GEOMETRIC SCHUR DUALITY OF CLASSICAL TYPE

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Abstract. This is a generalization of the classic work of Beilinson, Lusztig and MacPherson. We show that the quantum algebras obtained via a BLM-type stabilization procedure in the setting of partial flag varieties of type $B/C$ are two (modified) coideal subalgebras of the quantum general linear Lie algebra. These coideal algebras and a new variant of canonical bases arose in a recent approach to Kazhdan-Lusztig theory of type $B$ developed by two of the authors, where a Schur-type duality between such a coideal algebra and Iwahori-Hecke algebra of type $B$ was established. Here we provide a geometric realization of this Schur-type duality. The monomial bases and canonical bases of the modified coideal algebras are constructed for the first time. Moreover, precise connections between the two modified coideal algebras and their distinguished bases are developed. In an Appendix, a compatibility between canonical bases for modified coideal algebras and quantum Schur algebras is established.

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1. Introduction

1.1. In an influential 1990 paper [BLM], Beilinson, Lusztig and MacPherson (BLM) provided a geometric construction of the quantum Schur algebra and more importantly the (modified) quantum groups $\mathcal{U}(\mathfrak{gl}(N))$ and $\mathcal{U}(\mathfrak{gl}(N))$, using the partial flag varieties of type $A$. They further constructed a canonical basis for this modified quantum group (see Lusztig [Lu93] for generalization to all types). The BLM construction has played a fundamental role in categorification; see Khovanov-Lauda [KhL10]. The BLM construction is adapted subsequently by Grojnowski and Lusztig [GL92] to realize geometrically the Schur-Jimbo duality [Jim86]. These works raised an immediate question which remains open until now:

(Q1). What are the quantum algebras arising from flag varieties of classical type?

At the beginning, one was even tempted to hope that the corresponding Drinfeld-Jimbo quantum groups [Dr86] would provide the answer. However, the expectation for an answer being the quantum groups of classical type was somewhat diminished after Nakajima’s quiver variety construction [Na94] which has provided a powerful geometric realization of integrable modules of (quantum) Kac-Moody algebras of symmetric type. Nevertheless, there has been a successful generalization to the affine type $A$ in [GV93, Lu99, VV99, SV00] via affine flag varieties of type $A$.

In a seemingly unrelated direction two of the authors [BW13] recently developed a new approach to Kazhdan-Lusztig theory for the BGG category $\mathcal{O}$ of classical type by initiating a new theory of canonical basis arising from quantum symmetric pairs, and used it to solve the irreducible character problem for the ortho-symplectic Lie superalgebras. At a decategorification level, a duality was established in loc. cit. between a quantum algebra (denoted by $\mathcal{U}^i$ or $\mathcal{U}^j$) and the Iwahori-Hecke algebra of type $B$ acting on a tensor space, generalizing the Schur-Jimbo duality (we shall refer to this new duality as $\mathcal{i}$Schur duality, where $i$ partly stands for “involution”).

The aforementioned quantum algebras $\mathcal{U}^j$ and $\mathcal{U}^i$ are the so-called coideal subalgebras of the quantum group $\mathcal{U}(\mathfrak{gl}(N))$ and they form quantum symmetric pairs $(\mathcal{U}(\mathfrak{gl}(N)), \mathcal{U}^j)$ and $(\mathcal{U}(\mathfrak{gl}(N)), \mathcal{U}^i)$, depending on whether $N$ is odd or even. A general theory of quantum...
symmetric pairs was developed by Letzter [Le02]. The categorical significance of \(\ddagger\)Schur duality [BW13] raises the following natural question:

(Q2). Is there a geometric realization of \(\ddagger\)Schur duality and \(\ddagger\)canonical basis?

1.2. The goal of this paper is to settle the two questions (Q1) and (Q2) for type \(B/C\) completely by showing they provide answers to each other. The coideal algebras admit modified (i.e., idempotented) versions \(\hat{U}^j\) and \(\hat{U}^i\), following the by-now-standard algebraic construction. We show that \(\hat{U}^j\) and \(\hat{U}^i\) are the quantum algebras à la BLM using the geometry of partial flag varieties of type \(B/C\). As applications, the \(\ddagger\)canonical bases of the quantum algebras \(\hat{U}^j\) and \(\hat{U}^i\) are constructed for the first time, precise relations between \(\hat{U}^i\) and \(\hat{U}^j\) are established, and the \(\ddagger\)Schur dualities are realized geometrically à la Grojnowski-Lusztig.

1.3. Let us describe our constructions and results in detail.

We provide in Section 2 a geometric convolution construction of a \(\ddagger\)Schur \(\mathscr{A}\)-algebra \(S^j\) on pairs of \(N\)-step partial flags of type \(B_d\) as a convolution algebra, where \(\mathscr{A} = \mathbb{Z}[v, v^{-1}]\). This algebra and the Iwahori-Hecke algebra of type \(B_d\) (via a similar convolution construction on pairs of complete flags) satisfy a double centralizer property acting on an \(\mathscr{A}\)-module \(T_d\), which is also defined geometrically. We compute the geometric action of the Iwahori-Hecke algebra on \(T_d\) explicitly.

In Section 3, various basic multiplication formulas for \(S^j\), \(H_{B_d}\), and their commuting actions on \(T_d\) are worked out precisely (whose type \(A\) counterparts can be found in [BLM GL92]). We establish a generating set for \(S^j\) and several explicit relations satisfied by these generators. From geometry, we construct a standard basis, a monomial basis, and a canonical basis of \(S^j\), respectively. The signed canonical basis of \(S^j\) is shown to be characterized by an almost orthonormality similar to the more familiar Lusztig-Kashiwara canonical basis [Lu90, Ka91, Lu93] arising from the Drinfeld-Jimbo quantum groups.

In Section 4, we establish a remarkable stabilization property for \(S^j\) as \(d\) goes to infinity in a suitable sense, following the original approach of [BLM] (see [DDPW] for an exposition). This stabilization allows us to construct an \(\mathscr{A}\)-algebra \(K^j\). The algebra \(K^j\) is again naturally equipped with a standard basis, a monomial basis, and a canonical basis of \(S^j\), respectively. The signed canonical basis of \(S^j\) is shown to be characterized by an almost orthonormality similar to the more familiar Lusztig-Kashiwara canonical basis [Lu90, Ka91, Lu93] arising from the Drinfeld-Jimbo quantum groups.

In Section 5, we establish a remarkable stabilization property for \(S^j\) as \(d\) goes to infinity in a suitable sense, following the original approach of [BLM] (see [DDPW] for an exposition). This stabilization allows us to construct an \(\mathscr{A}\)-algebra \(K^j\). The algebra \(K^j\) is again naturally equipped with a standard basis, a monomial basis, and a canonical basis of \(S^j\), respectively. The signed canonical basis of \(S^j\) is shown to be characterized by an almost orthonormality similar to the more familiar Lusztig-Kashiwara canonical basis [Lu90, Ka91, Lu93] arising from the Drinfeld-Jimbo quantum groups.

In Section 6, we establish a remarkable stabilization property for \(S^j\) as \(d\) goes to infinity in a suitable sense, following the original approach of [BLM] (see [DDPW] for an exposition). This stabilization allows us to construct an \(\mathscr{A}\)-algebra \(K^j\). The algebra \(K^j\) is again naturally equipped with a standard basis, a monomial basis, and a canonical basis of \(S^j\), respectively. The signed canonical basis of \(S^j\) is shown to be characterized by an almost orthonormality similar to the more familiar Lusztig-Kashiwara canonical basis [Lu90, Ka91, Lu93] arising from the Drinfeld-Jimbo quantum groups.
leads to an integral $\mathcal{A}$-form of $\breve{U}^\dagger$. Among others, the geometric meaning of a distinguished generator $t$ in $\breve{U}^\dagger$ is made precise as sums of simple perverse sheaves, up to a shift. By the geometric constructions we have algebra embeddings $S^\dagger \subset S^\dagger$ and $K^\dagger \subset K^\dagger$, which are surprising from the algebraic definitions of $\breve{U}^\dagger$ and $\breve{U}^\dagger$. We then establish a remarkable property that the standard bases, the monomial bases, as well as the canonical bases are all compatible in a strong sense under such algebra embeddings.

The constructions of the previous sections are further adapted in the setting of flag varieties of type $C$ in Section 6. It is interesting to note a Langlands type duality phenomenon, namely, $\breve{U}^\dagger$ arises most naturally in the type $C$ setting as done in Section 4 (for type $B$ where $\breve{U}^\dagger$ arises), and then $\breve{U}^\dagger$ appears in a refined type $C$ construction.

In Appendix A written by three of the authors, we show that the surjective homomorphism $\phi_d : K^\dagger \to S^\dagger$ in Section 4 maps an arbitrary canonical (respectively, standard) basis element of $K^\dagger$ to a canonical (respectively, standard) basis element in $S^\dagger$ or zero; see Theorem 6.10. An analogous compatibility of standard and canonical bases for $K^\dagger$ and $S^\dagger$ under the homomorphism $\phi_d : K^\dagger \to S^\dagger$ in Section 5 also holds.

We caution that the convention for $U^\dagger$, $U^\dagger$, and $H_{B_d}$ used in this paper are chosen to fit most naturally to the convention from geometry, and it is not quite the same as in [BW13].

1.4. This paper forms an essential part of a larger program as outlined in [BW13 §0.5]. There are a few followup projects of [BW13] and this paper which will be pursued elsewhere. The type $D$ case will be treated separately. The coideal subalgebras of the quantum affine algebras of type $A$ as well as classical symmetric pairs will also be studied in depth. The constructions of canonical bases with (partly conjectural) positivity are leading to an categorification program, in which the geometric constructions in this paper will play a fundamental role.

Notations: $\mathbb{N}$ denotes the set of nonnegative integers, $[a, b]$ denotes the set of integers between $a$ and $b$.

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2. Geometric convolution algebras of type $B$

In this section, we construct a Schur algebra $S^\dagger$ and Iwahori-Hecke algebra $H_{B_d}$ via convolution products in the framework of partial flag varieties of type $B_d$. We also construct a $(S^\dagger, H_{B_d})$-bimodule $T_d$ geometrically.

2.1. Preliminaries in type $A$. Let us fix a pair $(N, D)$ of positive integers. Let $\mathbb{F}_q$ be a finite field of $q$ elements where $q$ is always assumed to be odd in this paper. We shall denote by $|U|$ the dimension of a vector space $U$ over $\mathbb{F}_q$. Consider the following data:

- The general linear group $GL(D)$ over $\mathbb{F}_q$ of rank $D$,
- The variety $\tilde{X}$ of $N$-step flags $V = (0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{N-1} \subseteq V_N = \mathbb{F}_q^D)$ in $\mathbb{F}_q^D$,
- The variety $\tilde{Y}$ of complete flags $F = (0 = F_0 \subset F_1 \subset \cdots \subset F_{D-1} \subset F_D = \mathbb{F}_q^D)$ in $\mathbb{F}_q^D$. 
Let $GL(D)$ act diagonally on the products $\tilde{X} \times \tilde{X}$, $\tilde{X} \times \tilde{Y}$ and $\tilde{Y} \times \tilde{Y}$. To a pair $(V, V') \in \tilde{X} \times \tilde{X}$, we can associate an $N \times N$ matrix $A = (a_{ij})$ by setting

$$a_{ij} = |(V_{i-1} + V_i \cap V'_j)/(V_{i-1} + V_i \cap V'_{j-1})|, \quad \forall 1 \leq i, j \leq N.$$  

The above assignment $(V, V') \mapsto A$ defines a bijection

$$GL(D) \backslash \tilde{X} \times \tilde{X} \leftrightarrow \Theta_D,$$

where $GL(D) \backslash \tilde{X} \times \tilde{X}$ is the set of $GL(D)$-orbits in $\tilde{X} \times \tilde{X}$ and

$$\Theta_D = \left\{ A = (a_{ij}) \in \text{Mat}_{N \times N}(\mathbb{N}) \mid \sum_{i,j \in [1,N]} a_{ij} = D \right\}.$$  

Here and below, $\text{Mat}_{k \times \ell}(R)$ denotes the set of $k \times \ell$ matrices with coefficients in $R$. Similar to (2.1), we have bijections

$$GL(D) \backslash \tilde{X} \times \tilde{Y} \leftrightarrow \tilde{\Pi}, \quad GL(D) \backslash \tilde{Y} \times \tilde{Y} \leftrightarrow \tilde{\Sigma},$$

where

$$\tilde{\Pi} = \left\{ B = (b_{ij}) \in \text{Mat}_{N \times D}(\mathbb{N}) \mid \sum_{i \in [1,N]} b_{ij} = 1, \forall j \in [1, D] \right\},$$

$$\tilde{\Sigma} = \left\{ \sigma = (\sigma_{ij}) \in \text{Mat}_{D \times D}(\mathbb{N}) \mid \sum_{i \in [1,D]} \sigma_{ij} = 1 = \sum_{j \in [1,D]} \sigma_{ij}, \forall i, j \in [1, D] \right\}.$$  

By [BLM] and [GL92], we have

$$\#\tilde{\Sigma} = D!, \quad \#\tilde{\Pi} = N^D, \quad \text{and} \quad \#\Theta_D = \binom{N^2 + D - 1}{D}.$$  

For any $N \times N$ matrix $A = (a_{ij})$, we define

$$\text{ro}(A) = \left( \sum_j a_{1j}, \sum_j a_{2j}, \ldots, \sum_j a_{Nj} \right),$$

$$\text{co}(A) = \left( \sum_i a_{i1}, \sum_i a_{i2}, \ldots, \sum_i a_{iN} \right).$$  

2.2. **Parametrizing $O(D)$-orbits.** We fix a pair $(n, d)$ of positive integers such that

$$N = 2n + 1, \quad D = 2d + 1,$$

where $(N, D)$ is a pair of positive integers considered in Section 2.1.

Let us fix a non-degenerate symmetric bilinear form $Q : \mathbb{F}_q^D \times \mathbb{F}_q^D \rightarrow \mathbb{F}_q$. Let $O(D)$ be the orthogonal subgroup of $GL(D)$ consisting of elements $g$ such that $Q(gu, gu') = Q(u, u')$ for any $u, u' \in \mathbb{F}_q^D$. If $U$ is a subspace of $\mathbb{F}_q^D$, we write $U^\perp$ for its orthogonal complement. Consider the following subsets of $\tilde{X}$ and $\tilde{Y}$:

- $X = \{ V = (V_k) \in \tilde{X} \mid V_i = V_j^\perp, \text{ if } i + j = N \};$
- $Y = \{ F = (F_i) \in \tilde{Y} \mid F_i = F_j^\perp, \text{ if } i + j = D \}.$

It is well known that $O(D)$ acts on $X$ and $Y$. Let $O(D)$ act diagonally on $X \times X$, $X \times Y$ and $Y \times Y$, respectively. Consider the following subsets of $\Theta_D$, $\tilde{\Pi}$ and $\tilde{\Sigma}$:

- $\Xi_d = \{ A = (a_{ij}) \in \Theta_D \mid a_{ij} = a_{N+1-i,N+1-j}, \forall i, j \in [1, N] \};$
\[ \Pi = \{ B = (b_{ij}) \in \tilde{\Pi} \mid b_{ij} = b_{N+1-i,D+1-j}, \forall i \in [1,N], j \in [1,D] \} \]

\[ \Sigma = \{ \sigma = (\sigma_{ij}) \in \tilde{\Sigma} \mid \sigma_{ij} = \sigma_{D+1-i,D+1-j}, \forall i, j \in [1,D] \}. \]

Note that \( a_{n+1,n+1} \) is odd for all \( A \in \Xi_d \), and similarly, \( b_{n+1,d+1} = 1 \) for all \( B \in \Pi \). Also note that \( ^tA \) denotes the transpose of \( A \).

**Lemma 2.1.** The bijections in (2.1) and (2.2) induce the following bijections:

\[ (2.4) \quad O(D) \setminus X \times X \longleftrightarrow \Xi_d, \quad O(D) \setminus X \times Y \longleftrightarrow \Pi, \quad \text{and} \quad O(D) \setminus Y \times Y \longleftrightarrow \Sigma. \]

(We shall denote the orbit corresponding to a matrix \( A \) by \( O_A \).)

**Proof.** The third bijection is the well-known Bruhat decomposition. We shall only prove the first one since the second one is similar.

Pick a pair \( (V, V') \in X \times X \). Let \( A \) be the associated matrix under the bijection (2.1). We must show that \( A \in \Xi_d \). Observe that we have

\[ a_{ij} = |V_i \cap V_j'|-|V_{i-1} \cap V_j'|-|V_i \cap V'_{j-1}|+|V_{i-1} \cap V'_{j-1}|. \]

Since \( V_i = V'_{N-i} \), we have

\[ |V_i \cap V_j'| = |V'_{N-i} \cap V'_{N-j}| = |(V_{N-i} + V'_{N-j})| \]

\[ = D - |V_{N-i} - V'_{N-j}| + |V_{N-i} \cap V'_{N-j}|. \]

This implies that \( a_{ij} = a_{N+1-i,N+1-j} \). So we have \( A \in \Xi_d \).

Then we need to show that the map \( (V, V') \mapsto A \) is surjective. If we take a pair \( (F, F') \in Y \times Y \) and throw away \( F_i \) and \( F'_j \) for some fixed \( i \) and \( j \), we get a pair \( (V, V') \in X \times X \) (with the choice of \( n = d - 1 \)). Suppose that \( \sigma \) is the associated matrix of \( (F, F') \), it is clear from [BLM, 1.1] that the associated matrix \( A \) of \( (V, V') \) is the one obtained from \( \sigma \) by merging, i.e., adding componentwise, the \( i \)-th row (resp. \( (D + 1 - i) \)-th) and \( (i + 1) \)-th row (resp. \( (D - i) \)-th row) and then merging the \( j \)-th column (resp. \( (D + 1 - j) \)-th) and \( (j + 1) \)-th column (resp. \( (D - j) \)-th column). So any matrix \( A \) in \( \Xi_d \) can be obtained from a not necessarily unique \( \sigma \in \Sigma \) by repetitively applying the above observation. From this, we see that there is a pair \( (V, V') \in X \times X \), obtained from a pair in \( Y \times Y \) by throwing away subspaces at appropriate steps, such that its associated matrix is \( A \). This shows that the map defined by \( (V, V') \mapsto A \) is surjective.

We are left to show that the assignment of each \( O(D) \)-orbit of \( (V, V') \) the matrix \( A \) is injective. Without loss of generality, we assume that \( n \leq d \) and \( V_i \neq V_j \) for any \( i \neq j \). Under such assumption, we have a nonzero entry at each row and each column and each flag in \( X \) can be obtained from a flag in \( Y \) by dropping certain steps of flags. Let \( (V, V') \) and \( (\tilde{V}, \tilde{V}') \) be two pairs in \( X \times X \) such that their associated matrices are the same, say \( A \). We further assume that all entries in \( A \) are either 0 or 1, except the entries \( (i_0, j_0) \) and \( (N + 1 - i_0, N + 1 - j_0) \) at which \( A \) takes value 2. The general case can be shown in a similar argument. Since \( O(D) \) acts transitively on \( X \) and \( Y \), we can assume that \( V = \tilde{V} \). We suppose that \( (V, V') \) and \( (\tilde{V}, \tilde{V}') \) are obtained from two pairs \( (F, F') \) and \( (\tilde{F}, \tilde{F}') \) in \( Y \times Y \), respectively. It is clear then that the associated matrices \( (F, F') \) and \( (\tilde{F}, \tilde{F}') \) differ only at a rank 2 submatrix whose upper left corner is \( (i_0, j_0) \). If the associated matrix of \( (F, F') \) is one of them, the associated matrix of the pair \( (\tilde{F}, \tilde{F}') \) is the other, where \( \tilde{F}' \) is a flag such that \( F'_i = \tilde{F}'_i \) for all \( i \neq j_0, D + 1 - j_0 \) and \( F'_{i_0} \neq \tilde{F}'_{i_0} \). From this, we see that there is a \( g \) in
the stabilizer of $F$ in $O(D)$ such that $g(F_i) = g(F'_i)$ for any $i \neq i_0, D + 1 - i_0$. This shows that $(V, V')$ and $(V, V'')$ are in the same $O(D)$-orbit. We are done. □

To each $B = (b_{ij}) \in \Pi$, we associate a sequence $\mathbf{r} = (r_1, \ldots, r_D)$ of integers where

(2.5) \[ r_c \text{ is the integer such that } b_{r_c,c} = 1 \text{ for each } c \in [1, D]. \]

Note that $\mathbf{r}$ is completely determined by the $d$-tuple $r_1 \cdots r_d$. This defines the following bijections

\[ \Pi \leftrightarrow \{ \text{sequences } \mathbf{r} = (r_1, \ldots, r_D) \text{ with each } r_c \in [1, N] \land r_c + r_{D+1-c} = N + 1 \} \]

(2.6) \[ \leftrightarrow \{ \text{sequences } r_1 \cdots r_d \text{ with each } r_c \in [1, N] \}. \]

Hence we can and shall denote the characteristic function of the $O(D)$-orbit $O_B$ by $e_{r_1 \cdots r_d}$.

The following is a counterpart of (2.3).

**Lemma 2.2.** We have $\#\Sigma = 2^d \cdot d!$, $\#\Pi = (2n + 1)^d$, and $\#\Xi_d = \binom{2n^2 + 2n + d}{d}$.

**Proof.** The first identity is well known and can be seen directly. The second identity follows from the bijection we defined before the lemma. We shall show the third identity.

