STRUCTURAL RESULTS ON CONVEXITY RELATIVE TO COST FUNCTIONS

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Abstract. Mass transportation problems appear in various areas of mathematics, their solutions involving cost convex potentials. Fenchel duality also represents an important concept for a wide variety of optimization problems, both from the theoretical and the computational viewpoints. We drew a parallel to the classical theory of convex functions by investigating the cost convexity and its connections with the usual convexity. We give a generalization of Jensen’s inequality for c-convex functions.

1. Introduction

Let I and J be two bounded intervals. Assume f is a real valued function defined on I such that there exists g a real valued function defined on J which satisfies

\[ f(x) = \sup_{y \in J} \{xy - g(y)\}. \]

The function f is called the Fenchel transform (conjugate) of g. It is known that \((1.1)\) characterizes convex functions (see [2]).

Throughout this paper the cost function \(c : I \times J \to \mathbb{R}\) is continuous (unless otherwise indicated); it represents the cost per unit mass for transporting material from \(x \in I\) to \(y \in J\).

A proper function \(f : I \to (-\infty, \infty]\) is said to be \(c\)-convex (see for instance [1], [10], [14]) if there exists \(g : J \to (-\infty, \infty]\) such that for all \(x \in I\) we have

\[ f(x) = \sup_{y \in J} \{c(x, y) - g(y)\}. \]

It adapts the notion of a convex function to the geometry of the cost function. Its \(c\)-transform (\(c\)-conjugate) is \(f^c\) defined by

\[ f^c(y) = \sup_{x \in I} \{c(x, y) - f(x)\}. \]

If for a fixed \(x_0\) the supremum is obtained at \(y_0\), then we say that \(c(x, y_0) - g(y_0)\) supports \(f\) (is tangent by below [1]) at \(x_0\). One has the double \(c\)-conjugate

\[ f^{cc}(x) = \sup_{y \in J} \inf_{z \in I} \{f(z) + c(x, y) - c(z, y)\} \]

for all \(x \in I\). This is the largest \(c\)-convex function majorized by \(f\), that is \(f^{cc} \leq f\) (see [9] pp. 125). We also recall that the condition \(f = f^{cc}\) is equivalent to the \(c\)-convexity of \(f\) (see [14] Proposition 5.8).

Replacing the supremum by the infimum one gets the definition of cost concavity.

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Before stating the results we establish the notation and recall some definitions from the literature (see [9]).

Given a function \( f : I \to \mathbb{R} \), we say that \( f \) admits a \( c \)--support curve at \( x_0 \in I \) if there exists \( y \in J \) such that

\[
    f(x) \geq f(x_0) + c(x, y) - c(x_0, y), \quad \text{for all } x \in I.
\]

The \( c \)--subdifferential \((c\--normal mapping [13]) of a real function \( f \) defined on an interval \( I \) is a multivalued function \( \partial_c f : I \to \mathcal{P}(J) \) given by

\[
    \partial_c f(x_0) = \{ y \in J : f(x) \geq f(x_0) + c(x, y) - c(x_0, y), \text{ for every } x \in I \}.
\]

The elements of \( \partial_c f(x) \) are called \( c \)--subgradients at \( x \).

We denote throughout the paper the effective domain of the \( c \)--subdifferential by

\[
    \text{dom} (\partial_c f) = \{ x_0 \in I : \partial_c f(x_0) \neq \emptyset \}.
\]

Every \( c \)--convex function admits a \( c \)--support curve at each interior point of its domain, that is \( f \) satisfies \( \text{dom} (\partial_c f) \supseteq \text{int}(I) \). Clearly, if the cost function is differentiable in its first variable at \( x_0 \), then we also have

\[
    \frac{\partial c}{\partial x} (x_0, \partial_c f(x_0)) \subseteq \partial f(x_0).
\]

The map \( x \to c(x, y) - f(x) \) is maximized at \( x_0 \), and so we have \( y \in \partial_c f(x_0) \) if and only if \( f^c(y) = c(x_0, y) - f(x_0) \). It follows that a \( c \)--convex function \( f \) can be represented as

\[
    f(x) = \sup_{y \in \partial_c f(x)} \{ c(x, y) - g(y) \}
\]

for every \( x \in \text{dom} (\partial_c f) \). Obviously then

\[
    f^{cc}(x) = \sup_{y \in \partial_c f(x)} \inf_{z \in I} \{ f(z) + c(x, y) - c(z, y) \}
\]

for all \( x \in I \).

