NON-GAUSSIANITY AND CONSTRAINTS FOR THE VARIANCE OF PERTURBATIONS IN THE CURVATON MODEL

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Recently, the primordial non-Gaussianity in the curvaton model has been predicted assuming sudden decay of the curvaton. We extend the calculation to non-instantaneous decay by employing \( \delta N \) formalism. The difference between the sudden-decay approximation and our numerical result is larger than 1% only if the non-linearity parameter is small, \(-1.16 < f_{NL} < 60\). Thus it is safe to use the sudden-decay approximation when deriving constraints for the curvaton model from WMAP3 (\( f_{NL} < 114 \)), but with the Planck forecast \(|f_{NL}| < 5\) one should employ the fully numerical result. Often, the curvaton perturbations \( \delta \sigma \) have been assumed to be small compared to the background value of the curvaton field \( \sigma_0 \). Consequently, the variance \( \Delta^2 = \langle \delta \sigma^2 \rangle / \sigma_0^2 \) has been assumed to be negligible. However, the measurements of CMB or large-scale structure perturbation amplitude do not constrain the variance if the main contribution to it comes from the ultraviolet (UV) scales, i.e., from smaller than observable scales. We discuss how, even in this case, observational constraints on non-Gaussianity set an upper bound to the small scale variance, \( \Delta^2_{UV} < 90 \).

1 Introduction

Most inflationary models give rise to nearly Gaussian primordial curvature perturbation. Typically, prediction for the non-linearity parameter \( f_{NL} \) in single-field models is of the order of \( \epsilon \) — the slow roll parameter which must be \( \lesssim 10^{-1} \) to guarantee the near scale-invariance of the primordial perturbations. In principle, measurement of \( f_{NL} \) would give valuable information on the inflaton potential, but unfortunately such a tiny non-Gaussianity is likely to remain unobservable. The current upper bound from the WMAP three-year data is \( |f_{NL}| < 114 \) while Planck is expected to bring this down to \(|f_{NL}| \lesssim 5\), which is still orders of magnitude larger than the typical inflationary prediction.

Nevertheless, there are classes of multi-field models that can lead to an observable non-Gaussianity. One well-motivated example is the curvaton model. In addition to the inflaton \( \phi \) there would be another, weakly coupled, light scalar field (e.g., MSSM flat direction), curvaton \( \sigma \), which was completely subdominant during inflation so that the inflaton drive the expansion of the universe. The potential could be as simple as \( V = \frac{1}{2} M^2 \phi^2 + \frac{1}{2} m^2 \sigma^2 \). At Hubble exit both fields acquire some classical perturbations that freeze in. However, the observed cosmic microwave (CMB) and large-scale structure (LSS) perturbations can result from the curvaton instead of the inflaton, if the inflaton perturbations are much smaller than \( 10^{-5} \). To simplify the analysis, in this talk we assume that the curvature perturbation from inflaton is completely negligible \( \zeta_\phi \ll \zeta_\sigma \), and the curvature perturbation from curvaton \( \zeta_\sigma \) is such that it leads to the observed amplitude of perturbations.

After the end of inflation the inflaton decays into ultra-relativistic particles (“radiation”) the curvaton energy density still being subdominant. At this stage the curvaton carries pure entropy (isocurvature) perturbation instead of the usual adiabatic perturbation. Namely, the entropy perturbation between radiation and curvaton is \( S_{r\sigma} = 3(\zeta_\sigma - \zeta_\phi) \approx -3\zeta_\sigma \). Since the observations have ruled out pure isocurvature primordial perturbation, a mechanism — curvaton decay into “radiation” — that converts the isocurvature perturbation to the adiabatic one is needed at some stage of the evolution before the primordial nucleosynthesis.

