Solution to a parabolic differential equation in Hilbert space via Feynman formula - parts I and II

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A parabolic partial differential equation $u'_t(t, x) = Lu(t, x)$ is considered, where $L$ is a linear second-order differential operator with time-independent coefficients, which may depend on $x$. We assume that the spatial coordinate $x$ belongs to a finite- or infinite-dimensional real separable Hilbert space $H$.

Assuming the existence of a strongly continuous resolving semigroup for this equation, we construct a representation of this semigroup by a Feynman formula, i.e. we write it in the form of the limit of a multiple integral over $H$ as the multiplicity of the integral tends to infinity. This representation gives a unique solution to the Cauchy problem in the uniform closure of the set of smooth cylindrical functions on $H$. Moreover, this solution depends continuously on the initial condition. In the case where the coefficient of the first-derivative term in $L$ vanishes we prove that the strongly continuous resolving semigroup exists (this implies the existence of the unique solution to the Cauchy problem in the class mentioned above) and that the solution to the Cauchy problem depends continuously on the coefficients of the equation.

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1 Introduction

Representation of a function by the limit of a multiple integral as multiplicity tends to infinity is called a Feynman formula, after the inventor of the equations of such type, R.P. Feynman, who was the first to use them (on the physical level of rigor) for the solution of the Cauchy problem for PDEs [24, 25]. The term ”Feynman formula” in this sense was introduced in 2002 by O.G. Smolyanov [30]. One can find out more about the Feynman formulas’ research up to 2009 in [31]. It is important to note that Feynman formulas are closely related to Feynman-Kac formulas [29], however the latter will not be studied in the present article. Usage of Feynman and Feynman-Kac formulas includes exact or numerical evaluation of integrals over Gaussian measures on spaces of high or infinite dimension; some useful approaches to this topic are developed in [6, 8].
Differential equations for functions of an infinite-dimensional argument arise in (quantum) field theory and string theory, theory of stochastic processes and financial mathematics. Evolutionary equations (i.e. PDEs in the form $u'_t(t,x) = \ldots$) in infinite-dimensional spaces have been studied since 1960s by O.G. Smolyanov, E.T. Shavgulidze, E. Nelson, A.Yu. Khrennikov, S. Albeverio, L.C.L. Botelho and others. We will mention just some of the publications, which are most recent and relevant for our study.

In [3] the Schrödinger equation in Hilbert space is studied. The equation includes the terms of second, first and zero order, the coefficient of the second order term is constant. The solution to the Cauchy problem is given by a Feynman-Kac-Ito formula.

In [28] a solution to a heat equation in Hilbert space without the terms of the first and zero order is discussed, the coefficient of the second-derivative term is constant. The solution is given in the form of a convolution with the Gaussian measure (analogous to the finite dimensional equation with constant coefficients), the existence of the resolving semigroup is proved. In [14] the solution to the same equation is given by a Feynman-Kac formula.

In [17] the parabolic equation in finite-dimensional space is studied for the case of variable coefficients. Under the assumption that a strongly continuous resolving semigroup exists for the Cauchy problem, Feynman and Feynman-Kac formulas were proven in [15] for the solution.

In [39], for a class of equations in an infinite-dimensional space, with a variable coefficient at the highest derivative (but without first- and zero-order derivatives’ terms), a Feynman formula was obtained and the existence of resolving semigroup was proven.

In spaces over the field of p-adic numbers, Feynman and Feynman-Kac formulas for the solutions of the Cauchy problem for evolutionary equations were given in [11, 12].

In [4, 16] Feynman formulas for perturbed semigroups are obtained.

The present article extends my first results in this area [28] to the case of non-zero coefficients at the first- and zero-order derivatives.

2 Notation and definitions

The symbol $H$ stands for the real separable Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$.

The self-adjoint, positive, non-degenerate (hence injective), linear operator $A: H \to H$ is assumed to be defined everywhere on $H$. The operator $A$ is assumed to be of trace class, which means that for every orthonormal basis $(e_k)$ in $H$ the sum $\sum_{k=1}^{\infty} \langle Ae_k, e_k \rangle = \text{tr} A$ is finite; this sum is called the trace of $A$ (it is independent of the choice of the basis $(e_k)$).

The symbol $\mathcal{X}$ below stands for any complex Banach space. The symbol $L_k(\mathcal{X}, \mathcal{X})$ stands for space of all linear bounded operators in $\mathcal{X}$, endowed with the classical operator norm.

Symbol $C(M, N)$ will mean the set of all continuous functions from $M$ to $N$, where $M$ and $N$ are topological spaces.

A function $f: H \to \mathbb{R}$ is called cylindrical [5, 10], if there exist vectors $e_1, \ldots, e_n$ from $H$ and function $f^n: \mathbb{R}^n \to \mathbb{R}$ such that for every $x \in H$ the equality $f(x) = f^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle)$ holds. In other words, the function $f: H \to \mathbb{R}$ is cylindrical if there exists a $n$-dimensional subspace $H_n \subset H$ and orthogonal projector $P: H \to H_n$ such that $f(x) = f(Px)$ for every $x \in H$. The cylindrical function $f$ can be imagined as a function, which is first defined on $H_n$ and then continued to the entire space $H$ in such a way that $f(x) = f(x_0)$ if $x_0 \in H_n$ and $x \in (x_0 + \ker P)$.
Symbol $D = C^\infty_{bc}(H, \mathbb{R})$ stands for the space of all continuous bounded cylindrical functions $H \to \mathbb{R}$ such that they have Fréchet derivatives \cite{17} of all positive integer orders at every point of $H$, and their Fréchet derivatives of any positive integer order are bounded and continuous.

If $f: H \to \mathbb{R}$ is twice Fréchet differentiable, then $f'(x)$ will stand for the first Fréchet derivative of $f$ at the point $x$, and $f''(x)$ will denote the second derivative. Riesz-Fréchet representation theorem allows us to assume $f'(x) \in H$ and $f''(x) \in L_0(H, H)$ for every $x \in H$.

Symbol $C_0(H, \mathbb{R})$ stands for the Banach space of all bounded continuous functions $H \to \mathbb{R}$, endowed with a uniform norm $\|f\| = \sup_{x \in H} |f(x)|$. It is regarded as a closed subspace of a complex Banach space $C_0(H, \mathbb{C})$.

Let $X = C^\infty_{bc}(H, \mathbb{R})$ be the closure of the space $D$ in $C_0(H, \mathbb{R})$. It is clear, that $X$ with the norm $\|f\| = \sup_{x \in H} |f(x)|$ is a Banach space, as it is a closed linear subspace of the Banach space $C_0(H, \mathbb{R})$. Function $f$ belongs to $X$ if and only if there is a sequence of functions $(f_j) \subset D$ such that $\lim_{j \to \infty} f_j = f$, i.e. $\lim_{j \to \infty} \sup_{x \in H} |f(x) - f_j(x)| = 0$.

Symbol $C_0(H, H)$ stands for a Banach space of all bounded continuous functions $B: H \to H$, endowed with the uniform norm $\|B\| = \sup_{x \in H} \|B(x)\|$.

Denote $D_H = \{B : H \to H \exists N \in \mathbb{N}, b_k \in H, B_k \in D : B(x) = B_1(x)b_1 + \cdots + B_N(x)b_N\}$.

Let $X_H$ be the closure of $D_H$ in $C_0(H, H)$.

If $x \in H$, and $R: H \to H$ is linear, trace class, positive, non-degenerate operator, then symbol $\mu^R_\varphi$ stands for the Gaussian probabilistic measure \cite{11, 15, 32} on $H$ with expectation $x$ and correlation operator $R$, i.e. the unique sigma-additive measure on Borel sigma-algebra in $H$ such that the equality $\int_H e^{i \langle z, x \rangle} \mu^R_\varphi(dy) = \exp \left( i \langle z, x \rangle - \frac{1}{2} \langle Rz, z \rangle \right)$ holds for every $z \in H$. To make it shorter, we will write $\mu_R$ instead of $\mu^0_\varphi$.

If $B: H \to H$ is a vector field, and $g: H \to \mathbb{R}$ and $C: H \to \mathbb{R}$ are real-valued functions, then symbol $L$ defines a differential operator on the space of functions $\varphi: H \to \mathbb{R}$

\[(L\varphi)(x) := g(x) \text{tr} A \varphi''(x) + \langle \varphi'(x), AB(x) \rangle + C(x) \varphi(x), \quad x \in H.\]

The pair $(\mathcal{L}, M)$ defines a linear operator $\mathcal{L}$ with the domain $M$. It will be shown in theorem \ref{4.2} that $L(D) \subset X$ when $A$, $B$, $g$ and $C$ have certain properties. So $(L, D)$ is a densely defined (on $D$) operator $L: X \supset D \to X$. Here the earlier defined spaces $D$ and $X$ are endowed with the uniform norm, induced from $C_0(H, \mathbb{R})$. Let $(\mathcal{L}, D_1)$ be the closure of $(L, D)$ in $X$. This means that

\[D_1 = \{f \in X \exists (f_j) \subset D : \lim_{j \to \infty} f_j = f, \exists \lim_{j \to \infty} Lf_j\},\]

and, if $f \in D_1$, then, by definition, $\mathcal{L}f = \lim_{j \to \infty} Lf_j$.

If for every fixed first argument $t > 0$ of the function $u: [0, +\infty) \times H \to \mathbb{R}$ we have $[x \mapsto u(t, x)] \in D_1$, then the expression $\mathcal{L}u(t, x)$ means the result of applying the operator $\mathcal{L}$ to the function $x \mapsto u(t, x)$ with the fixed $t > 0$.

Expression $(S_t)_{t \geq 0}$ defines the one-parameter family of linear operators in the space of functions $\varphi: H \to \mathbb{R}$

\[(S_t\varphi)(x) := e^{tc(x)} \int_H \varphi(x + y) e^{\langle \overline{y}, B(x,y) \rangle} \mu_{2t\varphi(x)A}(dy) \text{ for } t > 0, \text{ and } S_0\varphi := \varphi.\]

**Remark 2.1.** Further, in theorem \ref{4.1} we will prove that for every $t \geq 0$ and for $A$, $B$, $g$ and $C$ having certain properties the following holds i) $S_t(X) \subset X$, ii) operator $S_t$ is bounded, and iii) $\frac{d}{dt} S_t\varphi \big|_{t=0} = L\varphi$ for all $\varphi \in D$. This will allow us to use the Chernoff approximation (theorems \ref{3.1} \ref{3.2}) and prove the main result of the present article, theorem \ref{4.4}. 

3
3 Helpful facts and techniques

3.1 Integration in Hilbert space

Lemma 3.1. ([5], Chapter II, §2, 3°) If a function \( \varphi: H \to \mathbb{R} \) is cylindrical and measurable, i.e. \( \varphi(x) = \varphi^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle) \) for some \( n \in \mathbb{N} \), some measurable function \( \varphi^n: \mathbb{R}^n \to \mathbb{R} \), and some finite orthonormal family of vectors \( e_1, \ldots, e_n \) from space \( H \), then

\[
\int_H \varphi(y) \mu_A(dy) = \left( \frac{1}{\sqrt{2\pi}} \right)^n \frac{1}{\sqrt{\det M_Q}} \int_{\mathbb{R}^n} \varphi^n(z) \exp \left( -\frac{1}{2} \langle M_Q^{-1} z, z \rangle \right) dz,
\]

where \( H_n = \text{span}(e_1, \ldots, e_n) \), and \( P: H \ni h \mapsto \langle h, e_1 \rangle e_1 + \cdots + \langle h, e_n \rangle e_n \in H_n \), \( Q = PA \), \( Q: H_n \to H_n \), and \( M_Q \) is the matrix of the operator \( Q \) in basis \( e_1, \ldots, e_n \) of the space \( H_n \). If \( e_1, \ldots, e_n \) is a full set of eigenvectors of the operator \( Q \), and \( q_1, \ldots, q_n \) is the corresponding set of eigenvalues, then

\[
\int_H \varphi(y) \mu_A(dy) = \left( \frac{1}{\sqrt{2\pi}} \right)^n \frac{1}{\sqrt{\prod_{i=1}^n q_i}} \int_{\mathbb{R}^n} \varphi^n(z_1, \ldots, z_n) \exp \left( -\sum_{i=1}^n \frac{z_i^2}{2q_i} \right) dz_1 \ldots dz_n.
\]

Lemma 3.2. (Explicit form of some integrals over Gaussian measure)

Let \( H \) be a real separable Hilbert space of finite or infinite dimension, \( \tilde{A}: H \to H \) be a linear, trace class, symmetric, positive, non-degenerate operator, \( \mu_{\tilde{A}} \) be the centered Gaussian measure on \( H \) with the correlation operator \( \tilde{A} \), and \( G: H \to H \) be a bounded linear operator. Let \( w \) and \( z \) be non-zero vectors from \( H \).

