CONVERGENCE OF HARMONIC MAPS

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Abstract. In this paper we prove a compactness theorem for a sequence of harmonic maps which are defined on a converging sequence of Riemannian manifolds.

Harmonic maps are critical points of the energy functional defined on the space of maps between Riemannian manifolds. Their theory was developed by J. Eells and H. Sampson [ES64] in the 1960s. The notion of harmonic maps on smooth metric measure spaces has been introduced by Lichnerowicz in [Lic69] (see also [EL78]). Harmonic maps maps between singular spaces have been studied since the early 1990s in the works of Gromov-Schoen in [GS92] and Korevaar-Schoen in [KS93]. [EF01] by Eells-Fuglede contains a description of the application of the methods of [KS93] to the study of maps between polyhedra.

A smooth metric measure space is a triple \((M,g,\Phi \text{dvol})\), where \((M,g)\) is an \(n\)-dimensional Riemannian manifold, \(\text{dvol}\) denotes the corresponding Riemannian volume element on \(M\), and \(\Phi\) is a smooth positive function on \(M\). These spaces have been used extensively in geometric analysis and they arise as smooth collapsed measured Gromov-Hausdorff limits in the works of Cheeger-Colding [CC97, CC00a, CC00b], Fukaya [Fuk89] and Gromov [Gro07]. They have been studied recently by Morgan [Mor05]. See also works of Lott [Lot03], Qian [Qia97], Fang-Li-Zhang [FLZ09], Wei-Wylie [WW09], Wu [Wu10], Su-Zhang [SZ11] and Munteanu-Wang [MW11].

In this paper, we are going to study the behavior of harmonic maps under convergence. Let \(\mathcal{M}(n,D)\) denote the set of all compact Riemannian manifolds \(M\) such that \(\dim(M) = n\), \(\text{diam}(M) < D\), and the sectional curvature \(R_M\) satisfies, \(|R_M| \leq 1\), equipped with the measured Gromov-Hausdorff topology. Let \((M_i,g_i,\text{dvol}_i)\) in \(\mathcal{M}(n,D)\) be a sequence of manifolds which converge to a smooth metric measure space \((M,g,\Phi \text{dvol}_M)\). Suppose \(f_i : (M_i,g_i) \rightarrow (N,h)\) is a sequence of harmonic maps. We are interested in knowing under what circumstances the \(f_i\) converge to a harmonic map \(f\) on the smooth metric measure space \((M,g,\Phi)\).

When a sequence of manifolds \((M_i,g_i)\) in \(\mathcal{M}(n,D)\) converges to a metric space \(X\), according to Fukaya [Fuk88], \(X\) is a quotient space \(Y/O(n)\), where \(Y\) is a smooth manifold. Indeed \(Y\) is the limit point of the sequence of frame bundles, \(F(M_i)\), over the manifolds \(M_i\), and \(X\) has the structure of a Riemannian polyhedron \((X,g_X,\Phi_X\mu_g)\) where \(\mu_g\) is the Riemannian volume element related to the metric \(g_X\) on \(X\).

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We state the main result of this paper, which is a compactness theorem for sequences of harmonic maps.

**Theorem 0.1.** Let \((M_i, g_i)\) be a sequence of manifolds in \(\mathcal{M}(n, D)\) which converges to a metric measure space \((X, g, \Phi_{\mu_g})\) in the measured Gromov-Hausdorff Topology. Suppose \((N, h)\) is a compact Riemannian manifold. Let \(f_i : (M_i, g_i) \to (N, h)\) be a sequence of harmonic maps such that \(\|e_{g_i}(f_i)\|_{L^\infty} < C\), where \(\|e_{g_i}(f_i)\|_{L^\infty}\) is the \(L^\infty\)-norm of the energy density of the map \(f_i\) and \(C\) is a constant independent of \(i\). Then \(f_i\) has a subsequence which converges to a map \(f : (X, g, \Phi_{\mu_g}) \to (N, h)\), and this map is a harmonic map in \(\mathcal{H}^1((X, \Phi_{\mu_g}), N)\).

By \(\mathcal{H}^1(X, N)\) we mean
\[
\{ f \in \mathcal{H}^1(X, \mathbb{R}^q) \mid f(x) \in N \text{ for almost all } x \in M \},
\]
where \(\mathcal{H}^1(M, \mathbb{R}^q)\) is the standard Sobolev space and \(N\) is isometric embedded in \(\mathbb{R}^q\). In this work we use the notions \(\mathcal{H}^1\) and \(W^{1,2}\) interchangeably.

The rest of this paper is organized as follows. In the second section we introduce our main notations and preliminary results needed for the rest of this paper. In the third section, we prove Theorem 0.1. We divide the proof into three cases. In Subsection 2.1 we consider the non-collapsing case, Proposition 2.1. Moreover using the regularity results for harmonic maps in the work of Schoen and Lin [Sch84, Lin99] we study Theorem 0.1 under more restrictive assumption of uniform boundedness of the energy of the maps \(f_i\) (see Propositions 2.3, 2.4). In subsection 2.2 we consider the case of collapsing to a Riemannian manifold, Proposition 2.5. As a preliminary step we prove the result under some regularity assumption on the metric \(g_i\), see Proposition 2.6. The general case is considered in subsection 2.3. The Appendix, is where we study convergence of the tension field of the map \(f_i\) under the assumptions of Proposition 2.6.

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1. **Preliminaries**

1.1. **Weakly harmonic maps.** In this subsection, we first recall the definition of a weakly harmonic map on smooth metric measure spaces. We then briefly review this concept on Riemannian polyhedron. At the end we present some theorems and lemmas that we need in this paper. Let \((N, h)\) be a compact Riemannian manifold and \(i\) be an isometric embedding \(i : N \to \mathbb{R}^q\). Since \(i(N)\) is a smooth, compact submanifold, there exists a number \(\kappa > 0\) such that the neighborhood
\[
U_\kappa(N) = \{ y \in \mathbb{R}^q : \text{dist}(y, N) < \kappa \}\]
has the following property: for every \( y \in U_\kappa(N) \) there exists a unique point \( \pi_N(y) \in N \) such that
\[
|y - \pi_N(y)| = \text{dist}(y, N)
\]
The map \( \pi_N : U_\kappa(N) \to N \) defined as above is called the nearest point projection onto \( N \).

The Hess \( \pi_N \) defines an element in \( \Gamma(TN^* \otimes TN^* \otimes TN^\perp) \) which coincides with the second fundamental form of \( i : N \to \mathbb{R}^q \) up to a negative sign
\[
\langle \text{Hess} \pi_N(y)(X, Y), \eta \rangle = -\langle \nabla_Y \eta, X \rangle
\]
where \( X \) and \( Y \) are in \( TN \), \( y \) in \( N \) and \( \eta \) in \( TN^\perp \) (see §3 in Moser [Mos05]).

A map \( f : (M, g, \Phi) \to (N, h) \), belonging to \( H^1_{1\text{loc}}((M, \Phi \text{dvol}), N) \) is called weakly harmonic map if and only if
\[
\Delta i \circ f - \Pi(f)(df, df) + di \circ f(\nabla \ln(\Phi)) = 0
\]
in the weak sense. Here
\[
\Pi(f)(df, df) = \text{trace Hess}(\pi_N)(i \circ f)(di \circ f, di \circ f)
\]
and in coordinate
\[
\Pi(f)(df, df) = \sum g^{ij} \frac{\partial^2 \pi_N^A}{\partial z^B \partial z^C} \frac{\partial f^B}{\partial x^i} \frac{\partial f^C}{\partial x^j}
\]
For the map \( f : (M^n, g) \to (N^m, h) \) and \( \eta : M \to \mathbb{R}^q \), we define
\[
\Xi_g(f, \eta) = \langle di \circ f, d\eta \rangle - \langle \Pi(f)(df, df), \eta \rangle.
\]

We explain now the definition of harmonic maps on Riemannian polyhedra. Following Eells-Fuglede [EF01], on an admissible Riemannian polyhedron \( X \), a continuous weakly harmonic map \( u : (X, g_\mu) \to (N, h) \) is of class \( H^1_{1\text{loc}}(X, N) \) and satisfies: For any chart \( \eta : V \to \mathbb{R}^n \) on \( N \) and any open set \( U \subset u^{-1}(V) \) of compact closure in \( X \), the equation
\[
\int_U g(\nabla \lambda, \nabla u^k) \ d\mu_g = \int_U \lambda(\Gamma^k_{\alpha\beta} \circ u) g(\nabla u^\alpha, \nabla u^\beta) \ d\mu_g
\]
holds for every \( k = 1, \ldots, n \) and every bounded function \( \lambda \in \mathcal{H}^1_{0}(U) \). Here \( \Gamma^k_{\alpha\beta} \) denotes the Christoffel symbols on \( N \). Similarly on a polyhedron \( X \) with a measure \( \Phi \mu_g \) a continuous weakly harmonic map is a map in \( \mathcal{H}^1_{1\text{loc}}((X, \Phi \mu_g), N) \) which satisfies equation (5) with \( \Phi \mu_g \) in place of \( d\mu_g \).

**Theorem 1.1** (Moser [Mos05], Theorem 3.1). Let \( f \in \mathcal{H}^1(U, N) \cap C^0(U, N) \) be a weakly harmonic map, where \( U \) is an open domain in \( \mathbb{R}^n \). Then \( f \in C^\infty(U, N) \).

The energy functional is lower semi continuous, and we have

**Lemma 1.2** (Xin [Xin96]). Let \( S \subset \mathcal{H}^1(M, N) \) on which the energy is bounded and \( S \) is closed under weak limit, then \( S \) is sequentially compact.
We recall some regularity results for harmonic maps from \[\text{Sch84}\] and \[\text{Lin99}\]. Let \(M\) and \(N\) are compact manifolds. Define

\[
\mathcal{F}_\Lambda = \{ u \in C^\infty(M, N) : u \text{ is harmonic, } E(u) \leq \Lambda \}.
\]

We have the following theorem and remark.

