Multi-Attribute Decision-Making Based on Interval-Valued q-Rung Dual Hesitant Uncertain Linguistic Sets

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ABSTRACT The interval-valued q-rung dual hesitant fuzzy sets (IVq-RDHFSs) effectively model decision makers’ (DMs’) evaluation information as well as their high hesitancy in complicated multi-attribute decision-making (MADM) situations. Note that the IVq-RDHFSs only depict DMs’ evaluation values quantificationally but overlook their qualitative decision information. To improve the performance of IVq-RDHFSs in dealing with fuzzy information, we incorporate the concept of uncertain linguistic variables (ULVs) into them and propose a new tool, called interval-valued q-rung dual hesitant uncertain linguistic sets (IVq-RDHULSs). Then we investigate MADM approach with interval-valued q-rung dual hesitant uncertain linguistic (IVq-RDHUL) information. Afterwards, the concept of IVq-RDHULSs as well as their operations and ranking method are proposed. Further, we propose a set of IVq-RDHUL aggregation operators (AOs) on the basis of the powerful Muirhead mean, i.e., the IVq-RDHUL Muirhead mean operator, the IVq-RDHUL weighted Muirhead mean operator, the IVq-RDHUL dual Muirhead mean operator, and the IVq-RDHUL weighted dual Muirhead mean operator. The significant properties of the proposed AOs are also discussed in detail. Lastly, we try to introduce a new method to MADM issues in IVq-RDHUL context based on the newly developed AOs.

INDEX TERMS Interval-valued q-rung dual hesitant fuzzy sets, interval-valued q-rung dual hesitant uncertain linguistic sets, Muirhead mean, multi-attribute decision-making.

I. INTRODUCTION

Multi-attribute decision-making (MADM) refers to a collection of decision-making problems that aim to select or determine the optimal or most suitable alternative(s) under multiple attributes. In real life we always face MADM problems and hence, approaches to MADM have been a hot research topic in management sciences and operations research. A major obstacle in dealing with practical MADM problems is to describe decision makers’ (DMs’) evaluation information in a suitable and explainable method. It was Zadeh [1] who firstly described fuzzy information from the view of fuzzy sets (FSs) theory. The main contribution of FS is that it incorporates the idea of membership degree (MD) into ordinary sets, which describes the degree that an element to a given fixed set. Due to the good ability of FSs in dealing with fuzziness and uncertainty, they have received much attention [2]–[7]. In addition, FSs make it much easier to describe vague and uncertain DMs’ evaluation information, which also have been extensively employed in MADM [8]–[10]. Recently, Atanassov [11] extended the classical FS theory to intuitionistic FSs (IFSs) by taking both MDs and non-membership degrees (NMDs) into consideration. Compared with FSs, IFSs provide more sufficient decision information and more effectively handle DMs’ uncertain evaluation values. Therefore, IFSs based on MADM has soon become a new research topic and many decision-making methods have been proposed [12]–[16].

Although quite a few IFSs based MADM approaches have been proposed, with the increasing complexity of real
MADM problems, it has become more and more difficult to describe DMs’ evaluation information in the form of IFSs. In the framework of IFSs, the MD and NMD of an element are denoted by two single values. Nevertheless, in some real MADM problems DMs are likely to hesitate among several values when giving their assessment values. In other words, DMs often have high hesitancy in expressing their decision information, and they would like to use a set of single values to denote the MD and NMD. Therefore, Zhu et al. [17] generalized IFSs to dual hesitant fuzzy sets (DHFSs), which allow DMs to express their evaluation information through a collection of single values. After the introduction of DHFSs, Garg and Arora [18] studied dual hesitant fuzzy soft sets and their application in MADM. Arora and Garg [19] investigated the robust correlation coefficient measure of DHFSs and its application in MADM problems. Qu et al. [20] proposed a novel stochastic MADM method based on DHFSs, regret theory and group satisfaction degree. Zhang et al. [21] generalized DHFSs to probabilistic DHFSs by considering both randomness and impreciseness. 

Zhang et al. [22] studied the concept of dual hesitant fuzzy rough sets and applied them in medical diagnosis. Ren et al. [23] proposed the new method to rank dual hesitant fuzzy elements and extended the classical VIKOR method to MADM with dual hesitant fuzzy information. Zhang et al. [24] extended DHFSs to interval-valued DHFSs (IVDHFSs), proposed their operational laws and studied their applications in MADM. Some scholars studied MADM methods based on dual hesitant fuzzy aggregation operators (AOs) and we suggest readers to refer [25]–[35].

Recently, Wei [36] extended the IVDHFSs to the interval-valued dual hesitant fuzzy uncertain linguistic sets (IVDHFULSs) and studied their application to MADM. In IVDHFULSs, uncertain linguistic variables (ULVs) are utilized to describe DMs’ qualitative assessments, while interval-valued dual hesitant fuzzy MD and interval-valued dual hesitant fuzzy NMD depict DMs’ quantitative assessments. However, there are still some shortcomings of the MADM method proposed by Wei [36].

1) The MADM method proposed by Wei [36] is based on IVDHFULSs. The IVDHFULSs are constructed by interval-valued dual hesitant fuzzy uncertain linguistic numbers (IVDHFULNs), which satisfy the constraint that the sum of MD and NMD is less than or equal to one. However, such a constraint cannot be always strictly satisfied. In addition, if IVDHFULNs are employed to portray DMs’ evaluation information, some information may lose. For example, a group of DMs utilize an ULV \( s_{1,2} \) to express their evaluation and they are hesitant among \( (0.1, 0.2), (0.3, 0.6) \) and \( (0.3, 0.4), (0.6, 0.7) \) when determining the MD and NMD of the ULV \( s_{1,2} \). Then the group’s evaluation value can be denoted as \( \{1, 2, \}, \{1, 0.1, 0.2, 0.3, 0.6, 0.7\} \). Obviously, the above value cannot be handled by IVDHFULNs, as \( 0.6 + 0.7 = 1.3 > 1 \). In order words, if DMs utilize IVDHFULNs to express their evaluation value, then some information will lose, which will further lead to unreliable decision results.

2) Wei’s [36] MADM method employs simply weighted average operator to aggregate attributes. In other words, in Wei’s [36] opinions, attributes are independent and there is no interaction among attributes. However, more and more scholars and scientists have realized the existence of the interrelationship among attributes in MADM problems and quite a few AOs, which take the interrelationship among attributes into account when determining the optimal alternative(s), have been presented [37]–[39]. Based on the above analysis, Wei’s [36] decision-making method is flawed in dealing with practical MADM problems.

Based on the above analysis, the main motivation and purpose of this paper is to propose a novel MADM method, which overcomes the drawback of Wei’s [36] decision-making method. The main novelties and contributions of this paper are three-fold:

1) A new information representation tool, called interval-valued q-rung dual hesitant uncertain linguistic sets (IVq-RDHULSs), is proposed to depict DMs’ evaluation information. The IVq-RDHULS is a combination of interval-valued q-rung dual hesitant fuzzy sets (IVq-RDHFSs) [40] with ULVs. As an extension of q-rung orthopair fuzzy sets [41] and q-rung dual hesitant fuzzy sets [42], IVq-RDHFSs allow the MDs and NMDs to be represented by several interval values, such that the sum of \( q \)-th power of MD and \( q \)-th power of NMD is less than or equal to one. The IVq-RDHULSs inherit the advantage of IVq-RDHFSs. Compared with IVDHFULSs, the IVq-RDHULSs enlarge the describable information space and give DMs’ more freedom to fully express their evaluation opinions. Compared with IVq-RDHFSs, the IVq-RDHULSs comprehensively portray DMs’ evaluation opinions, as they can represent quantitative and qualitative evaluation information simultaneously.

2) Novel AOs to aggregate interval-valued q-rung dual hesitant uncertain linguistic (IVq-RDHUL) attribute values are proposed. To deal with the interrelationship among attributes, the powerful Muirhead mean (MM) [43] is extended to IVq-RDHUL environment to propose new AOs. The MM has gained much attention in the field of information fusions, due to its capacity of capturing the interrelationship among multiple attributes [44]–[50]. The main superiority of our proposed AOs is that they effectively take into account the interrelationships among arbitrary numbers of attributes, which overcomes the second drawback of Wei’s [36] decision-making method.

3) The main steps of a novel MADM method are clearly illustrated. The new method is applied to practical MADM problems to verify its validity and effectiveness. Comparative analysis is conducted to demonstrate
the advantages and superiorities of the proposed method.

In our proposed MADM method, the IVq-RDHULSs are employed to express DMs’ evaluation values and the MM operator is used to aggregate attributes values. Therefore, our method can more effectively solve practical MADM problems. To clearly present our works, we organize the remainder of this paper as follows. Section 2 reviews basic notions and proposes the concept of IVq-RDHULSs as well as their related notions. Section 3 presents some series of IVq-RDHUL operators and analyzes their properties. Section 4 presents the main steps of a new MADM method. Section 5 proves the efficiency of our method by conducting experimental examples. We make conclusions in Section 6.

II. RELATED CONCEPTS

A. THE INTERVAL-VALUED Q-RUNG DUAL HESITANT FUZZY SETS AND INTERVAL-VALUED Q-RUNG DUAL HESITANT UNCERTAIN LINGUISTIC SETS

Xu et al. [40] recently proposed the IVq-RDHFS and its definition is presented as follows.

**Definition 1** [40]: Let X be a fixed set, then an interval-valued q-rung dual hesitant fuzzy set (IVq-RDHFS) D defined on X is expressed as

\[
D = \{(x, h_D(x), g_D(x)) | x \in X\},
\]

where \(h_D(x) = \bigcup_{[r_D, r_D'] \in h_D(x)} \{[r_D, r_D']\}\) and \(g_D(x) = \bigcup_{[\eta_D, \eta_D'] \in g_D(x)} \{[\eta_D, \eta_D']\}\) are two sets of some interval values in \([0, 1]\), denoting the possible MD and NMD of the element \(x \in X\) to the set \(D\), respectively, with the conditions: \([r_D, r_D'] \cdot [\eta_D, \eta_D'] \subset \{0, 1\}\), and \(0 \leq (r_D')^q + (\eta_D')^q \leq 1\). If \(q = 1\), then \(D\) reduces to q-rung dual hesitant fuzzy set (q-RDHFS) [42]; if \(q = 1\), then \(D\) reduces to IVDHFULs [24]; if \(q = 2\), then \(D\) reduces to hesitant interval-valued Pythagorean fuzzy set [51].

