Correlations in few two-component quantum walkers on a tilted lattice

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We study the effect of inter-component interactions on the dynamical properties of quantum walkers. We consider the simplest situation of two indistinguishable non-interacting walkers on a tilted optical lattice interacting with a walker from a different component. The mediated effect of the third particle is then analyzed in the backdrop of various controlling parameters. The interaction-induced two-particle correlations are shown to be non-trivially affected by the particle statistics, choice of initial states, and tilting configurations of the lattice. Our analysis thus offers an overall picture and serves as a starting point of a study of interacting multi-component quantum walkers.

I. INTRODUCTION

Continuous-time quantum walk, where the particle moves on a lattice under the action of a time-independent Hamiltonian [1], has attracted a lot of attention in recent years, especially due to its application in quantum information processing and quantum computation [2]. Recently it has also generated interest as a possible candidate as a quantum simulator of dynamics of magnon excitations in ferro- or antiferromagnetic solid-state systems [3, 4]. Experimental realizations of quantum walks are therefore of great importance and form the basis of interesting real-world applications. For a single walker, this has been realized in various setups, such as spin systems [5, 6], photonic systems [7, 8], trapped ions [9], and with neutral cold atoms in optical lattices [5, 6, 10]. The last system is of pertinent interest as it provides a clean, coherent, and controllable way to investigate quantum many-body properties [11, 12]. The optical lattices are nowadays a viable candidate for simulating condensed matter systems [11–13] which have been instrumental in studies of strongly correlated many-body systems [14–21]. Microscopic control of the Hamiltonian parameters via external fields in these setups allows precise probing of more interesting situations when instead of a single quantum walker, a larger number of particles is considered [22, 23]. It was argued that in the presence of inter-particle interactions, a many-particle quantum walk may affect quantum interference depending on the particle statistics [24] and therefore it also may have some applicability in universal quantum computation schemes [25]. This path of exploration is even more intriguing when the quantum walk is realised in the tilted lattice, i.e., when local single-particle energies vary linearly with the site index. Starting from the famous theoretical paper by Bloch [26] it is known that on the single-particle level the lattice tilting may lead to a counter-intuitive and quite spectacular effect of spatial oscillations of a single-particle density (known as Bloch oscillations). This prediction was awaiting for experimental confirmation over 60 years and was first demonstrated for electrons moving in a semiconductor superlattice [27]. The role of inter-particle interactions between indistinguishable walkers in the tilted lattice has been deeply explored in the literature [28–38]. However, some questions still remain open and appropriate analysis is required. An important one of them is related to the problem of quantum walk realized in the tilted lattice by a multi-component system and to the non-obvious interplay between lattice geometry, initial state, quantum statistics, as well as intra- and inter-component interactions. To make the first step to fill this gap, in our work we focus on the simplest multi-component system capturing all these features, i.e., the case of three particles: two indistinguishable fermions or bosons belonging to the component $A$ and a third particle of fundamentally distinguishable flavor $B$. We study different dynamical properties of the system focusing mostly on quantum correlations between non-interacting $A$-particles induced by interactions with third $B$-walker.

This paper is organized in the following way. We provide the details of the system and the Hamiltonian in Sec. II. In Sec. III we analyse the dynamics of two identical particles localized initially in distinct sites and highlight the differences forced by quantum statistics. Then, in Sec. IV, we introduce an additional particle of different component and study its impact on the dynamics. Here, we also look for the effects of particle statistics in the two-particle sector and uncover a role of the initial state. Next, in Sec. V we show that inter-particle correlations may crucially depend on the lattice geometry. Finally, we summarize and conclude in Sec. VI.

