Finite difference schemes for the tempered fractional Laplacian

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Abstract. The second and all higher order moments of the β-stable Lévy process diverge, the feature of which is sometimes referred to as shortcoming of the model when applied to physical processes. So, a parameter λ is introduced to exponentially temper the Lévy process. The generator of the new process is tempered fractional Laplacian \((\Delta + \lambda)^{\beta/2}\) [W.H. Deng, B.Y. Li, W.Y. Tian, and P.W. Zhang, Multiscale Model. Simul., in press, 2017]. In this paper, we first design the finite difference schemes for the tempered fractional Laplacian equation with the generalized Dirichlet type boundary condition, their accuracy depending on the regularity of the exact solution on \(\bar{\Omega}\). Then the techniques of effectively solving the resulting algebraic equation are presented, and the performances of the schemes are demonstrated by several numerical examples.

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1. Introduction

The fractional Laplacian $\Delta^{\beta/2}$ is the generator of the $\beta$-stable Lévy process, in which the random displacements executed by jumpers are able to walk to neighboring or nearby sites, and also perform excursions to remote sites by way of Lévy flights \[4, 22, 23\]. The distribution of the jump length of $\beta$-stable Lévy process obeys the isotropic power-law measure $|x|^{-n-\beta}$, where $n$ is the dimension of the space. The extremely long jumps of the process make its second and higher order moments divergent, sometimes being referred to as a shortcoming when it is applied to physical model in which one expects regular behavior of moments \[28\]. The natural idea to damp the extremely long jumps is to introduce a small damping parameter $\lambda$ to the distribution of jump lengths, i.e., $e^{-\lambda|x|}|x|^{-n-\beta}$. With small $\lambda$, for short time, it displays the dynamics of Lévy process, while for sufficiently long time the dynamics will transit slowly from superdiffusion to normal diffusion. The generator of the tempered Lévy process is the tempered fractional Laplacian $(\Delta + \lambda)^{\beta/2}$ \[9\]. The tempered fractional Laplacian equation governs the probability distribution function of the position of the particles.

This paper focuses on developing the finite difference schemes for the tempered fractional Laplacian equation

\[
\begin{aligned}
\begin{cases}
-(\Delta + \lambda)^{\beta/2} u(x) = f(x), & x \in \Omega, \\
u(x) = g(x), & x \in \mathbb{R}\setminus\Omega,
\end{cases}
\end{aligned}
\]

(1.1)

where $\beta \in (0, 2)$, $\lambda \geq 0$, $\Omega = (a, b)$, and

\[
(\Delta + \lambda)^{\beta/2} u(x) := -c_\beta \text{ P.V.} \int_{\mathbb{R}} \frac{u(x) - u(y)}{e^{\lambda|x-y|}|x-y|^{1+\beta}} dy
\]

(1.2)

with

\[
c_\beta = \begin{cases}
\frac{\beta \Gamma\left(\frac{1+\beta}{2}\right)}{2^{\beta-1} \pi^{1/2} \Gamma(1-\beta/2)} & \text{for } \lambda = 0 \text{ or } \beta = 1, \\
\frac{\Gamma\left(\frac{1}{\beta}\right)}{2\pi^{1/2} \Gamma(-\beta/2)} & \text{for } \lambda > 0 \text{ and } \beta \neq 1,
\end{cases}
\]

(1.3)

and P.V. being the limit of the integral over $\mathbb{R}\setminus B_\epsilon(x)$ as $\epsilon \to 0$. The tempered operator in (1.2) is the generator of the tempered symmetric $\beta$-stable Lévy process $x(t)$ determined by the Lévy-Khintchine representation

\[
E(e^{i\omega x}) = \exp\left(t \int_{\mathbb{R}\setminus\{0\}}\left(e^{i\omega y} - 1 - i\omega y \chi_{\{|y|\leq 1\}}\right) \nu(dy)\right).
\]

(1.4)

Here $E$ is the expectation, $i = \sqrt{-1}$, $\chi_S := \begin{cases} 1, & x \in S, \\ 0, & s \notin S \end{cases}$, and $\nu(dy) = e^{-\lambda|y|}|y|^{-\beta-1}dye^{-\lambda|y|}|y|^{-\beta-1}dye^{-\lambda|y|}|y|^{-\beta-1}dy$ is the tempered Lévy measure. The parameter $\lambda (> 0)$ fixes the decay rate of big jumps, while $\beta$ determines the relative importance of smaller jumps in the path of the process. Model (1.1) corresponds to the one-dimensional case of the initial and boundary value
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problem in Eq. (49) recently proposed in [9], and the existence and uniqueness of its weak solution have been shown in [29]. Obviously, when \( \lambda = 0 \), (1.2) reduces to the fractional Laplacian [23]

\[
(\Delta)^{\beta/2}u(x) := -c_\beta \text{ P.V.} \int_\mathbb{R} \frac{u(x) - u(y)}{|x - y|^{1+\beta}} \, dy.
\]  

(1.5)

It is well known that for the proper classes of functions that decay quickly enough at infinity, the fractional Laplacian can be rewritten as the combination of the left and right Riemann-Liouville fractional derivatives \(-\infty D_\beta^x u(x)\) and \(x D_\infty^\beta u(x)\) (the so-called Riesz fractional derivative) [27], i.e.,

\[
-\Delta^{\beta/2}u(x) = -\infty D_\beta^x u(x) + x D_\infty^\beta u(x) \quad (2\cos(\beta \pi/2)), \quad \beta \neq 1.
\]  

(1.6)

The similar result also holds for the tempered fractional Laplacian. In fact, letting \(u(x) \in H^\beta(\mathbb{R})\) and \(\mathcal{F}[u(x)](\omega) := \int_\mathbb{R} u(x)e^{-i\omega x} \, dx\) be its Fourier transform, we have [29, Propositions 2.1 and 2.2]

\[
\mathcal{F}[(\Delta + \lambda)^{\beta/2} u(x)](\omega) = (-1)^{|\beta|} \left( \lambda^\beta - (\lambda^2 + |\omega|^2)^{\beta/2} \cos \left( \beta \arctan \left( \frac{|\omega|}{\lambda} \right) \right) \right) \mathcal{F}[u](\omega),
\]  

(1.7)

where \(|\beta| := \{z \in \mathbb{N} : 0 \leq \beta - z < 1\}\). Note that

\[
\frac{1}{2} (\lambda^2 + |\omega|^2)^{\beta/2} \cos \left( \beta \arctan \left( \frac{|\omega|}{\lambda} \right) \right) = (\lambda + i\omega)^\beta + (\lambda - i\omega)^\beta.
\]  

(1.8)

A simple calculation yields that

\[
-(\Delta + \lambda)^{\beta/2} u(x) = (-1)^{|\beta|} \left[ \frac{\partial_x^{\beta,\lambda} u(x) + \partial_{-x}^{\beta,\lambda} u(x)}{2} \right], \quad \beta \neq 1,
\]  

(1.9)

where

\[
\partial_x^{\beta,\lambda} u(x) := \begin{cases} 
\mathcal{F}^{-1} \left[ \left( (\lambda + i\omega)\beta - \lambda^\beta \right) \hat{u}(\omega) \right] (x), & \beta \in (0, 1), \\
\mathcal{F}^{-1} \left[ \left( (\lambda + i\omega)\beta - i\omega \beta \lambda^{\beta-1} - \lambda^\beta \right) \hat{u}(\omega) \right] (x), & \beta \in (1, 2)
\end{cases}
\]  

(1.10)

and

\[
\partial_{-x}^{\beta,\lambda} u(x) := \begin{cases} 
\mathcal{F}^{-1} \left[ \left( (\lambda - i\omega)\beta - \lambda^\beta \right) \hat{u}(\omega) \right] (x), & \beta \in (0, 1), \\
\mathcal{F}^{-1} \left[ \left( (\lambda - i\omega)\beta + i\omega \beta \lambda^{\beta-1} - \lambda^\beta \right) \hat{u}(\omega) \right] (x), & \beta \in (1, 2)
\end{cases}
\]  

(1.11)

are the left and right normalized tempered Riemann-Liouville fractional operations being given in [6, 24], and their representations in real space can be founded in [18].

Nowadays, many finite difference schemes have been proposed to solve equations with the Riemann-Liouville type fractional derivatives in (1.6) or (1.9) under zero boundary
conditions [6, 18, 20, 24, 30, 31], which usually are constructed based on the Grünwal formula or its variants; and there are also a lot of discussions on time-fractional operators or other numerical methods, e.g., [14, 26]. To the best of our knowledge, it seems that very few numerical schemes are based on the singular integral definition (1.5) to approximate the fractional Laplacian. In [10], to study the mean exit time and escape probability of the dynamical systems driven by non-Gaussian Lévy noises, the fractional Laplacian is approximated numerically by a “punched-hole” trapezoidal rule. The finite difference and finite element methods with systemically theoretical analysis for solving model (1.1) with $\lambda = 0$ are presented in [15] and [2], respectively. Usually, even for problems with $g(x) = 0$, to perform the convergence analysis, the finite element methods refer to the regularity of the exact solution on $\Omega [2, 29]$ while the finite difference methods require the regularity on the whole line $[6, 15, 18, 24, 24]$. The finite difference schemes provided in this paper for the tempered fractional Laplacian equation (1.1) just depend on the regularity of $u(x)$ on $\bar{\Omega}$. We give the detailedly theoretical analysis and effective algorithm implementation.

The rest of this paper is organized as follows. In Section 2 we discuss the numerical schemes of (1.1) with $\beta \in (0, 1)$. We first derive the finite difference discretizations of the tempered fractional Laplacian based on the singular integral definition (1.2), and then give convergence analysis and the related implementation techniques for solving the resulting algebraic equation with preconditioning. Two types of preconditioners are considered. In Section 3, we extend the suggested finite difference schemes to the case $\beta \in [1, 2)$, and most of the results and implementation techniques still hold. Numerical simulations are presented in Section 4 and we conclude the paper in Section 5.

