COMMUTING GRAPHS OF BOUNDEDLY GENERATED SEMIGROUPS

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Abstract. Araújo, Kinyon and Konieczny [3] pose several problems concerning the construction of arbitrary commuting graphs of semigroups. We observe that every star-free graph is the commuting graph of some semigroup. Consequently, we suggest modifications for some of the original problems, generalized to the context of families of semigroups with a bounded number of generators, and pose related problems.

We construct monomial semigroups with a bounded number of generators, whose commuting graphs have an arbitrary clique number. In contrast to that, we show that the diameter of the commuting graphs of semigroups in a wider class (containing the class of nilpotent semigroups), is bounded by the minimal number of generators plus two.

We also address a problem concerning knit degree.

1. Introduction

We denote the center of a semigroup $S$ by $Z(S)$. The commuting graph of $S$, denoted by $\Gamma(S)$, is the simple graph whose vertex set is $S - Z(S)$, where two vertices are connected by an edge if their corresponding elements in $S$ commute. For example, if $S$ is a commutative semigroup, then $\Gamma(S)$ is the empty graph. A study of these graphs in relation with certain algebraic structures can be found, for instance in [1, 2, 3, 4, 5].

Araújo, Kinyon and Konieczny [3] proved that for every natural number $n \geq 2$, there exists a semigroup whose commuting graph has diameter $n$. They also pose the following question:

Question 1 ([3], Section 6, (4)). Can every number $n \geq 3$ be the clique number (girth, chromatic number) of the commuting graph of a semigroup?

All semigroups constructed in this note are assumed to have a zero element, denoted 0. We call a semigroup monomial if it has a presentation with only monomial relations (i.e. relations of the form $w = 0$).

We now classify which graphs are realizable as commuting graphs of semigroups, thereby answering Question 1.

Definition 2. A vertex in a graph is called a star if it is connected to any other vertex. A graph is called star-free if none of its vertices is a star.

It is obvious that any commuting graph is star-free. The following result shows that the converse is true as well.
Proposition 3. Let $\Gamma = (V, E)$ be a star-free graph. Then there exists a (monomial) semigroup $S$ such that $\Gamma(S) = \Gamma$. Moreover, if $\Gamma$ has $n$ vertices, then there exists a (monomial) semigroup $S$, with $\Gamma(S) = \Gamma$, such that $S$ is generated by $n$ elements and its order is $n^2 + n + 1 - |E|$.

Proof. Label the vertices of $\Gamma$ to generate a free semigroup with zero. Consider its quotient semigroup $S = \left\langle v \in V \mid v_1v_2v_3 = 0 \quad \forall v_1, v_2, v_3 \right\rangle$. It is clear that the center $Z(S)$ consists of all elements of the form $vu$. It is now evident why $\Gamma(S) = \Gamma$. Also, an easy calculation shows that if $\Gamma$ has $n$ vertices then this semigroup has precisely $n^2 + n + 1 - |E|$ elements.

Note we could set the relations $uv = vu = 0$ whenever $(u, v) \in E$, resulting in a monomial semigroup with the same commuting graph. 

Let $\text{rank}(\Gamma)$ denote the minimal number of generators for a semigroup whose commuting graph is $\Gamma$. Note that the construction in Proposition 3 only shows that $\text{rank}(\Gamma) \leq |V|$. For a semigroup $S$, we call the minimal number of generators the rank of $S$.

Thus, one can modify Question 1 and ask for families of semigroups generated by a bounded number of generators, yet having arbitrary graph-theoretic invariants. We exhibit such families with arbitrary clique numbers (and leave open the same question about girth).

For a certain class of finite semigroups (which contains the class of nilpotent semigroups), we show that the diameter of the commuting graph is bounded by a linear function of the rank.

In Section 4 we provide a counterexample to another problem from [3] concerning the notion of knit degree (defined in the sequel).

2. Commuting Graphs with Arbitrary Clique Number

Denote the clique number of a graph $\Gamma$ by $\omega(\Gamma)$. In view of the previous section it is evident that there are semigroups whose commuting graphs have arbitrary clique number. However, the semigroups that arise from Proposition 3 have rank at least the prescribed clique number, thus unbounded.

