On the residual of a factorized group with widely supersoluble factors

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Abstract. Let \( \mathbb{P} \) be the set of all primes. A subgroup \( H \) of a group \( G \) is called \( \mathbb{P} \)-subnormal in \( G \), if either \( H = G \), or there exists a chain of subgroups \( H = H_0 \leq H_1 \leq \ldots \leq H_n = G \), \( |H_i : H_{i-1}| \in \mathbb{P} \), \( \forall i \). A group \( G \) is called widely supersoluble, w-supersoluble for short, if every Sylow subgroup of \( G \) is \( \mathbb{P} \)-subnormal in \( G \). A group \( G = AB \) with \( \mathbb{P} \)-subnormal w-supersoluble subgroups \( A \) and \( B \) is studied. The structure of its w-supersoluble residual is obtained. In particular, it coincides with the nilpotent residual of the \( A \)-residual of \( G \). Here \( A \) is the formation of all groups with abelian Sylow subgroups. Besides, we obtain new sufficient conditions for the w-supersolubility of such group \( G \).

Keywords. widely supersoluble groups, mutually \( sn \)-permutable subgroups, \( \mathbb{P} \)-subnormal subgroup, the \( X \)-residual.

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Introduction

Throughout this paper, all groups are finite and \( G \) always denotes a finite group. We use the standard notations and terminology of \([6]\). The formations of all nilpotent, supersoluble groups and groups with abelian Sylow subgroups are denoted by \( \mathfrak{N} \), \( \mathfrak{U} \) and \( \mathcal{A} \), respectively. The notation \( Y \leq X \) means that \( Y \) is a subgroup of a group \( X \) and \( \mathbb{P} \) be the set of all primes. Let \( \mathcal{X} \) be a formation. Then \( G^{\mathcal{X}} \) denotes the \( \mathcal{X} \)-residual of \( G \).
By Huppert’s Theorem [6, VI.9.5], a group $G$ is supersoluble if and only if for every proper subgroup $H$ of $G$ there exists a chain of subgroups

$$H = H_0 \leq H_1 \leq \ldots \leq H_n = G, \; |H_i : H_{i-1}| \in \mathbb{P}, \; \forall i.$$  

So naturally the following definition.

A subgroup $H$ of a group $G$ is called $\mathbb{P}$-subnormal in $G$, if either $H = G$, or there is a chain subgroups (1). We use the notation $H \mathbb{P}sn G$. This definition was proposed in [13] and besides, in this paper w-supersoluble (widely supersoluble) groups, i.e. groups with $\mathbb{P}$-subnormal Sylow subgroups, were investigated. Denote by wU the class of all w-supersoluble groups.

The factorizable groups $G = AB$ with w-supersoluble factors $A$ and $B$ were investigated in [8], [10], [11], [14]. There are many other papers devoted to study factorizable groups, and the reader is referred to the book [1] and the bibliography therein. A criteria for w-supersolvability was obtained by A. F. Vasil’ev, T. I. Vasil’eva and V. N. Tyutyunov [14].

**Theorem A.** [14, Theorem 4.7] Let $G = AB$ be a group which is the product of two w-supersoluble subgroups $A$ and $B$. If $A$ and $B$ are $\mathbb{P}$-subnormal in $G$ and $G^A$ is nilpotent, then $G$ is w-supersoluble.

We recall that two subgroups $A$ and $B$ of a group $G$ are said to be *mutually sn-permutable* if $A$ permutes with all subnormal subgroups of $B$ and $B$ permutes with all subnormal subgroups of $A$. If $A$ and $B$ are mutually sn-permutable subgroups of a group $G = AB$, then we say that $G$ is a *mutually sn-permutable product* of $A$ and $B$, see [4]. In soluble groups, mutually sn-permutable factors are $\mathbb{P}$-subnormal [14, Lemma 4.5]. The converse is not true, see the example 3.1 below.