Then the following equalities hold:

\[
\int_H \langle Gy, y \rangle \mu_{\tilde{A}}(dy) = \text{tr}(\tilde{A}G),
\]

\[
\int_H e^{\langle z, y \rangle} \mu_{\tilde{A}}(dy) = e^{\frac{1}{2} \langle \tilde{A}z, z \rangle},
\]

\[
\int_H \langle w, y \rangle e^{\langle z, y \rangle} \mu_{\tilde{A}}(dy) = \langle \tilde{A}w, z \rangle e^{\frac{1}{2} \langle \tilde{A}z, z \rangle},
\]

\[
\int_H \langle G, y \rangle e^{\langle z, y \rangle} \mu_{\tilde{A}}(dy) = (\text{tr} \tilde{A}G + \langle \tilde{A}z, \tilde{A}z \rangle) e^{\frac{1}{2} \langle \tilde{A}z, z \rangle}.
\]

**Proof.** Formulas (3) and (4) can be found in [5], chapter II, §2, 1°. Formula (5) can be derived from the fact that the function under the integral is cylindrical, so lemma 3.1 can be employed. For a proof of (6), one can make the change of variable in the integral, \( h = y - Aw \), then ([5], chapter II, §4, 2°, theorem 4.2) we have \( \mu_{\tilde{A}}(dh) = e^{-\frac{1}{2}(\tilde{A}w,w)}(h,w) \mu_{\tilde{A}}(dh) \), and the integral reduces to (3).

Lemma 3.3. (On a linear change of variable in the integral over Gaussian measure) Let \( H \) be a real separable Hilbert space. Suppose a linear operator \( A: H \to H \) is positive, non-degenerate, trace class, and self-adjoint. We will identify with the symbol \( \mu_A \) the centered Gaussian measure on \( H \) with the correlational operator \( A \). Let \( t > 0 \); the symbol \( tA \) denotes operator, that takes \( x \in H \) to \( tAx \in H \). Let \( f: H \to \mathbb{R} \) be a continuous integrable function.

Then

\[
\int_H f(x) \mu_A(dx) = \int_H f(\sqrt{tx}) \mu_A(dx).
\]

**Proof** uses the uniqueness of the Gaussian measure with a given Fourier transform, and the standard theorem of changing variable in the Lebesgue integral.
Lemma 3.4. (On integrability of a polynomial multiplied by an exponent) Let $H, A, \mu_A$ be as above, $P: \mathbb{R} \to \mathbb{R}$ be a polynomial, and $\beta \in \mathbb{R}$.

Then function $H \ni x \mapsto P(\|x\|)e^{\beta\|x\|} \in \mathbb{R}$ is integrable over $\mu_A$.

Proof is easy to construct by relying on Fernique’s theorem [23], which (applied to this case) says that there exists such $\alpha > 0$ that $\int_H e^{\alpha\|y\|^2} \mu_A(dy) < +\infty$.

3.2 Differentiation in Hilbert space

Proposition 3.1. Let $f$ be a cylindrical real-valued function on $H$, i.e. there is a number $n \in \mathbb{N}$ and a function $f^n: \mathbb{R}^n \to \mathbb{R}$ such that for every $x \in H$ the equality $f(x) = f^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle)$ holds. A set of vectors $e_1, \ldots, e_n$ can be considered orthonormal without loss of generality. Let’s complete this set to an orthonormal basis $(e_k)_{k \in \mathbb{N}}$ in $H$.

Then:

1. Function $f$ is differentiable in the direction $h$ if and only if the function $f^n$ is differentiable in the direction $(\langle h, e_1 \rangle, \ldots, \langle h, e_n \rangle) \in \mathbb{R}^n$, and

$$f'(x)h = \left(h, \left(\partial_1 f^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle), \ldots, \partial_n f^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle), 0, 0, 0, \ldots \right)\right),$$

where the symbol $\partial_1 f^n$ defines the partial derivative with respect to the $j$-th argument of the function $f^n$, and $(\alpha_1, \ldots, \alpha_n, 0, 0, 0, \ldots) = \alpha_1 e_1 + \cdots + \alpha_n e_n$. If the function $f$ has a Fréchet derivative at the point $x$, then $f'(x)$ is a vector whose first $n$ coordinates yield the gradient of the function $f^n$, and the other coordinates are zero:

$$f'(x) = \left(\partial_1 f^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle), \ldots, \partial_n f^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle), 0, 0, 0, \ldots \right).$$  \hfill (8)

2. Function $f$ has a Fréchet derivative in $H$ if and only if the function $f^n$ has a Fréchet derivative in $\mathbb{R}^n$.

3. Let $A: H \to H$ be a trace-class operator (i.e. let $\text{tr} A < \infty$). Then

$$\text{tr} A f^n(x) = \sum_{s=1}^n \sum_{k=1}^n \langle A e_s, e_k \rangle \left(\partial_k \partial_s f^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle)\right) = \text{tr} \left(A_n(f^n)^n(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle)\right),$$

where $A_n$ is the matrix of the operator $PA$ in the basis $e_1, \ldots, e_n$, where $P$ is the projector to the linear span of the vectors $e_1, \ldots, e_n$.

Proof is a straight-forward application of the derivative’s definition.

Proposition 3.2. For $(n + 1)$-times Fréchet differentiable function $f: H \to \mathbb{R}$ there is [9] a Taylor decomposition

$$f(x + h) = f(x) + \frac{1}{1!} f'(x)h + \frac{1}{2!} f''(x)(h, h) + \cdots + \frac{1}{n!} f^{(n)}(x)(h, \ldots, h) + R_n(x, h),$$

and

$$|R_n(x, h)| \leq \frac{\|h\|^{n+1}}{(n + 1)!} \sup_{z \in [x, x+h]} \left\| f^{(n+1)}(z) \right\|.$$  \hfill (11)

3.3 Differential operator on a finite-dimensional space

Lemma 3.5. ([7], theorems 4.3.1, 4.3.2. and Corollary 4.3.4) Suppose for every $i = 1, \ldots, n$ and $j = 1, \ldots, n$ functions $a^{ij}: \mathbb{R}^n \to \mathbb{R}$, $b^i: \mathbb{R}^n \to \mathbb{R}$, $c: \mathbb{R}^n \to \mathbb{R}$ from $C_b^\infty(\mathbb{R}^n, \mathbb{R})$ are given, where $C_b^\infty(\mathbb{R}^n, \mathbb{R})$ is the class of all bounded real-valued functions on $\mathbb{R}^n$, which have bounded partial derivatives of all orders. Suppose also that $c(x) \leq 0$ for all $x \in \mathbb{R}^n$. 


For \( u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}) \) we define a differential operator \( T \) by the formula

\[
(Tu)(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} u(x) + \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x_i} u(x) + c(x) u(x).
\]

Suppose that there exists a constant \( \kappa > 0 \) such that for every \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \) and all \( x \in \mathbb{R}^n \) the ellipticity condition is fulfilled: \( \sum_{i=1}^{n} \sum_{j=1}^{n} a^{ij}(x) \xi_i \xi_j \geq \kappa \| \xi \|^2 \). Take an arbitrary constant \( \lambda > 0 \) and function \( f \in C_0^\infty(\mathbb{R}^n, \mathbb{R}) \).

Then:

1. There is a unique function \( u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}) \), which is a solution of the equation

\[
(Tu)(x) - \lambda u(x) = f(x).
\]  

(12)

2. For every function \( v \in C_0^\infty(\mathbb{R}^n, \mathbb{R}) \) the following estimate is true

\[
\sup_{x \in \mathbb{R}^n} |(Tv)(x) - \lambda v(x)| \geq \lambda \sup_{x \in \mathbb{R}^n} |v(x)|.
\]  

(13)

Note that equation (12) can have unbounded solutions; this does not contradict the lemma.

### 3.4 Strongly continuous semigroups of operators and evolutionary equations

Let \( \mathcal{X} \) be a complex Banach space.

**Definition 3.1.** By a \( C_0 \)-semigroup, or a strongly continuous one-parameter semigroup \( (T_s)_{s \geq 0} \) of linear bounded operators in \( \mathcal{X} \) we (following [26, 18]) mean the mapping 

\[
T : [0, +\infty) \to L_b(\mathcal{X}, \mathcal{X})
\]

of the non-negative half-line into the space of all bounded linear operators on \( \mathcal{X} \), which satisfies the following conditions:

1. \( \forall \varphi \in \mathcal{X} : T_0 \varphi = \varphi \).
2. \( \forall t \geq 0, \forall s \geq 0 : T_{t+s} = T_t \circ T_s \).
3. \( \forall \varphi \in \mathcal{X} \) function \( s \mapsto T_s \varphi \) is continuous as a mapping \( [0, +\infty) \to \mathcal{X} \).

**Definition 3.2.** By the generator of a strongly continuous one-parameter semigroup \( (T_s)_{s \geq 0} \) of linear bounded operators on \( \mathcal{X} \) we mean a linear operator \( \mathcal{L} : \mathcal{X} \supset \text{Dom}(\mathcal{L}) \to \mathcal{X} \) given by the formula

\[
\mathcal{L} \varphi = \lim_{s \to +0} \frac{T_s \varphi - \varphi}{s}
\]

on its domain

\[
\text{Dom}(\mathcal{L}) = \left\{ \varphi \in \mathcal{X} : \exists \lim_{s \to +0} \frac{T_s \varphi - \varphi}{s} \right\},
\]

where the limit is understood in the strong sense, i.e. it is defined in terms of the norm in the space \( \mathcal{X} \).

The use of the symbol \( \mathcal{L} \) for the generator is related to the fact that the generator is always a closed operator:

**Proposition 3.3.** (Theorem 1.4 in [26], p. 51) The generator of a strongly continuous semigroup is a closed linear operator with a dense domain. The generator defines its semigroup uniquely.
Theorem 3.1. (P. R. Chernoff, 1968; see [33] and theorem 10.7.21 in [2]) Let $\mathcal{X}$ be Banach space, and $L_0(\mathcal{X}, \mathcal{X})$ be the space of all linear bounded operators in $\mathcal{X}$ endowed with the operator norm. Let $\overline{L}: \mathcal{X} \supset Dom(\overline{L}) \to \mathcal{X}$ be a linear dissipative operator $\mathcal{L}: \mathcal{X} \supset Dom(\mathcal{L}) \to \mathcal{X}$ in the Banach space $\mathcal{X}$ with the domain $Dom(\mathcal{L})$ dense in $\mathcal{X}$ is closable. The closure $\overline{L}: \mathcal{X} \supset Dom(\overline{L}) \to \mathcal{X}$ is also a dissipative operator.

The main tool for the construction of Feynman formulas for the solutions of the Cauchy problem is Chernoff’s theorem. For convenience we decompose its conditions into several blocks and give them separate names, as follows.

**Theorem 3.1.**

**Proposition 3.4.** (lemma 1.1 and definition 1.2. in [26], p. 48-49) The set $Dom(\overline{L})$ coincides with the set of those $\varphi \in \mathcal{X}$, for which the mapping $s \mapsto T_s \varphi$ is differentiable with respect to $s$ at every point $s \in [0, +\infty)$.

**Definition 3.3.**

1. The problem of finding a function $U: [0, +\infty) \to \mathcal{X}$ such that

$$
\begin{cases}
\frac{d}{dt} U(t) = \overline{L} U(t); & t \geq 0, \\
U(0) = U_0,
\end{cases}
$$

is called the abstract Cauchy problem, associated with the closed linear operator $\overline{L}: \mathcal{X} \supset Dom(\overline{L}) \to \mathcal{X}$ and a vector $U_0 \in \mathcal{X}$.

2. A function $U: [0, +\infty) \to \mathcal{X}$ is called a solution to abstract Cauchy problem (14) if, for every $t \geq 0$, the function $U$ has a continuous derivative $U': [0, +\infty) \to \mathcal{X}$, $U(t) \in Dom(\overline{L})$, and (14) holds.

3. A continuous function $U: [0, +\infty) \to \mathcal{X}$ is called a mild solution to abstract Cauchy problem (14) if for every $t \geq 0$ we have $\int_0^t U(s)ds \in Dom(\overline{L})$ and $U(t) = \overline{L} \int_0^t U(s)ds + U_0$.

**Proposition 3.5.** (proposition 6.2 in [26], p. 145) If the operator $\overline{L}$ is a generator of a strongly continuous semigroup $(T_t)_{t \geq 0}$, then:

1. For every $U_0 \in Dom(\overline{L})$ there is a unique classical solution to abstract Cauchy problem (14), which is given by the formula $U(t) = T(t)U_0$.

2. For every $U_0 \in \mathcal{X}$ there is a unique mild solution to abstract Cauchy problem (14), which is given by the formula $U(t) = T(t)U_0$.

**Definition 3.4.** Linear operator $\mathcal{L}: \mathcal{X} \supset Dom(\mathcal{L}) \to \mathcal{X}$ in Banach space $\mathcal{X}$ is called dissipative if for every $\lambda > 0$ and every $x \in Dom(\mathcal{L})$ the estimate $\|\mathcal{L}x - \lambda x\| \geq \lambda \|x\|$ holds.

