Structural properties of the optimal policy for dual-sourcing systems with general lead times

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This article considers a periodic-review inventory problem with two suppliers. The regular supplier has a longer lead time than the expedited supplier but has a lower unit cost. The structural properties of the optimal orders are characterized using the notion of $L^\natural$-convexity. Interestingly, the optimal regular order is more sensitive to the late-to-arrive outstanding orders, but the optimal expedited order is more sensitive to the soon-to-arrive outstanding orders. A heuristic policy is designed that provides an average cost saving of 1.02\% over the best heuristic policy in the literature.

**Keywords:** Control policy, dual-sourcing, lead time

1. Introduction

To reduce costs while maintaining customer service, many firms adopt a dual-sourcing strategy; i.e., they get the bulk of their materials from a cheaper regular supplier with a longer lead time but turn to a premium expedited channel at a higher cost and a shorter lead time when needed (Rao et al., 2000; Veeraraghavan and Scheller-Wolf, 2008; Allon and van Mieghem, 2010). For example, firms may have to decide whether to source from a global supplier that offers preferential pricing but is geographically distant (and is therefore associated with a longer lead time) or to source from a local supplier at a high unit cost.

In this article, we consider a periodic-review dual-sourcing inventory system with general lead times over a finite horizon or an infinite horizon. The firm replenishes its inventory either from a regular supplier with a long lead time but a low unit cost or from an expedited supplier with a short lead time but a high unit cost. Unmet demand is completely backlogged. The objective of the firm is to minimize the expected total discounted cost over the entire planning horizon.

To our knowledge, the optimal policy for the dual-sourcing system with general lead times has been characterized only when an exclusive source is used (Whittemore and Saunders, 1977) or when lead times of the two sources are consecutive (e.g., Yazlali and Erhun (2009) and Sethi et al. (2003)). It is pointed out by Veeraraghavan and Scheller-Wolf (2008, p. 863) that “The dual-sourcing decision under general lead times has been a challenging problem for over 40 years, despite its frequency in practice.”

Because of the complexity of the dual-sourcing decisions under general lead times, little is known to date about basic properties of the optimal policy. In this article, by using the notion of $L^\natural$-convexity, we characterize the structural properties of the optimal orders. We find that the optimal regular order is more sensitive to the late-to-arrive outstanding orders, but the optimal expedited order is more sensitive to the soon-to-arrive outstanding orders. To the best of our knowledge, we are the first to provide the structural properties of the optimal policy for dual-sourcing systems with general lead times. We also design a heuristic policy called the Best Weighted Bounds policy ($\text{BWB policy}$ for short). This policy provides an average cost-saving of 1.02\% over the Best Vector Base-Stock policy ($\text{BVBS policy}$ for short), which is the best heuristic in the literature. For products with very low profit margin products or high volume, multi-item systems, a 1.02\% reduction in cost can be considered significant. As pointed out by Uster et al. (2008), smaller percentages of cost reduction when converted to dollar amounts can also result in substantial annual savings especially for fast-moving items.

The remainder of this article is organized as follows: in Section 2, we review the related literature. In Section 3, we present the problem formulation. In Section 4, we characterize the structural properties of the optimal orders. In Section 5, we design a heuristic policy and numerically...
examine its performance. The paper ends with some concluding remarks in Section 6. All proofs of the results in this article can be found in Supplemental Files.

2. Literature review

The problem we consider is related to the growing literature dealing with dual-sourcing inventory problems. Many publications consider systems where the suppliers are different in terms of their unit costs, fixed costs, order size limits (e.g., Fox et al. (2006), Zhang et al. (2011), Zhang et al. (2012), and Zhang and Hua (2013)), or supply risks (e.g., Tomlin (2006) and Dada et al. (2007)). Here, we focus on the dual-sourcing literature where the suppliers are differentiated based on their unit costs and lead times.

Sheopuri et al. (2010) establish the connection between dual-sourcing systems with general lead times and single-sourcing lost-sales systems. They find that the optimal policy for a single-sourcing lost-sales system is also optimal for the dual-sourcing system where the unmet demand in any period can be fulfilled by an order from the additional supplier. Consequently, we also review the literature on single-sourcing lost-sales systems.

2.1. Single-sourcing lost-sales systems

As stated in Zipkin (2008a), the optimal policy for the single-sourcing lost-sales system is notoriously difficult to characterize. So far, only bounds on the optimal policy have been derived (see Karlin and Scarf (1958) and Morton (1969)). In particular, by using the notion of $L^2$-convexity, Zipkin (2008b) finds an easy way to recover the same bounds.

