On Products of Shifts in Arbitrary Fields

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Abstract

We adapt the approach of Rudnev, Shakan, and Shkredov presented in [2] to prove that in an arbitrary field \( F \), for all \( A \subset F \) finite with \( |A| < p^{1/4} \) if \( p := \text{Char}(F) \) is positive, we have

\[
|A(A + 1)| \gtrsim |A|^{11/9}, \quad |AA| + |(A + 1)(A + 1)| \gtrsim |A|^{11/9}.
\]

This improves upon the exponent of 6/5 given by an incidence theorem of Stevens and de Zeeuw.

1 Introduction and Main Result

For finite \( A \subseteq F \), we define the sumset and product set of \( A \) as

\[
A + A = \{a + b : a, b \in A\}, \quad AA = \{ab : a, b \in A\}.
\]

It is an active area of research to show that one of these sets must be large relative to \( A \). The central conjecture in this area is the following.

**Conjecture 1 (Erdős - Szemerédi).** For all \( \epsilon > 0 \), and for all \( A \subseteq \mathbb{Z} \) finite, we have

\[
|AA| + |A + A| \gg |A|^{2-\epsilon}.
\]

The notation \( A \ll B \) is used to hide absolute constants, and in addition the notation \( A \lesssim B \) is used to hide constant factors and factors of \( \log |A| \), i.e. \( A \lesssim B \) if and only if there exist absolute constants \( c > 0 \) and \( d \) such that \( A \leq cB(\log |A|)^d \). If \( A \lesssim B \) and \( B \lesssim A \) we write \( A \sim B \). Although Conjecture 1 is stated over the integers it can be considered over fields, the real numbers being
of primary interest. Current progress over \( \mathbb{R} \) places us at an exponent of \( \frac{4}{3} + c \) for some small \( c \), due to Shakan \cite{10}, building on works of Konyagin and Shkredov \cite{11} and Solymosi \cite{12}. Incidence geometry, and in particular the Szemerédi-Trotter Theorem, are the tools used to prove such results in the real numbers.

Conjecture \( \dagger \) can also be considered over arbitrary fields \( \mathbb{F} \). We will let \( p \) denote the characteristic of \( \mathbb{F} \) throughout. Due to the possible existence of subfields in \( \mathbb{F} \), extra restrictions on \( |A| \) relative to \( p \) must be imposed if \( p > 0 \). All such conditions can be ignored if \( p = 0 \). Over arbitrary fields we replace the Szemerédi-Trotter Theorem with a point-plane incidence theorem of Rudnev \cite{13}, which was used by Stevens and de Zeeuw to derive a point-line incidence theorem \cite{4}. The exponent of \( 6/5 \) was proved in 2014 by Roche-Newton, Rudnev, and Shkredov \cite{5}. An application of the Stevens - de Zeeuw Theorem also gives this exponent of \( 6/5 \) for Conjecture \( \dagger \) so that \( 6/5 \) became a threshold to be broken.

The \( 6/5 \) threshold has recently been broken, see \cite{1}, \cite{2}, and \cite{3}. The following theorem was proved in \cite{2} by Rudnev, Shakan, and Shkredov, and is the current state of the art bound.

**Theorem 1.** \cite{2} Let \( A \subset \mathbb{F} \) be finite with \( |A| < p^{18/35} \). Then
\[
|A + A| + |AA| \gtrsim |A|^{11/9}.
\]

Another way of considering the sum-product phenomenon is to consider the set \( A(A + 1) \), which we would expect to be quadratic in size. This encapsulates the idea that a translation of a multiplicatively structured set should destroy its structure, which is a main theme in sum-product questions. Study of growth of \( |A(A + 1)| \) began in \cite{6} by Garaev and Shen, see also \cite{7}, \cite{8}, and \cite{9}. Current progress for \( |A(A + 1)| \) comes from an application of the Stevens - de Zeeuw Theorem, giving the same exponent of \( 6/5 \). In this paper we use the multiplicative analogue of ideas in \cite{2} to prove the following theorem.

**Theorem 2.** Let \( A, B, C, D \subset \mathbb{F} \) be finite with the conditions
\[
|C(A + 1)||A| \leq |C|^3, \quad |C(A + 1)|^2 \leq |A||C|^3, \quad |B| \leq |D|, \quad |A||B||C||D| < p^{1/4}.
\]

Then we have
\[
|AB|^8|C(A + 1)|^2|D(B - 1)|^8 \gtrsim |B|^{13}|A|^5|C|^3|D|.
\]

In our applications of this theorem we have \( |A| = |B| = |C| = |D| \), so that the first three conditions are trivially satisfied. The conditions involving \( p \) could likely be improved, however for
sake of exposition we do not attempt to optimise these. The main proof closely follows [2] (in the multiplicative setting), the central difference being a bound on multiplicative energies in terms of products of shifts. An application of Theorem 2 beats the threshold of 6/5, matching the 11/9 appearing in Theorem 1. Specifically, we have

**Corollary 1.** Let \( A \subseteq \mathbb{F} \) be finite, with \( |A| < p^{1/4} \). Then
\[
|A(A + 1)| \gtrsim |A|^{11/9}, \quad |AA| + |(A + 1)(A + 1)| \gtrsim |A|^{11/9}.
\]

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2 Preliminary Results

We require some preliminary theorems. The first is the point-line incidence theorem of Stevens and de Zeeuw.