**Proposition 3.6.** (On the closability of a densely defined dissipative operator) (proposition 3.14 in [26]) A linear dissipative operator $\mathcal{L}: \mathcal{X} \supset Dom(\mathcal{L}) \to \mathcal{X}$ in the Banach space $\mathcal{X}$ with the domain $Dom(\mathcal{L})$ dense is closable. The closure $\overline{L}: \mathcal{X} \supset Dom(\overline{L}) \to \mathcal{X}$ is also a dissipative operator.

**Definition 3.5.** In the present article two mappings $F_1$ and $F_2$ are called Chernoff-equivalent if there exists a $C_0$-semigroup $(e^{\overline{L}})_{t \geq 0}$ such that $(F_1(n))_n f \to e^{\overline{L}} f$, $(F_2(n))_n f \to e^{\overline{L}} f$ for every $f \in \mathcal{X}$ as $n \to \infty$, and the limit is uniform with respect to $t$ from every segment $[0, t_0]$ for every fixed $t_0 > 0$. 


Remark 3.1. There are several slightly different definitions of the Chernoff equivalence, see e.g. [34, [36, [35]. We will just use this one not going into details. The only thing we need from this definition is that if $F$ satisfies all the conditions of Chernoff’s theorem, then by Chernoff’s theorem the mapping $F$ is Chernoff-equivalent to the mapping $F_1(t) = e^{t\overline{L}}$, i.e. the limit of $(F(t/n))^n$ as $n$ tends to infinity yields the $C_0$-semigroup $(e^{t\overline{L}})_{t \geq 0}$.

Definition 3.6. Let us follow [37] and call a mapping $F$ Chernoff-tangent to the operator $\mathcal{L}$ if it satisfies the conditions (CT1)-(CT4) of Chernoff’s theorem.

Remark 3.2. With these definitions the Chernoff-equivalence of $F$ to $(e^{t\overline{L}})_{t \geq 0}$ follows from: existence (E) of the $C_0$-semigroup + Chernoff-tangency (CT) + growth of the norm bound (N).

Theorem 3.2. (Chernoff-type theorem, [26], corollary 5.3 from theorem 5.2) Let $\mathcal{X}$ be a Banach space, and $L_b(\mathcal{X}, \mathcal{X})$ be the space of all linear bounded operators on $\mathcal{X}$ endowed with the operator norm. Suppose there is a function

$$V : [0, +\infty) \to L_b(\mathcal{X}, \mathcal{X}),$$

meeting the condition $V_0 = I$, where $I$ is the identity operator. Suppose there are numbers $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|(V_t)^k\| \leq Me^{\omega t}$ for every $t \geq 0$ and every $k \in \mathbb{N}$. Suppose the limit

$$\lim_{t \downarrow 0} \frac{V_t \varphi - \varphi}{t} =: \mathcal{L}\varphi$$

exists for every $\varphi \in \mathcal{D} \subset \mathcal{X}$, where $\mathcal{D}$ is a dense subspace of $\mathcal{X}$. Suppose there is a number $\lambda_0 > \omega$ such that $(\lambda_0 I - \mathcal{L})(\mathcal{D})$ is a dense subspace of $\mathcal{X}$.

Then the closure $\overline{\mathcal{L}}$ of the operator $\mathcal{L}$ is a generator of a strongly continuous semigroup of operators $(T_t)_{t \geq 0}$ given by the formula

$$T_t \varphi = \lim_{n \to \infty} \left( V_{\frac{t}{n}} \right)^n \varphi$$

where the limit exists for every $\varphi \in \mathcal{X}$ and is uniform with respect to $t \in [0, t_0]$ for every $t_0 > 0$. Moreover $(T_t)_{t \geq 0}$ satisfies the estimate $\|T_t\| \leq Me^{\omega t}$ for every $t \geq 0$.

Theorem 3.3. (Approximation of generator implies approximation of semigroup) (theorem 4.9 in [26])

Let $(e^{t\overline{L}_j})_{t \geq 0}$ be a sequence of strongly continuous semigroups of operators in a Banach space $\mathcal{X}$ with the generators $(\overline{L}_j$, Dom$(\overline{L}_j))$, which satisfies, for some fixed constants $M \geq 1, w \in \mathbb{R}$, the condition $\|e^{t\overline{L}_j}\| \leq Me^{wt}$ for all $t \geq 0$ and every $j \in \mathbb{N}$. Suppose there is a closed linear operator $(\mathcal{L}$, Dom$(\mathcal{L}))$ on $\mathcal{X}$ with a dense domain Dom$(\mathcal{L})$, such that $\overline{L}_j x \to Lx$ for every $x \in$ Dom$(\mathcal{L})$. Suppose the image of the operator $(\lambda_0 I - \mathcal{L})$ is dense in $\mathcal{X}$ for some $\lambda_0 > 0$.

Then the semigroups $(e^{t\overline{L}_j})_{t \geq 0}$, $j \in \mathbb{N}$ converge strongly (and uniformly in $t \in [0, t_0]$ for every fixed $t_0 > 0$) to a strongly continuous semigroup $(e^{t\overline{L}})_{t \geq 0}$ with the generator $\overline{L}$. In other words, for every $x \in \mathcal{X}$ there exists $\lim_{j \to \infty} e^{t\overline{L}_j} x = e^{t\overline{L}} x$ uniformly in $t \in [0, t_0]$ for every fixed $t_0 > 0$.

Remark 3.3. Below, the role of $\mathcal{X}$ will be played by space $X$, a closed real subspace of the complex Banach space $C_b(H, \mathbb{C})$. Because all the operators used in this paper below are real, and (as it will be proven further in theorems 1.1 and 1.2) $X$ is invariant with respect to them, the above theorems about $\mathcal{X}$ are applicable to $X$. 

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3.5 Properties of spaces $D$, $X$, $D_1$

**Remark 3.4.** It directly follows from the definitions of these spaces that
i) $D \subset D_1 \subset X \subset C_b(H, \mathbb{R}) \subset C_b(H, \mathbb{C})$;
ii) $D$ and $D_1$ are dense in $X$;
iii) $X$ is a Banach space.

**Proposition 3.7.** If $f \in D$, then $f$ is uniformly continuous.

**Proof.** It follows from the definition of the space $D$ that the function $D \ni f : H \to \mathbb{R}$ is bounded and its Fréchet derivatives of all orders exist and are bounded. In particular, there exists

$$\sup_{x \in H} \|f'(x)\| = M < \infty. \quad (15)$$

One can set $n = 0$ in Taylor’s formula (10) to ensure that for every $x \in H$ and every $y \in H$ there exists a real number $R_1(x, y)$ such that

$$f(x) - f(y) = R_1(x, y), \quad (16)$$

and the estimate holds

$$|R_1(x, y)| \leq \frac{|y - x|^1}{1!} \sup_{z \in [x, y]} \|f'(z)\| \leq M \|x - y\|. \quad (17)$$

Hence, for each $x \in H$ and each $y \in H$ we have

$$|f(x) - f(y)| \leq \|R_1(x, y)\| \leq M \|x - y\|, \quad (18)$$

which implies the uniform continuity of $f$.

**Proposition 3.8.** If $\varphi \in X$, then $\varphi$ is uniformly continuous.

**Proof.** Take any given $\varepsilon > 0$. Let us find $\delta > 0$ such that $\|x - y\| < \delta$ implies $|\varphi(x) - \varphi(y)| < \varepsilon$.

As $\varphi \in X$, there exists a sequence of functions $(f_j) \subset D$ converging to $\varphi$ uniformly. Hence, there exists a number $j_0$ such that (introducing the notation $f_{j_0} = f$) we have

$$\|\varphi - f_{j_0}\| = \|\varphi - f\| = \sup_{x \in H} |\varphi(x) - f(x)| < \frac{\varepsilon}{3}. \quad (19)$$

Moreover, as $f \in D$, proposition 3.7 implies estimate (18) with some $M > 0$.

Let us set $\delta = \frac{\varepsilon}{3M}$ and note that $\|x - y\| < \delta$. Then

$$|\varphi(x) - \varphi(y)| \leq |\varphi(x) - f(x)| + |f(x) - f(y)| + |f(y) - \varphi(y)| \leq \frac{\varepsilon}{3} + M \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} = \varepsilon. \quad (18)$$

**Proposition 3.9.** Suppose that a sequence of functions $(f_j)_{j=1}^\infty \subset X$ converges uniformly to a function $f_0 \in X$. Then the family $(f_j)_{j=0}^\infty$ is equicontinuous.

**Proof.** Suppose $\varepsilon > 0$ is given. Let us find $\delta > 0$ such that $\|x - y\| < \delta$ implies that $|\varphi(x) - \varphi(y)| < \varepsilon$. 

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By proposition 3.8, function $f_j$ is uniformly continuous for each $j = 0, 1, 2, \ldots$. Thus, for each $j = 0, 1, 2, \ldots$ there exists $\delta_j > 0$ such that $\|x - y\| < \delta_j$ implies

$$|f_j(x) - f_j(y)| < \frac{\varepsilon}{3}. \quad (20)$$

As $f_j \to f_0$ uniformly, there exists $j_0$ such that for all $j > j_0$

$$\sup_{x \in H} |f_0(x) - f_j(x)| < \frac{\varepsilon}{3}. \quad (21)$$

Let us set $\delta = \min(\delta_0, \delta_1, \ldots, \delta_{j_0})$. Then for $j > j_0$ we have that $\|x - y\| < \delta$ implies

$$|f_j(x) - f_j(y)| \leq |f_j(x) - f_0(x)| + |f_0(x) - f_0(y)| + |f_0(y) - f_j(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Now, if $0 \leq j \leq j_0$, then $\|x - y\| < \delta$ implies estimate (20), which is even stronger.

Remark 3.5. A number $a \in \mathbb{R}$ is called a limit at infinity of a function $f : H \to \mathbb{R}$ if

$$\lim_{R \to +\infty} \sup_{\|x\| \geq R} |f(x) - a| = 0.$$ 

It is shown in [28] that if $H$ is infinite-dimensional, then a non-constant function that belongs to $X$ cannot have a limit at infinity. For example, the function $x \mapsto -\exp(-\|x\|^2)$ belongs to $C_b(H, \mathbb{R})$ but not to $X$.

Remark 3.6. Suppose that $\alpha_k : \mathbb{R} \to \mathbb{R}$ is a family of infinitely-smooth functions, uniformly bounded with their first and second derivatives:

$$\sup_{p \in \{0, 1, 2\}} \sup_{k \in \mathbb{N}} \sup_{t \in \mathbb{R}} \left| \frac{d^p \alpha_k(t)}{dt^p} \right| \leq M \equiv \text{const.}$$

For example, $\alpha_k(t) = \sin(d_k(t - t_k))$, where $d_k$ and $t_k$ are constants and $0 < d_k \leq 1$. Suppose numerical series $\sum_{k=1}^{\infty} b_k$ converges absolutely. Let $(e_k)_{k=1}^{\infty}$ be an orthonormal basis in $H$.

Then function

$$f(x) = \sum_{k=1}^{\infty} b_k \alpha_k(\langle x, e_k \rangle)$$

belongs to the class $D_1$.

This statement can be easily extended to the case $\alpha_k : \mathbb{R}^{n_k} \to \mathbb{R}$.

Remark 3.7. Space $D$ is not separable (it does not have a countable dense subset). In the case of one-dimensional $H$ it can be shown similar to the standard proof of the nonseparability of $C_b(\mathbb{R}, \mathbb{R})$. If $\dim H > 1$, then $\mathbb{R}^1$ can be embedded into $H$ as a linear span of a non-zero vector $e \in H$. Using this, one can embed the set of cylindrical functions contributing to the non-separability of $D$ in the case of one-dimensional $H$, into the space $D$ in the general case.

Remark 3.8. By Remark 3.7 and the inclusion $D \subset D_1 \subset X$, one can see that $D_1$ and $X$ are not separable too.
4 Main results

4.1 Family $S_t$ provides a semigroup with generator $\overline{L}$

**Theorem 4.1.** (On the properties of family $(S_t)_{t \geq 0}$ and its connection to the operator $L$)

Suppose that $g \in X$, and for every $x \in H$ we have $g(x) \geq g_0 \equiv \text{const} > 0$. Suppose that $B \in X_H$ and $C \in X$. Suppose that $t > 0$, and $\mu_{2tg(x)}$ is the centered Gaussian measure on $H$ with the correlation operator $2tg(x)A$.

For $t \geq 0$ and $\varphi \in C_b(H, \mathbb{R})$ let us define

$$(S_t\varphi)(x) := e^{(C(x) - \frac{1}{2}AB(x), B(x))_{g(x)}} \int_H \varphi(x + y)e^{\left(\frac{1}{\sqrt{\pi t}}B(x), y\right)} \mu_{2tg(x)}(dy) \text{ for } t > 0, \text{ and } S_0\varphi := \varphi. \quad (22)$$

Then:

1. If $t \geq 0$ and $\varphi \in C_b(H, \mathbb{R})$, then $S_t\varphi \in C_b(H, \mathbb{R})$. For every $t \geq 0$ the operator $S_t$: $C_b(H, \mathbb{R}) \to C_b(H, \mathbb{R})$ is linear and bounded: its norm does not exceed $e^{\left(\frac{2|A||B|^2}{g_0} + ||C||\right)t}$.