Throughout the paper by the notation $A \lesssim B$ we mean that $A$ can be bounded by a multiple of $B$, independent of the parameters they may depend on, while the expression $A \simeq B$ means that $A \lesssim B \lesssim A$. We also use $C$ to denote a constant, which may be different for different lines.

2. Finite difference scheme for the case $\beta \in (0, 1)$

In this section, we discuss the finite difference scheme for the model (1.1) with $\beta \in (0, 1)$.
2.1. Derivation of the scheme

Let \( \Omega = (a, b) \) and \( s, s_1 \in \{0, 1\} \). We partition \( \Omega \) uniformly into \( a = x_0 < x_1 < \cdots < x_M < x_{M+1} = b \) with \( x_i = a + ih \), \( h = \frac{b-a}{M+1} \). Then for \( i = 1, 2, \cdots, M \), we have

\[
\int_{-\infty}^{x_i} \frac{u(x_i) - u(y)}{e^{\lambda|x_i-y|}|x_i-y|^{1+\beta}} dy = \int_a^{x_i-1} g_1(i, s, y) (x_i - y)^{s-1-\beta} dy + \int_{x_i+1}^{b} g_2(i, s, y) (x_i - y)^{s-1-\beta} dy + \int_0^h g_3(i, s, y) y^{s-1-\beta} dy + (B_1(i) + B_2(i)) u(x_i) - d_1(i) - d_2(i),
\]

where

\[
g_1(i, s, y) := \frac{(u(x_i) - u(y))}{(x_i - y)^{s}}, \quad y \in [a, x_i-1],
\]

\[
g_2(i, s, y) := \frac{(u(x_i) - u(y))}{(y - x_i)^{s}}, \quad y \in [x_i+1, b],
\]

\[
g_3(i, s, y) := \frac{(2u(x_i) - u(x_i - y) - u(x_i + y))}{y^s} \cdot e^{-\lambda y}, \quad y \in [0, h],
\]

and

\[
B_1(i) = \int_{-\infty}^{a} \frac{e^{\lambda(y-x_i)}}{(x_i-y)^{1+\beta}} dy, \quad B_2(i) = \int_{b}^{\infty} \frac{e^{\lambda(x_i-y)}}{(y-x_i)^{1+\beta}} dy,
\]

\[
d_1(i) = \int_{-\infty}^{a} \frac{g(y)}{e^{\lambda(x_i-y)}} \frac{e^{\lambda(x_i-y)}}{(x_i-y)^{1+\beta}} dy, \quad d_2(i) = \int_{b}^{\infty} \frac{g(y)}{e^{\lambda(x_i-y)}} \frac{e^{\lambda(x_i-y)}}{(y-x_i)^{1+\beta}} dy.
\]

Define an interpolation operator by

\[
I_{[a_1, a_2]}(p(y)) := \frac{y-a_1}{h} p(a_2) + \frac{a_2-y}{h} p(a_1), \quad y \in [a_1, a_2].
\]

We approximate the term \( \int_a^{x_i-1} g_1(i, y) (x_i - y)^{s-1-\beta} dy \) by

\[
\mathcal{A}_1(s, g_1) = \sum_{k=1}^{i-1} \int_{x_k}^{x_{k-1}} I_{[x_{k-1}, x_k]} (g_1(i, s, y)) (x_i - y)^{s-1-\beta} dy,
\]

\[
= \sum_{k=1}^{i-1} h^s (g_1(i, s, x_{k-1}) A_1(i, s, k) + g_1(i, s, x_k) A_2(i, s, k))
\]

with

\[
A_1(i, s, k) = \frac{1}{h^{s+1}} \int_{x_{k-1}}^{x_k} (x_k - y) (x_i - y)^{s-1-\beta} dy > 0,
\]

\[
A_2(i, s, k) = \frac{1}{h^{s+1}} \int_{x_{k-1}}^{x_k} (y - x_{k-1}) (x_i - y)^{s-1-\beta} dy > 0;
\]
the term $\int_{x_i+1}^{b} g_2(i, y)(y - x_i)^{s - 1 - \beta} dy$ is approximated by

$$\mathcal{A}_2(s, g_2) = \sum_{k=i+2}^{M+1} \int_{x_{k-1}}^{x_k} I_{[x_{k-1}, x_k]} (g_2(i, s, y)) (y - x_i)^{s - 1 - \beta} dy$$

$$= \sum_{k=i+2}^{M+1} h^t (g_2(i, s, x_{k-1}) A_3(i, s, k) + g_2(i, s, x_k) A_4(i, s, k)) \tag{2.11}$$

with

$$A_3(i, s, k) = \frac{1}{h^{s+1}} \int_{x_{k-1}}^{x_k} (x_k - y)(y - x_i)^{s - 1 - \beta} dy > 0, \tag{2.12}$$

$$A_4(i, s, k) = \frac{1}{h^{s+1}} \int_{x_{k-1}}^{x_k} (y - x_{k-1})(y - x_i)^{s - 1 - \beta} dy > 0; \tag{2.13}$$

and the term $\int_{0}^{h} g_3(i, s_1, y)y^{s_1 - 1 - \beta} dy$ by

$$\mathcal{A}_3(s_1, g_3) = \int_{0}^{h} I_{[0, h]} (g_3(i, s_1, y)) y^{s_1 - 1 - \beta} dy = \frac{h^{s_1 - \beta}}{s_1 - \beta + 1} g_3(i, s_1, h). \tag{2.14}$$

Assume $g_1(i, s, y) \in C^2[a, x_{i-1}], g_2(i, s, y) \in C^2[x_{i+1}, b]$, and $g_3(i, s_1, y) \in C^2[0, h]$. By the Lagrange interpolation error remainder, we have

$$\left| \int_{a}^{x_i-1} g_1(i, s, y)(x_i - y)^{s - 1 - \beta} dy - \mathcal{A}_1(s, g_1) \right| \leq \frac{\|g_1^{(2)}(i, s, y)\|_{L^{\infty}[a, x_{i-1}]} h^2}{2|s - \beta|} \left| h^{s - \beta} - (x_i - a)^{s - \beta} \right|, \tag{2.15}$$

$$\left| \int_{x_{i+1}}^{b} g_2(i, s, y)(y - x_i)^{s - 1 - \beta} dy - \mathcal{A}_2(s, g_2) \right| \leq \frac{\|g_2^{(2)}(i, s, y)\|_{L^{\infty}[x_{i+1}, b]} h^2}{2|s - \beta|} \left| h^{s - \beta} - (b - x_i)^{s - \beta} \right|, \tag{2.16}$$

and

$$\left| \int_{0}^{h} g_3(i, s_1, y)y^{s_1 - 1 - \beta} dy - \mathcal{A}_3(s_1, g_3) \right| \leq \frac{\|g_3^{(2)}(i, s_1, y)\|_{L^{\infty}[0, h]} h^{s_1 - \beta + 2}}{2(s_1 - \beta + 1)(s_1 - \beta + 2)}. \tag{2.17}$$
If \( u(y) \in C^2(\bar{\Omega}) \), it can be noted that
\[
\|g_1^{(2)}(i, 0, y)\|_{L^\infty[a, x_{i-1}]}, \|g_2^{(2)}(i, 0, y)\|_{L^\infty[x_{i-1}, b]}, \text{ and } \|g_3^{(2)}(i, 0, y)\|_{L^\infty[0, h]}
\]
are bounded, and the bounds may depend on the values of \( u^{(k)}(y), k = 0, 1, 2 \) on \( \bar{\Omega} \), but independent of \( h \); and if \( u \in C^3(\bar{\Omega}) \), let \( h(x) := \frac{u(x_i) - u(x)}{x_i - x} \) and using Taylor’s expansion,
\[
\|h(x)\|_{L^\infty[a, x_{i-1}]}, \|h^{(1)}(x)\|_{L^\infty[a, x_{i-1}]}, \text{ and } \|h^{(2)}(x)\|_{L^\infty[a, x_{i-1}]}
\]
also are bounded, and the bounds may depend on the values of \( u^{(k)}(x), k = 0, 1, 2, 3 \) on \( \bar{\Omega} \), but also independent of \( h \). Thus
\[
\|g_1^{(2)}(i, 1, y)\|_{L^\infty[a, x_{i-1}]} = \| (\lambda^2 h(x) + 2\lambda h^{(1)}(x) + h^{(2)}(x)) e^{\lambda(y-x)} \|_{L^\infty[a, x_{i-1}]} \leq C. \tag{2.19}
\]
Similarly, we have
\[
\|g_2^{(2)}(i, 1, y)\|_{L^\infty[a, x_{i-1}]} \leq C, \quad \|g_3^{(2)}(i, 1, y)\|_{L^\infty[0, h]} \leq C. \tag{2.20}
\]
Therefore, combining (2.8), (2.11), (2.14), and (2.15)-(2.20), for \( \beta \in (0, 1) \) and \( i = 1, 2, \ldots, M \), it follows that
\[
-(\Delta + \lambda)^{\beta/2} u(x_i) = \mathcal{A}_1(s, g_1) + \mathcal{A}_2(s, g_2) + \mathcal{A}_3(s_1, g_3) + (B_1(i) + B_2(i))u(x_i) - (d_1(i) + d_2(i)) + r_h^i,
\]
where
\[
|r_h^i| = \begin{cases} 
\Theta(h^{2-\beta}) & \text{for } u \in C^2(\bar{\Omega}), s = s_1 = 0, \\
\Theta(h^2) & \text{for } u \in C^3(\bar{\Omega}), s = s_1 = 1.
\end{cases} \tag{2.21}
\]

Remark 2.1. For \( \lambda = 0 \), the bounds of \( \|g_1^{(2)}(i, s, y)\|_{L^\infty[a, x_{i-1}]}, \|g_2^{(2)}(i, s, y)\|_{L^\infty[x_{i+1}, b]} \) and \( \|g_3^{(2)}(i, s_1, y)\|_{L^\infty[0, h]} \) only depend on the values of \( u^{(2)}(x) \) on \( \bar{\Omega} \) for \( s = s_1 = 0 \) and the values of \( u^{(3)}(x) \) on \( \bar{\Omega} \) for \( s = s_1 = 1 \).

