REFINED HEXAGONS FOR DIFFERENTIAL COHOMOLOGY

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Abstract. Cheeger-Simons differential characters and differential $K$-theory are refinements of ordinary cohomology theory and topological $K$-theory respectively, and they are examples of differential cohomology. Each of these differential cohomology theories fits into a hexagon on the cohomology level. We show that these differential cohomology theories fit into hexagons on the cocycle level, and the hexagons on the cocycle level induce the hexagons on the cohomology level.

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1. INTRODUCTION

Given a generalized cohomology theory $E$ its differential extension $\hat{E}$ [3], also known as differential cohomology, is a refinement of $E$ to the category of smooth manifolds by incorporating differential form information. Differential character $\hat{H}$ [4] and differential $K$-theory $\hat{K}$ [7, 2, 5, 15] are examples of differential extensions of ordinary cohomology theory $\hat{H}$ and topological $K$-theory $K$ respectively. $\hat{H}$ and $\hat{K}$ have been studied extensively in recent years due to its importance in geometry and topology, for example [2, 6, 5], and its applications in theoretical physics, for example quantum field theory and string theory [7, 8].

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Let $X$ be a smooth manifold and $A$ a proper subring of $\mathbb{R}$. Differential characters $\tilde{H}^k(X;\mathbb{R}/A)$ fits into the following hexagon, i.e., the diagonal sequences are exact and every square and triangle commutes [4] (the details is given in Example 1).

Moreover, this hexagon uniquely characterizes differential characters [14, Theorem 1.1], i.e., for any two functors from the category of smooth manifolds to the category of abelian groups fitting into the hexagon (1) there exists a unique natural transformation between the functors. Thus one can define differential characters $\tilde{H}$ to be a functor fitting into the hexagon (1).

In [5, 15] it is proved that differential $K$-theory $\tilde{K}$ fits into an analogous hexagon.

Here we use Freed-Lott’s model of differential $K$-theory [5].

The existence and uniqueness of differential extension of generalized cohomology theory is studied by Bunke-Schick [3]. They propose axioms for
which the differential extension of generalized cohomology theory must satisfy, and prove the uniqueness of the differential extension under some natural assumptions [3, Theorem 3.10].

In [7, Section 3] Hopkins-Singer define differential cohomology theory to be the cohomology of a cochain complex by using homotopic-theoretic method. In particular, they give concrete descriptions of differential characters and differential $K$-theory in terms of cochain complexes respectively. Thus a natural question is whether the hexagons (1) and (2) hold on the cocycle level in a certain sense. We show that these are true and the hexagons on the cocycle level implies the hexagons on the cohomology level respectively (Theorem 1 and Theorem 2). Note that the statements for the hexagon on the cocycle level are (expected to be) slightly weaker than the ones on the cohomology level as we have more non-zero elements in the group of cocycles.

In [13] differential characters admitting topological trivializations (i.e., $I(f) = 0$, where $f \in \hat{H}^k(X; \mathbb{R}/\mathbb{Z})$) are considered. [13, Proposition 3.12, Theorem 3.15] prove that the right-hand square of (7) with $\mathbb{R}/\mathbb{Z}$ coefficients commutes when the differential character admits topological trivialization, and is a torsor for the right-hand square of (1). Thus Theorem 1 can be viewed as a generalization of [13, Proposition 3.12, Theorem 3.15] to differential characters not necessarily admitting topological trivializations and to all coefficients.

The paper is organized as follow. In Section 2 we include the necessary background materials, including Bunke-Schick axioms of differential extension of generalized cohomology theory, the hexagons for differential characters and differential $K$-theory, and Hopkins-Singer’s differential cocycle model. The main results are proved in Section 3.

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2. BACKGROUND MATERIALS

2.1. Bunke-Schick’s axioms for differential cohomology. We recall the axiomatic definition of differential cohomology given in [3].

**Definition 1.** [3 Definition 1.1] Let $X$ be a smooth manifold, $E$ a generalized cohomology theory and $\text{ch} : E^\bullet(X) \to H^\bullet(X; V)$ a natural transformation of cohomology theories, where $V := E^\bullet(\ast) \otimes_\mathbb{Z} \mathbb{R}$ is a $\mathbb{Z}$-graded real vector space. A differential extension of $(E, \text{ch})$ is a quadruple $(\hat{E}, R, I, a)$ such that

(1) $\hat{E}$ is a contravariant functor from the category of smooth manifolds to $\mathbb{Z}$-graded abelian groups,
(2) $R$, $a$ and $I$ are natural transformations of $\mathbb{Z}$-graded abelian group-valued functors such that $R \circ a = d$. Moreover, the following diagram commutes

$$
\begin{array}{ccc}
\hat{E}^\bullet(X) & \xrightarrow{I} & E^\bullet(X) \\
\downarrow R & & \downarrow \text{ch} \\
\Omega_{d=0}(X; V) & \xrightarrow{\text{deR}} & H^\bullet(X; V)
\end{array}
$$

where $\Omega_{d=0}(X; V)$ is the group of closed forms on $X$ with values in $V$, and the following sequence is exact

$$
E^{\bullet-1}(X) \xrightarrow{\text{ch}} \Omega^{\bullet-1}(X; V) \xrightarrow{a} \hat{E}^\bullet(X) \xrightarrow{I} E^\bullet(X) \longrightarrow 0
$$

The following examples summarize differential characters and Freed-Lott differential $K$-theory. In this paper $C_k(X)$ denotes the group of smooth singular chains.

