Truth and Subjunctive Theories of Knowledge: No Luck?

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Withdrawn: The stabilizing ordinal of a sentence is ill-defined (Definition 14). I am not sure whether this can be fixed.

Abstract
The paper explores applications of Kripke’s theory of truth to semantics for anti-luck epistemology, that is, to subjunctive theories of knowledge. Subjunctive theories put forward modal or subjunctive conditions to rule out knowledge by mere luck as to be found in Gettier-style counterexamples to the analysis of knowledge as justified true belief. Because of the subjunctive nature of these conditions the resulting semantics turns out to be non-monotone, even if it is based on non-classical evaluation schemes such as strong Kleene or FDE. This blocks the usual road to fixed-point results for Kripke’s theory of truth within these semantics and consequently the paper is predominantly an exploration of fixed-point results for Kripke’s theory of truth within non-monotone semantics. Using the theory of quasi-inductive definitions we show that in case of the subjunctive theories of knowledge the so-called Kripke jump will have fixed points despite the non-monotonicity of the semantics: Kripke’s theory of truth can be successfully applied in the framework of subjunctive theories of knowledge.

1 Introduction
In contemporary epistemology a great amount of work has been devoted to singling out conditions that guarantee that an agent holds a true belief not merely by mere luck but on firmer grounds. Most importantly, so-called modal or subjunctive conditions have been employed to this effect. These conditions are intended to rule out cases in which an agent holds a true belief but in cases rather similar to the present, actual case the belief would turn out false. The most prominent of these conditions are Sensitivity, Adherence, and Safety:

(Sensitivity) In the closest ¬φ-worlds, the agent does not believe φ;

(Adherence) In the closest φ-worlds, the agent believes φ;

(Safety) was introduced to the debate by Sosa (1999).

1 Sensitivity and Adherence were discussed by, e.g., Nozick (1981). Safety was introduced to the debate by Sosa (1999). See Pritchard (2016) for an overview and discussion.
In all closest worlds in which the agent believes \( \varphi \), \( \varphi \) is true.

These modal conditions were introduced with the intention to block Gettier-style counterexamples to the definition of knowledge as justified true belief. In this paper, we do not enter the debate whether these conditions are successful in this particular endeavor. However, even theorists who hold that modal conditions will not enable us to resurrect a definition of knowledge in terms of belief acknowledge the importance of these conditions in contemporary epistemology. For example, Williamson (2000), the most prominent defender of knowledge-first epistemology, holds safe belief to be a necessary—though not sufficient—condition for knowledge. Thus, independently of ones particular take on the Sensitivity-, the Adherence-, and the Safety-condition, it is hard to deny that the modal conditions assume a prominent role in contemporary epistemology. A formal semantics of epistemological discourse needs to cater to this fact.

Holliday (2015) provides a systematic overview and development of the modal conditions in doxastic semantics. To this effect, Holliday extends more traditional doxastic possible world semantics (cf. Hintikka, 1962) by an ordering of the worlds relative to their comparative similarity along the lines of Lewis (1973). A striking, albeit unsurprising feature of these semantics is that independently of the use of classical negation they give rise to a non-monotone evaluation scheme, that is, even if we abandon classical logic in favor of a non-classical evaluation scheme such as strong Kleene or FDE, the resulting semantics will not be monotone. As observed by Banick (2019)—albeit in slightly different guises—this has the unwelcome effect that Kripke’s theory of truth (Kripke, 1975) may no longer be applicable in this semantic framework: we might not be able to find an interpretation of the truth predicate such that \( T^\varphi \) and \( \varphi \) receive the same semantic value for all sentences \( \varphi \) of the language. This in turn would imply that a coherent interpretation of the truth predicate could no longer be given, if the modal conditions were embraced.

The purpose of this paper is to show that situation is not as bleak as one may think. While, as correctly observed by Banick (2019), the so-called Kripke jump will no longer be monotone over the semantics proposed by Holliday (2015), we can still find fixed points of the jump operation, that is, we can find coherent and adequate interpretations of the truth predicate in the sense of Kripke (1975). Of course, we can no longer apply the Knaster-Tarski fixed-point theorem nor will it be possible—as we shall see—to construct fixed points using the theory of inductive definitions employed by Kripke (1975). However, we can construct fixed points by using the theory of quasi-inductive definitions (Herzberger, 1982; Gupta, 1982; Gupta and Belnap, 1993). For the purpose of this paper we shall focus on the safety-condition and put sensitivity and adherence aside, but our strategy will extend to semantics that embrace the latter conditions. More generally, the strategy extends to all doxastic semantics discussed in Holliday (2015) and which Banick (2019) observed to give rise to a non-monotone jump operation.

The paper is structured as follows. We first present the semantics for the language of truth

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2Of course, it would only mean that a coherent interpretation in the Kripkean sense could not be given. There can, and arguably will, be coherent interpretation of truth predicate based on the Revision Theory of Truth (Herzberger, 1982; Gupta, 1982; Gupta and Belnap, 1993) and Field’s theory of truth (Field, 2008, 2016). However, since Kripke’s theory is arguably the most popular and well entrenched theory of truth, this would still be a very disturbing consequence.
and safe belief and show how the truth predicate can be incorporated in this framework by following the outlines of Halbach and Welch (2009), Stern (2014b, 2015, 2016). We also show by means of a countermodel that the resulting semantics is non-monotone. As we discuss in the next section, Section 3, this implies that we can no longer establish the existence of fixed points, that is, attractive interpretations of the truth predicate, along the usual route. We also hint at the main idea of our fixed point construction. In a nutshell, while the Kripke jump will not be increasing, in some cases every sentence will stably be in the interpretation of the truth predicate from some ordinal onwards or it will stably not be in the interpretation of the truth predicate from from that ordinal onwards. As a consequence, the quasi-inductive definition will eventually yield at a fixed point of the Kripke jump. However, as we show this strategy will only work, if the notion of safe belief is taken to be a primitive notion of the language rather than formulated using a subjunctive conditional: if a standard ordering semantics for the subjunctive conditional is assumed, there will be no fixed points of the corresponding Kripke jump. In Section 4 we spell out the exact formal details of this idea and show how to construct fixed points of the Kripke jump using the theory of quasi-inductive definitions. However, the fixed points of the Kripke jump might not form a complete lattice and, in particular, there will not always be a unique minimal or maximal fixed point. In other words, while we can find fixed points despite the non-monotonicity of the Kripke jump, we loose the neat algebraic structure that is associated with the fixed points of a monotone operation.

2 Ordering Semantics for Truth and Safe Belief

Since, Hintikka’s seminal Knowledge and Belief (Hintikka, 1962) possible worlds semantics has been the standard semantics for knowledge and belief. The basic notion of possible world semantics is the notion of a frame, that is, a tuple consisting of a non-empty set of worlds and, in the case of a semantics for belief, a doxastic accessible relation. The semantics for safe belief, as spelled out by Holliday (2015), augments Hintikka’s semantics by an ordering relation or rather a function that assigns an ordering relation to each possible world. The ordering relation relative to a world \( w \) orders the possible worlds with respect to their similarity to \( w \). In contrast to other forms of so-called ordering semantics, e.g., ordering semantics for indicative conditionals or the semantics of counterfactuals proposed by Lewis (1973), we only assume weak centering and not strong centering. Weak centering allows for a world to be as similar to \( w \) on the ordering relative to \( w \) as \( w \) is to itself. Usually, the stronger condition of strong centering is assumed, which rules out the existence of worlds that are as similar to \( w \) as \( w \) itself. But if strong centering is assumed, safe belief reduces to true belief since the safety-condition is trivialized.

Before providing the details of the semantics we give a precise definition of the formal language and its syntax.

\[ \text{One consequence of assuming only weak centering instead of strong centering is that the ordering of the worlds should not be viewed as an ordering in terms of their comparative similarity in the sense of Lewis (1973). Rather a different account of the similarity ordering needs to be given by proponents of the safety or adherence condition. We shall not pursue this issue here.} \]
Definition 1 (Language). Let $\mathcal{L}$ be an arbitrary first-order language that is augmented by a syntax language $\mathcal{L}_S$, that is, $\mathcal{L}_S \subset \mathcal{L}$ and $\text{FC}_{\mathcal{L}_S} = \text{FC}_{\mathcal{L}}$. $\mathcal{L}_T$ extends $\mathcal{L}$ by the truth predicate $T$. $\mathcal{L}_B$ extends $\mathcal{L}_T$ by the belief operator $B$. Finally, $\mathcal{L}_K$ extends $\mathcal{L}_B$ by a safe-belief (or knowledge) operator $K$. The syntax of $\mathcal{L}_K$ is given by

$$\varphi ::=(t_1 = t_2) | P^n(t_1, \ldots, t_n) | Tt_1 | \neg\varphi | \varphi \land \varphi | B \varphi | K \varphi$$

where for $1 \leq i \leq n$, $t_i \in \text{Term}_\mathcal{L}$ and for all $n \in \omega$, $P^n \in \text{RC}_\mathcal{L}$.

Next we define the notion of an ordering frame, which, as mentioned, is the basic notion of ordering semantics. We keep things as simple as possible and adopt a constant domain semantics. Ultimately, a varying domain semantics may be more appropriate for belief but whether we adopt constant or varying domain semantics will have no bearing on the existence of fixed points.

Definition 2 (Ordering frame). An ordering frame $F$ is a tuple $\langle W, R, \preceq, D \rangle$ where $W \neq \emptyset$ is a set of worlds, $R \subset W \times W$ is the accessibility relation, and $\preceq: W \rightarrow W \times W$ a function that generates a partial ordering relation $\preceq_w$ for each $w \in W$. $W_w$ denotes the set $\{v \in W \mid w \preceq_v v\}$. We assume $\preceq_w$ to be a partial ordering and to be weakly centered, i.e., $\forall v \in W(w \preceq_w v)$. The doxastic accessibility relation $R$ is assumed to be a serial relation. Finally, $D$ is the domain of the ordering frame and contains all $\mathcal{L}_K$-expressions (or codes thereof), that is, $\text{Expr}_{\mathcal{L}_K} \subset D$.

According to the above definition all $\mathcal{L}_K$-expressions (or codes thereof) are contained in the domain of the frame. This is a requirement that arises because of the intended interpretation of the language $\mathcal{L}_S$, which possesses names for all $\mathcal{L}_K$-expressions. More generally, an interpretation function over an ordering frame fixes the entire non-logical vocabulary with the exception of the truth predicate.

