A NEW APPROACH TO CHARACTERISE ALL THE TRANSITIVE ORIENTATIONS FOR AN UNDIRECTED GRAPH

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Abstract

A new approach to find all the transitive orientations for a comparability graph (finite or infinite) is presented. This approach is based on the link between the notion of “strong” partitive set and the forcing theory (notions of simplices and multiplices). A mathematical algorithm is given for the case of a comparability graph which has only non limit sub-graphs.

Key-Words : Graph Theory, Forcing Theory, Comparability Graphs, Transitive Orientations, Partitive Sets.

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1 Introduction

The problem of the transitive orientation for a comparability graph was studied by Golumbic using the forcing theory [1]. The problem was solved for finite comparability graphs and an algorithm was given which gives one transitive orientation for a finite comparability graph.

The purpose of this paper is to study the transitive orientation for the case of infinite comparability graphs. The results for the finite case could not be extended to the infinite case because of the finite type like of the approach used in Ref.[1]. We were then obliged to consider a new approach, but remaining within the forcing theory. This was possible by introducing the notion of a “strong” partitive set. It happens that this idea permits to solve the problem of the transitive orientations by inducing a lot of characteristics of the forcing theory, from one hand and the undirected graphs, in general, from the other hand.

The paper is organized as follows: the section 2 is devoted to the definitions and notations used throughout the paper and also to recall the main results of the forcing theory which still valid for infinite graphs. In section 3, we establish narrow links between the notion of a “strong” partitive set and simplices (and multiplices) which are the principal tools of the forcing theory. In section 4, we used the results of section 3 to face the problem of transitive orientation of comparability graphs. We proved a theorem which is in fact a mathematical algorithm which gives all the transitive orientations for a comparability graph which has all its sub-graphs non limit.

2 Preliminairies

This section is devoted to the definitions and notations which will be used throughout this article. We also recall some results about the partitive sets and the implication classes. These results can be found with more details in Ref.[1].

We consider here any kind of graphs; finite or infinite. In what follows we denote by $G = (V, E)$ any graph, where $V$ is the set of vertices, and by $E(\subseteq V^2)$ the set of edges. Directed edge will be denoted by $(a, b)$ (for $a, b \in V$) and an undirected one is denoted by $ab, ab = \{(a, b), (b, a)\}$. We say that a graph $G = (V, E)$ is empty if $E = \emptyset$. 

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For any \( X \subseteq V \), \( G(X) = (X, E(X)) \) will denote the sub-graph of \( G \) induced on \( X \); where
\[
E(X) = \{(a, b) \in E \mid \{a, b\} \subseteq X\}
\]
we define also the set of vertices \( \tilde{A} \) spanned by a set of edges \( A \) as
\[
\tilde{A} = \{a \in V; \text{ there exists } b \in V \text{ so that } (a, b) \in A \text{ or } (b, a) \in A\}
\]

2.1 Partitive set and “strong” partitive set

Let \( \cong \) be a binary relation acting on \( V^2 \) defined by
\[
(a, b) \cong (c, d) \iff (\{(a, b), (c, d)\} \cap E = \emptyset \text{ or } \{(a, b), (c, d)\} \subseteq E)
\]
This means that the edges \( (a, b) \) and \( (c, d) \) are both belonging to \( E \) or are both out of \( E \).

Let \( X \subseteq V \) be a sub-set of \( V \). \( X \) is a partitive set of \( G \) (or \( V \)) if:

for every \( \{a, b\} \subseteq X \) and every \( c \in V - X \)

we have \( (a, c) \cong (b, c) \) and \( (c, a) \cong (c, b) \)

It means that the elements of \( X \) are related to any external element in the same manner. The notion of the partitive set is the analogue of the notion of interval in an ordered set. We will denote by \( I(G) \) the class of partitive sets of the graph \( G \). A partitive set is trivial if it is a singleton or equal to \( V \). By \( I^\ast(G) \) we will denote the class of non-trivial partitive sets.
A graph is indecomposable if all its partitive sets are trivial; otherwise, it is decomposable.

A partitive set \( X \in I(G) \) is called a “strong” partitive set of \( G \) (or \( V \)) if for every partitive set \( Y \in I(G) \) so that \( Y \cap X \neq \emptyset \), we have either \( X \subseteq Y \) or \( Y \subseteq X \). It means that the eventuality that \( X \) and \( Y \) shear only a common part is excluded.

We will denote by \( I_F(G) \) the class of “strong” partitive sets of \( G \) and by \( I_F^\ast(G) \) the class of the non-trivial “strong” partitive sets. We say that \( G \) is limit if \( I_F(G) \) do not contain any element different from \( V \) which is maximal for the inclusion(\( \subseteq \)); otherwise it is non limit.
**Isomorphism:** Two graphs \( G = (V, E) \) and \( G' = (V', E') \) are said to be isomorphic if:

(i) there is a bijection \( f : V \to V' \);

(ii) \( f \) preserves the edges, i.e., for every \( \{a, b\} \subseteq V : \)

\[
(a, b) \in E \iff (f(a), f(b)) \in E'.
\]

**Quotient graph:** Let \( G = (V, E) \) be a graph and \( P \) a partition of \( G \) made of partitive sets \( (P \subseteq I(G)) \). We define the quotient graph of \( G \) by \( P \), denoted by \( G/P \), as the isomorphic graph to \( G\bigl(f(P)\bigr) \), where \( f \) is a choice function from \( P \) to \( V \), i.e., \( X \in P \Rightarrow f(X) \in X \).

**Proposition 2.1** Let \( P \) be a partition of partitive sets (respectively of “strong” partitive sets) of \( G = (V, E) \) and \( X \) a sub-set of \( P \). We have then \( X \in I(G/P) \) (respectively \( X \in I_F(G/P) \)) if and only if: \( \cup X \in I(G) \) (respectively \( \cup X \in I_F(G) \)), where \( \cup X \) means the union of the vertices constituting the partitive sets (respectively the “strong” partitive sets) of \( X \).

