The Multiplicative Ideal Theory of Leavitt Path Algebras

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Abstract

It is shown that every Leavitt path algebra \( L \) of an arbitrary directed graph \( E \) over a field \( K \) is an arithmetical ring, that is, the distributive law \( A \cap (B + C) = (A \cap B) + (A \cap C) \) holds for any three two-sided ideals of \( L \).

It is also shown that \( L \) is a multiplication ring, that is, given any two ideals \( A, B \) in \( L \) with \( A \subseteq B \), there is always an ideal \( C \) such that \( A = BC \), an indication of a possible rich multiplicative ideal theory for \( L \). Existence and uniqueness of factorization of the ideals of \( L \) as products of special types of ideals such as prime, irreducible or primary ideals is investigated.

The irreducible ideals of \( L \) turn out to be precisely the primary ideals of \( L \). It is shown that an ideal \( I \) of \( L \) is a product of finitely many prime ideals if and only if the graded part \( gr(I) \) of \( I \) is a product of prime ideals and \( I/gr(I) \) is finitely generated with a generating set of cardinality no more than the number of distinct prime ideals in the prime factorization of \( gr(I) \). As an application, it is shown that if \( E \) is a finite graph, then every ideal of \( L \) is a product of prime ideals. The same conclusion holds if \( L \) is two-sided artinian or two-sided noetherian. Examples are constructed verifying whether some of the well-known theorems in the ideal theory of commutative rings such as the Cohen’s theorem on prime ideals and the characterizing theorem on ZPI rings hold for Leavitt path algebras.

1 Introduction and Preliminaries.

The ideals of a Leavitt path algebra \( L \) of an arbitrary directed graph \( E \) over a field \( K \) seem to possess many desirable properties. For instance every finitely generated ideal of \( L \) is a principal ideal (see [16]) and that the multiplication of ideals in \( L \) is commutative (see the forthcoming book [2]). The last statement is somewhat surprising since Leavitt path algebras are highly non-commutative. We shall give a direct proof of this important result. Using these as a starting point, the main theme of this paper is to explore the multiplicative ideal theory of Leavitt path algebras. We first show (Theorem 4.3)
that every Leavitt path algebra is an arithmetical ring, that is, the distributive law $A \cap (B + C) = (A \cap B) + (A \cap C)$ holds for any three ideals $A, B, C$ of $L$. Arithmetical rings were introduced by L. Fuchs [7] and commutative arithmetical rings possess interesting properties allowing representations of ideals as intersections/products of special types of ideals such as the prime, irreducible, primary or primal ideals (see, e.g., [8], [9], [10], [11]). Integral domains which are arithmetical rings have come to be known as Prüfer domains, a class of rings for which there are over a hundred interesting characterizing properties.

One consequence of Theorem 4.3 is that the Chinese remainder theorem holds in $L$. As another consequence, we show that $L$ is a multiplication ring (Theorem 4.4), a property, first introduced by W. Krull, that is useful in factorizing ideals (see [13]). Recall, a ring $R$ is a multiplication ring if given any two ideals $A, B$ of $R$ with $A \subseteq B$, there is an ideal $C$ such that $A = BC$. Both the above-mentioned results point to a potential rich theory of ideal factorizations in $L$. In this connection, factorization of graded ideals in $L$ seems to influence that of non-graded ideals in $L$ and, interestingly, this restricts the size of a generating set of the “non-graded part” of the ideals. Specifically, we show that a non-graded ideal $I$ in a Leavitt path algebra $L$ is a product of prime ideals if and only if its graded part $gr(I)$ is a product of graded prime ideals and $I/gr(I)$ is finitely generated with a generating set of cardinality at most the number of distinct prime ideals in the prime factorization of $gr(I)$ (Theorem 6.2). Products of primary ideals and irreducible ideals are also considered. It is shown that, unlike in the case of commutative rings where these two properties are independent, an ideal $I$ of the Leavitt path algebra $L$ is a primary if and only if $I$ is irreducible and this case $I = P^n$, a power of a prime ideal $P$. If further $I$ is a graded ideal, then $I$ turns out to be a prime ideal. As an application of the preceding results, we show that, if $E$ is a finite graph, then every ideal in the Leavitt path algebra $L_K(E)$ is a product of prime ideals. The same conclusion holds if $L_K(E)$ is two-sided artinian or two-sided noetherian. Examples are constructed illustrating ideal factorizations in $L$ and also examining whether some of the well-known theorems in the ideal theory of commutative rings such as the Cohen’s theorem on prime ideals, the characterizing theorem on ZPI rings etc. hold for Leavitt path algebras.

## 2 Preliminaries

For the general notation, terminology and results in Leavitt path algebras, we refer to [2], [15] and [18]. For basic results in associative rings and modules, see [12] and for commutative rings, we refer to [13]. We give below an outline of some of the needed basic concepts and results.

A (directed) graph $E = (E^0, E^1, r, s)$ consists of two sets $E^0$ and $E^1$ together with maps $r, s : E^1 \rightarrow E^0$. The elements of $E^0$ are called vertices and the elements of $E^1$ edges.

A vertex $v$ is called a sink if it emits no edges and a vertex $v$ is called a regular vertex if it emits a non-empty finite set of edges. An infinite emitter is
a vertex which emits infinitely many edges. For each $e \in E^1$, we call $e^*$ a ghost edge. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. A path $\mu$ of length $n > 0$ is a finite sequence of edges $\mu = e_1 e_2 \cdots e_n$ with $r(e_i) = s(e_{i+1})$ for all $i = 1, \cdots, n - 1$. In this case $\mu^* = e_n^* \cdots e_2^* e_1^*$ is the corresponding ghost path.

A vertex is considered a path of length 0. The set of all vertices on the path $\mu$ is denoted by $\mu^0$.

A path $\mu = e_1 \cdots e_n$ in $E$ is closed if $r(e_n) = s(e_1)$, in which case $\mu$ is said to be based at the vertex $s(e_1)$. A closed path $\mu$ as above is called simple provided it does not pass through its base more than once, i.e., $s(e_i) \neq s(e_1)$ for all $i = 2, \ldots, n$. The closed path $\mu$ is called a cycle if it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$.

An exit for a path $\mu = e_1 \cdots e_n$ is an edge $e$ such that $s(e) = s(e_i)$ for some $i$ and $e \neq e_i$.

If there is a path from vertex $u$ to a vertex $v$, we write $u \geq v$. A subset $D$ of vertices is said to be downward directed if for any $u, v \in D$, there exists a $w \in D$ such that $u \geq w$ and $v \geq w$. A subset $H$ of $E^0$ is called hereditary if, whenever $v \in H$ and $w, e \in E^1$ satisfy $v \geq w$, then $w \in H$. A hereditary set is saturated if, for any regular vertex $v$, $r(s^{-1}(v)) \subseteq H$ implies $v \in H$.

Given an arbitrary graph $E$ and a field $K$, the Leavitt path algebra $L_K(E)$ is defined to be the $K$-algebra generated by a set $\{v : v \in E^0\}$ of pair-wise orthogonal idempotents together with a set of variables $\{e, e^* : e \in E^1\}$ which satisfy the following conditions:

1. $s(e)e = e = er(e)$ for all $e \in E^1$.
2. $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.
3. (The "CK-1 relations") For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if $e \neq f$.
4. (The "CK-2 relations") For every regular vertex $v \in E^0$,

$$v = \sum_{e \in E^1, s(e) = v} ee^*.$$

Every Leavitt path algebra $L_K(E)$ is a $\mathbb{Z}$-graded algebra, namely, $L_K(E) = \bigoplus_{n \in \mathbb{Z}} L_n$ induced by defining, for all $v \in E^0$ and $e \in E^1$, $\deg(v) = 0$, $\deg(e) = 1$, $\deg(e^*) = -1$. Here the $L_n$ are abelian subgroups satisfying $L_m L_n \subseteq L_{m+n}$ for all $m, n \in \mathbb{Z}$. Further, for each $n \in \mathbb{Z}$, the homogeneous component $L_n$ is given by

$$L_n = \{ \sum k_i \alpha_i \beta_i^* \in L : |\alpha_i| - |\beta_i| = n \}.$$ 

An ideal $I$ of $L_K(E)$ is said to be a graded ideal if $I = \bigoplus_{n \in \mathbb{Z}} (I \cap L_n)$.

A breaking vertex of a hereditary saturated subset $H$ is an infinite emitter $w \in E^0 \backslash H$ with the property that $0 < |s^{-1}(w) \cap r^{-1}(E^0 \backslash H)| < \infty$. The set of all breaking vertices of $H$ is denoted by $B_H$. For any $v \in B_H$, $v^H$ denotes the element $v - \sum_{s(e) = v, r(e) \notin H} ee^*$. Given a hereditary saturated subset $H$ and a subset $S \subseteq B_H$, $(H, S)$ is called an admissible pair. Given an admissible pair $(H, S)$, the ideal generated by $H \cup \{v^H : v \in S\}$ is denoted by $I(H, S)$. It
was shown in [18] that the graded ideals of $L_K(E)$ are precisely the ideals of the form $I(H,S)$ for some admissible pair $(H,S)$. Moreover, $L_K(E)/I(H,S) \cong L_K(E\backslash(H,S))$. Here $E\backslash(H,S)$ is a Quotient graph of $E$ where $(E\backslash(H,S))^0 = (E^0\backslash H) \cup \{v' : v \in B_H\backslash S\}$ and $(E\backslash(H,S))^1 = \{e \in E^1 : r(e) \notin H\} \cup \{e' : e \in E^1 \text{ with } r(e) \in B_H\backslash S\}$ and $r,s$ are extended to $(E\backslash(H,S))^0$ by setting $s(e') = s(e)$ and $r(e') = r(e)$.

Every graded ideal $I(H,S)$ in $L$ is isomorphic to a Leavitt path algebra of some graph $F$ (see [17]) and hence contains local units, that is, to each $a \in I(H,S)$, there is an idempotent $u \in I(H,S)$ such that $ua = a = au$.

We will also be using the fact that the Jacobson radical (and in particular, the prime/Baer radical) of $L_K(E)$ is always zero (see [2]).

Let $\Lambda$ be an arbitrary (possibly, infinite) index set. For any ring $R$, we denote by $M_\Lambda(R)$ the ring of matrices over $R$ whose entries are indexed by $\Lambda \times \Lambda$ and whose entries, except for possibly a finite number, are all zero. It follows from the works in [11, 3] that $M_\Lambda(R)$ is Morita equivalent to $R$.

