An application of Mittag–Leffler-type Poisson distribution on certain subclasses of analytic functions associated with conic domains

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Abstract

In this paper our aim is to deduce some sufficient conditions and inclusion properties for Mittag-Leffler-type Poisson distribution series

\[ ψ^a((z)) = z + \sum_{m=2}^{\infty} \frac{Γ(β)m^{(p)}}{Γ(β)(m-1)+β} z^m \]

to be in the classes \( k-ST\{A, B\} \) and \( k-UC\{A, B\} \) of \( k \)-uniformly Janowski starlike and \( k \)-Janowski convex functions, respectively. Further, we obtain a condition for an integral operator \( G^a_{\nu}(z) = \int_{0}^{z} \psi^a(w) \, dw \) to be in the class \( k-UC\{A, B\} \). Several corollaries and consequences of the main results are also considered.

1. Introduction and definitions

The distributions of random variables have generated a great deal of interest in recent years. Their probability density functions, in a real variable \( x \) and a complex variable \( z \), have played an important role in statistics and probability theory. For this reason, distributions have been studied extensively. Many kinds of distributions appeared from real life situations like Binomial distribution, Poisson distribution, geometric distribution, hyper geometric distribution and negative binomial distribution.

A random variable \( x \) follows a Poisson distribution if its probability density function (PDF) is given by:

\[ f(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, \ldots \]  

and \( \lambda \) is the parameter of the distribution.

The most interesting thing in this distribution is the equality of its mean and variance. The Poisson distribution is also used as a counting process and is called Poisson process, which is the base of the queueing theory that considers the number of arrivals follows a Poisson process and the inter-arrival times follow an exponential distribution. Also, the Poisson process plays an important role in renewal theory.

In 2014, Porwal [1] introduced Poisson distribution whose coefficients are probabilities of the Poisson distribution, and then applied it on univalent functions. In 2017, Moments, and moment generating function are obtained to Mittag–Leffler-type Poisson distribution by Porwal and Dixit [2].

Let \( \mathcal{A} \) denote the class of the normalized functions of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.2) \]

which are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). Further, let \( \mathcal{F} \) be a subclass of \( \mathcal{A} \) consisting of functions of the form,

\[ f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in U. \]  

(1.3)

A function \( f \in \mathcal{A} \) is said to be in the class \( k \)-Janowski starlike functions, denoted by \( k-ST\{A, B\} \), \( k \geq 0, -1 \leq B < A \leq 1 \), if and only if,

\[ \Re \left\{ \frac{(B-1)/f'(z)}{f(z)} - \frac{(A-1)/f'(z)}{f(z)} \right\} > k \quad \left( \frac{(B-1)/f'(z)}{f(z)} - \frac{(A-1)/f'(z)}{f(z)} - (A-1) \right). \]  

(1.4)

Further, a function \( f \in \mathcal{A} \) is said to be in the class \( k \)-Janowski convex functions \( k-UC\{A, B\} \), \( k \geq 0, -1 \leq B < A \leq 1 \), if and only if,
\[ \Re \left( \frac{(B - 1)z f^{(\gamma)}(z)}{f(z)} - (A - 1) \right) > k \left( \frac{(B - 1)z f^{(\gamma)}(z)}{f(z)} - (A - 1) \right) - 1 \] \quad (1.5)

It can be easily seen that

\[ f(z) \in k-UC\mathcal{V}[A, B] \Leftrightarrow z f^{(\gamma)}(z) \in k-ST[A, B]. \]

The above classes \( k-ST[A, B] \) and \( k-UC\mathcal{V}[A, B] \) were introduced and studied by Noor and Malik [3].

Suitably specializing the parameters, we note that

1. \( k-ST[1, 1] = k-ST \) and \( k-UC\mathcal{V}[1, 1] = k-UC\mathcal{V} \), the well-known classes of \( k \)-starlike and \( k \)-convex functions respectively, introduced by Kanas and Wisniowska [4, 5] (see also, [6], [7]).

2. \( k-ST[1 - 2\gamma, 1] = k-SD[k, \gamma] \) and \( k-UC\mathcal{V}[1 - 2\gamma, 1] = k-KD[k, \gamma] \), the classes introduced by Shams et al. in [8].