### 2.2 Implication classes and simplices

**Comparability graph (or transitively orientable graph):** Let \( G = (V, E) \) be an undirected graph. \( G \) is a comparability graph if there exists an orientation of the edges of \( G \) so it constitutes a partial order on \( V \).

Comparability graphs are also known as transitively orientable graphs or partially orderable graphs.

**Implication classes:** Let us define the binary relation \( \Gamma \) on the edges of an undirected graph \( G = (V, E) \) as follows:

\[
(a, b) \Gamma (a', b') \Leftrightarrow \begin{cases} 
\text{either } a = a' \text{ and } bb' \notin E \\
\text{or } b = b' \text{ and } aa' \notin E.
\end{cases}
\]

We say that \( (a, b) \) directly forces \( (a', b') \) whenever \( (a, b) \Gamma (a', b') \). In graphical representation \( ab \) and \( a'b' \) will have only one common vertex and the orientation is so that arrows are both pointed to the extremities or are both pointed to the common vertex (see Figure 2.1).

Notice that \( \Gamma \) is not transitive. In Figure 2.1 \( (a, d) \) forces directly \( (a, b) \) and \( (a, c) \). The edges \( (b, c) \) and \( (c, b) \) are not forced by any other edge.
Let $\Gamma^*$ be the transitive closure of $\Gamma$. It is easy to verify that $\Gamma^*$ is an equivalence relation on $E$ and the equivalence classes related to $\Gamma^*$ are what one call the implication classes of $G$.

In what follows we will see that it is useful to define what one call color classes of $G$ (or shortly colors of $G$). If $A$ is an implication class, the color class associated to $A$ and denoted by $\hat{A}$, is the union of $A$ and $A^{-1}$ ($\hat{A} = A \cup A^{-1} \subseteq E$); where $A^{-1} = \{(a, b) \in E \text{ with } (b, a) \in A\}$.

**Theorem 2.2** *(Golumbic Ref.[1]*) Let $G = (V, E)$ be a comparability graph and $A$ an implication class of $G$. If $O = (V, E')$ is a partial order associated to $G$, we have necessarily either $E' \cap \hat{A} = A$ or $E' \cap \hat{A} = A^{-1}$ and, in either case, $A \cap A^{-1} = \emptyset$.

**Lemma 2.3** *(The Triangle Lemma).* Let $A, B$ and $C$ be implication classes of an undirected graph $G = (V, E)$ with $A \neq B$ and $A \neq C^{-1}$ and having edges $(a, b) \in C$, $(a, c) \in B$ and $(b, c) \in A$, we have then

(i) $(b', c') \in A \Rightarrow ((a, b') \in C \text{ and } (a, c') \in B)$;
(ii) $((b', c') \in A \text{ and } (a', b') \in C) \Rightarrow (a', c') \in B$;
(iii) $a \not\in \hat{A}$.

The following results are consequences of the triangle lemma.

**Figure 2.2 Illustration of the triangle lemma.**

**Theorem 2.4** *(Golumbic Ref.[1]*) Let $A$ be an implication class of an undirected graph $G = (V, E)$. Exactly one of the following alternatives holds:

(i) $A = A^{-1} = \hat{A}$ and $\hat{A}$ is not transitivity orientable;
(ii) $A \cap A^{-1} = \emptyset$ and then $A$ and $A^{-1}$ are two transitive orientations of $\hat{A}$.
Proposition 2.5 Let $X$ be a partitive set ($X \in I(G)$) and $\widehat{A}$ a color class of an undirected graph $G = (V, E)$ so that $E(X) \cap \widehat{A} \neq \emptyset$ we have then $\widehat{A} \subseteq E(X)$.

Proposition 2.6 If $\widehat{A}$ is a color class of $G = (V, E)$ then $\widetilde{A}$ is a partitive set of $G$ ($A \in I(G)$).

Simplex: Let $G = (V, E)$ be an undirected graph. A $K_{r+1}$ complete subgraph of $G$, $S = (V_S, E_S)$ on $r + 1$ vertices is called a simplex of rank $r$ if each undirected edge $ab$ of $E_S$ is contained in a different color class of $G$. A simplex is maximal if it is not properly contained in any larger simplex.

The multiplex $M$ generated by a simplex $S$ of rank $r$ is defined to be the part of $E$ constituted of all edges which their color classes are present in the simplex $S$ ($M(S) = \cup_{\hat{A} \subseteq E \neq \emptyset} \hat{A}$). $M$ is said also a multiplex of rank $r$. The multiplex $M$ is said to be maximal if $S$ is maximal. We will denote by $\hat{M}$ the collection of color classes present in the multiplex $M$.

3 Connection between multiplices and “strong” partitive sets

In this section we make connection between the notion of “strong” partitive set and multiplices. In our knowledge this was never made before. It happens that this connection allows us to recover results of Golumbic [1] for finite graphs and generalize them to the infinite graphs. The results presented in this section will be used in the following section to state a theorem on the decomposability of undirected graphs. In what follows $G$ will denote an undirected graph unless other mention is pointed out.

Proposition 3.1 If $\widehat{A}$ and $\widehat{B}$ are two color classes of $G = (V, E)$ we have then $(\widehat{A} = \widehat{B}) \iff (\widetilde{A} = \widetilde{B})$.

Proof. It is obvious that $\widehat{A} = \widehat{B} \Rightarrow \widetilde{A} = \widetilde{B}$. Let us suppose that $\widehat{A} = \widehat{B}$ and $\widehat{A} \neq \widehat{B}$ we have then for every $x \in \widehat{A}$ there exists $a \in \widehat{A}$ and $b \in \widehat{B}$ so that $ax \in \widehat{A}$ and $bx \in \widehat{B}$. Since $\widehat{A} \neq \widehat{B}$ we have necessarily $ab \in E$. Let $C$ be the color class which contains $ab$. We have then two alternatives:
Proposition 3.2 Let $\hat{G}$ be a color class of $G = (V, E)$ and $X$ a partitive set of $G$ so that $X \subset \hat{A}$, there exists then $a \in \hat{A} - X$ so that for every $x \in X$, $ax \in \hat{A}$.