**Example 1.** Let $X$ be a smooth manifold and $A$ a proper subring of $\mathbb{R}$. If $E = H$, differential characters $\hat{H}(X; \mathbb{R}/A)$ is the differential extension of $H(X; A)$. A differential character of degree $k \in \mathbb{N}$ is a homomorphism $f: Z_{k-1}(X) \rightarrow \mathbb{R}/A$ with a differential form $\omega \in \Omega^k(X)$ such that

$$
f(\partial c) = \int_c \omega \mod A,
$$

where $c \in C_k(X)$. It can be shown that $\omega$ is a closed form with periods in $A$. In [1], the maps are

$$
R(f) = \omega, \\
a(\eta)(z) = \int_z \eta \mod A, \\
I(f) = c_f \text{ which satisfies } r^*c_f = [\omega], \\
i([u]) = u|_{Z_{k-1}(X)}, \\
s([u]) = u, \\
\text{ch}([u]) = r^*[u], \\
\text{deR}(\omega) = [\omega],
$$

where the maps ($\sim, \beta, r$) are associated to the Bockstein sequence of the coefficient sequence $0 \rightarrow A \rightarrow \mathbb{R} \rightarrow \mathbb{R}/A \rightarrow 0$.

The map $s: H^{k-1}(X; \mathbb{R}) \rightarrow \Omega^{k-1}_A(X)$ is defined to be

$$
s([u]) = \omega \mod \Omega^{k-1}_{\text{exact}}(X),
$$

where $\text{deR}([\omega]) = [u]$ and $\text{deR}$ is the de Rham isomorphism between de Rham cohomology and singular cohomology.
**Example 2.** If $E = K$, differential $K$-theory $\hat{K}(X)$ is the differential extension of topological $K$-theory. In this paper we use Freed-Lott differential $K$-theory. Consider an abelian monoid $\text{Vect}(X)$ consists of elements of the form $E = (E, h, \nabla, \phi)$, where $E \to X$ is a Hermitian bundle, $h$ the Hermitian metric, $\nabla$ a connection compatible with $h$ and $\phi \in \Omega^{\text{odd}}(X)$. $E_1 = E_2$ if $E_1 \cong E_2$ and $\phi^E - \phi^F \equiv \text{CS}(\nabla^F, \nabla^E) \mod \Omega^{\text{odd}}(X)$. The only relation is $E_1 \sim E_2$ if and only if there exists $F = (F, h^F, \nabla^F, \phi^F) \in \text{Vect}(X)$ such that $E_1 \oplus F \equiv E_2 \oplus F$, i.e., $E_1 \oplus F \equiv E_2 \oplus F$ and $\phi_1 - \phi_2 = \text{CS}(\nabla^E_2 \oplus \nabla^F, \nabla^E_1 \oplus \nabla^F) \mod \Omega^{\text{odd}}(X)$. Define $\hat{K}(X) = \text{Vect}(X)/ \sim$. In [2], the maps are

\[
\begin{align*}
I(E, h, \nabla, \phi) &= [E], \\
R(E, h, \nabla, \phi) &= \text{ch}(\nabla) + d\phi, \\
a(\phi) &= (0, 0, d, \phi), \\
\alpha(\omega) &= ([\mathbb{C}^n], \nabla^{\text{flat}}, \omega) - ([\mathbb{C}^n], \nabla^{\text{flat}}, 0), \\
B(\mathcal{E} - \mathcal{F}) &= [E] - [F],
\end{align*}
\]

where $K^{-1}_L(X; \mathbb{R}/\mathbb{Z})$ is the geometric description of $\mathbb{R}/\mathbb{Z}$ $K$-theory [11]; $i : K^{-1}_L(X; \mathbb{R}/\mathbb{Z}) \to \hat{K}(X)$ is the canonical inclusion, and

\[
\Omega^*_{BU}(X) = \{ \omega \in \Omega^d_{d=0}(X) | [\omega] \in \text{Im}(\text{ch} : K^{-(\bullet \mod 2)} \to H^\bullet(X; \mathbb{Q})) \},
\]

where $\bullet \in \{ \text{even, odd} \}$. The maps $s$ and $dR$ are the same as in Example [1] and $\text{ch}$ is the Chern character composed with the inclusion of coefficients $\mathbb{Q} \hookrightarrow \mathbb{R}$.

### 2.2. Hopkins-Singer’s differential cocycle model

We recall Hopkins-Singer’s differential cocycle model of differential character [7 Section 3.2].

**Definition 2.** Fix $q \in \mathbb{N}$. Define a complex $\hat{C}^*(q)(X; A)$ by

\[
\hat{C}(q)^k(X; A) = \begin{cases} 
C^k(X; A) \times C^{k-1}(X; \mathbb{R}) \times \Omega^k(X), & \text{for } k \geq q \\
C^k(X; A) \times C^{k-1}(X; \mathbb{R}), & \text{for } k < q
\end{cases},
\]

with differential $\hat{d} : \hat{C}(q)^k(X; A) \to \hat{C}(q)^{k+1}(X; A)$ by

\[
\hat{d}(c, T, \omega) = (\delta c, \omega - c - \delta T, d\omega) \quad \text{for } k \geq q
\]

\[
\hat{d}(c, T) = \begin{cases} 
(\delta c, -c - \delta T, 0), & \text{for } k = q - 1 \\
(\delta c, -c - \delta T), & \text{for } k < q - 1
\end{cases}.
\]

One can easily check that $\hat{d}^2 = 0$. $(\hat{C}(q)^*(X; A), \hat{d})$ is the differential cocycle model of differential characters.

