Amenability and Weak Amenability of the Semigroup Algebra $\ell^1(S_T)$

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Abstract

Let $S$ be a semigroup with a left multiplier $T$ on $S$. A new product on $S$ is defined by $T$ related to $S$ and $T$ such that $S$ and the new semigroup $S_T$ have the same underlying set as $S$. It is shown that if $T$ is injective then $\ell^1(S_T) \cong \ell^1(S)\bar{T}$ where, $\bar{T}$ is the extension of $T$ on $\ell^1(S)$. Also, we show that if $T$ is bijective, then $\ell^1(S)$ is amenable if and only if $\ell^1(S_T)$ is so. Moreover, if $S$ completely regular, then $\ell^1(S_T)$ is weakly amenable.

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Introduction

Let $S$ be a semigroup and $T$ be a left multiplier on $S$. We present a general method of defining a new product on $S$ which makes $S$ a semigroup. Let $S_T$ denote $S$ with the new product. These two semigroups are sometimes different and we try to find conditions on $S$ and $T$ such that the semigroups $S$ and $S_T$ have the same properties. This idea has started by Birtel in [1] for Banach algebras and continued by Larsen in [11]. Recently, this notion developed by some authors, for more details see [1], [10], [11], [12] and [15]. One of the best result in this work expresses that $L^1(G)_T$ is Arens regular if and only if $G$ is a compact group [10]. We continue this direction on the regularity of $S$ and $S_T$ and the amenability of their semigroup algebras.

The term of semigroup will be a non-empty set $S$ endowed with an associative binary operation on $S$, defined by $(s, t) \rightarrow st$. If $S$ is also a Hausdorff topological space and the binary operation is jointly continuous, then $S$ is called a topological semigroup.

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Let \( p \in S \). Then \( p \) is an idempotent if \( p^2 = p \). The set of all idempotents of \( S \) is denoted by \( E(S) \).

An element \( e \) is a left (right) identity if \( es = s \) (resp. \( se = s \)) for all \( s \in S \). An element \( ecS \) is an identity if it is a left and a right identity. An element \( z \) is a left (resp. right) zero if \( zs = z \) (resp. \( sz = z \)) for all \( s \in S \). An element \( zcS \) is a zero if it is a left and a right zero. We denote any zero of \( S \) by \( 0_S \) (or \( z_S \)). An element \( pcS \) is a regular element of \( S \) if there exists \( tcS \) such that \( p = ptp \) and \( p \) is completely regular if it is regular and \( pt = tp \). We say that \( p \in S \) has an inverse if there exists \( tcS \) such that \( p = ptp \) and \( t = tpt \). Note that the inverse of element \( p \in S \) need not be unique. If \( pcS \) has an inverse, then \( p \) is regular and vise versa. Since, if \( p \in S \) is regular, there exists \( s \in S \) such that \( p = psp \). Let \( t = sps \). Then \( p = psp = (p(s)p)s = p = ptp \), \( t = sps = s(p)p(s) = (spsp)(spsp) = tpt \).

So \( p \) has an inverse. We say that \( S \) is a regular (resp. completely regular) semigroup if each \( pcS \) is regular (resp. completely regular). Also \( S \) is an inverse semigroup if each \( p \in S \) has a unique inverse. The map \( T : S \to S \) is called a left (resp. right) multiplier if

\[
T(st) = T(s)t \quad \text{(resp. } T(st) = sT(t) \text{)} \quad (s, tcS).
\]

The map \( T : S \to S \) is a multiplier if it is a left and right multiplier. Let \( S \) be a topological semigroup. The net \( (e_\alpha) \subseteq S \) is a left (resp. right) approximate identity if \( \lim _{\alpha} e_\alpha t = t \) (resp. \( \lim _{\alpha} t e_\alpha = t \)) \((tcS)\). The net \( (e_\alpha) \subseteq S \) is an approximate identity if it is a left and a right approximate identity.

Let \( S \) be a discrete semigroup. We denote by \( l^1(S) \) the Banach space of all complex function \( f : S \to \mathbb{C} \) having the form

\[
f = \sum_{s \in S} f(s) \delta_s ,
\]

such that \( \sum_{s \in S} |f(s)| = \| f \|_1 \) is finite, where \( \delta_s \) is the point mass at \( \{ s \} \). For \( f, g \in l^1(S) \) we define the convolution product on \( l^1(S) \) as follow:

\[
f * g(s) = \sum_{t_1, t_2 = s} f(t_1)g(t_2) \quad (scS),
\]

with this product \( l^1(S) \) becomes a Banach algebra and is called the semigroup algebra on \( S \).

