Three-body repulsive forces among identical bosons in one dimension

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I consider non-relativistic bosons interacting via pairwise potentials with infinite scattering length and supporting no two-body bound states. To lowest order in effective field theory, these conditions lead to non-interacting bosons, since the coupling constant of the Lieb-Liniger model vanishes identically in this limit. Since any realistic pairwise interaction is not a mere delta function, the non-interacting picture is an idealisation indicating that the effect of interactions is weaker than in the case of off-resonant potentials. I show that the leading order correction to the ground state energy for more than two bosons is accurately described by the lowest order three-body force in effective field theory that arises due to the off-shell structure of the two-body interaction. For natural two-body interactions with a short-distance repulsive core and an attractive tail, the emergent three-body interaction is repulsive and, therefore, three bosons do not form any bound states. This situation is analogous to the two-dimensional repulsive Bose gas, when treated using the lowest-order contact interaction, where the scattering amplitude exhibits an unphysical Landau pole. The avoidance of this state in the three-boson problem proceeds in a way that parallels the two-dimensional case. These results pave the way for the experimental realisation of one-dimensional Bose gases with pure three-body interactions using ultracold atomic gases.

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The theory of few-particle forces in quantum mechanics has a long history that dates back to the early studies of atomic nuclei [1]. It was soon realised that even highly sophisticated nucleon-nucleon potentials, which faithfully reproduced all experimental features of the deuteron [2] and the nucleon-nucleon scattering amplitudes [3], failed to account for the binding energy of the triton [4]. Tuning the short distance details of the nuclear potential, affecting the off-shell elements of the two-nucleon amplitude, moreover, is unnecessary, since these are not measurable and can be traded off in favour of on-shell three-body amplitudes [5]. This is where three-body forces come into play, as they can be used, in conjunction with accurate two-body interactions, to fit three-nucleon data [6], so that heavier nuclei can be investigated in this way.

The modern theory of few-body forces has evolved into a systematic, well controlled low-energy expansion of the interparticle interactions [7,8]. Based on the pioneering work of Weinberg on effective nuclear forces [9,10], model independent two- and higher-body interactions have been developed into what is now commonly known as (chiral) effective field theory (EFT). In essence, EFT considers all possible interactions that are consistent with the underlying symmetries of the problem at a given order in perturbation theory, and the bare coupling constants of the theory are renormalised in favour of low-energy, physical observables.

These effective theories, which are commonplace in nuclear physics, have slowly made their way into the ultracold atomic realm [11]. In fact, the lowest-order interactions were first introduced in the theory of Bose-Einstein condensates (BECs) using pseudopotentials back in 1957 [12]. Since the original motivation in ultracold gases was to produce BECs with alkali atoms [13,14], which interact very weakly, higher-order EFTs were unnecessary for a long time. Three-body interactions in three spatial dimensions, however, were shown to be needed in order to fix the energy of the lowest-lying Efimov state at or near unitarity and avoid the Thomas collapse [15,16], thereby generating great interest in few-body forces in the atomic physics community, which saw the first experimental evidence [17,18] for the elusive Efimov states [19]. Within this context, repulsive three-body forces have been proposed as a mechanism for the stabilisation of quantum atomic droplets [20] which, however, turned out to be stabilised by quantum fluctuations – a lowest-order effect in EFT – at least in actual experimental demonstrations [21,22].

The most promising candidate for the observation of effects due to three-particle forces is perhaps a system of ultracold bosons tightly confined to one spatial dimension. Recently, Guijarro et al. proposed using Bose-Bose and Fermi-Bose mixtures to engineer three-body repulsive interactions between dimers [23]. This proposal relies upon the ability to independently and simultaneously tune two different intraspecies interaction strengths, besides the interspecies scattering length. There are also several recent works focusing on attractive three-body forces in one dimension without [24,25] and with [26] a reference to a physical implementation, the latter requiring simultaneous tuning of several interaction strengths in a multicomponent Bose system on a tight-binding optical lattice. The trimers may also be observable with trapped ultracold atoms, as shown by Pricoupenko [27], who also developed the pseudopotential treatment of the three-body interaction in Ref. [28], which is most convenient for studies in the position representation.

