Uniqueness of Minimizers of Some Variational Problems Arising in Image Processing

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Abstract

We will study an open problem pertaining to the uniqueness of minimizers for a class of variational problems emanating from Meyer’s model for the decomposition of an image into a geometric part and a texture part. Mainly, we are interested in the uniqueness of minimizers of the problem:

\[
\inf \left\{ J(u) + J^*(\frac{u}{\mu}) : (u, v) \in L^2(\Omega) \times L^2(\Omega), f = u + v \right\}
\]

where the image \( f \) is a square integrable function on the domain \( \Omega \), the number \( \mu \) is a parameter, the functional \( J \) stands for the total variation and the functional \( J^* \) is its Legendre transform. We will consider Problem (1) as a special case of the problem:

\[
\inf \{ s(f - u) + s^*(u) : u \in \mathcal{X} \}
\]

where \( \mathcal{X} \) is a Hilbert space containing \( f \) and \( s \) is a continuous semi-norm on \( \mathcal{X} \). In finite dimensions, we will prove that Problem (2) reduces to a projection problem onto the polar of the unit ball associated to a given norm on an appropriate Euclidean space. We will also provide a characterization for the uniqueness of minimizers for a more general projection problem defined by using any norm and any nonempty, closed, bounded and convex set of an Euclidean space. Finally, we will provide numerical evidence in favor of the uniqueness of minimizers for the decomposition problem.

1 Introduction

An important problem in image processing consists of decomposing a given image \( f \in L^2(\Omega) \) into a sum of a regular geometric part \( u \) and an oscillating texture part \( v \). Most models in the literature use the total variation

\[
L^2(\Omega) \ni w \mapsto J(w) = \int_\Omega \| Dw \| = \sup \left\{ \int_\Omega w \text{div} \phi dx : \phi \in C^\infty_c(\Omega), \| \phi \|_{L^\infty(\Omega)} \leq 1 \right\}
\]

as well as its polar

\[
L^2(\Omega) \ni w \mapsto J^0(w) = \sup \left\{ \int_\Omega wgdx : g \in L^2(\Omega), J(g) \leq 1 \right\}.
\]

For instance, Rudin et al. proposed the following restoration model in [10]:

\[
\inf \left\{ \frac{1}{2\lambda} \| v \|_{L^2(\Omega)}^2 + \int_\Omega \| Du \| : (u, v) \in L^2(\Omega) \times L^2(\Omega), f = u + v \right\}.
\]
The study of the model in (3) led Meyer to propose in [8] the model:

$$\inf \{ J(u) + \alpha J^*(v) : (u, v) \in L^2(\Omega) \times L^2(\Omega), f = u + v \}. \quad (4)$$

In [11], we have a weighting parameter $\alpha$ and $J^*$ is the polar of $J$. Meyer’s model in turn is approximated in [2] and [11] by using the following functional depending on the parameters $\lambda$ and $\mu$:

$$F_{\lambda, \mu} : L^2(\Omega) \times L^2(\Omega) \ni (u, v) \mapsto J(u) + \frac{1}{2\lambda} \| f - u - v \|^2_{L^2(\Omega)} + \chi_\mu(v)$$

where the functional $\chi_\mu$ is defined over $L^2(\Omega)$ by

$$\chi_\mu(v) = \begin{cases} 0, & \text{if } J^*(v) \leq \mu \\ \infty, & \text{if } J^*(v) > \mu. \end{cases}$$

The approach taken by Aujol et al. in [2] consists of studying the problem:

$$\inf \{ F_{\lambda, \mu}(u, v) : (u, v) \in L^2(\Omega) \times L^2(\Omega) \}. \quad (5)$$

In this paper, we are interested in studying the asymptotic case of Problem (5), i.e., when $\lambda$ tends to zero. We are then concerned with the problem:

$$\inf \left\{ J(u) + J^* \left( \frac{v}{\mu} \right) : (u, v) \in L^2(\Omega) \times L^2(\Omega), f = u + v \right\} \quad (6)$$

with $J^*$ being the Legendre transform of $J$. Determining the uniqueness of minimizers in Problem (6) is open and also in its corresponding discrete version:

$$\inf \left\{ J_d(u) + J_d^* \left( \frac{v}{\mu} \right) : (u, v) \in X \times X, f = u + v \right\} \quad (7)$$

where $X$ is the set of $N \times N$ real matrices and $J_d$ is a discretization of $J$ that we will define shortly. First we will define a discrete gradient operator $\nabla : X \rightarrow X \times X; u \mapsto \nabla u$ defined by $(\nabla u)_{i,j} = ((\nabla u)_{1,i,j}, (\nabla u)_{2,i,j})$ for $i, j \in \{1, \ldots, N\}$ and where

$$(\nabla u)_{1,i,j} = \begin{cases} u_{i+1,j} - u_{i,j}, & \text{if } 1 \leq i < N \\ 0, & \text{if } i = N \end{cases}$$

and

$$(\nabla u)_{2,i,j} = \begin{cases} u_{i,j+1} - u_{i,j}, & \text{if } 1 \leq j < N \\ 0, & \text{if } j = N. \end{cases}$$

Next, for $u \in X$, its discrete total variation is defined by $J_d(u) = \sum_{i,j=1}^N |(\nabla u)_{i,j}|$. We refer the reader to [11] for more details on the decomposition problem. While the existence of a minimizer in Problems (6) and (7) follows directly from standard arguments in the calculus of variations, proving the uniqueness is a challenge that we will address in this paper.

