Borrowing From the Future: Addressing Double Sampling in Model-free Control

Yuhua Zhu  
Department of Mathematics  
Stanford University  
yuhuazhu@stanford.edu

Zach Izzo  
Department of Mathematics  
Stanford University  
zizzo@stanford.edu

Lexing Ying  
Department of Mathematics  
and  
Institute for Computational and Mathematical Engineering  
Stanford University  
lexing@stanford.edu

Abstract

In model-free reinforcement learning, the temporal difference method and its variants become unstable when combined with nonlinear function approximations. Bellman residual minimization with stochastic gradient descent (SGD) is more stable, but it suffers from the double sampling problem: given the current state, two independent samples for the next state are required, but often only one sample is available. Recently, the authors of Zhu et al. [2020] introduced the borrowing from the future (BFF) algorithm to address this issue for the prediction problem. The main idea is to borrow extra randomness from the future to approximately re-sample the next state when the underlying dynamics of the problem are sufficiently smooth. This paper extends the BFF algorithm to action-value function based model-free control. We prove that BFF is close to unbiased SGD when the underlying dynamics vary slowly with respect to actions. We confirm our theoretical findings with numerical simulations.

1 Introduction

Background The goal of reinforcement learning (RL) is to find an optimal policy which maximizes the return of a Markov decision process (MDP) [Sutton & Barto 2018]. One of the most common ways of finding an optimal policy is to treat it as the fixed point of the Bellman operator. Researchers have developed efficient iterative methods such as temporal difference (TD) [Sutton 1988], Q-learning [Watkins 1989], and SARSA [Rummery & Niranjan 1994] based on the contraction property of the Bellman operator.

Nonlinear function approximations have recently received a great deal of attention in RL. This follows the successful application of neural networks (NNs) to Atari games [Mnih. et al. 2013, 2015], as well as in Alpha Go and Alpha Zero [Silver et al. 2016, 2017]. However, when using a nonlinear approximation and off-policy data, the Bellman operator fails to retain the contraction property. The result is that training naive NN approximation may be unstable. Many variants and modifications have been proposed to stabilize training. For example, DQN [Mnih. et al. 2015] and A3C [Mnih. et al. 2016] stabilize Q-learning by using a slowly changing target network and replaying over past experiences or using parallel agents for exploration; double DQN reduces instability by using two...
separate $Q$ value estimators, one for choosing the action and the other for evaluating the action’s quality [van Hasselt et al. 2015].

Another way to stabilize RL with a nonlinear approximation is to formulate it as a minimization problem. This approach is known as Bellman residual minimization (BRM) [Baird 1995]. However, applying stochastic gradient descent (SGD) to BRM directly suffers from the so-called double sampling problem: at a given state, two independent samples for the next state are required in order to perform unbiased SGD. Such a requirement is often hard to fulfill in a model-free setting, especially for problems with a continuous state space.

**Contributions** In this paper, we revisit BRM for $Q$-value prediction and control problems in the model-free RL setting. The main assumption is that the underlying dynamics of the MDP can be written as $E[s_{m+1} - s_m | s_m, a_m] = \mu(s_m, a_m)\epsilon$, where $\epsilon$ is a small parameter. Note that knowledge of the dynamics is not required to implement the algorithm. We extend the borrowing-from-the-future (BFF) algorithm of Zhu et al. [2020] to action-value based RL. The key idea is to borrow extra randomness from the future by leveraging the smoothness of the underlying RL problem. We prove that when the underlying dynamics change slowly with respect to actions and the policy changes slowly with respect to states, the training trajectory of the proposed algorithm is statistically close to the training trajectory of unbiased SGD. The difference between the two algorithms will first decay exponentially and eventually stabilize at an error of $O(\epsilon \delta_*^* \epsilon)$, where $\delta_*$ is the smallest Bellman residual that unbiased SGD can achieve.

## 2 Models and key ideas

### 2.1 Continuous state space

In model-free RL, consider a discrete-time MDP with continuous state space $S \subseteq \mathbb{R}^d$. The action space $A \subseteq \mathbb{R}^d$ may be continuous or discrete. We denote the transition kernel of the MDP as

$$P^a(s, s') = P(s_{m+1} = s' | s_m = s, a_m = a).$$  \hfill (1)

The immediate reward function $r(s', s, a)$ specifies the reward if one takes action $a$ at state $s$ and ends up at state $s'$. A policy $\pi(a|s)$ gives the probability of taking action $a$ at state $s$, i.e., $P\{\text{take action } a \text{ at state } s\} = \pi(a|s)$. For a continuous state space, it is often convenient to rewrite the underlying transition in terms of the states:

$$s_{m+1} = s_m + \mu(s_m, a)\epsilon + \sqrt{\epsilon} Z_m,$$  \hfill (2)

where $Z_m$ is a mean-zero noise term. This form is particularly relevant when the MDP arises as a discretization of an underlying stochastic differential equation (SDE), with $\epsilon$ as its discretized time step. We remark that this SDE interpretation is not necessary; our theorems and algorithms apply to more general MDPs as long as the difference between the current and next state can be written as

$$E[s_{m+1} - s_m | s_m, a_m] = \mu(s_m, a_m)\epsilon.$$  \hfill (3)

Throughout the paper, we consider the case where for each state, the variation of the underlying drift $\mu(s, a)$ is a priori bounded in the action space.

The main object under study is the action-state pair value function $Q(s, a)$. There are two types of problems: $Q$-evaluation and $Q$-control. $Q$-evaluation refers to the prediction of the value function when the policy is given, while $Q$-control refers to finding the optimal policy through the maximization of $Q(s, a)$. For the $Q$-evaluation problem the state space and action space can be continuous or discrete, while for the $Q$-control problem we mainly consider the case of a (finite) discrete action space.

**$Q$-evaluation** Given a fixed policy $\pi$, the value function $Q^\pi(s, a)$ represents the expected return if one takes action $a$ at state $s$ and follows $\pi$ thereafter, i.e.,

$$Q^\pi(s, a) = \mathbb{E}\left[\sum_{t \geq 0} \gamma^t r(s_{m+t+1}, s_{m+t}, a_{m+t}) | s_m = s, a_m = a\right].$$
where \( \gamma \in (0, 1) \) is a discount factor. The value function \( Q^\pi \) satisfies the Bellman equation \cite{SuttonBarto2018}:

\[
T^\pi Q^\pi(s, a) = \mathbb{E}[r(s_{m+1}, a_m, a_{m+1}) + \gamma Q^\pi(s_{m+1}, a_{m+1}) | (s_m, a_m) = (s, a)].
\]  

(4)

In the nonlinear approximation setting, one seeks a solution to (4) from a family of functions \( Q^\pi(s, a; \theta) \) parameterized by \( \theta \in \mathbb{R}^{d_\theta} \). For example, the function approximation family could be the set of all NNs of a given architecture, and \( \theta \) specifies the network weights. One way to find \( Q^\pi(s, a; \theta) \) is to solve the following Bellman residual minimization (BRM) problem:

\[
\min_{\theta \in \mathbb{R}^{d_\theta}} \mathbb{E}_{(s,a) \sim \rho(s,a)} [\delta^2(s, a; \theta)]
\]  

(5)

where \( \rho(s, a) \) is a distribution over \( \mathcal{S} \times \mathcal{A} \) and

\[
\delta(s, a; \theta) = T^\pi Q^\pi(s, a; \theta) - Q^\pi(s, a; \theta).
\]  

(6)

Note that the expectation in (5) can be taken with respect to different distributions \( \rho \). For online learning, it is often the stationary distribution of the Markov chain. When \( \mathcal{S} \) and \( \mathcal{A} \) are discrete, it is also reasonable to choose a uniform distribution over \( \mathcal{S} \times \mathcal{A} \). Doing so often accelerates the rate of convergence compared to the stationary measure.

One approach for solving the Bellman minimization problem (5) is to directly apply SGD. The unbiased gradient estimate to the loss function is

\[
F = j(s_m, a_m, s_{m+1}; \theta_m) \nabla \theta j(s_m, a_m, s'_{m+1}; \theta_m),
\]  

(7)

where

\[
j(s_m, a_m, s_{m+1}; \theta_m) = r(s_{m+1}, s_m, a_m) + \gamma \int Q^\pi(s_{m+1}, a; \theta) \pi(a | s_{m+1}) da - Q^\pi(s_m, a_m; \theta).
\]  

(8)

Here \( s_{m+1} \) is the next state in the trajectory, while \( s'_{m+1} \) is an independent sample for the next state according to the transition process. However, in model-free RL, as the underlying dynamics are unknown, another independent sample \( s'_{m+1} \) of the next state is unavailable. Therefore, this unbiased SGD, referred to as uncorrelated sampling (US), is impractical. Even if one can store the whole trajectory, it is impossible to revisit a certain state multiple times when the state space is either continuous or discrete but of high dimension. This is the so-called double sampling problem.

One potential solution, called sample-cloning (SC), simply uses \( s_{m+1} \) as a surrogate for \( s'_{m+1} \), i.e. \( s'_{m+1} = s_{m+1} \). However, sample-cloning is not an unbiased algorithm for the BRM problem, and its bias grows rapidly with the conditional variance of \( s_{m+1} \) on \( s_m \).

To address the double sampling problem, \cite{Zhu2020} introduced the borrowing from the future (BFF) algorithm. The main idea of the BFF algorithm is to borrow the future difference \( \Delta s_{m+1} = s_{m+2} - s_{m+1} \) and approximate the second sample \( s'_{m+1} \) with \( s_m + \Delta s_{m+1} \). During SGD, the parameter \( \theta \) is updated based on the following estimate of the unbiased gradient:

\[
\hat{F} = j(s_m, a_m, s_{m+1}; \theta_m) \nabla \theta j(s_m, a_m, s_m + \Delta s_{m+1}; \theta_m).
\]  

(9)

where \( j \) is defined in (8). When the difference between \( \Delta s_m \) and \( \Delta s_{m+1} \) is small, the new \( s'_{m+1} \) is statistically close to the distribution of the true next state. Among the two versions (gradient based and loss function based) introduced in \cite{Zhu2020}, we adopt the gradient version, detailed in Algorithm \[\text{Algorithm 1}\] in Section 2.3 we comment on why the loss version is less accurate.

Due to the Markov property, the difference \( \Delta s_{m+1} \) is independent from the current difference \( \Delta s_m \), leading to two conditionally independent samples. Whether \( \hat{F} \) is a good approximation of the unbiased estimate \( F \) depends on three factors: 1) the variation of the drift \( \mu(s, a) \) over the action space; 2) the variation of the policy \( \pi(a | s) \) over the state space; 3) the size of \( \epsilon \). The smaller these three elements are, the closer BFF is to US.

In Algorithm \[\text{Algorithm 1}\] only one future step is used for generating a new sample of \( s_{m+1} \). In order to reduce the variance of the BFF gradient, it is useful to consider replacing the future step by a weighted average of multiple future steps. The estimate of the gradient then takes the form

\[
\hat{F}^m = j(s_m, a_m, s_{m+1}; \theta_m) \sum_{i=1}^n \alpha_i \nabla \theta j(s_m, a_m, s_{m+1} + \Delta s_{m+i}; \theta_m)
\]  

(10)

with \( \sum_i \alpha_i = 1 \). This comes at the cost of potentially increasing the estimate’s bias.
When the state space is discrete, one can view $Q(s, a; \theta)$ as a matrix. In this tabular form, one can directly use the previous function approximation framework by letting $Q(s, a; \theta) = \Phi(s, a)^\top \theta$, where $\Phi(s, a) \in \mathbb{R}^{[S] \times [A]}$ is the matrix with $(i, j)$-th entry equal to 1 and all other entries equal to 0. Equivalently, one can also derive the BFF algorithm directly by computing the gradient of the Bellman residual with respect to $Q$. An unbiased gradient is given by

$$
F(s_m, a_m) = -j_{eval}(s_m, a_m, s_{m+1})
$$

$$
F(s_{m+1}', a) = \pi(a | s_{m+1}^') \gamma j_{eval}(s_m, a_m, s_{m+1}), \quad \forall a \in \mathcal{A},
$$

where $s_{m+1}'$ is an independent sample of the next step in the trajectory given $s_m$ and $a_m$, $j_{eval}(s_m, a_m, s_{m+1}) = r(s_{m+1}, s_m, a_m) + \gamma \sum_a Q^\pi(s_{m+1}, a) \pi(a | s) - Q^\pi(s_m, a_m)$, and all other entries of $F$ are 0. By replacing the independent sample $s_{m+1}'$ with the BFF approximation $s_m + \Delta s_m$, we obtain the following BFF algorithm for the tabular case, summarized in Algorithm 2.

| Algorithm 1 BFF |
|-----------------|
| **Require:** $\eta$: Learning rate |
| **Require:** $Q^\pi(s, \theta) \in \mathbb{R}^{[A]}$ or $Q^\pi(s, a; \theta) \in \mathbb{R}$: Nonlinear approximation of $Q$ parameterized by $\theta$ |
| **Require:** $j_{eval}(s, a, \theta) := r(s', s, a) + \gamma \int Q^\pi(s', a; \theta)\pi(a | s')da - Q^\pi(s, a; \theta)$ |
| **Require:** $\theta_0$: Initial parameter vector |

1: $m \leftarrow 0$
2: while $\theta_m$ not converged do
3: $s_{m+1}' \leftarrow s_m + (s_{m+2} - s_m)$
4: $F_m \leftarrow j_{eval}(s_m, a_m, s_{m+1}; \theta_m) \nabla \theta j_{eval}(s_m, a_m, s_{m+1}; \theta_m)$
5: $\theta_{m+1} \leftarrow \theta_m - \eta F_m$
6: $m \leftarrow m + 1$
7: end while

$Q$-control The BFF algorithm mentioned above can be extended easily to $Q$-control, i.e., finding the value function $Q^\pi$ of the optimal policy $\pi^\star$. $Q^\pi$ satisfies the Bellman equation $Q^\pi(s, a) = T^\pi \cdot Q^\pi(s, a)$, where

$$
T^\pi \cdot Q^\pi(s, a) = \mathbb{E} \left[ r(s_{m+1}, s_m, a_m) + \gamma \max_{a'} Q^\pi(s_{m+1}, a'; \theta) \big| (s_m, a_m) = (s, a) \right].
$$

(11)

The BRM problem is the same as (5) but with the Bellman residual $\delta(s, a; \theta)$ given by $\delta(s, a; \theta) = T^\pi \cdot Q^\pi(s, a) - Q^\pi(s, a)$. Rather than generating a trajectory offline with a fixed policy, we instead generate a training trajectory online using an $\epsilon$-greedy policy. The algorithm for this case is identical to Algorithm 1 but with $j_{eval}$ replaced by $j_{eval}(s_m, a_m, s_{m+1}; \theta) = r(s_{m+1}, s_m, a_m) + \gamma \max_a Q^\pi(s_{m+1}, a; \theta) - Q^\pi(s_m, a_m; \theta)$. Refer to Appendix D for more details.