There is an extensive literature that designs heuristic policies for single-sourcing lost-sales systems. We do not attempt a comprehensive review here (see Zipkin (2008a) for such a review). Instead we focus on recent works. Levi et al. (2008) propose a heuristic policy called the dual-balancing policy, where the order equates the purchasing-plus-holding costs and penalty costs. They prove that the cost of the dual-balancing policy is at most twice the optimal cost. Zipkin (2008a) proposes a class of vector base-stock policies, which compare the system vector consisting of the partial sums of pipeline inventories to a fixed policy vector. This policy results in an order only when the system vector is less than the policy vector in every component; it then orders the minimum difference between the components of the system vector and those of the policy vector.

There is also a stream of literature that deals with the asymptotic optimality of heuristic policies for single-sourcing lost-sales systems. Huh et al. (2009) show that base-stock policies are optimal for single-sourcing lost-sales systems when the lost-sales penalty cost approaches infinity. Goldberg et al. (2012) show the asymptotic optimality of constant-order policies when the lead time approaches infinity.

The techniques and ideas used for the single-sourcing lost-sales systems can be borrowed to deal with dual-sourcing systems with general lead times. Similar to Zipkin (2008b), in this paper we characterize the structural properties of the optimal policy for dual-sourcing systems by using the notion of $L^2$-convexity.

2.2. Dual-sourcing systems with general lead times

Whittemore and Saunders (1977) consider an arbitrary lag between the lead times of two suppliers. They derive a complex stochastic dynamic programming model, and find that the optimal policy is no longer a simple base-stock policy. For multi-sourcing systems with consecutive lead times, Feng et al. (2006) show that the optimal orders from the fastest two suppliers have a base-stock structure, but the remaining ones do not. To date, for dual-sourcing systems with general lead times, little is known about the optimal policy or even the basic properties of the optimal policy.

Rather than finding the complex structure of optimal policies, recent literature has turned to exploring simple heuristic policies for infinite-horizon systems. Scheller-Wolf et al. (2007) propose a class of single-index policies, where both the regular order and the expedited order follow a base-stock policy by using the inventory position. By introducing the expedited inventory position, Veeraraghavan and Scheller-Wolf (2008) extend the work of Scheller-Wolf et al. (2007) and propose a class of dual-index policies, where the expedited order is placed by using the expedited inventory position instead. Sheopuri et al. (2010) generalize the dual-index policies and propose two classes of policies. One class consists of policies that have a base-stock structure for the expedited order and the other class for the regular order. Among the policies mentioned in Sheopuri et al. (2010), the BVBS policy has the best performance and provides an average cost saving of 1.1% over the dual-index policy. In the BVBS policy, the expedited order follows a base-stock policy, and the regular order follows a vector base-stock policy, which is borrowed from that for single-sourcing lost-sales systems (Zipkin, 2008b).

Allon and Van Mieghem (2010) propose a tailored base-surge sourcing policy for continuous-review dual-sourcing systems, where the expedited order follows a base-stock policy, and the regular order is a constant. This heuristic policy is equivalent to the kind of constant-order policies for the dual-sourcing systems studied in Janssen and de Kok (1999).

Based on the above review, we find that the existing literature mainly explores simple heuristic policies for dual-sourcing systems with general lead times. Contrary to the above, we provide the structural properties of the optimal policy for dual-sourcing systems with general lead times.

842
3. Problem formulation

We consider a periodic-review inventory problem of a single item over a finite horizon or an infinite horizon. Inventory can be replenished either from a regular supplier or from an expedited supplier. Denote the unit costs of the regular supplier and the expedited supplier as $c_r$ and $c_e$ ($c_e > c_r$), respectively. Without loss of generality, we assume that the lead times of the expedited order and the regular order are zero and $l$ ($l \geq 1$). By replacing the initial inventory level with the expedited inventory position (see Veeraraghavan and Scheller-Wolf 2008, p. 852), our problem can be easily extended to the case where the expedited lead time is positive.

Demands in different periods are identically and independently distributed. We use a non-negative random variable $D$ to represent the demand in each period. Unmet demand is fully backlogged but incurs a unit backlogging cost $b$ per period. Leftover inventory is carried over to the next period and incurs a unit inventory holding cost $h$. The objective of the firm is to minimize the expected total discounted cost over the entire planning horizon, where the cost in each period is the sum of the ordering cost and the holding/backlogging cost.

