The X-ray transform on 2-step nilpotent Lie groups of higher rank

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Dedicated to the memory of Sergio Console.

Abstract

We prove injectivity and a support theorem for the X-ray transform on 2-step nilpotent Lie groups with many totally geodesic 2-dimensional flats. The result follows from a general reduction principle for manifolds with uniformly escaping geodesics.

1 Background

The X-ray transform of a sufficiently rapidly decreasing continuous function $f$ on the Euclidean plane $\mathbb{R}^2$ is a function $\mathcal{X}f$ defined on the set of all straight lines via integration along these lines. More precisely, if $\xi$ is a straight line, given by a point $x \in \xi$ and a unit vector $\theta \in \mathbb{R}^2$ such that $\xi = x + \mathbb{R}\theta$, then

$$\mathcal{X}f(\xi) = \mathcal{X}f(x, \theta) = \int_{-\infty}^{\infty} f(x + s\theta) \, ds.$$ 

It is natural to ask about injectivity of this transform and, if yes, for an explicit inversion formula. If $f(x) = O(|x|^{-(2+\epsilon)})$ for some $\epsilon > 0$, the function $f$ can be recovered via the following inversion formula, going back to J. Radon [18] in 1917:

$$f(x) = -\frac{1}{\pi} \int_{0}^{\infty} \frac{F_x(t)}{t} \, dt,$$

where $F_x(t)$ is the mean value of $\mathcal{X}f(\xi)$ over all lines $\xi$ at distance $t$ from $x$:

$$F_x(t) = \frac{1}{2\pi} \int_{S^1} \mathcal{X}f(x + t\theta^\perp, \theta) \, d\theta,$$

where $(x, y) = (y, -x)$. Zalcman [29] gave an example of a non-trivial function $f \in C^\infty(\mathbb{R}^2)$ with $f(x) = O(|x|^{-2})$ and $\mathcal{X}f(\xi) = 0$ for all lines $\xi \subset \mathbb{R}^2$ and, therefore, the decay condition for the inversion formula is optimal.

Under stronger decay conditions, it is possible to prove the following support theorem (see [5] Thm. 2.1 or [7] Thm. I.2.6):
Theorem 1.1 (Support Theorem). Let $R > 0$ and $f \in C(\mathbb{R}^2)$ with $f(x) = O(|x|^{-k})$ for all $k \in \mathbb{N}$. Assume that $\mathcal{X}f(\xi) = 0$ for all lines $\xi$ with $d(\xi,0) > R$. Then we have $f(x) = 0$ for all $|x| > R$.

Again, the stronger decay condition is needed here by a counterexample of D.J. Newman given in Weiss [26] (see also [7, Rmk. I.2.9]). The Euclidean X-ray transform plays a prominent role in medical imaging techniques like the CT and PET (see, e.g., [12]).

The X-ray transform can naturally be generalized to other complete, simply connected Riemannian manifolds, by replacing straight lines by complete geodesics. Radon mentioned in [13] that there is an analogous inversion formula in the (real) hyperbolic plane $\mathbb{H}^2$, where the denominator in the integral of (1) has to be replaced by $\sinh(t)$ (see also [7, Thm. III.1.12(ii)]). There is also an analogue of the support theorem for the hyperbolic space (see [7, Thm. III.1.6]), valid for functions $f$ satisfying $f(x) = O(e^{-kd(x_0,x)})$ for all $k \in \mathbb{N}$ and $x_0 \in \mathbb{H}^n$.

