BOUNDS FOR TURÁNIANS OF MODIFIED BESSEL FUNCTIONS

ÁRPÁD BARICZ

Dedicated to Boróka and Koppány

Abstract. Motivated by some applications in applied mathematics, biology, chemistry, physics and engineering sciences, new tight Turán type inequalities for modified Bessel functions of the first and second kind are deduced. These inequalities provide sharp lower and upper bounds for the Turánian of modified Bessel functions of the first and second kind, and in most cases the relative errors of the bounds tend to zero as the argument tends to infinity. The chief tools in our proofs are some ideas of Gronwall [19], an integral representation of Ismail [28, 29] for the quotient of modified Bessel functions of the second kind, results of Hartman and Watson [24, 26, 59] and some recent results of Segura [52]. As applications of the main results some sharp Turán type inequalities are presented for the product of modified Bessel functions of the first and second kind and it is shown that this product is strictly geometrically concave.

1. Introduction

Let us denote by $I_\nu$ and $K_\nu$ the modified Bessel functions of the first and second kind of real order $\nu$, which are the linearly independent particular solutions of the second order modified Bessel differential equation. For definitions, recurrence formulas and many important properties of modified Bessel functions of the first and second kind we refer to the classical book of Watson [58]. Recall that the modified Bessel function $I_\nu$, called also sometimes as the Bessel function of the first kind with imaginary argument, has the series representation [58, p. 77]

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+\nu}}{n!\Gamma(n+\nu+1)},$$

where $\nu \neq -1, -2, \ldots$ and $u \in \mathbb{R}$. The modified Bessel function of the second kind $K_\nu$, called also sometimes as the MacDonald or Hankel function, is defined as [58, p. 78]

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu \pi},$$

where the right-hand side of this equation is replaced by its limiting value if $\nu$ is an integer or zero. We note that in view of the above series representation $I_\nu(x) > 0$ for all $\nu > -1$ and $x > 0$. Similarly, by using the familiar integral representation [58, p. 181]

$$K_\nu(x) = \int_0^\infty e^{-x \cosh t} \cosh(\nu t) dt,$$

which holds for each $x > 0$ and $\nu \in \mathbb{R}$, one can see that $K_\nu(x) > 0$ for all $x > 0$ and $\nu \in \mathbb{R}$. These functions are among the most important functions of the mathematical physics and have been used (for example) in problems of electrical engineering, hydrodynamics, acoustics, biophysics, radio physics, atomic and nuclear physics, information theory. These functions are also an effective tool for problem solving in areas of wave mechanics and elasticity theory. Modified Bessel functions of the first and second kind are an inexhaustible subject, there are always more useful properties than one knows. Recently, there has been a vivid interest on bounds for ratios of modified Bessel functions and on Turán type inequalities for these functions. For more details we refer the interested reader to the most recent papers in the subject [4, 5, 6, 10, 33, 34, 35, 52] and to the references therein.
Now, let us focus on the following Turán-type inequalities, which hold for all $\nu > -1$ and $x > 0$

\begin{equation}
0 < I_{\nu}^2(x) - I_{\nu-1}(x)I_{\nu+1}(x) < \frac{1}{\nu + 1} \cdot I_{\nu}^2(x) \tag{1.1}
\end{equation}

Note that their analogue hold for all $|\nu| > 1$ and $x > 0$

\begin{equation}
\frac{1}{1 - |\nu|} \cdot K_{\nu}^2(x) < K_{\nu}^2(x) - K_{\nu-1}(x)K_{\nu+1}(x) < 0. \tag{1.2}
\end{equation}

These inequalities have attracted the interest of many mathematicians, and were rediscovered by many times by different authors in different forms. To the best of author’s knowledge the Turán type inequality \[13\] for $\nu > -1$ was proved first by Thiruvenkatachar and Nanjundiah \[57\]. The left-hand side was proved also later by Amos \[2, p. 243\] for $\nu \geq 0$. Joshi and Bissu \[31\] proved also the left-hand side of \[13\] for $\nu > 0$, while Lorch \[34\] proved that this inequality holds for all $\nu \geq -1/2$. Recently, the author \[5\] reconsidered the proof of Joshi and Bissu \[31\] and pointed out that \[13\] holds true for all $\nu > -1$ and the constants 0 and $1/(\nu + 1)$ in \[1.1\] are best possible. Note that, as it was shown in \[7, 38\], the function $\nu \mapsto I_{\nu+\alpha}(x)/I_{\nu}(x)$ is decreasing for each fixed $\alpha \in (0, 2]$ and $x > 0$, where $\nu > -1$ and $\nu \geq -(\alpha + 1)/2$. Consequently, the function $\nu \mapsto I_{\nu}(x)$ is log-concave on $(-\infty, 1)$ for each fixed $x > 0$, as it was pointed out in \[7\]. See also the paper of Segura \[52\] for an alternative proof of \[1.1\]. For the sake of completeness it should be also mentioned here that the right-hand side of \[1.2\] was first proved independently by Ismail and Muldoon \[30\] and van Haeringen \[20\], and rediscovered later by Laforgia and Natalini \[33\]. Note that in \[30\] the authors actually proved that for all fixed $x > 0$ and $\beta > 0$, the function $\nu \mapsto K_{\nu+\beta}(x)/K_{\nu}(x)$ is increasing on $\mathbb{R}$. Another proof of the right-hand side of \[1.2\], which holds true for all $\nu \in \mathbb{R}$, was given in \[7\]. Recently, Baricz \[5\] and Segura \[52\], proved the two sided inequality in \[1.2\] by using different approaches. Note that in \[7\] the inequality \[1.2\] is stated only for $\nu > 1$, because of the well-known symmetry relation $K_{\nu}(x) = K_{-\nu}(x)$ we can change $\nu$ by $-\nu$.

See also \[10\] for more details on \[1.2\].

It is also worth to mention that according to the corresponding recurrence relations for the modified Bessel functions of the first and second kind the left-hand side of \[1.1\] is equivalent to

\begin{equation}
\frac{xI_{\nu}^2(x)}{I_{\nu}(x)} < \sqrt{x^2 + \nu^2}, \tag{1.3}
\end{equation}

while the right-hand side of \[1.2\] is equivalent to

\begin{equation}
\frac{xK_{\nu}^2(x)}{K_{\nu}(x)} < -\sqrt{x^2 + \nu^2}. \tag{1.4}
\end{equation}

Moreover, the inequalities \[1.3\] and \[1.4\] together imply that the function $x \mapsto P_{\nu}(x) = I_{\nu}(x)K_{\nu}(x)$ is strictly decreasing on $(0, \infty)$ for all $\nu > -1$. See \[4, 5\] for more details. Note that the above monotonicity property of $P_{\nu}$ was proved earlier by Penfold et al. \[10\] p. 142 by using a different approach. The study in \[46\] was motivated by a problem in biophysics. See also the paper of Grandison et al. \[15\] for more details. For the sake of completeness we recall also that the inequality \[1.3\] was deduced first\(^1\) by Gronwall \[19\] p. 277\] for $\nu > 0$, motivated by a problem in wave mechanics. This inequality was deduced also for $\nu \in \{1, 2, \ldots\}$ by Phillips and Malin \[47\] p. 407, and for $\nu > 0$ by Amos \[2\] p. 241\] and Paltsev \[45\] eq. (21)]. The inequality \[1.4\] was deduced first for $\nu \in \{1, 2, \ldots\}$ by Phillips and Malin \[47\] p. 407, and later for $\nu > 0$ by Paltsev \[45\] eq. (22)]. We note that the Turán type inequalities \[1.1, 1.2, 1.3\] and \[1.4\] as well as the monotonicity of the product of $P_{\nu}$ were used in various problems related to modified Bessel functions in various topics of applied mathematics, biology, chemistry and physics. For reader’s convenience we list here some of the related things:

1. The monotonicity of $P_{\nu}$ for $\nu > 1$ is used (without proof) in some papers about the hydrodynamic and hydromagnetic instability of different cylindrical models. See for example \[18, 49\]. See also the paper of Hasan \[27\], where the electrogravitational instability of onoscillating streaming fluid cylinder under the action of the selfgravitating, capillary and electrodynamic forces has been discussed. In these papers the authors use (without proof) the inequality

\[ P_{\nu}(x) < \frac{1}{2} \]
for all $\nu > 1$ and $x > 0$. We note that the above inequality readily follows from the fact that $P_\nu$ is strictly decreasing on $(0, \infty)$ for all $\nu > -1$. More precisely, for all $x > 0$ and $\nu > 1$ we have

$$P_\nu(x) < \lim_{x \to 0} P_\nu(x) = \frac{1}{2\nu} < \frac{1}{2}.$$  

2. The Turán type inequality (1.1) and the right-hand side of (1.2), together with the monotonicity of $P_\nu$ were used, among other things, by Klimek and McBride [32] to prove that a Dirac operator, subject to Atiyah-Patodi-Singer-like boundary conditions on the solid torus, has a bounded inverse, which is actually a compact operator.

3. Recently, Simitev and Biktashev [53] used the fact that the function $x \mapsto x I'_\nu(x)/I_\nu(x)$ is increasing on $(0, \infty)$ together with the inequality (1.3) in the study of asymptotic restitution curves in the caricature Noble model of electrical excitation in the heart. As it was pointed out above the inequality (1.3) is equivalent to the left-hand side of the Turán type inequality (1.1). Moreover, because of the relation [31, p. 339], [5, p. 256] \[\int_0^\infty x I'_\nu(x) - I_{\nu-1}(x) I_{\nu+1}(x) = I^2_\nu(x) \] (1.5) \[\frac{I^2_\nu(x)}{I_\nu(x)} \] the fact that the function $x \mapsto x I'_\nu(x)/I_\nu(x)$ is increasing is also equivalent to the left-hand side of the Turán type inequality (1.1). Thus, Simitev and Biktashev [53] actually used two times in their study exactly the left-hand side of the Turán type inequality (1.1). Here it is important to note that very recently, in order to prove that $\frac{\int_0^\infty x I_{\nu+1}(x)/I_\nu(x)}{\nu I_\nu(x)} > 0$ for all $\nu > 0$ and $x > 0$, Schlenk and Sicbaldi [51] p. 622 rediscovered the left-hand side of the inequality (1.1). They used the relation

$$\frac{x I_{\nu+1}(x)}{I_\nu(x)} = x \frac{I^2_\nu(x) - I_{\nu-1}(x) I_{\nu+1}(x)}{I^2_\nu(x)},$$

which in view of the recurrence relation $x I'_\nu(x) = \nu I_\nu(x) + x I_{\nu+1}(x)$, is actually the same as (1.3). We also mention that in [51] the authors rediscovered also the corresponding Turán type inequality for Bessel functions of the first kind. These results on Bessel and modified Bessel functions of the first kind were used in [51] to study bifurcating extremal domains for the first eigenvalue of the Laplacian. More precisely, in [51] the Turán type inequalities for Bessel and modified Bessel functions were used in the study of the monotonicity of the first eigenvalue of a linearized operator, in order to show that this operator satisfies the assumptions of the Crandall-Rabinowitz theorem, implying the main result of [51]. Finally, for a survey on the Turán type inequalities for Bessel functions of the first kind the interested reader is referred to [9].

