The Localization Principle of Spectral Expansions of Distributions on Closed Domain

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Abstract. The aim of this work is to investigate the problems of localization of spectral expansions of the distributions on closed domain, related to biharmonic operator. Using the mean value formula for the eigenfunctions of the biharmonic operator the Riesz means of the spectral function is estimated. Moreover, isomorphism of the generalized Holder spaces is applied.

1. Introduction
The first use of biharmonic operator $\Delta^2$ for the modeling of an elastic plate is attributed to a correction of Lagrange of a manuscript by Sophie Germain from 1811. For a more elaborate history of the biharmonic problem and the relation with elasticity from an engineering point of view one may consult a survey of Meleshko [8]. Mathematically interesting questions came up around 1900 when Almansi [2], [3], Boggio [5] and Hadamard [7] addressed existence and positivity questions.

In order to have physically meaningful and mathematically well-posed problems the plate equation $\Delta^2u = f$ has to be complemented with prescribing a suitable set of boundary data. The most commonly studied boundary value problems for second order elliptic equations are named Dirichlet, Neumann and Robin. These three types appear since they have a physical meaning. For fourth order differential equations such as the plate equation, the variety of possible boundary conditions is much larger.

We assume that the plate with the vertical projection of which is the planar region $\Omega \subset R^2$, is free to move horizontally at the boundary. Then a simple model for the elastic energy is

$$J(u) = \int_\Omega \left( \frac{1}{2} (\Delta u)^2 - fu \right) dx dy,$$

where $f(x, y)$ is the external vertical load and $u(x, y)$ is the deflection of the plate in vertical direction and, as above for the beam, first order derivatives are left out which indicates that the plate is free to move horizontally.

Minimizing the energy functional leads to the weak Euler-Lagrange equation $(dJ(u), v) = 0$. In the case of pure hinged plate the latter is equivalent to the following boundary value problem

$$\Delta^2 u(x, y) = f(x, y), \quad (x, y) \in \Omega$$
This boundary value problem is in general referred to as the one with homogeneous Navier boundary conditions. The differential equation \( \Delta^2 u = f \) is called the Kirchhoff-Love model for the vertical deflection of a thin elastic plate. In the current research we consider the problems of representing any function from Sobolev’s space by the eigenfunctions of the biharmonic operator.

2. The spectral expansions of the biharmonic operator

Let \( \Omega \subset R^2 \) be a domain with smooth boundary \( \partial \Omega \). We consider the biharmonic operator
\[
\Delta^2 = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2
\]
with domain \( C_0^\infty(\Omega) \).

In the present work we consider spectral expansions connected with the biharmonic operator on closed domain, and extend the localization properties of the spectral expansions to the distributions.

We define the space \( C^l(\Omega) \) as a set of functions in \( \Omega \) all of whose derivatives of order at most \( l \in Z_+ \) are continuous. The symbol \( C^\infty(\Omega) \) denotes the space of all infinitely differentiable functions on \( \Omega \). The subset of \( C^\infty(\Omega) \) which contains the functions from \( C^\infty(\Omega) \) with compact support on \( \Omega \) is denoted by \( C_0^\infty(\Omega) \).

The dual spaces (i.e., the spaces of continuous linear functionals on the sets of test functions) are denoted by
\[
(C_0^\infty(\Omega))' = D'(\Omega)
\]
and
\[
(C^\infty(\Omega))' = \varepsilon'(\Omega).
\]
The dual spaces are nested as follows:
\[
\varepsilon' \subset D'.
\]
Elements of the space \( D' \) are called distributions. Elements of the space \( \varepsilon' \) are distributions with compact support [6].

For \( \ell > 0 \) we define a functional \( \|\cdot\|_\ell \) as follows
\[
\|u\|_\ell = \left( \sum_{0 \leq |\alpha| \leq \ell} \|D^\alpha u\|_{L^2}^2 \right)^{1/2}
\] (1)
for any function \( u \) for which the right side makes sense.

