THE RELATIVE HELLER OPERATOR AND RELATIVE COHOMOLOGY FOR THE KLEIN 4-GROUP.

JONATHAN ELMER

Abstract. Let $G$ be the Klein Four-group and let $\mathbb{k}$ be an arbitrary field of characteristic 2. A classification of indecomposable $\mathbb{k}G$-modules is known. We calculate the relative cohomology groups $H^i_{\chi}(G, N)$ for every indecomposable $\mathbb{k}G$-module $N$, where $\chi$ is the set of proper subgroups in $G$. This extends work of Pamuk and Yalcin to cohomology with non-trivial coefficients. We also show that all cup products in strictly positive degree in $H^*_{\chi}(G, \mathbb{k})$ are trivial.

1. Introduction

Let $G$ be a finite group and $\mathbb{k}$ a field of characteristic $p > 0$. If $p \not|| |G|$, then every representation of $G$ over $\mathbb{k}$ is projective. Thus, by decomposing the regular module $\mathbb{k}G$ we can obtain all isomorphism classes of $\mathbb{k}G$-modules immediately.

From now on assume $p|| |G|$. Then the above is no longer true. However, it is well-known that, given a $\mathbb{k}G$-module $M$, we can find a projective module $P_0$ and a surjective $\mathbb{k}G$-morphism

$$\pi_0 : P_0 \rightarrow M.$$ 

If we choose $P_0$ and $\pi_0$ so that $P_0$ has smallest possible dimension, then this pair is unique, and known as the projective cover of $M$. The kernel of $\pi_0$ is denoted $\Omega(M)$. This is known as the Heller shift of $M$. $\Omega(-)$ can be viewed as an operation on the set of $\mathbb{k}G$-modules which takes indecomposable modules to indecomposable modules.

This construction can be iterated. For each $i > 0$, let $\pi_i : P_i \rightarrow \Omega_i(M)$ be the projective cover of $\Omega_i(M)$. By composing these maps with the inclusions $\Omega_i(M) \rightarrow P_{i-1}$, we obtain an exact sequence

$$\cdots P_i \rightarrow P_{i-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0.$$ 

This is an example of a projective resolution for $M$. If $N$ is any $\mathbb{k}G$-module, then the above induces a complex

$$0 \rightarrow \text{Hom}_{\mathbb{k}G}(P_0, N) \rightarrow \cdots \rightarrow \text{Hom}_{\mathbb{k}G}(P_i, N) \rightarrow \cdots$$

which is not exact in general. The homology groups of this complex are by definition the groups $\text{Ext}^i_{\mathbb{k}G}(M, N)$. A special case is

$$H^i(G, N) := \text{Ext}^i_{\mathbb{k}G}(\mathbb{k}, N).$$

We call this the cohomology of $G$ with coefficients in $N$.

There is a long and fruitful history of study of the cohomology groups $H^i(G, N)$ in modular representation theory. Further, one may define a pairing

$$\cdot : H^i(G, \mathbb{k}) \otimes H^j(G, \mathbb{k}) \rightarrow H^{i+j}(G, \mathbb{k})$$

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which gives $H^*(G, k)$ the structure of a graded-commutative graded ring. A celebrated theorem of Evens (see [3, Theorem 4.2.1]) states that, for any $G$, the ring $H^*(G, k)$ is finitely generated.

Now let $\chi$ be a set of proper subgroups of $G$. A $kG$-module $M$ is said to be projective relative to $\chi$ if $M$ is a direct summand of $\oplus_{X \in \chi} M \downarrow_X^{\chi^G}$. Other equivalent definitions will be given in section 2. It is less well-known, but still true, that every $kG$-module has a unique relative projective cover with respect to $\chi$. This is defined to be a $kG$-module $Q_0$ of smallest dimension such that

1. $Q_0$ is projective relative to $\chi$;
2. There is a surjective $kG$-morphism $\pi_0 : Q_0 \to M$ which splits on restriction to each $X \in \chi$.

The kernel of $\pi_0$ is denoted $\Omega_\chi(M)$ and called the relative Heller shift of $M$ with respect to $\chi$. We can mimic the construction of (1) to obtain a relative projective resolution of $M$, that is, an exact sequence

$\ldots Q_1 \to Q_0 \to \ldots \to Q_0 \to M \to 0.$

of $kG$ modules which are projective relative to $\chi$ and in which the connecting homomorphisms split over each $X \in \chi$. Given any $kG$-module $N$, the above induces a complex

$0 \to \text{Hom}_{kG}(Q_0, N) \to \ldots \to \text{Hom}_{kG}(Q_1, N) \to \ldots$

which is in general no longer exact. The homology groups of this complex are by definition the relative Ext-groups $\text{Ext}_{kG, \chi}^i(M, N)$. The relative cohomology of $G$ with respect to $\chi$ with coefficients in $N$ is the special case

$H^i_\chi(G, N) := \text{Ext}_{kG, \chi}^i(k, N).$

Further, one may define a pairing

$\sim : H^i_\chi(G, k) \otimes H^j_\chi(G, k) \to H^{i+j}_\chi(G, k)$

which gives $H^*_\chi(G, k)$ the structure of a graded-commutative graded ring.

Computations of $H^*_\chi(G, N)$ are rare in the literature. It is notable that the ring $H^*_{\chi_1}(G, k)$ is not finitely generated in general. This was first discovered by Blowers [4], who showed that if $G_1$ and $G_2$ are finite groups of order divisible by $p$, and $\chi_1, \chi_2$ are sets of subgroups of $G_1, G_2$ respectively with order divisible by $p$, then all products of elements of positive degree in $H^*_{\chi_i}(G, k)$ are zero, where $G = G_1 \times G_2$ and $\chi = \{G_1 \times X : X \in \chi_2\} \cup \{X \times G_2 : X \in \chi_1\}$. See also [5].

For the rest of this section, let $G = \langle \sigma, \tau \rangle$ denote the Klein four-group, and let $k$ be a field of characteristic 2. We set $\chi = \{H_1, H_2, H_3\}$, the set of all proper nontrivial subgroups of $G$, where $H_1 = \langle \sigma \rangle$, $H_2 = \langle \tau \rangle$, $H_3 = \langle \sigma \tau \rangle$.

The cohomology groups $H^i_\chi(G, k)$ were computed, by indirect means, by Pamuk and Yalcin [10]. In the present article we recover their result, and also compute $H^*_\chi(G, N)$ for any $kG$-module $N$. Our methods are more direct; we compute an explicit relative projective resolution for each $N$. Of course we are helped enormously by the fact that the representations of $G$ are completely classified. Our first main result is:

**Theorem 1.** Let $M$ be an indecomposable $kG$-module, which is not projective relative to $\chi$. Then we have

$\Omega_\chi(M) \cong \Omega^{-2}(M)$

if $M$ has odd dimension, and

$\Omega_\chi(M) \cong M$

otherwise.
The ring structure of \( H^*_\chi(G, k) \) was not considered in [10]. Note, however, that if \( \chi' \) is a subset of \( \chi \) with size 2, then all products in \( H^*_\chi(G, k) \) are zero, by a special case of Blowers’ result. It is perhaps not surprising, therefore, that we have

**Theorem 2.** Let \( \alpha_1, \alpha_2 \in H^*_\chi(G, k) \), where both have strictly positive degree. Then \( \alpha_1 \sim \alpha_2 = 0 \).