We number time periods in reverse order so that period $1$ is the last period of the planning horizon. The sequence of events in period $n$ is as follows:

1. Regular and expedited orders; i.e., $q_r^n$ and $q_e^n$ are placed based on the information in period $n$: the initial inventory level $x_n$ (i.e., the inventory on hand minus the backorder at the beginning of period $n$) and the outstanding orders, $q_{n,t}^r, q_{n,t}^e$.
2. The regular and expedited orders due in period $n$ (i.e., $q_{n+1}^r$ and $q_{n+1}^e$) arrive.
3. Demand $D$ occurs, and a holding or backlogging cost is incurred.

Let $x^0_n = x_n + q_{n+l}^r$ and $z_n = q_{n+l-i}^r$, $i = 1, \ldots, l-1$. Here $x^0_n$ is the sum of the initial inventory level and the regular order due in that period, and $z_n$ is the outstanding order that arrives in period $n - i$ (i.e., $l$ periods later). Then, the regular and expedited orders are decided based on the state of the system in period $n$: i.e., $z_n = (z_n^0, z_n^1, \ldots, z_n^{l-1})$. The state in the next period (i.e., period $n - 1$) is

$$
(z_n^0 + q_e^n - D + z_n^1, z_n^2, \ldots, z_n^{l-1}, q_e^n).
$$

The costs incurred in period $n$ are the ordering costs $c_r q_r^n$ and $c_e q_e^n$, plus the holding/backlogging cost $p(z_n^0 + q_e^n) = b E_D(z_n^0 + q_e^n - D)^+ + h E_D(D - z_n^0 - q_e^n)^+$, where $u^+ = \max(u, 0)$. Note that $p(\cdot)$ is a convex function. In the following analysis, we omit the subscript $n$ in the notation when there is no confusion.

For the finite-horizon discounted cost problem with $N + l$ periods, let $f_n(z)$ be the optimal (expected discounted) cost from period $n$ to the end of the planning horizon, given the initial state $z$. Without loss of generality, we assume that $f_0(z) \equiv 0$. It is easy to verify that $f_n(z)$ satisfies the following recursion equation:

$$
f_n(z) = \min_{q_r^n, q_e^n \geq 0} \left\{ c_r q_r^n + c_e q_e^n + p(z_n^0 + q_e^n) + \alpha E_D f_{n-1} \right\},
$$

where $\alpha \in (0, 1)$ is the discount factor, and the last term $f_{n-1}(\cdot)$ is the optimal cost in the following $n - 1$ periods.

For the infinite-horizon discounted cost problem, let $g(z)$ be the optimal stationary cost function. If $g(z)$ exists, then it satisfies the following functional equation:

$$
g(z) = \min_{q_r^n, q_e^n \geq 0} \left\{ c_r q_r^n + c_e q_e^n + p(z_n^0 + q_e^n) + \alpha E_D g \right\}.
$$

In Subsection 4.4, we will show that when $n$ approaches infinity, $f_n(z)$ converges absolutely and uniformly to a function that satisfies functional equation (3) when the demand over the entire planning horizon is finite.

Denote by $\hat{g}_n^r(z)$ ($\hat{g}_n^e(z)$) and $\hat{g}_n^{re}(z)$ the optimal regular and expedited orders for the finite (infinite) horizon problem, respectively. We assume that $\hat{g}_n^r(z)$ and $\hat{g}_n^{re}(z)$ are differentiable with respect to each component of the state $z$. Define “the sensitivity of the optimal regular (expedited) order” as the partial derivatives of the optimal regular (expedited) order with respect to the components of the state $z$ i.e., $\partial \hat{g}_n^r(z)/\partial z_i$ ($\partial \hat{g}_n^{re}(z)/\partial z_i$), $i = 0, \ldots, l - 1$. Intuitively, the sensitivity measures how the optimal regular (expedited) order is affected by the state $z$. In the following discussion, we characterize the sensitivities of and the bounds on the optimal orders. The sensitivities and bounds are helpful to construct a heuristic policy and reduce the decision space and the state space in computing the optimal solutions for the problem instances where the lead time is small and the support of demand is limited to a handful of integers (e.g., $\{0, 1, 2, 3, 4\}$).

4. Structural properties

In this section, we first introduce the notion of $L^2$-convexity, then characterize the structural properties of the optimal orders in the finite-horizon discounted cost problem, and finally extend our results to the infinite-horizon problem.

