WEAK LOCAL MONOMIALIZATION OF GERMS OF GENERICALLY FINITE COMPLEX ANALYTIC MORPHISMS

STEVEN DALE CUTKOSKY

1. INTRODUCTION

The purpose of this note is to prove a weak local monomialization theorem for germs of morphisms \( \varphi : X \to Y \) of reduced complex analytic spaces, with the condition that \( \dim X = \dim \varphi(X) \). The precise statement is given in the final section. This gives a “weak” generalization of the local monomialization theorem of [6] to generically finite morphisms of analytic spaces. It is also a generalization (in the local setting) to generically finite morphisms of complex analytic spaces of the weak toroidalization theorem for morphisms of characteristic zero algebraic varieties of Abramovich and Karu [2] and Abramovich, Denef and Karu [3]. This last result has been used by Denef in his new proof of \( p \)-adic quantifier elimination [9].

2. PRELIMINARIES ON COMPLEX ANALYTIC SPACES

In this section we recall some basic properties of analytic local rings and complex analytic spaces.

**Proposition 2.1.** Suppose that \( X \) is a complex analytic space and \( p \in X \). Then
1. \( \mathcal{O}_{X,p} \) is a Noetherian, Henselian, excellent local ring.
2. \( \mathcal{O}_{X,p} \) is equidimensional if and only if its completion \( \hat{\mathcal{O}}_{X,p} \) is equidimensional.
3. \( \mathcal{O}_{X,p} \) is reduced if and only if \( \hat{\mathcal{O}}_{X,p} \) is reduced.
4. \( \mathcal{O}_{X,p} \) is a domain if and only if \( \hat{\mathcal{O}}_{X,p} \) is a domain.

**Proof.** The fact that \( \mathcal{O}_{X,p} \) is Noetherian and Henselian is proven in Theorem 45.5, and by fact 43.4, [21]. Excellence is proven in Section 18 of [12] (or Theorem 102, page 291 [20]), and by (ii) of Scholie 7.8.3 [12]. Let \( A = \mathcal{O}_{X,p} \). Since \( A \) is a local ring, the natural map \( A \to \hat{A} \) is an inclusion. \( A \) and \( \hat{A} \) have the same Krull dimension (formula 1’ of page 175 [20]). Statements 2 and 3 follow from (vii) and (x) of Scholie 7.8.3 [12]. Further, \( \hat{\mathcal{O}}_{X,p} \) is a domain if and only if \( \mathcal{O}_{X,p} \) is a domain by Corollary 18.9.2 [12].

The dimension \( \dim E \) of a subset \( E \) of a complex analytic space \( X \) and the local dimension \( \dim_a E \) of \( E \) at a point \( a \in X \) are defined in II.1 and V.4.4 [19]. If \( E \) is an analytic space, then \( \dim_a E \) is the Krull dimension of \( \mathcal{O}_{E,a} \).

**Lemma 2.2.** Suppose that \( Y \) is a reduced complex analytic space, and \( \pi : B \to Y \) is the blow up of a closed complex analytic subspace \( E \) of \( Y \). Then \( Y \) is reduced. If \( Y \) is equidimensional, then \( B \) is equidimensional.

---

partially supported by NSF.
We have that by formula (1) of V.3.3 [19], Theorem 4, V.3.3 [19] and the assumption that
\[ \text{rank}(\varphi) \]
For \( p \in P \) of V.3.2 [19]). By the Cartan Remmert Theorem (Theorem 5, V.3.3 [19]), dim \( \text{Lemma 2.4.} \)
Suppose that \( \varphi : X \to Y \) is a morphism of complex analytic spaces. Suppose that \( p \in X \) and \( q = \varphi(p) \). Let \( \varphi^*_p : O_{Y,q} \to O_{X,p} \) be the induced homomorphism of germs of analytic functions, with associated homomorphism \( \varphi^*_p : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p} \) of complete local rings.
Suppose that \( X \) is reduced, irreducible, locally irreducible and \( Y \) is reduced, irreducible, locally irreducible, and \( \varphi : X \to Y \) is a morphism. For \( a \in \text{Reg}(X) \), define \( \text{rank}_a(\varphi) \) to be the rank of the map on tangent spaces \( d\varphi_a : T(X)_a \to T(Y)_{\varphi(a)} \), and
\[ \text{rank}(\varphi) = \max\{\text{rank}_a \varphi \mid a \in \text{reg}(X)\}, \]
and for \( p \in X \),
\[ \text{rank}_p(\varphi) = \min\{\text{rank}(\varphi|U)\} \]
For \( p \) in \( X \), we have that
\[ \text{rank}(\varphi) = \text{rank}_p(\varphi) = \dim_{\varphi(p)} \varphi(X) = \dim \varphi(X), \]
by Theorem 4 and Corollary 2 of V.3.3 [19].

Definition 2.3. Suppose that \( \varphi : X \to Y \) is a morphism of reduced, irreducible and locally irreducible complex analytic spaces. \( \varphi \) is said to be regular if \( \varphi(X) \) contains an open subset of \( Y \).

Lemma 2.4. Suppose that \( \varphi : X \to Y \) is a regular morphism of reduced, irreducible, locally irreducible complex analytic spaces. Then there exists a thin analytic subset \( G \) of \( X \) such that \( \varphi(X \setminus G) \) is an open subset of \( Y \), the restriction \( \varphi|X \setminus G \) is an open mapping and \( \dim \varphi(G) < \dim Y \).

Proof. For \( x \in X \), let \( \ell_x \varphi \) be the germ at \( x \) of the fiber of \( \varphi(x) \) by \( \varphi \) (defined on page 267 of V.3.2 [19]). By the Cartan Remmert Theorem (Theorem 5, V.3.3 [19]), \( \dim \ell_x \varphi \) is upper semi continuous on \( X \) in the analytic Zariski topology. Let
\[ t = \min\{\dim \ell_x \varphi \mid x \in X\}. \]
We have that
\[ t = \dim X - \text{rank}(\varphi) = \dim X - \dim \varphi(X) = \dim X - \dim Y, \]
by formula (1) of V.3.3 [19], Theorem 4, V.3.3 [19] and the assumption that \( \varphi \) is regular. Now
\[ G = \{x \in X \mid \dim \ell_x \varphi > t\} \]
is a proper subset of \( X \) which is closed in the analytic Zariski topology, so that it is a thin set (Proposition of II.3.5 [19]), and \( V = X \setminus G \) is an open subset of \( X \) on which \( \varphi \) has constant minimal fiber dimension \( t \). Further, by Remmert’s Rank Theorem (Theorem 1
of V.6 \cite{19}, for every $p \in X$ there exist arbitrarily small neighborhoods $U$ of $p$ in $X$ such that $\varphi(U)$ is locally analytic in $Y$, of dimension $\dim X - t$. We further have that
\[
\dim \varphi(G) \leq \dim G - (t + 1) < \dim X - t = \dim Y
\]
by Theorem 2, V.3.2 \cite{19}, since $\dim \ell_x(\varphi(G)) > t$ for all $x \in G$.

Finally, by Remmert’s Open Mapping Theorem (Theorem 2, V.6, \cite{19}), the restriction of $\varphi$ to $X \setminus G$ is an open mapping to $Y$, since $t = \dim X - \dim Y$.

**Proposition 2.5.** Suppose that $\varphi : X \to Y$ is a morphism of irreducible nonsingular complex analytic spaces, and $\varphi$ is regular. Then $\hat{\varphi} : \mathcal{O}_{Y,\varphi(p)} \to \mathcal{O}_{X,p}$ is 1-1 for all $p \in X$.

**Proof.** We have that for all $p \in X$, $\text{rank}_p(\varphi) = \text{rank}(\varphi) = \dim \varphi(X)$, by (1), so that $\hat{\varphi}_p : \mathcal{O}_{Y,\varphi(p)} \to \mathcal{O}_{X,p}$ is 1-1 for all $p \in X$ by Lemma 4.2 \cite{11}.

**Lemma 2.6.** Suppose that $A$ is an analytic local ring and $\mathfrak{p}$ is a prime ideal in $A$. Then there exists a field $K \subset A_\mathfrak{p}$ such that the induced map to the residue field $K \to (A/\mathfrak{p})_\mathfrak{p}$ is a finite field extension.

**Proof.** We have a representation $A \cong \mathcal{O}_n/I$ for some $n$ where $I$ is an ideal in the ring $\mathcal{O}_n$ of germs of analytic functions at the origin in $\mathbb{C}^n$. There is a prime ideal $P$ in $\mathcal{O}_n$ containing $I$, such that $P/I \cong \mathfrak{p}$. By the Proposition of III.2.5 \cite{19}, there exists a set of coordinates $z_1, \ldots, z_n$ in $\mathcal{O}_n$, so that $\mathcal{O}_n = \mathbb{C}\{z_1, \ldots, z_n\}$, and $k \leq n$ such that the induced map $\mathbb{C}\{z_1, \ldots, z_k\} \to \mathcal{O}_n/P$ is a 1-1 finite map. In particular, $\mathbb{C}\{z_1, \ldots, z_k\} \cap P = (0)$. Thus the induced map $\mathbb{C}\{z_1, \ldots, z_k\} \to \mathcal{O}_n/I \cong A$ is 1-1 and $\mathfrak{p} \cap \mathbb{C}\{z_1, \ldots, z_k\} = (0)$, so that we have an inclusion of the quotient field $K = \mathbb{C}\{z_1, \ldots, z_k\}$ into $A_\mathfrak{p}$, such that $(A/\mathfrak{p})_\mathfrak{p}$ is finite over $K$.

