Symmetry and Surface Symmetry Energies in Finite Nuclei

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A study of properties of the symmetry energy of nuclei is presented based on density functional theory. Calculations for finite nuclei are given so that the study includes isospin dependent surface symmetry considerations as well as isospin independent surface effects. Calculations are done at both zero and non-zero temperature. It is shown that the surface symmetry energy term is the most sensitive to the temperature while the bulk energy term is the least sensitive. It is also shown that the temperature dependence terms are insensitive to the force used and even more insensitive to the existence of neutron skin. Results for a symmetry energy with both volume and surface terms are compared with a symmetry energy with only volume terms along the line of β stability. Differences of several MeV are shown over a good fraction of the total mass range in A. Also given are calculations for the bulk, surface and Coulomb terms.

I. INTRODUCTION

Moderate to heavy nuclei have neutron excesses due to the growth of the Coulomb energy. The neutron/proton ratio N/Z drifts to higher values with increasing A with a value ∼ 1.5 near A = 200. The isospin index I = (N−Z)/A varies with neutron - proton difference and reaches a value ∼ 40/200 = 0.2 near Pb and somewhat higher for the very heaviest nuclei. For I ≠ 0 the binding energy of nuclei then acquires an isospin asymmetric part with a dependence that is I²A for the volume term. In finite nuclear systems a surface symmetry energy is also present and has a contribution with dependence I²A²/3. The energy per particle has a well known expansion in terms of a liquid droplet model known as the Weizsacker mass formula [1–7]. The nuclear symmetry energy along with the Coulomb energy determines the nuclear β stability line. Nuclei away from the valley of β stability will be explored in future rare isotope or exotic beam accelerator experiments. Systems with much higher neutron excess are encountered only in neutron stars which can have values of I approaching unity. In neutron stars only the volume part of the symmetry energy is important. However, extracting the volume part of the symmetry energy in finite nuclei requires an analysis of both the volume symmetry and surface symmetry energy terms. An analysis of the division of the symmetry energy into volume and surface terms should lead to a better extrapolation of the symmetry energy to the limits involved in neutron stars.

The isospin dependence of binding energies arises from the underlying isospin structure of the nuclear force between nucleons [7]. In particular the force has terms involving the isospin operator ⃗τ in the isospin conserving form ⃗τ · ⃗τ which arises from the exchange of isovector mesons. Isospin symmetry is broken basically by the Coulomb force [8, 9]. A small amount of isospin breaking is also present in the nuclear interaction. Nuclear properties explored with heavy ion collisions result in systems that are not at temperature T = 0. Non-zero temperatures bring in entropy considerations in hadronic systems. Specifically, the Helmholtz free energy F = E − TS becomes an important thermodynamic function with natural variables of volume V and temperature T. Nuclear phase transitions are governed by the Helmholtz free energy where energy factors and entropy terms compete to determine the cluster distribution seen experimentally [10–11]. Isospin properties appear in phase transitions in two component nuclear systems. An important example is isospin factorization [12,13] where the dilute gas phase prefers a much larger neutron excess than the denser liquid phase. The Coulomb force also plays a role in isospin factorization just as it does in determining the nuclear stability line [14–17]. Similarly, the phase diagram of nuclei is a function of temperature, density and proton fraction which are governed by the interplay of Coulomb and symmetry terms [15–17]. An extensive study of the role of isospin in heavy ion collisions can be found [18, 19]. Discussions of the symmetry and surface symmetry energy and how to place limits on the coefficients associated with them can be found in Ref. [20–22]. The role of isospin asymmetry and symmetry energies in nuclear astrophysics are extensively discussed in Ref. [22].

This paper is devoted to a study of nuclear energies using density functional theory based on a Skyrme interaction. The study involves the following features. A). The nuclear system is finite so that both volume and surface terms appear. B). An N ≠ Z asymmetry is present so that the more general case of isospin I ≠ 0 is considered with volume and surface symmetry energies both present. C). The proton component is charged generating a Coulomb interaction. D). The system is allowed to be at temperature T ≠ 0 and entropy features are present. The nuclear system is therefore non-degenerate. However, the temperature is low enough so that an expansion around the degenerate limit can be used. In Sect.2 the basic relations are summarized. The temperature dependence of nuclear energy of finite...
nuclei are discussed in Sect.3 and concluded in Sect.4. Appendix A gives some semi-analytic expressions for integrals used in the density functional theory.

II. FINITE TEMPERATURE DENSITY FUNCTIONAL THEORY AND THE SYMMETRY AND SURFACE SYMMETRY ENERGIES IN NUCLEI.

A density functional theory will be used to study properties of finite nuclei at non-zero temperature and also at temperature \( T = 0 \) as a limiting situation. We will limit the temperatures to the low temperature regime so that thermodynamic functions can be expanded about the degenerate limit and the \( T \to 0 \) can be done. The high temperature limit is an expansion about an ideal gas and is appropriate for studies of the liquid-gas phase transition [15,17,23]. In this density functional approach a Skyrme Hamiltonian is used which is

\[
H(\vec{r}) = H_B(\vec{r}) + H_S(\vec{r}) + H_C(\vec{r})
\]

\[
H_B = \frac{\hbar^2}{2m_p} \tau_p + \frac{\hbar^2}{2m_n} \tau_n + \frac{1}{4} \left[ t_1 \left( 1 + \frac{x_1}{2} \right) + t_2 \left( 1 + \frac{x_2}{2} \right) \right] \rho \tau - \frac{1}{4} \left[ t_1 \left( \frac{1}{2} + x_1 \right) - t_2 \left( \frac{1}{2} + x_2 \right) \right] \left( \rho_p \tau_p + \rho_n \tau_n \right)
\]

\[
+ \frac{t_0}{2} \left( 1 + \frac{x_0}{2} \right) \rho^2 - \left( \frac{1}{2} + x_0 \right) \left( \rho_p^2 + \rho_n^2 \right)
\]

\[
+ \frac{t_3}{12} \left( 1 + \frac{x_3}{2} \right) \rho^2 - \left( \frac{1}{2} + x_3 \right) \left( \rho_p^2 + \rho_n^2 \right) \rho^0
\]

