The Costabel-Stephan system of Boundary Integral Equations in the Time Domain

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Abstract

In this paper we formulate a transmission problem for the transient acoustic wave equation as a system of retarded boundary integral equations. We then analyse a fully discrete method using a general Galerkin semidiscretization-in-space and Convolution Quadrature in time. All proofs are developed using recent techniques based on the theory of evolution equations. Some numerical experiments are provided.

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1 Introduction

In this paper we study a boundary integral formulation for the problem of the scattering of a transient acoustic wave by a non-smooth obstacle in two or three space dimensions. The boundary integral formulation is the transient equivalent to Costabel and Stephan’s [13] system of integral equations for Helmholtz Transmission Problems. It is a direct formulation (with physical fields as unknowns) involving the four operators of the associated Calderón calculus (the entries of the Calderón projector) at two different wave speeds. In addition to studying the well-posedness of the problem, we give estimates for Galerkin semidiscretization-in-space, for any choice of the discrete spaces, and analyze the fully discrete method obtained by applying second order Convolution Quadrature to the associated system of semidiscrete delayed integral equations.

The paper will be written for a single obstacle. The case of multiple separate scatterers (with no common boundaries), each of them possibly with different material properties, is a straightforward extension of this work. The case of layered obstacles might raise new issues from the point of view of analysis, and is the object of future research. The more involved case of obstacles with piecewise constant material properties not organized into layers is due to be considerably more complicated, requiring the use of multitrace spaces following [19, 12]. Finally, the choice of modeling the interior of the scatterer using a variational formulation leads to BEM-FEM coupling strategies: [1, 16, 6].

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The analysis of the method follows recent techniques [26, 15, 5] entirely developed in
the time domain, instead of the Laplace domain analysis that stems from the original work
of Bamberger and Ha-Duong [2, 3]. To the best of our knowledge, it is the first time that
these techniques are used on a setting that does not involve a purely exterior boundary
value problem. We note that part of the Laplace domain analysis for transmission problems
using the Costabel-Stephan formulation was included in [22], and was extended recently
to the case of electromagnetic waves in [10]. As shown in [5], direct time domain analysis
provides bounds (for the operators and their discretization) that are sharper than the ones
obtained by Laplace domain techniques. The analysis in this paper (like the one in [26, 5])
is essentially independent of the spaces used for space discretization. Moreover, the results
for semidiscretization are also valid for full Galerkin discretization. For time-stepping we
have used a simple BDF2-based Convolution Quadrature method: the analysis for the
trapezoidal rule discretization follows very similar arguments and can be easily adapted
from what appears in Section 6 of this work and [5, Section 6]. Higher order much less
dispersive discretizations are easily attainable using RK-based CQ methods [4, 7, 8, 18].
In that case the analysis relies entirely on Laplace domain estimates and the behavior of
the bounds with respect to the time variable is still unclear.

A final point deserving clarification is the choice of the Costabel-Stephan format for
the formulation of the transmission problem as a system of integral equations. This is a
direct formulation using the interior Cauchy data as unknowns and leads to a system that
can be considered of equations of the first kind. If we revert to the realm of frequency
domain problems, there is a wealth of integral formulations with two goals in mind [9, 17,
20, 21, 27, 28, 29]: robustness with respect to frequency (so that there are no resonances
in the formulation) and good conditioning of the final system. Both goals can be achieved
for smooth scatterers using regularized combined integral formulations, but the analysis
of many of these methods is not applicable for non-smooth obstacles. In time domain
problems, and with general Lipschitz scatterers, there are additional issues. An important
one of them is the difficulty of showing that combined field formulations lead to well-posed
problems that stay well-posed after Galerkin discretization. An apparently lone step in
this direction appears in [11], by forcing a strong time-regularization in the combined field
that worsens the mapping properties of the operator in the time domain.

The paper is structured as follows. In Section 2 we present the transmission problem
and its associated boundary integral formulation. In Section 3 we introduce the general
Galerkin semidiscretization-in-space and state the two main theorems associated to this
part of the discretization: error for the semidiscretization-in-space and a stability theorem
that is needed for the study of time discretization. The proofs of these results appear in
Sections 4 and 5. Section 6 deals with full discretization of the problem and Section 7 has
some numerical experiments in two dimensions.

Background. Elementary results on Sobolev spaces (traces, weak normal derivatives,
etc) is assumed throughout. Some basic background results in operator-valued distributions
are used in the presentation of the equations (Section 2) and in the quite technical
reconciliation of the strong and weak solutions (Section 5). These aspects can be easily
learned from handbooks in advanced tools of applied analysis or, in a simplified version,
in [26]. All needed results from the theory of evolution equations are stated in clear terms
2 Prolegomena and definitions

Geometric setting and problem. This paper is concerned with a transmission problem for the wave equation in free space (in dimensions \(d = 2\) or \(d = 3\)), where a bounded obstacle with homogeneous isotropic material properties is surrounded by a medium with different homogeneous and isotropic properties. In order to set up the problem as soon as possible, let us describe the simplified model we will analyse. Let \(\Omega_- \subset \mathbb{R}^d\) be a bounded Lipschitz domain with boundary \(\Gamma\). The exterior domain will be denoted \(\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega}_-\) and \(\nu\) will be the unit normal vector field on \(\Gamma\), pointing from \(\Omega_-\) to \(\Omega_+\). Two positive parameters \(\kappa\) and \(c\) will be used to describe the material properties in \(\Omega_-\). For sufficiently smooth functions defined in \(\mathbb{R}^d \setminus \Gamma = \Omega_- \cup \Omega_+\), the restrictions to the boundary will be denoted \(\gamma^\pm u\). Similarly, \(\partial^n u\) will denote the normal derivatives from both sides of \(\Gamma\).

We will specify their weak definitions later on. We look for the solution to the following transient scattering problem:

\[
\begin{align*}
    c^{-2}\ddot{u}(t) &= \kappa \Delta u(t) \quad \text{in } \Omega_-, \quad \forall t \geq 0, \\
    \dot{v}(t) &= \Delta v(t) \quad \text{in } \Omega_+, \quad \forall t \geq 0, \\
    \gamma^- u(t) &= \gamma^+ v(t) + \beta_0(t) \quad \text{on } \Gamma, \quad \forall t \geq 0, \\
    \kappa \partial^n_- u(t) &= \partial^n_+ v(t) + \beta_1(t) \quad \text{on } \Gamma, \quad \forall t \geq 0, \\
    u(0) &= \dot{u}(0) = 0 \quad \text{in } \Omega_-, \\
    v(0) &= \dot{v}(0) = 0 \quad \text{in } \Omega_+.
\end{align*}
\]

Let us first clarify some notational aspects of (2.1). We need two Sobolev spaces

\[
H^1_\Delta(\Omega_-) := \{w \in H^1(\Omega_-) : \Delta w \in L^2(\Omega_-)\},
\]

endowed with their natural norms. We can then define the interior and exterior traces \(\gamma^\pm : H^1(\Omega_\pm) \to H^{1/2}(\Gamma)\) and normal derivatives \(\partial^n_\pm : H^1_\Delta(\Omega_\pm) \to H^{-1/2}(\Gamma)\). We look for sufficiently smooth functions \(u : [0, \infty) \to H^1_\Delta(\Omega_+)\) and \(v : [0, \infty) \to H^1_\Delta(\Omega_-)\) satisfying: the differential equations (2.1a)-(2.1b) in a classical sense in the time variable and with values in \(L^2(\Omega_-)\) in the space variables; the transmission condition (2.1c) in \(H^{1/2}(\Gamma)\); the condition (2.1d) in \(H^{-1/2}(\Gamma)\); and finally the initial conditions (2.1e)-(2.1f). The discontinuities across \(\Gamma\) are given by functions \(\beta_0 : [0, \infty) \to H^{1/2}(\Gamma)\) and \(\beta_1 : [0, \infty) \to H^{-1/2}(\Gamma)\). In principle, these functions are not related to each other, although in a physical setting there is an incident wave \(u_{\text{inc}}\) satisfying \(\ddot{u}_{\text{inc}}(t) = \Delta u_{\text{inc}}(t)\) in free space for all \(t\), and we take \(\beta_0(t) := \gamma u_{\text{inc}}(t)\) and \(\beta_1(t) := \partial^n u_{\text{inc}}(t)\).

