New permanent approximation inequalities via identities

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Abstract

The aim of this paper is to present new upper bounds for the distance between a properly normalized permanent of a rectangular complex matrix and the product of the arithmetic means of the entries of its columns. It turns out that the bounds improve on those from earlier work. Our proofs are based on some new identities for the above-mentioned differences and also for related expressions for matrices over a rational associative commutative unital algebra. Some of our identities are generalizations of results in Dougall (Proc. Edinburgh Math. Soc., 24:61–77, 1905). Second order results are also included.

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1 Introduction

It is well-known that computing the permanent of an \( n \times n \) matrix can be a difficult task, if \( n \) is a large natural number, see Valiant [26] and Minc [20, Chapter 7]. There are a couple of known explicit formulae, the most efficient of which seem to be due to Ryser [23, Theorem 4.1, page 26] or Glynn [15, Theorem 2.1] and require at least \( O(2^n n) \) arithmetic operations. Matrices with a special structure can sometimes be treated differently, e.g. see Minc [20, Section 3.4 or Lemma 1 on page 113], Bax and Franklin [2], Schwartz [24], Björklund et al. [5] and the references therein. On the other hand, there are approximation algorithms, e.g. see Jerrum et al. [18], Barvinok [1] and the references given there.

There are many upper and lower bounds for permanents, see e.g. Minc [20, Chapters 4–6]. But the literature seems to contain only a few explicit approximation inequalities. See

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Bhatia [3], Elsner [13], Bhatia and Elsner [4], and Friedland [14], for some upper bounds of the distance between two permanents of quadratic complex matrices; however, the inequalities given there are not easily comparable with those discussed below. Another approach is to approximate a permanent by more special expressions, which depend on the matrix under consideration. Here, for convenience, we consider properly normalized permanents, i.e. we divide by the number of summands.

In this paper, we consider the approximation of a normalized permanent of a rectangular complex matrix. If the rows of the matrix are approximately equal, a good approximant should be the product of the arithmetic means of the entries of the columns of this matrix. Some upper bounds for the approximation error can be found in Bobkov [6] and Roos [22].

To be more precise, we need the following notation. Let $N \in \mathbb{N}$, $n \in \mathbb{N}$ := $\{1, \ldots, N\}$ and $Z = (z_{j,r}) \in C^{N \times n}$ be an $N \times n$ matrix with complex entries. We set $\bar{z}_r = \frac{1}{N} \sum_{j=1}^{N} z_{j,r}$, $(r \in n)$ and assume that $|z_{j,r}| \leq 1$, $(j \in \mathbb{N}, r \in n)$. However, it is noteworthy that some of the results of Section 4 below do not require the latter boundedness assumption.

For arbitrary sets $A$ and $B$, let $A^B$, resp. $A^B_\varnothing$, be the set of all maps, resp. injective maps, $f : B \rightarrow A$. For $f \in A^B$ and $b \in B$, we write $f(b) = f_b$. Let $N^N = \mathbb{N}^n$ and $N^N_\varnothing = \mathbb{N}^n_\varnothing = \{(j_1, \ldots, j_n) \in N^n | j_r \neq j_s \text{ for all } r, s \in n \text{ with } r \neq s\}$. In particular, $N^N_\varnothing$ is the set of all permutations on the set $N$. The permanent of $Z$ can now be defined by

$$\text{Per}(Z) = \sum_{j \in N^N} \prod_{r=1}^{n} z_{j,r,r}.$$  

As indicated above,

$$\frac{(N - n)!}{N!} \text{Per}(Z) \approx \prod_{r=1}^{n} \bar{z}_r,$$

when

$$z_{1,r} \approx \cdots \approx z_{N,r} \text{ for all } r \in n. \quad (1)$$

We note that, if $Z$ has identical columns, i.e. $z_{j,1} = \cdots = z_{j,n}$ for all $j \in \mathbb{N}$, then we have $\prod_{r=1}^{n} \bar{z}_r = \bar{z}_1^n$, whereas

$$\frac{(N - n)!}{N!} \text{Per}(Z) = \frac{1}{(n)} \sum_{J \subseteq \mathbb{N}} \prod_{j \in J} z_{j,1}$$

is the normalized elementary symmetric polynomial of degree $n$ in the variables $z_{1,1}, \ldots, z_{N,1}$.

Here, for a finite set $J$, let $|J|$ be the number of its elements.

Let us give a review of some approximation inequalities from the literature. Bobkov [6, Theorem 2.1] showed by a somewhat complicated induction that

$$\left| \frac{(N - n)!}{N!} \text{Per}(Z) - \prod_{r=1}^{n} \bar{z}_r \right| \leq C_0 \frac{n}{N} \quad \text{with } C_0 = 16 \quad (2)$$
and used this inequality to study an approximate de Finetti representation for probability measures, on product measurable spaces, which are symmetric under permutations of coordinates. The upper bound in (2) is small if \( n \) is small in comparison with \( N \). But since it is independent of \( Z \), it is not good in the case (1).

A bound depending on \( Z \) was given in Roos \[22\]. From the more general Theorem 2.13 given there, it follows that

\[
\left| \frac{(N - n)!}{N!} \text{Per}(Z) - \prod_{r=1}^{n} \tilde{z}_r \right| \leq 3.57 \gamma, \tag{3}
\]

where

\[
\gamma = \gamma(1), \quad \gamma(x) = \frac{n \alpha}{N} \min \left\{ \frac{x n}{N}, \frac{1}{1 - \beta} \right\}, \quad (x \in [0, \infty)),
\]

\[
\alpha = \frac{1}{nN} \sum_{j=1}^{N} \sum_{r=1}^{n} |a_{j,r}|^2, \quad a_{j,r} = z_{j,r} - \tilde{z}_r, \quad \gamma = 1 + \frac{2}{n} \sum_{r=1}^{n} \tilde{z}_r^2.
\]

In Remark 2.9 of that paper, it was also shown that \( \gamma \leq \frac{n}{N} \). Consequently in (2), \( C_0 \) can be replaced with 3.57. However, inequality (3) is preferable to (2) with any constant \( C_0 \), since \( \gamma \) can be much smaller than \( \frac{n}{N} \). In fact, the right-hand side in (3) is small in the case (1).

The proof of (3) does not require an induction argument but instead is based on the representation (see [22, Theorem 2.8])

\[
\frac{(N - n)!}{N!} \text{Per}(Z) = H_n(Z),
\]

where \( H_\ell(Z) = \sum_{m=0}^{\ell} G_m(Z) \) for \( \ell \in \mathbb{N} \),

\[
G_m(Z) = \frac{(N - m)!}{(n - m)! N!} \text{Coeff} \left( x_1 \cdots x_n; \left( \sum_{r=1}^{n} \tilde{z}_r x_r \right)^{n-m} \prod_{j=1}^{N} \left( 1 + \sum_{r=1}^{n} a_{j,r} x_r \right) \right),
\]

for \( m \in \mathbb{N}_0 = \{0, \ldots, n\} \), and \( \text{Coeff} \) denotes the coefficient of \( x_1 \cdots x_n \) in the formal power series expansion of the expression given above. In particular, \( H_1(Z) = \prod_{r=1}^{n} \tilde{z}_r \), and, if \( n \geq 2 \),

\[
H_2(Z) = \prod_{r=1}^{n} \tilde{z}_r - \frac{1}{N(N - 1)} \sum_{R \subseteq \mathbb{N}} \left( \sum_{j=1}^{N} \prod_{r \in R} a_{j,r} \right) \prod_{r \in \mathbb{N} \setminus R} \tilde{z}_r.
\]

It turned out that \( \frac{(N - n)!}{N!} \text{Per}(Z) \) can be approximated by \( H_\ell(Z) \), \( (\ell \in \mathbb{N}) \), which we call the \( \ell \)th order approximant. In fact, the following estimate shows that the accuracy is increasing in \( \ell \): if \( \gamma < 1 \), then

\[
\left| \frac{(N - n)!}{N!} \text{Per}(Z) - H_\ell(Z) \right| \leq (\ell + 1)^{1/4} \tilde{C}_{\ell+1} \frac{\gamma^{(\ell+1)/2}}{(1 - \gamma)^{3/4}},
\]
where \( \tilde{C}_\ell = \left( \frac{\ell^\ell}{e^\ell \ell!} \right)^{1/2} \). We note that Corollary 2.12 in [22] gives in the case \( \ell \in \mathbb{Z} \) and \( \gamma < 1 \) the sometimes sharper bounds for the first and second order approximations:

\[
\left| \frac{(N-n)!}{N!} \text{Per}(Z) - \prod_{r=1}^{n} \tilde{z}_r \right| \leq \gamma (1/2) + \frac{2.12 \gamma^{3/2}}{(1-\gamma)^{3/4}},
\]

where, for \( \gamma \) small, then \( \left| \frac{(N-n)!}{N!} \text{Per}(Z) - \prod_{r=1}^{n} \tilde{z}_r \right| \) is bounded by \( C_1 \gamma (\frac{1}{2}) \) with \( C_1 \approx 1 \).

The results of the present paper imply that, in (3) or (4), not only the constants but also the form of the right-hand side can substantially be improved, see Theorems 4.1 and 4.2 below. In particular, Theorem 4.1 implies that, if \( 2 \leq n \leq N \) and

\[
\vartheta = \frac{1}{N(N-1) \sqrt{n(n-1)}} \left( \sum_{(r,s) \in \mathbb{Z}_2^n} \left( \sum_{(u,v) \in \mathbb{Z}_2^n} |z_{u,r} - z_{v,r}| |z_{u,s} - z_{v,s}| \right)^2 \right)^{1/2},
\]

then

\[
\left| \frac{(N-n)!}{N!} \text{Per}(Z) - \prod_{r=1}^{n} \tilde{z}_r \right| \leq \frac{n-1}{2N} \vartheta \frac{1 - \beta^{n/4}}{1 - \sqrt{\beta}},
\]

see (53). Here, the right-hand side of (6) can be further estimated by \( (1 + \sqrt{\beta}) \gamma (\frac{1}{2}) \leq 2 \gamma (\frac{1}{2}) \), see Remark 4.1 below. However, (6) can be much better than these alternative bounds, see Parts (b) and (c) of Example 4.1 on derangement and ménage numbers. Indeed, we obtain bounds of the order \( O(\frac{1}{n}) \) and \( O(\frac{1}{\sqrt{n}}) \) as \( n \to \infty \), whereas the upper bounds in (3) and (4) cannot be small, since they contain one of the terms \( \gamma \) or \( \gamma (\frac{1}{2}) \). The present paper also contains an improvement of (5), which however is more complicated, see Theorem 4.3.

Let us comment on the method used in this paper. Our approach consists of two steps. First, we develop some identities for the difference of \( \text{Per}(Z) \) and its approximant. After that, these identities together with the properties of the norm and further auxiliary inequalities for permanents (see Lemma 4.1) are applied. We do not use the methods of [6] or [22].

Our identities are not only valid for complex matrices, but also for matrices over a rational associative commutative unital algebra. In the theory of permanents one often considers matrices over a commutative ring (see Minc [20, page 1]), but this is not sufficient here, since we need to be able to multiply with rational numbers. Some of our identities are generalizations of old identities of Dougall [12], who considered, among other things, the difference \( \prod_{j=1}^{N} z_j - \bar{z}^N \), where \( z_1, \ldots, z_N \in \mathbb{C} \) and \( \bar{z} = \frac{1}{N} \sum_{j=1}^{N} z_j \). In fact, our first result is Theorem 3.1, which is a generalization of formula (3) in [12, page 65] concerning elementary symmetric polynomials, see Corollary 3.1 below. The latter result was a starting point for
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several other results in [12]. Similarly, our generalization implies the identity (see (25) below)

\[
\text{Per}(Z) - \frac{N!}{(N-n)!} \prod_{r=1}^{n} \tilde{z}_r = - \sum_{k=2}^{n} \frac{1}{2Nk(k-1)} \sum_{R \subseteq \mathbb{N}} \sum_{|R|=k} \sum_{(r,s) \in R^2} (z_{j,r} - z_{j,r}) (z_{j,s} - z_{j,s}) \times \left( \prod_{\ell \in R \setminus \{r,s\}} z_{j,\ell,\ell} \right) \prod_{\ell \in n \setminus R} \tilde{z}_{\ell},
\]

which, in turn, is a generalization of another identity in [12, page 77], see Corollary 3.3 below.

We note that, in (7), it is important to have the product \((z_{j,r} - z_{j,s})(z_{j,s} - z_{j,s})\) of two differences of certain entries of \(Z\). As a rule, an accurate approximation of \(\text{Per}(Z)\) should be reflected in a high number of such differences in the corresponding identity. Indeed, Theorem 3.3 contains an identity for the difference of \(\text{Per}(Z) - \frac{N!}{(N-n)!} H_2(Z)\), where the right-hand side consists of two expressions containing the product of three, resp. four, such differences.

The paper is structured as follows. Section 2 is devoted to the notation, which is needed to simplify the presentation. In Section 3, we derive some new identities for permanents and related expressions, some of which will be used in Section 4 to give refined upper bounds of \(|\frac{N!}{n!}(\text{Per}(Z) - H_\ell(Z))|\) for \(\ell \in 2\).