Similar concepts were developed for \( c \)--concave functions in [12]. Some authors (see for instance [5], [13], Section 6) consider by definition that a function \( f \) is \( c \)--concave if \( \text{dom} (\partial_c f) = I \), that is if it admits \( c \)--support curve at any point of its domain. For this they assume the function \( f \) to be upper semicontinuous.

For the particular case \( c(x, y) = xy \) we get from (1.2) the usual convexity of \( f \). Obviously then we recover the definitions of the usual subdifferential \( \partial f \) and of the support lines for convex functions. For the usual convex functions we will use the well-known notation \( f^c = f^* \) and \( f^{cc} = f^{**} \).

The aim of this paper is to investigate the cost convexity and to establish some connections with the usual convexity. See also [6] for more results on this topic. Before stating the results, since much of our attention here will be devoted to Jensen’s inequality (see [8]), we recall for the reader’s convenience its classical statement, both the discrete and integral forms:

**J1)** Let \( x_i \in I, \ p_i > 0, \ i = 1, ..., n, \ \sum p_i = 1 \). Then

\[
    f \left( \sum p_i x_i \right) \leq \sum p_i f \left( x_i \right)
\]

holds for every convex function \( f : I \to \mathbb{R} \).
J2) Let \( h : [a, b] \to I \) be an integrable function. Then
\[
f \left( \frac{1}{b-a} \int_a^b h(x) \, dx \right) \leq \frac{1}{b-a} \int_a^b f(h(x)) \, dx
\]
holds for every convex function \( f : I \to \mathbb{R} \), provided \( f \circ h \) is integrable.

2. Main results

2.1. Jensen’s inequality for \( c \)--convex functions. In what follows \( i \)--affine (convex, concave) stands for "affine (convex, concave) in the \( i \)--th variable". We firstly state and prove the discrete and continuous forms of Jensen’s inequality for \( c \)--convex functions.

**Theorem 1** (the discrete form of Jensen’s inequality). Let \( c : I \times J \to \mathbb{R} \) be a cost function. Assume \( f : I \to \mathbb{R} \) is a \( c \)--convex function. Let \( n \geq 2 \), \( x_i \in I \), \( p_i > 0 \), \( i = 1, \ldots, n \), \( \sum p_i = 1 \). Let \( y \in \partial_c f \left( \sum p_i x_i \right) \). Then
\[
\sum p_i f(x_i) - f \left( \sum p_i x_i \right) \geq \sum p_i c(x_i, y) - c \left( \sum p_i x_i, y \right).
\]

**Proof.** We consider the \( c \)--support curve at \( \sum p_i x_i \) corresponding to the \( c \)--gradient \( y \). It holds
\[
f(x) \geq f \left( \sum p_i x_i \right) + c(x, y) - c \left( \sum p_i x_i, y \right),
\]
for all \( x \in I \). Particularly we can write
\[
f(x_i) \geq f \left( \sum p_i x_i \right) + c(x_i, y) - c \left( \sum p_i x_i, y \right),
\]
for \( i = 1, \ldots, n \). By multiplying both sides by \( p_i \) and summing over \( i \) we get the claimed result. \( \square \)

**Corollary 1.** Let \( c : I \times J \to \mathbb{R} \) be a cost function and \( f : [a, b] \to \mathbb{R} \) be \( c \)--convex. Then
\[
(2.1) \quad \frac{f(a) + f(b)}{2} - f \left( \frac{a + b}{2} \right) \geq \frac{c(a, y) + c(b, y)}{2} - c \left( \frac{a + b}{2}, y \right)
\]
for all \( y \in \partial_c f \left( \frac{a + b}{2} \right) \).