To exhibit a family of boundedly generated semigroups with arbitrary clique number, we first introduce a family of monomial semigroups $S_n$ having arbitrary odd clique number $2n - 1$, all of which are generated by two elements.

Commuting graphs with even clique number are then constructed by the simple operations of null union of semigroups and join of graphs:

Definition 4. Let $S, T$ be two semigroups with zero. Their null union $S \bullet T$ is the semigroup $S \cup T$ after identifying together their zeros. For every pair of elements $s \in S$ and $t \in T$ their product in $S \bullet T$ is defined to be $st = ts = 0$.

Definition 5. The join of two simple graphs, denoted by $\vee$, is the graph union with all the edges that connect the vertices of the first graph with the vertices of the second graph.

Proposition 6. Let $n$ be a natural number. Then
(1) The semigroup

\[ S_n = \langle a, b \mid a^{n+1} = b^2 = 0, \ aba = ba^i b = 0 \forall i \geq 1 \rangle \]

satisfies \( \omega(\Gamma(S_n)) = 2n - 1 \).

(2) The semigroup \( S_n \cdot S_1 \) satisfies \( \omega(\Gamma(S_n \cdot S_1)) = 2n \).

Example 7. The graph \( \Gamma(S_2) \) is

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Proof of (1). Clearly \( S_n = \{0, b, a, a^2, \ldots, a^n, ab, a^2b, \ldots, a^n b, ba, ba^2, \ldots, ba^n\} \), so \( S_n \) has \( 3n + 2 \) elements. We calculate \( Z(S_n) = \{0, a^n b, ba^n\} \), so \( |\Gamma(S_n)| = 3n - 1 \). Take

\[ C = \{a^i b\}_{1 \leq i \leq n-1} \cup \{ba^j\}_{1 \leq j \leq n-1} \cup \{b\} \]

which is clearly a clique of size \( 2n - 1 \), and so \( \omega(\Gamma(S_n)) \geq 2n - 1 \).

Now, let \( C' \) be a clique of \( \Gamma(S_n) \). If \( b \in C' \), then \( a^i \notin C' \) for all \( i \) so \( |C'| \leq 2n - 1 \). If \( a^i \in C' \) with the minimal such \( i \), then \( b, a, \ldots, a^{i-1}, a^i b, ba^j \notin C' \) for all \( j \leq n - i \), which are together \( 2n - i \) vertices, so

\[ |C'| \leq (3n - 1) - [1 + (i - 1) + 2(n - i)] = n + i - 1 \leq 2n - 1 \]

If both \( b \notin C' \) and \( a^i \notin C' \) for all \( i \), then \( |C'| \leq 2n - 2 \). This completes the proof of the first claim that \( \omega(\Gamma(S_n)) = 2n - 1 \). \( \square \)

Proof of (2). In order to construct commuting graphs with arbitrary even clique number, we shall need the following properties of join of graphs and null union of semigroups.

(1) Let \( G, H \) be two graphs. Then \( \omega(G \lor H) = \omega(G) + \omega(H) \).

(2) Let \( S, T \) be two semigroups with zero. Then \( \Gamma(S \cdot T) \cong \Gamma(S) \lor \Gamma(T) \) (naturally), as \( Z(S \cdot T) = Z(S) \cdot Z(T) \).

(3) Let \( S, T \) be two semigroups with zero. Then \( \omega(\Gamma(S \cdot T)) = \omega(\Gamma(S)) + \omega(\Gamma(T)) \).

Thus, to exhibit boundedly generated semigroups whose commuting graph has arbitrary even clique number, it is enough to consider

\[ S_1 := \langle a, b \mid a^2 = b^2 = 0, aba = bab = 0 \rangle = \{0, a, b, ab, ba\} \].
Clearly we have that \( Z(S_1) = \{0, ab, ba\} \) and \( \Gamma(S_1) \cong K_2 \), the edgeless graph with two vertices.