\[
H_S = \frac{1}{16} \left[ 3t_1 \left( 1 + \frac{x_1}{2} \right) - t_2 \left( 1 + \frac{x_2}{2} \right) \right] (\nabla \rho)^2 - \frac{1}{16} \left[ 3t_1 \left( \frac{1}{2} + x_1 \right) + t_2 \left( \frac{1}{2} + x_2 \right) \right] (\nabla \rho_p)^2 + (\nabla \rho_n)^2
\]

\[
= \frac{1}{16} \left[ 3t_1 \left( 1 + \frac{x_1}{2} \right) - t_2 \left( 1 + \frac{x_2}{2} \right) \right] \rho \nabla^2 \rho + \frac{1}{16} \left[ 3t_1 \left( \frac{1}{2} + x_1 \right) + t_2 \left( \frac{1}{2} + x_2 \right) \right] \left( \rho_p \nabla^2 \rho_p + \rho_n \nabla^2 \rho_n \right)
\]

\[
H_C = \frac{\varepsilon^2}{2} \rho_p(\vec{r}) \int d^3r' \frac{\rho_p(\vec{r}')}{|\vec{r} - \vec{r}'|} - \frac{3e^2}{4} \left( \frac{3}{\pi} \right)^{1/3} \rho_p^{4/3}(\vec{r})
\]

(1)

The \( H(\vec{r}) \) has a bulk part \( H_B(\vec{r}) \), a surface part \( H_S(\vec{r}) \) with gradient terms and a Coulomb term \( H_C(\vec{r}) \). The gradient terms are important in finite nuclei and the Coulomb term is important for the charged proton component. The \( t_0, t_1, t_2, t_3 \) and \( x_0, x_1, x_2, x_3 \) are parameters. Different choices of these parameters give rise to different Skyrme interactions. Here, we consider two Skyrme interactions, SKM\( (m^* = m) \) and SLy4. These two Skyrme interaction have parameter sets given in Table I of Ref.[23]. The \( m^* \) is the effective mass which is given by

\[
\frac{m_q}{m_q} = 1 + \frac{2m_q}{\hbar^2} \left\{ \frac{1}{4} \left[ t_1 \left( 1 + \frac{x_1}{2} \right) + t_2 \left( 1 + \frac{x_2}{2} \right) \right] \rho - \frac{1}{4} \left[ t_1 \left( \frac{1}{2} + x_1 \right) - t_2 \left( \frac{1}{2} + x_2 \right) \right] \rho_q \right\}
\]

\[
= 1 + \frac{2m_q}{\hbar^2} \left\{ \frac{1}{16} \left[ 3t_1 + (5 + 4x_2) t_2 \right] \rho + \frac{1}{8} \left[ t_1 \left( \frac{1}{2} + x_1 \right) - t_2 \left( \frac{1}{2} + x_2 \right) \right] \rho_q (2y - 1) \right\}
\]

(2)

At low \( T \) or high density, the nearly degenerate proton and neutron Fermi gases have

\[
\tau_q(\vec{r}) = \frac{2m}{\hbar^2} \varepsilon_{K_q} = \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \frac{\rho_q^{5/3} + 5\pi^2 m_q^{*2}}{3\hbar^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} \rho_q^{1/3} T^2 + \cdots \right]
\]

(3)

The first term in square bracket is the degenerate limit and the \( T^2 \) term is the finite temperature correction. The \( \varepsilon_{K_q} \) is the kinetic energy density, where \( q = p \) for protons and \( q = n \) for neutrons.

Since our calculations are done at finite \( T \), entropy \( S \) and Helmholtz free energy \( F \) become important quantities with the connection to the energy \( E \) through \( F = E - TS \). The entropy density is

\[
TS = \sum_q \frac{h^2}{2m_q^*} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \frac{\pi^2 m_q^{*2}}{\hbar^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} \rho_q^{1/3} T^2 + \cdots \right]
\]

(4)

for low \( T \) or high density to first order in the expansion about the degenerate limit. The density distribution used in the evaluations presented in this work is

\[
\rho_q(\vec{r}) = \frac{\rho_{qc}}{1 + e^{(r-R)/a}}
\]

(5)
In our evaluation we will first take the proton and neutron radii \( R \) to be the same and also the diffuseness parameter \( a \) to be the same. The central density parameter \( \rho_{pc} \) and \( \rho_{nc} \) are determined to give the correct number of proton \( Z \) and neutron \( N \). With this choice of the density we can integrate the energy density of Eq. (11) explicitly (see Appendix) and thus the energy \( E(A, Z, T) \) becomes a function of nuclear size \( R \) for a fixed value of diffuseness parameter \( a \). For each nucleus with \( Z \) protons and \( N \) neutrons at a temperature \( T \) the nuclear size \( R \) can be determined by minimizing the Helmholtz free energy. We will also compare this situation with a case where the central densities \( \rho_{pc} \) are the same but the proton and neutron radii, \( R_p \) and \( R_n \), are different.

The main quantity of interest here will be

\[
E(A, Z, T) = -B(T)A + E_S(T)A^{2/3} + S_V(T)I^2 A + S_S(T)I^2 A^{2/3}
\]
\[
+ E_C A^{1/3} + E_{dif} + E_{ex} A^{1/3} + c \Delta A^{-1/2}
\]

where \( I = (N - Z)/A = (A - 2Z)/A \). The \( E_{dif} \) and \( E_{ex} \) are the coefficients for the diffuseness correction and the exchange correction to the Coulomb energy. For the pairing correction with constant \( \Delta \), \( c = +1 \) for odd-odd nuclei, 0 for odd-even nuclei, and -1 for even-even nuclei. The above formula at \( T = 0 \) is the well known Weizacker semiempirical mass formula [1, 24, 25] studied extensively by Myers and Swiatecki [26–29]. Early studies excluded the surface symmetry term \( S_S \) and only the surface term \( E_S \) was included. The values of the coefficients as found in textbooks such as Ref. [3] are \( B(0) \approx 16, E_S(0) \approx 17 \), and \( S_V(0) \approx 24 \) in MeV. The ratio \( E_S/B \) of surface to bulk energy at \( T = 0 \) is very close to unity.

