Error bound of critical points and KL property of exponent 1/2 for squared F-norm regularized factorization

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Abstract

This paper is concerned with the squared F(robenius)-norm regularized factorization form for noisy low-rank matrix recovery problems. Under a suitable assumption on the restricted condition number of the Hessian matrix of the loss function, we establish an error bound to the true matrix for the non-strict critical points with rank not more than that of the true matrix. Then, for the squared F-norm regularized factorized least squares loss function, we establish its KL property of exponent 1/2 on the global optimal solution set under the noisy and full sample setting, and achieve this property at its certain class of critical points under the noisy and partial sample setting. These theoretical findings are also confirmed by solving the squared F-norm regularized factorization problem with an accelerated alternating minimization method.

Keywords F-norm regularized factorization · Error bound · KL property of exponent 1/2

1 Introduction

Low-rank matrix recovery problems aim at recovering an unknown true low-rank matrix $M \in \mathbb{R}^{n_1 \times n_2}$ from as few observations as possible, and have wide applications in a host of fields such as statistics, control and system identification, signal and image processing, machine learning, quantum state tomography, and so on (see, e.g., [10,11,14,25]). Generally, when a tight upper estimation, say an integer $\kappa \geq 1$, is available for the rank of $M$, these problems can be formulated as the following rank constrained problem

$$\min_{X \in \mathbb{R}^{n_1 \times n_2}} \left\{ F(X) \text{ s.t. } \text{rank}(X) \leq \kappa \right\}$$

(1)
where \( F : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}_+ \) is an empirical loss function. Otherwise, one needs to solve a sequence of rank constrained optimization problems with an adjusted upper estimation for the rank of \( M \). For the latter scenario, one may consider the rank regularized model

\[
\min_{X \in \mathbb{R}^{n_1 \times n_2}} \left\{ F(X) + \lambda \text{rank}(X) \right\}
\]

(2)

with an appropriate \( \lambda > 0 \) to achieve a desirable low-rank solution. Model (1)–(2) reduce to the rank constrained and regularized least squares problem, respectively, when

\[
F(X) = \frac{1}{2} \| A(X) - y \|^2 \quad \forall X \in \mathbb{R}^{n_1 \times n_2},
\]

(3)

where \( A : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^m \) is the sampling operator and \( y \) is the noisy observation from

\[
y = A(M) + \omega.
\]

(4)

Due to the combinatorial property of the rank function, rank optimization problems are NP-hard and it is impossible to seek a global optimal solution with a polynomial-time algorithm. A common way to deal with them is to adopt the convex relaxation technique. For the rank regularized problem (2), the popular nuclear norm relaxation method (see, e.g., [6,11,25]) yields a desirable solution via a single convex minimization problem

\[
\min_{X \in \mathbb{R}^{n_1 \times n_2}} \left\{ F(X) + \lambda \| X \|_* \right\}.
\]

(5)

Over the past decade active research, this method has made great progress in theory (see, e.g., [6,7,20,25]). In spite of its favorable performance in theory, improving computational efficiency remains a challenge. In fact, almost all convex relaxation algorithms for (2) require an economic SVD of a full matrix in each iteration, which forms the major computational bottleneck and restricts their scalability to large-scale problems. Inspired by this, recent years have witnessed the renewed interest in the Burer-Monteiro factorization [3] for low-rank matrix optimization problems. By replacing \( X \) with its factored form \( UV^T \) for \( (U, V) \in \mathbb{R}^{n_1 \times \kappa} \times \mathbb{R}^{n_2 \times \kappa} \) with rank \( M \leq \kappa < \min(n_1, n_2) \), the factorization form of (5) is

\[
\min_{U \in \mathbb{R}^{n_1 \times \kappa}, \, V \in \mathbb{R}^{n_2 \times \kappa}} \left\{ \Phi_{\kappa}(U, \, V) := F(UV^T) + \frac{1}{2} \lambda \left( \| U \|_F^2 + \| V \|_F^2 \right) \right\}.
\]

(6)

Although the factorization form tremendously reduces the number of optimization variables since \( \kappa \) is usually much smaller than \( \min(n_1, n_2) \), the intrinsic bi-linearity makes the factored objective functions nonconvex and introduces additional critical points that are not global optimizers of factored optimization problems. A research line for factored optimization problems focuses on the nonconvex geometry landscape, especially the strict saddle property (see, e.g., [2,13,19,24,39,40]). Most of these works center around the factored optimization forms of problem (1) or their regularized forms with a balance term except [19], in which, under a restricted well-conditioned assumption on \( F \), the authors proved that factorization of \( X^* \) (i.e., \( X^* = UU^T \)), or is a strict saddle point (i.e., the critical point at which the Hessian matrix has a strictly negative eigenvalue). This, along with the equivalence between (5) and (6) (see also Lemma 1 in Appendix C), implies that many local search algorithms such as gradient descent and its variants can find a global optimal solution of (6) with a high probability if the parameter \( \lambda \) is chosen such that (6) has a solution with rank at most \( \kappa \). Another research line considers the (regularized) factorizations of rank optimization problems from a local view and aims to characterize the growth behavior of objective functions around the set of global optimal solutions (see, e.g., [16,23,30,32,36–38]).
For problem (1) associated to noisy low-rank matrix recovery, some researchers are also interested in the error bound to the true matrix $M$ for the local minima of the factorization form or its regularized form with a balance term. For example, for the noisy low-rank positive semidefinite matrix recovery, Bhojanapalli et al. [2] achieved the error bound for any local optimizer of the factorized exact or over-parameterization (i.e., $\kappa \geq \text{rank}(M)$) under a RIP condition on the sampling operator; and for a general noisy low-rank matrix recovery, Zhang et al. [35] established the error bound for any local optimizer of the factorized exact-parameterization with a balanced regularized term under a restricted strong convexity and smoothness condition of the loss function. However, there are few works to discuss error bounds for the critical point of the factorization associated to the rank regularized problem (2) or its convex relaxation (5) except [4] in which, for noisy matrix completion, the nonconvex Burer-Monteiro approach is used to demonstrate that the convex relaxation approach achieves near-optimal estimation errors.

This work is concerned with the error bound for the critical point of the nonconvex factorization (6) with $\kappa \geq \text{rank}(M)$ and the KL property of exponent $1/2$ of $\Phi_1\lambda$ associated to $\kappa = \text{rank}(M)$ under the noisy setting. Specifically, under a suitable assumption on the restricted condition number of the Hessian matrix $\nabla^2 F$, we derive an error bound to the true $M$ for those non-strict critical points with at most more than $\text{rank}(M)$, which is demonstrated to be optimal by using the exact characterization of global optimal solutions in [36] for the ideal noiseless and full sampling setting. Different from [4], our error bound result is obtained for a general smooth loss function by adopting a deterministic rather than a probability analysis technique. In addition, for the least squares loss function, we establish the KL property of exponent $1/2$ of $\Phi_1\lambda$ associated to almost all $\lambda > 0$ over its global minimizer set under the noisy and full sample setting, and also achieve this property of $\Phi_1\lambda$ at its certain class of critical points under the noisy and partial sample setting. This extends the result of [36, Theorem 2] to the noisy setting. Together with the strict saddle property of $\Phi_1\lambda$ established in [19], and the result of [19], this means that for problem (6) with the least squares loss function in the full sampling, many first-order methods with a good starting point can find a global optimal solution in a linear convergence rate with a high probability, provided that the parameter $\lambda$ is chosen such that (5) has an optimal solution with rank at most $\kappa$. Hence, it partly improves the convergence analysis results of the alternating minimization methods proposed in [15,25] for solving this class of problems. Although Li et al. mentioned in [19] that the explicit convergence rate for certain algorithms in [12,29] can be obtained by extending the strict saddle property with the similar analysis in [40], but to the best of our knowledge, there is no strict proof for this, and, the analysis in [40] is tailored to the factorization form with a balanced regularization term.

2 Notation and preliminaries

Throughout this paper, $\mathbb{R}^{n_1 \times n_2}$ represents the vector space of all $n_1 \times n_2$ real matrices, equipped with the trace inner product $\langle X, Y \rangle = \text{trace}(X^T Y)$ for $X, Y \in \mathbb{R}^{n_1 \times n_2}$ and its induced Frobenius norm, and we stipulate $n_1 \leq n_2$. The notation $\mathbb{O}^{n_1 \times n_2}$ denotes the set of matrices with orthonormal columns, and $\mathbb{O}^{n_1}$ stands for $\mathbb{O}^{n_1 \times n_1}$. Let $I$ and $e$ denote an identity matrix and a vector of all ones, respectively, whose dimensions are known from the context. For a matrix $X \in \mathbb{R}^{n_1 \times n_2}$, denotes $X_+ := \max(0, X)$, and $\sigma(X)$ denotes the singular value vector of $X$ arranged in a nonincreasing order, $\sigma^k(X)$ for an integer $k \geq 1$ means the
vector consisting of the first $\kappa$ entries of $\sigma(X)$, and $\mathbb{O}^{r_1,n_2}(X)$ is the set
\[
\mathbb{O}^{r_1,n_2}(X) := \left\{(P, Q) \in \mathbb{O}^{n_1} \times \mathbb{O}^{n_2} \mid X = P\text{Diag}(\sigma(X))Q^T\right\}
\]
where Diag($z$) represents a rectangular diagonal matrix with $z$ as the diagonal vector. We denote by $\|X\|$ and $\|X\|_*$ the spectral norm and the nuclear norm of $X$, respectively, by $X^\dagger$ the pseudo-inverse of $X$, and by col($X$) the column space of $X$. Let $\mathcal{P}_{on}$ and $\mathcal{P}_{off}$ denote the linear mappings from $\mathbb{R}^{(n_1+n_2) \times (n_1+n_2)}$ to itself, respectively, defined by
\[
\mathcal{P}_{on}\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} , \quad \mathcal{P}_{off}\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix} ,
\]
where $A_{11} \in \mathbb{R}^{n_1 \times n_1}$ and $A_{22} \in \mathbb{R}^{n_2 \times n_2}$. Unless otherwise stated, in the sequel we denote by $M$ the true matrix of rank $r$, and write
\[
\mathcal{E}^* := \left\{(P^* \Sigma^{*1/2} R, Q^* \Sigma^{*1/2} R) \mid (P^*, Q^*) \in \mathbb{O}^{r_1,n_2}(M^*), \ R \in \mathbb{O}^r\right\}
\]
where $\Sigma^* = \text{Diag}(\sigma_1(M), \ldots, \sigma_r(M)) \in \mathbb{R}^{r \times r}$, and $P_1^*$ and $Q_1^*$ are the matrix consisting of the first $r$ columns of $P^*$ and $Q^*$, respectively. With an arbitrary $(U^*, V^*) \in \mathcal{E}^*$, we always write
\[
W^* = (U^*; V^*) \quad \text{and} \quad \hat{W}^* = (U^*; -V^*).\]
For convenience, for a pair $(U, V) \in \mathbb{R}^{n_1 \times k} \times \mathbb{R}^{n_2 \times k}$, we denote by $\mathbb{B}((U, V), \delta)$ the closed F-norm ball of radius $\delta$ centered at $(U, V)$, and by dist$((U, V), \Gamma)$ the F-norm distance of $(U, V)$ from a set $\Gamma \subseteq \mathbb{R}^{n_1 \times k} \times \mathbb{R}^{n_2 \times k}$, and write
\[
W = (U; V) \quad \text{and} \quad \hat{W} = (U; -V).
\]
For a given $U \in \mathbb{R}^{n \times k}$, we also write $U = (U_1; U_2)$ where $U_1$ and $U_2$ are the matrix consisting of the first $r$ rows and the rest $n_1 - r$ rows of $U$. For any $A, B \in \mathbb{R}^{n \times k}$, define
\[
\text{dist}(A, B) = \min_{R \in \mathbb{O}^k} \|A - BR\|_F.
\]

### 2.1 Restricted strong convexity and smoothness

Restricted strong convexity (RSC) and restricted smoothness (RSS) are the common requirement for loss functions when handling low-rank matrix recovery problems (see, e.g., [19–21,35,40]). Now we recall the concepts of RSC and RSS used in this paper.

**Definition 2.1** A twice continuously differentiable function $\Psi : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}$ is said to satisfy the $(r, r)$-RSC of modulus $\alpha$ and the $(r, r)$-RSS of modulus $\beta$, respectively, if $0 < \alpha \leq \beta$ and for any $X, H \in \mathbb{R}^{n_1 \times n_2}$ with $\text{rank}(X) \leq r$ and $\text{rank}(H) \leq r$,
\[
\alpha\|H\|_F^2 \leq \nabla^2\Psi(X)(H, H) \leq \beta\|H\|_F^2 .
\]

For the least squares loss in (3), the $(r, r)$-RSC of modulus $\alpha$ and $(r, r)$-RSS of modulus $\beta$ reduces to requiring the $r$-restricted smallest and largest eigenvalue of $A^*A$ to satisfy
\[
0 < \alpha = \min_{\text{rank}(X) \leq r, \|X\|_F = 1} \|A(X)\|^2 \quad \text{and} \quad \beta = \max_{\text{rank}(X) \leq r, \|X\|_F = 1} \|A(X)\|^2 .
\]

Consequently, the $(r, r)$-RSC of modulus $\alpha = 1 - \delta_r$ along with the $(r, r)$-RSS of modulus $\beta = 1 + \delta_r$ for some $\delta_r \in (0, 1)$ reduces to the RIP condition for the operator $A$. Thus, from...
The least squares loss associated to many types of random sampling operators satisfies this property with a high probability. In addition, from the discussions in [19,39], some loss functions definitely have this property such as the weighted PCAs with positive weights, the noisy low-rank matrix recovery with noise matrix obeying Subbotin density [28, Example 2.13], or the one-bit matrix completion with full observations.

The following Lemma improves a little the result of [19, Proposition 2.1] that requires $\Psi$ to have the $(2r, 4r)$-RSC of modulus $\alpha$ and the $(2r, 4r)$-RSS of modulus $\beta$.

