NC Calabi-Yau Orbifolds in Toric Varieties with Discrete Torsion

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Abstract

Using the algebraic geometric approach of Berenstein et al (hep-th/005087 and hep-th/009209) and methods of toric geometry, we study non commutative (NC) orbifolds of Calabi-Yau hypersurfaces in toric varieties with discrete torsion. We first develop a new way of getting complex d mirror Calabi-Yau hypersurfaces $H^*_\Delta$ in toric manifolds $M^{*(d+1)}_\Delta$ with a $\mathbb{C}^r$ action and analyze the general group of the discrete isometries of $H^*_\Delta$. Then we build a general class of d complex dimension NC mirror Calabi-Yau orbifolds where the non commutativity parameters $\theta_{\mu \nu}$ are solved in terms of discrete torsion and toric geometry data of $M^{(d+1)}_\Delta$ in which the original Calabi-Yau hypersurfaces is embedded. Next we work out a generalization of the NC algebra for generic d dimensions NC Calabi-Yau manifolds and give various representations depending on different choices of the Calabi-Yau toric geometry data. We also study fractional D-branes at orbifold points. We refine and extend the result for NC $(T^2 \times T^2 \times T^2)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ to higher dimensional torii orbifolds in terms of Clifford algebra.

Key words: Toric geometry, mirror symmetry, orbifolds of Calabi-Yau hypersurfaces with discrete torsion, non commutative geometry, fractional D-branes.

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Contents

1 Introduction 2

2 Toric geometry of CY Manifolds 4
   2.1 Toric realization of CY manifolds 4
   2.2 Solving the mirror constraint eqs 7
   2.3 More on the mirror CY geometry 10

3 Discrete Symmetries and CY Orbifolds 11
   3.1 $P^{d+1}$ projective spaces 11
      3.1.1 Explicit construction of $b_\mu$ weights 12
      3.1.2 Complex Deformations 13
   3.2 $WP^{d+1}$ weighted projective spaces 14

4 NC Toric CY Manifolds 15
   4.1 Algebraic geometric approach for CY 15
   4.2 NC toric CY orbifolds 16
      4.2.1 Matrix representations for projective spaces 17
      4.2.2 Solution for weighted projective spaces 20
      4.2.3 Fractional Branes 22

5 Link with the BL Construction 23
   5.1 More on the NC quintic 25
   5.2 Comments on lower dimension CY manifolds 26

6 NC Elliptic Manifolds 27
   6.1 Solution I 30
   6.2 Solution II 31

7 Conclusion 31
1 Introduction

Non-commutative (NC) geometry plays an interesting role in the context of string theory [1] and in compactification of the Matrix model formulation of M-theory on NC torii [7], which has opened new lines of research devoted to the study NC quantum field theories [8]; see also [9-23]. In the context of string theory, NC geometry is involved whenever an antisymmetric $B$-field is turned on. For example, in the study of the ADHM construction $D(p-4)/Dp$ brane systems ($p > 3$) [24], the NC version of the Nahm construction for monopoles [25] and in the study of tachyon condensation using the so called GMS approach [26], see also[27-31].

More recently efforts have been devoted to go beyond the particular NC $R^d_\theta$, NC $T^d_\theta$ geometries [29-37]. A special interest has been given to build NC Calabi-Yau manifolds containing the commutative ones as subalgebras and a development has been obtained for the case of orbifolds of Calabi-Yau hypersurfaces. The key point of this construction, using a NC algebraic geometric method [38], see also [39, 40], is based on solving non commutativity in terms of discrete torsion of the orbifolds. In this regards, there are two ways one may follow to construct this extended geometry. (i) A constrained approach using purely geometric analysis, which we are interested in this paper. (ii) Crossed product algebra based on the techniques of the fibre bundle and the discrete group representations. For the first way, it has been shown that the $\frac{T^2 \times T^2 \times T^2}{\mathbb{Z}_2 \times \mathbb{Z}_2}$, orbifold of the product of three elliptic curves with torsion, embedded in the $C^6$ complex space, defines a NC Calabi-Yau threefolds [39] having a remarkable interpretation in terms of string states. Moreover on the fixed planes of this NC threefolds, branes fractionate and local complex deformations are no more trivial. This constrained method was also applied successfully to Calabi-Yau hypersurfaces described by homogeneous polynomials with discrete symmetries including $K3$ and the quintic as particular geometries [39, 40, 41, 42, 43]. NC algebraic geometric approach for building NC Calabi-Yau manifolds has very remarkable features and is suspected to have deep connections both with the intrinsic properties of toric varieties [44, 45, 46] and the R matrix of Yang-Baxter equations of quantum spaces [47, 48, 49].

In this study we extend the Berenstein and Leigh (BL for short) construction for NC Calabi-Yau manifolds with discrete torsion by considering $d$ dimensional complex Calabi-Yau orbifolds embedded in $(d+1)$ complex toric manifolds and using toric geometry method [50, 51, 52, 53]. In particular, we build a general class of $d$ complex dimension non commutative mirror Calabi-Yau orbifolds for which the non commutativity parameters $\theta_{\mu\nu}$ are solved in terms of discrete torsion and toric geometry data of dual polytopes $\Delta(M^d)$. To establish these results, we will proceed in three steps:
(i) We consider pairs of mirror Calabi-Yau hypersurfaces $H_{\Delta}^d$ and $H_{\Delta}^{*d}$ respectively embedded in the toric manifolds $M_{\Delta}^{d+1}$ and $M_{\Delta}^{(d+1)}$, where $\Delta$ is their attached polyhedron; and develop a manner of handling these spaces by working out the explicit solution for the so called $Y_{\alpha} = \prod_{i=1}^{k+1} x_i^{<V_i,V_{\alpha}^*>}$ invariants of the $C^{sr}$ actions and their mirrors $y_i = \prod_{f=1}^{k^*} z_f^{<V_{\alpha}^*,V_i>}$. The construction we will give here is a new one; it is based on pushing further the solving of the Calabi-Yau constraint eqs regarding the invariants under the $C^{sr}$ toric actions. Aspects of this analysis may be approached with the analysis of [53, 54], but the novelty is in the manner we treat the $C^{sr}$ invariants. Then we fix our attention on $H_{\Delta}^{*d}$ described by the zero of a homogeneous polynomial $P_{\Delta}(z)$ of degree $D$ and explore the general form of the group of discrete symmetries $\Gamma$ of $H_{\Delta}^{*d}$ using the toric geometry data $\{q_{ai}; V_i ; 1 \leq i \leq k+1; 1 \leq a \leq r; \quad d = (k - r)\}$ of the polyedron $\Delta$.

(ii) We show that for the special region in the moduli space where complex deformations are set to zero, the polynomials $P_{\Delta}$ defining the Calabi-Yau hypersurfaces have a larger group of discrete symmetries $\Gamma_0$ containing as a subgroup the usual $\Gamma_{cd}$ one; $\Gamma_{cd} \subset \Gamma_0$. We treat separately the two corresponding orbifolds $O_0$ and $O_{cd}$ and study their link to each other.

(iii) Finally we construct the NC extension of the Calabi-Yau hypersurfaces by first deriving the right constraint equations, then solve non commutativity in terms of discrete torsion and toric geometry data of the variety. This method can be applied to higher dimensional NC torii orbifolds extending the result of NC $(T^2 \times T^2 \times T^2)/(Z_2 \times Z_2)$ Calabi-Yau threefolds. In this case, the general solution is given in terms of $d$-dimensional Clifford algebra.

The organization of this paper is as follows: In section 2, we review the main lines of Calabi-Yau hypersurfaces using toric geometry methods. Then we develop a method of getting complex $d$ Calabi-Yau mirror coset manifolds $C^{k+1}/C^{sr}$, $k-r = d$, as hypersurfaces in $WP^{d+1}$, by solving the $y_i$ inavriants of mirror geometry in terms of invariants of the $C^*$ action of the weighted projective space and the toric geometry data of $C^{k+1}/C^{sr}$. In section 3, we explore the general form of discrete symmetries of the mirror hypersurface using their toric geometry data. Then we discuss orbifolds of toric Calabi-Yau hypersurfaces. In section 4, we build the corresponding NC toric Calabi-Yau algebras using the algebraic geometry approach of [38, 39]. Then we work out explicitly the matrix realizations of these algebras using toric geometry ideas. In section 5, we give the link with the BL construction while in section 6 we give the generalization of the NC $T^2\times T^2 \times T^2$ orbifold to $T^2 \otimes (2k+1)$, $k \geq 1$, where $(T^2) \otimes (2k+1)$ is realized by $(2k+1)$ elliptic curves embedded in $C^{(4k+2)}$ complex space. Our construction, which generalizes that of [39] given by $k = 1$, involves non commuting operators satisfying the
2k dimensional Clifford algebra. We end this paper by giving our conclusion.

2 Toric geometry of CY Manifolds

2.1 Toric realization of CY manifolds

The simplest \((d+1)\)-complex dimension toric manifold, which we denote as \(M^{d+1}_\Delta\), is given by the usual complex projective space \(P^{d+1} = \{C^{d+2}_\Delta - 0_{d+2}\}/C^5\) [55, 56, 57]. One can also build \(M^{d+1}_\Delta\) varieties by considering the \((k+1)\)-dimensional complex spaces \(C^{k+1}\), parameterized by the complex coordinates \(\{x = (x_1, x_2, x_3, ..., x_{k+1})\}\), and \(r\) toric actions \(T_a\) acting on the \(x_i\)'s as:

\[
T_a : x_i \rightarrow x_i \left(\lambda^a_i\right).
\]

(2.1)

Here the \(\lambda_a\)'s are \(r\) non zero complex parameters and \(q^a_i\) are integers defining the weights of the toric actions \(T_a\). Under these actions, the \(x_i\)'s form a set of homogeneous coordinates defining a \((d+1)\) complex dimensional coset manifold \(M^{d+1} = (C^{k+1})/C^r\) with dimension \(d = (k-r)\).

More generally, toric manifolds may be thought of as the coset space \((C^{k+1} - \mathcal{P})/C^r\) with \(\mathcal{P}\) a given subset of \(C^{k+1}\) defined by the \(C^r\) action and a chosen triangulation. \(\mathcal{P}\) generalizes the standard \(\{0_{k+1} = (0, 0, 0, ..., 0)\}\) singlet subset that is removed in the case of \(P^k\). One of the beautiful features of toric manifolds is their nice geometric realization known as the toric geometry representation. The toric data of this realization are encoded in a polyhedron \(\Delta\) generated by \((k+1)\) vertices carrying all geometric informations on the manifold. These data are stable under \(C^r\) actions and are useful in the geometric engineering method of \(4D\) \(\mathcal{N} = 2\) supersymmetric quantum field theory in particular in the building of the basic \((d+1)\) gauge invariant coordinates system \(\{u_I\}\) of the \((C^{k+1} - \mathcal{P})/C^r\) coset manifold in terms of the homogeneous coordinates \(x_i\) [50, 51, 52, 53, 55].

In toric geometry, \((d+1)\) complex manifolds \(M^{d+1}_\Delta\) are generally represented by an integral polytope \(\Delta\) spanned by \((k+1)\) vertices \(V_i\) of the standard lattice \(Z^{d+1}\). These vertices fulfill \(r\) relations given by:

\[
\sum_{i=1}^{k+1} q^a_i V_i = 0, \quad a = 1, ..., r,
\]

(2.2)

and are in one to correspondence with the \(r\) actions of \(C^r\) on the complex coordinates \(x_i\) eq(2.1). In the above relation, the \(q^a_i\) integers are the same as in eq(2.1) and are interpreted, in the \(\mathcal{N} = 2\) gauged linear sigma model language, as the \(U(1)^r\) gauge charges of the \(x_i\),
complex field variables of two dimensional $\mathcal{N} = 2$ chiral multiplets [56-62]. They are also known as the entries of the Mori vectors describing the intersections of complex curves $C_a$ and divisors $D_i$ of $M_{\Delta}^{d+1}$ [63, 64, 65].

