equation.** The sequences $(f_k)_{k \in \mathbb{N}}$, $(P_k)_{k \in \mathbb{N}}$, $(\tau_k)_{k \in \mathbb{N}}$, $(g_k)_{k \in \mathbb{N}}$, $(u_k)_{k \in \mathbb{N}}$, $(w_k)_{k \in \mathbb{N}}$ specified at the beginning of \S 3.3 lead to subsequences and limit functions $f, g, u, w$ as formulated in Lemma 3.6 and Corollaries 3.8, 3.9.

**Lemma 3.10.** The function $f : [0, T] \rightarrow L^p(\mathbb{R}^N)$ constructed in Corollary 3.8 is a weak solution of the initial value problem

$$
\partial_t f + \text{div}_x(f(g(t))) = u(t) \cdot f + w(t), \quad f(0) = f_0.
$$

In regard to this nonautonomous linear transport equation with the $\text{LEBESGUE}$ measurable coefficients $g, u, w$ given in Lemma 3.6, we conclude from Proposition B.1 (5.) immediately:

**Corollary 3.11.** $f : [0, T] \rightarrow L^p(\mathbb{R}^N)$ constructed in Corollary 3.8 is even the renormalized solution of the initial value problem in preceding Lemma 3.10 (in the sense of DiPerna and Lions [35]).

**Proof of Lemma 3.10.** For an arbitrary test function $\varphi \in C^2_c(\mathbb{R}^N)$ and any $0 \leq t_1 < t_2 \leq T$, we have to verify

$$
\int_{\mathbb{R}^N} \varphi \left( f(t_2) - f(t_1) \right) \, dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^N} f(s, x) \, g(s, x) \cdot \nabla_x \varphi(x) \, dx \, ds + \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \left( f(s, x) \, u(s, x) + w(s, x) \right) \varphi(x) \, dx \, ds.
$$
According to Corollaries A.9 and A.11, the convergence of

\[ \int_{\mathbb{R}^N} \phi \left( f_{(k)}(t_2) - f_{(k)}(t_1) \right) \, dx \]

implies

\[ \lim_{\ell \to \infty} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} f_{(k)}(s, x) \cdot \nabla \phi(x) \, dx \, ds + \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \left( f_{(k)}(s, x) u_{(k)}(s, x) + w_{(k)}(s, x) \right) \phi(x) \, dx \, ds. \]

The weak convergence of \((w_{(k)})_{k \in \mathbb{N}}\) to \(w \) in \( L^1(0, T; L^p(\mathbb{R}^N)) \) implies

\[ \int_{t_1}^{t_2} \int_{\mathbb{R}^N} w_{(k)}(s, x) \phi(x) \, dx \, ds \to \int_{t_1}^{t_2} \int_{\mathbb{R}^N} w(s, x) \phi(x) \, dx \, ds \quad (\ell \to \infty). \]

According to Corollaries A.9 and A.11, the convergence of \((f_{(k)}(t_2))_{k \in \mathbb{N}}\) to \(f(t_2) \in L^p(\mathbb{R}^N)\) w.r.t. \(d_{L^p}\) implies its weak convergence and so, we obtain in regard to the left-hand side

\[ \int_{\mathbb{R}^N} \phi \left( f_{(k)}(t_2) - f_{(k)}(t_1) \right) \, dx \to \int_{\mathbb{R}^N} \phi \left( f(t_2) - f(t_1) \right) \, dx \quad (\ell \to \infty). \]

Next, we focus on the following convergence for \(\ell \to \infty\) in detail

\[ \int_{t_1}^{t_2} \int_{\mathbb{R}^N} f_{(k)}(s) u_{(k)}(s) \phi \, dx \, ds \to \int_{t_1}^{t_2} \int_{\mathbb{R}^N} f(s) u(s) \phi \, dx \, ds \]

since essentially the same arguments can be applied to the remaining integrals of \((f_{(k)} g_{(k)} \cdot \nabla \phi)_{k \in \mathbb{N}}\).

The difference of the integrals is bounded in the following way:

\[
\left| \int_{t_1}^{t_2} \int_{\mathbb{R}^N} f_{(k)}(s) u_{(k)}(s) \phi \, dx \, ds - \int_{t_1}^{t_2} \int_{\mathbb{R}^N} f(s) u(s) \phi \, dx \, ds \right| \\
\leq \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \left( f_{(k)} u_{(k)} - f u_{(k)} \right) \phi \, dx \, ds + \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \left( f u_{(k)} - f u \right) \phi \, dx \, ds \\
\overset{(A.7)}{\leq} \int_{t_1}^{t_2} \max \left\{ \|u_{(k)}(s)\|_{L^1(\mathbb{R}^N)}, \|u_{(k)}(s)\|_{W^{1,\infty}(\mathbb{R}^N)} \right\} d_{L^p} \left( f_{(k)}(s), f(s) \right) \, ds + \int_{t_1}^{t_2} \int_{\mathbb{R}^N} f \left( u_{(k)} - u \right) \phi \, dx \, ds.
\]

The factor \(\max \left\{ \|u_{(k)}(s)\|_{L^1(\mathbb{R}^N)}, \|u_{(k)}(s)\|_{W^{1,\infty}(\mathbb{R}^N)} \right\}\) is bounded uniformly w.r.t. \(\ell \in \mathbb{N}\) and \(s \in [0, T]\) due to hypotheses 2.4 (i),(ii) and the compact support of \(\phi\) in \(\mathbb{R}^N\). Hence, \(L^1(\mathbb{R}^N)\) and \(L^1(0, T; L^p(\mathbb{R}^N))\). Then the weak convergence of \((u_{(k)})_{k \in \mathbb{N}}\) to \(u\) in \(L^1(0, T; L^p(\mathbb{R}^N))\) implies

\[ \lim_{\ell \to \infty} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (u_{(k)} - u) f \phi \, dx \, ds = 0. \]
3.6. Limits induce a solution of the partial differential inclusion (PDI). The links with the corresponding partial differential inclusion remains to be verified:

**Lemma 3.12.** The function \( f : [0, T] \to L^p(\mathbb{R}^N) \) constructed in Corollary 3.8 is a solution of partial differential inclusion (in the sense of Definition 2.1)

\[
\begin{cases}
\partial_t f \in -\text{div}_x(G(t, f) \cdot f) + U(t, f) \cdot f + \mathcal{W}(t, f) & \text{in } [0, T] \times \mathbb{R}^N \\
f(0) = f_0.
\end{cases}
\]

**Proof.** As a consequence of preceding Lemma 3.10, we are to prove for the functions \( g, u, w \) specified in Lemma 3.6 that \( g(t) \in G(t, f(t)), u(t) \in U(t, f(t)) \) and \( w(t) \in \mathcal{W}(t, f(t)) \) hold for \( L^1 \)-a.e. \( t \in [0, T] \).

Otherwise, the set of all \( t \in [0, T] \) with \( g(t) \notin G(t, f(t)), u(t) \notin U(t, f(t)) \) or \( w(t) \notin \mathcal{W}(t, f(t)) \) has positive outer Lebesgue measure denoted \( \lambda_e > 0 \). Consider the superset \( \Sigma \subset L^p(\mathbb{R}^N) \) of \( \mathcal{V} \) specified in Subsection 3.1. Then, the metric space \((\Sigma, d_{L^p})\) is complete due to Proposition A.14. Moreover, it is separable since so is \((L^p(\mathbb{R}^N), \| \cdot \|_{L^p})\). Now Scorza-Dragoni-type Corollary A.3 provides a closed subset \( I \subset [0, T] \) with \( L^1([0, T] \setminus I) < \frac{\lambda_e}{2} \) such that the three set-valued restrictions

\[
G|_{I \times \Sigma} : I \times (\Sigma, d_{L^p}) \rightrightarrows (L^q_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N), \delta_{L^q, \text{loc}}), \\
U|_{I \times \Sigma} : I \times (\Sigma, d_{L^p}) \rightrightarrows (L^q_{\text{loc}}(\mathbb{R}^N), \delta_{L^q, \text{loc}}), \\
\mathcal{W}|_{I \times \Sigma} : I \times (\Sigma, d_{L^p}) \rightrightarrows (L^p(\mathbb{R}^N), \| \cdot \|_{L^p})
\]

have closed graphs. There always exists an open subset \( \tilde{I} \subset \mathbb{R} \) of \( I \) with \( L^1([0, T] \setminus \tilde{I}) < \frac{\lambda_e}{2} \), because \( \mathbb{R} \setminus I \) is open and so, its boundary consists of (at most) countably many real numbers.