Throughout this paper, $E$ will denote an arbitrary graph (with no restriction on the number of vertices or on the number of edges emitted by each vertex) and $K$ will denote an arbitrary field. For convenience in notation, we will denote, most of the times, the Leavitt path algebra $L_K(E)$ by $L$. Finally, we write ideals to denote two-sided ideals and, by a product of ideals, we shall always mean a product of finitely many ideals. Also $<a>$ denotes the ideal generated by the element $a$.

The following two results will be used in our investigation.

**Theorem 2.1** (Proposition 3.5, [17]) Suppose $E$ is an arbitrary graph and $\{c_t : t \in T\}$ is the set of all cycles without exit in $E$. Let $M$ be the ideal of $L_K(E)$ generated by the vertices in all the cycles $c_t$ with $t \in T$. Then $M$ is a ring direct sum $M = \bigoplus_{t \in T} M_t$ where, for each $t$, $M_t$ is the ideal generated by the vertices on the cycle $c_t$ and $M_t \cong M_\Lambda(K[x,x^{-1}])$ with $\Lambda$ a suitable index set depending on $t$.

**Theorem 2.2** (Theorem 4, [17]) Let $I$ be a non-graded ideal of $L = L_K(E)$ with $H = I \cap E^0$ and $S = \{u \in B_H : uH \subseteq I\}$. Then

(a) $I = I(H,S) + \sum_{t \in T} < f_t(c_t) >$ where $T$ is some index set, for each $t \in T$, $c_t$ is a cycle without exit in $E \backslash (H,S)$, $c_t^0 \cap c_s^0 = \emptyset$ for $t \neq s$ and $f_t(x) \in K[x]$ is a polynomial of the smallest degree such that $f_t(c_t) \subseteq I$.

(b) $I(H,S)$ is the largest graded ideal inside $I$ and if $I$ is a prime ideal, then $I(H,S)$ is also a prime ideal.

Note that if $I$ is the non-graded ideal considered in Theorem 2.2, then $I/\text{gr}(I) = \bigoplus_{t \in T} < f_t(c_t) >$. This is because, in $L/I(H,S)$, $< f_t(c_t) > \subseteq M_t$, the ideal generated by the vertices on the cycle $c_t$ and by Theorem 2.1 $\sum M_t = \bigoplus M_t$.  

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Notation: For convenience, $I(H, S)$ will also be denoted by $gr(I)$ and we shall call it the graded part of $I$.

3 Some Properties of the Ideals in Leavitt Path Algebras

Let $E$ be an arbitrary graph. We begin by describing some special features of the graded ideals in the Leavitt path algebra $L = L_K(E)$. We first show that an ideal $A$ of $L$ is graded if and only if $A \cap B = AB$ for any ideal $B$ of $L$. It then follows easily that if $A$ is a graded ideal, then $AB = BA$ for any ideal $B$ of $L$. Actually, the commutativity property $AB = BA$ holds even for non-graded ideals. This interesting result is stated and proved in [2] by using a deep structure theorem on ideals of $L_K(E)$ and using a lattice isomorphism with the ideal lattice of $L_K(E)$. Since the book [2] is being written up and is yet to be published, and since this result is very relevant to the multiplicative ideal theory of Leavitt path algebras that we are investigating, we give below a direct proof of this important result using the ideas considered here in this paper.

A very useful observation is that if $c$ is a cycle based at a vertex $v$ in $E$, then $vLv \cong K[x, x^{-1}]$ under an isomorphism mapping $v$ to 1, $c$ to $x$ and $c^*$ to $x^{-1}$.

Our first Lemma points out some special characteristics of graded ideals of $L$.

Lemma 3.1 (i) An ideal $A$ of $L$ is a graded ideal if and only if for any ideal $B$ of $L$, $AB = A \cap B$ and $BA = B \cap A$. Thus a graded ideal $A$ satisfies $A^2 = A$ and $AB = BA$ for all ideals $B$.

(ii) Suppose $A$ is a graded ideal. Then $A = A_1 \cdots A_m$ is a product of ideals if and only if $A = A_1 \cap \cdots \cap A_m$ is an intersection of the ideals $A_i$. In this case, $A = \bigcap_{i=1}^m gr(A_i) = \prod_{i=1}^m gr(A_i)$.

(iii) If $A_1, \cdots, A_m$ are graded ideals of $L$, then $\prod_{i=1}^m A_i = \bigcap_{i=1}^m A_i$.

Proof. (i) Suppose $A$ is a graded ideal of $L$. Clearly $AB \subseteq A \cap B$. To prove the reverse inclusion, let $x \in A \cap B$. Since $A$ is a graded ideal, there is a local unit $u \in A$ satisfying $x = ux = xu$. Then $x = ux \in AB$. So $A \cap B = AB$.

Conversely, $B \cap A = BA$. Hence $AB = BA$. In particular, $A^2 = A \cap A = A$.

Similarly, $B \cap A = BA$. Hence $AB = BA$. In particular, $A^2 = A \cap A = A$.

Conversely, suppose $A \cap B = AB$ for all ideals $B$ in $L$. Suppose, on the contrary, $A$ is not graded. By Theorem 2.2, $A = I(H, S) + \sum_{i \in X} f_i(c_i)$, where each $c_i$ is a cycle without exits in $E \setminus (H, S)$ and $f_i(x) \in K[x]$. For convenience, write $N = \sum_{i \in X} f_i(c_i)$ and $gr(A) = I(H, S)$. Then $AN = (gr(A) + N)N = gr(A)N + N^2 = (gr(A) \cap N) + N^2$. Since $AN = A \cap N = N$, $A \cap N \subseteq A$.
\((gr(A) \cap N) + N^2 = N\) and so \([N^2 + gr(A)]/gr(A) = (N + gr(A))/gr(A)\). This means, in the Leavitt path algebra \(L = L/gr(A)\),

\[
\bigoplus_{i} < f_i(c_i) >^2 = \bigoplus_{i} < f_i(c_i) >.
\]

Since \(\bigoplus\) is a ring direct sum, we obtain \([< f_i(c_i) >]^2 = < f_i(c_i) >\). Since \(f_i(c_i) \in v_iLv_i \cong K[x, x^{-1}]\), we get \(< f_i(x) >^2 = < f_i(x) >\) in the integral domain \(K[x, x^{-1}]\), a contradiction. This contradiction shows that \(A\) is a graded ideal.

(ii) Suppose \(A = A_1 \cdots A_m\) is a product of ideals. Now, \(([A_1 \cap \cdots \cap A_m]/A)^m = 0\) in \(L/A\). On the other hand, if we write the graded ideal \(A\) as \(A = I(H, S)\), then \(L/A \cong L_K(E \setminus (H, S))\) and so \(L/A\) contains no non-zero nilpotent ideals.

Consequently, \([A_1 \cap \cdots \cap A_m]/A = 0\) or \(A = A_1 \cap \cdots \cap A_m\).

Conversely, suppose \(A = A_1 \cap \cdots \cap A_m\). Since \(A\) is graded, \(A = A^m\) by (i), and so we get

\[A = A^m \subseteq A_1 \cdots A_m \subseteq A_1 \cap \cdots \cap A_m = A.\]

Thus, \(A = A_1 \cdots A_m\).

Finally, if \(A = \bigcap_{i=1}^{m} A_i\), clearly \(\bigcap_{i=1}^{m} gr(A_i) \subseteq A\). On the other hand, since \(A\) is graded, \(A \subseteq gr(A_i)\) for all \(i\) and hence \(A \subseteq \bigcap_{i=1}^{m} gr(A_i)\). Thus \(A = \bigcap_{i=1}^{m} gr(A_i)\)

which is equal to \(\bigcap_{i=1}^{m} gr(A_i)\), by (i).

(iii) Now \(\bigcap_{i=1}^{m} A_i\) is a graded ideal and so (iii) follows from (ii). \(\blacksquare\)

**Lemma 3.2** Suppose \(A, B\) are ideals of \(L\) with \(A \subseteq B\). If \(A \subseteq gr(B)\), then \(AB = A = BA\). In particular, if \(A\) or \(B\) is a graded ideal, then \(AB = A = BA\).

**Proof.** If \(A \subseteq gr(B)\), then \(A = A \cap gr(B) = A gr(B) \subseteq AB \subseteq A\). So \(A = AB\).

Similarly, \(BA = A\). If \(A\) is graded, then \(A \subseteq gr(B)\). If \(B\) is graded, again \(A \subseteq B = gr(B)\). From the first part, we then conclude that \(A = AB = BA\). \(\blacksquare\)

Next we give a direct proof of the more general statement that the ideal multiplication in a Leavitt path algebra is indeed commutative.

We start with the following simple Lemma. In its proof we shall be using the fact that whenever \(p^*q \neq 0\), where \(p, q\) are paths in \(E\), then the CK-1 relation in \(L_K(E)\) implies that either \(p = qr\) or \(q = ps\) where \(r, s\) are suitable paths in \(E\).

**Lemma 3.3** Let \(E\) be an arbitrary graph. If \(c\) is a cycle without exits based at a vertex \(v\) in \(E\) and \(f(x), g(x) \in K[x]\), then

\[< f(c) >> g(c) >=< f(c) g(c) >=< g(c) >> f(c) >.\]
Proof. Now a typical element of \(< f(c) > < g(c) >\) is a \(K\)-linear combination of non-zero terms of the form \(\alpha \beta^* f(c) \gamma \delta^* g(c) \mu \nu^*\) where \(\alpha, \beta, \gamma, \delta, \mu, \nu\) are paths in \(E\). Here \(s(\gamma) = v = s(\delta)\) and \(r(\gamma) = r(\delta)\). Since \(c\) is a cycle without exits, \(\gamma \delta^*\) simplifies to an integer power of \(c\) or \(c^*\) which we denote by \(c^e\). Then \(\alpha \beta^* f(c) \gamma \delta^* g(c) \mu \nu^* = \alpha \beta^e c f(c) g(c) \mu \nu^* \in < f(c) g(c) >\). On the other hand, \(< f(c) g(c) > \subseteq < f(c) > \subseteq < g(c) >\) and so we get
\[
< f(c) > < g(c) > = < f(c) g(c) > = < g(c) f(c) > .
\]
In a similar fashion, we can show that \(< g(c) > < f(c) > = < g(c) f(c) >\). ■

**Theorem 3.4 ([2])** Let \(E\) be an arbitrary graph. Then for any two ideals \(A, B\) in \(L = L_K(E)\), we have \(AB = BA\).

Proof. In view of Lemma 3.1 (i), we may assume that both \(A\) and \(B\) are non-graded ideals. We distinguish two cases.

Case 1: Suppose \(A\) or \(B\) is contained in \(gr(A) + gr(B)\), say \(A \subseteq gr(A) + gr(B)\). By modular law, \(A = gr(A) + (A \cap gr(B))\). Then, by using Lemma 3.1(i) and Lemma 3.2, we get
\[
AB = gr(A)B + [A \cap gr(B)]B = (gr(A) \cap B) + (A \cap gr(B)).
\]
Similarly,
\[
BA = Bgr(A) + B[A \cap gr(B)] = (B \cap gr(A)) + (A \cap gr(B)).
\]
Thus \(AB = BA\).