**Proof.** We apply Theorem 1 taking \( x_1 = a \), \( x_2 = b \), \( p_1 = p_2 = \frac{1}{2} \). Then \( \frac{a+b}{2} \in (a, b) \subseteq \text{dom} (\partial_c f) \). \( \square \)

For \( c(x, y) = xy \) we recapture the inequality
\[
f \left( \frac{a + b}{2} \right) \leq \frac{f(a) + f(b)}{2}.
\]

Another straightforward consequence of Theorem 1 reads as follows.

**Corollary 2.** Let \( c : I \times J \to \mathbb{R} \) be a cost function and \( f : [a, b] \to \mathbb{R} \) be \( c \)--convex. Let \( y \in \partial_c f \left( \frac{a + b}{2} \right) \) and \( g : [a, b] \to \mathbb{R} \), \( g(x) = c(x, y) - f(x) \). Then
\[
g \left( \frac{a + b}{2} \right) \geq \frac{g(a) + g(b)}{2}.
\]

**Proof.** Directly from (2.1). \( \square \)

Under \( c \)--convexity conditions, the integral Jensen’s inequality is given by the following theorem.
Theorem 2 (the integral form of Jensen’s inequality). Let $c : I \times J \to \mathbb{R}$ be a cost function and $f : [a, b] \to \mathbb{R}$ be continuous and $c-$convex. Then

$$
(2.2) \quad f \left( \frac{a + b}{2} \right) (b - a) + \int_a^b \left[ c(x, y) - c \left( \frac{a + b}{2}, y \right) \right] \, dx \leq \int_a^b f(x) \, dx
$$

for all $y \in \partial_c f \left( \frac{a + b}{2} \right)$.

Proof. Let $y \in \partial_c f \left( \frac{a + b}{2} \right)$. We consider the $c-$support curve at $\frac{a + b}{2}$ corresponding to the $c-$gradient $y$. It holds

$$
f(x) \geq f \left( \frac{a + b}{2} \right) + c(x, y) - c \left( \frac{a + b}{2}, y \right)
$$

for all $x \in I$. To complete the proof, it remains to integrate the inequality on $[a, b]$. □

One can use the same recipe in order to obtain the weighted form of integral Jensen’s inequality, replacing the Lebesgue measure by a Borel probabilistic measure $\mu$ on $[a, b]$ with the barycenter $b_\mu \in (a, b)$. Thus

$$
f(b_\mu) + \int_a^b \left[ c(x, y) - c(b_\mu, y) \right] \, d\mu(x) \leq \int_a^b f(x) \, d\mu(x),
$$

for all $y \in \partial_c f (b_\mu)$.

Remark 1. Obviously (2.2) can be written in a more general form using another point $\xi \in \text{dom}(\partial_c f)$ instead of $\frac{a + b}{2}$. Then

$$
(2.3) \quad f(\xi)(b - a) + \int_a^b \left[ c(x, y) - c(\xi, y) \right] \, dx \leq \int_a^b f(x) \, dx,
$$

where $y \in \partial_c f (\xi)$.

From (2.3), for the particular case $c(x, y) = xy$ we recapture a result due to C.P. Niculescu and L.E. Persson [7, p. 668]:

Corollary 3. Let $f : [a, b] \to \mathbb{R}$ be a continuous, convex function, $\xi \in (a, b)$. It holds

$$
f(\xi) + y \left( \frac{a + b}{2} - \xi \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx,
$$

where $y \in \partial f (\xi)$.

Corollary 4. All continuous functions $f : [a, b] \to \mathbb{R}$, which are convex relative to 1-affine costs, satisfy

$$
f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx.
$$

Proof. Since the function $c(x, y)$ is 1-affine,

$$
c \left( \frac{a + b}{2}, y \right) (b - a) = \int_a^b c(x, y) \, dx.
$$

We use (2.2). This completes the proof. □
The 1-affine cost functions can be expressed as $c(x, y) = a(y)x + b(y)$ with $a, b : J \to \mathbb{R}$. The cost function $c(x, y) = xy$ is obviously 1-affine and Corollary 4 applies, hence the known Jensen’s inequality for convex functions becomes a particular case of Theorem 2. In the light of Jensen’s inequality it appears that the convexity relative to 1-affine cost functions implies the usual convexity.