4.1. Preliminaries

$L^2$-convexity is defined based on submodularity. A function $f : V \to \mathbb{R}$ is submodular if

$$
f(x) + f(y) \geq f(x \wedge y) + f(x \vee y)
$$

for all $x = (x^1, \ldots, x^d)$, $y = (y^1, \ldots, y^d) \in V$, where $x \wedge y = \min\{x^1, y^1\}, \ldots, \min\{x^d, y^d\}$ and $x \vee y = \max\{x^1, y^1\}, \ldots, \max\{x^d, y^d\}$. Let $e = (1, \ldots, 1)$ with an
appropriate length. According to Zipkin (2008b), $L^2$-convexity is defined as follows.

**Definition 1:** A function $f : V \rightarrow \mathbb{R}$ is $L^2$-convex if $g(x, \eta) = f(x - \eta e)$ is submodular on $V \times \mathbb{R}^l$.

Based on $L^2$-convexity, Zipkin (2008b) proposes the following helpful results (throughout this article, we use the terms increasing and decreasing both in the non-strict sense).

**Lemma 1** (Zipkin, 2008b). Let $\eta(x)$ be the largest value of $\eta \leq 0$ that minimizes a $L^2$-convex function $g(x, \eta)$. Then, $\eta(x)$ is increasing in $x$, but $\eta(x + u e) \leq \eta(x) + u$ for $u > 0$.

Lemma 1 shows that $\eta(x)$ is monotone in the vector $x$, with limited sensitivity. In the following Subsections 4.2 and 4.3, we will use it to characterize the sensitivities of the optimal regular orders.

Based on Lemma 1, we can derive the following lemma:

**Lemma 2.** For any $\tilde{x}$ and $x$, if we let $\delta_{\text{max}} = \max_{k=1,...,n}(\tilde{x}^k - x^k)$ and $\delta_{\text{min}} = \min_{k=1,...,n}(\tilde{x}^k - x^k)$, then $\eta(\tilde{x}) - \max(0, \delta_{\text{max}}) \leq \eta(x) \leq \eta(\tilde{x}) - \min(0, \delta_{\text{min}})$.

As shown in Lemma 2, the bounds on $\eta(x)$ can be derived based on the optimal $\eta$ at another vector $\tilde{x}$ and the maximum and minimum differences between the components of $x$ and $\tilde{x}$. In the following analysis, we will apply Lemma 2 to derive bounds on the optimal orders.

Based on $L^2$-convexity and Lemmas 1 and 2, we next characterize the structural properties of the optimal orders. To this end, we consider two kinds of state transformations in the subsequent analysis. Our sensitivities are derived in complete analogy to the approach taken in Zipkin (2008b). This is expected to be possible due to the connection between dual-sourcing systems and single-sourcing lost-sales systems established by Sheopuri et al. (2010).

### 4.2. Structural properties of the optimal regular order

To study the structural properties of the optimal regular order, we use the following state transformation. Let

$$v = (v^0, v^1, \ldots, v^{l-1})$$

where $v^i = \sum_{k=i}^{l-1} z^k, i = 0, \ldots, l-1$.

Here $v^0$ is the usual inventory position, and $v^i$ includes the orders scheduled to arrive in periods hence or later. There is a one-to-one correspondence between $v$ and $z$ because $z^i = v^i - v^{i+1}$ for $i = 0, \ldots, l-2$, and $z^l = v^l$.

Under this state transformation, Equation (1) implies that the state in the next period is

$$v^{*} = (v^0 + q^* - D, v^2, \ldots, v^{l-1}, 0 + q^* e).$$

Given the new state $v$, define $f^n(v)$ ($f^n(v) \equiv 0$) as the optimal cost when there are $n$ periods remaining. From Model (2), $f^n(v)$ satisfies the following recursion equation:

$$f^n(v) = \min_{q^* \geq 0, q^* \geq 0} \{c e q^* + c^r q^* + p(v^0 - v^1 + q^*) + \alpha E_D f_{n-1}^{v^*} \times [(v^0 + q^* - D, v^2, \ldots, v^{l-1}, 0 + q^* e)].$$

Let $\tilde{q}^*_n(v)$ be the optimal regular order, given the transformed state $v$. Based on Model (5) and Lemmas 1 and 2, we have the following results.