A fundamental theorem in complex analytic geometry is Hironaka’s theorem \cite{15} on the existence of a resolution of singularities of a reduced complex space $X$ (which is countable at infinity), by a sequence of blow ups of nonsingular subvarieties. The sequence is finite if $X$ is compact. In the case of a germ $(X, p)$, this already follows from Hironaka’s Theorem $I_{2}^N$ \cite{14}, since $\mathcal{O}_{X,p}$ is excellent and reduced. The general Theorem is proven in the monograph of Aroca, Hironaka and Vicente \cite{4}.

### 3. La Voûte Étoilée

In this section, we recall some definitions and results from \cite{17}.

**Definition 3.1.** (Definition 1.4 \cite{17}) A morphism $\pi : Y' \to Y$ of complex analytic spaces is called strict if there exists a complex analytic subspace $E'$ of $Y'$ such that $\pi$ is étale at all points of $Y' \setminus E'$ and $(Y', E')$ is minimal, in the sense that if $Z$ is a closed analytic subset of $Y'$ such that $Y' \setminus E' = Y \setminus E$, then $Y' = Y$.

Let $Y$ be a complex analytic space. A local blow up of $Y$ (page 148 \cite{17}) is the morphism $\pi : Y' \to Y$ determined by given $(U, E, \pi)$ where $U$ is an open subset of $Y$, $E$ is a Zariski closed subset of $U$ and $\pi$ is the composite of the blow up of $E$ with the inclusion of $U$ into $Y$.

A sequence of local blow ups of $Y$ is the composite of a finite sequence of local blow ups $(U_i, E_i, \pi_i)$. A sequence of local blow ups is strict (\cite{17}).

Let $Y$ be a complex analytic space. $\mathcal{E}(Y)$ will denote the category of morphisms $\pi : Y' \to Y$ which are sequence of local blow ups. For $\pi_1 : Y_1 \to Y \in \mathcal{E}(Y)$ and $\pi_2 : Y_2 \to Y \in \mathcal{E}(Y)$, $\text{Hom}(\pi_1, \pi_2)$ denotes the $Y$-morphisms $Y_2 \to Y_1$ (morphisms which factor $\pi_1$ and $\pi_2$). $\text{Hom}(\pi_1, \pi_2)$ has at most one element.
Definition 3.2. (Definition 2.1[17]) Let $Y$ be a complex analytic space. An étoile over $Y$ is a subcategory $e$ of $\mathcal{E}(Y)$ having the following properties:

1) If $\pi : Y' \to Y \in e$ then $Y' \neq \emptyset$.
2) If $\pi_1, \pi_2 \in e$ for $i = 1, 2$, then there exists $\pi_3 \in e$ which dominates $\pi_1$ and $\pi_2$; that is, $\text{Hom}(\pi_3, \pi_i) \neq 0$ for $i = 1, 2$.
3) For all $\pi_1 : Y_1 \to Y \in e$, there exists $\pi_2 : Y_2 \to Y \in e$ such that there exists $q \in \text{Hom}(\pi_2, \pi_1)$, and the image $q(Y_2)$ is relatively compact in $Y_1$.
4) (maximality) If $e'$ is a subcategory of $\mathcal{E}(Y)$ that contains $e$ and satisfies the above conditions 1) - 3), then $e' = e$.

$\mathcal{E}_Y$ denotes the set of all étoiles over $Y$.

Using property 3), Hironaka shows that for $e \in \mathcal{E}_Y$, and $\pi : Y' \to Y$, there exists a uniquely determined point $p_\pi(e) \in Y'$ which has the property that if $\alpha : Z \to Y \in e$ factors as $Z \xrightarrow{\beta} Y' \xrightarrow{\pi} Y$, then $\beta(p_{\alpha}(e)) = p_\pi(e)$. In particular, we have a natural map $p_Y : \mathcal{E}_Y \to Y$ defined by $p_Y(e) = p_{id}(e)$. Hironaka shows (in Theorem 3.4[17]) that $\mathcal{E}_Y$ has a natural topology so that $p_Y$ is continuous, surjective and proper.

$\mathcal{E}_Y$ with this topology is called “La voûte étoilée.”

4. BLOW UPS AND MORPHISMS ALONG AN ÉTOILE AND THE DISTINGUISHED IRREDUCIBLE COMPONENT

The join of $\pi_1, \pi_2 \in \mathcal{E}(Y)$ is defined in Proposition 2.9[17]. We will denote this join by $J(\pi_1, \pi_2) : Y_1 \to Y$. It is a morphism $J(\pi_1, \pi_2) : Y_1 \to Y$. It has the following universal property: Suppose that $f : Z \to Y$ is a strict morphism. Then there exists a $Y$-morphism $Z \to Y_f$ if and only if there exist $Y$-morphisms $Z \to Y_1$ and $Z \to Y_2$. It follows from 2.9.2[17] that if $\pi_1, \pi_2, e \in e \subset \mathcal{E}_Y$, then $J(\pi_1, \pi_2) \subseteq e$. We describe the construction of Proposition 2.9[17]. In the case when $\pi_1$ and $\pi_2$ are each local blowups, which are described by the data $(U_i, E_i, \pi_i) \to (\pi_1, \pi_2)$ is the blow up $J(\pi_1, \pi_2) : Y_f = B(I_{E_1}, I_{E_2} \mathcal{O}_Y|U_1 \cap U_2) \to Y$.

Now suppose that $\pi_1$ is a product $\alpha_0 \alpha_1 \cdot \alpha_r$, where $\alpha_i : Y_{i+1} \to Y_i$ are local blow ups defined by the data $(U_i, E_i, \alpha_i)$, and $\pi_2$ is a product $\alpha'_0 \alpha'_1 \cdots \alpha'_r$, where $\alpha'_i : Y'_{i+1} \to Y'_i$ are local blow ups defined by the data $(U'_i, E'_i, \alpha'_i)$. We may assume (by composing with identity maps) that the length of each sequence is a common value $r$. We define $J(\pi_1, \pi_2)$ by induction on $r$. Assume that $J_r = J(\alpha_0 \alpha_1 \cdots \alpha_{r-1}, \alpha'_0 \alpha'_1 \cdots \alpha'_{r-1})$ has been constructed, with projections $\gamma : Y_{J_r} \to Y_r$ and $\delta : Y_{J_r} \to Y_{J_r}'$. Then we define $J(\pi_1, \pi_2)$ to be the blow up $J(\pi_1, \pi_2) : Y_f = B(I_{E_1}, I_{E_2} \mathcal{O}_Y|\gamma^{-1}(U_r) \cap \delta^{-1}(U'_{J_r})) \to Y$.

Suppose that $e \in \mathcal{E}_Y$ is an étoile. By Lemma 2.3[17], there exists a point $p_\pi(e) \in Y'$ for all $\pi : Y' \to Y \in e$, such that if $\pi_1, \pi_2 \in e$ and $\varphi \in \text{Hom}(\pi_1, \pi_2)$, then

(2) $p_{\pi_2}(e) = \varphi(p_{\pi_1}(e))$.

(Condition 3) of Definition 3.2 is essential for this result.) Suppose that $Y$ is a reduced complex analytic space, $e \in \mathcal{E}_Y$ and $\pi : Y' \to Y \in e$. Suppose that $U$ is a neighborhood of $p_\pi(e) \in Y'$. We will define the distinguished irreducible component $\text{DC}_e(U)$ of $U$. Let $F_1, F_2, \ldots, F_s$ be the distinct irreducible components of $U$. Let $\pi' : U' \to U$ be a global blowup of a thin algebraic set, which separates out the irreducible components of $U$ into
distinct connected components \(Z_1, \ldots, Z_s\) such that \(\pi'(Z_i) \subset F_i\) for all \(i\), and \(Z_i \to Y_i\) is strict (such as a resolution of singularities of \(U\)). Then \(\pi' \pi \in e\) by Corollary 2.11.4 \([17]\).

There exists a unique component \(Z_i\) of \(U'\) such that \(p_{\pi \pi'}(e) \subset Z_i\). Define \(DC_e(U) = F_i\).

This definition is well defined, since if \(\pi'': U'' \to U\) is another global blowup of a thin analytic subset of \(U\) which separates the components of \(U\), then by 2) of Definition 3.2 there exists \(\lambda : W \to Y \in e\) and \(\alpha \in \text{Hom}(\lambda, \pi'), \beta \in \text{Hom}(\lambda, \pi'')\) such that \(\alpha(p_\lambda(e)) = p_{\pi \pi'}(e)\) and \(\beta(p_\lambda(e)) = p_{\pi''}(e)\). Since \(\pi'\) and \(\pi''\) are blow ups of thin analytic sets, there is an open subset of \(U'\) which intersects all components of \(U''\) non trivially which is isomorphic to an open subset of \(U''\) which intersects all components of \(U''\) non trivially. Thus the component of \(U''\) which contains \(p_{\pi''}(e)\) must map to \(DC_e(U)\).