**Lemma 2.1** Let $\Psi : \mathbb{R}^{n_1 \times n_2} \rightarrow \mathbb{R}$ be a twice continuously differentiable function satisfying the $(r, r)$-RSC of modulus $\alpha$ and the $(r, r)$-RSS of modulus $\beta$. Then, for any $X \in \mathbb{R}^{n_1 \times n_2}$ with $\text{rank}(X) \leq r$ and any $Y, Z \in \mathbb{R}^{n_1 \times n_2}$ with $\text{rank}([Y \ Z]) \leq r$,

$$\left| \frac{2}{\alpha + \beta} \nabla^2 \Psi(X)(Y, Z) - (Y, Z) \right| \leq \frac{\beta - \alpha}{\alpha + \beta} \| Y \|_F \| Z \|_F.$$

**Proof** Fix an arbitrary $(Y, Z) \in \mathbb{R}^{n_1 \times n_2} \times \mathbb{R}^{n_1 \times n_2}$ with $\text{rank}([Y \ Z]) \leq r$. Fix an arbitrary $X \in \mathbb{R}^{n_1 \times n_2}$ with $\text{rank}(X) \leq r$. If one of $Y$ and $Z$ is the zero matrix, the result is trivial. So, we assume that $Y \neq 0$ and $Z \neq 0$. Write $\overline{Y} := \frac{Y}{\| Y \|_F}$ and $\overline{Z} := \frac{Z}{\| Z \|_F}$. Notice that $\text{rank}([\overline{Y} \ \overline{Z}]) \leq r$ and $\text{rank}(\overline{Y} \pm \overline{Z}) \leq \text{rank}([\overline{Y} \ \overline{Z}])$. Then, we have

$$\alpha \| \overline{Y} + \overline{Z} \|_F^2 \leq \nabla^2 \Psi(X)(\overline{Y} + \overline{Z}, \overline{Y} + \overline{Z}) \leq \beta \| \overline{Y} + \overline{Z} \|_F^2,$$

$$\alpha \| \overline{Y} - \overline{Z} \|_F^2 \leq \nabla^2 \Psi(X)(\overline{Y} - \overline{Z}, \overline{Y} - \overline{Z}) \leq \beta \| \overline{Y} - \overline{Z} \|_F^2.$$

Along with $4|\nabla^2 \Psi(X)(\overline{Y}, \overline{Z})| = |\nabla^2 \Psi(X)(\overline{Y} + \overline{Z}, \overline{Y} + \overline{Z}) - \nabla^2 \Psi(X)(\overline{Y} - \overline{Z}, \overline{Y} - \overline{Z})|$, we have

$$4\nabla^2 \Psi(X)(\overline{Y}, \overline{Z}) \leq \beta \| \overline{Y} + \overline{Z} \|_F^2 - \alpha \| \overline{Y} - \overline{Z} \|_F^2 = 2(\beta - \alpha) + 2(\beta + \alpha)|\overline{Y}, \overline{Z}|,$$

$$-4\nabla^2 \Psi(X)(\overline{Y}, \overline{Z}) \leq \beta \| \overline{Y} - \overline{Z} \|_F^2 - \alpha \| \overline{Y} + \overline{Z} \|_F^2 = 2(\beta - \alpha) - 2(\beta + \alpha)|\overline{Y}, \overline{Z}|.$$

The last two inequalities imply the desired inequality. The proof is then completed. \qed

From the reference [33], we recall that a random variable $\xi$ is called sub-Gaussian if

$$K = \sup_{q \geq 1} q^{-1/2} (\mathbb{E}|\xi|^q)^{1/q} < \infty,$$

and $K$ is referred to as the sub-Gaussian norm of $\xi$. Equivalently, the sub-Gaussian random variable $\xi$ satisfies the following bound for a constant $\tau^2$:

$$\mathbb{P}\{|\xi| > t\} \leq 2e^{-t^2/(2\tau^2)} \text{ for all } t > 0. \ (12)$$

We call the smallest $\tau^2$ satisfying (12) the sub-Gaussian parameter. The tail-probability characterization in (12) enables us to define centered sub-Gaussian random vectors.

**Definition 2.2** (see [5]) A random vector $w = (w_1, \ldots, w_m)^T$ is said to be a centered sub-Gaussian random vector if there exists $\tau > 0$ such that for all $t > 0$ and all $\| v \| = 1$,

$$\mathbb{P}\{|v^T w| > t\} \leq 2e^{-t^2/(2\tau^2)}.$$

### 2.2 Properties of critical points to $\Phi_\lambda$

To give the gradient and Hessian matrix of $\Phi_\lambda$, define $\Xi : \mathbb{R}^{n_1 \times n_2} \rightarrow \mathbb{R}^{(n_1+n_2) \times (n_1+n_2)}$ by

$$\Xi(X) := \begin{pmatrix} \lambda I & \nabla F(X) \\ \nabla F(X)^T & \lambda I \end{pmatrix}. \quad (13)$$
For a given \((U, V) \in \mathbb{R}^{n_1 \times k} \times \mathbb{R}^{n_2 \times k}\), the gradient of \(\Phi_\lambda\) at \((U, V)\) takes the form of
\[
\nabla \Phi_\lambda(U, V) = \begin{bmatrix} \nabla F(X)V + \lambda U \\ \nabla F(X)^T U + \lambda V \end{bmatrix} = \Xi(X)W \quad \text{with} \quad X = U V^T;
\]
and for any \(\Delta = (\Delta_U; \Delta_V)\) with \(\Delta_U \in \mathbb{R}^{n_1 \times k}\) and \(\Delta_V \in \mathbb{R}^{n_2 \times k}\), it holds that
\[
\nabla^2 \Phi_\lambda(U, V)(\Delta, \Delta) = \nabla^2 F(X)(U \Delta_U^T + \Delta_U V^T, U \Delta_V^T + \Delta_U V^T) + 2(\nabla F(X), \Delta_U \Delta_V^T) + \lambda(\Delta, \Delta).
\]
By invoking (14), it is easy to get the balance property of the critical points of \(\Phi_\lambda\).

**Lemma 2.2** Fix an arbitrary \(\lambda > 0\). Any critical point of \(\Phi_\lambda\) belongs to the set
\[
\mathcal{E}_\lambda := \left\{(U, V) \in \mathbb{R}^{n_1 \times k} \times \mathbb{R}^{n_2 \times k} \mid U^T U = V^T V\right\},
\]
and consequently the set of local minimizers to the problem (6) is included in \(\mathcal{E}_\lambda\).

When \(F\) has a special structure, the critical points of \(\Phi_\lambda\) will have a favorable property.

**Lemma 2.3** Let \(F(X) := \frac{1}{2}\|X - D\|^2_F\) for \(X \in \mathbb{R}^{n_1 \times n_2}\), where \(D = \text{Diag}(d_1, \ldots, d_{n_1})\) is an \(n_1 \times n_2\) rectangular matrix for \(d_1 \geq \cdots \geq d_{n_1} \geq 0\) with \(d_r > 0\). Then,

(i) for any critical point \((U, V)\) of \(\Phi_\lambda\) associated to \(\lambda > 0\), it holds that \(U_1 = V_1\);

(ii) the critical point set of \(\Phi_\lambda\) associated to \(\lambda \geq d_r + 1\) takes the following form
\[
\text{crit } \Phi_\lambda = \left\{(U, V) \in \mathbb{R}^{n_1 \times k} \times \mathbb{R}^{n_2 \times k} \mid U_1 = V_1, U_2 = 0, V_2 = 0, (U_1 U_1^T - D_1 + \lambda I)U_1 = 0\right\}
\]
where \(D_1 = \text{Diag}(d_1, \ldots, d_r) \in \mathbb{R}^{r \times r}\) and \(D_2 = \text{Diag}(d_{r+1}, \ldots, d_{n_1}) \in \mathbb{R}^{(n_1-r) \times (n_2-r)}\).

**Proof** (i) Pick a critical point \((U, V)\) of \(\Phi_\lambda\) associated to \(\lambda > 0\). From (14), we get
\[
\begin{align*}
0 &= \nabla_1 \Phi_\lambda(U, V) = \begin{bmatrix} U_1(V^T V + \lambda I) - D_1 V_1 \\ U_2(V^T V + \lambda I) - D_2 V_2 \end{bmatrix}, \\
0 &= \nabla_2 \Phi_\lambda(U, V) = \begin{bmatrix} V_1(U^T U + \lambda I) - D_1^T U_1 \\ V_2(U^T U + \lambda I) - D_2^T U_2 \end{bmatrix}. 
\end{align*}
\]
By combining the two equations with Lemma 2.2, it is immediate to obtain that
\[(U_1 - V_1)(V^T V + \lambda I) + D_1(U_1 - V_1) = 0.\]
Let \(V^T V + \lambda I\) have the spectral decomposition as \(P \Lambda P^T\) with \(\Lambda = \text{Diag}(\mu_1, \ldots, \mu_\kappa)\) and \(P \in \mathbb{O}^k\). Clearly, \(\mu_i > 0\) for \(i = 1, \ldots, \kappa\). Then, the last equality can be rewritten as
\[(U_1 - V_1)P \Lambda + D_1(U_1 - V_1)P = 0.\]
Since \(\Lambda\) and \(D_1\) are positive definite diagonal matrix, the last equality implies that \((U_1 - V_1)P = 0\), which means that \(U_1 = V_1\).

(ii) Pick any \((U, V)\) of \(\text{crit } \Phi_\lambda\) with \(\lambda > d_r + 1\). By the second equality in (18a) and (18b),
\[
0 = \|U_2(V^T V + \lambda I) - D_2 V_2\|_F + \|U_2(U^T U + \lambda I) - D_2^T U_2\|_F \\
\geq \lambda \|U_2\|_F - d_{r+1} \|V_2\|_F + \lambda \|V_2\|_F - d_{r+1} \|U_2\|_F = (\lambda - d_{r+1}) (\|U_2\|_F + \|V_2\|_F).
\]
Since $\lambda > d_{r+1}$, this implies that $U_2 = 0$ and $V_2 = 0$. Together with part (i), we conclude that $(U, V)$ belongs to the set on the right hand side of (17). Conversely, for any $(U, V)$ from the set on the right hand side of (17), it is clear that $\nabla_1 \Phi_\lambda(U, V) = \nabla_2 \Phi_\lambda(U, V) = 0$, i.e., $(U, V) \in \text{crit} \Phi_\lambda$. Thus, we complete the proof. 

\[\square\]

2.3 KL property of an lsc function

**Definition 2.3** Let $f : \mathbb{R}^{n_1 \times n_2} \to (-\infty, \infty]$ be a proper function. The function $f$ is said to have the Kurdyka-Łojasiewicz (KL) property at $U$ and a neighborhood $\mathcal{U}$ of $T$ such that for all $x \in \mathcal{U} \cap \{ f(T) < f(x) + \eta \}$,

$$\psi(f(x) - f(T)) \text{dist}(0, \partial f(x)) \geq 1.$$

If $\psi$ can be chosen as $\psi(s) = c\sqrt{s}$ for some $c > 0$, then $f$ is said to have the KL property with an exponent of $1/2$ at $T$. If $f$ has the KL property of exponent $1/2$ at each point of $\text{dom} \partial f$, then $f$ is called a KL function of exponent $1/2$.

**Remark 2.1** To show that a proper function is a KL function of exponent $1/2$, it suffices to verify if it has the KL property of exponent $1/2$ at all critical points since, by [1, Lemma 2.1], it has this property at all noncritical points.

3 Error bound for critical points

For any critical point $(U, V)$ of $\Phi_\lambda$ with rank$(W) \leq r$, the following lemma implies a lower bound for $\| WW^T - W^*W^*^T \|_F$, whose proof is included in “Appendix A”.

**Lemma 3.1** Suppose $F$ has the $(2r, 4r)$-RSC of modulus $\alpha$ and the $(2r, 4r)$-RSS of modulus $\beta$. Fix an arbitrary $(U^*, V^*) \in \mathcal{E}^*$. Then, for any critical point $(U, V)$ of $\Phi_\lambda$ with rank$(W) \leq r$ and any column orthonormal $Q$ spanning col$(W)$,

$$\frac{1}{2} \langle (WW^T - W^*W^*^T)QQ^T \rangle_F^2 + \frac{\eta^2}{4} \langle (WW^T - W^*W^*^T)QQ^T \rangle_F \leq \beta - \alpha + \beta \langle UV^T - U^*V^*^T \rangle_F \| WW^T - W^*W^*^T \|_F \| QQ^T \|_F.$$

**Remark 3.1** Write $\Gamma := \langle WW^T - W^*W^*^T \rangle QQ^T$. The result of Lemma 3.1 implies that

$$\frac{1}{2} \| \Gamma \|_F^2 \leq \frac{2\sqrt{r}}{\alpha + \beta} \| \mathcal{E}(M) \|_F \| \Gamma \|_F + \beta - \alpha + \beta \| UV^T - U^*V^*^T \|_F \| \Gamma \|_F^2. \quad (19)$$

Recall that $2|ab| \leq \gamma a^2 + \gamma^{-1}b^2$ for any $a, b \in \mathbb{R}$ and any $\gamma > 0$. Then, it holds that

$$\frac{2\sqrt{r}}{\alpha + \beta} \| \mathcal{E}(M) \|_F \| \Gamma \|_F \leq \frac{64r}{(\alpha + \beta)^2} \| \mathcal{E}(M) \|_F^2 + \frac{1}{64} \| \Gamma \|_F^2,$$

$$\beta - \alpha + \beta \| UV^T - U^*V^*^T \|_F \| \Gamma \|_F \leq \left( \frac{\beta - \alpha}{\alpha + \beta} \right)^2 \| UV^T - U^*V^*^T \|_F^2 + \frac{1}{4} \| \Gamma \|_F^2.$$
Together with (19) and $\sqrt{2}\|UV^T - U^*V^*\|_F \leq \|WW^T - W^*W^*\|_F$, we have
\[
\frac{15}{64} \| \Gamma \|_F^2 \leq \frac{64r}{(\alpha + \beta)^2} \| \Xi(M) \|^2 + \frac{(\beta - \alpha)^2}{2(\alpha + \beta)^2} \| WW^T - W^*W^* \|_F^2. \tag{20}
\]
That is, $\| WW^T - W^*W^* \|_F^2$ is lower bounded by $\max \left( 0, \frac{15(\alpha + \beta)^2}{32(\beta - \alpha)^2} \| \Xi(M) \|^2 \right)$.

