Extension of Operators from Weak*-closed Subspaces of $\ell_1$
into $C(K)$ Spaces

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Abstract

It is proved that every operator from a weak*-closed subspace of $\ell_1$ into a space $C(K)$
of continuous functions on a compact Hausdorff space $K$ can be extended to an operator
from $\ell_1$ to $C(K)$.

Mathematics Subject Classification. Primary 46E15, 46E30; Secondary 46B15.

$^*$Supported in part by NSF DMS-9306376.
$†$Supported in part by a grant of the U.S.-Israel Binational Science Foundation.
$‡$Participant at Workshop in Linear Analysis and Probability, NSF DMS-9311902
1. Introduction.

This work is part of an effort to characterize those subspaces $E$ of a Banach space $X$ for which the pair $(E, X)$ has the following:

**Extension Property.** *(E.P., in short): Every (bounded, linear) operator $T$ from $E$ into any $C(K)$ space $Y$ has an extension $T: X \rightarrow Y$.*

There is a quantitative version of the E.P.: for any $\lambda \geq 1$ we say that the pair $(E, P)$ has the $\lambda$-EP if for every $T: E \rightarrow Y$ there is an extension $T: X \rightarrow Y$ with $\|T\| \leq \lambda\|T\|$. It is easy to see that if $(E, X)$ has the E.P., then it has the $\lambda$-E.P. for some $\lambda$.

It is known [Zip] that for each $1 < p < \infty$ and every subspace $E$ of $\ell_p$, $(E, \ell_p)$ has the 1-E.P., while for $F \subset c_0$, $(F, c_0)$ has the $(1 + \varepsilon)$-E.P. for every $\varepsilon > 0$ [LP]. However, there is a subspace $F$ of $c_0$ for which $(F, c_0)$ does not have the 1-E.P. [JZ2]. If $E$ itself is a $C(K)$ space then, clearly, $(E, X)$ has the E.P. if and only if $E$ is complemented in $X$. It follows from [Ami] that $C(K)$ has a subspace $E$ for which $(E, C(K))$ does not have the E.P. if $K$ is any compact metric space whose $\omega$-th derived set is nonempty (which is equivalent [BePe] to saying that $C(K)$ is not isomorphic to $c_0$).

Since every separable Banach space is a quotient of $\ell_1$, the following fact demonstrates the important rôle of the space $\ell_1$ in extension problems.

**Proposition 1.1.** Let $E$ be a subspace of a Banach space $X$ and let $Q$ be an operator from $Z$ onto $X$ so that $\|Q\| = 1$ and $Q$ Ball $Z \supset \delta$ Ball $X$. If $(Q^{-1}E, \ell_1)$ has the $\lambda$-E.P. then $(E, X)$ has the $\lambda/\delta$-E.P.

**Proof.** Let $T$ be an operator from $E$ into any $C(K)$ space $Y$. Consider the operator $S = TQ$: $Q^{-1}E \rightarrow Z$. If $S: Z \rightarrow Y$ extends $S$ then since $S$ vanishes on $\ker Q$, $S$ induces an operator $\tilde{S}$ from $X \sim Z/$ker $Q$ into $Y$ so that $\tilde{S}Q = S$ and $\|\tilde{S}\| \leq \|S\|/\delta$. □

An immediate consequence of Proposition 1 is that $\ell_1$ contains a subspace $F$ for which $(F, \ell_1)$ does not have the E.P. Indeed, if $E$ denotes an uncomplemented subspace of $C[0, 1]$ which is isomorphic to $C[0, 1]$ ([Ami]) and if $Q$: $\ell_1 \rightarrow C[0, 1]$ is a quotient map and $F = Q^{-1}E$, then $(F, \ell_1)$ does not have the E.P. The main purpose of this paper is to prove the following.
Theorem. Let $\{X_n\}_{n=1}^{\infty}$ be finite dimensional and let $E$ be a weak$^*$-closed subspace of $X = (\sum X_n)_1$, regarded as the dual of $X^* = (\sum X^*_n)_{c_0}$. Then $(E, X)$ has the E.P. Moreover, if $E$ has the approximation property, then $E$ has the $(1 + \varepsilon)$-E.P. for every $\varepsilon > 0$.

We know very little about the extension problem for general pairs $(E, X)$. However, the theorem makes the following small contribution in the general case.

**Corollary 1.1.** Let $E$ be a subspace of the separable space $X$. Assume that there is a weak$^*$-closed subspace $F$ of $\ell_1$ such that $X/E$ is isomorphic to $\ell_1/F$. Then $(E, X)$ has the E.P.

**Proof.** Let $Q: \ell_1 \to X$ and $S: X \to X/E$ be quotient maps. Theorem 2 of [LR] implies that there is an automorphism of $\ell_1$ which maps $Q^{-1}E = \ker(SQ)$ onto $F$. Since $(F, \ell_1)$ has the E.P. by the theorem, so does the pair $(Q^{-1}E, \ell_1)$. It follows from Proposition 1 that $(E, X)$ has the E.P. ■

We use standard Banach space theory notation and terminology, as may be found in [LT1], [LT2].
2. Preliminaries.

Let $E$ be a subspace of $X$, $\lambda \geq 1$, and $0 < \varepsilon < 1$. Given an operator $S: E \to Y$ we say that the operator $T: X \to Y$ is a $$(\lambda, \varepsilon)$$-approximate extension of $S$ if $\|T\| \leq \lambda\|S\|$ and

$$\|S - T|_E\| \leq \varepsilon\|S\|.$$ 

Our first observation is that the existence of approximate extensions implies the existence of extensions.