\textbf{Lemma 4.1.} Suppose that \(Y = Y_0\) is a reduced complex analytic space, \(e \in \mathcal{E}_Y\) and \(\pi : Y' \to Y \in e\). Suppose that \(\pi\) has a factorization \(\pi = \pi_0 \pi_1 \cdots \pi_r\) where \(\pi_i : Y_{i+1} \to Y_i\) are local blow ups determined by the data \((U_i, E_i, \pi_i)\). Then \(\pi_0 \cdots \pi_i \in e\), \(p_{\pi_0 \cdots \pi_{i-1}}(e) \in U_i\) and \(DC_e(U_i) \not\subset E_i\) for all \(i\).

\textbf{Proof.} We will first show that \(\pi_0 \cdots \pi_i \in e\) for all \(i\). We will use the criterion of Lemma 2.10 on page 431 of \([17]\). Suppose that \(\varphi_\alpha : Z_\alpha \to Y \in e\). We must show that there exists \(\varphi_\beta : Z_\beta \to Y \in e\) such that \(\text{Hom}(\varphi_\beta, \varphi_\alpha) \not= \emptyset\), and if \(J(\varphi_\beta, \pi_0 \cdots \pi_i) : Z_\beta \to Y\) is the join, then the natural image of \(Z_j\) in \(Y_{i+1}\) is relatively compact and non empty.

By 2) and 3) of Definition 3.2 there exists \(\varphi_\beta : Z_\beta \to Y \in e\) such that \(\text{Hom}(Z_\beta, Z_\alpha) \not= \emptyset\), \(\text{Hom}(Z_\beta, Y') \not= \emptyset\) and if \(q : Z_\beta \to Y'\) is the induced map, then \(q(Z_\beta)\) is relatively compact in \(Y'\). Let \(J(\varphi_\beta, \pi_0 \cdots \pi_i) : Z_\beta \to Y\) be the join.

Then \(Z_j = Z_\beta\) since \(\pi_\beta\) factors through \(\pi_0 \cdots \pi_i\). Since the image of \(Z_\beta\) is relatively compact in \(Y'\), the image of \(Z_\beta\) in \(Y_{i+1}\) is also relatively compact. The fact that \(p_{\pi_0 \cdots \pi_i}(e) \in U_{i+1}\) for all \(i\) now follows from \([12]\).

Let \(h = \pi_0 \cdots \pi_{i-1}\). Let \(\lambda : Z \to U_i\) be a global blow up which separates the irreducible components of \(U_i\). Then \(h \pi_i \in e\). Since \(h \pi_i \in e\), there exists (by 2) of Definition 3.2) \(\tau : W \to Y \in e\) with factorizations

\[
\begin{array}{ccc}
W & \xleftarrow{\alpha} & Z \\
\downarrow{Y_{i+1}} & & \downarrow{\beta} \\
\pi_i & \xleftarrow{U_i} & \tau \\
\downarrow{h} & & \downarrow{Y} \\
Y & & 
\end{array}
\]

Let \(H\) be the irreducible component of \(W\) which contains \(p_\tau(e)\). Then \(\lambda(\beta(H))\) must be dense in \(DC_e(U_i)\). Thus \(Y_{i+1}\) contains an irreducible component \(G\) such that \(\pi_i(G)\) is dense in \(DC_e(U_i)\), so that \(DC_e(U_i) \not\subset E_i\). \(\square\)

\textbf{Lemma 4.2.} Suppose that \(Y\) is a reduced complex analytic space, \(e \in \mathcal{E}_Y\), \(\pi_0 : Y_0 \to Y \in e\), and \((U, E, h)\) is a local blow up of \(Y_0\). Then \(\pi_0 h \in e\) if and only if \(p_{\pi_0}(e) \in U\) and \(DC_e(U) \not\subset E\).

\textbf{Proof.} The conditions \(p_{\pi_0}(e) \in U\) and \(DC_e(U) \not\subset E\) are certainly necessary for \(\pi_0 h\) to be in \(e\) by Lemma 4.1. Suppose that \(p_{\pi_0}(e) \in U\) and \(DC_e(U) \not\subset E\). We will verify the criterion of Lemma 2.10 on page 431 of \([17]\). Suppose that \(\varphi_\alpha : Y_\alpha \to Y \in e\). Let our map \(h\) be \(h : Y' \to Y_0\). We must show that there exists \(\pi_\beta : Y_\beta \to Y \in e\) such that \(\text{Hom}(\pi_\beta, \pi_\alpha) \not= \emptyset\), and if
$J(\pi_\beta, \pi_0 h) : Y_I \to Y$ is the join, then the natural image of $Y_J$ in $Y'$ is relatively compact and non empty.

We have that $U \to Y$ is in $e$ (by Corollary 2.11.4 [17]), so we can replace $Y_0$ with $U$, and assume that $U = Y_0$, and $E$ is closed in $Y_0$ with $DC_c(Y_0) \not\subseteq E$. By 2) and 3) of Definition 3.2, there exists $\pi_\beta : Y_\beta \to Y \in e$ and maps $\lambda \in \text{Hom}(\pi_\beta, \pi_0), \tau \in \text{Hom}(\pi_\beta, \pi_\alpha)$ such that $\lambda(Y_\beta)$ is relatively compact in $Y_0$. By the universal property, we have that the joins $J(\pi_\beta, \pi_0 h)$ and $J(\lambda, h)$ have a common domain, which we will denote by $Y'_\beta$. We have a commutative diagram:

$$
\begin{array}{c c c c c c c c c c c}
\gamma & h \downarrow & \delta \\
\downarrow & \hfill & \hfill & \hfill & \hfill & \hfill & \hfill & \hfill & \hfill & \hfill & \hfill & \hfill
\end{array}
$$

Let $K$ be the closure of $\lambda(Y_\beta)$ in $Y_0$, which is compact. $\delta(Y'_\beta) \subset h^{-1}(\lambda(K))$, which is compact since $h$ is a global blow up, so it is proper. Thus $\delta(Y'_\beta)$ is relatively compact.

It remains to show that $Y'_\beta \neq \emptyset$. We have that $Y_\beta \neq \emptyset$ (since $\pi_\beta \in e$). $\pi_\beta$ is strict, by Proposition 1.7 [17], so it is an open immersion on an open subset $W$ of $Y_\beta$ which intersects $DC(e(Y_\beta))$ nontrivially. $\lambda$ is thus necessarily also an open immersion on $W$. Thus $V = \lambda(W)$ is an open subset of $Y_0$ such that $DC_e(Y_0) \cap V \neq \emptyset$. By our assumption on $E$, we have that $E \cap DC_e(Y_0) \cap V$ is thin in $DC_e(Y_0) \cap V$. Let $F_1, \ldots, F_r$ be the irreducible components of $Y_0$, with $F_1 = DC_e(Y_0)$. $h$ is an isomorphism over the non trivial open set $V \setminus (E \cup F_2 \cup \cdots \cup F_r)$. Let $Z = Y_\beta|\lambda^{-1}(V \setminus (E \cup F_2 \cup \cdots \cup F_r))$. Let $\varepsilon : Z \to Y'$ be the morphism induced by $\lambda$ and $i : Z \to Y_\beta$ be the inclusion. Now $Z \neq \emptyset$, and since $\lambda i = h\varepsilon$, we have that $\text{Hom}(\varepsilon, \delta) \neq \emptyset$ be the universal property of the join. Thus $Y'_\beta \neq \emptyset$. \hfill \Box

Using resolution of singularities, and resolution of indeterminacy ([14], [15], [4]) we deduce the following Lemma.

**Lemma 4.3.** Suppose that $Y$ is a reduced complex analytic space and $e \in \mathcal{E}_Y$. Suppose that $\pi \in e$ factors as a sequence of local blow ups

$$
Y_n \to Y_{n-1} \to \cdots \to Y_1 \to Y
$$

where each $\pi_i : Y_{i+1} \to Y_i$ is a local blow up $(U_i, E_i, \pi_i)$. Then there exists $\pi' \in e$ which is a composition of local blow ups

$$
Y'_n \to Y'_{n-1} \to \cdots \to Y'_1 \to Y
$$

such that each $Y'_i$ is nonsingular, $\pi'_i : Y'_{i+1} \to Y'_i$ is a local blow up $(U'_i, E'_i, \pi'_i)$, and there exists a commutative diagram of strict morphisms

$$
\begin{array}{c c c c c c c c c c c}
Y'_n & \to & Y'_{n-1} & \cdots & \to & Y'_1 & \downarrow \hfill & \hfill & \hfill & \hfill & \hfill & \hfill
\downarrow & \hfill & \hfill & \hfill & \hfill & \hfill & \hfill & \hfill & \hfill & \hfill & \hfill & \hfill
Y_n & \to & Y_{n-1} & \cdots & \to & Y_1 & \to & Y.
\end{array}
$$

Suppose that $\varphi : X \to Y$ is a morphism of complex analytic spaces, and $\pi : Y' \to Y \in \mathcal{E}(Y)$. $\varphi^{-1}[\pi] : \varphi^{-1}[Y'] \to X$ will denote the strict transform of $\varphi$ by $\pi$ (Section 2 of [13]).

