Reciprocity relations between ordinary temperature and the Frieden-Soffer’s Fisher-temperature

F. Pennini and A. Plastino

Instituto de Física La Plata (IFLP)
Universidad Nacional de La Plata (UNLP) and Argentine National Research Council (CONICET)
C.C. 727, 1900 La Plata, Argentina

Frieden and Soffer conjectured some years ago the existence of a “Fisher temperature” $T_F$ that would play, with regards to Fisher’s information measure $I$, the same role that the ordinary temperature $T$ plays vis-a-vis Shannon’s logarithmic measure $S$. In a series of more recent publications, this conjecture was amply validated by showing that the Legendre transform structure of thermodynamics can be replicated without changes if ones substitutes $I$ for the Jaynes-Shannon entropy $S$ [3, 4]. In our present considerations we will play, with regards to Fisher’s information measure $I$, the same role that the ordinary temperature $T$ plays vis-a-vis Shannon’s logarithmic measure $S$. In this respect we will provide a first answer in this respect.

KEYWORDS: Fisher information, thermodynamics, temperature.

PACS numbers: 05.20.-y, 05.40.-a, 05.70.-a

INTRODUCTION

Frieden and Soffer conjectured some years ago [1, 2] the existence of a “Fisher temperature” $T_F$ that would play, with regards to Fisher’s information measure $I$, the same role that the ordinary temperature $T$ plays vis-a-vis Shannon’s logarithmic measure $S$ [3, 4]. In a series of more recent publications, this conjecture was amply validated by showing that the Legendre transform structure of thermodynamics can be replicated without changes if ones substitutes $I$ for the Jaynes-Shannon entropy $S$ [3, 4, 10, 11, 12], which yields then a “Fisher thermodynamics”. A question still lingers, though: what is the relation between $T$ and $T_F$? In this note we purport to provide an interpretation with reference to the meaning of the “best” estimator associated to $I$.

This “best” estimator is called the efficient estimator. Any other estimator must have a larger mean-square error. The only proviso to the above result is that all estimators be unbiased, i.e., satisfy $\langle \hat{θ}(x) \rangle = θ$. Thus, Fisher’s information measure has a lower bound, in the sense that, no matter what parameter of the system we choose to measure, $I$ has to be larger or equal than the inverse of the mean-square error associated with the concomitant experiment. This result, i.e.,

$$Ie^2 ≥ 1,$$

is referred to as the Cramer-Rao (CR) bound, and constitutes a very powerful statistical result [2].

BRIEF FISHER CONSIDERATIONS

Estimation theory [2] provides one with a powerful result with reference to a system that is specified by a physical parameter $θ$. Let $x$ be a stochastic variable and $p_θ(x)$ the probability density for this variable, which depends on the parameter $θ$. If an observer

• makes a measurement of $x$ and wishes to best infer $θ$ from this measurement, calling the resulting estimate $\hat{θ} = \hat{θ}(x)$ and
• wonders how well $θ$ can be determined,

then estimation theory asserts [2] that the best possible estimator $\hat{θ}(x)$, after a very large number of $x$-samples is examined, suffers a mean-square error $e^2$ from $θ$ that obeys a relationship involving Fisher’s $I$, namely, $Ie^2 = 1$, where the Fisher information measure $I$ is of the form

$$I = \int dx p_θ(x) \left( \frac{∂p_θ(x)}{∂θ} \right)^2 = \left( \frac{1}{p_θ} \frac{∂p_θ}{∂θ} \right)^2. \quad (1)$$

FORMALISM

We start by defining the well known density operator that describes a system at equilibrium [2, 4]

$$\hat{ρ} = \frac{1}{Z} e^{-\sum_{i=1}^{M} x_i \hat{A}_i}, \quad (3)$$

where the $x_i$ are Lagrangian multipliers associated to the $M$ observables $\hat{A}_i$ and

$$\langle A_i \rangle = \text{Tr} \hat{ρ} \hat{A}_i \quad (i = 1, \ldots, M), \quad (4)$$

where the partition function $Z$ is given by $Z(\chi_i) = \text{Tr} \left( e^{-\sum_{i=1}^{M} x_i \hat{A}_i} \right)$. In our present considerations we assume that these multipliers have already been determined.

Following Mandelbrot [13, 11, 12], we will associate the above Lagrange multipliers to parameters to be estimated via Fisher considerations. We write down now the Fisher information measure used in such an estimation...
procedure, here associated to the probability distribution
\[ \hat{\rho} = \{ \Gamma_i \} \]
\[ I = \left\langle \sum_{i=1}^{M} \Gamma_i \left( \frac{\partial \ln \hat{\rho}}{\partial \chi_i} \right)^2 \right\rangle, \]  
\[ (5) \]
where the \( \Gamma_i \) are suitable constants related to the need of expressing \( I \) as a dimensionless quantity, as discussed in \[ 12, 13 \]. After replacing (3) into (5) we then find that \( I \) is intimately connected to our observables’ fluctuations, as pointed out long ago by Mandelbrot \[ 10, 15 \].

