Abstract

We investigate the behaviour of a finite chain of Brownian particles, interacting through a pairwise potential $U$, with one end of the chain fixed and the other end pulled away, in the limit of slow pulling speed and small Brownian noise. We study the instant when and the place where the chain “breaks”, that is, the distance between two neighbouring particles becomes larger than a certain threshold.

We assume $U$ to be attractive and strictly convex up to the break distance, and three times continuously differentiable. We consider the regime, where both the pulling and the noise significantly influence the distribution of the break time and break position. It turns out that in this regime there is a universality of both the break time distribution and the break position distribution, in the sense that the limiting quantities do not depend on the details of $U$, but only on its curvature at the break distance.

Keywords: Interacting Brownian particles; stochastic differential equations, rupture of a molecular chain.

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1 Introduction

Interacting Brownian particles are a natural and popular model for physical systems such as crystals, soft matter, or interacting colloidal particles: the interaction models the force between the particles, while the noise models external influences on the system, such as the collision with much smaller particles that are not explicitly modelled, or thermal fluctuations. Some of the possible application scenarios are given e.g. in [20, 21].

If systems of interacting particles evolve in a smooth way (on large scales), it is natural to investigate their macroscopic behaviour via hydrodynamic limits. This is by no means an easy problem, but good progress has been made over the last decades. For systems at equilibrium, important
results include the seminal work [22] on Gaussian fluctuations around reversible equilibria, and the recent significant progress [8] on convergence to the KPZ equation for weakly asymmetric one-dimensional Ginzburg-Landau interface models. For the hydrodynamic limit of general non-equilibrium systems, the classical result [23] establishes the hydrodynamic limit for one-dimensional systems with repulsive interactions on the torus. In the case of Ginzburg-Landau models, significant progress (in particular, extension to higher dimensions) was achieved in [11].

A very different situation arises in cases where the system of interacting particles behaves in a way that is discontinuous on the macroscopic scale, such as when a material breaks under strain. The most significant, but also the most difficult instance of such a situation is the dynamics of the propagation of cracks through solids, which occurs on a different time scale than the standard macroscopic dynamics. Due to this fact there are, to our knowledge, no mathematical tools for rigorously investigating the problem, and all the activity is on numerical studies, see e.g. [5, 18]. Even for the one-dimensional case, i.e. the rupture of a molecular chain under strain, the majority of the activity is non-rigorous, such as [9, 10, 12, 19].

Indeed, one of very few\footnote{There is another model represented in [17]; see also [13, 14].} mathematically rigorous attempts on the problem of chain rupture that we are aware of is that started in [2], and then extended in different ways in [4, 3]. In the present paper, we significantly advance the understanding of that model; the most important new observation is that in the most relevant parameter regime (the intermediate regime), for a chain of finite but arbitrary length, the asymptotic distribution of both the break time and position are universal in the sense that they do not depend on most details of the intermolecular force.

Our model is mathematically equivalent to the one-dimensional Ginzburg-Landau model with a time-dependent boundary condition. To be precise, let $d \geq 2$ be an integer, and consider a chain of $d+1$ particles located on the real line, interacting via a nearest neighbour force given by the derivative of a potential $U$. The positions of the particles at time $t$ are denoted by $\lambda_t^0, \ldots, \lambda_t^d$. We assume that $\lambda_t^0 = i$, $\lambda_t^0 = 0$ for all $t$, and $\lambda_t^d = \lambda_0^d + \varepsilon t$ for $\varepsilon > 0$ and all $t$. This means that the leftmost particle is fixed and the rightmost one is pulled with speed $\varepsilon$ to the right. Altogether, the model is

\footnote{We always use the word ‘neighbour’ to mean consecutive indices, not neighbours in space. This is justified e.g. if we think of the chain as consisting of a string of molecules. On the other hand, in the applications we have in mind particles swap places with negligible probability, in which case there is no difference between spatial and index neighbours.}
thus described by the system of stochastic differential equations

\[
\begin{cases}
X_i^0 = i & i = 0, 1, \ldots, d; \\
X_i^0 = 0 & t \geq 0; \\
X_i^d = d + \varepsilon t & t \geq 0; \\
dX_i^t = \left( U'(X_i^{i+1}) - U'(X_i^i - X_{i-1}^i) \right) dt + \sigma dB_i^t, & i = 1, \ldots, d-1, t \geq 0,
\end{cases}
\]

where \((B_i^t)_{t \geq 0}\) are independent Brownian motions, \(i = 1, \ldots, d-1\), \(\sigma \geq 0\), \(\varepsilon \geq 0\), and \(U\) is a sufficiently regular function. We will often write \(\mathbf{X}_t := (X_0^t, \ldots, X_d^t)\).

We will be interested the asymptotic behaviour of the model as \(\varepsilon\) and \(\sigma\) vanish. Quantities of interest are the time and location (along the chain) of the chain rupture under the dynamics. The physically most desirable choice for \(U\) is a potential that is attractive at short distances but becomes flat at infinity. This then leads to a motion where at first the chain becomes more and more elongated, until at some point a fluctuation makes one of the gaps between two neighbouring particles so large that it energetically favourable for the chain to split into two disconnected pieces. It is not hard to see that e.g. for potentials \(U\) that are strictly convex on an interval containing the starting distance of 1 between two particles, this critical gap size corresponds to the first inflection point of \(U\) on \([1, \infty)\), i.e. the smallest value \(r\) for which \(U''(r) = 0\).

The investigation of a break at an inflection point poses some difficulties. The reason is that in the intermediate regime that we are interested in, the chain remains in a position where all particles are very nearly evenly spaced right up to the time when it breaks. In such a situation of almost equal distances \(s\), a Taylor expansion of \(U'\) around \(s\) shows that the effective force which prevents each particle from leaving the position in the middle of its neighbours is equal to \(U''(s)\). A first problem is that this vanishes when \(s\) approaches the inflection point \(r\), and higher order terms of the expansion take over and have to be dealt with. A second problem is that when some fluctuation eventually causes the distance between two particles to exceed the critical distance \(r\), another fluctuation may well bring them back closer together before the deterministic part of the dynamics has had enough time to pull the chain apart. It is therefore not even completely clear what the correct definition for a break time should be.

In [3], the first problem above was solved for the case when \(d = 3\) (one free particle) and \(U''\) is a suitable third order polynomial. Then, the random time when the size of one of the gaps first reaches the inflection point can be analysed using the asymptotics of Airy functions. No extension of this result to longer chains or to the second problem mentioned above exists. Since the problem has some similarities with a metastability situation, methods from [7] might work, but the time-dependence, the fact that all saddles of the total
potential energy are of the same height to leading order, and the detailed nature of the relevant questions (see below) mean that at least they would need to be extended in a non-trivial way.

In the present work, we proceed as in [2] and avoid both problems discussed above. We assume that there is a fixed distance $b > 1$ such that the chain breaks whenever two neighbouring particles are $b$ or more apart from each other. Physically, this can be justified by a sudden failure of the molecular bonds beyond a certain threshold. Mathematically, we introduce the stopping times

$$
\tau_{i}^{\epsilon, b}(\epsilon, \sigma) := \inf\{t \geq 0 \mid \mathcal{X}_{t}^{i} - \mathcal{X}_{t}^{i-1} = b\}
$$

for $i \in \{1, \ldots, d\}$, and

$$
\tau_{\mathcal{X}, b}(\epsilon, \sigma) := \min_{1 \leq q \leq d} \tau_{q}^{\epsilon, b} = \inf\{t \geq 0 : \exists i \in \{1, \ldots, d\} : \mathcal{X}_{t}^{i} - \mathcal{X}_{t}^{i-1} = b\},
$$

and investigate their distributions. Furthermore, we assume that $U$ is strictly convex and increasing up to $b$, thus guaranteeing that a particle configuration with equal distances between particles is a stable equilibrium of the no-noise dynamics. From the geometric point of view based on the observation of the process $(\mathcal{X}_{t})$, the break time simply means the exit time of $\mathcal{X}$ from a certain deterministic polytope. Accordingly, we call $\tau_{i}^{\epsilon, b}, \tau_{\mathcal{X}, b},$ and other similar variables exit times.

The case $d = 3$ of a single free particle with strictly convex $U$ was treated in [2]. There, methods from [6] together with some symmetry considerations (possible due to $d = 3$) yielded the following dichotomy: if $\sigma \ll \epsilon$ (fast pulling regime), the chain breaks deterministically at the right link, while for $\sigma \gg \epsilon$, it breaks at each link with equal probability. The threshold between both regimes was identified only up to a logarithmic factor, and a large deviation (slow pulling) regime where $1/\sigma$ is exponentially large in $1/\epsilon$ was not covered. These shortcomings were overcome in the recent work [4] for the case where $U$ is quadratic. In that work, a detailed analysis of the resulting Gaussian processes exhibited three regimes: the already mentioned fast pulling regime, where the limiting quantities are governed by the pulling force only; a slow pulling / large deviation regime where the rupture is caused essentially solely by the noise; and an interesting intermediate regime where both the pulling and the noise determine the limiting characteristics. In all of these regimes, [4] provides precise asymptotics of the break times and locations for chains of arbitrary length.

In the present paper, we extend the analysis beyond the case of quadratic potentials for the regime of intermediate pulling. This regime is the most interesting one, since the fast pulling case is anyway dominated by the deterministic dynamics, while the breakage in the slow pulling case relies on a large deviation event and results will therefore depend on the details of
the potential. In contrast, for the intermediate pulling regime, we observe a universal behaviour for the break time distribution, in the sense that it only depends on the curvature $U''(b)$ of $U$ at the breaking distance.

The reason for this universality is not hard to understand intuitively. Since the chain (of initial length $d$) must break once its total length exceeds $db$, there is the simple, but important bound

$$\tau_{X,b} \leq t_\epsilon = t_\epsilon(\epsilon, b) := d(b - 1)/\epsilon.$$  \hfill (4)

For the intermediate pulling regime, the break actually occurs when the chain is quite close to the maximal allowed elongation $db$, and is therefore initiated by rather small fluctuations around the stable equilibrium. Therefore, the quadratic approximation to the potential is a good one for this situation, and we obtain both the universality and the actual result by comparison with the relevant Gaussian processes. Our proof follows this intuition.

Let us end this introduction by briefly discussing possible future extensions of our results. First of all, the assumption of strict convexity of $U$ up to the break location is not harmless; in [3], where the situation with a break at an inflection point of $U$ is studied, it is shown there that the scaling of $\sigma$ with $\epsilon$ for the threshold between intermediate and fast pulling regime is different from the situation where we assume strict convexity. An extension of the results of [3] to longer chains in the spirit of the present paper would be very interesting, but would need to do without the theory of Ornstein-Uhlenbeck processes that we use crucially. Another route for improvement would be to investigate a chain of a length $d(\epsilon)$ that increases as $\epsilon \to 0$. Here, the obstacle to overcome is that in [4], the spectral gap of a certain discrete Laplacian plays a crucial role, which disappears as the chain gets infinitely long.

2 Main result

In this section we give our main result and an outline of its proof. We study the system (1) with the condition that the chain breaks when the distance between a pair of neighbouring particles reaches the value $b > 1$. The random time $\tau_{X,b}$ at which distance $b$ is reached by the $i$-th link is thus given by (2), and the break time $\tau_{X,b}$ of the chain by (3). Recall also the deterministic upper bound $t_\epsilon$ on the break time given in (4). We will make the following assumption for the potential $U$:

Assumption P. The function $U$ is three times continuously differentiable and $U''$ is strictly positive on $[1, b]$.

