Removal of UV cutoff for the Nelson model with variable coefficients

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Abstract

We consider the Nelson model with variable coefficients. Nelson models with variable coefficients arise when one replaces in the usual Nelson model the flat Minkowski metric by a static metric, allowing also the boson mass to depend on position. We study the removal of the ultraviolet cutoff.

1 The Nelson Hamiltonian with variable coefficients

1.1 Introduction

The Nelson model \cite{Ne} describes a spinless nonrelativistic particle linearly coupled to a scalar bose field. After adding an ultraviolet (UV) cutoff, this model can be defined as a self-adjoint operator on some Hilbert space. In \cite{Ne}, E. Nelson was able to remove the UV cutoff and to define the Hamiltonian as a self-adjoint operator without UV cutoff on the original Hilbert space.

We extend the Nelson model to the case with variable coefficients, which realizes the Nelson model defined on a static Lorentzian manifold. In a series of papers \cite{GHPS1}...
we show the existence or absence of ground states of the variable coefficients Nelson model $H(\rho)$ with a certain UV cutoff $\rho$. In this paper we consider the removal of UV cutoff for variable coefficients Nelson models. Denoting by $H^\kappa$ the Nelson Hamiltonian $H(\rho^\kappa)$ for the cutoff function $\rho^\kappa(x) = \kappa^3 \rho(\kappa x)$, we construct a particle potential $E^\kappa(X)$ such that $H^\kappa - E^\kappa(X)$ converge in strong resolvent sense to a bounded below selfadjoint operator $H^\infty$. The removal of the UV cutoff involves as in the constant coefficients case a sequence of unitary dressing operators $U^\kappa$. In contrary to the constant coefficients case, where all computations can be conveniently done in momentum space (after conjugation by Fourier transform), we have to use instead pseudodifferential calculus. Some of the rather advanced facts on pseudodifferential calculus which we will need are recalled in Appendix A.

1.2 Notation

We collect here some notation for reader’s convenience.

We denote by $x \in \mathbb{R}^3$ (resp. $X \in \mathbb{R}^3$) the boson (resp. electron) position. As usual we set $D_x = i^{-1} \nabla_x$, $D_X = i^{-1} \nabla_X$. If $x \in \mathbb{R}^d$, we set $\langle x \rangle = (1 + |x|^2)^{1/2}$. The domain of a linear operator $A$ on some Hilbert space will be denoted by $\text{Dom} A$, and its spectrum by $\sigma(A)$. If $\mathfrak{h}$ is a Hilbert space, the bosonic Fock space over $\mathfrak{h}$ denoted by $\Gamma_s(\mathfrak{h})$ is

$$\Gamma_s(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} \otimes^n \mathfrak{h}.$$

We denote by $a^*(h)$ and $a(h)$ for $h \in \mathfrak{h}$ the creation operator and the annihilation operator, respectively, which acts on $\Gamma_s(\mathfrak{h})$. The (Segal) field operators $\phi(h)$ are defined as

$$\phi(h) = \frac{1}{\sqrt{2}}(a^*(h) + a(h)).$$

If $\mathcal{K}$ is another Hilbert space and $v \in B(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$, one defines the operators $a^*(v)$ and $a(v)$ as unbounded operators on $\mathcal{K} \otimes \Gamma_s(\mathfrak{h})$ by

$$a^*(v) = \sqrt{n+1} \left( \mathbb{1}_{\mathcal{K}} \otimes \mathcal{S}_{n+1} \right) \left( v \otimes \mathbb{1} \otimes \otimes^n \mathfrak{h} \right),$$

$$a(v) = (a^*(v))^*,$$

$$\phi(v) = \frac{1}{\sqrt{2}}(a^*(v) + a(v)).$$

If $b$ is a selfadjoint operator on $\mathfrak{h}$ its second quantization $d\Gamma(b)$ is defined as

$$d\Gamma(b) = \sum_{j=1}^{n} \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes b \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}.$$

The number operator $N$ is defined by the second quantization of the identity operator $\mathbb{1}$: $N = d\Gamma(\mathbb{1})$. 


The annihilation operator and the creation operator satisfy the estimate:

\[(1.2) \quad \|a^\sharp(v)(N + 1)^{-\frac{1}{2}}\| \leq \|v\|,\]

where \(\|v\|\) is the norm of \(v\) in \(B(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})\).

### 1.3 Field Hamiltonian

Let

\[h_0 = -\sum_{1 \leq j, k \leq d} \frac{1}{c(x)\partial_j a^{jk}(x)\partial_k c(x)^{-1}},\]

\[h = h_0 + m^2(x),\]

with \(a^{jk}, c, m\) are real functions and

\[
\begin{align*}
C_0 \mathbb{I} &\leq [a^{jk}(x)] 
\leq C_1 \mathbb{I}, 
C_0 &\leq c(x) \leq C_1, 
C_0 &> 0, \\
\partial_\alpha^c a^{jk}(x) &\in O((x)^{-1}), \ |\alpha| \leq 1, \\
\partial_\alpha^c c(x) &\in O(1), \ |\alpha| \leq 2, \\
\partial_\alpha^c m(x) &\in O(1), \ |\alpha| \leq 1.
\end{align*}
\]

Clearly \(h\) is selfadjoint on \(H^2(\mathbb{R}^3)\) and \(h \geq 0\). The one-particle space is given by

\[\mathfrak{h} = L^2(\mathbb{R}^3, dx)\]

and one-particle energy by the selfadjoint operator:

\[\omega = h^{\frac{1}{2}}.\]

It can be easily seen that

1. \(\text{Ker}\omega = \{0\}\).
2. Assume in addition to (B) that \(\lim_{x \to \infty} m(x) = 0\). Then \(\inf \sigma(\omega) = 0\).

The field Hamiltonian is

\[d\Gamma(\omega),\]

acting on the bosonic Fock space \(\Gamma_s(\mathfrak{h})\).