A. Ballester-Bolinches, W. M. Fakieh and M. C. Pedraza-Aguilera [3] obtained the following results for the sn-permutable product of the w-supersoluble subgroups.

**Theorem B.** Let $G = AB$ be the mutually sn-permutable product of subgroups $A$ and $B$. Then the following hold:

1. if $A$ and $B$ are w-supersoluble and $N$ is a minimal normal subgroup of $G$, then both $AN$ and $BN$ are w-supersoluble, [3, Theorem 3];
2. if $A$ and $B$ are w-supersoluble and $(|A/A^A|, |B/B^A|) = 1$, then $G$ is w-supersoluble, [3, Theorem 5].

Present paper extends the Theorems A and B. We prove the following result.

**Theorem 1.** Let $A$ and $B$ be w-supersoluble $\mathbb{P}$-subnormal subgroups of $G$ and $G = AB$. Then the following hold:

1. $G^{wU} = (G^A)^B$;
2. if $N$ is a nilpotent normal subgroup of $G$, then both $AN$ and $BN$ are w-supersoluble;
if \(|A/A|, |B/B| = 1\), then \(G\) is \(w\)-supersoluble.

Theorem A follows from assertion (1) of Theorem 1. Theorem B follows from assertions (2) and (3) of Theorem 1 since the group \(G\) in Theorem B is soluble.

1 Preliminaries

In this section, we give some definitions and basic results which are essential in the sequel. A group whose chief factors have prime orders is called supersoluble. Recall that a \(p\)-closed group is a group with a normal Sylow \(p\)-subgroup and a \(p\)-nilpotent group is a group with a normal Hall \(p'\)-subgroup.

Denote by \(G'\), \(Z(G)\), \(F(G)\) and \(\Phi(G)\) the derived subgroup, centre, Fitting and Frattini subgroups of \(G\) respectively. We use \(E_p^t\) to denote an elementary abelian group of order \(p^t\) and \(Z_m\) to denote a cyclic group of order \(m\). The semidirect product of a normal subgroup \(A\) and a subgroup \(B\) is written as follows: \(A \rtimes B\).

Let \(\mathcal{F}\) be a formation. Recall that the \(\mathcal{F}\)-residual of \(G\), that is the intersection of all those normal subgroups \(N\) of \(G\) for which \(G/N \in \mathcal{F}\). We define \(\mathcal{X} \mathcal{Y} = \{G \in \mathcal{E} \mid G^\mathcal{Y} \in \mathcal{X}\}\) and call \(\mathcal{X} \mathcal{Y}\) the formation product of \(\mathcal{X}\) and \(\mathcal{Y}\). Here \(\mathcal{E}\) is the class of all finite groups.

If \(H\) is a subgroup of \(G\), then \(H^G = \bigcap_{x \in G} H^x\) is called the core of \(H\) in \(G\). If a group \(G\) contains a maximal subgroup \(M\) with trivial core, then \(G\) is said to be primitive and \(M\) is its primitivator.

A simple check proves the following lemma.

**Lemma 1.1.** Let \(\mathcal{F}\) be a saturated formation and \(G\) be a group. Assume that \(G \notin \mathcal{F}\), but \(G/N \in \mathcal{F}\) for all non-trivial normal subgroups \(N\) of \(G\). Then \(G\) is a primitive group.

**Lemma 1.2.** (\cite[Theorem II.3.2]{6}) Let \(G\) be a soluble primitive group and \(M\) is a primitivator of \(G\). Then the following statements hold:

1. \(\Phi(G) = 1\);
2. \(F(G) = C_G(F(G)) = O_p(G)\) and \(F(G)\) is an elementary abelian subgroup of order \(p^n\) for some prime \(p\) and some positive integer \(n\);
3. \(G\) contains a unique minimal normal subgroup \(N\) and moreover, \(N = F(G)\); 
4. \(G = F(G) \rtimes M\) and \(O_p(M) = 1\).

