Abstract. The eigenfunctions $e^{i(\lambda,x)}$ of the Laplacian on a flat torus have uniformly bounded $L^p$ norms. In this article, we prove that for every other quantum integrable Laplacian, the $L^p$ norms of the joint eigenfunctions blow up at least at the rate $\|\varphi_k\|_{L^p} \geq C(\varepsilon)\lambda_k^{p-2}4^{p-2}k$ when $p > 2$. This gives a quantitative refinement of our recent result [TZ1] that some sequence of eigenfunctions must blow up in $L^p$ unless $(M, g)$ is flat. The better result in this paper is based on mass estimates of eigenfunctions near singular leaves of the Liouville foliation.

0. Introduction

This paper, a companion to [TZ1], is concerned with the growth rate of the $L^p$-norms of $L^2$-normalized $\Delta$-eigenfunctions

$$\Delta \varphi_j = \lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle = \delta_{jk}$$

on compact Riemannian manifolds $(M, g)$ with completely integrable geodesic flow $G^t$ on $S^*M$. The motivating problem is to relate sizes of eigenfunctions to dynamical properties of its geodesic flow $G^t$ on $S^*M$. In general this is an intractable problem, but much can be understood by studying it in the framework of integrable systems. To be precise, we assume that $\Delta$ is quantum completely integrable or QCI in the sense that there exist $P_1, \ldots, P_n \in \Psi^1(M)$ ($n = \dim M$) satisfying $[P_i, P_j] = 0$ and such that their symbols $(p_1, \ldots, p_n)$ satisfy the standard assumption that $dp_1 \wedge dp_2 \wedge \cdots \wedge dp_n \neq 0$ on a dense open set $\Omega \subset T^*M - 0$ whose complement is contained in a hypersurface. Since $\{p_i, p_j\} = 0$, the $p_1, \ldots, p_n$ generate a Hamiltonian $\mathbb{R}^n$-action given by the joint Hamiltonian flow $\Phi_t(x, \xi) := \exp(t_1X_{p_1}) \circ \cdots \circ \exp(t_nX_{p_n})(x, \xi)$, where for $j = 1, \ldots, n$, $X_{p_j}$ denotes the Hamiltonian vector field of $p_j$. The associated moment map $P : T^*M \to \mathbb{R}^n$ is $P = (p_1, \ldots, p_n)$ and the orbits of this action foliate $T^*M - 0$ and since $S^*M$ is preserved they also foliate $S^*M$. See [TZ1], [TZ2] for a list of well-known examples. We will refer to this foliation as the Liouville foliation. Throughout, we will assume that the leaves of this foliation satisfy Eliasson’s non-degeneracy hypotheses (see [El] and Definition [L2]).

The main result of [TZ1] was that the $L^\infty$-norms of the $L^2$-normalized joint eigenfunctions $\{\varphi_\lambda\}$ of $(P_1, \ldots, P_n)$ are unbounded unless $(M, g)$ is flat. In this paper, we investigate
the blow-up rates of $L^p$-norms of sequences $|||\varphi_\lambda|||_{L^p}$, under the Eliasson non-degeneracy assumption.

**Theorem 1.** Suppose that $(M, g)$ is a compact Riemannian manifold with completely integrable geodesic flow satisfying Eliasson’s non-degeneracy condition. Then, unless $(M, g)$ is a flat torus, there exists for every $\epsilon > 0$, a sequence of eigenfunctions satisfying:

\[
\begin{align*}
  \|\varphi_k\|_{L^\infty} &\geq C(\epsilon) \lambda_k^{\frac{1}{4} - \epsilon}, \\
  \|\varphi_k\|_{L^p} &\geq C(\epsilon) \lambda_k^{\frac{p-2}{4p}-\epsilon}, \quad 2 < p < \infty.
\end{align*}
\]

This result is sharp in the setting of all completely integrable systems. It is based on the existence of a codimension one singular leaf of the Liouville foliation. Here, a singular leaf is an orbit of the $\mathbb{R}^n$ action with dimension $< n$, see Definition (1). When codimension $\ell$ leaves occur, the estimate becomes $\|\varphi_k\|_{L^\infty} \geq C(\epsilon) \lambda_k^{\frac{1}{4} - \epsilon}$.

Existence of a singular leaf is given in the following:

**Lemma 2.** Suppose that $(M, g)$ is a compact Riemannian manifold with completely integrable geodesic flow. Then the Liouville foliation of $(M, g)$ contains a singular leaf unless $(M, g)$ is a flat torus.

The proof only shows that some singular leaf occurs, but does not determine its codimension. As will be discussed in §4, it is quite plausible that codimension $n - 1$ singular leaves often occur for Hamiltonian $\mathbb{R}^n$ actions on cotangent bundles. Such leaves correspond to closed geodesics which are invariant under the $\mathbb{R}^n$ action. In such cases, the estimate improves to $\|\varphi_k\|_{L^\infty} \geq C(\epsilon) \lambda_k^{\frac{n-1}{4} - \epsilon}$. This estimate agrees with a recent result of Donnelly [D] on blow-up rates of $L^\infty$ norms of eigenfunctions on compact $(M, g)$ with isometric $S^1$ actions. He finds sequences of joint eigenfunctions which blow up at the rate $\lambda_k^{\frac{n-1}{4}}$; they are precisely the same kind of eigenfunctions which are associated to singular leaves, as we now describe.

The analytic ingredient of our proof gives an estimate on the growth rate of modes and quasimodes associated to a singular leaf of the Liouville foliation. It is here we assume that the singular leaf is Eliasson non-degenerate.

**Lemma 3.** Let $P^{-1}(c)$ with $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ be a singular level of the Lagrangian fibration associated with the quantum completely integrable system $P_1 = \sqrt{-\Delta}, \ldots, P_n$. Suppose $P^{-1}(c)$ contains a single compact, Eliasson non-degenerate orbit $\Lambda := \mathbb{R}^n \cdot (v_0)$ of dimension $\ell < n$ and that each connected component of $\pi(\Lambda) \subset M$ is an embedded $\ell$-dimensional submanifold. Then, there exists a sequence $\varphi_k$ of $L^2$-normalized joint eigenfunctions of $P_1, \ldots, P_n$ with $P_j \varphi_k = E_{jk} \varphi_k$, $E_{jk} = \lambda_k c_j + \mathcal{O}(1)$; $j = 1, \ldots, n$ such that for any $\epsilon > 0$,

\[
(i) \quad \|\varphi_k\|_{L^\infty} \geq C(\epsilon) \lambda_k^{\frac{n-\ell}{4} - \epsilon}.
\]

We also have that

\[
(ii) \quad \|\varphi_k\|_{L^p} \geq C(\epsilon) \lambda_k^{\frac{(n-\ell)(p-2)}{4p}-\epsilon}, \quad 2 < p < \infty.
\]
For simplicity we have stated Lemma (3) in the case where the level \( P^{-1}(c) \) contains a single Eliasson non-degenerate orbit, but the result extends in a straightforward fashion to the case of multiple singular orbits.

Some final words to close the introduction. First, in [TZ2] we give a different approach to eigenfunction blow-up in the completely integrable case which in some ways is closer in spirit to the approach of this paper. Namely, we relate norms of modes to norms of “quasi-modes,” i.e. approximate eigenfunctions associated to Bohr-Sommerfeld leaves of the Liouville foliation. In that paper, we also give a number of detailed examples of quantum completely integrable systems such as Liouville tori, surfaces of revolution, ellipsoids, tops and so on.

Second, we would like to emphasize that results on completely integrable systems are relevant to quite general classes of Riemannian manifolds. This is because one can approximate any geodesic flow by a completely integrable flow (its Birkhoff normal form) near a closed orbit or invariant torus. On the quantum level, one may approximate the Laplacian by its quantum Birkhoff normal form. The quasi-mode analysis of this paper is therefore of some relevance to general Riemannian manifolds. What is special about quantum integrable systems is the particularly close relations between modes and quasi-modes, which allows one to draw strong conclusions about the blow-up of sequences of modes from that of the corresponding quasi-modes. It is quite reasonable to suppose that the same kind of phenomenon occurs much more generally. In particular, we would conjecture that any compact \((M, g)\) with a stable elliptic orbit has a sequence of eigenfunctions whose \(L^\infty\) norms blow up at the rate \(\lambda_k^{-\frac{n-2}{4}}\). There surely exist quasi-modes with this property, and the difficulty is to show that this implies the existence of modes with this blow-up rate. We hope to pursue this direction in the future.