### 1.4 Electron Hamiltonian

We define the electron Hamiltonian as

\[K = K_0 + W(X),\]

where

\[K_0 = \sum_{1 \leq j, k \leq 3} D_{X_j} A^{jk}(X) D_{X_k},\]
acting on $\mathcal{K} = L^2(\mathbb{R}^3, dX)$, and

\[(E) \quad C_0 \mathbb{1} \leq [A^K(X)] \leq C_1 \mathbb{1}, \quad C_0 > 0.\]

We assume that $W(X)$ is a real potential such that $K_0 + W$ is essentially selfadjoint and bounded below. We denote by $\hat{K}$ the closure of $K_0 + W$.

### 1.5 Nelson Hamiltonian with variable coefficients

The constant

\[m = \inf \sigma(\omega) \geq 0\]

can be viewed as the mass of the scalar bosons. The Nelson Hamiltonian defined below will be called massive (resp. massless) if $m > 0$ (resp. $m = 0$). Let $\rho \in S(\mathbb{R}^3)$, with $\rho \geq 0$, $q = \int_{\mathbb{R}^3} \rho(y) dy \neq 0$. We set

\[\rho_X(x) = \rho(x - X)\]

and define the UV cutoff scalar bose fields as

\[(1.3) \quad \varphi_\rho(X) = \phi(\omega^{-\frac{1}{2}} \rho_X),\]

where $\phi(f)$ is the Segal field operator. The Nelson Hamiltonian with UV cutoff $\rho$ is given by

\[(1.4) \quad H(\rho) = K \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma(\omega) + \varphi_\rho(X),\]

acting on the Hilbert space:

\[\mathcal{H} = \mathcal{K} \otimes \Gamma_h(\hbar).\]

Set also

\[H_0 = K \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma(\omega),\]

which is selfadjoint on its natural domain. Moreover assume hypotheses (E) and (B). Then $H$ is selfadjoint and bounded below on $D(H_0)$.

### 2 Removal of the UV cutoff

#### 2.1 Nelson Hamiltonians with constant coefficients

In [Ne] Nelson considered the limit of $H(\rho)$ for

\[(2.1) \quad \omega = \omega(D_x) = (-\Delta_x + m^2)^{\frac{1}{2}}, \quad m \geq 0,\]

\[(2.2) \quad K = -\frac{1}{2} \Delta_X + W(X),\]

when $\rho$ tends to the Dirac mass $\delta$, equivalently, the Fourier transform $\hat{\rho}$ to $(2\pi)^{-3/2}$. 

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We quickly review the results in [Ne]. In the rest of this subsection we take the momentum representation for the field variables. Let $H_{\text{Stand}}$ be the constant coefficients Nelson model defined by $H(\rho)$ with (2.1) and (2.2). Suppose $m = 0$ and $\hat{\rho}_\Lambda(k) = 1$ for $|k| < \Lambda$ and $\hat{\rho}_\Lambda(k) = 0$ otherwise. We denote by $H_\Lambda$ the Hamiltonian $H_{\text{Stand}}$ with $\hat{\rho}$ replaced by $\hat{\rho}_\Lambda$. Let

\begin{equation}
(2.3) \quad \hat{T}(k) = \omega(k) + |k|^2/2
\end{equation}

and

\begin{equation}
(2.4) \quad E_\Lambda = -\frac{1}{2} \int_{\mathbb{R}^3} \omega(k)^{-1} \chi(\omega(k) > \sigma) \hat{T}(k)^{-1} |\hat{\rho}_\Lambda(k)|^2 dk,
\end{equation}

where $\sigma > 0$ is arbitrary and $\chi(\omega(k) > \sigma)$ is an IR cutoff function given by $\chi(\omega(k) > \sigma) = 0$ for $\omega(k) < \sigma$ and 1 for $\omega(k) \geq \sigma$. Define the dressing transformation by

\begin{equation}
(2.5) \quad U_\Lambda = e^{i\phi(i\beta_X)},
\end{equation}

where

\begin{equation}
(2.6) \quad \beta_X(k) = -\hat{T}(k)^{-1} \chi(\omega(k) > \sigma) \omega(k)^{-\frac{1}{2}} e^{-ikX} \hat{\rho}_\Lambda(k).
\end{equation}

It is easy to see that $U_\Lambda \to U_\infty$ strongly as $\Lambda \to \infty$, where $U_\infty$ is given by $U_\Lambda$ with $\hat{\rho}_\Lambda$ replaced by 1. Instead of $H_\Lambda$, Nelson considers the dressing-transformed Hamiltonian:

\begin{equation}
(2.7) \quad \tilde{H}_\Lambda = U_\Lambda H_\Lambda U_\Lambda^*.
\end{equation}

**Proposition 2.1** [Ne] There exists a bounded below self-adjoint operator $H_\infty$ such that $\tilde{H}_\Lambda - E_\Lambda$ converges to $H_\infty$ as $\Lambda \to \infty$ in the uniform resolvent sense, and $H_\Lambda - E_\Lambda$ to $U_\infty^* H_\infty U_\infty$ in the strong resolvent sense.

**Remark 2.2** Nelson [Ne] actually considered only the case of $m > 0$. It can be however extended to the case of $m = 0$.

In this section we study the same problem for the Nelson model with variable coefficients.

### 2.2 Preparations

In the constant coefficients Nelson model, the one-particle operator $\omega$ is diagonalized using the Fourier transform. In the variable coefficients Nelson Hamiltonian we will use instead the pseudodifferential calculus to define operators and constants corresponding to (2.4), (2.5), (2.6) and (2.7). In particular the renormalization constant $E_\Lambda$ will be changed to a function $E(X)$. 
We denote by \( S^0(\mathbb{R}^3) \) the space
\[
S^0(\mathbb{R}^3) = \{ f \in C^\infty(\mathbb{R}^3) \mid |\partial^\alpha f(x)| \leq C_\alpha, \ \alpha \in \mathbb{N}^3 \}.
\]
We will assume in addition to hypotheses (E) and (B) that
\[
(N) \ A_{jk}(X), \ a_{jk}(x), \ c(x), \ m^2(x) \in S^0(\mathbb{R}^3).
\]
It is easy to see that \( h \) can be rewritten as
\[
h = \sum_{jk} D_j c^{-2}(x) a^{jk}(x) D_j + v(x),
\]
where \( v \in S^0(\mathbb{R}^3) \), and that \( c^{-2}(x) a^{jk}(x) \in S^0(\mathbb{R}^3) \). Changing notation, we will henceforth assume that
\[
h = \sum_{jk} D_j a^{jk}(x) D_j + v(x),
\]
where \([a_{jk}](x)\) satisfies (B) and \( a^{jk}, \ v \in S^0(\mathbb{R}^3) \). We refer the reader to Appendix A for the notation and for some background on pseudodifferential calculus. It will be useful later to consider \( \omega = h^{\frac{1}{2}} \) as a pseudodifferential operator. Note first that
\[
h = h^w(x, D_x),
\]
for
\[
h(x, \xi) = \sum_{1 \leq j, k \leq 3} \xi_j a^{jk}(x) \xi_k + c(x).
\]
The symbol \( h(x, \xi) \) belongs to \( S((\xi)^2, g) \), for the standard metric
\[
g = dx^2 + (\xi)^{-2} d\xi^2,
\]
and is elliptic in this class. By Lemma A.1 and Theorem A.3 we know that if \( f \in S^p(\mathbb{R}) \), then the operator \( f(h) \) belongs to \( \Psi^w((\xi)^{2p}, g) \).

