Hamilton-Jacobi Theorems for Regular Controlled Hamiltonian System and Its Reductions

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Abstract. In this paper, we first prove a Hamilton-Jacobi theorem for regular controlled Hamiltonian (RCH) system on cotangent bundle of a configuration manifold, by using the symplectic form. This result is an extension of the geometric version of Hamilton-Jacobi theorem for Hamiltonian system given in Wang [25]. Next, we generalize the above result for regular reducible RCH systems with symmetry, and obtain the Hamilton-Jacobi theorems for regular point and orbit reduced RCH systems, by using the reduced symplectic forms. Moreover we prove that the RCH-equivalence for RCH system, and RpCH-equivalence and RoCH-equivalence for reducible RCH systems with symmetry, leave the solutions of corresponding Hamilton-Jacobi equations invariant. As an application of the theoretical results, we consider the regular point reducible RCH system on the generalization of a Lie group, and give the Hamilton-Jacobi theorem of the system. In particular, we show the Hamilton-Jacobi equations of rigid body and heavy top with internal rotors on the generalization of rotation group SO(3) and on the generalization of Euclidean group SE(3) by calculation in detail, respectively.

Keywords: regular controlled Hamiltonian system, Hamilton-Jacobi theorem, regular point reduction, regular orbit reduction, RCH-equivalence.

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1 Introduction

It is well-known Hamilton-Jacobi theory is an important part of classical mechanics. On the one hand, it provides a characterization of the generating functions of certain time-dependent canonical transformations, such that a given Hamiltonian system in such a form that its solutions are extremely easy to find by reduction to the equilibrium, see Abraham and Marsden [1], Arnold [2] and Marsden and Ratiu [17]. On the other hand, it is possible in many cases that Hamilton-Jacobi theory provides an immediate way to integrate the equation of motion of system, even when the problem of Hamiltonian system itself has not been or cannot be solved completely. In addition, the Hamilton-Jacobi equation is also fundamental in the study of the quantum-classical relationship in quantization, and it also plays an important role in the development of numerical integrators that preserve the symplectic structure and in the study of stochastic dynamical systems, see Ge and Marsden [7], Marsden and West [19] and Lázaro-Camí and Ortega [10]. For these reasons it is described as a useful tools in the study of Hamiltonian system theory, and has been extensively developed in past many years and become one of the most active subjects in the study of modern applied mathematics and analytical mechanics, which absorbed a lot of researchers to pour into it and a lot of deep and beautiful results have been obtained, see Carinena et al [5] and [6], Iglesias et al [8], León et al [11,12] Ohsawa and Bloch [20], Vitagliano [23], Wang [25] for more details.

On the other hand, we note that the theory of mechanical control systems has formed an important subject in recent years. Its research gathers together some separate areas of research such as mechanics, differential geometry and nonlinear control theory, etc., and the emphasis of this research on geometry is motivated by the aim of understanding the structure of equations of motion of the system in a way that helps both analysis and design. Thus, it is natural to study mechanical control systems by combining with the analysis of dynamic systems and the geometric reduction theory of Hamiltonian and Lagrangian systems. In particular, we note that in Marsden et al [18], the authors studied regular reduction theory of controlled Hamiltonian systems with symplectic structure and symmetry, as an extension of regular symplectic reduction theory of Hamiltonian systems under regular controlled Hamiltonian equivalence conditions. Wang in [24] generalized the work in [18] to study the singular reduction theory of regular controlled Hamiltonian systems, and Wang and Zhang in [26] generalized the work in [18] to study optimal reduction theory of controlled Hamiltonian systems with Poisson structure and symmetry by using optimal momentum map and reduced Poisson tensor (or reduced symplectic form), and Ratiu and Wang in [22] studied the Poisson reduction of controlled Hamiltonian system by controllability distribution. These research work not only gave a variety of reduction methods for controlled Hamiltonian systems, but also showed a variety of relationships of controlled Hamiltonian equivalence of these systems. Now, it is a natural problem how to study the Hamilton-Jacobi theory for controlled Hamiltonian system and a variety of reduced controlled Hamiltonian systems by combining with reduction theory and Hamilton-Jacobi theory of Hamiltonian systems. This is goal of our research.
Just as we have known that Hamilton-Jacobi theory from the variational point of view is originally developed by Jacobi in 1866, which state that the integral of Lagrangian of a system along the solution of its Euler-Lagrange equation satisfies the Hamilton-Jacobi equation. The classical description of this problem from the geometrical point of view is given by Abraham and Marsden in [1] as follows: Let $Q$ be a smooth manifold and $TQ$ the tangent bundle, $T^*Q$ the cotangent bundle with a canonical symplectic form $\omega$ and the projection $\pi_Q : T^*Q \rightarrow Q$.

**Theorem 1.1** Assume that the triple $(T^*Q, \omega, H)$ is a Hamiltonian system with Hamiltonian vector field $X_H$, and $W : Q \rightarrow \mathbb{R}$ is a given function. Then the following two assertions are equivalent:

(i) For every curve $\sigma : \mathbb{R} \rightarrow Q$ satisfying $\dot{\sigma}(t) = T\pi_Q(X_H(dW(\sigma(t))))$, $\forall t \in \mathbb{R}$, then $dW \cdot \sigma$ is an integral curve of the Hamiltonian vector field $X_H$.

(ii) $W$ satisfies the Hamilton-Jacobi equation $H(q^i, \frac{\partial W}{\partial q^i}) = E$, where $E$ is a constant.