Any matrix $A \in \Xi_d$ subject to the following condition:

\[ \sum_{i,j \in [1,N]} a_{ij} = 2 \sum_{i \in [1,N]} a_{ij} + 2 \sum_{j \in [1,N]} a_{n+1,j} + a_{n+1,n+1} = 2d + 1. \]

So $a_{n+1,n+1}$ must be a positive odd number in $[1, 2d + 1]$ and

\[ \sum_{i \in [1,N], j \in [1,N]} a_{ij} + \sum_{j \in [1,N]} a_{n+1,j} = \frac{2d + 1 - a_{n+1,n+1}}{2}. \]

Thus

\[ \#\Xi_d = \sum_{l=0}^{d} \binom{2n^2 + 2n - 1 + d - l}{d - l} = \binom{2n^2 + 2n + d}{d}. \]

The lemma follows. □

### 2.3. Convolution algebras in action

Let $v$ be an indeterminate and $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. We define

\[ \mathcal{S}^d = \mathcal{A}_{O(D)}(X \times X), \quad \mathcal{T}_d = \mathcal{A}_{O(D)}(X \times Y), \quad \mathcal{H}_{B_d} = \mathcal{A}_{O(D)}(Y \times Y) \]

to be the space of $O(D)$-invariant $\mathcal{A}$-valued functions on $X \times X$, $X \times Y$, and $Y \times Y$ respectively. (Note by Lemma 2.1 that the parametrizations of the $G$-orbits are independent of the finite fields $\mathbb{F}_q$.) For $A \in \Xi_d$, we denote by $e_A$ the characteristic function of the orbit $O_A$. Then $\mathcal{S}^d$ is a free $\mathcal{A}$-module with a basis $\{ e_A \mid A \in \Xi_d \}$. Similarly, $\mathcal{T}_d$ and $\mathcal{H}_{B_d}$ are free $\mathcal{A}$-modules with bases parameterized by $\Pi$ and $\Sigma$, respectively.

We define a convolution product $\ast$ on $\mathcal{S}^d$ as follows. For a triple of matrices $(A, A', A'')$ in $\Xi_d \times \Xi_d \times \Xi_d$, we choose $(f_1, f_2) \in \mathcal{O}_{A''}$, and we let $g_{A,A',A'';q}$ be the number of $f \in X$ such that $(f_1, f) \in \mathcal{O}_A$ and $(f, f_2) \in \mathcal{O}_{A'}$. A well-known property of the Iwahori-Hecke algebra implies that there exists a polynomial $g_{A,A',A''} \in \mathbb{Z}[v^2]$ such that $g_{A,A',A'';q} = g_{A,A',A''}|_{v=\sqrt{q}}$ for every odd prime power $q$. We define the convolution product on $\mathcal{S}^d$ by letting

\[ e_A \ast e_{A'} = \sum_{A''} g_{A,A',A''} e_{A''}. \]
Equipped with the convolution product, the $\mathcal{A}$-module $S^j$ becomes an associative $\mathcal{A}$-algebra. (A completely analogous convolution product gives us an $\mathcal{A}$-algebra structure on $H_{B_d}$, which is well known to be the Iwahori-Hecke algebra of type $B_d$.)

An analogous convolution product (by regarding the triples $(A, A', A'')$ as in $\Xi_d \times \Pi \times \Pi$ and $f \in Y$) gives us a left $S^j$-action on $T_d$: a suitably modified convolution gives us a right $H_{B_d}$-action on $T_d$. These two actions commute and hence we have obtained an $(S^j, H_{B_d})$-bimodule structure on $T_d$. Denote

$$\varrho S^j = Q(v) \otimes_{\mathcal{A}} S^j, \quad \varrho H_{B_d} = Q(v) \otimes_{\mathcal{A}} H_{B_d}, \quad \varrho T_d = Q(v) \otimes_{\mathcal{A}} T_d.$$ 

**Remark 2.3.** Let us write $A$ by

$$\begin{align*}
H_{B_d}\text{-bimodule structure on } S^j & \implies Equipped with the convolution product, the $\mathcal{A}$-module $S^j$ becomes an associative $\mathcal{A}$-algebra. (A completely analogous convolution product gives us an $\mathcal{A}$-algebra structure on $H_{B_d}$, which is well known to be the Iwahori-Hecke algebra of type $B_d$.)

An analogous convolution product (by regarding the triples $(A, A', A'')$ as in $\Xi_d \times \Pi \times \Pi$ and $f \in Y$) gives us a left $S^j$-action on $T_d$: a suitably modified convolution gives us a right $H_{B_d}$-action on $T_d$. These two actions commute and hence we have obtained an $(S^j, H_{B_d})$-bimodule structure on $T_d$. Denote

$$\varrho S^j = Q(v) \otimes_{\mathcal{A}} S^j, \quad \varrho H_{B_d} = Q(v) \otimes_{\mathcal{A}} H_{B_d}, \quad \varrho T_d = Q(v) \otimes_{\mathcal{A}} T_d.$$ 

**Remark 2.3.** Let us write $X_n$ for the variety $X$ of the $N$-step (isotropic) flags in $\mathbb{F}_q^D$ and write $^nS^j$ for $S^j$ for now (recall $N = 2n + 1$). Let $M = 2m + 1$ be another odd positive integer. Then via convolutions we can define a left action of $^nS^j$ and a commuting right $^mS^j$-action on $\mathcal{A}_{O(D)}(X_n \times X_m)$ which can be shown to form double centralizers. This is a type $B$ variant of the geometric symmetric Howe duality of the type $A$ considered in [W01].

**2.4. Geometric action of Iwahori-Hecke algebra.** For $1 \leq j \leq d$, we define $T_j \in H_{B_d}$ by

$$T_j(F, F') = \begin{cases} 
1, & \text{if } F_i = F'_i \ \forall i \in [1, d]\{j\} \text{ and } F_j \neq F'_j; \\
0, & \text{otherwise}.
\end{cases}$$

It is well known that $H_{B_d}$ is isomorphic to the Iwahori-Hecke algebra of type $B_d$, which is an $\mathcal{A}$-algebra generated by $T_i$ $(1 \leq i \leq d)$ subject to the relations: $(T_i - v^2)(T_i + 1) = 0$ for $1 \leq i \leq d$, $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for $1 \leq i < d - 1$, $T_i T_j = T_j T_i$ for $|i - j| > 1$, $T_d T_{d-1} T_d T_{d-1} = T_{d-1} T_d T_{d-1} T_d$.

**Lemma 2.4.** The right $H_{B_d}$-action on $T_d$,

$$T_d \times H_{B_d} \to T_d, \quad (e_{r_1 \ldots r_d}, T_j) \mapsto e_{r_1 \ldots r_d} T_j,$$

is given as follows. For $1 \leq j \leq d - 1$, we have

$$e_{r_1 \ldots r_d} T_j = \begin{cases} 
-e_{r_1 \ldots \widehat{r_j} \ldots r_d}, & \text{if } r_j < r_{j+1}; \\
v^2 e_{r_1 \ldots \widehat{r_j} \ldots r_d}, & \text{if } r_j = r_{j+1}; \\
(v^2 - 1)e_{r_1 \ldots r_d} + v^2 e_{r_1 \ldots r_{j-1} r_{j+1} r_j r_{j+2} \ldots r_d}, & \text{if } r_j > r_{j+1}.
\end{cases}$$

Moreover, (recalling $r_{d+2} = N + 1 - r_d$) we have

$$e_{r_1 \ldots r_d} T_d = \begin{cases} 
eq & \text{if } r_d < n + 1; \\
v^2 e_{r_1 \ldots r_d}, & \text{if } r_d = n + 1; \\
(v^2 - 1)e_{r_1 \ldots r_{d-1} r_d} + v^2 e_{r_1 \ldots r_{d-1} r_d}, & \text{if } r_d > n + 1.
\end{cases}$$

**Proof.** It suffices to prove the formula for $v = \sqrt{q}$; by definition we interpret the convolution product over $\mathbb{F}_q$ as $e_{r_1 \ldots r_d} T_j(V, F) = \sum_{F' \in Y} e_{r_1 \ldots r_d}(V, F') T_j(F', F)$.

The above formula (2.7) coincides with the one in [GL92, 1.12], whose proof is also the same as in the type $A$ case. We shall prove (2.8). By definitions, we have

$$e_{r_1 \ldots r_d} T_d(V, F) = \sum_{F' \in Y} e_{r_1 \ldots r_d}(V, F') T_d(F', F') = \sum_{F' \in Y : F' \neq F, \forall i \in [1, d-1]} e_{r_1 \ldots r_d}(V, F').$$
From the above expression, we see that the value of \( e_{r_1 \cdots r_d} T_d \) at \((V, F)\) is zero if the associated sequence of \((V, F)\) is not the one listed in the formula. It is also clear that the calculation is reduced to the case when \( D = 3 \) by comparing the pairs \((V, F)\) with the pair obtained from \((V, F)\) by intersecting with \( F_{d+2} \) and modulo \( F_{d-1} \), and in this case the formula can be derived by a direct computation. (Note that the restriction of \( Q \) to the quotient \( F_{d+2}/F_{d-1} \) is again a non-degenerate form.) □

We set

\[
e_{r_1 \cdots r_d} = v^{#\{(c, c')|c, c' \in [1, d], c < c', r < r'_c\}} + \epsilon e_{r_1 \cdots r_d},
\]

where

\[
\epsilon = \begin{cases} 
1, & \text{if } r_d < n + 1; \\
0, & \text{otherwise}. 
\end{cases}
\]

The formulas \((2.7)\) and \((2.8)\) can be rewritten as follows:

\[
(2.9) \quad \tilde{\epsilon}_{r_1 \cdots r_d} T_j = \begin{cases} 
\bar{v} e_{r_1 \cdots r_j-1 r_{j+1} r_{j+2} \cdots r_d}, & \text{if } r_j < r_{j+1}; \\
v^2 \tilde{e}_{r_1 \cdots r_d}, & \text{if } r_j = r_{j+1}; \\
(v^2 - 1) \tilde{e}_{r_1 \cdots r_d} + v \bar{v} e_{r_1 \cdots r_j-1 r_{j+1} r_{j+2} \cdots r_d}, & \text{if } r_j > r_{j+1},
\end{cases}
\]

for \(1 \leq j \leq d - 1\), and

\[
(2.10) \quad \tilde{\epsilon}_{r_1 \cdots r_d} T_d = \begin{cases} 
\bar{v} e_{r_1 \cdots r_d-1 r_{d+2}}, & \text{if } r_d < n + 1; \\
v^2 \tilde{e}_{r_1 \cdots r_d}, & \text{if } r_d = n + 1; \\
(v^2 - 1) \tilde{e}_{r_1 \cdots r_d} + v \bar{v} e_{r_1 \cdots r_d-1 r_{d+2}}, & \text{if } r_d > n + 1.
\end{cases}
\]

3. Structures of the Schur algebra \(S^\prime\)

In this section, we establish some fundamental multiplication formulas for the algebra \(S^\prime\) and its action on \(T_d\). Then we establish a monomial basis and a canonical basis for \(S^\prime\).

3.1. Relations for \(S^\prime\). We shall use the notation \(U \subset W\) to denote that \(U\) is a subspace of \(W\) of codimension \(a\) (for \(a = 1, 2\)). For \(i \in [1, n], a \in [1, n + 1]\), and for \(V, V' \in X\), we set

\[
e_i(V, V') = \begin{cases} 
v^{-|V_i+1/V'_{i+1}|}, & \text{if } V_i \subset V'_{i+1}, V_j = V_{j'}, \forall j \in [1, n]\{i\}; \\
0, & \text{otherwise},
\end{cases}
\]

\[
f_i(V, V') = \begin{cases} 
v^{-|V'_i/V'_{i+1}|}, & \text{if } V_i \supset V'_{i+1}, V_j = V_{j'}, \forall j \in [1, n]\{i\}; \\
0, & \text{otherwise},
\end{cases}
\]

\[
d_a^{\pm 1}(V, V') = \begin{cases} 
v^{\mp(|V'_a/V'_a-1|)}, & \text{if } V = V'; \\
0, & \text{otherwise}.
\end{cases}
\]

We set \(d_a = d_a^{\pm 1}\). Clearly, \(e_i, f_i, d_a, d_a^{-1}\) lie in the \(\mathcal{A}\)-algebra \(S^\prime\).

Let \(\bar{\cdot}\) be the bar involution on \(\mathcal{A}\) and \(\mathbb{Q}(v)\) by sending \(v \mapsto v^{-1}\). Let

\[
[r] = \frac{v^r - v^{-r}}{v - v^{-1}}, \quad \text{for } r \in \mathbb{Z},
\]

be the (bar-invariant) quantum integer \(r\). In the following proposition we adopt the convention of dropping the product symbol \(*\) to make the formulas more readable.
Proposition 3.1. The following relations hold in $S^j$: for $i, j \in [1, n]$ and $a \in [1, n + 1],$

$$d_id_i^{-1} = d_i^{-1}d_i = 1,$$
$$d_id_j = dijd_j,$$
$$d_je_jd_i^{-1} = v^{\delta_{i,j} - \delta_{i+1,j}}e_j,$$
$$d_if_fd_i^{-1} = v^{-\delta_{i,j} + \delta_{i+1,j}}f_j,$$
$$e_if_j - f_je_i = \delta_{i,j}d_id_i^{-1} - d_i^{-1}d_{i+1},$$  if $i, j \neq n,$
$$e_i^2e_j + e_je_i^2 = [2]e_ie_j,$$  if $|i - j| = 1,$
$$f_i^2f_j + f_jf_i^2 = [2]f_if_j f_i,$$  if $|i - j| = 1,$
$$e_i e_j = e_je_i,$$  if $|i - j| > 1,$
$$f_i f_j = f_j f_i,$$  if $|i - j| > 1,$

(a) $d_{n+1}d_i = d_id_{n+1},$ $d_{n+1}d_{n+1}^{-1} = d_{n+1}^{-1}d_{n+1} = 1,$

(a') $d_{n+1}e_id_{n+1}^{-1} = v^{-2\delta_{n+1,i}}e_i,$ $d_{n+1}f_fd_{n+1}^{-1} = v^{2\delta_{n+1,i}}f_i,$

(b) $e_i^2e_n + e_ne_i^2 = [2](e_ne_ne_n - e_n(vd_n d_{n+1}^{-1} + v^{-1}d_n^{-1}d_{n+1})),

(c) f_i^2e_n + e_nf_i^2 = [2](f_ne_nf_n - (vd_n d_{n+1}^{-1} + v^{-1}d_n^{-1}d_{n+1})f_n).$

Proof. It suffices to prove the formulas when we specialize $v$ to $v \equiv \sqrt{q}$ and then perform the convolution products over $F_q$.

The relations above except the labeled ones are identical to the type $A$ case and hence are verified as in $[BLM]$.

Let us verify (b). Without loss of generality, we assume that $n = 2$. We have

$$e_2e_2(V, V') = \begin{cases} v^{-2(D-3|V_1|-5)}(q + 1)^{\frac{4D-2|V_1|+1}{q-1}} & \text{if } V_1 \subset V_1', |V_1'| < d; \\
v^{-2(D-3|V_1|-5)}(q + 1) & \text{if } |V_1 \cap V_1'| = |V_1| - 1, |V_1'| \neq d; \\
0 & \text{otherwise.} \end{cases}$$

$$f_2e_2e_2(V, V') = \begin{cases} v^{-2(D-3|V_1|-3)}(q + 1)^{\frac{4|V_1|-1}{q-1}} & \text{if } V_1 \subset V_1'; \\
v^{-2(D-3|V_1|-3)}(q + 1) & \text{if } |V_1 \cap V_1'| = |V_1| - 1; \\
0 & \text{otherwise.} \end{cases}$$

$$e_2f_2e_2(V, V') = \begin{cases} v^{-2(D-3|V_1|-4)}\left(\frac{4D-2|V_1|-1}{q-1} + \frac{4|V_1|+1-q}{q-1}\right) & \text{if } V_1 \subset V_1'; \\
v^{-2(D-3|V_1|-4)}(q + 1) & \text{if } |V_1 \cap V_1'| = |V_1| - 1, |V_1'| \neq d; \\
v^{-2(D-3|V_1|-4)} & \text{if } |V_1 \cap V_1'| = |V_1| - 1, |V_1'| = d; \\
0 & \text{otherwise.} \end{cases}$$

Hence (b) follows.
The involution \((V, V') \mapsto (V', V)\) defines an \(\mathcal{A}\)-linear anti-automorphism \(\tau\) on \(S'\) such that \(\tau(d_a) = d_a\) (for \(a = 1, 2, \ldots, n + 1\)) and for any \(V, V' \in X\),
\[
\tau(e_2)(V, V') = v^{-(D-3|V|+1)}f_2(V, V'),
\]
\[
\tau(f_2)(V, V') = v^{D-3|V|+2}e_2(V, V').
\]

By applying the anti-automorphism \(\tau\) to (b), we obtain (c). The verifications of (a) and (a') are easy and will be skipped. \(\square\)

3.2. Multiplication formulas. For \(i, j \in [1, N]\), let \(E_{ij}\) be the standard elementary matrix in \(\text{Mat}_{N \times N}(\mathbb{N})\). Let
\[
E_{ij}^\theta = E_{ij} + E_{N+1-i,N+1-j}.
\]
The \((i, j)\)-entry of \(E_{ij}^\theta\) will be denoted by \(e_{ij}^\theta\). Note that
\[
e_{ij}^\theta = \begin{cases} 2, & \text{if } i = j = n + 1; \\ 1, & \text{otherwise}. \end{cases}
\]
Recall that the set \(\{e_A | A \in \Xi_d\}\) is an \(\mathcal{A}\)-basis for \(S'\). The following lemma is a counterpart of \([\text{BLM}]\) Lemma 3.2.

**Lemma 3.2.**

(a) For \(A, B \in \Xi_d\) such that \(\text{ro}(A) = \text{co}(B)\) and \(B - E_{h,h+1}^\theta\) is a diagonal matrix for some \(h \in [1, n]\), we have
\[
e_B \ast e_A = \sum_{p \in [1, N], a_{h+1,p} \geq e_{h+1,p}^\theta} v^{2\sum_{j>p} a_{h,j}} \frac{v^{2(1+a_{hp})} - 1}{v^2 - 1} e_{A+E_{hp}^\theta} + e_{A-E_{hp}^\theta + E_{h+1,p}^\theta}.
\]

(b) For \(A, C \in \Xi_d\) such that \(\text{ro}(A) = \text{co}(C)\) and \(C - E_{h+1,h}^\theta\) is a diagonal matrix for some \(h \in [1, n]\), we have
\[
e_C \ast e_A = \sum_{p \in [1, N], a_{hp} \geq 1} \frac{v^{2\sum_{j<p} a_{h+1,j}} v^{2(1+a_{hp})} - 1}{v^2 - 1} e_{A-E_{hp}^\theta + E_{h+1,p}^\theta}.
\]

**Proof.** The proof for case (a) with \(h \in [1, n]\) and for case (b) with \(h \in [1, n-1]\) is essentially the same as the proof of \([\text{BLM}]\) Lemma 3.2, and hence will not be repeated here. We shall prove the new case when \(h = n\) in (b) as follows. As before, the proof is further reduced to analogous results over finite fields by specializing \(v\) to \(v \equiv \sqrt{q}\). Under the assumption of (b) and \(h = n\), we have
\[
e_C \ast e_A = \sum_{p \in [1, N], a_{np} \geq 1} \#G_p e_{A-E_{np}^\theta + E_{n+1,p}^\theta},
\]
where the set \(G_p\) consists of all subspaces \(S\) in \(\mathbb{F}_q^D\) determined by the following conditions:

- \(S\) is isotropic;
- \(V_n \subset S\) and \(|S/V_n| = 1\);
- \(V_n \cap V_j' = S \cap V_j'\) for \(j < p\) and \(V_n \cap V_j' \neq S \cap V_j'\) for \(j \geq p\);
- \((V, V')\) is a fixed pair of flags in \(X\) whose associated matrix is \(A - E_{np}^\theta + E_{n+1,p}^\theta\).
This is obtained by an argument similar to [BLM] §3.1. So the problem is reduced to compute the number \( \#G_p \).

First, we consider the case when \( h = n \) and \( p \leq n \). The situation is the same as [BLM] when we observe that the subspace \( V_n + V_n' \cap V_j' \) is isotropic if \( V_n \) and \( V_j' \) are isotropic.

Next, we consider the case when \( h = n \) and \( p = n + 1 \). We set

\[
G_n' = \{ T \text{ isotropic } | \ V_n + V_n' \cap V_n' \subseteq T \subseteq V_n + V_n' \cap V_{n+1}' \}
\]

It is clear that

\[
\#G_{n+1}' = q^{a_{n+1,n+1} + 2 - 1} - q^{a_{n+1,n+1} + 1 - 1}.
\]

We define a map \( \Psi_n' : G_n' \to G_n' \) by \( \Psi_n'(S) = S + S' \cap V_n' \). For a fixed \( T \in G_n' \), we can identify \( W \) with the vector space \( V_n \oplus \frac{V_n + V_n' \cap V_n'}{V_n} \oplus L \) where \( L \) is an isotropic subspace of dimension 1 in \( V_n + V_n' \cap V_{n+1}' \). Under such an identification, we see that the fiber \( \Psi_n^{-1} \) is isomorphic to \( \frac{V_n + V_n' \cap V_n'}{V_n} \). So \( \Psi_n' \) gives us a vector bundle \( G_n' \) over \( G_n' \) with rank equal to \( \#G_{n+1}' = \#G_{n+1}' = q^{a_{n+1,n+1} + 1} \).

So the formula in (b) holds for \( h = n \) and \( p = n + 1 \).

Finally, we consider the case when \( h = n \) and \( p \geq n + 2 \). Let \( G_p' \) be the set of all flags \( W = (W_i)_{1 \leq i \leq n} \) in \( \mathbb{F}_q^D \) subject to the following conditions:

- \( W_i \) is isotropic for \( 1 \leq i \leq n \) and \( V_n \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_p \).
- \( V_n + V_n' \cap V_i' \subseteq W_i \) and \( \frac{W_i}{V_n + V_n' \cap V_i'} = 1 \) for \( 1 \leq i \leq N - p \).
- \( |V_n + V_n' \cap V_i'| = |W_i| \) and \( \frac{W_i}{V_n + V_n' \cap V_i'} = 1 \) for \( N - p + 1 \leq i \leq n \).
- \( W_1 \not\subseteq V_n + V_n' \cap V_p' \).

