ON THE MAXIMUM AREA OF INScribed POLYGONS

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Abstract. Given a convex \( n \)-gon \( P \) and a positive integer \( m \) such that \( 3 \leq m \leq n - 1 \), let \( Q \) denote the largest area convex \( m \)-gon contained in \( P \). We are interested in the minimum value of \( \Delta(Q)/\Delta(P) \), the ratio of the areas of these two polygons. More precisely, given positive integers \( n \) and \( m \), with \( 3 \leq m \leq n - 1 \), define

\[
f_n(m) = \min_{P \in \mathcal{P}_n} \max_{Q \subset P, \text{ } |Q| = m} \frac{\Delta(Q)}{\Delta(P)}
\]

where the maximum is taken over all \( m \)-gons contained in \( P \), and the minimum is taken over \( \mathcal{P}_n \), the entire class of convex \( n \)-gons. The values of \( f_4(3), f_5(4) \) and \( f_6(3) \) are known. In this paper we compute the values of \( f_5(3), f_6(5) \) and \( f_6(4) \).

In addition, we prove that for all \( n \geq 6 \) we have

\[
\frac{4}{n} \cdot \sin^2 \left( \frac{\pi}{n} \right) \leq 1 - f_n(n - 1) \leq \min \left( \frac{1}{n}, \frac{4}{n} \cdot \sin^2 \left( \frac{2\pi}{n} \right) \right).
\]

These bounds can be used to improve the known estimates for \( f_n(m) \).

1. Introduction

In 1940, Sás [11] proved the following:

**Theorem 1.1.** Let \( K \) be a compact convex body in the plane and let \( P_m \) be the largest area \( m \)-gon contained in \( K \). Then

\[
\frac{\Delta(P_m)}{\Delta(K)} \geq \frac{m}{2\pi} \cdot \sin \left( \frac{2\pi}{m} \right)
\]

where equality holds if and only if \( K \) is an ellipse.

Throughout the paper \( \Delta(\cdot) \) denotes the area. At about the same time, Fejes Tóth [5] proved an analogous theorem dealing with \( m \)-gons containing \( K \). A few years later, Lázár [10] proved a result involving both polygons that contain \( K \) and polygons that are contained in \( K \). The following year, John [9] published one of the major breakthroughs in the field of approximation of convex bodies. John’s ellipsoid theorem states that every \( d \)-dimensional convex body \( K \) lies between two concentric homothetic ellipsoids, whose ratio is no greater than \( d \).

The results mentioned above have various applications to packing and covering problems as well as in the design of numerous geometric algorithms - see [1, 6, 7, 12, 13].

In 1992, motivated by a problem in robot motion planning, Fleischer, Mehlhorn, Rote, Welzl, and Yap [5] raised the following version of Sás’ problem.

**Question 1.** Given a convex \( n \)-gon \( P \) and a positive integer \( m \) such that \( 3 \leq m \leq n - 1 \), let \( Q \) denote the largest area convex \( m \)-gon contained in \( P \). How small can \( \Delta(Q)/\Delta(P) \), the ratio of the areas of these two polygons, be?
More precisely, given positive integers \( n \) and \( m \), with \( 3 \leq m \leq n - 1 \), define

\[
f_n(m) = \min_{P \in \mathcal{P}_n} \max_{Q \subset P, |Q| = m} \frac{\Delta(Q)}{\Delta(P)}
\]

where the maximum is taken over all \( m \)-gons contained in \( P \), and the minimum is taken over \( \mathcal{P}_n \), the entire class of convex \( n \)-gons. It is easy to see that one can restrict the vertices of \( Q \) to an \( m \)-subset of the vertices of \( P \).

There are just a few known values of \( f_n(m) \). It is easy to see that \( f_4(3) = \frac{1}{2} \). Indeed, dividing a convex quadrilateral by a diagonal, we get two triangles at least one of which has area greater or equal than half of the area of the quadrilateral.

On the other hand, the case of parallelograms shows that the value \( \frac{1}{2} \) cannot be replaced by a larger one.

Du and Ding \[2, 3\] proved that \( f_5(4) = \frac{5 + \sqrt{5}}{10} \), while Fleischer et al. \[8\] showed that \( f_6(3) = 4/9 \). To the best of our knowledge, these are the only exact values of \( f_n(m) \) known at this time.

In this paper we show that

\[
(2) \quad f_5(3) = \frac{1}{\sqrt{5}}, \quad f_6(4) = \frac{2}{3}, \quad \text{and} \quad f_6(5) = \frac{5}{6}.
\]

Moreover, we find the exact order of magnitude of \( f_n(n - 1) \) by showing that for every \( n \geq 6 \)

\[
(3) \quad \frac{4}{n} \cdot \sin^2 \left( \frac{\pi}{n} \right) \leq 1 - f_n(n - 1) \leq \min \left( \frac{1}{n}, \frac{4}{n} \cdot \sin^2 \left( \frac{2\pi}{n} \right) \right).
\]

We use this inequality to prove that \( f_n(m) \geq m/n \) for all \( n - 1 \geq m \geq 5 \).

### 1.1. The Main Technique.

Throughout the entire paper we use the outer product of two vectors to express areas. This operation, also known as exterior product, is defined as follows:

For any two vectors \( \mathbf{v} = (a, b) \) and \( \mathbf{u} = (c, d) \), the outer product of \( \mathbf{v} \) and \( \mathbf{u} \) be given by

\[
\mathbf{v} \wedge \mathbf{u} := \frac{1}{2} \cdot (ad - bc).
\]

It is easy to see that the outer product represents the signed area of the triangle determined by the vectors \( \mathbf{v} \) and \( \mathbf{u} \), where the \( \pm \) sign depends on whether the angle between \( \mathbf{v} \) and \( \mathbf{u} \) - measured in the counterclockwise direction from \( \mathbf{v} \) towards \( \mathbf{u} \) - is smaller than or greater than 180°.

The following properties of the outer product are simple consequences of the definition and are going to be used extensively in the remaining part of the paper.