Let

\[
\hat{Z}(k)^k(X; A) : = \ker(\hat{d} : \hat{C}(k)^k(X; A) \to \hat{C}(k)^{k+1}(X; A)), \quad \text{and}
\]

\[
\hat{B}(k)^k(X; A) : = \hat{d}\hat{C}(k)^k(X; A)
\]
be the groups of cocycles and coboundaries respectively. Its cohomology group is denoted by \( \tilde{H}(k)_k^k(X; A) \). In [7, Section 3.2] it is proved that the map \( f : \tilde{H}(k)_k(X; A) \to \tilde{H}(k)_k(X; \mathbb{R}/A) \) given by

\[
f([c, T, \omega]) = T|_{Z_{k-1}(X)} \mod A
\]

is an isomorphism of graded commutative rings.

3. Main results

In this section we construct hexagons for differential characters and differential \( K \)-theory on the cocycle level and prove that each of these hexagons induces the corresponding hexagon on the cohomology level.

3.1. Refined hexagon for differential characters. We refine the hexagon for differential characters \( \tilde{H}(X; \mathbb{R}/\mathbb{Q}) \) with coefficients in \( \mathbb{Q} \) to the cocycle level. This case is of particular importance since it is the home for the differential Grothendieck-Riemann-Roch theorem for differential \( K \)-theory [2, 5]. An analogous but weaker result holds if \( \mathbb{Q} \) is replaced by any other proper subring of \( \mathbb{R} \).

**Theorem 1.** Let \( X \) be a smooth manifold. Consider the following diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & Z^{k-1}(X; \mathbb{R}/\mathbb{Q}) & \longrightarrow & B & \longrightarrow & Z^k(X; \mathbb{Q}) \times B^k(X; \mathbb{R}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & Z^{k-1}(X; \mathbb{R}) & \longrightarrow & \check{Z}(k)_k^k(X; \mathbb{Q}) & \longrightarrow & Z^k(X; \mathbb{R}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & \Omega^{k-1}(X) & \longrightarrow & \Omega^k(X) & \longrightarrow & \Omega^k_Q(X) & \longrightarrow & 0 \\
\end{array}
\]

(6)

where the maps will be defined in the proof. Every square and triangle commutes. The maps \( a \) is injective and \( R \) is surjective. The sequence

\[
0 \longrightarrow Z^{k-1}(X; \mathbb{R}/\mathbb{Q}) \longrightarrow \check{Z}(k)_k^k(X; \mathbb{Q}) \longrightarrow \Omega^k_Q(X) \longrightarrow 0
\]

is exact. Moreover, [6] induces [1], i.e., the commutativity of the squares and triangles and the exactness of the diagonal sequences of [1] can be deduced from (6). The analogous result holds for any proper subring \( A \) of \( \mathbb{R} \) if the group \( Z^{k-1}(X; \mathbb{R}/A) \) and the corresponding maps are not taken into consideration in this diagram.

The reason of the statement only works for \( \mathbb{Q} \) but not for any other proper subring \( A \) of \( \mathbb{R} \) will be clear from the proof.
Proof. The maps are defined as follows:
\[
I(c, T, \omega) = (c, \delta T),
\]
\[
R(c, T, \omega) = \omega,
\]
\[
a(\eta) = (0, \eta, d\eta),
\]
\[
ch(u, \delta S) = r(u) + \delta S,
\]
\[
deR(\omega)(c) = \int_c \omega \text{ for } c \in C_k(X),
\]
\[
s(u) = u,
\]
\[
-B(c) = (-\beta(c), 0),
\]
\[
i(c) = (-\beta(c), c, 0),
\]
where \(\alpha\) and \(r\) are the maps induced by the maps in the coefficient sequence
\[
0 \longrightarrow \mathbb{Q} \xrightarrow{r} \mathbb{R} \xrightarrow{\alpha} \mathbb{R}/\mathbb{Q} \longrightarrow 0
\]
The map \(i : Z^{k-1}(X; \mathbb{R}/\mathbb{Q}) \rightarrow \tilde{Z}(k)^k(X; \mathbb{Q})\) is defined as follows. By the axiom of choice there exists a well defined group homomorphism \(j : \mathbb{R}/\mathbb{Q} \rightarrow \mathbb{R}\) such that \(j([x]) \in [x]\) [11 p.54 and Remark 5.27]. By fixing such a map in the construction of the Bockstein homomorphism, it is well defined on the cocycle level (see Appendix A). The second entry \(c\) of the map \(i\) is considered to be \(c \in C^{k-1}(X; \mathbb{R})\) again by choosing the fixed map \(j : \mathbb{R}/\mathbb{Q} \rightarrow \mathbb{R}\). The map \(-B\) is defined in a similar way.