Note that as far as the uniqueness of minimizers is concerned, the study of Problems (6) and (7) may be reduced, respectively, to the study of the following problems:

$$\inf \left\{ J(u) + J^* \left( v \right) : (u, v) \in L^2(\Omega) \times L^2(\Omega), f = u + v \right\}, \quad (8)$$

$$\inf \left\{ J_d(u) + J_d^* \left( v \right) : (u, v) \in X \times X, f = u + v \right\} \quad (9)$$

and this could in turn bring valuable tools in the calculus of variations that may be helpful in determining the uniqueness of minimizers in problems involving the total variation (see, for instance, [3]). We will first consider Problem (3) as a special case of the more general problem:

$$\inf \{ s(f - x) + s^*(x) : x \in X \} \quad (10)$$
where $X$ is a Hilbert space containing $f$, the map $s$ is a semi-norm on $X$ and for all $x \in X$, $s^*(x) = \sup \{\langle x, y \rangle - s(y) : y \in X \}$. Let $Y$ be the orthogonal complement of the subspace $\{ x \in X : s(x) = 0 \}$, let $\rho$ be the norm obtained by restricting $s$ to $Y$. We will show in Lemma 3.2 that for every $f \in X$, we can find some $f_0 \in Y$ such that Problem (1) is equivalent to the problem:

$$ \inf \{ \rho(f_0 - x) + \rho^*(x) : x \in Y \} $$

with $\rho^*(x) = \sup \{ \langle x, y \rangle - s(y) : y \in Y \}$. Note that in the definition of $\rho^*$, the supremum is taken over $Y$ while for $s^*$, the supremum is taken over $X$. We can then proceed to study the problem:

$$ \inf \{ \rho(f_0 - x) + \rho^*(x) : x \in Y \} $$

(11)

where $Y$ is a Hilbert space containing $f$ and $\rho$ is a norm on $Y$. We may see Problem (11) as

$$ \inf \{ \rho(f - x) : x \in D \} $$

(12)

where $D = \{ x \in Y : \langle x, y \rangle \leq 1 \text{ whenever } y \in Y \text{ and } \rho(y) \leq 1 \}$ is the polar of the unit ball $\overline{B}_\rho(0, 1) = \{ x \in Y : \rho(x) \leq 1 \}$. We note that Problem (12) is a projection problem onto the polar of the unit ball $\overline{B}_\rho(0, 1)$.

Proving the uniqueness of minimizers in Problem (12) is far from being trivial. Obviously, if the unit ball of $\rho$ is strictly convex, then the minimizer is unique but this condition is not necessary to guarantee uniqueness. In fact if we consider $Y = \mathbb{R}^N$ and we let $\rho$ be the $l^1$-norm defined for $x \in \mathbb{R}^N$ by $||x||_1 = \sum_{i=1}^{N} |x_i|$, then $D = \{ x \in \mathbb{R}^N : ||x||_\infty = \sup \{|x_i| : i = 1, \ldots, N \} \leq 1 \}$. In this case, Problem (12) admits a unique solution despite the fact that the unit ball of the $l^1$-norm is not strictly convex (see Lemma 4.4).

We will also show that Problem (12) may have several minimizers. This is the case, for instance, when $\rho = \mathbb{R}^2$ and the norm is defined by $\rho(x, y) = |x| + \max(|x|, |y|)$ (see Proposition 4.4). We will reveal a strong connection between the uniqueness of minimizers in Problem (12) and the existence in the unit ball of $\rho$ of an edge that is orthogonal to a vertex. We will show in Proposition 4.5 that the existence in the unit ball of $\rho$ of an edge that is orthogonal to a vertex is a sufficient condition for nonuniqueness. In dimension two, this condition turns out to be also necessary (see Theorem 1.8). We conjecture that this result can be extended to any dimension greater than two.

Going back to Problem (8), the subspace $\{ u \in L^2(\Omega) : J(u) = 0 \}$ is the set of all constant functions and its orthogonal complement is the set $G = \{ u \in L^2(\Omega) : \int_\Omega u \, dx = 0 \}$. Call $T$ the norm obtained by restricting $J$ to $G$. To finish, we will provide numerical evidence that supports our conjecture that the minimizer in problem (8) is unique. The numerical simulations consist of taking a point $f$, choosing several random starting points and then using a projected subgradient algorithm to see if the iterates generated by the algorithm converge to the same solution or not.

2 Definitions and Notations

Before proceeding further, we will recall some definitions and also fix some notations:

**Definition 2.1** Let $\Omega \subset \mathbb{R}^N$ be an open set. Suppose $u \in L^1_{\text{loc}}(\Omega)$. The total variation of $u$ is given by

$$ \int_\Omega ||Du|| = \sup \left\{ \int_\Omega u \text{ div } \phi \, dx : \phi \in C^\infty_c(\Omega), \ |\phi|_{L^\infty(\Omega)} \leq 1 \right\} $$

and we will set $\|u\|_{BV(\Omega)} = \int_\Omega ||Du|| + \|u\|_{L^1(\Omega)}$.

We refer the reader to [7] for more details on the space $BV(\Omega)$. Let us point out that if $u : \mathbb{R} \to \mathbb{R}$ is a piecewise constant function which has finitely many jumps at $x_1 < x_2 < \cdots < x_k$, then

$$ \|u\|_{BV(\Omega)} = \sum_{i=1}^{k} |u(x_i^+) - u(x_i^-)| + \|u\|_{L^1(\Omega)} $$

with $u(x_i^-) = \lim_{x \to x_i^-} u(x)$ and $u(x_i^+) = \lim_{x \to x_i^+} u(x)$.