We will investigate the intermediate pulling regime characterized by the conditions

$$\sigma/\epsilon \to \infty \quad \text{and} \quad \sigma^2 |\ln \epsilon|^3 \to 0.$$  \hfill (5)
on the scaling parameters $\sigma$ and $\varepsilon$. For stating our result, we define the quantities

$$v^2 := \frac{d-1}{2d}, \quad \gamma := \sqrt{2d}v = \sqrt{d(d-1)},$$
$$A_1 := A_d := \frac{d}{d-1}, \quad A_i := \frac{2d}{d-1}, \quad i \in \{2, \ldots, d-1\},$$
$$a_i := vdA_i/\sqrt{2\pi} \quad \text{for} \quad i \in \{1, \ldots, d\},$$
$$a_0 := \sum_{i=1}^{d} a_i = 2vd^2/\sqrt{2\pi}, \quad b := \sqrt{2}/(vd).$$

Recall that a random variable $\chi$ is double exponential (or Gumbel) with parameters $a, b > 0$, if

$$\mathbb{P}(\chi \leq r) = \exp(-a \exp(-br)), \quad r \in \mathbb{R}.$$

**Theorem 1** Let $(X_i^t)_{i=0,\ldots,d}$ solve the system (1), where the potential $U$ satisfies Assumption P. Let $b > 1$, set $u := U''(b)$, and define $\tau_{X,b}^i$, $\tau_{X,b}$ and $t^*(\varepsilon)$ as in (2), (3) and (4), respectively. Then in the parameter regime described by (5), we have the following weak limit theorems for the break times as $\varepsilon, \sigma \to 0$:

$$\frac{\sqrt{u} \varepsilon}{\sigma} \sqrt{\ln(\sigma/\varepsilon)} \left( t^*(\varepsilon, b) - \gamma \frac{\sigma}{\sqrt{u} \varepsilon} \sqrt{\ln(\sigma/\varepsilon)} - \tau_{X,b}^i(\varepsilon, \sigma) \right) \overset{d}{\to} \chi_i(u), \quad i \in \{1, \ldots, d\},$$

and

$$\frac{\sqrt{u} \varepsilon}{\sigma} \sqrt{\ln(\sigma/\varepsilon)} \left( t^*(\varepsilon, b) - \gamma \frac{\sigma}{\sqrt{u} \varepsilon} \sqrt{\ln(\sigma/\varepsilon)} - \tau_{X,b}(\varepsilon, \sigma) \right) \overset{d}{\to} \chi_0(u),$$

where for each $i$, $\chi_i(u)$ is a double exponential random variable with parameters $\sqrt{u}a_i, b$, respectively.

Moreover, under the same assumptions, we have

$$\mathbb{P}(\tau_{X,b} = \tau_{X,b}^i) \to \begin{cases} \frac{1}{d-1} & i \in \{2, \ldots, d-1\}; \\ \frac{1}{2(d-1)} & i \in \{1, d\} \end{cases}$$

as $\sigma, \varepsilon \to 0$.

**Remark.** The above result is contained in [4] for the special case $U(x) = x^2/2$; in that case, the processes $X_i^t$ are Gaussian and can be analysed in great detail. Note that the break behaviour in the present result is indeed universal as it only depends on the potential via $u$. From the relevant formulae it is also apparent that $u > 0$ is important for them to make sense; this is another indication that in the case of break at an inflection point, as discussed in the introduction, a different asymptotic behaviour should be expected.
The first step in our proof consists in a slight generalization of the relevant result in [4]. Put very succinctly, it states that the statement of Theorem 1 holds in the special case

\[ U(x) = \frac{ux^2}{2} \]

with \( u > 0 \). Since we will need the notation in the proof later on anyway, we spell out the statement here.

Consider the linear system

\[
\begin{align*}
X^i_0 &= i & i &= 0, 1, \ldots, d; \\
X^0_t &= 0 & t & \geq 0; \\
X^d_t &= d + \varepsilon t & t & \geq 0; \\
\frac{dX^i_t}{dt} &= u(X^i_t + X^{i-1}_t)dt + \sigma dB^i_t & i &= 1, \ldots, d-1, t \geq 0,
\end{align*}
\]

where \( u > 0 \) is a constant and we will abbreviate \( X_t := (X^0_t, \ldots, X^d_t)^\top \). As before, define the break times as

\[
\tau^i = \tau^i(\varepsilon, \sigma) := \inf\{ t \geq 0 \mid X^i_t - X^{i-1}_t = b \}
\]

and

\[
\tau = \tau(\varepsilon, \sigma) := \min_{1 \leq i \leq d} \tau^i = \inf\{ t \geq 0 : \exists i \in \{1, \ldots, d\} : X^i_t - X^{i-1}_t = b \}.
\]

When we want to stress the dependence on the parameters \( u \) and \( b \), we will write \( X_{u,t}, X^i_{u,t}, \tau^i_{u,b}(\varepsilon, \sigma) \) and \( \tau_{u,b}(\varepsilon, \sigma) \).

We consider scaling regime where

\[
\frac{\sigma}{\varepsilon} \to \infty \quad \text{and} \quad \sigma^2|\ln \varepsilon| \to 0.
\]

Note that this regime is slightly wider than the one given in [5]; the small difference is due to the fact that close to the large deviation regime, the Gaussian and the non-Gaussian processes start to look differently. We have

**Theorem 2** Assume that (11) holds. Then, as \( \varepsilon, \sigma \to 0 \), the analogue of (9) holds for \( \tau_{u,b}(\varepsilon, \sigma) \) and \( \tau^i_{u,b}(\varepsilon, \sigma) \) and we have the following weak limit theorems for the break times:

\[
\frac{\sqrt{u\varepsilon}}{\sigma} \sqrt{\ln(\sigma/\varepsilon)} \left( t^*_s(\varepsilon, b) - \gamma \frac{\sigma}{\sqrt{u\varepsilon}} \sqrt{\ln(\sigma/\varepsilon)} - \tau^i_{u,b}(\varepsilon, \sigma) \right) \overset{d}{\to} \chi_i(u),
\]

\[
\text{for } i \in \{1, \ldots, d\},
\]

and

\[
\frac{\sqrt{u\varepsilon}}{\sigma} \sqrt{\ln(\sigma/\varepsilon)} \left( t^*_s(\varepsilon, b) - \gamma \frac{\sigma}{\sqrt{u\varepsilon}} \sqrt{\ln(\sigma/\varepsilon)} - \tau_{u,b}(\varepsilon, \sigma) \right) \overset{d}{\to} \chi_0(u),
\]

where \( \chi_i(u) \) is a double exponential random variable with parameters \( \sqrt{ua_i}, b \), and the values \( a_i, b \) are defined in (6).
contains Theorem 2 for the special case $b = 2$, $u = 1$. The passage to general $b$ and $u$ is made by standard scaling arguments, which we spell out in Section 3 for the convenience of the reader.

The second step is to replace $u$ in the system (10) by a time-dependent quantity $\phi(t)$. Again, the short version of the result is that the statement of Theorem 2 remains true with $u$ replaced by $\phi(t)$, but we spell out the result as we will need the notation later in the proofs anyway.

Let $Z_t = (Z^0_t, \ldots, Z^d_t)^\top$ solve the linear system with time-dependent coefficient

$$
\begin{align*}
Z^i_0 &= i, & i = 0, 1, \ldots, d; \\
Z^i_t &= 0, & t \geq 0; \\
Z^i_t &= d + \varepsilon t, & t \geq 0; \\
dZ^i_t &= \phi(t)(Z^i_{t+1} + Z^i_{t-1} - 2Z^i_t)dt + \sigma dB^i_t, & i = 1, \ldots, d-1, t \geq 0,
\end{align*}
$$

Here we immediately take $\phi(t) := U''(q_t)$ with

$$q_t := 1 + \varepsilon \frac{t}{d},$$

where $U$ fulfills Assumption P. This is the correct linearization of (11) in the following sense: by the assumptions on $U$, the potential energy $U(x) = \sum_{i=1}^d U(x_i - x_{i-1})$ is minimized by the vector $(i(1 + \varepsilon \frac{t}{d}))_{0 \leq i \leq d}$ at time $t$. Since the pulling is slow and the noise is small, the system will be close to that energy minimum at all times, and as terms of order zero and one cancel when we Taylor expand each $U(x_i - x_{i-1})$ around $1 + \varepsilon \frac{t}{d}$, so that $\phi(t) = U''(1 + \varepsilon \frac{t}{d})$ is the dominant term. Notice in particular that for quadratic potentials $U(x) = u x^2 / 2$, we have $\phi(\cdot) \equiv u$ and $(Z_t)$ coincides with $(X_{u,t})$.

Since $Z = (Z^0, \ldots, Z^d)^\top$ is still a Gaussian process, we will be able to analyse it in great detail. We work in the scaling regime

$$\sigma/\varepsilon \to \infty \quad \text{and} \quad \sigma^2 |\ln \varepsilon|^{3/2} \to 0,$$

which is larger than the one for Theorem 1 but smaller than the one for Theorem 2 due to our need to accommodate the fact that the coefficient of the linear force now depends on time. As above, we introduce the break times

$$\tau^i_{Z,b} = \tau^i_{Z,b}(\varepsilon, \sigma) := \inf\{t \geq 0 \mid Z^i_t - Z^i_{t-1} = b\}$$

and

$$\tau_{Z,b} = \tau_{Z,b}(\varepsilon, \sigma) := \min_{1 \leq i \leq d} \tau^i_{Z,b} = \inf\{t \geq 0 : \exists i \in \{1, \ldots, d\} : Z^i_t - Z^i_{t-1} = b\}.$$

We have
Theorem 3 Assume that (15) holds. Fix $b > 1$ and set $u := U''(b)$. Then, as $\varepsilon, \sigma \to 0$, the analogue of (9) holds for $\tau_{Z,b}(\varepsilon, \sigma)$ and $\tau_{i,Z,b}(\varepsilon, \sigma)$ and we have the following weak limit theorems for the break times:

$$
\frac{\sqrt{u} \varepsilon}{\sigma} \sqrt{\ln(\sigma/\varepsilon)} \left( t_*(\varepsilon, b) - \gamma \frac{\sigma}{\sqrt{u} \varepsilon} \sqrt{\ln(\sigma/\varepsilon)} - \tau_{Z,b}(\varepsilon, \sigma) \right) \xrightarrow{d} \chi_i(u), \quad i \in \{1, \ldots, d\},
$$

and

$$
\frac{\sqrt{u} \varepsilon}{\sigma} \sqrt{\ln(\sigma/\varepsilon)} \left( t_*(\varepsilon, b) - \gamma \frac{\sigma}{\sqrt{u} \varepsilon} \sqrt{\ln(\sigma/\varepsilon)} - \tau_{Z,b}(\varepsilon, \sigma) \right) \xrightarrow{d} \chi_0(u),
$$

where $\chi_i(u)$ is a double exponential random variable with parameters $\sqrt{u} a_i, b$, and $a_i, b$ are defined in (10).

The rest of this paper is structured as follows. As already mentioned, Theorem 2 is proved in Section 3, while Theorem 3 is proved in Section 4. It is used to prove Theorem 1 in Section 5. Proofs of various technical results used in these proofs are collected in Section 6.

