Non-cyclic graphs of (non)orientable genus one

XUANLONG MA

Sch. Math. Sci. & Lab. Math. Com. Sys.,
Beijing Normal University, 100875, Beijing, China.

Abstract

Let $G$ be a finite non-cyclic group. The non-cyclic graph $\Gamma_G$ of $G$ is the graph whose vertex set is $G \setminus \text{Cyc}(G)$, two distinct vertices being adjacent if they do not generate a cyclic subgroup, where $\text{Cyc}(G) = \{a \in G : \langle a, b \rangle \text{ is cyclic for each } b \in G\}$. In this paper, we classify all finite non-cyclic groups $G$ such that $\Gamma_G$ has (non)orientable genus one.

Keywords: Non-cyclic graph, finite non-cyclic group, genus.
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1 Introduction

All graphs in this paper are undirected, with no loops or multiple edges. A graph $\Gamma$ is called a planar graph if $\Gamma$ can be drawn in the plane so that no two of its edges cross each other, and in this case we say that $\Gamma$ can be embedded in the plane. For a non-planar graph, it can be embedded in some surface obtained from the sphere by attaching some handles or crosscaps. We denote by $S_k$ a sphere with $k$ handles and by $N_k$ a sphere with $k$ crosscaps. Note that both $S_0$ and $N_0$ are the sphere itself, and $S_1$ and $N_1$ are the torus and the projective plane, respectively. The smallest non-negative integer $k$ such that a graph $\Gamma$ can be embedded on $S_k$ (resp. $N_k$) is called the orientable genus or genus (resp. nonorientable genus) of $\Gamma$, and is denoted by $\gamma(\Gamma)$ (resp. $\overline{\gamma}(\Gamma)$).

The problem of finding the graph genus is NP-hard [9]. The (non)orientable genera of some graphs constructed from some algebraic structures have been studied, for instance, see [3–5, 7, 10].

E-mail addresses: xuanlma@mail.bnu.edu.cn.
All groups considered in this paper are finite. Denote by $\mathbb{Z}_n$ and $D_{2n}$ the cyclic group of order $n$ and the dihedral group of order $2n$, respectively. Let $G$ be a non-cyclic group. The cyclicizer $\text{Cyc}(G)$ of $G$ is
\[ \{ a \in G : \langle a, b \rangle \text{ is cyclic for each } b \in G \} . \]
and is a normal subgroup of $G$ (see [6]). The non-cyclic graph $\Gamma_G$ of $G$ is the graph whose vertex set is $G \setminus \text{Cyc}(G)$, and two distinct vertices being adjacent if they do not generate a cyclic subgroup. The non-cyclic graph $\Gamma_G$ was first considered by Abdollahi and Hassanabadi [1] and they studied the properties of the graph and established some graph theoretical properties (such as regularity) of this graph in terms of the group ones. In [2], Abdollahi and Hassanabadi classified all non-cyclic groups $G$ such that $\Gamma_G$ is planar.

A natural question is the following: Which finite non-cyclic groups have their non-cyclic graphs have (non)orientable genus one? The goal of the paper is to find all non-cyclic graphs of (non)orientable genus one. Our main results are the following theorems.

**Theorem 1.1.** Let $G$ be a finite non-cyclic group. Then $\Gamma_G$ has genus one if and only if $G$ is isomorphic to one of the following groups:
\[ \mathbb{Z}_3^2, \mathbb{Z}_2^3, \mathbb{Z}_2 \times \mathbb{Z}_4, \ D_8, \ \mathbb{Z}_2 \times \mathbb{Z}_6. \] (1)

**Theorem 1.2.** Let $G$ be a finite non-cyclic group. Then $\Gamma_G$ has nonorientable genus one if and only if $G$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $D_8$.

2 Preliminaries

An element of order 2 in a group is called an *involution*. Let $G$ be a group and $g$ be an element of $G$. Denote by $|G|$ and $|g|$ the orders of $G$ and $g$, respectively. We denote the symmetric group on $n$ letters and the quaternion group of order 8 by $S_n$ and $Q_8$, respectively. Also $\mathbb{Z}_n^m$ is used for the $m$-fold direct product of the cyclic group $\mathbb{Z}_n$ with itself. In the following, we state some results which we need in the sequel.