Definition 3 (Interpretation, Belief Model). An interpretation function $I$ provides an interpretation of $\mathcal{L}$-vocabulary over $D$ relative to each world such that the interpretation of the $\mathcal{L}_S$-vocabulary assigns the intended $\mathcal{L}_K$-expressions to the terms of the language. Moreover, the interpretation is constant (rigid) across worlds for the entire $\mathcal{L}_S$-vocabulary and the individual constants of $\mathcal{L}$, that is, $I(w, c) = I(v, c)$ for all $c \in \text{Const}_\mathcal{L}$ and $w, v \in W$. For all $P^n \in \text{RC}_\mathcal{L}$, $I$ provides an extension and an antiextension relative to every possible world $w$:

$$I(w, P^n) = (I(w, P^n)^+, I(w, P^n)^-) \subseteq D^n \times D^n.$$  

If $P^n \in \text{RC}_{\mathcal{L}_S}$, then $I(w, P^n)^+ \cup I(w, P^n)^- = D^n$ and $I(w, P^n)^+ \cap I(w, P^n)^- = \emptyset$. A tuple $M = (F, I)$ is called a belief model.

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4See Stern (2020) for a varying domain semantics for truth and belief.

5The seriality of $R$ is commonly thought to be the minimal requirement on a doxastic accessibility relation. Nothing hinges on the choice of properties assumed for $R$.

6We assume the individual constant to be interpreted rigidly merely for sake of convenience. See Stern (2020) for a doxastic possible world semantics which allows for non-rigid designators. Allowing non-rigid designators into the picture will not affect the main results of our paper. For sake of simplicity we also assume that there are no function symbols in $\mathcal{L}_K - \mathcal{L}_S$. 

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In a belief model the interpretation of the truth predicate is left unspecified. The latter is provided by an evaluation function that provides a subset of the domain for each possible world, which serves as the interpretation of the truth predicate at that world. At this point there is no adequacy condition on the interpretation of the truth predicate: any subset of the domain may serve as the interpretation of the truth predicate at a world.

**Definition 4** (Evaluation function). Let $F$ be an ordering frame. An evaluation function relative to $F$ is a function $f : W \to P(D)$ that assigns to each possible world a subset of the domain—the interpretation of the truth predicate. The set of all evaluation functions relative to a frame $F$ is denoted by $\text{Val}_F$.

We move on to the notion of truth in a belief model relative to an evaluation function $f$. It is well known that in classical logic we cannot hope to define a transparent truth predicate for a semantically closed language with a sufficiently rich syntax theory. Following Kripke (1975) we therefore move to non-classical logic and define the notion of truth in a belief model according to the FDE-scheme in a four-valued semantics.

**Definition 5** (Strong Kleene Truth in a Belief Model). Let $F$ be an ordering frame and $f \in \text{Val}_F$ an evaluation function $f$. The notion of truth in a model induced by the frame $F$ and the evaluation function $f$ at a world $w$ according to the strong Kleene scheme, $\mathcal{V}_k$, for formula of $\mathcal{L}_k$ is defined by the following clauses:

1. $M, w \models^f_k s = t$ $\iff$ $I(w, s) = I(v, t)$
2. $M, w \models^f_k s \neq t$ $\iff$ $I(w, s) \neq I(v, t)$
3. $M, w \models^f_k P^n(t_1, \ldots, t_n) \iff \langle I(w, t_1), \ldots, I(w, t_n) \rangle \in I(w, P^n)^+$
4. $M, w \models^f_k \neg P^n(t_1, \ldots, t_n) \iff \langle I(w, t_1), \ldots, I(w, t_n) \rangle \in I(w, P^n)^-$
5. $M, w \models^f_k \top$ $\iff$ $I(t, w) \in f(w)$
6. $M, w \models^f_k \neg \top$ $\iff$ $(\neg I(w, t)) \in f(w)$ or $I(w, t) \not\in \text{Sent}_{\mathcal{L}_k}$
7. $M, w \models^f_k \neg \psi$ $\iff$ $M, w \models^f_k \psi$
8. $M, w \models^f_k \psi \land \chi$ $\iff$ $M, w \models^f_k \psi$ and $M, w \models^f_k \chi$
9. $M, w \models^f_k \neg(\psi \land \chi)$ $\iff$ $M, w \models^f_k \neg \psi$ or $M, w \models^f_k \neg \chi$
10. $M, w \models^f_k \forall x \psi$ $\iff$ $\forall c \in \text{Con}_{\mathcal{L}}(M, w \models^f_k \psi(c/x))$
11. $M, w \models^f_k \forall x \psi$ $\iff$ $\exists c \in \text{Con}_{\mathcal{L}}(M, w \models^f_k \psi(c/x))$
12. $M, w \models^f_k B\psi$ $\iff$ $\forall v (wRu \implies M, v \models^f_k \psi)$
13. $M, w \models^f_k \neg B\psi$ $\iff$ $\exists v (wRu \land M, v \models^f_k \neg \psi)$
14. $M, w \models^f_k K\psi$ $\iff$ $M, w \models^f_k B\psi \land$

\[
\forall v \in W_{\mathcal{L}}(\exists u (u <_w v) \land M, v \models^f_k B\psi \implies M, v \models^f_k \psi)\]

15. $M, w \models^f_k \neg K\psi$ $\iff$ $M, w \models^f_k \neg B\psi \lor [M, w \models^f_k B\psi \land$

\[
\exists v \in W_{\mathcal{L}}(\exists u (u <_w v) \land M, v \models^f_k B\psi \land M, v \models^f_k \psi)]$
If \( M, w \models^f_k \varphi \) for all \( w \in W \) we say that \( \varphi \) is true in the belief model \( M \) and the evaluation function \( f \) and write \( M \models^f_k \varphi \).

This completes our presentation of ordering semantics for truth and safe belief and the language \( \mathcal{L}_k \). It is not difficult to see that the semantics is not monotone. For interpretation functions \( I, J \) on a frame \( F \) and relative to an arbitrary evaluation function \( f \) we set \( I \sqsubseteq J \) iff for all literals \( \varphi \) of \( \mathcal{L}_k \) and all \( w \in W \):

\[
(F, I), w \models^f_k \varphi \Rightarrow (F, J), w \models^f_k \varphi.
\]

Then, if the semantics were monotone \( I \sqsubseteq J \), would imply that whenever an arbitrary formula \( \varphi \) is true at a world in the belief model based on \( I \), \( \varphi \) is also true in the belief model based on \( J \). Alas this does not hold good in our semantics on many frames as we now illustrate by means of the following example.

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**Example 6** (Non-monotonicity). Let \( F \) be as specified in Figure 1. Let \( I, J \) be interpretations functions on \( F \) such that \( I \sqsubseteq J \). In particular, let \( I(x, c) \in I(x, P)^+ \) for \( x \in \{ w, v \} \) and \( J(x, c) \in J(x, P)^+ \) for \( x \in \{ w, v, y \} \) but \( I(x, c) \notin I(x, P)^+ [I(x, P)^-] \) for \( x \in \{ z, y \} \) and \( J(z, c) \notin J(z, P)^+ [J(z, P)^-] \). Then \( (F, I), w \models^f_k \text{KPt} \) but \( (F, J), w \models^f_k \text{KPt} \) for all \( f \in \text{Val}_f \).

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The non-monotonicity of the semantics is due to the non-positive occurrences of \( \models^f_k \) in clauses (14) and (15) of Definition 5 and, as we shall explain shortly, it is the non-monotonicity of the semantics that prevents the application of Kripke’s theory of truth along the lines of, e.g., Halbach and Welch (2009) or Stern (2014a, 2015, 2016, 2020). Recall that up to now the interpretation of the truth predicate was left unconstrained in our semantics. Ultimately, we would like to single out suitable evaluation functions, that is, evaluation functions \( f \) such that for all \( \varphi \in \text{Sent}_{\mathcal{L}_k} \) and \( w \in W \):

\[
M, w \models^f_k T^\gamma \varphi \iff M, w \models^f_k \varphi,
\]

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Figure 1: An ordering frame with an ordering \( \succeq_w \) relative to \( w \) represented by dashed arrows, plain arrows represent the doxastic accessibility relation \( R \).
where \( \tau \phi \) is a name of \( \phi \). But, as illustrated by Banick (2019) and as we shall now explain, finding such evaluation functions is not as smooth sailing as in monotone semantics.

## 3 Kripke’s Theory of Truth, Monotonicity and Ordering Semantics

The fundamental idea of Kripke’s theory of truth is to find interpretations of the truth predicate such that the set of sentences which are true in the model under the given interpretation of truth predicate coincide with the interpretation of the truth predicate. In other words adequate interpretations of the truth predicate are fixed points of the operation—the Kripke jump—that takes interpretations of the truth predicate, i.e. sets of sentences, as input and outputs the set sentences that are true in the model under the given interpretation of the truth predicate.

In possible world semantics this idea needs to be generalized so to simultaneously apply to interpretations of the truth predicate at every world \( w \in W \). As a consequence, the Kripke jump will not be defined on sets of sentences but on evaluation functions (Halbach and Welch, 2009; Stern, 2014a, 2016).

**Definition 7 (\( \mathcal{K} \)-jump).** Let \( \mathcal{M} = (F, I) \) be a belief model. Then \( \mathcal{K}_M : \text{Val}_F \rightarrow \text{Val}_F \) is an operation on \( \text{Val}_F \) relative to \( M \) such that for all \( w \in W \)

\[
[\mathcal{K}_M(f)](w) = \{ \phi \in \text{Sent}_k \ | \ M, w \Vdash^f_k \phi \}.
\]

We frequently write \( \mathcal{K} \) instead of \( \mathcal{K}_M \) if the belief model is clear from context.

**Definition 8 (Ordering).** The ordering \( \leq \) on \( \text{Val}_F \) is defined by

\[
f \leq h : \iff \forall w(f(w) \subseteq h(w))
\]

for all \( f, h \in \text{Val}_F \).

In standard possible world semantics for modal and doxastic notions, that is, possible world semantics where truth in a model is defined without clauses (14) and (15) of Definition 5 the resulting Kripke jump, call it \( B \), will be a monotone operation on \( \text{Val}_F \), that is, for \( f, g \in \text{Val}_F \)

\[
f \leq g \Rightarrow B(f) \leq B(g).
\]

The monotonicity of \( B \) on \( \text{Val}_F \) will guarantee the existence of a minimal and a maximal fixed point by well rehearsed algebraic arguments. We refer to Visser (1989) and Fitting (1986) for lucid overviews of these arguments and results.