**Theorem 3** (Stevens - de Zeeuw, [4]). Let \( A \) and \( B \) with \( |A| \geq |B| \) be finite subsets of \( \mathbb{F} \) a field, and let \( L \) be a set of lines. Assuming \( |L||B| \ll p^2 \) and \( |B||A|^2 \leq |L|^3 \), we have
\[
I(A \times B, L) \ll |A|^{1/2} |B|^{3/4} |L|^{3/4} + |L|
\]

Note that as \( A \) is the larger of \( A \) and \( B \), we may swap the powers of \( |A| \) and \( |B| \) in this result. Before stating the next two theorems we require some definitions. For \( x \in \mathbb{F} \) we define the representation function
\[
r_{A/D}(x) = \left| \left\{ (a, d) \in A \times D : \frac{a}{d} = x \right\} \right|.
\]

The set \( A/D \) in this definition can be changed to any other combination of sets, changing the fraction \( \frac{a}{d} \) in the definition to match. We also define the \( n \)th moment multiplicative energy of sets \( A, D \subseteq \mathbb{F} \) as
\[
E_n(A, D) = \sum_x r_{A/D}(x)^n.
\]

We use Theorem 3 to prove two further results. The first is a bound on the fourth order multiplicative energy relative to products of shifts.
Theorem 4. For all finite non-empty $A, C, D \subset F$ with $|A|^2|C(A + 1)| \leq |D||C|^3$, $|A||C(A + 1)|^2 \leq |D|^2|C|^3$, and $|A||C||D|^2 \ll p^2$, we have

$$E^*_4(A, D) \lesssim \min \left\{ \frac{|C(A + 1)|^2|D|^3}{|C|}, \frac{|C(A + 1)|^3|D|^2}{|C|^2} \right\}.$$ 

The second result is similar, but for the second moment multiplicative energy.

Theorem 5. For all finite and non-empty $A, C, D \subset F$ with $|A|^2|C(A + 1)| \leq |D||C|^3$, $|A||C(A + 1)|^2 \leq |D|^2|C|^3$, and $|A||C||D|^2 \min(|C|, |D|) \ll p^2$, we have

$$E^*(A, D) \lesssim \frac{|C(A + 1)|^{3/2}|D|^{3/2}}{|C|^{3/2}}.$$ 

The $A + 1$ appearing in these theorems can be changed to any $A + \lambda$ for $\lambda \neq 0$, by noting that $|C(A + 1)| = |C(\lambda A + \lambda)|$ and renaming $A' = \lambda A$. For our purposes, we will use $\lambda = \pm 1$.

Proof of Theorem 4. WLOG we can assume that $0 \notin A, C, D$. We begin by proving that

$$E^*_4(A, D) \lesssim \frac{|C(A + 1)|^2|D|^3}{|C|}.$$ 

Define the set

$$S_\tau := \{x \in A/D : \tau \leq r_{A/D}(x) < 2\tau\}.$$ 

By a dyadic decomposition, there is some $\tau$ with

$$|S_\tau|^4 \sim E^*_4(A, D).$$

Take an element $t \in S_\tau$. It has $\tau$ representations in $A/D$, so there are $\tau$ ways to write $t = a/d$ with $a \in A, d \in D$. For all $c \in C$, we have

$$t = \frac{a}{d} = \frac{1}{d} \left( \frac{ac + c - c}{c} \right) = \frac{1}{d} \left( \frac{p}{c} - 1 \right)$$

where $p = c(a + 1) \in C(A + 1)$. This shows that we have $|S_\tau|\tau|C|$ incidences between the lines

$$L = \{l_{d,c} : d \in D, c \in C\}, \quad l_{d,c} \text{ given by } y = \frac{1}{d} \left( \frac{x}{c} - 1 \right)$$
and the point set \( P = C(A + 1) \times S_r \). Under the conditions \(|D||C| \min\{|S_r|, |C(A + 1)|\} \ll p^2\) and \(|S_r||C(A + 1)| \max\{|S_r|, |C(A + 1)|\} \leq |D|^3|C|^3\), we have that

\[
|S_r|r|C| \leq I(P, L) \ll |C(A + 1)|^{1/2} |S_r|^{3/4} |C|^{3/4} |D|^{3/4} + |D||C|.
\]