The map \(s\) is defined as follows. By [11 Theorem 10.6] there exists a smooth triangulation \((K, f)\) [11 Definition 8.3] of \(X\). In [12] a chain map \(\eta_k : C^\text{simp}_k(K) \rightarrow C^\text{sing}_k(X)\) is defined from simplicial chains into singular chains which induces isomorphisms for both homology [12 Theorem 34.3] and cohomology [12 Theorem 44.2] for every \(k \in \mathbb{N}\). Denote by \(I_k : \Omega^k(X) \rightarrow C^k_\text{simp}(K)\) the chain map given by \(I_k(\omega)(c) = \int_c \omega\) for \(c \in C^k_\text{simp}(K)\). It is shown in [16 §27, Chapter IV] that there exists a chain map \(\phi_k : C^k_\text{simp}(K) \rightarrow \Omega^k(X)\) such that \(I_k \circ \phi_k = \text{id}_{C^k_\text{simp}(K)}\) for every \(k \in \mathbb{N}\).

Define the map \(s : Z^{k-1}_\text{sing}(X; \mathbb{R}) \rightarrow \Omega^{k-1}(X)\) to be the composition
\[
Z^{k-1}_\text{sing}(X; \mathbb{R}) \xrightarrow{\eta_k^{-1}} Z^{k-1}_\text{simp}(K; \mathbb{R}) \xrightarrow{\phi_k^{-1}} \Omega^{k-1}(X)
\]
Henceforth we suppress the subscripts simp and sing. \textit{A priori} the map \(s : Z^{k-1}(X; \mathbb{R}) \rightarrow \Omega^{k-1}(X)\) depends on the choice of the smooth triangulation of \(X\). However, if \((L, g)\) is another triangulation of \(X\), then there exist \(\delta\)-approximations [11 Definition 8.5] \(f' : K' \rightarrow X\) of \(f : K \rightarrow X\) and \(g' : L' \rightarrow X\) of \(g : L \rightarrow X\), where \((K', f')\) and \((L', g')\) are also smooth triangulations of \(X\), such that \(L'\) is linearly isomorphic to \(K'\) via \((f')^{-1} \circ g'\) [11 Theorem 10.5]. By replacing \(K\) by \(K'\) if necessary, we see that \(s\) is independent of the choice of the triangulation of \(X\).

For the injectivity of \(a\), if \(a(\eta) = (0, \eta, d\eta) = (0, 0, 0)\), then \(\eta = 0\).
For the injectivity of $i$, if $i(u) = (-\beta(u), u, 0) = (0, 0, 0)$, then $u = 0 \in C^{k-1}(\mathbb{R})$, and therefore $u \in Z^{k-1}(X; \mathbb{R}/\mathbb{Q})$.

For the surjectivity of $R$, let $\omega \in \Omega^k_Q(X)$. Then there exists a unique $[c] \in H^k(X; \mathbb{Q})$ such that $[\omega] = r[c]$. Thus $\omega = r(c) + \delta T \in Z^k(X; \mathbb{R})$ for some $T \in C^{k-1}(X; \mathbb{R})$. Note that $(c, T, \omega) \in \tilde{Z}^k(X; \mathbb{Q})$ and $R(c, T, \omega) = \omega$.

For the surjectivity of $I$, let $(c, \delta T) \in Z^k(X; \mathbb{Q}) \times B^k(X; \mathbb{R})$. Since there exists a unique de Rham class $[\omega] \in H^k_{\text{dR}}(X)$ such that $[rc + \delta T] = [\omega]$, it follows $\delta T = \omega + \delta T' - c$ for some $T' \in \Omega^{k-1}(X)$. Thus $(c, \delta T, \omega + dT') \in \tilde{Z}(k)^k(X; \mathbb{Q})$ such that $I(c, \delta T, \omega + dT') = (c, \delta T)$.

To prove the exactness at $\tilde{Z}(k)^k(X; \mathbb{Q})$, note that $R \circ i = 0$ follows directly from the definitions of $R$ and $i$. Let $(c, T, \omega) \in \ker(R)$. Then $\delta T = -c$, which implies $\delta T = 0$. Thus $\tilde{T} \in Z^{k-1}(\mathbb{R}/\mathbb{Q})$. By the construction of the Bockstein homomorphism (see Appendix A), we have $\beta(\tilde{T}) = -c$. Thus

$$(c, T, 0) = (-\beta(\tilde{T}), T, 0) = i(\tilde{T}).$$

Therefore $\text{Im}(i) = \ker(R)$.

For the commutativity of the right square, since $(c, T, \omega) \in \tilde{Z}(k)^k(X; \mathbb{Q})$ implies $\delta T = \omega - c$, we have

$$\text{ch}(I(c, T, \omega)) = \text{ch}(c, \delta T) = r(c) + \delta T = \int \omega = \text{deR}(R(\omega)).$$

Note that $R \circ a = d$ follows directly from definitions of $R$ and $a$. To prove $I \circ i = -B$, take $u \in Z^{k-1}(X; \mathbb{R}/\mathbb{Q})$. Then

$$I(i(u)) = I(-\beta(c), c, 0) = (-\beta(c), \delta c) = (-\beta(c), 0) = -B(c).$$

For the commutativity of the left square, take $u \in Z^{k-1}(\mathbb{R})$. Then

$$i(\alpha(u)) = i(\bar{u}) = (-\beta(u), u, 0) = (0, u, 0) = (0, u, d\bar{u}) = a(u) = a(s(u)).$$

We prove that (6) induces (1), i.e., when $\tilde{Z}(k)^k(X; \mathbb{Q})$ is quotient out by $\bar{B}(k)^k(X; \mathbb{Q})$, an appropriate subgroup will be quotient out in every other entry in (6), and the resulting diagram and the corresponding induced maps, for which we will denote by the same symbols, coincide with (1) under the isomorphism (5).