In the case of a continuous function $f$ on a closed Riemannian manifold $X$, the domain of $\mathcal{X}f$ is the set of all closed geodesics. Continuous functions $f$ can only be recovered from their X-ray transform $\mathcal{X}f$ if the union of all closed geodesics is dense in $X$. But this condition is not sufficient as the following simple example of the two-sphere $\mathbb{S}^2$ shows. Every even continuous function $f$ on $\mathbb{S}^2$ (i.e., $f(-x) = f(x)$) can be recovered by its integrals over all great circles. This fact and a solution similar to (1) goes back to Minkowski 1911 and Funk 1913 (see [7, Section II.4.A] and the references therein). But, on the other hand, it is easy to see that $\mathcal{X}f$ vanishes for all odd functions, so the restriction to even functions is essential. For injectivity and support theorems of the X-ray transform on compact symmetric spaces $X$ other than $\mathbb{S}^n$ see, e.g., [7, Section IV.1]. Injectivity properties of the extended X-ray transform for symmetric $k$-tensors on closed manifolds (with respect to the solenoidal part) play an important role in connection with spectral rigidity (see [1]) and were proved for closed manifolds with Anosov geodesic flows (see [3, Thms 1.1 and 1.3] for $k = 0, 1$) or strictly negative curvature (see [2] for arbitrary $k \in \mathbb{N}$).

Another class of manifolds for which the X-ray transform and its extension to symmetric $k$-tensors has been studied are simple manifolds, i.e., manifolds $X$ with strictly convex boundary and without conjugate points (see [23]). An application is the boundary rigidity problem, i.e., whether it is possible to reconstruct the metric of $X$ (modulo isometries fixing the boundary) from the knowledge of the distance function between points on the boundary $\partial X$. Solenoidal injectivity is known for $k = 0, 1$ for all simple manifolds (see [14] and [1]), and for all $k \in \mathbb{N}$ for surfaces [16] and for negatively curved manifolds [14]. There are also support type theorem for the X-ray transform on simple manifolds (see [10, 11] and [23] and the references therein). A very recommendable survey with a list of open problems is [17].
2 A reduction principle for manifolds with uniformly escaping geodesics

In this article, we will only consider complete Riemannian manifolds $X$ whose geodesics escape in the sense of e.g. [27], [28], [9], in a uniform way. Simply connected manifolds without conjugate points have this property, but we like to stress that the main examples in this article will be manifolds with conjugate points. Geodesics will always be parametrized by arc length.

Definition 2.1. A Riemannian manifold $X$ has uniformly escaping geodesics if for each $r \in \mathbb{R}_0^+$ there is $P(r) \in \mathbb{R}_0^+$ such that for every geodesic $\gamma: \mathbb{R} \to X$ and every $t > P(r)$, we have $d(\gamma(t), \gamma(0)) > r$. We call $P$ an escape function of $X$.

The smallest such function $P$,

$$P(r) := \sup \{t \geq 0 \mid \exists \text{ geodesic } \gamma: \mathbb{R} \to X, d(\gamma(0), \gamma(t)) \leq r\}$$

is thus required to be finite for all $r$. After time $P(r)$ every geodesic has left a closed ball $B_r(p)$ of radius $r \in \mathbb{R}_0^+$ around its center $p \in X$. The function $P$ increases and satisfies $P(r) \geq r$. Note that $P$ may not be continuous.

Manifolds with this property must be simply connected and non-compact. As mentioned earlier, simply connected Riemannian manifolds without conjugate points have this property with escape function $P(r) = r$.

The class of compactly supported continuous functions on such a manifold is preserved under restriction to totally geodesic immersed submanifolds. Thus if $f$ is a compactly supported continuous function on $X$, say $\text{supp}(f) \subset B_r(p)$ for some $p \in X$ and $r > 0$, and $\phi: Y \to X$ a totally geodesic isometric immersion, then $f$ has compact support on $Y$ and $\text{supp}(f \circ \phi) \subset B_{Y,r}(p)$. In particular, this holds for geodesics (as 1-dimensional immersions) and the integral of $f$ over any geodesic in $X$ is thus defined.