If the model is massive, then picking a function \( f \in S^\frac{1}{2}(\mathbb{R}) \) equal to \( \lambda^\frac{1}{2} \) in \( \{ \lambda \geq m/2 \} \), we see that \( \omega = f(h) \in \Psi^w((\xi), g) \). If the model is massless, we fix \( \sigma > 0 \) (\( \sigma = 1 \) will do) and pick \( f \in C^\infty(\mathbb{R}) \) such that
\[
f(\lambda) = \begin{cases} \lambda^{\frac{1}{2}} & \text{if } |\lambda| \geq 4\sigma^2, \\ \sigma & \text{if } |\lambda| \leq \sigma^2. \end{cases}
\]
We set
\[
\omega_\sigma = f(h).
\]
Again by Theorem A.3 we know that \( \omega_\sigma \) belongs to \( \Psi^w((\xi), g) \). In the massive case \( \omega = f(h) \) will also be denoted by \( \omega_\sigma \). Consider now the operator
\[
T = K_0 \otimes 1 + 1 \otimes \omega_\sigma,
\]
acting on \( L^2(\mathbb{R}^3, dx) \otimes L^2(\mathbb{R}^3, dx) \). Clearly \( T \) is selfadjoint on its natural domain and \( T \geq \sigma \).
Lemma 2.3 Set
\[ M(\Xi, \xi) = \langle \Xi \rangle^2 + \langle \xi \rangle, \quad G = dX^2 + dx^2 + \langle \Xi \rangle^{-2}d\Xi^2 + \langle \xi \rangle^{-2}d\xi^2. \]

Then \( T^{-1} \) belongs to \( \Psi^w(M^{-1}, G) \).

Proof. By Lemma A.2 the metric \( G \) and weight \( M \) satisfy all the conditions in Subsect. A.1. Clearly \( T \in \Psi^w(M, G) \). By Theorem A.3 \( T^{-1} = f(T) \in \Psi^w(M^{-1}, G) \). \( \Box \)

Let us fix another cutoff function \( F(\lambda \geq \sigma) \in C^\infty(\mathbb{R}) \) with \( F(\lambda \geq \sigma) = \)
\[ F(\lambda \geq \sigma) = \begin{cases} 1 & \text{for } |\lambda| \geq 4\sigma, \\ 0 & \text{for } |\lambda| \leq 2\sigma, \end{cases} \]
and set \( F(\lambda \leq \sigma) = 1 - F(\lambda \geq \sigma) \).

Lemma 2.4 Set
\[ \beta(X, x) = \beta_X(x) = -T^{-1}F(\omega \geq \sigma)\omega^{-\frac{1}{2}}\rho_X = -T^{-1}F(\omega \geq \sigma)\omega^{-\frac{1}{2}}\rho_X. \]

Then
1. \( \beta \in C^\infty(\mathbb{R}^6) \).
2. Let \( 0 \leq \alpha < 1 \). Then \( \omega^\alpha \beta_X \in L^2(\mathbb{R}^3, dx) \) and there exists \( s > 3/2 \) such that
\[ \|\omega^\alpha \beta_X\|_{L^2(\mathbb{R}^3, dx)} \leq C\|\rho\|_{H^{-s}(\mathbb{R}^3)}, \]
uniformly in \( X \).
3. Let \( \alpha > 0 \). Then \( \omega^{-\alpha}\nabla_X \beta_X \in L^2(\mathbb{R}^3, dx) \) and there exists \( s > 3/2 \) such that
\[ \|\omega^{-\alpha}\nabla_X \beta_X\|_{L^2(\mathbb{R}^3, dx)} \leq C\|\rho\|_{H^{-s}(\mathbb{R}^3)}, \]
uniformly in \( X \).
4. One has
\[ \omega^{-\frac{1}{2}}\rho_X + (K_0 \otimes 1 + 1 \otimes \omega)\beta_X = \omega^{-\frac{1}{2}}F(\omega \leq \sigma)\rho_X. \]

Proof. The function \( \rho_X(x) \) is clearly \( C^\infty \) in \((X, x)\), so (1) follows from the fact that \( T^{-1} \) and \( \omega^{-\frac{1}{2}}F(\omega \geq \sigma) \) are pseudodifferential operators.

We claim that there exists a symbol \( b_X(x, \xi) = b(X, x, \xi) \) such that
\[ b(X, x, \xi) \in S((\langle \xi \rangle)^{-5/2}, dX^2 + dx^2 + \langle \xi \rangle^{-2}d\xi^2), \]
\[ \beta_X = b_X^{(1,0)}(x, D_x)\rho_X. \]
Let us prove our claim. Let $B(X, x, \Xi, \xi) \in S(M^{-1}, G)$ be the (1, 0) symbol of $T^{-1}$. Applying Lemma 2.3 and (A.10), we know that $T^{-1} \in \Psi^{(1,0)}(M^{-1}, G)$. Setting $w(X, x) = T^{-1} \rho_X$, this yields

$$w(X, x) = (2\pi)^{-3} \int e^{i(x \cdot \Xi + x \cdot \xi)} B(X, x, \Xi, \xi) \delta(\xi + \Xi) \hat{\rho}(\xi) d\xi d\Xi$$

(2.9)

for

$$b_X(x, \xi) = B(X, x, -\xi, \xi).$$

(2.10)

This implies that

$$b_X \in S((\langle \xi \rangle)^{-2}, dX^2 + dx^2 + \langle \xi \rangle^{-2} d\xi^2).$$

Applying once again Theorem [A.3] we know that $F(\omega \geq \sigma, \omega^\sigma \gamma^{-1/2} \in \Psi^{(1,0)}((\langle \xi \rangle)^{-1/2}, g)$. By the composition property (A.11), we obtain our claim.