When the critical point $(U, V)$ in Lemma 3.1 satisfies $\nabla^2 \Phi(\lambda)(U, V)(\Delta, \Delta) \geq 0$, we can provide a lower bound for $\|W \Delta^T\|_F^2$, where $\Delta$ is a special direction defined by
\[
\Delta := W - [W^* 0]R \quad \text{with} \quad R \in \arg\min_{R^T \in \Omega^*} \| W - [W^* 0]R^T \|_F. \tag{21}
\]
This result is stated in the following lemma, whose proof is included in “Appendix B”.

**Lemma 3.2** Suppose that $F$ has the $(2r, 4r)$-RSC of modulus $\alpha$ and the $(2r, 4r)$-RSS of modulus $\beta$. Fix an arbitrary $\lambda > 0$ and an arbitrary $(U^*, V^*) \in \mathcal{E}^*$. Let $(U, V)$ be a critical point of (6) with $\text{rank}(W) \leq r$. Then, for the direction $\Delta = (\Delta_U; \Delta_V)$ defined by (21),
\[
\nabla^2 \Phi(\lambda)(U, V)(\Delta, \Delta) = \nabla^2 F(X)(U \Delta^T_V + \Delta_U V^T, U \Delta^T_U + \Delta_U V^T) - \langle \Xi(X), WW^T - W^*(W^*)^T \rangle.
\]
\[
\|W \Delta^T\|_F^2 \geq \max \left( 0, \frac{\alpha}{2\beta} \| WW^T - W^*(W^*)^T \|_F^2 \right.
\]
\[
\left. + \frac{1}{\beta} \langle \Xi(M), WW^T - W^*(W^*)^T \rangle \right). \tag{22}
\]

Now we state one of the main results which provides an upper bound to the true $M$ for those non-strict critical points of $\Phi_\lambda$ with rank at most $r$.

**Theorem 3.1** Suppose that $F$ satisfies the $(2r, 4r)$-RSC of modulus $\alpha$ and the $(2r, 4r)$-RSS of modulus $\beta$, respectively, with $\beta/\alpha \leq 1.38$. Fix an arbitrary $\lambda > 0$ and an arbitrary $(U^*, V^*) \in \mathcal{E}^*$. Then, for any critical point $(U, V)$ of (6) with $\text{rank}(W) \leq r$ and $\nabla^2 \Phi(\lambda)(U, V)$ being PSD, there exists $\gamma_0 > 0$ (depending only on $\alpha$ and $\beta$) such that the following inequality holds
\[
2 \| UV^T - M \|_F^2 \leq \| WW^T - W^*W^* \|_F^2 \leq \gamma_0 \| \Xi(M) \|^2 \leq 2\gamma_0 r(\lambda^2 + \| \nabla F(M) \|^2). \tag{24}
\]

**Proof** Let $R$ be given by (21) with $W$ and $W^*$. Note that $R \in \arg\max_{R^T \in \Omega^*} \| [W^* 0]^T W, \tilde{R} \|$. It is easy to check that $W^T W^* R_1 = (W^* R_1)^T W$ is PSD. By [19, Lemma 3.6], we have
\[
\| W \Delta^T \|_F^2 \leq \frac{1}{8} \| WW^T - W^*W^* \|_F^2 + \left( 3 + \frac{1}{2\sqrt{2} - 2} \right) \| WW^T - W^*W^* \|_F \cdot Q \bar{Q}^T \|_F^2. \tag{25}
\]
In addition, from the inequality (23) in Lemma 3.2, it follows that
\[
\| W \Delta^T \|_F^2 \geq \frac{\alpha}{2\beta} \| WW^T - W^*W^* \|_F^2 + \frac{1}{\beta} \langle \Xi(M), WW^T - W^*W^* \rangle \geq \frac{\alpha}{2\beta} \| WW^T - W^*W^* \|_F^2 - \frac{\sqrt{2}r}{\beta} \| \Xi(M) \| \| WW^T - W^*W^* \|_F.
\]
Combining this inequality with the inequality (20) yields that

$$\geq \frac{\alpha}{2\beta} \|WW^T - W^*W^*\|_F^2 - \frac{64r}{\alpha\beta} \|\Sigma(M)\|_F^2 - \frac{\alpha}{128\beta} \|WW^T - W^*W^*\|_F^2$$

$$= \frac{63\alpha}{128\beta} \|WW^T - W^*W^*\|_F^2 - \frac{64r}{\alpha\beta} \|\Sigma(M)\|_F^2$$

where the third inequality is due to $2|ab| \leq \gamma a^2 + \gamma^{-1}b^2$ for any $a, b \in \mathbb{R}$ and any $\gamma > 0$. From the last two inequalities, it is not hard to obtain that

$$\frac{63\alpha}{128\beta} \|WW^T - W^*W^*\|_F^2 \leq \frac{1}{8} \|WW^T - W^*W^*\|_F^2 + \frac{64r}{\alpha\beta} \|\Sigma(M)\|_F^2$$

$$+ \frac{7+\sqrt{2}}{2} \|\Sigma(M)\|_F^2.$$

Combining this inequality with the inequality (20) yields that

$$\left( \frac{63\alpha}{128\beta} - \frac{1}{8} - \frac{16(7+\sqrt{2})(\beta - \alpha)^2}{15(\alpha + \beta)^2} \right) \|WW^T - W^*W^*\|_F^2$$

$$\leq \frac{2048(7+\sqrt{2})r}{15(\alpha + \beta)^2} \|\Sigma(M)\|_F^2 + \frac{64r}{\alpha\beta} \|\Sigma(M)\|_F^2.$$

Since $\beta/\alpha \leq 1.38$, we have $\gamma_1 = \frac{63\alpha}{128\beta} - \frac{1}{8} - \frac{16(7+\sqrt{2})(\beta - \alpha)^2}{15(\alpha + \beta)^2} > 0$. So, the desired inequality holds with $\gamma_0 = \gamma_1 \gamma_1$ for $\gamma_2 = \frac{2048(7+\sqrt{2})}{15(\alpha + \beta)^2} + \frac{64r}{\alpha\beta}$ and $\|\Sigma(M)\|_F^2 \leq 2(\lambda^2 + \|\nabla F(M)\|_F^2)$. $\square$

**Remark 3.2** (i) For the least squares loss (3) with a noiseless and full sampling $A$, the error bound in Theorem 3.1 becomes $\|UV^T - M\|_F \leq \sqrt{\gamma_0 r} \lambda$ for each local minimizer of (6) with rank($[U; V]$) $\leq r$. By the characterization of the global optimal solution set in [36] for (6) with $\kappa = r$, each global optimal solution $(\tilde{U}, \tilde{V})$ with rank([$\tilde{U}; \tilde{V}$]) $\leq r$ satisfies $\|\tilde{U}V^T - M\|_F \leq \sqrt{\gamma_0 \lambda}$. This shows that the obtained error bound is optimal.

(ii) From [19] many kinds of first-order methods for problem (6) can find a critical point $(U, V)$ with a PSD Hessian $\nabla^2 \Phi_{\lambda}(U, V)$ in a high probability. Along with Theorem 3.1, when solving (6) with one of these first-order methods, if the obtained critical point has rank at most $r$, it is highly possible for it to have a desirable error bound to the true $M$.

(iii) By combining Theorem 3.1 with Lemma 1 in Appendix C, it follows that each optimal solution $\tilde{X}_{cr}$ of the convex problem (5) with rank($\tilde{X}_{cr}$) $\leq r$ satisfies

$$\|\tilde{X}_{cr} - M\|_F \leq \gamma_0 r \left[ \lambda + \|\nabla F(M)\|_F \right],$$

which is consistent with the one in [20, Corollary 1] for the optimal solution (though it is unknown whether its rank is less than $r$ or not) of the convex relaxation approach. This implies that the error bound of the convex relaxation approach is near optimal.

Next we illustrate the result of Theorem 3.1 via two specific observation models.

### 3.1 Matrix sensing

The matrix sensing problem aims to recover the true matrix $M$ via the observation model (4), where the sampling operator $A$ is defined by $[A(Z)]_i := (A_i, Z)$ for $i = 1, \ldots, m$, and the entries $\omega_1, \ldots, \omega_m$ of the noise vector $\omega$ are assumed to be i.i.d. sub-Gaussian of parameter $\sigma^2_\omega$. By Definition 2.2 and the discussions in [9, Page 24], for every $u \in \mathbb{R}^m$, there exists an
absolute constant $\tilde{c} > 0$ such that with probability at least $1 - \frac{1}{n_{1n_2}}$,

$$\left| \sum_{i=1}^{m} u_i \omega_i \right| \leq \tilde{c} \sigma_\omega \sqrt{\ln(n_{1n_2})} \|u\|. \quad (27)$$

**Assumption 3.1** The sampling operator $A$ has the $4r$-RIP of constant $\delta_{4r} \in (0, \frac{19}{119})$.

Take $F(Z) := \frac{1}{2m} \|A(Z) - y\|^2$ for $Z \in \mathbb{R}^{n_1 \times n_2}$. Then, under Assumption 3.1, the loss function $F$ satisfies the conditions in Theorem 3.1 with $\beta = \frac{1 + \delta_{4r}}{m}$ and $\alpha = \frac{1 - \delta_{4r}}{m}$. We next upper bound $\|A^*(\omega)\|$. Let $\mathcal{S}^{n_1-1} = \{u \in \mathbb{R}^{n_1} \mid \|u\| = 1\}$ denote the Euclidean sphere in $\mathbb{R}^{n_1}$. From the variational characterization of the spectral norm of matrices,

$$\|A^*(\omega)\| = \sup_{u \in \mathcal{S}^{n_1-1}, v \in \mathcal{S}^{n_2-1}} \langle u, A^*(\omega)v \rangle = \sup_{u \in \mathcal{S}^{n_1-1}, v \in \mathcal{S}^{n_2-1}} \langle A(uv^T), \omega \rangle = \sup_{u \in \mathcal{S}^{n_1-1}, v \in \mathcal{S}^{n_2-1}} \sum_{i=1}^{m} \omega_i \langle A_i, uv^T \rangle.$$

By invoking (27) and the RIP of $A$, with probability at least $1 - \frac{1}{n_{1n_2}}$ it holds that

$$\|A^*(\omega)\| \leq \tilde{c} \sigma_\omega \sqrt{\ln(n_{1n_2})} \sup_{u \in \mathcal{S}^{n_1-1}, v \in \mathcal{S}^{n_2-1}} \|A(uv^T)\|$$

$$\leq \tilde{c} \sigma_\omega \sqrt{\ln(n_{1n_2})} \sqrt{1 + \delta_{4r}} \sup_{u \in \mathcal{S}^{n_1-1}, v \in \mathcal{S}^{n_2-1}} \|uv^T\|_F$$

$$\leq \tilde{c} \sigma_\omega \sqrt{\ln(n_{1n_2})} \sqrt{1 + \delta_{4r}}.$$ 

Notice that $\nabla F(M) = \frac{1}{m} A^*(\omega)$. By Theorem 3.1, we obtain the following conclusion.

**Corollary 3.1** Suppose that the sampling operator $A$ satisfies Assumption 3.1. Then, for any critical point $(U, V)$ with rank($W$) $\leq r$ and $\nabla^2 \Phi_\lambda(U, V)$ being PSD of problem (6) for $F(\cdot) = \frac{1}{2m} \|A(\cdot) - y\|^2$,

$$\sqrt{2}\|UV^T - M\|_F \leq \|WW^T - W^*W^*\|_F \leq \phi(\delta_{4r}) \sqrt{r} \left(\lambda + \sigma_\omega \sqrt{\ln(n_{1n_2})}/m\right). \quad (28)$$

holds w.p. at least $1 - \frac{1}{n_{1n_2}}$, where $\phi(\delta_{4r})$ is a nondecreasing positive function of $\delta_{4r}$. When $\lambda = c \sigma_\omega \frac{\sqrt{\ln(n_{1n_2})}}{m}$ for an absolute constant $c > 0$, w.p. at least $1 - \frac{1}{n_{1n_2}}$ we have

$$\|UV^T - M\|_F \asymp O(\sigma_\omega \sqrt{r \ln(n_{1n_2})}/m).$$

### 3.2 Weighted principle component analysis

The weighted PCA problem aims to recover an unknown true matrix $M \in \mathbb{R}^{n_1 \times n_2}$ from an elementwise weighted observation $Y = H \circ (M + E)$, where $H$ is the positive weight matrix, $E$ is the noise matrix, and ”$\circ$” denotes the Hadamard product of matrices. This corresponds to the observation model (4) with $A(Z) := \text{vec}(H \circ Z)$ and $\omega = A(E)$. We assume that the entries $E_{ij}$ of $E$ are i.i.d. sub-Gaussian random variables of parameter $\sigma_E^2$. By Definition 2.2 and the discussions in [9, Page 24], for every $H \in \mathbb{R}^{n_1 \times n_2}$, there exists an absolute constant $\tilde{c} > 0$ such that with probability at least $1 - \frac{1}{n_{1n_2}},$

$$\left| \sum_{i,j} H_{ij} E_{ij} \right| \leq \tilde{c} \sigma_E \sqrt{\ln(n_{1n_2})} \|H\|_F. \quad (29)$$

Take $F(X) := \frac{1}{2} \|H \circ [X - (M + E)]\|_F^2$ for $X \in \mathbb{R}^{n_1 \times n_2}$. Then, for each $X, \Delta \in \mathbb{R}^{n_1 \times n_2}$,

$$\nabla F(X) = H \circ H \circ [X - (M + E)]$$

and

$$\nabla^2 F(X)[\Delta, \Delta] = \|H \circ \Delta\|_F^2.$$
Clearly, $F$ satisfies the $(2r, 4r)$-RSC of modulus $\alpha = \|H\|_{\text{min}}^2$ and $(2r, 4r)$-RSS of modulus $\beta = \|H\|_{\text{max}}^2$, where $\|H\|_{\text{min}} := \min_{i,j} H_{ij}$ and $\|H\|_{\text{max}} := \max_{i,j} H_{ij}$. Notice that

$$\|\nabla F(M)\| = \|H \circ H \circ E\| = \sup_{u \in S^{n^2-1}, v \in S^{n^2-1}} \langle u, (H \circ H \circ E)v \rangle$$

$$= \sup_{u \in S^{n^2-1}, v \in S^{n^2-1}} \sum_{i,j} u_i H_{ij}^2 E_{ij} v_j.$$ 

By invoking (29) and the $(2r, 4r)$-RSS of $F$, with probability at least $1 - \frac{1}{n^2}$ we have

$$\|\nabla F(M)\| = \|H \circ H \circ E\| \leq \tilde{c}\sigma_E \sqrt{\ln(n/\nu_2)} \sup_{u \in S^{n^2-1}, v \in S^{n^2-1}} \|H \circ H \circ (uv^T)\|_F$$

$$\leq \tilde{c}\sigma_E \sqrt{\ln(n/\nu_2)} \|H\|_{\text{max}}^2.$$ 

By invoking Theorem 3.1 with this loss function, we have the following conclusion.