We define a map \( \Psi : G_p' \to G_p' \) by \( \Psi(S) = (S + S' \cap V_n')_{1 \leq i \leq n} \). Let us fix a flag \( W = (W_i)_{1 \leq i \leq n} \) in \( G_p' \). Then the subspace \( W_1 \) can be rewritten as

\[
W_1 \simeq V_n \oplus \frac{V_n + V_n' \cap V_i'}{V_n} \oplus \langle w_1 \rangle,
\]

where \( w_1 \) is a vector not contained in \( V_n + V_n' \cap V_p' \) and \( \langle w_1 \rangle \) is the subspace spanned by \( w_1 \). One can check that

\[
\Psi^{-1}((W_i)_{1 \leq i \leq n}) \simeq \{ V_n + \langle w_1 + x \rangle | x \in V_n + V_n' \cap V_i' \} \simeq \mathbb{F}_q^{a_{n+1,1}},
\]

if the two vector spaces in (3.5) are identified. This implies that \( \Psi \) is surjective and a vector bundle of fiber dimension \( a_{n+1,1} \).

Let \( I_p \) be the set of all flags \( U = (U_{N-p+1} \subseteq \cdots \subseteq U_n) \) subject to the following conditions:

- \( V_n + V_n' \cap V_{N-p} \subseteq U_{N-p+1} \subseteq V_n + V_n' \cap V_{N-p+1} \) and \( \frac{V_n + V_n' \cap V_{N-p+1}}{U_{N-p+1}} = 1 \);
- \( V_n + V_n' \cap V_{i-1} \not\subseteq U_i \subseteq V_n + V_n' \cap V_i \) and \( \frac{V_n + V_n' \cap V_i}{U_i} = 1 \) for \( N - p + 2 \leq i \leq n \).
We stratify \( G'_p \) as

\[
G'_p = \bigcup_{U \in I_p} G'_{p,U}, \quad G'_{p,U} = \{ W \in G'_p | W_i \cap (V_n + V_n^\perp \cap V'_j) = U_i, \forall N - p + 1 \leq i \leq n \}.
\]

Inside \( G'_{p,U} \), the subspace \( W_n \) is subject to the conditions:

\[
U_n \subset W_n \subset U_n^\perp, \quad W_n \neq V_n + V_n^\perp \cap V'_n \quad \text{and} \quad W_n \not\subset V_n + V_n^\perp \cap V'_{n+1}.
\]

The number of choices for such a \( W_n \) is

\[
\frac{q^{a_{n+1,n+1}+1} - 1}{q - 1} - 1 - q^{a_{n+1,n+1}+1} - 1 = q^{a_{n+1,n+1}}.
\]

Fixing \( W_n \), we see that the number of choices for \( W_{n-1} \) is \( q^{1_{U_n/U_{n-1}}} = q^{a_{n+1,n+1}} \). Inductively, we have

\[ (3.7) \quad \#Z''_{p,U} = \prod_{1 \leq i \leq n} q^{a_{n+1,i+1}}. \]

We now consider the index set \( I_p \). The number of choices for \( U_{N-p+1} \) is \( q^{1_{a_{n+1,n+1+N-p+1}}} \).

Fixing \( U_{N-p+1} \), we see that the number of choices for \( U_{N-p+2} \) is \( q^{a_{n+1,n+1+N-p+2}} = q^{a_{n+1,p}} \). Inductively, we conclude that

\[ (3.8) \quad \#I_p = \frac{q^{1_{a_{n+1,p}+1}} - 1}{q - 1} \prod_{n+2 \leq i \leq p} q^{a_{n+1,i}}. \]

By putting together (3.6), (3.7) and (3.8), we see immediately that

\[
\#G_p = q^{a_{n+1,1}} \#G'_p = q^{a_{n+1,1}} \#I_p \#G'_{p,U} = \frac{q^{1_{a_{n+1,p}+1}} - 1}{q - 1} \prod_{1 \leq i < p} q^{a_{n+1,i}}.
\]

This finishes the proof of the lemma. \( \square \)

We set, for \( a \in \mathbb{Z} \) and \( b \in \mathbb{N} \),

\[
\begin{bmatrix} a \\ b \end{bmatrix} = \prod_{1 \leq i \leq b} \frac{v^{2(a-i+1)} - 1}{v^{2i} - 1}, \quad \text{and} \quad [a] = \begin{bmatrix} a \\ 1 \end{bmatrix}.
\]

We have the following multiplication formulas for the algebra \( S^\gamma \), which is an analogue of [BLM] Lemma 3.4(a1),(b1)].

**Proposition 3.3.** Suppose that \( h \in [1,n] \) and \( R \in \mathbb{N} \).

(a) For \( A, B \in \Xi_d \) such that \( B - RE_{h+1}^\theta \) is diagonal and \( \text{ro}(A) = \text{co}(B) \), we have

\[ (3.9) \quad e_B * e_A = \sum_t v^{2 \sum_{j>u} a_{h,j} t_u} \prod_{u=1}^N \begin{bmatrix} a_{hu} + t_u \\ t_u \end{bmatrix} e_{A+\sum_{u=1}^N t_u(E_{h,u}^\theta - E_{h+1,u}^\theta)}, \]

where \( t = (t_1, \ldots, t_N) \in \mathbb{N}^N \) with \( \sum_{u=1}^N t_u = R \) such that

\[
\begin{cases} 
    t_u \leq a_{h+1,u}, & \text{if } h < n, \\
    t_u + t_{N+1-u} \leq a_{h+1,u}, & \text{if } h = n.
\end{cases}
\]
(b) For \(A, C \in \Xi_d\) such that \(C - R E^\theta_{h+1,h}\) is diagonal and \(\text{co}(C) = \text{ro}(A)\), we have

\[
e_C \ast e_A = \sum_t v^2 \sum_{j<u} a_{h+1,j} t_u v^2 \sum_{j<u<j'} t_u t_j + \sum_{j<u} t_u \left[ \frac{t_u(t_u-1)}{2} \right] \prod_{u < n+1} \left[ a_{n+1,u} + t_u \right]
\]

\(e_C \ast e_A \ast A = \sum_t \left[ a_{n+1,u} + t_u + t_{N-1-u} \right] \prod_{i=0}^{t_u-1} \left[ a_{n+1,n+1} + 1 + 2i \right]^{\frac{(n+1)}{2}} = e_A - \sum_{u=1}^N t_u (E^\theta_{n,u} - E^\theta_{n+1,u}) \),

for \(h < n\), and for \(h = n\),

\[
e_C \ast e_A = \sum_t v^2 \sum_{j<u} a_{n+1,j} t_u v^2 \sum_{j<u<j'} t_u t_j + \sum_{j<u} t_u \left[ \frac{t_u(t_u-1)}{2} \right] \prod_{u < n+1} \left[ a_{n+1,u} + t_u \right]
\]

\(e_C \ast e_A \ast A = \sum_t \left[ a_{n+1,u} + t_u + t_{N-1-u} \right] \prod_{i=0}^{t_u-1} \left[ a_{n+1,n+1} + 1 + 2i \right]^{\frac{(n+1)}{2}} = e_A - \sum_{u=1}^N t_u (E^\theta_{n,u} - E^\theta_{n+1,u}) \),

where \(t = (t_1, \ldots, t_N) \in \mathbb{N}^N\) such that \(\sum_{u=1}^N t_u = R\) and \(t_u \leq a_{h,u}\).

(Note that the above coefficients are in \(A\) since \(a_{n+1,n+1}\) is an odd integer.)

**Proof.** We shall prove (3.11) for \(h = n\) by induction on \(R\) in detail. It is clear that (3.11) holds for \(R = 1\) by Lemma 3.2(b). Let us write \(C_R\) instead of \(C\) in (b) to indicate the dependence on \(R\). Similarly we let \(C_{R+1} \in \Xi_d\) be such that \(C_{R+1} - (R+1) E^\theta_{h+1,h}\) is diagonal and \(\text{co}(C_{R+1}) = \text{ro}(A)\), and let \(C_1\) be such that \(C_1 - E^\theta_{h+1,h}\) is diagonal and \(\text{co}(C_1) = \text{ro}(C_R)\). We have by Lemma 3.2(b) again that

\[
e_C \ast e_C \ast e_A = [R+1] e_{C_{R+1}}.
\]

We write \(A(t) = A - \sum_{u=1}^N t_u (E^\theta_{n,u} - E^\theta_{n+1,u})\) and \(G_{A,t}\) the coefficient of \(A(t)\) in (3.11) for \(t = (t_1, \ldots, t_N)\). So we have

\[
e_C \ast e_C \ast e_A = \sum_{t,s} G_{A,t} G_{A(t),s} e_{A(t+s)},
\]

where the sum runs over all \((s,t)\) such that \(\sum t_u = R\) and \(\sum s_u = 1\). By a direct computation, for any \(r\) such that \(\sum r_u = R + 1\), we have

\[
\frac{1}{[R+1]} \sum_{t+s=r} G_{A,t} G_{A(t),s} = \frac{1}{[R+1]} \sum_s v^2 \sum_{j<s'} t_j [t_{s'}] G_{A,r} = G_{A,r},
\]

where \(s'\) is the unique nonzero position in \(s\). The formula (3.11) follows. The proofs of (3.9) and (3.10) are similar and will be skipped.

### 3.3. The \(S^d\)-action on \(T_d\)

Recall we have an \(A\)-basis \(\{e_{r_1 \cdots r_d} \mid 1 \leq r_1, \ldots, r_d \leq N\}\) for \(T_d\), and there is a bijection (see (2.6)) between these \(d\)-tuples \(r_1 \cdots r_d\) and \(r = (r_1, \ldots, r_D)\) subject to \(r_c + r_{D+1-c} = N + 1\). A (simpler) variant of the proof of Lemma 3.2 gives us the following proposition.

**Proposition 3.4.** The left \(S^d\)-action on \(T_d\) via the convolution product

\[
S^d \times T_d \longrightarrow T_d
\]

is given as follows: for \(1 \leq i \leq n\),

\[
e_i e_{r_1 \cdots r_d} = v^{-\# \{1 \leq k \leq D \mid r_k = i+1\}} \sum_{1 \leq p \leq D \mid r_p = i} v^{2^\# \{1 \leq j < p \mid r_j = i+1\}} e_{r_1' \cdots r_d'},
\]
where \( r' = (r'_1, \ldots, r'_{p}) \) for each \( p \) with \( r_p = i \) satisfies \( \lambda_s' = \lambda_s \) (for \( s \neq p, D + 1 - p \)), \( r'_p = i + 1 \), and \( r'_{D+1-p} = N - i \);

\[
(3.15) \quad f^0_{e_{r_1} \cdots e_{r_d}} = v^{-\# \{1 \leq k \leq D | r_k = i \}} \sum_{1 \leq r \leq D | r_p = i + 1} v^{2 \# \{ p \leq D | r_p = i \}} e_{r''_{1} \cdots r''_{d}},
\]

where \( r'' = (r''_1, \ldots, r''_{D}) \) for each \( r_p = i + 1 \) satisfies \( \lambda_s'' = \lambda_s \) (for \( s \neq p, D + 1 - p \)), \( r''_p = i \), and \( r''_{D+1-p} = N + 1 - i \); and \( d_a^0 e_{r_1 \cdots r_d} = v^{\# \{1 \leq j \leq D | r_j = a \}} e_{r_1 \cdots r_d}. \)

3.4. **A standard basis.** Recall that \( O_A \) is the associated \( O(D) \)-orbit of \( A \). We are interested in computing its dimension over the algebraic closure \( \overline{F}_q \). We first recall that the dimension of \( O(D) \) is \( D(D - 1)/2 \). Next we shall compute the stabilizer of a point \((V, V')\) in \( O_A \). We decompose the vector space \( \overline{F}_q^D \) into \( \overline{F}_q^D = \bigoplus_{1 \leq i,j \leq N} Z_{ij} \) such that

\[
\begin{align*}
V_a &= \bigoplus_{i \leq a, j \leq [1, N]} Z_{ij}, \\
V_b &= \bigoplus_{i \in [1, N], j \leq b} Z_{ij}.
\end{align*}
\]

With respect to the decomposition and the lexicographic order for the set \( \{(i, j) | i, j \in [1, N]\} \), we can choose the bilinear form \( Q \) to be anti-diagonal and identity block matrix on each anti-diagonal position. The Lie algebra of the stabilizer \( G_{V,V'} \) of the point \((V, V')\) in \( O(D) \) is then the space of all linear maps \( x_{(i,j),(k,l)} : Z_{ij} \to Z_{kl} \) satisfying the following conditions.

- \((a)\) \( x_{(i,j),(k,l)} = 0 \) unless \( i \geq k \) and \( j \geq l \).
- \((b)\) \( x_{(i,j),(k,l)} = -tx_{(N+1-k,N+1-l),(N+1-i,N+1-j)} \) \( \forall i, j, k, l \in [1, N] \).

Note that the condition \((a)\) is obtained as in \( [BLM, 2.1] \), while the condition \((b)\) is from the choice of \( Q \). From \((b)\), we see that \( x_{(i,j),(k,l)} = -x_{(i,j),(k,l)} \) if and only if \( i + k = N + 1 \) and \( j + l = N + 1 \). So the dimension of the stabilizer \( G_{V,V'} \) is

\[
\sum_{i+k \leq N+1} a_{ij} a_{kl} + \sum_{i+k = N+1, j+l \leq N+1} a_{ij} a_{kl} + \sum_{i \leq n+1, j \leq n+1} a_{ij} (a_{ij} - 1)/2.
\]

Summarizing, we have proved the following.

**Lemma 3.5.** The dimension of \( O_A \), denoted by \( d(A) \), is given by

\[
d(A) = \sum_{i<k \text{ or } j<l} a_{ij} a_{kl} + \sum_{i+k = N+1, j+l \leq N+1} a_{ij} a_{kl} + \sum_{i \leq n+1 \text{ or } j \leq n+1} a_{ij} (a_{ij} - 1)/2.
\]

(Here the condition \( i<k \text{ or } j<l \) means that \( i < k, i+k < N+1 \) or \( i \geq k, j < l, i+k < N+1 \).)

Denote by \( r(A) \) the dimension of the image of \( O_A \) under the first projection \( X \times X \to X \). Note that \( r(A) = d(B) \), the dimension of the orbit \( O_B \), where \( B \) is a diagonal matrix such that \( b_{ii} = \sum_j a_{ij} \). By applying Lemma 3.5 to the matrix \( B \), we have

\[
(3.15) \quad r(A) = \sum_{i<k, i+k < N+1} a_{ij} a_{kl} + \sum_{i \leq n+1} a_{ij} a_{kl}/2 - \sum_{i \leq n+1} a_{ij}/2.
\]

By Lemma 3.5 and (3.15), we have

\[
(3.16) \quad d(A) - r(A) = \sum_{i<k, j<l} a_{ij} a_{kl} + \sum_{i \leq n+1 \text{ or } j \leq n+1} a_{ij} a_{kl} + \sum_{i \geq n+1} a_{ij} (a_{ij} - 1)/2.
\]
We set
\begin{equation}
[A] = [A]_d = v^{-d(A) + r(A)} e_A, \quad \forall A \in \Xi_d.
\end{equation}
(The notation $[A]_d$ will only be used when it is necessary to indicate the dependence on $d$.) Then \{ $[A] \mid A \in \Xi_d$ \} forms an $\mathcal{A}$-basis for $S^\theta$, which we call a standard basis of $S^\theta$.

**Remark 3.6.** It follows by the same argument as for [BLM, Lemma 3.10] that the assignment
\begin{equation}
[A] \mapsto [\tau A]
\end{equation}
defines an $\mathcal{A}$-linear anti-automorphism on $S^\theta$.

The following is a reformulation of the multiplication formulas for $S^\theta$ in Proposition 3.3.

**Proposition 3.7.** (a) Under the assumptions in Proposition 3.3(a), we have
\begin{equation}
[B] \ast [A] = \sum_t v^\beta(t) \prod_{u=1}^N \left[ a_{hu} + t_u \right] \left[ A + \sum_u t_u (E^\theta_{hu} - E^\theta_{h+1,u}) \right],
\end{equation}
where $t$ is summed over as in Proposition 3.3(a) and
\begin{equation}
\beta(t) = \sum_{j \leq l} a_{hl} t_j - \sum_{j \leq l} a_{hl+1,t_j} + \sum_{j \leq l} t_j t_l + \theta_{h,n} \left( \sum_{j+l \leq N+1} t_j t_l + \sum_{j \leq n+1} t_j (t_j + 1) \right).
\end{equation}

(b) Under the assumptions in Proposition 3.3(b), we have
\begin{equation}
[C] \ast [A] = \sum_t v^\beta(t) \prod_{u=1}^N \left[ a_{hl+1,u} + t_u \right] \left[ A - \sum_u t_u (E^\theta_{hu} - E^\theta_{h+1,u}) \right], \forall h < n,
\end{equation}
where $t$ is summed over as in Proposition 3.3(b) and
\begin{equation}
\beta'(t) = \sum_{j \leq l} a_{hl+1} t_j - \sum_{j \leq l} a_{hl} t_j + \sum_{j \leq l} t_j t_l.
\end{equation}

For $h = n$, we have
\begin{equation}
[C] \ast [A] = \sum_t v^{\beta''(t)} \prod_{u < n+1} \left[ a_{n+1,u} + t_u + t_{N+1-u} \right] \prod_{u > n+1} \left[ a_{n+1,u} + t_u \right] \left[ A - \sum_{u=1}^N t_u (E^\theta_{nu} - E^\theta_{n+1,u}) \right],
\end{equation}
\begin{equation}
\beta''(t) = \sum_{j \geq l} a_{hl+1} t_j - \sum_{j \geq l} a_{hl} t_j + \sum_{j \geq l} t_j t_l - \sum_{l \leq j < n+1} t_j t_l - \sum_{j \leq n+1} t_j (t_j - 1) + R(R - 1).
\end{equation}

**Proof.** By Proposition 3.3, we have
\begin{equation}
\beta(t) = d(X) - r(X) - (d(A) - r(A)) - (d(B) - r(B)) + 2 \sum_{j > u} a_{hu} t_u + 2 \sum_{u} a_{hu} t_u,
\end{equation}
where
\begin{equation}
X = A + \sum_u t_u (E^\theta_{hu} - E^\theta_{h+1,u}).
\end{equation}
By direct computations, we have \( d(B) - r(B) = \sum_{j,u} a_{hj} t_u \). Then by a lengthy calculation, we have

\[
d(X) - r(X) - (d(A) - r(A)) = \sum_{j<l} a_{hj} t_j - \sum_{j<l} a_{h+1,j} t_j + \sum_{j<l} t_j t_l + \delta_{h,n} \left( \sum_{j<l,j+1<N+1} t_j t_l + \sum_{j<n+1} \frac{t_j(t_j+1)}{2} \right).
\]

So we obtain the formula of \( \beta(t) \). The computations for \( \beta'(t) \) and \( \beta''(t) \) are similar. \( \square \)

3.5. A monomial basis. We say that \( A \leq B \) if \( \mathcal{O}_A \subseteq \mathcal{O}_B \) over \( \mathbb{F}_q \). This defines a partial order \( \preceq \) in \( \Xi_d \). Following [BLM 3.5], we define a second partial order \( \preceq \) on \( \Xi_d \) by declaring \( A \preceq B \) if and only if

\[
\sum_{r \leq i \leq j} a_{rs} \leq \sum_{r \leq i \leq j} b_{rs}, \quad \forall i < j,
\]

\[
\sum_{r \geq i \leq j} a_{rs} \leq \sum_{r \geq i \leq j} b_{rs}, \quad \forall i > j.
\]

Note that (3.23) is redundant, since it can be deduced from (3.22) and \( a_{ij} = a_{N+1-i,N+1-j} \). Since the Bruhat orders on Weyl groups of type \( A \) and \( B \) are compatible with each other, the next result follows immediately from [BLM 3.5].

**Lemma 3.8.** If \( A \preceq B \) for \( A, B \in \Xi_d \), then we have \( A \preceq B \).

We introduce a partial order \( \preceq \) on \( \Xi_d \) as follows: for \( A, A' \in \Xi_d \), we say that

\[
A' \preceq A \text{ if and only if } A' \preceq A, \text{ ro}(A') = \text{ro}(A) \text{ and } \text{co}(A') = \text{co}(A).
\]

We write \( A' \sqsubset A \) if \( A' \preceq A \) and \( A' \neq A \).

In the expression “\( M \) + lower terms” below, the “lower terms” represents a linear combination of elements strictly less than \( M \) with respect to the partial order \( \preceq \).

**Lemma 3.9.** Let \( R \) be a positive integer.

(a) Suppose that \( A \in \Xi_d \) satisfies one of the following conditions:

\[
a_{hj} = 0, \quad \forall j \geq k; \quad a_{h+1,k} = R, \quad a_{h+1,j} = 0, \quad \forall j > k, \quad \text{if } h \in [1, n]; \text{ or}
\]

\[
a_{nj} = 0, \quad \forall j \geq k; \quad a_{n+1,k} = R, \quad a_{n+1,j} = 0, \quad \forall j > k, \quad \text{if } h = n, k \in (n + 1, N]; \text{ or}
\]

\[
a_{n+1,j} = 0, \quad \forall j > n + 1; \quad a_{n+1,n+1} = 2R + a, \quad a_{n+1,j} = 0, \quad \forall j > n + 1, \quad \text{if } h = n, k = n + 1,
\]

for some odd integer \( a \). Let \( B \) be the matrix such that \( B - RE^0_{h,h+1} \) is diagonal and \( \text{co}(B) = \text{ro}(A) \). Then

\[
[B] \ast [A] = [M] + \text{lower terms}, \quad \text{where } M = A + R(E^0_{h,k} - E^0_{h+1,k}).
\]

(b) Suppose that \( A \in \Xi_d \) satisfies one of the following conditions:

\[
a_{hj} = 0, \quad \forall j < k; \quad a_{nk} = R; \quad a_{h+1,j} = 0, \quad \forall j \leq k, \quad \text{if } h \in [1, n]; \text{ or}
\]

\[
a_{nj} = 0, \quad \forall j < k; \quad a_{nk} = R; \quad a_{n+1,j} = 0, \quad \forall j \leq k, \quad \text{if } h = n, k \in [1, n].
\]

Let \( C \in \Xi_d \) be a matrix such that \( C - RE^0_{h+1,h} \) is diagonal and \( \text{co}(C) = \text{ro}(A) \). Then

\[
[C] \ast [A] = [M] + \text{lower terms}, \quad \text{where } M = A - R(E^0_{h,k} - E^0_{h+1,k}).
\]
Proof. Observe that \( \beta(t) = \beta'(t) = 0 \) in Proposition 3.3 for \( t \) such that \( t_k = R \) and 0, otherwise. The lemma follows from the same argument as that of [BLM, 3.8] by using again that the partial order \( \preceq \) is compatible with the analogous one in [BLM, 3.5]. \( \square \)

Theorem 3.10. For any \( A \in \Xi_d \), we have
\[
\prod_{1 \leq j \leq h < i \leq N} [D_{i,h,j} + a_{ij} E_{h+1,h}^\theta] = [A] + \text{lower terms}
\]
(3.25)
\[
(\text{this element in } S^j \text{ will be denoted by } m_A),
\]
where the product in \( (S^j,*) \) is taken in the following order: \( (i,h,j) \) proceeds \( (i',h',j') \) if and only if \( i < i' \), or \( i = i', j < j' \), or \( i = i', j = j', h > h' \); the diagonal matrices \( D_{i,h,j} \in \text{Mat}_{N \times N}(\mathbb{N}) \) are uniquely determined by \( \text{ro}(A) \) and \( \text{co}(A) \). Moreover, the product has \( N(N^2-1)/6 \) terms.