Remark 1.1. If \( f \in l^1(S) \) then \( f = 0 \) on \( S \) except at most on a countable subset of \( S \). In other words, the set \( A = \{ scS : \| f(s) \| = 0 \} \) is at most countable. Since, if \( A_n = \{ scS : \| f(s) \| \geq \frac{1}{n} \} \), \( A = \bigcup _{n \in \mathbb{N}} A_n \). Set \( \| f \|_1 = M \) and \( m \in \mathbb{N} \) is fixed. Then we have

\[
M = \sum_{s \in S} |f(s)| \geq \sum_{s \in A_n} |f(s)| \geq \sum_{s \in A_n} \frac{1}{n} = \frac{1}{n} |A_n| ,
\]
where \(|A_n|\) is the cardinality of \(A_n\). So \(|A_n| \leq nM\). Hence \(A_n\) is a finite subset of \(S\) and thus \(A\) is at most countable.

**Semigroup \(S_T\)**

Let \(T \in \text{Mul}_1(S)\). Then we define a new binary operation "\(o\)" on \(S\) as follow:

\[
s \circ t = s \cdot T(t) \quad (s,t \in S).
\]

The set \(S\) equipped with the new operation "\(o\)" is denoted by \(S_T\) and sometimes called "induced semigroup of \(S\)". Now we have the following results.

**Theorem 2.1.** Let \(S\) be a Semigroup. Then (i) if \(T \in \text{Mul}_1(S)\) then \(S_T\) is a semigroup.

The converse is true if \(S\) is left cancellative and \(T\) is surjective.

(ii) If \(S_T\) is left cancellative and \(T\) is surjective, then \(T^{-1} \in \text{Mul}_1(S)\).

(iii) If \(S\) is a topological semigroup and \(S_T\) has a left approximate identity then \(T^{-1} \in \text{Mul}_1(S)\).

**Proof.**

i) Let \(T \in \text{Mul}_1(S)\) and take \(r,s,t \in S\). Then

\[
r \circ (s \circ t) = r \cdot T(s \circ t) = r \cdot T(s) \cdot T(t) = (r \cdot T(s)) \cdot T(t) = (r \circ s) \circ t.
\]

So, \(S_T\) is a semigroup.

Conversely, suppose that \(S\) is left cancellative and take \(r,s,t \in S\). Since \(T\) is surjective, there exists \(u \in S\) such that \(T(u) = t\). Then

\[
r \cdot T(st) = r \cdot T(sT(u)) = r \circ (s \circ u) = (r \circ s) \circ u = (r \cdot T(s)) \cdot T(u) = r \cdot T(st).
\]

By the left cancellativity of \(S\), we have \(T(st) = T(s) \cdot T(t) = (r, s \in S)\). So, \(T\) is a left multiplier.

ii) We must prove that \(T\) is injective. To do this end, take \(r,s,u \in S\) and let \(T(r) = T(s)\).

Then \(u \circ r = u \cdot T(r) = u \cdot T(s) = u \circ s\). So \(r = s\), since \(S_T\) is left cancellative. Hence \(T^{-1}\) exists.

Now, we show that \(T^{-1} \in \text{Mul}_1(S)\). Take \(r,s \in S\). Then

\[
T^{-1}(rs) = T^{-1}[TT^{-1}(r)s] = T^{-1}[T(T^{-1}(r)s)] = (T^{-1}T)[T^{-1}(r)s] = T^{-1}(r)s.
\]

iii) It is enough to show that \(T\) is injective. Take \(r,s \in S\) and suppose that \(T(r) = T(s)\).

Then

\[
r = \lim_\alpha e_\alpha \circ r = \lim_\alpha e_\alpha T(r) = \lim_\alpha e_\alpha T(s) = \lim_\alpha e_\alpha \circ s = s.
\]
There are many properties that induced from $S$ to semigroup $S_T$. But sometimes they are different.

**Theorem 2.2.** Let $S$ be a Hausdorff topological semigroup and $\text{TeMul}_1(S)$. If $S$ is commutative then so is $S_T$. The converse is true if $\overline{T(S)} = S$.

**Proof.** Suppose $S$ is commutative and take $r, s \in S$. Then

$$r \circ s = r \ T(s) = T(s)r = T(sr) = T(rs) = T(r)s = s \ T(r) = s \circ r.$$

So, $S_T$ is commutative.

Conversely, Let $S_T$ be commutative and take $r, s \in S$. Then there exist nets $(r_\alpha)$ and $(s_\beta)$ in $S$ such that $\lim_\alpha T(r_\alpha) = r$ and $\lim_\beta T(s_\beta) = s$.

So, we have

$$rs = \lim_\alpha \lim_\beta T(r_\alpha \circ s_\beta) = \lim_\alpha \lim_\beta T(s_\beta \circ r_\alpha) = \lim_\alpha \lim_\beta \ T(s_\beta) \ T(r_\alpha) = s \ r.$$

Thus $S$ is commutative.

In the sequel, we investigate some relations between two semigroup $S$ and $S_T$ according to the role of the left multiplier $T$.