Strong and weak causality. Given a function \(f : \mathbb{R} \to X\), where \(X\) is any function or operator space, we will say that \(f\) is causal when \(f \equiv 0\) in \((-\infty, 0]\). Given a distribution \(f\) in \(\mathbb{R}\) with values in a Banach space \(X\), we will say that \(f\) is causal when its support is contained in \([0, \infty)\). Note that if \(X\) and \(Y\) are two Banach spaces, \(A : X \to Y\) is linear and bounded, and \(f\) is a causal \(X\)-valued distribution, then \(Af\) defines a causal
Y-valued distribution. In particular if \( X \subset Y \) with bounded embedding, every \( X \)-valued distribution can be read as a \( Y \)-valued distribution. For equality of two distributions \( f \) and \( g \) with values in a space \( X \) we will use the notation \( f = g \) (in \( X \)), as a reminder of where the equality takes place. Instead of thinking of the system \((2.1)\) with strong differentiation and strongly imposed initial conditions, we can write it in the sense of vector-valued distributions: we then look for a causal \( H^1_\Delta(\Omega_\pm) \)-valued distribution \( u \) and a causal \( H^1_\Delta(\Omega_\pm) \)-valued distribution \( v \) satisfying

\[
\begin{align*}
\mathcal{D} \mathcal{B} u &= \kappa \Delta u & (\text{in } L^2(\Omega_-)), \\
\mathcal{D} \mathcal{B} v &= \Delta v & (\text{in } L^2(\Omega_+)), \\
\mathcal{D} \mathcal{B} \gamma^- u &= \gamma^+ v + \beta_0 & (\text{in } H^{1/2}(\Gamma)), \\
\kappa \mathcal{D} \mathcal{B} \gamma^- u &= \mathcal{D} \mathcal{B} \gamma^+ v + \beta_1 & (\text{in } H^{-1/2}(\Gamma)).
\end{align*}
\]

Now \( \beta_0 \) and \( \beta_1 \) are causal \( H^{1/2}(\Gamma) \)- and \( H^{-1/2}(\Gamma) \)-valued distributions respectively, and differentiation is understood in the sense of distributions in \( \mathbb{R} \). The initial conditions are now implicit in the fact that we assume the distributions to be causal and in the fact that differentiation is taken over the entire real line, and not only on \((0, \infty)\).

### Weak form of the Huygens’ Potentials.

Given a causal \( H^{1/2}(\Gamma) \)-valued distribution \( \varphi \) and a causal \( H^{-1/2}(\Gamma) \)-valued distribution \( \eta \), and assuming that both of them are Laplace transformable, we consider the following transmission problem: find a causal \( H^1_\Delta(\Omega_\pm) \)-valued distribution \( w \) such that

\[
\begin{align*}
\mathcal{D} \mathcal{B} \mathcal{D} \mathcal{B} w &= \Delta w & (\text{in } L^2(\Omega_\pm)), \\
\mathcal{D} \mathcal{B} \gamma^- w - \gamma^+ w &= \varphi & (\text{in } H^{1/2}(\Gamma)), \\
\mathcal{D} \mathcal{B} \partial_\nu w - \partial_\nu^+ w &= \eta & (\text{in } H^{-1/2}(\Gamma)).
\end{align*}
\]

As shown in \([26]\) Chapters 2 and 3, problem \((2.3)\) admits a unique solution that can be written using two convolution operators

\[
w = \mathcal{S}_m * \eta - \mathcal{D}_m * \varphi. \tag{2.4}
\]

Here \( \mathcal{S}_m \) and \( \mathcal{D}_m \) are causal distributions with values in \( \mathcal{L}(H^{-1/2}(\Gamma), H^1_\Delta(\mathbb{R}^d \setminus \Gamma)) \) and \( \mathcal{L}(H^{1/2}(\Gamma), H^1_\Delta(\mathbb{R}^d \setminus \Gamma)) \) respectively, that can be described using their Laplace transforms. Neither the expressions of the Laplace transforms of \( \mathcal{S}_m \) and \( \mathcal{D}_m \) nor the expression of the convolutional formula \((2.4)\) for smooth enough (in time) input \( \varphi \) and \( \eta \) is relevant for what follows. The convolution operators in \((2.3)\) are called the single and double layer (retarded) potentials. The Laplace transforms will be used later on for the time discretization of the problem using Lubich’s Convolution Quadrature techniques. Associated to the layer potentials there are four boundary integral operators, formed by convolution with the following distributions

\[
\begin{align*}
\mathcal{V}_m := (\frac{1}{2} \gamma^- + \frac{1}{2} \gamma^+) \mathcal{S}_m &= \gamma^\pm \mathcal{S}_m, & \text{valued in } \mathcal{L}(H^{-1/2}(\Gamma), H^{1/2}(\Gamma)), \\
\mathcal{K}_m := (\frac{1}{2} \gamma^- + \frac{1}{2} \gamma^+) \mathcal{D}_m &= \gamma^\pm \mathcal{D}_m, & \text{valued in } \mathcal{L}(H^{1/2}(\Gamma), H^{1/2}(\Gamma)), \\
\mathcal{J}_m := (\frac{1}{2} \partial_\nu^- + \frac{1}{2} \partial_\nu^+) \mathcal{S}_m &= -\partial_\nu^\pm \mathcal{D}_m, & \text{valued in } \mathcal{L}(H^{-1/2}(\Gamma), H^{-1/2}(\Gamma)).
\end{align*}
\]

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Note that $[\gamma \cdot ] S_m = 0$ and $[\partial_\nu \cdot ] D_m = 0$, which justifies the three possible ways of defining $V_m$ and $W_m$. When $m = 1$ in (2.3) we will omit it in the potential expressions (2.4) and in the operators (2.5).

**The time domain Costabel-Stephan system.** We return to the weak transmission problem (2.2), and choose two unknowns on the boundary

$$\phi := \gamma - v, \quad \lambda := \partial_\nu v$$

(2.6)
to represent the solution of that problem. Using uniqueness arguments tied to the definition of the layer potentials, and thinking of $u$ as extended by zero to $\Omega_+$ and $v$ to be extended by zero to $\Omega_-$, we represent

$$u = S_m * \lambda - D_m * \phi, \quad v = -S * (\kappa \lambda - \beta_1) + D * (\phi - \beta_0), \quad \text{with } m := c\sqrt{\kappa}.$$  

(2.7)

Then, the quantities $\lambda$ and $\phi$ satisfy the system of convolutional boundary integral equations

$$\begin{bmatrix} V_m + \kappa V & -K_m - \kappa J_m + \frac{1}{\kappa} W_m + \frac{1}{\kappa} W \end{bmatrix} \ast \begin{bmatrix} \lambda \\ \phi \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{\kappa} \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \frac{1}{\kappa} V \\ \frac{1}{\kappa} J \\ \frac{1}{\kappa} W \end{bmatrix} \ast \begin{bmatrix} \beta_1 \\ \beta_0 \end{bmatrix}. $$

(2.8)

We make these arguments more precise in the next result. Its proof follows from [22, Section .].

**Proposition 2.1.** Let $(u, v)$ be a causal solution of (2.2). Then the distributions (2.6) are a causal solution of the system of convolution equations (2.8) and $(u, v)$ can be represented using the potential expressions (2.7). Reciprocally, if $(\lambda, \phi)$ is a causal solution of the system (2.8) and we define the pair $(u, v)$ using (2.7), then $(u, v)$ is a causal solution of (2.2) and (2.6) holds.

### 3 Semidiscretization in space

**Galerkin semidiscretization in space.** We next address the semidiscretization in space of (2.8) using a Galerkin scheme. Let then $X_h \subset H^{-1/2}(\Gamma)$ and $Y_h \subset H^{1/2}(\Gamma)$ be finite dimensional spaces. We will tag the spaces in the parameter $h$, with no geometric meaning. In order to say that a constant is independent of the choice of the finite dimensional spaces $X_h$ and $Y_h$ we will just say that it is independent of $h$. We will also assume that $\mathbb{P}_0(\Gamma) \subset X_h$, i.e., the space of constant functions is a subspace of $X_h$. The polar sets of $X_h$ and $Y_h$ are

$$X_h^\circ := \{ \varphi \in H^{1/2}(\Gamma) : \langle \mu^h, \varphi \rangle_\Gamma = 0 \; \forall \mu^h \in X_h \},$$

$$Y_h^\circ := \{ \eta \in H^{-1/2}(\Gamma) : \langle \eta, \varphi^h \rangle_\Gamma = 0 \; \forall \varphi^h \in Y_h \},$$

where the angled bracket will be used for the $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ duality product. Following [22], we will use polar sets to give shorthand forms of Galerkin-type identities. We will thus write

$$f - g \in X_h^\circ \quad \text{to mean} \quad \langle \mu^h, f \rangle_\Gamma = \langle \mu^h, g \rangle_\Gamma \quad \forall \mu^h \in X_h.$$
The Galerkin semidiscrete version of (2.8), associated to the spaces \( X_h \) and \( Y_h \) looks for a causal distribution \( \lambda^h \) with values in \( X_h \) and a causal distribution \( \phi^h \) with values in \( Y_h \) such that

\[
\begin{bmatrix}
\mathcal{V}_m + \kappa \mathcal{V} \\
\mathcal{J}_m + \mathcal{J} \\
\mathcal{W}_m + \frac{\kappa}{2} \mathcal{W}
\end{bmatrix} \ast \begin{bmatrix}
\lambda^h \\
\phi^h \\
\frac{1}{\kappa} \beta_0
\end{bmatrix} - \frac{1}{2} \begin{bmatrix}
\mathcal{V} \\
\mathcal{J} \\
\frac{\kappa}{2} \mathcal{W}
\end{bmatrix} \ast \begin{bmatrix}
\beta_1 \\
\beta_0
\end{bmatrix} \in X_h^* \times Y_h^*. \tag{3.1}
\]

