Integrable systems associated with the Bruhat Poisson structures.

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June 25, 2001

Abstract

The purpose of this note is to give a simple description of a (complete) family of functions in involution on certain hermitian symmetric spaces. This family obtained via the bi-hamiltonian approach using the Bruhat Poisson structure is especially simple for projective spaces, where the formulas in terms of the momentum map coordinates are presented. We show how these functions are related to the Gelfand-Tsetlin coordinates. We also show how the Lenard scheme can be applied.

1 Introduction.

Let $K$ be a compact real form of a complex semi-simple Lie group $G$ and let $H \subset K$ be a subgroup of $K$ defined by $H = K \cap P$, where $P$ is a parabolic subgroup of $G$ containing a Borel subgroup $B \subset G$. The Bruhat Poisson structure $\pi_\infty$ on $X = K/H$, first introduced by Soibelman [14] and Lu-Weinstein [8], has the property that its symplectic leaves are precisely the Bruhat cells in $X$. If $T = K \cap B$ is a maximal torus of $K$, then $\pi_\infty$ is $T$-invariant. Let $\omega_s$ (respectively, $\pi_s$) stand for a $K$-invariant symplectic form (respectively, dual bi-vector field) on $X$, which we assume now to be a compact hermitian symmetric space. It was shown by Khoroshkin-Radul-Rubtsov in [6] that the two Poisson structures, $\pi_\infty$ and $\pi_s$ are compatible, meaning that the Schouten bracket of $\pi_\infty$ and $\pi_s$ vanishes, $[\pi_\infty, \pi_s] = 0$. In particular, any bi-vector field of the form $\alpha \pi_\infty + \beta \pi_s$, $(\alpha, \beta) \in \mathbb{R}^2$, is Poisson. In this situation, one can introduce the following family $\{f_k\}$ of functions:

$$f_k := (\pi_\infty \wedge^k, \omega_s \wedge^k),$$

obtained by the duality pairing of exterior powers of $\omega$ and $\pi_\infty$. If the (real) dimension of $X$ is equal to $2n$, then we have $n$ functions, $f_1, ..., f_n$ which may carry some useful information about $X$. These functions are in involution with

\footnote{Research is partially supported by NSF grant DMS-0072520.}
respect to either of the two Poisson structures. We make explicit computations for $\mathbb{CP}^n$, since this is the only case, where we can present an explicit coordinate approach. We show how the function that we have obtained are related to the Gelfand-Tsetlin integrable systems studied by Guillemin and Sternberg \cite{5}. Analogous statements for other hermitian symmetric spaces will appear elsewhere \cite{1}. In the last part of the paper we make explicit computations using the Lenard scheme \cite{9}.

Acknowledgments. I would like to thank Lu Jiang-Hua and Sam Evens for answering many questions regarding Bruhat-Poisson structures. Lu Jiang-Hua also provided simple proofs of Propositions 2.1 and 2.2. I thank Hermann Flaschka for conversations about integrable systems. I thank Yan Soibelman for historical remarks, and Ping Xu for discussions about Poisson-Nijenhuis manifolds.

2 Families of functions in involution.

Multi-hamiltonian structures are very important in the theory of integrable systems. Starting with the fundamental works of Magri \cite{11}, bi- and multi-Hamiltonian structures found many interesting and fundamental applications, as in \cite{7}, \cite{2}, \cite{13} and references therein.

Let $M$ be a manifold and let $\pi_6$ and $\pi_s$ be two Poisson structures on $M$ such that

1. The Poisson structure $\pi_s$ is non-degenerate (so the subscript $s$ stands for symplectic).
2. The Poisson structures $\pi_s$ and $\pi_b$ are compatible, meaning that the Schouten bracket $[\pi_s, \pi_b]$ vanishes. Or, equivalently, for any two real numbers $\alpha$ and $\beta$, the bi-vector field $\alpha \pi_s + \beta \pi_b$ defines a Poisson structure on $M$.