**Lemma 2.1.** Let $E$ be a subspace of $X$ and assume that each operator $S: E \to Y$ has a $$(\lambda, \varepsilon)$$-approximate extension. Then the pair $(E, X)$ has the $\mu$-E.P. with $\mu \leq \lambda(1 - \varepsilon)^{-1}$.

**Proof.** Put $S_1 = S$ and let $T_1$ be a $$(\lambda, \varepsilon)$$-approximate extension of $S_1$. Then $\|T_1\| \leq \lambda\|S_1\| = \lambda\|S\|$ and $\|S_1 - T_1|_E\| \leq \varepsilon\|S\|$. Construct by induction sequences of operators $\{S_n\}_{n=1}^\infty$ from $E$ into $Y$ and $\{T_n\}_{n=1}^\infty$ from $X$ into $Y$ such that for each $n \geq 1$ $S_{n+1} = S_n - T_n|_E$ and $T_{n+1}$ is a $$(\lambda, \varepsilon)$$-approximate extension of $S_{n+1}$. Then, by definition, $\|T_n\| \leq \lambda\|S_n\|$ and $\|S_{n+1}\| \leq \varepsilon\|S_n\|$ for every $n \geq 1$. It follows that $\|S - \sum_{i=1}^n T_i|_E\| \leq \varepsilon^n\|S\|$ and $\|T_n\| \leq \lambda\varepsilon^{n-1}\|S\|$ for all $n \geq 1$. Hence the operator $T = \sum_{i=1}^\infty T_i$ extends $S$ and $\|T\| \leq \lambda(1 - \varepsilon)^{-1}\|S\|$. 

Given a finite dimensional decomposition (FDD, in short) $\{Z_n\}_{n=1}^\infty$ of a space $Z$, we will be interested in subspaces of $Z$ with FDD’s which are particularly well-positioned with respect to $\{Z_n\}_{n=1}^\infty$.

**Definition.** Let $F \subset Z$ and let $\{F_n\}_{n=1}^\infty$ be an FDD for $F$. We say that $\{F_n\}_{n=1}^\infty$ is **alternately disjointly supported** with respect to $\{Z_n\}_{n=1}^\infty$ if there exist integers $1 = k(1) < k(2) < \cdots$ such that for each $n \geq 1$, $F_n \subset Z_{k(n)} + Z_{k(n)+1} + \cdots + Z_{k(n+2)-1}$.

An important property of an alternatively disjointly supported FDD is that if $\{n(j)\}_{j=1}^\infty$ is any increasing sequence of integers and if we drop $\{F_{n(j)}\}_{j=1}^\infty$, then the remaining $F_n$’s can be grouped into blocks $\widetilde{F}_j = \sum_{i=n(j)+1}^{n(j+1)-1} F_i$ which form an FDD that is disjointly supported on the $\{Z_n\}_{n=1}^\infty$; more precisely, with the above notation,

$$\widetilde{F}_j \subset \sum_{m=k(n(j)+1)}^{k(n(j)+1)+1} Z_m \quad \text{for all } j \geq 1.$$
We will show that for certain subspaces of a dual space with an FDD, a given FDD can be replaced by one which is alternately disjointly supported.

We first need the following main tool:

**Proposition 2.1.** Let \( \{X_n\}_{n=1}^{\infty} \) be a shrinking FDD for \( X \), let \( Q \) be a quotient mapping of \( X \) onto \( Y \) and suppose that \( \{\tilde{E}_n\}_{n=1}^{\infty} \) is an FDD for \( Y \). Then there are a blocking \( \{E'_n\}_{n=1}^{\infty} \) of \( \{\tilde{E}_n\}_{n=1}^{\infty} \), an FDD \( \{W_n\}_{n=1}^{\infty} \) of \( X \) which is equivalent to \( \{X_n\}_{n=1}^{\infty} \), and \( 1 = k(1) < k(2) < \cdots \) so that for each \( n \) and each \( k(n) \leq j < k(n + 1) \), \( QW_j \subset E'_n + E'_{n+1} \). Moreover, given \( \varepsilon > 0 \), \( \{E'_n\}_{n=1}^{\infty} \) and \( \{W_n\}_{n=1}^{\infty} \) can be chosen so that there is an automorphism \( T \) on \( X \) with \( \|I - T\| < \varepsilon \) and \( TX_n = W_n \) for all \( n \).

**Proof.** In order to avoid complicated notation we shall prove the statement for the case where, for every \( n \geq 1 \), \( X_n \) (and hence also \( W_n \)) is one dimensional. The same arguments, with only obvious modifications yield the FDD case. (Actually, in the proof of the theorem, only the basis case of Proposition 2 is needed. Indeed, in Step 3 of the proof of the theorem, one can replace \( E \) by \( E_1 \equiv E \oplus_1 (\sum G_n)_1 \) and \( X \) by \( X_1 = X \oplus_1 (\sum G_n)_1 \), where \( \{G_n\}_{n=1}^{\infty} \) is a sequence which is dense in the sense of the Banach-Mazur distance in the set of all finite dimensional spaces, and use the fact [JRZ], [Pel] that \( E_1 \) has a basis. In fact, this trick is used in a different way for the proof of the “moreover” statement in the theorem.)