A solution of (3.1) is then used to build approximations of \((u, v)\) using the same potential expressions as in (2.7):

\[
u^h := S_m * \lambda^h - D_m * \phi^h, \quad v^h := -S * (\kappa \lambda^h - \beta_1) + D * (\phi^h - \beta_0). \tag{3.2}
\]

**An exotic transmission problem.** There is a form, based on [22, 25, 15], of writing the Galerkin semidiscretization (3.1)-(3.2) by focusing directly on \((u^h, v^h)\). Note that the potentials are defined on both sides of the interface \(\Gamma\). They also satisfy the following double transmission problem (with two functions on each side of the boundary and six transmission conditions): \((u^h, v^h)\) are causal \(H^1_* (\mathbb{R}^d \setminus \Gamma)\)-valued distributions such that

\[
\begin{align*}
(-\kappa^2)^{-1} u^h &= \kappa \Delta u^h \quad \text{(in } L^2(\mathbb{R}^d \setminus \Gamma)), \tag{3.3a} \\
v^h &= \Delta v^h \quad \text{(in } L^2(\mathbb{R}^d \setminus \Gamma)), \tag{3.3b} \\
[\gamma u^h] + [\gamma v^h] &= \beta_0 \quad \text{(in } H^{1/2}(\Gamma)), \tag{3.3c} \\
\kappa [\partial_\nu u^h] + [\partial_\nu v^h] &= \beta_1 \quad \text{(in } H^{-1/2}(\Gamma)), \tag{3.3d} \\
[\gamma u^h] &\in Y_h \quad \text{(in } H^{1/2}(\Gamma)), \tag{3.3e} \\
[\partial_\nu u^h] &\in X_h \quad \text{(in } H^{-1/2}(\Gamma)), \tag{3.3f} \\
\gamma^+ u^h - \gamma^- v^h &\in X_h^* \quad \text{(in } H^{1/2}(\Gamma)), \tag{3.3g} \\
\kappa [\partial_\nu^+ u^h] - [\partial_\nu^- v^h] &\in Y_h^* \quad \text{(in } H^{-1/2}(\Gamma)). \tag{3.3h}
\end{align*}
\]

The fact that there are six transmission conditions (instead of only four) is related to the fact that (3.3g) and (3.3h) can be understood as incomplete transmission conditions (only some moments of function are imposed to vanish), and they have to be complemented with dual conditions (3.3e)-(3.3f) to make the problem well-posed. As emphasized in [22], there is no need for \(X_h\) and \(Y_h\) to be finite dimensional. Any pair of closed spaces would do. In particular, when we choose \(X_h = H^{-1/2}(\Gamma)\) and \(Y_h = H^{1/2}(\Gamma)\), the conditions (3.3e)-(3.3f) are void (the jumps are in those spaces by definition) and the conditions (3.3g)-(3.3h) can be read as \(\gamma^+ u^h - \gamma^- v^h = 0\) and \(\kappa [\partial_\nu^+ u^h] - [\partial_\nu^- v^h] = 0\) respectively. The following result gives a precise relationship between (3.3) and (3.1) and states unique solvability of both. The proof of the result follows from [22, Section 8].

**Proposition 3.1.** Let \((u^h, v^h)\) be a causal solution of (3.3), and let

\[
(\lambda^h, \phi^h) := ([\partial_\nu u^h], [\gamma u^h]). \tag{3.4}
\]

Then \((\lambda^h, \phi^h)\) is a causal \((X_h \times Y_h)\)-valued distribution solving (3.1). Reciprocally, if \((\lambda^h, \phi^h)\) is a causal \((X_h \times Y_h)\)-valued distribution satisfying (3.3) and \((u^h, v^h)\) are defined using (3.2), then \((u^h, v^h)\) are causal \(H^1(\mathbb{R}^d \setminus \Gamma)\)-valued distributions satisfying (3.3). Finally, problem (3.3) admits a unique causal Laplace transformable solution.
The Galerkin solver and the Galerkin error operator. The Galerkin solver is the linear operator that for given $\beta_0, \beta_1$, solves the problem (3.1)-(3.2) or equivalently solves (3.3)-(3.4). The Galerkin error operator is the operator that given a causal distribution $(\lambda, \phi)$ with values in $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$, looks for an $X_h \times Y_h$-valued causal distribution $(\lambda^h, \phi^h)$ satisfying

\[
\begin{bmatrix}
V_m + \kappa V \\
J_m + J
\end{bmatrix}
- \begin{bmatrix}
-K_m - K \\
W_m + \frac{1}{\kappa} W
\end{bmatrix}
* \begin{bmatrix}
\lambda^h - \lambda \\
\phi^h - \phi
\end{bmatrix}
\in X_h^o \times Y_h^o,
\]

and then outputs

\[
(\varepsilon^h, \varepsilon^h) := (\lambda^h - \lambda, \phi^h - \phi),
\]

as well as the potentials

\[
e^u_h := S_m * \varepsilon^h - D_m * \varepsilon^h, \quad e^v_h := -\kappa S * \varepsilon^h + D * \varepsilon^h.
\]

The potentials $e^u_h$ and $e^v_h$ are causal solutions of the following problem:

\[
e^{-2} e^v_h = \kappa \Delta e^v_h \quad \text{(in $L^2(\mathbb{R}^d \setminus \Gamma)$)},
\]

\[
e^v_h = \Delta e^v_h \quad \text{(in $L^2(\mathbb{R}^d \setminus \Gamma)$)},
\]

\[
[\gamma e^h_u] + [\gamma e^h_v] = 0 \quad \text{(in $H^{1/2}(\Gamma)$)},
\]

\[
[\gamma e^h_u] + [\gamma e^h_v] = 0 \quad \text{(in $H^{1/2}(\Gamma)$)},
\]

\[
[\gamma e^h_u] + \phi \in Y_h \quad \text{(in $H^{1/2}(\Gamma)$)},
\]

\[
[\gamma e^h_u] + \lambda \in X_h \quad \text{(in $H^{1/2}(\Gamma)$)},
\]

\[
\gamma^+ e^h_u - \gamma^- e^h_v \in X^o_h \quad \text{(in $H^{1/2}(\Gamma)$)},
\]

\[
\kappa \partial^+ e^u_h - \partial^- e^v_h \in Y^o_h \quad \text{(in $H^{1/2}(\Gamma)$)}.
\]

(The problem is very similar to (3.3). The only modification is which of the transmission conditions are non-homogeneous.) Using Laplace domain techniques as in [26, Chapter 5], the Galerkin solver and the Galerkin error operator can be described as convolution operators. We next state the two main theorems of this part of the paper. Their proofs will be the goal of the next two sections. Some notation is first needed. For a given Hilbert space $X$, we consider the spaces

\[
C^k_+(X) := \{ f \in C^k(\mathbb{R}; X) : \text{supp } f \subset [0, \infty) \},
\]

\[
W^k_+(X) := \{ f \in C^{k-1}(\mathbb{R}; X) : \text{supp } f \subset [0, \infty), f^{(k)} \in L^1_{\text{loc}}(\mathbb{R}; X) \}.
\]

We also consider the cumulative seminorms

\[
H_k(f, t; X) := \sum_{j=0}^k \int_0^t \| f(\tau) \|_X d\tau
\]

and the antidifferentiation operator

\[
(\partial^{-1} f)(t) := \int_0^t f(\tau) d\tau.
\]
Theorem 3.2. Let \( \beta_0 \in \mathcal{W}_0^2(H^{1/2}(\Gamma)) \) and \( \beta_1 \in \mathcal{W}_+^1(H^{-1/2}(\Gamma)) \), and let \((u^h, v^h, \lambda^h, \phi^h)\) be the solution of (3.1)–(3.2). Then
\[
u^h, v^h \in C_+^1(L^2(\mathbb{R}^d)) \cap C_+^0(H^1(\mathbb{R}^d \setminus \Gamma)), \quad \phi^h \in C_+^0(H^{1/2}(\Gamma))
\]
and
\[
\|u^h(t)\|_{1,\mathbb{R}^d \setminus \Gamma} + \|v^h(t)\|_{1,\mathbb{R}^d \setminus \Gamma} + \|\phi^h(t)\|_{1/2,\Gamma}
\leq C\left(H_3(\partial^{-1}\beta_0, t; H^{1/2}(\Gamma)) + H_2(\partial^{-1}\beta_1, t; H^{-1/2}(\Gamma))\right).
\]
If \( \beta_0 \in \mathcal{W}_+^2(H^{1/2}(\Gamma)) \) and \( \beta_1 \in \mathcal{W}_+^2(H^{-1/2}(\Gamma)) \), then \( \lambda^h \in C_+^0(H^{1/2}(\Gamma)) \) and
\[
\|\lambda^h(t)\|_{-1/2,\Gamma} \leq C\left(H_3(\beta_0, t; H^{1/2}(\Gamma)) + H_2(\beta_1, t; H^{-1/2}(\Gamma))\right).
\]

