We study the energy distribution of maxima and minima of a simple one-dimensional disordered Hamiltonian. We find that in systems with short range correlated disorder there is energy separation between maxima and minima, such that at fixed energy only one kind of stationary points is dominant in number over the other. On the other hand, in the case of systems with long range correlated disorder maxima and minima are completely mixed.

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\[ G_L(x) = \frac{1}{\pi} \int_{1/L}^{\infty} dq \hat{G}(q) e^{iqx}. \]  

(1)

The function \( \hat{G}(q) \) must be positive and for the LR case must be not integrable in zero. In order to avoid any ultraviolet divergence we can assume both for the SR and LR cases \( \hat{G}(q) \) to decay at infinity faster than any power. We can thus define these two classes of models simply in terms of the behavior of \( \hat{G} \) at zero momentum. In this way the LR case is well defined: \( \Delta_L(d) \) increases indefinitely with \( d \) and \( \tilde{V}_L(x)^2 = G_L(0) \) diverges with \( L \), as expected, since in a LR random potential the uncertainty on the height \( V \) of one single point \( x \) increases with the size \( L \) of the system, while it remains finite in a SR potential. Our analysis will not depend on the explicit form of \( \hat{G} \).

Let us denote by \( \mathcal{N}_k'(E,m) dE \) the number of stationary points of \( H(x) \) with degree of instability equal to \( k \) (\( k = 0 \) for minima, \( k = 1 \) for maxima), which have energy between \( E \) and \( E + dE \), for a given mass \( m \). The superscript \( V \) indicates that this distribution corresponds to the sample \( V \). Eventually we shall average over \( V \). The distribution \( \mathcal{N}_k'(E,m) \) is given by,

\[ \mathcal{N}_k'(E,m) = \int dx \delta(H') \delta(H - E) |H''| \delta[\theta(-H'') - k]. \]

Using an integral representation for the \( \delta \)-function, we can write,

\[ I \equiv |H''| \delta[\theta(-H'') - k] = \int d\mu e^{i\mu H'' + i\mu'} \frac{1}{2\pi} e^{i\mu' \frac{x}{\mu}} \frac{d\mu}{\sqrt{2\pi \mu}}. \]

The last two factors can be rewritten using the identity,

\[ (H'' + i\mu)^{n^+} = \int d\bar{\chi}^{n^+} d\chi^{n^+} \exp \left( -\sum_{b=1}^{n^+} \chi^{n^+}_b (H'' + i\mu) \bar{\chi}^{n^+}_b \right), \]

where \( \bar{\chi}^{n^+} \) and \( \chi^{n^+} \) are Grassmann variables and the analytic continuation \( n^+ \to (1/2 + \mu/2\pi) \) must be done. As a next step we define the Grassmann vector \( \psi \): \( \psi(a) = (\chi^{1+}_a, \ldots, \chi^{n+}_a, \chi^{1-}_a, \ldots, \chi^{n-}_a) \), which allows us to write,

\[ I = \int d\mu e^{i\mu H'' + i\mu'} \int d\psi d\bar{\psi} \exp \left( -\sum_{a=1}^{n^+} \bar{\psi}_a (H'' + i\epsilon_a) \psi_a \right), \]

where the vector \( \epsilon_a \) is split into two parts: \( \epsilon_a = \epsilon \) for \( a \leq n_+ \), \( \epsilon_a = -\epsilon \) for \( a > n_+ \), and \( n = (n_+ + n_-) \to 1 \).

Note that this replica approach can be easily generalized to \( N \) dimensions.

Let us introduce in the expression for \( \mathcal{N}_k' \) the Lagrange multiplier \( \lambda \) and \( \omega \), to represent respectively \( \delta(H') \) and \( \delta(H - E) \). The \( V \)-dependent part then becomes: \( \exp[(i\omega + i\lambda \partial_x - \psi_a \partial_x \partial_y) V(x)] \), which can be averaged over the Gaussian distribution of \( V \). This produces a quartic term \( (\sum_\alpha \psi_\alpha \bar{\psi}_\alpha)^2 \), that can be made quadratic by means of a Hubbard-Stratonovich transformation, introducing an auxiliary variable \( y \). It is now possible to perform all the Gaussian integrals over \( (\lambda, x, \psi) \). This gives a term \( (m + y + i\epsilon)^{1/2 + \mu/2\pi} (m + y - i\epsilon)^{1/2 - \mu/2\pi} \), which, using back relations \( \Psi \), can be written as \( |m + y| \exp[-i\mu \theta(m - y)] \). Integrating over \( \mu \) we finally obtain the average distributions \( \mathcal{N}_k(E,m) \equiv \mathcal{N}_k'(E,m) \),