2. If $g \in D, C \in D, B \in D_H$, then the space $D$ for every $t \geq 0$ is invariant with respect to the operator $S_t$.

3. If $g \in X, C \in X, B \in X_H$, then the space $X$ for every $t \geq 0$ is invariant with respect to the operator $S_t$.

4. For every function $\varphi \in D$, for $g \in X, C \in X, B \in X_H$ there exists (uniformly with respect to $x \in H$) a limit

$$\lim_{t \to 0} \frac{(S_t\varphi)(x) - \varphi(x)}{t} = g(x)\text{tr}A\varphi''(x) + \langle \varphi'(x), AB(x) \rangle + C(x)\varphi(x) = (L\varphi)(x).$$

5. If $\varphi \in X, g \in X, C \in X, B \in X_H$, then the function $[0, +\infty) \ni t \mapsto S_t\varphi \in X$ is continuous, i.e. if $t_0 \geq 0, t_n \geq 0$ and $t_n \to t_0$, then $\sup_{x \in H} |(S_{t_n}\varphi)(x) - (S_{t_0}\varphi)(x)| \to 0$.

**Proof.**

1. Function $\varphi$ is bounded, so integral $(22)$ exists by lemma 3.4. Suppose a number $t > 0$ and a function $\varphi \in C_b(H, \mathbb{R})$ are fixed. Recalling lemma 3.3 one can see that

$$\int_H \varphi(x + y)e^{\left(\frac{1}{\sqrt{\pi t}}B(x), y\right)} \mu_{2tg(x)}(dy) = \int_H \varphi(x + \sqrt{2tg(x)y})e^{\left(\frac{1}{\sqrt{\pi t}}B(x), \sqrt{2tg(x)y}\right)} \mu_{A}(dy).$$

Introducing the notation $\|B\| = \sup_{x \in H} \|B(x)\|$, we obtain the estimate

$$\|S_t\varphi\| = \sup_{x \in H} \left| e^{(C(x) - \frac{1}{2}AB(x), B(x))_{g(x)}} \int_{\mathbb{R}} \varphi(x + \sqrt{2tg(x)y})e^{\left(\frac{1}{\sqrt{\pi t}}B(x), \sqrt{2tg(x)y}\right)} \mu_{A}(dy) \right| \leq \sup_{x \in H} \left| e^{(C(x) - \frac{1}{2}AB(x), B(x))_{g(x)}} \right| \sup_{x \in H} |\varphi(x)| \sup_{x \in H} \int_{\mathbb{R}} e^{\left(\frac{1}{\sqrt{\pi t}}B(x), y\right)} \mu_{A}(dy) \leq e^{\left(\|C\| + \left\|\frac{(AB(x), B(x))_{g(x)}}{g(x)}\right\|\|\varphi\|} \sup_{x \in H} e^{\frac{1}{2}\left(\frac{2}{\sqrt{\pi t}}\right)^2 \left\|\frac{(AB(x), B(x))_{g(x)}}{g(x)}\right\|^2} \left\|\varphi\right\| \leq e^{\left(\frac{2|A||B|^2}{g_0} + ||C||\right)t}\|\varphi\|, \quad (23)$$

which implies that the function $x \mapsto (S_t\varphi)(x)$ is bounded. Let us prove that this function is continuous. Suppose $x_j \to x$, then for every $y \in H$

$$\varphi\left(x_j + \sqrt{2tg(x_j)y}\right)e^{\left(\frac{1}{\sqrt{\pi t}}B(x_j), y\right)} \to \varphi\left(x + \sqrt{2tg(x)y}\right)e^{\left(\frac{1}{\sqrt{\pi t}}B(x), y\right)}.$$
Moreover, \( \phi \left( x_j + \sqrt{2t}g(x_j) y \right) e^{\left\langle \frac{\partial L}{\partial x_j} B(x_j), y \right\rangle} \leq \| \phi \| e^{\sqrt{\frac{2t}{g_0}} \| B \| \| y \|} \) and, in a similar way,
\[
\phi \left( x_j + \sqrt{2t}g(x) y \right) e^{\left\langle \frac{\partial L}{\partial x_j} \cdot B(x), y \right\rangle} \leq \| \phi \| e^{\sqrt{\frac{2t}{g_0}} \| B \| \| y \|}.
\]

Lemma 3.4 implies that the function \( y \mapsto e^{\sqrt{\frac{2t}{g_0}} \| B \| \| y \|} \) is integrable over the measure \( \mu_A \). Therefore by Lebesgue dominated convergence theorem
\[
\lim_{j \to \infty} \int_H \phi \left( x_j + \sqrt{2t}g(x_j) y \right) e^{\left\langle \frac{\partial L}{\partial x_j} B(x_j), y \right\rangle} \mu_A = \int_H \phi \left( x + \sqrt{2t}g(x) y \right) e^{\left\langle \frac{\partial L}{\partial x_j} \cdot B(x), y \right\rangle} \mu_A.
\]

Because the functions \( C, B, g \) are continuous and \( g(x) \geq g_0 > 0 \), we have \( e^{tC(\cdot, g(\cdot)B(\cdot))} \to e^{tC(x) - \frac{1}{g_0}(AB(x), B(x))} \). Therefore, \( (S_t \phi)(x_j) \to (S_t \phi)(x) \). So we have proved that \( S_t \phi \in C^\infty(H, \mathbb{R}) \).

Estimate (25) shows that \( \| S_t \| \leq e^{\sqrt{\frac{2t}{g_0}} \| B \|^2 + \| C \|} \).

2. We fix \( t > 0 \) and prove that \( S_t \phi \in D \).

i) First of all, if \( g \in D, C \in D, B \in D_H \), then the operator \( S_t \) maps the cylindrical function \( \phi \) into a cylindrical function \( S_t \phi \). This follows from the fact that (22) is a cylindrical function of cylindrical functions, which are functions of finite number of linear functionals of \( x \). Therefore, the number \( (S_t \phi)(x) \) depends on \( x \) only via a finite number of linear functionals, hence \( x \mapsto (S_t \phi)(x) \) is a cylindrical function, see (25) for the exact formula.

ii) Let us introduce some notation. As \( \phi \) is cylindrical, then for every \( x \in H \) the equality \( \phi(x) = \phi^n(x, e_1, \ldots, x, e_n) \) holds for some \( n \in \mathbb{N} \), some function \( \phi^n: \mathbb{R}^n \to \mathbb{R} \) and some set of vectors \( e_1, \ldots, e_n \) of the space \( H \). Functions \( g, C, B \) are also cylindrical, and without loss of generality we can accept that the set of vectors \( e_1, \ldots, e_n \) is so large that the following holds:
\( g(x) = g^n((x, e_1), \ldots, (x, e_n)), C(x) = C^n((x, e_1), \ldots, (x, e_n)), B(x) = B^n((x, e_1), \ldots, (x, e_n)) e_1 + \cdots + B_n((x, e_1), \ldots, (x, e_n)) e_n. \)

At this moment vectors \( e_1, \ldots, e_n \) can be in arbitrary position with respect to the eigenvectors of the operator \( A \). Without loss of generality the set \( e_1, \ldots, e_n \) can be considered orthonormalized.

Let us introduce the definitions: \( \Psi_n: H \ni x \mapsto ((h, e_1), \ldots, (h, e_n)) \in \mathbb{R}^n \) — a projector, \( H_n = \text{span}(e_1, \ldots, e_n) \) — a subspace in \( H \), \( I_n: H_n \ni (h, e_1) \mapsto (h, e_1, \ldots, (h, e_n)) \in \mathbb{R}^n \) — an isomorphism, \( P_n: H \ni h \mapsto (h, e_1)e_1 + \cdots + (h, e_n)e_n = h \) — a projector. Next, denote \( x_1 = (x, e_1, \ldots, x_n) \in \mathbb{R}^n \) and \( B_1^{-1} (x_1^n) = (B_1(x_1), \ldots, B_n(x_1, \ldots, x_n)) \in \mathbb{R}^n \). With these definitions we have \( \Psi_n = I_n P_n \) and \( \phi(x) = \phi^n(\Psi_n x), g(x) = g^n(\Psi_n x), C(x) = C^n(\Psi_n x), B(x) = B_1(\Psi_n x). \)

Let us introduce the function \( \Phi: \mathbb{R}^n \to \mathbb{R} \) in the following way:
\[
\Phi(x_1^n) = \int_H \phi^n(x_1^n + \sqrt{2t}g^n(x_1^n) \Psi_n(y)) e^{\sqrt{\frac{2t}{g_0}}(B_1^{-1}(x_1^n) \Psi_n(y))} \mu_A(dy). \tag{24}
\]

Then for every \( x \in H \) we have
\[
(S_t \phi)(x) = \Phi(\Psi_n x) \exp \left( tC^n(\Psi_n x) - \frac{\left\langle A^{-1} B_1^{-1}(\Psi_n x), B_1^{-1}(\Psi_n x) \right\rangle}{g^n(\Psi_n x)} \right). \tag{25}
\]

iii) Now let us prove that \( S_t \phi \) has bounded Fréchet derivatives of all orders employing the proposition 3.4. To do this we need to prove that the functions \( \mathbb{R}^n \to \mathbb{R} \) have bounded...
Fréchet derivatives of all orders. The exponent in (25) has this property because the exponent is obtained by composition and arithmetical operations from the functions with this property.

Let us show that $\Phi$ has Fréchet derivatives of all orders. The product of differentiable functions under the sign of integral in (24) is differentiable, so the problem is reduced to the verification of the differentiability of the integral. To do this, we apply lemma 3.1 and arrive from an integral over $H$ to an integral over $\mathbb{R}^n$ in the expression for $\Phi$ (this is possible because the integrand is cylindrical).

Operator $A$ is non-degenerate and symmetric on $H$, therefore the operator $P_n A$ is non-degenerate and symmetric on $H_n$, and therefore it can be diagonalized in some orthonormal basis $b_1, \ldots, b_n$. Without loss of generality we can assume that the vectors $e_1, \ldots, e_n$ form such a basis. Indeed, changing the basis in the space $H_n$ will just produce linear non-degenerate change of variables in the functions $\mathbb{R}^n \to \mathbb{R}$ used to define cylindrical functions $H \to \mathbb{R}$. This will give us new functions $\mathbb{R}^n \to \mathbb{R}$, but all their properties that we need will be preserved.

The matrix of the operator $P_n A$ in $H_n$ coincides with the matrix of the operator $Q_n = I_n P_n A I_n^{-1}$ in $\mathbb{R}^n$. Next, let $q_1, \ldots, q_n$ be the eigenvalues of the operator $Q_n$, corresponding to the eigenvectors $\Psi_n e_1, \ldots, \Psi_n e_n$. Note, that $q_i > 0$ and $g^n \left( \frac{x^2}{1} \right) \geq g_0 \equiv \text{const} > 0$ for every $x^2 \in \mathbb{R}^n$. Then by (2) we have

$$
\Phi \left( \frac{x^2}{1} \right) = \left( \frac{1}{\sqrt{2\pi}} \right)^n \frac{1}{\sqrt{\prod_{i=1}^n q_i}} \int_{\mathbb{R}^n} \phi^n \left( \frac{x^2}{1} + 2 t g^n \left( \frac{x^2}{1} \right) \right) e^{\sqrt{\phi^n (x^2)}} e^{\phi^n \left( \frac{x^2}{1}, \frac{x^2}{1} \right)} \exp \left( - \sum_{i=1}^n \frac{z_i^2}{2q_i} \right) \nu \left( dz \right).
$$

Now we introduce a measure $\nu$ on $\mathbb{R}^n$ given by its density with respect to the Lebesgue measure: for every measurable set $A \subset \mathbb{R}^n$ we set

$$
\nu(A) := \left( \frac{1}{\sqrt{2\pi}} \right)^n \frac{1}{\sqrt{\prod_{i=1}^n q_i}} \int_{A} \exp \left( - \sum_{i=1}^n \frac{z_i^2}{2q_i} \right) \nu \left( dz \right).
$$

It follows from the definitions given above that

$$
\Phi \left( \frac{x^2}{1} \right) = \int_{\mathbb{R}^n} \phi^n \left( \frac{x^2}{1} + 2 t g^n \left( \frac{x^2}{1} \right) \right) e^{\sqrt{\phi^n (x^2)}} e^{\phi^n \left( \frac{x^2}{1}, \frac{x^2}{1} \right)} \nu \left( dz \right). \tag{26}
$$

The integrand in (26) is a composition of mappings with the continuous bounded Fréchet derivative. Thus, it has a continuous bounded Fréchet derivative. The Fréchet derivative of the integrand is uniformly bounded (the estimate is obtained from the chain rule formula), and $(\mathbb{R}^n, \nu)$ is locally compact, countable at infinity, linear normed space with the non-negative Radon measure. Therefore we can apply theorem 115 from [13] on the Fréchet differentiation under the Lebesgue integral. Repeating this reasoning for every $k \in \mathbb{N}$, we conclude that as the integrand has, everywhere in $\mathbb{R}^n$, continuous Fréchet derivatives of $k$-th order, then the function $\Phi$ has, everywhere in $\mathbb{R}^n$, continuous bounded Fréchet’s derivatives of $k$-th order. So the functions $\mathbb{R}^n \to \mathbb{R}$ in the right-hand side of (25) all have continuous bounded Fréchet’s derivatives of $k$-th order.