Case 2: Suppose \(A \not\subseteq gr(A) + gr(B)\) and \(B \not\subseteq gr(A) + gr(B)\). We write \(gr(A) + gr(B) = I(H, S)\) where \(H = (gr(A) + gr(B)) \cap E^0\) and \(S = \{u \in B_H : u^H \in gr(A) + gr(B)\}\). By Theorem 2.1, we can write
\[
A = gr(A) + \sum_{j \in X} < f_j(c_j) > \quad \text{and} \quad B = gr(B) + \sum_{k \in Y} < g_k(c_k) >
\]
where \(X, Y\) are some index sets, for each \(j \in X\) and \(k \in Y\), \(f_j(x), g_k(x) \in K[x]\). \(c_j\) and \(c_k\) are cycles without exits in \(E^0 \setminus gr(A)\) and \(E^0 \setminus gr(B)\), respectively. In \(L/I(H, S) \cong L_K(E \setminus (H, S))\), let \(M\) denote the ideal of generated by the vertices in all the cycles \(c_t (t \in T)\) without exits in \(E \setminus (H, S)\). By Theorem 2.1, \(M\) is a ring direct sum \(M = \bigoplus_{t \in T} M_t\) where \(M_t\) is the ideal generated by the vertices on the cycle \(c_t\) in \(L/I(H, S)\). Now \(\tilde{A} = (A + I(H, S))/I(H, S)\) (and likewise \(\tilde{B} = (B + I(H, S))/I(H, S)\)) is an epimorphic image of \(A/gr(A)\) (of \(B/gr(B)\)) and so \(\tilde{A}, \tilde{B} \subseteq M\). Consequently, we can write \(\tilde{A} = \bigoplus_{r \in X' \subseteq T} < f_r(c_r) > \) and \(\tilde{B} = \bigoplus_{s \in Y' \subseteq T} < g_s(c_s) >\) where \(c_r, c_s\) are cycles without exits in \(E \setminus (H, S)\). Since the product \(M_r M_s = 0\) for all \(r \neq s\) in \(T\), we have, using Lemma 3.3,
\[
\tilde{A} \tilde{B} = \bigoplus_{k \in X' \cap Y'} < f_k(c_k) > < g_k(c_k) > = \bigoplus_{k \in X' \cap Y'} < g_k(c_k) > < f_k(c_k) > = \tilde{B} \tilde{A}.
\]
Then \( AB = BA + gr(A) + gr(B) \). Now \( A \cap B \) contains both \( AB, BA \) and so, using modular law,

\[
AB = (A \cap B) \cap AB = BA + (A \cap B) \cap [gr(A) + gr(B)]
\]

\[ = BA, \text{ as } BA \text{ contains the second term.} \]

\[ \square \]

### 4 Leavitt Path Algebras as Arithmetical Rings and Multiplication Rings

The main theorem of this section shows that the ideals of a Leavitt path algebra \( L \) form a distributive lattice. As a consequence, the Chinese remainder theorem holds in \( L \). We begin with two propositions which are useful in the proof of the main theorem (Theorem 4.3) and which show how things work out nicely for graded ideals. Using Theorem 4.3, we then show that every Leavitt path algebra \( L \) is a multiplication ring, a useful property in the ideal theory of rings.

**Proposition 4.1** Let \( E \) be an arbitrary graph. If \( A, B, C \) are ideals of \( L \) and if one of them is a graded ideal, then we have

\[ A \cap (B + C) = (A \cap B) + (A \cap C). \]

**Proof.** Suppose \( A \) is a graded ideal. By Lemma 3.1, we then obtain

\[ A \cap (B + C) = A(B + C) = AB + AC = (A \cap B) + (A \cap C). \]

Next, suppose one of \( B \) and \( C \), say \( B \), is a graded ideal. We need only to show that \( A \cap (B + C) \subseteq (A \cap B) + (A \cap C) \). Let \( a = b + c \in A \cap (B + C) \). Since \( B \) is graded, \( b = ubu \) for some local unit \( u \in B \). So \( b = ubu = uau - ucu \in (A \cap B) + (B \cap C) \). Then \( c = a - b \in C \cap [A + (A \cap B) + (B \cap C)] = C \cap [A + (B \cap C)] = (C \cap A) + (B \cap C) \), by modular law. It then follows that

\[
\begin{align*}
    a &= b + c \\
    &= (A \cap B) + (A \cap C) + (A \cap B \cap C) \\
    &= (A \cap B) + (A \cap C).
\end{align*}
\]

Similar argument holds when \( C \) is a graded ideal. \[ \square \]

**Proposition 4.2** Let \( A, B, C \) be non-graded ideals of \( L \). If one of \( A, B, C \) is contained in \( gr(B) + gr(C) \), then \( A \cap (B + C) = (A \cap B) + (A \cap C) \).

**Proof.** Suppose \( A \subseteq gr(B) + gr(C) \). Then

\[
A \cap (B + C) = A \cap [gr(B) + gr(C)] \cap (B + C)
\]

\[ = A \cap [gr(B) + gr(C)]
\]

\[ = (A \cap gr(B)) + (A \cap gr(C)), \text{ by Proposition 4.1}
\]

\[ \subseteq (A \cap B) + (A \cap C). \]

\[ 8 \]
Suppose one of $B$ and $C$, say, $B \subseteq \text{gr}(B) + \text{gr}(C)$. Then $B = \text{gr}(B) + (B \cap \text{gr}(C))$, by modular law. Consequently,

$$A \cap (B + C) = A \cap ([\text{gr}(B) + (B \cap \text{gr}(C))] + C)$$

$$= A \cap (\text{gr}(B) + C)$$

$$= (A \cap \text{gr}(B)) + (A \cap C), \text{ by Proposition 4.1}$$

$$\subseteq (A \cap B) + (A \cap C).$$

Similar argument works when $C \subseteq \text{gr}(B) + \text{gr}(C)$.

We are now ready to show that every Leavitt path algebra is an arithmetical ring.

**Theorem 4.3** The ideals of any Leavitt path algebra form a distributive lattice. Specifically, if $A, B, C$ are any three ideals of a Leavitt path algebra $L$ of an arbitrary graph $E$, then

$$A \cap (B + C) = (A \cap B) + (A \cap C).$$

**Proof.** In view of Propositions 4.1 and 4.2 we may assume that $A, B, C$ are all non-graded ideals such that none of $A, B$ or $C$ is contained in $\text{gr}(B) + \text{gr}(C)$. By Theorem 2.2

$$B = I(H_1, S_1) + \sum_{i \in X} < f_i(c_i) > \text{ and } C = I(H_2, S_2) + \sum_{j \in Y} < g_j(c_j) >$$

where $X, Y$ are some index sets, $I(H_1, S_1) = \text{gr}(B)$, $I(H_2, S_2) = \text{gr}(C)$, for all $i \in X$ and $j \in Y$, $f_i(x), g_j(x) \in K[x]$ and the $c_i, c_j$ are cycles without exits in $E \setminus (H_1, S_1)$ and $E \setminus (H_2, S_2)$ respectively. Let $I(H, S)$ denote the graded ideal $\text{gr}(B) + \text{gr}(C)$. Consider $L/I(H, S)$ which we identify with $L_K(E \setminus (H, S))$. Let $A, B, C$ denote the images of $I(H, S)$ in $A, B, C$ respectively. Let $M$ be the ideal of $L/I(H, S)$ generated by the vertices in all the cycles without exits in $E \setminus (H, S)$. Since $\bar{B} = [B + I(H, S)]/I(H, S)$ (and likewise, $\bar{C} = [C + I(H, S)]/I(H, S)$) is a non-zero homomorphic image of $B/I(H_1, S_1)$ (respectively, of $C/I(H_2, S_2)$), $\bar{B}, \bar{C} \subseteq M$.

If $\bar{A} \cap (\bar{B} + \bar{C}) = 0$, then $A \cap (B + C) \subseteq \text{gr}(B) + \text{gr}(C)$ and we get,

$$A \cap (B + C) = A \cap [A \cap (B + C)] \subseteq A \cap \text{gr}(B) + \text{gr}(C), \text{ by Proposition 4.1}$$

$$\subseteq (A \cap B) + (A \cap C).$$

Suppose $\bar{A} \cap (\bar{B} + \bar{C}) \neq 0$, so $A' = \bar{A} \cap M \neq 0$. Now, by Theorem 2.3 $M$ is a ring direct sum of matrix rings of the form $M_\Lambda(K[x, x^{-1}])$ whose ideal lattice is distributive (as $M_\Lambda(K[x, x^{-1}])$ is Morita equivalent to the principal ideal domain $K[x, x^{-1}]$). Hence the ideal lattice of $M$ is also distributive and we obtain

$$A \cap (B + C) = A' \cap (B + C) = (A' \cap B) + (A' \cap C) = (A \cap B) + (A \cap C).$$
Then
\[
A \cap (B + C) \subseteq (A \cap B) + (A \cap C) + gr(B) + gr(C).
\]

Intersecting with \( A \) on both sides of (*) and using modular law and Proposition 4.1 we obtain
\[
A \cap (B + C) \subseteq (A \cap B) + (A \cap C) + A \cap [gr(B) + gr(C)]
= (A \cap B) + (A \cap C) + (A \cap (gr(B)) + (A \cap gr(C))
= (A \cap B) + (A \cap C).
\]

This proves that the ideal lattice of \( L \) is distributive.

Remark: As an interesting consequence of Theorem 4.3, one can show, using well-known arguments (see e.g. Theorem 18, ch IV, [19]) that the Chinese remainder theorem holds in Leavitt path algebras.

We next use Theorem 4.3, to show that every Leavitt path algebra is a multiplication ring, a useful property in the multiplicative ideal theory of Leavitt path algebras.

**Theorem 4.4** The Leavitt path algebra \( L = L_K(E) \) of an arbitrary graph \( E \) is a multiplication ring, that is, for any two ideals \( A, B \) of \( L \) with \( A \subseteq B \), there is an ideal \( C \) of \( L \), such that \( A = BC = CB \).