\[ \mathcal{N}_0(E,m) = \int_{-m}^{+\infty} dy F(y,E,m), \]
\[ \mathcal{N}_1(E,m) = \int_{-\infty}^{-m} dy F(y,E,m), \]

with,

\[ F(y,E,m) = \frac{|m + y|}{\sqrt{m}} \frac{e^{-\frac{y^2}{2m}}}{\sqrt{2\pi a_2}} \times \]
\[ \times \int \frac{d\omega}{2\pi} \frac{e^{-\frac{1}{2}(a_0 - a_0^2/a_2)\omega^2 + i\omega E + i\omega a_1/a_2}}{\sqrt{m + a_1\omega}}. \]

By integrating equations \( 2 \) and \( 3 \) over the energy we get the total number of minima and maxima, \( \mathcal{N}_0(m) \) and \( \mathcal{N}_1(m) \), at a given value of the mass. Note that, as required by the Morse theorem \( \mathcal{T} \), the total number of minima minus maxima is equal to one, that is \( \mathcal{N}_0(m) - \mathcal{N}_1(m) = \int dE \mathcal{D}(E,m) = 1 \). The explicit expression for the total number of minima is,

\[ \mathcal{N}_0(m) = \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{m}{\sqrt{2a_2}} \right) + \frac{m}{a_2} \text{erf} \left( \frac{-m^2}{2a_2} \right). \]

\( \mathcal{N}_0(m) \) is a smooth function of \( m \) which goes to one when \( m \) goes to infinity and starts increasing very steeply at masses smaller than \( m \sim \sqrt{a_2} \). This value of the mass marks a crossover from the region where only one minimum exists to the region where many different minima (and maxima) appear. As expected this mass is the same critical mass as the \( N \)-dimensional mean-field case, \( m_c = \sqrt{a_2}/3 \), where a glassy transition occurs \( \mathcal{T} \). In the following we will always consider \( m < m_c \).

We analyze now our results, starting with the SR potential. In Fig.\( \mathcal{T} \) we plot \( \mathcal{N}_0(E,m), \mathcal{N}_1(E,m) \) and \( \mathcal{D}(E,m) \) functions of the energy \( E \), for \( m < m_c \).
The first thing we notice is that the two curves of minima and maxima are quite separated one with respect to the other, so that their peaks do not overlap. As a consequence $D(E,m)$ gives by itself a rather clear picture of the distribution of the different stationary points and at low energies it approximates well $N_0(E,m)$. This is important because $D$ is always very simple to compute, being $D(E,m) = \int dx \delta(H') H'' \delta(H-E)$. Thus, the computation of this quantity does not require the modulus, nor the $\theta$-function, which are in general very difficult to treat. In other words, in the SR case there is a partial decoupling between maxima and minima, which is sharper the lower is the energy. As a consequence, at fixed energy only one kind of stationary points is dominant over the other and $N_0(E,m) \sim D(E,m)$, for low enough $E$. It is remarkable that this holds for the SR potential. Indeed, it has been proved in [12] that in the $N$-dimensional $p$-spin spherical spin glass, which belongs to the SR class [12], an identical phenomenon occurs: in the limit $N \to \infty$ from the SR case) only in the limit $N \to \infty$ we know that

$$\lim_{N \to \infty} \frac{N}{N} = 1.$$  

and study the distributions of maxima and minima as functions of $E$. Denoting these new distributions by $P_0$ and $P_1$, we have

$$P_k(E,m) dE = N_k(E,m) dE.$$  

We stress that $E$ is the only variable we can sensibly regard as the energy for the LR potential. Note, on the other hand, that this rescaling is irrelevant for the SR case, where $a_0$ is finite. Taking the limit $L \to \infty$ in Eq. (4) we find

$$P_{0,1}(E,m) \to N_{0,1}(m) \frac{1}{\sqrt{2\pi}} e^{-\frac{E^2}{2}}.$$  

This equation shows that in the LR case the two distributions are just the same function, scaled by the total number $N_0(m)$ or $N_1(m)$. Maxima and minima are no longer separated in energy. Indeed, for $m < m_c$ we have $N_0(m)/N_1(m) \sim 1$ and the two curves collapse one on to the other. The conclusion is that when the total number of stationary points is large in a LR system, maxima and minima are completely mixed together, so that at each given energy they are equally numerous. Thus, in stark contrast with the SR case, no decoupling of the stationary points occurs, no matter how low is the energy.

