Nested algebraic Bethe ansatz for open $GL(N)$ spin chains with projected $K$-matrices

Rafael I. Nepomechie

Physics Department, P.O. Box 248046, University of Miami
Coral Gables, FL 33124 USA

Abstract

We consider an open spin chain model with $GL(N)$ bulk symmetry that is broken to $GL(M) \times GL(N - M)$ by the boundary, which is a generalization of a model arising in string/gauge theory. We prove the integrability of this model by constructing the corresponding commuting transfer matrix. This construction uses operator-valued “projected” $K$-matrices. We solve this model for general values of $N$ and $M$ using the nested algebraic Bethe ansatz approach, despite the fact that the $K$-matrices are not diagonal. The key to obtaining this solution is an identity based on a certain factorization property of the reduced $K$-matrices into products of $R$-matrices. Numerical evidence suggests that the solution is complete.

1nepomechie@physics.miami.edu
1 Introduction

As shown long ago by Sklyanin [1], the construction of an integrable open spin chain model requires two main ingredients: an \( R \)-matrix (solution of the Yang-Baxter equation) which determines the bulk terms in the Hamiltonian, and right/left \( K \)-matrices (solutions of the boundary Yang-Baxter equation [2, 3]) which determine the right/left boundary terms in the Hamiltonian. By now it is well understood how to solve models with diagonal \( K \)-matrices. However, solving models with \( K \)-matrices which are not diagonal in general remains a challenging problem. Such \( K \)-matrices can have matrix elements which are either \( c \)-numbers or operators. While models with non-diagonal \( c \)-number-valued \( K \)-matrices have received considerable attention (see, for example, [4]-[17]), much less is known about models with operator-valued \( K \)-matrices.

An interesting class of non-diagonal operator-valued \( K \)-matrices consists of so-called projected \( K \)-matrices found by Frahm and Slavnov [18]. Integrable open spin chains constructed with \( K \)-matrices of this type have found applications in condensed matter physics [19, 20, 21] and string/gauge theory [22, 23, 24].

We consider here an integrable open spin chain model constructed with such projected \( K \)-matrices. The chain has \( L + 2 \) sites, labeled \( X, 1, \ldots, L, Y \). The space of states is

\[
C^M \otimes C^N \otimes \cdots \otimes C^N \otimes C^M, \tag{1.1}
\]

where \( 1 < M < N \). That is, the vector spaces of the “bulk” sites (labeled \( 1, \ldots, L \)) all have dimension \( N \), while the vector spaces of left and right “boundary sites” (labeled \( X \) and \( Y \), respectively) have a lower dimension \( M \). The Hamiltonian is given by

\[
H = Q_X^{(M)} h_X Q_X^{(M)} + \sum_{l=1}^{L-1} h_{l,l+1}^{(M)} + Q_Y^{(M)} h_{L,Y} Q_Y^{(M)}, \tag{1.2}
\]

where the two-site Hamiltonian \( h_{l,l+1} \) is given by

\[
h_{l,l+1} = \mathbb{I}_{l,l+1} - P_{l,l+1}, \tag{1.3}
\]

where \( \mathbb{I} \) and \( P \) are the identity and permutation matrices on \( C^N \otimes C^N \), respectively; and \( Q^{(M)} \) is a diagonal \( N \times N \) matrix which projects \( C^N \) to \( C^M \),

\[
Q^{(M)} = \text{diag}(1,1,\ldots,1,0,0,\ldots,0). \tag{1.4}
\]

\(^1\)For related work in string/gauge theory, see e.g. [25]-[28] and references therein.
We drop the null rows and columns of the left and right boundary terms in the Hamiltonian, which therefore should be understood as $MN \times MN$ matrices acting on $C^M \otimes C^N$ and $C^N \otimes C^M$, respectively.

Although the bulk terms have $GL(N)$ symmetry, the boundary terms reduce the symmetry to $GL(M) \times GL(N - M)$. We shall refer to this model as the $GL(N)/(GL(M) \times GL(N - M))$ model. The case $(N, M) = (3, 2)$ was recently studied (following [22, 23]) in [24].

Within the quantum inverse scattering method, the standard approach for solving integrable spin chains with higher-rank symmetry is nested algebraic Bethe ansatz (ABA) [32, 33]. This approach has been adapted to open spin chains with diagonal $K$-matrices in [34, 35, 36]. We further adapt this method, along the lines in [19, 20, 21] for a related model with $M = 2$, to solve the $GL(N)/(GL(M) \times GL(N - M))$ model for general values of $N$ and $M$. The identity (4.3), which relies on a certain factorization property of the “reduced” $K$-matrices into products of $R$-matrices, plays an essential role in obtaining a solution. Numerical evidence suggests that the solution is complete.

The outline of this paper is as follows. In Sec. 2 we construct the transfer matrix corresponding to the Hamiltonian (1.2), thereby proving the integrability of the latter. In Sec. 3 we consider, as a warm-up, the special case $M = 2$. We establish our notation, present the nested ABA solution, and provide some evidence of its completeness. We then treat the general case $M \geq 2$ in Sec 4. Finally, in Sec. 5 we present our conclusions and list some interesting unresolved questions. Appendix A contains our proof of the important identity (4.3).

## 2 Transfer matrix

As already mentioned in the Introduction, the transfer matrix is constructed from an $R$-matrix and right/left $K$-matrices. The former is a solution $R(u)$ of the Yang-Baxter equation (YBE)

$$R_{12}(u_1 - u_2) R_{13}(u_1) R_{23}(u_2) = R_{23}(u_2) R_{13}(u_1) R_{12}(u_1 - u_2). \quad (2.1)$$

\footnote{The idea of breaking a symmetry down to a subgroup by boundary interactions has recently been explored a great deal in the $O(N)$ case at the critical point in [29–31].}
In view of the $GL(N)$ symmetry of the bulk terms of the Hamiltonian, we take the well-known rational solution

$$R(u) = uI + i\mathcal{P} = a(u)\sum_{a=1}^{N} e^{(N)}_{aa} \otimes e^{(N)}_{aa} + b(u)\sum_{a,b=1 \atop a \neq b}^{N} e^{(N)}_{ab} \otimes e^{(N)}_{bb} + i\sum_{a,b=1 \atop a \neq b}^{N} e^{(N)}_{ab} \otimes e^{(N)}_{ba},$$

where

$$a(u) = u + i, \quad b(u) = u,$$

and $e^{(N)}_{ab}$ is the standard elementary $N \times N$ matrix whose $(a, b)$ matrix element is 1, and all others are zero; i.e., $[e^{(N)}_{ab}]_{ij} = \delta_{ai}\delta_{bj}$.

The right $K$-matrix $K^-(u)$, which here acts on $C^N \otimes C^M$, is a solution of the right boundary Yang-Baxter equation (BYBE) \[1\] \[2\] \[3\]

$$R_{12}(u_1 - u_2) K^-_{13}(u_1) R_{12}(u_1 + u_2) K^-_{23}(u_2) = K^-_{23}(u_2) R_{12}(u_1 + u_2) K^-_{13}(u_1) R_{12}(u_1 - u_2).$$

We take the solution \[18\] \[3\]

$$K^-(u) = a_1(u)\sum_{a,b=1}^{M} e^{(N)}_{aa} \otimes e^{(M)}_{bb} + a_2(u)\sum_{a,b=1}^{M} e^{(N)}_{ab} \otimes e^{(M)}_{ba} + a_3(u)\sum_{a=M+1}^{N} \sum_{b=1}^{M} e^{(N)}_{a a} \otimes e^{(M)}_{b b},$$

where

$$a_1(u) = 1 - u^2, \quad a_2(u) = -2iu, \quad a_3(u) = 1 + u^2.$$

For the case $(N, M) = (3, 2)$, this matrix coincides with the one we used earlier in \[3\]. (There we called the left and right $K$-matrices $K^L$ and $K^R$ instead of $K^+$ and $K^-$; and we labeled the left and right spaces 0 and $L + 1$ instead of $X$ and $Y$, respectively.) Unfortunately, we were unaware of \[18\] at that time.

Here we define the left $K$-matrix $K^+(u)$ to also act on $C^N \otimes C^M$. \[4\] It satisfies the left BYBE \[1\]

$$R_{12}(-u_1 + u_2) K^+_{13}(u_1)^{t_1} R_{12}(-u_1 - u_2 - \eta) K^+_{23}(u_2)^{t_2} = K^+_{23}(u_2)^{t_2} R_{12}(-u_1 - u_2 - \eta) K^+_{13}(u_1)^{t_1} R_{12}(-u_1 + u_2),$$

\[3\]See Eq. (3.12) for $\pi_1 K^-(u)\pi_1$ in \[18\]. Here we take the constant $c = 0$ (in order to match with the Hamiltonian \[1\] \[2\]), we rescale the rapidity $u \mapsto -iu$ (in order to match with our conventions for the $R$-matrix \[2\]), and we clear the denominators by performing an overall rescaling.

\[4\]In \[3\], we instead defined the left $K$-matrix to act on $C^M \otimes C^N$ (with $(N, M) = (3, 2)$); i.e., the two $K$-matrices are related by permutation.
where \( t_i \) denotes transposition in the \( i \)th space, and \( \eta = iN \) appears in the crossing-unitarity relation

\[
R_{12}(u)^{t_i} \ R_{12}(-u - \eta)^{t_i} \propto \mathbb{I}, \quad (2.8)
\]

where the proportionality factor is some scalar function of \( u \). A solution is provided by the “less obvious” isomorphism \([1]\)

\[
K_{13}^+(u) = \text{tr}_2 \mathcal{P}_{12} R_{12}(-2u - \eta) K_{23}^-(u), \quad (2.9)
\]

which gives (up to an irrelevant overall factor)

\[
K^+(u) = b_1(u) \sum_{a,b=1}^{M} e^{(N)}_{aa} \otimes e^{(M)}_{bb} + b_2(u) \sum_{a,b=1}^{M} e^{(N)}_{ab} \otimes e^{(M)}_{ba} + b_3(u) \sum_{a=M+1}^{N} \sum_{b=1}^{M} e^{(N)}_{aa} \otimes e^{(M)}_{bb}, \quad (2.10)
\]

where

\[
b_1(u) = u^2 + i(N - M)u, \quad b_2(u) = 2iu - N, \quad b_3(u) = -u^2 - iMu. \quad (2.11)
\]

Again, for the case \((N, M) = (3, 2)\), this solution agrees with the one used in \([5]\).