In the case of a single local blowup $(U, E, \pi)$ of $Y$, $\varphi^{-1}[Y']$ is the blow up $B(I_{E\pi}\mathcal{O}_X|\varphi^{-1}(U))$. In the case when $\pi = \pi_0 \pi_1 \cdots \pi_r$, with $\pi_i : Y_{i+1} \to Y_i$ given by local blow ups $(U_i, E_i, \pi_i)$, we inductively define $\varphi^{-1}[\pi]$. Assume that $\pi_0^{-1}[\pi_0 \cdots \pi_{r-1}]$ has been constructed. Let $h = \pi_0 \cdots \pi_{r-1}$, so that $\pi = h\pi_r$. Let $\varphi' : \varphi^{-1}[Y_r] \to Y_r$ be the natural morphism. Then define $\varphi^{-1}[Y_{r+1}]$ to be the blow up $B(I_{E_r}\mathcal{O}_{\varphi^{-1}[Y_r]}((\varphi')^{-1}(U_r)))$. 

6
Lemma 4.4. Suppose that $\pi_1, \pi_2 \in \mathcal{E}(Y)$. Then
\[ J(\varphi^{-1}[\pi_1], \varphi^{-1}[\pi_2]) = \varphi^{-1}[J(\pi_1, \pi_2)]. \]

Proof. The fact that these two constructions are canonically isomorphic can be realized by comparing the explicit constructions given above. The essential case is that of the strict transform of the join of two local blow ups $\pi_1 : Y_1 \to Y$ and $\pi_2 : Y_2 \to Y$ given by local data $(U_1, E_1, \pi_1)$ and $(U_2, E_2, \pi_2)$. The join $J(\pi_1, \pi_2)$ is then the blow up
\[ J(\pi_1, \pi_2) : B(I_{E_1}I_{E_2}|U_1 \cap U_2) \to Y; \]
and $\varphi^{-1}[J(\pi_1, \pi_2)]$ is the blow up
\[ (3) \quad \varphi^{-1}[J(\pi_1, \pi_2)]B(I_{E_1}I_{E_2}O_X|\varphi^{-1}(U_1 \cap U_2)) \to X. \]
However, $\varphi^{-1}_i[\pi_i]$ are the blow ups $\varphi^{-1}_i[\pi_i] : B(I_{E_i}O_X\varphi^{-1}(U_i)) \to X$ Thus the construction of $J(\varphi^{-1}[\pi_1], \varphi^{-1}[\pi_2])$ described at the beginning of this section gives us again the blow up (3). □

Lemma 4.5. Suppose that $\varphi : X \to Y$ is a morphism of complex analytic spaces and $e \in \mathcal{E}_X$. Let
\[ S(\varphi, e) = \{ \pi \in \mathcal{E}(Y) \mid \varphi^{-1}[\pi] \in e \}. \]
Then $S(\varphi, e)$ satisfies properties 1), 2) and 3) of Definition 8.2.

Proof. This follows from Lemma 4.4 and 2.9.2 of [17]. □

Lemma 4.6. Suppose that $\varphi : X \to Y$ is a morphism of reduced complex analytic spaces and $e \in \mathcal{E}_X$. Suppose that $f \in \mathcal{E}_Y$ contains $S(\varphi, e)$, and $\pi : Y' \to Y \in S(\varphi, e)$. Then
\[ p_\pi(f) = \varphi'(p_{\varphi^{-1}[\pi]}(e)) \]
and
\[ \varphi'(DC_e(\varphi^{-1}[Y'])) \subset DC_f(Y'), \]
where $\varphi' : \varphi^{-1}[Y'] \to Y'$ is the induced morphism.

Proof. Suppose that $U$ is any neighborhood of $\varphi'(p_{\varphi^{-1}[\pi]}(e))$ in $Y'$. Then $\pi|U \in S(\varphi, e)$ (by Lemma 4.2). Thus
\[ p_\pi(f) = \varphi'(p_{\varphi^{-1}[\pi]}(e)). \]
Suppose that $V = \varphi'(DC_e(\varphi^{-1}[Y'])) \not\subset DC_f(Y')$. Then the Zariski closure of $V$ in $Y'$ is not contained in $DC_f(Y')$, since $DC_f(Y')$ is an irreducible component of $Y'$. There exists a closed analytic subspace $E$ of $Y'$, which is a proper subset of $DC_f(Y')$, such that if $\alpha : Z \to Y'$ is the blow up of $E$, then $\pi\alpha \in f$ (by Lemma 4.2), $DC_f(Z)$ is a connected component of $Z$, and the strict transform $W$ in $Z$ of $V$ is disjoint from $DC_f(Z)$. $\varphi^{-1}[Z]$ is the blow up of $\varphi^{-1}[Y']$ at $F = (\varphi')^{-1}(E)$. $F$ does not contain $DC_e(\varphi^{-1}[Y'])$. Thus $\varphi^{-1}[\pi\alpha] : \varphi^{-1}[Z] \to X \in e$ by Lemma 4.2. Now $DC_e(\varphi^{-1}[Z])$ is the strict transform of $DC_e(\varphi^{-1}[Y'])$. Thus $p_{\pi\alpha}(f) = \varphi''(p_{\varphi^{-1}[\pi\alpha]}(e)) \in W$, where $\varphi'' : \varphi^{-1}[Z] \to Z$ is the natural map, which is impossible since $W$ is disjoint from $DC_f(Z)$. Thus
\[ \varphi'(DC_e(\varphi^{-1}[Y'])) \subset DC_f(Y'). \]
□

Proposition 4.7. Suppose that $\varphi : X \to Y$ is a morphism of reduced complex analytic spaces. Then $S(\varphi, e) \in \mathcal{E}_Y$ if and only if for all $\pi : Y' \to Y \in S(\varphi, e)$, with associated morphism $\varphi' : \varphi^{-1}[Y'] \to Y'$, $\varphi'(DC_e(\varphi^{-1}[Y']))$ is not contained in a proper analytic subset of an irreducible component of $Y'$. 7
Proof. Suppose that \( f \in \mathcal{E}_Y \) contains \( S(\varphi,e) \) and there exists \( \pi : Y' \to Y \in S(\varphi,e) \) such that \( \varphi'(DC_e(\varphi^{-1}[Y'])) \) is contained in a proper analytic subset \( E \) of an irreducible component of \( Y' \). Let \( \alpha : Z \to Y' \) be the blow up of \( E \). Then \( \pi \alpha \in f \) by Lemma 4.6. We have a commutative diagram of morphisms

\[
\begin{array}{ccc}
\varphi^{-1}[Z] & \xrightarrow{\varphi'} & Z \\
\varphi^{-1}[\alpha] \downarrow & & \downarrow \alpha \\
\varphi^{-1}[Y'] & \xrightarrow{\alpha} & Y'.
\end{array}
\]

\( DC_e(\varphi^{-1}[Y']) \) is a subspace of \( (\varphi')^{-1}(E) \) and \( \varphi^{-1}[\alpha] : \varphi^{-1}[Z] \to \varphi^{-1}[Y'] \) is the blow up of \( (\varphi')^{-1}(E) \). Thus \( \varphi^{-1}[\pi \alpha] = \varphi^{-1}[\pi] \varphi^{-1}[\alpha] \notin e \) by Lemma 4.2.

Now suppose that for all \( \pi : Y' \to Y \in S(\varphi,e) \), with associated morphism \( \varphi' : \varphi^{-1}[Y'] \to Y' \), \( \varphi'(DC_e(\varphi^{-1}[Y'])) \) is not contained in a proper analytic subset of an irreducible component of \( Y' \). Suppose that \( f \in \mathcal{E}_Y \) contains \( S(\varphi,e) \). Suppose that \( \pi \in f \).

We will show that \( \pi \in S(\varphi,e) \).

We prove this by induction on the length \( r \) of a factorization \( \pi = h_0 h_1 \cdots h_{r-1} h_r \) where \( (U_i, E_i, h_i) \) are local blow ups \( h_i : Y_{i+1} \to Y_i \). By Lemma 4.1 \( h_0 \cdots h_r \in f \), \( p_{h_0 \cdots h_r}(f) \in U_r \) and \( DC_f(U_r) \notin E_r \).