1. \( \mathbf{v} \wedge \mathbf{u} = -\mathbf{u} \wedge \mathbf{v} \), anticommutativity. In particular, \( \mathbf{v} \wedge \mathbf{v} = 0 \).
2. \( (\alpha \mathbf{v} + \beta \mathbf{u}) \wedge \mathbf{w} = \alpha (\mathbf{v} \wedge \mathbf{w}) + \beta (\mathbf{u} \wedge \mathbf{w}) \), linearity.

### 2. Setup for pentagons

In this section we are going to prove that \( f_5(4) = (5 + \sqrt{5})/10 \) and \( f_6(3) = 1/\sqrt{5} \).

As mentioned before, the first result has already been proved in \[2, 3\]. We believe that our approach leads to simpler, more transparent proofs.

Let \( ABCDE \) be an arbitrary convex pentagon. After an eventual relabeling of the vertices we may assume that

\[
(4) \quad \Delta(DEA) = \min \{ \Delta(ABC), \Delta(BCD), \Delta(CDE), \Delta(DEA), \Delta(EAB) \}
\]
In the literature, the triangles formed by three consecutive vertices of a convex polygon are sometimes called ears. Assumption (4) above fixes the ear of least area. Denote the intersection of $AC$ and $BD$ by $O$. Then define $\mathbf{u} = \overrightarrow{OD}$, $\mathbf{v} = \overrightarrow{OA}$. After an appropriate scaling, we may assume that $\mathbf{u} \wedge \mathbf{v} = \Delta(\text{AOD}) = 1$. Since $A$, $O$, and $C$ are collinear and $D$, $O$, and $B$ are collinear, we can write $\overrightarrow{BO} = a \cdot \overrightarrow{OD} = a \mathbf{u}$ and $\overrightarrow{CO} = b \cdot \overrightarrow{OA} = b \mathbf{v}$, with $a > 0$, $b > 0$ (see figure 1).

Without loss of generality we may assume that $a \leq b$. Using the triangle rule, we obtain that $\overrightarrow{AB} = -a \mathbf{u} - \mathbf{v}$, $\overrightarrow{BC} = a \mathbf{u} - b \mathbf{v}$, and $\overrightarrow{CD} = \mathbf{u} + b \mathbf{v}$. We know that every vector in the plane can be written as a linear combination of any two independent vectors.

Set $\overrightarrow{OE} = \mathbf{w} = c \mathbf{u} + d \mathbf{v}$, with $c, d > 0$. It follows that $\overrightarrow{DE} = \mathbf{w} - \mathbf{u}$ and $\overrightarrow{AE} = \mathbf{v} - \mathbf{w}$. We have

$$\Delta(ODE) = \mathbf{u} \wedge \mathbf{w} = \mathbf{u} \wedge (c \mathbf{u} + d \mathbf{v}) = c(\mathbf{u} \wedge \mathbf{u}) + d(\mathbf{u} \wedge \mathbf{v}) = d,$$
$$\Delta(OAE) = \mathbf{w} \wedge \mathbf{v} = (c \mathbf{u} + d \mathbf{v}) \wedge \mathbf{v} = c(\mathbf{u} \wedge \mathbf{v}) + d(\mathbf{v} \wedge \mathbf{v}) = c.$$

After similar calculations, we can write the areas of various triangles in pentagon $ABCDE$ in terms of the positive constants $a, b, c, d$ as shown below:

$$\Delta(OAB) = -\mathbf{v} \wedge a \mathbf{u} = a,$$
$$\Delta(OBC) = -a \mathbf{u} \wedge -b \mathbf{v} = ab,$$
$$\Delta(OCD) = -b \mathbf{v} \wedge (b \mathbf{v} + \mathbf{u}) = b.$$
We can now compute the total area of the pentagon
\( \Delta(ABCDE) = \Delta(OAB) + \Delta(OBC) + \Delta(OCD) + \Delta(ODE) + \Delta(OEA) \), that is,
\[
\Delta(ABCDE) = a + b + c + d + ab. \tag{5}
\]
Next, we compute the areas of the ears of the pentagon.
\[
\Delta(ABC) = AB \wedge BC = (-au - v) \wedge (au - bv) = a + ab,
\Delta(BCD) = BC \wedge CD = (au - bv) \wedge (bv + u) = b + ba,
\]
\[
\Delta(CDE) = CD \wedge DE = (bv + u) \wedge ((c - 1)u + dv) = b + d - bc,
\Delta(DEA) = DE \wedge EA = ((e - 1)u + dv) \wedge (-cu + (1 - d)v) = c + d - 1,
\Delta(EAB) = EA \wedge AB = (-cu + (1 - d)v) \wedge (-au - v) = a + c - ad.
\]
Translating assumption (6) in terms of \(a, b, c, \) and \(d\), we obtain
\[
\Delta(DEA) \leq \Delta(ABC) \iff c + d - 1 \leq a + ab \iff c + d \leq 1 + a + ab
\]
\[
\Delta(DEA) \leq \Delta(CDE) \iff c + d - 1 \leq b + d - bc \iff (c - 1)(1 + b) \leq 0 \iff c \leq 1,
\Delta(DEA) \leq \Delta(EAB) \iff c + d - 1 \leq a + c - ad \iff (d - 1)(1 + a) \leq 0 \iff d \leq 1.
\]
Therefore,
\[
\Delta(DEA) \leq \Delta(CDE) \iff c \leq 1, d \leq 1, \text{ and } c + d \leq 1 + a + ab
\]
We are now in position to find the values of \(f_5(4)\) and \(f_5(3)\).

