Entropy-power inequality for weighted entropy

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Abstract

We analyse an analog of the entropy-power inequality for the weighted entropy. In particular, we discuss connections with weighted Lieb’s splitting inequality and an Gaussian additive noise formula. Examples and counterexamples are given, for some classes of probability distributions.

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1 Introduction. The weighted entropy-power inequality

Let $x \in \mathbb{R} \rightarrow \phi(x) \geq 0$ be a given (measurable) function. The weighted differential entropy (WDE) $h^w_\phi(Z)$ of a real-valued random variable (RV) $Z$ with a probability density function (PDF) $f_Z$ is defined by the formula

$$h^w_\phi(Z) = h^w_\phi(f_Z) := -E \phi(Z) \ln f_Z(Z) = - \int_{\mathbb{R}^n} \phi(x)f_Z(x) \ln f_Z(x)dx,$$  \hspace{1cm} (1.1)

assuming that the integral is absolutely convergent (with the usual agreement that $0 \cdot \ln 0 = 0$). Cf. [1], [2], [7]. For $\phi(x) \equiv 1$, the definition yields the standard (Shannon) differential entropy (SDE). Furthermore, $\phi$ is called a weight function (WF). When we say that $h^w_\phi(Z)$ is finite we mean that RV $Z$ has a PDF $f_Z$, and the integral in (1.1) absolutely converges.

We propose the following bound which we call the weighted entropy-power inequality (WEPI): for two independent RVs $X_1$ and $X_2$, with $X = X_1 + X_2$,

$$\exp \left[ \frac{2 h^w_\phi(X_1)}{E \phi(X_1)} \right] + \exp \left[ \frac{2 h^w_\phi(X_2)}{E \phi(X_2)} \right] \leq \exp \left[ \frac{2 h^w_\phi(X)}{E \phi(X)} \right],$$  \hspace{1cm} (1.2)

assuming that the integral is absolutely convergent.
assuming that the WDEs \( h^w_{\phi}(X) \), \( h^w_{\phi}(X_1) \) and \( h^w_{\phi}(X_2) \) are finite, as well as the expected values \( \mathbb{E} \phi(X), \mathbb{E} \phi(X_1), \mathbb{E} \phi(X_2) \) (the latter means that \( \mathbb{E} \phi(X), \mathbb{E} \phi(X_1), \mathbb{E} \phi(X_2) \in (0, \infty) \)). Again, for \( \phi(x) \equiv 1 \), this yields the famous EPI put forward by Shannon; see [3], [8], [4], [5]. In this note we offer a sufficient condition for (1.2) (see Eqns (1.5) and (1.6) below); the origins of bound (1.6) go back to Ref. [6]. We set:

\[
\alpha = \tan^{-1}\left( \exp \left[ \frac{h^w_{\phi}(X_2)}{\mathbb{E} \phi(X_2)} - \frac{h^w_{\phi}(X_1)}{\mathbb{E} \phi(X_1)} \right] \right), \quad Y_1 = \frac{X_1}{\cos \alpha}, \quad Y_2 = \frac{X_2}{\sin \alpha},
\]

(1.3)

and

\[
\kappa = \exp \left[ \frac{2h^w_{\phi}(X_1)}{\mathbb{E} \phi(X_1)} \right] + \exp \left[ \frac{2h^w_{\phi}(X_2)}{\mathbb{E} \phi(X_2)} \right].
\]

(1.4)

**Theorem 1:** Given independent RVs \( X_1, X_2 \) and a WF \( \phi \), set \( X = X_1 + X_2 \) and make the following suppositions.

(i) The expected values obey

\[
\mathbb{E} \phi(X_1) \geq \mathbb{E} \phi(X) \quad \text{and} \quad \mathbb{E} \phi(X_2) \geq \mathbb{E} \phi(X) \quad \text{if} \quad \kappa \geq 1, \\
\mathbb{E} \phi(X_1) \leq \mathbb{E} \phi(X) \quad \text{and} \quad \mathbb{E} \phi(X_2) \leq \mathbb{E} \phi(X) \quad \text{if} \quad \kappa \leq 1.
\]

(1.5)

(ii) With \( \alpha \) and \( Y_2, Y_2 \) as in Eqn (1.3),

\[
(\cos \alpha)^2 h^w_{\phi_C}(Y_1) + (\sin \alpha)^2 h^w_{\phi_S}(Y_2) \leq h^w_{\phi}(X).
\]

(1.6)

Here

\[
\phi_C(x) = \phi(x \cos \alpha), \quad \phi_S(x) = \phi(x \sin \alpha)
\]

(1.7)

and we assume finite WDEs \( h^w_{\phi}(X) \) and

\[
h^w_{\phi_C}(Y_2) = -\mathbb{E}\phi_C(Y_2) \ln f_{Y_2}(Y_2), \quad h^w_{\phi_S}(Y_2) = -\mathbb{E}\phi_S(Y_2) \ln f_{Y_2}(Y_2).
\]

Then WEPI (1.2) holds true.

**Proof:** We can write

\[
h^w_{\phi}(X_1) = h^w_{\phi_C}(Y_2) + \mathbb{E} \phi(X_1) \log \cos \alpha, \quad h^w_{\phi}(X_2) = h^w_{\phi_S}(Y_2) + \mathbb{E} \phi(X_2) \log \sin \alpha.
\]

Using (1.6), we have the following inequality:

\[
\begin{align*}
\frac{h^w_{\phi}(X)}{\mathbb{E} \phi(X)} & \geq (\cos \alpha)^2 \left[ \frac{h^w_{\phi}(X_1) - \mathbb{E} \phi(X_1) \log \cos \alpha}{\mathbb{E} \phi(X_1)} \right] \\
& \quad + (\sin \alpha)^2 \left[ \frac{h^w_{\phi}(X_2) - \mathbb{E} \phi(X_2) \log \sin \alpha}{\mathbb{E} \phi(X_2)} \right].
\end{align*}
\]
Furthermore, recalling (1.4) we obtain:

\[ h^w_\phi(X) \geq \frac{1}{2\kappa} \left[ \mathbb{E}\phi(X_1) \log \kappa \right] \exp \left[ \frac{2h^w_\phi(X_1)}{\mathbb{E}\phi(X_1)} \right] + \frac{1}{2\kappa} \left[ \mathbb{E}\phi(X_2) \log \kappa \right] \exp \left[ \frac{2h^w_\phi(X_2)}{\mathbb{E}\phi(X_2)} \right]. \]

By virtue of assumption (1.5), we derive:

\[ h^w_\phi(X) \geq \frac{1}{2} \mathbb{E}\phi(X) \log \kappa. \]

The definition of \( \kappa \) in Eqn (1.4) leads directly to the result. \( \square \)

Paying homage to Ref. [6], we call the bound (1.6) the WLSI (weighted Lieb’s splitting inequality). In the spirit of [6], the following Theorem 2 can be offered. (The notation used in Theorem 2 is self-explanatory; the proof of Theorem 2 is one-line and omitted.)