This paper is organised as follows. In section 2 we define relative projectivity and derive the results we will need to do the computations in later sections. This section follows [9, Section 2] fairly closely. As most proofs can be constructed by adapting familiar results on projectivity to the relative case, they are omitted. In section 3 we describe the classification of modules for the Klein-four group and prove Theorem 1. We also compute \( H^*_\chi(G, N) \) for every \( kG \)-module \( N \) and prove Theorem 2.

1.1. **Notation.** All groups under consideration are finite groups, and for any group \( G \), by a \( kG \)-module we mean a finitely-generated \( k \)-vector space with compatible \( G \) action. The one-dimensional trivial \( kG \)-module will be denoted by \( kG \) or simply \( k \) when the group acting is obvious, and for \( n \in \mathbb{N} \) and \( M \) a \( kG \)-module we write \( nM \) for the direct sum of \( n \) copies of \( M \).

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## 2. Relative projectivity

In this section, let \( p > 0 \) be a prime and let \( G \) be a finite group of order divisible by \( p \). Let \( k \) be a field of characteristic \( p \) and let \( \chi \) be a set of subgroups of \( G \). Now let \( M \) be a finitely generated \( kG \)-module. \( M \) is said to be \emph{projective relative to} \( \chi \) if the following holds: let \( \phi : M \to Y \) be a \( kG \)-homomorphism and \( j : X \to Y \) a surjective \( kG \)-homomorphism which splits on restriction to any subgroup of \( H \in \chi \), then there exists a \( kG \)-homomorphism \( \psi \) making the following diagram commute.

\[
\begin{array}{ccc}
X & \xrightarrow{j} & Y \\
\downarrow & & \downarrow \phi \\
M & \xrightarrow{\psi} & 0
\end{array}
\]

Dually, one says that \( M \) is \emph{injective relative to} \( \chi \) if the following holds: given an injective \( kG \)-homomorphism \( i : X \to Y \) which splits on restriction to each \( H \in \chi \) and a \( kG \)-homomorphism \( \phi : X \to M \), there exists a \( kG \)-homomorphism \( \psi \) making the following diagram commute.

\[
\begin{array}{ccc}
0 & \xrightarrow{i} & X \\
\downarrow & & \downarrow \phi \\
M & \xrightarrow{\psi} & Y
\end{array}
\]

These notions are equivalent to the usual definitions of projective and injective \( kG \)-modules when we take \( \chi = \{1\} \). We will say a \( kG \)-homomorphism is \( \chi \)-split if it splits on restriction to each \( H \in \chi \). Since a \( kG \)-module is projective relative to \( H \) if and only if it is also projective relative to the set of all subgroups of \( H \), we often assume \( \chi \) is closed under taking subgroups.
We denote the set of $G$-fixed points in $M$ by $M^G$. For any $H \leq G$ there is a $kG$-map $M^H \to M^G$ defined as follows:

$$\text{Tr}_H^G(x) = \sum_{\sigma \in S} \sigma x$$

where $x \in M$ and $S$ is a left-transversal of $H$ in $G$. This is called the relative trace or transfer. It is clear that the map is independent of the choice of $S$. If $H = 1$ we usually write this as $\text{Tr}_G^G$ and call it simply the trace or transfer. For any set of subgroups $\chi$ of $G$ we define the subspace

$$M^G,\chi := \sum_{H \in \chi} \text{Tr}_H^G(M^H)$$

and quotient

$$M^G_{\chi} := \frac{M^G}{M^G,\chi}.$$ 

Now let $N$ be another $kG$-module. We can define an action of $G$ on $\text{Hom}_k(M,N)$:

$$(g \cdot \phi)(x) = g\phi(g^{-1}x) \text{ for } g \in G, x \in M.$$ 

Notice that with this action we have $\text{Hom}_k(M,N)^G = \text{Hom}_{kG}(M,N)$. Further, the transfer construction gives a map

$$\text{Tr}_H^G : \text{Hom}_{kH}(M,N) \to \text{Hom}_{kG}(M,N).$$

There are various ways to characterize relative projectivity:

**Proposition 3.** Let $G$ be a finite group of order divisible by $p$, $\chi$ a set of subgroups of $G$ and $M$ a $kG$-module. Then the following are equivalent:

(i) $M$ is projective relative to $\chi$;

(ii) Every $\chi$-split epimorphism of $kG$-modules $\phi : N \to M$ splits;

(iii) $M$ is injective relative to $\chi$;

(iv) Every $\chi$-split monomorphism of $kG$-modules $\phi : M \to N$ splits;

(v) $M$ is a direct summand of $\oplus_{H \in \chi} M^H \uparrow^G$;

(vi) $M$ is a direct summand of a direct sum of modules induced from subgroups in $\chi$;

(vii) There exists a set of homomorphisms $\{\beta_H : H \in \chi\}$ such that $\beta_H \in \text{Hom}_{kH}(M,M)$ and $\sum_{H \in \chi} \text{Tr}_H^G(\beta_H) = \text{id}_M$.

The last of these is called Higman’s criterion.

**Proof.** The proof when $\chi$ consists of a single subgroup of $G$ can be found in [2, Proposition 3.6.4]. This can easily be generalised. □

For homomorphisms $\alpha \in \text{Hom}_{kG}(M,N)$ we have the following:

**Lemma 4.** Let $M$, $N$ be $kG$-modules, $\chi$ a collection of subgroups of $G$, and $\alpha \in \text{Hom}_{kG}(M,N)$. Then the following are equivalent:

(i) $\alpha$ factors through $\oplus_{H \in \chi} M^H \downarrow^G$.

(ii) $\alpha$ factors through some module which is projective relative to $\chi$.

(iii) There exist homomorphisms $\{\beta_H : H \in \chi\} \in \text{Hom}_{kH}(M,N)$ such that $\alpha = \sum_{H \in \chi} \text{Tr}_H^G(\beta_H)$.

**Proof.** This is easily deduced from [2, Proposition 3.6.6]. □

The above tells us that $\text{Hom}_k(M,N)^{G,\chi}$ consists of the $kG$-homomorphisms which factor through a module which is projective relative to $\chi$. We write

$$\text{Hom}^\chi_{kG}(M,N) := \text{Hom}_k(M,N)^G_{\chi}.$$
Let $M$ be a $kG$-module and let $X$ be a $kG$-module that is projective relative to $\chi$. It is easily shown, using Proposition 3, that $M \otimes X$ is projective relative to $\chi$. For example, the module $M \otimes X$ where $X = \bigoplus_{H \in \chi} k_H \uparrow^G$ is projective relative to $\chi$. Moreover, with $X$ as defined above, the natural map $\sigma : M \otimes X \to M$ given by
\[\sigma(m \otimes x) = m\]
is a $\chi$-split $kG$-epimorphism (to see the splitting, use the Mackey Theorem). It follows that for each $M$, there exists a $kG$-module $Q_0$ which is projective relative to $\chi$ and a $\chi$-split $kG$-epimorphism $\pi_0 : Q_0 \to M$.

