Unified \((\alpha, \beta)\)-Flows on Triangulated Manifolds with Two and Three Dimensions

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Abstract

In this paper, we introduce a framework of \((\alpha, \beta)\)-flows on triangulated manifolds with two and three dimensions, which unifies several discrete curvature flows previously defined in the literature.

1 Background and Preliminaries

In his famous book [15], Thurston introduced the circle packing metric, which was used to study low dimensional topology. In [12], Hamilton introduced the well-known Ricci flow, which is a powerful tool to deform Riemannian metrics. More important to our subject, as a pioneering work, Chow and Luo [1] established the intrinsic connection between Hamilton’s surface Ricci flow and Thurston’s circle patterns. They first introduced the combinatorial (discrete) Ricci flows on triangulated surface. Inspired by their work, Glickenstein [10] first introduced the combinatorial Yamabe flows on triangulated 3-dimensional manifolds. Since then, discrete curvature flow has been becoming popular for its usefulness in engineering fields, especially in the Graphics and Image Processing areas.

Suppose \(M^n\) is a closed manifold with dimension \(n = 2\) or \(3\). Given a triangulation \(\mathcal{T} = \{\mathcal{T}_0, \mathcal{T}_1, \cdots, \mathcal{T}_n\}\) on \(M\), where \(\mathcal{T}_i\) represents the set of \(i\)-simplices (\(0 \leq i \leq n\)). In what follows, \((M^n, \mathcal{T})\) will be referred to as a triangulated \(n\)-manifold. All the vertices are ordered one by one, marked by \(v_1, \cdots, v_N\), where \(N = \mathcal{T}_0^\sharp\) is the number of vertices. We use \(i \sim j\) to denote that the vertices \(i\) and \(j\) are adjacent if there is an edge \(\{ij\} \in \mathcal{T}_1\).

Throughout this paper, all functions \(f : \mathcal{T}_0 \to \mathbb{R}\) will be regarded as column vectors in \(\mathbb{R}^N\). And we denote \(C(\mathcal{T}_0)\) as the set of functions defined on \(\mathcal{T}_0\).

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The most natural way to define a discrete metric on a triangulated manifold \((M^n, \mathcal{T})\) is to evaluate a length \(l_{ij}\) for each edge \(i \sim j\) directly. Alternatively, discrete metrics can also be defined on all vertices, based on which the length \(l_{ij}\) can be derived indirectly. Thurston introduced the circle packing metric for \(n = 2\), while Cooper and Rivin considered the sphere packing metric for \(n = 3\). Now we review these two definitions:

**Definition 1.1.** (Thurston’s circle packing metric) Given a triangulated surface \((M^2, \mathcal{T})\). Let \(\Phi : \mathcal{T} \rightarrow [0, \frac{\pi}{2}]\) be a function assigning each edge \(\{ij\}\) a weight \(\Phi_{ij}\). Each map \(r : \mathcal{T}_0 \rightarrow (0, +\infty)\) is called a circle packing metric.

For given \((M^2, \mathcal{T}, \Phi)\), we attach each edge \(\{ij\}\) a length

\[
l_{ij} = \sqrt{r_i^2 + r_j^2 + 2r_i r_j \cos \Phi_{ij}}. \tag{1.1}
\]

Thurston proved \([13]\) that the lengths \(\{l_{ij}, l_{jk}, l_{ik}\}\) satisfy the triangle inequality on each face \(\{ijk\} \in \mathcal{T}_2\), which ensures that the face \(\{ijk\}\) could be realized as an Euclidean triangle with lengths \(\{l_{ij}, l_{jk}, l_{ik}\}\). Thus the space of all circle packing metrics is exactly

\[\mathbb{R}^N_{>0} \triangleq (0, +\infty)^N.\]

The triangulated surface \((M^2, \mathcal{T}, \Phi)\) can be seen as gluing many Euclidean triangles coherently. However, this gluing produces singularities at vertices, which is described as discrete curvature.

**Definition 1.2.** (Discrete Gaussian curvature) Suppose \(\theta_{ijk}\) is the inner angle of triangle \(\triangle v_i v_j v_k\) at vertex \(i\), the classical discrete Gauss curvature at \(i\) is defined as

\[
K_i = 2\pi - \sum_{\{ijk\} \in \mathcal{T}_2} \theta_{ijk}, \tag{1.2}
\]

where the sum is taken over all the triangles with \(v_i\) as one of their vertices.