Define
\[
F_1 := f(x_1) + d_1(1) + d_2(1) + \frac{h^{\beta} e^{-\lambda h}}{s_1 + 1 - \beta} u(a) + A_4(1, s, M + 1) \frac{e^{-\lambda M h}}{M^s} u(b), \tag{2.22}
\]
\[
F_M := f(x_M) + d_1(M) + d_2(M) + \frac{h^{\beta} e^{-\lambda h}}{s_1 - \beta + 1} u(b) + A_1(M, s, 1) \frac{e^{-\lambda M h}}{M^s} u(a), \tag{2.23}
\]
\[
F_i := f(x_i) + d_1(i) + d_2(i) + A_1(i, s, 1) \frac{e^{-\lambda i h}}{i^s} u(a) + A_4(i, s, M + 1) \frac{e^{-\lambda (M+1-i) h}}{(M + 1 - i)^s} u(b), \quad i = 2, 3, \ldots, M - 1, \tag{2.24}
\]
\[
F_i := f(x_i) + d_1(i) + d_2(i) + A_1(i, s, 1) \frac{e^{-\lambda i h}}{i^s} u(a) + A_4(i, s, M + 1) \frac{e^{-\lambda (M+1-i) h}}{(M + 1 - i)^s} u(b), \quad i = 2, 3, \ldots, M - 1, \tag{2.24}
\]
\[ h_{i,j} := \begin{cases} 
-(A_1(i,s,j + 1) + A_2(i,s,j)) \frac{e^{-\lambda(j-i)h}}{(j-i)^{\beta}}, & 1 \leq j \leq i - 2, \\
\frac{h^{-\beta}e^{-\lambda h}}{s_1+1-\beta} - A_2(i,s,i-1)e^{-\lambda h}, & j = i - 1, \\
\frac{h^{-\beta}e^{-\lambda h}}{s_1+1-\beta} - A_3(i,s,i+2)e^{-\lambda h}, & j = i + 1, \\
-(A_3(i,s,j + 1) + A_4(i,s,j)) \frac{e^{-\lambda(j-i)h}}{(j-i)^{\beta}}, & i + 2 \leq j \leq M, 
\end{cases} \quad (2.25) \]

with the \( h_{i,i} \) satisfying

\[ h_{i,i} + \sum_{j=1,j\neq i}^{M} h_{i,j} - B_1(i) - B_2(i) \\
= \begin{cases} 
\frac{h^{-\beta}e^{-\lambda h}}{s_1+1-\beta} + A_4(i,s,M + 1) \frac{e^{-\lambda(M + 1 - i)h}}{(M + 1 - i)^{\beta}}, & i = 1, \\
A_1(i,s,1) \frac{e^{-\lambda h}}{i^{\beta}} + A_4(i,s,M + 1) \frac{e^{-\lambda(M + 1 - i)h}}{(M + 1 - i)^{\beta}}, & 2 \leq i \leq M - 1, \\
\frac{h^{-\beta}e^{-\lambda h}}{s_1+1-\beta} + A_1(i,s,1) \frac{e^{-\lambda h}}{i^{\beta}}, & i = M. 
\end{cases} \quad (2.26) \]

Let

\[ H := (h_{i,i})_{i,j=1}^{M}, \quad U := (u(x_1), u(x_2), \cdots, u(x_M))^T, \]
\[ F := (F_1, F_2, \cdots, F_M)^T, \quad R_h := (r^1_h, r^2_h, \cdots, r^M_h)^T. \quad (2.27) \]

Then making use of (2.21), it holds that

\[ HU = F + R_h. \quad (2.28) \]

We denote the numerical solution of \( u \) at \( x_i \) as \( U_i \) and define \( U_h := (U_1, U_2, \cdots, U_M)^T \). By discarding the truncation error \( R_h \) and replacing \( U \) by \( U_h \), the numerical scheme of (1.1) can be given as

\[ HU_h = F. \quad (2.29) \]

### 2.2. Error Estimates

By a simple calculation, for \( \beta \in (0, 1) \), we have

\[ A_1(i,s,j + 1) + A_2(i,s,j) = A_3(i,s,j + 1) + A_4(i,s,j) = C_{\beta,s}(2|2i-j|^{1-\beta+s} - (|i-j|-1)^{1-\beta+s} - (|i-j| + 1)^{1-\beta+s}), \]
\[ A_2(i,s,i - 1) = C_{\beta,s}(2 - \beta + s - 2^{1-\beta+s}), \]
\[ A_1(i,s,1) = C_{\beta,s}(1^{1-\beta+s} - (i - 1)^{1-\beta+s} - (1 - \beta + s)^{1-\beta+s}), \]
\[ A_4(i,s,M + 1) = C_{\beta,s}((M + 1 - i)^{1-\beta+s} - (M - i)^{1-\beta+s} - (1 - \beta + s)(M + 1 - i)^{1-\beta+s}), \quad (2.30) \]
where \( C_{\beta,i} := \frac{h^{-\beta}}{(\beta-\delta)(1-\beta+\delta)} \). By (2.25) and (2.30), it holds that \( h_{i,j} = h_{j,i} \), i.e., matrix \( H \) is symmetric. As for \( B_1(i) \) and \( B_2(i) \), when \( \lambda = 0 \), we have

\[
B_1(i) = \frac{(x_i-a)^{-\beta}}{\beta}, \quad B_2(i) = \frac{(b-x_i)^{-\beta}}{\beta};
\]

(2.31)

when \( \lambda > 0 \), it holds that

\[
B_1(i) = \frac{e^{-\lambda(x_i-a)}}{\beta(x_i-a)^{\beta}} + \frac{\lambda}{\beta(1-\beta)} \frac{e^{-\lambda(x_i-a)}}{(x_i-a)^{\beta-1}}
+ \lambda^\beta \Gamma(-\beta) + \frac{\lambda^2}{\beta(1-\beta)} \int_{x_i-a}^{x_i-a} e^{-\lambda t} t^{\beta-1} dt,
\]

(2.32)

\[
\int_{0}^{x_i-a} e^{-\lambda t} t^{\beta-1} dt = \left( \frac{x_i-a}{2} \right)^{2-\beta} \int_{-1}^{1} e^{-\lambda(1+\xi)}(1+\xi)^{1-\beta} d\xi,
\]

(2.33)

and

\[
B_2(i) = \frac{e^{-\lambda(b-x_i)}}{\beta(b-x_i)^{\beta}} + \frac{\lambda}{\beta(1-\beta)} \frac{e^{-\lambda(b-x_i)}}{(b-x_i)^{\beta-1}}
+ \lambda^\beta \Gamma(-\beta) + \frac{\lambda^2}{\beta(1-\beta)} \int_{0}^{b-x_i} e^{-\lambda t} t^{\beta-1} dt,
\]

(2.34)

\[
\int_{0}^{b-x_i} e^{-\lambda t} t^{\beta-1} dt = \left( \frac{b-x_i}{2} \right)^{2-\beta} \int_{-1}^{1} e^{-\lambda(1+\xi)}(1+\xi)^{1-\beta} d\xi.
\]

(2.35)

The integrals in (2.33) and (2.35) can be calculated by the Jacobi-Gauss quadrature with the weight function \( (1-\xi)^{\beta}(1+\xi)^{1-\beta} \) [13, Appendix A, p. 447] and [25]. Since \( e^{-\lambda(1+\xi)}(1+\xi)^{1-\beta} \) is sufficiently smooth in \([-1,1]\), these calculations yield the spectral accuracy. We assume that \( B_1(i) \) and \( B_2(i) \) are exact in the following analysis. By (2.9), (2.10), (2.12), (2.13), (2.25), and (2.26), it holds that

**Lemma 2.1.** The entries of matrix \( H \) satisfies

\[
h_{i,j} < 0 \quad (j \neq i), \quad h_{i,i} > 0, \quad h_{i,i} + \sum_{j=1,j\neq i}^{M} h_{i,j} > B_1(i) + B_2(i).
\]

(2.36)

According to the Gersgorin theorem [5, Theorem 4.4], the minimum eigenvalue of \( H \) satisfies

\[
\lambda_{\min}(H) > \min_{1 \leq i \leq M} (B_1(i) + B_2(i)) > 0.
\]

(2.37)

Thus \( H \) is a strictly diagonally dominant \( M \)-matrix [5, Lemma 6.2] and a symmetric positive definite (s.p.d.) matrix. Therefore, the scheme (2.29) has a unique solution. Define the discrete \( L_2 \) inner product and norms:

\[
(v,w) = \sum_{i=1}^{M} v_i w_i, \quad \|v\| = \sqrt{(v,v)}, \quad \|v\|_\infty = \max_{1 \leq i \leq M} |v_i|.
\]

(2.38)
Theorem 2.1. For the scheme (2.29), the following hold.

1. Let $\beta \in (0, 1), s = s_1 = 0$, and $u(x) \in C^2(\bar{\Omega})$. Then

$$
\|U - U_h\| \leq C_1 h^{2-\beta}, \quad \|U - U_h\|_{\infty} \leq C_2 h^{2-\beta},
$$

(2.39)

where $C_1$ and $C_2$ may depend on the values of $u^{(k)}(x), k = 0, 1, 2$ on $\bar{\Omega}$, but independent of $h$.

2. Let $\beta \in (0, 1), s = s_1 = 1$, and $u(x) \in C^3(\bar{\Omega})$. Then

$$
\|U - U_h\| \leq C_1 h^2, \quad \|U - U_h\|_{\infty} \leq C_2 h^2,
$$

(2.40)

where $C_1$ and $C_2$ may depend on the values of $u^{(k)}(x), k = 0, 1, 2, 3$ on $\bar{\Omega}$, but independent of $h$.