3. \( 0-ST[A, B] = S^\gamma[A, B] \) and \( 0-UC\mathcal{V}[A, B] = C[A, B] \), the well-known classes of Janowski starlike and Janowski convex functions respectively, introduced by Janowski [9].

4. \( 0-ST[1 - 2\gamma, 1] = S^\gamma(\gamma) \) and \( 0-UC\mathcal{V}[1 - 2\gamma, 1] = C(\gamma) \), the well-known classes of starlike functions of order \( \gamma \) (\( 0 \leq \gamma < 1 \)) and convex functions of order \( \gamma \) (\( 0 \leq \gamma < 1 \)) respectively, (see [10]).

Geometrically, if a function \( f(z) \in k-ST[A, B] \) then

\[ w = \frac{(B - 1)z f^{(\gamma)}(z) - (A - 1)}{(B + 1)z f^{(\gamma)}(z) - (A + 1)} \]

takes all values from the domain \( \Omega_k \), \( k \geq 0 \) as

\[ \Omega_k = \{ w : \Re w > k |w - 1| \} = \left\{ a + i \beta : a > k \sqrt{(a - 1)^2 + \beta^2} \right\} \]

The domain \( \Omega_k \) represents the right half plane for \( k = 0 \); a hyperbola for \( 0 < k < 1 \); a parabola for \( k = 1 \) and an ellipse for \( k > 1 \), (see [3]).

A function \( f \in A \) is said to be in the class \( R^\gamma(C, D) \), \( r \in C \setminus \{0\} \), \( -1 \leq D < C \leq 1 \), if it satisfies the inequality

\[ \left| \frac{f'(z) - 1}{(C - D)z f'(z) - (C - D)} \right| < 1, \quad z \in U. \]

This class was introduced by Díaz and Pal [11] and they proved the following result.

**Lemma 1.1.** [11] If \( f \in R^\gamma(C, D) \) is of the form (1.2), then

\[ |a_n| \leq (C - D)^{-\gamma}, \quad n \in \mathbb{N} \setminus \{1\}. \]

The result is sharp for the function

\[ f(z) = \int_0^z \frac{1}{(1 + (C - D)^{-\gamma}zt^{\gamma})} dt, \quad (z \in U; n \in \mathbb{N} \setminus \{1\}). \]

Now we recall the well Mittag-Leffler function \( E_\alpha(z) \) studied by Mittag-Leffler [12] and given by

\[ E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (z \in \mathbb{C}, |\Re(\alpha)| > 0). \]

A more general function \( E_{\alpha, \beta}(z) \) generalizing \( E_\alpha(z) \) was introduced by Wiman [13, 14] and defined by

\[ E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (z, \alpha, \beta \in \mathbb{C}, |\Re(\beta)| > 0, \Re(\alpha) > 0). \]

The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems (see for example, [15, 16, 17, 18, 19, 20, 21, 22]). Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found e.g. in [23, 24, 25, 26, 27, 28].

Observe that Mittag-Leffler function \( E_{\alpha, \beta} \) does not belong to the family \( A \). Therefore, we consider the following normalization of the Mittag-Leffler function (see, [24, 29])

\[ E_{\alpha, \beta}(z) = \frac{\Gamma(\beta) z E_{\alpha, \beta}(z)}{\Gamma(\alpha(n - 1) + \beta)} z^n, \quad (1.6) \]

where \( z, \alpha, \beta \in C; \beta \neq 0, -1, -2, -3, \ldots \) and \( |\Re(\beta)| > 0, |\Re(\alpha)| > 0 \).

Whilst formula (1.6) holds for complex-valued \( \alpha, \beta \) and \( z \in C \), however in this paper, we shall restrict our attention to the case of real-valued \( \alpha, \beta \) and \( z \in U \). Observe that the function \( E_{\alpha, \beta} \) contains many well-known functions as its special case, for example, \( E_{1,1}(z) = z \sinh \sqrt{z} \), \( E_{1,2}(z) = z \cosh \sqrt{z} \), \( E_{2,1}(z) = 2(\cosh \sqrt{z}) \) and \( E_{2,2}(z) = 6(\sinh \sqrt{z} / \sqrt{z}) \).