3 Scaling of the standard model

Here we prove Theorem 2. Recall that the case where $u = 1, b = 2$ has been done in [4]. Below, we will refer to this case as the standard problem.

Step 1. Scaling of $u$. We first consider the scaling of $u$ with fixed $b$. Let $\tilde{X}$ be a solution of the system (10) with parameters $u = 1, \tilde{\varepsilon}$ and $\tilde{\sigma}$. We make the time change $\hat{X}_i^t := \tilde{X}_i^{u t}$. This family solves the system with parameters $u, \varepsilon := u \tilde{\varepsilon}$ and $\sigma := \sqrt{u} \tilde{\sigma}$, because $\hat{X}_i^0 = \tilde{X}_i^0 = 0$, $\hat{X}_i^d = \tilde{X}_i^{d, u t} = d + \tilde{\varepsilon}(u t) = d + (u \tilde{\varepsilon}) t$, and an easy calculation shows that

$$
d\hat{X}_i^t = u (\hat{X}_i^{t+1} + \hat{X}_i^{t-1} - 2\hat{X}_i^t) dt + \sqrt{u} \tilde{\sigma} d\hat{B}_i^t, \quad i = 1, \ldots, d - 1, t \geq 0,
$$

with appropriate independent Brownian motions $\hat{B}_i^t$.

The above argument shows that for given $\varepsilon, \sigma$, we have

$$
X_{u,t}(\varepsilon, \sigma) \overset{d}{=} X_{1,u t} \left( \frac{\varepsilon}{u} \cdot \frac{\sigma}{\sqrt{u}} \right),
$$

the latter being a solution of the standard problem, yet with modified parameters.
Next, having a relation between the processes, we obtain the relations between the break times for arbitrary $b > 1$,

$$\tau_{u,b}^i(\varepsilon, \sigma) = u^{-1} \tau_{u,1,b}^i(\varepsilon/u, \sigma/\sqrt{u}),$$  \hspace{1cm} (18)

$$\tau_{u,b}(\varepsilon, \sigma) = u^{-1} \tau_{1,b}(\varepsilon/u, \sigma/\sqrt{u}).$$  \hspace{1cm} (19)

Step 2. Scaling of $b$. Now we fix $u = 1$ and scale $b$. Let us fix a break position $i \in \{1, \ldots, d\}$. The break condition is

$$X_i^t - X_{i-1}^t = b.$$  

This is equivalent to

$$\sigma V_i^t + \varepsilon t/d + \varepsilon R_i^t = b - 1,$$

where $V_i$ is an appropriate asymptotically stationary process, and $R_i$ is a bounded deterministic function, both defined in [4]. Dividing by $b - 1$ yields the equivalent form

$$\frac{\sigma}{b - 1} V_i^t + \varepsilon t/d + \varepsilon R_i^t = 1,$$

which coincides with the break condition for the standard case $b = 2$ with parameters $\tau_{1,2}^i(\varepsilon)$. We conclude that $\tau_{u,b}^i(\varepsilon, \sigma)$ coincides in distribution with $\tau_{1,2}^i(\varepsilon) = \tau_{1,2}^i(\varepsilon/u, \sigma/\sqrt{u}).$

Step 3. Combining two scalings. By combining the results of two scalings we see that the vector $\tau_{u,b}^i(\varepsilon, \sigma)$, $i = 1, \ldots, d$, has the same distribution as the vector $u^{-1} \tau_{u,1,b}^i(\varepsilon/u, \sigma/\sqrt{u})$, $i = 1, \ldots, d$. Subsequently, this also holds for $\tau_{u,b} = \min_i \tau_{u,b}^i$ and $\tau_{1,2} = \min_i \tau_{1,2}^i$.

Step 4. Weak convergence. From [4] we know that Theorem 2 is valid in the standard problem. We apply this version of the theorem to the modified pair of parameters $\left(\frac{\varepsilon}{u(b-1)}, \frac{\sigma}{\sqrt{u(b-1)}}\right)$ and obtain

$$\frac{\varepsilon}{\sigma \sqrt{u}} \sqrt{\ln \left(\frac{\sigma \sqrt{u}}{\varepsilon}\right) \left(\frac{du(b-1)}{\varepsilon} - \frac{\sigma \sqrt{u}}{\varepsilon} \sqrt{\ln \left(\frac{\sigma \sqrt{u}}{\varepsilon}\right)} - u \tau_{u,b}^i(\varepsilon, \sigma)\right)} \overset{d}{\to} \chi_i.$$  

or, equivalently,

$$\frac{\varepsilon \sqrt{u}}{\sigma} \sqrt{\ln \left(\frac{\sigma \sqrt{u}}{\varepsilon}\right) \left(\frac{d(b-1)}{\varepsilon} - \frac{\sigma}{\varepsilon \sqrt{u}} \sqrt{\ln \left(\frac{\sigma \sqrt{u}}{\varepsilon}\right)} - \tau_{u,b}^i(\varepsilon, \sigma)\right)} \overset{d}{\to} \chi_i.$$  

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Next, using expansion
\[
\sqrt{\ln \left( \frac{\sigma \sqrt{u}}{\varepsilon} \right)} = \sqrt{\ln \left( \frac{\sigma}{\varepsilon} \right)} + \frac{\ln u}{2} = \sqrt{\ln \left( \frac{\sigma}{\varepsilon} \right)} \sqrt{1 + \frac{\ln u}{2 \ln \left( \frac{\sigma}{\varepsilon} \right)}}
\]
\[
= \sqrt{\ln \left( \frac{\sigma}{\varepsilon} \right)} \left( 1 + \frac{\ln u}{4 \ln \left( \frac{\sigma}{\varepsilon} \right)} (1 + o(1)) \right)
\]
\[
= \sqrt{\ln \left( \frac{\sigma}{\varepsilon} \right)} + \frac{\ln u}{4 \sqrt{\ln \left( \frac{\sigma}{\varepsilon} \right)}} (1 + o(1))
\]
we obtain
\[
\frac{\varepsilon \sqrt{u}}{\sigma} \sqrt{\ln \left( \frac{\sigma}{\varepsilon} \right)} \left( \frac{d(b - 1)}{\varepsilon} - \gamma \frac{\sigma}{\varepsilon \sqrt{u}} \sqrt{\ln \left( \frac{\sigma}{\varepsilon} \right)} - \tau_{u,b}^i (\varepsilon, \sigma) \right) \overset{d}{\to} \chi_i + \gamma \ln u \frac{1}{4}.
\]

Notice that a shifted variable having a double exponential distribution belongs to the same class: If \( \kappa \in \mathbb{R} \) and \( \xi \) is double exponential with parameters \( a, b \), then \( \xi + \kappa \) is also double exponential with parameters \( a \exp \{ b \kappa \}, b \).

In our case, \( \kappa = \frac{\gamma \ln u}{4} \), hence \( \exp \{ b \kappa \} = u^{b \gamma / 4} \). Furthermore, an easy calculation shows that \( b \gamma = \sqrt{2} (vd)^{-1} \cdot \sqrt{2} dv = 2 \). Therefore, \( \exp \{ b \kappa \} = \sqrt{u} \). We finally obtain
\[
\frac{\varepsilon \sqrt{u}}{\sigma} \sqrt{\ln \left( \frac{\sigma}{\varepsilon} \right)} \left( \frac{d(b - 1)}{\varepsilon} - \gamma \frac{\sigma}{\varepsilon \sqrt{u}} \sqrt{\ln \left( \frac{\sigma}{\varepsilon} \right)} - \tau_{u,b}^i (\varepsilon, \sigma) \right) \overset{d}{\to} \chi_i \left( u \right), \quad (20)
\]
where \( \chi_i (u) \) is double exponential with parameters \( \sqrt{ua}, b \).

The latter equation coincides with the claim \( (12) \). The proof of \( (13) \) is exactly the same. Relation \( (9) \) follows as a by-product of the scaling reductions. Therefore, the proof of Theorem \( 2 \) is finished.

**Remark 4** We see from the proof that the weak convergence in \( (12) \) and \( (13) \) is locally uniform in \( b \) (as long as \( b \) is bounded away from 1) for every fixed \( u \). We will use this fact later on.

## 4 Comparing the Gaussian processes \( X_{u,b} \) and \( Z \)

Before we start with the proof of Theorem \( 3 \) we need two auxiliary results. The first one shows that \( Z \) has no early breaks. Recall the notation \( t_* = t_*(\varepsilon, b) := d(b - 1)/\varepsilon \).

**Lemma 5** There exists \( \gamma_1 = \gamma_1 (d, U(\cdot)) \) such that
\[
P \left( \tau_{Z,b} (\varepsilon, \sigma) < t_* - \gamma_1 \frac{\sigma}{\varepsilon} \sqrt{\ln \left( \frac{\sigma}{\varepsilon} \right)} \right) \to 0, \quad \text{as } \varepsilon, \sigma \to 0. \quad (21)
\]
From Theorem 2 we know that the same statement is true for $X_u$, namely, for any $\gamma_1 > \gamma/\sqrt{u}$,

$$
\mathbb{P}\left(\tau_{u, b}(\varepsilon, \sigma) < t_* - \gamma_1 \frac{\sigma}{\varepsilon} \sqrt{\ln \left(\frac{\sigma}{\varepsilon}\right)}\right) \rightarrow 0. \tag{22}
$$

But we also need (21) for $Z$. The proof of Lemma 5 is given in Section 6.2.

The second auxiliary lemma shows that $Z$ and $X_u$ are close on an important time interval.

**Lemma 6** Let $\gamma_1 > 0$ and $\theta > 0$. Assume (15). Denote $\delta = \delta(\varepsilon, \sigma) := \frac{\theta \sigma}{\sqrt{\ln(\frac{\sigma}{\varepsilon})}}$. Then, as $\varepsilon, \sigma \rightarrow 0$, for every $i = 0, 1, \ldots, d$, we have

$$
\mathbb{P}\left(\sup_{t \in [t_* - \gamma_1 \frac{\sigma}{\varepsilon} \sqrt{\ln(\frac{\sigma}{\varepsilon})}, t_*]}|Z_i t - X_i u, t| > \delta\right) \rightarrow 0. \tag{23}
$$

The proof of Lemma 6 is given in Section 6.3.