To prove the induced map $R : \tilde{H}(k)^k(X; \mathbb{Q}) \to \Omega^k_Q(X)$ is well defined, note that $R(\bar{B}(k)^k(X; \mathbb{Q})) = 0$, which follows from that fact that if $\bar{d}(c, T) = (\delta c, -c - \delta T, 0) \in \bar{B}(k)^k(X; \mathbb{Q})$, then $R(\delta c, -c - \delta T, 0) = 0$.

Note that $I(\bar{d}(c, T)) = I(\delta c, -c - \delta T, 0) = (\delta c, -\delta c) \in B^k(X; \mathbb{Q}) \times B^k(X; \mathbb{R})$. Thus the induced map

$$I : \tilde{H}(k)^k(X; \mathbb{Q}) \to \frac{Z^k(X; \mathbb{Q}) \times B^k(X; \mathbb{R})}{B^k(X; \mathbb{Q}) \times B^k(X; \mathbb{R})} \cong H^k(X; \mathbb{Q})$$

is well defined.

Let $\eta \in \Omega^{k-1}_Q(X)$. Then

$$a(\eta) = (0, \eta, d\eta) = (0, \eta, 0) = \bar{d}(\eta, 0) \in \bar{B}(k)^k(X; \mathbb{Q}).$$
Thus the induced map \( a : \frac{\Omega^{k-1}(X)}{\Omega^k_{\mathbb{Q}}(X)} \to \tilde{H}(k)^k(X; \mathbb{Q}) \) is well defined.

Finally, note that for \( \delta u \in B^{k-1}(X; \mathbb{R}/\mathbb{Q}) \),

\[
i(\delta u) = (-\beta(\delta u), \delta u, 0) = (0, \delta u, 0) = d(0, u) \in \tilde{B}(k)^k(X; \mathbb{Q}),
\]

where \( u \in C^{k-2}(X; \mathbb{R}/\mathbb{Q}) \) is regarded as \( j(u) \in C^{k-2}(X; \mathbb{R}) \) through (6). Thus the induced map \( i : H^{k-1}(X; \mathbb{R}/\mathbb{Q}) \to \tilde{H}(k)^k(X; \mathbb{Q}) \) is well defined. We quotient \( Z^k(X; \mathbb{R}) \) out by \( \tilde{B}(k)^k(X; \mathbb{R}) \) to get \( H^k(X; \mathbb{R}) \) in the right-most entry of (6) and similarly for the left-most entry in (6) to get \( H^{k-1}(X; \mathbb{R}) \).

It is easy to see that all the induced maps of (6) coincide with the maps in (1) by the isomorphism (5). Thus (6) induces (1). \( \square \)

**Remark 1.** Note that the hexagon on the cocycle level in Theorem 1 is valid for each fixed choice of triangulation of \( X \) and each fixed map \( \mathbb{R}/\mathbb{Q} \to \mathbb{R} \) coming from the axiom of choice. Of course it is independent of the choices when hexagon is passed to the cohomology level.

### 3.2. Refined hexagon for differential \( K \)-theory

In this subsection we construct a hexagon for differential \( K \)-theory on the cocycle level and prove that it induces the hexagon on the cohomology level.

Instead of using Hopkins-Singer’s definition of differential \( K \)-theory [7, Section 4.4], we use Freed-Lott differential \( K \)-theory, a geometric model given by vector bundles with connections and differential forms.

Let \( X \) be a compact manifold. Recall call from Example 2 that \( \hat{K}(X) := \hat{\text{Vect}}(X)/\sim \). Let \( \hat{\text{Vect}}_{L}^{-1}(X) \) be the sub semi-group of \( \hat{\text{Vect}}(X) \) whose elements \((E, h, \nabla, \phi)\) satisfy the relation \( \text{ch}(\nabla) - \text{rank}(E) = -d\phi \).

Define an abelian monoid

\[
\hat{\text{Vect}}'(X) := \{(E, \omega) \in \text{Vect}(X) \times \Omega_{d=0}^{\text{even}}(X) | [\omega] = r \text{ch}(E)\}.
\]

Two elements of the form \((E, \omega)\) and \((F, \beta)\) are equal if \( E \cong F \) and \( \omega = \beta \mod \Omega_{\text{exact}}^{\text{even}}(X) \). Define \( K'(X) := K(\hat{\text{Vect}}'(X)) \) to be the symmetrization of \( \hat{\text{Vect}}'(X) \). We write the class of \((E, \omega) \in \hat{\text{Vect}}'(X)\) in \( K'(X) \) as \([E, \omega]\).

**Lemma 1.** For any compact manifold \( X \), \( K'(X) \cong K(X) \).

**Proof.** Note that every element of \( K'(X) \) can be written as \([E, \omega] - [F, \beta]\). \( [E, \omega] - [F, \beta] = [E', \omega'] - [F', \beta'] \) if and only if there exists \((G, \eta) \in \hat{\text{Vect}}'(X)\) such that \( E \oplus F' \oplus G \cong E' \oplus F \oplus G \) and \( \omega - \beta = \omega' - \beta' \mod \Omega_{\text{exact}}^{\text{even}}(X) \).