Let $\pi_0 : Q_0 \to M$ and $\pi'_0 : Q'_0 \to M$ be two such pairs. The proof of Schanuel’s Lemma (see [2, Lemma 1.5.3, Lemma 3.9.1]) extends more or less verbatim to the relative case; if $K_0 = \ker(\pi_0)$ and $K'_0 = \ker(\pi'_0)$ then $K_0 \oplus Q'_0 \cong K'_0 \oplus Q_0$.

If we choose among all such pairs, one in which the dimension of $Q_0$ is minimal, the kernel $K_0$ is defined uniquely. This pair $(Q_0, \pi_0)$ is called the relative projective cover of $M$. For this choice we set $\Omega_\chi(M) = K_0$. We can iterate this construction, setting $\Omega_\chi(M) = \Omega_\chi(\Omega^{-1}_\chi(M))$. Minimality implies that if $K'_i$ is the kernel of any other $\chi$-split $kG$-epimorphism $Q'_0 \to M$, then $K'_i \cong \Omega_\chi(M) \oplus (\text{rel. proj})$, where (rel. proj) is some module which is projective relative to $\chi$.

Dually, we always have that $M$ is a submodule of $M \otimes X$ with $X = \bigoplus_{H \in \chi} k_H \uparrow^G$, and the inclusion $\rho : M \to M \otimes X$ splits on restriction to each $H \in \chi$. It follows that for each $M$, there exists a $kG$-module $J_0$ and a $\chi$-split $kG$-monomorphism $\rho_0 : M \to J_0$.

Let $\rho_0 : M \to J_0$ and $\rho'_0 : M \to J'_0$ be two such pairs. Again, by the relative version of Schanuel’s Lemma, if $C_0 = \coker(\pi)$ and $C'_0 = \coker(\pi'_0)$ then $C_0 \oplus J'_0 \cong C'_0 \oplus J_0$.

If we choose among all such pairs, one in which the dimension of $J_0$ is minimal, the cokernel $C_0$ is defined uniquely. The pair $(J_0, \rho_0)$ is called a relative injective hull of $M$ with respect to $\chi$. For this choice we set $\Omega^-_\chi(M) = C_0$. We can iterate this construction, setting $\Omega^{-1}_\chi(M) = \Omega^{-1}_\chi(\Omega^{-1}(M))$. Minimality implies that if $C'_i$ is the kernel of any other $\chi$-split $kG$-monomorphism $M \to J_0$, then $C'_i \cong \Omega^{-1}_\chi(M) \oplus (\text{rel. proj})$, where (rel. proj) is some module which is projective relative to $\chi$.

The following gives some properties of the operators $\Omega^i_\chi$.

**Proposition 5.** Let $M_1, M_2$ be $kG$-modules without summands which are projective relative to $\chi$, and $i, j$ nonzero integers. Then:

(i) $\Omega^i_\chi(M_1 \oplus M_2) \cong \Omega^i_\chi(M_1) \oplus \Omega^i_\chi(M_2)$;
(ii) $\Omega^i_\chi(M)^* \cong \Omega^{-1}(M^*)$;
(iii) $M \cong \Omega_\chi(\Omega^{-1}_\chi(M)) \oplus (\text{rel. proj}) \cong \Omega^{-1}_\chi(\Omega_\chi(M)) \oplus (\text{rel. proj})$.

**Proof.** (i) is obvious. (ii,iii) are easily deduced from the relative version of Schanuel’s Lemma. \(\square\)

(i) above shows that $\Omega^i_\chi$ is a well-defined operator on the set of indecomposable $kG$-modules which are not relatively projective to $\chi$. Note that (iii) does not say that $\Omega_\chi \circ \Omega^{-1}_\chi$ is the identity in general. If we define $\Omega^0_\chi(M)$ to be the direct sum of all summands of $M$ which are not projective relative to $\chi$, then we have $\Omega^{i+j} = \Omega^i_\chi \circ \Omega^j_\chi$ for all $i$ and $j$.

The following result is sometimes useful.

**Lemma 6.** Let $M$ be a $kG$-module which is projective relative to a set $\chi$ of subgroups of $G$. Then $M^G = \bigoplus_{H \in \chi} \text{Tr}_H^G(MH)$.

**Proof.** See [9, Lemma 2.9] \(\square\)
As a consequence of the above, if \( M = N \oplus (\text{rel. proj.}) \), we get that \( M^G = N^G \).

The operators \( \Omega^1 \) extend in a natural way to homomorphisms between modules. Let \( f \in \text{Hom}_k(M,N) \). Let \((Q, \pi), (Q', \pi')\) be the relative projective covers of \( M, N \). Then the relative projectivity of \( Q \) ensures the existence of a homomorphism \( \bar{f} \in \text{Hom}_k(Q, Q') \) making the following diagram commute

\[
\begin{array}{c}
\Omega_\chi(M) \quad \rightarrow \quad Q \quad \pi \quad \rightarrow \quad M \quad \rightarrow \quad 0 \\
\Omega_\chi(f, \bar{f}) \quad \quad \downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \\
\Omega_\chi(N) \quad \rightarrow \quad Q' \quad \pi' \quad \rightarrow \quad N \quad \rightarrow \quad 0 \\
\end{array}
\]

and an easy diagram chase shows that the image of \( \Omega_\chi(f, \bar{f}) := \bar{f}|_{\ker(\pi)} \) is contained in \( \ker(\pi') \). In this way, \( f \) induces a homomorphism

\[
\Omega_\chi(f, \bar{f}) \in \text{Hom}_k(\Omega_\chi(M), \Omega_\chi(N)).
\]

Moreover, this homomorphism factors through a relative projective if and only if \( f \) does so.

The homomorphism \( \Omega_\chi(f, \bar{f}) \) depends, as the notation suggests, on the choice of \( \bar{f} \) in general. However, if \( \bar{f} \) and \( \tilde{f} \in \text{Hom}_k(Q, Q') \) are both homomorphisms making the diagram commute, then one can show that

\[
\Omega_\chi(f, \bar{f}) - \Omega_\chi(f, \tilde{f})
\]

factors through a relative projective.

For a given homomorphism \( f : M \rightarrow N \), denote by \([f]\) its equivalence class in \( \text{Hom}_k^\chi(M, N) \). By the discussion following Lemma 4, the equivalence class

\[
[[\Omega_\chi(f, \bar{f})]] \in \text{Hom}_k^\chi(\Omega_\chi(M), \Omega_\chi(N))
\]

does not depend on \( \bar{f} \), so we write this as \( \Omega_\chi[f] \). In this way, we obtain a well-defined homomorphism

\[
\Omega_\chi : \text{Hom}_k^\chi(M, N) \rightarrow \text{Hom}_k^\chi(\Omega_\chi(M), \Omega_\chi(N)).
\]

In a similar fashion, let \((J, \rho), (J', \rho')\) be the relative injective hulls of \( M, N \) respectively. Then relative injectivity of \( J' \) ensures the existence of a homomorphism \( \bar{f} \in \text{Hom}(J, J') \) making the following diagram commute,

\[
\begin{array}{c}
\Omega^{-1}_\chi(M) \quad \rightarrow \quad J \quad \leftarrow \quad M \quad \rightarrow \quad 0 \\
\Omega^{-1}_\chi(f, \bar{f}) \quad \quad \downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \\
\Omega^{-1}_\chi(N) \quad \rightarrow \quad J' \quad \leftarrow \quad N \quad \rightarrow \quad 0 \\
\end{array}
\]

and a diagram chase shows that \( \bar{f} \) induces a homomorphism

\[
\Omega^{-1}_\chi(f, \bar{f}) \in \text{Hom}(\Omega^{-1}_\chi(M), \Omega^{-1}_\chi(N)).
\]