Theorem 3.3. Let \( \phi \in \mathcal{W}_0^2(H^{1/2}(\Gamma)) \) and \( \lambda \in \mathcal{W}_+^1(H^{-1/2}(\Gamma)) \), and let \((e_u^h, e_v^h, \varepsilon_\lambda^h, \varepsilon_\phi^h)\) be the solution of (3.5). Then
\[
e_u^h, e_v^h \in C_+^1(L^2(\mathbb{R}^d)) \cap C_+^0(H^1(\mathbb{R}^d \setminus \Gamma)), \quad \varepsilon_\lambda^h, \varepsilon_\phi^h \in C_+^0(H^{1/2}(\Gamma))
\]
and
\[
\|e_u^h(t)\|_{1,\mathbb{R}^d \setminus \Gamma} + \|e_v^h(t)\|_{1,\mathbb{R}^d \setminus \Gamma} + \|\varepsilon_\lambda^h(t)\|_{1/2,\Gamma}
\leq C\left(H_3(\partial^{-1}\phi, t; H^{1/2}(\Gamma)) + H_2(\partial^{-1}\lambda, t; H^{-1/2}(\Gamma))\right).
\]
If \( \phi \in \mathcal{W}_+^3(H^{1/2}(\Gamma)) \) and \( \lambda \in \mathcal{W}_+^2(H^{-1/2}(\Gamma)) \), then \( \varepsilon_\lambda^h \in C_+^0(H^{-1/2}(\Gamma)) \) and
\[
\|\varepsilon_\lambda^h(t)\|_{-1/2,\Gamma} \leq C\left(H_3(\phi, t; H^{1/2}(\Gamma)) + H_2(\lambda, t; H^{-1/2}(\Gamma))\right).
\]

Error estimate for Galerkin semidiscretization. Note that Theorem 3.3 is de facto an error estimate for Galerkin semidiscretization in space. The reason is simple: if \( \phi \) and \( \lambda \) take values in \( Y_h \) and \( X_h \), respectively, then all error quantities \((e_u^h, e_v^h, \varepsilon_\lambda^h, \varepsilon_\phi^h)\) vanish. Therefore, if \( \Pi_{Y_h} : H^{1/2}(\Gamma) \rightarrow Y_h \) and \( \Pi_{X_h} : H^{-1/2}(\Gamma) \rightarrow X_h \) are orthogonal projectors, we can place \( \phi - \Pi_{Y_h}\phi \) and \( \lambda - \Pi_{X_h}\lambda \) instead of \( \phi \) and \( \lambda \) in the right-hand-sides of the bounds in Theorem 3.3.

4 Short term analysis in cut-off domains

The cut-off problems. Let \( B_0 := B(0; R) = \{x \in \mathbb{R}^d : |x| < R\} \) be such that \( \overline{\Omega}_- \subset B_0 \). Let then \( B_T := B(0; R + \max\{1, m\}T) \).

We will now need two more trace operators
\[
\gamma_T : H^1(B_T \setminus \Gamma) \rightarrow H^{1/2}(\partial B_T), \quad \partial_T^\nu : H^1_\lambda(B_T \setminus \Gamma) \rightarrow H^{-1/2}(\partial B_T),
\]
and the space
\[
H^1_{\partial B_T}(B_T \setminus \Gamma) := \{w \in H^1(B_T \setminus \Gamma) : \gamma_Tw = 0\}.
\]
We next present cut-off strong versions of problems (3.3) and (3.6). To unify notation of both problems and to make comparisons with [22] simpler, the pairs \((u^h, v^h)\) in (3.3) and \((e_u^h, e_v^h)\) are renamed \((u, u^*)\). From this moment on starred variables will always correspond to the original exterior problem. The data for the cut-off Galerkin solver are smooth enough (this will be precised later on) functions \(\beta_0 : [0, \infty) \rightarrow H^{1/2}(\Gamma)\) and \(\beta_1 : [0, \infty) \rightarrow H^{-1/2}(\Gamma)\). We look for two functions \(u, u^* : [0, \infty) \rightarrow H^1_B(B_T \setminus \Gamma)\) satisfying, for all \(t \geq 0\),

\[
\begin{align*}
    c^{-2}\ddot{u}(t) &= \kappa \Delta u(t) \quad \text{in } B_T \setminus \Gamma, \\
    \gamma_T u(t) &= 0, \\
    \gamma^+ u(t) - \gamma^- u^*(t) &= X^o_h, \\
    \kappa \partial_\nu u(t) - \partial_\nu^- u^*(t) &= Y^o_h, \\
\end{align*}
\]

vanishing initial conditions

\[
\begin{align*}
    u(0) &= 0, \quad \dot{u}(0) = 0, \quad u^*(0) = 0, \quad \dot{u}^*(0) = 0, \\
\end{align*}
\]

and two more transmission conditions

\[
\begin{align*}
    [\gamma u](t) &= \beta_0(t), \\
    [\gamma u^*](t) &= \beta_1(t), \\
    \kappa [\partial_\nu u](t) + [\partial_\nu^- u^*](t) &= \beta_1(t), \\
    [\gamma u](t) &= 0, \\
    [\gamma u^*](t) &= 0, \\
    \kappa [\partial_\nu u](t) + [\partial_\nu^- u^*](t) &= 0, \\
    [\gamma u](t) + \phi(t) &= 0, \\
    [\gamma u^*](t) + \phi(t) &= 0, \\
\end{align*}
\]

The cut-off Galerkin error operator equations, corresponding to (3.6), substitute (4.2) by

\[
\begin{align*}
    [\gamma u](t) + [\gamma u^*](t) &= 0, \\
    \kappa [\partial_\nu u](t) + [\partial_\nu^- u^*](t) &= 0, \\
    [\gamma u](t) + \phi(t) &= 0, \\
    [\gamma u^*](t) + \phi(t) &= 0, \\
\end{align*}
\]

where \(\phi : [0, \infty) \rightarrow H^{1/2}(\Gamma)\) and \(\lambda : [0, \infty) \rightarrow H^{-1/2}(\Gamma)\).

The plan of this section and the next is very similar to what appears in [26, Chapters 7 & 8] and in [3]. We will just sketch the different results, pointing out novelties and new difficulties that did not arise in simpler situations. The goal is simple. We first study problems (4.1), completed with (4.2) or (4.3), and the quantities \([\gamma u]\) and \([\partial_\nu u]\) as \(t\) grows, showing clear dependence on the cut-off section \(T\) in all bounds. We next identify the solution of these two problems with the solution of (3.3) or (3.6) (and therefore of the integral system (3.1) or (3.6)) in the time interval \([0, T]\). Finally, we use this identification to provide bounds for the solution of the system (3.1) or (3.3) as a function of \(t\).

**Spaces, norms, and integration by parts formulas.** We need to introduce some spaces and operators in order to describe the solution of (4.1). Let

\[
H := L^2(B_T)^2
\]

be endowed with the inner product

\[
((u, u^*), (v, v^*))_H := c^{-2}(u, v)_{B_T} + (u^*, v^*)_{B_T}.
\]

Consider also its subspace

\[
V := \{(v, v^*) \in H^1_{\partial B_T}(B_T \setminus \Gamma)^2 : [\gamma v] = -[\gamma v^*] \in Y_h, \quad \gamma^+ v - \gamma^- v^* \in X^o_h\},
\]

\[
(4.4c)
\]
endowed with the inner product

\[ [(u, u^*), (v, v^*)] := \kappa(\nabla u, \nabla v)_{B_T \setminus \Gamma} + (\nabla u^*, \nabla v^*)_{B_T \setminus \Gamma}. \]  

(4.4d)

The proof that (4.4d) is an inner product in \( V \) is an easy exercise. Consider next the space

\[ D(A) := \left\{ (v, v^*) \in V : \Delta v, \Delta v^* \in L^2(B_T \setminus \Gamma), \right. \]

\[ \left[ \partial_v v \right] = -\kappa^{-1} [\partial_v v^*] \in X_h, \kappa \partial_v v^* - \partial_v v^* \in Y^0 \right\}, \]  

(4.4e)

and the operator \( A : D(A) \to H \) given by \( A(v, v^*) := (c^2 \kappa \Delta v, \Delta v^*). \)

**Proposition 4.1.** The following properties hold:

(a) The inclusion of \( V \) into \( H \) is dense and compact.

