A proof of the Kochen–Specker theorem can always be converted to a state-independent noncontextuality inequality

Xiao-Dong Yu¹, Yan-Qing Guo¹ and D M Tong²

¹ Department of Physics, Shandong University, Jinan 250100, People’s Republic of China
² Department of Physics, Dalian Maritime University, Dalian 116026, People’s Republic of China

E-mail: tdm@sdu.edu.cn

Keywords: Kochen-Specker theorem, noncontextuality inequality, quantum contextuality

Abstract

Quantum contextuality is one of the fundamental notions in quantum mechanics. Proofs of the Kochen–Specker theorem and noncontextuality inequalities are two means for revealing the contextuality phenomenon in quantum mechanics. It has been found that some proofs of the Kochen-Specker theorem, such as those based on rays, can be converted to a state-independent noncontextuality inequality, but it remains open whether this is true in general, i.e., whether any proof of the Kochen-Specker theorem can always be converted to a noncontextuality inequality. In this paper, we address this issue. We prove that all kinds of proofs of the Kochen-Specker theorem, based on rays or any other observables, can always be converted to state-independent noncontextuality inequalities. Besides, our constructive proof also provides a general approach for deriving a state-independent noncontextuality inequality from a proof of the KS theorem.

1. Introduction

Quantum contextuality [1] as a natural generalization of Bell nonlocality [2] is one of the fundamental notions in quantum mechanics, and has drawn a lot of interest recently. Proofs of the Kochen-Specker (KS) theorem and noncontextuality inequalities are two different methods for revealing the contextuality phenomenon in quantum mechanics. Both of them have been widely used in many fields of quantum mechanics, including quantum computation [3–5] and quantum information [6–11].

KS theorem was found from the inconsistency of the noncontextual hidden variable (NCHV) model with quantum mechanics. The NCHV model [1, 2] consists of two basic assumptions: that every observable A has a definite value v(A) at all time, and that v(A) does not depend on whether A is measured alone or together with B or C if A is compatible with B and C. The first assumption is at the core of the hidden variable theory while the latter is the exhibition of noncontextuality. An observable corresponds to a Hermitian operator in quantum mechanics, and two observables being compatible in the NCHV model corresponds to the operators being commutative in quantum mechanics. A measurement context in the NCHV model, defined as a set of mutually compatible observables, corresponds to a set of mutually commutative operators in quantum mechanics. To examine the inconsistency of the NCHV model with quantum mechanics, Kochen and Specker assumed that the algebraic structure of compatible observables in quantum mechanics is also preserved in the NCHV model, which leads to the sum rule and the product rule [12, 13]. The sum rule means that if A and B are two compatible observables and C = A + B, then v(C) = v(A) + v(B). Similarly, the product rule means that if A and B are two compatible observables and C = A · B, then v(C) = v(A) · v(B). It was found that there always exists a finite set of observables for any n-dimensional Hilbert space with n ≥ 3, such that all elements in the set cannot simultaneously have values satisfying the sum rule and product rule. This finding is usually called the KS theorem, which shows that the NCHV model is not compatible with quantum mechanics.

A proof of the KS theorem is accomplished by finding an observable set in which no value assignment of the observables can satisfy the sum rule and product rule, i.e., it inevitably leads to a logical contradiction if the value assignments of the observables are required to satisfy the sum rule and product rule. Such sets of observables that
can lead to a logical contradiction between the value assignments and the algebraic structure are not unique, and each of them provides a proof of the KS theorem. Early proofs of the KS theorem are all based on rays, i.e. rank-1 projectors, in the Hilbert space. The original proof of the KS theorem proposed by Kochen and Specker involves 117 rays. The size of the ray sets is reduced step by step [14–18] and the simplest proofs based on rays are with 31 rays for a 3-dimensional Hilbert space [16] and 18 rays for a 4-dimensional Hilbert space [18]. Generalization of proofs of the KS theorem started with the work of Mermin and Peres [14, 19], who found that the number of observables involved in a proof of the KS theorem can be significantly reduced if general observables are used, instead of only using rays. The Mermin–Peres square proposed by them involves only 9 observables in a 4-dimensional Hilbert space. Since then, a great number of proofs of the KS theorem based on general observables have been proposed [20–32].

Noncontextuality inequalities are an alternative tool for demonstrating the inconsistency between the NCHV model and quantum mechanics. A noncontextuality inequality is an expression which is fulfilled for all value assignments of observables in the NCHV model but is violated for some states in quantum mechanics. A noncontextuality inequality is said to be state-independent if it is violated by all quantum states. Proofs of the KS theorem and noncontextuality inequalities are two ways of showing the inconsistency between the NCHV model and quantum mechanics. Compared with proofs of the KS theorem, noncontextuality inequalities are more experiment-friendly, and therefore have drawn a lot of interest recently [33–34].