**Theorem 1.3** (Schoen \[\text{Sch84}\]). Let \(M\) and \(N\) be compact manifolds. Any map \(u\) in the weak closure \(\mathcal{F}_\Lambda\) is smooth and harmonic outside a relatively closed singular set of locally finite Hausdorff \((n - 2)\)-dimensional measure.

and

**Remark 1** (Schoen \[\text{Sch84}\], Lin \[\text{Lin99}\]). Let \(u_i\) be a sequence in \(\mathcal{F}_\Lambda\), then there exists a subsequence which converges weakly to some \(u\) in \(H^1(M, N)\). Define

\[
\Sigma = \bigcap_{r > 0} \left\{ x \in M, \liminf_{i \to \infty} \frac{r^{2-n}}{2} \int_{B_r(x)} e(u_i) \geq \epsilon_0 \right\}
\]

where \(\epsilon_0 = \epsilon_0(n, N) > 0\) is a constant independent of \(u_i\) as in Theorem 2.2 in \[\text{Sch84}\]. If we consider a sequence of Radon measure \(\mu_i = |du_i|^2 dx\), without loss of generality we may assume \(\mu_i \rightharpoonup \mu\) weakly as Radon measures. By Fatou’s lemma, we may write

\[
\mu = |du|^2 dx + \nu
\]

for some non-negative Radon measure \(\nu\). We can show that \(\Sigma = \text{spt} \nu \cup \text{sing} u\) and \(\nu\) is absolute continuous with respect to \(H^{n-2}|_\Sigma\). Therefore \(u_i\) strongly converges in \(H^1(M, N)\) to \(u\) if and only if \(|du_i|^2 dx \rightharpoonup |du|^2 dx\) weakly if and only if \(\nu = 0\) if and only if \(H^{n-2}(\Sigma) = 0\) if and only if there is no smooth non-constant harmonic map from \(S^2\) (2-sphere) into \(N\) (e.g. negatively curved manifolds). See Lemma 3.1 in \[\text{Lin99}\] for the complete discussion on the mentioned results.

The following reduction theorem shows the relation between the tension field of equivariant harmonic maps under Riemannian submersions.

**Theorem 1.4** (Xin \[\text{Xin96}\], Theorem 6.4). Let \(\pi_1 : E_1 \to M_1\) and \(\pi_2 : E_2 \to M_2\) be Riemannian submersions, \(H_1\) the mean curvature vector of the submanifold \(F_1\) in \(E_1\) and \(B_2\) the second fundamental form of the fiber submanifold \(F_2\) in \(E_2\). Let \(f : E_1 \to E_2\) be a horizontal equivalent map, \(\bar{f}\) its induced map from \(M_1\) to \(M_2\) with tension field \(\tau(\bar{f})\). \(f^\perp\) denotes the restriction of \(f\) on the fibers \(F_1\). Then we have following formula,

\[
\tau(f) = \tau^*(\bar{f}) + B_2(f_*(e_t), f_*(e_t)) - f_*(H_1) + \tau(f^\perp)
\]

where \(\{e_t\}, t = n_1 + 1, \ldots, m_1\) is local orthonormal frame field of fibers \(F_1\) and \(\tau^*(\bar{f})\) denotes the horizontal lift of \(\tau(\bar{f})\).
1.2. Hölder spaces on manifolds. Let $(M, g)$ be a Riemannian manifold and let $\nabla$ be the Levi-Civita connection on $M$. Let $V \to M$ be a vector bundle on $M$ equipped with the Euclidean metric on its fibers. Let $\nabla$ be a connection on $V$ preserving these metrics. Let $C^k(M)$ be the space of continuous, bounded function $f$ that have $k$ continuous, bounded derivatives, and define the norm $\| \cdot \|_{C^k}$ on $C^k(M)$ by, $\|f\|_{C^k} = \sum_{j=0}^k \sup_M |\nabla^j f|.$

Now we define the Hölder space $C^{0, \alpha}(M)$, for $\alpha \in (0, 1)$. The function $f$ on $M$ is said to be Hölder continuous with exponent $\alpha$, if

$$[f]_\alpha = \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}$$

is finite. The vector space $C^{0, \alpha}(M)$ is the set of continuous, bounded functions on $M$ which are Hölder continuous with exponent $\alpha$ and the norm $C^{0, \alpha}(M)$ is $\|f\|_{C^{0, \alpha}} = \|f\|_{C^0} + [f]_\alpha$.

In the same way, we shall define Hölder norms on spaces of sections $\nu$ of a vector bundle $V$ over $M$, equipped with Euclidean metrics in the fibers as above. Let $\delta(g)$ be the injectivity radius of the metric $g$ on $M$, which we suppose to be positive and set

$$[\nu]_\alpha = \sup_{x \neq y \in M, d(x, y) < \delta(g)} \frac{|\nu(x) - \nu(y)|}{d(x, y)^\alpha}$$

We now interpret $|\nu(x) - \nu(y)|$. When $x \neq y \in M$, and $d(x, y) \leq \delta(g)$, there is unique geodesic $\gamma$ of length $d(x, y)$ joining $x$ and $y$ in $M$. Parallel translation along $\gamma$ using $\nabla$ identifies the fibres of $V$ over $x$ and $y$ and the metrics on the fibres. With this understanding, the expression $|\nu(x) - \nu(y)|$ is well defined.

So define $C^{k, \alpha}(M)$ to be the set of $f$ in $C^k(M)$ for which the supremum $[\nabla^k f]_\alpha$ defined by $[\nabla]$ exists, working in the vector bundle $\bigotimes^k T^*M$ with its natural metric and connection. The Hölder norm on $C^{k, \alpha}(M)$ is $\|f\|_{C^{k, \alpha}} = \|f\|_{C^k} + [\nabla^k f]_\alpha$.

Lemma 1.5. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose that $F : \Omega \to \mathbb{R}^q$ is bounded and Hölder continuous. Let $Q : \mathbb{R}^q \to \mathbb{R}^p$ be a quadratic function, then $Q \circ F : \Omega \to \mathbb{R}^p$ is also Hölder continuous and

$$[Q \circ F]_\alpha \leq A \sup_{\Omega} \|F\|_{\mathbb{R}^q} \|F\|_{\mathbb{R}^q} [\nu]_\alpha$$

where $A$ is a constant.

In the above lemma by a Quadratic function we mean

$$Q(y) = \sum_{i,j=1}^q Q_{ij} y_i y_j \quad Q_{ij} \in C^1(\overline{\Omega}).$$

We have

Corollary 1.6. Let $f \in C^{1, \alpha}(M, N)$, then

$$[\Pi(f)(df, df)]_{C^\alpha} \leq A \cdot \|d(f)\|_{L^\infty} \cdot [df]_{C^\alpha}$$
Proof. Let \( \{ \Omega_j \} \) be an atlas of \( M \), such that \( \text{diam}(\Omega_j) \leq \text{injrad}(M) \) and set \( F_j = df|_{\Omega_j} \) and \( Q = \text{Hess} \pi_N(X,X) \), for an smooth vector field \( X \). Then using the previous lemma and an appropriate partition of unity we will have the result. \( \square \)

**Schauder Estimate.** In this part, we give a quick review on the Schauder estimate of solutions to linear elliptic partial differential equations. Suppose \((M, g)\) is compact and \( L \) is an elliptic operator, \( L = a^{ij} \nabla_i \nabla_j + b_i \nabla_i + c \), where \( a \) symmetric and positive definite tensor, \( b \) is a \( C^0, \alpha \) vector field on \( M \) and \( c \in C^0, \alpha (M) \), and that \( L \) satisfies the conditions
\[
\|a\|_{C^0, \alpha} + \|b\|_{C^0, \alpha} + \|c\|_{C^0, \alpha} \leq \Lambda
\]
\[
\lambda \|\xi\|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda \|\xi\|^2, \quad \text{for all } x \in M, \text{ and } \xi \in \mathbb{R}^n.
\]

Consider the following problem,
\[
Lu = f \quad \text{in } M,
\]
if \( \partial M = \emptyset \) and
\[
\begin{cases}
Lu = f & \text{in } M \\
u = g & \text{on } \partial M.
\end{cases}
\]
if \( \partial M \neq \emptyset \). Then we have (c.f. Gilbarg-Trudinger [GT83]),

**Theorem 1.7 (Schauder Estimate).** If \( f \in C^{0, \alpha}(M) \) and \( u \in C^2(M) \), then \( u \in C^{2, \alpha}(M) \) and we have
\[
\|u\|_{C^1, \alpha} \leq C(\|f\|_{L^\infty} + \|u\|_{L^\infty}),
\]
\[
\|u\|_{C^{2, \alpha}} \leq C(\|f\|_{C^0, \alpha} + \|u\|_{L^\infty}),
\]
where \( C \) depends on \( M, \lambda, \Lambda \).

Hereafter we present an introduction to the convergence and collapsing theory. Most of the materials in this part was gathered from the work of Rong [Ron10].

1.3. **Convergence.** Here we recall the definition of Gromov-Hausdorff distance between two metric spaces. Let \( X \) and \( Y \) be two compact metric spaces. The Gromov-Hausdorff distance between \( X \) and \( Y \) is defined as
\[
d_{GH}(X,Y) = \inf_Z \{ d^Z_H(\phi(X), \psi(Y)) : \exists \text{ isometric embedding } \phi : X \hookrightarrow Z, \psi : Y \hookrightarrow Z \}
\]
where \( Z \) runs over all such metric spaces and \( \phi \) and \( \psi \) runs over all possible isometric embedding and \( d_H \) is Hausdorff distance.

Let \( \mathcal{MET} \) denote the set of all isometry class of nonempty compact metric spaces, then \( (\mathcal{MET}, d_{GH}) \) is a complete metric space.

**An alternative definition of Gromov-Hausdorff distance.** Let \( X \) and \( Y \) be two elements of \( \mathcal{MET} \), a map \( \phi : X \to Y \) is said to be an \( \epsilon \)-Hausdorff approximation, if the following two conditions satisfied,

i) \( \epsilon \)-onto: \( B_\epsilon(\phi(X)) = Y \).

ii) \( \epsilon \)-isometry: \( |d(\phi(x), \phi(y)) - d(x,y)| < \epsilon \) for all \( x, y \in X \).
The Gromov-Hausdorff distance $\hat{d}_{\text{GH}}(X,Y)$, between $X$ and $Y$ defined to be the infimum of the positive number $\epsilon$ such that there exist $\epsilon-$Hausdorff approximation from $X$ to $Y$ and form $Y$ to $X$. In fact $\hat{d}_{\text{GH}}$ doesn’t satisfy triangle inequality and $\hat{d}_{\text{GH}} \neq d_{\text{GH}}$ but we can show that
\[ \frac{2}{3}d_{\text{GH}} \leq \hat{d}_{\text{GH}} \leq 2d_{\text{GH}} \]

because a sequence in $\mathcal{M}\mathcal{E}\mathcal{T}$ converges with respect to $d_{\text{GH}}$ if and only if it converges with respect to $\hat{d}_{\text{GH}}$, we will not distinguish $\hat{d}_{\text{GH}}$ from $d_{\text{GH}}$.

For the notions of pointed Gromov-Hausdorff convergence and equivariant Gromov-Hausdorff convergence we refer the reader to [Ron10]. Also for the notion of Lipschitz distance see [Fuk88] from Fukaya.