Why BFF works We prove in Lemma C.1 and C.2 that the difference between the SC and US gradients is $O(\epsilon)$, while the difference between the BFF and US gradients is $O(\mathbb{E}[\delta^2])$ (see Lemma 3.1). Although both differences are $O(\epsilon)$, BFF depends on the Bellman residual $\delta$ while SC does not. As the algorithm proceeds, $\delta$ approaches 0, causing the difference between BFF and US to further decrease. On the other hand, the difference between SC and unbiased SGD is always $O(\epsilon)$. This is the high-level reason why BFF outperforms SC. (See Section 4 for numerical comparisons.)

2.2 Discrete state space

When the state space is discrete, one can view $Q \in \mathbb{R}^{[S] \times [A]}$ as a matrix. In this tabular form, one can directly use the previous function approximation framework by letting $Q^\pi(s, a; \theta) = \Phi(s, a)^\top \theta$, where $\Phi(s, a) \in \mathbb{R}^{[S] \times [A]}$ is the matrix with $(i, j)$-th entry equal to 1 and all other entries equal to 0. Equivalently, one can also derive the BFF algorithm directly by computing the gradient of the Bellman residual with respect to $Q$. An unbiased gradient is given by

$$
F(s_m, a_m) = -j_{eval}(s_m, a_m, s_{m+1})
$$

$$
F(s_{m+1}', a) = \pi(a | s_{m+1}^') \gamma j_{eval}(s_m, a_m, s_{m+1}), \quad \forall a \in \mathcal{A},
$$

where $s_{m+1}'$ is an independent sample of the next step in the trajectory given $s_m$ and $a_m$, $j_{eval}(s_m, a_m, s_{m+1}) = r(s_{m+1}, s_m, a_m) + \gamma \sum_a Q^\pi(s_{m+1}, a) \pi(a | s) - Q^\pi(s_m, a_m)$, and all other entries of $F$ are 0. By replacing the independent sample $s_{m+1}'$ with the BFF approximation $s_m + \Delta s_m$, we obtain the following BFF algorithm for the tabular case, summarized in Algorithm 2.

The $Q$-control algorithm for the tabular case can be found in Appendix D. As in equation (10), one can use multiple future steps to reduce the variance of the gradient in the tabular case as well. Refer to Appendix D for more details.

2.3 Related work

There is another version of the BFF algorithm proposed in Zhu et al. [2020] for value function evaluation. One applies the same idea to the loss function instead of the gradient by minimizing a biased Bellman residual:

$$
\min_{\theta \in \mathbb{R}^{d\theta}} \mathbb{E} [ \mathbb{E} [ j(s_m, a_m, s_{m+1}; \theta) j(s_m, a_m, s_m + \Delta s_m; \theta) | s_m, a_m ] ].
$$

(12)
Algorithm 2 BFF (tabular case)

Require: \( \eta \): Learning rate
Require: \( Q^\pi \in \mathbb{R}^{S \times A} \): matrix of \( Q^\pi(s, a) \) values
Require: \( j^{\text{eval}}(s_m, a_m, s_{m+1}) = r(s_{m+1}, s_m, a_m) + \gamma \sum_a Q^\pi(s_{m+1}, a) \pi(a|s) - Q^\pi(s_m, a_m) \)

1: \( m \leftarrow 0 \)
2: while \( Q^\pi \) not converged do
3: \( s'_{m+1} \leftarrow s_m + (s_{m+2} - s_{m+1}) \)
4: \( \hat{F}_m \leftarrow 0 \in \mathbb{R}^{[S] \times [A]} \)
5: \( \hat{F}_m(s_m, a_m) \leftarrow -j^{\text{eval}}(s_m, a_m, s_{m+1}) \)
6: for \( a \in A \) do
7: \( \hat{F}_m(s'_{m+1}, a) \leftarrow \pi(a|s'_{m+1}) j^{\text{eval}}(s_m, a_m, s_{m+1}) \)
8: end for
9: \( Q^\pi \leftarrow Q^\pi - \eta \hat{F}_m \)
10: \( m \leftarrow m + 1 \)
11: end while

For state value function evaluation, this loss version performs better than the sample-cloning algorithm because it has a difference of only \( O(\epsilon^2) \) from US while SC has an \( O(\epsilon) \) difference. However, this loss version does not work for \( Q \)-evaluation. The reason is that the gradient of the above loss function contains two parts, \( j(s_{m+1}) \nabla_\theta r(s_m, \Delta s_{m+1}) + \nabla_\theta j(s_{m+1}) j(s_m, \Delta s_{m+1}) \), so the difference between the loss version of BFF and US is \( O(\epsilon^2 + \epsilon \nabla \delta) \), and \( \nabla \delta \) does not necessarily decrease as the algorithm proceeds. (For example, if \( \delta^2 = \theta^2 \), then \( \nabla \delta = 1 \) is a constant.) Therefore, the error is still dominated by an \( O(\epsilon) \) term, which means that the loss version behaves similarly to SC.

In [Wang et al. 2017, 2016], the stochastic compositional gradient method (SCGD), a two-step scale algorithm, is proposed to address the double sampling problem. However, it is not clear how to apply SCGD to BRM with a continuous state space.

Another way to avoid the double sampling problem in BRM is to consider the primal-dual (PD) formulation of the minimization problem and view it as a saddle point of a minimax problem. Such methods include GTD and its variants [Sutton 2008, Sutton et al. 2009, Bhatnagar et al. 2009, Mahadevan et al. 2011, Liu et al. 2015], and SBEED [Dai et al. 2018]. However, when a nonlinear function approximation is used, the maximum is taken over a non-concave function. This can be significantly more difficult than solving the minimization problem directly. (See Section 4 for details.)

3 Theoretical results

This section states the main theoretical results which bound the difference between BFF and US on a continuous state space. Recall that the one-step transition is governed by the state dynamics

\[
s_{m+1} = s_m + \mu(s_m, a) \epsilon + \sigma \sqrt{\epsilon} Z_m,
\]

where \( \mu(s, a) \) is the drift, \( Z_m \) is assumed to be normal \( N(0, I_{d}s \times d_a) \), and \( \sigma \) is the diffusion coefficient. It is convenient to introduce \( \Delta s_m := s_{m+1} - s_m = \mu(s_m, a) \epsilon + \sigma \sqrt{\epsilon} Z_m \). For a discrete action space \( A \), the drift term \( \{\mu(s, a)\}_{a \in A} \) is a family of continuous functions, while for a continuous action space, \( \mu(s, a) \) is a continuous function in both state and action. We choose to work with a discretized stochastic differential equation (SDE) in order to simplify the presentation of the algorithms and the theorems. Our lemmas and theorems can be extended to the more general case specified by (3).

3.1 Differences at each step

The following lemma bounds the difference between BFF and US at each step. That is, assuming the current parameters \( \theta_m \) are the same, Lemma 3.1 bounds the expected difference between BFF and US for \( Q \)-evaluation and \( Q \)-control after one step. See Appendix A for a more detailed version of Lemma 3.1 and its proof.

Lemma 3.1 (short version). For \( Q \)-evaluation, assume

\[
\sup_{s \in S, \theta \in \mathbb{R}^{d\theta}} |\partial_s E_{\theta}[\nabla_\theta Q^\pi(s, a; \theta)]| \leq C, \quad \sup_{s \in S, a \in A} \|E_{a'}[\mu(s, a)|s] - \mu(s, a)\| \leq C, \quad \text{a.s.}
\]
For Q-control, let \( f(s; \theta) = \max_{a' \in A} Q^*(s, a'; \theta) \) and assume

\[
\sup_{s \in S, \theta \in \mathbb{R}^{d\theta}} |\partial_s \nabla_\theta f(s_m; \theta)| \leq C, \quad \sup_{s \in S, a \in A} |E_{\theta}[\mu(s, a')|s] - \mu(s, a)| \leq C, \quad a.s. \tag{15}
\]

The difference between the BFF gradient \( \hat{F} \) and the unbiased gradient \( F \) is bounded by

\[
\left| E[\hat{F}] - E[F] \right| \leq \gamma C^2 E[|\delta(e + O(\epsilon))|].
\]

Note that the upper bounds in the assumptions (14) and (15) that affect the magnitude of the constant \( C \) in front of \( E[|\delta|] \) can be translated to assumptions on \( Q^* \), \( \pi \), and \( \mu \). For instance, the first inequality in (14) is satisfied if \( |\partial_s \nabla_\theta Q| \) and \( |\partial_\theta \pi(a|s)| \) are bounded because \( |\partial_s E_\theta[\nabla_\theta Q^*(s, a; \theta)|s] = |\partial_s \int (\nabla_\theta Q^*(s, a; \theta))\pi(a|s)| \leq C \). The magnitude of \( |\partial_s \nabla_\theta Q| \) can be controlled through the function space used to approximate \( Q^* \). Similarly, the first equation in (15) is related to \( |\partial_\theta \nabla_\theta Q^*| \), which can be controlled through the approximating function space as well.

The second inequality in (14), (15), \( |E_\theta[\mu(s, a)|s] - \mu(s, a)| \), is satisfied if \( \forall s \in S \),

\[
\text{discrete } A: \quad \max_{a, b \in A} |\mu(s, a) - \mu(s, b)| \leq C';
\]

\[
\text{continuous } A: \quad |\partial_\theta \mu(s, a)| \leq C' \tag{16}
\]

In summary, the crucial elements that affect the difference between BFF and US are 1) the magnitude of the change in the behavior policy \( |\partial_\theta \pi(a|s)| \) and 2) the variation of the drift term \( \mu(s, a) \) over the action space. Therefore, when the policy changes more slowly with respect to the state and the drift changes more slowly with respect to the action, the difference is smaller and BFF performs better.

### 3.2 Differences of density evolutions

This subsection compares the probability density functions (p.d.f.) for the parameters over the course of the complete BFF and US algorithms. To simplify the analysis, the p.d.f.s of the two algorithms are modeled with the p.d.f.s of the continuous stochastic processes. The updates of the parameter \( \theta_k \) by SGD can be viewed as a discretization of a function in time \( \Theta_t \equiv \Theta(t) \). It is shown in [Li et al. (2017), Hu et al. (2017)] that when the learning rate \( \eta \) is small, the dynamics of SGD can be approximated by a continuous time SDE

\[
d\Theta_t = -E[F(\Theta_t)]dt + \sqrt{\nabla E[F(\Theta_t)]}dB_t \tag{17}
\]

with \( \Theta_{t=k\eta} \approx \theta_k \), where \( E \) and \( \nabla \) are expectation and variance taken over \( \rho(s, a) \). Here \( E[F(\Theta_t)] \) denotes the true gradient of population loss function in the case of US, or the biased gradient of the population loss in the case of BFF. For simplicity, we assume \( \nabla E[F] \equiv \xi \) is constant. Let \( p(t, \theta) \) and \( \hat{p}(t, \theta) \) be the p.d.f. of the parameter \( \theta \) at step \( k = t/\eta \) for US and BFF, respectively, and define \( \hat{d}(t, \theta) = p - \hat{p} \) to be their difference. We introduce the following weighted norm to measure the difference between the p.d.f.s:

\[
\left\| \hat{d} \right\|_\star := \int \hat{d}^2/p^\infty d\theta, \quad p^\infty = e^{-\frac{2}{\eta^2}E[\delta^2]}/Z,
\]

where \( p^\infty \) is the limiting p.d.f. for \( p(t, \theta) \) as \( t \to \infty \), \( Z = \int e^{-\beta E[\delta^2]}d\theta \) is a normalizing constant, and \( \delta(s; a, \theta) = T^a Q - Q \) is the Bellman residual \( T^\pi \) defined in (4) for Q-evaluation and in (11) for Q-control. Here the expectation \( E \) is taken over \( \rho(s, a) \).

**Theorem 3.2** (short version). For small \( \eta \), the difference \( \hat{d}(t) \) of the p.d.f.s for US and BFF is bounded by

\[
\left\| \hat{d}(t) \right\|_\star \leq C_1 e^{-C_2 t} + O \left( \epsilon \sqrt{E[\delta^2]} \eta^C_3 \right) \sqrt{1 - e^{-C_2 t}}, \tag{18}
\]

where \( E[\delta^2] = \min \mathbb{E}[\delta^2] \) and \( C_1, C_2, C_3 \) are all positive constants.

The precise version of Theorem 3.2 and its proof are given in Appendix B. This theorem implies that as the algorithm moves on, the difference between BFF and US will decay exponentially. After running the algorithm for sufficiently many steps, the difference will eventually be \( O \left( \epsilon \sqrt{E[\delta^2]} \eta^C_3 \right) \).
As long as $\mathbb{E}[\delta_{a}^2]$ is small, BFF will achieve a minimizer close to US with an error much smaller than $O(\epsilon)$. Note that if $\mathbb{E}[\delta_{a}^2] = 0$, the difference still does not vanish. Instead, the leading order term of the last term in (18) becomes $O(\epsilon \eta^{-1/2})$, which is shown in Corollary B.2 of Appendix B.