The conditions are satisfied under the assumptions \(|D||C| \min\{|D|, |C|\} \ll p^2\), \(|A|^2|C(A + 1)| \leq |D||C|^3\), and \(|A||C(A + 1)|^2 \leq |D|^2|C|^3\). Assuming that the leading term is dominant, we have

\[
|S_r|r^4|C| \ll |C(A + 1)|^2 |D|^3
\]

so that as \( |S_r|r^4 \sim E_4^s(A, D) \), we have

\[
E_4^s(A, D) \lesssim \frac{|C(A + 1)|^2 |D|^3}{|C|}.
\]

We therefore assume the leading term is not dominant. Suppose \(|D||C|\) is dominant, so that

\[
|D||C| \gg |C(A + 1)|^{1/2} |S_r|^{3/4} |C|^{3/4} |D|^{3/4}.
\]  

(1)

Raising to the power four and multiplying through by \( r^{12} \) we get the bound

\[
|D||C|r^{12} \gtrsim |C(A + 1)|^2 |S_r|^3 |C|^{3/4} |D|^{3/4} \implies E_4^s(A, D) \gtrsim \frac{|D|^{1/3} |C|^{1/3} r^4}{|C(A + 1)|^{2/3}}.
\]

We now assume that the result doesn’t hold, that is

\[
\frac{|C(A + 1)|^2 |D|^3}{|C|} \lesssim \frac{|D|^{1/3} |C|^{1/3} r^4}{|C(A + 1)|^{2/3}}
\]

which gives

\[
|D|^8 |C|^4 |A|^4 \ll |D|^8 |C(A + 1)|^8 \lesssim r^{12} |C|^4 \ll |A|^{12} |C|^4
\]

so that we have \( |D| \lesssim |A| \). We return to equation (1) and simplify, to find

\[
|A|^{1/4} |C|^{1/4} \gtrsim |D|^{1/4} |C|^{1/4} \gg |C(A + 1)|^{1/2} |S_r|^{3/4} \geq |A|^{1/4} |C|^{1/4} |S_r|^{3/4}
\]

so that \( |S_r| \sim 1 \). We then have

\[
|D||C| \gg |C(A + 1)|^{1/2} |S_r|^{3/4} |C|^{3/4} |D|^{3/4} \sim |C(A + 1)|^{1/2} |C|^{3/4} |D|^{3/4} \gg |A|^{1/4} |C||D|^{3/4} \gg |D||C|
\]

so that the two terms are in fact balanced and the result follows.

Secondly, we prove that

\[
E_4^s(A, D) \lesssim \frac{|C(A + 1)|^3 |D|^2}{|C|}.
\]
To do this, we swap the roles of \( D \) and \( S_\tau \) from above. We define the line set and point set by

\[
L = \{ l_{t,c} : t \in S_\tau, c \in C \}, \quad P = C(A + 1) \times D.
\]

Any incidence from the previous point and line set remains an incidence for the new ones, via

\[
t = \frac{1}{d} (\frac{2}{p} - 1) \iff d = \frac{1}{t} (\frac{2}{p} - 1).
\]

Under the conditions

\[
|S_\tau|C| \ll I(P, L) \ll |C(A + 1)|^{3/4}|S_\tau|^{3/4}|C|^{3/4}|D|^{1/2} + |S_\tau||C|.
\]

If the leading term dominates, the result follows from \( |S_\tau| \sim E^*_4(A, D) \). Assume the leading term is not dominant, that is,

\[
|C(A + 1)|^{3/2}|D|^2 \ll |S_\tau||C|.
\]

Then by using \( |S_\tau| \ll |A||D| \), we have

\[
|A||C|^2|D|^2 \leq |C(A + 1)|^{3/2}|D|^2 \ll |A||D||C|
\]

so that \(|C| \sim |D| \sim 1 \) and the result is trivial.

We now check the conditions \( 2 \) for using Theorem \( 3 \). The first condition in \( 2 \) is satisfied if \( |A||C||D|^2 \ll p^2 \), which is true under our assumptions. The second condition depends on \( \max\{ |D|, |C(A + 1)| \} \), which we assume is \(|D| \) (if not the first term in Theorem \( 4 \) gives stronger information, which we have already proved). Assuming the second condition does not hold, we have

\[
|D|^2|C(A + 1)| \gg |S_\tau||C|^3.
\]

Multiplying by \( \tau^{12} \) on both sides and bounding \( \tau \ll |A| \), we get

\[
E^*_4(A, D) \lesssim \frac{|A|^4|D|^{2/3}|C(A + 1)|^{1/3}}{|C|}.
\]