**Definitions and Notations from Convex Analysis**
1. In what follows, the set $E$ stands for a Euclidean space.

2. Let $\rho$ be a norm on $E$, then the associated open unit ball will be denoted by $B_{\rho}(0,1)$.

3. Let $\Omega \subset E$ be convex and symmetric with respect to the origin (i.e., if $x \in \Omega$, then $-x \in \Omega$). Suppose also that $\Omega$ has a nonempty interior. To $\Omega$ we will associate the functional

$$
\rho_\Omega(x) = \inf\{t > 0 : t^{-1}x \in \Omega\}.
$$

The functional $\rho_\Omega$ is called the Minkowski functional or the gauge of $\Omega$. The functional $\rho_\Omega$ is a norm and its unit ball is

$$
B_{\rho_\Omega}(0,1) = \Omega = \{x \in \Omega : \rho_\Omega(x) < 1\}.
$$

4. To a norm $\rho$ on $E$, we will associate the dual norm $\rho^\circ : E \to [0,\infty)$ defined by

$$
\rho^\circ(y) = \max\{x \cdot y : x \in E, \rho(x) \leq 1\} = \inf\{\lambda > 0 : x \cdot y \leq \lambda \rho(x) \text{ for all } x \in E\}.
$$

Clearly for all $x, y \in E$, we have $x \cdot y \leq \rho(x)\rho^\circ(y)$ (see [5] for further properties of the gauge and its polar).

5. The polar of a convex set $\Omega \subset E$ is defined by

$$
\Omega^\circ = \{y : x \cdot y \leq 1 \text{ for all } x \in \Omega\}.
$$

6. A point $x \in \Omega \subset E$ is said to be an extreme point of $\Omega$ if $2x = x_1 + x_2$ for $x_1, x_2 \in \Omega$ implies that $x = x_1 = x_2$. We say that $\Omega \subset E$ is strictly convex when it is convex and every point on the boundary is an extreme point of $\Omega$.

7. Let $T : E \to \bar{\mathbb{R}}$ be a function. The Legendre transform of $T$ is $T^* : E \to \bar{\mathbb{R}}$ defined by

$$
T^*(y) = \sup\{x \cdot y - T(x) : x \in E\}.
$$

8. The characteristic function of the set $\Omega \subset E$ is defined by

$$
\chi_\Omega(x) = \begin{cases} 
0, & \text{if } x \in \Omega \\
\infty, & \text{if } x \notin \Omega.
\end{cases}
$$

**Remark 2.2** We have $\rho^* = \chi_{B_{\rho^\circ}(0,1)}$, i.e., for all $y \in E$

$$
\rho^*(y) = \begin{cases} 
0, & \text{if } \rho^\circ(x) \leq 1 \\
\infty, & \text{if } \rho^\circ(x) > 1.
\end{cases}
$$

The results in this subsection can be found, for instance, in [4], [5], [6] and [9]. We will finish this subsection with the following definition.

**Definition 2.3** A subset $D$ of a normed vector space $X$ is said to be a proximinal (respectively, a Chebyshev) set with respect to the norm $\rho$ if for every $x_0 \in X$ the problem:

$$
\inf\{\rho(x - x_0) : x \in D\}
$$

admits a solution (respectively, a unique solution).

**Image Modelization:**

1. We will denote by $X$, the space $\mathbb{R}^{N \times N}$ of all $N \times N$ real matrices. We will endow $X$ with the scalar product $\langle u, v \rangle_X = \sum_{i,j=1}^{N} u_{i,j} v_{i,j}$ and the norm $\|u\|_X = \sqrt{\langle u, u \rangle_X}$. 

2. The space $Y$ is defined to be $X \times X$. For $g = (g^1, g^2) \in Y$, we will define

$$
\|g\|_{\infty} = \max \left\{ \sqrt{(g^1_{i,j})^2 + (g^2_{i,j})^2} : i, j = 1, \ldots, N \right\}.
$$

3. The discrete gradient operator $\nabla : X \to Y; \ u \mapsto \nabla u$ is defined for $i, j = 1, \ldots, N$ by $(\nabla u)_{i,j} = ((\nabla u)^1_{i,j}, (\nabla u)^2_{i,j})$ where

$$(\nabla u)^1_{i,j} = \begin{cases}
    u_{i+1,j} - u_{i,j}, & \text{if } i < N \\
    0, & \text{if } i = N
\end{cases}$$

and

$$(\nabla u)^2_{i,j} = \begin{cases}
    u_{i,j+1} - u_{i,j}, & \text{if } j < N \\
    0, & \text{if } j = N.
\end{cases}$$

4. For $u \in X$, its total variation is defined by $J_d(u) = \sum_{i,j=1}^N \| (\nabla u)_{i,j} \|$.

5. The discrete divergence operator $\text{div} : Y \to X$ is defined by the relation

$$
\langle -\text{div}(p), u \rangle_X = \langle p, \nabla u \rangle_Y \quad \text{for all } u \in X, \ p \in Y.
$$

Note that:

$$(\text{div } p)_{i,j} = (\text{div } p)^1_{i,j} + (\text{div } p)^2_{i,j}$$

where

$$(\text{div } p)^1_{i,j} = \begin{cases}
    p^1_{i,j}, & \text{if } i = 1 \\
    p^1_{i,j} - p^1_{i-1,j}, & \text{if } 1 < i < N \\
    -p^1_{i-1,j}, & \text{if } i = N
\end{cases}$$

and

$$(\text{div } p)^2_{i,j} = \begin{cases}
    p^2_{i,1}, & \text{if } j = 1 \\
    p^2_{i,j} - p^2_{i,j-1}, & \text{if } 1 < j < N \\
    -p^2_{i,j-1}, & \text{if } j = N.
\end{cases}$$

3 Minimization Problems Involving a Semi-Norm and its Legendre Transform

In this section, we will consider two minimization problems, one involving a semi-norm and the other a norm. We will show that these problems are equivalent or in other words, they have the same set of minimizers.