We use notation \( W_2^\ell(\Omega) \) to denote the Sobolev’s space:
\[
W_2^\ell(\Omega) = \{ u \in L_2(\Omega) : D^\alpha u \in L_2(\Omega), 0 \leq |\alpha| \leq \ell \}
\]
Equipped with the norm (1) the space \( W_2^\ell(\Omega) \) is called the Sobolev’s space. Next we define the dual space as follows
\[
(W_2^\ell(\Omega))' = W_2^{-\ell}(\Omega).
\]
Let \( \hat{A} \) be a self-adjoint extension of the biharmonic operator in \( L_2(\Omega) \). If \( E_\lambda \) is the corresponding spectral resolution unity, then the spectral expansion of any element \( f \in L_2(\Omega) \) has the form
\[
E_\lambda f(x) = \sum_{\lambda_n < \lambda} f_n u_n(x),
\] (2)
where \( f_n \) is the Fourier coefficients of the function \( f \), with respect to the eigenfunctions \( \{u_n(x)\}, n = 1, 2, ... \).
It is well known that the operators $E_\lambda$ are integral operators whose kernels $\Theta(x, y, \lambda)$ belong to the $C^\infty(\Omega)$ with respect to both variables $x = (x_1, x_2)$ and $y = (y_1, y_2)$ and for any $\lambda > 0$. The spectral decomposition of an arbitrary function $g \in L_2(\Omega)$ is defined by the following

$$E_\lambda g(x) = \int_\Omega g(y) \Theta(x, y, \lambda) dy.$$  

Since $\Theta(x, y, \lambda) \in C^\infty(\Omega \times \Omega)$, it follows that for any $f \in \mathcal{E}'(\Omega)$ we can define the spectral decomposition of $f$ by the formula

$$E_\lambda f(x) = \langle f, \Theta(x, y, \lambda) \rangle,$$

where the functional $f$ acts on $\Theta(x, y, \lambda)$ with respect to the second argument. While the Riesz means of the spectral decomposition of $f$ can be defined as follows

$$E_s^\lambda f(x) = \langle f, \Theta^s(x, y, \lambda) \rangle, \quad s \geq 0$$

where

$$\Theta^s(x, y, \lambda) = \int_0^\lambda \left(1 - \frac{t}{\lambda}\right)^s d_t \Theta(x, y, t).$$

Note that for any $f \in \mathcal{E}'(\Omega)$ its spectral expansions $E_s^\lambda f(x)$ belongs to the space $C^\infty(\Omega)$ for all $\lambda > 0$.

The problems of convergent of the eigenfunction expansions of the differential operators have a lot of practical applications in physics: in gas dynamics, in hydrodynamics, in elasticity theory and in other areas of mechanics, an important role is played by the theory of discontinuous solutions of the differential equations. In this work we investigate the principles of the localization of spectral decompositions and their Riesz means: a dependence of the convergence of spectral decompositions or their Riesz means on the behavior of the function in a small neighborhood of the given point.

Studies of the spectral expansions of distributions in the spaces with negative smoothness were carried out first in the work of Alimov (see [1]). He obtained the precise conditions for localization of the spectral expansions of distributions in Hilbert spaces, related to the Laplace operator. This work of Alimov gave impetus to conducting in-depth research into the spectral expansions of distributions, related to differential operators, in the development of new methods of investigation of the spectral expansions of distributions and in obtaining the final conditions for decomposability in different functional spaces with negative smoothness. Spectral decompositions of distributions associated with elliptic differential operators of order $2m$ were considered in the works [10], [11]. It is proved that the localization principle holds for the Riesz means of the spectral expansions of the distribution from the Sobolev space $W_p^{-\ell}(\Omega)$, $\Omega \subset \mathbb{R}^N$ under the following conditions: $1 < p \leq 2$, $\ell > 0$, $s > (N - 1)/2 + \ell$. In the present work we consider spectral expansions connected with the biharmonic operator on the closed domain, and extend the localization properties of the spectral expansions related to distributions.