Moreover, there exist some relatively compact subsets \( \mathcal{G} \subset (L^q_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N), \delta_{L^q, \text{loc}}), \mathcal{U} \subset (L^q_{\text{loc}}(\mathbb{R}^N), \delta_{L^q, \text{loc}}) \) and \( \mathcal{W} \subset (L^p(\mathbb{R}^N), \| \cdot \|_{L^p}) \) containing all the values of \( G, U, \mathcal{W} \) respectively. Indeed, using the a priori bounds specified in Subsection 3.1 and fixing the radius \( \rho > 0 \) arbitrarily, the sets

\[
\mathcal{G} := \{ g \in W^{1, \infty}(\mathbb{R}^N, \mathbb{R}^N) : \| g \|_{W^{1, \infty}} \leq \gamma_b \}, \\
\mathcal{U} := \{ u \in W^{1, \infty}(\mathbb{R}^N) : \| u \|_{L^{\infty}(\mathbb{R}^N)} \leq \gamma_a, \| \nabla_x u \|_{L^{\infty}(\mathbb{R}^N, \mathbb{R}^N)} \leq \gamma_b \}
\]

are compactly embedded in \( L^q(\mathbb{B}_\rho, \mathbb{R}^N) \) and \( L^q(\mathbb{B}_\rho) \) respectively according to the embedding theorem by Rellich and Kondrachov (see, e.g., [21, Theorem 9.16]). CANTOR’s diagonal method w.r.t. \( \rho \in \mathbb{N} \) and Remark A.16 (2.) reveal that \( \mathcal{G} \) and \( \mathcal{U} \) are relatively (sequentially) compact w.r.t. \( \delta_{L^q, \text{loc}} \).

Due to hypothesis 2.4 (i), all values of \( \mathcal{W}|_{[0, T] \times \Sigma} \) are contained in the closed norm ball

\[
\mathcal{W} := \{ w \in L^p(\mathbb{R}^N) : \| w \|_{L^p(\mathbb{R}^N)} \leq \gamma_a \}
\]

which is weakly compact as a consequence of KAKUTANI’s theorem (see, e.g., [21, Theorem 3.17]).

Sequence properties (b), (c), (f) (formulated at the beginning of \( \S \) 3.3) state for every \( t \in [0, T] \) and \( k \in \mathbb{N} \) that \( d_{L^p}(f_k(t), P_k(t)) \leq \frac{1}{k}, P_k(t) \in \mathcal{V} \subset \Sigma, \max \{0, t - \frac{1}{k} \} \leq \tau_k(t) \leq t \) and

\[
\begin{cases}
f_k(t) \in \mathcal{G}, \\
g_k(t) \in G(\tau_k(t), P_k(\tau_k(t))) \quad (\text{if } t \in J_k), \\
u_k(t) \in U(\tau_k(t), P_k(\tau_k(t))) \quad (\text{if } t \in J_k), \\
w_k(t) \in \mathcal{W}(\tau_k(t), P_k(\tau_k(t))) \quad (\text{if } t \in J_k).
\end{cases}
\]
Fix $\rho \in \mathbb{N}$ arbitrarily. Lemma 3.6 guarantees for $\ell \to \infty$

\[
\left\{
\begin{array}{l}
g(k_\ell)_{\mathbb{B}_\rho} \rightharpoonup g_{\mathbb{B}_\rho} \text{ weakly in } L^1(0,T; L^q(\mathbb{B}_\rho, \mathbb{R}^N)), \\
u(k_\ell)_{\mathbb{B}_\rho} \rightharpoonup u_{\mathbb{B}_\rho} \text{ weakly in } L^1(0,T; L^q(\mathbb{B}_\rho)), \\
(u(k_\ell))_{\mathbb{B}_\rho} \rightharpoonup w \text{ weakly in } L^1(0,T; L^p(\mathbb{R}^N)),
\end{array}
\right.
\]

For a moment, we focus on the coefficients $u(k_\ell)$, $u$ as an example. Mazur’s lemma provides a sequence $(\bar{u}_t)_{t \in \mathbb{N}}$ of functions $[0,T] \to W^{1,\infty}(\mathbb{R}^N)$ satisfying

\[
\left\{
\begin{array}{l}
\bar{u}_t \in \text{co} \{u(k_\ell), u(k_{\ell+1}), u(k_{\ell+2}), \ldots \} \text{ for every } \ell \in \mathbb{N}, \\
\bar{u}_t|_{\mathbb{B}_\rho} \to u|_{\mathbb{B}_\rho} \text{ strongly in } L^1(0,T; L^q(\mathbb{B}_\rho)) \text{ for } \ell \to \infty
\end{array}
\right.
\]

(see, e.g., [21, Corollary 3.8] or [80, § V.1 Theorem 2]). In addition, we observe the same form of weak convergence as formulated for $(u(k_\ell))_{\ell \in \mathbb{N}}$ in Lemma 3.6, i.e.,

\[
\bar{u}_t|_{\mathbb{B}_r} \to u|_{\mathbb{B}_r} \text{ weakly in } L^1(0,T; L^q(\mathbb{B}_r)) \text{ for } \ell \to \infty \text{ and each fixed } r > 0.
\]

Next, this construction of sequences is repeated inductively for every radius $\rho \in \mathbb{N}$. Finally, CANTOR’s diagonal method leads to a sequence $(\bar{u}_t)_{t \in \mathbb{N}}$ with the following properties

- $\bar{u}_t \in \text{co} \{u(k_\ell), u(k_{\ell+1}), u(k_{\ell+2}), \ldots \} \text{ for every } \ell \in \mathbb{N}$,
- $\bar{u}_t(t) \in \text{co} \bigcup_{m \geq \ell} \mathcal{U}(\tau_{(k_m)}(t), P_{(k_m)}(\tau_{(k_m)}(t))) \text{ for every } \ell \in \mathbb{N} \text{ and all } t \in J_{(k_m)} \cap [0,T]$,
- for every $\rho > 0$ fixed, $u(t)_{\mathbb{B}_\rho} \to \bar{u}_t|_{\mathbb{B}_\rho} \text{ strongly in } L^1(0,T; L^q(\mathbb{B}_\rho)) \text{ for } \ell \to \infty$.