4. Note that, as it was pointed out in [52], the left-hand side of the Turán type inequality (1.2) provides actually an upper bound for the effective variance of the generalized Gaussian distribution. More precisely, Alexandrov and Lacis [3] used (without proof) the inequality $0 < v_\text{eff} < 1/(\mu - 1)$ for $\mu = \nu + 4$, where

$$v_\text{eff} := \left[ \int_0^\infty r^2 f_\nu(r)dr \right] \left[ \int_0^\infty r^4 f_\nu(r)dr \right] \left[ \int_0^\infty r^3 f_\nu(r)dr \right]^2 - 1 = \frac{K_{\mu-1}(1/w)K_{\mu+1}(1/w)}{[K_\mu(1/w)]^2} - 1$$

is the effective variance of the generalized Gaussian distribution and

$$f_\nu(r) = \frac{1}{2\nu K_{\nu+1}(1/w)} \frac{r^\nu}{s^{\nu+1}} \exp \left[ -\frac{1}{2w} \left( \frac{s}{r} + \frac{r}{s} \right) \right]$$

is the generalized inverse Gaussian particle size distribution function, $w$ represents the width of the distribution, $s$ is an effective size parameter, and $\nu$ is the order of the distribution.

5. Simon [34] used the Turán type inequality (1.3) for $\nu = 1/3$ to prove that the positive 1/3-stable distribution with density

$$f_{1/3}(x) = \frac{1}{3\pi x^{1/2}} K_{1/3} \left( \frac{2}{3\sqrt{3}x} \right)$$

An alternative proof for this result is as follows: according to Watson [59] the function $x \mapsto I_{\nu+1}(x)/I_\nu(x)$ is increasing on $(0, \infty)$ for all $\nu \geq -1/2$. Thus, $[x I_{\nu+1}(x)/I_\nu(x)]' = I_{\nu+1}(x)/I_\nu(x) + x [I_{\nu+1}(x)/I_\nu(x)]' > 0$ for all $\nu \geq -1/2$ and $x > 0$. 

is multiplicative strongly unimodal in the sense of Cuculescu-Theodorescu, that is, \( t \mapsto f_{1/3}(e^t) \) is log-concave in \( \mathbb{R} \). Here for \( \alpha \in (0, 1) \) the positive \( \alpha \)-stable density is normalized such that

\[
\int_0^\infty e^{-\lambda t} f_\alpha(t) dt = E\left[e^{-\lambda Z_\alpha}\right] = e^{-\lambda^\alpha},
\]

where \( \lambda \geq 0 \) and \( Z_\alpha \) is the corresponding random variable.

6. Recently, motivated by some results in finite elasticity, Laforgia and Natalini \[55\] proved that for \( x > 0 \) and \( \nu \geq 0 \) the following inequality is valid

\[
\frac{I_\nu(x)}{I_{\nu-1}(x)} > -\nu + \sqrt{x^2 + \nu^2} \frac{1}{x},
\]

We note an alternative proof of (1.6) was given recently by Kokologiannaki \[33\] eq. (2.1)]. Moreover, as it was pointed out in \[10\], (1.6) was proved already by Amos \[2\] eq. (9)] for \( \nu \geq 1 \) and \( x > 0 \). It is also worth to mention that the authors showed in \[10\] that the inequality (1.6) is equivalent to (1.3), which is equivalent to the left-hand side of (1.1). Observe that the inequality (1.6) can be rewritten in the form

\[
\frac{1}{x} \frac{I_\nu(x)}{I_{\nu-1}(x)} > \frac{1}{\nu + \sqrt{x^2 + \nu^2}},
\]

where \( x > 0 \) and \( \nu \geq 0 \). In \[52\] it was pointed out that the inequality (1.7), which is actually equivalent to the left-hand side of (1.1), appears in a problem of chemistry. More precisely, in \[39\] the authors considered the mean number of molecules of a given class dissolved in a water droplet and compared the so-called classical and stochastic approaches. If \( n_c \) and \( n_s \) are the respective mean numbers of molecules by using the classical and stochastic approaches, then according to Segura \[52\], after the redefinition of the variables it can be shown that

\[
n_c = \frac{x^2}{4} \frac{1}{\nu + 1 + \sqrt{x^2 + (\nu + 1)^2}} \quad \text{and} \quad n_s = \frac{x}{4} \frac{I_{\nu+1}(x)}{I_\nu(x)}
\]

and by using the inequality (1.7) for all \( x > 0 \) and \( \nu \geq -1 \) we have \( n_s > n_c \). Note that this inequality was known before only for small or large values of \( x \).

7. The analogue of (1.6) for modified Bessel functions of the second kind, that is,

\[
\frac{K_\nu(x)}{K_{\nu-1}(x)} < \frac{\nu + \sqrt{x^2 + \nu^2}}{x}
\]

was proved recently by Laforgia and Natalini \[35\] for \( x > 0 \) and \( \nu \in \mathbb{R} \). Note that in \[10\] the authors pointed out that in fact (1.8) is equivalent to (1.4), which is equivalent to the right-hand side of (1.2). The inequality (1.8) was used recently by Fabrizi and Trivisano \[15\] to deduce an upper bound for the expected value of a random variable which has a generalized inverse Gaussian distribution, while Lechleiter and Nguyen \[36\] used the inequality (1.8) to deduce an error estimate for an approximation to the waveguide Green’s function.

8. It is also interesting to note that the Turánian \( K_\nu^2(x) - K_{\nu-1}(x) K_{\nu+1}(x) \) appears in the variance of the non-central \( F \)-Bessel distribution defined by Thabane and Dreikic \[53\], and it appears also in \[13\] eq. (37), related with the variance of a different distribution. Moreover, in \[11\] the authors investigated the convexity with respect to power means of the modified Bessel functions \( I_\nu \) and \( K_\nu \) by using the Turán type inequalities presented above. We note that the left-hand side of the inequality (1.4) was used also by Milenkovic and Compton \[43\], and for \( \nu = 1 \) by Bertini et al. \[12\]. The property that \( x \mapsto x I_\nu(x)/I_\nu(x) \) is increasing on \( (0, \infty) \) for all \( \nu \geq 0 \) was used by Giorgi and Smits \[16\] p. 237], \[17\] p. 610], and also by Lombardo et al. \[37\] together with its analogue that \( x \mapsto x K_\nu(x)/K_\nu(x) \) is decreasing on \( (0, \infty) \) for all \( \nu \geq 0 \). The later property is used in \[37\] without proof, however, this is actually equivalent to the right-hand side of the Turán type inequality (1.2), according to relation \[3\] p. 259]

\[
x \left[ K_\nu^2(x) - K_{\nu-1}(x) K_{\nu+1}(x) \right] = K_\nu^2(x) \left[ \frac{x K_\nu'(x)}{K_\nu(x)} \right].
\]

Motivated by the above applications, in this paper our aim is to reconsider the Turán type inequalities for modified Bessel functions of the first and second kind. By using some ideas of Gronwall \[19\], an integral representation of Ismail \[28\] \[29\] for the quotient of modified Bessel functions of the second kind, results of Hartman and Watson \[24\] \[26\] \[59\] and some recent results of Segura \[52\], in the present paper we
make a contribution to the subject and we deduce some new tight Turán type inequalities for modified Bessel functions of the first and second kind. These inequalities, studied in details in Sections 2 and 3, provide sharp lower and upper bounds for the Turánian of modified Bessel functions of the first and second kind, and in most cases the relative errors of the bounds tend to zero as the argument tends to infinity. In addition, in Section 2 we point out some mathematical errors in the papers of Gronwall [19], Hamsici and Martinez [23] and of Joshi and Bissu [31], and we also correct these errors. Moreover, we present new proofs for the right-hand sides of (1.1) and (1.2), and also for some of the results of Hartman and Watson [26]. At the end of Section 2 an open problem is discussed in details, which may be of interest for further research. Finally, in Section 4 we present some applications of the main results of Section 2 and 3. Here we prove that the product of modified Bessel functions of the first and second kind is strictly geometrically concave and we deduce some sharp Turán type inequalities for this product.

2. Turán type inequalities for modified Bessel functions of the first kind

In this section our aim is to study the Turán type inequalities for modified Bessel functions of the first kind motivated by the above applications and by the paper of Hamsici and Martinez [23]. Joshi and Bissu [31] proved for \( x > 0 \) and \( \nu \geq 0 \) the following two Turán type inequalities

\[
I^2_\nu(x) - I_{\nu-1}(x)I_{\nu+1}(x) < \frac{4}{J_{\nu,1}} \cdot I^2_\nu(x)
\]

and

\[
I^2_\nu(x) - I_{\nu-1}(x)I_{\nu+1}(x) < \frac{1}{x + \nu} \cdot I^2_\nu(x),
\]

where \( j_{\nu,1} \) is the first positive zero of the Bessel function \( J_\nu \). Observe that, in view of the Rayleigh inequality [55, p. 502] \( j_{\nu,1} > 4(\nu + 1) \), the first inequality would be an improvement of the right-hand side of (1.1). However, based on numerical experiments, unfortunately both of the above inequalities from [31] are not valid for all \( x > 0 \) and \( \nu \geq 0 \). The reason for that the first inequality is not true for all \( x > 0 \) and \( \nu \geq 0 \) is that in the right-hand side of (1.1) the constant \( 1/(\nu + 1) \) is best possible, according to [3 p. 257], and consequently cannot be improved by other constant (independent of \( x \)). On the other hand, the inequality (2.1) is not valid because its proof is not correct. By using only the so-called Nasell inequality [31, p. 253]

\[
1 + \frac{\nu}{x} < \frac{I_\nu(x)}{I_{\nu+1}(x)}
\]

it is not possible to prove the inequality (2.1). Because of this, the proofs of the extensions of (2.1) in [31] cannot be correct too. We note that actually by using some recent results of Segura [52] the inequality (2.1) can be corrected. More precisely, let us focus on the inequalities [52, eqs. (45), (54)]

\[
(2.2) \quad \frac{1}{\nu + \frac{1}{2} + \sqrt{\nu^2 + (\nu + \frac{1}{2})^2}} \cdot I^2_\nu(x) < I^2_{\nu-1}(x) - I_{\nu-1}(x)I_{\nu+1}(x) < \frac{2}{\nu + 1 + \sqrt{\nu^2 + (\nu + 1)^2}} \cdot I^2_\nu(x),
\]

where \( x > 0 \) and \( \nu \geq -1 \). By using the inequality (2.2) clearly we have