What all of the above works on the one-dimensional three-body interaction agree upon is the important fact that the three-body problem with pure three-body forces...
in one dimension is kinematically equivalent to a two-
dimensional two-body problem at low energies. Indeed,
the former exhibits the same quantum anomaly as the
latter, which has recently been investigated experi-
mentally in two different works \[31, 32\], a fact that was the
focus of Refs. \[25, 20\] in the present case. The most im-
mmediate consequence of this is that, while for attractive
interactions three- and many-body bound states appear,
for repulsive interactions one needs to deal with an un-
physical bound state, i.e. a Landau pole in the scattering
amplitude. Fortunately, it is possible to deal with it in
the same way as for the two-dimensional problem with
two-body interactions thanks to the kinematic equiva-

In this work, I consider non-relativistic identical bosons
interacting via two-body forces exhibiting a zero-energy
resonance (infinite scattering length \[33\]). The model
interactions I use have a soft repulsive core at short
distances, and an attractive finite-range tail. This
type of interactions are justified for effectively reduced-
dimensional systems after integrating out the transversal
dimensions \[34\]. The particular form of the
two-body interactions \(V(x) = V(x_i - x_j)\) between parti-
cles \(i\) and \(j\) that I will use is given by

\[
V(x) = V_0 e^{-\lambda_0 x^2} + V_1 e^{-\lambda_1 x^2},
\]

where \(V_0\) \((< 0)\) and \(V_1\) \((> 0)\) give, respectively, the
strength of the attractive tail and soft core of the in-
teraction, \(\lambda_0\) and \(\lambda_1\) determine their spatial spread, and
I shall denote by \(x_0\) \((x_0^2 = \log |\lambda_1 V_1/\lambda_0 V_0|/(\lambda_1 - \lambda_0))\)
the length scale determining the potential minimum. In
what follows, I choose these parameters in such a way
that the two-boson scattering length diverges \((1/a = 0)\),
i.e., such that the zero-energy solution to the stationary
two-body Schrödinger equation in the relative coordinate

\[
-k^2/m \psi''(x) + V(x) \psi(x) = 0,
\]

has the asymptotic form \(\psi(x) \propto 1\) as \(x \to \pm \infty\), and such
that there are no two-body bound states. The effective
two-body interaction, to lowest order and in the momen-
tum representation, is given by a vanishing Lieb-Liniger
coupling constant \(g = -2\hbar^2/ma = 0\) \[36\].

In the two-boson sector, the next-order interaction in-
volves the effective range \(r\) \[37\], whose effect is identi-
cally zero at zero energy. To see this, and to analyse the
three-body problem, it is most convenient to abandon
the collision-theoretical approach and instead place the
few-body systems on a finite line of length \(L\) with per-
odic boundary conditions. The analysis of the finite-size
spectrum can be used to extract low-energy scattering amplitudes \[38, 32\], and has come to be the method of
choice in modern studies of scattering processes, from
low-energy nuclear physics \[40\] to lattice QCD \[41\]. In
1D, the eigenenergies \(E = \hbar^2 k^2/m\) at zero total momen-
tum for two-bosons can be calculated from the equation

\[
k = \frac{2\pi n}{L} - \frac{2}{L} \theta(k), \quad n \in \mathbb{Z},
\]

where \(\theta(k)\) is the even-wave scattering phase shift in 1D
\[42\]. For the ground state \((n = 0)\), since \(1/a = 0\), we ob-
tain the solution \(k = 0\) and therefore, as claimed, the
effective range has no effect on it. For the first excited state
\((n = 1)\), however, using \(k \tan \theta(k) = 1/a + \pi k^2/2 + O(k^4)
\[42\], the energy shift with respect to the non-interacting
energy \(E_1^{(0)}\) is given by \(\Delta E_1 \approx -2E_1^{(0)}r/L = O(L^{-3})\).
Therefore, the lowest-order correction for \(N \geq 3\) parti-
cles is given by the contribution of effective three-body
forces which naively scales as \(O(L^{-2})\), for both ground
and low-lying excited states.