Assuming that the result does not hold, we have

\[
\frac{|C(A + 1)|^{3/2}|D|}{|C|} \lesssim \frac{|A|^4|D|^{2/3}|C(A + 1)|^{1/3}}{|C|}
\]

giving

\[
|A|^8|D|^4 \ll |C(A + 1)|^8|D|^4 \ll |A|^{12}.
\]
So that $|D| \lesssim |A|$. In turn, this implies $|A| \gtrsim |D| \geq |C(A+1)| \gg |A|$, so that $|A| \sim |C(A+1)| \sim |D|$. Returning to equation\textsuperscript{3}, this gives

$$E^*_\alpha(A, D) \lesssim \frac{|A|^4|D|^{2/3}|C(A+1)|^{1/3}}{|C|} \sim \frac{|C(A+1)|^3|D|^2}{|C|}$$

and the result is proved. \hfill \blacksquare

**Proof of Theorem 5.** The proof follows similarly to that of Theorem 4. We again define the lines and points

$$L = \{l_{d,c} : d \in D, c \in C\}, \quad l_{d,c} \text{ given by } y = \frac{1}{d} \left( \frac{x}{c} - 1 \right), \quad P = C(A + 1) \times S_\tau,$$

where in this case the set $S_\tau$ is rich with respect to $E^*(A, D)$, so that $|S_\tau|\tau^2 \sim E^*(A, D)$. With the conditions $|A||C||D| \min\{|D|, |C|\} \ll \rho^2$ and $|S_\tau||C(A+1)| \max\{|S_\tau|, |C(A+1)|\} \leq |D|^3|C|^3$, (which are satisfied under our assumptions) we have by Theorem\textsuperscript{3}

$$|S_\tau||C| \ll I(P, L) \ll |S_\tau|^{1/2}|C(A+1)|^{3/4}|D|^{3/4}|C|^{3/4} + |D||C|.$$

If the leading term dominates, we have

$$|S_\tau|\tau^2 \ll \frac{|C(A+1)|^{3/2}|D|^{3/2}}{|C|^{1/2}}$$

and the result follows from $|S_\tau|\tau^2 \sim E^*(A, D)$. We therefore assume that the leading term does not dominate, that is,

$$|S_\tau|^{1/2}|C(A+1)|^{3/4}|D|^{3/4}|C|^{3/4} \ll |D||C|.$$

Multiplying through by $\tau$ and squaring, we get the bound

$$E^*(A, D) \lesssim \frac{|D|^{1/2}|C|^{1/2}\tau^2}{|C(A+1)|^{3/2}}.$$

Assuming the result does not hold, we have

$$\frac{|D|^{3/2}|C(A+1)|^{3/2}}{|C|^{1/2}} \lesssim \frac{|D|^{1/2}|C|^{1/2}\tau^2}{|C(A+1)|^{3/2}} \implies |D||C(A+1)|^3 \lesssim |C|\tau^2.$$

Bounding $\tau \leq |A|$ and $|C||A|^2 \ll |C(A+1)|^3$ we have $|D| \sim 1$. Similarly, bounding $\tau^2 \leq |A||D|$ and $|C(A+1)|^3 \geq |C|^2|A|$, we find $|C| \lesssim 1$, so that the result is trivial. \hfill \blacksquare
3 Proof of Theorem 2

We follow a multiplicative analogue of the argument in [2]. For $A$ and $B$ finite subsets of $\mathbb{F}$, define a popular set of products

$$P := \left\{ x \in AB : r_{AB}(x) \geq \frac{|A||B|}{\log |A||AB|} \right\}.$$ 

Note that by writing

$$|\{(a, b) \in A \times B : ab \in P\}| + |\{(a, b) \in A \times B : ab \notin P\}| = \frac{|A||B|}{\log |A||AB|}$$

and noting that

$$|\{(a, b) \in A \times B : ab \notin P\}| = \sum_{x \notin P} r_{AB}(x) < |P| \frac{|A||B|}{\log |A||AB|} \leq \frac{|A|}{|A|}$$

we have

$$|\{(a, b) \in A \times B : ab \in P\}| \geq \left(1 - \frac{1}{\log |A|}\right) \frac{|A||B|}{|A|}.$$ 

We also define a popular subset of $A$ with respect to $P$, as

$$A' := \left\{ a \in A : \{|b \in B : ab \in P\} \geq \frac{2}{3}|B| \right\}.$$ 

We have

$$|\{(a, b) \in A \times B : ab \in P\}| = \sum_{a \in A'} |\{b : \{ab \in P\}| + \sum_{a \in A \setminus A'} |\{b : ab \in P\}| \geq \left(1 - \frac{1}{\log |A|}\right) |A||B|$$

Suppose that $|A \setminus A'| = c|A|$ for some $c \geq 0$, so that also $|A'| = (1-c)|A|$. Noting that

$$\sum_{a \in A'} |\{b : ab \in P\}| \leq (1-c)|A||B|, \quad \sum_{a \in A \setminus A'} |\{b : ab \in P\}| \leq \frac{2c}{3} |A||B|,$$

we have

$$(1-c)|A||B| + \frac{2c}{3} |A||B| \geq (1 - \frac{1}{\log |A|})|A||B| \Rightarrow c < \frac{3}{\log |A|},$$

so that $|A'| \geq \left(1 - \frac{3}{\log |A|}\right) |A|$.