(2) follows from (2.8), if we note that $\omega^\sigma F(\omega \geq \sigma, \omega^\sigma \gamma^{-1/2} \in \Psi^{(1,0)}((\langle \xi \rangle)^{-1/2}, g)$ and use the mapping property of pseudodifferential operators between Sobolev spaces recalled in (A.13). (3) is proved similarly, using that

$$\nabla_X b_X(x, D_x) \rho_X = \partial_X b_X(x, D_x) \rho_X - b_X(x, D_x) \nabla_x \rho_X.$$

Finally (4) follows from the fact that $(\omega - \omega^\sigma) F(\omega \geq \sigma) = 0$. □

### 2.3 Dressing transformation

Let $\rho$ be a charge density as above. We set for $\kappa \gg 1$

$$\rho^\kappa(x) = \kappa^3 \rho(\kappa x), \quad \rho_X^\kappa(x) = \rho^\kappa(x - X),$$

so that

$$\lim_{\kappa \to \infty} \rho_X^\kappa = q \delta_X \text{ in } H^{-s}(\mathbb{R}^3), \quad \forall \ s > 3/2,$$

where $q = \int_{\mathbb{R}^3} \rho(y) dy$. This implies

$$\|\rho_X^\kappa\|_{H^{-s}(\mathbb{R}^3)} \leq C, \text{ uniformly in } X, \kappa, \text{ for all } s > 3/2.$$

We set

$$H^\kappa = H(\rho^\kappa),$$

and as in [Ne]

$$U^\kappa = e^{i\phi(\beta_X^\kappa)},$$

which is a unitary operator on $\mathcal{H}$. (Recall that $\beta_X^\kappa$ is defined in Lemma 2.4).
Proposition 2.5 \textit{Set}
\[
a_j^\kappa(X) = \frac{1}{\sqrt{2}}a(\nabla X_j \beta_X^\kappa),
\]
\[
R^\kappa = 2 \sum_{j,k} \nabla X_j A_{jk}(X)a_k^\kappa(X) - a_j^{\kappa*}(X)A_{jk}(X)\nabla X_k
+ \sum_{j,k} 2a_j^{\kappa*}(X)A_{jk}(X)a_k^\kappa(X) - a_j^{\kappa*}(X)A_{jk}(X)a_k^\kappa(X) - a_j^\kappa(X)A_{jk}(X)a_k^\kappa(X),
\]
\[
V^\kappa(X) = -\rho^\kappa_X |\omega^{-1}F(\omega \geq \sigma)T^{-1}\rho^\kappa_X| + \frac{1}{2}(T^{-1}\rho^\kappa_X|F^2(\omega \geq \sigma)T^{-1}\rho^\kappa_X|)
+ \frac{1}{2} \sum_{j,k} A_{jk}(X)(\nabla X_j T^{-1}\rho^\kappa_X|\omega^{-1}F^2(\omega \geq \sigma)\nabla X_k T^{-1}\rho^\kappa_X|).
\]
Then
\[
U^\kappa H^\kappa U^{\kappa*} = K + d\Gamma(\omega) + \phi(\omega^{-\frac{1}{2}}F(\omega \leq \sigma)\rho^\kappa_X)
+ R^\kappa + V^\kappa(X).
\]

\textbf{Proof.} We recall some well-known identities
\[
(2.13) \quad U^\kappa(d\Gamma(\omega) + \phi(\omega^{-\frac{1}{2}}\rho_{\kappa,X}))U^{\kappa*} = d\Gamma(\omega) + \phi(\omega\beta_{\kappa}^X + \omega^{-\frac{1}{2}}\rho_{\kappa_X}) + \text{Re}(\omega \beta_{\kappa}^X + \omega^{-\frac{1}{2}}\rho_{\kappa X})|\beta_{\kappa}^X).
\]
Note that the scalar product in the rhs is real valued, since \(\rho_{\kappa_X}, \beta_{\kappa}^X\) and \(\omega\) are real vectors and operators. Using once more that \(\beta_{\kappa}^X\) is real, we see that the operators \(\phi(i\beta_{\kappa}^X)\) for different \(X\) commute, which yields
\[
U_{\kappa}D_{X_j}U^{\kappa*} = D_{X_j} - \phi(i\nabla X_j \beta_{\kappa}^X),
\]
and hence
\[
U^\kappa KU^{\kappa*} = \sum_{j,k} \left(D_{X_j} - \phi(i\nabla X_j \beta_{\kappa}^X)\right)A_{jk}(X) (D_{X_k} - \phi(i\nabla X_k \beta_{\kappa}^X)) + W(X).
\]