**Theorem 2:** Let \( f \) and \( g \) be PDFs on \( \mathbb{R} \) and \( \phi \) a given WF. Assume that the WDEs \( h^w_\phi(f \ast g) \), \( h^w_\phi(f) \) and \( h^w_\phi(g) \) are finite, as well as expected values \( \mathbb{E}_f \phi, \mathbb{E}_g \phi \). Set

\[ \tau = \exp \left[ \frac{2h^w_\phi(f)}{\mathbb{E}_f \phi} \right] + \exp \left[ \frac{2h^w_\phi(g)}{\mathbb{E}_g \phi} \right]. \]

Also suppose that

\[ \mathbb{E}_f \phi \geq \mathbb{E}_{f \ast g} \phi \quad \text{and} \quad \mathbb{E}_g \phi \geq \mathbb{E}_{f \ast g} \phi \quad \text{if} \quad \tau \geq 1, \]

\[ \mathbb{E}_f \phi \leq \mathbb{E}_{f \ast g} \phi \quad \text{and} \quad \mathbb{E}_g \phi \leq \mathbb{E}_{f \ast g} \phi \quad \text{if} \quad \tau \leq 1. \]

(1.8)

and the following inequality holds:

\[ 2h^w_\phi(f \ast g) \geq 2\lambda h^w_\phi(f) + 2(1 - \lambda)h^w_\phi(g) - \mathbb{E}_f \phi \log \lambda - \mathbb{E}_g \phi (1 - \lambda) \log(1 - \lambda), \]

(1.9)

where \( \lambda \in [0, 1] \) is given by

\[ \lambda = \tau^{-1} \exp \left[ \frac{2h^w_\phi(f)}{\mathbb{E}_f \phi} \right]. \]

(1.10)

Then Eqn (1.2) holds true for independent RVs \( X_1 \) and \( X_2 \) where \( X_1 \sim f, X_2 \sim g \).

**Remark.** The arguments developed in Section 1 do not use the fact that RVs \( X_1 \) and \( X_2 \) possess PDFs. The question of whether the WEPI (as it is presented in Eqn (1.2) or in a modified form) may hold for cases of discrete distributions requires a separate investigation. However, constructions used in Section 3 demand existence of PDFs \( f_{X_1} \) and \( f_{X_2} \) although some of their technical parts are valid in a more general situation.
2 Examples and counterexamples

In this section we give several examples where the above inequalities hold or do not hold true.

2.1 Examples. First, let us discuss specific conditions equivalent to (1.2), (1.5) or (1.6), for various pairs of RVs. In the next subsection we present results of numerical simulations showing domains of parameters where Eqns (1.2), (1.5) and (1.6) are fulfilled or violated.

2.1.1. (Normal distributions) Let $X_1$, $X_2$ be two independent normal RVs: $X_1 \sim N(0, \sigma_1^2)$, $X_2 \sim N(0, \sigma_2^2)$ and $X \sim N(0, \sigma_1^2 + \sigma_2^2)$. Recall (see [7], Example 3.1), the WDE $h^w_\phi(Z)$ of a normal random variable $Z \sim N(0, \sigma^2)$ reads

$$h^w_\phi(Z) = \frac{\log(2\pi \sigma^2)}{2} \mathbb{E}\phi(Z) + \frac{\log e}{2\sigma^2} \mathbb{E}Z^2\phi(Z);$$

we will use it for $Z = X, X_1, X_2$. The condition $\kappa \geq (\leq) 1$ is re-written as

$$\sigma_i^2 \exp \left\{ \frac{\mathbb{E}[X_i^2\phi(X_i)]}{\sigma_i^2 \mathbb{E}\phi(X_i)} \right\} + \sigma_j^2 \exp \left\{ \frac{\mathbb{E}[X_j^2\phi(X_j)]}{\sigma_j^2 \mathbb{E}\phi(X_j)} \right\} \geq (\leq) (2\pi)^{-1}. \hspace{1cm} (2.1)$$

We have to match it with inequalities

$$\mathbb{E}\phi(X_1), \mathbb{E}\phi(X_2) \geq (\leq) \mathbb{E}\phi(X)$$

to fulfill (1.5).

To specify the WLSI (1.5), we write:

$$h^w_{\phi_C}(Y_1) = \frac{1}{2} \left[ \log \left( \frac{2\pi \sigma_1^2}{(\cos \alpha)^2} \right) \mathbb{E}\phi_C(Y_1) + \frac{(\cos \alpha)^2 \log e}{2\sigma_1^2} \mathbb{E}[Y_1^2\phi_C(Y_1)] \right]. \hspace{1cm} (2.2)$$

Plugging-in the definition of $\phi_C$:

$$\mathbb{E}\phi_C(Y_1) = \mathbb{E}\phi(X_1), \mathbb{E}[Y_1^2\phi_C(Y_1)] = \frac{\mathbb{E}[X_1^2\phi(X_1)]}{(\cos \alpha)^2}.$$

Similar equations hold for $h^w_{\phi_S}(Y_2)$. Then Eqn (1.6) takes the form

$$\left[ \log \left( \frac{2\pi(\sigma_1^2 + \sigma_2^2)}{(\cos \alpha)^2} \right) \mathbb{E}\phi(X) + \frac{\log e}{\sigma_1^2 + \sigma_2^2} \mathbb{E}[X^2\phi(X)] \right] \geq (\cos \alpha)^2 \left[ \frac{\log \left( \frac{2\pi \sigma_1^2}{(\cos \alpha)^2} \right)}{\sigma_1^2 \mathbb{E}\phi(X_1)} \right] \mathbb{E}\phi(X_1) + \frac{(\cos \alpha)^2 \log e}{\sigma_1^2} \mathbb{E}[X_1^2\phi(X_1)] \hspace{1cm} (2.3)$$

$$+ (\sin \alpha)^2 \left[ \frac{\log \left( \frac{2\pi \sigma_2^2}{(\sin \alpha)^2} \right)}{\sigma_2^2 \mathbb{E}\phi(X_2)} \right] \mathbb{E}\phi(X_2) + \frac{(\sin \alpha)^2 \log e}{\sigma_2^2} \mathbb{E}[X_2^2\phi(X_2)].$$

2.1.2. (Gamma-distributions) Let $X_1$ and $X_2$ have Gamma distributions, with PDFs $f_{X_i}(x) = \frac{\lambda_i^\beta}{\Gamma(\beta)} x^{\beta_i-1} e^{-\lambda x}$, $i = 1, 2$, and $f_X(x) = \frac{\lambda^{\beta}}{\Gamma(\beta)} x^{\beta-1} e^{-\lambda x}$, $x > 0$ where $\beta = \beta_1 + \beta_2$. The WDEs are

$$h^w_\phi(X_i) = (1 - \beta_i)\mathbb{E}[\phi(X_i) \log X_i] + \lambda_i \mathbb{E}[X_i\phi(X_i)] + \log \left( \frac{\Gamma(\beta_i)}{\lambda_i} \right) \mathbb{E}\phi(X_i)$$

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and similarly for \( X \) (with \( \beta \) instead of \( \beta_i \)). The condition \( \kappa \geq (\leq)1 \) reads

\[
\left( \frac{\Gamma(\beta)}{\lambda^{\beta}} \right) \exp \left\{ \frac{\lambda \mathbb{E}[X_1 \phi(X_1)] - (\beta - 1) \mathbb{E}[\phi(X_1) \log X_1]}{\mathbb{E}[\phi(X_1)]} \right\} + \left( \frac{\Gamma(\beta)}{\lambda^{\beta}} \right) \exp \left\{ \frac{\lambda \mathbb{E}[X_2 \phi(X_2)] - (\beta - 1) \mathbb{E}[\phi(X_2) \log X_2]}{\mathbb{E}[\phi(X_2)]} \right\} \geq (\leq)1; \tag{2.4}
\]

as above, it has to be in conjunction with \( \mathbb{E}[\phi(X_1)], \mathbb{E}[\phi(X_2)] \geq (\leq) \mathbb{E}[\phi(X)] \). The WLSI (1.6) takes the following form:

\[
\log \frac{\Gamma(\beta)}{\lambda^{\beta}} \mathbb{E}[\phi(X)] - (\beta - 1) \mathbb{E}[\phi(X) \log X] + \lambda \mathbb{E}[X \phi(X)] \\
\geq (\cos \alpha)^2 \left[ \lambda \mathbb{E}[X_1 \phi(X_1)] - (\beta - 1) \mathbb{E}[\phi(X_1) \log X_1] \right] + \mathbb{E}[\phi(X_1)] \log \frac{\Gamma(\beta)}{\lambda^{\beta} \cos \alpha} \tag{2.5} \\
+ (\sin \alpha)^2 \left[ \lambda \mathbb{E}[X_2 \phi(X_2)] - (\beta - 1) \mathbb{E}[\phi(X_2) \log X_2] \right] + \mathbb{E}[\phi(X_2)] \log \frac{\Gamma(\beta)}{\lambda^{\beta} \sin \alpha}.
\]