**Corollary 3.2** For any critical point $(U, V)$ with rank($W$) $\leq r$ of problem (6) with $F(\cdot) = \frac{1}{2}\|H \circ [\cdot - (M + E)]\|_F^2$, with probability at least $1 - \frac{1}{n^2}$ it holds that

$$\sqrt{2}\|UV^T - M\|_F \leq \|WW^T - W^*W^*\|_F$$

$$\leq \phi\left(\|H\|_{\text{max}}, \|H\|_{\text{min}}\right) \sqrt{r\left(\lambda + \sigma_E \ln(n/\nu_2)\right) \|H\|_{\text{max}}^2}$$

where $\phi(\|H\|_{\text{max}}, \|H\|_{\text{min}}) := \tilde{\phi}(\|H\|_{\text{max}}/\|H\|_{\text{min}}) / \|H\|_{\text{min}}^2$ with $\tilde{\phi}(\|H\|_{\text{max}}/\|H\|_{\text{min}})$ is a nondecreasing positive function of $\|H\|_{\text{max}}/\|H\|_{\text{min}}$. For $\lambda = c\sigma_E \ln(n/\nu_2) \|H\|_{\text{max}}^2$ with an absolute constant $c > 0$, it holds with probability at least $1 - \frac{1}{n^2}$ that

$$\|UV^T - M\|_F \asymp O\left(\sigma_E \sqrt{r \ln(n/\nu_2)} \|H\|_{\text{max}}^2\right).$$

### 4 KL property of exponent 1/2

In this section, for $\kappa = r$, we focus on the KL property of exponent $1/2$ of $\Phi_\lambda$ under the full sampling and partial sample setting, respectively, with noise.

#### 4.1 Noisy and full sample setting

Now $F(X) := \frac{1}{2}\|X - \widetilde{M}\|_F^2$ for $X \in \mathbb{R}^{n_1 \times n_2}$, where $\widetilde{M} = M + E$ is a noisy observation on the true $M$. By the unitary invariance of the F-norm, problem (6) is equivalent to

$$\min_{U \in \mathbb{R}^{n_1 \times r}, V \in \mathbb{R}^{n_2 \times r}} \left\{ \tilde{\Phi}_\lambda(U, V) := \frac{1}{2}\|UV^T - \text{Diag}(\sigma(\widetilde{M}))\|_F^2 + \frac{1}{2}\lambda\left(\|U\|_F^2 + \|V\|_F^2\right) \right\}$$

(30)

in the sense that if $(\bar{U}^*, \bar{V}^*)$ is a global optimal solution of (30), then $(P\bar{U}^*, Q\bar{V}^*)$ with $(P, Q) \in (n_1, n_2) (\bar{M})$ is globally optimal to problem (6); and conversely, if $(U^*, V^*)$ is a global optimal solution of (6), then $(P^T\bar{U}^*, Q^T\bar{V}^*)$ is globally optimal to problem (30). In fact, if $(\bar{U}, \bar{V})$ is a critical point of (30), then $(P\bar{U}, Q\bar{V})$ with $(P, Q) \in (n_1, n_2) (\bar{M})$ is a critical point to problem (6); and conversely, if $(\bar{U}, \bar{V})$ is a critical point of (6), then $(P^T\bar{U}, Q^T\bar{V})$ is also a critical point to problem (30).

By Definition 2.3 and the relation between the critical point sets of (30) and (6), it is not difficult to obtain the following result for the KL property of $\Phi_\lambda$ and $\Phi_\lambda$. 
Lemma 4.1 Fix an arbitrary \( \lambda > 0 \). If \( \tilde{\Phi}_\lambda \) has the KL property of exponent \( 1/2 \) at a critical point \((\tilde{U}, \tilde{V})\) of problem \((30)\), then \( \Phi_\lambda \) has the KL property of exponent \( 1/2 \) at \((P\tilde{U}, Q\tilde{V})\) for every \((P, Q) \in \mathbb{O}^{n_1,n_2}(\tilde{M})\). Conversely, if \( \Phi_\lambda \) has the KL property of exponent \( 1/2 \) at a critical point \((U, V)\) of problem \((6)\), then \( \tilde{\Phi}_\lambda \) has the KL property of exponent \( 1/2 \) at \((P^TU, Q^TV)\) for every \((P, Q) \in \mathbb{O}^{n_1,n_2}(\tilde{M})\).

Inspired by Lemma 4.1, to achieve the KL property of exponent \( 1/2 \) of \( \Phi_\lambda \) at a global optimal solution of \((6)\), it suffices to establish the KL property of exponent \( 1/2 \) of \( \tilde{\Phi}_\lambda \) at a global optimal solution of \((30)\). For convenience, we write \( \hat{\Sigma} = \text{Diag}(\sigma(\tilde{M})) \) and \( \hat{\Sigma}_1 = \text{Diag}(\sigma_1(\tilde{M}), \ldots, \sigma_r(\tilde{M})) \in \mathbb{R}^{r \times r}, \hat{\Sigma}_2 = \text{Diag}(\sigma_{r+1}(\tilde{M}), \ldots, \sigma_{n_1}(\tilde{M})) \in \mathbb{R}^{(n_1-r) \times (n_2-r)}\).

The following lemma characterizes the global optimal solution set of \((30)\) with \( \lambda > 0 \).

Lemma 4.2 Suppose that \( \sigma_1(\hat{\Sigma}) > \sigma_{r+1}(\hat{\Sigma}) \). Then, the global optimal solution set of problem \((30)\) associated to any given \( \lambda > 0 \) takes the following form

\[
\mathcal{W}_\lambda = \left\{ \left[ \begin{array}{c} \tilde{U} \\ \tilde{V} \end{array} \right] \mid \tilde{U}_1 = \tilde{V}_1 = \tilde{P}(\hat{\Sigma}_2 - \lambda I_{n_2})^{1/2} R \text{ for } \tilde{P} \in \mathbb{O}^r(\hat{\Sigma}_1), R \in \mathbb{O}^r \right\}.
\]

(31)

Proof By using the expression of \( \tilde{\Phi}_\lambda \) and the von Neumann’s trace inequality, for any \((U, V) \in \mathbb{R}^{n_1 \times r} \times \mathbb{R}^{n_2 \times r}\), it holds that

\[
\tilde{\Phi}_\lambda(U, V) \geq \frac{1}{2} \| \sigma(UV^T) - \sigma(\hat{\Sigma}) \|^2 + \lambda \| \sigma(UV^T) \|_1
\]

\[
= \frac{1}{2} \| \sigma(UV^T) - \sigma(\hat{\Sigma}) \|^2 + \lambda \| \sigma(UV^T) \|_1 + \frac{1}{2} \sum_{i=r+1}^n [\sigma_i(\hat{\Sigma})]^2
\]

\[
\geq \min_{z \in \mathbb{R}^r_+} \frac{1}{2} \| z - \sigma(UV^T) \|^2 + \lambda \| z \|_1 + \frac{1}{2} \sum_{i=r+1}^n [\sigma_i(\hat{\Sigma})]^2
\]

(32)

where the equality is due to rank\((UV^T) \leq r\). Clearly, \( x^* = (\sigma^*(\hat{\Sigma}) - \lambda)_+ \) is the unique optimal solution of the minimization problem in \((32)\). In particular, for any \( R \in \mathbb{O}^r \), by taking \( \tilde{U} = [\tilde{P}(\hat{\Sigma}_2 - \lambda I_{22})^{1/2} R; 0] \) and \( \tilde{V} = [\tilde{P}(\hat{\Sigma}_1 - \lambda I_{11})^{1/2} R; 0] \) with \( \tilde{P} \in \mathbb{O}^r(\hat{\Sigma}_1) \), it follows that \( \tilde{\Phi}_\lambda(\tilde{U}, \tilde{V}) \) is equal to the optimal value of \((32)\). This means that the set on the right hand side of \((31)\) is included in \( \mathcal{W}_\lambda \).

For the inverse inclusion, pick a global optimal solution \((\tilde{U}, \tilde{V})\) of \((30)\). Then, the inequalities in \((32)\) necessarily become equalities (if not, taking \( \tilde{U}^* = P^* \text{Diag}(\sqrt{x^*}) \) and \( \tilde{V}^* = Q^* \text{Diag}(\sqrt{x^*}) \) with \((P^*, Q^*) \in \mathbb{O}^{n_1,n_2}(UV^T)\) yields that \( \tilde{\Phi}_\lambda(U^*, V^*) < \tilde{\Phi}_\lambda(U, V) \)). Thus, \((UV^T, \hat{\Sigma}) = (\sigma(UV^T), \sigma(\hat{\Sigma}))\), which means that \( UV^T = \hat{\Sigma} \) and \( \hat{\Sigma} \) have the same ordered SVD, i.e., there exist \( P \in \mathbb{O}^{n_1} \) and \( Q \in \mathbb{O}^{n_2} \) such that \( UV^T = PD\text{Diag}(\sigma(UV^T))Q^T \) and \( \hat{\Sigma} = P\hat{\Sigma}Q^T \). From \( \tilde{P} = P\tilde{\Sigma}Q^T \), one may obtain \( P = \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} \) and \( Q = \begin{pmatrix} P_{11} & 0 \\ 0 & Q_{22} \end{pmatrix} \) with \( P_{11} \in \mathbb{O}^r \) such that \( P_{11}\hat{\Sigma}_1 P_{11}^T = \hat{\Sigma}_1 \). Together with \( \sigma(UV^T) = x^* \), it holds that

\[
\begin{pmatrix} U_1 \tilde{V}_1 \cr U_2 \tilde{V}_2 \end{pmatrix} = U\tilde{V} = PD\text{Diag}(x^*)Q^T = \begin{pmatrix} P_{11} \text{Diag}(x^*) P_{11}^T & 0 \\ 0 & 0 \end{pmatrix},
\]

which implies that \( U_1 \tilde{V}_1 = P_{11} \text{Diag}(x^*) P_{11}^T \). When \( \lambda \geq \sigma_r(\hat{\Sigma}) \), by Lemma 2.3 (ii), we have \( U_2 = 0 \) and \( \tilde{V}_2 = 0 \). When \( \lambda < \sigma_r(\hat{\Sigma}) \), the matrices \( U_1 \) and \( \tilde{V}_1 \) are nonsingular, which
along with $U_2V_1^T = 0$ and $U_1V_2^T = 0$ imply $U_2 = 0$ and $V_2 = 0$. Together with Lemma 2.2, it follows that $U_1^TV_1 = V_1^TU_1$. This, along with $U_1V_1^T = P_1\text{Diag}(x^*)P_1^T$, implies that $V_1 = P_1\text{Diag}(x)^{1/2}R$ for some $R \in \mathcal{O}^r$. Thus, $(U, V)$ belongs to the set on the right hand side of (31).

Inspired by Lemma 4.2 and the proof of [36, Theorem 2(a),] we establish the following conclusion, which extends the result of [36, Theorem 2(a)] to the noisy setting. In fact, as will be shown by Remark 4.1, Theorem 4.2 actually implies that $\Phi_\lambda$ associated to almost all $\lambda > 0$ has the KL property of exponent $1/2$ at its global optimal solutions.