**Proof.** If \( A = B \), then, as \( L \) is a ring with local units, \( A = B = BL \). So assume that \( A \not\subseteq B \). If \( A \) or \( B \) is a graded ideal or if \( A \subseteq gr(B) \), then, as shown in Lemma 3.2, \( C = A \) satisfies \( A = BC = CB \). So we may assume that both \( A \) and \( B \) are non-graded ideals with \( A \not\subseteq B \) but \( A \not\subseteq gr(B) \). By Theorem 2.2, \( A = I(H_1, S_1) + \sum_{i \in X} < f_i(c_i) > \) and \( B = I(H_2, S_2) + \sum_{j \in Y} < g_j(c_j) > \) where \( I(H_1, S_1) = gr(A) \subseteq I(H_2, S_2) = gr(B) \), the \( c_i \) and \( c_j \) are cycles without exits in \( E \backslash H_1 \) and in \( E \backslash H_2 \) respectively, based at vertices \( v_i, v_j \) and \( f_i(x), g_j(x) \) \in K[x]. Let \( T = \{ i \in X : v_i \in gr(B) \} \) so that \( \sum_{i \in T} < f_i(c_i) > \subseteq gr(B) \). Moding out \( gr(A) \) and denoting \( \bar{A} = A / gr(A) \) and \( \bar{B} = B / gr(A) \), we have in \( \bar{L} = L / gr(A) \cong L_K(E \backslash (H_1, S_1)) \),
\[
\bar{A} = \sum_{i \in X} < f_i(c_i) > \subseteq \bar{B} = [gr(B) / gr(A)] + \sum_{j \in Y} < g_j(c_j) >.
\]

Now \( B / gr(B) = \bigoplus_{j \in Y} < g_j(c_j) > \) is a ring direct sum of ideals and the ideal
\[
[A + gr(B) / gr(B)] = \bigoplus_{j \in X \backslash T} < f_j(c_j) > \subseteq B / gr(B).
\]

It is then easy to see that, for each \( j \in X \backslash T \), \( < f_j(c_j) > \subseteq < g_j(c_j) > \). From Theorem 2.2 (a), \( g_j(x) \) is a polynomial of the smallest degree in \( K[x] \) such that
$g_j(c_j) \in B$. Consequently, $g_j(x)$ must be a divisor of $f_j(x)$ in $K[x]$. Hence, for each $j \in X \setminus T$, $f_j(c_j) = g_j(c_j)f'_j(c_j)$ where $f'_j(x) \in K[x]$ satisfies $f_j(x) = g_j(x)f'_j(x)$.

Our goal is to show that $A = BC$, where

$$C = gr(A) + \sum_{i \in T} < f_i(c_i) > + \sum_{j \in X \setminus T} < f'_j(c_j) > .$$

We first prove the following claim where $gr(B) = gr(B)/gr(A)$.

**Claim:** In $\bar{L} = L/gr(A)$,

$$gr(B) \cap [ \sum_{j \in X \setminus T} < h_j(c_j) > ] = 0$$

where, for each $j \in X \setminus T$, $h_j(x)$ is any arbitrarily chosen polynomial belonging to $K[x]$.

**Proof of the claim:** We identify $\bar{L}$ with $L_K(E\setminus(H_1,S_1))$. By Theorem 4.3, distributive law holds and so we have

$$gr(B) \cap [ \sum_{j \in X \setminus T} < h_j(c_j) > ] = \sum_{j \in X \setminus T} [gr(B) \cap < h_j(c_j) >] .$$

Now, by Lemma 3.1, $gr(B) \cap < h_\lambda(c_\lambda) > = gr(B) \cdot < h_\lambda(c_\lambda) >$ where $\cdot$ denotes the product. Hence it is enough to show that $gr(B) \cdot < h_\lambda(c_\lambda) > = 0$, for any given $\lambda \in X \setminus T$. For convenience in writing, denote the graded ideal $gr(B)$ by $I(H,S)$ where $H = H_2 \setminus H_1 \subseteq E\setminus(H_1,S_1)$. By Lemma 5.6 in [18], the elements of the graded ideal $gr(B)$ are of the form $\sum_{j=1}^m l_j \alpha_j \beta_j^* + \sum_{i=1}^n l_i \gamma_i v_i^H \delta_i^*$ where $l_j, t_i \in K$, $\alpha_j = r(\alpha_j) \in H$ and $\gamma_i = r(\gamma_i) = u_i \in S$.

Let $x \in I(H,S)$: $< h_\lambda(c_\lambda) >$. Then $x$ is a finite sum of products of the form

$$\left[ \sum_{j=1}^m l_j \alpha_j \beta_j^* + \sum_{i=1}^n t_i \gamma_i v_i^H \delta_i^* \right] \cdot \left[ \sum_{k=1}^{n'} l'_k p_k q_k^* h_\lambda(c_\lambda) r_k s_k^* \right],$$

where $l'_k \in K$ and $p_k, q_k, r_k, s_k$ are all paths in $E$. A typical term in this product is of the form

$$l_j l'_k \alpha_j \beta_j^* p_k q_k^* h_\lambda(c_\lambda) r_k s_k^* + t_i l'_k \gamma_i u_i^H \delta_i^* p_k q_k^* h_\lambda(c_\lambda) r_k s_k^*.$$

Now $s(q_k) = r(q_k^*) \in c^0_\lambda$ and since $c_\lambda$ has no exits, $r(p_k) = r(q_k^*) \in c^0_\lambda$.

If $\beta_j^* p_k \neq 0$, then either $\beta_j = p_k \beta'$ or $p_k = \beta_j p'$. If $\beta_j = p_k \beta'$, then $\beta'$ will be path from a vertex in $c_\lambda$ to the vertex $r(\beta_j) \in H$ contradicting the fact that $c_\lambda$ has no exits. Similarly, if $p_k = \beta_j p'$, $p'$ will be a path from $r(\beta_j) \in H$ to $r(p') \in c^0_\lambda$ and this implies, as $H$ is hereditary, $c^0_\lambda \subseteq H$, again a contradiction. So $l_j l'_k \alpha_j \beta_j^* p_k q_k^* h_\lambda(c_\lambda) r_k s_k^* = 0$.

Likewise, if $\delta_i^* p_k \neq 0$, then either $\delta_i = p_k \delta'$ or $p_k = \delta_i p'$. If $\delta_i = p_k \delta'$, $\delta'$ will be a path from $r(p_k) \in c^0_\lambda$ to the vertex $r(\delta_i) \in S$, contradicting that $c_\lambda$ has no
exits. If \( p_k = \delta p' \), then \( p' \) will be a path from \( r(\delta_i) = u_i \) to \( r(p_k) \in \mathcal{O}_L \). Observe that in this case, \( p' = fp'' \) where \( f \) is the initial edge of \( p' \) satisfying \( r(f) \notin H \). Thus

\[
u^H t_i^* p_k = u^H t_i^* p_k = (v_i - \sum_{s(e) = v_i, r(e) \notin H} ee^*)fp'' = fp'' - fp'' = 0.
\]

So \( t_i^* p_k \neq 0 \). Hence \( I(H, S) \cdot < h_\lambda(c) >= 0 \), thus proving our claim.

Let \( C \) be the ideal in \( L \) such that

\[
C/\text{gr}(A) = \bar{C} = \sum_{i \in T} < f_i(c_i) > + \sum_{j \in X \setminus T} < f'_j(c_j) > \subseteq \bar{L}.
\]

Then, using the claim (*) in \( \bar{L} \), Lemma 3.2 and the fact that the index set \( Y = (Y \setminus X) \cup (X \setminus T) \), we have

\[
\bar{B}C = \left[ \text{gr}(\bar{B}) + \sum_{j \in Y \setminus X} < g_j(c_j) > + \sum_{j \in X \setminus T} < g_j(c_j) > \right] \times \\
\left[ \sum_{i \in T} < f_i(c_i) > + \sum_{j \in X \setminus T} < f'_j(c_j) > \right] \\
= \text{gr}(\bar{B})[\sum_{i \in T} < f_i(c_i) > + [\sum_{j \in X \setminus T} < g_j(c_j) >][\sum_{i \in T} < f_i(c_i) >] \\
+ [\sum_{j \in X \setminus T} < g_j(c_j) >][\sum_{j \in X \setminus T} < f'_j(c_j) >], \text{ by claim (*)}
\]

\[
= \sum_{i \in T} < f_i(c_i) > + [\sum_{j \in X \setminus T} < g_j(c_j) >][\sum_{i \in T} < f_i(c_i) >] + [\sum_{j \in X \setminus T} < f'_j(c_j) > \\
= \sum_{i \in X} < f_i(c_i) > + 0 = \bar{A},
\]

as \( [\sum_{j \in X \setminus T} < g_j(c_j) >][\sum_{i \in T} < f_i(c_i) >] \subseteq [\sum_{j \in X \setminus T} < g_j(c_j) >]\text{gr}(\bar{B}) = 0 \), by our claim (*). It then follows that \( A = BC \).

From Theorem 4.3, we obtain the following interesting property of prime ideals in \( L \).

**Corollary 4.5** If \( P \) is a prime ideal of \( L \), then for any ideal \( A \) with \( P \subseteq A \), we have \( P = AP \).

**Proof.** Since \( L \) is a multiplication ring, there is an ideal \( C \) of \( L \) such that \( P = AC \). Since \( P \) is a prime ideal and \( A \nsubseteq P \), we conclude that \( C \subseteq P \). Then

\[
P = AC \subseteq AP \subseteq P.
\]

Hence \( P = AP \).
5 Prime, Irreducible and Primary Ideals of a Leavitt Path Algebra

In this section, we investigate special types of ideals in $L$ such as the prime, the irreducible and the primary ideals. While these three concepts are independent for ideals in a commutative ring, we show that they coincide for graded ideals in the Leavitt path algebra $L$. We also show that a non-graded ideal $I$ of $L$ is irreducible if and only if $I$ is a primary ideal if and only if $I = P^n$, a power of a prime ideal $P$. This is useful in the factorization of ideals in the next section. We also point out some interesting properties of the prime ideals in $L$.

The following description of prime ideals of $L$ was given in [15].

**Theorem 5.1** (Theorem 3.2, [15]) An ideal $P$ of $L = L_K(E)$ with $P \cap E^0 = H$ is a prime ideal if and only if $P$ satisfies one of the following properties:

(i) $P = I(H, B_H)$ and $E^0 \setminus H$ is downward directed;

(ii) $P = I(H, B_H \{u\})$, $v \geq u$ for all $v \in E^0 \setminus H$ and the vertex $u'$ that corresponds to $u$ in $E(H, B_H \{u\})$ is a sink;

(iii) $P$ is a non-graded ideal of the form $P = I(H, B_H) + < p(c) >$, where $c$ is a cycle without exits based at a vertex $u$ in $E(H, B_H)$, $v \geq u$ for all $v \in E^0 \setminus H$ and $p(x)$ is an irreducible polynomial in $K[x, x^{-1}]$ such that $p(c) \in P$.

We shall use the observation that if $E^0 \setminus H$ contains a cycle without exits, then in Theorem 5.1 the case (ii) will not occur.