Proof. Firstly, taking an inner product of (2.29) with $U_h$, and using the Cauchy-Schwarz inequality, we have

$$
\lambda_{\min}(H) \|U_h\|^2 \leq (HU_h, U_h) \leq \|F\| \|U_h\|.
$$

(2.41)

Since

$$
B_1(i) + B_2(i) = \int_0^a \frac{e^{\lambda(y-x_i)}}{(x_i - y)^{1+\beta}} dy + \int_b^\infty \frac{e^{\lambda(x_i - y)}}{(y - x_i)^{1+\beta}} dy,
$$

$$
\geq \int_{a-\delta}^a \frac{e^{\lambda(y-x_i)}}{(x_i - y)^{1+\beta}} dy + \int_b^{b+\delta} \frac{e^{\lambda(x_i - y)}}{(y - x_i)^{1+\beta}} dy
$$

$$
\geq e^{-\lambda(b-a+\delta)} \left( \int_{a-\delta}^a \frac{1}{(b-y)^{1+\beta}} dy + \int_b^{b+\delta} \frac{1}{(y-a)^{1+\beta}} dy \right)
$$

$$
= \frac{2e^{-\lambda(b-a+\delta)}(b-a)^{-\beta}}{\beta},
$$

(2.42)

where $\sigma$ is a given positive real number. Thus, by (2.37), it holds that

$$
\|U_h\| \leq e^{\lambda(b-a+\delta)}(b-a)^{\beta} \|F\|.
$$

(2.43)

Secondly, assume $\|U_h\|_{\infty} = |U_{i_0}|$ with $1 \leq i_0 \leq M$. By Lemma 2.1, we have

$$
U_{i_0} \left( \sum_{j=1}^{i_0-1} h_{i_0,j} U_j + \left( h_{i_0,i_0} - (B_1(i_0) + B_2(i_0)) \right) U_{i_0} + \sum_{j=i_0+1}^M h_{i_0,j} U_j \right)
$$

$$
\geq \sum_{j=1}^{i_0-1} h_{i_0,j} U_{i_0}^2 + \left( h_{i_0,i_0} - (B_1(i_0) + B_2(i_0)) \right) U_{i_0}^2 + \sum_{j=i_0+1}^M h_{i_0,j} U_{i_0}^2
$$

$$
\geq 0.
$$

(2.44)
Then
\[
(B_1(i_0) + B_2(i_0)) \|U_h\|_{\infty} = (B_1(i_0) + B_2(i_0)) |U_{i_0}|
\]
\[
\leq \left| \sum_{j=1}^{M} h_{i_0,j} U_j \right| = |F_{i_0}|. \tag{2.45}
\]

Therefore,
\[
\|U_h\|_{\infty} \leq \frac{1}{B_1(i_0) + B_2(i_0)} |F_{i_0}|
\]
\[
\leq e^{\lambda(b-a+\delta)}(b-a)\beta \frac{\|F\|_{\infty}}{2}. \tag{2.46}
\]

Finally, by (2.28) and (2.29), it holds that
\[
H(U - U_h) = R_h. \tag{2.47}
\]

Then the desired results follow from (2.43), (2.46), and (2.21). \hfill \Box

**Remark 2.2.** If \( g(x) \neq 0 \), then \( d_1(i) \) and \( d_2(i) \) usually should also be calculated by numerical integrations. Since \( g(x) \) is known, with some of today’s well-developed algorithms and software [1, 3, 12, 25], they may be calculated directly or adaptively with the accuracy not less than the order of the local truncation errors. Thus, Theorem 2.1 still holds.

### 2.3. Algorithm implementation

This section focuses on the effective algorithm implementation.

#### 2.3.1. Structure of the stiffness matrix

A symmetric matrix \( T_M \) is called a symmetric Toeplitz matrix if its entries are constant along each diagonal, i.e.,

\[
T_M = \begin{bmatrix}
    t_0 & t_1 & \cdots & t_{M-2} & t_{M-1} \\
    t_1 & t_0 & t_1 & \cdots & t_{M-2} \\
    \vdots & t_1 & \ddots & \cdots & \vdots \\
    t_{M-2} & \cdots & \cdots & t_0 & t_1 \\
    t_{M-1} & t_{M-2} & \cdots & t_1 & t_0 
\end{bmatrix}. \tag{2.48}
\]

The Toeplitz matrix \( T_M \) is circulant if \( t_k = t_{M-k} \) for all \( 1 \leq k \leq M-1 \). It should be noticed that a symmetric Toeplitz matrix is determined by its first column (or first row). Therefore, we can store \( T_M \) with \( M \) entries. Moreover, the product of matrix \( T_M \) with a vector \( V \in \mathbb{R}^M \) can be performed by FFT in \( O(M \log M) \) arithmetic operations [7, pp. 11-12] and [16, 17, 19].
From (2.25) and (2.30), it is easy to see that except the main diagonal, the entries of matrix $H$ are constant along each diagonal. Let $D$ denotes the main diagonal matrix of $H$. Then

$$H = D + (H - D), \quad (2.49)$$

and $H - D$ is a symmetric Toeplitz matrix. Therefore, we can store $H$ with $2M$ entries, and calculate $HV$ by $DV + (H - D)V$ with the cost $\mathcal{O}(M \log M)$.

### 2.3.2. The fast conjugate gradient method

The matrix $H$ is fully dense due to the nonlocal property of the tempered fractional Laplacian. The $\mathcal{O}(M^3)$ operations are required to solve the linear system (2.29) by a direct method. Since the product $HV$ can be effectively computed in $\mathcal{O}(M \log M)$, the Krylov subspace iterative methods such as the conjugate gradient (CG) method naturally provide feasible and economical choices for solving such linear systems. These iterative methods only require a few matrix-vector products at each step, so they can be conveniently accomplished in $\mathcal{O}(M \log M)$ operations if the total number of iteration steps needed for achieving their convergence is not too large.

It is well known that the convergence speed of the CG method is influenced by the condition number, or more precisely, the eigenvalue distribution of $H$; the more clustered around the unity the eigenvalues are, the faster the convergence rate will be [7, pp. 8-10]. By the Gersgorin theorem, (2.36), and (2.42), it holds that

$$\lambda_{\min}(H) > B_1(i) + B_2(i) \geq \frac{2e^{-\lambda(b-a+\delta)(b-a)^{-\beta}}}{\beta} \quad (2.50)$$

and

$$\lambda_{\max}(H) \leq \max_{1 \leq i \leq M} \left( h_{i,i} - \sum_{j=1, j\neq i}^{M} h_{i,j} \right) \leq \max_{1 \leq i \leq M} \left( 2h_{i,i} - B_1(i) - B_2(i) \right)$$

$$\leq \max_{1 \leq i \leq M} \left( 2\mathcal{O}_1(s, e^{\lambda(y-x_i)}) + 2\mathcal{O}_2(s, e^{\lambda(x_i-y)}) \right) + \frac{4h^{-\beta}e^{-\lambda h}}{s_1 + 1 - \beta} + B_1(i) + B_2(i) \quad (2.51)$$
Noting that
\[ \mathcal{A}_1 \left( s, \frac{e^{\lambda(x-y)}}{(x-y)^s} \right) \leq \int_{x_1}^{x_{i-1}} (x_i - y)^{-1-\beta} \, dy + \int_{x_{i-2}}^{x_{i-1}} (x_i - x_{i-1})^{-s} (x_i - y)^{-1-\beta} \, dy \]
\[ \leq h^{-\beta} \left( \frac{1 - (i-1)^{-\beta}}{\beta} + 1 \right) \leq Ch^{-\beta}, \]
\[ \mathcal{A}_2 \left( s, \frac{e^{\lambda(x-y)}}{(y-x_i)^s} \right) \leq \int_{x_{i+1}}^{x_{i+2}} (x_{i+1} - x_i)^{-s} (y - x_i)^{-1-\beta} \, dy + \int_{x_{i+1}}^{x_M} (y - x_i)^{-1-\beta} \, dy \]
\[ \leq h^{-\beta} \left( \frac{1 - (M-i)^{-\beta}}{\beta} + 1 \right) \leq Ch^{-\beta}, \]
and \( B_1(i) \leq \frac{h^\beta}{\beta}, B_2(i) \leq \frac{h^\beta}{\beta}, \) we have
\[ \lambda_{\text{max}}(H) \leq Ch^{-\beta}. \] (2.52)

Furthermore, defining
\[ V_1 := \left( \underbrace{0, 0, \ldots, 0, 1, 0, \ldots, 0}_{i-1}, \underbrace{1, 1, \ldots, 1}_{M-i} \right), \quad V_2 := \left( \underbrace{1, 1, \ldots, 1}_{M} \right) \]
and using the Courant-Fischer theorem [7, Theorem 1.5]
\[ \lambda_{\text{min}}(H) = \min_{V \neq 0} \frac{(HV, V)}{(V, V)}, \quad \lambda_{\text{max}}(H) = \max_{V \neq 0} \frac{(HV)}{(V, V)}, \] (2.53)

it follows that
\[ \lambda_{\text{max}}(H) \geq \frac{(HV, V)}{(V, V)} = h_{i,i} \geq \frac{2h^{-\beta} e^{-\lambda}}{s + 1 - \beta} \geq Ch^{-\beta} \] (2.54)