The constant $C_1$ depends on the initial p.d.f. of the algorithm. The constant $C_2$ is related to the shape of $\mathbb{E}[\delta_{a}^2](\theta)$ in the parameter space. If the shape at the minimizer is flatter, then $C_2$ is smaller. The constant $C_3$ decreases as $\eta$ decreases, so the first term increases as $\eta$ decreases, while the last term $O(\epsilon^2 \mathbb{E}[\delta_{a}^2]]^2)$ does the opposite. This suggests that one should set the learning rate $\eta$ large at first, making the exponential decay faster. As the training progresses, $\eta$ should be reduced to make the final error smaller.

4 Numerical examples

Code for reproducing these experiments can be found in the supplementary material. Due to space constraints, full details of the experiments can be found in Appendix F.

In each of the settings below, we test the efficacy of learning $Q^\pi$ via SC and BFF. We test the generalized version of BFF specified by equation (10). The label nBFF in the plots corresponds to using the estimate $\hat{F}_m$ from equation (10); 1BFF corresponds to the standard BFF algorithm (algorithms 1 and 2). In each case, we use the uniform weights $\alpha_i = 1/n$. When applicable, we also compare to US and PD. (For the full definition of the PD algorithm, see Appendix F.3).

4.1 Continuous state space

We consider an MDP with continuous state space $S = [0, 2\pi)$. The transition dynamics are

$$\Delta s_m = a_m c + \sigma Z_m \sqrt{\epsilon},$$

where $a_m \in \mathbb{A} = \{\pm 1\}$ is drawn from policy $\pi$ to be defined later and $Z_m \sim N(0, 1)$. We set $\epsilon = \frac{4\pi}{10}$ and $\sigma = 0.2$. The reward function is $r(s_{m+1}, s_m, a_m) = \sin(s_{m+1}) + 1$.

In the first two experiments, we approximate $Q^\pi$ with a neural network with two hidden layers. Each hidden layer contains 50 neurons and cosine activations. The NN takes a state as input and outputs a vector in $\mathbb{R}^{|\mathbb{A}|}$; the $i$-th entry of the output vector corresponds to $Q^\pi(s, a_i)$. The CartPole experiments use a larger network with ReLU activations.

**Q-evaluation** We first estimating $Q^\pi$ for the fixed policy $\pi(a|s) = 1/2 + a \sin(s)/5$. The results are plotted in Figure 1. BFF exhibits superior performance compared to SC and PD, with only slightly worse performance than the (impractical) US algorithm.

![Figure 1: Results of each method for fixed-policy Q-evaluation. We plot the best result out of 10 runs for PD. The BFF algorithm performs better than both SC and PD. Changing the number of future steps used to compute the BFF approximation does not have a large impact on its performance in this case.](image)

**Q-control** In the control case, we use a fixed behavior policy to generate the training trajectory. At each step, the behavior policy samples an action uniformly at random, i.e. $\pi(a|s) = 1/2$ for all $a \in \mathbb{A}$ and $s \in S$. The results are shown in Figure 2. Again, BFF has comparable performance to SC and outperforms both SC and PD.

**CartPole** We tested the BFF algorithm on the CartPole environment from OpenAI gym Brockman et al. (2016). It is straightforward to modify BFF for use in conjunction with adaptive SGD
algorithms such as Adam [Kingma et al. 2014], and we use BFF with Adam for this experiment. The results are plotted in Figure 3. BFF reaches the max reward (200) faster than SC and achieves it with greater regularity throughout the training process. In contrast to both of these methods, the PD method fails to converge even after an extensive hyperparameter search.

Figure 3: Reward per training episode for the CartPole experiment. BFF is the first to reach the maximum reward and achieves it more consistently than sample-cloning. It achieves slightly better performance using 2 future steps (2BFF in the plot). Despite an extensive hyperparameter search, PD was not able to learn an effective policy.

4.2 Tabular case

We next consider an MDP with a discrete state space $\mathbb{S} = \{\frac{2\pi k}{n}\}_{k=0}^{n-1}$ and $n = 32$. The transition dynamics are given by

$$\Delta s_m = \frac{2\pi}{n} a_m \epsilon + \sigma Z_m \sqrt{\epsilon},$$

(19)

where $a_m \in \mathbb{A} = \{\pm 1\}$ is drawn from the policy $\pi(a|s) = 1/2 + a \sin(s)/5$ and $Z_m \sim N(0, 1)$. We then set $s_{m+1} = \arg\min_{s \in \mathbb{S}} |s_m + \Delta s_m - s|$. For the experiment below, $\sigma = 1$ and $\epsilon = 1$. The results are plotted in Figure 4. In this case, BFF is nearly indistinguishable from training via US. Due to space constraints and its similarity to the previous experiments, we defer the case of tabular $Q$-control to the appendix.

Figure 4: Results of each method for fixed-policy $Q$-evaluation in the tabular case. BFF gives a better estimate for the gradient than SC, leading to improved performance. BFF’s performance does not change significantly with the number of future steps in this case. Note that the PD method does not apply to this case.
5 Conclusion

In this paper, we show that BFF has an advantage over other BRM algorithms for model-free RL problems with continuous state spaces and smooth underlying dynamics. We also prove that the difference between the BFF algorithm and the uncorrelated sampling algorithm first decays exponentially and eventually stabilizes at an error of $O(\epsilon \delta_*)$, where $\delta_*$ is the smallest Bellman residual that US can achieve.

6 Broader Impact

The main societal impact of deep reinforcement learning has been its ability to automate ever more complicated tasks. Recent advances in driverless vehicles [Sallab et al. [2017]] and automated control of robots [Gu et al. [2017]] use deep Q-function approximations to learn an optimal policy. Our work on BFF contributes directly to improving the capabilities of automation.

Automation provides clear economic and utilitarian benefits. Well-designed robot or AI workers make fewer mistakes, produce greater output, and, in the long run, may cost less than their human counterparts. This leads to greater economic productivity and technological advances [Carlsson [2012]].

Increased automation is not without its risks. As AI capabilities improve, large sections of the population may face unemployment [Leontief et al. [1986]]. Members of underprivileged classes will likely be disproportionately affected by the decreased availability of low-skill jobs, while simultaneously having less access to the benefits automation provides. As the power of AI increases, so too does our understanding of the unintended consequences. For instance, the recent work of [Bissell et al. [2020]] studies these effects in the case of autonomous vehicles.

BFF is a tool which can facilitate advances in science and technology, and the exacerbation of social inequality is an inherent risk of any new technology. It is via an ethical application of these new discoveries that society can realize the greatest benefit.
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**Appendices**

### A Extension and Proof of Lemma 3.1

**Lemma A.1** (Extension of Lemma 3.1 for $Q$-evaluation). If

$$
\sup_{s \in \mathcal{S}, \theta \in \mathbb{R}^d} |\partial_s \mathbb{E}_\theta \nabla Q^\pi(s, a; \theta) | s | \leq C
$$

and

$$
\sup_{s \in \mathcal{S}, a \in \mathcal{A}} |\mathbb{E}_\theta [\mu(s, a) | s] - \mu(s, a) | \leq C \text{ a.s.,}
$$

then the difference between the gradients of the US and BFF algorithms for $Q$-evaluation is bounded by

$$
\left| \mathbb{E}[\hat{F}] - \mathbb{E}[F] \right| \leq \gamma C^2 \mathbb{E} |\delta(\epsilon + o(\epsilon))|;
$$

In addition, if

$$
|\mathbb{E}_\theta [Q^\pi(s, a; \theta) - Q(s, a; \theta), |\mathbb{E}_\theta [\nabla^2 Q^\pi(s, a; \theta)] - \nabla Q(s, a; \theta), |\mu(s, a) - \mu(s, a')|, |r(s, s, a)| \leq C \text{ a s.} \forall s \in \mathcal{S}, a \in \mathcal{A}, \theta \in \mathbb{R}^d,
$$

then the difference between the variances can also be bounded by

$$
\left| \mathbb{V}[\hat{F}] - \mathbb{V}[F] \right| \leq O(\epsilon),
$$

where $\mathbb{V}$ stands for the variance and

$$
F = j(s_m, a_m, s_{m+1}; \theta) \nabla Q(s_m, a_m, s'_m; \theta),
$$

$$
\hat{F} = j(s_m, a_m, s_{m+1}; \theta) \nabla Q(s_m, a_m, s_m + \Delta s_{m+1}; \theta),
$$

$$
j(s_m, a_m, s_{m+1}; \theta_m) = r(s_{m+1}, s_m, a_m) + \gamma \int Q^\pi(s_{m+1}, a; \theta) \pi(a|s_{m+1}) da - Q^\pi(s_m, a_m; \theta),
$$

$$
\delta(s_m, a_m; \theta) = \mathbb{E} \left[ j(s_m, a_m, s_{m+1}; \theta_m) | s_m, a_m \right].
$$
Note that the above form also works for the discrete action spaces. Specifically, \( \pi(a|s) da = \sum_{a_i \in A} \pi(a_i|s) \delta_{a_i}(a) da \) in the discrete action space, where \( \delta_{a_i}(a) \) is the Dirac delta function.

**Proof.** The expectation of the US gradient is

\[
\mathbb{E}[F] = \mathbb{E}[\mathbb{E}[j|m, a_m] \mathbb{E}[\nabla_{\theta} j|m, a_m]] = \mathbb{E}[\delta(s, a; \theta) \nabla_{\theta} \delta(s, a; \theta)],
\]

(A.1)

with \( j' = j(m, a_m, m + 1; \theta) \). The expectation of the BFF gradient is

\[
\mathbb{E}[\hat{F}] = \mathbb{E}[\mathbb{E}[j|m, a_m] \mathbb{E}[\nabla_{\theta} j|m, a_m]] = \mathbb{E}[\delta(s, a; \theta) \nabla_{\theta} \delta(s, a; \theta)],
\]

(A.2)

with \( \hat{j} = j(s, a, m, m + 1; \theta) \). By subtracting the two gradients in (A.1) and (A.2), we see that the difference between the BFF and US gradients is

\[
\mathbb{E}[\hat{F}] - \mathbb{E}[F] = \mathbb{E}\left[\delta(s, a; m) \mathbb{E}\left[\nabla_{\theta} j - \nabla_{\theta} j' \bigg| s, a, m\right]\right]
\]

(A.3)

For notational convenience, in what follows we drop the explicit dependence of \( Q^\pi \) on \( \theta \). All of the gradients \( \nabla \) are taken with respect to \( \theta \). For ease of exposition, we consider a one-dimensional state space \( S \). It is straightforward to generalize to the multi-dimensional case. Using a Taylor expansion, we can expand \( \nabla Q^\pi(m + 1, a) \pi(a|m + 1) \) around \( \nabla Q^\pi(m, a) \pi(a|m) \) by

\[
\nabla Q^\pi(m + 1, a) \pi(a|m + 1) = \nabla Q^\pi(m, a) \pi(a|m) + \partial_s(\nabla Q^\pi(m, a) \pi(a|m)) \Delta s_m + \frac{1}{2} \partial^2_s(\nabla Q^\pi(m, a) \pi(a|m)) \Delta s^2_m.
\]

Substituting \( \Delta s_m = \mu(s, m, a_m) \epsilon + \sigma Z_m \sqrt{\epsilon} \) yields

\[
\nabla_{\theta} j' = \gamma \int_{f_0}^{f_1} \nabla Q^\pi(s, m, a) \pi(a|s, m) da - \nabla Q^\pi(s, m, a)
\]

\[
+ \left( \gamma \int_{f_0}^{f_1} \partial_s(\nabla Q^\pi(s, m, a) \pi(a|s, m)) da \right) \mu(s, m, a_m) \epsilon
\]

\[
+ \left( \gamma \int_{f_2}^{f_3} \partial_s(\nabla Q^\pi(s, m, a) \pi(a|s, m)) da \right) \sigma Z_m \sqrt{\epsilon}
\]

\[
+ \left( \gamma \int_{f_4}^{f_5} \partial^2_s(\nabla Q^\pi(s, m, a) \pi(a|s, m)) da \right) \sigma^2 Z^2_m \epsilon + o(\epsilon).
\]

(A.4)

Similarly, we can expand \( \nabla Q^\pi(s + \Delta s, m, a) \pi(a|s + \Delta s) \) around \( \nabla Q^\pi(s, m, a) \pi(a|m) \). This yields

\[
\nabla Q^\pi(m + 1, a) \pi(a|m + 1) = \nabla Q^\pi(m, a) \pi(a|m) + \partial_s(\nabla Q^\pi(m, a) \pi(a|m)) \Delta s_m + \frac{1}{2} \partial^2_s(\nabla Q^\pi(m, a) \pi(a|m)) \Delta s^2_m.
\]

By Taylor expanding \( \mu(s, a + 1) \) around \( \mu(s, a + 1) \) and using the fact that \( \Delta s_m = O(\sqrt{\epsilon}) \), we see that

\[
\mu(s, a + 1) = \mu(s, a + 1) + \partial_s \mu(s, a + 1) \Delta s_m + O(\Delta s^2_m)
\]

\[
= \mu(s, a + 1) + o(1).
\]

Substituting this into the expression for \( \Delta s_{m+1} \) yields

\[
\Delta s_{m+1} = \mu(s, a + 1) \epsilon + \sigma Z_m \sqrt{\epsilon} + o(\epsilon).
\]
Combining this expression for $\Delta s_{m+1}$ with the Taylor expansion of $\nabla Q^\pi$, we conclude that
\[
\nabla \theta j = \gamma \int \nabla Q^\pi (s_m + \Delta s_{m+1}, a) \pi(a|s_m + \Delta s_{m+1}) da - \nabla Q^\pi (s_m, a_m) \\
= f_0 + \left( \gamma \int \partial_s (\nabla Q^\pi (s_m, a) \pi(a|s_m)) da \right) \mu(s_m, a_m) + f_2 Z_{m+1} \epsilon + f_3 Z_m^2 \epsilon + o(\epsilon).
\]

(A.5)

It follows that
\[
\mathbb{E} \left[ \nabla j - \nabla j' \right| s_m, a_m] = \mathbb{E}[(f_1 - f_1) \epsilon|s_m, a_m] + f_2 \mathbb{E}[Z_{m+1} - Z_m|s_m, a_m] \sqrt{\epsilon} + f_3 \mathbb{E}[Z_m^2 - Z_m^2|s_m, a_m] \epsilon + o(\epsilon) \\
= \mathbb{E}[(f_1 - f_1) \epsilon|s_m, a_m] + o(\epsilon)
\]

Recall the assumptions of the lemma:
\[
|\partial_s \mathbb{E}_a [\nabla \theta Q^\pi (s, a; \theta)|s]| \leq C, \quad \forall s \in S, \theta \in \mathbb{R}^d; \\
|\mathbb{E}_a [\mu(s, a)|s] - \mu(s, a)| \leq C, \quad \forall s \in S, a \in A,.
\]

Using these inequalities, we find
\[
\mathbb{E} \left[ \nabla j - \nabla j' \right| s_m, a_m] \leq \gamma C^2 \epsilon + o(\epsilon).
\]

Substituting the above inequality into (A.3) finally yields
\[
\mathbb{E}[^{\hat{F}} - F] = \gamma C^2 \mathbb{E} \left[ \| \delta(\epsilon + o(\epsilon)) \| \right]
\]
which completes the proof for the first part of the lemma.

