Deformation quantization of covariant fields

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Abstract

After sketching recent advances and subtleties in classical relativistically co-
variant field theories, we give in this short Note some indications as to how the
deformation quantization approach can be used to solve or at least give a better
understanding of their quantization.

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1 Introduction

We would like to discuss here a possible application of deformation quantization, viz.
a nonperturbative construction of star-products (along the lines indicated in [Di93]),
to the quantization of interacting field theory. It is based on recent advances in the
linearization programme for covariant wave equations proposed in [FS80a, FS80b] that
uses nonlinear representation theory [FPS77]. We shall present this approach on an
example. It can now be formulated in a more general setting thanks to results obtained
by Flato, Simon, and Taflin concerning the existence of global solutions, scattering
theory, and linearization of several covariant wave equations relevant to physics [ST85,
FST87, ST93, FST97] (see [ST00] for a recent nontechnical survey of these results).

2 A sketch of the linearization programme

We shall give only the most basic facts and describe a few results, relevant for our dis-
cussion of the linearization programme [FS80a, FS80b] for covariant nonlinear equa-
tions. We skip the technical details: the reader can find them in the references listed
at the end of the paper. At this stage we shall only briefly present the context. For the
reader’s, convenience, we have collected in an appendix the basic notions on nonlinear
representation theory of Lie groups and algebras introduced and developed in [FPS77]
and some later works.
Let \( G \) be a Lie group and \( E \) be some vector space of functions (endowed with a Banach or Fréchet topology). We assume that there is an analytic action of \( G \) on a neighborhood \( \mathcal{O} \) of the origin in \( E \) and that \( 0 \in E \) is a fixed point for the action. Let \( H(\mathcal{O}, \mathcal{O}) \) be the space of analytic maps from \( \mathcal{O} \) to \( \mathcal{O} \). Hence there is a map \( g \in G \rightarrow S_g \in H(\mathcal{O}, \mathcal{O}) \) such that \( S_g(\mathcal{O}) \subset \mathcal{O}, S_{g_1g_2} = S_{g_1}S_{g_2}, S_{g^{-1}} = S_g^{-1} \), and \( S_g(0) = 0 \). The pair \((S, \mathcal{O})\) is called here an analytic nonlinear representation of \( G \) in \( \mathcal{O} \).

At the infinitesimal level, by differentiating the action of \( S_{\exp(tX)} \), where \( X \) is an element of the Lie algebra \( \mathfrak{g} \) of \( G \), one would expect to find “vector fields” \( T_\mathcal{O} \) on \( \mathcal{O} \) (or on a open subset of it) satisfying \( T_{[X,Y]} = T_X Y - T_Y X \). The pair \((T, \mathcal{O})\) is called an analytic nonlinear Lie algebra representation of \( \mathfrak{g} \) on \( \mathcal{O} \).

One may think of the maps \( S \) and \( T \) be given by their Taylor coefficients: \( S = \sum_{n \geq 1} S^n \) and \( T = \sum_{n \geq 1} T^n \). Then the first order term \( S^1 \) (resp. \( T^1 \)) is a linear representation of \( G \) (resp. \( \mathfrak{g} \)). Nonlinear representations are built by successive extensions of a linear representation by its successive tensor powers: they can be viewed as a kind of deformation of their linear part. Interactions described by covariant nonlinear equations (see below) can then be seen, as they should, as deformations of the free (noninteracting) situation, in line with Flato’s deformation philosophy \([\text{Fl82}]\).