Proof. The proof is a slight modification of the proof of [BLM, Proposition 3.9] by using Lemma 3.9. One just needs to be cautious when \( h > n \). In this case, \( E_{h+1,h}^\theta = E_{N-h,N+1-h}^\theta \), from which one uses Lemma 3.9(a).

Let us explain the proof in more details for the \( n = 2 \) (i.e., \( N = 5 \)) case. We start with a diagonal matrix \( D \) such that \( \text{ro}(D) = \text{co}(D) = \text{co}(A) \). We must fill in each off diagonal entries with the desired number. Since all matrices involved satisfy the property that the entries of \( (i,j) \) and \( (N+1-i,N+1-j) \) are the same, we only need to fill in all the entries below the diagonal. We do it by multiplying repetitively from the left a certain matrix. Of course, we always need to have a leading term in each step. To make this work, we use Lemma 3.9 and fill the entries below the diagonal from bottom to top and from right to left, which is exactly the order stated in the theorem.

To this end, we shall first fill in the \((5,4)\)-entry. We multiply \( D_{5,4,4} + a_{54} E_{54}^\theta \) by \( D \) where \( a_{ij} \) is the \((i,j)\)-entry of \( A \) and \( D_{5,4,4} \) is a diagonal matrix such that \( \text{co}(D_{5,4,4} + a_{54} E_{54}^\theta) = \text{ro}(D) \). In particular
\[
[D_{5,4,4} + a_{54} E_{54}^\theta] * [D] = [D_{5,4,4} + a_{54} E_{54}^\theta],
\]
with the right hand side of the desired number at the \((5,4)\)-entry. Next, we fill the \((5,3)\)-entry. We multiply the above product by \( [D_{5,3,3} + a_{53} E_{33}^\theta] \) from the left. Since \( E_{33}^\theta = E_{23}^\theta \), we obtain by Lemma 3.9(a) that the leading term of the resulting product is of the form:
\[
\begin{pmatrix}
* & a_{12} & 0 & 0 & 0 \\
0 & * & a_{13} & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
0 & 0 & a_{53} & * & 0 \\
0 & 0 & 0 & a_{54} & *
\end{pmatrix}
\]
To bring down \( a_{53} \) to the desired position, we multiply the above matrix by \( [D_{5,2,2} + a_{53} E_{32}^\theta] \) from the left. By Lemma 3.9(b), the leading term for the resulting product is
\[
\begin{pmatrix}
* & a_{12} & a_{13} & 0 & 0 \\
0 & * & 0 & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
0 & 0 & 0 & a_{53} & a_{54} \\
0 & 0 & a_{53} & a_{54} & *
\end{pmatrix}
\]
Summarizing, in order to put $a_{53}$ in the $(5, 3)$-entry, we need the piece

$$[D_{5,4,3} + a_{53}E_{54}^\theta] \ast [D_{5,3,3} + a_{53}E_{43}^\theta].$$

We can now apply repetitively the above procedure to the rest of the entries in the prescribed order. In particular, for $a_{52}$, we need the piece

$$[D_{5,4,2} + a_{52}E_{54}^\theta] \ast [D_{5,3,2} + a_{52}E_{43}^\theta] \ast [D_{5,2,2} + a_{52}E_{32}^\theta].$$

For $a_{51}$, we need the piece

$$[D_{5,4,1} + a_{51}E_{54}^\theta] \ast [D_{5,3,1} + a_{51}E_{43}^\theta] \ast [D_{5,2,1} + a_{51}E_{32}^\theta] \ast [D_{5,1,1} + a_{51}E_{21}^\theta].$$

For $a_{43}$, we need the piece

$$[D_{4,3,3} + a_{43}E_{43}^\theta].$$

For $a_{42}$, we need the piece

$$[D_{4,3,2} + a_{42}E_{43}^\theta] \ast [D_{4,2,2} + a_{42}E_{32}^\theta].$$

For $a_{41}$, we need the piece

$$[D_{4,3,1} + a_{41}E_{43}^\theta] \ast [D_{4,2,1} + a_{41}E_{32}^\theta] \ast [D_{4,1,1} + a_{41}E_{21}^\theta].$$

For $a_{32}$, we need

$$[D_{3,2,2} + a_{32}E_{32}^\theta].$$

For $a_{31}$, we need

$$[D_{3,2,1} + a_{31}E_{32}^\theta] \ast [D_{3,1,1} + a_{31}E_{11}^\theta].$$

For $a_{21}$, we need

$$[D_{2,1,1} + a_{21}E_{21}^\theta].$$

By putting the pieces together, we have the theorem for $n = 2$ and the general case follows in the same pattern.

It follows by Remark 3.11 below that the number of the terms in the product (3.25) is half of what is in [BLM 3.9(a)].

**Remark 3.11.** The ordering of the product (3.25) coincides with the one used in [DDPW Theorem 13.24]. Namely, if the superscript $\theta$ in (3.25) is dropped, the left hand side becomes exactly the second half of a similar product in [DDPW Theorem 13.24] (which is basically [BLM 3.9(a)]). As explained in [DDPW Notes for §13.7, pp.589] and the reference therein, the ordering of the products adopted in [DDPW Theorem 13.24] is not the same as the one used in [BLM 3.9(a)], but the resulting products are the same.

Then by Theorem 3.10 the transition matrix from $\{m_A \mid A \in \Xi_d\}$ to the standard basis $\{[A] \mid A \in \Xi_d\}$ is unital triangular, and hence $\{m_A \mid A \in \Xi_d\}$ forms an $\mathcal{A}$-basis of $S'$, which we call a monomial basis of $S'$.

The following lemma follows by definitions.

**Lemma 3.12.** For $i \in [1, n], a \in [1, n + 1]$, we have the following identities in $S'$:

$$f_i = \sum_B [B], \quad e_i = \sum_C [C], \quad d_a = \sum_D v^{-D_{aa}}[D]$$

where the sums are over $B, C, D \in \Xi_d$ such that $B = E_{i,i+1}^\theta$, $C = E_{i+1,i}^\theta$, and $D$ are diagonal, respectively.
The following corollary of Theorem 3.10 is now immediate by applying Lemma 3.12 and a standard Vandermonde-determinant-type argument.

**Corollary 3.13.** The $e_i$, $f_i$, $d_i^{\pm 1}$ and $d_{i+1}^{\pm 1}$ for $i \in [1, n]$ generate the $\mathbb{Q}(v)$-algebra $\mathscr{S}$.  

Bearing in mind the presentations of the standard $v$-Schur algebras of type $A$ (cf. [DDPW]), we expect a presentation of the $\mathbb{Q}(v)$-algebra $\mathscr{S}$ with generators given in the above corollary subject to relations in Proposition 3.1 together with the following additional relations:

$$
(d_{n+1} - v^{-1}) \cdots (d_{n+1} - v^{-D}) = 0, \quad \forall i \in [1, n],
$$

$$
(d_i - 1)(d_i - v^{-1})(d_i - v^{-2}) \cdots (d_i - v^{-d}) = 0, \quad \forall i \in [1, n],
$$

3.6. **A canonical basis.** Let $IC_A$, for $A \in \Xi_d$, be the shifted intersection complex associated with the closure of the orbit $O_A$ such that the restriction of $IC_A$ to $O_A$ is the constant sheaf on $O_A$. Since $IC_A$ is $O(D)$-equivariant, the stalks of the $i$-th cohomology sheaf of $IC_A$ at different points in $O_A$ are isomorphic. Let $H^i_{O_{A'}}(IC_A)$ denote the stalk of the $i$-th cohomology group of $IC_A$ at any point in $O_{A'}$. We set

$$
P_{A', A} = \sum_{i \in \mathbb{Z}} \dim \mathcal{H}^i_{O_{A'}}(IC_A) v^{i - d(A) + d(A')},
$$

(3.26)

$$
\{A\} = \sum_{A' \leq A} P_{A', A}[A'].
$$

When it is necessary to indicate the dependence on $d$ of $\{A\}$, we will sometimes write $\{A\}_d$ for $\{A\}$. By the properties of intersection complexes, we have

$$
P_{A, A} = 1, \quad P_{A', A} \in v^{-1}[v^{-1}] \text{ if } A' < A.
$$

As in [BLM 1.4], we have an anti-linear bar involution $\bar{\cdot}: \mathbb{S} \to \mathbb{S}$ such that

$$
\bar{v} = v^{-1}, \quad \bar{\{A\}} = \{A\}, \quad \forall A \in \Xi_d.
$$

In particular, we have

$$
\overline{[A]} = \sum_{A' \leq A} c_{A', A}[A'], \quad \text{where } c_{A, A} = 1, c_{A', A} \in \mathbb{Z}[v, v^{-1}].
$$

Then $B_d := \{\{A\} \mid A \in \Xi_d\}$ forms an $\mathscr{A}$-basis for $\mathbb{S}_d$, called a *canonical basis*.  

The approach to the canonical basis for $\mathbb{S}_d$ above follows [BLM], and it can also be done following an alternative algebraic approach developed by Du (see [Du92]).

3.7. **An inner product.** We set

$$
d_A = d(A) - r(A).
$$

Then, recalling $^tA$ denotes the transpose of $A$ we have

$$
2(d_A - d_tA) = \frac{1}{2} \left( \sum_{i=1}^{N} \text{ro}(A)_i^2 - \text{co}(A)_i^2 \right) - \frac{1}{2} (\text{ro}(A)_{n+1} - \text{co}(A)_{n+1}).
$$

(3.28)
Given $A, A' \in \Xi_d$, fix any element $L'$ in $X_{\text{col}(A)}$ and set $X_{A}' = \{L \mid (L', L) \in \mathcal{O}_{\text{col}}\}$. A standard argument shows that $\# X_{A}'$ is the specialization at $v = \sqrt{q}$ of a Laurent polynomial $f_{A,A'}(v) \in \mathbb{Z}[v, v^{-1}]$. Following McGerty [Mc12], we define a bilinear form
\[(\cdot, \cdot)_D : S' \times S' \longrightarrow \mathbb{Q}(v)\]
by
\[(e_A, e_{A'})_D = \delta_{A,A'}v^{2(d_A-d_A)}f_{A,A'}(v).\]
By using (3.28) and arguing in exactly the same manner as that of [Mc12 Proposition 3.2], we have the following.

**Proposition 3.14.** $(\{A\}e_{A_1}, e_{A_2})_D = v^{d_A-d_A}(e_{A_1}, [^tA]e_{A_2})_D$, for all $A, A_1, A_2 \in \Xi_d$.

We record the following useful consequence of Proposition 3.14.

**Corollary 3.15.** Let $i \in [1, n]$, $a \in [1, n + 1]$, and let $A_1, A_2 \in \Xi_d$. Then we have
(a) $(e_ie_{A_1}, e_{A_2})_D = (e_{A_1}, v^{-1}d_i-d_{i+1}e_{A_2})_D$;
(b) $(f_ie_{A_1}, e_{A_2})_D = (e_{A_1}, ve_i d_i d_i^{-1}e_{A_2})_D$;
(c) $(d_ie_{A_1}, e_{A_2})_D = (e_{A_1}, d_i e_{A_2})_D$.

The same argument as for [Mc12 Lemma 3.5] now gives us the following.

**Proposition 3.16.**
(a) $(\{A\}, \{A\})_D \in 1 + v^{-1}\mathbb{Z}[v^{-1}]$, for any $A \in \Xi_d$.
(b) $(\{A\}, \{A'\})_D = 0$ if $A \neq A'$.

The next proposition follows from Proposition 3.16 together with the definition and property of the canonical basis given in (3.26) and (3.27).

**Proposition 3.17.**
(a) The canonical basis $B^d_i$ satisfies the almost orthonormality, i.e.,
$(\{A\}, \{A'\})_D \in \delta_{A,A'} + v^{-1}\mathbb{Z}[v^{-1}]$, for any $A, A' \in \Xi_d$.
(b) The signed canonical basis $(-B^d_i) \cup B^d_i$ of the $\mathcal{A}$-module $S'$ is characterized by the almost orthonormal property (a) together with the bar-invariance.

4. **THE ALGEBRA $K^d$ AND ITS IDENTIFICATION AS A COIDEAL ALGEBRA**

In this section we construct an $\mathcal{A}$-algebra $K^d$ out of $S'$ from a stabilization procedure. We then show that $K^d$ is isomorphic to an integral form of a modified coideal algebra $\tilde{\mathcal{U}}^d$. The canonical bases for $K^d$ and $\tilde{\mathcal{U}}^d$ are constructed. A geometric realization of the $(\mathcal{U}^d, H_{B^d})$-duality is established.

4.1. **Stabilization.** Below we shall imitate [BLM §4] to develop a stabilization procedure to construct a limit $\mathcal{A}$-algebra $K^d$ out of $S'$ as $d$ goes to $\infty$. As the constructions are largely the same as loc. cit., we will be sketchy.

Recall $N = 2n + 1$. We introduce the set
\[
\tilde{\Xi} = \{A = (a_{ij}) \in \text{Mat}_{N \times N}(\mathbb{Z}) \mid a_{ij} \geq 0 (i \neq j), a_{n+1,n+1} \in 2\mathbb{Z} + 1, a_{ij} = a_{N+1-i,N+1-j} (\forall i, j)\}.
\]
Let $K^d$ be the free $\mathcal{A}$-module with an $\mathcal{A}$-basis given by the symbols $[A]$, for $A \in \tilde{\Xi}$. Also set
\[
\tilde{\Xi}^{\text{diag}} = \{A \in \tilde{\Xi} \mid A \text{ is diagonal}\}, \quad \Xi := \bigcup_{d \geq 0} \tilde{\Xi}_d.
\]
For $A \in \tilde{\Xi}$, setting $2pA := A + 2pI$ we have $2pA \in \Xi$ for integers $p \gg 0$. Given matrices $A_1, A_2, \ldots, A_f \in \tilde{\Xi}$ (with the same total sum of entries), one shows exactly as in [BLM, 4.2] that there exists $Z_i \in \tilde{\Xi}$ ($1 \leq i \leq m$) such that

$$\frac{[2pA_1] \ast [2pA_2] \ast \cdots \ast [2pA_f]}{\ast} = \sum_{i=1}^{m} G_i(v, v^{-2p})[2pZ_i],$$

where $G_i(v, v')$ lies in a subring $\mathcal{R}_1$ of $\mathbb{Q}(v)[v']$ as defined in [BLM, 4.1]. This allows us to define a unique structure of associative $\mathcal{A}$-algebra (without unit) on $K^j$ (with product denoted by $\cdot$) such that

$$[A_1] \cdot [A_2] \cdots [A_f] = \sum_{i=1}^{m} G_i(v, 1)[Z_i].$$

From the above stabilization procedure, the multiplication formula in Proposition 3.7(a) leads to the following. For any $A, B \in \tilde{\Xi}$ such that $\text{ro}(A) = \text{co}(B)$ and $B - RE^\theta_{h,h+1}^\theta$ is diagonal for some $h \in [1, n]$, we have

$$[B] \cdot [A] = \sum_t v^{\beta(t)} \prod_{u=1}^{N} \left[ a_{hu} + t_u \over t_u \right] \left[ A + \sum_u t_u(E^\theta_{hu} - E^\theta_{h+1,u}) \right],$$

where $\beta(t)$ is defined in (3.18) and $t = (t_1, \ldots, t_N) \in \mathbb{N}^N$ such that $\sum_{u=1}^{N} t_u = R$ and $t_u \leq a_{h+1,u}$ for $u \neq h + 1$ and $h < n$ or $t_u + t_{N+1-u} \leq a_{n+1,u}$ for $u \neq n + 1$ and $h = n$.

Similarly, we obtain the following multiplication formula from the one in Proposition 3.7(b) via the above stabilization procedure. For any $A, C \in \tilde{\Xi}$ such that $\text{ro}(A) = \text{co}(C)$ and $C - RE^\theta_{h,h+1}^\theta$ is diagonal for some $h \in [1, n - 1]$, we have

$$[C] \cdot [A] = \sum_t v^{\beta'(t)} \prod_{u=1}^{N} \left[ a_{h+1,u} + t_u \over t_u \right] \left[ A - \sum_u t_u(E^\theta_{hu} - E^\theta_{h+1,u}) \right],$$

where $\beta'(t)$ is defined in (3.19) and $t = (t_1, \ldots, t_N) \in \mathbb{N}^N$ such that $\sum_{u=1}^{N} t_u = R$ and $0 \leq t_u \leq a_{h,u}$ for $u \neq h$. For $h = n$, we have

$$[C] \cdot [A] = \sum_t v^{\beta''(t)} \prod_{u=n+1}^{t_n} \left[ a_{n+1,u} + t_u + t_{N+1-u} \over t_u \right] \prod_{u=n+1}^{t_n} \left[ a_{n+1,u} + t_u \over t_u \right] \prod_{i=0}^{t_n+1} \left[ a_{n+1,n+1} + i + 2 \over i + 1 \right] \left[ A - \sum_{u=1}^{N} t_u(E^\theta_{nu} - E^\theta_{n+1,u}) \right],$$

where $\beta''(t)$ is defined in (3.21) and $t = (t_1, \ldots, t_N) \in \mathbb{N}^N$ such that $\sum_{u=1}^{N} t_u = R$ and $0 \leq t_u \leq a_{n,u}$ for $u \neq n$.

We extend the partial order $\sqsubseteq$ on $\Xi_d$ to $\tilde{\Xi}$ by the same definition (3.24) with $A, A' \in \tilde{\Xi}$. Now it follows by Theorem 3.10 that

$$\prod_{1 \leq j \leq h < i \leq N} [D_{i,h,j} + a_{ij}E^\theta_{h+1,h}] = [A] + \sum_{A' \subseteq A} \gamma_{A',A}[A'], \quad \text{for } \gamma_{A',A} \in \mathcal{A}$$

(this element in $K^j$ will be denoted by $M_A$),
where the product is taken in the same order as specified in Theorem \[3.10\] and the notation

\[D_{i,h,j}\] can be found therein. Then \[\{M_A \mid A \in \mathbb{Z}\}\] forms an \(\mathcal{A}\)-basis (called the monomial basis) of \(K^j\). Summarizing, we have established the following.

**Proposition 4.1.** The formula (4.1) endows \(K^j\) a structure of associative \(\mathcal{A}\)-algebra (without unit). Moreover, this \(\mathcal{A}\)-algebra structure on \(K^j\) is characterized by the multiplication formulas (4.5)–(4.7).

### 4.2. A canonical basis of \(K^j\).

Just as in \[BLM\ 4.3, 4.5(b)\], we have an anti-linear bar involution on \(K^j\) induced from the ones on \(S^j\) (as \(d\) goes to \(\infty\)). More explicitly, one shows the following stabilization phenomenon of the bar involutions on \(S^j\):

\[
\overline{[2p]A} = \sum_{i=1}^{m} H_i(v, v^{-2p})[2p]T_i \quad \forall p \gg 0.
\]

where \(H_i(v, v') \in \mathbb{Q}(v)[v', v'^{-1}]\). Following loc. cit., we obtain an anti-linear bar involution \(\overline{\cdot} : K^j \to K^j\) defined by

\[
\overline{[A]} = \sum_{i=1}^{m} H_i(v, 1)[T_i].
\]

In particular, we have

\[
\overline{[A] = [A] + \sum_{A': A' \subseteq A, A' \neq A} \tau_{A', A}[A'], \quad \text{for } \tau_{A', A} \in \mathcal{A}}.
\]

By a standard argument (see, e.g., [Lu93 24.2.1]), we have the following.

**Proposition 4.2.** There exists a unique \(\mathcal{A}\)-basis \(B^j = \{\{A\} \mid A \in \mathbb{Z}\}\) for \(K^j\) such that

\[
\{A\} = [A],
\]

\[
\{A\} = [A] + \sum_{A' \subseteq A} \pi_{A', A}[A'], \quad \text{for } \pi_{A', A} \in v^{-1} \mathbb{Z}[v^{-1}].
\]

The basis \(B^j\) is called the canonical basis of \(K^j\).

