EXACTNESS OF THE FOCK SPACE REPRESENTATION
OF THE $q$-COMMUTATION RELATIONS

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Abstract. We show that for all $q$ in the interval $(-1, 1)$, the Fock representation of the $q$-commutation relations can be unitarily embedded into the Fock representation of the extended Cuntz algebra. In particular, this implies that the C*-algebra generated by the Fock representation of the $q$-commutation relations is exact. An immediate consequence is that the $q$-Gaussian von Neumann algebra is weakly exact for all $q$ in the interval $(-1, 1)$.

1. Introduction

The $q$-commutation relations provide a $q$-analogue of the bosonic ($q = 1$) and the fermionic ($q = -1$) commutation relations from quantum mechanics. These relations have a natural representation on a deformed Fock space which was introduced by Bozejko and Speicher in [1], and was subsequently studied by a number of authors (see e.g. [2], [3], [4], [5], [6], [7], [8], [9], [10]).

For the entirety of this paper, we fix an integer $d \geq 2$. Consider the usual full Fock space $\mathcal{F}$ over $\mathbb{C}^d$,

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n$$

(orthogonal direct sum),

where $\mathcal{F}_0 = \mathbb{C}\Omega$ and $\mathcal{F}_n = (\mathbb{C}^d)^{\otimes n}$ for $n \geq 1$.

Corresponding to the vectors in the standard orthonormal basis of $\mathbb{C}^d$, one has left creation operators $L_1, \ldots, L_d \in B(\mathcal{F})$. Define the C*-algebra $\mathcal{C}$ by

$$\mathcal{C} := C^*(L_1, \ldots, L_d) \subseteq B(\mathcal{F}).$$

It is well known that $\mathcal{C}$ is isomorphic to the extended Cuntz algebra. (Although it is customary to denote the extended Cuntz algebra by $\mathcal{E}$, we use $\mathcal{C}$ here to emphasize that we are working with a concrete C*-algebra of operators.)

Now let $q \in (-1, 1)$ be a deformation parameter. We consider the $q$-deformation $\mathcal{F}^{(q)}$ of $\mathcal{F}$ as defined in [1]. Thus

$$\mathcal{F}^{(q)} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n^{(q)}$$

(orthogonal direct sum),

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where every $F^{(q)}_n$ is obtained by placing a certain deformed inner product on $(\mathbb{C}^d)^\otimes n$. (The precise definition will be reviewed in Subsection 2.1 below.) For $q = 0$, one obtains the usual non-deformed Fock space $F$ from above.

In this deformed setting, one also has natural left creation operators $L^{(q)}_1, ..., L^{(q)}_d \in B(F^{(q)})$, which satisfy the $q$-commutation relations

$$L^{(q)}_i(L^{(q)}_j)^* = \delta_{ij}I + q(L^{(q)}_i)^*L^{(q)}_j, \quad 1 \leq i, j \leq d.$$  

Define the $C^\ast$-algebra $C^{(q)}$ by

$$C^{(q)} := C^\ast(L^{(q)}_1, ..., L^{(q)}_d) \subseteq B(F^{(q)}).$$

For $q = 0$, this construction yields the extended Cuntz algebra $C$ from above.

It is widely believed that the algebra $C$ and the deformed algebra $C^{(q)}$ are actually unitarily equivalent. In fact, this is known for sufficiently small $q$. In [5], a unitary $U : F^{(q)} \to F$ was constructed which embeds $C$ into $C^{(q)}$ for all $q \in (-1, 1)$, i.e. $C \subseteq UC^{(q)}U^\ast$, and it was shown that for $|q| < 0.44$ this embedding is actually surjective, i.e. $C = UC^{(q)}U^\ast$.

The main purpose of the present paper is to show that it is possible to unitarily embed $C^{(q)}$ into $C$ for all $q \in (-1, 1)$. Specifically, we construct a unitary operator $U_{opp} : F^{(q)} \to F$ such that $U_{opp}C^{(q)}U_{opp}^\ast \subseteq C$. The unitary $U_{opp}$ is closely related to the unitary $U$ from [5], as we will now see.

**Definition 1.1.** Let $J : F \to F$ be the unitary conjugation operator which reverses the order of the components in a tensor in $(\mathbb{C}^d)^\otimes n$, i.e.

$$J(\eta_1 \otimes \cdots \otimes \eta_n) = \eta_n \otimes \cdots \otimes \eta_1, \quad \forall \eta_1, \ldots, \eta_n \in \mathbb{C}^d.$$  

Note that for $n = 0$, Equation (1.5) says that $J(\Omega) = \Omega$.

Let $J^{(q)} : F^{(q)} \to F^{(q)}$ be the operator which acts as in Equation (1.5), where the tensor is now viewed as an element of the space $F^{(q)}_n$. It is known that $J^{(q)}$ is also unitary operator (see the review in Subsection 2.1).

**Definition 1.2.** Let $q \in (-1, 1)$ be a deformation parameter and let $U : F^{(q)} \to F$ be the unitary defined in [5]. Define a new unitary $U_{opp} : F^{(q)} \to F$ by

$$U_{opp} = JUJ^{(q)}.$$  

The following theorem is the main result of this paper.

**Theorem 1.3.** For every $q \in (-1, 1)$ the unitary $U_{opp}$ from Definition 1.2 satisfies

$$U_{opp}C^{(q)}U_{opp}^\ast \subseteq C.$$  

The following corollary follows immediately from Theorem 1.3.

**Corollary 1.4.** For every $q \in (-1, 1)$ the $C^\ast$-algebra $C^{(q)}$ is exact.

To prove Theorem 1.3 we first consider the more general question of how to verify that an operator $T \in B(F)$ belongs to the algebra $C$. It is well
known that a necessary condition for \( T \) to be in \( \mathcal{C} \) is that it commutes modulo the compact operators with the C*-algebra generated by right creation operators on \( \mathcal{F} \). Unfortunately, this condition isn’t sufficient (and wouldn’t be sufficient even if we were to set \( d \) equal to 1, cf. \cite{4}). Nonetheless, by restricting our attention to a \(*\)-subalgebra of “band-limited operators” on \( \mathcal{F} \) and considering commutators modulo a suitable ideal of compact operators in this algebra, we do obtain a sufficient condition for \( T \) to belong to \( \mathcal{C} \). This bicommutant-type result is strong enough to help in the proof of Theorem 1.3.

In addition to this introduction, the paper has four other sections. In Section 2, we provide a brief review of the requisite background material. In Section 3, we prove the above-mentioned bicommutant-type result, Theorem 3.8. In Section 4, we establish the main results, Theorem 1.3 and Corollary 1.4. In Section 5, we apply these results to the family of \( q \)-Gaussian von Neumann algebras, showing in Theorem 5.1 that these algebras are weakly exact for every \( q \in (-1, 1) \).

2. Review of background

2.1. Basic facts about the \( q \)-deformed Fock space. As explained in the introduction, there is a fairly large body of research devoted to the \( q \)-deformed Fock framework and its generalizations. Here we provide only a brief review of the terminology and facts which will be needed in Section 4.

2.1.1. The \( q \)-deformed inner product. As mentioned above, the integer \( d \geq 2 \) will remain fixed throughout this paper. Also fixed throughout this paper will be an orthonormal basis \( \xi_1, \ldots, \xi_d \) for \( \mathbb{C}^d \). For every \( n \geq 1 \) this gives us a preferred basis for \( (\mathbb{C}^d)^\otimes n \), namely

\[
\{ \xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \mid 1 \leq i_1, \ldots, i_n \leq d \}.
\]

This basis is orthonormal with respect to the usual inner product on \( (\mathbb{C}^d)^\otimes n \) (obtained by tensoring \( n \) copies of the standard inner product on \( \mathbb{C}^d \)). As in the introduction, we will use \( \mathcal{F}_n \) to denote the Hilbert space \( (\mathbb{C}^d)^\otimes n \) endowed with this inner product. The full Fock space over \( \mathbb{C}^d \) is then the Hilbert space \( \mathcal{F} \) from Equation (1.1), with the convention that \( \mathcal{F}_0 = \mathbb{C} \Omega \) for a distinguished unit vector \( \Omega \), referred to as the “vacuum vector”.