**Lemma 2.1.** ([2, Proposition 4.3]) $\Gamma_G$ is planar if and only if $G$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, $S_3$ or $Q_8$.

Let $\Gamma$ be a graph. Denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and the edge set of $\Gamma$, respectively. We use the notation $\lceil x \rceil$ to denote the least integer that is greater than or equal to $x$. Denote by $K_n$ and $K_{m,n}$ the complete graph of order $n$ and the complete bipartite graph, respectively. The following result from [11] gives the (non)orientable genus of a complete graph and a complete multipartite graph.
Lemma 2.2. ([11]) Let \( n \) be an integer at least 3. Then

(a) \( \gamma(K_n) = \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil \).

(b) \( \tau(K_n) = \left\lceil \frac{1}{3}(n-3)(n-4) \right\rceil \) if \( n \neq 7 \) and \( \tau(K_7) = 3 \).

(c) \( \gamma(K_{m,n}) = \left\lceil \frac{1}{4}(m-2)(n-2) \right\rceil \).

(d) \( \tau(K_{m,n}) = \left\lceil \frac{1}{2}(m-2)(n-2) \right\rceil \).

(e) \( \gamma(K_{n,m,n}) = \frac{1}{2}(n-1)(n-2) \).

(f) \( \gamma(K_{n,n,m,n}) = (n-1)^2 \) for \( n \neq 3 \) and \( \gamma(K_{3,3,3,3}) = 5 \).

Lemma 2.3. ([8, pp. 252, Theorem 9.7.3]) Suppose that \( G \) is a \( p \)-group for some prime \( p \) and has a unique subgroup of order \( p \). If \( p = 2 \), then \( G \) is cyclic or generalized quaternion. If \( p > 2 \), then \( G \) is cyclic.

The following result is one of Sylow theorems.

Theorem 2.4. Suppose that \( G \) is a group and \( p \) is a prime divisor of \( |G| \). Then the number of Sylow \( p \)-subgroups is congruent to 1 modulo \( p \). In particular, the number of subgroups of order \( p \) is congruent to 1 modulo \( p \).

Denote by \( \varphi \) the Euler’s totient function.

Lemma 2.5. Let \( G \) be a non-cyclic group, \( p, q \) two distinct primes and \( m \) a positive integer at least 1. If \( \gamma(\Gamma_G) = 1 \), the each of the following statements does not hold:

(a) \( G \) has 4 cyclic subgroups of order \( p^m \) and an element of order \( q \), where \( \varphi(p^m) \geq 2 \).

(b) \( G \) has 4 cyclic subgroups of order 3 and \( |G| \geq 10 \).

(c) \( G \) has 3 cyclic subgroups of order 4 and an element of order \( q^m \), where \( \varphi(q^m) \geq 3 \) and \( q \neq 2 \).

(d) \( G \) has 7 cyclic subgroups of order 2 and an element of order \( q \), where \( q \neq 2 \).

(e) \( G \) has 3 cyclic subgroups of order \( p^m \), where \( \varphi(p^m) \geq 4 \).

(f) \( G \) has 2 cyclic subgroups of order \( p^m \), where \( \varphi(p^m) \geq 5 \).

Proof. (a) Suppose, for a contradiction, that (a) holds. Let \( \langle a \rangle, \langle b \rangle, \langle c \rangle, \langle d \rangle \) be 4 cyclic subgroups of order \( p^m \) of \( G \) and \( g \) be an element of order \( q \). If \( g \) and each element of \( \{a, a^{-1}, b, b^{-1}, c, c^{-1}, d, d^{-1}\} \) cannot generate a cyclic subgroup, then the induced subgraph by \( \{a, a^{-1}, b, b^{-1}, c, c^{-1}, d, d^{-1}, g\} \) has a subgraph isomorphic to \( K_{4,5} \) that has partition sets \( \{c, c^{-1}, d, d^{-1}\} \) and \( \{a, a^{-1}, b, b^{-1}, g\} \) and so \( \gamma(\Gamma_G) \geq \gamma(K_{4,5}) = 2 \), a contradiction. Thus, we may suppose that \( g \) and an element of \( \{a, a^{-1}, b, b^{-1}, c, c^{-1}, d, d^{-1}\} \) can generate a cyclic subgroup \( \langle h \rangle \). Without loss of generality, let \( \langle a, g \rangle = \langle h \rangle \). Then \( h \in V(\Gamma_G) \) and thereby, the induced subgraph by \( \{a, a^{-1}, b, b^{-1}, c, c^{-1}, d, d^{-1}, h\} \) has a subgraph isomorphic to \( K_{4,5} \) that has partition
sets \( \{c, c^{-1}, d, d^{-1}\} \) and \( \{a, a^{-1}, b, b^{-1}, h\} \). So \( \gamma(\Gamma_G) \geq \gamma(K_{4,5}) > 1 \), also a contradiction.