2.1.3. (Exponential distributions) Let \( X_1 \) and \( X_2 \) be two independent exponential RVs, with means \( \lambda_1^{-1} \) and \( \lambda_2^{-1} \), and WDEs \( h^w_{\phi}(X_i) = (\lambda_i \log \lambda_i) \phi(X_i) + \mathbb{E}X_i \phi(X_i), i = 1, 2 \). See [7], Example 3.2. Then \( f_X(x) = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} (e^{-\lambda_1 x} - e^{-\lambda_2 x}) \). The inequality \( \kappa \geq (\leq)1 \) becomes

\[
\frac{\lambda_2^2}{\lambda_1 - \lambda_2} \exp \left\{ \frac{2 \lambda_1 \mathbb{E}[X_1 \phi(X_1)]}{\mathbb{E}[\phi(X_1) \log X_1]} \right\} + \frac{\lambda_1^2}{\lambda_2 - \lambda_1} \exp \left\{ \frac{2 \lambda_2 \mathbb{E}[X_2 \phi(X_2)]}{\mathbb{E}[\phi(X_2) \log X_2]} \right\} \geq (\leq) \lambda_1^2 \lambda_2^2; \tag{2.6}
\]

and to fulfill Eqn (1.5) it has to be combined with

\[
\mathbb{E}[\phi(X_1)], \mathbb{E}[\phi(X_2)] \geq (\leq) \frac{\lambda_1 \mathbb{E}[\phi(X_2)] - \lambda_2 \mathbb{E}[\phi(X_1)]}{\lambda_1 - \lambda_2}. \tag{2.7}
\]

In this example, the WLSI (1.6) reads

\[
\frac{\log \lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \left[ \lambda_2 \mathbb{E}[\phi(X_1)] - \lambda_1 \mathbb{E}[\phi(X_2)] \right] + \frac{\lambda_2}{\lambda_1 - \lambda_2} \mathbb{E} \left[ \phi(X_1) \log \left( \frac{e^{-\lambda_2 X_1} - e^{-\lambda_1 X_1}}{\lambda_1 - \lambda_2} \right) \right] \\
- \frac{\lambda_1}{\lambda_1 - \lambda_2} \mathbb{E} \left[ \phi(X_2) \log \left( \frac{e^{-\lambda_2 X_2} - e^{-\lambda_1 X_2}}{\lambda_1 - \lambda_2} \right) \right] \geq \lambda_1 (\cos \alpha)^2 \mathbb{E}[X_1 \phi(X_1)] - (\cos \alpha)^2 \mathbb{E}[\phi(X_1)] \log (\lambda_1 \cos \alpha) \tag{2.8} \\
+ \lambda_2 (\sin \alpha)^2 \mathbb{E}[X_2 \phi(X_2)] - (\sin \alpha)^2 \mathbb{E}[\phi(X_2)] \log (\lambda_2 \sin \alpha).
\]

2.1.4. (Uniform distributions) Set \( \Phi(x) = \int_0^x \phi(u)du \), \( \Phi^r(x) = \int_0^x u \phi(u)du \). Let \( X_1 \sim U(a_1, b_1) \) and \( X_2 \sim U(a_2, b_2) \), independently, where \( L_i := b_i - a_i > 0, i = 1, 2 \). The WDE \( h^w_{\phi}(X_i) = \frac{\Phi(b_i) - \Phi(a_i)}{L_i} \log L_i \). Suppose for definiteness that \( L_2 \geq L_1 \) or, equivalently, \( C_1 := a_2 + b_1 \leq \)
\[ a_1 + b_2 =: C_2. \] Then PDF \( f_X \) for \( X = X_1 + X_2 \) has a trapezoidal form with corner points at \( A = a_1 + a_2, C_1, C_2 \) and \( B = b_1 + b_2 \):

\[
f_X(x) = \frac{1}{L_1 L_2} \times \begin{cases} 
0, & \text{if } x < A \text{ or } x > B, \\
x - A, & \text{if } A < x < C_1, \\
L_1, & \text{if } C_1 < x < C_2, \\
B - x, & \text{if } C_2 < x < B.
\end{cases}
\]

The condition \( \kappa \geq (\leq) 1 \) takes the form

\[
L_1^2 + L_2^2 \geq (\leq) 1. \tag{2.9}
\]

Consider the quantity \( \Lambda \) (which may be as positive as well as negative):

\[
\Lambda = \frac{\log L_1}{L_2} \left[ \Phi(C_1) - \Phi(C_2) \right] + \frac{1}{L_1 L_2} \left[ \int_A^{C_1} \phi(x)(x - A) \log(x - A) \, dx + \int_{C_2}^{B} \phi(x)(B - x) \log(B - x) \, dx \right]. \tag{2.10}
\]

Then

\[
\mathbb{E} \phi(X) = \frac{1}{L_1 L_2} \left\{ \left[ \Phi^*(C_1) - \Phi^*(A) - \Phi^*(B) + \Phi^*(C_2) \right] \\
- A \left[ \Phi(C_1) - \Phi(A) \right] + L_1 \left[ \Phi(C_2) - \Phi(C_1) \right] + B \left[ \Phi(B) - \Phi(C_2) \right] \right\}.
\]

To satisfy condition (1.5), we have to assume that

\[
L_2 \left[ \Phi(b_1) - \Phi(a_1) \right], \quad L_1 \left[ \Phi(b_2) - \Phi(a_2) \right] \geq (\leq) \mathbb{E} \phi(X), \tag{2.11}
\]

in conjunction with bound (2.9).

