A three generation oscillation analysis of the
Super-Kamiokande atmospheric neutrino data beyond one
mass scale dominance approximation

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Abstract
In this paper we do a three-generation oscillation analysis of the latest (1144 days) Super-
Kamiokande (SK) atmospheric neutrino data going beyond the one mass scale dominance
(OMSD) approximation. We fix $\Delta_{12} = \Delta_{13}$ ($\Delta_{LSND}$) in the range $\text{eV}^2$ as allowed by the
results from LSND and other accelerator and reactor experiments on neutrino oscillation and
keep $\Delta_{23}$ ($\Delta_{ATM}$) and the three mixing angles as free parameters. We incorporate the matter
effects, indicate some new allowed regions with small $\Delta_{23}$ ($< 10^{-4} \text{ eV}^2$) and $\sin^2 2\theta_{23}$ close to
0 and discuss the differences with the two-generation and OMSD pictures. In our scenario,
the oscillation probabilities for the accelerator and reactor neutrinos involve only two of the
mixing angles $\theta_{12}$ and $\theta_{13}$ and one mass scale. But the atmospheric neutrino oscillation is in
general governed by both mass scales and all the three mixing angles. The higher mass scale
gives rise to $\Delta m^2$ independent average oscillations for atmospheric neutrinos and does not
enter the $\chi^2$ analysis as an independent parameter. The $\Delta_{23}$ and the three mixing angles on
the other hand appear as independent parameters in the $\chi^2$ analysis and the best-fit values of
these are determined from an analysis of a) the SK data, b) the SK and CHOOZ data. The
allowed values of the mixing angles $\theta_{12}$ and $\theta_{13}$ from the above analysis are compared with
the constraints from all accelerator and reactor experiments including the latest results from
LSND and KARMEN2. Implications for future long baseline experiments are discussed.

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1 Introduction

The Super-Kamiokande results on atmospheric neutrino flux measurement show a deficit of the $\nu_\mu$ flux [1, 2]. Two generation analyses of the SK data show that the $\nu_\mu - \nu_\tau$ oscillation hypothesis provides a very good fit to the SK data [3, 4, 5]. The high statistics of SK also makes it possible to study the zenith-angle dependence of the neutrino flux from which one can conclude that the $\nu_\mu$'s show signs of oscillation but the $\nu_e$ events are consistent with the no-oscillation hypothesis. Independently the results from the reactor experiment CHOOZ [6] disfavors the $\nu_\mu - \nu_e$ oscillation hypothesis in a two-generation analysis. It is important however to see the implications of these results in a three-generation picture. The most popular three-generation picture in the context of the SK data is the scenario shown in fig. 1a, where one of the mass squared differences is in the solar neutrino range and the other is suitable for atmospheric neutrino oscillations [3, 7]. In such a scheme one mass scale dominance applies for atmospheric neutrinos and the relevant probabilities are functions of two of the mixing angles and one mass squared difference. This picture however cannot explain the LSND results [8]. In this paper we perform a three flavor $\chi^2$-analysis of the SK atmospheric neutrino data assuming a mass pattern with $\Delta_{12} \simeq \Delta_{13}$ fixed in the eV$^2$ range and allowing the other mass scale to vary arbitrarily. This mass pattern is shown in fig. 1b. Apart from being suitable to explain the SK atmospheric neutrino data this spectrum is also interesting for the laboratory based neutrino oscillation experiments as the higher mass scale is exploriable in the short base line experiments, whereas the lower mass scale can be probed in the long base line experiments. In this scheme to a good approximation, neutrino oscillation in the short-base line accelerators and reactors will be governed by one (the higher) mass scale [4, 10] – and only two of the mixing angles appear in the expressions for the oscillation probabilities. For the atmospheric and the long baseline experiments the characteristic energy and length scales are such that in general both mass differences are of relevance and the probabilities involve all

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$^4$The $\nu_\mu - \nu_\tau$ solution is now ruled out at 99% C.L. by the SK collaboration [2].
the three mixing angles. However the higher mass scale gives rise to $\Delta m^2$ independent average oscillations and it does not enter the $\chi^2$ fit directly. We determine the best-fit values of $\Delta_{23}$ and the three mixing angles by performing a $\chi^2$ analysis of

- the SK atmospheric neutrino data
- SK atmospheric and CHOOZ data

Finally we compare the allowed values of the mixing angles as obtained from the above analysis with those allowed by the other accelerator and reactor neutrino oscillation data including LSND and KARMEN2.

The mass scheme of this paper was first considered in [11, 12] after the declaration of the LSND result. These papers performed a combined three generation analysis of accelerator and reactor results as well as the Kamiokande atmospheric neutrino data. Three-generation picture with the higher mass difference in the eV$^2$ range and the lower mass difference in the atmospheric range has also been considered in [13, 14] (pre-SK) and [13, 16, 17, 18] (post-SK). These papers attempted to explain both solar and atmospheric neutrino anomalies mainly by maximal $\nu_\mu \leftrightarrow \nu_e$ oscillations driven by $\Delta_{ATM} \sim 10^{-3}$ eV$^2$. Although it was claimed in [15, 16] that this scenario can provide a good fit to all the available data on neutrino oscillations, it was shown in [17] and also later in [18] that this scenario cannot reproduce the zenith angle dependence of the SK atmospheric neutrino data.

In this paper our aim is to determine the allowed oscillation parameter ranges consistent with SK atmospheric, CHOOZ, LSND and other accelerator and reactor experiments. The solar neutrino problem can be explained by invoking a sterile neutrino. We discuss in the conclusions how the solar neutrino flux suppression can be explained in our scenario.

The plan of the paper is as follows. In section 2 we discuss very briefly the atmospheric neutrino code employed for the analysis of the SK data. In section 3 we present the formalism for three-generation oscillation analysis and calculate the required probabilities including the earth matter effects. In section 4 we present the $\chi^2$ analysis of
only SK atmospheric neutrino data. In section 5 we present the combined \(\chi^2\) analysis of SK and CHOOZ data. In section 6 we compare the allowed values of mixing angles from the above analyses with those allowed by the other accelerator and reactor data including the latest results from LSND and KARMEN2. In section 7 we discuss the implications of our results for the future long baseline experiments and end in section 8 with some discussions and conclusions.

2 The Atmospheric Neutrino Code

We define the quantities \(N_{\mu\text{osc}}\) and \(N_{e\text{osc}}\) as

\[
N_{\mu\text{osc}} = N_{\mu\mu} + N_{e\mu}
\]

\[
N_{e\text{osc}} = N_{ee} + N_{\mu e}
\]

\(N_{e,\mu\text{osc}}\) are the numbers of \(e\)-like and \(\mu\)-like events in the detector and \(N_{l'}\) is defined as

\[
N_{l'} = n_T \int_0^\infty dE \int_{(E_{l'})_{\text{min}}}^{(E_{l'})_{\text{max}}} dE_{l'} \int_{-1}^{+1} d\cos \psi \int_{-1}^{+1} d\cos \xi \frac{1}{2\pi} \int_{0}^{2\pi} d\phi \times \frac{d^2 F_l(E,\xi)}{dE \, d\cos \xi} \cdot \frac{d^2 \sigma_{l'}(E, E_{l'}, \cos \psi)}{dE_{l'} \, d\cos \psi} \epsilon(E_{l'}) \cdot P_{\nu_l \nu_{l'}}(E, \xi).
\]

\(n_T\) denotes the number of target nucleons, \(E\) is the neutrino energy, \(E_{l'}\) is the energy of the final charged lepton, \(\psi\) is the angle between the incoming neutrino \(\nu_l\) and the scattered lepton \(l'\), \(\xi\) is the zenith angle of the neutrino and \(\phi\) is the azimuthal angle corresponding to the incident neutrino direction. The zenith angle of the charged lepton is given by

\[
\cos \Theta = \cos \xi \cos \psi + \sin \xi \cos \phi \sin \psi
\]

\(d^2 \sigma_{l'}/dE_{l'}d\cos \psi\) is the differential cross section for \(\nu_{l'} N \to l'X\) scattering, \(\epsilon(E_{l'})\) is the detection efficiency for the 1 ring events in the detector and \(P_{\nu_l \nu_{l'}}\) is the probability of a neutrino flavour \(l\) to convert to a neutrino of flavour \(l'\). We use the atmospheric neutrino fluxes \(\frac{d^2 F_l(E,\xi)}{dE \, d\cos \xi}\) from [19]. For further details regarding the calculation of number of events we refer to [20].
3 Three-Flavor Analysis