2.2. The $c$–convexity and the role of the $c$–subdifferential. We establish next some new connections between the usual convexity and the cost convexity. Due to its dependence on the cost function, the concept of cost subdifferential is providing conceptual clarity and plays a crucial role in what follows.

Every continuous $c$–convex function is the upper envelope of its $c$–support curves. More precisely:

**Proposition 1.** Let $c : I \times J \to \mathbb{R}$ be uniformly continuous and $f : I \to \mathbb{R}$ be continuous and $c$–convex. Assume $y$ is a selection of $\partial c f$, that is $y(t) \in \partial c f(t)$ for all $t \in \text{dom}(\partial c f)$. Then

$$f(x) = \sup_{t \in \text{int}(I)} \{f(t) + c(x, y(t)) - c(t, y(t))\}$$

for all $x \in I$.

**Proof.** The case of interior points is clear. Let $x$ be an endpoint, say the leftmost one. By the continuity at $x$, for each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for all $t$ with $|t - x| < \delta_\varepsilon$ we have $|f(t) - f(x)| < \frac{\varepsilon}{2}$ and $|c(t, y(t)) - c(x, y(t))| < \frac{\varepsilon}{2}$. This shows that $f(x) + \varepsilon > f(t) + c(x, y(t)) - c(t, y(t))$ for $t \in (x, x + \delta_\varepsilon)$. We also have

$$\lim_{t \to x^+} [c(x, y(t)) - c(t, y(t))] = 0$$

and the result follows. $\square$

In the context of usual convexity, Proposition 1 has the following known corollary:

**Corollary 5.** (Theorem 1.5.2). Let $f : I \to \mathbb{R}$ be continuous and convex. Assume $y$ is a selection of $\partial f$, that is $y(t) \in \partial f(t)$ for all $t \in I$. Then

$$f(x) = \sup_{t \in \text{int}(I)} \{f(t) + (x - t) y(t)\}$$

for all $x \in I$.

The following proposition lets us see the way the $c$–subdifferential and the subdifferential are connected.

**Proposition 2.** (relating $c$–subdifferentials to subdifferentials). Let $c : I \times J \to \mathbb{R}$ be a cost function and $f : I \to \mathbb{R}$. It holds

$$(x, y) \in \partial c f \Rightarrow \partial c_y(x) \subseteq \partial f(x),$$

where $c_y(x) = c(x, y)$. Moreover if $f$ is differentiable and $c$ is differentiable in its first variable, then $\frac{\partial c}{\partial x}(x, y) = f'(x)$.
Proof. For \((x, y) \in \partial_c f\), \(\alpha \in \partial_c g(x)\) we have
\[f(z) - f(x) \geq c(z, y) - c(x, y) \geq \alpha (z - x),\]
for all \(z \in I\). It leads to \(\alpha \in \partial f(x)\). Under the differentiability assumptions we also have \(\partial g(x) = \left\{ \frac{\partial g}{\partial x}(x, y) \right\}\) and \(\partial f(x) = \left\{ f'(x) \right\}\).

The proof is completed. \(\square\)

The counterpart of Proposition 2, for \(c\)-superdifferentials, can be read in \(3\) Lemma 3.1, Lemma C.7\], for the particular case \(c = h(x - y)\).

**Proposition 3.** Let \(c : I \times J \to \mathbb{R}\) be a cost function. For all \(x \in I\) and \(y \in J\) we have \((x, y) \in \partial_c c\).

*Proof.* The proof is an immediate consequence of the definition of the \(c\)-subdifferential. \(\square\)

**Proposition 4.** Suppose that \(c : I \times J \to \mathbb{R}\) is a cost function and \(f, g : I \to \mathbb{R}\) are \(c\)-convex. It holds
\[\partial_c f(x) \cap \partial_c g(x) \subseteq \partial_c ((1 - \lambda) f + \lambda g)(x)\]
for all \(\lambda \in [0, 1]\).