The transfer matrix \( t(u) \) is given by \([1]\)

\[
t(u) = \text{tr}_a K_{aX}^+(u) T_{a1\ldots L}(u) K_{aY}^-(u) \hat{T}_{a1\ldots L}(u), \quad (2.12)
\]

where the trace (\( \text{tr} \)) is over an \( N \)-dimensional auxiliary space denoted by \( a \). The argument of the trace acts on

\[
\downarrow C^N \otimes C^M \otimes C^N \otimes \cdots \otimes C^N \otimes C^M, \quad (2.13)
\]

and therefore \( t(u) \) acts on \((1.1)\), as does the Hamiltonian. The monodromy matrices \( T \) and \( \hat{T} \) are given by

\[
T_{a1\ldots L}(u) = R_{a1}(u) \cdots R_{aL}(u), \quad \hat{T}_{a1\ldots L}(u) = R_{aL}(u) \cdots R_{a1}(u). \quad (2.14)
\]

Indeed, it can be shown that the transfer matrix \((2.12)\) obeys the fundamental commutativity property

\[
[t(u), t(v)] = 0. \quad (2.15)
\]

It can also be shown that this transfer matrix contains the Hamiltonian \((1.2)\),

\[
H = c_1 \left. \frac{d}{du} t(u) \right|_{u=0} + c_2 \mathbb{I}, \quad (2.16)
\]
where
\[ c_1 = \frac{i}{2N}(-1)^L, \quad c_2 = L + 1 + \frac{1}{N}. \] (2.17)

The relations (2.15) - (2.17) demonstrate the integrability of the Hamiltonian.

The transfer matrix has the \( GL(M) \times GL(N - M) \) symmetry
\[ [t(u), h \otimes g \otimes L \otimes h] = 0, \] (2.18)
where
\[ g = \begin{pmatrix} h & 0 \\ 0 & h' \end{pmatrix}, \quad h \in GL(M), \quad h' \in GL(N - M). \] (2.19)

3 Nested ABA for \( M = 2 \)

We now proceed to diagonalize the transfer matrix of the \( GL(N)/(GL(M) \times GL(N - M)) \) model via nested ABA for the special case \( M = 2 \).

3.1 Preliminaries

We begin by assembling the ingredients needed to carry out the ABA analysis: suitable operators, pseudovacuum states and commutation relations. For the \( M = 2 \) case, the left \( K \)-matrix (2.10) has the form (as an \( N \times N \) matrix in the auxiliary space)
\[
K^+_{ax}(u) = \begin{pmatrix}
\alpha_{11}(u) & \alpha_{12}(u) & 0 & 0 & \cdots & 0 \\
\alpha_{21}(u) & \alpha_{22}(u) & 0 & 0 & \cdots & 0 \\
0 & 0 & \beta(u) & 0 & \cdots & 0 \\
0 & 0 & 0 & \beta(u) & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \beta(u)
\end{pmatrix},
\] (3.1)
where \( \alpha_{jk}(u) \) and \( \beta(u) \) are operators on the two-dimensional quantum space \( X \). For future reference, we now introduce a “down” pseudovacuum state for this space,
\[
|0\rangle_X = \begin{pmatrix}
0 \\
1
\end{pmatrix},
\] (3.2)
and note that it is an eigenstate of the diagonal operators,
\[
\alpha_{11}(u)|0\rangle_X = b_1(u)|0\rangle_X, \\
\alpha_{22}(u)|0\rangle_X = (b_1(u) + b_2(u))|0\rangle_X, \\
\beta(u)|0\rangle_X = b_3(u)|0\rangle_X,
\] (3.3)
and is annihilated by $\alpha_{12}(u)$,

$$\alpha_{12}(u)|0\rangle_X = 0.$$  \hfill (3.4)

The right $K$-matrix \[2.5\] has a similar structure. Introducing a “down” pseudovacuum state also for the quantum space $Y$,

$$|0\rangle_Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$  \hfill (3.5)

we see that it is an eigenstate of the diagonal operators\footnote{We denote by $[K^-(u)]_{jk}$ the $(j,k)$ element of $K_{aY}^-(u)$ considered as an $N \times N$ matrix in the auxiliary space, analogously to \[3.1\].}

$$[K^-(u)]_{11} |0\rangle_Y = a_1(u)|0\rangle_Y,$$
$$[K^-(u)]_{22} |0\rangle_Y = (a_1(u) + a_2(u))|0\rangle_Y,$$
$$[K^-(u)]_{jj} |0\rangle_Y = a_3(u)|0\rangle_Y, \quad j = 3, \ldots, N,$$  \hfill (3.6)

and is annihilated by $[K^-(u)]_{12}$,

$$[K^-(u)]_{12} |0\rangle_Y = 0.$$  \hfill (3.7)

The transfer matrix \[2.12\] can be reexpressed as

$$t(u) = \text{tr}_a K_{aX}^+(u) \mathcal{T}_{a1\ldots L Y}^-(u),$$  \hfill (3.8)

where $\mathcal{T}_{a1\ldots L Y}^-(u)$, defined by

$$\mathcal{T}_{a1\ldots L Y}^-(u) = T_{a1\ldots L}(u) K_{aY}^-(u) \hat{T}_{a1\ldots L}(u),$$  \hfill (3.9)

also obeys the right BYBE \[2.4\]. It is from this object that we must identify suitable operators (among them, creation-like operators). In view of the form \[3.1\] of the left $K$-matrix, we follow \[19, 20, 21\] (see also \[10, 34, 35, 36\] and references therein) and write $\mathcal{T}_{a1\ldots L Y}^-(u)$ as follows (as an $N \times N$ matrix in the auxiliary space)

$$\mathcal{T}_{a1\ldots L Y}^-(u) = \begin{pmatrix} A_{11}^1(u) & \cdots & A_{1,N-1}^1(u) & B_1^1(u) \\ \vdots & \ddots & \vdots & \vdots \\ A_{N-1,1}^1(u) & \cdots & A_{N-1,N-1}^1(u) & B_{N-1}^1(u) \\ C_1^1(u) & \cdots & C_{N-1}^1(u) & D^1(u) \end{pmatrix},$$  \hfill (3.10)
where $A_{jk}^{(1)}(u), B_j^{(1)}(u), C_j^{(1)}(u), D^{(1)}(u)$ are operators on the quantum spaces
\[
\downarrow 1 \quad \downarrow L \quad \downarrow Y \quad \downarrow 2 \quad \downarrow C^N \otimes \cdots \otimes C^N \otimes C^2 .
\]

With respect to the all “down” pseudovacuum state
\[
|0\rangle_{1\cdots L} = Y |0\rangle_{1\cdots L} Y, \quad |0\rangle_{1\cdots L} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \otimes L,
\]

$B_j^{(1)}(u)$ and $C_j^{(1)}(u)$ are annihilation and creation operators, respectively,
\[
B_j^{(1)}(u)|0\rangle_{1\cdots L} = 0, \quad C_j^{(1)}(u)|0\rangle_{1\cdots L} \neq 0,
\]

and $D^{(1)}(u)$ is diagonal,
\[
D^{(1)}(u)|0\rangle_{1\cdots L} = a(u)^{2L} a_3(u)|0\rangle_{1\cdots L}.
\]

Moreover, defining the operators $\tilde{A}_{jk}^{(1)}(u)$ by
\[
\tilde{A}_{jk}^{(1)}(u) = A_{jk}^{(1)}(u) - \frac{i}{a(2u)^2} \delta_{jk} D^{(1)}(u),
\]

we find that
\[
\tilde{A}_{jk}^{(1)}(u)|0\rangle_{1\cdots L} = b(u)^{2L} \left[ K^{-1}^{(1)}(u) \right]_{jk} |0\rangle_{1\cdots L},
\]

where $\left[ K^{-1}^{(1)}(u) \right]_{jk}$ are operators on the two-dimensional quantum space $Y$ defined by
\[
\left[ K^{-1}^{(1)}(u) \right]_{jk} = \left[ K^{-1}(u) \right]_{jk} - \frac{ia_3(u)}{a(2u)} \delta_{jk} \overline{\mathbb{I}}, \quad j, k = 1, \ldots, N - 1.
\]

The trace over the auxiliary space in the expression (3.8) for the transfer matrix can now be performed, resulting in the more explicit expression
\[
t(u) = \sum_{j,k=1}^{N-1} \left[ K^{(1)}(u) \right]_{jk} \tilde{A}_{kj}^{(1)}(u) + F^{(1)}(u) D^{(1)}(u),
\]

where $\left[ K^{(1)}(u) \right]_{jk}$ are operators on the two-dimensional quantum space $X$ defined by
\[
\left[ K^{(1)}(u) \right]_{jk} = \left[ K^{(u)} \right]_{jk} = \begin{cases} \alpha_{jk}(u), & j, k = 1, 2 \\ \delta_{jk} \beta(u), & j, k = 3, \ldots, N - 1 \end{cases}
\]
and

\[
F^{(1)}(u) = \beta(u) + \frac{i}{a(2u)} \left[ \sum_{j=1}^{2} \alpha_{jj}(u) + (N - 3)\beta(u) \right].
\] (3.20)

Note that the expression (3.18) for \( t(u) \) does not involve either annihilation or creation operators, which is necessary for carrying out the nested ABA analysis.

The operators obey the following commutation relations,

\[
\tilde{A}_{ik}^{(1)}(u) C_j^{(1)}(v) = \frac{1}{b(u-v)a(u+v)} \left[ R^{(1)}(u + v + i) \right]_{ij'} [ R^{(1)}(u - v) ]_{k'j} C_j^{(1)}(v) A_{k'k}(u) - \frac{i}{a(2u)b(u-v)} \left[ R^{(1)}(2u + i) \right]_{ij';ik} C_j^{(1)}(u) \tilde{A}_{k'k}(v)
\]

\[
+ \frac{ib(2v)}{a(2u)a(2v)a(u+v)} \left[ R^{(1)}(2u + i) \right]_{ij';jk} C_j^{(1)}(u) D^{(1)}(v),
\]

\[
D^{(1)}(u) C_j^{(1)}(v) = \frac{a(v-u)b(v+u)}{b(v-u)a(u+v)} \left[ C_j^{(1)}(v) D^{(1)}(u) \right] - \frac{i}{a(u+v)} C_j^{(1)}(u) \tilde{A}_{k'k}(v),
\]

\[
C_j^{(1)}(u) C_k^{(1)}(v) = \frac{1}{a(u-v)} \left[ R^{(1)}(u - v) \right]_{jk';kj} C_k^{(1)}(v) C_j^{(1)}(u),
\] (3.21)

where \( R^{(1)}(u) \) is the \( GL(N-1) \) \( R \)-matrix, with matrix elements

\[
\left[ R^{(1)}(u) \right]_{jjjj} = a(u), \quad \left[ R^{(1)}(u) \right]_{jk;jk} = b(u), \quad \left[ R^{(1)}(u) \right]_{jk;kj} = i, \quad k \neq j,
\]

\[
j, k = 1, \ldots, N - 1,
\] (3.22)

with \( a(u) \) and \( b(u) \) as before (2.3). Summation over repeated indices is understood in the commutation relations.