II. THE SYSTEM STUDIED

In our work we consider the system of three quantum walkers experiencing the same one-dimensional tilted lattice potential. We assume that two of them are indistinguishable (bosons or fermions) and belong to the component $A$, while the third one belongs to the other component $B$. In the simplest case, when particles move in the lowest band of the lattice, the Hamiltonian of the system can be written in the tight-binding approximation as the following single-band Hubbard Hamilto-
\[
\hat{H} = \sum_i \left[ -J_i \left( \hat{a}_i^\dagger \hat{a}_{i+1} + \hat{b}_i^\dagger \hat{b}_{i+1} + \text{h.c.} \right) + E_i (\hat{n}_{A_i} + \hat{n}_{B_i}) \right] + U \sum_i \hat{n}_{A_i} \hat{n}_{B_i} + \frac{U_A}{2} \sum_i \hat{n}_{A_i} (\hat{n}_{A_i} - 1),
\]

where \(J_i\) and \(E_i\) characterize lattice geometry and they determine single-particle tunnelings and on-site energies, respectively. Since we consider the simplest scenario of tilted lattice, in the following we set tunneling amplitudes \(J_i = J\) as site-independent. On the contrary, the local energies, although component-independent, are linear in the site index, \(E_i = F \cdot i\). The tilting parameter \(F\) being under experimental control can be viewed as a constant force acting along the lattice. By definition, operators \(\hat{a}_i\) and \(\hat{b}_i\) annihilate particles at site \(i\) belonging to the component \(A\) and \(B\), respectively. Depending on the quantum statistics, they obey intra-component commutation or anti-commutation relations. At the same time, any two operators acting in subspaces of different components do commute. Since in our work we consider systems containing only a single \(B\)-particle, all the results presented are insensitive to the intra-component quantum statistics of operators \(\hat{b}_i\). For convenience, we introduced the local number operators \(\hat{n}_{A_i} = \hat{a}_i^\dagger \hat{a}_i\) and \(\hat{n}_{B_i} = \hat{b}_i^\dagger \hat{b}_i\).

In our work, we assume that the interaction part of the Hamiltonian is dominated by local terms. This assumption is particularly well justified for ultra-cold atomic systems interacting mainly via short-range intra- and inter-component interactions. Note, however, that in the case of fermionic \(A\)-particles, a double occupation at any site cannot occur due to the Pauli exclusion principle \((\hat{n}_{A_i} \in \{0, 1\} \text{ for any } i)\). Thus, intra-component interaction terms controlled by \(U_A\) rigorously vanish and only the inter-component terms (controlled by \(U\)) remain in play. In the case of bosonic \(A\)-particles additional intra-component interactions controlled by \(U_A\) are possible. In most of the cases studied here, we are interested in the case of non-interacting \(A\)-particles \((U_A = 0)\). In view of these remarks, the Hamiltonian (1) is appropriately written for bosonic as well as for fermionic particles.

### III. DYNAMICS OF TWO PARTICLES

To make further analysis clearer, let us first make some observations on the dynamics of two identical particles belonging to the component \(A\) initially occupying two different sites near the center of the lattice, i.e., as the initial state we take \(|\Psi_0\rangle = \hat{a}_l^\dagger \hat{a}_j^\dagger |\text{vac}\rangle\), where \(d \neq 0\) denotes the distance between occupied sites. In this case the Hamiltonian (1) is reduced to the single component and has the following form

\[
\hat{H} = \sum_i \left[ -J \left( \hat{a}_i^\dagger \hat{a}_{i+1} + \hat{a}_{i+1}^\dagger \hat{a}_i \right) + F i \hat{n}_{A_i} \right]
\]

![Image](image.png)

**FIG. 1.** Evolution of the single-particle density profile \(\rho_{kl}(t)\) for two different tilting of the lattice and initial distance between occupied sites \(d = 2\). Horizontal dashed lines indicate the half-time of the Bloch oscillation period \(T_{1/2}\). As argued in the main text, these results are independent on the quantum statistics of \(A\)-particles. Time is measured in natural unit of the problem \(\hbar/J\).