**Lemma 3.1** Let $X$ be a Hilbert space with the scalar product $\phi : X \times X \to \mathbb{R}$. Let $\rho : X \to [0, \infty)$ be a continuous semi-norm. Define the set $G$ by

$$
G = \{ x \in X : \phi(x, y) = 0 \text{ whenever } y \in X \text{ and } \rho(y) = 0 \}.
$$

Call $\rho_G$ the norm obtained by restricting $\rho$ to $G$ and let $\rho_G^*$ be its Legendre transform, i.e., for all $y \in G$ we have

$$
\rho_G^*(y) = \sup \{ \phi(w, y) - \rho_G(w) : w \in G \}.
$$

Let $K_\rho = \{ x \in X : \rho(x) = 0 \}$, then the following claims hold:

1. For all $(x, y) \in X \times K_\rho$, we have $\rho(x + y) = \rho(x)$.
2. We have $\rho^*(x) = \infty$ whenever $x \notin G$.
3. If $x \in G$, then $\rho_G^*(x) = \rho^*(x)$. 

5
Proof. Firstly note that $G$ is the orthogonal complement of $K_{\rho}$:

$$G = K_{\rho}^\perp = \{ x \in \mathcal{X} : \phi(x, y) = 0 \text{ for all } y \in K_{\rho} \}.$$ 

For all $x \in \mathcal{X}$ we will denote by $\hat{x}$, the orthogonal projection of $x$ onto the subspace $K_{\rho}$. We then have

$$\rho(\hat{x}) = 0 \text{ for all } x \in \mathcal{X}. \quad (13)$$

1. Let $x \in \mathcal{X}$ and $y \in K_{\rho}$. Using the fact that $\rho$ is subadditive and relation $(13)$, we have

$$\rho(x + y) \leq \rho(x) + \rho(y) = \rho(x) \leq \rho(x + y) + \rho(-y) = \rho(x + y),$$

from which we deduce that:

$$\rho(x + y) = \rho(y) \text{ for all } x \in \mathcal{X}, y \in K_{\rho}. \quad (14)$$

2. Suppose $x \notin G$. It follows that $\hat{x} \neq 0$ and we now use the definition of $\rho^*$ and $(13)$ to obtain for all $t \in \mathbb{R}$:

$$\rho^*(x) \geq \phi(t\hat{x}, x) - \rho(t\hat{x}) = t\phi(\hat{x}, \hat{x}).$$

We let $t$ go to infinity to obtain $\rho^*(x) = \infty$. Whence,

$$\rho^*(x) = \infty \text{ whenever } x \notin G.$$

3. Suppose $x \in G$. We have

$$\rho^*(x) = \sup\{\phi(x, y) - \rho(y) : y \in \mathcal{X}\}$$

$$= \sup\{\phi(x, y - \hat{y}) - \rho(y) : y \in \mathcal{X}\} \quad \text{(as } x \in G\text{)}$$

$$= \sup\{\phi(x, y - \hat{y}) - \rho(y - \hat{y}) : y \in \mathcal{X}\} \quad \text{(by } (14)\text{)}$$

$$= \sup\{\phi(x, w) - \rho(w) : w \in G\}$$

$$= \rho^*_G(x).$$

□

Lemma 3.2 Let $\mathcal{X}$ be a Hilbert space with scalar product $\phi : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$. Let $\rho : \mathcal{X} \to [0, \infty)$ be a continuous semi-norm. Define the set $G$ by

$$G = \{ x \in \mathcal{X} : \phi(x, y) = 0 \text{ whenever } y \in \mathcal{X} \text{ and } \rho(y) = 0 \}.$$ 

Call $\rho_G$ the norm obtained by restricting $\rho$ to $G$ and let $\rho^*_G$ be its Legendre transform, i.e., for all $y \in G$ we have

$$\rho^*_G(y) = \sup\{\phi(w, y) - \rho_G(w) : w \in G\}.$$ 

For every $f \in \mathcal{X}$, there exists some $f_0 \in G$ such that the following problems are equivalent (i.e., they have the same set of minimizers):

$$\inf\{\rho(f - x) + \rho^*(x) : x \in \mathcal{X}\}, \quad (15)$$

$$\inf\{\rho_G(f_0 - x) + \rho^*_G(x) : x \in G\}. \quad (16)$$

Proof. We will use the second claim in Lemma 3.1 to deduce that Problem $(15)$ is equivalent to the problem:

$$\inf\{\rho(f - x) + \rho^*(x) : x \in G\}. \quad (17)$$

Let $K_{\rho} = \{ x \in \mathcal{X} : \rho(x) = 0 \}$ and call $\hat{f}$ the orthogonal projection of $f$ on $K_{\rho}$. We will choose $f_0 = f - \hat{f}$. Using the first claim in Lemma 3.1 it follows that Problem $(17)$ is equivalent to the problem:

$$\inf\{\rho(f_0 - x) + \rho^*(x) : x \in G\}. \quad (18)$$

Finally, we will use the fact that for all $x \in G$ we have $f_0 - x \in G$ and the third claim in Lemma 3.1 to deduce that Problem $(18)$ is equivalent to Problem $(16)$. We have now established that Problem $(15)$ is equivalent to Problem $(16)$.
Remark 3.3 Let us make the following observations:

1. We do not have uniqueness of \( f_0 \) in Lemma 3.2. An example of this is the case when \( X = G = \mathbb{R}^2 \) and \( \rho \) is the Euclidean norm. If \( f = (1,0) \), then any \( f_0 \) of the form \( f_0 = (t,0) \) with \( t \geq 1 \) will satisfy the conclusion of Lemma 3.2.

2. It is apparent from the proof of Lemma 3.3 that \( f_0 \) may be taken, for instance, to be the orthogonal projection of \( f \) onto \( G \).