3.1 The vacuum oscillation probabilities

The general expression for the probability that an initial $\nu_\alpha$ of energy $E$ gets converted to a $\nu_\beta$ after traveling a distance $L$ in vacuum is given by,

$$P(\nu_\alpha, 0; \nu_\beta, t) = \delta_{\alpha\beta} - 4 \sum_{j>i} U_{\alpha i} U_{\beta i} U_{\alpha j} U_{\beta j} \sin^2 \left( \frac{\pi L}{\lambda_{ij}} \right)$$  \hspace{1cm} (3)

where $\lambda_{ij}$ is defined to be the neutrino vacuum oscillation wavelength given by,

$$\lambda_{ij} = (2.47 \text{ m}) \left( \frac{E}{\text{MeV}} \right) \left( \frac{\text{eV}}{\Delta m^2} \right)$$ \hspace{1cm} (4)

which denotes the scale over which neutrino oscillation effects can be significant and $\Delta m^2 = | m_j^2 - m_i^2 |$. The actual forms of the various survival and transition probabilities depend on the spectrum of $\Delta m^2$ assumed and the choice of the mixing matrix $U$ relating the flavor eigenstates to the mass eigenstates. We choose the flavor states $\alpha = 1, 2, \text{ and } 3$ to correspond to e, $\mu$ and $\tau$ respectively. The most suitable parameterization of $U$ for the mass spectrum chosen by us is $U = R_{13} R_{12} R_{23}$ where $R_{ij}$ denotes the rotation matrix in the $ij$-plane. This yields:

$$U = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} c_{23} - s_{13} s_{23} & c_{13} s_{12} s_{23} + s_{13} c_{23} \\ -s_{12} & c_{12} c_{23} & c_{12} s_{23} \\ -s_{13} c_{12} & -s_{13} s_{12} c_{23} - c_{13} s_{23} & -s_{13} s_{12} s_{23} + c_{13} c_{23} \end{pmatrix}$$ \hspace{1cm} (5)

where $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$ here and everywhere else in the paper. We have assumed CP-invariance so that $U$ is real. The above choice of $U$ has the advantage that $\theta_{23}$ does not appear in the expressions for the probabilities for the laboratory experiments \[12\].

The probabilities relevant for atmospheric neutrinos are

$$P_{\nu_e, \nu_e} = 1 - 2 c_{13}^2 c_{12}^2 + 2 c_{13}^4 c_{12}^4 - 4 (c_{13} s_{12} c_{23} - s_{13} s_{23})^2 (c_{13} s_{12} s_{23} + s_{13} c_{23})^2 S_{23}$$ \hspace{1cm} (6a)

$$P_{\nu_\mu, \nu_e} = 2 c_{13}^2 c_{12}^2 s_{12}^2 - 4 c_{13}^4 c_{12}^4 c_{23} s_{23} (c_{13} s_{12} c_{23} - s_{13} s_{23}) (c_{13} s_{12} s_{23} + s_{13} c_{23}) S_{23}$$ \hspace{1cm} (6b)

$$P_{\nu_\mu, \nu_\mu} = 1 - 2 c_{12}^2 s_{12}^2 - 4 c_{12}^4 c_{23}^2 s_{23}^2 S_{23}$$ \hspace{1cm} (6c)
where $S_{23} = \sin^2(\pi L/\lambda_{23})$. Apart from the most general three generation regime, the following limits are of interest, as we will see later in the context of the SK data:

1. The two-generation limits

Because of the presence of more parameters as compared to the one mass scale dominance picture there are twelve possible two-generation limits [21] with the oscillations driven by either $\Delta_{LSND}$ or $\Delta_{ATM}$. Below we list these limits specifying the mass scales that drive the oscillations:

- $s_{12} \to 0, s_{13} \to 0$ \quad ($\nu_{\mu} - \nu_{\tau}$, $\Delta_{ATM}$)
- $s_{12} \to 1, s_{13} \to 0$ \quad ($\nu_{e} - \nu_{\tau}$, $\Delta_{ATM}$)
- $s_{12} \to 0, s_{13} \to 1$ \quad ($\nu_{\mu} - \nu_{e}$, $\Delta_{ATM}$)
- $s_{12} \to 1, s_{13} \to 1$ \quad ($\nu_{e} - \nu_{\tau}$, $\Delta_{ATM}$)
- $s_{13} \to 0, s_{23} \to 0$ \quad ($\nu_{\mu} - \nu_{e}$, $\Delta_{LSND}$)
- $s_{13} \to 0, s_{23} \to 1$ \quad ($\nu_{\mu} - \nu_{e}$, $\Delta_{LSND}$)
- $s_{13} \to 1, s_{23} \to 0$ \quad ($\nu_{\mu} - \nu_{\tau}$, $\Delta_{LSND}$)
- $s_{13} \to 1, s_{23} \to 1$ \quad ($\nu_{\mu} - \nu_{\tau}$, $\Delta_{LSND}$)
- $s_{12} \to 0, s_{23} \to 0$ \quad ($\nu_{e} - \nu_{\tau}$, $\Delta_{LSND}$)
- $s_{12} \to 0, s_{23} \to 1$ \quad ($\nu_{e} - \nu_{\tau}$, $\Delta_{LSND}$)
- $s_{12} \to 1, s_{23} \to 0$ \quad ($\nu_{e} - \nu_{\tau}$, $\Delta_{ATM}$)
- $s_{12} \to 1, s_{23} \to 1$ \quad ($\nu_{e} - \nu_{\tau}$, $\Delta_{ATM}$)

2. $s_{12}^2 = 0.0$

In this limit the relevant probabilities become

\[
\begin{align*}
P_{\nu_{e}\nu_{e}} &= 1 - 2c^2_{13}s^2_{13} + 4s^2_{13} c^2_{23} s^2_{23} S_{23} \\ 
P_{\nu_{e}\nu_{\mu}} &= 4s^2_{13} c^2_{23} S_{23} \\ 
P_{\nu_{\mu}\nu_{\mu}} &= 1 - 4c^2_{23} s^2_{23} S_{23}
\end{align*}
\]

Thus $P_{\nu_{e}\nu_{\mu}}$ is the same as the two generation limit, $P_{\nu_{e}\nu_{e}}$ is governed by two of the mixing angles and one mass scale and $P_{\nu_{\mu}\nu_{e}}$ is governed by two mixing angles and both mass scales.
3. $s_{13}^2 = 0.0$

For this case the probabilities take the form

$$P_{\nu_e \nu_e} = 1 - 2c_{12}^2 s_{12}^2 - 4s_{12}^4 c_{23}^2 s_{23}^2 S_{23} \quad (8a)$$

$$P_{\nu_e \nu_\mu} = 2c_{12}^2 s_{12}^2 - 4c_{12}^2 s_{12}^2 c_{23}^2 S_{23} \quad (8b)$$

$$P_{\nu_\mu \nu_\mu} = 1 - 2c_{12}^2 s_{12}^2 - 4c_{12}^4 c_{23}^2 s_{23}^2 S_{23} \quad (8c)$$

In this case the probabilities are governed by two mass scales and two mixing angles.

We note that for cases (2) and (3) the probabilities are symmetric under the transformation $\theta_{23} \rightarrow \pi/2 - \theta_{23}$. The probabilities for these cases are functions of at most two mixing angles as in the OMSD case but they are governed by both mass scales making these limits different from the OMSD limit.

### 3.2 Earth matter effects

Since on their way to the detector the upward going neutrinos pass through the earth, it is important in general to include the matter effect in the atmospheric neutrino analysis.

The matter contribution to the effective squared mass of the electron neutrinos:

$$A = 2\sqrt{2} G_F E n_e \quad (9)$$

where $E$ is the neutrino energy and $n_e$ is the ambient electron density. Assuming a typical density of 5 gm/cc and $E = 10$ GeV, the matter potential $A \simeq 3.65 \times 10^{-3}$ eV$^2$ and since this is of the same order as $\Delta_{23}$ in our case, matter effects should be studied carefully.

The mass matrix in the flavor basis in presence of matter is given by

$$M_F^2 = U M^2 U^\dagger + M_A \quad (10)$$

where $M^2$ is the mass matrix in the mass eigenbasis, $U$ is the mixing matrix and

$$M_A = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (11)$$
Since $\Delta_{12} \sim \Delta_{13} \gg \Delta_{23} \sim A$, one can solve the eigenvalue problem using the degenerate perturbation theory, where the $\Delta_{23}$ and $A$ terms are treated as a perturbation to the dominant $\Delta_{12}$ and $\Delta_{13}$ dependent terms. The mixing angle in matter is then given by

$$
\tan 2\theta_{23}^{M} = \frac{\Delta_{23} \sin 2\theta_{23} - As_{12} \sin 2\theta_{13}}{\Delta_{23} \cos 2\theta_{23} - A(s_{13}^{2} - c_{13}^{2}s_{12}^{2})} \tag{12}
$$

while the mass squared difference in matter turns out to be

$$
\Delta_{23}^{M} = \left[ (\Delta_{23} \cos 2\theta_{23} - A(s_{13}^{2} - c_{13}^{2}s_{12}^{2}))^{2} + (\Delta_{23} \sin 2\theta_{23} - As_{12} \sin 2\theta_{13})^{2} \right]^{1/2} \tag{13}
$$

The mixing angles $\theta_{12}$ and $\theta_{13}$ as well as the larger mass squared difference $\Delta_{12}$ remain unaltered in matter. From eq. (12) and (13) we note the following:

- In the limit of both $s_{12} \to 0$ and $s_{13} \to 0$, the matter effect vanishes and we recover the two-generation $\nu_{\mu} - \nu_{\tau}$ limit.