The WLSI (1.6) becomes

\[
h_\phi^w(X) = -\Lambda + \left[ \log \left( L_1 L_2 \right) \right] \mathbb{E} \phi(X) \\
\geq (\cos \alpha)^2 \frac{\Phi(b_1) - \Phi(a_1)}{L_1} \log \frac{L_1}{\cos \alpha} + (\sin \alpha)^2 \frac{\Phi(b_2) - \Phi(a_2)}{L_2} \log \frac{L_2}{\sin \alpha}. \tag{2.12}
\]

**2.1.5.** (A mixed case) Let \( X_1 \) be a Gamma RV with PDF \( f_{X_1}(x) = \frac{\lambda^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\lambda x} \) and the cumulative distribution function \( F_{X_1}(x) \). The WDE \( h_\phi^w(X_1) \) has been specified in Example 2.1.3. Take RV \( X_2 \) from the uniform distribution \( U(a,b) \), where \( L := b - a > 0 \), independent of \( X_1 \). The WDE \( h_\phi^w(X_1) \) has been specified in Example 2.1.4. We can write

\[
f_X(t) = \frac{1}{L} \left[ F_{X_1}(t-a) - F_{X_1}(t-b) \right].
\]
As in Example 2.1.3, let $\Phi(x) = \int_0^x \phi(u) du$. Next, set:

$$\Theta = \int_0^\infty \phi(x) [F_{X_1}(x-a) - F_{X_1}(x-b)] \log [F_{X_1}(x-a) - F_{X_1}(x-b)] \, dx.$$  (2.13)

Then

$$h_{\phi}^w(X) = \frac{\log(L \Gamma(\beta))}{L} \times \mathbb{E} \left[ \Phi(X_1 + b) 1(X_1 > -b) - \Phi(X_1 + a) 1(X_1 > -a) \right] - \frac{\Theta}{L}.$$  (2.14)

The quantity $\kappa$ is specified by

$$\kappa = \left( \frac{\Gamma(\beta)}{\lambda^\beta} \right)^2 \exp \left[ \frac{\lambda \mathbb{E}[X_1 \phi(X_1)] - (\beta - 1) \mathbb{E}[\phi(X_1) \log X_1]}{\mathbb{E}[\phi(X_1)]} \right] + L^2.$$  (2.15)

Note that if $b - a \geq 1$ we always have $\kappa > 1$. To fulfill condition (1.5), we have to assume that

$$\left\{ \begin{array}{l}
L \mathbb{E} \phi(X_1) \geq (\leq) \mathbb{E}[\Phi(X_1 + b) - \Phi(X_1 + a)] \\
\Phi(b) - \Phi(a) \geq (\leq) \mathbb{E}[\Phi(X_1 + b) - \Phi(X_1 + a)]
\end{array} \right.$$  depending on $\kappa \geq (\leq) 1$.  (2.16)

The WLSI (1.6) takes the form:

$$h_{\phi}^w(X) := \log[L \Gamma(\beta)] \mathbb{E} \left[ \Phi(X_1 + b) 1(X_1 > -b) - \Phi(X_1 + a) 1(X_1 > -a) \right] - \Theta \geq L (\cos \alpha)^2 \left[ (1 - \beta) \mathbb{E}[\phi(X_1) \log X_1] + \lambda \mathbb{E}[X_1 \phi(X_1)] + \log \left( \frac{\Gamma(\beta)}{\lambda^\beta \cos \alpha} \right) \mathbb{E}[\phi(X_1)] \right]$$

$$+ (\sin \alpha)^2 \frac{\Phi(b) - \Phi(a)}{b - a} \log \left( \frac{L}{\sin \alpha} \right).$$  (2.17)

2.1.6. (Cauchy distributions) Let $X_1, X_2$ be independent, with PDFs $f_{X_j}(x) = (\pi \theta_j)^{-1} x \left[ 1 + (x - \mu_j)^2 / \theta_j^2 \right]^{-1}, x \in \mathbb{R}, j = 1, 2$. Then $f_{X_j}(x)$ is the same form, with parameters $\mu = \mu_1 + \mu_2$ and $\theta = \theta_1 + \theta_2$. For the WDEs $h_{\phi}^w(X)$ we have the formula

$$h_{\phi}^w(X) = \mathbb{E} \phi(X) \log(\pi \theta) + \mathbb{E} [\phi(X) \log(1 + (X - \mu)^2 / \theta^2)]$$

and similarly for $h_{\phi}^w(X_j)$. The condition $\kappa \geq (\leq) 1$ is re-written as

$$\theta_1^2 \exp \left\{ \frac{2 \mathbb{E}[\phi(X_1) \log(1 + (X_1 - \mu_1)^2 / \theta_1^2)]}{\mathbb{E}[\phi(X_1)]} \right\} + \theta_2^2 \exp \left\{ \frac{2 \mathbb{E}[\phi(X_2) \log(1 + (X_2 - \mu_2)^2 / \theta_2^2)]}{\mathbb{E}[\phi(X_2)]} \right\} \geq (\leq) \pi^{-2};$$  (2.18)

to satisfy (1.6) we have to match it with $\mathbb{E}[\phi(X_1)]$, $\mathbb{E}[\phi(X_2)] \geq (\leq) \mathbb{E}[\phi(X)]$. The WLSI (1.6) reads

$$\mathbb{E} \phi(X) \log(\pi \theta) + \mathbb{E} [\phi(X) \log(1 + (X - \mu)^2 / \theta^2)]$$

$$\geq (\cos \alpha)^2 \mathbb{E} [\phi(X_1) \log(1 + (X_1 - \mu_1)^2 / \theta_1^2)] + (\cos \alpha)^2 \mathbb{E} \phi(X_1) \log \frac{\pi \theta_1}{\cos \alpha}$$

$$+ (\sin \alpha)^2 \mathbb{E} [\phi(X_2) \log(1 + (X_2 - \mu_2)^2 / \theta_2^2)] + (\sin \alpha)^2 \mathbb{E} \phi(X_2) \log \frac{\pi \theta_2}{\sin \alpha},$$  (2.19)
2.1.7. (A distribution with an infinite SDE) Here we take independent $X_1, X_2 \sim g$ where $g$ is a PDF on $(1, \infty)$:

$$g(x) = \frac{1}{x[(\ln x)^2 + 1]}, \quad x > 1.$$  

Here we have that

$$f_X(t) = \int g(t - s)g(s)ds = \int_1^{t-1} \frac{1}{(t - s)[1 + (\ln(t - s))^2][1 + (\ln s)^2]} ds.$$  

Therefore

$$h^w_\phi(X) = -\int_2^\infty \frac{\phi(x)}{(x - s)[1 + (\ln(x - s))^2][1 + (\ln s)^2]} ds \times \log \left( \int_1^{x-1} \frac{1}{(x - w)[1 + (\ln(x - w))^2][1 + (\ln w)^2]} dw \right) dx,$$

assuming that WF $\phi$ decreases fast enough so that the integral in (2.20) absolutely converges. The bound $\kappa \geq (\leq) 1$ now reads:

$$\exp \left\{ \frac{2E[\phi(X_1)\log \{X_1[1 + (\ln X_1)^2 + 1]\}]}{E\phi(X_1)} \right\} \geq (\leq) \frac{1}{2},$$

and we again have to match it with inequality $E\phi(X_1), \ E\phi(X_2) \geq (\leq) E\phi(X)$.

The WLSI (1.6) takes the form:

$$-\int_2^\infty \phi(x) \int_1^{x-1} \frac{1}{(x - s)[1 + (\ln(x - s))^2][1 + (\ln s)^2]} ds \times \log \left( \int_1^{x-1} \frac{1}{(x - w)[1 + (\ln(x - w))^2][1 + (\ln w)^2]} dw \right) dx$$

$$\geq E[\phi(X_1)\log X_1] + E\left\{ \phi(X_1)\log [1 + (\ln X_1)^2] \right\} - E\phi(X_1) \left[ (\cos \alpha)^2 \log \cos \alpha + (\sin \alpha)^2 \log \sin \alpha \right].$$

2.2. Numerical results. As was said, in this subsection we comment on some numerical evidence that (i) Eqn (1.2) does not always hold true, and (ii) assumptions (1.5) (1.6) in Theorem 1 are not necessary for the WEPI (1.2). Our observations are of a preliminary character, and we think that further numerical simulations are needed here, to build a detailed picture.
2.2.1. (Normal distributions) Assume that $X_1$ and $X_2$ are normal RVs as in Example 2.1.1. Choose $\phi(x) = |x^2 - 2|$. The graph in Figure 2.2.1 presents the difference between the RHS and the LHS in (1.2): when this difference is non-negative, the WEPI is satisfied, otherwise the WEPI fails. The graph shows that there is a domain of parameter $(\sigma_1, \sigma_2)$ where Eqn (1.2) does not hold. Additional simulations state that there is a domain where (1.2) holds and only one of Eqns (1.5), (1.6) is satisfied. For example, in the square where $0.55 < \sigma_1, \sigma_2 < 0.551$ the condition (1.6) is violated but (1.5), (1.2) hold true.