Now let \( q \in (-1, 1) \) be a deformation parameter. It was shown in \cite{1} that there exists a positive definite inner product \( \langle \cdot, \cdot \rangle_q \) on \( (\mathbb{C}^d)^\otimes n \), uniquely determined by the requirement that for vectors in the natural basis \( (2.1) \), one has the formula

\[
\langle \xi_{i_1} \otimes \cdots \otimes \xi_{i_n}, \xi_{j_1} \otimes \cdots \otimes \xi_{j_n} \rangle_q = \sum_{\sigma} q^{\text{inv}(\sigma)} \delta_{i_1, \sigma(j_1)} \cdots \delta_{i_n, \sigma(j_n)}.
\]

The sum on the right-hand side of Equation (2.2) is taken over all permutations \( \sigma \) of \( \{1, \ldots, n\} \), and \( \text{inv}(\sigma) \) denotes the number of inversions of \( \sigma \), i.e.

\[
\text{inv}(\sigma) := |\{(i, j) \mid 1 \leq i < j \leq n, \ \sigma(i) > \sigma(j)\}|.
\]
Note that under this new inner product, the natural basis (2.1) will typically no longer be orthogonal.

We will use $\mathcal{F}_n^{(q)}$ to denote the Hilbert space $(\mathbb{C}^d)^{\otimes n}$ endowed with this deformed inner product. In addition, we will use the convention that $\mathcal{F}_0^{(q)}$ is the same as $\mathcal{F}_0$, i.e. it is spanned by the same vacuum vector $\Omega$. The $q$-deformed Fock space over $\mathbb{C}^d$ is then the Hilbert space $\mathcal{F}_n^{(q)}$ from Equation (1.3). For $q = 0$, the construction of $\mathcal{F}_n^{(q)}$ yields the usual non-deformed Fock space $\mathcal{F}$ from Equation (1.1).

2.1.2. The deformed creation and annihilation operators. For every $1 \leq j \leq d$, one has deformed left creation operators $L_j^{(q)} \in \mathcal{B}(\mathcal{F}_n^{(q)})$ and deformed right creation operators $R_j^{(q)} \in \mathcal{B}(\mathcal{F}_n^{(q)})$, which act on the natural basis of $\mathcal{F}_n^{(q)}$ by $L_j^{(q)}(\Omega) = R_j^{(q)}(\Omega) = \xi_j$ and

\[
L_j^{(q)}(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \xi_j \otimes \xi_{i_1} \otimes \cdots \otimes \xi_{i_n}, \\
R_j^{(q)}(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \otimes \xi_j.
\]

Their adjoints are the deformed left annihilation operators $(L_j^{(q)})^*$ and the deformed right annihilation operators $(R_j^{(q)})^*$, which act on the natural basis of $\mathcal{F}_n^{(q)}$ by

\[
(L_j^{(q)})^*(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \sum_{m=1}^n q^{m-1}\delta_{j,m} \xi_{i_1} \otimes \cdots \otimes \hat{\xi}_{i_m} \otimes \cdots \otimes \xi_{i_n}, \\
(R_j^{(q)})^*(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \sum_{m=1}^n q^{n-m}\delta_{i,m} \hat{\xi}_{i_1} \otimes \cdots \otimes \xi_{i_{m-1}} \otimes \cdots \otimes \xi_{i_n},
\]

where the “hat” symbol over the component $\xi_{i_m}$ means that it is deleted from the tensor (e.g. $\xi_{i_1} \otimes \hat{\xi}_{i_2} \otimes \xi_{i_3} = \xi_{i_1} \otimes \xi_{i_3}$).

It’s clear from these formulas that the left creation (left annihilation) operators commute with the right creation (right annihilation) operators. For the commutator of a left annihilation operator and a right creation operator, a direct calculation (see also Lemma 3.1 from [10]) gives the formula

\[
[(L_i^{(q)})^*, R_j^{(q)}] |_{\mathcal{F}_n^{(q)}} = \delta_{ij}q^n I_{\mathcal{F}_n^{(q)}}, \quad \forall n \geq 1.
\]

Taking adjoints gives the formula for the commutator of a left creation operator and a right annihilation operator.

When we are working on the non-deformed Fock space $\mathcal{F}$ corresponding to the case when $q = 0$, it will be convenient to suppress the superscripts and write $L_j$ and $R_j$ for the left and right creation operators respectively.
Note that in this case, Equation (2.3) and Equation (2.4) imply that
\[ \sum_{j=1}^{d} L_j L_j^* = \sum_{j=1}^{d} R_j R_j^* = 1 - P_0, \]
where \( P_0 \) is the orthogonal projection onto \( F_0 \).

2.1.3. The unitary conjugation operator. For every \( n \geq 1 \), let \( J^{(q)}_n : F^{(q)}_n \to F^{(q)}_n \) be the operator which reverses the order of the components in a tensor in \((\mathbb{C}^d)^{\otimes n}\), i.e., \( J^{(q)}_n \) acts by the formula in Equation (1.5) of the Introduction. A consequence of Equation (2.2), which defines the inner product \( \langle \cdot, \cdot \rangle^{(q)} \), is that \( J^{(q)}_n \) is a unitary operator in \( B(F^{(q)}_n) \). Indeed, this is easily seen to follow from Equation (2.2) and the following basic fact about inversions of permutations: if \( \theta \) denotes the special permutation which reverses the order on \( \{1, \ldots, n\} \), then one has \( \text{inv}(\theta \tau \theta) = \text{inv}(\tau) \) for every permutation \( \tau \) of \( \{1, \ldots, n\} \).

Therefore, we can speak of the unitary operator \( J^{(q)} \in B(F^{(q)}) \) from Definition 1.2, which is obtained as \( J^{(q)} := \bigoplus_{n=0}^{\infty} J^{(q)}_n \). Note that \( J^{(q)} \) is an involution, i.e. \( (J^{(q)})^2 = I_{F^{(q)}} \), and that it intertwines the left and right creation operators, i.e.
\[ R^{(q)}_j = J^{(q)} L^{(q)}_j J^{(q)}, \quad 1 \leq j \leq d. \]

2.2. The original unitary operator. In this subsection, we review the construction of the unitary \( U : F^{(q)} \to F \) from [5], which appears in Definition 1.2. An important role in the construction of this unitary is played by the positive operator
\[ M^{(q)} := \sum_{j=1}^{d} L^{(q)}_j (L^{(q)}_j)^* \in B(F^{(q)}). \]
Clearly \( M^{(q)} \) can be written as a direct sum \( M^{(q)} = \bigoplus_{n=0}^{\infty} M^{(q)}_n \), where \( M^{(q)}_n \) is a positive operator on \( F^{(q)}_n \), for every \( n \geq 0 \). Using Equation (2.3) and Equation (2.4), one can show that \( M^{(q)}_n \) acts on the natural basis of \( F^{(q)}_n \) by
\[ M^{(q)}_n (\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \sum_{m=1}^{n} q^{m-1} \xi_{i_m} \otimes \xi_{i_1} \otimes \cdots \otimes \hat{\xi}_{i_m} \otimes \cdots \otimes \xi_{i_n}. \]
(Recall that the “hat” symbol over the component \( \xi_{i_m} \) means that it is deleted from the tensor.)

With the exception of \( M^{(q)}_0 \) (which is zero), the operators \( M^{(q)}_n \) are invertible. This is implied by Lemma 4.1 of [5], which also gives the estimate
\[ \|(M^{(q)}_n)^{-1}\| \leq (1 - |q|) \prod_{k=1}^{\infty} \frac{1 + |q|^k}{1 - |q|^k} < \infty, \quad \forall n \geq 1. \]
An important thing to note about Equation (2.9) is that the upper bound on the right-hand side is independent of $n$.

The unitary operator $U$ is defined as a direct sum, $U := \oplus_{n=0}^{\infty} U_n$, where the unitaries $U_n : \mathcal{F}_n^{(q)} \to \mathcal{F}_n$ are defined recursively as follows: we first define $U_0$ by $U_0(\Omega) = \Omega$, and for every $n \geq 1$ we define $U_n$ by

\begin{equation}
U_n := (I \otimes U_{n-1})(M_n^{(q)})^{1/2}.
\end{equation}

In Proposition 3.2 of [5] it was shown that $U_n$ as defined in Equation (2.10) is actually a unitary operator, and hence that $U$ is a unitary operator. Moreover, in Section 4 of [5] it was shown that $C \subseteq U \mathcal{C}^{(q)} U^*$ for every $q \in (-1,1)$.