(b) There exists \( C_T \) independent of \( h \) such that

\[ \| (v, v^*) \|_H \leq C_T \| (v, v^*) \|_V \quad \forall (v, v^*) \in V. \]  

(4.5)

(c) The following Green identity holds:

\[ (A(u, u^*), (v, v^*))_H + [(u, u^*), (v, v^*)] = 0 \quad \forall (u, u^*) \in D(A), \quad (v, v^*) \in V. \]  

(4.6)

(d) The operator \( I - A : D(A) \to H \) is surjective.

**Proof.** The proof of (a) is straightforward. To verify (b), note that there exists \( C_T > 0 \) such that (4.5) holds for all \( (v, v^*) \) in the space

\[ \{ (v, v^*) \in H^1_{\partial B_T}(B_T \setminus \Gamma)^2 : [\gamma v] = -[\gamma v^*], \langle 1, \gamma^+ v - \gamma^- v^* \rangle = 0 \}. \]

This can be proved with a simple compactness argument. To prove (c) it is enough to note that some simple algebraic manipulations yield:

\[ \kappa \left( (\partial_v u, \gamma^- v) - (\partial_v^+ u, \gamma^+ v) \right) + (\partial_v^+ u^*, \gamma^- v^*) - (\partial_v^+ u^*, \gamma^+ v^*) \]

\[ = \kappa \langle [\partial_v u], \gamma^+ v - \gamma^- v^* \rangle + \langle \kappa \partial_v^+ u - \partial_v^+ u^*, [\gamma v] \rangle 
+ \langle \partial_v^- u^*, [\gamma v] + [\gamma v^*] \rangle + \langle \kappa [\partial_v u] + [\partial_v u^*], \gamma^+ v^* \rangle. \]  

(4.7)

Let us finally sketch the proof of (d). Given \( (f, f^*) \in H \), we solve the coercive variational problem

\[ (u, u^*) \in V, \]

\[ ((u, u^*), (v, v^*))_H + [(u, u^*), (v, v^*)] = ((f, f^*), (v, v^*))_H \quad \forall (v, v^*) \in V. \]  

(4.8a)

(4.8b)

Testing equations (4.8) with \( (v, v^*) \in D(B_T \setminus \Gamma)^2 \) (the space of smooth compactly supported functions), it follows that

\[ -c^2 \kappa \Delta u + u = f \quad \text{in } B_T \setminus \Gamma, \quad -\Delta u^* + u^* = f^* \quad \text{in } B_T \setminus \Gamma. \]  

(4.9)
Plugging (4.9) in (4.8), using the definition of weak normal derivatives and (4.7), we then show that
\[
\kappa[\partial_t u], \gamma^+ v - \gamma^- v^* + (\kappa \partial_t^+ u - \partial_t^- u^*, [\gamma v]) + (\kappa[\partial_t u] + [\partial_t u^*], \gamma^- v^*) = 0 \quad \forall (v, v^*) \in V.
\]
Finally, the map from \(H^{1/2}_\partial(B_T \setminus \Gamma)^2 \rightarrow H^{1/2}(\Gamma)^4\) given by
\[
(v, v^*) \mapsto \begin{bmatrix}
\gamma^+ v - \gamma^- v^* \\
[\gamma v] \\
[\gamma v^*]
\end{bmatrix} = \begin{bmatrix}
0 & 1 & -1 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & -1 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
\gamma^- v \\
\gamma^+ v \\
\gamma^+ v^* \\
\gamma^- v^*
\end{bmatrix}
\]
is surjective, and therefore that the map
\[
V \ni (v, v^*) \mapsto (\gamma^+ v - \gamma^- v^*, [\gamma v], \gamma^+ v^*) \in X^0_h \times Y_h \times H^{1/2}(\Gamma)
\]
is also surjective. This implies that the condition (4.10) yields what is necessary to ensure that \((u, u^*) \in D(A)\) and therefore \((u, u^*) - A(u, u^*) = (f, f^*)\) by (4.9).

**Proposition 4.2** (Second order differential equations). Let \(D(A) \subset V \subset H\) be Hilbert spaces and \(A : D(A) \rightarrow H\) be a bounded linear operator. Assume that (a)-(d) in Proposition 4.1 are satisfied. If \(f \in C([0, \infty); V)\), then the initial value problem
\[
\dot{u}(t) = Au(t) + f(t), \quad t \geq 0, \quad u(0) = \dot{u}(0) = 0.
\]
has a unique solution
\[
u \in C([0, \infty); D(A)) \cap C^1([0, \infty); V) \cap C^2([0, \infty); H).
\]
For all \(t \geq 0\), this solution satisfies
\[
C_T^{-1}\|u(t)\|_H \leq \|u(t)\|_V \leq \int_0^t \|f(\tau)\|_H d\tau,
\]
\[
\|\dot{u}(t)\|_V \leq \int_0^t \|f(\tau)\|_V d\tau,
\]
\[
\|\ddot{u}(t)\|_H \leq \int_0^t \|f(\tau)\|_H d\tau,
\]
\[
\|Au(t)\|_H \leq \int_0^t \|f(\tau)\|_V d\tau.
\]
If \(f \in C^1([0, \infty); H)\) and \(f(0) = 0\), then problem (4.11) has a unique solution in (4.12) satisfying, for all \(t\)
\[
\|\dot{u}(t)\|_V \leq \int_0^t \|\dot{f}(\tau)\|_H d\tau,
\]
\[
\|Au(t)\|_H \leq 2\int_0^t \|\dot{f}(\tau)\|_H d\tau,
\]
as well as (4.13a) and (4.13c).

**Proof.** The proofs for these results can be found in [26, Chapter 6] (see also [15, Appendix A] for an earlier version). Note that those results might be attainable with more abstract tools of functional calculus and semigroup theory and that no claim is made to their originality.
Proof. This follows from an easy (but careful) variational argument. Consider the bilinear spaces $H$ while in $H$

Additional notation. In the space $H^1(B_T \setminus \Gamma)$, we consider the usual Sobolev norm

while in $H^1(B_T \setminus \Gamma)$, we can consider the seminorm

which is equal to the $H(\text{div}, B_T \setminus \Gamma)$ norm of $\nabla u$. For a Hilbert space $X$, we will also consider the spaces

$$C^k_0([0,\infty);X) := \{ f \in C^k([0,\infty);X) : f^{(j)}(0) = 0, \quad j \leq k - 1 \}.$$
Proposition 4.4. Let $\beta_0 \in C^3_0([0, \infty); H^{1/2}(\Gamma))$ and $\beta_1 \in C^2_0([0, \infty); H^{-1/2}(\Gamma))$. Then the solution of (4.1) and (4.2) is in the space
\[
C^2([0, \infty); L^2(B_T)^2) \cap C^1([0, \infty); H^1(B_T \setminus \Gamma)^2) \cap C([0, \infty); H^1_\Delta(B_T \setminus \Gamma)^2)
\]
and satisfies for all $t \geq 0$
\[
|u(t)|_{\Delta, B_T \setminus \Gamma} + |u^*(t)|_{\Delta, B_T \setminus \Gamma} \leq C(H_3(\beta_0, t; H^{1/2}(\Gamma)) + H_2(\beta_1, t; H^{-1/2}(\Gamma))),
\]
\[
\|\dot{u}(t)\|_{1, B_T \setminus \Gamma} + \|\dot{u}^*(t)\|_{1, B_T \setminus \Gamma} \leq C(H_3(\beta_0, t; H^{1/2}(\Gamma)) + H_2(\beta_1, t; H^{-1/2}(\Gamma))).
\]
Proof. The key is the decomposition of the solution in the form
\[
\hat{u} = (u, u^*) = u_0 + u_1 + \tilde{w}_0 + \tilde{w}_1,
\]
where
\[
u_0 := L(\beta_0, 0, 0, 0) \in C^3_0([0, \infty); H), \quad \nu_1 := L(0, 0, \beta_1, 0) \in C^2_0([0, \infty); V)
\]
(see Proposition 4.3), and
\[
\tilde{w}_0 = Au_0 + f_0, \quad \tilde{w}_0(0) = \tilde{w}_0(0) = 0, \quad \tilde{f}_0 := u_0 - \nu_0 = L(\beta_0 - \beta_0, 0, 0, 0), \quad \tilde{f}_1 := \nu_1 - \nu_1 = L(0, 0, \beta_1 - \beta_1, 0).
\]
It is clear that $f_0 \in C_0^1([0, \infty); H)$ and $f_1 \in C([0, \infty); V)$. The solution of (4.16a) and (4.16b) fits as a particular instance of Proposition 4.2. The bounds follow from Proposition 4.3 and the inequalities (4.13) in Proposition 4.2 (Note that Proposition 4.3 is used to bound different norms of $f_0$, in terms of norms of $\beta_j$).

