Scalar bosons with Coulomb potentials in a cosmic string background: Scattering and bound states

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Abstract. The relativistic quantum motion of scalar bosons under the influence of a full vector (minimal $A^\mu$ and nonminimal $X^\mu$) and scalar ($V_s$) interactions embedded in the background of a cosmic string is explored in the context of the Klein-Gordon equation. Considering Coulomb interactions, the effects of this topological defect in equation of motion, phase shift and S-matrix are analyzed and discussed. Bound-state solutions are obtained from poles of the S-matrix and it is shown that bound-state solutions are possible only for a restrict range of coupling constants.

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1 Introduction

Scalar bosons usually are represented by the Klein-Gordon (KG) equation. Vector and scalar interactions are considering by replacing (usual substitutions)

\begin{align}
\rho^\mu &\to \rho^\mu - A^\mu, \\
M &\to M + V_s.
\end{align}

With this substitutions, a variety of relativistic and nonrelativistic effects can be studied. The vector interaction $A^\mu$ refers to a kind of coupling that behaves like a four-vector under a Lorentz transformation and it can be associated to electromagnetic interaction. On the other hand, the scalar interaction $V_s$ refers to a kind of coupling that behaves like a scalar (invariant) under a Lorentz transformation. Though the scalar interaction finds many of their applications in nuclear and particle physics, it could also simulate an effective mass in solid state physics. Due to weak potentials, relativistic effects are considered to be small in solid state physics, but the relativistic wave equations can give relativistic corrections to the results obtained from the nonrelativistic wave equation, therefore the relativistic extension of this problem is also of interest and remains unexplored.

An additional form of interaction is achieved through the substitution

\begin{equation}
p^2 \to (p + iM \omega \hat{r}) \cdot (p - iM \omega \hat{r}),
\end{equation}

such interaction is dubbed the KG oscillator. This alternative form of interaction furnishes a Schrödinger equation with the harmonic oscillator potential in a nonrelativistic scheme \cite{1} and spurred great deal of research in last years \cite{2,3,4,5,6,7,8,9,10,11,12,13}. Considering the KG oscillator as a particular case of other four-vector $X^\mu$, we can generalize this interaction through the substitution \cite{14}

\begin{equation}
p^\mu p_\mu \to (p^\mu - iX^\mu) \cdot (p_\mu + iX_\mu).
\end{equation}

Note that in contrast to $A^\mu$, $X^\mu$ is not minimally coupled and for this reason $X^\mu$ is called of nonminimal vector interaction. Thus, the KG equation with interactions (the most general Lorentz structure) consists in vectors ($A^\mu$ and $X^\mu$), and scalar ($V_s$) potentials.

Scalar particles in the background space-time generated by a cosmic string are very interesting systems that have been studied extensively in the literature in recent years \cite{15,16,17,18,19,20}. A natural question arises, whether the presence of such kind of topological defect can also influence the behavior of scattering states and bound states of a given quantum system. In \cite{19}, the quantum dynamics of scalar bosons embedded in the background of a topological defect were considered. Notably, in that work it was shown that the energy spectrum associated with the scalar sector of the Duffin-Kemmer-Petiau (DKP) equation in a cosmic string space depends on the deficit angle of the conical space-time. The scattering S-matrix for a fermionic system is considered in \cite{21}, where it was shown that phase shifts and the normalization factor are influenced by the presence of the string.

Solutions of relativistic wave equations in curved spaces with these potentials have been obtained for many systems. For instance, the Cornell potential \cite{22} is a linear combination of the Coulomb and linear potentials. It provides a good description of the heavy quark–antiquark system. In Ref. \cite{23}, relativistic Landau levels in the rotating cosmic string space-time with the Cornell potential, is considered. It was shown that the Landau levels of the spinning cosmic string remain the same.
even when the internal magnetic flux vanishes. Solutions of the KG equation with Coulomb-type scalar potential in the cosmic string space-time are considered in Ref [17]. Another example of vector–scalar Coulomb potentials has been developed in [15] considering the presence of a dyon and Aharonov-Bohm magnetic field. In this paper the energy spectra and the scattering states of the KG equation have been analyzed, it was shown that the phase shifts depend on the geometry of the space-time. Recently, solutions of the D-dimensional KG equation via mapping onto the nonrelativistic one-dimensional Morse potential have been considered [14]. This is a straightforward procedure for finding solutions of the scalar equation with scalar, vector, and nonminimal coupling potentials. The influence of a Coulomb-type potential on the KG oscillator has been investigated by Bakke and Furtado [12], the authors have determined bound-state solutions to the KG equation for both attractive and repulsive Coulomb-type potentials.