Submanifolds $\mathcal{N}$ of $M_{\Delta}^{d+1}$ may be also studied by using the $\Delta$ toric data $\{q_a^s, V_i^s\}$ of the original manifold. An interesting example of $M_{\Delta}^{d+1}$ subvarieties is given by the $d$ complex dimension Calabi-Yau manifolds $H^d_{\Delta}$ defined as hypersurfaces in $M_{\Delta}^{d+1}$ as follows [53]:

$$p(x_1, x_2, x_3, \ldots, x_{k+1}) = \sum_I b_I^{k+1} \prod_{i=1}^{k+1} x_i^{<V_i, V^*_I>} = 0,$$

(2.3)

together with the Calabi-Yau condition

$$\sum_{i=1}^{k+1} q_i^a = 0, \quad a = 1, \ldots, r. \quad (2.4)$$

The $V^*_I$'s appearing in the relation (2.3) are vertices in the dual polytope $\Delta^*$ of $\Delta$; their scalar product with the $V_i$'s is positive, $< V_i, V^*_I > \geq 0$. For convenience, we will set from now on $< V_i, V^*_I > = n_i^I$. The $b_I$ coefficients are complex moduli describing the complex structure of $H^d_{\Delta}$; their number is given by the Hodge number $h^{(d-1,1)}(H^d_{\Delta})$. Using the $n_i^I$ integers, the $d$ dimensional hypersurfaces $H^d_{\Delta}$ in $M_{\Delta}^{d+1}$ eq(2.3) read as

$$\sum_I b_I^{k+1} \prod_{i=1}^{k+1} x_i^{n_i^I} = 0. \quad (2.5)$$

At this stage it is interesting to make some remarks regarding the above relation. At first sight, one is tempted to make a correspondence between this relation and the hypersurface eq used in [39] and take it as the starting point to build NC Calabi-Yau manifolds a la Berenstein et al. However this is not so obvious; first because the polynomial (2.5) is not a homogeneous one and second even though one wants to try to bring it to a homogeneous form, one has to specify the toric data $\{q_I^s; V_I^s\}$ of the polyhedron $\Delta^*$; mirror to $\{q_i^a; V_i\}$ data of $\Delta$. The mirror data satisfy similar relations as (2.2) and (2.4) namely:

$$\sum_{I=1}^{k^*+1} q_I^{sA} = 0 \quad A = 1, \ldots, r^*, \quad (2.6)$$

$$\sum_{I=1}^{k^*+1} q_I^{sA} V_I^* = 0 \quad A = 1, \ldots, r^*$$

together with $k+1-r = k^*+1-r^* = d$. Moreover setting $Y_I = \prod_{i=1}^{k+1} x_i^{n_i^I}$, the above polynomial becomes a linear combination of the $Y_I$ gauge invariants as $\sum_I b_I Y_I = 0$. This relation can
however be rewritten in terms of the \((d+1)\) dimensional generator basis \(\{Y_\alpha; 1 \leq \alpha \leq (d+1)\}\) as follows

\[
1 + \sum_{\alpha=1}^{d+1} b_\alpha Y_\alpha + \sum_{I=d+2}^{k^*+1} b_I Y_I = 0, \tag{2.7}
\]

where the remaining \(Y_I\) invariants, that is the set \(\{Y_I; (d + 2) \leq I \leq (k^* + 1)\}\) are determined by solving the following Calabi-Yau constraint eqs

\[
\prod_{I=1}^{k^*+1} Y_I^{q_i A'} = 1; \quad A = 1, ..., r^*. \tag{2.8}
\]

To realize the relation (2.7) as a homogeneous polynomial describing the hypersurfaces \(H^d_\Delta\) with the desired properties, in particular the Calabi-Yau condition, one has to solve the above constraint eqs. Though this derivation can apriori be done using (2.8), we will not proceed in that way. What we will do instead is to use the so called mirror Calabi-Yau manifolds \(H^*_{\Delta}\) and derive its homogeneous description. The point is that the mirror geometry has some specific features and constraint eqs that involve directly the toric data \(\{q_i^a; V_i^\star\}\) of the \(\Delta\) polyhedron contrary to the original hypersurfaces \(H^d_\Delta\) which involve the \(\{q_i^a; V_i^\star\}\) data of \(\Delta^\star\). Once the rules of getting the \(H^d_{\Delta}\) homogeneous hypersurfaces are defined, one can also reconsider the analysis of \(H^d_\Delta\) by starting from the relations (2.7-8), use the \(\Delta^\star\) toric data and perform similar analysis to that we will be developing herebelow.

Under mirror symmetry, toric manifolds \(M^{(d+1)}_\Delta\) and Calabi-Yau hypersurfaces \(H^d_\Delta\) are mapped to \(M^{(d+1)^\star}_\Delta\) and \(H^{d^\star}_{\Delta}\) respectively. They are obtained by exchanging the roles of complex and Kahler structures in agreement with the Hodge relations

\[
h^{(d-1,1)}(H^d_\Delta) = h^{(1,1)}(H^{d^\star}_{\Delta}), \tag{2.9}
\]

and similarly for \(M^{(d+1)}_\Delta\) and \(M^{(d+1)^\star}_\Delta\) [64-67]. In practice the building of \(M^{(d+1)^\star}_\Delta\) and so \(H^{d^\star}_{\Delta}\) is achieved by using the vertices \(V_i^\star\) of the convex hull spanned by the \(V_i^\star\). Following [66, 67, 68, 69, 70, 71, 72], mirror Calabi-Yau manifolds \(H^{d^\star}_{\Delta}\) is given by the zero of the polynomial

\[
p(z_1, z_2, ..., z_{k^*+1}) = \sum_{i=1}^{k+1} a_i \prod_{I=1}^{k^*+1} \left( z_I^{q_i}\right), \tag{2.10}
\]

where the \(z_I\)’s are the mirror coordinates. The \(C^{*^\star}\) actions of \(M^{(d+1)^\star}_\Delta\) act on the \(z_I\)’s as

\[
z_I \rightarrow z_I^{\lambda_i A'}, \tag{2.11}
\]
with $q_i^* A$ as in eq(2.6). The $a_i$’s are the complex structure of the mirror Calabi-Yau manifold $H_\Delta^*$; they describe also the Kahler deformations of $H_\Delta^*$. An interesting feature of the relation (2.10) is its representation in terms of the $(k+1)$ invariants $y_i = \prod_{I=1}^{k+1} \left( z^m_I \right)$ under the $C^* r^*$ actions of $M_\Delta^*$; i.e:

$$\sum_{i=1}^{k+1} a_i y_i = 0,$$

(2.12)

together with the $r$ following constraint eqs of the mirror geometry

$$\prod_{i=1}^{k+1} (y_i^{q_a}) = 1, \quad a = 1, ..., r.\quad\quad\quad (2.13)$$

These eqs involve $(k+1)$ variables $y_i$, not all of them independent since they are subject to $(r+1)$ conditions ($r$ from eqs(2.13) and one from (2.12)) leading indeed to the right dimension of $H_\Delta^*$. Eqs(2.12-13) will be our starting point towards building NC alabi-Yau manifolds using the Berenstein et al approach. Before that let us put these relations into a more convenient form.

2.2 Solving the mirror constraint eqs

As shown on the above eqs, not all the $y_i$’s are independent variables, only $(d+1)$ of them do. In what follows we shall fix this redundancy by using a coordinate patch of the $(d+1)$ weighted projective spaces $WP^{d+1}$ parameterized by the system of variables $\{u_\alpha, 1 \leq \alpha \leq (d+1); u_{d+2}\}$. In the coordinate patch $u_{d+2} = 1$, the $u_\alpha$ variables behave as $(d+1)$ independent gauge invariants parameterizing the coset manifold \[(C^{d+2})/C^* \sim [(C^{k+1})/C^*].\] The $r$ remaining $y_i$’s are given by monomials of the $u_\alpha$’s. A nice way of getting the relation between $y_i$’s and $u_\alpha$’s is inspired from the analysis [53, 54]; it is based on introducing the following system $\{N_i; 1 \leq i \leq (k+1)\}$ of $(d+1)$ dimensional vectors of integer entries $(N_i)_\alpha = < V_i, V_\alpha^* > \equiv n^\alpha_i$. From eq(2.2), it is not difficult to see that:

$$\sum_{i=1}^{k+1} q_i^a N_i = 0, \quad a = 1, ..., r; \quad \alpha = 1, ..., d+1,$$

(2.14)

or equivalently:

$$\sum_{i=1}^{k+1} q_i^a n^\alpha_i = 0, \quad a = 1, ..., r; \quad \alpha = 1, ..., d+1.$$

(2.15)

Note that the introduction of the system $\{(N_i)_\alpha \equiv n^\alpha_i; 1 \leq i \leq (k+1)\}$ has a remarkable interpretation; it describes the complex deformations of $H_\Delta^*$ and by the correspondence (2.9)
the Kahler ones of $H_{\Delta}^d$. Observe also that shifting the $N_i$’s by a constant vector, say $t_0$, eq(2.14) remains invariant due to the Calabi-Yau condition (2.4). Therefore the $V_i$ vertices of eqs(2.2) can be solved by a linear combination of $N_i$ and $t_0$; $V_i = N_i + at_0$. Having these relations in mind, we can use them to reparametrize the $y_i$ in terms of the $(d + 2)$ generators $u_\mu$ ($u_{d+2}$ arbitrary) as follows

$$
y_i = u_1^{(n_1^i - 1)} u_2^{(n_2^i - 1)} \cdots u_{d+1}^{(n_{d+1}^i - 1)} u_{d+2}^{(n_{d+2}^i - 1)} = \prod_{\mu=1}^{d+2} u_\mu^{(n_\mu^i - 1)},$$

$$y_0 = 1 \Leftrightarrow (n_0^\alpha - 1) = 0, \quad \forall \alpha = 1, \ldots, d + 2. \tag{2.17}$$

Note that $\prod_{i=1}^{k+1}(y_i^{d_i}) = 1$ is automatically satisfied due to eqs(2.14) and(2.15). Note also the $n_i^{d+2}$ integers are extra quantities introduced for later use; they should not be confused with the $\{n_i^\alpha; 1 \leq \alpha \leq d + 1\}$ entries of $N_i$. Putting the relations (2.16-17) back into eq(2.12), we get an equivalent way of writing eq(2.10), namely:

$$a_0 1 + \sum_{i=1}^{k+1} a_i u_1^{(n_1^i - 1)} u_2^{(n_2^i - 1)} \cdots u_{d+1}^{(n_{d+1}^i - 1)} u_{d+2}^{(n_{d+2}^i - 1)} = 0. \tag{2.18}$$