So assume that \( X \) has a normalized shrinking basis \( \{x_n\}_{n=1}^{\infty} \) with biorthogonal functionals \( \{f_n\}_{n=1}^{\infty} \); we are looking for an equivalent basis \( \{w_n\}_{n=1}^{\infty} \) of \( X \) for which the statement holds. First we perturb the basis for \( X \) to get another basis whose images under \( Q \) are supported on finitely many of the \( \tilde{E}_n \)'s. This step does not require the hypothesis that \( \{x_n\}_{n=1}^{\infty} \) be shrinking.

For each \( n \geq 1 \) let \( \tilde{Q}_n \) be the FDD’s natural projection from \( Y \) onto \( \tilde{E}_1 + \tilde{E}_2 + \cdots + \tilde{E}_n \). Let \( 1 > \varepsilon > 0 \) and set \( C = \sup_n \|f_n\| \). Choose \( p_1 < p_2 < \cdots \) so that for each \( n \), \( \|Qx_n - \tilde{Q}_{p_n}Qx_n\| < \varepsilon C^{-1}2^{-n} \). Since \( Q \) is a quotient mapping, there is for each \( n \) a vector \( z_n \) in \( X \) with \( \|z_n\| < \varepsilon C^{-1}2^{-n} \) and \( Qz_n = Qx_n - \tilde{Q}_{p_n}Qx_n \). Let \( y_n = x_n - z_n \), so that \( Qy_n \) is in \( \tilde{E}_1 + \cdots + \tilde{E}_{p_n} \). It is standard to check that \( \{y_n\}_{n=1}^{\infty} \) is equivalent to \( \{x_n\}_{n=1}^{\infty} \). Indeed, define an operator \( S \) on \( X \) by \( Sx = \sum_{n=1}^{\infty} f_n(x)z_n \). Then \( \|S\| < \varepsilon \) and \( Sx_n = z_n \), so \( I - S \) is an isomorphism from \( X \) onto \( X \) which maps \( x_n \) to \( y_n \).
Define a blocking \( \{ E_n \}_{n=1}^{\infty} \) of \( \{ \tilde{E}_n \}_{n=1}^{\infty} \) by \( E_n = \tilde{E}_{p_n-1} + \cdots + \tilde{E}_{p_n} \) (where \( p_0 \equiv 0 \)). Then for each \( n \), \( Qy_n \) is in \( E_1 + \cdots + E_n \).

Let \( Q_n \) be the basis projection from \( Y \) onto \( E_1 + \cdots + E_n \), \( P_n \) the basis projection from \( X \) onto \( \text{span} \{ y_1, \ldots, y_n \} \), and set \( C_1 = \sup_n \| P_n \| \). Since \( \{ y_n \}_{n=1}^{\infty} \) is shrinking, \( \lim_{n \to \infty} \| Q_n Q(I - P_n) \| = 0 \). Since \( Q \) is a quotient mapping, for each \( n \) there exists a mapping \( T_n \) from \( E_1 + \cdots + E_n \) into \( X \) so that \( QT_n \) is the identity on \( E_1 + \cdots + E_n \). Set \( M_n = \| T_n \| \), let \( 1 > \epsilon > 0 \), and recursively choose \( 0 = k(0) < k(1) < k(2) < \cdots \) so that for each \( n \), \( \| Q_{k(n)} Q(I - P_{k(n+1)-1}) \| < (2C_1 M_{k(n)})^{-1} 2^{-n} \epsilon \). Setting \( w_j = y_j - T_{k(n)} Q_{k(n)} Q y_j \) for \( k(n+1) \leq j < k(n+2) \), we see that \( Q w_j \) is in \( E_{k(n)+1} + \cdots + E_{k(n+2)} \) when \( k(n+1) \leq j < k(n+2) \).

The desired blocking of \( \{ \tilde{E}_n \}_{n=1}^{\infty} \) is defined by \( E'_n = E_{k(n)-1+1} + E_{k(n)-1+2} + \cdots + E_{k(n)} \), but it remains to be seen that \( \{ w_n \}_{n=1}^{\infty} \) is a suitably small perturbation of \( \{ y_n \}_{n=1}^{\infty} \).

The inequality \( \| Q_{k(n)} Q(I - P_{k(n+1)-1}) \| < (2C_1 M_{k(n)})^{-1} 2^{-n} \epsilon \) implies, by composing on the right with \( P_{k(n+2)-1} \), that \( \| Q_{k(n)} (P_{k(n)} - P_{k(n+1) - 1}) \| < (2M_{k(n)})^{-1} 2^{-n} \epsilon \). Thus if we define an operator \( V \) on \( X \) by \( V x = \sum_{n=0}^{\infty} T_{k(n)} Q_{k(n)} Q (P_{k(n+2)-1} - P_{k(n+1)-1}) x \), we see that \( \| V \| < \epsilon \) and hence \( T \equiv I - V \) is invertible. But for \( k(n+1) \leq j < k(n+2) \), \( V y_j = T_{k(n)} Q_{k(n)} Q x_j \); that is, \( T y_j = w_j \).

Using a duality argument we get from Proposition 2.1 the following.