We use a multiplicative version of Lemma 8 in [2]. The proof we present is an expanded version of the proof present in [2].

**Lemma 1.** For all $A \subset \mathbb{F}$, there exists $A_1 \subseteq A$ with $|A_1| \gg |A|$, such that

$$E^*_{4/3}(A_1') \gg E^*_{4/3}(A_1).$$
Proof. We give an algorithm which shows such a subset exists, as otherwise we have a contradiction. We recursively define

$$A_i = A_{i-1}', \quad A_0 = A, \quad i < \log |A|$$

where $A'_i$ is defined relative to $A_i$. Using the same arguments as above, we have $|A'_i| \geq \left(1 - \frac{3}{\log |A|}\right) |A_{i-1}|$, so that $|A_i| \gg |A|$ for all $i$, by following the chain $|A_i| = |A'_{i-1}| \geq \left(1 - \frac{3}{\log |A|}\right) |A_{i-1}| \geq \left(1 - \frac{3}{\log |A|}\right)^i |A_0| \geq \frac{|A|}{e^3}$. We assume that at all steps, we have

$$E_{4/3}^*(A'_i) < \frac{E_{4/3}^*(A_i)}{4}$$

as otherwise we have $E_{4/3}^*(A'_i) \gg E_{4/3}^*(A_i)$ and we are done. After $\log |A|$ steps, we have a set $A_k$ with

$$|A_k| \gg |A|, \quad E_{4/3}^*(A_k') < \frac{E_{4/3}^*(A_k)}{4} < \frac{E_{4/3}^*(A_{k-1})}{16} < \ldots < \frac{E_{4/3}^*(A)}{4^\log |A|}.$$ 

But then we have

$$E_{4/3}^*(A) > E_{4/3}^*(A'_k)^{4^\log |A|} \gg |A|^{4/3 + 2} = |A|^{10/3}$$

which is a contradiction. Therefore at some step we have an $A_i$ satisfying the lemma. \qed

We apply this lemma at the outset, redefining the subset $A_i$ found by Lemma 1 as $A$ to ensure WLOG that we have

$$E_{4/3}^*(A') \gg E_{4/3}^*(A).$$

We pigeonhole the ratio set $A'/A'$ in relation to the energy $E_{4/3}^*(A')$ to find a set $Q \subset A'/A'$ with $|Q|\Delta^{4/3} \sim E_{4/3}^*(A')$ for some $\Delta > 0$.

We will bound the number of solutions to the trivial equation

$$\frac{a}{a'} = \frac{ab}{a'b} = \frac{ab'}{a'b'} \quad (4)$$

such that $(a, a', b, b') \in A^2 \times B^2$, $\frac{a}{a'} \in Q$, $ab, ab', a'b, a'b' \in P$. We have

$$N = \sum_{a, a' \in A', a/a' \in Q} |\{b \in B : ab, a'b \in P\}|^2$$

and we see that as for all $a \in A'$, $|\{b \in B : ab \in P\}| \geq \frac{2}{\Delta} |B|$, by considering the intersection of $\{b \in B : ab \in P\}$ and $\{b \in B : a'b \in P\}$, we have that for all $a, a' \in A'$, $|\{b \in B : ab, a'b \in P\}| \geq \frac{1}{\Delta} |B|$. Therefore $N \geq \frac{4}{\Delta} |B|^2|Q|\Delta$.  

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Define an equivalence relation on $A^2 \times B^2$ via

$$(a, a', b, b') \sim (c, c', d, d') \iff \exists \lambda \text{ s.t. } a = \lambda c, a' = \lambda c', b = \frac{d}{\lambda}, b' = \frac{d'}{\lambda}.$$  

Note that the conditions $\frac{a}{a'} \in Q, \frac{ab}{a'b'} \in P$ are invariant in the class (i.e. if one class element satisfies these conditions, then they all do). Call the number of equivalence classes satisfying these conditions $|X|$. Also note that any quadruple satisfying these conditions gives a solution to (4). We can therefore write the number of solutions $N$ as the sum over each equivalence class;

$$N = \sum_{\frac{a}{a'}, \frac{ab}{a'b'} \in P} |[a, a', b, b']|^2.$$  

By Cauchy-Schwarz and completing the sum over all equivalence classes, we have

$$|Q|^2 \Delta^2 |B|^4 \ll N^2 \leq |X| \sum_{[a, a', b, b']} |[a, a', b, b']|^2.$$  

We must now bound the two quantities on the right hand side of this equation. We first claim that

$$\sum_{[a, a', b, b']} |[a, a', b, b']|^2 \leq \sum_x r_{A/A}(x)^2 r_{B/B}(x)^2.$$  