### 2.3. Summable band-limited operators

Throughout this section, we fix a Hilbert space $\mathcal{H}$, and in addition we fix an orthogonal direct sum decomposition of $\mathcal{H}$ as

\begin{equation}
\mathcal{H} = \oplus_{n=0}^{\infty} \mathcal{H}_n.
\end{equation}

We will study certain properties an operator $T \in B(\mathcal{H})$ can have with respect to this decomposition of $\mathcal{H}$. We would like to emphasize that the concepts considered here depend not only on $\mathcal{H}$, but also on the orthogonal decomposition for $\mathcal{H}$ in Equation (2.11).

**Definition 2.1.** Let $T$ be an operator in $B(\mathcal{H})$. If there exists a non-negative integer $b$ such that

\begin{equation}
T(\mathcal{H}_n) \subseteq \bigoplus_{m \geq 0} \mathcal{H}_m, \quad \forall n \geq 0,
\end{equation}

then we will say that $T$ is band-limited. A number $b$ as in Equation (2.12) will be called a band limit for $T$. The set of all band-limited operators in $B(\mathcal{H})$ will be denoted by $\mathcal{B}$.

**Definition 2.2.** Let $T$ be an operator in $\mathcal{B}$. We will say that $T$ is summable when it has the property that

$$
\sum_{n=0}^{\infty} \|T \mid \mathcal{H}_n\| < \infty,
$$

where we have used $T \mid \mathcal{H}_n \in B(\mathcal{H}_n, \mathcal{H})$ to denote the restriction of $T$ to $\mathcal{H}_n$. The set of all summable band-limited operators in $B(\mathcal{H})$ will be denoted by $\mathcal{S}$.

**Proposition 2.3.** With respect to the preceding definitions,

1. $\mathcal{B}$ is a unital $*$-subalgebra of $B(\mathcal{H})$ and
2. $\mathcal{S}$ is a two-sided ideal of $\mathcal{B}$ which is closed under taking adjoints.

**Proof.** The proof of (1) is left as an easy exercise for the reader. To verify (2), we first show that $\mathcal{S}$ is closed under taking adjoints. Suppose $T \in \mathcal{S}$, and let $b$ be a band limit for $T$. By examining the matrix representations of
T and of $T^*$ with respect to the orthogonal decomposition (2.11), it is easily verified that
\[
\|T^* |_{\mathcal{H}_n}\| \leq \sum_{m \geq 0} \|T |_{\mathcal{H}_m}\|, \quad \forall n \geq 0.
\]
This implies that
\[
\sum_{n=0}^{\infty} \|T^* |_{\mathcal{H}_n}\| \leq (2b + 1) \sum_{m=0}^{\infty} \|T |_{\mathcal{H}_m}\| < \infty,
\]
which gives $T^* \in \mathcal{S}$. Next, we show that $\mathcal{S}$ is a two-sided ideal of $\mathcal{B}$. Since $\mathcal{S}$ was proved to be self-adjoint, it will suffice to show that it is a left ideal. It is clear that $\mathcal{S}$ is closed under linear combinations. The fact that $\mathcal{S}$ is a left ideal now follows from the simple observation that for $T \in \mathcal{B}$ and $S \in \mathcal{S}$ we have
\[
\sum_{n=0}^{\infty} \|TS |_{\mathcal{H}_n}\| \leq \|T\| \sum_{n=0}^{\infty} \|S |_{\mathcal{H}_n}\| < \infty,
\]
which implies $TS \in \mathcal{S}$. \hfill \QED

In the following definition, we identify some special types of band-limited operators.

**Definition 2.4.** Let $T$ be an operator in $\mathcal{B}$.

1. If $T$ satisfies $T(\mathcal{H}_n) \subseteq \mathcal{H}_n$ for all $n \geq 0$, then we will say that $T$ is block-diagonal.
2. If there is $k \geq 0$ such that $T$ satisfies $T(\mathcal{H}_n) \subseteq \mathcal{H}_{n+k}$ for $n \geq 0$, then we will say that $T$ is $k$-raising.
3. If there is $k \geq 0$ such that $T$ satisfies $T(\mathcal{H}_n) \subseteq \mathcal{H}_{n-k}$ for $n \geq k$ and $T(\mathcal{H}_n) = \{0\}$ for $n < k$, then we will say that $T$ is $k$-lowering.

Note that a block-diagonal operator is both 0-raising and 0-lowering.

The following proposition gives a Fourier-type decomposition for band-limited operators.

**Proposition 2.5.** Let $T$ be an operator in $\mathcal{B}$ with a band-limit $b \geq 0$, as in Definition 2.1. Then we can decompose $T$ as
\[
T = \sum_{k=0}^{b} X_k + \sum_{k=1}^{b} Y_k,
\]
where each $X_k$ is a $k$-raising operator for $0 \leq k \leq b$, and each $Y_k$ is a $k$-lowering operator for $1 \leq k \leq b$. This decomposition is unique. Moreover, if $T$ is summable in the sense of Definition 2.2, then each of the $X_k$ and $Y_k$ are summable.

**Proof.** First, fix an integer $k$ satisfying $0 \leq k \leq b$. For each $n \geq 0$, consider the linear operator $P_{n+k}T |_{\mathcal{H}_n} \in B(\mathcal{H}_n, \mathcal{H}_{n+k})$ which results from composing the orthogonal projection $P_{n+k}$ onto $\mathcal{H}_{n+k}$ with the restriction...
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Clearly $\| P_{n+k} T \|_{\mathcal{H}_n} \leq \| T \|$. This allows us to define an operator $X_k \in B(\mathcal{H})$ which acts on $\mathcal{H}_n$ by

$$X_k \xi = P_{n+k} T \xi, \quad \forall \xi \in \mathcal{H}_n.$$  

(2.14)

It follows from this definition that $X_k$ is a $k$-raising operator.

Similarly, for an integer $k$ satisfying $1 \leq k \leq b$, we can define a $k$-lowering operator $Y_k \in B(\mathcal{H})$ which acts on $\xi \in \mathcal{H}_n$ by

$$Y_k \xi = \begin{cases} P_{n-k} T \xi & \text{if } k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$  

(2.15)

It’s clear that Equation (2.13) holds with each $X_k$ and $Y_k$ defined as above. Conversely, if Equation (2.13) holds, then it’s clear that each $X_k$ and $Y_k$ is completely determined as in Equation (2.14) and Equation (2.15) respectively. This implies the uniqueness of this decomposition.

Finally, suppose $T$ is summable. The fact that each $X_k$ and $Y_k$ is summable then follows from the observation that Equation (2.14) and Equation (2.15) imply $\| X_k \|_{\mathcal{H}_n} = \| P_{n+1} T \|_{\mathcal{H}_n} - \| P_n T \|_{\mathcal{H}_n}$ for every $n \geq 0$.

The following result about commutators will be needed in Section 4.

**Proposition 2.6.** Let $T \in B$ be a positive block-diagonal operator, and let $V \in B$ be a 1-raising operator. Suppose that the commutator $[T, V]$ satisfies

$$\sum_{n=0}^{\infty} \| ([T, V] |_{\mathcal{H}_n})^{1/2} < \infty.$$  

Then the commutator $[T^{1/2}, V]$ is a summable 1-raising operator.