**Lemma 1.3.** (\cite[Proposition 2.2.8, Proposition 2.2.11]{2}) Let \(\mathcal{F}\) and \(\mathcal{H}\) be formations, \(K\) be normal in \(G\). Then the following hold:

1. \((G/K)^\mathcal{H} = G^\mathcal{H}K/K\).
(2) \( G^{\delta \circ} = (G^\delta)^\delta \);
(3) if \( H \subseteq \delta \), then \( G^\delta \leq G^\circ \);
(4) if \( G = HK \), then \( H^\delta K = G^\delta K \).

Recall that a group \( G \) is said to be siding if every subgroup of the derived subgroup \( G' \) is normal in \( G \), see [12, Definition 2.1]. Metacyclic groups, t-groups (groups in which every subnormal subgroup is normal) are siding. The group \( G = (Z_6 \times Z_2) \rtimes Z_2 \) (IdGroup(G)=[24,8]) is siding, but not metacyclic and a t-group.

**Lemma 1.4.** Let \( G \) be siding. Then the following hold:
1. If \( N \) is normal in \( G \), then \( G/N \) is siding;
2. If \( H \) is a subgroup of \( G \), then \( H \) is siding;
3. \( G \) is supersoluble

**Proof.**
1. By [6, Lemma I.8.3], \( (G/N)' = G'N/N \). Let \( A/N \) be an arbitrary subgroup of \( (G/N)' \). Then
\[
A \leq G'N, \ A = A \cap G'N = (A \cap G')N.
\]
Since \( A \cap G' \leq G' \), we have \( A \cap G' \) is normal in \( G \). Hence \( (A \cap G')N/N \) is normal in \( G/N \).
2. Since \( H \leq G \), it follows that \( H' \leq G' \). Let \( A \) be an arbitrary subgroup of \( H' \). Then \( A \leq G' \) and \( A \) is normal in \( G \). Therefore \( A \) is normal in \( H \).
3. We proceed by induction on the order of \( G \). Let \( N \leq G' \) and \( |N| = p \), where \( p \) is prime. By the hypothesis, \( N \) is normal in \( G \). By induction, \( G/N \) is supersoluble and \( G \) is supersoluble.

**Lemma 1.5.** ( [9, Lemma 3]) Let \( H \) be a subgroup of \( G \), and \( N \) be a normal subgroup of \( G \). Then the following hold:
1. If \( H \leq H/N \) \( P \)sn \( G/N \), then \( H \ P \)sn \( G \);
2. If \( H \ P \)sn \( G \), then \( (H \cap N) \ P \)sn \( N \), \( HN/N \ P \)sn \( G/N \) and \( HN \ P \)sn \( G \);
3. If \( H \leq K \leq G \), \( H \ P \)sn \( K \) and \( K \ P \)sn \( G \), then \( H \ P \)sn \( G \);
4. If \( H \ P \)sn \( G \), then \( H^g \ P \)sn \( G \) for any \( g \in G \).

**Lemma 1.6.** ( [9, Lemma 4]) Let \( G \) be a soluble group, and \( H \) be a subgroup of \( G \). Then the following hold:
1. If \( H \ P \)sn \( G \) and \( K \leq G \), then \( (H \cap K) \ P \)sn \( K \);
2. If \( H_i \ P \)sn \( G \), \( i = 1, 2 \), then \( (H_1 \cap H_2) \ P \)sn \( G \).

**Lemma 1.7.** ( [9, Lemma 5]) If \( H \) is a subnormal subgroup of a soluble group \( G \), then \( H \) is \( P \)-subnormal in \( G \).
Lemma 1.8. ([13, Theorem 2.7]) The class \( wU \) is a hereditary saturated formation.

Lemma 1.9. (1) If \( G \in wU \), then \( G^A \) is nilpotent, [13, Theorem 2.13].

(2) \( G \in wU \) if and only if every metanilpotent subgroup of \( G \) is supersoluble, [7, Theorem 2.6].

(3) \( G \in wU \) if and only if \( G \) has a Sylow tower of supersoluble type and every biprimary subgroup of \( G \) is supersoluble, [9, Theorem B].