Proof. Let $x \in X \subset \hat{A}$, there exists $a \in \hat{A}$ so that $ax \in \hat{A}$. Necessarily $a \in \hat{A} - X$ (otherwise $(ax \in \hat{A} \cap E(X)) \Rightarrow \hat{A} \subseteq E(X)$ [P.2.5]) then for every $x \in X$ we have $xa \in E$, since $X \in I(G)$. Let us assume that there exists $y \in X$ and a color $B \neq \hat{A}$ so that $ay \in B$, then $xy \in E$. Let $\hat{C}$ be the color which contains $xy$. We have $\hat{C} \neq \hat{A}$ and $\hat{C} \neq \hat{B}$ (otherwise: ($\hat{C} = \hat{A} \Rightarrow \hat{A} \subseteq X)$ and ($\hat{C} = \hat{B} \Rightarrow \hat{B} \subseteq X)$) which contradicts the fact that $a \in (\hat{A} \cap \hat{B}) - X$. Then using the triangle lemma we will have $y \notin \hat{A}$ which is absurd. Finally we have for every $x \in X, ax \in \hat{A}$.

Theorem 3.3 Let $\hat{A}$ and $\hat{B}$ be two color classes of $G = (V, E)$ so that $\hat{A} - \hat{B} \neq \phi$ and $\hat{B} - \hat{A} \neq \phi$. We have then that $X = \hat{A} \cap \hat{B}$ is a “strong” partitive set of $G$.

Proof. Since $\hat{A}$ and $\hat{B}$ are partitive sets[P.2.6] we have that $X = \hat{A} \cap \hat{B}$ is a partitive set. If $X = \phi$ or $X$ is a singleton ($|X| = 1$) then $X$ is a “strong” partitive set.

Let us suppose now that $|X| > 1$. Let $Y$ be a partitive set ($Y \in I(G)$) so that $X \cap Y \neq \phi$ and $Y - X \neq \phi$ and let $z \in X \cap Y$ and $y \in Y - X$. We have to show that $X \subset Y$. Applying [P.2.2] we have:

- $X \subset \hat{A} \Rightarrow$ there exists $a \in \hat{A} - X$ so that for every $x \in X$, $ax \in \hat{A}$;
- $X \subset \hat{B} \Rightarrow$ there exists $b \in \hat{B} - X$ so that for every $x \in X$, $bx \in \hat{B}$.

Thus $za \in \hat{A}$ and $zb \in \hat{B}$.

- If $a \in Y$ then $az \in E(Y) \cap \hat{A} \neq \phi$, thus $X \subseteq \hat{A} \subseteq Y$ [P.2.5].
- If $b \in Y$ then $bz \in E(Y) \cap \hat{B} \neq \phi$, thus $X \subseteq \hat{B} \subseteq Y$ [P.2.5].

Let us now suppose that $\{a, b\} \cap Y = \phi$, we have then:

(i) $\hat{C} \neq \hat{A} \neq \hat{B} \Rightarrow b \notin \hat{A}$ (Triangle Lemma) which is absurd since $b \in \hat{B} = \hat{A}$;

(ii) $\hat{C} = \hat{A} \neq \hat{B} \Rightarrow a \notin \hat{B}$ (Triangle Lemma) which is also absurd, since $a \in \hat{A} = \hat{B}$.
Let us suppose that $\hat{x} \cup \hat{y}$ holds for $X \subset \hat{A}, yx \in E$.

Let $\hat{K}$ be the color class of $ya$ and $\hat{C}$ that one of $yz$. We have then:

(i) If $y \notin \hat{A}$ then: $(\{y, z\} \subseteq Y \in I(G)$ and $za \in E) \Rightarrow ya \in E$;

(ii) If $y \in (\hat{K} \cap \hat{C}) - \hat{A}$, $a \in \hat{K} - Y$ and $yz \in \hat{C} \cap E(Y) \neq \phi$) implies that $(\hat{K} \neq \hat{A} \neq \hat{C}) \Leftrightarrow (\hat{K} \neq \hat{A} \neq \hat{C})$, where at the last step we have applied [P.3.1].

Let $v \in X$ (with $v \neq z$) and $\hat{D}$ be the color of $yz$. We have then:

$\hat{K} \neq \hat{A} \neq \hat{D}$ (since $\hat{K} \neq \hat{A}, yv \in \hat{D} \cap E(Y \cup X) \neq \phi$ and $\hat{A} \in (\hat{A} \cap \hat{K}) - (Y \cup X)$). The two tricolor triangles $(a, y, z; \hat{K}, \hat{A}, \hat{C})$ and $(a, y, v; \hat{K}, \hat{A}, \hat{D})$ have two common colors, thus $\hat{D} = \hat{C}$, thus for every $x \in X, yx \in \hat{C}$ which implies that $X \subset \hat{C} \subseteq Y$ (since $yz \in \hat{C} \cap E(Y) \neq \phi$) (see Figure 3.1).

(ii) If $y \in \hat{A}$ then $(y \notin \hat{A} \cap \hat{B}$ and $y \in \hat{A}) \Rightarrow y \notin \hat{B}$. By replacing $a$ by $b$ and $\hat{A}$ by $\hat{B}$ up here in (i), we get $X \subset \hat{C} \subseteq Y$.

**Figure 3.1:** We have: $\hat{D} = \hat{C}$ [Triangle Lemma].

**Theorem 3.4** Let $X$ and $Y$ be two “strong” partitive sets of $G = (V, E)$ so that $X \cap Y = \phi$. $X$ and $Y$ can be related to each other by only one color at most.