For discrete Gaussian curvature \(K_i\), there is a discrete version of Gauss-Bonnet identity

\[
\sum_{i \in \mathcal{T}_0} K_i = 2\pi \chi(M). \tag{1.3}
\]

**Definition 1.3.** (Cooper & Rivin’s ball packing metric) Given a triangulated surface \((M^3, \mathcal{T})\). Each map \(r : \mathcal{T}_0 \rightarrow (0, +\infty)\) derives a length

\[
l_{ij} = r_i + r_j. \tag{1.4}\]

for each \(i \sim j\). If for each \(\{i, j, k, l\} \in \mathcal{T}_3\), \(l_{ij}, l_{ik}, l_{il}, l_{jk}, l_{jl}, l_{kl}\) determines an Euclidean tetrahedron, then call \(r\) a (non-degenerate) ball packing metric.
It is pointed out [10] that a tetrahedron \( \{i, j, k, l\} \in T_3 \) generated by four positive radii \( r_i, r_j, r_k, r_l \) can be realized as an Euclidean tetrahedron if and only if

\[
Q_{ijkl} = \left( \frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_k} + \frac{1}{r_l} \right)^2 - 2 \left( \frac{1}{r_i^2} + \frac{1}{r_j^2} + \frac{1}{r_k^2} + \frac{1}{r_l^2} \right) > 0.
\]

(1.5)

Thus the space of all ball packing metrics is exactly

\[
\mathcal{M}_T = \left\{ r \in \mathbb{R}_>^N \mid Q_{ijkl} > 0, \forall \{i, j, k, l\} \in T_3 \right\}.
\]

Cooper and Rivin [2] proved that \( \mathcal{M}_T \) is a simply connected open subset of \( \mathbb{R}_>^N \), but not convex. The ball packing metric induces a piecewise linear metric which makes the curvature flat everywhere on \( M^3 - T_1 \) while singular on \( T_1 \). Thus Cooper and Rivin defined a discrete scalar curvature concentrated on all vertices.

**Definition 1.4.** (Cooper & Rivin’s discrete scalar curvature) Denote \( \alpha_{ijkl} \) as the solid angle at a vertex \( i \) in a single Euclidean tetrahedron \( \{i, j, k, l\} \in T_3 \), then the discrete scalar curvature at vertex \( i \) is

\[
K_i = 4\pi - \sum_{\{i, j, k, l\} \in T_3} \alpha_{ijkl},
\]

(1.6)

where the sum is taken over all \( \{j, k, l\} \in T_2 \) such that \( \{i, j, k, l\} \in T_3 \).

In what follows we call Cooper & Rivin’s discrete curvature as CR-curvature for short. To study the CR-curvature \( K_i \), the first of the authors and Xu [9] introduced the following 3-dimensional \( \alpha \)-flow,

\[
\frac{dr_i}{dt} = s_\alpha r_i^\alpha - K_i,
\]

(1.7)

where \( s_\alpha = \frac{\sum_i K_i r_i}{\sum_i r_i^{\alpha+1}} \). They proved that if the \( \alpha \)-flow (1.7) converges, there exists a metric \( r^* \) whose \( \alpha \)-order curvature \( K_i/r_i^\alpha \) is a constant for each \( i \in V \). On the contrary, assume \( r^* \) is a metric whose \( \alpha \)-order curvature is a constant, and the first positive eigenvalue of \( -\Delta_\alpha \) (see (4.2) for a definition) at \( r^* \) is bigger than \( \alpha s_\alpha^* \), then \( r^* \) is a asymptotically stable point of the \( \alpha \)-flow (1.7). In this paper, we shall generalize the \( \alpha \)-flow (1.7) to one with the following more universal form

\[
\frac{dg_i}{dt} = s_\alpha r_i^\alpha - K_i,
\]

where \( g_i = \ln r_i \) or \( g_i = r_i^\sigma \), \( \sigma \in \mathbb{R} \). We explain the idea behind this generalization. If we consider the conformal deformation of the discrete metric \( r_i \), we may take \( g_i = \ln r_i \). This is inspired by Chow and Luo’s pioneer work [1]. If we consider the deformation of
the conical metric, we may take take $g_i = r_i^\sigma$ as a metric (of $\sigma$-order). This is inspired by the viewpoint of Riemannian geometry. A piecewise flat metric is a singular Riemannian metric on $M$, which produces conical singularities at all vertices. For any $\sigma \in \mathbb{R}$, a metric $g$ with conical singularity at a point can be expressed as

$$g(z) = e^{f(z)} \frac{dz \wedge d\bar{z}}{|z|^{2(1-\sigma)}}$$

locally. Choosing $f(z) = -\ln \sigma^2$, then $g(z) = |dz|^2$. Comparing $r^\sigma$ with $|dz|$, the $\sigma$-order metric $r^\sigma$ may be considered as a discrete analogue of conical metric to some extent.

