Some Recent Advances in Loop Quantum Cosmology

Abhay Ashtekar
E-mail: ashtekar@gravity.psu.edu
Institute for Gravitation and the Cosmos & Physics Department, The Pennsylvania State University, University Park PA 16802, USA

Abstract. In my talk I discussed three recent advances in loop quantum cosmology: i) Path integral formulation and its WKB approximation; 2) Cosmological spin foams and lessons they provide; and 3) Probability of a slow roll inflationary phase compatible with the 7 year WMAP data. In addition to presenting an overview, this discussion also provides the necessary background for a number of talks in the parallel sessions.

1. Path Integrals and the WKB approximation in LQC
The key difference between the Wheeler DeWitt theory (WDW ) theory and Loop Quantum Cosmology (LQC) is that, thanks to the quantum geometry effects inherited from Loop Quantum Gravity, LQC has a novel, in-built repulsive force. While it is completely negligible when curvature is less than, say, 1% of the Planck scale, it grows dramatically once curvature becomes stronger, overwhelming the classical gravitational attraction and causing a quantum bounce that resolves the big-bang singularity. (For details, see e.g. [1]). From a path integral viewpoint (see, e.g., [2]), on the other hand, this stark departure from classical solutions seems rather surprising at first. For, in the path integral formulation quantum effects usually become important when the action is small, comparable to the Planck’s constant $\hbar$, while the Einstein-Hilbert action along classical trajectories that originate in the big-bang is generically very large. Thus, there is an apparent conceptual tension. Recent detailed analyses [3, 4] have resolved this issue in the $k=0$ and $k=1$ Friedman Lemaître Robertson Walker (FLRW) model. I will summarize the main ideas and results using the $k=0$ model.

1.1. Strategy
Since LQC uses a Hilbert space framework, it is most natural to return to the original derivation of path integrals, where Feynman began with the expressions of transition amplitudes in the Hamiltonian theory and reformulated them as an integral over all kinematically allowed paths [5]. But in non-perturbative quantum gravity, there is a twist: at a fundamental level, one deals with a constrained system without external time whence the notion of a transition amplitude does not have an a priori meaning. It is replaced by an extraction amplitude—a Green’s function which extracts physical quantum states from kinematical ones and also provides the physical inner product between them. (For a discussion in the cosmological context, see, e.g., [6, 7, 8, 9]). If the theory can be deparameterized it inherits a relational time variable, and then the extraction amplitude should reduce to the standard transition amplitude with respect to that time. In LQC with a massless scalar field, a natural deparametrization is indeed available [10, 11, 12, 1]
and this expectation is borne out [7]. However, more generally —e.g. if one were to introduce a potential for the scalar field— it is difficult to find a global time variable. Since the conceptual tension between the LQC results and the path integral intuition is generic, it is best not to have to rely heavily on deparametrization. Therefore, in [3] the comparison was carried out in the timeless framework. This is also the setting of spin foams, the path integral approach to full LQG; see e.g. [13]. The idea then is to start with the expression of the extraction amplitude in the Hilbert space framework of LQC, cast it as a path integral, and re-examine the tension between the path integral intuition and the singularity resolution.

Recall that solutions to the quantum constraint equation, as well as inner product between them, can be obtained through a group averaging procedure [14, 15, 16]. Let us work in the basis $|\nu, \phi\rangle$ in which the volume and the scalar field operators, $\hat{V}$ and $\hat{\phi}$, on the kinematical Hilbert space $\mathcal{H}_{\text{kin}}^{\text{total}}$ are diagonal. The extraction amplitude $\mathcal{E}(\nu, \phi; \nu', \phi')$ is a Green’s function that results from this group averaging:

$$\mathcal{E}(\nu, \phi; \nu', \phi_i) := \int_{-\infty}^{\infty} d\alpha \bra{\nu, \phi}| e^{i\alpha\hat{C}} \ket{\nu_i, \phi_i},$$