### 4.3. Definition of \(\hat{U}^j\).

The algebra \(U^j\) is defined to be the associative algebra over \(\mathbb{Q}(v)\) generated by \(e_i, f_i, d_a, d_a^{-1}, i = 1, 2, \ldots, n, a = 1, 2, \ldots, n + 1\) subject to the following relations, for \(i, j = 1, 2, \ldots, n, a, b = 1, 2, \ldots, n + 1\):

\[
\begin{align*}
&d_a d_a^{-1} = d_a^{-1} d_a = 1, \\
d_a d_b &= d_b d_a, \\
d_a e_j d_a^{-1} &= v^{-\delta_{a,j+1}} d_a e_j + \delta_{a,j} e_j, \\
d_a f_j d_a^{-1} &= v^{-\delta_{a,j+1}} + \delta_{a,j+1} + \delta_{n+1-a, j+1} f_j, \\
e_i f_j - f_j e_i &= \delta_{i,j} \frac{d_a d_a^{-1} - d_a^{-1} d_a}{v - v^{-1}}, \quad \text{if } i, j \neq n, \\
e_i^2 e_j + e_j e_i^2 &= [2] e_i e_j e_i, \\
f_i^2 f_j + f_j f_i^2 &= [2] f_i f_j f_i, \\
e_i e_j &= e_j e_i, \\
f_i f_j &= f_j f_i, \\
f_n^2 e_n + e_n f_n^2 &= [2] (f_n e_n f_n - (v d_n d_n^{-1} + v^{-1} d_n^{-1} d_n) f_n), \\
e_n^2 f_n + f_n e_n^2 &= [2] (e_n f_n e_n - e_n (v d_n d_n^{-1} + v^{-1} d_n^{-1} d_n)).
\end{align*}
\]
We also write $e_i = f_{N+1-i}$ and $f_i = e_{N+1-i}$ for $n + 1 < i \leq N$. Denote by $^0 \mathcal{U}^j$ the $\mathbb{Q}(v)$-subalgebra of $\mathcal{U}^j$ generated by $d_a^{\pm 1}$ for all $a = 1, 2, \ldots, n + 1$.

**Remark 4.3.** The $\mathbb{Q}(v)$-subalgebra of $\mathcal{U}^j$ generated by $e_i, f_i, d_i^{\pm 1}, d_{i+1}^{\pm 1}, i = 1, 2, \ldots, n - 1$ is naturally isomorphic to the quantum group $\mathcal{U}(\mathfrak{gl}(n))$. The algebra $\mathcal{U}^j$ and the quantum group $\mathcal{U}(\mathfrak{gl}(N))$ form the quantum symmetric pair $(\mathcal{U}(\mathfrak{gl}(N)), \mathcal{U})$. This is a variant of the quantum symmetric pair associated with the quantum group $\mathcal{U}(\mathfrak{gl}(N))$; see [Le02], [K12] and [BW13, Section 6]. The algebra $\mathcal{U}^j$ (and $\mathcal{U}^i$ in later sections) also appeared independently in [ES13]. The relation between the algebra $\mathcal{U}^j$ defined here and the algebra of the same notation defined in [BW13, Section 6] is just the usual $\mathfrak{sl}$ vs $\mathfrak{gl}$ relation and can be described as follows: the generators $e_i$ and $f_i$ match the generators with $e_{\alpha_{n-i+\frac{1}{2}}}$ and $f_{\alpha_{n-i+\frac{1}{2}}}$ respectively, while

$$d_a d_a^{-1} = \begin{cases} k_{\alpha_{n-a+\frac{1}{2}}} & \text{if } 1 \leq a < n; \\ v k_{\alpha_{\frac{1}{2}}} & \text{if } a = n. \end{cases}$$

The convention on the generators $d_a$ in particular $d_{n+1}$ is made to better match the geometric construction.

**Lemma 4.4.** (cf. [BW13, Lemma 6.1]) The algebra $\mathcal{U}^j$ has an anti-linear bar involution, denoted by $\bar{\cdot}$, such that $\bar{d_a} = d_a^{-1}$, $\bar{e_i} = e_i$, and $\bar{f_i} = f_i$ for $i = 1, \ldots, n$ and $a = 1, \ldots, n + 1$.

Denote by $\mathcal{U}(\mathfrak{gl}(N))$ the quantum general linear Lie algebra of rank $N$, which is $\mathbb{Q}(v)$-algebra generated by $E_i, F_i, K_i^{\pm 1}$, and $K_{i+1}^{\pm 1}$, for $i \in [1, N - 1]$ subject to a standard set of relations which can be found as part of the relations [1.9] in different notations. (Recall from Remark 4.3 that the quantum group $\mathcal{U}(\mathfrak{gl}(n))$ is naturally a subalgebra of $\mathcal{U}^j$ with the corresponding subset of relations from [1.9].)

**Proposition 4.5.** There is an injective $\mathbb{Q}(v)$-algebra homomorphism $\mathcal{J} : \mathcal{U}^j \to \mathcal{U}(\mathfrak{gl}(N))$ given by, for all $i = 1, \ldots, n$,

$$d_i \mapsto K_i^{-1} K_{N+1-i}, \quad e_i \mapsto F_i + K_i^{-1} K_{i+1} E_{N-i}, \quad f_i \mapsto E_i K_{N-i} K_{N+1-i} + F_{N-i}.$$

**Proof.** This is a $\mathfrak{gl}(N)$-variant of [BW13, Proposition 6.2]; also see [K12], [ES13].

Following [BLM] and [Lu93, §23.1], we shall similarly define the modified quantum algebra $\breve{\mathcal{U}}^j$ from $\mathcal{U}^j$, where the unit of $\mathcal{U}^j$ is replaced by a collection of orthogonal idempotents. We shall denote by $\lambda = \text{diag}(\lambda_1, \ldots, \lambda_N)$ a matrix in $\Xi_{\text{diag}}$. For $\lambda, \lambda' \in \Xi_{\text{diag}}$, we set

$$\lambda \mathcal{U}^j_{\lambda'} = \mathcal{U}^j / \left( \sum_{a=1}^{n+1} (d_a - v^{-\lambda_a}) \mathcal{U}^j + \sum_{a=1}^{n+1} \mathcal{U}^j (d_a - v^{-\lambda'_a}) \right).$$

Let $\pi_{\lambda, \lambda'} : \mathcal{U}^j \to \lambda \mathcal{U}^j_{\lambda'}$ be the canonical projection. Set

$$\breve{\mathcal{U}}^j = \bigoplus_{\lambda, \lambda' \in \Xi_{\text{diag}}} \lambda \mathcal{U}^j_{\lambda'}.$$

Let $D_{\lambda} := \pi_{\lambda, \lambda}(1)$. Following [Lu93, 23.1], $\breve{\mathcal{U}}^j$ is naturally an associative $\mathbb{Q}(v)$-algebra containing $D_{\lambda}$ as orthogonal idempotents, and $\mathcal{U}^j$ is naturally a $\mathcal{U}^j$-bimodule. In particular,
we have
\[ \hat{U}^j = \sum_{\lambda \in \Xi^{\text{diag}}} U^j D_\lambda = \sum_{\lambda \in \Xi^{\text{diag}}} D_\lambda U^j. \]

By construction, the following relations hold:
\[ D_\lambda D_{\lambda'} = \delta_{\lambda,\lambda'} D_\lambda, \]
\[ d_a D_\lambda = D_\lambda d_a = v^{-\lambda_a} D_\lambda. \]

For any \( I = (i_1, i_2, \ldots, i_j) \) with \( i_k \in \{1, 2, \ldots, n, n+2, \ldots, N\} \), we set \( \mathcal{E}_I = e_{i_1} e_{i_2} \cdots e_{i_j} \), where \( e_{i_0} = 1 \). Following [12 Proposition 6.2], there exists a collection of such indices \( I \), denoted by \( \mathcal{I} \), such that \( \{ \mathcal{E}_I \mid I \in \mathcal{I} \} \) forms a basis of the free \( 0 \)-module \( U^j \). Therefore the elements \( \mathcal{E}_I D_\lambda (I \in \mathcal{I}, \lambda \in \Xi^{\text{diag}}) \) form a basis of the \( \mathbb{Q}(v) \)-vector space \( \hat{U}^j \).

### 4.4. A presentation of \( \hat{U}^j \)

Given \( \lambda \in \Xi^{\text{diag}} \), we introduce the short-hand notation \( \lambda - \alpha_i = \lambda + E_{i_1+1, i_1+1}^\theta - E_{i_1, i_1}^\theta \) and \( \lambda + \alpha_i = \lambda - E_{i_1+1, i_1+1}^\theta + E_{i_1, i_1}^\theta \), for \( 1 \leq i \leq n \). We define \( \mathbf{A} \) to be the \( \mathbb{Q}(v) \)-algebra generated by the symbols \( D_\lambda, e_i D_\lambda, D_\lambda e_i, f_i D_\lambda \) and \( D_\lambda f_i \), for \( i = 1, \ldots, n \) and \( \lambda \in \Xi^{\text{diag}} \), subject to the following relations (4.10), for \( i, j = 1, \ldots, n \) and \( \lambda, \lambda' \in \Xi^{\text{diag}} \):

\[
\begin{align*}
    xD_\lambda x' &= \delta_{\lambda, \lambda'} xD_\lambda x', \quad \text{for} \ x, x' \in \{ e_i, e_j, f_i, f_j \}, \\
    e_i D_\lambda &= D_{\lambda - \alpha_i} e_i, \\
    f_i D_\lambda &= D_{\lambda + \alpha_i} f_i, \\
    e_i D_{\lambda f_j} &= f_j D_{\lambda - \alpha_i} e_i, \\
    e_i D_{\lambda f_i} &= f_i D_{\lambda - 2\alpha_i} e_i + \left[ \lambda_{i+1} - \lambda_i \right] D_{\lambda - \alpha_i}, \quad \text{if} \ i \neq j, \\
    e_i D_{\lambda f_i} &= f_i D_{\lambda - 2\alpha_i} e_i, \quad \text{if} \ i = j, \\
    (e_i^2 + e_r e_i^2) D_\lambda &= \left[ 2 \right] e_i e_i e_i D_\lambda, \quad \text{if} \ i = j, \\
    (f_i^2 + f_j f_i^2) D_\lambda &= \left[ 2 \right] f_i f_j f_i D_\lambda, \quad \text{if} \ i = j, \\
    f_i f_j f_i D_\lambda &= f_j f_i f_i D_\lambda, \quad \text{if} \ i = j, \\
    (f_n^2 - \left[ 2 \right] f_n e_n f_n + e_n f_n^2) D_\lambda &= -\left[ 2 \right] \left( v^{\lambda_{n+1} - \lambda_n - 2} + v^{\lambda_n - \lambda_{n+1} + 2} \right) f_n D_\lambda, \\
    (e_n^2 f_n - \left[ 2 \right] e_n f_n e_n + f_n e_n^2) D_\lambda &= -\left[ 2 \right] \left( v^{\lambda_{n+1} - \lambda_n + 1} + v^{\lambda_n - \lambda_{n+1} - 1} \right) e_n D_\lambda.
\end{align*}
\]

To simplify the notation, we shall write \( x_1 D_{\lambda_1} x_2 D_{\lambda_2} \cdots x_l D_{\lambda_l} = x_1 x_2 \cdots x_l D_{\lambda_l} \), if the product is not zero; in this case such \( \lambda^1, \lambda^2, \ldots, \lambda^l \) are unique.

**Proposition 4.6.** We have an isomorphism of \( \mathbb{Q}(v) \)-algebras \( \hat{U}^j \cong \mathbf{A} \) by identifying the generators in the same notation. That is, the relations (4.10) are the defining relations of the \( \mathbb{Q}(v) \)-algebra \( \hat{U}^j \).

**Proof.** By definitions of \( \hat{U}^j \) and \( U^j \) with relations (4.9) in (4.3) we see that \( \hat{U}^j \) satisfies the same relations (4.10) as for \( \mathbf{A} \). Hence there is a surjective algebra homomorphism \( \mathbf{A} \to \hat{U}^j \), sending generators to generators in the same notation. Following the presentation of the algebra \( \mathbf{A} \) in (4.10), we see that the algebra \( \mathbf{A} \) has a natural \( U^j \)-bimodule structure, where the actions of \( e_i, f_i \in U^j \) are defined in the obvious way, and the action of \( d_a \) on the idempotents \( D_\lambda \) is given by \( D_\lambda d_a = d_a D_\lambda = v^{-\lambda_a} D_\lambda \). As a left or right \( U^j \)-module, \( \mathbf{A} \) is generated by the idempotents \( D_\lambda \), for \( \lambda \in \Xi^{\text{diag}} \). Hence we have \( \mathbf{A} = \sum_{\lambda \in \Xi^{\text{diag}}} U^j D_\lambda \).

Recall that \( \{ \mathcal{E}_I \mid I \in \mathcal{I} \} \) forms a basis of the free \( 0 \)-module \( U^j \). Therefore we have \( \mathbf{A} = \sum_{I \in \mathcal{I}, \lambda \in \Xi^{\text{diag}}} \mathbb{Q}(v) \mathcal{E}_I D_\lambda \). Since \( \{ \mathcal{E}_I D_\lambda \mid I \in \mathcal{I}, \lambda \in \Xi^{\text{diag}} \} \) forms a \( \mathbb{Q}(v) \)-basis of \( \hat{U}^j \),
these elements in the same notation in $A$ must be linearly independent, and hence the homomorphism $A \to \hat{A}^0$ is an isomorphism.

### 4.5. Isomorphism $\mathcal{A}\hat{U}^0 \cong K^\lambda$.

The following theorem provides a geometric realization of $\hat{U}^0$ thanks to the geometric nature of $K^\lambda$.

**Theorem 4.7.** We have an isomorphism of $\mathbb{Q}(v)$-algebras $\Phi : \hat{U}^0 \to \mathcal{A}K^\lambda$ which sends

\[
e_iD_\lambda \mapsto [D_\lambda - E_{i,i}^\theta + E_{i,i+1}^\theta], \quad f_iD_\lambda \mapsto [D_\lambda - E_{i+1,i+1}^\theta + E_{i+1,i}^\theta],
\]

and $D_\lambda \mapsto [D_\lambda]$, for all $i = 1, \ldots, n$, and $\lambda \in \Xi_{\text{diag}}$.

**Proof.** Via a direct computation we can check using the multiplication formulas (4.5)–(4.7) that the relations (4.10) are satisfied by the images of $D_\lambda, e_iD_\lambda, f_iD_\lambda$ as specified in the lemma. Since the relations (4.10) are defining relations for $\hat{U}^0$ by Proposition 4.6, we conclude that $\Phi$ is an algebra homomorphism.

It remains to show $\Phi$ is a linear isomorphism. Set $\varepsilon_{n+1} = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^{2n+1}$, where 1 is in the $(n+1)$st position. Note that $\varepsilon_{n+1}$, when regarded as in $\Xi_{\text{diag}}$, is simply the diagonal matrix $E_{n+1,n+1}$. Also set

$$\tilde{\Theta} = \{ A = (a_{ij}) \in \text{Mat}_{n \times n}(\mathbb{Z}) \mid a_{ij} \geq 0 \ (i \neq j) \},$$

$$\tilde{\Theta}^- = \{ A \in \tilde{\Theta} \mid a_{ij} = 0 \ (i < j), \co(A) = 0 \},$$

$$\tilde{\Xi}^- = \{ A \in \tilde{\Xi} \mid \co(A) = \varepsilon_{n+1} \}.$$

The diagonal entries of a matrix $A' \in \tilde{\Theta}^-$ (respectively, $A \in \tilde{\Xi}^-$) are completely determined by its strictly lower triangular entries. Hence, there is a natural bijection $\tilde{\Theta}^- \leftrightarrow \tilde{\Xi}^-$, which sends $A' \in \tilde{\Theta}^-$ to $A \in \tilde{\Xi}^-$ such that the strictly lower triangular parts of $A$ and $A'$ are identical.

Denote by $K$ the $\mathcal{A}$-algebra in the type $A$ stabilization of $[BLM]$, and $\mathcal{A}K = \mathbb{Q}(v) \otimes_\mathcal{A} K$. The quantum group $U = U(\mathfrak{gl}(N))$ has a triangular decomposition $U = U^-U^0U^+$. Denote by $\hat{U} = \hat{U}(\mathfrak{gl}(N))$ the modified quantum group of $\mathfrak{gl}(N)$ with idempotents $1_\lambda$ and denote its $\mathcal{A}$-form by $\mathcal{A}\hat{U}$. It was shown in $[BLM]$ that there exists a $\mathbb{Q}(v)$-algebra isomorphism $\Phi^a : \hat{U} \to \mathcal{A}K$ (entirely parallel to the homomorphism $\Phi : \hat{U}^0 \to \mathcal{A}K^\lambda$ above). Here and below we will add superscript $a$ to indicate type $A$ and distinguish from the notations already used in the type $B$ setting of this paper. Recall $K$ has a monomial $\mathcal{A}$-basis $\{a_M | A' \in \tilde{\Theta}\}$ which is given in [BLM] 4.6(c) (without such notation or terminology), entirely parallel to the monomial basis for $K^\lambda$ which we defined in (4.1). Via the isomorphism $\Phi^a$, the monomial $\mathcal{A}$-basis $\{a_M | A' \in \tilde{\Theta}\}$ for the $\mathcal{A}$-submodule $\Phi^a(\mathcal{A}U^-1_0)$ of $K$ corresponds to a monomial $\mathcal{A}$-basis $\{a_M | A' \in \tilde{\Theta}\}$ for the $\mathcal{A}$-submodule $\mathcal{A}U^-1_0$ of $\mathcal{A}\hat{U}$, where $\Phi^a(a_M) = a_M$.

Note that we may regard that $e_i, f_i \in \hat{U}^0$ have “leading terms” $F_i, F_{N-i}$ by Proposition 1.3. By [Le02] (see [K12] Proposition 6.2), replacing the leading terms $F_i, F_{N-i}$ by $e_i$ and $f_i$ respectively, we obtain a monomial $\mathbb{Q}(v)$-basis $\{\tilde{M}_A | A \in \tilde{\Xi}^-\}$ for $\hat{U}^0D_{n+1}$ from the monomial basis $\{a_M | A' \in \tilde{\Theta}\}$ for $U^-1_0$. The homomorphism $\Phi : \hat{U}^0 \to \mathcal{A}K^\lambda$ restricts to a $\mathbb{Q}(v)$-linear map $\Phi|_{\tilde{M}_{n+1}} : \hat{U}^0D_{n+1} \to \mathcal{A}K[D_{n+1}]$, which sends $\tilde{M}_A$ to $M_A$ for $A \in \tilde{\Xi}^-$ (using Remark 3.11 and the definition (4.8) of a monomial basis element as a product of generators). Hence $\Phi|_{\tilde{M}_{n+1}}$ is a $\mathbb{Q}(v)$-linear isomorphism. This leads to a $\mathbb{Q}(v)$-linear isomorphism
Corollary 4.8. The homomorphism $\Phi|_\lambda : \hat{U}^jD_\lambda \to \mathcal{K}^j[D_\lambda]$ (which is a restriction of $\Phi$), for any $\lambda \in \tilde{\Xi}^{\text{diag}}$, via the following commutative diagram:

$$
\begin{array}{ccc}
\hat{U}^jD_{e_{n+1}} & \xrightarrow{\Phi|_{e_{n+1}}^j} & \mathcal{K}^j[D_{e_{n+1}}] \\
\downarrow \Phi|_\lambda & & \downarrow \Phi|_\lambda \\
\hat{U}^jD_\lambda & \xrightarrow{\Phi|_\lambda} & \mathcal{K}^j[D_\lambda]
\end{array}
$$

Here $\Phi|_\lambda : \mathcal{K}^j[D_{e_{n+1}}] \to \mathcal{K}^j[D_\lambda]$ is a $\mathbb{Q}(v)$-linear isomorphism which sends a monomial basis element $m_{A+e_{n+1}}$ to $m_{A+\lambda}$, and $\Phi|_\lambda : \hat{U}^jD_{e_{n+1}} \to \hat{U}^jD_\lambda$ is defined accordingly.

Putting $\Phi|_\lambda$ together, we have shown that $\Phi : \hat{U}^j \to \mathcal{K}^j$ is an isomorphism.

The bar involution on $\hat{U}^j$ (given in Lemma 4.4) induces a compatible bar involution on $\hat{U}^j$, denoted also by $\bar{\cdot}$, which fixes all the generators $D_\lambda, e_iD_\lambda, f_iD_\lambda$.

**Corollary 4.8.** The homomorphism $\Phi$ intertwines the bar involutions on $\hat{U}^j$ and on $\mathcal{K}^j$, i.e., $\Phi(\bar{u}) = \bar{\Phi(u)}$, for $u \in \hat{U}^j$.

**Proof.** The corollary follows by checking when $u$ is a generator of $\hat{U}^j$. □

We define an $\mathcal{A}$-subalgebra of $\hat{U}^j$ by $\mathcal{A}\hat{U}^j := \Phi^{-1}(K^j)$. Clearly we have $\mathcal{A}\hat{U}^j \otimes_{\mathcal{A}} \mathbb{Q}(v) = \hat{U}^j$.

**Corollary 4.9.** The integral form $\mathcal{A}\hat{U}^j$ is a free $\mathcal{A}$-submodule of $\hat{U}^j$ and it is stable under the bar involution.

The isomorphism $\Phi : \mathcal{A}\hat{U}^j \to K^j$ allows us to transport the canonical basis $B^j$ for $K^j$ to a canonical basis for $\mathcal{A}\hat{U}^j$ (and for $\hat{U}^j$). Introduce the divided powers $e_i^{(r)} = e_i^r/[r]!$ and $f_i^{(r)} = f_i^r/[r]!$, for $r \geq 1$, where $[r]! = [r] \cdots [1]$.

**Proposition 4.10.** For $r \geq 1$, we have

$$
\Phi(e_i^{(r)}D_\lambda) = [D_\lambda - rE^\theta_{i,i} + rE^\theta_{i+1,i}] \quad \text{and} \quad \Phi(f_i^{(r)}D_\lambda) = [D_\lambda - rE^\theta_{i,i+1} + rE^\theta_{i,i+1}].
$$

Moreover, the $\mathcal{A}$-algebra $\mathcal{A}\hat{U}^j$ is generated by $e_i^{(r)}D_\lambda$ and $f_i^{(r)}D_\lambda$ for various $i, r$ and $\lambda$.