\[
(2.3) \quad \frac{1}{x + 2\nu + 1} \cdot I^2_\nu(x) < I^2_{\nu-1}(x) - I_{\nu-1}(x)I_{\nu+1}(x) < \frac{2}{x + \nu + 1} \cdot I^2_\nu(x),
\]

with the same range of validity as in (2.2). The right-hand side of the above inequality actually implies that (2.1) can be corrected as

\[
(2.4) \quad I^2_\nu(x) - I_{\nu-1}(x)I_{\nu+1}(x) < \frac{2}{x + \nu} \cdot I^2_\nu(x),
\]

where \( x > 0 \) and \( \nu \geq 0 \). Recall that, according to [5, p. 257], for

\[
\varphi_\nu(x) = 1 - \frac{I_{\nu-1}(x)I_{\nu+1}(x)}{I^2_\nu(x)}
\]

we have \( \lim_{x \to \infty} \varphi_\nu(x) = 0 \) and \( \lim_{x \to 0} \varphi_\nu(x) = 1/(\nu + 1) \). Thus, all the inequalities in (2.2) and (2.3) are sharp as \( x \to \infty \), while the right-hand side of (2.2) is also sharp as \( x \to 0 \). Observe that clearly the left-hand sides of (2.2) and (2.3) improve the left-hand side of (1.1), and the right-hand side of (2.2)
improves the right-hand side of (1.1) for all $x > 0$ and $\nu > -1$. The right-hand side of (2.3) also improves the right-hand side of (1.1) for all $x > \nu + 1 > 0$. We note that in view of the left-hand side of (2.2) it can be proved that the inequality (2.1) is reversed for all $1/2 \leq x \leq \nu(\nu + 1)$ and $\nu \geq 0$, which also shows that (2.1) cannot be correct for all $x > 0$ and $\nu \geq 0$. Moreover, by using the relation (1.5), the Turán type inequalities (2.3) can be rewritten as

$$\frac{x}{x + 2\nu + 1} < \frac{[xI_\nu'(x)]'}{I_\nu(x)} < \frac{2x}{x + \nu + 1},$$

or

$$[x - (2\nu + 1)\ln(x + 2\nu + 1)]' < \frac{[xI_\nu'(x)]'}{I_\nu(x)} < [2x - 2(\nu + 1)\ln(x + \nu + 1)]'.$$

Now, having in mind the fact that $xI_\nu'(x)/I_\nu(x)$ tends to $\nu$ as $x \to 0$, according to

$$\frac{xI_\nu'(x)}{I_\nu(x)} = \nu + \frac{x^2}{2(\nu + 1)} - \frac{x^4}{8(\nu + 1)^2(\nu + 2)} + \ldots,$$

we obtain the new inequalities

$$\int_0^x [t - (2\nu + 1)\ln(t + 2\nu + 1)]' dt < \int_0^x \left[\frac{tI_\nu'(t)}{I_\nu(t)}\right]' dt < \int_0^x [2t - 2(\nu + 1)\ln(t + \nu + 1)]' dt,$$

that is,

$$x + \nu + (2\nu + 1)\ln\left(\frac{2(\nu + 1)}{x + 2\nu + 1}\right) < \frac{xI_\nu'(x)}{I_\nu(x)} < 2x + \nu + 2(\nu + 1)\ln\left(\frac{\nu + 1}{x + \nu + 1}\right),$$

where $x > 0$ and $\nu > -1$. Note that both of the above inequalities are sharp as $x \to 0$.

It is important to note here that recently Hamsici and Martinez [23, p. 1595] used the Turán type inequality (2.1) and concluded that for all $x > 0$ and $\nu > 0$ we have

$$\hat{b}_2(x) = \frac{xI_\nu(x)}{I_{\nu-1}(x)I_{\nu+1}(x)} < \frac{x + \nu}{x} < -1.$$

See also [21, p. 70] and [22, p. 36]. Since the inequality (2.1) is not valid for all $x > 0$ and $\nu \geq 0$ we can see that the left-hand side of the above inequality is not valid too for all $x > 0$ and $\nu > 0$. In view of (2.4) the above inequality should be written as

$$\hat{b}_2(x) < \frac{x + \nu}{2x} < -\frac{1}{2}.$$

This implies that the bias of the hyperplane in [23 Proposition 4] does not have the property that its absolute value is greater than 1, at least according to the proof given in [23]. In view of the above correct inequality the absolute value of the bias will be just greater than 1/2 and this means that proof of the assertion [23 Proposition 4] “that the hyperplane given in (12) does not intersect with the sphere and can be omitted for classification purposes” is not complete. All the same, by using the right-hand side irrational bound in (2.2) we can prove that $\hat{b}_2(x) < -1$, but only for $0 < x \leq 4(\nu + 1)/3$ and $\nu > -1$.

Moreover, by using a result of Gronwall [19], it is possible to show that the claimed inequality $\hat{b}_2(x) < -1$, that is,

$$I_\nu(x) - I_{\nu-1}(x)I_{\nu+1}(x) < \frac{1}{x} \cdot I_\nu^2(x),$$

is actually valid for all $\nu \geq 1/2$ and $x > 0$. This corrects the proof of [23 Proposition 4]. More precisely, observe that in view of (1.5) the inequality (2.6) is equivalent to

$$y''_\nu(x) < 1,$$

where $y_\nu(x) = xI_\nu'(x)/I_\nu(x)$. In other words, to prove (2.5) we just need to show that $|y_\nu(x) - x|^2 < 0$ for all $\nu \geq 1/2$ and $x > 0$. However, the proof of this monotonicity property was given by Gronwall [19, p. 276] and is based on the inequality [19, p. 275]

$$\frac{xI_\nu'(x)}{I_\nu(x)} > x - \frac{1}{2},$$

which is valid for all $x > 0$ and $\nu \geq 1/2$. We note that (2.7) can be improved as [52, p. 526]

$$\frac{xI_\nu'(x)}{I_\nu(x)} > \sqrt{x^2 + \left(\nu - \frac{1}{2}\right)^2} - \frac{1}{2},$$




where \( x > 0 \) and \( \nu \geq 1/2 \). Next, let us mention that the inequalities (2.7) and (2.8) of Gronwall and Segura can be improved too by using a result in the proof of [26, Proposition 7.2], due to Hartman and Watson [26, p. 606]. Namely, in the proof of [26, Proposition 7.2] it is stated that

\[
\nu(x) = [\ln(\sqrt{x}I_\nu(x))]' > \nu(x) = \sqrt{1 + \frac{x^2 - \frac{1}{4}}{\nu^2}}.
\]

that is,

\[
xI'_\nu(x) = I_\nu(x) \sqrt{x^2 + \nu^2 - \frac{1}{4}} - \frac{1}{2}
\]

is valid for all \( \nu \geq 1/2 \) and \( x > 0 \). It is interesting that an alternative proof of this inequality follows from (2.9) or (2.10). More precisely, by using the notation \( \mu = \nu^2 - 1/4 \), the inequality (2.10) implies that \( x^2(x) < \sqrt{x^2 + \mu} \) for all \( \nu \geq 1/2 \) and \( x > 0 \). On the other hand, since \( I_\nu \) satisfies the modified Bessel differential equation, the function \( y_\nu \) satisfies

\[
y_\nu(x) = x^2 + \nu^2 - y_\mu^2(x)
\]

and consequently

\[
y_\nu^2(x) > x^2 + \nu^2 - \sqrt{x^2 + \mu} = \left( \sqrt{x^2 + \mu} - \frac{1}{2} \right)^2,
\]

or equivalently

\[
(y_\nu(x) - \sqrt{x^2 + \mu} + \frac{1}{2}) \left( y_\nu(x) + \sqrt{x^2 + \mu} - \frac{1}{2} \right) > 0,
\]

which implies (2.9). Here we used the fact that the function \( x \mapsto y_\nu(x) + \sqrt{x^2 + \mu} \), as a sum of two strictly increasing functions, is strictly increasing on \((0, \infty)\) for all \( \nu \geq 1/2 \), and consequently

\[
y_\nu(x) + \sqrt{x^2 + \mu} > \nu + \sqrt{\mu} \geq 1/2
\]

for all \( \nu \geq 1/2 \) and \( x > 0 \).

Now, by using the inequality (2.9) we can prove the following theorem, which improves (2.5).

**Theorem 1.** If \( \nu \geq 1/2 \) and \( x > 0 \), then the next Turán type inequalities are valid

\[
\frac{\nu + \frac{1}{2}}{\nu + 1} \cdot \left( I^2_\nu(x) - I^2_\nu(x) - I^-_\nu(x)I^+_{\nu + 1}(x) \right) > \frac{1}{\sqrt{x^2 + \nu^2 - \frac{1}{4}}} \cdot I^2_\nu(x).
\]

Moreover, the left-hand side of (2.11) holds true for all \( \nu \geq -1/2 \) and \( x > 0 \). Each of the above inequalities are sharp as \( x \to \infty \), and the left-hand side of (2.11) is sharp as \( x \to 0 \).

Clearly, the right-hand side of (2.11) is better than the inequality (2.5) for all \( x > 0 \) and \( \nu \geq 1/2 \). Moreover, observe that the left-hand type inequality (2.5) is better than the right-hand side of (2.2) for \( x \geq 4(\nu + 1)/3 + \nu \geq 1/2 \), and is better than the right-hand side of (2.3) for \( x \geq \nu + 1 \) and \( \nu \geq 1/2 \). Note also that the left-hand side of (2.11) improves the left-hand side of (2.2) for all \( \nu > -1/4 \) and \( x > 0 \) such that \( x^2 \leq (4\nu + 1)(4\nu + 3)(\nu + 1/2)^2 \). It is worth to mention here that the relative errors of the bounds for the Turánian of the modified Bessel function of the first kind in the left-hand side of the inequalities (2.2) and (2.3), in inequality (2.5) and in the right-hand side of (2.11) have the property that tend to zero as the argument tends to infinity. For example, the inequality (2.5) can be rewritten as \( \varphi_\nu(x) < 1/x = r(x) \), and if we use the asymptotic formula [1] p. 377

\[
I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \left[ 1 - \frac{4\nu^2 - 1}{11(8x)} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{2(8x)^2} - \ldots \right],
\]

which holds for large values of \( x \) and for fixed \( \nu \), one has \( \lim_{x \to \infty} \varphi_\nu(x)/r(x) = 1 \) and consequently for the relative error we have the limit \( \lim_{x \to \infty} \left[ r(x) - \varphi_\nu(x) \right]/\varphi_\nu(x) = 0 \), as we required. In other words, the lower bounds in the Turán type inequalities (2.2) and (2.3), and the upper bounds in (2.5) and (2.11) for large values of \( x \) are quite tight. This is illustrated also on Fig. 1. We note that in this figure the bounds in (2.2) are considered as bounds for \( \varphi_\nu(x) \), that is, they are understood in the sense that the lower bound is

\[
\frac{1}{\nu + \frac{1}{2} + \sqrt{x^2 + \nu^2 - \frac{1}{4}}}.
\]
while the upper bound is

\[
\frac{2}{\nu + 1 + \sqrt{x^2 + (\nu + 1)^2}}
\]

The bounds in (2.5) and (2.11) in Fig. 1 have the same meaning.