**Theorem 4.1** Suppose that $\sigma_r(\widetilde{\Sigma}) > \sigma_{r+1}(\widetilde{\Sigma})$. Let $\tilde{\sigma}_1 > \tilde{\sigma}_2 > \ldots > \tilde{\sigma}_s$ for $1 \leq s \leq r$ be the distinct singular values of $\tilde{\Sigma}_1$. Fix any $\lambda \in (\tilde{\sigma}_{k+1}, \tilde{\sigma}_k)$ for some $0 \leq k \leq s$ with $\tilde{\sigma}_0 = +\infty$ and $\tilde{\sigma}_{s+1} = 0$. Consider any point $(\overline{U}, \overline{V}) \in \overline{\mathcal{W}}_\lambda$. Then, there exists a constant $\eta > 0$ such that for all $(U, V) \in \mathbb{B}((\overline{U}, \overline{V}), \delta)$ with $\delta \leq \min\\{\frac{\sqrt{\tilde{\sigma}_2-\lambda}}{2}, \frac{\lambda-\tilde{\sigma}_{s+1}}{2\sqrt{\tilde{\sigma}_1}}, \frac{\tilde{\sigma}_s-\sigma_{r+1}}{4\sqrt{\tilde{\sigma}_1}}\}$,

$$\|\nabla \Phi_\lambda(U, V)\|_F^2 \geq \eta[\overline{\Phi}_\lambda(U, V) - \overline{\Phi}_\lambda(\overline{U}, \overline{V})].$$

**Proof** Fix any $(U, V) \in \mathbb{B}((\overline{U}, \overline{V}), \delta).$ Clearly, $\text{dist}((U, V), \overline{\mathcal{W}}_\lambda) \leq \delta.$ Pick an arbitrary $R^* \in \arg\min_{R \in \mathcal{O}^r} \|\langle U, V \rangle - (\overline{U}R, \overline{V}R)\|_F.$ Clearly, it holds that

$$(\overline{U}R^*, \overline{V}R^*) \in \mathbb{B}((\overline{U}, \overline{V}), \delta) \text{ and } \overline{\Phi}_\lambda(U, V) = \overline{\Phi}_\lambda(U, V).$$

(33)

Notice that $\nabla \overline{\Phi}_\lambda$ is globally Lipschitz continuous on $\mathbb{B}((\overline{U}, \overline{V}), \delta).$ From the descent lemma and the relation in (33), it follows that

$$\overline{\Phi}_\lambda(U, V) - \overline{\Phi}_\lambda(U, \overline{V}) \leq \frac{L}{2} (\|U - \overline{U}R^*\|_F^2 + \|V - \overline{V}R^*\|_F^2) \leq \frac{L}{2} \text{dist}^2((U, V), \overline{\mathcal{W}}_\lambda).$$

For each $j = 1, 2, \ldots, s$, write $a_j := \{i \mid \sigma_i(\tilde{\Sigma}_1) = \tilde{\sigma}_j\}.$ Let $(U_1)_J$ and $(\overline{U}_J)_J$ be the matrix consisting of the first $\tilde{r} = \sum_{i=1}^s |a_i|$ rows of $U_1$ and $\overline{U}_1$, respectively, and let $(U_1)_J$ and $(\overline{U}_1)_J$ be the last $r - \tilde{r}$ rows of $U_1$ and $\overline{U}_1$, respectively. By Lemma 4.2, it follows that

$$\overline{\mathcal{W}}_\lambda = \left\{ \left( \begin{bmatrix} U_1 \\ 0 \end{bmatrix}, \begin{bmatrix} V_1 \\ 0 \end{bmatrix} \right) \mid U_1 = \begin{bmatrix} \tilde{P} \sqrt{\tilde{\Sigma}_{11} - \lambda I} \\ 0 \end{bmatrix} \overline{V}_1 \right\} \text{ for } \tilde{P} \in \mathcal{O}^r(\tilde{\Sigma}_{11}), R \in \mathcal{O}^r,$$

(34)

where $\tilde{\Sigma}_{11} = \text{Diag}(\sigma_1(\tilde{\Sigma}), \ldots, \sigma_r(\tilde{\Sigma})) \in \mathbb{R}^{\tilde{r} \times \tilde{r}}.$ By the expression of $\overline{\mathcal{W}}_\lambda$, it follows that

$$\text{dist}^2((U, V), \overline{\mathcal{W}}_\lambda) = \|U_2\|_F^2 + \|V_2\|_F^2 + \min_{R \in \mathcal{O}^r} \{\|U_1 - \overline{U}_1R\|_F^2 + \|V_1 - \overline{V}_1R\|_F^2\}$$

$$\leq \|U_2\|_F^2 + \|V_2\|_F^2 + 2\|U_1 - \overline{U}_1\|_F^2 + 3\text{dist}^2(U_1, \overline{U}_1),$$

$$\leq \|U_2\|_F^2 + \|V_2\|_F^2 + 2\|U_1 - \overline{U}_1\|_F^2 + 3\text{dist}^2(U_1, \overline{U}_1)_J + 3\text{dist}^2((U_1)_J, [(\tilde{\Sigma}_{11} - \lambda I)^{1/2} 0])$$

where the first inequality is since $\|V_1 - \overline{V}_1R\|_F^2 \leq 2\|V_1 - \overline{V}_1\|_F^2 + 2\|U_1 - \overline{U}_1R\|_F^2,$ and the second is due to $\text{dist}((U_1)_J, (\overline{U}_1)_J) = \text{dist}((U_1)_J, [(\tilde{\Sigma}_{11} - \lambda I)^{1/2} 0]).$ Thus, we obtain

$$\overline{\Phi}_\lambda(U, V) - \overline{\Phi}_\lambda(U, \overline{V}) \leq \|U_2\|_F^2 + \|V_2\|_F^2 + 2\|U_1 - \overline{U}_1\|_F^2 + 3\text{dist}^2(U_1, \overline{U}_1)_J + 3\text{dist}^2((U_1)_J, [(\tilde{\Sigma}_{11} - \lambda I)^{1/2} 0]).$$

Next we bound $\|\nabla \overline{\Phi}_\lambda(U, V)\|_F.$ By the expressions of $\overline{\Phi}_\lambda$, it is easy to obtain that
\[
\begin{align*}
\nabla_1 \tilde{\Phi}_\lambda(U, V) &= (UV^T - \tilde{\Sigma})V + \lambda U = \begin{bmatrix} U_1(V^T V + \lambda I) - \tilde{\Sigma}_1 V_1 \\ U_2(V^T V + \lambda I) - \tilde{\Sigma}_2 V_2 \end{bmatrix}, \\
\nabla_2 \tilde{\Phi}_\lambda(U, V) &= (UV^T - \tilde{\Sigma})^T U + \lambda V = \begin{bmatrix} V_1(U^T U + \lambda I) - \tilde{\Sigma}_1 U_1 \\ V_2(U^T U + \lambda I) - \tilde{\Sigma}_2^T U_2 \end{bmatrix}.
\end{align*}
\]

By using the second equality of (36a) and (36b), it is not difficult to obtain that
\[
\begin{align*}
\| \nabla_1 \tilde{\Phi}_\lambda(U, V) \|_F + \| \nabla_2 \tilde{\Phi}_\lambda(U, V) \|_F &
\geq \| U_2(V^T V + \lambda I) - \tilde{\Sigma}_2 V_2 \|_F + \| V_2(U^T U + \lambda I) - \tilde{\Sigma}_2^T U_2 \|_F \\
&\geq (\sigma_1^2(U) + \lambda)\| U_2 \|_F - \| \tilde{\Sigma}_2 \|_F \| V_2 \|_F + (\sigma_2^2(U) + \lambda)\| V_2 \|_F - \| \tilde{\Sigma}_2 \|_F \| U_2 \|_F \\
&\geq \frac{\min(\sigma_1^2(U), \sigma_2^2(V)) + \lambda - \| \tilde{\Sigma}_2 \|_F}{2} (\| U_2 \|_F + \| V_2 \|_F),
\end{align*}
\]
where the last inequality is \(\min(\sigma_1^2(U), \sigma_2^2(V)) + \lambda - \| \tilde{\Sigma}_2 \|_F \geq \tilde{\sigma}_s - \sigma_{r+1}(\Sigma) > 0\) implied by \(\lambda > \tilde{\sigma}_s\) for \(k < s\), and for \(k = s\),
\[
\frac{\tilde{\sigma}_s - \sigma_{r+1}(\Sigma)}{2},
\]
implied by \(\sigma_s(U) \geq \sigma_r(U_1) \geq \sqrt{\tilde{\sigma}_s} - \lambda - \delta \leq \frac{\tilde{\sigma}_s - \sigma_{r+1}(\Sigma)}{4\sqrt{\sigma_1}}\). In addition, from the first equality of (36a) and (36b), we can obtain that
\[
\begin{align*}
\| \nabla_1 \tilde{\Phi}_\lambda(U, V) \|_F + \| \nabla_2 \tilde{\Phi}_\lambda(U, V) \|_F &
\geq \| U_1(V^T V + \lambda I) - \tilde{\Sigma}_1 V_1 \|_F + \| V_1(U^T U + \lambda I) - \tilde{\Sigma}_1 U_1 \|_F \\
&\geq \| U_1(V^T V + \lambda I) - \tilde{\Sigma}_1 V_1 \|_F + \| V_1(U^T U + \lambda I) - \tilde{\Sigma}_1 U_1 \|_F - \| \tilde{\Sigma}_1 \|_F \| U_1 \|_F - \| \tilde{\Sigma}_1 \|_F \| V_1 \|_F \\
&\geq \frac{\tilde{\sigma}_s - \sigma_{r+1}(\Sigma)}{2} (\| V_2 \|_F + \| U_2 \|_F)
\end{align*}
\]
where the second one is using \(\max(\| U_2 \|_F, \| V_2 \|_F) \leq \text{dist}((U, V), \mathcal{W}_k) \leq \delta\), and the last one is due to \(\max(\| U_1 \|_F, \| V_1 \|_F) \leq \| U_1 \| F + \| V_1 \| F \leq 2\sqrt{\sigma_1} \) and \(\delta \leq \min\left\{ \frac{\lambda - \tilde{\sigma}_k + 1}{2\sqrt{\sigma_1}}, \frac{\tilde{\sigma}_s - \sigma_{r+1}(\Sigma)}{4\sqrt{\sigma_1}} \right\}\). Then
\[
2(\| \nabla_1 \tilde{\Phi}_\lambda(U, V) \|_F + \| \nabla_2 \tilde{\Phi}_\lambda(U, V) \|_F)
\geq \| U_1(V^T V + \lambda I) - \tilde{\Sigma}_1 V_1 \|_F + \| V_1(U^T U + \lambda I) - \tilde{\Sigma}_1 U_1 \|_F.
\]
Next we shall employ (38) to establish the following two inequalities
\[
\begin{align*}
\left(\frac{8\tilde{\sigma}_1}{\lambda} + 2\right)(\| \nabla_1 \tilde{\Phi}_\lambda(U, V) \|_F + \| \nabla_2 \tilde{\Phi}_\lambda(U, V) \|_F) &\geq \tilde{\sigma}_s \| U_1 - V_1 \|_F, \\
\left[\frac{4\tilde{\sigma}_1 + \lambda}{\tilde{\sigma}_s} \left(\frac{8\tilde{\sigma}_1}{\lambda} + 2\right)\right](\| \nabla_1 \tilde{\Phi}_\lambda(U, V) \|_F + \| \nabla_2 \tilde{\Phi}_\lambda(U, V) \|_F) &\geq \| U_1 U_1^T - (\tilde{\Sigma}_1 - \lambda I) \| U_1 \| F.
\end{align*}
\]
Firstly, from inequality (38), it is not difficult to obtain that
\[
4\sqrt{\sigma_1}(\| \nabla_1 \tilde{\Phi}_\lambda(U, V) \|_F + \| \nabla_2 \tilde{\Phi}_\lambda(U, V) \|_F)
\]
The last two inequalities imply that (39) holds. In addition, using (38) again yields that
\[\|U_1 U_1^T V_1 + \lambda I - U_1^T \Sigma_1 V_1 + \lambda I\|_F \geq \|V_1 \|_F \|U_1 U_1^T U_1 + \lambda I - V_1^T \Sigma_1 U_1\|_F\]
and on the other hand,
\[2(\| \nabla_1 \Phi_{\lambda}(U, V) \|_F + \| \nabla_2 \Phi_{\lambda}(U, V) \|_F) \geq \| U_1 (V_1^T V_1 + \lambda I - \Sigma_1 V_1) - U_1^T \Sigma_1 U_1 \|_F\]
\[\geq \| U_1 - V_1 \|_F (V_1^T V_1 + \lambda I) + \Sigma_1 (U_1 - V_1) + V_1 (V_1^T V_1 - U_1^T U_1)\|_F\]
\[\geq \| U_1 - V_1 \|_F (V_1^T V_1 + \lambda I) + \Sigma_1 (U_1 - V_1)\|_F - \| V_1 \|_F \| V_1^T V_1 - U_1^T U_1 \|_F\]
\[\geq \sigma_R \| U_1 - V_1 \|_F - 2\sqrt{\sigma_1} \| V_1^T V_1 - U_1^T U_1 \|_F.\]

The last two inequalities imply that (39) holds. In addition, using (38) again yields that
\[2(\| \nabla_1 \Phi_{\lambda}(U, V) \|_F + \| \nabla_2 \Phi_{\lambda}(U, V) \|_F) \geq \| V_1 (U_1^T U_1 + \lambda I) - \Sigma_1 U_1 \|_F\]
\[= \| [U_1 U_1^T - (\Sigma - \lambda I)] U_1 + (V_1 - U_1^T U_1 + \lambda I) \|_F\]
\[\geq \| [U_1 U_1^T - (\Sigma - \lambda I)] U_1 - (\Sigma_{12} - \lambda I) (U_1)\|_F\]
\[\geq \| (U_1) - (\Sigma_{12} - \lambda I)\|_F - \| \Sigma_{12} (U_1)\|_F\]
\[\geq \| (U_1) - (\Sigma_{12} - \lambda I)\|_F - \| \Sigma_{12} (U_1)\|_F - \| \Sigma_{12} (U_1)\|_F\]
\[\geq \lambda \| (U_1)\|_F - \| \Sigma_{12} (U_1)\|_F\]
which together with (39) implies that (40) holds. Next we bound the right hand side of (40).