(1)

where $\hat{C} = -\hbar^2 (\partial_\phi^2 + \hat{C}_{\text{grav}})$ is the full Hamiltonian constraint and $\alpha$ is a parameter (with dimensions $[L^{-2}]$). The integral averages the ket (or the bra) over the group generated by the constraint. Since the integrated operator is heuristically ‘$\delta(\hat{C})$,’ the amplitude $\mathcal{E}(\nu_f, \phi_f; \nu_i, \phi_i)$ satisfies the Hamiltonian constraint in both of its arguments. Consequently, it serves as a Green’s function that extracts physical states $\Psi_{\text{phys}}(\nu, \phi)$ in $\mathcal{H}_{\text{phys}}$ from kinematical states $\Psi_{\text{kin}}(\nu, \phi)$ in $\mathcal{H}_{\text{kin}}^{\text{total}}$ through a convolution

$$\Psi_{\text{phys}}(\nu, \phi) = \sum_{\nu'} \int d\phi' \mathcal{E}(\nu, \phi; \nu', \phi') \Psi_{\text{kin}}(\nu', \phi').$$

(2)

This is why $\mathcal{E}$ is referred to as the extraction amplitude. Since it is the group averaging Green’s function, $\mathcal{E}$ also enables us to write the physical inner product in terms of the kinematical:

$$(\Phi_{\text{phys}}, \Psi_{\text{phys}}) := \sum_{\nu, \nu'} \int d\phi d\phi' \mathcal{E}(\nu, \phi; \nu', \phi') \Psi_{\text{kin}}(\nu', \phi').$$

(3)

Thus, in the ‘timeless’ framework without any deparametrization, all the information in the physical sector of the quantum theory is neatly encoded in the extraction amplitude $\mathcal{E}(\nu, \phi; \nu', \phi')$. Therefore to relate the Hilbert space and path integral frameworks, it suffices to recast the expression (1) of this amplitude as a path integral.

1.2. Path integral for the extraction amplitude

Let us begin by recalling the procedure Feynman used to arrive at the path integral expression of the transition amplitude in quantum mechanics. He began with the Hamiltonian framework, wrote the unitary evolution as a composition of $N$ infinitesimal ones, inserted a complete basis between these infinitesimal evolution operators to arrive at a ‘discrete time’ path integral, and finally took the limit $N \to \infty$. In the timeless framework, we need to adapt this procedure to the extraction amplitude $\mathcal{E}(\nu_f, \phi_f; \nu_i, \phi_i)$, using $e^{i\alpha\hat{C}}$ in the integrand of Eq. (1) in place of the evolution operator, and then performing the $\alpha$ integration in a final step.

More precisely, the integrand of (1) can be thought of as a matrix element of a fictitious evolution operator $e^{i\alpha\hat{C}}$. One can regard $\alpha\hat{C}$ as playing the role of a (purely mathematical) Hamiltonian, the evolution over an unit time interval. We can then decompose this fictitious evolution into $N$ evolutions of length $\epsilon = 1/N$ and insert a complete basis at suitable intermediate steps to write the extraction amplitude as a ‘skeletonized’ phase space path integral and finally
take the limit $N \to \infty$ (or $\epsilon \to 0$). As is standard in the path integral literature, the result can be expressed as a formal infinite dimensional integral,

$$\mathcal{E}(\nu, \phi; \nu_i, \phi_i) = \int \mathcal{D}\nu(T) [\mathcal{D}b(T)] [\mathcal{D}p(\phi)(T)] [\mathcal{D}\phi(T)] e^{iS}$$  \hspace{1cm} (4)

where $b$ and $p(\phi)$ are momenta conjugate to $\nu$ and $\phi$ respectively. A detailed calculation [3] shows that the action $S$ in this expression is given by

$$S = \int_0^1 dT \left( p(\phi) \dot{\phi} - \frac{1}{2} b^2 - \alpha \left( p^2(\phi) - 3\pi G \nu^2 \frac{\sin^2 \lambda b}{\lambda^2} \right) \right)$$  \hspace{1cm} (5)