Moreover \( \Omega^{-1}_\chi(f, \bar{f}) \) factors through a projective if and only if \( f \) does so, and although \( \Omega^{-1}_\chi(f, \bar{f}) \) depends on the choice of \( \bar{f} \) in general, the equivalence class
[Ω\(^{-1}\chi\)](f, \tilde{f}) depends only on \(f\), so we write it as \(Ω\(^{-1}\chi\)[f]\). Thus, we obtain a well-defined homomorphism

\[ \Omega^{-1}_\chi : \text{Hom}_{kG}(M, N) \to \text{Hom}_{kG}(Ω^{-1}_\chi(M), Ω^{-1}_\chi(N)). \]

One can show further that, for \([f] \in \text{Hom}_{kG}(M, N)\) we have

\[ [f] = Ω^{-1}_\chi Ω_\chi [f] = Ω_\chi Ω^{-1}_\chi [f], \]

which justifies the following:

**Proposition 7.** For all \(i \in \mathbb{Z}\), \(Ω^i_\chi(−)\) induces an isomorphism

\[ \text{Hom}^X_{kG}(M, N) \cong \text{Hom}^X_{kG}(Ω^i_\chi(M), Ω^i_\chi(N)). \]

As explained in the introduction, the idea of a relatively projective cover can be extended to a relatively projective resolution; that is, an exact complex

\[ \ldots \to Q_i \to Q_{i-1} \to \ldots \to Q_0 \to M \to 0 \]

of relatively projective modules in which the connecting homomorphisms split over \(χ\). If

\[ \ldots \to Q'_i \to Q'_{i-1} \to \ldots \to Q'_0 \to M \to 0 \]

is another relatively projective resolution, then it turns out that any two chain maps between them are chain homotopic (see [2, Theorem 3.9.3] for the version with \(χ\) consisting of one subgroup - the proof of the more general version is the same). Consequently, for any \(kG\)-module \(N\), the homology groups of the induced complex

\[ 0 \to \text{Hom}_{kG}(Q_0, N) \to \ldots \to \text{Hom}_{kG}(Q_i, N) \to \ldots \]

are independent of the choice of resolution. The homology groups of this complex are by definition the relative Ext-groups \(\text{Ext}^i_{kG,χ}(M, N)\). The relative cohomology of \(G\) with respect to \(χ\) with coefficients in \(N\) is the special case

\[ H^i_\chi(G, N) := \text{Ext}^i_{kG,χ}(k, N). \]

We will use a minimal relative projective resolution of the trivial module to compute relative cohomology; that is, a relatively projective resolution

\[ \ldots \to Q_i \xrightarrow{∂_i} Q_{i-1} \to \ldots \to Q_0 \to k \to 0 \]

in which \(\ker(∂_{i-1}) = Ω^i_\chi(k)\). We can construct this by taking for each \(i\) a short exact sequence

\[ 0 \to Ω^{i+1}_\chi(k) \xrightarrow{ρ_i} Q_i \xrightarrow{π_i} Ω^i_\chi(k) \to 0 \]

and setting \(∂_i := ρ_iπ_{i+1}\). For each \(i\) let

\[ δ_i : \text{Hom}_{kG}(Q_i, k) \to \text{Hom}_{kG}(Q_{i+1}, k) \]

denote the map induced by \(∂_i\).

Our main tool will be the following:

**Proposition 8.** Let \(N\) be a \(kG\)-module. Then we have

(i) \(H^0_\chi(G, N) = N^G;\)

(ii) \(H^1_\chi(G, N) \cong \text{Hom}^X_{kG}(Ω^0_χ(k), N).\)

The proof is the same as in the case \(χ = \{1\}\), but we give a sketch for lack of a good reference to this proof.
Proof. We first show that for each \( i \geq 0 \),
\[
\ker(\delta_i) \cong \text{Hom}_{k[G]}(\Omega^i_k(k), N).
\]
To see this, let \( \phi \in \ker(\delta_i) \subseteq \text{Hom}_{k[G]}(Q_i, N) \). For \( x \in \Omega_k^i(k) \), choose \( q \in Q_i \) such that \( \pi_i(q) = x \) and define \( \hat{\phi}(x) = \phi(q) \). Then \( \hat{\phi} \in \text{Hom}_{k[G]}(\Omega_k^i(k), N) \). The assignment \( \phi \mapsto \hat{\phi} \) is well-defined: for if \( q' \in Q_i \) with \( \pi_i(q') = x \) and \( \hat{\phi}(x) = \phi(q') \), then since \( q - q' \in \ker(\pi_i) \) we get \( q - q' \in \ker(\delta_i) \) and \( \hat{\phi}(q - q') = 0 \) since \( \phi \in \ker(\delta_i) \).

Conversely, given \( \phi \in \text{Hom}_{k[G]}(\Omega_k^i(k), N) \) we can define \( \hat{\phi} = \phi \circ \pi_i \in \ker(\delta_i) \). It’s easy to see that the two assignments are inverse to each other.

This in particular shows that (i) holds, since \( \text{Hom}_{k[G]}(k, N) \cong N^G \). We now show that \( \text{im}(\delta_{i-1}) \) consists of the homomorphisms in \( \text{Hom}_{k[G]}(\Omega_k^i(k), N) \) which factor through a module which is projective relative to \( \chi \). To see this, first suppose \( \phi \in \text{im}(\delta_{i-1}) \subseteq \text{Hom}_{k[G]}(Q_i, N) \), say \( \phi = \psi \circ \partial_{i-1} \) where \( \psi \in \text{Hom}_{k[G]}(Q_i, N) \). Then with \( x \in \Omega_k^i(k) \) and \( q, \phi \) as before we note that
\[
\psi \circ \rho_{i-1}(x) = \psi \circ \rho_{i-1} \circ \pi_i(q) = \psi \circ \partial_i(q) = \hat{\phi}(q) = \hat{\phi}(x)
\]
which shows that \( \hat{\phi} \) factors through the module \( Q_{i-1} \) which is projective relative to \( \chi \). Conversely, if \( \phi \in \text{Hom}_{k[G]}(\Omega_k^i(k), N) \) factors through any module which is projective relative to \( \chi \), then it factors through \( Q_{i-1} \), because \( \rho_{i-1} \) is injective and \( Q_{i-1} \) is also an injective module with respect to \( \chi \) by Lemma 3.