The bare lowest-order three-body interaction \(V_3^{LO}\) is
obtained by expanding a 1D hyperspherically symmetric
three-body potential to zero-th order in the hyperspheri-
cal momentum, and corresponds to a contact interaction
in the position representation of the form

\[
V_3^{LO}(x_1, x_2, x_3) = g_3 \delta(x_1 - x_2) \delta(x_2 - x_3),
\]

where \(g_3\) is the bare three-body interaction strength.
For a pure three-body interaction, the three-body scat-
ttering amplitude can be obtained directly through the
Lippmann-Schwinger equation since the Faddeev de-
composition is unnecessary. The three-body T-mat-
rix \(T_3(z)\) for the interaction \[1\] is readily obtained as \(\langle k_1, k_2', k_3' | T_3(z) | k_1, k_2, k_3 \rangle = 2\pi \delta(K - K') t_3(z)\),
with \(K = k_1 + k_2 + k_3\) and \(K' = k_1' + k_2' + k_3'\) the conserved
total momentum. The constant \(t_3(z)\), after setting the
total momentum to zero, is given by

\[
t_3(z) = \left[ g_3^{-1} - \mathcal{I}(z) \right]^{-1},
\]

where

\[
\mathcal{I}(z) = \int \frac{dq_1 dq_2 dq_3}{(2\pi)^2} \frac{\delta(q_1 + q_2 + q_3)}{z - \frac{\Lambda^2}{2m} (q_1^2 + q_2^2 + q_3^2)}
\]

In order to calculate the coupling constant \(g_3\), the in-
tegral \(\mathcal{I}(z)\), Eq. \[6\], must be regularised. I use a
hard cutoff \(\Lambda\) in the hyperradial integral, by chang-
ing variables to Jacobi coordinates \(x = (q_1 - q_2)/\sqrt{2},
\(y = \sqrt{2/3} (q_3 - (q_1 + q_2)/2)\), and defining the hyperrad-
ial momentum \(\rho = \sqrt{x^2 + y^2}\). The real part of \(\mathcal{I}(z)\) for
\(z = E + i0^+\) \((E > 0)\) is given, in the limit \(\Lambda \to \infty\), by

\[
\text{Re} \mathcal{I}(z) = -\frac{m}{2\pi\sqrt{3}\hbar^2} \log \left| \frac{\Lambda^2}{2mE/\hbar^2} \right|
\]

For attractive interactions, the T-matrix is renormalised
by fixing the three-body binding energy \(E_B = -|E| = \hbar^2 Q_s^2/2m\) while, for repulsion, \(E_B\) marks the location of a (unphysical) Landau pole, completely equivalent to
its two-body two-dimensional counterpart \[43\]. Here, \(Q_s\)
plays the role of a momentum scale beyond which the EFT description breaks down. As noted by Beane in Ref. [43] for the 2D case, the three-body scattering length \(a_0 = 1.076\ldots\), \(\lambda_1/\lambda_0 = 2\), \(mV_0/\hbar^2\lambda_0 = -5\), \(V_1/V_0 = -1.59151239\), corresponding to inverse scattering length \(x_0/a \approx -3.5 \cdot 10^{-7}\). Small blue dots correspond to the numerical solution of the three-body Schrödinger equation with potential [43], so do the large blue dots, using a larger basis set for convergence; the red dashed line is the fit of Eq. (14) to the data for \(L\) in [4, 5, 10], including the effective range correction (see text). Inset: same as the main figure, but for \(mL^2E/\hbar^2\).

\[
E_0 = \frac{4\pi^2\hbar^2}{mL^2} \left[ g_R - \sigma_1 g_R^3 + (\sigma_1^2 - \sigma_2) g_R^3 + O(g_R^5) \right].
\]

As seen above, the naïve scaling of the energy \(\propto L^{-2}\) is modified by the quantum anomaly [29, 20] in the form of logarithmic corrections.