It is worthy of note that if we take that $\gamma = dW$ in the above theorem, then $\gamma$ is a closed one-form on $Q$, and the equation $d(H \cdot dW) = 0$ is equivalent to the Hamilton-Jacobi equation $H(q^i, \frac{\partial W}{\partial q^i}) = E$, where $E$ is a constant. This result is used the formulation of a geometric version of Hamilton-Jacobi theorem for Hamiltonian system, see Cariñena et al [5] and Iglesias et al [8]. On the other hand, this result is developed in the context of time-dependent Hamiltonian system by Marsden and Ratiu in [17]. The Hamilton-Jacobi equation may be regarded as a nonlinear partial differential equation for some generating function $S$, and the problem is become how to choose a time-dependent canonical transformation $\Psi : T^*Q \times \mathbb{R} \rightarrow T^*Q \times \mathbb{R}$, which transforms the dynamical vector field of a time-dependent Hamiltonian system to equilibrium, such that the generating function $S$ of $\Psi$ satisfies the time-dependent Hamilton-Jacobi equation. In particular, for the time-independent Hamiltonian system, we may look for a symplectic map as the canonical transformation. This work offers a important idea that one can use the dynamical vector field of a Hamiltonian system to describe Hamilton-Jacobi equation. Moreover, assume that $\gamma : Q \rightarrow T^*Q$ is a closed one-form on $Q$, and define that $X_{H}^{\gamma} = T\pi_Q \cdot X_H \cdot \gamma$, where $X_H$ is the dynamical vector field of Hamiltonian system $(T^*Q, \omega, H)$. Then the fact that $X_{H}^{\gamma}$ and $X_H$ are $\gamma$-related, that is, $T\gamma \cdot X_{H}^{\gamma} = X_H \cdot \gamma$ is equivalent that $d(H \cdot \gamma) = 0$, which is given in Cariñena et al [5] and Iglesias et al [8]. Motivated by the above research work, Wang in [25] used the dynamical vector field of Hamiltonian system and the regular reduced Hamiltonian system to describe the Hamilton-Jacobi theory for these systems.

Since the Hamilton-Jacobi theory is developed based on the Hamiltonian picture of dynamics, it is natural idea to extend the Hamilton-Jacobi theory to the regular controlled Hamiltonian system and its regular reduced systems, which are introduced in Marsden et al [18], and it is possible to describe the relationship between the RCH-equivalence for RCH systems and the solutions of corresponding Hamilton-Jacobi equations. The main contributions in this paper is given as follows. (1) We prove a Hamilton-Jacobi theorem for RCH system on cotangent bundle of a configuration manifold, by using the symplectic form under a weaker condition; (2) We generalize the above result to regular reducible RCH systems with symmetry by using the reduced symplectic forms, and obtain the Hamilton-Jacobi theorems for regular point reduced RCH system and regular orbit reduced RCH system (see Theorem 3.4 and Theorem 4.4); (3) We prove that the RCH-equivalence for RCH system, and RpCH-equivalence and RoCH-equivalence for reducible RCH systems with symmetry, leave the solutions of corresponding Hamilton-Jacobi equations invariant (see Theorem 2.7, Theorem 3.6 and Theorem 4.6); (4) As an application,
we give the Hamilton-Jacobi theorem of the regular point reducible RCH system on the generalization of a Lie group, and show the Hamilton-Jacobi equations of rigid body and heavy top with internal rotors by calculation in detail, respectively.

A brief of outline of this paper is as follows. In the second section, we first review some relevant definitions and basic facts about RCH systems and RCH-equivalence, then give a key lemma, which is obtained by a careful modification for the corresponding result of Abraham and Marsden in [1], then prove a Hamilton-Jacobi theorem for RCH system on cotangent bundle of a configuration manifold, by using the symplectic form and above lemma. This result is an extension of the geometric version of Hamilton-Jacobi theorem for Hamiltonian system given in Wang [25]. From the third section we begin to discuss the regular reducible RCH systems with symmetry by combining with the Hamilton-Jacobi theory and regular symplectic reduction theory for RCH system. The regular point and regular orbit reducible RCH systems with symmetry are considered respectively in the third section and the fourth section, and give the Hamilton-Jacobi theorems of regular point and regular orbit reduced RCH systems by using the reduced symplectic forms. As the applications of the theoretical results, in fifth section, we consider the regular point reducible RCH system on the generalization of a Lie group, and give the Hamilton-Jacobi theorem of the reduced system. In particular, we show the Hamilton-Jacobi equations of rigid body and heavy top with internal rotors on the generalization of rotation group SO(3) and on the generalization of Euclidean group SE(3) by calculation in detail, respectively. These research work develop the reduction and Hamilton-Jacobi theory of RCH systems with symmetry and make us have much deeper understanding and recognition for the structure of Hamiltonian systems and RCH systems.

2 Hamilton-Jacobi Theorem of RCH System

In this paper, our goal is to study Hamilton-Jacobi theory of RCH systems with symplectic structure and symmetry. We shall prove the Hamilton-Jacobi theorems for RCH system and regular reducible RCH systems, and describe the relationship between the RCH-equivalence for RCH systems and the solutions of corresponding Hamilton-Jacobi equations. In order to do these, in this section, we first review some relevant definitions and basic facts about RCH systems and RCH-equivalence, which will be used in subsequent sections. Then we prove the Hamilton-Jacobi theorem of RCH system by using symplectic form, and state that the solution of Hamilton-Jacobi equation for RCH system leaves invariant under the conditions of RCH-equivalence. We shall follow the notations and conventions introduced in Abraham and Marsden [1], Marsden and Ratiu [17], Libermann and Marle [13], Ortega and Ratiu [21], and Marsden et al [18], Wang [25]. In this paper, we assume that all manifolds are real, smooth and finite dimensional and all actions are smooth left actions.