### 3.2 First level

The pseudovacuum state for the full space of states is given by

\[
|0\rangle_{X1\ldots LY} = |0\rangle_X |0\rangle_{1\ldots LY} = |0\rangle_X |0\rangle_{1\ldots L} |0\rangle_Y.
\] (3.23)

It is an eigenstate of the transfer matrix (3.18) by virtue of (3.3), (3.4), (3.6), (3.7), (3.14) - (3.17). This state is not the lowest-energy state. Indeed, it is an eigenstate of the Hamiltonian (1.2) with energy eigenvalue 2, while there are eigenstates (such as the all “up” state) with energy 0.
We make the ansatz that the eigenstates $|\Omega^{(1)}\rangle$ of the transfer matrix (which are independent of the spectral parameter $u$ by virtue of the commutativity property (2.13)) can be obtained by acting on the pseudovacuum state with the creation operators, namely,

$$|\Omega^{(1)}\rangle = C_{i_1}^{(1)}(u_{1,1}) \ldots C_{i_{m_1}}^{(1)}(u_{1,m_1})|0\rangle_{X_1 \ldots Y_1} F^{(1)}_{i_1 \ldots i_{m_1}}, \quad (3.24)$$

where again summation over repeated indices is understood.

By acting with the expression (3.18) for the transfer matrix on this state, and using the commutation relations (3.21) to repeatedly move $\hat{A}^{(1)}(u)$ and $D^{(1)}(u)$ past consecutive creation operators until arriving at the pseudovacuum state, two types of terms are generated. The “wanted” terms are those generated by the first terms in the commutation relations; the remaining terms are “unwanted”. The “wanted” terms give

$$t(u)|\Omega^{(1)}\rangle = \Lambda(u)|\Omega^{(1)}\rangle, \quad (3.25)$$

with

$$\Lambda(u) = f_0(u) a(u)^{2L} \prod_{j=1}^{m_1} \frac{a(u_{1,j} - u)b(u_{1,j} + u)}{b(u_{1,j} - u)a(u_{1,j} + u)}$$

$$+ b(u)^{2L} \prod_{j=1}^{m_1} \frac{1}{b(u - u_{1,j})a(u + u_{1,j})} \Lambda^{(1)}(u; \{u_{1,j}\}), \quad (3.26)$$

where

$$f_0(u) = a_3(u) \left\{ b_3(u) + \frac{i}{a(2u)} [2b_1(u) + b_2(u) + (N - 3)b_3(u)] \right\}$$

$$= - \frac{(2u + iN)(u^2 + 1)^2}{2u + i}. \quad (3.27)$$

Moreover, $\Lambda^{(1)}(u; \{u_{1,j}\})$ is a solution of the eigenvalue problem

$$t^{(1)}(u; \{u_{1,j}\})_{j_1 \ldots j_{m_1}; i_1 \ldots i_{m_1}} F^{(1)}_{i_1 \ldots i_{m_1}} = \Lambda^{(1)}(u; \{u_{1,j}\}) F^{(1)}_{i_1 \ldots i_{m_1}}, \quad (3.28)$$

where the level-one inhomogeneous transfer matrix $t^{(1)}(u; \{u_{1,j}\})$ is defined by

$$t^{(1)}(u; \{u_{1,j}\}) = tr_{a^{(1)}} K_{a^{(1)}X}^{(1)}(u) T_{a^{(1)}1 \ldots m_1 Y}^{-}(u; \{u_{1,j}\}) \quad (3.29)$$

where now the auxiliary space, denoted by $a^{(1)}$, has dimension $N - 1$; and

$$T_{a^{(1)}1 \ldots m_1}^{-}(u; \{u_{1,j}\}) = T^{(1)}_{a^{(1)}1 \ldots m_1}(u; \{u_{1,j}\}) K_{a^{(1)}Y}^{-}(u) T^{(1)}_{a^{(1)}1 \ldots m_1}(u; \{u_{1,j}\}) \quad (3.30)$$

where the level-one inhomogeneous monodromy matrices are given by

$$T_{a^{(1)}1 \ldots m_1}(u; \{u_{1,j}\}) = R_{a^{(1)}1}(u + u_{1,1} + i) \ldots R_{a^{(1)}m_1}(u + u_{1,m_1} + i),$$

$$T_{a^{(1)}1 \ldots m_1}^{-}(u; \{u_{1,j}\}) = R_{a^{(1)}m_1}(u - u_{1,m_1}) \ldots R_{a^{(1)}1}(u - u_{1,1}). \quad (3.31)$$
By virtue of the fact that the level-one $K$-matrices satisfy shifted BYBEs

\begin{align}
R_{12}^{(1)}(u_1 - u_2) K_{13}^{(1)}(u_1) R_{12}^{(1)}(u_1 + u_2 + i) K_{23}^{(1)}(u_2) \\
= K_{23}^{(1)}(u_2) R_{12}^{(1)}(u_1 + u_2 + i) K_{13}^{(1)}(u_1) R_{12}^{(1)}(u_1 - u_2),
\end{align}

(3.32)

\begin{align}
R_{12}^{(1)}(-u_1 + u_2) K_{13}^{(1)}(u_1) t_1 R_{12}^{(1)}(-u_1 - u_2 - \eta - i) K_{23}^{(1)}(u_2) t_2 \\
= K_{23}^{(1)}(u_2) t_2 R_{12}^{(1)}(-u_1 - u_2 - \eta - i) K_{13}^{(1)}(u_1) t_1 R_{12}^{(1)}(-u_1 + u_2), \quad \eta = i(N - 1),
\end{align}

(3.33)

(cf. Eqs. (2.4), (2.7), respectively), the level-one transfer matrix (3.29) has the commutativity property

\begin{align}
[t^{(1)}(u; \{u_{1,j}\}), t^{(1)}(v; \{u_{1,j}\})] = 0.
\end{align}

(3.34)

Although for the level-one transfer matrix the auxiliary space and the “bulk” quantum spaces (i.e., those labeled 1, \ldots, $m_1$) have dimension one lower compared with the original transfer matrix, the “boundary” quantum spaces (i.e., those labeled $X, Y$) remain unchanged.

### 3.3 Iterating

We continue to iterate the above procedure. We define

\begin{align}
T_{a(0)1\ldots m_1Y}^{(-l)}(u; \{u_{1,j}\}) = T_{a(0)1\ldots m_1}^{(l)}(u; \{u_{1,j}\}) \cdot T_{a(0)Y}^{(-l)}(u) \cdot \hat{T}_{a(0)1\ldots m_1}^{(l)}(u; \{u_{1,j}\}),
\end{align}

(3.35)

where the auxiliary space, denoted by $a^{(l)}$, has dimension $N - l$, and

\begin{align}
T_{a(0)1\ldots m_1}^{(l)}(u; \{u_{1,j}\}) &= R_{a(0)1}^{(l)}(u + u_{1,1} + il) \cdots R_{a(0)m_1}^{(l)}(u + u_{l,m_1} + il), \\
\hat{T}_{a(0)1\ldots m_1}^{(l)}(u; \{u_{1,j}\}) &= R_{a(0)m_1}^{(l)}(u - u_{l,m_1}) \cdots R_{a(0)1}^{(l)}(u - u_{1,1}),
\end{align}

(3.36)

where $R^{(l)}(u)$ is the $GL(N - l)$ $R$-matrix,

\begin{align}
[R^{(l)}(u)]_{jjjk} = a(u), \quad [R^{(l)}(u)]_{jkjk} = b(u), \quad [R^{(l)}(u)]_{jkjk} = i, \quad k \neq j, \\
j, k = 1, \ldots, N - l.
\end{align}

(3.37)

We set

\begin{align}
T_{a(0)1\ldots m_1Y}^{(-l)}(u; \{u_{1,j}\}) &= \begin{pmatrix}
A_{11}^{(l+1)}(u) & \cdots & A_{1,N-l-1}^{(l+1)}(u) & B_1^{(l+1)}(u) \\
\vdots & \ddots & \vdots & \vdots \\
A_{N-l-1,1}^{(l+1)}(u) & \cdots & A_{N-l-1,N-l-1}^{(l+1)}(u) & B_{N-l-1}^{(l+1)}(u) \\
C_1^{(l+1)}(u) & \cdots & C_{N-l-1}^{(l+1)}(u) & D^{(l+1)}(u)
\end{pmatrix}, \\
l = 1, \ldots, N - 3, \quad N \geq 4.
\end{align}

(3.38)
The above equations are valid also for \( l = 0 \) if we identify

\[
u_{0,j} = 0, \quad m_0 = L, \quad (3.39)
\]

and also \( T^{−(0)} = T^{−} \), etc., see \( (3.9), (3.10) \). We define

\[
\tilde{A}^{(l+1)}_{jk}(u) = A^{(l+1)}_{jk}(u) - \frac{i}{a(2u + il)} \delta_{jk} D^{(l+1)}(u), \quad (3.40)
\]

and find that

\[
D^{(l+1)}(u)|0\rangle_{1 \cdots m_Y} = \frac{b(2u)a_3(u)}{b(2u + il)} \prod_{j=1}^{m_l} a(u - u_{l,j}) a(u + u_{l,j} + il)|0\rangle_{1 \cdots m_Y},
\]

\[
\tilde{A}^{(l+1)}_{jk}(u)|0\rangle_{1 \cdots m_Y} = \prod_{j=1}^{m_l} b(u - u_{l,j}) b(u + u_{l,j} + il) \left[K^{−(l+1)}(u)\right]_{jk} |0\rangle_{1 \cdots m_Y}, \quad (3.41)
\]

where

\[
\left[K^{−(l+1)}(u)\right]_{jk} = \left[K^{−(l)}(u)\right]_{jk} - \frac{ib(2u)a_3(u)}{b(2u + il)a(2u + il)} \delta_{jk}, \quad j, k = 1, \ldots, N - l - 1. \quad (3.42)
\]