- The resonance condition now becomes $\Delta_{23} \cos 2\theta_{23} = A(s_{13}^{2} - c_{13}^{2}s_{12}^{2})$. So that for $\Delta_{23} > 0$, one can have resonance for both neutrinos – if $s_{13}^{2} > c_{13}^{2}s_{12}^{2}$ – as well as for antineutrinos – if $s_{13}^{2} < c_{13}^{2}s_{12}^{2}$. This is different from the OMSD picture where for $\Delta m^{2} > 0$ only neutrinos can resonate [22].

- In the limit of $s_{12} \to 0$

$$
\tan 2\theta_{23}^{M} = \frac{\Delta_{23} \sin 2\theta_{23}}{\Delta_{23} \cos 2\theta_{23} - As_{13}^{2}} \tag{14}
$$

Here one gets resonance for neutrinos only (if $\Delta_{23} > 0$) and this is similar to the OMSD case.

- In the limit $s_{13} \to 0$

$$
\tan 2\theta_{23}^{M} = \frac{\Delta_{23} \sin 2\theta_{23}}{\Delta_{23} \cos 2\theta_{23} + As_{12}^{2}} \tag{15}
$$

For this case for $\Delta_{23} > 0$, there is no resonance for neutrinos but antineutrinos can resonate.
In the limit where $\Delta_{23} \rightarrow 0$

$$\tan 2\theta_{23}^M = \frac{s_{12} \sin 2\theta_{13}}{s_{13}^2 - c_{13}^2 s_{12}^2}, \quad \Delta_{23}^M = A(s_{13}^2 + c_{13}^2 s_{12}^2)$$  \hspace{1cm} (16)

Thus even for small values of $\Delta_{23} < 10^{-4}$ the mass squared difference in matter is $\sim A$ and one may still hope to see oscillations for the upward neutrinos due to matter effects. The other point to note is that the mixing angle in matter $\theta_{23}^M$ depends only on $\theta_{12}$ and $\theta_{13}$ and is independent of the vacuum mixing angle $\theta_{23}$ and $\Delta_{23}$. Contrast this with the OMSD case (where the expressions for $\tan 2\theta_{23}^M$ is given by an expression similar to eq. [14] [22]) and the two-generation $\nu_\mu - \nu_e$ oscillations. For both the two-generation $\nu_\mu - \nu_e$ as well as the three-generation OMSD case, for $\Delta_{23} \rightarrow 0$, the mixing angle $\tan 2\theta_{23}^M \rightarrow 0$, but for the mass spectrum considered in this paper the $\tan 2\theta_{23}^M$ maybe large depending on the values of $s_{12}^2$ and $s_{13}^2$. Hence we see that the demixing effect which gives the lower bound on allowed values of $\Delta m^2$ in the two generation $\nu_\mu - \nu_e$ or the three-generation OMSD case, does not arise here and we hope to get allowed regions even for very low values of $\Delta_{23}$. On the other hand even small values of $\theta_{23}$ in vacuum can get enhanced in matter. This special case where $\Delta_{23} \sim 0$ was considered in an earlier paper [23].

In the limit of $s_{23}^2 \rightarrow 0$

$$\tan 2\theta_{23}^M = \frac{-A s_{12} \sin 2\theta_{13}}{\Delta_{23} - A(s_{13}^2 - c_{13}^2 s_{12}^2)}$$  \hspace{1cm} (17)

While for $s_{23}^2 \rightarrow 1$

$$\tan 2\theta_{23}^M = \frac{-A s_{12} \sin 2\theta_{13}}{-\Delta_{23} - A(s_{13}^2 - c_{13}^2 s_{12}^2)}$$  \hspace{1cm} (18)

For the last two cases, corresponding to $\sin^2 2\theta_{23} \rightarrow 0$, again the mixing angle $\theta_{23}$ in matter is independent of its corresponding value in vacuum and hence for appropriate choices of the other three parameters, $\Delta_{23}$, $s_{12}^2$ and $s_{13}^2$, one can get large values for $\sin^2 2\theta_{23}^M$ even though the vacuum mixing angle is zero.
The amplitude that an initial $\nu_\alpha$ of energy $E$ is detected as $\nu_\beta$ after traveling through the earth is

$$A(\nu_\alpha, t_0, \nu_\beta, t) = \sum_{\sigma, \lambda, \rho} \sum_{i,j,k,l} \left[ (U_{J\sigma}^{M_m} e^{-iE_{J}^{M_m}(t-t_3)} U_{M_m}^{\sigma l}) (U_{M_m}^{\sigma k} e^{-iE_{K}^{M_m}(t_3-t_2)} U_{M_m}^{\lambda k}) \right] \times \left[ (U_{\lambda j}^{M_m} e^{-iE_{J}^{M_m}(t_2-t_1)} U_{M_m}^{\rho j}) (U_{\rho i} e^{-iE_{i}(t_1-t_0)} U_{\alpha i}) \right]$$

(19)

where we have considered the earth to be made of two slabs, a mantle and a core with constant densities of 4.5 gm/cc and 11.5 gm/cc respectively and include the non-adiabatic effects at the boundaries. The mixing matrix in the mantle and the core are given by $U^{M_m}$ and $U^{M_c}$ respectively. $E_i^X \approx m_{iX}^2/2E$, $X = \text{core( mantle)}$ and $m_{iX}$ is the mass of the $i^{th}$ neutrino state in the core( mantle). The neutrino is produced at time $t_0$, hits the earth mantle at $t_1$, hits the core at $t_2$, leaves the core at $t_3$ and finally hits the detector at time $t$. The Greek indices $(\sigma, \lambda, \rho)$ denote the flavor eigenstates while the Latin indices $(i, j, k, l)$ give the mass eigenstates. The corresponding expression for the probability is given by

$$P(\nu_\alpha, t_0, \nu_\beta, t) = |A(\nu_\alpha, t_0, \nu_\beta, t)|^2$$

(20)

For our calculations of the number of events we have used the full expression given by eq.(19) and (20).

4 $\chi^2$-analysis of the SK data

We minimize the $\chi^2$ function defined as [3, 4]

$$\chi^2 = \sum_{i,j=1,40} \left( N_i^{th} - N_i^{exp} \right) (\sigma_{ij}^{-2}) \left( N_j^{th} - N_j^{exp} \right)$$

(21)

where the sum is over the sub-GeV and multi-GeV electron and muon bins. The experimentally observed number of events are denoted by the superscript “exp” and the theoretical predictions for the quantities are labeled by “th”. The element of the error matrix $\sigma_{ij}$ is calculated as in [3], including the correlations between the different
bins. For contained events there are forty experimental data points. The probabilities for the atmospheric neutrinos are explicit functions of one mass-squared difference and three mixing angles making the number of degrees of freedom (d.o.f) 36. The other mass squared difference gives rise to $\Delta m^2$ independent average oscillations and hence does not enter the fit as an independent parameter.

For two-flavour $\nu_\mu - \nu_\tau$ oscillation the 1144 days of data gives the following best-fits and $\chi^2_{\text{min}}$:

- $\chi^2_{\text{min}}/\text{d.o.f.} = 36.23/38, \Delta m^2 = 0.0027 \text{ eV}^2, \sin^2 2\theta = 1.0$

This corresponds to a goodness of fit of 55.14%.

For the general three-generation scheme the $\chi^2_{\text{min}}$ and the best-fit values of parameters that we get are

- $\chi^2_{\text{min}}/\text{d.o.f.} = 34.65/36, \Delta_{23} = 0.0027 \text{ eV}^2, s_{23}^2 = 0.51, s_{12}^2 = 0.04$ and $s_{13}^2 = 0.06$

This solution is allowed at 53.28% C.L.

The solid(dashed) lines in fig. 2 present the variation of the $\Delta \chi^2 = \chi^2 - \chi^2_{\text{min}}$ for the SK data, with respect to one of the parameters keeping the other three unconstrained, when we include(exclude) the matter effect. In fig. 2(a) as we go towards smaller values of $\Delta_{23}$ around $10^{-3} \text{ eV}^2$ the effect of matter starts becoming important as the matter term is now comparable to the mass term. If matter effects are not there then for values of $\Delta_{23} \lesssim 10^{-4} \text{ eV}^2$ the $S_{23}$ term in eq.(6) is very small and there is no up-down asymmetry resulting in very high values of $\chi^2$ as is evident from the dashed curve. If the matter effects are included, then in the limit of very low $\Delta_{23}$ the matter term dominates and $\Delta_{23}^M$ is given by eq.(16). Since this term $\sim 10^{-3} \text{ eV}^2$ there can be depletion of the neutrinos passing through the earth causing an updown asymmetry. For $\Delta_{23}$ around $10^{-4} \text{ eV}^2$, there is cancellation between the two comparable terms in the numerator of eq. (12) and the mixing angle becomes very small and hence the $\chi^2$ around these values of $\Delta_{23}$ comes out to be very high.