**Measured Gromov-Hausdorff Convergence.** Let $\mathcal{MM}$ denotes the class of all pairs $(X,\mu)$ of compact metric space $X$ and a Borel measure $\mu$ on it such that $\mu(X) = 1$. Let $(X_i,\mu_i)$ be a sequence in $\mathcal{MM}$. We say that $(X_i,\mu_i)$ converges to an element $(X,\mu)$ in $\mathcal{MM}$ with respect to measured Gromov-Hausdorff topology if there exist Borel measurable $\epsilon$-Hausdorff approximation $f_i : (X_i,\mu_i) \to (X,\mu)$ and $f_i(\mu_i)$ converges to $\mu$ with respect to weak* topology. When $M$ is a Riemannian manifold with finite volume, we put $\mu_M = \frac{d\text{vol}_M}{\text{vol}(M)}$, where $d\text{vol}_M$ denotes the volume element of $M$ and regard $(M,\mu_M)$ as an element in $\mathcal{MM}$.

**Convergence of maps.** [GP91] Let $(X_i,p_i), (X,p), (Y_i,q_i)$ and $(Y,q)$ be pointed metric spaces, such that $(X_i,p_i)$ converges to $(X,p)$ in the pointed Gromov-Hausdorff topology (resp. $(Y_i,q_i)$ converges to $(Y,q)$). We say that a sequence of maps $f_i : (X_i,p_i) \to (Y_i,q_i)$ converges to a map $f : (X,p) \to (Y,q)$, if there exists a subsequence $X_{i_k}$ such that if $x_{i_k} \in X_{i_k}$ and $x_{i_k}$ converges to $x$ (in $\coprod X_{i_k} \coprod X$ with the admissible metric), then $f_{i_k}(x_{i_k})$ converges to $f(x)$ and we have

**Lemma 1.8.** i) If $f_i$s are equicontinuous, then there is uniformly continuous map $f$ and a converging subsequence $X_{i_k}$ such that $f_i \to f$.

ii) If $f_i$s are isometries, then the limit map $f : (X,p) \to (Y,q)$ is also an isometry.

For the proof of the above lemma see [Ron10] Lemma 1.6.12.

1.4. **Convergence Theorems, Non-Collapsing.** This subsection is devoted to the theory of convergence of manifolds in non-collapsing case. A sequence of $n$-manifolds $M_i$ converging to a metric space $X$ is called non-collapsing, if $\text{vol}(M_i) \geq v > 0$. Otherwise it is called collapsing. For a non-collapsing sequence of manifolds, there is a uniform lower bound on the injectivity radius of $M_i$ and thus $M_i$s are diffeomorphic for large $i$. This result is due to Cheeger-Gromov (Cheegr [Che70], Peters [Pet84], Greene-Wu [GW88]) and is formulated as following.

**Theorem 1.9.** Let $M_i$ be a sequence of closed $n$-manifolds such that $|\text{sec}_{M_i}| \leq 1$ and $\text{vol}(M_i) > v > 0$, and $M_i$ converges to the metric space $X$. Then $X$ is homeomorphic
to manifold $M$, and for large $i$, there are diffeomorphisms, $\phi_i : M \to M_i$ such that the pullback metric converges to a $C^{1,\alpha}$-metric on $M$ in the $C^{1,\alpha}$-norm.

The following smoothing result concerns the approximation of Riemannian manifolds uniformly by the smooth one.

**Theorem 1.10** (Bemelmans-Oo-Ruh [BMOR84]). Let $(M, g)$ be a compact $n$-manifold with $|\text{sec}_M| < 1$. For any $\epsilon > 0$, there is a metric $g_\epsilon$ such that

$$|g_\epsilon - g|_{C^1} < \epsilon, \quad |\text{sec}_{(M,g_\epsilon)}| \leq 1, \quad |\nabla^k R_{g_\epsilon}| \leq C(n,k) \cdot \epsilon^k.$$  

In particular

$$\begin{cases} 
  e^{-\epsilon} \text{injrad}(M, g) \leq \text{injrad}(M, g_\epsilon) \leq e^{\epsilon} \text{injrad}(M, g) \\
  e^{-\epsilon} \text{diam}(M, g) \leq \text{diam}(M, g_\epsilon) \leq e^{\epsilon} \text{diam}(M, g) \\
  e^{-\epsilon} \text{vol}(M, g) \leq \text{vol}(M, g_\epsilon) \leq e^{\epsilon} \text{vol}(M, g)
\end{cases}$$

1.5. **Convergence Theorems-Collapsing.** This subsection is devoted to the theory of convergence of manifolds, in the collapsing case. Here we state some of the main results in this context.

**Theorem 1.11** (Fibration theorem, Fukaya [Fuk87b]). Let $M^n$ and $N^m$ be compact manifolds satisfying

$$\text{sec}_{M^n} \geq -1, \quad |\text{sec}_{N^m}| \leq 1 \quad (m \geq 2), \quad \text{injrad}(N^m) \geq i_0 > 0$$

There exist a constant $\epsilon(n, i_0)$ such that if $d_{GH}(M^n, N^m) < \epsilon \leq \epsilon(n, i_0)$, then there is a $C^1$-fibration map $f : M^n \to N^m$ with connected fibre such that

i) The diameter of any $f$-fibres is at most $c \cdot \epsilon$, where $c = c(n, \epsilon)$ is such that $c \to 1$ as $\epsilon \to 0$.

ii) $f$ is an almost Riemannian submersion, that is for any vector $\xi \in TM$ orthogonal to a fibre,

$$e^{-\tau(\epsilon)} \leq \frac{|df(\xi)|}{|\xi|} \leq e^{\tau(\epsilon)},$$

where $\tau(\epsilon) \to 0$ as $\epsilon \to 0$.

iii) If in addition, $\text{sec}_{M^n} \leq 1$, then $f$ is smooth and the second fundamental form of any fiber satisfies $|H_{f^{-1}(y)}| \leq c(n)$.

We have the following complement of the previous theorem.

**Theorem 1.12** (Equivariant affine fibration theorem, Fukaya [Fuk88]). Let $M^n$ and $N^m$ be compact manifolds satisfying

$$\text{sec}_{M^n} \geq -1, \quad |\text{sec}_{N^m}| \leq 1 \quad (m \geq 2), \quad \text{injrad}(N^m) \geq i_0 > 0.$$

Assume $M^n$ and $N^m$ admit isometric compact Lie group $G$-actions. There exist a constant $\epsilon(n, i_0) > 0$ such that if $d_{eqGH}((M^n, G), (N^m, G)) < \epsilon \leq \epsilon(n, i_0)$, then there is a $C^1$-fibration $G$-map, $f : M^n \to N^m$ satisfies i-iii above and the following.

iv) The fibers are diffeomorphic to an infranilmanifold, $\Gamma \backslash N$, where $N$ is simply connected nilpotent group, $\Gamma \subset N \rtimes \text{Aut}(N)$, such that $[\Gamma, N \cap \Gamma] \leq \omega(n)$. 
v) There are canonical flat connections on fibres that vary continuously and the $G$-action preserves flat connections.

vi) The structure group of the fibration is contained in $\frac{\text{cent}(N)}{\text{cent}(N) \cap \Gamma} \times \text{Aut}(N \cap \Gamma)$.

By the Fibration Theorem, each fiber is an almost flat manifold and so is diffeomorphic to an infranilmanifold $\Gamma \backslash N$. First any Lie group has a canonical flat connection $\nabla^{\text{can}}$, for which left invariant fields are parallel. The $N$-action on $\Gamma \backslash N$ from the right generate the left invariant fields on $\Gamma \backslash N$ and thus defines the canonical flat connection. By Malcev’s rigidity theorem, any affine structure on $\Gamma \backslash N$ are affine equivalent to $(\Gamma \backslash N, \nabla^{\text{can}})$. If the flat connection on the fibre can be chosen smoothly, then the structure group of the fibration reduces to a subgroup of the affine transformation on $\Gamma \backslash N$. The flat connection on $\Gamma \backslash N$ may depend on the choice of base point on the fiber. By averaging among all flat connections from the various choices of base points, a continuous family of flat connections on each fibers are constructed. In this way we can construct a canonical invariant metric for the left action of $N$.

When a sequence of $n$-manifolds with bounded curvature collapses, the limit space can be a singular space. We have

Theorem 1.13 (Singular fibration theorem, Fukaya [Fuk88]). Let $M_i$ be a sequence of closed $n$-manifolds with $|\text{sec}_{M_i}| \leq 1$ and $\text{diam}(M_i) \leq D$, which converges to the closed metric space $(X,d)$ in $\mathcal{MET}$. Then

i) The frame bundles equipped with canonical metrics converge, $(F(M_i), O(n)) \to (Y, O(n))$, where $Y$ is a manifold.

ii) There is an $O(n)$-invariant fibration $\tilde{f}_i : F(M_i) \to Y$ satisfying the conditions in Theorem 1.11 which becomes for $\epsilon > 0$, a nilpotent Killing structure with respect to an $\epsilon C^1$-closed metric. Moreover each fibre on $M_i$ has positive dimension.

iii) For any $\bar{x} \in X$, a fibre $f_i^{-1}(\bar{x})$ is singular if and only if $p^{-1}(\bar{x})$ is a singular $O(n)$-orbit in $Y$.

In the above theorem, the fibration map $\tilde{f}_i$ descends to a (singular) fibration map $f_i : M_i \to X = Y/O(n)$ such that the following diagram commutes

$$
\begin{array}{ccc}
F(M_i) & \xrightarrow{\tilde{f}_i} & Y \\
\downarrow{p_i} & & \downarrow{p} \\
M_i & \xrightarrow{f_i} & X
\end{array}
$$

metric for the left action of $N$. This leads to the following refinement of Fibration Theorem. A pure nilpotent Killing structure on $M^n$, is a fibration $N \to M^n \to N^m$, with fibre $N$ a nilpotent manifold (equipped with flat connection) on which parallel fields are Killing fields and the $G$-action preserves affine fibration. The underlying $G$-invariant affine bundle structure is called a pure $N$-structure. When we have a pure nilpotent Killing structure on $M^n$ as before, we can construct an invariant metric (invariant under the left action of $N$), and therefore the fibration map $f$ is a Riemannian submersion considering the induced
Let $(\cdot,\cdot)$ denote the original metric and $(\cdot,\cdot)$, the invariant one and suppose $M^n$ and $N^m$ are $A$-regular, which means that for some sequence $A = \{A_k\}$ of real non-negative numbers, we have
\[ |\nabla^k R| \leq A_k, \tag{7} \]
then we have
\[ |\nabla^k((\cdot,\cdot)-(\cdot,\cdot))| \leq c(n, A) \cdot \epsilon \cdot \text{injrad}(N)^{-k+1}, \tag{8} \]
where $c(n, A)$ is a generic constant depending to finitely many $A_k$ and $n$ (see Proposition 4.9 in [CPC92]). Here the GH-distance $M^n$ and $N^m$ is less than $\epsilon$.