We expand the squares in the r.h.s. using the definition of \(a_j^\kappa(X)\) in the proposition. After rearranging the various terms, we obtain
\[
U^\kappa KU^{\kappa*} = K + \phi(K_0\beta_{\kappa}^X)
+ 2 \sum_{j,k} \nabla X_j A_{jk}(X)a_k^\kappa(X) - a_j^{\kappa*}(X)A_{jk}(X)\nabla X_k
+ \sum_{j,k} 2a_j^{\kappa*}(X)A_{jk}(X)a_k^\kappa(X) - a_j^{\kappa*}(X)A_{jk}(X)a_k^\kappa(X) - a_j^\kappa(X)A_{jk}(X)a_k^\kappa(X)
+ \frac{1}{2} \sum_{j,k} A_{jk}(X)(\nabla X_j \beta_{\kappa}^X|\nabla X_k \beta_{\kappa}^X).
\]
This yields
\[
U^\kappa H^\kappa U^{\kappa*} = K + d\Gamma(\omega)
+ 2 \sum_{j,k} \nabla X_j A_{jk}(X)a_k^\kappa(X) - a_j^{\kappa*}(X)A_{jk}(X)\nabla X_k
+ \sum_{j,k} 2a_j^{\kappa*}(X)A_{jk}(X)a_k^\kappa(X) - a_j^{\kappa*}(X)A_{jk}(X)a_k^\kappa(X) - a_j^\kappa(X)A_{jk}(X)a_k^\kappa(X)
+ \phi(\omega^{-\frac{1}{2}}\rho_{\kappa_X} + (K_0 + \omega)\beta_{\kappa}^X)
+ (\omega^{-\frac{1}{2}}\rho_{\kappa_X} + \frac{1}{2}\omega\beta_{\kappa}^X|\beta_{\kappa}^X) + \frac{1}{2} \sum_{j,k} A_{jk}(X)(\nabla X_j \beta_{\kappa}^X|\nabla X_k \beta_{\kappa}^X).
\]
The sum of the second and third lines equals \(R^\kappa\). By Lemma 2.3, the fourth line equals \(\phi(\omega^{-\frac{1}{2}}F(\omega \leq \sigma)\rho_{\kappa_X})\). The fifth line equals \(V^\kappa(X)\), using the definition of \(\beta_{\kappa}^X\). \(\square\)
2.4 Removal of the ultraviolet cutoff

Set

\[ h_0(x, \xi) = \sum_{1 \leq j,k \leq 3} \xi_j a_{jk}(x) \xi_k, \quad K(X, \xi) = \sum_{1 \leq j,k \leq 3} \xi_j A_{jk}(X) \xi_k. \]

and

(2.14) \[ E^\kappa(X) = -\frac{1}{2} (2\pi)^{-3} \int (h_0(X, \xi) + 1)^{-\frac{1}{2}} K(X, \xi)(K(X, \xi) + 1)^{-\frac{1}{2}} |\hat{\rho}|^2(\xi^{-1}) d\xi. \]

Lemma 2.6 Then there exists a bounded continuous potential \( V_{\text{ren}} \) such that

\[ \lim_{\kappa \to +\infty} V^\kappa(X) - E^\kappa(X) = V_{\text{ren}}(X), \]

in \( L^\infty(\mathbb{R}^3) \).

We will prove this lemma later. We are in the position to state the main theorem.

Theorem 2.7 Assume hypotheses (E), (B), (N). Then the family of selfadjoint operators

\[ H^\kappa - E^\kappa(X) \]

converges in strong resolvent sense to a bounded below selfadjoint operator \( H^\infty \).

Proof. By Prop. 2.8 below, \( U^\kappa(H^\kappa - E^\kappa(X))U^\kappa \) converges in norm resolvent sense to \( H^\infty \). Moreover by Lemma 2.4 (2), \( \beta^\kappa \) converges in \( B(K, K \otimes \mathfrak{h}) \) when \( \kappa \to \infty \), hence \( U^\kappa \) converges strongly to some unitary operator \( U^\infty \). It follows that \( H^\kappa \) converges in strong resolvent sense to

\[ H^\infty = U^\infty H^\infty U^\infty. \]

Proof of Lemma 2.6. For simplicity we will assume that the model is massive \( m > 0 \), which allows to remove the cutoffs \( F(\omega \geq \sigma) \) in the various formulas. The massless case can be treated similarly. Recall that

\[ T^{-1} \rho_X^\kappa = b_X(x, D_x) \rho_X^\kappa, \]

\[ \partial_X T^{-1} \rho_X^\kappa = \partial_X b_X(x, D_x) \rho_X^\kappa - b_X(x, D_x) \partial_X \rho_X^\kappa, \]

where \( b_X(x, \xi) \) is defined in (2.10). Plugging the second identity in (2.15) into the formula giving \( V_\kappa(X) \) we get

\[ V^\kappa(X) = V_1^\kappa(X) + V_2^\kappa(X), \]

for

\[ V_1^\kappa(X) = \frac{1}{2} \|b_X(x, D_x) \rho_X^\kappa\|^2 + \frac{1}{2} \sum_{jk} A_{jk}(X)(\partial_X b_X(x, D_x) \rho_X^\kappa |\omega^{-1} \partial_X b_X(x, D_x) \rho_X^\kappa) \]

\[ - \sum_{jk} A_{jk}(X)(\partial_X b_X(x, D_x) \rho_X^\kappa |\omega^{-1} b_X(x, D_x) \partial_X \rho_X^\kappa), \]

\[ V_2^\kappa(X) = -(\rho_X^\kappa |\omega^{-1} b_X(x, D_x) \rho_X^\kappa) + \frac{1}{2} \sum_{jk} A_{jk}(X)(b_X(x, D_x) \partial_X \rho_X^\kappa |\omega^{-1} b_X(x, D_x) \partial_X \rho_X^\kappa). \]
We will use that
\begin{align}
\rho_X^\kappa &\to q\delta_X \text{ in } H^s(\mathbb{R}^3), \; \forall s < -\frac{3}{2}, \\
\partial_x \rho_X^\kappa &\to q\partial_x \delta_X \text{ in } H^s(\mathbb{R}^3), \; \forall s < -\frac{5}{2}, \text{ uniformly in } X \in \mathbb{R}^3,
\end{align}
where we recall that \( q = \int_{\mathbb{R}^3} \rho(y) dy \). Using that \( b_X(x, \xi) \in S((\xi)^{-2}, g) \) and the mapping properties of pseudodifferential operators between Sobolev spaces, we obtain that
\[ \lim_{\kappa \to \infty} V_1^\kappa(X) = V_1^\infty(X) \] exists uniformly for \( X \in \mathbb{R}^3 \), and \( V_1^\infty(X) \) is a bounded continuous function, whose exact expression is obtained by replacing \( \rho_X^\kappa \) by \( q\delta_X \) in the formula giving \( V_1^\kappa(X) \).

We now consider the potential \( V_2^\kappa(X) \), which will be seen to be logarithmically divergent when \( \kappa \to \infty \). To extract its divergent part, we use symbolic calculus. We will use only the \((1,0)\) quantization and omit the corresponding superscript. We first use Prop. \( A.4 \) for the metric \( G \) defined in Lemma 2.3. Note that the 'Planck constant' for the metric \( G \) is \( \lambda(X, x, \Xi, \xi) = \min(\langle \Xi \rangle, \langle \xi \rangle) \).