With regard to the volume symmetry term about \( \frac{1}{2} \) the numerical value of the coefficient \( S_V(T = 0) \) comes from kinetic energy considerations for two degenerate Fermi gases of proton and neutrons. Specifically, the kinetic energy contribution is simply related to the Fermi energy \( E_F \) as

\[
S_V(0)_{kin} = \frac{1}{3} E_F \approx 12 \text{MeV}
\]

The other \( \frac{1}{2} \) of the symmetry energy coefficient arises from interaction terms. For systems with a neutron excess, the neutrons and protons experience interaction potentials. In an independent particle model the average potential, (called a Lane potential, see Ref. [7]), felt by a neutron or proton differ. The potential is written in the form

\[
V = V_0 + \frac{1}{2} N - Z A V_1
\]

with \( t_z = 1/2 \) for neutrons and \( t_z = -1/2 \) for protons. A total potential energy can be obtained by summing the one body potential energy and taking \( \frac{1}{2} \) of the result to give

\[
V_{pot} = \frac{1}{2} \left( N + Z \right) V_0 + \frac{1}{8} \frac{N - Z}{A} V_1 N - \frac{1}{8} \frac{N - Z}{A} V_1 Z = \frac{1}{2} A V_0 + \frac{1}{8} \frac{(N - Z)^2}{A} V_1
\]

A depth \( V_1 = 96 \text{MeV} \) gives \( S_V(0)_{pot} = V_1/8 = 12 \text{MeV} \), the remaining part of the symmetry energy coefficient \( S_V(0) = 24 \text{MeV} \). More recently, the importance of a surface symmetry term has been noted. Several calculations at \( T = 0 \) have been presented regarding this term [21–22]. Considerable variation in the ratio of the \( T \) independent surface to volume symmetry energy with the difference \( R_n - R_p \) were noted [22]. Here our focus will be on the \( T \)-dependent features which arises from the kinetic energy terms and effective mass terms.

The Coulomb energy \( E_{Coul} \) for a proton distribution of Eq. (5) (see p.145 and p.160 of Ref. [7]) is

\[
E_{Coul} = \int d^3r H_C(r) = \frac{3}{5} \frac{e^2 Z^2}{R} \left[ 1 - \frac{\left( 7\pi^2 \right)^{1/2}}{6} \frac{\left( a/R \right)^2}{\pi^2} \right] - \frac{3}{4} \frac{\pi^{1/3}}{\rho_p^{4/3}} \frac{4\pi}{3} R^3
\]

\[
= \frac{3}{5} \frac{e^2}{r_0} \frac{Z^2}{A^{1/3}} - \frac{7}{10} \frac{\pi^2}{a^2} \frac{Z^2}{r_0 A} - \frac{3}{4} \frac{e^2}{r_0} \frac{\left( 3 \frac{Z^{2/3}}{2} \right)}{A^{1/3}}
\]

with \( R = r_0 A^{1/3} \). For \( a = 0.53 \text{fm} \) and \( r_0 = 1.25 \text{fm} \), \( E_{Coul} = 0.6912, E_{dif} = -1.43081 \), and \( E_{ex} = -0.5278064 \) in MeV unit.

**III. TEMPERATURE DEPENDENCE OF ENERGY OF FINITE NUCLEI**

To obtain the temperature dependence of the various coefficients, the Helmholtz free energy is minimized for various nuclei along \( \beta \) stability line with \( Z \) protons and \( N \) neutrons from \( ^{20}\text{Ne} \) to \( ^{208}\text{Pb} \). The Eq. (10) is then used to obtain the
expansion coefficients of energy. The results for two different Skyrme interactions, SKM($m^* = m$) with no effective mass and SLy4 with a density dependent effective mass of $m^*/m = 0.7$ at nuclear matter density [23, 30], are presented in Table I.

The central density $\rho_c$ for SKM($m^* = m$) has a range of $0.14595 \sim 0.16697$ fm$^{-3}$ at $T = 0$, $0.14556 \sim 0.16618$ fm$^{-3}$ at $T = 1$ MeV, $0.14446 \sim 0.16369$ fm$^{-3}$ at $T = 2$ MeV, $0.14254 \sim 0.15944$ fm$^{-3}$ at $T = 3$ MeV, $0.13973 \sim 0.15290$ fm$^{-3}$ at $T = 4$ MeV, and $0.13586 \sim 0.14327$ fm$^{-3}$ at $T = 5$ MeV with maximum for $^{208}$Pb at $30\%$ and minimum for $^{40}$Ca at $1\%$. The central density $\rho_c$ for SLy4 has a range of $0.14286 \sim 0.14886$ fm$^{-3}$ at $T = 0$, $0.14223 \sim 0.14799$ fm$^{-3}$ at $T = 1$ MeV, $0.14017 \sim 0.14542$ fm$^{-3}$ at $T = 2$ MeV, $0.13664 \sim 0.14091$ fm$^{-3}$ at $T = 3$ MeV, $0.13117 \sim 0.13392$ fm$^{-3}$ at $T = 4$ MeV with minimum for $^{208}$Pb and maximum for $^{64}$Zn at $3\%$. At $T = 5$ MeV, it has a range of $0.12302 \sim 0.12519$ with maximum for $^{116}$Sn and minimum for $^{208}$Pb at $30\%$ and $\rho_c = 0.09748$ for $^{208}$Pb and $\rho_c = 0.11736$ for $^{40}$Ca. As the temperature $T$ increases, the central density $\rho_c$ which minimizes the free energy $F$ decreases and thus the nuclear size $R$ becomes larger. This indicates that no nucleus can be bound with a minimum value of $F$ at high temperature due to the entropy which increases as $R$ increases. For $^{208}$Pb with SLy4 parameter, there was no $\rho_c$ or $R$ value which minimizes $F$ at $T \leq 5.3$ MeV. The density $\rho_c$ which minimizes the free energy $F$ is sensitive to the interaction used.