We now bound the difference of the variance. By the definition of $F, ^{\hat{F}}$ in (7), (8), we have,
\[
\left| \nabla [^\hat{F}] - \nabla [F] \right| = \mathbb{E}[(j^2 ((\nabla \theta j)^2 - (\nabla \theta j')^2))] - \left( \mathbb{E}[j^2 (\nabla \theta j)^2 - \mathbb{E}[j \nabla \theta j]^2] \right) \\
= \mathbb{E} \left[ \mathbb{E}_j \left[ j^2 |s_m, a_m| \mathbb{E} \left[ (\nabla \theta j)^2 - (\nabla \theta j')^2 |s_m, a_m] \right] \right] \\
- \left( \mathbb{E} \left[ \mathbb{E}_j \left[ j^2 |s_m, a_m| \mathbb{E} \left[ (\nabla \theta j)^2 |s_m, a_m] \right] \right] - \mathbb{E} \left[ \mathbb{E}_j \left[ j^2 |s_m, a_m| \mathbb{E} \left[ (\nabla \theta j')^2 |s_m, a_m] \right] \right] \right).
\]

Using the same approximations of $\nabla \theta j$, $\nabla \theta j'$ as in (A.4), (A.5) gives
\[
\nabla \theta j - \nabla \theta j' = (f_1 - f_1) \epsilon + f_2 (Z_{m+1} - Z_m) \sqrt{\epsilon} + f_3 (Z_m^2 - Z_m^2) \epsilon + o(\epsilon).
\]

It follows that
\[
\mathbb{E}[(\nabla \theta j)^2 - (\nabla \theta j')^2 |s_m, a_m] = \mathbb{E}[(\nabla \theta j - \nabla \theta j')(\nabla \theta j + \nabla \theta j')|s_m, a_m] \\
= \mathbb{E}[2 f_0 (f_1 - f_1) \epsilon + f_2 f_2 (Z_{m+1} - Z_m) \sqrt{\epsilon} + f_3 (Z_m^2 - Z_m^2) \epsilon + f_3 (Z_{m+1} - Z_m^2) \epsilon + o(\epsilon)|s_m, a_m] \\
= 2 f_0 (f_1 - f_1) \epsilon + o(\epsilon),
\]

Again, using a Taylor expansion, we can approximate $j$ by
\[
\begin{align*}
\hat{j} &= r + \gamma \int Q^\pi \pi da - Q^\pi \\
&\quad + \left( \partial_s r + \gamma \int \partial_s (Q^\pi) da \right) \sigma Z_m \sqrt{\epsilon} + \left( \partial_s^2 r + \gamma \int \partial_s^2 (Q^\pi) da \right) \sigma^2 Z_m^2 \epsilon + o(\epsilon).
\end{align*}
\]

(A.6)
where we abbreviate \( r(s_m, s_m, a_m), Q^\pi(s_m, a), \pi(a|s_m), \) and \( \mu(s_m, a_m) \) by \( r, Q^\pi, \pi, \) and \( \mu \), respectively. It follows that
\[
I = \mathbb{E}
\left[
\frac{\partial^2}{\partial \theta^2} E[(\nabla_{\theta^2})^2 (s_m, a_m)]
\right]
= 2\mathbb{E}[g_0^2 f_0(f_1 - f_1) + o(\epsilon)].
\]

Furthermore, we have
\[
\mathbb{E}[\nabla_{\theta^2}(s_m, a_m)] = f_0 + f_1 + f_3 + o(\epsilon),
\]
\[
\mathbb{E}[\nabla_{\theta^2}(s_m, a_m)] = f_0 + f_1 + f_3 + o(\epsilon),
\]
\[
\mathbb{E}[g(s_m, a_m)] = g_0 + g_1 + g_3 + o(\epsilon).
\]

Combining these expressions shows
\[
\mathbb{E}[\nabla_{\theta^2}]^2 = \mathbb{E}[f_0 g_0] + \mathbb{E}[f_0(g_1 + g_3 + g_0 f_3)] + \mathbb{E}[g_0 f_1] + o(\epsilon) = \mathbb{E}[f_0 g_0] + \mathbb{E}[f_0(g_1 + g_3 + g_0 f_3)] + \mathbb{E}[g_0 f_1] + o(\epsilon)
\]
\[
\mathbb{E}[\nabla_{\theta^2}]^2 = \mathbb{E}[f_0 g_0] + 2\mathbb{E}[f_0 g_0] + 2\mathbb{E}[f_0 g_0] + 2\mathbb{E}[f_0 g_0] + \mathbb{E}[g_0 f_1] + o(\epsilon)
\]
which in turn yields
\[
II = \mathbb{E}[\nabla_{\theta^2}]^2 - \mathbb{E}[\nabla_{\theta^2}]^2 = \mathbb{E}[f_0 g_0] + \mathbb{E}[g_0 f_1] + o(\epsilon).
\]
Combining \( I \) and \( II \), we see that
\[
\mathbb{E}[\nabla_{\theta^2} - \nabla_{\theta^2}] = I - II \leq 2\mathbb{Cov}(f_0 g_0, g_0 f_1 - f_1) + o(\epsilon) \leq O(\epsilon)
\]
as long as the covariance of \( f_0 g_0 \) and \( g_0 f_1 - f_1 \) is bounded. But since \( f_0, g_0, f_1 - f_1 \) are all bounded by the conditions in the second part of the lemma, \( \mathbb{Cov}(f_0 g_0, g_0 f_1 - f_1) \) must be bounded as well. This concludes the proof.

**Lemma A.2** (Extension of Lemma A.1 for Q-control). Let \( f(s; \theta) = \max_{a' \in A} Q^*(s, a'; \theta) \). Suppose that \( f(s; \theta) \) is continuous in \( s \in \mathbb{S} \) and that \( \partial_s f(s; \theta), \partial_{a} f(s; \theta) \) exist almost surely. Further assume that \( \sup_{s \in \mathbb{S}, \theta \in \mathbb{R}^d} |\partial_s f(s; \theta)|, \sup_{s \in \mathbb{S}, a \in A} |\mu(s, a)| \leq C a.s. \) Then the difference between the gradients in US and BFF for Q-control is bounded by
\[
\left| \nabla_{\theta} \mathbb{E}[\hat{F}] - \nabla_{\theta} [F] \right| \leq \gamma C^2 \mathbb{E} \left[ \delta(\sqrt{\epsilon} + O(\epsilon)) \right],
\]
In addition, if \( \max_{a} Q(s, a; \theta) - Q(s, a; \theta) \leq \nabla_{\theta} \max_{a} Q^*(s, a; \theta) - \nabla_{\theta} Q(s, a; \theta) \leq C \) almost surely over \( s \in \mathbb{S}, a \in A, \theta \in \mathbb{R}^d \), then
\[
\left| \nabla_{\theta} \mathbb{E}[\hat{F}] - \nabla_{\theta} [F] \right| \leq O(\sqrt{\epsilon}).
\]

**Here \( F, \hat{F} \) are the same as in Lemma A.1 with \( j \) and \( \delta \) replaced by**
\[
\tilde{j}(s_m, a_m, s_{m+1}; \theta) = r(s_{m+1}, s_m, a_m) + \gamma \max_{a} Q^*(s_{m+1}, a; \theta) - Q^*(s_m, a_m; \theta),
\]
\[
\delta = \mathbb{E}
\left[
\frac{\partial^2}{\partial \theta^2} E[r(s_{m+1}, s_m, a_m) + \gamma \max_{a} Q^*(s_{m+1}, a; \theta) - Q^*(s_m, a_m; \theta)] s_m, a_m
\right].
\]

**Proof.** The difference between the two algorithms is
\[
\mathbb{E}[\hat{F}] - \mathbb{E}[F] = \mathbb{E}
\left[
\delta(s_m, a_m) \mathbb{E} \left[ \nabla_{\theta^2} j - \nabla_{\theta^2} j \right] s_m, a_m
\right]
\]
where
\[
\nabla_{\theta^2} j = \nabla_{\theta} j(s_m, a_m, s_{m+1}; \theta), \quad \nabla_{\theta^2} j = \nabla_{\theta} j(s_m, a_m, s + \Delta s_{m+1}; \theta)
\]
and \[ \nabla_{\theta} j(s_m, a_m, s_{m+1}; \theta) = \nabla_{\theta} \max_{a' \in \mathcal{A}} Q^*(s_m, a'; \theta) - \nabla_{\theta} Q^*(s_m, a_m; \theta). \]

Since we have assumed that \( f(s; \theta) = \max_{a' \in \mathcal{A}} Q^*(s, a; \theta) \) is continuous in \( s \in \mathbb{S} \) and that \( \partial_s f(s; \theta), \partial_{ss}^2 f(s; \theta) \) exist almost surely, we can write \( \nabla_{\theta} j \) as

\[ \nabla_{\theta} j(s_m, a_m, s_{m+1}; \theta) = \nabla_{\theta} f(s_{m+1}; \theta) - \nabla_{\theta} Q^*(s_m, a_m; \theta). \]

Similarly to the proof of Lemma [A.1], we use a Taylor expansion:

\[
\nabla_{\theta} j = \gamma \nabla f(s_m) - \nabla Q^*(s_m, a_m) + \gamma \partial_s \nabla f(s_m) \mu(s_m, a_m) \epsilon
+ \gamma \partial_s \nabla f(s_m) \sigma Z_m \sqrt{\epsilon} + \gamma \partial_{ss}^2 \nabla f(s_m) \sigma^2 Z_m^2 \epsilon + o(\epsilon),
\]

\[(A.8)\]

Using the expressions from \( (A.8) \), we see that

\[
E[\hat{F}] - E[F] = E \left[ \delta \hat{E}[f_1 - f_1|s_m, a_m] \right] \epsilon + o(\epsilon)
= E \left[ \delta \gamma \partial_s \nabla f(s_m)(\mathbb{E}[\mu(s_m, a_m+1)|s_m, a_m] - \mu(s_m, a_m)) \right] \epsilon + o(\epsilon).
\]

Since

\[
\mathbb{E}[\mu(s_m, a_m+1)|s_m, a_m] - \mu(s_m, a_m) = \int \mu(s, a) \pi(s|s_m)a - \mu(s_m, a_m)
= \int \mu(s, a) \left( \pi(a|s_m) + \partial_s \pi(a|s_m) \right) \epsilon Z_m \sqrt{\epsilon} \right) da - \mu(s_m, a_m) + O(\epsilon)
= \int \mu(s, a) \pi(a|s_m) da - \mu(s_m, a_m) + O(\epsilon),
\]

we have

\[
E[\hat{F}] - E[F] = E \left[ \delta \gamma \partial_s \nabla f(s_m) \left( \int \mu(s, a) \pi(a|s_m) da - \mu(s_m, a_m) \right) \right] \epsilon + o(\epsilon).
\]

Since we have additionally assumed that

\[
\sup_{s \in \mathbb{S}, \theta \in \mathbb{R}^d} |\partial_s \nabla f(s_m; \theta)|, \sup_{s \in \mathbb{S}, a \in \mathcal{A}} |\mathbb{E}_a[\mu(s, a)|s] - \mu(s, a)| \leq C,
\]

it follows that

\[
E[\hat{F}] - E[F] \leq \gamma C^2 E[\delta(\epsilon + o(\epsilon))]
\]
as desired.