The probability mass function of the Mittag-Leffler-type Poisson distribution is given by

\[ P(x = r) = \frac{m^r}{\Gamma(\alpha(n - 1) + \beta)} E_{\alpha, \beta}(m), \quad r = 0, 1, 2, 3, \ldots, \]

where \( m > 0, \alpha > 0 \) and \( \beta > 0 \). Using the normalized form of Mittag-Leffler function as assumed in (1.6), Frasin et al. [30] define a power series whose coefficients are probabilities of Mittag-Leffler-type Poisson distribution series, as below:

\[ \psi^{\alpha, \beta}_{m}(z) = z + \sum_{n=1}^{\infty} \frac{\Gamma(\beta) m^{n-1}}{\Gamma(\alpha(n - 1) + \beta)} E_{\alpha, \beta}(m) z^n, \quad z \in U. \]

Further, we define the series

\[ \Phi^{\alpha, \beta}_{m}(z) = 2z - \psi^{\alpha, \beta}_{m}(z) = z + \sum_{n=1}^{\infty} \frac{\Gamma(\beta) m^{n-1}}{\Gamma(\alpha(n - 1) + \beta)} E_{\alpha, \beta}(m) z^n, \quad z \in U. \] (1.7)

Using the concept convolution or Hadamard product of two series, we introduce the convolution operator

\[ I_{\alpha, \beta}^{\gamma} f(z) = \psi^{\alpha, \beta}_{m}(z) * f(z) = z + \sum_{n=1}^{\infty} \frac{\Gamma(\beta) m^{n-1}}{\Gamma(\alpha(n - 1) + \beta)} E_{\alpha, \beta}(m) a_n z^n, \quad z \in U, \]

where \( * \) denotes the convolution or Hadamard product of two series.

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions, using hypergeometric functions (see for example, [31, 32, 33, 34]), generalized Bessel functions (see for example, [35, 36]), Struve functions (see for example, [37, 38, 39]), Poisson distribution series (see for example, [1, 40, 41, 42, 43, 44]) and Pascal distribution series (see for example, [45, 46, 47, 48, 49, 50, 51]), in this paper we determine conditions for \( \Phi^{\alpha, \beta}_{m}(z) \) to be in the classes \( k-ST[A, B] \) and \( k-UC\mathcal{V}[A, B] \). Furthermore, we estimate certain inclusion relations between the classes \( R^\gamma(C, D) \) and \( k-UC\mathcal{V}[A, B] \). Finally, we give a condition for an integral operator \( \mathcal{G}_{\alpha, \beta, m}(z) = \int_0^z \frac{\Phi^{\alpha, \beta}_{m}(t)}{t} dt \) to be in the class \( k-UC\mathcal{V}[A, B] \).

### 2. Main results

To establish our main results, we shall require the following lemmas.

**Lemma 2.1.** [3] A function \( f \) of the form (1.2) is in the class \( k-ST[A, B] \), if it satisfies the condition

\[ \sum_{n=1}^{\infty} \frac{\Gamma(\beta) m^{n-1}}{\Gamma(\alpha(n - 1) + \beta)} E_{\alpha, \beta}(m) a_n \leq |B - A| \] (2.1)

where \( -1 \leq B < A \leq 1 \) and \( k \geq 0 \).
Lemma 2.2. [3]A function $f$ of the form (1.2) is in the class $k-U\mathcal{CV}[A,B]$, if the condition
\[ \sum_{n=2}^{\infty} n[2(k+1)(n-1) + n(B+1) - (A+1)] |a_n| \leq |B-A| \quad (2.2) \]
where $-1 \leq B < A \leq 1$ and $k \geq 0$.

Unless otherwise mentioned, we shall assume in this paper that $a, m > 0$, $k \geq 0$ and $-1 \leq B < A \leq 1$.

Firstly, we obtain the necessary and sufficient conditions for $\Phi^m_{\alpha,\beta}$ to be in the class $k-S[T][A,B]$.