2 Factorizable groups with \( \mathbb{P} \)-subnormal \( w \)-supersoluble subgroups

Lemma 2.1. ([14, Theorem 4.4]) Let \( A \) and \( B \) be \( \mathbb{P} \)-subnormal subgroups of \( G \), and \( G = AB \). If \( A \) and \( B \) have an ordered Sylow tower of supersoluble type, then \( G \) has an ordered Sylow tower of supersoluble type.

Proof of Theorem 1 (1). If \( G \) is \( w \)-supersoluble, then \( G^{wU} = 1 \) and \( G^A \) is nilpotent by Lemma 1.9(1). Consequently \( G^{wU} = 1 = (G^A)^{\mathfrak{N}} \) and the statement is true. Further, we assume that \( G \) is non-\( w \)-supersoluble. Since \( wU \subseteq \mathfrak{N}A \), it follows that

\[
G^{(\mathfrak{N}A)} = (G^A)^{\mathfrak{N}} \leq G^{wU}
\]

by Lemma 1.3(2-3). Next we check the converse inclusion. For this we prove that \( G/(G^A)^{\mathfrak{N}} \) is \( w \)-supersoluble. By Lemma 1.3(1),

\[
(G/(G^A)^{\mathfrak{N}})^A = G^A(G^A)^{\mathfrak{N}}/(G^A)^{\mathfrak{N}} = G^A/(G^A)^{\mathfrak{N}}
\]

and \((G/(G^A)^{\mathfrak{N}})^A\) is nilpotent. The quotients

\[
G/(G^A)^{\mathfrak{N}} = (A(G^A)^{\mathfrak{N}}/(G^A)^{\mathfrak{N}})(B(G^A)^{\mathfrak{N}}/(G^A)^{\mathfrak{N}}),
\]

\[
A(G^A)^{\mathfrak{N}}/(G^A)^{\mathfrak{N}} \cong A/A \cap (G^A)^{\mathfrak{N}},
\]

\[
B(G^A)^{\mathfrak{N}}/(G^A)^{\mathfrak{N}} \cong B/B \cap (G^A)^{\mathfrak{N}},
\]

hence the subgroups \( A(G^A)^{\mathfrak{N}}/(G^A)^{\mathfrak{N}} \) and \( B(G^A)^{\mathfrak{N}}/(G^A)^{\mathfrak{N}} \) are \( w \)-supersoluble by Lemma 1.8 and by Lemma 1.5(2), they are \( \mathbb{P} \)-subnormal in \( G/(G^A)^{\mathfrak{N}} \). By Theorem A, \( G/(G^A)^{\mathfrak{N}} \) is \( w \)-supersoluble. \( \square \)

Lemma 2.2. Let \( G \) be a group, and \( A \) be a subgroup of \( G \) such that \( |G : A| = p^\alpha \), where \( p \in \pi(G) \) and \( \alpha \in \mathbb{N} \). Suppose that \( A \) is \( w \)-supersoluble and \( \mathbb{P} \)-subnormal in \( G \). If \( G \) is \( p \)-closed, then \( G \) is \( w \)-supersoluble.
Proof. Let $P$ be a Sylow $p$-subgroup of $G$. Since $P$ is normal in $G$ and $G = AP$, we have $G/P \cong A/A \cap P \in w\mathfrak{U}$, in particular, $G$ is soluble. Because $G$ is soluble, it follows that $P$ is $\mathfrak{P}$-subnormal in $G$ by Lemma 1.7. Let $Q$ be a Sylow $q$-subgroup of $G$, $q \neq p$. Then $Q \leq A^x$ for some $x \in G$. By Lemma 1.5(4), $A^x$ is $\mathfrak{P}$-subnormal in $G$. Since $A^x \in w\mathfrak{U}$, it follows that $Q$ is $\mathfrak{P}$-subnormal in $A^x$ and $Q$ is $\mathfrak{P}$-subnormal in $G$ by Lemma 1.5(3). So, $G$ is $w$-supersoluble.