Cabello may be the first who noted that some proofs of the KS theorem can be used to construct state-independent noncontextuality inequalities, and proposed the first state-independent noncontextuality inequality [35]. Soon after, Badziag et al showed that every proof of the KS theorem based on rays can be converted to a state-independent noncontextuality inequality [36]. Indeed, for a proof of the KS theorem based on the ray set \( \mathcal{S} = \{ P_1, P_2, \ldots, P_j \} \) in an \( n \)-dimensional Hilbert space, one can construct the function

\[
F = \sum_{\alpha=1}^{N} (\sum_{i=1}^{n} A_{k_i}^\alpha - \prod_{i=1}^{n} \left(1 + A_{k_i}^\alpha\right)),
\]

where \( A_{k_i}^\alpha = 1 - 2P_{k_i}^\alpha \), and \( \{ P_{k_1}^\alpha, P_{k_2}^\alpha, \ldots, P_{k_n}^\alpha \} \in \mathcal{S} \) is the \( \alpha \)-th basis of the space, \( \alpha = 1, 2, \ldots, N \) with \( N \) being the total number of the bases. If \( \mathcal{S} \) satisfies the condition that every pair of orthogonal rays belongs to some \( \{ P_{k_1}^\alpha, P_{k_2}^\alpha, \ldots, P_{k_n}^\alpha \} \), then one can immediately obtain the noncontextuality inequality for the NCHV model, \( F \leq N(n-2) - 2 \), which is violated for all quantum states since \( F = N(n-2) \) in quantum mechanics. If \( \mathcal{S} \) does not satisfy this condition, it should be enlarged by adding new rays to satisfy the condition and an inequality can be derived from the enlarged set. Most recently, the result of Badzia et al was improved by Yu and Tong in [49], where the authors put forward an alternative method, which need not enlarge the given ray set \( \mathcal{S} \) and can give a simpler inequality from a proof of the KS theorem based on rays.

These previous results show that some proofs of the KS theorem, such as those based on rays, can be converted to state-independent noncontextuality inequalities. However, it remains open whether such correspondence is true in general, i.e., whether any proof of the KS theorem, not limited to those based on rays, can always be converted to a state-independent noncontextuality inequality. This is a fundamental issue related to the relation between the proofs of the KS theorem and noncontextuality inequalities. In this paper, we address this issue. Our results show that every proof of the KS theorem, based on rays or other observables, can always be converted to a state-independent noncontextuality inequality. Besides, our constructive proof also provides a simple approach for deriving a state-independent noncontextuality inequality from a general proof of the KS theorem. Proofs of the KS theorem based on general observables are usually simpler than those only based on rays, and therefore the inequalities derived from them are expected to be simpler and more useful.

This paper is organized as follows. In section 2, we demonstrate the existence of a noncontextuality inequality in any proof of the KS theorem and put forward a general approach to derive the inequality. In sections 3 and 4, as examples of showing its application, we apply our approach to proofs of the KS theorem based on rays and proofs of the KS theorem based on parity arguments, respectively, and derive the corresponding noncontextuality inequalities. Section 4 is the conclusion and remarks.

2. A general approach of converting a proof of the KS theorem to a noncontextuality inequality

We first specify some notation and relations. \( \mathcal{S} \) is used to denote a set of observables which carries out a proof of the KS theorem, \( \mathcal{S} = \{ A_1, A_2, \ldots, A_m \} \), where the observables \( A_i \) may be rays or other observables. A measurement context is defined as a subset of \( \mathcal{S} \), denoted as \( \mathcal{S}_\alpha \), in which all observables are mutually compatible, where the subscript \( \alpha \) is used to label different measurement contexts, \( \alpha = 1, 2, \ldots, L \). The \( \alpha \)-th measurement context can be written as \( \mathcal{S}_\alpha = \{ A_{k_1}^\alpha, A_{k_2}^\alpha, \ldots, A_{k_m}^\alpha \} \), where the subscripts

\[
1 \leq k_1^\alpha < k_2^\alpha < \ldots < k_m^\alpha \leq \mu
\]