**Proof of Theorem 3:** Let $\gamma_1$ be chosen large enough to satisfy Lemma 5. Let $t \in [t_* - \gamma_1 \frac{\sigma}{\varepsilon} \sqrt{\ln(\frac{\sigma}{\varepsilon})}, t_*]$ be such that

$$
\max_{1 \leq i \leq d} (Z_i t - Z_i t^{-1}) \geq b.
$$

If, additionally,

$$
\max_{0 \leq i \leq d} |Z_i t - X_i u, t| < \delta, \tag{24}
$$

then

$$
\max_{1 \leq i \leq d} (X_i u, t - X_i u, t^{-1}) \geq b - 2\delta.
$$

It follows that $\tau_{u, b-2\delta} \leq t$. By using Lemma 6 and Lemma 6 we obtain

$$
\mathbb{P}(\tau_{Z, b} \leq t) \leq \mathbb{P}(\tau_{u, b-2\delta} \leq t) + o(1)
$$

with $o(1)$ uniformly over $t \in [t_* - \gamma_1 \frac{\sigma}{\varepsilon} \sqrt{\ln(\frac{\sigma}{\varepsilon})}, t_*]$.
Let \( r \in \mathbb{R} \) be fixed. The previous equation and the definition of \( \delta \) yield

\[
P_r := P \left( \sqrt{u} \frac{\varepsilon}{\sigma} \ln(\sigma/\varepsilon) \left( t_* - \frac{\sigma}{\sqrt{u} \varepsilon} \sqrt{\ln(\sigma/\varepsilon)} - \tau_{Z,b}(\varepsilon, \sigma) \right) \geq r \right)
\]

\[
= P \left( \tau_{Z,b}(\varepsilon, \sigma) \leq t_* - \frac{\sigma}{\sqrt{u} \varepsilon} \sqrt{\ln(\sigma/\varepsilon)} - \frac{r \sigma}{\sqrt{u} \varepsilon \sqrt{\ln(\sigma/\varepsilon)}} \right)
\]

\[
\leq P \left( \tau_{u,b-2\delta}(\varepsilon, \sigma) \leq t_* - \frac{\sigma}{\sqrt{u} \varepsilon} \sqrt{\ln(\sigma/\varepsilon)} - \frac{r \sigma}{\sqrt{u} \varepsilon \sqrt{\ln(\sigma/\varepsilon)}} \right) + o(1)
\]

\[
= P \left( \tau_{u,b-2\delta}(\varepsilon, \sigma) \leq t_* - \frac{d(b - 2\delta - 1)}{\varepsilon} + \frac{2d\delta}{\varepsilon} - \frac{\sigma}{\sqrt{u} \varepsilon} \sqrt{\ln(\sigma/\varepsilon)} - \frac{r \sigma}{\sqrt{u} \varepsilon \sqrt{\ln(\sigma/\varepsilon)}} \right) + o(1),
\]

\[
= P \left( \tau_{u,b-2\delta}(\varepsilon, \sigma) \leq t_*(\varepsilon, b - 2\delta) - \frac{\sigma}{\sqrt{u} \varepsilon} \sqrt{\ln(\sigma/\varepsilon)} - \frac{r \sigma}{\sqrt{u} \varepsilon \sqrt{\ln(\sigma/\varepsilon)}} \right) + o(1),
\]

\[
= P \left( \sqrt{u} \frac{\varepsilon}{\sigma} \ln(\sigma/\varepsilon) \left( t_*(\varepsilon, b - 2\delta) - \frac{\sigma}{\sqrt{u} \varepsilon} \sqrt{\ln(\sigma/\varepsilon)} - \tau_{u,b-2\delta}(\varepsilon, \sigma) \right) \geq r + 2d\theta \sqrt{u} \right) + o(1),
\]

By applying Theorem 2 with \( \tilde{b} := b - 2\delta \) instead of \( b \), we obtain

\[
\limsup_{\varepsilon, \sigma \to 0} P_r \leq P(\chi_0(u) \geq r + 2d\theta \sqrt{u}).
\]

Finally, by letting \( \theta \to 0 \) we obtain the desired upper bound

\[
\limsup_{\varepsilon, \sigma \to 0} P_r \leq P(\chi_0(u) \geq r).
\]

The lower bound follows by the same lines. If for some \( t \in [t_* - \gamma_1 \sqrt{\ln(\sigma/\varepsilon)}, t_*] \) we have

\[
\max_{1 \leq i \leq d} \left( X_{i,t}^* - X_{i,t-1}^* \right) \geq b + 2\delta
\]

and (24) holds, then

\[
\max_{1 \leq i \leq d} \left( Z_{i,t}^* - Z_{i,t-1}^* \right) \geq b.
\]

In other words, \( \tau_{Z,b} \leq t \).

By using (22) and Lemma 6 we obtain

\[
P(\tau_{Z,b} \leq t) \geq P(\tau_{u,b+2\delta} \leq t) - o(1),
\]

and the rest of the derivation leading to

\[
\liminf_{\varepsilon, \sigma \to 0} P_r \geq P(\chi_0(u) \geq r).
\]

continues as above, \( b - 2\delta \) being replaced with \( b + 2\delta \).

\[\square\]

---

\[^3\text{We stress that } \tilde{b} \text{ slightly depends of } \varepsilon, \sigma \text{ through } \delta. \text{ Therefore, we need a uniform version of the theorem, cf. Remark 4.}\]
5 Break times for the non-linear system

In order to obtain information about the process $X$ from the analysis of $Z$, we must investigate the difference between $Z$ and $X$. We first need some definitions and notations, assuming that $Z$ and $X$ are defined on the same probability space, and are driven by the same Brownian motions. On this probability space we define the stochastic processes

$$S_t^* := \sup_{0 \leq s \leq t} \|Z_s - X_s\| = \sup_{0 \leq s \leq t} \left( \sum_{i=0}^{d} |Z_s^i - X_s^i|^2 \right)^{1/2}$$

and

$$M_t^* := \sup_{0 \leq s \leq t} \left( \sum_{i=1}^{d} |Z_s^i - Z_s^{i-1} - q_s|^2 \right)^{1/2},$$

where we recall that $q_s = 1 + \frac{\varepsilon_s}{d}$.

The key approximation result is as follows.

**Proposition 7** Let $r$ be as in Assumption P. Then there exists a large constant $C$ depending on $d$ and on the potential $U$ such that for all $t \leq t'$ and all $\delta > 0$ we have

$$\mathbb{P}(S_{t'}^* \geq \delta) \leq \mathbb{P}(M_{t'}^* \geq \sqrt{\delta/C}) + \mathbb{P}(M_{t'}^* + 4C(M_{t'}^*)^2 \geq r) + \mathbb{P}(CM_{t'}^* \geq 1). \quad (25)$$

The advantage of this proposition is that it evaluates the difference of the two processes in terms of the Gaussian process alone. Its proof is given in Section 6.4.

Based on this general estimate, we obtain a specific bound suited for the theorem's proof.

**Proposition 8** Assume that (5) holds. Let $\theta > 0$ and $\delta = \delta(\varepsilon, \sigma) := \frac{\theta \sigma}{\sqrt{\ln(\frac{\sigma}{\varepsilon})}}$. Then

$$\lim_{\varepsilon, \sigma \to 0} \mathbb{P}(S_{t'}^* \geq \delta) = 0. \quad (26)$$

The proof of this proposition is given in Section 6.5.

**Proof of Theorem 1**

**Lower bound.** Assume that for some $s$ we have

$$\max_{1 \leq i \leq d} (X_{u,s}^i - X_{u-1,s}^i) \geq b + 4\delta,$$

$$\max_{0 \leq i \leq d} |X_{u,s}^i - Z_s^i| < \delta,$$

$$\max_{0 \leq i \leq d} |Z_s^i - X_{u,s}^i| < \delta.$$

Then

$$\max_{1 \leq i \leq d} (X_{u,s}^i - X_{u,s-1}^i) \geq b.$$
It follows that for every $\gamma_1 > \gamma / \sqrt{u}$ and every $t \in [t_s - \gamma_1 \sigma \sqrt{\ln (\sigma / \varepsilon)}, t_s]$

$$\mathbb{P}(\tau_{X,b} \leq t) \geq \mathbb{P}(\tau_{u,b+4\delta} \leq t) - \mathbb{P}(\tau_{u,b} \leq t_s - \gamma_1 \sigma \sqrt{\ln (\sigma / \varepsilon)})$$

$$- \sum_{i=1}^{d-1} \mathbb{P} \left( \sup_{s \in [t_s - \gamma_1 \sigma \sqrt{\ln (\sigma / \varepsilon)}, t_s]} |X^i_{u,s} - X^i_{u,t}| > \delta \right)$$

$$- \mathbb{P}(S^*_t \geq \delta)$$

$$= \mathbb{P}(\tau_{u,b+4\delta} \leq t) - o(1),$$

where we used Theorem 2, Lemma 6 and Proposition 8. The rest of the proof goes along the same lines as in Theorem 3.

Upper bound. Assume that for some $s$ we have

$$\max_{1 \leq i \leq d} (X^i_{u,s} - X^i_{u,t}) \geq b,$$

$$\max_{0 \leq i \leq d} |Z^i_s - X^i_{u,s}| < \delta,$$

Then

$$\max_{1 \leq i \leq d} (Z^i_s - Z^i_{t}) \geq b - 2\delta.$$

If, additionally

$$\max_{0 \leq i \leq d} |X^i_{u,s} - Z^i_s| < \delta,$$

then

$$\max_{1 \leq i \leq d} (X^i_{u,s} - X^i_{u,t}) \geq b - 4\delta.$$

It follows that for every $t \in [t_s - \gamma_1 \sigma \sqrt{\ln (\sigma / \varepsilon)}, t_s]$

$$\mathbb{P}(\tau_{X,b} \leq t) = \mathbb{P}(\tau_{X,b} \leq t_s - \gamma_1 \sigma \sqrt{\ln (\sigma / \varepsilon)}) + \mathbb{P}(\tau_{X,b} \in [t_s - \gamma_1 \sigma \sqrt{\ln (\sigma / \varepsilon)}, t]) + \mathbb{P}(\tau_{u,b+4\delta} \leq t)$$

$$\leq \mathbb{P}(\tau_{Z,b-2\delta} \leq t_s - \gamma_1 \sigma \sqrt{\ln (\sigma / \varepsilon)}) + \mathbb{P}(\tau_{u,b-4\delta} \leq t)$$

$$+ \sum_{i=1}^{d-1} \mathbb{P} \left( \sup_{s \in [t_s - \gamma_1 \sigma \sqrt{\ln (\sigma / \varepsilon)}, t_s]} |X^i_{u,t} - Z^i_t| > \delta \right) + \mathbb{P}(S^*_t \geq \delta)$$

$$= \mathbb{P}(\tau_{u,b-4\delta} \leq t) + o(1),$$

where we used Lemma 5, Lemma 6 and Proposition 8. The rest of the proof goes along the same lines as in Theorem 3. □
6 Auxiliary technical results

6.1 Some properties of the process \( Z \)

Scalar analogues of \( Z \)

We work under the assumptions (15) and \( P \). Recall that under assumption \( P \) our potential \( U \) is convex on \([1, b]\) and there exist finite positive constants \( \kappa_{\text{min}}, \kappa_{\text{max}} \) and \( K \) such that

\[
\kappa_{\text{min}} \leq U''(x) \leq \kappa_{\text{max}}, \quad x \in [1, b],
\]

\[
|U'''(x)| \leq K, \quad x \in [1, b].
\]

**Lemma 9** Let \( U : [1, b] \to \mathbb{R} \) be a function satisfying Assumption \( P \). Set \( \phi(t) := U''(1 + \frac{\varepsilon t}{d}) \), \( t \in [0, t_*] \) and \( u := \phi(t_*) = U''(b) \), where \( t_* := d(b - 1)/\varepsilon \).