2.1 Regular Controlled Hamiltonian Systems and RCH-equivalence

In order to describe uniformly RCH systems defined on a cotangent bundle and on the regular reduced spaces, in this subsection we first define a RCH system on a symplectic fiber bundle. Then we can obtain the RCH system on the cotangent bundle of a configuration manifold as a special case, and discuss RCH-equivalence. In consequence, we can regard the associated Hamiltonian system on the cotangent bundle as a special case of the RCH system without external force and control, such that we can study the RCH systems with symmetry by combining with regular symplectic reduction theory of Hamiltonian systems. For convenience, we assume that
Let \((E, M, N, \pi, G)\) be a fiber bundle and \((E, \omega_E)\) be a symplectic fiber bundle. If for any function \(H : E \to \mathbb{R}\), we have a Hamiltonian vector field \(X_H\) by \(i_{X_H} \omega_E = \text{d}H\), then \((E, \omega_E, H)\) is a Hamiltonian system. Moreover, if considering the external force and control, we can define a kind of regular controlled Hamiltonian (RCH) system on the symplectic fiber bundle \(E\) as follows.

**Definition 2.1** (RCH System) A RCH system on \(E\) is a 5-tuple \((E, \omega_E, H, F, W)\), where \((E, \omega_E, H)\) is a Hamiltonian system, and the function \(H : E \to \mathbb{R}\) is called the Hamiltonian, a fiber-preserving map \(F : E \to E\) is called the (external) force map, and a fiber submanifold \(W\) of \(E\) is called the control subset.

Sometimes, \(W\) also denotes the set of fiber-preserving maps from \(E\) to \(W\). When a feedback control law \(u : E \to W\) is chosen, the 5-tuple \((E, \omega_E, H, F, u)\) denotes a closed-loop dynamic system. In particular, when \(Q\) is a smooth manifold, and \(T^*Q\) its cotangent bundle with a symplectic form \(\omega\) (not necessarily canonical symplectic form), then \((T^*Q, \omega)\) is a symplectic vector bundle. If we take that \(E = T^*Q\), from above definition we can obtain a RCH system on the cotangent bundle \(T^*Q\), that is, 5-tuple \((T^*Q, \omega, H, F, W)\). Where the fiber-preserving map \(F : T^*Q \to T^*Q\) is the (external) force map, that is the reason that the fiber-preserving map \(F : E \to E\) is called an (external) force map in above definition.

In order to describe the dynamics of the RCH system \((E, \omega_E, H, F, W)\) with a control law \(u\), we need to give a good expression of the dynamical vector field of RCH system. At first, we introduce a notations of vertical lift maps of a vector along a fiber. For a smooth manifold \(E\), its tangent bundle \(TE\) is a vector bundle, and for the fiber bundle \(\pi : E \to M\), we consider the tangent mapping \(T\pi : TE \to TM\) and its kernel \(\ker(T\pi) = \{\rho \in TE | T\pi(\rho) = 0\}\), which is a vector subbundle of \(TE\). Denote by \(VE := \ker(T\pi)\), which is called a vertical bundle of \(E\). Assume that there is a metric on \(E\), and we take a Levi-Civita connection \(A\) on \(TE\), and denote by \(HE := \ker(A)\), which is called a horizontal bundle of \(E\), such that \(TE = HE \oplus VE\).

For any \(x \in M\), \(a_x, b_x \in E_x\), any tangent vector \(\rho(b_x) \in T_{b_x}E\) can be split into horizontal and vertical parts, that is, \(\rho(b_x) = \rho^H(b_x) \oplus \rho^V(b_x)\), where \(\rho^H(b_x) \in H_{b_x}E\) and \(\rho^V(b_x) \in V_{b_x}E\). Let \(\gamma\) be a geodesic in \(E_x\) connecting \(a_x\) and \(b_x\), and denote by \(\rho^\gamma_x(a_x)\) a tangent vector at \(a_x\), which is a parallel displacement of the vertical vector \(\rho^\gamma_x(b_x)\) along the geodesic \(\gamma\) from \(b_x\) to \(a_x\). Since the angle between two vectors is invariant under a parallel displacement along a geodesic, then \(T\pi(\rho^\gamma_x(a_x)) = 0\), and hence \(\rho^\gamma_x(a_x) \in V_{a_x}E\). Now, for \(a_x, b_x \in E_x\) and tangent vector \(\rho(b_x) \in T_{b_x}E\), we can define the vertical lift map of a vector along a fiber given by

\[
\text{vlift} : TE_x \times E_x \to TE_x; \quad \text{vlift}(\rho(b_x), a_x) = \rho^\gamma_x(a_x).
\]

It is easy to check from the basic fact in differential geometry that this map does not depend on the choice of \(\gamma\). If \(F : E \to E\) is a fiber-preserving map, for any \(x \in M\), we have that \(F_x : E_x \to E_x\) and \(TF_x : TE_x \to TE_x\), then for any \(a_x \in E_x\) and \(\rho \in TE_x\), the vertical lift of \(\rho\) under the action of \(F\) along a fiber is defined by

\[
(\text{vlift}(F_x)\rho)(a_x) = \text{vlift}((TF_x\rho)(F_x(a_x)), a_x) = (TF_x\rho)^\gamma_x(a_x),
\]