We have a commutative diagram of morphisms

\[
\begin{array}{ccc}
\varphi^{-1}[Y'] & \xrightarrow{\varphi_1} & Y' \\
\varphi_{r-1}^{-1}[h_r] \downarrow & & \downarrow h_r \\
\varphi_1^{-1}[Y_r] & \xrightarrow{\varphi_1} & Y_r \\
\varphi^{-1}[h_0 \cdots h_{r-1}] \downarrow & & \downarrow h_0 \cdots h_{r-1} \\
X & \xrightarrow{\varphi} & Y.
\end{array}
\]

By our induction assumption, \( h_0 \cdots h_{r-1} \in S(\varphi,e) \), so that \( \varphi^{-1}[h_0 \cdots h_{r-1}] \in e \). \( p_{h_0 \cdots h_{r-1}}(f) \in U_r \) by Lemma 4.1. Thus \( \alpha = h_0 \cdots h_{r-1} U_r \in f \) by Lemma 4.2. \( \varphi^{-1}[\alpha] : \varphi^{-1}[U_r] \to X \) is in \( e \) by Lemma 4.6 since \( p_{\varphi^{-1}[h_0 \cdots h_{r-1}]}(f) \in \varphi^{-1}(p_{h_0 \cdots h_{r-1}}(f)) \). Thus \( \alpha \in S(\varphi,e) \), so that \( \varphi_{r-1}(DC_e(\varphi_{r-1}^{-1}(U_r))) \) is not contained in a proper analytic subset of an irreducible component of \( U_r \), by assumption. Since \( \varphi_{r-1}(DC_e(\varphi_{r-1}^{-1}(U_r))) \subset DC_f(U_r) \) by Lemma 4.6 and \( DC_f(U_r) \notin E_r \) (by Lemma 4.1), we have that \( \varphi_{r-1}(DC_e(\varphi_{r-1}^{-1}(U_r))) \notin E_r \), so \( DC_e(\varphi_{r-1}^{-1}(U_r)) \notin \varphi_{r-1}^{-1}(E_r \cap U_r) \). Thus \( \varphi^{-1}[\pi] = \varphi^{-1}[h_0 \cdots h_{r-1}] \varphi_{r-1}^{-1}[h_r] \in e \) by Lemma 4.2 so that \( \pi \in S(\varphi,e) \).

**Proposition 4.8.** Suppose that \( \varphi : X \to Y \) is a morphism of reduced, irreducible, locally irreducible complex analytic spaces, and \( e \in \mathcal{E}_X \). Suppose that \( \varphi \) is regular. Then \( S(\varphi,e) \in \mathcal{E}_Y \).

**Proof.** This follows from Proposition 4.7 and since a local blow up is strict.

**Theorem 4.9.** Suppose that \( \varphi : X \to Y \) is a morphism of reduced complex analytic spaces, and \( e \in \mathcal{E}_X \). Then there exists \( \pi : Y' \to Y \in S(\varphi,e) \) (so that \( \varphi^{-1}[\pi] \in e \)) such that either \( \varphi' : \varphi^{-1}[Y'] \to Y' \) is flat at \( p_{\varphi^{-1}[\pi]}(e) \) or \( \varphi'(DC_e(\varphi^{-1}[Y'])) \) is contained in a proper analytic subset of an irreducible component of \( Y' \).

**Proof.** Let \( f \in \mathcal{E}_Y \) be such that \( S(\varphi,e) \subset f \). By Theorem 3 [18 or 10], there exists \( \pi \in f \) such that \( \varphi' : \varphi^{-1}[Y'] \to Y' \) is flat at points of \( (\varphi')^{-1}(p_{\pi}(f)) \cap (\varphi^{-1}[\pi])^{-1}(p_{\text{id}}(e)) \).

If \( \pi \in S(\varphi,e) \), then \( \varphi^{-1}[\pi] \in e \), so that \( p_{\varphi^{-1}[\pi]}(e) \in (\varphi')^{-1}(p_{\pi}(f)) \cap (\varphi^{-1}[\pi])^{-1}(p_{\text{id}}(e)) \) by Lemma 4.6 so that \( \varphi' \) is flat at \( p_{\varphi^{-1}[\pi]}(e) \).
Now suppose that $\pi \not\in S(\varphi, e)$. We can factor $\pi = h_0 h_1 \cdots h_r$ where $(U_i, E_i, h_i)$ are local blow ups $h_i : Y_{i+1} \to Y_i$. By Lemma 4.11 $h_0 \cdots h_s \in f$, $p_{h_0 \cdots h_s}(f) \in U_r$ and $\text{DC}_f(U_s) \not\in E_s$ for all $s$. There exists a largest $s$ such that $h_0 \cdots h_{s-1} \in S(\varphi, e)$, but $h_0 \cdots h_s \not\in S(\varphi, e)$. $U_s \subset Y_s$ contains $p_{h_0 \cdots h_{s-1}}(f)$, so that $U_s \to Y \in S(\varphi, e)$ by Lemma 1.2.

Let $\lambda = (h_0 \cdots h_{s-1})|U_s$, and $\varphi'' : \varphi^{-1}U_s \to U_s$ be the induced morphism. Then $\varphi''(\text{DC}_e(\varphi^{-1}[U_s])) \subset \text{DC}_f(U_s)$ by Lemma 4.6. Since $\lambda h_s \not\in S(\varphi, e)$, we have that $\varphi''(\text{DC}_e(\varphi^{-1}[U_s])) \subset E_s \cap \text{DC}_f(U_s)$, which is a proper analytic subset of the irreducible component $\text{DC}_f(U_s)$ of $U_s$. Now replacing $\pi$ with $\lambda$, we have obtained the conclusions of the theorem.

**Corollary 4.10.** Suppose that $\varphi : X \to Y$ is a morphism of reduced complex analytic spaces, and $e \in \mathcal{E}_X$. Then there exists a commutative diagram of morphisms

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\varphi}} & \tilde{Y} \\
\gamma \downarrow & & \downarrow \delta \\
X & \xleftarrow{\varphi} & Y
\end{array}
$$

such that $\gamma \in e$, $\delta$ is sequence of morphisms consisting of local blow ups and inclusions of proper analytic subsets, $\tilde{X}$ is reduced, $\tilde{Y}$ is reduced, and $\tilde{\varphi}$ is flat at $p_\gamma(e)$.

**Proof.** The proof is by induction on the dimension of $Y$. If dim $Y = 0$, then $Y$ is a finite union of points, so $\varphi$ is necessarily flat, since $\mathcal{O}_{Y,q}$ is a field for all $q \in Y$. Suppose that the Corollary is true for all reduced complex analytic spaces of dimension less than dim $Y$.

By Theorem 4.9 there exists $\varphi : Y' \to Y \in S(\alpha, e)$ such that either

$$(4) \quad \text{the induced morphism } \varphi' : \varphi^{-1}[Y'] \to Y' \text{ is flat at } p_{\varphi^{-1}[\pi]}(e),$$

or

$$(5) \quad \varphi'(\text{DC}_e(\varphi^{-1}[Y'])) \subset \text{DC}_f(Y').$$

If (4) holds then we have achieved the conclusions of the Corollary. Suppose that (5) holds. There exists an irreducible analytic subset $F$ of $Y'$ such that $\varphi'(\text{DC}_e(\varphi^{-1}[Y'])) \subset F$ and $F$ is not an irreducible component of $Y'$ (so that dim $F < \text{dim } Y$).

Let $\tau : X'' \to \varphi^{-1}[Y']$ be a resolution of singularities, obtained by blowing up a thin analytic subset of $\varphi^{-1}[Y']$. Then $\tau \in e$. Then $X^* = \text{DC}_e(X'')$ is a connected component of $X''$, so the composition of inclusion of $X^*$ into $X''$ and the morphism $\varphi^{-1}[\tau]$ $\tau$ is in $e$. We have an induced morphism of $X^*$ to $F$. By induction on the dimension of $Y$, the conclusions of the Corollary hold.

**Proposition 4.11.** Suppose that $\varphi : X \to Y$ is a morphism of reduced, irreducible, locally irreducible complex analytic spaces and $\varphi$ is regular. Further suppose that $\alpha : X' \to X$, $\beta : Y' \to Y$ are sequences of local blow ups of thin analytic subsets such that $X'$ and $Y'$ are reduced, irreducible, locally irreducible, and there is a commutative diagram of morphisms

$$
\begin{array}{ccc}
X' & \xrightarrow{\varphi'} & Y' \\
\alpha \downarrow & & \downarrow \beta \\
X & \xrightarrow{\varphi} & Y
\end{array}
$$

Then $\varphi'$ is regular.

**Proof.** There exists an analytic subset $F$ of $Y'$ such that dim $F < \text{dim } Y = \text{dim } Y$, dim $\beta(F) < \text{dim } Y$, $Y' \setminus F = Y' \setminus \beta^{-1}(\beta(F))$, and $\beta(|Y' \setminus F) : Y' \setminus F \to Y$ is an isomorphism onto an open subset of $Y$.
There exists an analytic subset $H$ of $X'$ such that $\dim H < \dim X' = \dim X$, $\dim \alpha(H) < \dim X$, $V = X' \setminus H = X' \setminus \alpha^{-1}(\alpha(H))$ is an open subset of $X'$ and $\alpha|V : V \to X$ is an isomorphism onto an open subset of $X$.