Consider two scalar stochastic differential equations

\[
d\bar{Y}_t = -u \bar{Y}_t dt + \sigma dB_t, \quad t \geq 0, \quad \bar{Y}_0 = 0,
\]

and

\[
d\bar{Z}_t = -\phi(t) \bar{Z}_t dt + \sigma dB_t, \quad t \geq 0, \quad \bar{Z}_0 = 0. \tag{27}
\]

Assume that \( \varepsilon, \sigma \to 0 \) with (15) being true. Let

\[
T = T(\sigma, \varepsilon, \gamma) := \left[ t_* - (\gamma \sigma / \varepsilon) \sqrt{\ln(\sigma / \varepsilon)}, t_* \right].
\]

Then, for any \( \gamma_1 > 0 \) and \( \theta > 0 \), we have

\[
P \left( \sup_{t \in T} |\bar{Y}_t - \bar{Z}_t| > \frac{\theta \sigma}{\sqrt{\ln(\sigma / \varepsilon)}} \right) \to 0.
\]

**Proof:** We will use the abbreviations \( h := \varepsilon / \sigma \to 0 \) and \( \psi := \ln(h^{-1}) \to \infty \).

**Step 1:** We derive formulas for the variances of \( \bar{Y}, \bar{Z} \) and their covariances. Note that the explicit solution of (27) is given by

\[
\bar{Z}_t = \sigma \exp(-\Phi(t)) \int_0^t \exp(\Phi(s)) dB_s, \quad \tag{28}
\]

where \( \Phi(t) := \int_0^t \phi(u) du \). Therefore, for the covariance we have

\[
\text{cov}(\bar{Z}_{t_1}, \bar{Z}_{t_2}) = \sigma^2 \int_0^{\min(t_1, t_2)} \exp(2\Phi(s) - \Phi(t_1) - \Phi(t_2)) ds.
\]

As a special case, for the variance we obtain

\[
\text{var} \bar{Z}_t = \sigma^2 \int_0^t \exp(2(\Phi(s) - \Phi(t))) ds. \tag{29}
\]
Replacing \( \phi(t) \) by the constant \( u \) in these formulas yields

\[
\mathcal{Y}_t = \sigma \int_0^t \exp(u(s-t)) \, dB_s; \quad (30)
\]

\[
\text{cov}(\mathcal{Y}_{t_1}, \mathcal{Y}_{t_2}) = \sigma^2 \int_0^{\min(t_1,t_2)} e^{u(2s-t_1-t_2)} \, ds
= \frac{\sigma^2}{2u} \left( e^{-u|t_1-t_2|} - e^{-u(t_1+t_2)} \right); \quad (31)
\]

\[
\text{var} \mathcal{Y}_t = \frac{\sigma^2}{2u} \left( 1 - e^{-2ut} \right). \quad (32)
\]

For the covariance between \( \mathcal{Y} \) and \( \mathcal{Z} \), one obtains from (28) and (30) that

\[
\text{cov}(\mathcal{Y}_t, \mathcal{Z}_t) = \sigma^2 \int_0^t e^{u(s-t)+\Phi(s) - \Phi(t)} \, ds. \quad (33)
\]

We shall show that for \( t \in T \)

\[
\text{var} \mathcal{Y}_t = \frac{\sigma^2}{2u} \left( 1 + O(\varepsilon \ln \varepsilon) \right), \quad (34)
\]

\[
\text{var} \mathcal{Z}_t = \frac{\sigma^2}{2\phi(t)} \left( 1 + O(\varepsilon \ln \varepsilon) \right), \quad (35)
\]

\[
\text{cov}(\mathcal{Y}_t, \mathcal{Z}_t) = \frac{\sigma^2}{\phi(t) + u} \left( 1 + O(\varepsilon \ln \varepsilon) \right), \quad (36)
\]

and

\[
E [ (\mathcal{Y}_{t_2} - \mathcal{Y}_{t_1})^2 ] \leq C \sigma^2 |t_2 - t_1|, \quad t_1, t_2 \leq t_s, |t_2 - t_1| \leq 1. \quad (38)
\]

**Step 2:** Variance analysis: proof of (34) and (35).

Under assumption (15) we have \( T \subseteq [t_s/2, t_s] \).

Relation (34) follows now directly from (32), as

\[
e^{-2ut} \leq e^{-2ut_s/2} = e^{-ud(b-1)e^{-1}} \ll \varepsilon \ln \varepsilon \quad \text{for} \quad t \in T.
\]

Now we move to the proof of (35). Recall that under Assumption P the coefficient function \( \phi \) is bounded away from zero on \([0, t_s] \), namely \( \phi(t) \geq \kappa_{\text{min}} > 0 \), \( t \in [0, t_s] \). This implies that for \( 0 \leq s \leq t \)

\[
\Phi(t) - \Phi(s) = \int_s^t \phi(u) \, du \geq \kappa_{\text{min}} (t-s).
\]

Therefore, for all \( 0 \leq w \leq t \), we have

\[
\int_0^{t-w} e^{2(\Phi(s) - \Phi(t))} \, ds \leq \int_0^{t-w} e^{2\kappa_{\text{min}}(s-t)} \, ds
\leq \int_{-\infty}^{t-w} e^{2\kappa_{\text{min}}(s-t)} \, ds = (2\kappa_{\text{min}})^{-1} e^{-2\kappa_{\text{min}}w}. \]

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We use this for \( w := \kappa_{\text{min}}^{-1} \ln |\varepsilon| \) to get that
\[
\int_0^{t-w} e^{2(\Phi(s) - \Phi(t))} \, ds \leq (2\kappa_{\text{min}})^{-1} \varepsilon^2 \ll \varepsilon \ln |\varepsilon|.
\] (39)

In the essential zone \( s \in [t-w, t] \) we use the Taylor expansion of \( \Phi \):
\[
\Phi(s) - \Phi(t) = \Phi'(t)(s-t) + \frac{1}{2} \Phi''(\tilde{s})(s-t)^2
\]
\[
= \phi(t)(s-t) + \frac{1}{2} \phi'(\tilde{s})(s-t)^2
\]
\[
= \phi(t)(s-t) + \frac{\varepsilon}{2d} U''(\tilde{x})(s-t)^2,
\] (40)

with some \( \tilde{s} \in [s, t] \) and \( \tilde{x} := 1 + \frac{\tilde{s}}{d} \in [1, b] \).

Recall that under Assumption P we have \( |U''(\tilde{x})| \leq K \). Further, using that \( |s-t| \leq w = \kappa_{\text{min}}^{-1} \ln |\varepsilon| \), we obtain
\[
\Phi(s) - \Phi(t) \leq \phi(t)(s-t) + \frac{K}{2d \kappa_{\text{min}}} \varepsilon |\ln |\varepsilon|| (t-s).
\]

It follows that
\[
\int_{t-w}^{t} \exp(2(\Phi(s) - \Phi(t))) \, ds \leq \int_{t-w}^{t} \exp \left( -2 \left( \phi(t) - \frac{K}{2d \kappa_{\text{min}}} |\ln |\varepsilon|| \right) (s-t) \right) \, ds
\]
\[
\leq \int_{0}^{\infty} \exp \left( -2 \left( \phi(t) - \frac{K}{2d \kappa_{\text{min}}} |\ln |\varepsilon|| \right) u \right) \, du
\]
\[
= \frac{1}{2(\phi(t) - \frac{K}{2d \kappa_{\text{min}}} |\ln |\varepsilon||)}
\]
\[
\leq \frac{1}{2\phi(t)} (1 + O(\varepsilon |\ln |\varepsilon|)),
\]

having used again that \( U'' \) (and so \( \phi \)) is bounded away from zero.

Putting together the last relation with (39) and (29) shows the upper bound in (35).

For the lower bound in (35), we argue similarly. Again we set \( w := \kappa_{\text{min}}^{-1} \ln |\varepsilon| \). Using the Taylor expansion of \( \Phi \) from (40), we obtain for \( s \in [t-w, t] \)
\[
\Phi(s) - \Phi(t) \geq \phi(t)(s-t) - \frac{\varepsilon}{d} K (s-t)^2 \geq \left[ \phi(t) + \frac{\varepsilon}{2d} Kw \right] (s-t).
\]

Using this estimate, we get that
\[
\int_{t-w}^{t} e^{2(\Phi(s) - \Phi(t))} \, ds \geq \int_{t-w}^{t} e^{-2(t-s)[\phi(t)+\varepsilon wK/(2d)]} \, ds
\]
\[
= \int_{0}^{w} e^{-2u[\phi(t)+\varepsilon wK/(2d)]} \, du
\]
\[
= \frac{1}{2[\phi(t)+\varepsilon wK/(2d)]} \cdot (1 - e^{-2w[\phi(t)+\varepsilon wK/(2d)]}).
\]
The last term is lower bounded by
\[
\frac{1}{2\phi(t)} \left( 1 - \frac{\varepsilon w K}{2d\phi(t)} \right) \cdot (1 - e^{-2\varepsilon w_{\min}}) \geq \frac{1}{2\phi(t)} \left( 1 - \frac{\varepsilon |\ln \varepsilon| K}{2d\kappa_{\min}^2} \right) \cdot (1 - \varepsilon^2) = \frac{1}{2\phi(t)} \left( 1 - O(\varepsilon |\ln \varepsilon|) \right).
\]

This shows (recall (29)) that \( \text{var} Z_t \geq \frac{1}{2\phi(t)} \left( 1 - O(\varepsilon |\ln \varepsilon|) \right) \), as required.

**Step 3:** Covariance analysis: proof of (36) and (37).

Relation (36) follows in the same way as we proved (35), because the integral in (33) has the same structure as the integral in (29), the only difference being that \( \Phi(t) \) is replaced by \( \Phi(t) + ut \) and all properties used in the above proof carry over.

We now show (37). Using the Taylor expansion of \( \phi \), we get that for \( t \in T \),
\[
\phi(t) = U''(1 + \frac{\varepsilon t}{d}) = u + U''(\tilde{x}) \frac{\varepsilon}{d} (t - t^*_s),
\]
with some \( \tilde{x} \in [1, b] \). Therefore, for \( t \in T \) we have
\[
|\phi(t) - u| \leq \frac{|U''(\tilde{x})|}{d} \cdot \varepsilon |t - t^*_s| \leq C\varepsilon \gamma_1 h^{-1} \sqrt{\psi} \sim C\gamma_1 \sigma \sqrt{|\ln \varepsilon|} \rightarrow 0,
\]
by the assumption (15) and Assumption P. Using this in the following computation, we get:
\[
\mathbb{E} \left[ (\mathcal{G}_t - 3_t)^2 \right] = \text{var} \mathcal{G}_t + \text{var} 3_t - 2 \text{cov} (\mathcal{G}_t, 3_t)
\]
\[
= \sigma^2 \left( \frac{1}{2u} + \frac{1}{2\phi(t)} - \frac{2}{u + \phi(t)} + O(\varepsilon |\ln \varepsilon|) \right)
\]
\[
= \sigma^2 \left( \frac{\phi(t) u + \phi(t)^2 + u^2 + \phi(t) u - 4\phi(t) u}{2u\phi(t)(u + \phi(t))} + O(\varepsilon |\ln \varepsilon|) \right)
\]
\[
= \sigma^2 \left( \frac{(\phi(t) - u)^2}{2u\phi(t)(u + \phi(t))} + O(\varepsilon |\ln \varepsilon|) \right)
\]
\[
= \sigma^2 \left( O((\phi(t) - u)^2) + O(\varepsilon |\ln \varepsilon|) \right)
\]
\[
= \sigma^2 \left( O(\sigma^2 \psi) + O(\varepsilon |\ln \varepsilon|) \right),
\]
as stated.