Proposition 4.5. Let $\phi \in C^3_0([0, \infty); H^{1/2}(\Gamma))$ and $\lambda \in C^2_0([0, \infty); H^{-1/2}(\Gamma))$. Then the solution of (4.1) and (4.3) is in the space
\[
C^2([0, \infty); L^2(B_T)^2) \cap C^1([0, \infty); H^1(B_T \setminus \Gamma)^2) \cap C([0, \infty); H^1_\Delta(B_T \setminus \Gamma)^2)
\]
and satisfies for all $t \geq 0$
\[
|u(t)|_{\Delta, B_T \setminus \Gamma} + |u^*(t)|_{\Delta, B_T \setminus \Gamma} \leq C(H_3(\phi, t; H^{1/2}(\Gamma)) + H_2(\lambda, t; H^{-1/2}(\Gamma))),
\]
\[
\|\dot{u}(t)\|_{1, B_T \setminus \Gamma} + \|\dot{u}^*(t)\|_{1, B_T \setminus \Gamma} \leq C(H_3(\phi, t; H^{1/2}(\Gamma)) + H_2(\lambda, t; H^{-1/2}(\Gamma))).
\]
Proof. The proof is very similar to that of Proposition 4.4 using the sum of the liftings $L(0, \phi, 0, 0)$ and $L(0, 0, \lambda, \lambda)$, plus the solution of two non-homogeneous differential equations of the second order associated to the operator $A$. Details are omitted.

5 Long term analysis in free space

Causality arguments. Assume that $f$ is a causal $L(X, Y)$-valued distribution whose Laplace transform satisfies
\[
||F(s)|| \leq e^{-\mu \text{Re}s} C_F(\text{Re}s)|s|^\mu \quad \forall s \in \mathbb{C}, \quad \text{Re}s > 0,
\]
where $\mu \in \mathbb{R}$ and $C_F : (0, \infty) \rightarrow (0, \infty)$ is a non-increasing function that can be bounded by a rational function close to zero. Then, if $g$ is an $X$-valued causal Laplace transformable distribution, $f \ast g$ is a $Y$-valued distribution whose support is contained in $[M, \infty)$. This is proved in [26, Proposition 3.6.1].
Extension operators. Consider a function $f : [0, \infty) \to X$ such that $\|f(t)\|_X$ is polynomially bounded in $t$. We can then define

$$(Ef)(t) := \begin{cases} f(t), & t \geq 0, \\ 0, & t < 0, \end{cases}$$

which can be considered as a causal $X$-valued distribution. If $u : [0, \infty) \to L^2(B_T)$, we define

$$u(t) := \begin{cases} u(t), & \text{in } B_T, \\ 0, & \text{in } \mathbb{R}^d \setminus B_T. \end{cases}$$

We consider the quantity

$$\delta := \frac{\text{dist}(\Gamma, \partial B_0)}{\max\{1, m\}},$$

which measures the waiting time for a wave starting at $\Gamma$ and with speed equal to the maximum of the speeds of waves in (4.1), to reach the boundary of the first cut-off ball $B_0$.

**Proposition 5.1.** Let

$$u \in C^2([0, \infty); L^2(B_T)) \cap C^1([0, \infty); H^1(B_T \setminus \Gamma)) \cap C([0, \infty); H^1_\Delta(B_T \setminus \Gamma))$$

be polynomially bounded in time and satisfy

$$\ddot{u}(t) = \Delta u(t) \quad t \geq 0,$$  

$$\gamma_T u(t) = 0 \quad t \geq 0,$$  

$$u(0) = \dot{u}(0) = 0.$$  

Then

$$u(t) = S * E[\partial_\nu u](t) - D * E[\gamma u](t) \quad \forall t \in [0, T + \delta]$$

and

$$u(t) \in C([0, T + \delta]; H^1_\Delta(\mathbb{R}^d \setminus \Gamma)).$$

The same result holds if waves propagate at speed $m$ instead of unit speed.

**Proof.** Let

$$w_p := (S * E[\partial_\nu u] - D * E[\gamma u])|_{B_T},$$  

$$w := Eu - w_p,$$  

$$\xi := \gamma_T w = -\gamma_T w_p,$$  

$$\mu := \partial_\nu^\tau w = E\partial_\nu^\tau u - \partial_\nu^\tau w_p,$$

where $\partial_\nu^\tau$ is used to denote the interior normal derivative on $\partial B_T$. If $S := \mathcal{L}\{S\}$ and $D := \mathcal{L}\{D\}$, then $\gamma_T S$ and $\gamma_T D$ can be shown to satisfy (5.1) with $M = T + \delta$. This can be done using a direct argument on the kernels of these operators, and proves (see above) that

$$\text{supp } \xi \subset [T + \delta, \infty).$$

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Similarly
\[ \text{supp } \partial_T^\nu w_\rho \subset [T + \delta, \infty). \] (5.6)

The distribution \( w \) can then be shown to be a causal solution of
\[
\begin{align*}
\ddot{w} = \Delta w & \quad \text{(in } L^2(B_T \setminus \Gamma)), \\
[\gamma w] &= 0 \quad \text{(in } H^{1/2}(\Gamma)), \\
[\partial_\nu w] &= 0 \quad \text{(in } H^{-1/2}(\Gamma)), \\
\gamma_T w &= \xi \quad \text{(in } H^1(\partial B_T)).
\end{align*}
\] (5.7a) (5.7b) (5.7c) (5.7d)

However, (5.7) can be considered as a uniquely solvable distributional evolution equation (this is done using Laplace transforms and elementary arguments) and therefore \( \xi \mapsto \partial_T^\nu w \) can be shown to be a causal convolution operator. This and (5.5) imply that \( \text{supp } \mu \subset [T + \delta, \infty) \), and then (5.6) implies that \( \text{supp } \partial_T^\nu u \subset [T + \delta, \infty) \). Therefore
\[ \partial_T^\nu u(t) = 0 \quad \forall t \in [0, T + \delta]. \] (5.8)

Consider now the spatial extension \( u(t) \). Since the trace and normal derivative vanish on both sides of \( \partial B_T \) up to \( t = T + \delta \), it follows that
\[ u(t) \in H^1_\Delta(\mathbb{R}^d \setminus \Gamma) \quad t \in [0, T + \delta], \]
and in that time interval, the Laplacian operators in \( \mathbb{R}^d \setminus \Gamma \) and \( \mathbb{R}^d \setminus (\Gamma \cup \partial B_T) \) applied to \( u(t) \) give the same result. In the entire time interval
\[ u \in C^2([0, \infty); L^2(\mathbb{R}^d)) \cap C^1([0, \infty); H^1(\mathbb{R}^d)) \cap C([0, \infty); H^1_\Delta(\mathbb{R}^d \setminus (\Gamma \cup \partial B_T))) \]
and
\[ \ddot{u}(t) = \Delta u(t), \quad u(0) = \dot{u}(0) = 0, \]
which makes \( Eu \) a causal solution of
\[ \frac{d^2}{dt^2}(Eu) = \Delta Eu \quad \text{(in } H^1(\mathbb{R}^d \setminus (\Gamma \cup \partial B_T))). \]

Using the distributional Kirchhoff formula, we can write
\[ Eu = S * E[\partial_\nu u] - D * E[\gamma u] - S_{\partial B_T} * E\partial_T^\nu u, \]
where \( S_{\partial B_T} \) is a single layer potential emanating from \( \partial B_T \). Finally, by (5.8), it follows that
\[ \text{supp } (Eu - S * E[\partial_\nu u] + D * E[\gamma u]) \subset [T + \delta, \infty), \]
which proves (5.3). \( \square \)
Recovery of densities. So far the solution of (4.1) and (4.2), or of (4.1) and (4.3) has been denoted without explicit reference to $T$. We now have to show that these solutions coincide on finite time intervals: they start being different once the outgoing wave hits the boundary of the cut-off domain. This will be the final step in reconciling the solutions of the cut-off problems with the distributional solutions of the problems in Section 3.

Proposition 5.2. Let $(u_T, u_T^\ast)$ be the solution of (4.1)-(4.2). Let then
\[ \psi(T) := [\gamma u_T](T), \quad \psi^\ast(T) := [\gamma u_T^\ast](T), \quad \mu(T) := [\partial_v u_T](T), \quad \mu^\ast(T) := [\partial_v u_T^\ast](T). \]
Then for all $t \in [0, T + \delta]$
\[ [\gamma u_T](t) = \psi(t), \quad [\gamma u_T^\ast](t) = \psi^\ast(t), \quad [\partial_v u_T](t) = \mu(t), \quad [\partial_v u_T^\ast](t) = \mu^\ast(t). \]
The same result holds for the solution of (4.1) and (4.3).