One can define a pairing \( \sim : H^1_k(G, k) \otimes H^1_k(G, k) \to H^{1+1}_k(G, k) \) in a few different ways. On the one hand, elements of \( H^1_k(G, k) = \text{Ext}_{k[G]}^1(k, k) \) can be viewed as equivalence classes of extensions of \( k \) by \( k \) split over \( \chi \), and the usual Yoneda splice gives the required pairing; see [2, Section 2.6.3.9] for details in the case \( \chi = \{1\} \) are given in [6], and all of these extend in a natural way to arbitrary \( \chi \). Happily, all these methods give the same construction. In the present article we will use the following construction: recall that
\[
H^1_k(G, k) \cong \text{Hom}_{k[G]}(\Omega^1_k(k), k).
\]
Similarly
\[
H^1_k(G, k) = \text{Hom}_{k[G]}(\Omega^1_k(k), k) \cong \text{Hom}_{k[G]}(\Omega^{1+1}_k(k), \Omega^1_k(k))
\]
with the second isomorphism arising from Proposition 7. Therefore we may define a product as follows: for \( \alpha \in H^1_k(G, k) \) and \( \beta \in H^1_k(G, k) \) choose \( f \in \text{Hom}_{k[G]}(\Omega^1_k(k), k) \), \( g \in \text{Hom}_{k[G]}(\Omega^1_k(k), k) \) representing \( \alpha, \beta \) respectively. Then \( \Omega^1_k(g) \in \text{Hom}_{k[G]}(\Omega^{1+1}_k(k), \Omega^1_k(k)) \), so that
\[
f \circ \Omega^1_k(g) \in \text{Hom}_{k[G]}(\Omega^{1+1}_k(k), k).
\]
We take \( \alpha \sim \beta \) to be the cohomology class represented by \( f \circ \Omega^1_k(g) \). This is called the cup product of \( \alpha \) and \( \beta \).

3. Representations of \( C_2 \times C_2 \)

In this section, let \( G = \langle \sigma, \tau \rangle \) denote the Klein four-group, and let \( k \) be a field of characteristic 2 (not necessarily algebraically closed). We set \( \chi = \{H_1, H_2, H_3\} \), the set of all proper non-trivial subgroups of \( G \), where \( H_1 = \langle \sigma \rangle, H_2 = \langle \tau \rangle, H_3 = \langle \sigma \tau \rangle \).

Let \( X : = \sigma - 1 \in kG, Y : = \tau - 1 \in kG \). Then \( X^2 = Y^2 = 0, \langle X, Y \rangle \) is isomorphic to the quotient ring
\[
R : = k[X, Y]/(X^2, Y^2),
\]
and \( kG \)-modules can be viewed as \( R \)-modules. We will describe \( R \)-modules by means of the diagrams for modules popularised by Alperin in [1]. In these diagrams,
nodes represent basis elements, and two nodes labelled \( a \) and \( b \) are joined by a south-west directed arrow if \( Xa = b \), and by a south-east directed arrow if \( Ya = b \). If no south-west arrow begins at \( a \) then it is understood that \( Xa = 0 \), similarly for \( Y \).

Our statement of the classification of \( kG \)-modules resembles that found in [7], which is based on calculations first found in [8]. We recommend the former reference as an easily accessible proof.

**Proposition 9.** Let \( M \) be an indecomposable \( kG \)-module. Then \( M \) is isomorphic to one of the following:

1. The module \( V_{2n+1} \) (\( n \geq 0 \)), with odd dimension \( 2n + 1 \) and diagram

2. The module \( V_{-(2n+1)} \) (\( n \geq 0 \)), with odd dimension \( 2n + 1 \) and diagram

Note that \( V_1 \cong V_{-1} \cong k \), with trivial \( G \)-action, but otherwise these modules are pairwise non-isomorphic.

3. The module \( V_{2n, \infty} \) (\( n \geq 1 \)), with even dimension \( 2n \) and diagram

4. The module \( V_{2n, \theta} \) (\( n \geq 1 \)), with even dimension \( 2n \) and diagram,

Here, \( \theta(x) = \sum_{i=0}^{n} \lambda_i x^{n-i} \) is a power of an irreducible monic polynomial with coefficients in \( k \) and the dotted line labelled by \( \theta \) indicates that \( Xa_1 = \sum_{i=1}^{n} \lambda_i b_1 \).

5. The projective indecomposable module \( P \), with dimension 4 and diagram

The following, also taken from [7], may be proved directly from the classification above.

**Proposition 10.** Let \( M \) be an indecomposable \( kG \)-module. Then we have

1. \( M \cong M^* \) if \( M \) is even-dimensional.
2. \( M^* \cong V_{-(2n+1)} \) if \( M \cong V_{2n+1} \) is odd dimensional.
(3) \( M^* \cong V_{2n+1} \) if \( M \cong V_{-(2n+1)} \) is odd-dimensional.

Clearly (3) follows from (2) above, but we include it for completeness. In addition,

**Proposition 11.** Let \( M \) be an indecomposable \( \mathbb{k}G \)-module. Then we have

1. \( \Omega(M) \cong M \) if \( M \) is even-dimensional.
2. \( \Omega^{-1}(M) \cong V_{-(2n+3)} \) if \( M \cong V_{-(2n+1)} \) is odd dimensional.
3. \( \Omega(M) \cong V_{2n+3} \) if \( M \cong V_{2n+1} \) is odd-dimensional.

Again (3) follows from (2) when we take into account that \( \Omega(M)^* \cong \Omega^{-1}(M^*) \) in general.

### 3.1. Relative shifts

The goal of this subsection is to prove Theorem 1.

Among the indecomposable \( \mathbb{k}G \)-modules listed in the previous section, only four are projective relative to \( \chi \). These are the projective indecomposable \( P \), and the three modules \( V_{2,\infty}, V_{2,x} \) and \( V_{2,x+1} \). Here the last two are the indecomposable modules \( V_{2,0} \) where \( \theta(x) \) is the monic irreducible \( x \) or \( x+1 \in \mathbb{k}[x] \). Note that \( \tau \) acts trivially on \( V_{2,\infty} = \mathbb{k}_{H_2} \oplus \mathbb{k}^G \), while \( \sigma \) acts trivially on \( V_{2,x} = \mathbb{k}_{H_3} \oplus \mathbb{k}^G \) and \( \sigma \tau \) acts trivially on \( V_{2,x+1} = \mathbb{k}_{H_3} \oplus \mathbb{k}^G \). As these three play an important role in what follows, we denote them by \( Q_\tau, Q_\sigma \) and \( Q_{\sigma \tau} \) respectively. We set \( Q = Q_\sigma \oplus Q_\tau \oplus Q_{\sigma \tau} \).

We begin by considering odd-dimensional modules.

**Lemma 12.** Let \( n \geq 0 \):

1. The relative projective cover of \( V_{-(2n+1)} \) is \( Q \oplus nP \).
2. We have \( \Omega(\chi(V_{-(2n+1)})) \cong V_{-(2n+5)} \).

**Proof.** Let \( M \cong V_{-(2n+1)} \) and let \( \pi : N \to M \) be its relative projective cover with respect to \( \chi \). \( N \) must decompose as a direct sum of modules of the form \( P, Q_\sigma, Q_\tau \) and \( Q_{\sigma \tau} \).

Let \( a_1, a_2, \ldots, a_n, b_0, b_1, \ldots, b_n \) be a basis of \( M \), with action given by the diagram as in Proposition 9. Since \( \pi \) is a surjective \( \mathbb{k}G \)-map and no \( a_i \) is fixed by any element of \( G \), the same must be true of their unique pre-images. The modules \( Q_\sigma, Q_\tau \) and \( Q_{\sigma \tau} \) all have non-trivial kernels. Therefore \( N \) contains at least \( n \) copies of \( P \).