In particular, \( L^1(0,T) \setminus \bigcap_{m \geq \ell} J_{(k_m)} \leq \sum_{m = \ell}^{\infty} L^1([0,T] \setminus J_{(k_m)}) \leq \sum_{m = \ell}^{\infty} \frac{1}{(k_m)^2} \)

\[
\leq \sum_{m = \ell}^{\infty} \frac{1}{m^2} \quad \ell \to \infty \to 0.
\]

Now we are going to conclude from the closed graph of $\mathcal{U}_{|I \times \mathcal{L}}$ that for every $t \in \bar{I}$,

\[
\bigcap_{\varepsilon > 0} \text{co } \mathcal{U}([0,T] \cap \mathbb{B}_\varepsilon(t), \{ \phi \in \mathcal{L} \mid d_{L^p}(f(t), \phi) < \varepsilon \}) \subset \mathcal{U}(t, f(t)) \quad (7)
\]

where the closure is understood w.r.t. the metric $\delta_{L^q,\text{loc}}$ because it leads to a contradiction: Inclusion (7) and Corollary 3.9 ensure $u(t) \in \mathcal{U}(t, f(t))$ for $L^1$-a.e. $t \in \bar{I}$. The corresponding arguments also imply $\mathcal{G}(t, f(t)) \subset W(t, f(t))$ for $L^1$-a.e. $t \in \bar{I}$, i.e., in a subset of $[0,T]$ of LEBESGUE measure $\geq T - \frac{2}{3} \lambda_e$ – contracting the initial choice of the outer LEBESGUE measure $\lambda_e > 0$.

Similarly to the proof of [63, Lemma 5.20], we introduce the auxiliary set for arbitrary $\delta, \rho > 0$

\[
\mathcal{E}_\delta(\mathcal{U}(t, f(t)); \mathbb{B}_\rho) := \left\{ \nu \in L^q_{\text{loc}}(\mathbb{R}^N) \mid \delta \geq \text{dist}_{L^q(\mathbb{B}_\rho)}(\nu, \mathcal{U}(t, f(t))) \right\}
\]

It is closed w.r.t. $\delta_{L^q,\text{loc}}$ and convex since so is $\mathcal{U}(t, f(t)) \subset W^{1,\infty}(\mathbb{R}^N) \subset L^q_{\text{loc}}(\mathbb{R}^N)$ by assumption 2.4 (iv).
For all $t \in I$, $\delta > 0$ and $\rho > 0$, there exists a radius $\varepsilon = \varepsilon(t, \delta, \rho) > 0$ with

$$U([0, T] \cap B_\varepsilon(t), \{\phi \in L \mid d_{L^p}(f(t), \phi) < \varepsilon\}) \subset B_\delta(U(t,f(t)); B_\rho).$$

Indeed, we can easily adapt the indirect conclusion concerning the well-known step from the closedness of graph of a set-valued mapping to its upper semicontinuity (a.k.a. outer semicontinuity) (see, e.g., [14, Proposition 1.4.8] and [51, Ch. 1, Prop. 2.23]) – for three reasons: First, the graph of $U$ is closed. Second, all its values are contained in the relatively compact subset $U \subset (L^q_{\text{loc}}(\mathbb{R}^N), \delta_{L^q,\text{loc}})$. Third, $I$ is specified as an open subset of $\mathbb{R}$ and so, $[0, T] \cap B_\varepsilon(t) \subset I$ holds for all $\varepsilon > 0$ sufficiently small. Thus, we obtain

$$\bigcap_{\varepsilon > 0} \bigcap_{\rho > 0} \bigcap_{\delta > 0} B_\delta(U(t,f(t))); B_\rho$$

$$\bigcap_{\delta > 0} \bigcap_{\rho > 0} B_\delta(U(t,f(t))); B_\rho$$

where the closed ball mentioned last, i.e., $B_\delta(U(t,f(t))) \subset L^q_{\text{loc}}(\mathbb{R}^N)$, refers to the metric $\delta_{L^q,\text{loc}}$ (introduced in Definition A.15). According to hypothesis 2.4 (iv), $U$ has closed values in $(L^q_{\text{loc}}(\mathbb{R}^N), \delta_{L^q,\text{loc}})$ and so, we obtain the claimed inclusion (7):

$$\bigcap_{\varepsilon > 0} U([0, T] \cap B_\varepsilon(t), \{\phi \in L \mid d_{L^p}(f(t), \phi) < \varepsilon\}) \subset \bigcap_{\delta > 0} B_\delta(U(t,f(t)))$$

Essentially the same arguments can be used for $W_{I \times E} : I \times (L, d_{L^p}) \rightrightarrows (L^p(\mathbb{R}^N), \| \cdot \|_{L^p})$ (and its closed graph) because all values are convex and so, we are free to supply $L^p(\mathbb{R}^N)$ with the weak topology in combination with Mazur’s lemma.

\section*{Appendix A. Tools from metric spaces and set-valued analysis.}

\subsection*{A.1. An extension of Scorza-Dragoni theorem to metric spaces and closed graphs.}

There are many versions of the Scorza-Dragoni theorem available in the literature (see, e.g., [24, 50, 51, 54, 78]). Inspired by the results of Frankowska, Plaskacz and Rzeźuchowski in [41], we prefer the following formulation concerning set-valued maps with closed graphs on metric spaces:

**Lemma A.1** ([72, Theorem 1]). Let $X$ and $Y$ be separable metric spaces. Suppose for a set-valued map $F : [0, T] \times X \rightrightarrows Y$ that for $L^1$-a.e. $t \in [0, T]$, the graph of $F(t, \cdot) : X \rightrightarrows Y$ is a closed subset of $X \times Y$. Then there exists a set-valued map $\hat{F} : [0, T] \times X \rightrightarrows Y$ with (possibly empty) closed values and the subsequent properties:

(i) $\hat{F}(t, x) \subset F(t, x)$ holds for $L^1$-a.e. $t \in [0, T]$ and every $x \in X$.

(ii) For every Lebesgue measurable set $J \subset [0, T]$ and all measurable maps $u : J \to X$ and $v : J \to Y$ with $v(t) \in F(t, u(t))$ for $L^1$-a.e. $t \in J$, it holds $v(t) \in \hat{F}(t, u(t))$ for $L^1$-a.e. $t \in J$. 
(iii) For any \( \varepsilon > 0 \), there is a closed subset \( J_\varepsilon \subset [0, T] \) with \( L^1([0, T] \setminus J_\varepsilon) < \varepsilon \) such that the set-valued restriction \( \hat{F}|_{J_\varepsilon \times X} : J_\varepsilon \times X \rightrightarrows Y \) has a closed graph.

In regard to the Viability Theorem 2.4 and its set-valued coefficient maps \( G, U, W \), however, there are three obstacles of the auxiliary map \( \hat{F} \). First, the set values of \( \hat{F} \) might be empty. Second, it is not guaranteed that its set values are convex if so are the values of \( F \). But then we are free to consider the pointwise convex hull of \( \hat{F} \) instead. The third obstacle concerns contingent condition 2.4 (viii). It is not clear in which form this property holds for the “modified” coefficients (i.e., after the Scorza-Dragoni argument is applied to \( G, U \) and \( W \)).

Hence, we prefer the following extensions of Lemma A.1. Indeed, the arguments proving the established formulations of Scorza-Dragoni-type theorems in, e.g., [51, 54, 72] imply some more properties than usually listed.

**Proposition A.2.** Suppose for the nonempty sets \( X, Y \) and the set-valued map \( F : [0, T] \times X \rightrightarrows Y \):

(a) \( X \) is a separable metric space.
(b) \( Y \) is a Polish metric space, i.e., a complete separable metric space.
(c) \( F \) has nonempty closed values.
(d) For \( L^1\)-a.e. \( t \in [0, T] \), the graph of \( F(t, \cdot) : X \rightrightarrows Y \) is a closed subset of \( X \times Y \).
(e) For every \( x \in X \), the map \( F(\cdot, x) : [0, T] \rightrightarrows Y \) is Lebesgue measurable.