**Proof.** For every $n \geq 0$, let $T_n = T |_{\mathcal{H}_n} \in B(\mathcal{H}_n)$ and let $V_n = V |_{\mathcal{H}_n} \in B(\mathcal{H}_n, \mathcal{H}_{n+1})$. Since $T$ is block-diagonal and $V$ is 1-raising, it’s clear that $[T, V]$ and $[T^{1/2}, V]$ are 1-raising operators which satisfy

$$[T, V] |_{\mathcal{H}_n} = T_{n+1} V_n - V_n T_n, \quad \forall n \geq 0,$$

and

$$[T^{1/2}, V] |_{\mathcal{H}_n} = T_{n+1}^{1/2} V_n - V_n T_n^{1/2}, \quad \forall n \geq 0.$$  

It follows that the hypothesis (2.16) can be rewritten as

$$\sum_{n=0}^{\infty} \| T_{n+1} V_n - V_n T_n \|^{1/2} < \infty,$$

while the required conclusion that $[T^{1/2}, V] \in \mathcal{S}$ is equivalent to

$$\sum_{n=0}^{\infty} \| T_{n+1}^{1/2} V_n - V_n T_n^{1/2} \| < \infty.$$  

We will prove that this holds by showing that for every $n \geq 0$,

$$\| T_{n+1}^{1/2} V_n - V_n T_n^{1/2} \| \leq \frac{1}{4} \| V \|^{1/2} \| T_{n+1} V_n - V_n T_n \|^{1/2}.$$  

(2.17)
For the rest of the proof, fix \( n \geq 0 \). Consider the operators \( A, B \in B(H_n \oplus H_{n+1}) \) which, written as \( 2 \times 2 \) matrices, are given by
\[
A := \begin{bmatrix} T_n & 0 \\ 0 & T_{n+1} \end{bmatrix}, \quad B := \begin{bmatrix} 0 & V_n^* \\ V_n & 0 \end{bmatrix}.
\]
Since \( T \) is positive, it follows that \( A \) is positive, with
\[
A^{1/2} = \begin{bmatrix} T^{1/2}_n & 0 \\ 0 & T^{1/2}_{n+1} \end{bmatrix}.
\]
A well-known commutator inequality (see e.g. [8]) gives
\[
\| [A^{1/2}, B] \| \leq \frac{5}{4} \| B \|^{1/2} \| [A, B] \|^{1/2}.
\]
From the definitions of \( A \) and \( B \), we compute
\[
[A, B] = \begin{bmatrix} 0 & (T_{n+1}V_n - V_nT_n)^* \\ T_{n+1}V_n - V_nT_n & 0 \end{bmatrix},
\]
and this implies \( \| [A, B] \| = \| T_{n+1}V_n - V_nT_n \| \). Similarly, \( \| [A^{1/2}, B] \| = \| T^{1/2}_{n+1}V_n - V_nT^{1/2}_n \| \), and it’s clear that \( \| B \| = \| V_n \| \). By substituting these equalities into (2.18) we obtain
\[
\| T^{1/2}_{n+1}V_n - V_nT^{1/2}_n \| \leq \frac{5}{4} \| V_n \|^{1/2} \| T_{n+1}V_n - V_nT_n \|^{1/2}.
\]
Since \( \| V_n \| \leq \| V \| \), this clearly implies that (2.17) holds. \( \Box \)

3. An inclusion criterion

In this section, we work exclusively in the framework of the (non-deformed) extended Cuntz algebra \( \mathcal{C} \). We will use the terminology of Subsection 2.3 with respect to the natural decomposition \( \mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n \). In particular, we will refer to the unital \( * \)-subalgebra \( B \subseteq B(\mathcal{F}) \) which consists of bandlimited operators as in Definition 2.1 and to the ideal \( \mathcal{S} \) of \( B \) which consists of summable band-limited operators as in Definition 2.2.

The main result of this section is Theorem 3.8. This is an analogue in the \( \mathcal{C}^* \)-framework of the bicommutant theorem from von Neumann algebra theory, where we restrict our attention to the \( * \)-algebra \( B \) and consider commutators modulo the ideal \( \mathcal{S} \). In this framework, the role of “commutant” is played by the \( \mathcal{C}^* \)-algebra generated by right creation operators on \( \mathcal{F} \).

For clarity, we will first consider the special case of a block-diagonal operator.

**Definition 3.1.** Let \( T \in B \) be a block-diagonal operator. The sequence of \( \mathcal{C} \)-approximants for \( T \) is the sequence \( (A_n)_{n=0}^{\infty} \) of block-diagonal elements of \( \mathcal{C} \) defined recursively as follows: we first define \( A_0 \) by \( A_0 = (T(\Omega), \Omega)I_{\mathcal{F}} \),
and for every $n \geq 0$ we define $A_{n+1}$ by
\begin{equation}
A_{n+1} := A_n + \sum_{1 \leq i_1, \ldots, i_{n+1} \leq d} c_{i_1, \ldots, i_{n+1}, j_1, \ldots, j_{n+1}} (L_{i_1} \cdots L_{i_{n+1}})(L_{j_1} \cdots L_{j_{n+1}})^* ,
\end{equation}
where the coefficients $c_{i_1, \ldots, i_{n+1}, j_1, \ldots, j_{n+1}}$ are defined by
\begin{equation}
c_{i_1, \ldots, i_{n+1}, j_1, \ldots, j_{n+1}} := \langle T(\xi_{j_1} \otimes \cdots \otimes \xi_{j_{n+1}}), \xi_{i_1} \otimes \cdots \otimes \xi_{i_{n+1}} \rangle - \delta_{i_{n+1}, j_{n+1}} \cdot \langle T(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n}), \xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \rangle.
\end{equation}

The main property of the approximant $A_n$ is that it agrees with the operator $T$ on each subspace $F_m$ for $m \leq n$. More precisely, we have the following lemma.

**Lemma 3.2.** Let $T \in B$ be a block-diagonal operator, and let $(A_n)_{n=0}^\infty$ be the sequence of $C$-approximants for $T$, as in Definition 3.1. Then for every $m \geq 0$,
\begin{equation}
A_n \mid_{F_m} = \begin{cases} T \mid_{F_m} & \text{if } m \leq n, \\ (T \mid_{F_n}) \otimes I_{m-n} & \text{if } m > n. \end{cases}
\end{equation}

**Proof.** We will show that for every fixed $n \geq 0$, Equation (3.3) holds for all $m \geq 0$. The proof of this statement will proceed by induction on $n$. The base case $n = 0$ is left as an easy exercise for the reader. The remainder of the proof is devoted to the induction step. Fix $n \geq 0$ and assume that Equation (3.3) holds for this $n$ and for all $m \geq 0$. We will prove the analogous statement for $n + 1$.

From Equation (3.1), it is immediate that
\[ A_{n+1} \mid_{F_m} = A_n \mid_{F_m} = T \mid_{F_m}, \forall m \leq n. \]
Thus it remains to fix $m \geq n + 1$ and verify that
\[ A_{n+1} \mid_{F_m} = (T \mid_{F_{n+1}}) \otimes I_{m-n-1} \in B(F_m). \]

In light of how $(T \mid_{F_{n+1}}) \otimes I_{m-n-1}$ acts on the canonical basis of $F_m$, this amounts to showing that for every $1 \leq k_1, \ldots, k_m, \ell_1, \ldots, \ell_m \leq d$, one has
\begin{equation}
\langle A_{n+1} (\xi_{\ell_1} \otimes \cdots \otimes \xi_{\ell_m}), \xi_{k_1} \otimes \cdots \otimes \xi_{k_m} \rangle = \delta_{k_{n+2}, \ell_{n+2}} \cdots \delta_{k_m, \ell_m} \langle T(\xi_{\ell_1} \otimes \cdots \otimes \xi_{\ell_{n+1}}), \xi_{k_1} \otimes \cdots \otimes \xi_{k_{n+1}} \rangle.
\end{equation}

On the left-hand side of Equation (3.4) we substitute for $A_{n+1}$ using the recursive definition given by Equation (3.1). This gives
\begin{align}
\langle A_{n+1} (\xi_{\ell_1} \otimes \cdots \otimes \xi_{\ell_m}), & \xi_{k_1} \otimes \cdots \otimes \xi_{k_m} \rangle \\
= \langle A_n \xi_{i_1} \otimes \cdots \otimes \xi_{i_m}, & \xi_{k_1} \otimes \cdots \otimes \xi_{k_m} \rangle \\
+ & \sum_{i_1, \ldots, i_{n+1}, j_1, \ldots, j_{n+1}} c_{i_1, \ldots, i_{n+1}, j_1, \ldots, j_{n+1}, \alpha(i_1, \ldots, i_{n+1}, j_1, \ldots, j_{n+1})},
\end{align}
where for every $1 \leq i_1, \ldots, i_{n+1}, j_1, \ldots, j_{n+1} \leq d$, we have written

$$
\alpha(i_1, \ldots, i_{n+1}; j_1, \ldots, j_{n+1}) = \langle (L_{i_1} \cdots L_{i_{n+1}}) (L_{j_1} \cdots L_{j_{n+1}})^* (\xi_{i_1} \otimes \cdots \otimes \xi_{\ell_m}), (\xi_{k_1} \otimes \cdots \otimes \xi_{k_m}) \rangle.
$$