### 2.1 Covariant nonlinear equations

We shall be concerned with relativistic equations on Minkowski spacetime with direct physical relevance and “covariant” will always mean covariant under the action of the Poincaré group \( \mathcal{P} = \text{SL}(2, \mathbb{C}) \ltimes \mathbb{R}^4 \). The Lie algebra of \( \mathcal{P} \) will be denoted by \( \mathfrak{p} = \text{sl}(2, \mathbb{C}) + \mathbb{R}^4 \). We use standard conventions: Einstein summation convention is used throughout for upper and lower indices (Latin indices run from 1 to 3 and Greek ones from 0 to 3). The metric tensor \( g_{\mu \nu} \) has signature \((+ - - -)\). For convenience of notation we shall write \( x \in \mathbb{R}^4 \) as \( x = (t, \vec{x}) \), where \( t = x^0, \vec{x} = (x^1, x^2, x^3) \). For a function \( \phi : \mathbb{R}^4 \rightarrow V \) where \( V \) is a vector space, we set \( \phi(t) : \mathbb{R}^3 \rightarrow V \) by \( \phi(t)(\vec{x}) = \phi(t, \vec{x}) \equiv \phi(x) \). An element \( g \in \mathcal{P} \) is written as \( g = (\Lambda, a) \) where \( \Lambda \in \text{SL}(2, \mathbb{C}) \) and \( a \in \mathbb{R}^4 \). Let \( \Lambda \rightarrow \tilde{\Lambda} \) be the canonical homomorphism of \( \text{SL}(2, \mathbb{C}) \) onto the group \( \text{SO}_0(1,3) \). Then the group law in \( \mathcal{P} \) is \((\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1 \Lambda_2, a_1 + \tilde{\Lambda}_1 a_2) \). The natural action of the Poincaré group on spacetime will be denoted by \( g \cdot x = \tilde{\Lambda} x + a \). Finally, the standard basis of \( \mathfrak{p} \) is \( \{P_\mu, M_{\alpha \beta}\} \) with \( \alpha < \beta \).

Most of the classical wave equations met in physics can be cast into evolution equations governed by some Hamiltonian operator (see e.g. \([\text{RS79}]\)). Suppose that one starts with a vector space \( E \) of initial data (usually a Banach or Fréchet space). For simplicity, elements of \( E \) are functions defined on the whole of \( \mathbb{R}^3 \) and take their values in a finite-dimensional vector space \( V \). Let \( H \) be an operator on \( E \) given by a (non)linear differential operator such that \( H(0) = 0 \). We are interested in the solutions of the following evolution equation:

\[
\frac{d\psi(t)}{dt} = H(\psi(t)), \quad \psi(0) = \psi_0 \in E. \tag{2.1}
\]

\( ^1 \)This kind of holomorphic mappings space between topological vector spaces can be given several topologies and the theory is well developed, see \([\text{Tsu95}]\).
Suppose that the previous evolution equation is associated to some covariant nonlinear (wave) equation. So one has a finite-dimensional representation of \( SL(2, \mathbb{C}) \) in \( V \), \( \Lambda \rightarrow A(\Lambda) \), and given a solution \( \psi \) of Eq. (2.1), then \( \psi^g(x) \equiv A(\Lambda)\psi(\Lambda^{-1}(x-a)) \) is also a solution for any \( g = (\Lambda, a) \in P \). There is a local action \( S \) of \( P \) on \( E \) having the origin in \( E \) as fixed point and given by \( S^{g_1}_g(\psi_0) = \psi_{g_1}(t) \), where \( \psi_0(x) = \psi(0, x) \), i.e., \( S^{g_2}_g = S^{g_1}_g S^{g_2}_g \). In particular, if \( t \mapsto U_t \) is the evolution operator of Eq. (2.1), \( U_{t+t'} = U_t U_{t'} \), \( \psi(t) = U_t(\psi_0) \) and we have \( U(t) = S_{\exp(tP_0)} \), where \( P_0 \in \mathfrak{p} \) is the generator of time translations. Hence we can associate to the covariant equation (2.1) what looks very much like a nonlinear representation of \( P \) in \( E \). Conversely, if one has a nonlinear representation \( (S, E) \) of \( P \), by formally differentiating \( \psi(t) = S_{\exp(tP_0)}(\psi(0)) \) with respect to \( t \), one would expect to recover a covariant evolution equation as in Eq. (2.1).

The linearization programme [FS80a, FS80b] is based on the idea to take into account the full covariance of a nonlinear equation by associating to it a nonlinear representation, and then study its linearizability. In the case where the nonlinear representation is linearizable, i.e., equivalent to its linear part which corresponds to the associated linear (free) equation, then one can directly deduce global existence in time of the solutions of the nonlinear equation.

As one would expect from deformation theory, the linearization involves the computation of some cohomology spaces. It is remarkable that formal nonlinear massive representations of the Poincaré group are always formally linearizable [Ta84]. The covariance provides the right spaces where to perform the study (spaces of differentiable or analytic vectors for the linear part) and then to look for convergence of the expressions involved when going to the analytic setting.