1. **Geometry of completely integrable systems**

This section is devoted to the geometric aspects of our problem. We first prove Lemma (4) and then give background on Eliasson non-degeneracy of singular orbits. We begin with some preliminary background on completely integrable systems, in part repeated from [TZ1].

As mentioned in the introduction, the moment map of a completely integrable system is defined by

\[
\mathcal{P} = (p_1, \ldots, p_n) : T^*M \to B \subset \mathbb{R}^n.
\]

The Hamiltonians \(p_j\) generate the \(\mathbb{R}^n\)-action

\[
\Phi_t = \exp t_1 \Xi_{p_1} \circ \exp t_2 \Xi_{p_2} \cdots \circ \exp t_n \Xi_{p_n}.
\]

We denote \(\Phi_t\)-orbits by \(\mathbb{R}^n \cdot (x, \xi)\).

By the Liouville-Arnold theorem, the orbits of the joint flow \(\Phi_t\) are diffeomorphic to \(\mathbb{R}^k \times T^m\) for some \((k, m), k + m \leq n\). By the properness assumption on \(\mathcal{P}\), each connected component of a regular level is a Lagrangean torus, i.e.

\[
\mathcal{P}^{-1}(b) = \Lambda^{(1)}(b) \cup \cdots \cup \Lambda^{(m_{\text{cl}})}(b), \quad (b \in B_{\text{reg}})
\]

where each \(\Lambda^{(l)}(b) \simeq T^n\) is a Lagrangian torus.
We are particularly interested in singular orbits. To clarify the notation and terminology, we repeat the definition from [TZ1]:

**Definition:**

- $b \in B_{\text{sing}}$ if $\mathcal{P}^{-1}(b)$ is a singular level of the moment map, i.e. if there exists a point $(x, \xi) \in \mathcal{P}^{-1}(b)$ with $dp_1 \wedge \cdots \wedge dp_n(x, \xi) = 0$. Such a point $(x, \xi)$ is called a singular point of $\mathcal{P}$.
- A connected component of $\mathcal{P}^{-1}(b)$ ($b \in B_{\text{sing}}$) is a singular component if it contains a singular point.
- An orbit $R_n \cdot (x, \xi)$ of $\Phi_t$ is singular if it is non-Lagrangean, i.e. has dimension $< n$.
- $b \in B_{\text{reg}}$ and that $\mathcal{P}^{-1}(b)$ is a regular level if all points $(x, \xi) \in \mathcal{P}^{-1}(b)$ are regular, i.e. if $dp_1 \wedge \cdots \wedge dp_n(x, \xi) \neq 0$.
- A component of $\mathcal{P}^{-1}(b)$ ($b \in B_{\text{sing}} \cup B_{\text{reg}}$) is regular if it contains no singular points.

When $b \in B_{\text{sing}}$ we first decompose $\mathcal{P}^{-1}(b) = \bigcup_{j=1}^r \Gamma_{\text{sing}}^{(j)}(b)$ the singular level into connected components $\Gamma_{\text{sing}}^{(j)}(b)$ and then decompose $\Gamma_{\text{sing}}^{(j)}(b) = \bigcup_{k=1}^p \mathbb{R}^n \cdot (x_k, \xi_k)$ each component into orbits. Both decompositions can take a variety of forms. The regular components $\Gamma_{\text{sing}}^{(j)}(b)$ must be Lagrangean tori by the properness assumption. A singular component consists of finitely many orbits by the Eliasson non-degeneracy assumption (see below). The orbit $\mathbb{R}^n \cdot (x, \xi)$ of a singular point is necessarily singular, hence has the form $\mathbb{R}^k \times T^m$ for some $(k, m)$ with $k + m < n$. Regular points may also occur on a singular component, whose orbits are Lagrangean and can take any one of the forms $\mathbb{R}^k \times T^m$ for some $(k, m)$ with $k + m = n$.

1.1. **Proof of Lemma (2).** In this section, we prove Lemma (2): the Liouville foliation of metrics with completely integrable geodesic flows always contain singular leaves unless $(M, g)$ is a flat torus.

**Proof.** The hypothesis that the Liouville foliation is non-singular has two immediate geometric consequences:

- (i) By Mane’s theorem [M], $(M, g)$ is a manifold without conjugate points;
- (ii) The (homogeneous) moment map $\mathcal{P} : T^*M - 0 \to \mathbb{R}^n - 0$ is a torus fibration by $T^n$.

Statement (ii) follows from the Liouville-Arnold theorem. On a non-singular leaf, we must have $dp_1 \wedge \cdots \wedge dp_n \neq 0$. Since this holds everywhere, $\mathcal{P}$ is a submersion; and since it is proper, it is a fibration. The fiber must be $T^n$, again by the Liouville-Arnold theorem. Since $\mathcal{P}$ is homogeneous, the image $\mathcal{P}(T^*M - 0) = \mathbb{R}_+ \cdot \mathcal{P}(S^*M)$. Since $\mathcal{P}$ is a submersion, the image is a smooth submanifold of $S^{n-1}$ hence must be all of $S^{n-1}$.

By (i) it follows that $M = \tilde{M}/\Gamma$ where $(\tilde{M}, \tilde{g})$ is the universal Riemannian cover of $(M, g)$ (diffeomorphic to $\mathbb{R}^n$), and where $\Gamma \cong \pi_1(M)$ is the group of covering transformations.
We claim that (ii) implies $\pi_1(M) = \mathbb{Z}^n$. Indeed, $T^*M - 0$ is a double fibration (with indicated fibers):

$$T^*M - 0$$

$$\pi \searrow (\mathbb{R}^n - 0) \quad (T^n) \searrow \mathcal{P} \quad \nearrow M \quad \mathbb{R}^n - 0$$

By the homotopy sequence of a fibration $\pi : E \to B$,

$$\cdots \pi_q(F) \to \pi_q(E) \to \pi_q(B) \to \pi_{q-1}(F) \cdots \to \pi_0(E) \to \pi_0(B) \to 0$$

and using that $\pi_2(T^n) = 1$ and that $\pi_2(M) = 1$ by (i), we obtain

(5) $$1 \to \pi_1(S^{n-1}) \to \pi_1(S^*M) \to \pi_1(M) \to \pi_0(S^{n-1}) \to \pi_0(S^*M) \to \pi_0(M) \to 1$$

$$1 \to \pi_2(S^*M) \to \pi_2(S^{n-1}) \to \pi_1(T^n) \to \pi_1(S^*M) \to \pi_1(S^{n-1}) \to \pi_0(T^n) \to \pi_0(S^*M) \to \pi_0(S^{n-1}) \to 1$$

Since $\pi_2(N) = \pi_2(\tilde{N})$ (where $\tilde{N}$ is the universal cover, and since $S^*M = S^\mathbb{R}^n - 1 \times \mathbb{R}^n$, we have $\pi_2(S^*M) = \pi_2(S^{n-1})$. From its definition, we see that the homomorphism $\pi_2(S^*M) \to \pi_2(S^{n-1})$ is an isomorphism, hence the second sequence simplifies to

(6) $$1 \to \pi_1(T^n) \to \pi_1(S^*M) \to \pi_1(S^{n-1}) \to \pi_0(T^n) \to \pi_0(S^*M) \to \pi_0(S^{n-1}) \to 1$$

Let us first assume that $n \geq 3$. Then $\pi_1(S^*M) = \pi_1(M)$ and $\pi_1(S^{n-1}) = 1$, so we get

$$1 \to \pi_1(T^n) \to \pi_1(M) \to 1,$$

i.e. $\pi_1(T^n) \cong \pi_1(M)$.

We now consider dimension 2. Actually, it follows by a classic result of Kozlov [K] that the only surfaces that can possibly have a completely integrable geodesic flow (even with singularities) are $M = S^2, T^2$. We can easily disqualify $M = S^2$ since $S^*S^2 \equiv \mathbb{R}P^3$ does not fiber over $S^1$.

It follows then that $M$ is diffeomorphic to $\mathbb{R}^n/\mathbb{Z}^n$, i.e. $M$ is a torus. Since $g$ has no conjugate points, the proof is concluded by Burago-Ivanov’s theorem [B].

\[ \square \]

1.1.1. **Remarks on the geometry.** We would like to add two remarks on the geometry. We will not use them in the proof of our results, and therefore only sketch the issues. Our second remark could lead to a substantial improvement of our main result.

**Remark (i): On flat manifolds.**

In our previous article [TZ1], we showed that eigenfunction blow-up occurs in the integrable setting unless $(M, g)$ is flat. Here, we are showing that it occurs unless $(M, g)$ is a flat torus. We briefly explain why Liouville foliations of other flat manifolds must have singular leaves.