In the three-body problem under the resonant and no bound state conditions, the contribution of the lowest-order effective three-body force to the ground state energy, Eq. (14), is dominant. However, higher order effects are present, and in order to extract the three-body momentum scale \(Q_*\) accurately, a next-order term of \(O(L^{-4})\) must be included. To see what this term corresponds to, I write the three-body effective range correction to the scattering amplitude by simply replacing

\[
\frac{1}{g_R} \rightarrow \frac{1}{g_R} - r_3^2 k^2,
\]

which is completely analogous to the problem of 2D two-body scattering [11]. The correction to the energy due to the effective range is given by \(\Delta E_{r_3} = 16\pi r_3^2 g_R^3 \hbar^2/mL^4\). This results in a two-parameter fit that needs at least two numerical or experimental data points. In Fig. 1 I plot the ground state energy of three particles in a periodic box as a function of the system’s size for a resonant interaction. I extract the ultraviolet (UV) scale \(Q_*\) by fitting Eq. (14), including the next-to-leading order correction.

FIG. 1: Ground state energy of three particles with pairwise interactions in Eq. (14), with \(\lambda_0 = 1.076\ldots\), \(\lambda_1/\lambda_0 = 2\), \(mV_0/\hbar^2\lambda_0 = -5\), \(V_1/V_0 = -1.59151239\), corresponding to inverse scattering length \(x_0/a \approx -3.5 \cdot 10^{-7}\). Small blue dots correspond to the numerical solution of the three-body Schrödinger equation with potential [43], so do the large blue dots, using a larger basis set for convergence; the red dashed line is the fit of Eq. (14) to the data for \(L\) in [4, 5, 10], including the effective range correction (see text). Inset: same as the main figure, but for \(mL^2E/\hbar^2\).
are small, one can estimate the central density and the particle numbers in the experiment of Ref. [49]. Other choices of the particular functional form of the potential \( \kappa \), and other particular values of the potential’s parameters that keep the scattering length divergent yield qualitatively identical results.

I now move on to discuss the possible experimental demonstration of the three-body force in one dimension under the two conditions mentioned throughout this work. It must be noted that the requirement of no two-body bound states is given for theoretical convenience. Experimentally, this is justified when there are no shallow bound states, yet with sizeable binding energies, as deep bound states that generically exist in ultracold atomic systems are far in energy from the continuum and are therefore not populated in typical experimental time scales. The resonant condition can be satisfied by either using transversal confinement with anharmonic, anisotropic traps [45, 46], which can reach effectively infinite scattering lengths [48], for example for \( ^{133}\text{Cs} \), which can set the 3D scattering length to zero [49], and therefore the effective 1D scattering length to infinity via dimensional reduction [50], or a combination thereof.

As for the observation of tangible effects of the three-body forces, I shall consider a realistic experimental scenario, in particular Ref. [49]. There, they effectively confine a BEC of \( ^{133}\text{Cs} \) atoms to one dimension by applying a transversal 2D optical lattice with effective harmonic length \( a_\perp = 1440a_0 \), with \( a_0 \) the Bohr radius, and the system consists of an array of quasi-1D tubes. The longitudinal harmonic length is given by \( a_\parallel = 8310a_0 \). The central tube, in the repulsive weak coupling regime relevant to this work, has only a few atoms, \( N = 8 - 11 \). A 1D resonance (\( 1/a = 0 \)) is obtained for a magnetic field \( B = 17.119 \text{ G} \), at which point they measure the lowest longitudinal breathing mode and show that indeed it corresponds to the non-interacting limit within experimental uncertainty. In the model interaction of Eq. (1), the most relevant length scale is given by \( x_0 \), which marks the position of the potential minimum. Typical interatomic interactions have \( x_0 \sim 5 - 10\text{Å} \) [51]. Using these values for \( x_0 \), the example given above, which shows a sizeable effect of the three-body force, yet remaining in the weak coupling limit, corresponds to a UV three-body momentum scale \( Q_\parallel \sim 42 - 84\text{Å}^{-1} \) \( (Q_\parallel x_0 \sim 420) \). With the density in the three-body sector \( \rho_{3b} = 3/L \sim 10^{-2} - 10^{-1}\text{Å}^{-1} \), the relevant dimensionless constant \( \kappa = Q_\parallel /\rho_{3b} \) takes on values in the range \( \kappa \sim 420 - 8400 \). Since the three-body force is weak and the particle numbers in the experiment of Ref. [49] are small, one can estimate the central density \( \rho(0) \) in the central tube by using the non-interacting ground state in a harmonic well. This gives \( \rho(0) \sim N/\sqrt{\pi a_0^2} \), with values in the range \( \rho(0) \sim 3.2\text{μm}^{-1} - 4.4\text{μm}^{-1} \), implying that \( Q_\parallel /\rho(0) \sim 10^5 - 3 \cdot 10^5 \), corresponding to a three-body coupling constant reduced by a factor of about 2 – 5 (see Eq. (13)), very similar in magnitude to the example shown here. It would be, nevertheless, beneficial to have higher particle numbers in the tube, of the order of 100, which would mildly increase the three-body coupling constant. Since the focus of Ref. [49] was the strongly interacting limit, it would be interesting to explore the 1D resonant regime in more detail, and study the shift in breathing mode frequency due to the residual three-body interactions. Another possible experimental observation of the effects of three-body forces would be through measurements of the speed of sound, which can be probed using magnetic field gradients [52] or Bragg spectroscopy [53] in quasi-1D ultracold atomic systems.