The results of Table I, from the values of “$T$-indep.” and “$T^2$ term” which are approximately $T$ independent, can be summarized approximately at low $T$ as

\[
\begin{align*}
B(T) &= 15.310 - 0.0394T^2 \\
E_S(T) &= 18.303 + 0.501T^2 \\
S_V(T) &= 19.685 + 0.423T^2 \\
S_S(T) &= -33.176 - 2.055T^2
\end{align*}
\]

in MeV unit for SKM($m^* = m$) and

\[
\begin{align*}
B(T) &= 15.308 - 0.01157T^2 \\
E_S(T) &= 20.008 + 0.550T^2 \\
S_V(T) &= 31.113 + 0.548T^2 \\
S_S(T) &= -41.035 - 2.595T^2
\end{align*}
\]

in MeV unit for SLy4. When the Helmholtz free energy $F$ is minimized, the $T$ dependence of energy $E(A, Z, T)$ at

| TABLE I: Energy coefficient minimizing free energy in MeV unit. The rows labeled with “$T^2$ term” are the expansion of the coefficient of the explicit $T^2$-dependent term of the kinetic energy coming from the $T^2$ term in Eq. (11), the kinetic energy term, and the rows labeled with “$T$-indep.” are the expansion of the remainders in $E(A, Z, T)$. The rows labeled with “$T^2(R_n \neq R_p)$” are for $R_n \neq R_p$ and are explained in text. All other rows are for $R_n = R_p$. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $T$ (MeV) | $B(T)$ | $E_S(T)$ | $S_V(T)$ | $S_S(T)$ | $B(T)$ | $E_S(T)$ | $S_V(T)$ | $S_S(T)$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0               | 15.310          | 18.303          | 19.685          | -33.176         | 15.308          | 20.008          | 31.113          | -41.035         |
| 1               | 15.270          | 18.003          | 19.655          | -33.167         | 15.296          | 20.024          | 31.064          | -41.095         |
| 2               | 15.152          | 17.851          | 19.604          | -33.220         | 15.264          | 20.049          | 31.015          | -41.155         |
| 3               | 14.955          | 17.685          | 19.532          | -33.430         | 15.217          | 20.099          | 31.065          | -41.215         |

For $T$-indep. | $B(T)$ | $E_S(T)$ | $S_V(T)$ | $S_S(T)$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| $T^2$ term | 0.039416 | 0.50052 | 0.42287 | -0.04200 |
| $T^2(R_n \neq R_p)$ | 0.039465 | 0.50314 | 0.43643 | 0.43658 |
| $E_S(T)$ | 18.303 | 18.304 | 18.311 | 18.340 |
| $T^2$ term | 0.50052 | 0.50314 | 0.50729 | 0.50738 |
| $T^2(R_n \neq R_p)$ | 0.50037 | 0.50240 | 0.50738 | 0.50738 |
| $S_V(T)$ | 19.685 | 19.685 | 19.699 | 19.745 |
| $T^2$ term | 0.42287 | 0.42898 | 0.43643 | 0.43658 |
| $T^2(R_n \neq R_p)$ | 0.43200 | 0.43658 | 0.44579 | 0.44579 |
| $S_S(T)$ | -33.176 | -33.167 | -33.220 | -33.430 |
| $T^2$ term | -0.20551 | -0.20858 | -0.21230 | -0.21460 |
| $T^2(R_n \neq R_p)$ | -2.0790 | -2.1088 | -2.1460 | -2.1460 |
low $T$ becomes

$$E(A, Z, T) = -(15.31 - 0.04T^2)A + (18.30 + 0.50T^2)A^{2/3} + (19.69 + 0.42T^2)I^2A$$
$$- (33.18 + 2.06T^2)I^2A^{2/3} + E_c Z^2 A^{1/3} + E_{dif} Z^2 A + E_{ex} Z^4 A^{1/3} + c\Delta A^{-1/2}$$

(13)

$$E(A, Z, T) = -(15.31 - 0.11T^2)A + (20.01 + 0.55T^2)A^{2/3} + (31.11 + 0.55T^2)I^2A$$
$$- (41.04 + 2.59T^2)I^2A^{2/3} + E_c Z^2 A^{1/3} + E_{dif} Z^2 A + E_{ex} Z^4 A^{1/3} + c\Delta A^{-1/2}$$

(14)

for SKM($m^*=m$) and SLy4 respectively in MeV unit.

The temperature dependence of energy Eq. (5) and Eqs. (11) - (14) comes from the $T$ dependent second term in Eq. (3), the kinetic energy. Comparing Eqs. (2) and (11), we can see the entropy $TS$ term is twice that of the $T$ dependent part of the kinetic energy with the effective mass $c = \sqrt{\frac{k}{m_T}}$. This means that the $T$ dependence of energy originate mainly from the $T$ dependence of the entropy and the $T$ dependence of the free energy $F = E - TS$ is the same as the $T$ dependence in $E$ with an opposite sign at low $T$. The entropy can be approximated as

$$S = T \left( 0.079A + 1.00A^{2/3} + 0.85I^2A - 4.11I^2A^{2/3} \right)$$

(15)

for SKM($m^*=m$) and

$$S = T \left( 0.22A + 1.10A^{2/3} + 1.10I^2A - 5.19I^2A^{2/3} \right)$$

(16)

for SLy4 in MeV unit. The volume term of Eqs. (13) - (16) has the smallest coefficient and the surface symmetry term has the largest coefficient among four terms. The minimum free energy $F = E - TS$ of finite nuclei at low $T$ becomes

$$F(A, Z, T) = -(15.31 + 0.04T^2)A + (18.30 - 0.50T^2)A^{2/3} + (19.69 - 0.42T^2)I^2A$$
$$- (33.18 - 2.06T^2)I^2A^{2/3} + E_c Z^2 A^{1/3} + E_{dif} Z^2 A + E_{ex} Z^4 A^{1/3} + c\Delta A^{-1/2}$$

(17)

$$F(A, Z, T) = -(15.31 + 0.11T^2)A + (20.01 - 0.55T^2)A^{2/3} + (31.11 - 0.55T^2)I^2A$$
$$- (41.04 - 2.59T^2)I^2A^{2/3} + E_c Z^2 A^{1/3} + E_{dif} Z^2 A + E_{ex} Z^4 A^{1/3} + c\Delta A^{-1/2}$$

(18)

for SKM($m^*=m$) and SLy4 respectively in MeV unit.