In the following $\mathcal{M}(n, D)$ denotes the set of all compact Riemannian manifold $M$ such that, $\dim(M) = n$, $\dim(M) < D$ and sectional curvature $|\text{sec}| \leq 1$, and $\mathcal{M}(n, D, v)$ the set of manifolds in $\mathcal{M}(n, D)$, with volume $\geq v$. In the following remark we collect the main points that we need from the theorems above and explain the classification in the proof of Theorem 0.1.

**Remark 2.** When a sequence of manifolds $M_i$ converges in $\mathcal{M}(n, D)$ to a metric space $X$, then according to Theorem 1.13 the frame bundles over $M_i$ equipped with canonical metrics $\hat{g}_i$ converge to a manifold $Y$, and $\tilde{f}_i : (F(M_i), \hat{g}_i, O(n)) \to (Y, O(n))$ is an $O(n)$ invariant fibration map. To see this, let $\hat{g}_{i\epsilon}$ be the smooth metric as it appeared in Theorem 1.10 then $(F(M_i), \hat{g}_{i\epsilon})$ converges to smooth manifold $Y_{\epsilon}$. For a small fixed $\epsilon_0$ and $\epsilon < \epsilon_0$, the sectional curvature on $(F(M_i), \hat{g}_{i\epsilon})$ is uniformly bounded and we can apply Theorem 1.12 to conclude that there exists an $O(n)$-invariant smooth fibration map $\tilde{f}_{i\epsilon}$. By the continuity $(F(M_i), \hat{g}_{i\epsilon})$ is conjugate to $(F(M_i), \hat{g}_{i\epsilon_0})$. This implies that $Y_{\epsilon} \to Y$ is equivalent to a convergent sequence of metrics on $Y_{\epsilon_0}$ and so $(Y, O(n))$ conjugate $(Y_{\epsilon_0}, O(n))$

\[ (F(M_i), O(n)) \simeq (F(M_i), \hat{g}_{i\epsilon_0}, O(n)) \xrightarrow{\tilde{f}_{i\epsilon_0}} (Y_{\epsilon_0}, O(n)) \simeq (Y, O(n)), \]

and it induces a fibration map $(F(M_i), \hat{g}_i, O(n)) \xrightarrow{\tilde{f}_i} (Y, O(n))$ (see proof of Theorem 4.1.3 in [Ron10]). Note that $(F(M_i), \hat{g}_{i\epsilon}, O(n))$ is a pure nilpotent Killing structure and so there exists an invariant Riemannian metric close to $\hat{g}_{i\epsilon}$ which satisfies the inequality (8). Consequently the fibration map $\tilde{f}_{i\epsilon}$ is a Riemannian submersion (considering the induced Riemannian metric on $Y_{\epsilon}$ by this map).

### 1.6. Density function.

Let $\mathcal{DM}(n, D)$ denote the closure $\mathcal{M}(n, D)$ in $\mathcal{MM}$ with respect to the measured Hausdorff topology. Then $\mathcal{DM}(n, D)$ is compact with respect to measured Hausdorff topology. Let $(M_i, g_i, \frac{\text{dvol}_M}{\text{vol}(M_i)}) \in \mathcal{M}(n, D)$, be a sequence of manifolds which converges to a manifold $(M, g, \mu)$ in measured Gromov-Hausdorff topology. Then there exists a fibration map $\psi_i : M_i \to M$ satisfying the following: for $x \in M$, we put
\[ \Phi_i^t(x) = \text{vol}(\psi_i^{-1}(x)), \quad \Phi_i = \frac{\Phi_i^t}{\text{vol}(M_i)}. \]

Then there exists $\Phi$ such that $\Phi = \lim_{t \to \infty} \Phi_i$ and $\mu$ is absolutely continuous with respect to $\text{dvol}_M$, $\mu = \Phi \cdot \text{dvol}_M$ (see §3 in [Fuk87a]).
In the general case, for \((X, \mu) \in D\mathcal{M}(n, D)\), we recall first a remark on quotient spaces.

**Remark 3** (Besse [Bes08]). Let \((M, g)\) be a Riemannian manifold and \(G\) a closed subgroup of isometries of \(M\). Assume that the projection \(\pi : M \to M/G\) is a smooth submersion. Then there exist one and only one Riemannian metric \(\tilde{g}\) on \(B = M/G\) such that \(\pi\) is a Riemannian submersion (see Subsection 9.12).

We recall that using the general theory of slices for the action of a group of isometries on a manifold, one may show that there always exists an open dense submanifold \(U\) of \(M\) (the union of the principle orbits), such that the restriction \(\pi|_U : U \to U/G\) is a smooth submersion.

Considering now \(M/G\) as a Riemannian polyhedron and \(\mu_g\) as its Riemannian volume, the restriction of \(\mu_g\) on \(U/G\) is equal to \(d\text{vol}_{U/G} = d\text{vol}_{B - S(B)}\).

Suppose \(M_i\) in \(\mathcal{M}(n, D)\) converges to the metric space \(X\). We may assume that \(FM_i\) with the induced \(O(n)\)-invariant metric \(\tilde{g}_i\), converges to \((Y, g, \Phi_Y \cdot d\text{vol}_Y)\) with respect to the \(O(n)\)-measured Hausdorff topology, \(g, \Phi_Y\) are \(C^{1,\alpha}\)-regular. Moreover, since \(\pi_i : F(M_i) \to M_i\) is a Riemannian submersion with totally geodesic fibres and since the fibres are isometric to each other, it follows that \((FM_i, d\text{vol}_{FM_i})/O(n) = (M_i, d\text{vol}_{M_i})\). Hence by equivariant Gromov-Hausdorff convergence \(M_i\) converges to \((X, \nu) = (Y, \Phi_Y \cdot d\text{vol}_Y)/O(n)\) (see Theorem 0.6 in [Fuk89]) and by Remark 3 \(\nu(S(X)) = 0\).

For all \(x\) in \(X\), we put
\[
\Phi_X(x) = \int_{y \in \pi^{-1}(x)} \Phi_Y(y) \ d\text{vol}_{\nu^{-1}(x)}
\]
where \(\pi : Y \to X\) is the natural projection. For each open set \(U\)
\[
\nu(U) = \int_U \Phi_X(x) \ d\text{vol}_{X - S(X)}
\]
Now we are ready to start the proof of Theorem 0.1.

### 2. Proof of the Convergence Theorem

In this section we are going to prove Theorem 0.1. We split the proof in three cases:

**Case I: Non-collapsing.** \(M_i\) converge to \(M\) in \(\mathcal{M}(n, D, v)\). We first consider the situation where \(M_i = M\) and \(g_i\) converges to a metric \(g\) in \(\mathcal{M}(n, D, v)\). Then we study the problem in the general case using Theorem 1.9.

**Case II: Collapsing to a manifold.** \((M_i, g_i)\) converge to manifold \((M, g)\) in \(\mathcal{M}(n, D)\) with \(g\) a \(C^{1,\alpha}\)-metric. We first consider the situation when \((M_i, g_i)\) satisfies some regularity assumption (see Assumption 3 below). Then we discuss the general case using the fact that there is always a sequence of metric \(g_i(\epsilon)\) on \(M_i\), \(C^1\)-close to the the metric \(g_i\) which...
satisfies Assumption 1 as it is explained in Remark 2.

**Case III: Collapsing to a singular space.** \( M_i \) converge to a metric space \((X,d)\) in \( \mathcal{M}(n,D) \). When a sequence of manifolds \( M_i \) converges in \( \mathcal{M}(n,D) \) to a metric space \( X \), the frame bundles over \( M_i \) converge to a Riemannian manifold \( Y \), with a \( C^{1,\alpha} \)-metric and \( X = Y/O(n) \). The harmonic maps over \( M_i \), induce harmonic maps over \( F(M_i) \) and this case followed from the study of harmonic maps on quotient spaces.

We fix an isometric embedding \( i : N \to \mathbb{R}^q \) and we often denote the composition \( i \circ f \) simply by \( f \), unless we need to explicitly distinguish these two maps.

### 2.1. Case I: Non-collapsing.

In this subsection we prove

**Proposition 2.1.** Let \((M_i,g_i)\) be a sequence of Riemannian manifolds in \( \mathcal{M}(n,D,v) \) which converges to Riemannian manifold \( (M,g,\Phi \text{dvol}_g) \) in the measured Gromov-Hausdorff Topology. Suppose \((N,h)\) is a compact Riemannian manifold. Let \( f_i : (M_i,g_i) \to (N,h) \) be a sequence of harmonic maps such that \( \|e_{g_i}(f_i)\|_{L^\infty} < C \), where \( C \) is a constant independent of \( i \). Then \( f_i \) has a subsequence which converges to a map \( f : (M,g,\Phi \text{dvol}_g) \to (N,h) \), and this map is a smooth harmonic map.

To go through the proof in this case, we consider first the situation when a sequence of metrics \( g_i \) on a manifold \( M \), converges to a metric \( g \).

**Lemma 2.2.** Let \( g_i \) be a sequence of metrics on smooth Riemannian manifold \( M \) and \( (M,g_i) \) converge to \( (M,g) \) in \( \mathcal{M}(n,D,v) \). Suppose \( f_i : M \to N \) is a sequence of harmonic maps such that

\[
\|e_{g_i}(f_i)\|_{L^\infty} < C
\]

where \( C \) is a constant independent of \( i \). Then there exists a subsequence of \( f_i \) which converges to some \( f \) in \( C^k \)-topology for \( k \geq 0 \) and \( f \) is also harmonic.