**Step 4:** Proof of (38).
Let \(0 \leq t_1 \leq t_2 \leq t^*_\). Then, using (28), we have

\[
\begin{align*}
\mathbb{E} \left[ (Z_{t_2} - Z_{t_1})^2 \right] &= \sigma^2 \left( e^{-\Phi(t_2)} - e^{-\Phi(t_1)} \right)^2 \int_0^{t_1} e^{2\Phi(s)} ds + \sigma^2 e^{-2\Phi(t_2)} \int_{t_1}^{t_2} e^{2\Phi(s)} ds \\
&= \sigma^2 \left( e^{\Phi(t_1)} - e^{\Phi(t_2)} - 1 \right)^2 \int_0^{t_1} e^{2\Phi(s)} ds + \sigma^2 \int_{t_1}^{t_2} e^{2\Phi(s)} ds \\
&\leq \sigma^2 \left( \Phi(t_2) - \Phi(t_1) \right)^2 \int_0^{t_1} e^{2\Phi(s)} ds + \sigma^2 \int_{t_1}^{t_2} e^0 ds \\
&\leq \sigma^2 \left( \frac{1}{2\kappa_{\min}} + \sigma^2 (t_2 - t_1) \right) \leq C \sigma^2 (t_2 - t_1),
\end{align*}
\]

if \(t_2 - t_1 \leq 1\), as required in (38).

**Step 5: Probability evaluation.** Define \(D_t := \sigma^{-1}(\mathcal{G}_t - Z_t)\). The claim of the lemma is to show that

\[
\mathbb{P} \left( \sup_{t \in \mathcal{T}} |D_t| > \frac{\theta}{\sqrt{\psi}} \right) \to 0.
\]

Using the union bound and the fact that the length of \(\mathcal{T}\) is \(\gamma h^{-1} \sqrt{\psi}\), it is sufficient to show that

\[
h^{-1} \sqrt{\psi} \sup_{I \subseteq \mathcal{T}, |I| = 1} \mathbb{P} \left( \sup_{t \in I} |D_t| > \frac{\theta}{\sqrt{\psi}} \right) \to 0. \tag{41}
\]

In order to estimate the last probability, we will use the standard techniques from the theory of Gaussian processes, see [13] and [14]. Essentially, we have to estimate the maximal variance of \((D_t)\) as well as the compactness properties of \(I\) w.r.t. the distance induced by \((D_t)\).

Fix \(I \subseteq \mathcal{T}\) with \(|I| = 1\) and recall from (37) that

\[
V_I := \max_{t \in I} \text{var} D_t \leq C (\sigma^2 \psi + \varepsilon |\ln \varepsilon|). \tag{42}
\]

Let \(\bar{E}\) and \(\bar{m}\) denote the expectation and the median of the random variable \(\sup_{t \in I} D_t\), respectively. It is known from the general Gaussian theory that \(\bar{m} \leq \bar{E}\), see e.g. [13, Lemma 12.2].

Assume that we have shown that

\[
\bar{E} \leq \frac{\theta}{2\sqrt{\psi}}. \tag{43}
\]

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Then by concentration principle, see [13, Theorem 12.2], we have

\[
\mathbb{P}(\sup_{t \in I}|D_t| > \frac{\theta}{\sqrt{\psi}}) \leq 2 \mathbb{P}(\sup_{t \in I}|D_t| > \frac{\theta}{\sqrt{\psi}}) \\
\leq 2 \exp \left( -\frac{(\frac{\theta}{\sqrt{\psi}} - \bar{m})^2}{2 V_I} \right) \\
\leq 2 \exp \left( -\frac{(\frac{\theta}{\sqrt{\psi}} - \bar{E})^2}{2 V_I} \right) \\
\leq 2 \exp \left( -\frac{\theta^2}{8 \psi V_I} \right) \\
\leq 2 \exp \left( -\frac{\theta^2}{8 \psi C (\max \{\sigma^2 \psi, \varepsilon \ln \varepsilon\})} \right) \\
\leq 2 \exp \left( -\frac{\theta^2}{16 \psi C \max \{\sigma^2 \psi, \varepsilon \ln \varepsilon\}} \right),
\]

which yields (41) because on the one hand (using (15))

\[
h^{-1} \sqrt{\psi} e^{-c \psi^{-1} \varepsilon^{-1} |\ln \varepsilon|^{-1}} \ll \varepsilon^{-1} \sqrt{|\ln \varepsilon| e^{-c \varepsilon^{-1} |\ln \varepsilon|^{-2}}} \to 0,
\]

while on the other hand – using that we can choose \( \sigma^2 \psi^3 < \delta \) due to (15) – we obtain

\[
h^{-1} \sqrt{\psi} e^{-c \varepsilon^{-1} |\ln \varepsilon|^{-1}} = h^{-1} \sqrt{\psi} e^{-c \psi^{-1} \varepsilon^{-1} |\ln \varepsilon|^{-1}} \\
\leq h^{-1} e^{-c \psi^{-1} \varepsilon^{-1} |\ln \varepsilon|^{-1} / (25)} \cdot \sqrt{\psi} e^{-c \psi^{-1} \varepsilon^{-1} |\ln \varepsilon|^{-2} / (25)} \\
= h^{-1} h^{c / (25)} \cdot \sqrt{\psi} e^{-c \psi^{-1} \varepsilon^{-1} |\ln \varepsilon|^{-2} / (25)} \to 0,
\]

for \( \delta \) chosen small enough.

**Step 6:** We finally show (43).

We shall use the Dudley bound, see [13, Theorem 14.1],

\[
\bar{E} \leq 4 \sqrt{2} \int_0^{\sqrt{V_I}} (\ln N(\rho))^{1/2} d\rho,
\]

where \( N(\rho) \) is the minimal number of \( \rho \)-balls that is needed to cover \( I \) in the process-induced distance

\[
\Delta(s, t) := \mathbb{E}\|D_t - D_s\|^{2^{1/2}}
\]

From the result (58) (and the corresponding result for \( Y \), which is simple to show) one obtains that

\[
\mathbb{E}\|D_t - D_s\|^{2^{1/2}} \leq C|t - s|^{1/2}, \quad t, s \in \mathcal{T},
\]

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showing $N(\rho) \leq C\rho^{-2}$ for $\rho > 0$. This implies that the Dudley integral in (44) is upper bounded by a constant times $\sqrt{V_I|\ln V_I|}$. The claim in (43) follows, if this quantity is of lower order compared to $\theta/\sqrt{\psi}$, i.e. we need to show that $V_I|\ln V_I| \ll \psi^{-1}$. Taking relation (42) into account, this is obtained from the following two relations:

$$
\psi^{-1} \gg \sigma^2 \psi |\ln(\sigma^2 \psi)|, \quad (45)
$$

$$
\psi^{-1} \gg \varepsilon |\ln \varepsilon| |\ln(\varepsilon |\ln \varepsilon)|. \quad (46)
$$

Finally, using (15), we obtain

$$
\sigma^2 \psi |\ln(\sigma^2 \psi)| \ll (\sigma^2 \psi)^{2/3} = (\sigma^2 \psi^3)^{2/3} \psi^{-4/3} \ll \psi^{-4/3} \ll \psi^{-1},
$$

thus proving (45). Furthermore, from

$$
\psi = \ln(\sigma/\varepsilon) \leq |\ln \varepsilon|
$$

inequality (46) follows trivially.

\[ \square \]

A representation of $Z$

Let us return to the SDE system (14) with a scalar function $\phi(\cdot)$, where $\phi(t) = U''(1 + \varepsilon t)$. We will now connect (14) to the scalar processes treated in Lemma 9.

The system (14) can be written in the vector form

$$(Z_1^1, \ldots, Z_{d-1}^1, {\ldots})^\top = \phi(t)A(Z_1^1, \ldots, Z_{d-1}^1, {\ldots})^\top dt + \sigma dB_t,$$

where the $(d - 1) \times (d - 1)$-matrix $A$ is defined by $A_{i,i} = -2$, $A_{i,i+1} = A_{i+1,i} = -1$, and $A_{i,j} = 0$ otherwise. Consider a diagonalization of $A$ in the form $A = Q^\top DQ$, where $D = \text{diag}(\lambda_1, \ldots, \lambda_{d-1})$ and $Q$ being a unitary operator. We only need that all eigenvalues $\lambda_j$, $1 \leq j \leq d - 1$, are negative.

Further, consider the scalar SDEs:

$$
d\bar{Z}_j^i = \lambda_j \phi(t)\bar{Z}_j^i dt + \sigma dB_t^j, \quad t \geq 0, \quad \bar{Z}_0^j = 0, \quad j = 1, \ldots, d, \quad (47)
$$

with independent Brownian motions $B^j$ and the same scalar function $\phi$ as above. Note that up to the prefactors $\lambda_j$, these are the processes treated in Lemma 9. The system of these equations can be rewritten in the vector form with $\bar{Z}_t = (\bar{Z}_1^1, \ldots, \bar{Z}_{d-1}^1)$

$$
d\bar{Z}_t = D\bar{Z}_t \phi(t) dt + \sigma dB_t, \quad \bar{Z}_0 = 0. \quad (48)
$$

\[ \text{Recall that $Z$ has $d+1$ components, namely $(Z^0, \ldots, Z^d)$ with the trivial parts $Z^0 \equiv 0$ and $Z^d_t = d + \varepsilon t$, while in the last equation we only want to represent the non-trivial components of $Z$.} \]
Set
\[ g_t := (g_1^t, \ldots, g_{d-1}^t) := - \int_0^t \exp(\mathcal{A}(\Phi_t - \Phi_s))ds \cdot \nu, \]
and further \( g_0^t \equiv g_d^t \equiv 0 \), where \( \nu^j := j/d, \ j = 1, \ldots, d - 1 \), and \( \Phi_t := \int_0^t \phi(s)ds \). It is simple to check that
\[ A g_t \phi(t) = g_t^i + \nu. \]
This yields the following representation of \((Z_1^t, \ldots, Z_{d-1}^t)\) in terms of processes \((Z_1^t, \ldots, Z_{d-1}^t)\).

**Lemma 10** Assume that the \( Z \) solves \((48)\). Then the following is a solution to \((14)\):
\[ Z_i^t := \frac{i}{d}(\varepsilon t + d) + \varepsilon g_i^t + (Q^\top Z)^i_t, \quad i = 1, \ldots, d - 1, \] (49)
where \((g_i^t)\) are bounded deterministic functions defined above.