**Theorem 2.3.** Let $S$ be a semigroup and $\text{TeMul}_1(S)$ .Then

(i) If $T$ is surjective and $S_T$ is an inverse semigroup then $S$ is an inverse semigroup and $T(s^{-1}) = T(s)^{-1}$ for all $s \in S$.

(ii) If $S_T$ is an inverse semigroup and $T$ is injective then $T(S)$ is an inverse subsemigroup of $S$.

(iii) If $T$ is bijective then $S_T$ is an inverse semigroup if and only if $S$ is an inverse semigroup.

**Proof.** i) Suppose that $S_T$ is an inverse semigroup and $T$ is surjective. Define the map $\varphi: S_T \to S$ by $\varphi(s) = T(s)$. Take $r, s \in S$, then

$$\varphi(r \circ s) = T(r \circ s) = T(r)T(s) = \varphi(r)\varphi(s).$$

So, $\varphi$ is an epimorphism from $S_T$ onto $S$, since $T$ is surjective. By theorem 5.1.4[7], $S$ is an inverse semigroup and $T(s^{-1}) = T(s)^{-1}$ for all $s \in S$.

ii) Suppose that $T$ is injective and $S_T$ is an inverse semigroup. Evidently, $T(S)$ is a subsemigroup of $S$. We show that it is an inverse semigroup. Take $s \in T(S)$. There exists $t \in S$ such that $s = T(t)$. Also, there exists a unique element $u \in S$ such that $t = t_0 u_0 t$, since $S_T$ is an inverse semigroup. Therefore, $T(t) = T(t)T(u)T(t)$, or $s = s_0 T(u) s$. Of course, $T(u)$ is unique because $u \in S$ is unique and $T$ is injective. Hence $T(S)$ is an inverse subsemigroup of $S$.
iii) Suppose that \( T \) is bijective and let \( S_T \) be an inverse semigroup. Since \( T \) is injective and surjective, by (i) and (ii), \( S = T(S) \) is an inverse semigroup.

Conversely, suppose that \( S \) is an inverse semigroup. Since \( T \) is bijective, by theorem 2.1(ii), \( T^{-1} \in Mul(S) \). So \( \varphi^{-1}: S \rightarrow S_T \) defined by \( \varphi^{-1}(s) = T^{-1}(s) \) is an epimorphism. Hence by (i) \( S_T \) is an inverse semigroup.

We say that \( T \in Mul(S) \) is an inner left multiplier if it has the form \( T = L_s \) for some \( s \in S \) where \( L_s(t) = st \) \((teS)\).

If \( T \in Mul(S) \) is inner, then each ideal of \( S \) is permanent under \( T \); that is \( T(I) \subseteq I \) for all ideal \( I \) of \( S \). It is easily to see that if \( S \) has an identity, then each \( T \in Mul(S) \) is inner.

Let \( S \) be a semigroup. Then \( S \) is called semisimple if \( I^2 = I \) for all ideal \( I \) of \( S \) (see [9], page 95 for more details).

**Theorem 2.4.** Let \( S \) be a semigroup whit an identity and \( T \in Mul(S) \). If \( S_T \) is semisimple, then \( S \) is so. The converse is true if \( S_T \) is left cancellative and \( T \) is surjective.

**Proof.** Since \( S \) is unital there exists \( \mu \in S \) such that \( T = L_\mu \). Suppose that \( S_T \) is semisimple and \( I \) is an ideal of \( S \). Then

\[ I \circ S = I T(S) \subseteq IS \subseteq I. \]

Similarly, \( S \circ I \subseteq I \). It follows that \( I \) is an ideal of \( S_T \). By the hypothesis \( (I_T)^2 = I \circ I = I \). Now, take \( r \in I \) then there are \( s, t \in I \) such that

\[ r = s \circ t = sT(t) = s(\mu t) \in I^2. \]

So we show that \( I^2 = I \) and hence \( S \) is semisimple.

Conversely, assume that \( S_T \) is left cancellative and \( T \in Mul(S) \) is surjective then by theorem 2.1(ii), \( T^{-1} \in Mul(S) \). So, there exists \( b \in S \) such that \( T^{-1} = L_b \). Suppose that \( \bar{S} = S_{T^{-1}} \). Then we have:

\[ S = S_{TT^{-1}} = (S_T)_{T^{-1}} = \bar{S}_{T^{-1}}. \]

By hypothesis and above the proof, \( \bar{S} = S_{T^{-1}} \) is semisimple.