and
\[ \lambda_{\text{min}}(H) \leq \frac{(HV_2, V_2)}{(V_2, V_2)} = \frac{1}{M} \sum_{i=1}^{M} \left( h_{ii} + \sum_{j \neq i}^{M} h_{ij} \right) \]
\[ \leq \frac{1}{M} \left( \sum_{i=1}^{M} \int_{x_0}^{x_1} \frac{1}{(x_i - y)^{1+\beta}} \, dy + \sum_{i=1}^{M} \int_{x_i}^{x_{i+1}} \frac{1}{((M+i-1)y)^{1+\beta}} \, dy \right) + \frac{1}{M} \sum_{i=1}^{M} \int_{x_i}^{x_{i+1}} \frac{1}{(y-x_i)^{1+\beta}} \, dy \]
\[ \leq \frac{4h}{b-a} \left( \int_{h}^{Mh} y^{-1-\beta} \, dy + \frac{h^{-\beta}}{2s} + \sum_{i=1}^{M} \frac{1}{\beta} + \frac{h^{-\beta}}{(s+1-\beta)} \right), \]
\[ \leq \frac{4h}{b-a} \left( \frac{h^{-\beta} - (Mh)^{-\beta}}{\beta} + \frac{h^{-\beta}}{2s} + \frac{1}{\beta} \int_{0}^{Mh} y^{-\beta} \, dy + \frac{h^{-\beta}}{(s+1-\beta)} \right), \]
\[ \leq \frac{4}{b-a} \left( \frac{1}{\beta} + \frac{1}{2s} + \frac{(b-a)^{-\beta}}{(1-\beta)} + \frac{1}{s + 1 - \beta} \right), \] (2.55)
where

\[
\sum_{i=2}^{M} \int_{x_0}^{x_1} \frac{(x_i - y)^{s-1-\beta}}{(ih)^{i}} dy \\
\leq (2h)^{-s} \int_{x_0}^{x_1} (x_2 - y)^{s-1-\beta} dy + \sum_{i=2}^{M-1} \int_{x_0}^{x_1} (x_i - y)^{s-1-\beta} dy \\
\leq \frac{h^{-\beta}}{2^s} + \int_{h}^{Mh} y^{-1-\beta} dy
\]  

(2.56)

and

\[
\sum_{i=1}^{M-1} \int_{x_M}^{x_{M+1}} \frac{(y - x_i)^{s-1-\beta}}{((M + 1 - i)h)^{i}} dy \\
\leq \sum_{i=2}^{M-1} \int_{x_M}^{x_{M+1}} (y - x_i)^{s-1-\beta} dy + (2h)^{-s} \int_{x_M}^{x_{M+1}} (y - x_{M-1})^{s-1-\beta} dy \\
\leq \frac{h^{-\beta}}{2^s} + \int_{h}^{Mh} y^{-1-\beta} dy
\]  

(2.57)

have been used. Thus, \(\lambda_{\min} \sim 1\) and \(\lambda_{\max} \sim h^{-\beta}\). As \(h\) becomes small, the eigenvalues of \(H\) distribute in a very large interval of length \(Ch^{-\beta}\). Therefore, efficient preconditioning is required to speed up the convergence of CG iterations, that is, instead of solving the original system \(HU_h = F\), we find a s.p.d. matrix \(B = LL^T\) and solve the preconditioned system

\[H^*U^* = F^*,\]

(2.58)

where \(H^* = L^{-1}HL^{-T}, U^* = L^TU_h\) and \(F^* = L^{-1}F\). We require that \(B\) ‘near’ to \(H\) in some sense, such that the eigenvalue distributions of \(H^*\) is clustered compared to \(H\). In the following, we consider two types of preconditioners:

Firstly, since for \(j \leq i - 2\), we have

\[
|h_{i,j}| = (A_1(i, j + 1) + A_2(i, j)) e^{-\lambda(i-j)h} (i - j)^{-s} = \\
e^{-\lambda(i-j)h} h^{s+1(i-j)^3} \left( \int_{x_{j+1}}^{x_{j+1}} (x_{j+1} - y)(x_i - y)^{s-1-\beta} dy \\
+ \int_{x_{j-1}}^{x_{j-1}} (y - x_{j-1})(x_i - y)^{s-1-\beta} dy \right) \\
\leq C(i - j)^{-s} (i - j - 1)^{s-1-\beta} h^{-\beta} e^{-\lambda(i-j)h},
\]

(2.59)

if \(|i - j|\) is sufficiently large, the entries \(h_{i,j}\) are very small relative to the one near the main diagonal (with the order \(h^{-\beta}\)). Hence, similar to [19, 30], we define a symmetric
and expect $\mathbf{G}$ to be a reasonable approximation of $\mathbf{H}$. Here $\mathbf{O}$ is a diagonal matrix satisfying $\mathbf{H}\mathbf{e} = \mathbf{G}\mathbf{e}$, $\mathbf{e} = (1, 1, \cdots, 1)$, which is a common technique (i.e., the so-called diagonal compensation) in designing preconditioners for $M$-matrices [5, Sections 6 and 7]. By Lemma 2.1 and (2.37), $\mathbf{G}$ is a s.p.d. $M$-matrix. Thus, we can perform its incomplete Cholesky (ichol) factorization to generate a banded matrix $\mathbf{L}$. We desire that matrix $\mathbf{B} = \mathbf{L}\mathbf{L}^T$ serves as an effective preconditioner of $\mathbf{H}$.

Secondly, T. Chan’s (optimal) circulant preconditioner has been widely used in solving the Toeplitz systems [7,17]. For the Toeplitz matrix $\mathbf{T}_M$ defined in (2.48), the entries in the first column of the T. Chan circulant preconditioner $C_F(\mathbf{T}_M)$ are given by

$$c_k = \frac{(M - k)t_k + kt_{M-k}}{M}, \quad 0 \leq k \leq M - 1. \quad (2.61)$$

However, matrix $\mathbf{H}$ here may not be a Toeplitz matrix (due to the entries on the main diagonal), we can not construct the T. Chan circulant preconditioner directly. Recalling that the generation process of $h_{i,i}$, it holds that $h_{i,i} = h_{M+1-i,M+1-i}$ for $i = 1, 2, \cdots, \lfloor \frac{M}{2} \rfloor$, and

$$h_{i+1,i+1} - h_{i,i} = B_1(i + 1) + B_2(i + 1) - B_1(i) - B_2(i)$$
$$\quad + \int_a^{x_1} I_{[a,x_1]} (e^{-\lambda(x_{i+1}-y)}(x_{i+1} - y)^{-\beta}) (x_{i+1} - y)^{1-\beta} dy$$
$$\quad - \int_{x_M}^b I_{[x_M,b]} (e^{-\lambda(y-x_1)}(y-x_1)^{-\beta}) (y-x_1)^{1-\beta} dy$$
$$\quad = \int_{x_1-a}^{x_{i+1}-a} (I_{[x_1-a,x_{i+1}-a]} (e^{-\lambda t} t^{-\beta}) - e^{-\lambda t} t^{-\beta}) t^{1-\beta} dy$$
$$\quad - \int_{b-x_i}^{b-x_{i+1}} (I_{[b-x_{i+1},b-x_i]} (e^{-\lambda t} t^{-\beta}) - e^{-\lambda t} t^{-\beta}) t^{1-\beta} dy \quad (2.62)$$
for $i = 1, 2, \cdots, \lfloor \frac{M}{2} \rfloor - 1$, where $\lfloor \beta \rfloor := \{ z \in \mathbb{N} : 0 < z - \beta \leq 1 \}$, and the definition of $I_{[a,b]}$ is given in (2.7). We have the following observations: when $\lambda = s = 0$, then $h_{i,i} = h_{i+1,i+1}$, which means that matrix $\mathbf{H}$ actually is a Toeplitz matrix; when $s = 0$ and $\lambda > 0$, by $\left| (e^{-\lambda t}) \right| = \lambda^2 e^{-\lambda t} \leq \lambda^2 (t \geq 0)$, it follows that

$$\left| h_{i+1,i+1} - h_{i,i} \right| \leq C h^2 \beta \ll h_{i,i} \sim h^{-\beta} \quad \text{for } h \rightarrow 0. \quad (2.63)$$
Though it is not easy to prove that the changes of the entries of the main diagonal of $H$ are slow for the cases $s = 1$, the numerical results show they actually do. These inspire us to construct a Toeplitz matrix $G$ as
\[
G = \sum_{i=1}^{M} h_i i + (H - D) \quad (I \text{ is identity matrix})
\]
and expect the corresponding matrix $C_\beta(G)$ to be an effective preconditioner of $H$.

The algorithms of the CG method and the preconditioned CG (PCG) method can be founded in [5, pp. 470-473]. At each iteration, the required product of $H$ with a vector $V \in \mathbb{R}^M$ can be performed with the cost $\mathcal{O}(M \log M)$. Note that in the PCG algorithm, the matrices $L$ and $L^T$ do not appear explicitly, to perform the preconditioning, we only need to calculate $B^{-1}V$ or to solve the corresponding equation $BX = V$ effectively. For the ichol factorization preconditioner, the bandwidth characteristic of matrix $L$ allows us to solve $BX = V$ with the cost $\mathcal{O}(kM)$; for the T. Chan circulant preconditioner $B = C_\beta(G)$, $B^{-1}V$ can be calculated by the FFT with the cost $\mathcal{O}(M \log M)$ [7, pp. 11-12]. Thus the total cost for each iteration still is $\mathcal{O}(M \log M)$.