**Proof.** By Theorem 1.9, the metric \( g_i \) converges to \( g \) in \( \mathcal{M}(n,D,v) \) in \( C^{1,\alpha} \)-topology. Using Schauder estimate, \( f_i \)'s have bounded norm in \( C^k(M,g) \) for every \( k \geq 0 \) and so they are converging to a map \( f \in C^k(M,g) \). We have

\[
\lim_{i \to \infty} \Delta_{g_i}f_i = \Delta_g f
\]

and

\[
\lim_{i \to \infty} \Pi(f_i)(df_i,df_i) = \Pi(f)(df,df)
\]

The above limits lead to harmonicity \( f \). \( \square \)

Using the above lemma we can prove Proposition 2.1.
Proof of Proposition 2.2. When \( M_i \) converges to \( M \) in \( \mathcal{M}(n,D,v) \), by Theorem 1.9 there is a \( C^{1,\beta} \)-diffeomorphism \( \phi_n : M_n \to M \), such that the push forward of \( \phi_n \) of the metrics \( g_n \) on \( M_n \) converges to the metric \( g \). Since the map \( \Phi_n : (M_n, g_n) \to (M, \phi_n^*(g)) \) is an isometry

\[
e_{g_n}(f_n) = e_{\phi_n^*(g_n)}(\bar{f}_n)
\]

where \( \bar{f}_n \) is the map \( f_n \circ \phi_n^{-1} \). The map \( f_n \) is harmonic and so \( \bar{f}_n \). So all the assumptions of Lemma 2.2 are satisfied here and the proof of theorem 0.1 in this case is complete. \( \square \)

In Lemma 2.2 if we replace the assumption uniform bound on the energy density \( \|e_{g_i}(f_i)\|_{L\infty} < C \), with the assumption uniform bound on the energy \( E_{g_i}(f_i) < C \), then the limiting map is not necessarily harmonic (see Theorem 1.3 and Remark 1).

Proposition 2.3. Let \((M_i, g_i)\) be a sequence of manifolds in \( \mathcal{M}(n,D,v) \) which converges to a Riemannian manifold \((M, g)\) in the measured Gromov-Hausdorff topology. Suppose \((N,h)\) is a compact Riemannian manifold which doesn’t carry any harmonic 2-sphere \( S^2 \). Let \( f_i : (M_i, g_i) \to (N,h) \) be a sequence of harmonic maps such that \( \|E_{g_i}(f_i)\| < C \), where \( C \) is a constant independent of \( i \). Then \( f_i \) has a subsequence which converges to a map \( f : (M, g) \to (N,h) \), and this map is a weakly harmonic map.

Proof. When we have a sequence of manifolds \((M_i, g_i)\) converges in \( \mathcal{M}(n,d,v) \), then the injectivity radius is bounded from below and \( \text{dvol}_{g_i} \) converges to \( \text{dvol}_g \) weakly. Therefore if \( E_{g_i}(f_i) < C \), \( C \) independent of \( i \), then \( E_g(f_i) \) is uniformly bounded. By Remark 1 \( f_i \) converges strongly in \( \mathcal{H}^1 \) to the map \( f \). Also \( \text{Hess}(\pi_N) \) restricted to a neighborhood of \( N \) is Lipschitz and by Lemma 6.4 in Taylor’s book [Tay00], \( \text{Hess}(\pi_N) \circ f_i \) converges to \( \text{Hess}(\pi_N) \circ f \) in \( \mathcal{H}^1 \)-norm and so we have \( \Pi(f_i)(df_i, df_i) \to \Pi(f)(df, df) \) weakly. We have the same for \( \Delta f_i \) and so \( f \) is a weakly harmonic map. \( \square \)

Under some other assumptions on \( N \) or on the image of \( f \), we can show that the limit map \( f \) is strongly harmonic. These results are direct consequences of some of the theorems in [Sch84] and here we avoid to repeat statement of these results.

Proposition 2.4. Let \((M_i, g_i)\) and \( f_i \) be as in Proposition 2.3. Then the map \( f \) is strongly harmonic, providing that \( N \) is a compact manifold and we have one of the following conditions:

a) \((N,h)\) is a non-positively curved manifold.

b) There is no strictly convex bounded function on \( f(M) \).

Proof. a) See Proposition 2.1 in [Sch84].

b) See Corollary 2.4 in [Sch84] \( \square \)

2.2. Case II: Collapsing to a manifold. In this subsection we prove

Proposition 2.5. Let \((M_i, g_i)\) be a sequence of Riemannian manifolds in \( \mathcal{M}(n,D) \) which converges to Riemannian manifold \((M,g, \Phi \text{dvol}_g)\) in the measured Gromov-Hausdorff Topology with \( C^{1,\alpha} \)-metric \( g \). Suppose \((N,h)\) is a compact Riemannian manifold. Let \( f_i : (M_i, g_i) \to (N,h) \) be a sequence of harmonic maps such that \( \|e_{g_i}(f_i)\|_{L\infty} < C \), where
$C$ is a constant independent of $i$. Then $f_i$ has a subsequence which converges to a map $f : (M, g, \Phi \, d\text{vol}_g) \to (N, h)$, and this map is a weakly harmonic map.

Before we prove the proposition in general, we will prove the following proposition which has an additional regularity assumption. Then at the end of this subsection, we will apply this proposition to prove case II. Consider the following assumption,

**Assumption 1.** Let the metric $g_i$ be regular, i.e. there exists a sequence $C = \{C_k\}$ of positive number $C_k$ independent of $i$, such that

\[ |\nabla^k \text{R}(M, g_i)| < C_k. \]  

(10)

Suppose also that the Riemannian metric $g_i$ is an invariant metric.

We have

**Proposition 2.6.** Let $(M_i, g_i)$ be a converging sequence of manifolds in $M(n, D)$ (with respect to the measured Gromov-Hausdorff Topology) such that $g_i$ satisfies the Assumption 1. Let $(M, g, \Phi)$ be the limit manifold. Suppose $(N, h)$ is a compact Riemannian manifold. Let $f_i : (M_i, g_i) \to (N, h)$ be a sequence of harmonic maps such that $\|e_{g_i}(f_i)\|_{L^\infty} < C$, where $C$ is a constant independent of $i$. Then $f_i$ has a subsequence which converges to a map $f : (M, g, \Phi) \to (N, h)$, and this map is a smooth harmonic map.

Before we prove the proposition, we first remark on some results of Fukaya. Then we prove Lemma 2.7 which is the main element in the proof of Proposition 2.6.

**Remark 4.** In [Fuk88, Fuk89] he proves that with extra regularity assumption \[\text{(10)}\] $(M_i, g_i, \frac{d\text{vol}_M}{\text{vol}(M_i)})$ converges to a smooth Riemannian manifold $(M, g, \Phi)$, with the smooth pair $(g, \Phi)$ (see Lemma 2.1 in [Fuk89]). By Theorem 1.11, we know that for $i$ large enough there is a fibration map, $\psi_i : M_i \to M$. Since $g_i$ is an invariant metric (see 2), therefore there exist metrics $g_i^M$ on $M$ such that the maps $\psi_i : (M_i, g_i) \to (M, g_i^M)$ is a Riemannian submersion. By Theorem 8 $(M, g_i^M)$ converges to $(M, g)$ in $C^{1,\alpha}$-topology.

**Remark 5** (Fukaya [Fuk88, Fuk89]). Take an arbitrary point $p_0$ in $M$ and choose $p_i \in \psi_i^{-1}(p_0)$. By $|\text{sec}_M| \leq 1$, we know at point $p_i$ on $M_i$, the conjugate radius is greater than some constant name it $\rho$. If we consider the pullback metric by exponential map at $p_i$, $\exp_{p_i}$, on the conjugate domain on the tangent space at $p_i$, then the injectivity radius at $0$ is at least the conjugate radius at $p_i$ (see Corollary 2.2.3 in Ron10).

Consider the ball $B = B(0, \rho)$ in $T_{p_i}M_i$ with the metric $g_i$ induced by the exponential map. By virtue of assumption (10), $g_i$ will converge to some $g_0$ in the $C^\infty$-topology. There are local groups $G_i$ converging to a Lie group germ $G$ such that

1. $G_i$ acts by isometries on the pointed metric space $((B, g_i), 0)$.
2. $((B, g_i), 0)/G_i$ is isometric to a neighborhood of $p_i$ in $M_i$.
3. $G$ acts by isometries on the pointed metric space $((B, g_0), 0)$.
4. $((B, g_0), 0)/G$ is isometric to a neighborhood of $p_0$ in $M$ and the action of $G$ is free. It follows that there is a neighborhood $U$ of $p_0$ in $M$ and a $C^\infty$ map $s : U \to B$ such that $s(p_0) = 0$. 

2. \( P \circ s = 1d \), where \( P \) denotes the composition of the projection map and the above mentioned isometry in 4.

3. \( d_{\mathcal{B}(\mathcal{M})}(s(q),0) = d_{\mathcal{N}}(q,p_0) \) holds for \( q \in \mathcal{N} \).

Therefore there is some \( \rho \) independent of \( i \) such that \( \mathcal{M} = \bigcup_{j=1}^{m} B_{\rho}(x_j, \mathcal{M}) \) and \( B_{\rho}(x_j, \mathcal{M}) \) satisfies the preceding conditions. Also we can construct a \( C^\infty \) section \( s_{i,j} : B_{1}(x_j, \mathcal{M}) \to \mathcal{M}_i \) of \( \psi_i \), such that

\[
\frac{|(s_{i,j})_*(v)|}{|v|} < C \tag{11}
\]

for each \( v \in TB_{1}(x_j, \mathcal{M}) \) and \( C \) is a constant independent of \( i \). Hereafter we put \( p_{i,j} = \psi_i^{-1}(x_j) \) and by \( B(p_{i,j}) \) we mean a ball centered at \( p_{i,j} \), with radius \( \rho \) in \( T_{p_{i,j}} \mathcal{M}_i \). See section 3 in \([\text{Fuk88}]\) and section 2 in \([\text{Fuk89}]\).

Now we show that \( f_i \)'s are almost constant on the fibers of \( \mathcal{M}_i \). The following lemma is similar to Lemma 4.3 in \([\text{Fuk87a}]\). Recall that if \( v \) is a tangent vector to \( \mathcal{M}_i \) and \( h \) is a function on \( \mathcal{M}_i \), then \( v \cdot h = dh(v) \) denotes the derivative of \( h \) in the direction of \( v \). In the following Lemma \((\mathcal{M}_i, g_i) \) is a converging sequence in \( \mathcal{M}(n, D) \) such that \( g_i \) satisfies \([\text{10}]\) and \( N \) and \( N \) is a compact Riemannian manifold.

**Lemma 2.7.** Let \( h_i : \mathcal{M}_i \to i(N) \subset \mathbb{R}^q \) be a smooth maps which satisfies the Euler-Lagrange equation \([2]\). Suppose \( v_i \in T_{p_i}(\mathcal{M}_i) \), satisfies \((\psi_i)_*(v_i) = 0 \), and \( v'_i, v''_i \in T_{p_i}(\mathcal{M}_i) \) \((p \in B_{2\rho/3}(p_{i,j}, \mathcal{M}_i))\). Then we have

\[
|v_i \cdot h_i| \leq C_1 \cdot |v_i| \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}) \tag{12}
\]

\[
|v'_i \cdot h_i| \leq C_2 \cdot |v'_i| \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}) \tag{13}
\]

where \( C_1 \) and \( C_2 \) are some constants independent of \( i \) and \( \epsilon'_i \) is a sequence converging to zero.