Notice that the Maxwell-Dirac system in \((1+3)\) dimensions (i.e., classical electrodynamics) presents many subtleties due to the zero mass of the photon. Even at the formal level, one cannot get a (local) nonlinear representation of Lie algebra by using the techniques of [FPS77]. Further technical refinements are needed and it is a real tour de force that was achieved in [FST97] by solving the long standing problem of global solutions for classical electrodynamics and asymptotic completeness. It turns out that Maxwell-Dirac system cannot be linearized completely because of the long-range nature of the vector potential. The asymptotic fields do not transform according to a linear representation of the Poincaré group and consequently they are not asymptotically free. Nevertheless, one gets tractable expressions which provide new insights on the infrared problem in Quantum Electrodynamics.

### 2.2 Nonlinear Klein-Gordon equation

We present an example that we shall need for our discussion on deformation quantization. It is based on results obtained in [ST93]. Consider the massive nonlinear Klein-Gordon equation:

\[
\Box \Phi + m^2 \Phi = P(\Phi, \partial_t \Phi, \nabla \Phi), \quad m > 0,
\]

(2.2)

where \( \Box = \partial^\mu \partial_{\mu} - \Delta \), and \( P \) is a \( C^\infty \) function covariant under \( \mathcal{P} \) and vanishes at 0 along with its first derivatives.
Simon and Taflin [ST85, ST93] solved the Cauchy problem for Eq. (2.2), in \((1+n)\)-dimensional spacetime with \(n \geq 2\), by using linearization techniques. They have shown global existence and asymptotic completeness for Cauchy data in some neighborhood of the origin of an Hilbert space.

Equation (2.2) can be put in the form of an evolution equation (2.1) by taking:

\[
\psi = \begin{pmatrix} \Phi \\ \Pi \end{pmatrix}, \quad H(\psi) = \begin{pmatrix} 0 & I \\ \Delta - m^2 & 0 \end{pmatrix} \begin{pmatrix} \Phi \\ \Pi \end{pmatrix} + \begin{pmatrix} 0 \\ P(\Phi, \Pi, \nabla \Phi) \end{pmatrix}. \tag{2.3}
\]

We shall use the following decomposition into Fourier modes:

\[
\Phi(t)(\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2} \omega(k)} (\Pi(t)(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} + a(t)(\vec{k}) e^{i\vec{k} \cdot \vec{x}}),
\]

\[
\Pi(t)(\vec{x}) = i \int \frac{d^3k}{(2\pi)^{3/2}} (\Pi(t)(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} - a(t)(\vec{k}) e^{i\vec{k} \cdot \vec{x}}), \tag{2.4}
\]

where \(\omega(\vec{k}) = \sqrt{|\vec{k}|^2 + m^2}\), along with:

\[
a_+(t)(\vec{x}) = i \int \frac{d^3k}{(2\pi)^{3/2}} \Pi(t)(\vec{k}) e^{-i\vec{k} \cdot \vec{x}}, \quad a_-(t)(\vec{x}) = -i \int \frac{d^3k}{(2\pi)^{3/2}} a(t)(\vec{k}) e^{i\vec{k} \cdot \vec{x}}. \tag{2.5}
\]

In terms of \(a = (a_+, a_-)\), the evolution equation for the nonlinear Klein-Gordon equation reads:

\[
\frac{d}{dt} \begin{pmatrix} a_+(t) \\ a_-(t) \end{pmatrix} = i \omega(-i\nabla) \begin{pmatrix} a_+(t) \\ -a_-(t) \end{pmatrix} + \begin{pmatrix} F(a(t)) \\ F(a(t)) \end{pmatrix}, \tag{2.6}
\]

where \(F(a(t)) = P(\Phi(t), \Pi(t), \nabla \Phi(t))\) through the substitutions (2.4) and (2.5) and \(\omega(-i\nabla)\) is the operator acting as multiplication by \(\omega(k)\) after Fourier transformation.

The free Klein-Gordon equation (when \(F = 0\)) induces a Lie algebra linear representation \(T^1\) of \(p\) in \(E_\infty = \mathcal{S}(\mathbb{R}^3, \mathbb{C}) \oplus \mathcal{S}(\mathbb{R}^3, \mathbb{C})\). For \((f_+, f_-) \in E_\infty\) it is given by:

\[
T^1_p(f_+, f_-) = \begin{cases} i\omega(-i\nabla)(f_+, -f_-); \\
\partial_j(f_+, f_-); \\
(x_j \partial_j - x_j \partial_i)(f_+, f_-); \\
i\omega(-i\nabla)(x_j f_+, -x_j f_-). \end{cases}
\]

The representation \(T^1\) exponentiates to a continuous linear representation of \(\mathcal{S}\) in the Hilbert space \(E = L^2(\mathbb{R}^3, \mathbb{C}) \oplus L^2(\mathbb{R}^3, \mathbb{C})\), for which \(E_\infty\) is the set of \(C^\infty\)-vectors.