\*We note in passing that even in the presence of adequate interpretation of the truth predicate the semantics might be deemed to be an inadequate semantics for truth and (safe) belief, since \( K \phi (B \phi) \) and \( K^\tau \phi \) \( \land B^\tau \phi \) will be semantically equivalent for all sentence \( \phi \). See Stern (2020) for a discussion of this issue and a semantics that blocks this equivalence.\*
However, while the monotonicity of the Kripke jump is a sufficient condition for the existence of fixed points it is not a necessary condition. The strategy used by Kripke to establish the existence of fixed points appeals to the monotonicity of the Kripke jump to inductively define the minimal fixed point of the jump operation via a stage by stage process starting from the empty set. That is, Kripke uses the monotonicity of the jump to show that iterative applications of the jump operation over the empty set lead to an increasing sequence of interpretations of the truth predicate. Then, as there are more ordinals than sentences of the language we are guaranteed to eventually reach a fixed point of the jump operation. By reflecting on the strategy we may infer that it is possible to establish the existence of fixed points of any strategy used by Kripke to establish the existence of fixed points cannot, in general, be carried out in this setting: we cannot inductively construct the minimal fixed point of \( \mathcal{K} \) over \( \varnothing \). In the proof of (ii) we only show that \( \mathcal{K}(g) \) is not increasing for \( \alpha = 1 \) and one may wonder whether \( \mathcal{K} \) is increasing for \( \alpha > 1 \). Unfortunately, neither strategy seems promising:

**Lemma 9.** Let \( F \) be a frame as specified in Figure 1. Then

(i) if \( M \) is an arbitrary belief model, then there exists \( f, j \in \text{Val}_F \) such that \( f \leq j \) but \( \mathcal{K}(f) \neq \mathcal{K}(j) \)

(ii) there exists a belief model based on \( F \) such that \( \mathcal{K}^\alpha(g) \neq \mathcal{K}^{\alpha+1}(g) \) for some ordinal \( \alpha > 0 \) and the evaluation function \( g \) with \( g(w) = \varnothing \) for all \( w \in W \)

**Proof.** For (i), let \( \tau \) be a truth-teller sentence and choose an arbitrary belief model on \( F \). Then \( f \leq j \) holds for evaluation functions \( f, j \in \text{Val}_F \) such that \( \tau \in f(x) \) for \( x \in \{ w, v \}, \tau \in j(x) \) for \( x \in \{ w, v, y \} \), and \( \tau \notin f(x) \) and \( \tau \notin j(x) \) otherwise. However, if \( K \tau \in [\mathcal{K}(f)](w) \) but \( K \tau \notin [\mathcal{K}(j)](w) \), which implies \( \mathcal{K}(f) \neq \mathcal{K}(j) \).

For (ii), let \( P, Q \in \text{RC}_L - \text{RC}_{L^*} \), \( c \in \text{Const}_L \), and \( M \) be a belief model such that \( M, a \models_k^L P \) for \( a \in \{ w, v \} \) and \( M, y \models_k^L Qc \). Set \( \sigma := Pc \lor (Qc \land T^s = s^\top) \). It follows that \( M, w \models_k^L B \sigma \) since \( \forall u(wRu \Rightarrow M, u \models_k^L Pc) \). Also \( M, y \models_k^L \sigma \) since neither \( M, y \models_k^L P \) nor \( M, y \models_k^L T^s = s^\top \). This implies that \( M, z \models_k^L B \sigma \) and hence \( M, w \models_k^L K_\sigma \) since the safety condition is trivially satisfied. We have \( \sigma \in [\mathcal{K}(g)](w) \). However, \( s = s \in [\mathcal{K}(g)](w) \) likewise, which implies \( M, y \models_k^{\mathcal{K}(g)} T^s = s^\top \) and thus \( M, y \models_k^{\mathcal{K}(g)} \sigma \). This, in turn, yields \( M, z \models_k^{\mathcal{K}(g)} B \sigma \) and, as consequence, \( M, w \models_k^{\mathcal{K}(g)} K_\sigma \), as the safety condition is no longer satisfied under the evaluation function \( \mathcal{K}(g) \). Therefore, \( K_\sigma \models_k [\mathcal{K}(\mathcal{K}(g))](w) \), which shows that \( \mathcal{K}(g) \neq \mathcal{K}(\mathcal{K}(g)) \).

While (i) of Lemma 9 establishes the non-monotonicity of \( \mathcal{K} \) relative to every belief model, the relevance of (ii) may be slightly more opaque. Lemma 9(ii) shows that for a broad class of belief models, iterative application of \( \mathcal{K} \) over the minimal evaluation function do not lead to an increasing sequence of evaluation functions. This shows that the Kripkean argument for the existence of fixed points cannot, in general, be carried out in this setting: we cannot inductively construct the minimal fixed point of \( \mathcal{K} \) over \( \varnothing \). In the proof of (ii) we only show that \( \mathcal{K} \) is not increasing for \( \alpha = 1 \) and one may wonder whether \( \mathcal{K} \) is increasing for \( \alpha > 1 \). Unfortunately,

\footnote{This is basically Proposition 4 in Banick 2019.}

\footnote{See Definition 13 for a rigorous definition of \( \mathcal{K}^\alpha \) for \( \alpha \in \text{ON} \).}

\footnote{A truth-teller sentence \( \tau \) is a sentence \( \text{TT} \) such that \( F \models_k t = \text{TT}^\top \).}
this will not be the case: let \( T^\alpha s = s^\gamma \) denote the sentence where \( T s = s^\gamma \) is preceded by \( \alpha \)-many iterations of the truth predicate and set \( \sigma^\alpha = \sigma(T^\alpha s = s^\gamma) \) \( / T^\alpha s = s^\gamma \) where \( \sigma \) is as in the proof of Lemma \( (ii) \). Then, whenever \( \sigma^\alpha \) is a formula of \( L_k \), it follows that \( K^\alpha(g) \not= K^{\alpha+1}(g) \) relative to the belief model employed in the proof of Lemma \( (ii) \). The observation shows that we should not hope to establish a fixed-point result by looking at increasing sequences of evaluation functions in the Kripkean fashion.\(^{11}\)

These observations show that the standard arguments for establishing the existence of fixed points will not work for \( K \), but by inspecting the behavior \( \sigma^\alpha \) in the sequence of evaluation functions obtained by iterative applications of \( K \) to the evaluation function \( g \) of Lemma \( (ii) \) we can see that \( \sigma^\alpha \not\in [K^\beta(g)](u) \) for all \( \beta > \alpha \) and \( u \in W \). In other words, the semantic value of \( \varphi^\alpha \) remains stable from \( \alpha + 1 \) onwards. This suggest an alternative road to a fixed point result and to construct fixed points using the theory of quasi-inductive definition as opposed to inductive definitions: if we can show that for some evaluation function \( f \) and every sentence \( \varphi \) of \( L_k \), there exists an ordinal \( \alpha \)—call this the stabilizing ordinal of \( \varphi \)—such that \( \varphi \) is either in \([K^\beta(f)](\omega)\) for all \( \beta > \alpha \) or \( \varphi \) is not in \([K^\beta(f)](\omega)\) for all \( \beta > \alpha \), then we know that by transfinitely iterating \( K \) over \( f \) every sentence of \( \text{Sent}_{L_k} \) will eventually stabilize, that is, we will eventually reach a fixed point of \( K \). In Section \( 4 \) we show that for every belief model we can find such evaluation functions, that is, \( K \) has fixed points. In particular, over every belief model iterative applications of \( K \) over the minimal evaluation function \( \varphi \) with \( \varphi^\alpha(e) = \varnothing \) will lead to a fixed point, yet, this fixed point will not be the minimal fixed point of \( K \)—indeed, there will not always be a minimal fixed point.

3.1 No Luck: Safety and Subjunctive Conditionals

As we have just argued we can find fixed points for the operator \( K \) despite the non-monotonicity of semantics for safe belief. In the semantics presented in Section \( 2 \) we have treated safe belief (or knowledge) as a primitive expression of the language. However, the safety condition was originally introduced by Sosa (1999) using a subjunctive conditional as

\[
B\varphi \Box \varphi,
\]

that is, if the agent were to believe \( \varphi \), \( \varphi \) would be the case. In this case safe belief (or knowledge) can be defined in the object language by

\[
K\varphi : \leftrightarrow B\varphi \land (B\varphi \Box \varphi).
\]

This way of spelling out safe belief and the safety condition will require a non-standard semantics for the subjunctive conditional since within standard ordering semantics for subjunctive conditionals the ordering relation is usually taken to respect strong centering and not only weak centering. More importantly, if the antecedent of the conditional is true, the conditional will be true on most semantics for subjunctive conditionals whenever the consequent of the

\(^{11}\)Of course, if \( K \) has fixed points, iterative application of \( K \) over specific evaluations function, that is, the fixed points themselves, will lead to increasing—not strictly increasing—sequences of evaluation functions. However, this will hardly help in establishing the existence of fixed points.
conditional is true. As discussed at the beginning of Section 2, it may thus be preferable to introduce the notion of safe belief directly, that is, without appealing to a subjunctive conditional in the framework of an ordering semantics with an appropriate ordering relation.\(^{12}\)

Be that as it may, in the context of Kripke’s theory of truth there is some further reason why it is preferable to introduce the notion of safe belief as a primitive rather than appealing to a subjunctive conditional to this effect: the Kripke jump defined relative to an ordering semantics for the subjunctive conditional that assumes some form of centering, i.e. strong or weak centering, will not have fixed points. Hence, there will be no fixed points of the Kripke jump in a semantics for safe belief, if the latter notion is defined using the subjunctive conditional.

**Definition 10** (Subjunctive Conditionals). Let \( \mathcal{L}_{\Box \rightarrow} \) be \( \mathcal{L}_B \) extended by the two-place connective \( \Box \rightarrow \). The semantics of \( \mathcal{L}_{\Box \rightarrow} \) is just like the semantics specified in Section 2 except that clauses \(^{14}\) and \(^{15}\) in Definition 5 are replaced by

\[
\begin{align*}
(16) \quad & M, w \models^f_k \Box \psi \rightarrow \chi \iff \\
& \neg \exists v \in W_w(M, v \models^f_k \psi) \text{ or } \exists z \in W_w(M, z \models^f_k \psi) \& \\
& \forall u (u \preceq_w z \& M, u \models^f_k \psi \Rightarrow M, u \models^f_k \chi) ; \\
(17) \quad & M, w \models^f_k \neg (\psi \rightarrow \chi) \iff \\
& \exists v \in W_w(M, v \models^f_k \psi) \& \forall z \in W_w(M, z \models^f_k \psi) \Rightarrow \\
& \exists u (u \preceq_w z \& M, u \models^f_k \psi \& M, u \models^f_k \neg \chi) ;
\end{align*}
\]

In alignment with (16) and (17) we convey that a formula \( \varphi \) is true in a counterfactual belief model at a world \( w \) and an evaluation function \( f \) by \( M, w \models^f_k \varphi \).

Ordering semantics for the subjunctive conditional, like the semantics introduced in Section 2, is non-monotone. However, in contrast we know that the Kripke jump defined relative to the semantics for subjunctive conditionals will not have fixed points.

**Definition 11** (Counterfactual Kripke Jump). Let \( F \) be a frame and \( M \) an arbitrary counterfactual belief model. Then \( C : \operatorname{Val}_F \rightarrow \operatorname{Val}_F \) is the counterfactual Kripke jump with

\[
\begin{align*}
[C_M(f)](w) := \{ \varphi \mid M, w \models^f_k \varphi \}.
\end{align*}
\]

We drop the reference to \( M \) if the model is clear from context.