2.2.2. (Gamma distributions) Assume that $X_1$ and $X_2$ are gamma RVs as in Example 2.1.2. Here we choose $\phi(x) = xe^{-x}$. The graph in Figure 2.2.2 again shows the difference between the RHS and the LHS in (1.2): Here the WEPI is satisfied (in the presented range of parameters $(\beta_1, \beta_2)$). As in Example 2.2.1, additional simulations assert that there is a domain where (1.2) holds while none of (1.5), (1.6) is satisfied. For example, in the square $0.01 < \beta_1 < 0.1$ and $5 < \beta_2 < 6.1$ both conditions (1.5), (1.6) are violated but (1.2) holds true.

3 WDE and an additive Gaussian noise

3.1. Integral representations for WDEs. Following [8], [4], the WLSI (1.6) can be re-written (under certain conditions on $f_{X_2}$, $f_{X_2}$ and $\phi$) in terms of integral representations of the entropies $h^w_{\phi}(X)$, $h^w_{\phi_C}(Y_1)$ and $h^w_{\phi_S}(Y_2)$: cf. Eqns (3.4), (3.5) below. Despite its cumbersome appearance, formulas (3.4) and (3.5) have an advantage: they does not include logarithms. (However, note condition (3.2).) Throughout the presentation in this section, the reader can notice persistent
similarities with [4].

In this section we work with a two-variable WF \((x, y) \in \mathbb{R} \times \mathbb{R} \mapsto \rho(x, y) \geq 0\) and a number of reduced WDEs involving various integrals of \(\rho\). Let \(Z\) and \(N\) be two independent RVs, where \(N \sim N(0, 1)\) with standard normal PDF \(f^{No}\), while \(Z\) has a PDF \(f_Z\). Following [5] and [4], RV \(Z\) will represent a signal and \(N\) an (additive) Gaussian noise; RV \(Z\) will be a pre-cursor for \(X = X_1 + X_2, Y_1\) and \(Y_2\). Given \(\gamma > 0, y \in \mathbb{R},\) set:

\[
\begin{align*}
\xi_Z(y, \gamma) &= \int (y - t\sqrt{\gamma}) y f^{No}(y - t\sqrt{\gamma}) f_Z(t) dt, \\
\eta_Z(y, \gamma) &= \int (y - z\sqrt{\gamma}) w f^{No}(y - w\sqrt{\gamma}) f_Z(w) dw, \\
\zeta_Z(x, y, \gamma) &= \int_{-\infty}^{y} \rho(x, v)(v - x\sqrt{\gamma}) x f_Z(x, y) f_Z(x, v) f_Y(x, v) d v.
\end{align*}
\]

**Theorem 3.** Let \(X_1, X_2, N\) and \(N'\) be independent RVs, with \(X = X_1 + X_2\), where (i) \(X_j\) are with bounded and continuous PDFs \(f_{X_j}\) such that \(\mathbb{E}[\log f_X(X)] < \infty\) and \(\mathbb{E}[\log f_{X_j}(X_j)] < \infty, j = 1, 2,\) and (ii) \(N, N' \sim N(0, 1)\). Assume that for \(Z = X_1, Z = X_2\) and \(Z = X\), the conditional expectation \(\mathbb{E}\left[f_Z\left(Z + \frac{N - N'}{\sqrt{\gamma}}\right) \mid Z, N\right]\) is such that, for some integrable RV \(\chi(Z, N) \geq 0\)

\[
\left| \log \mathbb{E}\left[f_Z\left(Z + \frac{N - N'}{\sqrt{\gamma}}\right) \mid Z, N\right] \right| \leq \chi(Z, N).
\]

Next, consider a WF \((x, y) \in \mathbb{R} \times \mathbb{R} \mapsto \rho(x, y)\). Suppose that \(\rho\) is continuous and bounded, and \(\forall \ x \in \mathbb{R}, \exists \ \text{a limit } \phi(x) = \lim_{y \to \pm \infty} \rho(x, y)\). Introduce additional WFs

\[
\rho_C(x, y) = \rho(x \cos \vartheta, y), \ \rho_S(x, y) = \rho(x \sin \vartheta, y), \ \text{with } \phi_C/S(x) = \lim_{y \to \pm \infty} \rho_C/S(x, y), \\
\phi^*_1(v) = \int \rho(x, v)f_X(x)dx, \phi^*_2(v) = \int \rho_C(x, v)f_Y(x)dx, \phi^*_2(v) = \int \rho_S(x, v)f_{Y_2}(x)dx
\]

where \(Y_2, Y_1\) are as in (1.3).

Then, the WDE \(h^w_\phi(X)\) of the sum \(X = X_1 + X_2\) in the RHS of (1.6) admits the representation

\[
\int_{0}^{\infty} \frac{1}{2\sqrt{\gamma}} \mathbb{E}\left[\xi_X(X, X \sqrt{\gamma} + N, \gamma) \xi_X(X \sqrt{\gamma} + N, \gamma) \right. \\
- \rho(X, X \sqrt{\gamma} + N) \eta_X(X \sqrt{\gamma} + N, \gamma) \left. \right] d \gamma + h^w_{\phi^*_*}(N).
\]

On the other hand, for \(h^w_{\phi_C}(Y_1)\) and \(h^w_{\phi_S}(Y_2)\) we have the respective formulas

\[
\int_{0}^{\infty} \frac{1}{2\sqrt{\gamma}} \mathbb{E}\left[\xi_{Y_j}(Y_j, Y_j \sqrt{\gamma} + N, \gamma) \xi_{Y_j}(Y_j \sqrt{\gamma} + N, \gamma) \right. \\
- \rho(Y_j, Y_j \sqrt{\gamma} + N) \eta_{Y_j}(Y_j \sqrt{\gamma} + N, \gamma) \left. \right] d \gamma + h^w_{\phi^*_j}(N), \ j = 1, 2.
\]
The proof of Theorem 3 uses two technical assertions, Lemmas 3.1 and 3.2. They address the cases $\gamma = 0$ and $\gamma \to \infty$ (that is, the integration endpoints in (3.4) and (3.5)). For the definition of the weighted conditional and mutual entropies, see Eqns (1.11), (1.12) in [7].