**Semigroup Algebra** \( \ell^1(S_T) \)

We say that a discrete semigroup \( S \) is amenable if there exists a positive linear functional on \( \ell^\infty(S) \) called a mean such that \( m(1) = 1 \) and \( m(l_s) = m(f) \), \( m(r_s) = m(f) \) for each \( s \in S \), where \( l_s(f)(t) = f(st) \) and \( r_s(f)(t) = f(ts) \) for all \( t \in S \). The definition of amenable group is similar to semigroup case. Refer to [12] for more details.
Let \( \mathfrak{A} \) be a Banach algebra and let \( X \) be a Banach \( \mathfrak{A} \)-bimodule. A derivation from \( \mathfrak{A} \) to \( X \) is a linear map \( D: \mathfrak{A} \to X \) such that
\[
D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathfrak{A}).
\]
A derivation \( D \) is inner if there exists \( x \in X \) such that
\[
D(a) = a \cdot x - x \cdot a \quad (a \in \mathfrak{A}).
\]
The Banach algebra \( \mathfrak{A} \) is amenable if every bounded derivation \( D: \mathfrak{A} \to X^* \) is inner for all Banach \( \mathfrak{A} \)-bimodule \( X \). Where \( X^* \) is the dual space of \( X \). We say that the Banach algebra \( \mathfrak{A} \) is weakly amenable if any bounded derivation \( D \) from \( \mathfrak{A} \) to \( \mathfrak{A}^* \) is inner. For more details see [12], [16].

If \( S \) is a commutative semigroup, by theorem 5.8 of [8] \( \ell^1(S) \) is called semisimple if and only if for all \( x, y \in S \), \( x^2 = y^2 = xy \) implies \( x = y \).

**Theorem 3.1.** Let \( S \) be a commutative semigroup and let \( T \in \text{Mul}_l(S) \) be injective. Then \( \ell^1(S) \) is semisimple if and only if \( \ell^1(S_T) \) is semisimple.

**Proof.** Take \( r, s \in S \). Then \( r^2 = s^2 = rs \) if and only if \( T(r^2) = T(s^2) = T(r)T(s) \) or equivalently \( r \circ r = s \circ s = r \circ s \), because \( T \) is injective. So, by theorem 5.8 [8], \( \ell^1(S) \) is semisimple if and only if \( \ell^1(S_T) \) is semisimple.

**Theorem 3.2.** Let \( S \) be a discrete semigroup and \( T \in \text{Mul}_l(S) \). Then
(i) The left multiplier \( T \) has an extension \( \tilde{T} \in \text{Mul}_l(\ell^1(S)) \) with the norm decreasing.
(ii) The left multiplier \( T \) is injective if and only if so is \( T \).
(iii) If \( T \) is injective then \( \tilde{T} \) is an isometry and also \( \ell^1(S_T) \) and \( (\ell^1(S))_T \) are isomorphic.

**Proof.** (i) An arbitrary element \( f \in \ell^1(S) \) is of the form \( f: S \to \mathbb{C} \) such that \( f(x) = 0 \) except at the most countable subset \( A \) of \( S \). If \( A \) is a finite subset of \( S \) then \( f = \sum_{k=1}^{n} f(x_k) \delta_{x_k} \) for some fixed \( n \in \mathbb{N} \). So in general we have
\[
f = \sum_{x \in S} f(x) \delta_x = \sum_{x \in A} f(x) \delta_x + \sum_{k=1}^{\infty} f(x_k) \delta_{x_k}.
\]
Now, for each \( n \in \mathbb{N} \), let \( f_n = \sum_{k=1}^{n} f(x_k) \delta_{x_k} \) and define \( \tilde{T}: \ell^1(S) \to \ell^1(S) \) by
\[
\tilde{T}(\delta_x) = \delta_{T(x)} \quad (x \in S),
\]
\[
\tilde{T}(f_n) = \sum_{k=1}^{n} f(x_k) \tilde{T}(\delta_{x_k}) = f_n.
\]
For each \( m, n \in \mathbb{N} \) where \( n \geq m \), we have
\[
\| \tilde{T}(f_n) - \tilde{T}(f_m) \|_1 = \| f_n - f_m \|_1 = \| \sum_{k=m}^{n} f(x_k) \tilde{T}(\delta_{x_k}) \|_1 = \| \sum_{k=m}^{n} f(x_k) \delta_{T(x_k)} \|_1 \leq \sum_{k=m}^{n} |f(x_k)| = \| f_n - f_m \|_1.
\]
So \( \{ \tilde{T}(f_n) \}_{n} \) is a Cauchy sequence and it is convergent. Now, we define \( \tilde{T}(f) = \lim_n \tilde{T}_n \). Then the definition is well defined. Hence
\[
\tilde{T}(f) = \sum_{k=1}^{\infty} f(x_k) \tilde{T}(\delta_{x_k}) = \tilde{f},
\]
also
\[
\|\tilde{f}\|_1 \leq \sum_{x \in A} |f(x_k)| = \|f\|_1 \quad \text{or} \quad \|\tilde{T}(f)\|_1 \leq \|f\|_1.
\]
It shows that \( \tilde{T} \) is norm decreasing.