Due to the analogy between disclinations in solids and cosmic strings [24], several results obtained from the analysis of cosmic strings can be useful in the study of condensed matter systems. This fact is associated to the metric which describes a disclination corresponding to the spatial part of the line element of the cosmic string. The study of system of fermions in low dimensions plays an important role in the context of integer quantum Hall effect in graphene [25] and topological insulators [26], among others. It is worthwhile to mention that such problems also have their version for bosonic systems: the integer quantum Hall effect for bosons [27] and symmetry–protected topological (SPT) phase [28]. In this context, bosonic systems has been employed on the study of novel topological semimetals [29]. Other potential applications can be found in Bose–Einstein (BE) condensates and neutral atoms [30, 31]. Therefore, we believe that the Klein–Gordon equation embedded in a background of cosmic string under external interactions deserves to be more explored.

The purpose of this paper is to study scattering and bound-state solutions of scalar bosons with a Coulomb potential with the most general Lorentz structure embedded in a cosmic string background. In this case, the problem is mapped into a Schrödinger-like equation with a effective Coulomb-like potential. The Coulomb phase shift and scattering S-matrix are calculated from a Whittaker differential equation via partial wave analysis. The bound-state solutions are obtained from the poles of the S-matrix and the restriction on the potential parameters are discussed in detail. We investigate how the topological and geometric features of the defect affect the energy levels as compared with the flat space-time (α = 1) and also we are able to reproduce all cases already discussed in the literature as particular cases. Beyond its intrinsic interest, the Coulomb potential (hydrogen atom) in a curved space-time can give us a heuristic look of a plausible mechanism for the detection of gravitational radiation [32]. Therefore, ours results shed some light on better understanding of energy level shifts due to topological and geometric features, which in principle enables one to emulate a radiation detector.

This paper is structured as follows: In Section 2, we give a brief review on a cosmic string background. In Section 3, we consider the generalized Klein-Gordon (KG) equation in a cosmic string background. We also consider a generalized Coulomb potential (Section 3.1) and we analyze scattering and bound-state solutions (Sections 3.2 and 3.3, respectively). Additionally, we discuss some particular cases (Section 3.4). Finally, in Section 4 we present our conclusions.

2 Cosmic string background

The cosmic strings are systems that are supposed to be formed during a symmetry breaking phase transition in the early universe [33–39]. They are 1-dimensional topological defects and are candidates for the generation of observable astrophysical phenomena such as gravitational waves and high energy cosmic rays [34]. Furthermore, they would also be associated with galaxy evolution and gravitational lens [34, 35]. They have high mass density, of the order of 10^{23} g/cm, and very small thickness, equivalent to the Compton wavelength 10^{-29} cm. Cosmic strings can be straight and have infinite length or can form a closed loop. Nowadays, there are not a direct proof of their existence but, we have some indirect evidences that cosmic string can really exist [36]. The high-frequency gravitational waves emitted by cusps and kinks of cosmic strings might be detectable by the gravitational waves detectors LIGO/VIRGO and LISA [40]. The space-time of a cosmic string has cylindrical symmetry such that its line element is given by [37]

\[ ds^2 = -A^2(\rho) dt^2 + d\rho^2 + C^2(\rho) d\varphi^2 + D^2(\rho) d\zeta^2, \]  

(5)

where \(-\infty < t < +\infty, 0 \leq \rho < +\infty, 0 \leq \varphi \leq 2\pi\) and \(-\infty < \z < +\infty\). Let us admit that our space-time is invariant by boosts in the \(z\) direction, so that \(A^2(\rho)\) must be equal to \(D^2(\rho)\). To determine the remaining functions, \(A(\rho)\) and \(C(\rho)\), we should make our metric to satisfy the Einstein field equations. By making this, we will obtain

\[ ds^2 = -dt^2 + d\rho^2 + \alpha^2 \rho^2 d\varphi^2 + d\zeta^2. \]  

(6)