**Lemma 3.1.** (Cf. Lemma 2.4 in [1].) Let $Z$, $U$ be independent RVs. Assume that $U$ has a bounded and continuous PDF $f_U \in C^0(\mathbb{R}^d)$: $\int f_U(x)dx = 1$ and $\text{ess sup}[f_U(x), x \in \mathbb{R}^d] < +\infty$. The distribution of $Z$ may have discrete and continuous parts; we refer to the PMF $f_Z(x)$ relative to a reference measure $\nu(dx)$. Next, suppose that a bounded WF $(x, y) \in \mathbb{R} \times \mathbb{R} \mapsto \rho(x, y)$ has been given and assume that $\mathbb{E} \log f_Z(Z) < +\infty$. Consider the weighted mutual entropy (WME) $i^w_\rho(Z : \sqrt{\gamma}Z + U)$ between $Z$ and $\sqrt{\gamma}Z + U$ where $\gamma > 0$ is a parameter. Then

$$\lim_{\gamma \to 0} i^w_\rho(Z : \sqrt{\gamma}Z + U) = 0. \quad (3.6)$$

**Proof.** According to the definition of the WME, for a pair of RVs $Z,V$ with a conditional PDF $f_{V|Z}(y,x)$ we have an equality involving a weighted conditional entropy (WCE)

$$i^w_\rho(Z : V) = h^w_\rho(Z) - h^w_\rho(Z|V)$$

where $\psi_Z(x) = \int \rho(x, y)f_{V|Z}(y|x)dy$. Setting $V = \sqrt{\gamma}Z + U$, we can write a representation for the WCE:

$$h^w_\rho(Z|\sqrt{\gamma}Z + U) = \int \rho(x, y)f_U(y - \sqrt{\gamma}x)f_Z(x)\times \log \left[ \frac{\int f_U(y - \sqrt{\gamma}z)f_Z(z)dz}{f_U(y - \sqrt{\gamma}x)f_Z(x)} \right] dy\nu(dx).$$

Using that $\rho$ and $f_U$ are bounded, with the help of the Lebesgue dominated convergence theorem we have that as $\gamma \to 0$, the ratio under the log converges to $[f_Z(x)]^{-1}$. Consequently,

$$\lim_{\gamma \to 0} h^w_\rho(Z|\sqrt{\gamma}Z + U) = h^w_{\psi_Z}(Z)$$

where $\psi_Z(x) = \int \rho(x, y)f_U(y)dy$. Moreover, with $\psi_{Z,\gamma}(x) = \int \rho(x, y)f_U(y - \sqrt{\gamma}x)dy$, we introduce:

$$h^w_{\psi_{Z,\gamma}}(Z) = - \int \rho(x, y)f_U(y - \sqrt{\gamma}x)f_Z(x)\log f_Z(x)dy\nu(dx).$$

At this stage we again apply the Lebesgue dominated convergence theorem and deduce that

$$\lim_{\gamma \to 0} h^w_{\psi_{Z,\gamma}}(Z) = h^w_{\psi_Z}(Z).$$

This leads to (3.6). \qed

**Lemma 3.2.** (Cf. Lemma 4.1 in [4].) Let $Z$, $U$, and $U'$ be independent RVs, $Z$ with a PDF $f_Z$ and $U, U'$ with a PDF $f_U$. As before, consider a WF $(x, y) \in \mathbb{R} \times \mathbb{R} \mapsto \rho(x, y)$. Suppose that
$f_Z$ and $\rho$ are continuous and bounded, and there exists a limit $\overline{\gamma}(x) = \lim_{y \to \infty} \rho(x,y)$. Next, assume that for some RV $\chi(Z, U) \geq 0$ with $E\chi(Z, U) < \infty$, we have $\left| \log E \left[ f_Z \left( Z + \frac{U - \chi}{\sqrt{\gamma}} \right) \right] \right| \leq \chi(Z, U)$. Then

$$h^w_\rho(Z) = \lim_{\gamma \to \infty} \left[ i^w_\rho(Z : \sqrt{\gamma}Z + U) + h^w_{\psi^*_\gamma}(U) \right]$$

$$h^w_{\psi^*\gamma}(U) = -E\psi^*_\gamma(U) \log f_U(U) \text{ and } \psi^*_\gamma(u) = \int \rho(x, u + \sqrt{\gamma}x) f_Z(x) dx. \tag{3.7}$$

**Proof.** We can write

$$i^w_\rho(Z : \sqrt{\gamma}Z + U) + h^w_{\psi^*\gamma}(U)$$

$$= -\int \rho(x, u + \sqrt{\gamma}x) f_U(u) f_Z(x) \log \left[ \int f_U(v)f_Z \left( x + \frac{u - v}{\sqrt{\gamma}} \right) dv \right] dudx. \tag{3.8}$$

Passing to the limit $\gamma \to \infty$, Eqn (3.8) yields (3.7), again owing to the Lebesgue dominated convergence theorem. \hfill \square

**Proof of Theorem 3.** We again use $Z$ as a substitute for RVs $Y_1$, $Y_2$ and $X = X_1 + X_2$. Given $\gamma > 0$, write the joint WDE for $Z$ and $Z\sqrt{\gamma} + N$:

$$h^w_\rho(Z, Z\sqrt{\gamma} + N) = -\int \rho(x, x\sqrt{\gamma} + v) f^{N_0}(v) f_Z(x) \log f^{N_0}(v) dx dv$$

$$-\int \rho(x, x\sqrt{\gamma} + v) f^{N_0}(v) f_Z(x) \log f_Z(x) dx dv = h^w_{\psi^{(1)}_\gamma}(Z) + h^w_{\psi^{(2)}_\gamma}(N). \tag{3.9}$$

Here and below, $\forall \gamma, \theta > 0$,

$$\psi^{(1)}_{Z,\gamma}(x) = \int \rho(x, x\sqrt{\gamma} + v) f^{N_0}(v) dv, \quad \psi^{(2)}_{N,\theta}(v) = \int \rho(x, x\sqrt{\gamma} + v) f_Z(x) dx, \tag{3.10}$$

with $h^w_{\psi^{(2)}_\gamma}(N) = -\int \left[ \int \rho(x, x\sqrt{\gamma} + v) f_Z(x) dx \right] f^{N_0}(v) \log f^{N_0}(v) dv$.

Moreover, according to Lemma 3.2 (with $U = N$, $U' = N'$), $\forall \epsilon > 0$ we have

$$h^w_\rho(Z) = \int_\epsilon^\infty \frac{d}{d\gamma} \left[ i^w_\rho(Z : Z\sqrt{\gamma} + N) + h^w_{\psi^{(2)}_\gamma}(N) \right] d\gamma + i^w_\rho(Z : Z\sqrt{\epsilon} + N) + h^w_{\psi^{(2)}_\gamma}(N).$$

To analyze the WDE $h^w_{\psi^{(2)}_\gamma}(N)$, we use Lebesgue’s dominated convergence theorem. This yields

$$\lim_{\epsilon \to 0} h^w_{\psi^{(2)}_\gamma}(N) = h^w_{\rho_N}(N) \quad \text{where} \quad \rho^*_N(v) = \int \rho(x, v) f_Z(x) dx.$$

In addition we get that the WME $i^w_\rho(Z : Z\sqrt{\gamma} + N)$ is represented as the difference

$$h^w_{\psi}(Z\sqrt{\gamma} + N) - h^w_{\psi^{(2)}_\gamma}(N) \text{ with } \psi(x) = \int \rho(x, v) f^{N_0}(v - \sqrt{\gamma}x) dv. \tag{3.11}$$
Note that $\psi_{N,\gamma}^{(2)} = \psi_{U,\gamma}$; see (3.7). Furthermore, owing to Lemma 3.1 (with $U = N$), we write:

$$h^w_N(Z) = \int_0^\infty \frac{d}{d\gamma} h^w_{\psi}(Z \sqrt{\gamma} + N) d\gamma + h^w_{\psi}(N),$$

still with $\varphi(x) = \lim_{y \to \infty} \rho(x, y)$. \hspace{1cm} (3.12)