Therefore, according to point 2 of proposition 3.1 the function $x \mapsto (S_k \varphi)(x)$ also has, for every $k \in \mathbb{N}$, Fréchet derivatives of order $k$, continuous and bounded everywhere on $H$. Therefore $S_k \varphi \in D$.

3. i) Now suppose $\varphi \in X$, which means that $\varphi \in C_0(H, \mathbb{R})$ and there exists a sequence $(\varphi_j) \subset D$ such that $\varphi_j(x) \to \varphi(x)$ uniformly with respect to $x \in H$. Suppose also that $g \in X,$
so \( g \in C_b(H, \mathbb{R}) \) and there exists a sequence \((g_j) \subset D\) such that \(g_j \to g\) uniformly. It follows from \(g(x) \geq g_0 = \text{const} > 0\) for all \(x \in H\) that there exists a number \(j_0 \in \mathbb{N}\) such that for all \(j > j_0\) and for all \(x \in H\) the inequality \(g_j(x) \geq \frac{g_0}{2}\) holds. Therefore, we will not restrict the generality when assuming that for the sequence \((g_j)\) the inequality \(g_j(x) \geq \frac{g_0}{2}\) already holds for all \(j \in \mathbb{N}\) and for all \(x \in H\).

Also suppose \(C \subset X\), so \(C \subset C_b(H, \mathbb{R})\) and there exists a sequence \((C_j) \subset D\) such that \(C_j \to C\) uniformly. Finally, suppose \(B \in X_H\), so \(B \in C_b(H, H)\) and there exists a sequence \((B_j) \subset D_H\) such that \(B_j \to B\) uniformly. Let \(t > 0\) be fixed as before.

Let us denote the operator \(S_t\) constructed with functions \(g_j, B_j, C_j\) by \((S_j)_t\). According to (just proven above) item 2 of the theorem, \((S_j)_t \varphi_j \in D\) for all \(j \in \mathbb{N}\). Next, in ii) and iii) we will prove that \(((S_j)_t \varphi_j)(x) \to (S_t \varphi)(x)\) uniformly with respect to \(x \in H\); by the definition of the space \(X\) this will mean that \(S_t \varphi \in X\).

ii) First of all let us prove that for every fixed \(y \in H\) the sequence of functions \(x \mapsto \varphi_j(x + \sqrt{2t g_j(x)} y)\) converges to the function \(x \mapsto \varphi(x + \sqrt{2t g(x)} y)\) uniformly with respect to \(x \in H\). Indeed, suppose \(\varepsilon > 0\) is given. Let us find \(j^* \in \mathbb{N}\) such that for all \(j > j^*\) the following estimate holds

\[
\sup_{x \in H} \left| \varphi_j \left( x + \sqrt{2t g_j(x)} y \right) - \varphi \left( x + \sqrt{2t g(x)} y \right) \right| \leq \varepsilon. \tag{27}
\]

Notice that \(\varphi, \varphi_j, g, g_j\) are elements of \(X\) by the hypothesis made above, and all the functions in \(X\) are uniformly continuous, according to proposition \[3.8\]. According to proposition \[3.9\] the family of functions \(\{\varphi_j : j \in \mathbb{N}\}\) is equicontinuous. Thus, there exists \(\delta > 0\) such that, for all \(j \in \mathbb{N}\), if \(||x_1 - x_2|| < \delta\), then

\[
|\varphi_j(x_1) - \varphi_j(x_2)| < \frac{\varepsilon}{2}. \tag{28}
\]

Function \([0, +\infty) \ni a \mapsto \sqrt{2at} \in \mathbb{R}\) is uniformly continuous, so the uniform (with respect to \(x \in H\)) convergence \(g_j(x) \to g(x)\) implies the uniform (with respect to \(x \in H\)) convergence \([H \ni x \mapsto \sqrt{2t g_j(x)} \in \mathbb{R}] \to [H \ni x \mapsto \sqrt{2t g(x)} \in \mathbb{R}]\). This shows that for every fixed \(y \in H\) there exists \(j_1 \in \mathbb{N}\) such that for all \(j > j_1\) and all \(x \in H\) we have

\[
\left\| \left( x + \sqrt{2t g_j(x)} y \right) - \left( x + \sqrt{2t g(x)} y \right) \right\| < \delta. \tag{29}
\]

Besides, because \(\varphi_j(z) \to \varphi(z)\) uniformly with respect to \(z \in H\), there exists a number \(j_2\) such that for all \(j > j_2\) and all \(z \in H\) we have

\[
|\varphi_j(z) - \varphi(z)| < \frac{\varepsilon}{2}. \tag{30}
\]

For each fixed \(y \in H\), for all \(j \in \mathbb{N}\) and for all \(x \in H\), we have

\[
\left| \varphi_j \left( x + \sqrt{2t g_j(x)} y \right) - \varphi \left( x + \sqrt{2t g(x)} y \right) \right| \leq \left| \varphi_j \left( x + \sqrt{2t g_j(x)} y \right) - \varphi_j \left( x + \sqrt{2t g(x)} y \right) \right| + \left| \varphi_j \left( x + \sqrt{2t g(x)} y \right) - \varphi \left( x + \sqrt{2t g(x)} y \right) \right|.
\]

Now, we define \(j^* = \max(j_1, j_2)\). For all \(j > j^*\) the first summand is less than \(\frac{\varepsilon}{2}\) due to \[28\] and \[29\]; the easiest way to see that is to set \(x_1 = x + \sqrt{2t g_j(x)} y\) and \(x_2 = x + \sqrt{2t g(x)} y\). As for the second summand, it is less than \(\frac{\varepsilon}{2}\) on account of \[30\]; the easiest way to see that is to set \(z = x + \sqrt{2t g(x)} y\).

So, for each fixed \(y \in H\) we have found a number \(j^* \in \mathbb{N}\) such that for all \(j > j^*\) and for all \(x \in H\) the following inequality holds:

\[
\left| \varphi_j \left( x + \sqrt{2t g_j(x)} y \right) - \varphi \left( x + \sqrt{2t g(x)} y \right) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
By taking \( \sup_{x \in H} \) we obtain the needed estimate \([27]\), as the right hand side of the inequality above does not depend on \( x \).

iii) A reasoning similar to ii) shows that for fixed \( y \) we have \( e^{\left\langle \frac{2t}{\sqrt{g_j(x)}} B_j(x), y \right\rangle} \to e^{\left\langle \frac{2t}{\sqrt{g(x)}} B(x), y \right\rangle} \) uniformly with respect to \( x \in H \). Let us omit the detailed proof. Uniting this with the results of ii) and keeping in mind that for fixed \( y \) all the sequences of functions are bounded collectively, we obtain that for each fixed \( y \) we have

\[
\varphi_j \left( x + \sqrt{2tg_j(x)} y \right) e^{\left\langle \frac{2t}{\sqrt{g_j(x)}} B_j(x), y \right\rangle} \to \varphi \left( x + \sqrt{2tg(x)} y \right) e^{\left\langle \frac{2t}{\sqrt{g(x)}} B(x), y \right\rangle}
\]

uniformly with respect to \( x \in H \).

iv) For fixed \( y \), functions in \((31)\) are bounded, therefore the sequence of functions \( Y_j : H \to \mathbb{R} \)

\[
Y_j = \left[ y \mapsto \sup_{x \in H} \left| \varphi_j \left( x + \sqrt{2tg_j(x)} y \right) e^{\left\langle \frac{2t}{\sqrt{g_j(x)}} B_j(x), y \right\rangle} - \varphi \left( x + \sqrt{2tg(x)} y \right) e^{\left\langle \frac{2t}{\sqrt{g(x)}} B(x), y \right\rangle} \right] \right]
\]

is well defined. It follows from iii), that \( Y_j(y) \) converges to zero pointwise, in other words for each \( y \). Functions \( Y_j \) are non-negative and bounded collectively by an integrable (due to Lemma 3.4) function \( y \mapsto \alpha e^{\|y\|} + \gamma \) with appropriate constants \( \alpha, \beta \) and \( \gamma \). The Lebesgue dominated convergence theorem gives us that \( \int_H Y_j(y) \mu_A(dy) \to 0 \). As the number sequence \( \int_H Y_j(y) \mu_A(dy) \) converges, it is bounded. For brevity, let us denote

\[
\Psi(x, y) = \varphi \left( x + \sqrt{2tg(x)} y \right) e^{\left\langle \frac{2t}{\sqrt{g(x)}} B(x), y \right\rangle}, \quad \Psi_j(x, y) = \varphi_j \left( x + \sqrt{2tg_j(x)} y \right) e^{\left\langle \frac{2t}{\sqrt{g_j(x)}} B_j(x), y \right\rangle},
\]

\[
E(x) = \exp \left( tC(x) - t \frac{\langle AB(x), B(x) \rangle}{g(x)} \right), \quad E_j(x) = \exp \left( tC_j(x) - t \frac{\langle AB_j(x), B_j(x) \rangle}{g_j(x)} \right).
\]

A reasoning similar to ii) shows that \( E_j(x) \to E(x) \) uniformly with respect to \( x \in H \). We obtain the estimate

\[
\left\| (S_j)_t \varphi - S_t \varphi \right\| = \sup_{x \in H} \left| E_j(x) \int_H \Psi_j(x, y) \mu_A(dy) - E(x) \int_H \Psi(x, y) \mu_A(dy) \right| \leq
\]

\[
\sup_{x \in H} \left| E_j(x) \int_H \Psi_j(x, y) \mu_A(dy) - E_j(x) \int_H \Psi(x, y) \mu_A(dy) \right| + \sup_{x \in H} \left| E_j(x) \int_H \Psi(x, y) \mu_A(dy) - E(x) \int_H \Psi(x, y) \mu_A(dy) \right| \leq
\]

\[
\sup_{x \in H} \left| E_j(x) \right| \int_H Y_j(y) \mu_A(dy) + \sup_{x \in H} \left| E_j(x) - E(x) \right| \int_H \Psi(x, y) \mu_A(dy).
\]

Finally let us note that \( \sup_{x \in H} \left| E_j(x) - E(x) \right| \to 0 \) and \( \int_H Y_j(y) \mu_A(dy) \to 0 \), and the multipliers of these terms in the above estimate are bounded, which implies \( \left\| (S_j)_t \varphi - S_t \varphi \right\| \to 0 \).

4. Suppose \( \varphi \in D, \ t > 0 \) and \( x \in H \) are fixed. Let us consider the integral

\[
\int_H \varphi(x + y) e^{\left\langle \frac{2t}{\sqrt{g(x)}} B(x), y \right\rangle} \mu_{2tg(x)}(A(dy)).
\]

We will work with the approximation of the function \( \varphi \) by its Taylor polynomial \((10)\) of the second order with the center at the point \( x \). Before we start, it is important to note that the
remainder term $R(x, y)$ will not be small, as the vector $y$ ranges over the whole space $H$, and vector $x$ is fixed. However as $\varphi$ is three times Fréchet differentiable on $H$, for each $x \in H$ and each $y \in H$ there exists a real number $R(x, y)$ such that

$$\int_H \varphi(x + y)e^{\frac{1}{(\pi t)^2}B(x,y)} \mu_{2t g(x)A}(dy) = \int_H \left\{ \varphi(x) + \langle \varphi'(x), y \rangle + \frac{1}{2!} \langle \varphi''(x)y, y \rangle + R(x, y) \right\} e^{\frac{1}{(\pi t)^2}B(x,y)} \mu_{2t g(x)A}(dy),$$

and, according to (31), the estimate

$$|R(x, y)| \leq \frac{||y||^3}{(3)!} \sup_{z \in [x, x-y]} ||\varphi^{(3)}(z)|| \leq \frac{1}{3!} ||\varphi''|| ||y||^3 \quad (32)$$

holds, where we define $||\varphi''|| = \sup_{z \in H} ||\varphi^{(3)}(z)||$. Also, let us denote $||g|| = \sup_{z \in H} |g(z)|$.