**Figure 1.** The graph of the function \( \varphi_1 \) and of the bounds in (2.2), (2.5) and (2.11) for \( \nu = 1 \) on \([0, 10]\).

**Proof of Theorem 1.** First we prove the left-hand side of (2.11). For this recall the fact that the function \( x \mapsto \frac{I_{\nu+1}(x)}{I_\nu(x)} \) is increasing and concave on \((0, \infty)\) for all \( \nu \geq -1/2 \). By using the Mittag-Leffler expansion \[14, eq. 7.9.3\]

\[
\frac{I_{\nu+1}(x)}{I_\nu(x)} = \sum_{n \geq 1} \frac{2x}{x^2 + j_{\nu,n}^2}
\]

and the Rayleigh formula \[58, p. 502\]

\[
\sum_{n \geq 1} \frac{1}{j_{\nu,n}^2} = \frac{1}{4(\nu + 1)}
\]

where \( j_{\nu,n} \) is the \( n \)th positive zero of the Bessel function \( J_\nu \), in order to prove that for all \( x > 0 \) and \( \nu \geq -1/2 \)

\[
y''_\nu(x) = 2 \left[ \frac{I_{\nu+1}(x)}{I_\nu(x)} \right]' + x \left[ \frac{I_{\nu+1}(x)}{I_\nu(x)} \right]'' < 2 \lim_{x \to 0} \left[ \frac{I_{\nu+1}(x)}{I_\nu(x)} \right]' = \frac{1}{\nu + 1}.
\]

Here we used the Mittag-Leffler expansion \[14, eq. 7.9.3\]

\[
\frac{I_{\nu+1}(x)}{I_\nu(x)} = \sum_{n \geq 1} \frac{2x}{x^2 + j_{\nu,n}^2}
\]

and the Rayleigh formula \[58, p. 502\]

\[
\sum_{n \geq 1} \frac{1}{j_{\nu,n}^2} = \frac{1}{4(\nu + 1)}
\]

where \( j_{\nu,n} \) is the \( n \)th positive zero of the Bessel function \( J_\nu \), in order to prove that

\[
2 \lim_{x \to 0} \left[ \frac{I_{\nu+1}(x)}{I_\nu(x)} \right]' = 2 \lim_{x \to 0} \sum_{n \geq 1} \frac{2(j_{\nu,n}^2 - x^2)}{(x^2 + j_{\nu,n}^2)^2} = \sum_{n \geq 1} \frac{4}{j_{\nu,n}^2} = \frac{1}{\nu + 1}.
\]

Now, differentiating both sides of (2.10) we obtain

\[
xy''_\nu(x) = 2x - (2y_\nu(x) + 1)y'_\nu(x),
\]

4For reader's convenience we note that this result of Watson was used also by Robert \[50\], Marchand and Perron \[10, 11\], Marchand and Najafabadi \[42\] in different problems of statistics and probability.
and consequently in view of (2.12) we have for all \( x > 0 \) and \( \nu \geq -1/2 \)
\[
\frac{2\nu + 1}{\nu + 1} x < (2y_\nu(x) + 1)y'_\nu(x).
\]
Combining this with the inequality [52, p. 526]
\[
y_\nu(x) < \sqrt{x^2 + \left(\nu + \frac{1}{2}\right)^2} - \frac{1}{2}
\]
we arrive at
\[
\frac{1}{x}y'_\nu(x) > \frac{\nu + \frac{1}{2}}{\nu + 1} \frac{1}{\sqrt{x^2 + \left(\nu + \frac{1}{2}\right)^2}}
\]
and taking into account the relation (1.5) the proof of the left-hand side of (2.11) is complete.

To prove the right-hand side of (2.11) we use the idea of Gronwall [19, p. 277]. Let \( \mu = \nu^2 - 1/4 \). We prove that the function \( x \mapsto u_\nu(x) = \sqrt{x^2 + \mu} - y_\nu(x) \) satisfies \( u'_\nu(x) > 0 \) for all \( \nu \geq 1/2 \) and \( x > 0 \). For this observe that
\[
\sqrt{x^2 + \mu} = \sqrt{\mu} + \frac{1}{2\sqrt{\mu}} \frac{x^2}{\mu} - \frac{3}{8\mu\sqrt{\mu}} + \ldots
\]
and
\[
\sqrt{x^2 + \mu} - y_\nu(x) = \sqrt{\nu} - \nu + \left[ \frac{1}{\sqrt{\nu}} - \frac{1}{\nu + 1} \right] \frac{x^2}{2} - \left[ \frac{1}{\mu\sqrt{\nu}} - \frac{1}{(\nu + 1)^2} \right] \frac{x^4}{8} + \ldots,
\]
which implies that for small values of \( x \) the function \( u_\nu \) is strictly increasing. Thus the first extreme of this function, if any, is a maximum. However, when \( u'_\nu(x) = 0 \), that is,
\[
y'_\nu(x) = \frac{x}{\sqrt{x^2 + \mu}},
\]
by using (2.13) and (2.9) we have for all \( x > 0 \) and \( \nu \geq 1/2 \)
\[
u^2 - 1/4 + \frac{1}{2\sqrt{\mu}} \frac{x^2}{\mu} - \frac{3}{8\mu\sqrt{\mu}} + \ldots \]
which is a contradiction. Consequently, the derivative of the function \( u_\nu \) does not vanish, and then \( u'_\nu(x) > 0 \) for all \( \nu \geq 1/2 \) and \( x > 0 \), as we required. This in turn implies that for all \( x > 0 \) and \( \nu \geq 1/2 \) we have
\[
\frac{1}{x}y'_\nu(x) < \frac{1}{\sqrt{x^2 + \mu}},
\]
which in view of (1.3) is equivalent to the right-hand side of (2.11).

Finally, let us discuss the sharpness of the inequalities. Observe that (2.11) can be rewritten as
\[
\frac{\nu + \frac{1}{2}}{\nu + 1} \frac{1}{\sqrt{x^2 + \left(\nu + \frac{1}{2}\right)^2}} < \varphi_\nu(x) < \frac{1}{\sqrt{x^2 + \nu^2 - \frac{1}{4}}}.
\]
Now, since [5, p. 257] \( \lim_{x \to \infty} \varphi_\nu(x) = 0 \), each of the above inequalities are sharp as \( x \to \infty \). Moreover, since [5, p. 257] \( \lim_{x \to 0} \varphi_\nu(x) = 1/(\nu + 1) \), the left-hand side of the above Turán type inequality is sharp as \( x \to 0 \).

We note that the inequality (2.12) can be used also to prove the right-hand side of the Turán type inequality (1.11) for \( \nu \geq -1/2 \). More precisely, by using (2.12), for all \( x > 0 \) and \( \nu \geq -1/2 \) we get
\[
\int_0^x y''_\nu(t)dt < \int_0^x \left( \frac{t}{\nu + 1} \right)'dt,
\]
that is,
\[
y'_\nu(x) < \frac{x}{\nu + 1},
\]
which in view of (1.3) is equivalent to the right-hand side of the Turán type inequality (1.11). It is worth to mention also here that the proof of the right-hand side of (2.11) was motivated by Gronwall’s proof [19, p. 277] of the fact that the function \( x \mapsto w_\nu(x) = \sqrt{x^2 + \nu^2} - y_\nu(x) \) is increasing on \((0, \infty)\) for all \( \nu > 0 \). Unfortunately, Gronwall’s proof is not correct since the equation [19, p. 277]
\[
\frac{d^2w}{dz^2} = \frac{\nu^2}{(\nu^2 + z^2)^{3/2}} + \frac{2\nu^2w}{z^2(\nu^2 + z^2)^{1/2}}
\]
should be rewritten as
\[ \frac{d^2 w}{dz^2} = \frac{\nu^2}{(\nu^2 + z^2)^{3/2}} + \frac{1 - 2w}{(\nu^2 + z^2)^{1/2}}, \]
which is not necessarily positive for all \( z > 0 \) and \( \nu > 0 \). Moreover, it can be proved that the function
\[ x \mapsto w_{1/2}(x) = \sqrt{x^2 + \frac{1}{4} - \frac{x I_{\nu/2}(x)}{I_{\nu/2}(x)} = \sqrt{x^2 + \frac{1}{4} - \frac{x \cosh(x)}{\sinh(x)} + \frac{1}{2}} \]
is increasing \((0, x_{1/2}]\) and decreasing on \([x_{1/2}, \infty)\), where \( x_{1/2} \simeq 3.57784794 \) is the unique root of the equation \( w'_{1/2}(x) = 0 \). Thus, Gronwall’s statement that \( w_\nu \) is increasing on \((0, \infty)\) for all \( \nu > 0 \) is not valid. However, observe that to correct Gronwall’s proof we would need to show that for all \( x > 0 \) and \( \nu > 0 \) the following inequality is valid
\[ \nu^2 + (1 - 2w_\nu(x))(x^2 + \nu^2) > 0, \]
that is,
\[ y_\nu(x) > \sqrt{x^2 + \nu^2} - \frac{x^2 + 2\nu^2}{2x^2 + 2\nu^2}. \]
By using the inequality [15, p. 572]
\[ -\frac{x^2}{2(x^2 + \nu^2)^{3/2}} < y_\nu(x) - \sqrt{x^2 + \nu^2} + \frac{x^2}{2(x^2 + \nu^2)} \]
we can prove that (2.14) is valid for all \( x > 0 \) and \( \nu \geq 1/2 \) such that \( x^2 \leq 2\nu^3(\nu + \sqrt{\nu^2 + 1}) \). All the same, we were not able to prove that the function \( w_\nu \) is increasing on \((0, \infty)\) for all \( \nu > 0 \). Computer experiments suggest that the graph of \( w_\nu \) intersects once the straight line \( y = 1/2 \), and because \( w_\nu(x) \) tends to 1/2 as \( x \) tends to infinity, there exists an \( x_\nu > 0 \) (depending on \( \nu \)) such that \( w_\nu \) is increasing on \((0, x_\nu]\) and decreasing on \([x_\nu, \infty)\). Here we used the asymptotic formula [19, p. 276]
\[ \frac{x I_{\nu/2}'(x)}{I_{\nu/2}(x)} \sim x - \frac{1}{2} + \frac{4\nu^2 - 1}{8x} - \ldots, \]
which holds for large values of \( x \) and fixed \( \nu \), to prove that \( \lim_{x \to \infty} w_\nu(x) = 1/2 \).

Now, let us consider the function \( x \mapsto \lambda_\nu(x) = y_\nu(x) - \sqrt{x^2 + (\nu + 1)^2} \). Based on numerical experiments we believe, but are unable to prove the following result: if \( \nu \geq -1/2 \) and \( x > 0 \), then \( \lambda_\nu(x) > 0 \), and equivalently the Turán type inequality
\[ (2.15) \quad \frac{1}{\sqrt{x^2 + (\nu + 1)^2}} \cdot I_{\nu/2}'(x) < I_{\nu/2}'(x) - I_{\nu-1}(x)I_{\nu+1}(x) \]
is valid.