We next bound the difference of the variance. We have

\[
\mathbb{V}[\hat{F}] - \mathbb{V}[F] = \mathbb{E}[j^2((\nabla_{\theta} j)^2 - (\nabla_{\theta} j')^2)] - \left( \mathbb{E}[j \nabla_{\theta} j]^2 - \mathbb{E}[j \nabla_{\theta} j']^2 \right).
\]

Substituting the Taylor expansions of \( \nabla_{\theta} j', \nabla_{\theta} j \) from \( (A.8) \), we obtain

\[
j = r(s_m, s_m, a_m) + \gamma f(s_m) - Q^*(s_m, a_m) + \partial_s r(s_m) + \partial_{ss}^2 f(s_m) \mu \epsilon
+ \left( \partial_s r(s_m) + \gamma \partial_s f(s_m) \right) \sigma Z_m \sqrt{\epsilon} + \left( \partial_{ss}^2 r(s_m) + \gamma \partial_{ss}^2 f(s_m) \right) \sigma^2 Z_m^2 \epsilon + o(\epsilon).
\]

(A.9)

Following steps similar to the proof of Lemma [A.1], we arrive at

\[
\mathbb{V}[\hat{F}] - \mathbb{V}[F] = O(\epsilon),
\]

provided that \( g_0, f_0, \hat{f}_1 - f_1 \) are all bounded. The boundedness of these quantities is precisely the second set of assumptions in the lemma, so we are done.
B Extension and proof of Theorem 3.2

Since the continuous evolution of the parameters satisfies (17), the p.d.f. of the parameters in the optimization process satisfies the following two equations:

Uncorrelated: \[ \partial_t p = \nabla \cdot \left[ \mathbb{E}[F]p + \frac{n}{2} \nabla \cdot (\mathbb{V}[F]p) \right] ; \] (B.1)

BFF: \[ \partial_t \hat{p} = \nabla \cdot \left[ \mathbb{E}[\hat{F}]\hat{p} + \frac{n}{2} \nabla \cdot (\mathbb{V}[\hat{F}]\hat{p}) \right] ; \] (B.2)

Since \( \mathbb{E}[F] = \nabla \theta \mathbb{E}[\delta^2] \) (\( \delta \) denotes the Bellman residual) and we have assumed \( \mathbb{V}[F] \equiv \xi \), it is easy to check that the steady state of (B.1) is

\[ p^\infty = \frac{1}{Z} e^{-\beta \mathbb{E}[\delta^2]} , \quad \beta = \frac{2}{\eta \xi} , \] (B.3)

where \( Z = \int e^{-\beta \mathbb{E}[\delta^2]} d\theta \) is a normalizing constant. The difference of the p.d.f. \( \hat{d} = p - \hat{p} \) satisfies

\[ \partial_t \hat{d} = \nabla \cdot \left[ \mathbb{E}[\hat{F}]\hat{d} + \frac{n}{2} \nabla \cdot (\mathbb{V}[\hat{F}]\hat{d}) \right] + \nabla \cdot \left[ (\mathbb{E}[F] - \mathbb{E}[\hat{F}]) \hat{p} + \frac{n}{2} \nabla \cdot (\mathbb{V}[F] - \mathbb{V}[\hat{F}]) \hat{p} \right] . \] (B.4)

The proof of Theorem 3.2 is based on Corollary B.2 and Lemma B.4, as well as the following assumptions on \( \mathbb{E}[\delta^2] \) and \( \theta \in \mathbb{R}^{d_{\theta}} \). We assume that either

1) \( \lim_{|\theta| \to \infty} \mathbb{E}[\delta^2] \to \infty \) and \( \int e^{-\mathbb{E}[\delta^2]} < \infty \),

2) \( \lim_{|\theta| \to \infty} \left( \frac{|\nabla \mathbb{E}[\delta^2]|}{2} - \mathbb{E}[\Delta \delta^2] \right) = +\infty \),

or \( \theta \in \Omega \subset \mathbb{R}^{d_{\theta}} \) is in a compact set. These assumptions ensure that the probability measure \( p^\infty \) satisfies the Poincare inequality

\[ \int f^2 p^\infty d\theta \leq \lambda(\beta) \int (\nabla f)^2 p^\infty d\theta , \quad \forall \int f d\theta = 0 , \] (B.6)

where \( \lambda(\beta) \) is the Poincare constant depending on \( \beta \). Typically \( \lambda(\beta) \) becomes smaller as \( \beta \) becomes larger.

The following two lemmas hold for any function \( \delta(\theta)^2 \) on a compact domain \( \theta \in \Omega \subset \mathbb{R}^{d_{\theta}} \), or on an unbounded domain \( \theta \in \mathbb{R}^{d_{\theta}} \) if \( \lim_{|\theta| \to \infty} \delta(\theta)^2 \to +\infty \).

**Lemma B.1.** Let \( f(\theta) = \mathbb{E}[\delta^2] \) and define \( f_* = \min f(\theta) \). Suppose that \( f \) has only finitely many discrete minimizers, and that all of the minima are strict. Then there exists a constant \( C \) (depending on the Hessian \( \nabla^2 f \) of \( f \) at each of the minimizers) such that for \( \beta \) large enough,

\[ \int f(\theta) e^{-\beta f(\theta)} d\theta \leq C \left( f_* \beta^{-\frac{d_{\theta}}{2}} \right) + C \left( \beta^{-\frac{d_{\theta}+2}{2}} \right) , \]

where \( d_{\theta} \) is the dimension of \( \theta \).

**Proof.** See Appendix [B.1] \( \square \)

**Corollary B.2.** Suppose that \( f(\theta) \) has non-strict minima, i.e. minima at which the Hessian is not strictly positive definite. Define \( d_{\theta_*} \) as

\[ d_{\theta_*} = \min \{ \text{number of positive eigenvalues of } \nabla^2 f(\theta_*) : \theta_* = \text{argmin } f(\theta) \} . \] (B.7)

Then there exists a constant \( C \) (depending on the Hessian \( \nabla^2 f \) of \( f \) at each of the minimizers) such that

\[ \int f(\theta) e^{-\beta f(\theta)} d\theta \leq C \left( f_* \beta^{-\frac{d_{\theta_*}}{2}} \right) + C \left( \beta^{-\frac{d_{\theta_*}+2}{2}} \right) . \] (B.8)

**Proof.** The proof of the corollary is similar to the proof of Lemma [B.1], so we omit it here. \( \square \)
Remark B.3. Note that the bound (B.8) depends on \( d_{a_s} \), not the parameter dimension \( d_\theta \). When the dimension of the parameter space is high (i.e. when \( d_\theta \) is large), it is more likely that there are many minima which are flat in some direction (i.e. \( \nabla^2 \tilde{f}(\theta_*) \) is a positive semi-definite matrix at these minima). In the above, \( d_{a_s} \) denotes the smallest number of positive eigenvalues of \( \nabla^2 \tilde{f}(\theta_*) \) among all minima. As a result, when the dimension \( d_\theta \) becomes larger, the upper bound \( \beta^{-d_{a_s}^2/2} \) does not necessarily become smaller.

Furthermore, if \( f_* = 0 \) (i.e. there exists \( \theta_* \) such that \( Q^\pi(s,a;\theta_*) \) exactly satisfies the Bellman equation) then
\[
\int f(\theta) e^{-\beta f(\theta)} d\theta \leq C \left( \beta^{-d_{a_s}^2/2} \right).
\]

Lemma B.4. The solution to (B.1) (i.e. the approximate p.d.f. of US)
\[
\frac{\mathbb{E}[\delta^2]}{p^\infty} (p(t, \theta) - p^\infty)^2 d\theta \leq C_0 e^{-b(\beta)t},
\]
where \( C_0 \) is a constant depending on the initial data \((p(0, \theta) - p^\infty)\), \( b(\beta) = \frac{2\lambda(\beta)^2}{C_0 + \lambda(\beta)} \) with Poincare constant \( \lambda(\beta) \) and \( C = \sup \{ |\nabla \delta(a, s)|^2 \} \). In addition,
\[
\int \mathbb{E}[\delta^2] p^2(t, \theta) d\theta \leq C_0 e^{-b(\beta)t} + O \left( \mathbb{E}[\delta^2] \right),
\]
where \( \lambda(\beta) \) is the Poincare constant defined in (B.6), \( \mathbb{E}[\delta^2] = \min_{a,s} \mathbb{E}[\delta^2] \), and \( d_{a_s} \) is defined in (B.7) for \( f = \mathbb{E}[\delta^2] \).

Proof. See Appendix B.2.

We remark that all the above lemmas and theorems work for both \( \Omega = \mathbb{R}^{d_\theta} \) and compact domains \( \Omega \). However, the following theorem holds only for compact \( \Omega \). Therefore, we need a reflective boundary condition for the PDEs (B.1), (B.2), i.e.
\[
\left[ \mathbb{E}[F] \right] p + \frac{\eta}{2} \nabla \left[ \mathbb{E}[F] \right] p \cdot n_{\partial \Omega} = 0,
\]
and similar boundary conditions for \( \tilde{p} \). It is not clear whether the compactness assumption can be removed; we leave it for future study. In practice, the BFF algorithm still works for unconstrained domains.

Theorem B.5. The difference \( \hat{d} \) between the p.d.f.s of the US and BFF algorithms is bounded by
\[
\| \hat{d}(t) \|_* \leq \| p(0) - p^\infty \|_* e^{-\lambda(\beta)t} + O(\epsilon) e^{-\frac{\lambda(\beta)}{2}t} + O \left( \epsilon \sqrt{\mathbb{E}[\delta^2]} \beta^{-d_{a_s}} \right) \sqrt{1 - e^{-\lambda(\beta)t}},
\]
where \( \lambda(\beta) \) is the Poincare constant defined in (B.6), \( b(\beta) \) is the same constant as in Lemma B.4, \( \mathbb{E}[\delta^2] = \min_{a,s} \mathbb{E}[\delta^2] \), and \( d_{a_s} \) is defined in (B.7) for \( f = \mathbb{E}[\delta^2] \).

Proof. Observe that
\[
\mathbb{E}[\hat{F}] \hat{p} - \mathbb{E}[F] p = \mathbb{E}[F] \hat{d} + (\mathbb{E}[\hat{F}] - \mathbb{E}[F]) \hat{d} + (\mathbb{E}[\hat{F}] - \mathbb{E}[F]) p
\]
\[
= \mathbb{E}[F] \hat{d} + \mathbb{E}[\delta O(e)] \hat{d} + \mathbb{E}[\delta O(e)] p.
\]
Similarly, we have
\[
\nabla[\hat{F}] \hat{p} - \nabla[F] p = \nabla[F] \hat{d} + O(\epsilon) \hat{d} + O(\epsilon) p,
\]
Multiplying (B.4) by \( \frac{\hat{d}}{p^\infty} \), then integrating with respect to \( \theta \), we have
\[
\frac{1}{2} \| \hat{d} \|_*^2 = - \int \left( \frac{p^\infty \nabla \left( \frac{\hat{d}}{p^\infty} \right)}{p^\infty} \right) \cdot \nabla \left( \frac{\hat{d}}{p^\infty} \right) d\theta - \int \left[ \mathbb{E}[\delta O(e)] \hat{d} + O(\epsilon) \nabla \hat{d} \right] \cdot \nabla \left( \frac{\hat{d}}{p^\infty} \right) d\theta
\]
\[
- \int \left[ \mathbb{E}[\delta O(e)] p + O(\epsilon) \nabla p \right] \cdot \nabla \left( \frac{\hat{d}}{p^\infty} \right) d\theta.
\]
We proceed by bounding the terms $I - III$ separately. First, note that

$$I = - \int \left[ \nabla \left( \frac{\dot{d}}{p^\infty} \right) \right]^2 p^\infty d\theta \leq - \frac{1}{2} \int \left[ \nabla \left( \frac{\dot{d}}{p^\infty} \right) \right]^2 p^\infty d\theta - \frac{\lambda}{2} \| \dot{d} \|_*^2,$$

where we have used the Poincare inequality \eqref{B.6}. For the second term, we have

$$II \leq O(\epsilon) \int \| \dot{d} \|_*^2 \left[ \nabla \left( \frac{\dot{d}}{p^\infty} \right) \right]^2 p^\infty d\theta + O(\epsilon \eta) \int \left| \nabla \left( \frac{\dot{d}}{p^\infty} \right) \right| \left| \nabla \left( \frac{\dot{d}}{p^\infty} \right) \right| d\theta \quad \text{(boundedness of $E \delta$)}
$$

$$\leq O(\epsilon) \| \dot{d} \|_*^2 + \frac{1}{8} \int \left[ \nabla \left( \frac{\dot{d}}{p^\infty} \right) \right]^2 p^\infty d\theta + O(\epsilon \eta) \int \left| \nabla \left( \frac{\dot{d}}{p^\infty} \right) \right| \left| \nabla \left( \frac{\dot{d}}{p^\infty} \right) \right| \left( \frac{\dot{d}}{p^\infty} \right) d\theta.
$$

(Cauchy-Schwartz Inequality)

Since $\eta^2 \beta^2 = O(1), O(\epsilon^2 \eta^2 \beta^2) = O(\epsilon^2).$ This yields

$$II \leq O(\epsilon) \| \dot{d} \|_*^2 + \left( \frac{1}{4} + O(\epsilon \eta) \right) \int \left| \nabla \left( \frac{\dot{d}}{p^\infty} \right) \right|^2 p^\infty d\theta.$$

For the last term, by the Cauchy Schwartz inequality,

$$III \leq O(\epsilon^2) \int E[\delta^2] \frac{\dot{p}^2}{p^\infty} d\theta + \frac{1}{16} \int \left| \nabla \left( \frac{\dot{p}}{p^\infty} \right) \right|^2 p^\infty d\theta + O(\epsilon \eta) \int \left| \nabla \left( \frac{\dot{p}}{p^\infty} \right) \right| \left| \nabla \left( \frac{\dot{p}}{p^\infty} \right) \right| \left( \frac{\dot{p}}{p^\infty} \right) d\theta
$$

$$\leq O(\epsilon^2) \int E[\delta^2] \frac{\dot{p}^2}{p^\infty} d\theta + O(\epsilon^2 \eta^2) \int \left| \nabla \left( \frac{\dot{p}}{p^\infty} \right) \right|^2 p^\infty d\theta + \frac{2}{16} \int \left| \nabla \left( \frac{\dot{p}}{p^\infty} \right) \right|^2 p^\infty d\theta
$$

$$+ O(\epsilon^2 \eta^2 \beta^2) \int \frac{E[\delta^2]}{p^\infty} \frac{\dot{p}^2}{p^\infty} d\theta + \frac{1}{16} \int \left| \nabla \left( \frac{\dot{p}}{p^\infty} \right) \right|^2 p^\infty d\theta
$$

$$\leq O(\epsilon^2) \int E[\delta^2] \frac{\dot{p}^2}{p^\infty} d\theta + O(\epsilon^2 \eta^2) \int \left| \nabla \left( \frac{\dot{p}}{p^\infty} \right) \right| \left| \nabla \left( \frac{\dot{p}}{p^\infty} \right) \right| \left( \frac{\dot{p}}{p^\infty} \right) d\theta + \frac{3}{16} \int \left| \nabla \left( \frac{\dot{p}}{p^\infty} \right) \right|^2 p^\infty d\theta.
$$

Combining the above three terms, we have

$$\frac{1}{2} \partial_t \| \dot{d} \|_*^2 \leq \left( \frac{1}{16} - O(\epsilon \eta) \right) \int \left| \nabla \left( \frac{\dot{d}}{p^\infty} \right) \right|^2 p^\infty d\theta - \left( \frac{\lambda}{2} - O(\epsilon) \right) \| \dot{d} \|_*^2
$$

$$+ O(\epsilon^2) \int E[\delta^2] \frac{\dot{p}^2}{p^\infty} d\theta + O(\epsilon^2 \eta^2) \int \left| \nabla \left( \frac{\dot{p}}{p^\infty} \right) \right|^2 p^\infty d\theta.
$$