If $\dim(M) = 2n$, then we can define $n$ functions $f_1, ..., f_n$ as follows:

$$f_j = \frac{\pi^j_b \wedge \pi^{n-j}_s}{\pi^n_s}.$$ 

The operation of division by the top degree bi-vector field makes perfect sense, since $\pi_s$ is non-degenerate, and thus in any local coordinate system $(x_1, ..., x_{2n})$ the $2n$-vector field $\pi^n_s$ looks like

$$\pi^n_s = h(x_1, ..., x_{2n}) \partial_{x_1} \wedge \cdots \wedge \partial_{x_{2n}},$$

for a non-vanishing function $h(x_1, ..., x_{2n})$. Equivalently, if $\omega_s$ is the symplectic form dual to $\pi_s$, then one can define

$$f_k := (\pi^\omega_{\infty} \wedge^k, \omega^\omega_s \wedge^k),$$
where we use the duality pairing
\[ \Gamma(M, \wedge^{2k} T^* M) \otimes \Gamma(M, \wedge^{2k} T M) \to C^\infty(M). \]

It turns out that this family of functions has the following property.

**PROPOSITION 2.1** The family of functions \( f_i \) defined above are in involution with respect to either Poisson structure, \( \pi_b \) or \( \pi_s \).

**Proof.** (J.-H. Lu) Let \( X_i = id f_i \pi_b \) and let \( Y_j = id f_j \pi_s \). Consider the equality \( f_k \pi^n_s = \pi^k_b \wedge \pi^{n-k}_s \) and compute \( L_{X_l} \) of both sides to arrive to the following identity:
\[ \frac{n-k}{k+1} \{ f_{k+1}, f_l \}_s = -\{ f_k, f_l \}_b + n f_k \{ f_1, f_l \}_s, \]
where the subscripts \( s \) or \( b \) indicate with respect to which Poisson structure the Poisson bracket is taken. Finally, use the induction on \( l \). \( \Box \)

**Remark.** The approach that we have followed here is intimately related to the Poisson-Nijenhuis structures, that were studied by Magri and Morosi [12], Kosmann-Schwarzbach and Magri [7], Vaisman [15] and others. The set of our functions \( \{ f_j \} \) can be expressed, polynomially, through the traces of powers of the intertwining operator corresponding to the Nijenhuis tensor.

Now let us take \( M = X \) to be a coadjoint orbit in \( \mathfrak{k}^* \), which we assume to be a compact hermitian symmetric space. We take \( \pi_s = \pi \) - the Kirillov-Kostant-Souriau symplectic structure and \( \pi_b = \pi_\infty \) - the Bruhat-Poisson structure, which is obtained via an identification of \( X \) with \( K/(P \cap K) \) as in Introduction. Under this identification, \( \pi \) is \( K \)-invariant. The following was first proved in [5].

**PROPOSITION 2.2** If \( X \) is a hermitian symmetric space as above, then the Poisson structures \( \pi \) and \( \pi_\infty \) are compatible.

**Proof.** (J.-H. Lu) Let \( X \) be a generating vector field for the \( K \)-action. Clearly, the \( K \)-invariance of \( \pi \) implies that \( L_X \pi = 0 \). Since \( \pi_\infty \) came from \( \mathfrak{k} \wedge \mathfrak{k} \) by applying left and right actions of \( K \), \( L_X \pi_\infty \) is obtained from \( \delta(X) \) by applying the \( K \)-action. Here, \( \delta(X) \) is the co-bracket of \( X \), which is an element of \( \mathfrak{k} \otimes \mathfrak{k} \), since we can view \( X \) as an element of \( \mathfrak{k} \). Therefore, \( L_X \pi_\infty \) is a sum of wedges of generating vector fields for the action of \( K \). Accordingly,
\[ [L_X \pi_\infty, \pi] = 0, \]
which in turn implies that
\[ L_X [\pi_\infty, \pi] = 0, \]
and thus \( [\pi_\infty, \pi] \) is a \( K \)-invariant 3-vector field on \( X \). When \( X \) is a hermitian symmetric space, there are none such (since the nil-radical of the corresponding parabolic group is abelian), so it must be zero. \( \Box \)
Therefore, we have the following

**PROPOSITION 2.3** Let $X$ be a coadjoint orbit in $K$. Assume that $X$ is a hermitian symmetric space of complex dimension $n$. The above recipe yields $n$ functions $(f_1, \ldots, f_n)$ on $X$, which are in involution with respect to either $\pi_\infty$ or $\pi$.