3.1 The Case of the Total Variation

We will study the problem:

\[
\inf \{ J(f - v) + J^*(v) : v \in L^2(\Omega) \}.
\]

Let

\[
G = \left\{ v \in L^2(\Omega) : \int_\Omega ud\lambda = 0 \right\} \quad \text{and} \quad G_d = \left\{ u \in X : \sum_{i,j=1}^N u_{i,j} = 0 \right\}.
\]

Call \( T : G \to [0,\infty) \) the restriction of \( J \) to \( G \) and \( T_d : G_d \to [0,\infty) \) the restriction of \( J_d \) to \( G_d \). The next Corollary follows from Lemma 3.2.

Corollary 3.4 For every \( f \in L^2(\Omega) \), there exists some \( f_0 \in G \) such that the following problems are equivalent (i.e., they have the same set of minimizers):

\[
\inf \{ J(f - x) + J^*(x) : x \in L^2(\Omega) \},
\]

\[
\inf \{ T(f_0 - x) + T^*(x) : x \in G \}.
\]

Remark 3.5 A result analogous to Corollary 3.4 also holds for the discrete version, i.e., \( L^2(\Omega) \), \( G \) and \( J \) replaced with \( X \), \( G_d \) and \( J_d \), respectively.

Proposition 3.6 Neither of the unit balls of the norms \( T \) and \( T_d \) is strictly convex.

Proof. For \((a,b) \in \mathbb{R}^2\) define the matrix \( \bar{M}(a,b) \) by \( [\bar{M}(a,b)]_{i,j} = a \) for \( j = 1,\ldots,N \) and \( [\bar{M}(a,b)]_{i,j} = b \) for \( i = 2,\ldots,N \) and \( j = 1,\ldots,N \). Let \( \bar{M}(a,b) \) be the constant matrix with coefficient \( Na + N(N-1)b \) and \( M(a,b) = \bar{M}(a,b) - \bar{M}(a,b) \). It follows that \( M(a,b) \in G_d \) and \( T_d(M(a,b)) = N|b-a| \). One can now verify that the following equations hold:

\[
2M\left(\frac{1}{2}, -\frac{1}{2}\right) = M(1,0) + M(0,-1),
\]

\[
2T_d\left(\frac{1}{2}, -\frac{1}{2}\right) = T_d(1,0) + T_d(0,-1).
\]

Hence the unit ball of \( T_d \) is not strictly convex. A similar proof also holds for the other case.

\[
\square
\]

Lemma 3.7 Suppose that:

1. The matrix \( f_0 = f - \hat{f} \) where \( \hat{f} \) is the \( N \times N \) constant matrix with coefficient \( \sum_{i,j=1}^N f_{ij} \).

2. The set \( D_d = \{ y \in G_d : \rho_T(y) \leq 1 \} = \hat{B}_{T_d}(0,1) \).

Then Problem (19) is equivalent to the problem:

\[
\inf \{ T_d(f_0 - v) : v \in D_d \}.
\]

Proof. As \( T_d \) is a norm on \( G_d \), we have \( T_d = \chi_{\hat{B}_{T_d}(0,1)} \). We will next use Corollary 3.4 to deduce Lemma 3.7.

\[
\square
\]

Remark 3.8 Problem (19) is a projection problem with respect to the norm \( T_d \) onto the set \( D_d \) which is the dual of the unit ball associated to \( T_d \).
The Projection Problem onto the Dual Unit Ball

Let \( \rho \) be a norm on \( E \) and \( x_0 \in E \). We will study the problem:

\[
\inf \{ \rho(x_0 - x) + \rho^*(x) : x \in E \}
\]

where the map \( \rho^* \) is the Legendre transform of \( \rho \) and is defined by

\[
\rho^*(y) = \sup \{ x \cdot y - \rho(x) : x \in E \} \text{ for all } y \in E.
\]

Problem (20) is a projection problem with respect to the norm \( \rho \) onto the dual of the unit ball associated to \( \rho \). To see this, recall that for all \( x, y \in E \), one has \( x \cdot y \leq \rho(x)\rho^*(y) \) and it holds that \( \rho^* = \chi_{B_{\rho^*}(0,1)} \). Let \( D = \{ x \in E : \rho^*(x) = 0 \} \), then \( D = \{ x \in E : \rho^*(x) \leq 1 \} = \{ x : x \cdot y \leq \rho(y) \text{ for all } y \in \mathbb{R}^N \} \) and it follows that:

\[
\inf \{ \rho(x_0 - x) + \rho^*(x) : x \in E \} = \inf \{ \rho(x_0 - x) : x \in D \}.
\]

Furthermore, as \( D \subset E \) is closed and bounded and \( \rho \) is a norm, we read from (21) that Problem (20) admits a minimizer.

**Lemma 4.1** The following claims hold:

1. If \( x_0 \in D \), then Problem (20) admits a unique solution which is \( x_0 \).
2. If \( x_0 \notin D \), then any minimizer of Problem (20) lies on the boundary of \( D \).
3. If \( x_0 \notin D \) and \( D \) is strictly convex (i.e., every point on the boundary of \( D \) is an extreme point of \( D \)), then Problem (20) admits a unique minimizer.

Proof. We will prove the claims of the lemma one by one:

1. Suppose \( x_0 \in D \). As \( \rho(x_0 - x) \geq 0 = \rho(x_0 - x_0) \) for all \( x \in E \), it holds that \( x_0 \) is a minimizer of Problem (20). Let \( x_1 \) be another minimizer of Problem (20). Then \( \rho(x_0 - x_1) = 0 \), this implies that \( x_0 = x_1 \). Hence the minimizer is unique.