It is noted that IVq-RDHFSs portray DMs’ evaluations from both MD and NMD, which are quantitative information. However, to comprehensively express DMs’ evaluation information, we have to depict both the quantitative and qualitative. In order to do this, scholars usually combine the fuzzy sets theory and linguistic variables or ULVs to propose some hybrid tools. The representatives are intuitionistic linguistic sets [52], intuitionistic uncertain linguistic sets [53], Pythagorean fuzzy linguistic sets [54], Pythagorean fuzzy uncertain linguistic sets [55], etc. Similarly, we combine IVq-RDHFSs with ULVs to propose the IVq-RDHULSs.

**Definition 2**: Let \(X\) be a fixed set and \(\tilde{S}\) be a continuous linguistic term set, then an IVq-RDHULS defined on \(X\) can be expressed as

\[
D = \{(x, [s_{\theta(x)}, s_{\tau(x)}], h_D(x), g_D(x)) | x \in X\},
\]

where \([s_{\theta(x)}, s_{\tau(x)}] \in S\) is an ULV, \(h_D(x) = \bigcup_{[r_D, r_D'] \in h_D(x)} \{[r_D, r_D']\}\) and \(g_D(x) = \bigcup_{[\eta_D, \eta_D'] \in g_D(x)} \{[\eta_D, \eta_D']\}\) are two sets of some interval values in \([0, 1]\), denoting the possible MD and NMD of the element \(x \in X\) to the set \(D\), respectively, such that \([r_D, r_D'] \cdot [\eta_D, \eta_D'] \subset \{0, 1\}\), and \(0 \leq \left((r_D')^q + (\eta_D')^q\right)^r \leq 1\). If \([r_D, r_D'] \in h_D(x), [\eta_D, \eta_D'] \in g_D(x)\), then \((r_D')^q + (\eta_D')^q \leq 1\). Hence, \(D\) reduces to q-rung dual hesitant uncertain linguistic variable (IVq-RDHULV), which can be denoted by \(D = \{\langle s_{\theta}, s_{\tau}, (h, g)\rangle\}\) for simplicity.

**Remark 1**: From the above definition, we know that the IVq-RDHULS is characterized by some interval-valued MDs and NMDs, with the constraint that the sum of qth power of MD and qth power of NMD is no more than one. This characteristic makes IVq-RDHULSs powerful and flexible to express DMs’ evaluation information comprehensively. For example, a group of DMs prefer to use an ULV \([s_1, s_3]\) to express their evaluation. Additionally, they would like to use \([0, 1, 0.2, [0.5, 0.7]]\) and \([0, 3, 0.4, [0.7, 0.8]]\) to denote the possible MDs and NMDs of the ULV. Then, the group’s evaluation value can be denoted as \(d = \langle[s_1, s_3], \{[[0, 1, 0.2, [0.5, 0.7]], [0, 3, 0.4, [0.7, 0.8]]]\}\). Obviously, as \(0.7 + 0.8 = 1.5 > 1\), \(d\) cannot be handled by IVDHFULNs. In addition, we notice that \(0.7^3 + 0.8^3 = 0.855 < 1\). Hence, \(d\) can be represented by IVq-RDHULV \((q = 3)\). This example illustrates the wider information range that IVq-RDHULSs can describe. Moreover, some existing fuzzy sets are special cases of the IVq-RDHULS. If \(s_1 = s_3\), then the IVq-RDHULSs reduce to the interval-valued q-rung dual hesitant linguistic sets; If \(r_D = r_D'\) and \(\eta_D = \eta_D'\), then the IVq-RDHULSs reduce to the q-rung dual hesitant uncertain linguistic sets; If \(s_1 = s_3\) and \(r_D = r_D'\), then the IVq-RDHULSs reduce to the q-rung dual hesitant uncertain linguistic sets; If \(q = 2\), then the IVq-RDHULSs reduce to the interval-valued Pythagorean dual hesitant uncertain linguistic sets; If \(q = 1\), then the IVq-RDHULSs reduce to the IVDHFULVs [36].

Motivated by the operations of the IVq-RDHFSs [31] and ULVs [56], we propose the operational laws of IVq-RDHULVs.

**Definition 3**: Let \(d_1 = \langle[s_{\theta_1}, s_{\tau_1}], (h_1, g_1)\rangle\) and \(d_2 = \langle[s_{\theta_2}, s_{\tau_2}], (h_2, g_2)\rangle\) be any three IVq-RDHULVs,
To compare any two IVq-RDHULVs, we propose a ranking method of IVq-RDHULVs.

**Theorem 1:** Let \( d_1 = \left[ s_{q_1}, s_{r_1} \right], \ (h_1, g_1) \) and \( d_2 = \left[ s_{q_2}, s_{r_2} \right], \ (h_2, g_2) \) be any three IVq-RDHULVs, then

1. \( d_1 \oplus d_2 = d_2 \oplus d_1; \)
2. \( d_1 \otimes d_2 = d_2 \otimes d_1; \)
3. \( \lambda (d_1 \oplus d_2) = \lambda d_1 \oplus \lambda d_2, \lambda \geq 0; \)
4. \( \lambda_1 d_1 \oplus \lambda_2 d_2 = (\lambda_1 \oplus \lambda_2) d_1, \lambda_1, \lambda_2 \geq 0; \)
5. \( d_1^\lambda \otimes d_2^\kappa = (d_1 \otimes d_2)^{\lambda \kappa}, \lambda \geq 0; \)
6. \( d_1^{h_1} \otimes d_2^{h_2} = (d_1 \otimes d_2)^{h_1 + h_2}, \lambda_1, \lambda_2 \geq 0; \)

and the accuracy function of \( d \) as

\[
p(g) = \frac{1}{2} \left( \left( \frac{1}{\#_h} \sum_{[r^l, q^l] \in h} r^l \right)^q + \left( \frac{1}{\#_r} \sum_{[q^r, q^l] \in h} r^l \right)^q \right)
- \left( \frac{1}{\#_g} \sum_{[q^g, q^r] \in g} \eta^r \right)^q - \left( \frac{1}{\#_g} \sum_{[q^g, q^r] \in g} \eta^l \right)^q \times (\theta + \tau),
\]

where \( \#_h \) and \( \#_g \) represent the numbers of interval values in \( h \) and \( g \), respectively. Then, let \( d_i = \left[ s_{q_i}, s_{r_i} \right], \ (h_i, g_i) \) \( (i = 1, 2) \) be any two IVq-RDHULVs, we can get

1. if \( f(d_1) > f(d_2) \), then \( d_1 > d_2; \)
2. if \( f(d_1) = f(d_2) \), then
   - if \( p(d_1) = p(d_2) \), then \( d_1 = d_2; \)
   - if \( p(d_1) > p(d_2) \), then \( d_1 > d_2. \)

**B. THE MUIRHEAD MEAN OPERATOR**

**Definition 3.** Let \( a_j (j = 1, 2, \ldots, n) \) be a collection of crisp numbers and \( T = (t_1, t_2, \ldots, t_n) \) \( T^n \) a vector of parameters, then the MM is expressed as

\[
MM^T (a_1, a_2, \ldots, a_n) = \left( \frac{1}{n^n} \sum_{\theta \in S_n} \prod_{j=1}^{n} a_{j}^{t_j} \right)^{\frac{1}{n}},
\]

where the permutation of \( (1, 2, \ldots, n) \), \( S_n \) is the collection of \( \theta \) \( (j = 1, 2, \ldots, n). \)

Yang and Pang [50] proposed the dual form of MM, called dual Muirhead mean (DMM) operator.

**Definition 4.** Let \( a_j (j = 1, 2, \ldots, n) \) be a collection of crisp numbers and \( T = (t_1, t_2, \ldots, t_n) \) \( T^n \) a vector of parameters. If

\[
DMM^T (a_1, a_2, \ldots, a_n) = \left( \frac{1}{n^n} \sum_{\theta \in S_n} \prod_{j=1}^{n} t_j^{a_{j}^{t_j}} \right)^{\frac{1}{n}},
\]

then \( DMM^T \) is called the DMM, where the permutation of \( (1, 2, \ldots, n) \), \( S_n \) is the collection of \( \theta \) \( (j = 1, 2, \ldots, n). \)

**III. AGGREGATION OPERATORS FOR INTERVAL-VALUED Q-RUNG DUAL HESITANT UNCERTAIN LINGUISTIC INFORMATION**

In this section, we extend MM and DMM to IVq-RDHUL environment and develop some IVq-RDHUL Muirhead mean AOs. Properties of these operators are also discussed in this section.

**A. THE INTERVAL-VALUED Q-RUNG DUAL HESITANT UNCERTAIN LINGUISTIC MUIRHEAD MEAN OPERATOR**

**Definition 7.** Let \( d_j = \left[ s_{q_j}, s_{r_j} \right], \ (h_j, g_j) \) \( (j = 1, 2, \ldots, n) \) be a collection of IVq-RDHULVs and \( T = (t_1, t_2, \ldots, t_n) \) \( T^n \)
be a vector of parameters. If
\[
IVq - RDHULMM^T (d_1, d_2, \ldots, d_n)
\]
\[
= \left( \frac{1}{n^!} \bigoplus_{j \in S_n} \otimes d_{\vartheta(j)}^{(j)} \right) \frac{1}{\sum_{j=1}^{n} y_j}. \tag{17}
\]
than \( IVq - RDHULMM^T \) is called the interval-valued q-rung dual hesitant uncertain linguistic Muirhead mean (IVq-RDHULMM) operator, where \( \vartheta (j) (j = 1, 2, \ldots, n) \) is any permutation of \( (1, 2, \ldots, n) \), \( S_n \) is the collection of \( \vartheta (j) (j = 1, 2, \ldots, n) \).

Based on the operational laws of the IVq-RDHULVs shown in Definition 3, the following theorem can be derived.