The results of Table II and Eqs. (11) - (14) show the $T$ dependence of the bulk, surface, symmetry and surface symmetry energy from $T = 0$ MeV to $T = 3$ MeV. At $T = 0$ the $F$ and $E$ are the same and thus the minimum of each of them are also the same. Eq. (3) shows a characteristic $T^2$ dependence coming from a nearly degenerate Fermi gases of protons and neutrons. The bulk and surface terms are similar for the two Skyrme interactions. However, a comparison of the symmetry term and surface symmetry term at $T = 0$ are quite different showing a sensitivity to the force used. The large variation of symmetry energy depends on the force used, which is also shown in Ref. 31 at zero temperature. (In Ref. 31 $a_s = E^2 / E_S$ is given instead of $E_S$.) The SLy4 interaction has a much larger coefficients than the SKM($m^*=m$) interaction. Most of the difference comes from the $T$ independent part of the $T^2$ dependent term of symmetry energy is much less sensitive to the force used. The volume symmetry and surface symmetry terms at $T = 0$ are also sensitive to the difference in neutron - proton radii [22]. However, in Ref. 32 an investigation of the neutron - proton radii difference in $^{208}$Pb using a Skyrme-Hartree-Fock model shows a small difference of $R_n - R_p = 0.16$ fm compared to nuclear size of about 7 fm.

The results show that the magnitude of $B$ becomes smaller as $T$ increases while the other coefficients become larger. The bulk energy $B$ is insensitive to the temperature having a small coefficient for the $T^2$ term. The surface energy $E_S$ and the volume symmetry energy $S_V$ have larger temperature dependences which are of the same order. The surface symmetry energy $S_S$ is the most sensitive to the temperature having a large coefficient in front of $T^2$. The $T$ dependent terms are insensitive to the force used. Most of the dependence on the force used appears in the $T$ independent part of energy (see Eq. (11)). This fact shows that the $T$ dependence comes from the kinetic energy including an effective mass term, and only the effective mass part has a dependence on force used. The interaction part which is sensitive to the force used is insensitive to the temperature. The central density $\rho_c$ or the nuclear size $R$ which minimizes the free energy $F$ is sensitive to the interaction used. From Table I we can see there is a small extra $T$ dependence beside the dominant $T^2$ dependence given by Eqs. (11) and (12). Including this small extra $T$
dependence, Eq. (11) and Eq. (12) have small corrections that are as follows. By extracting $T$ dependence up to $T^3$ directly from $B(T)$, $E_S(T)$, $S_V(T)$, and $S_S(T)$ in Table I itself, for SLy4, the $T$ dependences of Eq. (12) are now

\[
B(T) = 15.30754 + 0.00122333T - 0.0138600T^2 + 0.00114667T^3
\]
\[
E_S(T) = 20.00770 + 0.0226783T + 0.507110T^2 + 0.02095167T^3
\]
\[
S_V(T) = 31.11260 + 0.0619383T + 0.434450T^2 + 0.05360167T^3
\]
\[
S_S(T) = -41.03514 - 0.304690T - 2.059045T^2 - 0.2404950T^3
\] (19)

The coefficients for SKM($m^* = m$) have the similar $T$ dependence but less sensitive than for SLy4 case. Specifically, Eq. (11) is now modified to

\[
B(T) = 15.30963 - 0.0004933T - 0.0386350T^2 - 0.000201667T^3
\]
\[
E_S(T) = 18.30316 + 0.0063483T + 0.487620T^2 + 0.007201667T^3
\]
\[
S_V(T) = 19.68525 + 0.0024133T + 0.408910T^2 + 0.01111667T^3
\]
\[
S_S(T) = -33.17567 - 0.01930T - 1.977970T^2 - 0.056400T^3
\] (20)

This extra $T$ dependence is due to the different density (central density $\rho_c$) which minimizes $F$ for different $T$ and causes the small variation of the coefficients in Table I as $T$ changes and the differences between Eqs. (12) and (11) and between Eqs. (11) and (20). The density $\rho_c$ is sensitive to the interaction used as mentioned before.

Here we used the same size of proton and neutron distribution, $R_p = R_n$. In this case the system has no neutron skin. The existence of neutron skin may affects the surface dependence of energy. However since we are interested in the temperature dependence of energy we examined the $T^2$ dependent term in kinetic energy (Eq. (3)) by evaluating with different values of $R_p$ and $R_n$ which are determined by requiring the correct number of $Z$ and $N$ with the same central density of $\rho_{pc} = \rho_{nc} = \rho_c/2$ where $\rho_c$ is the total central density of the results minimizing $F$ with $R_p = R_n$. The results are shown in Table III labeled with “$T^2(R_p \neq R_n)$” which are very close to the values of “$T^2$ term”. This shows that the effect of the different size of neutron and proton distribution is much smaller than the effect of different force parameter set. For the case of $R_p = R_n$ the most asymmetry occurs in the central region while it occurs at the surface region for the case of $\rho_{pc} = \rho_{nc}$. If the overall asymmetry effects of these two extreme cases are the same then the total symmetry energy is insensitive to the existence of a neutron skin. The Weizacker expansion of Eq. (4) cannot distinguish between effects coming from different central densities and different radii. This result is shown by comparing the values in the rows of “$T^2$ term” and “$T^2(R_p \neq R_n)$” for each case in Table III. Of course the $T$ independence part may have a larger effect for different values of $R_p$ and $R_n$ than the $T^2$ dependent part. A discussion of the surface to volume symmetry energy at $T = 0$ can be found in Ref. 22.