2. Suppose \( x_0 \notin D \) and suppose also that a minimizer \( x_1 \in D \) lies in the interior of \( D \). We may find \( \epsilon \in (0,1) \) such that the ball \( B_\rho(x_1, \epsilon) \subset D \). Since \( x_0 \notin D \) and \( x_1 \in D \), we have \( \rho(x_0 - x_1) > 0 \). Let \( t \in (0,1) \) be such that \( 0 < t \rho(x_0 - x_1) < \epsilon \) and set \( x_t = x_1 + t(x_0 - x_1) \). We have that \( x_t \in D \) and

\[
\rho(x_t - x_t) = \rho(x_0 - x_1 - t(x_0 - x_1)) = \rho((1-t)(x_0 - x_1)) = (1-t)\rho(x_0 - x_1).
\]

Thus

\[
(1-t)\rho(x_0 - x_1) \geq \rho(x_0 - x_1).
\]

The inequality (22) reads \( 1-t \geq 1 \), which is absurd as \( t > 0 \) and hence \( x_t \in \partial D \).

3. Let \( x_1, x_2 \) be two distinct minimizers of Problem (20). Then \( x_1, x_2 \) lie on the boundary of \( D \) and are extreme points. It holds that \( \frac{x_1 + x_2}{2} \) is also a minimizer and must be an extreme point. Hence \( x_1 = x_2 \) and the minimizer is unique.

\[ \Box \]

### 4.1 A Case where the Minimizer is Unique: The \( l^p(\mathbb{R}^N) \)-Norms

If \( p \in (1, \infty) \), then the unit ball of the \( l^p(\mathbb{R}^N) \)-norm is strictly convex, hence any nonempty, closed, bounded and convex set is a Chebyshev set. In particular, the closed dual unit ball \( \bar{B}_p(0,1) \) is also a Chebyshev set. We will study next the case of the \( l^1(\mathbb{R}^N) \)-norm. Consider for this purpose the map \( \alpha : \mathbb{R} \to \mathbb{R} \) defined by

\[
\alpha(t) = \begin{cases} 
-1, & \text{if } t \leq -1 \\
t, & \text{if } -1 < t < 1 \\
1, & \text{if } t \geq 1.
\end{cases}
\]

Call \( P : \mathbb{R}^N \to \mathbb{R}^N \), the map defined for \( x \in \mathbb{R}^N \) by \( (P(x))_i = \alpha(x_i) \) for \( i = 1, \ldots, N \).
Lemma 4.2 Let $f \in \mathbb{R}^N$, then the unique solution of the problem:

$$\inf \{ \| f - u \|_1 : u \in \mathbb{R}^N, \| u \|_\infty \leq 1 \}$$

is given by $P(f)$.

Proof. Observe that for $t \in \mathbb{R}$, $\alpha(t)$ is the unique solution of the problem:

$$\inf \{ |t - s| : s \in \mathbb{R}, |s| \leq 1 \}.$$ 

For $f, u \in \mathbb{R}^N$ and $\|u\|_\infty \leq 1$, one has

$$\| f - u \|_1 = \sum_{i=1}^{N} |f_i - u_i| \geq \sum_{i=1}^{N} |f_i - \alpha_i|$$

with equality if and only if $u_i = \alpha(f_i)$ for all $i = 1, \ldots, N$. We deduce that the unique minimizer of Problem 23 is given by $P(f)$.

\[ \square \]

4.2 A Case where the Minimizer is not Unique: A Norm for which the Dual Unit Ball is not Chebyshev

Let $N = 2$ and consider the polygon $A \subset \mathbb{R}^N$ with vertices located at $(0, 1), (0.5, 0.5), (0.5, -0.5), (0, -1), (-0.5, 0.5)$ and $(-0.5, -0.5)$. The polygon $A$ is also characterized by the following inequalities:

$$x + y \leq 1, \quad 2x \leq 1, \quad x - y \leq 1,$$

$$-x + y \leq 1, \quad -2x \leq 1, \quad -x - y \leq 1.$$ 

It follows that the polar of $A$ is the polygon $A^\circ$ with vertices located at $(1, 1), (2, 0), (1, -1), (-1, -1), (-2, 0)$ and $(-1, 1)$ (see Figure 1). To $A$ we will associate the norm $\rho_A(x) = \inf \{ t : t^{-1}x \in A \}$ which is the Minkowski functional or gauge function associated to $A$. As a consequence

$$\rho^\circ(x) = \inf \{ t : t^{-1}x \in A^\circ \}.$$ 

Note that:

$$\rho(x, y) = |x| + \max(|x|, |y|) = \begin{cases} 2|x|, & \text{if } |x| > |y| \\ |x| + |y|, & \text{if } |x| \leq |y| \end{cases}$$

while

$$\rho^\circ(x, y) = \frac{1}{2}|y| + \frac{1}{2}\max(|x|, |y|) = \begin{cases} |y|, & \text{if } |y| > |x| \\ \frac{1}{2}(|x| + |y|), & \text{if } |y| \leq |x| \end{cases}.$$ 

Lemma 4.3 The dual unit ball $A^\circ$ of $\rho$ is not a Chebyshev set.

Proof. Consider $f = (2, 2)$, then the minimizers of the problem:

$$\inf \{ \rho(f - u) : u \in A^\circ \}$$

are points of the form $(1 + t, 1 - t)$ with $t \in [0, 1]$. Hence $A^\circ$ is not a Chebyshev set with respect to the norm $\rho$.

\[ \square \]

The following result is a consequence of Lemma 4.3.