Before we formulate the reduction principle, let us first fix some notation. The unit tangent bundle of $X$ is denoted by $SX$. For a Riemannian manifold $X$ let $C_c(X)$ be the space of all continuous functions $f: X \to \mathbb{C}$ with compact support. By $G(X)$ we denote the set of (unparametrized oriented) geodesics, i.e.

$$G(X) = \{\gamma(\mathbb{R}) \mid \gamma: \mathbb{R} \to X \text{ geodesic }\}$$

The X-ray transform of $f \in C_c(X)$ is the function $\mathcal{X} f: G(X) \to \mathbb{C}$ with

$$\mathcal{X} f(L) = \int_L f = \int_{-\infty}^{+\infty} f(\gamma(t)) dt$$

if $L = \gamma(\mathbb{R})$ and $\gamma$ a unit speed geodesic.

Definition 2.2. Let $r_0 \geq 0$ and $\sigma: [r_0, \infty) \to \mathbb{R}_0^+$ be a function. We say that the $\sigma$-support theorem holds on $X$ if for $p \in X$ and $f \in C_c(X)$, $r \in [r_0, \infty)$ we have that $\mathcal{X} f_{|G(X)\setminus B_{r_0}(p)}(p) = 0$ implies $f_{|X\setminus B_r(p)} = 0$. We say that $X$ has a support theorem if this holds for a function $\sigma$ with $\lim_{r \to \infty} \sigma(r) = \infty$. 

Remark 2.3. If $X$ has a $\sigma$-support theorem, then $X$ has a support theorem for all smaller functions as well. Moreover, we can always modify $\sigma : [r_0, \infty) \to \mathbb{R}_0^+$ to be monotone non-decreasing. If $r_0 = 0$, i.e., $\sigma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, the $\sigma$-support theorem implies injectivity of the X-ray transform.

Then we have the following reduction principle.

**Theorem 2.4.** Let $X$ be a complete, Riemannian manifold which has uniformly escaping geodesics with escape function $P$.

(i) Assume there exists, for every $x \in X$, a closed totally geodesic immersed submanifold $Y \subset X$ through $x$ such that the X-ray transform on $Y$ is injective. Then the X-ray transform on $X$ is also injective.

(ii) Let $\mu : [r_0, \infty) \to \mathbb{R}_0^+$ be a function with $\mu \geq P(0)$. Assume there exists, for every $v \in S_X$, a closed totally geodesic immersed submanifold $Y \subset X$ with $v \in SY$ such that the $\mu$-support theorem holds on $Y$. Then a $\sigma$-support theorem holds on $X$ for any function $\sigma : [r_0, \infty) \to \mathbb{R}_0^+$ with $P(\sigma(r)) \leq \mu(r)$ for all $r \geq r_0$. In particular, we can choose $\sigma$ to be unbounded if $\mu$ is unbounded.

**Proof.** (i) is obviously true by restriction since all geodesics in $Y$ are also geodesics in $X$.

For (ii), let $f \in C_c(X)$ and $r \geq r_0$. We fix a point $p \in X$ and let $\mathcal{Y}_p$ be a set of closed totally geodesic immersed submanifolds $Y$ with $\mu$-support theorem and so that each geodesic through $p$ lies in one of the $Y \in \mathcal{Y}_p$.

We then have

$$f|_{X\setminus B^X_r(p)} = 0$$

if

$$\forall Y \in \mathcal{Y}_p : f|_{Y \setminus B^Y_r(p)} = 0,$$

since, by assumption, each geodesic in $X$ is contained in some $Y$. Now, by the $\mu$-support theorem in $Y \in \mathcal{Y}_p$, we have

$$f|_{Y \setminus B^Y_r(p)} = 0$$

if

$$\mathcal{X}f|_{G(Y \setminus B^Y_{\mu(r)}(p))} = 0.$$ 

Since $X$ has uniformly escaping geodesics property, this is guaranteed if

$$\mathcal{X}f|_{G(X \setminus B^X_{\mu(r)}(p))} = 0$$

for any $s \geq 0$ with $P(s) \leq \mu(r)$. Thus $X$ has a $\sigma$-support theorem for any function $\sigma : [r_0, \infty) \to \mathbb{R}_0^+$ satisfying $P(\sigma(r)) \leq \mu(r)$.