\[ I = \sum_{i=1}^{M} \Gamma_i \left( \hat{A}_i - \langle \hat{A}_i \rangle \right)^2, \]  
\[ (6) \]
It is well known (and straightforwardly verified) that the statistical fluctuations of an observable obey the relation\[ 10, 15 \]
\[ \left\langle \left( \hat{A}_i - \langle \hat{A}_i \rangle \right)^2 \right\rangle = -\frac{\partial \langle \hat{A}_i \rangle}{\partial \chi_i}. \]  
\[ (7) \]
which allows us to recast the Fisher measure in the fashion
\[ I = -\sum_{i=1}^{M} \Gamma_i \frac{\partial \langle \hat{A}_i \rangle}{\partial \chi_i}, \]  
\[ (8) \]
As stated above, thermodynamics’ Legendre structure can be neatly re-obtained if one uses Fisher’s information instead of Boltzmann’ entropy \[ 12, 13, 14 \]. We will be dealing with the same mean values \( \langle \hat{A}_i \rangle \) used above, but different Lagrange multipliers will ensue. Let us call the Fisher multipliers \( \gamma_i \) and borrow from \[ 8 \] the well known thermodynamic relation that links information measure, Lagrange multipliers (here the Fisher ones), and expectation values \[ 3, 11, 12 \]
\[ \gamma_i = \frac{\partial I}{\partial \langle \hat{A}_i \rangle}. \]  
\[ (9) \]
It is now clear that, introducing the above result into \[ 8 \], we get an expression for the Fisher multipliers \( \gamma_i \) in terms of the Shannon ones
\[ \gamma_i = -\sum_{j=1}^{M} \Gamma_j \frac{\partial }{\partial \langle \hat{A}_j \rangle} \frac{\partial \langle \hat{A}_i \rangle}{\partial \chi_j}, \]  
\[ (10) \]
a relation which could be used to determine them. It might seem at this point natural to ask: what happens if we consider a canonical distribution in which the Lagrange multipliers are the \( \gamma_i \) instead of the \( \chi_i \)? We discuss this question show below for classical systems within the strictures of the canonical ensemble.

**EQUIPARTITION THEOREM**

In classical statistical mechanics there exists a useful general result concerning the energy \( E \) of a system expressed as a function of \( N \) generalized coordinates \( q_i \) and momenta \( p_i \). The result holds in the case of the following (frequent) occurrence

1. the energy splits additively into the form \( E = \epsilon_i(r_i) + E'(q_1, \ldots, p_N) \), where \( \epsilon_i(r_i) \) involves only the degree of freedom \( i \) (the variable \( r_i \)) and the remaining part \( E' \) does not depend on \( r_i \).

2. the function \( \epsilon_i(r_i) \) is quadratic in \( r_i \).

In these circumstances \( \langle \epsilon_i \rangle = kT/2 \), with \( k \) the Boltzmann’s constant and \( T \) the temperature. This is the equipartition theorem \[ 10 \]. Its demonstration assumes that the thermal equilibrium Boltzmann-Gibbs equilibrium probability distribution
\[ f = \frac{1}{Z} e^{-\beta E}, \]  
\[ (11) \]
where \( \beta = 1/kT = \lambda/k \) is the Lagrange multiplier associated with the the mean-energy constraint \( \langle E \rangle = \int dr fE \), with \( dr \) the phase-space volume element. Setting \( \Gamma = 1/k \) yields a dimensionless Fisher information measure \[ 8 \] for the canonical ensemble
\[ I = -\frac{1}{k} \frac{\partial \langle E \rangle}{\partial \lambda}. \]  
\[ (12) \]

**RECIPROCITY**

Since we assume equipartition we immediately find \[ 10 \]
\( E = (Nk)\lambda^{-1} \). We have then
\[ \frac{\partial \langle E \rangle}{\partial \lambda} = -Nk\lambda^{-2} = -\frac{E^2}{kN}, \]  
\[ (13) \]
entailing that the Fisher multiplier \( \gamma \) is
\[ \gamma = \frac{1}{kT_F} = -(1/k) \frac{\partial \langle E \rangle}{\partial \lambda} = \frac{2}{k\lambda}. \]  
\[ (14) \]
Since the multipliers are inverse temperatures we obtain the interesting relationship
\[ T_F = \frac{1}{2T}, \]  
\[ (15) \]
our main result here, which, on reflection, should not surprise anyone since it is a well known fact that whenever \( I \) grows Shannon’s \( S \) decreases, and viceversa \[ 3 \].

As stated above, we introduce now \( \gamma = 2/\lambda \) as the multiplier entering the canonical probability distribution \( f \) in
and repeat the above discussion. Now $E = Nk/\gamma$ and
\[ \frac{\partial \langle E \rangle}{\partial \gamma} = -Nk\gamma^{-2} = \frac{E^2}{kN}. \] (16)

We ask ourselves what is now
\[ \gamma_2 = \frac{1}{k} \frac{\partial}{\partial \langle E \rangle} \frac{\partial \langle E \rangle}{\partial \gamma}? \] (17)
Obviously,
\[ \gamma_2 = \frac{2}{k\gamma} = \frac{2}{k} = \frac{k\lambda}{2} = \lambda = \frac{1}{T}. \] (18)

Here we encounter reciprocity! The “Fisher multiplier” $\gamma_2$ is the inverse Boltzmann-Gibbs-Shannon temperature that verifies
\[ \frac{1}{kT} = \frac{1}{kT_F} = \frac{\partial I(\gamma)}{\partial \langle E \rangle\gamma} = \frac{\partial I(\beta)}{\partial \langle E \rangle\beta}, \] (19)

in self-explanatory notation.

CONCLUSIONS

We have in this note provided two results that we deem important for the Fisher practitioners, namely
- $T_F = 1/2T$
- the reciprocity relations (19).

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