In this work we will use spherical coordinates, in this way, we consider the coordinate transformation \(\rho = r \sin \theta\) and \(z = r \cos \theta\), the result is the cosmic string space-time, described by the line element

\[ ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + \alpha^2 r^2 \sin^2 \theta d\varphi^2, \]  

(7)

in spherical coordinates \((t, r, \theta, \varphi)\), where \(-\infty < t < +\infty, r \geq 0, 0 \leq \theta \leq \pi/2\) and \(0 \leq \varphi \leq 2\pi\). The parameter \(\alpha\) is associated with the linear mass density \(\tilde{m}\) of the string by \(\alpha = 1 - 4\tilde{m}\) and runs in the interval \([0, 1]\) and corresponds to a deficit angle \(\gamma = 2\pi(1 - \alpha)\). Note that, in the limit as \(\alpha \to 1\) we obtain the line element of spherical coordinates.

3 Generalized Klein-Gordon equation in a cosmic string background

Scalar bosons are represented by the usual Klein-Gordon (KG) equation which can be generalized to a curved space-time framework. Incorporating the Klein-Gordon oscillator as a particular case, a generalized Lorentz-covariant KG equation for a particle of mass \(M\) in a curved space-time under the influence of external vectors \((A^\mu \text{ and } X^\mu)\), and scalar \((V_i)\) fields reads

\[ -\frac{1}{\sqrt{-g}} D_{\mu}^{(+) g} g^{\mu \nu} \sqrt{-g} D_{\nu}^{(+) g} + (M + V_i) \Psi = 0, \]  

(8)
where
\[ D_{\mu}^\pm = \partial_{\mu} \pm iA_{\mu}. \]
We can note that in contrast to \( A^{\mu}\), the vector potential \( X^{\mu}\) is not minimally coupled [14]. Furthermore, invariance under the time-reversal transformation demands that \( A^{\mu}\) and \( X^{\mu}\) have opposite behaviours. Similarly to the scalar potential, the non-minimal vector potential does not couple to the charge (invariance under the charge-conjugation operation), i.e. \( X^{\mu}\) and \( V_\alpha\) do not distinguish particles from antiparticles. At this stage, without loss of generality we make \( A = 0\) due to the space component of the minimal coupling can be gauged away for spherically symmetric potentials.

Considering only time-independent and spherically symmetric potentials \( A_0(r) = V_\alpha(r) = V_\beta(r)\), it is reasonable to write the solution as
\[ \Psi(t, r, \theta, \phi) = \frac{u(r)}{r} f(\theta) e^{-iEt + im\phi}, \] (10)
where \( m = 0, \pm 1, \pm 2, \ldots \) and \( E\) is the energy of the scalar boson. Substituting (10) into Eq. (8), we obtain the following equation
\[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} f(\theta) \right) - \frac{m^2}{\sin^2 \theta} f(\theta) = \lambda f(\theta), \] (11)
for the angular part. The solution \( f(\theta)\) can be expressed as a generalized Legendre functions \( P_{l \alpha}^m(\cos \theta)\) and thus we get
\[ \lambda = \lambda_a = (l_a + 1), \] (12)
where \( l_a = n + |m_a| = l + |m|(1/\alpha - 1) \) with \( n\) a non-negative integer, \( l = n + |m|\) and \( m_a = m/\alpha\). Note that \( |m_a|\) lies in the range \(-l_a \leq m_a \leq l_a\). Here, \( l\) and \( m\) are the orbital angular momentum and the magnetic quantum numbers in the flat space, respectively.

For \( r \neq 0\) the radial function \( u(r)\) obeys the radial equation
\[ \frac{d^2u}{dr^2} + \left[ K^2 - V_{\text{eff}} - \frac{l_a(l_a + 1)}{r^2} \right] u = 0, \] (13)
where the effective energy \( K^2\) and the effective potential \( V_{\text{eff}}\) are expressed by
\[ K^2 = E^2 - M^2, \] (14)
\[ V_{\text{eff}} = V_s^2 - V_v^2 + 2(MV_s + EV_v) + \frac{dV_s}{dr} + \frac{V_v}{r} + V_s^2 - V_0^2. \] (15)
Therefore, the equation (13) describes the quantum dynamics of scalar bosons under external interactions in a cosmic string background, whose solution can be found by solving a Schrödinger-like equation. Note that, if the potentials \( V_s, V_v, V_r\) and \( V_0\) go to zero at \( r \to \infty\) the solution \( u(r)\) has the asymptotic behavior \( \sim e^{iK r}\) and so we can see that scattering states only occur if \( K \in \mathbb{R}\) (\(|E| > M\)), whereas bound states might occur only if \( K = \pm i|K|\) (\(|E| < M\)).