where \(\gamma\) is a geodesic in \(E_x\) connecting \(F_x(a_x)\) and \(a_x\).
In particular, when $\pi : E \rightarrow M$ is a vector bundle, for any $x \in M$, the fiber $E_x = \pi^{-1}(x)$ is a vector space. In this case, we can choose the geodesic $\gamma$ to be a straight line, and the vertical vector is invariant under a parallel displacement along a straight line, that is, $\rho^\nu_\gamma(a_x) = \rho^\nu(b_x)$. Moreover, when $E = T^*Q$, $M = Q$, by using the local trivialization of $TT^*Q$, we have that $TT^*Q \cong TQ \times T^*Q$. Because of $\pi : T^*Q \rightarrow Q$, and $T\pi : TT^*Q \rightarrow TQ$, then in this case, for any $\alpha_x, \beta_x \in T^*_xQ$, $x \in Q$, we know that $(0, \beta_x) \in V_{\beta_x}T^*_xQ$, and hence we can get that

$$\text{vlift}((0, \beta_x)(\beta_x), \alpha_x) = (0, \beta_x)(\alpha_x) = \frac{d}{ds}\bigg|_{s=0} (\alpha_x + s\beta_x),$$

which is consistent with the definition of vertical lift map along fiber in Marsden and Ratiu [17].

For a given RCH System $(T^*Q, \omega, H, F, W)$, the dynamical vector field of the associated Hamiltonian system $(T^*Q, \omega, H)$ is that $X_H = (dH)^\sharp$, where, $\sharp : T^*T^*Q \rightarrow TT^*Q : dH \mapsto (dH)^\sharp$, such that $i_0(dH)^\sharp \gamma = dH$. If considering the external force $F : T^*Q \rightarrow TQ$, by using the above notation of vertical lift map of a vector along a fiber, the change of $X_H$ under the action of $F$ is that

$$\text{vlift}(F)X_H(\alpha_x) = \text{vlift}((TFX_H)(F(\alpha_x)), \alpha_x) = (TFX_H)^\sharp(\alpha_x),$$

where $\alpha_x \in T^*_xQ$, $x \in Q$ and $\gamma$ is a straight line in $T^*_xQ$ connecting $F_x(\alpha_x)$ and $\alpha_x$. In the same way, when a feedback control law $u : T^*Q \rightarrow W$ is chosen, the change of $X_H$ under the action of $u$ is that

$$\text{vlift}(u)X_H(\alpha_x) = \text{vlift}((TuX_H)(u(\alpha_x)), \alpha_x) = (TuX_H)^\sharp(\alpha_x).$$

In consequence, we can give an expression of the dynamical vector field of RCH system as follows.

**Proposition 2.2** The dynamical vector field of a RCH system $(T^*Q, \omega, H, F, W)$ with a control law $u$ is the synthetic of Hamiltonian vector field $X_H$ and its changes under the actions of the external force $F$ and control $u$, that is,

$$X_{(T^*Q, \omega, H, F, u)}(\alpha_x) = X_H(\alpha_x) + \text{vlift}(F)X_H(\alpha_x) + \text{vlift}(u)X_H(\alpha_x),$$

for any $\alpha_x \in T^*_xQ$, $x \in Q$. For convenience, it is simply written as

$$X_{(T^*Q, \omega, H, F, u)} = (dH)^\sharp + \text{vlift}(F) + \text{vlift}(u). \hspace{1cm} (2.1)$$

We also denote that $\text{vlift}(W) = \bigcup\{\text{vlift}(u)X_H \mid u \in W\}$. For the RCH system $(E, \omega_E, H, F, W)$ with a control law $u$, we have also a similar expression of its dynamical vector field. It is worthy of note that in order to deduce and calculate easily, we always use the simple expression of dynamical vector field $X_{(T^*Q, \omega, H, F, u)}$. Moreover, we also use the simple expressions for $R_P$-reduced vector field $X_{((T^*Q)_{\mu}, \omega_{\mu}, h_{\mu}, f_{\mu}, u_{\mu})}$ and $R_O$-reduced vector field $X_{((T^*Q)_{\nu}, \omega_{\nu}, h_{\nu}, f_{\nu}, u_{\nu})}$ in §3 and §4.

Next, we note that when a RCH system is given, the force map $F$ is determined, but the feedback control law $u : T^*Q \rightarrow W$ could be chosen. In order to describe the feedback control law to modify the structure of RCH system, the Hamiltonian matching conditions and RCH-equivalence are induced as follows.

**Definition 2.3** (RCH-equivalence) Suppose that we have two RCH systems $(T^*Q_i, \omega_i, H_i, F_i, W_i)$, $i = 1, 2$, we say them to be RCH-equivalent, or simply, $(T^*Q_1, \omega_1, H_1, F_1, W_1) \sim_{RCH}$
(T^*Q_i, \omega_i, H_i, F_i, W_i), \text{ if there exists a diffeomorphism } \varphi : Q_1 \to Q_2, \text{ such that the following Hamiltonian matching conditions hold:}

**RHM-1:** The cotangent lift map of \varphi, that is, \varphi^* = T^* \varphi : T^*Q_2 \to T^*Q_1 \text{ is symplectic, and } W_1 = \varphi^*(W_2).