**Corollary 2.1.** Let \( \{ Z_n \}_{n=1}^{\infty} \) be an \( \ell_1 \)-FDD for a space \( Z \). Regard \( Z \) as the dual of the space \( Z_* = (\sum Z_n^*)_{c_0} \) and let \( F \) be a weak*-closed subspace of \( Z \) with an FDD. Then \( Z \) and \( F \) have \( \ell_1 \)-FDD’s \( \{ V_n \}_{n=1}^{\infty} \) and \( \{ U_n \}_{n=1}^{\infty} \), respectively, so that \( \{ U_n \}_{n=1}^{\infty} \) is alternately disjointly supported with respect to \( \{ V_n \}_{n=1}^{\infty} \). Moreover, given \( \epsilon > 0 \), \( \{ V_n \}_{n=1}^{\infty} \) can be chosen so that for some blocking \( \{ Z'_n \}_{n=1}^{\infty} \) of \( \{ Z_n \}_{n=1}^{\infty} \), there is an automorphism \( T \) of \( Z_* \) with \( \| I - T \| < \epsilon \) and \( T Z'_n = V_n \) for all \( n \geq 1 \).

**Proof.** Being weak*-closed, \( F \) has a predual \( F_* = Z_* / F_\perp \) which is a quotient space of \( Z_* \). By [JRZ], \( F_* \) has a shrinking FDD and consequently, by Theorem 1 of [JZ2], \( F_* \) has a shrinking \( c_0 \)-FDD \( \{ \tilde{E}_n \}_{n=1}^{\infty} \). Let \( Q \): \( Z_* \to F_* \) be the quotient mapping. By Proposition 2.1 there are a blocking \( \{ E'_n \}_{n=1}^{\infty} \) of \( \{ \tilde{E}_n \}_{n=1}^{\infty} \), an FDD \( \{ W_n \}_{n=1}^{\infty} \) of \( Z_* \) which is equivalent to \( \{ Z'_n \}_{n=1}^{\infty} \), even the image of \( \{ Z'_n \}_{n=1}^{\infty} \) under some automorphism on \( Z_* \) which is arbitrarily
close to \( I_{Z_*} \), and 1 = \( k(1) < k(2) < \cdots \) so that for each \( n \) and \( k(n) \leq j < k(n + 1) \), \( QW_j \subset E'_n + E'_{n+1} \). The equivalence implies that \( \{W_n\}_{n=1}^\infty \) is a \( c_0 \)-FDD and, being a blocking of a \( c_0 \)-FDD, \( \{E'_n\}_{n=1}^\infty \) is a \( c_0 \)-FDD. Let \( \{V_n\}_{n=1}^\infty \) (resp. \( \{U_n\}_{n=1}^\infty \)) be the dual FDD of \( \{W_n\}_{n=1}^\infty \) (resp. \( \{E'_n\}_{n=1}^\infty \)) for \( Z \) (resp. \( F \)). Then \( \{V_n\}_{n=1}^\infty \) is an \( \ell_1 \)-FDD for \( Z \) and \( \{U_n\}_{n=1}^\infty \) is an \( \ell_1 \)-FDD for \( F \). Moreover, suppose that \( u \) is in \( U_n \) and \( w_j \) is in \( W_j \), where either \( j < k(n) \) or \( j \geq k(n + 2) \). Let \( m \) be the integer for which \( k(m) \leq j < k(m + 1) \). Then either \( m < n \) or \( m > n + 1 \) hence \( n \neq m \) and \( n \neq m + 1 \). Then \( Qw_j \in E'_m + E'_{m+1} \), hence \( u(w_j) = \langle u, Qw_j \rangle = 0 \). This proves that \( U_n \) is supported on \( \sum_{j=k(n)}^{k(n+2)-1} V_j \). \( \blacksquare \)
3. Proof of the Theorem.

The proof consists of four parts, the first three of which are essentially simple special cases of the theorem.

Step 1. $E$ has an FDD $\{E_n\}_{n=1}^{\infty}$ with $E_n \subset X_n$ for all $n$.

Proof. Let $Y = C(K)$ and let $S: E \to Y$ be any operator. Using the $L_{\infty,1+\varepsilon}$-property of $Y$ (or see Theorem 6.1 of [Lin]), one sees that the finite rank operator $S|_{E_n}$ has an extension $S_n: X_n \to Y$ with $\|S_n\| \leq (1 + \varepsilon)\|S_n\|$. Define the extension $S$ of $S$ by $S\left(\sum_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} S_n x_n$. Since $\{X_n\}_{n=1}^{\infty}$ is an exact $\ell_1$-decomposition, it follows that $\|S\| \leq (1 + \varepsilon)\|S\|$. 

Step 2. $E$ has an $\ell_1$-FDD $\{E_n\}_{n=1}^{\infty}$ which is alternately disjointly supported with respect to $\{X_n\}_{n=1}^{\infty}$.

Proof. Given $\delta > 0$, let $1 < (1 + \varepsilon)(1 - \varepsilon)^{-1} < 1 + \delta$ and choose an integer $N > (1 + \varepsilon)M\varepsilon^{-1}$ where $M$ is the constant of the $\ell_1$-FDD $\{E_n\}_{n=1}^{\infty}$; that is, the constant of equivalence of $\{E_n\}_{n=1}^{\infty}$ to the natural $\ell_1$-FDD for $(\sum E_n)_1$. Let $Y = C(K)$ and let $S: E \to Y$ be an operator with $\|S\| = 1$. For each $1 \leq j \leq N$ let

$$Z_j = \text{span}\{E_i : i \neq kN + j, k = 0, 1, 2, \ldots\}.$$ 

Each subspace $Z_j$ has a natural $\ell_1$-FDD which is disjointly supported with respect to $\{X_n\}_{n=1}^{\infty}$ because $\{E_n\}_{n=1}^{\infty}$ is alternately disjointly supported with respect to $\{X_n\}_{n=1}^{\infty}$. By Step 1, $S|_{Z_j}$ has an extension $T_j: X \to Y$ with

$$\|T_j\| \leq (1 + \varepsilon)\|S_j\| \leq (1 + \varepsilon)\|S\| = 1 + \varepsilon.$$