**Remark 2.5.** If the escape function $P : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is left-continuous, i.e. $\lim_{s \to r} P(s) = P(r)$, we can choose $\sigma(r) = \sup\{s \geq 0 \mid P(s) \leq \mu(r)\}$. 

4
3 Applications of the reduction principle

In this section we demonstrate that many interesting examples can be derived by the reduction principle from $\mathbb{R}^2$ and $\mathbb{H}^2$. The X-ray transform on the euclidean and on the hyperbolic plane is injective and both have a $\mu$-support theorem with $\mu(r) = r$. This follows directly from the euclidean or hyperbolic version of Radon’s classical inversion formula \([1]\), or Theorem 1.1.

If $X = X_1 \times X_2$ is the product of two Riemannian manifolds of positive dimension with uniformly escaping geodesics, with escape functions $P_1$ and $P_2$ respectively, then $X$ has uniformly escaping geodesics with function $P$ satisfying

$$\max\{P_1(r), P_2(r)\} \leq P(r) = \sup\left\{ \sqrt{P_1(r_1)^2 + P_2(r_2)^2} \mid r_1^2 + r_2^2 = r^2 \right\} \leq P_1(r) + P_2(r).$$

Each vector $v \in S(X_1 \times X_2)$ lies in a 2-flat $F \subset X_1 \times X_2$, i.e. a totally geodesic immersed flat submanifold. By the reduction principle, the $\sigma$-support theorem holds on $X_1 \times X_2$ for any function $\sigma$ with $P(\sigma(r)) \leq r$ for all $r \in [P(0), \infty)$. Note that this result does not require that there are support theorems for the X-ray transforms on the factors $X_1$ and $X_2$.

The reduction principle can also be applied to symmetric spaces of noncompact type. These spaces have no conjugate points and each of their geodesics is contained in a flat of dimension at least 2 if their rank is at least 2. In non-compact rank-1 symmetric spaces each geodesic is contained in a real hyperbolic plane. Therefore, the reduction principle yields injectivity of the X-ray transform and a support theorem with $\sigma(r) = r$ ([6], also [7, Cor. IV.2.1]).

Another interesting family are noncompact harmonic manifolds, which do not have conjugate points. Prominent examples in this family are Damek-Ricci spaces. In [21], Rouviere used the fact that each geodesic of a Damek-Ricci space is contained in a totally geodesic complex hyperbolic plane $\mathbb{CH}^2$ to obtain a support theorem with $\sigma(r) = r$ for Damek Ricci spaces.

The main result in this article is about injectivity of the X-ray transform and a support theorem for a certain class of 2-step nilpotent Lie groups with a left invariant metric and higher rank introduced in [22]. By [13] these spaces have conjugate points. Therefore, the methods of [10] do not immediately apply to these spaces. The spaces in [22] differ also significantly from Heisenberg-type groups which do not even infinitesimally have higher rank.

3.1 2-step nilpotent Lie groups have uniformly escaping geodesics.

The Lie algebra of a 2-step nilpotent Lie algebra $\mathfrak{n}$ splits orthogonally as $\mathfrak{n} = \mathfrak{h} \oplus \mathfrak{z}$, $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$ the commutator and $\mathfrak{h} = \mathfrak{z}^\perp$ its orthogonal complement. We
can thus view $\mathfrak{z} \subset \mathfrak{so}(\mathfrak{h})$ as a vectorspace of skew symmetric endomorphisms of $\mathfrak{h}$. We have
\[ ([h, k] | z) = (zh | k) \]
for $h, k \in \mathfrak{h}$, $z \in \mathfrak{z}$. We show that 2-step nilpotent Lie groups have uniformly escaping geodesics, hence the X-ray transform for all functions with compact support is defined.