In this paper, we will express the flow (2.2) as

$$\frac{dr_i}{dt} = s_\alpha r_i^\beta - K_i r_i^{\beta-\alpha}$$

by leaving out a constant $\sigma$ and taking $\beta = 1 + \alpha - \sigma$ (see Definition 2.1 below). We will prove that the flow (2.2) exhibits similar convergence properties as the $\alpha$-flow (1.7) in section 4.

\section{Definition of unified ($\alpha, \beta$)-flow}

For any $\alpha \in \mathbb{R}$, the first of the authors and Xu \cite{8, 9} studied discrete circle (ball) packing metrics with all curvature $K_i/r_i^\alpha$ equal to a constant. It is easy to see that, if $K_i/r_i^\alpha \equiv s_\alpha$ for each vertex $i$, where $s_\alpha$ is a constant, then

$$s_\alpha = \frac{\sum K_i r_i^{\alpha+n-2}}{||r||^{\alpha+n-2}}.$$  \hspace{1cm} (2.1)

Following the definition in \cite{6}, we call this type of metric as “discrete $\alpha$-quasi-Einstein metric” or “constant $\alpha$-curvature metric”.

One of the most important problem we concern is to understand whether there are discrete $\alpha$-quasi-Einstein metrics in $\mathcal{M}_T$ for $n = 3$, or in $\mathbb{R}^N_0$ for $n = 2$. For the $n = 2$ case, this problem is basically resolved in \cite{8, 9}. Especially for triangulated surfaces with $\alpha \chi(M) \leq 0$, both the combinatorial-topological conditions and the analytical conditions are given for the existence of discrete $\alpha$-quasi-Einstein metrics. Furthermore, discrete $\alpha$-quasi-Einstein metrics can be obtained by evolving discrete curvature flows or minimizing discrete Ricci potentials. For the $n = 3$ case, very few combinatorial-topological conditions are known for the existence of discrete $\alpha$-quasi-Einstein metrics. We want to study discrete $\alpha$-quasi-Einstein metrics by introducing an unified ($\alpha, \beta$)-flow:

\textbf{Definition 2.1.} Let $s_\alpha$ be defined in (2.1). If $n = 2$, let $K_i$ be defined in (1.2). If $n = 3$, let $K_i$ be defined in (1.6). For any $\alpha, \beta \in \mathbb{R}$, the unified ($\alpha, \beta$)-flow is defined as

$$\frac{dr_i}{dt} = s_\alpha r_i^\beta - K_i r_i^{\beta-\alpha}.$$  \hspace{1cm} (2.2)
The prototype of the unified \((\alpha, \beta)\)-flow \(\text{(2.2)}\) is Chow-Luo’s combinatorial Ricci flow \([1]\) in dimension two, which can be expressed as \(\frac{dr_i}{dt} = K_{av}r_i - K_i r_i\). Take \((\alpha, \beta) = (0, 1)\), and \(n = 2\), then the unified \((\alpha, \beta)\)-flow \(\text{(2.2)}\) becomes Chow-Luo’s flow. The unified \((\alpha, \beta)\)-flow \(\text{(2.2)}\) modeled its form after that of the well known smooth Yamabe flow \(\frac{\partial g}{\partial t} = (s - R)g\), where \(g\) is a smooth Riemannian metric tensor, \(R\) is the scalar curvature and \(s\) is the average curvature. If we express the unified \((\alpha, \beta)\)-flow \(\text{(2.2)}\) as \(\frac{dr_i}{dt} = (s_{\alpha} - K_i) r_i^\beta\) and further take \(\beta = 1\), then it’s easy to see that the term \(K_i r_i^\alpha\) plays similar role as the term \(R\) plays in the smooth Yamabe flow. To some extent, \(K_i r_i^\alpha\) is the discrete analogy of the smooth scalar curvature \(R\).

Now it’s the time to say a bit more about the motivations to introduce the unified \((\alpha, \beta)\)-flow. The first motivation is to unify the various discrete curvature flows previously defined in the literature, see Section 3 for more details. The second motivation is to approach the following combinatorial \(\alpha\)-Yamabe problem, which is modeled after the smooth Yamabe problem and was previously raised by the first of the authors and Xu \([9]\).

**The Combinatorial \(\alpha\)-Yamabe Problem.** Given a 2-dimensional (or 3-dimensional) manifold \(M^2\) (or \(M^3\)) with triangulation \(T\), find a circle (or ball) packing metric with constant combinatorial \(\alpha\)-curvature in the combinatorial conformal class \(R_\alpha^N > 0\) (or \(\mathcal{M}_T\)).