It is clear that an inner product like the one just written simplifies as follows:

$$
= \langle (L_{j_1} \cdots L_{j_{n+1}})^* (\xi_{\ell_{i_1}} \otimes \cdots \otimes \xi_{\ell_m}), (L_{i_1} \cdots L_{i_{n+1}})^* (\xi_{k_1} \otimes \cdots \otimes \xi_{k_m}) \rangle
$$

$$
= \delta_{i_1, k_1} \cdots \delta_{i_{n+1}, k_{n+1}} \delta_{j_1, \ell_1} \cdots \delta_{j_{n+1}, \ell_{n+1}} \langle \xi_{\ell_{i_1+1}} \otimes \cdots \otimes \xi_{\ell_m}, \xi_{k_{n+2}} \otimes \cdots \otimes \xi_{k_m} \rangle
$$

Thus in the sum on the right-hand side of Equation (3.5), the only term that survives is the one corresponding to $i_1 = k_1, \ldots, i_{n+1} = k_{n+1}$ and $j_1 = \ell_1, \ldots, j_{n+1} = \ell_{n+1}$, and we obtain that

$$
(3.6) \quad \langle A_{n+1}(\xi_{\ell_1} \otimes \cdots \otimes \xi_{\ell_m}), \xi_{k_1} \otimes \cdots \otimes \xi_{k_m} \rangle
$$

$$
= \langle A_n(\xi_{\ell_1} \otimes \cdots \otimes \xi_{\ell_m}), \xi_{k_1} \otimes \cdots \otimes \xi_{k_m} \rangle
$$

$$
+ \delta_{\ell_{n+2}, \ell_{n+2}} \cdots \delta_{\ell_m, \ell_m} \langle T(\xi_{\ell_1} \otimes \cdots \otimes \xi_{\ell_m}), \xi_{k_1} \otimes \cdots \otimes \xi_{k_m} \rangle.
$$

Finally, we remember our induction hypothesis, which gives

$$
(3.7) \quad \langle A_n(\xi_{\ell_1} \otimes \cdots \otimes \xi_{\ell_m}), \xi_{k_1} \otimes \cdots \otimes \xi_{k_m} \rangle
$$

$$
= \delta_{\ell_{n+1}, \ell_{n+1}} \cdot \cdots \delta_{\ell_m, \ell_m} \langle T(\xi_{\ell_1} \otimes \cdots \otimes \xi_{\ell_m}), \xi_{k_1} \otimes \cdots \otimes \xi_{k_m} \rangle.
$$

A straightforward calculation shows that if we substitute Equation (3.7) into Equation (3.6) and use Formula (3.2) which defines the coefficient $c_{k_1, \ldots, k_{n+1}; \ell_1, \ldots, \ell_{n+1}}$, then we arrive at the right-hand side of Equation (3.4). This completes the induction argument.

Lemma 3.3. Let $T \in \mathcal{B}$ be a block-diagonal operator, and let $(A_n)_{n=1}^\infty$ be the sequence of $C$-approximants for $T$, as in Definition 3.1. Then for every $n \geq 1$,

$$
(3.8) \quad \| A_{n+1} - A_n \| = \| T | F_{n+1} - (T | F_n) \otimes I \|.
$$

Proof. Note that since $A_{n+1} - A_n$ is block-diagonal,

$$
\| A_{n+1} - A_n \| = \sup_{m \geq 0} \| A_{n+1} | F_m - A_n | F_m \|.
$$

To compute this supremum, there are three cases to consider. In each case we apply Lemma 3.2. First, for $m \leq n$,

$$
\| A_{n+1} | F_m - A_n | F_m \| = 0.
$$

Next, for $m = n + 1$,

$$
\| A_{n+1} | F_{n+1} - A_n | F_{n+1} \| = \| T | F_{n+1} - (T | F_n) \otimes I \|.
$$
Finally, for \( m > n + 1 \),
\[
\| A_{n+1} \mid \mathcal{F}_m - A_n \mid \mathcal{F}_m \| = \| (T \mid \mathcal{F}_{n+1}) \otimes I_{m-n-1} - (T \mid \mathcal{F}_n) \otimes I_{m-n} \|
\]
\[
= \| (T \mid \mathcal{F}_{n+1}) - (T \mid \mathcal{F}_n) \otimes I \| \otimes I_{m-n-1}
\]
\[
= \| T \mid \mathcal{F}_{n+1} - (T \mid \mathcal{F}_n) \otimes I \|.
\]
This makes it clear that the supremum over all \( m \geq 0 \) is equal to the right hand side of Equation (3.8), as required. □

**Lemma 3.4.** Let \( T \) be a block-diagonal operator. If \( T \) satisfies
\[
\sum_{n=1}^{\infty} \| (T \mid \mathcal{F}_{n+1}) - (T \mid \mathcal{F}_n) \otimes I \| < \infty,
\]
then \( T \in \mathcal{C} \).

**Proof.** Let \( (A_n)_{n=1}^{\infty} \) be the sequence of \( \mathcal{C} \)-approximants for \( T \), as in Definition 3.1. In view of Lemma 3.3, the hypothesis of the present lemma implies that the sum \( \sum_{n=1}^{\infty} \| A_{n+1} - A_n \| \) is finite. This in turn implies that the sequence \( (A_n)_{n=1}^{\infty} \) converges in norm to an operator \( A \). Since each \( A_n \) belongs to \( \mathcal{C} \), it follows that \( A \) belongs to \( \mathcal{C} \). But we must have \( A = T \), as Lemma 3.2 implies that
\[
A \mid \mathcal{F}_m = \lim_{n \to \infty} A_n \mid \mathcal{F}_m = T \mid \mathcal{F}_m, \quad \forall m \geq 0.
\]
Hence \( T \in \mathcal{C} \), as required. □

**Proposition 3.5.** Let \( T \) be a block-diagonal operator. If the block-diagonal operator \( T - \sum_{i=1}^{d} R_i T R_i^* \) belongs to the ideal \( \mathcal{S} \), then \( T \in \mathcal{C} \).

**Proof.** The hypothesis is equivalent to
\[
\sum_{n=1}^{\infty} \| (T - \sum_{i=1}^{d} R_i T R_i^*) \mid \mathcal{F}_n \| < \infty.
\]
It’s easy to verify that for \( n \geq 1 \),
\[
(\sum_{i=1}^{d} R_i T R_i^*) \mid \mathcal{F}_n = (T \mid \mathcal{F}_{n-1}) \otimes I,
\]
which gives
\[
\| (T - \sum_{i=1}^{d} R_i T R_i^*) \mid \mathcal{F}_n \| = \| T \mid \mathcal{F}_n - (T \mid \mathcal{F}_{n-1}) \otimes I \|.
\]
Therefore, (3.9) implies that the hypothesis of Lemma 3.4 holds, and the result follows by applying the said lemma. □

**Corollary 3.6.** Let \( T \in \mathcal{B} \) be a block-diagonal operator such that \( [T, R_i^*] \in \mathcal{S} \) for \( 1 \leq i \leq d \). Then \( T \in \mathcal{C} \).

**Proof.** By Proposition 3.5 it suffices to show that \( T - \sum_{i=1}^{d} R_i T R_i^* \in \mathcal{S} \). We can write
\[
T - \sum_{i=1}^{d} R_i T R_i^* = (P_0 + \sum_{i=1}^{d} R_i R_i^*) T - \sum_{i=1}^{d} R_i T R_i^* = P_0 T - \sum_{i=1}^{d} R_i [T, R_i^*],
\]
where \( P_0 \) is the orthogonal projection onto \( \mathcal{F}_0 \), and where we have used Equation (2.6). Since \( P_0 \) and \([T, R^*_j]\) belong to \( \mathcal{S} \), and since \( T \) and \( R_i \) belong to \( \mathcal{B} \), the result follows from the fact that \( \mathcal{S} \) is a two-sided ideal of \( \mathcal{B} \).

We now apply the above results on block-diagonal operators in order to bootstrap the case of general band-limited operators. It is convenient to first consider the case of \( k \)-raising/lowering operators, which were introduced in Definition 2.4.

**Proposition 3.7.** Let \( T \in \mathcal{B} \) be a \( k \)-raising or \( k \)-lowering operator for some \( k \geq 0 \). If \( T \) satisfies \([T, R^*_j]\) \( \in \mathcal{S} \) for \( 1 \leq j \leq d \), then \( T \in \mathcal{S} \).