**Theorem 2:** Let \( d_j = (\vartheta_0(j), \vartheta_1(j), (h_j, g_j)) (j = 1, 2, \ldots, n) \) be a collection of IVq-RDHULVs, then the aggregated value by using the IVq-RDHULM operator is still an IVq-RDHULV and
\[
IVq - RDHULMM^T (d_1, d_2, \ldots, d_n)
\]
\[
= \left\{ \left[ \sum_{j=1}^{n} \left( \prod_{\vartheta \in S_n} (r_{\vartheta(j)}^{(j)}) \right)^{\frac{1}{n}} \right] \left( \prod_{\vartheta \in S_n} (r_{\vartheta(j)}^{(j)}) \right) \right\}^{\frac{1}{n^!}} \frac{1}{\sum_{j=1}^{n} y_j},
\]
\[
\bigcup_{\vartheta \in S_n} \left( \left( \prod_{\vartheta \in S_n} (r_{\vartheta(j)}^{(j)}) \right)^{\frac{1}{n}} \right) \frac{1}{\sum_{j=1}^{n} y_j},
\]
\[
\left\{ \left[ \left( \prod_{\vartheta \in S_n} (r_{\vartheta(j)}^{(j)}) \right)^{\frac{1}{n}} \right] \frac{1}{\sum_{j=1}^{n} y_j} \right\}^{\frac{1}{n}},
\]
\[
\left\{ \left( \prod_{\vartheta \in S_n} (r_{\vartheta(j)}^{(j)}) \right)^{\frac{1}{n}} \right\}^{\frac{1}{n}},
\]
\[
\left\{ \left( \prod_{\vartheta \in S_n} (r_{\vartheta(j)}^{(j)}) \right)^{\frac{1}{n}} \right\}^{\frac{1}{n}}.
\]

**Proof:** According to the Definition 3, we have
\[
d_{\vartheta(j)}^{(j)} \\
= \left\{ \left[ \prod_{\vartheta \in S_n} (r_{\vartheta(j)}^{(j)}) \right] \right\}^{\frac{1}{n}},
\]
\[
\left\{ \left( \prod_{\vartheta \in S_n} (r_{\vartheta(j)}^{(j)}) \right) \right\}^{\frac{1}{n}},
\]
\[
= \left\{ \left[ \prod_{\vartheta \in S_n} (r_{\vartheta(j)}^{(j)}) \right] \right\}^{\frac{1}{n}},
\]
\[
\left\{ \left( \prod_{\vartheta \in S_n} (r_{\vartheta(j)}^{(j)}) \right) \right\}^{\frac{1}{n}}.
\]
\[
\left\{ \left( \prod_{\vartheta \in \mathcal{S}_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( \eta_{\vartheta(j)}^{q_j} \right)^{\frac{1}{q_j}} \right) \right) \right)^{\frac{1}{\sum_{j=1}^{q_j}}} \right\},
\]

Moreover,
\[
\left( \frac{1}{n!} \bigotimes_{\vartheta \in \mathcal{S}_n}\sum_{j=1}^{q_j} d_{\vartheta(j)}^{\vartheta} \right)^{\frac{1}{\sum_{j=1}^{q_j}}}
\]

\[
= \left\{ \left( \prod_{\vartheta \in \mathcal{S}_n} \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( \eta_{\vartheta(j)}^{q_j} \right)^{\frac{1}{q_j}} \right) \right) \right)^{\frac{1}{\sum_{j=1}^{q_j}}} \right\},
\]

\[
\bigcup_{r_{\vartheta(j)} \in \mathcal{K}_{\vartheta(j)}, \delta_{\vartheta(j)} \in \mathcal{G}_{\vartheta(j)}} \left\{ \left( 1 - \prod_{\vartheta \in \mathcal{S}_n} \left( 1 - \prod_{j=1}^{n} \left( r_{\vartheta(j)}^{q_j} \right)^{\frac{1}{q_j}} \right) \right) \right\}.
\]

Evidently,
\[
0 \leq r_{\vartheta(j)}^l \leq 1, \quad 0 \leq \left( r_{\vartheta(j)}^u \right)^{\frac{1}{q_j}} \leq 1,
\]

and
\[
0 \leq \prod_{j=1}^{n} \left( r_{\vartheta(j)}^l \right)^{\frac{1}{q_j}} \leq 1, \quad 0 \leq \left( \prod_{j=1}^{n} \left( r_{\vartheta(j)}^u \right)^{\frac{1}{q_j}} \right)^{q_j} \leq 1.
\]

Then,
\[
0 \leq 1 - \left( \prod_{j=1}^{n} \left( r_{\vartheta(j)}^l \right)^{\frac{1}{q_j}} \right)^{q_j} \leq 1,
\]

and
\[
0 \leq \prod_{\vartheta \in \mathcal{S}_n} \left( 1 - \prod_{j=1}^{n} \left( r_{\vartheta(j)}^l \right)^{\frac{1}{q_j}} \right)^{\frac{1}{q_j}} \leq 1.
\]

Further,
\[
0 \leq \left( \prod_{\vartheta \in \mathcal{S}_n} \left( 1 - \prod_{j=1}^{n} \left( r_{\vartheta(j)}^u \right)^{\frac{1}{q_j}} \right) \right)^{\frac{1}{q_j}} \leq 1,
\]

and
\[
0 \leq 1 - \left( \prod_{\vartheta \in \mathcal{S}_n} \left( 1 - \prod_{j=1}^{n} \left( r_{\vartheta(j)}^u \right)^{\frac{1}{q_j}} \right) \right)^{\frac{1}{q_j}} \leq 1.
\]

Therefore, \( 0 \leq r^l \leq 1 \). Similarly, we can get \( 0 \leq r^u \leq 1 \), \( 0 \leq \eta^l \leq 1 \), \( 0 \leq \eta^u \leq 1 \).

Since \( 0 \leq \left( \left( r_{\vartheta(j)}^u \right)^{\frac{1}{q_j}} \right)^{q_j} \leq 1 \), then \( 0 \leq \left( \left( r_{\vartheta(j)}^u \right)^{\frac{1}{q_j}} \right)^{q_j} \leq 1 \), so we have \( \left( \left( r_{\vartheta(j)}^u \right)^{\frac{1}{q_j}} \right)^{q_j} \leq 1 \), \( \left( \left( \eta_{\vartheta(j)}^u \right)^{\frac{1}{q_j}} \right)^{q_j} \leq 1 \).
which means that the aggregated value is still an IVq-RDHULV. The proof of Theorem 2 is completed.

Moreover, the IVq-RDHULMM operator has the following properties.

**Theorem 3 (Idempotency):** If all the $d_j$ ($j = 1, 2, \ldots, n$) are equal, i.e., $d_j = d = ([s_h, s_g], (h, g))$ for all $j$, and there is only one interval-valued element in $h$ and $g$, respectively, then

$$\text{IVq} - \text{RDHULMM}^T (d_1, d_2, \ldots, d_n) = d. \quad (19)$$

**Proof:** According to Theorem 2, we can get

$$\text{IVq}-\text{RDHULMM}^T (d_1, d_2, \ldots, d_n)$$

$$= \left[ s \left( \frac{1}{\pi} \sum_{\theta \in S_h} \frac{1}{\eta_j} \right)^{\gamma} - s \left( \frac{1}{\pi} \sum_{\theta \in S_h} \frac{1}{\eta_j} \right)^{\gamma} \right], \quad \cup_{r \in h, \eta \in g}$$

$$= \left[ \left( 1 - \left( \prod_{\theta \in S_h} \left( 1 - \left( \prod_{j=1}^{n} (1 - \eta_j^u) \right)^{\frac{1}{\pi}} \right) \right) \right) \left( \frac{1}{\pi} \right) \right]^\frac{1}{\gamma},$$

$$= \left[ \left( 1 - \left( \prod_{\theta \in S_h} \left( 1 - \left( \prod_{j=1}^{n} (1 - \eta_j^u) \right)^{\frac{1}{\pi}} \right) \right) \right) \right]^\frac{1}{\gamma}.$$

$$= \left[ \left( 1 - \left( \prod_{\theta \in S_h} \left( 1 - \left( \prod_{j=1}^{n} (1 - \eta_j^u) \right)^{\frac{1}{\pi}} \right) \right) \right) \right]^\frac{1}{\gamma}.$$

$$= \left[ \left( 1 - \left( \prod_{\theta \in S_h} \left( 1 - \left( \prod_{j=1}^{n} (1 - \eta_j^u) \right)^{\frac{1}{\pi}} \right) \right) \right) \right]^\frac{1}{\gamma}.$$

$$= \left[ \left( 1 - \left( \prod_{\theta \in S_h} \left( 1 - \left( \prod_{j=1}^{n} (1 - \eta_j^u) \right)^{\frac{1}{\pi}} \right) \right) \right) \right]^\frac{1}{\gamma}.$$

$$= \left[ \left( 1 - \left( \prod_{\theta \in S_h} \left( 1 - \left( \prod_{j=1}^{n} (1 - \eta_j^u) \right)^{\frac{1}{\pi}} \right) \right) \right) \right]^\frac{1}{\gamma}.$$

$$= \left[ \left( 1 - \left( \prod_{\theta \in S_h} \left( 1 - \left( \prod_{j=1}^{n} (1 - \eta_j^u) \right)^{\frac{1}{\pi}} \right) \right) \right) \right]^\frac{1}{\gamma}.$$

$$= \left[ \left( 1 - \left( \prod_{\theta \in S_h} \left( 1 - \left( \prod_{j=1}^{n} (1 - \eta_j^u) \right)^{\frac{1}{\pi}} \right) \right) \right) \right]^\frac{1}{\gamma}.$$

$$= \left[ \left( 1 - \left( \prod_{\theta \in S_h} \left( 1 - \left( \prod_{j=1}^{n} (1 - \eta_j^u) \right)^{\frac{1}{\pi}} \right) \right) \right) \right]^\frac{1}{\gamma}.$$

$$= \left[ \left( 1 - \left( \prod_{\theta \in S_h} \left( 1 - \left( \prod_{j=1}^{n} (1 - \eta_j^u) \right)^{\frac{1}{\pi}} \right) \right) \right) \right]^\frac{1}{\gamma}.$$
and
\[
\prod_{j=1}^{n} (r_{\delta(j)})^y \geq \prod_{j=1}^{n} (r_{\delta(j)})^y,
\]
then
\[
\left(\prod_{j=1}^{n} (r_{\delta(j)})^y\right)^q \geq \left(\prod_{j=1}^{n} (r_{\delta(j)})^y\right)^q,
\]
\[
1 - \left(\prod_{j=1}^{n} (r_{\delta(j)})^y\right)^q \leq 1 - \left(\prod_{j=1}^{n} (r_{\delta(j)})^y\right)^q,
\]
and
\[
\prod_{\vartheta \in S_n} \left(1 - \left(\prod_{j=1}^{n} (r_{\delta(j)})^y\right)^q\right) \leq \prod_{\vartheta \in S_n} \left(1 - \left(\prod_{j=1}^{n} (r_{\delta(j)})^y\right)^q\right).
\]
Further,
\[
\left(\prod_{\vartheta \in S_n} \left(1 - \left(\prod_{j=1}^{n} (r_{\delta(j)})^y\right)^q\right)\right)^{\frac{1}{q}} \leq \left(\prod_{\vartheta \in S_n} \left(1 - \left(\prod_{j=1}^{n} (r_{\delta(j)})^y\right)^q\right)\right)^{\frac{1}{q}}.
\]
and
\[
1 - \left(\prod_{\vartheta \in S_n} \left(1 - \left(\prod_{j=1}^{n} (r_{\delta(j)})^y\right)^q\right)\right) \geq 1 - \left(\prod_{\vartheta \in S_n} \left(1 - \left(\prod_{j=1}^{n} (r_{\delta(j)})^y\right)^q\right)\right).
\]
Therefore,
\[
\left(1 - \left(\prod_{\vartheta \in S_n} \left(1 - \left(\prod_{j=1}^{n} (r_{\delta(j)})^y\right)^q\right)\right)\right)^{\frac{1}{q}} \geq \left(\prod_{\vartheta \in S_n} \left(1 - \left(\prod_{j=1}^{n} (r_{\delta(j)})^y\right)^q\right)\right)^{\frac{1}{q}}.
\]
Then we can get \(r^l \geq r^l\).
Similarly, we also have \(r^u \geq r^u, \eta^l \leq \eta^l, \eta^u \leq \eta^u\).
Obviously, \(s_0 \geq s_0^t, \eta \geq s_\).