It is very instructive to note that for the initial state considered here, the time evolution of a whole single-particle density matrix

\[
\rho_{kl}(t) = \langle \Psi(t) | \hat{a}_k^\dagger \hat{a}_l | \Psi(t) \rangle,
\]

where \(|\Psi(t)\rangle = e^{-iHt/\hbar} |\Psi_0\rangle\), is independent on the statistics, i.e., it is exactly the same for non-interacting bosons and fermions. Indeed, the commutation relation

\[
[\hat{H}, \hat{a}_k^\dagger \hat{a}_l] = J \left( \hat{a}_l^\dagger (\hat{a}_{l+1} + \hat{a}_{l-1}) - (\hat{a}_{k+1}^\dagger + \hat{a}_{k-1}^\dagger) \hat{a}_l \right)
\]

is the same for non-interacting bosonic and fermionic particles. The expectation values of this commutator and all the higher order commutators with the Hamiltonian in the localized initial state \(|\Psi_0\rangle\) are insensitive to the particle statistics. Therefore, the single-particle density matrix \(\rho_{kl}(t)\) evolves identically for bosons as well as for fermions. This means that any single-particle measurement is not able to capture any dynamical difference between both statistics if at initial moment particles do not occupy the same site. This result is quite counterintuitive since it is clear that tunnelling to an already occupied neighboring site is blocked for fermions. In Fig. 1 we display the time evolution of the single-particle density profile (the diagonal part of \(\rho_{kl}(t)\)) for two different tilting of the lattice and initial distance \(d = 2\). Clearly visible characteristic spatial oscillations are exactly the same for bosons and fermions and their frequency depends only on the strength of the tilting. The half of the oscillation period is given by \(T_{1/2} = \pi \hbar/F\), and this is when the particles reach the farthest part of the lattice.

The dynamical difference between bosonic and fermionic systems is however well-captured by higher-order correlations. In the case of two-particle systems one of them is particularly important since it can be obtained directly by repeated instant-time measurements.
of particles’ positions – the two-particle density profile
\[ \Gamma_{kl}(t) = \langle \Psi(t) | \hat{n}_{Ak} \hat{n}_{Al} | \Psi(t) \rangle. \] (5)

In this case the commutator \([\hat{H}, \hat{n}_{Ak} \hat{n}_{Al}]\) would again have the identical operator form for both bosons and fermions. However, the initial expectation values are not the same anymore and they give rise to different dynamical behaviors. This is clearly visible when two-particle profiles are compared at the half-time of a Bloch oscillation period, \(T_{1/2}\) (see Fig. 2). Independent of the initial distance \(d\), two-particle densities are essentially different not only close to the diagonal of the doublon occupancy which is forbidden for fermions, but also in a whole spatial range accessible to particles.

This significant difference between bosonic and fermionic particles can be quantified with another quantity that is important from a measurement point of view – the mean distance between particles, \(D(t) = \sum_{k|l \neq k} |k - l| \Gamma_{kl}(t)\). The bosonic enhancement results in a smaller mean distance between non-interacting bosons when compared to the fermionic case. In Fig. 3 we present the mean distance \(D(t)\) for the particular initial state with \(d = 2\). For a complete picture, we display also the bosonic distance \(D(t)\) when the on-site repulsion term in the Hamiltonian controlled by \(U_A\) is present. It is clear, that as the interaction increases the mean distance for the bosons approaches the fermionic case. In the limiting scenario of hard-core bosons \((U_A/J \to \infty)\) both statistics display the same behavior.

### IV. INFLUENCE OF THE THIRD PARTICLE

After discussing the dynamical consequences of statistics for indistinguishable particles we focus on a system of three particles. In the following, we assume that two particles belonging to the \(A\)-component (bosons or fermions) do not interact with each other but their dynamics is affected by a third \(B\)-particle via on-site interaction term of the form \(U \sum \hat{n}_{Ai} \hat{n}_{Bi}\). In this way, we want to figure out how the dynamical features of two-particle correlations described above are affected by the presence of additional, fundamentally different, particle.

The role played by the third particle crucially depends not only on interaction strength \(U\) but also on its initial state. Therefore, we consider a whole family of initial states of the form of Gaussian wave packets localized exactly between \(A\)-particles. The initial configuration of the two \(A\)-particles is same as in the previous case. All it means is that the initial state of the whole system reads

\[ |\Psi_0\rangle = \mathcal{N} \hat{a}_0^{+} \hat{a}_d^{+} \left( \sum_i e^{-\left(i - d/4\right)^2/2\sigma^2 i b_i^2} \right) |\text{vac}\rangle, \] (6)

where \(\mathcal{N}\) is the normalization factor and \(\sigma\) determines width of the component \(B\) wave packet. The limit of ideally localized \(B\)-particle \((\sigma = 0)\) is well-defined only...
Different correlations are enhanced. First, when the distance \(d\) between them, \(\sigma = 0\) (top) and delocalized with \(\sigma = 5\) (bottom). Note that, for delocalized \(B\)-particle a specific 'cross-like' structure is enhanced together with appearance of partial pairing. Compare with Fig. 5 for other initial distances \(d\).