The commutation relations are generalizations of \( (3.21) \); in particular, the terms which generate the “wanted” terms are given by

\[
\tilde{A}^{(l+1)}_{ik}(u) C^{(l+1)}_{j}(v) = \frac{1}{b(u - v)a(u + v + il)} \left[R^{(l+1)}(u + v + il + 1)\right]_{ij',i'j} \times \left[R^{(l+1)}(u - v)\right]_{k'k;j} C^{(l+1)}_{j'}(v) \tilde{A}^{(l+1)}_{ik'}(u) + \ldots
\]

\[
D^{(l+1)}(u) C^{(l+1)}_{j}(v) = \frac{a(v - u)b(v + u + il)}{b(v - u)a(v + u + il)} C^{(l+1)}_{j}(v) D^{(l+1)}(u) + \ldots. \quad (3.43)
\]

The level-\( l \) transfer matrix is given by

\[
t^{(l)}(u; \{u_{l,j}\}) = \text{tr}_{a^{(l)}} K^{+(l)}_{a^{(l)X}}(u) T^{−(l)}_{a^{(l)1 \cdots m_Y}}(u; \{u_{l,j}\})
\]

\[
= \sum_{j,k=1}^{N-l-1} \left[K^{+(l+1)}(u)\right]_{kj} \tilde{A}^{(l+1)}_{kj}(u) + F^{(l+1)}(u) D^{(l+1)}(u), \quad (3.44)
\]

where

\[
\left[K^{+(l+1)}(u)\right]_{jk} = \left[K^{+(l)}(u)\right]_{jk}, \quad j, k = 1, \ldots, N - l - 1, \quad (3.45)
\]

and

\[
F^{(l+1)}(u) = \beta(u) + \frac{i}{a(2u + il)} \left[\sum_{j=1}^{2} \alpha_{jj}(u) + (N - l - 3)\beta(u)\right]. \quad (3.46)
\]
The $K$-matrices satisfy the shifted BYBEs

$$R_{12}^{(l)}(u_1-u_2)K_{13}^{(l)}(u_1)R_{12}^{(l)}(u_1+u_2+il)K_{23}^{(l)}(u_2) = K_{23}^{(l)}(u_2)R_{12}^{(l)}(u_1+u_2+il)K_{13}^{(l)}(u_1)R_{12}^{(l)}(u_1-u_2), \quad (3.47)$$

$$R_{12}^{(l)}(-u_1+u_2)K_{13}^{(l)}(u_1)^{t_1}R_{12}^{(l)}(-u_1-u_2-\eta-il)K_{23}^{(l)}(u_2)^{t_2} = K_{23}^{(l)}(u_2)^{t_2}R_{12}^{(l)}(-u_1-u_2-\eta-il)K_{13}^{(l)}(u_1)^{t_1}R_{12}^{(l)}(-u_1+u_2), \quad \eta = i(N-l), \quad (3.48)$$

and therefore the level-$l$ transfer matrix also has the commutativity property.

Acting with the transfer matrix (3.44) on the Bethe state

$$|\Omega^{(l+1)}\rangle = C_{i_1}^{(l+1)}(u_{l+1,1}) \cdots C_{i_{m_l+1}}^{(l+1)}(u_{l+1,m_{l+1}})|0\rangle_X \cdots_{m_{l+1}} \mathcal{F}^{(l+1)i_1 \cdots i_{m_{l+1}}} , \quad (3.49)$$

the “wanted” terms give

$$t^{(l)}(u;\{u_{l,j}\})|\Omega^{(l+1)}\rangle = \Lambda^{(l)}(u;\{u_{l,j}\})|\Omega^{(l+1)}\rangle , \quad (3.50)$$

with

$$\Lambda^{(l)}(u;\{u_{l,j}\}) = f_l(u) \prod_{j=1}^{m_l} a(u-u_{l,j})a(u+u_{l,j}+il) \prod_{j=1}^{m_{l+1}} \frac{a(u_{l+1,j}-u)b(u_{l+1,j}+u-il)}{b(u_{l+1,j}-u)a(u_{l+1,j}+u-il)}$$

$$+ \prod_{j=1}^{m_l} b(u-u_{l,j})b(u+u_{l,j}+il) \prod_{j=1}^{m_{l+1}} \frac{1}{b(u-u_{l+1,j})a(u+u_{l+1,j}+il)} \Lambda^{(l+1)}(u;\{u_{l+1,j}\}) , \quad (3.51)$$

where

$$f_l(u) = f_l^+(u)f_l^-(u) , \quad (3.52)$$

and

$$f_l^-(u) = \frac{b(2u)a_3(u)}{b(2u-il)} = \frac{2u(u^2 + 1)}{2u + il} ,$$

$$f_l^+(u) = b_3(u) + \frac{i}{a(2u+il)}[2b_1(u) + b_2(u) + (N-l-3)b_3(u)]$$

$$= -\frac{(2u + iN)(u^2 + 1)}{2u + il(l+1)} , \quad l = 0, \ldots, N-3 . \quad (3.53)$$
3.4 Final level

We iterate the recursion relation \((3.51)\) until we reach \(l = N - 3\). At that stage we need the eigenvalues of the transfer matrix \(t^{(N-2)}(u; \{u_{N-2,j}\}) = \text{tr}_a^{(N-2)} K^{+(N-2)}(u \ T^{-}(N-2)(u; \{u_{N-2,j}\})),\) where the auxiliary space \(a^{(N-2)}\) has only two dimensions. The \(K\)-matrices are given by

\[
K^{-(N-2)}(u) = \begin{pmatrix}
a_1(u) + a_2(u) & a_1(u) & a_2(u) \\
a_2(u) & a_1(u) & a_2(u) \\
a_1(u) & a_2(u) & a_1(u)
\end{pmatrix} - \frac{i(N - 2)a_3(u)}{2u + i(N - 2)} I,
\]

\[
K^{+(N-2)}(u) = \begin{pmatrix}
b_1(u) + b_2(u) & b_1(u) & b_2(u) \\
b_2(u) & b_1(u) & b_2(u) \\
b_1(u) & b_2(u) & b_1(u)
\end{pmatrix},
\]

where matrix elements which are zero are left empty. They obey the shifted BYBEs \((3.47), (3.48)\), respectively, with \(l = N - 2\).

A priori, one would expect to encounter serious difficulty in diagonalizing this transfer matrix, since both \(K\)-matrices (in particular, the left one) are not diagonal. Remarkably, this is not the case. Indeed, we note the identity

\[
t^{(N-2)}(u; \{u_{N-2,j}\}) = -\frac{2u}{2u + i(N - 2)} \text{tr}_a S^{(N-2)}_{aX1 \cdots m_{N-2}Y}(u; \{u_{N-2,j}\}) \hat{S}^{(N-2)}_{aX1 \cdots m_{N-2}Y}(u; \{u_{N-2,j}\}),
\]

where

\[
S^{(N-2)}_{aX1 \cdots m_{N-2}Y}(u; \{u_{N-2,j}\}) = R^{(N-2)}_{aX}(u + i(N - 2)) R^{(N-2)}_{a1}(u + u_{N-2,1} + i(N - 2)) \cdots \\
\times R^{(N-2)}_{am_{N-2}}(u + u_{N-2,m_{N-2}} + i(N - 2)) R^{(N-2)}_{aY}(u + i(N - 2)),
\]

\[
\hat{S}^{(N-2)}_{aX1 \cdots m_{N-2}Y}(u; \{u_{N-2,j}\}) = R^{(N-2)}_{aY}(u) R^{(N-2)}_{am_{N-2}}(u - u_{N-2,m_{N-2}}) \cdots \\
\times R^{(N-2)}_{a1}(u - u_{N-2,1}) R^{(N-2)}_{aX}(u),
\]

and \(a \equiv a^{(N-2)}\) is the two-dimensional auxiliary space. That is, the transfer matrix \(t^{(N-2)}(u; \{u_{N-2,j}\})\) is the same as the transfer matrix of an open inhomogeneous spin-1/2 \(GL(2)\)-invariant chain of length \(2 + m_{N-2}\) with trivial \(K\)-matrices (i.e., equal to the identity matrix)

\[\text{\textsuperscript{[3]}}\] A proof for

\[\text{\textsuperscript{6}}\] A similar observation (although without proof and only for the case \(M = 2\)) has been made for related models in \([19, 20, 21]\).
general values of \( M \) is given in Appendix A. The corresponding eigenvalues can therefore be determined by standard methods such as [1], and we obtain

\[
\Lambda^{(N-2)}(u; \{u_{N-2,j}\}) = f_{N-2}(u) \prod_{j=1}^{m_{N-2}} (u - u_{N-2,j} + i)(u + u_{N-2,j} + i(N - 1))
\]

\[
\times \prod_{j=1}^{m_{N-1}} \left( \frac{u - u_{N-1,j} - i}(u + u_{N-1,j} + i(N - 2)) \right)
\]

\[
+ f_{N-1}(u) \prod_{j=1}^{m_{N-2}} (u - u_{N-2,j})(u + u_{N-2,j} + i(N - 2))
\]

\[
\times \prod_{j=1}^{m_{N-1}} \left( \frac{u - u_{N-1,j} + i}(u + u_{N-1,j} + iN) \right),
\]

where

\[
f_{N-2}(u) = \frac{-2u(u + i)^2(u + i(N - 1))^2(2u + iN)}{(2u + i(N - 2))(2u + i(N - 1))},
\]

\[
f_{N-1}(u) = \frac{-2u^3(u + i(N - 2))^2}{2u + i(N - 1)}. \tag{3.58}
\]

Combining the above results, we conclude that the eigenvalues of the original transfer matrix \( (2.12) \) with \( M = 2 \) and \( N \geq 3 \) are given by

\[
\Lambda(u) = f_0(u)(u + i)^{2L} \frac{Q_1(u - \frac{i}{2})}{Q_1(u + \frac{i}{2})} + u^{2L} \left\{ \sum_{l=1}^{N-2} f_l(u) \frac{Q_l(u + \frac{i}{2}(l + 2))}{Q_l(u + \frac{i}{2}l)} \frac{Q_{l+1}(u + \frac{i}{2}(l + 1))}{Q_{l+1}(u + \frac{i}{2}(l + 1))} \right\},
\]

\[
+ f_{N-1}(u) \frac{Q_{N-1}(u + \frac{i}{2}(N - 1))}{Q_{N-1}(u + \frac{i}{2}(N - 1))}, \tag{3.59}
\]

where

\[
Q_l(u) = \prod_{j=1}^{m_l} (u - u_{l,j})(u + u_{l,j}), \quad l = 1, \ldots, N - 1,
\]

and we have made the shifts \( u_{l,j} \rightarrow u_{l,j} - \frac{i}{2}l \). We recall that the functions \( f_l(u) \) are given by \( (3.52), (3.53), (3.58) \).