2.1. Large Quadrilaterals in Pentagons.

**Theorem 2.1.** \((\ref{2} \tag{3})\) \(f_5(4) = \frac{5 + \sqrt{5}}{10}\).

**Proof.** It is easy to check that \(f_5(4) \leq \frac{5 + \sqrt{5}}{10}\) as the case of the regular pentagon shows. From \(\ref{1}\) it follows that \(ABCD\) is the largest quadrilateral contained in the pentagon \(ABCDE\). It remains to prove the opposite inequality, that is,
\[
\frac{\Delta(ABCD)}{\Delta(ABCDE)} \geq \frac{5 + \sqrt{5}}{10}. \tag{8}
\]
Since \(\Delta(ABCDE) = \Delta(ABCD) + \Delta(DEA)\), \(\ref{8}\) is equivalent to
\[
\frac{\Delta(DEA)}{\Delta(ABCDE)} \leq \frac{5 - \sqrt{5}}{10} = \frac{2}{5 + \sqrt{5}}. \tag{9}
\]
Since \(\Delta(DEA) = c + d - 1\) and \(\Delta(ABCD) = a + b + c + d + ab\), relation \(\ref{9}\) translates to \((5 + \sqrt{5})(c + d - 1) \leq 2a + 2b + 2c + 2d + 2ab\), that is,
\[
(3 + \sqrt{5})(c + d) \leq 5 + \sqrt{5} + 2a + 2b + 2ab. \tag{10}
\]
To prove \(\ref{10}\) we will proceed in two cases.

**Case 1:** \(a \leq \frac{\sqrt{5} - 1}{2}\).

Using \(\ref{1}\), we have \(c + d \leq 1 + a + ab\), so it would be sufficient to show that
\[
(3 + \sqrt{5})(1 + a + ab) \leq 5 + \sqrt{5} + 2a + 2b + 2ab \iff
\[\iff (1 + \sqrt{5})(a + ab) \leq 2 + 2b \iff (1 + \sqrt{5})a \leq 2 \iff a \leq \frac{\sqrt{5} - 1}{2}.
\]

**Case 2:** \(a \geq \frac{\sqrt{5} - 1}{2}\).
In this case, using (7) we have $c + d \leq 2$, so it would suffice to prove that

$$2(3 + \sqrt{5}) \leq 5 + \sqrt{5} + 2a + 2b + 2ab \leq 1 + \sqrt{5} \leq 4a + 2a^2 \quad \iff \quad a \geq \frac{\sqrt{5} - 1}{2}.$$  

This completes the proof of Theorem 2.1. \qed

2.2. Large Triangles in Pentagons. We will now compute the areas of three of the remaining five triangles determined from the vertices of $ABCDE$ - refer to figure 1.

$$\Delta(ABD) = \Delta(ABCDE) - \Delta(DEA) - \Delta(BCD) = a + 1$$  

(11) $\Delta(ACD) = \Delta(ABCDE) - \Delta(DEA) - \Delta(ABC) = b + 1$  

$\Delta(BCE) = \Delta(ABCDE) - \Delta(BAE) - \Delta(CDE) = ab + ad + bc.$

Recall that $\Delta(BCD) = ab + a$.

Theorem 2.2. $f_5(3) = \frac{1}{\sqrt{5}}$.

Proof. It is easy to verify that $f_5(3) \leq \frac{1}{\sqrt{5}}$ as the case of the regular pentagon shows. It would be sufficient to show that, with the notations above,

$$\max\{\Delta(ACD), \Delta(BCE), \Delta(BCD)\} \geq \frac{1}{\sqrt{5}} \Delta(ABCDE).$$  

(12) $\Delta(ACD) \geq \Delta(BCD) \Rightarrow \sqrt{5}(b + 1) \geq a + b + (c + d) + ab \iff \sqrt{5}(b + 1) \geq a + b + (1 + a + ab) + ab \iff (\sqrt{5} - 1)(b + 1) \geq 2a(1 + b) \iff a \leq \frac{\sqrt{5} - 1}{2}$.

This settles Case 1.

Case 2: $\frac{\sqrt{5} - 1}{2} \leq a \leq \frac{\sqrt{5} + 1}{2}$.

Subcase 2(i): $\Delta(ACD) \geq \Delta(BCE)$.

In this case, $b + 1 \geq ab + ad + bc \iff b + 1 \geq ab + ad + ac \iff c + d \leq \frac{b + 1}{a} - b$.

Then, $\Delta(ACD) \geq \Delta(BCD) \iff \sqrt{5}(b + 1) \geq a + b + (c + d) + ab \iff \sqrt{5}(b + 1) \geq a + b + \left(\frac{b + 1}{a} - b\right) + ab \iff \sqrt{5}(b + 1) \geq a(b + 1) + \frac{b + 1}{a} \iff \sqrt{5} \geq a + \frac{1}{a} \iff \frac{\sqrt{5} - 1}{2} \leq a \leq \frac{\sqrt{5} + 1}{2}$.

Subcase 2(ii): $\Delta(BCE) \geq \Delta(ACD)$.

In this case, $ab + ad + bc \geq 1 + b$ which gives

$$d \geq \frac{1 + b - bc}{a} - b.$$  

(13) $d \geq \frac{1 + b - bc}{a} - b.$
Then, \( \frac{\Delta(BCE)}{\Delta(ABCDE)} \geq \frac{1}{\sqrt{5}} \iff \sqrt{5}(ab + ad + bc) \geq a + b + c + d + ab \iff \\
\iff (\sqrt{5} - 1)ab + d(\sqrt{5}a - 1) + \sqrt{5}bc \geq a + b + c \quad \text{(13)}
\iff \sqrt{5}(1 + b) - \frac{1 + b}{a} \geq a(1 + b) + c \left(1 - \frac{b}{a}\right) \iff \\
\iff \sqrt{5}(1 + b) \geq a(1 + b) + \frac{1 + b}{a} \iff \\
\iff \sqrt{5} \geq a + \frac{1}{a} \iff \frac{\sqrt{5} - 1}{2} \leq a \leq \frac{\sqrt{5} + 1}{2}.

This ends the proof of Case 2.

Case 3: \( a \geq \frac{\sqrt{5} + 1}{2} \).