Since $\varphi$ is regular, by Lemma 2.3, there exists a thin analytic subset $G$ of $X$ such that $\varphi(X \setminus G)$ is an open subset of $Y$, $\dim \varphi(G) < \dim Y$, and $\varphi|(X \setminus G)$ is an open mapping.

$\dim G \leq \dim X - 1$ implies $W := (X \setminus G) \cap \alpha(V)$ is a nonempty open subset of $X$. $\varphi(W)$ is an open subset of $Y$. $\varphi(W) \subset \beta(Y') = \beta(Y' \setminus F) \cup \beta(F)$. Since $\dim \beta(F) < \dim Y$, we have that $\varphi(W) \cap \beta(Y' \setminus F)$ is a nonempty open subset of $Y$. Since $\alpha$ is an isomorphism over $W$ and $\beta$ is an isomorphism over $\beta(Y' \setminus F)$, we have that $\varphi'(V)$ contains the nonempty open set $\beta^{-1}((\varphi(W) \cap \beta(Y' \setminus F))$. Thus $\varphi'$ is regular.

**Theorem 4.12.** Suppose that $\varphi : X \to Y$ is a morphism of reduced complex analytic spaces, and $e \in \mathcal{E}_X$. Then there exists a commutative diagram of morphisms

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\varphi}} & \tilde{Y} \\
\gamma & & \downarrow \delta \\
X & \xrightarrow{\varphi} & Y
\end{array}
$$

such that $\gamma \in e$, $\delta$ is sequence of morphisms consisting of local blow ups and inclusions of proper analytic subsets, $\tilde{X}$ is nonsingular and irreducible, $\tilde{Y}$ is nonsingular and irreducible and $\tilde{\varphi}$ is regular.

**Proof.** By Corollary 4.10 we may assume that $\varphi$ is flat. Let $p = p|_{\tilde{\varphi}}(e)$.

There exists an open subset $V$ of $Y$ which contains $\varphi(p)$, such that all irreducible components of $V$ are locally irreducible.

There exists an open subset $U$ of $\varphi^{-1}(V)$ containing $p$ such that $\text{DC}_e(U)$ is locally irreducible. Let $G$ be the analytic Zariski closure of $U \setminus \text{DC}_e(U)$ in $U$. Let $W = \text{DC}_e(U) \setminus G$. $W$ is a nonempty open subset of $X$, so $\varphi(W)$ is an open subset of $V$, since $\varphi$ is flat [10]. Let $V^*$ be the irreducible component of $V$ containing $\varphi(\text{DC}_e(U))$. By definition, the induced map $\varphi : \text{DC}_e(U) \to V^*$ is regular at $p$.

Let $\tau : V' \to V$ be a resolution of singularities. $\tau$ is the blow up of a thin analytic set $E$, and $H = \varphi^{-1}(E)$ is thin in $X$ since $\varphi$ is flat. Let $\mathcal{I}$ be the ideal sheaf of $H$ in $X$.

Let $\pi : X' \to U$ be a resolution of singularities, obtained by a sequence of global blow ups of thin analytic sets, so that $\mathcal{I}O_{X'}$ is invertible. The composition of $\pi$ with the inclusion of $U$ into $X$ is in $e$. Since $\text{DC}_e(X')$ is a connected component of $X'$, the induced morphism $\text{DC}_e(X') \to X$ is in $e$.

$\text{DC}_e(X')$ is necessarily the strict transform of $\text{DC}_e(U)$ in $X'$. Thus $\text{DC}_e(X') \to \text{DC}_e(U)$ is a product of blow ups. Thus the induced morphism $\tilde{X} = \text{DC}_e(X') \to \tilde{Y} = (V')^*$ is regular by Proposition 4.11, where $(V')^*$ is the connected component of $V'$ which contains the image of $\text{DC}_e(X')$.

\[\Box\]

5. THE VALUATION ASSOCIATED TO AN ÉTOILE

Suppose that $Y$ is a reduced complex analytic space, $e \in \mathcal{E}_Y$ and $\pi \in e$. We will call $\pi$ nonsingular if $\pi$ is a composition of local blow ups

$$Y_{n-1} \to Y_{n-1} \to \cdots \to Y_1 \to Y$$

such that each $Y_i$ is nonsingular.

We associate to a nonsingular $\pi \in e$ the local ring $A_\pi = O_{X,\varphi(e)}$. The set

$$\{A_\pi \mid \pi \in e \text{ is nonsingular}\}$$
is then a directed set, by Lemma 4.3 and Definition 3.2. The set of quotient fields \( K_\pi \) of the \( A_\pi \) also form a directed set. Let
\[
\Omega_e = \lim_{\rightarrow} K_\pi \text{ and } V_e = \lim_{\rightarrow} A_\pi.
\]
\( \Omega_e \) is a field, and \( V_e \) is a local ring with quotient field \( \Omega_e \).

**Lemma 5.1.** \( V_e \) is a valuation ring.

**Proof.** Suppose that \( f \in K_e \). Then there exists \( \pi \in e \) such that \( f \in K_\pi \). \( f = \frac{g}{h} \) with \( g, h \in A_\pi \), where \( A_\pi \) is the local ring associated to \( \pi : X_\pi \to Y \). Let \( U \subset X_\pi \) be an open neighborhood of \( p_\pi(e) \) on which \( g \) and \( h \) are holomorphic. There exists an ideal sheaf \( \mathcal{I} \subset \mathcal{O}_U \) such that the blow up \( X' = B(\mathcal{I}) \) of \( \mathcal{I} \) is nonsingular, and \( (g,h)\mathcal{O}_{X'} \) is locally principal. Let \( \lambda : X' \to X_\pi \) be the induced local blow up. \( \pi \lambda \in e \) by Lemma 4.2. We have that either \( g \) or \( h \) is in \( A_\pi \). Thus \( f \) or \( \frac{1}{\lambda} \in A_\pi \subset V_e \).

**Proposition 5.2.** Suppose that \( X \) and \( Y \) are nonsingular complex analytic spaces, \( \varphi : X \to Y \) is a regular morphism, and \( e \in \mathcal{E}_X \). Then \( f = S(\varphi, e) \in \mathcal{E}_Y, \Omega_f \subset \Omega_e \) and \( V_f = V_e \cap \Omega_f \).

**Proof.** \( f \in \mathcal{E}_Y \) by Proposition 4.8.

Suppose that \( \pi : Y' \to Y \in f \) is nonsingular. By Lemma 4.3, there exists a nonsingular \( \alpha : Z \to X \in e \) such that \( \text{Hom}(Z, \varphi^{-1}[Y']) \neq \emptyset \). We have associated local homomorphisms
\[
O_{Y', p_\pi(f)}(\varphi') \to O_{\varphi^{-1}[Y'], p_{\varphi^{-1}[\pi]}(e)} \to \mathcal{O}_Z, p_\alpha(e)
\]
where \( \varphi' : \varphi^{-1}[Y'] \to Y' \) is the natural morphism. By Proposition 4.11, the homomorphism of the sequence of complete local rings
\[
\tilde{\mathcal{O}}_{Y', p_\pi(f)}(\varphi') \to \tilde{\mathcal{O}}_{\varphi^{-1}[Y'], p_{\varphi^{-1}[\pi]}(e)} \to \tilde{\mathcal{O}}_Z, p_\alpha(e)
\]
is 1-1. Thus the homomorphism in (6) is 1-1. We have an associated inclusion of rings \( A_\pi \to A_\alpha \) with induced inclusion of quotient fields \( K_\pi \to K_\alpha \). This gives us 1-1 homomorphisms \( A_\pi \to V_e \) and \( K_\pi \to \Omega_e \).

Taking the limit over the nonsingular elements of \( f \), we have natural 1-1 homomorphisms \( V_f \to V_e \) and \( \Omega_f \to \Omega_e \).

Suppose that \( h \in \Omega_f \cap \Omega_e \). Then there exist nonsingular \( \alpha : Y_1 \to Y \in f \) and \( \beta : X_1 \to X \in e \) such that \( h \in A_\beta \cap A_\alpha \). \( h \) has an expression \( h = \frac{\alpha a}{\beta b} \) with \( a, b \in A_\alpha \). Let \( U \) be a neighborhood of \( p_\alpha(f) \) on which \( a \) and \( b \) are analytic. There exists \( \gamma : Y_2 \to Y \) such that \( Y_2 \) is the blow up of an ideal sheaf of \( \mathcal{O}_U \), \( Y_2 \) is nonsingular and \( (f,g)\mathcal{O}_{Y_2} \) is locally principal. Thus \( \gamma \in f \) is nonsingular, and either \( h \) or \( \frac{1}{h} \in A_\gamma \). There exists \( \delta : X_2 \to X \in e \) which is nonsingular, such that \( \text{Hom}(X_2, \varphi^{-1}[Y_2]) \neq \emptyset \) and \( \text{Hom}(X_2, X_1) \neq \emptyset \). We have constructed a commutative diagram:

\[
\begin{array}{ccc}
X_2 & \xrightarrow{\delta} & X_1 \\
\downarrow & & \downarrow \\
Y_2 & \xleftarrow{\varphi^{-1}[Y_2]} & Y_1 \\
\downarrow & & \downarrow \\
X & \xleftarrow{\gamma} & Y
\end{array}
\]

If \( h \in A_\gamma \) then \( h \in V_f \). Suppose that \( h \notin A_\gamma \). Then \( \frac{1}{h} \in m \) where \( m \) is the maximal ideal of \( A_\gamma \). Now \( A_\gamma \to A_\delta \) is a local homomorphism, so \( \frac{1}{h} \) is in the maximal ideal \( n \) of \( A_\delta \). But this is impossible since \( h \notin A_\delta \). Thus we must have \( h \in A_\gamma \subset V_e \). □
Suppose that $Y$ is a reduced complex space and $e \in \mathcal{E}_Y$. Let $V_e$ be the valuation ring associated to $e$. We have a directed system $\{A_\pi\}$ for $\pi : Y' \to Y \in e$, where we define $A_\pi = \mathcal{O}_{DC_e(Y'),p_e(e)}$. In the case when $\pi$ is nonsingular, $Y' = DC_e(Y')$, so this agrees with our earlier definition. Taking the limit over this larger directed system again gives us the same limit $V_e$, by Lemma 5.3.