A further step is necessary to prove that this mixing in the LR case is a typical behavior and not simply an artifact coming from the average. Indeed, it is possible to think of a system where sample by sample maxima and minima are well separated, but where the mixing described above appears only after averaging over different samples. As an example, we consider the family of Hamiltonians $H_w(x) = \sin(x) + w$, where $w$ is a random variable with zero average and variance $\sigma$. It is clear that for each sample maxima and minima are perfectly separated, since $N_0^w(E) = \delta(E \pm 1 - w)$. However, averaging over $w$ we obtain two distributions with separation between their averages equal to 2 and variance $\sigma$. Thus, if we rescale the energy by a factor $\sqrt{\sigma}$ and take the limit $\sigma \to \infty$, we would conclude that there is mixing between maxima and minima, which is sample by sample false.

In order to prove that the LR potential does not correspond to such an artifact, we consider the statistics of the extreme values of the Hamiltonian. Let us define the two distributions $A_0(E) = \delta(E - E_{\text{MIN}})$ and $A_1(E) = \delta(E - E_{\text{MAX}})$, where $E_{\text{MIN}}$ and $E_{\text{MAX}}$ are the energies of the absolute minimum and maximum of $H(x)$ (we consider as absolute maximum the highest local maximum). The separation between these two distributions is $\Delta A = E_{\text{MAX}} - E_{\text{MIN}} = \langle E \rangle_{A_1} - \langle E \rangle_{A_0}$, and let $S$ be their variance. Consider now the ratio $\Delta A / \sqrt{S}$. It is easy to see that in the artificial case described above this ratio goes to zero when the variance of the disorder $\sigma$ goes to infinity, since $\Delta A$ is finite, whereas $S$ diverges as $\sigma$. On the other hand, for the LR potential the ratio $\Delta A / \sqrt{S}$ remains finite in the limit $L \to \infty$. This is simple to prove exactly for $\gamma = 1/2$, which corresponds to
a Brownian random potential of size $L \left[ \frac{3}{4} \right]$. For the general case the idea is that in a LR potential both $\Delta A$ and $\sqrt{S}$ diverge as $L^4 \left[ \frac{5}{4} \right]$, as can also be checked by means of numerical simulations. The divergence of $\Delta A$ implies that the variance of the energy distribution of maxima and minima is sample by sample diverging as well with $L$. Therefore, unlike what happens in the case of the artifact, the average scenario of the long range potential, where maxima and minima are completely mixed in energy, is the typical one.

In this Letter we have shown that the energy distribution of maxima and minima in a one-dimensional random system is radically different whether the disorder is long range or short range correlated. An important issue is the extension of this investigation to $N$ dimensions, where we expect to find a qualitative similar behavior, at least at the mean-field level.

Indeed, as mentioned above, the $p$-spin spherical spin glass is a clear example of an $N$-dimensional mean-field SR system where at sufficiently low energies a decoupling between stationary points of different nature does occur. Moreover, a crucial feature of this SR model is that the asymptotic dynamical energy reached by the system is exactly the same energy where the stationary points decouple and is larger than the equilibrium one, so that the dynamics never gets to the equilibrium landscape.

On the other hand, let us assume that for a LR $N$-dimensional model a generalization of equation (3) is valid, so that all the stationary points collapse on one single distribution, since saddles of any degree of instability will be bounded in energy by maxima and minima. This allows us to put forward the following hypothesis: if, as indicated by our results, in a LR system there is no separation at all between different stationary points, then no decoupling energy can exist, which means no energy level capable of trapping the system. In such a situation we would expect the dynamics to reach the minimum available energy, that is the equilibrium energy.

An evidence of this conjecture can be found in the context of mean-field models for spin glasses. Here two very different classes of systems exist, a first class where the dynamical energy is larger than the equilibrium one and a second class where these two energies are the same. What has been noted in [3], is that the first class corresponds to systems with SR disorder (as the $p$-spin model), whereas to the second class belong LR disordered systems. This correspondence finds its natural explanation in the framework we have depicted above: the inability of a SR system to dynamically reach its equilibrium energy is due to the existence of an energy level below which stationary points of different nature are decoupled, while this cannot happen in the LR case. As we have tried to show in this Letter, whether this decoupling occurs or not, is an information encoded in the energy distribution of all the stationary points.

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