To conclude, I have studied the three-body problem with identical bosons in one spatial dimension interacting via semi-realistic pairwise interactions and found that, on resonance, the leading order contribution to the three-body scattering amplitude corresponds to an effective, repulsive three-body interaction. I have analyzed the problem in a finite box with periodic boundary conditions, which allows for the extraction of the three-body collisional parameters, and hence the scattering amplitude, via the ground state energy of the system. I have also shown that under rather usual experimental conditions, the effects of the three-body force should be observable. It is also worth noticing that the resonant condition is not stringent, and these effects are sizeable slightly away from the resonance, even on the slightly attractive side of it. Moreover, the energy shifts due to four- and five-body forces naively scale as \( L^{-3} \) and \( L^{-4} \), respectively, and may play a non-trivial role in the equation of state of the resonant Bose gas. Many-body physics under these conditions are yet to be explored, and open up a plethora of new possibilities with one-dimensional quantum many-particle systems beyond the Lieb-Liniger model.

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[1] H. Yukawa, Proc. Phys. Math. Soc. Japan 17, 48 (1935).
[2] K. Erkelenz, Phys. Rep. 13C, 191 (1974).
[3] R. A. Bryan and B. L. Scott, Phys. Rev. 135, B434 (1964).
[4] A. Bömelburg, Phys. Rev. C 28, 403 (1983).
[5] H. -W. Hammer, A. Nogga and A. Schwenk, Rev. Mod. Phys. 85, 197 (2013).
[6] A. Nogga, D. Hüber, H. Kamada and W. Glöcke, Phys. Lett. B 409, 19 (1997).
[7] E. Epelbaum, H. -W. Hammer and U. -G. Meißner, Rev. Mod. Phys. 81, 1773 (2009).
[8] E. Epelbaum, H. Krebs and U. G. Meißner, Phys. Rev. Lett. 115, 122301 (2015).
[9] S. Weinberg, Phys. Lett. B 251, 288 (1990).
[10] S. Weinberg, Nucl. Phys. B 631, 447 (1998).
[11] E. Braaten, M. Kusunoki and D. Zhang, Ann. Phys. (NY) 323, 1770 (2008).
[12] K. Huang and C. N. Yang, Phys. Rev. A 105, 767 (1957).
[13] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman and E. A. Cornell, Science 269, 288 (1990).
[14] S. Weinberg, Nucl. Phys. B 631, 447 (1998).
[15] E. Braaten, M. Kusunoki and D. Zhang, Ann. Phys. (NY) 323, 1770 (2008).
[16] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995).
[17] P. F. Bedaque, H. -W. Hammer and U. van Kolck, Phys. Rev. Lett. 82, 463 (1999).
[18] P. F. Bedaque, H. -W. Hammer and U. van Kolck, Nucl. Phys. A 646, 444 (1999).
[19] T. Kraemer et al., Nature 440, 315 (2006).
[20] S. Knoop et al., Nature Phys. 5, 227 (2009).
[21] M. Zaccanti et al., Nature Phys. 5, 586 (2009).
[22] V. Efimov, Phys. Lett. B 33, 563 (1970).