Let us formulate the main results of the work

**Theorem 2.1** Let $f(x) \in \mathcal{E}'(\Omega) \cap W_2^{-\ell}(\Omega)$, $\ell > 0$ and $s \geq \ell + \frac{1}{2}$. Then the Riesz means $E_s^\lambda f(x)$ uniformly converge to 0 on any compact $K \subset \overline{\Omega} \setminus \text{supp}(f)$.

The following is essential in proving the Theorem 2.1.
Theorem 2.2 For $u_n(x)$ and $\lambda_n$, respectively, eigenfunctions and eigenvalues of the biharmonic operator $\Delta^2$ on $\Omega \subset \mathbb{R}^2$, corresponding to the boundary conditions $u|_{\partial \Omega} = \Delta u|_{\partial \Omega} = 0$ one has

$$\sum_{\mu < \sqrt{\lambda_n} < \mu + 1} u_n^2(x) \leq C(\mu + 1)\ln^2(\mu + 1),$$

(3) for all $x \in \bar{\Omega} = \Omega \cup \partial \Omega$ and $\mu \to +\infty$.

3. Mean value formula for the eigenfunctions of the biharmonic operator

Let $\Omega \subset \mathbb{R}^2$ be a domain with smooth boundary $\partial \Omega$. We denote the eigenfunctions and eigenvalues of the biharmonic operator $\Delta^2$ by $u_n(x)$ and $\lambda_n$ respectively where $\Delta = \partial^2_{x_1} + \partial^2_{x_2}$ is the Laplace operator. Then we have

$$\Delta^2 u_n(x) - \lambda_n u_n(x) = 0,$$

$$u_n|_{\partial \Omega} = 0 = \Delta u_n|_{\partial \Omega}.$$ 

Let $u_n(x) = \varphi(\sqrt{\lambda_n}x)$.

$$\Delta^2 u_n(x) - \lambda_n u_n(x) = \lambda_n \Delta^2 \varphi - \lambda_n \varphi = \lambda_n (\Delta^2 \varphi_n - \varphi_n) = 0.$$ 

We use notations

$$\psi_n(x) = (\Delta + 1)\varphi_n$$

(4)

and

$$\phi_n(x) = (\Delta - 1)\varphi_n$$

(5)

Note that,

$$\Delta \psi_n(x) = \Delta^2 \varphi_n + \Delta \varphi_n = \varphi_n + \Delta \varphi_n = \psi_n(x),$$

$$\Delta \phi_n(x) = \Delta^2 \varphi_n - \Delta \varphi_n = \varphi_n - \Delta \varphi_n = -\phi_n(x).$$

Therefore,

$$\Delta \psi_n(x) - \psi_n(x) = 0,$$

and

$$\Delta \phi_n(x) + \phi_n(x) = 0.$$

For the eigenfunctions $v_n(x)$ of the Laplace operator

$$\Delta v_n + \lambda_n v_n = 0,$$

one has the following mean value formula
\[ \int_{0}^{2\pi} v_n(x_1 + r\cos \theta, x_2 + r\sin \theta) d\theta = 2\pi J_0(r \sqrt{\lambda_n}) v_n(x_1, x_2). \]  

(6)

by using the latter, we obtain

for \( \Delta \phi_n(x) + \phi_n(x) = 0 \),

\[ \int_{0}^{2\pi} \phi_n(x + r\theta) d\theta = 2\pi J_0(r) \phi_n(x_1, x_2), \]

for \( \Delta \psi_n(x) - \psi_n(x) = 0 \),

\[ \int_{0}^{2\pi} \psi_n(x + r\theta) d\theta = 2\pi J_0(r \sqrt{-1}) \psi_n(x_1, x_2). \]