Applying Prop. \( A.4 \) we obtain that the symbol \( b_X(x, \xi) \) in (2.9) equals
\[ b_X(x, \xi) = (K(X, \xi) + (h_0(x, \xi) + 1)\frac{i}{2})^{-1} + S((\xi)^{-3}, g) \]
\begin{align}
&= (K(X, \xi) + 1)^{-1} + S((\xi)^{-3}, g).
\end{align}
The same argument for the metric \( g \) shows that \( \omega^{-1} = d(x, D_x) \) for
\[ d(x, \xi) = (h_0(x, \xi) + 1)^{-\frac{i}{2}} + S((\xi)^{-2}, g). \]
Combining (2.17) and (2.18) we get that
\[ \omega^{-1}b_X(x, D_x) = c_X(x, D_x) + r_X(x, D_x), \]
\[ b_X^\kappa(x, D_x)\omega^{-1}b_X(x, D_x) = d_X(x, D_x) + s_X(x, D_x), \]
where
\begin{align}
c_X(x, \xi) &= (h_0(x, \xi) + 1)^{-\frac{i}{2}}(K(X, \xi) + 1)^{-1}, \\
d_X(x, \xi) &= (h_0(x, \xi) + 1)^{-\frac{i}{2}}(K(X, \xi) + 1)^{-2}, \\
r_X(x, \xi) &\in S((\xi)^{-1}, g), \; s_X(x, \xi) \in S((\xi)^{-6}, g), \text{ uniformly in } X \in \mathbb{R}^3.
\end{align}
Setting
\[ \tilde{V}_2^\kappa(X) = -(\rho_X^\kappa|c_X(x, D_x)\rho_X^\kappa) + \frac{1}{2} \sum_{jk} A_{jk}(X)(\partial_x j \rho_X^\kappa|d_X(x, D_x)\partial_x j \rho_X^\kappa), \]
we see using again (2.16) that

\begin{equation}
\lim_{\kappa \to \infty} V^\kappa_2(X) - \tilde{V}^\kappa_2(X) = V^\infty_2(X) \text{ exists uniformly for } X \in \mathbb{R}^3
\end{equation}

and is a bounded continuous function. The potential \( \tilde{V}^\kappa_2(X) \) can be explicitly evaluated. In fact

\begin{equation}
(\rho^\kappa_\chi | c_\chi(x, D_x) \rho^\kappa_\chi) = (2\pi)^{-3} \int e^{i(x-x_\xi)} c_\chi(x, \xi) \rho^\kappa_\chi(x) \hat{\rho} (\kappa^{-1} \xi) dx d\xi
\end{equation}

Similarly

\begin{equation}
(\partial_{x_j} \rho^\kappa_\chi | d_\chi(x, D_x) \partial_{x_k} \rho^\kappa_\chi) = (2\pi)^{-3} \int e^{i(x-x_\xi)} \partial_{x_j} \rho^\kappa_\chi(x) d_\chi(x, \xi) i \xi_j \hat{\rho} (\kappa^{-1} \xi) dx d\xi
\end{equation}

The second term in the rhs has a finite limit when \( \kappa \to \infty \). By the same argument as above, we have

\begin{equation}
(2\pi)^{-3} \int e^{i(x-x_\xi)} \rho^\kappa_\chi(x) d_\chi(x, \xi) i \xi_j \hat{\rho} (\kappa^{-1} \xi) dx d\xi
\end{equation}

Using the definition of \( c_\chi(x, \xi) \) and \( d_\chi(x, \xi) \) in (2.20), we get that

\begin{equation}
-c_\chi(X, \xi) + \frac{1}{2} \sum_{jk} A_{jk}(X) \xi_j \xi_k d_\chi(X, \xi)
\end{equation}

Using the definition of \( E^\kappa(X) \) and (2.22), (2.23) and (2.24) it follows that

\begin{equation}
\lim_{\kappa \to \infty} \tilde{V}^\kappa_2(X) - E^\kappa(X) \text{ exists uniformly for } X \in \mathbb{R}^3.
\end{equation}

This completes the proof of the lemma. \( \square \)

**Proposition 2.8** Let

\[ \hat{H}^\kappa = U^\kappa H^\kappa U^{\kappa*} - E^\kappa(X). \]

Then there exists a bounded below selfadjoint operator \( \hat{H}^\infty \) such that

1. \( \hat{H}^\kappa \) converges to \( \hat{H}^\infty \) in norm resolvent sense;
(2) $D(|\tilde{H}^\infty|^\frac{1}{2}) = D(H_0^\frac{1}{2})$.

**Proof.** The proof is analogous to the one in [Ne], using Theorem A.6 so we will only sketch it. The important point is the convergence of $R^\kappa$ as quadratic form on $D(|H_0|^{\frac{1}{2}})$ when $\kappa \to \infty$. The various terms in $R^\kappa$ are estimated with the help of Lemma A.5 applied to the coupling operator $v^\kappa = \nabla X_j \beta_X^\kappa$. From Lemma 2.4 (3), we obtain that $\omega^{-\alpha} \nabla X_j \beta_X^\kappa$ converges in $B(K, K \otimes h)$ when $\kappa \to \infty$. The only remaining point to consider is the fact that powers of the number operator $N$ appear in Lemma A.5. This is sufficient in the massive case since $H_0$ dominates $N$. In the massless case, we use the fact that $\beta_X^\kappa = F(\omega \geq \sigma/2) \beta_X^\kappa$. Therefore if we apply Lemma A.5, we can replace $N$ by $d\Gamma(1 \ll [\sigma/2, +\infty)(\omega))$, which is dominated by $H_0$. The rest of the proof is standard. \[\Box\]

### A Background on pseudodifferential calculus

In this section we recall various standard results on pseudodifferential calculus that will be needed in the sequel. It is convenient to use the language of the Weyl-Hörmander calculus.