2 Notation

From now on, unless stated otherwise, our notation is as follows. Let \(Z\) be a rational associative commutative unital algebra, \(N \in \mathbb{N}\), \(n \in \mathbb{N} = \{1, \ldots, N\}\), \(Z = (z_{j,r}) \in \mathbb{Z}^{N \times n}\),

\[
\tilde{z}_r = \frac{1}{N} \sum_{j=1}^{N} z_{j,r} \quad \text{for} \quad r \in \mathbb{N} \quad \text{and} \quad y_{j,k,r} = z_{j,r} - z_{k,r} \quad \text{for} \quad j, k \in \mathbb{N}, \ r \in \mathbb{N}.
\]

Let \(p_{j,R} = \prod_{r \in R} z_{j,r}\) for \(R \subseteq \mathbb{N}\) and \(j \in \mathbb{N}^S\), whenever \(S \subseteq \mathbb{N}\) with \(R \subseteq S\). For \(R \subseteq \mathbb{N}\), set

\[
\bar{p}_R = \sum_{j \in \mathbb{N}^S} p_{j,R} \quad \text{and} \quad \bar{p}_R = \prod_{r \in R} \tilde{z}_r.
\]

In particular, we have

\[
\bar{p}_\emptyset = \frac{N!}{(N-n)!}, \quad \bar{p}_{\{r\}} = \frac{N!}{(N-n)!} \tilde{z}_r \quad \text{for} \quad r \in \mathbb{N}, \quad \bar{p}_n = \text{Per}(Z), \quad \bar{p}_\emptyset = 1, \quad \bar{p}_{\{r\}} = \tilde{z}_r \quad \text{for} \quad r \in \mathbb{N}, \quad \bar{p}_n = \prod_{r=1}^{n} \tilde{z}_r.
\]

For a set \(A\), let \(1_A(x) = 1\), when \(x \in A\), and \(1_A(x) = 0\) otherwise. We always set \(0^0 = 1\), \(\frac{1}{0} = \infty\), and \(1^\infty = 1\). As usual, empty sums, resp. empty products, are defined to be zero, resp. one.
3 Some identities for permanents

Our first main result is Theorem 3.1 below, the proof of which requires the following lemma. For \( r, s \in \mathbb{N} \), let \( \tau_{r,s} \in \mathbb{N}_N^N \) be the transposition, which interchanges \( r \) with \( s \), i.e.

\[
\tau_{r,s}(\ell) = \begin{cases} \ell & \text{for } \ell \in \mathbb{N} \setminus \{r, s\}, \\ s & \text{for } \ell = r, \\ r & \text{for } \ell = s. \end{cases}
\]

**Lemma 3.1** Let \( r \in \mathbb{N} \), \( T_r = (T_{r,1}, T_{r,2}) : (\mathbb{N}_N^N)^2 \rightarrow (\mathbb{N}_N^N)^2 \), \( T_r(j, k) = (T_{r,1}(j, k), T_{r,2}(j, k)) \) with \( T_{r,1}(j, k) = j \circ \tau_{r,(j-1)ok}(r), T_{r,2}(j, k) = k \circ \tau_{r,(k-1)oj}(r) \) for \( (j, k) \in (\mathbb{N}_N^N)^2 \). Here \( \circ \) means composition of functions. Then \( T_r \circ T_r \) is the identity map on \( (\mathbb{N}_N^N)^2 \). In particular, \( T_r \) is bijective and we have \( T_{r,1}(j, k)(r) = k_r, T_{r,2}(j, k)(r) = j_r \).

**Proof.** For \( (j, k) \in (\mathbb{N}_N^N)^2 \), set \( \tilde{j} = T_{r,1}(j, k) \) and \( \tilde{k} = T_{r,2}(j, k) \). Then

\[
(\tilde{j}^{-1} \circ \tilde{k})(r) = ((j \circ \tau_{r,(j-1)ok}(r))^{-1} \circ k \circ \tau_{r,(k-1)oj}(r))(r) = (\tau_{r,(j-1)ok}(r) \circ j^{-1} \circ k)((k^{-1} \circ j)(r)) = \tau_{r,(j-1)ok}(r)(r) = (j^{-1} \circ k)(r)
\]

and therefore

\[
T_{r,1}(T_r(j, k)) = \tilde{j} \circ \tau_{r,(j-1)ok}(r) = (j \circ \tau_{r,(j-1)ok}(r)) \circ \tau_{r,(j-1)ok}(r) = j.
\]

Similarly \( (\tilde{k}^{-1} \circ \tilde{j})(r) = (k^{-1} \circ j)(r) \) and \( T_{r,2}(T_r(j, k)) = k \). \( \square \)

**Remark 3.1** (a) Another way of describing \( T_r \) is the following: For \( j, k \in \mathbb{N}_N^N \), we obtain \( T_{r,1}(j, k) \) and \( T_{r,2}(j, k) \), if in both tuples \( j \) and \( k \), we replace \( j_r \) with \( k_r \). More precisely, if \( r \in \mathbb{N}, j, k \in \mathbb{N}_N^N, a, b \in \mathbb{N}, u, v \in \mathbb{N} \) with \( j_r = u, j_a = v, k_r = v, k_b = u \), then

\[
T_{r,1}(j, k)(s) = \begin{cases} j_s & \text{for } s \in \mathbb{N} \setminus \{r, a\}, \\ v & \text{for } s = r, \\ u & \text{for } s = a, \end{cases} \quad T_{r,2}(j, k)(s) = \begin{cases} k_s & \text{for } s \in \mathbb{N} \setminus \{r, b\}, \\ u & \text{for } s = r, \\ v & \text{for } s = b. \end{cases}
\]

For example, if \( N = 3, r = 2, j = (j_1, j_2, j_3) = (2, 1, 3) \) and \( k = (k_1, k_2, k_3) = (3, 2, 1) \), then \( T_{r,1}(j, k) = (1, 2, 3) \) and \( T_{r,2}(j, k) = (3, 1, 2) \).

(b) According to Lemma 3.1, we have \( (\mathbb{N}_N^N)^2 = \{T_r(j, k) \mid (j, k) \in (\mathbb{N}_N^N)^2 \} \) for all \( r \in \mathbb{N} \).

Hence, for an arbitrary function \( f : (\mathbb{N}_N^N)^2 \rightarrow \mathbb{Z} \),

\[
\sum_{j \in \mathbb{N}_N^N} \sum_{k \in \mathbb{N}_N^N} f(j, k) = \sum_{j \in \mathbb{N}_N^N} \sum_{k \in \mathbb{N}_N^N} f(T_r(j, k)),
\]

which is the main idea in the proof of the next theorem.
Theorem 3.1 Let \( R, S \subseteq \mathbb{N} \) and \( r \in \mathbb{N} \setminus (R \cup S) \neq \emptyset \). Then
\[
\overline{p}_{R \cup \{r\}} - \overline{p}_R \overline{p}_{S \cup \{r\}} = \frac{1}{2} \sum_{s \in \mathbb{N} \setminus \{r\}} \sum_{(u,v) \in \mathbb{N}^2_p} y_{u,v,r} y_{u,v,s} \sum_{j \in \mathbb{N}^p_{f_j}} \sum_{k \in \mathbb{N}^p_{f_k}} (1_S(s)p_{j,S \setminus \{s\}}p_{k,R} - 1_R(s)p_{j,R \setminus \{s\}}p_{k,S}).
\]
(8)

If \( n = 1 \), then the right-hand side of the equality in (8) is defined to be zero.

Proof. We have
\[
((N - n)!^2(\overline{p}_{R \cup \{r\}} - \overline{p}_R \overline{p}_{S \cup \{r\}}) = ((N - n)!^2 \sum_{j \in \mathbb{N}^N_p} \sum_{k \in \mathbb{N}^N_p} (p_{j,R \cup \{r\}} p_{k,S} - p_{j,R} p_{k,S \cup \{r\}})
= \sum_{j \in \mathbb{N}^N_p} \sum_{k \in \mathbb{N}^N_p} y_{j,r,k} p_{j,R} p_{k,S}.
\]
(9)

Now we use the decompositions \( \mathbb{N} \setminus \{r\} = (\mathbb{N} \setminus (R \cup \{r\})) \cup R = (\mathbb{N} \setminus (S \cup \{r\})) \cup S \) and obtain
\[
((N - n)!^2(\overline{p}_{R \cup \{r\}} - \overline{p}_R \overline{p}_{S \cup \{r\}})) = A_1 + A_2 + A_3 + A_4,
\]
(10)

where
\[
A_1 = \sum_{a \in R} \sum_{b \in S} \sum_{(u,v) \in \mathbb{N}^2_p} \sum_{j \in \mathbb{N}^N_p} \sum_{k \in \mathbb{N}^N_p} y_{u,v,r} z_{v,a} P_{j,R \setminus \{a\}} p_{k,S},
\]

\[
A_2 = \sum_{a \in \mathbb{N} \setminus (R \cup \{r\})} \sum_{b \in S} \sum_{(u,v) \in \mathbb{N}^2_p} \sum_{j \in \mathbb{N}^N_p} \sum_{k \in \mathbb{N}^N_p} y_{u,v,r} z_{r,b} p_{j,R} p_{k,S \setminus \{b\}},
\]

\[
A_3 = \sum_{a \in R} \sum_{b \in \mathbb{N} \setminus (S \cup \{r\})} \sum_{(u,v) \in \mathbb{N}^2_p} \sum_{j \in \mathbb{N}^N_p} \sum_{k \in \mathbb{N}^N_p} y_{u,v,r} z_{v,a} u_{a,b} p_{j,R \setminus \{a\}} p_{k,S \setminus \{b\}},
\]

\[
A_4 = \sum_{a \in \mathbb{N} \setminus (R \cup \{r\})} \sum_{b \in \mathbb{N} \setminus (S \cup \{r\})} \sum_{(u,v) \in \mathbb{N}^2_p} \sum_{j \in \mathbb{N}^N_p} \sum_{k \in \mathbb{N}^N_p} y_{u,v,r} p_{j,R} p_{k,S}.
\]

A representation similar to (10) can be shown by using (9), Lemma 3.1 and the fact that \( y_{v,u,r} = -y_{u,v,r} \) for \( u, v \in \mathbb{N} \). Indeed, if \( T_r = (T_{r,1}, T_{r,2}) \) is defined as in that lemma, then
\[
((N - n)!^2(\overline{p}_{R \cup \{r\}} - \overline{p}_R \overline{p}_{S \cup \{r\}}) = \sum_{j \in \mathbb{N}^N_p} \sum_{k \in \mathbb{N}^N_p} y_{k,r,j} r_p T_{r,1}(j,k), P_{T_{r,2}(j,k)}, S

= -\sum_{a \in \mathbb{N} \setminus \{r\}} \sum_{b \in \mathbb{N} \setminus \{r\}} \sum_{(u,v) \in \mathbb{N}^2_p} \sum_{j \in \mathbb{N}^N_p} \sum_{k \in \mathbb{N}^N_p} y_{u,v,r} T_{r,1}(j,k), P_{T_{r,2}(j,k)}, S

= -(A'_1 + A'_2 + A'_3 + A'_4),
\]
(11)
Now we write
\[ A_1' = \sum_{a \in R} \sum_{b \in \mathbb{N}\backslash(S \cup \{r\})} \sum_{(u,v) \in N_\varphi^2} \sum_{j \in N_\varphi} \sum_{k \in N_\varphi} y_{u,v,r} z_{u,a} P_j, R \backslash \{a\} P_k, S, \]
\[ A_2' = \sum_{a \in \mathbb{N}\backslash(R \cup \{r\})} \sum_{b \in S} \sum_{(u,v) \in N_\varphi^2} \sum_{j \in N_\varphi} \sum_{k \in N_\varphi} y_{u,v,r} z_{v,b} P_j, R P_k, S \backslash \{b\}, \]
\[ A_3' = \sum_{a \in R} \sum_{b \in S} \sum_{(u,v) \in N_\varphi^2} \sum_{j \in N_\varphi} \sum_{k \in N_\varphi} y_{u,v,r} z_{u,a} z_{v,b} P_j, R \backslash \{a\} P_k, S \backslash \{b\} \]
and \( A_4' = A_4 \). Adding the right-hand sides of (10) and (11) and dividing by two, we get the
identity
\[ ((N - n)!^2 (\mathcal{P}_{R \cup \{r\}} P'_S - \mathcal{P}_{R} P_{S \cup \{r\}}) = \frac{1}{2} (-B_1 + B_2 + B_3), \quad (12) \]
where
\[ B_1 = -A_1 + A_1' = \sum_{a \in R} \sum_{b \in \mathbb{N}\backslash(S \cup \{r\})} \sum_{(u,v) \in N_\varphi^2} \sum_{j \in N_\varphi} \sum_{k \in N_\varphi} y_{u,v,r} y_{u,v,a} P_j, R \backslash \{a\} P_k, S, \]
\[ B_2 = A_2 - A_2' = \sum_{a \in \mathbb{N}\backslash(R \cup \{r\})} \sum_{b \in S} \sum_{(u,v) \in N_\varphi^2} \sum_{j \in N_\varphi} \sum_{k \in N_\varphi} y_{u,v,r} y_{u,v,b} P_j, R P_k, S \backslash \{b\}, \]
\[ B_3 = A_3 - A_3' = \sum_{a \in R} \sum_{b \in S} \sum_{(u,v) \in N_\varphi^2} \sum_{j \in N_\varphi} \sum_{k \in N_\varphi} y_{u,v,r} (z_{v,a} z_{u,b} - z_{u,a} z_{v,b}) P_j, R \backslash \{a\} P_k, S \backslash \{b\}. \]