To see this, note that the left hand side counts pairs of elements of equivalence classes. Take any two elements $(a, a', b, b'), (c, c', d, d') \in A^2 \times B^2$ from the same equivalence class, so that we may write $(c, c', d, d') = (\lambda a, \lambda a', \frac{b}{\lambda}, \frac{b'}{\lambda})$. The 8-tuple $(a, a', b, b', c, c', d, d')$ satisfies

$$\lambda = \frac{c}{a} = \frac{c'}{a'} = \frac{b}{d} = \frac{b'}{d'}$$

and thus contributes to the sum $\sum_x r_{A/A}(x)^2 r_{B/B}(x)^2$. We also see that different pairs from equivalence classes give different 8-tuples, and so the claim is proved. We use Cauchy-Schwarz on the right hand side of equation (5) to transform it into a pair of fourth energies.

$$\sum_x r_{A/A}(x)^2 r_{B/B}(x)^2 \leq E_4^*(A)^{1/2} E_4^*(B)^{1/2}.$$  

We use Theorem (4) to bound these energies. We bound via

$$E_4^*(A) \lesssim \frac{|C(A + 1)|^2 |A|^3}{|C|}, \quad E_4^*(B) \lesssim \frac{|D(B - 1)|^2 |B|^3}{|D|}$$

with conditions

$$|C(A + 1)||A| \leq |C|^3, \quad |C(A + 1)|^2 \leq |A||C|^3, \quad |A|^3 |C| \ll p^2$$

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which are all satisfied under our assumptions. This gives us

\[ |Q|^2 \Delta^2 |B|^4 \lesssim |X| \frac{|C(A+1)||A|^{3/2}|D(B-1)||B|^{3/2}}{|C|^{1/2}|D|^{1/2}}. \]

We now bound \(|X|\), the number of equivalence classes. Note that any \((a, a', b, b')\) a solution to equation (3) with the relevant conditions as above transforms into a solution to the equation

\[ w = \frac{s}{t} = \frac{u}{v} \]

with \(w \in Q, s, t, u, v \in P\), by taking \(w = \frac{a}{r}, s = ab, t = a'b, u = ab', v = a'b'\). Note that taking two solutions \((a, a', b, b')\) and \((c, c', d, d')\) that are not from the same equivalence class necessarily gives us two different solutions to equation (3) via the map above. Thus we have

\[ |X| \leq \left| \left\{(w, s, t, u, v) \in Q \times P^4 : w = \frac{s}{t} = \frac{u}{v} \right\} \right| \]

The popularity of \(P\) allows us to bound this by

\[ |X| \lesssim \frac{|AB|^4}{|A|^4|B|^4} \left| \left\{(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4) \in A^4 \times B^4 : \frac{a_1 b_1}{a_2 b_2} = \frac{a_3 b_3}{a_4 b_4} \in Q \right\} \right|. \]

We dyadically pigeonhole the set \(\frac{BA}{A}\) in relation to the number of solutions to \(r/a = r'/a' \in Q\) with \(r, r' \in \frac{BA}{A}, a, a' \in A\) to find popular subsets \(R_1, R_2 \subseteq \frac{BA}{A}\) in terms of these solutions. Specifically, we have

\[ |X| \lesssim \frac{|AB|^4}{|A|^4|B|^4} \sum_{i} \sum_{x \in \frac{BA}{A}} \left[ \frac{r}{r'}(x) \left| \left\{(a_3, a_4, b_1, b_3, b_4) \in A^2 \times B^3 : \frac{x}{b_1} = \frac{a_3 b_3}{a_4 b_4} \in Q \right\} \right| \right] \]

to give us a \(\Delta_1 > 0\) and an \(R_1 \subseteq \frac{AB}{A}\) such that

\[ |X| \lesssim \frac{|AB|^4}{|A|^4|B|^4} \frac{\Delta_1}{A^2} \left| \left\{(r_1, a_3, a_4, b_2, b_3, b_4) \in R_1 \times A^2 \times B^3 : \frac{r_1}{b_2} = \frac{a_3 b_3}{a_4 b_4} \in Q \right\} \right|. \]

We perform a similar dyadic decomposition to get a \(\Delta'_1 > 0\) and \(R_2 \subseteq \frac{AB}{A}\) with

\[ |X| \lesssim \frac{|AB|^4}{|A|^4|B|^4} \Delta_1 \Delta_1 \left| \left\{(r_1, r_2, b_2, b_4) \in R_1 \times R_2 \times B^2 : \frac{r_1}{b_2} = \frac{r_2}{b_4} \in Q \right\} \right|. \]
We use these decompositions to get the bound

\[ |X| \lesssim \frac{|AB|^4}{|A||B|} \Delta_1 \Delta'_1 \left| \left\{ (r_1, r_2, b_2, b_4) \in R_1 \times R_2 \times B^2 : \frac{r_1}{b_2} = \frac{r_2}{b_4} \in Q \right\} \right| \]