**RHM-2:** Im[(\text{d}H_1)^{\sharp} + \text{vlift}(F_1) - ((\varphi_{\ast})^{\sharp} \text{d}H_2)^{\sharp} - \text{vlift}(\varphi_{\ast}^*F_2\varphi_{\ast}))] \subset \text{vlift}(W_1), \text{ where the map } \varphi_{\ast} = (\varphi^{-1})^{\ast} : T^*Q_1 \to T^*Q_2 \text{, and } (\varphi_{\ast})^{\ast} = T^*\varphi_{\ast} : T^*T^*Q_2 \to T^*T^*Q_1, \text{ and Im means the pointwise image of the map in brackets.}

It is worthy of note that our RCH system is defined by using the symplectic structure on the cotangent bundle of a configuration manifold, we must keep with the symplectic structure when we define the RCH-equivalence, that is, the induced equivalent map \varphi^* is symplectic on the cotangent bundle. In the same way, for the RCH systems on the symplectic fiber bundles, we can also define the RCH-equivalence by replacing \T^*Q_i \text{ and } \varphi : Q_1 \to Q_2 \text{ by } E_i \text{ and } \varphi : E_2 \to E_1, \text{ respectively. Moreover, the following Theorem 2.4 explains the significance of the above RCH-equivalence relation, its proof is given in Marsden et al [18].}

**Theorem 2.4** Suppose that two RCH systems \(T^*Q_i, \omega_i, H_i, F_i, W_i), \text{ i} = 1, 2, \text{ are RCH-equivalent, then there exist two control laws } u_i : T^*Q_i \to W_i, \text{ i} = 1, 2, \text{ such that the two closed-loop systems produce the same equations of motion, that is, } X_{(T^*Q_i, \omega_i, H_i, F_i, u_i)}(\varphi_{\ast}) = T^*(\varphi^*)X_{(T^*Q_2, \omega_2, H_2, F_2, u_2)}(\varphi_{\ast}), \text{ where the map } T(\varphi_{\ast}) : T^*Q_2 \to T^*Q_1 \text{ is the tangent map of } \varphi_{\ast}. \text{ Moreover, the explicit relation between the two control laws } u_i, i = 1, 2 \text{ is given by}

\[ \text{vlift}(u_1) - \text{vlift}(\varphi_{\ast}u_2\varphi_{\ast}) = -(\text{d}H_1)^{\sharp} - \text{vlift}(F_1) + ((\varphi_{\ast})^{\ast} \text{d}H_2)^{\sharp} + \text{vlift}(\varphi_{\ast}^*F_2\varphi_{\ast}) \]  

(2.2)

**2.2 Hamilton-Jacobi Theorem of RCH System**

In this subsection, we shall prove the Hamilton-Jacobi theorem of RCH system, and state that the solution of Hamilton-Jacobi equation for RCH system leaves invariant under the conditions of RCH-equivalence. We first give the following Lemma 2.5, which is the key to the proof of the Hamilton-Jacobi theorems of RCH system and regular reduced RCH system. It is worthy of note that this Lemma 2.5 is obtained by a careful modification for the corresponding result of Abraham and Marsden in [1], also see Wang [25]. Let \(Q \text{ be a smooth manifold and } TQ \text{ the tangent bundle, } T^*Q \text{ the cotangent bundle with a (canonical) symplectic form } \omega \text{ and the projection } \pi_Q : T^*Q \to Q.\)

**Lemma 2.5** Assume that \(\gamma : Q \to T^*Q \text{ is a one-form on } Q, \text{ and } \gamma \text{ is closed with respect to } T\pi_Q : TT^*Q \to TQ, \text{ that is, } \text{d}\gamma(T\pi_Q(v), T\pi_Q(w)) = 0, \forall v, w \in TT^*Q. \text{ Then we have that}

(i) for any \(v, w \in TT^*Q, \pi_Q^{\ast}\omega(T(\gamma \cdot \pi_Q) \cdot v, w) = \pi_Q^{\ast}\omega(v, w - T(\gamma \cdot \pi_Q) \cdot w);

(ii) \(T \gamma : TQ \to TT^*Q \text{ is injective with respect to } T\pi_Q : TT^*Q \to TQ.

**Proof:** We first prove the conclusion (i). For any \(v, w \in TT^*Q, \text{ note that } v - T(\gamma \cdot \pi_Q) \cdot v \text{ is vertical, because}

\[ T\pi_Q(v - T(\gamma \cdot \pi_Q) \cdot v) = T\pi_Q(v) - T(\pi_Q \cdot \gamma \cdot \pi_Q) \cdot v = T\pi_Q(v) - T\pi_Q(v) = 0, \]

where we used the relation \(\pi_Q \cdot \gamma \cdot \pi_Q = \pi_Q. \text{ Thus,}

\[ \pi_Q^{\ast}\omega(v - T(\gamma \cdot \pi_Q) \cdot v, w - T(\gamma \cdot \pi_Q) \cdot w) = \omega(T\pi_Q(v - T(\gamma \cdot \pi_Q) \cdot v), T\pi_Q(w - T(\gamma \cdot \pi_Q) \cdot w)) = 0, \]
and hence,

\[ \pi_Q^* \omega(T(\gamma \cdot \pi_Q) \cdot v, w) = \pi_Q^* \omega(v, w - T(\gamma \cdot \pi_Q) \cdot w) + \pi_Q^* \omega(T(\gamma \cdot \pi_Q) \cdot v, T(\gamma \cdot \pi_Q) \cdot w). \]

However, the second term on the right-hand side vanishes, that is,

\[ \pi_Q^* \omega(T(\gamma \cdot \pi_Q) \cdot v, T(\gamma \cdot \pi_Q) \cdot w) = \gamma^* \pi_Q^* \omega(T\pi_Q(v), T\pi_Q(w)) = -d\gamma(T\pi_Q(v), T\pi_Q(w)) = 0, \]

where we used the fact that for one-form \( \gamma \) on \( Q \), \( \gamma^* \pi_Q^* \omega = -d\gamma \), see Abraham and Marsden [1] in detail, and the assumption that \( \gamma \) is closed with respect to \( T\pi_Q : TT^*Q \to TQ \). It follows that the conclusion (i) holds.