Then there exists a set-valued map \( \hat{F} : [0, T] \times X \rightrightarrows Y \) with the subsequent properties:

(i) \( \hat{F}(t, x) \in F(t, x) \) holds for \( L^1\)-a.e. \( t \in [0, T] \) and all \( x \in X \).
(ii) For every Lebesgue measurable set \( J \subset [0, T] \) and all measurable functions \( u : J \to X, v : J \to Y \) with \( v(t) \in \hat{F}(t, u(t)) \) for \( L^1\)-a.e. \( t \in J \), it holds \( v(t) \in \hat{F}(t, u(t)) \) for \( L^1\)-a.e. \( t \in J \).
(iii) For any \( \varepsilon > 0 \), there is a closed subset \( J_\varepsilon \subset [0, T] \) with \( L^1([0, T] \setminus J_\varepsilon) < \varepsilon \) such that the set-valued restriction \( \hat{F}|_{J_\varepsilon \times X} \) has a closed graph.
(iv) There exists a Lebesgue measurable subset \( J_0 \subset [0, T] \) of full measure (i.e., \( L^1([0, T] \setminus J_0) = 0 \)) such that for every \( t \in J_0 \), all set values of \( \hat{F}(t, \cdot) : X \rightrightarrows Y \) are nonempty.

**Corollary A.3.** Let both \( X \) and \( Y \) be Polish metric spaces. For \( F : [0, T] \times X \rightrightarrows Y \) suppose:

(c) \( F \) has nonempty closed values.
(d) For \( L^1\)-a.e. \( t \in [0, T] \), the graph of \( F(t, \cdot) : X \rightrightarrows Y \) is a closed subset of \( X \times Y \).
(e) The set-valued map \( F \) is (jointly) measurable, i.e., for every open set \( \Omega \subset Y \), the inverse image \( F^{-1}(\Omega) \) belongs to the \( \sigma \)-algebra \( L^1([0, T]) \otimes B(X) \).

Then, \( F \) has all the properties of \( \hat{F} \) specified in Proposition A.2. In particular, for each \( \varepsilon > 0 \), there exists a closed subset \( J_\varepsilon \subset [0, T] \) with \( L^1([0, T] \setminus J_\varepsilon) < \varepsilon \) such that the set-valued restriction \( F|_{J_\varepsilon \times X} \) has a closed graph.

**Proof of Proposition A.2.** For constructing \( \hat{F} \) and verifying the claimed properties (i) – (iii), we essentially follow the arguments presented by Rzeżuchoowski in [72] (similarly to the proof of [51, Ch. 2, Proposition 7.20]).

First, the set-valued map \( \hat{F} : [0, T] \times X \rightrightarrows Y \) is constructed. The product \( X \times Y \) supplied with the sum of the componentwise metrics of \( X \) and \( Y \) is separable. Let
\{a_k \in X \times Y \mid k \in \mathbb{N}\} denote an arbitrary dense subset of \(X \times Y\) and consider \(\varphi_k : [0, T] \to [0, \infty), t \mapsto \text{dist}(a_k, \text{Graph} \mathcal{F}(t, \cdot))\) for each \(k \in \mathbb{N}\) with \(\text{Graph} \mathcal{F}(t, \cdot) \overset{\text{def}}{=} \{(x, y) \in X \times Y \mid y \in \mathcal{F}(t, x)\} \subset X \times Y\). Then, for each index \(k \in \mathbb{N}\), there exists a \textit{Lebesgue} measurable function \(\bar{\varphi}_k : [0, T] \to [0, \infty]\) satisfying

- \(\bar{\varphi}_k \geq \varphi_k\) \(L^1\)-a.e. in \([0, T]\) and
- every measurable function \(\psi : [0, T] \to [0, \infty]\) with \(\varphi_k \leq \psi\) \(L^1\)-a.e. in \([0, T]\)

due to [72, Lemma 1] or [51, proof of Ch.2, Prop. 7.20]. Define the set-valued mapping \(\tilde{\mathcal{F}} : [0, T] \times X \rightrightarrows Y\) by

\[
\text{Graph} \tilde{\mathcal{F}}(t, \cdot) = \bigcap_{k \in \mathbb{N}} \{ (x, y) \in X \times Y \mid d_{X \times Y}(a_k, (x, y)) \geq \bar{\varphi}_k(t) \} \subset X \times Y
\]

for each \(t \in [0, T]\). \(\tilde{\mathcal{F}}\) has the claimed properties (the first three ones are already proved in [72]):

(i) The general inequality \(\bar{\varphi}_k \geq \varphi_k\) \(L^1\)-a.e. in \([0, T]\) for every index \(k \in \mathbb{N}\) implies \(\text{Graph} \tilde{\mathcal{F}}(t, \cdot) \subset \text{Graph} \mathcal{F}(t, \cdot)\) for \(L^1\)-a.e. \(t \in [0, T]\) and thus, we obtain \(\tilde{\mathcal{F}}(t, x) \subset \mathcal{F}(t, x)\) for all \(x \in [0, T]\) and \(L^1\)-a.e. \(t \in [0, T]\).

(ii) Consider any \textit{Lebesgue} measurable set \(J \subset [0, T]\) and measurable functions \(u : J \to X, v : J \to Y\) with \(v(t) \in \mathcal{F}(t, u(t))\) for \(L^1\)-a.e. \(t \in J\). Then, \(z := (u, v) : J \to X \times Y\) is measurable with \(z(t) \in \mathcal{F}(t, \cdot)\) for \(L^1\)-a.e. \(t \in J\). Hence, for every index \(k \in \mathbb{N}\), the function \(d_{X \times Y}(a_k, z(\cdot)) : J \to \mathbb{R}\) is \textit{Lebesgue} measurable with \(\varphi_k(t) \leq d_{X \times Y}(a_k, z(t))\) for \(L^1\)-a.e. \(t \in J\). The choice of \(\bar{\varphi}_k\) implies \(\bar{\varphi}_k \leq d_{X \times Y}(a_k, z(\cdot))\) \(L^1\)-a.e. in \(J\) and so, we conclude \((u(t), v(t)) = z(t) \in \text{Graph} \tilde{\mathcal{F}}(t, \cdot)\) for \(L^1\)-a.e. \(t \in J\).

(iii) For \(\varepsilon > 0\) fixed arbitrarily and each index \(k \in \mathbb{N}\), \textit{Lusin’s} theorem guarantees a closed subset \(A_{\varepsilon, k} \subset [0, T]\) with \(L^1([0, T] \setminus A_{\varepsilon, k}) < \frac{\varepsilon}{k}\) such that the restriction \(\bar{\varphi}_k|_{A_{\varepsilon, k}}\) is continuous (see, e.g., [17, Theorems 2.2.10, 7.1.13]). Then, \(J_{\varepsilon} := \bigcap_{k \in \mathbb{N}} A_{\varepsilon, k}\) is a closed subset of \([0, T]\) with \(L^1([0, T] \setminus J_{\varepsilon}) < \varepsilon\). In particular, the graph of the restriction \(\tilde{\mathcal{F}}|_{J_{\varepsilon} \times X} : J_{\varepsilon} \times X \rightrightarrows Y\) is closed. In other words, it contains all its accumulation points since the construction of \(\tilde{\mathcal{F}}\) is based on the sequences of scalar functions \(d_{X \times Y}(a_k, \cdot) : X \times Y \to \mathbb{R}\) and \(\bar{\varphi}_k|_{J_{\varepsilon}} : J_{\varepsilon} \to \mathbb{R}\) (\(k \in \mathbb{N}\)) each of which is (sequentially) continuous.