Observe that, if the inequality (2.15) would be valid, then it would improve the left-hand side of (2.2) for \( x > 0 \) and \( \nu \geq -1/2 \) such that \( \nu^2 + 2\sqrt{x^2 + (\nu + 1)^2} \geq 1/2 \). Observe also that (2.15) is better than the left-hand side of (2.11) for all \( \nu \geq -1/2 \) and \( x > 0 \). Moreover, (2.15) is sharp as \( x \to 0 \) or as \( x \to \infty \), and it can be shown that the relative error of the bound \( 1/\sqrt{x^2 + (\nu + 1)^2} \) in (2.15) tends to zero as \( x \) tends to infinity. On the other hand, by using the inequalities [52, eq. (72)]
\[ (2.16) \quad \sqrt{x^2 + (\nu + 1)^2} - 1 < \frac{x I_{\nu/2}'(x)}{I_{\nu/2}(x)} < \sqrt{x^2 + \left(\nu + \frac{1}{2}\right)^2} - \frac{1}{2}, \]
where \( \nu \geq -1 \) on the left-hand side and \( \nu \geq -1/2 \) on the right-hand side, it is clear that \( \lambda_\nu \) maps \((0, \infty)\) into \((-1, -1/2)\) when \( \nu \geq -1/2 \). Moreover, by using the power series representation of \( y_\nu \) and the above asymptotic formula for \( y_\nu \), we obtain that \( \lim_{x \to 0} \lambda_\nu(x) = -1 \) and \( \lim_{x \to \infty} \lambda_\nu(x) = -1/2 \). Observe that, if the inequality (2.15) is true, then for all \( \nu \geq -1/2 \) and \( x > 0 \) we have
\[ \left[ \sqrt{x^2 + (\nu + 1)^2} \right]' < y_\nu(x) \]
and consequently
\[ \int_0^x \left[ \sqrt{t^2 + (\nu + 1)^2} \right]' dt < \int_0^x y_\nu(t) dt, \]
which is equivalent to the left-hand side of (2.10). We also mention here that the left-hand side of (2.10) actually can be proved also by using the properties of the function \( \lambda_\nu \). More precisely, in view of the power series representation of \( y_\nu(x) \) and of \( \sqrt{x^2 + (\nu + 1)^2} \), we obtain

\[
\lambda_\nu(x) = -1 + \frac{x^4}{8(\nu + 1)^3(\nu + 2)} - \ldots
\]

and then clearly the function \( \lambda_\nu \) is strictly increasing and convex for small values of \( x \). Now, let \( x_1 \) be the smallest positive value of \( x \) for which \( \lambda_\nu(x) = -1 \). Then \( \lambda'_\nu(x_1) \leq 0 \), that is, in view of (2.10)

\[
x_1 \lambda'_\nu(x_1) = -\lambda'_\nu(x_1) - 2(\nu + 1) \sqrt{x_1^2 + (\nu + 1)^2} \leq 0
\]

or equivalently \( x^2 + 2(\nu + 1)^2 \leq 2(\nu + 1) \sqrt{x_1^2 + (\nu + 1)^2} \). The above inequality can be rewritten as \( \sqrt{x_1^2 + (\nu + 1)^2} \leq \nu + 1 \) or \( x^2 \leq 0 \), which is a contradiction. Thus, the graph of \( \lambda_\nu \) does not intersect the straight line \( y = -1 \) and hence \( \lambda'_\nu(x) > -1 \) for all \( \nu > -1 \) and \( x > 0 \).

Finally, observe that to prove (2.15) it would enough to show that the inequality

\[
(2.17) \quad \frac{x I'_\nu(x)}{I_\nu(x)} < \frac{x}{\nu - 1 + \sqrt{x^2 + (\nu - 1)^2} - \frac{1}{2} x^2 + 2(\nu + 1)^2}
\]

is valid for all \( x > 0 \) and \( \nu \geq 1/2 \). Namely, since \( \lambda_\nu \) is increasing for small values of \( x \), the first extreme, if any, should be a maximum. But, according to (2.10), (2.13) and (2.17), for such values of \( x \) when \( \lambda'_\nu(x) = 0 \), that is,

\[
y'_\nu(x) = \frac{x}{\sqrt{x^2 + (\nu + 1)^2}}
\]

we would have

\[
(x^2 + (\nu + 1)^2)^{3/2} \lambda''_\nu(x) = 2(x^2 + (\nu + 1)^2)^{3/2} - 2(x^2 + (\nu + 1)^2) y_\nu(x) - (x^2 + 2(\nu + 1)^2) > 0,
\]

which would be a contradiction.

3. Turán type inequalities for modified Bessel functions of the second kind

This section is devoted to the study of Turán type inequalities for modified Bessel functions of the second kind, and our aim is to obtain analogous results to those given in Section 2. Recently, in order to prove (1.2), Segura proved the next Turán type inequalities [52, eqs. (50), (56)]

\[
(3.1) \quad -\frac{2K^2_\nu(x)}{\nu - 1 + \sqrt{x^2 + (\nu - 1)^2}} < K^2_\nu(x) - K_{\nu-1}(x) K_{\nu+1}(x) < -\frac{K^2_\nu(x)}{\nu - \frac{1}{2} + \sqrt{x^2 + (\nu - \frac{1}{2})^2}},
\]

where \( x > 0 \) and \( \nu \geq 1/2 \). Observe that by changing \( \nu \) with \(-\nu\) in (3.1), and using (3.1) we obtain

\[
(3.2) \quad -\frac{2K^2_\nu(x)}{|\nu| - 1 + \sqrt{x^2 + (|\nu| - 1)^2}} < K^2_\nu(x) - K_{|\nu|-1}(x) K_{\nu+1}(x) < -\frac{K^2_\nu(x)}{|\nu| - \frac{1}{2} + \sqrt{x^2 + (|\nu| - \frac{1}{2})^2}},
\]

where \( x > 0 \) and \( |\nu| \geq 1/2 \). These inequalities are analogous to (2.2). Observe that from (3.2) the following inequalities can be obtained, which are analogous to (2.2)

\[
(3.3) \quad -\frac{2}{x + |\nu| - 1} \cdot K^2_\nu(x) < K^2_\nu(x) - K_{\nu-1}(x) K_{\nu+1}(x) < -\frac{1}{x + 2|\nu| - 1} \cdot K^2_\nu(x),
\]

where \( x > 0 \) and \( |\nu| \geq 1/2 \). Recall that for [5] p. 260

\[
\phi_\nu(x) = 1 - \frac{K_{\nu-1}(x) K_{\nu+1}(x)}{K^2_\nu(x)}
\]

we have \( \lim_{x \to \infty} \phi_\nu(x) = 0 \), where \( \nu \geq 0 \), and \( \lim_{x \to 0} \phi_\nu(x) = 1/(1 - \nu) \), provided \( \nu > 1 \). Thus, the inequalities (3.2) and (3.3) are sharp as \( x \) approaches infinity, while for \( |\nu| > 1 \) the left-hand side of (3.2) is also sharp as \( x \to 0 \). Moreover, by using the relation (1.9), the Turán type inequalities (3.3) can be rewritten as

\[
-\frac{2x}{x + |\nu| - 1} < \left[ \frac{x K'_\nu(x)}{K_\nu(x)} \right] < -\frac{x}{x + 2|\nu| - 1}
\]

or

\[
[-2x + 2(|\nu| - 1) \ln(x + |\nu| - 1)]' \left[ \frac{x K'_\nu(x)}{K_\nu(x)} \right] < [-x + 2|\nu| - 1] \ln(x + 2|\nu| - 1)'.
\]
Now, having in mind the fact\footnote{To prove this observe that it is enough to consider the case when $\nu \geq 0$. When $\nu = 0$ by using the asymptotic relations \[^{11}\text{p. 375}\] $K_0(x) \sim -\ln x$ and $2K_\nu(x) \sim \Gamma(\nu)(x/2)^{-\nu}$, where $\nu > 0$ and $x \to 0$, we obtain that $xK'_0(x)/K_0(x) = -xK'_1(x)/K_0(x)$ tends to zero as $x \to 0$. Similarly, for $\nu > 0$ by using the asymptotic relations \[^{11}\text{p. 375}\] $\Gamma(\nu + 1)I_\nu(x) \sim (x/2)^\nu$ and $2K_\nu(x) \sim \Gamma(\nu)(x/2)^{-\nu}$, where $\nu > 0$ and $x \to 0$, we obtain that $P_\nu(x)$ tends to $1/(2\nu)$ as $x \to 0$, and consequently $1/P_\nu(x)$ tends to $2\nu$ as $x \to 0$. Now, by using the Wronskian recurrence relation $xI'_\nu(x)/I_\nu(x) \sim -xK'_\nu(x)/K_\nu(x) = 1/P_\nu(x)$ and the fact that $I'_\nu(x)/I_\nu(x)$ tends to $\nu$ as $x \to 0$, we obtain that $xK'_\nu(x)/K_\nu(x)$ tends to $-\nu$ as $x \to 0$, as required. Alternatively, this can be proved directly by using the asymptotic formula $2K_\nu(x) \sim \Gamma(\nu)(x/2)^{-\nu}$ and the recurrence relation $xK'_\nu(x)/K_\nu(x) = \nu - xK'_{\nu+1}(x)/K_{\nu}(x)$.} that $xK'_\nu(x)/K_\nu(x)$ tends to $-|\nu|$ as $x \to 0$, we obtain the new inequalities
\[
\int_0^x [-2t + 2|\nu| - 1] \ln(t + |\nu| - 1)]' dt < \int_0^\infty \left[\frac{tK'_\nu(t)}{K_\nu(t)}\right]' dt < \int_0^x [-t + 2|\nu| - 1] \ln(t + 2|\nu| - 1)]' dt,
\]
or equivalently,
\[
-2x - |\nu| + 2(|\nu| - 1) \ln \left(1 + \frac{x}{|\nu| - 1}\right) < \frac{xK'_\nu(x)}{K_\nu(x)} < -x - |\nu| + 2(|\nu| - 1) \ln \left(1 + \frac{x}{2|\nu| - 1}\right),
\]
where $|\nu| \geq 1/2$ and $x > 0$. Observe that both inequalities are sharp as $x \to 0$.

The next result is analogous to (3.5).