**Lemma 12.** Let \( F \) be a frame and \( M \) an arbitrary counterfactual belief model. Then there exists no \( f \in \operatorname{Val}_F \) such that

\[
C(f) = f,
\]

that is, \( C \) does not have fixed points.

\(^{12}\)Similar remarks apply to the adherence condition, at least, if used for distinguishing knowledge from true belief. The sensitivity condition on its own is compatible with strong centering, yet it is typically used in tandem with the adherence condition. Clearly, the similarity ordering at play is understood in the same way in these two conditions, which suggests that sensitivity also ought to be understood in the context of a non-standard ordering semantics for subjunctive conditional. Of course, if the modal conditions are not spelled out using a subjunctive conditional, this ultimately requires an alternative specification of the similarity ordering since Lewis-style comparative similarity as uses in the semantics of conditional is thought to satisfy strong centering.

\(^{13}\)The truth conditions (16) and (17) are due to Lewis (1973) with the modicum that he works with classical logic.
Proof. Let $\lambda$ be the sentence

$$T^\top \lambda \top \rightarrow \bot$$

obtained by some form of diagonalization. Now assume for reductio that $f \in \forall a \in W$ is a fixed point. For all $w \in W$ we know that either $\lambda \in f(w)$ or $\lambda \notin f(w)$. If $\lambda \in f(w)$ then—since, by assumption, $f$ is a fixed point—$M, w \models_k T^\top \lambda \top \rightarrow \bot$. The latter implies either (i) $\neg \exists u \in W_w(M, v \models_k T^\top \lambda^\top)$ or (ii)

$$\exists z \in W_w[M, z \models_k T^\top \lambda \top & \forall u(u \subseteq_w z \& M, u \models_k T^\top \lambda \top \Rightarrow M, u \models_k \bot)].$$

(ii) is equivalent to

$$\exists z \in W_w[M, z \models_k T^\top \lambda \top & \forall u(u \subseteq_w z \Rightarrow M, u \models_k T^\top \lambda \top)],$$

which is unsatisfiable since $z \subseteq_w z$ for all $z \in W_w$. Hence, (i) needs to be the case, but (i) holds iff $\forall u \in W_w(\lambda \notin f(v))$. Since $w \in W_w$, we conclude $\lambda \notin f(w)$. Contradiction. We may conclude that if $f$ is a fixed point $\lambda \notin f(w)$ for all $w \in W$.

Suppose then that $f$ is a fixed point and $\lambda \notin f(w)$ for all $w \in W$. Then $M, w \models_k T^\top \lambda \top \rightarrow \bot$, that is, (a) $\exists v \in W_w(M, v \models_k T^\top \lambda \top)$ and (b)

$$\forall z \in W_w[M, z \models_k T^\top \lambda \top \Rightarrow \exists u(u \subseteq_w z \& M, u \models_k T^\top \lambda \top \Rightarrow M, u \models_k \bot)].$$

But (a) implies that there exists a world $v \in W_w$ such that $\lambda \notin f(v)$. Contradiction. $C$ does not have fixed points.

Lemma [12] establishes that, if we assume standard Lewis-Stalnaker semantics for counterfactuals, the Kripke jump defined relative to this semantics will not have fixed points. This implies that if we seek to spell out the safety condition by making explicit appeal to subjunctive conditionals, we will not be able to give a satisfactory semantics for truth and safe belief.

One may wonder why there is such stark contrast between the two semantics: why, as we shall see, can we find fixed points of $K$ while we cannot find fixed points of $C$. First and foremost, $\top \rightarrow$, as opposed to $K$, is a binary connective. The trick we used in the proof of Lemma [12] was that conditionals typically allows us to define the classical negation of a formula $\varphi$ (or something close enough) via the conditional: if $\varphi$, then $\bot$. Accordingly, we can obtain a Liar sentence that will flip flop in and out of the interpretation of the truth predicate. Since $K$ is a unary connective—the implicit conditional is fully determined by the single argument of $K$—we cannot apply the same trick and it is not generally possible to define an operation resembling classical negation.

Second and related, a formula $K \psi$ only embeds one very specific conditional statement, i.e., $\Box \varphi \top \rightarrow \varphi$. In this conditional, the atomic formulas that are subformulas of the antecedent are precisely the atomic subformulas of the consequent of the conditional. Moreover, if an atomic formula is a subformula of the conditional, it occurs positively (negatively) in both the antecedent and the consequent. This means that if we can make sure that the consequent $\varphi$ has stabilized on some semantic value in every world of the frame, then we are guaranteed that
the antecedent $B \phi$ will also have stabilized on a semantic value and, as a consequence, we can also assign a stable semantic value to $B \phi \rightarrow \varphi$, that is, to $K \varphi$.

4 Constructing Fixed Points

In the previous section we have seen that we cannot hope to obtain fixed points of $\mathcal{K}$ by exploiting the monotonicity of $\mathcal{K}$—for $\mathcal{K}$ is not monotone—or by constructing increasing sequences of interpretations of the truth predicate. Rather it seemed more promising to turn to quasi-inductive processes and to show the existence of fixed points of the quasi-inductive definition associated with $\mathcal{K}$. Our strategy of constructing these fixed points will consist of two steps. First, we identify general properties of an evaluation function $f$ that guarantee that every sentence of the language will eventually stabilize on some semantic value in the quasi-inductive process relative to $\mathcal{K}$. Second, we show that such evaluation functions exist.

We start by giving a precise definition of the quasi-inductive process, that is, of iterative applications of $\mathcal{K}$.

**Definition 13 (Iterated $\mathcal{K}$-jump).** Let $F$ be a frame and $M = (F, I)$ be a belief model. Then $\mathcal{K} : \text{Val}_F \rightarrow \text{Val}_F$ is an operation on $\text{Val}_F$ relative to $M$ such that for all $w \in W$

$$[\mathcal{K}_M(f)](w) = \{ \varphi \in \mathcal{L}_k \mid M, w \Vdash \varphi \}.$$ 

Iterative applications of $\mathcal{K}$ are defined by recursion

$$[\mathcal{K}_M^\alpha(f)](w) := \begin{cases} f(w), & \text{if } \alpha = 0; \\ \{ \varphi \in \mathcal{L}_k \mid M, w \Vdash \varphi \}, & \text{if } \alpha = \beta + 1; \\ \{ \varphi \in \mathcal{L}_k \mid \varphi \in \bigcup_{\beta < \alpha} \bigcap_{\gamma < \alpha} [\mathcal{K}_M^\gamma(f)](w) \}, & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

It is the definition of $\mathcal{K}_M^\alpha$ for limit ordinals $\alpha$ that distinguishes the present set up from Kripke’s construction (cf. Kripke, 1975) and the theory of inductive definitions. If we were to work with inductive definitions, $\mathcal{K}_M^\alpha$ for limit ordinals $\alpha$ could be taken to be the union over all $\mathcal{K}_M^\beta$ with $\beta < \alpha$. If we were to take the union at limit ordinals as customary in the case of inductive definitions, sentences that had previously settled on a semantic value might become unsettled again and we would need to start afresh at each limit ordinal. Instead, we use to the so-called liminf-rule (cf. Herzberger, 1982) in Definition 13 to handle the case of limit ordinals. According to the liminf-rule a sentence is in the interpretation of the truth predicate at a world relative to an evaluation function $f$ at a limit ordinal $\alpha$ if the sentence has stabilized on a semantic value at some ordinal smaller than $\alpha$. We could have used other limit rules, e.g., the so-called limsup-rule. A sentence is true at a limit ordinal according to the limsup-rule iff the sentence is not stably untrue from some $\beta < \alpha$ onwards, that is, iff for every ordinal $\beta \leq \alpha$ we can find an ordinal $\gamma$ at which the sentence is true. The point of difference between the two limit rules is that the limsup-rule will turn sentences that have not stabilized by the relevant
ordinal into truth-value gluts, while the liminf-rule will turn such sentences into truth-value gaps.

We now begin with the first step of our construction, that is, we specify general properties of evaluations functions that guarantee that every sentence of the language will eventually stabilize on a semantic value. To this effect we assign a stabilizing ordinal to sentences of $\mathcal{L}_k$ relative to an evaluation function $f$. The stabilizing ordinal of a sentence tells us after how many applications of $\mathcal{K}$ the sentence should stabilize on a semantic value. Intuitively, each application of the Kripke jump corresponds to an application of the truth predicate to a sentence of the language and a sentence $\varphi$ is thus expected to have settled on a semantic value at level $\alpha$, if $\alpha$ is the maximum number of iterations or embeddings of the truth predicate in $\varphi$. We therefore take the stabilizing ordinal of a sentence to be the maximum depth of $T$-embeddings in the sentence. However, not all sentences can be assigned a stabilizing ordinal because it is not always possible to determine the maximum depth of $T$-embeddings in the sentence, viz. self-referential, ungrounded sentences like liar or truth-teller sentences. As a consequence, such ungrounded sentences will not be assigned a stabilizing ordinal, unless they are deemed true (false) at the outset of the construction. Assigning stabilizing ordinals to sentences of the language thus amounts to defining a partial function relative to a frame $\mathcal{F}$ and evaluation function $f$ that, if defined, assigns an ordinal—the stabilizing ordinal—to a sentence of $\mathcal{L}_k$. To this effect we first determine the stabilizing ordinal of a sentence $\varphi$ at a world relative to an evaluation function $f$ and then define the stabilizing ordinal of $\varphi$ relative to $f$ as the supremum of former.

**Definition 14** (Stabilizing Ordinal). Let $F$ be a doxastic ordering frame, $f \in \text{Val}_F$ and

$$f^-(w) := \{ \varphi \mid \varphi \equiv \neg \psi & \psi \in f(w) \}$$

for all $w \in W$. For all $w \in W$ set

$$\text{Base}_w := f(w) \cup f^-(w) \cup \{ (\neg)Tt \mid I(w, t) \notin \text{Sent}_{\mathcal{L}_k} \} \cup \text{Sent}_{\mathcal{L}_k}$$

Then $\rho_{f(w)} : \text{Sent}_{\mathcal{L}_k} \rightarrow \text{ON}$ is a partial function such that

---

13We assume that the doxastic accessibility relation is right unbounded (serial). Otherwise we would need to add $\neg \exists t (wRt)$ to $\text{Base}_w$ for all sentences $\varphi$ whenever $\neg \exists t (wRt)$.