It is similar to the proof of (a), we can prove (b), (c) and (d).

(e) Assume, to the contrary, that (e) holds. Take 4 generators in every cyclic subgroup of order \( p^m \). Then it is easy to see that the induced subgraph by the generators has a subgraph isomorphic to \( K_{4,4,4} \) that has genus 3 by Lemma 2.2, a contradiction.

(f) It is similar to the proof of (e).

\[ \square \]

**Lemma 2.6.** Let \( G \) be a non-cyclic \( p \)-group, where \( p \) is a prime. Then \( \gamma(\Gamma_G) = 1 \) if and only if \( G \) is isomorphic to one of the following groups:

\[ \mathbb{Z}_3^2, \mathbb{Z}_2^3, \mathbb{Z}_2 \times \mathbb{Z}_4, D_8. \]  

(2)

**Proof.** Note that \( \Gamma_{\mathbb{Z}_3^2} \cong K_{2,2,2,2}, \Gamma_{\mathbb{Z}_2^3} \cong K_7 \) and each of \( \Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_4} \) and \( \Gamma_{D_8} \) is a subgraph of \( K_7 \). By Lemma 2.2, we see that \( \Gamma_G \) has genus one for each group \( G \) in (2). We next assume that \( \gamma(\Gamma_G) = 1 \).

Suppose that \( p \geq 3 \). Then, by Lemma 2.3 and Theorem 2.4 we have that \( G \) has at least 4 subgroups of order \( p \). It follows from (e) and (b) of Lemma 2.5 that \( |G| \leq 9 \). This implies that \( G \cong \mathbb{Z}_3^2 \), as desired.

Now suppose that \( p = 2 \). If \( |G| \leq 8 \), then \( G \) is isomorphic to \( \mathbb{Z}_2^2, Q_8, \mathbb{Z}_2^3, \mathbb{Z}_2 \times \mathbb{Z}_4 \) or \( D_8 \) and by Lemma 2.1 we get the desired result. Thus, we may suppose that \( |G| \geq 16 \). If \( G \) is generalized quaternion, then \( G \) has a subgroup \( \langle x \rangle \) of order 8 and contains at least 4 elements \( y_1, \ldots, y_4 \) of order 4 that do not belong to \( \langle x \rangle \), and so \( \Gamma_G \) has a subgraph isomorphic to \( K_{6,4} \) that has partition sets \( \{x^i : 0 < i < 8, i \neq 4\} \) and \( \{y_j : j = 1, \ldots, 4\} \), a contradiction by Lemma 2.2. Therefore, by Lemma 2.3 we may assume that \( G \) has at least 3 involutions.

**Case 1.** \( G \) has an element \( g \) of order 8.

Suppose that \( G \) has two distinct subgroups \( \langle g \rangle, \langle h \rangle \) of order 8. Then we may pick an involution \( a \) in \( G \setminus (\langle g \rangle \cup \langle h \rangle) \). Now we get a subgraph of \( \Gamma_G \) isomorphic to \( K_{5,4} \) that has partition sets \( \{g, g^3, g^5, a\} \) and \( \{h, h^3, h^5, h^7\} \), a contradiction by Lemma 2.2.