In the following, we extend \( \tilde{T} \) by linearity. Let \( f, g \in \ell^1(S) \). Then there are two at most countable sub set \( A, B \) of \( S \) such that
\[
f = \sum_{x \in A} f(x) \delta_x, \quad g = \sum_{x \in B} g(x) \delta_x.
\]
Suppose that \( D = A \cup B \). So we have \( f + g = \sum_{x \in D} (f(x) + g(x)) \delta_x \).

Then, it follows that
\[
\tilde{T}(f + g) = \tilde{f} + \tilde{g} = \sum_{x \in D} (f(x) + g(x)) \tilde{T}(\delta_x) = \sum_{x \in A} f(x) \tilde{T}(\delta_x) + \sum_{x \in B} g(x) \tilde{T}(\delta_x)
\]
which shows that \( \tilde{T} \) is norm decreasing.

Therefore, \( \tilde{T} \) is a bounded linear isometry.

Now, we prove that \( \tilde{T} \in \text{Mul}_1 \left( \ell^1(S) \right) \). Take \( x, y \in S \). Then
\[
\tilde{T}(\delta_x * \delta_y) = \tilde{T}(\delta_{xy}) = \delta_{T(xy)} = \delta_{T(x)y} = \delta_{T(x)} * \delta_y = \tilde{T}(\delta_x) * \delta_y.
\]

Let \( y \in S \) be fixed and \( f, g \in \ell^1(S) \). Then
\[
\tilde{T}(f * \delta_y) = \tilde{T} \left( \sum_{x \in A} f(x) \delta_{xy} \right) = \sum_{x \in A} f(x) \tilde{T}(\delta_{xy}) = \left( \sum_{x \in A} \tilde{T}(\delta_x) \right) * \delta_y = \tilde{T}(f) * \delta_y.
\]

In the general case, we have
\[
\tilde{T}(f * g) = \tilde{T} \left( \sum_{x \in A} f(x) \sum_{y \in B} g(y) \delta_{xy} \right) = \sum_{x \in A} f(x) \sum_{y \in B} g(y) \tilde{T}(\delta_x) * \delta_y = \sum_{x \in A} f(x) \tilde{T}(\delta_x) * \sum_{y \in B} g(y) \delta_y = \tilde{T}(f) * g.
\]
This shows that \( \tilde{T} \) is a multiplier on \( \ell^1(S) \).

(ii) Let \( T \) be injective. Take \( x, y \in S \) and suppose that \( \tilde{T}(\delta_x) = \tilde{T}(\delta_y) \). Then \( \delta_{T(x)} = \tilde{T}(\delta_x) = \tilde{T}(\delta_y) = \delta_{T(y)} \).

Therefore, \( T(x) = T(y) \). Since \( T \) is injective, we have \( x = y \). It follows that \( \delta_x = \delta_y \), consequently \( \tilde{T} \) is injective.

Conversely, the same argument shows that the converse holds.

(iii) Let \( T \) be injective and \( f \in \ell^1(S) \). Then there exists at most a countable subset \( A \subseteq S \) such that
Since $A$ and $T(A)$ have the same cardinal number, \( \|\tilde{T}(f)\| = \|\sum_{x \in A} f(x) \delta_x\| = \sum_{x \in A} |f(x)| = \|f\|_1 \), so $\tilde{T}$ is an isometry.

Now, we can define a new multiplication "$\boxtimes$" on $\ell^1(S)$ as follow
\[
f \boxtimes g = f \ast \tilde{T}(g) \quad (f, g \in \ell^1(S)).
\]

By a similar argument in theorem 1.31 [10], $\ell^1(S)$ with the new product is a Banach algebra that is denoted it by $\ell^1(S)_{\tilde{T}}$. We define the map $\Psi: \ell^1(S_T) \rightarrow \ell^1(S)_{\tilde{T}}$, by
\[
\Psi(\delta_x) = \delta_x \quad (x \in S).
\]

Take $x, y \in S$. Then
\[
\Psi(\delta_x \boxtimes \delta_y) = \Psi(\delta_{x \ast y}) = \delta_x \ast \delta_T(y) = \delta_x \ast \delta_T(y)
\]
\[
= \delta_x \ast \tilde{T}(\delta_y) = \delta_x \boxtimes \delta_y
\]
\[
= \Psi(\delta_x) \boxtimes \Psi(\delta_y).
\]

So, in general case, we have
\[
\Psi(f \ast g) = \Psi(f) \boxtimes \Psi(g) \quad (f, g \in \ell^1(S)).
\]

Thus, $\Psi$ is an isomorphism. Therefore $\ell^1(S_T)$ and $\ell^1(S)_{\tilde{T}}$ are isomorphic.

**Theorem 3.3.** Let $S$ be a semigroup and $T \in \text{Mul}_f(S)$ be bijective. Then $\ell^1(S)$ is amenable if and only if $\ell^1(S_T)$ is amenable.