We have thus far ignored all the contributions from “unwanted” terms in the commutation relations. Such contributions vanish provided the parameters \( \{u_{l,j}\} \) satisfy the Bethe ansatz
equations
\[
e_1(u_{1,k})^{2L} = \Theta_1(u_{1,k}) \prod_{j=1 \atop j \neq k}^{m_1} e_2(u_{1,k} - u_{1,j}) e_2(u_{1,k} + u_{1,j}) \\
\times \prod_{j=1}^{m_2} e_{-1}(u_{1,k} - u_{2,j}) e_{-1}(u_{1,k} + u_{2,j}), \quad k = 1, \ldots, m_1,
\]

\[
1 = \Theta_l(u_{l,k}) \prod_{j=1 \atop j \neq k}^{m_l} e_2(u_{l,k} - u_{l,j}) e_2(u_{l,k} + u_{l,j}) \prod_{j=1}^{m_l-1} e_{-1}(u_{l,k} - u_{l-1,j}) e_{-1}(u_{l,k} + u_{l-1,j}) \\
\times \prod_{j=1}^{m_{l+1}} e_{-1}(u_{l,k} - u_{l+1,j}) e_{-1}(u_{l,k} + u_{l+1,j}), \quad k = 1, \ldots, m_l, \quad l = 2, \ldots, N - 2,
\]

\[
1 = \Theta_{N-1}(u_{N-1,k}) \prod_{j=1 \atop j \neq k}^{m_{N-1}} e_2(u_{N-1,k} - u_{N-1,j}) e_2(u_{N-1,k} + u_{N-1,j}) \\
\times \prod_{j=1}^{m_{N-2}} e_{-1}(u_{N-1,k} - u_{N-2,j}) e_{-1}(u_{N-1,k} + u_{N-2,j}), \quad k = 1, \ldots, m_{N-1},
\]

where

\[
\Theta_l(u) = \begin{cases} 
  e_N(u)^2 & l = N - 2 \\
  \frac{e_{N-3}(u)^2}{e_{N-1}(u)^2} & l = N - 1 \\
  1 & \text{otherwise}
\end{cases}
\]

and we have used the standard notation

\[
e_n(u) = \frac{u + in/2}{u - in/2}.
\]

Finally, from the relation (2.16) between the transfer matrix and the Hamiltonian, we find that the energy eigenvalues are given by

\[
E = c_1 \frac{d}{du} \Lambda(u) \bigg|_{u=0} + c_2 = 2 + \sum_{k=1}^{m_1} \frac{1}{u_{1,k}^2 + 1/4}.
\]

3.5 The case \(N = 3, M = 2\)

For the case \((N, M) = (3, 2)\), the above results do not coincide with those in our previous work [24]. Indeed, there we found that the eigenvalues are given by the same expression
but with different functions \( f_i(u) \), namely,

\[
\begin{align*}
\mathcal{f}_0^{\text{previous}}(u) &= -\frac{(2u + 3i)(u + i)^4}{2u + i} = \left( \frac{u + i}{u - i} \right)^2 \mathcal{f}_0(u), \\
\mathcal{f}_1^{\text{previous}}(u) &= -\frac{u^3(2u + 3i)(u + i)}{2u + i} = \left( \frac{u}{u + 2i} \right)^2 \mathcal{f}_1(u), \\
\mathcal{f}_2^{\text{previous}}(u) &= -u^3(u + i) = \mathcal{f}_2(u). 
\end{align*}
\]

(3.65)

(See Eqs. (2.33) and (2.36) in \[24\].) Equivalently, the two sets of results can instead be related by

\[
\begin{align*}
Q_1(u) &= g(u) \mathcal{Q}_1^{\text{previous}}(u), \quad g(u) = (u + \frac{i}{2})^2(u - \frac{i}{2})^{-2}, \\
Q_2(u) &= \mathcal{Q}_2^{\text{previous}}(u), 
\end{align*}
\]

(3.66)

since

\[
\begin{align*}
\frac{g(u - \frac{i}{2})}{g(u + \frac{i}{2})} = \frac{\mathcal{f}_0^{\text{previous}}(u)}{\mathcal{f}_0(u)}, \\
\frac{g(u + \frac{3i}{2})}{g(u + \frac{i}{2})} = \frac{\mathcal{f}_1^{\text{previous}}(u)}{\mathcal{f}_1(u)}. 
\end{align*}
\]

(3.67)

The discrepancy in the two sets of results arises from different choices of pseudovacua. In \[24\] we chose the pseudovacuum to be a ground state \((E = 0)\), while here we have taken the pseudovacuum to be an excited state \((E = 2)\). (Notice the additive constant in the expression (3.64) for the energy.)

We have performed a numerical analysis of completeness of the new solution for small values of \( L \) along the lines discussed in Appendix B of \[24\]. The results for the case \( L = 3 \), for which case there are \( M^2 N^L = 108 \) states, are displayed in Table 1. Although we find some levels for which \( m_2 > m_1 \) (which we did not find with our previous solution), this solution also appears to be complete, at least for small values of \( L \). Note that the Bethe roots for the ground \((E = 0)\) state have a rather complicated structure. Comparing this table with Table 2 in Ref. \[24\], we see little apparent relation between the two sets of Bethe roots describing a given energy level.

It would be interesting to re-derive our previous solution \[24\] (obtained by analytic Bethe ansatz, which is a heuristic approach) by the more rigorous nested ABA approach considered here. Unfortunately, we have so far not succeeded. Indeed, if we try to use the all “up” state as the pseudovacuum, then the creation and annihilation operators seem to be \( A_{12}^{(1)}(u), B_1^{(1)}(u) \) and \( A_{21}^{(1)}(u), C_1^{(1)}(u) \), respectively; hence the transfer matrix seems to involve creation and annihilation operators.
3.6 The cases $N > 3$, $M = 2$

For $N > 3$, $M = 2$, the solution also seems to be complete. For example, we display in Table 2 our results for $(N, M) = (4, 2)$ and $L = 2$, for which case there are $M^2 N^L = 64$ states.

4 Nested ABA for general values of $M$

For $M \geq 2$, the left $K$-matrix has the form (cf. Eq. (3.1))

$$K_{aX}^+(u) = \begin{pmatrix}
\alpha_{11}(u) & \cdots & \alpha_{1M}(u) & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\alpha_{M1}(u) & \cdots & \alpha_{MM}(u) & 0 & \cdots & 0 \\
0 & \cdots & 0 & \beta(u) & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \beta(u)
\end{pmatrix}.$$  (4.1)

We can therefore iterate the recursion relation (3.51) until we reach $l = N - M - 1$. At that stage we need the eigenvalues of the transfer matrix $t^{(N-M)}(u; \{u_{N-M,j}\})$, for which the auxiliary space $a^{(N-M)}$ has dimension $M$. The corresponding $K$-matrices are given by

$$K^{- (N-M)}(u) = \left[ a_1(u) - \frac{i(N-M)a_3(u)}{2u + i(N-M)} \right] \mathbb{I} + a_2(u) \mathcal{P},$$

$$K^{+ (N-M)}(u) = b_1(u) \mathbb{I} + b_2(u) \mathcal{P},$$  (4.2)

where $\mathbb{I}$ and $\mathcal{P}$ are the identity and permutation matrices on $C^M \otimes C^M$, respectively. They obey the shifted BYBEs (3.47), (3.48), respectively, with $l = N - M$.

Since these $K$-matrices (in particular, the left one) are not diagonal, it is not evident how to diagonalize the transfer matrix. Fortunately, there is an identity generalizing (3.55), (3.56), namely

$$t^{(N-M)}(u; \{u_{N-M,j}\}) = -\frac{2u}{2u + i(N-M)} \text{tr}_a S_{aX1\cdots m_{N-M}Y}^{(N-M)}(u; \{u_{N-M,j}\}) \tilde{S}_{aX1\cdots m_{N-M}Y}^{(N-M)}(u; \{u_{N-M,j}\}),$$  (4.3)

where

$$S_{aX1\cdots m_{N-M}Y}^{(N-M)}(u; \{u_{N-M,j}\}) = R_{aX}^{(N-M)}(u + i(N-M)) R_{a1}^{(N-M)}(u + u_{N-M,1} + i(N-M)) \cdots$$

$$\times R_{am_{N-M}}^{(N-M)}(u + u_{N-M,m_{N-M}} + i(N-M)) R_{aY}^{(N-M)}(u + i(N-M)),$$

$$\tilde{S}_{aX1\cdots m_{N-M}Y}^{(N-M)}(u; \{u_{N-M,j}\}) = R_{aY}^{(N-M)}(u) R_{am_{N-M}}^{(N-M)}(u - u_{N-M,m_{N-M}}) \cdots$$

$$\times R_{a1}^{(N-M)}(u - u_{N-M,1}) R_{aX}^{(N-M)}(u),$$  (4.4)
and \( a \equiv a^{(N-M)} \) is the \( M \)-dimensional auxiliary space. That is, the transfer matrix is the same as that of an open inhomogeneous \( GL(M) \)-invariant chain of length \( 2 + m_{N-M} \) with trivial \( K \)-matrices and spins in the vector \((M\text{-dimensional})\) representation. See Appendix A for a proof. The corresponding eigenvalues can be found by the “ordinary” nested ABA

\[
\Lambda^{(N-M)}(u; \{u_{N-M,j}\}) = f_{N-M}(u) \prod_{j=1}^{m_{N-M}} (u - u_{N-M,j} + i)(u + u_{N-M,j} + i(N - M + 1)) \]

\[
\times \prod_{j=1}^{m_{N-M}+1} \frac{(u - u_{N-M+1,j} - i)(u + u_{N-M+1,j} + i(N - M))}{(u - u_{N-M+1,j})(u + u_{N-M+1,j} + i(N - M + 1))} \]

\[+ \prod_{j=1}^{m_{N-M}} (u - u_{N-M,j})(u + u_{N-M,j} + i(N - M))\]

\[
\times \left[ \sum_{l=N-M+1}^{N-2} f_l(u) \prod_{j=1}^{m_l} \frac{(u - u_{l,j} + i)(u + u_{l,j} + i(l + 1))}{(u - u_{l,j})(u + u_{l,j} + il)} \prod_{j=1}^{m_{l+1}} \frac{(u - u_{l+1,j} - i)(u + u_{l+1,j} + il)}{(u - u_{l+1,j})(u + u_{l+1,j} + i(l + 1))} \right],
\]

where

\[
f_l(u) = \begin{cases} 
-\frac{2u(u+i)^2(u+i(N-M+1)^2)(2u+iN)}{(2u+i(N-M))(2u+i(N-M+1))} & l = N - M \\
-\frac{2u^3(u+i(N-M))^2(2u+iN)}{(2u+iN)(2u+i(N+1))} & l = N - M + 1, \ldots, N - 1
\end{cases}
\]