Thus, we may suppose that \( G \) has a unique subgroup \( \langle u \rangle \) of order 8, which is normal in \( G \). Take an involution \( b \) that does not belong to \( \langle u \rangle \). Then \( \langle u, b \rangle \) is a subgroup of order 16 and has precisely one subgroup of order 8. Since \( b \notin \langle u \rangle, \langle u, b \rangle \) is not cyclic. Note that \( G \) is not generalized quaternion. By verifying the groups of order 16, we get that \( G \cong D_{16} \) or \( QD_{16} \), where \( QD_{16} = \langle a, b : a^8 = b^2 = 1, bab = a^3 \rangle \). If \( G \cong D_{16} \), then \( G \) has 9 involutions which induce a subgraph isomorphic to \( K_9 \), \( \gamma(\Gamma_G) \geq \gamma(K_9) = 3 \) by Lemma 2.2, a contradiction. Note that \( QD_{16} \) has only 6
elements of order 4. If $G \cong QD_{16}$, then the subgraph induced by 6 elements of order 4 and 4 elements of order 8 has a subgraph isomorphic to $K_{6,4}$, also a contradiction.

**Case 2.** $G$ has no elements of order 8.

If $G$ has no elements of order 4, then $\Gamma_G$ is isomorphic to $K_{|G|-1}$, a contradiction as $\gamma(K_{|G|-1}) > 1$ for $|G| \geq 16$. Thus, in this case we may assume that $\pi_e(G) = \{1, 2, 4\}$. Note that $|G| \geq 16$. Since all involutions induce a complete graph and $\gamma(K_8) \geq 2$, $G$ has at least 8 elements of order 4. Since a power graph induced by 10 elements of order 4 of a group has a subgraph isomorphic to $K_{6,4}$ that has genus two, $G$ has precisely 8 elements of order 4 and 7 involutions. Take an involution $a$ in $G$ that does not belong to any subgroup of order 4. Then it is easy to see that the subgraph induced by all elements of order 4 and $a$ has a subgraph isomorphic to $K_{5,4}$ that has genus two, a contradiction. \qed

## 3 Proof of the main theorems

**Proof of Theorem 1.1.** Note that $\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}$ has a subgraph isomorphic to $K_{3,3}$. Hence $\gamma(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}) \geq 1$. On the other hand, we can embed $\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}$ into the tours as shown in Figure 1. This implies that $\gamma(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}) = 1$. Now by Lemma 2.6 we see that $\Gamma_G$ has genus one for any group $G$ in (1).

Now we assume that $\gamma(\Gamma_G) = 1$. If $G$ is a p-group, the desired result follows from Lemma 2.6. Thus, we may assume that $G$ is not a p-group. Let $q$ be an odd prime divisor of $|G|$. If the number of subgroups of order $q$ is not 1, then by Theorem 2.4 $G$ has at least 4 subgroups of order $q$ and by (a) of Lemma 2.5, we have a contradiction. Thus, $G$ has a unique subgroup of order $q$ and thereby, every Sylow $q$-subgroup of $G$ is cyclic by Lemma 2.3. Similarly, we can get that $G$ has a unique Sylow $q$-subgroup.
Note that $G$ is not cyclic. Thus, we may assume that $G = P \times Q$, where $P$ is a 2-group and $Q$ is a cyclic group of odd order. We next prove that $P$ is not cyclic.

Suppose to the contrary that $P$ is cyclic. Suppose that $|P| = 2$. Then $G$ is dihedral. By Lemma 2.1, we see that $Q \not\cong \mathbb{Z}_3$ and so $\varphi(|Q|) \geq 4$ and $G$ has at least 5 involutions. This implies that $\Gamma_G$ has a subgraph isomorphic to $K_{5,4}$ that has two partition sets consisting of 5 involutions and 4 generators of $Q$, a contradiction. Suppose now that $P \cong \mathbb{Z}_4$. Note that $G$ has at least 3 cyclic subgroups of order 4. By (c) of Lemma 2.5, we get $Q \cong \mathbb{Z}_3$. By checking the groups of order 12, $G \cong \langle a, b : a^6 = b^4 = 1, b^2 = a^3, b^{-1}ab = a^5 \rangle$. It is easy to check that $\gamma(\Gamma_G)$ has a subgraph isomorphic to $K_{4,5}$, a contradiction if $|P| \geq 8$, since $P$ is not normal in $G$, $G$ has at least 3 Sylow 2-subgroups and since $\varphi(|P|) \geq 4$, a contradiction by (e) of Lemma 2.5. This means that $P$ is not cyclic.