#### A.1 Symbol classes

We start by recalling the definition of symbol classes and weights. Let $g$ be a Riemannian metric on $\mathbb{R}^d$, i.e. a map

$$g : \mathbb{R}^d \ni X \mapsto g_X,$$

with values in positive definite quadratic forms on $\mathbb{R}^d$. If $M : \mathbb{R}^d \to ]0, +\infty]$ is a strictly positive function called a *weight*, one denotes by $S(M, g)$ the symbol class of functions in $C^\infty(\mathbb{R}^d)$ such that

$$|\prod_{i=1}^k (v_i \cdot \nabla_X) a(X)| \leq C_k M(X) \prod_{i=1}^k |g_X(v_i)|^{\frac{1}{2}},$$

uniformly for $X \in \mathbb{R}^d$, $v_1, \ldots, v_k \in \mathbb{R}^d$ and $k \in \mathbb{N}$. The best constants $C_k$ are seminorms on $S(M, g)$.

Usually $d = 2n$ and one sets $\mathbb{R}^d \ni X = (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. If

$$g_X = dx^2 + \langle \xi \rangle^{-2} d\xi^2$$

and $M(X) = \langle \xi \rangle^m$, the symbol class $S(M, g)$ is the usual symbol class

$$S^m_{1,0} = \{ a \ | \partial_x^\alpha \partial_\xi^\beta a(x, \xi) | \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}, \ \alpha, \beta \in \mathbb{N}^n \}.$$ 

For simplicity we will also denote by $S^p(\mathbb{R})$, $p \in \mathbb{R}$, the space

$$S^p(\mathbb{R}) = \{ f \ | f^{(k)}(\lambda) | \leq C_k(\lambda)^{p-k}, \ k \in \mathbb{N} \},$$
ie \( S^p(\mathbb{R}) = S(\langle \lambda \rangle^p, \langle \lambda \rangle^{-2}d\lambda^2) \).

If one equips \( \mathbb{R}^{2n} \) with the usual symplectic form \( \sigma \), one can consider the dual metric \( g_X^\sigma \). Diagonalising \( g_X \) in (linear) symplectic coordinates on \( \mathbb{R}^{2n} \), one can write

\[
g_X(dx, d\xi) = \sum_{i=1}^{n} \frac{dx_i^2}{a_i^2(X)} + \frac{d\xi_i^2}{\alpha_i^2(X)},
\]

and

\[
g_X^\sigma(dx, d\xi) = \sum_{i=1}^{n} \alpha_i^2(X) dx_i^2 + a_i^2(X) d\xi_i^2.
\]

One introduces also the two functions \( \lambda(X), \Lambda(X) \) which are the best functions such that

\[
\lambda(X)^2 g_X \leq g_X^\sigma \leq \Lambda(X)^2 g_X,
\]
equal to

\[
\lambda(X) = \min_i a_i(X) \alpha_i(X), \quad \Lambda(X) = \max_i a_i(X) \alpha_i(X).
\]

The function \( \lambda(X) \) plays the role of the Planck constant.

One says that \( g \) is a Hörmander metric, if the following conditions are satisfied

1. **uncertainty principle**: \( \lambda(X) \geq 1 \);
2. **slowness**: there exists \( C > 0 \) such that

   \[
   g_Y(X - Y) \leq C^{-1} \quad \Rightarrow \quad (g_Y(\cdot)/g_X(\cdot))^{\pm 1} \leq C;
   \]
3. **temperateness**: there exist \( C > 0, N \in \mathbb{N} \) such that

   \[
   (g_Y(\cdot)/g_X(\cdot))^{\pm 1} \leq C(1 + g_Y^\sigma(Y - X))^N.
   \]

One says that a weight \( M \) is admissible for \( g \) if there exist \( C > 0, N \in \mathbb{N} \) such that

\[
(M(Y)/M(X))^{\pm 1} \leq \begin{cases} C, & \text{for } g_Y(X - Y) \leq C^{-1}, \\ C(1 + g_Y^\sigma(X - Y))^N, & \text{for } X, Y \in \mathbb{R}^{2n}. \end{cases}
\]

The metric \( g \) is geodesically temperate if \( g \) is temperate and if there exist \( C > 0 \) and \( N \in \mathbb{N} \) such that

\[
(g_Y(\cdot)/g_X(\cdot))^{\pm 1} \leq C(1 + d^\sigma(X,Y))^N,
\]

where \( d^\sigma \) is the geodesic distance for the metric \( g^\sigma \).

The metric \( g \) is strongly slow if there exists \( C > 0 \) such that

\[
g_Y^\sigma(X - Y) \leq C^{-1} \Lambda(Y)^2 \quad \Rightarrow \quad (g_Y(\cdot)/g_X(\cdot))^{\pm 1} \leq C.
\]

**Lemma A.1** The metric \( dx^2 + \langle \xi \rangle^{-2}d\xi^2 \) and weight \( \langle \xi \rangle^\alpha \) for \( \alpha \in \mathbb{R} \) satisfy all the above conditions.
Proof. Most conditions are immediate, except the last two. To check (A.6), we note that $d^s(X,Y) \leq |\xi - \eta|$, from which (A.6) follows. (A.7) follows from the fact that $\Lambda(X) = \langle \xi \rangle$.

\textbf{Lemma A.2} Assume that $(g_i, M_i), i = 1, 2$ are two metrics and weights on $\mathbb{R}^{2n}$ satisfying all the above conditions. Then $(g, M)$ on $\mathbb{R}^n$ satisfy all the above conditions for $n = n_1 + n_2$ and

$$g_X(dx) = g_{X_1}(dx_1) + g_{X_2}(dx_2), \ M(X) = M_1(X_1) + M_2(X_2).$$

\textbf{A.2 Pseudodifferential calculus}

To a symbol $a \in S'(\mathbb{R}^{2n})$, one can associate the operator defined by

$$(A.8) \quad a^w(x, D)u(x) = (2\pi)^{-n} \int e^{i(x-y)\cdot \xi} a(\frac{x+y}{2}, \xi) u(y) dyd\xi,$$

called the Weyl quantization of $a$, which is well defined as a bounded operator from $S(\mathbb{R}^n)$ into $S'(\mathbb{R}^n)$. Let $(g, M)$ be a metric and weight satisfying (A.3), (A.4), (A.5). We set

$$\Psi^w(M, g) = \{ a^w \ a \in S(M, g) \}.$$

If $a \in S(M, g)$ then $a^w$ sends $S(\mathbb{R}^n)$ into itself. Moreover as quadratic forms on $S(\mathbb{R}^n)$

$$(a^w)^* = \overline{a^w}.$$ 

One often uses also the $(1, 0)$ quantization defined by

$$(A.9) \quad a^{1,0}(x, D)u(x) = (2\pi)^{-n} \int e^{i(x-y)\cdot \xi} a(x, \xi) u(y) dyd\xi,$$