and \( m = m(\alpha) \) is the number of observables in the \( \alpha \)-th measurement context. Note that \( m \) in the subscript of \( k_m^\alpha \) depends on \( \alpha \), i.e., the number of observables in different measurement contexts may be different. We use \( \mathbb{R}[A_{k_1}^\alpha, A_{k_2}^\alpha, \ldots, A_{k_m}^\alpha] \) to denote the set of all polynomials of \( A_{k_1}^\alpha, A_{k_2}^\alpha, \ldots, A_{k_m}^\alpha \),
with real coefficients. \( I_0 \) is a subset of \( \mathbb{R}[A^x_k, A^{k-1}_k, ..., A^1_k] \), defined as
\[
I_0 = \{ r \in \mathbb{R}[A^x_k, A^{k-1}_k, ..., A^1_k] | r(\hat{A}^x_k, \hat{A}^{k-1}_k, ..., \hat{A}^1_k) = 0 \},
\]
where \( \hat{A}^x \) is the operator of observable \( A^x \). The terminology, 'value assignment' \( v \) to \( S \) means that a value, denoted as \( v(A_i) \), is assigned to each observable \( A_i \) in \( S \), where \( v(A_i) \) is an eigenvalue of the operator \( \hat{A}^x_i \). If a value assignment \( v \) satisfies the sum rule and product rule, then \( r \) satisfies \( r(v(A_{i1}), v(A_{i2}), ..., v(A_{iL})) = 0 \) for \( r \in I_0, \alpha = 1, 2, ..., L \), since the observables \( A^x_k, A^{k-1}_k, ..., A^1_k \) are mutually compatible.

We now demonstrate the existence of a noncontextuality inequality in a proof of the KS theorem expressed by \( S \). The fact that the observable set \( S \) provides a proof of the KS theorem means that no value assignment \( v \) can simultaneously fulfill all the equations in \( \{ r \mid v \} = 0 \) \( r \in \bigcup_{i=1}^{L} I_0 \}, \) which is an infinite set since each \( I_0 \) is infinite. Note that \( I_0 \) is an ideal of the polynomial ring \( \mathbb{R}[A^x_k, A^{k-1}_k, ..., A^1_k] \). According to Hilbert's basis theorem, \( \mathbb{R}[A^x_k, A^{k-1}_k, ..., A^1_k] \) is a Noetherian ring, and therefore \( I_0 \) can be finitely generated. For each \( I_0 \), we can always find a finite subset of \( I_0 \), \( \{ n^1, n^2, ..., n^L \} \), such that all \( r \in I_0 \) can be written as \( r = \sum_{i=1}^{L} f^i n^i \) with \( f^i \in \mathbb{R}[A^x_k, A^{k-1}_k, ..., A^1_k] \). This implies that for all \( r \in I_0 \) if and only if \( n^i | v = 0 \) for \( i = 1, 2, ..., L \), we denote as \( v \). Hence, the observable set \( S \) provides a proof of the KS theorem if and only if no value assignment \( v \) can simultaneously fulfill the equations, \( \{ n^i | v = 0 \mid i = 1, 2, ..., L \}, \) which include a finite number of polynomials \( n^i \). By using these polynomials, we can define a function,
\[
F = -\sum_{i=1}^{L} \sum_{j=1}^{L} r^i_j (A^x_k, A^{k-1}_k, ..., A^1_k) \bigg|_{v = 0}^2.\]
It is easy to verify that \( F < 0 \) defines a state-independent noncontextuality inequality. In fact, since at least one \( n^i | v = 0 \) is nonzero for any value assignment \( v \) and therefore \( F = -\sum_{i=1}^{L} \sum_{j=1}^{L} (r^i_j | v = 0)^2 < 0, \) the average value \( \langle F \rangle \) must be less than zero. On the other hand, by definition, \( r^i_j (\hat{A}^x_k, \hat{A}^{k-1}_k, ..., \hat{A}^1_k) = 0 \). We have \( \langle F \rangle = \text{Tr}(\rho^i_j) = -\sum_{i=1}^{L} \sum_{j=1}^{L} \text{Tr}(\rho^i_j \rho^j_i) = 0 \) for all quantum states \( \rho \), which means that all quantum states violate the inequality \( \langle F \rangle < 0 \).

With the help of the above demonstration, we may now develop a general approach of converting a proof of the KS theorem to a state-independent noncontextuality inequality. The above discussions show that the key is to find a set of observable polynomials \( \{ n^1, n^2, ..., n^L \} \), which satisfies the two conditions: that \( n \) is a polynomial of some measurement context \( S_0 = \{ A^x_k, A^{k-1}_k, ..., A^1_k \} \) satisfying \( n (\hat{A}^x_k, \hat{A}^{k-1}_k, ..., \hat{A}^1_k) = 0 \), and that for any value assignment \( v \), at least one of \( n_i | v = 0 \) is nonzero. For convenience, hereafter, we call such a set of observable polynomials that satisfies the two conditions as a complete set of polynomials. Clearly, for a given set of observables \( S \), there may be many different complete sets of polynomials and the number \( N \) for different sets may be different too. We only need to find one of them to construct a noncontextuality inequality in \( S \).