Fig 2(b) illustrates the corresponding variation of $\Delta \chi^2$ with $s_{23}^2$ while the other three parameters are allowed to vary arbitrarily. For small and large values of $s_{23}^2$
the inclusion of matter effect makes a difference. For $s_{23}^2$ either very small or large ($\sin^2 2\theta_{23} \rightarrow 0$) the overall suppression of the $\nu_\mu$ flux is less than that required by the data if vacuum oscillation is operative and so it is ruled out. If we include matter effects then in the limit of $s_{23}^2 = 0$ and $s_{23}^2 = 1$ the matter mixing angle is given by eqs. (17) and (18), which can be large for suitable values of $s_{13}^2$ and $s_{12}^2$ and hence one gets lower $\chi^2$ even for these values of $s_{23}^2$.

In figs 2(c) and 2(d) we show the effect of $s_{12}^2$ and $s_{13}^2$ respectively on $\Delta \chi^2$. From the solid and the dashed lines it is clear that matter effects do not vary much the allowed ranges of $s_{12}^2$ and $s_{13}^2$.

The dashed-dotted line in the figure shows the 99% C.L. (= 13.28 for 4 parameters) limit. In Table 1 we give the allowed ranges of the mixing parameters, inferred from fig. 2 at 99% C.L. for the SK atmospheric data, with and without matter effects.

Table 1: The allowed ranges of parameters for the SK data.

|                  | $\Delta_{23}$ in eV$^2$ | $s_{23}^2$       | $s_{12}^2$ | $s_{13}^2$ |
|------------------|--------------------------|------------------|------------|------------|
| with matter effects | $1.6 \times 10^{-4} \leq \Delta_{23} \leq 7.0 \times 10^{-3}$ | $0.26 \leq s_{23}^2 \leq 0.77$ | $s_{12}^2 \leq 0.21$ | $s_{13}^2 \leq 0.55$ |
|                  | $\Delta_{23} \leq 6.5 \times 10^{-5}$ | $s_{23}^2 \geq 0.85$ |            |            |
| without matter effects | $5 \times 10^{-4} \leq \Delta_{23} \leq 7.0 \times 10^{-3}$ | $0.27 \leq s_{23}^2 \leq 0.74$ | $s_{12}^2 \leq 0.21$ | $s_{13}^2 \leq 0.6$ |

4.1 Zenith-Angle distribution

Since the probabilities in our case are in general governed by two mass scales and all three mixing angles it is difficult to understand the allowed regions. To facilitate the qualitative understanding we present in fig. 3 the histograms which describe the zenith angle distribution. The event distributions in these histograms are approximately given by,

$$\frac{N_\mu}{N_{\mu_0}} \approx P_{\nu_\mu \nu_\mu} + \frac{N_{\nu_0}}{N_{\mu_0}} P_{\nu_e \nu_\mu}$$  \hspace{1cm} (22)

$$\frac{N_e}{N_{e_0}} \approx P_{\nu_e \nu_e} + \frac{N_{\mu_0}}{N_{e_0}} P_{\nu_\mu \nu_e}$$ \hspace{1cm} (23)
where the quantities with suffix 0 indicates the no-oscillation values. For the sub-GeV data $N_{\mu_\text{e}}/N_{\mu_0} \approx 2$ to a good approximation however for the multi-GeV data this varies in the range 2 (for $\cos \Theta = 0$) to 3 (for $\cos \Theta = \pm 1$) \cite{3}.

In fig. 3a we study the effect of varying $s_{12}^2$ and $s_{13}^2$ for fixed values of $\Delta_{23} = 0.002$ eV$^2$ and $s_{23}^2 = 0.5$. From eq. (12), (13) and from fig. 2 we see that for the values of the $\Delta_{23}$ and $s_{23}^2$ considered in this figure the matter effects are small and we can understand the histograms from the vacuum oscillation probabilities. The thick solid line shows the event distribution for $s_{12}^2 = 0$ and $s_{13}^2 = 0.1$. As $s_{13}^2$ increases from 0, keeping $s_{12}^2$ as 0, from eqs. (9) $P_{\nu_e\nu_e}$ decreases from 1 and $P_{\nu_e\nu_\mu}$ increases from zero resulting in a net electron depletion according to eq. (23). The long dashed line corresponds to $s_{13}^2 = 0.3$ for which the electron depletion is too high as compared to data. The muon events are also affected as $P_{\nu_e\nu_e}$ increases with increasing $s_{13}^2$ even though $P_{\nu_e\nu_\mu}$ is independent of $s_{13}^2$. On the other hand for $s_{13}^2 = 0.0$, the effect of increasing $s_{12}^2$ is to increase the number of electron events and decrease the number of muon events according to eqs. (10), (23) and (22). This is shown by the short-dashed and dotted lines in fig. 3a. For $s_{12}^2 = 0.2$ the electron excess and muon depletion both becomes too high as compared to the data. For the case when both $s_{12}^2$ and $s_{13}^2$ are 0.1 the electron depletion caused by increasing $s_{12}^2$ and the excess caused by increasing $s_{13}^2$ gets balanced and the event distributions are reproduced quite well, shown by the dashed-dotted line.

In fig. 3b we study the effect of varying $s_{23}^2$ and $\Delta_{23}$ in the limit of $s_{12}^2 = 0$ with $s_{13}^2$ fixed at 0.1. Although we use the full probabilities including the matter effect, for 0.004 eV$^2$ this is not so important and one can understand the histograms from the vacuum oscillation probabilities. For fixed $\Delta_{23}$ as $s_{23}^2$ increases, $P_{\nu_e\nu_e}$ decreases, making the muon depletion higher. This is shown in the figure for two representative values of $\Delta_{23}$. The electron events are not affected much by change of $s_{23}^2$. The slight increase with $s_{23}^2$ is due to increase of both $P_{\nu_e\nu_e}$ and $P_{\nu_\mu\nu_\mu}$. To understand the dependence on $\Delta_{23}$ we note that for $s_{23}^2 = 0.2$, if one looks at the vacuum oscillation probabilities, $N_\mu/N_{\mu_0} \approx 1 - 0.65S_{23}$. For 0.004 eV$^2$ the contribution of $S_{23}$ is more resulting in a lower number of muon events. For the electron events however the behavior with
\(\Delta_{23}\) is opposite, with \(N_e/N_{e0} = 0.82 + 0.12S_{23}\). Thus with increasing \(\Delta_{23}\) the number of electron events increase. Also note that since the contribution of \(S_{23}\) comes with opposite sign the zenith-angle distribution for a fixed \(\Delta_{23}\) is opposite for the muon and the electron events.

In fig. 3c we show the histograms in the limit of \(s_{13}^2 = 0.0\), keeping \(s_{12}^2 = 0.1\) and varying \(\Delta_{23}\) and \(s_{23}^2\). As \(s_{23}^2\) increases all the relevant probabilities decrease and therefore both \(N_\mu/N_{\mu0}\) and \(N_e/N_{e0}\) decrease giving less number of events for both. For this case the \(S_{23}\) term comes with the same sign (negative) in both \(N_\mu/N_{\mu0}\) and \(N_e/N_{e0}\). Therefore the depletion is more for higher \(\Delta_{23}\) for both muon and electron events.

Finally, the long dashed line in fig. 3d represent the histograms for the best-fit value for two-generation \(\nu_\mu - \nu_\tau\) oscillations, for which \(P_{\nu_e\nu_e} = 1\). The short dashed line gives the histograms for the three-generation best-fit values. Both give comparable explanation for the zenith angle distribution of the data. The dotted line gives the event distribution for \(\Delta_{23} = 10^{-5}\) eV\(^2\). As discussed in section 3.2 even for such low value of \(\Delta_{23}\), we find that due to the unique feature of the beyond OMSD neutrino mass spectrum, earth matter effects ensure that both the sub-GeV as well as the multi-GeV upward muon events are very well reproduced, as are the electron events. But since \(s_{12}^2\) is high, the downward \(\nu_\mu\) are depleted more than the data requires (eq. (6)).