*Proof.* Assume \(\partial_c f(x) \cap \partial_c g(x) \neq \emptyset\). Let \(y \in \partial_c f(x) \cap \partial_c g(x)\). Then
\[f(y) \geq f(x) + c(z, y) - c(x, y),\]
\[g(y) \geq g(x) + c(z, y) - c(x, y),\]
for all \(z \in I\). Let \(\lambda \in [0, 1]\). We infer
\[((1 - \lambda) f + \lambda g)(y) \geq ((1 - \lambda) f + \lambda g)(x) + c(z, y) - c(x, y)\]
therefore \(y \in \partial_c ((1 - \lambda) f + \lambda g)(x)\). \(\square\)

Our next result can be seen as a counterpart in the framework of \(c\)-convexity, for \(3\) Lemma 4.1.

**Proposition 5.** Assume \(c : I \times J \to \mathbb{R}\) is a cost function and \(f, g : I \to \mathbb{R}\) are \(c\)-convex. Let \(X = \{ x : f(x) < g(x) \}\). If there exists \(u \in X\) and \(v \in I\) such that
\[
\partial_c g(u) \cap \partial_c f(v) \neq \emptyset
\]
then \(v \in X\).

*Proof.* Let \(y \in \partial_c g(u) \cap \partial_c f(v)\). One has
\[g(v) \geq g(u) + c(v, y) - c(u, y),\]
\[f(u) \geq f(v) + c(u, y) - c(v, y),\]
which implies \(g(v) \geq f(v) + [g(u) - f(u)] > f(v)\).

Hence \(v \in X\). \(\square\)

The remaining results of this subsection were obtained by imposing some additional conditions to the cost function in order to get a nicer shaped graph of the set-valued function \(\partial_c f\).

**Proposition 6.** Let \(f : I \to \mathbb{R}\) be convex relative to a \(2\)-affine cost function \(c\). Then, for all \(x \in I\), the set \(\partial_c f(x)\) is convex, possibly empty at the endpoints of \(I\).
Proof. Let \( y_1, y_2 \in \partial_c f(x) \). Then
\[
f(z) \geq f(x) + c(z, y_i) - c(x, y_i), \text{ for all } z \in I, \ i = 1, 2.
\]
By direct computation, we obtain
\[
f(z) \geq f(x) + (1 - \lambda) [c(z, y_1) - c(x, y_1)] + \lambda [c(z, y_2) - c(x, y_2)]
\]
\[
= f(x) + [c(z, (1 - \lambda) y_1 + \lambda y_2) - c(x, (1 - \lambda) y_1 + \lambda y_2)],
\]
that is \((1 - \lambda) y_1 + \lambda y_2 \in \partial_c f(x)\). \(\square\)

Remark 2. This result represents a counterpart (in the framework of \(c\)-convexity) of the assertion that for every convex function \(f\), the sets \(\partial f(x)\) are convex, possibly empty at the endpoints of the domain. It makes sense to us to denote the upper and lower bounds of \(\partial_c f(x)\) (if the set is nonempty and convex) by \(f^c(x), f^c_+(x)\) and call them lateral \(c\)-derivatives.

The set
\[
Y = \{ y \in J : \exists x_1 \neq x_2 \in I \text{ such that } y \in \partial_c f(x_1) \cap \partial_c f(x_2) \}
\]
has the Lebesgue measure zero (see [5, Lemma 3.1]) when \(f\) is lower semicontinuous. Combining this result with Proposition 6 we derive the following remark.

Remark 3. For a continuous (hence lower semicontinuous) and \(c\)-convex function \(f\), when dealing with \(2\)-affine costs, the intersections \(\partial_c f(x_1) \cap \partial_c f(x_2)\), \(x_1 \neq x_2 \in I\) can have at most one element. This agrees with the case of usual convex functions.

Proposition 7. Suppose that the cost function \(c\) is concave and \(2\)-affine. Let \(f : I \to \mathbb{R}\) be convex and \(c\)-convex. Then \(\partial_c f\) is a convex set-valued function, i.e. for \(x_1, x_2 \in \text{dom}(\partial_c f)\) it holds
\[
(1 - \lambda) \partial_c f(x_1) + \lambda \partial_c f(x_2) \subseteq \partial_c f((1 - \lambda) x_1 + \lambda x_2)
\]
for all \(\lambda \in [0, 1]\).