**Figure 3.2** If $X$ and $Y$ are “strong” partitive sets we have necessarily $\hat{A} = \hat{B}$.

**Proof.** Let $\hat{A}$ and $\hat{B}$ be two colors connecting $X$ and $Y$. We have then $X \subset \hat{A} \cap \hat{B}$ and $Y \subset \hat{A} \cap \hat{B}$ (applying [P.2.6] and the definition of the “strong” partitive set). Let us suppose that $\hat{A} \neq \hat{B}$ and define the following sets

$I_A = \{Z \in I_F(G) \text{ with } Z \subset \hat{A}\}$ and $I_B = \{Z \in I_F(G) \text{ with } Z \subset \hat{B}\}$

we have $\cup I_A = \hat{A}$ and $\cup I_B = \hat{B}$ since: $\cup I_A \subseteq \hat{A}$ and if $x \in \hat{A}$, then $x \in I_F(G) \cap I_A$, thus $x \in \cup I_A$ and then $\hat{A} \subseteq \cup I_A$. The same situation holds for $I_B$.

Since $\{X, Y\} \subseteq I_A \cap I_B$ we have $\hat{A} \cap \hat{B} \neq \phi$, $\hat{A} \cap E(\hat{A} \cap \hat{B}) \neq \phi$ and $\hat{B} \cap E(\hat{A} \cap \hat{B}) \neq \phi$ we have then only two alternatives:
(i) either $\bar{A} = \bar{B} \Rightarrow \hat{A} = \hat{B}$ [P.3.1]

(ii) or $\bar{A} \neq \bar{B} \Rightarrow (\hat{A} - (\bar{A} \cap \bar{B}) \neq \phi$ or $\bar{B} - (\bar{A} \cap \bar{B}) \neq \phi$) which is absurd since it contradicts with \( \hat{A} \cap E(\bar{A} \cap \bar{B}) \neq \phi \) and $\hat{B} \cap E(\bar{A} \cap \bar{B}) \neq \phi$ [P.2.5].

**Corollary 3.5** Let $P$ be a partition of “strong” partitive sets of $G = (V, E)$ and $f$ a choice mapping from $P$ to $V$, i.e., $f : X \in P \rightarrow f(X) \in X \subseteq V$. We have then that the isomorphism from $G/P$ to $G(f(P))$ conserves the color classes.

**Corollary 3.6** Let $X$ be a “strong” partitive set of $G = (V, E)$ with $X \neq V$ and let $u \in V - X$. We have then:

(i) either for every $x \in X, ux \notin E$;

(ii) or there exists a color $\hat{A}$ of $G$ so that for every $x \in X, ux \in \hat{A}$.

*Proof.* This is true because for every $u \in V$ the singleton $\{u\}$ is a “strong” partitive set of $G$.

**Theorem 3.7** Let $X$ be a “strong” partitive set of $G = (V, E)$ and $M(S)$ a multiplex of $G$ generated by $S = (V_S, E_S)$. We have then the following implication: $M \cap E(X) \neq \phi \Rightarrow M \subseteq E(X)$.

*Proof.* If $E_S \subseteq E(X)$ then for every color $\hat{A} \subseteq M$ we have $\hat{A} \cap E(X) \neq \phi$ and thus for every $\hat{A} \subseteq M, \hat{A} \subseteq E(X)$ which implies that $M \subseteq E(X)$.

In the other hand if $E_S \cap E(X) = \phi$ then $M \cap E(X) = \phi$. Let us assume now that $E_S - E(X) \neq \phi$ and $E_S \cap E(X) \neq \phi$, then $V_S - X \neq \phi$. Let $u \in V_S - X$ and $ab \in E_S \cap E(X)$, then $\{a, b\} \subseteq X \cap V_S$ and $u$ is related to $a$ and $b$ by two different colors (applying the definition of a simplex). This is absurd because it contradicts with the corollary [3.5].

This result is the analogue for multiplices and ”strong” partitive sets of [P.2.5] which deals with colors and partitive sets.

**Lemma 3.8** Let $G = (V, E)$ be an undirected graph with the number of vertices greater than 2 ($|V| > 2$). If $G$ is decomposable and has a color $\hat{A}$ so that $\hat{A} = V$, then $G$ has a non-trivial maximal “strong” partitive set.
Proof. Let $X$ be a non-trivial partitive set of $G$ ($X \in I^*(G)$), thus $X \neq \tilde{A}$. Let us define on $I(G)$ the following binary relation $R$:

$$XRY \iff ((X = Y) \text{ or } (X \cap Y \neq \emptyset, X - Y \neq \emptyset \text{ and } Y - X \neq \emptyset)).$$

And let $R^*$ be the transitive closure of $R$.

It is easy to verify that $R^*$ is an equivalence relation on $I(G)$. Let $X^*$ be the equivalence class of $X$ modulo $R^*$. By the definition itself of $R^*$, $\cup X^*$ is a “strong” partitive set of $G$.

In the other hand $X \neq \tilde{A}$ ⇒ for every $Y \in X^*$, $E(Y) \cap \tilde{A} = \emptyset$ (otherwise $\tilde{A} = Y$ and for every $Z \in X^*$, $Z \subset Y$ which contradicts with the definition of $R^*$). We have then $\cup X^* \subset V$. There exists then a $a \in V - \cup X^*$ so that for every $x \in \cup X^*$, $ax \in \tilde{A}$[P.3.2]. But $\{a\} \in I^*_f(G)$ and if $Y \in I^*_f(G)$ so that $\{a\} \subset Y$ then $Y \subseteq V - \cup X^*$ (otherwise $\tilde{A} \cap E(Y) \neq \emptyset \Rightarrow \tilde{A} = V \subseteq Y$[P.2.5]), consequently $a$ is contained in a maximal ”strong” partitive set different from $V$. Thus $\cup X^*$ is contained in a non trivial maximal ”strong” partitive set.