The functions $(f_1, \ldots, f_n)$ that we have constructed turn out to be related to the Gelfand-Tsetlin coordinates in the case when $K = SU(n)$, as we will see later on. In the next section we will carry explicit computations of these functions on the projective spaces.

### 3 Computations for the projective spaces.

Let $\mathbb{CP}^n$ be a complex projective space of (complex) dimension $n$, and let $[Z_0 : Z_1 : \ldots : Z_n]$ be a homogeneous coordinate system on it. We use the standard Fubini-Study form $\omega$ for $\omega_s$ and the following description of $\pi_\infty$ obtained by Lu Jiang-Hua in [9] and [10]. First, we need Lu’s coordinates on the largest Bruhat cell, where $Z_0 \neq 0$ and we let $z_i = Z_i/Z_0$:

$$y_i := \frac{z_i}{\sqrt{1 + |z_{i+1}|^2 + \cdots + |z_n|^2}}, \quad 1 \leq i \leq n.$$  

Lu’s coordinates are not holomorphic, but convenient for the Bruhat Poisson structure, which now assumes the following form

$$\pi_\infty = \sqrt{-1} \sum_{i=1}^n (1 + |y_i|^2) \partial_{y_i} \wedge \partial_{\bar{y}_i}.$$  

In order to be able to compute with $\omega$ and $\pi_\infty$, we need to move to the polar variables $r_i, \phi_j$ defined by $z_i = r_i e^{\sqrt{-1} \phi_i}$ and eventually to the momentum map variables $x_i, \phi_j$ defined by

$$x_i = \delta_{1,i} - \frac{r_i^2}{1 + r_1^2 + \cdots + r_n^2}.$$  

These variable are just a slight distortion (for later convenience) of the standard coordinates on $\mathbb{R}^n$ for the momentum map associated with the maximal compact torus action on $\mathbb{CP}^n$. One of the advantages of using this coordinate system is that the Fubini-Study symplectic structure has the following simple form:

$$\omega = \sum_{i=1}^n dx_i \wedge d\phi_i.$$
In fact, the simplest form for the Bruhat Poisson structure is also achieved in this coordinate system.

**Proposition 3.1** The Bruhat Poisson structure $\pi_\infty$ on $\mathbb{C}P^n$ can be written in the coordinate system $(x_j, \phi_i)$ as

$$\pi_\infty = \sum_{i=1}^n \Theta_i \wedge \partial_{\phi_i},$$

where

$$\Theta_i = (x_1 + \cdots + x_i)\partial_{x_i} + \sum_{j=i+1}^n x_j \partial_{x_j}.$$

**Proof.** The proof of this statement is purely computational. One can introduce auxiliary variables $q_i = \log(1 + |y_i|^2)$, and use those to write

$$\pi_\infty = \sum_{i=1}^n \partial_{q_i} \wedge \partial_{\phi_i}$$

- the action-angle form for $\pi_\infty$. Eventually, one can establish the following relations: $x_1 = e^{-q_1}$, and for $j > 1$,

$$x_j = e^{-(q_1 + \cdots + q_j)} - e^{-(q_1 + \cdots + q_{j-1})}.$$

The rest is straightforward. ☐

Now, one can see that the simple linear and triagonal form of $\pi_\infty$ makes the computation of the functions $\{f_i\}$ extremely simple. We will introduce the following linear change of variables on $\mathbb{R}^n$:

$$c_k = \sum_{i=1}^k x_i.$$

In these variables, the set of functions $\{f_i\}$ looks as follows.