The energy expansion coefficients for some other Skyrme parameter sets with various effective masses are also shown in Table III. In Tables I and III we compared the results for the parameter sets with wide range of effective masses from $m^* / m = 1.0$ to 0.577. Eqs. (11) - (3) show the temperature dependence of energy comes mostly from the effective mass. Tables I and III show that the above discussions about the qualitative behavior of the temperature dependence of energy coefficients are independent to the force parameter sets used. Of course the quantitative values of the $T$ dependences depend on the Skyrme parameters used. These tables show that the effect of the effective mass on the $T$ dependence of the energy is most visible in the $T^2$ term of the volume energy $B(T)$. For smaller effective mass the $T^2$ dependent term of $B(T)$ has the tendency of becoming smaller which can be understood by Eq. (3). The other energy coefficients do not show any specific pattern of the dependence on the effective mass.

It is of interest to compare the total symmetry energy using the coefficients given in Table III for the two Skyrme interactions along the line of $\beta$ stability. In particular we calculate the symmetry energy $E_{sym} = S_V I^2 A + S_S I^2 A^{2/3}$ along a simplified stability line given by

\[
\frac{Z_{\beta}}{A} = \frac{1}{2} + \frac{1}{3} \frac{4}{a_{sym}} \frac{a_{sym}}{A} \frac{a_{sym}}{Z_{\beta}}
\] (21)

which is determined by minimizing $E(Z,A) = E(A) + a_{sym} F^2 A + a_z Z^2 A^{1/3}$ with $a_z = 0.72$ MeV and $a_{sym} = 24$ MeV at $T = 0$. The $I_{\beta} = (N_{\beta} - Z_{\beta})/A = (A - 2Z_{\beta})/A$ which is substituted into $E_{sym}(T) = S_V(T) I^2 A + S_S(T) I^{2/3} A^{2/3}$ to give the symmetry energy along the line of $\beta$ stability. The energy is labeled $E_{sym,\beta} = (S_V(T) A + S_S(T) A^{2/3}) I_{\beta}$ where $I_{\beta} = (Z_{\beta} - a_z) / (A - 2Z_{\beta})$ in Fig I. Fig I shows that the $T$ dependence for the symmetry energy is more visible for larger $A$ and the symmetry energy is larger for higher $T$. It shows also that the SKM($m^* = m$) parameter set has a symmetry energy which is too small.

In Fig 2 the difference $\Delta E_{sym,\beta}$ between the symmetry energy $E_{sym,\beta}(T)$ for SLy4 parameter set and the volume symmetry energy with $S_V = 24$ MeV and $S_S = 0$ (dotted curve in Fig 1). This figure shows that the symmetry
TABLE II: Same as Table I but for different Skyrme interaction sets of SkT8 [33], SkM* [30], and SkI3 [34] with the effective mass $m^*/m$ of 0.833, 0.79, and 0.577 respectively.

| $T$ (MeV) | SkT8 | SkM* | SkI3 |
|-----------|------|------|------|
| $B(T)$    | 14.940 | 14.922 | 14.869 | 15.127 | 15.108 | 15.051 | 15.102 | 15.102 | 15.104 |
| $T$-indep. | 14.940 | 14.940 | 14.942 | 15.127 | 15.127 | 15.127 | 15.102 | 15.102 | 15.104 |
| $T^2$ term | 0.01851 | 0.01809 | 0.01810 | 0.01921 | 0.01911 | 0.01911 | 0.00048 | 0.00048 | 0.000002 |
| $T^2(\langle R_n \rangle \neq \langle R_p \rangle)$ | 0.01851 | 0.01809 | 0.01810 | 0.01921 | 0.01911 | 0.01911 | 0.00048 | 0.00048 | 0.000002 |
| $E_S(T)$ | 21.684 | 22.266 | 24.056 | 18.756 | 18.926 | 20.947 | 22.386 | 22.638 | 24.735 |
| $J$-indep. | 21.684 | 21.685 | 21.704 | 18.756 | 18.757 | 18.770 | 22.386 | 22.387 | 22.404 |
| $T^2$ term | 0.5809 | 0.5881 | 0.5881 | 0.5388 | 0.5442 | 0.5442 | 0.5762 | 0.5892 | 0.5832 |
| $T^2(\langle R_n \rangle \neq \langle R_p \rangle)$ | 0.5810 | 0.5882 | 0.5882 | 0.5389 | 0.5443 | 0.5443 | 0.5763 | 0.5831 | 0.5831 |
| $S_V(T)$ | 24.940 | 25.506 | 27.294 | 29.655 | 30.210 | 31.956 | 27.382 | 27.980 | 29.873 |
| $T$-indep. | 24.940 | 24.942 | 24.981 | 29.655 | 29.657 | 29.691 | 27.382 | 27.385 | 27.427 |
| $T^2$ term | 0.5631 | 0.5783 | 0.5783 | 0.5527 | 0.5662 | 0.5662 | 0.5954 | 0.6115 | 0.6115 |
| $T^2(\langle R_n \rangle \neq \langle R_p \rangle)$ | 0.5719 | 0.5871 | 0.5871 | 0.5723 | 0.5859 | 0.5859 | 0.5982 | 0.6145 | 0.6145 |
| $S_S(T)$ | -20.661 | -23.383 | -31.920 | -43.596 | -46.185 | -54.325 | -27.382 | -30.042 | -38.438 |
| $T$-indep. | -20.661 | -20.674 | -20.832 | -43.596 | -43.606 | -43.760 | -27.382 | -27.393 | -27.571 |
| $T^2$ term | -2.708 | -2.772 | -2.772 | -2.579 | -2.639 | -2.639 | -2.649 | -2.717 | -2.717 |
| $T^2(\langle R_n \rangle \neq \langle R_p \rangle)$ | -2.717 | -2.780 | -2.780 | -2.627 | -2.687 | -2.687 | -2.643 | -2.710 | -2.710 |

This paper presents an investigation of the energy of finite nuclei using density functional theory. A Skyrme approach for the nuclear interaction was used in this study. Bulk, surface, Coulomb and both volume and surface symmetry energies are calculated for a nucleus at both zero and non-zero temperatures. It is shown that the surface symmetry energy term is the most sensitive to the temperature while the bulk energy term is the least sensitive. Understanding features associated with the volume and surface symmetry energy was the main part of this study. Specifically, the volume and surface symmetry energy coefficients $S_V$ and $S_S$ were calculated for two different Skyrme interactions, SLy4 and SKM, and the results compared. The SLy4 interaction with a density dependent effective mass $m^*$ is in better agreement with known features of the symmetry energy of finite nuclei. The results suggest the importance of a density dependent effective mass. The results also show that the temperature $T$ dependence of the energy is insensitive to the force used and even more insensitive to the difference of $R_p$ and $R_n$. These results are quite different than the $T = 0$ results which are sensitive to the force used and the difference of $R_n$ and $R_p$.