**Proposition 1.** For the finite-horizon discounted cost problem, $f^n(v)$ is $L^2$-convex for all $n$, and the following results hold:

i. $\tilde{q}^*_n(v)$ is decreasing in $v$, but $\tilde{q}^*_n(v + u e) \geq \tilde{q}^*_n(v) - u$ for $u > 0$. The sensitivity of the regular order is $-1 \leq \partial \tilde{q}^*_n(v)/\partial z^l - 1 \leq -\partial \tilde{q}^*_n(v)/\partial z^l \leq \partial \tilde{q}^*_n(v)/\partial z^l \leq 0$.

ii. There exist parameters $s^r_0, \ldots, s^r_l$ and $s^l_0 \geq 0$ so that for all $v$:

$$\min[s^r_0, \min_{i=0,\ldots,l-1}(s^r_i - v^i) + s^l_0] \leq \tilde{q}^*_n(v) \leq \max(s^r_l, \max_{i=0,\ldots,l-1}(s^r_i - v^i)).$$

In words, Proposition 1(i) states that the optimal regular order is a decreasing function of each component of the state $z$, and that the rate of the decrease is smaller than one. Furthermore, the optimal regular order is more sensitive to the late-to-arrive outstanding orders. Proposition 1(ii) shows that the lower (upper) bound on the optimal regular order can be described by the minimum (maximum) of the respective differences between the components of $(v, 0)$ and a vector $(s^r_0, \ldots, s^r_l)$. Note that the optimal regular order $\tilde{q}^*_n(v)$ at $v = (s^r_0 - s^r_1, \ldots, s^r_l - s^r_0)$ is $s^r_0$.

The sensitivity of the regular order can be explained as follows. The optimal regular order $\tilde{q}^*_n(z)$ is placed to meet the demands after period $n - l$ (i.e., $l$ periods later). Clearly, the initial inventory level in period $n - l$ is increasing in not only $z^0, \ldots, z^l$ but also in the expedited orders placed from period $n$ to period $n - l + 1$. Thus, $z^0, \ldots, z^l$ have only limited negative impact on the optimal regular order. Due to the expedited orders placed from period $n - i$ (in which the outstanding order $z^i$ arrives) to period $n - l + 1$, $\tilde{q}^*_n(z)$ is more sensitive to the late-to-arrive outstanding orders.

### 4.3. Structural properties of the optimal expedited order

We next define another kind of state transformation to study the structural properties of the optimal expedited order. Let

$$w = (w^0, w^1, \ldots, w^{l-1}),$$

where

$$w^i = \sum_{k=0}^{i} z^k, i = 0, \ldots, l-1.$$\n
Here $w^0 = z^0$ is the sum of the initial inventory level and the regular order due in period $n$, and $w^{l-1}$ is the usual inventory position. There is a one-to-one correspondence between $w$ and $z$ because $z^0 = w^0$ and $z^i = w^i - w^{i-1}$ for $i = 1, \ldots, l-1$.

With the new state $w$, Equation (1) suggests that the state in the next period is

$$w^{*} = (w^1, w^2, \ldots, w^{l-1} + q^* - (\xi + D)e),$$

where $\xi = -q^*$.\n

Optimal policy for dual-sourcing systems

4.4. Infinite-horizon problem

In this subsection, we assume that the demand in the infinite-horizon systems is finite i.e., \( \Pr(D \leq d_{\text{max}}) = 1 \), where \( d_{\text{max}} \) is the maximum of the demand realizations over the entire planning horizon. Under this mild assumption, we have the following results.

**Proposition 3.** There exists a stationary policy that is optimal for the infinite-horizon discounted cost problem with finite demand; the limit of any convergent subsequence of the optimal finite-horizon policies is such a policy.

Here we sketch the proof of this proposition. We first prove that the optimal finite-horizon cost function \( f_n(z) \) converges monotonically and uniformly to a stationary function \( f(z) \) as \( n \) approaches \( \infty \) and then show that \( f(z) \) satisfies the functional equation (3). This implies that there exist some convergent subsequences in the optimal finite-horizon policies, and the limit of any convergent subsequence is optimal for the infinite-horizon discounted cost problem.

In addition, it is easy to verify that the stationary function \( f(z) \) is also \( L^i \)-convex. Therefore, the optimal orders in the infinite-horizon discounted cost problem; i.e., \( q^e(z) \) and \( q^c(z) \) have the same structural properties as shown in Propositions 1 and 2.

It is known that the optimal policies under the infinite-horizon discounted cost model converge to an optimal policy under the infinite-horizon average cost model as the discount factor \( \alpha \) approaches one (Schäl, 1993; Sheopuri et al., 2010). Hence, Proposition 3 also holds for the infinite-horizon average cost problem.