Now we write \( B_3 = B_3' - B_3'' \), where
\[ B_3' = \sum_{a \in R} \sum_{b \in S} \sum_{(u,v) \in N_\varphi^2} \sum_{j \in N_\varphi} \sum_{k \in N_\varphi} y_{u,v,r} z_{v,a} y_{u,v,b} P_j, R \backslash \{a\} P_k, S \backslash \{b\} \]
\[ = \sum_{a \in R} \sum_{b \in S} \sum_{(u,v) \in N_\varphi^2} \sum_{j \in N_\varphi} \sum_{k \in N_\varphi} y_{u,v,r} y_{u,v,b} P_j, R P_k, S \backslash \{b\} \quad (13) \]
and
\[ B_3'' = \sum_{a \in R} \sum_{b \in S} \sum_{(u,v) \in N_\varphi^2} \sum_{j \in N_\varphi} \sum_{k \in N_\varphi} y_{u,v,r} y_{u,v,a} z_{v,b} P_j, R \backslash \{a\} P_k, S \backslash \{b\} \]
\[ = \sum_{a \in R} \sum_{b \in S} \sum_{(u,v) \in N_\varphi^2} \sum_{j \in N_\varphi} \sum_{k \in N_\varphi} y_{u,v,r} y_{u,v,a} P_j, R \backslash \{a\} P_k, S. \quad (14) \]
Indeed, (15) can be derived from (14) by interchanging \( u \) with \( v \) for \((a, b) \in R \times S\) being fixed. We note that here \( y_{u,v,r}y_{u,v,a} = y_{v,u,r}y_{v,u,a} \) and

\[
\sum_{j \in N_N^r \setminus \{a\}} p_{j,R \setminus \{a\}} = \sum_{j \in N_N^r \setminus \{a\}} p_{j,R \setminus \{a\}}, \quad \sum_{k \in N_N^s \setminus \{b\}} z_{u,b}p_{k,S \setminus \{b\}} = \sum_{k \in N_N^s \setminus \{b\}} p_{k,S}
\]

for \((u, v) \in N^2_N\). Combining (12), (13) and (15), we get

\[
((N - n)!)^2 (p_{R,\{r\}}p_{S} - p_{R}p_{S,\{r\}}) = \frac{1}{2} (C_1 - C_2),
\]

where \( C_1 = B_2 + B'_3 \), \( C_2 = B_3 + B''_3 \). We have

\[
C_1 = \sum_{a \in N_N \setminus \{r\}} \sum_{b \in S} \sum_{(u,v) \in N^2_N} \sum_{j \in N_N^r : j_r = u, j_a = v} \sum_{k \in N_N^s} y_{u,v,r}y_{u,v,a}p_{j,R \setminus \{a\}}k_{k,R}
\]

\[
= \sum_{s \in N \setminus \{r\}} \sum_{(u,v) \in N^2_N} y_{u,v,r}y_{u,v,s} \sum_{j \in N_N^r : j_r = u} \sum_{k \in N_N^s} \mathbb{I}_S(s)p_{j,R \setminus \{a\}}k_{k,R}
\]

\[
= \sum_{s \in N \setminus \{r\}} \sum_{(u,v) \in N^2_N} y_{u,v,r}y_{u,v,s} \sum_{j \in N_N^r : j_r = u} \sum_{k \in N_N^s} \mathbb{I}_S(s)p_{j,S \setminus \{s\}}k_{k,R}
\]

\[
= \sum_{s \in N \setminus \{r\}} \sum_{(u,v) \in N^2_N} y_{u,v,r}y_{u,v,s} \sum_{j \in N_N^r : j_r = u} \sum_{k \in N_N^s} \mathbb{I}_S(s)p_{j,S \setminus \{s\}}k_{k,R}.
\]

Here, (17) follows from the definitions of \( B_2 \) and \( B'_3 \). To get (18), we replace \( b \) by \( s \) and note that \( \sum_{a \in N_N \setminus \{r\}} \sum_{j \in N_N^r : j_r = u, j_a = v} = \sum_{j \in N_N^r : j_r = u} \) for fixed \((u, v) \in N^2_N\). For (19), we interchanged \( j \) with \( k \). Finally, (20) follows by interchanging \( j_r \) with \( j_s \) and noting that \( \sum_{j \in N_N^r : j_r = v, j_s = u} p_{j,S \setminus \{s\}} = \sum_{j \in N_N^r : j_r = u, j_s = v} p_{j,S \setminus \{s\}} \) for fixed \((u, v) \in N^2_N\). Similarly,

\[
C_2 = \sum_{a \in R} \sum_{b \in N_N \setminus \{r\}} \sum_{(u,v) \in N^2_N} \sum_{j \in N_N^r : j_r = u, j_a = v} \sum_{k \in N_N^s} y_{u,v,r}y_{u,v,a}p_{j,R \setminus \{a\}}k_{k,R}
\]

\[
= \sum_{s \in N \setminus \{r\}} \sum_{(u,v) \in N^2_N} y_{u,v,r}y_{u,v,s} \sum_{j \in N_N^r : j_r = u} \sum_{k \in N_N^s} \mathbb{I}_R(s)p_{j,R \setminus \{a\}}k_{k,R}
\]

\[
= \sum_{s \in N \setminus \{r\}} \sum_{(u,v) \in N^2_N} y_{u,v,r}y_{u,v,s} \sum_{j \in N_N^r : j_r = u} \sum_{k \in N_N^s} \mathbb{I}_R(s)p_{j,R \setminus \{a\}}k_{k,R}
\]

In particular, (21) follows by interchanging \( u \) with \( v \) for fixed \( s \in N \setminus \{r\} \). Combining (16), (20), and (22) the assertion is shown.

The next result on elementary symmetric polynomials is due to Dougall [12, formula (3) on page 65]. We now show that it is a consequence of Theorem 3.1.
Corollary 3.1 Let $N \in \mathbb{N}$, $z_j \in \mathbb{Z}$ for $j \in N$. $E_{A,k} = \sum_{|J|=k} \prod_{j \in J} z_j$ for $A \subseteq N$ and $k \in \mathbb{Z}$, $a, b \in \sum_{j=0} = \{0, 1, \ldots, N\}$. In particular, $E_{A,0} = 1$ and $E_{A,k} = 0$ if $k < 0$ or $k > |A|$. Then
\[(a + 1)(N - b)E_{N,a+1}E_{N,b} - (b + 1)(N - a)E_{N,a}E_{N,b+1} = \frac{1}{2} \sum_{(u,v) \in \sum_{j=2}} (z_u - z_v)^2 (E_{N\{i,v\},b-1}E_{N\{i,v\},a} - E_{N\{i,v\},a-1}E_{N\{i,v\},b})\].

Proof. For $a = N$ or $b = N$, the assertion is trivial. Let now $a < N$ and $b < N$ and consider the assumptions of Theorem 3.1, where $n = N$ and $Z$ has identical columns, i.e. $z_{j,1} = \cdots = z_{j,n} = z_j$ for all $j \in N$. Further, let $|R| = a$ and $|S| = b$. Then the assertion follows from (8) and
\[\overline{p}_R = a!(N - a)E_{N,a}, \quad \overline{p}_S = b!(N - b)E_{N,b}, \quad \sum_{\ell \in S} \sum_{j \in \mathbb{N}_n} p_{j,S\{\ell\}} = b!(N - b - 1)E_{N\{i,v\},b-1}, \quad \sum_{k \in \mathbb{N}_n} p_{k,R} = a!(N - a - 1)E_{N\{i,v\},a} \ \text{and} \ \sum_{k \in \mathbb{N}_n} p_{k,S} = b!(N - b - 1)E_{N\{i,v\},b}\]
and $E_{N\{i,v\},a} = E_{N\{i,v\},a} + z_v E_{N\{i,v\},a-1}$, $E_{N\{i,v\},b} = E_{N\{i,v\},b} + z_v E_{N\{i,v\},b-1}$, where $(u, v) \in \sum_{j=2}$.

Corollary 3.2 Let $R \subseteq n$ and $r \in n \setminus R \neq \emptyset$. Then
\[\overline{p}_{R\backslash\{r\}} - \overline{z}_r \overline{p}_R = -\frac{1}{2N} \sum_{s \in R} \sum_{r \in \mathbb{N}_n} y_{j,r,s} p_{j,r,s} \overline{p}_{R\backslash\{s\}}. \quad (23)\]

Proof. In Theorem 3.1, set $S = \emptyset$.

The next result follows from Corollary 3.2 and is the main argument in the proof of our inequalities in Section 4.1.

Theorem 3.2 If $r \in \mathbb{n}_0$, $R_k := R_k(r) := \{r_\ell \mid \ell \in k\}$ for $k \in \mathbb{n}_0 = \{0, \ldots, n\}$, then $\emptyset = R_0 \subseteq R_1 \subseteq \cdots \subseteq R_n = n$ is a maximal chain of subsets of $n$ and
\[\overline{p}_n - \frac{N! \overline{p}_n}{(N - n)!} = -\frac{1}{2N} \sum_{k=2}^n \sum_{R \subseteq R_k, |R| = k} \sum_{J \subseteq \mathbb{N}_n} y_{j,k,r} p_{j,k-1,r-1} \overline{p}_n \overline{p}_R. \quad (24)\]

On the other hand,
\[\overline{p}_n - \frac{N! \overline{p}_n}{(N - n)!} = -\sum_{(u,v) \in \sum_{j=2}} \sum_{(r,s) \in \sum_{j=2}} y_{u,v,r} y_{u,v,s} \frac{1}{2N} \sum_{k=2}^n \sum_{R \subseteq \mathbb{N}_n \setminus \{r,s\}, |R| = k-2} \overline{p}_{n \setminus (R \cup \{r,s\})} \overline{p}_{R \setminus \{r,s\}}. \quad (25)\]

\[= -\sum_{(u,v) \in \sum_{j=2}} \sum_{(r,s) \in \sum_{j=2}} \frac{1}{2N} \sum_{k=2}^n \sum_{R \subseteq \mathbb{N}_n \setminus \{r,s\}, |R| = k-2} \overline{p}_{n \setminus (R \cup \{r,s\})} \overline{p}_{R \setminus \{r,s\}}. \quad (26)\]
If \( n = 1 \), the right-hand sides of (24), (25) and (26) are defined to be zero.

**Proof.** In view of Corollary 3.2 and the identities \( \overline{p}_0 = \frac{N!}{(N-n)!} \) and \( \overline{p}_0 = 1 \), we see that

\[
\overline{p}_n - \frac{N! \overline{p}_n}{(N-n)!} = \sum_{k=1}^{n} \left( \overline{p}_{n-k} \right)_{R_k} \overline{p}_{R_k} - \overline{p}_{n-k-1} \overline{p}_{R_{k-1}} \\
= \sum_{k=1}^{n} \left( \overline{p}_{R_k} - z_{R_k} \overline{p}_{R_k-1} \right) \overline{p}_{R_k} \\
= - \frac{1}{2N} \sum_{k=2}^{n} \sum_{s \in R_{k-1}} j \in \mathbb{N}_n \sum_{y_{j,r}, j \in R_{k-1} \setminus \{s\}} y_{j,r}^2 \overline{p}_{R_k} \overline{p}_{R_k}
\]

giving (24). Hence,

\[
\overline{p}_n - \frac{N! \overline{p}_n}{(N-n)!} = - \frac{1}{2N} \sum_{k=2}^{n} \frac{1}{n!} \sum_{r \in \mathbb{N}_n^k} \sum_{t \in R_{k-1} \setminus \{r\}} j \in \mathbb{N}_n \sum_{y_{j,r}, j \in R_{k-1} \setminus \{t\}} y_{j,r}^2 \overline{p}_{R_k} \overline{p}_{R_k}
\]

since for \( k \in \mathbb{N} \setminus \{1\} \), \( R \subseteq n \) with \( |R| = k \) and \( t \in R \), the number of \( r \in \mathbb{N}_n^k \) with \( R_k(r) = R \) and \( r_k(t) \) is equal to \((k-1)! (n-k)!\). This shows (25). Furthermore, (26) is clear. \( \square \)

**Remark 3.2** (a) If \( Z \) has identical rows, i.e. \( z_{1,r} = \cdots = z_{N,r} \) for all \( r \in \mathbb{n} \), then \( \overline{p}_R = \frac{N! \overline{p}_n}{(N-n)!} \) for all \( R \subseteq \mathbb{N} \) and therefore both sides in each identity (8) and (23)–(26) give zero. For identities in the case, when \( Z \) has identical columns, see Corollary 3.3 below.