The nonlinear representation \(T = \sum_{n \geq 1} T^n\) of \(p\) in \(E_\infty\) associated to Eq. (2.3) is determined by the interaction term \(F\). By writing \(T = T^1 + \tilde{T}\), on \(f = (f_+, f_-) \in E_\infty\), it
is defined by:

\[ \tilde{T}_0(f_+, f_-) = (F(f), F(f)); \]
\[ T_p(f_+, f_-) = 0; \]
\[ \tilde{T}_{M_i}(f_+, f_-) = 0; \]
\[ \tilde{T}_{M_0}(f_+, f_-) = (x_i F(f), x_j F(f)). \]

The coefficients \( T^n, n \geq 2 \), are the homogeneous terms of degree \( n \) in \( \tilde{T} \). With these notations, Eq. (2.6) now reads:

\[ \frac{df(t)}{dt} = (T^1_0 + \tilde{T}_0)(f(t)). \]

It is shown in [ST93] that this nonlinear representation of \( p \) can be exponentiated to a nonlinear representation \( U = \sum_{n \geq 1} U^n \) of \( \mathcal{P} \) on some neighborhood of \( 0 \in E \) and that \( U \) is linearizable. We shall be interested in the particular case where the interaction term \( P \) in (2.2) is a covariant polynomial of the field \( \Phi \) only, without constant term or terms of degree 1. Also we require that the classical Hamiltonian associated to Eq. (2.2) is a positive functional. In that particular case, it follows directly from more general results proved in [ST93] that:

1. \( U \) is an analytic map \( \mathcal{P} \times E_\infty \to E_\infty \) and \( g \mapsto U^1_g \) is a continuous map from \( \mathcal{P} \) to the space of Banach analytic maps \( \mathcal{H}(E_\infty, E_\infty) \).

2. There exists a unique invertible analytic \( \Omega_\epsilon : E_\infty \to E_\infty, \epsilon = \pm, \) such that \( U_\epsilon \Omega_\epsilon = \Omega_\epsilon U^1_\epsilon \) for \( g \in \mathcal{P} \).

3. \( \lim_{t \to \pm \infty} ||U_{\exp(tP)}(\Omega_\pm(\psi)) - U_{\exp(tP)}(\psi)||_E = 0, \) \( \psi \in E_\infty \).

4. If the Cauchy data \( (\Phi(0), \partial_t \Phi(0)) \) are in \( E_\infty \), then \( \Phi(t) \) is in \( \mathcal{S}(\mathbb{R}^3, \mathbb{C}) \) for all \( t \in \mathbb{R} \) and \( \Phi \in C^\infty(\mathbb{R}^4) \).

### 3 Application of deformation quantization

Consider a finite-dimensional manifold \( M \) with a Poison bracket \( \{\cdot, \cdot\} \) defined on it. Recall that a deformation quantization of \( M \) is given by a formal associative deformation \( \star \) (star-product) of the usual multiplication of smooth functions on \( M \):

\[ f \star g = fg + \sum_{n \geq 1} \lambda^n C_n(f, g), \tag{3.7} \]

such that \([f, g]_\lambda = \frac{1}{2\lambda}(f \star g - g \star f)\) is a formal Lie algebra deformation of the Poisson bracket on \( M \). All the series involved are formal series in the deformation parameter \( \lambda \) with coefficients in the space of smooth functions on \( M \) (see e.g. [DS] for details). The cochains \( C_n \) are usually required to be bidifferential operators vanishing on constants.
In quantum-mechanical models, one takes $\lambda = \frac{i\hbar}{2}$ and the formal series often converge in the distribution sense.