The Riemannian connection of $(M, g)$ gives a distribution of horizontal planes in the unit sphere bundle, i.e. $T(S^*M) = H \oplus V$ where $V$ is the tangent bundle to the fibers of $\pi : S^*M \to M$. If $g$ is a flat metric, then $H$ is involutive and hence $S^*M$ is foliated by smooth $n$-manifolds $L$ (which we will call horizontal leaves). $(M, g)$ is also the Riemannian
moment map, $P$ the existence of codimension 1.2. Eliasson non-degeneracy.

| quotient of a torus $T^n$ by a finite group $G$ of isometries, and the derivative $p_*$ of the projection $p : T^n \to M$ takes the horizontal distribution of $T^n$ to that of $M$. Since the horizontal leaves of $T^n$ are compact Lagrangean tori, it follows that the horizontal leaves of $(M, g)$ are compact Lagrangean submanifolds of $T^* M - 0$. Thus, the geodesic flow of $(M, g)$ leaves invariant a non-singular Lagrangean foliation.

However, the horizontal leaves are not orbits of a Hamiltonian $\mathbb{R}^n$ action in general. (We thank W. Goldman for several helpful discussions on this point). In fact, the leaves have the form $L(G) \backslash \mathbb{R}^n / \mathbb{Z}^n$ where $L(G)$ is the linear holonomy group (a finite group). So the leaves are quotients of tori, but $L(G)$ is generally not a subgroup of $T^n$ so the leaves are finite quotients of tori. If $L(G)$ acts freely on $T^* M - 0$, then the leaves are tori, but there is no $\mathbb{R}^n$ action generating them.

The natural ‘Hamiltonians’ on $T^* M - 0$ belong to the algebra $\mathcal{I}^G$ where $\mathcal{I} = \langle I_1, \ldots, I_n \rangle$ is the algebra generated by the action variables $I_j(x, \xi) = \xi_j$ on the covering torus, and where $\mathcal{I}^G$ denotes the $G$-invariant elements. Since we require that the $\mathbb{R}^n$ action be homogeneous, we only consider functions $q_j(I_1, \ldots, I_n)$ which are homogeneous of degree one relative to the natural $\mathbb{R}_+$ action on $T^* M - 0$. A completely integrable system on $T^* M - 0$ is given by a choice of $n$ generators of this algebra. The $\mathbb{R}^n$ action they generate will leave invariant the horizontal foliation, but the leaves of that foliation are not in general the orbits of the action. What the proof above shows is that any such action must have some singular orbits, of lower dimension than the leaf it is contained in.

**Remark (ii): On existence of high codimension singular leaves.**

The simple proof above on existence of singular leaves does not give any information on the codimension of such leaves. As mentioned in the introduction, our lower bounds $||\varphi_k||_{L^\infty} \geq C(\epsilon) \lambda_k^{\frac{2}{k+1} - \epsilon}$ improve as this codimension $q$ increases. Can a better argument prove the existence of codimension $n - 1$ leaves in generic cases?

Here is heuristic existence argument: the image $\mathcal{C}$ of $T^* M - 0$ under the moment map $\mathcal{P}$ of a homogeneous $\mathbb{R}^n$-action is a convex polyhedral cone in $\mathbb{R}^n$. In Lemma (2), we observed that $\mathcal{P}$ cannot be a torus bundle over $\mathbb{R}^n - 0$ unless $(M, g)$ is a flat torus. In all other cases, $\mathcal{C}$ will have a boundary over which the singular orbits ‘fiber’.

The geometric problem is to determine the geometry of $\mathcal{C}$. In the case of Hamiltonian torus actions, the moment polyhedra have been studied by Lerman et al. [4, 23]. They do not seem to have been studied for general Hamiltonian $\mathbb{R}^n$ actions. However, it is natural to conjecture that as long as $\mathcal{C}$ has a non-empty boundary, there will boundary faces of each codimension and in particular there will be one-dimensional edges. If this is true, then our lower bound improves to $>> \lambda_k^{\frac{2}{k+1} - \epsilon}$.

1.2. **Eliasson non-degeneracy.** Let $c := (c_1, \ldots, c_n) \in B_{sing}$ be a singular value of the moment map, $\mathcal{P}$. Suppose that:

$$\text{rank } (dp_1, \ldots, dp_n) = \text{rank } (dp_1, \ldots, dp_k) = k < n$$

at some point $v_0 \in \mathcal{P}^{-1}(c)$. Denote the orbit through $v_0$ by $\mathbb{R}^n \cdot (v_0) := \{ \exp t_1 \Xi_{p_1} \circ \cdots \circ \exp t_k \Xi_{p_k}(v_0); \ t = (t_1, \ldots, t_k) \in \mathbb{R}^n \}$, which we henceforth assume is compact. Our aim is to
define the notion of Eliasson non-degeneracy of such a singular orbit; this is a straightforward extension of the notion defined in [E] for fixed points of symplectic maps.

By the Liouville-Arnold theorem, the orbits of the joint flow \( \Phi_t \) are diffeomorphic to \( \mathbb{R}^k \times T^m \) for some \((k, m), k + m \leq n\). Let us first consider regular levels where \( m = n \). By the properness assumption on \( \mathcal{P} \), a regular level has the form

\[
\mathcal{P}^{-1}(b) = \Lambda^{(1)}(b) \cup \cdots \cup \Lambda^{(m,cl)}(b), \quad (b \in B_{reg})
\]

where each \( \Lambda^{(i)}(b) \simeq T^n \) is an \( n \)-dimensional Lagrangian torus. The classical (or geometric) multiplicity function \( m_{cl}(b) = \#\mathcal{P}^{-1}(b) \), i.e. the number of orbits on the level set \( \mathcal{P}^{-1}(b) \), is constant on connected components of \( B_{reg} \) and the moment map \( \mathcal{P} \) is a fibration over each component with fiber (8). In sufficiently small neighbourhoods \( \Omega^{(i)}(b) \) of each component torus, \( \Lambda^{(i)}(b) \), the Liouville-Arnold theorem also gives the existence of local action-angle variables \((I_1^{(i)}, \ldots, I_n^{(i)}, \theta_1^{(i)}, \ldots, \theta_n^{(i)})\) in terms of which the joint flow of \( \Xi_{p_1}, \ldots, \Xi_{p_n} \) is linearized [AM]. For convenience, we henceforth normalize the action variables \( I_1^{(i)}, \ldots, I_n^{(i)} \) so that \( I_j^{(i)} = 0; \ j = 1, \ldots, n \) on the torus \( \Lambda^{(i)}(b) \).

Now let us consider singular levels of rank \( k \). We first observe that \( dp_1, \ldots, dp_k \) (in the notation of (3)) are linearly independent everywhere on \( \mathbb{R}^n \cdot (v_0) \). Indeed, the Liouville-Arnold theorem in the singular case [AM] states that there exists local canonical transformation

\[
\psi = \psi(I, \theta, x, y) : \mathbb{R}^{2n} \to T^*M = 0,
\]

where

\[
I = (I_1, \ldots, I_k), \theta = (\theta_1, \ldots, \theta_k) \in \mathbb{R}^k, \quad x = (x_1, \ldots, x_{n-k}), \quad y = (y_1, \ldots, y_{n-k}) \in \mathbb{R}^{n-k}
\]

defined in an invariant neighbourhood of \( \mathbb{R}^n \cdot (v_0) \) such that

\[
p_i \circ \psi = I_i \quad (i = 1, \ldots, k),
\]

and such that the symplectic form \( \omega \) on \( T^*M \) takes the form

\[
\psi^*\omega = \sum_{j=1}^{k} dI_j \wedge d\theta_j + \sum_{j=1}^{n-k} dx_j \wedge dy_j.
\]

As a consequence of the normal form (10), it follows that there exist constants \( c_{ij} \) with \( i = k + 1, \ldots, n \) and \( j = 1, \ldots, k \), such that at each point of the orbit, \( \mathbb{R}^n \cdot (v_0) \),

\[
dp_i = \sum_{j=1}^{k} c_{ij} dp_j.
\]

Since \( dp_1, \ldots, dp_k \) are linearly independent in a sufficiently neighbourhood \( U \) of \( v_0 \in \mathcal{P}^{-1}(c) \), the action of the flows corresponding to the Hamilton vector fields, \( \Xi_{p_1}, \ldots, \Xi_{p_k} \) generates a symplectic \( \mathbb{R}^k \) action on \( \mathcal{P}^{-1}(c_0) \cap U \). An application of Marsden-Weinstein reduction yields an open \( 2(n - k) \)-dimensional symplectic manifold:

\[
\Sigma_k := \mathcal{P}^{-1}(c_0) \cap U/\mathbb{R}^k,
\]

with the induced symplectic form, \( \sigma \). We will denote the canonical projection map by:

\[
\pi_k : \mathcal{P}^{-1}(c_0) \cap U \longrightarrow \Sigma_k.
\]
Since \( \{p_i, p_j\} = 0 \) for all \( i, j = 1, ..., n \), it follows that \( p_{k+1}, ..., p_n \) induce \( C^\infty \) functions on \( \Sigma_k \), which we will, with some abuse of notation, continue to write as \( p_{k+1}, ..., p_n \). From (14), it follows that

\[
dp_i(p_k(v_0)) = 0; \quad i = k + 1, ..., n.
\]

Here, we denote the single point \( p_k(\mathbb{R}^n \cdot (v_0)) \) by \( p_k(v_0) \).