### 4.2 Allowed parameter region

In fig. 4a the solid lines give the 99% C.L. allowed area from SK data in the \(\Delta_{23}-s_{23}^2\) plane keeping the values of \(s_{13}^2\) and \(s_{12}^2\) fixed in the allowed range from fig. 2 and Table 1. The first panel represents the two-generation \(\nu_\mu - \nu_\tau\) oscillation limit modulo the difference in the definition of the C.L. limit as the number of parameters are different. We have seen from the histograms in fig. 3a that raising \(s_{12}^2\) results in electron excess and muon depletion. On the other hand increase in \(s_{13}^2\) causes electron depletion. The above features are reflected in the shrinking and disappearance of the allowed regions in the first row and column. In the panels where both \(s_{12}^2\) and \(s_{13}^2\) are nonzero one may
get allowed regions only when the electron depletion due to increasing $s_{13}^2$ is replenished by the increase in $s_{12}^2$.

In fig. 4b we present the 99% C.L. allowed areas in the bilogarithmic $\tan^2\theta_{12} - \tan^2\theta_{13}$ plane for various fixed values of the parameters $\Delta_{23}$ and $s_{23}^2$. We use the log(tan) representation which enlarges the allowed regions at the corners and the clarity is enhanced. The four corners in this plot refer to the two-generation limits discussed in section 3. The extreme left corner ($\theta_{12} \to 0, \theta_{13} \to 0$) correspond to the two generation $\nu_\mu - \nu_\tau$ oscillation limit. As we move up increasing $\theta_{13}$, one has $\nu_e - \nu_\mu$ and $\nu_e - \nu_\tau$ mixing in addition and for $s_{13}^2 \to 1$ one goes to the two generation $\nu_\mu - \nu_e$ oscillation region. For the best-fit values of $\Delta_{23}$ and $s_{23}^2$ if we take $s_{12}^2$ and $s_{13}^2$ to be 0 and 1 respectively, then the $\chi^2_{min}$ is 66.92 which is therefore ruled out. Both the right hand corners in all the panels refer to pure $\nu_e - \nu_\tau$ oscillations and therefore there are no allowed regions in these zones. For the panels in the first row, $\Delta_{23} = 0.006$ eV$^2$ and the 2-generation $\nu_\mu - \nu_\tau$ oscillation limit is just disallowed (this can also be seen in the first panel of fig. 4a). The small area allowed for the middle panel of first row (between the solid lines) is due to the fact that for non-zero $s_{12}^2$ and $s_{13}^2$ the electron events are better reproduced, while $s_{23}^2 = 0.5$ takes care of the muon events. Hence for this case slight mixture of $\nu_\mu - \nu_e$ and $\nu_e - \nu_\tau$ oscillations is favoured. This feature was also reflected in the fact that in the fig. 4a, the panel for $s_{12}^2 = 0.1$ and $s_{13}^2 = 0.3$ has more allowed range for $\Delta_{23}$ than the panel for the 2-generation $\nu_\mu - \nu_\tau$ limit. For the panels with $\Delta_{23} = 0.002$ eV$^2$, both the pure $\nu_\mu - \nu_\tau$ limit as well as full three-generation oscillations, give good fit. For the last two rows with $\Delta_{23} = 0.0007$ eV$^2$ and $0.0004$ eV$^2$ the matter effects are important in controlling the shape of the allowed regions. Infact the allowed region that one gets for $0.0004$ eV$^2$ and $s_{23}^2 = 0.5$ is the hallmark of the matter effect in this particular three-generation scheme. As can be seen from fig. 2a and Table 1, if one does not include the matter effect, then there are no allowed regions below $\Delta_{23} = 0.0005$ eV$^2$ for any arbitrary combination of the other three parameters. Even for the first and the last panels with $\Delta_{23} = 0.0007$ eV$^2$, one gets allowed areas solely due to matter effects.
In fig. 4c the solid lines show the 99% C.L. allowed regions from SK data in the $s_{23}^2 - s_{12}^2$ plane for fixed values of $\Delta_{23}$ and $s_{13}^2$. In contrast to the previous figure, here (and in the next figure) we use the sin$-\sin$ representation because the allowed regions are around $\theta_{23} = \pi/4$ and this region gets compressed in the log($\tan$)$-\log(\tan)$ representation. For explaining the various allowed regions we separate the figures in two sets:

- For $s_{13}^2 = 0.0$, the four corners of the panels represent the no-oscillation limits inconsistent with the data. Also as discussed in section 3 for $s_{23}^2 = 0.0$ or 1.0 one goes to the limit of pure $\nu_\mu - \nu_e$ conversions driven by $\Delta_{LSND}$, which is not consistent with data. One obtains allowed regions only when $s_{23}^2$ is close to 0.5 with $s_{12}^2$ small, so that $\nu_\mu - \nu_\tau$ conversions are dominant. The allowed range of $s_{12}^2$ is controlled mainly by the electron excess as has been discussed before while the allowed range of $s_{23}^2$ is determined mostly by the muon depletion.

- For $s_{13}^2 \neq 0$, the four corners represent the two-generation $\nu_e - \nu_\tau$ oscillation limit discussed in section 3 and hence these corners are not allowed. For $s_{23}^2 = 0.0$ or 1.0 and $s_{12}^2 \neq 0$ or 1 one has $\Delta_{LSND}$ driven $\nu_\mu - \nu_e$ and $\nu_\mu - \nu_\tau$ conversion and $\Delta_{ATM}$ driven $\nu_e - \nu_\tau$ conversions. This scenario is not allowed as it gives excess of electron events and also fails to reproduce the correct zenith angle dependence. For a fixed $\Delta_{23}$ as $s_{13}^2$ increases the electron depletion increases which can be balanced by increasing $s_{12}^2$ which increases the number of electron events. Hence for a fixed $\Delta_{23}$ the allowed regions shift towards higher $s_{12}^2$ values.

As in fig. 4b the allowed area in the middle panel of the last row is due to the inclusion of the matter effect.

In fig. 4d the solid contours refer to the 99% C.L. allowed areas from SK atmospheric neutrino data in the $s_{13}^2 - s_{23}^2$ plane for various values of $\Delta_{23}$ and $s_{12}^2$.

- For $s_{12}^2 = 0.0$ the corners represent no oscillation limits. In the limit $s_{23}^2 \rightarrow 0$ or 1, one gets $\nu_e - \nu_\tau$ oscillation driven by $\Delta_{LSND}$ which is also not allowed. For $s_{13}^2$...
= 0.0 and $s_{23}^2 \sim 0.5$ one has maximal two-flavour $\nu_\mu - \nu_\tau$ oscillation limit which is therefore allowed (not allowed for $\Delta_{23} = 0.006$ eV$^2$ as discussed before). As $s_{13}^2$ increases the electron depletion becomes higher and that restricts higher $s_{13}^2$ values.

- For $s_{12}^2 \neq 0$, the four corners represent two-generation limits driven by $\Delta_{LSND}$. This is the regime of average oscillations and cannot explain the zenith angle dependence of the data. For a fixed $\Delta_{23}$ the allowed region first expands and then shrinks in size and also shifts towards higher $s_{13}^2$ values as $s_{12}^2$ increases just as in fig. 4c.

Matter effect is important for the last two rows and the increase in the allowed areas for the last two panels of $\Delta_{23} = 0.0004$ eV$^2$ are typical signatures of matter effect.

In fig. 4e we present the allowed range in the $\Delta_{23} - s_{23}^2$ plane with $\Delta_{23}$ in the $10^{-5} - 10^{-4}$ eV$^2$ range and $s_{12}^2$, $s_{13}^2$ fixed at 0.185 and 0.372 respectively. We get allowed regions in this range of small $\Delta_{23}$ and small mixing due to matter effects – a feature unique to the mass spectrum considered in this paper.

5 $\chi^2$ analysis of the SK + CHOOZ data

The CHOOZ experiment can probe upto $10^{-3}$ eV$^2$ and hence it can be important to cross-check the atmospheric neutrino results. In particular a two-generation analysis shows that CHOOZ data disfavours the $\nu_\mu - \nu_e$ solution to the atmospheric neutrino problem. The general expression for the survival probability of the electron neutrino in presence of three flavours is

$$P_{\nu_e\nu_e} = 1 - 4U_{e1}^2(1 - U_{e1}^2)\sin^2(\pi L/\lambda_{12}) - 4U_{e2}^2U_{e3}^2\sin^2(\pi L/\lambda_{23})$$

(24)
This is the most general expression without the one mass scale dominance approximation. We now minimize the $\chi^2$ defined as

$$\chi^2 = \chi^2_{\text{ATM}} + \chi^2_{\text{CHOOZ}}$$

(25)

where we define $\chi^2_{\text{CHOOZ}}$ as

$$\chi^2_{\text{CHOOZ}} = \sum_{j=1,15} (x_j - y_j)^2$$

(26)

where $x_j$ are the experimental values, $y_j$ are the corresponding theoretical predictions and the sum is over 15 energy bins of data of the CHOOZ experiment. For the CHOOZ experiment the $\sin^2(\pi L/\lambda_{12})$ term does not always average out to 0.5 (for SK this term always averages to 0.5) and one has to do the energy integration properly. For our analysis we keep the $\Delta_{12}$ fixed at 0.5 eV$^2$ and do a four parameter fit as in SK. The $\chi^2_{\text{min}}$ and the best-fit values of parameters that we get are

- $\chi^2_{\text{min}}/\text{d.o.f.} = 42.22/51$, $\Delta_{23} = 0.0023$ eV$^2$, $s^2_{23} = 0.5$, $s^2_{12} = 0.0022$ and $s^2_{13} = 0.0$.