14We generally assume that $\{ \rho_{f(w)}(\psi), \rho_{f(w)}(\chi) \}$, $\{ \rho_{f(w)}(\psi(t)) \mid t \in \text{Term}_{\mathcal{L}_k} \} \in \text{ON}$, that is, if either $\rho_{f(w)}(\psi), \rho_{f(w)}(\chi)$ or $\rho_{f(w)}(\psi(t))$ are undefined, then they do not contribute to the relevant set.
Lemma 16. Let $F$ be a frame. Then the frame supremum $\xi_F$ exists.
Proof. For \( f \in \text{Val}_f \), \( \rho_f \) will only assign countable ordinals to the sentences of \( \text{Sent}_{\text{Ke}} \). Hence, there exists a \( \xi_f \in \text{ON} \) such that \( \xi_f > \alpha \) for every \( \alpha \in \text{ON} \) for which there exists \( \varphi \in \text{Sent}_{\text{Ke}} \) and \( f \in \text{Val}_f \) with \( \rho_f(\varphi) = \alpha \). \( \square \)

We can now lay out precise condition when iterative applications of \( \mathcal{K} \) over an evaluation function \( f \) will eventually lead to a fixed point of \( \mathcal{K} \). We shall call evaluation functions that meet these conditions prefixed points.

**Definition 17 (Prefix\(_M\)).** Let \( M \) be a belief model on a frame \( F \). Then Prefix\(_M \) \( \subseteq \text{Val}_f \) is the set of all those evaluation functions \( f \in \text{Val}_f \) such that

(i) if \( \rho_f(\varphi) \) is undefined, then \( \varphi \in [\mathcal{K}^\alpha(f)](w) \) for all \( \alpha \in \text{ON} \) and \( w \in W \);

(ii) if \( \rho_f(\varphi) = \alpha \) for \( \alpha \in \text{ON} \), then for all \( \gamma > \alpha \) and \( w \in W \):

(a) \( \varphi \in [\mathcal{K}^\gamma(g)](w) \Rightarrow \forall \delta \geq \gamma(\varphi \in [\mathcal{K}^\delta(g)](w); \)

(b) \( \varphi \in [\mathcal{K}^\gamma(g)](w) \Rightarrow \forall \delta \geq \gamma(\varphi \in [\mathcal{K}^\delta(g)](w).

To sum up, an evaluation function is a prefixed-point iff (i) sentences that have not been assigned a stabilizing ordinal will not enter the interpretation of the truth predicate at any stage and (ii) sentences that have been assigned a stabilizing ordinal will settle on a semantic value from that ordinal onwards. It is immediate to show that if an evaluation function satisfies (i) and (ii), iterated applications of \( \mathcal{K} \) will eventually reach a fixed point.

**Lemma 18 (Fixed Point Lemma).** Let \( M \) be a belief model on a frame \( F \) and \( f \in \text{Prefix}_M \). Then \( \mathcal{K}^\xi(f) \) is a fixed point of \( \mathcal{K} \), i.e., \( \mathcal{K}(\mathcal{K}^\xi(f)) = \mathcal{K}^\xi(f) \).

**Proof.** By Definition\(_{17} \) if \( f \in \text{Prefix}_M \), then for all \( \eta \geq \xi_f \) and \( w \in W \)

- if \( \varphi \in [\mathcal{K}^\eta(f)](w) \), then \( \varphi \in [\mathcal{K}^\xi(f)](w); \)

- if \( \varphi \notin [\mathcal{K}^\eta(f)](w) \), then \( \varphi \in [\mathcal{K}^\xi(f)](w). \)

It follows that \( \mathcal{K}(\mathcal{K}^\xi(f)) = \mathcal{K}^\xi(f) \). \( \square \)

The fixed point lemma shows that if there are prefixed points, \( \mathcal{K} \) will also have fixed points. This leads to the second part of our construction, namely, of showing that for arbitrary frames \( \text{Prefix}_M \neq \emptyset \).

**Lemma 19.** Let \( M \) be a belief model on a frame \( F \). Then \( \text{Prefix}_M \) is non-empty. In particular the function \( g \in \text{Val}_f \) with \( g(w) = \emptyset \) for all \( w \in W \) is in \( \text{Prefix}_M \).

**Proof.** Via a tedious induction on \( \alpha \) one shows that \( g \) satisfies properties (i) and (ii) of Definition\(_{17} \) See Lemmas\(_{31} \)and\(_{32} \)of the appendix. \( \square \)

The existence of (consistent) fixed points of \( \mathcal{K} \), which is the principal result of the paper, is now an immediate by Lemmas\(_{18} \)and\(_{19} \)

**Proposition 20 (Existence of Fixed Points).** Let \( M \) be a belief model on a frame \( F \) and \( \text{Fix}_M = \{ f \in \text{Val}_f \mid \mathcal{K}_M(f) = f \} \). Then
(i) \( \text{Fix}_M \neq \emptyset \);

(ii) Let \( M \) be a consistent belief model on \( M \). Then \( \exists f \in \text{Fix}_M (\forall \varphi \in \text{Sent}_L (\forall w \in W (\varphi \notin f(w) \text{ or } \varphi \notin f(w)))) \);

(iii) \( \text{Fix}_M \subseteq \text{Prefix}_M \);

(iv) \( \text{Fix}_M = \{ \mathcal{K}^\mathcal{F} (f) \mid f \in \text{Prefix}_M \} \).

\( \text{Proof.} \) (i) is a direct corollary of Lemmas 18 and 19; (ii) is a consequence of Corollary 33 and Lemma 31 in the appendix; (iii) follows from Definition 17. The inclusion is strict since \( g \in \text{Prefix}_M \) but \( g \notin \text{Fix}_M \), if \( g \) is the minimal evaluation function; (iv) by (iii) and Lemma 18.

One may wonder whether with \( \mathcal{K}^\mathcal{E}_L (g) \) is the minimal fixed point of \( \mathcal{K} \). However, this will not be the case unless special restriction are imposed on the underlying ordering frame and/or the interpretation function \( I \) of the belief model at stake. This follows from a variation on the arguments given in the proof of Lemma 9.

**Lemma 21.** Let \( F \) be the frame specified in Figure 1 and let \( g \in \text{Val}_I \). Then there exists a belief model such that

(i) \( \exists f \in \text{Fix}_M (\mathcal{K}^\mathcal{E}_L (g) \neq f) \);

(ii) \( \neg \exists f' \in \text{Fix}_M \forall f \in \text{Fix}_M (f' \neq f) \).

\( \text{Proof.} \) Let \( I \) be an interpretation function such that \( (F, I), x \models_P t \) for \( x \in \{ w, v \} \) but \( (F, I), x \not\models_P P t \) if \( x \notin \{ w, v \} \). Furthermore, let \( \tau \) be a truth-teller sentence (cf. Footnote 10) and \( j \in \text{Val}_I \) such that \( \tau \in j(y) \) but \( \tau \notin j(x) \) if \( x \neq y \). By adopting the proof of Lemmas 31 and 32 it is not hard to show that \( j \in \text{Prefix} \) and that \( \tau \in [\mathcal{K}^\mathcal{E}_L (j)](y) \) but \( \tau \notin [\mathcal{K}^\mathcal{E}_L (j)](x) \) if \( x \neq y \). But then \( K(P \tau \lor \tau) \in [\mathcal{K}^\mathcal{E}_L (g)](w) \) and \( K(P \tau \lor \tau) \in [\mathcal{K}^\mathcal{E}_L (j)](w) \), i.e., \( \mathcal{K}^\mathcal{E}_L (g) \neq \mathcal{K}^\mathcal{E}_L (h) \). This establishes claim (i).

For (ii), suppose that there exists an \( f' \in \text{Fix}_M \) such that \( f' \neq f \) for all \( f \in \text{Fix}_M \). Then, in particular, \( f \neq \mathcal{K}^\mathcal{E}_L (g) \) and thus \( f \notin f'(w) \) for all \( w \in W \). But hence \( K(P \tau \lor \tau) \in f'(w) \) and \( f' \neq j \).

While \( \mathcal{K}^\mathcal{E}_L (g) \) is not the minimal fixed point of \( \mathcal{K} \) it is of course minimal in the sense that the it consists of only those sentences that are true if no sentence is arbitrarily declared true at the beginning of the construction. In the construction of all other fixed points “ungrounded” sentences like the truth teller will be taken to be true at the outset of the fixed point construction. 

As a consequence \( \mathcal{K}^\mathcal{E}_L (g) \) will coincide with the minimal fixed point of the \( B \)-jump on the sentences \( L_B \): let \( I_B \) be the minimal fixed point of \( B \). Then \( \mathcal{K}^\mathcal{E}_L (g) \cap \text{Sent}_B = I_B [17] \)

As a corollary of the previous lemma, or rather the proof of Lemma 21 we know that relative to the belief model discussed in the proof, \( \mathcal{K} \) will not only lack a unique minimal fixed point but have uncountable -incomparable minimal fixed-points.

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\(^{17}\)This shows that \( \mathcal{K}^\mathcal{E}_L (g) \) is \( \Pi^1_1 \)-hard for all frames \( F \) and belief models \( M \). Relative to specific belief models \( \mathcal{K}^\mathcal{E}_L (g) \) will be \( \Pi^1_1 \)-complete, e.g., if the modal distinction is collapsed. But we lack a precise characterization of the upper bound of \( \mathcal{K}^\mathcal{E}_L (g) \) relative to arbitrary belief models, e.g., belief models of the kind we appealed to in the proof of Lemma 24.
Corollary 22. Let $F$ be the frame specified in Figure $\square$. Then there exists a belief model $M$ such that $\mathcal{K}_M$ has $2^{\aleph_0}$-many, incomparable minimal fixed points.

Proof sketch. For each natural number $n$ we can find a distinct truth-teller sentence $\tau_n$. Hence, by using the reasoning in the proof of Lemma 21 for all $X \subseteq \omega$ we can find a fixed point $f_X$ such that $Pt \lor \tau_i \in f_X(y)$ whenever $i \in X$. Moreover, we have $f_X \nsubseteq f_Y$ and $f_Y \nsubseteq f_X$ for all $X, Y \subseteq \omega$, again by following the reasoning in the proof of Lemma 21.

Lemma 21 and Corollary 22 show that $\text{Fix}_M$ at least for some frames $F$ and belief models $M$ does not display the same neat algebraic structure as the set of fixed points of monotone Kripke jumps. In particular, the set $\text{Fix}_M$ does not always form a complete (semi)lattice: by Lemma 21 we know that there are belief models $M$ such that there is no infimum of $X$ in $\text{Fix}_M$ for some $X \subseteq \text{Fix}_M$. Similarly, by Lemma 27 below, we also know that there are belief models $M$ such that there is no supremum of $X$ in $\text{Fix}_M$ for some $X \subseteq \text{Fix}_M$. In sum, for some belief models $M$ the structure $(\text{Fix}_M, \leq)$ will look very different to the fixed-point structure induced by monotone operations.