(iv) Now we benefit from assumption (c) (i.e., the measurability of \(\mathcal{F}\) w.r.t. the first variable) for the first time. Indeed, it lays the basis for adapting the conclusions from properties (ii), (iii) presented in [54, Theorem 3.1] to our metric setting:

Otherwise, there exists a subset \(E \subset [0, T]\) with outer \textit{Lebesgue} measure \(L^*(E) > 0\) such that for each \(t \in E\), there is an element \(x = x(t) \in X\) with \(\tilde{\mathcal{F}}(t, x) = 0\). Property (iii) provides a \textit{Lebesgue} measurable set \(A \subset [0, T]\) with \(L^1([0, T] \setminus A) < L^*(E)\) such that the restriction \(\tilde{\mathcal{F}}|_{A \times X} : A \times X \rightrightarrows Y\) has a closed graph. In particular, we conclude \(L^*(A \cap E) > 0\). Let \(B \subset [0, T]\) denote the subset of all density points of \(A\) (see, e.g., [17, \S 5.8 (ii))]. Then, \(L^1(A \setminus B) = 0\) implies \(L^*(B \cap E) > 0\).
Select any $\tau \in B \cap E$. By definition of $E$, there exists an element $x_\tau \in X$ with $\hat{F}(\tau, x_\tau) = \emptyset$. The closed graph of $\hat{F}|_{A \times X}$ provides a radius $\rho = \rho(\tau, x_\tau) > 0$ such that $\hat{F}(t, x) = \emptyset$ holds for all $(t, x) \in A \times X$ with $|t - \tau| + d_X(x, x_\tau) < \rho$.

The set-valued map $F(\cdot, x_\tau) : [0, T] \Rightarrow Y$ is assumed to be Lebesgue measurable with nonempty closed values. Hence, the well-known selection theorem of Kuratowski and Ryll-Nardzewski [59] guarantees a Lebesgue measurable selection $v : [0, T] \rightarrow Y$ of $F(\cdot, x_\tau)$ (see, e.g., [7, Theorem 8.1.4] or [25, Theorem III.6]). The tuple consisting of the constant curve selection $F(\cdot, x_\tau)$ and the restriction $v|_A : A \rightarrow Y$ satisfies the conditions in property (ii) and so, we obtain $v(t) \in \hat{F}(t, u(t)) = \hat{F}(t, x_\tau)$ for every $t \in A$ – contradicting the empty values of $\hat{F}(\cdot, x_\tau)$ concluded before.

The next step formulated in Corollary A.3 is based on the following consequence of Lemma A.3.

**Lemma A.4.** Under the assumptions of Corollary A.3 about $X, Y$ and $F$, the function $[0, T] \rightarrow \mathbb{R}$, $t \mapsto \text{dist}(\xi, \text{Graph } F(t, \cdot))$ is Lebesgue measurable for each $\xi \in X \times Y$.

**Proof of Lemma A.4.** The assumed joint measurability of $F$ implies its so-called graph measurability, i.e., that the graph of $F$ belongs to the $\sigma$-algebra $(\mathcal{L}^1([0, T]) \otimes \mathcal{B}(X)) \otimes \mathcal{B}(Y)$ (see, e.g., [25, Proposition III.13] or [51, Ch. 2, Proposition 1.7]). The definition of product $\sigma$-algebras leads to $(\mathcal{L}^1([0, T]) \otimes \mathcal{B}(X)) \otimes \mathcal{B}(Y) = \mathcal{L}^1([0, T]) \otimes (\mathcal{B}(X) \otimes \mathcal{B}(Y))$ and, we conclude $\mathcal{B}(X) \otimes \mathcal{B}(Y) = \mathcal{B}(X \times Y)$ from the assumptions about $X, Y$ (see, e.g., [17, Lemma 6.4.2] or [51, Ch. 2, Proposition 1.48]). Hence, the graph of $F$ belongs to $\mathcal{L}^1([0, T]) \otimes \mathcal{B}(X \times Y)$.

Now we consider $X \times Y$ a complete metric space, and $[0, T]$ supplied with the Lebesgue measure $\mathcal{L}^1$ is a complete finite measure space. Due to assumption (d), the set-valued map $[0, T] \Rightarrow X \times Y$, $t \mapsto \text{Graph } F(t, \cdot)$ has nonempty values which are closed for $\mathcal{L}^1$-a.e. $t$. Thus, it is Lebesgue measurable (see, e.g., [14, Characterization Theorem 8.1.4] or [51, Ch. 2, Theorem 2.4 (d)]).

Finally, for arbitrary $\xi \in X \times Y$, we conclude the claimed Lebesgue measurability of $[0, T] \rightarrow \mathbb{R}$, $t \mapsto \text{dist}(\xi, \text{Graph } F(t, \cdot))$ from the same well-known theorems about measurable set-valued maps (see, e.g., [25, Theorem III.9] complementarily).

**Proof of Corollary A.3.** Consider a dense sequence $(a_k)_{k \in \mathbb{N}}$ in $X \times Y$ and the functions $\varphi_k : [0, T] \rightarrow [0, \infty)$, $t \mapsto \text{dist}(a_k, \text{Graph } F(t, \cdot))$ ($k \in \mathbb{N}$) as in the proof of Proposition A.2. According to Lemma A.4, each $\varphi_k$ is Lebesgue measurable. Hence, essentially the same conclusions as in the proof of Proposition A.2 can now be drawn for $\varphi_k$ (instead of $\tilde{\varphi}_k$). In particular, we obtain property A.2 (iii) for $F$: For any $\varepsilon > 0$, there is a closed subset $J_\varepsilon \subset [0, T]$ with $\mathcal{L}^1([0, T] \setminus J_\varepsilon) < \varepsilon$ such that the set-valued restriction $F|_{J_\varepsilon \times X} : J_\varepsilon \times X \Rightarrow Y$ has a closed graph.

**Remark A.5.** These extensions A.2 and A.3 are tools for set-valued maps of quite general interest. Corollary A.3, for example, leads to shorter complete proofs of [64, Lemma 4.33] “(a) $\Rightarrow$ (b)” and thus of [64, Proposition 2.17] – under the additional assumption that the respective coefficient mapping is jointly measurable.

**A.2. The metrics $d_{LR}, \ell_{LP}$ of the state space $L^p(\mathbb{R}^N)$.** Now we specify the metrics of the state space $L^p(\mathbb{R}^N)$ and summarize their essential features for the
sake of a self-contained presentation. The details are verified in, e.g., [58, § 4.1]. In a word, the metric \( d_{L^p} \) introduced in Definition 2.3 is usually used for comparing two states in \( L^p(\mathbb{R}^N) \) at the same time instant whereas the further metrics \( e_{L^p}, \hat{e}_{L^p} \) are preferred for describing the regularity of weak solutions with respect to time.

**Definition A.6** ([58, Definition 10]). Fix \( 1 < p < \infty \) and \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and define \( \hat{e}_{L^p}, e_{L^p} : L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N) \to [0, \infty) \) as

\[
\begin{align*}
\hat{e}_{L^p}(f, g) &:= \sup \left\{ \int_{\mathbb{R}^N} \varphi \cdot (f - g) \, dx \mid \varphi \in C^1_c(\mathbb{R}^N), \|\varphi\|_{W^{1,q}} \leq 1 \right\} \\
e_{L^p}(f, g) &:= \hat{e}_{L^p}(f, g) + \|f\|_{L^p(\mathbb{R}^N)} + \|g\|_{L^p(\mathbb{R}^N)}.
\end{align*}
\]

**Remark A.7.** Obviously the following inequalities hold for all \( f, g \in L^p(\mathbb{R}^N) \) and \( \varphi \in C^1_c(\mathbb{R}^N) \) using the abbreviations \( \alpha_1 := \max \{\|\varphi\|_{W^{1,q}}, \|\varphi\|_{W^{1,\infty}}\} \) and \( \alpha_2 := \max \{\|\varphi\|_{L^p}, \|\varphi\|_{W^{1,\infty}}\} \)

\[
\begin{align*}
\hat{e}_{L^p}(f, g) &\leq d_{L^p}(f, g) \leq \|f\|_{L^p} + \alpha_1 \cdot \hat{e}_{L^p}(f, g), \\
e_{L^p}(f, g) &\leq 2 \cdot \|f\|_{L^p(\mathbb{R}^N)} + \alpha_2 \cdot d_{L^p}(f, g).
\end{align*}
\]