Thus the best-fit values shift towards the two-generation limit when we include the CHOOZ result. This provides a very good fit to the data being allowed at 80.45% C.L.

The dotted lines in fig. 2 give the combined SK+CHOOZ $\Delta\chi^2(= \chi^2 - \chi^2_{\text{min}})$ given by eq. (23), as a function of one of the parameters, keeping the other three unconstrained. We find that the CHOOZ data severely restricts the allowed ranges for the parameters $s^2_{12}$ and $s^2_{13}$ to values $\lesssim 0.047$, while $\Delta_{23}$ and $s^2_{23}$ are left almost unaffected. Since CHOOZ is consistent with no oscillation one requires $P_{\nu_e\nu_e}$ close to 1. So the second and the third terms in eq. (24) should separately be very small. The second term implies $U^2_{e1}$ to be close to either 0 or 1. $U^2_{e1}$ close to zero implies either $s^2_{12}$ or $s^2_{13}$ close to 1 which is not consistent with SK. Therefore $U^2_{e1}$ is close to 1. Then from unitarity both $U^2_{e2}$ and $U^2_{e3}$ are close to 0 and so the third term goes to zero irrespective of the value of $\Delta_{23}$ and $s^2_{23}$. Hence contrary to expectations, CHOOZ puts almost no restriction on the allowed values of $s^2_{23}$ and $\Delta_{23}$, although $\Delta_{23} \sim 10^{-3}$ eV$^2$ – in the regime in which CHOOZ is sensitive. On the other hand it puts severe constraints on
the allowed values of $s_{12}^2$ and $s_{13}^2$ in order to suppress the average oscillations driven by
$\Delta_{12}$. Because of such low values of $s_{12}^2$ and $s_{13}^2$ the matter effects for the atmospheric
neutrinos are not important and the additional allowed area with low $\Delta_{23}$ and high
$s_{23}^2$ obtained in the SK analysis due to matter effects are no longer allowed. The 99%
C.L. regions allowed by a combined analysis of SK and CHOOZ data is shown by the
dotted lines in figs. 4a-d. It is seen that most of the regions allowed by the three-
flavour analysis of the SK data is ruled out when we include the CHOOZ result. None
of the allowed regions shown in fig. 4a are allowed excepting the two-generation $\nu_{\mu} - \nu_{\tau}$
oscillation limit because CHOOZ does not allow such high values of either $s_{13}^2$ or $s_{12}^2$.
Hence we present again in fig. 5 the allowed regions in the $\Delta_{23} - s_{23}^2$ plane for various
fixed values of $s_{12}^2$ and $s_{13}^2$, determined from the dotted lines in fig. 2. The solid lines
in fig. 5 give the 99% C.L. area allowed by the SK data while the dotted lines give the
corresponding allowed region from the combined analysis of SK+CHOOZ. We find that
for the combined analysis we get allowed regions in this plane only for much smaller
values of $s_{12}^2$ and $s_{13}^2$, which ensures that the electron events are neither less nor more
than expectations.

6 Combined allowed area from short baseline ac-

celerator and reactor experiments

As mentioned earlier the higher mass scale of this scenario can be explored in the accel-
erator based neutrino oscillation search experiments. For the mass-pattern considered
the most constraining accelerator experiments are LSND [8], CDHSW [24], E531 [25]
and KARMEN [26]. Among these only LSND reported positive evidence of oscillation.
Other experiments are consistent with no-oscillation hypothesis. Also important in this
mass range are the constraints from the reactor experiment Bugey [27]. The relevant
probabilities are [12]
We note that the probabilities are functions of one of the mass scales and two mixing angles. Thus the one mass scale dominance approximation applies. There are many analyses in the literature of the accelerator and reactor data including LSND under this one mass scale dominance assumption [12, 28]. These analyses showed that when one considers the results from the previous (prior to LSND) accelerator and reactor experiments there are three allowed regions in the $\theta_{12} - \theta_{13}$ plane [12, 28]

- low $\theta_{12}$ - low $\theta_{13}$
- low $\theta_{12}$ - high $\theta_{13}$
- high $\theta_{12}$ - $\theta_{13}$ unconstrained

When the LSND result was combined with these results then only the first and the third zones remained allowed in the mass range $0.5 \leq \Delta_{12} \leq 2 \text{ eV}^2$. In these earlier analyses of the accelerator and reactor data [12, 28, E776, 29] was more constraining than KARMEN. But with the new data KARMEN2 gives stronger constraint than E776. Also the results from the KARMEN2 experiment now rule out most of the region allowed by the LSND experiment above 1 eV$^2$ [26]. The LSND collaboration has also now reduced the value of the transition probability that they see [30].
repeated the analysis with the latest LSND and KARMEN results for one representative value of $\Delta_{12} = 0.5 \text{ eV}^2$ and present the allowed region in fig. 6.

The light-shaded area in fig. 6 shows the 90% C.L. allowed area in the bilogarithmic $\tan^2 \theta_{12} - \tan^2 \theta_{13}$ plane from the observance of no-oscillation in all the other above mentioned accelerator and reactor experiments except KARMEN2. The inclusion of the KARMEN2 results as well gives the 90% C.L. region shown by the area shaded by asterix. The 90% allowed region by the LSND experiment is within the dashed lines. The KARMEN2 data severely restricts the LSND allowed regions. The solid line shows the 90% C.L. ($\chi^2 \leq \chi^2_{\text{min}} + 7.78$) region allowed by the combined $\chi^2$ analysis of the SK+CHOOZ data keeping $\Delta_{23}$ and $s_{23}^2$ at 0.002 eV$^2$ and 0.5 respectively. The combined SK atmospheric and the CHOOZ reactor data rule out the third zone (high $\theta_{12}$ with $\theta_{13}$ unconstrained) allowed from LSND and other accelerator and reactor experiments. Thus if one takes into account constraints from all experiments only a small region in the first zone (small $\theta_{12}, \theta_{13}$) remains allowed. This common allowed region is shown as a dark-shaded area in the fig. 6. As evident from the expression of the probabilities for the accelerator and reactor experiments the combined allowed area of all the accelerator reactor experiments remains the same irrespective of the value of $\Delta_{23}$ and $s_{23}^2$. Even though the combined area in fig. 6 shows that in the first zone (small $\theta_{12}, \theta_{13}$), SK+CHOOZ data allows more area in the $\theta_{12} - \theta_{13}$ plane for $\Delta_{23} = 0.002 \text{ eV}^2$ and $s_{23}^2 = 0.5$, from fig. 4b we see that for some other combinations of $\Delta_{23}$ and $s_{23}^2$ one does not find any allowed zones from the SK+CHOOZ analysis, even at 99% C.L.. For those sets of values of $\Delta_{23}$ and $s_{23}^2$ the SK+CHOOZ analysis is more restrictive than the LSND and other accelerator reactor data.

7 Implications

From our analysis of the SK atmospheric data the explicit form for the $3 \times 3$ mixing matrix $U$ at the best-fit values of parameters is
From the combined SK+CHOOZ analysis the mixing matrix at the best-fit values of the parameters is

$$U = \begin{pmatrix} 0.95 & -0.039 & 0.31 \\ -0.2 & 0.686 & 0.7 \\ -0.24 & -0.727 & 0.644 \end{pmatrix}$$ (31)

From the combined allowed area of fig. 6 the mixing matrix at $\Delta_{12} = 0.5$ eV$^2$, $\Delta_{23} = 0.0028$ eV$^2$, $s_{12}^2 = 0.005$, $s_{13}^2 = 0.001$ and $s_{23}^2 = 0.5$, is

$$U = \begin{pmatrix} 0.999 & 0.033 & 0.033 \\ -0.047 & 0.706 & 0.706 \\ -0.0 & -0.707 & 0.707 \end{pmatrix}$$ (32)

Thus the allowed scenario corresponds to the one where $\langle \nu_1 | \nu_e \rangle$ is close to 1 while the states $\nu_2$ and $\nu_3$ are combinations of nearly maximally mixed $\nu_\mu$ and $\nu_\tau$.