\[ \leq \frac{|AB|^4}{|A||B|} \Delta_1 \Delta'_1 \left( \sum_{q \in Q} r_{R_1/B}(q) r_{R_2/B}(q) \right)^{1/2} \left( \sum_{q \in Q} r_{R_2/B}(q)^2 \right)^{1/2} \]

\[ \leq \frac{|AB|^4}{|A||B|} \Delta_1 \Delta'_1 \left( \sum_{q \in Q} r_{R_1/B}(q)^2 \right)^{1/2} \left( \sum_{q \in Q} r_{R_2/B}(q)^2 \right)^{1/2} \]

\[ \leq \frac{|AB|^4}{|A||B|} \Delta_1 \Delta'_1 |Q|^{1/2} E_4^*(B, R_1)^{1/4} E_4^*(B, R_2)^{1/4} \]

We will now show that given \(|B||D| |R_i|^2 \ll p^2\) and \(|B| \leq |D|\) (which are true under our assumptions), we have

\[ E_4^*(B, R_i) \lesssim \frac{|D(B - 1)|^3 |R_i|^2}{|D|}. \quad (7) \]

Firstly, with the additional conditions

\[ |B|^2 |D(B - 1)| \leq |R_i||D|^3, \quad |B||D(B - 1)|^2 \leq |R_i|^2 |D|^3 \]

we may bound these fourth energies by Theorem 4 to get (7). We can therefore assume one of these conditions does not hold.

Suppose that \(|B|^2 |D(B - 1)| \geq |R_i||D|^3 \). We have

\[ E_4^*(B, R_i) \leq |R_i|^4 |B|. \]

Note that we want to have

\[ E_4^*(B, R_i) \leq \frac{|D(B - 1)|^3 |R_i|^2}{|D|} \]

which would follow from

\[ |R_i|^4 |B| \leq \frac{|D(B - 1)|^3 |R_i|^2}{|D|} \]

which is true if and only if \(|R_i|^2 |B||D| \leq |D(B - 1)|^3 \). Using our assumed bound for \(|R_i|\), we know that

\[ |R_i|^2 |B||D| \leq \frac{|B|^4 |D(B - 1)|^2}{|D|^5} \]

Noting that we certainly have

\[ |B| \leq |D| \Rightarrow \frac{|B|^4 |D(B - 1)|^2}{|D|^5} \leq |D(B - 1)|^3 \]
so that we must have
\[ |R_i|^2 |B||D| \leq |D(B - 1)|^3 \]
and so the bound on the fourth energy holds.

Now assume the second condition from (8) does not hold, that is, \(|B||D(B - 1)|^2 \geq |R_i|^2 |D|^3\).
Again, we have
\[ E_4^*(B, R_i) \leq |R_i|^4 |B|. \]

We have
\[ |R_i|^4 |B| \leq \frac{|D(B - 1)|^3 |R_i|^2}{|D|} \iff |R_i|^2 |B||D| \leq |D(B - 1)|^3 \]
so that it is enough to prove \(|R_i|^2 |B||D| \leq |D(B - 1)|^3\), as before. Using the assumption to bound \(|R_i|\), we have the information that
\[ |R_i|^2 |B||D| \leq \frac{|B|^2 |D(B - 1)|^2}{|D|^2} \leq |D(B - 1)|^3. \]
Therefore we have that \(|R_i|^2 |B||D| \leq |D(B - 1)|^3\) and so the bound on the fourth energy holds.

Plugging this in, we get
\[ |X| \lesssim \frac{|AB|^4}{|A|^4 |B|^2} \Delta_1 \Delta_1' |Q|^{1/2} E_4^*(B, R_1) E_4^*(B, R_2)^{1/4} \]
\[ \lesssim \frac{|Q|^{1/2} |AB|^4 |D(B - 1)|^{3/2}}{|A|^4 |B|^4 |D|^{1/2}} |R_1|^4 |R_2|^{1/2} \Delta_1 \Delta_1' \]
The product \(|R_1|^{1/2} |R_2|^{1/2} \Delta_1 \Delta_1'\) can be bounded by
\[ \left[ |R_1|^{1/2} |R_2|^{1/2} \Delta_1 \Delta_1' \right]^2 \leq \sum_{x \in R_1} r_{\Delta_1}(x)^2 \sum_{x \in R_2} r_{\Delta_1'}(x)^2 \]
where it is important to note that \(r_{\Delta_1}(x)\) gives a triple \((b, a, a')\). For \(i = 1, 2\), we have
\[ \sum_{x \in R_i} r_{\Delta_1}(x)^2 \leq \left| \left\{ (a_1, a_2, a_3, a_4, b_1, b_2) \in A^4 \times B^2 : \frac{b_1 a_1}{a_2} = \frac{b_2 a_3}{a_4} \right\} \right|. \]
Following the process as before, we find a pair of subsets $S_1, S_2 \subseteq A/A$ with respect to these solutions, and some $\Delta_2, \Delta'_2 > 0$ with

$$\sum_{x \in R_i} r_{S_1}(x)^2 \lesssim \Delta_2 \Delta'_2 \sum_{x} \sum_{S_1, B(x) \cap S_2 B(x)} \lesssim \Delta_2 \Delta'_2 E^*(B, S_1)^{1/2} E^*(B, S_2)^{1/2}.$$