Proof. Given $(u_T, u_T^\ast)$ and $M > 0$, we consider the extensions
\[ u(t) := \begin{cases} u_T(t), & \text{in } B_T, \\ 0, & \text{in } B_{T+M} \setminus B_T \end{cases}, \quad u^\ast(t) := \begin{cases} u_T^\ast(t), & \text{in } B_T, \\ 0, & \text{in } B_{T+M} \setminus B_T. \end{cases} \]
With the argument given in the proof of Proposition 5.1 we can show that
\[ u, u^\ast : [0, T + \delta] \to H^1_\Delta(B_{T+M} \setminus \Gamma) \]
and then the pairs $(u_{T+M}, u_{T+M}^\ast)$ and $(u, u^\ast)$ solve equations (4.1)-(4.2) in the time interval $[0, T + \delta]$. A simple energy argument can then be shown to show that they are equal, which implies that
\[ u_{T+M}|_{B_T}(t) \equiv u_T(t), \quad u_{T+M}^\ast|_{B_T}(t) \equiv u_T^\ast(t), \quad \forall t \in [0, T + \delta], \quad \forall T, M \geq 0. \]
The result is a straightforward consequence of this fact.

Proof of Theorem 3.2. Assume first that $\beta_0|_{[0, \infty)} \in C^2_0([0, \infty); H^{1/2}(\Gamma))$ and $\beta_1|_{[0, \infty)} \in C^2_0([0, \infty); H^{-1/2}(\Gamma))$. Because of Proposition 4.4, we have enough regularity to apply Proposition 5.1 to $u_T$ and $u_T^\ast$ (changing the wave velocity in the latter case). We use then the jumps defined in Proposition 5.2 and define the distributions
\[ u^h := S \ast E\mu - D \ast E\psi, \quad v^h := S_m \ast E\mu^\ast - D_m \ast E\psi^\ast. \]
Then the pair $(u^h, v^h)$ is the solution of (3.3), as follows from comparing the different transmission conditions in (3.3) with those of (4.1)-(4.2). Then Proposition 5.1 proves that
\[ u_T(t) = u^h(t), \quad u_T^\ast(t) = v^h(t), \quad \forall t \in [0, T + \delta] \]
and by Proposition 4.2
\[ u^h, v^h \in C^2([0, \infty); L^2(\mathbb{R}^d)) \cap C^1([0, \infty); H^1(\mathbb{R}^d \setminus \Gamma)) \cap C([0, \infty); H^1_\Delta(\mathbb{R}^d \setminus \Gamma)) \]
and
\[ \|u^h(t)\|_{\Delta, R \setminus \Gamma} + \|v^h(t)\|_{\Delta, R \setminus \Gamma} \leq C(H_2(\beta_0; t; H^{1/2}(\Gamma)) + H_2(\beta_1; t; H^{-1/2}(\Gamma))) \]
\[ \|\dot{u}^h(t)\|_{1, R \setminus \Gamma} + \|\dot{v}^h(t)\|_{1, R \setminus \Gamma} \leq C(H_2(\beta_0; t; H^{1/2}(\Gamma)) + H_2(\beta_1; t; H^{-1/2}(\Gamma))). \]
The remainder of the proof follows from simple shifting and density arguments. Details are identical to similar proofs in [25, Section 7.5].

Proof of Theorem 3.3. It is a slight variant of the previous proof.
6 Time discretization with Convolution Quadrature

Full discretization. The final step for discretization of our problem consists of applying one of the possible Convolution Quadrature techniques to the semidiscretized integral system (3.1) (this includes using CQ in the convolution operators acting on the data functions) and to the potential postprocessing (3.2). We will apply a BDF2-based CQ discretization. Other multistep techniques (based on the implicit Euler scheme or on the trapezoidal rule) can be presented and analyzed with similar tools. Finally, multistage CQ is also available, using implicit Runge-Kutta methods as background ODE solvers in the numerical scheme. Multistep CQ originated in [23], applied to parabolic problems, and was extended in [24] to hyperbolic problems, including a time domain boundary integral equation in acoustics. A modern introduction to computational uses of CQ for wave propagation problems is given in [8]. Algorithmic details and several possible interpretations of CQ can be found in [18]. In this section we will follow the plan developed in [26, Section 10.3], based on [5, Section 6]. We note that while a time domain analysis of multistep CQ discretizations for wave propagation problems is known, at the current stage of research multistage CQ methods require a Laplace domain analysis of the associated Galerkin solver (see [22]).

Followed by the computation of

\[ \lambda_k^h := [\partial_v u_k^h], \quad \phi_k^h := [\gamma v_k^h]. \]
Theorem 6.2. The difference between the semidiscrete and the fully discrete solutions follows. The result in the statement is then a direct consequence of this inequality and the bound

Error equations. Consider the errors for (6.1) as a discretization of (3.3):
\[ e_u := u^h - u^h_k, \quad e_v := v^h - v^h_k, \]
The distributions \((e_u, e_v)\) are a causal solution to the error equations

\[ c^{-2} \partial_t^2 e_u = \kappa \Delta e_u + c^{-2} \left( \partial_t^2 u^h - \dot{u}^h \right) \quad \text{(in } L^2(\mathbb{R}^d \setminus \Gamma)), \]
\[ \partial_t^2 e_v = \Delta e_v + \left( \partial_t^2 v^h - \dot{v}^h \right) \quad \text{(in } L^2(\mathbb{R}^d \setminus \Gamma)), \]
\[ [\gamma e_u] = -[\gamma e_v] \in Y_h^0 \quad \text{(in } H^{1/2}(\Gamma)), \]
\[ \kappa [\partial_t e_u] = -[\partial_t e_v] \in X_h \quad \text{(in } H^{-1/2}(\Gamma)), \]
\[ \gamma^+ e_u - \gamma^- e_v \in X_h^0 \quad \text{(in } H^{1/2}(\Gamma)), \]
\[ \kappa \partial_t^+ e_u - \partial_t^- e_v \in Y_h^0 \quad \text{(in } H^{-1/2}(\Gamma)). \]

Proposition 6.1. The errors \(e_u\) and \(e_v\) can be bounded for all \(t \geq 0\) as follows

\[ \| c^{1/2} \nabla e_u(t) \|_{\mathbb{R}^d \setminus \Gamma} + \| \nabla e_v(t) \|_{\mathbb{R}^d} + \| c^{-1} \partial_t e_u(t) \|_{\mathbb{R}^d} + \| \partial_t e_v(t) \|_{\mathbb{R}^d} \leq C k^2 t^2 \left( \max_{0 \leq \tau \leq t} \| (P_2 u^h)(\tau) \|_{\mathbb{R}^d} + \max_{0 \leq \tau \leq t} \| (P_2 v^h)(\tau) \|_{\mathbb{R}^d} \right), \]
where \(P_2 f := f + \dot{f} + \ddot{f} \).

Proof. Let \(E_u, E_v, U^h, V^h\) be the respective Laplace transforms of \(e_u, e_v, u^h,\) and \(v^h\). Taking the Laplace transform of (6.2), and using an integration by parts argument (with the same format as in the proof of Proposition 4.1(c)), it is easy to prove that for all \(s \in \mathbb{C}\) with \(\Re s > 0\), it holds

\[ \Re s_k \left( |s_k|^2 \| c^{-1} E_u(s) \|_{\mathbb{R}^d}^2 + \kappa \| \nabla E_u(s) \|_{\mathbb{R}^d \setminus \Gamma}^2 + |s_k|^2 \| E_v(s) \|_{\mathbb{R}^d}^2 + \| \nabla E_v(s) \|_{\mathbb{R}^d \setminus \Gamma}^2 \right) = \Re \left( s_k \left( (c^{-2}(s_k^2 - s^2)^h(s), E_u(s))_{\mathbb{R}^d} + ((s_k^2 - s^2)^h(s), E_v(s))_{\mathbb{R}^d} \right) \right), \]
where \(s_k = \frac{3}{2} - 2e^{-s_k} + \frac{1}{2} e^{-2s_k}\). Using that

\[ |s_k| \leq C |s| \quad |s_k^2 - s^2| \leq C k^2 |s|^4, \quad \Re s_k \geq C \min\{1, \Re s\} \forall k \leq 1, \quad \Re s > 0, \]
the bound

\[ \| s_k c^{-1} E_u(s) \|_{\mathbb{R}^d} + \| c^{1/2} \nabla E_u(s) \|_{\mathbb{R}^d \setminus \Gamma} + \| s_k E_v(s) \|_{\mathbb{R}^d} + \| \nabla E_v(s) \|_{\mathbb{R}^d \setminus \Gamma} \leq C k^2 \frac{t}{\min\{1, \Re s\}} \left( \| s^4 U^h(s) \|_{\mathbb{R}^d} + \| s^4 V^h(s) \|_{\mathbb{R}^d} \right) \]
follows. The result in the statement is then a direct consequence of this inequality and [15] Theorem 7.1.