Lemma 2.3. Let $A$ and $B$ be $w$-supersoluble $\mathfrak{P}$-subnormal subgroups of $G$, and $G = AB$. Suppose that $|G : A| = p^a$, where $p \in \pi(G)$. If $p$ is the greatest in $\pi(G)$, then $G$ is $w$-supersoluble.

Proof. Since every $w$-supersoluble group has an ordered Sylow tower of supersoluble type, then by Lemma 2.1, $G$ has an ordered Sylow tower of supersoluble type. Hence $G$ is $p$-closed. By Lemma 2.2 we have that $G$ is $w$-supersoluble.

Theorem 2.1. Let $A$ be a $w$-supersoluble $\mathfrak{P}$-subnormal subgroup of $G$, and $G = AB$. Then $G$ is $w$-supersoluble in each of the following cases:

1. $B$ is nilpotent and normal in $G$;
2. $B$ is nilpotent and $|G : B|$ is prime;
3. $B$ is normal in $G$ and is a siding group.

Proof. We prove all three statements at the same time using induction on the order of $G$. Note that $G$ is soluble in any case. By Lemma 1.7, $B$ is $\mathfrak{P}$-subnormal in $G$ and $G$ has an ordered Sylow tower of supersoluble type by Lemma 2.1. If $N$ is a non-trivial normal subgroup of $G$, then $AN/N$ is $\mathfrak{P}$-subnormal in $G/N$ by Lemma 1.5(2) and $AN/N \cong A/A \cap N$ is $w$-supersoluble by Lemma 1.8. The subgroup $BN/N \cong B/B \cap N$ is nilpotent or a siding group by Lemma 1.4(1). Hence $G/N = (AN/N)(BN/N)$ is $w$-supersoluble by induction. Since the formation of all $w$-supersoluble groups is saturated by Lemma 1.8, we have $G$ is a primitive group by Lemma 1.1. By Lemma 1.2, $F(G) = N = G_p$ is a unique minimal normal subgroup of $G$ and $N = C_G(N)$, where $p$ is the greatest in $\pi(G)$.

Since $A$ is $\mathfrak{P}$-subnormal in $G$, it follows that $G$ has a subgroup $M$ such that $A \leq M$ and $|G : M|$ is prime. By Dedekind’s identity, $M = A(M \cap B)$. The subgroup $A$ is $\mathfrak{P}$-subnormal in $M$. The subgroup $M \cap B$ satisfies the requirements (1)–(3). By induction, $M$ is $w$-supersoluble.

1. If $B$ is nilpotent and normal in $G$, then $B = N$. Hence $G = AN$ and $A$ is a maximal subgroup of $G$. Since $A$ is $\mathfrak{P}$-subnormal in $G$, we have $|G : A| = p = |N|$ and $G$ is supersoluble. Therefore $G$ is $w$-supersoluble. So, in (1), the theorem is proved.
2. Let $B$ be nilpotent and $|G : B| = q$, where $q$ is prime. Besides, let $|G : M| = r$, where $r$ is prime. If $q \neq r$, then $(|G : M|, |G : B|) = 1$. Since $G = MB$, $M$ and $B$ are $P$-subnormal in $G$ and w-supersoluble, it follows obviously that $G$ is w-supersoluble. Hence $q = r$. If $q = p$, then $N$ is not contained in $M$. Thus $G = N \rtimes M$ and $|N|$ is prime. Consequently $G$ is supersoluble and therefore $G$ is w-supersoluble. So, $q \neq p$. Then $G_p = N \leq M \cap B$. Since $B$ is nilpotent, $G_p = B \leq M$. Because $G = MB$, we have $G = M$, a contradiction. So, in (2), the theorem is proved.