Then a comparison was made between the SLy4 Skyrme interaction results with $S_V(T)$ and $S_S(T)$ of Table I ($S_V = 31$ MeV, $S_S = -41$ MeV at $T = 0$) and a model without surface effects $S_S = 0$ MeV and with a volume coefficient $S_V = 24$ MeV. The results were compared in Figs.1 and 2. Differences of up to 4 MeV were found to be present in the mid mass range of finite nuclei. However for small nuclear masses and for a larger nuclear mass ($A \approx 200$ for $T = 0$) the two choices agree for the total symmetry energy for these coefficients. The SLy4 interaction has the total symmetry energy $E_{sym}(T)$ similar to the value for $E_{sym} = 24T^2A$ MeV up to $A \approx 200$ for $T < 3$ MeV.

IV. CONCLUSIONS

energy is approximately $T$ independent at around $A = 100$ with the total symmetry energy of 43 MeV for SLy4 set compared to 46 MeV for pure volume symmetry energy only. The symmetry energy for higher $T$ is smaller than the symmetry energy for lower $T$ at smaller $A$ than $A = 100$ while this result is opposite at larger $A$. Differences of up to 4 MeV were found to be present in the mid mass range of finite nuclei. However for small nuclear masses and for a larger nuclear mass ($A \approx 200$ for $T = 0$) the two choices agree for the total symmetry energy for these coefficients. This means SLy4 interaction has the total symmetry energy $E_{sym}(T)$ similar to the value for $E_{sym} = 24T^2A$ MeV up to $A \approx 200$ for $T < 3$ MeV.

"The results also show that the temperature $T$ dependence of the energy is insensitive to the force used and even more insensitive to the difference of $R_p$ and $R_n$. These results are quite different than the $T = 0$ results which are sensitive to the force used and the difference of $R_n$ and $R_p$.

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FIG. 1: The total symmetry energy \(E_{\text{sym}}(T) = S_V(T)I^2A + S_S(T)I^2A^{2/3}\) in MeV versus mass number \(A\) along the line of stability. The upper curves are the interaction SLy4 (the solid curve for \(T = 0\) and the dashed curve for \(T = 3\) MeV) and the lower curves are SKM\((m^* = m)\) (the dash-dotted curve for \(T = 0\) and the dash-dot-dot-dotted curve for \(T = 3\) MeV). The dotted curve is a pure volume term symmetry energy with \(S_V = 24\) MeV and \(S_S = 0\) MeV. The \(E_{\text{sym},\beta}\) is in MeV.

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Appendix A

For a density \(\rho(\vec{r}) = \rho_c/[1 + e^{(r-R)/a}]\), using partial integral,

\[
\int d^3r \rho(\vec{r})^n = 4\pi \rho_c^n \int_0^\infty \frac{r^2 dr}{[1 + e^{(r-R)/a}]^n} = \frac{4\pi}{3} \rho_c^n \int_0^\infty \frac{r^3 dr}{a [1 + e^{(r-R)/a}]^{n+1}} = \frac{4\pi}{3} \rho_c^n \int_{-\infty}^\infty dy (ay + R)^3 \frac{ne^{(r-R)/a}}{(1 + e^y)^{n+1}} \quad (A1)
\]

For the last expression, the lower limit of the integration of \(y\) is extended from \(-R/a\) to \(-\infty\) which is a good approximation for \(n \geq 1\) since the symmetric factor \(e^y/(1 + e^y)^2\) peaks at \(y = 0\) and becomes zero at \(\pm\infty\). The integral part for various \(n\) are

\[
\int_{-\infty}^\infty dy (ay + R)^3 \frac{e^y}{(1 + e^y)^2} = \left(\pi^2 a^2 R + R^3\right) \quad (A2)
\]

\[
\int_{-\infty}^\infty dy (ay + R)^3 \frac{2e^y}{(1 + e^y)^3} = \left(-\pi^2 a^3 + \pi^2 a^2 R - 3a R^2 + R^3\right) \quad (A3)
\]

\[
\int_{-\infty}^\infty dy (ay + R)^3 \frac{3e^y}{(1 + e^y)^4} = \left(-\frac{3\pi^2}{2} a^3 + (3 + \pi^2)a^2 R - \frac{9}{2} a R^2 + R^3\right) \quad (A4)
\]
FIG. 2: Comparison of a symmetry energy with both volume and surface terms and a symmetry energy with just a volume term for SLy4 interaction. The symmetry energy is calculated along the line of $\beta$ stability. The upper and lower figures are the difference $\Delta E_{\text{sym,\,}\beta}$ and $\Delta E_{\text{sym,\,}\beta}/A$ in MeV between the two symmetry energies with both volume and surface symmetry energy terms for SLy4 interaction and with volume symmetry energy term only (dotted curve in Fig.1). The solid curve is at $T = 0$, the dashed curve at $T = 1$, the dash-dotted curve at $T = 2$, and the dash-dot-dot-dotted curve at $T = 3$ MeV. At $A = 100$ and $T = 0$, the SLy4 interaction gives a total symmetry energy of 43.09 MeV, while the pure volume calculation has 46.44 MeV.