**Theorem 2.** Let $\mu = \nu^2 - 1/4$. If $|\nu| \geq 1/2$ and $x > 0$, then the next Turán type inequalities are valid
\[
-\frac{1}{x} \cdot K'_\nu(x) \leq K^2_\nu(x) - K_{\nu-1}(x)K_{\nu+1}(x) \leq -\left(1 - \frac{\mu}{x^2}\right) \frac{1}{x} \cdot K^2_\nu(x).
\]

Moreover, if $|\nu| < 1/2$ and $x > 0$, then the above inequalities are reversed, that is,
\[
-\left(1 - \frac{\mu}{x^2}\right) \frac{1}{x} \cdot K^2_\nu(x) < K^2_\nu(x) - K_{\nu-1}(x)K_{\nu+1}(x) < -\frac{1}{x} \cdot K^2_\nu(x).
\]

In (3.4) we have equality for $\nu = 1/2$. The left-hand side of (3.4) is sharp as $x \to 0$ when $1/2 \leq |\nu| \leq 1$, while (3.5) is sharp as $x \to 0$ for all $|\nu| < 1/2$. Each of the above inequalities are sharp as $x \to \infty$.

Observe that the left-hand side of the Turán type inequality (3.4) for $x \geq |\nu| - 1 > 0$ is better than the left-hand side of the inequality (1.2), while for $x \geq |\nu| - 1 \geq -1/2$ is better than the left-hand side of (3.3). For $x \geq 4(|\nu| - 1)/3 \geq -2/3$ the left-hand side of (3.4) is also better than the left-hand side of (3.2). We also note that the right-hand side of (3.4) is better than the right-hand side of (1.2) for $x \geq \sqrt{\mu}$ and $|\nu| \geq 1/2$. The upper bound in (3.4) is also better than the upper bound in (3.3) when $20x \geq \mu + \sqrt{\mu^2 + 4|\nu|^2}$ for $\alpha = 2|\nu| - 1 > 0$. Because of their different nature, it is not easy to compare the upper bounds in (3.2) and (3.4). However, numerical experiments suggest that for large values of $x$ the upper bound in (3.4) is better than the upper bound in (3.2). This is illustrated also on Fig. 2.

We note that in this figure the bounds in (3.1) are considered as bounds for $\phi_\nu(x)$, that is, they are understood in the sense that the lower bound is
\[
-\frac{2}{\nu - 1 + \sqrt{x^2 + (\nu - 1)^2}}.
\]
while the upper bound is
\[
-\frac{1}{\nu - \frac{1}{2} + \sqrt{x^2 + (\nu - \frac{1}{2})^2}}.
\]

The bounds in (3.4) in Fig. 2 have the same meaning.

Now, let us discuss the tightness of the bounds in (3.5). Having in mind from the introduction the fact that $\phi_\nu(x) < 0$ for all $|\nu| < 1/2$ and $x > 0$ and in view of the notations
\[
r_\nu(x) = -\frac{1}{x} + \frac{\mu}{x^2} \quad \text{and} \quad s(x) = -\frac{1}{x},
\]
the inequality (3.5) can be rewritten as
\[
\frac{x^2}{x^2 - \mu} = \frac{s(x)}{r_\nu(x)} < \frac{s(x)}{\phi_\nu(x)} < 1 \quad \text{or} \quad \frac{x^2 - \mu}{x^2} = \frac{r_\nu(x)}{s(x)} > \frac{r_\nu(x)}{\phi_\nu(x)} > 1.
\]

These inequalities actually imply that $s(x)/\phi_\nu(x)$ and $r_\nu(x)/\phi_\nu(x)$ tend to 1 as $x \to \infty$, and consequently the relative errors
\[
\frac{s(x) - \phi_\nu(x)}{\phi_\nu(x)} \quad \text{and} \quad \frac{r_\nu(x) - \phi_\nu(x)}{\phi_\nu(x)}
\]
and if we use the asymptotic formula \[1, p. 378\] the asymptotic relation for \(K\) of \(\nu\) also have the same property that tend to zero as the argument approaches infinity. Moreover, the relative errors of the bounds for the Turánian of the modified Bessel function of the second kind in the right-hand side of inequalities (3.2) and (3.3) have the same property. Observe that these properties of the bounds in (3.2), (3.3) and (3.4) can be proved also by using the corresponding asymptotic formula for \(K\). For example, the inequality (3.4) can be rewritten as \(\phi_\nu(x) > -1/x = s(x)\), and if we use the asymptotic formula \[11, p. 378\]

\[
K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \left[ 1 + \frac{4\nu^2 - 1}{1!(8x)} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{2!(8x)^2} + \ldots \right],
\]

which holds for large values of \(x\) and for fixed \(\nu\), one has \(\lim_{x \to \infty} \phi_\nu(x)/s(x) = 1\) and consequently for the relative error we have \(\lim_{x \to \infty} [s(x) - \phi_\nu(x)]/\phi_\nu(x) = 0\), as we required. In other words, the upper bounds in the Turán type inequalities (3.2) and (3.3), and also the lower and upper bounds in (3.4) for large values of \(x\) are quite tight.

**Proof of Theorem 2.** First recall that the function \(\nu \mapsto K_\nu(x)\) is even, that is, we have \[58, p. 79\] \(K_{-\nu}(x) = K_\nu(x)\). Because of this, without loss of generality, it is enough to prove the inequality (3.4) for \(\nu \geq 1/2\) and the inequality (3.3) for \(0 \leq \nu < 1/2\). Recall also that by using Ismail’s formula \[28, p. 583\], \[29, p. 356\]

\[
\frac{K_{\nu-1}(\sqrt{x})}{\sqrt{\pi K_\nu(\sqrt{x})}} = 4 \frac{\pi}{2^2} \int_{0}^{\infty} \frac{\gamma_\nu(t) dt}{x + t^2}, \quad \text{where} \quad \gamma_\nu(t) = \frac{t^{-1}}{J_5^2(t) + Y_5^2(t)},
\]

where \(x > 0\), \(\nu \geq 0\) and \(J_\nu\) and \(Y_\nu\) stand for the Bessel function of the first and second kinds, it can be shown that \[5, p. 260\] \[6\]

\[
(3.6) \quad \phi_\nu(x) = \frac{1}{x} \left[ x K'_\nu(x) / K_\nu(x) \right] = -\frac{8}{\pi^2} \int_{0}^{\infty} \frac{t^2 \gamma_\nu(t) dt}{(x^2 + t^2)^2}.
\]

\[6\] It should be mentioned here that in \[3, p. 260\] the expressions

\[
\phi_\nu(x) = -\frac{4}{\pi^2} \int_{0}^{\infty} \frac{t^2 (x^2 + t^2)^2 + \gamma_\nu(t) dt}{(x^2 + t^2)^2}, \quad \phi'_\nu(x) = \frac{8}{\pi^2} \int_{0}^{\infty} \frac{x (x^2 + t^2)^2 + \gamma_\nu(t) dt}{(x^2 + t^2)^2}
\]

are not correct and should be rewritten as

\[
\phi_\nu(x) = -\frac{8}{\pi^2} \int_{0}^{\infty} \frac{t^2 \gamma_\nu(t) dt}{(x^2 + t^2)^2}, \quad \phi'_\nu(x) = \frac{32}{\pi^2} \int_{0}^{\infty} \frac{x t^2 \gamma_\nu(t) dt}{(x^2 + t^2)^3}.
\]

See also \[10\] for more details.
On the other hand, it is known that \([58, p. 446]\) the function \(t \mapsto 1/\gamma_\nu(t)\) is decreasing on \((0, \infty)\) for all \(\nu > 1/2\) and is increasing on \((0, \infty)\) for all \(0 \leq \nu < 1/2\). Consequently, we obtain that \(\gamma_\nu(t) < \pi/2\) for all \(t > 0\) and \(\nu > 1/2\). Moreover, \(\gamma_\nu(t) > \pi/2\) for all \(t > 0\) and \(0 \leq \nu < 1/2\). Thus, we have

\[
\phi_\nu(x) > -\frac{4}{\pi} \int_0^\infty \frac{t^2 dt}{(x^2 + t^2)^2} = -\frac{1}{x},
\]

where \(\nu > 1/2\) and \(x > 0\). The same proof works in the case \(0 \leq \nu < 1/2\). The only difference is that the above inequality is reversed. Now, by using for \(\nu = 1/2\) the relations \([58, p. 79]\)

\[
K_{\nu+1}(x) - K_{\nu-1}(x) = \frac{2\nu}{x} K_\nu(x), \quad K_\nu(x) = K_{-\nu}(x),
\]

we obtain \(\phi_{1/2}(x) = -1/x\). This completes the proof of the left-hand side of \((3.3)\) and of the right-hand side of \((3.5)\). We note that there is another proof for these results. Namely, in view of the Nicholson formula \([58]\)

\[
J_\nu^2(t) + Y_\nu^2(t) = \frac{8}{\pi^2} \int_0^\infty K_0(2t \sin s) \cosh(2\nu s) ds,
\]

the function \(\nu \mapsto \gamma_\nu(t)\) is decreasing on \([0, \infty)\) for all \(t > 0\) fixed. This in turn implies that the function \(\nu \mapsto \phi_\nu(x)\) is increasing on \([0, \infty)\) for all \(x > 0\) fixed. Consequently, \(\phi_\nu(x) \geq \phi_{1/2}(x) = -1/x\) for all \(x > 0\) and \(\nu \geq 1/2\), and \(\phi_\nu(x) < \phi_{1/2}(x) = -1/x\) for all \(x > 0\) and \(0 \leq \nu < 1/2\).

Now, let us focus on the right-hand side of \((3.4)\) and on the left-hand side of \((3.5)\). Observe that the inequality \([24, eq. (4.6)]\)

\[
t \left(1 - \frac{\mu}{t^2}\right) [J_\nu^2(t) + Y_\nu^2(t)] < \frac{2}{\pi},
\]

where \(t > 0\) and \(\nu > 1/2\), is equivalent to

\[
\gamma_\nu(t) > \left(1 - \frac{\mu}{t^2}\right) \frac{\pi}{2}.
\]

Since

\[
t \left[ J_{1/2}^2(t) + Y_{1/2}^2(t) \right] = t \left[ \frac{2}{\pi t^2} \sin^2 t + \frac{2}{\pi t^2} \cos^2 t \right] = \frac{2}{\pi},
\]

for \(\nu = 1/2\) in inequalities \((3.7)\) and \((3.8)\) we have equality. These in turn imply that for all \(x > 0\) and \(\nu \geq 1/2\) we have

\[
\phi_\nu(x) < -\frac{4}{\pi} \int_0^\infty \frac{t^2 dt}{(x^2 + t^2)^2} + \frac{4\mu}{\pi} \int_0^\infty \frac{dt}{(x^2 + t^2)^2} = -\frac{1}{x} + \frac{\mu}{x^3},
\]

with equality when \(\nu = 1/2\), that is, \(\mu = 0\). The same proof works in the case \(0 \leq \nu < 1/2\). The only difference is that the inequality \((3.7)\) is reversed, according to \([24, eq. (4.7)]\), and then \((3.8)\) is reversed too.