**Theorem 5 (Boundedness):** Let \(d_j = \{s_{0,j}, s_{1,j}, \ldots, s_{n,j}, h_{j}, \ldots, g_{j}\}\) \((j = 1, 2, \ldots, n)\) be a collection of IVq-RDHULVs, and
\[
d^- = \left(\left(\prod_{j=1}^{n} (s_{0,j})^y\right)^{\frac{1}{q}} , \left(\prod_{j=1}^{n} (r_{\delta(j)})^y\right)^{\frac{1}{q}} \right),
\]
\[
d^+ = \left(\left(\prod_{j=1}^{n} (s_{\delta(j)})^y\right)^{\frac{1}{q}} , \left(\prod_{j=1}^{n} (r_{\delta(j)})^y\right)^{\frac{1}{q}} \right).
\]
\[d^- \leq IVq - \text{RDHULMM}^T (d_1, d_2, \ldots, d_n) \leq d^+.
\]

**Proof:** Based on Theorem 4, we have
\[
IVq - \text{RDHULMM}^T (d_1, d_2, \ldots, d_n) \geq IVq - \text{RDHULMM}^T (d^-, d^-, \ldots, d^-),
\]
and
\[
IVq - \text{RDHULMM}^T (d_1, d_2, \ldots, d_n) \leq IVq - \text{RDHULMM}^T (d^+, d^+, \ldots, d^+).
\]
In addition, both \(d^-\) and \(d^+\) only have one MD and one NMD. Therefore,
\[
IVq - \text{RDHULMM}^T (d^-, d^-, \ldots, d^-) = d^-,
\]
\[
IVq - \text{RDHULMM}^T (d^+, d^+, \ldots, d^+) = d^+.
\]
So, we can get \(d^- \leq IVq - \text{RDHULMM}^T (d_1, d_2, \ldots, d_n) \leq d^+\).

Evidently, the parameter \(q\) and parameter vector \(T\) play important roles in the aggregation results. In the followings, we will discuss some special cases of the IVq-RDHULMM operator with respect to \(T\) and \(q\).

**Special Case 1:** if \(T = (1, 0, \ldots, 0)\), then the IVq-RDHULMM operator reduces to the following
\[
IVq - \text{RDHULMM}^{(1,0,\ldots,0)} (d_1, d_2, \ldots, d_n)
\]
\[
= \left[\left(\frac{1}{n} \sum_{j=1}^{n} s_{0,j}^y \right)^{\frac{1}{q}} , \left(\frac{1}{n} \sum_{j=1}^{n} s_{1,j}^y \right)^{\frac{1}{q}} \right], \cup_{\eta_j \in h_j, \eta_j \in g_j} \left[\left(\prod_{j=1}^{n} (1 - (r_{\delta(j)})^y)^{\frac{1}{q}} \right)^{\frac{1}{q}}, \left(\prod_{j=1}^{n} (1 - (r_{\delta(j)})^y)^{\frac{1}{q}} \right)^{\frac{1}{q}} \right],
\]
\[
\left\{\left(\prod_{j=1}^{n} (s_{0,j}^y)^{\frac{1}{q}}, \left(\prod_{j=1}^{n} \eta_j^u \right)^{\frac{1}{q}} \right)\right\} = \frac{1}{n} \prod_{j=1}^{n} d_j.
\]
\[
\left\{ \left[ \left( 1 - \left( 1 - \frac{n}{ \sum_{i,j=1}^{n, n} \left( \theta_{ij} \times \tau_{ij} \right) } \right)^{q} \right)^{\frac{1}{q}} \right] \frac{1}{n(n-1) \bigoplus n \bigoplus \ldots \bigoplus n \bigoplus \ldots \bigoplus n} \right\} \]
\]
where \( \theta_{ij} \) and \( \tau_{ij} \) are the linguistic terms.

\[\text{IVq-RDHULMM}^{(1,1,0,...,0)} (d_1, d_2, \ldots, d_n) = \left( \frac{1}{n(n-1)} \bigoplus \ldots \bigoplus \frac{1}{n(n-1)} \right)^{1/2} \]

which is the interval-valued q-rung dual hesitant uncertain linguistic Bonferroni mean operator.

**Special Case 4:** If \( T = (1, 1, \ldots, 1, 0, 0, \ldots) \), then the IVq-RDHULMM operator reduces to the following

\[\text{IVq-RDHULMM}^{(1,1,1,\ldots,1)} (d_1, d_2, \ldots, d_n) = \left( \frac{1}{n(n-1)} \bigoplus \ldots \bigoplus \frac{1}{n(n-1)} \right)^{1/2} \]

which is the interval-valued q-rung dual hesitant uncertain linguistic geometric (IVq-RDHULG) operator.

**Special Case 6:** if \( q = 1 \), then the IVq-RDHULMM operator reduces to the following

\[\text{IVq-RDHULM}^{R} (d_1, d_2, \ldots, d_n) = \left( \left( \frac{1}{n} \bigoplus \ldots \bigoplus \frac{1}{n} \right)^{1/2} \right) \]

which is the interval-valued q-rung dual hesitant uncertain linguistic geometric (IVq-RDHULG) operator.
which is the interval-valued dual hesitant uncertain linguistic Muirhead mean operator.

**Special Case 7:** if \( q = 2 \), then the IVq-RDHULMM operator reduces to the following

\[
IVq - RDHULMM^T (d_1, d_2, \ldots, d_n)
= \left[ \sum_{\tau_{(j)}} \phi_{(j)}^{\tau_{(j)}} \sum_{j=1}^{n} (\tau_{(j)})^{\frac{1}{n}} \right]^{\frac{1}{n}}
\]

(27)

which is the interval-valued Pythagorean dual hesitant uncertain linguistic Muirhead mean operator.

**Remark 2:** We can get more special cases of the IVq-RDHULMM operator. For example, when \( q = 1 \), if \( T = (1, 0, \ldots, 0) \), then IVq-RDHULMM reduces to the interval-valued dual hesitant uncertain linguistic average operator; if \( T = (\lambda, 0, 0, \ldots, 0) (\lambda > 0) \), then IVq-RDHULMM reduces to the interval-valued dual hesitant uncertain linguistic generalized average operator; if \( T = (1, 0, \ldots, 0) \), then IVq-RDHULMM reduces to the interval-valued dual hesitant uncertain linguistic Bonferroni mean operator; if \( T = (1, 1, \ldots, 1) \), then IVq-RDHULMM reduces to the interval-valued dual hesitant uncertain linguistic Maclaurin symmetric mean operator; if \( T = (1/n, 1/n, \ldots, 1/n) \), then the IVq-RDHULMM operator reduces to the interval-valued dual hesitant uncertain linguistic geometric operator. Similarly, if \( q = 2 \), more special cases of the proposed IVq-RDHULMM operator are obtained.

**B. THE INTERVAL-VALUED Q-RUNG DUAL HESITANT UNCERTAIN LINGUISTIC WEIGHTED MUIRHEAD MEAN OPERATOR**

**Definition 8:** Let \( d_j = [s_{(j)}, s_{(x)}] \), \( (h_j, g_j) \) \( j = 1, 2, \ldots, n \) be a collection of IVq-RDHULVs and \( T = (t_1, t_2, \ldots, t_n) \in T^n \) be a vector of parameters. Let \( w = (w_1, w_2, \ldots, w_n)^T \) be the weight vector, such that \( 0 \leq w_j \leq 1 \) and \( \sum_{j=1}^{n} w_j = 1 \). Then

\[
IVq - RDHULWM^T (d_1, d_2, \ldots, d_n)
= \left( \frac{1}{n!} \sum_{w_1, w_2, \ldots, w_n} (nw_{(j)} w_{(j)})^{\frac{1}{n}} \right)
\]

(29)

then \( IVq - RDHULWM^T \) is called the interval-valued q-rung dual hesitant uncertain linguistic weighted Muirhead mean (IVq-RDHULWM) operator, where \( \vartheta (j) (j = 1, 2, \ldots, n) \) is any permutation of \( (1, 2, \ldots, n) \). \( S_q \) is the collection of \( \vartheta (j) (j = 1, 2, \ldots, n) \).

Based on the Definition 3, the following theorem can be gained.

**Theorem 6:** Let \( d_j = [s_{(j)}, s_{(x)}], (h_j, g_j) \) \( j = 1, 2, \ldots, n \) be a collection of IVq-RDHULVs, then the aggregated value by using the IVq-RDHULWM operator is still an IVq-RDHULV and

\[
IVq - RDHULWM^T (d_1, d_2, \ldots, d_n)
= \left[ \sum_{\tau_{(j)}} \phi_{(j)}^{\tau_{(j)}} \sum_{j=1}^{n} (\tau_{(j)})^{\frac{1}{n}} \right]^{\frac{1}{n}}
\]

(28)
The proof of Theorem 6 is similar to that of Theorem 2. So, we omitted it here. Moreover, the IVq-RDHULWMM also has the properties of monotonicity and boundedness.