### 3. Numerical scheme for the case $\beta \in [1, 2)$

By choosing $s = 0, s_1 = 1$ or $s = s_1 = 1$ in (2.2)-(2.4), the numerical schemes introduced in Subsection (2.1) can be easily extended to the cases $\beta \in [1, 2)$. If $\beta \in (1, 2)$, the estimates (2.15)-(2.17) still hold and we have
\[
|r_h^i| = \begin{cases} 
\mathcal{O}(h^{2-\beta}) & \text{for } u \in C^3(\Omega), \ s = 0, s_1 = 1, \\
\mathcal{O}(h^{3-\beta}) & \text{for } u \in C^3(\overline{\Omega}), \ s = s_1 = 1.
\end{cases}
\]
If $\beta = 1$, (2.17) still holds for $s_1 = 1$ and (2.15) and (2.16) are also true for $s = 0$, however, for $s = 1$, we have the following estimates
\[
\left| \int_a^{x_{i-1}} g_1(i, s, y)(x_i - y)^{s-1-\beta} dy - \mathcal{O}(s, g_1) \right| 
\leq \frac{\|g_1^{(2)}(i, s, y)\|_{L^\infty[a, x_{i-1}]}}{2} h^2 \left| \ln(h) - \ln(x_i - a) \right| \quad (3.2)
\]
and
\[
\left| \int_{x_{i+1}}^{b} g_2(i, s, y)(y - x_i)^{s-1-\beta} dy - \mathcal{O}(s, g_2) \right| 
\leq \frac{\|g_2^{(2)}(i, s, y)\|_{L^\infty[x_{i+1}, b]}}{2} h^2 \left| \ln(h) - \ln(b - x_i) \right|; \quad (3.3)
\]
then we have that if $\beta = 1$,
\[
|r_h^i| = \begin{cases} 
\mathcal{O}(h^{2-\beta}) & \text{for } u \in C^3(\Omega), \ s = 0, s_1 = 1, \\
\mathcal{O}(h^2 \ln(h)) & \text{for } u \in C^3(\overline{\Omega}), \ s = s_1 = 1.
\end{cases}
\]

\[
(2.64)
\]
With the same proof process as in Theorem 2.1, it holds that

**Theorem 3.1.** For the scheme (2.29), we have the following estimates.

1. Let $\beta \in [1,2), s = 0, s_1 = 1$, and $u(x) \in C^3(\Omega)$. Then

$$
\|U - U_h\| \leq C_1 h^{2-\beta}, \quad \|U - U_h\|_{\infty} \leq C_2 h^{2-\beta},
$$

(3.5)

where $C_1$ and $C_2$ may depend on the values of $u^{(k)}(x), k = 0,1,2,3$ on $\Omega$, but independent of $h$.

2. Let $\beta \in [1,2), s = s_1 = 1$, and $u(x) \in C^3(\Omega)$. Then

$$
\|U - U_h\| \leq C_1 h^{3-\beta}, \quad \|U - U_h\|_{\infty} \leq C_2 h^{3-\beta}
$$

(3.6)

for $\beta \in (1,2)$, and

$$
\|U - U_h\| \leq C_1 |\ln(h)| h^2, \quad \|U - U_h\|_{\infty} \leq C_2 |\ln(h)| h^2
$$

(3.7)

for $\beta = 1$, where $C_1$ and $C_2$ may depend on the values of $u^{(k)}(x), k = 0,1,2,3$ on $\Omega$, but independent of $h$.

As for generating the stiffness matrix $H$, the results in (2.30) and (2.31)-(2.35) still hold for $\beta \in (1,2)$. While for $\beta = 1$ and $s = 0$, one has

$$
A_1(i,0,j+1) + A_2(i,0,j) = A_3(i,0,j+1) + A_4(i,0,j)
$$

$$
= \frac{1}{6} (2 \ln(|i-j| - \ln(|i-j|+1) - \ln(|i-j|-1)),
$$

$$
A_2(i,0,i-1) = A_3(i,i+2) = \frac{1}{6} (1 - \ln(2)),
$$

$$
A_1(i,0,1) = \frac{1}{6} (\ln(i+1) - \frac{1}{2}),
$$

$$
A_4(i,0,M+1) = \frac{1}{6} (\ln(M+1-i) - \frac{1}{M+1-i});
$$

(3.8)

and for $\beta = 1$ and $s = 1$, one has

$$
A_1(i,1,j+1) + A_2(i,1,j) = A_3(i,1,j+1) + A_4(i,1,j)
$$

$$
= \frac{1}{6} \left( -2i-j + \ln(|i-j|) + (i-j+1) \ln(|i-j|+1)
$$

$$
+ (i-j-1) \ln(|i-j|-1) \right),
$$

$$
A_2(i,1,i-1) = A_3(i,i+2) = \frac{1}{6} (2 \ln(2) - 1),
$$

$$
A_1(i,1,1) = \frac{1}{6} ((1-i) \ln(i+1) + 1),
$$

$$
A_4(i,1,M+1) = \frac{1}{6} ((1-M) \ln(M+1-i) + 1).
$$

(3.9)

The calculations for $B_1(i)$ with $\beta = 1$ are given below: when $\lambda = 0$, the results in (2.31) still hold; when $\lambda > 0$ and $b - x_i \geq \frac{1}{\lambda}$, we first rewrite $B_1(i)$ as

$$
\int_{b-x_i}^{\infty} e^{-\lambda t} t^{-2} dt \approx \int_{\lambda/K}^{\infty} e^{-\frac{\lambda}{t}} t dt = \left( \frac{1}{2(\lambda/K)} - \frac{\lambda}{2K} \right) \int_{-1}^{1} e^{-\lambda/\eta} d\xi
$$

(3.10)
with $\eta(\xi, x_i) =: \frac{\xi^2}{2(b-x_i)} - \frac{\lambda(\xi-1)}{2b}$ and then calculate $\int_{-1}^{1} e^{-\lambda/\eta(\xi, x_i)} d\xi$ by the Jacobi-Gauss quadrature with the weight function $(1-\xi)^{\alpha}(1+\xi)^{\beta}$ [13,25] (In our calculation, $K$ is chosen as 80); when $\lambda > 0$ and $b - x_i < \frac{1}{2\lambda}$, we first rewrite $B_1(i)$ as

$$B_1(i) = \frac{e^{-\lambda(b-x_i)}}{b-x_i} - \lambda \int_{\lambda(b-x_i)}^{\infty} \frac{e^{-t}}{t} dt,$$  
(3.11)

and then use the series expansion representation in [1, Eq. 5.1.11], i.e.,

$$\int_{\lambda(b-x_i)}^{\infty} \frac{e^{-t}}{t} dt = -\gamma - \ln(z) - \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n\Gamma(n+1)}, \quad z = \lambda(b-x_i),$$  
(3.12)

where $\gamma$ is the Euler constant (in our calculations, the series is truncated with the first 26 items).

Since matrix $H$ has the same structure as the case $\beta \in (0, 1)$, the implementation techniques developed in Section 2.3 can also be used here to solve the corresponding algebraic equation, and the numerical results show they still work well.

**4. Numerical results**

In this section, we make some numerical experiments to show the performance of numerical schemes above. All are run in MATLAB 7.11 on a PC with Intel(R) Core (TM)i7-4510U 2.6 GHz processor and 8.0 GB RAM. For the CG and PCG iterations, we adopt the initial guess $U_0 = 0$ and the stopping criterion

$$\frac{\|r(k)\|_2}{\|r(0)\|_2} \leq 1e^{-9},$$

where $r(k)$ denotes the residual vector after $k$ iterations. Let $h_1 = (b-a)/(M_1+1)$ and $h_2 = (b-a)/(M_2+1)$. The convergence rates at $M = M_1$ are computed by

$$\text{rate} = \left\{ \begin{array}{ll}
\frac{\ln((\text{the error at } h_1)) / \ln(h_1/h_2)}{\ln((\text{the error at } h_2)) / \ln(h_1/h_2)}, & s = s_1 = 1, \beta = 1, \\
\frac{\ln((\text{the error at } h_1 / \text{the error at } h_2))}{\ln(h_1/h_2)}, & \text{otherwise}.
\end{array} \right.$$  
(4.1)

**Example 4.1.** Consider model (1.1) with $g(x) = 0$, and the force term $f(x)$ being derived from the exact solution $u(x) = x^2(1-x)$ for $x \in \Omega$.

Note that $u \in C^3(\bar{\Omega})$. If $\lambda = 0$, the explicit form of $f(x)$ is given in [29, Example 1]. If $\lambda \neq 0$, the value of $f(x)$ at $x_i$ should be calculated numerically. More specifically, for
Consider model (1.1) in $\Omega = (0, 1)$ with the boundary condition

$$g(x) = (-2x)\chi_{[-\frac{1}{2}, 0]} + (2x - 2)\chi_{[1, \frac{3}{2}]}$$  \tag{4.4}$$

and source term $f(x)$ being derived from the exact solution

$$p(x) = (-2x)\chi_{[-\frac{1}{2}, 0]} + (x - x^2)^2\chi_{(0, 1)} + (2x - 2)\chi_{[1, \frac{3}{2}]}.$$  \tag{4.5}$$

Obviously, $u(x)$ is discontinuous at $x = \frac{1}{2}$ and $x = -\frac{1}{2}$. In the numerical simulation, for $\lambda = 0$, the $f(x_i)$, $d_1(i)$, $d_2(i)$ are obtained exactly; for $\lambda \neq 0$, they are calculated numerically with the techniques as in Example 4.1. The numerical results are listed in Table 4.
Table 1: Errors and convergence rates in Example 4.1 with $M = 2^J - 1$.

| $(\beta, s, \gamma)$ | $J$ | $\lambda = 0.5$ | $\lambda = 3$ |
|----------------------|-----|----------------|----------------|
|                      | $L^2$-Err | Rate | $L^\infty$-Err | Rate | $L^2$-Err | Rate | $L^\infty$-Err | Rate |
| (0.5, 0, 0)          | 12   | 2.0468e-06 | – | 3.1760e-06 | – | 7.0941e-06 | – | 1.0474e-05 | – |
| (0.5, 0, 0)          | 14   | 3.0234e-07 | 1.50 | 1.3177e-06 | 1.50 | 2.5222e-06 | 1.49 | 3.7238e-06 | 1.49 |
| (0.5, 0, 0)          | 12   | 2.6157e-09 | – | 4.1651e-09 | – | 2.7141e-08 | – | 3.9151e-08 | – |
| (0.5, 0, 0)          | 13   | 6.5490e-10 | 2.00 | 1.0428e-09 | 2.00 | 6.8022e-09 | 2.00 | 9.8127e-09 | 2.00 |
| (0.5, 0, 0)          | 14   | 1.6391e-10 | 2.00 | 2.6096e-10 | 2.00 | 1.7036e-09 | 2.00 | 2.4577e-09 | 2.00 |

Table 2: Performance of the CG and PCG methods in Example 4.1 with $\lambda = 0.5$ and $M = 2^J - 1$.