**Theorem 3.1.** Let $N$ be a simply connected 2-step nilpotent Lie group with Lie algebra $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{h}$, $\mathfrak{z} \subset \mathfrak{so}(\mathfrak{h})$. Then $N$ has uniformly escaping geodesics with a continuous escape function $P$.

**Proof.** We will prove that for each $r \in \mathbb{R}_0^+$ there is $P(r) \in \mathbb{R}_0^+$ such that every geodesic $\gamma$ with $\gamma(0) = e$ (the neutral element of $N$) we have that $d(\gamma(t), e) \leq r$ implies $t \leq P(r)$.

We denote by $\exp^\mathfrak{n}: \mathfrak{n} \to N$ the exponential map of the Lie group. Since $N$ is simply connected nilpotent this is a diffeomorphism. In particular, $(\exp^\mathfrak{n})^{-1}(B_r(e)) \subset B_{\rho(r)}^\mathfrak{n}(0)$ for some increasing continuous function $\rho(r): \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\rho(0) = 0$. We will show that there is $P(r)$ such that for every geodesic $\gamma$ in $N$ with $\gamma(0) = e$, the curve $(\exp^\mathfrak{n})^{-1} \circ \gamma$ has left $B_{\rho(r)}^\mathfrak{n}(0)$ after time $P(r)$.

From [8] for a geodesic $\gamma(t) = \exp^\mathfrak{n}(z(t) + h(t))$ with $z(t) \in \mathfrak{z}$, $h(t) \in \mathfrak{h}$, $\gamma'(0) = z_0 + h_0$, we have
\[ h''(t) = z_0 h'(t), \]
\[ z'(t) = z_0 + \frac{1}{2}[h(t), h'(t)], \]
which we need to solve subject to the initial conditions
\[ \gamma(0) = \exp^\mathfrak{n}(z(0) + h(0)) = e \text{ hence } z(0) = 0 = h(0), \]
\[ \gamma'(0) = z_0 + h_0 = z'(0) + h'(0), \]
so that $\|z_0\|^2 + \|h_0\|^2 = 1$. The solution to the first equation is
\[ h(t) = (e^{t z_0} - 1)z_0^{-1} h_0. \]
Note that this is well defined even if $z_0$ is not invertible. Inserting this into the second equation gives
\[ z'(t) = z_0 + \frac{1}{2} \left[ (e^{t z_0} - 1)z_0^{-1} h_0, e^{t z_0} h_0 \right]. \]
Taking the scalar product of this with $z_0$ gives
\[ (z'(t) | z_0) = \|z_0\|^2 + \frac{1}{2} \langle z_0 | \left[ (e^{t z_0} - 1)z_0^{-1} h_0, e^{t z_0} h_0 \right] \rangle \]
\[ = \|z_0\|^2 + \frac{1}{2} \langle z_0(e^{t z_0} - 1)z_0^{-1} h_0 | e^{t z_0} h_0 \rangle \]
\[ = \|z_0\|^2 + \frac{1}{2} \|h_0\|^2 - \frac{1}{2} \langle h_0 | e^{t z_0} h_0 \rangle. \]
since $e^{tz_0}$ is orthogonal. In order to compute $\langle z(t) \mid z_0 \rangle$, we integrate,

$$\langle z(t) \mid z_0 \rangle = t\|z_0\|^2 + \frac{t}{2}\|h_0\|^2 + \frac{1}{2}(h_0 \mid (1 - e^{tz_0})z_0^{-1}h_0).$$