C. THE INTERVAL-VALUED Q-RUNG DUAL HESITANT UNCERTAIN LINGUISTIC DUAL MIURHEAD MEAN OPERATOR

**Definition 9.** Let \( d_j = \{[s_0, s_1], [h_0, h_1]\} (j = 1, 2, \ldots, n) \) be a collection of IVq-RDHULVs and \( T = (t_1, t_2, \ldots, t_n) \) be a vector of parameters. If

\[
IVq - RDHULDMM^T (d_1, d_2, \ldots, d_n) = \frac{1}{n} \sum_{j=1}^{n} \left( \bigotimes_{\theta \in S_n} (t_j d_{\theta(j)}) \right) \prod_{j=1}^{n} t_j, \tag{31}
\]

then \( IVq - RDHULDMM^T \) is called the interval-valued q-rung dual hesitant uncertain linguistic dual Muirhead mean (IVq-RDHULDM) operator, where \( \theta(j) (j = 1, 2, \ldots, n) \) is any a permutation of \( (1, 2, \ldots, n) \), and \( S_n \) is the collection of \( \theta(j) (j = 1, 2, \ldots, n) \).

**Theorem 7:** Let \( d_j = \{[s_0, s_1], [h_0, h_1]\} (j = 1, 2, \ldots, n) \) be a collection of IVq-RDHULVs, then the aggregated value by using the IVq-RDHULDM operator is still an IVq-RDHULV and

\[
IVq - RDHULDMM^T (d_1, d_2, \ldots, d_n) = \left[ \frac{s}{\sum_{j=1}^{n} t_j}, \frac{s}{\sum_{j=1}^{n} t_j} \right] \prod_{\theta \in S_n} \left( \bigotimes_{j=1}^{n} t_j d_{\theta(j)} \right) \prod_{j=1}^{n} t_j, \tag{32}
\]

The proof is similar to that of Theorem 2, so we omit it here.

Similarly, the IVq-RDHULDM operator has the properties of idempotency, monotonicity, and boundedness. The proofs are omitted here. In the followings, we investigate special cases of the IVq-RDHULDM operator.

**Special Case 1:** If \( T = (1, 0, \ldots, 0) \), then the IVq-RDHULDM operator reduces to the following

\[
IVq - RDHULDMM^{(1,0,\ldots,0)} (d_1, d_2, \ldots, d_n) = \left[ \frac{s}{\prod_{j=1}^{n} r_j^{l_j} \prod_{j=1}^{n} r_j^{h_j}}, \frac{s}{\prod_{j=1}^{n} r_j^{l_j} \prod_{j=1}^{n} r_j^{h_j}} \right] \prod_{\theta \in S_n} \left( \bigotimes_{j=1}^{n} (1 - (h_j^q))^\frac{1}{q} \right) \prod_{j=1}^{n} r_j^{l_j}, \tag{33}
\]

which is the IVq-RDHULG operator.

**Special Case 2:** If \( T = (\lambda, 0, 0, \ldots, 0) (\lambda > 0) \), the IVq-RDHULDM operator reduces to the following

\[
IVq - RDHULDMM^{(\lambda,0,\ldots,0)} (d_1, d_2, \ldots, d_n) = \left[ \frac{s}{\prod_{j=1}^{n} \lambda r_j^{h_j}}, \frac{s}{\prod_{j=1}^{n} \lambda r_j^{h_j}} \right] \prod_{\theta \in S_n} \left( \bigotimes_{j=1}^{n} (1 - (h_j^q))^\frac{1}{q} \right) \prod_{j=1}^{n} r_j^{l_j}. \tag{34}
\]
which is the interval-valued q-rung dual hesitant uncertain linguistic geometric Bonferroni mean operator.

**Special Case 3:** If \( T = (1, 1, 0, \ldots, 0) \), then the IVq-RDHULDMM operator reduces to the following

\[
IVq - RDHULDMM^{(1, 1, 0, \ldots, 0)}(d_1, d_2, \ldots, d_n) = \left\{ \left[ \frac{1}{k} \sum_{j=1}^{n} (\tau_j \eta_j)^{\frac{1}{q}} \right] \right\}
\]

which is the interval-valued q-rung dual hesitant uncertain linguistic dual Maclaurin symmetric mean (IVq-RDHULDMSM) operator.

**Special Case 5:** If \( T = (1, 1, \ldots, 1) \) or \( T = (1/n, 1/n, \ldots, 1/n) \), then the IVq-RDHULDMM operator reduces to the following

\[
IVq - RDHULDMM^{(1, 1, 1, \ldots, 1) \text{ or } (1/n, 1/n, \ldots, 1/n)}(d_1, d_2, \ldots, d_n) = \left\{ \left[ \frac{1}{n} \sum_{j=1}^{n} (\tau_j \eta_j)^{\frac{1}{q}} \right] \right\}
\]

which is the IVq-RDHULA operator.

**Special Case 6:** If \( q = 1 \), then the IVq-RDHULDMM operator reduces to the following

\[
IVq - RDHULDMM^T(d_1, d_2, \ldots, d_n) = \left\{ \left[ \frac{1}{n} \sum_{j=1}^{n} (\tau_j \eta_j)^{\frac{1}{q}} \right] \right\}
\]
If \( T = (1, 0, \ldots, 0) \), then the IVq-RDHULDMM operator reduces to the interval-valued dual hesitant uncertain linguistic geometric operator; if \( T = (\lambda, 0, 0, \ldots, 0) (\lambda > 0) \), then the IVq-RDHULDMM operator reduces to the interval-valued dual hesitant uncertain linguistic generalized geometric operator; if \( T = (1, 1, 0, \ldots, 0) \), then the IVq-RDHULDMM operator reduces to the interval-valued q-rung dual hesitant uncertain linguistic geometric Bonferroni mean operator; if \( T = (1, 1, \ldots, 1, 0, 0, \ldots, 0) \), then the IVq-RDHULDMM operator reduces to the interval-valued q-rung dual hesitant uncertain linguistic weighted dual Muirhead mean operator. If \( q = 2 \), then we can get more special cases of the IVq-RDHULDMM operator.

**D. THE INTERVAL-VALUED Q-RUNG DUAL HESITANT UNCERTAIN LINGUISTIC WEIGHTED DUAL MUIRHEAD MEAN OPERATOR**

**Definition 10:** Let \( d_j = \{ [s_{j1}, s_{j2}], \ldots, [h_{j1}, h_{j2}] \} \) \((j = 1, 2, \ldots, n)\) be a collection of IVq-RDHULVs and \( T = (t_1, t_2, \ldots, t_k) \in T^n \) be a vector of parameters. Let \( w = (w_1, w_2, \ldots, w_n) \) be the weight vector, such that \( 0 \leq w_j \leq 1 \) and \( \sum_{j=1}^{n} w_j = 1 \). If

\[
IVq - RDHUDLWDM^{T} (d_1, d_2, \ldots, d_n) = \left[ \begin{array}{c}
\frac{1}{\sum_{j=1}^{n} t_j} 
\left( \prod_{j=1}^{n} t_{j} \right)^{\frac{1}{n}} \\
\sum_{j=1}^{n} t_j 
\end{array} \right]^{\frac{1}{n}},
\]

then \( IVq - RDHUDLWDM^{T} \) is called the interval-valued q-rung dual hesitant uncertain linguistic weighted dual Muirhead mean (IVq-RDHULDWM) operator, where \( \theta(j) (j = 1, 2, \ldots, n) \) is any a permutation of \((1, 2, \ldots, n)\), and \( S_{i\theta(j)} \) is the collection of all \( \theta(j) \) \((j = 1, 2, \ldots, n)\).

**Theorem 8:** Let \( d_j = \{ [s_{j1}, s_{j2}], \ldots, [h_{j1}, h_{j2}] \} \) \((j = 1, 2, \ldots, n)\) be a collection of IVq-RDHULVs, then the aggregated value by using the IVq-RDHULDWM operator is still an IVq-RDHULV, and

\[
IVq - RDHUDLWDM^{T} (d_1, d_2, \ldots, d_n) = \left[ \begin{array}{c}
\frac{1}{\sum_{j=1}^{n} t_j} 
\left( \prod_{j=1}^{n} t_{j} \right)^{\frac{1}{n}} \\
\sum_{j=1}^{n} t_j 
\end{array} \right]^{\frac{1}{n}},
\]

which is the interval-valued Pythagorean dual hesitant uncertain linguistic dual Muirhead mean operator.

**Remark 3:** We can get more special cases of the IVq-RDHULDWM operator. For example, when \( q = 1 \),
The comprehensive preference scores can be obtained.

or the IVq-RDHULWDMM operator

\[ C j \in \Omega_B, \quad C j \in \Omega_C. \] (42)

where \( s_{ij} = [s_{0j}, s_{1j}], \) and \( h_{ij} = \bigcup_{q_j \in I_{0j}} \left\{ \left[ r_{ij}^q, r_{ij}^q \right] \right\}, g_{ij} = \bigcup_{h_j \in G_j} \left\{ \left[ h_{ij}^q, h_{ij}^q \right] \right\}, \] \( \Omega_B \) and \( \Omega_C \) are the collections of benefit attributes and cost attributes, respectively.

Step 2: For alternative \( A_i (i = 1, 2, \ldots, m), \) utilize the IVq-RDHULWMM operator

\[ d_i = IVq - RDHULWMM^T (d_{i1}, d_{i2}, \ldots, d_{in}), \] (43)

or the IVq-RDHULWDMM operator

\[ d_i = IVq - RDHULWDMM^T (d_{i1}, d_{i2}, \ldots, d_{in}), \] (44)

to aggregate all attribute values, so that a series of comprehensive preference scores can be obtained.

Step 3: Rank the overall values \( d_i (i = 1, 2, \ldots, m) \) based on their scores according to Definition 4.

Step 4: Rank the corresponding alternatives according to the result of Step 3, and then select the best alternative.

V. AN APPLICATION OF THE PROPOSED MADM METHOD IN CLINICIAN PERFORMANCE ASSESSMENT

Clinician performance assessment aims to qualitatively and quantitatively measure and calculate the achievement of clinician’s tasks over a certain period of time, which is closely related to their salary, rewards and punishments, title promotion and so forth. Therefore, it is very significant to develop a scientific and effective performance evaluation system. For clinicians working in a teaching hospital, their tasks involve not only medical services but also the work of a university teacher, which makes the assessment of their performances more complex. Suppose that there are four clinicians \( A_i (i = 1, 2, 3, 4), \) and their performances need to be evaluated from three aspects \( C_j (j = 1, 2, 3): \) clinical services \( (C_1), \) teaching quality \( (C_2) \) and scientific research level \( (C_3), \) of which the weighted vector is \( w = (0.4, 0.2, 0.4)^T. \) To give the DMs more freedom in decision-making process, they are allowed to give interval-valued q-rung dual hesitant fuzzy uncertain linguistic information and the decision matrix is presented in Table 1. The proposed method is utilized to obtain their scores, and the higher the score, the better the working achievement of the clinician. Note that DMs can express their judgments on clinician’ performance through linguistic set \( S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}, \) and the degree of “Good” becomes stronger and stronger from \( s_1 \) to \( s_7.\)

A. THE DECISION-MAKING PROCESS

Step 1: As all attributes are benefit type, the original decision matrix does not need to be normalized.