Next we point out some interesting properties related to prime ideals of $L$.

**Lemma 5.2** Suppose $P$ is a prime ideal of $L$ and $A$ is an ideal such that, for some integer $n > 1$, $P^n \subseteq A \subseteq P$. Then $A = P^r$ for some $1 \leq r \leq n$.

**Proof.** If $P$ is graded, then $P = P^n = A$. So assume that $P$ is a non-graded prime ideal. By Theorem 5.1 $P = gr(P) + < p(c) >$ with $gr(P) = I(H, B_H)$, where $c$ is a cycle without exits in $E(H, B_H)$ and $p(x)$ is an irreducible polynomial in $K[x]$. By Lemma 3.1 $gr(P^n) = gr(P)$. As

$$P^n / gr(P) = (P / gr(P))^n = < p(c) >^n$$

in $L / gr(P)$, $P^n = gr(P) + < p(c) >^n$. Now, $A$ must be a non-graded ideal. Because if $A$ is graded, then $A \subseteq gr(P) \subseteq P^n$ and this implies $A = P^n$, a contradiction as $P^n$ is not graded. Hence $A = gr(P) + < f(c) >$ where $f(x) \in K[x]$. In $L / gr(P)$, we have $< p(c) >^n \subseteq < f(c) > \subseteq < p(c) >$ and hence $f(x)$ is a divisor of $p^n(x)$ in $K[x]$. So $f(x) = p^r(x)$ for some $r \leq n$ and $A = gr(P) + < p(c) >^r = P^r$. ■

**Proposition 5.3** Let $P$ be a prime ideal of a Leavitt path algebra $L$. Then for any ideal $A$ with $P \subseteq A$, either $P = A$ or $P \subseteq gr(A)$.

**Proof.** If $P$ is graded, then clearly $P$ possess the desired properties. So assume that $P$ is a non-graded ideal. Suppose there is an ideal $A$ such that $P \subseteq A$.
and $P \not\subseteq gr(A)$. Clearly $A$ is non-graded and so we have, by Theorem 2.2, $A = I(H', S') + \sum_{i \in X} < f_i(c_i)$ where $H' = A \cap E^0$, $X$ is some index set, for each $i$, $f_i(x) \in K[x]$ and $c_i$ is a cycle without exits in $E^0 \setminus H'$. By Theorem 5.1 (iii), $P = I(H, B_H) + < p(c) >$, where $H = P \cap E^0$, $c$ is a cycle without exits based at a vertex $u$ in $E \setminus (H, B_H)$, $v \geq u$ for all $v \in E^0 \setminus H$ and $p(x)$ is an irreducible polynomial in $K[x]$. Clearly, $c$ is the only cycle without exits in $E^0 \setminus H$ based at a vertex $u$. As $P \not\subseteq gr(A) = I(H', S')$, $p(c) \notin I(H', S')$ and $c$ is a cycle without exits in $E^0 \setminus H'$ also. Since $v \geq u$ for all $v \in E^0 \setminus H'$, $c_i = c$ for all $i$. Moreover, $H' = H$ because if there is a $v \in H' \setminus H$, then $v \geq u$ implies that $u \in H'$, a contradiction. In this case $S' = B_H$. Thus $A$ is of the form $A = I(H, B_H) + < f(c) >$ where $f(x) \in K[x]$. Clearly, in $L/I(H, B_H)$, $< p(c) > \subseteq < f(c) >$. Since, by Theorem 2.2, $f(x)$ is a polynomial of smallest degree in $K[x]$ such that $f(c) \in A$ and since $p(x)$ is irreducible in $K[x]$, $< p(c) > = < f(c) >$ in $L/I(H, B_H)$. Then $A = I(H, B_H) + < p(c) > = P$, as desired. \[\Box\]

Remark: Corollary 1.3 (that, for a prime ideal $P$, $P = PA$ for any ideal $A \supsetneq P$) can also be derived from Proposition 5.3 as $P = P \cap gr(A) = P gr(A) \subseteq PA \subseteq P$.

Recall, an ideal $I$ of a ring $R$ is called an irreducible ideal if, for ideals $A, B$ of $R$, $I = A \cap B$ implies that either $I = A$ or $I = B$. Given an ideal $I$, the radical $Rad(I)$ of $I$ is the intersection of all prime ideals containing $I$. A useful property is that if $a \in Rad(I)$, then $a^n \in I$ for some integer $n \geq 0$ (see [12]). An ideal $I$ of $R$ is said to be a primary ideal if, for any two ideals $A, B$, if $AB \subseteq I$ and $A \nsubseteq I$, then $B \subseteq Rad(I)$.

Lemma 5.4 For any ideal $I$ of $L$, $gr(Rad(I)) = gr(I)$.

Proof. Clearly $gr(I) \subseteq gr(Rad(I))$. On the other hand, if $a \in Rad(I)$, then $a^n \in I$ for some integer $n \geq 0$. This means that every idempotent element in $Rad(I)$ belongs to $I$. Since the graded ideal $gr(Rad(I))$ is generated by idempotents, $gr(Rad(I)) \subseteq I$ and hence $gr(Rad(I)) = gr(I)$. \[\Box\]

Remark: We note in passing that for any graded ideal $I$ of $L$, say $I = I(H, S)$, $Rad(I) = I$. Because, $Rad(I)/I$ is a nil ideal in $L/I$ and $L/I \cong L_K(E \setminus (H, S))$ has no non-nil ideals.

Lemma 5.5 Let $I$ be a primary or an irreducible ideal of $L$. Then $gr(I)$ is a prime ideal.

Proof. Suppose $I$ is a primary ideal. In view of Proposition II.1.4, Chapter II in [14], we need only to show that $gr(I)$ is graded prime. Consider two graded ideals $A, B$ such that $AB \subseteq gr(I)$ and $A \nsubseteq gr(I)$. As $A$ is graded, $A \nsubseteq I$. Since $I$ is primary, $B \subseteq Rad(I)$. As $B$ is a graded ideal, we have $B \subseteq gr(Rad(I)) = gr(I)$, by Lemma 5.4. Hence $gr(I)$ is a prime ideal.

Suppose now that $I$ is an irreducible ideal. As before, we need only to show that $gr(I)$ is graded prime. Suppose $A, B$ are graded ideals of $L$ such that
Let \( I \subseteq \text{gr}(I) \). If both \( A \not\subseteq \text{gr}(I) \) and \( B \not\subseteq \text{gr}(I) \), then again \( A \not\subseteq I \) and \( B \not\subseteq I \). By distributive law (Theorem 1.3) we then have,

\[
(I + A) \cap (I + B) = (I \cap I) + (I \cap B) + (A \cap I) + (A \cap B) = I + (AB), \text{ as } (A \cap B) = AB, \text{ by Lemma 3.1(i)}
\]

This contradicts the fact that \( I \) is irreducible. Hence \( \text{gr}(I) \) is a prime ideal. □

**Corollary 5.6** Suppose \( I \) is a graded ideal of \( L \). Then the following are equivalent:

(a) \( I \) is a primary ideal;
(b) \( I \) is a prime ideal;
(c) \( I \) is an irreducible ideal.

We are now ready to prove the main result of this section which extends the above corollary to arbitrary ideals of \( L \).

**Theorem 5.7** Let \( L = L_K(E) \) be the Leavitt path algebra of an arbitrary graph \( E \). Then the following properties are equivalent for an ideal \( I \) of \( L \):

(i) \( I \) is an irreducible ideal;
(ii) \( I = P^n \), a power of a prime ideal \( P \) for some \( n \geq 1 \);
(iii) \( I \) is a primary ideal.

**Proof.** (i) \( \Rightarrow \) (ii). Suppose \( I \) is an irreducible ideal. If \( I \) is a graded ideal, then \( I \) must be a prime ideal, by Corollary 5.6.

Suppose \( I \) is a non-graded irreducible ideal. By Theorem 2.2, \( I = I(H, S) + \sum_{i \in X} < f_i(c_i) > \), where for each \( i \), \( c_i \) is a cycle without exits based at a vertex \( v_i \) in \( E \setminus (H, S) \) and \( f_i(x) \in K[x] \). From Lemma 5.6, \( \text{gr}(I) \) is a graded prime ideal. By Theorem 5.1, we then have \( \text{gr}(I) = I(H, B_H) \) and \( E \setminus (H, B_H) \) is downward directed. Hence there can be only one cycle, say \( c \) based at a vertex \( v \) and without exits in \( E \setminus (H, B_H) \). Thus \( I \) is of the form \( I = I(H, B_H) + < f(x) > \) for some polynomial \( f(x) \in K[x] \). Let \( f(x) = p_1^{k_1}(x) \cdots p_m^{k_m}(x) \) be a factorization of \( f(x) \) as a product of powers of distinct irreducible polynomials \( p_i(x) \) in \( K[x] \). We claim \( m = 1 \), that is, \( f(x) \) is a power of a single irreducible polynomial. Assume, on the contrary, \( m > 1 \). Let \( g(x) = p_1^{k_1}(x) \) and \( h(x) = p_2^{k_2}(x) \cdots p_m^{k_m}(x) \). Note that each \( p_i(x) \) is still irreducible in \( K[x, x^{-1}] \). Clearly, \( < f(x) > = < g(x) > \cap < h(x) > \) in \( K[x, x^{-1}] \). If \( M \) is the (graded) ideal generated by \( c^0 \) in \( L/I(H, B_H) \), then \( M \) contains the ideals \( < f(c) >, < g(c) >, < h(c) > \) and, by Theorem 2.4, \( M \cong M_\Lambda(K[x, x^{-1}]) \) which is Morita equivalent to \( K[x, x^{-1}] \). As the ideal lattices of \( M \) and \( K[x, x^{-1}] \) are isomorphic, we then conclude that, in \( M \) and hence in \( L/I(H, B_H) \), \( < f(c) > = < g(c) > \cap < h(c) > \). Let \( A = I(H, B_H) + < g(c) > \) and \( B = I(H, B_H) + < h(c) > \). Then \( [A/I(H, B_H)] \cap [B/I(H, B_H)] = I/I(H, B_H) = < f(c) > \) and so \( A \cap B = I \). Since \( A \neq I \) and \( B \neq I \), this contradicts that \( I \) is irreducible. Hence \( I = I(H, B_H) + < p^n(c) > \) where \( p(x) \) is an irreducible polynomial and \( c \) is a cycle without exits in \( E^{0\cap}H \) and \( E^{0\cap}H \) is downward directed. It is then clear that \( P = I(H, B_H) + < p(c) > \) is a prime
Let \( A, B \) be factorizations of \( f \) have in \( \text{Rad} \) an irreducible polynomial. Suppose, on the contrary, \( I \) is a primary ideal. Hence \( I \) is a primary ideal. Consequently, without loss of generality, that both \( A, B \) such that \( AB \subseteq I \subseteq P \). If \( A \nsubseteq P \), then \( B \subseteq P \). But \( P = \text{Rad}(I) \). Hence \( I \) is an irreducible polynomial. Therefore, \( I = A \cap B \) for some ideals \( A, B \) in \( L \). Since \( I = P \cap A = (P \cap A) \cap (P \cap B) \), we may assume, without loss of generality, that both \( P \) is a primary ideal. Consider two ideals \( A, B \) such that \( AB \subseteq I \subseteq P \). If \( A \nsubseteq P \), then \( B \subseteq P \). But \( P = \text{Rad}(I) \). Hence \( I \) is a primary ideal.