**Theorem 3.2** The integrals $f_i$ (up to constant multiples) arising from the bihamiltonian structure $(\pi_s, \pi_\infty)$ on $\mathbb{C}P^n$ are given by the elementary polynomials in $(c_1, \ldots, c_n)$:

$$f_1 = c_1 + \cdots + c_n,$$

$$f_j = \sum_{i_1 < \cdots < i_j} c_{i_1} \cdots c_{i_j},$$

$$f_n = c_1 \cdots c_n.$$
The explicit nature of these integrals is essential in looking at the relation with the certain natural flows \[6\]. The hamiltonian $f_1$ in terms of the momentum map variables is given by

$$f_1 = nx_1 + (n-1)x_2 + \cdots + 2x_{n-1} + x_n.$$  

Then the gradient in the momentum simplex has coordinates $\lambda_i = n + 1 - i$. Those numbers also are the weights assigned to the vertices (which correspond to the centers of the Bruhat cells). Thus we arrive to

**THEOREM 3.3** The above flow on $\mathbb{CP}^n$ with eigenvalues consecutive integers from 1 to $n$ determines the standard Bruhat cell decomposition.

4 Relation with Gelfand-Tsetlin coordinates.

When $X = Gr(k)$ - the grassmannian of $k$-planes in $\mathbb{C}^{n+1}$, we have obtained $k(n-k+1)$ functions in involution on $X$. Let us recall the standard embedding

$$\Psi : F_n \hookrightarrow Gr(1) \times \cdots \times Gr(n),$$

where $F_n$ is the manifold of full flags in $\mathbb{C}^{n+1}$, and the locus of the embedding is given by the incidence relations. This embedding respects the KKS Kähler structures on the manifolds involved, if we would like to view them as coadjoint orbits in $\mathfrak{k}^*$. Moreover, this embedding is equivariant with respect to the $K = SU(n+1)$-action.

Recall the Gelfand-Tsetlin system on $F_n$. We fix the orbit type of $F_n$, i.e. we fix the eigenvalues $\sqrt{-1}\lambda_i$ and order them, so $\lambda_1 > \lambda_2 > \cdots > \lambda_{n+1}$. For convinience and easier visualization, we will assume that $\lambda_1 = \cdots = \lambda_k > 0$, and other $\lambda$'s equal to zero. When $k$ varies from 1 to $n$, the picture above acquires more and more non-zero elements. At
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each step, while going from level $k$ to $k+1$, we will get new integrals on $Gr(k+1)$, which we can pull back to $F_n$ using $\Psi$.

Our goal is to relate the integrals $f_j$, that we obtained in Section 3 using the bi-hamiltonian approach on hermitian symmetric spaces, and the Gelfand-Tsetlin coordinates. We will start working with $M = \mathbb{CP}^n$, the complex projective space.

Let $B$ be the $(n+1) \times (n+1)$ matrix, representing an element of $u(n+1)^*$ such that the only non-zero element of $B$ is $\sqrt{-1}\lambda$, located in the very left upper place. The coadjoint orbit $O_B$ of $B$ is isomorphic to $\mathbb{CP}^n$, where the identification goes as follows. Any element in the coadjoint orbit of $B$ can be viewed as $ABA^{-1}$, where $A \in U(n+1)$. Let $(a_{ij})$ be the entries of $A$. Then the identification

$$w: O_B \rightarrow \mathbb{CP}^n$$

is given by

$$w(ABA^{-1}) = [a_{11} : a_{21} : \cdots : a_{n+1,1}],$$

in terms of a homogeneous coordinate system $[Z_0 : \cdots : Z_n]$ on $\mathbb{CP}^n$. We suspect that the following is well-known, and in any case, is not hard to compute, that the Gelfand-Tsetlin coordinates are:

$$\mu^k_r = 0 \quad \text{for} \quad r \neq 1,$$

$$\mu^k_1 = \lambda(x_1 + \cdots + x_{n-k+1}),$$

where $(x_1, \ldots, x_n)$ are the momentum map coordinates that we used in the previous section. We arrive to the conclusion that the Gelfand-Tsetlin coordinates $\{\mu^k_1\}$ coincide (up to the multiple of $\lambda$, which we can assume equal to one) with the coordinates $\{c_k\}$ introduced in the previous section. Now, it remains to notice that the Theorem 3.2 from the previous Section immediately yields

**Theorem 4.1** The complete family of integrals in involution $\{f_i\}$ on $\mathbb{CP}^n$ obtained using the bi-hamiltonian approach with respect to the Bruhat Poisson structure and an invariant symplectic structure are expressed by the elementary polynomials in the Gelfand-Tsetlin coordinates.