**Remark A.8.** The metrics \( d_{L^p} \) and \( \hat{e}_{L^p} \) are constructed in a very similar way, namely in terms of a supremum for all test functions in a unit ball. What differs, however, is the class of test functions and the norm underlying the unit ball. From a more general point of view, they both modify the \( L^p \) norm. Indeed, the well-known Hahn-Banach theorem implies for every \( f, g \in L^p(\mathbb{R}^N) \)

\[
\|f - g\|_{L^p(\mathbb{R}^N)} = \sup \left\{ \int_{\mathbb{R}^N} \varphi \cdot (f - g) \, dx \mid \varphi \in L^q(\mathbb{R}^N), \|\varphi\|_{L^q} \leq 1 \right\}
\]

\[
= \sup \left\{ \int_{\mathbb{R}^N} \varphi \cdot (f - g) \, dx \mid \varphi \in C^1_c(\mathbb{R}^N), \|\varphi\|_{L^q} \leq 1 \right\}
\]

since \( L^q(\mathbb{R}^N) \) represents the dual space of \( L^p(\mathbb{R}^N) \) and \( C^1_c(\mathbb{R}^N) \) is dense in the Banach space \( L^q(\mathbb{R}^N) \). The main difference concerns the class of test functions (for more details see, e.g., [58, 63, 66]).

**Lemma A.9** ([58, Lemma 13]). \( \hat{e}_{L^p} \) metrizes the weak topology on norm-bounded tight balls in \( L^p(\mathbb{R}^N) \) in the following sense: Suppose \( f \in L^p(\mathbb{R}^N) \) and let \((f_k)_{k \in \mathbb{N}}\) be any sequence in \( L^p(\mathbb{R}^N) \) such that \( (|f_k|^p) \) is tight in \( \mathbb{R}^N \), i.e.,

\[
\lim_{\rho \to \infty} \sup_{k \in \mathbb{N}} \|f_k\|_{L^p(\mathbb{R}^N \setminus B_\rho(0))} = 0.
\]

Then,

\[
f_k \rightharpoonup f \text{ weakly in } L^p(\mathbb{R}^N) \quad (k \to \infty) \iff \left\{ \begin{array}{l}
\sup_{k \in \mathbb{N}} \|f_k\|_{L^p(\mathbb{R}^N)} < \infty \quad \text{and} \\
\lim_{k \to \infty} \hat{e}_{L^p}(f_k, f) = 0.
\end{array} \right\}
\]

**Corollary A.10** ([58, Corollary 14]). Every norm-bounded closed tight subset of \( L^p(\mathbb{R}^N) \) is (sequentially) compact w.r.t. \( \hat{e}_{L^p} \).

**Corollary A.11** ([58, Corollary 15]). Let \((f_k)_{k \in \mathbb{N}}\) and \((g_k)_{k \in \mathbb{N}}\) be two bounded sequences in \( L^p(\mathbb{R}^N) \) such that both \( (|f_k|^p)_{k \in \mathbb{N}} \) and \( (|g_k|^p)_{k \in \mathbb{N}} \) are tight in \( \mathbb{R}^N \). Then the following equivalence holds

\[
\lim_{k \to \infty} \hat{e}_{L^p}(f_k, g_k) = 0 \iff \lim_{k \to \infty} d_{L^p}(f_k, g_k) = 0.
\]
Corollary A.12 ([58, Corollary 16]). Norm-bounded closed convex tight subsets of \( L^p(\mathbb{R}^N) \) are relatively compact with respect to \( d_{L^p} \) in the following sense: Let \( M \subset L^p(\mathbb{R}^N) \) be any norm-bounded closed convex subset with \( \lim_{\rho \to \infty} \sup_{f \in M} \| f \|_{L^p(\mathbb{R}^N \setminus B_\rho(0))} = 0 \). Then every sequence in \( M \) has a subsequence converging in \( (L^p(\mathbb{R}^N), d_{L^p}) \).

Proposition A.13 ([58, Proposition 17]). This equivalence holds for any sequence \((f_k)_{k \in \mathbb{N}} \) in \( L^p(\mathbb{R}^N) \):

\[
\lim_{k \to \infty} \| f_k - f \|_{L^p(\mathbb{R}^N)} = 0 \iff \begin{cases} 
\lim_{k \to \infty} e_{L^p}(f_k, f) = 0 \\
(\| f_k \|_{L^p})_{k \in \mathbb{N}} \text{ is tight in } \mathbb{R}^N 
\end{cases}
\]

\[
\iff \begin{cases} 
\lim_{k \to \infty} \| f_k \|_{L^p} = \| f \|_{L^p} \\
\lim_{k \to \infty} d_{L^p}(f_k, f) = 0 \\
(\| f_k \|_{L^p})_{k \in \mathbb{N}} \text{ is tight in } \mathbb{R}^N.
\end{cases}
\]

Proposition A.14 ([58, Proposition 18]). Norm-bounded closed convex tight subsets of \( L^p(\mathbb{R}^N) \) are complete with respect to \( d_{L^p} \) in the following sense: Let \( M \subset L^p(\mathbb{R}^N) \) be any norm-bounded closed convex subset with \( \lim_{\rho \to \infty} \sup_{f \in M} \| f \|_{L^p(\mathbb{R}^N \setminus B_\rho(0))} = 0 \). Then every Cauchy sequence w.r.t. \( d_{L^p} \) in \( M \) has a limit in \( M \) w.r.t. \( d_{L^p} \).

A.3. The metrics of local \( L^q \) norm convergence for coefficient functions of space. Several conclusions in Section 3 concern sequences of coefficient functions \([0, T] \to L^2_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N)\) or \([0, T] \to L^2_{\text{loc}}(\mathbb{R}^N)\) and their converging subsequences due to an appropriate form of compactness. The standard compactness results, however, concern bounded domains with sufficiently smooth boundaries. We are going to restrict our considerations to open balls with center 0 and arbitrary radius \( \rho > 0 \). This notion and hypotheses 2.4 (iv), (v) motivate the following notation:

Definition A.15. For \( p \in (1, \infty) \), \( q := \frac{p}{p-1} \), set \( \delta_{L^q, \text{loc}} : L^q_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N) \times L^q_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N) \to [0, \infty) \)

\[
\delta_{L^q, \text{loc}}(u, v) := \sum_{\rho = 1}^{\infty} 2^{-\rho} \frac{\| u - v \|_{L^q(B_\rho)}}{1 + \| u - v \|_{L^q(B_\rho)}}
\]

and define \( \delta_{L^p, \text{loc}} : L^p_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N) \times L^p_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N) \to [0, \infty) \) by the corresponding formula (just applied to vector-valued functions).

Remark A.16. (1.) \( \delta_{L^q, \text{loc}} \) is a Fréchet metric on \( L^q_{\text{loc}}(\mathbb{R}^N) \) and \( L^q_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N) \) respectively (as characterized in [2, § 2.7], for example). In particular, the triangle inequality results from the same arguments as in [80, page 27].