(ii) \( = \) (iii) Suppose \( I = P^n \) where \( P \) is a prime ideal. Consider two ideals \( A, B \) such that \( AB \subseteq I \subseteq P \). If \( A \nsubseteq P \), then \( B \subseteq P \). But \( P = \text{Rad}(I) \). Hence \( I \) is a primary ideal.

(iii) \( = \) (ii) Suppose \( P \) is a primary ideal. By Lemma 5.3 and Corollary 5.6, we may assume that \( I \) is a non-graded ideal such that \( gr(I) = I(H, B_H) \) is a prime ideal. Consequently, \( E \) is downward directed and hence can have no more than one cycle, say \( c \) without exits in \( E^0 \). Then \( I \) will be of the form \( I = gr(I)^+ < f(c) > \) where \( f(x) \in K[x] \). We claim that \( f(x) \) is a power of an irreducible polynomial. Suppose, on the contrary, \( f(x) = p_1^{k_1}(x) \cdots p_m^{k_m}(x) \) is a factorization of \( f(x) \) as a product of powers of distinct irreducible polynomials \( p_1(x), \ldots, p_m(x) \) where \( n > 1 \). Let \( g(x) = p_1^{k_1}(x) \) and \( h(x) = p_2^{k_2}(x) \cdots p_m^{k_m}(x) \). Let \( A = gr(I)^+ < g(c) > \) and \( B = gr(I)^+ < h(c) > \). Using Lemma 5.3, we have in \( L/\text{gr}(I) \),

\[
(A/\text{gr}(I))(B/\text{gr}(I)) = < g(c) >= < h(c) >=< f(c) >= I/\text{gr}(I).
\]

Consequently, \( AB = I \). But neither \( A/\text{gr}(I) \) nor \( B/\text{gr}(I) \) is contained in \( \text{Rad}(I)/\text{gr}(I) \) which is the ideal generated by the product \( p_1(c) \cdots p_m(c) \). This contradicts that \( I \) is a primary ideal. Hence, \( I = gr(I)^+ < p^n(c) > \) where \( p(x) \) is an irreducible polynomial in \( K[x] \) and \( n \geq 0 \). Then \( I = P^n \) where \( P = gr(I)^+ < p(c) > \) is a prime ideal. This proves (ii).

To prove (i) we must show that if \( P^n \subseteq I \subseteq P \) and \( P^n \) is a primary ideal. Let \( I = A \cap B \) for some ideals \( A, B \) in \( L \). Since \( I = P \cap I = (P \cap A) \cap (P \cap B) \), we may assume, without loss of generality, that both \( A, B \subseteq P \). Thus \( P^n \subseteq A, B \subseteq P \) and so, by Lemma 5.3, \( A = P^r \) and \( B = P^s \) for some \( r, s \leq n \). Then \( P^r \cap P^s = P^n \) implies one of \( r \) or \( s \) must be \( n \). Thus \( I = A \) or \( B \). Hence \( I \) is irreducible.

6 Factorization of Ideals in \( L \)

As noted in the Introduction, ideals in an arithmetical ring admit interesting representations as products of special types of ideals. In this section, we explore the factorization of ideals in a Leavitt path algebra \( L \) as products of prime ideals and as products of irreducible/primary ideals. The prime factorization of graded ideals of \( L \) seems to influence that of the non-graded ideals in \( L \). Indeed, an ideal \( I \) is a product of prime ideals in \( L \) if and only its graded part \( gr(I) \) has the same property and, moreover, \( I/gr(I) \) is finitely generated with a generating set of cardinality no more than the number of distinct prime ideals in an irredundant factorization of \( gr(I) \). We also show that \( I \) is an intersection of irreducible ideals if and only if \( I \) is an intersection of prime ideals. If \( L \) is the Leavitt path algebra of a finite graph or, more generally, if \( L \) is two-sided
noetherian or two-sided artinian, then every ideal of $L$ is shown to be a product of prime ideals. The uniqueness of such factorizations was discussed in [6].

We begin with the following useful proposition.

Proposition 6.1 Suppose $I$ is a non-graded ideal of $L$. If $gr(I)$ is a prime ideal, then $I$ is a product of prime ideals.

Proof. By Theorem 2.2 $I = I(H, S) + \sum_{t \in T} < f_t(c_t) >$, where $T$ is some index set, for each $t \in T, c_t$ is a cycle without exits in $E(H, S), c_t \cap c_s = \emptyset$ for $t \neq s$ and $f_t(x) \in K[x]$. Now $gr(I) = I(H, S)$. If $I(H, S)$ is a prime ideal, then $E^0(H)\downarrow$ is downward directed (Theorem 5.1) and so there can be only one cycle $c$ without exits in $E(H, S)$ based at some vertex $v$. This means that $I(H, S) = I(H, B_H)$ (Theorem 5.1) and $I$ must be of the form $I = I(H, B_H) + < f(c) >$ where $f(x) = p_1(x) \cdots p_n(x)$ be a factorization of $f(x)$ as a product of irreducible polynomials $p_i(x)$ in $K[x]$. Note that each $p_i(x)$ is irreducible in $K[x, x^{-1}]$. Now, for each $j$, $P_j = I(H, B_H) + < p_j(c) >$ is a prime ideal (Theorem 5.1). Clearly $\prod_{j=1}^n P_j \supseteq I$. Using Lemma 3.3 and a simple induction on $n$, we have, in $L/I(H, B_H)$,

$$\prod_{j=1}^n P_j/I(H, B_H) = < p_1(c) > \cdots < p_n(c) >$$

$$= < p_1(c) \cdots p_n(c) > = < f(c) > = I/I(H, B_H).$$

Hence $I = \prod_{j=1}^n P_j$ is a product of prime ideals.

Theorem 6.2 Let $E$ be an arbitrary graph. For a non-graded ideal $I$ of $L := L_k(E)$, the following are equivalent:

(a) $I$ is a product of prime ideals;
(b) $I$ is a product of primary ideals;
(c) $I$ is a product of irreducible ideals;
(d) $gr(I)$ is a product of (graded) prime ideals;
(e) $gr(I) = P_1 \cap \cdots \cap P_m$ is an irredundant intersection of $m$ graded prime ideals $P_j$ and $I/gr(I)$ is generated by at most $m$ elements and is of the form $I/gr(I) = \bigoplus_{r=1}^k < f_r(c_r) >$ where $k \leq m$ and, for each $r = 1 \cdots k, c_r$ is a cycle without exits in $E^0 \setminus I$ and $f_r(x) \in K[x]$ is a polynomial of smallest degree such that $f_r(c_r) \in I$.

Proof. Now (a) $\Rightarrow$ (b) $\Rightarrow$ (c), since every prime ideal is primary and primary ideals in $L$ are irreducible, by Theorem 5.7.
(c) => (d). Suppose $I = P_1 \cdots P_n$ is a product of irreducible ideals. Clearly $gr(I) \subseteq gr(P_j)$ for all $j = 1, \ldots, n$ and so $gr(I) \subseteq \bigcap_{j=1}^{n} gr(P_j)$. On the other hand, by Lemma 5.1, $\bigcap_{j=1}^{n} gr(P_j) = \prod_{j=1}^{n} gr(P_j) \subseteq I$ and is a graded ideal. So

$$\bigcap_{j=1}^{n} gr(P_j) \subseteq gr(I),$$

by Theorem 2.2. Thus $gr(I) = \bigcap_{j=1}^{n} gr(P_j) = \prod_{j=1}^{n} gr(P_j)$. Now, by Lemma 5.5, each $gr(P_j)$ is a prime ideal. Thus $gr(I)$ is a product/intersection of (graded) prime ideals.

(d) => (e). Suppose $gr(I) = P_1 \cdots P_n$ is a product of graded prime ideals of $L$. By Lemma 5.1, $gr(I) = P_1 \cap \cdots \cap P_n$. If needed, remove appropriate ideals $P_j$ and assume that $gr(I) = P_1 \cap \cdots \cap P_m$ is an irredundant intersection of graded prime ideals $P_j$ which are thus all distinct and none contains the other ideals. Let $H = E^0 \cap I$ and $S = \{v \in B_H : v^H \in I \}$ so that $gr(I) = I(H, S)$. By Theorem 2.2, the non-graded ideal $I$ is of the form $I = I(H, S) + \sum_{t \in T} < f_t(c_t) >$, where $T$ is some index set and, for each $t$, $c_t$ is a cycle without exits in $E^0 \backslash (H, S)$ based at a vertex $v_t$ and $e_t^0 \cap e_t^0 = \emptyset$ if $t \neq s$. Now, for each $t \in T$, there must exist an index $j_t$ depending on $t$, such that $c_t \notin P_{j_t}$. Because, otherwise, $c_t \in \bigcap_{i=1}^{m} P_i = gr(I) = I(H, S)$, a contradiction. Let $P_{j_t} \cap E^0 = H_{j_t}$. Then $E^0 \backslash H_{j_t}$ is downward directed, as $P_{j_t}$ is a prime ideal. Now $v_t \in E^0 \backslash H_{j_t}$ and, since $c_t$ is a cycle without exits in $E^0 \backslash H_{j_t}$, we have $u \geq v_t$ for all $u \in E^0 \backslash H_{j_t}$. From the description of the prime ideals in Theorem 5.1 we then conclude that $P_{j_t} = I(H_{j_t}, B_{H_{j_t}})$. We claim that $v_t \notin P_j$ for all $j \neq j_t$. Suppose, on the contrary, $v_t \notin P_j$ for some $j \neq j_t$. Let $H_j = P_j \cap E^0$. Since $E^0 \backslash H_j$ is downward directed, we have $u \geq v_t$ for every $u \in E^0 \backslash H_j$ and $P_j = I(H_j, B_{H_j})$. If $P' = P_j \cap P_j$ with $H' = P' \cap E^0$, then every $u \in E^0 \backslash H' = E^0 \backslash (H_j \cap H_j) = (E^0 \backslash H_{j_t}) \cup (E^0 \backslash H_j)$ satisfies $u \geq v_t$. This means that $E^0 \backslash H'$ is downward directed and $B_{H'} = B_{H_{j_t}} \cap B_{H_j}$. Hence $P' = I(H', B_{H'})$ is a prime ideal. But then $P_j \cdot P_j = P_j \cap P_j = P'$ implies $P_j \subseteq P'$ or $P_j \subseteq P'$. This means either $P_j \subseteq P_j$ or $P_j \subseteq P_j$, contradicting the fact that $P_j \cap \cdots \cap P_m$ is an irredundant intersection. Hence, for each $t \in T$, $v_t \notin P_j$ but $v_t \in P_j$ for all $j \neq j_t$. Also, if $s \in T$ with $s \neq t$ (so $c_s \neq c_t$), then $P_j \neq P_j$. Thus $|T| \leq m$, the number of prime ideals $P_j$. Hence $T$ must be a finite set. Thus $I/g(I)$ is generated by the finite set $\{f_j(c_j) : j \in T \subseteq \{1, \ldots, m\}\}$. Now each $< f_j(c_j) >$ is an ideal in the ideal $A_j$ generated by the vertices on the cycle $c_j$. It was shown in the proof of Theorem 2.2 that $\sum A_j = \bigoplus A_j$. Hence $I/g(I) = \bigoplus_{r=1}^{k} < f_r(c_r) >$, where $k \leq m$. This proves (e).