By the above properties and the fact that \( \omega(\Gamma(S_1)) = 1 \) one obtains that \( \omega(\Gamma(S_n \cdot S_1)) = 2n \) and that \( S_n \cdot S_1 \) can be generated by 4 elements.

Notice that the commuting graph of the semigroup \( S'_n := S_1 \cdot \ldots \cdot S_1 \) has clique number \( n \). However, the semigroup \( S'_n \) can not be generated by less than \( 2n \) elements, which is obviously unbounded. \( \square \)

3. Diameter of Commuting Graphs

Recall that the diameter of a (connected) graph is the maximal distance between two vertices. In \([3]\), the authors introduce constructions of semigroups with connected commuting graphs having arbitrary diameter. The constructions there have unbounded rank that grows linearly with the diameter. By Proposition 3 it is again possible to obtain such examples, now with rank equal to the number of vertices (hence bigger than the prescribed diameter).

Proposition 4 exhibits commuting graphs of monomial semigroups with at most 4 generators having arbitrary clique number. For a wider family of semigroups, we show next that the diameter of their commuting graphs is effectively bounded by the rank of the semigroups.

We now show that there is a tight connection between the diameter and the rank of a semigroup in a wide class of semigroups. For a semigroup \( S \) and \( m \in \mathbb{N} \) denote

\[
S^m := \{a_1a_2\cdots a_m | a_i \in S \ \forall i\} \subseteq S
\]

In particular we have \( S^m \subseteq S^{m-1} \subseteq \cdots \subseteq S \).

**Proposition 8.** Let \( S \) be a semigroup generated by \( d \) elements. Suppose it has a non-central ideal \( I \) such that \( IS, SI \subseteq Z(S) \). If \( \Gamma(S) \) is connected, then its diameter \( D \) satisfies \( D \leq d + 2 \).

**Proof.** Pick \( u \in I \setminus Z(S) \). We claim that for every \( v \in S^2 \), we have that \( uv = vu \).

Indeed, write \( v = t_1t_2 \), and compute

\[
uv = ut_1t_2 = t_2ut_1 = t_1t_2u = vu
\]

since \( ut_1 \in IS \) and \( t_2u \in SI \) are central.

Let \( x, y \in \Gamma(S) \) be two vertices. If \( x, y \in S^2 \) then a path of length (at most) 2 can be found between them, as \( u \) commutes with both of them. Suppose only one of \( x, y \), say \( x \), is not contained in \( S^2 \) (but \( y \in S^2 \)). Note that \( |S \setminus S^2| \leq d \), because \( S \) is generated by \( d \) elements. Then by connectivity a shortest path can be found between \( x \) and some vertex \( z \in S^2 \). As the path is minimal, its length does not exceed \( d \), which is the number of generators. We can now concatenate this path to \( z - u - y \), resulting in a path of length at most \( d + 2 \), connecting \( x \) and \( y \).

We now assume that both \( x, y \notin S^2 \). We can find shortest paths \( \gamma_x, \gamma_y \) from \( x, y \) to some \( x', y' \in S^2 \), respectively and denote the lengths of these paths by \( d_x, d_y \). As before, minimality implies that \( \gamma_x \) consists of precisely \( d_x \) vertices from \( S \setminus S^2 \) (and likewise \( \gamma_y \) consists of \( d_y \) such vertices). But \( d_x + d_y \leq d \), for otherwise the pigeonhole principle would imply that some vertex from \( S \setminus S^2 \) appears in both \( \gamma_x, \gamma_y \), so a path between \( x \) and \( y \) can be found of length at most \( d - 1 \). Hence the path \( x - \cdots - x' - u - y' - \cdots - y \) obtained by concatenating \( \gamma_x, x' - u - y', \gamma_y \) has length at most \( d + 2 \). \( \square \)
Let $S$ be a non-commutative nilpotent semigroup of nilpotency degree $c$, and let $m$ be the minimal exponent for which $S^m \subseteq Z(S)$. Then a possible ideal satisfying the premise of the previous Proposition is $I = S^{m-1}$. The nilpotency of $S$ guarantees $m \leq c$ because $S^n = \{0\} \subseteq Z(S)$. As mentioned, if $S$ is commutative (e.g. $c \leq 2$), then its commuting graph is empty and the claim holds trivially.