Although it is difficult to prove whether an observable set provides a proof of the KS theorem, it is trivial to find a complete set of polynomials from the observable set that has been proved to be a proof of the KS theorem. Indeed, if a set of observables is a proof of the KS theorem, it means that no value assignment to the set can satisfy the sum rule and product rule. This is verified either by finding, for each value assignment, a polynomial of compatible observables that is equal to zero in quantum mechanics but nonzero for the value assignment in the NCHV model, or by finding several polynomials of compatible observables that are equal to zero in quantum mechanics but lead to a logical contradiction if they are required to be zero in the NCHV model. In each of the two cases, the set of polynomials used in the proof is just what we want to find. They comprise a complete set of polynomials \( \{ n^1, n^2, ..., n^L \} \), and can be used to construct a noncontextuality inequality, \( \langle F \rangle < 0 \) with \( F = -\sum_{i=1}^{N} r^i \). If we would like to obtain an inequality with an explicit upper bound, we may normalize each polynomial by \( |n_i | v = 0 \), where \( c_i = \text{min}_{|v = 0} n_i | v = 0 \) is a constant. In this case, \( r^i | v = 0 \) is violated. By using these normalized polynomials \( n_i / \sqrt{c_i} \) to take the place of \( n \), the inequality can be expressed as \( \langle F \rangle \leq -1 \). Besides, the state-independent inequality obtained can be simplified by reducing \( n^i | v = 0 \) with \( (A_i - a_i) (A_i - a_i) \) \( (A_j - a_j) = 0 \), where \( a_0, a_2, ..., a_d \) are the \( d \) different eigenvalues of \( A_i \).

So far, we have demonstrated the existence of a noncontextuality inequality in any proof of the KS theorem and have shown how to convert a proof of the KS theorem to a noncontextuality inequality. To summarize our approach briefly, one may derive a state-independent noncontextuality inequality from a proof of the KS theorem expressed by a set of observables \( S = \{ A_1, A_2, ..., A_m \} \) via the following three steps.

1) Find a complete set of normalized polynomials, denoted as
\[
\mathcal{P}_c = \{ n^1, n^2, ..., n^L \}.\tag{1}
\]
This can be done by first finding a complete set of polynomials and then normalizing them by multiplying each by a suitable constant.
2) Define a function of observables by using the normalized polynomials,

\[ F = - \sum_{i=1}^{N} r_i^2, \]  

and simplify \( F = F(A_1, A_2, ..., A_n) \) by using the relation \((A_i - a_i)(A_i - a_2)\cdots(A_i - a_d) = 0\).

3) Write out the expression,

\[ \left\{ F(A_1, A_2, ..., A_n) \right\} < -1, \]  

which is a state-independent noncontextuality inequality, i.e., violated by all the quantum states.

It is worth noting that in the above discussions, the polynomials \( r_i \) have been assumed to be real for simplification. Although this assumption is applicable for almost all known proofs of the KS theorem, there are a few proofs of the KS theorem in which complex polynomials are involved [24, 29]. Our approach is also applicable to these cases. When complex polynomials are involved in a proof of the KS theorem, the only modification is that equation (2) is replaced by

\[ F = - \sum_{i=1}^{N} r_i^4. \]

In the following sections, we will take two well-known kinds of proofs of the KS theorem, proofs based on rays and proofs based on parity arguments, as examples to illustrate the approach.

3. Application to proof of the KS theorem based on rays

Let \( \mathcal{S} \) be a set of rays that provides a proof of the KS theorem, \( \mathcal{S} = \{ P_1, P_2, ..., P_n \} \), where \( P_i \) are rays in an \( n \)-dimensional Hilbert space. A ray set provides a proof of the KS theorem if and only if no value assignment \( \rho \) can simultaneously satisfy the KS rules: (i) \( \nu(P_i)\nu(P_j) = 0 \) if \( P_i \) and \( P_j \) are orthogonal; and (ii) \( \sum_{i=1}^{n} \nu(P_k^\alpha) = 1 \), if \( \{P_k^\alpha\} \) forms an orthogonal basis for the \( n \)-dimensional Hilbert space. In other words, at least one of these equations is invalid for each value assignment.