In order to treat field theoretic problems, one would naturally go over to infinite-dimensional manifolds of fields. Then one would consider a family of functionals of the fields and try to deform it via a star-product. Very quickly, one will be faced with the problem of giving a meaning to (3.7) as a formal series, i.e., to have well-defined cochains $C_n$ on the family of functionals considered. Of course it is always possible to take a small enough family of functionals such that the star-product is well defined. For example, if the manifold $M = H \oplus H$ where $H$ is a Hilbert space, take as algebra of functionals $A^\star$ the one generated by (finite) tensor products of continuous linear forms on $H$, then it is easy to write down a well-defined star-product such that $A^\star \star A^\star \subset A^\star[\lambda]$. But this is of no interest if one has in mind applications to field quantization: even free Hamiltonians do not belong to this family of functionals. So our very first concern is to construct star-products defined on a sufficiently rich family of functionals which should at least includes the Poincaré generators.

For more than a half-century, physicists have known how to handle divergences appearing in physical interacting quantum models, especially in QED, and developed perturbative renormalization methods. The idea underlying most of the renormalization methods is to first regularize singular expressions (e.g., with a cut-off), then identify and extract terms becoming singular when the regularization is removed. This is done perturbatively with respect to various relevant parameters of the model considered.

In the deformation quantization framework, one will obviously meet divergences in the cochains of the star-product and try to remove them in a more controlled way. The idea here is to identify the diverging terms as singular cocycles or coboundaries of the Hochschild cohomology and get rid of them by passing to an equivalent star-product. The perturbative renormalization methods. The idea underlying most of the renormalization methods is to first regularize singular expressions (e.g., with a cut-off), then identify and extract terms becoming singular when the regularization is removed. This is done perturbatively with respect to various relevant parameters of the model considered.

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### 3.1 Normal star-product

Consider the free massive real scalar field $\Phi$. It satisfies the linear Klein-Gordon equation $(\Box + m^2)\Phi(t, \vec{x}) = 0$, $m > 0$. Let $E_\infty = \mathcal{S}(\mathbb{R}^3, \mathbb{R}) \oplus \mathcal{S}'(\mathbb{R}^3, \mathbb{R})$. The initial data $(f, \pi) \in E_\infty$ are decomposed into (definite energy) Fourier modes by:

$$\begin{align*}
\phi(\vec{x}) &= \int \frac{d^3k}{2(2\pi)^{3/2}\omega(k)} (\pi(k) e^{-ik\cdot x} + a(k) e^{ik\cdot x}), \\
\pi(\vec{x}) &= i\int \frac{d^3k}{2(2\pi)^{3/2}} (\pi(k) e^{-ik\cdot x} - a(k) e^{ik\cdot x}),
\end{align*}
$$

where as usual $\omega(k) = (|\vec{k}|^2 + m^2)^{1/2}$. Since $m > 0$ and $(f, \pi) \in E_\infty$ it follows that $(\pi, a)$ is in $W_\infty \equiv \mathcal{S}'(\mathbb{R}^3, \mathbb{C}) \oplus \mathcal{S}'(\mathbb{R}^3, \mathbb{C})$. We look at $W_\infty$ as a real vector space.

Denote by $A = \mathcal{H}_2(W_\infty)$ the space of analytic $\mathbb{C}$-valued mappings on $W_\infty$ having semiregular kernels as tempered distributions. An element $F \in \mathcal{H}_2(W_\infty)$ can be written...
where the kernel $F^{mn} \in \mathcal{S}'(\mathbb{R}^{3m}, \mathbb{C}) \oplus \mathcal{S}'(\mathbb{R}^{3n}, \mathbb{C})$ is regular with respect to either its first $3m$ variables or its last $3n$ variables. Semiregularity implies that usual functional derivatives of $F$ are in $\mathcal{S}$ as long as one does not consider mixed derivatives. Notice that the free Hamiltonian $H_0(\vec{\pi}, a) = \int \frac{d^3k}{2} A(\vec{\pi}) a(\vec{k})$ belongs to $\mathcal{A}$ along with all of the free Poincaré generators.

We define normalized functional derivatives as:

$$D_{a(\vec{k})} = (2\omega(\vec{k}))^{1/2} \delta \overline{\alpha(\vec{k})} \quad D_{\vec{\pi}(\vec{k})} = (2\omega(\vec{k}))^{1/2} \delta \overline{\pi(\vec{k})},$$

and the standard Poisson bracket reads:

$$\{F, G\} = \frac{2}{i} \int d^3k (D_{a(\vec{k})} F) D_{\vec{\pi}(\vec{k})} (G) - D_{\vec{\pi}(\vec{k})} (F) D_{a(\vec{k})} (G).$$

Clearly $\mathcal{A}$ endowed with this bracket is a Poisson algebra (i.e., it is closed for the bracket operation), and $(W_0, \mathcal{A}, \{\cdot,\cdot\})$ is a Poisson space.