Next, we prove the conclusion (ii). In fact, we can prove that if \( \gamma \) is closed one-form on \( Q \), then \( T\gamma : TQ \to TT^*Q \) is injective. We take a local coordinates \( q^i \), \( i = 1, \cdots, n = \dim Q \), on \( Q \), and assume that \( \gamma = \sum_{i=1}^n \gamma_i(q) dq^i \). Then \( d\gamma = \frac{1}{2} \sum_{i < j} (\partial_{q^i} \gamma_j - \partial_{q^j} \gamma_i) dq^i \wedge dq^j \). Since \( \gamma \) is closed one-form on \( Q \), we have that \( \partial_{q^i} \gamma_j = \partial_{q^j} \gamma_i \), \( i = j, 1, \cdots, n \). Notice that \( \gamma : Q \to T^*Q \), \( \gamma(q^i) = (q^i, \gamma_i(q)) \), and \( T\gamma(q^i) = \frac{\partial \gamma}{\partial q^i} + \sum_j \frac{\partial \gamma_i \gamma_j}{\partial q^i} \partial_{q^j} \), hence, \( T\gamma(q^i) \neq T\gamma(q^j) \), \( i \neq j \). Thus, \( T\gamma : TQ \to TT^*Q \) is injective. In the same way, if \( \gamma \) is closed with respect to \( T\pi_Q : TT^*Q \to TQ \), then \( T\gamma : TQ \to TT^*Q \) is injective with respect to \( T\pi_Q : TT^*Q \to TQ \). ■

Now, for any given RCH system \( (T^*Q, \omega, H, F, W) \), by using the above Lemma 2.5, we can follow the previous Hamilton-Jacobi theorem for RCH system. For convenience, the maps involved in the following theorem and its proof are shown in Diagram-1.

Diagram-1

**Theorem 2.6 (Hamilton-Jacobi Theorem of RCH System)** For a RCH system \( (T^*Q, \omega, H, F, W) \), assume that \( \gamma : Q \to T^*Q \) is a one-form on \( Q \), and \( \gamma \) is closed with respect to \( T\pi_Q : TT^*Q \to TQ \), and \( \tilde{X}^\gamma = T\pi_Q \cdot \tilde{X} \cdot \gamma \), where \( \tilde{X} = X_{(T^*Q, \omega, H, F, W)} \) is the dynamical vector field of the RCH system \( (T^*Q, \omega, H, F, W) \) with a control law \( u \). Then the following two assertions are equivalent:

(i) \( \tilde{X}^\gamma \) and \( \tilde{X} \) are \( \gamma \)-related, that is, \( T\gamma \cdot \tilde{X}^\gamma = \tilde{X} \cdot \gamma \);

(ii) \( X_{H, \gamma} + \text{vlift}(F \cdot \gamma) + \text{vlift}(u \cdot \gamma) = 0 \), or \( \tilde{X}^\gamma \).

Where the equation that \( X_{H, \gamma} + \text{vlift}(F \cdot \gamma) + \text{vlift}(u \cdot \gamma) = 0 \), is called a Hamilton-Jacobi equation for the RCH system \( (T^*Q, \omega, H, F, W) \) with a control law \( u \), and \( \gamma \) is called a solution of the Hamilton-Jacobi equation.

**Proof:** We first prove that (i) implies (ii). We take that \( v = \tilde{X} \cdot \gamma \in TT^*Q \), for any \( w \in TT^*Q \), from Lemma 2.5(i) we have that

\[ \pi_Q^* \omega(T\gamma \cdot \tilde{X}^\gamma, w) = \pi_Q^* \omega(T\gamma \cdot \pi_Q) \cdot \tilde{X} \cdot \gamma, w) = \pi_Q^* \omega(\tilde{X} \cdot \gamma, w - T(\gamma \cdot \pi_Q) \cdot w) \]

\[ = \pi_Q^* \omega(\tilde{X} \cdot \gamma, w) - \pi_Q^* \omega(\tilde{X} \cdot \gamma, T(\gamma \cdot \pi_Q) \cdot w). \]

By assuming (i), \( T\gamma \cdot \tilde{X}^\gamma = \tilde{X} \cdot \gamma \), we can obtain that \( \pi_Q^* \omega(\tilde{X} \cdot \gamma, T(\gamma \cdot \pi_Q) \cdot w) = 0 \). On the other hand, note that \( X_{H, \gamma} = T\gamma \cdot X_{H, \gamma} \) and \( \text{vlift}(F)X_{H, \gamma} = T\gamma \cdot \text{vlift}(F \cdot \gamma)X_{H, \gamma} \), and hence
\( \dot{X} \cdot \gamma = X_H \cdot \gamma + \operatorname{vlift}(F) \cdot \gamma + \operatorname{vlift}(u) \cdot \gamma = T\gamma \cdot X_H,\gamma + T\gamma \cdot \operatorname{vlift}(F) \cdot \gamma + T\gamma \cdot \operatorname{vlift}(u) \cdot \gamma \). Then we have that

\[
0 = \pi^*_Q \omega(\dot{X} \cdot \gamma, T(\gamma \cdot \pi_Q) \cdot w)
= \pi^*_Q \omega(T\gamma \cdot (X_H \cdot \gamma + \operatorname{vlift}(F) \cdot \gamma + \operatorname{vlift}(u) \cdot \gamma), T\gamma \cdot (T\pi_Q \cdot w))
= \gamma^* \pi^*_Q \omega(X_H \cdot \gamma + \operatorname{vlift}(F) \cdot \gamma + \operatorname{vlift}(u) \cdot \gamma), T\pi_Q \cdot w)
= -d\gamma(X_H \cdot \gamma + \operatorname{vlift}(F) \cdot \gamma + \operatorname{vlift}(u) \cdot \gamma, T\pi_Q \cdot w).
\]