It follows that

$$z(t) = tz_0 + \frac{t\|h_0\|^2 + (h_0 \mid (1 - e^{tz_0})z_0^{-1}h_0)}{2\|z_0\|^2}z_0 + w(t)$$

with $w(t) \in \mathfrak{g}$ perpendicular to $z_0$. Hence, in the norm $\| \cdot \|$ of $\mathfrak{n}$, we can estimate

$$\|z(t) + h(t)\|^2 \geq \|(e^{tz_0} - 1)z_0^{-1}h_0\|^2 + \frac{1}{4\|z_0\|^2}(2\|z_0\|^2t + \|h_0\|^2 + \langle h_0 \mid (1 - e^{tz_0})z_0^{-1}h_0 \rangle)^2.$$

We split $h = \oplus_{\lambda \in \mathbb{R}}E(z_0, i\lambda)$ into the eigenspaces of $z_0$ and let $h_{\text{max}} \in E(z_0, i\lambda) = \mathbb{C}$ be the largest component of $h_0$, $i\lambda$ the corresponding eigenvalue. Thus $|h_{\text{max}}|^2 \geq \frac{1}{\dim \mathfrak{h}}\|h_0\|^2$. Disregarding all other components, we estimate

$$\|z(t) + h(t)\|^2 \geq \left|\frac{e^{it\lambda} - 1}{i\lambda}\right|^2 |h_{\text{max}}|^2 + \frac{1}{4\|z_0\|^2}(2\|z_0\|^2t + \|h_{\text{max}}\|^2 + \text{Re}\left(\frac{1 - e^{it\lambda}}{i\lambda}\right) |h_{\text{max}}|^2)^2
= \frac{2 - 2\cos(\lambda t)}{\lambda^2} |h_{\text{max}}|^2 + \frac{1}{4\|z_0\|^2}(2\|z_0\|^2t + \left(t - \frac{\sin(\lambda t)}{\lambda}\right) |h_{\text{max}}|^2)^2
= \|z_0\|^2t^2 + |h_{\text{max}}|^2 \left(\frac{2 - 2\cos(\lambda t)}{\lambda^2} + t - \frac{\sin(\lambda t)}{\lambda}\right) + \frac{|h_{\text{max}}|^2}{4\|z_0\|^2} \left(t - \frac{\sin(\lambda t)}{\lambda}\right)^2.$$

We now consider the cases:

If $\|z_0\|^2 \geq \frac{1}{2}$: Then $\|z(t) + h(t)\|^2 \geq \frac{1}{2}t^2$.

If $\|z_0\|^2 \leq \frac{1}{2}$, then $\|h_0\|^2 = 1 - \|z_0\|^2 \geq \frac{1}{2}$, hence $|h_{\text{max}}|^2 \geq \frac{1}{2\dim \mathfrak{h}}$. We can therefore estimate

$$\|z(t) + h(t)\|^2 \geq \frac{1}{2\dim \mathfrak{h}} \left(\frac{2 - 2\cos(\lambda t)}{\lambda^2} + t - \frac{\sin(\lambda t)}{\lambda}\right) + \frac{1}{4\dim \mathfrak{h}} \left(t - \frac{\sin(\lambda t)}{\lambda}\right)^2.$$

If $\lambda = 0$ the bracket evaluates to $t^2$, hence $\|z(t) + h(t)\|^2 \geq \frac{1}{2\dim \mathfrak{h}}t^2$.

If $0 \leq t \leq \frac{1}{2\lambda\pi}$ then $\cos(\lambda t) \leq 1 - \frac{1}{2}(\lambda t)^2$. The other two summands are always nonnegative. Hence in this case,

$$\|z(t) + h(t)\|^2 \geq \frac{t^2}{2\dim \mathfrak{h}}.$$
If \( t > \frac{\pi}{2} \) then \( t - \frac{\sin(\lambda t)}{\lambda} \geq (\frac{\pi}{2} - 1) t \). Observing that the rightmost and the leftmost summand are nonnegative, we get in this case that

\[
\|z(t) + h(t)\|^2 \geq \frac{(\pi - 2)t^2}{4 \dim \mathfrak{h}}.
\]