Define a map \( f : K'(X) \to K(X) \) by

\[
f([E - F, \omega - \beta]) = [E] - [F].
\]

We prove that \( f \) is a well defined group isomorphism. Suppose \([E - F, \omega - \beta] = [E' - F', \omega' - \beta'] \in K'(X)\), then there exists \((G, \eta) \in \hat{\text{Vect}}'(X)\) such that \( E \oplus F' \oplus G \cong E' \oplus F \oplus G \) and \( \omega - \beta = \omega' - \beta' \mod \Omega_{\text{exact}}^{\text{even}}(X) \). Thus \([E] - [F] = [E'] - [F']\), which implies that \( f \) is well defined. Obviously \( f \) is a surjective group homomorphism. For the injectivity of \( f \), suppose \( f([E, \omega] - [F, \beta]) = [E] - [F] = 0 \). Then there exists a \( G \in \text{Vect}(X) \) such
that $E \oplus G \cong F \oplus G$. Since $[\omega] - [\beta] = r(\text{ch}(E) - \text{ch}(F)) = 0$, it follows that $\omega = \beta \mod \Omega^\text{even}_{\text{exact}}(X)$. Thus $[E, \omega] - [F, \beta] = 0 \in K'(X)$. \hfill $\square$

**Theorem 2.** Let $X$ be a compact manifold. Consider the following diagram

\[
\begin{array}{cccc}
\hat{\text{Vect}}_l^{-1}(X) & \xrightarrow{B} & \text{Vect}'(X) \\
\downarrow \alpha & & \downarrow \chi' \\
Z^{\text{odd}}(X; \mathbb{R}) & \xrightarrow{I'} & \text{Vect}(X) & \xrightarrow{R} & Z^{\text{even}}(X; \mathbb{R}) \\
\downarrow s & & \downarrow d & & \downarrow \text{deR} \\
\Omega^{\text{odd}}(X) & \xrightarrow{a} & \Omega^{\text{even}}_{\text{BU}}(X) \\
\end{array}
\]

where all the maps will be defined in the proof. Every square and triangle commutes. The maps $I'$ and $R$ are surjective and $I' \circ a = 0$. Moreover, $(7)$ induces $(2)$.

**Proof.** The maps are defined as follows:

$I'(E, h, \nabla, \phi) = (E, \text{ch}(\nabla) + d\phi),$

$R(E, h, \nabla, \phi) = \text{ch}(\nabla) + d\phi,$

$a(\omega) = (0, 0, d, \omega),$

$i(E, h, \nabla, \phi) = (E, h, \nabla, \phi),$

$\text{deR}(\omega)(c) = \int_c \omega$ \text{ where } c \in C^{\text{even}}(X),

$B(E, h, \nabla, \phi) = (E, \text{ch}(\nabla) + d\phi),$

$s(u) = u, \text{ (as in the proof of Theorem 1)},$

$\alpha(u) = (0, 0, d, u),$

$\chi'(E, \omega)(c) = \int_c \omega, \text{ where } c \in C^{\text{even}}(X).$

The map $R$ is clearly surjective. For the surjectivity of $I'$, let $[E, \omega] \in K'(X)$. Since $\omega = \text{ch}(\nabla) + d\phi$ for some connection $\nabla$ on $E \to X$ and for some $\phi \in \Omega^{\text{odd}}(X)$, it follows that $I'(E, h, \nabla, \phi) = [E, \text{ch}(\nabla) + d\phi] = [E, \omega]$.

From the definitions, we have $R \circ a = d$ and $I' \circ i = -B$. To prove the commutativity of the right square, let $(E, h, \nabla, \phi) \in \hat{\text{Vect}}(X)$. Then

\[
\chi'(I'(E, h, \nabla, \phi)) = \chi'(E, \text{ch}(\nabla) + d\phi) = \int (\text{ch}(\nabla) + d\phi) = \text{deR}(\text{ch}(\nabla) + d\phi) = \text{deR}(R(E, h, \nabla, \phi)).
\]

For the commutativity of the left square, let $u \in Z^{\text{odd}}(X; \mathbb{R})$. Then

\[
i(\alpha(u)) = i(0, 0, d, u) = (0, 0, d, u) = a(u) = a(s(u)).
\]

We prove that $(7)$ induces $(2)$. As in the proof of Theorem 1, we show that once we quotient out $\text{Vect}(X)$ by the equivalence relation $\sim$, every
other entry in (9) is quotient out by a suitable subgroup so that the induced hexagon coincides with (2), i.e., all the entries and the corresponding maps are equal.

Let $\mathcal{E}_i = (E^i, h^i, \nabla^i, \phi^i) \in \text{Vect}(X)$, where $i = 1, 2$, be such that $\mathcal{E}_1 - \mathcal{E}_2 = 0 \in \widehat{K}(X)$, i.e., there exists $(F, h^F, \nabla^F, \phi^F) \in \text{Vect}(X)$ such that $E^1 \oplus F \cong E^2 \oplus F$ and
\[
\phi^1 - \phi^2 = \text{CS}(\nabla^2 \oplus \nabla^F, \nabla^1 \oplus \nabla^F) \mod \Omega_{\text{exact}}^{\text{odd}}(X).
\]
Then
\[
R(\mathcal{E}_1 - \mathcal{E}_2) = \text{ch}(\nabla^1) + d\phi^1 - \text{ch}(\nabla^2) - d\phi^2
= d(\text{CS}(\nabla^1 \oplus \nabla^F, \nabla^2 \oplus \nabla^F) + \phi^1 - \phi^2) = 0 \text{ from (9)}.
\]
Thus the induced map $R : \widehat{K}(X) \to \Omega_{\text{BU}}^{\text{even}}(X)$ is well defined.