(b) The identities of Theorem 3.2 can be rewritten as expansions for the permanent \( \overline{p}_n \). Further such formulas can be found in the literature, e.g. see Minc [20, Chapter 7]. For instance, Ryser [23, Theorem 4.1, page 26] proved that

\[
\overline{p}_n = \sum_{k=1}^{n} (-1)^{n-k} \binom{N-k}{n-k} \prod_{J \subseteq \mathbb{N}} \prod_{r=1}^{n} \left( \sum_{j \in J} z_{j,r} \right).
\]

In the case \( n = N \), this implies that \( \overline{p}_n = \frac{N! \overline{p}_n}{(N-n)!} \), which however is not comparable with the identities of Theorem 3.2 under the present assumption. We note that a second order expansion for \( \overline{p}_n \) can be found in Theorem 3.3 below.

(c) Let us assume that \( Z = (z_{j,r}) \in [0, \infty)^{N \times n} \) has decreasing columns, i.e. \( z_{j,r} \geq z_{j+1,r} \) for all \( j \in \mathbb{n}-1 \) and \( r \in \mathbb{n} \). Then \( y_{j_1,j_2,r} y_{j_1,j_2,s} \geq 0 \) for all \( j_1, j_2 \in \mathbb{N} \) and \( r, s \in \mathbb{n} \). Therefore, Corollary 3.2 implies in this case that \( \overline{p}_{R \cup \{r\}} \leq \overline{z} \overline{p}_R \) for \( R \subseteq \mathbb{n} \) and \( r \in \mathbb{n} \setminus R \neq \emptyset \). Further, Theorem 3.2 gives in this case that \( \overline{p}_n \leq \frac{N! \overline{p}_n}{(N-n)!} \). Both inequalities above also follow from the more general Corollary 4.9 in Brändén et al. [7], which was shown with the help of the monotone column permanent theorem.
Corollary 3.3 Let $N \in \mathbb{N}$, $n \in \mathbb{N}$, $z_j \in \mathbb{Z}$ for $j \in \mathbb{N}$, and $\bar{z} = \frac{1}{N} \sum_{j=1}^{N} z_j$. For $A \subseteq \mathbb{N}$ and $k \in \mathbb{Z}$, let $E_{A,k} = \sum_{|j|=k}^{A} \prod_{j \in A} z_j$. Then

$$
\frac{1}{(N)_n} E_{N,n} - \bar{z}^n = -\frac{1}{2N} \sum_{(u,v) \in \mathbb{N}_2} (z_u - z_v)^2 \sum_{k=2}^{n} \bar{z}^{n-k} \sum_{(r,s) \in \mathbb{N}_2} \sum_{|R|=k}^{\mathbb{N}\{u,v\}} \sum_{j \in \mathbb{N}\{u,v\}} \prod_{j \in \mathbb{N}\{u,v\}} z_j.
$$

In particular for $n = N$, we get Dougall’s [12, page 77] identity

$$
\prod_{j=1}^{n} z_j - \bar{z}^n = -\frac{1}{2N} \sum_{(u,v) \in \mathbb{N}_2} (z_u - z_v)^2 \sum_{k=2}^{n} \bar{z}^{n-k} \sum_{(r,s) \in \mathbb{N}_2} \sum_{|R|=k}^{\mathbb{N}\{u,v\}} \sum_{j \in \mathbb{N}\{u,v\}} \prod_{j \in \mathbb{N}\{u,v\}} z_j.
$$

Proof. Identity (27) follows from (26) in the case that $Z$ has identical columns. Indeed, letting $z_{j,1} = \cdots = z_{j,n} = z_j$ for all $j \in \mathbb{N}$, then $\bar{z}_1 = \cdots = \bar{z}_n = \bar{z}$ and $\bar{p}_n = n! E_{N,n}$, $\bar{p}_n = \bar{z}^n$ and

$$
\bar{p}_n - \frac{N! \bar{p}_n}{(N-n)!} = -\sum_{(u,v) \in \mathbb{N}_2} \sum_{k=2}^{n} \bar{z}^{n-k} \sum_{(r,s) \in \mathbb{N}_2} \sum_{|R|=k}^{\mathbb{N}\{u,v\}} \sum_{j \in \mathbb{N}\{u,v\}} \prod_{j \in \mathbb{N}\{u,v\}} z_j.
$$

Identity (28) follows from (27), if $n = N$. □

We note that the right-hand side of (27) gives an expansion for the difference between the normalized elementary symmetric polynomial $\frac{1}{(N)_n} E_{N,n}$ and $\bar{z}^n$. Further, identities similar to (27) or (28) have been proved by Hurwitz [17] and Dinghas [11].

The next lemma is needed in the proof of our last main result of this section.

Lemma 3.2 If $n \geq 3$, $R \subseteq \mathbb{n}$ with $|R| \leq n - 3$ and $(r,s,t) \in (\mathbb{n} \setminus R)^3$, then

$$
\sum_{j \in \mathbb{N}_2} y_{j,r,s} y_{j,r,s}(p_{j,R}(t) - \bar{z}_j p_{j,R}) = D_1 - D_2,
$$

where

$$
D_1 := D_1(r,s,t,R) := \frac{2}{N} \sum_{j \in \mathbb{N}_2} y_{j,r,s} y_{j,r,s} y_{j,r,s} y_{j,r,s} p_{j,R},
$$

$$
D_2 := D_2(r,s,t,R) := \frac{1}{2N} \sum_{q \in R} \sum_{j \in \mathbb{N}_2} y_{j,r,s} y_{j,r,s} y_{j,r,s} y_{j,r,s} y_{j,q} y_{j,q} p_{j,R} p_{j,R} \setminus \{q\},
$$

Proof. Let $D_0 := D_0(r,s,t,R)$ denote the left-hand side of the equation in (29). For $j \in \mathbb{N}_2$, we have $\bar{z}_j = \frac{1}{N} \sum_{q=1}^{N} z_{j,q}$ and therefore

$$
(N-n)! D_0 = \sum_{j \in \mathbb{N}_2} y_{j,r,s} y_{j,r,s} (z_{j,t} - \bar{z}_j) p_{j,R} = \frac{1}{N} \sum_{q \in R \setminus \{t\}} \sum_{j \in \mathbb{N}_2} y_{j,r,s} y_{j,r,s} y_{j,q} y_{j,q} p_{j,R}.
$$
where
\[
\sum_{q \in \mathbb{N} \setminus \mathbb{N}_p} \sum_{j \in \mathbb{N}_p} y_{j_r, s} y_{j_r, j_q} y_{j_r, j_t} p_{j, R} = 0,
\]
which follows by interchanging \( j_t \) with \( j_q \). Hence
\[
D_0 = \frac{1}{N} \sum_{q \in \mathbb{N} \setminus \{t\}} \sum_{j \in \mathbb{N}_p} y_{j_r, s} y_{j_r, j_q} y_{j_r, j_t} p_{j, R} = D_1 - D_2 + D_3,
\]
where
\[
D_1 = \frac{1}{N} \sum_{q \in \{r, s\}} \sum_{j \in \mathbb{N}_p} y_{j_r, s} y_{j_r, j_q} y_{j_r, j_t} p_{j, R},
\]
\[
D_2 = -\frac{1}{N} \sum_{q \in R} \sum_{j \in \mathbb{N}_p} y_{j_r, s} y_{j_r, j_q} y_{j_r, j_t} p_{j, R},
\]
\[
D_3 = \frac{1}{N} \sum_{q \in \mathbb{N} \setminus \{R \cup \{r, s, t\}\}} \sum_{j \in \mathbb{N}_p} y_{j_r, s} y_{j_r, j_q} y_{j_r, j_t} p_{j, R}.
\]
Let us consider the term \( D_1 \). Interchanging \( j_s \) with \( j_r \) in the summand for \( q = s \), we obtain
\[
D_1 = \frac{2}{N} \sum_{j \in \mathbb{N}_p} y_{j_r, s} y_{j_r, j_s} y_{j_r, j_t} p_{j, R}.
\]
The term \( D_2 \) can be treated similarly. By interchanging \( j_q \) with \( j_t \) in the second sum, we derive
\[
D_2 = -\frac{1}{N} \sum_{q \in R} \sum_{j \in \mathbb{N}_p} y_{j_r, s} y_{j_r, j_q} y_{j_r, j_t} z_{j_t, q} p_{j, R \setminus \{q\}} = \frac{1}{N} \sum_{q \in R} \sum_{j \in \mathbb{N}_p} y_{j_r, s} y_{j_r, j_q} y_{j_r, j_t} z_{j_t, q} p_{j, R \setminus \{q\}}.
\]
Now, adding the right-hand sides of (32), (33) and dividing by two we get
\[
D_2 = \frac{1}{2N} \sum_{q \in R} \sum_{j \in \mathbb{N}_p} y_{j_r, s} y_{j_r, j_q} y_{j_r, j_t} y_{j_r, j_q} p_{j, R \setminus \{q\}}.
\]
Finally, we have \( D_3 = 0 \), since
\[
D_3 = \frac{1}{N} \sum_{q \in \mathbb{N} \setminus \{R \cup \{r, s, t\}\}} \sum_{j \in \mathbb{N}_p} y_{j_r, s} y_{j_r, j_q} y_{j_r, j_t} p_{j, R} = -D_3,
\]
which follows by interchanging \( j_t \) with \( j_q \) in the second sum. This completes the proof. □

The next result contains a second order expansion for \( f_n \) and is the main argument in the proof of Theorem 4.3.
Theorem 3.3 Let $2 \leq n \leq N$ and $\bar{p}^{(2)} = \sum_{R \subseteq \mathbb{R}^k} \bar{p}_R \sum_{j=1}^N (z_j, r - \bar{z}_r)$. Then

$$
\bar{p}_n - \frac{N! \bar{p}_n}{(N-n)!} + \frac{(N-2)! \bar{p}_n^{(2)}}{(N-n)!} = \frac{1}{2N^2} \sum_{k \in \mathbb{N}^2} h_{k,n} \sum_{|R| = k} \sum_{(r,s,t) \in R^3} \bar{p}_R \sum_{j \in N^2} y_{j,r} y_{j,s} y_{j,t} \bar{p}_j \bar{R}_{\{r,s,t}\}},
$$

(34)

where $h_{k,n} = \frac{(n+k-2)(n-k+1)}{k(k-1)(k-2)(n-k)}$ for $k \in \mathbb{N} \setminus \frac{n}{2}$. If $n = 2$ the right-hand side of the equality in (34) is defined to be zero.

Proof. We have

$$
\bar{p}^{(2)} = \frac{1}{2} \sum_{(r,s) \in \mathbb{N}^2} \sum_{j=1}^N (z_j, r - \bar{z}_r)(z_j, s - \bar{z}_s) \bar{p}_n \{r,s\}.
$$

(35)

where, for $(r, s) \in \mathbb{N}^2$,

$$
\sum_{j=1}^N (z_j, r - \bar{z}_r)(z_j, s - \bar{z}_s) = \sum_{j \in N} z_{j,r} z_{j,s} - N \bar{z}_r \bar{z}_s = \frac{1}{2N} \sum_{j \in N^2} y_{j,1,2,r} y_{j,1,2,s}.
$$

(36)

Furthermore

$$
\frac{1}{2} = \sum_{k=2}^n \frac{k(k-1)}{kn(n-1)} = \sum_{k=2}^n \frac{(n-2)k}{n^2 k}.
$$

(37)

In view of (35), (36) and (37), we see that

$$
\frac{(N-2)! \bar{p}_n^{(2)}}{(N-n)!} = \frac{(N-2)! \bar{p}_n}{(N-n)!} \frac{1}{4N} \sum_{(r,s) \in \mathbb{N}^2} \sum_{j \in N^2} y_{j,1,2,r} y_{j,1,2,s} \bar{p}_n \{r,s\}
$$

$$
= \sum_{k=2}^n \frac{(n-2)k}{2N k} \sum_{(r,s) \in \mathbb{N}^2} \sum_{j \in N^2} y_{j,1,2,r} y_{j,1,2,s} \bar{p}_n \{r,s\}
$$

$$
= \sum_{k=2}^n \frac{1}{2N k} \sum_{|R| = k} \sum_{(r,s) \in R^2} \sum_{j \in N^2} y_{j,1,2,r} y_{j,1,2,s} \bar{p}_n \{r,s\}.
$$

(38)

Combining (25) and (38), we obtain

$$
\bar{p}_n - \frac{N! \bar{p}_n}{(N-n)!} + \frac{(N-2)! \bar{p}_n^{(2)}}{(N-n)!} = -\sum_{k=2}^n \frac{1}{2N k} \sum_{|R| = k} \sum_{(r,s) \in R^2} \bar{p}_R \sum_{j \in N^2} y_{j,1,2,r} y_{j,1,2,s} (\bar{p}_j \bar{R} \{r,s\} - \bar{p}_R \{r,s\}).
$$

(39)
In particular, we see that, for \( n = 2 \), (34) is true. From now on, let \( n \geq 3 \). For \( k \in \mathbb{n} \setminus \{1\} \), \( R \subseteq \mathbb{n} \) with \( |R| = k \), \((r, s) \in R^2 \) and \( j \in \mathbb{N}^n \), we have

\[
p_{j,R \setminus \{r,s\}} - \tilde{p}_{R \setminus \{r,s\}} = p_{j,R \setminus \{r,s\}} - \tilde{p}_R = p_{j,R \setminus \{r,s\}} - \tilde{p}_{R \setminus \{r,s\}}
\]

\[
= \sum_{\ell=1}^{n-2} \left( \frac{1}{\ell} \right) \sum_{L \subseteq \mathbb{R} \setminus \{r,s\}, |L| = \ell} \tilde{p}_{j,L \cup \{r,s\}} - \frac{1}{\ell} \sum_{L \subseteq \mathbb{R} \setminus \{r,s\}, |L| = \ell} \tilde{p}_{L \cup \{r,s\}}
\]

\[
= \sum_{\ell=1}^{n-2} \left( \frac{1}{\ell} \right) \sum_{L \subseteq \mathbb{R} \setminus \{r,s\}, |L| = \ell} \tilde{p}_{R \setminus \{r,s\}}(p_{j,L \cup \{r,s\}} - \tilde{p}_{j,R \setminus \{r,s\}}).
\]