3. Let $B$ is normal in $G$ and is a siding group. If $B$ is nilpotent, then $G$ is w-supersoluble by (1). Hence $B' \neq 1$. Because $B'$ is normal in $G$ and nilpotent, we have $N = B'$. If $N$ is not contained in $M$, then $G = N \rtimes M$ and $|N|$ is prime. Consequently $G$ is supersoluble and therefore $G$ is w-supersoluble. Let $N$ be contained in $M$ and $N_1$ be a subgroup of prime order of $N$ such that $N_1$ is normal in $M$. Then $N_1$ is normal in $B$ by definition of siding group. Hence $N_1$ is normal in $G$. Consequently $G$ is w-supersoluble. So, in (3), the theorem is proved.

Proof of Theorem 1 (2).

Note that by the Lemma 2.1, $G$ is soluble. By Theorem 2.1 (1), Theorem 1 (2) is true.

Proof of Theorem 1 (3). Assume that the claim is false and let $G$ be a minimal counterexample. By Lemma 2.1 $G$ has an ordered Sylow tower of supersoluble type. If $N$ is a non-trivial normal subgroup of $G$, then $AN/N$ and $BN/N$ are $P$-subnormal in $G/N$ by Lemma 1.5 (2). Besides, $AN/N \simeq A/A \cap N$ and $BN/N \simeq B/B \cap N$ are w-supersoluble by Lemma 1.8. By Lemma 1.3 we have

$$(|(AN/N)/(AN)^A|, |(BN/N)/(BN)^A|) =$$

$$= (|AN/(AN)^A|, |BN/(BN)^A|) =$$

$$= (|AN/A^A|, |BN/B^A|) = \left( \frac{|A/A^A|}{|S_1|}, \frac{|B/B^A|}{|S_2|} \right),$$

$S_1 = (A \cap N)/(A \cap N)$, $S_2 = (B \cap N)/(B \cap N)$.

Since $(|A/A^A|, |B/B^A|) = 1$, it follows that

$$(|(AN/N)/(AN)^A|, |(BN/N)/(BN)^A|) = 1.$$ The quotient $G/N = (AN/N)(BN/N)$ is w-supersoluble by induction.

Since the formation of all w-supersoluble groups is saturated by Lemma 1.8, we have $G$ is a primitive group by Lemma 1.1. By Lemma 1.2, $F(G) = N =
G_p is a unique minimal normal subgroup of G and N = C_G(N), where p is the greatest in π(G).

By Lemma 2.2, AN is w-supersoluble. If AN = G, then G is w-supersoluble, a contradiction. Hence in the future we consider that AN and BN are proper subgroups of G.

By Lemma 1.9 (1), (AN)^A is nilpotent. Since N = C_G(N), we have (AN)^A is a p-group. Because AN/(AN)^A ∈ A, it follows that all Sylow r-subgroups of A are abelian, r ≠ p. Since A_p ≤ G_p, where A_p is a Sylow p-subgroup of A, we have A ∈ A. Similarly, B ∈ A. Hence A^A = 1 = B^A and (|A|, |B|) = (|A/A^A|, |B/B^A|) = 1. It is clear that G is w-supersoluble, a contradiction. □

3 Examples

The following example shows that for a soluble group G = AB the mutually sn-permutability of subgroups A and B doesn’t follow from P-subnormality of these factors.

Example 3.1. The group G = S_3 ⋊ Z_3 (IdGroup=[18,3]) has P-subnormal subgroups A ≃ E_3^2 and B ≃ Z_2. However A and B are not mutually sn-permutable.

The following example shows that we cannot omit the condition «G is p-closed» in Lemma 2.2.

Example 3.2. The group G = (S_3 × S_3) ⋊ Z_2 (IdGroup=[72,40]) has P-subnormal supersoluble subgroups A ≃ Z_3 × S_3. Besides |G : A| = 2^2 and Sylow 2-subgroup is maximal in G. Hence G is non-w-supersoluble.

The following example shows that in Theorem 2.1 (1) the normality of subgroup B cannot be weakened to P-subnormality.