For even \(d\) (\(B\)-particle is localized exactly at site \(i = d/2\)). Therefore, to capture also this limiting case, we focus mainly on the \(d = 2\) case in the following.

At this point, it should be noted that in the absence of interactions, the dynamics of the Gaussian wave packet is reduced to simple oscillations without change of the shape, provided that width \(\sigma\) is larger than the periodicity of the lattice [30, 36]. On the contrary, when inter-component interactions are present \((U \neq 0)\), any non-zero \(\sigma\) introduces finite initial interaction energy to the system.

Exactly as in the case of two particles, we inspect the dynamics of the system during one Bloch oscillation. For clarity, we capture the differences caused by interactions and quantum statistics at moment when they become the largest, i.e., at the half-time of the Bloch oscillation period \(T_{1/2}\) with the tilting strength fixed at \(F/J = 0.2\).

In Fig. 4 we display the two-particle density profile (5) at \(T_{1/2}\) in the case of strong inter-component interactions \(U/J = 10\). Depending on the quantum statistics of \(A\)-particles and the width of Gaussian packet \(\sigma\) different correlations are enhanced. First, when the \(B\)-particle is precisely localized \((\sigma = 0)\), there is no significant difference between bosonic and fermionic particles (left panel). It means that strong interaction with the \(B\)-particle localized exactly between the \(A\)-particles changes the behavior considerably, even in the absence of initial interaction energy. Significant differences caused by the statistics, which was observed previously (compare with the panels in the second row in Fig. 2), are almost completely blurred. In both cases, the probability of finding particles on opposite sides of the central site is the largest. On top of this clear correlation, one finds also a very weak ‘cross-like’ structure emerging in the density profile. It uncovers additional enhancement of probability of finding one particle near the center site with the second particle smeared over a whole available space.

The situation is significantly different when the \(B\)-particle is initially delocalized over several sites. As an example, in the bottom panel of Fig. 4 we display the two-particle density profile for \(A\)-particles at half-time \(T_{1/2}\) when the third particle is initially distributed with \(\sigma = 5\). For both statistics of \(A\)-particles we observe a noticeable enhancement of the ‘cross-like’ structure in the distribution. Moreover, inter-component repulsions support the pair-like formation of \(A\)-particles towards the left edge of the lattice (direction favored by the tilting of the lattice). We checked that this effect remains the same if the sign of inter-component interactions is changed to attractions. It is also independent of the initial distance between particles \(d\) provided that the width \(\sigma\) is large enough to make mediation between \(A\)-particles possible (see Fig. 5). This result evidently shows that a specific attraction between non-interacting particles is dynamically induced when particles do interact with other particle which necessarily needs to be sufficiently delocalized.