From the definitions of $J$ and $\tilde{J}$, it follows that
\[\|U_1 U_1^T - (\Sigma - \lambda I)\|_F \geq \| (U_1) - (\Sigma_{12} - \lambda I)\|_F - \| \Sigma_{12} (U_1)\|_F\]
\[\geq \| (U_1)\|_F - \| \Sigma_{12} (U_1)\|_F\]
\[\geq (\lambda - \sigma_{k+1}) \| (U_1)\|_F.\]
(42)

In addition, from (41) and $\| (U_1)\|_F = \| (U_1)\|_F - \| (U_1)\|_F \leq \text{dist}(U, V), \Sigma_{12} \| \leq \delta$, we get
\[\|U_1 U_1^T - (\Sigma - \lambda I)\|_F \geq \| (U_1)\|_F - \| \Sigma_{12} (U_1)\|_F\]
\[\geq \| (U_1)\|_F - \| \Sigma_{12} (U_1)\|_F - \| \Sigma_{12} (U_1)\|_F\]
\[\geq \sigma_J (U_1)\| (U_1)\|_F - 2\sqrt{\sigma_1} \| (U_1)\|_F\]
(43)

where the last inequality is using $\| A B \|_F \geq \sigma_J (B) \| A \|_F$ for $A \in \mathbb{R}^{k \times r}$ and $B \in \mathbb{R}^{r \times r}$, and $\| (U_1)\|_F \leq \| (U_1)\|_F + \delta \leq 2\sqrt{\sigma_1}$. Now we bound $\sigma_J (U_1)\|_F$ from below. Let $\tilde{R} \in \mathbb{R}^{r \times r}$ be such that $\text{dist}(U_1) = (\tilde{R})^{1/2} \| \tilde{R} \|_F = (\tilde{R})^{1/2} \| \tilde{R} \|_F$. Then,
\[\sigma_J (U_1) = \min_{\| x \|_1 = 1, \| y \|_1 = 1} \| x^T (U_1) y \|_2\]
\[\geq \min_{\| x \|_1 = 1, \| y \|_1 = 1} \| x^T (\tilde{R})^{1/2} \|_2\]
\[= \max_{\| x \|_1 = 1, \| y \|_1 = 1} \| x^T (U_1) - (\tilde{R})^{1/2} \|_2\]
\[ \geq \sqrt{\sigma_k - \lambda} - \|(U_1)_{ij} - [A((\Sigma_{11} - \lambda I)^{1/2} 0)]R\|_F \]
\[ = \sqrt{\sigma_k - \lambda} - \text{dist}((U_1)_{ij}, [(\Sigma_{11} - \lambda I)^{1/2} 0]) \geq \frac{1}{2} \sqrt{\sigma_k - \lambda} \quad (44) \]

where the last inequality is due to \( \text{dist}((U_1)_{ij}, [(\Sigma_{11} - \lambda I)^{1/2} 0]) \leq \delta \leq \frac{\lambda - \sigma_{k+1}}{2\sqrt{\sigma_1}} \). Now by invoking equation (42)–(44) and \( \delta \leq \frac{\lambda - \sigma_{k+1}}{2\sqrt{\sigma_1}} \), it follows that

\[ 2\|U_1U_1^T - (\Sigma_{11} - \lambda I)\|_F \geq \frac{1}{2} \sqrt{\sigma_k - \lambda} \| (U_1)_{ij}(U_1)_{ij}^T - (\Sigma_{11} - \lambda I) \|_F \]
\[ \geq \frac{1}{2} \{\sigma_k - \lambda\}^{3/2} \| (\Sigma_{11} - \lambda I)^{-1/2}(U_1)_{ij}(\Sigma_{11} - \lambda I)^{-1/2} - I \|_F \]

Let \( (\Sigma_{11} - \lambda I)^{-1/2}(U_1)_{ij} \) have the SVD as \( \Sigma_{11} - \lambda I)^{-1/2}(U_1)_{ij} = L[\Lambda \ 0]H^T \), where \( L \in \mathbb{O}^r \) and \( H \in \mathbb{O}^r \). Take \( L = \begin{pmatrix} L & 0 \\ 0 & I \end{pmatrix} \in \mathbb{R}^{r \times r} \). Clearly, \( L^T L = L H^T = I \). Then, it holds that

\[ \| (\Sigma_{11} - \lambda I)^{-1/2}(U_1)_{ij}(\Sigma_{11} - \lambda I)^{-1/2} - I \|_F \]
\[ = \| \Lambda^2 - I \|_F = \| (\Lambda + I)(\Lambda - I) \|_F \]
\[ \geq \| [\Lambda \ 0] - [I \ 0] \|_F = \| (\Sigma_{11} - \lambda I)^{-1/2}(U_1)_{ij}H - L[\ 0 \] \|_F \]
\[ \geq \sigma_T (\Sigma_{11} - \lambda I)^{-1/2} \| (U_1)_{ij}H - [(\Sigma_{11} - \lambda I)^{1/2} L \ 0] \|_F \]
\[ = \sigma_T (\Sigma_{11} - \lambda I)^{-1/2} \| (U_1)_{ij}H - [(\Sigma_{11} - \lambda I)^{1/2} L \ 0] \|_F \]
\[ \geq \frac{1}{\sqrt{\sigma_1 - \lambda}} \| (U_1)_{ij} - [(\Sigma_{11} - \lambda I)^{1/2} \ 0]LH^T \|_F \]
\[ \geq \frac{1}{\sqrt{\sigma_1 - \lambda}} \text{dist}((U_1)_{ij}, [(\Sigma_{11} - \lambda I)^{1/2} \ 0]) \].

By combining the last two inequalities and (40), it is immediate to obtain that

\[ 2\left[ \frac{4\sigma_1 + \lambda}{\sigma_s} \left( \frac{8\sigma_1}{\lambda} + 2 \right) \right] \| \nabla_1 \Phi_\lambda(U, V) \|_F + \| \nabla_2 \Phi_\lambda(U, V) \|_F \]
\[ \geq \frac{(\sigma_k - \lambda)^{3/2}}{2\sqrt{\sigma_1 - \lambda}} \text{dist}((U_1)_{ij}, [(\Sigma_{11} - \lambda I)^{1/2} \ 0]) \].

Now, combining the last inequality with (37), (39), (40) and (42), and comparing with (35), we obtain the conclusion. The proof is completed. \( \square \)

**Remark 4.1** (i) When \( \lambda > \sigma_r (\Sigma) \), inequality (37) can be replaced by the following one

\[ \| \nabla_1 \Phi_\lambda(U, V) \|_F + \| \nabla_2 \Phi_\lambda(U, V) \|_F \geq (\lambda - \sigma_{r+1}(\Sigma))(\| U_2 \|_F + \| V_2 \|_F) \],

and the conditions \( \sigma_r (\Sigma) > \sigma_{r+1}(\Sigma) \) and \( \delta \leq \frac{\sigma_1 - \sigma_{r+1}(\Sigma)}{4\sqrt{\sigma_1}} \) in Theorem 4.1 can be removed.

(ii) When \( k = s \) and \( E = 0 \), Theorem 4.1 implies that \( \Phi_\lambda \) with \( 0 < \lambda < \sigma_r (\Sigma) \) has the KL property of exponent 1/2 at every global minimizer, which is precisely the result of [36, Theorem 2(a)]. When \( k = 0 \), Theorem 4.1 implies that \( \Phi_\lambda \) with \( \lambda > \sigma_1 (\Sigma) \) has the KL property of exponent 1/2 at every global minimizer. Together with Lemma 4.1, \( \Phi_\lambda \) associated to the corresponding \( \lambda \) also has the KL property of exponent 1/2 at its global minimizers.
4.2 Noisy and partial sample setting

Now the function $F$ is given by (3). For each $\lambda > 0$, we denote by $S_\lambda$ the critical point set of $\Phi_\lambda$. Define $\Upsilon_1 : \mathbb{R}^{n_1 \times r} \times \mathbb{R}^{n_2 \times r} \to \mathbb{R}^{n_1 \times r}$ and $\Upsilon_2 : \mathbb{R}^{n_1 \times r} \times \mathbb{R}^{n_2 \times r} \to \mathbb{R}^{n_2 \times r}$ by

$$\Upsilon_1(U, V) := A^\ast(A(UV^T - M)V)$$

and

$$\Upsilon_2(U, V) := (A^\ast(A(UV^T - M))^T U).$$

(45)

The following theorem states that $\Phi_\lambda$ has the KL property of exponent $1/2$ at a critical point $(\overline{U}, \overline{V})$ if the calmness modulus of $\Upsilon_1$ and $\Upsilon_2$ at $(\overline{U}, \overline{V})$ is not too greater than $\lambda$.

**Theorem 4.2** Fix an arbitrary $\lambda > 0$. Consider a critical point $(\overline{U}, \overline{V}) \in S_\lambda$. Suppose that there exists $\varepsilon > 0$ such that the calmness modulus of $\Upsilon_1$ and $\Upsilon_2$ on $\mathbb{B}((\overline{U}, \overline{V}), \varepsilon)$, say $c_1$ and $c_2$, satisfies $2\varepsilon + \|A^\ast(\omega)\| + \sqrt{2\varepsilon + \|A^\ast(\omega)\|^2 + 4\varepsilon\|A^\ast(\omega)\|} < \lambda$, where $\varepsilon = \max(c_1, c_2)$. Then $\Phi_\lambda$ has the KL property of exponent $1/2$ at $(\overline{U}, \overline{V})$.

**Proof** By the definition of calmness in [26, Chapter 8F], for any $(U, V) \in \mathbb{B}((\overline{U}, \overline{V}), \varepsilon)$,

$$\|\Upsilon_1(U, V) - \Upsilon_1(\overline{U}, \overline{V})\|_F \leq c_1 \|(U, V) - (\overline{U}, \overline{V})\|_F, \tag{46a}$$

$$\|\Upsilon_2(U, V) - \Upsilon_2(\overline{U}, \overline{V})\|_F \leq c_2 \|(U, V) - (\overline{U}, \overline{V})\|_F. \tag{46b}$$

Observe that $\nabla \Phi_\lambda$ is globally Lipschitz continuous on $\mathbb{B}((\overline{U}, \overline{V}), \varepsilon)$. Then, there exists a constant $L > 0$ such that for all $(U, V), (U', V') \in \mathbb{B}((\overline{U}, \overline{V}), \varepsilon)$,

$$\Phi_\lambda(U, V) - \Phi_\lambda(U', V') < L \left(\frac{1}{2}(\|U - \overline{U}\|_F^2 + \|V - \overline{V}\|_F^2)\right).$$

(47)

Pick any $(U, V)$ from $\mathbb{B}((\overline{U}, \overline{V}), \varepsilon)$. Notice that $\nabla U \Phi_\lambda(U, V) = 0$. Then, it holds that

$$\|\nabla U \Phi_\lambda(U, V)\|_F^2 = \|A^\ast(A(UV^T - M) - \omega)V + \lambda(U - A^\ast(A(\overline{U}V^T - M) - \omega)V - \lambda\overline{U})\|_F^2$$

$$= \|A^\ast(A(UV^T - M)V - A^\ast(A(\overline{U}V^T - M))V - A^\ast(\omega)(V - \overline{V}) + \lambda(U - \overline{U})\|_F^2$$

$$= \|A^\ast(A(UV^T - M)V - A^\ast(A(UV^T - M))V)\|_F^2 + \|A^\ast(\omega)(V - \overline{V})\|_F^2$$

$$+ \lambda^2\|U - \overline{U}\|_F^2 - 2\lambda(A^\ast(\omega)(V - \overline{V}), (U - \overline{U}))$$

$$- 2(\Upsilon_1(U, V) - \Upsilon_1(\overline{U}, \overline{V}), A^\ast(\omega)(V - \overline{V}) - \lambda(U - \overline{U})).$$

Similarly, by the expression of $\nabla V \Phi_\lambda(U, V)$ and the fact that $\nabla V \Phi_\lambda(U, V) = 0$, we have

$$\|\nabla V \Phi_\lambda(U, V)\|_F^2 = \|A^\ast(A(UV^T - M))\|_F^2 - \|A^\ast(A(\overline{U}V^T - M))\|_F^2$$

$$+ \|A^\ast(\omega)\|_F^2(U - \overline{U})^2 + \lambda^2\|V - \overline{V}\|_F^2 - 2\lambda(A^\ast(\omega)(V - \overline{V}), U - \overline{U})$$

$$- 2(\Upsilon_2(U, V) - \Upsilon_2(\overline{U}, \overline{V}), A^\ast(\omega)(V - \overline{V}) - \lambda(V - \overline{V})).$$

From the above two equalities, it immediately follows that

$$\|\nabla \Phi_\lambda(U, V)\|_F^2 = \|\nabla U \Phi_\lambda(U, V)\|_F^2 + \|\nabla V \Phi_\lambda(U, V)\|_F^2$$

$$\geq \lambda^2\|U - \overline{U}\|_F^2 + \lambda^2\|V - \overline{V}\|_F^2 - 4\lambda(A^\ast(\omega)(V - \overline{V}), U - \overline{U})$$

$$- 2(\Upsilon_1(U, V) - \Upsilon_1(\overline{U}, \overline{V}), A^\ast(\omega)(V - \overline{V}) - \lambda(U - \overline{U}))$$

$$\geq \lambda^2\|U - \overline{U}\|_F^2 + \lambda^2\|V - \overline{V}\|_F^2 - 4\lambda\|A^\ast(\omega)\|_F\|U - \overline{U}\|_F\|V - \overline{V}\|_F$$

$$+ 2\lambda\|A^\ast(\omega)\|_F\|V - \overline{V}\|_F - 2\lambda\|A^\ast(\omega)\|_F\|U - \overline{U}\|_F.$$
Similarly, for the term $I_2$ in (48), following the same analysis and using (46b) yield that
\[
I_2 \leq 2c_2\|A^*(\omega)\|\|V - \bar{V}\|^2_F + 2\lambda c_2\|V - \bar{V}\|^2_F + 2c_2(\lambda + \|A^*(\omega)\|)\|U - \bar{U}\|_F\|V - \bar{V}\|_F.
\]

By combining the last two inequalities with (48), we obtain that
\[
\|\nabla \Phi_\lambda(U, V)\|^2_F \geq \Gamma_1(\lambda)\|U - \bar{U}\|^2_F + \Gamma_2(\lambda)\|V - \bar{V}\|^2_F
\]
where
\[
\Gamma_1(\lambda) := \lambda^2 - 2(\lambda c_1 + c_2\|A^*(\omega)\|) - 2(\lambda c_1 + c_2)(\lambda + \|A^*(\omega)\|);
\]
\[
\Gamma_2(\lambda) := \lambda^2 - 2(\lambda c_2 + c_1\|A^*(\omega)\|) - 2\|A^*(\omega)\| - (c_1 + c_2)(\lambda + \|A^*(\omega)\|).
\]

Recall that $2\bar{c} + \|A^*(\omega)\| + \sqrt{(2\bar{c} + \|A^*(\omega)\|)^2 + 4\bar{c}\|A^*(\omega)\|} < \lambda$. We have $\Gamma_1(\lambda) > 0$ and $\Gamma_2(\lambda) > 0$. By comparing (49) with (47), there exists a constant $\eta > 0$ such that for all $(U, V) \in \mathbb{B}((\bar{U}, \bar{V}), \varepsilon)$, $\|\nabla \Phi_\lambda(U, V)\|_F \geq \eta\sqrt{\Phi_\lambda(U, V) - \Phi_\lambda(\bar{U}, \bar{V})}$. \hfill \Box

**Remark 4.2** Suppose that $(\bar{U}, \bar{V})$ is a global minimizer of $\Phi_\lambda$ with $\bar{c} \ll \|A^*(\omega)\|$. Then, the condition that $2\bar{c} + \|A^*(\omega)\| + \sqrt{(2\bar{c} + \|A^*(\omega)\|)^2 + 4\bar{c}\|A^*(\omega)\|} < \lambda$ approximately requires $\lambda > 2\|A^*(\omega)\|$. Such $\lambda$ is close to the optimal one obtained in [20, Corollary 1] for the error bound of the optimal solution to the nuclear norm regularized problem.