Combining this result with those from the recursion relation (3.51), we conclude that the eigenvalues of the original transfer matrix (2.12) are given by

\[
\Lambda(u) = f_0(u)(u + i)^{2l} \frac{Q_1(u - \frac{i}{2})}{Q_1(u + \frac{i}{2})} + u^{2l} \left\{ \sum_{l=1}^{N-2} f_l(u) \frac{Q_l(u + \frac{i}{2}(l + 2))}{Q_l(u + \frac{i}{2}l)} \frac{Q_{l+1}(u + \frac{i}{2}(l + 1))}{Q_{l+1}(u + \frac{i}{2}l)} \right\} + f_{N-1}(u) \frac{Q_{N-1}(u + \frac{i}{2}(N + 1))}{Q_{N-1}(u + \frac{i}{2}(N - 1))},
\]

where

\[
f_l(u) = \begin{cases} 
-\frac{2u(u^2+1)^2(2u+iN)}{(2u+iN)(2u+i(N+l+1))} & l = 0, \ldots, N - M - 1 \\
-\frac{2u(u+i)^2(u+i(N-M+1))^2(2u+iN)}{(2u+i(N-M))(2u+i(N-M+1))} & l = N - M \\
-\frac{2u^3(u+i(N-M))^2(2u+iN)}{(2u+iN)(2u+i(N+1))} & l = N - M + 1, \ldots, N - 1
\end{cases}
\]
\[ Q_l(u) = \prod_{j=1}^{m_l} (u - u_{l,j})(u + u_{l,j}), \quad l = 1, \ldots, N - 1, \quad (4.9) \]

and (as before) we have made the shifts \( u_{l,j} \mapsto u_{l,j} - \frac{i}{2} l \).

The corresponding Bethe ansatz equations are given by

\[
e_1(u_{1,k})^{2L} = \Theta_1(u_{1,k}) \prod_{j=1, j \neq k}^{m_1} e_2(u_{1,k} - u_{1,j}) e_2(u_{1,k} + u_{1,j})
\times \prod_{j=1}^{m_2} e_{-1}(u_{1,k} - u_{2,j}) e_{-1}(u_{1,k} + u_{2,j}), \quad k = 1, \ldots, m_1,
\]

\[ 1 = \Theta_l(u_{l,k}) \prod_{j=1, j \neq k}^{m_l} e_2(u_{l,k} - u_{l,j}) e_2(u_{l,k} + u_{l,j}) \prod_{j=1}^{m_{l+1}} e_{-1}(u_{l,k} - u_{l+1,j}) e_{-1}(u_{l,k} + u_{l+1,j}), \quad k = 1, \ldots, m_l, \quad l = 2, \ldots, N - 2, \]

\[ 1 = \Theta_{N-1}(u_{N-1,k}) \prod_{j=1, j \neq k}^{m_{N-1}} e_2(u_{N-1,k} - u_{N-1,j}) e_2(u_{N-1,k} + u_{N-1,j})
\times \prod_{j=1}^{m_{N-2}} e_{-1}(u_{N-2,j}) e_{-1}(u_{N-2,j}), \quad k = 1, \ldots, m_{N-1}, \quad (4.10) \]

where now

\[
\Theta_l(u) = \begin{cases} 
  e_{N-M+2}(u)^2 & l = N - M \\
  \frac{e_{N-M+1}(u)^2}{e_{N-M-1}(u)^2} & l = N - M + 1 \\
  1 & \text{otherwise}
\end{cases}
\quad (4.11)
\]

and \( e_n(u) \) is defined in (3.63). The energy eigenvalues are given by the same formula (3.64).

The identity (4.3), the expression (4.7)-(4.9) for the eigenvalues of the transfer matrix and the corresponding Bethe ansatz equations (4.10), (4.11) are the main results of this paper.

For \( M > 2 \), this solution also seems to be complete, as is the case for \( M = 2 \) discussed in Sec. 3.6. For example, we display in Table 3 our results for \((N, M) = (4, 3)\) and \(L = 2\), for which case there are \( M^2 N^L = 144 \) states.
5 Conclusions

We have considered the $GL(N)/(GL(M) \times GL(N - M))$ model with Hamiltonian \[1.2\], which is a generalization of a model arising in string/gauge theory. We have proved the integrability of this model by constructing the corresponding commuting transfer matrix. The latter makes use of the non-diagonal operator-valued $K$-matrices found in \[18\].

We have found a Bethe ansatz solution of this model for general values of $N$ and $M$ using the nested ABA approach, despite the fact that the $K$-matrices are not diagonal. The main results are the eigenvalues \((4.7)-(4.9)\) and Bethe ansatz equations \((4.10), (4.11)\). The key to obtaining this solution is the identity \((4.3)\), which relies on the factorization property \((A.1)\) of the “reduced” (level $N - M$) $K$-matrices into products of $R$-matrices. In hindsight, this property is not too surprising, since the projected $K$-matrices originate from “dressed” diagonal $K$-matrices \[18\]. For the case $(N, M) = (3, 2)$, this solution is not the same as the one found in \[24\] using analytic Bethe ansatz, as the two solutions are based on different pseudovacua. Nevertheless, numerical evidence suggests that both $N = 3$ solutions are complete. Moreover, the nested ABA solution appears to be complete for general values of $N$ and $M$.

Many interesting questions remain unanswered. It is unusual for an integrable model with a non-graded symmetry algebra to have more than one Bethe ansatz solution. (Models with graded symmetry algebras are known to have more than one Bethe ansatz solution, corresponding to the non-uniqueness of the associated Dynkin diagrams. See e.g. \[37\] and references therein.) This underscores the question of whether the two proposed solutions for the case $(N, M) = (3, 2)$ (namely, the one found in \[24\] by analytic Bethe ansatz, and the one found here by nested ABA) are equivalent. As noted in Sec. 3.5, one would like to have a more rigorous derivation of the solution found in \[24\]. Similarly, for general values of $N$ and $M$, there may be additional equivalent solutions based on different pseudovacua. Perhaps Bethe ansatz equations for generic open spin chains (or at least for open chains constructed with projected $K$-matrices) can be formulated in terms of group theory data (namely, the “bulk” symmetry algebra and the unbroken “boundary” symmetry subalgebra); and the multiplicity of Bethe ansatz solutions reflects the various ways of choosing the boundary symmetry subalgebra. We hope to be able to address these and related questions in the future.
Acknowledgments

I am grateful to A. Lima-Santos, J. Links and M. Martins for helpful correspondence. This work was supported in part by the National Science Foundation under Grants PHY-0554821 and PHY-0854366.

A Proof of the transfer-matrix identity

Our proof of the transfer-matrix identity (3.55), (4.3) is based on the following remarkable factorization property of the “reduced” $K$-matrices (i.e., the $(N, M)$ projected $K$-matrices at level $N - M$) into products of $R$-matrices,

\[
K_{aY}^{(N-M)}(u) = \frac{-2u}{2u + i(N - M)} R_{aY}^{(N-M)}(u + i(N - M)) R_{aY}^{(N-M)}(u),
\]

\[
K_{aX}^{(N-M)}(u) = \text{tr}_b \mathcal{P}_{ab} R_{aX}^{(N-M)}(u + i(N - M)) R_{aX}^{(N-M)}(u),
\]

which can be verified from the expressions (3.54), (4.2). Omitting the quantum-space indices and denoting the $M$-dimensional auxiliary space by $a$ in order to streamline the notation, we have

\[
t^{(N-M)}(u) = \text{tr}_a K_{a}^{+(N-M)}(u) T_{a}^{-(N-M)}(u)
\]

\[
= \text{tr}_a \mathcal{P}_{ab} R_{a}^{(N-M)}(u + i(N - M)) R_{b}^{(N-M)}(u) T_{a}^{-(N-M)}(u)
\]

\[
= \text{tr}_a \mathcal{P}_{ab} R_{a}^{(N-M)}(u + i(N - M)) T_{a}^{-(N-M)}(u) R_{b}^{(N-M)}(u)
\]

\[
= \text{tr}_a R_{a}^{(N-M)}(u + i(N - M)) T_{a}^{(N-M)}(u) K_{a}^{-(N-M)}(u) T_{a}^{-(N-M)}(u) R_{a}^{(N-M)}(u)
\]

\[
= -\frac{2u}{2u + i(N - M)} \text{tr}_a R_{a}^{(N-M)}(u + i(N - M)) T_{a}^{(N-M)}(u)
\]

\[
\times R_{a}^{(N-M)}(u + i(N - M)) R_{a}^{(N-M)}(u) T_{a}^{(N-M)}(u) R_{a}^{(N-M)}(u)
\]

\[
= -\frac{2u}{2u + i(N - M)} \text{tr}_a S_{a}^{(N-M)}(u) \hat{S}_{a}^{(N-M)}(u),
\]