We now derive a state-independent noncontextuality inequality from the set \( \mathcal{S} \) by using our approach described in section 2. First, \( \{P_i^\alpha | \sum_{i=1}^{n} P_i^\alpha = 0, P_i \in \mathcal{S} \} \cup \{\sum_{i=1}^{n} P_i^\alpha = 1 | \sum_{i=1}^{n} P_i^\alpha = 1, P_i \in \mathcal{S} \} \) comprise a complete set of polynomials, and each of the polynomials is already normalized since \( \nu(P_i)\nu(P_j) = 1 \) if \( \nu(P_i)\nu(P_j) = 0 \) and \( \sum_{i=1}^{n} \nu(P_k^\alpha) = 1 \). Therefore, we may take this set of polynomials as \( \mathcal{P} \), a complete set of normalized polynomials defined in equation (1). Second, substituting all the polynomials for \( r_i \) in equation (2), and using the relation \( P_i^2 = P_i \), we obtain

\[ F = - \sum_{i,j} P_i P_j - \sum_{\alpha} \left( 2 \sum_{i<j} P_i^\alpha P_j^\alpha - \sum_{i=1}^{n} P_i^\alpha + 1 \right). \]

Third, the state-independent noncontextuality inequality reads \( \langle F \rangle < -1 \) and the quantum violation is given by \( \langle F \rangle = 0 \) for all quantum states \( \rho \). It is interesting to note that the inequality obtained by using the general approach is equivalent to the one in [49], which is given by trial and error.

For the case considered by Badziag et al. [36], where \( \mathcal{S} \) includes \( N \) bases, \( \mathcal{S}_\alpha = \{ P_{k_1}^\alpha, P_{k_2}^\alpha, ..., P_{k_N}^\alpha \} \), \( \alpha = 1, 2, ..., N \), and every pair of orthogonal rays belongs to some \( \mathcal{S}_\alpha \), the complete set of normalized polynomials can be taken as \( \mathcal{P}_\alpha = \{ \sum_{i=1}^{n} P_i^\alpha - 1 = 0 | \alpha = 1, 2, ..., N \} \). In this special case, equation (5) is reduced to

\[ F = - \sum_{\alpha} \left( 2 \sum_{i<j} P_i^\alpha P_j^\alpha - \sum_{i=1}^{n} P_i^\alpha + 1 \right), \]

and the state-independent noncontextuality inequality reads

\[ \sum_{\alpha=1}^{N} \left( \sum_{i=1}^{n} P_i^\alpha - 2 \sum_{i<j} P_i^\alpha P_j^\alpha \right) \leq N - 1, \]

which is valid for the NCHV model, but is violated by all quantum states \( \rho \) since the expectation is \( N \) in quantum mechanics. In the previous work on noncontextuality inequalities, observables were usually chosen to be

\[ If the ray set is denoted by a graph, in which each vertex corresponds to a ray and two vertices corresponding to two orthogonal rays are connected by an edge, then these polynomials just correspond to the edges and \( n \)-vertex cliques in the graph.\]
\{-1, 1\}-dichotomic. If we let \( A_i = 1 - 2P_i \), the expressions (5), (6) and (7) can be rewritten with observables \( A_i \). For example, the inequality (7) becomes an equivalent form,

\[
\sum_{\alpha=1}^{N} \left( 2(n - 2) \sum_{i=1}^{n} \langle A_{k_i}^\alpha \rangle - 2 \sum_{i<j} \langle A_{k_i}^\alpha A_{k_j}^\alpha \rangle \right) \leq \left( n^2 - 3n + 4 \right) N - 4,
\]

which is valid for the NCHV model, but is violated by all quantum states \( \rho \) since the expectation is \( (n^2 - 3n + 4) N \) in quantum mechanics. Compared with the noncontextuality inequality constructed by Badziąg et al, in which correlations of \( n \) compatible observables are involved, the inequalities derived by our approach, which only involve correlations of two compatible observables, are simpler and should be more feasible experimentally.

4. Application to proof of the KS theorem based on parity arguments

Let \( \mathcal{S} = \{ A_1, A_2, \ldots, A_n \} \) be a set of observables that provides a proof of the KS theorem based on parity arguments, i.e., a parity proof, in an \( n \)-dimensional Hilbert space. The observables in \( \mathcal{S} \) are \{-1, 1\} -dichotomic, and there are \( N \) measurement contexts in \( \mathcal{S} \), denoted as \( \mathcal{S}_a = \{ A_{k_1}^\alpha, A_{k_2}^\alpha, \ldots, A_{k_n}^\alpha \}, \alpha = 1, 2, \ldots, N \), which satisfy the expressions \( \hat{A}_{k_1}^\alpha \hat{A}_{k_2}^\alpha \cdots \hat{A}_{k_n}^\alpha = \delta_0 I_n \), where \( \delta_0 \) equals 1 or \(-1\) and \( m = m(\alpha) \) is the number of observables in \( \mathcal{S}_a \). \( \mathcal{S} \) being a parity proof means that these \( N \) measurement contexts satisfy the relations: (i) \( \delta_1 \delta_2 \cdots \delta_N = -1 \), and (ii) each \( A_i \) appears an even number of times in total in all \( N \) measurement contexts, i.e., the number of measurement contexts that contain any \( A_i \) is even.