| $(\beta, s, \gamma)$ | $J$ | CG | PCG (Chol) | PCG (T) | Gauss |
|----------------------|-----|----------------|----------|--------|-------|
|                      |     | # iter | CPU(s) | # iter | CPU(s) | # iter | CPU(s) | # iter | CPU(s) | # iter | CPU(s) |
| (0.5, 0, 0)          | 12   | 97    | 0.8174 | 40     | 0.1472 | 11     | 0.0182 | 1.1997 |
| (0.5, 0, 0)          | 13   | 115   | 0.3248 | 44     | 0.1789 | 11     | 0.0627 | 6.0118 |
| (0.5, 0, 0)          | 14   | 138   | 2.7400 | 49     | 2.1894 | 11     | 0.1691 | 56.6159 |
| (0.5, 0, 0)          | 12   | 74    | 0.1622 | 39     | 0.0851 | 0      | 0.0159 | 0.7742 |
| (0.5, 0, 0)          | 13   | 88    | 0.3222 | 43     | 0.1619 | 11     | 0.0609 | 7.5780 |
| (0.5, 0, 0)          | 14   | 105   | 3.1492 | 49     | 1.1515 | 11     | 0.1355 | 57.7863 |
| (0.5, 0, 0)          | 12   | 329   | 1.3938 | 47     | 0.1632 | 15     | 0.0215 | 0.8483 |
| (0.5, 0, 0)          | 13   | 468   | 1.3587 | 58     | 0.2424 | 16     | 0.0888 | 6.0763 |
| (0.5, 0, 0)          | 14   | 664   | 7.3204 | 71     | 3.3773 | 17     | 0.2877 | 53.1450 |
| (0.5, 0, 0)          | 12   | 337   | 0.6903 | 47     | 0.7371 | 15     | 0.0715 | 0.8431 |
| (0.5, 0, 0)          | 13   | 479   | 1.6629 | 58     | 0.2146 | 16     | 0.0872 | 6.1244 |
| (0.5, 0, 0)          | 14   | 680   | 7.5843 | 71     | 3.3687 | 17     | 0.2486 | 53.5024 |
| (0.5, 0, 0)          | 12   | 1363  | 1.8886 | 30     | 0.2089 | 29     | 0.0356 | 0.8860 |
| (0.5, 0, 0)          | 13   | 2300  | 8.8463 | 35     | 0.1220 | 34     | 0.1779 | 6.7129 |
| (0.5, 0, 0)          | 14   | 3880  | 25.9750 | 42     | 0.4814 | 41     | 0.4571 | 55.0814 |
| (0.5, 0, 0)          | 12   | 1383  | 1.5639 | 30     | 0.0609 | 29     | 0.0448 | 0.9285 |
| (0.5, 0, 0)          | 13   | 2333  | 7.2669 | 35     | 0.1538 | 33     | 0.1960 | 7.5258 |
| (0.5, 0, 0)          | 14   | 3935  | 30.7326 | 42     | 1.0987 | 39     | 0.4440 | 57.1891 |

Since $u(x)$ is smooth enough on $\widehat{\Omega}$, the convergence rates are consistent with the theoretical predictions in Theorems 2.1 and 3.1. In fact, for $\lambda = 0$ and $\beta \in (1, 2)$, the numerical schemes obtained with $s_1 = s = 1$ seem to have a slightly bigger convergence rate than $3 - \beta$. 
### Table 3: Performance of the CG and PCG methods in Example 4.1 with $\lambda = 3$ and $M = 2^J - 1$.

| $(\beta, s, \gamma)$ | $J$ | \(\text{CG} \) | \(\text{PCG (Chol)} \) | \(\text{PCG (T)} \) | \(\text{Gauss} \) |
|----------------------|-----|-----------------|-----------------|-----------------|-----------------|
|                      |     | \# iter | CPU(s) | \# iter | CPU(s) | \# iter | CPU(s) | \# iter | CPU(s) |
| $(0.5, 0, 0)$        | 12  | 127     | 0.2231 | 70     | 0.1079 | 12      | 0.0207 | 0.8877 |
|                      | 14  | 182     | 5.2566 | 108    | 0.9474 | 12      | 1.0048 | 49.5857 |
| $(0.5, 1, 1)$        | 12  | 97      | 0.5242 | 70     | 0.0897 | 11      | 0.0199 | 0.9509 |
|                      | 14  | 116     | 0.3499 | 88     | 0.2889 | 12      | 0.0525 | 6.8463 |
| $(1.0, 1, 0)$        | 12  | 376     | 0.3920 | 57     | 0.0727 | 17      | 0.0269 | 0.8913 |
|                      | 14  | 534     | 1.7202 | 74     | 0.3233 | 17      | 0.1141 | 7.2649 |
| $(1.0, 1, 1)$        | 12  | 758     | 5.2566 | 108    | 0.9474 | 12      | 1.0048 | 50.5869 |
|                      | 14  | 116     | 0.3499 | 88     | 0.2889 | 12      | 0.0525 | 6.8463 |
| $(1.5, 1, 0)$        | 12  | 1423    | 1.9613 | 29     | 1.1896 | 30      | 0.0516 | 0.8078 |
|                      | 14  | 2400    | 6.6799 | 35     | 0.2333 | 37      | 0.2130 | 6.2326 |
| $(1.5, 1, 1)$        | 12  | 2435    | 9.4863 | 36     | 0.1528 | 37      | 0.1880 | 6.8698 |
|                      | 14  | 2435    | 9.4863 | 36     | 0.1528 | 37      | 0.1880 | 6.8698 |

### Table 4: Errors and convergence rates in Example 4.2 with $M = 2^J$, solved by the PCG method with T. Chan’s preconditioner.

| $(\beta, s, \gamma)$ | $J$ | $L^2$-Err | Rate | $L^\infty$-Err | Rate | iter | $L^2$-Err | Rate | $L^\infty$-Err | Rate | iter |
|----------------------|-----|-----------|------|-----------------|------|------|-----------|------|-----------------|------|------|
| $(0.5, 0, 0)$        | 11  | 1.1146e-06| -    | 1.6944e-06      | -    | 8    | 5.9414e-06| -    | 9.6168e-06      | -    | 9    |
|                      | 12  | 3.9593e-07| 1.49 | 6.0100e-07      | 1.50 | 8    | 2.1173e-06| 1.49 | 3.4273e-06      | 1.49 | 9    |
| $(0.5, 1, 1)$        | 11  | 4.5687e-09| -    | 6.7988e-09      | -    | 8    | 4.8132e-08| -    | 7.1820e-08      | -    | 9    |
|                      | 12  | 1.1421e-09| 2.00 | 1.6999e-09      | 2.01 | 8    | 1.2059e-08| 2.00 | 1.7999e-08      | 2.00 | 9    |
| $(1.0, 1, 0)$        | 11  | 3.7770e-06| -    | 6.2802e-06      | -    | 12   | 4.5179e-05| -    | 6.3557e-05      | -    | 13   |
|                      | 12  | 1.8917e-06| 1.00 | 3.1450e-06      | 1.00 | 12   | 22.2684e-05| 0.99 | 3.1930e-05      | 0.99 | 13   |
| $(1.0, 1, 1)$        | 11  | 9.4646e-07| 1.00 | 1.5737e-06      | 1.00 | 13   | 1.1368e-05| 1.00 | 1.6007e-05      | 1.00 | 14   |
|                      | 12  | 9.4496e-09| 51.3552e-08 | - | 12   | 1.6933e-07| - | 2.6506e-07 | - | 13   |
| $(1.5, 1, 0)$        | 11  | 5.5961e-05| -    | 9.0153e-05      | -    | 19   | 9.7334e-05| -    | 1.5346e-04      | -    | 20   |
|                      | 12  | 3.9500e-05| 0.50 | 6.3783e-05      | 0.50 | 21   | 6.9986e-05| 0.48 | 1.1035e-04      | 0.48 | 22   |
| $(1.5, 1, 1)$        | 10  | 6.2783e-08| -    | 8.7275e-08      | -    | 16   | -        | -    | -                | -    | -    |
|                      | 11  | 1.5532e-08| 2.01 | 2.1605e-08      | 2.01 | 19   | 2.3015e-06| 1.49 | 3.6244e-06      | 1.49 | 20   |
|                      | 12  | 3.8558e-09| 2.01 | 5.3661e-09      | 2.01 | 21   | 8.1576e-07| 1.50 | 1.2851e-06      | 1.50 | 23   |
| $(1.5, 1, 1)$        | 13  | 9.5433e-10| 2.01 | 1.3282e-09      | 2.01 | 24   | 2.8888e-07| 1.50 | 4.5524e-07      | 1.50 | 25   |

**Example 4.3.** Consider model (1.1) in $\Omega = (-r, r)$ with source term $f(x) = 1$ and absorbing boundary condition $g(x) = 0$. 
When $\lambda = 0$, the exact solution is [9, Subsection 3.1]
\[
  u(x) = \frac{\sqrt{\pi}(r^2 - x^2)^{\beta/2}}{2^{\beta} \Gamma(1 + \beta/2)\Gamma(1/2 + \beta/2)} \quad \text{for } x \in \Omega. \tag{4.6}
\]
It is easy to see that $u(x)$ has a poor regularity at the boundaries of $\Omega$. When $\lambda \neq 0$, $u(x)$ cannot be obtained explicitly; the errors (i.e., the data under $L^2$-Err and $L^\infty$-Err) under stepsize $h$ in Table 5 with the $L^2$ and $L^\infty$ norms are, respectively,
\[
  \|U_{h/2} - U_h\| \quad \text{and} \quad \max_{1 \leq i \leq M} |U_{h/2} - U_h|, \tag{4.7}
\]
and the convergence rates are measured by using these errors, where $h = \frac{2r}{M+1}$ with $M = 2^J - 1$. The numerical results in Table 5 show that the convergence rates are small for two cases, being consistent with the results in [29, Example 2].