The main difference between this relation and eq(2.10) is that the above one involve $(d + 2)$ variables only, contrary to the case of eq (2.10) which rather involve $(d + r^* + 1)$ coordinates; that is $r^*$ variables in more. Eq (2.18) is then a relation where the $C^{r^*}$ symmetries on the $z_i$’s eq(2.11) are completely fixed. Indeed starting from eq (2.10), it is not difficult to rederive eq (2.18) by working in the remarkable coordinate patch $\mathcal{U} = \{(z_1, z_2, \ldots, z_{d+2}, 1, 1, \ldots, 1)\}$, which is isomorphic to a weighted projective space $WP^{d+1}_{(\delta_1, \ldots, \delta_{d+2})}$ with a weight vector $\delta_\mu = (\delta_1, \ldots, \delta_{d+2})$. In this way of viewing things, the $y_i$ variables may be thought of as gauge invariants under the projective action $WP^{d+1}_{(\delta_1, \ldots, \delta_{d+2})}$ and consequently the Calabi-Yau manifold (2.18) as a hypersurface in $WP^{d+1}_{(\delta_1, \ldots, \delta_{d+2})}$ described by a homogeneous polynomial $p(u_1, \ldots, u_{d+2})$ embedded of degree $D = \sum_{\mu=1}^{d+2} \delta_\mu$. Thus, under the projective action $u_\mu \rightarrow \lambda^{\delta_\mu} u_\mu$, the monomials $y_i = \prod_{\mu=1}^{d+2} u_\mu^{(n_\mu^i - 1)}$ transform as $y_i \lambda^{\sum_\mu (\delta_\mu (n_\mu^i - 1))}$ and so the following constraint eqs should hold,

$$\sum_{\mu=1}^{d+2} \delta_\mu = D, \tag{2.19}$$

$$\sum_{\mu=1}^{d+2} \delta_\mu n_\mu^i = D. \tag{2.20}$$

These relations show that the $n_\mu^i$ integers can be solved in terms of the partitions $d_\mu^i$ of the degree $D$ of the homogeneous polynomial $p(u_1, \ldots, u_{d+2})$. Indeed from $\sum_{\mu=1}^{d+2} d_\mu^i = D$, one sees
that $n_i^\mu = \frac{d_\mu}{\delta_\mu}$; among which we have the following remarkable ones

$$n_i^\mu = \frac{D}{\delta_\mu} \quad i f \quad i = \mu \quad \text{for} 1 \leq \mu \leq d + 2. \quad (2.21)$$

To get the $V_i$ vertices, we keep the \{ $n_i^\alpha$; 1 $\leq$ $\alpha$ $\leq$ d + 1 \} entries and substruct the trivial monomial associated with \{(t_0^\alpha) = (1, 1, ..., 1)\}. So the $V_i$ vertices are

$$V_i^\alpha = n_i^\alpha - t_0^\alpha = \frac{d_\mu}{\delta_\alpha} - t_0^\alpha. \quad (2.22)$$

For the $(d + 3)$ leading vertices, we have:

$$V_0 = (0, 0, 0, ..., 0, 0)$$
$$V_1 = \left( \frac{D}{\delta_1} - 1, -1, -1, ..., -1, -1 \right),$$
$$V_2 = (-1, \frac{D}{\delta_2} - 1, -1, ..., -1, -1) \quad (2.23)$$
$$V_3 = (-1, -1, \frac{D}{\delta_3} - 1, ..., -1, -1)$$
$$...$$
$$V_d+1 = (-1, -1, -1, ..., \frac{D}{\delta_{d+1}} - 1, -1),$$
$$V_{d+2} = (-1, -1, -1, ..., -1, \frac{D}{\delta_{d+2}} - 1).$$

Before going ahead, let us give some remarks: (a) the integrality of the entries of these vertices requires that the $D$ degree should be a commun multiple of the weights $\delta_\mu$. Moreover the number of partitions of $D$ should be less than $(k + 2)$. (b) As far the $(d + 3)$ leading vertices are concerned, the corresponding homogeneous monomials are

\[ N_0 \rightarrow \prod_{\mu=1}^{d+2} u_\mu, \quad (2.24) \]
\[ N_\mu \rightarrow u_\mu^{\mu}, \quad \mu = 1, ..., d + 2. \quad (2.25) \]

So the corresponding mirror polynomial takes the form:

\[ \sum_{\mu=1}^{d+2} u_\mu^{\mu} + a_0 \prod_{\mu=1}^{d+2} (u_\mu) = 0. \quad (2.26) \]

More generally the mirror polynomial $P_\Delta(u)$ describing $H_{\Delta}^{d^*_x}$ reads as

$$P_\Delta(u) = \sum_{\mu=1}^{d+2} u_\mu^{\mu} + a_0 \prod_{\mu=1}^{d+2} (u_\mu) + \sum_{i=d+3}^{k+1} a_i \prod_{\mu=1}^{d+2} (u_\mu^{\mu_i}) = 0, \quad (2.27)$$

where the $a_i$’s are complex moduli of the mirror Calabi-Yau hypersurface.
2.3 More on the mirror CY geometry

Here we further explore the relations between the realizations (2.10) and (2.27) of the mirror Calabi-Yau manifolds. In particular, we give an explicit derivation of the weights $\delta_{\mu}$ involved in the polynomials (2.27) in terms of the Calabi-Yau $q_i^a$ charges. To do so, first of all recall that under the projective action

$$u_{\mu} \rightarrow \lambda^{\delta_{\mu}} u_{\mu},$$

(2.28)

the polynomial $P_\Delta(u)$ behaves as $P_\Delta(\lambda^{\delta_{\mu}} u) = \lambda^D P_\Delta(u)$ leaving the zero locus invariant. Using the identity $\sum_{\mu=1}^{d+2} \delta_{\mu} = D$, one may reinterpret the Calabi-Yau condition (2.4) or equivalently by introducing $r$ integers $p_a$

$$\sum_{\mu=1}^{d+2} \sum_{a=1}^r p_a q_{\mu}^a = - \sum_{i=d+3}^{k+1} \sum_{a=1}^r p_a q_i^a,$$

by thinking about it as

$$\delta_{\mu} = \sum_{a=1}^r p_a q_{\mu}^a$$

(2.29)

$$D = \sum_{i=d+3}^{k+1} \delta_i = - \sum_{i=d+3}^{k+1} \sum_{a=1}^r p_a q_i^a.$$

(2.30)

For instance, for ordinary projective spaces $P^k$, we can use the generalization of the transformation introduced in [39]; namely

$$u_{\mu} \rightarrow \omega^{Q_{\mu}^a} u_{\mu},$$

$$u_{d+2} \rightarrow u_{d+2},$$

(2.31)

where, roughly speaking, $\omega$ is a $D-th$ root of unity. This transformation leaves $P_\Delta(u)$ invariant as far as the $Q_{\mu}^a$'s obey the Calabi-Yau condition $\sum_{\mu=1}^{d+1} Q_{\mu}^a = 0$ and $Q_{d+2}^a = 0$, in agreement with the choice of the coordinate patch $u_{d+2} = 1$. Next by appropriate choice of $\lambda$, we can compare both the transformations (2.28) and (2.31) as well as their actions on the monomials $y_i = \Pi_{\mu=1}^{d+2} \left( u_{\mu}^{(\mu_i^a-1)} \right)$ respectively given by $y_i \rightarrow y_i \omega^{\Sigma_{\mu}^a \delta_{\mu}(\mu_i^a-1)}$ and $y_i \rightarrow y_i \omega^{\Sigma_{\mu}^a Q_{\mu}^a (\mu_i^a-1)}$. Invariance under these actions lead to eqs (2.19-20), and their toric geometry equations analogue

$$\sum_{\mu=1}^{d+2} Q_{\mu}^a = 0 \; \text{modulo} \; (D)$$

(2.32)

$$\sum_{\mu=1}^{d+2} Q_{\mu}^a \mu_i^a = 0; \; \text{modulo} \; (D).$$

(2.33)
Comparing these eqs with eqs(2.32-33) and (2.19-20), one gets the following relation between the $Q^a_{\mu}$ and $q_i^a$ charges of the original manifold

$$Q^a_{\mu} = \left( q^a_{\mu} + \frac{1}{d+2} \sum_{i=d+3}^{k} q_i^a \right); \mod (D)$$  \hspace{1cm} (2.34)

As the isometries of eqs(2.26-27) will be involved in the study of the NC hypersurface Calabi-Yau orbifolds, let us derive general form of these isometries using geometry toric data. We will distinguish between two cases: (i) the group of isometries $\Gamma_0$ leaving eq(2.26) invariant. (ii) its subgroup $\Gamma_{cd}$ of discrete symmetries of eq(2.27), commuting with complex deformations.

### 3 Discrete Symmetries and CY Orbifolds

To determine the discrete symmetries of the Calabi-Yau homogeneous hypersurfaces, let us derive the general groups $\Gamma_0$ and $\Gamma_{cd}$ of transformations leaving eqs(2.26) and (2.27) invariants:

$$\Gamma = \{ g_\omega \mid g_W : u_\mu \rightarrow g_\omega (u_\mu) = u'_\mu = u_\mu (W)^{b_\mu}, \quad P_\Delta(u') = P_\Delta(u) \}, \hspace{1cm} (3.1)$$

where

$$W^{b_\mu} = \Pi_{\nu=1}^{d+2} \left[ (\omega_\nu)^{a_\nu^b} \right]$$

and where $\{b_\mu\}_{1 \leq \mu \leq d+2}$ is a $(d + 2)$ dimensional vector weight and $a_\mu^b$ are their entries. They will be determined by symmetry requirements and the Calabi-Yau toric geometry data. As the solutions we will build depend on the weights $\delta_\mu$, we will distinguish hereafter the $P^{d+1}$ and $WP^{d+1}$ spaces; a matter of illustrating the idea and the techniques we will be using.

#### 3.1 $P^{d+1}$ projective spaces

The crucial point to note here is that because of the equality $\delta_1 = \delta_2 = ... = \delta_{d+2} = 1$, the $D$ degree of the polynomials $P_\Delta(u)$ is equal to $(d + 2)$ and so the constraint eq (2.20) reduces to: $\sum_{\mu=1}^{d+2} n_i^\mu = (d + 2)$ for any value of the $i$ index. Putting back $\delta_\mu = 1$ in eqs (2.26), one sees that invariance under $\Gamma_0$ of the first terms $u_\mu^{d+2}$ shows that a natural solution is given by taking $\omega_1 = \omega_2 = ... = \omega_{d+1} = \omega = \exp i (\frac{2\pi}{d+2})$ and then $\omega^{b_\mu} = \exp i \frac{2\pi}{d+2} b_\mu$. However, invariance of the term $\prod_{\mu=1}^{d+2} (u_\mu)$ under the change (3.1), implies that $b_\mu$ should satisfy the following constraint equation

$$\sum_{\mu=1}^{d+2} b_\mu = 0, \mod (d+2). \hspace{1cm} (3.2)$$
In what follows, we shall give an explicit class of special solutions for the constraint \[ \sum_{\mu=1}^{d+2} b_\mu = 0, \] by using the toric geometry data of the \( H^d_\Delta \) Calabi-Yau manifold eqs (2.2) and (2.4). The solutions, modulo \( d+2 \), are obtained by making appropriate shifts.

### 3.1.1 Explicit construction of \( b_\mu \) weights

The solution for \( b_\mu \) we will construct herebelow contains two terms which are intimately linked to toric geometry eqs (2.2) and (2.4). To have an idea on the explicit derivation of the \( b_\mu \)'s, let us first introduce the two following \( Q_\mu \) and \( \xi_\mu \) quantities. They will be used in realizing \( b_\mu \).

**The \( Q_\mu \) Weights:**

This is a quantity defined as:

\[
Q_\mu = Q_\mu (p_1, ..., p_r) = \sum_{a=1}^{r} p_a Q^a_\mu, \quad 1 \leq \mu \leq d+2, \tag{3.3}
\]

where the \( p_a \)'s are given integers and where \( Q^a_\mu \) are a kind of shifted Calabi-Yau charges; they are given in terms of the \( q^a_\mu \) Mori vectors of the toric manifold shifted by constant numbers \( \tau^a \); as shown on the following relation

\[
Q^a_\mu = q^a_\mu + \tau^a. \tag{3.4}
\]

The \( \tau^a \)'s are determined by requiring that the \( Q^a_\mu \) shifted charged have to satisfy the Calabi-Yau condition \( \sum_{\mu=1}^{d+2} Q^a_\mu = 0 \). Using (2.4), we find

\[
\tau^a = \frac{1}{d+2} \sum_{i=d+3}^{k+1} q^a_i. \tag{3.5}
\]

Replacing \( Q^a_\mu \) by its explicit expression in terms of the Mori vector charges, we get

\[
Q_\mu = \sum_{a=1}^{r} p_a \left( q^a_\mu + \frac{1}{d+2} \sum_{i=d+3}^{k} q^a_i \right). \tag{3.6}
\]

It satisfies identically the property \( \sum_{\mu=1}^{d+2} Q_\mu = 0 \), which we will interpret as the Calabi-Yau condition because of its link with the original relation \( \sum_{i=1}^{k+1} q^a_i = 0 \).