To describe the Eliasson construction, let \((M^{2n}, \sigma)\) be any symplectic manifold and \( C \), the Lie algebra of Poisson commuting functions, \( f_i \in C^\infty(M) \), with the property that:

\[
f_i(m) = df_i(m) = 0; \quad i = 1, ..., n,
\]

for a fixed \( m \in M \). Let \( Q(2n) \) denote the Lie algebra of quadratic forms and consider the Lie algebra homomorphism, \( \mathcal{H} : C \to Q(2n) \) given by

\[
\mathcal{H}(f) := d^2 f(m).
\]

The result of Eliasson [El] says that, if \( \mathcal{H}(f_1, ..., f_n) \) is a Cartan subalgebra of \( Q(2n) \), there exists a locally-defined canonical mapping, \( \kappa : U \to U_0 \), from a neighbourhood, \( U \), of \( m \in M \) to a neighbourhood, \( U_0 \), of \( 0 \in \mathbb{R}^n \), with the property that:

\[
\forall i, j \{ f_i \circ \kappa^{-1}, I_{ij}^e \} = \{ f_i \circ \kappa^{-1}, I_j^h \} = \{ f_i \circ \kappa^{-1}, I_j^{ch} \} = 0.
\]

Here \( I_i, I_{ij}, I_{ij}^e, I_{ij}^{ch}, I_{ij}^h, I_{ij}^{h+L+1}, I_{ij}^{h+L+2}, ..., I_{ij}^{2n}, I_{ij}^{ch} \) denote the standard basis of the Cartan subalgebra

\[
\mathcal{H}(f_1, ..., f_n), \text{ where } I_i^h = x_i \xi_i \text{ in the case of a hyperbolic summand, } I_i^e = x_i^2 + \xi_i^2 \text{ for an elliptic summand, and } I_i^{ch} = x_i \xi_{i+1} - x_{i+1} \xi_i + \sqrt{-1}(x_i \xi_i + x_{i+1} \xi_{i+1}) \text{ in the complex-hyperbolic case.}
\]

By making a second-order Taylor expansion about \( I^e = I^{ch} = I^h = 0 \), it follows from (13) that \( \forall i = 1, ..., n \), there locally exist \( F_{ij} \in C^\infty(U_0) \) with \( \{ I_i^e, F_{ij} \} = \{ I_i^h, F_{ij} \} = \{ I_i^{ch}, F_{ij} \} = 0 \) such that:

\[
(14) \quad f_i \circ \kappa^{-1} = \sum_{j=1}^H F_{ij} \cdot I_j^h + \sum_{j=H+1}^{H+L+1} F_{ij} \cdot I_j^{ch} + \sum_{j=H+L+1}^n F_{ij} \cdot I_j^e.
\]

The non-degeneracy of \( \mathcal{H}(f_1, ..., f_n) \) implies that

\[
(F_{ij})(0) \in Gl(n; \mathbb{R}).
\]

There is also a parameter-dependent version of this result [El] that is valid in a neighbourhood of the orbit, \( \mathbb{R}^n \cdot (v_0) \), and combines the normal form in (13) with that in (10). More precisely, if \( \mathcal{H}(dp_{k+1}, ..., dp_n) \) is a Cartan subalgebra, then there exists a neighbourhood, \( \Omega \), of the orbit, \( \mathbb{R}^n \cdot (v_0) \), and a canonical map \( \kappa : \Omega \to \mathbb{T}^k \times D \) with the property that, for all \( i = 1, ..., n \),

\[
(15) \quad p_i \circ \kappa^{-1} = \sum_{j=1}^H F_{ij} \cdot I_j^h + \sum_{j=H+1}^{H+L+1} F_{ij} \cdot I_j^{ch} + \sum_{j=H+L+1}^{n-k} F_{ij} \cdot I_j^e + \sum_{j=n-k+1}^n F_{ij} \cdot I_{n+1-j}.
\]

Here \( I^h := (I_1^h, ..., I_H^h), I^{ch} := (I_{H+1}^{ch}, ..., I_{H+L+1}^{ch}) \) and \( I^e := (I_{H+L+2}^e, ..., I_{n-k}^e) \) denote the standard generators of the Cartan algebra on \( \Sigma_k \), \( I := (I_1, ..., I_k) \) the regular action coordinates coming from the normal form in (10) and \( D \subset \mathbb{R}^{n-k} \) a small ball containing \( 0 \in \mathbb{R}^{n-k} \). The \( F_{ij} \) Poisson-commute with all the action functions.
Definition: (i) We say that the rank $\ell < n$ orbit, $\mathbb{R}^n \cdot (v_0)$, is Eliasson non-degenerate provided it is compact and the algebra $\mathcal{H}(p_{\ell+1}, \ldots, p_n)$ is Cartan.

(ii) The rank $\ell < n$ orbit $\mathbb{R}^n \cdot (v_0)$ is said to be strongly Eliasson non-degenerate if it is Eliasson non-degenerate and in addition, $\pi(\mathbb{R}^n \cdot (v_0)) \subset M$ is an embedded $\ell$-dimensional submanifold of $M$.

2. Quantum integrable systems and Birkhoff normal forms

Our purpose in this section is to construct a microlocal Birkhoff normal form for a QCI (quantum completely integrable) system near a singular orbit. We first recall the definition of a QCI system.

2.1. Quantum integrable systems. Quantum completely integrable Hamiltonians (e.g. Laplacians) or those which commute with a maximal family of observables. We now make precise the kind of observables we will need to use.

Given an open $U \subset \mathbb{R}^n$, we say that $a(x, \xi; \hbar) \in C^\infty(U \times \mathbb{R}^n)$ is in the symbol class $S^{m,k}(U \times \mathbb{R}^n)$, provided

$$|\partial^\alpha_x \partial^\beta_\xi a(x, \xi; \hbar)| \leq C_{\alpha\beta} \hbar^{-m} (1 + |\xi|)^{k-|\beta|}.$$ 

We say that $a \in S^{m,k}_{cl}(U \times \mathbb{R}^n)$ provided there exists an asymptotic expansion:

$$a(x, \xi; \hbar) \sim \hbar^{-m} \sum_{j=0}^\infty a_j(x, \xi) \hbar^j,$$

valid for $|\xi| \geq \frac{1}{\hbar} > 0$ with $a_j(x, \xi) \in S^{0,k-j}(U \times \mathbb{R}^n)$ on this set. We denote the associated $\hbar$ Kohn-Nirenberg quantization by $Op_\hbar(a)$, where this operator has Schwartz kernel given locally by the formula:

$$Op_\hbar(a)(x, y) = (2\pi \hbar)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\xi/\hbar} a(x, \xi; \hbar) d\xi.$$ 

By using a partition of unity, one constructs a corresponding class, $Op_\hbar(S^{m,k})$, of properly-supported $\hbar$-pseudodifferential operators acting on $C^\infty(M)$. Moreover, this calculus of operators is independent of the particular choice of partition of unity. There exists a natural pseudodifferential calculus for such operators with the usual local symbolic composition formula: Given $a \in S^{m_1,k_1}$ and $b \in S^{m_2,k_2}$, the composition $Op_\hbar(a) \circ Op_\hbar(b) = Op_\hbar(c) + O(\hbar^\infty)$ in $L^2(M)$ where locally,

$$c(x, \xi; \hbar) \sim \hbar^{-(m_1+m_2)} \sum_{|\alpha|=0}^\infty \frac{(-i\hbar)^{|\alpha|}}{\alpha!} (\partial^\alpha_x a) \cdot (\partial^\alpha_\xi b).$$

Definition: We say that the operators $Q_j \in Op_\hbar(S^{m,k}_{cl})$; $j = 1, \ldots, n$, generate a semiclassical quantum completely integrable system if

$$[Q_i, Q_j] = 0; \quad \forall 1 \leq i, j \leq n,$$

and the respective semiclassical principal symbols $q_1, \ldots, q_n$ generate a classical integrable system with $dq_1 \wedge dq_2 \wedge \cdots \wedge dq_n \neq 0$ on a dense open set $\Omega \subset T^*M - 0$. 