Long baseline (LBL) experiments can be useful to confirm if the atmospheric neutrino anomaly is indeed due to neutrino oscillations, using well monitored accelerator neutrino beams. Some of the important LBL experiments are K2K (KEK to SK, $L \approx 250$ km), MINOS (Fermilab to Soudan, $L \approx 730$ km) and the proposed CERN to Gran Sasso experiments ($L \approx 730$ km). In this section we explore the sensitivity of the LBL experiment K2K in probing the parameter spaces allowed by the SK+CHOOZ and other accelerator and reactor experiments including LSND. K2K will look for $\nu_\mu$ disappearance as well as $\nu_e$ appearance. In fig. 7 we show the regions in the $\Delta_{23} - s_{23}^2$ plane that can be probed by K2K using their projected sensitivity from [31]. The top left panel is for the two-generation $\nu_\mu - \nu_\tau$ limit. The other panels are for

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5. Thus this scenario is the same as the one termed 3a in Table VI in the pre-SK analysis of [14]. In their notation the states 2 and 3 were 1 and 2. It was disfavoured from solar neutrino results.

6. K2K has already presented some preliminary results.
different fixed values of $s^2_{12}$ and $s^2_{13}$ while $\Delta_{12}$ is fixed at 0.5 eV$^2$. For LBL experiments the term containing $\Delta_{12}$ averages to 0.5 as in the atmospheric case. The solid lines in the panels show the region that can be probed by K2K using the $\nu_\mu$ disappearance channel while the dotted lines give the 90% C.L. contours allowed by SK+CHOOZ. One finds that for for $\Delta_{23} \geq 2 \times 10^{-3}$ eV$^2$, the whole region allowed by SK+CHOOZ can be probed by the $\nu_\mu$ disappearance channel in K2K. The dashed lines show the 90% C.L. area that K2K can probe by the $\nu_e$ appearance mode. As $s^2_{12}$ increases the constraint from the $P_{\nu_\mu\nu_\mu}$ channel becomes important as is seen in the top right panel of fig. 7. However such high values of $s^2_{12}$, although allowed by SK+CHOOZ, is not favoured when one combines LSND and other accelerator and reactor results. For lower $s^2_{12}$ values allowed by all the accelerator, reactor and SK atmospheric neutrino experiment the projected sensitivity in the $\nu_\mu - \nu_e$ channel of K2K is not enough to probe the allowed regions in the $\Delta_{23} - s^2_{23}$ plane as is shown by the absence of the dashed curves in the lower panels.

In fig. 8 we show the regions in the bilogarithmic $\tan^2 \theta_{12} - \tan^2 \theta_{13}$ plane which can be probed by K2K. For drawing these curves we fix $\Delta_{23} = 0.002$ eV$^2$, $s^2_{23} = 0.5$ and $\Delta_{12} = 0.5$ eV$^2$. Shown is the area that can be explored by the $\nu_\mu - \nu_\mu$ (left of the solid line) and $\nu_\mu - \nu_e$ (hatched area) channels in K2K at 90% C.L.. The light-shaded area is allowed by SK+CHOOZ and the dark shaded area is allowed by the combination of all the accelerator, reactor and SK atmospheric neutrino data at 90% C.L.. It is clear from the figure that even though the sensitivity of the $\nu_e$ appearance channel is not enough, K2K can still probe the combined allowed region in the $\theta_{12} - \theta_{13}$ plane from $\nu_\mu$ disappearance.

The projected sensitivities of MINOS and the CERN to ICARUS proposals are lower than K2K and it will be interesting to check if one can probe the regions allowed in this picture better in these experiments. However since in our case the OMSD approximation is not applicable one has to do the energy averaging properly to get the corresponding contours in the three-generation parameters space, and one cannot merely scale the allowed regions from the two-generation plots. For K2K we could use
An important question in this context is whether one can distinguish between the OMSD three generation and this mass scheme. In both pictures the SK atmospheric neutrino data can be explained by the dominant $\nu_\mu - \nu_\tau$ oscillations mixed with little amount of $\nu_e - \nu_\mu(\nu_\tau)$ transition. However the mixing matrix $U$ is different. A distinction can be done if one can measure the mixing angles very accurately.

What is the prospect in LBL experiments to distinguish between these pictures? We give below a very preliminary and qualitative discussion on this. If we take $s_{12}^2 = 0.02$, $s_{13}^2 = 0.02$ and $s_{23}^2 = 0.5$, $P_{\nu_\mu\nu_e}$ would be $(0.038 + 0.0004 \langle S_{23} \rangle)$. As the second term is negligible one has average oscillations. This is different from the OMSD limit where $P_{\nu_\mu\nu_e} = 4U_{\mu 3}^2U_{e 3}^2S_{23}$ is energy dependent. If one combines the other accelerator and reactor experiments including LSND then the allowed values of of $s_{12}^2$ and $s_{13}^2$ are even less and choosing $s_{12}^2 = 0.005$, $s_{13}^2 = 0.001$ and $s_{23}^2 = 0.5$ we get $P_{\nu_e\nu_\mu} = 0.01 - 0.004\langle S_{23} \rangle$. Here also the term involving $\langle S_{23} \rangle$ is one order of magnitude smaller and the oscillations will be averaged. Thus this channel has different predictions for the OMSD limit and beyond the OMSD limit.

8 Discussions and Conclusions

In this paper we have done a detailed $\chi^2$ analysis of the SK atmospheric neutrino data going beyond the OMSD approximation. The mass spectrum chosen is such that $\Delta_{12} = \Delta_{13} \sim \text{eV}^2$ to explain the LSND data and $\Delta_{23}$ is in the range suitable for the atmospheric neutrino problem. We study in details the implications of the earth matter effects and bring out the essential differences of our mass pattern with the OMSD scenario and the two-generation limits.

We first examine in detail what are the constraints obtained from only SK data considering its overwhelming statistics. The allowed regions include

the fig. 5 of [31] to circumvent this problem. However since the analogous information for MINOS and CERN-Gran Sasso proposals is not available to us we cannot check this explicitly.
• the two-generation $\nu_\mu - \nu_\tau$ limit (both $s_{12}^2$ and $s_{13}^2$ zero)

• regions where either $s_{12}^2$ or $s_{13}^2$ is zero; in this limit the probabilities are functions in general of two mixing angles and two mass scales.

• the three-generation regions with all three mixing angles non-zero and the probabilities governed by both mass scales.

The last two cases correspond to dominant $\nu_\mu - \nu_\tau$ oscillation with small admixture of $\nu_\mu - \nu_e$ and $\nu_e - \nu_\tau$ oscillation.

• regions with very low $\Delta_{23}$ ($< 10^{-4}$ eV$^2$) and $s_{23}^2$ close to 1, for which the earth matter effects enhance the oscillations of the upward neutrinos and cause an up-down flux asymmetry. This region is peculiar to the mass spectrum considered by us and is absent in the two-generation and the OMSD pictures.

We present the zenith angle distributions of the events in these cases. With the inclusion of the CHOOZ result the allowed ranges of the mixing angles $s_{12}^2$ and $s_{13}^2$ is constrained more ($\lesssim 0.047$), however the allowed ranges of $\Delta_{23}$ and $s_{23}^2$ do not change much (see fig. 2) except that the low $\Delta_{23}$ region allowed by SK due to matter effects is now disallowed. The inclusion of the constraints from LSND and other accelerator and reactor experiments may restrict the allowed area in the $\theta_{12} - \theta_{13}$ plane for certain values of $\Delta_{23}$ and $s_{23}^2$, but for some other combinations of $\Delta_{23}$ and $s_{23}^2$, SK+CHOOZ turns out to be more constraining. We have included the latest results from LSND and KARMEN2 in our analysis.

In order to explain the solar neutrino problem in this picture one has to add an extra light sterile neutrino. With the new LSND results the allowed 4 neutrino scenarios are

• the (2+2) picture where two degenerate mass states are separated by the LSND gap [21, 34, 35, 36].

• the (3+1) scheme with three neutrino states closely degenerate in mass and the fourth one separated from these by the LSND gap [36, 37]. In [36] the separated
state is predominantly a sterile state. In [37], on the other hand, the state separated by the LSND gap has a very small sterile component.

The extension of our scenario to the 2+2 picture is straightforward. One has to add an extra sterile state 4 close to the state 1 such that $\Delta_{14}$ is in the solar range. Then we have two almost decoupled two-generation pictures in which the atmospheric neutrino problem is mainly due to $\nu_\mu - \nu_\tau$ oscillation and the solar neutrino problem is explained by $\nu_e - \nu_s$ oscillation. The SMA MSW solution for two-generation $\nu_e - \nu_s$ picture is allowed at $\sim 15\%$ C.L. [38]. A detailed global fit of solar and atmospheric neutrino data under this picture would tell us how much this will change due to the small admixture with the other generations. If on the other hand we assume the 4th state to be close to the 2nd and the third state then we will have a (3+1) picture where the 1 state, separated by the LSND gap, is predominantly $\nu_e$. This picture will have difficulties in solving the solar neutrino problem as because of the CHOOZ constraints $U_{ei}$ (i=2,3,4) are small so that $P_{\nu_e\nu_e} \approx 1$ indicating very small suppression of the solar neutrino flux.