where the prime denotes derivative with respect to the (fictitious) time $T$ and $\lambda^2 = 4\sqrt{3}\pi \gamma \ell_{Pl}^2$ is the area gap of LQC. Note that the final integration is over all paths in the classical phase space, including the ones which go through the point $\nu = 0$, normally regarded as the singularity. Therefore, the tension between the Hamiltonian LQC and the path integral formulation is brought to forefront: How can we see the singularity resolution in the path integral setting? The answer is that the paths are not weighted by the standard Einstein-Hilbert action but by a ‘polymerized’ version (5) of it which still retains the memory of the quantum geometry underlying the Hamiltonian theory. As we will see, this action is such that a path going through the classical singularity has negligible contribution whereas bouncing trajectories give the dominant contribution.

Note that the new action (5) retains memory of quantum geometry through the area gap of LQC which depends on $\hbar$. This means that the Einstein-Hilbert action itself has received quantum corrections. This may seem surprising at first. However, this occurs also in some familiar examples if one systematically arrives at the path integral starting from the Hilbert space framework. Perhaps the simplest such example is that of a non-relativistic particle on a curved Riemannian manifold for which the standard Hamiltonian operator is simply $H = -(\hbar^2/2m)\nabla_\alpha \nabla_\alpha b$. When the quantum dynamics generated by this $H$ is recast as a path integral, the integrand turns out to be $e^{iS_{corr}}$ where $S_{corr} = S_{cl} + \int dt (\frac{\lambda^2}{2m} R)$ is a quantum corrected action [17]. Note that the extrema of this action are not the geodesics one obtains in the classical theory but rather particle trajectories in a $\hbar$-dependent potential; the two can be qualitatively different. Thus, this elementary example shows that one cannot always assume that the correct quantum theory will result from the path integral, where each path is weighted just by $e^{iS_{cl}}$.

1.3. The steepest descent approximation

We can now use the steepest descent approximation to understand the singularity resolution in LQC from a path integral perspective. However there are two subtleties that need to be addressed. First, the standard WKB analysis refers to unconstrained systems and we have to adapt it to the constrained one, replacing the Schrödinger equation with the quantum constraint. This is not difficult (see, e.g., sections 3.2 & 5.2 of [18], or Appendix A in [3]). However, as in the standard WKB approximation for the transition amplitude, the procedure assumes that the action that features in the path integral has no explicit $\hbar$ dependence. The second subtlety arises because, in our case, the action does depend on $\hbar$ through $\lambda^2 \sim \gamma^2 \hbar$. Therefore to explore the correct semi-classical regime of the theory now one has to take the limit $\hbar \to 0$, keeping $\gamma^2 \hbar$ fixed [3]. One can then evaluate the action in the saddle point approximation. One finds that it is precisely the bouncing solutions—rather than the classical FLRW metrics—that dominate the WKB approximation.

To summarize, in the timeless framework all the physical information is contained in the extraction amplitude which reduces to the standard transition amplitude if a global
deparametrization can be found. Following Feynman, one can start with the Hilbert space expression of the extraction amplitude and recast it as a phase space path integral. Quantum geometry effects of LQC leave their trace on the weight associated with each path: The action functional is modified. This in turn implies that, in the WKB approximation, the extraction amplitude is dominated by universes that undergo a bounce. Thus, from the LQC perspective, it would be incorrect to simply define the theory starting with the Einstein-Hilbert action because this procedure completely ignores the quantum nature of the underlying Riemannian geometry. For a satisfactory treatment of ultraviolet issues such as the singularity resolution, it is crucial that the calculation retains appropriate memory of its quantum nature. Indeed, this is why in the spin foam models one sums over quantum geometries, not smooth metrics. For further details, see [1, 3, 8]