The induced map $\text{ch}' : K'(X) \to H^{\text{even}}(X; \mathbb{R})$ is well defined since for $[E - F, \omega - \beta] = 0 \in K'(X)$ and for any $z \in Z_{\text{even}}(X)$, we have
\[
\text{ch}'([E, \omega] - [F, \beta])(z) = \text{ch}'(E, \omega)(z) - \text{ch}'(F, \beta)(z) = \int_z \omega - \beta = \int_z d\alpha = 0,
\]
where $\alpha \in \Omega^{\text{odd}}(X)$.

We claim that the induced map $I' : \widehat{K}(X) \to K'(X)$ is well defined. To see this, suppose $\mathcal{E} = F \in \widehat{K}(X)$. Then there exists $(G, h^G, \nabla^G, \phi^G) \in \text{Vect}(X)$ such that $E \oplus G \cong F \oplus G$ and $\phi^E - \phi^F = \text{CS}(\nabla^E \oplus \nabla^G, \nabla^F \oplus \nabla^G)$, which implies that $d(\phi^F - \phi^E) = \text{ch}(\nabla^E) - \text{ch}(\nabla^F)$. Then
\[
I'([E - F, \omega - \beta]) = [E, \text{ch}(\nabla^E)] + d\phi^E - [F, \text{ch}(\nabla^F)] = 0 \in K'(X).
\]
Define the map $I : \widehat{K}(X) \to K(X)$ to be the composition
\[
\widehat{K}(X) \stackrel{I'}{\longrightarrow} K'(X) \stackrel{f}{\longrightarrow} K(X),
\]
where $f$ is given in Lemma 2. Thus
\[
I([E - F]) = f(I'([E - F])) = f([([E - F, \omega - \beta])]) = [E] - [F],
\]
which coincides with the map $I : \widehat{K}(X) \to K(X)$ in (2).

The induced map $\text{deR} : \Omega_{\text{BU}}^{\text{even}}(X) \to H^{\text{even}}(X; \mathbb{R})$ is defined to be the composition
\[
\Omega_{\text{BU}}^{\text{even}}(X) \stackrel{\text{deR}}{\longrightarrow} Z^{\text{even}}(X; \mathbb{R}) \stackrel{[1]}{\longrightarrow} H^{\text{even}}(X; \mathbb{R})
\]
which coincide with the map $\text{deR}$ in (2). The induced map $\alpha : H^{\text{odd}}(X; \mathbb{R}) \to K_{L}^{-1}(X; \mathbb{R}/\mathbb{Z})$ is well defined and coincides with the map $\alpha$ in (2) by the definition of $K_{L}^{-1}(X; \mathbb{R}/\mathbb{Z})$. Since $i : \text{Vect}_{L}^{-1}(X) \to \text{Vect}(X)$ is the canonical inclusion map, its induced map $i : K_{L}^{-1}(X; \mathbb{R}/\mathbb{Z}) \to \widehat{K}(X)$ is well defined and coincides with the map $i$ in (2).

We prove that the induced map $a : \Omega_{\text{BU}}^{\text{odd}}(X) \to \widehat{K}(X)$ is well defined. Let $\omega \in \Omega_{\text{BU}}^{\text{odd}}(X)$. Then $\omega = \text{ch}^{\text{odd}}([g]) + d\alpha$, where $g : X \to U(n)$ is a smooth
map for some \( n \in \mathbb{N}, \alpha \in \Omega^\text{even}(X) \) and \( \text{ch}^{\text{odd}}(\lceil g \rceil) := \text{CS}(g^{-1}dg, d) \). Here we have identified \( K^{-1}(X) \cong [X, U] \). Since \( a(\omega) = a(\text{ch}^{\text{odd}}(\lceil g \rceil)) + a(\alpha) \) and \( a(d\alpha) = (0, 0, d, d\alpha) = 0 \), it suffices to show that \( a(\text{ch}^{\text{odd}}(\lceil g \rceil)) = 0 \). Since \( R \circ a = d \), it follows from the exactness of the following sequence

\[
0 \longrightarrow K^{-1}_L(X; \mathbb{R}/\mathbb{Z}) \overset{i}{\longrightarrow} \hat{K}(X) \overset{R}{\longrightarrow} \Omega^\text{even}_\text{BU}(X) \longrightarrow 0
\]

that \( a(\text{ch}^{\text{odd}}(\lceil g \rceil)) \in K^{-1}_L(X; \mathbb{R}/\mathbb{Z}) \). Consider the following sequence

\[
K^{-1}(X) \overset{\text{ch}^{\text{odd}}}{\longrightarrow} H^\text{odd}(X; \mathbb{R}) \overset{i}{\longrightarrow} \hat{K}(X) \overset{\alpha}{\longrightarrow} \Omega^\text{odd}(X) \begin{array}{c} \Omega^\text{odd}_\text{BU}(X) \end{array}
\]

Since the upper sequence is exact [9, §7.21] and the square commutes, it follows that

\[
a(\text{ch}^{\text{odd}}(\lceil g \rceil)) = a(s(\text{ch}^{\text{odd}}(\lceil g \rceil))) = i(\alpha(\text{ch}^{\text{odd}}(\lceil g \rceil)))) = i(0) = 0.
\]

Thus the map \( a : \Omega^\text{odd}_\text{BU}(X) \rightarrow \hat{K}(X) \) is well defined.