According to the expressions \( \hat{A}_{k_1}^\alpha \hat{A}_{k_2}^\alpha \cdots \hat{A}_{k_n}^\alpha = \delta_0 I_n \), we have the equations \( v(A_{k_1}^\alpha) v(A_{k_2}^\alpha) \cdots v(A_{k_n}^\alpha) = \delta_0 \) if the value assignment \( v \) satisfies the sum rule and product rule. However, these equations, for any value assignment, cannot be simultaneously valid, because there is a contradiction if they are used to calculate

\[
\prod_{\alpha=1}^{N} v(A_{k_1}^\alpha A_{k_2}^\alpha \cdots A_{k_n}^\alpha). \quad \text{On one hand,} \quad \prod_{\alpha=1}^{N} v(A_{k_1}^\alpha A_{k_2}^\alpha \cdots A_{k_n}^\alpha) = \prod_{\alpha=1}^{N} \delta_0 = -1, \quad \text{but on the other hand,} \quad \prod_{\alpha=1}^{N} v(A_{k_1}^\alpha A_{k_2}^\alpha \cdots A_{k_n}^\alpha) = \prod_{\alpha=1}^{N} v(A_{k_1}^\alpha) v(A_{k_2}^\alpha) \cdots v(A_{k_n}^\alpha) = 1, \quad \text{since the number of} \ A_i \text{in the expression is even. This is the main idea of a proof of the KS theorem based on parity arguments.}
\]

We now derive a state-independent noncontextuality inequality from the parity proof. The above discussion shows that \( A_{k_1}^\alpha A_{k_2}^\alpha \cdots A_{k_n}^\alpha = \delta_0 \) and that at least one of the equations, \( v(A_{k_1}^\alpha) v(A_{k_2}^\alpha) \cdots v(A_{k_n}^\alpha) = \delta_0 \), is invalid for each value assignment \( v \). Therefore, \( \{ A_{k_1}^\alpha A_{k_2}^\alpha \cdots A_{k_n}^\alpha = \delta_0 | \alpha = 1, 2, \ldots, N \} \) comprise a complete set of polynomials. The normalized forms of the \( N \) polynomials are

\[
\{ \frac{1}{2}(A_{k_1}^\alpha A_{k_2}^\alpha \cdots A_{k_n}^\alpha = \delta_0) \ | \alpha = 1, 2, \ldots, N \}, \quad \text{since} \delta_0 = \pm 1 \text{ and} \quad v(A_{k_1}^\alpha) v(A_{k_2}^\alpha) \cdots v(A_{k_n}^\alpha) = -\delta_0 \text{ if} \quad A_{k_1}^\alpha A_{k_2}^\alpha \cdots A_{k_n}^\alpha = \delta_0. \quad \text{We then obtain the complete set of normalized polynomials. Substituting them into equation (2) and using the relations} \ (A_1 - 1)(A_1 + 1) = 0 \text{, i.e.,} \ A_1^2 = 1, \text{we obtain}
\]

\[
F = -\frac{N}{2} + \frac{1}{2} \sum_{\alpha=1}^{N} \delta_0 \prod_{i=1}^{m(\alpha)} A_{k_i}^\alpha.
\]

The state-independent noncontextuality inequality can be written as

\[
\sum_{\alpha=1}^{N} \delta_0 \left( \prod_{i=1}^{m(\alpha)} A_{k_i}^\alpha \right) \leq N - 2,
\]

which is violated by all quantum states \( \rho \) since \( \sum_{\alpha=1}^{N} \delta_0 \left( \prod_{i=1}^{m(\alpha)} A_{k_i}^\alpha \right) = N \) in quantum mechanics.

By applying our approach to proofs of the KS theorem based on parity arguments, we have obtained a general expression of the state-independent noncontextuality inequality. With this result, all known parity proofs, for example, those in \([14, 17–20, 22, 23, 25–28, 30]\) can be easily converted to state-independent noncontextuality inequalities, and the inequalities in \([35]\) and \([51]\) are special cases of equation (10).

5. Conclusions

We have shown that a proof of the KS theorem can always be converted to a state-independent noncontextuality inequality. This conclusion is true for all kinds of proofs of the KS theorem, based on rays or any other observables. It is interesting to note that our constructive proof actually provides a general approach for deriving a noncontextuality inequality from a proof of the KS theorem. By following the steps described by equations (1) to (3), one can derive a state-independent noncontextuality inequality from any proof of the KS theorem. As examples, we have applied our approach to two kinds of well-known proofs of the KS theorem, i.e., proofs based
on rays and proofs based on parity arguments. Certainly, our approach is applicable to all kinds of proofs of the KS theorem, including those in [21, 26, 31, 32], and not limited to the two examples.