In this case we have
\( \frac{\Delta(BCD)}{\Delta(ABCDE)} \geq \frac{1}{\sqrt{3}} \iff \sqrt{5}(ab + b) \geq a + b + c + d + ab \iff \\
\iff \sqrt{5}b(a + 1) \geq a + b + 2 + ab \iff (\sqrt{5} - 1)b(a + 1) \geq a + 2 \iff \\
\iff a(a + 1) \geq a + 2 \iff \\
\iff a^2 \geq 2 \iff a \geq \sqrt{2} \iff a \geq \frac{\sqrt{5} + 1}{2} > \sqrt{2}.

This completes the proof of Theorem 2.2. \( \square \)

3. Setup for hexagons

In this section we present proofs of two new results: \( f_6(5) = 5/6 \) and \( f_6(4) = 2/3 \).

Let \( ABCDEF \) be an arbitrary convex hexagon. Suppose that the long diagonals, \( AD, BE, \) and \( CF \) are not concurrent. If these diagonals do have a common point, then perturb the position of one of the vertices by an arbitrarily small amount so that the diagonals are not concurrent anymore. By continuity, any inequality which is valid in the latter case is also valid in the former. Let \( M = AD \cap BE, N = AD \cap CF, \) and \( P = BF \cap CE \). Denote \( u = MN, v = MP \) as shown in figure 2.

It follows that \( \overrightarrow{NP} = v - u \). After an appropriate scaling we may assume that \( \Delta(MNP) = u \land v = 1 \). Since \( A, M, N, D \) are collinear, \( 
\overrightarrow{AM} = au, 
\overrightarrow{ND} = du 
\) with \( a, d > 0 \).

Similarly, \( \overrightarrow{BM} = bv, \overrightarrow{CN} = c(v - u), \overrightarrow{PE} = ev, \overrightarrow{PF} = f(v - u) \) with \( b, c, e, f \) being positive constants. By the symmetry of the figure, we may assume that \( a = \min\{a, b, c, d, e, f\} \).
Using the outer product, we obtain the following area formulas.

\[ \Delta(ANF) = (a + 1)u \wedge (f + 1)(v - u) = (a + 1)(f + 1). \]

\[ \Delta(BPC) = (b + 1)v \wedge (c + 1)(v - u) = (b + 1)(c + 1). \]

\[ \Delta(DME) = (d + 1)u \wedge (e + 1)v = (d + 1)(e + 1). \]

\[ \Delta(AMB) = au \wedge bv = ab. \]

\[ \Delta(CND) = dv \wedge c(v - u) = cd. \]

\[ \Delta(EPF) = ev \wedge f(v - u) = ef. \]
We are now in position to compute the area of the hexagon \( ABCDEF \).

\[
\Delta(ABCDEF) = \Delta(ANF) + \Delta(BPC) + \Delta(DME) + \Delta(AMB) + \\
\Delta(CNF) + \Delta(EPF) - 2\Delta(MNP)
\]
which implies that

\[
\Delta(ABCDEF) = 1 + a + b + c + d + e + f + ab + bc + cd + de + ef + fa.
\]

3.1. **Large Pentagons in Hexagons.** The following result is optimal. A lower bound of \( f_6(5) \geq \frac{10}{15 - \sqrt{5}} \approx 0.78345 \ldots \) had been given by Du and Ding [2].

**Theorem 3.1.** \( f_6(5) = \frac{5}{6} \).

**Proof.** It is easy to check that \( f_6(5) \leq \frac{5}{6} \), as the case of the regular hexagon shows. The proof will be complete if we show that \( \frac{\Delta(ABCDEF)}{\Delta(BCDEF)} \geq \frac{5}{6} \). Proving this is equivalent to proving that \( \frac{\Delta(FAB)}{\Delta(ABCDEF)} \leq \frac{1}{6} \). From figure 2, we have that \( \overrightarrow{AB} = au - bv \) and \( \overrightarrow{AF} = (f - a)u - (1 + f)v \). It follows that

\[
\Delta(FAB) = \overrightarrow{AB} \wedge \overrightarrow{AF} = (au - bv) \wedge ((a - f)u + (1 + f)v).
\]
After simplifying, we obtain

\[
\Delta(FAB) = a(1 + f) + b(a - f) = a(1 + b + f) - bf.
\]
From the assumption that \( a = \min\{a, b, c, d, e, f\} \), it follows that there exist nonnegative numbers \( x_i, i = 1, \ldots, 5 \) such that

\[
b = a + x_1, c = a + x_2, d = a + x_3, e = a + x_4, f = a + x_5.
\]
Substituting the above equalities into (14) and (15), we obtain that

\[
\Delta(ABCDEF) - 6\Delta(FAB) = 1 + (2a + 1)(x_1 + x_2 + x_3 + x_4 + x_5) + \\
x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + 6x_5x_1.
\]
Thus, for any hexagon with nonconcurrent long diagonals, \( \Delta(ABCDEF) - 6\Delta(FAB) > 0 \). By continuity, the non-strict inequality holds for all convex hexagons. This completes the proof of Theorem 3.1. \( \square \)

3.2. **Large Quadrilaterals in Hexagons.** The following result is optimal. A lower bound of \( f_6(4) \geq \frac{10}{15 - \sqrt{5}} \approx 0.566915 \ldots \) had been proved by Du and Ding [3].