**Proof:** Note that
\[
\begin{align*}
(Z_{i+1}^t - 2Z_i^t + Z_{i-1}^t)\phi(t)dt + \sigma d\hat{B}_i^t &= \frac{i + 1 - 2i + i - 1}{d}(\varepsilon t + d)\phi(t)dt + \varepsilon(g_i^{i+1} - 2g_i^i + g_i^{i-1})\phi(t)dt \\
&+ ((Q^\top Z)^{i+1}_t - 2(Q^\top Z)^i_t + (Q^\top Z)^{i-1}_t)\phi(t)dt + \sigma d\hat{B}_i^t \\
&= \varepsilon(\mathcal{A} g_i \phi(t))^{i+1}_t + (\mathcal{A} Q^\top Z_i \phi(t))^{i}_t dt + \sigma d\hat{B}_i^t \\
&= \varepsilon(g_i^t + \nu)^i dt + (Q^\top D Z_i \phi(t))^i dt + \sigma d( Q^\top B)^i_t \\
&= \varepsilon(g_i^t + \nu)^i dt + (Q^\top Z_i)^i dt \\
&= dZ_i^t,
\end{align*}
\]
where we used \((48)\) in the last but one step. Also the initial condition is verified:
\[ Z_0^i = i + \varepsilon g_0^i + (Q^\top Z)^i_0 = i. \]
Let us finally show that the functions \((g_i^t)\) are bounded. Indeed, let \( \mu := \)
\[ \min |\lambda_j| > 0. \] Then
\[
\| g_t \|_\infty \leq \| g_t \|_2 \\
\leq \int_0^t \| \exp(\mathcal{A}(\Phi_t - \Phi_s))\nu \|_2 ds \\
= \int_0^t \| Q^\top \exp(D(\Phi_t - \Phi_s))Q\nu \|_2 ds \\
= \int_0^t \| \exp(D(\Phi_t - \Phi_s))Q\nu \|_2 ds \\
= \int_0^t \left( \sum_{j=1}^{d-1} \exp(-2\lambda_j(\Phi_t - \Phi_s))(Q\nu)^j \right)^2 ds \\
\leq \int_0^t \left( \sum_{j=1}^{d-1} \exp(-2\lambda_j(\Phi_t - \Phi_s))(Q\nu)^j \right)^2 ds \\
= \int_0^t \exp(-\lambda(\Phi_t - \Phi_s)) \cdot \|Q\nu\|_2 ds \\
= \int_0^t \exp(-\lambda) \int_s^t \phi(r)dr \cdot \|\nu\|_2 ds \\
\leq \int_0^t \exp(-\mu) \int_s^t \kappa_{\min} dr \|\nu\|_2 ds \\
= (1 - \exp(-\mu t)) \frac{\|\nu\|_2}{\mu \kappa_{\min}}.
\]

\[ \square \]

**Remark 11** We proved in Lemma \[27\] that the solution to \[27\] satisfies
\[
\text{var} \ Z^j_t \leq C_1 \sigma^2, \quad t \leq t_*, \quad \mathbb{E} [(Z^j_{t_1} - Z^j_{t_2})^2] \leq C_2 \sigma^2 |t_1 - t_2|, \quad t_1, t_2 \leq t_*, |t_1 - t_2| \leq 1,
\]
which carries over to linear combinations of the \( Z^j \), e.g. to the processes \((Q^\top Z)^i\) in the representation \[49\].

### 6.2 Proof of Lemma \[5\]

Let us fix a break position \( i \in \{1, \ldots, d\} \). Our starting point is a representation from Lemma \[10\]
\[
Z^i_t - Z^i_{t-1} = q_t + \varepsilon(g^i_t - g^{i-1}_t) + \sum_{j=1}^{d-1} c_{ij} Z^j_t =: q_t + \varepsilon \Delta^i_t + V^i_t, \quad i = 0, \ldots, d-1,
\]

(50)
where $\Delta_i$ is a bounded deterministic part, and $\mathcal{Z}^j$ are the independent processes from (47) (for this proof, independence is irrelevant). Let $D^i := \sup_{t>0} |\Delta_i^i|$. The exit condition of $i$-th component at a time $s$ is now equivalent to $q_s + \varepsilon \Delta_i^i + V_s^i = b$: in other words, $V_s^i = \frac{q_f}{\sigma}(t_s - s) - \varepsilon \Delta_i^i$. A necessary condition for the break is $V_s^i \geq \frac{\varepsilon}{\sigma}(t_s - s) - \varepsilon D^i$. We may restate it as
\[
P(\tau_{Z,b}^i \leq t) \leq \mathbb{P} \left( \exists s \leq t : V_s^i \geq \frac{\varepsilon}{\sigma}(t_s - s) - \varepsilon D^i \right).
\]

By Remark 11, the centered Gaussian process $V^i$, being a linear combination of the $\mathcal{Z}^j$, inherits their following properties:
\[
\text{var} V_t^i \leq C_1 \sigma^2, \quad t \leq t_s,
\]
\[
\mathbb{E} \left( (V_{t_2}^i - V_{t_1}^i)^2 \right) \leq C_2 \sigma^2 |t_1 - t_2|, \quad t_1, t_2 \leq t_s, |t_1 - t_2| \leq 1,
\]
with the constants $C_1, C_2$ depending on dimension $d$ and potential $U$. By standard arguments of Gaussian process theory these bounds yield
\[
\mathbb{P} \left( \sup_{s \in [t-1,t]} |V_s^i| \geq \sigma R \right) \leq \exp \left( -C_3 R^2 \right), \quad R > 0, 1 \leq t \leq t_s, \quad (51)
\]
with some $C_3 = C_3(C_1, C_2)$.

By using (51), it follows that
\[
P(\tau_{Z,b}^i \leq t) \leq \sum_{k=0}^{\infty} \mathbb{P} \left( \sup_{s \in [t-k-1,t-k] \cap \mathbb{N}} |V_s^i| \geq \frac{\varepsilon}{\sigma}(t_s - t + (k - dD_i)) \right)
\]
\[
\leq \sum_{k=0}^{\infty} \exp \left( -C_3 \frac{\varepsilon}{\sigma}(t_s - t + (k - dD_i))^2 / \sigma^2 \right)
\]
\[
\leq \exp \left( -C_3 \frac{\varepsilon}{\sigma}(t_s - t)^2 / \sigma^2 \right) \sum_{k=0}^{\infty} \exp \left( -2C_3 \frac{\varepsilon^2}{d^2 \sigma^2}(t_s - t)(k - dD_i) \right)
\]
\[
= \exp \left( -C_3 \frac{\varepsilon}{\sigma}(t_s - t)^2 / \sigma^2 \right) \left[ 1 - \exp \left( -2C_3 \frac{\varepsilon^2}{d^2 \sigma^2}(t_s - t) \right) \right]^{-1}
\]
\[
\times \exp \left( 2C_3 \frac{\varepsilon^2}{d^2 \sigma^2}(t_s - t) D_i \right).
\]
Letting here $t = t_s - \gamma_1 \frac{\sigma}{\varepsilon} \sqrt{\ln \left( \frac{\sigma}{\varepsilon} \right)}$, as in the assertion of Lemma 5, we obtain
\[
P(\tau_{Z,b}^i \leq t) \leq \left( \frac{\sigma}{\varepsilon} \right)^{-C_3 \gamma_1^2 / d^2} \left[ \frac{2C_3 \gamma_1 \varepsilon}{d^2 \sigma} \sqrt{\ln \left( \frac{\sigma}{\varepsilon} \right)} \right]^{-1} (1 + o(1))
\]
\[
\leq \left( \frac{\sigma}{\varepsilon} \right)^{-1-C_3 \gamma_1^2 / d^2} \frac{d^2}{2C_3 \gamma_1} \left( \ln \left( \frac{\sigma}{\varepsilon} \right) \right)^{-1/2} (1 + o(1)) \to 0,
\]

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if we choose $\gamma_1$ so large that $C_3^2 \gamma_1^2/d^2 > 1$. This finishes the proof of Lemma 5.

6.3 Proof of Lemma 6

As above, we set $T := [t_* - \gamma_1 \frac{e}{2} \sqrt{\ln \sigma / \varepsilon}, t_*]$.

By Lemma 10 the solutions to (10) and (14) can be represented by

$$X_{u,t} = i d(\varepsilon t + d) + \varepsilon g_i + (Q^\top \mathcal{Y})_i$$

and

$$Z_{i,t} = i d(\varepsilon t + d) + \varepsilon \tilde{g}_i + (Q^\top \mathcal{Z})_i,$$

respectively, where $g$ and $\tilde{g}$ are bounded deterministic functions and $\mathcal{Y}$ and $\mathcal{Z}$ are the vectors of the solutions to (27) with constant prefactor $u$ and varying prefactor $\phi$, respectively.

For the differences, we have

$$X_{u,t} - Z_{i,t} = \varepsilon (g_i - \tilde{g}_i) + (Q^\top (\mathcal{Y} - 3))_i,$$

so that (using the boundedness of $g_t$ and $\tilde{g}_t$) for some $K > 0$ and all $t \geq 0$

$$||X_{u,t} - Z_{i,t}||_\infty \leq \varepsilon K + ||Q^\top (\mathcal{Y} - 3)||_\infty$$

$$\leq \varepsilon K + ||Q^\top (\mathcal{Y} - 3)||_2$$

$$= \varepsilon K + ||\mathcal{Y} - 3||_2$$

$$\leq \varepsilon K + \sqrt{d - 1}||\mathcal{Y} - 3||_\infty.$$

Fix $\theta > 0$. Using that $\varepsilon \ll \sqrt{\ln \frac{\sigma}{\varepsilon}}$, we have, as $\varepsilon, \sigma \to 0$,

$$\mathbb{P} \left( \sup_{t \in T} ||X_{u,t} - Z_{i,t}||_\infty > \frac{\theta \sigma}{\sqrt{\ln \frac{\sigma}{\varepsilon}}} \right)$$

$$\leq \mathbb{P} \left( \sqrt{d - 1} \sup_{t \in T} ||\mathcal{Y}_t - 3||_\infty > \frac{1}{2} \frac{\theta \sigma}{\sqrt{\ln \frac{\sigma}{\varepsilon}}} \right)$$

$$\leq \sum_{i=1}^{d-1} \mathbb{P} \left( \sup_{t \in T} ||\mathcal{Y}_t - 3||_i > \frac{(\theta / (2\sqrt{d - 1})) \sigma}{\sqrt{\ln \frac{\sigma}{\varepsilon}}} \right)$$

by Lemma 9. This finishes the proof of Lemma 6.
6.4 Proof of Proposition 7

Let, as above, \(A\) denote the discrete Laplace operator in one dimension with \(d\) supporting points, i.e. the \((d-1)\)-dimensional square matrix \(A\) with \(A_{i,j} = -2\) when \(i = j\), \(A_{i,j} = 1\) when \(|i-j|=1\), and \(A_{i,j} = 0\) otherwise. The largest eigenvalue \(\lambda_1\) of \(A\) is strictly negative, namely \(\lambda_1 = -2(1 - \cos(\pi/d))\).

By Assumption P, \(U''(x)\) is continuous and strictly positive on \([1,b]\). Therefore, there exist some \(r > 0\) and \(u_2 = u_2(r) > 0\) such that \(U''(x) > u_2(r)\) for all \(x \in [1,b+r]\). By the continuity of the third derivative of \(U\), we have \(u_3(r) := \sup\{|U'''(x)|: |x| \leq b+r\} < \infty\). We define the constant

\[ c_r := c(U,d,r) := \frac{6(d-1)u_3(r)}{|\lambda_1|u_2(r)} \]

and the stopping time

\[ t_r(\omega) := \inf\{t \in \mathbb{R} : \exists i \leq n \text{ with } |X^i_t(\omega) - X^{i-1}_t(\omega)| \geq b+r\} \wedge t_s. \]

**Proposition 12** Let \(r, c_r\) be as above. For all \(\omega \in \Omega\) and all \(t \leq t_r(\omega)\) we have

\[ S^*_r(\omega) \leq c_r((S^*_r(\omega))^2 + \frac{1}{2}(M^*_r(\omega))^2). \]

**Proof:** Let \(t\) be arbitrary at first. We define \(\mathbf{W}_t = \mathbf{X}_t - \mathbf{Z}_t\). Since \(\mathbf{Z}\) and \(\mathbf{X}\) are driven by the same Brownian motions, the process \(\mathbf{W}\) fulfills

\[ d\mathbf{W}^i_t = d\mathbf{X}^i_t - d\mathbf{Z}^i_t = U'(X^{i+1}_t - X^i_t)\,dt - U'(X^i_t - X^{i-1}_t)\,dt \]
\[ - U''(q_t)(Z^{i+1}_t - Z^i_t)\,dt. \]

By adding and subtracting the term \(U''(q_t)(X^{i+1}_t + X^{i-1}_t - 2X^i_t)\,dt\), and making the definition

\[ (\psi_t(x))^i := U'(x^{i+1} - x^i) - U'(x^i - x^{i-1}) - U''(q_t)(x^{i+1} + x^{i-1} - 2x^i), \]

we obtain

\[ d\mathbf{W}^i_t = (\psi_t(X^i_t))^i + U''(q_t)(\mathbf{W}^{i+1}_t - \mathbf{W}^i_t - 2\mathbf{W}^i_t)\,dt \]

In matrix notation, using the discrete Laplacian, this reads as

\[ d\mathbf{W}_t = \psi_t(\mathbf{X}_t) + U''(q_t)A\mathbf{W}_t\,dt, \]

and we get

\[ \mathbf{W}_t = \int_0^t e^{\int_s^t U''(q_u)\,du} \psi_s(\mathbf{X}_s)\,ds. \]
and the second order remainder terms give the estimate \( q(52) \) around the point and thus \( s \) for all \( \tilde{t} \) and since \( Z \) for all \( \tilde{t} > t_0 \) and all \( \omega \in \Omega \).