On the other hand, we have, for any \( i \),

\[
M \downarrow H_i \cong \mathbb{k}_{H_i} \oplus n \mathbb{k}H_i
\]

The restrictions to \( H_1 \) of \( P, Q_\tau \) and \( Q_{\sigma \tau} \) contain no trivial \( H_1 \)-summands. So \( N \) must contain a direct summand isomorphic to \( Q_\sigma \) if \( \pi \) is to split on restriction to \( H_1 \). A similar argument (restricting to \( H_2, H_3 \)) shows that \( N \) must contain summands isomorphic to \( Q_\tau \) and \( Q_{\sigma \tau} \).

We will construct a surjective \( \mathbb{k}G \)-homomorphism \( Q \oplus nP \to M \). The following diagrams label the basis elements:

\[
\begin{array}{cccc}
& Q_\sigma & Q_\tau & Q_{\sigma \tau} \\
Q_\sigma & & & \\
\cdots & x_1 & \cdots & x_n \\
\cdots & y_1 & \cdots & y_n \\
& & & \\
Q_\tau & & & \\
Q_{\sigma \tau} & & & \\
\end{array}
\]

The diagram for \( Q_{\sigma \tau} \) is not as described in Proposition 9, but makes sense, because \( Xa_1 = Ya_1 = b_1 \) in this case. We now define a linear map \( \pi : Q \oplus nP \to M \) by

- \( \pi(w_i) = a_i \) for \( i = 1, \ldots, n \).
- \( \pi(x_i) = b_{i-1} \) for \( i = 1, \ldots, n \).
- \( \pi(y_i) = b_i \) for \( i = 1, \ldots, n \).
The reader should check that \( \pi \) is a \(*G\)-homomorphism. The kernel of \( \pi \) is spanned by
\[
\{ z_i : i = 1,\ldots, n \} \cup \{ s_1, s_2, s_3 \} \cup \{ x_i + y_i : i = 2,\ldots, n \} \cup \{ x_1 + r_1, x_1 + r_3, y_2 + r_2 \}.
\]
It has dimension \( 2n + 5 \), and the fixed-point space within this module is spanned by \( \{ z_1, z_2, \ldots, z_n, s_1, s_2, s_3 \} \), so it has dimension \( n + 3 \). It is easily checked that no element of the kernel outside of the fixed-point space is fixed by any subgroup \( H_i \).

Therefore
\[
\ker(\pi) \downarrow_{H_i} \cong k_{H_i} \oplus (n + 2)k_{H_i}
\]
for any \( i \). This, combined with (6) and the fact that
\[
(Q \oplus nP) \downarrow_{H_i} \cong 2k_{H_i} \oplus (2n + 2)k_{H_i}
\]
shows that \( \pi \) splits on restriction to any \( H_i \). The construction ensures the minimality of \( Q \oplus nP \), so \( Q \oplus nP = N \), proving (1). Further, \( \Omega_\chi(M) = \ker(\pi) \), and the classification of \(*G\)-modules, together with the fact that \( \ker(\pi) \) must be indecomposable, implies that \( \ker(\pi) \cong V_{-(2n+5)} \), proving (2).

The following follows immediately from the above using Propositions 10 and 5(3).

**Lemma 13.** Let \( n \geq 0 \): Then we have \( \Omega_\chi(V_{(2n+5)}) \cong V_{(2n+1)} \).

To complete the picture for odd-dimensional modules, it remains only to show that

**Lemma 14.** Let \( M \cong V_3 \). Then:
\begin{enumerate}
  
  \item The relative projective cover of \( M \) is \( Q \);
  
  \item We have \( \Omega_\chi(M) \cong V_{-3} \).
\end{enumerate}

**Proof.** We have \( M \downarrow_{H_i} \cong k_{H_i} \oplus k_{H_i} \), for \( i = 1, 2, 3 \), so once more the projective cover must contain a summand isomorphic to \( Q \). We shall construct a \(*G\)-homomorphism \( \pi : Q \to M \). We retain the notation for a basis of \( Q \) used in Lemma 12; a basis for \( M \) is \( \{ a_0, a_1, b_1 \} \) with action given as in the classification.

Define:
\[
\begin{align*}
\pi(r_1) &= a_0 \\
\pi(r_2) &= a_1 \\
\pi(r_3) &= a_0 + a_1 \\
\pi(s_1) &= \pi(s_2) = \pi(s_3) &= b_1.
\end{align*}
\]

The reader should check this is a \(*G\)-homomorphism. The kernel of \( \pi \) is spanned by \( \{ s_1 + s_2, s_2 + s_3, r_1 + r_2 + r_3 \} \), and the fixed-point space of the kernel is two-dimensional, spanned by \( \{ s_1 + s_3, s_2 + s_3 \} \). Noting that
\[
X(r_1 + r_2 + r_3) = s_2 + s_3, Y(r_1 + r_2 + r_3) = s_1 + s_3,
\]
we see that the kernel of \( \pi \) is indecomposable, and as a \(*G\)-module is isomorphic to \( V_{-3} \). Therefore
\[
\ker(\pi)H_i \oplus k_{H_i} \oplus k_{H_i}
\]
for all \( i \), from which we deduce that \( \pi \) splits on restriction to each \( H_i \). Our construction ensures the minimality of \( Q \), so \( Q \) is indeed the relative projective cover of \( M \), proving (1), and \( \ker(\pi) = \Omega_\chi(M) \cong V_{-3} \), proving (2).

We now turn to even dimensional modules. Note that \( V_{2,\infty} = Q_\tau \) is already projective relative to \( \chi \), so \( \Omega_\chi(V_{2,\infty}) \) is not defined.
Lemma 15. Let \( n \geq 2 \) and \( M \cong V_{2n, \infty} \). Then:

1. The relative projective cover of \( M \) is \( 2Q_\tau \oplus (n - 1)P \);
2. We have \( \Omega_\chi(M) \cong M \).

Proof. Let \( \pi : N \to M \) be the relative projective cover of \( M \). Notice that

\[
M \downarrow_{H_i} = nkH_i
\]

for \( i = 1, 3 \) whereas

\[
M \downarrow_{H_2} = 2kH_2 \oplus (n - 1)kH_2.
\]

So if \( \pi : N \to M \) is to split on restriction to \( H_2 \), \( N \) must contain a pair of direct summands isomorphic to \( Q_\tau \). On the other hand, retaining the notation from Proposition 9, the basis elements \( a_1, \ldots, a_{n-1} \) are not fixed by any element of \( G \), so the same must be true of their unique pre-images in \( N \). From this it follows that \( N \) must contain \( n - 1 \) direct summands isomorphic to \( P \).

We will construct a \( kG \)-homomorphism \( 2Q_\tau \oplus (n - 1)P \to M \). The following diagram gives the labelling for a basis of the domain:

\[
\begin{array}{ccc}
\bullet & r_1 & \bullet \\
& s_1 & \\
\bullet & r_2 & \bullet \\
& s_2 &
\end{array}
\quad
\begin{array}{ccc}
\bullet & \cdots & \bullet \\
x_1 & \cdots & x_{n-1} \\
\bullet & \cdots & \bullet \\
y_1 & \cdots & y_{n-1}
\end{array}
\]

We define:

- \( \pi(w_1) = a_i \) for \( i = 1, \ldots, n - 1 \).
- \( \pi(x_i) = b_i \) for \( i = 1, \ldots, n - 1 \).
- \( \pi(y_i) = b_{i+1} \) for \( i = 1, \ldots, n - 1 \).
- \( \pi(z_i) = 0 \) for \( i = 1, \ldots, n - 1 \).
- \( \pi(r_1) = b_1 \).
- \( \pi(s_1) = 0 \).
- \( \pi(r_2) = a_n \).
- \( \pi(s_2) = b_n \).