**Theorem 3.2.** \( f_6(4) = \frac{2}{3} \).

**Proof.** It is easy to check that \( f_6(4) \leq 2/3 \), as the case of the regular hexagon shows. It remains to be shown that \( f_6(4) \geq 2/3 \). Let us define \( Q_1 := \Delta(BCDF), Q_2 := \Delta(ACDE), Q_3 := \Delta(BDEF), Q_4 := \Delta(BCEF), \) and \( H := \Delta(ABCDEF) \). We will show that at least one of \( Q_1/H, Q_2/H, Q_3/H, \) and \( Q_4/H \) is greater than or equal to \( 2/3 \). To this end, we will show that at least one of \( 3Q_1 - 2H, 3Q_2 - 2H, 3Q_3 - 2H, \) and \( 3Q_4 - 2H \) is nonnegative.
First we will compute the areas of all six ears of the hexagon $ABCDEF$.

\[ \Delta(ABC) = \overrightarrow{AB} \wedge \overrightarrow{BC} = (au - bv) \wedge ((c + 1)u + (b - c)v) = b(a + c + 1) - ac, \]

\[ \Delta(BCD) = \overrightarrow{BC} \wedge \overrightarrow{CD} = ((c + 1)u + (b - c)v) \wedge ((-c - d)u + cv) = c(b + d + 1) - bd, \]

\[ \Delta(CDE) = \overrightarrow{CD} \wedge \overrightarrow{DE} = ((-c - d)u + cv) \wedge ((1 - d)u + (e - 1)v) = d(c + e + 1) - ce, \]

\[ \Delta(DEF) = \overrightarrow{DE} \wedge \overrightarrow{EF} = ((1 - d)u + (e - 1)v) \wedge (-fu + (f - e)v) = e(d + f + 1) - df, \]

\[ \Delta(EFA) = \overrightarrow{EF} \wedge \overrightarrow{FA} = (-fu + (f - e)v) \wedge ((-a)u + (-f - 1)v) = f(e + a + 1) - ea, \]

\[ \Delta(FAB) = \overrightarrow{FA} \wedge \overrightarrow{AB} = ((-a)u + (-f - 1)v) \wedge (au - bv) = a(f + b + 1) - fb. \]

With this information, we now compute $Q_1$, $Q_2$, $Q_3$, and $Q_4$ in terms of $a$, $b$, $c$, $d$, $e$, and $f$.

**Case 1:** $a \leq c \leq \min\{b, d, e, f\}$.

It follows that there exist nonnegative numbers $x_i$, $i = 1, \ldots, 5$ such that

\[ c = a + x_1, b = a + x_1 + x_2, d = a + x_1 + x_3, e = a + x_1 + x_4, f = a + x_1 + x_5. \]

Substituting the above equalities into (14), (17), (18), and (19), we obtain that

\[ T_2 + T_3 + T_4 = 3 + (3 + 6a + 3x_1)(x_1 + x_4) + 3x_1^2 + 6x_1(x_2 + x_3) + 3x_2(x_3 + 2x_5), \]

which is clearly nonnegative. Therefore, at least one of the numbers $T_2$, $T_3$, $T_4$ is nonnegative.

**Case 2:** $a \leq d \leq \min\{b, c, e, f\}$.

It follows that there exist nonnegative numbers $x_i$, $i = 1, \ldots, 5$ such that

\[ d = a + x_1, b = a + x_1 + x_2, c = a + x_1 + x_3, e = a + x_1 + x_4, f = a + x_1 + x_5. \]

Substituting the above equalities into (14), (16), (18), and (19), we obtain that

\[ T_1 + T_3 + T_4 = 3 + (3 + 6a + 6x_1)(2x_1 + x_2 + x_5) + 6x_1(x_2 + x_3) + 9x_2x_5 + 3x_3x_4, \]

which is obviously nonnegative. Hence, at least one of the numbers $T_1$, $T_3$, $T_4$ is nonnegative.

**Case 3:** $a \leq e \leq \min\{b, c, d, f\}$.

In this case there exist nonnegative numbers $x_i$, $i = 1, \ldots, 5$ such that

\[ e = a + x_1, b = a + x_1 + x_2, c = a + x_1 + x_3, d = a + x_1 + x_4, f = a + x_1 + x_5. \]

Substituting the above equalities into (14), (16), (17), and (18), we obtain that

\[ T_1 + T_2 + T_4 = 3 + (3 + 6a + 3x_1)(x_1 + x_3) + 3x_1^2 + 6x_1(x_2 + x_3) + 3x_5(2x_2 + x_4), \]

which is nonnegative. It follows that at least one of the numbers $T_1$, $T_2$, $T_4$ is nonnegative.

**Case 4:** $a \leq b \leq \min\{c, d, e, f\}$.

We investigate four separate subcases.
Subcase 4(i): \( a \leq b \leq c \leq \min\{d, e, f\} \).
In this case there exist nonnegative numbers \( x_i, i = 1, \ldots, 5 \) such that
\[
b = a + x_1, c = a + x_1 + x_2, d = a + x_1 + x_2 + x_3, e = a + x_1 + x_2 + x_4, f = a + x_1 + x_2 + x_5.
\]
Substituting the above equalities into (14), (16), (17), and (19), we obtain that
\[
T_1 + 2T_3 = 3 + (3 + 6a + 6x_1)(2x_1 + x_2 + x_3 + x_5) + 6x_1(x_2 + x_5) + 3x_2(x_2 + x_3),
\]
which is nonnegative. Thus, at least one of \( T_1 \) and \( T_3 \) must be nonnegative.

Subcase 4(ii): \( a \leq b \leq d \leq \min\{c, e, f\} \).
In this case there exist nonnegative numbers \( x_i, i = 1, \ldots, 5 \) such that
\[
b = a + x_1, d = a + x_1 + x_2, c = a + x_1 + x_2 + x_3, e = a + x_1 + x_2 + x_4, f = a + x_1 + x_2 + x_5.
\]
Substituting the above equalities into (14), (16), (18), and (19), we obtain that
\[
T_1 + T_3 + T_4 = 3 + (3 + 6a + 6x_1)(2x_1 + x_2 + x_3 + 2x_5) + 6x_1(x_2 + x_5) + 3x_2x_5 + 3x_3x_4,
\]
which is nonnegative. Hence, at least one of \( T_1, T_3, \) and \( T_4 \) must be nonnegative.