**Proof.** First, suppose that \( T \) is \( k \)-raising. For every \( 1 \leq i_1, \ldots, i_k \leq d \), the fact that the left and right annihilation operators commute implies that

\[
[(L_{i_1} \cdots L_{i_k})^*T, R^*_j] = (L_{i_1} \cdots L_{i_k})^*[T, R^*_j], \quad \forall 1 \leq j \leq d.
\]

Since \([T, R^*_j]\) \( \in \mathcal{S} \) by hypothesis, and since \( \mathcal{S} \) is a two-sided ideal of \( \mathcal{B} \), it follows that \([(L_{i_1} \cdots L_{i_k})^*T, R^*_j]\) \( \in \mathcal{S} \). The operator \((L_{i_1} \cdots L_{i_k})^*T\) is block-diagonal, hence Corollary 3.6 gives \((L_{i_1} \cdots L_{i_k})^*T \in \mathcal{C}\).

Since \( T \) is \( k \)-raising, the range of \( T \) is orthogonal to the subspace \( \mathcal{F}_\ell \) whenever \( \ell < k \). This implies that

\[
\left( I - \sum_{1 \leq i_1, \ldots, i_k \leq d} L_{i_1} \cdots L_{i_k} (L_{i_1} \cdots L_{i_k})^* \right) T = 0.
\]

Hence

\[
T = \sum_{1 \leq i_1, \ldots, i_k \leq d} L_{i_1} \cdots L_{i_k} ((L_{i_1} \cdots L_{i_k})^*T),
\]

and it follows that \( T \in \mathcal{C} \).

The case when \( T \) is \( k \)-lowering is handled in a similar way by considering the operators \( T L_{i_1} \cdots L_{i_k} \) for every \( 1 \leq i_1, \ldots, i_k \leq d \).

**Theorem 3.8.** Let \( T \in \mathcal{B} \) be an operator such that either \([T, R^*_j]\) \( \in \mathcal{S} \) for all \( 1 \leq j \leq d \), or \([T, R_j]\) \( \in \mathcal{S} \) for all \( 1 \leq j \leq d \). Then \( T \in \mathcal{C} \).

**Proof.** First, suppose that \( T \) satisfies \([T, R^*_j]\) \( \in \mathcal{S} \) for every \( 1 \leq j \leq d \). Let \( b \geq 0 \) be a band-limit for \( T \). By Proposition 2.5 we can decompose \( T \) as

\[
T = \sum_{k=0}^{b} X_k + \sum_{k=1}^{b} Y_k,
\]

where each \( X_k \) is a \( k \)-raising operator, and each \( Y_k \) is a \( k \)-lowering operator. We will prove that each \( X_k \in \mathcal{C} \) and each \( Y_k \in \mathcal{C} \).
Fix for the moment $1 \leq j \leq d$. We have

$$\left[T, R_j^*\right] = \sum_{k=0}^{b} [X_k, R_j^*] + \sum_{k=1}^{b} [Y_k, R_j^*]$$

$$= \sum_{k=0}^{b+1} X'_k + \sum_{k=0}^{b+1} Y'_k,$$

(3.10)

where

$$X'_k = \begin{cases} [X_{k+1}, R_j^*] & \text{if } 0 \leq k \leq b - 1, \\ 0 & \text{if } k = b \text{ or } k = b + 1, \end{cases}$$

and

$$Y'_k = \begin{cases} [X_0, R_j^*] & \text{if } k = 1, \\ [Y_{k-1}, R_j^*] & \text{if } 2 \leq k \leq b + 1. \end{cases}$$

It is clear that each $X'_k$ is a $k$-raising operator, and that each $Y'_k$ is a $k$-lowering operator. Hence Equation (3.10) provides the (unique) Fourier-type decomposition for $\left[T, R_j^*\right]$, as in Proposition 2.5. Since it is given that $\left[T, R_j^*\right] \in S$, Proposition 2.5 implies that each $X'_k \in S$ and each $Y'_k \in S$. This in turn implies that $[X_k, R_j^*] \in S$ for every $0 \leq k \leq b$, and that $[Y_k, R_j^*] \in S$ for every $1 \leq k \leq b$.

Now let us unfix the index $j$ from the preceding paragraph. For every $0 \leq k \leq b$, we have proved that $[X_k, R_j^*] \in S$ for all $1 \leq j \leq d$, hence Proposition 3.7 implies that $X_k \in C$. The fact that $Y_k \in C$ for every $1 \leq k \leq b$ is obtained in the same way. This concludes the proof in the case when the hypothesis on $T$ is that $\left[T, R_j^*\right] \in S$ for all $1 \leq j \leq d$.

If $T$ satisfies $[T, R_j] \in S$ for all $1 \leq j \leq d$, then since the ideal $S$ is closed under taking adjoints, it follows that $[T^*, R_j^*] \in S$ for all $1 \leq j \leq d$. The above arguments therefore apply to $T^*$, and lead to the conclusion that $T^* \in C$, which gives $T \in C$. \(\square\)

4. Construction of the embedding

In this section we fix a deformation parameter $q \in (-1, 1)$ and consider the $C^*$-algebra $C(q) = C^*([L_1(q), \ldots, L_d(q)] \subseteq B(F(q))$ from Equation (1.4). The main result of this section (and also this paper), Theorem 3.3, shows that it is possible to unitarily embed $C(q)$ into the $C^*$-algebra $C = C^*([L_1, \ldots, L_d] \subseteq B(F)$ from Equation (1.2).

We will once again utilize the terminology of Subsection 2.3 with respect to the natural decomposition $F = \oplus_{n=0}^{\infty} F_n$. In particular, we will refer to the unital $*$-algebra $B \subseteq B(F)$ consisting of band-limited operators, and to the ideal $S$ of $B$ consisting of summable band-limited operators.

The deformed Fock space $F(q)$ also has a natural decomposition $F(q) = \oplus_{n=0}^{\infty} F_n(q)$, and we will also need to utilize the terminology of Subsection 2.3 with respect to this decomposition. We will let $B(q) \subseteq B(F(q))$ denote the
unital *-algebra consisting of band-limited operators, and we will let \(S^{(q)}\) denote the ideal of \(B^{(q)}\) which consists of summable band-limited operators.

**Remark 4.1.** Recall the positive block-diagonal operator \(M^{(q)} = \oplus_{n=0}^{\infty} M_n^{(q)} \in B^{(q)}\), which was reviewed in Subsection 2.2. It was recorded there that for \(n \geq 1\), \(M_n^{(q)}\) is an invertible operator on \(F_n^{(q)}\). Moreover, for every \(n \geq 1\), one has the upper bound (2.9) for the norm \(\| (M_n^{(q)})^{-1} \|\), and this upper bound is independent of \(n\).

Therefore, the only obstruction to the operator \(M^{(q)}\) being invertible on \(F^{(q)}\) is the fact that \(M_0^{(q)} = 0\). We can overcome this obstruction by working instead with the operator \(\tilde{M}^{(q)}\) defined by

\[
\tilde{M}^{(q)} := P_0^{(q)} + M^{(q)},
\]

where \(P_0^{(q)} \in B(F^{(q)})\) is the orthogonal projection onto the subspace \(F_0^{(q)}\).

It’s clear that \(\tilde{M}^{(q)}\) is invertible, and that the bound from (2.9) applies to \(\| (\tilde{M}^{(q)})^{-1} \|\).

**Lemma 4.2.** The operator \(\tilde{M}^{(q)}\) satisfies \([ (\tilde{M}^{(q)})^{-1/2}, R_j^{(q)} ] \in S^{(q)}\) for all \(1 \leq j \leq d\).