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As the Hubble rate, \( H \), decreases with time, eventually \( H^2 \lesssim m^2 \), and the curvaton starts to oscillate about the minimum of its potential. Then it behaves like pressureless dust ("matter", \( \rho_\sigma \propto a^{-3} \)) so that its relative energy density starts to grow with respect to radiation (\( \rho_r \propto a^{-4} \)). Finally, the curvaton decays into ultra-relativistic particles leading to the standard radiation dominated adiabatic primordial perturbations. However, this mechanism may create from the initially Gaussian curvaton field perturbation a strongly non-Gaussian primordial curvature perturbation. The more subdominant the curvaton is during its decay the more non-Gaussianity results in. Since the time of the decay depends on the model parameters (such as the curvaton mass \( m \) and decay rate \( \Gamma_\sigma \)), the observational upper bounds on non-Gaussianity provide a method to constrain these parameters.

In the simplest case the (possible) non-Gaussianity results from the second-order correction to the linear result

\[
\zeta = \zeta_1 + \frac{1}{2} \zeta_2 = \zeta_1 + \frac{3}{5} f_{\text{NL}} \zeta_1^2.
\]

Here \( \zeta_1 \) is proportional to the Gaussian field perturbation at the Hubble exit, so it is Gaussian, and \( \zeta_2^2 \) is \( \chi^2 \) distributed. In this talk we briefly derive \( f_{\text{NL}} \) in the sudden-decay approximation using a slightly different and more general approach than in [12], and then, for the first time, present the results in the case of non-instantaneous decay of curvaton. For the full derivation and discussion of our results see [11], where we, in addition to the second-order calculation, go to the third order, and finally derive fully non-linear results and full probability density function (pdf) of the primordial \( \zeta \) in the long-wavelength limit. Since, in the early universe, all today’s observable scales are super-Hubble, we take advantage of the separate universe assumption throughout the calculations.

## 2 Sudden-decay approximation

In the absence of interactions, fluids with a barotropic equation of state, such as radiation (\( P_r = \rho_r/3 \)) or the non-relativistic curvaton (\( P_\sigma = 0 \)), have a conserved curvature perturbation [11]

\[
\zeta_i(t, \vec{x}) = \delta N(t, \vec{x}) + \frac{1}{3} \int \frac{\rho_i(t, \vec{x})}{\rho_i(\bar{t})} \frac{d\rho_i'}{\rho_i'} d\mathbf{t},
\]

where \( \delta N = N(t, \vec{x}) - \bar{N}(t) \) with \( N(t, \vec{x}) \) being the perturbed (i.e., local) number of e-folds of expansion until time \( t \) and \( \bar{N}(t) \) being the average expansion. At the first order this fully non-linear definition reduces to the usual definition: \( \zeta_1 = -\psi_1 - H \frac{\rho_\sigma}{\rho_r} \) with \( \psi_1 = -\delta_1 N \).

Applying (2) for the curvaton during its oscillation but before the decay, we have

\[
\zeta_\sigma(t, \vec{x}) = \frac{1}{3} \ln \left( \frac{\rho_\sigma(t, \vec{x})}{\rho_\sigma(\bar{t})} \right)_{\delta N=0},
\]

where \( \rho_\sigma(t, \vec{x}) = \frac{1}{4} m^2 \sigma^2(\vec{x}) \) is evaluated on spatially flat (\( \delta N = 0 \)) hypersurface, and \( \sigma(\vec{x}) \) is the local amplitude of oscillation. The time of the beginning of the curvaton oscillation, \( t_{\text{in}} \), is defined in terms of the local Hubble rate as \( H(t_{\text{in}}, \vec{x}) = m \). Since \( H^2 = (8\pi G/3)\rho_{\text{tot}} \), the constant time \( t = t_{\text{in}} \) surface is a uniform-total density hypersurface.