We are now going to analyze the derivative $\frac{d}{d\gamma} h^w_{\psi}(Z \sqrt{\gamma} + N)$ representing it as

$$-\frac{d}{d\gamma} \int \int \rho(x, y) f^{No}(y - x \sqrt{\gamma}) f_Z(x) \log \left[ \int f^{No}(y - t \sqrt{\gamma}) f_Z(t) dt \right] dxdy$$

$$= -\frac{1}{2\sqrt{\gamma}} \int \int \rho(x, y) f_Z(x) f^{No}(y - x \sqrt{\gamma}) (y - x \sqrt{\gamma}) x$$

$$\times \log \left[ \int f^{No}(y - t \sqrt{\gamma}) f_Z(t) dt \right] dxdy$$

$$-\frac{1}{2\sqrt{\gamma}} \int \int \rho(x, y) f_Z(x) f^{No}(y - x \sqrt{\gamma})$$

$$\times \left[ \frac{\int \int w(y - w \sqrt{\gamma}) f^{No}(y - w \sqrt{\gamma}) f_Z(w) dw}{\int f^{No}(y - z \sqrt{\gamma}) f_Z(z) dz} \right] dxdy.$$ \hspace{1cm} (3.13)

The first integral in the RHS of (3.13) is done by parts. This leads to the following expression:

$$\frac{1}{2\sqrt{\gamma}} \int \int \rho(x, v) f_Z(x) f^{No}(v - x \sqrt{\gamma})(v - x \sqrt{\gamma}) x$$

$$\times \left[ \frac{\int \int (y - t \sqrt{\gamma}) y f^{No}(y - t \sqrt{\gamma}) f_Z(t) dt}{\int f^{No}(y - z \sqrt{\gamma}) f_Z(z) dz} \right] dy$$

$$-\frac{1}{2\sqrt{\gamma}} \int \int \rho(x, y) f_Z(x) f^{No}(y - x \sqrt{\gamma})$$

$$\times \left[ \frac{\int \int (y - w \sqrt{\gamma}) w f^{No}(y - w \sqrt{\gamma}) f_Z(w) dw}{\int f^{No}(y - z \sqrt{\gamma}) f_Z(z) dz} \right] dxdy.$$

Then, taking into account Eqn (3.1),

$$\frac{d}{d\gamma} h^w_{\psi}(Z \sqrt{\gamma} + N) = \frac{1}{2\sqrt{\gamma}} \int \int \zeta(x, y, \gamma) \xi(y, \gamma) f_{Z, Z, \sqrt{\gamma} + N} f_Z(x, y) dxdy$$

$$-\frac{1}{2\sqrt{\gamma}} \int \int \rho(x, y) \eta(y, \gamma) f_{Z, Z, \sqrt{\gamma} + N} f_Z(x, y) dxdy$$

$$= \frac{1}{2\sqrt{\gamma}} \mathbb{E} \left\{ \zeta_{\rho}(Z, Z, \sqrt{\gamma} + N) \xi(Z \sqrt{\gamma} + N, \gamma) - \rho(Z, Z, \sqrt{\gamma} + N) \eta(Z \sqrt{\gamma} + N, \gamma) \right\}. \hspace{1cm} (3.14)$$

3.2. WLSI for a WF close to a constant. Concluding this section, we analyze the WLSI \(\Psi\) when the WF $\phi$ lies in vicinity of a constant $\bar{\phi}$ (and hence, is bounded). Given independent RVs $X_1$, $X_2$ with PDFs $f_{X_1}$ and $f_{X_2}$, we refer to $Y_1$ and $Y_2$ as $Y_1 = X_1 / \cos \alpha$, $Y_2 = X_2 / \sin \alpha$ where $\alpha \in [-\pi, \pi]$ is as in Eqn (3.3). For $Z = Y_1, Y_2$ or $X_1 + X_2 = Y_1 \cos \alpha + Y_2 \sin \alpha$, set:

$$M(Z; \gamma) = \mathbb{E} \left[ |Z - \mathbb{E}(Z|Z \sqrt{\gamma} + N)|^2 \right] \hspace{1cm} (3.15)$$

and suppose that for the above choices of RV $Z$:

$$\mathbb{E}[\ln f_Z(Z)] < \infty. \hspace{1cm} (3.16)$$
We also need to assume a uniform integrability condition (3.2): for an independent triple \( Z, N, N' \) where \( N, N' \sim N(0, 1) \), there exists an integrable RV \( \chi(Z, N) \geq 0 \) such that

\[
\left| \log \mathbb{E} \left[ f_Z \left( Z + \frac{N - N'}{\sqrt{\gamma}} \right) | Z, N \right] \right| \leq \chi(Z, N). \quad (3.17)
\]

According to formula (4.5) in [4], for these choices of \( Z \), the standard SDE \( h(Z) \) is represented as

\[
h(Z) = h(N) + \frac{1}{2} \int \left[ M(Z, \gamma) - 1(\gamma > 1) \frac{1}{\gamma} \right] d\gamma. \quad (3.18)
\]

Furthermore, as follows from the proof of Theorem 4.1 in [4] (see [4], Eqn (4.8)), for any \( \tilde{\alpha} \in [-\pi, \pi] \) (including \( \tilde{\alpha} = \alpha \), the value from (1.3)),

\[
M(Y_1 \cos \tilde{\alpha} + Y_2 \sin \tilde{\alpha}, \gamma) \geq M(Y_1, \gamma)(\cos \tilde{\alpha})^2 + M(Y_2, \gamma)(\sin \tilde{\alpha})^2.
\]

For \( \tilde{\alpha} = \alpha \), this becomes

\[
M(X_1 + X_2, \gamma) \geq M(Y_1, \gamma_0)(\cos \alpha)^2 + M(Y_2, \gamma_0)(\sin \alpha)^2. \quad (3.19)
\]

Now we are in position to establish Theorem 4 below. As before, we refer to \( Z = Y_1, Y_2 \) or \( X_1 + X_2 = Y_1 \cos \alpha + Y_2 \sin \alpha \).

**Theorem 4.** Let \( \gamma_0 > 0 \) be is a point of continuity of \( M(Z, \gamma) \), \( Z = Y_1, Y_2, X_1 + X_2 \). Suppose that there exists \( \delta > 0 \) such that

\[
M(X_1 + X_2, \gamma_0) \geq M(Y_1, \gamma_0)(\cos \alpha)^2 + M(Y_2, \gamma_0)(\sin \alpha)^2 + \delta. \quad (3.20)
\]

Also assume (3.16) and (3.17).

Then there exists \( \epsilon = \epsilon(\gamma_0, \delta, f_{X_1}, f_{X_2}) \) with the following property. Let function \( x \in \mathbb{R} \mapsto \phi(x) \geq 0 \) be such that \( |\phi(x) - \overline{\phi}| \leq \epsilon, \forall x \), for a constant \( \overline{\phi} > 0 \). Then the WLSI (1.6) with the WF \( \phi \) holds true.