(40)

For \( R \subseteq \mathbb{n} \) with \( |R| \geq 3 \), \((r, s, t) \in R^3 \), Lemma 3.2 implies that

\[
F(r, s, t, R) := \sum_{j \in \mathbb{N}^n} y_{j_r, j_s, j_t} y_{j_r, j_s, j_t} (p_{j,R \setminus \{r,s\}} - \tilde{p}_{j,R \setminus \{r,s\}})
\]

\[
= D_1(r, s, t, R \setminus \{r, s, t\}) - D_2(r, s, t, R \setminus \{r, s, t\}),
\]

(41)

where \( D_\nu(r, s, t, R \setminus \{r, s, t\}) \) for \( \nu \in \mathbb{Z}^+ \) are defined as in (30) and (31). Using (39) and (40), we get

\[
\bar{p}_n - \frac{N! \bar{p}_n}{(N-n)!} + \frac{(N-2)!}{(N-n)!} \tilde{p}^{(2)}
\]

\[
= - \sum_{k=2}^{n} \sum_{\ell=1}^{k-2} \frac{1}{2Nk((\ell)\ell)} \sum_{L \subseteq \mathbb{R} \setminus \{r,s\}, |L| = \ell} \tilde{p}_{L \cup \{r,s\}}(p_{j,L \cup \{r,s\}} - \tilde{p}_{j,R \setminus \{r,s\}})
\]

\[
= - \sum_{k=2}^{n} \sum_{\ell=1}^{k-2} \frac{(n-\ell-2)!}{2Nk((\ell)\ell)(n-k)!(k-\ell-2)!} \sum_{L \subseteq \mathbb{R} \setminus \{r,s\}, |L| = \ell} \tilde{p}_{L \cup \{r,s\}}(p_{j,L \cup \{r,s\}} - \tilde{p}_{j,R \setminus \{r,s\}})
\]

where, for \( \ell \in \mathbb{n} \),

\[
\sum_{k=\ell+2}^{n} \frac{(n-\ell-2)!}{k((\ell)\ell)(n-k)!(k-\ell-2)!} = \frac{(\ell-1)!}{n!} \sum_{k=\ell+2}^{n} (k-1)
\]

\[
= \frac{2}{(\ell-1)!}\frac{(n+\ell)(n+\ell-1)}{n!} = \frac{2}{(\ell+1)(\ell+2)(\ell+2)}.
\]

Hence

\[
\bar{p}_n - \frac{N! \bar{p}_n}{(N-n)!} + \frac{(N-2)!}{(N-n)!} \tilde{p}^{(2)} = - \frac{1}{4N} \sum_{k=3}^{n} h_{k,n} \sum_{L \subseteq \mathbb{R} \setminus \{r,s,t\}, |L| = k} \tilde{p}_{L \cup \{r,s\}}(p_{j,L \cup \{r,s\}} - \tilde{p}_{j,R \setminus \{r,s\}}).
\]

Using this in combination with (41), the assertion is shown.

As a corollary of Theorem 3.3, we give a second order expansion for the normalized elementary symmetric polynomials.
Corollary 3.4 Let \( n, N \in \mathbb{N} \) with \( 2 \leq n \leq N \), \( z_j \in \mathbb{Z} \) for \( j \in \mathbb{N} \) and \( \bar{z} = \frac{1}{N} \sum_{j=1}^{N} z_j \). For \( A \subseteq \mathbb{N} \) and \( k \in \mathbb{Z} \), let \( E_{A,k} = \sum_{\{j\in A\}} j \prod_{j \in j} z_j \). Then, we have

\[
\frac{1}{(N)!} E_{N,n} - \bar{z}^n + \frac{n(n-1)}{2N(N-1)} \sum_{j=1}^{N} (z_j - \bar{z})^2 \bar{z}^{n-2} = \frac{1}{2N^2} \sum_{(r,s,t) \in \mathbb{N}^3} (z_r - z_s)^2 (z_r - z_t) \sum_{k \in \mathbb{N} \setminus \{2\}} \tilde{h}_{k,n,N} \bar{z}^{n-k} E_{N,\{r,s,t\},k-3}
\]

\[
+ \frac{1}{8N^2} \sum_{(q,r,s,t) \in \mathbb{N}^4} (z_q - z_r)^2 (z_q - z_t)^2 \sum_{k \in \mathbb{N} \setminus \{2\}} \tilde{h}_{k,n,N} \bar{z}^{n-k} E_{N,\{q,r,s,t\},k-4},
\]

(42)

where \( \tilde{h}_{k,n,N} = \frac{(n+k-2)(n-k+1)}{k(k-1)(k-2)(n-k)} \) for \( k \in \mathbb{N} \setminus \{2\} \). If \( n = 2 \), the right-hand side of the equality in (42) is defined to be zero.

Proof. Similarly as in the proof of Corollary 3.3, Identity (42) follows from Theorem 3.3 in the case that \( Z \) has identical columns. Indeed, letting \( z_{j,1} = \cdots = z_{j,n} = z_j \) for all \( j \in \mathbb{N} \), then \( \bar{z}_1 = \cdots = \bar{z}_n = \bar{z} \) and

\[
\mathbf{p}_n = n! E_{N,n}, \quad \bar{p}_n = \bar{z}^n, \quad \bar{p}^{(2)} = \left( \frac{n}{2} \right) \sum_{j=1}^{N} (z_j - \bar{z})^2 \bar{z}^{n-2}.
\]

Therefore

\[
\frac{1}{(N)!} E_{N,n} - \bar{z}^n + \frac{n(n-1)}{2N(N-1)} \sum_{j=1}^{N} (z_j - \bar{z})^2 \bar{z}^{n-2} = \frac{(N-n)!}{N!} \left( \frac{N! \bar{p}_n}{(N-n)!} + \frac{(N-2)! \bar{p}^{(2)}}{(N-n)!} \right)
\]

\[
= \frac{(N-n)!}{N!} \left( \frac{M_1}{2N^2} + \frac{M_2}{8N^2} \right),
\]

(43)

where

\[
M_1 = \sum_{k \in \mathbb{N} \setminus \{2\}} \sum_{R \subseteq \mathbb{N} \setminus \{k\}} \left| R \right| = k \sum_{(r,s,t) \in \mathbb{R}_p} \tilde{p}_{\mathcal{G}(R)} \sum_{j \in \mathbb{N}_p^3} \prod_{y_{j,r,j,s} \in \mathcal{G}(R) \setminus \{r,s,t\}} y_{j,r,j,s} \sum_{j \in \mathbb{N}_p^3} y_{j,r,j,s} \sum_{j \in \mathbb{N}_p^3} y_{j,s,j,t} \sum_{j \in \mathbb{N}_p^3} y_{j,t,j,r}
\]

\[
M_2 = \sum_{k \in \mathbb{N} \setminus \{2\}} \sum_{R \subseteq \mathbb{N} \setminus \{k\}} \left| R \right| = k \sum_{(q,r,s,t) \in \mathbb{R}_p} \tilde{p}_{\mathcal{G}(R)} \sum_{j \in \mathbb{N}_p^3} \prod_{y_{j,q,j,r} \in \mathcal{G}(R) \setminus \{q,r,s,t\}} y_{j,q,j,r} \sum_{j \in \mathbb{N}_p^3} y_{j,q,j,r} \sum_{j \in \mathbb{N}_p^3} y_{j,r,j,s} \sum_{j \in \mathbb{N}_p^3} y_{j,s,j,t} \sum_{j \in \mathbb{N}_p^3} y_{j,t,j,r}
\]

Here

\[
M_1 = \sum_{k \in \mathbb{N} \setminus \{2\}} \sum_{R \subseteq \mathbb{N} \setminus \{k\}} \left| R \right| = k \sum_{(r,s,t) \in \mathbb{R}_p} \tilde{z}^{n-k} \sum_{j \in \mathbb{N}_p^3} (z_j - z_s)^2 (z_j - z_t) \prod_{\ell \in \mathbb{R}_p \setminus \{r,s,t\}} z_{j\ell}
\]

\[
= \sum_{(u,v,w) \in \mathbb{N}^3} (z_u - z_v)^2 (z_u - z_w) \sum_{k \in \mathbb{N} \setminus \{2\}} \sum_{R \subseteq \mathbb{N} \setminus \{k\}} \left| R \right| = k \sum_{j \in \mathbb{N}_p^3} \prod_{\ell \in \mathbb{R}_p \setminus \{r,s,t\}} z_{j\ell},
\]
where, for \((u, v, w) \in \mathcal{N}_\mathbb{P}^3\) and \(k \in \mathbb{N} \setminus 2\),

\[
\sum_{\substack{R \subseteq \mathbb{N} \\ |R| = k}} \sum_{(r, s, t) \in R^3} \sum_{j \in \mathcal{N}_\mathbb{P}^3} \prod_{j_k = u, j_s = v, j_t = w} z_{j_k} = \frac{N! \binom{n}{k}}{(N - n)! \binom{n}{k}} E_{N \setminus \{u,v,w\}, k - 3}
\]

and \(h_{k,n} \left( \frac{n}{k} \right) = \tilde{h}_{k,n,N}\). This implies that

\[
M_1 = \frac{N!}{(N - n)!} \sum_{(r, s, t) \in \mathcal{N}_\mathbb{P}^3} (z_r - z_s)^2(z_r - z_t) \sum_{k \in \mathbb{N} \setminus 2} \tilde{h}_{k,n,N} z_{n-k} E_{N \setminus \{r, s, t\}, k - 3}. \tag{44}
\]

Furthermore

\[
M_2 = \sum_{k \in \mathbb{N} \setminus 3} h_{k,n} \sum_{\substack{R \subseteq \mathbb{N} \\ |R| = k}} \sum_{(q, r, s, t) \in R^4} \sum_{j \in \mathcal{N}_\mathbb{P}^3} (z_{j_q} - z_{j_r}) \sum_{j \in \mathcal{N}_\mathbb{P}^3} (z_{j_s} - z_{j_t}) \prod_{\ell \in R \setminus \{q, r, s, t\}} z_{j_\ell}
\]

\[
\times \sum_{k \in \mathbb{N} \setminus 3} \tilde{h}_{k,n,N} z_{n-k} \sum_{\substack{R \subseteq \mathbb{N} \\ |R| = k}} \sum_{(q, r, s, t) \in R^4} \sum_{j \in \mathcal{N}_\mathbb{P}^3} \prod_{\ell \in R \setminus \{q, r, s, t\}} z_{j_\ell},
\]

where, for \((u, v, w, x) \in \mathcal{N}_\mathbb{P}^4\) and \(k \in \mathbb{N} \setminus 3\),

\[
\sum_{\substack{R \subseteq \mathbb{N} \\ |R| = k}} \sum_{(q, r, s, t) \in R^4} \sum_{j \in \mathcal{N}_\mathbb{P}^3} \prod_{\ell \in R \setminus \{q, r, s, t\}} z_{j_\ell} = \frac{N! \binom{n}{k}}{(N - n)! \binom{n}{k}} E_{N \setminus \{u,v,w,x\}, k - 4}.
\]

Hence

\[
M_2 = \frac{N!}{(N - n)!} \sum_{(q, r, s, t) \in \mathcal{N}_\mathbb{P}^4} (z_q - z_r)^2(z_s - z_t)^2 \sum_{k \in \mathbb{N} \setminus 3} \tilde{h}_{k,n,N} z_{n-k} E_{N \setminus \{q,r,s,t\}, k - 4}. \tag{45}
\]

Combining (43)–(45), the assertion is shown. \qed

4 Approximation of normalized permanents

In this section, we employ the notation of Section 2 with \(Z := \mathbb{C}\). It should be mentioned that, unless stated otherwise, we do not assume that the numbers \(|z_{j,r}|\) for \(j \in \mathbb{N}, r \in \mathbb{N}\) are bounded by one.

4.1 Main approximation results

The first results in this section are Theorems 4.1 and 4.2 below, the proof of which require the following lemma.
Lemma 4.1 (a) (Hadamard type permanent inequality) Without any further restrictions, we have

\[ |\text{Per}(Z)| \leq \frac{N!}{(N-n)!} \prod_{r=1}^{n} \left( \frac{1}{N} \sum_{j=1}^{N} |z_{j,r}|^2 \right)^{1/2}. \]

(b) (Brégman-Minc permanent inequality) If \( Z \in \{0,1\}^{N \times n} \), then

\[ \text{Per}(Z) \leq \frac{N!}{(N-n)!} \prod_{r=1}^{n} \zeta(N \tilde{z}_r) \left( \frac{N!}{N} \right)^{1/N}, \]

where \( \zeta(k) = (k!)^{1/k} \) for \( k \in \mathbb{N} \) and \( \zeta(0) = 0 \).