**Proof.** We shall only show the first identity, as the second one is entirely similar. Following the multiplication formula (4.5), we have

$$
[D_\lambda - E^\theta_{i,i} + E^\theta_{i+1,i}][D_{\lambda''} - (r - 1)E^\theta_{i,i} + (r - 1)E^\theta_{i+1,i}] = [r][D_\lambda - rE^\theta_{i,i} + rE^\theta_{i+1,i}],
$$

where $\text{ro}(D_\lambda - E^\theta_{i,i} + E^\theta_{i+1,i}) = \text{ro}(D_\lambda - rE^\theta_{i,i} + rE^\theta_{i+1,i})$ and $\text{co}(D_\lambda - (r - 1)E^\theta_{i,i} + (r - 1)E^\theta_{i+1,i}) = \text{co}(D_\lambda - rE^\theta_{i,i} + rE^\theta_{i+1,i})$. The first statement follows.

The second statement follows from Theorem 3.10 and the definition of $\mathcal{A}\hat{U}^j$. □

4.6. Homomorphism from $K^j$ to $S^j$. Now we establish a precise relation between the algebras $K^j$ and $S^j$. 
Proposition 4.11. There exists a unique surjective \(\mathcal{A}\)-algebra homomorphism \(\phi_d : K^d \to S^j\) such that, for \(R \geq 0, i \in [1, n]\) and \(D \in \Xi_{\text{diag}}\),
\[
\phi_d([D + RE^{\theta}_{i,i+1}]) = \begin{cases} 
[D + RE^{\theta}_{i,i+1}], & \text{if } D + RE^{\theta}_{i,i+1} \in \Xi_d; \\
0, & \text{otherwise}; 
\end{cases}
\]
\[
\phi_d([D + RE^{\theta}_{i+1,i}]) = \begin{cases} 
[D + RE^{\theta}_{i+1,i}], & \text{if } D + RE^{\theta}_{i+1,i} \in \Xi_d; \\
0, & \text{otherwise}. 
\end{cases}
\]

Hence we have a surjective \(\mathcal{A}\)-algebra homomorphism \(\phi_d \circ \Phi : \hat{U}^j \to S^j\).

Proof. The existence of a homomorphism of \(\mathcal{Q}(v)\)-algebras \(\phi_d : \hat{\mathcal{A}}K^d \to \hat{\mathcal{A}}S^j\) (or equivalently, a homomorphism \(\phi_d \circ \Phi : \hat{\mathcal{A}} \hat{U}^j \to \hat{\mathcal{A}}S^j\)) given by the above formulas with \(R = 0, 1\) follows immediately from Proposition 3.1 and the presentation of \(\hat{U}^j\) (and of \(\hat{\mathcal{A}}K^d\)) from Proposition 4.10 and Theorem 4.17. The surjectivity and uniqueness of such \(\Phi\) are clear.

It follows by the multiplication formulas for \(K^d\) and for \(S^j\) that \(\phi_d\) matches the “divides powers”, as indicated by the formulas in the proposition. Now the proposition follows by noting that these “divided powers” (which corresponds to the divided powers in \(\mathcal{A}\hat{U}^j\) by Proposition 4.10) generate \(K^d\) and \(S^j\).

\[\square\]

4.7. A geometric duality. Recall the embedding \(j : U^j \to U(\mathfrak{gl}(N))\) in Proposition 4.5

Let \(V\) be the natural representation of \(U(\mathfrak{gl}(N))\) with the standard basis \(\{v_1, \ldots, v_N\}\). Then \(V^{\otimes d}\) becomes a \(U(\mathfrak{gl}(N))\)-module via the coproduct:
\[
\Delta : U(\mathfrak{gl}(N)) \longrightarrow U(\mathfrak{gl}(N)) \otimes U(\mathfrak{gl}(N)),
\]
\[
E_i \mapsto 1 \otimes E_i + E_i \otimes K_i K_{i+1}^{-1},
\]
\[
F_i \mapsto F_i \otimes 1 + K_i^{-1} K_{i+1} \otimes F_i,
\]
\[
K_a \mapsto K_a \otimes K_a,
\]
for \(i = 1, \ldots, n\) and \(a = 1, \ldots, n+1\). Then \(V\) and \(V^{\otimes d}\) are naturally \(U^j\)-modules via the embedding \(j : U^j \to U(\mathfrak{gl}(N))\). Following [Lu93, 23.1], \(V^{\otimes d}\) becomes a \(U^j\)-module as well.

We write \(v_{r_1, \ldots, r_d} = v_{r_1} \otimes \cdots \otimes v_{r_d}\). There is a right action of the Iwahori-Hecke algebra \(H_{B^j}\) on the \(\mathcal{Q}(v)\)-vector space \(V^{\otimes d}\) as follows:
\[
v_{r_1, \ldots, r_d} T_j = \begin{cases} 
v v_{r_1, \ldots, r_j-1, r_{j+1}, \ldots, r_d}, & \text{if } r_j < r_{j+1}; \\
v^2 v_{r_1, \ldots, r_d}, & \text{if } r_j = r_{j+1}; \\
(v^2 - 1) v_{r_1, \ldots, r_d} + v v_{r_1, \ldots, r_j-1, r_{j+1}, \ldots, r_d}, & \text{if } r_j > r_{j+1},
\end{cases}
\]
for \(1 \leq j \leq d-1\), and
\[
v_{r_1, \ldots, r_d-1, r_d} T_d = \begin{cases} 
v v_{r_1, \ldots, r_d-1, r_{d+2}}, & \text{if } r_d < n + 1; \\
v^2 v_{r_1, \ldots, r_d-1, r_d}, & \text{if } r_d = n + 1; \\
(v^2 - 1) v_{r_1, \ldots, r_d-1, r_d} + v v_{r_1, \ldots, r_d-1, r_{d+2}}, & \text{if } r_d > n + 1,
\end{cases}
\]
where \(r_{d+2} = N + 1 - r_d\). The \((\hat{U}^j, H_{B^j})\)-duality established in [BW13, Theorem 6.26], states that we have commuting actions of \(\hat{U}^j\) and \(\hat{\mathcal{A}}H_{B^j}\) on \(V^{\otimes d}\) which form double centralizers. We caution the reader that the conventions of the algebras \(\hat{U}^j\) and \(\hat{\mathcal{A}}H_{B^j}\) formulated...
above are chosen to fit with the geometric counterpart (see Proposition 4.12 below) and they differ from those in loc. cit.

Recall the right action of $H_{B_d}$ on $T_d$ from Lemma 2.4. The left action of $S^j$ on $T_d$ given in §2.3 is lifted to a left action of $K^j$ on $T_d$ via the homomorphism $\phi_d : K^j \to S^j$. We define a $\mathbb{Q}(v)$-vector space isomorphism:

$$\Omega : \mathbb{V}^\otimes d \longrightarrow \mathcal{L}T_d$$

$$v_{r_1} \otimes v_{r_2} \otimes \cdots \otimes v_{r_d} \mapsto \tilde{e}_{r_1, r_2, \ldots, r_d}.$$  

The following provides a geometric realization of the $(\dot{U}, H_{B_d})$-duality established in [BW13, Theorem 6.26].

**Proposition 4.12.** We have the following commutative diagram of double centralizing actions under the identification $\Omega : \mathbb{V}^\otimes d \longrightarrow \mathcal{L}T_d$:

$$\begin{array}{c}
\mathcal{L}K^j \uparrow \mathcal{L}T_d \uparrow \mathcal{L}H_{B_d} \\
\Phi \uparrow \Omega \uparrow \parallel
\end{array}$$

A general consideration (cf. [P09]) also shows that the actions of $\mathcal{L}S^j$ and $\mathcal{L}H_{B_d}$ on $\mathcal{L}T_d$ satisfy the double centralizer property (under the assumption that $n \geq d$).

**Remark 4.13.** The $\mathcal{L}$canonical basis of the $U^j$-module $\mathbb{V}^\otimes d$ was constructed in [BW13], and it does not coincide with Lusztig’s canonical basis of $\mathbb{V}^\otimes d$ (regarded as a $U(\mathfrak{gl}(N))$-module). An $\mathcal{L}$canonical basis on $T_d$ can also be defined via a standard intersection complex construction (similar to the one on $S^j$). These two $\mathcal{L}$canonical bases coincide under the isomorphism $\Omega$.

**Remark 4.14.** The (geometric) symmetric Howe duality in Remark 2.3 can also afford an algebraic formulation. Note that an (algebraic) skew Howe duality was formulated in [ES13] though the double centralizer property was not proved therein (actually this property can be proved easily using $\mathcal{L}$Schur duality and a deformation argument).

5. A second geometric construction in type $B$

There is also a $(\dot{U}', H_{B_d})$-duality for the other coideal algebra $U'$ established in [BW13]. In this section, we show that the modified algebra $\dot{U}'$ appears also naturally in the geometric setting of type $B$, and we provide further a geometric realization of the $(\dot{U}', H_{B_d})$-duality. Connections between distinguished bases for $\dot{U}'$ and $\dot{U}$ are established.

5.1. **The setup.** Recall $\Xi_d, \Pi$ from [2.2] and $\tilde{\Xi}$ from (4.1). Let $'\Xi_d$ (and respectively, $'\tilde{\Xi}$) be a subset of $\Xi_d$ (and respectively, $\tilde{\Xi}$) consisting of matrices $A$ in $\Xi_d$ (and respectively, in $\tilde{\Xi}$) such that all the entries in the $(n+1)$st row and in the $(n+1)$st column are 0 except $a_{n+1,n+1} = 1$. Also set

$$'\tilde{\Xi}_{d}^{\text{diag}} = \{ A \in '\tilde{\Xi} \mid A \text{ is diagonal} \}, \quad '\Xi_{d}^{\text{diag}} = \{ A \in '\Xi_d \mid A \text{ is diagonal} \}.$$ 

Similarly, we denote by $'\Pi$ the subset of $\Pi$ consisting of matrices $B = (b_{ij}) \in \Pi$ such that all the entries in the $(n+1)$st row and in the $(d+1)$st column are 0 except $b_{n+1,d+1} = 1$. This
extra condition makes a matrix in \(\mathcal{\Xi}_d\), \(\mathcal{\Xi}\) or \(\Pi\) look like
\[
\begin{pmatrix}
* & \cdots & * & 0 & \cdots & * \\
* & \cdots & * & 0 & \cdots & * \\
\vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
\vdots \\
* & \cdots & * & 0 & \cdots & * \\
* & \cdots & * & 0 & \cdots & * 
\end{pmatrix}.
\]

Recall \(X\) from [2.2] We consider the following subset of \(X\):
\[\mathcal{\Xi}_d = \{V = (V_i) \in X \mid V_n\text{ is maximal isotropic}\}.
\]
The products \(\mathcal{\Xi}_d \times \mathcal{\Xi}_d\) and \(\mathcal{\Xi}_d \times \mathcal{\Xi}\) are invariant under the diagonal action of \(O(D)\). Note that \(V_n\) being maximal isotropic for \(V = (V_i) \in \mathcal{\Xi}_d\) (and \(V_{n+1} = V_n^\perp\)) implies that \(V_{n+1}/V_n\) is one-dimensional. Therefore Lemma [2.1] readily leads to the following.

**Lemma 5.1.** The bijections in Lemma [2.1] induce the following bijections:
\[\mathcal{\Xi}_d \leftrightarrow \mathcal{\Xi}_d, \quad \mathcal{\Xi}_d \times \mathcal{\Xi} \leftrightarrow \mathcal{\Xi}_d \times \mathcal{\Xi}_d, \quad \mathcal{\Xi} \leftrightarrow \mathcal{\Xi}_d \times \mathcal{\Xi}.
\]
The following is a counterpart of Lemma [2.2] and we skip the similar argument.

**Lemma 5.2.** We have
\[\# \mathcal{\Xi}_d = \binom{2n^2 + d - 1}{d}, \quad \# \mathcal{\Xi} = (2n)^d.
\]
We define
\[\mathcal{S}^d = \mathcal{A}_{O(D)}(\mathcal{\Xi}_d), \quad \mathcal{T}_d = \mathcal{A}_{O(D)}(\mathcal{\Xi}_d \times \mathcal{\Xi}_d)
\]to be the space of \(O(D)\)-invariant \(\mathcal{A}\)-valued functions on \(\mathcal{\Xi}_d\) and \(\mathcal{\Xi}_d \times \mathcal{\Xi}_d\), respectively. Following [2.3] under the convolution product \(\mathcal{S}^d\) is an \(\mathcal{A}\)-subalgebra of \(\mathcal{S}^d\), \(\mathcal{T}_d\) is a left \(\mathcal{S}^d\)-submodule and also a right \(H_{B_d}\)-submodule of \(\mathcal{T}_d\), and we obtain commuting actions of \(\mathcal{S}^d\) and \(H_{B_d}\) on \(\mathcal{T}_d\). In particular, \(\mathcal{S}^d\) is a free \(\mathcal{A}\)-module with a basis \(\{e_A \mid A \in \mathcal{\Xi}_d\}\) and with a standard basis \(\{[A] \mid A \in \mathcal{\Xi}_d\}\) (inherited from their counterparts in \(\mathcal{S}^d\) by restriction).

5.2. **Relations for \(\mathcal{S}^d\).** We can still define the elements \(e_i, f_i\) and \(d_a\) in \(\mathcal{S}^d\) for \(i \in [1, n-1]\) and \(a \in [1, n]\) as done for \(\mathcal{S}^d\) in [3.1] However, the elements \(e_n\) and \(f_n\) defined for \(\mathcal{S}^d\) will no longer make sense here. Instead, we define a new element \(t \in \mathcal{S}^d\) by setting, for \(V, V' \in \mathcal{\Xi}_d\),
\[v^{- ||V''_n||-|V'_n-1|}, \quad \text{if } |V_n \cap V'_n| \geq d - 1, V_j = V'_j, \forall j \in [1, n-1];
\[0, \quad \text{otherwise}.
\]

**Remark 5.3.** One checks that, for \(V, V' \in \mathcal{\Xi}_d\), (and recall that \(|V_{n+1}/V_n^\perp| = 1\)),
\[t(V, V') = f_ne_n(V, V') - \delta_{V', V} \left[|V_n'/V'_n| - |V_{n+1}/V_n^\perp|\right].
\]
Proposition 5.4. The elements $e_i, f_i, d_i, d_i^{\pm 1}, d_{i+1}^{\pm 1}$ for $i \in [1, n-1]$, and $t$ in $S'$ satisfy the following relations:

(a) the defining relations of $U(\mathfrak{gl}(n))$ for $e_i, f_i, d_i, d_i^{\pm 1}, d_{i+1}^{\pm 1}, \forall i$ (see Remark 4.3);
(b) $d_i t = t d_i, \quad \forall i \in [1, n]$;
(c) $e_i t = t e_i, \quad f_i t = t f_i, \quad \forall i \in [1, n-2]$;
(d) $e_{n-1}^2 t - [2] e_{n-1} t e_{n-1} + t e_{n-1}^2 = 0$;
(e) $f_{n-1}^2 t - [2] f_{n-1} t f_{n-1} + t f_{n-1}^2 = 0$;
(f) $t^2 e_{n-1} - [2] t e_{n-1} t + e_{n-1} t^2 = e_{n-1}$;
(g) $t^2 f_{n-1} - [2] t f_{n-1} t + f_{n-1} t^2 = f_{n-1}$.

Proof. It suffices to prove the formulas when we specialize $v$ to $v \equiv \sqrt{q}$ and then perform the convolution products over $\mathbb{F}_q$.

The relations (a), (b) and (c) are clear. The remaining relations can be reduced to Proposition 3.1 by using (5.2). Below let us give a direct proof.

We now prove (d). Without loss of generality, we may and shall assume that $t \in \mathbb{F}_q$. The relations (a), (b) and (c) are clear. The remaining relations can be reduced to Proposition 3.1 by using (5.2). Below let us give a direct proof.

We now prove (d). Without loss of generality, we may and shall assume that $n = 2$. A direct computation yields

$$e_1^2 t(V, V') = \begin{cases} v^{-3(d-|V_1|)+6} (v + v^{-1}), & \text{if } V_1 \subseteq V'_1 \subseteq V_2, |V_2 \cap V'_2| \geq d - 1, \\ 0, & \text{otherwise}; \end{cases}$$

$$t e_1^2 (V, V') = \begin{cases} v^{-3(d-|V_1|)+4} (v + v^{-1}), & \text{if } V_1 \subseteq V'_1, |V_2 \cap V'_2| \geq d - 1, \\ 0, & \text{otherwise}; \end{cases}$$

$$e_1 t e_1 (V, V') = \begin{cases} v^{-3(d-|V_1|)+5} (v + v^{-1}), & \text{if } V_1 \subseteq V'_1 \subseteq V_2 \cap V'_2, |V_2 \cap V'_2| \geq d - 1, \\ v^{-3(d-|V_1|)+4}, & \text{if } V_1 \subseteq V'_1 \not\subseteq V_2 \cap V'_2, |V_2 \cap V'_2| \geq d - 1, \\ 0, & \text{otherwise}. \end{cases}$$

The relation (d) follows.

Similarly, the relation (f) is obtained by the following computations:

$$t^2 e_1 (V, V') = \begin{cases} v^{-3(d-|V_1|)+1} \frac{q^{d-|V_1|+1-1}}{q-1}, & \text{if } V_1 \subseteq V'_1, V_2 = V'_2, \\ v^{-3(d-|V_1|)+1} (q + 1), & \text{if } V_1 \subseteq V'_1, |V_2 \cap V'_2| = d - 1 \text{ or } d - 2, \\ 0, & \text{otherwise}; \end{cases}$$

$$e_1 t^2 (V, V') = \begin{cases} v^{-3(d-|V_1|)+1} \frac{q^{d-|V_1|-1}}{q-1}, & \text{if } V_1 \subseteq V'_1, V_2 = V'_2, \\ v^{-3(d-|V_1|)+1} (q + 1), & \text{if } V_1 \subseteq V'_1 \subseteq V_2, |V_2 \cap V'_2| = d - 1 \text{ or } d - 2, \\ 0, & \text{otherwise}. \end{cases}$$
Finally, the involution \((V, V') \mapsto (V', V)\) on \('X \times 'X\) induces an anti-involution \(\sigma\) on \(S^t\) such that 
\[
\sigma(e_1) = v^{d+1}f_1, \quad \sigma(d_1) = d_1, \quad \sigma(d_2) = d_2, \quad \text{and} \quad \sigma(t) = t.
\]
This implies that the relations (e) and (g) follow from (d) and (f), respectively. \(\square\)

5.3. A sheaf-theoretic description of \(t\). We shall now give a geometric interpretation of the element \(t\). To do this, we need to divert from the previous setting over a finite field to a setting over the complex field \(\mathbb{C}\). This permutation does not contain any bad pattern in \([BL00, 13.3.3]\), hence implies that each connected component in \('\tau\) is rationally smooth (see \([KL79, A1]\)) by \([BL00, 8.3.16]\).

Let \(S(\tau)\) be the closed subvariety in \('X(\mathbb{C}) \times 'X(\mathbb{C})\) defined by
\[
S(\tau) = \{(V, V') \in 'X(\mathbb{C}) \times 'X(\mathbb{C}) \mid |V_n \cap V'_n| \geq d - 1, V_j = V'_j, \forall j \in [1, n - 1]\}.
\]
It is clear that \(S(\tau)\) is the subvariety corresponding to the support of \(t\).

**Lemma 5.5.** The variety \(S(\tau)\) is rationally smooth. In particular, the constant sheaf on \(S(\tau)\) is a semisimple complex.

**Proof.** The trick is to reduce the proof to a problem in the Schubert varieties in \(Y(\mathbb{C})\) and then apply the known results of their singular loci. Without loss of generality, we assume that \(n = 1\). We only need to show that each connected component in \(S(\tau)\) is rationally smooth. Let us fix a connected component, say \(C\), in \(S(\tau)\). This is further reduced to show the rational smoothness of the inverse image, say \(C'\), in \(Y(\mathbb{C}) \times Y(\mathbb{C})\) of \(C\) under the projection from \(Y(\mathbb{C}) \times Y(\mathbb{C})\) to a suitable connected component of \('X(\mathbb{C}) \times 'X(\mathbb{C})\). Fix a flag \(F\) in \(Y(\mathbb{C})\), and let \(C'' = C' \cap \{F\} \times Y(\mathbb{C})\). It is finally reduced to show that \(C''\) is rationally smooth. Observe that \(C''\) is a Schubert variety in \(Y(\mathbb{C})\). It is indexed by the permutation \(\tau = (d - 1, \ldots, 1, d)\) in the Weyl group of type \(B_d\) in the notation in \([BL00]\) and \([B98]\). This permutation does not contain any bad pattern in \([BL00, 13.3.3]\), hence implies that \(C''\) is rationally smooth (see \([KL79, A1]\)) by \([BL00, 8.3.16]\). \(\square\)

A direct consequence of this lemma is that \(t\) is a shadow of a semisimple complex. Its idempotent components are shadows of simple perverse sheaves, up to shifts.