As long as $\epsilon, \eta$ are small enough, and using the fact that $\nabla \left( \frac{\dot{p}}{p^\infty} \right) = 0,$ we have

$$\frac{1}{2} \partial_t \| \dot{d} \|_*^2 \leq - \frac{\lambda}{4} \| \dot{d} \|_*^2 + O(\epsilon^2) \int E[\delta^2] \frac{\dot{p}^2}{p^\infty} d\theta + O(\epsilon^2 \eta^2) \int \left| \nabla \left( \frac{\dot{p}}{p^\infty} \right) \right|^2 p^\infty d\theta. \quad \text{(B.10)}
$$

Setting $d = p - p^\infty,$ it is easy to see that $d$ also satisfies \eqref{B.1}. Multiplying \eqref{B.1} by $\frac{\dot{d}}{p^\infty}$ and integrating with respect to $\theta,$ we have

$$\frac{1}{2} \partial_t \| \dot{d} \|_*^2 = - \int \left| \nabla \left( \frac{\dot{d}}{p^\infty} \right) \right|^2 p^\infty d\theta \leq - \frac{1}{2} \int \left| \nabla \left( \frac{\dot{d}}{p^\infty} \right) \right|^2 p^\infty d\theta - \frac{\lambda}{2} \| \dot{d} \|_*^2. \quad \text{(B.11)}$$
Adding equations (B.10) and (B.11) gives
\[
\frac{1}{2} \partial_t \left( \left\| \hat{d} \right\|_\ast^2 + \left\| d \right\|_\ast^2 \right) \leq - \frac{\lambda}{4} \left( \left\| \hat{d} \right\|_\ast^2 + \left\| d \right\|_\ast^2 \right) - \frac{\lambda}{4} \left\| d \right\|_\ast^2 + O(c^2) \int \mathbb{E}[\delta^2] \frac{\beta^2}{p \infty} d\theta.
\]

Applying Lemma B.4, we have
\[
\partial_t \left[ e^{\frac{1}{2}t} \left( \left\| \hat{d} \right\|_\ast^2 + \left\| d \right\|_\ast^2 \right) \right] \leq e^{\frac{1}{2}t} O(c^2) \int \mathbb{E}[\delta^2] \frac{\beta^2}{p \infty} d\theta.
\]

We then integrate the above inequality on both sides to obtain
\[
\left( \left\| \hat{d}(t) \right\|_\ast^2 + \left\| d(t) \right\|_\ast^2 \right) \leq e^{\frac{1}{2}t} \left( \left\| \hat{d}(0) \right\|_\ast^2 + \left\| d(0) \right\|_\ast^2 \right) + O(c^2)(e^{-bt} + e^{-\frac{1}{2}t}) + O\left( e^2 \mathbb{E}[\delta^2] \beta \frac{d_\ast}{\tilde{F}} \right) \left( 1 - e^{-\frac{1}{2}t} \right).
\]
Since \( \hat{d}(0) = 0 \), the above inequality is equivalent to
\[
\left\| \hat{d}(t) \right\|_\ast^2 \leq e^{\frac{1}{2}t} \left\| d(0) \right\|_\ast^2 + O(c^2) e^{-bt} + O\left( e^2 \mathbb{E}[\delta^2] \beta \frac{d_\ast}{\tilde{F}} \right) \left( 1 - e^{-\frac{1}{2}t} \right)
\]
as desired. \( \square \)

### B.1 Proof of Lemma B.1

**Proof.** For the unbounded domain, since \( \lim_{|\theta| \to \infty} f(\theta) = +\infty \) and \( \lim_{f \to +\infty} f e^{-\beta f} = 0 \), there always exists a compact domain \( \Omega = \{ |\theta| \leq M \} \) such that
\[
\int_{\mathbb{R}^d \setminus \Omega} f(\theta) e^{-\beta f(\theta)} d\theta \leq O\left( f(\theta_*) \beta \frac{d_\ast}{\tilde{F}} \right) + O\left( \beta \frac{d_\ast}{\tilde{F}} \right).
\]

We can divide \( \Omega \) into \( \{ \Omega_i \}_{i=1}^k \) such there is only one minimizer \( \theta_* \) in each \( \Omega_i \), or else \( f(\Omega_i) \equiv 0 \).

For this latter case, it is trivial to see that \( \int_{\Omega_i} f(\theta) e^{-\beta f(\theta)} = 0 \).

For the former case, notice that the integral can be separated into two parts,
\[
\int_{\Omega_i} f(\theta) e^{-\beta f(\theta)} d\theta = \int_{|\theta - \theta_*| \leq \epsilon} f(\theta) e^{-\beta f(\theta)} d\theta + \int_{\Omega_i \setminus \{ \theta - \theta_* \leq \epsilon \}} f(\theta) e^{-\beta f(\theta)} d\theta.
\]

For any \( \epsilon > 0 \), we can choose \( \beta \) large enough that the second integral will be smaller than \( O\left( \beta^{-\frac{d_\ast + 2}{\tilde{F}}} \right) \).

Since \( \theta_* \) is a minimizer, \( \nabla f(\theta_*) = 0 \) and we have
\[
\int_{\Omega_i} f(\theta) e^{-\beta f(\theta)} d\theta = \int_{|\theta - \theta_*| \leq \epsilon} f(\theta) \nabla^2 f(\theta_*) (\theta - \theta_*) + O\left( |\theta - \theta_*|^3 \right) \exp\left( -\beta f(\theta_*) - \beta (\theta - \theta_*)^T \nabla^2 f(\theta_*) (\theta - \theta_*) - \beta O\left( |\theta - \theta_*|^3 \right) \right) d\theta + O\left( \beta^{-\frac{d_\ast + 1}{\tilde{F}}} \right)
\]
\[
= f(\theta_*) \exp\left( -\beta f(\theta_*) \right) \int_{|\theta - \theta_*| \leq \epsilon} \exp\left( -\beta (\theta - \theta_*)^T \nabla^2 f(\theta_*) (\theta - \theta_*) \right) d\theta
\]
\[
+ \exp\left( -\beta f(\theta_*) \right) \int_{|\theta - \theta_*| \leq \epsilon} (\theta - \theta_*)^T \nabla^2 f(\theta_*) (\theta - \theta_*) \exp\left( -\beta (\theta - \theta_*)^T \nabla^2 f(\theta_*) (\theta - \theta_*) \right) d\theta
\]
\[
+ (\text{higher order terms in } \beta).
\]

(B.12)

We will prove later the higher order terms are all smaller than \( O\left( \beta^{-\frac{d_\ast + 2}{\tilde{F}}} \right) \). Without loss of generality, we assume \( \nabla^2 f(\theta_*) \) is a diagonal matrix. (If it is not, we can simply perform a change of basis.)
Since \( \theta_* \) is a local minimum, \( \partial^2 \theta, f(\theta_*) > 0 \). Then after making the change of variables \( \tilde{\theta} = \theta - \theta_* \), we have

\[
\int_{\Omega_1} f(\theta) e^{-\beta f(\theta)} d\theta = f(\theta_*) \exp(-\beta f(\theta_*)) \int_{\Omega_1} \prod_i \exp \left(-\beta \partial^2_{\theta_i} f(\theta_*) \tilde{\theta}_i^2 \right) d\tilde{\theta}_1 \cdots d\tilde{\theta}_{d_\theta} + \exp(-\beta f(\theta_*)) \int_{\Omega_1} \left( \sum_i \partial^2_{\theta_i} f(\theta_*) \tilde{\theta}_i^2 \right) \prod_i \exp \left(-\beta \partial^2_{\theta_i} f(\theta_*) \tilde{\theta}_i^2 \right) d\tilde{\theta}_1 \cdots d\tilde{\theta}_{d_\theta} + O(\beta^{-\frac{d_\theta+1}{2}}).
\]

(B.13)

Since

\[
\int_{\mathbb{R}} \exp \left(-\beta \partial^2_{\theta_i} f(\theta_*) \tilde{\theta}_i^2 \right) d\tilde{\theta}_i = \sqrt{2\pi(\beta \partial^2_{\theta_i} f(\theta_*)} \right)^{-1/2},
\]

\[
\int_{\mathbb{R}} \tilde{\theta}_i^2 \exp \left(-\beta \partial^2_{\theta_i} f(\theta_*) \tilde{\theta}_i^2 \right) d\tilde{\theta}_i = \sqrt{2\pi(\beta \partial^2_{\theta_i} f(\theta_*)} \right)^{-3/2},
\]

we have,

\[
\int_{\mathbb{R}^{d_\theta}} \prod_i \exp \left(-\beta \partial^2_{\theta_i} f(\theta_*) \tilde{\theta}_i^2 \right) d\tilde{\theta}_1 \cdots d\tilde{\theta}_{d_\theta} = (2\pi)^{d_\theta/2} \beta^{-d_\theta/2} \prod_i (\partial^2_{\theta_i} f(\theta_*))^{-1/2} = O \left( \beta^{-\frac{d_\theta}{2}} \right);
\]

\[
\int_{\mathbb{R}^{d_\theta}} \prod_i \exp \left(-\beta \partial^2_{\theta_i} f(\theta_*) \tilde{\theta}_i^2 \right) d\tilde{\theta}_1 \cdots d\tilde{\theta}_{d_\theta} = (2\pi)^{d_\theta/2} \beta^{-d_\theta/2} (\partial^2_{\theta_i} f(\theta_*))^{-1} \prod_j (\partial^2_{\theta_j} f(\theta_*))^{-1/2} = O \left( \beta^{-\frac{d_\theta+2}{2}} \right).
\]

Plugging the above estimate back to (B.13) and recalling that \( \theta_* \) is the only minimizer in \( \Omega_1 \), we have

\[
\int_{\Omega_1} f(\theta) e^{-\beta f(\theta)} d\theta \leq O \left( f(\theta_*) \beta^{-\frac{d_\theta}{2}} \right) + O \left( \beta^{-\frac{d_\theta+2}{2}} \right).
\]

(B.14)

Now we will estimate the higher order terms in (B.12).

(higher order terms in \( \beta \))

\[
f(\theta_*) \exp(-\beta f(\theta_*)) \int_{|\theta| \leq \varepsilon} \left[ \frac{\tilde{\theta}^T \nabla^2 f(\theta_*) \tilde{\theta} \left( e^{-\beta O(\tilde{\theta}^3)} - 1 \right)}{\leq 1} \right] \left[ \frac{\exp \left(-\beta \tilde{\theta}^T \nabla^2 f(\theta_*) \tilde{\theta} \left( e^{-\beta O(\tilde{\theta}^3)} - 1 \right) \right)}{\leq \exp \left(-\beta \tilde{\theta}^T \nabla^2 f(\theta_*) \tilde{\theta} \left( e^{-\beta O(\tilde{\theta}^3)} - 1 \right) \right)} \right] d\theta
\]

\[
+ \exp(-\beta f(\theta_*)) \int_{|\theta| \leq \varepsilon} \left[ \frac{\tilde{\theta}^T \nabla^2 f(\theta_*) \tilde{\theta} \left( e^{-\beta O(\tilde{\theta}^3)} - 1 \right) \right] \left[ \frac{\exp \left(-\beta \tilde{\theta}^T \nabla^2 f(\theta_*) \tilde{\theta} - \beta O(\tilde{\theta}^3) \right)}{\leq \exp \left(-\beta \tilde{\theta}^T \nabla^2 f(\theta_*) \tilde{\theta} - \beta O(\tilde{\theta}^3) \right)} \right] d\theta
\]

\[
+ \exp(-\beta f(\theta_*)) \int_{|\theta| \leq \varepsilon} O(\tilde{\theta}^3) \left[ \frac{\tilde{\theta}^T \nabla^2 f(\theta_*) \tilde{\theta} \left( e^{-\beta O(\tilde{\theta}^3)} - 1 \right) \right] \left[ \frac{\exp \left(-\beta \tilde{\theta}^T \nabla^2 f(\theta_*) \tilde{\theta} - \beta O(\tilde{\theta}^3) \right)}{\leq \exp \left(-\beta \tilde{\theta}^T \nabla^2 f(\theta_*) \tilde{\theta} - \beta O(\tilde{\theta}^3) \right)} \right] d\theta
\]

\[
\leq O \left( \frac{\text{vol} \left( \{ |\theta| \leq \varepsilon \} \right) \left( e^{\beta \tilde{\theta}^3} - 1 \right) + \varepsilon^3 }{\leq 1} \right).
\]

From the above estimates, we see that as long as \( \varepsilon \) is small enough, the higher order terms are smaller than \( O \left( \beta^{-\frac{d_\theta+2}{2}} \right) \). Since we assumed that the number of discrete minimizers is finite, this completes the proof.