(A.2)

where in the last line we have used the fact (see Eqs. (4.1), (3.36))

\[
S_{a}^{(N-M)}(u) = R_{a}^{(N-M)}(u + i(N - M)) T_{a}^{(N-M)}(u) R_{a}^{(N-M)}(u + i(N - M)),
\]

\[
\hat{S}_{a}^{(N-M)}(u) = R_{a}^{(N-M)}(u) T_{a}^{(N-M)}(u) R_{a}^{(N-M)}(u).
\]

(A.3)
| $E$ | $s$    | $\left\{ u_{1,k} \right\}$                                      | $\left\{ u_{2,k} \right\}$ |
|-----|--------|---------------------------------------------------------------|-----------------------------|
| 0   | 5/2    | $1.11803i , 0.442686 \pm 1.0936i$                             | --                          |
| 0.381966 | 3/2    | $1.2944i , 0.375279 \pm 1.36374i$                            | 0                           |
| 0.585786 | 2      | $0.204205 \pm 1.22426i$                                      | --                          |
| 0.82259 | 1/2, 1 | $0.15313 \pm 1.36461i$                                       | 0                           |
| 1.07919 | 0      | $1.36676i , 1.88488i$                                         | $0 , 1.56857i$              |
| 1.26795 | 3/2    | --                                                            | $1.27123i$                  |
| 1.38197 | 3/2    | $0.936268 , 0.180565 \pm 1.20371i$                           | 0                           |
| 1.38197 | 1/2    | $1.36676i$                                                   | 0                           |
| 1.58579 | 1/2, 1 | $1.91214 , 1.31987i$                                         | 0                           |
| 1.69722 | 1/2    | $1.88488i$                                                   | 0                           |
| 2    | 2      | $0.866025 , 1.11803i$                                         | --                          |
| 2    | 1, 3/2 | --                                                            | --                          |
| 2    | 0, 1   | --                                                            | 0                           |
| 2.58579 | 1      | $0.639467 , 1.15027i$                                         | $1.0505$                    |
| 2.61803 | 3/2    | $0.322878 \pm 0.50042i , 1.04607i$                            | 0                           |
| 3    | 0      | $0.606658 , 1.36676i$                                         | $0 , 0.707107i$             |
| 3    | 1/2    | $0.866025$                                                   | 1                           |
| 3.31526 | 0      | $0.606658 , 1.88488i$                                         | $1.15861i$                  |
| 3.32164 | 1/2, 1 | $0.451092 , 1.17552i$                                         | 0                           |
| 3.41421 | 2      | $0.479032 \pm 0.521886i$                                     | --                          |
| 3.61803 | 1/2    | $0.606658$                                                   | 0                           |
| 3.61803 | 3/2    | $0.331608 , 0.404442 \pm 0.90768i$                            | 0                           |
| 4    | 1      | $\pm 0.5i$                                                   | 1                           |
| 4.41421 | 1/2, 1 | $0.077447i , 0.959277i$                                       | 0                           |
| 4.68474 | 0      | $0.229729 , 1.36676i$                                         | $0 , 0.810943i$             |
| 4.73205 | 3/2    | $0.340625$                                                   | --                          |
| 5    | 1/2    | $0.288675$                                                   | $0.745356$                  |
| 5    | 0      | $0.229729 , 1.88488i$                                         | $0 , 1.22474i$              |
| 5.30278 | 1/2    | $0.229729$                                                   | 0                           |
| 5.41421 | 1      | $0.301797 , 1.35023$                                         | $0.62964$                   |
| 5.85577 | 1/2, 1 | $0.248411 , 1.13757$                                         | 0                           |
| 6.92081 | 0      | $0.229729 , 0.606658$                                        | $0 , 0.678531$              |

Table 1: Energy, spin, and Bethe roots for $N = 3 , M = 2 , L = 3$. 
Table 2: Energy, degeneracy, and Bethe roots for $N = 4, M = 2, L = 2$.

| $E$       | $deg$ | $\{u_{1,k}\}$                  | $\{u_{2,k}\}$                  | $\{u_{3,k}\}$                  |
|-----------|-------|---------------------------------|---------------------------------|---------------------------------|
| 0         | 5     | $0.238862 \pm 0.986773i$        | $0.240994 \pm 1.54642i$         | –                               |
| 0.585786  | 3     | $0.204205 \pm 1.22426i$         | $\pm 1.84776i$                 | $1.39897i$                      |
| 1         | 8     | $1.11803i$                      | $1.63299i$                      | –                               |
| 1.26795   | 5     | $1.27123i$                      | $1.79779i$                      | $1.01915i$                      |
| 2         | 9     | –                               | –                               | –                               |
| 2         | 7     | –                               | –                               | $0.866025i$                     |
| 2         | 3     | $0.866025, 1.11803i$            | $0.8556, 1.65289i$              | $0.866025i$                     |
| 3         | 8     | $0.866025$                      | 0                               | –                               |
| 3.41421   | 3     | $0.479032 \pm 0.521886i$        | $\pm 0.765367i$                | $0.736813i$                     |
| 4         | 7     | $0.5$                           | –                               | –                               |
| 4         | 1     | $0.5$                           | –                               | $0.866025i$                     |
| 4.73205   | 5     | $0.340625$                      | $0.481717$                      | $0.679209i$                     |

Table 3: Energy, degeneracy, and Bethe roots for $N = 4, M = 3, L = 2$.

| $E$       | $deg$ | $\{u_{1,k}\}$                  | $\{u_{2,k}\}$                  | $\{u_{3,k}\}$                  |
|-----------|-------|---------------------------------|---------------------------------|---------------------------------|
| 0         | 15    | $0.238862 \pm 0.986773i$        | –                               | –                               |
| 0.585786  | 15    | $0.204205 \pm 1.22426i$         | $0$                             | –                               |
| 1         | 10    | $1.11803i$                      | –                               | –                               |
| 1.26795   | 14    | $1.27123i$                      | $0$                             | –                               |
| 2         | 6     | –                               | –                               | –                               |
| 2         | 11    | –                               | $0$                             | –                               |
| 2         | 15    | $0.866025, 1.11803i$            | $0$                             | –                               |
| 2.58579   | 3     | $0.639467, 1.15027i$            | $0, 1.0505$                    | $0.89542$                       |
| 3         | 10    | $0.866025$                      | –                               | –                               |
| 3         | 1     | $0.866025$                      | $0, 1$                         | $0.866025$                      |
| 3.41421   | 15    | $0.479032 \pm 0.521886i$        | $0$                             | –                               |
| 4         | 8     | $0.5$                           | $0.816497$                      | –                               |
| 4         | 3     | $\pm 0.5i$                      | $0, 1$                         | $0.866025$                      |
| 4.73205   | 14    | $0.340625$                      | $0$                             | –                               |
| 5         | 1     | $0.288675$                      | $0, 0.745356$                  | $0.726483$                      |
| 5.41421   | 3     | $0.301797, 1.35023$             | $0, 0.62964$                   | $0.669495$                      |
References

[1] E.K. Sklyanin, “Boundary conditions for integrable quantum systems,” *J. Phys.* A21, 2375 (1988).

[2] I.V. Cherednik, “Factorizing particles on a half line and root systems,” *Theor. Math. Phys.* 61, 977 (1984).

[3] S. Ghoshal and A.B. Zamolodchikov, “Boundary S-Matrix and Boundary State in Two-Dimensional Integrable Quantum Field Theory,” *Int. J. Mod. Phys.* A9, 3841 (1994) [arXiv:hep-th/9306002].

[4] J. Cao, H.-Q. Lin, K.-J. Shi and Y. Wang, “Exact solutions and elementary excitations in the XXZ spin chain with unparallel boundary fields,” [cond-mat/0212163]; J. Cao, H.-Q. Lin, K.-J. Shi and Y. Wang, “Exact solution of XXZ spin chain with unparallel boundary fields,” *Nucl. Phys.* B663, 487 (2003).

[5] R.I. Nepomechie, “Functional relations and Bethe ansatz for the XXZ chain,” *J. Stat. Phys.* 111, 1363 (2003) [hep-th/0211001]; R.I. Nepomechie, “Bethe ansatz solution of the open XXZ chain with nondiagonal boundary terms,” *J. Phys.* A37, 433 (2004) [hep-th/0304092].

[6] R.I. Nepomechie and F. Ravanini, “Completeness of the Bethe ansatz solution of the open XXZ chain with nondiagonal boundary terms,” *J. Phys.* A36, 11391 (2003); Addendum, *J. Phys.* A37, 1945 (2004) [hep-th/0307095].

[7] A. Doikou, “Fused integrable lattice models with quantum impurities and open boundaries,” *Nucl. Phys.* B668, 447 (2003) [arXiv:hep-th/0303205]; A. Doikou, “A note on the boundary spin s XXZ chain,” *Phys. Lett.* A366, 556 (2007) [hep-th/0612268].

[8] J. de Gier and P. Pyatov, “Bethe Ansatz for the Temperley-Lieb loop model with open boundaries,” *J. Stat. Mech.* 0403, P002 (2004) [hep-th/0312235]; A. Nichols, V. Rittenberg and J. de Gier, “One-boundary Temperley-Lieb algebras in the XXZ and loop models,” *J. Stat. Mech.* 0505, P003 (2005) [cond-mat/0411512]; J. de Gier, A. Nichols, P. Pyatov and V. Rittenberg, “Magic in the spectra of the XXZ quantum chain with boundaries at $\Delta = 0$ and $\Delta = -1/2$,” *Nucl. Phys.* B729, 387 (2005) [hep-th/0505062];
J. de Gier and F.H.L. Essler, “Bethe Ansatz Solution of the Asymmetric Exclusion Process with Open Boundaries,” *Phys. Rev. Lett.* **95**, 240601 (2005) [cond-mat/0508707]; J. de Gier and F.H.L. Essler, “Exact spectral gaps of the asymmetric exclusion process with open boundaries,” *J. Stat. Mech.* **0612**, P011 (2006) [cond-mat/0609645].

[9] D. Arnaudon, J. Avan, N. Crampé, A. Doikou, L. Frappat and E. Ragoucy, “General boundary conditions for the $sl(N)$ and $sl(M|N)$ open spin chains,” *J. Stat. Mech.* **0408**, P005 (2004) [math-ph/0406021].

[10] W. Galleas and M. J. Martins, “Solution of the $SU(N)$ vertex model with non-diagonal open boundaries,” *Phys. Lett.* **A335**, 167 (2005) [arXiv:nlin/0407027].