Finally, let us discuss the sharpness of inequalities. Observe that \((3.4)\) and \((3.5)\) can be rewritten as

\[
-\frac{1}{x} \leq \phi_\nu(x) \leq -\frac{1}{x} + \frac{\mu}{x^3} \quad \text{and} \quad -\frac{1}{x} + \frac{\mu}{x^3} < \phi_\nu(x) < -\frac{1}{x}.
\]

Since for all \(\nu \geq 0\) we have \([5, p. 260]\) \(\lim_{x \to \infty} \phi_\nu(x) = 0\), clearly both of the above inequalities are sharp as \(x \to \infty\). Moreover, because \([5, p. 260]\) \(\lim_{x \to 0} \phi_\nu(x) = 1/(1 - \nu)\), provided \(\nu > 1\), the inequality \((3.4)\) is not sharp as \(x \to 0\). But using the asymptotic relation \([58, p. 375]\)

\[
2K_\nu(x) \sim \Gamma(\nu)(x/2)^{-\nu} \text{ as } x \to 0
\]

and the using the asymptotic relation \([58, p. 375]\)

\[
\phi_\nu(x) \sim 1 - \frac{\Gamma(1 - \nu)\Gamma(1 + \nu)}{\Gamma(2\nu)} \left(\frac{x}{2}\right)^{2\nu - 2},
\]

and then we have \(\lim_{x \to 0} \phi_\nu(x) = -\infty\). Combining the above asymptotic relation with \([58, p. 375]\)

\[
K_\nu(x) \sim -\ln x, \text{ we obtain } \phi_1(x) \sim 1 + \ln x, \text{ and thus } \lim_{x \to 0} \phi_1(x) = -\infty. \text{ These show that the left-hand side of the inequality } (3.3) \text{ is sharp as } x \to 0 \text{ when } 1/2 \leq |\nu| \leq 1, \text{ while } (3.5) \text{ is sharp as } x \to 0 \text{ for all } |\nu| < 1/2. \]
In what follows, without loss of generality, we assume that $0 < \nu \leq 1/2$ and in (3.10) for $\nu = 1/2$ we have equality, and by using the symmetry with respect to $\nu$, we conclude that (3.10) is valid for all $|\nu| \geq 1/2$ and $x > 0$. Moreover, it is worth to note here that the left-hand side of the Turán type inequality (3.4) implies the inequality (3.11). More precisely, in view of (1.9) the left-hand side of (3.4) is equivalent to $z_\nu(x) \geq -1$ for all $|\nu| \geq 1/2$ and $x > 0$. This implies that $x z_\nu'(x) \geq -\sqrt{x^2 + \mu}$ for all $\mu = \nu^2 - 1/4 \geq 0$ and $x > 0$. On the other hand, since $K_\nu$ satisfies the modified Bessel differential equation, the function $z_\nu$ satisfies

$$z_\nu(x) = \frac{x K_\nu'(x)}{K_\nu(x)} > -\sqrt{x^2 + \nu^2 - \frac{1}{4} - \frac{1}{2}},$$

and consequently

$$z_\nu^2(x) \leq x^2 + \nu^2 + \sqrt{x^2 + \mu} = \left(\sqrt{x^2 + \mu} + \frac{1}{2}\right)^2,$$

which implies (3.11).

Similar bounds to (3.10) for the logarithmic derivative of $K_\nu$ were given also in [45, 52] for $\nu \geq 0$ and $x > 0$. For $\nu \geq 1/2$ the inequality (3.10) improves [52, eq. (74)]

$$\frac{x K_\nu'(x)}{K_\nu(x)} > -\sqrt{x^2 + \nu^2 - \frac{1}{2}},$$

and also improves [45, eq. (22)]

$$\frac{x K_\nu'(x)}{K_\nu(x)} > -\sqrt{x^2 + \nu^2 - \frac{1}{2}}.$$

In addition, for $\nu \geq 1/2$ and $x^2 \geq 3\nu^2 - 4\nu + 5/4$ the inequality (3.10) improves [52, eq. (75)]

$$\frac{x K_\nu'(x)}{K_\nu(x)} > -\sqrt{x^2 + (\nu - 1)^2} - 1.$$

Now, we are going to improve the left-hand side of the inequality (3.5). Observe that (3.11) improves the reversed form of (3.8) and hence the left-hand side of (3.11) improves the left-hand side of (3.5). We note that the expression on the left-hand side of (3.13) divided by $K^{2}_\nu(x)$ provides a tight lower bound for $\phi_\nu(x)$, its relative error tends to zero as $x$ approaches infinity.

**Theorem 3.** If $\mu = \nu^2 - 1/4 \leq 0$ and $x > \sqrt{-\mu}$, then the next Turán type inequality is valid

$$\gamma_\nu(t) \leq \frac{2\nu}{\nu - 1}.$$
Theorem 4. If $|\nu| \geq 1/2$ and $x > 0$, then the following Turán type inequality holds

\begin{equation}
K_{\nu}^2(x) - K_{\nu - 1}(x)K_{\nu + 1}(x) \leq - \frac{1}{\sqrt{x^2 + \mu^2} - \frac{1}{4}} \cdot K_{\nu}^2(x).
\end{equation}

In (3.15), we have equality for $\nu = 1/2$. This inequality is sharp as $x \to \infty$.

Observe that (3.15) improves the right-hand side of (3.1) for all $|\nu| \geq 3/2$ and $x > 0$, and it is clearly better than the right-hand side of (3.4) for all $|\nu| > 1/2$ and $x > 0$. Moreover, by using the asymptotic formula for $K_{\nu}(x)$ for large $x$, as above, it can be proved that the relative error of the bound in (3.15) has the property that tends to zero as $x$ tends to infinity. Finally, observe that by using (1.9), the inequality (3.15) can be rewritten as

\[ \left[ xK_{\nu}'(x) \right]' \leq - \left[ \sqrt{x^2 + \mu} \right]', \]

which implies

\[ \int_0^x \left[ tK_{\nu}'(t) \right]' dt \leq - \int_0^x \left[ \sqrt{t^2 + \mu} \right]' dt, \]

that is,

\[ \frac{xK_{\nu}'(x)}{K_{\nu}(x)} \leq - \sqrt{x^2 + \mu} + \sqrt{\mu} - \nu, \]

where $\mu = \nu^2 - 1/4 \geq 0$ and $x > 0$. Observe that for all $|\nu| \geq 1/2$ and $x > 0$ this inequality is better than (3.4), however, it is weaker than the left-hand side of the inequality [52, eq. (75)]

\begin{equation}
(3.16)
\end{equation}

Proof of Theorem 4 Since $\phi_{1/2}(x) = -1/x$, in (3.15) for $\nu = 1/2$ we have equality. Thus, without loss of generality, we suppose that $\nu > 1/2$. Because of (1.9) to prove (3.15) we need to show that the function $x \mapsto q_{\nu}(x) = z_{\nu}(x) + \sqrt{x^2 + \mu}$, where $\mu = \nu^2 - 1/4$, satisfies $q_{\nu}'(x) < 0$ for all $\nu > 1/2$ and $x > 0$. By using (3.16) it results that

\[ q_{\nu}(x) \leq \sqrt{x^2 + \mu} - \sqrt{x^2 + \left( \nu - \frac{1}{2} \right)^2 - \frac{1}{2}} \leq \sqrt{\mu} - \nu = \lim_{x \to 0} q_{\nu}(x) \]

for all $\nu \geq 1/2$ and $x > 0$. On the other hand, according to (3.11) we have $q_{\nu}(x) > -1/2$ for all $\nu > 1/2$ and $x > 0$. Moreover, in view of the asymptotic relation [55, eq. (20)]

\[ \frac{xK_{\nu}'(x)}{K_{\nu}(x)} \sim -x - \frac{1}{2} + \frac{4\nu^2 - 1}{8x} + \frac{4\nu^2 - 1}{8x^2} + \ldots, \]

which holds for large values of $x$ and fixed $\nu$, we obtain $\lim_{x \to 0} q_{\nu}(x) = -1/2$. In other words, for all $x > 0$ and $\nu > 1/2$ we have

\[ \lim_{x \to 0} q_{\nu}(x) > q_{\nu}(x) > \lim_{x \to \infty} q_{\nu}(x). \]

It is also clear that by using (3.8) we have $\lim_{x \to 0} q_{\nu}(x) = 0$. Thus, for small values of $x$ the function $q_{\nu}$ is decreasing. Now, suppose that $q_{\nu}'(x)$ vanish for some $x > 0$. Since $\lim_{x \to \infty} q_{\nu}(x) = -1/2$ and $\lim_{x \to 0} q_{\nu}(x) > -1/2$ for $\nu > 1/2$ it follows that $q_{\nu}'(x)$ will vanish at least one more time, and then the second extreme, if any, should be a local maximum. However, for $x$ such that $q_{\nu}'(x) = 0$, that is,

\[ z'_{\nu}(x) = -\frac{x}{\sqrt{x^2 + \mu}} \]

we have

\[ q_{\nu}'(x) = \frac{\mu}{(x^2 + \mu)^{3/2}} + \frac{2q_{\nu}(x) + 1}{\sqrt{x^2 + \mu}} > 0, \]

according to (3.10) and the relation $xz_{\nu}'(x) = 2x - (2z_{\nu}(x) + 1)z_{\nu}''(x)$, which follows from (3.11). But, this is a contradiction. Consequently, the derivative of $q_{\nu}$ does not vanish on $(0, \infty)$ and then $q_{\nu}'(x) < 0$ for all $\nu > 1/2$ and $x > 0$, as we required.

\[ \square \]

\[ \text{We note that in the left-hand side of [52, eq. (75)] it is assumed that } \nu \geq 1. \text{ However, because of [52, eq. (30)], we can suppose that } \nu \geq 1/2 \text{ in the above inequality.} \]
We note that following the steps of the above proof it can be proved that, if \( \nu \in \mathbb{R} \) and \( x > 0 \), then
\[
K_\nu^2(x) - K_{\nu-1}(x)K_{\nu+1}(x) \leq -\frac{1}{\sqrt{x^2 + \nu^2}} K_\nu^2(x). \tag{3.17}
\]
More precisely, if we suppose that \( \nu > 0 \) and consider the function \( x \mapsto t_\nu(x) = z_\nu(x) + \sqrt{x^2 + \nu^2} \), then according to \( 1.4 \) and \( 3.12 \) we have
\[
0 = \lim_{x \to 0} t_\nu(x) > t_\nu(x) > \lim_{x \to \infty} t_\nu(x) = -\frac{1}{2}.
\]
Moreover, \( \lim_{x \to 0} t_\nu'(x) = 0 \). Thus, for small values of \( x \) the function \( t_\nu \) is decreasing. Now, if we suppose that \( t_\nu'(x) \) vanish for some \( x > 0 \), then \( t_\nu'(x) \) will vanish at least one more time, and then the second extreme, if any, should be a local maximum. However, for \( x \) such that \( t_\nu'(x) = 0 \), that is,
\[
z_\nu'(x) = -\frac{x}{\sqrt{x^2 + \nu^2}}
\]
we have
\[
t_\nu''(x) = \frac{\nu^2}{(x^2 + \nu^2)^{3/2}} + \frac{2t_\nu(x)}{\sqrt{x^2 + \nu^2}} > 0,
\]
which is a contradiction. Consequently, the derivative of \( t_\nu \) does not vanish on \( (0, \infty) \) and then \( t_\nu'(x) < 0 \) for all \( \nu > 1/2 \) and \( x > 0 \). Note however, that the Turán type inequality \( 3.17 \) is weaker than \( 3.15 \) for \( |\nu| \geq 1/2 \) and \( x > 0 \), and it is also weaker than the right-hand side of the inequality \( 3.5 \) for \( |\nu| < 1/2 \) and \( x > 0 \). All the same, this result can be used to prove \( 1.4 \). Namely, in view of \( 1.9 \) the inequality \( 3.17 \) is equivalent to
\[
\left[ \frac{xK_\nu'(x)}{K_\nu(x)} \right]' < -\left[ \sqrt{x^2 + \nu^2} \right]',
\]
which implies
\[
\int_0^x \left[ \frac{tK_\nu'(t)}{K_\nu(t)} \right] dt < -\int_0^x \left[ \sqrt{t^2 + \nu^2} \right]' dt,
\]
that is, the inequality \( 1.4 \).