The unified \((\alpha, \beta)\)-flow provides a natural way to approach the combinatorial \(\alpha\)-Yamabe problem. We want to deform an arbitrary metric \(r(0)\) to a discrete \(\alpha\)-quasi-Einstein metric \(r^*\), one effective way is to evolve it according to an ODE system \(r'_i(t) = f_i(r(t))\) with \(r^*\) as its critical point. Thus \(f_i(r^*)\) equals to zero. The easiest way is to choose \(f_i(r) = s_\alpha - K_i r_i^\beta\). In fact, we found this selection of \(f_i(r)\) is too restrictive, this fact motivates us to relax it as \(f_i(r) = s_\alpha r_i^\alpha - K_i r_i^\beta - s_\alpha\). In the following Theorem 4.2 we will show that if the solution to the \((\alpha, \beta)\)-flow converges, then the combinatorial \(\alpha\)-Yamabe problem is solvable.

We shall show that the unified \((\alpha, \beta)\) flow \(\text{(2.2)}\) is generally not a “normalization” of the following flow

\[
\frac{dr_i}{dt} = -K_i r_i^{\beta - \alpha} \tag{2.3}
\]

except for \((\alpha, \beta) = (0, 1)\), where the word “normalization” means that the solutions to this two flows are related by a change of the scale in space and a change of the parametrization in time. Assume the solution of the unified \((\alpha, \beta)\) flow \(\text{(2.2)}\) differs from the solution of the flow \(\text{(2.3)}\) only by a change of the scale in space and a change of the parametrization in time. Let \(t, r, K, s_\alpha\) denote the variables for the flow \(\text{(2.3)}\), and \(\hat{t}, \hat{r}, \hat{K}, \hat{s}_\alpha\) for the flow \(\text{(2.2)}\). Let \(r(t), t \in [0, T]\) be a solution of the flow \(\text{(2.3)}\), while \(\hat{r}(\hat{t}), \hat{t} \in [0, \hat{T}]\) be a solution of the flow \(\text{(2.2)}\). Set \(\hat{r}(\hat{t}) = \varphi(t)r(t)\), where \(\varphi(t) > 0\) is a scaling factor and is independent
of the vertices. Obviously, \( \tilde{K}(\tilde{t}) = K(t) \), \( \tilde{s}_\alpha(\tilde{t}) = s_\alpha(t) \varphi^{-\alpha}(t) \), and \( \tilde{s}_\alpha \tilde{r}_i^\alpha = s_\alpha r_i^\alpha \). Hence

\[
\frac{d\tilde{r}_i}{d\tilde{t}} = \frac{d(\varphi r_i)}{dt} \frac{dt}{d\tilde{t}} = \left( r_i \frac{d\varphi}{dt} - \varphi \tilde{K}_i \tilde{r}_i^{\beta-\alpha} \right) \frac{dt}{d\tilde{t}}.
\]

On the other hand, \( \tilde{r}_i \) satisfies the equation (2.2), i.e.,

\[
\frac{d\tilde{r}_i}{dt} = \tilde{s}_\alpha \tilde{r}_i^{\beta} - \tilde{K}_i \tilde{r}_i^{\beta-\alpha} = s_\alpha r_i^\beta \varphi^{\beta-\alpha} - K_i r_i^{\beta-\alpha} \varphi^{\beta-\alpha}.
\]

Comparing the above two expressions about \( \frac{d\tilde{r}_i}{d\tilde{t}} \), we obtain \( \beta - \alpha = 1 \), \( dt/d\tilde{t} = 1 \) and \( \frac{1}{\varphi} \frac{d\varphi}{dt} = s_\alpha r_i^\alpha \) for every \( i \in V \). Since \( \varphi \) is independent of \( i \in V \), we further get \( \alpha = 0 \) and \( \beta = 1 \). Hence the unified \((\alpha, \beta)\) flow (2.2) is a normalization of the flow (2.3) if and only if \( (\alpha, \beta) = (0, 1) \).

### 3 Some examples

There are several discrete curvature flows on two and three dimensional triangulated manifolds. We list some of them below. Firstly let us take a look at the 2-dimensional examples.