In this article, we are only concerned with Laplacians on compact Riemannian manifolds. Hence, we only consider the homogeneous case where the operators $P_1 = \sqrt{\Delta}, P_2, ..., P_n$ are classical pseudodifferential operators of order one. For fixed $b = (b_1, b_2, ..., b_n) \in \mathbb{R}^n$, we define a ladder of joint eigenvalues of $P_1 = \sqrt{\Delta}, P_2, ..., P_n$ by:

$$\{(\lambda_{1k}, ..., \lambda_{nk}) \in \text{Spec}(P_1, ..., P_n); \forall j = 1, ..., n, \lim_{k \to \infty} \frac{\lambda_{jk}}{|\lambda_k|} = b_j\},$$

where $|\lambda_k| := \sqrt{\lambda_{1k}^2 + ... + \lambda_{nk}^2}$. We denote the corresponding joint eigenfunctions by $\varphi_\mu$. It is helpful to rewrite the spectral problem in semiclassical notation. The rescaled operators $Q_j := hP_j \in Op(S^{0,1})$ clearly generate a semiclassical quantum integrable system in the sense of Definition 2.1. With no loss of generality we may restrict $h$ to the sequence $h_k = 1/|\lambda_k|$, and then the ladder eigenvalue problem (16) has the form

$$Q_j \varphi_\mu = \mu_j(h)\varphi_\mu, \text{ where } \mu(h) = b + o(1) \text{ as } h \to 0.$$  

The semi-classical ladder will be denoted by:

$$\Sigma_h(h) := \{(\mu(h) := (\mu_1(h), ..., \mu_n(h)) \in \text{Spec}(Q_1, ..., Q_n); |\mu_j(h) - b_j| \leq Ch, j = 1, ..., n\}.$$  

2.2. Model cases. Quantum Birkhoff normal forms are microlocal expressions of a given QCI system in terms of certain model system. Model quantum completely integrable systems are direct sums of the quadratic Hamiltonians:

- $\hat{I}^h := h(D_y y + y D_y)$ (hyperbolic Hamiltonian),
- $\hat{I}^e := h^2 D_y^2 + y^2$, (elliptic Hamiltonian),
- $\hat{I}^{ch} := h[(y_1 D_{y_1} + y_2 D_{y_2}) + \sqrt{-1}(y_1 D_{y_2} - y_2 D_{y_1})]$ (complex hyperbolic Hamiltonian),
- $\hat{I}^{reg} := h D_\theta$, (regular Hamiltonian).

The corresponding model eigenfunctions are:

- $u_k(y; \lambda, h) = |\log h|^{-1/2} [c_+(h)Y(y) |y|^{-1/2 + i\lambda/h} + c_-(h)Y(-y) |y|^{-1/2 + i\lambda/h}]; |c_-(h)|^2 + |c_+(h)|^2 = 1; \lambda(h) \in \mathbb{R}$.
- $u_{ch}(r, \theta; t_1, t_2, h) = |\log h|^{-1/2} e^{(-1+i\lambda(t_1+\epsilon))/h} e^{it_2(h)\theta}; t_1(h), t_2(h) \in \mathbb{R}$.
- $u_e(y; n, h) = h^{-1/4}\exp(-y^2/h) \Phi_n(h^{-1/2}y); n \in \mathbb{N}$.
- $u_{reg}(\theta; m, h) = e^{im\theta}; m \in \mathbb{Z}$.

Here, $Y(x)$ denotes the Heaviside function, $\Phi_n(y)$ the $n$-th Hermite polynomial and $(r, \theta)$ polar variables in the $(y_1, y_2)$ complex hyperbolic plane. We restrict to those model eigenfunctions with $\lambda(h), t_1(h), t_2(h) = O(h)$ as $h \to 0$, since these are the ones which localize properly. The important part of a model eigenfunctions is its microlocalization to a neighborhood of $x = \xi = 0$, so we put:

$$\psi(x; \hbar) := Op_h(\chi(x) \chi(y) \chi(\xi)) \cdot u(y; \hbar),$$

where $\epsilon > 0$ and $\chi \in C_0^\infty([-\epsilon, \epsilon])$. In the hyperbolic, complex hyperbolic, elliptic and regular cases, we write $\psi_{ch}(y; \hbar), \psi_{ch}(y; \hbar), \psi_e(y; \hbar)$ and $\psi_{reg}(y; \hbar)$ respectively. A straightforward
2.3. Following quantum analogue (see also [VN]) of the classical Eliasson normal form in (15): for a quantum completely integrable system near a singular orbit. The main result is the

\[ F_n \text{ functions, the microlocalizations are } C^\infty \text{ and supported near the origin.} \]

**Proof:** The proof is essentially the same as in ([VN] Theorem 3.6). The only complication here is that since \( \mathbb{R}^n \cdot v_0 \) is a rank \( k < n \) torus and not a point, \( hD_{\theta_1}, \ldots, hD_{\theta_k} \) must be added to the space of model operators. The proof can be reduced to that in [VN] by making Fourier series decompositions in the \( (\theta_1, \ldots, \theta_k) \) variables (see, for instance [T2] Theorem 3).

Consider the microlocal (quasi-)eigenvalue problem in \( \Omega \):

\[ Q_j \psi_{\nu_j} = h(\hat{\nu}_j)\psi_{\nu_j}; j = 1, \ldots, n. \]

We say that \( \nu_k(h) \) is a quasi-classical eigenvalue if there exists a non-trivial solution of (21). The set of quasi-classical eigenvalues around \( c \) is thus:

\[ Q_j \Sigma_c(h) := \{ \nu(h) : (21) \text{ holds, with } |\nu(h) - c| \leq Ch, j = 1, \ldots, n \}. \]

The solution space of (21) can be characterized uniquely (up to a \( C(h) \)-multiple) in terms of the model quasimodes \( \psi_{e}, \psi_{h}, \psi_{ch} \) and \( \psi_{reg} \). In the following, we use the abbreviation

\( (u_e(y; n, h) \cdot u_h(y; \lambda_k(h), h) \cdot u_{ch}(y; t_{1,k}(h), t_{2,k}(h), h) \prod_{j=1}^k e^{i m_j \theta_j}) \) for the expression

\[ \Pi_{j=1}^H \psi_{y_j} \otimes Q^{H+L}_{j=H+1} \psi_{o_j}(y_j; h) \otimes Q^{H+L+E}_{j=H+L+1} \psi_{t_j}(y_j; h) \otimes Q^{H+E+1}_{j=H+L+E+1} \psi_{r_j}(y_j; h) \]

in which the \( (m, n, \lambda, t_1, t_2) \) parameters are put in.

**Proposition 5.** For any admissible solution \( \psi_{\nu}(h) \) of (21), there exist \( t_1(h), t_2(h), \lambda(h) \in \mathbb{R} \) and \( n, m \in \mathbb{N} \) and an \( h \)-dependent constant \( c(h) \) such that

\[ \psi_{\nu} = \Omega c(h) F (u_e(y; n, h) \cdot u_h(y; \lambda(h), h) \cdot u_{ch}(y; t_1(h), t_2(h), h) \prod_{j=1}^k e^{i m_j \theta_j} ), \]

where \( F_h \) is the microlocally unitary \( h \)-Fourier integral operator in Lemma (4).
Proof. The proposition follows from the uniqueness of microlocal solutions of the model eigenfunction equations (see \cite{CP} and \cite{VN}) up to $C(\hbar)$-multiples and Lemma (4).

3. Blow-up of eigenfunctions attached to singular leaves of the Lagrangian fibration

The purpose of this section is to prove Theorem (1) and Lemma (3). The key estimate is a small scale $L^2$ mass estimate near a singular leaf $\Lambda$. Roughly speaking, it says that the $L^2$-mass of the eigenfunctions $\varphi_\mu$ with $\mu(\hbar) \in \Sigma_\epsilon(\hbar)$ in a tube of radius $\hbar^\delta$ around $\pi(\Lambda)$ satisfies

$$\langle Op_\hbar(\chi_1^\delta(x; h))\varphi_\mu, \varphi_\mu \rangle \geq C > 0,$$

where $C$ is a constant independent of $\hbar$ and where $\chi_1^\delta(x; h) := \chi_1(h^{-\delta}x)$ with $\chi_1(x)$ a cutoff supported near $\pi(\Lambda)$.