### 3.1 Generalized Coulomb potential

Let us consider the potentials in the form
\[ V_s = \frac{\beta_s}{r}, \quad V_v = \frac{\beta_v}{r}, \quad V_r = \frac{\alpha_r}{r}, \quad V_t = \frac{\alpha_t}{r}. \] (16)
Substituting (16) in (13), we obtain
\[ \frac{d^2u}{dr^2} + \left[ K^2 - \frac{\alpha_t}{r} - \frac{\alpha_v}{r^2} + l_a(l_a + 1) \right] u = 0, \] (17)
with
\[ \alpha_1 = 2(\alpha_M + \alpha_v E), \] (18)
\[ \alpha_2 = \beta_s(\beta_v + 1) + \alpha_r^2 - \beta_0^2 - \alpha_t^2. \] (19)
The equation of motion (17) is precisely the time-independent Schrödinger equation for a Coulomb-like potential. This effective potential has well structure when \( \alpha_t < 0\), which implies that \( E < E_c = -M \left( \frac{\beta_0}{\alpha_t}\right)\). It is worthwhile to mention that presence of \( \alpha_t\) or \( \alpha_r\) (or both) is necessary for the existence of bound-state solutions. Additionally, bound states are expected for \(|E| < M\). Therefore, we can conclude that bound-state solutions are possible only for \( \alpha_t < |\alpha_r|\), corresponding to energies in the interval \(-M < E < E_c\).

On the other hand, using the abbreviations
\[ \eta = \sqrt{\left( l_a + \frac{1}{2} \right)^2 + \alpha_2}, \] (20)
\[ \eta = \frac{\alpha_1}{2K}, \] (21)
and the change \( z = -2iKr\), the equation (17) becomes
\[ \frac{d^2u}{dz^2} + \left( -\frac{1}{4} - \frac{i\eta}{z} + \frac{1/4 - \eta^2}{z^2} \right) u = 0. \] (22)
This second-order differential equation is the called Whittaker equation, which have two linearly independent solutions \( M_{-\eta, \frac{k}{2}}(z)\) and \( W_{-\eta, \frac{k}{2}}(z)\) behaving like \( z^{1/2 + \eta}\) and \( z^{1/2 - \eta}\) close to the origin, respectively. Owing to \( u(0) = 0\), one has to consider the solution proportional to
\[ u(z) = A e^{-z/2} z^{1/2 + \eta} M(1/2 + \eta + i\eta, 1 + 2\eta; z). \] (23)
where \( A\) is a arbitrary constant,
\[ \alpha_2 > -1/4, \] (24)
and \( M(a, b; z)\) is the confluent hypergeometric function (Kummer’s function) [42]. From (24) one can obtain an upper limit for \( \alpha_t\), given by
\[ |\alpha_t| < (\alpha_t)_{\max} = \frac{1}{\sqrt{4} + \alpha_2^2 + \beta_s(\beta_v + 1) - \beta_0^2}. \] (25)

The asymptotic behavior for large \(|z|\) with a purely imaginary \( z = -i|\tilde{z}|\), where \( |\tilde{z}| = 2Kr\) is given by [43]
\[ M(a, b; z) \sim \frac{\Gamma(b)}{\Gamma(b-a)} e^{-\frac{z}{2}|\tilde{z}|^{1/2}} e^{-iz|\tilde{z}|^{1/2}} e^{-i\left(\frac{z}{2} - \frac{\eta}{2}\right)|\tilde{z}|^{1/2}}. \] (26)
3.2 Scattering states

We can show that for $|z| \gg 1$ and $K \in \mathbb{R}$ the asymptotic behavior dictated by (26) implies

$$u(r) \approx \sin \left( K r - \frac{l \pi}{2} + \delta_l \right), \quad (27)$$

where the relativistic Coulomb phase shift $\delta_l = \delta_l(\eta)$ is given by

$$\delta_l = \frac{\pi}{2} (l + 1/2 - \gamma) + \arg(1/2 + \gamma + i\eta). \quad (28)$$