5.4. Generators for \(S^t\). Recall \(S^t\) is an \(\mathcal{A}\)-subalgebra of \(S^l\). Fix any \(A \in \Psi_d\). By \((3.25)\), \(m_A \in S^l\) can be generated by elements of the form \([D_{i,h_j} + a_{ij}E_{h+1,h}]\). Note that elements of the form \([D_{i,h,j} + a_{ij}E_{n,n+1}^\theta]\) or \([D_{i,h,j} + a_{ij}E_{n+1,n}^\theta]\) in general do not lie in \(S^t\). However, the “twin product” \([D_{i,h,j} + a_{ij}E_{n,n+1}^\theta] \ast [D_{i,h,j} + a_{ij}E_{h+1,h}^\theta]\) (where \(D_{i,h,j} + a_{ij}E_{n,n+1}^\theta \in \Psi_d\)) is naturally an element in \(S^t\) thanks to Remark \(5.3\) and we observe that \(m_A\) is always a product of such twin products together with \([D_{i,h,j} + a_{ij}E_{h+1,h}^\theta]\) is \(S^t\) for \(h \neq n, n + 1\). Thus, we always have \(m_A \in S^t\). For example, for \(n = 2\), \(m_A\) is a product of 20 terms in \((3.25)\).
which simplifies to 14 terms thanks to $A \in \Xi_d$ as follows, and it actually lies in $S'$:

$$
[D_{2,1,1} + a_{21} E_{21}^q] \ast ([D_{4,3,1} + a_{41} E_{43}^q] \ast [D_{4,2,1} + a_{41} E_{32}^q] \ast [D_{4,1,1} + a_{41} E_{21}^q])
$$

$$
\ast \left( ([D_{4,3,2} + a_{42} E_{43}^q] \ast [D_{4,2,2} + a_{42} E_{32}^q] \ast [D_{4,1,1} + a_{41} E_{21}^q])
\ast \left( ([D_{5,3,1} + a_{51} E_{53}^q] \ast [D_{5,2,1} + a_{51} E_{52}^q] \ast [D_{5,1,1} + a_{51} E_{51}^q]) \ast [D_{5,4,2} + a_{52} E_{54}^q]
\ast \left( ([D_{5,3,2} + a_{52} E_{53}^q] \ast [D_{5,2,2} + a_{52} E_{52}^q] \ast [D_{5,4,4} + a_{54} E_{54}^q])
\right) \right)
\right)
$$

We summarize the above discussions as follows.

Proposition 5.6. For any $A \in \Xi_d$, we have $m_A \in S'$. Moreover, $\{ m_A \mid A \in \Xi_d \}$ forms a monomial $\mathcal{A}$-basis for $S'$.

Assume that $D + RE_{n,n+1}^q \in \Xi_d$. It follows by Proposition 3.7 that

$$
[D + RE_{n,n+1}^q] \ast [D + RE_{n+1,n}^q] = [D + RE_{n,n+2}^q] + \text{lower terms}
$$

and they lie in $S'$. By Theorem 3.10, Proposition 5.6 and the above analysis, we have established the following.

Proposition 5.7. The $\mathcal{A}$-algebra $S'$ is generated by the elements $[D + RE_{n,n+2}^q]$, $[D + RE_{i,i+1}^q]$, $[D + RE_{i,i+1}^q]$, where $1 \leq i \leq n-1$, $R \in [0,d]$ and $D \in \Xi_d^{diag}$. $\Xi_{d-R}$.

By using Lemma 5.9, we have

$$
[D + RE_{n,n}^q + E_{n,n+2}^q] \ast [D + E_{n,n}^q + RE_{n,n+2}^q] = [D + (R+1)E_{n,n+2}^q] + \text{lower terms}.
$$

Hence Proposition 5.7 implies the following.

Corollary 5.8. The $Q(v)$-algebra $\mathcal{S}S'$ is generated by the elements $[D']'$ with $D' \in \Xi_d^{diag}$, and the elements $[D + E_{i,i+1}^q]$, $[D + E_{i,i+1}^q]$, $[D + E_{n,n+2}]$, where $1 \leq i \leq n-1$ and $D \in \Xi_d^{diag}$. $\Xi_{d-1}$.

Recall in §3.6 we have defined the canonical basis $\mathcal{B}_d = \{ \{ A \mid A \in \Xi_d \}$ for $S'$. The bar involution on $S'$ is identified with the restriction from the bar involution on $S$ via the inclusion $S' \subset S$ by the geometric construction. It follows from Proposition 5.6 (and recall every monomial basis element is bar invariant) that $\{ A \} \in S'$ for $A \in \Xi_d$. In particular, we have the following.

Proposition 5.9. The canonical basis for $S'$ is given by $\mathcal{B}_d \cap S' = \{ \{ A \} \mid A \in \Xi_d \}$.

Hence all the results on the inner product and canonical basis for $S'$ in §3.7 and §3.8 make sense by restriction to the subalgebra $S'$.

5.5. The algebra $K'$. We define $K'$ to be the $\mathcal{A}$-subalgebra of $K'$ with an $\mathcal{A}$-basis $\{ [A] \mid A \in \Xi \}$. We have shown in §5.2 that the multiplications in $S$ and in $S'$ are remarkably compatible with the inclusion $S' \subset S$ and their respective standard (as well as monomial) bases. By the definition of the stabilization procedure in §4.1, we conclude the following.

Theorem 5.10. The standard basis, monomial basis, and canonical basis of $K'$ restrict to the corresponding basis of $K'$. That is, the $\mathcal{A}$-algebra $K'$ has a standard basis $\{ [A] \mid A \in \Xi \}$, a monomial basis $\{ m_A \mid A \in \Xi \}$, and a canonical basis $\{ [A] \mid A \in \Xi \}$.

Similar arguments as for Proposition 5.7 and Corollary 5.8 now using (4.8) where it is understood that $A, A' \in \Xi$, give us the following.
Corollary 5.11.  
(a) The $\mathcal{A}$-algebra $K'$ is generated by the elements $[D + RE_{\theta}^{\lambda}D_{i+n+2}]$, $[D + RE_{i+n+1}^{\theta}]$, $[D + RE_{i+n+1}^{\theta+1}]$, for $1 \leq i \leq n - 1$, $R \in \mathbb{N}$, and $D \in \mathcal{E}_{\text{diag}}$.
(b) The $\mathcal{Q}(v)$-algebra $\mathcal{A}K'$ is generated by $[D]$, $[D + E_{\theta}^{\lambda}]$, $[D + E_{i+n+2}^{\theta}]$, $[D + E_{i+n+1}^{\theta}]$, $[D + E_{i+n+1}^{\theta+1}]$, for $1 \leq i \leq n - 1$ and $D \in \mathcal{E}_{\text{diag}}$.

5.6. Algebras $K'$ vs $\hat{U}^\dagger$. Below we formulate the counterparts of Sections 4.3, 4.4, 4.5, 4.6, and 4.7. The proofs are very similar and hence will be omitted.

The algebra $U'$ is defined to be the associative algebra over $\mathcal{Q}(v)$ generated by $e_i$, $f_i$, $d_a$, $a_i^{-1}$, $t$, $i = 1, 2, \ldots, n - 1$, $a = 1, 2, \ldots, n$, subject to the following relations:

$$d_a d_a^{-1} = d_a^{-1} d_a = 1,$$
$$d_a d_b = d_b d_a,$$
$$d_a e_j d_a^{-1} = v^{-\delta_{a,j+1} - \delta_{N+1-a,j+1} + \delta_{a,j} e_j},$$
$$d_a f_j d_a^{-1} = v^{-\delta_{a,j} + \delta_{a,j+1} + \delta_{N+1-a,j+1} f_j},$$
$$e_i f_j - f_j e_i = \delta_{ij} (d_a d_a^{-1} d_a^{-1} d_a^{-1}),$$
$$e_i e_j = \begin{cases} [2] e_i e_j e_i & \text{if } |i - j| = 1, \\ e_j e_i & \text{if } |i - j| > 1, \end{cases}$$
$$f_i f_j + f_j f_i = \begin{cases} [2] f_i f_j f_i & \text{if } |i - j| = 1, \\ f_j f_i & \text{if } |i - j| > 1, \end{cases}$$
$$f_i t = t f_i,$$
$$t^2 f_i = \begin{cases} [2] t f_i t + f_i & \text{if } i \neq n - 1, \\ f_i t & \text{if } i = n - 1, \end{cases}$$
$$t^2 e_i = \begin{cases} [2] t e_i t + e_i & \text{if } i \neq n - 1, \\ e_i t & \text{if } i = n - 1, \end{cases}$$

This algebra is a coideal subalgebra of $U(\mathfrak{gl}(N - 1))$, and hence it is a $\mathfrak{g}l$-version of the coideal algebra in the same notation defined in [BW13 §2.1] (which is a coideal subalgebra of $U(\mathfrak{sl}(N + 1))$).

Similar to the construction of $\hat{U}^\dagger$ in [4.3] we can define the modified quantum algebra $U'$ for $U'$, where the unit of $U'$ is replaced by a collection of orthogonal idempotents $D_{\lambda}$ for $\lambda \in \mathcal{E}_{\text{diag}}$. Moreover, $U'$ is naturally a $U'$-bimodule. By introducing similarly $\lambda U'_{\lambda'}$, for $\lambda, \lambda' \in \mathcal{E}_{\text{diag}}$, we have

$$\hat{U}' = \bigoplus_{\lambda, \lambda' \in \mathcal{E}_{\text{diag}}} \lambda U'_{\lambda'},$$
$$= \sum_{\lambda \in \mathcal{E}_{\text{diag}}} U' D_{\lambda},$$
$$= \sum_{\lambda \in \mathcal{E}_{\text{diag}}} D_{\lambda} U'.$$

In the same way establishing the presentation for $\mathcal{A}U'$ given in Proposition 4.6 we can show that the $\mathcal{Q}(v)$-algebra $\mathcal{A}U'$ is isomorphic to the $\mathcal{Q}(v)$-algebra generated by the symbols $D_{\lambda}, e_i D_{\lambda}, D_{\lambda} e_i, f_i D_{\lambda}, D_{\lambda} f_i, t D_{\lambda},$ and $D_{\lambda} t$, for $i = 1, \ldots, n - 1$ and $\lambda \in \mathcal{E}_{\text{diag}}$, subject to the following relations (5.4):
for $i, j = 1, \ldots, n - 1$, $\lambda, \lambda' \in \mathfrak{t}_{\text{diag}}$ and for $x, x' \in \{e_i, e_j, f_i, f_j, t\}$,

$$\begin{align*}
x D_{\lambda} D_{\lambda'} x' &= \delta_{\lambda, \lambda'} x D_{\lambda} x', \\
e_i D_{\lambda} &= D_{\lambda - a_i} e_i, \\
f_i D_{\lambda} &= D_{\lambda + a_i} f_i, \\
t D_{\lambda} &= D_{\lambda t}, \\
e_i D_{\lambda} f_j &= f_j D_{\lambda - a_i - a_j} e_i, \quad \text{if } i \neq j, \\
e_i D_{\lambda} f_i &= f_i D_{\lambda - 2a_i} e_i + [\lambda_{i+1} - \lambda_i] D_{\lambda - a_i}, \\
(e_i^2 + e_j e_i^2 + e_j^2) D_{\lambda} &= [2] e_i e_j e_i D_{\lambda}, \quad \text{if } |i - j| = 1, \\
(f_i^2 f_j + f_j f_i^2) D_{\lambda} &= [2] f_i f_j f_i D_{\lambda}, \quad \text{if } |i - j| = 1, \\
e_i e_j D_{\lambda} &= e_j e_i D_{\lambda}, \quad \text{if } |i - j| > 1, \\
f_i f_j D_{\lambda} &= f_j f_i D_{\lambda}, \quad \text{if } |i - j| > 1, \\
t f_i D_{\lambda} &= f_i t D_{\lambda}, \quad \text{if } i \neq n - 1, \\
(t^2 f_{n-1} + f_{n-1} t^2) D_{\lambda} &= \left(2 f_{n-1} t + f_{n-1}\right) D_{\lambda}, \\
(f_{n-1}^2 + tf_{n-1}^2) D_{\lambda} &= \left(2 f_{n-1} t f_{n-1}\right) D_{\lambda}, \\
te_i D_{\lambda} &= e_i t D_{\lambda}, \quad \text{if } i \neq n - 1, \\
(t^2 e_{n-1} + e_{n-1} t^2) D_{\lambda} &= \left(2 t e_{n-1} + e_{n-1}\right) D_{\lambda}, \\
(e_{n-1}^2 t + e_{n-1}^2 t) D_{\lambda} &= \left(2 e_{n-1} t e_{n-1}\right) D_{\lambda}.
\end{align*}$$

(5.4)

To simplify the notation, we shall write $x_1 D_{\lambda_1} x_2 D_{\lambda_2} \cdots x_i D_{\lambda_i} = x_1 x_2 \cdots x_i D_{\lambda_i}$, if the product is not zero.

**Theorem 5.12.** We have an isomorphism of $\mathbb{Q}(v)$-algebras $\Phi^i : \tilde{U}^i \to \mathfrak{g} K^i$ which sends

$$D_{\lambda} \mapsto [D_{\lambda}], \quad t D_{\lambda} \mapsto [D_{\lambda} - E_{n,n}^\theta + E_{n,n+2}^\theta + v^{-\lambda_n}[D_{\lambda}]],$$

$$e_i D_{\lambda} \mapsto [D_{\lambda} - E_{i,i}^\theta + E_{i+1,i}^\theta], \quad f_i D_{\lambda} \mapsto [D_{\lambda} - E_{i+1,i+1}^\theta + E_{i,i+1}^\theta],$$

for all $1 \leq i \leq n - 1$ and $\lambda \in \mathfrak{t}_{\text{diag}}$.

**Proof.** This follows by a parallel argument to the one for Theorem 4.7. The somewhat unusual formula for $t D_{\lambda}$ has its origin in the formulas (5.1)-(5.2). \hfill \Box

The isomorphisms in Theorems 4.7 and 5.12 allow us to transfer the algebra embedding $K^i \subset K^j$ in Theorem 5.10 to an algebra embedding $\tilde{U}^i \to \tilde{U}^j$.

**Proposition 5.13.** There is a $\mathbb{Q}(v)$-algebra embedding $\tilde{U}^i \to \tilde{U}^j$, which sends the generators $D_{\lambda}, e_i D_{\lambda}, f_i D_{\lambda}$ (for $1 \leq i \leq n - 1$ and $\lambda \in \mathfrak{t}_{\text{diag}}$) to the generators in the same notation, and sends $t D_{\lambda}$ to $f_n e_n D_{\lambda} - [\lambda_n - \lambda_{n+1}][D_{\lambda}]$; here we recall $\lambda_{n+1} = 1$.

**Proof.** By using Proposition 3.7 and keeping in mind $D_{n+1,n+1} = 1$ for $D \in \mathfrak{t}_{\text{diag}}$, we have that

$$[D + E_{n,n+1}^\theta] * [D + E_{n+1,n}^\theta] = [D + E_{n,n+2}^\theta] + v D_{n,n+1}^{D_{n,n+2}^\theta} [D_{n,n+1}^\theta + 1][D + E_{n,n}^\theta],$$

Setting $D = D_{\lambda} - E_{n,n}^\theta$ leads to the equivalent formula in the proposition. \hfill \Box

The bar involution on $U^i$ (see [BW13]) induces a compatible bar involution on $\tilde{U}^i$, denoted also by $\bar{\cdot}$, which fixes all the generators $D_{\lambda}, t D_{\lambda}, e_i D_{\lambda}, f_i D_{\lambda}$. The isomorphism $\Phi^i$ intertwines the bar involutions on $\tilde{U}^j$ and on $\mathfrak{g} K^j$, i.e., $\Phi^i(\bar{u}) = \Phi^i(u)$, for $u \in U^i$.

We define an $\mathfrak{g}$-subalgebra of $\tilde{U}^i$ by $\mathfrak{g} \tilde{U}^i := (\Phi^i)^{-1}(\mathfrak{g}^i)$ so that $\mathfrak{g} \tilde{U}^i \otimes_{\mathfrak{g}} \mathbb{Q}(v) = \tilde{U}^i$. The integral form $\mathfrak{g} \tilde{U}^i$ is a free $\mathfrak{g}$-submodule of $\tilde{U}^i$ and it is stable under the bar involution.
The isomorphism $\Phi^i : \mathcal{U}^i \to K^i$ allows us to transport the canonical basis for $K^i$ to a canonical basis for $\mathcal{U}^i$. Introduce the divided powers $e_i^{(r)} = e_i^{r}/[r]!$ and $f_i^{(r)} = f_i^{r}/[r]!$, for $r \geq 1$. Then we have

$$\Phi^i(e_i^{(r)} D) = [D - rE_{i,i} + rE_{i,i+1}], \quad \Phi^i(f_i^{(r)} D) = [D - rE_{i+1,i,i+1} + rE_{i+1,i+1}].$$

### 5.7. Homomorphism from $K^i$ to $S^i$

The following is a counterpart of Proposition 4.11, which is proved in the same way, now by applying Corollary 5.11.

**Proposition 5.14.** There exists a unique surjective $\mathcal{A}$-algebra homomorphism $\phi_d^i : K^i \to S^i$ such that for $R \geq 0$, $i \in [1, n - 1]$ and $D \in \mathcal{I}^{\text{diag}}$,

$$\phi_d^i([D + RE_{n,n+2}]) = \begin{cases} [D + RE_{n,n+2}], & \text{if } D + RE_{n,n+2} \in \mathcal{I}^d; \\ 0, & \text{otherwise}; \end{cases}$$

$$\phi_d^i([D + RE_{i,i+1}]) = \begin{cases} [D + RE_{i,i+1}], & \text{if } D + RE_{i,i+1} \in \mathcal{I}^d; \\ 0, & \text{otherwise}; \end{cases}$$

$$\phi_d^i([D + RE_{i+1,i,i+1}]) = \begin{cases} [D + RE_{i+1,i,i+1}], & \text{if } D + RE_{i+1,i,i+1} \in \mathcal{I}^d; \\ 0, & \text{otherwise}. \end{cases}$$

There is an algebra embedding $U^i \to U(gl(N - 1))$ (cf. [BW13] Proposition 2.2), with convention and notation adjusted in a way similar to the embedding $U^j \to U(gl(N))$ in Proposition 1.5. Denote by $\mathcal{V}^d$ the natural representation of $U(gl(N - 1))$. Then the tensor space $\mathcal{V}^d \otimes \mathcal{V}^d$ is naturally a $U(gl(N - 1))$-module, which becomes a $U^i$-module by restriction, and hence a $\mathcal{U}^i$-module. The right action of the Iwahori-Hecke algebra $H_{B_d}$ on $T_d$ is similar to the one on $T_d$ in Lemma 2.13 (and is actually the same as the one given in Lemma 6.2 below).

The action of the Iwahori-Hecke algebra $H_{B_d}$ on $\mathcal{V}^d \otimes \mathcal{V}^d$ is very similar to its action on $\mathcal{V}^d \otimes \mathcal{V}^d$ given in (4.11)-(4.12). The $(U^i, H_{B_d})$-duality established in [BW13] Theorem 5.4] states that the actions of $U^i$ and $H_{B_d}$ on $\mathcal{V}^d \otimes \mathcal{V}^d$ commute and they form double centralizers. We note that the (algebraic) Schur duality using the Schur algebra instead of the coideal algebra $\mathcal{U}^i$ has appeared in [G97]. Similar to the isomorphism $\Omega : \mathcal{V}^d \otimes \mathcal{V}^d \to T_d^i$ in (4.13), now we have an isomorphism $\Omega^i : \mathcal{V}^d \otimes \mathcal{V}^d \to T_d^i$ given by an analogous formula. As a counterpart of Proposition 4.12 we have the following geometric realization of the $(\mathcal{U}^i, H_{B_d})$-duality.

**Proposition 5.15.** We have the following commutative diagram of double centralizing actions under the identification $\Omega^i : \mathcal{V}^d \otimes \mathcal{V}^d \to T_d^i$.

$$\begin{array}{ccc}
\mathcal{V}^d \otimes \mathcal{V}^d & \xrightarrow{\Phi^i} & \mathcal{U}^i \\
\Omega^i \uparrow & & \uparrow \Phi^i \\
\mathcal{V}^d \otimes \mathcal{V}^d & \xrightarrow{\mathcal{H}_{B_d}} & H_{B_d} \\
\end{array}$$

6. **Convolution algebras from geometry of type C**

In this section, we formulate analogous constructions and results in type $C$. This could have been done in a completely analogous way as before, but to avoid much repetition we choose some short cuts to reduce the considerations quickly to the type $B$ counterparts.
6.1. A first formulation. We fix the following data in this subsection:

- A pair of positive integers \((n, d)\) such that \(N = 2n\) and \(D = 2d\) in Section 2.1
- A non-degenerate skew-symmetric bilinear form \(Q : \mathbb{F}_q^D \times \mathbb{F}_q^D \rightarrow \mathbb{F}_q\)

Let \(\text{Sp}(D)\) be the symplectic subgroup of \(\text{GL}(D)\) consisting of all elements \(g\) such that \(Q(gu, gu') = Q(u, u')\), for \(u, u' \in \mathbb{F}_q^D\). We can define the sets \(X_{C_d}, Y_{C_d}, \Xi_{C_d}, \Pi_{C_d}\) and \(\Sigma_{C_d}\) in formally the same way (in notations \(N\) and \(D\)) as the sets \(X, Y, \Xi, \Pi\) and \(\Sigma\) in Section 2.2 respectively. For example, \(X_{C_d}\) is the set of \(N\)-step flags \((V_i)_{0 \leq i \leq N}\) in \(\mathbb{F}_q^D\) satisfying \(V_{N-i} = V_i^\perp\), and in particular, \(V_n = V_n^\perp\) is Lagrangian.

We have the following analogue of Lemmas 2.1 and 2.2 whose proof is skipped.

Lemma 6.1.  
(a) We have
\[
\#\Sigma_{C_d} = 2^d \cdot d!, \quad \#\Pi_{C_d} = (2n)^d, \quad \text{and} \quad \#\Xi_{C_d} = \binom{2n^2 + d - 1}{d}.
\]
(b) We have canonical bijections \(\text{Sp}(D) \setminus X_{C_d} \times X_{C_d} \leftrightarrow \Xi_{C_d}\), \(\text{Sp}(D) \setminus X_{C_d} \times Y_{C_d} \leftrightarrow \Pi_{C_d}\), and \(\text{Sp}(D) \setminus Y_{C_d} \times Y_{C_d} \leftrightarrow \Sigma_{C_d}\).

Following [2.3] we define
\[
^CS^j = \mathcal{A}_{Sp(D)}(X_{C_d} \times X_{C_d}), \quad ^CT^d = \mathcal{A}_{Sp(D)}(X_{C_d} \times Y_{C_d}), \quad H_{B_d} = \mathcal{A}_{Sp(D)}(Y_{C_d} \times Y_{C_d})
\]
to be the space of \(\text{Sp}(D)\)-invariant \(\mathcal{A}\)-valued functions on \(X_{C_d} \times X_{C_d}, X_{C_d} \times Y_{C_d},\) and \(Y_{C_d} \times Y_{C_d}\) respectively. (Note the superscript \(j\) here instead of \(j\) is used!) As before, \(^CS^i\) is endowed with an \(\mathcal{A}\)-algebra structure via a convolution product. Also, the convolution algebra \(\mathcal{A}_{Sp(D)}(Y_{C_d} \times Y_{C_d})\) is known to be canonically isomorphic to the Iwahori-Hecke algebra \(H_{B_d}\) of type \(B_d\), and so there is no ambiguity of notation above.