It follows that either \( X_H \cdot \gamma + \operatorname{vlift}(F) \cdot \gamma + \operatorname{vlift}(u) \cdot \gamma = 0 \), or there is someone \( \tilde{v} \in TT^*Q \), such that \( X_H \cdot \gamma + \operatorname{vlift}(F) \cdot \gamma + \operatorname{vlift}(u) \cdot \gamma = T\pi_Q \cdot \tilde{v} \). But from assuming (i), \( T\gamma \cdot \dot{X} \cdot \gamma = X \cdot \gamma = T\gamma \cdot (X_H \cdot \gamma + \operatorname{vlift}(F) \cdot \gamma + \operatorname{vlift}(u) \cdot \gamma) \), and Lemma 2.5(ii), \( T\gamma : TQ \to TT^*Q \) is injective with respect to \( T\pi_Q : TT^*Q \to TQ \), hence, \( X_H \cdot \gamma + \operatorname{vlift}(F) \cdot \gamma + \operatorname{vlift}(u) \cdot \gamma = \dot{X} \gamma \). Thus, (i) implies (ii).

Conversely, from the above arguments we have that for any \( w \in TT^*Q \),

\[
\pi^*_Q \omega(T\gamma \cdot \dot{X} \gamma, w) - \pi^*_Q \omega(\dot{X} \cdot \gamma, w) = d\gamma(X_H \cdot \gamma + \operatorname{vlift}(F) \cdot \gamma + \operatorname{vlift}(u) \cdot \gamma, T\pi_Q \cdot w).
\]

Thus, since \( \pi^*_Q \omega \) is nondegenerate, the proof that (ii) implies (i) follows from these arguments in the same way. ■

In particular, if both the external force and control of the RCH system \((T^*Q, \omega, H, F, u)\) are zero, in this case the RCH system is just a Hamiltonian system \((T^*Q, \omega, H)\), and from the above Theorem 2.6 we can obtain the geometric version of Hamilton-Jacobi theorem of Hamiltonian system, which is given in Wang [25]. Thus, Theorem 2.6 can be regarded as an extension of Hamilton-Jacobi theorem for Hamiltonian system to the system with external force and control. Moreover, if considering the RCH-equivalence of RCH systems, we can obtain the following Theorem 2.7, which states that the solution of Hamilton-Jacobi equation for RCH system leaves invariant under the conditions of RCH-equivalence.

**Theorem 2.7** Suppose that two RCH systems \((T^*Q_i, \omega_i, H_i, F_i, W_i)\), \(i = 1, 2\), are RCH-equivalent with an equivalent map \( \varphi : Q_1 \to Q_2 \).

(i) If \( \gamma_2 : Q_2 \to T^*Q_2 \) is a solution of the Hamilton-Jacobi equation for RCH system \((T^*Q_2, \omega_2, H_2, F_2, W_2)\), and \( \gamma_2 \) is closed with respect to \( T\pi_{Q_2} : TT^*Q_2 \to TQ_2 \). Then \( \gamma_1 = \varphi^* \cdot \gamma_2 \cdot \varphi : Q_1 \to T^*Q_1 \) is a solution of the Hamilton-Jacobi equation for RCH system \((T^*Q_1, \omega_1, H_1, F_1, W_1)\);

(ii) If \( \gamma_1 : Q_1 \to T^*Q_1 \) is a solution of the Hamilton-Jacobi equation for RCH system \((T^*Q_1, \omega_1, H_1, F_1, W_1)\), and \( \gamma_1 \) is closed with respect to \( T\pi_{Q_1} : TT^*Q_1 \to TQ_1 \). Then \( \gamma_2 = (\varphi^{-1})^* \cdot \gamma_1 \cdot \varphi^{-1} : Q_2 \to T^*Q_2 \) is a solution of the Hamilton-Jacobi equation for RCH system \((T^*Q_2, \omega_2, H_2, F_2, W_2)\).

**Proof:** We first prove the conclusion (i). If two RCH systems \((T^*Q_i, \omega_i, H_i, F_i, W_i)\), \(i = 1, 2\), are RCH-equivalent with an equivalent map \( \varphi : Q_1 \to Q_2 \), from Theorem 2.4 we know that there exist two control laws \( u_i : T^*Q_i \to W_i \), \(i = 1, 2\), such that \( X_{(T^*Q_1, \omega_1, H_1, F_1, u_1)} \cdot \varphi^* = T(\varphi^*)X_{(T^*Q_2, \omega_2, H_2, F_2, u_2)}, \) that is, \( \dot{X} \cdot \varphi^* = T(\varphi^*) \dot{X} \). From the following commutative Diagram-2:

\[
\begin{array}{ccccccc}
Q_1 & \xrightarrow{\gamma_1} & T^*Q_1 & \xrightarrow{\dot{X}_1} & TT^*Q_1 & \xrightarrow{T\pi_{Q_1}} & TQ_1 \\
\varphi \downarrow & & & & & & T\varphi \\
Q_2 & \xrightarrow{\gamma_2} & T^*Q_2 & \xrightarrow{\dot{X}_2} & TT^*Q_2 & \xrightarrow{T\pi_{Q_2}} & TQ_2 \\
\end{array}
\]