Subcase 4(iii): \( a \leq b \leq e \leq \min\{c, d, f\} \).
In this case there exist nonnegative numbers \( x_i, i = 1, \ldots, 5 \) such that
\[
b = a + x_1, e = a + x_1 + x_2, c = a + x_1 + x_2 + x_3, d = a + x_1 + x_2 + x_4, f = a + x_1 + x_2 + x_5.
\]
Substituting the above equalities into (14), (17), (18), and (19), we obtain that
\[
T_1 + T_2 + T_3 = 3 + (3 + 6a + 3x_1)(x_1 + x_2 + x_3) + 3x_1(x_1 + x_2 + 2x_5) + 3x_2x_3 + 3x_4x_5,
\]
which is clearly nonnegative. It follows that at least one of \( T_1, T_2, \) and \( T_4 \) must be nonnegative.

Subcase 4(iv): \( a \leq b \leq f \leq \min\{c, d, e\} \).
In this case there exist nonnegative numbers \( x_i, i = 1 \ldots 5 \) such that
\[
b = a + x_1, f = a + x_1 + x_2, c = a + x_1 + x_2 + x_3, d = a + x_1 + x_2 + x_4, e = a + x_1 + x_2 + x_5.
\]
Substituting the above equalities into (14), (17), and (19), we obtain that
\[
2T_2 + T_1 = 3 + (3 + 6a)(x_2 + x_3 + x_5) + 3x_3(x_2 + x_5),
\]
which is a sum of nonnegative numbers, hence, at least one of \( T_2 \) and \( T_4 \) must be nonnegative. This completes the proof of Case 4.

Case 5: \( a \leq f \leq \min\{b, c, d, e\} \). Again, we study four different subcases.

Subcase 5(i): \( a \leq f \leq b \leq \min\{c, d, e\} \).
It follows that there exist nonnegative numbers \( x_i, i = 1, \ldots, 5 \) such that
\[
f = a + x_1, b = a + x_1 + x_2, c = a + x_1 + x_2 + x_3, d = a + x_1 + x_2 + x_4, e = a + x_1 + x_2 + x_5.
\]
Substituting the above equalities into (14), (17), and (19), we obtain that
\[
2T_2 + T_3 = 3 + (3 + 6a + 3x_1)(x_1 + x_2 + x_3) + 3x_1(x_1 + x_2 + 2x_3) + 3x_2x_5 + 3x_3x_4,
\]
which is certainly nonnegative numbers. It follows that either \( T_2 \) or \( T_4 \) is nonnegative.

Subcase 5(ii): \( a \leq f \leq c \leq \min\{b, d, e\} \).
In this case there exist nonnegative numbers \( x_i, i = 1, \ldots, 5 \) such that
\[
f = a + x_1, c = a + x_1 + x_2, b = a + x_1 + x_2 + x_3, d = a + x_1 + x_2 + x_4, e = a + x_1 + x_2 + x_5.
\]
Substituting the above equalities into (14), (17), (18), and (19), we obtain that
\[
T_2 + T_3 + T_4 = 3 + (3 + 6a + 3x_1)(x_1 + x_2 + x_3 + x_5) + 3x_1(x_1 + x_2 + 2x_3) + 3x_2x_5 + 3x_3x_4,
\]
which is nonnegative. Hence, at least one of \( T_2, T_3, T_4 \) must be nonnegative.
Subcase 5(iii): $a \leq f \leq d \leq \min \{b, c, e\}$.

It follows that there exist nonnegative numbers $x_i$, $i = 1, \ldots, 5$ such that

\[ f = a + x_1, \quad d = a + x_1 + x_2, \quad b = a + x_1 + x_2 + x_3, \quad c = a + x_1 + x_2 + x_4, \quad e = a + x_1 + x_2 + x_5. \]

Substituting the above equalities into (14), (16), (18), and (19) we obtain that

\[ T_1 + T_3 + T_4 = 3 + (3 + 6a + 6x_1)(2x_1 + x_2 + x_3) + 6x_1(x_2 + x_3) + 3x_2x_3 + 3x_4x_5, \]

which is nonnegative. Hence, at least one of $T_1$, $T_3$, $T_4$ must be nonnegative.

Subcase 5(iv): $a \leq f \leq e \leq \min \{b, c, d\}$

In this case there exist nonnegative numbers $x_i$, $i = 1, \ldots, 5$ such that

\[ f = a + x_1, \quad e = a + x_1 + x_2, \quad b = a + x_1 + x_2 + x_3, \quad c = a + x_1 + x_2 + x_4, \quad d = a + x_1 + x_2 + x_5. \]

Substituting the above equalities into (14), (16), (18), and (19) we obtain that

\[ 2T_1 + T_3 = 3 + (3 + 6a + 6x_1)(2x_1 + x_2 + x_3) + 6x_1(x_2 + x_3) + 3x_2x_3 + 3x_3(x_2 + x_5), \]

which is nonnegative, therefore, at least one of $T_1$ and $T_3$ must be nonnegative, which completes the analysis of Case 5, and with it, the proof of Theorem 3.2. \qed

4. A Closer Look at $f_n(n - 1)$

As previously noticed, finding the largest $(n-1)$-gon contained in a given convex $n$-gon $A_1A_2 \ldots A_n$ is equivalent to finding the smallest ear of the original $n$-gon. Define

\[ g_n := \max \min_{i \leq k \leq n} \frac{\Delta(A_{k-1}A_kA_{k+1})}{\Delta(A_1A_2 \ldots A_n)} \]

where the maximum is taken over all convex $n$-gons $A_1A_2 \cdots A_n$. Clearly, $g_n = 1 - f_n(n - 1)$. Theorems (2.1) and (3.1) can therefore be restated as $g_5 = (5 - \sqrt{5})/10$ and $g_6 = 1/6$. In general,

\[ g_n \geq \frac{4}{n} \cdot \sin^2 \left( \frac{\pi}{n} \right) \]

as it can easily be seen by checking the case when the $n$-gon is regular. In this section we provide two upper bounds for $g_n$, the second of which shows that the order of magnitude of the lower bound in (21) is the correct one.