From (4) and (5),

\[ \varphi_n(x) = \frac{1}{2} (-\phi_n(x) + \psi_n(x)), \]

so

\[ \int_{0}^{2\pi} \varphi_n(x + r\theta) d\theta = \frac{1}{2} \left( -2\pi J_0(r) \phi_n(x_1, x_2) + 2\pi J_0(r i) \psi_n(x_1, x_2) \right). \]

by using \( u_n(x) = \varphi(\sqrt[n]{\lambda_n}x) \) then from (4), \( \phi(\sqrt[n]{\lambda_n}x) = \sqrt[n]{\lambda_n} \Delta - u_n \) and from (5), \( \psi(\sqrt[n]{\lambda_n}x) = \sqrt[n]{\lambda_n} \Delta + u_n \).

For the eigenfunctions of the biharmonic operator, we get

\[ \int_{0}^{2\pi} u_n(x + r\theta) d\theta = \frac{1}{2} \left( -2\pi J_0(r \sqrt{\lambda_n}) (\sqrt{\lambda_n} \Delta - u_n) + 2\pi J_0(r i \sqrt{\lambda_n}) (\sqrt{\lambda_n} \Delta + u_n) \right). \]

Finally,

\[ \int_{0}^{2\pi} u_n(x + r\theta) d\theta = \pi J_0(r \sqrt{\lambda_n}) u_n(x) - \pi J_0(r \sqrt{\lambda_n}) \sqrt{\lambda_n} \Delta u_n + \pi J_0(r i \sqrt{\lambda_n}) (\sqrt{\lambda_n} \Delta u_n + u_n). \]

We denote

\[ S_r(\lambda_n) = \pi [J_0(r \sqrt{\lambda_n}) - (r \sqrt{\lambda_n})] \sqrt{\lambda_n} \Delta + u_n + \pi J_0(r i \sqrt{\lambda_n}) u_n. \]
We note here that
\[
\int_0^{2\pi} u_n(x_1 + r\cos\theta, x_2 + r\sin\theta) d\theta = \pi J_0(r \sqrt{\lambda_n}) u_n(x_1, x_2) + O(e^{-(r_0 - r) \sqrt{\lambda_n}}),
\]
where \(0 < r_0 < r\).

Proof of Theorem 2.1

Let \(\{u_n\}\) be the set of eigenfunctions of Biharmonic operator corresponding to eigenvalues \(\{\lambda_n\}\) as the following
\[
\Delta^2 u_n(x) - \lambda_n u_n(x) = 0.
\]

For eigenfunctions \(u_n(x)\) of the biharmonic operator, we have
\[
\int_0^{2\pi} u_n(x_1 + r\cos\theta, x_2 + r\sin\theta) d\theta = 2\pi J_0(|x - y| \sqrt{\lambda_n}) u_n(x) + O(e^{-\beta \alpha (r_0 - r) \sqrt{\lambda_n}}),
\]
where \(0 < r < r_0 < \text{dist}(x, \partial \Omega)\).

Let fix \(\delta > 0\) and \(R > \delta\). Let \(x \in \Omega_\delta\), the radial function
\[
V(r) = \begin{cases} 
\Gamma(s + 1) 2^s (2\pi)^{-1} \lambda_n^{-s} \times \frac{J_{s+1}(r \sqrt{\lambda_n})}{(r \sqrt{\lambda_n})^{s+1}}, & r \leq R \\
0, & r > R 
\end{cases}
\]

We use Parseval’s formula to establish representation for Riesz means
\[
\sum_{n=1}^{\infty} f_n d_n = \int f(x) g(x) dx \quad (7)
\]
where
\[
f_n = \int_\Omega f_n u_n(x) dx, \\
d_n = \int_\Omega g(x) u_n(x) dx.
\]