One has with obvious notations

$$(A.10) \quad \Psi^w(M, g) = \Psi^{(1,0)}(M, g).$$

Moreover

$$(A.11) \quad \Psi^w(M_1, g) \times \Psi^w(M_2, g) \subset \Psi^w(M_1M_2, g),$$

where $\ast = w$ or $(1, 0)$ and if $a \in S(M, g)$

$$(A.12) \quad a^w(x, D_x) = b^{(1,0)}(x, D_x), \text{ where } a - b \in S(M\lambda^{-1}, g).$$

Let now $g$ be the standard metric defined in (A.1) and $\mathcal{H}^s(\mathbb{R}^d)$ be the Sobolev space of order $s \in \mathbb{R}$. Then

$$(A.13) \quad \Psi^w(\langle \xi \rangle^p, g) \subset B(\mathcal{H}^s(\mathbb{R}^d), \mathcal{H}^{s-p}(\mathbb{R}^d)),$$

and the norm of $a^w$ in $B(\mathcal{H}^s(\mathbb{R}^d), \mathcal{H}^{s-p}(\mathbb{R}^d))$ is controlled by a finite number of seminorms of $a$ in $S(\langle \xi \rangle^p, g)$. 

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A.3 Functional calculus for pseudodifferential operators

Assume that the weight \( M \) satisfies

\[
M(X) \leq C(1 + \lambda(X))^N, \quad C > 0, \quad N \in \mathbb{N}.
\]

(A.14)

A symbol \( a \in S(M, g) \) is elliptic if

\[
1 + |a(X)| \geq C^{-1}M(X).
\]

(A.15)

The following result can easily be obtained.

The following theorem is shown in [Bo, Cor. 4.5]. Assume moreover that \( M \geq 1, a \in S(M, g) \) is real and elliptic, and \( a^w \) is essentially selfadjoint on \( S(\mathbb{R}^n) \). Then if \( f \in \mathcal{S}(\mathbb{R}) \), the operator \( f(a^w) \) belongs to \( \Psi^w(M^p, g) \).

The following result can easily be obtained.

Proposition A.4 Assume the hypotheses of Theorem A.3. Then

\[
f(a^w) - f(a)^w \in \Psi^w(M^p\lambda^{-1}, g),
\]

where the function \( \lambda(X) \) is defined in Subsect. A.7.

Note that the same result holds for the \((1, 0)\) quantization, thanks to (A.12).

Proof. one first proves the result for \( f(\lambda) = (\lambda - z)^{-1}, z \in \mathbb{C}\setminus\mathbb{R} \), which amounts to construct a so-called parametrix for \( a^w - z \). From symbolic calculus it follows that if \( b_z(x, \xi) = (a(x, \xi) - z)^{-1} \), then \( b_z^w(a^w - z) - \mathbb{I} \in \Psi^w(\lambda^{-1}, g) \). To extend the result to arbitrary functions one expresses \( f(a^w) \) in terms of \( (a^w - z)^{-1} \) using the well known functional calculus formula based on an almost analytic extension of \( f \) (see eg [DG, Prop. C.2.2]). \( \square \)

A.4 Various estimates

The following lemma is proved in [A, Lemma 3.3].

Lemma A.5 For \( s \in [0, 1] \), and \( v_i \in B(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h}), i = 1, 2 \) we have

1) \( \| (N + 1)^{-\frac{s}{2}} a(v_1) (H_0 + 1)^{-\frac{1-s}{2}} \| \leq \| \omega^{-\frac{s}{2}} v_1 \|_{B(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})} \),

2) \( \| (H_0 + 1)^{-\frac{s}{2}} a^*(v_1) (N + 1)^{-\frac{1-s}{2}} \| \leq \| \omega^{-\frac{s}{2}} v_1 \|_{B(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})} \),

3) \( \| (N + 1)^{-s} a(v_1) a(v_2) (H_0 + 1)^{1+s} \| \leq \| \omega^{-\frac{1+s}{2}} v_1 \|_{B(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})} \| \omega^{-\frac{1+s}{2}} v_2 \|_{B(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})} \),

4) \( \| (H_0 + 1)^{-s} a^*(v_1) a^*(v_2) (N + 1)^{-1+s} \| \leq \| \omega^{-\frac{s}{2}} v_1 \|_{B(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})} \| \omega^{-\frac{s}{2}} v_2 \|_{B(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})} \).
**Theorem A.6** Let $H_0$ be a positive selfadjoint operator on a Hilbert space $\mathcal{H}$. Let for $\kappa < \infty$, $B_\kappa$ be quadratic forms on $D(H_0^{\frac{1}{2}})$ such that

$$|B_\kappa(\psi, \psi)| \leq a \|H_0^{\frac{1}{2}} \psi\|^2 + b \|\psi\|^2,$$

where $a < 1$ uniformly in $\kappa$ and $B_\kappa \to B_\infty$ on $D(H_0^{\frac{3}{2}})$.

Then

1. there exists a selfadjoint operator $H_\kappa$ with $D(H_\kappa) \subset D(H_0^{\frac{3}{2}})$ and

$$\langle H_\kappa \psi, \psi \rangle = B_\kappa(\psi, \psi) + \langle H_0^{\frac{3}{2}} \psi, H_0^{\frac{3}{2}} \psi \rangle, \psi \in D(H_\kappa) \text{ for } \kappa \leq \infty.$$

2. the resolvent $(z - H_\kappa)^{-1}$ converges in norm to $(z - H_\infty)^{-1}$.

3. $e^{itH_\kappa}$ converges strongly to $e^{itH_\infty}$ when $\kappa \to +\infty$.

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