Step 2: Utilize the IVq-RDHULWMM operator to aggregate DMs’ assessments so that the overall assessments of alternatives can be derived (assume \( T = (1, 1, 1) \) and \( q = 3). \) As the results are too complicated, we omit them here.

Step 3: Compute the score values of the overall assessments, and we can get

\[ f(d_1) = 2.1719, \quad f(d_2) = 2.3319, \]
\[ f(d_3) = 3.0040, \quad f(d_4) = 2.2526. \]

Step 4: According to Definition 4, we can obtain the ranking order

\[ f(d_3) > f(d_4) > f(d_2) > f(d_1), \]

which means the order of the job performances of clinicians is \( A_3 > A_4 > A_2 > A_1. \) Therefore, \( A_3 \) deserves the best reward.

In Step 2, if IVq-RDHULWDM operator is utilized to aggregate attribute values (assume \( T = (1, 1, 1) \)) and \( q = 3), \) the score values of alternatives are

\[ f(d_1) = 2.2670, \quad f(d_2) = 2.7740, \]
\[ f(d_3) = 3.4059, \quad f(d_4) = 2.4765. \]
TABLE 1. The interval-valued q-rung dual hesitant uncertain linguistic decision matrix.

| A | C1 | C2 | C3 |
|---|---|---|---|
| \(A_1\) | \([[[0.5, 0.9], [0.0, 0.5]], [[[0.1, 0.3], [0.2, 0.4]], [0.6, 0.7]], [0.1, 0.3]]\) | \([[[0.5, 0.9], [0.0, 0.5]], [[[0.1, 0.3], [0.2, 0.4]], [0.6, 0.7]], [0.1, 0.3]]\) | \([[[0.5, 0.9], [0.0, 0.5]], [[[0.1, 0.3], [0.2, 0.4]], [0.6, 0.7]], [0.1, 0.3]]\) |
| \(A_2\) | \([[[0.5, 0.9], [0.0, 0.5]], [[[0.1, 0.3], [0.2, 0.4]], [0.6, 0.7]], [0.1, 0.3]]\) | \([[[0.5, 0.9], [0.0, 0.5]], [[[0.1, 0.3], [0.2, 0.4]], [0.6, 0.7]], [0.1, 0.3]]\) | \([[[0.5, 0.9], [0.0, 0.5]], [[[0.1, 0.3], [0.2, 0.4]], [0.6, 0.7]], [0.1, 0.3]]\) |
| \(A_3\) | \([[[0.5, 0.9], [0.0, 0.5]], [[[0.1, 0.3], [0.2, 0.4]], [0.6, 0.7]], [0.1, 0.3]]\) | \([[[0.5, 0.9], [0.0, 0.5]], [[[0.1, 0.3], [0.2, 0.4]], [0.6, 0.7]], [0.1, 0.3]]\) | \([[[0.5, 0.9], [0.0, 0.5]], [[[0.1, 0.3], [0.2, 0.4]], [0.6, 0.7]], [0.1, 0.3]]\) |
| \(A_4\) | \([[[0.5, 0.9], [0.0, 0.5]], [[[0.1, 0.3], [0.2, 0.4]], [0.6, 0.7]], [0.1, 0.3]]\) | \([[[0.5, 0.9], [0.0, 0.5]], [[[0.1, 0.3], [0.2, 0.4]], [0.6, 0.7]], [0.1, 0.3]]\) | \([[[0.5, 0.9], [0.0, 0.5]], [[[0.1, 0.3], [0.2, 0.4]], [0.6, 0.7]], [0.1, 0.3]]\) |

Thus, the order of performances is \(A_3 > A_2 > A_4 > A_1\), which means that during the assessment period, \(A_3\) achieved the best performance.

**B. THE INFLUENCE OF THE PARAMETERS ON THE RANKING RESULTS**

As aforementioned, the parameter vector \(T\) and parameter \(q\) play significant roles in final results. Therefore, it is crucial to further investigate the influence of parameters on the score values and ranking results. Firstly, we assign different parameter vectors to \(T\) in IVq-RDHULWMM and IVq-RDHULWDMM operators (suppose \(q = 3\)), respectively. The decision-making outcomes are presented in Tables 2 and 3.

From Tables 2 and 3, we know that the score values obtained by IVq-RDHULWMM and IVq-RDHULWDMM operators are different based on different \(T\), which indicates \(T\) does have significant influence on the score values. In detail, if only the first parameter of vector \(T\) is real number and the others are 0, for IVq-RDHULWMM, we can find that the larger the real number is, the greater the score value of each alternative is. In contrast, for IVq-RDHULWDMM, the larger the real number, the smaller the score value of each alternative. Therefore, different values of \(T\) can reflect DMs’ risk preferences of experts. In addition, although there is a little difference among the ranking orders, the clinician with highest score is always \(A_3\), which reveals the efficiency and stability of the proposed method. On the other hand, it is well-known that MM and DMM are characterized by taking into account the interrelationship among multiple input arguments. In real management environment, there are often some relationships among decision attributes. For the above-mentioned clinician performance assessment, due to the limited personal energy of the clinician, more investigation in clinical services (\(C_1\)) may led to the decrease of teaching quality (\(C_2\)) and scientific research level (\(C_3\)), and eventually his/her comprehensive score may change. So, it is definitely important to consider the interrelationship among these attributes, and users who are confronted with kinds of management environment can choose appropriate \(T\) according to practical needs.

**TABLE 2. Scores and ranking results with different \(R\) in the IVq-RDHULWMM (\(q = 3\)).**

| \(T\) | Score values \(f(d_i)\) \((i = 1, 2, 3, 4)\) | Ranking orders |
|---|---|---|
| \(T = (1, 0, 0)\) | \(f(d_1) = 2.2983\), \(f(d_2) = 2.5687\), \(f(d_3) = 3.5315\), \(f(d_4) = 2.5631\) | \(A_1 > A_3 > A_2 > A_4\) |
| \(T = (2, 0, 0)\) | \(f(d_1) = 2.4192\), \(f(d_2) = 2.7727\), \(f(d_3) = 3.9260\), \(f(d_4) = 2.7959\) | \(A_1 > A_3 > A_2 > A_4\) |
| \(T = (3, 0, 0)\) | \(f(d_1) = 2.5188\), \(f(d_2) = 2.9388\), \(f(d_3) = 4.2322\), \(f(d_4) = 2.9821\) | \(A_1 > A_3 > A_2 > A_4\) |
| \(T = (1, 1, 1)\) | \(f(d_1) = 2.2255\), \(f(d_2) = 2.4194\), \(f(d_3) = 3.2342\), \(f(d_4) = 2.4101\) | \(A_1 > A_4 > A_2 > A_3\) |

**TABLE 3. Scores and ranking results with different \(R\) in the IVq-RDHULWDMM (\(q = 3\)).**

| \(T\) | Score values \(f(d_i)\) \((i = 1, 2, 3, 4)\) | Ranking orders |
|---|---|---|
| \(T = (1, 0, 0)\) | \(f(d_1) = 2.0131\), \(f(d_2) = 2.1063\), \(f(d_3) = 2.7943\), \(f(d_4) = 2.1885\) | \(A_1 > A_2 > A_3 > A_4\) |
| \(T = (2, 0, 0)\) | \(f(d_1) = 1.9556\), \(f(d_2) = 1.9554\), \(f(d_3) = 2.7677\), \(f(d_4) = 2.1687\) | \(A_1 > A_2 > A_3 > A_4\) |
| \(T = (3, 0, 0)\) | \(f(d_1) = 1.9152\), \(f(d_2) = 1.8646\), \(f(d_3) = 2.7447\), \(f(d_4) = 2.1541\) | \(A_1 > A_2 > A_3 > A_4\) |
| \(T = (1, 1, 1)\) | \(f(d_1) = 2.1737\), \(f(d_2) = 2.4378\), \(f(d_3) = 3.2976\), \(f(d_4) = 2.3977\) | \(A_1 > A_2 > A_3 > A_4\) |
| \(T = (1, 1, 1)\) | \(f(d_1) = 2.2670\), \(f(d_2) = 2.7740\), \(f(d_3) = 3.4059\), \(f(d_4) = 2.4765\) | \(A_1 > A_2 > A_3 > A_4\) |

In the followings, by assigning different values to the parameter \(q\) in the IVq-RDHULWMM and IVq-RDHULWDMM operators (suppose \(T = (1, 1, 1)\)), the effects on the score values and ranking orders are discussed. The decision results are shown in Tables 4 and 5, respectively.
As we can see from Table 4 and Table 5, different score values and ranking results are produced with different values of \( q \). However, the clinician who has the highest score is always \( A_3 \), which means that he/she achieves the best job performance and should get more rewards. In general, the parameter \( q \) can make the proposed method more flexible. As for the numerical selection of \( q \), we recommend DMs choose the minimum integer that can make the sum of \( q \)th power of maximum element in membership degree set and in non-membership degree set no larger than one. For example, evaluation value is \( \langle [s_4, s_5], \{[0.5, 0.6], [0.7, 0.8] \}, \{[0.1, 0.2], [0.2, 0.3] \} \rangle \), we can set \( q = 2 \) as \( 0.8 + 0.3 = 1.1 > 1 \) and \( 0.8^2 + 0.3^2 = 0.73 < 1 \).

### C. Effectiveness Analysis

In order to verify the effectiveness of our method, we compare the proposed method based on IVq-RDULWMM operator with Wei’s [36] method based on interval-valued dual hesitant fuzzy uncertain linguistic weighted average (IVDUFULWA) operator.

#### Example 1: Our method based on IVq-RDULWMM operator and Wei’s [36] method based on IVDUFULWA operator, respectively, are utilized to assess these clinicians’ performances (the above example) based on the decision matrix showed in Table 1. The results are presented in Table 6.

As displayed in Table 6, the ranking orders of the alternatives produced by the two methods are the same, i.e., \( A_3 > A_4 > A_2 > A_1 \), which indicates the effectiveness of our proposed method.