For scattering states in spherically symmetric scatterers, the scattering amplitude can be written as partial wave series

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) f_l P_l(\cos \theta), \quad (29)$$

where $\theta$ is the angle of scattering, $P_l$ is the Legendre polynomial of order $l$ and the partial scattering amplitude is $f_l = (i^{2l} - 1)/(2iK)$. From this last expression one can recognize the scattering S-matrix as $S_l = e^{2i\delta_l}$. With the phase shift (28), up to a logarithmic phase inherent to the Coulomb field, we find

$$S_l = e^{i\pi(l+1/2-\gamma)} \frac{\Gamma(1/2 + \gamma + i\eta)}{\Gamma(1/2 + \gamma - i\eta)}. \quad (30)$$

Information about the energies of the bound-state solutions can be obtained from poles of the S-matrix when one considers $K$ imaginary.

3.3 Bound states

If $K = i|K|$, the S-matrix becomes infinite when $1/2 + \gamma + i\eta = -N$, where $N = 0, 1, 2, \ldots$, due to the poles of the gamma function in the numerator of (30), and (26) implies that $u(r)$ tends to $r^{1/2 + \gamma + N} e^{-|K|r}$ for large $r$. Therefore, bound-state solutions are possible only for $\alpha_s < 0$ and the spectrum is expressed as

$$E = \frac{M}{1 + \left( \frac{\alpha_s}{\mu} \right)^2} \left[ -\frac{\alpha_s}{\mu} \pm \Delta \right], \quad (31)$$

where

$$\Delta = \sqrt{1 + \left( \frac{\alpha_s}{\mu} \right)^2 - \left( \frac{\alpha_s}{\mu} \right)^2}, \quad (32)$$

with $\mu = N + 1/2 + \gamma$. Note that the condition $\alpha_s < |\alpha_s|$ guarantees $E$ real. Considering the expression (25), we can conclude that bound-state solutions are possible only if the following constrain on potential parameters is satisfied

$$|\alpha_s| < \sqrt{\frac{1}{4} + \alpha_s^2 + \beta_\tau (\beta_\tau + 1) - \beta_\tau^2}, \quad (33)$$

In this case, $M(\beta_\tau, \delta_\tau)$ is proportional to the generalized Laguerre polynomial $L_N^{(\beta_\tau-1)}(\tau)$ [42], and so one can write the solution as

$$u(r) \propto r^{\gamma + 1/2} e^{-\frac{\alpha_s r}{2\mu}} L_N^{(\beta_\tau)} \left( \frac{|\alpha_s| r}{\mu} \right). \quad (34)$$

Figure 1 shows the numerical values of the negative energy spectrum as a function of the quantum numbers $N$ and $l$. It is noticeable that for a given set of potential parameters one finds that the lowest quantum numbers correspond to the highest energy levels and are to be identified with antiparticle levels. In this case, energy levels for bound-state solutions can be found in the interval $-M < E < -0.875M$. Also, one sees that the energy levels tend to $-M$ as quantum numbers increase. The energy levels could sink into the negative continuum but this couldn’t menace the single-particle interpretation of KG equation since one has antiparticle levels plunging into the antiparticle continuum. In Fig. 2, we illustrate the behavior of the energy as a function of $\alpha$ for three different values of $N$. Figure 2 clearly shows the effects of $\alpha$ on the energy levels for fixed values of $N, l$ and $m$, one can see that the energy $|E|$ decreases as $\alpha$ increases.

3.4 Particular cases

Here, we reproduce some well-known particular cases, already discussed in the literature. The case of a pure nonminimal vector potential $X^\mu$ is not considered because it does not furnish bound-state solutions.
3.4.1 Minimal vector Coulomb potential

For $\alpha_0 = \beta_0 = \beta_0 = 0$, bound-state solutions are possible only for $\alpha_1 = 2 \alpha_0 M < 0$, i.e., when $\alpha_0 < 0$ and $\alpha_1 < 0$, and so the expression (31) reduces to

$$E = -\frac{\text{sgn}(\alpha_0) M}{1 + \frac{\alpha_0^2}{(N + 1/2 + \gamma_\nu)^2}}, \quad (35)$$

where $\gamma_\nu = \sqrt{(l_\nu + \frac{1}{2})^2 - \alpha_0^2}$. For $\alpha = 1$ (flat space-time) this last result is exactly the expression for the energies for the KG equation with minimal vector Coulomb potential [44].