In general, \( \sigma_{\text{in}} \) depends non-linearly on the field value at Hubble exit \( \sigma_*. \) Thus we write [11]

\[
\sigma_{\text{in}} = g(\sigma_*) = g(\bar{\sigma}_*) + g' \delta \sigma_* + \frac{1}{2} g''(\delta \sigma_*)^2 + \ldots,
\]

where \( \delta = \partial / \partial \sigma_* \). (For exactly quadratic potential

\[\text{Hadn’t we assumed negligible inflaton curvature perturbation, } \zeta_0 \approx 0 \), some “residual” isocurvature would have resulted if the curvaton was sub-dominant during its decay. This would have lead to an interesting mixture of correlated adiabatic and isocurvature perturbations which was studied in [11]. Following the guidelines of [11], our calculation should be straightforward to generalise. It should be noted that observations do not rule out a correlated isocurvature component if it is less than 20% of the total primordial perturbation amplitude.\[12\]
$g''$ and higher derivatives vanish, but even a slight deviation from quadratic potential can change the resulting $f_{NL}$ considerably.\[12\] via $g''$. Substituting this into \[13\] and expanding up to second order we obtain $\zeta_\sigma = \zeta_\sigma + \frac{1}{2} \zeta_2 + \ldots$ with $\zeta_\sigma = \frac{2}{3} \frac{g''}{g} \delta \sigma$ and $\zeta_2 = -\frac{3}{2} \left( 1 - g g'' / g'^2 \right) \zeta_5^2$, where $\delta \sigma$ is well described by a Gaussian random field as we assume the inflaton and curvaton to be uncoupled (or only weakly coupled), see e.g. Refs. \[13,14\]. We can express this result in terms of the effective non-linearity parameter for the curvaton perturbation, analogous to Eq. \[10\],

\[ f_{NL}^\sigma = -\frac{5}{3} \left( 1 - \frac{g''}{g'} \right). \]

Hence we find $f_{NL} = -5/4$ for the curvaton $\zeta_\sigma$ in the absence of any non-linear evolution ($g'' = 0$). If the curvaton comes to dominate the total energy density in the universe before it decays, so that $\zeta = \zeta_\sigma$, then this is the generic prediction for the primordial $f_{NL}$ in the curvaton model, as emphasised by \[15\].

Assume now that the curvaton decays instantaneously at time $t_{dec}$ (before it has become completely dominant) on a uniform-total density hypersurface corresponding to $H \simeq \Gamma_\sigma$, i.e., when the local Hubble rate equals the decay rate for the curvaton (assumed constant). Hence

\[ \rho_r(t_{dec}, \vec{x}) + \rho_\sigma(t_{dec}, \vec{x}) = \bar{\rho}(t_{dec}), \]

where we use a bar to denote the homogeneous, unperturbed quantity. Note that from Eq. \[2\] we have $\zeta = \delta N$ on the decay surface, and we can interpret $\zeta$ as the perturbed expansion, or “$\delta N$”. Assuming all the curvaton decay products are relativistic ($P = \rho/3$), we have that $\zeta$ is conserved after the curvaton decay. The local curvaton and radiation densities on this decay surface may be inhomogeneous. Indeed we have from Eq. \[2\] $\zeta_r = \zeta + \frac{1}{4} \ln (\rho_r / \bar{\rho}_r)$ and $\zeta_\sigma = \zeta + \frac{1}{4} \ln (\rho_\sigma / \bar{\rho}_\sigma)$ or, equivalently, $\rho_r = \bar{\rho}_r e^{4(\zeta_r - \zeta)}$ and $\rho_\sigma = \bar{\rho}_\sigma e^{3(\zeta_\sigma - \zeta)}$. Requiring that the total density is uniform on the decay surface, Eq. \[11\], then gives the simple relation $(1 - \Omega_{\sigma,dec} e^{4(\zeta_r - \zeta)}) + \Omega_{\sigma,dec} e^{3(\zeta_\sigma - \zeta)} = 1$, where $\Omega_{\sigma,dec} = \bar{\rho}_\sigma / (\bar{\rho}_r + \bar{\rho}_\sigma)|_{t_{dec}}$ is the dimensionless density parameter for the curvaton at the decay time. This equation can be rewritten in the form