The normal star-product $\ast_N$ on $(W_0, \mathcal{A})$ is defined by

$$F \ast_N G = FG + \sum_{n\geq 1} \hbar^n C^N_n (F, G)$$

where:

$$C^N_n (F, G) = \frac{1}{n!} \int d^3k_1 \cdots d^3k_n (D_{a(\vec{k}_1)} \cdots D_{a(\vec{k}_n)} (F) D_{\vec{\pi}(\vec{k}_1)} \cdots D_{\vec{\pi}(\vec{k}_n)} (G)).$$

Notice that the deformation parameter is $\frac{\hbar}{2}$; the factor $\frac{1}{2}$ has been absorbed in the definition of the cochains, so the star-bracket is $[f, g]_{\ast_N} = \frac{\hbar}{2} (F \ast_N G - G \ast_N F)$.

The normal star-product realizes a deformation quantization of $(W_0, \mathcal{A}, \{\cdot,\cdot\})$ and it will be used in the next section as starting point for the construction of other star-products.

### 3.2 Deformation quantization for interacting fields

For our discussion of the applications of the linearization programme to deformation quantization of interacting fields, we shall consider a massive real scalar field theory in $(1+3)$-dimensions with polynomial interaction such that the Hamiltonian is positive:

$$\square + m^2 \Phi(t, \vec{x}) + V'(\Phi(t, \vec{x})) = 0 \quad m > 0,$$

where $V'$ is the derivative of the potential in the Hamiltonian. $V$ is assumed here to fulfill $V(0) = V'(0) = V''(0) = 0$. We denote by $\{P_\mu, M_{\alpha\beta}\}$ the Poincaré generators for Eq. (3.12) as seen as functionals of the Cauchy data $(\phi, \vec{\pi})$ or their Fourier modes $(\vec{\pi}, a)$ defined above. Also $\{P'_\mu, M'_{\alpha\beta}\}$ will be the corresponding free generators.
For initial data \( \Phi(0) = \phi \) and \( \partial_t \Phi(0) = \pi \) in \( E_\infty = \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}) \oplus \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}) \), it follows from the results of \([ST93]\), reviewed in Section 2.3, that there exists a unique global solution \( \Phi \) with free asymptotic fields \( \Phi^\pm \):

\[
\lim_{t \to \pm \infty} \| \Phi(t) - \Phi^\pm(t) \|_E = 0,
\]

where \( \| \cdot \|_E \) denotes the Hilbertian norm of \( E = L^2(\mathbb{R}^3, \mathbb{R}) \oplus L^2(\mathbb{R}^3, \mathbb{R}) \).

Denote by \((\phi^\pm, \pi^\pm)\) the initial data of \( \Phi^\pm \). The wave operators \( \Omega^\pm : (\phi^\pm, \pi^\pm) \to (\phi, \pi) \) are invertible analytic mappings from \( E_\infty \) onto \( E_\infty \), they preserve the Poisson bracket (Poisson maps), and the classical scattering operator \( S = (\Omega^+)^{-1}\Omega^- \) is also analytic on \( E_\infty \). Moreover, and more importantly for us, the wave operators linearize the Poincaré generators:

\[
P_\mu \circ \Omega^\pm = P^f_\mu, \quad M_{\alpha \beta} \circ \Omega^\pm = M^f_{\alpha \beta}.
\]

(3.13)

Notice that in general the generators \( \{ P_\mu, M_{\alpha \beta} \} \) do not belong to the space \( \mathcal{A} \) (e.g. for \( \Phi^4 \)-theory). However for the kind of potential we are considering, the first cochain of the normal star-product \( C^N_1 \) makes still sense on the Poincaré generators, the remark applies to the Poisson bracket as well. However, the second cochain \( C^N_2 \) is not defined on the Poincaré generators \( \{ P_\mu, M_{\alpha \beta} \} \) in general.