**Proof.** By theorem 3.2, we have $\ell^1(S_T) \cong \ell^1(S)_{\tilde{T}}$. Suppose that $\ell^1(S_T)$ is amenable and define $\phi: \ell^1(S_T) \rightarrow \ell^1(S)$ by $\phi(f) = \tilde{T}(f)$. Take $x, y \in S$. Then
\[
\phi(\delta_x \boxtimes \delta_y) = \tilde{T}(\delta_x \boxtimes \delta_y) = \tilde{T}(\delta_{x \ast T(y)}) = \tilde{T}(\delta_x \ast \delta_T(y)) = \tilde{T}(\delta_x) \ast \delta_T(y)
\]
\[
= \tilde{T}(\delta_x) \ast \tilde{T}(\delta_y) = \phi(\delta_x) \ast \phi(\delta_y).
\]

Now, by induction and continuity of $\tilde{T}$, we have
\[
\phi(f \boxtimes g) = \phi(f) \ast \phi(g).
\]

If $T$ is bijective, $\tilde{T}$ is bijective. Therefore $\phi$ is an epimorphism of $\ell^1(S_T)$ onto $\ell^1(S)$.

Hence, by proposition 2.3.1 [16], $\ell^1(S)$ is amenable.

Conversely, suppose that $\ell^1(S)$ is amenable. Since $T$ is bijective, $\tilde{T}$ is bijective. Therefore $\tilde{T}^{-1}$ exists. Now define $\theta: \ell^1(S) \rightarrow \ell^1(S_T) \cong \ell^1(S)_{\tilde{T}}$ by $\theta(f) = \tilde{T}^{-1}(f)$.

Take $x, y \in S$. Then
\[
\theta(\delta_x \ast \delta_y) = \tilde{T}^{-1}(\delta_{xy}) = \tilde{T}^{-1}(\delta_x) \tilde{T}^{-1}(\delta_y) = \tilde{T}^{-1}(\delta_x) \boxtimes \tilde{T}^{-1}(\delta_y)
\]
\[
= \theta(\delta_x) \boxtimes \theta(\delta_y).
\]
Similarly \( \theta \) is an epimorphism from \( \ell^1(\mathcal{S}) \) onto \( \ell^1(\mathcal{S}_\tau) \). By proposition 2.3.1 [16] \( \ell^1(\mathcal{S}_\tau) \) is amenable.

Note that, in general, it is not known when \( \ell^1(\mathcal{S}) \) is weakly amenable. For more details see [2].

**Theorem 3.4.** Let \( \mathcal{S} \) be a semigroup and \( T \in \text{Mul}_\ell(\mathcal{S}) \) be bijective. Then, if \( \mathcal{S} \) is completely regular then \( \ell^1(\mathcal{S}_\tau) \) is weakly amenable.

**Proof.** It is enough to prove that \( \mathcal{S}_\tau \) is completely regular, then by theorem 3.6 [2], \( \ell^1(\mathcal{S}_\tau) \) can be weakly amenable. Take \( s \in \mathcal{S} \). Then there exists \( r \in \mathcal{S} \) such that \( T(s) = T(s)T(r)T(s) \), \( T(r)T(s) = T(s)T(r) \), since \( T \) is bijective and \( \mathcal{S} = T(\mathcal{S}) \) is completely regular. So we have \( T(s) = T(s \circ r \circ s) \) and \( T(r \circ s) = T(s \circ r) \). Hence \( s = s \circ r \circ s \) and \( r \circ s = s \circ r \) for some \( r \in \mathcal{S} \), since \( T \) is injective. Therefore \( \mathcal{S}_\tau \) is completely regular.

**Corollary 3.5.** Suppose that \( \mathcal{S} \) is a commutative completely regular semigroup and \( T \in \text{Mul}_\ell(\mathcal{S}) \) is injective. Then \( \ell^1(T(\mathcal{S})_\tau) \) is weakly amenable.

**Proof.** [2, theorem 3.6] \( \ell^1(\mathcal{S}) \) is weakly amenable. Define \( \varphi: \mathcal{S} \rightarrow \ell^1(\mathcal{S})_\tau \) by 

\[
\varphi(s) = T^{-1}(s) \quad (s \in \mathcal{S}).
\]

We show that \( \varphi \) is a homomorphism. Take \( s \in \mathcal{S} \), then we have 

\[
\varphi(rs) = T^{-1}(rs) = T^{-1}(r)s = T^{-1}(r) \circ (T^{-1}s).
\]

So \( \varphi \) is a homomorphism. Then by proposition 2.1[7], \( \ell^1(T(\mathcal{S})_\tau) \) is weakly amenable.