Proof. Let us first consider the case \( n = N \). For (a), see Carlen et al. [9, Theorem 1.1] or Cobos et al. [10, Theorem 5.1]. As stated in [9, Introduction], this can also be obtained from Theorem 9.1.1 in Appendix 1 of Nesterov and Nemirovskii [21]. Part (b) was conjectured by Minc [19] and proved by Brégman [8]. The general case of \( n \in \mathbb{N} \) follows from the above and the simple observation that \( \text{Per}(Z) = \text{Per}(Z') \frac{(N-n)!}{N!} \), where \( Z' = (z'_{j,r}) \in \mathbb{C}^{N \times N} \) with \( z'_{j,r} = z_{j,r} \) for \( j \in \mathbb{N}, r \in n \) and \( z'_{j,r} = 1 \) for \( j \in \mathbb{N}, r \in N \setminus n \). \[ \square \]

We note that in [22, Lemma 2.2], a Hadamard type inequality for the permanent of a matrix with zero column sums was shown, which is uniformly better than the general bound in Lemma 4.1(a).

The next theorem contains an improvement of the inequalities (2)–(4) from the introduction.

Theorem 4.1 Let us assume that \( 2 \leq n \leq N \) and set

\[ \beta = \frac{1}{n} \sum_{r=1}^{n} |\tilde{z}_r|^2, \quad \vartheta = \frac{1}{N(N-1)\sqrt{n(n-1)}} \left( \sum_{(r,s) \in \mathbb{N}^2} \left( \sum_{(u,v) \in \mathbb{N}^2} |y_{u,v,r,y_{u,v,s}}|^2 \right)^2 \right)^{1/2}, \]

\[ \kappa = \begin{cases} \frac{1}{(n-2)(N-2)} \sum_{(u,v) \in \mathbb{N}^2} \sum_{(r,s) \in \mathbb{N}^2} |z_{j,l}|^2, & \text{if } n \geq 3, \\ 1, & \text{if } n = 2, \end{cases} \]

\[ f_n(x_1, x_2) = \sum_{k=2}^{n} \frac{(k-1)x_1^{n-k}x_2^{k-2}}{(x_1-x_2)^2} = \frac{(n-1)x_1^n - nx_1x_2^{n-1} + x_2^n}{(x_2-x_1)^2}, \quad (x_1, x_2 \in \mathbb{R}). \]

Then

\[ \left| \frac{(N-n)!}{N!} \tilde{p}_n - \tilde{p}_n \right| \leq \frac{\vartheta}{2N} f_n(\sqrt{\beta}, \sqrt{\kappa}). \tag{46} \]

Proof. In the case \( n = 2 \), (26) gives

\[ \frac{(N-n)!}{N!} \tilde{p}_n - \tilde{p}_n = -\frac{1}{2N^2(N-1)} \sum_{(u,v) \in \mathbb{N}^2} y_{u,v,1}y_{u,v,2}, \tag{47} \]
which together with the identities $\vartheta = \frac{1}{N(N-1)} \sum_{(u,v) \in N_2^n} |y_{u,v}|^2 |y_{u,n,2}|$ and $f_n(\sqrt{\beta}, \sqrt{\kappa}) = 1$ implies (46). Let us now assume that $3 \leq n \leq N$. From (25), we get
\[
\left| \frac{(N-n)!}{N!} \tilde{p}_n - \tilde{p}_n \right| \leq \frac{1}{N!} \sum_{k=2}^n \frac{(N-n)!}{2Nk(n)_k} \left| \sum_{R \subseteq_n |R|=k} \sum_{(r,s) \in R^2} \sum_{j \in N_n^R} |y_{ji,j,s}p_{j,R \{r,s\}} \tilde{p}_n R| \right|
\]
\[
= \frac{1}{N!} \sum_{k=2}^n \frac{(N-k)!}{2Nk(n)_k} \left| \sum_{R \subseteq_n |R|=k} \sum_{(r,s) \in R^2} \sum_{(u,v) \in N_n^2} |y_{u,v,r}y_{u,v,s} R| \left| \sum_{j \in (N \{u,v\})^R \{r,s\}} |p_{j,R \{r,s\}}| \right| \right|. \tag{48}
\]
Lemma 4.1(a) implies that, for $R \subseteq_n |R|=2$, $(r,s) \in R^2$, $(u,v) \in N_n^2$, and $j \in (N \{u,v\})^R \{r,s\}$, we obtain
\[
\sum_{j \in (N \{u,v\})^R \{r,s\}} |p_{j,R \{r,s\}}| \leq \frac{(N-2)!}{(N-k)!} \prod_{\ell \in R \{r,s\}} \kappa_{u,v,\ell}^{1/2}
\]
where $\kappa_{u,v,\ell} = \frac{1}{N-2} \sum_{j \in (N \{u,v\})^R \{r,s\}} |z_{j,\ell}|^2$ for $(u,v) \in N_n^2$ and $\ell \in n$. For $k \in n \setminus 1$, the Cauchy-Schwarz inequality gives
\[
\sum_{R \subseteq_n |R|=k} \sum_{(r,s) \in R^2} \sum_{(u,v) \in N_n^2} |\tilde{p}_n R| \left| y_{u,v,r}y_{u,v,s} \prod_{\ell \in R \{r,s\}} \kappa_{u,v,\ell}^{1/2} \right|^2 \leq \left( \sum_{R \subseteq_n |R|=k} \sum_{(r,s) \in R^2} |\tilde{p}_n R|^2 \right) \left( \sum_{R \subseteq_n |R|=k} \sum_{(u,v) \in N_n^2} \left( \sum_{(u,v) \in N_n^2} \left| y_{u,v,r}y_{u,v,s} \prod_{\ell \in R \{r,s\}} \kappa_{u,v,\ell}^{1/2} \right|^2 \right) \right)^{1/2}
\]
Using Maclaurin’s inequality (see Hardy et al. [16, Theorem 52, page 52]), we obtain
\[
\sum_{R \subseteq_n |R|=k} \sum_{(r,s) \in R^2} |\tilde{p}_n R|^2 = k(k-1) \sum_{R \subseteq_n |R|=n-k} |\tilde{p}_R|^2 \leq k(k-1) \binom{n}{n-k} \beta^{n-k}.
\]
Furthermore, for $(r,s) \in n^2$, we have
\[
\sum_{R \subseteq_n |R|=k} \left( \sum_{(u,v) \in N_n^2} \left| y_{u,v,r}y_{u,v,s} \prod_{\ell \in R \{r,s\}} \kappa_{u,v,\ell}^{1/2} \right|^2 \right)^{1/2}
\]
\[
= \sum_{(u,v) \in N_n^2} \sum_{(u,v') \in N_n^2} \left| y_{u,v,r}y_{u,v,s}y_{u',v',r}y_{u',v',s} \sum_{R \subseteq_n |R|=k-2} \prod_{\ell \in R \{r,s\}} (\kappa_{u,v,\ell} \kappa_{u',v',\ell})^{1/2} \right|^2
\]
where, for $(u,v), (u',v') \in N_n^2$, the Cauchy-Schwarz inequality gives
\[
\sum_{R \subseteq_n |R|=k-2} \prod_{\ell \in R \{r,s\}} (\kappa_{u,v,\ell} \kappa_{u',v',\ell})^{1/2} \leq \left( \sum_{R \subseteq_n |R|=k-2} \prod_{\ell \in R \{r,s\}} \kappa_{u,v,\ell} \right)^{1/2} \left( \sum_{R \subseteq_n |R|=k-2} \prod_{\ell \in R \{r,s\}} \kappa_{u',v',\ell} \right)^{1/2}
\]
and Maclaurin’s inequality implies that
\[
\sum_{R \subseteq_n |R|=k-2} \prod_{\ell \in R \{r,s\}} \kappa_{u,v,\ell} \leq \binom{n-2}{k-2} \left( \frac{1}{n-2} \sum_{\ell \in n \setminus \{r,s\}} \kappa_{u,v,\ell} \right)^{k-2} \leq \binom{n-2}{k-2} \kappa^{k-2}.
\]
Hence
\[
\sum_{(r,s) \in \mathbb{Z}^{2}} \sum_{R \subseteq \mathbb{Z}^{2} \setminus \{r,s\}, |R| = k-2} \left( \sum_{(u,v) \in \mathbb{Z}^{2}} |y_{u,v,r}y_{u,v,s}| \prod_{\ell \in R} \kappa_{u,v,\ell}^{1/2} \right)^{2} \leq \left( \frac{n - 2}{k - 2} \right)^{k-2} n(n - 1)(N(N - 1)\vartheta)^{2}.
\]
Combining the inequalities above, we obtain
\[
\left| \frac{(N - n)!}{N!} \mathbf{p}_{n} - \tilde{\mathbf{p}}_{n} \right| \leq \frac{1}{N!} \sum_{k=2}^{n} \frac{(N - 2)!}{2Nk^{2}} \left( \sum_{R \subseteq \mathbb{Z}^{2} \setminus \{r,s\}, |R| = k-2} \left( \sum_{(u,v) \in \mathbb{Z}^{2}} |y_{u,v,r}y_{u,v,s}| \prod_{\ell \in R} \kappa_{u,v,\ell}^{1/2} \right)^{2} \right)^{1/2}
\times \left( \sum_{(r,s) \in \mathbb{Z}^{2}} \sum_{R \subseteq \mathbb{Z}^{2} \setminus \{r,s\}, |R| = k-2} \left( \sum_{(u,v) \in \mathbb{Z}^{2}} |y_{u,v,r}y_{u,v,s}| \prod_{\ell \in R} \kappa_{u,v,\ell}^{1/2} \right)^{2} \right)^{1/2}
\leq \frac{\vartheta}{2N} \sum_{k=2}^{n} (k - 1)\beta^{(n-k)/2} \kappa^{(k-2)/2}
= \frac{\vartheta}{2N} f_{n}(\sqrt{\beta}, \sqrt{\kappa}),
\]
which implies the assertion. \[\Box\]

The proof of Theorem 4.1 requires only Part (a) of Lemma 4.1. But if \(Z\) is a \((0, 1)\)-matrix, then Part (b) can also be applied, as is shown in the following theorem.

**Theorem 4.2** Let the assumptions of Theorem 4.1 be valid, where we assume that \(Z \in \{0, 1\}^{N \times n}\). Further, let \(\zeta\) be as in Lemma 4.1(b), \(\eta_{u,v,\ell} = \sum_{j \in \mathbb{N} \setminus \{u,v\}} z_{j,\ell}, \tilde{\kappa}_{u,v,\ell} = \frac{(\eta_{u,v,\ell})^{2}}{(N - 2)!\beta^{2}(N - 2)}\) for \((u, v) \in \mathbb{Z}^{2}, \ell \in n\) and set
\[
\tilde{\kappa} = \begin{cases} \frac{1}{n - 2} \sum_{(r,s) \in \mathbb{Z}^{2} \setminus \{r,s\}, (u,v) \in \mathbb{Z}^{2}} \tilde{\kappa}_{u,v,\ell}, & \text{if } n \geq 3, \\ 1, & \text{if } n = 2. \end{cases}
\]
Then
\[
\left| \frac{(N - n)!}{N!} \mathbf{p}_{n} - \tilde{\mathbf{p}}_{n} \right| \leq \frac{\vartheta}{2N} f_{n}(\sqrt{\beta}, \sqrt{\tilde{\kappa}}).
\](49)

**Proof.** If \(n = 2\), the assertion directly follows from Theorem 4.1. In the case \(3 \leq n \leq N\), the proof is very similar to the one of Theorem 4.1. Indeed, we use (48) in combination with Lemma 4.1(b), which implies that, for \(R \subseteq \mathbb{Z}^{2}\) with \(|R| = k \geq 2\), \((r, s) \in \mathbb{Z}^{2}, (u, v) \in \mathbb{Z}^{2}\),
\[
\sum_{j \in \mathbb{N} \setminus \{u,v\}} |\mathbf{p}_{j,R \setminus \{r,s\}}| \leq \frac{(N - 2)!}{(N - k)!} \prod_{\ell \in R \setminus \{r,s\}} \tilde{\kappa}_{u,v,\ell}^{1/2},
\]
where \(\frac{1}{n-2} \sum_{\ell \in \mathbb{Z}^{2} \setminus \{r,s\}} \tilde{\kappa}_{u,v,\ell} \leq \tilde{\kappa}.\) \[\Box\]

The right-hand sides of (46) and (49) can be further estimated by using the following lemma. The inequalities given there can be used for the comparison with the bounds given in the introduction.
Lemma 4.2 Let us assume that $2 \leq n \leq N$. Let $\beta$, $\vartheta$, $f_n$ be as in Theorem 4.1 and set $\alpha = \frac{1}{nN} \sum_{r \in \mathbb{N}} \sum_{j \in \mathbb{N}} |z_{j,r} - \tilde{z}_r|^2$. Then

$$\alpha = \frac{1}{nN} \sum_{j=1}^{N} \sum_{r=1}^{n} |z_{j,r}|^2 - \beta, \quad (50)$$

and

$$\vartheta \leq \frac{2N\alpha}{N-1}, \quad (51)$$

$$f_n(x,1) \leq (n-1) \frac{1-x^{n/2}}{1-x} \leq (n-1) \min \left\{ \frac{n}{2}, \frac{1}{1-x} \right\}, \quad (x \in [0,1]). \quad (52)$$