Theorem 6.2. The difference between the semidiscrete and the fully discrete solutions of (2.7)-(2.8) can be bounded for all \(t \geq 0\)

\[ t \| \nabla u^h(t) - \nabla u_k^h(t) \|_{\mathbb{R}^d \setminus \Gamma} + t \| \nabla v^h(t) - \nabla v_k^h(t) \|_{\mathbb{R}^d \setminus \Gamma} + \| \phi^h(t) - \phi_k^h(t) \|_{1/2, \Gamma} \leq C k^2 t^3 \left( H_0(\beta_0, t; H^{1/2}(\Gamma)) + H_7(\beta_1, t; H^{-1/2}(\Gamma)) \right), \]
\[ \| \lambda^h(t) - \lambda_k^h(t) \|_{-1/2, \Gamma} \leq C k^2 t^2 \left( H_0(\beta_0, t; H^{1/2}(\Gamma)) + H_7(\beta_1, t; H^{-1/2}(\Gamma)) \right). \]
Proof. The bounds for the gradients follows from Proposition 6.1 and Theorem 3.2. In order to bound the traces we use the fact that for a causal $X$-valued function $f$

$$\|f(t)\|_X \leq k \sum_{j=0}^{\infty} \|\partial_k f(t-t_j)\|_X \leq t \max_{0 \leq \tau \leq t} \|f(\tau)\|_X$$

and then use this to obtain $L^2(\mathbb{R}^d)$ estimates of $e_v(t)$ from those of $\partial_k e_v(t)$ given in Proposition 6.1. Finally, in order to bound normal derivatives of $e_v$, we use the error equation $\Delta e_v = \partial^2_k e_v + (\ddot{v}^h - \partial^2_k v^h)$ to bound $\Delta e_v(t)$. The details are very similar to those of the proof of [5, Theorem 6.7] (see also [26, Section 10.3]) and are therefore omitted.

7 Numerical experiments

A smooth obstacle. In this experiment we choose $\Gamma \subset \mathbb{R}^2$ to be the smooth closed curve parametrized by the function

$$z \mapsto ((1 + (\cos z)^2) \cos z, (1 + (\sin z)^2) \sin z) \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix},$$

which is shaped like a smoothened square parallel to the coordinate axes. We choose $\kappa = 0.8$ and $c^2 \kappa = 1.2^2$. The data in (2.1) are taken so that the exterior solution is zero and the interior solution is a plane wave $u(x, t) = \sin(c \sqrt{\kappa}(t - t_0) - x \cdot d) h(c \sqrt{\kappa}(t - t_0) - x \cdot d)$, where $t_0 = 2.2$, $d = (1/\sqrt{2}, -1/\sqrt{2})$ and $h$ is a smoothened version of the Heaviside function, namely a polynomial of degree ten that connects the points $(0,0)$ and $(1,1)$. We will integrate in the time interval $[0, 4]$ using the BDF2-based CQ scheme.

For spatial discretization we use a division of $\Gamma$ into $N$ elements, by choosing a uniform grid in parametric space. The space $X_h$ is composed of piecewise constant functions in this mesh. For $Y_h$ we choose continuous piecewise linear functions in the parameter $z$, mapped to $\Gamma$ and defined on a uniform grid with $N$ elements which is staggered with respect to the grid that is used to define $X_h$. With adequately chosen reduced integration, it is possible to rewrite all the elements of the matrices in the language of the fully discrete Calderón calculus of [14]. The interior solution will be computed in the center and corners of the square $[-0.5, 0.5]^2 \subset \Omega_-$: these five points will be denoted $x^\text{obs}_\ell$ for $\ell = 1, \ldots, 5$. We then measure relative errors associated to the absolute errors

$$e^\lambda := \|\lambda(T) - \Pi_0 \kappa \partial_\nu u(T)\|_{L^\infty(\Gamma)},$$

$$e^\varphi := \|\varphi(T) - \Pi_1 \gamma u(T)\|_{L^\infty(\Gamma)},$$

$$e^u := \max_{\ell} |u^h(x^\text{obs}_\ell, T) - u(x^\text{obs}_\ell, T)|.$$

(The corresponding relative errors will be denoted with a capital $E$ in the tables.) Here $\Pi_0$ is the midpoint interpolation operator on $X_h$ and $\Pi_1$ is the natural Lagrange interpolation on $Y_h$. The results are reported in Table 1 and Figure 1.
\( N = M \)

| \( N \) | \( E^\phi \) | e.c.r. | \( E^\lambda \) | e.c.r. | \( E^u \) | e.c.r. |
|-------|--------|-------|--------|-------|--------|-------|
| 50    | 3.0159E(-01) |   | 1.0925E(+00) |   | 1.6530E(-01) |   |
| 100   | 2.2310E(-01) | 0.4349 | 7.8330E(-01) | 0.4800 | 4.1091E(-02) | 2.0082 |
| 150   | 1.7356E(-01) | 0.6192 | 6.8195E(-01) | 0.3418 | 1.8436E(-02) | 1.9767 |
| 200   | 1.4065E(-01) | 0.7308 | 6.1521E(-01) | 0.3580 | 1.0385E(-02) | 1.9952 |
| 300   | 9.7445E(-02) | 0.9052 | 5.2374E(-01) | 0.3970 | 4.5692E(-03) | 2.0248 |
| 500   | 5.0972E(-02) | 1.2686 | 3.0757E(-01) | 1.0420 | 1.7325E(-03) | 1.8984 |
| 700   | 2.9493E(-02) | 1.6260 | 1.7386E(-01) | 1.6954 | 8.5386E(-04) | 2.1029 |
| 900   | 1.8488E(-02) | 1.8584 | 1.0393E(-01) | 2.0475 | 4.8647E(-04) | 2.2386 |
| 1200  | 1.0462E(-02) | 1.9791 | 5.9202E(-02) | 1.9560 | 2.6762E(-04) | 2.0774 |
| 1600  | 5.8992E(-03) | 1.9917 | 3.2909E(-02) | 2.0412 | 1.4956E(-04) | 2.0226 |

Table 1: Relative errors at final time \((T = 4)\) for trace, normal derivative and interior solution in the case of a smooth obstacle.

![Error graphs corresponding to Table 1](image)

A polygonal obstacle. We now consider an obstacle whose boundary is the quadrilateral with vertices \((0, 0), (1, 0), (0.8, 0.8), \) and \((0.2, 1)\). The same physical data \((\kappa \text{ and } c)\) and the exact solution are the same as in the previous experiment, and we also integrate up to time \(T = 4\) with the BDF2-CQ scheme. Each of the edges of \(\Gamma\) is subdivided into an equal number of elements (the partition is thus piecewise uniform) to a total of \(N\) elements. The spaces \(X_h\) and \(Y_h\) are respectively composed of piecewise constant and continuous piecewise linear functions on this grid. The interior solution will be observed in the points

\[
x_1^{\text{obs}} := (0.3, 0.4), \quad x_2^{\text{obs}} := (0.5, 0.7), \quad x_3^{\text{obs}} := (0.65, 0.4), \quad x_4^{\text{obs}} := (0.5, 0.2).
\]
We measure relative errors corresponding to
\[ e^\lambda := \|\lambda(T) - \kappa \partial_n u(T)\|_{L^2(\Gamma)}, \]
\[ e^\varphi := \|\varphi(T) - \gamma u(T)\|_{L^2(\Gamma)}, \]
\[ e^u := \max_{\ell} |u^h(x_{\ell}^{obs},T) - u(x_{\ell}^{obs},T)|. \]

| N   | M   | \( E^\varphi \) | e.c.r | \( E^\lambda \) | e.c.r | \( E^u \)  | e.c.r |
|-----|-----|-----------------|-------|-----------------|-------|-------------|-------|
| 4   | 150 | 4.9350E(-02)    |       | 6.0959E(-02)    |       | 1.5495E(-02)|       |
| 8   | 300 | 4.8955E(-03)    | 3.3335| 4.1232E(-02)    | 0.5641| 2.8475E(-03)| 2.4441|
| 16  | 600 | 1.2233E(-03)    | 2.0007| 1.3734E(-02)    | 1.5860| 8.5962E(-04)| 1.7279|
| 32  | 1200| 8.6127E(-05)    | 3.8281| 1.7186E(-03)    | 2.9984| 9.4632E(-05)| 3.1833|
| 64  | 2400| 1.9057E(-05)    | 2.1762| 4.1594E(-04)    | 2.0468| 2.6609E(-05)| 1.8304|
| 128 | 4800| 4.6711E(-06)    | 2.0284| 1.0252E(-04)    | 2.0205| 6.7798E(-06)| 1.9726|

Table 2: Relative errors at final time for trace, normal derivative and interior solution in the case of a non-smooth scatterer.

Figure 2: Error graphs corresponding to Table 2

A multiple scatterer illustration. We finally show several snapshots of the scattering of a plane wave by four circular obstacles with different material properties. In all four obstacles \( \kappa = 1 \). The wave velocity is set to be \( c = 2 \) in the obstacles placed in NE and SW positions (see Figure 3) and \( c = 0.5 \) in the other two obstacles.

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Figure 3: Six images of the scattering of a short-pulse plane wave by four penetrable obstacles. The obstacle in the upper right corner propagates waves at twice the speed of the surrounding medium, giving a head start to the part of the wave that traverses the obstacle with respect to the incident wave.