The reader should check that \( \pi \) is a \( kG \)-homomorphism. The kernel of \( \pi \) is spanned by

\[
\{ z_i : i = 1, \ldots, n - 1 \} \cup \{ x_i + y_{i-1} : i = 2, \ldots, n - 1 \} \cup \{ s_1, x_1 + r_2, y_{n-1} + s_2 \}.
\]

This has dimension \( 2n \). The fixed points within this module are spanned by

\[
\{ z_i : i = 1, \ldots, n - 1 \} \cup \{ s_1 \}.
\]

These span the fixed points of \( H_1 \) and \( H_3 \), while \( H_2 \) has a fixed point space of dimension \( n + 1 \), spanned by the above and \( y_{n+1} + s_2 \). Therefore we have

\[
\ker(\pi) \downarrow_{H_i} \cong nkH_i
\]

for \( i = 1, 3 \) and

\[
\ker(\pi) \downarrow_{H_2} \cong 2kH_2 \oplus (n - 1)kH_2.
\]

Note that

\[
(2Q_\tau \oplus (n - 1)P) \downarrow_{H_i} \cong 2nkH_i
\]

for \( i = 1, 3 \) and

\[
(2Q_\tau \oplus (n - 1)P) \downarrow_{H_2} \cong 4kH_2 \oplus (2n - 2)kH_i.
\]

Thus, \( \pi \) splits on restriction to each \( H_i \). The construction ensures the minimality of \( 2Q_\tau \oplus (n - 1)P \), so this is equal to \( N \) and we have (1). Further, \( \ker(\pi) = \Omega_\chi(M) \) must be indecomposable. By the classification (looking at the dimension of the
fixed point space of each subgroup of \( G \) to distinguish among modules of even
dimension) we must have \( \Omega_\chi(M) \cong M \) as required for (2).

Notice that if \( \theta(x) = x^n \), then \( V_{2n,\theta} \) can be obtained from \( V_{2n,\infty} \) by applying
the automorphism of \( G \) which swaps \( \sigma \) and \( \tau \). Similarly if \( \theta(x) = (x+1)^n \), then \( V_{2n,\theta} \)
can be obtained from \( V_{2n,\infty} \) by applying the automorphism of \( G \) which swaps \( \sigma \tau \)
and \( \tau \). We therefore obtain immediately from Lemma 15 above that \( \Omega_\chi(M) = M \)
if \( M \) is one of these.

It remains only to prove the following:

**Lemma 16.** Let \( n \geq 1 \) and let \( M \cong V_{2n,\theta} \), where \( \theta \) is neither \( x^n \) nor \( (x+1)^n \).
Then:

1. The relative projective cover of \( M \) is \( nP \);
2. \( \Omega_\chi(M) \cong M \).

**Proof.** Observe that \( M / H_i = nkH_i \) for each \( i \). The proof of [7, Proposition 3.1]
shows that the projective (as opposed to relatively projective) cover of \( M \) is \( nP \) and
\( \Omega(M) \cong M \), so there is a surjective \( kG \)-homomorphism \( \pi : nP \to M \) with kernel
isomorphic to \( M \). Noting that \( nP / H_i \cong 2nkH_i \) for each \( i \), we see that \( \pi \) splits on
restriction to each \( H_i \). On the other hand, if \( N \) is a \( kG \)-module having \( Q_\tau \) (resp.
\( Q_\sigma, Q_{\sigma\tau} \)) as a direct summand then \( N / H_i \) contains a pair of trivial \( kH_i \)-modules as
direct summand, and no surjective homomorphism \( N \to M \) may split. This shows the
minimality of the dimension of \( nP \) among relatively projective modules with a
\( \chi \)-split epimorphism to \( M \), i.e. we have proved (1). We also have

\[
\Omega_\chi(M) = \ker(\pi) = \Omega(M) \cong M
\]
as required for (2). \( \square \)

**Remark 17.** Combining all the Lemmas in this section with Proposition 11, we
obtain Theorem 1.

### 3.2. Computing Cohomology

In this subsection we will determine \( H^i(G, N) \) for all \( i \geq 0 \) and for all indecomposable \( kG \)-modules \( N \). First observe that if \( N \) is
projective relative to \( \chi \), then \( H^i(G, N) = 0 \) for all \( i > 0 \): this is an immediate
consequence of Proposition 8(ii). Further, recall from part (i) of the same that
\( H^0_\chi(G, N) = N^G \) for any \( kG \)-module. It follows that:

**Proposition 18.** Let \( N \in \{ P, Q_\sigma, Q_\tau, Q_{\sigma\tau} \} \). Then,

\[
\dim(H^i_\chi(G, N)) = \begin{cases} 
1 & \text{if } i = 0, \\
0 & \text{otherwise.}
\end{cases}
\]

Now we consider even-dimensional modules which are not relatively projective.
Recall that for \( i > 0 \) we have

\[
H^i_\chi(G, N) = \text{Hom}^i_{kG}(\Omega^{-i}_\chi(k), N) \cong \text{Hom}^i_{kG}(k, \Omega^{-i}_\chi(N)) \cong \text{Hom}^i_{kG}(k, N) \cong N^G,
\]

using the fact that, for these modules \( N \), we have \( \Omega^{-i}_\chi(N) \cong N \).

We obtain by direct calculation:

**Proposition 19.** Let \( N \) be an even-dimensional \( kG \)-module which is not projective
relative to \( \chi \). Then,

\[
\dim(H^i_\chi(G, N)) = \begin{cases} 
n & \text{if } N \cong V_{2n,\infty} \text{ or } N \cong V_{2n,\sigma} \text{ where } \theta(x) = x^n \text{ or } \theta(x) = (x+1)^n, \text{ for any } i, \\
n - 1 & \text{otherwise.}
\end{cases}
\]

if \( N \cong V_{2n,\infty} \) or \( N \cong V_{2n,\sigma} \) where \( \theta(x) = x^n \) or \( \theta(x) = (x+1)^n \), while

\[
\dim(H^i_\chi(G, N)) = n
\]

for any \( i \), if \( V \cong V_{2n,\theta} \) for some other choice of \( \theta \).
For odd-dimensional modules we proceed as follows. Let \( N \) be an odd-dimensional indecomposable module and let \( i > 0 \). Then
\[
H^i_N(G, N) = \text{Hom}_G(\Omega^i_G(k), N) \cong \text{Hom}_G(k, \Omega^{-i}_G(N)) \cong \text{Hom}_G(k, \Omega^{2i}(N)) \cong \Omega^{2i}(N)^G
\]
using Theorem 1. Suppose \( N \cong V_{2n+1} \) where \( n \geq 0 \). Then \( \Omega^{2i}(N) \cong V_{2(n+2i)+1} \). A basis for \( V_{2(n+2i)+1} \) is given by \( \{ a_0, a_1, \ldots, a_{n+2i}, b_1, b_2, \ldots, b_{n+2i} \} \), with action given by the diagram in Proposition 9. The \( b_i \) are all fixed points, and in addition \( a_0 \) is fixed by \( H_1, a_{n+2i} \) by \( H_2 \) and \( a_0 + a_1 + \ldots + a_{n+2i} \) by \( H_3 \). Therefore \( b_1, b_{n+2i} \) and \( b_1 + b_2 + \ldots + b_{n+2i} \) lie in \( \Omega^{2i}(N)^G \). We therefore have

**Proposition 20.** Let \( N \cong V_{2n+1} \) for some \( n \geq 0 \). Then
1. \( \dim(H^i_N(G, N)) = n \) if \( n > 0 \), and 1 if \( n = 0 \).
2. \( \dim(H^i_N(G, N)) = \max(0, n+2i-3) \) for \( i > 0 \).