\( \square \)
B.2 Proof of Lemma B.4

Proof. Setting \( d = p - p^\infty \), it is easy to see that \( d \) also satisfies (B.1). Multiplying (B.1) by \( \mathbb{E}[\delta^2] \frac{d}{p^\infty} \) and then integrating with respect to \( \theta \), we have

\[
\frac{1}{2} \partial_t \int \mathbb{E}[\delta^2] \frac{d^2}{p^\infty} d\theta \leq - \int p^\infty \nabla \left( \frac{d}{p^\infty} \right) \nabla \left( \mathbb{E}[\delta^2] \frac{d}{p^\infty} \right) \]

\[
= - \int p^\infty \left[ \nabla \left( \frac{d}{p^\infty} \right) \nabla \left( \delta^2 \frac{d}{p^\infty} \right) \right] d\theta d\mu(s, a)
\]

\[
= - \int p^\infty \left[ \left( \nabla \left( \frac{d}{p^\infty} \right) \right)^2 - \left( \frac{d}{p^\infty} \right)^2 (\nabla \delta)^2 \right] d\theta d\mu(s, a)
\]

\[
\leq - \int p^\infty \left( \nabla \delta \frac{d}{p^\infty} \right)^2 d\theta d\mu(s, a) + C \int \int \left( \frac{d}{p^\infty} \right)^2 p^\infty d\theta d\mu(s, a)
\]

\[
\leq - \lambda \int \int \left( \frac{d}{p^\infty} \right)^2 d\theta d\mu(s, a) + C \int d^2 \left( \frac{d}{p^\infty} \right)^2 d\theta d\mu(s, a)
\]

Using the fact that \( \frac{1}{2} \partial_t \|d\|^2 \leq -\lambda \|d\|^2 \), we have

\[
\frac{1}{2} \partial_t \left[ \int \mathbb{E}[\delta^2] \frac{d^2}{p^\infty} d\theta + \left( \frac{C}{\lambda} + 1 \right) \|d\|^2 \right] \leq - \lambda \left( \int \mathbb{E}[\delta^2] \frac{d^2}{p^\infty} d\theta + \|d\|^2 \right)
\]

\[
\leq - \frac{\lambda^2}{C + \lambda} \left[ \int \mathbb{E}[\delta^2] \frac{d^2}{p^\infty} d\theta + \left( \frac{C}{\lambda} + 1 \right) \|d\|^2 \right].
\]

By Grownwall’s inequality,

\[
\left[ \int \mathbb{E}[\delta^2] \frac{d(t)^2}{p^\infty} d\theta + \left( \frac{C}{\lambda} + 1 \right) \|d(t)\|^2 \right] \leq e^{-\frac{\lambda t}{C + \lambda}} \left[ \int \mathbb{E}[\delta^2] \frac{d(0)^2}{p^\infty} d\theta + \left( \frac{C}{\lambda} + 1 \right) \|d(0)\|^2 \right],
\]

\[
\int \mathbb{E}[\delta^2] \frac{d(t)^2}{p^\infty} d\theta \leq e^{-\frac{\lambda t}{C + \lambda}} \int \mathbb{E}[\delta^2] \frac{d(0)^2}{p^\infty} d\theta + \left( \frac{C}{\lambda} + 1 \right) \|d(0)\|^2.
\]

which completes the first part of the proof.

While the second part of the Lemma is obtained by inserting \( p^2 = (d + p^\infty)^2 \leq 2d^2 + 2(p^\infty)^2 \) into the following equation,

\[
\int \mathbb{E}[\delta^2] \frac{p(t)^2}{p^\infty} d\theta \leq 2 \int \mathbb{E}[\delta^2] \frac{d(t)^2}{p^\infty} d\theta + 2 \int \mathbb{E}[\delta^2] p^\infty d\theta = C_0 e^{-\frac{\lambda t}{C + \lambda}} + O(\beta^{-\frac{d+2}{2}}),
\]

where Lemma B.1 is applied to the last equality.

\[ \square \]

C Difference between SC and US

The SC parameter update is given by

\( \theta_{m+1} = \theta_m - \eta \tilde{F} \), \( \tilde{F} = j(s_m, a_m, s_{m+1}; \theta_m) \nabla_{\theta} j(s_m, a_m, s_{m+1}; \theta_m) \).

The definition of \( j \) depends on whether we are doing Q-evaluation or Q-control:

Q-evaluation: \( j(s_m, a_m, s_{m+1}; \theta_m) = r(s_{m+1}, s_m, a_m) + \gamma \int Q^\pi(s_{m+1}, a; \theta) \pi(a|s_{m+1}) da \)  

\( Q^\pi(s_m, a_m; \theta) \);

Q-control: \( j(s_m, a_m, s_{m+1}; \theta_m) = r(s_{m+1}, s_m, a_m) + \gamma \max_a Q^\pi(s_{m+1}, a; \theta) - Q^\pi(s_m, a_m; \theta) \).

The expectation of the SC gradient \( \tilde{F} \) at each step is

\[
\mathbb{E}[\tilde{F}] = \mathbb{E}[j \nabla_{\theta} j | s_m = s, a_m = a],
\]

(C.1)
which is the gradient of the following loss function

\[ \dot{J}(\theta) = \frac{1}{2} \mathbb{E} \left[ \mathbb{E} \left[ \frac{1}{2} s_m = s, a_m = a \right] \right]. \]  

(C.2)

Note that this is not the same as the desired objective function \( J(\theta) = \frac{1}{2} \mathbb{E}[(\mathbb{E}[s_m, a_m])^2] \).

\[ \text{C.1 Difference at each step} \]

In Lemmas C.1 and C.2, we prove that the difference between the gradients used in US and SC is \( O(\epsilon) \). The constants hidden by the big-\( O \) depend on the square of the diffusion \( \sigma^2 \). In practice, this means that SC will not converge to a good approximation for \( Q^\pi \).

**Lemma C.1.** Suppose that

\[ \sup_{s \in S, \theta \in \mathbb{R}^d} |\partial_s \mathbb{E}_\theta[Q^\pi(s, a; \theta)]|, \sup_{s \in S, \theta \in \mathbb{R}^d} |\partial_s \mathbb{E}_\theta[\nabla_\theta Q^\pi(s, a; \theta)]|, \sup_{s \in S, a \in A} |\partial_a r(s, s, a)| \leq C \]

almost surely. Then the difference between the gradients in the US and SC algorithms is bounded by

\[ \mathbb{E}[\dot{F}] - \mathbb{E}[F] \leq 2\sigma^2 C^2 \epsilon + o(\epsilon); \]

In addition, if \( |\mathbb{E}_\theta[Q^\pi(s, a; \theta)] - Q(s, a; \theta)|, |\mathbb{E}_\theta[\nabla_\theta Q^\pi(s, a; \theta)] - \nabla_\theta Q(s, a; \theta)|, |r(s, s, a)| \leq C \)

almost surely in \( \forall s \in S, a \in A, \theta \in \mathbb{R}^d \), then the difference between the variances is bounded by

\[ \mathbb{V}[\dot{F}] - \mathbb{V}[F] \leq O(\epsilon), \]

where \( \dot{F}, F, \sigma \) are defined in (C.1) (A.1) and (A.2) respectively.

**Proof.** The proof is similar to the proof of Lemma A.1. Subtracting (A.1) from (C.1) yields

\[ \mathbb{E}[\dot{F}] = \mathbb{E} \left[ j \left( \nabla j - \mathbb{E} \left[ \nabla j | s_m, a_m \right] \right) | s_m, a_m \right]. \]  

(C.3)

By the approximation of \( \nabla j \) in (A.4), we have

\[ \nabla j - \mathbb{E} \left[ \nabla j | s_m, a_m \right] = f_2 Z_m \sqrt{\epsilon} + f_3 (Z_m^2 - 1) \epsilon + o(\epsilon). \]

Combining this with the approximation of \( j \) in (A.6) gives

\[ \mathbb{E}[\dot{F}] = \mathbb{E}[g_2 f_2 \epsilon] + o(\epsilon) \]

\[ \gamma \mathbb{E} \left[ \partial_s r \left( \nabla Q^\pi \right) da \right] \sigma^2 \epsilon + \gamma^2 \mathbb{E} \left[ \partial_s (Q^\pi \pi) da \right] \sigma^2 \epsilon. \]

Therefore, as long as

\[ \partial_s (\mathbb{E}_\theta \nabla Q^\pi(s, a; \theta)) \leq C, \quad \forall s \in S, \theta, \]
\[ \partial_s (\mathbb{E}_\theta Q^\pi(s, a; \theta)) \leq C, \quad \forall s \in S, \theta, \]
\[ \partial_a r(s', s, a) \leq C, \quad \forall s', s \in S, a \in A, \]

then the difference between the gradients is bounded by

\[ \mathbb{E} \left| \dot{F} - F \right| \leq 2\gamma^2 C^2 \epsilon + o(\epsilon) \]

as desired.

Next, we bound the difference of the variance. We have

\[ \mathbb{V} \left[ \dot{F} - F \right] = \mathbb{E} \left[ j^2 \left( (\nabla j)^2 - (\nabla j')^2 \right) \right] - \mathbb{E} \left[ j^2 \right] \sigma^2 \epsilon + o(\epsilon) \]

\[ = \mathbb{E} \left[ j^2 \left( (\nabla j)^2 - \mathbb{E} \left[ (\nabla j)^2 | s_m, a_m \right] \right) | s_m, a_m \right] \]

\[ - \mathbb{E} \left[ j^2 \mathbb{E} \left[ (\nabla j)^2 | s_m, a_m \right] \mathbb{E} \left[ (\nabla j)^2 | s_m, a_m \right] \right]. \]  

(C.5)
Using the approximations for \( \nabla j, j \) in (A.4), (A.6), we have

\[
\mathbb{E}(|\nabla j|^2|s_m, a_m] - (\nabla j)^2
\]

\[
= \mathbb{E}[f_0^2 + 2f_0f_2Z_\infty \sqrt{\epsilon} + 2f_0f_1\epsilon + (2f_0f_3 + f_2^2)Z_m^2\epsilon + o(\epsilon)|s_m, a_m]\n+ (f_0^2 + 2f_0f_2Z_\infty \sqrt{\epsilon} + 2f_0f_1\epsilon + (2f_0f_3 + f_2^2)Z_m^2\epsilon + o(\epsilon))\n= -2f_0f_2Z_\infty \sqrt{\epsilon} + (2f_0f_3 + f_2^2)(1 - Z_m^2)\epsilon + o(\epsilon);\n\]

\[
\mathbb{E}[j^2|s_m, a_m]\n= \mathbb{E}[(g_0^0 + 2g_0g_2Z_\infty \sqrt{\epsilon} + 2g_0g_1\epsilon + (2g_0g_3 + g_2^2)Z_m^2\epsilon + o(\epsilon))|s_m, a_m]\n= -4g_0g_2f_0f_2\epsilon + o(\epsilon).\n\]

It follows that

\[
I = -4\mathbb{E}[j^2|s_m, a_m] = 4\epsilon\mathbb{E}[g_0g_2f_0f_2] + o(\epsilon).\quad (C.6)
\]

Furthermore, we have

\[
\mathbb{E}[j|s_m, a_m] = g_0 + (g_1 + g_3)\epsilon + o(\epsilon),\quad (2)
\]

\[
\mathbb{E}[\nabla j|s_m, a_m] = f_0 + (f_1 + f_3)\epsilon + o(\epsilon),\quad (3)
\]

\[
\mathbb{E}[j^2|s_m, a_m] = (\mathbb{E}[g_0f_0] + \mathbb{E}[(g_0 + (g_1 + g_3))\epsilon + o(\epsilon)]^2 + 2\mathbb{E}[g_0f_0]\mathbb{E}[(g_0 + (g_1 + g_3))\epsilon + o(\epsilon)],\quad (4)
\]

and

\[
\mathbb{E}[j\nabla j|s_m, a_m] = \mathbb{E}[f_0g_0 + f_0g_1\epsilon + f_0g_2Z_\infty \sqrt{\epsilon} + f_0g_3Z_m^2\epsilon + f_1g_0\epsilon + (2g_0g_1 + g_2^2)Z_m^2\epsilon + g_0g_3Z_m^2\epsilon|s_m, a_m]\n= f_0g_0 + (f_0g_1 + g_3) + g_0(f_1 + f_3)\epsilon + f_2g_2\epsilon + o(\epsilon).
\]

\[
\mathbb{E}[j^2|s_m, a_m] = \mathbb{E}[f_0g_0]^2 + 2\mathbb{E}[f_0g_0]\mathbb{E}[(g_0 + (g_1 + g_3))\epsilon + o(\epsilon)],\quad (5)
\]

\[
\mathbb{E}[j\nabla j|s_m, a_m] = \mathbb{E}[f_0g_0]^2 + 2\mathbb{E}[f_0g_0]\mathbb{E}[(g_0 + (g_1 + g_3))\epsilon + o(\epsilon)],\quad (6)
\]

Combining these shows that

\[
II = \mathbb{E}[j^2|s_m, a_m] - \mathbb{E}[j^2|s_m, a_m] = 2\epsilon\mathbb{E}[f_0g_0]\mathbb{E}[f_g2g] + o(\epsilon).\quad (C.7)
\]

Finally, substituting (C.6) and (C.7) into (C.5) gives,

\[
\left|\nabla[\hat{F}] - \nabla[F]\right| = 2\epsilon\left(2\mathbb{E}[f_0g_0f_2g_2] - \mathbb{E}[f_0g_0]\mathbb{E}[f_g2g]\right) + o(\epsilon).\quad (C.8)
\]

A sufficient condition for \( 2\mathbb{E}[f_0g_0f_2g_2] - \mathbb{E}[f_0g_0]\mathbb{E}[f_g2g] \) to be bounded is that \( f_0, g_0, f_2, g_2 \) are all bounded. The condition (C.4) guarantees \( f_2, g_2 \) are bounded, and since \( f_0 = \gamma \mathbb{E}_{a}[\nabla \theta^Q(s_m, a)] - Q^s(s_m, a_m), g(0) = r(s_m, s_m, a_m), \gamma \mathbb{E}_{a}[\nabla \theta^Q(s_m, a)] - Q^s(s_m, a_m), \) as long as

\[
\left|\mathbb{E}_{a}[\nabla \theta^Q(s_m, a)] - Q^s(s_m, a)\right| \leq C, \quad \forall s_m, a_m, \theta \in \mathbb{R}^d;
\]

\[
\left|\mathbb{E}_{a}[\nabla \theta^Q(s_m, a)] - Q^s(s_m, a)\right| \leq C, \quad \forall s_m, a_m, \theta \in \mathbb{R}^d;
\]

\[
|r(s, s, a)| \leq C, \quad \forall s, a, \theta \in \mathbb{R}^d;
\]

then the coefficient of \( \epsilon \) is bounded. This implies that \( \left|\nabla[\hat{F}] - \nabla[F]\right| = O(\epsilon) \), completing the proof. \( \square \)
Lemma C.2. Let \( f(s; \theta) = \max_{a' \in A} Q^*(s, a'; \theta) \). Suppose that \( f(s; \theta) \) is continuous in \( s \in \mathbb{R} \) and \( \partial_s f(s; \theta), \partial_{ss} f(s; \theta) \) exist almost surely. Further assume that \( \sup_{s \in \mathbb{R}, \theta \in \mathbb{R}^d} |\partial_s f(s, \theta)| \), \( \sup_{s \in \mathbb{R}, \theta \in \mathbb{R}^d} |\partial_{ss} f(s, \theta)| \), \( \sup_{s \in \mathbb{R}, \theta \in \mathbb{R}^d} |\partial_s r(s, s, a)| \) exist almost surely. Then the difference of the p.d.f.s of the parameters during the SC algorithm satisfies the equation

\[
\frac{\partial}{\partial t} \tilde{p} = \nabla \cdot \left( \mathbb{E}[\tilde{F}] \tilde{p} + \frac{\eta}{2} \nabla \cdot \left( \nabla \left[ \tilde{V} - \mathbb{E}[\tilde{V}] \right] \tilde{p} + \frac{\eta}{2} \nabla \left( \left[ \mathbb{E}[\tilde{F}] - \mathbb{E}[\tilde{F}] \right] \tilde{p} + \frac{\eta}{2} \nabla \left( \left[ \mathbb{E}[\tilde{F}] - \mathbb{E}[\tilde{F}] \right] \tilde{p} \right) \right) \right) .
\]  

(C.9)

Therefore, the difference of the p.d.f.s \( \tilde{d} = p - \tilde{p} \) satisfies

\[
\frac{\partial}{\partial t} \tilde{d} = \nabla \cdot \left[ \mathbb{E}[\tilde{F}] \tilde{d} + \frac{\eta}{2} \nabla \cdot \left( \nabla \left[ \tilde{V} - \mathbb{E}[\tilde{V}] \right] \tilde{d} \right) + \nabla \cdot \left( \left[ \mathbb{E}[\tilde{F}] - \mathbb{E}[\tilde{F}] \right] \tilde{d} \right) + \frac{\eta}{2} \nabla \cdot \left( \left[ \mathbb{E}[\tilde{F}] - \mathbb{E}[\tilde{F}] \right] \tilde{d} \right) \right) .
\]  

(C.10)

Using this observation, we can prove the following theorem.