5. Heuristic policy

In this section, we develop a heuristic policy called the *BWB* policy and numerically examine its performance. We use the infinite-horizon average cost as our performance measure because it is just a number, whereas the discounted cost is a function that depends on the starting state. We next formulate the infinite-horizon average cost. Let \( I_\infty \) be the stationary distribution of the inventory level after receiving the regular and expedited orders but before demand realization, and \( Q^c_\infty \) be the stationary distribution of the expedited order. The infinite-horizon average cost (we assume without loss of generality that \( c^e = 0 \) as in Sheopuri et al. (2010)) is computed as

\[
c^e E(Q^c_\infty) + hE(I_\infty - D) + bE(D - I_\infty). \tag{8}
\]

The first term is the average ordering cost of expedited orders, and the last two terms are the average holding and backlogging costs, respectively.

In the following discussion, we first construct the *BWB* policy, and then numerically examine its performance.

5.1. *BWB* policy

In the *BWB* policy, we will assume that the expedited orders follow a base-stock policy with a target level of \( S^e \); i.e., \( q^e = (S^e - s^o)_+ \) and will use a weighted average of the upper
and lower bounds on the regular orders to determine the regular orders.

Before introducing the BWB policy, we first show why we assume a base-stock policy for the expedited orders. If we were to apply Proposition 2 to develop a heuristic policy with some parameters for the expedited order, then we would have to first simulate \( I_{\infty} \) under all possible values of these parameters and then find the optimal values of these parameters to minimize the cost in Equation (8). Due to the computational complexity, we assume that \( q^e = (S^e - z^0)^+ \). In this case, the optimal value of \( S^e \) can be directly computed by applying a newsvendor formula. The details are shown as follows.

Given that \( q^e = (S^e - z^0)^+ \), \( I_{\infty} \) can be expressed as \( S^e + O V_{\infty} \), where \( O V_{\infty} \) represents the stationary distribution of the overshoot \((z^0 - S^e)^+ \) (\( z^0 \) may exceed \( S^e \) due to the regular order). Therefore, the infinite-horizon average cost in Equation (8) can be rewritten as

\[
c^e E[\max(0, D - Y_{\infty})] + h E(S^e + O V_{\infty} - D)^+ + b E(D - S^e - O V_{\infty})^+, \tag{9}
\]

where \( Y_{\infty} \) is the stationary distribution of \( Y = O V + z^1 \). Sheo puri et al. (2010) find that \( Y_{\infty} \) and \( O V_{\infty} \) do not depend on the choice of parameter \( S^e \). Consequently, the optimal \( S^e \) can be directly computed by applying the following newsvendor formula:

\[
\Pr(S^e \geq D - O V_{\infty}) = b/(b + h). \tag{10}
\]

We next study the structural properties of the optimal regular order given that \( q^e = (S^e - z^0)^+ \). When \( q^e = (S^e - z^0)^+ \), it is sufficient to consider the policies where the regular order depends on the vector \( z = (O V, z^1, \ldots, z^{l-1}) \) (Sheo puri et al., 2010). Denote by \( q^n(z) \) the optimal regular order, given that \( q^e = (S^e - z^0)^+ \). We also assume that \( q^n(z) \) is differentiable with respect to each component of the vector \( z \). By applying the technique used in Section 4, we have the following proposition.

**Proposition 4.** For the infinite-horizon (discounted cost or average cost) problem given that \( q^e = (S^e - z^0)^+ \), the following results hold:

i. the sensitivity of \( q^n(z) \) is \(-1 \leq \partial q^n(z)/\partial z_i \leq \partial q^n(z)/\partial z_{i-1} \leq \cdots \leq \partial q^n(z)/\partial z_0 \leq 0 \).

ii. there exist parameters \( \bar{s}^0, \ldots, \bar{s}^{l-1} \), and \( \bar{s}^l \geq 0 \), so that for all \( z \):

\[
\begin{align*}
\min(\bar{s}^i, \min_{i=0, \ldots, l-1}(\bar{s}^{i+1} - \bar{v}^i)^+) & \leq q^n(z) \leq \max(\bar{s}^l, \max_{i=0, \ldots, l-1}(\bar{s}^{i+1} - \bar{v}^i)),
\end{align*}
\]

where \( \bar{v}^i = \sum_{k=i}^{l-1} \bar{s}^k, i = 1, \ldots, l-1 \) and \( \bar{v}^0 = \bar{v}^l = 1 + O V \).