Thus we have shown that

\[
\|z(t) + h(t)\|^2 \geq t^2 \min \left\{ \frac{1}{2}, \frac{1}{2 \dim \mathfrak{h}}, \frac{\pi - 2}{4 \dim \mathfrak{h}} \right\} = t^2 \frac{\pi - 2}{4 \dim \mathfrak{h}}.
\]

Thus the curve \((\exp^n)^{-1}(\gamma(t)) = z(t) + h(t)\) has left \( B^0_{\rho(r)}(0) \) after time \( t = P(r) := \rho(r) \sqrt{\frac{4 \dim \mathfrak{h}}{\pi - 2}} \).

\[\Box\]

### 3.2 X-ray transform on certain 2-step nilpotent Lie groups

Let \( \mathfrak{h} = \mathbb{R}^{2q} = \mathbb{C}^q \) and \( \mathfrak{z} = \mathfrak{t}_{q-1} \subset \mathfrak{su}(q) \subset \mathfrak{so}(2q) \) be the Lie algebra of the maximal torus of \( SU(q) \) and consider the 2-step nilpotent Lie group \( N_q \) with Lie algebra \( \mathfrak{n}_q = \mathfrak{z} \oplus \mathfrak{h} = \mathfrak{t}_q \oplus \mathbb{R}^{2q} \) endowed with a left invariant metric. In [22], it was shown that for every \( q \in \mathbb{N}, q \geq 3 \), the Lie group \( N_q \) has the property that each geodesic is contained in a totally geodesic immersed 2-dimensional flat submanifold. The reduction principle and Theorem 3.1 yield

**Theorem 3.2.** The X-ray transform on \( N_q \) is injective and has a support theorem.

**Remark 3.3.** Since the escape function \( P \), defined at the end of the proof of Theorem 3.1, is continuous, \( N_q \) admits a \( \sigma \)-support theorem with \( \sigma(r) = \sup \{ s \geq 0 \mid P(s) \leq r \} \), due to Remark 2.5.

Moreover, the \( \sigma \)-support theorem can be extended to general compact sets (not only metric balls) in \( N_q \). This extension is based on the following direct consequence of the classical support theorem (Theorem 1.1) for the X-ray transform on \( \mathbb{R}^2 \): Let \( K_0 \subset \mathbb{R}^2 \) be a compact set and \( \text{conv}(K_0) \subset \mathbb{R}^2 \) be its convex hull. Let \( f \in C(\mathbb{R}^2) \) with decay conditions as in Theorem 1.1. Then \( \mathcal{X} f|_{G(\mathbb{R}^2 \setminus K_0)} = 0 \) implies \( f|_{\mathbb{R}^2 \setminus \text{conv}(K_0)} = 0 \) (see [7, Cor. 1.2.8]). Using this fact, we conclude for any compact set \( K \subset N_q \), any point \( p \in N_q \), and any \( f \in C_c(N_q) \) with \( \mathcal{X} f|_{G(N_q \setminus K)} = 0 \) that

\[
f|_{N_q \setminus \text{conv}_p(K)} = 0,
\]

where

\[
\text{conv}_p(K) = \{ x \in X \mid \forall Y \in \mathcal{Y}_p \text{ with } x \in Y : x \in \text{conv}_Y(K \cap Y) \}
\]

and

- \( \mathcal{Y}_p \) is a set of totally geodesic immersions of submanifolds isometric to \( \mathbb{R}^2 \) such that each geodesic through \( p \) lies in one of the \( Y \in \mathcal{Y}_p \).
conv\(Y(Z)\) denotes the convex hull of a subset \(Z\) of \(Y \cong \mathbb{R}^2\).

The proof is a straightforward modification of the proof of Theorem 2.4. The \(\sigma\)-support theorem is then just the special case \(K = B_{\sigma(r)}(p)\), since then\(\text{conv}_p(K) \subset B_r(p)\).

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