Note that $V(\Gamma_P) \subseteq V(\Gamma_G)$. Then $\gamma(\Gamma_P) = 0$ or 1 and by Lemmas 2.1 and 2.6, $P$ is isomorphic to one of the following groups:

$$\mathbb{Z}_2^3, \mathbb{Z}_2 \times \mathbb{Z}_4, D_8, Q_8, \mathbb{Z}_2^2.$$

First by (d) of Lemma 2.5, we conclude $P \not\cong \mathbb{Z}_2^3$.

**Case 1.** $P \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

If $P$ is not normal in $G$, then $G$ has at least 4 cyclic subgroups of order 4, a contradiction by (a) of Lemma 2.5. Thus, $G \cong P \times Q$ and so $G$ has precisely three involutions and at least two cyclic subgroups of order $4k$ for some odd prime $k$. Considering the generators of the two cyclic subgroups of order $4k$ and some involution, we have that $\Gamma_G$ has a subgraph isomorphic to $K_{4,5}$, a contradiction.

**Case 2.** $P \cong D_8$.

If $P$ is not normal in $G$, then $G$ has at least 7 involutions in the union of all Sylow 2-subgroups, a contradiction by (d) of Lemma 2.5. Therefore, we may assume that $G \cong P \times Q$. Let $g$ be an element of odd order. Then $G$ has a cyclic subgroup of order $4|g|$, which has at least 4 generators $\{g_1, \cdots, g_4\}$. Now it is easy to see that $\Gamma_G$ has a subgraph isomorphic to $K_{4,5}$ that has two partition sets 4 involutions and $\{g_1, \cdots, g_4, a\}$ for some involution $a$, a contradiction.

**Case 3.** $P \cong Q_8$.

Note that $Q_8$ has 3 cyclic subgroups of order 4. By (a) of Lemma 2.5, we may assume that $G \cong P \times Q$. So $G$ has at least 3 cyclic subgroups of order $4k$ for some odd prime $k$. By (e) of Lemma 2.5, a contradiction.

**Case 4.** $P \cong \mathbb{Z}_2^2$. 

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If $P$ is not normal in $G$, then $G$ has at least 7 involutions, a contradiction by (d) of Lemma 2.5. Now we assume that $G \cong \mathbb{Z}_2^3 \times Q$. If $Q \cong \mathbb{Z}_3$, then as desired. Thus, we may assume that $|Q| \geq 5$. Then it is easy to see that $G$ has at least 3 cyclic subgroups of order $2k$ for some odd number $k \neq 3$. By (e) of Lemma 2.5, a contradiction.

**Proof of Theorem 1.2.** Since $\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_4}$ and $\Gamma_{D_8}$ all have some subgraphs isomorphic to $K_{3,3}$, one has that $\gamma(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_4}) \geq 1$ and $\gamma(\Gamma_{D_8}) \geq 1$. On the other hand, we may embed $\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_4}$ and $\Gamma_{D_8}$ into $N_1$ as shown in Figures 2 and 3, respectively. So we have $\gamma(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_4}) = \gamma(\Gamma_{D_8}) = 1$.

Now we assume that $\gamma(\Gamma_G) = 1$. By Lemma 2.2 we see that $\gamma(K_{4,4}) = 2$ and $\gamma(K_n) \geq 2$ for $n \geq 7$. Thus, $\Gamma_G$ has no subgraphs isomorphic to $K_{4,4}$ and $K_n$ for
$n \geq 7$. By the proof of Theorem 1.1, it is easy to see that $G$ is one group of (1). Since $\Gamma_{\mathbb{Z}_3^4}$ has a subgraph isomorphic to $K_{4,4}$ and $\Gamma_{\mathbb{Z}_3^3}$ has a subgraph isomorphic to $K_7$, one has that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, $D_8$ or $\mathbb{Z}_2 \times \mathbb{Z}_6$. In order to complete our proof, we next prove $\gamma(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}) \geq 2$.

Clearly, $\gamma(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}) \geq 1$. Suppose for a contradiction that $\gamma(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}) = 1$. Note that $|V(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6})| = 9$ and $|E(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6})| = 27$. Thus, by the Euler characteristic formulas, if $\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}$ is embedded into the surface of nonorientable genus $\gamma(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6})$, resulting in $f$ faces, then

$$|V(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6})| - |E(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6})| + f = 2 - \gamma(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}).$$

This implies that $2|E(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6})| \geq 3f$, which is a contradiction as $f = 19$. \hfill \Box

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