For $\pi : Y_0 \to Y \in e$, let $V_\pi = V_e \cap K_\pi$, which is a valuation ring of $K_\pi$, which dominates $A_\pi$. Let $m_\pi$ be the maximal ideal of $A_\pi$. Let $\nu_\pi$ be a valuation of $\Omega_e$ whose valuation ring is $V_\pi$, and let $\nu_\pi$ be the restriction of $\nu$ to $K_\pi$, so that $V_\pi$ is the valuation ring of $\nu_\pi$.

**Lemma 5.3.** $V_e$ has finite rational rank, which is less than or equal to $\dim Y$.

**Proof.** Suppose that $V_e$ has rational rank larger than $n = \dim Y$. Choose $t_1, \ldots, t_{n+1} \in V_e$ such that their values are rationally independent. There exists a nonsingular $\pi : Y' \to Y \in e$ such that $t_1, \ldots, t_{n+1} \in V_\pi$. Thus $\nu_\pi$ has rational rank $> \dim Y$. But $\nu_\pi$ dominates the noetherian local domain $\mathcal{O}_{Y',p_\pi(e)}$, which has dimension $\leq \dim Y$. This is a contradiction to Abhyankar’s inequality [1], Appendix 2 of [22].

Thus the rank $r$ of $V_e$ is finite (by Lemma 5.3), with $r = \operatorname{rank}(V_e) \leq \operatorname{ratrank}(V_e) \leq \dim Y$.

6. **Weak Local Monomialization When $\dim \varphi(X) = \dim X$**

**Theorem 6.1.** Suppose that $\varphi : X \to Y$ is a germ of a morphism of reduced, irreducible, locally irreducible complex analytic spaces, such that $\dim \varphi(X) = \dim X$. Suppose that $e \in \mathcal{E}_X$ is an étale. Then there exists a sequence of local blow ups $\alpha : X_1 \to X$ and $\beta : Y_1 \to Y$ such that $\alpha \in e$, making a commutative diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\varphi_1} & Y_1 \\
\downarrow\alpha & & \downarrow\beta \\
X & \xrightarrow{\varphi} & Y
\end{array}
$$

such that $X_1$ and $Y_1$ are nonsingular, and $\varphi_1$ is a monomial mapping at the center $p_1 = p_\alpha(e)$ of $e$ on $X_1$; that is, there exist local coordinates $y_1, \ldots, y_n$ in $Y_1$ at $\varphi_1(p_1)$ and local coordinates $x_1, \ldots, x_m$ in $X_1$ at $p_1$, and an $m \times m$ matrix $(a_{ij})$ of nonnegative integers with $\operatorname{Det}(a_{ij}) \neq 0$, such that under the map $\varphi_1^* : \mathcal{O}_{Y_1,p_1} \to \mathcal{O}_{X_1,p}$, we have that

$$
\varphi_1^*(y_i) = \left\{ \begin{array}{ll}
\prod_{j=1}^m x_j^{a_{ij}} & \text{if } 1 \leq i \leq m \\
0 & \text{if } i > m
\end{array} \right.
$$

**Proof.** By Theorem 4.10 there exists a commutative diagram of morphisms

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\varphi}} & \tilde{Y} \\
\downarrow\gamma & & \downarrow\delta \\
X & \xrightarrow{\varphi} & Y
\end{array}
$$

such that $\gamma \in e$, $\delta$ is a sequence of morphisms consisting of local blow ups, $\tilde{Y}$ is an analytic subset of $Y^*$, with the properties that $\tilde{X}$, $\tilde{Y}$ and $Y^*$ are reduced and equidimensional, and $\tilde{\varphi}$ is flat at $p_\gamma(e)$.

$\tilde{\varphi} : \tilde{X} \to \tilde{Y}$ is an open mapping in a neighborhood $U$ of $p := p_\gamma(e)$ (by 10). Let $q = \tilde{\varphi}(p)$. Since $\tilde{\varphi}$ is open, we have that $\dim \tilde{\varphi}(U) = \dim \tilde{Y}$. We have that

$$
\dim U \geq \dim \tilde{\varphi}(U) \geq \dim \delta \tilde{\varphi}(U) = \dim \varphi(X) = \dim(X) = \dim(U),
$$

so that $\dim U = \dim \tilde{\varphi}(U)$.
By Remmert’s open mapping theorem (Theorem 2 of V.6 [19]) or in fact by Lemma 4 of V.6 [19], we have that \( \dim \ell_a \tilde{\varphi} = \dim U - \dim \tilde{Y} = 0 \) for each \( a \in U \). Thus \( \tilde{\varphi}: U \to \tilde{Y} \) is “light” at \( p \), and by the discussion on page 300 of V.7.2 [19], we have that the extension \( \mathcal{O}_{\tilde{Y},p} \to \mathcal{O}_{U,p} \) is finite.

Let \( \mathfrak{p} \) be the prime ideal in \( \mathcal{O}_{U,p} \) of the germ \( DC_e(U) \) at \( p \) and let \( \mathfrak{q} \) be the prime ideal in \( \mathcal{O}_{Y^*,q} \) of the germ of the Zariski closed set \( \tilde{\varphi}(DC_e(U)) \) (this set is Zariski closed by Remmert’s Rank Theorem, Theorem 1, V.6 [19]).

Let \( \nu_e \) be a valuation associated to \( e \), which has the valuation ring \( V_e \) of \( \Omega_e \) constructed in the previous section. Let

\[
S = \mathcal{O}_{DC_e(U),p} = \mathcal{O}_{\tilde{Y},q}/\mathfrak{q},
\]

and

\[
R = \mathcal{O}_{\tilde{\varphi}(DC_e(U)),p} = \mathcal{O}_{\tilde{X},p}/\mathfrak{p},
\]

with inclusion \( \tilde{\varphi}^*: R \to S \), and so that the induced inclusion of quotient fields \( K \to L \) is a finite algebraic extension.

In the notation of the previous section, we have that \( S = A_\gamma \) and \( L = K_\gamma \). Let \( V_\nu = V_e \cap L \) be the valuation ring of the restriction of \( \nu_e \) to a valuation \( \nu \) of \( L = K_\gamma \) as described in the previous section. We will work for a while in the category of algebraic schemes (instead of complex analytic spaces).

Let \( r \) be the rank of \( \nu \), and let

\[
(0) = P_0 \subset P_1 \subset \cdots \subset P_r
\]

be the chain of prime ideals in the valuation ring \( V_\nu = V_\gamma \) in \( L \) of \( \nu \). We have inclusions of fields

\[
(R/P_i \cap R)_{P_i \cap R} \to (S/P_i \cap S)_{P_i \cap S}
\]

for \( 0 \leq i \leq r \) which are all finite algebraic, since \( R \to S \) is finite.

By Lemma 2.3 there exist subfields \( k_i \) of \( R_{P_i \cap R} \), with \( 0 \leq i \leq n \), such that the induced maps \( k_i \to (R/P_i \cap R)_{P_i \cap R} \) are finite.

By [14], there exists a blow up \( Y_1 = \text{Proj}(\bigoplus_{s \geq 0} b^s) \to \text{Spec}(\mathcal{O}_{\tilde{Y},q}) \) of an ideal \( b \) in \( \mathcal{O}_{\tilde{Y},q} \), such that \( Y_1 \) is nonsingular, and the strict transform \( \overline{Y}_1 \) of \( \text{Spec}(\mathcal{O}_{\tilde{Y},q}/\mathfrak{q}) \) in \( Y_1 \) is nonsingular. There also exists a blow up \( X_1 = \text{Proj}(\bigoplus_{s \geq 0} a^s) \to \text{Spec}(\mathcal{O}_{\tilde{X},p}) \) of an ideal \( a \) in \( \mathcal{O}_{\tilde{X},p} \), such that the strict transform \( \overline{X}_1 \) of \( \text{Spec}(\mathcal{O}_{\tilde{X},p}/\mathfrak{p}) \) in \( X_1 \) is nonsingular and is a connected component of \( X_1 \), and such that \( b\mathcal{O}_{\overline{X}_1} \) is an invertible ideal sheaf, so that we have an induced morphism \( \overline{X}_1 \to \overline{Y}_1 \). Let \( S_1 \) be the local ring of the center of \( \nu \) on \( X_1 \), and let \( R_1 \) be the local ring of the center of \( \nu \) on \( \overline{Y}_1 \).