In the proof, we will need to use additional pseudo-differential operators belonging to a more refined semi-classical calculus, containing cutoffs such as $\chi_1(\hbar^{-\delta}x)$, which involve smaller length scales.

3.1. Eigenfunction mass near non-degenerate, singular orbits. We first give a lower bound for the microlocal mass on ‘large’ length scales of joint eigenfunctions with joint eigenvalues $\mu(\hbar) \in \Sigma_\epsilon(\hbar)$ near $\Gamma_{\text{sing}}(c)$. Let $\chi_{\text{reg}} \in C^\infty_0(T^*M)$ and $\chi_{\text{sing}} \in C^\infty_0(T^*M)$ be cutoff functions supported near $\Gamma_{\text{reg}}(c)$ and $\Gamma_{\text{sing}}(c)$ respectively. We claim that for any $f \in S(R)$ with $\tilde{f} \in C^\infty_0(R)$ with sufficiently small support and $\hbar \in (0, \hbar_0]$, we have

$$\sum_{j=1}^\infty \langle Op_\hbar(\chi_{\text{sing}})\varphi_\mu, \varphi_\mu \rangle f(h^{-1}(\mu_j(\hbar) - c)) \geq \frac{1}{C_0} > 0.$$  

Here, the sum runs over the entire joint spectrum. The estimate (24) follows readily from a local (singular) Weyl law near a singular rank-$k$ orbit $\mathbb{R}^n \cdot v_0$. The local Weyl law in turn follows from the Eliasson normal form and standard wave trace computations (\cite{BPU} Theorems 1.1-1.3). When the singular orbit has only hyperbolic summands, the RHS in (24) can be improved to $|\log \hbar|^k$ (see \cite{BPU}).

Consequently, for each $\hbar$, there exists at least one joint eigenfunction $\varphi_\mu$ of the $Q_j$’s with joint eigenvalue $\mu(\hbar) \in \Sigma_\epsilon(\hbar)$ satisfying:

$$\langle Op_\hbar(\chi_{\text{sing}})\varphi_\mu, \varphi_\mu \rangle \geq \frac{1}{C_0} > 0,$$

provided we choose the constant $C > 0$ in (22) large enough. By the local Weyl law in the regular case, again for $C > 0$ large enough in (22), there also exist joint eigenfunctions $\varphi_\mu$ with $\mu = \mu(\hbar) \in \Sigma(\hbar)$ satisfying

$$\langle Op_\hbar(\chi_{\text{reg}})\varphi_\mu, \varphi_\mu \rangle \geq \frac{1}{C_0}.$$  

Here, $C_0 > 0$ denotes possibly different constants from line to line.
Definition: Let \( V_{sing}(h) \) (resp. \( V_{reg}(h) \)) denote the set of \( L^2 \)-normalized joint eigenfunctions of \( Q_1, \ldots, Q_n \) with \( \mu(h) \in \Sigma_c(h) \) and satisfying (25) (resp. 24).

An easy argument involving invariant measures (see for example [TZ1]) shows that the connected component \( \Gamma_{sing}(c) \) must contain a rank-\( k < n \) compact orbit \( \mathbb{R}^n \cdot (v) \). Again, to simplify the writing we can, without loss of generality that there is precisely one such compact orbit in \( \Gamma_{sing}(c) \). As the next Lemma shows, the mass much concentrate on at least one of these singular orbits \( \Lambda \).

Lemma 6. Let \( \Lambda := \mathbb{R}^n \cdot (v_0) \) be an Eliasson non-degenerate orbit, and let \( \chi_\Lambda(x, \xi) \in C_0^\infty(\Omega; [0, 1]) \) be a cutoff function supported in an invariant neighbourhood \( \Omega \) of \( \Lambda \) and identically equal to one on a smaller neighbourhood \( \bar{\Omega} \) of \( \Lambda \). Then there exists a constant \( C_0 > 0 \) such that for \( \hbar \in (0, \hbar_0] \) and \( \varphi_\mu \in V(h) \),

\[
(\operatorname{Op}_h(\chi_\Lambda)\varphi_\mu, \varphi_\mu) \geq \frac{1}{C_0}.
\]

Proof. With no essential loss of generality, we will assume for notational simplicity that

\[
\mathcal{P}^{-1}(c) = \Gamma_{sing}(c) \cup \Gamma_{reg}(c)
\]

and that \( \Gamma_{sing}(c) \) contains a single compact rank-\( k \) orbit, \( \Lambda := \mathbb{R}^n \cdot (v_0) \). The argument easily extends to the case of multiple orbits and components.

Let \( \chi_{sing} \in C_0^\infty(T^* M) \) be the cutoff functions defined above (cf. [24 - 25]). Since \( \chi_\Lambda \chi_{sing} = \chi_\Lambda \), hence \( \operatorname{Op}_h(\chi_\Lambda) \circ \operatorname{Op}_h(\chi_{sing}) = \operatorname{Op}_h(\chi_\Lambda) + O(\hbar) \), it follows from (23) that

\[
1/C \leq (\operatorname{Op}_h(\chi_\Lambda)\varphi_\mu, \varphi_\mu) + ((1 - \operatorname{Op}_h(\chi_\Lambda)) \circ \operatorname{Op}_h(\chi_{sing})\varphi_\mu, \varphi_\mu).
\]

Since non-compact orbits do not support probability measures which are invariant under the joint flow \( \Phi_t \) (see [TZ1]), by the compactness of \( \mathcal{P}^{-1}(c) \) it follows that \( \Lambda = \mathbb{R}^n \cdot (v_0) \), the compact rank \( k < n \) orbit of \( v_0 \in \mathcal{P}^{-1}(c) \), is the forward limit set for the joint flow on \( \mathcal{P}^{-1}(c) \). The semiclassical Egorov theorem then gives:

\[
((1 - \operatorname{Op}_h(\chi_\Lambda)) \circ \operatorname{Op}_h(\chi_{sing})\varphi_\mu, \varphi_\mu) = (\operatorname{Op}_h(\Phi_t(1 - \chi_\Lambda)) \circ \operatorname{Op}_h(\chi_{sing})\varphi_\mu, \varphi_\mu) + O(\hbar).
\]

Moreover, since \( \Lambda = \mathbb{R}^n \cdot (v_0) \) is an \( \omega \)-limit set for the joint flow \( \Phi_t \) on \( \mathcal{P}^{-1}(c) \), there exists \( t_0 \in \mathbb{R}^n \) such that:

- \( \operatorname{supp}(\Phi_{t_0}^*(1 - \chi_\Lambda)) \cap \operatorname{supp}(\chi_{sing}) \subset \Omega \), where the QBNF is valid in \( \Omega \).
- \( \operatorname{supp}(\Phi_{t_0}^*(1 - \chi_\Lambda)) \cap \mathbb{R}^n \cdot (v_0) = \emptyset \).

In ([T2] Lemma 6), it is shown that microlocal eigenfunctions \( \varphi_\mu \) given by the QBNF in Lemma (4) near the orbit \( \Lambda \) satisfy

\[
(\operatorname{Op}_h(\Phi_{t_0}^*(1 - \chi_\Lambda)) \circ \operatorname{Op}_h(\chi_{sing})\varphi_\mu, \varphi_\mu) = O(|\log \hbar|^{-1}).
\]

The lemma then follows from (27), (28). \( \square \)
3.2. Localization on singular orbits. We claim that the joint eigenfunctions \( \varphi_\mu \in V(h) \) must blow up along \( \pi(\Lambda) \). The first way of quantifying this blowup involves computing the asymptotics for the expected values \( (O\!p_h(q)\varphi_\mu, \varphi_\mu) \) where \( q \in C_0^\infty(T^*M) \).

**Lemma 7.** Let \( \varphi_\mu \in V(h) \). Then:

\[
(29) \quad (O\!p_h(q)\varphi_\mu, \varphi_\mu) = |c(h)|^2 \int_{\mathbb{R}^n(\nu_0)} q \, d\mu + O(\log h)^{-1/2},
\]

again with \( |c(h)| \geq \frac{1}{c_0} \).