[11] C.S. Melo, G.A.P. Ribeiro and M.J. Martins, “Bethe ansatz for the XXX-S chain with non-diagonal open boundaries,” *Nucl. Phys. B711*, 565 (2005) [arXiv:nlin/0411038].

[12] W.-L. Yang, Y.-Z. Zhang and M. Gould, “Exact solution of the XXZ Gaudin model with generic open boundaries,” *Nucl. Phys. B698*, 503 (2004) [hep-th/0411048]; W.-L. Yang and Y.-Z. Zhang, “Exact solution of the $A_{n-1}^{(1)}$ trigonometric vertex model with non-diagonal open boundaries,” *JHEP* **01**, 021 (2005) [hep-th/0411190]; W.-L. Yang, Y.-Z. Zhang and R. Sasaki, “$A_{n-1}$ Gaudin model with open boundaries,” *Nucl. Phys. B729*, 594 (2005) [hep-th/0507148].

[13] R. Murgan and R.I. Nepomechie, “Bethe Ansatz derived from the functional relations of the open XXZ chain for new special cases,” *J. Stat. Mech.* **0505**, P007 (2005); Addendum, *J. Stat. Mech.* **0511**, P004 (2005) [hep-th/0504124]; R. Murgan and R.I. Nepomechie, “Generalized T-Q relations and the open XXZ chain,” *J. Stat. Mech.* **0508**, P002 (2005) [hep-th/0507139]; R. Murgan, “Bethe ansatz of the open spin-s XXZ chain with nondiagonal boundary terms,” *JHEP* **0904**, 076 (2009) [arXiv:0901.3558].

[14] P. Baseilhac and K. Koizumi, “A deformed analogue of Onsager’s symmetry in the XXZ open spin chain,” *J. Stat. Mech.* **0510**, P005 (2005) [hep-th/0507053]; P. Baseilhac, “The q-deformed analogue of the Onsager algebra: beyond the Bethe ansatz approach,” *Nucl. Phys. B754*, 309 (2006) [math-ph/0604036]; P. Baseilhac and K. Koizumi, “Exact spectrum of the XXZ open string chain from the q-Onsager algebra representation theory,” *J. Stat. Mech.* **0709**, P006 (2007) [hep-th/0703106].

[15] W.-L. Yang, R.I. Nepomechie and Y.-Z. Zhang, “$Q$-operator and T-Q relation from the fusion hierarchy,” *Phys. Lett.* **B633**, 664 (2006) [hep-th/0511134];
W.-L. Yang and Y.-Z. Zhang, “T-Q relation and exact solution for the XYZ chain with general nondiagonal boundary terms,” *Nucl. Phys. B* **744**, 312 (2006) [hep-th/0512154]; W.-L. Yang and Y.-Z. Zhang, “On the second reference state and complete eigenstates of the open XXZ chain,” *JHEP* **04**, 044 (2007) [hep-th/0703222].

[16] Z. Bajnok, “Equivalences between spin models induced by defects,” *J. Stat. Mech.* **0606**, P010 (2006) [hep-th/0601107].

[17] L. Frappat, R. Nepomechie and E. Ragoucy, “Complete Bethe Ansatz solution of the open spin-s XXZ chain with general integrable boundary terms,” *J. Stat. Mech.* **0709**, P009 (2007) [arXiv:0707.0653].

[18] H. Frahm and N.A. Slavnov, “New solutions to the reflection equation and the projecting method,” *J. Phys.* **A32**, 1547 (1999) [arXiv:cond-mat/9810312].

[19] H.-Q. Zhou and M.D. Gould, “Algebraic Bethe ansatz for integrable Kondo impurities in the one-dimensional supersymmetric t-J model,” *Phys. Lett. A251*, 279 (1999) [arXiv:cond-mat/9809055].

[20] H.-Q. Zhou, X.-Y. Ge, J. Links and M.D. Gould, “Graded reflection equation algebras and integrable Kondo impurities in the one-dimensional t-J model,” *Nucl. Phys. B546*, 779 (1999) [arXiv:cond-mat/9809056].

[21] H.-Q. Zhou, X.-Y. Ge, J. Links and M.D. Gould, “Integrable Kondo impurities in one-dimensional extended Hubbard models,” *Phys. Rev. B62*, 4906 (2000) [arXiv:cond-mat/9908036].

[22] D. Berenstein and S. E. Vázquez, “Integrable open spin chains from giant gravitons,” *JHEP* **0506**, 059 (2005) [arXiv:hep-th/0501078].

[23] D.M. Hofman and J.M. Maldacena, “Reflecting magnons,” *JHEP* **0711**, 063 (2007) [arXiv:0708.2272].

[24] R. I. Nepomechie, “Bethe ansatz equations for open spin chains from giant gravitons,” *JHEP* **0905**, 100 (2009) [arXiv:0903.1646].

[25] O. DeWolfe and N. Mann, “Integrable open spin chains in defect conformal field theory,” *JHEP* **0404**, 035 (2004) [arXiv:hep-th/0401041]; T. Erler and N. Mann, “Integrable open spin chains and the doubling trick in $\mathcal{N} = 2$ SYM with fundamental matter,” *JHEP* **0601**, 131 (2006) [arXiv:hep-th/0508064].

[26] A. Agarwal, “Open spin chains in super Yang-Mills at higher loops: Some potential problems with integrability,” *JHEP* **0608**, 027 (2006) [arXiv:hep-th/0603067].
[27] K. Okamura and K. Yoshida, “Higher loop Bethe ansatz for open spin-chains in AdS/CFT,” *JHEP* **0609**, 081 (2006) [arXiv:hep-th/0604100].

[28] W. Galleas, “The Bethe Ansatz Equations for Reflecting Magnons,” *Nucl. Phys.* **B820**, 664 (2009) [arXiv:0902.1681].

[29] J. L. Jacobsen and H. Saleur, “Conformal boundary loop models,” *Nucl. Phys.* **B788**, 137 (2008) [arXiv:math-ph/0611078];
J. L. Jacobsen and H. Saleur, “Combinatorial aspects of boundary loop models,” *J. Stat. Mech.* **0801**, P01021 (2008) [arXiv:0709.0812];
J. Dubail, J. L. Jacobsen and H. Saleur, “Conformal two-boundary loop model on the annulus,” *Nucl. Phys.* **B813**, 430 (2009) [arXiv:0812.2746];
J. Dubail, J. L. Jacobsen and H. Saleur, “Conformal boundary conditions in the critical $O(n)$ model and dilute loop models,” *Nucl. Phys.* **B827**, 457 (2010) [arXiv:0905.1382].

[30] J. de Gier and A. Nichols, “The two-boundary Temperley-Lieb algebra,” *J. Algebra* **321**, 1132 (2009) [arXiv:math/0703338].

[31] I. Kostov, “Boundary loop models and 2D quantum gravity,” *J. Stat. Mech.* **0708**, P08023 (2007) [arXiv:hep-th/0703221];
J. E. Bourgine and K. Hosomichi, “Boundary operators in the $O(n)$ and RSOS matrix models,” *JHEP* **0901**, 009 (2009) [arXiv:0811.3252];
J. E. Bourgine, “Boundary changing operators in the $O(n)$ matrix model,” *JHEP* **0909**, 020 (2009) [arXiv:0904.2297];
J. E. Bourgine, K. Hosomichi and I. Kostov, “Boundary transitions of the $O(n)$ model on a dynamical lattice,” [arXiv:0910.1581].

[32] P.P. Kulish and N.Yu. Reshetikhin, “Generalized Heisenberg ferromagnet and the Gross-Neveu model,” *Sov. Phys. JETP* **53**, 108 (1981);
P.P. Kulish and N.Y. Reshetikhin, “Diagonalization of $GL(N)$ invariant transfer matrices and quantum $N$-wave system (Lee model),” *J. Phys.* **A16**, L591 (1983).

[33] O. Babelon, H. J. de Vega and C. M. Viallet, “Exact solution of the $Z_{n+1} \times Z_{n+1}$ symmetric generalization of the XXZ model,” *Nucl. Phys.* **B200**, 266 (1982).

[34] A. Foerster and M. Karowski, “The supersymmetric t - J model with quantum group invariance,” *Nucl. Phys.* **B 408**, 512 (1993).

[35] H. J. de Vega and A. González-Ruiz, “Exact solution of the $SU_q(n)$ invariant quantum spin chains,” *Nucl. Phys.* **B 417**, 553 (1994) [arXiv:hep-th/9309022];
H. J. de Vega and A. González-Ruiz, “Exact Bethe ansatz solution for $A_{n-1}$ chains with
non-$SU_q(n)$ invariant open boundary conditions,” *Mod. Phys. Lett.* A 9, 2207 (1994) [arXiv:hep-th/9404141].

[36] R.-H. Yue, H. Fan and B.-Y. Hou, “Exact diagonalization of the quantum supersymmetric $SU_q(n|m)$ model,” *Nucl. Phys.* B462, 167 (1996);
M.T. Batchelor, X.-W. Guan, A. Foerster, A.P. Tonel and H.-Q. Zhou, “Thermodynamic properties of an integrable quantum spin ladder with boundary impurities,” *Nucl. Phys.* B669, 385 (2003) [arXiv:cond-mat/0305196];
G. L. Li and K. J. Shi, “The algebraic Bethe ansatz for open vertex models,” *J. Stat. Mech.* 0701, 018 (2007) [arXiv:hep-th/0611127].

[37] F. Woynarovich, “Low-energy excited states in a Hubbard chain with on-site attraction,” *J. Phys.* C16, 6593 (1983);
P.-A. Bares, J.M.P. Carmelo, J. Ferrer and P. Horsch, “Charge-spin recombination in the one-dimensional supersymmetric t-J model,” *Phys. Rev.* B46, 14624 (1992);
F.H.L. Essler and V.E. Korepin, “Higher conservation laws and algebraic Bethe Ansätze for the supersymmetric t-J model,” *Phys. Rev.* B46, 9147 (1992) [hep-th/9207007];
F. Göhmann and A. Seel, “A note on the Bethe ansatz solution of the supersymmetric t-J model,” *Czech. J. Phys.* 53, 1041 (2003) [arXiv:cond-mat/0309138];
N. Beisert, V. A. Kazakov, K. Sakai and K. Zarembo, “Complete Spectrum of Long Operators in $\mathcal{N}=4$ SYM at One Loop,” *JHEP* 0507 030 (2005) [arXiv:hep-th/0503200].