4. A Semigroup with Knit Degree 3

Another question that was posed in [3] is related to the notion of knit degree:

\textbf{Definition 9 ([3])}. Let $S$ be a semigroup. A path $a_1 - a_2 - \cdots - a_m$ in $\Gamma(S)$ is called a \textit{left path} (abbreviated in [3] as $l$-path) if $a_1 \neq a_m$ and $a_1 a_i = a_m a_i$ for every $1 \leq i \leq m$. If $\Gamma(S)$ contains a left path, then the \textit{knit degree} of $S$ is defined to be the length of a shortest left path in $\Gamma(S)$.

For $n = 2$ and every $n \geq 4$ a band with knit degree $n$ was constructed in [3]. In [3, Section 6(1)] it was guessed that a semigroup with knit degree 3 does not exist.

We now provide a counterexample.

\textbf{Example 10}. The following semigroup has knit degree 3:

$$S = \langle x_1, x_2, x_3, x_4 | R \rangle$$

with the relations

$$R = \left\{ \begin{array}{l}
  x_1 x_j x_k = 0 \quad \forall i, j, k \\
  x_1^2 = x_4 x_1 \\
  x_4^2 = x_1 x_4 \\
  x_1 x_2 = x_4 x_2 = x_1 x_3 = x_4 x_3 = x_2 x_1 = x_2 x_3 = x_3 x_2 = x_3 x_4 = 0 
\end{array} \right\}$$

One may verify that the graph $\Gamma(S)$ is a path graph on four vertices:

![Graph Image]

The relations (except for $x_1 x_2 x_3 = 0$) ensure that $x_1 - x_2 - x_3 - x_4$ is a left path.

We are left to show that a shorter left path does not exist in the graph. There is an automorphism transposing $x_1 \leftrightarrow x_4$ and $x_2 \leftrightarrow x_3$, so it suffices to check that $x_1 - x_2$, $x_2 - x_3$ and $x_1 - x_2 - x_3$ are not left paths, which is an easy exercise.

5. Further Problems

\textbf{Question 11}. Can $\Gamma(S)$ be connected with arbitrarily large diameter for $S$ with bounded rank?

We would like to remark that in general, there is no uniform bound on the rank of finite graphs. In fact, we have:

\textbf{Claim 12}. Let $G$ be a $(2n - 2)$-regular graph on $2n$ vertices. Then $\operatorname{rank}(G) = 2n$.

\textbf{Proof}. By the generic construction in Proposition [3], we have that $\operatorname{rank}(G) \leq 2n$. To show that $\operatorname{rank}(G) = 2n$, consider any subset $X$ of at most $2n - 1$ vertices of $G$. Then $X$ lacks some vertex $v$, so there exists a vertex $u$ in $G$ (the one not connected to $v$) which is connected to all vertices of $X$ except $u$ itself. Let $S$ be a semigroup such that $\Gamma(S) = G$. Suppose that the elements of $X$ plus some (perhaps none) elements of $Z(S)$ generate $S$. Then $u$ must lie in $Z(S)$. We conclude that $\operatorname{rank}(G) = 2n$. \hfill $\Box$
Following that, we restate another question that was posed in [3].

**Question 13.** *Is there a family of graphs with bounded rank and unbounded girth?*

We remark that for the class of semigroups satisfying the condition in Proposition 8, the answer to the above question is negative.

It is natural to specialize and ask about $C_n$, the cyclic graph with $n$ vertices.

**Question 14.** *What is the rank of $C_n$?*

For example, here is an argument showing that the rank of $C_4$ is 4. Clearly every generating subset must contain at least 2 non-central generators, say $x_1$ and $x_2$. They must lie on non-adjacent vertices of $C_4$. If the rank were 2, then every other vertex would have corresponded to a central element (as it commutes with both $x_1$ and $x_2$), a contradiction. Hence, there must be at least 3 generators, but then the third generator again commutes with all generators, so 4 generators are needed. A similar reasoning shows that the rank of $C_5$ is at least 3.

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