**Proof.** We put \( \Phi_{i,j} = \exp_{p_{i,j}} : B(p_{i,j}) \to \mathcal{M}_i \), \( \tilde{g}_{i,j} = \Phi_{i,j*}(g_i) \) and \( a = \Phi_{i,j}^{-1} \). We also denote \( h_i \circ \Phi_{i,j} \) by \( h_{i,j} \).

From the Schauder estimate for elliptic equations (see Theorem \([1.7]\)) we have,

\[
\|h_{i,j}\|_{C^{1,\alpha}} \leq C' \cdot (\|\Delta h_{i,j}\|_{L^\infty} + \|h_{i,j}\|_{L^\infty}) \tag{14}
\]

and hence

\[
\|v'_i \cdot h_{i,j}\|_{C^{\alpha}} \leq C'' \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}) \tag{15}
\]

where \( C' \) depends on the metric \( \tilde{g}_{i,j} \). Since \( \Phi_{i,j} \) is an isometry, by composition formula (see formula 1.4.1 in \([\text{Xin96}]\)), we have \( \Delta h_{i,j}(x) = \Delta h_i(\Phi_{i,j}(x)) \). Also from \((14)\), and the fact that \( \tilde{g}_{i,j} \) converges in \( C^{\alpha} \),

\[
\|\Pi(h_{i,j})(dh_{i,j}, dh_{i,j})\|_{C^{\alpha}} \leq C'' \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty})
\]

where \( C'' \) is a constant independent of \( i \). By equation \((2)\), we have

\[
\|\Delta h_{i,j}\|_{C^{\alpha}} \leq C'' \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty})
\]
Using the Schauder estimate for second derivative, we have
\[ \| h_{i,j} \|_{C^{2,\alpha}} \leq C \cdot (\| \Delta h_i \|_{L^\infty} + \| h_i \|_{L^\infty}) \] 
for some $C$ independent of $i$ and we have (13).

Now we will prove (12) by contradiction. Assume $|v_i| = 1$. Let $\sigma^i(t) = \exp_{p_i}^F(tv_i)$ be a 
geodesic in the fiber containing $p_i$, $F_i \subset M_i$ such that $\frac{d}{dt}_{|t=0} \sigma^i(t) = v_i$. For $0 \leq t \leq \frac{L}{\epsilon}$ this curve has a lift $l^i(t) \subset B(p_{i,j})$ such that $\Phi_{i,j}(l^i(t)) = \sigma^i(t)$. We have
\[ d(\sigma_i(t), p) \leq \text{diam}(F_i) \leq \epsilon_i \]
By contradiction we assume that there is subsequence of $h_i$ and a positive number $A$ such that
\[ |v_i \cdot h_{i,j}| > A \cdot (\| \Delta h_i \|_{L^\infty} + \| h_i \|_{L^\infty}) \]
We know that
\[ v_i \cdot h_i = v_i \cdot h_{i,j} = \frac{d}{dt}_{|t=0} h_{i,j} \circ l^i(t). \]
There exist $\beta > 0$ and $\delta > 0$ independent of $i$ such that for any $t < \delta$, we have
\[ |h_{i,j} \circ l^i(t) - h_{i,j}(a)| > \beta \cdot t \cdot (\| \Delta h_i \|_{L^\infty} + \| h_i \|_{L^\infty}). \] (17)
To explain this, let $h_{i,j} \circ l^i(t) = q_{i,j}(t)$. We know from (16) that
\[ \left| \frac{d}{dt}_{|t=0} q_{i,j}^i(t) \right| \leq C(\| \Delta h_i \|_{L^\infty} + \| h_i \|_{L^\infty}), \]
so for some fix $\delta$ and $0 < t < \delta$ we have
\[ |q_{i,j}^i(t) - q_{i,j}^i(0)| \leq C' \cdot t \cdot (\| \Delta h_i \|_{L^\infty} + \| h_i \|_{L^\infty}). \]
On the other hand we have
\[ |q_{i,j}^i(0)| > A \cdot (\| \Delta h_i \|_{L^\infty} + \| h_i \|_{L^\infty}), \]
so for $\delta$ small enough and $t < \delta$ we have
\[ |q_{i,j}^i(t)| > \beta \cdot (\| \Delta h_i \|_{L^\infty} + \| h_i \|_{L^\infty}). \]
Therefore
\[ |q_{i,j}(t) - q_{i,j}(0)| = |q_{i,j}^i(\theta_i) \cdot t| > \beta \cdot t \cdot (\| \Delta h_i \|_{L^\infty} + \| h_i \|_{L^\infty}). \]
from which (17) follows.

There exists $b \in B(p_{i,j})$, such that $d(a, b) < \epsilon_i$ and $\Phi_{i,j}(l_i(\delta')) = b$, for a fixed $\delta' < \delta$ we have
\[ |h_{i,j}(b) - h_{i,j}(a)| > \beta \cdot \delta' \cdot (\| \Delta h_i \|_{L^\infty} + \| h_i \|_{L^\infty}). \]
If we fix \( \{\xi_k\}_{k=0}^{k=n} \) as a coordinate system at the point \( a \in B(p_{i,j}) \), for some \( b' \in B(p_{i,j}) \) we have
\[
\sum_{k=0}^{k=n} \frac{\partial h_{i,j}}{\partial \xi^k} > C \cdot \beta \cdot \frac{\delta'}{\epsilon_i} \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}),
\]
and this contradicts (15). \qed

Now we prove Proposition 2.6.

Proof of Proposition 2.6. As we assumed \( \|e(f_i)\|_{L^\infty} < c \) and by the Euler-Lagrange equation and Corollary 1.6 we have that \( \|\Delta f_i\|_{L^1} \) is uniformly bounded. Moreover, \( \|f_i\|_{L^\infty} \) is uniformly bounded, therefore by the above lemma (12), the maps \( f_i \) are equicontinuous. By Lemma 1.8 there is a limit map \( f : M \to N \) which is continuous.

We consider the following maps on \( M \),
\[
\tilde{f}_i = \sum \beta_j \cdot (i \circ f_i) \circ s_{i,j}, \quad (18)
\]
\( \beta_j \) is an arbitrary \( C^\infty \) partition of unity associated to \( B_{e_i}(x_j, M) \) and \( s_{i,j} \) is the section associated to \( \psi_i \) as mentioned in Remark 5 and \( i : N \to \mathbb{R}^q \) is an isometric embedding. Along a subsequence which we again denote by \( f_i \), we have
\[
\lim_{i \to \infty} f_i(s_{i,j}(x)) = f(x) \quad \text{for} \ x \in B_{e_i}(x_j, M),
\]
and also
\[
\lim_{i \to \infty} \tilde{f}_i(x) = i \circ f(x) \quad \text{for} \ x \in B_{e_i}(x_j, M).
\]

Since the energy density of \( f_i \) is bounded and also \( s_{i,j} \) satisfies in (11), we have \( \|e(\tilde{f}_i)\|_{L^\infty} \) is uniformly bounded. Indeed, we have \( \|\tilde{f}_i\|_{L^1} \) is bounded, \( \tilde{f}_i \) converge uniformly to \( i \circ f \). Moreover \( \psi_i \) has bounded second fundamental form (see Theorem 2.6 in [CFG92]) and the same is true for \( s_{i,j} \). So \( \tilde{f}_i \) has bounded \( C^2 \)-norm and there is a subsequence of \( \tilde{f}_i \) which converges to \( i \circ f \) in \( C^1 \)-topology.

Choose a local orthonormal frame \( \{e_k\}_{k=1}^m \) on \( (M_i, g_i^M) \). Denote its horizontal lift on \( (M_i, g_i) \) by \( \{e_k\}_{k=1}^m \). Suppose \( \{e_i\}_{i=m+1} \) is a local orthonormal frame field of the fiber \( F_i \) in \( M_i \). Thus \( \{e_k, e_l\} \) form a local orthonormal frame field in \( M_i \). (Note that we omit put the index \( i \) for the orthonormal frame fields on \( (M_i, g_i) \) and \( (M, g_i^M) \).) Our aim is to show that \( f \) is also weakly harmonic.

Lemma 2.8. We have
\[
\lim_{i \to \infty} \left| \langle di \circ f_i, d\eta_i \rangle(p) - \langle d\tilde{f}_i, d\tilde{\eta}(\psi_i(p)) \rangle \right| = 0,
\]
where \( \tilde{\eta} : M \to \mathbb{R}^q \), is a \( C^\infty \)-map and \( \eta_i = \tilde{\eta} \circ \psi_i \) and \( p \) in \( M_i \).

Proof. By inequality (12),
\[
|\langle di \circ f_i, d\eta_i \rangle(p) - \sum_{k=1}^m \langle di \circ f_i(e_k), d\eta_i(e_k) \rangle(p) | \leq C \cdot \epsilon_i
\]
for \( i \) large enough and \( C \), a constant. Let \( F_i \) denote the fibre containing \( p \) and choose a point \( q \) in \( F_i \). By (13) and \( \text{diam}(F_i) \leq \epsilon_i \)
\[ |d_i \circ f_i(e_k)(p) - d_i \circ f_i(e_k)(q)| \leq C \cdot \epsilon_i, \]
and so
\[ |d_i \circ f_i(e_k)(p) - d_i \circ f_i(e_k)(s_{i,j} \circ \psi_i(p))| \leq C \cdot \epsilon_i. \]
Because \( \psi_i \circ s_{i,j} = \text{Id} \), we have for \( x \in M \)
\[ \psi_i^*(e_k(s_{i,j}(x)) - s_{i,j}^*(\tilde{e}_k(x))) = 0. \]
By the inequality (11), we have
\[ |e_k(s_{i,j}(x)) - s_{i,j}^*(\tilde{e}_k(x))| \leq C, \]
for some constant \( C \), therefore by (12),
\[ |d_i \circ f_i(e_k)(p) - d_i \circ f_i(e_k)(\bar{f}_i)(\psi_i(p))| \leq C \cdot \epsilon_i. \]
From the convergence of \( f_i \circ s_{i,j} \) to \( f \), we have
\[ \lim_{i \to \infty} \sum d\beta_j \cdot (i \circ f_i) \circ s_{i,j} - \sum d\beta_j \cdot (i \circ f) = 0, \]
So
\[ \lim_{i \to \infty} |d\tilde{f}_i - \sum \beta_j \cdot (i \circ f_i) \circ s_{i,j}| = 0, \]
since \( \sum_j \beta_j = 1 \), we finally have
\[ \lim_{i \to \infty} |\langle d_i \circ f_i, d\eta \rangle(p) - \langle d\tilde{f}_i, d\tilde{\eta} \rangle(\psi_i(p))| = 0. \]