**Theorem 3.9** Let $M(S)$ be a multiplex of $G = (V, E)$ generated by the simplex $S = (V_S, E_S)$. $M(S)$ is maximal if and only if the set of vertices $\tilde{M}$ spanned by $M$ is a “strong” partitive set.

**Proof.** Let us suppose that $M(S)$ is maximal, then $S$ is maximal. $\tilde{M}$ is a partitive set of $G$. Let $Y$ be a partitive set of $G$ so that $Y \cap \tilde{M} \neq \emptyset$ and $Y - \tilde{M} \neq \emptyset$. Assume that $\tilde{M} - Y \neq \emptyset$ and let us show that it is absurd. Let $y \in Y - \tilde{M}$. $G(\tilde{M})$ is connected and then there exists $u \in Y \cap \tilde{M}$ and $v \in \tilde{M} - Y$ so that $uv \in M$. The following statements hold:

$(Y \in I(G), u \in Y \text{ and } v \in \tilde{M} - Y) \Rightarrow \text{ For any } x \in Y, xv \in E \Rightarrow yv \in E$;

$(\tilde{M} \in I(G), v \in \tilde{M} \text{ and } y \in Y - \tilde{M}) \Rightarrow \text{ for any } x \in \tilde{M}, xy \in E$

(see Figure 3.3).

**Figure 3.3** For every $x \in \tilde{M}$, $xy \in E$.

The colors connecting $y$ to the summits of $S$ can not be all different, otherwise, $M$ will not be maximal. Let us suppose that there exists $\{a, b\} \in S$ and a color $\hat{A}$ so that $\{ya, yb\} \subseteq \hat{A}$. Since $y \in \hat{A} - \tilde{M}$ and $\tilde{M} \in I(G)$, then we have $\hat{A} \cap M = \emptyset$[P.2.5]. If rank$(M) = 1$ then there exists a color $\hat{K}$ so
that $\bar{K} = M$. But $(ab \in \bar{K} \cap E(\bar{A}) \neq \phi) \Rightarrow \bar{K} \subseteq \bar{A}$ : \{uv, ab\} \subseteq \bar{K} \Rightarrow \{yu, yv\} \in \bar{A}$[Triangle Lemma], then we have $(yu \in \bar{A} \cap E(\bar{Y})) \Rightarrow \bar{M} = \bar{K} \subseteq \bar{A} \subseteq \bar{Y}$[P.2.5] (see Figure 3.4) which is in contradiction with our proposition.

**Figure 3.4** $yu \in \bar{A} \cap E(\bar{Y}) \neq \phi$ and $ab \in \bar{K} \cap E(\bar{A}) \neq \phi$.

Let us assume now that rank($M$) $\geq 2$. Let $c \in V_S - \{a, b\}$, $\hat{B}$ the color of $yc$, $\hat{C}$ the color of $ac$ and $\hat{D}$ the color of $bc$. If $\hat{A} \neq \hat{B}$ then $S$ contains two tricolor triangles: $(y, a, c ; ay \in \hat{A}, cy \in \hat{B}, ac \in \hat{C})$ and $(y, b, c ; by \in \hat{A}, cy \in \hat{B}, bc \in \hat{D})$ having two common colors, then $\hat{C} = \hat{D}$ [Triangle lemma] which is absurd because $S$ is a simplex, then $\hat{A} = \hat{B}$ (see Figure 3.5).

**Figure 3.5** $\hat{B} \neq \hat{A} \Rightarrow \hat{C} = \hat{D}$.

Thus for every $x \in V_S$, $yx \in \hat{A}$ and by consequence $V_S \subset \hat{A}$ which implies that for every color $\hat{H} \subset M$ we have $\hat{H} \cap E(\bar{A}) \neq \phi$ then $\bar{M} \subset \bar{A}$. But $(yu \in \bar{A} \cap E(\bar{Y}) \neq \phi) \Rightarrow \bar{A} \subseteq \bar{Y}$ hence $\bar{M} \subset \bar{A} \subseteq \bar{Y}$, which also contradicts the first assumption ($\bar{M} - \bar{Y} \neq \phi$).

Finally we have showed that $\bar{M} \subset \bar{Y}$ which means that $\bar{M}$ is a “strong” partitive set. Let us now prove the converse. Let us assume that $\bar{M}$ is a “strong” partitive set of $G$ and let us show that $M$ is a maximal. If the summits of $\bar{M}$ are related to $y \in V - \bar{M}$, then they are related by the same color which achieves the proof.

**Corollary 3.10** Let $M(S)$ be a multiplex of $G = (V, E)$ generated by the simplex $S = (V_S, E_S)$ and let $a \in V - M$. The simplex $S$ is extensible to a simplex $S' = (V_{S'}, E_{S'})$ (with $S$ a sub-graph of $S'$) so that $V_{S'} = V_S \cup \{a\}$ if and only if there exists $\{b, c\} \subset \bar{M}$ and two colors $\hat{A}$ and $\hat{B}$ of $E - M$ so that $ab \in \hat{A}$ and $ac \in \hat{B}$.

**Proof.** Use [C.3.6] and [T.3.9].
4 Transitive orientations of an undirected graph

In this section, using the results of the previous section we prove the existence of a partition of maximal multiplices for the set of edges of an undirected graph. Therefore, the transitive orientations of a comparability graph turn up to the transitive orientations of their multiplices. These orientations are independent to each other. A theorem of decomposability for a non-limit undirected graph is proved.

Lemma 4.1 Let $G = (V, E)$ be any graph. Let $X$ and $Y$ be two partitive sets of $G$ so that $X \subseteq Y$. The following statements hold:

(i) $X \in I_F(G) \Rightarrow X \in I_F(G(Y))$;

(ii) $Y \in I_F(G) \Rightarrow (X \in I_F(G) \iff X \in I_F(G(Y))$.