### D. Advantages of Our Method

In order to illustrate the superiorities of the proposed method, we compare our method with that proposed by Wei [36] based on IVDUFULWA operator, and that introduced by Lu and Wei [57] based on dual hesitant fuzzy uncertain linguistic weighted average/geometric (DHFULWA/DHFULWGE) operator. We employ these methods to solve some practical numerical examples and conduct comparative analysis to discuss the advantages of our proposed method. Basically, our method has the following four advantages.
TABLE 7. The dual hesitant fuzzy uncertain linguistic decision matrix of Example 2.

|    | C₁       | C₂       | C₃       |
|----|----------|----------|----------|
| A₁ | [s₄, s₅], [0.3, 0.4], [0.6] > | [s₂, s₃], [0.4, 0.5], [0.3, 0.4] > | [s₃, s₂], [0.2, 0.3], [0.7] > |
| A₂ | [s₁, s₂], [0.6], [0.4] >    | [s₄, s₅], [0.2, 0.4, 0.5], [0.4] > | [s₃, s₂], [0.2], [0.6, 0.7, 0.8] > |
| A₃ | [s₂, s₃], [0.5, 0.7], [0.2] > | [s₁, s₃], [0.2], [0.7, 0.8] >    | [s₃, s₅], [0.2, 0.3, 0.4], [0.6] > |
| A₄ | [s₂, s₃], [0.7], [0.3] >    | [s₄, s₅], [0.6, 0.7, 0.8], [0.2] >| [s₃, s₅], [0.1, 0.2], [0.3] >    |
| A₅ | [s₁, s₄], [0.6, 0.7], [0.2] >| [s₂, s₃], [0.2, 0.3, 0.4], [0.5] >| [s₃, s₄], [0.4, 0.5], [0.2] >    |

TABLE 8. The interval-valued dual hesitant uncertain linguistic decision matrix for Example 2.

|    | C₁       | C₂       | C₃       |
|----|----------|----------|----------|
| A₁ | [s₄, s₅], [0.3, 0.3], [0.4, 0.4], [0.6, 0.6] > | [s₂, s₃], [0.4, 0.4], [0.5, 0.5], [0.3, 0.3], [0.4, 0.4] > | [s₃, s₂], [0.2, 0.2], [0.3, 0.3], [0.7, 0.7] > |
| A₂ | [s₁, s₂], [0.6, 0.6], [0.4, 0.4] >    | [s₄, s₅], [0.2, 0.2], [0.4, 0.4], [0.5, 0.5], [0.4, 0.4] > | [s₃, s₂], [0.2, 0.2], [0.6, 0.6], [0.7, 0.7], [0.8, 0.8] > |
| A₃ | [s₂, s₃], [0.5, 0.5], [0.7, 0.7], [0.2, 0.2] > | [s₁, s₃], [0.2, 0.2], [0.7, 0.7], [0.8, 0.8] >    | [s₃, s₅], [0.2, 0.2], [0.3, 0.3], [0.4, 0.4], [0.6, 0.6] > |
| A₄ | [s₄, s₅], [0.7, 0.7], [0.3, 0.3] >    | [s₂, s₃], [0.6, 0.6], [0.7, 0.7], [0.8, 0.8] >    | [s₃, s₅], [0.1, 0.1], [0.2, 0.2], [0.3, 0.3] > |
| A₅ | [s₁, s₄], [0.6, 0.6], [0.7, 0.7], [0.2, 0.2] >| [s₂, s₃], [0.2, 0.2], [0.3, 0.3], [0.4, 0.4], [0.5, 0.5] >| [s₃, s₄], [0.4, 0.4], [0.5, 0.5], [0.2, 0.2] > |

TABLE 9. Score functions and ranking orders by different methods for Example 2.

| Method                          | Score values                           | Ranking orders   |
|--------------------------------|----------------------------------------|-----------------|
| Lu and Wei’s [56] method based on DHFULWA operator | f(d₁) = 1.0994, f(d₂) = 1.1154, f(d₃) = 2.0060, f(d₄) = 3.7044, f(d₅) = 1.8792 | A₁ > A₄ > A₃ > A₂ > A₅ |
| Lu and Wei’s [56] method based on DHFULWG operator | f(d₁) = 1.0745, f(d₂) = 0.9792, f(d₃) = 1.6352, f(d₄) = 2.9943, f(d₅) = 1.5303 | A₁ > A₄ > A₃ > A₂ > A₅ |
| The proposed method based on IVq-RDHUDLWM operator (T = 1, 1, 1, q = 3) | f(d₁) = 1.0605, f(d₂) = 1.0251, f(d₃) = 1.2583, f(d₄) = 2.9878, f(d₅) = 1.3060 | A₁ > A₄ > A₃ > A₂ > A₅ |
| The proposed method based on IVq-RDHUDLWDM operator (T = 1, 1, 1, q = 3) | f(d₁) = 1.2609, f(d₂) = 1.0898, f(d₃) = 2.1305, f(d₄) = 3.4413, f(d₅) = 1.6722 | A₁ > A₄ > A₃ > A₂ > A₅ |

1) ITS ABILITY OF PORTORYING DM’S EVALUATION INFORMATION MORE ACCURATELY

In our proposed MADM method, IVq-RDHULSs are employed to depict DMs’ evaluation information. IVq-RDHULSs allow DMs to express the MD and NMD of an ULV by a series of interval values rather than crisp numbers. Evidently, the proposed IVq-RDHULS is more powerful and flexible than DHFULS proposed by Lu and Wei [56] which utilizes some crisp numbers to denote the possible MD and NMD of corresponding ULV. Furthermore, the dual hesitant fuzzy uncertain linguistic element (DHFULE) is a special case of IVq-RDHULV (q = 1) in which the upper bound and lower bound of each interval value in the collections of MD and NMD are equal. Therefore, the decision-making problems in which attribute values in the form of DHFULEs can also be solved by our proposed method. We provide the following example to better demonstrate this characteristic.

Example 2 (Revised From [56]): There are five possible emerging technology enterprises Aᵢ (i = 1, 2, 3, 4, 5). The experts are required to evaluate the five alternatives under three attributes Cⱼ (j = 1, 2, 3), where C₁ represents the technical advancement, C₂ represents the potential market opportunity, and C₃ represents the industrialization infrastructure, human resources and financial conditions. The weight vector of the attributes is w = (0.35, 0.25, 0.40)ᵀ. DMs are required to evaluate these alternatives with dual hesitant fuzzy uncertain linguistic information and the revised decision matrix is shown in Table 7. In Example 2, DMs employ DHFULEs to express their evaluations. As mentioned above, DHFULE is a special case of IVq-RDHULV, so we can transform a DHFULE into an IVq-RDHULV.

For example, let ([s₄, s₅], [0.3, 0.4], [0.6]) be a DHFULE, then we can transform it into ([s₄, s₅], [0.3, 0.3], [0.4, 0.4], [0.6, 0.6]) which is an IVq-RDHULV. Hence, we can transform the original dual hesitant fuzzy uncertain linguistic decision matrix into an IVq-RDHUL decision matrix (please see Table 8). Following-ly, the proposed method based on IVq-RDHUDLWM and IVq-RDHUDLWDM operators and Lu and Wei’s [56] method based on DHFULWA and DHFULWG operators are applied to cope with the problem, and outcomes are presented in Table 9.

As we see from Table 9, the ranking orders derived by Lu and Wei’s [56] method and our method are slightly different, but the first is always A₃. This also demonstrates the effectiveness of our proposed method. However, our proposed method is still more powerful and flexible than Lu and Wei’s [56] method. In reality, due to the high complicacy of decision-making problem, it is usually difficult for DMs to express their judgments by crisp numbers. Instead, in order to comprehensively provide their evaluation
TABLE 10. The interval-valued dual hesitant uncertain linguistic decision matrix $R$ of Example 3.

| $A_i$ | $C_j$ | $C_k$ | $C_l$ |
|-------|-------|-------|-------|
| $[s_4, s_5], [0.6, 0.7], [0.7, 0.8]$ | $[s_4, s_5], [0.6, 0.7], [0.7, 0.8]$ | $[s_4, s_5], [0.6, 0.7], [0.7, 0.8]$ | $[s_4, s_5], [0.6, 0.7], [0.7, 0.8]$ |

*The original interval-value dual hesitant fuzzy decision matrix does not satisfy the constraint of IVDHFULs. We revised the original decision matrix and new one is listed in Table 10.*

TABLE 11. Score functions and ranking orders derived by different methods of Example 3.

| Method | Score values $f(d_{ij})$ $(i=1,2,3,4)$ | Ranking orders |
|--------|--------------------------------------|----------------|
| Wei’s [36] method based on IVDHFULWA operator | $f(d_{11}) = 1.3881$, $f(d_{12}) = 1.2045$, $f(d_{13}) = 2.1709$, $f(d_{14}) = 5.1459$, $f(d_{15}) = 2.4528$ | $A_1 > A_4 > A_3 > A_2$ |
| Wei’s [36] method based on IVDHFULWG operator | $f(d_{11}) = 1.6960$, $f(d_{12}) = 0.8404$, $f(d_{13}) = 1.6223$, $f(d_{14}) = 3.5157$, $f(d_{15}) = 1.7724$ | $A_1 > A_3 > A_4 > A_2$ |
| The proposed method based on IVq-RDHULWMM operator ($\eta = (1, 1, 1), q = 3$) | $f(d_{11}) = 1.2111$, $f(d_{12}) = 0.9222$, $f(d_{13}) = 1.0997$, $f(d_{14}) = 3.7542$, $f(d_{15}) = 1.5256$ | $A_1 > A_3 > A_4 > A_2$ |
| The proposed method based on IVq-RDHULWDM operator ($\eta = (1, 1, 1), q = 3$) | $f(d_{11}) = 1.3737$, $f(d_{12}) = 1.0865$, $f(d_{13}) = 2.9099$, $f(d_{14}) = 3.9820$, $f(d_{15}) = 1.8316$ | $A_1 > A_3 > A_4 > A_2$ |

information, DMs usually employ interval numbers to express their assessments.

Our proposed method can not only solve MADM problems where the MDs and NMDs are expressed by crisp numbers, but also deal with decision-making situations wherein the possible MDs and NMDs are expressed by interval values. But, Lu and Wei’s [57] method can only solve MADM problems in which the MDs and NMDs of ULVs are expressed by crisp numbers. For example, if DMs use $\{s_4, s_5\}$, $[0.2, 0.3], [0.3, 0.4], [0.4, 0.6]$ to express their evaluation of attribute $C_1$ of alternative $A_1$, then Lu and Wei’s [57] method is powerless to deal with this case, whereas our proposed method can still determine the ranking order of alternatives.