2. Cosmological Spin foams

2.1. General Setting

Spin foams were discussed by several speakers at this conference. Let us recall the key ideas. This is a sum over histories approach whose distinguishing feature relative to the older path integral approaches such as those of [2] is that now the sum is over specific quantum geometries that originate in loop quantum gravity. In the simplest version one can consider a simplicial decomposition of a 4-manifold of interest and consider the 2-complex dual to it. Each vertex of the 2-complex lies in a simplex, each face is labeled by a spin and each edge by an ‘intertwiner’. Thus, a 2-complex can be thought of as the ‘time evolution’ of spin a network, with a genuine change/evolution occurring at the vertices. As explained in section 1, the basic object determining dynamics is the extraction amplitude which, in full quantum gravity, is written out as a sum over partial amplitudes, each associated with a 2-complex. Since the 2-complex joining the ‘in’ and the ‘out’ spin network is characterized by the number of vertices it contains, this formal expression of the extraction amplitude is called the vertex expansion.

Over the last three years, there have been significant advances in the area of spin foams (see, e.g. [13, 19]). However, key issues still remain: Can the vertex expansion be summed to yield a meaningful answer? Does it provide the full extraction amplitude or does one have to take a ‘continuum limit’ of the 2-complex as in the standard Regge approach? Is the vertex amplitude necessarily local or can there be Planck scale non-localities that become invisible on coarse graining? What should the summand/integrand reduce to in the classical limit? There is a debate on whether the integrand should yield $e^{iS}$ where $S$ is the Einstein-Hilbert action or $\cos S$. This is related to the question: What paths should one sum over? Should they be restricted to be suitably time oriented [20]?

2.2. Insights from LQC

LQC models provide a particularly convenient setting to probe these conceptual issues [6, 7, 8, 9, 21]. Because the Hamiltonian theory in these models is fully under control, as in section 1, we can again use the procedure followed by Feynman [5] to rewrite the well-defined extraction amplitude as a sum over histories. To obtain a sum analogous to that used in spin foams, however, one needs to discard the 2-complex as in the configuration space, without introducing a further sum over momentum basis in the intermediate steps as was done in section 1.

For simplicity let us consider the FLRW model where, in the timeless framework, the configuration variables are volume $\nu$ of the fiducial cell and the scalar field $\phi$. (For details on the Bianchi I model, see [21]). Then the extraction amplitude $\mathcal{E}(\nu_f, \phi_f; \nu_i, \phi_i)$ can be expressed as
\[
\mathcal{E}(\nu_f, \phi_f; \nu_i, \phi_i) = \int d\alpha \langle \nu_f, \phi_f | e^{i\alpha \hat{C}} | \nu_i, \phi_i \rangle \\
= \int d\alpha \langle \nu_f, e^{i\alpha \hat{C}_{\text{grav}}} | \nu_i \rangle \langle \phi_f | e^{i\alpha \hat{C}_{\text{mat}}} | \phi_i \rangle \\
= \int d\alpha \ A_{\text{grav}}(v_f, v_i; \alpha) \ A_{\phi}(\phi_f, \phi_i; \alpha)
\] (6)

It is completely straightforward to calculate the matter amplitude \( A_{\phi} \). Mathematically, we can regard the gravitational amplitude \( A_{\text{grav}} \) as the ‘transition amplitude’ between \( |v_i \rangle \) and \( |v_f \rangle \) in a ‘unit time interval’, caused by a ‘Hamiltonian operator’ \( \alpha \hat{C}_{\text{grav}} \). Although ‘time’ and the ‘Hamiltonian’ are mathematical constructs without direct physical significance, this strategy lets us follow, step by step, Feynman’s construction [5] to express \( A_{\text{grav}} \) as a path integral. The resulting sum can be rearranged by grouping together all paths in which there are precisely \( M \) volume transitions between \( \nu_i \) and \( \nu_f \) in ‘unit time interval’. Finally, the integral over \( \alpha \) can be carried out explicitly to express the full extraction amplitude \( \mathcal{E}(\nu_f, \phi_f; \nu_i, \phi_i) \) as a sum over amplitudes \( A(v_M, \ldots, v_0; \phi_f, \phi_i) \) associated with quantum histories in which the universe makes ordered transitions\(^1 \) \( \nu_1 \to \nu_2 \to \ldots \to \nu_{M-1} \to \nu_f \) at increasing but otherwise arbitrary values of \( \phi \in (\phi_f, \phi_i) \):