**Proof.** Since \( \varphi_\mu \) solves the equation (21) exactly (and a fortiori microlocally on \( \Omega \)), we may express it by Proposition (3) in the form:

\[
(30) \quad \varphi_\mu = \Omega c(h) F u_\mu,
\]

for some constant \( c(h) \). Here,

\[
(31) \quad u_\mu = (u_c(y; n, h) \cdot u_h(y; \lambda(h), h) \cdot u_{ch}(y; t_1(h), t_2(h), h)) \prod_{j=1}^k e^{im_j \theta_j}.
\]

Here, by applying the operators on both sides of the QBNF in Lemma (4) to the model eigendistributions \( u_\mu \) and using the uniqueness result in Lemma (5), it follows that for some \( n \times n \) matrix \( M \) with \( M(0) \in GL_n \),

\[
M(mh, nh, \lambda(h), t_1(h), t_2(h)) \cdot (mh, nh, \lambda(h), t_1(h), t_2(h)) = \mu(h).
\]

By the inverse function theorem, the \( (mh, nh, \lambda(h), t_1(h), t_2(h)) \) are uniquely determined (modulo \( O(h^\infty) \)) by the joint eigenvalues \( \mu(h) \) and moreover, when \( \mu(h) \in \Sigma(h) \) it follows that \( mh, nh, \lambda(h), t_1(h), t_2(h) = O(h) \). By Lemma (6), by (19) and by (30), it follows that for \( h \in (0, h_0] \),

\[
(32) \quad \frac{1}{c_0} \leq (O\!p_h(\chi_\Lambda)\varphi_\mu, \varphi_\mu) = c(h)(F^*O\!p_h(\chi_\Lambda)F u_\mu, u_\mu) \leq |c(h)|.
\]

Granted this lower bound on \( |c(h)| \), the Lemma reduces to estimating matrix elements of model eigenfunctions. We now evaluate the matrix elements case by case. The most interesting case is where the orbit \( \Lambda \) is strictly real or complex hyperbolic. Then, given any \( q \in C_0^\infty(T^*M) \), it follows from (28) that

\[
(33) \quad (O\!p_h(q)\varphi_\mu, \varphi_\mu) = (O\!p_h(q) \circ O\!p_h(\chi_\Lambda)\varphi_\mu, O\!p_h(\chi_\Lambda)\varphi_\mu) + O(\log h)^{-1}.
\]

We now use (30) to conjugate to the model setting. The function \( q \) goes to \( q \circ \chi \) where \( \chi \) is the canonical transformation underlying \( F \). The model \( \mathbb{R}^n \)-action locally reduces to a compact torus \( T^k \)-action, so we can average the function \( q \circ \chi \) over the action to obtain a smooth invariant function. We then Taylor expand this averaged function in the directions \( (y, \eta) \) transverse to the action. We obtain:

\[
(34) \quad (O\!p_h(q) \circ O\!p_h(\chi_\mu)\varphi_\mu, O\!p_h(\chi_\Lambda)\varphi_\mu) = |c(h)|^2 \int_{\mathbb{R}^n(\nu_0)} q \, d\mu + (O\!p_h(r_h)u_\mu, u_\mu) + (O\!p_h(r_{ch})u_\mu, u_\mu) + O(h).
\]
where, $r_h, r_{ch} \in C_0^\infty(\Omega)$ are the Taylor remainders with $r_h, r_{ch} = O(|y| + |\eta|)$. A direct computation for the model distributions, $u_\mu$ (see [T2] Lemma 5 and Proposition 3) shows that:

$$
(\text{Op}_h(r_h)u_\mu, u_\mu) = O(|\log h|^{-1/2}), \quad (\text{Op}_h(r_{ch})u_\mu, u_\mu) = O(|\log h|^{-1/2}).
$$

The remaining cases are where elliptic (i.e. Hermite factors). Each such factor satisfies

$$
(\text{Op}(r_e)u_\mu, u_\mu) = O(h)
$$

so is better than what is claimed. \hfill \Box

3.3. Lower bounds on $L^p$ norms: Proof of Lemma 0.3. Our objective in this section is to refine the argument in Lemma 3.3 to actually produce pointwise lower bounds for eigenfunctions attached to Eliasson non-degenerate leaves of the Lagrangian fibration. To do this, we study matrix elements on much smaller length scales. That requires the use of small scale pseudodifferential operators. We pause to define these objects.

3.3.1. Small scale semiclassical pseudo-differential calculus. The more refined symbols are defined as follows: Given an open set $U \subset \mathbb{R}^n$ and $0 \leq \delta < \frac{1}{2}$, we say that $a(x, \xi; h) \in S^m(U \times \mathbb{R}^n)$ if

$$
|\partial_x^\alpha \partial_{\xi}^\beta a(x, \xi; h)| \leq C_{\alpha\beta} h^{-|\alpha|+|\beta|}.
$$

Model symbols include cutoffs of the form $\chi(h^{-\delta}x, h^{-\delta}\xi)$ with $\chi \in C_0^\infty(\mathbb{R}^{2n})$. There is a pseudodifferential calculus $\text{Op}_h S^m(U \times \mathbb{R}^n)$ associated with such symbols with the usual symbolic composition formula and Calderon-Vaillancourt $L^2$-boundedness theorem [Sj]. Composition with operators in our original class $\text{Op}_h S^{m,0}(U \times \mathbb{R}^n)$ preserves $\text{Op}_h S^m(U \times \mathbb{R}^n)$.

3.3.2. Outline of proof. Let us now outline the proof. If $\chi_1^\delta(x, \xi; h) \in C_0^\infty(T^*M)$ is a second cutoff supported in a radius $h^\delta$ tube around $\Lambda$ then clearly

$$
\chi_1^\delta \gtrsim \chi_2^\delta
$$

where we view both functions in (38) as being defined on $T^*M$ and so (38) is just a pointwise estimate. Modulo small errors (see (39)), inequality (38) implies the corresponding operator bound for the matrix elements:

$$
(\text{Op}_h(\chi_1^\delta)\varphi_\mu, \varphi_\mu) \gg (\text{Op}_h(\chi_2^\delta)\varphi_\mu, \varphi_\mu).
$$

Now, take $\varphi_\mu \in V(h)$. From the expressions (40) and (41), it follows that the matrix elements on the RHS of (39) can be estimated from below by simply computing the masses of the model distributions $u(y, \theta; h) = \prod_{j=1}^k e^{im_\theta y} u_e(y) \cdot u_h(y) \cdot u_{ch}(y)$. Our purpose now is to compute these masses on shrinking neighbourhoods of diameter $h^\delta$ centered around a singular rank $\ell < n$ orbit, $\Lambda$. We show that in such neighbourhoods of diameter $h^\delta$, these model distributions have finite mass bounded from below by a positive constant independent of $h \in (0, h_0]$ provided we choose $\delta < 1/2$. It follows that, for $\delta < 1/2$,

$$
(\text{Op}_h(\chi_2^\delta)\varphi_\mu, \varphi_\mu) \geq \frac{1}{C} > 0.
$$
Then, combined with (39), the lower bound (10) implies the \( L^\infty \) lower bound in Lemma 0.3. We now turn to the details of the proof:

**Proof.** By choosing a sufficiently small tubular neighbourhood, \( \tilde{M} \supset M \) inside \( T^* M \), we can extend the Riemannian metric on \( M \) to a Riemannian metric on the entire tube, \( \tilde{M} \). Moreover, by possibly rescaling \( h \) by an appropriate constant, we can assume that \( \mathcal{P}^{-1}(c) \subset \tilde{M} \). We will make this assumption without further comment. We will continue to denote by \( A \) and \( A_e \) the tubular neighbourhoods defined above. In addition, we define

\[
A_{h^\delta} = \exp \{ (x, v) \in E(\delta); |v| \leq h^\delta \}.
\]

Let \( \chi_1^\delta(x; h) \in \mathcal{C}_0^\infty (A_{h^\delta}; [0, 1]) \) be a cutoff function which is identically equal to 1 in \( A_{h^\delta/2} \). Clearly,

\[
\chi_1^\delta(x; \tilde{h}) \in \mathcal{C}_0^0.
\]

Now, choose another cutoff function, \( \chi_2^\delta(x, \xi; \tilde{h}) \in \mathcal{C}_0^\infty (T^* M) \), which is supported in neighbour- 

\[
(O_{ph}(\chi_2^\delta)\varphi_{\mu}, \varphi_{\mu}) \geq (O_{ph}(\chi_2^\delta)\varphi_{\mu}, \varphi_{\mu}) - C_1 \tilde{h}^{1-2\delta}.
\]