Compared with the methods proposed in [36] and [49], which are applicable only to proofs of the KS theorem based on rays, the present approach is applicable to all kinds of proofs of the KS theorem. The inequalities obtained by our approach are simpler than the previous results in [36] and as simple as the ones in [49] when it is applied to proofs based on rays, and new state-independent noncontextuality inequalities are given if they are applied to proofs based on other observables.

Acknowledgments

This work was supported by NSF China through Grant No. 11175105 and the National Basic Research Program of China through Grant No. 2015CB921004. DMT acknowledges support from the Taishan Scholarship Project of Shandong Province.

References

[1] Kochen S and Specker E 1967 The problem of hidden variables in quantum mechanics J. Math. Mech. 17 59
[2] Bell J S 1966 On the problem of hidden variables in quantum mechanics Rev. Mod. Phys. 38 447–52
[3] Raussendorf R 2013 Contextuality in measurement-based quantum computation Phys. Rev. A 88 022322
[4] Howard M, Wallman J, Veitch V and Emerson J 2014 Contextuality supplies the ‘magic’ for quantum computation Nature 510 351–5
[5] Dellos N, Guerin P A, Jian R and Raussendorf R 2015 Wigner function negativity and contextuality in quantum computation on rebits Phys. Rev. X 5 021003
[6] Horodecki K, Horodecki M, Horodecki P, Horodecki R, Pawlowski M and Bourennane M Contextuality offers device-independent security (arXiv:1006.0468)
[7] Cabello A, D’Ambrosio V, Nagali E and Scarrino F 2011 Hybrid quasqrt-encoded quantum cryptography protected by kochen-specker contextuality Phys. Rev. A 84 030302
[8] Nagali E, D’Ambrosio V, Scarrino F and Cabello A 2012 Experimental observation of impossible-to-beat quantum advantage on a hybrid photonic system Phys. Rev. Lett. 108 090501
[9] Um M, Zhang X, Zhang J, Wang Y, Yanghao S, Deng D-L, Duan L-M and Kim K 2013 Experimental certification of random numbers via quantum contextuality Sci. Rep. 3 1627
[10] Guhne O, Budroni C, Cabello A, Kleinmann M and Larsson J-Å 2014 Bounding the quantum dimension with contextuality Phys. Rev. A 89 062107
[11] Cañas G, Arias M, Etcheverry S, Gómez E S, Cabello A, Xavier G B and Lima G 2014 Applying the simplest kochen-specker set for quantum information processing Phys. Rev. Lett. 113 090404
[12] Fine A and Teller P 1978 Algebraic constraints on hidden variables Found. Phys. 8 629–36
[13] Redhead M 1987 Incompleteness, Nonlocality, and Realism: A Prelegomenon to the Philosophy of Quantum Mechanics (Oxford: Oxford University Press)
[14] Peres A 1991 Two simple proofs of the kochen-specker theorem J. Phys. A 24 11175
[15] Bab J 1996 Schütte’s tautology and the kochen-specker theorem Found. Phys. 26 787–806
[16] Conway J H and Kochen J 1993 First reported in A. Peres Quantum Theory: Concepts and Methods (Dordrecht: Kluwer) p 114
[17] See also Conway J H and Kochen S 2002 Quantum Uncontextual hidden variables: From Bell to Quantum Information ed R A Bertlmann and A Zeilinger (Berlin: Springer) p 257
[18] Kernaghan M 1994 Bell-kochen-specker theorem for 20 vectors J. Phys. A 27 L829
[19] Cabello A, Estebanitz J and Garcia-Alcaine G 1996 Bell-kochen-specker theorem: A proof with 18 vectors Phys. Lett. A 212 183–187
[20] Mermin N D 1990 Simple unified form for the major no-hidden-variables theorems Phys. Rev. Lett. 65 3733–6
[21] Pagonis C, Redhead M L and Clifton R K 1991 The breakdown of quantum non-locality in the classical limit Phys. Lett. A 155 441
[22] Kernaghan M and Peres A 1995 Kochen-specker theorem for eight-dimensional space Phys. Lett. A 198 1–5
[23] Cabello A 1999 Quantum correlations are not contained in the initial state Phys. Rev. A 60 877–80
[24] Cabello A 2001 Multiparty multilevel greenberger-horne-zeilinger states Phys. Rev. A 63 022104
[25] Cerf N J, Massar S and Pironio S 2002 Greenberger-horne-zeilinger paradoxes for many qudits Phys. Rev. Lett. 89 080402
[26] Plant M 2012 On small proofs of the bell-kochen-specker theorem for two, three and four qudits Euro. Phys. J. Plus 127 1