**Example 1.** Given a triangulated surface \((M^2, \mathcal{T})\), consider Thurston’s circle packing metric \( r \) with fixed weight \( \Phi \). Denote the discrete \( \alpha \)-curvature as \( R_{\alpha,i} = K_i/r_i^\alpha \). Notice that \( s_\alpha = 2\pi \chi(M)/\|r\|_\alpha^2 \) by discrete Gauss-Bonnet formula. Then we have the following six different discrete flows on triangulated surfaces:

1. \( \dot{u}_i = K_{av} - K_i \), where \( u_i = \ln r_i \), \( K_{av} = 2\pi \chi(M)/N \), see [1];
2. \( \dot{u}_i = s_\alpha r_i^\alpha - K_i \), where \( u_i = \ln r_i \), \( \alpha \in \mathbb{R} \), see [9];
(2)’ \( \dot{u}_i = s_\alpha r_i^\alpha - K_i \), where \( u_i = \ln r_i \), \( \alpha \in \mathbb{R} \) and \( \alpha \neq 0 \), differs from (2) by a constant \( \alpha \);
3. \( \dot{g}_i = (R_{av} - R_i)g_i \), where \( g_i = r_i^2 \), \( R_{av} = s_2 \) and \( R_i = R_{2,i} \), see [8];
4. \( \dot{u}_i = s_\alpha - R_{\alpha,i} \), where \( u_i = \ln r_i \), \( \alpha \in \mathbb{R} \), see [8];
(4)’ \( \dot{u}_i = s_\alpha - R_{\alpha,i} \), where \( u_i = \ln r_i \), \( \alpha \in \mathbb{R} \) and \( \alpha \neq 0 \), differs from (4) by a constant \( \alpha \).

Next we see the cases with dimension three.

**Example 2.** Given a triangulated 3-manifold \((M^3, \mathcal{T})\), consider Cooper and Rivin’s sphere packing metric \( r \) and \( \alpha \)-curvature \( R_{\alpha,i} = K_i/r_i^\alpha \). Then we have the following nine different discrete flows on \( M^3 \):

1. \( \dot{u}_i = -K_i \), where \( u_i = \ln r_i \), see [10];
\( \dot{r}_i = (s_0 - K_i)r_i, \) a normalization of the flow in (1), see [4];

\( \dot{r}_i = \lambda r_i - K_i, \) with \( \lambda = s_1, \) see [5];

\( \dot{r}_i = s_\alpha r_i^\alpha - K_i, \alpha \in \mathbb{R}, \) see [9];

\( \dot{u}_i = s_\alpha - R_{\alpha,i} (or \dot{r}_i = s_\alpha r_i - \frac{K_i}{r_i^\alpha}), \) where \( u_i = \ln r_i, \alpha \in \mathbb{R}, \) see [5];

\( \dot{u}_i = s_\alpha r_i^\alpha - K_i, \) where \( u_i = r_i^\alpha, \alpha \in \mathbb{R} \) and \( \alpha \neq 0, \) differs from (5) by a constant \( \alpha; \)

\( \dot{g}_i = (R_{av} - R_i)g_i, \) where \( g_i = r_i^2, R_{av} = s_2, R_i = K_i/r_i^2, \) see [8];

\( \dot{u}_i = s_\alpha r_i^\alpha - K_i, \) where \( u_i = \ln r_i, \alpha \in \mathbb{R}, \) see [6];

\( \dot{u}_i = s_\alpha r_i^\alpha - K_i, \) where \( u_i = \ln r_i^\alpha, \alpha \in \mathbb{R} \) and \( \alpha \neq 0, \) differs from (7) by a constant \( \alpha. \)

It is remarkable that all the discrete flows presented in Examples 1 and 2 can be unified by the \((\alpha, \beta)-flow\)

- For the cases in Example 1 If \( \alpha = 0 \) and \( \beta = 1, \) then the \((0, 1)\)-flow is just the flow (1). If \( \beta = \alpha + 1, \) then the \((\alpha, \alpha + 1)\)-flow is just the flow (2). If \( \beta = 1, \) then the \((\alpha, 1)\)-flow is just the flow (5). Notice that the flow (3) differs from the \((\alpha, \alpha + 1)\)-flow by a constant \( \alpha. \) Moreover, the flow (4) is a special case of the flow (6) with \( \alpha = 2. \) The flow (6) differs from the \((\alpha, 1)\)-flow by a constant \( \alpha. \)

- For the cases in Example 2 If \( \beta = \alpha = 1, \) then the \((1, 1)\)-flow is just the flow (2). If \( \beta = \alpha, \) then the \((\alpha, \alpha)\)-flow is just the flow (3). If \( \beta = 1, \) then the \((\alpha, 1)\)-flow is just the flow (4). If \( \beta = \alpha + 1, \) then the \((\alpha, \alpha + 1)\)-flow is just the flow (7). Notice that the flow (1) is in fact a type of \((\alpha, \alpha + 1)\)-flow without normalization. The flow (5) differs from the \((\alpha, 1)\)-flow by a constant \( \alpha. \) The flow (6) is a special case of the flow (5) with \( \alpha = 2. \) The last flow (8) differs from the \((\alpha, \alpha + 1)\)-flow (7) by a constant \( \alpha. \)