We will use a similar argument as above to prove that with the two conditions $|B||D||S_i| \min\{|D|, |S_i|\} \ll p^2$ and $B \leq D$ (which are satisfied under our assumptions), we have

$$E^*(B, S_i) \lesssim \frac{|S_i|^{3/2} D(B - 1)^{3/2}}{|D|^{1/2}}.$$

(9)

Under the extra conditions

$$|B|^2 D(B - 1) | \leq |S_i| |D|^3, \quad |B||D(B - 1)|^2 \leq |S_i|^2 |D|^3$$

(10)

we can bound this energy by Theorem 5 to get (9). We therefore assume the first condition from (10) does not hold, that is, $|B|^2 D(B - 1) \geq |S_i||D|^3$. We bound the energy via

$$E^*(B, S_i) \leq |B||S_i|^2.$$

We wish to show that

$$|B||S_i|^2 \leq \frac{|S_i|^{3/2} D(B - 1)^{3/2}}{|D|^{1/2}} \quad \text{which is true iff} \quad |B||D|^{1/2} |S_i|^{1/2} \leq |D(B - 1)|^{3/2}.$$

Using our assumption on $|S_i|$, we have that

$$|B||D|^{1/2} |S_i|^{1/2} \leq \frac{|B|^2 D(B - 1)^{1/2}}{|D|}.$$

Our assumption that $|B| \leq |D|$ gives

$$\frac{|B|^2 D(B - 1)^{1/2}}{|D|} \leq |B||D(B - 1)|^{1/2} \leq |D(B - 1)|^{3/2}$$

so that $|B||D|^{1/2} |S_i|^{1/2} \leq |D(B - 1)|^{3/2}$, and the bound (11) holds. Next we assume that the second condition in (10) does not hold, that is, $|B||D(B - 1)|^2 \geq |S_i|^2 |D|^3$. We again have

$$E^*(B, S_i) \leq |B||S_i|^2.$$
Comparing this bound to our desired bound, we have
\[ |B||S_i|^2 \leq \frac{|S_i|^{3/2}|D(B - 1)|^{3/2}}{|D|^{1/2}} \implies |B||D|^{1/2}|S_i|^{1/2} \leq |D(B - 1)|^{3/2}. \]

So that the bound we want follows from the second inequality above. Using our assumption on \(|S_i|\), we know that
\[ |B||D|^{1/2}|S_i|^{1/2} \leq \frac{|B|^{5/4}|D(B - 1)|^{1/2}}{|D|^{1/4}} \]
and by our assumption that \( |B| \leq |D| \), we have
\[ \frac{|B|^{5/4}|D(B - 1)|^{1/2}}{|D|^{1/4}} \leq |D(B - 1)|^{3/2} \]
so that we have \(|B||D|^{1/2}|S_i|^{1/2} \leq |D(B - 1)|^{3/2}\) as needed.

In all cases the bound on \(E^*(B, S_i)\) holds, so that we find
\[
\left( |R_1|^{1/2}|R_2|^{1/2} \Delta_1 \Delta_1^* \right)^2 \leq \Delta_2^2 \Delta_3^2 \frac{E^*(B, S_1)E^*(B, S_2)}{|D|} \leq \frac{E_{4/3}^*(A)^3|D(B - 1)|^3}{|D|}.\]

Putting all these bounds together, we have
\[ |Q|^{3/2} \Delta^2 |B|^{13/2}|A|^{5/2}|C|^{3/2}|D|^{1/2} \lesssim |AB|^4 |C(A + 1)||D(B - 1)|^4 E_{4/3}^*(A)^{3/2} \]
which simplifies to
\[ E_{4/3}^*(A')^3 |B|^{13}|A|^5|C|^3|D| \lesssim |AB|^8 |C(A + 1)|^2|D(B - 1)|^8 E_{4/3}^*(A)^3. \]

We use our assumption \(E_{4/3}(A') \gg E_{4/3}(A)\) to conclude that
\[ |B|^{13}|A|^5|C|^3|D| \lesssim |AB|^8 |C(A + 1)|^2|D(B - 1)|^8. \]

The first part of Corollary \[\text{II}\] can be seen by setting \(B = A + 1, C = A\) and \(D = A + 1\) to give
\[ |A(A + 1)| \gtrsim |A|^{11/9}. \]
Alternatively, setting \(B = -A, D = C = A + 1\) gives the second part,
\[ |AA| + |(A + 1)(A + 1)| \gtrsim |A|^{11/9}. \]
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