**Proof.** First, we will show that \(\tilde{M}^{(q)}\) and \(R^{(q)}\) satisfy the hypotheses of Proposition 2.6. It’s clear that \(\tilde{M}^{(q)}\) is block-diagonal and that \(R^{(q)}\) is 1-raising, but it will require a bit of work to check that

\[
\sum_{n=0}^{\infty} \| [\tilde{M}^{(q)}, R_j^{(q)} ] \|_{F_n^{(q)}}^{1/2} < \infty, \quad \forall 1 \leq j \leq d.
\]

In order to show that (4.2) holds, fix \(1 \leq j \leq d\). Using Equation (4.1), which defines \(\tilde{M}^{(q)}\), we can write

\[
[\tilde{M}^{(q)}, R_j^{(q)} ] = [P_0^{(q)}, R_j^{(q)} ] + \sum_{i=1}^{d} [L_i^{(q)} (L_i^{(q)})^*, R_j^{(q)} ] = [P_0^{(q)}, R_j^{(q)} ] + \sum_{i=1}^{d} L_i^{(q)} [(L_i^{(q)})^*, R_j^{(q)} ],
\]

where the last equality follows from the fact that \(L_i^{(q)}\) and \(R_j^{(q)}\) commute. The sum in this equation has only a single non-zero term. Indeed, as a consequence of Equation (2.5), we have \([ (L_i^{(q)})^*, R_j^{(q)} ] = 0\) whenever \(i \neq j\). Thus we arrive at the following formula:

\[
[\tilde{M}^{(q)}, R_j^{(q)} ] = [P_0^{(q)}, R_j^{(q)} ] + L_j^{(q)} [(L_j^{(q)})^*, R_j^{(q)} ].
\]

We next restrict the operators on both sides of (4.3) to a subspace \(F_n^{(q)}\), for \(n \geq 1\). Noting that \([P_0^{(q)}, R_j^{(q)} ] = -R_j^{(q)} P_0^{(q)}\) vanishes on \(F_n^{(q)}\), we obtain
that
\[ [\hat{M}^{(q)}, R_j^{(q)}] \big|_{\mathcal{F}_n^{(q)}} = L_j^{(q)}[(L_j^{(q)})^*, R_j^{(q)}] \big|_{\mathcal{F}_n^{(q)}}, \quad \forall n \geq 1. \]

Finally, we take norms in Equation (4.4) and invoke Equation (2.5) once more to obtain that
\[ \| [\hat{M}^{(q)}, R_j^{(q)}] \big|_{\mathcal{F}_n^{(q)}} \| \leq |q|^n \| L_j^{(q)} \|, \quad \forall n \geq 1. \]

The conclusion that (4.2) holds follows from here, since \( \sum_{n=1}^{\infty} |q|^n < \infty \).

Therefore, we can apply Proposition 2.6 to \( \hat{M}^{(q)} \) and \( R_j^{(q)} \), and conclude that \( [(\hat{M}^{(q)})^{1/2}, R_j^{(q)}] \in \mathcal{S}(q) \). Note that the operator \( (\hat{M}^{(q)})^{-1/2} \) is bounded and block-diagonal, meaning in particular that it belongs to the *-algebra \( \mathcal{B}(q) \). The desired result now follows from the obvious identity
\[ (\hat{M}^{(q)})^{-1/2}, R_j^{(q)}] = -[(\hat{M}^{(q)})^{1/2}, (\hat{M}^{(q)})^{-1/2}, R_j^{(q)}], \]
and the fact that \( \mathcal{S}(q) \) is a two-sided ideal of \( \mathcal{B}(q) \). \qed

**Lemma 4.3.** For \( 1 \leq j \leq d \), the unitary \( U = \oplus_{n=0}^{\infty} U_n \) from Subsection 2.2 satisfies
\[ U^*_n L_j^* U_n = (L_j^{(q)})^* (M_n^{(q)})^{-1/2}, \quad \forall n \geq 1. \]
(Note that on the left-hand side of Equation (4.5), we view \( L_j^* \) as an operator in \( B(\mathcal{F}_n, \mathcal{F}_{n-1}) \). On the right-hand side of Equation (4.5), we view \( (L_j^{(q)})^* \) as an operator in \( B(\mathcal{F}_n^{(q)}, \mathcal{F}_{n-1}^{(q)}) \).)

**Proof.** Consider the operator \( A_j^{(q)} : \mathcal{F}_n^{(q)} \to \mathcal{F}_{n-1}^{(q)} \) which acts on the natural basis of \( \mathcal{F}_n^{(q)} \) by
\[ A_j^{(q)}(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = \delta_{i,j} \xi_{i_2} \otimes \cdots \otimes \xi_{i_n}, \quad \forall 1 \leq i_1, \ldots, i_n \leq d. \]

We claim that \( A_j^{(q)} \) satisfies
\[ A_j^{(q)} = (L_j^{(q)})^* (M_n^{(q)})^{-1} \]
To see this, note that for \( 1 \leq i_1, \ldots, i_n \leq d \),
\[ A_j^{(q)} M_n^{(q)}(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}) = A_j^{(q)} \sum_{m=1}^{n} q^{m-1} \xi_{m} \otimes \xi_{i_1} \otimes \cdots \otimes \xi_{i_m} \otimes \cdots \otimes \xi_{i_n} = \sum_{m=1}^{n} q^{m-1} \delta_{j,m} \xi_{i_1} \otimes \cdots \otimes \xi_{i_m} \otimes \cdots \otimes \xi_{i_n} = (L_j^{(q)})^* (\xi_{i_1} \otimes \cdots \otimes \xi_{i_n}), \]
where the first and last equalities follow from Equation (2.8) and Equation (2.4) respectively. Hence \( A_j^{(q)} = (L_j^{(q)})^* \big|_{\mathcal{F}_n^{(q)}} \), so multiplying on the right by \( (M_n^{(q)})^{-1} \) establishes the claim.
Now, from Equation (2.10), which defines $U_n$, we see that
\[ U^*_n L_j^* U_n = U^*_n L_j^* (I \otimes U_{n-1}) (M^{(q)}_n)^{1/2}, \]
and from the definition of $A_j^{(q)}$ it’s immediate that
\[ L_j^* (I \otimes U_{n-1}) = U_{n-1} A_j^{(q)}. \]
Together, this allows us to write
\[ U^*_n L_j^* U_n = U^*_n U_{n-1} A_j^{(q)} (M^{(q)}_n)^{1/2} = A_j^{(q)} (M^{(q)}_n)^{1/2}. \]
Applying Equation (4.6) now gives Equation (4.5), as required. \hfill \Box

**Proposition 4.4.** For $1 \leq i, j \leq d$, the unitary $U$ from Subsection 2.2 satisfies $[U^* L_j^* U, R_i^{(q)}] \in S^{(q)}$.

**Proof.** Fix $i$ and $j$ and let $C$ denote the commutator $C = [U^* L_j^* U, R_i^{(q)}]$. It’s clear that $C$ is a block-diagonal operator on $\mathcal{F}^{(q)}$. In order to show that $C \in S^{(q)}$, we will need to estimate the norm of its diagonal blocks.

For $n \geq 1$, Lemma 4.3 gives
\[
C \mid \mathcal{F}_n^{(q)} = U^*_n L_j^* U_{n+1} R_i^{(q)} - R_i^{(q)} U^*_n L_j^* U_n
\]
\[
= (L_j^{(q)})^* (M^{(q)}_{n+1})^{-1/2} R_i^{(q)} - R_i^{(q)} (L_j^{(q)})^* (M^{(q)}_n)^{-1/2}
\]
\[
= (L_j^{(q)})^* ((M^{(q)}_{n+1})^{-1/2} R_i^{(q)} - R_i^{(q)} (M^{(q)}_n)^{-1/2})
\]
\[
+ ((L_j^{(q)})^* R_i^{(q)} - R_i^{(q)} (L_j^{(q)})^*) (M^{(q)}_n)^{-1/2}. \]
Since $C$ is block-diagonal, this gives
\[
C = (L_j^{(q)})^* [(M^{(q)}_n)^{-1/2}, R_i^{(q)}] + [(L_j^{(q)})^*, R_i^{(q)}] (M^{(q)}_n)^{-1/2}. \]

Now, $[(M^{(q)}_n)^{-1/2}, R_i^{(q)}] \in S^{(q)}$ by Lemma 4.2. By Equation (2.5),
\[
[(L_j^{(q)})^*, R_i^{(q)}] \mid \mathcal{F}_n^{(q)} = \delta_{ij} q^n I_{\mathcal{F}_n^{(q)}},
\]
and since the operator $[(L_j^{(q)})^*, R_i^{(q)}]$ is block-diagonal, this implies that it also belongs to $S^{(q)}$. Since $(L_j^{(q)})^*$ and $(M^{(q)}_n)^{-1/2}$ both belong to $\mathcal{B}^{(q)}$, and since $S^{(q)}$ is a two-sided ideal of $\mathcal{B}^{(q)}$, it follows that $C \in S^{(q)}$. \hfill \Box

We are now able to complete the proof of the embedding theorem.