By the discussion before the assumptions required for application of Theorem 5.3 [6], there exist sequences of monoidal transforms \( R_1 \to R_2 \) and \( S_1 \to S_2 \) along \( \nu \), such that we have an inclusion \( R_2 \to S_2 \) and

\[
\text{trdeg}_{(R_2/P_i \cap R_2)_{P_i \cap R_2}}(V_\nu/P_i)_{P_i} = 0
\]

for \( 0 \leq i \leq r \). Since \( R_2 \) is essentially of finite type over \( R \) and \( S_2 \) is essentially of finite type over \( S \), we have that \( (R_2/P_i \cap R_2)_{P_i \cap R_2} \) is a finitely generated extension field of \( (R/P_i \cap R)_{P_i \cap R} \), and \( (S_2/P_i \cap S_2)_{P_i \cap S_2} \) is a finitely generated extension field of \( (S/P_i \cap S)_{P_i \cap S} \). Thus \( (S_2/P_i \cap S_2)_{P_i \cap S_2} \) is a finitely generated extension field of \( k_i \) for \( 0 \leq i \leq r \).

We have thus realized the assumptions required for application of Theorem 5.3 [6] (the statement of the existence of \( k_i \), which is necessary for the application of Theorem 5.1 [6] in the course of the proof Theorem 5.3, was omitted from the statement of Theorem 5.3 in
Thus, there exists a sequence of monoidal transforms \( R_2 \to R_3 \) and \( S_2 \to S_3 \) along \( \nu \) such that we have an inclusion \( R_3 \to S_3 \), such that \( R_3 \) has regular parameters \( z_1, \ldots, z_m \), \( S_3 \) has regular parameters \( w_1, \ldots, w_m \), there exist units \( \delta_i \in S_3 \) for \( 1 \leq i \leq m \) and an \( m \times m \) matrix \( (a_{ij}) \) of non negative integers with \( \text{Det}(a_{ij}) \neq 0 \) such that there are expressions

\[
z_i = \prod_{j=1}^{m} w_j^{a_{ij}} \delta_i
\]

for \( 1 \leq i \leq m \). Here \( m = \dim X = \dim R_2 = \dim S_2 \), since the residue field of \( V_\nu \) is \( \mathbb{C} \), which is the residue field of \( S \), and \( S \) has dimension \( m \).

By Resolution of Singularities [14], there exist ideals \( I \subset O_{X,p} \) and \( J \subset O_{Y^*,q} \) such that there is a morphism of schemes \( \Phi : Z = \text{Proj}(\bigoplus_{s \geq 0} I^s) \to W = \text{Proj}(\bigoplus_{s \geq 0} J^s) \) such that \( S_3 = O_{Z,p'} \) is the local ring of a closed point \( p' \) in \( Z \), \( O_{W,q'} \) is a regular local ring, where \( q' = \Phi(p') \), and there exists a nonsingular closed subscheme \( W' \) of \( W \) such that \( R_3 = O_{W',q'} \) is the local ring of \( q' \) in \( W' \). Thus there exist regular parameters \( z_1, \ldots, z_m, z_{m+1}, \ldots, z_n \) in \( O_{W,q} \) such that the stalk of the ideal sheaf \( I_{W'} \) of \( W' \) at \( q' \) is \( I_{W',q'} = (z_{m+1}, \ldots, z_n) \), \( I_{W',q'} \) is the kernel of \( \Phi_{p'} : O_{W',q'} \to O_{Z,p'} \), and the induced map \( O_{W',q'} \to O_{Z,p'} \) is the extension of (5) constructed above.

Let \( V \) be an analytic neighborhood of \( q \) in \( Y^* \) such that \( J \) extends to a coherent sheaf of ideals \( J \) on \( V \). Let \( \beta : Y_1 \to Y^* \) be the local blow up of \( J \) above \( V \). Let \( U \) be an analytic neighborhood of \( p \) in \( X \), such that \( J \) extends to a coherent sheaf of ideals \( I \) on \( U \). Let \( \alpha : X_1 \to X \) be the local blow up of \( I \) above \( U \). We have that \( \gamma \alpha \in e \). Taking \( U \) to be a sufficiently small neighborhood of \( p \), we have that \( J \) is an invertible ideal sheaf (since \( J \) is invertible), so that we have an induced morphism of analytic spaces \( \varphi_1 : X_1 \to Y_1 \). Let \( p_1 = p_0(e) \) and \( q_1 = \varphi_1(p_1) \). Since the restriction of the valuation \( \nu_e \) to \( L \) is \( \nu \), we have that \( O_{X_1,p_1} \) is the analytic local ring constructed from \( R_3 \), \( O_{Y_1,q_1} \) is the analytic local ring constructed from \( S_3 \), and \( \varphi_1^* : O_{Y_1,q_1} \to O_{X_1,p_1} \) is the analyzation of the homomorphism \( \Phi_{p'} \). We thus have that \( p_1 \) is a nonsingular point of \( X_1 \) and \( q_1 \) is a nonsingular point of \( Y_1 \). Since \( \text{Det}(a_{ij}) \neq 0 \), and \( O_{Y_1,q_1} \) is Henselian, we can make a change of variables at \( q_1 \) to get an expression (3) such that \( \delta_i = 1 \) for all \( i \), giving the desired monomial form of the statement of the theorem. \( \square \)

References

[1] S. Abhyankar, On the valuations centered in a local domain, Amer. J. Math. 78, 321 - 348 (1956).
[2] D. Abramovich and K. Karu, Weak semistable reduction in characteristic 0, Invent. Math. 139 (2000), 241 - 273.
[3] D. Abramovich, J. Denef and K. Karu, Weak toroidalization over non closed fields, preprint.
[4] J. M. Aroca, H. Hironaka and J. L. Vicente, Introduction to the theory of infinitely near singular points, The theory of maximal contact, Desingularization theorems, Memorias de matemática del Instituto “Jorge Juan” 28 (1974), 29 (1975), 30 (1977).
[5] E. Bierstone and P. Milman Subanalytic Geometry, in Model Theory, Algebra and Geometry, MSRI Publications 39 (2000).
[6] S.D. Cutkosky, Local monomialization and factorization of morphisms, Astérisque 260, Société mathématique de France, (1999).
[7] S.D. Cutkosky, Errata of Local monomialization and factorization of morphisms, www.math.missouri.edu/~dale
[8] S.D. Cutkosky, Local monomialization of transcendental extensions, Annales e L’Institut Fourier, 1517 - 1586, (2005).
[9] J. Denef, Monomialization of morphisms and \( p \)-adic quantifier elimination, preprint.
[10] Douady, Le problème des modules pour les sous-espaces analytiques compacts d’un espace analytique donné, Ann. Inst. Fourier (1966), 1-95.
[11] A.M. Gabrielov, Formal relations between analytic functions, Math USSR Izv. 7 (1973) 1056 - 1088.
[12] A. Grothendieck, and A. Dieudonné, Éléments de géométrie algébrique IV, vol. 2, Publ. Math. IHES 24 (1965), vol. 4, Publ. Math. IHES 32 (1967).
[13] R. Hartshorne, Algebraic Geometry, Springer Verlag (1977).
[14] H. Hironaka, Resolution of Singularities of an algebraic variety over a field of characteristic zero, Annals of Math., 79 (1964), 109-326.
[15] H. Hironaka, Desingularization of complex-analytic varieties, in Actes du Congrès International des mathématicains (Nice, 1970) Tome 2, 627 - 631, Gauthier-Cillars, Paris, 1971.
[16] H. Hironaka, Introduction to real-analytic sets and real-analytic maps, Quaderni dei Gruppi di Recerca Matematica del Consiglio Nazionale delle Ricerche, Istituto Matematico “L. Tonelli” dell’Università di Pisa, Pisa, 1973.
[17] H. Hironaka, La Voute étoilée, in Singularités à Cargèse, Astérisque 7 and 8, (1973).
[18] H. Hironaka, M. Lejeune-Jalabert and B. Teissier, Platificateur local en géométrie analytique et aplatissement local, in Singularités à Cargèse, Astérisque 7 and 8, (1973).
[19] S. Lojasiewicz, Introduction to complex analytic geometry, Birkhauser, (1991).
[20] H. Matsumura, Commutative Algebra, 2nd edition, Benjamin/Cummings (1980).
[21] M. Nagata, Local Rings, Wiley Interscience (1962).
[22] O. Zariski and P. Samuel, Commutative Algebra Vol II, Van Nostrand (1960).

Steven Dale Cutkosky, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: cutkoskys@missouri.edu