**The \( \xi_\mu \) Weights:**

These weights carry informations on the data of the polytope \( \Delta \) of the toric varieties and so on their Calabi-Yau submanifolds. They are defined as

\[
\xi_\mu = \xi_\mu (s_1, ..., s_{d+1}) = \sum_{a=1}^{d+1} s_a \xi^a_\mu \tag{3.7}
\]
where the $s_{\alpha}$'s are integers and where $\xi_{\alpha}^\mu$ are defined in terms of the toric data of $M_{d+1}^\Delta$ as follows

$$\xi_{\alpha}^\mu = \sum_{a=1}^r p_a \left( q_{\alpha}^a n_{\alpha}^a + \frac{1}{d+2} \sum_{i=d+3}^{k+1} q_{\alpha}^i n_{\alpha}^i \right).$$

(3.8)

Like for the $Q_{\mu}$ weights, one can check here also that the sum $\sum_{\mu=1}^{d+2} \xi_{\mu}$ vanishes identically due to the constraint equation (2.15).

The $b_{\mu}$ Weights :

A class of solutions for $b_{\mu}$ based on the Calabi-Yau toric geometry data (2.2) and (2.4), may be given by a linear combination of the $Q_{\mu}$ and $\xi_{\mu}$ weights as shown herebelow:

$$b_{\mu} = m_1 Q_{\mu} + m_2 \xi_{\mu},$$

(3.9)

where $m_1$ and $m_2$ are integers modulo $(d + 2)$. Moreover setting $b_{\mu} = \sum_{\nu=1}^{d+2} a_{\nu}^\mu$ and

$$Q_{\mu}^\alpha = Q_{\mu}^\alpha \quad for \quad \alpha = 1, ..., r;$$

$$Q_{\mu}^\alpha = 0 \quad for \quad \alpha = (r + 1), ..., (d + 2),$$

(3.10)

while $Q_{\mu}^\alpha = Q_{\mu}^\alpha$ for $r \geq d + 1$, we can rewrite the above solutions as follows:

$$a_{\mu}^\nu = m_1 Q_{\mu}^\nu + m_2 \xi_{\mu}^\nu.$$

(3.11)

Therefore, the general transformations of the $\Gamma_0(P^{d+1})$ group of discrete isometries are given by the change (3.1) with $b_{\mu}$ vector weights depending on $(r + d + 1) = k$ integers; namely $r$ integers $p_a$ and $(d + 1)$ integers $s_{\alpha}$.

3.1.2 Complex Deformations

To get the discrete symmetries of the full Calabi-Yau homogeneous complex hypersurface including the complex deformations eq(2.27), one should solve more complicated constraint relations which we give hereafter. Under $\Gamma_{cd}$ of transformations eq(2.27), the complex deformations of the Calabi-Yau manifold $P_{\Delta}(u)$ are stable provided the $b_{\mu}$ weights satisfy eq(3.2) but also the following constraint eqs:

$$\sum_{\mu=1}^{d+2} b_{\mu} n_{\mu}^\nu = 0,$$

(3.12)

where the $n_{\mu}^\nu$'s are as in eq(2.27). A particular solution of these constraint eqs is given by taking $b_{\mu} = Q_{\mu}$ that is $m_1 = 1$ and $m_2 = 0$. Indeed replacing $b_{\mu}$ by its expression(3.9) and
putting back into the above relation, we get by help of the identity (2.20),
\[
\left[ \sum_{\mu=1}^{d+2} \sum_{a=1}^{r} p_a (q^a_{\mu} + \tau^a n_{\mu}^\nu) \right] = \sum_{a=1}^{r} p_a \left[ \sum_{\mu=1}^{d+2} q^a_{\mu} n_{\mu}^\nu + (d+2) \tau^a \right] = \sum_{a=1}^{r} p_a \left[ \sum_{\mu=1}^{d+2} q^a_{\mu} n_{\mu}^\nu + \sum_{i=d+3}^{k} q^a_i n_{\mu}^\nu \right] = 0. \tag{3.13}
\]
For \( m_1, m_2 \neq 0 \), the relation \( b_{\mu} = m_1 Q_{\mu} + m_2 \xi_{\mu} \) cease to be a solution of the constraint eq(3.12). Therefore \( \Gamma_{cd} \) is a subgroup of \( \Gamma_0 \). It depends on the \( p_a \) integers and involves the Calabi-Yau condition only.

### 3.2 \( WP^{d+1} \) weighted projective spaces

The previous analysis made for the case of \( P^{d+1} \) applies as well for \( WP^{d+1} \). Starting from eq(2.26) and making the change (3.1), invariance requirement leads to take the \( \omega_{\mu} \) group parameters as \( \omega_{\mu} = \exp i \frac{2\pi \delta_{\mu}}{D} \) and the \( a_{\nu}^\mu \) coefficients constrained as
\[
\sum_{\nu=1}^{d+2} \delta_{\nu} a_{\nu}^\mu = 0, \quad \text{modulo } \delta_{\mu} \\
\sum_{\mu=1}^{d+2} a_{\nu}^\mu = 0. \tag{3.14}
\]
Following the same reasoning as before, one can write down a class of solution, with integer entries, in terms of the previous weights as follows
\[
a_{\nu}^\mu = (\delta_{\nu})^{-1} \left[ m_1 Q_{\mu}^\nu + m_2 \xi_{\mu}^\nu \right], \tag{3.15}
\]
where \( Q_{\mu}^\nu \) and \( \xi_{\mu}^\nu \) are as in eq (3.11). In case where the complex deformations of eq(2.27) are taken into account, the discrete symmetry group is no longer the same since the constraint eq(3.13) is now replaced by the following one
\[
\sum_{\mu=1}^{d+2} a_{\nu}^\mu n_{\mu}^i = 0, \forall \quad \nu = 1, ..., (d+2). \tag{3.16}
\]
Like in the projective case where the \( \delta_{\nu} \)'s are equal to one, the solutions for the \( a_{\nu}^\mu \) integers are given by eq(3.15) with \( m_1 \neq 0 \) and \( m_2 = 0 \). To conclude this section, one should note that the group of discrete isometries \( \Gamma_{cd} \subset \Gamma_0 \) of the Calabi-Yau hypersurfaces including complex deformations is intimately related to the Calabi-Yau condition.
4 NC Toric CY Manifolds

Before exposing our results regarding NC toric Calabi-Yau’s, let us begin this section by reviewing briefly the BL idea of building NC orbifolds of Calabi-Yau hypersurface.

4.1 Algebraic geometric approach for CY

Roughly speaking, given a $d$ dimensional a Calabi-Yau manifold $X^d$ described algebraically by a complex equation $p(z_i) = 0$ with a group $\Gamma$ of discrete isometries. Taking the quotient of $X^d$, by the action of the finite group $\Gamma$

$$\Gamma : z_i \rightarrow gz_ig^{-1}, \ g \in \Gamma \quad (4.1)$$

such that the two following conditions are fulfilled $p(z_i)$ polynomial and the $(d, 0)$ holomorphic from are invaraints. The latter condition is equivalently to the vanishing the first Chern class $c_1 = 0$. Using the discrete torsion, one can build the NC extensions of the orbifold, $(\frac{X^d}{\Gamma})_{nc}$ as follows. The coordinate $z_i$’s are replaced by matrix operators $Z_i$ satisfying

$$Z_iZ_j = \theta_{ij}Z_jZ_i, \quad (4.2)$$

Invariance of $p(z_i)$ requires that the parameter $\theta_{ij}$’s to be in the discrete group $\Gamma$. Moreover, the Calabi-Yau condition imposes the extra constraint equation

$$\prod_i \theta_{ij} = 1, \ \forall j \neq i, \quad (4.3)$$

In this case of the quintic, embedded in a $P^5$ projective space described by the homogeneous polynomial $p(z_1, ..., z_5)$ of degree 5:

$$p(z_i) = z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 + a_0 \prod_{i=1}^{5} z_i = 0. \quad (4.4)$$

The group $\Gamma$ acts as $z_i \rightarrow z_i \omega^{Q_i^a}$ where $\omega^5 = 1$ and where the $Q_i^a$ vectors are

$$Q_i^1 = (1, -1, 0, 0, 0),$$
$$Q_i^2 = (1, 0, -1, 0, 0),$$
$$Q_i^3 = (1, 0, 0, -1, 0). \quad (4.5)$$

In the coordinate patch $\mathcal{U} = \{(z_1, z_2, z_3, z_4); z_5 = 1\}$, eq(4.5) reduces to

$$1 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + a_0 \prod_{j=1}^{4} z_j = 0. \quad (4.6)$$
The local NC algebra $A_{nc}$ describing the NC version of equation (4.5) is obtained by associating to $z_5$ the matrix $z_5 I_5$ and to each holomorphic variable $z_i$ a $5 \times 5$ matrix $Z_i$ satisfying the BL algebra

\begin{align}
Z_1 Z_2 &= \alpha Z_2 Z_1, & Z_1 Z_3 &= \alpha^{-1} \beta Z_3 Z_1, \\
Z_1 Z_4 &= \beta^{-1} Z_4 Z_1, & Z_2 Z_3 &= \alpha \gamma Z_3 Z_2, \\
Z_2 Z_4 &= \gamma^{-1} Z_4 Z_2, & Z_3 Z_4 &= \beta \gamma Z_4 Z_3
\end{align}

(4.7)

where $\alpha, \beta$ and $\gamma$ are fifth roots of unity. The centre of this algebra $Z(A_{nc}) = \{I_5, Z_5^\nu, \Pi_{\nu=1}^4 Z_\nu\}$, that is,

\begin{align}
[Z_\mu, Z_5^\nu] &= 0, \\
[Z_\mu, \Pi_{\nu=1}^4 Z_\nu] &= 0
\end{align}

(4.8)

According to the Schur lemma, one can set $Z_5^\nu = I_5 z_5^\nu$ and $\Pi_{\nu=1}^4 Z_\nu = I_5 \Pi_{\nu=1}^4 z_\nu$ and so the centre coincide with the equation of the quintic. In what follows we extend this analysis to NC toric Calabi-Yau orbifolds.