We now introduce two star-products from the normal one:

\[
(F \star^\pm G) \circ \Omega^\pm = (F \circ \Omega^\pm) \star_{M} (G \circ \Omega^\pm).
\]

(3.14)

They are defined on \( \mathcal{A}^\pm \) and push-forward of \( \mathcal{A} \) by \( \Omega^\pm \). Since \( \Omega^\pm \) are Poisson maps the star-brackets of \( \star^\pm \) are also deformations of the standard Poisson bracket (3.10). Since \( \Omega^\pm \) depends (in a non-trivial way!) on the parameters (coupling constants) that appear in the potential \( V \) in (3.12), so will the cochains \( C^N_\mu \) of \( \star^\pm \). One should also remark that (3.14) does not necessarily imply the equivalence of the star-products.

From Eq. (3.13), we deduce in particular for the \( k \)-th \( \star^\pm \)-power of the interacting Hamiltonian \( H = P_0^f \):

\[
(\star^\pm H)^k = (\star^N H_0)^k \circ (\Omega^\pm)^{-1},
\]

(3.15)

where \( H_0 = P_0^f \) is the free Hamiltonian. It is easy to check that \( (\star^N H_0)^k \) belongs to \( \mathcal{A}^\pm \) for all \( k \geq 1 \), and, since \( (\Omega^\pm)^{-1} \) sends \( W_\infty \) onto \( W_\infty \), the right-hand side of (3.15) is well defined for all \( n \geq 0 \). Consequently all the powers \( (\star^\pm H)^n \) are defined on \( W_\infty \). Obviously, the same holds for any finite \( \star^\pm \)-products of the \( \{ P_\mu, M_{\alpha \beta} \} \). Therefore \( \mathcal{A}^\pm \) contains a \( \star^\pm \)-subalgebra isomorphic to the universal enveloping algebra of the Poincaré Lie algebra.

A fundamental tool in deformation quantization is the star-exponential:

\[
\text{Exp}_\star(\frac{tF}{i\hbar}) = \sum_{n \geq 0} \frac{1}{n!} (\frac{t}{i\hbar})^n (\star F)^n.
\]

(3.16)

In many quantum-mechanical models, this series is convergent as a distribution, and allows to determine the spectrum and eigenstates of some Hamiltonian or any physically relevant observable. This is accomplished by performing a Fourier analysis of
However due to the presence of the factor \( \frac{1}{\hbar} \) one cannot conclude that the series is a formal series in both \( \hbar \) and \( \hbar^{-1} \), because their coefficients involve infinite sums. The linearization maps would be useful for these considerations.

The star-products \( \star^\pm \) are best suited than the normal product as they are free of a large class of divergences by construction. It is not clear how the fields behave in the framework we have presented. A deeper study of these constructions should open more avenues to treat divergences problem in field theory and make more explicit the interplay between classical and quantum field theories.

We end this Note by making some remarks about Electrodynamics. Maxwell-Dirac equations cannot be linearized and the asymptotic fields are not free (the vector potential is free, but the Dirac spinor is not). The existence of modified wave operators has been established in [FST97] on a space of small initial data in neighborhood of the origin in some Hilbert space. These modified operators do linearize the Poincaré generators (seen as functional of the fields) and the electric current, but before applying a procedure similar to the one we have sketched here, one will first have to solve the deformation quantization of the asymptotic fields themselves.

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Appendix: Notions on nonlinear representations

Let \( E \) be a Fréchet space. Denote by \( \mathcal{L}^n(E, E) \) the space of continuous symmetric \( n \)-linear maps on \( E \) taking their values in \( E \). The natural identification \( \mathcal{L}^n(E, E) \sim \mathcal{L}(\hat{\otimes}^n E, E) \), where \( \hat{\otimes} \) is the completed symmetric projective tensor product, will be implicitly assumed. For \( f \in \mathcal{L}^n(E, E) \), we shall denote by \( f \) the associated homogeneous polynomial on \( E \), i.e., \( f(x) = f(x, \ldots, x) \). Let \( \mathcal{F}(E) \) be the vector space of formal series \( F = \sum_{n \geq 1} f^n \), where \( f^n \in \mathcal{L}^n(E, E) \). There is an associative product on \( \mathcal{F}(E) \), denoted \( \circ \), given by the composition of formal series. For \( F = \sum_{n \geq 1} f^n \) and \( H = \sum_{n \geq 1} h^n \), the associated homogeneous polynomial of the term of degree \( n \) in \( F \circ H \) reads:

\[
(F \circ H)^n(x) = \sum_{1 \leq p \leq n} f^p \left( \sum_{n_1 + \cdots + n_p = n} \hat{h}^{n_1}(x) \otimes \cdots \otimes \hat{h}^{n_p}(x) \right), \quad x \in E. \tag{A.1}
\]

Let \( \mathcal{F}^{\text{inv}}(E) \) be the group of invertible elements in \( (\mathcal{F}(E), \circ) \).