We now conjugate the right side to the model by the \( \tilde{h} \) Fourier integral operator \( F \) of Lemma (11). With no loss of generality, we may assume the cutoff function \( \chi_1^\delta(y, \eta; I; \tilde{h}) \) to be of product type:

\[
\chi_1^\delta(y, \eta; I; \tilde{h}) = \prod_{j=1}^{L+N} \chi(h^{-\delta}y_j) \chi(h^{-\delta}\eta_j) \cdot \prod_{j=L+N+1}^{L+M+N+1} \chi(h^{-\delta}\rho_j) \chi(h^{-\delta}\alpha_j) \cdot \prod_{j=n-l+1}^n \chi(h^{-\delta}I_{n+1-j}).
\]

Here \( (r_j, \alpha_j) \) denote radial variables in the \( j \)-th complex hyperbolic summand. Since \( F \) is a microlocally elliptic \( \tilde{h} \)-Fourier integral operator associated to a canonical transformation \( \kappa \), it follows by Egorov’s theorem

\[
(O_{ph}(\chi_2^\delta)\varphi_{\mu}, \varphi_{\mu}) = |c(\tilde{h})|^2 (O_{ph}(\chi_2^\delta \circ \kappa)u_{\mu}, u_{\mu}) - C_3 \tilde{h}^{1-2\delta}
\]

where \( c(\tilde{h})u_{\mu}(y, \theta; \tilde{h}) \) is the microlocal normal form (11) for the eigenfunction \( \varphi_{\mu} \). To simplify the notation a little, we will write \( \chi^\delta(x; \tilde{h}) := \chi(h^{-\delta}x) \) below. Now, \( (\chi^\delta u, u) \) consists of products of four types of terms. The first three are:

\[
M_e = \int_{-\infty}^{\infty} \chi^\delta(\eta; \tilde{h}) |\chi^\delta u_e(\eta; \tilde{h})|^2 d\eta,
\]

\[
M_h = \int_{-\infty}^{\infty} \chi^\delta(\eta; \tilde{h}) |\chi^\delta u_h(\eta; \tilde{h})|^2 d\eta,
\]

and finally,

\[
M_{ch} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi^\delta(\eta_1, \eta_2; \tilde{h}) |\chi^\delta u_{ch}(\eta_1, \eta_2; \tilde{h})|^2 d\eta_1 d\eta_2.
\]

To estimate \( M_e \), we note that, since \( \varphi_{\mu} \in V(\tilde{h}) \), and

\[
\mathcal{F}(e^{-|\eta|^2/h}\Phi_n(\eta h^{-1/2}))(\eta) = e^{-|\eta|^2/h}\Phi_n(\eta h^{-1/2}),
\]
it follows that,
\[
M_\epsilon \gg \int_{-\infty}^{\infty} e^{-2|\eta|^2/h} |\Phi_n(h^{-1/2}\eta)|^2 d\eta + \mathcal{O}(h^\infty)
\]
and so for \( h \in (0, \tilde{h}_0] \), \( M_\epsilon(h) \gg 1 \).

To estimate \( M_h \), we write:
\[
M_h = \frac{|c_+(h)|^2 + |c_-(h)|^2}{\log h} \left( \int_0^\infty \chi(h\xi/h^k) \left| \int_0^\infty e^{-ix x^{-1/2+i\lambda/h}\chi(x/h^\delta \xi)} dx \right|^2 d\xi \right).
\]
Since, \( \varphi_\mu \in V(h) \), it follows that \( |c_+(h)|^2 + |c_-(h)|^2 \gg 1 \) and the integral in \((13)\) is bounded from below by
\[
\frac{1}{C_0} (\log h)^{-1} \int_0^{h^{\delta-1}} \frac{d\xi}{\xi} \left| \int_0^{h^{\delta} \xi} e^{-ix x^{-1/2+i\lambda/h} dx} \right|^2 + \mathcal{O}(\log h^{-1}).
\]
To estimate this last integral, assume first that \( \xi \in [0, h^{-\delta}] \). Then, by an integration by parts,
\[
\int_0^{h^{\delta} \xi} e^{-ix x^{-1/2+i\lambda/h} dx} = \mathcal{O}(|h^{\delta} \xi|^{1/2}) + \mathcal{O}(|h^{\delta} \xi|^{3/2}),
\]
and so,
\[
M_h \gg |\log h|^{-1} \int_{h^{-\delta}}^{h^{\delta-1}} \frac{d\xi}{\xi} \left| \int_0^{h^{\delta} \xi} e^{-ix x^{-1/2+i\lambda/h} dx} \right|^2 + \mathcal{O}(\log h^{-1}).
\]
From \((17)\), it follows that:
\[
M_h \gg |\Gamma(1/2 + i\lambda/h)|^2 (1 - 2\delta) + \mathcal{O}(\log h^{-1}).
\]
Since, \( \delta = 1/2 - \epsilon \), there exists a constant \( C(\epsilon) \) such that, \( M_h \geq C(\epsilon) > 0 \) uniformly for \( h \in (0, \tilde{h}_0(\epsilon)] \).

Finally, we are left with the integral \( M_{ch} \) corresponding to a loxodromic subspace. Since \( |\mathcal{J}_k(\rho)| \leq 1 \) for all \( k \in \mathbb{Z} \) and \( \rho \in \mathbb{R} \), it follows that:
\[
M_{ch} \gg |\log h|^{-1} \int_{h^{-\delta}}^{h^{\delta-1}} \frac{d\alpha}{\alpha} \left| \int_0^{h^{\delta} \alpha} \mathcal{J}_k(\rho) \rho^{it/h} d\rho \right|^2 + \mathcal{O}(\log h^{-1}).
\]
Here, \( \mathcal{J}_k(\rho) \) denotes the \( k \)-th integral Bessel function of the first kind \([AS]\). For \( \alpha \geq h^{-\delta} \),
\[
\left| \int_0^{h^{\delta} \alpha} \mathcal{J}_k(\rho) \rho^{it/h} d\rho \right| = \frac{2\pi h^{\delta/2} \Gamma(k+1+it/h)}{\Gamma(k+1-it/h)} |h^{\delta} \alpha|^{-1/2} = 1 + \mathcal{O}(|h^{\delta} \alpha|^{-1/2}),
\]
and so,
\[
M_{ch} \gg |\log h|^{-1} \int_{h^{-\delta}}^{h^{\delta-1}} \frac{d\alpha}{\alpha} + \mathcal{O}(\log h^{-1}) = 1 - 2\delta + \mathcal{O}(\log h^{-1}).
\]
Consequently, given \( \delta = 1/2 - \epsilon \) it again follows that \( M_{ch} \geq C(\epsilon) > 0 \) uniformly for \( h \in (0, \tilde{h}_0(\epsilon)] \).
The final step involves estimating \((\Op_h(\chi^\delta(I))e^{im\theta}, e^{im\theta})\). An integration by parts in the \(I_1, \ldots, I_\ell\) variables shows that:

\[
\Op_h(\chi^\delta(I))e^{im\theta}, e^{im\theta}) = 1 + \mathcal{O}(h^{1-\delta}).
\]

(52)

As a consequence of the estimates above for \(M_h, M_{ch}, M_e\) and the bound in (44) it follows that for any \(\epsilon > 0\) and \(\delta = 1/2 - \epsilon\), there exists a constant \(C(\epsilon) > 0\) such that for all \(\varphi_\mu \in V(h)\),

\[
(\Op_h(\chi^\delta_1)\varphi_\mu, \varphi_\mu) \geq C(\epsilon) > 0.
\]

(53)

Thus,

\[
\|\varphi_\mu\|_{L^\infty}^2 \cdot \left(\int_M \chi^\delta_1(x; h) \, dvol(x)\right) \geq C(\epsilon),
\]

(54)

uniformly for \(h \in (0, h_0(\epsilon)]\). Since

\[
\int_M \chi^\delta_1(x; h) \, dvol(x) = \mathcal{O}(h^{\delta(n-\ell)})
\]

with \(h^{-1} \in \text{Spec } -\sqrt{\Delta}\), the lower bound coming from (53) is:

\[
\|\varphi_\lambda\|_{L^\infty} \geq C(\epsilon)\lambda^{\frac{n-\ell}{4}-\epsilon}.
\]

The proof of Lemma 3 (i) is complete. Lemma 3 (ii) then follows by applying the Hölder inequality in the estimate (23).

\[\square\]

**Remark:** From our earlier results in the case of the quantum Euler top [T1], it seems likely that in fact \(\|\varphi_\lambda\|_{L^\infty} \geq C\lambda^{\frac{n-\ell}{4}}\) in the elliptic case and \(\|\varphi_\lambda\|_{L^\infty} \geq C\lambda^{\frac{n-\ell}{4}}(\log \lambda)^{-\alpha}\) for an appropriate \(\alpha > 0\) in the hyperbolic case.

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