Proof. Let $X$ be a “strong” partitive set of $G$ ($X \in I_F(G)$). We have for every $Z \in I(G(Y))$, $Z \in I(G)$. Hence $X \in I_F(G(Y))$. Let $X \in I_F(G(Y))$, $Y \in I_F(G)$ and $Z \in I(G)$ so that $Z \cap X \neq \emptyset$ and $Z \setminus X \neq \emptyset$. Then $Z \cap X \neq \emptyset \Rightarrow Z \cap Y \neq \emptyset$. But $Y \in I_F(G)$, thus either $Z \subseteq Y$ or $Y \not\subseteq Z$. But ($Y \subseteq Z \Rightarrow X \subseteq Z$) and ($X \subseteq Y \Rightarrow Z \in I(G(Y)) \Rightarrow X \subseteq Z$).

Lemma 4.2 Let $G = (V, E)$ be any graph. Let $F$ and $F'$ be two partitions of $G$ constituted of maximal “strong” partitive sets. We have then $F = F'$.

Proof. Let us assume that $F \neq F'$. Since $F$ and $F'$ are partitions of $V$, if $X \in F$ then there exists $X' \in F'$ so that $X \cap X' \neq \emptyset$. Thus either $X \subseteq X'$ or $X' \subseteq X$. But $X$ and $X'$ are maximal “strong” partitive sets, thus $X = X'$. Hence $F = F'$.

Proposition 4.3 Let $M(S)$ a multiplex of $G = (V, E)$ generated by the simplex $S$ with rank$(M) \geq 2$. We have then for every $x \in \tilde{M}$ there exists two colors $\{\tilde{A}, \tilde{B}\} \subseteq \tilde{M}$ with $\tilde{A} - \tilde{B} \neq \emptyset$ and $\tilde{B} - \tilde{A} \neq \emptyset$ so that $x \in \tilde{A} \cap \tilde{B}$.

Proof. Let $x \in M$, then there exists $\tilde{A} \subseteq M$ and $y \in \tilde{A}$ so that $xy \in \tilde{A}$. Moreover ($\tilde{A} \subseteq M$ and rank $(M) \geq 2) \Rightarrow S$ contains one tricolor triangle: $(a, b, c ; \tilde{A}, \tilde{B}, \tilde{C})$ so that $bc \in \tilde{A}$, $ac \in \tilde{B}$ and $ab \in \tilde{C}$. Thus $a \not\in \tilde{A}$, $b \not\in \tilde{B}$ and $c \not\in \tilde{C}$ [Triangle lemma]. Hence $\tilde{A} - \tilde{B} \neq \emptyset$ and $\tilde{B} - \tilde{A} \neq \emptyset$. In the other hand: $xy \in \tilde{A} \Rightarrow ((ax \in \tilde{B} \text{ and } ay \in \tilde{C}) \text{ or } (ax \in \tilde{C} \text{ and } ay \in \tilde{B}))$ [Triangle lemma]. If one suppose $ax \in \tilde{B}$ then $x \in \tilde{A} \cap \tilde{B}$.
Theorem 4.4 Let $M(S)$ be a multiplex of $G = (V, E)$ generated by the simplex $S = (V_S, E_S)$. We have then the following statements:

(i) $G(\tilde{M})$ has a partition of maximal “strong” partitive sets $F_M \neq \{\tilde{M}\}$;

(ii) If rank$(M) = 1$, we have either $G(\tilde{M})/F_M$ is isomorphic to $S$ or $G(\tilde{M})/F_M$ is indecomposable and isomorphic to a sub-graph $G' = (V', E')$ of $G(M)$ so that $E' \subseteq M$;

(iii) If rank$(M) \geq 2$, then $G(\tilde{M})/F_M$ is isomorphic to $S$.

Proof. 1) If rank$(M) = 1$ then $M$ contains only one color $\hat{A} = M$. Moreover if $G(\tilde{M})$ is indecomposable then $F_M = \{\{x\} : x \in \tilde{M}\}$. Hence $G(\tilde{M})/F_M$ is isomorphic to $G(\tilde{M})$ and thus $G(\tilde{M})/F_M$ is indecomposable.

Let us assume now that $G(\tilde{M})$ is decomposable. Thus $G(\tilde{M})$ contains a non-trivial partitive set $X$. After [L.4.2] $F_M$ exists. If $|F_M| = 2$ then $G(\tilde{M})/F_M$ is complete and isomorphic to $S$.

Let us suppose that $|F_M| \geq 3$. If $G(\tilde{M})/F_M$ has a non-trivial partitive set $X$, $G(\tilde{M})/F_M$ has a non-trivial maximal “strong” partitive set $Y$ [L.4.2] and [C.3.5].

Hence $\cup Y \in I^*_F(G(M))[P.21]$ which is absurd because $F_M$ is already maximal. Thus $G(\tilde{M})/F_M$ is indecomposable.

2) Let us now assume that rank$(M) \geq 2$. Using the proposition [4.3] one gets that for every $x \in \tilde{M}$ there exists two colors $\{\hat{A}, \hat{B}\} \subseteq \tilde{M}$ with $\hat{A} - \hat{B} \neq \emptyset$ and $\hat{B} - \hat{A} \neq \emptyset$ so that $x \in \hat{A} \cap \hat{B}$.

Applying [T.3.3] we have that the intersection $\hat{A} \cap \hat{B} \neq \tilde{M}$ is a “strong” partitive set of $G$. Thus $\hat{A} \cap \hat{B}$ is a “strong” partitive set of $G(\tilde{M})$ itself since $\tilde{M} \in I(G)$ [L.4.1]. Let $F_M$ be the set of the intersections two by two of colors of $M$. Then for every $a \in V_S$ there exists $X \in F_M$ so that $a \in X$. Let $\{a, b\} \subset V_S$ so that $a \in X \in F_M$ and $b \in Y \in F_M$. Let us assume that there exists a “strong” partitive set $Z$ of $G(\tilde{M})$ so that $X \cup Y \subseteq Z$. $S$ will contain a tricolor triangle $(a, b, c : \hat{A}, \hat{B}, \hat{C})$ with $bc \in \hat{A}$, $ac \in \hat{B}$ and $ab \in \hat{C}$. This is absurd since $\{c\}$ and $Z$ are two “strong” partitive sets of $G(\tilde{M})$ and can be related at most by only one color[T.3.4]. Hence $F_M$ is a maximal partition of ”strong” partitive sets and separates the summits of $S$. Finally we get that $G(\tilde{M})/F_M$ is isomorphic to $S$. 