**Definition A.1.** A formal nonlinear representation \((S, E)\) of a Lie group \( G \) in \( E \) is a homomorphism \( S: G \rightarrow \mathcal{F}^{\text{inv}}(E) \) such that \((g; x_1, \ldots, x_n) \mapsto S^1_g(x_1, \ldots, x_n)\) are continuous maps from \( G \times E^n \) to \( E \), \( \forall n \geq 1 \).

As a consequence of Eq. (A.1), we have \( S^1_g(x) = S^1_g(S^1_{g'}(x)) \) for \( g, g' \in G, x \in E \). Thus the first order term of \( S_g = \sum_{n \geq 1} S^1_g \) is a linear representation of \( G \) in \( E \), denoted \( (S^1, E) \) and called the free part of \((S, E)\).
Definition A.2. Two formal nonlinear representations $(S,E)$ and $(S',E')$ of a Lie group $G$ in $E$ are said to be formally equivalent, if there exists an element $A \in \mathcal{F}^{inv}(E)$ such that $S_g = A \circ S'_g \circ A^{-1}$, $\forall g \in G$. A formal nonlinear representation $(S,E)$ is called formally linearizable, if it is formally equivalent to the linear representation $(S^1,E)$.

Roughly speaking, a formal nonlinear representation can be seen as a formal action of $G$ in $E$ having $0 \in E$ as a fixed point. By looking at the infinitesimal action, one would expect to encounter formal vector fields and their Lie bracket. It is indeed the case. For $F,H \in \mathcal{F}(E)$, define $F \bullet H \in \mathcal{F}(E)$ by:

$$ (\hat{F} \hat{H})^p(x) = \sum_{1 \leq p \leq n} f^p \left( \sum_{1 \leq i \leq p} (x^{\otimes i-1}) \otimes \hat{h}_{n-p+1}(x) \otimes (x^{\otimes p-i}) \right), \quad x \in E. \quad (A.2) $$

The bracket $[F,H] = F \bullet H - H \bullet F$ endows $\mathcal{F}(E)$ with a Lie algebra structure, and we have the corresponding of Def. [A.1] for formal nonlinear representations of a Lie algebra.

Definition A.3. A formal nonlinear representation $(T,E)$ of a Lie algebra $\mathfrak{g}$ in $E$ is a Lie algebra homomorphism $T : \mathfrak{g} \to \mathcal{F}(E)$.

By writing $T = \sum_{n \geq 1} T^n$, one can check that $T^1 : \mathfrak{g} \to \mathcal{L}(E,E)$ is a linear Lie algebra representation of $\mathfrak{g}$ in $E$, and $(T^1,E)$ is called the free part of $(T,E)$.

There are analytic counterparts to the previous definitions. One would speak of analytic nonlinear representations of Lie groups and algebras, analytic equivalence of analytic nonlinear representations, etc. if the formal series involved are analytic in some neighborhood of the origin in $E$.

If one has a continuous linear representation $\pi$ of a real Lie group $G$ in a Banach space $E$, then one has a linear representation $d\pi$ of the Lie algebra $\mathfrak{g}$ of $G$ in $E_m$, the Fréchet space of $C^m$-vectors of $\pi$. It is a remarkable fact that a similar statement holds for nonlinear representations. More precisely [FPS77, ST95]: any (formal or analytic) nonlinear representation $(S,E)$ of a real Lie group $G$ on a Banach space $E$ can be differentiated and gives a nonlinear representation $(T,E_m)$ of the Lie algebra $\mathfrak{g}$ on the space of $C^m$-vectors for the free part $(S^1,E)$ of $(S,E)$. Then $T^1$ is equal to $dS^1$. Conversely, if $G$ is moreover connected and simply connected, any nonlinear representation $(T,E_m)$ of $\mathfrak{g}$ such that its free part $(T^1,E_m)$ is the differential of a linear representation $\pi$ of $G$ in a Banach space $E$, can be exponentiated to a unique nonlinear representation $(S,E_m)$ of $G$ such that $S^1 = \pi$ (under some regularity assumptions on $(T,E_m)$, $(S,E_m)$ can be extended to a nonlinear representation $(S,E)$).

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