\[ e^{3\zeta_\sigma} = \frac{3 + e^r}{4} e^{\zeta} + \frac{3r - 3}{4} e^{-\zeta}, \]

where $r = \frac{3\Omega_{\sigma,dec}}{2 - \Omega_{\sigma,dec}}$. Recalling Eq. \[3\] the LHS of Eq. \[5\] is $\rho_\sigma(t_{in}, \vec{x}) / \rho_\sigma(t_{in}) = g^2[\sigma_*(\vec{x})] / g^2[\bar{\sigma}_*]$. As Eq. \[5\] is a fourth degree equation for $e^\zeta$, the primordial curvature perturbation, $\zeta$, as a function of Gaussian $\sigma_*$ can be solved exactly. Remarkably, Eq. \[4\] was derived using fully non-linear definitions. Hence we have found an exact fully non-linear solution for $\zeta$ (as opposed to only second-order calculations in the literature). Expanding this solution in $\delta \sigma$, up to second order we then find $f_{NL}$, up to third order the so called $g_{NL}$, etc. Since the exact solution is quite long, it turns out to be easier to directly expand RHS of Eq. \[3\] up to any wanted order, and then equate it to the LHS, i.e., to $g^2[\sigma_*(\vec{x})] / g^2[\bar{\sigma}_*]$ order by order. Up to second order the solution is $\zeta_1 = r \zeta_{11} = r \frac{2g'}{3g} \delta \sigma_*$ and $\zeta_2 = [\frac{2g'}{3g} (1 + gg'' / g'^2) - 2 - r] \zeta_1$. These give the non-linearity parameter \[14\] in the sudden-decay approximation \[17\]

\[ f_{NL} = \frac{5}{16} \left( 1 + \frac{gg''}{g'^2} \right) - \frac{5}{3} - \frac{5r}{6}. \]

In the limit $r \to 1$, when the curvaton dominates the total energy density before it decays, we recover the non-linearity parameter $f_{NL,\sigma}^\sigma$ of the curvaton $f_{NL} \to f_{NL,\sigma}^\sigma = -\frac{5}{4} \left( 1 - g g'' / g'^2 \right)$. On the other hand we may get a large non-Gaussianity ($|f_{NL}| \gg 1$) in the limit $r \to 0$, where we have $f_{NL} \to \frac{5}{16} \left( 1 + gg'' / g'^2 \right)$.

### 3 Non-instantaneous decay

After the curvaton has decayed the universe is dominated by “radiation”, with equation of state $P = \rho/3$, and hence the curvature perturbation is non-linearly conserved on large scales.
To achieve this, we need to start from different non-trivial mapping between \( \Gamma > H \). The energy density is continually decaying once the curvaton begins oscillating until finally (when \( \Gamma > H \)) its density becomes negligible, and during this process \( \zeta_\sigma \) does evolve\( ^{15} \). Another problem with results derived from the sudden-decay approximation is that the final amplitude of the primordial curvature perturbation, and its non-linearity, are given in terms of the density of the curvaton at the decay time which is not simply related to the initial curvaton density, especially as the decay time, \( H \approx \Gamma_\sigma \), itself is somewhat ambiguous.

In the non-instantaneous case we can still define the first order transfer efficiency “\( r \)” of the initial curvaton perturbation to the output radiation perturbation, but now it must be calculated numerically from the definition \( r = \zeta_{1r,\text{out}}/\zeta_{1\sigma,\text{in}} \). It turns out to be a function solely of the parameter \( p_{\text{in}} = [\Omega_\sigma(H/\Gamma_\sigma)^{1/2}]_{\text{in}} = \frac{\delta^2}{3M_1^2} \left( \frac{m}{M_1^2} \right)^{1/2} \), where \( M_1^2 \equiv 1/\left(8\pi G\right) \).