**Lemma 2.9.** We have
\[ \lim_{i \to \infty} \left| \Pi(f_i)(p)(d_i \circ f_i, d_i \circ f_i) - \Pi(f_i)(\psi_i(p))(d\tilde{f}_i, d\tilde{f}_i) \right| = 0, \]

*Proof.* By the above Lemma, we have
\[ \lim_{i \to \infty} |df_i(p) - d\tilde{f}_i(\psi_i(p))| = 0 \]
By the same argument as Lemma 2.8 we can conclude
\[ \left| \Pi(f_i)(p)(d_i \circ f_i, d_i \circ f_i) - \Pi(f_i)(\psi_i(p))(d\tilde{f}_i, d\tilde{f}_i) \right| \leq C \cdot |df_i(p) - d\tilde{f}_i(\psi_i(p))| \]
and so we have the result \( \square \)
The map $\tilde{f}_i : (M, g_i^M, \text{dvol}_{g_i^M}) \to \mathbb{R}^q$, converges in $C^1$ to the map $i \circ f$, and $\Phi_i$ converges to $\Phi$ in $C^\infty$ topology. Also $(M, g_i^M)$ converges to $(M, g)$ in $\mathcal{M}(n, D, v)$. Therefore, we have

$$\left| \int_M \Xi_M(\eta, \tilde{f}_i) \Phi_i \text{dvol}_{g_i^M} - \int_M \Xi(\eta, f) \Phi \text{dvol}_g \right| \leq C \cdot \epsilon_i,$$

where $\Xi(\cdot, \cdot)$ is defined in \[Fuk88\]. By Lemma 2.3 and 2.9, we have

$$\lim_{i \to \infty} \left| \int_{M_i} \Xi_{g_i}(\eta_i, f_i) \frac{\text{dvol}_{M_i}}{\text{vol}(M_i)} - \int_M \Xi_{g_i^M}(\eta, \tilde{f}_i) \psi_i \left( \frac{\text{dvol}_{M_i}}{\text{vol}(M_i)} \right) \right| = 0,$$

and so we have

$$\lim_{i \to \infty} \int_{M_i} \Xi_{g_i}(\eta_i, f_i) \frac{\text{dvol}_{M_i}}{\text{vol}(M_i)} = \int_M \Xi_g(\eta, f) \Phi \text{dvol}_M.$$

So we have $f$ is weakly harmonic and since it is continuous, it is also a harmonic map. \qed

Now we prove Case II without considering Assumption 1.

**Proof of Proposition 2.5.** By Theorem 1.13 we can obtain $C^1$-close metric $g_i(\epsilon)$ which satisfies $\Xi$ and such that the map $\psi_i : (M_i, g_i(\epsilon)) \to (M, \psi_i^*(g_i(\epsilon)))$ is a Riemannian submersion.

For small $\epsilon$, let $M(\epsilon)$ be the Gromov-Hausdorff limit of some subsequence $(M_i, g_i(\epsilon))$. By Lemma 2.3 in \[Fuk88\], $(M_i, g_i(\epsilon))$ and $(M(\epsilon), g(\epsilon))$ converge to $(M_i, g_i)$ and $(M, g)$ in $\mathcal{M}(n, D, v)$ respectively.

$f_i : (M_i, g_i) \to (N, h)$ is harmonic and since $g_i(\epsilon)$ is $C^1$-close to $g$, we have

$$|\Xi_{g_i}(f_i, \eta_i) - \Xi_{g_i(\epsilon)}(f_i, \eta_i)| \leq C \cdot \epsilon_i.$$

By (19), we have

$$\lim_{i \to \infty} \left| \int_{M_i} \Xi_{g_i(\epsilon)}(f_i, \eta_i) \frac{\text{dvol}_{M_i, g_i(\epsilon)}}{\text{vol}(M_i, g_i(\epsilon))} - \int_{M(\epsilon)} \Xi_g(f, \eta) \Phi(\epsilon) \text{dvol}_{M(\epsilon)} \right| = 0,$$

and finally since $g(\epsilon)$ converges to $g$ in $C^{1,\alpha}$-topology, we have the desired result. \qed

2.3. **Case III: Collapsing to a singular space.** Now we are going to investigate the general case when the sequence converges to a singular space. This means that $M_i \in \mathcal{M}(n, D)$ converges to some metric space $(X, d)$. First we recall the following remark from \[Fuk87a\].

**Remark 6 (Fuk87a, §7).** Let $Y$ be a Riemannian manifold on which $O(n)$ acts as isometries, and $\theta : Y \to [0, \infty)$ be an $O(n)$-invariant smooth function. Put $X = Y/O(n)$. Let $\pi : Y \to X$ be the natural projection, $\theta : X \to [0, \infty)$ the function induced from $\theta$, and $S(X)$ the set of all singular points of $X$. The set $S(X) \subset X$ has a well defined normal bundle on the codimension 2 strata ($X = Y/O(n)$ is a Riemannian polyhedra and $S(X)$ is a subset of $n-2$-skeleton of $X$). Set

$$\text{Lip}(X, S(X)) = \{ u \in \text{Lip}(X) \mid v \cdot u = 0 \text{ if } v \text{ is perpendicular to } S(X) \}.$$
Define $Q_1 : \text{Lip}(Y) \times \text{Lip}(Y) \to [0, \infty)$ and $Q_2 : \text{Lip}(X,S(X)) \times \text{Lip}(X,S(X)) \to [0, 1)$ by

$$Q_1(\tilde{k}, \tilde{h}) = \int_Y \theta \cdot \langle \nabla \tilde{k}, \nabla \tilde{h} \rangle \, d\text{vol}_Y$$

$$Q_2(k, h) = \int_X \bar{\theta} \cdot \langle \nabla k, \nabla h \rangle \, d\mu_g$$

It is easy to see that $f \circ \pi \in \text{Lip}(Y)$ for each $f$ contained in $\text{Lip}(X,S(X))$. Define $\pi^* : \text{Lip}(X,S(X)) \to \text{Lip}(Y)$ by $\pi^*(f) = f \circ \pi$. Let $\text{Lip}_{O(n)}(Y)$ be the set of all $O(n)$-invariant elements of $\text{Lip}(Y)$. Then, we can easily prove the following:

**Lemma 2.10.** $\pi^*$ is a bijection between $\text{Lip}(X,S(X))$ and $\text{Lip}_{O(n)}(Y)$. For elements $f$, $k$ of $\text{Lip}(X,S(X))$, we have

$$Q_1(f, k) = Q_2(\pi^*(f), \pi^*(k)), \quad (19)$$

and

$$\int_Y \theta \cdot \pi^*(f) \pi^*(k) \, d\text{vol}_Y = \int_X \bar{\theta} \cdot fk \, d\mu_g. \quad (20)$$

and finally

**Proof of Theorem 0.1.** The frame bundle $\pi_i : F(M_i) \to M_i$ is a Riemannian submersion with totally geodesic fibres. So using the reduction formula, the map $\hat{f}_i = f_i \circ \pi_i$ is harmonic on $F(M_i)$ and it is invariant under the action of $O(n)$. Furthermore $\|\epsilon(\hat{f}_i)\|_\infty$ is bounded ($\pi_i$ is a Riemannian submersion). Using Case II, $\hat{f}_i$ converge to some map $\hat{f}$ on $(Y, g, \Phi_Y \text{dvol}_Y)$. The map $\hat{f}$ satisfies

$$\int_Y \Xi_g(\hat{f}, \eta) \, \Phi_Y \text{dvol}_Y = 0,$$

where $\eta$ is a test function. The map $\hat{f}$ is also $O(n)$ invariant and continuous. Consider the quotient map $f$ such that $\hat{f} = \pi^*(f)$. First we show that $f$ is in $H^1((X, \nu), N)$. By the argument in Case II, $\hat{f}$ is in $H^1((Y, \Phi_Y \text{dvol}_Y), N)$ and so by equation (19), $f$ is of finite energy.

Now, we show that $f$ is weakly harmonic on $(X, \nu)$. By equation (19), for $\eta$ in $\text{Lip}(X,S(X))$

$$\int_Y \langle \nabla i \circ \hat{f}, \nabla \pi^*(\eta) \rangle \Phi_Y \text{dvol}_Y = \int_X \langle \nabla i \circ f, \nabla \eta \rangle \Phi_X d\mu_g.$$

We have

$$\int_Y \langle \Pi(\hat{f})(\nabla^g(i \circ \hat{f}), \nabla^g(i \circ \hat{f})), \pi^*(\eta) \rangle \Phi_Y \text{dvol}_Y$$

$$= \int_X \langle \Pi(f)(\nabla(i \circ f), \nabla(i \circ f)), \pi(\eta) \rangle \Phi_X d\mu_g$$

and since $\Phi_Y = \pi^*(\Phi_X)$

$$\int_Y \Xi_g(\hat{f}, \pi^*(\eta)) \Phi_Y \text{dvol}_Y = \int_X \Xi(f, \eta) \Phi_X d\mu_g.$$
which shows that \( f : X \to N \) is a weakly harmonic map. \( \square \)

3. Appendix: Further Discussion on Proposition 2.6

In this section we study convergence of the tension field of the map \( f_i, \tau(f_i) \) under the assumptions of Proposition 2.6.

Assume \((M_i, g_i), f_i, N\) be as in Proposition 2.6. Moreover consider the following assumption

**Assumption 2.** The section \( s_{i,j} \) is almost harmonic,

\[
|\tau(s_{i,j})| \leq C \cdot \epsilon_i, \tag{21}
\]

and also

\[
|\nabla_X ds_{i,j}(X)| \leq C \cdot \epsilon_i. \tag{22}
\]

where \( X \) is a smooth vector field on \( M \) and \( \bar{X} \) is its horizontal lift.

Using Assumption 1 and by Theorem 1.4 we have

\[
\tau(f_i) = (\nabla_{e_k} df_i) e_k + (\nabla_{e_t} df_i) e_t \tag{23}
\]

\[
= (\nabla_{e_k} df_i) e_k + \nabla_{f_i*(e_t)} f_i*(e_t)
- f_i*(\nabla_{e_t} e_t) H - f_i*(\nabla_{e_t} e_t) V
= (\nabla_{e_k} df_i) e_k - f_i*(H_i) + \tau(f_i^\perp)
\]

where \( \{e_k, e_t\} \) and \( \bar{e}_k \) are as in the proof of Proposition 2.6 and \( f_i^\perp \) denotes the restriction of \( f_i \) to the fibers \( F_i \), and \( H_i \) is the mean curvature vector of the submanifold \( F_i \).

Now we investigate how each term of the equation above behave as \( f_i \to f \).