[27] Waegell M and Aravind P K 2012 Proofs of the kochen-specker theorem based on a system of three qudits J. Phys. A 45 405301
[28] Waegell M and Aravind P K 2013 Proofs of the kochen-specker theorem based on the ℓ -qudit pauli group Phys. Rev. A 88 012102
[29] Ruuge A E 2012 New examples of kochen-specker-type configurations on three qudits J. Phys. A 45 465304
[30] Tang W, Yu S and Oh C H 2013 Greenberger-horne-zeilinger paradoxes from qului graph states Phys. Rev. Lett. 110 100403
[31] Lisoněk P, Raussendorf R and Singh V Generalized parity proofs of the kochen-specker theorem (arXiv:1401.3033)
[32] Toh S 2013 Kochen-specker sets with a mixture of 16 rank-1 and 14 rank-2 projectors for a three-qubit system Chin. Phys. Lett. 30 100302
[33] Toh S 2013 State-independent proof of kochen-specker theorem with thirty rank-two projectors Chin. Phys. Lett. 30 100303
[34] Klyachko A A, Can M A, Bincioglu S and Shumovsky A S 2008 Simple test for hidden variables in spin-1 systems Phys. Rev. Lett. 101 020403
[35] Cabello A, Filipp S, Rauch H and Hasegawa Y 2008 Proposed experiment for testing quantum contextuality with neutrons Phys. Rev. Lett. 100 100403
[36] Cabello A 2008 Experimentally testable state-independent quantum contextuality Phys. Rev. Lett. 101 210401
[37] Badziȩg P, Bengtsson I, Cabello A and Pitowsky I 2009 Universality of state-independent violation of correlation inequalities for noncontextual theories Phys. Rev. Lett. 103 050401
[38] Guhne O, Kleinmann M, Cabello A, Larsson J-Å, Kirchmair G, Zähringer F, Gerritsma R and Roos C F 2010 Compatibility and noncontextuality for sequential measurements Phys. Rev. A 81 022121
[38] Liang Y-C, Spekkens R W and Wiseman H M 2011 Specker’s parable of the overprotective seer: A road to contextuality, nonlocality and complementarity Phys. Rep. 506 1–39
[39] Cabello A State-independent quantum contextuality and maximum nonlocality arXiv:1112.5149
[40] Yu S and Oh C.H 2012 State-independent proof of kochen-specker theorem with 13 rays Phys. Rev. Lett. 108 030402
[41] Abramsky S and Brandenburger A 2011 The sheaf-theoretic structure of non-locality and contextuality New J. Phys. 13 113036
[42] Bengtsson I, Blanchfield K and Cabello A 2012 A kochen-specker inequality from a sic Phys. Lett. A 376 374–6
[43] Kleinmann M, Budroni C, Larsson J-Å, Gühne O and Cabello A 2012 Optimal inequalities for state-independent contextuality Phys. Rev. Lett. 109 250402
[44] Kurzyński P and Kaszlikowski D 2012 Contextuality of almost all qutrit states can be revealed with nine observables Phys. Rev. A 86 042125
[45] Kurzyński P, Ramanathan R and Kaszlikowski D 2012 Entropic test of quantum contextuality Phys. Rev. Lett. 109 020404
[46] Chaves R and Fritz T 2012 Entropic approach to local realism and noncontextuality Phys. Rev. A 85 032113
[47] Amselem E, Danielsen L E, López-Tarrida A J, Portillo J R, Bourennane M and Cabello A 2012 Experimental fully contextual correlations Phys. Rev. Lett. 108 200405
[48] Araújo M, Quintino M T, Budroni C, Cunha M T and Cabello A 2013 All noncontextuality inequalities for the $n$-cycle scenario Phys. Rev. A 88 022118
[49] Yu X-D and Tong D M 2014 Coexistence of kochen-specker inequalities and noncontextuality inequalities Phys. Rev. A 89 010101
[50] Ramanathan R and Horodecki P 2014 Necessary and sufficient condition for state-independent contextual measurement scenarios Phys. Rev. Lett. 112 040404
[51] Lisoněk P, Badziąg P, Portillo J and Cabello A 2014 Kochen-specker set with seven contexts Phys. Rev. A 89 042101
[52] Winter A 2014 What does an experimental test of quantum contextuality prove or disprove? J. Phys. A 47 424031
[53] Su H-Y, Chen J-L and Liang Y-C 2015 Demonstrating quantum contextuality of indistinguishable particles by a single family of noncontextuality inequalities Sci. Rep. 5 11657
[54] Acín A, Fritz T, Leverrier A and Sainz A B 2015 A combinatorial approach to nonlocality and contextuality Commun. Math. Phys. 334 553