\[
\mathcal{E}(\nu_f, \phi_f; \nu_i, \phi_i) = \sum_{M=0}^{\infty} \left[ \sum_{v_{M-1}, \ldots, v_1} A(v_M, \ldots, v_0; \phi_f, \phi_i) \right] = \sum_{M=0}^{\infty} A(M). 
\] (7)

Here in the last step \( A(M) \) denotes the contribution to the extraction amplitude obtained by summing over all quantum histories in which there are precisely \( M \) volume transitions between the initial and the final configurations. This is the analog of the \( M \)th term in the spin foam vertex expansion obtained by summing over all colorings for a given 2-complex with precisely \( M \) vertices. (The analogy becomes even closer when anisotropies are included [21, 1].) These partial amplitudes are explicitly given by matrix elements of the gravitational part \( \hat{C}_{\text{grav}} \) of the quantum Hamiltonian constraint.

2.3. Lessons

This analysis in the cosmological models has shed light on several issues that have been open in the spin-foam models. First, since the Hamiltonian theory is fully under control, in these models one can start with a well-defined expression of the extraction amplitude \( \mathcal{E}(\nu_f, \phi_f; \nu_i, \phi_i) \) which we know to be an exact (i.e. not just a formal) expression from the Hamiltonian theory. Thus we are simply asking if this correct expression can be recast as a ‘vertex expansion’. One finds that this is possible and — modulo one precise assumption that an infinite sum can be interchanged with an integral — the sum is convergent. In particular, since the histories summed over are ‘quantum’, the allowed values of \( \nu \) are discrete and remain discrete in the final sum; a continuum limit is not necessary. This provides a strong support for the general paradigm underlying the spin foam program. Of course in the cosmological models one freezes out all but a finite number of degrees of freedom whence new difficult issues will arise in testing convergence in the full theory. But the analysis supports the underlying conceptual viewpoint that, thanks to quantum geometry, a continuum limit is not necessary.

There are also a number of other insights. Since the scalar field serves as a good clock in the cosmological models, it can be regarded as a relational time variable to deparameterize the classical as well as the quantum theory. In the deparameterized quantum description, we have a Hamiltonian operator \( \hat{H} \) generating evolution. Therefore, we can construct a path integral following Feynman [5] not just mathematically but physically. The result is the transition

\(^1\) Dynamics of LQC implies that the relevant part of the spectrum of the volume operator is discrete; all \( \nu \) that feature in these transitions are given by \( 4n \nu_o \) where \( \nu_o \) is a fixed number.
amplitude \( \langle \nu_f | e^{i\hat{H}(\phi_f - \phi_i)} | \nu_i \rangle \) for the universe to pass from a initial state \(| \nu_i \rangle\) at time \( \phi_i \) to the final state \(| \nu_f \rangle\) at time \( \phi_f \). This transition amplitude in the deparameterized theory can be shown to be exactly equal to the physical inner product \( \langle [\nu_f, \phi_f] | | [\nu_i, \phi_i] \rangle \) between physical states \([\nu_f, \phi_f]\) and \([\nu_i, \phi_i]\) \(^{[7]}\). Thus, one of the implicit assumptions on the relation between the sum over histories in the timeless and deparameterized frameworks is borne out in detail in these models. Secondly, the analysis has also provided a perturbative expansion for the extraction amplitude a la group field theory \(^{[22]}\) and shown in detail that an early suggestion of Oriti that the coupling constant in the group field theory may be related to the cosmological constant is realized in a precise fashion \(^{[7]}\). Furthermore, it is possible to spell out the precise meaning of the approximation in which one truncates the sum just to a finite number of terms.