**Lemma 3.1.** We have

\[
\lim_{i \to \infty} \left| di((\nabla_{e_k} df_i) e_k(p) - \left( \Delta^{g^M_i} \tilde{f}_i - \Pi(\tilde{f}_i)(d\tilde{f}_i, d\tilde{f}_i) \right) (\psi_i(p)) \right| = 0 \tag{24}
\]

**Proof.** By the discussion in the proof of Proposition 2.6, we know \( \tilde{f}_i \) converges to \( f \) in \( C^1 \)-topology. Using composition formula, we have

\[
di(B f_i(X_1, X_2)) = B(i \circ f_i)(X_1, X_2) - B(\pi_N)(d(i \circ f_i)(X_1), d(i \circ f_i)(X_2))
\]

and so for \( k = 1, \ldots, n \)

\[
di((\nabla_{e_k} df_i) e_k) = (\nabla_{e_k} d(i \circ f_i)) e_k - B(\pi_N)(d(i \circ f_i)(e_k), d(i \circ f_i)(e_k))
\]

First we show that

\[
|\nabla_{e_k} d(i \circ f_i)(e_k(p)) - \Delta^{g^M_i} \tilde{f}_i(\psi_i(p))| \leq o(\epsilon_i)
\]
By definition of \( \tilde{f}_i \),
\[
(N_{e_k} d\tilde{f}_i)\bar{e}_k = \sum_j (d\beta_j(\bar{e}_k) \cdot df_i(s_{i,j,*}(\bar{e}_k)) + \beta_j \cdot (N_{e_k} d(f_i \circ s_{i,j}))\bar{e}_k + \Delta \beta_j \cdot f_i \circ s_{i,j}).
\]
and again by composition formula, we have,
\[
\tau(f_i \circ s_{i,j}) = B_{s_{i,j,*}(\bar{e}_k), s_{i,j,*}(\bar{e}_k)} f_i + df_i(\tau(s_{i,j})). \tag{25}
\]
Since \( f_i \circ s_{i,j} \) converges in \( C^1 \) to \( f \), we have
\[
\lim_{i \to \infty} \left| \sum_j d\beta_j(\bar{e}_k) \cdot df_i(s_{i,j,*}(\bar{e}_k)) \right| = 0,
\]
\[
\lim_{i \to \infty} \sum_j \Delta \beta_j \cdot f_i \circ s_{i,j}(x) = \sum \Delta \beta_j \cdot f(x) = 0.
\]
Also, \( \psi_i*(e_k - s_{i,j,*}(\bar{e}_k)) = 0 \) and so \( e_k - s_{i,j,*}(\bar{e}_k) \) is vertical. On the other hand we have
\[
|e_k - s_{i,j,*}(\bar{e}_k)| \leq \epsilon_i.
\]
By inequality (12) and assumption (21), the second term on the right hand side (25) converges to zero. Again by inequality (13) and the assumption (22), we have
\[
\lim_{i \to \infty} |(N_{e_k} d(f_i))(e_k - s_{i,j,*}(\bar{e}_k))| = 0,
\]
\[
\lim_{i \to \infty} |(N(e_k - s_{i,j,*}(\bar{e}_k))df_i)ek| = 0.
\]
Finally
\[
\lim_{i \to \infty} |(N_{e_k} d(i \circ f_i))ek(p) - (N_{e_k} d\tilde{f}_i)\bar{e}_k(\psi(p))| = 0.
\]
We have the same for the second term
\[
\lim_{i \to \infty} |\Pi(f_i)(p)(df_i, df_i) - \Pi(\tilde{f}_i)(\psi(p))(d\tilde{f}_i, d\tilde{f}_i)| = 0.
\]

By the above lemma and \( \psi_i*(\frac{d\text{vol}_M}{\text{vol}(M_i)}) = \Phi_i \text{vol}^p_M \), we have
\[
\lim_{i \to \infty} \left| \int_{M_i} \langle di((N_{e_k} df_i)ek), \eta_i \rangle \frac{d\text{vol}_M}{\text{vol}(M_i)} - \int_M \langle \Delta \Phi_i \tilde{f}_i - \Pi(\tilde{f}_i)(d\tilde{f}_i, d\tilde{f}_i)\rangle \Phi_i \text{vol}^p_M \right| = 0
\]
and we conclude
\[
\lim_{i \to \infty} \int_{M_i} \langle di((N_{e_k} df_i)ek), \eta_i \rangle \frac{d\text{vol}_M}{\text{vol}(M_i)} = \int_M \langle df, d\eta \rangle + \langle df(\nabla \ln \Phi) - \Pi(f)(df, df), \eta \rangle \Phi \text{vol}_M. \tag{26}
\]
Now we will consider the second and third term in the decomposition of \( \tau(f_i) \).

**Lemma 3.2.** With the same assumptions as above, we have
\[
(a) \lim_{i \to \infty} \int_{M_i} \langle df_i(H^i), \eta_i \rangle \frac{d\text{vol}_M}{\text{vol}(M_i)} = - \int_M \langle df(\nabla \Phi), \eta \rangle \Phi \text{vol}_M \tag{27}
\]
\[(b) \lim_{i \to \infty} \|\tau(f_i^{-1})\| = 0 \quad (28)\]

here \(H^F_i\) denotes the mean curvature vector of the fiber \(F_i^x = \psi_i^{-1}(x)\). As before \(\eta\) is a test map on \(M\) and \(\eta_i = \eta \circ \psi_i\).

**Proof.** To prove equality (27), we need the following.

**Sublemma 1.** We have

\[\nabla \ln \Phi(x) = -\lim_{i \to \infty} \psi_i^* (H^F_i) \quad \text{weakly.} \quad (29)\]

**proof.** Suppose \(X\) is a smooth vector field on \(M\) and \(X_i\) its horizontal lift on \(M_i\). The flow \(\theta^i_t\) of \(X_i\) sends fibers to fibers diffeomorphically. By the first variation formula

\[\frac{d}{dt} \bigg|_{t=0} \theta^i_t (\text{dvol}_{F_i^x}) = -\int_{F_i^x} \langle X_i, H^F_i \rangle \text{ dvol}_{F_i^x} \quad (30)\]

Also

\[\Phi_i(x) = \frac{\text{vol}(\psi_i^{-1}(x))}{\text{vol}(M_i)}\]

and by (30),

\[d\Phi_i(X)(x) = -\int_{F_i^x} \langle X_i, H^F_i \rangle \frac{\text{dvol}_{M_i}}{\text{vol}(M_i)},\]

For an arbitrary \(\eta\) in \(C^\infty(M)\), we prove

\[\int_M \eta d\Phi_i(X) \text{ dvol}^M_i = -\int_{M_i} \eta_i \langle X_i, H_i \rangle \frac{\text{dvol}_{M_i}}{\text{vol}(M_i)}. \quad (31)\]

If we consider \((U_\gamma, h_\gamma)\) as a local trivialization of the fibration \(\psi_i\), then

\[\int_M \chi_{U_\gamma} d\Phi_i(X) \text{ dvol}^M_i = -\int_{U_\gamma} \chi_{U_\gamma} \langle X_i, H_i \rangle \frac{\text{dvol}_{M_i}}{\text{vol}(M_i)} \text{ dvol}^M_i\]

and so

\[\int_M \chi_{U_\gamma} d\Phi_i(X) \text{ dvol}^M_i = -\int_{\psi_i^{-1}(U_\gamma)} \langle X_i, H_i \rangle \frac{\text{dvol}_{M_i}}{\text{vol}(M_i)}\]

where \(\chi_{U_\gamma}\) denotes the characteristic function on \(U_\gamma\) and so we have (31). \(\Phi_i\) tends to \(\Phi\) in \(C^\infty\) and also \(\text{dvol}^M_i\) tends to \(\text{dvol}_{M}\) as \(i\) goes to infinity. Letting \(i \to \infty\) on both hand side of (31) and by the definition of weak derivatives

\[\int_M \eta \nabla \text{ln} \Phi(X) \Phi \text{ dvol}^M = -\lim_{i \to \infty} \int_M \eta \langle X, \psi_i^* (H_i) \rangle \Phi_i \text{ dvol}^M_i.\]

This proves the Sublemma 1.

Now recall that by definition of \(\tilde{f}_i\), we have

\[d\tilde{f}_i(x) = \sum d\beta_j(x) \cdot f_i \circ s_{i,j}(x) + \sum \beta_j(x) \cdot d(f_i \circ s_{i,j})(x)\]
and so
\[
\lim_{i \to \infty} \int_{M_i} \langle df_i(H_{x}^i) \circ \eta_i \rangle \frac{d\text{vol}_{M_i}}{\text{vol}(M_i)}
\]
\[
= \lim_{i \to \infty} \int_{M_i} \langle \sum [\beta_j \circ s_{i,j}] \circ \psi_i \rangle \frac{d\text{vol}_{M_i}}{\text{vol}(M_i)}
\]
\[
+ \lim_{i \to \infty} \int_{M_i} \langle \sum [\beta_j \cdot d(f_i \circ s_{i,j})] \circ \psi_i \rangle \frac{d\text{vol}_{M_i}}{\text{vol}(M_i)}
\]
\[
= \lim_{i \to \infty} \int_{M_i} \langle df_i(H_{x}^i) \circ \eta_i \rangle \frac{d\text{vol}_{M_i}}{\text{vol}(M_i)}.
\]

We have
\[
\lim_{i \to \infty} \sum d\beta_j(\psi_i(H_{x}^i)) \cdot (f_i \circ s_{i,j}) = 0,
\]
and by inequality (12) and the fact that \(s_{i,j} \circ \psi_i(H_{x}^i) - H_{x}^i\) is vertical, we obtain the last equality. By the above sublemma,
\[
\lim_{i \to \infty} \int_{M_i} \langle df_i(H_{x}^i), \eta_i \rangle \frac{d\text{vol}_{M_i}}{\text{vol}(M_i)} = - \int_M \langle df(\nabla \ln \Phi), \eta \rangle \cdot \Phi d\text{vol}_M
\]

To prove (b) start with
\[
\tau(f_i^{-1}) = \nabla_{f_i^{-1}} f_i \cdot f_i \cdot (\nabla e_t e_t)\]

From (12) and (13)
\[
|\nabla_{f_i^{-1}} f_i| < C \cdot \epsilon_i
\]

where \(C\) is a constant independent of \(i\). Again from (12), we have
\[
\|f_i^{-1} \cdot (\nabla e_t e_t)\| < C \cdot \epsilon_i \|f_i^{-1} \cdot (\nabla e_t e_t)\|
\]

and so we see that
\[
\lim_{i \to \infty} \|\tau(f_i^{-1})\| = 0.
\]

\[\square\]

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