Theorem 4.2 states that $\Phi_\lambda$ has the KL property of exponent 1/2 only at its partial critical points. In fact, even in the noiseless and full sample setup, $\Phi_\lambda$ with some $\lambda$ does not have the KL property of exponent 1/2 at its critical points; see Example 4.1.

**Example 4.1** Consider $r = 2$ and $\tilde{\Sigma} = aI$ for any constant $a > 0$.

Fix any $\lambda < a$. Take $(\bar{U}, \bar{V}) \in \mathbb{R}^{n_1 \times 2} \times \mathbb{R}^{n_2 \times 2}$ with $\bar{U}_1 = \bar{V}_1 = \text{Diag}(0, \sqrt{d})$ and $\bar{U}_2 = \bar{V}_2 = (0, 0)$ for $d = a - \lambda$. It is easy to check that $(\bar{U}, \bar{V})$ is a critical point of $\Phi_\lambda$ with $\tilde{\Phi}_\lambda(\bar{U}, \bar{V}) = \frac{\lambda^2 + \lambda^2}{2} + \lambda(a - \lambda)$. For each $k \in \mathbb{N}$, let $(U^k, V^k) \in \mathbb{R}^{n_1 \times 2} \times \mathbb{R}^{n_2 \times 2}$ with
\[ U_1^k = V_1^k = \left( \frac{0}{k_2^2 \sqrt{d} + \frac{1}{k_2}} \right) \] and \((U_2^k, V_2^k) = 0\). Clearly, \(\|U^k, V^k\) \(−\)(\(\overline{U}, \overline{V}\)\)\)\(||_F = O\left(\frac{1}{k^2} \right)\).

An elementary calculation yields
\[
(U_1^k)^T U_1^k - \overline{\Xi} = \left( \frac{k^2}{k^2} - a \cdot \frac{2 d + \sqrt{d}}{k^2} + \frac{1}{k^5} \right),
\]
and consequently, \(\tilde{\Phi}_\lambda(U^k, V^k) = \frac{a^2 + d^2}{2} + \lambda (a - \lambda) + O\left(\frac{1}{k^4}\right)\). Thus, we obtain that
\[
\tilde{\Phi}_\lambda(U^k, V^k) - \tilde{\Phi}_\lambda(\overline{U}, \overline{V}) = O\left(\frac{1}{k^4}\right).
\]

On the other hand, by the expression of \(\nabla \Phi_\lambda(U, V)\), it is not difficult to calculate that
\[
\|
\nabla \tilde{\Phi}_\lambda(U^k, V^k)\|^2_F = 2 \left\| U_1^k ((U_1^k)^T U_1^k - d I_2) \right\|^2_F
\]
\[
= 2 \left\| \left( \frac{2 d + \sqrt{d}}{k^2} + \frac{1}{k^5} \right) \right\|^2_F = O\left(\frac{1}{k^8}\right).
\]

The last two equations show that \(\tilde{\Phi}_\lambda\) associated to \(\lambda < a\) does not have the KL property of exponent 1/2 at the critical point \((\overline{U}, \overline{V})\).

For Example 4.1, after a simple calculation, the calmness modulus of the corresponding \(\Upsilon_1\) is at least \(2a - \lambda\). Clearly, the condition of Theorem 4.2 is not satisfied.

### 5 Numerical experiments

In this section, we confirm the previous theoretical findings by applying an accelerated alternating minimization (AAL) method for solving the nonconvex problem (6). Let \(\tilde{F}(U, V) := F(U V^T)\) for \((U, V) \in \mathbb{R}^{n_1 \times k} \times \mathbb{R}^{n_2 \times k}\), and denote by \(\nabla_1 \tilde{F}(U', V')\) and \(\nabla_2 \tilde{F}(U', V')\) denote the partial gradient of \(F\) with respect to \(U\) and \(V\), respectively, at \((U', V')\). Fix an arbitrary \(\lambda > 0\) and an arbitrary \((U^0, V^0) \in \mathbb{R}^{n_1 \times k} \times \mathbb{R}^{n_2 \times k}\). Write
\[
\mathcal{L}_{\lambda,0} := \left\{ (U, V) \in \mathbb{R}^{n_1 \times k} \times \mathbb{R}^{n_2 \times k} | \Phi_\lambda(U, V) \leq \Phi_\lambda(U^0, V^0) \right\}.
\]

Since the function \(\Phi_\lambda\) is coercive, the set \(\mathcal{L}_{\lambda,0}\) is nonempty and compact. Clearly, for each \((U, V) \in \mathcal{L}_{\lambda,0}\), the functions \(\nabla_1 \tilde{F}(\cdot, V)\) and \(\nabla_2 \tilde{F}(U, \cdot)\) are globally Lipschitz continuous on \(\mathcal{L}_{\lambda,0}\), and we denote by \(L_U\) and \(L_V\) their Lipschitz constants. Notice that \(L_U\) and \(L_V\) are determined by \(\|\nabla^2 F(U, V)\|\), which is bounded on the set \(\mathcal{L}_{\lambda,0}\). Hence, there exists a constant \(L_F > 0\) such that \(\max(L_U, L_V) \leq L_F\) for all \((U, V) \in \mathcal{L}_{\lambda,0}\). Now we describe the iterate steps of the AAL method for solving the problem (6).

**Remark 5.1** (i) Algorithm 1 is the special case of [34, Algorithm 1] with \(s = 2\). Since \(\Phi_\lambda\) is semialgebraic, by following the analysis technique there, one may achieve its global convergence. Moreover, together with the strict saddle property of \(\Phi_\lambda\) established in [19] and the equivalence relation between (5) and (6) (see Lemma 1 in Appendix C), the sequence generated by Algorithm 1 with \(\kappa = r\) for the least squares loss converges linearly to a global optimal solution \((\overline{U}, \overline{V})\) of (6) under the assumption of Theorem 4.2 with \(\beta/\alpha \leq 1.5\), and the error bound of \(\overline{X} = \overline{U} \overline{V}^T\) to the true \(M\) is \(O(\sqrt{\varphi(\lambda + \|A^*(\omega)\|)})\).
Algorithm 1 (AAL method for solving the problem (6))

Initialization: Choose an appropriate $\lambda > 0$, an integer $\kappa \geq 1$, a starting point $(U^0, V^0) \in \mathbb{R}^{n_1 \times \kappa} \times \mathbb{R}^{n_2 \times \kappa}$ and a parameter $\beta_0 \in [0, \sqrt{\frac{L}{L + L_F}}]$ with $L \geq L_F$, where $L_F$ is an upper bound of $\max(L_U, L_V)$ on $\mathcal{L}_{\lambda, 0}$. Set $(U^{-1}, V^{-1}) = (U^0, V^0)$ and $k = 0$.

while the stopping conditions are not satisfied do

- Set $\tilde{U}^k = U^k + \beta_k (U^k - U^{k-1})$ and $\tilde{V}^k = V^k + \beta_k (V^k - V^{k-1})$;
- Solve the following two minimization problems
  \[ U^{k+1} \in \arg \min_{U \in \mathbb{R}^{n_1 \times \kappa}} \left\{ \langle \nabla_1 F(\tilde{U}^k, V^k), U - \tilde{U}^k \rangle + \frac{L_F}{2} \| U - \tilde{U}^k \|_F^2 + \frac{\lambda}{2} \| U \|_F^2 \right\}, \]
  \[ V^{k+1} \in \arg \min_{V \in \mathbb{R}^{n_2 \times \kappa}} \left\{ \langle \nabla_2 F(\tilde{U}^{k+1}, \tilde{V}^k), V - \tilde{V}^k \rangle + \frac{L_F}{2} \| V - \tilde{V}^k \|_F^2 + \frac{\lambda}{2} \| V \|_F^2 \right\}. \]
- Update $\beta_k$ to be $\beta_{k+1}$ such that $\beta_{k+1} \in [0, \sqrt{\frac{L}{L + L_F}}]$.

end while

(ii) When the parameter $\beta_k$ is chosen by the formula $\beta_k = \frac{\theta_{k-1} - 1}{\theta_k}$ with $\theta_k$ updated by

\[ \theta_{k+1} := \frac{1}{2} \left( 1 + \sqrt{1 + 4 \theta_k^2} \right) \quad \text{for} \quad \theta_{-1} = \theta_0 = 1, \]

the accelerated strategy in Algorithm 1 is Nesterov’s extrapolation technique [22]. Unless otherwise stated, all numerical results are computed by this accelerated strategy.

(iii) By comparing the optimal conditions of the two subproblems with that of (6), when

\[ \frac{\| \nabla_1 F(\tilde{U}^k, V^k) - \nabla_1 F(U^{k+1}, V^{k+1}) + L_F (U^{k+1} - \tilde{U}^k) \|}{1 + \| y \|} \leq \epsilon, \]
\[ \frac{\| \nabla_2 F(U^{k+1}, \tilde{V}^k) - \nabla_2 F(U^{k+1}, V^{k+1}) + L_F (V^{k+1} - \tilde{V}^k) \|}{1 + \| y \|} \leq \epsilon \]

holds for a pre-given tolerance $\epsilon > 0$, we terminate Algorithm 1 at the iterate $(U^{k+1}, V^{k+1})$.

We take the least squares loss (3) for example to confirm our theoretical results. For the subsequent testing, the starting point $(U^0, V^0)$ of Algorithm 1 is always chosen as $(P \text{Diag}(\sqrt{\sigma^k(X^0)}), Q \text{Diag}(\sqrt{\sigma^k(X^0)}))$ with $(P, Q) \in \mathcal{O}^{n_1, n_2}(X^0)$ for $X^0 = A^*(y)$, where $\sigma^k(X^0) \in \mathbb{R}^k$ is the vector consisting of the first $\kappa$ components of $\sigma(X^0)$. It should be emphasized that such a starting point is not close to the bi-factors of $M$ unless $\kappa = r$. All numerical tests are done by a desktop computer running on 64-bit Windows Operating System with an Intel(R) Core(TM) i7-7700 CPU 3.6GHz and 16 GB memory.

5.1 RMSE comparison with convex relaxation method

We compare the relative RMSE (root-mean-square-error) yielded by Algorithm 1 for solving (6) with those yielded by the accelerated proximal gradient (APG) method for solving (5) (see [31]). Let $X_f$ be the final output of a solver. The RMSE is defined as

\[ \text{RMSE} := \frac{\| X_f - M \|_F}{\| M \|_F}. \]
We generate the vector $y \in \mathbb{R}^m$ via the model (4), where the true $M$ is generated by $M = U^* (V^*)^T$ with $U^* \in \mathbb{R}^{n_1 \times r}$ and $V^* \in \mathbb{R}^{n_2 \times r}$, the sampling operator $A$ is defined by $(A(X))_i = \langle A_i, X \rangle$ for $i = 1, 2, \ldots, m$ with $A_1, \ldots, A_m$ being i.i.d random Gaussian matrix whose entries follow the normal distribution $N(0, 1)$, and the entries of $\omega$ are i.i.d. and follow the normal distribution $N(0, \sigma^2_\omega)$ with $\sigma_\omega = 0.1 \|A^* (\omega)\|$ and follow the normal distribution $\mathcal{N}(0, \frac{1}{m})$ for $\xi \sim \mathcal{N}(0, I_m)$. We take $n_1 = n_2 = 100$, $r = 5$ and $m = 1950$ for testing. Figure 1 plots the relative RMSE of Algorithm 1 for solving (6) with $\kappa = 3r$ and $\lambda = v\|A^* (\omega)\|$ and that of APG for solving the convex problem (5) with the same $\lambda$. The stopping tolerance for the two solvers is chosen as $10^{-5}$. For each $v$, we conduct 5 tests and calculate the average relative RMSE of the total tests. We see that the relative RMSE of the two solvers has very little difference, but for $\lambda \leq 1.5 \|A^* (\omega)\|$ the solutions given by Algorithm 1 have lower ranks. This is not only consistent with the discussion in Remark 3.2(iii) but also implies that the factorization approach yields a lower rank solution with the same relative error.