The sum can be integrated termwise as every term can be bounded by a polynomial of $||y||$, multiplied by an exponent of $||y||$ and such functions are integrable due to lemma 3.4. Let us calculate integrals in the sum and for fixed $\varphi$ evaluate the decay rate as $t \to 0$ uniformly with respect to $x \in H$.

$$\int_H \varphi(x)e^{\frac{1}{(\pi t)^2}B(x,y)} \mu_{2t g(x)A}(dy) = \varphi(x) \int_H e^{\frac{1}{(\pi t)^2}B(x,y)} \mu_{2t g(x)A}(dy) =$$

(set $z = \frac{B(x)}{g(x)}$ and $\tilde{A} = 2t g(x)A$ in (41))

$$\varphi(x) \exp \left( \frac{1}{2} \left\langle 2t g(x)A \frac{B(x)}{g(x)}, B(x) \right\rangle \frac{1}{g(x)} \right) = \varphi(x) \exp \left( \frac{\langle AB(x), B(x) \rangle}{g(x)} t \right) =$$

$$\varphi(x) \left( 1 + \frac{\langle AB(x), B(x) \rangle}{g(x)} t + o(t) \right) = \varphi(x) + \varphi(x) \frac{\langle AB(x), B(x) \rangle}{g(x)} t + o(t).$$

Next, let $z$ and $\tilde{A}$ be as before, and set $w = \varphi'(x)$ in (3).

$$\int_H \langle \varphi'(x), y \rangle e^{\frac{1}{(\pi t)^2}B(x,y)} \mu_{2t g(x)A}(dy) \leq$$

$$2t \langle A \varphi'(x), B(x) \rangle \exp \left( \frac{t}{g(x)} \left\| \sqrt{AB(x)} \right\|^2 \right) = 2t \langle A \varphi'(x), B(x) \rangle + o(t).$$

For the term with $\varphi''$ we have

$$\int_H \langle \varphi''(x)y, y \rangle e^{\frac{1}{(\pi t)^2}B(x,y)} \mu_{2t g(x)A}(dy) \leq$$

$$\left( 2t g(x) \text{tr} A \varphi''(x) + 4t^2 \langle \varphi''(x)AB(x), AB(x) \rangle \right) \exp \left( \frac{\langle AB(x), B(x) \rangle}{g(x)} t \right) = 2t g(x) \text{tr} A \varphi''(x) + o(t).$$

Finally,

$$\left| \int_H R(x, y)e^{\frac{1}{(\pi t)^2}B(x,y)} \mu_{2t g(x)A}(dy) \right| \leq \int_H \left| R(x, \sqrt{2t g(x)}y) \right| e^{\frac{1}{(\pi t)^2}B(x,\sqrt{2t g(x)}y)} \mu_A(dy) \leq$$
\[
\int_H \frac{1}{3!} \|\phi''\| \left( \sqrt{2t g(x)} \right)^3 \|y\|^3 e^{\left(\sqrt{\frac{\mu}{\phi'}} B(x), y\right)} \mu_A(dy) \leq \\
\frac{\sqrt{2}}{3} \|\phi''\| \|g\|^\frac{3}{2} \int_H \|y\|^3 e^{\left(\sqrt{\frac{\mu}{\phi'}} B(x), y\right)} \mu_A(dy) \leq \\
(t^\frac{1}{2}) \left( \frac{\sqrt{2}}{3} \|\phi''\| \|g\|^\frac{3}{2} \int_H \|y\|^3 e^{\left(\sqrt{\frac{\mu}{\phi'}} B(x), y\right)} \mu_A(dy) \right) \leq \\
\frac{1}{\|B(x)\|} \leq B_0, g_0 \leq g(x) \) and monotonicity of the exponent function.
\]

Due to the inequality \(\langle z, y \rangle \leq \|z\|\|y\|\) and monotonicity of the exponent function

\[
t^\frac{1}{2} \left( \frac{\sqrt{2}}{3} \|\phi''\| \|g\|^\frac{3}{2} \int_H \|y\|^3 e^{\left(\sqrt{\frac{\mu}{\phi'}} B(x), y\right)} \mu_A(dy) \right) = o(t) \text{ uniformly with respect to } x \in H.
\]

Summing everything up, one can see that

\[
\int_H \varphi(x+y)e^{\left(\frac{1}{\phi'} B(x), y\right)} \mu_{2\Omega(x)A}(dy) = \varphi(x)+t\varphi'(x), AB(x)+tg(x)trA\varphi''(x)-t^2\varphi(x)\frac{\langle AB(x), B(x)\rangle}{g(x)}+o(t), \\
\text{uniformly with respect to } x \in H.
\]

Consider the term exp \(tC(x) - \frac{\langle AB(x), B(x)\rangle}{g(x)} t\). As \(\|C\|, \|A\|, \|B\|\) are bounded from infinity and \(g\) is bounded from zero, we have \(e^{tC(x)} - \frac{\langle AB(x), B(x)\rangle}{g(x)} t = 1+tC(x) - \frac{\langle AB(x), B(x)\rangle}{g(x)} t + o(t) \) uniformly with respect to \( x \in H \). Therefore, uniformly with respect to \( x \in H \) for \( t \to 0 \) one obtains

\[
(S_t\varphi)(x) = e^{tC(x)} - \frac{\langle AB(x), B(x)\rangle}{g(x)} t \int_H \varphi(x+y)e^{\left(\frac{1}{\phi'} B(x), y\right)} \mu_{2\Omega(x)A}(dy) = \\
\varphi(x)+t\varphi'(x), AB(x)+tg(x)trA\varphi''(x)+tC(x)\varphi(x)+o(t).
\]

This implies that uniformly with respect to \( x \in H \)

\[
\lim_{t \to 0} \frac{(S_t\varphi)(x) - \varphi(x)}{t} = g(x)trA\varphi''(x)+\langle \varphi'(x), AB(x)\rangle+C(x)\varphi(x) = (L\varphi)(x).
\]

5. i) First let us consider the case \( t_0 \neq 0 \). If \( t_n \to t_0 \), then \( 2t_n g(x) \to 2t_0 g(x) \) uniformly with respect to \( x \in H \). Because the function \( z \to \sqrt{z} \) is uniformly continuous, it follows that \( \sqrt{2t_n g(x)} \to \sqrt{2t_0 g(x)} \) uniformly with respect to \( x \in H \).

According to proposition 3.3, the function \( \varphi \) is uniformly continuous. Therefore, for every fixed \( y \in H \) \( \varphi(x + \sqrt{2t_n g(x)} y) \to \varphi(x + \sqrt{2t_0 g(x)} y) \) uniformly with respect to \( x \in H \).

Next, for every \( y \in H \) the sequence \( \left( \frac{1}{g(x)} B(x), \sqrt{2t_n g(x)} y \right) \) converges to \( \left( \frac{1}{g(x)} B(x), \sqrt{2t_0 g(x)} y \right) \) uniformly with respect to \( x \in H \) because of the linearity of the scalar product. Since the number sequence \( t_n \) converges, it is bounded; besides, \( g(x) \geq g_0 \equiv \text{const} > 0 \) and functions \( x \to g(x) \) and \( x \to \|B(x)\| \) are bounded. Therefore, there exists a constant \( K > 0 \) such that for fixed \( y \in H \) for all \( k = 0, 1, 2, 3, \ldots \) and all \( x \in H \) the inequality \( \left| \left( \frac{1}{g(x)} B(x), \sqrt{2t_k g(x)} y \right) \right| \leq K \) holds. Function \( z \to e^z \) is uniformly continuous with respect to \( z \in [-K, K] \), therefore for every \( y \in H \) we have the convergence \( e^{\left(\frac{1}{\phi'} B(x), \sqrt{2t_n g(x)} y\right)} \to e^{\left(\frac{1}{\phi'} B(x), \sqrt{2t_0 g(x)} y\right)} \), uniformly with respect to \( x \in H \).
The product of two collectively bounded uniformly converging sequences is a sequence, uniformly converging to the product of the limits of these sequences. Therefore, from the two last paragraphs, it follows that for every $y \in H$ we have

$$\varphi(x + \sqrt{2t_n}g(x)y)e^{\left(\frac{2t_n}{\pi^2}B(x)\right)\sqrt{2t_n}g(x)y} \to \varphi(x + \sqrt{2t_0}g(x)y)e^{\left(\frac{2t_0}{\pi^2}B(x)\right)\sqrt{2t_0}g(x)y}$$

(33) uniformly with respect to $x \in H$.

Since for fixed $y$ the functions from (33) are bounded, the sequence of the number functions is well defined

$$Y_n = \left[y \mapsto \sup_{x \in H} \varphi(x + \sqrt{2t_n}g(x)y)e^{\left(\frac{2t_n}{\pi^2}B(x)\right)\sqrt{2t_n}g(x)y} - \varphi(x + \sqrt{2t_0}g(x)y)e^{\left(\frac{2t_0}{\pi^2}B(x)\right)\sqrt{2t_0}g(x)y}\right].$$

According to above results, $Y_n(y)$ converges to zero pointwise, i.e. for every $y$. Functions $Y_n$ are bounded by an integrable function (see lemma 3.4), therefore, by the Lebesgue dominated convergence theorem, we have $\int_H Y_n(y)\mu_A(dy) \to 0$.

Finally, $C \in X, B \in X_H$, and therefore $\sup_{x \in H} |C(x)| < \infty, \sup_{x \in H} \|B(x)\| < \infty$. United with $g(x) \geq g_0$ this implies $e^{t_n C(x) - \frac{(AB)(x)B(x)}{\pi^2}t_n} \to e^{t_0 C(x) - \frac{(AB)(x)B(x)}{\pi^2}t_0}$ uniformly with respect to $x \in H$.

So, the sequence $S_{t_n}\varphi(x)$ is a product of two collectively bounded uniformly converging sequences; thus it converges to the product of the limits of these sequences, i.e. $S_{t_0}\varphi(x)$.

ii) Now let us consider the case $t_0 = 0$. Recall that $S_0 = I$, so we need for each fixed $\varphi$ prove that if $t_n \to 0$ then $S_{t_n}\varphi \to \varphi$, i.e. $\|S_{t_n}\varphi - \varphi\| \to 0$. Without loss of generality we can assume that $0 < t_n \leq 1$.

ia) Let us first consider the case $\varphi \in D$. Then by item 4 of the present theorem $S_{t}\varphi = \varphi + tL\varphi + o(t)$, so $t_n \to 0$ implies $S_{t_n}\varphi \to \varphi$.

ib) Now suppose that $\varphi \in X$, so there exists a sequence $(\varphi_k) \subset D$ such that $\|\varphi_k - \varphi\| \to 0$. Item 1 of the present theorem implies that there exists such a constant $\omega \geq 1$ such that for all $t \in [0, 1]$ we have $\|S_t\| \leq \omega$. Now for arbitrary $\varepsilon > 0$ we apply ”$\varepsilon/3$-method” based on the estimate

$$\|S_{t_n}\varphi - \varphi\| \leq \|S_{t_n}\varphi - S_{t_n}\varphi_k\| + \|S_{t_n}\varphi_k - \varphi_k\| + \|\varphi_k - \varphi\|.$$

We can select such $k$ that $\|\varphi_k - \varphi\| < \varepsilon/(3\omega)$. For fixed $k$ we can thanks to iia) find such a number $n_0$ that for all $n > n_0$ one has $\|S_{t_n}\varphi_k - \varphi_k\| < \varepsilon/3$. So for all $n > n_0$ we see that

$$\|S_{t_n}\varphi - \varphi\| \leq \|S_{t_n}\varphi - \varphi - \varphi_k\| + \|S_{t_n}\varphi_k - \varphi_k\| + \|\varphi_k - \varphi\| < \omega \frac{\varepsilon}{3\omega} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3\omega} \leq \varepsilon.$$

\[\Box\]

**Theorem 4.2. (On the properties of the operator $L$)** Suppose for each $x \in H$ the inequalities $g(x) \geq g_0 \equiv \text{const} > 0$ and $C(x) \leq 0$ hold. As $C \in X$, there exists a sequence $(C_j) \subset D$ converging to $C$ uniformly; let us additionally require that this sequence can be selected in such a way that $C_j(x) \leq 0$ for all $j \in \mathbb{N}$ and all $x \in H$. The operator $L: D \to X$ is defined by the equation

$$(L\varphi)(x) = g(x)\text{tr}A\varphi''(x) + \langle \varphi'(x), AB(x) \rangle + C(x)\varphi(x).$$

Symbol $I$ stands for the identity operator.

Then:

1. If $g \in D, C \in D, B \in D_H$ and $\varphi \in D$, then $L\varphi \in D$. If $g \in X, C \in X, B \in X_H$ and $\varphi \in D$, then $L\varphi \in X$.

2. If $g \in D, C \in D, B \in D_H$, then for each $\lambda > 0$ the operator $\lambda I - L$ is surjective on $D$, therefore $(\lambda I - L)(D) = D$ is a dense subspace in $X$.  

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3. If \( g \in D, C \in D, B \in D_H \), then the operator \((L, D)\) is dissipative and closable.
4. If \( g \in X, C \in X, B = 0 \), then for each \( \lambda > 0 \) the space \((\lambda I - L)(D)\) is dense in \( X \).
5. If \( g \in X, C \in X, B \in X_H \), then the operator \((L, D)\) is dissipative and has the closure \((\overline{L}, D_1)\). This operator is also dissipative.