In the case that \( \mathcal{S} \) is a group, it is easy to see that the amenability of \( \mathcal{S} \) implies the amenability of \( \ell^1(\mathcal{S}_\tau) \). Indeed, when \( \mathcal{S} \) is a group, by theorem 2.1, \( \mathcal{S}_\tau \) is a semigroup and one can easily prove that \( \mathcal{S}_\tau \) is also a group. On the other hand, \( \text{Mul}_\ell(\mathcal{S}) \cong \mathcal{S} \) because \( \mathcal{S} \) is a unital semigroup, so each \( T \in \text{Mul}_\ell(\mathcal{S}) \) is inner and of the form \( T = L_s \) for some \( s \in \mathcal{S} \). Also \( T^{-1} = L_{a^{-1}} \) exists, since \( \mathcal{S} \) is a group. Then the map \( \theta: \mathcal{S}_\tau \rightarrow \mathcal{S} \) defined by \( \theta(s) = T(s) \) is an isomorphism; that is \( \mathcal{S} \cong \mathcal{S}_\tau \). Thus we have the following result:

**Corollary 3.6.** Let \( \mathcal{S} \) be a cancellative regular discrete semigroup. Then \( \ell^1(\mathcal{S}) \) is amenable if and only if \( \ell^1(\mathcal{S}_\tau) \) is amenable.

**Proof.** By [9, Exercise 2.6.11] \( \mathcal{S} \) is a group. So the assertion holds by [15, theorem 2.1.8]
Examples

In this section we present some examples which either comment on our results or indicate necessary condition in our theorems.

4.1. There are semigroups $S$ and $T \in \text{Mul}_t(S)$ such that the background semigroups $S$ are not commutative but their induced semigroups $S_T$ are commutative.

This example shows that the condition $\overline{T(S)} = S$, in theorem 2.2, can not be omitted.

Let $S$ be the set \{a, b, c, d, e\} with operation table given by

\[
\begin{array}{cc|ccc}
 & a & b & c & d \\
\hline
a & a & a & a & d \\
b & a & b & c & d \\
c & a & c & b & d \\
d & d & d & d & a \\
e & d & e & e & a \\
\end{array}
\]

Clearly $(S, \cdot)$ is a non-commutative semigroup. Now, put $T = L_a$ where $L_a(x) = ax$ for all $x \in S$. One can get easily the operation table of $S_T$ as follow:

\[
\begin{array}{cc|ccc}
 & a & b & c & d \\
\hline
a & a & a & a & d \\
b & a & a & a & d \\
c & a & a & a & d \\
d & d & d & d & a \\
e & d & d & d & a \\
\end{array}
\]

The operation table shows that the induced semigroup $S_T$ is commutative and $T(S) \neq S$. Also the other induced semigroup $S_T$ is commutative for $T = L_d$ analogously.

Now we present some important theorems from[14] that we need in the following examples:

**Theorem 4.2.** Let $S$ be a semigroup. Suppose that $\ell^1(S)$ is amenable. Then

(i) $S$ is amenable

(ii) $S$ is regular.

(iii) $E(S)$ is finite.

(iv) $\ell^1(S)$ has an identity.

**Proof.** (i) That is lemma 3 in [5].

(ii) and (iii) See theorem 2 in[6].
(iv) That is corollary 10.6 in [4].

**Theorem 4.3.** Let \( S \) be a finite semigroup. Then the following statements are equivalent:

(i) \( \ell^1(S) \) is amenable.

(ii) \( S \) is regular and \( \ell^1(S) \) is nuital.

(ii) \( S \) is regular and \( \ell^1(S) \) is semisimple.

**Proof.** Refer to [3].

**4.4.** There are semigroups \( S \) and \( T \in \text{Mul}(S) \) such that \( S \) and \( \ell^1(S) \) are amenable but \( S_T \) is not regular and also, \( \ell^1(S_T) \) is not amenable.

This example shows that two semigroup algebras \( \ell^1(S) \) and \( \ell^1(S_T) \) can be different in some properties. Also, it notifies that the bijectivity of \( T \) in the theorem 3.3 is essential. Put \( S = \{x_0, x_1, x_2, \ldots, x_n\} \) with the operation \( x_i x_j = x_{\max(i,j)} \) \((0 \leq i, j \leq n, n \geq 2)\).

Then \( S \) is a semigroup. Since

\[
\max\{i, \max\{j, k\}\} = \max\{\max\{i, j\}, k\} = \max\{i, j, k\}.
\]

We denote it by \( S_V \). This semigroup is commutative. So by (0.18) in [12], it is amenable. \( S_V \) is a unital semigroup and has a zero; indeed, \( e_s = x_0 \) and \( e_s = x_n \). Also, it is a regular semigroup and \( \text{Mul}(S_V) \cong S_V \) because \( S_V \) has an identity.

Evidently, \( S_V \) is regular since each \( s \in S_V \) is idempotent. The semigroup algebra \( \ell^1(S_V) \) is a unital algebra because \( S_V \) has an identity. So by theorem 4.3 (ii) \( \ell^1(S_V) \) is amenable.