2) THE GREATER FREEDOM IT PROVIDES FOR DMs

Our proposed method is based on IVq-RDHULs which satisfy the condition that the sum of $q$th power of MD and NMD of a ULV is less than or equal to one. In practical decision-making problems, DMs can choose a proper value of $q$ to make $(r^q)^+ + (\eta^q)^+ \leq 1$ hold.

The MADM method proposed by Wei [36] is based on IVDHFULs which satisfy the constraint that the sum of MD and NMD is less than or equal to one, i.e. $(r^q)^+ + (\eta^q)^+ \leq 1$. Compared with IVq-RDHULs, the constraint of IVDHFULs is more rigorous. Hence, our proposed method gives DMs more freedom to express their fuzzy judgments. For instance, if DMs utilize $\{s_4, s_5\}$, $[0.3, 0.5], [0.5, 0.6], [0.7, 0.9]$ to denote their evaluation value with respect to an attribute of a certain alternative. Then, as $0.6 + 0.9 = 1.5 > 1$, Wei’s [36] method is unable to handle this case. However, as $0.6^3 + 0.9^3 = 0.945 < 1$, we can set $q = 3$, and then our proposed method can still deal with this case. Therefore, compared with Wei’s [36] IVDHFULs our proposed IVq-RDHULs describe a larger information space and provide more freedom for DMs to express their evaluation information. Moreover, IVDHFULs is regarded as a special case of IVq-RDHULs (when $q = 1$). To better demonstrate this characteristic, we provide the following examples.

Example 3 (Revised From [36]): There is a panel with five possible service outsourcing providers of communications industry $A_i (i = 1, 2, 3, 4, 5)$ to select. The expert team selects three attributes $C_j (j = 1, 2, 3)$ to evaluate the five candidates, i.e., business reputation ($C_1$), technical ability ($C_2$), and management ability ($C_3$). The weight vector of the attributes is $w = (0.35, 0.25, 0.4)^T$. The DMs are required to evaluate the five possible providers under the above attributes with the interval-valued dual hesitant fuzzy uncertain linguistic information. The original decision matrix $D = (d_{ij})_{5 \times 3}$ is presented in Table 10.

First, we use our developed method based on IVq-RDHULM and IVq-RDHULDM operators, and Wei’s [36] method based on IVDHFULWA and IVDHFULWG operators to determine the most desirable outsourcing providers. Their outcomes are then shown in Table 11. It is clear from Table 11 that all the four methods can be utilized to solve this problem, and although score values derived by four methods are slightly different, the best choice is always $A_4$. Note that although our proposed method is based on the IVq-RDHULs, it still can be employed in solving Example 3. This is because IVDHFULs is a special case of IVq-RDHULs and we can always find a value of $q$ which makes $(r^q)^+ + (\eta^q)^+ \leq 1$ hold. However, Wei’s [36] method will fail if not all attribute values satisfy the condition $(r^q)^+ + (\eta^q)^+ \leq 1$. We provide the following example to better explain this characteristic.

Example 4: Given the DMs’ subjective bias in real world, some evaluation values in Table 10 are revised, i.e., let $d_{31} = \{s_4, s_5\}, \{(0.5, 0.6), (0.7, 0.9), (0.2, 0.4)\}$, $d_{32} = \{s_4, s_5\}, \{(0.2, 0.5), (0.7, 0.8), (0.8, 0.9)\}$ and $d_{33} = \{s_4, s_5\}, \{(0.2, 0.3), (0.3, 0.4), (0.4, 0.5), (0.6, 0.7)\}$,
TABLE 12. Score functions and ranking orders derived by different methods of Example 4.

| Method                           | Score values $f(d_i) i \in \{1, 2, 3, 4, 5\}$ | Ranking orders |
|----------------------------------|-----------------------------------------------|----------------|
| Wei's [36] method based on IVDHFULWA operator | Cannot be calculated                          | —              |
| Wei's [36] method based on IVDHFULWG operator | Cannot be calculated                          | —              |
| The proposed method based on IVq-RDHFULWM operator ($\hat{r} = (1, 1, 1), q = 3$) | $f(d_1) = 2.0816, f(d_2) = 0.6025, f(d_3) = 0.8943, f(d_4) = 4.0048, f(d_5) = 1.8245$ | $A_2 > A_1 > A_4 > A_3 > A_5$ |
| The proposed method based on IVq-RDHFULWDM operator ($\hat{r} = (1, 1, 1), q = 3$) | $f(d_1) = 0.9757, f(d_2) = 0.7925, f(d_3) = 0.9396, f(d_4) = 3.7081, f(d_5) = 1.6858$ | $A_2 > A_1 > A_4 > A_3 > A_5$ |

respectively. And all other values remain unchanged. The score values and ranking orders derived by different methods are displayed in Table 12.

It is clear that Wei’s [36] MADM method is powerless to deal with Example 4, while our proposed method can still get the ranking orders of alternatives. This is because none of $d_{31}, d_{32}$ or $d_{33}$ satisfy the constraint $(r_1^+)^q + (\eta_1^+)^q + 1 \leq 1$ so that they cannot be represented by IVDHFULSs. However, we can always find a proper value of $q$ that makes $d_{31}, d_{32}$ and $d_{33}$ satisfy the condition $(r_1^+)^q + (\eta_1^+)^q + 1 \leq 1$. For example, we can set $q = 3$, then $0.93 + 0.43 = 0.79 < 1$, $0.53 + 0.93 = 0.854 < 1$, and $0.53 + 0.73 = 0.468 < 1$. Hence, our proposed method can provide more freedom for DMs to fully express their evaluation values.

3) THE ABILITY TO CONSIDER THE INTERRELATIONSHIP AMONG MULTIPLE ATTRIBUTES

Generally, in practical MADM problems there usually are interrelationship among multiple attributes. When calculating the comprehensive evaluation values of alternatives, not only the attributes values and their corresponding weight information but also the complicated interrelationships among attributes should be taken into account.

As we know from Tables 9 and 11, the ranking orders derived by our proposed method are slightly different from those obtained by Lu and Wei’s [57] and Wei’s [36] methods. This is because Lu and Wei’s [57] and Wei’s [36] methods are based on the simply weighted average/geometric AOs which do not consider the interrelationship between attributes. Our proposed method is based on MM (DMM) so that it has the ability to capture the interrelationships among interacted attributes. Basically, the interrelationship among attributes widely exists, while Lu and Wei’s [57] and Wei’s [36] methods assume that attributes are always independent, which is inconsistent with the reality. Our method based on the MM operator not only deals with the interrelationships among attributes, but also has the ability to manipulate the number of interacted attributes. Hence, our proposed method is more suitable for real MADM problems. For example, if $T = (1, 1, 0)$, then our method captures the interrelationships between any two attributes; If $T = (1, 1, 1)$, then our method can reflect the interrelationship among all the three attributes; If there is indeed no interrelationship between the attributes, then we can set $T = (1, 0, 0)$. Therefore, our proposed method is more powerful and flexible than Lu and Wei’s [57] and Wei’s [36] methods.

4) THE EFFICIENCY IN PORTRAYING DMS’ EVALUATION INFORMATION BOTH QUANTIFICATIONALLY AND QUALITIVELY

Basically, to appropriately express their evaluation information, DMs usually prefer using ULVs. In addition, the MDs and NMDs of ULVs provided by DMs can represent the decision judgments more accurately. Xu et al.’s [40] method is based on the IVq-RDHFSs which employ some interval-valued values to denote the possible MDs and NMDs. Obviously, the MADM method proposed by Xu et al. [40] cannot fully express DMs’ evaluation information because it ignores the qualitative decision information. Our method is based on IVq-RDHFULS which is a combination of Xu et al.’s [40] IVq-RDHFS with ULVs. It not only effectively depicts DMs’ quantitative decision information (the same as Xu et al.’s method [40]), but also portrays the qualitative evaluation information by ULVs.

E. SUMMARY

Table 13 is provided to better demonstrate the characteristics of different MADM methods. In the following, we summarize the defects of some existing MADM methods and the superiorities of our proposed method.

(1) First, Lu and Wei’s [57] MADM method is based on DHFULS in which crisp numbers are employed to denote the
possible MDs and NMDs of ULV. Our proposed method is based on IVq-RDHULSs where the MDs and NMDs of ULVs are expressed as interval values. Obviously, interval values can take more information into account than crisp numbers. Furthermore, the DHFULS requires the sum of MD and NMD to be less than or equal to one. If the DHFULS is used to describe DMs’ evaluation information, in order to meet its constraint DMs may provide some anamorphic evaluation values, which further leads to unreasonable decision results. Compared with DHFULSs, our IVq-RDHULSs have laxer constraint and so that DMs can express their preference information freely. Hence, our proposed method is better than Lu and Wei’s [57] method in information expression form.

(2) Wei’s [36] method is based on IVDHFULSs which also use interval values to denote the possible MDs and NMDs of ULVs. This characteristic is same as our proposed method. However, the constraint of IVDHFULSs is so rigorous that it may incur information distoration to some extent. Our method is based on IVq-RDHULSs and DMs have enough freedom to express their preference information. Moreover, IVDHFULSs is a special case of IVq-RDHULS. Hence, our proposed method is more powerful than Wei’s [36] MADM method.

(3) The main defect of Xu et al.’s [40] method is that DMs’ qualitative evaluation information is ignored in the whole process of MADM. In contrast, our method reflects both DMs’ quantitative and qualitative evaluation values. Therefore, our method is more suitable than Xu et al.’s [40] method in solving practical MADM problems.

(4) Lu and Wei’s [57] and Wei’s [36] MADM methods are based on the weighted average/geometric operators which fail to consider the inherent interrelationship between attributes. Our proposed MADM method is based on the MM and DMM operators so that it has the ability to capture the interrelationships among attributes. Furthermore, the information aggregation process of our method is more flexible. As a result, our method is more powerful and flexible than Wei’s [36] and Lu and Wei’s [57] MADM methods.

VI. CONCLUSION REMARKS
In this paper, we introduced a new MADM method based on IVq-RDHULWMM and IVq-RDHULWMM operators. First, the concept of IVq-RDHULSs were put forward by extending the IVq-RDHFS to linguistic environment. Then in order to appropriately aggregate IVq-RDHUL attribute values, we extended the powerful MM and DMM to IVq-RDHULSs and proposed a series of new AOs of IVq-RDHULVs. In the manuscript, some important properties of these AOs were also discussed. Based on the IVq-RDHULSs and their AOs we further introduced a new MADM method. Finally, in order to demonstrate the superiorities of this method over some recently presented methods, several numerical examples were provided, and comparative analysis was conducted. In future works, we shall investigate more applications of IVq-RDHULSs in MADM problems.

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