Proof. Let \(z \in (1 - \lambda) \partial_c f(x_1) + \lambda \partial_c f(x_2)\) for an arbitrary fixed \(\lambda \in [0, 1]\). Then we can write \(z = (1 - \lambda) a + \lambda b\), for some \(a \in \partial_c f(x_1), b \in \partial_c f(x_2)\).

Since
\[
f(x) \geq f(x_1) + c(x, a) - c(x_1, a),
\]
\[
f(x) \geq f(x_2) + c(x, b) - c(x_2, b),
\]
we get
\[
f(x) \geq (1 - \lambda) f(x_1) + \lambda f(x_2) + c(x, (1 - \lambda) a + \lambda b) - (1 - \lambda) c(x_1, a) - \lambda c(x_2, b)
\]
\[
\geq f((1 - \lambda) x_1 + \lambda x_2) + c(x, (1 - \lambda) a + \lambda b)
\]
\[
- c((1 - \lambda) x_1 + \lambda x_2, (1 - \lambda) a + \lambda b).
\]
Therefore
\[
f(x) \geq f((1 - \lambda) x_1 + \lambda x_2) + c(x, z) - c((1 - \lambda) x_1 + \lambda x_2, z),
\]
hence \(z \in \partial_c f((1 - \lambda) x_1 + \lambda x_2)\).

This completes the proof. \(\square\)

Our next result reads as follows.
Proposition 8. Let the cost function \( c : I \times J \to \mathbb{R} \) be 1-concave. Assume \( f : I \to \mathbb{R} \) is convex and \( c \)-convex. Then

\[
\partial_c f(x_1) \cap \partial_c f(x_2) \subset \partial_c f((1 - \lambda) x_1 + \lambda x_2)
\]

for all \( \lambda \in [0, 1] \) and \( x_1, x_2 \in \text{dom} (\partial_c f) \).

Proof. We focus on the case \( \partial_c f(x_1) \cap \partial_c f(x_2) \neq \emptyset \). Let \( z \in \partial_c f(x_1) \cap \partial_c f(x_2) \). Then

\[
\begin{align*}
f(x) & \geq f(x_1) + c(x, z) - c(x_1, z), \\
f(x) & \geq f(x_2) + c(x, z) - c(x_2, z),
\end{align*}
\]

for all \( x \in I \). Let \( \lambda \in [0, 1] \). Consequently

\[
f(x) \geq (1 - \lambda) f(x_1) + \lambda f(x_2) + c(x, z) - (1 - \lambda) c(x_1, z) - \lambda c(x_2, z)
\]

\[
\geq f((1 - \lambda) x_1 + \lambda x_2) + c(x, z) - c((1 - \lambda) x_1 + \lambda x_2),
\]

which helps us to deduce \( z \in \partial_c f((1 - \lambda) x_1 + \lambda x_2) \).

Thus the proof is completed. \( \square \)

Example 1. The cost function \( c(x, y) = -\log (1 - xy) \), which appears in the reflector antenna design problem (the far field case \([4, 15]\)) is 1-convex. Since the problem deals with \( c \)-concave functions, mutatis mutandis Proposition 8 applies.

If we apply Proposition 8 for a cost function which is 1-concave and 2-affine, we have via Remark 4.

Remark 4. For a continuous and \( c \)-convex function \( f \) the set \( \partial_c f((1 - \lambda) x_1 + \lambda x_2) \) has exactly one element for all \( \lambda \in (0, 1) \), \( x_1 \neq x_2 \in I \) such that \( \partial_c f(x_1) \cap \partial_c f(x_2) \neq \emptyset \). Particularly, this means when \( c(x, y) = xy \) that if there exist two points \( x < y \in I \) such that \( f'_+(x) = f'_-(y) \), then the function is affine on \([x, y]\).

Corollary 6. Let the cost function \( c : I \times J \to \mathbb{R} \) be 1-concave. Assume \( f \) is convex on \( I \). If there exist \( x_1 < x_2 \) such that \( \partial_c f(x_1) \cap \partial_c f(x_2) \neq \emptyset \), then \( [x_1, x_2] \subset \text{dom} (\partial_c f) \).