**Proof of Theorem 4.** According to Theorem 1, to prove the WLSI (1.6), we only need to check that

\[
(cos \alpha)^2 h_{\phi C}^w(Y_1) + (sin \alpha)^2 h_{\phi S}^w(Y_2) \leq h_{\phi}^w(Y_1 \cos \alpha + Y_2 \sin \alpha).
\]

For a constant WF \( \overline{\phi} \), the following inequality is valid (see Ref [4], Lemma 4.2 or Ref [8], Eqns (9) and (10))

\[
(cos \alpha)^2 h_{\overline{\phi} C}^w(Y_1) + (sin \alpha)^2 h_{\overline{\phi} S}^w(Y_2) \leq h_{\overline{\phi}}^w(Y_1 \cos \alpha + Y_2 \sin \alpha).
\]
Next, Theorem 4.1 from [4] (applicable because of (3.16) and (3.17)) implies that under condition (3.20), for $\epsilon$ small enough

$$(\cos \alpha)^2 h_\phi^w(Y_1) + (\sin \alpha)^2 h_\phi^w(Y_2) + \epsilon \leq h_\phi^w(Y_1 \cos \alpha + Y_2 \sin \alpha).$$

Define $\varphi(x) = |\phi(x) - \overline{\phi}|$. It remains to check that

$$h_\varphi^w(Y_1) \leq \epsilon/3, h_\varphi^w(Y_2) \leq \epsilon/3, h_\varphi^w(Y_1 \cos \alpha + Y_2 \sin \alpha) \leq \epsilon/3.$$ 

But this inequality immediately follows, owing to (3.16). This completes the proof of Theorem 4. \qed

The statement of Theorem 4 can be made more efficient for given PDFs $f_{X_1}$ and $f_{X_1}$. As an example, consider the case where RVs $X_1, X_2$ are normal and WF $\phi \in C^2$.

**Lemma 3.3.** Let RVs $X_i \sim N(\mu_i, \sigma_i^2), i = 1, 2$ be independent, and $X = X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$. Suppose that WF $x \in \mathbb{R} \mapsto \phi(x) \geq 0$ is twice continuously differentiable and slowly varying in the sense that $\forall x, |\phi''(x)| \leq \epsilon \phi(x), |\phi(x) - \overline{\phi}| < \epsilon$, (3.21)

where $\epsilon > 0$ and $\overline{\phi} > 0$ are constants. Then there exists $\epsilon_0 = \epsilon_0(\mu_0, \mu_1, \sigma_0^2, \sigma_1^2) > 0$ such that for any $0 < \epsilon \leq \epsilon_0$, the WLSI (2.3) with the WF $\phi$ holds true.

**Proof of Lemma 3.3.** Let $\alpha$ be as in Eqn (1.3); to check (2.3), we use Stein's formula: for $Z \sim N(\mu, \sigma^2)$,

$$E\left[Z^2\phi(Z)\right] = \sigma^2 E\left[\phi(Z)\right] + \sigma^4 E\left[\phi''(Z)\right].$$ (3.22)

Owing to the inequality $|\phi(x) - \overline{\phi}| < \epsilon$,

$$\alpha < \alpha_0 = \tan^{-1}\left(\exp\left\{\overline{\phi} + \epsilon\right\}^{2[h_+(X_2) - h_-(X_1)]} - (\overline{\phi} - \epsilon)^2[h_+(X_1) - h_-(X_2)]\right)\right).$$

Here

$$h_\pm(X_i) = -E\left[1(X_i \in A^i_\pm) \log f_{X_i}^{No}(X_i)\right], \ i = 1, 2,$$

and

$$A^i_+ = \left\{x \in \mathbb{R} : f_{X_i}^{No}(x) < 1\right\}, \ A^i_- = \left\{x \in \mathbb{R} : f_{X_i}^{No}(x) > 1\right\}.$$

Evidently, under conditions $|\phi'(x)|, |\phi''(x)| \leq \epsilon \phi(x)$ we have that $\alpha_0 < \frac{\pi}{2} - \epsilon$ and $0 < \epsilon < (\sin \alpha)^2, (\cos \alpha)^2 < 1 - \epsilon < 1$. We claim that inequality (2.3) is satisfied with $\phi$ replaced by $\overline{\phi}$.
and added $\delta > 0$:

$$
\left[ \log \left( 2\pi (\sigma_1^2 + \sigma_2^2) \right) \right] \overline{\phi} + \frac{\log e}{\sigma_1^2 + \sigma_2^2} \overline{\phi E}[X^2]
\geq (\cos \alpha)^2 \left[ \log \frac{2\pi \sigma_1^2}{(\cos \alpha)^2} \right] \overline{\phi} + \frac{(\cos \alpha)^2 \log e}{\sigma_1^2} \overline{\phi E}[X_1^2]
+(\sin \alpha)^2 \left[ \log \frac{2\pi \sigma_2^2}{(\sin \alpha)^2} \right] \overline{\phi} + \frac{(\sin \alpha)^2 \log e}{\sigma_2^2} \overline{\phi E}[X_2^2] + \delta.
$$

Here $\delta > 0$ is calculated through $\epsilon$ and increases to a limit $\delta_0 > 0$ as $\epsilon \to 0$.

Indeed, strict concavity of $\log y$ for $y > 0$ implies that

$$
\left[ \log \left( 2\pi (\sigma_1^2 + \sigma_2^2) \right) \right] \overline{\phi} + (\cos \alpha)^2 \left[ \log \frac{2\pi \sigma_1^2}{(\cos \alpha)^2} \right] \overline{\phi} + (\sin \alpha)^2 \left[ \log \frac{2\pi \sigma_2^2}{(\sin \alpha)^2} \right] \overline{\phi}
\geq (\cos \alpha)^2 \left[ \log \frac{2\pi \sigma_1^2}{(\cos \alpha)^2} \right] \overline{\phi} + (\sin \alpha)^2 \left[ \log \frac{2\pi \sigma_2^2}{(\sin \alpha)^2} \right] \overline{\phi} + \delta.
$$

On the other hand,

$$
\frac{1}{\sigma_1^2 + \sigma_2^2} \overline{\phi E}[X^2] = \frac{(\cos \alpha)^2}{\sigma_1^2} \overline{\phi E}[X_1^2] + \frac{(\sin \alpha)^2}{\sigma_2^2} \overline{\phi E}[X_2^2].
$$

Now, if we look at Eqn (2.3) with WF $\phi$ then, owing to Eqn (3.23) it suffices to verify that

$$
\left[ \log \left( 2\pi (\sigma_1^2 + \sigma_2^2) \right) \right] \overline{\phi E}\left(\phi(X) - \overline{\phi}\right) + \frac{\log e}{\sigma_1^2 + \sigma_2^2} \overline{\phi E}[X^2(\phi(X) - \overline{\phi})]
-(\cos \alpha)^2 \left[ \log \frac{2\pi \sigma_1^2}{(\cos \alpha)^2} \right] \overline{\phi E}\left(\phi(X_1) - \overline{\phi}\right) + \frac{(\cos \alpha)^2 \log e}{\sigma_1^2} \overline{\phi E}[X_1^2(\phi(X_1) - \overline{\phi})]
-(\sin \alpha)^2 \left[ \log \frac{2\pi \sigma_2^2}{(\sin \alpha)^2} \right] \overline{\phi E}\left(\phi(X_2) - \overline{\phi}\right) + \frac{(\sin \alpha)^2 \log e}{\sigma_2^2} \overline{\phi E}[X_2^2(\phi(X_2) - \overline{\phi})] < \delta.
$$

We check this by a brute force, claiming that each term in (3.24) has the absolute value $< \delta/6$ when $\epsilon$ is small enough. For the terms containing $\overline{\phi E}(\phi(Z) - \overline{\phi})$, $Z = X, X_1, X_2$, this follows since $|\phi(x) - \overline{\phi}| < \epsilon$.

The terms containing factor $\overline{\phi E}[Z^2(\phi(Z) - \overline{\phi})]$ we use Stein’s formula (3.22) and the condition that $|\phi''(x)| \leq \epsilon \phi(x)$.

Similar assertions can be established for other examples of PDFs $f_{X_1}$ and $f_{X_2}$.

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