3.4.2 Scalar Coulomb potential

For $\alpha_0 = \beta_0 = \beta_0 = 0$, bound-state solutions are possible only for $\alpha_1 = 2 \alpha_0 M < 0$, i.e., when $\alpha_0 < 0$ and so the expression (31) reduces to

$$E = \pm M \sqrt{1 - \frac{\alpha_0^2}{(N + 1/2 + \gamma_\nu)^2}}, \quad (36)$$

where $\gamma_\nu = \sqrt{(l_\nu + \frac{1}{2})^2 + \alpha_0^2}$. For $\alpha = 1$ (flat space-time) this last result is exactly the expression for the energies for the KG equation with scalar Coulomb potential [44].

3.4.3 Mixed scalar-vector Coulomb potential

For $\beta_0 = \beta_0 = 0$, bound-state solutions are possible only for $\alpha_1 = 2 (\alpha_0 M + \alpha_0 E) < 0$, i.e., when $\alpha_0 < |\alpha_0|$ and so the expression (31) reduces to

$$E = M \frac{-\alpha_0 \alpha_1}{\alpha_0 - \alpha_1} \pm \sqrt{1 + \left(\frac{\alpha_0}{\alpha_1}\right)^2 - \left(\frac{\alpha_0}{\alpha_1}\right)^2}, \quad (37)$$

where $\nu = N + \frac{1}{2} + \sqrt{(l_\nu + \frac{1}{2})^2 + \alpha_0^2 - \alpha_0^2}$. For $\alpha = 1$ (flat space-time) this last result is exactly the Eq. (22) of Ref.[45].

4 Conclusions

We studied the relativistic quantum motion of scalar bosons with a potential with the most general Lorentz structure embedded in the background of a cosmic string via Klein–Gordon equation. Considering Coulomb interactions for the full vector ($A^\mu$ and $X^\mu$) and scalar ($V_\nu$), this problem was mapped into a Schrödinger-like equation with an effective Coulomb-like potential and we showed that the scattering and bound-state solutions can be studied by solving a Whittaker differential equation.

For scattering solutions ($|E| > M$), the Coulomb phase shift and scattering S-matrix was calculated as a function of the potential parameters and the angular deficit of the cosmic string background $\alpha$. In this case, the minimal vector interaction satisfies the constraint $|\alpha_0| < (\alpha_0)_{\text{max}}$ [Eq. (25)]. The poles of the S-matrix when $K \rightarrow |K|$ provided bound-state solutions only for $|\alpha_0| < 0$. From this condition, we concluded that the presence of $\alpha_0$ or $\alpha_0$ (or both) is necessary for the existence of bound-state solutions. The presence of a pure nonminimal vector potential $X^\mu$ does not furnish bound-state solutions. Furthermore, we found that bound-state solutions are possible only for $\alpha_0 < |\alpha_0| < (\alpha_0)_{\text{max}}$, corresponding to energies in the interval $-M < E < -M \left(\frac{\alpha_0}{\alpha_1}\right)$ and also we found that the eigenfunctions are expressed in terms of the generalized Laguerre polynomials. We also showed that the discrete set of energies $|E|$ for this background decreases as $\alpha$ increases. The results obtained in this work are consistent in the limit of flat space-time ($\alpha = 1$) with those found in the literature.

Indeed, it is expected that the energy of quantum systems in this space-time carries information about the local features of the background space-time in which the system is placed [46]. In this way, it was suggested that the hydrogen atom in a curved space-time can be used as a radiation detector, i.e., the existence of atomic hydrogen in the vicinity of sources of gravitational radiation carry a signature on the energy level shifts and could thus provide a plausible mechanism for detection of gravitational radiation [32]. Beyond investigating how the topological and geometric features of the defect affect the energy levels as compared with the flat space-time ($\alpha = 1$) our results can be seen as a first step (toy model) for a better understanding of energy level shifts, which enables one to emulate a detector of gravitational waves. This future application is currently under study and will be reported elsewhere.

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