We start the numerical integration of the background Friedmann eqn together with radiation and curvaton continuity eqns (with \( \pm \Gamma_\rho \) as a source term) from uniform-density surface at \( t_{\text{in}} \) and end it when curvaton has completely decayed at some suitable uniform-density hypersurface \( H(t_{\text{end}}, \vec{x}) = H_{\text{end}} \). Then the fully non-linear primordial curvature perturbation will be \( \zeta = \zeta_{r,\text{out}} = \delta N(t_{\text{end}}, \vec{x}) \). Repeating the calculation with different initial values \( \sigma_{\text{in}} \) we find the function \( N(\sigma_{\text{in}}) \). From this we then calculate the first and second order quantities by employing \( \delta N = N' \delta \sigma_s + \frac{1}{2} N'' (\delta \sigma_s)^2 + \ldots \equiv \zeta_1 + \frac{1}{2} \zeta_2 + \ldots \), and finally \( f_{\text{NL}} = \frac{5}{6} N''/(N')^2 \). From Fig. 1(a) we see that if \( f_{\text{NL}} > 60 \) (\( r < 0.02 \)) or \( f_{\text{NL}} < -1.16 \) (\( r > 0.95 \)), the sudden-decay result differs from the non-instantaneous decay result less than \( 1\% \). In the both cases, as \( r \to 1 \) we have \( f_{\text{NL}} \to -\frac{5}{4} \). As \( r \to 0 \), the sudden-decay result is \( f_{\text{NL}} \to \frac{5}{4}\pi(1+g''/g'^2) - 1.67 \), whereas the non-instantaneous decay gives \( f_{\text{NL}} \to \frac{5}{4}\pi(1+g''/g'^2) - 2.27 \). We can write the full result in a form \( f_{\text{NL}} = \frac{5}{4}\pi \left( 1 + \frac{g''}{g'^2} \right) + \frac{3}{2} h(r)/r^2 \), where the function \( h(r) \) is defined by equation \( r' = [2r + h(r)]g'/g \), and determined numerically in the non-instantaneous case. For the sudden decay we have from \( ^6 \) \( h(r) = -\frac{4}{3}r^2 - \frac{3}{2}r^3 \).

\(^4\) After this talk the non-instantaneous decay calculation was done\( ^{10} \) employing second order perturbation equations instead of \( \delta N \) formalism which we use. The results agree after taking into account that the authors of\( ^{13} \) compare \( f_{\text{NL}} \)s resulting from the fixed \( p_{\text{in}} \) while we prefer to compare \( f_{\text{NL}} \) when the sudden decay and non-instantaneous decay produce same \( r \), or, in other words, the same observable first order curvature perturbation. To achieve this, we need to start from different \( p_{\text{in}} \) for sudden decay than for non-instantaneous decay. The non-trivial mapping between \( p_{\text{in}} \) and \( r \) is demonstrated in Fig. 1(b) for the non-instantaneous decay.
4 Large variance on small scales

Finally, we consider the possibility of a large small-scale variance \( \Delta^2 \equiv \langle (\delta \sigma/\sigma_{in})^2 \rangle \). Let us name the observable CMB scales as infrared (IR) and smaller scales as ultraviolet (UV) so that \( \lambda_{IR} \gg \lambda_{UV} \gg H^{-1} \). As the observations require \( \langle \zeta^2 \rangle_{IR} \lesssim 10^{-9} \), they set constraint for \( \frac{1}{5} r^2 \Delta^2_{IR} \). However, observations do not directly constrain \( \Delta^2_{UV} \). Allowing large \( \Delta^2_{UV} \) we find \( f_{NL} = \frac{5}{4} \left( 1 + \Delta^2_{UV} \right)^{1/2} \left( 1 + g g''/g'g^2 \right) + \frac{5}{4} h(r)/r^2 \), where \( h(r) \) remains same as without large variance. Thus, large homogeneous small-scale variance modifies the first term of \( f_{NL} \) only. Recalling that \( r \leq 1 \), and \(-54 < f_{NL} < 114 \) (from WMAP3), we find an upper bound \( 1 + \Delta^2_{UV} < \frac{1}{5} \times 114 = 91 \).

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