**Proof.**

1. First part of the statement is obvious: the result of the applying a differential operator with smooth cylindrical coefficients to a smooth cylindrical function is a smooth cylindrical function. As it follows from the chain rule, all the derivatives are bounded. Let the coefficients of the operator \( L \) be uniform limits \( g, B, C \) of cylindrical functions \( g_j, B_j, C_j \). We denote the operator \( L \) that corresponds to the functions \( g_j, B_j, C_j \) as \( L_j \). Then, as \( j \to \infty \) the sequence \( L_j \phi \) converges uniformly to \( L \phi \), therefore \( L \phi \in X \), because \( L \phi \in D \) as we proved above.

2. Suppose \( g \in D, B \in D_H, C \in D \). Recall that all these functions are cylindrical and thus they are closely related to functions on \( \mathbb{R}^n \); this will be the core idea of the proof that follows. Let us fix \( \lambda > 0 \), choose arbitrary function \( \varphi \in D \) and then show, that there exists a function \( f \in D \) satisfying the equation

\[
\lambda f(x) - g(x) \text{tr} Af^n(x) - \langle f'(x), AB(x) \rangle - C(x)f(x) = \varphi(x). \tag{34}
\]

Let the vectors \( e_1, \ldots, e_n \) be such that for every \( x \in H \) we have

\[
C(x) = C^n((x, e_1), \ldots, (x, e_n)), B(x) = \sum_{k=1}^{n} B^n_k((x, e_1), \ldots, (x, e_n))e_k, \tag{35}
\]

\[
\varphi(x) = \varphi^n((x, e_1), \ldots, (x, e_n)) \quad \text{and} \quad g(x) = g^n((x, e_1), \ldots, (x, e_n)), \tag{36}
\]

where \( \varphi^n : \mathbb{R}^n \to \mathbb{R}, g^n : \mathbb{R}^n \to \mathbb{R}, C^n : \mathbb{R}^n \to \mathbb{R} \) and \( B^n_k : \mathbb{R}^n \to \mathbb{R} \) are continuously differentiable functions, bounded along with all the derivatives. Let us find a solution of the equation \( f^n = (x, e_1), \ldots, (x, e_n)) \),

\[
f^n(x, e_1), \ldots, (x, e_n)), \tag{37}
\]

where \( f^n : \mathbb{R}^n \to \mathbb{R} \) is a continuously differentiable function, bounded along with all its derivatives.

According to item 1 of proposition \(33 \),

\[
f(x) = f^n((x, e_1), \ldots, (x, e_n)), \tag{37}
\]

we have that

\[
\langle f'(x), AB(x) \rangle = \left( \sum_{s=1}^{n} \partial_s f^n((x, e_1), \ldots, (x, e_n))e_s, A \sum_{k=1}^{n} B^n_k((x, e_1), \ldots, (x, e_n))e_k \right) = \sum_{s=1}^{n} \left( \sum_{k=1}^{n} B^n_k((x, e_1), \ldots, (x, e_n)) \langle Ae_s, e_k \rangle \right) \partial_s f^n((x, e_1), \ldots, (x, e_n)). \tag{38}
\]

One can see, by taking into account \(9), (36), (37) and (38), that equation \(34 \) for the unknown function \( f \) of form \(37 \) is equivalent to the equation for an unknown function \( f^n \):

\[
g^n(x_1, \ldots, x_n) \sum_{s=1}^{n} \sum_{k=1}^{n} \langle Ae_s, e_k \rangle \partial_s \partial_k f^n(x_1, \ldots, x_n)
\]
of the operator

The quadratic form given by the equation (37) belongs to $\phi$ is elliptic, and (39) has form (12), i.e.

$$L^n f^n(x_1, \ldots, x_n) - \lambda f^n(x_1, \ldots, x_n) = -\varphi^n(x_1, \ldots, x_n),$$

where we denote $x_j = \langle x, e_j \rangle$.

Thus, for every $\lambda > 0$ the operator $\lambda I - L$ is surjective in $D$, because in the preimage of the function $\varphi \in D$ there is at least the function $f(x) = f^n((x, e_1), \ldots, (x, e_n))$, where $f^n: \mathbb{R}^n \to \mathbb{R}$ is the solution of the equation (39).

3. Suppose $g \in D, C \in D, B \in D_H$. Let us prove that the operator $L$ is dissipative. Let $f \in D$ and $\lambda > 0$ be fixed.

As in the proof of item 2 of this theorem, the value of the function $Lf$ at the point $x \in H$ is equal to the value of the function $L^n f^n$ at the point $(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle) \in \mathbb{R}^n$. Again, the operator $A$ is positive, and the function $g$ satisfies the inequality $g(x) \geq g_0 \equiv \text{const} > 0$, so we can apply item 2 of lemma 3.5 to the finite-dimensional operator. This gives us that the finite-dimensional operator is dissipative, which implies that the operator $L$ is dissipative too. This idea can be expressed in a more formal way:

$$\||L^f - \lambda f|| = \sup_{x \in H} |g(x)\text{tr}(Af^n(x)) + \langle f'(x), AB(x) \rangle + C(x)f^n(x) - \lambda f^n(x)| =$$

$$\sup_{(x_1, \ldots, x_n) \in \mathbb{R}^n} \left| g^n(x_1, \ldots, x_n) \sum_{s=1}^{n} \sum_{k=1}^{n} \langle Ae_s, e_k \rangle \partial_k \partial_s f^n(x_1, \ldots, x_n) + \right.$$}

$$\sum_{s=1}^{n} \left( \sum_{k=1}^{n} B^n_k(x_1, \ldots, x_n) \langle Ae_s, e_k \rangle \right) \partial_s f^n(x_1, \ldots, x_n) +$$

$$C^n(x_1, \ldots, x_n)f^n(x_1, \ldots, x_n) - \lambda f^n(x_1, \ldots, x_n) \right| \geq$$

$$\lambda \sup_{(x_1, \ldots, x_n) \in \mathbb{R}^n} |f^n(x_1, \ldots, x_n)| = \lambda \sup_{x \in H} |f(x)| = \lambda \|f\|,$$

where $x_j = \langle x, e_j \rangle$. The inequality

$$\||L^f - \lambda f|| \geq \lambda \|f\|$$

means that the operator $L$ is dissipative.
Finally, according to proposition 3.6, the closability of $L$ follows from the fact $L$ is dissipative and densely defined.

4. i) Suppose $g \in X$ and $C \in X$, i.e. there exists sequences $(g_j) \subset D$ and $(C_j) \subset D$ such that $\|g - g_j\| \to 0$ and $\|C - C_j\| \to 0$. Suppose that $B(x) = 0$ for each $x \in H$. The images of the operators $(\lambda I - L)$ and $(L - \lambda I)$ are equal. As $D$ is dense in $X$, it is enough to show that the image of the operator $(L - \lambda I)$ is dense in $D$, then it will be dense in $X$. Let the number $\lambda > 0$ and function $\psi \in D$ be fixed. We will approximate $f$ by the values of the operator $(L - \lambda I)$.

Since $g_j \to g$ and, for every $x \in H$, the estimate $g(x) \geq g_0$ holds, it follows that there is a number $j'_0$ such that for every $x \in H$ and all $j > j'_0$ we have

$$g_j(x) \geq \frac{g_0}{2}.$$  \hspace{1cm} (41)

We denote the operator $L$ corresponding to the functions $g_j$ and $C_j$ by the symbol $L_j$. Note that $C_j(x) \leq 0$ and $g_j(x) \geq \frac{g_0}{2}$, so we can apply item 2 of this theorem to the operator $L_j$. Item 2 says that image of the operator $(L_j - \lambda I)$ is equal to $D$, so for every $j > j'_0$ there exists a function $f_j \in D$ such that

$$L_j f_j - \lambda f_j = \psi.$$ \hspace{1cm} (42)

The goal is to prove that $L f_j - \lambda f_j \to \psi$ as $j \to \infty$. This will imply that the image of the operator $(\lambda I - L)$ is dense in $D$.

ii) Let us prepare several estimates. First, since $C_j \to C$, there is a number $j_0$ such that for every $j > j_0$

$$\|C_j\| \leq 2\|C\|.$$ \hspace{1cm} (43)

Second, it follows from (42) and (40) that for every $j > j'_0$ we have

$$\lambda \|f_j\| \leq \|L_j f_j - \lambda f_j\| \leq \|\psi\|,$$

i.e. for every $j > j'_0$

$$\|f_j\| \leq \frac{\|\psi\|}{\lambda}.$$ \hspace{1cm} (44)

Finally, expressing the term $\text{tr}(A f_j')$ by the use of the equation

$$\psi = L_j f_j - \lambda f_j = g_j \text{tr}(A f_j') + (C_j - \lambda) f_j,$$

we find that for every $j > \max(j_0, j'_0)$

$$\|\text{tr}(A f_j')\| = \left\| \frac{\psi + (\lambda - C_j) f_j}{g_j} \right\| \leq \frac{\|\psi\| + (\lambda + 2\|C\|)\|\psi\|/\lambda}{g_0/2}.$$ \hspace{1cm} (45)

iii) Now let us prove that if $j \to \infty$, then $L f_j - \lambda f_j \to \psi$. Indeed, for every $j > \max(j_0, j'_0)$ one obtains

$$\|L f_j - \lambda f_j - \psi\| \leq \|L f_j - \lambda f_j - (L_j f_j - \lambda f_j)\| = \|(g - g_j)\text{tr}(A f_j') + (C - C_j) f_j\|$$

$$\leq \|(g - g_j)\|\|\psi\| + (\lambda + 2\|C\|)\|\psi\|/\lambda \rightarrow 0.$$
because \(\|(g - g_j)\| \to 0\) and \(\|(C - C_j)\| \to 0\). Item 4 is proven.

5. Let the coefficients of the operator \(L\) be uniform limits \(g, B, C\) of the continuously differentiable cylindrical functions \(g_j, B_j, C_j\). As \(g_j \to g\) and, for all \(x \in H\), we have \(g(x) \geq g_0\), it follows that there exists a number \(j_0\) such that for all \(x \in H\) and all \(j > j_0\) we have \(g_j(x) \geq \frac{g_0}{2}\).

Also recall that \(C_j(x) \leq 0\). This all allows us to use item 3 of this theorem.

According to (10), for every function \(\varphi \in D\) and every \(\lambda > 0\) we have

\[
\|g_j \text{tr}(A\varphi''') + \langle \varphi', AB_j \rangle + C_j \varphi - \lambda \varphi \| \geq \lambda \|\varphi\|.
\]

Taking the limit as \(j \to \infty\), we obtain the estimate \(\|L \varphi - \lambda \varphi\| \geq \lambda \|\varphi\|\), which means that \(L\) is dissipative. According to proposition 3.6, the dissipative operator \((L, D)\) with the domain \(D\) dense in \(X\) is closable. Let us denote its closure with \((\overline{L}, D_1)\). Note that by proposition 3.6 the closure also is a dissipative operator.

\[ \square \]

The constructions above were built to prove the following result:

**Theorem 4.3.** (On the connection between the family \((S_t)_{t \geq 0}\) and the semigroup with the generator \(\overline{L}\))

Suppose that \(g \in X, B \in X_H, C \in X\), and for every \(x \in H\) we have \(g(x) \geq g_0 \equiv \text{const} > 0\) and \(C(x) \leq 0\). As \(C \in X\), there exists a sequence \((C_j) \subset D\), converging to \(C\) uniformly; let us additionally claim that this sequence can be selected in such a way that \(C_j(x) \leq 0\) for all \(j \in \mathbb{N}\) and all \(x \in H\). Then the following holds:

1. If the closure \((\overline{L}, D_1)\) of the operator \((L, D)\) is a generator of a strongly continuous semigroup \((e^{\overline{L}t})_{t \geq 0}\) of linear continuous operators on the space \(X\), then

\[
e^{\overline{L}t} \varphi = \lim_{n \to \infty} (S_+^t)^n \varphi, \tag{46}
\]

where limit exists for every \(\varphi \in X\) and is uniform with respect to \(t \in [0, t_0]\) for every \(t_0 > 0\).

2. If \(B = 0\), then the operator \((\overline{L}, D_1)\) is a generator of a strongly continuous semigroup \((e^{\overline{L}t})_{t \geq 0}\) of linear continuous operators on the space \(X\). Moreover for every \(t \geq 0\) we have

\[
\|e^{\overline{L}t}\| \leq 1, \text{ i.e. the semigroup } (e^{\overline{L}t})_{t \geq 0} \text{ is contractive.}
\]

3. Suppose \(B = 0\), and for all \(j \in \mathbb{N}\) the functions \(g_j \in X, B_j \in X_H\) and \(C_j \in X\) are given. Suppose \(B_j = 0\) for all \(j \in \mathbb{N}\). Suppose there exists a number \(\varepsilon_0 > 0\) such that for all \(j \in \mathbb{N}\) and all \(x \in H\) we have \(g_j(x) \geq \varepsilon_0\) and \(C_j(x) \leq 0\). Let us denote by the symbol \(L_j\) the operator \(L\), which corresponds to the functions \(g_j, B_j\) and \(C_j\), and the operator \(L\) corresponding to the functions \(g, B\) and \(C\) will be denoted by \(L_0\). Suppose also that \(g_j(x) \to g(x)\) and \(C_j(x) \to C(x)\), uniformly with respect to \(x \in H\).