**Proof of Theorem 1.3.** It suffices to show that $U_{opp} L_i^{(q)} U_{opp}^*$ belongs to $C$, for $1 \leq i \leq d$. Since $U_{opp} L_i^{(q)} U_{opp}^*$ belongs to the algebra $\mathcal{B}$ of all band-limited operators, by Theorem 3.3 it will actually be sufficient to verify that
\[
[U_{opp} L_i^{(q)} U_{opp}^* R_j^{(q)}] \in S, \quad \forall 1 \leq i, j \leq d.
\]
By Definition 1.1 we can write
\[ U_{\text{opp}} L_i^{(q)} U_{\text{opp}}^* = J U J_l^{(q)} L_i^{(q)} J U^* J \]
where the last equality follows from Equation (2.7). This gives
\[ [U_{\text{opp}} L_i^{(q)} U_{\text{opp}}^*, R_j^*] = [J U J_l^{(q)} U J^*, R_j^*] = J U [R_i^{(q)}, U J R_j^* U J^*] U^* J = J U [R_i^{(q)}, U^* L_j^* U] U^* J, \]
and we know from Proposition 4.4 that \([R_i^{(q)}, U J R_j^* U] \in S^{(q)}\). It is clear that conjugation by the unitary \(J U\) takes \(S^{(q)}\) onto \(S\), so this gives the desired result. \(\square\)

The proof that \(\mathcal{C}^{(q)}\) is exact now follows from some simple observations about nuclear and exact \(\mathcal{C}^*-\)algebras (see e.g. [3]).

Proof of Corollary 1.4. The extended Cuntz algebra \(\mathcal{C}\) is (isomorphic to) an extension of the Cuntz algebra. Since the Cuntz algebra is nuclear, this implies that \(\mathcal{C}\) is nuclear, and in particular that \(\mathcal{C}\) is exact. Since exactness is inherited by subalgebras (see e.g. Chapter 2 of [3]), it follows from Theorem 1.3 that \(U_{\text{opp}} \mathcal{C}^{(q)} U_{\text{opp}}^*\) is exact, and hence that \(\mathcal{C}^{(q)}\) is exact. \(\square\)

Remark 4.5. Since Theorem 1.3 holds for all \(q \in (-1, 1)\), a natural thought is that the methods used above could also be applied to establish the inclusion \(U \mathcal{C}^{(q)} U^* \subseteq \mathcal{C}\) for all \(q \in (-1, 1)\), and hence (since the opposite inclusion was shown in [5]) that \(U \mathcal{C}^{(q)} U^* = \mathcal{C}\). To do this, it would be necessary to establish that
\[ [UL^{(q)} U^*, R_j^*] \in S, \quad \forall 1 \leq i, j \leq d. \] This condition looks superficially similar to the condition from Proposition 4.4 but this is deceptive. We believe that establishing (4.7) will require a deeper understanding of the combinatorics which underlie the \(q\)-commutation relations.

The algebra \(\mathcal{C}^{(q)}\) arises as a representation of the univeral algebra \(\mathcal{E}^{(q)}\) corresponding to the \(q\)-commutation relations. It was shown in [6] that for \(|q| < \sqrt{2} - 1\), \(\mathcal{C}^{(q)}\) and \(\mathcal{E}^{(q)}\) are isomorphic (and in particular that they are both isomorphic to the extended Cuntz algebra). It is believed that this is the case for all \(q \in (-1, 1)\).

5. An application to the \(q\)-Gaussian von Neumann algebras

The \(q\)-Gaussian von Neumann algebra \(\mathcal{M}^{(q)}\) is the von Neumann algebra generated by \(\{L_i^{(q)} + (L_i^{(q)})^* \mid 1 \leq i \leq d\}\). This algebra can be considered as a type of deformation of \(L(F_d)\), the von Neumann algebra of the free group on \(d\) generators. Indeed, for \(q = 0\), a basic result in free probability states
that $\mathcal{M}^{(q)}$ is precisely the realization of $L(\mathbb{F}_d)$ as the von Neumann algebra generated by a free semicircular family (see e.g. Section 2.6 of [11] for the details).

For general $q \in (-1, 1)$ it is known that $\mathcal{M}^{(q)}$ is a von Neumann algebra in standard form, with $\Omega$ being a cyclic and separating trace-vector. The commutant of $\mathcal{M}^{(q)}$ is the von Neumann algebra generated by $\{R_i^{(q)} + (R_i^{(q)})^* \mid 1 \leq i \leq d\}$ (see Section 2 of [2]).

Not much is known about the isomorphism class of the algebras $\mathcal{M}^{(q)}$ for $q \neq 0$. The major open problem is to determine the extent to which they behave like $L(\mathbb{F}_d)$. The best results to date show that $\mathcal{M}^{(q)}$ does share certain properties with $L(\mathbb{F}_d)$. Nou showed in [7] that $\mathcal{M}^{(q)}$ is non-injective, and Ricard showed in [9] that it is a $II_1$ factor. Shlyakhtenko showed in [10] that if we assume $|q| < 0.44$, then the results in [6] and [5] can be used to obtain that $\mathcal{M}^{(q)}$ is solid in the sense of Ozawa.

Based on the results in Section 4, we show here that $\mathcal{M}^{(q)}$ is weakly exact. For more details on weak exactness, we refer the reader to Chapter 14 of [3].

**Theorem 5.1.** For every $q$ in the interval $(-1, 1)$, the $q$-Gaussian von Neumann algebra $\mathcal{M}^{(q)}$ is weakly exact.

**Proof.** It is known that a von Neumann algebra is weakly exact if it contains a weakly dense C$^*$-algebra which is exact (see e.g. Theorem 14.1.2 of [3]). Consider the C$^*$-algebra $\mathcal{A}^{(q)}$ generated by $\{L_i^{(q)} + (L_i^{(q)})^* \mid 1 \leq i \leq d\}$. It is clear that $\mathcal{A}^{(q)}$ is weakly dense in $\mathcal{M}^{(q)}$, while on the other hand, we have $\mathcal{A}^{(q)} \subseteq C^{(q)}$. Therefore, the exactness of $\mathcal{A}^{(q)}$ follows from Corollary 1.4 combined with the fact that exactness is inherited by subalgebras. □

**References**

[1] M. Bozejko, R. Speicher. An example of a generalized Brownian motion, Communications in Mathematical Physics 137 (1991), 519-531.

[2] M. Bozejko, B. Kümmerer, R. Speicher. q-Gaussian processes: Non-commutative and classical aspects, Communications in Mathematical Physics 185 (1997), 129-154.

[3] N. Brown, N. Ozawa. C$^*$-algebras and finite dimensional approximations, Graduate Studies in Mathematics, vol. 88, American Mathematical Society (2008).

[4] K. Davidson. On operators commuting with Toeplitz operators modulo the compact operators, Journal of Functional Analysis 24 (1977), 291-302.

[5] K. Dykema, A. Nica. On the Fock representation of the $q$-commutation relations, Journal für Reine und Angewandte Mathematik 440 (1993), 201-212.

[6] P.E.T. Jorgensen, L.M. Schmitt, R.F. Werner. q-canonical commutation relations and stability of the Cuntz algebra, Pacific Journal of Mathematics 165 (1994), 131-151.

[7] A. Nou. Non-injectivity of the $q$-deformed von Neumann algebras, Mathematische Annalen 330 (2004) 17-38.

[8] G.K. Pedersen. A commutator inequality, Operator Algebras, Mathematical Physics and Low-Dimensional Topology (Istanbul 1991), Research Notes in Mathematics 5, AK Peters, Wellesley, MA (1993), 233-235.

[9] E. Ricard. Factoriality of q-gaussian von Neumann algebras, Communications in Mathematical Physics 257 (2005), 659-665.
[10] D. Shlyakhtenko. Some estimates for non-microstates free dimension, with applications to q-semicircular families, International Math Research Notices 51 (2004), 2757-2772.

[11] D. Voiculescu, K. Dykema, A. Nica. Free random variables, CRM Monograph Series, vol. 1, American Mathematical Society, Providence, RI (1992).

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