Substituting \(g(x)\) with the radial function \(V(r)\), one has
\[
d_n = \int_\Omega V(r) u_n(y) dy \\
= \int_\Omega \int_{|x-y| \leq R} V(r) u_n(y_1, y_2) dy_1 dy_2
\]
\[ \int_{0}^{R} V(r) \int_{0}^{2\pi} u_n(x_1 + r\cos\theta, x_2 + r\sin\theta) r dr d\theta = \int_{0}^{R} V(r) r(J_0(\sqrt{\lambda}n)u_n(x_1, x_2) + S_n) d\theta dr. \]

Applying mean value formula, we get

\[ d_n = u_n(x) \int_{0}^{R} \Gamma(s + 1)2^s(2\pi)^{-1}\lambda^{\frac{1-s}{2}} \times \frac{J_{1+s}(r\sqrt{\lambda})}{r^{1+s}} r J_0(\sqrt{\lambda}n) dr + \]

\[ u_n(x) \int_{0}^{R} \Gamma(s + 1)2^s(2\pi)^{-1}\lambda^{\frac{1-s}{2}} \times \frac{J_{1+s}(r\sqrt{\lambda})}{r^{1+s}} r S_n dr. \]

Now, we will consider the integration part from the first term of the following

\[ d_n = \Gamma(s + 1)2^s(2\pi)^{-1}\lambda^{\frac{1-s}{2}} u_n(x) \int_{0}^{R} J_{1+s}(r\sqrt{\lambda}) J_0(\sqrt{\lambda}n) r^{-s} dr + u_n(x) B_n^\lambda(R), \]

where

\[ B_n^\lambda(R) = \int_{0}^{R} \Gamma(s + 1)2^s(2\pi)^{-1}\lambda^{\frac{1-s}{2}} \times \frac{J_{1+s}(r\sqrt{\lambda})}{r^{1+s}} r S_n dr. \]

Note that, (see [12])

\[ \int_{0}^{\infty} J_{1+s}(r\sqrt{\lambda}) J_0(\sqrt{\lambda}n) r^{-s} dr = \begin{cases} \frac{2^{-s}}{\lambda^{\frac{1-s}{2}}\Gamma(s+1)} \times (1 - \sqrt{\lambda}n)^s, & \sqrt{\lambda}n \leq \lambda \\ 0, & \sqrt{\lambda}n > \lambda \end{cases} \]  

(8)

Separating the integral into two parts i.e. \( \int_{0}^{\infty} = -\int_{0}^{R} + \int_{R}^{\infty} \), we obtain

\[ d_n = \Gamma(s + 1)2^s(2\pi)^{-1}\lambda^{\frac{1-s}{2}} u_n(x) \int_{0}^{R} J_{1+s}(r\sqrt{\lambda}) J_0(\sqrt{\lambda}n) r^{-s} dr + u_n B_n^\lambda(R). \]
Substituting the value of integral (8), we have

\[ d_n = u_n(x)(1 - \frac{\sqrt{\lambda n}}{\lambda})^s \delta_n^\lambda + \Gamma(s + 1)2^s(2\pi)^{-1}\lambda^{\frac{1-s}{2}} u_n(x) \int_R J_{1+s}(r\sqrt{\lambda}) J_0(r\sqrt{\lambda}) r^{-s} dr + u_n B_n^\lambda(R), \]

where

\[ \delta_n^\lambda = \begin{cases} 1, & \text{if } \sqrt{\lambda n} \leq \lambda \\ 0, & \text{if } \sqrt{\lambda n} > \lambda \end{cases} \]

by using the Parseval’s formula, we obtain

\[
\Gamma(s + 1)2^s(2\pi)^{-1} \int_\Omega f(x) \lambda^{\frac{1-s}{2}} x \frac{J_{1+s}(r\sqrt{\lambda})}{r^{1+s}} dx \\
= \sum_{n=1}^\infty f_n(u_n(x))(1 - \frac{\sqrt{\lambda n}}{\lambda})^s - \sum_{n=1}^\infty f_n \Gamma(s + 1)2^s(2\pi)^{-1}\lambda^{\frac{1-s}{2}} u_n(x) \\
\times \int_R J_{1+s}(r\sqrt{\lambda}) J_0(r\sqrt{\lambda}) r^{-s} dr + \sum_{n=1}^\infty f_n u_n B_n^\lambda(R).
\]