Finally, the analysis also suggests directions for future work in spin foams. First, in the cosmological context, the correct extraction amplitude of the Hamiltonian theory is recovered in the spin foam type expansion only if one restricts the sum to paths which satisfy an appropriate positive frequency requirement. When this is done, in the classical limit the amplitude is given by \( e^{iS} \) rather than \( \cos S \). This suggests that it may be necessary to restrict the sum in the general spin foam program to quantum histories that are ‘time oriented’ in a suitable sense. For further details, see \(^{[20, 7, 23]}\).

3. Probability of inflation

3.1. The question

The inflationary paradigm has been extremely successful in accounting for the observed inhomogeneities in the CMB which serve as seeds for the subsequent formation of the large scale structure in the universe. Consequently, it is widely regarded as the leading candidate to describe the very early universe. However, it faces two conceptual issues. First, Borde, Guth and Vilenkin \(^{[24]}\) have shown that the inflationary space-times are necessarily past incomplete; even with eternal inflation one cannot avoid the initial big-bang. In LQC, on the other hand, thanks to the quantum geometry effects, the singularity is resolved and inflationary space-times are past complete.

The second issue is that of ‘naturalness.’ While a given theory —such as general relativity— may admit solutions with an appropriate inflationary phase, is the occurrence of such a phase generic, or, does it require a careful fine tuning? For definiteness, let us suppose we have an inflaton \( \phi \) with a quadratic potential, \( V(\phi) = (1/2)m^2\phi^2 \). Then the WMAP data \(^{[25]}\) provides us with a remarkably detailed picture of the conditions at the onset of inflation \(^{[26]}\). More precisely, the data refers to the time \( t(k_*) \) at which a reference mode \( k_* \) used by WMAP exited the Hubble horizon in the early universe. (Today, this mode has a wave-length about 12\% of the radius of the Hubble horizon.) Within error bars of less than 4.5\%, the data tells us that the field configurations were:

\[
\begin{align*}
\phi(t(k_*)) &= \pm 3.15 \text{ } m_{\text{Pl}}, \\
\dot{\phi}(t(k_*)) &= \mp 1.98 \times 10^{-7} \text{ } m_{\text{Pl}}^2, \\
H(t(k_*)) &= 7.83 \times 10^{-6} \text{ } m_{\text{Pl}},
\end{align*}
\]

where, the dot denotes derivative with respect to the proper (or cosmic) time and \( H = \dot{a}/a \) is the Hubble parameter \(^{[26]}\). Thus, for the quadratic potential, the 7 year WMAP data requires the dynamical trajectory of the universe to have entered a tiny neighborhood of the phase-space point given by (8). One’s first reaction would be that this condition would be met only by a very small fraction of all dynamical trajectories. If so, a priori the required inflationary phase would seem very implausible and the theory would be left with a heavy burden of ‘explaining’ why it actually occurred.

3.2. The Liouville measure

Issue of ‘naturalness’ has, of necessity, a subjective element and one could just say that our universe simply happened to pass through this tiny region of phase space and, since we have
only one universe, the issue of likelihood is irrelevant. However, the broader community did not adopt this viewpoint. Rather, it has sought to sharpen the question of naturalness by introducing a measure on the space of solutions of the given theory: the required probability is then given by the fractional volume occupied by those solutions which do pass through configurations specified by the WMAP data at some time during their evolution.

To find the measure, the following general strategy was introduced over twenty years ago [27, 28, 29]. Recall that solutions to the field equations are in 1-1 correspondence with phase space trajectories, and the natural Liouville measure $d\mu_L$ on the phase space is preserved by the dynamical evolution. On the one hand, this measure is natural because it is constructed using just the phase space structure. On the other hand, precisely for the same reason, it does not encode additional information that may be important for the physics of the specific system. Therefore, it only provides a priori probabilities, i.e. 'bare' estimates. Further physical input can and should be used to provide sharper probability estimates and a more reliable likelihood. However, a priori probabilities themselves can be directly useful if they are very low or very high. In these cases, it would be an especially heavy burden on the fundamental theory to come up with the physical input that significantly alters them.