### 4.2 NC toric CY orbifolds

Following the same lines as [38, 39, 40, 41, 42] and using the discrete symmetry group $\Gamma$, one can build the orbifolds $O = H^{d*}_\Delta / \Gamma$ of the Calabi-Yau hypersurface and work out their non commutative extensions $O_{nc}$. The main steps in the building of $O_{nc}$ may summarized as follows: First start from the Calabi-Yau hypersurfaces $H^{d*}_\Delta$ eqs (2.26-27) and fix a coordinate patch of $WP^{d+1}$, say $u_{d+2} = 1$. Then impose the identification under the discrete automorphisms (3.1) defining $H^{d*}_\Delta / \Gamma$. The NC extension of this orbifold is obtained as usual by extending the commutative algebra $A_c$ of functions on $H^{d*}_\Delta / \Gamma$ to a NC one $A_{nc} \sim O_{nc}$. In this algebra, the $u_\mu$ coordinates are replaced by matrix operators $U_\mu$ satisfying the algebraic relations

\begin{align}
U_\mu U_\nu = \theta_{\mu\nu} U_\nu U_\mu, \quad \nu > \mu = 1, ..., d + 1
\end{align}

(4.9)

where the $\theta_{\mu\nu}$ non commutativity parameters obey the following constraint relations

\begin{align}
\theta_{\mu\nu} \theta_{\nu\mu} &= 1, \\
(\theta_{\mu\nu})^D &= 1, \\
\prod_{\mu=1}^{d+1} (\theta_{\mu\nu}) &= 1
\end{align}

(4.10-4.12)
as far as eq(2.26) is concerned that is in the region of the moduli space where the complex moduli \( a_i \) are zero \( (i = 1, \ldots) \). However, in the general case where the \( a_i \)'s are non zero we should have moreover:

\[
\prod_{\mu=1}^{d+1} \left( \theta_{\mu \nu}^{n_{\mu}} \right) = 1; \quad \alpha = 1, \ldots, d + 1.
\] (4.13)

Let us comment briefly these constraint relations. Eq(4.11) reflects that the parameters \( \theta_{\nu \mu} \) are just the inverse of \( \theta_{\mu \nu} \) and can be viewed as describing deformations away from the identity suggesting by the occasion that they may be realized as

\[
\theta_{\mu \nu} = \exp \eta_{\mu \nu}
\]

, where \( \eta_{\mu \nu} = -\eta_{\nu \mu} \) is the infinitesimal version of the non commutativity parameter. The constraint (4.12-13) reflect just the remarkable property according to which \( U_{\nu}^{\mu} \) and \( \Pi_{\nu=1}^{d+2} (U_{\nu}) \) are elements in the centre \( \mathcal{Z}(\mathcal{A}_{nc}) \) of the non commutative algebra \( \mathcal{A}_{nc} \), i.e;

\[
\left[ U_{\mu}, U_{\nu}^{\mu} \right] = 0,
\] (4.14)

\[
\left[ U_{\mu}, \Pi_{\nu=1}^{d+2} (U_{\nu}) \right] = 0.
\] (4.15)

Finally, the constraint eqs(4.14), obtained by requiring \( \left[ U_{\mu}, \Pi_{\nu=1}^{d+2} (U_{\nu}^{\mu}) \right] = 0 \), describe the compatibility between non commutativity and deformations of the complex structure of the Calabi-Yau hypersurfaces.

In what follows we shall solve the above constraint equations (4.11-14) in terms of toric geometry data of the toric variety in which the mirror geometry is embedded. Since these solutions depend on the weight vector \( \delta \) we will consider two cases; \( \delta_{\mu} = 1 \) for all values of \( \mu \) and \( \delta_{\mu} \) taking general numbers eqs(2.20).

### 4.2.1 Matrix representations for projective spaces

The analysis we have developed so far can be made more explicit by computing the NC algebras associated to the Calabi-Yau hypersurface orbifolds with discrete torsion. In this regards, a simple and instructive class of solutions of the above constraint eqs may be worked in the framework of the \( P^{d+1} \) ordinary projective spaces. To do this, consider a \( d \) complex dimension Calabi-Yau homogeneous hypersurfaces in \( P^{d+1} \) namely,

\[
u_{1}^{d+2} + \nu_{2}^{d+2} + \nu_{3}^{d+2} + \nu_{4}^{d+2} + \ldots + \nu_{d+2}^{d+2} + \nu_{0} \prod_{\mu=1}^{d+2} u_{\mu} = 0,
\] (4.16)
with the discrete isometries (2.31) and Calabi-Yau charges $Q^a_\mu$ satisfying
\begin{equation}
\sum_{\mu=1}^{d+2} Q^a_\mu = 0, \quad a = 1, \ldots, d.
\end{equation}
(4.17)

From constraint eq (4.12), it is not difficult to see that $\theta_{\mu\nu}$ is an element of the $\mathbb{Z}_{d+2}$ group and so can be written as
\begin{equation}
\theta_{\mu\nu} = \omega^{L_{\mu\nu}},
\end{equation}
(4.18)

where $\omega = \exp \frac{2\pi i}{d+2}$ and where $L_{\mu\nu}$ is a $(d+1) \times (d+1)$ antisymmetric matrix, i.e $L_{\mu\nu} = -L_{\nu\mu}$, as required by eq(4.11). Putting this solution back into eq(4.13), one discovers that this tensor should satisfy
\begin{equation}
\sum_{\mu=1}^{d+1} L_{\mu\nu} = 0, \mod (d+2).
\end{equation}
(4.19)

Using the toric data of the Calabi-Yau manifold $\sum_{\mu=1}^{d+1} Q^a_\mu = 0$ and $\sum_{\mu=1}^{d+1} \xi^\alpha_\mu = 0$, namely
\begin{align*}
Q_\mu &= \sum_{a=1}^r p_a \left( q^a_\mu + \frac{1}{d+1} \sum_{i=d+2}^k q^a_i \right), \\
\xi^\alpha_\mu &= \sum_{a=1}^r p_a \left( q^a_\mu n^\alpha_\mu + \frac{1}{d+1} \sum_{i=d+2}^{k+1} q^a_i n^\alpha_i \right),
\end{align*}
(4.20)

one sees that the $L_{\mu\nu}$'s can be solved as bilinear forms of $Q^a_\mu$ and $\xi^\alpha_\mu$ namely:
\begin{equation}
L_{\mu\nu} = L_1 \Omega_{ab} Q^a_\mu Q^b_\nu + L_2 \Omega_{\alpha\beta} \xi^\alpha_\mu \xi^\beta_\nu.
\end{equation}
(4.22)

Here $L_1$ and $L_2$ are numbers modulo $(d+2)$ and $\Omega_{ab}$ and $\Omega_{\alpha\beta}$ are respectively the antisymmetric $r \times r$ and $(d+2) \times (d+2)$ for even integer values of $r$ and $d$ or their generalized expressions otherwise. Moreover, $L_{\mu\nu}$ can also be rewritten in terms of the $a^\nu_\mu$ components of $b_\mu$. For the particular case $L_2=0$, eq(4.23) reduces to:
\begin{equation}
L_{\mu\nu} = -L_{\nu\mu} = m_{ab} Q^{[a}_\mu Q^{b]}_\nu,
\end{equation}
(4.23)

where $m_{ab}$ is an antisymmetric $d \times d$ matrix of integers modulo $(d+2)$. It satisfies
\begin{equation}
\sum_{\mu=1}^{d+2} L_{\mu\nu} = 0.
\end{equation}
(4.24)

The NC extension of eq(4.17) is given by the following algebra, to which we refer to as $\mathcal{A}_{nc}(d+2)$:
\begin{align*}
U_\mu U_\nu &= \omega_{\mu\nu} \omega_{\nu\mu} U_{\nu} U_{\mu}, \quad \mu, \nu = 1, \ldots, (d+1), \\
U_\mu U_{d+2} &= U_{d+2} U_{\mu}, \quad \mu = 1, \ldots, (d+1),
\end{align*}
(4.25)
where $\varpi_{\mu\nu}$ is the complex conjugate of $\omega_{\mu\nu}$. The latters are realized in terms of the Calabi-Yau charges data as follows:

$$
\omega_{\mu\nu} = \exp i \left( \frac{2\pi}{d+2} m_{ab} Q^a_{\mu} Q^b_{\nu} \right) = \omega^{m_{ab} Q^a_{\mu} Q^b_{\nu}}.
$$

(4.26)

Using the propriety $\varpi_{\mu\nu}^{d+2} = 1$ and $\prod_{\mu} \varpi_{\mu} = 1$, one can check that the center of the algebra (4.26) is given by the

$$
\mathcal{Z}(\mathcal{A}_{nc}) = \lambda_1 U_1^{d+2} + \lambda_2 U_2^{d+2} + \ldots + \lambda_{d+1} U_{d+1}^{d+2} + \lambda_{d+2} I_{d+2} + \prod_{\mu=1}^{d+1} U_{\mu}.
$$

(4.27)

Schur lemma implies that this matrix equation can be written

$$
\mathcal{Z}(\mathcal{A}_{nc}) = p(u_1, u_2, \ldots, u_{d+1}) I_{d+2}.
$$

(4.28)

To determine the explicit expression of $p(u_1, u_2, \ldots, u_{d+1})$, let us discuss in what follow the matrix irreducible representations of the NC Calabi-Yau algebra for a regular point. In the next subsection we will give the representation for the fixed points, where the representation becomes reducible and corresponds to fractional branes.

Finite dimensional representations of the algebra (4.26) are given by matrix subalgebras $\text{Mat}[n(d+2), C]$, the algebra of $n(d+2) \times n(d+2)$ complex matrices, with $n = 1, 2, \ldots$.

Computing the determinant of both sides of eqs (4.26)

$$
det (U_{\mu} U_{\nu}) = (\omega_{\mu\nu} \varpi_{\nu\mu})^D det (U_{\nu} U_{\mu}) = det (U_{\nu} U_{\mu}),
$$

(4.29)

the dimension $D$ of the representation to be such that:

$$
(\omega_{\mu\nu} \varpi_{\nu\mu})^D = 1.
$$

(4.30)

Using the identity (4.19), one discovers that $D$ is a multiple of $(d + 2)$. We consider the fundamental $(d+2) \times (d+2)$ matrix representation obtained by introducing the following set $\{ Q; P_{\alpha ab}; a, b = 1, \ldots, d \}$ of matrices:

$$
P_{\alpha ab} = \text{diag}(1, \alpha_{ab}, \alpha_{ab}^2, \ldots, \alpha_{ab}^{d+1}); \quad Q = 
\begin{pmatrix}
0 & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix}
$$

(4.31)
where $\alpha_{ab} = u^{m_{ab}}$ satisfying $\alpha_{ab}^{d+2} = 1$. From these expressions, it is not difficult to see that the $\{Q; P_{\alpha_{ab}}; a, b = 1, \ldots, d\}$ matrices obey the algebra:

\[
P_{\alpha}P_{\beta} = P_{\alpha\beta},
\]

\[
P_{\alpha}^{d+2} = 1,
\]

\[
Q_{\alpha}^{d+2} = 1.
\]

Using the following identities

\[
P_{\alpha_{\mu}}Q_{\mu}^{m_{\mu}} = \alpha_{\mu}^{m_{\mu}}Q_{\mu}^{m_{\mu}}P_{\alpha_{\mu}},
\]

\[
(P_{\alpha_{\mu}}Q_{\mu}^{m_{\nu}})(P_{\alpha_{\nu}}Q_{\mu}^{m_{\mu}}) = \alpha_{\mu}^{m_{\mu}}\alpha_{\nu}^{m_{\nu}}Q_{\mu}^{m_{\nu}}(P_{\alpha_{\mu}}Q_{\mu}^{m_{\nu}})(P_{\alpha_{\nu}}Q_{\mu}^{m_{\mu}}),
\]

one can check that the $U_{\mu}$ operators can be realized as

\[
U_{\mu} = u_{\mu} \prod_{a,b=1}^{d} (P_{\alpha_{ab}}Q_{\mu}^{a}),
\]

where $u_{\mu}$ are $C$-number which should be thought of as in (4.17). From the Calabi-Yau condition, one can also check that the above representation satisfy

\[
U_{\mu}^{d+2} = u_{\mu}^{d+2}I_{d+2},
\]

\[
\prod_{\mu=1}^{d+1} U_{\mu} = I_{d+2} \left( \prod_{\mu=1}^{d+1} u_{\mu} \right).
\]

Putting these relations back into (4.29), one finds that the polynomial $p(u_{\mu})$ is nothing but the eq (4.17) of the Calabi-Yau hypersurface.