Example 3.3. The group G = (Z_2 × (E_3^2 × Z_4)) ⋊ Z_2 (IdGroup=[144,115]) is non-w-supersoluble and factorized by subgroups A = D_{12} and B = Z_{12}. The subgroup A has the chain of subgroups A < S_3 × S_3 < Z_2 × S_3 × S_3 < G and B has the chain of subgroups B < Z_3 × (Z_3 × Z_4) < (Z_3 × (Z_3 × Z_4)) ⋊ Z_2 < G. Therefore A and B are P-subnormal in G.

The following example shows that in Theorem 2.1 (2) it is impossible to weak the restrictions on the index of subgroup B.
Example 3.4. The alternating group $G = A_4$ is non-w-supersoluble and factorized by subgroups $A = E_2$ and $B = Z_3$. It is clear that $A$ is supersoluble and $\mathbb{P}$-subnormal in $G$, and $B$ is nilpotent and $|G : B| = 2^2$. The group $G = E_2 \rtimes Z_3$ is non-w-supersoluble and has a nilpotent subgroup $Z_3$ of index $5^2$. Therefore even for the greatest $p$ of $\pi(G)$, the index of $B$ cannot be equal $p^\alpha$, $\alpha \geq 2$.

The following example shows that in Theorem 2.1 (3) the normality of subgroup $B$ cannot be weakened to subnormality.

Example 3.5. The group $G = Z_3 \times ((S_3 \times S_3) \rtimes Z_2)$ (IdGroup=[216,157]) is non-w-supersoluble and factorized by $\mathbb{P}$-subnormal supersoluble subgroup $A \simeq S_3 \times S_3$ and subnormal siding subgroup $B \simeq Z_3 \times Z_3 \times S_3$.

References

[1] Ballester-Bolinches, A., Esteban-Romero, R., Asaad, M. (2010). Products of finite groups. Berlin; New York: Walter de Gruyter.

[2] Ballester-Bolinches, A., Ezquerro, L.M. (2006). Classes of Finite Groups. Dordrecht: Springer.

[3] Ballester-Bolinches, A., Fakieh, W.M., Pedraza-Aguilera, M.C. (2019). On Products of Generalised Supersoluble Finite Groups. Mediterr. J. Math. 16:46.

[4] Carocca, A. (1998). On factorized finite groups in which certain subgroups of the factors permute. Arch. Math. 71:257–262.

[5] GAP (2019) Groups, Algorithms, and Programming, Version 4.10.2. www.gap-system.org.

[6] Huppert, B. (1967). Endliche Gruppen. Berlin: Springer-Verlag.

[7] Monakhov, V.S. (2016). Finite groups with abnormal and $\mathfrak{A}$-subnormal subgroups. Siberian Math. J. 57:352–363.

[8] Monakhov, V.S., Chirik, I.K. (2017). On the supersoluble residual of a product of subnormal supersoluble subgroups. Siberian Math. J. 58:271–280.

[9] Monakhov, V. S., Kniahina, V. N. (2013). Finite group with $\mathbb{P}$-subnormal subgroups. Ricerche Mat. 62:307–323.
[10] Monakhov, V. S., Trofimuk, A. A. (2019). Finite groups with two supersoluble subgroups. *J. Group Theory.* 22:297–312.

[11] Monakhov, V. S., Trofimuk, A. A. (2020). On supersolubility of a group with seminormal subgroups. *Siberian Math. J.* 61:118–126.

[12] Perez, E. R. (1999). On products of normal supersoluble subgroups. *Algebra Colloq.* 6:341–347.

[13] Vasil’ev, A. F., Vasil’eva, T. I., Tyutyanov, V. N. (2010). On the finite groups of supersoluble type. *Siberian Math. J.* 51:1004–1012.

[14] Vasil’ev, A. F., Vasil’eva, T. I., Tyutyanov, V. N. (2012). On the products of P-subnormal subgroups of finite groups. *Siberian Math. J.* 53:47–54.