We will need the following simple

**Lemma 4.1.** Let $ABCDE$ be a convex pentagon in which

\[ \Delta(DEA) \leq \min(\Delta(CDE), \Delta(EAB)) \]

Then

\[ \Delta(DEA) \leq \min(\Delta(ABD), \Delta(ACD)) \]

**Proof.** Using equalities (9), we have $c + d - 1 \leq \min(b + d - bc, a + c - ad)$ which imply that $c \leq 1$ and $d \leq 1$, that is, $c + d - 1 \leq 1$. On the other hand, by (11),

\[ \min(\Delta(ABD), \Delta(ACD)) = \min(1 + a, 1 + b) > 1. \]

The conclusion follows. \qed

The following theorem was proved by Du, Feng, and Tan [4]. We present a different argument below.

**Theorem 4.2.** [4] For all $n \geq 4$ we have

\[ g_{n+1} \leq \frac{g_n}{1 + g_n}. \]
Proof. Let $P = A_1A_2\ldots A_nA_{n+1}$ be an extremal convex $(n + 1)$-gon for which

$$\max(\Delta(Q) : Q \text{ is an } n-gon \text{ contained in } P) = f_{n+1}(n) = 1 - g_{n+1}. \tag{23}$$

Without loss of generality we can assume that $Q = A_1A_2\ldots A_n$, that is, $A_nA_{n+1}A_1$ is the triangle of smallest area determined by three consecutive vertices of $P$. Denote $\Delta(A_nA_{n+1}A_1) = \alpha$. Then,

$$1 - g_{n+1} = f_{n+1}(n) = \frac{\Delta(Q)}{\Delta(P)} = \frac{\Delta(Q)}{\alpha + \Delta(Q)} \implies \frac{\alpha}{\Delta(Q)} = \frac{g_{n+1}}{1 - g_{n+1}}.$$ 

Let $R$ be the largest area $(n - 1)$-gon contained in $Q$. Clearly, $R$ is obtained by removing from $Q$ one of the following $n$ triangles: $A_1A_2A_3, A_2A_3A_4, \ldots, A_{n-2}A_{n-1}A_n, A_{n-1}A_nA_1$, or $A_nA_1A_2$.

Each of the first $n - 2$ triangles has area $\geq \alpha$; this is because these triangles are ears of $P$ and we assumed that $A_nA_{n+1}A_1$ is the smallest area ear of $P$.

In particular, $\alpha = \Delta(A_nA_{n+1}A_1) \leq \min(\Delta(A_nA_{n+1}A_1), \Delta(A_nA_1A_2))$.

Using Lemma [11] for the convex pentagon $ABCDE = A_1A_2A_3A_4A_5$ it follows that

$$\alpha = \Delta(A_nA_{n+1}A_1) \leq \min(\Delta(A_nA_{n+1}A_1), \Delta(A_nA_1A_2))$$

This proves that each of the triangles $A_1A_2A_3, A_2A_3A_4, \ldots, A_{n-2}A_{n-1}A_n, A_{n-1}A_nA_1, A_nA_1A_2$ have area $\geq \alpha$. It follows that $\Delta(R) \leq \Delta(Q) - \alpha$ from which

$$1 - g_n = f_n(n - 1) \leq \frac{\Delta(R)}{\Delta(Q)} \leq 1 - \frac{\alpha}{\Delta(Q)} = 1 - \frac{g_{n+1}}{1 - g_{n+1}}.$$ 

Comparing the first and last terms in the inequality chain above implies [22], as claimed.

**Corollary 4.3.**

(a) For every $n \geq 6$, $g_n = 1 - f_n(n - 1) \leq \frac{1}{n}$.

(b) For every $m, n$ such that $5 \leq m < n$, $f_n(m) \geq \frac{m}{n}$.

Proof. We proved in Theorem [3] then $f_6(5) = 5/6$, so the first statement is true for $n = 6$. Using inequality [22], it follows by induction that $g_n \leq 1/n$, as claimed.

In particular, $f_n(n - 1) \geq (n - 1)/n$ for all $n \geq 6$.

For proving the second part, given $5 \leq m < n$, let $P_n$ be an extremal convex $n$-gon for which

$$\max(\Delta(Q) : Q \text{ is an } m-gon \text{ contained in } P_n) = f_n(m). \tag{26}$$

Consider a finite sequence of convex polygons, $P_{n-1}, P_{n-2}, \ldots, P_m, P_m$, constructed as follows: for each $m \leq k \leq n - 1$, $P_k$ is the convex $k$-gon obtained by removing the smallest area ear of $P_{k+1}$. It follows that

$$\frac{\Delta(P_k)}{\Delta(P_{k+1})} \geq f_{k+1}(k),$$

and therefore by using part (a) we obtain

$$f_n(m) \geq \frac{\Delta(P_m)}{\Delta(P_n)} = \prod_{k=m}^{n-1} \frac{\Delta(P_k)}{\Delta(P_{k+1})} \geq \prod_{k=m}^{n-1} f_{k+1}(k) \geq \prod_{k=m}^{n-1} \frac{k}{k + 1} = \frac{m}{n}.$$
In particular, $f_7(5) \geq 5/7 = 0.714\ldots$ improving the bound $f_7(5) \geq 5/(10 - \sqrt{5}) = 0.644\ldots$ due to Du and Ding [2].

Inequalities (21) and (24) provide bounds for $g_n$. However, the upper bound is rather weak as we expect that $g_n = \Theta(n^{-3})$. We prove that this is indeed the case.

**Theorem 4.4.** For every $n \geq 4$

$$g_n \leq \frac{4}{n} \cdot \sin^2 \left(\frac{2\pi}{n}\right).$$

**Proof.** By a celebrated result of John [9], it is known that every convex polygon lies between two concentric homothetic ellipses of ratio 2 - see figure 3.

After an appropriate affine transformation $T$, these ellipses are mapped into a pair of concentric circles, $C$ and $2C$. Since such a transformation preserves ratios, it would be sufficient to prove that the result holds for the polygon $T(P) = A_1A_2\ldots A_n$.