Proposition 4(i) shows that the partial derivatives of the optimal regular order \( q^n(z) \), with respect to \( O V \) and \( z^1, \ldots, z^{l-1} \), are negative but greater than \(-1 \). The more recent an outstanding order is, the more sensitive \( q^n(z) \) is to that order. Sheo puri et al. (2010) use this sensitivity without proof to design the weighted dual-index policy.

Proposition 4(ii) shows that the lower and upper bounds on \( q^n(z) \) can be described based on an \((l + 1)\)-dimensional vector \((\bar{s}^0, \ldots, \bar{s}^l)\). Similar to Zipkin (2008a), we approximate \( \bar{s}^i \) (\( i = 1, \ldots, l \)) by \( F_{1+i-1}(\theta) \) (i.e., the \((1 - \theta)\) fractile of the distribution of \((l + 1 - i)\) periods' demand), where \( \theta \) is a parameter. This reduces the \((l + 1)\)-dimensional search over \( \bar{s}^0, \ldots, \bar{s}^l \) to a one-dimensional search over \( \theta \). Then,
the bounds on the regular order for the regular order. That is, 
\[ q^r = (S^r - z^0)^+ \] and 
\[ q^r = \omega LB(\theta) + (1 - \omega) UB(\theta), \]
where \( \omega \in [0,1] \) is a weight. According to Lemma 1 of Loynes (1962), \( \gamma(= OV + z^i) \) and \( OV \) in the \( BWB \) policy converge to stationary distributions. \( UB(\theta) \) and \( LB(\theta) \) by applying Equation (10) and then find the optimal \( \theta \) that minimizes the cost in Equation (9). By comparing the minimum costs under the \( m \) values of \( \omega \), we finally obtain the optimal \( \omega \).

In the literature, the best heuristic policy for the dual-sourcing problem with general lead times is the \( BWB \) policy. Using our notation, the expedited order and regular order in the \( BWB \) policy are \( q^c = (S^c - z^0)^+ \) and \( q^r = LB(\theta) \) with optimal parameters \( S^c \) and \( \theta \), respectively. It is expected that the \( BWB \) policy performs as good as or better than the \( BWB \) policy because the \( BWB \) policy just covers the lower bound in the \( BWB \) policy and thus is a special case of our \( BWB \) policy with \( \omega = 1 \).

5.2. Numerical experiments

In this subsection, we conduct numerical experiments to compare the performance of the \( BWB \) policy with the \( BWB \) policy. Demand distributions and parameters of our numerical experiments are set as follows. We consider the same demand distributions as in Sheopuri et al. (2010); i.e., the Geometric (0.5), Geometric (0.4), and Normal (3, 1) distributions. The lead time \( l \) of regular orders varies from two to four in increments of one. By setting \( h = 5 \), we use five values of \( b \): 15, 20, 85/3, 45, and 95, corresponding to \( b/(h + b) = 75, 80, 85, 90, \) and 95%. As stated earlier, we assume without loss of generality that \( c^c = 0 \) from Sheopuri et al. (2010). We use three values of \( c^c \): 10, 20, and 30.

If \( c^c \geq b \), then backlogging the demands for \( l \) periods is better than placing expedited orders, and the dual-sourcing system degenerates into the single-sourcing system with the regular supplier. Thus, for each demand distribution, the problem instance with \((l, c^c, b) = (2, 30, 15)\) is excluded from \(3 \times 5 \times 3 = 45\) possible problem instances because \( c^c = b \).

The computational time required to find the \( BWB \) policy strongly depends on the discretization of the weight \( \omega \). By optimizing \( \omega \) over six levels (i.e., 0.0, 0.2, 0.4, 0.6, 0.8, and 1) in each problem instance, the computational time for the
The BVBS policy is about 120 seconds, which is six times that required for the BVBS policy. Recall that the BVBS policy is a special case of the BWB policy with \( \omega = 1 \). If one uses a finer scheme, such as using 11 levels, the computational time increases.