Then the (existing by item 2) strongly continuous semigroups \((e^{L_jt})_{t \geq 0}\) converge strongly (and uniformly with respect to \(t \in [0, t_0]\) for every fixed \(t_0 > 0\)) to the (existing by item 2) strongly continuous semigroup \((e^{L_0t})_{t \geq 0}\) with the generator \(L_0\). In other words for every \(t_0 > 0\) and every \(\varphi \in X\) there exists a limit

\[
\lim_{j \to \infty} (e^{L_jt_0} \varphi)(x) = (e^{L_0t_0} \varphi)(x), \tag{47}
\]

uniformly with respect to \(x \in H\) and \(t \in [0, t_0]\).

**Proof.**
1. Recall theorem \textit{3.1} and set $F(t) = S_t$, $\omega = \frac{2\|A\|\|B\|^2 + \|C\|}{90}$, $\mathcal{X} = X$, $\mathcal{D} = D$, $F'(0) = L$, $G = T$. One can see that according to items 1, 4 and 5 of theorem \textit{4.1} and item 5 of theorem \textit{4.2} all the conditions of theorem \textit{3.1} are fulfilled.

2. Note that $C(x) \leq 0$, so $\sup_{x \in B} e^{C(x)} \leq 1$ and for $B = 0$ one obtains the estimate $\|S_t\| \leq 1$. Conditions of theorem \textit{3.2} are fulfilled if one sets $\mathcal{X} = X$, $\mathcal{D} = D$, $L = L$, $V_t = S_t$, $M = 1$, $\omega = 0$. Indeed, according to item 1 of theorem \textit{1.1} for all $t \geq 0$ the estimate $\|S_t\| \leq e^{\omega t} = 1$ holds true, therefore $\| (S_t)^k \| \leq 1 \cdots 1 = 1$. Other conditions of theorem \textit{3.2} follow from item 4 of theorem \textit{4.1} and items 4 and 5 of theorem \textit{4.2}.

3. Recall theorem \textit{3.3} and set $\mathcal{X} = X$, $\mathcal{D} = D$, $\mathcal{L} = L_0$, $\mathcal{L}_n = L_j$. One can see that item 2 of this theorem and items 4 and 5 of theorem \textit{4.2} imply all the conditions of theorem \textit{3.3} except for the following one: if $\varphi \in D$, then $\lim_{j \to \infty} L_j \varphi = L_0 \varphi$. A simple check shows that this condition is also fulfilled.

$\Box$

4.2 Feynman formula solves the Cauchy problem for the parabolic equation

We want to find a function $u: [0, +\infty) \times H \rightarrow \mathbb{R}$ satisfying the following conditions (we call them Cauchy problem for the parabolic differential equation):

$$
\begin{align*}
\left\{\begin{array}{ll}
u'_t(t, x) = Lu(x, t); & t \geq 0, x \in H, \\
u(0, x) = u_0(x); & x \in H.
\end{array}\right.
\end{align*}
$$

(48)

To this Cauchy problem, we relate the so-called abstract Cauchy problem (see Definition \textit{3.3}), which we define as the following system of conditions upon the function $U: [0, +\infty) \rightarrow X$:

$$
\begin{align*}
\left\{\begin{array}{ll}
d U(t) = \mathcal{L}U(t); & t \geq 0, \\
U(0) = u_0,
\end{array}\right.
\end{align*}
$$

(49)

Remark 4.1. Problem \textit{(48)} can be considered as problem \textit{(49)} in the following sense. Function $u: (t, x) \mapsto u(t, x)$ of two variables $(t, x)$ can be considered as a function $u: t \mapsto [x \mapsto u(t, x)]$ of one variable $t$, with values in the space of functions of variable $x$. Then

$$
u(t, x) = (U(t))(x), \quad t \geq 0, x \in H.
$$

Using this correspondence, we start from Definition \textit{3.3} and define the solution of problem \textit{(48)}.

Definition 4.1. We call a function $u: [0, +\infty) \times H \rightarrow \mathbb{R}$ a strong solution of problem \textit{(48)} if it satisfies the following conditions:

$$
\begin{align*}
\left\{\begin{array}{ll}
u(t, \cdot) \in D_t; & t \geq 0, \\
function t \mapsto u(t, \cdot) \text{ is continuous; } & t \geq 0, \\
Uniformly \text{ for } x \in H \exists \lim_{\varepsilon \to 0} \frac{u(t+\varepsilon, x) - u(t, x)}{\varepsilon} = u'_t(t, x); & t \geq 0, \\
u'_t(t, \cdot) \in X; & t \geq 0, \\
Function t \mapsto u'_t(t, \cdot) \text{ is continuous; } & t \geq 0, \\
u'_t(t, x) = Lu(t, x); & t \geq 0, x \in H, \\
u(0, x) = u_0(x); & x \in H.
\end{array}\right.
\end{align*}
$$

(50)

Definition 4.2. We call a function $u: [0, +\infty) \times H \rightarrow \mathbb{R}$ a mild solution of problem \textit{(48)} if it satisfies the following conditions:
\[ \begin{cases} u(t, \cdot) \in X; \\ \text{Function } t \mapsto u(t, \cdot) \text{ is continuous}; \\ \int_0^t u(s, \cdot)ds \in D_1; \\ u(t, x) = L \int_0^t u(s, x)ds + u_0(x); \\ u_0 \in X. \end{cases} \tag{51} \]

**Definition 4.3.** Let us use the symbol \( C([0, +\infty), X) \) for the class of all functions \( u: [0, +\infty) \times H \to \mathbb{R} \) such that for every \( t \geq 0 \) the function \( x \mapsto u(t, x) \) belongs to the class \( X \), and the mapping \( t \mapsto u(t, \cdot) \in X \) is continuous for every \( t \geq 0 \).

Finally, let us state and prove the main result of the article. We use definitions and notation from Section 2.

**Theorem 4.4.** (On the solution of the Cauchy problem for a parabolic differential equation in Hilbert space)

Suppose \( g \in X, C \in X, B \in X_H \). Suppose there is a number \( g_0 > 0 \) such that for all \( x \in H \) we have \( g(x) \geq g_0 \) and \( C(x) \leq 0 \). As \( C \in X \), there exists a sequence \( (C_j) \subset D \), converging to \( C \) uniformly; let us additionally require that this sequence can be selected in such a way that \( C_j(x) \leq 0 \) for all \( j \in \mathbb{N} \) and all \( x \in H \).

Then the following holds:

1. If there exists a strongly continuous semigroup with the generator \( \mathcal{L} \), then for every \( u_0 \in D_1 \) there exists a solution \( u \) of problem (51), unique in the class \( C([0, +\infty), X) \). The solution depends continuously on \( u_0 \), and is given by the formula \( u(t, x) = \lim_{n \to \infty} \left( \left( S_n^2 \right)^n u_0 \right)(x) \), where the limit is uniform with respect to \( t \in [0, t_0] \) for every \( t_0 > 0 \).

2. If there exists a strongly continuous semigroup with the generator \( \mathcal{L} \), then for every \( u_0 \in X \) there exists a solution \( u \) of problem (51), unique in the class \( C([0, +\infty), X) \). It depends continuously on \( u_0 \), and is given by the formula \( u(t, x) = \lim_{n \to \infty} \left( \left( S_n^2 \right)^n u_0 \right)(x) \), where the limit is uniform with respect to \( t \in [0, t_0] \) for every \( t_0 > 0 \).

3. If \( B = 0 \), then there exists a strongly continuous semigroup with the generator \( \mathcal{L} \). The formula \( u(t, x) = \lim_{n \to \infty} \left( \left( S_n^2 \right)^n u_0 \right)(x) \) becomes simpler than in the case \( B \neq 0 \). Namely, for \( B = 0 \) we have

\[
\begin{align*}
\left. u(t, x) = \lim_{n \to \infty} \int_{H} \int_{H} \cdots \int_{H} & \, e^{-t\left(C(x) + \sum_{k=1}^{n-1} C(y_k) \right)} u_0(y_1) \mu_{y_2}^{\frac{\gamma_2}{g(y_2)}} A(dy_1) \mu_{y_3}^{\frac{\gamma_3}{g(y_3)}} A(dy_2) \cdots \\
& \cdots \mu_{y_n}^{\frac{\gamma_n}{g(y_n)}} A(dy_{n-1}) \mu_{y_n}^{\frac{\gamma_n}{g(y_n)}} A(dy_n). \end{align*}
\tag{52}
\]

In this case the solution \( u \) for all \( t > 0 \) satisfies the estimate \( \sup_{x \in H} |u(t, x)| \leq \sup_{x \in H} |u_0(x)| \).

4. Let \( B = 0 \), and let the functions \( g_j \in X, B_j \in X_H \) and \( C_j \in X \) be given for all \( j \in \mathbb{N} \). Let \( B_j = 0 \) for all \( j \in \mathbb{N} \). Suppose there exists \( \varepsilon_0 > 0 \) such that \( g_j(x) \geq \varepsilon_0 \) and \( C_j(x) \leq 0 \) for all \( j \in \mathbb{N} \) and all \( x \in H \). Let us use the symbol \( L_j \) for the operator \( L \) that corresponds to the functions \( g_j \), \( B_j \), and \( C_j \), and the symbol \( L_0 \) for the operator \( L \) that corresponds to the functions \( g \), \( B \) and \( C \). Suppose also that \( g_j(x) \to g(x) \) and \( C_j(x) \to C(x) \), uniformly with respect to \( x \in H \). We denote as \( u_j \) the solution of problems (50) and (51) for the operator \( L_j \). For solution of problems (50) and (51) with the operator \( L \), we use the symbol \( u \).

Then \( u_j(t, x) \) converges to \( u(t, x) \) as \( j \to \infty \), uniformly with respect to \( x \in H \) and uniformly with respect to \( t \in [0, t_0] \) for every fixed \( t_0 > 0 \).
Remark 4.2. Note that if $B = 0$, then solution depends continuously on the data of the Cauchy problem: the coefficients of the equation (item 4) and the initial condition (items 1 and 2).

Remark 4.3. Analogous theorems for $\mathbb{C}$- or $\mathbb{R}^n$-valued functions $u$ can be formulated mutatis mutandis. The result will hold true due to the theorem above and the linearity of $L$ and $S_t$. The only additional condition will be that the coefficients of the equation must be real-valued. The same remark is applicable to all the key theorems of this article.

Proof of the theorem.

1. Suppose that there exists a strongly continuous semigroup with the generator $\mathcal{L}$. Then by item 1 of proposition 3.5 we obtain the existence of a strong solution (definition 3.3) to Cauchy problem (49), and the solution is unique in the class $C([0, +\infty), X)$. By item 1 of theorem 4.3 the semigroup is given in the form described. Using the relation between problems (48) and (49) explained in remark 4.1, we obtain the solution for problem (50). The solution is unique in the class $C([0, +\infty), X)$, as follows from remark 4.1.

2. The proof is similar to that in item 1. The only difference is that in proposition 3.5 we use item 2 instead of item 1.

3. The existence of the sought semigroup follows from item 2 of theorem 4.3. The estimate for the supremum of the absolute value of the solution follows from the fact that the semigroup is contractive.

Let us explain how the equality $u(t, x) = \lim_{n \to \infty} \left( (S_n t)^n u_0 \right)(x)$ implies formula (52). For a continuous bounded function $\psi: H \to \mathbb{R}$ and a point $x \in H$, the following change of variables rule in the integral is correct:

$$\int_H \psi(y) \mu_A(dy) = \int_H \psi(y - x) \mu^x_A(dy).$$

Applying this rule, and changing $A$ to $2tg(x)A$, we come to the equality

$$(S_t \varphi)(x) = e^{tC(x)} \int_H \varphi(x + y) \mu_{2tg(x)A}(dy) = e^{tC(x)} \int_H \varphi(y) \mu^x_{2tg(x)A}(dy).$$

For $n = 2$ in formula (52) we get the expression

$$\left( \left( S_{\frac{t}{2}} \right)^2 \varphi \right)(x) = \left( S_{\frac{t}{2}} \left( S_{\frac{t}{2}} \varphi \right) \right)(x) = \int_H \left( \int_H e^{\frac{t}{2}(C(x) + C(y_1))} \varphi(y_1) \mu_{y_2(y_2)}^{y_2(y_2)A}(dy_1) \right) \mu^x_{2tg(x)A}(dy_2).$$

In the same way expressions for $n > 2$ are derived. Thus, the formula (52) is proven.

4. The proof follows immediately from item 3 of theorem 4.3.

□

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