The first term is equal to Riesz means, so we have

\[
\Gamma(s + 1)2^s(2\pi)^{-1} \int_\Omega f(x) \lambda^{\frac{1-s}{2}} x \frac{J_{1+s}(r\sqrt{\lambda})}{r^{1+s}} dx = E_x^\lambda f(x) - \sum_{n=1}^\infty f_n u_n(x) I_n(\lambda, \lambda_n) + \sum_{n=1}^\infty f_n u_n B_n^\lambda,
\]

where

\[ I_n(\lambda, \lambda_n) = \Gamma(s + 1)2^s(2\pi)^{-1}\lambda^{\frac{1-s}{2}} x \int_R J_{1+s}(r\sqrt{\lambda}) J_0(r\sqrt{\lambda}) r^{-s} dr. \]

Thus, for Riesz means we obtain

\[
E_x^\lambda f(x) = \Gamma(s + 1)2^s(2\pi)^{-1} \int_{r \leq \delta} f(x) \lambda^{\frac{1-s}{2}} x \frac{J_{1+s}(r\sqrt{\lambda})}{r^{1+s}} dx \\
+ \sum_{n=1}^\infty f_n u_n(x) I_n(\eta, \lambda, \lambda_n) - \sum_{n=1}^\infty f_n u_n B_n^\lambda(R).
\]

The first part is equal to zero, so

\[
E_x^\lambda f(x) = \sum_{n=1}^\infty f_n u_n(x) I_n(\eta, \lambda, \lambda_n) - \sum_{n=1}^\infty f_n u_n B_n^\lambda(R).
\]
Lemma 3.1 For any compact $K \subset \Omega$ there exists $C = C(k)$, such that uniformly for all $X \in K$ one has

$$\sum_{n=1}^{\infty} \frac{u_n^2(x)}{1 + (\sqrt{\lambda_n} - \lambda)^2} \times (\sqrt{\lambda_n})^{-1} \leq C(k).$$

For the proof we refer the readers to [13].

Let denote $A$ by a self-adjoint extension of Biharmonic operator in $L_2(\Omega)$. Then we have

$$E_s^{s}f(x) = \langle A^{-\frac{\ell}{2}} f, A^{\frac{\ell}{2}} \theta^s(x, y, \lambda) \rangle$$

We prove that

$$||A^{\frac{\ell}{2}} \theta^s(x, y, \lambda)||_{L_2} \leq C(\sqrt{\lambda})^{\ell-s+\frac{1}{2}}.$$

where

$$|E_s^{s}f(x)| \leq |\sum f_n(1 + \lambda_n)^{-\frac{\ell}{2}}(1 + \lambda_n)^{\frac{\ell}{2}} u_n(x) I_n(\lambda, \lambda_n)| \leq (\sum f_n(1 + \lambda_n)^{-\frac{\ell}{2}})^{\frac{1}{2}}(\sum (1 + \lambda_n)^{\ell} u_n^2(x) I_n(\lambda, \lambda_n))^\frac{1}{2} \leq C||f||_{-\ell}(\sqrt{\lambda})^{\ell-s+\frac{1}{2}}.$$

So we have

$$|E_s^{s}f(x)| \leq C(\sqrt{\lambda})^{\ell-s+\frac{1}{2}}||f||_{W^{-\ell}_2}.$$

For any function from $C^\infty_0(\Omega)$, the Riesz means, $E_s^{s}f(x)$ uniformly convergence to $f$ on closed domain $\Omega$. From density of $C^\infty_0(\Omega)$ in $W^{-\ell}_2(\Omega)$ and using techniques from the work [13], we obtain uniformly convergence of $E_s^{s}f(x)$ to $f(x)$ on closed domain.

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