Figure 3. A convex polygon lying between two concentric circles, $C$ and $2C$
Denote the lengths of the sides $A_1A_2, A_2A_3, \ldots, A_nA_1$ by $l_1, l_2, \ldots, l_n$, respectively, and denote by $L = l_1 + l_2 + \ldots + l_n$ the perimeter of $A_1A_2\ldots A_n$. We have that
\[
\sum_{k=1}^{n} \sqrt{2\Delta(A_{k-1}A_kA_{k+1})} = \sum_{k=1}^{n} \sqrt{l_kl_{k+1}} \cdot \sqrt{\sin(\angle A_k)} \leq \left( \sum_{k=1}^{n} \sqrt{l_kl_{k+1}} \right)^{1/2} \left( \sum_{k=1}^{n} \sqrt{\sin(\angle A_k)} \right)^{1/2} \leq \left( \sum_{k=1}^{n} \frac{l_k + l_{k+1}}{2} \right)^{1/2} \cdot \left( \frac{n}{\sqrt{n}} \sqrt{\frac{\sum_{k=1}^{n} \angle A_k}{n}} \right)^{1/2} = \left( \sum_{k=1}^{n} l_k \right)^{1/2} \cdot \sqrt{n} \cdot 4 \sqrt{\sin \left( \frac{(n-2)\pi}{n} \right)} = L^{1/2} \cdot \sqrt{n} \cdot 4 \sqrt{\sin \left( \frac{2\pi}{n} \right)},
\]
where we used Cauchy-Schwarz inequality, the geometric-arithmetic mean inequality and Jensen’s inequality for the concave function $\sqrt{\sin x}$ on the interval $(0, \pi)$. It follows that
\[
\min_{1 \leq k \leq n} \Delta(A_{k-1}A_kA_{k+1}) \leq \frac{L^2 \cdot \sin \left( \frac{2\pi}{n} \right)}{2n^2}, \quad \text{and after dividing by } \Delta = \Delta(A_1A_2\ldots A_n)
\]
(27) \[
\min_{1 \leq k \leq n} \frac{\Delta(A_{k-1}A_kA_{k+1})}{\Delta} \leq \frac{L^2}{\Delta} \cdot \sin \left( \frac{2\pi}{n} \right) \frac{2n^2}{2n^2}
\]
The polygon $A_1A_2\ldots A_n$ contains a circle of radius $r$, hence its area is at least as large as the area of the regular $n$-gon circumscribed about a circle of radius $r$, that is,
\[
\Delta \geq nr^2 \tan \left( \frac{\pi}{n} \right).
\]
On the other hand, the polygon $A_1A_2\ldots A_n$ is contained in a circle of radius $2r$, hence its perimeter is no greater than the perimeter of the regular $n$-gon inscribed in a circle of radius $2r$, that is
\[
L \leq 4nr \sin \left( \frac{\pi}{n} \right).
\]
Combining (27), (28) and (29) we obtain
\[
\min_{1 \leq k \leq n} \frac{\Delta(A_{k-1}A_kA_{k+1})}{\Delta} \leq \frac{16n^2r^2 \sin^2 \left( \frac{\pi}{n} \right)}{nr^2 \tan \left( \frac{\pi}{n} \right)} \cdot \sin \left( \frac{2\pi}{n} \right) \frac{2n^2}{2n^2} = 4 \frac{n}{n} \sin^2 \left( \frac{2\pi}{n} \right).
\]
This completes the proof.}

\[\square\]

\textbf{REFERENCES}

[1] Ahn, H. K., Bae S. W., Cheong, O., Gudmundsson, J.: Aperture-angle and Hausdorff-approximation of convex figures. \textit{Comput. Geom. (SCG'07)}, 37–45, ACM, New York (2007).
[2] Du, Y., Ding, R.: On the maximum area pentagon in a planar point set, \textit{Applied Mathematics Letters}, 19, 1228-1236 (2006).
[3] Du, Y., Ding, R.: On maximum area polygons in a planar point set, \textit{Elemente der Mathematik}, 63, 88-96 (2008).
[4] Du, Y., Feng, H., Tan, H.: More on the maximum area polygons in a planar point set, \textit{International Journal of Applied Mathematics}, 26, 701-712 (2013).
[5] Fejes, L.: Über die Approximation konvexer Kurven durch Polygonfolgen. (German) \textit{Compositio Math.}, 6, 456–467 (1939).
[6] Fejes Tóth, L.: "Lagerungen in der Ebene, auf der Kugel und im Raum." (German) Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band LXV. Springer-Verlag, Berlin-Göttingen-Heidelberg (1953).

[7] Fejes Tóth, G., Kuperberg, W.: Packing and covering with convex sets. In: Handbook of convex geometry, Vol. A, B, pp 799-860, North-Holland, Amsterdam (1993).

[8] Fleischer, R., Mehlhorn, K., Rote, G., Welzl, E., Yap, C.K.: Simultaneous inner and outer approximation of shapes. Algorithmica 8, 365-389 (1992).

[9] John, F.: Extremum problems with inequalities as subsidiary conditions. Studies and Essays Presented to R. Courant on his 60th Birthday, pp 187-204, Interscience Publishers, Inc., New York, NY (1948).

[10] Lázár, D.: Sur l’approximation des courbes convexes par des polygones. (French) Acta Universitatis Szegediensis 11, 129–132 (1947).

[11] Sás, E.: On a certain extremum-property of the ellipse. (German) Compositio Math. 7, 474–476 (1940).

[12] Schwarzkopf, O., Fuchs, U., Rote, G., Welzl, E.: Approximation of convex figures by pairs of rectangles. In: Proc. Seventh Annual Symposium on Theoretical Aspects of Computer Science, Lecture Notes in Comput. Sci., vol. 415, 240-249 Springer, Berlin (1990).

[13] Yap, C. K.: Algorithmic motion planning. In: Advances in Robotics, Vol. 1, eds. J.T. Schwartz and C.K. Yap, Chapter 3, Erlbaum, Hillsdale, NJ (1987).

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