In statistical physics, the solution $u(x)$ denotes the mean first exit time of a particle starting at $x$ away from the given domain $\Omega$ [8, 11]. For $\lambda = 0, 0.5$ and $3$, the numerical solutions obtained with $(s, s_1) = (1, 1)$ and different values of $\beta = 0.5, 1, 1.5, r = 1, 2, 5$ are listed in Figures 2, which show: for the same domain $\Omega$, the mean first exit time increases with the increases of the value of $\lambda$; and when $\lambda > 0$, for any fixed value of the starting point $x$, the mean exit times are shorter for larger values of $\beta$. 

Figure 1: Eigenvalue distribution of the systems (2.29) or (2.58) in Example 4.1 with $J = 13$ and $\lambda = 3$. First column: without preconditioning; second column: preconditioned with the ichol factorization; third column: preconditioned with the T. Chan circulant matrix. The horizontal and vertical axes are respectively the real and imaginary axis.
Table 5: Errors and convergence rates in Example 4.3 with $M = 2^j - 1$, solved by the PCG method with T. Chan's preconditioner. The data in the parentheses denote the rates obtained with the errors in (4.7).

| $(\beta, s, \tau)$ | $M$ | $L^2$-Err Rates | $L^\infty$-Err rates | $\lambda = 0$ |
|-------------------|-----|------------------|----------------------|-------------|
| \((0.5, 0, 0)\)   | 11  | 3.5668e-03       | -                    | 3.3886e-02  |
|                   | 12  | 2.1205e-03 0.25 (0.25) | 4.5975e-02 0.25 (0.25) | 2.117e-01 |
|                   | 13  | 1.2608e-03 0.25 (0.25) | 3.8658e-02 0.25 (0.25) | 8.5734e-03 |
| \((0.5, 1, 0)\)   | 11  | 1.4675e-03       | -                    | 1.3158e-02  |
|                   | 12  | 8.7272e-04 0.25 (0.25) | 2.0163e-02 0.25 (0.25) | 8.406e-02 |
|                   | 13  | 5.1896e-04 0.25 (0.25) | 1.4254e-02 0.25 (0.25) | 6.5320e-02 |
| \((1.0, 1, 0)\)   | 11  | 6.3631e-04       | -                    | 1.6940e-03  |
|                   | 12  | 3.3183e-04 0.45 (0.45) | 3.2322e-03 0.45 (0.45) | 4.2848e-03 |
|                   | 13  | 1.7249e-04 0.45 (0.45) | 2.3497e-03 0.45 (0.45) | 3.0161e-03 |
| \((1.0, 1, 1)\)   | 11  | 6.3631e-04       | -                    | 1.6940e-03  |
|                   | 12  | 3.3183e-04 0.45 (0.45) | 3.2322e-03 0.45 (0.45) | 4.2848e-03 |
|                   | 13  | 1.7249e-04 0.45 (0.45) | 2.3497e-03 0.45 (0.45) | 3.0161e-03 |
| \((1.5, 0, 0)\)   | 11  | 2.2426e-03       | -                    | 2.7348e-03  |
|                   | 12  | 8.6880e-04 0.56 (0.56) | 2.1375e-03 0.56 (0.56) | 4.2848e-03 |
|                   | 13  | 4.2029e-04 0.56 (0.56) | 1.7159e-03 0.56 (0.56) | 3.0161e-03 |
| \((1.5, 1, 0)\)   | 11  | 1.1260e-04       | -                    | 1.8108e-04  |
|                   | 12  | 6.9126e-05 0.75 (0.75) | 1.4278e-04 0.75 (0.75) | 7.5640e-04 |
|                   | 13  | 3.6486e-05 0.75 (0.75) | 8.406e-05 0.75 (0.75) | 2.0489e-04 |

5. Conclusion

The tempered fractional Laplacian is a recently introduced operator for treating the weaknesses of the fractional Laplacian in some physical processes. This paper provides several classes of finite difference schemes for the tempered fractional Laplacian equation with $\beta \in (0, 2)$. According to the regularity of the solution, one can choose the more appropriate numerical schemes. The detailed numerical analyses are performed, and the effective preconditioning techniques are provided. The implementation details are also discussed, including that the entries of the stiffness matrix can be explicitly and conveniently calculated and the stiffness matrix has Toeplitz-like structure. The efficiencies of the algorithms are verified by extensive numerical experiments and the desired convergence rates are confirmed.

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References

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover Publications, 1965, Chapter 5.
Figure 2: Dependence of the mean exit time $u(x)$ on the size of domain $\Omega$, and the values of $\beta$ and $\lambda$.

[2] G. Acosta and J. P. Borthagaray, *A fractional Laplace equation: regularity of solutions and finite element approximations*, SIAM J. Numer. Anal., 55 (2017), pp. 472-495.

[3] Advanpix Multiprecision Computing Toolbox for MATLAB, http://www.advanpix.com/, accessed 2017-10-24.

[4] D. Applebaum, *Lévy Process and Stochastic Calculus*, 2nd ed., Cambridge University Press, Cambridge, UK, 2009.

[5] O. Axelsson, *Iterative Solution Methods*, Cambridge University Press, UK, 1996.

[6] B. Baeumer and M. M. Meerschaert, *Tempered stable Lévy motion and transient super-diffusion*, J. Comput. Appl. Math., 233 (2010), pp. 2438-2448.

[7] R. H. Chan and X. Q. Jin, *An Introduction to Iterative Toeplitz Solvers*, SIAM, 2007.

[8] W. H. Deng, X. C. Wu, and W. L. Wang, *Mean exit time and escape probability for the anomalous processes with the tempered power-law waiting times*, EPL, 117 (2016).

[9] W. H. Deng, B. Y. Li, W. Y. Tian, and P. W. Zhang, *Boundary problems for the fractional and tempered fractional operators*, Multiscale Model. Simul., in press, 2017.

[10] T. Gao, J. Duan, X. Li, and R. Song, *Mean exit time and escape probability for dynamical systems driven by Lévy noises*, SIAM J. Sci. Comput., 36 (2014), pp. A887-A906.

[11] R. K. Getoor, *First passage times for symmetric stable processes in space*, Trans. Amer. Math.
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Soc., 101 (1961), pp. 75-90.

[12] I. S. Gradshteyn, I. M. Ryzhik, V. Y. Grigoriev, and M. Y. Tseytlin, *Table of Integrals, Series, and Products*, A. Jeffrey, ed., Translated by Scripta Technica, Academic Press, USA, 1980.

[13] J. S. Hesthaven and T. Warburton, *Nodal Discontinuous Galerkin Methods: Algorithms, Analysis, and Applications*, Springer, Berlin, 2008.

[14] J. F. Huang, Y. F. Tang, and L. Vázquez, Convergence analysis of a block-by-block method for fractional differential equations, Numer. Math. Theor. Meth. Appl. 5 (2012), pp. 229-241.

[15] Y. H. Huang and A. Oberman, Numerical methods for the fractional Laplacian: a finite difference-quadrature approach, SIAM. J. Numer. Anal. 52 (2014), pp. 3056-3084.

[16] J. H. Jia and H. Wang, Fast finite difference methods for space-fractional diffusion equations with fractional derivative boundary conditions, J. Comput. Phys., 293 (2015), pp. 359-369.

[17] J. F. Jiang and T. Wang, A circulant preconditioner for fractional diffusion equations, J. Comput. Phys., 242 (2013), 715-725.

[18] C. Li and W. H. Deng, High order schemes for the tempered fractional diffusion equations, Adv. Comput. Math., 42 (2016), pp. 543-572.

[19] F. R. Lin, S. W. Yang, and X. Q. Jin, Preconditioned iterative methods for fractional diffusion equation, J. Comput. Phys., 256 (2014), pp. 109-117.

[20] M. M. Meerschaert and C. Tadjeran, Finite difference approximations for fractional advection-dispersion flow equations, J. Comput. Appl. Math., 172 (2004), pp. 65-77.

[21] J. Y. Pan, R. Ke, M. K. Ng, and H. W. Sun, Preconditioning techniques for diagonal-times-Toeplitz matrices in fractional diffusion equations, SIAM J. Sci. Comput., 36 (2014), pp. A2698-A2719.

[22] S. Peszat and J. Zabczyk, *Stochastic Partial Differential Equations with Levy Processes*, Cambridge University Press, Cambridge, UK, 2007.

[23] C. Pozrikidis, *The Fractional Laplacian*, CRC Press, London, 2016.

[24] F. Sabzikar, M. M. Meerschaert, and J. H. Chen, Tempered fractional calculus, J. Comput. Phys., 293 (2015), pp. 14-28.

[25] J. Shen, T. Tang, and L. L. Wang, *Spectral Methods: Algorithms, Analysis and Applications*, Springer, New York, 2011.

[26] C. T. Sheng and J. Shen, A hybrid spectral element method for fractional two-point boundary value problems, Numer. Math. Theor. Meth. Appl. 10 (2017), pp. 437-464.

[27] Q. Yang, F. Liu, and I. Turner, Numerical methods for fractional partial differential equations with Riesz space fractional derivatives, Appl. Math. Model., 34 (2010), pp. 200-218.

[28] V. Zuburadiev, S. Demidov, and J. Klafter, Lévy walks, Rev. Modern Phys., 87 (2015), pp. 483-530.

[29] Z. J. Zhang, W. H. Deng, and G. E. Karniadakis, A Riesz basis Galerkin method for the tempered fractional Laplacian, arXiv:1709.10415, 2017.

[30] Z. Zhao, X. Q. Jin, and M. M. Lin, Preconditioned iterative methods for space-time fractional advection-diffusion equations, J. Comput. Phys., 319 (2016), pp. 266-279.

[31] X. Zhao, Z. Z. Sun, and Z. H. Hao, A fourth-order compact ADI scheme for two-dimensional nonlinear space fractional Schrödinger equation, SIAM J. Sci. Comput., 36 (2014), pp. A2865-A2886.