4. Inequalities for product of modified Bessel functions of the first and second kind

In this section we present some applications of the main results of Section 2 and 3. By definition a function \( f : [a, b] \subseteq \mathbb{R} \to (0, \infty) \) is log-convex if \( \ln f \) is convex, i.e. if for all \( x, y \in [a, b] \) and \( \lambda \in [0, 1] \) we have
\[
f(\lambda x + (1 - \lambda)y) \leq [f(x)]^\lambda [f(y)]^{1-\lambda}.
\]
Similarly, a function \( g : [a, b] \subseteq (0, \infty) \to (0, \infty) \) is said to be geometrically (or multiplicatively) convex if \( g \) is convex with respect to the geometric mean, i.e. if for all \( x, y \in [a, b] \) and \( \lambda \in [0, 1] \) we have
\[
g(x^\lambda y^{1-\lambda}) \leq [g(x)]^\lambda [g(y)]^{1-\lambda}.
\]
We note that if the functions \( f \) and \( g \) are differentiable then \( f \) is (strictly) log-convex if and only if the function \( x \mapsto f'(x)/f(x) \) is (strictly) increasing on \( [a, b] \), while \( g \) is (strictly) geometrically convex if and only if the function \( x \mapsto xg'(x)/g(x) \) is (strictly) increasing on \( [a, b] \). A similar definition and characterization of differentiable (strictly) log-concave and (strictly) geometrically concave functions also holds. Observe that the left-hand side of \( 1.1 \) together with \( 1.3 \), and the right-hand side of \( 1.2 \) together with \( 1.3 \) imply that \( I_\nu \) is strictly geometrically convex on \( (0, \infty) \) for all \( \nu > -1 \), while \( K_\nu \) is strictly geometrically concave on \( (0, \infty) \) for all \( \nu \in \mathbb{R} \), respectively. Moreover, summing up the corresponding parts of the right-hand sides of Turán type inequalities \( 2.11 \) and \( 3.15 \) and taking into account the relations \( 1.3 \) and \( 1.9 \) we obtain
\[
\left[ \frac{xP_\nu'(x)}{P_\nu(x)} \right]' = \left[ \frac{xI_\nu'(x)}{I_\nu(x)} \right]' + \left[ \frac{xK_\nu'(x)}{K_\nu(x)} \right]' < 0
\]
for all \( \nu \geq 1/2 \) and \( x > 0 \).

Consequently, the following result is valid.

**Corollary 1.** If \( \nu \geq 1/2 \), then the function \( P_\nu \) is strictly geometrically concave on \( (0, \infty) \). In particular, for all \( x, y > 0 \) and \( \nu \geq 1/2 \) we have
\[
P_\nu(\sqrt{xy}) > \sqrt{P_\nu(x)P_\nu(y)}.
\]
It is also important to note here that since for \( \omega_\nu(x) = xP_\nu(x) = xI_\nu(x)K_\nu(x) \) we have
\[
\frac{x\omega'_\nu(x)}{\omega_\nu(x)} = 1 + \frac{xP'_\nu(x)}{P_\nu(x)},
\]
the above result implies that the function \( \omega_\nu \) is also strictly geometrically concave on \((0, \infty)\) for all \( \nu \geq 1/2 \). On the other hand, since the function \( 2\omega_\nu \) is a continuous cumulative distribution function, according to [26, Proposition 7.2], it follows that the \( \omega_\nu \) is strictly log-concave on \((0, \infty)\) for all \( \nu \geq 1/2 \). This result is similar to the result of Hartman [25], who proved that \( \omega_\nu \) is strictly concave on \((0, \infty)\) for all \( \nu > 1/2 \). Since \( x \mapsto 2\omega_{1/2}(x) = 1 - e^{-2x} \) is strictly concave on \((0, \infty)\), we conclude that in fact the function \( \omega_\nu \) is strictly concave, and hence strictly log-concave on \((0, \infty)\) for all \( \nu \geq 1/2 \). We also mention here that recently in [8] it was shown that surprisingly the most common continuous univariate distributions, like the standard normal, standard log-normal (or Gibrat), Student’s t, Weibull (or Rosin-Rammler), Kumaraswamy, Fisher-Snedecor’s \( F \), gamma and Sichel (or generalized inverse Gaussian) distributions, have the property that their probability density functions are geometrically concave and consequently their cumulative distribution functions and survival functions are also geometrically concave. Taking into account the above discussion, the distribution of which cumulative distribution function \( 2\omega_\nu \) was considered by Hartman and Watson [26, Proposition 7.2] belongs also to the class of geometrically concave univariate distributions.

Observe that if we combine the inequality (1.1) with the Wronskian recurrence relation
\[
\frac{xI'_\nu(x)}{I_\nu(x)} \cdot \frac{xK'_\nu(x)}{K_\nu(x)} = \frac{1}{P_\nu(x)},
\]
then we obtain the following chain of inequalities
\[
2 \left( \frac{xK'_\nu(x)}{K_\nu(x)} \right)' < \frac{P'_\nu(x)}{P_\nu(x)} < -2 \left( \frac{xI'_\nu(x)}{I_\nu(x)} \right)',
\]
where \( \nu \geq 1/2 \) and \( x > 0 \). In other words, by using (1.5) and (1.9), for \( \nu \geq 1/2 \) and \( x > 0 \) the left-hand side of the Turán type inequality (1.1) implies that the product of modified Bessel functions of the first and second kind is strictly decreasing, which implies the right-hand side of the Turán type inequality (1.2). Thus, when \( \nu \geq 1/2 \) the left-hand side of the Turán type inequality (1.1) is stronger than the right-hand side of (1.2).

Now, let us show some Turán type inequalities for the product of modified Bessel functions of the first and second kind.

**Corollary 2.** Let \( \mu = \nu^2 - 1/4 \). If \( \nu \geq 1/2 \) and \( x > 0 \), then the next Turán type inequality is valid
\[
(4.2) \quad \frac{x^2 + \mu}{\sqrt{x^2 + \mu}} \cdot P^2_\nu(x) < P^2_\nu(x) - P_{\nu-1}(x)P_{\nu+1}(x) < \frac{P^2_\nu(x)}{x\sqrt{x^2 + \mu}}.
\]
Both of the inequalities are sharp as \( x \to \infty \).

We note that by using the inequalities (1.1) and (1.2) clearly we can deduce some Turán type inequalities for the product of modified Bessel functions. However, the inequalities obtained in this way are far from being sharp. Now, the bounds in (4.2) are sharp for large values of \( x \) and it can be shown by using the asymptotic formula for the product of modified Bessel functions of the first and second kind that the relative errors of the bounds in (4.2) tend to zero as \( x \) approaches infinity. Thus, the bounds in (4.2) are tight for large values of \( x \). This is illustrated in Fig. 3

**Proof of Corollary 2.** By using the left-hand sides of (2.2) and (3.1) we obtain
\[
\varphi_\nu(x) + \phi_\nu(x) > \frac{1}{\nu + \frac{1}{2} + \sqrt{x^2 + (\nu + \frac{1}{2})^2}} - \frac{1}{x}.
\]
Similarly, by using the left-hand side of (2.2) and the right-hand side of (3.1), one has
\[
-\varphi_\nu(x)\phi_\nu(x) > \frac{1}{x\sqrt{x^2 + \mu}} - \frac{1}{\nu + \frac{1}{2} + \sqrt{x^2 + (\nu + \frac{1}{2})^2}}.
\]
On the other hand
\[
1 - \frac{P_{
u-1}(x)P_{
u+1}(x)}{P^2_{
u}(x)} = \varphi_{\nu}(x) + \phi_{\nu}(x) - \varphi_{\nu}(x)\phi_{\nu}(x),
\]
and summing up the corresponding parts of the above inequalities the proof of the left-hand side of (4.2) is done. Now, by using the right-hand sides of (2.11) and (3.15) we obtain that
\[
\varphi_{\nu}(x) + \phi_{\nu}(x) < 0
\]
for all \(\nu \geq 1/2\) and \(x > 0\). Similarly, by using the right-hand side of (2.11) and the left-hand side of (3.4), we get
\[
-\varphi_{\nu}(x)\phi_{\nu}(x) < \frac{1}{x\sqrt{x^2 + \mu}}.
\]
These inequalities imply the right-hand side of (4.2). Now, let us focus on the sharpness when \(x \to \infty\). Clearly (4.2) can be rewritten as
\[
\frac{x - (\nu + \frac{1}{2}) - \sqrt{x^2 + (\nu + \frac{1}{2})^2}}{x\sqrt{x^2 + \mu}} < 1 - \frac{P_{\nu-1}(x)P_{\nu+1}(x)}{P^2_{\nu}(x)} < \frac{1}{x\sqrt{x^2 + \mu}}.
\]
In view of the asymptotic relation (11] p. 378]
\[
I_{\nu}(x)K_{\nu}(x) \sim \frac{1}{2x} \left[ 1 - \frac{1}{2} \frac{4\nu^2 - 1}{(2x)^2} + \frac{1 \cdot 3 (4\nu^2 - 1)(4\nu^2 - 9)}{(2x)^4} + \ldots \right],
\]
which holds for large values of \(x\) and for fixed \(\nu\), one has
\[
\lim_{x \to \infty} \left[ 1 - \frac{P_{\nu-1}(x)P_{\nu+1}(x)}{P^2_{\nu}(x)} \right] = 0,
\]
and thus the lower and upper bounds in (4.3) are sharp as \(x\) approaches infinity. \(\square\)

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Department of Economics, Babes-Bolyai University, Cluj-Napoca 400591, Romania

E-mail address: bariczoci@yahoo.com