**Proof.** As already mentioned in [22, Remark 2.9], (50) is true. A repeated application of the Cauchy-Schwarz inequality yields

$$\alpha = \frac{1}{2nN^2} \sum_{r \in \mathbb{N}} \sum_{(u,v) \in \mathbb{N}_2^2} |y_{u,v,r}|^2 \leq \frac{1}{2N^2} \sqrt{n} \left( \sum_{r \in \mathbb{N}} \left( \sum_{(u,v) \in \mathbb{N}_2^2} |y_{u,v,r}|^2 \right)^2 \right)^{1/2}$$

and

$$N^2(N-1)^2 n(n-1) \vartheta^2 \leq \sum_{(r,s) \in \mathbb{N}_2^2} \left( \sum_{(u,v) \in \mathbb{N}_2^2} |y_{u,v,r}|^2 \right) \sum_{(u,v) \in \mathbb{N}_2^2} |y_{u,v,s}|^2$$

$$= \left( \sum_{r \in \mathbb{N}} \sum_{(u,v) \in \mathbb{N}_2^2} |y_{u,v,r}|^2 \right)^2 - \sum_{r \in \mathbb{N}} \left( \sum_{(u,v) \in \mathbb{N}_2^2} |y_{u,v,r}|^2 \right)^2$$

$$\leq 4N^4 n^2 \alpha^2 - 4N^4 n \alpha^2 = 4N^4 n(n-1) \alpha^2,$$

giving (51). Finally, (52) follows from the Jensen inequality. Indeed, since $x \in [0,1]$,

$$f_n(x,1) = \frac{n-1 - nx + x^n}{(1-x)^2} = \frac{n-1}{1-x} \left( 1 - \frac{1}{n-1} \sum_{m=1}^{n-1} x^m \right) \leq (n-1) \frac{1-x^{n/2}}{1-x},$$

where, for $n \geq 2$, $\frac{1-x^{n/2}}{1-x}$ is increasing in $x \in [0,1]$, so that $\frac{1-x^{n/2}}{1-x} = \sum_{m=0}^{n-1} x^{m/2} \leq \frac{n}{2} \leq \frac{1}{2}$. \(\square\)

**Corollary 4.1** If $2 \leq n \leq N$, $|z_{j,r}| \leq 1$ for all $j \in \mathbb{N}$, $r \in \mathbb{N}$ and $\alpha$, $\beta$, $\vartheta$ are as in Theorem 4.1 and Lemma 4.2, respectively, then

$$\left| \frac{(N-n)!}{N!} \tilde{p}_n - \tilde{p}_m \right| \leq \frac{n-1}{2N^{\beta}} \frac{1-\beta^{n/4}}{1-\sqrt{\beta}} \quad (53)$$

$$\leq \frac{n-1}{N-1} \min \left\{ \frac{n}{2}, \frac{1}{1-\sqrt{\beta}} \right\} \quad (54)$$

$$\leq (1+\sqrt{\beta}) \frac{n-1}{N-1}. \quad (55)$$

**Proof.** This can easily be derived from Theorem 4.1, Lemma 4.2, and the trivial fact that $\kappa \leq 1$, i.e. $f_n(\sqrt{\beta}, \sqrt{\kappa}) \leq f_n(\sqrt{\beta}, 1)$. \(\square\)
Remark 4.1 Inequality (54) implies that, in (6), the right-hand side can be replaced with the expression \((1 + \sqrt{\beta})\gamma(\frac{1}{2})\).

We now discuss the sharpness of some of the inequalities above.

Remark 4.2 Let us assume that \(2 = n \leq N\) and that \(Z = (z_{j,r}) \in \mathbb{R}^{N \times 2}\). Then, in (52), equality holds. Below, we will additionally assume the validity of some of the following conditions:

\[
z_{j,1} = z_{j,2} \text{ for all } j \in \mathbb{N}, \quad (56)
\]
\[
Z \text{ has decreasing columns.} \quad (57)
\]

(a) If (56) is satisfied, then, in (51), equality holds.

(b) The right-hand side of (46) is equal to \(\frac{\varrho}{2N} = \frac{1}{2N(N-1)} \sum (u,v) \in \mathbb{N}^2 |y_{u,v,1}y_{u,v,2}|\). In view of (47), we see that, if one of the conditions (56) or (57) is satisfied, then \(y_{u,v,1}y_{u,v,2} \geq 0\) such that, in (46), equality holds.

(c) Let us assume that \(Z \in [-1,1]^{N \times 2}\). If one of the conditions (56) or (57) is satisfied, then, in (53), equality holds. If (56) holds, then, in (54), equality holds. If additionally \(\alpha = 1\) and \(\beta = 0\), then, in (55), equality holds.

Corollary 4.2 Let \(n, N \in \mathbb{N}\) with \(2 \leq n \leq N\), \(z_j \in \mathbb{C}\) with \(|z_j| \leq 1\) for all \(j \in \mathbb{N}\). Set \(\tilde{z} = \frac{1}{N} \sum_{j=1}^{N} z_j\), \(\kappa = \frac{1}{N-2} \max_{(u,v) \in \mathbb{N}^2} \sum_{j \in \mathbb{N}\setminus\{u,v\}} |z_j|^2\) if \(n \geq 3\), and \(\kappa = 1\) otherwise. Let \(E_{N,n} = \sum_{|J|=n} \prod_{j \in J} z_j\). Then

\[
\left| \frac{1}{N!} E_{N,n} - \tilde{z}^n \right| \leq \frac{f_n(|\tilde{z}|, \sqrt{\kappa})}{N(N-1)} \sum_{j=1}^{N} |z_j - \tilde{z}|^2 \quad (58)
\]
\[
\leq \frac{n(n-1)}{N(N-1)} \sum_{j=1}^{N} |z_j - \tilde{z}|^2 \min \left\{ \frac{1}{2}, \frac{1}{n(1-|\tilde{z}|)} \right\}. \quad (59)
\]

Proof. Let us consider the matrix \(Z\) with \(z_{j,r} = z_j\) for all \(j \in \mathbb{N}\) and \(r \in \mathbb{N}\), i.e. \(Z\) has identical columns. Using the notation in Theorem 4.1, we obtain

\[
\frac{1}{(n)} E_{N,n} = \frac{(N-n)!}{N!} p_{\tilde{u}} = \tilde{z}^n = p_{\tilde{u}}, \quad \beta = |\tilde{z}|^2,
\]
\[
\frac{(N-1)!}{2} = \frac{1}{2N} \sum_{(u,v) \in \mathbb{N}^2} |z_u - z_v|^2 = \sum_{j=1}^{N} |z_j - \tilde{z}|^2
\]

and hence

\[
\left| \frac{1}{(n)} E_{N,n} - \tilde{z}^n \right| = \left| \frac{(N-n)!}{N!} p_{\tilde{u}} - \tilde{z}^n \right| \leq \frac{\varrho}{2N} f_n(|\tilde{z}|, \sqrt{\kappa}) = \frac{f_n(|\tilde{z}|, \sqrt{\kappa})}{N(N-1)} \sum_{j=1}^{N} |z_j - \tilde{z}|^2,
\]
which proves (58). Inequality (59) follows with the help of (52).

It should be mentioned that, in the case $2 \leq n = N$, Corollary 4.2 gives bounds for the Euclidean distance between the product $\prod_{j=1}^{n} z_j$ and $(\frac{1}{n} \sum_{j=1}^{n} z_j)^n$.

Let us now discuss the benefit of the bounds (46) and (49) in the next example.

**Example 4.1** Let $2 \leq n = N$ and let the notation of Theorem 4.1 be valid.

(a) We consider the case of $Z \in \{0, 1\}^{n\times n}$, where $\sum_{\ell=1}^{n} z_{j,\ell} = \sum_{k=1}^{n} z_{k,r} = n\tilde{z}_1$ for all $j, r \in \mathbb{N}$, i.e. the row and column sums of $Z$ are identical. In particular, $\beta = \tilde{z}_1^2$. For $(u, v) \in \mathbb{N}_n^2$ and $r \in \mathbb{N}_n$ we have $\tilde{z}_{u,r} = z_{u,r}$ and and $y_{u,v,r}^2 = |y_{u,v,r}| \in \{0, 1\}$. Hence

$$(n(n - 1))^3 \vartheta^2 = \sum_{(r,s) \in \mathbb{N}_n^2} \left( \sum_{(u,v) \in \mathbb{N}_n^2} y_{u,v,r}^2 y_{u,v,s}^2 \right)^2 = \sum_{(r,s) \in \mathbb{N}_n^2} \left( \sum_{(u,v) \in \mathbb{N}_n^2} (z_{u,r} - 2z_{u,r}z_{v,r} + z_{v,r}) \right)^2 = \sum_{(r,s) \in \mathbb{N}_n^2} \left( \sum_{(u,v) \in \mathbb{N}_n^2} (z_{u,r} - 2z_{u,r}z_{v,r} + z_{v,r}) \right)^2 = \sum_{(r,s) \in \mathbb{N}_n^2} (n(2 - 8\tilde{z}_1)) \sum_{u \in \mathbb{N}_n} z_{u,r} z_{u,s} + 4 \left( \sum_{u \in \mathbb{N}_n} z_{u,r} z_{u,s} \right)^2 + 2n^2 \tilde{z}_1^2) \right)^2,$$

from which a formula for $\vartheta$ can be derived. Furthermore, for $(u, v), (r, s) \in \mathbb{N}_n^2$, we have

$$\sum_{j \in \mathbb{N} \setminus \{u,v\}} \sum_{\ell \in \mathbb{N} \setminus \{r,s\}} |z_{j,\ell}|^2 = \sum_{j \in \mathbb{N} \setminus \{u,v\}} \left( \sum_{\ell \in \mathbb{N} \setminus \{u,v\}} z_{j,\ell} - z_{j,r} - z_{j,s} \right) = \sum_{j \in \mathbb{N} \setminus \{u,v\}} \left( \sum_{\ell \in \mathbb{N} \setminus \{u,v\}} z_{j,\ell} - z_{j,r} - z_{j,s} \right) = \left( \sum_{u \in \mathbb{N}_n} \tilde{z}_1 - 4n\tilde{z}_1 + z_{u,r} + z_{u,s} + z_{v,r} + z_{v,s} \leq n^2 \tilde{z}_1 - 4n\tilde{z}_1 + 4 = (n - 2)^2 \tilde{z}_1 + 4(1 - \tilde{z}_1) \right)^2,$$

such that

$$\kappa \leq \min \left\{ 1, \tilde{z}_1 + \frac{4(1 - \tilde{z}_1)}{(n - 2)^2} \right\}.$$

The calculations given here will be used in the subsequent parts of this example.

(b) (Derangement numbers) Let us now assume that $Z = J - I_n \in \{0, 1\}^{n\times n}$, where $J$ denotes the matrix all of whose entries are 1 and $I_n$ is the identity matrix. Then $\overline{\text{Per}}(Z)$ is the $n$th derangement number, i.e. the number of permutations in $\mathbb{N}_n^2$ without fixed points. It satisfies

$$\text{Per}(Z) = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!}.$$
and can also be interpreted as the number of ways a dance can be arranged for \( n \) married couples, so that no one dances with his or her partner, e.g. see Minc [20, page 44]. Under the present assumptions, we have \( \bar{z}_1 = \frac{n-1}{n} \), \( \beta = \bar{z}_1^2 = \left( \frac{n-1}{n} \right)^2 \), and \( \sum_{u \in \mathbb{N}} z_{u,r} z_{u,s} = n - 2 \) for all \( (r, s) \in n^2 \). Therefore (a) gives \( \vartheta = \frac{2}{n(n-1)} \) and \( \kappa \leq \min \{ 1, \frac{n-1}{n} + \frac{4}{n(n-2)} \} \).

Theorems 4.1, 4.2 and (52) imply that
\[
\left| \frac{\overline{P}_n}{n!} - \overline{p}_n \right| \leq \frac{f_n(\sqrt{\beta}, \sqrt{\min \{ \kappa, \tilde{\kappa} \}})}{n^2(n-1)} \leq \frac{1}{2n}.
\]

In particular, \( \left| \frac{\overline{P}_n}{n!} - \overline{p}_n \right| = O\left( \frac{1}{n^2} \right) \) as \( n \to \infty \). Here, using that \( (k!)^{1/k} \) is increasing in \( k \in \mathbb{N} \), it is easily shown that, for \( n \geq 4 \), \( \tilde{\kappa} \) from Theorem 4.2 satisfies
\[
\tilde{\kappa} = \frac{1}{n-2} \left( (n-4) \frac{(n-3)!}{(n-2)!} \right)^2 (n-2) + 2.
\]

We note that (54) only gives the bad bound \( \left| \frac{\overline{P}_n}{n!} - \overline{p}_n \right| \leq \frac{n-1}{2n} \). Furthermore, the upper bounds in (3), (4) and (5) from the introduction cannot be small, since \( \gamma = \frac{n-1}{2(n-1)} \), where \( \gamma \) is defined there.