To conclude, one can get allowed regions from the SK atmospheric neutrino data where both the mass scales and all the three mixing angles are relevant. The beyond one mass scale dominance spectrum considered in this paper allows new regions in the low mass – low mixing regime due to the earth matter effects. With the inclusion of the CHOOZ, LSND and other accelerator reactor results, the allowed regions are constrained severely. It is, in principle, possible to get some signatures in the LBL experiments to distinguish this picture from the OMSD limit.

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Figure Captions

Fig. 1: The two possible neutrino mass spectra in a three generation scheme.

Fig. 2: The variation of $\Delta \chi^2 = \chi^2 - \chi^2_{\text{min}}$ with one of the parameters keeping the other three unconstrained. The solid (dashed) line corresponds to only SK data when matter effects are included (excluded) while the dotted curve gives the same for SK+CHOOZ. The dashed-dotted line shows the 99% C.L. limit for 4 parameters.

Fig. 3a: The zenith angle distribution of the lepton events with $\Delta_{23} = 0.002$ eV$^2$ and $s^2_{23} = 0.5$ for various combinations of $s^2_{12}$ and $s^2_{13}$. $N$ is the number of events as given by eq. (1) and $N_0$ is the corresponding number with survival probability 1. The panels labelled $SG_\alpha$ and $MG_\alpha$ ($\alpha$ can be $e$ or $\mu$) give the histograms for the sub-GeV and multi-GeV $\alpha$-events respectively. Also shown are the SK experimental data points with $\pm$ 1$\sigma$ error bars.

Fig. 3b: Same as in fig. 3a for fixed $s^2_{12} = 0.1$ and $s^2_{13} = 0.0$ varying $\Delta_{23}$ and $s^2_{23}$.

Fig. 3c: Same as in fig. 3a fixing $s^2_{12} = 0.0$ and $s^2_{13} = 0.1$ for different $\Delta_{23}$ and $s^2_{23}$ values.

Fig. 3d: The long-dashed (short-dashed) line gives the zenith angle distribution of the lepton events for the best-fit cases of the two-generation (three-generation) oscillation solutions for SK. The dotted line gives the corresponding distribution for $\Delta_{23} = 10^{-5}$ eV$^2$, $s^2_{12} = 0.2$, $s^2_{13} = 0.4$ and $s^2_{23} = 1.0$.

Fig. 4a: The allowed parameter regions in the $\Delta_{23} - s^2_{23}$ plane for various fixed values of $s^2_{12}$ and $s^2_{13}$, shown at the top of each panel. The solid lines corresponds to the 99% C.L. contours from the SK data alone, while the dotted line gives the 99% contour from the combined analysis of the SK+CHOOZ data.
Fig. 4b: Same as 4a but in the bilogarithmic $\tan^2 \theta_{12} - \tan^2 \theta_{13}$ plane for fixed values of $\Delta_{23}$ and $s_{23}^2$.

Fig. 4c: Same as 4a but in the $s_{12}^2 - s_{23}^2$ plane for fixed values of $s_{12}^2$ and $\Delta_{23}$.

Fig. 4d: Same as 4a but in the $s_{13}^2 - s_{23}^2$ plane for various fixed values of $s_{12}^2$ and $\Delta_{23}$.

Fig. 4e: The allowed parameter space in the $\Delta_{23} - s_{23}^2$ plane with $\Delta_{23}$ in the range $10^{-5} - 10^{-4}$ eV$^2$ and with fixed values of $s_{12}^2 = 0.185$ and $s_{13}^2 = 0.372$.

Fig. 5: Same as 4a but for smaller values of $s_{12}^2$ and $s_{13}^2$, chosen from the range determined by the SK+CHOOZ dashed line in fig. 3.

Fig. 6: The area between the dashed lines is the 90% C.L. region allowed by LSND while the light shaded zone gives the 90% C.L. allowed region from the non-observance of neutrino oscillation in the other short baseline accelerator and reactor experiments except KARMEN2. The corresponding area which includes KARMEN2 as well is shown by the region shaded by asterix. The 90% C.L. allowed region from SK+CHOOZ analysis is within the dotted line. The dark shaded area corresponds to the combined allowed region.

Fig. 7: 90% C.L. regions in the $\Delta_{23} - s_{23}^2$ plane that can be explored by the $\nu_{\mu} - \nu_{\mu}$ (solid line) and $\nu_{\mu} - \nu_{e}$ (dashed line) oscillation channels in the K2K experiment. The area inside the dotted line shows the 90% C.L. region allowed by SK+CHOOZ. The curves are presented for fixed values of $s_{12}^2$ and $s_{13}^2$ with $\Delta_{12} = 0.5$ eV$^2$.

Fig. 8: Sensitivity of the K2K experiment in the $\tan^2 \theta_{12} - \tan^2 \theta_{13}$ plane for $\Delta_{23} = 0.002$ eV$^2$, $s_{23}^2 = 0.5$ and $\Delta_{12} = 0.5$ eV$^2$. The area that can be explored by the $\nu_{\mu} - \nu_{\mu}$ (left of solid line) and $\nu_{\mu} - \nu_{e}$ (hatched area) channels in K2K at 90% C.L. is shown. The light-shaded area is allowed by SK+CHOOZ and the dark-shaded region is the combined area allowed by all accelerator and reactor data at 90% C.L..
Fig. 1
Fig. 2
\begin{tabular}{|c|c|c|c|c|}
\hline
\textbf{line type} & $\Delta_{23}$ & $s_{23}^2$ & $s_{12}^2$ & $s_{13}^2$ \\
\hline
thick solid & 0.002 eV$^2$ & 0.5 & 0.0 & 0.1 \\
long dashed & 0.002 eV$^2$ & 0.5 & 0.0 & 0.3 \\
short dashed & 0.002 eV$^2$ & 0.5 & 0.1 & 0.0 \\
dotted & 0.002 eV$^2$ & 0.5 & 0.2 & 0.0 \\
dashed-dotted & 0.002 eV$^2$ & 0.5 & 0.1 & 0.1 \\
\hline
\end{tabular}

\textbf{Fig. 3a}
Fig. 3b
Fig. 3c
Fig. 3d
\[ \sin^2 \theta_{13} = 0.0, \quad \sin^2 \theta_{12} = 0.0 \]

\[ \sin^2 \theta_{13} = 0.1, \quad \sin^2 \theta_{12} = 0.1 \]

\[ \sin^2 \theta_{13} = 0.2, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 0.3, \quad \sin^2 \theta_{12} = 0.1 \]

\[ \sin^2 \theta_{13} = 0.4, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 0.5, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 0.6, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 0.7, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 0.8, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 0.9, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 1.0, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 1.1, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 1.2, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 1.3, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 1.4, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 1.5, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 1.6, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 1.7, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 1.8, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 1.9, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 2.0, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 2.1, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 2.2, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 2.3, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 2.4, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 2.5, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 2.6, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 2.7, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 2.8, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 2.9, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 3.0, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 3.1, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 3.2, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 3.3, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 3.4, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 3.5, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 3.6, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 3.7, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 3.8, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 3.9, \quad \sin^2 \theta_{12} = 0.2 \]

\[ \sin^2 \theta_{13} = 4.0, \quad \sin^2 \theta_{12} = 0.2 \]

Fig. 4a
Fig. 4b
\begin{align*}
\sin^2 \theta_{13} &= 0.0 \\
\Delta m^2_{23} &= 6 \times 10^{-3} \text{eV}^2 \\
\sin^2 \theta_{13} &= 0.3 \\
\Delta m^2_{23} &= 2 \times 10^{-4} \text{eV}^2 \\
\sin^2 \theta_{13} &= 0.0 \\
\Delta m^2_{23} &= 2 \times 10^{-4} \text{eV}^2 \\
\sin^2 \theta_{13} &= 0.3 \\
\Delta m^2_{23} &= 7 \times 10^{-4} \text{eV}^2 \\
\sin^2 \theta_{13} &= 0.5 \\
\Delta m^2_{23} &= 4 \times 10^{-4} \text{eV}^2 \\
\sin^2 \theta_{13} &= 0.3 \\
\Delta m^2_{23} &= 4 \times 10^{-4} \text{eV}^2 \\
\sin^2 \theta_{13} &= 0.5
\end{align*}

\textbf{Fig. 4c}
Fig. 4d
\[ \sin^2 \theta_{12} = 0.185, \quad \sin^2 \theta_{13} = 0.372 \]

Fig. 4e
Fig. 5
Fig. 6
\begin{align*}
\sin^2 \theta_{13} = 0.00 \\
\sin^2 \theta_{12} = 0.00 \\
\sin^2 \theta_{13} = 0.00 \\
\sin^2 \theta_{12} = 0.03 \\
\sin^2 \theta_{13} = 0.03 \\
\sin^2 \theta_{12} = 0.00 \\
\sin^2 \theta_{13} = 0.001 \\
\sin^2 \theta_{12} = 0.005
\end{align*}

Fig. 7
Fig. 8