Define $T: Z \to Y$ by $T = N^{-1}\sum_{j=1}^{N} T_j$. Then $\|T\| \leq (1 + \varepsilon)\|S\| = 1 + \varepsilon$. Moreover, if $e \in E_i$ and $i = kN + h$ for some $1 \leq h \leq N$, then $T_h e = S_j e = S e$ for all $j \neq h$ hence $T$ is “almost” an extension of $S$. Indeed, $\|Te - Se\| = \frac{1}{N}\|T_h e - Se\| \leq \frac{2+\varepsilon}{N}\|e\|$ whenever $e \in E_i$ for some $i$. Recalling that the $\ell_1$-FDD $\{E_n\}_{n=1}^{\infty}$ has constant $M$, we have that

$$\|T|_E - S\| \leq M \sup_n \|T|_{E_n} - S|_{E_n}\| \leq \frac{M(2 + \varepsilon)}{N} < \varepsilon.$$ 

This proves that $T$ is an $(1 + \varepsilon, \varepsilon)$-approximate extension of $S$ and therefore, by Lemma 2.1, $(E, Z)$ has the $(1 + \varepsilon)(1 - \varepsilon)^{-1}$-E.P.
Step 3. $E$ has an FDD.

Proof. By Corollary 2.1, $X$ and $E$ have $\ell_1$-FDD’s $\{Z_n\}_{n=1}^\infty$ and $\{E_n\}_{n=1}^\infty$, respectively, with $\{E_n\}_{n=1}^\infty$ is alternately disjointly supported with respect to $\{Z_n\}_{n=1}^\infty$, and, by Remark 2.1, $\{Z_n\}_{n=1}^\infty$ has constant of equivalence to $(\sum Z_n)_1$ arbitrarily close to one. Hence, by Step 2, $(E, X)$ has the $(1 + \delta)$-E.P. for every $\delta > 0$.

This gives the “moreover” statement when $E$ has an FDD. When $E$ just has the approximation property, we enlarge $X$ to $X_1 \equiv X \oplus_1 C_1$, where $C_1 = (\sum G_n)_1$ and $\{G_n\}_{n=1}^\infty$ is a sequence of finite dimensional spaces which is dense (in the sense of the Banach-Mazur distance) in the set of all finite dimensional spaces; and we enlarge $E$ to $E_1 \equiv E \oplus_1 C_1$. $X_1$ is again an exact $\ell_1$-sum of finite dimensional spaces and $E_1$ is weak$^*$-closed in $X_1$. Moreover, since $E$ is a dual space which has the approximation property, $E$ has the metric approximation property [LT1], and hence by [Joh], $E_1$ is a $\pi$-space, whence, since $E_1$ is a dual space, $E_1$ has an FDD by [JRZ]. Thus by Step 3, $(E_1, X_1)$ has the $(1 + \delta)$-E.P. for each $\delta > 0$, and, therefore, so does $(E, X)$.

Step 4. The general case.

We start with a lemma.

Lemma 3.1. Let $Z$ be a Banach space and let $E$ be a subspace of $Z$. Suppose that $E$ has a subspace $F$ such that $(F, Z)$ has the $\lambda$-E.P. and $(E/F, Z/F)$ has the $\mu$-E.P. Then $(E, Z)$ has the $(\lambda + \mu(1 + \lambda))$-E.P.

Proof. Let $Y = C(K)$ and let $S: E \to Y$ be any operator. Let $S_1: Z \to Y$ be an extension of $S|_F$ with $\|S_1\| \leq \lambda\|S\|$. The operator $W = S - S_1|_E$ from $E$ into $Y$ vanishes on $F$ and so induces an operator $\widetilde{W}: E/F \to Y$ in the usual way, and $\|\widetilde{W}\| = \|W\| \leq \|S\| + \|S_1\| \leq (1 + \lambda)\|S\|$. By our assumptions, $\widetilde{W}$ extends to an operator $W_1: Z/F \to Y$ with $\|W_1\| \leq \mu\|\widetilde{W}\| \leq \mu(1 + \lambda)\|S\|$. Let $Q: Z \to Z/F$ denote the quotient map. Then $T = S_1 + W_1Q$ is the desired extension of $S$. Indeed, for every $e \in E$

$$Te = S_1e + W_1Qe = S_1e + We = S_1e + (S - S_1)e = Se$$

and $\|T\| \leq \|S_1\| + \|W_1\| \leq (\lambda + \mu(1 + \lambda))\|S\|$.

$\blacksquare$
Let us now return to the proof of the general case. Being a weak$^*$-closed subspace of $\ell_1$, $E$ is the dual of the quotient space $E_* = (\sum X_n^*)_c/E$. Our main tool in this part of the proof is Theorem IV.4 of [JR] and its proof. This theorem states that $E_*$ has a subspace $V$ so that both $V$ and $E_*/V$ have shrinking FDD’s. Under these circumstances, Theorem 1 of [JZ1] implies that both $V$ and $E_*/V$ have $c_0$-FDD’s. In order to prove the theorem it suffices, in view of Lemma 3.1, to show that both pairs $(V_\perp, X)$ and $(E/V_\perp, X/V_\perp)$ have the E.P. Now $(V_\perp, X)$ has the $(1+\delta)$-E.P. for all $\delta > 0$ by Step 3, so it remains to discuss the pair $(E/V_\perp, X/V_\perp)$. This discussion requires some preparation and some minor modification in the proof of Theorem IV.4. of [JR].