4 Basic properties of \((\alpha, \beta)\)-flow

Since the two most representative flows (2) and (4) in Example 1 are have intensively been studied in [8, 9], we mainly study 3-dimensional unified \((\alpha, \beta)\)-flow in the remaining of this section. It's obviously to prove

**Proposition 4.1.** Let \( \delta = \alpha - \beta + n - 1. \) If \( \delta \neq 0, \) then \( \sum_{i=1}^{N} r_i^\delta(t) \) is invariant along the \((\alpha, \beta)\)-flow. If \( \delta = 0, \) then \( \prod_{i=1}^{N} r_i(t) \) is invariant along the flow.
Theorem 4.2. The critical points of the \((\alpha, \beta)\)-flow is a constant \(\alpha\)-curvature metric, and the solution \(r(t)\) to this flow always exists locally. Moreover, if the solution \(r(t)\) converges, then the combinatorial \(\alpha\)-Yamabe problem is solvable, that is, there exists a constant \(\alpha\)-curvature metric in the combinatorial conformal class.

Proof. Note that, in \(\mathcal{M}_T\), \(K_i\) as a function of \(r = (r_1, \ldots, r_N)^T\) is smooth and hence locally Lipschitz continuous. By Picard theorem in classical ODE theory, flow has a unique solution \(r(t), t \in [0, \epsilon)\) for some \(\epsilon > 0\). The convergence of \(r(t)\) means that there exists a metric \(r^* \in \mathcal{M}_T\), such that \(r(t) \to r^*\) according to the Euclidean topology. By the classical ODE theory, \(r^*\) should be the critical point of flow, which implies the conclusion above. □

It follows natural to know when the unified \((\alpha, \beta)\)-flow converges. The following lemma is very useful:

Lemma 4.3. \([2, 11, 14]\) Suppose \((M, T)\) is a triangulated 3-manifold with sphere packing metric \(r, S = \sum K_i r_i\) is the Einstein-Hilbert-Regge functional. Then we have

\[
\nabla_r S = K. \tag{4.1}
\]

If we set

\[
\Lambda = \text{Hess}_r S = \pdv{(K_1, \ldots, K_N)}{(r_1, \ldots, r_N)} = \begin{pmatrix}
\pdv{K_1}{r_1} & \cdots & \pdv{K_1}{r_N} \\
\vdots & \ddots & \vdots \\
\pdv{K_N}{r_1} & \cdots & \pdv{K_N}{r_N}
\end{pmatrix},
\]

then \(\Lambda\) is positive semi-definite with rank \(N - 1\) and the kernel of \(\Lambda\) is the linear space spanned by the vector \(r\).

We recall the definition of \(\alpha\)-order combinatorial Laplacian.

Definition 4.4. \([8, 9]\) Given a triangulated \(n\)-manifold \((M, T)\) with \(n = 2\) or \(3\). For any \(\alpha \in \mathbb{R}\), the \(\alpha\)-order combinatorial Laplacian ("\(\alpha\)-Laplacian" for short) \(\Delta_\alpha : C(T_0) \to C(T_0)\) is defined as

\[
\Delta_\alpha f_i = \frac{1}{r_i^\alpha} \sum_{j \sim i} \left(-\pdv{K_j}{r_j} r_j\right) (f_j - f_i) \tag{4.2}
\]

for \(f \in C(T_0)\).

The \(\alpha\)-Laplacian \((4.2)\) can also be written in a matrix form

\[
\Delta_\alpha = -\Sigma^{-\alpha} \Lambda \Sigma \tag{4.3}
\]

with \(\Delta_\alpha f = -\Sigma^{-\alpha} \Lambda \Sigma f\) for each \(f \in C(T_0)\), where \(\Sigma = \text{diag}\{r_1, \ldots, r_N\}\).
Theorem 4.5. Given a triangulated manifold \((M^3, T)\). Suppose \(r^* \in S^3\) is a constant \(\alpha\)-curvature metric satisfying \(\lambda_1(\Delta_\alpha) > \alpha s^*_\alpha\), or more specifically, \(r^* \in S^3\) is a constant \(\alpha\)-curvature metric with \(\alpha s^*_\alpha \leq 0\). If \(\|r(0) - r^*\|\) is small enough, then the solution of the unified \((\alpha, \beta)\)-flow \((2.2)\) exists for all time and converges to \(r^*\).