(2.) A sequence \((u_j)_{j \in \mathbb{N}} \) in \( L^q_{\text{loc}}(\mathbb{R}^N) \) converges to \( u \in L^q_{\text{loc}}(\mathbb{R}^N) \) w.r.t. \( \delta_{L^q, \text{loc}} \) if and only if for every radius \( \rho > 0 \), the restrictions \( u_j|_{B_\rho} : B_\rho \to \mathbb{R} \) (\( j \in \mathbb{N} \)) tend to \( u|_{B_\rho} \) w.r.t. the \( L^q(B_\rho) \) norm. Hence, assumption 2.4 (v) can be (re-) formulated in the following way: For \( \mathcal{L}^1 \text{-a.e. } t \in [0, T] \), the graphs of the set-valued maps

\[
\mathcal{G}(t, \cdot) : (L^p(\mathbb{R}^N), d_{L^p}) \ni (L^q_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N), \delta_{L^q, \text{loc}}), \\
\mathcal{U}(t, \cdot) : (L^p(\mathbb{R}^N), d_{L^p}) \ni (L^q_{\text{loc}}(\mathbb{R}^N), \delta_{L^q, \text{loc}}), \\
\mathcal{W}(t, \cdot) : (L^p(\mathbb{R}^N), d_{L^p}) \ni (L^p(\mathbb{R}^N), \| \cdot \|_{L^p}).
\]

are closed.
(3.) The metric spaces \( L^q_{loc}(\mathbb{R}^N), \delta_{L^q_{loc}} \) and \( L^q_{loc}(\mathbb{R}^N), \delta_{L^q_{loc}} \) are complete and separable because \( L^q(\Omega) \) is a separable BANACH space for every nonempty subset \( \Omega \subset \mathbb{R}^N \) (according to the theorem of FISCHER-RIESZ in, e.g., [21, Theorem 4.8] and due to [21, Theorem 4.13]).

(4.) Many conclusions about Lebesgue-Bochner integrals can still be drawn if the values of the integrated functions are not in a BANACH space, but in a Fréchet space like \( L^q_{loc}(\mathbb{R}^N), \delta_{L^q_{loc}} \). More details about this generalization can be found in, e.g., [43, Chapitre VI].

Appendix B. Results about solutions to nonautonomous linear transport equations. The following statement is a restriction of [58, Theorem 4] to nonautonomous linear transport equations and, it provides additional aspects of regularity concluded from [58, Proposition 34 & Lemma 38]:

Proposition B.1. Consider the initial value problem

\[
\partial_t f + \text{div}_x(f(x)g) = u(t) \cdot f + w(t), \quad f(0) = f_0 \tag{8}
\]

with the single-valued coefficient functions

\[
g : [0, T] \rightarrow W^{1, \infty}(\mathbb{R}^N, \mathbb{R}^N) \cap L^q(\mathbb{R}^N, \mathbb{R}^N),
\]

\[
u : [0, T] \rightarrow W^{1, \infty}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N),
\]

\[
w : [0, T] \rightarrow L^p(\mathbb{R}^N)
\]

under the following assumptions:

(i) (Global a priori bounds)

(a) \( \gamma_a := \sup_{t \in [0, T]} \left( \|\text{div}_x g(t)\|_{L^\infty(\mathbb{R}^N)} + \|u(t)\|_{L^\infty(\mathbb{R}^N)} + \|w(t)\|_{L^p(\mathbb{R}^N)} \right) < \infty \)

(b) \( \gamma_b := \sup_{t \in [0, T]} \left( \|g(t)\|_{W^{1, \infty}(\mathbb{R}^N)} + \|\nabla_x u(t)\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^N)} \right) < \infty \)

(ii) (Locally uniform choice of function dominating values of \( w \)) There exist \( \tilde{w} \in L^p(\mathbb{R}^N) \) and a compact set \( K_x \subset \mathbb{R}^N \) such that \( |w(t, x)| \leq \tilde{w}(x) \) for all \( t \in [0, T] \) and \( L^N \)-a.e. \( x \in \mathbb{R}^N \setminus K_x \).

(iii) \( g, u : [0, T] \rightarrow \left( L^q, \|\cdot\|_{L^q} \right) \) and \( w : [0, T] \rightarrow \left( L^p, \|\cdot\|_{L^p} \right) \) are Lebesgue measurable.

Then for every \( f_0 \in L^p(\mathbb{R}^N) \), the initial value problem (8) has a unique weak solution \( f : [0, T] \rightarrow L^p(\mathbb{R}^N) \), i.e., \( f(0) = f_0 \) and

\[
\int_{\mathbb{R}^N} \varphi(f(t_2) - f(t_1)) \, dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^N} f(s, x) \, g(s, x) \cdot \nabla_x \varphi(x) \, dx \, ds + \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (f(s, x) \, u(s, x) + w(s, x)) \, \varphi(x) \, dx \, ds \tag{9}
\]

holds for any \( 0 \leq t_1 < t_2 \leq T \) and \( \varphi \in C^1(\mathbb{R}^N) \). Furthermore, \( f \) has the following properties:

(1.) \( \|f(t)\|_{L^p(\mathbb{R}^N)} \leq (\|f_0\|_{L^p(\mathbb{R}^N)} + \gamma_a t) \cdot e^{c_{\text{const}}(N, \gamma_a) t} \) for every \( t \in [0, T] \),

(2.) \( f : [0, T] \rightarrow \left( L^p(\mathbb{R}^N), \|\cdot\|_{L^p} \right) \) is \( \Lambda \)-Lipschitz continuous with a constant \( \Lambda \) depending on \( N, p, \gamma_a, \|f_0\|_{L^p}, \sup_t \|g(t)\|_{L^\infty(\mathbb{R}^N, \mathbb{R}^N)} \),

(3.) \( f : [0, T] \rightarrow \left( L^p(\mathbb{R}^N), \text{weak} \right) \) is continuous,

(4.) for \( L^1 \)-a.e. \( t \in [0, T] \), \( \lim_{h \downarrow 0} \frac{1}{h} \cdot d_{L^p} \left( f(t + h), \varphi(f(t)), u(t), w(t) \right) (h, f(t)) = 0 \).
(5.) $f$ is a renormalized solution of (8) (in the sense of DiPerna and Lions \[35\]).

(6.) $\{ |f(t)|^p \mid t \in [0, T] \} \subset L^1(\mathbb{R}^N)$ is tight in $\mathbb{R}^N$, i.e.,
$$\lim_{\rho \to \infty} \sup_{t \in [0, T]} \|f(t)\|^p_{L^p(\mathbb{R}^N \setminus B_\rho(0))} = 0.$$  

It is worth mentioning that property (6.) of tightness can be formulated in more detail since the limit is uniform w.r.t. $f_0$, $\tilde{\omega}$, $N$, $p$, $T$, $\gamma_a$, $\gamma_b$. Indeed, the arguments proving [58, Lemma 31 and Proposition 32] lead to the following property in terms of an auxiliary function $\omega$ used in \S 3.2:

**Corollary B.2.** Under the hypotheses of Proposition B.1, there exist a constant $\kappa = \kappa(p, N, \gamma_a) \geq 1$ and a function $\omega : [0, \infty) \to [0, \infty)$ depending on $f_0$, $\tilde{\omega} \in L^p(\mathbb{R}^N)$ and $N$, $p$, $\gamma_a$, $\gamma_b$ only such that

- for every $t \in [0, T]$ and $\rho > 0$,
  $$\|f(t)|^p_{L^p(\mathbb{R}^N \setminus B_\rho(0))} \overset{\text{Def.}}{=} \int_{\mathbb{R}^N \setminus B_\rho(0)} |f(t, x)|^p \, dx \leq e^{\kappa t} \cdot \omega(\rho),$$
- $\lim_{\rho \to \infty} \omega(\rho) = 0$.