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Corollary 4.5 The only multiplices $M(S)$ which might be not transitively orientable are those of rank $= 1$ and so that $G(\tilde{M})/F_{\tilde{M}}$ is non isomorphic to $S$.

Proof. Because complete graphs are orientable.

Corollary 4.6 An undirected graph $G = (V, E)$ can have at most one multiplex which spanned all its summits.

Proof. Let $M(S)$ and $M(S')$ be two multiplices of $G$ so that $\tilde{M} = V$ and $\tilde{M'} = V$, we have then $G(\tilde{M}) = G(\tilde{M'}) = G$. Let $F$ and $F'$ be two partitions of maximal “strong” partitive sets of $G$ related to $M$ and $M'$. Thus $G/F$ is isomorphic to $S$ and $G/F'$ is isomorphic to $S'$. Moreover the two isomorphisms conserve the colors [C.3.5]. But after [L.4.2] we have $F = F'$. Hence $S = S'$ and $M = M'$.

Corollary 4.7 Let $G = (V, E)$ be an undirected graph and $M(S), M'(S')$ two maximal multiplices of $G$. We have then $M \cap M' \neq \phi \Rightarrow M = M'$.

Proof. Since $M$ and $M'$ are “strong” partitive sets we have $M \cap M' \neq \phi \Rightarrow \tilde{M} \cap \tilde{M'} \neq \phi \Rightarrow (\tilde{M} \subseteq \tilde{M'}$ or $\tilde{M'} \subseteq \tilde{M}$).

Let us assume that $\tilde{M} \subseteq \tilde{M'}$ and let $F, F'$ be two partitions of maximal “strong” partitive sets related respectively to $G(\tilde{M})$ and $G(\tilde{M'})$. If there exists $X \in F'$ so that $M \cap E(X) \neq \phi$ then $M \subseteq E(X) [T.3.7]$ and $M \cap M' = \phi$ (since $G(M')/F'$ is isomorphic to $S'$ and $S$ would be a sub-graph of $G(X)$. Hence $E_S \cap E_S' = \phi$.

Finaly we have that for every $X \in F'$, $E(X) \cap M = \phi$. Thus $S$ is isomorphic to a sub-graph of $S'$. But $S$ is maximal. Hence $S = S'$ and $M = M'$.

Corollary 4.8 Let $G = (V, E)$ be an undirected graph with $E \neq \phi$. $E$ has then a partition of maximal multiplices.

The theorem 4.4 tell us that a multiplex has the same number of transitive orientations as the simplex which generated this multiplex. The simplex itself has a number of transitive orientations equal to the number of the possible permutations of its summits. Moreover [C.4.3] asserts that the only multiplices $M(S)$ which might be not transitively orientable are those with rank $1$ so that $G(\tilde{M})/F_{\tilde{M}}$ is not isomorphic to $S$. Thus using [C.4.3] the
problem of transitive orientation for a comparability graph come down to
the transitive orientation of its multiplices. But the following problem is
rised: if we orientate in any way and at certain step a given multiplex, will
this orientation influence or not the orientations of the other multiplices at
the following steps ? The response is not and we will prove this statement
using a theorem [T.4.12] which is known and for which we propose a new
proof outcoming from the forcing theory.

Before announcing [T.4.12], we will announce a theorem [T.4.10] which, in
fact, is a mathematical algorithm permitting to find all the transitive orien-
tations for a comparability graph which has only non limit sub-graphs, e.g.,
case of finite graphs.

**Lemma 4.9** Let $G = (V, E)$ be a connected undirected graph. Then $G$
can not contain a multiplex $M$ so that both $\tilde{M} \neq V$ and $\tilde{M}$ is maximal for the
inclusion among the $\tilde{N}$, where $N$ is any multiplex of $G$

*Proof.* Let us assume that such a multiplex $M$ exists and show that it is
absurd. Since $\tilde{M}$ is maximal for the inclusion it implies that $M$ is maximal.
After [T.3.9], $\tilde{M}$ is a ”strong” partitive set. Since $\tilde{M} \neq V$ and $G$ is connected,
we have :
there exists $x \in V - \tilde{M}$ and $y \in \tilde{M}$ so that $xy \in E$. Let $\tilde{A}$ the color containing
$xy$. Then $\tilde{A} \not\subset M$ and $x \in (\tilde{A} \cap \tilde{M}) \neq \phi$. But $\tilde{M}$ is a ”strong” partitive set
of $G$ and $\tilde{A}$ is a partitive set of $G$, thus $\tilde{M} \subset \tilde{A}$, which is absurd because $\tilde{A}$
is a multiplex of rank 1.

**Theorem 4.10** Let $G = (V, E)$ be an undirected graph having a partition
of maximal “strong” partitive sets $F_G \neq \{V\}$. We have then that $G/F_G$
satisfies one of the following exclusive assertions:

(i) $G/F_G$ is empty.

(ii) $G/F_G$ is indecomposable and there exists a maximal multiplex $M_G$ of $G$
with $\text{rank}(M_G) = 1$, $\tilde{M}_G = V$ and $G/F_G$ is isomorphic to a sub-graph
$G' = (V', E')$ of $G$ so that $E' \subseteq M_G$.

(iii) $G/F_G$ is complete and isomorphic to a maximal simplex $S$ generating
a maximal multiplex $M_G$ so that $\tilde{M}_G = V$. 