(e) => (a). Suppose $gr(I) = P_1 \cap \cdots \cap P_m$ is an irredundant intersection
of graded prime ideals $P_j$ and $I/gr(I) = \bigoplus_{r=1}^k < f_r(c_r) >$ where $k \leq m$ and, for each $r = 1 \cdots k$, $c_r$ is a cycle without exits based at a vertex $v_r$ in $E^0 \setminus (I \cap E^0)$ and $f_r(x) \in K[x]$. Thus $I = (P_1 \cap \cdots \cap P_m) + \sum_{r=1}^k < f_r(c_r) >$. From the proof of (d) = > (c), we can assume, after re-indexing, that for each $r$, $v_r \notin P_r$ and that $v_r \in P_s$ for all $s \neq r$. For each $r = 1, \cdots, k$, define $Q_r = P_r + < f_r(c_r) >$. By Proposition 6.1, each ideal $Q_r$ is a product of prime ideals. So we are done if we show that

$$(P_1 \cap \cdots \cap P_m) + \sum_{r=1}^k < f_r(c_r) > = Q_1 \cdots Q_k P_{k+1} \cdots P_m.$$ 

We prove this by induction on $k$. Suppose $k = 1$. Consider $Q_1 P_2 \cdots P_m$. Now $A = P_2 \cdots P_m = P_2 \cap \cdots \cap P_m$ is a product of prime ideals and $v_1 \in A$. Using Lemma 3.1 and the fact that $< f_1(c_1) > \subseteq A$, we get

$$Q_1 P_2 \cdots P_m = Q_1 A = Q_1 \cap A = (P_1 + < f_1(c_1) >) \cap A = (P_1 \cap A) + < f_1(c_1) > = (P_1 \cap \cdots \cap P_m) + < f_1(c_1) >.$$ 

Suppose $k > 1$ and assume that the statement holds for $k - 1$ so that

$$Q_1 \cdots Q_{k-1} P_k \cdots P_m = (P_1 \cap \cdots \cap P_m) + \sum_{r=1}^{k-1} < f_r(c_r).$$

Then

$$Q_1 \cdots Q_k P_{k+1} \cdots P_m = Q_1 \cdots Q_{k-1}(P_k + < f_k(c_k) >)P_{k+1} \cdots P_m = Q_1 \cdots Q_{k-1}P_k \cdots P_m + Q_1 \cdots Q_{k-1}(< f_k(c_k) >)P_{k+1} \cdots P_m = (P_1 \cap \cdots \cap P_m) + \sum_{r=1}^{k-1} < f_r(c_r) > + (< f_k(c_k) >),$$

due to the fact that $< f_k(c_k) > \subseteq P_j$ for all $j \neq k$ and that

$$< f_k(c_k) > P_j = P_j < f_k(c_k) >= < f_k(c_k) >,$$

by Lemma 3.1(i). This shows that $I$ is a product of prime ideals, thus proving (a). ■

**Remark:** It is clear from the above theorem that if a graded ideal $I(H, S)$ is a product of prime ideals, then there will necessarily be at most finitely many cycles without exits in $E \setminus (H, S)$.

As an application of the above theorem, we obtain the following propositions.
Proposition 6.3 Let \( E \) be a finite graph, or more generally, let \( E^0 \) be finite. Then every ideal of \( L = L_K(E) \) is a product of prime ideals.

Proof. In view of Theorem 6.2 we need only to show that every graded ideal \( I \) of \( L \) is an intersection of finitely many prime ideals of \( L \). Let \( I \cap E^0 = H \).

Since \( L/I \cong L_K(E\setminus H) \) is a Leavitt path algebra, its prime radical is 0, and so \( I = \cap\{P : P \) prime ideal \( \supseteq I\} = \cap\{P : P \) graded prime ideal \( \supseteq I\} \). Since \( (E\setminus H)^0 \) is finite, there are only finitely many hereditary saturated subsets of \( (E\setminus H)^0 \) and so there are only finitely many graded ideals in \( L/I \). This means that \( I \) is an intersection of finitely many graded prime ideals. \( \square \)

Proposition 6.4 Suppose \( L \) is (a) two-sided artinian or (b) two-sided noetherian. Then every ideal of \( L \) is a product of prime ideals.

Proof. Assume (a). We first show that \( \{0\} \) is an intersection of finitely many prime ideals. Suppose, on the contrary, no intersection of finitely many prime ideals of \( L \) is 0. In particular, \( \{0\} \) is not a prime ideal. Let \( F = \{Q : Q \) is a non-zero intersection of finitely many prime ideals\} \) is non-empty since, as observed in Remark 2.9 in [6] (also from the proof of Proposition 6.5 below), every Leavitt path algebra always contains a prime ideal. Let \( M \) be a minimal element of \( F \), say \( M = P_1 \cap \cdots \cap P_m \neq \{0\} \). Now, for any prime ideal \( P \), \( M \cap P \neq \{0\} \), by our supposition. So by the minimality of \( M \), \( M \cap P = M \) for every prime ideal \( P \). This means \( M = \bigcap_{P \text{ prime ideal of } L} P \) which is \( \{0\} \) as the prime radical of \( L \) is zero. This contradiction shows that \( \{0\} \) is an intersection of finitely many prime ideals. Now for every graded ideal \( I = I(H,S), L/I \cong L_K(E\setminus(H,S)) \) is a Leavitt path algebra which is also two-sided artinian and, by the above argument, there are finitely many prime ideals in \( L/I \) whose intersection is zero. This means that every graded ideal \( I \) is an intersection (and, by Lemma 3.1 a product) of finitely many prime ideals. By Theorem 6.2 every ideal in \( L \) is then a product of prime ideals.

Assume (b), so the ideals of \( L \) satisfy the ascending chain condition. In view of Theorem 6.2 we need to only to show that every graded ideal \( I \) in \( L \) is a product of prime ideals. Since any graded homomorphic image of \( L \) is also a two-sided noetherian Leavitt path algebra, it is enough to show that \( \{0\} \) is a product of finitely many prime ideals in \( L \). If \( \{0\} \) is a prime ideal, we are done. Otherwise, we wish to show that \( \{0\} \) is a product of finitely many minimal prime ideals in \( L \). We shall use the usual argument given in such situations in the study of commutative rings. Assume the contrary. Consider the set \( X = \{A : A \neq 0, A \) a product of finitely many minimal prime ideals in \( L\} \). Let \( C = \{J : J \) is an ideal of \( L \) such that \( A \not\subseteq J \) for all \( A \in X\} \). Now \( C \neq \emptyset \), as \( \{0\} \in C \). Since the ascending chain condition holds, we appeal to Zorn’s Lemma to get a maximal element \( M \) of \( C \). We claim that \( M \) is a
prime ideal. To see this, suppose \( a \notin M \) and \( b \notin M \) so that \( M + LaL \supseteq A \) and \( M + LbL \supseteq B \) for some \( A, B \in X \). Then \( AB \subseteq (M + LaL)(M + LbL) \subseteq M + LbL. \) Now, by our assumption, \( AB \in X \) and so \( AB \notin M \). Consequently, \( aL \notin M \). This shows that \( M \) is a prime ideal of \( L \). If \( P \) is a minimal prime ideal inside \( M \), then \( P \neq \{0\} \) (as \( \{0\} \) is not a prime ideal) and so \( P \) satisfies \( P \in X \). Since we also have \( P \subseteq M \), we reach a contradiction. Thus \( \{0\} \) is a product of finitely many (minimal) prime ideals.

Note: (i) In [5], it was shown that the Leavitt path algebra of a finite graph is two-sided noetherian. Using this, one can also derive Proposition 6.3 from Proposition 6.4.

(ii) The reader might wonder if the descending chain condition on ideals of \( L \) implies the ascending chain condition in which case Proposition 6.4 (b) will follow from Proposition 6.4 (a). However, these two concepts are independent for Leavitt path algebras. Indeed, for the graph \( F \) in Example 7.2 below, \( L_K(F) \) is two-sided artinian, but is not two-sided noetherian. Likewise, if \( E \) is the graph with a single vertex \( v \) and a loop \( e \) based at \( v \), then \( L_K(E) \cong K[x, x^{-1}] \) is a (commutative) noetherian ring, but is not an artinian ring.

In the case when \( E \) is an arbitrary graph, we have the following description of the Leavitt path algebra \( L = L_K(E) \) in which every ideal is a product of prime ideals.

**Proposition 6.5** Let \( E \) be an arbitrary graph and let \( L = L_K(E) \). Then every proper ideal of \( L \) is a product of prime ideals if and only if, every homomorphic image of \( L \) is either a prime ring or contains only finitely many minimal prime ideals.