### 4.2.2 Solution for weighted projective spaces

In the case of weighted projective spaces with a weight vector $\delta = (\delta_1, \ldots, \delta_{d+2})$; the degree $D$ of the Calabi-Yau polynomials and the corresponding $N_i$ vertices are respectively given by eqs(2.19-20) and (2.24-25). Note that integrality of the vertex entries requires that $D$ should be the smallest common multiple of the weights $\delta_{\mu}$; that is $D/\delta_{\mu}$ an integer. Following the same reasoning as for the case of the projective space, one can work out a class of solutions of the constraint eqs(4.11-13) in terms of powers of $\omega_{\mu}$. We get the result

\[
\theta_{\mu\nu} = \exp i2\pi \left[ \frac{(\delta_{\mu})L_{\mu\nu}}{D} \right],
\]

\[20\]
where $L_{\mu\nu}$ is as in eq(4.23). Instead of being general, let us consider a concrete example dealing with the analogue of the quintic in the weighted projective space $\mathbb{W}^4_{\{\delta_1,\delta_2,\delta_3,\delta_4,\delta_5\}}$. In this case the Calabi-Yau hypersurface, $\sum_{\mu=1}^5 u_{\mu}^5 + a_0 \prod_{\mu=1}^5 (u_{\mu}) = 0$; which for the example $\delta_1 = 2$ and $\delta_2 = \delta_3 = \delta_4 = \delta_5 = 1$, reduces to:

$$u_1^3 + u_2^6 + u_3^6 + u_4^6 + u_5^6 + a_0 \prod_{\mu=1}^5 (u_{\mu}) = 0. \quad (4.38)$$

This polynomial has discrete isometries acting on the homogeneous coordinates $u_{\mu}$ as:

$$u_{\mu} \rightarrow u_{\mu} \zeta_{\mu}^\nu \quad \mu = 1, \ldots, 5, \quad (4.39)$$

with $\zeta_1^3 = 1$ while $\zeta_\mu^6 = \omega^6 = 1$; i.e $\zeta_1 = \omega^2$ and $\zeta_\mu = \omega$ and where the $a_{\mu}^\nu$ are consistent with the Calabi-Yau condition. In the coordinate patch $\{u_{\mu}\}_{1 \leq 4}$ with $u_5 = 1$, the equations defining the NC geometry of the Calabi-Yau (4.39) with discrete torsion, upon using the correspondence $u \rightarrow U$, are given by the algebra (5.1) where the $\theta_{\mu\nu}$ parameters should obey now the following constraint eqs:

$$\begin{align*}
\theta_{\mu\nu}^{31} &= 1, \quad \mu = 2, 3, 4, \\
\theta_{\mu\nu}^6 &= 1, \quad \nu \neq 1, \mu \\
\prod_{\mu=1}^4 \theta_{\nu\mu} &= 1, \quad \forall \nu \\
\theta_{\mu\nu}\theta_{\nu\mu} &= 1, \quad \forall \mu, \nu.
\end{align*} \quad (4.40)$$

Setting $\theta_{\mu\nu} = \omega^{L_{\mu\nu}}$ the constraints on $L_{\mu\nu}$ read as:

$$\begin{align*}
L_{\mu\nu} &= -L_{\nu\mu} \quad \text{integers modulo 6}, \\
L_{\mu1} &= \text{even modulo 6}. \quad (4.41)
\end{align*}$$

Particular solutions of this geometry may be obtained by using antisymmetric bilinears of $a_{\mu}^\nu$. Straightforward calculations show that, for $p_{\mu} = 1$, $L_{\mu\nu}$ is given by the following $4 \times 4$ matrix:

$$L_{\mu\nu} = \begin{pmatrix}
0 & k_1 - k_3 & -k_1 + k_2 & k_3 - k_2 \\
-k_1 + k_3 & 0 & k_1 & -k_3 \\
k_1 - k_2 & -k_1 & 0 & k_2 \\
-k_3 + k_2 & k_3 & -k_2 & 0
\end{pmatrix} \quad (4.42)$$

where the $k_{\mu}$ integers are such that $k_{\mu} - k_{\nu} \equiv 2r_{\mu\nu} \in 2\mathbb{Z}$.

The NC algebra associated with eq(4.39) reads, in terms of $\omega_{\mu} = \omega^{k_{\mu}}$ and $\omega_{\mu} = \omega^{-k_{\mu}}$, as
follows:

\begin{align*}
U_1 U_2 &= \omega_1 \overline{w}_3 U_2 U_1, \\
U_1 U_3 &= \omega_1 \omega_2 U_3 U_1, \\
U_1 U_4 &= \omega_3 \overline{w}_2 U_4 U_1, \\
U_2 U_3 &= \omega_1 U_3 U_2, \\
U_2 U_4 &= \omega_3 U_4 U_2, \\
U_3 U_4 &= \omega_2 U_4 U_3. \\
\end{align*}

Furthermore taking \( \alpha = \omega_1 \overline{w}_3, \beta = \omega_2 \overline{w}_3 \) and \( \gamma = \omega_3 \), one discovers an extension of the BL NC algebra (4.4); the difference is that now the deformation parameters are such that:

\[ \alpha^3 = \beta^3 = \gamma^6 = 1. \] (4.44)

### 4.2.3 Fractional Branes

Here we study the fractional branes corresponding to reducible representations at singular points. To illustrate the idea, we give a concrete example concerning the mirror geometry in terms of the \( \mathbb{P}^{d+1} \) projective space. First note that the \( A_{nc}(d+2) \) (4.37) corresponds to regular points of NC Calabi-Yau. This solution is irreducible and the branes do not fractionate. A similar solutions may be worked out as well for fixed points where we have fractional branes.

We focus our attention on the orbifold of the eight-tic, namely,

\[ u_1^8 + u_2^8 + \ldots + u_8^8 + a_0 \prod_{\mu=1}^{8} u_\mu = 0, \] (4.45)

with the discrete isometries \( \mathbb{Z}_8^{\delta} \) and Calabi-Yau charges \( Q_\mu^a \)

\[ Q_\mu^1 = (1, -1, 0, 0, 0, 0, 0, 0) \]
\[ Q_\mu^2 = (1, 0, -1, 0, 0, 0, 0, 0) \]
\[ Q_\mu^3 = (1, 0, 0, -1, 0, 0, 0, 0) \]
\[ Q_\mu^4 = (1, 0, 0, 0, -1, 0, 0, 0) \] (4.46)
\[ Q_\mu^5 = (1, 0, 0, 0, 0, -1, 0, 0) \]
\[ Q_\mu^6 = (1, 0, 0, 0, 0, 0, -1, 0). \]

The corresponding NC algebra is deduced from the general one given in (4.26). At regular points, the matrix theory representation of this algebra is irreducible as shown on eqs (4.37). However, the situation is more subtle at fixed points where representations are reducible. One way to deal with the singularity of the orbifold with respect to \( \mathbb{Z}_8^{\delta} \) is to interpret the algebra as describing a \( \mathbb{Z}_8^{\delta} \) orbifold with \( \mathbb{Z}_8^{\delta} \) discrete torsions having singularities in codimension four.
Starting from eqs(4.26) and choosing matrix coordinates $U_5$, $U_6$ and $U_7$ in the centre of the algebra by setting

$$(\omega_{\mu \nu} c_{\nu \mu}) = 1, \quad \text{for } \mu = 5, 6, 7, 8; \quad \forall \nu = 1, \ldots, 8,$$

the algebra reduces to

\begin{align*}
U_1 U_2 &= \alpha_1 \alpha_2 U_2 U_1 \\
U_1 U_3 &= \alpha_1^{-1} \alpha_3 U_3 U_1 \\
U_1 U_4 &= \alpha_2^{-1} \alpha_3^{-1} U_4 U_1 \\
U_2 U_3 &= \alpha_1 U_3 U_2 \\
U_2 U_4 &= \alpha_2 U_4 U_2, \\
U_3 U_4 &= \alpha_3 U_4 U_3
\end{align*}

and all remaining other relations are commuting. In these equations, the $\alpha_\mu$’s are such that $\alpha_8^\mu = 1$; these are the phases $Z_3^{15}$. At the singularity where the $u_1$, $u_2$, $u_3$, and $u_4$ moduli of eq(4.37) go to zero, one ends with the familiar result for orbifolds with discrete torsion. Therefore the D-branes fractionate in the codimension four singularities of the eight-tic geometry.

5 Link with the BL Construction

In this section we want to rederive the results of [39] concerning NC quintic using the analysis developed in section 3 and 4. Recall that in the coordinate patch $\{u_\mu\}_{1 \leq 4}$ and $u_5 = 1$, the defining equations of NC geometry of the quintic with discrete torsion, upon using the correspondence $u \rightarrow U$, are given by the following operators algebra.

$$U_\mu U_\nu = \theta_{\mu \nu} U_\nu U_\mu, \quad \nu > \mu = 1, \ldots, 4,$$

where the $\theta_{\mu \nu}$’s are non zero complex parameters. As the monomials $U_\mu^5$ and $\prod_{\mu=1}^5 (U_\mu)$ are commuting with all the $U_\mu$’s, we have also

\begin{align*}
\left[ U_\nu, U_\mu^5 \right] &= 0, \\
\left[ U_\nu, \prod_{\mu=1}^4 U_\mu \right] &= 0.
\end{align*}

23
Compatibility between eqs (5.1-2) gives constraint relations on $\theta_{\mu\nu}$'s namely

$$\theta^5_{\nu\mu} = 1, \quad \text{(5.3)}$$
$$\prod_{\mu=1}^4 \theta_{\nu\mu} = 1, \quad \forall \nu \quad \text{(5.4)}$$
$$\theta_{\mu\nu} \theta_{\nu\mu} = 1; \quad \theta_{5\mu} = 1, \quad \forall \mu, \nu. \quad \text{(5.5)}$$

To establish the link between our way of doing and the construction of [41], it is interesting to note that the analysis of [41] corresponds in fact to a special representation of the formalism we developed so far. The idea is summarized as follows: First start from eq(3.1), which reads for the quintic as:

$$u_\mu \rightarrow u_\mu \omega^{b_\mu}, \quad \text{(5.6)}$$

where the $b_\mu$ weights, $b_\mu = \sum_{\nu=1}^5 a_\nu^\mu, \mu = 1, \ldots, 5$, are such that

$$\sum_{\nu=1}^5 b_\mu = 0. \quad \text{(5.7)}$$

This relation, interpreted as the Calabi-Yau condition, can be solved in different ways. A way to do is to set the $b_\mu$ weights as

$$b_\mu = (p_1 + p_2 + p_3, -p_1, -p_2, -p_3, 0), \quad \text{(5.8)}$$

or equivalently by taking the weight components $b_\nu^\mu$ as:

$$a^{\nu}_{\mu} = \begin{pmatrix}
p_1 & p_2 & p_3 & 0 & 0 \\
-p_1 & 0 & 0 & 0 & 0 \\
0 & -p_2 & 0 & 0 & 0 \\
0 & 0 & -p_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \text{(5.9)}$$

where $p_\mu$ are integers modulo 5. More general solutions can be read from eqs (4.23) by following the same method. The next step is to take $\theta_{\mu\nu} = \exp(\frac{2\pi}{5} L_{\mu\nu})$ with $L_{\mu\nu}$ as follows:

$$L_{\mu\nu} = m_{12} (a^1_\mu a^2_\nu - a^1_\nu a^2_\mu) - m_{23} (a^2_\mu a^3_\nu - a^2_\nu a^3_\mu) + m_{13} (a^1_\mu a^3_\nu - a^1_\nu a^3_\mu), \quad \text{(5.10)}$$

where $m_{12} = k_1$, $m_{23} = k_2$ and $m_{13} = k_3$ are integers modulo 5. For $p_\mu = 1$, we get

$$L_{\mu\nu} = \begin{pmatrix}
0 & k_1 - k_3 & -k_1 + k_2 & k_3 - k_2 \\
-k_1 + k_3 & 0 & k_1 & -k_3 \\
k_1 - k_2 & -k_1 & 0 & k_2 \\
-k_3 + k_2 & k_3 & -k_2 & 0
\end{pmatrix}, \quad \text{(5.11)}$$

24
and so the NC quintic algebra reads:

$$
\begin{align*}
U_1 U_2 &= \omega^{k_1 - k_2} U_2 U_1, & U_1 U_3 &= \omega^{-k_1 + k_2} U_3 U_1, \\
U_1 U_4 &= \omega^{k_1 - k_2} U_4 U_1, & U_2 U_3 &= \omega^{k_1} U_3 U_2, \\
U_2 U_4 &= \omega^{-k_3} U_4 U_2, & U_3 U_4 &= \omega^{k_2} U_4 U_3.
\end{align*}
$$

(5.12)

Setting $\omega_\mu = \omega^{k_\mu}$ and $\varpi_\mu = \omega^{-k_\mu}$, the above relations become:

$$
\begin{align*}
U_1 U_2 &= \omega_1 \varpi_3 U_2 U_1, & U_1 U_3 &= \varpi_1 \omega_2 U_3 U_1, \\
U_1 U_4 &= \omega_3 \varpi_2 U_4 U_1, & U_2 U_3 &= \omega_1 U_3 U_2, \\
U_2 U_4 &= \varpi_3 U_4 U_2, & U_3 U_4 &= \omega_2 U_4 U_3.
\end{align*}
$$

(5.13)

Now taking $\alpha = \omega_1 \varpi_3$, $\beta = \omega_2 \varpi_3$ and $\gamma = \omega_3$, one discovers exactly the BL algebra eqs(4.8).