4.1 The Top-Down Construction

The preceding fixed-point construction proceeded via a bottom-up procedure. However, Kripke’s construction can also be turned upside down. Indeed, Martin and Woodruff (1975) construct fixed points and, in particular, the maximal fixed-point via a top-down process. We now show that fixed points of $\mathcal{K}$ can also be constructed via a top down process. In this case rather than adding more and more true sentences to a valuation function one eliminates more and more untrue sentences until one has reached a fixed point. We illustrate the top-down construction by constructing a fixed point of $\mathcal{K}$—call it $\mathcal{K}^t(h)$—that is maximal in the same way $\mathcal{K}^p(g)$ is minimal: it is the fixed point that arises if all sentences are deemed true at the outset of the construction. No sentence is arbitrarily dismissed at the beginning of the construction. Yet, $\mathcal{K}^t(h)$ cannot be assumed to be the maximal fixed point of $\mathcal{K}$, as there might not be a maximal fixed point in the same way as there might not be a minimal fixed point.

In contrast to bottom-up construction, we will not be looking for prefixed points in the sense of Definition 17 but for postfixed points. The major point of difference between the two is that for a postfixed point $f$ the relevant stabilizing ordinal is not computed with respect to $f$ itself but relative to its complement, i.e., the evaluation function $\overline{f}$ such that $\overline{f}(w) = \text{Sent}_{\mathcal{L}} - f(w)$ for all $w \in W$. Intuitively, the complement $\overline{f}(w)$ contains those sentences that we wish to explicitly rule out at the outset of the top-down construction. Notice that this also means that $\rho(\varphi)$ is undefined for every sentence $\varphi$ such that $\varphi, \lnot \varphi \in f(w)$ for all $w \in W$. While in the case of prefixed points sentences that do not have a stabilizing ordinal will never enter the interpretation of the truth predicate, the opposite is the case for postfixed points: if $\rho(\varphi)$ is undefined, then $\varphi \in [\mathcal{K}^p(f)](w)$ for all $w \in W$ and $\alpha \in \text{ON}$. Intuitively, postfixed points turn the gaps (gluts) of a corresponding prefixed point into gluts (gaps).

\footnote{We leave the investigation into the precise properties of the structure $(\text{Fix}_M, \leq)$ for another occasion.}
Definition 23 (Postfix$_M$). Let $M$ be a belief model on a frame $F$ and $f \in \text{Val}_F$. Moreover, let $\overline{f} \in \text{Val}_F$ be the evaluation function such that for all $w \in W$, 
\[ \overline{f}(w) = \text{Sent}_{\mathcal{L}_k} - f(w). \]
Then Postfix$_M$ is the set of those evaluation functions $f \in \text{Val}_F$ such that for all $\varphi \in \text{Sent}_{\mathcal{L}_k}$

(i) if $\rho_{\overline{f}}(\varphi)$ is undefined, then $\varphi \in [\mathcal{K}^\gamma(f)](w)$ for all $\alpha \in \text{ON}$ and $w \in W$;

(ii) if $\rho_{\overline{f}}(\varphi) = \alpha$ for $\alpha \in \text{ON}$, then for all $\gamma > \alpha$ and $w \in W$:

(a) $\varphi \in [\mathcal{K}^\gamma(g)](w) \implies \forall \delta \geq \gamma(\varphi \in [\mathcal{K}^\delta(g)](w);$

(b) $\varphi \in [\mathcal{K}^\gamma(g)](w) \implies \forall \delta \geq \gamma(\varphi \in [\mathcal{K}^\delta(g)](w).

The counterpart of Lemma 18 holds for postfixed points, but since $\text{Fix}_M \subset \text{Postfix}_F$ by Definition 23 we know by Lemma 20 that $\text{Postfix} \neq \emptyset$.

Lemma 24 (Downwards Fixed Point Lemma). Let $M$ be a belief model on a frame $F$ and. Then

\[ \{\mathcal{K}^\gamma(f) \mid f \in \text{Postfix}_M\} = \text{Fix}_M \neq \emptyset. \]

While the evaluation function $g$ is always a member of Prefix$_M$, the maximal evaluation function $h$ with $h(w) = \text{Sent}_{\mathcal{L}_k}$ is always a member of Postfix$_M$.

Lemma 25. Let $M$ be a belief model on a frame $F$ and $h \in \text{Val}_F$ such that $h(w) = \text{Sent}_{\mathcal{L}_k}$ for all $w \in W$. Then $h \in \text{Postfix}_M$.

Proof. The proof amounts to a straightforward modification of Lemmas 31 and 32 of the appendix and is left to the reader. □

As a direct corollary of Lemma 25 we obtain that $\mathcal{K}^\gamma(h)$ is a fixed point. This shows that there are not only consistent but also complete fixed points.

Corollary 26. Let $M$ be a belief model on a frame $F$ and let $h \in \text{Val}_F$ be defined as in Lemma 25. Then

(i) $\text{Fix}_M \subset \text{Postfix}_M$;

(ii) $\mathcal{K}^\gamma(h) \in \text{Fix}_M$;

(iii) If $M$ is a complete belief model, then $\exists f \in \text{Fix}_M(\forall \varphi \in \text{Sent}_{\mathcal{L}_k}(\forall w \in W(\phi \in f(w) \text{ or } \neg \varphi \in f(w)))$.

Proof. (i) and (ii) are immediate by Definition 23, Lemma 25 and the fact that the maximal evaluation function $h \notin \text{Fix}_F$; (iii) is obtained similarly as (ii) of Lemma 20 that is, as a corollary of the suitably modified Lemmas 31 and 32 □

We end this section by showing that in the same way that $\mathcal{K}^\gamma(g)$ is not the minimal fixed point of $\mathcal{K}$, $\mathcal{K}^\gamma(h)$ is not its maximal fixed point.
Lemma 27. Let $F$ be a frame as specified in Figure 1 and let $h \in Val_F$. Then there exists a belief model such that

(i) $\exists f \in Fix_M(f \not= K^{\triangledown}(h))$;
(ii) $\neg \exists f' \in Fix_M \forall f \in Fix_M(f \leq f')$.

Proof. Let $I$ be an interpretation function such that $(F, I), x \vdash_k Pt$ for $x \in \{w, v, y\}$ but $(F, I), z \not\vdash_k Pt$. Furthermore, let $\tau$ be a truth-teller sentence (cf. Footnote 10) and $j \in Val_\tau$ such that $\tau \in j(x)$ for $x \in \{w, v\}$ but $\tau \notin j(x)$ for $x \in \{y, z\}$. By adopting the proof of Lemmas 31 and 32 it is not hard to show that $j \in Postfix_M$ and that $\tau \in [K^{\triangledown}(j)](x)$ for $x \in \{w, v\}$ but $\tau \notin [K^{\triangledown}(j)](x)$ for $x \in \{y, z\}$. Then we have $K(Pt \land \tau) \in [K^{\triangledown}(j)](w)$ but $K(Pt \land \tau) \notin [K^{\triangledown}(h)](w)$ since $\tau \in [K^{\triangledown}(h)](u)$ for all $u \in W$, i.e., $K^{\triangledown}(j) \not= K^{\triangledown}(h)$. This establishes claim (i).

For (ii), suppose that there exists an $f' \in Fix_M$ such that $f \leq f'$ for all $f \in Fix_M$. Then, in particular, $K^{\triangledown}(h) \leq f'$ and thus $\tau \in f'(u)$ for all $u \in W$. But hence $K(Pt \land \tau) \not\in f'(w)$ and $K^{\triangledown}(j) \not= f'$.

Corollary 28. Let $F$ be the frame specified in Figure 1. Then there exists a belief model $M$ such that $K_M$ has $2^{\aleph_0}$-many, incomparable maximal fixed points.

Proof sketch. By generalizing the argument of Lemma 27 to parametrized truth tellers.

4.2 Further Fixed Points

We have seen that we can construct fixed points by iterating $K$-operation over the minimal and the maximal evaluation function. Are there any other evaluation function that lead to fixed points? In the proof of Lemmas 21, 27, and Corollaries 22, 23 we have already alluded to the fact that we can construct fixed points such that truth-teller sentences are true in some specific worlds but not in others. This observation generalizes to the extent that, for every belief model based on a frame $F$, whenever an evaluation function $f$ is a fixed point of $B$ in $L_B$, then $f \in Prefix_M$ and thus $K^{\triangledown}(f) \in Fix_M$. This follows from the fact that the truth of the sentences $\varphi$ such that $\varphi \in f(w)$ for some $w$ will not depend on clauses (14) and (15) of Definition 5 that is, the truth of these sentences depends only on the monotone clauses of Definition 5.

One might wonder whether we can find fixed points that contain specific K-self-referential sentences as so far we have only discussed truth-teller sentences which are sentence of $L_T$ and do not depend on the structure of the frame $F$ for their evaluation. We now outline the construction of a consistent fixed point such that an “K-teller” is true in the fixed point at some world $w$. We take the K-teller to be a sentence $KT_1$ such that for every frame $F$

$$F \vdash_k I = ^*KT_1$$

10Recall that $B$ is the monotone operation that arises if the jump operation is based on the notion of truth in a belief model as specified in Definition 5 but without clauses (14) and (15).
20The proof of the fact that for all $f \in Val_\tau$,

$$B(f) = f \Rightarrow f \in Prefix$$

follows the outlines of the proof of Lemma 19 via Lemmas 31 and 32 in the appendix.
and thus for all \( f \in \text{Fix}_F \) and \( w \in W \)

\[
F, w \vdash_k T_i \iff F, w \vdash_k KT_i.
\]

**Example 29** (K-teller fixed points). Let \( M \) be a consistent belief model on a frame \( F \) and let \( w \in W \). We show that there is a fixed point \( f \) such that \( KT_i \in f(w) \). To this effect we need to make sure that \( T_i \) is true at the appropriate worlds in \( F \).

1. Let \( R^+ \) be the transitive closure of \( R \) and let \( TC : \mathcal{P}(W) \to \mathcal{P}(W) \) such that for all \( Y \subseteq W \)

\[
TC(Y) = \{ v \in W \mid \exists w \in Y(wR^+v) \} \cup \{ w \}.
\]

2. Set \( Sim : \mathcal{P}(W) \to \mathcal{P}(W) \) such that for all \( Y \subseteq W \)

\[
Sim(Y) = \{ v \in W \mid \exists w \in W(w =_w v) \}
\]

3. Set \( F := Sim \circ TC \). Then for \( \alpha \in \text{ON} \) and \( Y \subseteq W \)

\[
F^\alpha(Y) = \begin{cases} 
Y & \text{if } \alpha = 0, \\
F(F^\beta(Y)) \cup F^\beta(Y) & \text{if } \alpha = \beta + 1, \\
\bigcup_{\beta < \alpha} F^\beta(Y) & \text{if } \alpha \text{ is limit.}
\end{cases}
\]

4. Now let \( \mu \) be such that \( F^\beta(\{ w \}) = F^\mu(\{ w \}) \) for all \( \beta > \mu \) (since \( F \) is monotone there must be such a \( \mu \)). Then define an evaluation function \( f \) such that

\[
f(v) = \begin{cases} 
\{ KT_i \} & \text{if } v \in F^\mu(\{ w \}), \\
\emptyset & \text{otherwise.}
\end{cases}
\]

5. By adopting the strategy of Lemma 19 one can then show that \( f \in \text{Prefix}_M \) and, moreover, that \( KT_i \in [K^{\leq f}](\nu) \) for all \( \alpha \in \text{ON} \) iff \( \nu \in F^\alpha(\{ w \}) \) and thus, in particular, \( KT_i \in [K^{\leq f}](\nu) \).