Proposition 4.4 Let $\rho$ be the norm on $\mathbb{R}^2$ defined by $\rho(x, y) = |x| + \max(|x|, |y|)$. There exists some $f_0 \in \mathbb{R}^2$ such that the problem:

$$\inf \{ \rho(f_0 - x) + \rho^\circ(x) : x \in \mathbb{R}^2 \}$$

admits infinitely many solutions. In particular, the closed dual unit ball associated to $\rho$ is not a Chebyshev set.
Figure 1: A norm with a non-Chebyshev dual. The polygons with dotted, dashed and solid boundaries are the unit ball, dual unit ball and the ball of radius two centered at \( f = (2, 2) \), respectively.

4.3 Characterization for the Uniqueness of Minimizers

We will first provide a characterization for the uniqueness of minimizers for a projection problem defined by using any norm \( \rho \) on \( E \) and any nonempty, closed, bounded and convex set \( \Omega \) of \( E \). In order to state subsequent results, we will have the need to fix some additional notations:

1. For \( a \in E \setminus \{0\} \) and \( \alpha \in \mathbb{R} \), we will define the hyperplane:
   \[
   H(a, \alpha) = \{ x \in E : x \cdot a = \alpha \}
   \]
   and the closed halfspaces:
   \[
   H^+(a, \alpha) = \{ x \in E : x \cdot a \geq \alpha \},
   H^-(a, \alpha) = \{ x \in E : x \cdot a \leq \alpha \}.
   \]

2. The line segment between \( a, b \in E \) is denoted by
   \[
   [a, b] = \{ a + t(b - a) : t \in [0, 1] \}.
   \]

3. For a convex set \( \Omega \subset E \), we will denote by \( \text{extr}(\Omega) \) the set of its extreme points.

4. The unit sphere associated with the norm \( \rho \) on \( E \) will be denoted by \( \partial \bar{B}_\rho(0, 1) \).

**Proposition 4.5** Let \( \Omega \) be a nonempty, closed, bounded and convex subset of \( E \) and assume \( \rho \) is a norm on \( E \). Then the following statements are equivalent:

1. There exists some \( x_0 \in E \) for which the problem:
   \[
   \inf \{ \rho(x_0 - x) : x \in \Omega \}
   \]
   has more than one minimizer.
2. There exists some vector $a \in E$, distinct points $w_1, w_2 \in \Omega$, $u_1, u_2 \in \partial B_\rho(0,1)$ and $r \neq 0$ such that the following three conditions hold:

(a) We have $\Omega \subset H^+(a, w_1 \cdot a)$ and $[w_1, w_2] \subset H(a, w_1 \cdot a)$.
(b) We have $B_\rho(0,1) \subset H^-(a, u_1 \cdot a)$ and $[u_1, u_2] \subset H(a, u_1 \cdot a)$.
(c) We have $w_1 - w_2 = r(u_1 - u_2)$.

Proof. We will first prove that the first statement implies the second one. Suppose $w_1, w_2$ are two distinct minimizers of the problem:

$$\inf\{\rho(x_0 - x) : x \in \Omega\}$$

and let $r := \inf\{\rho(x_0 - x) : x \in \Omega\} > 0$. Then $[w_1, w_2]$ is a set of minimizers and $[w_1, w_2] \subset \partial \Omega$. Since $\Omega \cap B_\rho(x_0, r) = 0$, we can then use the theorem on the separation of convex sets to find $a \in E$ and $\alpha \in \mathbb{R}$ such that $\Omega \subset H^+(a, \alpha)$ and $B_\rho(x_0, r) \subset H^-(a, \alpha)$. Next, as $\partial B_\rho(x_0, r) \subset H^-(a, \alpha)$ and $[w_1, w_2] \subset \Omega \cap \partial B_\rho(x_0, r)$, we deduce that $[w_1, w_2] \subset H(a, \alpha)$ and $\alpha = w_1 \cdot a = w_2 \cdot a$. Furthermore, let us consider the affine map $T : E \to E$ defined for $x \in E$ by $T(x) = r^{-1}(x - x_0)$ and let $u_1 = T(w_1)$ and $u_2 = T(w_2)$. For $t \in [0,1]$, we have

$$(u_1 + t(u_2 - u_1)) \cdot a = (u_1 + tr^{-1}(w_1 - w_2)) \cdot a = u_1 \cdot a.$$ 

Thus $[u_1, u_2] \subset H(a, a \cdot u_1)$ and also $u_1, u_2 \subset B_\rho(0,1)$. Let $x \in B_\rho(0,1)$ and set $y = T^{-1}(x)$. We have $y \in B_\rho(x_0, r)$ and then $y \cdot a \leq w_1 \cdot a$. Since $y = rx + x_0$, we have

$$(rx + x_0) \cdot a \leq w_1 \cdot a$$

$$rx \cdot a \leq (w_1 - x_0) \cdot a$$

$$x \cdot a \leq r^{-1}(w_1 - x_0) \cdot a$$

$$= u_1 \cdot a.$$ 

Therefore, $B_\rho(0,1) \subset H^-(a, u_1 \cdot a)$. Finally, $u_1 - u_2 = r^{-1}(w_1 - w_2)$.