\[
\int_{-\infty}^{\infty} dy (ay + R)^3 \frac{4e^y}{(1 + e^y)^5} = \left(-\frac{(6 + 11\pi^2)}{6}a^3 + (6 + \pi^2)a^2R - \frac{11}{2}aR^2 + R^3\right)
\]
(A5)

\[
\int_{-\infty}^{\infty} dy (ay + R)^3 \frac{5e^y}{(1 + e^y)^6} = \left(-\frac{5(6 + 5\pi^2)}{12}a^3 + \frac{(35 + 4\pi^2)}{4}a^2R - \frac{25}{4}aR^2 + R^3\right)
\]
(A6)

\[
\int_{-\infty}^{\infty} dy (ay + R)^3 \frac{(1/3)e^y}{(1 + e^y)^{4/3}} = (159.3784a^3 + 54.8029a^2R + 7.66445aR^2 + R^3)
\]
(A7)
\[ \int_{-\infty}^{\infty} dy (ay + R)^3 \frac{(2/3)e^y}{(1 + e^y)^{5/3}} = (15.855575a^3 + 15.77375a^2R + 2.2230575aR^2 + R^3) \quad (A8) \]

\[ \int_{-\infty}^{\infty} dy (ay + R)^3 \frac{(4/3)e^y}{(1 + e^y)^{7/3}} = (-5.030325a^3 + 8.81615625a^2R - 1.335546875aR^2 + R^3) \quad (A9) \]

\[ \int_{-\infty}^{\infty} dy (ay + R)^3 \frac{(5/3)e^y}{(1 + e^y)^{8/3}} = (-7.80506a^3 + 9.10458a^2R - 2.276943aR^2 + R^3) \quad (A10) \]

\[ \int_{-\infty}^{\infty} dy (ay + R)^3 \frac{(7/3)e^y}{(1 + e^y)^{10/3}} = (-11.6424a^3 + 10.81947a^2R - 3.58554aR^2 + R^3) \quad (A11) \]

\[ \int_{-\infty}^{\infty} dy (ay + R)^3 \frac{(8/3)e^y}{(1 + e^y)^{11/3}} = (-13.26781a^3 + 11.836907a^2R - 4.0769333aR^2 + R^3) \quad (A12) \]

\[ \int_{-\infty}^{\infty} dy (ay + R)^3 \frac{(7/6)e^y}{(1 + e^y)^{13/6}} = (-3.00561a^3 + 9.07054a^2R - 0.7352638484aR^2 + R^3) \quad (A13) \]

\[ \int_{-\infty}^{\infty} dy (ay + R)^3 \frac{(13/6)e^y}{(1 + e^y)^{19/6}} = (-10.7804a^3 + 10.331a^2R - 3.30669aR^2 + R^3) \quad (A14) \]

For \( n = 1 \), we get the constraint

\[ A = \int d^3r \rho(\vec{r}) = \frac{4\pi}{3} \rho_c \int_{-\infty}^{\infty} dy (ay + R)^3 \frac{e^y}{(1 + e^y)^2} = \frac{4\pi}{3} R^3 \rho_c \left[ 1 + \pi^2 \left( \frac{a}{R} \right)^2 \right] \quad (A15) \]

This relation expresses the central density \( \rho_c \) and \( \rho_c \) in terms of the size \( R \) and a given diffuseness parameters \( a \) for a given number of protons \( Z \) and neutrons \( N \). Similarly for the gradient dependent term,

\[ \int d^3r \rho(\vec{r}) \nabla^2 \rho(\vec{r}) = 4\pi \int_0^\infty r^2 dr \left[ \frac{\rho_c}{1 + e^{(r-R)/a}} \right] \frac{1}{r} \frac{dr}{d^2 r^2} \left[ \frac{\rho_c}{1 + e^{(r-R)/a}} \right]^2 \]

\[ = 4\pi \rho_c^2 \int_{-\infty}^{\infty} dy \frac{1}{(1 + e^y)^2} \left[ \frac{2(ay + R)^2}{a} \left( \frac{e^y}{1 + e^y} \right)^2 - \frac{(ay + R)^2}{a} \left( \frac{e^y}{1 + e^y} \right)^2 \right] \]

\[ - \frac{2(ay + R)}{a} \left( \frac{e^y}{1 + e^y} \right) \]

\[ = -\frac{4\pi}{3} R^3 \rho_c^2 \frac{2}{2Ra} \left[ 1 + \left( \frac{\pi^2}{3} - 2 \right) \left( \frac{a}{R} \right)^2 \right] \quad (A16) \]
[21] P. Danielewicz and J. Lee, Int. J. Mod. Phys. E18, 892 (2009)
[22] A. W. Steiner, M. Prakash, J. M. Lattimer and P. J. Ellis, Phys. Reports 411, 325 (2005)
[23] S. J. Lee and A. Z. Mekjian, Phys. Rev. C79, 044323 (2009)
[24] H. A. Bethe, R. F. Bacher, Rev. Mod. Phys. 8, 82 (1936).
[25] M. A. Preston, R. K. Bhaduri, Structure of the Nucleus, Addison Wesley, Reading, MA, 1975.
[26] W. D. Myers, W. J. Swiatecki, Nucl. Phys. A 81, 1 (1966).
[27] W. D. Myers, W. J. Swiatecki, Ann. Phys. 55, 395 (1969).
[28] W. D. Myers, W. J. Swiatecki, Ann. Phys. 84, 186 (1974).
[29] W. D. Myers, Droplet Model of Atomic Nuclei, IFI/Plenum, New York, 1975.
[30] E. Chabanat, P. Bonche, P. Haensel, J. Meyer, and R. Schaeffer, Nucl. Phys. A635, 231 (1998).
[31] P. Danielewicz and J. Lee, Nucl. Phys. A818, 36 (2009).
[32] B. A. Brown, Phys. Rev. Letts. 85, 5296 (2000).
[33] F. Tondeur, M. Brack, M. Farine, and J. M. Pearson, Nucl. Phys. A420, 297 (1984).
[34] P.-G. Reinhard and H. Flocard, Nucl. Phys. A584, 467 (1995).