Associated to \(B \in \Pi_{C_d}\), we have a sequence of integers \(r_1, \ldots, r_D\) as defined in (2.5) which satisfies the same bijections (2.6). We shall denote the characteristic function of the \(\text{Sp}(D)\)-orbit \(O_B\) by \(e_{r_1 \ldots r_d}\). The following is an analogue of Lemma 2.4 whose similar proof is skipped.

Lemma 6.2. The right \(H_{B_d}\)-action on \(^CT^d\)
\[
^CT^d \times H_{B_d} \rightarrow ^CT^d, \quad (e_{r_1 \ldots r_d}, T_j) \mapsto e_{r_1 \ldots r_d} T_j,
\]
is given as follows. For \(1 \leq j \leq d - 1\), we have
\[
e_{r_1 \ldots r_d} T_j = \begin{cases} e_{r_1 \ldots r_j-1r_j+1r_j+2 \ldots r_d}, & \text{if } r_j < r_{j+1}; \\
^{v^2}e_{r_1 \ldots r_d}, & \text{if } r_j = r_{j+1}; \\
(v^2 - 1)e_{r_1 \ldots r_d} + v^2 e_{r_1 \ldots r_{j-1}r_{j+1}r_{j+2} \ldots r_d}, & \text{if } r_j > r_{j+1}.
\end{cases}
\]
Moreover, (recalling \(r_{d+1} = N + 1 - r_d\)) we have
\[
e_{r_1 \ldots r_d-1r_d} T_d = \begin{cases} e_{r_1 \ldots r_d-1r_{d+1}}, & \text{if } r_d < r_{d+1}; \\
^{v^2}e_{r_1 \ldots r_{d-1}r_d}, & \text{if } r_d = r_{d+1}; \\
(v^2 - 1)e_{r_1 \ldots r_{d-1}r_d} + v^2 e_{r_1 \ldots r_{d-1}r_{d+1}}, & \text{if } r_d > r_{d+1}.
\end{cases}
\]
We set
\[
\bar{e}_{r_1 \ldots r_d} = v^{\#\{(c, c') | c, c' \in [1, d+1], c < c' \text{ or } c < r_d\}} e_{r_1 \ldots r_d}.
\]
The formulas (6.1) and (6.2) can be rewritten as follows. For \(1 \leq j \leq d - 1\),

\[
\tilde{e}_{r_1 \ldots r_d} T_j = \begin{cases} 
  v \tilde{e}_{r_1 \ldots r_{j-1} r_{j+1} r_{j+2} \ldots r_d}, & \text{if } r_j < r_{j+1}; \\
  v^2 \tilde{e}_{r_1 \ldots r_d}, & \text{if } r_j = r_{j+1}; \\
  (v^2 - 1) \tilde{e}_{r_1 \ldots r_d} + v \tilde{e}_{r_1 \ldots r_{j-1} r_{j+1} r_{j+2} \ldots r_d}, & \text{if } r_j > r_{j+1}; 
\end{cases}
\]

(6.3)

\[
\tilde{e}_{r_1 \ldots r_{d-1} r_d} T_d = \begin{cases} 
  v \tilde{e}_{r_1 \ldots r_{d-1} r_{d+1}}, & \text{if } r_d < r_{d+1}; \\
  v^2 \tilde{e}_{r_1 \ldots r_{d-1} r_d}, & \text{if } r_d = r_{d+1}; \\
  (v^2 - 1) \tilde{e}_{r_1 \ldots r_{d-1} r_d} + v \tilde{e}_{r_1 \ldots r_{d-1} r_{d+1}}, & \text{if } r_d > r_{d+1}. 
\end{cases}
\]

(6.4)

**Remark 6.3.** The formulas (6.3) and (6.4) essentially coincide with the one given in [G97] in an opposite ordering. (Note that the presentation of Iwahori-Hecke algebras in [G97] is somewhat different from ours via \(T_1\).) This shows that the \(A\)-algebra \(C S^i\) is isomorphic to the hyperoctahedral Schur algebra in [G97].

6.2. **A variation.** We fix the following data in this subsection:

- A pair of positive integers \((n, d)\) such that \(N = 2n + 1\) and \(D = 2d\) in Section 6.1.
- A non-degenerate skew-symmetric bilinear form \(Q : \mathbb{F}_q^D \times \mathbb{F}_q^D \rightarrow \mathbb{F}_q\).

(Note the only difference from the data in Section 6.1 is that \(N = 2n + 1\) now.) We can define analogous sets as those in Section 6.1. Let \('X_{C_d}\) be the set defined formally as \(X_{C_d}\) in Section 6.1 with now \(N = 2n + 1\). We keep exactly the same \(Y_{C_d}\) as in Section 6.1. Following [2.3], we can define an \(A\)-algebra \(C S^i := \mathcal{A}_{\text{Sp}(D)} \rtimes (X_{C_d} \times 'X_{C_d})\) with a convolution product; Similarly, we have a commuting left \(C S^i\)-action and a right \(H_{B_d}\)-action on \(C T_d := \mathcal{A}_{\text{Sp}(D)} \rtimes (X_{C_d} \times Y_{C_d})\). As before (see (2.5) and (2.6)), \(C T_d')\) has a basis given by the characteristic functions \(e_{r_1 \ldots r_d}\), where the \(d\)-tuples \(r_1 \cdots r_d\) are in bijection with the \(Sp(D)\)-orbits in \('X_{C_d} \times Y_{C_d}\).

There is a natural inclusion \(X_{C_d} \subset 'X_{C_d}\), which identifies a flag \((\cdots \subseteq V_n \subseteq \cdots)\) with \((\cdots \subseteq V_n \subseteq V_n \subseteq \cdots)\).

**Lemma 6.4.** The \(A\)-algebra \(C S^i\) is naturally a subalgebra of \(C S^j\) induced by the inclusion \(X_{C_d} \subset 'X_{C_d}\).

The next lemma follows by similar arguments for Lemmas 2.2 and 2.4.

**Lemma 6.5.** (a) We have

\[
\#(Sp(D) \backslash 'X_{C_d} \times 'X_{C_d}) = \binom{2n^2 + 2n + d}{d}, \quad \#(Sp(D) \backslash 'X_{C_d} \times Y_{C_d}) = (2n + 1)^d.
\]

(b) The right \(H_{B_d}\)-action on \(C T_d')\) is given by the formulas (6.1)-(6.3).

6.3. **Type C vs type B.** By Lemmas 2.2, 2.4 and 6.5, we have a right \(H_{B_d}\)-module isomorphism \(\mathcal{A}_{T_d} \xrightarrow{\phi_1} \mathcal{A}_{C T_d}\) by sending \(e_{r_1 \ldots r_d}\) to the element in the same notation. This isomorphism induces an algebra isomorphism

\[
\phi_1 : \text{End}_{H_{B_d}}(T_d) \xrightarrow{\sim} \text{End}_{H_{B_d}}(C T_d').
\]

Earlier (see Proposition 4.12 and [P09]) we have obtained an \(A\)-algebra isomorphism

\[
\phi_2 : S^j \rightarrow \text{End}_{H_{B_d}}(T_d).
\]
Similarly, we have an $\mathcal{A}$-algebra isomorphism  
\[ \phi_3 : C \mathcal{S}^j \rightarrow \text{End}_{H_{\beta_d}}(C T'_d). \]

The following proposition allows us to reduce the type $C$ case to the type $B$ case.

**Proposition 6.6.** We have natural $\mathcal{A}$-algebra isomorphisms $S^j \cong C S^j$ and $S^j \cong C \mathcal{S}^j$.

**Proof.** The first isomorphism is given by $\phi^{-1}_3 \phi_2 \phi_1$, and the second isomorphism can be obtained similarly. $\square$

Actually, the above isomorphisms are canonical in the sense they match various bases; we treat one case below in some detail.

**Proposition 6.7.** The isomorphism $\phi^{-1}_3 \phi_2 \phi_1 : S^j \rightarrow C S^j$ sends the basis element $e_A$, for $A \in \Xi$, to the element $e_A - E_n^{n+1}$, for $A \in \mathcal{S}^j$. 

**Proof.** When $A$ is a diagonal matrix, the statement holds by definition. When $A = E_{h,h+1}^\theta$ is diagonal, the statement holds by checking directly that the action of $e_A - E_{n+1,n+1}^\theta$ on $e_{r_1 \cdots r_d}$ is the same as the action of $e_A \in S^j$ on $e_{r_1 \cdots r_d}$. Note that the counterpart of Lemma 3.2 for $C S^j$ holds with the same structure constants. The proof is basically the same as that of Lemma 3.2 (with $e_n^{n+1}$ defined to be 1) with the extra care that the number of isotropic lines in $F D_q$ with respect to the skew-symmetric form is $\frac{q^D - 1}{q-1}$. The proposition for general $A$ follows from this by induction. $\square$

Consequently, the sequence of $\mathcal{A}$-algebras $C \mathcal{S}^j$ (as $d$ goes to infinity) leads to the $\mathcal{A}$-algebra $K^j$ introduced in Section 5, and the sequence of $\mathcal{A}$-algebras $C S^j$ leads to the $\mathcal{A}$-algebra $K^j$ defined in Section 4. In this way, we obtain geometric realizations of the modified coideal algebras $\check{U}^j$ and $\check{V}^j$ in the framework of type $C$ flag varieties. All the main results in earlier sections for type $B$ afford similar formulations in the type $C$ setting.

**Appendix A. Compatibility of canonical bases**

(by H. Bao, Y. Li, and W. Wang)

In this Appendix we work in the setting of Sections 3 and 4. We shall use the notations with subscripts, $\{A\}_d$ and $\{A\}$, to denote a standard and canonical basis element in $S^j$, and as before use $[A]$ and $\{A\}$ to denote a standard and canonical basis element in $K^j$. Recall from Proposition 4.11 the surjective $\mathcal{A}$-algebra homomorphism $\phi_d : K^j \rightarrow S^j$. 

**Lemma 6.8.** The bar involutions on $K^j$ (as well as on $\check{U}^j$) and $S^j$ commute with the homomorphism $\phi_d : K^j \rightarrow S^j$. 

**Proof.** The lemma follows by checking on the generators with the help of Proposition 4.11. $\square$

A type $A$ version of the following lemma appears in [DF14, Lemma 6.4], and the proof below follows the one in [Fu12, Proposition 6.3]. We thank Jie Du and Qiang Fu for clarifying our misunderstanding of their crucial lemma.

**Lemma 6.9.** Let $A \in \tilde{\Xi}$. Then $\phi_d : K^j \rightarrow S^j$ sends 
\[ \phi_d([A]) = \begin{cases} [A]_d, & \text{if } A \in \Xi_d; \\ 0, & \text{otherwise}. \end{cases} \]
Proof. We define an $\mathcal{A}$-linear map $\phi'_d : K^j \to S^j$ by sending $[A]$ to $[A]_d$ for $A \in \Xi_d$ and to 0 otherwise. We shall show that $\phi'_d = \phi_d$. Observe that $\phi'_d$ coincides with $\phi_d$ given in Proposition 4.11 on the generators of $K^j$. So it suffices to show that $\phi'_d$ is an algebra homomorphism. To that end, by the description of generators for $K^j$ in Proposition 4.10 it suffices to check that

\begin{equation}
\phi'_d([B] \cdot [A]) = \phi'_d([B]) \ast \phi'_d([A]),
\end{equation}

for $B = (b_{ij}) = [D + RE^\theta_{h,h+1}]$ or $[D + RE^\theta_{h,h+1}]$ (for $R \geq 0$, $h \in [1,n]$ and $D \in \Xi_{\text{diag}}$) and for all $A = (a_{ij}) \in \tilde{\Xi}$. We can further assume without loss of generality that $\co(B) = \ro(A)$ and $\sum_{i,j} a_{ij} = 2d + 1 = \sum_{i,j} b_{ij}$. We will treat the case for $B = [D + RE^\theta_{h,h+1}]$ in detail. The verification of (6.5) is divided into 3 cases.

(1) Assume both $B, A \in \Xi_d$. The identity (6.5) follows directly from the multiplication formulas in Proposition 3.7 and (4.5)–(4.7).

(2) Assume $A \not\in \Xi_d$. Then there exists $i_0$ such that $a_{i_0,i_0} < 0$. It follows by definition that $\phi'_d([B]) \ast \phi'_d([A]) = 0$. On the other hand, the matrices $A + \sum_u t_u(E^\theta_{hu} - E^\theta_{h+1,u})$ in the product $[B] \cdot [A]$ (see (4.5)) have negative $(i_0, i_0)$-entry with possible exceptions when $i_0 = h$ and $t_{hh} > t_{hh} + a_{hh} \geq 0$. In such exceptional cases, the coefficient of such a matrix is a product with one factor $a_{hh} + t_h$, which is 0. Therefore $\phi'_d([B] \cdot [A]) = 0$ by definition of $\phi'_d$ and (6.5) holds.

(3) Assume $B \not\in \Xi_d$. By definition we have $\phi'_d([B]) \ast \phi'_d([A]) = 0$. By assumption, there exists $i_0$ such that $b_{i_0,i_0} < 0$. We separate the proof that $\phi'_d([B] \cdot [A]) = 0$ into 2 subcases (i)–(ii) below. (i) Assume $i_0 \neq h$. Then the $i_0$-component of $\ro(B)$ is negative and all the resulting new matrices in the product $[B] \cdot [A]$ (see (4.5)) have $\ro(B)$ as their row vectors and so they all contain some negative entry. Thus $\phi'_d([B] \cdot [A]) = 0$. (ii) Assume $i_0 = h$. Then $\co(B)$ has a negative $h$th component. Since $\ro(A) = \co(B)$, we have $A \not\in \Xi_d$. Hence we are back to Case (2) above, and so $\phi'_d([B] \cdot [A]) = 0$.

Summarizing (1)-(3), we have verified (6.5) for $B = [D + RE^\theta_{h,h+1}]$. The completely analogous remaining case for $B = [D + RE^\theta_{h,h+1}]$ will be skipped. The lemma is proved. \(\square\)

The next theorem shows that the canonical bases of $K^j$ and $S^j$ are compatible under the homomorphism $\phi_d$. (A similar result holds in the type $A$ setting; cf. [SV00] and [DF14]).

**Theorem 6.10.** Let $A \in \tilde{\Xi}$. Then $\phi_d : K^j \to S^j$ sends the canonical basis for $K^j$ to the corresponding canonical basis for $S^j$ or zero. More precisely, we have

\[ \phi_d(\{A\}) = \begin{cases} \{A\}_d, & \text{if } A \in \Xi_d; \\ 0, & \text{otherwise}. \end{cases} \]

**Proof.** By Lemma 6.8 we have $\overline{\phi_d(\{A\})} = \phi_d(\{A\})$, that is, $\phi_d(\{A\})$ is bar invariant. By Proposition 4.2 we have

\[ \{A\} = [A] + \sum_{A' \subseteq A, A' \in \Xi} \pi_{A',A}[A'], \quad \pi_{A',A} \in v^{-1}Z[v^{-1}]. \]
First assume that \( A \in \Xi_d \). Then \( \phi_d([A]) = [A]_d \) by Lemma 3.9 and we have
\[
\phi_d(\{ A \}) = [A]_d + \sum_{A' \subseteq A, A' \in \Xi_d} \pi_{A', A}[A']_d.
\]
By Lemma 3.8 the (geometric) partial order \( \leq \) is stronger than \( \subseteq \), and note that the canonical basis element \( \{ A \}_d \) is characterized by the bar-invariance and the property that \( \{ A \}_d \in [A]_d + \sum_{A' \subseteq A, A' \in \Xi_d} v^{-1}Z[v^{-1}][A']_d \); compare (3.26)–(3.27). By a comparison with (6.6) above, we must have \( \phi_d(\{ A \}) = \{ A \}_d \).

Now assume that \( A \not\in \Xi_d \). Then \( \phi_d([A]) = 0 \) by Lemma 6.9 and we have
\[
\phi_d(\{ A \}) = \sum_{A' \subseteq A, A' \in \Xi_d} \pi_{A', A}[A']_d,
\]
which lies in \( \sum_{A' \in \Xi_d} v^{-1}Z[v^{-1}][A''_d] \) by applying an inverse version of (3.26). Since \( \phi_d(\{ A \}) \) is bar invariant, it must be 0. The theorem is proved. \( \square \)

**Remark 6.11.** The standard bases as well as canonical bases on \( K^i \) and \( S^i \) are also compatible under the \( \mathcal{A} \)-algebra homomorphism \( \phi_d : K^i \to S^i \) from Proposition 5.14 as follows. We denote by \([A]_d \) and \( \{ A \}_d \) (with subscripts added) the standard and canonical basis element in \( S^i \) defined in Section 5. Then by Theorem 5.10 Lemma 6.9 and Theorem 6.10 the homomorphism \( \phi_d : K^i \to S^i \) sends \([A] \) to \([A]_d \) (and respectively, \( \{ A \} \) to \( \{ A \}_d \)) for \( A \in \Xi_d \), and to 0 otherwise.

**References**

[DDPW] B. Deng, J. Du, B. Parshall and J. Wang, *Finite dimensional algebras and quantum groups*. Mathematical Surveys and Monographs, 150. American Mathematical Society, Providence, RI, 2008.

[BW13] H. Bao and W. Wang, *A new approach to Kazhdan-Lusztig theory of type B via quantum symmetric pairs*, arXiv:1310.0103.

[BLM] A. Beilinson, G. Lusztig and R. McPherson, *A geometric setting for the quantum deformation of GL_n*, Duke Math. J., 61 (1990), 655–677.

[B98] S. Billey, *Pattern avoidance and rational smoothness of Schubert varieties*, Advances in Mathematics 139 (1998), 141–156.

[BL00] S. Billey and V. Lakshmibai, *Singular loci of Schubert varieties*, Progress in Mathematics 182, Birkhäuser Boston, Inc., Boston, MA, 2000.

[Du92] J. Du, *Kazhdan-Lusztig bases and isomorphism theorems for q-Schur algebras*, Contemp. Math. 139 (1992), 121–140.

[DF14] J. Du and Q. Fu, *The integral quantum loop algebra of \( \hat{\mathfrak{gl}}_n \)*, arXiv:1404.5679.

[Dr86] V. Drinfeld, *Quantum groups*, Proc. Int. Congr. Math. Berkeley 1986, vol. 1, Amer. Math. Soc. 1988, 798–820.

[ES13] M. Ehrig and C. Stroppel, *Nazarov-Wenzl algebras, coideal subalgebras and categorified skew Howe duality*, arXiv:1310.1972.

[Fu12] Q. Fu, *BLM realization for \( \mathcal{U}_\mathbb{Z}(\hat{\mathfrak{gl}}_n) \)*, arXiv:1204.3142.

[G97] R.M. Green, *Hyperoctahedral Schur algebras*, J. Algebra 192 (1997), 418–438.

[GL92] I. Grojnowski and G. Lusztig, *On bases of irreducible representations of quantum GL_n*. In: Kazhdan-Lusztig theory and related topics (Chicago, IL, 1989), 167-174, Contemp. Math., 139, Amer. Math. Soc., Providence, RI, 1992.

[GV93] V. Ginzburg and E. Vasserot, *Langlands reciprocity for affine quantum groups of type A_n*, Internat. Math. Res. Notices 3 (1993), 67–85.

[Jim86] M. Jimbo, *A q-analogue of \( U(\mathfrak{gl}(N+1)) \), Hecke algebra, and the Yang-Baxter equation*, Lett. Math. Phys. 11 (1986), 247–252.
[Ka91] M. Kashiwara, *On crystal bases of the Q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 456–516.

[K12] S. Kolb, *Quantum symmetric Kac-Moody pairs*, arXiv:1207.6036.

[KL79] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.

[KL80] D. Kazhdan and G. Lusztig, *Schubert varieties and Poincaré duality*. Geometry of the Laplace operator (Univ. Hawai, Honolulu, Hawaii, 1979), pp. 185–203, Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980.

[KhL10] M. Khovanov and A. Lauda, *A categorification of quantum \( \mathfrak{sl}(n) \)*, Quantum Topology **1** (2010), 1–92.

[Le02] G. Letzter, *Coideal subalgebras and quantum symmetric pairs*, New directions in Hopf algebras (Cambridge), MSRI publications, vol. **43**, Cambridge Univ. Press, 2002, pp. 117–166.

[Lu90] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 447–498.

[Lu93] G. Lusztig, *Introduction to Quantum Groups*, Modern Birkhäuser Classics, Reprint of the 1993 Edition, Birkhäuser, Boston, 2010.

[Lu99] G. Lusztig, *Aperiodicity in quantum affine \( \mathfrak{gl}_n \)*, Asian J. Math. **3** (1999), 147–177.

[Mc12] K. McGerty, *On the geometric realization of the inner product and canonical basis for quantum affine \( \mathfrak{sl}_n \)*, Alg. and Number Theory **6** (2012), 1097–1131.

[Na94] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), 365–416.

[P09] G. Pouchin, *A geometric Schur-Weyl duality for quotients of affine Hecke algebras*, J. Algebra **321** (2009), 230–247.

[SV00] O. Schiffmann and E. Vasserot, *Geometric construction of the global base of the quantum modified algebra of \( \hat{\mathfrak{gl}}_n \)*, Transform. Groups **5** (2000), 351–360.

[VV99] M. Varagnolo and E. Vasserot, *On the decomposition matrices of the quantized Schur algebra*, Duke Math. J. **100** (1999), 267–297.

[W01] W. Wang, *Lagrangian construction of the \((\mathfrak{gl}_n, \mathfrak{gl}_m)\)-duality*, Commun. in Contemp. Math. **3** (2001), 201–214.

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