### 5.1 More on the NC quintic

As we mentioned, the solution given by eqs(5.8-9) is in fact a special realization of the BL algebra (4.8). One can also write down other representations of the NC quintic; one of them is based on taking $a_\mu^\nu$ as:

$$
a_\mu^\nu = \begin{pmatrix}
    p_1 & 0 & p_3 & 0 & 0 \\
    -2p_1 & p_2 & 0 & 0 & 0 \\
    p_1 & -2p_2 & p_3 & 0 & 0 \\
    0 & p_2 & -2p_3 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

(5.14)

The corresponding $b_\mu$ weight vector is then:

$$
b_\mu = (p_1 + p_3, -2p_1 + p_2, p_1 - 2p_2 + p_3, p_2 - 2p_3; 0),
$$

(5.15)

with $p_\mu$'s are integers modulo 5. As one sees this is a different solution from that given in eq(5.8-9) as the corresponding $\Gamma$ group of isometries acts differently on the $u_\mu$ variables leading then to a different orbifold with discrete torsion. Note that setting $p_\mu = 1$, the $a_\mu^\nu$ weights are nothing but the Mori vectors of the blow up of the the $\tilde{A}_2$ affine singularity of K3, used in the geometric engineering method of $4D N = 2$ superconformal theories embedded in type II superstrings.

Setting $p_\mu = 1$ and using equations (5.10) and (5.14), the anti-symmetric $L_{\mu \nu}$ matrix reads as:

$$
L_{\mu \nu} = \begin{pmatrix}
    0 & k_1 + k_2 + 2k_3 & -2k_1 - 2k_2 & k_1 + k_2 - 2k_3 \\
    -k_1 - k_2 - 2k_3 & 0 & 3k_1 - k_2 - 2k_3 & -2k_1 + 2k_2 + 4k_3 \\
    2k_1 + 2k_2 & -3k_1 + k_2 + 2k_3 & 0 & k_1 - 3k_2 - 2k_3 \\
    -k_1 - k_2 + 2k_3 & 2k_1 - 2k_2 - 4k_3 & -k_1 + 3k_2 + 2k_3 & 0
\end{pmatrix}
$$

(5.16)
where the $k_1, k_2$ and $k_3$ are integers modulo 5. The new algebra describing the NC quintic reads, in terms of the $\omega_\mu$ and $\varpi_\nu$ generators of the $\mathbb{Z}_5^3$, as follows:

\begin{align*}
U_1 U_2 &= \omega_1 \omega_2 \omega_3^2 U_2 U_1, \\
U_1 U_4 &= \omega_1 \omega_2 \omega_3^3 U_4 U_1, \\
U_2 U_4 &= \omega_1^2 \omega_2^3 \omega_3^4 U_4 U_2, \\
U_3 U_4 &= \omega_1^3 \omega_2^3 \omega_3^4 U_4 U_3.
\end{align*}

(5.17)

Setting $\alpha = \omega_1 \omega_2 \omega_3^2$, $\beta = \varpi_1 \varpi_2 \omega_3^2$ and $\gamma = \omega_1^2 \varpi_2 \omega_3^4$, one discovers, once again, the $BL$ algebra (4.7). Therefore eq(5.9) and eq(5.14) give two representations of the $BL$ algebra.

### 5.2 Comments on lower dimension CY manifolds

The analysis we developed so far applies to complex $d$ dimension homogeneous hypersurfaces with discrete torsion; $d \geq 2$. We have discussed the cases $d \geq 3$; here we want to complete this study for lower dimension Calabi-Yau manifolds namely $K3$ and the elliptic curve. These are very special cases which deserves some comments. For the $K3$ surface in $CP^3$, we have

\begin{equation}
 u_1^4 + u_2^4 + u_3^4 + u_4^4 + a_0 \prod_{\mu=1}^{4} u_\mu = 0,
\end{equation}

(5.18)

This is a quartic polynomial with a $\mathbb{Z}_4 \times \mathbb{Z}_4$ symmetry acting on the $u_i$ variables as:

\begin{equation}
 u_\mu \rightarrow w^{Q_\mu} u_\mu,
\end{equation}

(5.19)

where $w^4 = 1$ and where $a_\mu$ are integers satisfying the Calabi-Yau condition $\sum_{\mu=1}^{4} Q_\mu = 0$. Choosing $Q_\mu$ as,

\begin{align*}
 Q_1 &= (1, -1, 0, 0) \\
 Q_2 &= (1, 0, -1, 0)
\end{align*}

(5.20)

the $3 \times 3$ matrix $L_{\mu\nu}$ reads as

\begin{equation}
 L_{\mu\nu} = \begin{pmatrix}
 0 & k & -k \\
 -k & 0 & k \\
 k & -k & 0
\end{pmatrix}.
\end{equation}

(5.21)

Therefore the NC $K3$ algebra reads as:

\begin{align*}
 U_1 U_2 &= U_2 U_1 e^{\frac{2\pi k}{4}}, \\
 U_1 U_3 &= U_3 U_1 e^{-\frac{2\pi k}{4}},
\end{align*}

(5.22)
where \( k \) is an integer modulo 4. Note that one gets similar results by making other choices of \( Q_i^a \) such as,

\[
Q_1 = (1, -2, 1, 0) \\
Q_2 = (1, 1, -2, 0).
\]

More general results may also be written down for \( K3 \) embedded in \( WP(\delta_1, \delta_2, \delta_3, \delta_4) \). In the case of the one dimensional elliptic fiber given by a cubic in \( P^2 \)

\[
u_1^3 + \nu_2^3 + \nu_3^3 + a_0 \prod_{\mu=1}^3 \nu_\mu = 0,
\]

the constraint equations defining non-commutativity are trivially solved. They show that \( L_{\mu\nu} = 0 \) and so \( \theta_{12} = 1 \) leading then to a commutative geometry. NC geometries involving elliptic curves can be constructed; the idea is to consider orbifolds of products of elliptic curves. More details are exposed in the following section. Related ideas with fractional branes will be considered as well.

### 6 NC Elliptic Manifolds

In this section we want refine the study of the NC Calabi-Yau hypersurface defined in terms of orbifolds of elliptic curves. The original idea of this construction was introduced first in [39]; see also [73], in connection with the NC orbifold \( T^2/Z_2 \). The method is quite similar to that discussed for the quintic and generalized Calabi-Yau geometries in sections 4 and 5. To start consider the following elliptic realization of \( T^{2n+2}_{Z_2} \); that is \( T^{2n+2} \) is represented by the product of \( (n+1) \) elliptic curves \( (T^2)^{(2k+1)} \) where \( n = 2k \). Each elliptic curve is given in Weierstrass form as:

\[
y_\mu^2 = x_\mu(x_\mu - 1)(x_\mu - a_\mu), \quad \mu = 1, \ldots, n+1,
\]

with a point added at infinity \( \mu = 1, \ldots, n+1 \). The system \( \{(x_\mu, y_\mu); \quad \mu = 1, \ldots, n+1\} \) defines the complex coordinates of \( O^{2n+2} \) space and \( a_\mu \) are \( (n+1) \) complex moduli. For later use, we introduce the algebra \( A_c \) of holomorphic functions on \( T^{2n+2} \). This is a commutative algebra
generated by monomials in the $x_{\mu}$'s and $y_{\mu}$'s with the conditions eqs(6.1). The discrete group $\Gamma$ acts on $x_{\mu}$'s and $y_{\mu}$'s as:

\[
\begin{align*}
x_{\mu} & \rightarrow x'_{\mu} = x_{\mu}, \\
y_{\mu} & \rightarrow y'_{\mu} = y_{\mu} \omega^{Q_{\mu}},
\end{align*}
\]

(6.2)

where $\omega$ is an element of the discrete group $\Gamma$ and where $Q_{\mu}$ are integers which should be compared with eq(4.24). Note that if one is requiring that eqs(6.1) to be invariant under $\Gamma$, then $\omega^2$ should be equal to one that is $\omega = \pm 1$. If one requires moreover that the monomial $\prod_{\mu=1}^{n+1} y_{\mu}$ or again the holomorphic $((n + 1), 0)$ form $dy_1 \wedge dy_2 \ldots \wedge dy_{n+1}$, to be invariants under the orbifold action, it follows then that $\prod_{\mu=1}^{n+1} \omega^{Q_{\mu}} = \omega^{\Sigma_{\mu} Q_{\mu}} = 1$. This is equivalent to

\[
\sum_{\mu=1}^{n+1} Q_{\mu} = 0, \quad \text{modulo 2} \quad (6.3)
\]

defines the Calabi-Yau condition for the orbifold $\mathbb{T}^{2n+2}_{\Gamma}$. Therefore the $\Gamma$ discrete group is given by $\Gamma = (\mathbb{Z}_2)^{\otimes n}$. Following the discussion we made in section 4, this equation can also be rewritten as

\[
\sum_{\mu=1}^{n+1} Q_{\mu}^a = 0, \quad \text{modulo 2; } \quad a = 1, \ldots, n. \quad (6.4)
\]

The four fixed points of the orbifold for each two torus $T^2$ are located at $(x_{\mu} = 0, 1, a_{\mu}; y_{\mu} = 0)$ and the point at infinity; i.e $(x_{\mu} = \infty; y_{\mu} = \infty)$. This later can be brought to a fixed finite point by working in another coordinate patch related to the old one by using the change of variables:

\[
\begin{align*}
y_{\mu} & \rightarrow y'_{\mu} = \frac{y_{\mu}}{x_{\mu}^2}, \\
x_{\mu} & \rightarrow x'_{\mu} = \frac{1}{x_{\mu}}.
\end{align*}
\]

(6.5)

The NC version of the orbifold $\mathbb{T}^{2n+2}_{\Gamma}$ is obtained by substituting the usual commuting $x_{\mu}$ and $y_{\mu}$ variables by the matrix operators $X_{\mu}$ and $Y_{\mu}$ respectively. These matrix operators satisfy the following NC algebra structure:

\[
\begin{align*}
Y_{\mu} Y_{\nu} & = \theta_{\mu \nu} Y_{\nu} Y_{\mu}, \\
X_{\mu} X_{\nu} & = X_{\nu} X_{\mu}, \\
X_{\mu} Y_{\nu} & = Y_{\nu} X_{\mu}, \\
X_{\mu} Y_{\nu}^2 & = Y_{\nu}^2 Y_{\mu},
\end{align*}
\]