Theorem 2.3. Let $\beta > 1$. Then $\Phi^m_{\alpha,\beta} \in k-S[T][A,B]$ if
\[ \frac{\Gamma(\beta)}{E_{\alpha,\beta}(m)} \left\{ \begin{array}{l}
\left[ \frac{2k + B + 3}{a} \left( E_{\alpha,\beta}(m) - \frac{1}{\Gamma(\beta-1)} \right) \\
+ \left[ \frac{2k + B + 3}{a} \right] (1 - \beta) + (B + A + 2) \right]
\end{array} \right\} \leq |B-A| \]
\[ \leq |B-A| \quad (2.3) \]

Proof. In view of Lemma 2.1 and (2.1) it suffices to show that
\[ Q_1 := \sum_{n=2}^{\infty} \left\{ 2(k+1)(n-1) + n(B+1) - (A+1) \right\} \frac{\Gamma(\beta)m^{n-1}}{\Gamma(a(n-1) + \beta) E_{\alpha,\beta}(m)} \leq |B-A| \]
Now
\[ Q_1 \leq \sum_{n=2}^{\infty} \left\{ 2(k+1)(n-1) + n(B+1) - (A+1) \right\} \frac{\Gamma(\beta)m^{n-1}}{\Gamma(a(n-1) + \beta) E_{\alpha,\beta}(m)} \]
\[ = \sum_{n=2}^{\infty} \left\{ 2(k+1)(n-1) + n(B+1) - (A+1) \right\} \frac{\Gamma(\beta)m^{n-1}}{\Gamma(a(n-1) + \beta) E_{\alpha,\beta}(m)} \]
\[ = \sum_{n=2}^{\infty} \left\{ 2(k+1)(n-1) + n(B+1) - (A+1) \right\} \frac{\Gamma(\beta)m^{n}}{\Gamma(a(n+1) + \beta) E_{\alpha,\beta}(m)} \]
\[ = \sum_{n=2}^{\infty} \left\{ 2(k+1)(n-1) + n(B+1) - (A+1) \right\} \frac{\Gamma(\beta)m^{n}}{\Gamma(a(n+1) + \beta) E_{\alpha,\beta}(m)} \]
\[ = \frac{2k + B + 3}{a} \sum_{n=2}^{\infty} \left\{ (n+1 - (1 - \beta)) \right\} \frac{\Gamma(\beta)m^{n}}{\Gamma(a(n+1) + \beta) E_{\alpha,\beta}(m)} \]
\[ + (B + A + 2) \sum_{n=2}^{\infty} \left\{ \Gamma(\beta)m^{n-1} \right\} \frac{\Gamma(\beta)m^{n}}{\Gamma(a(n+1) + \beta) E_{\alpha,\beta}(m)} \]
\[ = \frac{2k + B + 3}{a} \sum_{n=2}^{\infty} \left\{ (n+1 - (1 - \beta)) \right\} \frac{\Gamma(\beta)m^{n}}{\Gamma(a(n+1) + \beta) E_{\alpha,\beta}(m)} \]
\[ + (B + A + 2) \sum_{n=2}^{\infty} \left\{ \Gamma(\beta)m^{n-1} \right\} \frac{\Gamma(\beta)m^{n}}{\Gamma(a(n+1) + \beta) E_{\alpha,\beta}(m)} \]
\[ \leq |B-A| \]
by the given hypothesis (2.3). This completes the proof of Theorem 2.3.

Theorem 2.4. Let $\beta > 2$. Then $\Phi^m_{\alpha,\beta} \in k-U\mathcal{CV}[A,B]$ if
\[ \frac{\Gamma(\beta)}{E_{\alpha,\beta}(m)} \left\{ \begin{array}{l}
\left[ \frac{2k + B + 3}{a^2} \left( E_{\alpha,\beta-2}(m) - \frac{1}{\Gamma(\beta-1)} \right) \\
+ \left[ \frac{(2k + B + 3)(3(2 - \beta) + a(2B + A + 2k + 5)}{a^2} \right] \right)
\end{array} \right\} \leq |B-A| \]
\[ \leq |B-A| \quad (2.4) \]