### 4.3 Comparison to Gupta’s fixed-point construction

Interestingly, the construction used by Gupta (1982) to establish the existence of a classical model of truth can be seen as a special case of our fixed point construction. Gupta showed that if names of sentences are not given via a full-fledged syntax theory but by an expressively weak set of quotation names, then it is possible to construct a (unique) classical truth fixed point. To this effect let

\[
T(S) := \{ \varphi \in \text{Sent}_T \mid (M, S) \models \varphi \}
\]

the non-modal, classical truth jump. Then for suitable base models \( M \), we can find a unique interpretation \( Tr \) such that \( T(Tr) = Tr \). In other words, the T-scheme holds true in \( (M, Tr) \):

\[
(M, Tr) \models T \iff \varphi.
\]

---

\(^{21}\)See Visser (1989) for a concise presentation of Gupta’s construction.
For this to work it is important that the model has no means of ascribing any internal structure to the quotation name \( \psi \). For example, a suitable model will not be able to “see” that \( \psi \) names a sentence that results from the sentence ‘Ps’ by substituting ‘t’ for ‘s’. The blindness of the model to the internal structure of the quotation names blocks vicious and non-vicious forms of self-reference, even if we allow for a self-applicable truth predicate in the language. In this case there are no self-referential or ungrounded sentences like the truth-teller sentence or the liar sentence from the perspective of the model: every sentence can be constructed by applying the truth predicate to a sentences that has already been constructed and subsequently closing the novel set of sentences under the boolean operations. It is then easy to see that every sentence of the language will, according to Definition 14, receive a stabilizing ordinal. Moreover, in our construction, if we were to start with a classical model and then the empty interpretation of the truth predicate, the resulting model of the language with the truth predicate would be classical as well and a fixed point of \( \mathcal{C} \) (or its non-modal counterpart). Indeed, there will only be one fixed point relative to \( \mathcal{M} \) and that fixed point can be reached independently of the initial interpretation of the truth predicate, as the truth (falsehood) of every sentence is fully determined by the model.

Our strategy thus generalizes Gupta’s fixed point construction to a more general setting: if we work with a full-fledged syntax theory, then, as we have seen, not every sentence will have a stabilizing ordinal and whether a sentence receives a stabilizing ordinal or not will depend on the initial interpretation of the truth predicate. The increased expressive power that arises due to ability of the language or model to express syntactic concepts like substitution or instantiation, has two immediate consequences. First, fixed points can only be found if we work with non-classical evaluation schemes. Second, there will be more than one fixed point, as the truth (falsehood) of some sentences is not fully determined by the underlying model \( \mathcal{M} \). In many cases, the use of a non-classical evaluation scheme allows sidetracking a Gupta-style fixed-point construction due to the monotonicity of the evaluation scheme. However, if even the non-classical semantics turns out to be non-monotone, we need to resort to the more cumbersome route for establishing the existence of fixed points, that is, if such fixed points exist.

5 Conclusion

In this paper we developed a strategy for proving the existence of fixed points for certain quasi-inductive definitions. Using this strategy, we showed that one can find fixed points of Kripke’s theory of truth, that is, attractive interpretations of the truth predicate, in a semantics for safe belief and, more generally, in semantics for anti-luck epistemology. However, this result depends crucially on introducing the notion of sensitive, adherent, and safe belief as primitives into the language rather than formulating them via a subjunctive conditional. If a subjunctive

\[\text{Definition 14 is a more general version of Gupta’s definition of the quotation degree of a sentence (Gupta, 1982, Definition 3).}\]
conditional is added to the language we run out of luck: in this case there will be no fixed points of Kripke’s theory of truth in the semantics of anti-luck epistemology.

References

Banick, K. (2019). Epistemic logic, monotonicity, and the Halbach-Welch rapprochement strategy. *Studia Logica*, 107(4):669–693.

Field, H. (2008). *Saving Truth from Paradox*. Oxford University Press.

Field, H. (2016). Indicative conditionals, restricted quantification, and naive truth. *The Review of Symbolic Logic*, 9(1):181–208.

Fitting, M. (1986). Notes on the mathematical aspects of Kripke’s theory of truth. *Notre Dame Journal of Formal Logic*, 27(1):75–88.

Gupta, A. (1982). Truth and paradox. *Journal of Philosophical Logic*, 11:1–60.

Gupta, A. and Belnap, N. (1993). *The revision theory of truth*. The MIT Press.

Halbach, V. and Welch, P. (2009). Necessities and necessary truths: A prolegomenon to the use of modal logic in the analysis of intensional notions. *Mind*, 118:71–100.

Herzberger, H. (1982). Naive semantics and the liar paradox. *The Journal of Philosophy*, 79:479–497.

Hintikka, J. (1962). *Knowledge and Belief*. Cornell University Press, Ithaca and London.

Holliday, W. H. (2015). Epistemic Closure and Epistemic Logic I: Relevant Alternatives and Subjunctivism. *Journal of Philosophical Logic*, 44(1):1–62.

Kripke, S. (1975). Outline of a theory of truth. *The Journal of Philosophy*, 72:690–716.

Lewis, D. (1973). *Counterfactuals*. Wiley-Blackwell.

Martin, R. L. and Woodruff, P. W. (1975). On Representing “true-in-L” in L. *Philosophia. Philosophical Quarterly of Israel*, 5:213–217.

Nozick, R. (1981). *Philosophical Explanations*. Harvard University Press, Cambridge, MA.

Pritchard, D. (2016). *Epistemology*. Springer.

Sosa, E. (1999). How to defeat opposition to moore. *Noûs*, 33(s13):141–153.

Stern, J. (2014a). Modality and Axiomatic Theories of Truth II: Kripke-Feferman. *The Review of Symbolic Logic*, 7(2):299–318.

Stern, J. (2014b). Montague’s Theorem and Modal Logic. *Erkenntnis*, 79(3):551–570.

Stern, J. (2015). Necessities and Necessary Truths. Proof-theoretically. *Ergo*, 2(10):207–237.
A Prefix\textsubscript{M} is non-empty

Lemma 30. Let \((F, I)\) be a belief model and \(g \in \text{Val}_F\) an evaluation function such that \(g(w) = \emptyset\) for all \(w \in W\). Then for all \(\varphi \in \text{Sent}_C\), \(\rho_g(\varphi)\) is defined iff \(\rho_g(\neg \varphi)\) is defined and \(\rho_g(\varphi) = \rho_g(\neg \varphi)\).

Lemma 31. Let \((F, I)\) be a belief model and \(g \in \text{Val}_F\) an evaluation function such that \(g(w) = \emptyset\) for all \(w \in W\). Then for all \(\varphi \in \text{Sent}_C\), if \(\rho_g(\varphi)\) is undefined, \(\varphi \notin [K^\beta(g)](w)\) for all \(\alpha \in \text{ON}\) and \(w \in W\).

Proof. The proof is by transfinite induction on \(\alpha\). As I.H. we assume that for all \(\beta < \alpha\) and \(w \in W\) if \(\rho_g(\varphi)\) is undefined, then \(\varphi \notin [K^\beta(g)](w)\). We now run a secondary induction on the positive complexity of \(\varphi\). We may ignore the case for literals \(\varphi \in \text{Sent}_C\) and whenever \(\varphi \equiv \neg \text{TT}\) and \(I(w, t) \in \text{Sent}_C\) for \(w \in W\).

- \(\varphi \equiv \neg \text{TT}\). By definition of \(\rho\) we may assume that there exists a \(\psi\) such that \(I(w, t) = \psi \in \text{Sent}_C\) for all \(w \in W\) and \(\rho_g(\psi)\) is undefined. For reductio we now assume \(\varphi \in [K^\alpha(g)](w)\). If \(\alpha = \beta + 1\), then \(\varphi \in [K^\alpha(g)](w)\) iff \(M, w \Vdash K^\beta(g) \neg \text{TT} \iff \neg \psi \in [K^\beta(g)](w)\). By IH the latter implies that \(\rho_g(\psi)\) is defined. Contradiction.

Similarly, if \(\alpha\) is a limit ordinal, then there must be some \(\beta < \alpha\) such that for all \(\gamma\) with \(\beta \leq \gamma < \alpha\), \((\neg \psi) \in [K^\gamma(g)](w)\), which by IH also contradicts the fact that \(\rho_g(\psi)\) must be undefined.

- \(\varphi \equiv \neg \neg \psi\). Immediate by the induction hypothesis of the secondary induction.

- \(\varphi \equiv \neg \text{BB}\). By Lemma 30 both \(\rho_g(\psi)\) and \(\rho_g(\neg \psi)\) must be undefined. Hence, by IH of the secondary induction we obtain \(\psi \notin [K^\beta(g)](w)\) and \(\neg \psi \notin [K^\beta(g)](w)\) for all \(\forall \in W\) and, in particular, for all worlds accessible from \(w\).

- \(\varphi \equiv \neg \text{KK}\). By the definition of \(\rho\) we know that \(\rho_g(\psi)\) and \(\rho_g(\neg \psi)\) must be undefined. Using the reasoning of the previous case \(B \psi \notin [K^\beta(g)](w)\) and \(\neg B \psi \notin [K^\beta(g)](w)\) for all \(\forall \in W\) and hence also \(K \psi \notin [K^\beta(g)](w)\) and \(\neg K \psi \notin [K^\beta(g)](w)\) for all \(\forall \in W\).