Proof. Denote \(\Gamma_i(r) = s_\alpha r^\beta_i - K_i r^\beta_i - \alpha, 1 \leq i \leq N\), then the \((\alpha, \beta)\)-flow \((2.2)\) can be written as \(\dot{r} = \Gamma(r)\), which is an autonomous ODE system. Differentiating \(\Gamma\) with respect to \(r\), we can get

\[
D_r \Gamma(r) = \Sigma^{\beta - \alpha} \left(-\Lambda + \alpha s_\alpha \left(\Sigma^{\alpha - 1} - \frac{r^\alpha (r^\alpha)^T}{\|r\|^\alpha_{\alpha + 1}}\right) - H\right),
\]

where

\[
H = (\beta - \alpha) \Sigma^{-1} \begin{pmatrix}
K_1 - s_\alpha r^\alpha_1 \\
\vdots \\
K_N - s_\alpha r^\alpha_N
\end{pmatrix} + \frac{r^\alpha (K - s_\alpha r^\alpha)^T}{\|r\|^\alpha_{\alpha + 1}}.
\]

At constant \(\alpha\)-curvature metric points \(r^*\), \(H = 0\), thus we have

\[
D_r \Gamma|_{r^*} = \Sigma^{\beta - \alpha} \left(-\Lambda + \alpha s_\alpha \left(\Sigma^{\alpha - 1} - \frac{r^\alpha (r^\alpha)^T}{\|r\|^\alpha_{\alpha + 1}}\right)\right)|_{r^*}.
\]

(4.4)

We follow the tricks in \([7, 9]\) to derive the conclusion. Denote \(\bar{\Lambda} = \Sigma^{-\frac{1}{2}} \Lambda \Sigma^{-\frac{1}{2}}\). Then

\[-\Delta_\alpha = \Sigma^{-\alpha} \Lambda \Sigma = \Sigma^{-\frac{1}{2}} \bar{\Lambda} \Sigma^{-\frac{1}{2}},\]

which implies that

\[\lambda_1(\Delta_\alpha) = \lambda_1(\bar{\Lambda}).\]

Choose a matrix \(P \in O(N)\) such that \(P^T \bar{\Lambda} P = diag\{\lambda_1(\bar{\Lambda}), \cdots, \lambda_{N-1}(\bar{\Lambda})\}\). Suppose \(P = (e_0, e_1, \cdots, e_{N-1})\), where \(e_i\) is the \((i + 1)\)-column of \(P\). Then \(\bar{\Lambda} e_0 = 0\) and \(\bar{\Lambda} e_i = \lambda_i e_i\), \(1 \leq i \leq N - 1\), which implies \(e_0 = \frac{r^\alpha_{\alpha + 1}}{\|r\|^\alpha_{\alpha + 1}}\) and \(r^\alpha_{\alpha + 1} \perp e_i, 1 \leq i \leq N - 1\). Hence

\[
\left(I - \frac{\alpha_{\alpha + 1}}{\|r\|^\alpha_{\alpha + 1}} \frac{(\alpha_{\alpha + 1})^T}{\|r\|^\alpha_{\alpha + 1}}\right) e_0 = 0 \quad \text{and} \quad \left(I - \frac{\alpha_{\alpha + 1}}{\|r\|^\alpha_{\alpha + 1}} \frac{(\alpha_{\alpha + 1})^T}{\|r\|^\alpha_{\alpha + 1}}\right) e_i = e_i, 1 \leq i \leq N - 1.
\]

Furthermore,

\[
-D_r \Gamma|_{r^*} = \Sigma^{\beta - \alpha} \Sigma^{-\frac{\alpha - 1}{2} P \text{diag} \{0, \lambda_1(\bar{\Lambda}) - \alpha s^*_\alpha, \cdots, \lambda_{N-1}(\bar{\Lambda}) - \alpha s^*_\alpha\} P^T \Sigma^{-\frac{\alpha - 1}{2}}

= \Sigma^{\frac{\beta - \alpha}{2}} \Sigma^{\frac{\beta - 1}{2} P \text{diag} \{0, \lambda_1(\bar{\Lambda}) - \alpha s^*_\alpha, \cdots, \lambda_{N-1}(\bar{\Lambda}) - \alpha s^*_\alpha\} P^T \Sigma^{\frac{\beta - 1}{2}} \Sigma^{-\frac{\beta - \alpha}{2}}