Proof. The inclusion \( (2.4) \) still holds and combined with our assumption yields

\[
\partial_c f((1 - \lambda) x_1 + \lambda x_2) \neq \emptyset
\]

for all \( \lambda \in [0, 1] \). \( \square \)

2.3. Local and global \( c \)-convexity. Let \( I \) be a bounded open interval and \( f : I \to \mathbb{R} \). We introduce the local \( c \)-subdifferential by

\[
\partial_c^I f(x_0) = \{ y \in J : \exists \varepsilon > 0 \text{ such that } f(x) \geq f(x_0) + c(x, y) - c(x_0, y) \text{ for } x \in U_{\varepsilon} \}.
\]

Here the set \( U_{\varepsilon} = \{ x : |x - x_0| < \varepsilon \} \). The function \( h^I_{x_0} : U_{\varepsilon} \to \mathbb{R}, \)

\[
h^I_{x_0}(x) = f(x_0) + c(x, y) - c(x_0, y)
\]

is called local \( c \)-support curve. Note that \( \partial f(x_0) \subseteq \partial_c^I f(x_0) \).

We call a proper function \( f : I \to (-\infty, \infty] \) locally \( c \)-convex at \( x_0 \) if there exists \( \varepsilon > 0 \) and \( g : J \to (-\infty, \infty] \) such that

\[
f(x) = \sup_{y \in \partial_c f(x)} \{ c(x, y) - g(y) \}
\]
for $x \in U_\varepsilon$. Then one has
\begin{equation*}
 f_i^c(y) = \sup_{x \in U_\varepsilon} \{ c(x, y) - f(x) \}
\end{equation*}
for all $y \in \partial f(x)$ and
\begin{equation*}
 f_i^{cc}(x) = \sup_{y \in \partial^2 f(x)} \inf_{z \in U_\varepsilon} \{ f(z) + c(x, y) - c(z, y) \}.
\end{equation*}
Obviously the condition $f = f_i^{cc}$ on $U_\varepsilon$ is equivalent to the local $c$-convexity of $f$ at $x_0$.

**Proposition 9.** Let $f : I \to \mathbb{R}$, $\alpha \in I$. The function $f$ admits a local $c$--support curve at $\alpha$ if and only if $f(\alpha) = f_i^{cc}(\alpha)$.

**Proof.** We assume that $f$ admits a local $c$--support curve at $\alpha$. Let $y \in \partial f(\alpha)$. Then there exists $\varepsilon > 0$ such that
\begin{equation*}
 f(z) \geq f(\alpha) + c(z, y) - c(\alpha, y) \text{ for all } z \in U_\varepsilon.
\end{equation*}
Thus, since
\begin{equation*}
 \inf_{z \in U_\varepsilon} \{ f(z) + c(\alpha, y) - c(z, y) \} = f(\alpha),
\end{equation*}
we have
\begin{equation*}
 f_i^{cc}(\alpha) = \sup_{y \in \partial f(\alpha)} \inf_{z \in U_\varepsilon} h(f(z) + c(\alpha, y) - c(z, y)) = f(\alpha).
\end{equation*}
Conversely, let $\varepsilon > 0$. The function $f_i^{cc}$ is $c$-convex on $U_\varepsilon$, hence it admits a $c$–support curve at $\alpha$, that is there exists $y \in \partial^2 f(\alpha)$ such that
\begin{equation*}
 f_i^{cc}(\alpha) + c(z, y) - c(\alpha, y) \text{ for all } z \in U_\varepsilon.
\end{equation*}
Also we know that $f_i^{cc} \leq f$ on $U_\varepsilon$, which yields
\begin{equation*}
 f(z) \geq f_i^{cc}(\alpha) + c(z, y) - c(\alpha, y)
 = f(\alpha) + c(z, y) - c(\alpha, y) \text{ for all } z \in U_\varepsilon.
\end{equation*}
Summarizing the above discussion, there exists $y \in \partial^2 f(\alpha)$ such that
\begin{equation*}
 f(z) \geq f(\alpha) + c(z, y) - c(\alpha, y) \text{ for all } z \in U_\varepsilon
\end{equation*}
and the claim follows. □

**Remark 5.** This agrees with the known fact that the function $f$ admits a supporting line at $\alpha$ if and only if $f(\alpha) = f^{**}(\alpha)$ (see [11]).

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