For each problem instance, we examine the percentage improvement in the infinite-horizon average cost of the BWB policy over that of the BVBS policy,

\[
P_I = \frac{(BVBS - BWB)}{BVBS} \times 100%.
\]

To evaluate whether the BWB policy is valuable, it is best to compare the cost of the BWB policy and the optimal cost. However, the optimal cost is computationally feasible only for the problem instances when the lead time \( l \) is small and the support of the demand is limited to a handful of integers. Similar to Sheopuri et al. (2010), we use the optimal cost of the dual-sourcing system with \( l = 1 \) as a cost lower bound (denoted by CLB) to examine the gap between the cost of the BWB policy and CLB:

\[
GAP = \frac{(BWB - CLB)}{CLB} \times 100%.
\]

Table 3. Performance of BWB policy for the Normal (3, 1) distribution

| \( l \) | \( c^e \) | \( b \) | BVBS | BWB | PI% | CLB | GAP% | VR |
|---|---|---|---|---|---|---|---|---|
| 2 | 10 | 15 | 10.1 | 10.0 | 0.03 | 8.7 | 15.11 | 0.40 |
| 3 | 10 | 15 | 11.3 | 11.2 | 0.77 | 8.7 | 28.00 | 0.22 |
| 4 | 10 | 15 | 12.0 | 11.8 | 1.11 | 8.7 | 35.52 | 0.13 |
| 2 | 10 | 20 | 10.9 | 10.9 | 0.30 | 9.5 | 14.28 | 0.40 |
| 3 | 10 | 20 | 12.0 | 11.9 | 1.18 | 9.5 | 25.07 | 0.22 |
| 4 | 10 | 20 | 12.6 | 12.4 | 1.71 | 9.5 | 30.46 | 0.04 |
| 2 | 10 | 85/3 | 11.9 | 11.8 | 0.47 | 10.4 | 13.43 | 0.24 |
| 3 | 10 | 85/3 | 12.8 | 12.5 | 2.50 | 10.4 | 19.98 | 0.10 |
| 4 | 10 | 85/3 | 13.4 | 13.1 | 2.10 | 10.4 | 25.89 | 0.04 |
| 2 | 10 | 45 | 13.2 | 13.0 | 0.89 | 11.6 | 12.17 | 0.23 |
| 3 | 10 | 45 | 14.1 | 13.7 | 2.60 | 11.6 | 18.04 | 0.10 |
| 4 | 10 | 45 | 14.9 | 14.6 | 2.38 | 11.6 | 23.17 | 0.04 |
| 2 | 10 | 30 | 12.2 | 12.0 | 1.72 | 9.6 | 11.84 | 0.27 |
| 3 | 10 | 30 | 13.2 | 12.9 | 2.99 | 9.6 | 17.78 | 0.17 |
| 4 | 10 | 30 | 15.8 | 15.6 | 1.40 | 9.6 | 23.04 | 0.10 |
| 2 | 10 | 85/3 | 16.7 | 16.4 | 1.73 | 13.3 | 22.86 | 0.04 |
| 3 | 10 | 85/3 | 17.4 | 17.1 | 1.52 | 13.3 | 28.35 | 0.17 |
| 4 | 10 | 85/3 | 19.5 | 19.1 | 1.73 | 13.5 | 33.04 | 0.10 |

It can be observed from Tables 1 to 3 that the percentage improvement is not monotone in the lead time. Interestingly, we find that, as the lead time increases, almost all of the regular orders become less volatile (please see the column VR in Tables 1 to 3). This is expected from the asymptotic optimality of constant-order policies as the lead time approaches infinity (Goldberg et al., 2010).
6. Concluding remarks

In this article, we study a periodic-review dual-sourcing inventory problem with general lead times. Procuring from the regular supplier involves a long lead time but a low unit cost, whereas procuring from the expedited supplier involves a negligible lead time but a high unit cost. Unmet demand is backlogged. The objective of the firm is to minimize the expected total discounted cost over the entire planning horizon.

By using the notion of $L^1$-convexity, we characterize the sensitivities of, and the bounds on, the optimal orders. We find that the optimal regular order is more sensitive to late-to-arrive outstanding orders, but the optimal expedited order is more sensitive to soon-to-arrive outstanding orders. We also propose a heuristic policy called the BWB policy, which provides an average cost saving of 1.02% over the BVBS policy proposed in Sheopuri et al. (2010).

Our results can be further extended to systems with stochastic lead times and/or Markov-modulated demand. When one or two suppliers are capacitated, the results on the sensitivities of the optimal regular and expedited orders in this paper remain valid but the bounds on optimal orders and the heuristic policy should be modified. However, the dual-sourcing inventory problem when one or two suppliers have non-zero fixed ordering costs will have different structural properties, which deserves future research effort.

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Supplemental Material

Supplemental data for this article can be accessed on the publisher's website.

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