\sim (\Sigma^{\frac{\beta - 1}{2} P} \text{diag} \{0, \lambda_1(\bar{\Lambda}) - \alpha s^*_\alpha, \cdots, \lambda_{N-1}(\bar{\Lambda}) - \alpha s^*_\alpha\}) (\Sigma^{\frac{\beta - 1}{2} P})^T.
\]

This shows that the eigenvalue of \(D_r \Gamma|_{r^*}\) are all negative when restricted to the hypersurface \(\sum_{i=1}^{N} r_i(t)\) when \(\delta \neq 0\), or to the hypersurface \(\prod_{i=1}^{N} r_i(t)\) when \(\delta = 0\). Then the theorem is a consequence of the Lyapunov Stability Theorem in classical ODE theory.
Remark 1. The $\alpha=0$ case is of special interest. The unified $(0, \beta)$-flow always converges to a metric $r^*$ with Cooper and Rivin’s curvature $K_i \equiv \text{constant}$, if $\|r(0) - r^*\|$ is small enough. Specifically, there exists a $\epsilon > 0$, such that if the initial metric $r(0)$ satisfies $\|r(0) - r^*\| < \epsilon$, then the solution $r(t)$ to the unified $(0, \beta)$-flow exists for all time $t \geq 0$ and converges to $r^*$ as $t \to +\infty$. This means that constant $K$-curvature metric $r^*$ is locally stable. More specifically, it seems that the $(0, 1)$-flow, which lies at the intersection of $(\alpha, 1)$-flow and $(\alpha, \alpha+1)$-flow (see Figure 1), shows much better convergence properties than other types of unified $(\alpha, \beta)$-flows. For more details, see [4, 5, 6].

Figure 1: $(\alpha, \beta)$ flows

5 Calculating discrete constant $\alpha$-curvature metrics

The $(\alpha, \beta)$-flow provides an efficient way to find constant $\alpha$-curvature metrics on triangulated manifolds. To see the power of this method, we take the 16-cell triangulation of $S^3$ as an example, and see how the $(\alpha, \beta)$-flow can be used in concrete calculations.

Consider a standard geometric triangulation $T_s$ of $S^3$. Let $A_1 = (1, 0, 0, 0)$, $A_2 = (-1, 0, 0, 0)$, $B_1 = (0, 1, 0, 0)$, $B_2 = (0, -1, 0, 0)$, $C_1 = (0, 0, 1, 0)$, $C_2 = (0, 0, -1, 0)$, $D_1 = (0, 0, 0, 1)$, $D_2 = (0, 0, 0, -1)$ be the vertices of $\mathcal{T}_s$, $P_iQ_j\{\{P, Q\} \in \{A, B, C, D\}, i, j = 1, 2\}$ be the edges of $\mathcal{T}_s$, $P_iQ_jR_k\{\{P, Q, R\} \subset \{A, B, C, D\}, i, j, k = 1, 2\}$ be the faces of $\mathcal{T}_s$, and the regular tetrahedrons $A_iB_jC_kD_l(i, j, k, l = 1, 2)$ be the tetrahedrons of $\mathcal{T}_s$.

We then consider a topological triangulation $\mathcal{T}$ of $S^3$, which has the same combinatorial structure with $\mathcal{T}_s$. $\mathcal{T}$ is often called the 16-cell triangulation of $S^3$ in previous literature. It is easy to see that $\mathcal{T}$ carries a trivial constant $\alpha$-curvature metric for each $\alpha$. In fact, let $r_i = 1$, then $r$ is exactly a constant $\alpha$-curvature metric. We want to know whether there are other (up to scaling) constant $\alpha$-curvature metrics? By evolving the unified $(\alpha, \beta)$-flow
with appropriate initial value, we can easily get many constant $\alpha$-curvature metrics (up to scaling). For example, when $\alpha = 1$, we denote the metrics on $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$, $D_1$, $D_2$ by $r_1^+, r_1^-, r_2^+, r_2^-, r_3^+, r_3^-, r_4^+, r_4^-$ respectively. Then

$$r_1^+ = r_2^+ = \frac{1}{12}, \quad r_3^+ = r_4^+ = \frac{1}{6}$$

is exactly a constant $1$-curvature metric. For this case,

$$K_1^+ = K_2^+ = 8\pi - 16 \arccos \frac{1}{\sqrt{10}},$$

$$K_3^+ = K_4^+ = 12\pi - 8 \arccos \frac{3}{5} - 16 \arccos \frac{1}{\sqrt{10}}.$$ 

Furthermore, we have $\frac{K_i^+}{r_i^+} = 12(8\pi - 16 \arccos \frac{1}{\sqrt{10}}) \approx -61.8$ for each $1 \leq i \leq 4$.

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