We will now prove that the second statement implies the first one. Let $x_0 = w_1 - ru_1$ and we will now show that $w_1$ and $w_2$ are two distinct minimizers of the problem:

$$\inf\{\rho(x_0 - x) : x \in \Omega\}.$$ 

Firstly note that $x_0 = w_2 - ru_2$ and for $i = 1, 2$, we have

$$\rho(w_i - x_0) = \rho(ru_i) = r \rho(u_i) = r$$

which implies $w_1, w_2 \in \partial B_\rho(x_0, r)$. Suppose $\rho(x - x_0) < r$, then $r^{-1}(x - x_0) \in B_\rho(0,1)$. As $B_\rho(0,1) \subset H^-(a, u_1 \cdot a)$, we have

$$r^{-1}(x - x_0) \cdot a \leq u_1 \cdot a$$

$$r^{-1}(x - w_1 + ru_1) \cdot a \leq u_1 \cdot a$$

$$r^{-1}(x - w_1) \cdot a \leq 0$$

$$x \cdot a \leq w_1 \cdot a.$$ 

Thus $B_\rho(x_0, r) \subset H^-(a, w_1 \cdot a)$ and as $w_1 \in \partial B_\rho(x_0, r)$, we deduce that $H(a, w_1 \cdot a)$ is a supporting hyperplane of $B_\rho(x_0, r)$. Thus $x \in H^+(a, u_1 \cdot a)$ is equivalent to $\rho(x - x_0) \geq r$. As $\Omega \subset H^+(a, w_1 \cdot a)$, we deduce that $w_1$ and $w_2$ are two distinct minimizers of the problem:

$$\inf\{\rho(x_0 - x) : x \in \Omega\}$$ 

since by assumption $w_1$ and $w_2$ are distinct. \hfill \square
Lemma 4.6 Let $\rho$ be a norm on the $N$-dimensional Euclidean space $E$ such that its closed unit ball is the convex hull of a finite number of points. If the problem:

$$\inf \{\rho(x_0 - x) + \rho^*(x) : x \in E\}$$

admits a unique minimizer for all $x_0 \in E$, then the following set:

$$W = \{(x_1, x_2, x_3) \in \text{extr}(B_\rho(0, 1)) \times \text{extr}(B_\rho(0, 1)) \times \text{extr}(B_\rho(0, 1)),$$

$$[x_2, x_3] \subset \partial B_\rho(0, 1), \ x_2 \neq x_3, \ x_1 \cdot (x_2 - x_3) = 0\}$$

is empty.

Proof. Assume $(x_1, x_2, x_3) \in W$ and let $D$ be the polar of $B_\rho(0, 1)$. As $x_1$ is an extreme point of $B_\rho(0, 1)$, there exists some $y \in \partial D$ such that $K = D \cap H(x_1, y \cdot x_1)$ is a $(N - 1)$-dimensional face of $D$. We may find $u_1, u_2 \in K$ and $r \neq 0$ such that $u_1 - u_2 = r(x_3 - x_2)$. We will use Proposition 4.3 to deduce that the problem:

$$\inf \{\rho(x_0 - x) : x \in D\}$$

admits more than one minimizer.

Remark 4.7 When the set $W$ in Lemma 4.6 is nonempty, it means that an edge of the closed unit ball associated to $\rho$ is orthogonal to one of its vertices.

Theorem 4.8 Let $\rho$ be a norm on the 2-dimensional Euclidean space $E$ such that its closed unit ball is the convex hull of a finite number of points. The problem:

$$\inf \{\rho(x_0 - x) + \rho^*(x) : x \in E\}$$

admits a unique solution for all $x_0 \in E$ if and only if the following set:

$$W = \{(x_1, x_2, x_3) \in \text{extr}(B_\rho(0, 1)) \times \text{extr}(B_\rho(0, 1)) \times \text{extr}(B_\rho(0, 1)),$$

$$[x_2, x_3] \subset \partial B_\rho(0, 1), \ x_2 \neq x_3, \ x_1 \cdot (x_2 - x_3) = 0\}$$

is empty.

Proof. If $W$ is nonempty, by Lemma 4.6 we have more than one minimizers. Suppose now that we have more than one minimizers and let $D$ be the polar of $B_\rho(0, 1)$. By Proposition 4.3 we can find distinct points $u_1, u_2 \in \partial B_\rho(0, 1)$ and $w_1, w_2 \in \partial D$ such that $[u_1, u_2] \subset \partial B_\rho(0, 1)$, $[w_1, w_2] \subset \partial D(0, 1)$ and $w_1 - w_2$ is parallel to $u_1 - u_2$. Using the characterization of the polar of a set and the fact that the underlying space is of dimension two, we can find an extreme point $x_1 \in B_\rho(0, 1)$ that is orthogonal to $w_1 - w_2$. We will use again the fact that the underlying space is of dimension two, to find two distinct extreme points $x_2, x_3 \in B_\rho(0, 1)$ such that $x_2 - x_3$ is parallel to $u_1 - u_2$ and $[x_1, x_2] \subset \partial B_\rho(0, 1)$. It holds that $x_1 \cdot (x_2 - x_3) = 0$ and hence $W$ is nonempty.

Remark 4.9 We conjecture that Theorem 4.8 can be generalized to dimensions greater than two.

5 Numerical Simulations

One shows that the the problem:

$$\min \{J_d(u_0 - u) + J_d^*(u) : u \in X\}$$

admits a unique minimizer for all $u$ in $X$ if and only if for all $g_0 \in Y$ the operator $\text{div}$ is constant on the set:

$$S(g_0) = \text{argmin} \{J(\text{div}(h - g_0)) : h \in Y, \ |h|_\infty \leq 1\}.$$

We start by picking $g_0$ outside the set $\{y \in Y : \|y\|_\infty \leq 1\}$ but close to it and then choosing randomly a number of points to initialize a projected subgradient algorithm. Next, we check if the iterates generated by the algorithm converge to the same solution or not. Table I shows the results for the C++ implementation of the algorithm with $N = 16$ and the choice of 500 initialization points and 200000 iterations of the algorithm.
Table 1: Diameters of solution sets obtained using the distance induced by the norm $\| \cdot \|_X$.

| Experiment Number | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
|-------------------|------|------|------|------|------|------|------|------|------|------|
| Diameter          | 0.00123793 | 0.00122184 | 0.00150921 | 0.00126712 | 0.00139588 | 0.00155337 | 0.00131911 | 0.00135451 | 0.00119512 | 0.00100439 |

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