5.2 Illustration of linear convergence

We take a matrix completion problem for example to illustrate the linear convergence of Algorithm 1 without accelerated strategy when solving problem (6) under the noisy and full sample setting, i.e., $F(X) = \frac{1}{2} \|X - \tilde{M}\|^2$ for $X \in \mathbb{R}^{n_1 \times n_2}$, where $\tilde{M} = M + E$ is a noisy observation. The true $M \in \mathbb{R}^{n_1 \times n_2}$ is generated in the same way as in Sect. 5.1, and the noise matrix $E$ is randomly generated by $E = 0.1 \frac{M}{\|M\|_F} B$, where every entry of $B$ obeys the standard normal distribution $\mathcal{N}(0, 1)$. We take $n_1 = n_2 = 3000$ and $r = 15$.

Figure 2 plots the iteration error curve $\| (U^k, V^k) - (U^f, V^f) \|_F$ of Algorithm 1, where $(U^f, V^f)$ is the final output of Algorithm 1 for solving (6) with $\kappa = r$, $\epsilon = 10^{-10}$ and the number of max iteration $k_{\text{max}} = 5000$. We see that the sequence $\{(U^k, V^k)\}$ displays the linear convergence behavior. By the strict saddle property in [19], there is a high probability for the limit point of $\{(U^k, V^k)\}$ to be a global optimal solution of (6). Consequently, the convergence behavior is consistent with the result of Theorem 4.2.
6 Conclusion

For the factorization form (6) of the nuclear norm regularized problem, we have derived the error bound to the true $M$ for the non-strict critical points with rank not more than $r$ under a restricted condition number assumption on $\nabla^2 F$, which is demonstrated to be optimal in the ideal noiseless and full sampling setting. In addition, for the factorized form of the squared $F$-norm regularized least squares loss function, we have established its KL property of exponent $1/2$ associated to almost all $\lambda > 0$ at its global minimizers under the noisy and full sampling setting, and also achieved this property at its certain class of critical points under the noisy and partial sample setting. This result, along with the strict saddle property in [19], partly improves the convergence analysis result of some first-order methods for solving the nonconvex problem (6) such as the alternating minimization methods in [15,25]. It is interesting to consider the error bound of critical points for other equivalent or relaxed factorization form of the rank regularized model (2). We will leave them as our future research topics.

Acknowledgements The authors are deeply indebted to Professor Bhojanapalli from Toyota Technological Institute at Chicago for providing some helpful comments on [2, Lemma 4.4]. The research of Shaohua Pan and Shujun Bi is supported by the National Natural Science Foundation of China under project No.11971177 and No.11701186.

Declarations

Funding Funding was provided by National Natural Science Foundation of China (Grant Nos. 11971177, 11701186).
Appendix A: The proof of Lemma 3.1.

Fix an arbitrary critical point \((U, V)\) of \(\Phi_\lambda\). Then, for any \((Z_U, Z_V) \in \mathbb{R}^{n_1 \times k} \times \mathbb{R}^{n_2 \times k}\),

\[
\langle \nabla \Phi_\lambda(U, V), Z \rangle = 0 \quad \text{with} \quad Z = (Z_U; Z_V).
\]

Recall that \(M = U^*V^\top\) with \((U^*, V^*) \in \mathcal{E}^*.\) From (14), this equality is equivalent to

\[
\left( \begin{bmatrix} 0 & \nabla F(X) - \nabla F(M) \\ \nabla F(M)^\top & 0 \end{bmatrix} \right) Z W^\top + \langle \Xi(M), Z W^\top \rangle = 0
\]

\[
\iff \langle \nabla F(X) - \nabla F(M), Z_U V^\top + U Z_V^\top \rangle + \langle \Xi(M), Z W^\top \rangle = 0
\]

\[
\iff \int_0^1 \nabla^2 F(tX + (1 - t)M)(X - M, Z_U V^\top + U Z_V^\top) \, dt + \langle \Xi(M), Z W^\top \rangle = 0.
\]

Since \(\text{rank}([X - M \quad Z_U V^\top + U Z_V^\top]) \leq 4r\), invoking Lemma 2.1 yields that

\[
\left| \frac{2}{\alpha + \beta} \nabla^2 F(tX + (1 - t)M)(X - M, Z_U V^\top + U Z_V^\top) - (X - M, Z_U V^\top + U Z_V^\top) \right|
\]

\[
\leq \frac{\beta - \alpha}{\alpha + \beta} \left( U V^\top - U^*V^\top \right) \|F\|_{Z_U V^\top + U Z_V^\top}.
\]

Combining this inequality with equality (52), we have

\[
\left| \frac{2}{\alpha + \beta} \langle \Xi(M), Z W^\top \rangle + \langle U V^\top - U^*(V^*)^\top, Z_U V^\top + U Z_V^\top \rangle \right|
\]

\[
\leq \beta - \alpha \left( U V^\top - U^*(V^*)^\top \|F\|_{Z_U V^\top + U Z_V^\top} \right).
\]

Take \(Z = (W W^\top - W^*(W^*)^\top) W^\top\) where \(W^*\) is defined as in (8) with \((U^*, V^*).\) Since the column orthonormal matrix \(Q\) spans the subspace \(\text{col}(W)\), it is not hard to check that \((W^\top)^\top W^\top = Q Q^\top\). Then, it follows that \(Z W^\top = (W W^\top - W^*(W^*)^\top) Q Q^\top\). Next we bound the terms \(I_1, I_2\) and \(I_3\) successively. First, for the term \(I_1\), it holds that

\[
I_1 = \frac{2}{\alpha + \beta} \langle \Xi(M), Z W^\top \rangle = \frac{2}{\alpha + \beta} \langle \Xi(M), (W W^\top - W^*(W^*)^\top) Q Q^\top \rangle.
\]

For the term \(I_2\), by recalling the definition of the linear operator \(P_{\text{off}}\), we have

\[
I_2 = \langle U V^\top - U^*(V^*)^\top, Z_U V^\top + U Z_V^\top \rangle = \langle P_{\text{off}}(W W^\top - W^*(W^*)^\top), Z W^\top \rangle.
\]

By the expressions of \(W, W^*\) and \(\hat{W}, \hat{W}^*\), it is not hard to check that

\[
P_{\text{off}}(W W^\top - W^*(W^*)^\top) = \frac{1}{2} [W W^\top - W^*(W^*)^\top - \hat{W} \hat{W}^\top + \hat{W}^*(\hat{W}^*)^\top].
\]

which along with \(Z W^\top = (W W^\top - W^*(W^*)^\top) Q Q^\top\) implies that

\[
I_2 = \frac{1}{2} \langle (W W^\top - W^*(W^*)^\top, (W W^\top - W^*(W^*)^\top) Q Q^\top \rangle
\]

\[
- \frac{1}{2} \langle \hat{W} \hat{W}^\top - \hat{W}^*(\hat{W}^*)^\top, (W W^\top - W^*(W^*)^\top) Q Q^\top \rangle.
\]
Since \((U, V)\) is a critical point of \(\Phi\), then by Lemma 2.2 and \((U^*, V^*) \in \mathcal{E}^*\) we have
\[
\langle \hat{W} \hat{W}^T - \hat{W}^* (\hat{W}^*)^T, (WW^T - W^* (W^*)^T)QQ^T \rangle = -\langle \hat{W} \hat{W}^T, W^* (W^*)^T QQ^T \rangle - \langle \hat{W}^* (\hat{W}^*)^T, WW^T QQ^T \rangle = -\langle \hat{W}^* (\hat{W}^*)^T, WW^T \rangle \leq 0
\]
where the second equality is using \(QQ^T = (W^T)^T W^T\) and the inequality is due to \((A, B) \geq 0\) for positive semidefinite \(A, B\). Then,
\[
I_2 \geq \frac{1}{2} \| (WW^T - W^* (W^*)^T)QQ^T \|_F^2 = \frac{1}{2} \| (WW^T - W^* (W^*)^T)QQ^T \|_F^2. \tag{55}
\]
For the term \(I_3\), recalling that \(Z = (WW^T - W^* (W^*)^T)W^T\), we calculate that
\[
I_3 = \| ZU V^T + UZV \|_F \leq \sqrt{2} \| \mathcal{P}_{off}(ZW^T) \|_F = \| ZW^T \|_F \tag{56}
\]
Now combining inequalities (54), (55) and (56) with (53) yields the desired result.

**Appendix B: The proof of Lemma 3.2.**

Let \(\Delta_U\) and \(\Delta_V\) denote the matrix consisting of the first \(n_1\) rows and the last \(n_2\) rows of \(\Delta\), respectively. Since \(\Delta \Delta^T = WW^T - W(W^* R_1)^T - W^* R_1 W^T + W^*(W^*)^T\) and \(W\) satisfies the first-order optimality condition \(\Xi(X) W = 0\) with \(\Xi(X) = \Xi(X)^T\), it follows that
\[
2 \langle \nabla F(X), \Delta_U \Delta_U^T \rangle + \lambda \langle \Delta, \Delta \rangle = \langle \begin{pmatrix} I & \nabla F(X) \\ \nabla F(X)^T & I \end{pmatrix}, \Delta \Delta^T \rangle
\]
\[
= \langle \Xi(X), WW^T - W(W^* R_1)^T - W^* R_1 W^T + W^*(W^*)^T \rangle
\]
\[
= \langle \Xi(X), W^*(W^*)^T - WW^T \rangle.
\]
Together with (15), we obtain the equality (22). We next show that the inequality (23) holds. From \(\nabla^2 \Phi(U, V)(\Delta, \Delta) \geq 0\) and equality (22), it follows that
\[
\nabla^2 F(X)(UU^T + \Delta_U V^T, U \Delta_V^T + \Delta_U V^T) \geq \langle \Xi(X), WW^T - W^*(W^*)^T \rangle. \tag{57}
\]
According to the given assumption on \(F\), it is immediate to have that
\[
\nabla^2 F(X)(UU^T + \Delta_U V^T, U \Delta_V^T + \Delta_U V^T) \leq \beta \| U \Delta_U^T + \Delta_U V^T \|_F^2
\]
\[
\leq 2 \beta (\| U \Delta_U^T \|_F^2 + \| \Delta_U V^T \|_F^2). \tag{58}
\]
In addition, by the restricted strong convexity of \(F\), it holds that
\[
\langle \Xi(X), WW^T - W^*(W^*)^T \rangle
\]
\[
= \langle \Xi(X) - \Xi(M), WW^T - W^*(W^*)^T \rangle + \langle \Xi(M), WW^T - W^*(W^*)^T \rangle
\]
\[
= 2 \langle \nabla F(X) - \nabla F(M), X - M \rangle + \langle \Xi(M), WW^T - W^*(W^*)^T \rangle
\]
\[
= 2 \int_0^1 \nabla^2 F(M + t(X - M))(X - M, X - M) dt + \langle \Xi(M), WW^T - W^*(W^*)^T \rangle
\]
\[
\geq 2 \alpha \| X - M \|_F^2 + \langle \Xi(M), WW^T - W^*(W^*)^T \rangle.
\]
Together with inequalities (57) and (58), we obtain
\[
2\beta(\| U \Delta^T \|^2_F + \| \Delta U \|^2_F) \geq 2\alpha \| X - M \|^2_F + \langle \Sigma(M), WW^T - W^*(W^*)^T \rangle
\]
\[
\iff \beta \| U \Delta^T \|^2_F \geq 2\alpha \| X - M \|^2_F + \langle \Sigma(M), WW^T - W^*(W^*)^T \rangle
\]
where the equivalence is due to \( \| U \Delta^T \|^2_F = \| V \Delta^T \|^2_F \) and \( \| V \Delta^T \|^2_F = \| U \Delta^T \|^2_F \), implied by \( U^T U = V^T V \). From [19, Lemma 4.5] it follows that
\[
\| WW^T - W^*(W^*)^T \|^2_F = \| \mathcal{P}_{on}(WW^T - W^*(W^*)^T) \|^2_F + \| \mathcal{P}_{off}(WW^T - W^*(W^*)^T) \|^2_F
\]
\[
\leq 2\| \mathcal{P}_{off}(WW^T - W^*(W^*)^T) \|^2_F = 4\| X - M \|^2_F,
\]
which implies that \( 2\alpha \| X - M \|^2_F \geq \frac{\beta}{2} \| WW^T - W^*(W^*)^T \|^2_F \). Together with (59), we obtain the desired inequality (23). The proof is completed.

**Appendix C**

The following lemma states the relation between the optimal solution set of (5) and the global optimal solution set of (6), whose proof is easy by the following result in [27]:

\[
\| X \|_* = \min_{R \in \mathbb{R}^{n_1 \times k}, L \in \mathbb{R}^{n_2 \times k}} \left\{ \frac{1}{2}(\| R \|^2_F + \| L \|^2_F) \right\} \text{ s.t. } X = RL^T. \tag{60}
\]

**Lemma 1** Fix an arbitrary \( \lambda > 0 \). If \((\overline{U}, \overline{V})\) is globally optimal to (6), then \( \overline{X} = \overline{U}\overline{V}^T \) is an optimal solution of (5) over the set \( \{ X \in \mathbb{R}^{n_1 \times n_2} \mid \text{rank}(X) \leq \kappa \} \); and conversely, if \( \overline{X} \) is an optimal solution of (5) with \( \text{rank}(\overline{X}) \leq \kappa \), then \((\overline{R}, \overline{L})\) with \( \overline{R} = \overline{P}[\text{Diag}(\sigma^k(\overline{X}))]^{1/2} \) and \( \overline{L} = \overline{Q}[\text{Diag}(\sigma^k(\overline{X}))]^{1/2} \) for \((\overline{P}, \overline{Q})\) is a global optimal solution to (6).

**Remark 1** Combining this lemma with [19, Theorem 4.1], we conclude that every critical point of (6) is either a global optimal solution or a strict saddle provided that \( F \) has the \((2\kappa, 4\kappa)\)-RSC of modulus \( \alpha \) and the \((2\kappa, 4\kappa)\)-RSS of modulus \( \beta \) with \( \beta/\alpha \leq 1.5 \).

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