Now, take \( T = L_{x_k} \) for a fixed \( x_k \in S \) where \( k \geq 1 \). By theorem 2.2, \( (S_V)_T \) is commutative so is amenable. We show that \( T \) is neither injective and nor surjective.

Take \( x_i \in S_V \), then \( Tx_i = x_k x_i = x_{\max(k, i)} \). So

\[
T(S_V) = \{x_k, x_{k+1}, \ldots, x_n\} \neq S_V. \]

Hence, \( T \) is not surjective.

Again, take distinct elements \( x_i, x_j \) in \( S_V \) for some \( i, j < k \) such that \( T(x_i) = T(x_j) \). Then we have \( x_{\max(k, i)} = x_{\max(k, j)} \) but \( x_i \neq x_j \). So \( T \) is not injective.

We prove that \( (S_V)_T \) is not regular. If \( (S_V)_T \) is regular, then for \( x_{k-1} \in S_V \) there exists an element \( x_j \in S_V \) such that

\[
x_{k-1} = x_{k-1} o x_j o x_{k-1} = x_{\max(k, j)}.
\]
That implies that $\max\{k, j\} = k - 1$; which is impossible. Consequently, by theorem 4.2 (ii) or 4.3 (ii), $\ell^1((S_\upsilon)_T)$ is not amenable.

Also, the inequality $S_\upsilon \circ S_\upsilon = \{x_k, x_{k+1}, ..., x_n\} \neq S_\upsilon$ shows that $\ell^1((S_\upsilon)_T)$ is not weakly amenable. In the next example we show that in the theorem 3.2 (iii) the condition "injectivity of " cannot be omitted.

**4.5** There are a semigroup $S$ and $T \in \text{Mul}_1(S)$ such that $T \in \text{Mul}_1(S)$ is not injective and the corresponding $\tilde{T} \in \text{Mul}_1\left(\ell^1\left( S_T \right) \right)$ is not an isometry.

Suppose that $S_\upsilon$ is a semigroup as in example 4.4 and $T = L_{x_k}$ for some fixed $1 < k < n$. If $f \in \ell^1(S_\upsilon)$ then $f = \sum_{i=0}^{n} f(x_i) \delta_{x_i}$ and also $T(f) = \sum_{i=0}^{n} f(x_i) \delta_{T(x_i)}$. But

$$T(x_i) = \begin{cases} x_i & k < i \leq n \\ x_k & 0 \leq i \leq k \end{cases}$$

so

$$T(f) = \left( \sum_{i=0}^{k} f(x_i) \right) \delta_{x_k} + \sum_{i=k+1}^{n} f(x_i) \delta_{T(x_i)}.$$

Hence

$$\|\tilde{T}(f)\| = \left| \sum_{i=0}^{k} f(x_i) \right| + \sum_{i=k+1}^{n} |f(x_i)|$$

$$\leq \sum_{i=0}^{k} |f(x_i)| + \sum_{i=k+1}^{n} |f(x_i)| = \|f\|_1,$$

It shows that $\tilde{T}$ is not an isometry.

**4.6**. There are semigroups $S$ and $T \in \text{Mul}_1(S)$ such that $\ell^1(S)$ is semisimple. But $\ell^1(S_T)$ is not semisimple. This example remind that, in theorem 3.1 the multiplier $T$ must be injective.

Let $S$ be a set $\{x_0, x_1, ..., x_n\}$ where $n \in \mathbb{N}$ and $n \geq 3$ is fixed. by operation given by $xy = x_{\min\{i, j\}}$, $S$ is a commutative semigroup. Since

$$\min \{i, \min\{j, k\}\} = \min \{\min\{i, j\}, k\} = \min\{i, j, k\} \quad (i, j, k \in \mathbb{N}).$$

We denote it briefly by $S_\Lambda$. For each $x, y \in S$ the equality $x^2 = y^2 = xy$ implies $x = y$. So by Theorem 5.8 [8] $\ell^1(S_\Lambda)$ is semisimple.

Now, let $T = L_{x_k}$ for a fixed $1 \leq k < n - 1$. It is easy to see that $T(x_k) = T(x_n)$ but $x_k \neq x_n$. So the multiplier $T$ is not injective.

We show that neither $S_\Lambda$ nor $\ell^1(S_\Lambda)_T$ is semisimple.

Each ideal of $S$ is of the form

$$I_m = \{x_0, x_1, ..., x_m\} \quad (m \leq n).$$

We claim that $S_T$ is not semisimple. Since for each $m \in \mathbb{N}$ we have
On the other hand, for each $x_i, x_j \in S$ where $i \neq j$ and $i, j > k$, we have $x_i \cdot x_i = x_j \cdot x_j = x_i \cdot x_j = x_k$, while $x_i \neq x_j$. Thus, Theorem 5.8 [8] shows that $\ell^1(S_n)_T$ is not semisimple.

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