Proof. In view of Lemma 2.2 and (2.2) it suffices to show that
\[ Q_2 := \sum_{n=2}^{\infty} \left\{ 2(k+1)(n-1) + n(B+1) - (A+1) \right\} \frac{\Gamma(\beta)m^{n-1}}{\Gamma(a(n-1) + \beta) E_{\alpha,\beta}(m)} \]
\[ \leq |B-A| \]
Now
\[ Q_2 \leq \sum_{n=2}^{\infty} \left\{ 2(k+1)(n-1) + n(B+1) - (A+1) \right\} \frac{\Gamma(\beta)m^{n-1}}{\Gamma(a(n-1) + \beta) E_{\alpha,\beta}(m)} \]
\[ = \sum_{n=2}^{\infty} \left\{ 2(k+1)(n-1) + n(B+1) - (A+1) \right\} \frac{\Gamma(\beta)m^{n}}{\Gamma(a(n+1) + \beta) E_{\alpha,\beta}(m)} \]
\[ = \sum_{n=2}^{\infty} \left\{ 2(k+1)(n-1) + n(B+1) - (A+1) \right\} \frac{\Gamma(\beta)m^{n}}{\Gamma(a(n+1) + \beta) E_{\alpha,\beta}(m)} \]
\[ = \frac{(2k + B + 3)(3 - \beta) + a(2B + A + 2k + 5)}{a^2} \sum_{n=2}^{\infty} \left\{ \Gamma(\beta)m^{n-1} \right\} \frac{\Gamma(\beta)m^{n}}{\Gamma(a(n+1) + \beta) E_{\alpha,\beta}(m)} \]
\[ \leq |B-A| \]
by the given hypothesis (2.4). This completes the proof of Theorem 2.4.

3. Inclusion properties

Making use of Lemma 1.1 we will study the action of the Mittag-Leffler-type Poisson distribution series on the class $k-U\mathcal{CV}[A,B]$. 
Corollary 5.2. Let $\beta > 1$. Then $\Phi_{a,\beta}^m \in k$-$KD[k,\gamma]$ if
\[
\frac{\Gamma(\beta)}{E_{a,\beta}(m)} \left[ \frac{k+1}{a^2} \left( E_{a,\beta-1}(m) - \frac{1}{\Gamma(\beta-1)} \right) \right]
+ \left( \frac{k+1}{a^2} (1 - \beta) + 1 - \gamma \right) \left( E_{a,\beta}(m) - \frac{1}{\Gamma(\beta)} \right)
\leq 1 - \gamma.
\] (5.2)

Corollary 5.3. Let $\beta > 1$. If $f \in R^* (C, D)$, then $T_{a,\beta}^m f \in k$-$KD[k,\gamma]$ if
\[
\frac{(C-D)|r| \Gamma(\beta)}{E_{a,\beta}(m)} \left[ \frac{k+1}{a^2} \left( E_{a,\beta-1}(m) - \frac{1}{\Gamma(\beta-1)} \right) \right]
+ \left( \frac{k+1}{a^2} (1 - \beta) + 1 - \gamma \right) \left( E_{a,\beta}(m) - \frac{1}{\Gamma(\beta)} \right)
\leq 1 - \gamma.
\]

Corollary 5.4. Let $\beta > 1$. Then the integral operator given by (4.1) is in the class $k$-$KD[k,\gamma]$ if the inequality (5.1) is satisfied.

6. Conclusion

The Mittag-Leffler function has attracted the increasing attention of researchers because of its key role in treating problems related to integral and differential equations of fractional order. The main purpose of this article is to find sufficient conditions and inclusion relations for Mittag-Leffler-type Poisson distribution series to be in two subclasses of analytic functions $k$-$ST[A, B]$ and $k$-$U'CV[A, B]$ of $k$-uniformly Janowski starlike and $k$-Janowski convex functions, respectively. Finally, we give a condition for an integral operator $G_{a,\beta}^m(z) = \int_0^z \frac{\Phi_{a,\beta}^m(t)}{t} dt$ to be in the class $k$-$U'CV[A, B]$. Several corollaries and consequences of the main results are also considered.

Declarations

Author contribution statement

M. Ahmad: Conceived and designed the experiments; Analyzed and interpreted the data; Wrote the paper. B. Frasin: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data. G. Murugusundaramoorthy: Performed the experiments; Analyzed and interpreted the data. A. Alkhazaleh: Analyzed and interpreted the data.

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