[23] A. Bulgac, Phys. Rev. Lett. 89, 050402 (2002).
[24] C. R. Cabrera, L. Tanzi, J. Sanz, B. Naylor, P. Thomas, P. Cheiney and L. Tarruell, Science 359, 301 (2018).
[25] G. Semeghini et al., Phys. Rev. Lett. 120, 235301 (2018).
[26] G. Guijarro, A. Pricoupenko, G. E. Astrakharchik, J. Boronat and D. S. Petrov, Phys. Rev. A 97, 061605 (2018).
[27] J. E. Drut, J. R. McKenney, W. S. Daza, C. L. Lin and C. R. Ordóñez, Phys. Rev. Lett. 120, 243002 (2018).
[28] W. S. Daza, J. E. Drut, C. L. Lin and C. R. Ordóñez, e-print arXiv:1808.0711v1.
[29] Y. Nishida, Phys. Rev. A 97, 061603 (2018).
[30] Y. Sekino and Y. Nishida, Phys. Rev. A 97, 011602 (2018).
[31] L. Pricoupenko, Phys. Rev. A 97, 061604 (2018).
[32] L. Pricoupenko, Phys. Rev. A 99, 012711 (2019).
[33] M. Holten, L. Bayha, A. C. Klein, P. A. Murthy, P. M. Preiss and S. Jochim, Phys. Rev. Lett. 121, 120401 (2018).
[34] A. Del Maestro, M. Boninsegni and I. Affleck, Phys. Rev. Lett. 106, 105303 (2011).
[35] A. Del Maestro, Int. J. Mod. Phys. B 26, 1244002 (2012).
[36] P. F. Duc, M. Savard, M. Petrescu, B. Rosenow, A. Del Maestro and G. Gervais, Science Adv. 1, e1400222 (2015).
[37] E. H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963).
[38] M. Valiente and N. T. Zinner, Few-body syst. 56, 845 (2015).
[39] M. Lüscher, Commun. Math. Phys. 104, 177 (1986).
[40] M. Lüscher, Commun. Math. Phys. 105, 153 (1986).
[41] S. R. Beane, Phys. Lett. B 585, 106 (2004).
[42] R. A. Briceno, J. J. Dudek and R. D. Young, Rev. Mod. Phys. 90, 025001 (2018).
[43] V. E. Barlette, M. M. Leite and S. Adhikari, Eur. J. Phys. 21, 435 (2000).
[44] M. Valiente and N. T. Zinner, Sci. China-Phys. Mech. & Astr. 59, 114211 (2016).
[45] S. Sala, P. I. Schneider and A. S. Saenz, Phys. Rev. Lett. 109, 073201 (2012).
[46] M. Valiente and K. Mølmer, Phys. Rev. A 84, 053628 (2011).
[47] S. Adhikari, Am. J. Phys. 54, 362 (1986).
[48] C. Chiu, R. Grimm, P. Julienne and E. Tiesinga, Rev. Mod. Phys. 82, 1225 (2010).
[49] E. Haller et al., Science 325, 1224 (2009).
[50] M. Obshani, Phys. Rev. Lett. 81, 938 (1998).
[51] R. A. Aziz, V. P. S. Nain, J. S. Carley, W. L. Taylor and G. T. McConville, J. Chem. Phys. 70, 4330 (1979).
[52] B. Yang et al., Phys. Rev. Lett. 119, 165701 (2017).
[53] T. L. Yang et al., Phys. Rev. Lett. 121, 103001 (2018).
[54] In the literature, there is no general consensus in the terminology for infinite scattering length. This situation is sometimes referred to as "zero crossing". I shall, however, call this a zero-energy resonance, in analogy with the three-dimensional case.
[55] As in the two-body problem in two dimensions, there is freedom in choosing the renormalised coupling constant, i.e. by choosing different scales. This amounts to a mere reparametrisation of the weak-coupling expansion.