When we Taylor expand the first two terms on the right hand side of around the point \( q_0 \), the terms of order one cancel the third term there, and the second order remainder terms give the estimate

\[
|\psi_s(x)| \leq \frac{1}{2} |U''(\xi_+)(x^{i+1} - x^i - q_s)| + \frac{1}{2} |U''(\xi_-)(x^{i} - x^{i-1} - q_s)|
\]

where \( \xi_+ \) lies between \( q_s \) and \( x^{i+1} - x^i \), and where \( \xi_- \) lies between \( q_s \) and \( x^{i-1} \).

With \( s \leq t_r(\omega) \) and \( x^i = X_s(\omega) \), the definition of \( t_r(\omega) \) and the fact that \( 0 \leq q_s \leq b \) for \( s \leq t \leq t_0 \leq t_s \) yield \( |\xi_\pm| < b + r \). Therefore,

\[
|\psi_s(X_s(\omega))| \leq \frac{u_3(r)}{2} \left( (X_s^{i+1}(\omega) - X_s^i(\omega) - q_s)^2 + (X_s^i(\omega) - X_s^{i-1}(\omega) - q_s)^2 \right),
\]

and thus

\[
\|\psi_s(X_s(\omega))\|_2 \leq (d-1) \sum_{i=1}^{d-1} |\psi_s(X_s(\omega))| \leq (d-1) u_3(r) \sum_{i=1}^{d} \left( (X_s^i(\omega) - X_s^{i-1}(\omega) - q_s)^2 \right).
\]

for all \( s \leq t_r(\omega) \). By the inequality

\[
(X_s^i - X_s^{i-1} - q_s)^2 \leq 3(Z_s^i - X_s^i)^2 + 3(Z_s^{i-1} - X_s^{i-1})^2 + 3(Z_s^i - Z_s^{i-1} - q_s)^2,
\]

and since \( Z^0 \equiv X^0 \) and \( Z^d \equiv X^d \), we obtain

\[
\|\psi_s(X_s(\omega))\|_2 \leq 6(d-1) u_3(r) \left( \|X_s(\omega) - Z_s(\omega)\|_2^2 + \frac{1}{2} \sum_{i=1}^{d} |Z_s^i - Z_s^{i-1} - q_s|^2 \right).
\]

Inserting this into the inequality \( \|Z_{\tilde{t}} - X_{\tilde{t}}\|_2 \leq \frac{6(d-1) u_3(r)}{u_2(\rho)} \lambda_1 \left( (S_{\tilde{t}}^s(\omega))^2 + \frac{1}{2} (M_{\tilde{t}}^s(\omega))^2 \right) \) for all \( \tilde{t} \leq t_r(\omega) \). The maps \( \tilde{t} \mapsto (S_{\tilde{t}}^s(\omega)) \) and \( \tilde{t} \mapsto (M_{\tilde{t}}^s(\omega)) \) are monotone increasing. Therefore for \( t \leq t_r(\omega) \), we obtain the result by taking the supremum over \( \tilde{t} \) for both sides of the above inequality. \( \square \)
Proof of Proposition 7. We decompose

\[ P(S_t^* \geq R) = P(S_t^* \geq R, t \leq t_r, \sqrt{2}c_r M_t^* < 1) \]  
(54)

\[ + P(S_t^* \geq R, t > t_r, \sqrt{2}c_r M_t^* < 1) \]  
(55)

\[ + P(S_t^* \geq R, \sqrt{2}c_r M_t^* \geq 1). \]  
(56)

The term (56) is simply estimated by \( P(\sqrt{2}c_r M_t^* \geq 1) \), giving the third term on the right hand side of the claim.

Turning to the term (55), we will show that for all \( \omega \in \Omega \) with \( S_t(\omega) \geq R \), \( t \leq t_r(\omega) \) and \( \sqrt{2}c_r M_t^*(\omega) < 1 \), the inequality

\[ M_t^*(\omega)^2 \geq \frac{S_t^*(\omega)}{2c_r} \geq \frac{R}{2c_r} \]  
(57)

holds, which then gives the first term on the right hand side of the claim. To see (57), recall first that by Proposition 12 we have

\[ c_r(S_t^*(\omega))^2 - S_t^*(\omega) + \frac{1}{2}c_r(M_t^*(\omega))^2 \geq 0 \]

for all \( s \leq t_r(\omega) \). Since \( t \leq t_r(\omega) \), this inequality holds for all \( s \leq t \). Let \( M > 0 \). The equation \( y^2 + M^2/2 \) has two nonnegative solutions if and only if \( c_r^2 M^2 < 1/2 \), and in this case the smaller one of those is given by

\[ \frac{1 - \sqrt{1 - 2c_r^2 M^2}}{2c_r} < 2c_r M^2. \]

By the condition \( \sqrt{2}c_r M_t^*(\omega) < 1 \) and monotonicity we have \( \sqrt{2}c_r M_t^*(\omega) < 1 \) for all \( s \leq t \); by the above considerations (with \( M = M_t^*(\omega) \)), we find that for all \( s \leq t \) and all \( \omega \) fulfilling the relevant conditions in (54), the value of \( S_t^*(\omega) \) can not be in the interval between the two solutions of the quadratic equation for any \( s \leq t \). Since the function \( s \mapsto S_t^*(\omega) \) is continuous and has the value 0 for \( s = 0 \), it therefore has to stay to the left of the smaller root, and is therefore for \( s = t \) bounded by \( 2c_r M_t^*(\omega)^2 \). We thus arrive at (57).

Finally we discuss the term (55). The considerations of the previous paragraph still apply, but only up to \( s = t_r(\omega) < t \). Therefore while we do not have (57), we still know that

\[ S_{t_r(\omega)}^*(\omega) \leq 2c_r M_{t_r(\omega)}^*(\omega) \]

for all \( \omega \) relevant to (55). Since \( t_r(\omega) < t \leq t_s \), by definition of \( t_r(\omega) \), there is at least one \( i \leq d \) with \( |X_{t_r(\omega)}^i(\omega) - X_{t_r(\omega)}^{i-1}(\omega)| = b + r \), and we get

\[ b + r = |X_{t_r(\omega)}^i(\omega) - X_{t_r(\omega)}^{i-1}(\omega)| \leq |X_{t_r(\omega)}^i(\omega) - Z_{t_r(\omega)}^i(\omega)| \]

\[ + |Z_{t_r(\omega)}^i(\omega) - Z_{t_r(\omega)}^{i-1}(\omega)| \leq |Z_{t_r(\omega)}^i(\omega) - Z_{t_r(\omega)}^{i-1}(\omega)| + 2S_{t_r(\omega)}^*(\omega) \]

\[ \leq |Z_{t_r(\omega)}^i(\omega) - Z_{t_r(\omega)}^{i-1}(\omega)| + 4c_r(M_{t_r(\omega)}^*(\omega))^2. \]
Therefore,
\[
M_t^*(\omega) \geq |Z_{i,t}(\omega) - Z_{i-1,t}(\omega)| \geq |Z_{i,t}(\omega) - Z_{i-1,t}(\omega)| - b \geq r - 4c_t(M_{i,t}(\omega))^2 \geq r - 4c_t(M_t^*(\omega))^2.
\]
Therefore, \(M_t^*(\omega) + 4c_t(M_t^*)(\omega)^2 \geq r\) for all \(\omega\) relevant to (55), and we obtain the second term on the right hand side of the claim. \(\Box\)

6.5 Proof of Proposition 8

By Proposition 7, it is enough to prove that for \(t = t^* = d(b - 1)\) the right hand side of (25) tends to zero, as \(\varepsilon, \sigma \to 0\).

Under assumption (5) we have \(\delta \to 0\). Therefore, the first probability in the right hand side of (25) dominates two others. It remains to prove that for every \(\theta > 0\) we have
\[
\lim_{\varepsilon, \sigma \to 0} \mathbb{P}(M_{i,t^*} \geq \sqrt{\delta}/C) = 0.
\]
Since \(\theta > 0\) is arbitrary, we may drop \(C\) here. Furthermore, by using the union bound, it is sufficient to prove that
\[
\lim_{\varepsilon, \sigma \to 0} \mathbb{P}(\sup_{0 \leq s \leq t^*} |Z_s^i - Z_{s-1}^i - q_s| \geq \sqrt{\delta}) = 0, \quad i = 1, \ldots, d.
\]
We fix \(i\). By representation (50) we have
\[
\mathbb{P}(\sup_{0 \leq s \leq t^*} |Z_s^i - Z_{s-1}^i - q_s| \geq \sqrt{\delta}) \leq \mathbb{P}(\sup_{0 \leq s \leq t^*} |V_s^i| \geq \sqrt{\delta} - \varepsilon D_i) \leq (t^* + 1) \max_{0 \leq k \leq [t^*]} \mathbb{P}(\sup_{k \leq s \leq k+1} |V_s^i| \geq \sqrt{\delta} - \varepsilon D_i).
\]
Under (5) we have
\[
\varepsilon \delta = \varepsilon \theta \sigma \sqrt{\ln \left(\frac{\sigma}{\varepsilon} \right)} \to 0.
\]
Hence, \(\varepsilon \ll \delta \ll \sqrt{\varepsilon}\), and eventually \(\sqrt{\delta} - \varepsilon D_i \geq \sqrt{\delta}/2\). By using (41), we obtain
\[
\mathbb{P}(\sup_{0 \leq s \leq t^*} |Z_s^i - Z_{s-1}^i - q_s| \geq \sqrt{\delta}) \leq (t^* + 1) \exp \left\{ -\frac{C_3 \delta}{4 \sigma^2} \right\}.
\]
Notice that \(t^* \approx \varepsilon^{-1}\), while under assumption (5)
\[
\sigma^{-2} \delta = \theta \sigma^{-1} \sqrt{\ln \left(\frac{\sigma}{\varepsilon} \right)} \geq \theta \sigma^{-1} / \sqrt{\ln \varepsilon} = \theta \sigma (\ln \varepsilon)^{1/2} |\ln \varepsilon| \gg |\ln \varepsilon|,
\]
which completes the proof of Proposition 8.

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