**Theorem C.3.** The difference \( \tilde{d} \) of the p.d.f. between US and SC satisfies,

\[
\left\| \tilde{d}(t) \right\|_s \leq e^{-\frac{\Lambda(t)}{4}} \left\| p(0) - p^\infty \right\|_s + O(\epsilon) \sqrt{1 - e^{-\frac{\Lambda(t)}{2}}}.
\]

Unlike the evolution of \( \tilde{d} \) in Theorem [B.5] the difference between SC and US will eventually decay to \( O(\epsilon) \) instead of \( O(\epsilon \sqrt{\mathbb{E}[\sigma^2] \eta^{-\frac{d}{4}}} \). As a result, the error of SC is much larger than that of BFF.

**Proof.** The analysis of \( \left\| \tilde{d} \right\|_s^2 \) is similar to the analysis of \( \left\| \tilde{d} \right\|_s^2 \) before applying Lemma [B.4]. Therefore, similar to [B.10], we have

\[
\frac{1}{2} \frac{\partial}{\partial t} \left( \left\| \tilde{d} \right\|_s^2 + \left\| \tilde{d} \right\|_s^2 \right) + O(\epsilon^2) \int \frac{\tilde{d}^2}{p^\infty} d\theta + O(\epsilon^2 \eta^2) \int \left\| \nabla \left( \frac{\tilde{d}^2}{p^\infty} \right) \right\|^2 p^\infty d\theta 
\]

\[
\leq - \frac{\Lambda(t)}{4} \left( \left\| \tilde{d} \right\|_s^2 + \left\| \tilde{d} \right\|_s^2 \right) + O(\epsilon^2) \int \left\| \nabla \left( \frac{\tilde{d}^2}{p^\infty} \right) \right\|^2 p^\infty d\theta,
\]

where \( d = p - p^\infty \). Combining the above equation with [B.11] and taking \( d = p - p^\infty \), we have

\[
\frac{1}{2} \frac{\partial}{\partial t} \left( \left\| \tilde{d} \right\|_s^2 + \left\| \tilde{d} \right\|_s^2 \right) \leq - \frac{\Lambda(t)}{4} \left( \left\| \tilde{d} \right\|_s^2 + \left\| \tilde{d} \right\|_s^2 \right) + O(\epsilon^2)
\]

\[
\frac{1}{2} \frac{\partial}{\partial t} \left( \left\| \tilde{d} \right\|_s^2 + \left\| \tilde{d} \right\|_s^2 \right) \leq O(\epsilon^2) e^{\frac{\Lambda(t)}{2}}.
\]
Integrating the above inequality on both sides leads to

\[
\left( \|\hat{d}(t)\|_*^2 + \|d(t)\|_*^2 \right) \leq e^{-\frac{\lambda}{2} t} \left( \|\hat{d}(0)\|_*^2 + \|d(0)\|_*^2 \right) + O(\epsilon^2) \left( 1 - e^{-\frac{\lambda}{2} t} \right).
\]

Since \(\hat{d}(0) = 0\), the above inequality is equivalent to,

\[
\|\hat{d}(t)\|_*^2 \leq e^{-\frac{\lambda}{2} t} \|d(0)\|_*^2 + O(\epsilon^2) \left( 1 - e^{-\frac{\lambda}{2} t} \right).
\]

\[\square\]

### D BFF algorithm for Q-control

#### Algorithm 3 BFF

**Require:** \(\eta\): learning rate

**Require:** \(Q^* (s; \theta) \in \mathbb{R}^{|A|}\): nonlinear function approximation of \(Q\) parameterized by \(\theta\)

**Require:** \(j_{\text{cal}}(s_m, a_m, s_{m+1}; \theta_m) = r(s_{m+1}, s_m, a_m) + \gamma \max_a Q^*(s_{m+1}, a) - Q^*(s_m, a_m; \theta)\)

**Require:** \(\theta_0\): Initial parameter vector

1: \(m \leftarrow 0\)
2: while \(\theta_m\) not converged do
3: \(s_{m+1}' \leftarrow s_m + (s_m + 2 - s_{m+1})\)
4: \(\hat{F}_m \leftarrow -j_{\text{cal}}(s_m, a_m, s_{m+1}; \theta_m)\)
5: \(\theta_{m+1} \leftarrow \theta_m - \eta \hat{F}_m\)
6: \(m \leftarrow m + 1\)
7: end while

#### Algorithm 4 BFF (tabular case)

**Require:** \(\eta\): Learning rate

**Require:** \(Q \in \mathbb{R}^{|S| \times |A|}\): matrix of \(Q(s, a)\) values

**Require:** \(j_{\text{cal}}(s_m, a_m, s_{m+1}) = r(s_{m+1}, s_m, a_m) + \gamma \max_a Q(s_{m+1}, a) - Q^*(s_m, a_m)\)

1: \(m \leftarrow 0\)
2: while \(Q^*\) not converged do
3: \(s_{m+1}' \leftarrow s_m + (s_m + 2 - s_{m+1})\)
4: \(F_m \leftarrow 0 \in \mathbb{R}^{|S| \times |A|}\)
5: \(F_m(s_m, a_m) \leftarrow -j_{\text{cal}}(s_m, a_m, s_{m+1})\)
6: \(a^*_{m+1} \leftarrow \arg\max_a Q(s_{m+1}', a)\)
7: \(F_m(s_{m+1}', a^*_{m+1}) \leftarrow \gamma j_{\text{cal}}(s_m, a_m, s_{m+1})\)
8: \(Q^* \leftarrow Q^* - \eta F_m\)
9: \(m \leftarrow m + 1\)
10: end while

### E Multiple future steps, tabular case

Algorithm 5 details the multiple-future-step version of BFF for the tabular control case.
Algorithm 5 BFF (Multiple-future-step, tabular case)

Require: \( \eta \): Learning rate
Require: \( Q \in \mathbb{R}^{[S] \times [A]} \): matrix of \( Q(s, a) \) values
Require: \( \{ \alpha_k \}_{k=1}^n \)
Require: \( j^{\text{arr}}(s_m, a_m, s_{m+1}) = r(s_{m+1}, s_m, a_m) + \gamma \max_a Q(s_{m+1}, a) - Q^\pi(s_m, a_m) \)

1: \( m \leftarrow 0 \)
2: while \( Q^\pi \) not converged do
3: \( \hat{F}_m \leftarrow 0 \in \mathbb{R}^{[S] \times [A]} \)
4: for \( k = 1 \ldots n \) do
5: \( s'_{m+k} \leftarrow s_{m+k-1} + (s_{m+k+1} - s_{m+k}) \)
6: \( \hat{F}_m(s_m, a_m) \leftarrow \hat{F}_m(s_m, a_m) - j^{\text{arr}}(s_m, a_m, s_{m+1}) \)
7: \( a^*_{m+k} \leftarrow \arg\max_a Q(s'_{m+k}, a) \)
8: \( \hat{F}_m(s'_{m+k}, a^*_{m+k}) \leftarrow \hat{F}_m(s'_{m+k}, a^*_{m+k}) + \alpha_k \gamma j^{\text{arr}}(s_m, a_m, s_{m+1}) \)
9: end for
10: \( Q^\pi \leftarrow Q^\pi - \eta \hat{F}_m \)
11: \( m \leftarrow m + 1 \)
12: end while

F Experiment details

F.1 Tabular evaluation case

The training procedure is as follows. We generate a long trajectory of length \( T = 10^7 \) from the MDP dynamics using a fixed policy \( \pi(a|s) = \frac{1}{2} + a \sin(s) \). We use a learning rate of \( \eta = 0.5 \) and a batch size of 50 for each of the methods. We find the exact matrix \( Q^* \) by first forming a Monte Carlo estimate of the transition matrix \( P \) based on 50,000 repetitions per entry, then forming the expected reward vector \( R \) and solving the Bellman equation based on this estimate for \( P \).

F.2 Tabular control case

We find the exact \( Q^* \) by running US on a trajectory of length \( 10^8 \) with batch size 1000 and learning rate 0.5 to obtain an approximation \( Q^1 \). We then refine \( Q^1 \) by training via US on a trajectory of length \( 10^7 \) with batch size 10000 and a learning rate of 0.1 to obtain the true \( Q^* \). We confirm the correctness of \( Q^* \) via Monte Carlo (not shown).

We test each of the methods (US, SC, and BFF) on a trajectory of length \( 5 \times 10^7 \) with a learning rate of 0.5 and a batch size of 100. The results are shown in Figure 5. BFF outperforms SC by a wide margin and has performance comparable to US. Using a greater number of future steps to approximate the BFF gradient improved its performance marginally.

![Figure 5: Results of each method for Q-control in the tabular case. SC is unable to learn an accurate approximation for Q, while BFF’s performance is almost indistinguishable from US. Using 5 future steps to compute the BFF approximation helped improve its performance slightly.](image)

F.3 The PD algorithm

The primal dual method transfer the minimization problem to a mimimax problem, that is,

\[
\min_\theta \max_\omega \mathbb{E}(s_m, a_m) [\delta(s_m, a_m; \theta) y(s_m, a_m; \omega) - \frac{1}{2} y(s_m, a_m; \omega)^2]
\]
Therefore SGD applied to the above minimax problem does not have the double sampling problem anymore. The algorithm updates the parameters in the following way,

\[
\omega_{k+1} = \omega_k + \beta(s_m, a_m; \theta_k)) \nabla_\omega y(s_m, a_m; \omega_k) - y(s_m, a_m; \omega_k) \nabla_\omega y(s_m, a_m; \omega_k);
\]

\[
\theta_{k+1} = \theta_k - \eta(\nabla_\theta \delta(s_m, a_m; \theta_k)y(s_m, a_m; \omega_{k+1})).
\]

We usually set \(y(s, a; \omega)\) to be the same model as \(Q(s, a; \theta)\).

### F.4 Q-evaluation, continuous case

We use a neural network with two hidden layers to approximate \(Q\). Each hidden layer has 50 neurons. The activations are \(\cos(x)\) for the hidden layers and identity for the output layer.

The training procedure is similar to the tabular case. We generate a trajectory of length \(10^6\) and run BFF, SC, and US with batch size \(M = 50\) and learning rate \(\eta = 0.1\). We also train via PD with \(\beta = \eta = 0.1\) and all other hyperparameters identical. We compute the exact \(Q\) by running US on a trajectory of length \(10^7\).

As discussed previously, PD has unstable performance. In Figure 6 we plot the results of the 10 different runs of PD. We compare to the error of BFF and the exact \(Q\) function for reference.

![Figure 6: Results for each of the 10 runs of PD for fixed-policy Q-evaluation. We include the error plot for BFF as well as the exact Q function for reference. The shapes of the Q function learned by PD are quite unstable with large variation between runs.](image)

### F.5 Q-control, continuous case

The training procedure is identical to the continuous Q-evaluation experiment (with the same hyperparameters, trajectory length, etc.), but we generate the trajectory with the fixed behavior policy which samples an action uniformly at random. PD is again unstable and we report the results of the 10 runs below.

![Figure 7: Results for each of the 10 runs of PD for Q-control. We include the error plot for BFF as well as the exact Q function for reference. As in the case of Q-evaluation, there is large variation in the quality of the learned Q function.](image)

### F.6 CartPole

We approximate \(Q\) with a neural network with a single hidden layer of size 100. The hidden layer has ReLU activations. For both BFF and sample-cloning, we train using Adam with the default settings for \(\beta_1\) and \(\beta_2\) (\(\beta_1 = 0.9, \beta_2 = 0.999\)). For all of the methods, we use batch size 50 and experience replay storing the 10,000 most recent experiences in the training trajectory. We train for 200 episodes. We also use an \(\epsilon\)-greedy approach to generate the trajectory. Initially, we set \(\epsilon = 1\) (so the agent acts completely randomly at the beginning of training), and decay \(\epsilon\) by 0.99 after each parameter update.
We stop decaying $\epsilon$ when it reaches $0.1$, so there is always some randomness in our training actions to prevent getting stuck on an ineffective policy.

For the PD algorithm, We tried fixed values for $\beta$ and $\eta$, as well as decaying $\beta$ and $\eta$ with different starting values and with the decay recommended in Wang et al. [2017]. The results in Figure 3 have $\beta_k = 0.1 \times k^{-3/4}$ and $\eta_k = 0.1 \times k^{-1/2}$, where $\beta_k$ and $\eta_k$ denote the parameters used for the $k$-th step.