We first need a known perturbation lemma:

**Lemma 3.2.** Suppose $E, F$ are subspaces of $X^*$ with $F$ norm dense in $X^*$ and $X^*$ is separable. Then for each $\varepsilon > 0$ there is an automorphism $T$ on $X$ so that $\|I - T\| < \varepsilon$ and $T^* E \cap F$ is norm dense in $T^* E$.

**Proof.** Let $(x_n, x_n^*)$ be a biorthogonal sequence in $X \times E$ with $\text{span } x_n^* = E$ (see, e.g., [Mac]) and take $y_n^* \in F$ so that $\sum \|x_n^* - y_n^*\| \|x_n\| < \varepsilon$. Define $T: X \to X$ by

$$Tx = x - \sum_{n=1}^{\infty} (x_n^* - y_n^*)x_n.$$  

Returning to the proof of the theorem, we may assume, in view of Lemma 3.2, that $E \cap \text{span } \bigcup_{n=1}^{\infty} X_n$ is norm dense in $E$. The standard back-and-forth technique [Mac] for producing biorthogonal sequences yields a biorthogonal sequence $\{(x_n, x_n^*)\}_{n=1}^{\infty} \subset X^* \times E$ with $\text{span } \{Qx_n\}_{n=1}^{\infty} = \text{span } \bigcup_{n=1}^{\infty} QX_n^*$, $\text{span } \{x_n^*\}_{n=1}^{\infty} = E \cap \text{span } \bigcup_{n=1}^{\infty} X_n$, and where $Q$ is the quotient mapping from the predual $X_* = (\sum X_n^*)_c$ of $X$ onto the predual $E_*$ of $E$.

This means that for any $N$, $x_j^*$ is in $\text{span } \bigcup_{n=N}^{\infty} X_n$ if $j$ is sufficiently large.

We now refer to the construction in Theorem IV.4 of [JR] and the finite sets $\Delta_1 \subset \Delta_2 \subset \cdots$ of natural numbers defined there. From that construction, it is clear that, having defined $\Delta_n$, the smallest element, $k(n)$, in $\Delta_{n+1} \setminus \Delta_n$ can be as large as we desire. In particular, if $\{x_j^*\}_{j=1}^{\max \Delta_n}$ is a subset of $\text{span } \bigcup_{n=1}^{m(n)} X_i$, then we choose $k(n)$ large enough so that for $j \geq k(n)$, $x_j^*$ is in $\text{span } \bigcup_{i=1}^{\infty} X_i$. Thus setting

$$Z_n = \text{span } \{x_j^*: j \in \Delta_n \setminus \Delta_{n+1}\}$$

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(where $\Delta_0 \equiv \emptyset$), we have that $\{Z_n\}_{n=1}^{\infty}$ is disjointly supported relative to $\{X_n\}_{n=1}^{\infty}$. (In the notation above and setting $m(0) = 0$, we have for each $n$ that

\[
(*) \quad Z_n \subset \text{span}\{X_j\}_{j=m(n^{-1})+1}^{m(n)}.
\]

The subspace $V$ of $E_*$ is defined to be the annihilator of $\left\{ x_j^*: j \in \bigcup_{n=1}^{\infty} \Delta_n \right\}$ and, as mentioned earlier, it follows from [JR] and [JZ1] that $V$ has a $c_0$-FDD and thus $V^* = E/V^\perp$ has an $\ell_1$-FDD. It is also proved in [JR], but is obvious from the “extra” we have added here, that $\overline{\text{span}}\{Z_j\}_{j=1}^{\infty}$ is weak$^*$-closed and hence equals $V^\perp$. It is also obvious from (*) that $X/V^\perp$ has an $\ell_1$-FDD. Therefore, by Step 3 ($E_*/V^\perp$, $X/V^\perp$) has the E.P. \[\blacksquare\]

**Remark.** Under the hypotheses of the theorem, we do not know whether $(E, X)$ has the $(1 + \varepsilon)$-E.P. for every $\varepsilon > 0$ when $E$ fails the approximation property. The proof we gave yields only that $(E, X)$ has the $(3 + \varepsilon)$-E.P. for all $\varepsilon > 0$. 

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4. Concluding Remarks and Problems.

Very little is known about the Extension Property, so there is no shortage of problems.

**Problem 4.1.** If $E$ is a subspace of $X$ and $X$ is reflexive, does $(E, X)$ have the E.P.? What if $X$ is superreflexive? What if $X$ is $L_p$, $1 < p \neq 2 < \infty$?

**Problem 4.2.** If $E$ is a reflexive subspace of $X$, does $(E, X)$ have the E.P.? What if $E$ is just isomorphic to a conjugate space? In the latter case, what if, in addition, $X$ is $\ell_1$?

If $E$ is a subspace of $c_0$, then $(E, c_0)$ has the $(1 + \varepsilon)$-E.P. for every $\varepsilon > 0$ [LP] but need not have the 1-E.P. [JZ2]. We do not know if this phenomenon can occur in the setting of reflexive spaces:

**Problem 4.3.** If $X$ is reflexive and $(E, X)$ has the $(1 + \varepsilon)$-E.P. for every $\varepsilon > 0$, does $(E, X)$ have the 1-E.P.?