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Proof. If $G$ is non connected, $F_G$ is the class of the connected components and $G/F_G$ is empty. In the other hand, it is obvious that if $G/F_G$ is empty then $G$ is non connected. Hence $G/F_G$ is empty if and only if $G$ is non connected.

Let us assume now that $G$ is connected. Since $G$ is non limit, it implies that $G$ has maximal multiplex $M$ so that $\tilde{M}$ is maximal for the inclusion. Thus using the [L.4.9] we have $\tilde{M} = V$. Hence $G = G(\tilde{M})$ and using [T.4.4] we get the result.

**Corollary 4.11** Let $G = (V, E)$ be a connected and undirected graph. $G$ has then a partition of maximal “strong” partitive sets $F_G \neq \{V\}$ if and only if $G$ has a multiplex $M_G$ so that $\tilde{M} = V$.

**Proof.** Applying [T.4.10] we have: $F_G$ exists $\Rightarrow M_G$ exists. In the other hand applying [T.4.4], we have the other implication: $M_G$ exists $\Rightarrow F_G$ exists.

**Theorem 4.12** Let $O = (V, E)$ be a partial order and $G = (V, E)$ be its comparability graph. $O$ and $G$ have then the same “strong” partitive sets ($I_F(G) = I_F(O)$).

Before giving the proof of this theorem, we present some preleminary results which will be used for the proof.

**Lemma 4.13** Let $O = (V, E')$ be a partial order and $G = (V, E)$ its comparability graph. Then every partitive set of $O$ is a partitive set of $G$ ($I(O) \subseteq I(G)$).

**Proof.** Let $Y \in I(O)$. If $Y$ is a trivial partitive set, we have $Y \in I(G)$. Let us suppose that $Y$ is non trivial, then if $a \in V - Y$ we have one of the following statements:

- for every $y \in Y$, $((a, y) \in E') \Rightarrow ay \in E$.
- or for every $y \in Y$, $((y, a) \in E') \Rightarrow ay \in E$.
- or for every $y \in Y$, $\{(a, y), (y, a)\} \cap E' = \emptyset \Rightarrow ay \notin E$.

Therefore $Y \in I(G)$. 

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Proposition 4.14 Let $O = (V, E')$ be a partial order and $G = (V, E)$ its comparability graph. Then every “strong” partitive set of $G$ is a “strong” partitive set of $O$ ($I_F(G) \subseteq I_F(O)$).

Proof. Let $X$ be a “strong” partitive set, i.e., $X \in I_F(G)$. If $X$ is trivial then $X \in I_F(O)$. Let us suppose that $X$ is non trivial. We have then one of the following statements:

- for every $x \in X$, $xa \not\in E$, then for every $\{x, y\} \subseteq X$, $(x, a) \cong (y, a)$ (cf. §2.1).

- or there exists a color $\hat{A}$ of $G$ so that for every $x \in X$, $xa \in \hat{A}$, then for every $\{x, y\} \subseteq X$, $(x, a) \cong (y, a)$ (since $G$ is a comparability graph [T.2.2]).

Therefore $X \in I(O)$. Let $Y \in I(O)$ so that $Y \cap X \neq \emptyset$ and $Y - X \neq \emptyset$. Then using the previous lemma [4.13] we have $Y \in I(G)$ and then $X \subset Y$. Thus $X \in I_F(O)$.

Proposition 4.15 If $X$ is a maximal “strong” partitive set of $G$ then $X$ is a maximal “strong” partitive set of $O$.

Proof. If $X \in I_F(G)$ is maximal, it implies that $G$ is non limit. Thus $G$ has a partition $F$ of maximal “strong” partitive sets and $X \in F \subseteq I_F(O)$. After [T.4.10] we have either $G/F$ is empty and then $X$ is a connecting class. Thus $X$ is maximal in $I_F(O)$; or $G/F$ is complete and then $O/F$ is a chain, therefore $G/F$ do not has a non trivial “strong” partitive sets, thus $X$ is maximal in $I_F(O)$; or $G/F$ is indecomposable and the partitive sets of $G/F$ are trivial, therefore, after [L.4.13] the partitive sets of $O/F$ are trivial, thus $X$ is maximal in $I_F(O)$.

Proposition 4.16 Let $X$ be a non trivial “strong” partitive set of $O$. There exists then a non trivial “strong” partitive set $Y$ of $G$ so that $X \subseteq Y$.

Proof. If $G$ is non limit then it has a partition $F$ of maximal “strong” partitive sets which is also a partition of maximal “strong” partitive sets of $O$[P.4.15]. Thus there exists $Y \in F$ so that $X \subseteq Y$. If $G$ is limit then since $X \in I(O) \Rightarrow X \in I(G)$ [L.4.13], therefore $Y$ exists.
In what follows, we present the proof of the theorem [4.12].

**Proof.** If $X \in I_F(G)$ then after [P.4.15] $X \in I_F(O)$. Let $X \in I_F^*(O)$. Then after [P.4.16] there exists $Y \in I_F^*(G)$ so that $X \subseteq Y$. Let us suppose that $X \notin I_F(G)$. Thus $X \subset Y$. $Y \in I_F^*(G) \Rightarrow Y \in I_F^*(O)$. Therefore after [L.4.1] $X \in I_F^*(O(Y))$. Thus there exists $Y_1 \in I_F^*(G(Y))$ so that $X \subset Y_1$. So we have constructed a strictly decreasing suite of elements of $I_F^*(G) \subseteq I_F^*(O)$ (after [P.4.14]).

The intersection $\cap Y_i$ of this suite is a “strong” partitive set of $O$ and it is the smallest element of $I_F^*(O)$ which contains $X$. Therefore $X = \cap Y_i$. But $\cap Y_i$ is also a “strong” partitive set of $G$. Thus $X \in I_F^*(G)$ and then $I_F(O) = I_F(G)$.

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