(6.6, 6.7, 6.8, 6.9)
as it is required by eq(6.1) and
\[
\left[ Y_\mu, \prod_{\nu=1}^{n+1} Y_\nu \right] = 0. \tag{6.10}
\]
Like for the case of the homogeneous hypersurfaces we considered in sections 4 and 5, here also the Calabi-Yau condition is fulfilled by imposing that the \( \prod_{\nu=1}^{n+1} Y_\nu \) belongs to the centre of the NC algebra \( \mathcal{A}_{nc} \). Now using eqs\((6.6\text{-}10)\), one gets the explicit expression of the \( \theta_{\mu\nu} \)'s by solving the following constraint eqs:

\[
\begin{align*}
\theta_{\mu\nu} \theta_{\mu\nu} &= 1, \quad \tag{6.11} \\
\prod_{\mu=1}^{n+1} \theta_{\mu\nu} &= 1, \quad \tag{6.12} \\
\theta_{\mu\nu} \theta_{\nu\mu} &= 1, \quad \theta_{\mu\mu} = 1. \quad \tag{6.13}
\end{align*}
\]

Note that eq\((6.11)\) is a strong constraint which will have a drastic consequence on the solving of noncommutativity. Comparing this relation to eq\((6.12)\), one can write

\[
\theta_{\mu\nu} = (-1)^L_{\mu\nu},
\]

\[
\sum_{\mu=1}^{n+1} L_{\mu\nu} = 0, \quad \text{modulo 2,} \tag{6.14}
\]

where \( L_{\mu\nu} \) is antisymmetric matrix, \( L_{\mu\nu} = -L_{\nu\mu} \), of integer entries given by

\[
L_{\mu\nu} = \Omega_{ab} Q^a_\mu Q^b_\nu; \tag{6.15}
\]

where \( \Omega_{ab} = -\Omega_{ba} \), and \( \Omega_{ab} = 1 \) for \( a < b \). This relation should be compared to eq\((4.25)\). Moreover, one learns from eq\((6.14)\) that two cases should be distinguished. The first one corresponds to the case \( \theta_{\mu\nu} = -1 \forall \mu \neq \nu \); that is;

\[
L_{\mu\nu} = 1; \quad \text{modulo 2.} \tag{6.16}
\]

In this case, the constrained \((6.12)\) is fulfilled provided \( n \) is even; i.e \( n = 2k \). So the group \( \Gamma \) is given by \( \Gamma = (\mathbb{Z}_2)^{\otimes 2k} \). The second case corresponds to the situation where some \( \theta_{\mu\nu} \)'s are equal to one:

\[
\begin{align*}
L_{\mu\nu} &= 1; \quad \text{modulo 2} \quad \mu = 1, ..., (n + 1 - r); \quad \mu \neq \nu \tag{6.17} \\
L_{\mu\nu} &= 0; \quad \text{modulo 2}; \quad \mu = (n - r + 2), ..., n + 1, \tag{6.18}
\end{align*}
\]

where we have rearranged the variables so that the matrix takes the form

\[
L_{\mu\nu} = \begin{pmatrix}
I_{\mu'\nu'} & 0 \\
0 & 0
\end{pmatrix} \tag{6.19}
\]

In this case eq\((6.17)\) shows that \( n \) is even if \( r \) is even and odd if \( r \) is odd. In what follows we build the solutions of the NC algebra\((6.6\text{-}8)\) using finite dimensional matrices.
6.1 Solution I

Replacing the relation (6.16) back into eqs(6.6-8), the non commutativity algebra, which reads as:

\[ Y_\mu Y_\nu = -Y_\nu Y_\mu, \quad (6.20) \]
\[ Y_\mu Y_\nu^2 = Y_\nu^2 Y_\mu, \quad (6.21) \]
\[ X_\mu X_\nu = X_\nu X_\mu, \quad (6.22) \]
\[ X_\mu Y_\nu = Y_\nu X_\mu, \quad (6.23) \]

may be realized in terms of \(2^k \times 2^k\) matrices of the space of matrices \(M(2^k, \mathbb{C})\). These are typical relations naturally solved by using the \(2k\) dimensional Clifford algebra generated by the basis system \(\{\Gamma^i, \mu = 1, ..., 2k\}\):

\[ \{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu}, \]
\[ \{\Gamma^i, \Gamma^{2k+1}\} = 0, \quad (6.24) \]

where \(\Gamma^{2k+1} = \prod_{i=1}^{2k} \Gamma^i\). We therefore have:

\[ Y_\mu = b_\mu \Gamma^\mu, \quad \mu = 1, ..., 2k, \quad (6.25) \]
\[ Y_{2k+1} = b_0 \Gamma^{2k+1}, \quad (6.26) \]
\[ X_\mu = x_\mu I_{2^k} \quad (6.27) \]

where the \(b_\mu\)'s are complex scalars. This solution has remarkable features: (i) After choosing a hermitian \(\Gamma\) matrices representation, one can see at the fixed planes, where \(2k\) variables among the \((2k+1)\) \(y_\mu\)'s act by zero and all others zero, that there exists a multiplicity of inequivalent representations for each set of roots \(x_\mu\) of the Weierstrass forms. Therefore, one can get \(2^k\) distinct NC points, as there are \(2^k\) irreducibles representations corresponding to \(2^k\) eigenvalues of the non zero matrix variable and so the branes fractionate on the singularity.

(ii) The non-commutative points of the singular planes are then seen to be a \(2k\) cover of the commutative singular plane, which is a \((CP^1)^{\odot k}\). The \(2k\) cover is branched around the four points \(x_k = 0, 1, a_k, \infty\) and hence the NC points form an elliptic manifold \(T^{2k}\) of the form eq(6.1). Around each of these four points, there is a \((Z_2)\) monodromy of the representations, which is characteristic of the local singularity as measuring the effect of discrete torsion.
6.2 Solution II

Putting the relations (6.17) back into eqs(6.6-8), the resulting NC algebra depends on the integer \( r \) and reads as:

\[
Y_\mu Y_\nu = -Y_\nu Y_\mu, \quad \mu, \nu = 1, \ldots, (n+1-r). \tag{6.28}
\]

\[
Y_\mu Y_\nu = Y_\nu Y_\mu, \quad \mu, \nu = (n+2-r), \ldots, (n+1), \tag{6.29}
\]

\[
Y_\mu Y^2_\nu = Y^2_\nu Y_\mu, \quad \mu = 1, \ldots, (n+1), \tag{6.30}
\]

\[
X_\mu X_\nu = X_\nu X_\mu, \tag{6.31}
\]

\[
X_\mu Y_\nu = Y_\nu X_\mu. \tag{6.32}
\]

For \( r = 2s \) even, irreducible representations of this algebra are given, in terms of \( 2^{k-s} \times 2^{k-s} \) matrices of the space \( M(2^{k-s}, \mathbb{C}) \), by the \( 2(k-s) \) dimensional Clifford algebra. The result is:

\[
Y_\mu = b_\mu \Gamma^\mu, \quad i = 1, \ldots, 2(k-s), \tag{6.33}
\]

\[
Y_{2(k-s)+1} = b_0 \prod_{\mu=1}^{2(k-s)} \Gamma^\mu, \tag{6.34}
\]

\[
Y_\mu = y_\mu I_{2^{k-s}}, \quad i = 2(k-s), \ldots, (2k+1), \tag{6.35}
\]

\[
X_\mu = x_\mu I_{2^{k-s}}. \tag{6.36}
\]

In the end of this section, we would like to note that this analysis could be extended to a general case initiated in [73], where the elliptic curves are replaced by \( K3 \) surfaces. This might be applied to the resolution of orbifold singularities in the moduli space of certain models, describing a \( D2 \) brane wrapped \( n \) times over the fiber of an elliptic \( K3 \), as follows [74]

\[
\mathcal{M}_{1,n} = Sym(K3) = \frac{K3^{\otimes n}}{S_n}. \tag{6.37}
\]

Here \( \mathcal{M}_{1,n} \) denotes the moduli space of a \( D2 \)-brane with charges \( (1, n) \) and \( S_n \) is the group of permutation of \( n \) elements.

7 Conclusion

In this paper we have studied the NC version of Calabi-Yau hypersurface orbifolds using the algebraic geometry approach of [40,41] combined with toric geometry method of complex
manifolds. Actually this study extends the analysis on the NC Calabi-Yau manifolds with discrete torsion initiated in [41] and expose explicitly the solving of non commutativity in terms of toric geometry data. Our main results may be summarized as follows:

(1) First we have developed a method of getting \(d\) complex Calabi-Yau mirror coset manifolds \(C^{k+1}/C^* r\), \(k - r = d\), as hypersurfaces in \(WP^{d+1}\). The key idea is to solve the \(y_i\) inavriants (2.12-13) of mirror geometry in terms of invariants of the \(C^*\) action of the weighted projective space and the toric data of \(C^{k+1}/C^* r\). As a matter of facts, the above mentioned mirror Calabi-Yau spaces are described by homogeneous polynomials \(P_\Delta(u)\) of degree \(D = \sum_{\mu=1}^{d+2} \delta_\mu = \sum_{\mu=1}^{d+2} \sum_{a=1}^{r} p_\alpha q_\mu^a\), where \(\delta_\mu\) are projective weights of the \(C^*\) action, \(q_\mu^a\) entries of the well known Mori vectors and the \(p_\alpha\)'s are given integers. Then we have determined the general group \(\Gamma\) of discrete isometries of \(P_\Delta(u)\). We have shown by explicit computation that in general one should distinguish two cases \(\Gamma_0\) and \(\Gamma_{cd}\). First \(\Gamma_0\) is the group of isometries of the hypersurface \(\sum_{\mu=1}^{d+2} u_\mu^{b_\mu} + a_0 \prod_{\mu=1}^{d+2} (u_\mu) = 0\), generated by the changes \(u_\mu' = u_\mu (W)^{b_\mu}\) where the weight vector \(b_\mu\) is given by the sum of \(Q_\mu\) and \(\xi_\mu\) respectively associated with the Calabi-Yau charges and the vertices data of the toric manifold \(M_{d+1}\). In case where the complex deformations are taken into account (see eq(2.27)), the symmetry group reduces to the subgroup \(\Gamma_{cd}\) generated by the changes \(u_\mu' = u_\mu (W)^{b_\mu}\) where now \(b_\mu\) has no \(\xi_\mu\) factor.

(2) Using the above results and the algebraic geometry approach, we have developed a method of building NC Calabi-Yau orbifolds in toric manifolds. Non commutativity is solved in terms of the discrete torsion and bilinears of the weight vector \(a_\mu';\) see eq(3.11). Among our results, we have worked out several matrix representations of the NC quintic algebra obtained in [41] by using various Calabi-Yau toric geometry data. We have also given the generalization of these results to higher dimensional Calabi-Yau hypersurface orbifolds and derived the explicit form of the non commutative \(D\)-tic orbifolds.

(3) We have extended to higher complex dimension Calabi-Yau’s realized as toric orbifold of type \(T_{2k+2}/\Gamma\) with discrete torsion. Due to constraint eqs on non commutativity, we have shown that in the elliptic realization of the two torii factors, \(\Gamma\) is constrained to be equal to \(Z_{2k}^2\), the real dimension should be \(2k + 2\) and non commutativity is solved in terms of the \(2k\) dimensional Clifford algebra. We have also discussed the fractional brane which correspond to reducible representations of toric Calabi-Yau algebras.

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