The following observation gives an affirmative answer to Problem 4.3 in a special case.

**Proposition 4.1.** If $X$ is uniformly smooth and $(E, X)$ has the $(1 + \varepsilon)$-E.P. for every $\varepsilon > 0$, then $(E, X)$ has the 1-E.P.

**Proof.** In preparation for the proof, we recall Proposition 2 of [Zip], which says:

$(E, X)$ has the $\lambda$-E.P. if and only if there exists a weak$^*$-continuous extension mapping from Ball $E^*$ to $\lambda$Ball $X^*$; that is, a continuous mapping $\phi : (\text{Ball } E^*, \text{weak }^*) \to (\lambda \text{Ball } X^*, \text{weak }^*)$ for which $(\phi e^*)|_E = e^*$ for every $e^*$ in Ball $E^*$.

Since $X$ is uniformly smooth, given $\varepsilon > 0$ there exists $\delta > 0$ so that if $x^*, y^*$ in $X^*$ and $x$ in $X$ satisfy $\|x^*\| = \|x\| = 1 = \langle x^*, x \rangle = \langle y^*, x \rangle$ with $\|y^*\| < 1 + \delta$, then $\|x^* - y^*\| < \varepsilon$. Letting $\phi_n : \text{Ball } E^* \to (1 + \frac{1}{n})\text{Ball } X^*$ be a weakly continuous extension mapping and letting $f : \text{Sphere } E^* \to \text{Sphere } X^*$ be the (uniquely defined, by smoothness) Hahn-Banach extension mapping, we conclude that

$$\lim_{n \to \infty} \sup \{\|\phi_n(x^*) - f(x^*)\| : x^* \in \text{Sphere } E^*\} = 0.$$ 

That is, $\{\phi_n|_{\text{Sphere } E^*}\}_{n=1}^\infty$ is uniformly convergent to $f|_{\text{Sphere } E^*}$. Since each $\phi_n$ is weakly continuous, so is $f|_{\text{Sphere } E^*}$. 

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If $E$ is finite dimensional, then clearly the positively homogeneous extension of $f$ to a mapping from $\text{Ball } E^*$ into $\text{Ball } X^*$ is a weakly continuous extension mapping. So assume that $E$ has infinite dimension. But then $\text{Sphere } E^*$ is weakly dense in $\text{Ball } E^*$, so by the weak continuity of the $\phi_n$’s and the weak lower semicontinuity of the norm, we have

$$\sup \{ \| \phi_n(x^*) - \phi_m(x^*) \| : x^* \in \text{Ball } E^* \} = \sup \{ \| \phi_n(x^*) - \phi_m(x^*) \| : x^* \in \text{Sphere } E^* \}$$

which we saw tends to zero as $n, m$ tend to infinity. That is, $\{ \phi_n \}_{n=1}^\infty$ is a uniformly Cauchy sequence of weakly continuous functions and hence its limit is also weakly continuous.

It is apparent from the proof of Proposition 4.1 that the 1-E. P. is fairly easy to study in a smooth reflexive space $X$ because every extension mapping from $\text{Ball } E^*$ to $\text{Ball } X^*$ is, on the unit sphere of $E^*$, the unique Hahn-Banach extension mapping. Let us examine this situation a bit more in the general case. Suppose $E$ is a subspace of $X$ and let $A(E)$ be the collection of all norm one functionals in $E^*$ which attain their norm at a point of $\text{Ball } E^*$. The Bishop-Phelps theorem [BP], [Die] says that $A(E)$ is norm dense in $\text{Sphere } E^*$, hence, if $E$ has infinite dimension, $A(E)$ is weak$^*$ dense in $\text{Ball } E^*$. Therefore $(E, X)$ has the 1-E.P. if and only if there is a weak$^*$ continuous Hahn-Banach selection mapping $\phi : A(E) \to \text{Ball } X^*$ which has a weak$^*$ continuous extension to a mapping $\phi$ from $\overline{A(E)}^{w^*} = \text{Ball } E^*$ to $\text{Ball } X^*$, since clearly $\phi$ will then be an extension mapping. The existence of $\phi$ is equivalent to saying that whenever $\{x^*_\alpha\}$ is a net in $A(E)$ which weak$^*$ converges in $E^*$, then $\{\phi x^*_\alpha\}$ weak$^*$ converges in $X^*$ (see, for example, [Bou I.8.5]). Now when $X$ is smooth, there is only one mapping $\phi$ to consider, and in this case the above discussion yields the next proposition when $\dim E = \infty$ (when $\dim E < \infty$ one extends from $\overline{\text{Sphere } E^*} = \overline{A(E)}^{w^*}$ to $\text{Ball } E^*$ by homogeneity).

**Proposition 4.2.** Let $E$ be a subspace of the smooth space $X$. The pair $(E, X)$ fails the 1-E.P. if and only if there are nets $\{x^*_\alpha\}, \{y^*_\alpha\}$ of functionals in $\text{Sphere } X^*$ which attain their norm at points of $\overline{\text{Sphere } E}$ and which weak$^*$ converge to distinct points $x^*$ and $y^*$, respectively, which satisfy $x^*|_E = y^*|_E$.