Define \( \text{ch} : K(X) \rightarrow H^\text{even}(X; \mathbb{R}) \) to be the composition

\[
K(X) \overset{f^{-1}}{\longrightarrow} K'(X) \overset{\text{ch}'}{\longrightarrow} H^\text{even}(X; \mathbb{R})
\]

Note that this map \( \text{ch} : K(X) \rightarrow H^\text{even}(X; \mathbb{R}) \) coincides with the map \( \text{ch} : K(X) \rightarrow H^\text{even}(X; \mathbb{R}) \) defined in [2].

The commutativity of the following square

\[
\begin{array}{c}
\hat{K}(X) \overset{I}{\longrightarrow} K(X) \\
\Omega^\text{even}_\text{BU}(X) \overset{\text{deR}}{\longrightarrow} H^\text{even}(X; \mathbb{R})
\end{array}
\]

follows from the fact that \( \text{ch} \circ I = \text{ch}' \circ f^{-1} \circ f \circ I' = \text{ch}' \circ I' = \text{deR} \circ R \), where the last equality follows from (8). Thus (7) induces (2). \( \square \)

**Remark 2.** In Theorem 1 we have the exact sequence

\[
0 \longrightarrow Z^{k-1}(X; \mathbb{R}/\mathbb{Q}) \overset{i}{\longrightarrow} \hat{Z}(k)^b(X; \mathbb{R}/\mathbb{Q}) \overset{R}{\longrightarrow} \Omega^b_{\mathbb{Q}}(X) \longrightarrow 0
\]

while in Theorem 2 the “corresponding” sequence

\[
0 \longrightarrow \hat{\text{Vect}}^{-1}_L(X) \overset{i}{\longrightarrow} \hat{\text{Vect}}(X) \overset{R}{\longrightarrow} \Omega^\text{even}_{\text{BU}}(X) \longrightarrow 0
\]

is not exact and not even \( R \circ i = 0 \). However, when these entries are quotient out by the coboundaries we have the exactness and \( R \circ i = 0 \). This
is due to the fact that $\widehat{\text{Vect}}_{L}^{-1}(X)$ is a sub semi-group of $\widehat{\text{Vect}}(X)$ but not a sub-monoid and, after symmetrizing, $K_{L}^{-1}(X;\mathbb{R}/\mathbb{Z})$ is a proper subgroup of $K(\widehat{\text{Vect}}_{L}^{-1}(X))$ consisting of elements with virtual rank zero.

**Appendix A. Bockstein homomorphism**

In this appendix we recall the construction of the Bockstein homomorphism $\beta : H^{k-1}(X;\mathbb{R}/\mathbb{Q}) \to H^{k}(X;\mathbb{Q})$ associated to the coefficient sequence $0 \to \mathbb{Q} \to \mathbb{R} \to \mathbb{R}/\mathbb{Q} \to 0$, and show that $\beta(\delta u) = 0 \in C^{k}(X;\mathbb{Q})$ for $u \in C^{k-2}(X;\mathbb{R}/\mathbb{Q})$.

Consider the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & C^{k}(X;\mathbb{Q}) & \overset{r}{\longrightarrow} & C^{k}(X;\mathbb{R}) & \overset{\sim}{\longrightarrow} C^{k}(X;\mathbb{R}/\mathbb{Q}) & \longrightarrow 0 \\
0 & \longrightarrow & C^{k-1}(X;\mathbb{Q}) & \overset{r}{\longrightarrow} & C^{k-1}(X;\mathbb{R}) & \overset{\sim}{\longrightarrow} C^{k-1}(X;\mathbb{R}/\mathbb{Q}) & \longrightarrow 0 \\
 \end{array}
\]

Let $u \in Z^{k-1}(X;\mathbb{R}/\mathbb{Q})$. There exists a lift $b \in C^{k-1}(X;\mathbb{R})$ such that $\tilde{b} = u$. \hspace{1cm} (10)

Since $0 = \delta u = \delta \tilde{b} = \delta \tilde{b}$, there exists a unique $a \in C^{k}(X;\mathbb{Q})$ such that $\delta b = ra$. \hspace{1cm} (11)

The Bockstein homomorphism is defined by $\beta(u) = a$. It is well defined on the cohomology level.

Since we have a well defined group homomorphism $j : \mathbb{R}/\mathbb{Q} \to \mathbb{R}$ from the axiom of choice [1, p.54 and Remark 5.27], we can take the lift of $u$ to be $j(u)$, where we denote the homomorphism $j : \text{Hom}(C_{k-1}(X),\mathbb{R}/\mathbb{Q}) \to \text{Hom}(C_{k-1}(X),\mathbb{R})$ by the same symbol as $j : \mathbb{R}/\mathbb{Q} \to \mathbb{R}$. Thus the Bockstein homomorphism $\beta$ is well defined on the cocycle level. If $u = \delta \mu$, where $\mu \in C^{k-2}(X;\mathbb{R}/\mathbb{Q})$, then there exists $\gamma \in C^{k-2}(X;\mathbb{R})$ such that $\gamma = \mu$. Thus $u = \delta \mu = \delta \tilde{\gamma} = \tilde{\gamma}$, and therefore we can take the $b$ in (10) to be $\delta \gamma$. From (11) we have $ra = \delta \gamma = 0$. Since $r$ is injective, it implies that $\beta(\delta \mu) = a = 0 \in C^{k}(X;\mathbb{Q})$.

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