**Remark 21.** This includes [10, Theorem 1.2] as a special case \((n = 0)\).

For the remaining odd dimensional modules things are a little more complicated, since \( \Omega^{2i}(N) \) eventually moves into the “positive” part of the spectrum. We begin by noting that if \( n \geq 0 \), then \( V^H_{-(2n+1)} = V^G_{-(2n+1)} \) for all \( i \). Therefore \( (V^G_{-(2n+1)})^G = 0 \).

Now let \( N \cong V_{-(2n+1)} \) where \( n \geq 1 \). For \( i \leq n/2 \) we have \( \Omega^{2i}(N) \cong V_{-(2n-2i)+1} \). Therefore
\[
H^i_N(G, N) = \text{Hom}_G(\Omega^i_G(k), N) \cong \text{Hom}_G(k, \Omega^{-i}_G(N)) \cong \text{Hom}_G(k, \Omega^{2i}(N)) \cong \Omega^{2i}(N)^G.
\]
For \( i > n/2 \) we have \( \Omega^{2i}(N) \cong V^{2i-n}_{-(n-1)+1} \). We therefore obtain the following:

**Proposition 22.** Let \( N \cong V_{-(2n+1)} \) where \( n \geq 1 \). Then
\[
dim(H^i_N(G, N)) = \begin{cases} n + 1 - 2i & i \leq n/2 \\ \max(0, 2i - n - 3) & i > n/2. \end{cases}
\]

### 3.3. Calculating cup products.

The aim of this section is to prove Theorem 2. We begin with a lemma:

**Lemma 23.** Let \( M \cong V_{-(2m+1)} \) and \( N \cong V_{-(2n+1)} \) for some \( m > n \geq 0 \). Let \( \phi \in \text{Hom}_G(M, N) \). Then
1. \( \text{im}(\phi) \subseteq N^G \);
2. \( M^G \subseteq \text{ker}(\phi) \).

**Proof.** Note first that \( \phi(M^G) \subseteq N^G \) for arbitrary \( G \) and \( kG \)-modules \( M \) and \( N \). Let \( a_1, a_2, \ldots, a_m, b_1, b_1, \ldots, b_m \) and \( a'_1, a'_2, \ldots, a'_m, b'_1, b'_1, \ldots, b'_n \) be bases of \( M \) and \( N \) respectively, with action given by the diagrams in proposition 9. Note that if \( n = 0 \), then (1) is immediate. So suppose \( n > 0 \) and (1) does not hold: then we can find a maximal \( k \geq 1 \) such that \( \phi(a_k) \notin N^G \).

We claim that \( k = m \). To see this, write
\[
\phi(a_k) = \sum_{i=1}^{n} \lambda_i a'_i \mod N^G.
\]
Then
\[
\phi(b_k) = \phi(Ya_k) = Y\phi(a_k) = \sum_{i=1}^{n} \lambda_i b'_i.
\]
If \( k < m \) then also
\[
\phi(b_k) = \phi(Xa_{k+1}) = X\phi(a_{k+1}) = 0
\]
since \( \phi(a_{k+1}) \in N^G \). So \( \lambda_i = 0 \) for all \( i \) and \( \phi(a_k) \in N^G \), a contradiction.
Now we claim that, for all $0 \leq j \leq n$, we have
\begin{equation}
\phi(a_{m-j}) = \sum_{i=j+1}^{n} \lambda_i a_{i-j} \mod N^G
\end{equation}
and $\lambda_i = 0$ for $i = 1, \ldots, j$. We prove this by induction on $j$. The base case $j = 0$ is true by definition. Assuming the above for some $0 \leq j < n$ and noting that $n < m$, we have
\[
\phi(b_{m-j-1}) = \phi(Xa_{m-j}) = X\phi(a_{m-j}) = \sum_{i=j+1}^{n} \lambda_i b_{i-j-1}.
\]
But
\[
\phi(b_{m-j-1}) = \phi(Ya_{m-j-1}) = Y\phi(a_{m-j-1}) = b_1, \ldots, b_n
\]
which shows that $\lambda_{j+1} = 0$. Therefore
\[
\phi(b_{m-j-1}) = \sum_{i=j+1}^{n} \lambda_i b_{i-j-1},
\]
which shows that
\[
\phi(a_{m-j-1}) = \sum_{i=j+1}^{n} \lambda_i a_{i-j-1} \mod N^G
\]
proving our claim. Taking $j = n$ in (9) shows that $\phi(a_m) \in N^G$, a contradiction. This proves (1).

For (2), let $x \in M^G$. We may write
\[
x = \sum_{i=0}^{m} \mu_i b_i
\]
for some coefficients $\mu_i$. Then
\[
\phi(x) = \sum_{i=0}^{m} \mu_i \phi(b_i) = \mu_0 \phi(Xa_0) + \sum_{i=1}^{m} \mu_i \phi(Ya_{i-1}) = \mu_0 \phi(a_0) + Y\phi(\sum_{i=1}^{n} \mu_i a_i) = 0
\]
by (1).

The following is immediate:

**Corollary 24.** Let $L \cong V_{-(2l+1)}$, $M \cong V_{-(2m+1)}$ and $N \cong V_{-(2n+1)}$ for some $l > m > n \geq 0$. Let $\phi \in \text{Hom}_{kG}(M, N)$ and $\psi \in \text{Hom}_{kG}(L, M)$. Then $\phi \circ \psi = 0$.

We may now proceed with the proof of Theorem 2:

**Proof.** Let $i, j > 0$. Let $\alpha \in H^i(G, k)$ and $\beta \in H^j(G, k)$. Choose $\phi \in \text{Hom}_{kG}(\Omega^i_G(k), k)$ and $\psi \in \text{Hom}_{kG}(\Omega^j_G(k), k)$, such that the equivalence classes
\[
[\phi] \in \text{Hom}_{kG}(\Omega^i_G(k), k), [\psi] \in \text{Hom}_{kG}(\Omega^j_G(k), k)
\]
represent $\alpha$ and $\beta$ respectively. By definition, $\alpha \sim \beta$ is represented by $[\phi \circ \Omega^i_G(\psi)]$.

By Lemma 12 we have
\[
\phi \in \text{Hom}(V_{-(2i+1)}, V_{-(i-1)}), \Omega^i_G(\psi) \in \text{Hom}(V_{-(2i+2j+1)}, V_{-(2i+1)})
\]
and by Corollary 24 the composition of these two is the trivial map.
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Middlesex University, The Burroughs, Hendon, London, NW4 4BT UK
Email address: j.elmer@mdx.ac.uk