[58, Propositions 6 and 27] formulate sufficient conditions for the continuous dependence of the solution on the given data and lead to the following estimate:

**Lemma B.3.** Under the assumptions of Proposition B.1 consider the system of two nonautonomous linear transport equations
\[
\begin{align*}
\partial_t f^{(1)} + \text{div}_x(f^{(1)} g_1(t)) &= u_1(t) \cdot f^{(1)} + w_1(t), \\
\partial_t f^{(2)} + \text{div}_x(f^{(2)} g_2(t)) &= u_2(t) \cdot f^{(2)} + w_2(t).
\end{align*}
\]

Then there exists a constant $\hat{C} = \hat{C}(\gamma_a, \gamma_b, N, T) > 0$ such that for every $t \in [0, T]$,
\[
\begin{align*}
d_{L^p}(f^{(1)}(t), f^{(2)}(t)) \\
\leq e^{\hat{C} t} \cdot \left( d_{L^p}(f^{(1)}(0), f^{(2)}(0)) + \hat{C} \cdot (\|f^{(1)}(0)\|_{L^p} + \gamma_a T) \cdot \int_0^t \left( \|g_1(s) - g_2(s)\|_{L^p} + \|u_1(s) - u_2(s)\|_{L^p} + \|w_1(s) - w_2(s)\|_{L^p} \right) \, ds \right).
\end{align*}
\]

This estimate can be regarded as the key purpose why we assume all coefficient values $g(t)$, $u(t)$ to be in $L^q$. Indeed, the hypotheses $g(t) \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$, $u(t) \in W^{1,\infty}(\mathbb{R}^N)$ imply that they all belong to $L^q_{\text{loc}}$ anyway. A further step of approximation will lay the foundations for dispensing with the additional $L^q$ assumptions:

**Corollary B.4.** Consider the initial value problem (8) with the single-valued coefficient functions
\[
\begin{align*}
g : [0, T] &\to W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N) \subset L^q_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N), \\
u : [0, T] &\to W^{1,\infty}(\mathbb{R}^N) \subset L^q_{\text{loc}}(\mathbb{R}^N), \\
w : [0, T] &\to L^p(\mathbb{R}^N)
\end{align*}
\]
under the assumptions (i), (ii) of Proposition B.1 and

(iii') $g, u : [0, T] \to (L^q_{\text{loc}}, \delta_{L^q_{\text{loc}}})$ and $w : [0, T] \to (L^p, \|\cdot\|_{L^p})$ are Lebesgue measurable.
Then for every \( f_0 \in L^p(\mathbb{R}^N) \), the initial value problem (8) has a unique weak solution \( f : [0, T] \rightarrow L^p(\mathbb{R}^N) \). Moreover, \( f \) has the properties (1.) – (3.), (5.), (6.) in Proposition B.1.

Proof. The spatial regularity of the coefficients implies that every weak solution of transport equation (8) is even a renormalized solution (in the sense of Di PERNA and LIONS). This is formulated in [58, Proposition 19] and results from smoothing arguments presented in [35, 60].

Choose any smooth cut-off function \( \theta \in C^\infty(\mathbb{R}, [0, 1]) \) with \( \theta = 1 \) in \((-\infty, 1]\) and support in \((-\infty, 2]\). For each radius \( \rho > 0 \), we consider the modified coefficient functions

\[
\gamma_\rho := \theta(\frac{\|\cdot\|}{\rho}) \cdot g : [0, T] \rightarrow W^{1, \infty}(\mathbb{R}^N, \mathbb{R}^N) \cap L^q(\mathbb{R}^N, \mathbb{R}^N), t \mapsto \theta(\frac{\|\cdot\|}{\rho}) \cdot g(t, \cdot),
\]

\[
\gamma_\rho := \theta(\frac{\|\cdot\|}{\rho}) \cdot u : [0, T] \rightarrow W^{1, \infty}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N), t \mapsto \theta(\frac{\|\cdot\|}{\rho}) \cdot u(t, \cdot)
\]

whose values have compact support in \( B_{2\rho}(0) \subset \mathbb{R}^N \). Proposition B.1 ensures a renormalized solution \( f_\rho : [0, T] \rightarrow L^p(\mathbb{R}^N) \) of the nonautonomous linear transport equation

\[
\partial_t f_\rho + \text{div}_x (f_\rho \gamma_\rho(t)) = u_\rho(t) \cdot f_\rho + w(t), \quad f_\rho(0) = f_0.
\]

Its \( L^p \) norm is bounded uniformly w.r.t. \( \rho > 0 \) due to Proposition B.1 (1.). As a consequence of Corollary B.2, the subset \( \{\|f_\rho(t)\| \mid t \in [0, T], \rho > 0\} \subset L^1(\mathbb{R}^N) \) is tight. Corollaries A.11 and A.12 imply that \( \{f_\rho(t) \mid t \in [0, T], \rho > 0\} \subset L^p(\mathbb{R}^N) \) is relatively compact w.r.t. \( \text{d}_{L^p} \) and \( \tilde{\varepsilon}_{L^p} \). Furthermore, there is a constant \( \Lambda = \Lambda(N, p, \gamma_\alpha, \|f_0\|_{L^p}, \sup_{t} \|u(t)\|_{L^p(\mathbb{R}^N, \mathbb{R}^N)}) \) such that for every \( \rho > 0 \), the function \( f_\rho : [0, T] \rightarrow L^p(\mathbb{R}^N) \) is \( \Lambda \)-LIPSCHITZ continuous w.r.t. \( \varepsilon_{L^p} \) due to Prop. B.1 (2.).

As a consequence of the ARZELÀ-ASCOLI theorem in metric spaces (formulated in Lemma 3.7, see, e.g., [6, Theorem 4.4.3], [46]), there exist a sequence \( \rho_k \nearrow \infty \) and \( f : [0, T] \rightarrow L^p(\mathbb{R}^N) \) with \( \sup_{t \in [0, T]} \varepsilon_{L^p}(f_{\rho_k}(t), f(t)) \rightarrow 0 \) for \( k \rightarrow \infty \).

This function \( f \) proves to be a weak solution of initial value problem (8). Indeed, fix the test function \( \phi \in C^\infty_c(\mathbb{R}^N) \) arbitrarily and select a radius \( r = r(\phi) > 0 \) with \( \text{supp} \phi \subset B_r \subset \mathbb{R}^N \). By construction, we have \( \gamma_\rho = g \) and \( u_\rho = u \) in \( B_r \) for all \( \rho > r \) and thus, the weak solution property of \( f_\rho \) states for all \( t_1, t_2 \in [0, T] \) (\( t_1 < t_2 \)) and \( \rho > r \)

\[
\int_{\mathbb{R}^N} \phi (f(t_2) - f(t_1)) \, dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^N} f_\rho(s, x) \cdot \nabla_x \phi(x) \, dx \, ds
\]

\[
+ \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (f_\rho(s, x) u_\rho(s, x) + w(s, x)) \phi(x) \, dx \, ds
\]

\[
= \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (f_\rho \cdot \nabla_x \phi(x) + (f_\rho u + w) \phi(x)) \, dx \, ds.
\]

For \( \rho = \rho_k \) (\( k \in \mathbb{N} \)) in particular, the limit for \( k \rightarrow \infty \) reveals

\[
\int_{\mathbb{R}^N} \phi (f(t_2) - f(t_1)) \, dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (f \cdot \nabla_x \phi(x) + (f u + w) \phi(x)) \, dx \, ds
\]

due to \( g \cdot \nabla_x \phi, u \phi \in L^q \cap W^{1, \infty} \) with compact support, the pointwise convergence of \( (f_{\rho_k})_{k \in \mathbb{N}} \) to \( f \) w.r.t. \( \text{d}_{L^p} \) and Remark A.7.
Properties B.1 (1.) – (3.), (6.) of \( f \) result from its approximation by \( (f_\rho)_k \) \( \in \mathbb{N} \) and the equivalence between \( d_{L^p} \) and weak convergence in \( L^p(\mathbb{R}^N) \) (specified in Lemma A.9).

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Received December 2017; revised May 2018.

*E-mail address: thomas.lorenz@hs-rm.de*