**Proof.** Assume that every homomorphic image of \( L \) is a prime ring or contains only finite number of minimal prime ideals. In view of Theorem 6.2, we need only to show that every graded ideal is a product/intersection of finitely many prime ideals. Suppose \( I = I(H, S) \) is a graded ideal of \( L \). If \( L = L/I \) is a prime ring, then \( I \) is a prime ideal and we are done. Suppose, \( I \) is not a prime ideal. Now \( L = L/I \cong L_K(E \setminus (H, S)) \) and by hypothesis, contains only finitely many minimal prime ideals \( P_1, \ldots, P_n \) which clearly must all be graded as \( gr(P_j) \) is also a prime ideal for each \( j \) (Theorem 2.2(b)). We claim that their intersection must be zero. Suppose, on the contrary, \( A = P_1 \cap \cdots \cap P_n \neq 0 \). Now \( A \) is a non-zero graded ideal of \( L_K(E \setminus (H, S)) \) and so \( A \) contains a vertex \( u \in E \setminus (H, S) \). Let \( P \) be an ideal of \( L \) maximal with respect to the property that \( u \notin P \). We claim that \( P \) is a prime ideal. To see this, suppose \( a \notin P \) and \( b \notin P \) are elements of \( L \). So \( u \in LaL + P \) and \( u \in LbL + P \). Then \( u = u^2 \in (LaL + P)(LbL + P) = LaLbL + P \). Since \( u \notin P \), this implies \( aLb \notin P \). Hence \( P \) is a prime ideal of \( L \). As \( P \) must contain one of the minimal prime ideals \( P_i \), we have \( u \in P_i \subseteq P \), a contradiction. Thus \( P_1 \cap \cdots \cap P_n = 0 \) and we conclude that \( I \) is the intersection and hence a product of the pre-images of the \( P_1, \cdots, P_n \) in \( L \) all of which are prime ideals in \( L \).

\footnote{I thank Zak Mesyan for pointing out this argument}
Conversely, suppose every ideal \( I \) of \( L \) is a product of prime ideals. Consider the factor ring \( L/I \). If \( I \) is a prime ideal, then \( L/I \) is a prime ring. Otherwise, by hypothesis, \( I = P_1 \cdots P_n \) where each \( P_j \) is a prime ideal. Then, in \( L/I \), \( P_1 \cdots P_n = 0 \), where \( P_j = P_j/I \). If \( Q \) is a minimal prime ideal in \( L/I \), then \( P_1 \cdots P_n = 0 \). By minimality, \( Q = P_j \). This shows that \( L/I \) contains only finitely many minimal prime ideals.

### 7 Examples

We next construct various graphs illustrating the results obtained in the preceding sections. These examples are also used to examine whether some of the well-known theorems in commutative rings, such as the Cohen’s theorem on prime ideals and theorems on ZPI rings, hold for Leavitt path algebras.

**Example 7.1** Consider the following “\( \mathbb{N} \times \mathbb{N} \)-Lattice” graph \( E \) where the vertices in \( E \) are points in the first quadrant of the coordinate plane whose coordinates are integers \( \geq 0 \). Specifically, \( E^0 = \{(m,n) : m,n \in \mathbb{Z} \text{ with } m,n \geq 0 \} \). Every vertex \((m,n)\) emits two edges connecting \((m,n)\) with \((m+1,n)\) and \((m,n+1)\).

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\uparrow & \uparrow & \uparrow & \uparrow \\
\bullet(0,3) & \rightarrow & \bullet(1,3) & \rightarrow & \bullet(2,3) & \rightarrow & \bullet(3,3) & \rightarrow \cdots \\
\uparrow & \uparrow & \uparrow & \uparrow \\
\bullet(0,2) & \rightarrow & \bullet(1,2) & \rightarrow & \bullet(2,2) & \rightarrow & \bullet(3,2) & \rightarrow \cdots \\
\uparrow & \uparrow & \uparrow & \uparrow \\
\bullet(0,1) & \rightarrow & \bullet(1,1) & \rightarrow & \bullet(2,1) & \rightarrow & \bullet(3,1) & \rightarrow \cdots \\
\uparrow & \uparrow & \uparrow & \uparrow \\
\bullet(0,0) & \rightarrow & \bullet(1,0) & \rightarrow & \bullet(2,0) & \rightarrow & \bullet(3,0) & \rightarrow \cdots \\
\end{array}
\]

Now \( E \) is a row finite graph and and contains no cycles. So the ideals of \( L = L_K(E) \) are generated by vertices in \( E \). If \( A \) is a non-zero proper ideal of \( L \), then \( A \) is a principal ideal. To see this, suppose \((i,j) \in A\) such that \( i+j = k \) is the smallest integer, then it is easy to check that \( A \) contains the entire quadrant \( Q_{ij} = \{(m,n) : m \geq i, n \geq j \} \). It is also easy to verify that in this case, no other vertex \((m,n)\) satisfying \( m+n = k \) will be in \( A \). Thus \( A = Q_{ij} \) is the principal ideal generated by the single vertex \((i,j)\). Moreover, if \((i,j) \neq (m,n)\) are different vertices, then the ideals \(<(i,j)> \neq <(m,n)>\). We list below the various properties of \( L \):

- \( L \) is a prime ring.
- All the ideals of \( L \) are graded.
- Every non-zero proper ideal of \( L \) is a principal ideal generated by a single vertex.
- An ideal \( P \) is a prime ideal of \( L \) if and only \( P \) is generated by a vertex "on the axis ", that is, is generated by a vertex of the form \((0,m)\) or \((m,0)\).
Every ideal of $L$ is either a prime ideal or is a product of two prime ideals. Indeed, if $A = (m, n)$ with $m \neq 0$ and $n \neq 0$, then $A = PQ = P \cap Q$, where $P = (m, 0)$ and $Q = (0, n)$.

**Example 7.2** Consider the following row-finite graph $F$ in which, for each of the infinitely many $i$, there are two loops at the vertex $v_i$ and this indicated by $\bigcirc$.

\[
\begin{array}{cccc}
  \cdots & \leftarrow \bullet & \leftarrow \bullet & \leftarrow \bullet \\
  & \downarrow & \downarrow & \downarrow \\
  \cdots & \rightarrow \bigcirc & \rightarrow \bigcirc & \rightarrow \bigcirc \\
  & \uparrow & \uparrow & \uparrow \\
  \cdots & \leftarrow \bullet & \leftarrow \bullet & \leftarrow \bullet \\
  \end{array}
\]

Now $F$ satisfies Condition (K) and so all the ideals of $L_K(F)$ are graded. Also $F^0$ is downward directed and so $\{0\}$ is a prime ideal. For each $n \geq 1$, $H_n = \{v_1, \ldots, v_n\}$ is a hereditary saturated set and $E^0 \setminus H_n$ is downward directed. Hence $P_n = (H_n)$ is a prime ideal and we get an ascending chain of prime ideals

\[0 \subset P_1 \subset \cdots \subset P_n \subset \cdots \tag{**}\]

Let $P_\omega = \bigcup_{n \in \mathbb{N}} P_n$. Let $M_1 = (u_1)$ and $M_2 = (w_1)$ be the ideals generated by $u_1, w_1$ respectively in $L_K(F)$. It is straightforward to verify the following:

(a) The non-zero proper ideals of $L_K(F)$ are precisely the (graded) ideals $P_n (n \geq 1)$, $P_\omega$, $M_1$ and $M_2$.

(b) All the non-zero ideals of $L_K(F)$, other than $P_\omega$, are prime ideals. The ideal $P_\omega$ is not a prime ideal since $E^0 \setminus H$ is not downward directed, where $H = P_\omega \cap E^0 = \{v_n : n \geq 1\}$. However, $P_\omega = M_1 \cap M_2 = M_1M_2$ is a product/intersection of two prime ideals. Thus $L_K(F)$ is a prime ring in which every ideal is a product of at most two prime ideals.

(c) For each $n$, $P_n = (v_n)$ is a principal ideal generated by the vertex $v_n$. Thus all the non-zero ideals of $L_K(F)$, other than $P_\omega$, are principal ideals. $P_\omega$ is not finitely generated.

(d) $L_K(F)$ is a two-sided artinian ring, but is not two-sided noetherian.

Example 7.2 is also an example to illustrate the following statements.

I) A well-known theorem of Cohen states that if $R$ is a commutative ring and if every prime ideal of $R$ is finitely generated, then $R$ is a noetherian ring. Example 7.2 shows that Cohen’s theorem does not hold for the two-sided ideals in Leavitt path algebras (as every prime ideal of $L_K(F)$ is a principal ideal, but the ideal $P_\omega$ is not finitely generated). Also the chain of ideals (**) shows that the ascending chain condition does not hold in $L_K(F)$.

II) It is known ([13]) that if $R$ is a commutative ring in which every ideal is a product of prime ideals, then $R$ must be a noetherian ring. Such rings are known as generalized ZPI rings and have been completely characterized. Example 7.2 shows the Leavitt path algebra $L_K(F)$ is a generalized ZPI ring, but it is not two-sided noetherian.
III) In a commutative ring, the union of an ascending chain of prime ideals is again a prime ideal. In the Example 7.2, $P_\omega$ is the union of a countable ascending chain of prime ideals, but $P_\omega$ is not a prime ideal of $L_K(F)$.

IV) As a passing remark, we point out that in the case of a commutative ring $R$ with identity, the ascending chain condition on ideals of $R$ is equivalent to every ideal of $R$ being finitely generated. Example 7.2 shows that this no longer holds in Leavitt path algebras as is clear by considering $L'_K = L_K(F')$, the Leavitt path algebra of the graph $F' = F\setminus\{u_n : n \geq 1\} \cup \{w_n : n \geq 1\}$, where every proper ideal of $L'$ is a principal ideal, but the ascending chain condition for ideals does not hold in $L'$.

V) Also, in a commutative ring, a product of two finitely generated ideals is again finitely generated. But in the Leavitt path algebra $L_K(F)$ in Example 7.2, $M_1$ and $M_2$ are principal ideals, but the product $M_1M_2 = P_\omega$ is not finitely generated.

We next give an example of an ideal in a Leavitt path algebra which cannot be factored as a product of finitely many prime ideals.

**Example 7.3** Let $E$ be a graph with $E^0 = \{v, v_1, \ldots, v_n, \cdots\}$ and $E^1 = \{e_1, \cdots, e_n, \cdots\} \cup \{e_1, \cdots, e_n, \cdots\}$. Further, for each $i$,

$s(e_i) = v_i$ and $r(e_i) = v$. Also each $c_i$ is a loop at $v_i$, $s(c_i) = r(c_i) = v_i$.

Now $L_K(E)$ is a prime ring. The prime ideals of $L_K(E)$ are ${\{0\}}$ and, for each $i = 1, 2, \cdots$, the graded ideal $P_i = \langle \{v_j : j \neq i\} \rangle$ and the non-graded ideal $Q_i^{(x)} = P_i + \langle p(c_i) \rangle$, for each irreducible polynomial $p(x) \in K[x]$, . Now the graded ideal $A = \langle v \rangle$ is not an intersection of finitely many prime ideals. Then, by Theorem 6.2, we have, for any irreducible polynomial $p(x) \in K[x]$ and for any subset $S \subseteq \{c_1, \cdots, c_n, \cdots\}$, the ideal

$I = A + \sum_{c_i \in S} \langle p(c_i) \rangle$ is then not a product of finitely many prime ideals.

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