To get a better understanding of these non-trivial correlations appearing in the system, besides computing the two-particle density profiles \(\Gamma_{kl}\), we also consider the density profile of doublons \(\eta_i(t)\). This is given by the probability density of finding at least one \(A\)-particle and \(B\)-particle together at the same site. In the case of fermionic particles it can be calculated straightforwardly as \(\eta_i(t) = \langle \Psi(t)|\hat{n}_{A_i}\hat{n}_{B_i}|\Psi(t)\rangle\). For bosons, due
FIG. 6. Density distributions for the strongly interacting system \((U/J = 10)\) initially prepared in the state (6) with \(\sigma = 2\) (top row) \(\sigma = 5\) (bottom row) and exactly at the half of the Bloch period \(T_{1/2}\). Dashed green line and red solid lines correspond to the single-particle density of \(A\)-particles \(\rho_{ii}\) and the doublon density \(\eta_i\), respectively. To possible double-occupancy of \(A\)-particles, the definition is slightly modified and in the case studied can be expressed as \(\eta_i(t) = \langle \Psi(t) | n_{A1} n_{B1} (3 - n_{A1})/2 | \Psi(t) \rangle\). It is important to note, that due to a well-known halving of the time period of the Bloch oscillation for doublons [31, 32], the doublon density is particularly very straightforward at time \(T_{1/2}\). Exactly at this moment, when the single-particle density peaks are maximally distant from the origin, the doublon density \(\eta_i\) is non-zero only in the vicinity of the central site. In Fig. 6 we illustrate this effect for different initial widths \(\sigma\) of the \(B\)-particle wave packet and different statistics of \(A\)-particles (solid red lines for doublon density \(\eta_i\) and dashed green lines for the single-particle density \(\rho_{ii}\) of \(A\)-particle). This specific composition of the doublon density and the single-particle profile results in the final two-particle correlation of \(A\)-particles. The ‘cross-like’ structure is triggered mostly by the doublon, while the remaining part at the edges of the system comes from the distribution of the remaining \(A\)-particle.

V. ROLE OF THE LATTICE GEOMETRY

As we have shown, the two-body correlations between non-interacting \(A\)-particles forced by the presence of the additional particle depends crucially on the initial state. Interestingly and quite counterintuitively these correlations are less sensitive to the initial distance between particles and their statistics provided that inter-component interactions are strong enough. For completeness, we will now focus on the role of lattice geometry in these correlations. To provide a concrete example, let us consider a one-dimensional lattice that is not homogeneously tilted but the tilting becomes inverted around the center of the initial state, i.e., a lattice having local energies of the form \(E_i = -F|i - i_0|\). Such an exotic tilting, although simple for theoretical consideration, may be demanding for experimental manufacturing. However, in the view of recent progress in creating very different lattice configurations, it is not impossible. At this point it worth to point out that our arguments on the dynamics of two non-interacting particles presented in Sec. III remain valid: at any moment, any single-particle quantity persists insensitive to the quantum statistics provided that initially particles are localized in two different sites of the lattice.

As the simplest example, we focus on the three-particle initial state (6) with \(d = 2\) and \(\sigma = 5\) centered exactly around the middle point of the lattice \((i_0 = 1)\) with inter-component interaction \(U/J = 10\) and tilt strength
The single-particle density profiles, respectively for A and B columns correspond to different statistics of configuration is presented in Fig. 7. Exactly as previously, columns A and B correspond to different statistics of configuration exactly in the center of the system. This increased concentration of A-particles in the center is also clearly visible in the single-particle density profile (the fourth row, dashed green line). Note that in this case it is not triggered by the existence of doublons which is highly suppressed (red line). As the two-particle correlations are substantially different from the previous case (see Figs. 4 and 5) we systematically change the tilting configuration from the uniform structure (considered in the preceding sections) to the ‘broken’ configuration (reported in this section) and compute the correlations. As the correlation pattern changes, we see the doublon peak gradually diminishing at the center of the lattice, where the single-particle density consequently increases. Although not displayed here, this behavior is confirmed in our calculations.

VI. CONCLUSION

We have analyzed the dynamics of the simplest system of two-component quantum walkers on a tilted optical lattice and studied the effect of particle statistics, interaction, choice of initial states, and lattice geometry, which lead to three main findings. Firstly, single-particle measurements cannot distinguish the quantum statistical nature of two non-interacting walkers during evolution from initially localized states. One must perform two-particle measurements to differentiate between the particle-statistics. Secondly, interaction with the third walker from a different distinguishable component can induce non-trivial correlations between two walkers which can be controlled by changing the initial states. As our third and final result, we show that the two-particle correlations can also be significantly modified by changing the tilting structure of the lattice. Implementation of a non-trivial tilting is a challenging task, however, with the ever-growing standard of ongoing optical lattice experiments with a controlled number of cold atoms, all the results presented in this work can be realized and bench-marked. This study can be further expanded to probe the effect of intra-component interactions, single-particle state preparations, and a generalized study of the dynamics of multi-component quantum walkers, to name a few.

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