The question of probability of inflation along these lines has received a considerable attention in the literature (see for eg. [32, 33, 30, 31, 34, 36, 26, 35]). In the general relativity literature, conclusions have been vastly different: For the slow roll associated with configurations (8)), they range from the probability being close to 1 [30] to its being exponentially suppressed [31] (by factors of $\sim e^{-195}$). It turns out that these vast differences arise because procedure is intrinsically ambiguous because of the big bang singularity. As I will now sketch the ambiguity can be naturally resolved in LQC by working with the initial data at the big bounce [26, 35].

3.3. General relativity versus effective LQC

The subtleties involved can be summarized as follows. It turns out that the Liouville measure of the full phase space is infinite. Recall that, to calculate a priori probabilities one has to compute the relative volumes occupied by regions of interest in the phase space. If one were to proceed directly, this amounts to comparing one infinity with another, whence the answer is ambiguous. However, this infinity is physically spurious because it is associated with the gauge freedom $a \rightarrow \lambda a$ in rescaling the scale factor by a positive number $\lambda$. Thus, given a solution $(a(t), \phi(t))$ to the field equations, the solution $(\lambda a(t), \phi(t))$ is mathematically distinct but cannot be physically distinguished from the original one. Under this rescaling, both the symplectic structure and hence the Liouville volume is rescaled (but so is the Hamiltonian with the consequence that the Hamiltonian vector field is left invariant). Therefore, the Liouville measure does not project down to the quotient space. To remove this spurious infinity, then, one has to gauge fix. In the space-time picture, this can be interpreted as working with the initial data $(\phi, \dot{\phi})$ at a fixed time instant (fixing $a$, say $a = 1$, there). In general relativity, the only natural ‘instant’ is the big bang where everything diverges. Thus, there is no natural gauge fixing. In LQC, by contrast, one can fix gauge by working at the big bounce where all fields are regular. Then it is possible to ask and answer a well-defined and physically interesting question: What is the fractional volume of the initial data at the big bounce which, when evolved, yields a dynamical trajectory that will eventually pass through the tiny region of the phase space selected by the WMAP data? To answer it, one needed not only analytical calculations that address the conceptual subtleties correctly, but also non-trivial numerical simulations, carried out by Sloan [26]. They showed that the probability is very close to one; greater than $1 - 3 \times 10^{-6}$. In this precise sense, provided there is a scalar field with a suitable potential, the desired inflationary phase occurs almost always in LQC. The analysis also provides detailed information on dynamics between the bounce and the onset of the slow roll compatible with WMAP which is now being used to investigate the seeds of structure formation within LQC.
Let us summarize. The recent investigations [26, 35] asked a sharp question: given a quadratic potential, what is the a priori probability of obtaining slow roll inflation that is compatible with the 7 year WMAP data? This is a much more refined than asking just for a the probability of obtaining a certain number of slow roll e-foldings [30, 31]. To address it unambiguously, one has to take into account certain subtleties associated with the Liouville measure: The procedure requires one to fix a time instant, consider all permissible initial data at that time, and ask for the fractional $d\mu_L$-volume of the subset of these configurations which, upon evolution, enter the small region of phase space singled out by WMAP. In general relativity, the only natural choice of the time instant is the big bang where fields of interest diverge. In LQC, there is a natural choice of the initial time—the bounce time. During the passage from the big bounce to the onset of the WMAP slow roll, the density decreases by some 12 orders of magnitude. The detailed analysis showed that, in this passage, the trajectories starting out at essentially any initial data at the bounce get funneled into the tiny region of the phase space allowed by the WMAP data. This impressive focussing of the effective LQC trajectories is much more than the qualitative statement that inflationary trajectories are attractors in general relativity.

For further details, see [1, 26, 35].

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