An immediate, but surprising to us, corollary to Proposition 4.2 is:

**Corollary 4.1.** Let $E$ be a subspace of the smooth space $X$. If the pair $(E, X)$ fails
the 1-E.P., then there is a subspace \( F \) of \( X \) of codimension one which contains \( E \) so that \( (F, X) \) fails the 1-E.P.

**Proof.** Get \( x^*, y^* \) from Proposition 4.2 and set \( F = \text{span} \, E \cup (\ker x^* \cap \ker y^*) \).

**Problem 4.4.** Is Corollary 4.1 true for a general space \( X \)?

**Corollary 4.2.** For \( 1 < p \neq 2 < \infty \), \( L_p \) has a subspace \( E \) for which \( (E, L_p) \) fails the 1-E.P.

**Proof.** We regard \( L_p \) as \( L_p(0, 2) \) and make the identifications \( L_p^* = L_q = L_q(0, 2) \), where \( q = \frac{p}{p-1} \) is the conjugate index to \( p \). Let

\[
  f = 1_{(0, \frac{1}{2})} - 1_{(\frac{1}{2}, 1)}, \quad g = -2 \cdot 1_{(\frac{1}{2}, 1)} - 1_{(1, 2)},
\]

regarded as elements of \( L_q \), and define

\[
  E = (f - g)^\perp = \{ x \in L_p(0, 2) : \int_0^2 x = 0 \}.
\]

Notice that \(|f|^q - 1 \text{sign } f\) is in \( E \), which implies that \( 1 = \|f\|_q = \|f\|_{L_p^*} = \|f|_E\|_{E^*} \). So \( f \) and \( g \) induce the same linear functional on \( E \) (we write \( f|_E = g|_E \)), and \( f \) is the unique Hahn-Banach extension of this functional to a functional in \( L_p^* = L_q \).

**Claim.** There exists \( h \) in \( L_q \) supported on \([0, \frac{1}{2}]\) so that \( \int_0^2 h = 0 = \int_0^2 |g+h|^q - 1 \text{sign } (g+h) \).

Assume the claim. Set \( \lambda = \|g + h\|_q \) and let \( \{h_n\}_{n=1}^\infty \) be a sequence of functions which have the same distribution as \( h \), are supported on \([0, \frac{1}{2}]\), and are probabilistically independent as random variables on \([0, \frac{1}{2}]\) with normalized Lebesgue measure. Then \( g_n \equiv \lambda^{-1}(g + h_n) \) defines a sequence on the unit sphere of \( L_q(0, 2) \) which converges weakly to \( \lambda^{-1}g \). Moreover, \(|g_n|^q - 1 \text{sign } g_n\) is in \( E \), which means that as a linear functional on \( L_p \), \( g_n \) attains its norm at a point on the unit sphere of \( E \). In view of Proposition 4.2, to complete the proof it suffices to find a sequence \( \{f_n\}_{n=1}^\infty \) on the unit sphere of \( L_q \) which converges weakly in \( L_q \) to \( \lambda^{-1}f \) so that \(|f_n|^q - 1 \text{sign } f_n\) is in \( E \). This is easy: take \( w \) supported on \([1, 2]\) so that

\[
  \int_0^2 w = 0 = \int_0^2 |w|^q - 1 \text{sign } w \left( = \int_0^2 |f + w|^q - 1 \text{sign } (f + w) \right)
\]
and \( \|f + w\|_q^q = 1 = 1 + \|w\|_q^q = \lambda^q \) (so \( w \) can be a multiple of \( \mathbf{1}_{(1, \frac{3}{2})} - \mathbf{1}_{(\frac{3}{2}, 2)} \)). Let \( \{w_n\}_{n=1}^\infty \) be a sequence of functions which have the same distribution as \( w \), are supported on \([1, 2]\), and are probabilistically independent as random variables on \([1, 2]\). Now set \( f_n = \lambda^{-1}(f + w_n) \).

We turn to the proof of the claim. Fix any \( 0 < \varepsilon < \frac{1}{4} \). For appropriate \( d \), the choice

\[
h = d(4\varepsilon \mathbf{1}_{(0, \frac{1}{4})} - \mathbf{1}_{(\frac{1}{4} - \varepsilon, \frac{1}{4})})
\]

works. Indeed, \( \int_0^2 h = 0 \) no matter what \( d \) is, and \( gh = 0 \), so we need choose \( d \) to satisfy

\[
-\int_0^2 |g|^{q-1}\text{sign } g = \int_0^2 |h|^{q-1}\text{sign } h.
\]

The left side of (*) is \( 2^{q-1} + 1 > 0 \), while the right side is \( |d|^{q-1}\text{sign } g \varepsilon^{q-1}[(\frac{1}{4})^{2-q} - \varepsilon^{2-q}] \), so such a choice of \( d \) is possible for \( p \neq 2 \).

\[\blacksquare\]

**Problem 4.5.** If \( E \) is a weak*-closed subspace of \( \ell_1 \), does \((E, \ell_1)\) have the \( 1 + \varepsilon \)-E.P. for every \( \varepsilon > 0 \)?

A negative answer to Problem 4.5 would be particularly interesting, because it would justify the weird approach we used to prove the Theorem. However, we do not even know a counterexample to:

**Problem 4.6.** If \( E \) is a weak*-closed subspace of \( \ell_1 \), does \((E, \ell_1)\) have the 1-E.P.?

The answer to Problem 4.6 is known to be yes for finite dimensional \( E \), [Sam1], [Sam2].
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