We prove a similar result for other hermitian symmetric spaces in a forthcoming paper [1].

5 **Comparison to the Lenard scheme.**

Recall the following result [11]: If $\alpha\pi_0 + \beta\pi_1$ is a Poisson pencil on a manifold $M$, and $V$ a vector field, preserving this pencil, then there exists a sequence of smooth functions $\{g_i\}$ on $M$, such that $g_1$ is the Hamiltonian of $V$ with respect
to $\pi_0$ and the vector field of the $\pi_0$-hamiltonian $f_j$ is the same as the vector field of the $\pi_1$-hamiltonian $f_{j+1}$:

$$i_{df_j}\pi_0 = i_{df_{j+1}}\pi_1.$$ 

Moreover, the functions in the family $\{f_j\}$ are in involution with respect to both $\pi_0$ and $\pi_1$.

Our goal in this section is to show that if we start with $M = \mathbb{C}P^n$, and take the pencil $(\pi_s, \pi_\infty)$ as before, then there is a natural choice of $V$ on $\mathbb{C}P^n$ leading to a completely integrable systems, and the integrals $\{g_j\}$ in question can be easily expressed in terms of the coordinates $(c_1, ..., c_n)$ that we introduced in Section 3.

It is a matter of a simple computation that if we start with a hamiltonian $g_1 = a_1 x_1 + \cdots + a_n x_n$, where $(x_1, ..., x_n)$ are the momentum map coordinates as before, then the corresponding initial vector field $V$ is given by

$$V = i_{dg_1}\pi_\infty = \sum_j [(a_j x_1 + \cdots + a_j x_j) + a_{j+1} x_{j+1} + \cdots + a_n x_n] \partial_{\phi_j}.$$ 

From this, one can compute $g_2 = \sum_j \frac{a_j}{2} x_j^2 + \sum_{l<k} a_k x_l x_k$, etc. An interesting choice for $g_1$ turns out to be

$$g_1 = c_1 + \cdots + c_n = nx_1 + (n-1)x_2 + \cdots + x_n,$$

which coincides with $f_1$ from Section 3. The reason for this choice is

**PROPOSITION 5.1** The Lenard scheme associated to the Poisson pencil $(\pi_\infty, \pi_s)$ on $\mathbb{C}P^n$ which starts with $g_1 = c_1 + \cdots + c_n$ and

$$V = \sum_j [(n-j+1)(x_1 + \cdots + x_j) + (n-j)x_{j+1} + \cdots + 2x_{n-1} + x_n] \partial_{\phi_j},$$ 

yields

$$g_k = c_1^k + c_2^k + \cdots + c_n^k,$$

which determines a completely integrable bi-hamiltonian system on $\mathbb{C}P^n$.

**Proof.** With all the explicit formulas that we have presented in this paper, the proof is a simple computation. ∎

We should remark, that the constants $(a_1, ..., a_n)$ for the first hamiltonian in the Lenard scheme have to be chosen with care for two reasons. First, the computations are not simple for an arbitrary choice. Second, as the next example shows, we do not always arrive to a completely integrable system.
Example. If one takes \( g_1 = x_1 + \cdots + x_n \), and \( V = \sum_j (x_1 + \cdots + x_n) \partial_{x_j} \), then applying the above scheme, one would obtain

\[
g_k = (x_1 + \cdots + x_n)^k = (g_1)^k.
\]

The differentials of all functions in this family are clearly linearly dependent.

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*AMS subj. class.:* primary 58F07, secondary 53B35, 53C35.