- \(\varphi \equiv \neg (\psi \land \chi)\). If \(\rho_g(\varphi)\) is undefined, then both \(\rho_g(\psi)\) and \(\rho_g(\chi)\) are undefined and the claim is immediate by the IH of the secondary induction.
Lemma 32 (Main Lemma). Let \((F, I)\) be a belief model and \(g \in \text{Val}_F\) an evaluation function such that \(g(w) = \emptyset\) for all \(w \in W\). Then for all \(\varphi \in \text{Sent}_{L_k}\) and \(\alpha \in \text{On}\), if \(\rho_I(\varphi) = \alpha\), then for all \(\gamma > \alpha\) and \(w \in W\):

(i) \(\varphi \in [\mathcal{K}^\gamma(g)](w) \Rightarrow \forall \delta \geq \gamma (\varphi \in [\mathcal{K}^\delta(g)](w));\)

(ii) \(\varphi \notin [\mathcal{K}^\gamma(g)](w) \Rightarrow \forall \delta \geq \gamma (\varphi \notin [\mathcal{K}^\delta(g)](w)).\)

Proof of Main Lemma. The proof is again by transfinite induction on \(\alpha\). As induction hypothesis we assume that for all \(\varphi\) with \(\rho_I(\varphi) = \beta < \alpha\), all \(\gamma > \beta\) and all \(w \in W\)

(i) \(\varphi \in [\mathcal{K}^\gamma(g)](w) \Rightarrow \forall \delta \geq \gamma (\varphi \in [\mathcal{K}^\delta(g)](w));\)

(ii) \(\varphi \notin [\mathcal{K}^\gamma(g)](w) \Rightarrow \forall \delta \geq \gamma (\varphi \notin [\mathcal{K}^\delta(g)](w)).\)

We then conduct a secondary induction on the positive complexity of \(\varphi\).

(a) \(\varphi \equiv \neg P_{t_1}, \ldots, t_n\). For \(\varphi \in \text{Sent}_{L_k}\) the semantic value of \(\varphi\) depends solely on the interpretation function \(I\) and is independent of the evaluation function assumed.

(b) \(\varphi \equiv \text{TT}\). There are two cases

1. \(I(w, t) \notin \text{Sent}_{L_k}\) and the claim follows trivially by Definition 5.

2. \(I(w, t) = \psi\) for some sentence \(\psi\) and all \(w \in W\) and \(\rho_k(\psi) < \rho_k(\text{TT})\). For claim (i), we assume \(\text{TT} \in [\mathcal{K}^\gamma(g)](w)\) and infer \(\psi \notin [\mathcal{K}^\gamma(g)](w)\) by Definition 13. Since \(\rho_k(\psi) < \rho_k(\text{TT})\) we may use the IH to conclude \(\psi \notin [\mathcal{K}^\delta(g)](w)\) for all \(\delta \geq \gamma\) and thus \(\text{TT} \notin [\mathcal{K}^\delta(g)](w)\) for all \(\delta \geq \gamma\).

For claim (ii), we assume \(\text{TT} \notin [\mathcal{K}^\gamma(g)](w)\) and infer \(\psi \notin [\mathcal{K}^\gamma(g)](w)\) by Definition 13. Since \(\rho_k(\psi) < \rho_k(\text{TT})\) we may use the IH to conclude \(\psi \notin [\mathcal{K}^\delta(g)](w)\) for all \(\delta \geq \gamma\) and thus \(\text{TT} \notin [\mathcal{K}^\delta(g)](w)\) for all \(\delta \geq \gamma\).

(c) \(\varphi \equiv \neg \text{TT}\). Again there are two cases

1. \(I(w, t) \notin \text{Sent}_{L_k}\) and the claim follows trivially by Definition 5.

2. \(I(w, t) = \psi\) for some sentence \(\psi\) and all \(w \in W\) and \(\rho_k(\neg \psi) \leq \rho_k(\neg \text{TT})\). Analogous to case 2. of (b)

(d) \(\varphi \equiv \neg \neg \psi\) immediate by secondary induction hypothesis as \(\rho_k(\psi) \leq \rho_k(\phi)\).

(e) \(\varphi \equiv \text{B} \psi\). We first claim (i) and assume \(\text{B} \psi \in [\mathcal{K}^\gamma(g)](w)\) for \(\gamma > \alpha\). By Definitions 13 and 5 we infer

\[\forall u (wRu \Rightarrow \psi \in [\mathcal{K}^\gamma(g)](u))\]
for $\gamma > \alpha$. Then by the IH of our secondary induction

$$\forall u \ (wRv \Rightarrow \forall \delta \geq \gamma (\widehat{\psi} \in [\mathcal{K}^\gamma (g)](v)))$$

and hence $\forall \delta \geq \gamma (B \widehat{\psi} \in [\mathcal{K}^\gamma (g)](w))$. 

To show claim (ii) we assume $B \widehat{\psi} \in [\mathcal{K}^\gamma (g)](w)$, that is Definitions [13] and [5], $3v(wRv \& \psi \in [\mathcal{K}^\gamma (g)](v))$ for $\gamma > \alpha$. By the the IH of the secondary induction we infer

$$3v(wRv \& \forall \delta \geq \gamma (\widehat{\psi} \in [\mathcal{K}^\gamma (g)](v)))$$

and hence $\forall \delta \geq \gamma (B \widehat{\psi} \in [\mathcal{K}^\gamma (g)](w))$.

(f) $\phi \equiv \lnot B \psi$. Analogous to case [e].

(g) $\phi \equiv K \psi$. By Lemma [30] we may assume that both $\rho_k(\psi)$ and $\rho_k(\lnot \psi)$ are defined and that $\rho_k(\psi), \rho_k(\lnot \psi) \equiv \alpha$. For claim (i) assume $K \psi \in [\mathcal{K}^\gamma (g)](w)$ for $\gamma > \alpha$. By Definitions [13] and [5] this implies $B \widehat{\psi} \in [\mathcal{K}^\gamma (g)](w)$ and

$$\forall u \in W_w(\exists u (u <_w v) \& B \widehat{\psi} \in [\mathcal{K}^\gamma (g)](v) \Rightarrow \psi \in [\mathcal{K}^\gamma (g)](v))$$

so by using the IH of the secondary induction together with the reasoning of case [e] we know that

(†) $B \widehat{\psi} \in [\mathcal{K}^\gamma (g)](w) \iff \forall \delta \geq \gamma (B \widehat{\psi} \in [\mathcal{K}^\delta (g)](w)))$ for $\gamma > \alpha$. (†) in combination with the IH of the secondary induction allows us to conclude

$$\forall \delta \geq \gamma (B \widehat{\psi} \in [\mathcal{K}^\delta (g)](w) \& \forall u \in W_w(\exists u (u <_w v) \& B \widehat{\psi} \in [\mathcal{K}^\delta (g)](v) \Rightarrow \psi \in [\mathcal{K}^\delta (g)](v)))$$

that is, $\forall \delta \geq \gamma (K \psi \in [\mathcal{K}^\delta (g)](w)$. For claim (ii), we assume $K \psi \not\in [\mathcal{K}^\gamma (g)](w)$ for $\gamma > \alpha$. By Definitions [13] and [5] this implies either $B \widehat{\psi} \not\in [\mathcal{K}^\gamma (g)](w)$ or

$$\exists u \in W_w(\exists u (u <_w v) \& B \widehat{\psi} \in [\mathcal{K}^\gamma (g)](v) \& \psi \in [\mathcal{K}^\gamma (g)](v)).$$

We can then reason analogously as for (i), using the fact that

(‡) $B \widehat{\psi} \in [\mathcal{K}^\gamma (g)](w) \iff \forall \delta \geq \gamma (B \widehat{\psi} \in [\mathcal{K}^\delta (g)](w)))$, to establish $\forall \delta \geq \gamma (K \psi \not\in [\mathcal{K}^\delta (g)](w)$.

(h) $\phi \equiv \lnot K \psi$. By Lemma [30] we may assume that both $\rho_k(\psi)$ and $\rho_k(\lnot \psi)$ are defined and that $\rho_k(\psi), \rho_k(\lnot \psi) \equiv \alpha$. For claim (i) and assume $\lnot K \psi \in [\mathcal{K}^\gamma (g)](w)$ for $\gamma > \alpha$. By Definitions [13] and [5] this implies that $\lnot B \psi \not\in [\mathcal{K}^\gamma (g)](w)$ or

$$B \widehat{\psi} \in [\mathcal{K}^\gamma (g)](w) \& \exists u \in W_w(\exists u (u <_w v) \& B \widehat{\psi} \in [\mathcal{K}^\gamma (g)](v) \& \psi \in [\mathcal{K}^\gamma (g)](v)).$$

If the former, the claim follows by carrying out the reasoning of case [f]. If the latter, the IH of the secondary induction in combination with (†) does the job.
(i) $\varphi \equiv \psi \land \chi$. We may assume that $\rho_{\varphi}(\psi), \rho_{\varphi}(\chi) \leq \alpha$ and that $\rho_{\varphi}(\psi)$ or $\rho_{\varphi}(\chi)$ is defined. For claim (i), we assume that $\psi \land \phi \in [K^\gamma(g)](w)$ and infer by Definitions 13 and 5 that $\psi \in [K^\gamma(g)](w)$ and $\chi \in [K^\gamma(g)](w)$. Then by Lemma 31 both $\rho_{\varphi}(\psi)$ and $\rho_{\varphi}(\chi)$ are defined and by IH of the secondary induction we infer $\forall \delta \geq \gamma(\psi \in [K^{\delta}(g)](w)$ and $\forall \delta \geq \gamma(\chi \in [K^{\delta}(g)](w)$. This implies $\forall \delta \geq \gamma(\varphi \in [K^{\delta}(g)](w)$.

For claim (ii), we assume that $\psi \land \phi \in [K^\gamma(g)](w)$ and infer by Definitions 13 and 5 that either $\psi \not\in [K^\gamma(g)](w)$ or $\chi \not\in [K^\gamma(g)](w)$. Suppose we have the former. Then either $\rho_{\varphi}(\psi)$ is undefined and $\forall \delta \geq \gamma(\psi \not\in [K^{\delta}(g)](w)$ by Lemma 31 or $\rho_{\varphi}(\psi) \leq \rho_{\varphi}(\phi)$ and $\forall \delta \geq \gamma(\psi \not\in [K^{\delta}(g)](w)$ follows by the IH of the secondary induction. Since $\forall \delta \geq \gamma(\psi \not\in [K^{\delta}(g)](w)$ implies $\forall \delta \geq \gamma(\phi \not\in [K^{\delta}(g)](w)$ we are done. If $\chi \not\in [K^\gamma(g)](w)$ the reasoning is the same.

(j) $\varphi \equiv \lnot(\psi \land \chi)$. Analogous to case (i).

(k) The quantifier cases are just infinitary versions of cases (i) and (j) and can be proved analogously.

Corollary 33. Let $(F,I)$ be a consistent belief model and $g \in \text{Val}_F$ the evaluation function such that $g(w) = \emptyset$ for all $w \in W$. Then for all $\varphi \in \text{Sent}_{\text{\Omega}_F}$, if $\rho_{\varphi}(\varphi) = \alpha$, then $\varphi \not\in [K^\gamma(g)](w)$ for all $\gamma > \alpha$ or $\lnot\varphi \not\in [K^\gamma(g)](w)$ for all $\gamma > \alpha$ for all $w \in W$.

Proof sketch. By Lemma 32 a transfinite induction on $\alpha$ and a secondary induction on the positive complexity of $\varphi$. □