(c) (Ménage numbers) We now consider the matrix \( Z = J - I_n - P \in \{0, 1\}^{n \times n} \), where \( n \geq 3 \) and \( P \) is the matrix with 1’s in positions \( (1, 2), (2, 3), \ldots, (n-1, n), (n, 1) \) and 0’s otherwise. Then \( \overline{P}_n = \text{Per}(Z) \) is the \( n \)th ménage number, which can be described as the number of ways of seating a set of married couples at a circular table so that men and women alternate and nobody sits next to his or her partner, see Minc [20, page 44]. The following explicit formula is due to Touchard [25]:
\[
\text{Per}(Z) = \sum_{j=0}^{n} (-1)^j \frac{2n}{2n-j} \binom{2n-j}{j} (n-j) !.
\]

In this situation, we have \( \bar{z}_1 = \frac{n-2}{n} \), \( \beta = \bar{z}_1^2 = \left( \frac{n-2}{n} \right)^2 \), and, for \( (r, s) \in n^2 \),
\[
\sum_{u \in \mathbb{N}} z_{u,r} z_{u,s} = \begin{cases} 
    n - 3, & \text{if } |\{ (r, s), (s, r) \} \cap \{ (1, 2), \ldots, (n-1, n), (n, 1) \}| = 1, \\
    n - 4, & \text{otherwise}.
\end{cases}
\]

Therefore (a) gives \( \vartheta = \frac{\sqrt{8(n^2+4n-20)}}{n(n-1)^{3/2}} \) and \( \kappa \leq \min \{ 1, \frac{n-2}{n} + \frac{8}{n(n-2)^2} \} \). Theorems 4.1, 4.2 and (52) imply that
\[
\left| \frac{\overline{P}_n}{n!} - \overline{p}_n \right| \leq \frac{\sqrt{2(n^2+4n-20)}}{n^2(n-1)^{3/2}} f_n(\sqrt{\beta}, \sqrt{\min \{ \kappa, \tilde{\kappa} \}}) \leq \frac{\sqrt{n^2+4n-20}}{2(n-1)^{3/2}},
\]
i.e. \( \left| \frac{\overline{P}_n}{n!} - \overline{p}_n \right| = O\left( \frac{1}{\sqrt{n}} \right) \), as \( n \to \infty \). As in Part (b), (54) only gives a bad bound, namely \( \left| \frac{\overline{P}_n}{n!} - \overline{p}_n \right| \leq \frac{n-2}{n} \). Again, the bounds in (3), (4) and (5) from the introduction cannot be small, since \( \gamma = \frac{n-2}{2(n-1)} \). We finally note that, for \( n \geq 5 \), \( \tilde{\kappa} \) can easily be evaluated as
\[
\tilde{\kappa} = \frac{1}{n-2} \left( (n-5) \frac{(n-4)!}{(n-2)!} \right)^{2/(n-2)} + 2 \left( \frac{(n-3)!}{(n-2)!} \right)^{2/(n-2)} + 1.
\]
4.2 Second order approximation

The next theorem contains an improvement of inequality (5) from the introduction.

**Theorem 4.3** Let $2 \leq n \leq N$, $\tilde{p}^{(2)} = \sum_{R \subseteq \mathbb{N}, |R|=2} \tilde{p}_R \sum_{j=1}^{N} \prod_{r \in R}(z_{j,r} - \bar{z}_r)$ and $\beta = \frac{1}{n} \sum_{r=1}^{n} |\bar{z}_r|^2$. Let

\[
\vartheta_3 = \frac{(N-3)!}{N!} \sqrt{\frac{(n-3)!}{n!}} \left( \sum_{(r,s,t) \in \mathbb{N}_n^3} \left( \sum_{(u,v,w) \in \mathbb{N}_n^3} |y_{u,v,r}y_{u,v,s}y_{u,w,t}| \right)^2 \right)^{1/2}, \quad \text{if } n \geq 3,
\]
\[
\vartheta_4 = \frac{(N-4)!}{N!} \sqrt{\frac{(n-4)!}{n!}} \left( \sum_{(q,r,s,t) \in \mathbb{N}_n^3} \left( \sum_{(u,v,w,x) \in \mathbb{N}_n^3} |y_{u,v,q}y_{u,v,r}y_{u,w,x}y_{u,w,x}| \right)^2 \right)^{1/2}, \quad \text{if } n \geq 4.
\]

If $n = 2$, then $\vartheta_3 = 0$; further, if $n \in \{2, 3\}$, then $\vartheta_4 = 0$. Let

\[\kappa^{(\nu)} = \begin{cases} \frac{1}{(n-\nu)(N-\nu)} \max_{J \subseteq \mathbb{N}, |J|=|R|=\nu} \sum_{j \in J} \sum_{r \in \mathbb{N}_n^3} |z_{j,r}|^2, & \text{if } n \geq \nu + 1, \\
1, & \text{if } n \leq \nu, \end{cases} \quad (\nu \in \{3, 4\})\]

then

\[f_n(x_1, x_2) = \sum_{k \in \mathbb{N}_n^3} (n + k - 2)(n - k + 1)x_1^{n-k}x_2^{k-3},
\]
\[g_n(x_1, x_2) = \sum_{k \in \mathbb{N}_n^3} (k - 3)(n + k - 2)(n - k + 1)x_1^{n-k}x_2^{k-4},
\]

for $x_1, x_2 \in \mathbb{R}$. Then

\[\left| \frac{(N-n)!}{N!} \tilde{p}_R - \tilde{p}_R + \frac{\tilde{p}^{(2)}}{N(N-1)} \right| \leq \frac{\vartheta_3}{2N^2} f_n(\sqrt{\beta}, \sqrt{\kappa^{(3)}}) + \frac{\vartheta_4}{8N^2} g_n(\sqrt{\beta}, \sqrt{\kappa^{(4)}}). \tag{60}
\]

**Proof.** Theorem 3.3 implies that

\[\left| \frac{(N-n)!}{N!} \tilde{p}_R - \tilde{p}_R + \frac{\tilde{p}^{(2)}}{N(N-1)} \right| \leq \frac{M_1}{2N^2} + \frac{M_2}{8N^2},
\]

where

\[M_1 = \frac{(N-n)!}{N!} \sum_{k \in \mathbb{N}_n^3} h_{k,R} \sum_{(r,s,t) \in \mathbb{N}_n^3} \left| \tilde{p}_{R\setminus k} \right| \sum_{j \in \mathbb{N}_n^3} \left| y_{j,r,s}y_{j,r,t} \right|,
\]
\[M_2 = \frac{(N-n)!}{N!} \sum_{k \in \mathbb{N}_n^3} h_{k,R} \sum_{(q,r,s,t) \in \mathbb{N}_n^3} \left| \tilde{p}_{R\setminus k} \right| \sum_{j \in \mathbb{N}_n^3} \left| y_{j,s,r,q}y_{j,s,t} \right|.
\]

Similarly as in the proof of Theorem 4.1, one can apply Lemma 4.1(a), the Cauchy-Schwarz inequality in combination with Maclaurin’s inequality to show that $M_1 \leq \vartheta_3 f_n(\sqrt{\beta}, \sqrt{\kappa^{(3)}})$ and $M_2 \leq \vartheta_4 g_n(\sqrt{\beta}, \sqrt{\kappa^{(4)}}).$ \hfill \Box

A theorem for $(0,1)$-matrices $Z$ similar to the above can be proved with the help of Lemma 4.1(b). Further, the auxiliary inequalities contained in the next lemma can be used...
in combination with (51) to prove upper bounds of the right-hand side of (60). For instance, it is possible to give an estimate, which is of the same order as the right-hand side of (5), if \( \gamma \) is bounded away from 1. Since (51) and (61) are based on the Cauchy-Schwarz inequality, the form of (60) is better than that of (5). We omit the details here.

**Lemma 4.3** Let us assume that \( 2 \leq n \leq N \). Let \( \beta, f_n \) and \( g_n \) be as in Theorem 4.3. Let \( \vartheta \) as in Theorem 4.1. Then, for \( x \in [0, 1] \),

\[
\vartheta_3 \leq \frac{(N - 2)!}{N!} \left( \frac{(n - 3)!}{n!} \right)^{3/2} \left( \sum_{r \in \mathbb{N}_x} |y_{u,v,r}|^2 \right)^{3/2},
\]

(61)

\[
\vartheta_4 \leq \sqrt{\frac{n(n - 1)}{(n - 2)(n - 3)(N - 2)(N - 3)}} \vartheta^2,
\]

(62)

\[
f_n(x, 1) \leq 2(n - 1) \min \left\{ \frac{n(n - 2)}{3}, \frac{1}{(1 - x)^2} \right\},
\]

(63)

\[
g_n(x, 1) \leq 2(n - 1)(n - 3) \min \left\{ \frac{n(n - 2)}{8}, \frac{1}{(1 - x)^2} \right\}.
\]

(64)

**Proof.** Using that

\[
\sum_{(r,s,t) \in \mathbb{N}_x^3} \left( \sum_{(u,v,w) \in \mathbb{N}_x^3} |y_{u,v,r}y_{u,v,s}y_{u,w,t}| \right)^2
\]

\[
= \sum_{(u,v,w) \in \mathbb{N}_x^3} \sum_{(u',v',w') \in \mathbb{N}_x^3} \sum_{(r,s,t) \in \mathbb{N}_x^3} |y_{u,v,r}y_{u,v,s}y_{u,w,t}y_{u',v',r}y_{u',v',s}y_{u',w',t}|
\]

together with the Cauchy-Schwarz inequality, we get

\[
\left( \sum_{(r,s,t) \in \mathbb{N}_x^3} \left( \sum_{(u,v,w) \in \mathbb{N}_x^3} |y_{u,v,r}y_{u,v,s}y_{u,w,t}| \right)^2 \right)^{1/2} \leq \sum_{(u,v,w) \in \mathbb{N}_x^3} \left( \sum_{(r,s,t) \in \mathbb{N}_x^3} |y_{u,v,r}y_{u,v,s}y_{u,w,t}|^2 \right)^{1/2}
\]

and this, in turn, can be estimated with the help of the Hölder inequality by

\[
\sum_{(u,v,w) \in \mathbb{N}_x^3} \left( \sum_{r \in \mathbb{N}} |y_{u,v,r}|^2 \right) \left( \sum_{r \in \mathbb{N}} |y_{u,w,r}|^2 \right)^{1/2}
\]

\[
\leq \left( \sum_{(u,v,w) \in \mathbb{N}_x^3} \left( \sum_{r \in \mathbb{N}} |y_{u,v,r}|^2 \right)^{3/2} \right)^{2/3} \left( \sum_{(u,v,w) \in \mathbb{N}_x^3} \left( \sum_{r \in \mathbb{N}} |y_{u,w,r}|^2 \right)^{3/2} \right)^{1/3}
\]

\[
= (N - 2) \sum_{(u,v) \in \mathbb{N}_x^3} \left( \sum_{r \in \mathbb{N}} |y_{u,v,r}|^2 \right)^{3/2},
\]

and this establishes the bounds given in (61) and (62). The bounds for \( f_n(x, 1) \) and \( g_n(x, 1) \) follow from a similar analysis.
which implies (61). Further,
\[
\sum_{(q,r,s,t) \in \mathbb{N}_p^4} \left( \sum_{(u,v,w,x) \in \mathbb{N}_p^4} |y_{u,v,q} y_{u,v,r} y_{w,x,s} y_{w,x,t}| \right)^2 \\
\leq \sum_{(q,r) \in \mathbb{N}_p^2} \sum_{(s,t) \in \mathbb{N}_p^2} \left( \sum_{(u,v) \in \mathbb{N}_p^2} |y_{u,v,q} y_{u,v,r}| \sum_{(w,x) \in \mathbb{N}_p^2} |y_{w,x,s} y_{w,x,t}| \right)^2 \\
= \left( \sum_{(q,r) \in \mathbb{N}_p^2} \left( \sum_{(u,v) \in \mathbb{N}_p^2} |y_{u,v,q} y_{u,v,r}| \right)^2 \right)^2,
\]
from which (62) follows. For (63), we note that
\[
f_n(x, 1) \leq \sum_{k \in \mathbb{N} \setminus 2} (n + k - 2)(n - k + 1) = \frac{2}{3} n(n - 1)(n - 2)
\]
and
\[
f_n(x, 1) = \sum_{k \in \mathbb{N} \setminus 2} (n + k - 2)(n - k + 1)x^{n-k} \leq 2(n - 1) \sum_{k \in \mathbb{N} \setminus 2} (n - k + 1)x^{n-k},
\]
where
\[
\sum_{k \in \mathbb{N} \setminus 2} (n - k + 1)x^{n-k} \leq \sum_{k=0}^{\infty} (k + 1)x^k = \frac{1}{(1-x)^2}.
\]
Furthermore,
\[
g_n(x, 1) = \sum_{k \in \mathbb{N} \setminus 3} (k - 3)(n + k - 2)(n - k + 1)x^{n-k} \leq (n - 3)f_n(x, 1) \leq \frac{2(n - 1)(n - 3)}{(1-x)^2}
\]
and
\[
g_n(x, 1) \leq \sum_{k \in \mathbb{N} \setminus 2} (k - 3)(n + k - 2)(n - k + 1) = \frac{n!}{4(n-4)!},
\]
which implies (64).
\[\square\]

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