Diagram-2
we have that \( \gamma_1 = \varphi^* \cdot \gamma_2 \cdot \varphi \), and \( T \varphi \cdot T \pi_{Q_1} \cdot T \varphi^* = T \pi_{Q_2} \). Note that \( \tilde{X}_{\gamma_i} = T \pi_{Q_2} \cdot \tilde{X}_i \cdot \gamma_i \), \( i = 1, 2 \), and \( \varphi : Q_1 \to Q_2 \) is a diffeomorphism, and \( \varphi^* \), \( T \varphi \) and \( T \varphi^* \) are isomorphism, then we have that
\[
T \varphi \cdot \tilde{X}_{\gamma_1} = T \varphi \cdot T \pi_{Q_1} \cdot \tilde{X}_1 \cdot \gamma_1 = T \varphi \cdot T \pi_{Q_1} \cdot T \varphi^* \cdot (T \varphi^*)^{-1} \cdot \tilde{X}_1 \cdot \varphi^* \cdot \gamma_2 \cdot \varphi
= T \pi_{Q_2} \cdot (T \varphi^*)^{-1} \cdot T \varphi \cdot \tilde{X}_2 \cdot \gamma_2 \cdot \varphi = T \pi_{Q_2} \cdot \tilde{X}_2 \cdot \gamma_2 \cdot \varphi = \tilde{X}_{\gamma_2} \cdot \varphi.
\]
Thus, \( \tilde{X}_{\gamma_2} = 0 \), \( \iff \tilde{X}_{\gamma_1} = 0 \). Since \( \gamma_2 \) is closed with respect to \( T \pi_{Q_2} : TT^*Q_2 \to TQ_2 \), note that \( d\gamma_1 = \varphi^* \cdot d\gamma_2 \cdot \varphi \), and \( T \varphi \cdot T \pi_{Q_1} \cdot T \varphi^* = T \pi_{Q_2} \), then \( \gamma_1 \) is closed with respect to \( T \pi_{Q_1} : TT^*Q_1 \to TQ_1 \). In consequence, from Lemma 2.5(ii), we know that maps \( T \gamma_i : TQ_i \to TT^*Q_i \) are injective with respect to \( T \pi_{Q_i} : TT^*Q_i \to TQ_i \) for \( i = 1, 2 \). Moreover, from the proof of Theorem 2.6, we know that
\[
T \gamma_i(X_{H_{\gamma_i}} + \text{vlift}(F_i \cdot \gamma_i) + \text{vlift}(u_i \cdot \gamma_i)) = T \gamma_i \cdot \tilde{X}_{\gamma_i}, \quad i = 1, 2.
\]
Thus,
\[
X_{H_{\gamma_2}} + \text{vlift}(F_2 \cdot \gamma_2) + \text{vlift}(u_2 \cdot \gamma_2) = 0, \quad \iff \tilde{X}_{\gamma_2} = 0,
\]
\[
\iff \tilde{X}_{\gamma_1} = 0, \quad \iff X_{H_{\gamma_1}} + \text{vlift}(F_1 \cdot \gamma_1) + \text{vlift}(u_1 \cdot \gamma_1) = 0.
\]
It follows that the conclusion (i) of Theorem 2.7 holds. Conversely, by using the same way, we can also prove the conclusion (ii) of Theorem 2.7.

In the following we shall generalize the above results to regular point and regular orbit reducible RCH systems with symmetry, and give a variety of Hamilton-Jacobi theorems for regular reduced RCH systems.

3 Hamilton-Jacobi Theorem of Regular Point Reduced RCH System

In this section, we first review some relevant definitions and basic facts about regular point reducible RCH systems and RpCH-equivalence, then we prove the Hamilton-Jacobi theorem of regular point reduced RCH system, by using reduced symplectic form and Lemma 2.5, and state the relationship between the solutions of Hamilton-Jacobi equations and regular point reduction, as well as that the solution of Hamilton-Jacobi equation for RCH system with symmetry leaves invariant under the conditions of RpCH-equivalence. We shall follow the notations and conventions introduced in Marsden et al [18], Wang [25].

3.1 Regular Point Reducible RCH System

Let \( Q \) be a smooth manifold and \( T^*Q \) its cotangent bundle with the symplectic form \( \omega \). Let \( \Phi : G \times Q \to Q \) be a smooth left action of the Lie group \( G \) on \( Q \), which is free and proper. Then the cotangent lifted left action \( \Phi^{T^*} : G \times T^*Q \to T^*Q \) is symplectic, free and proper, and admits a \( \text{Ad}^* \)-equivariant momentum map \( J : T^*Q \to g^* \), where \( g \) is a Lie algebra of \( G \) and \( g^* \) is the dual of \( g \). Let \( \mu \in g^* \) be a regular value of \( J \) and denote by \( G_\mu \) the isotropy subgroup of the coadjoint \( G \)-action at the point \( \mu \in g^* \), which is defined by \( G_\mu = \{ g \in G | \text{Ad}^*_g \mu = \mu \} \). Since \( G_\mu (\subset G) \) acts freely and properly on \( Q \) and on \( T^*Q \), then \( Q_\mu = Q/G_\mu \) is a smooth manifold and that the canonical projection \( \rho_\mu : Q \to Q_\mu \) is a surjective submersion. It follows that \( G_\mu \)
acts also freely and properly on \( J^{-1}(\mu) \), so that the space \( (T^*Q)_\mu = J^{-1}(\mu)/G_\mu \) is a symplectic manifold with symplectic form \( \omega_\mu \) uniquely characterized by the relation

\[
\pi^*_\mu \omega = i^*_\mu \omega .
\] (3.1)

The map \( i_\mu : J^{-1}(\mu) \to T^*Q \) is the inclusion and \( \pi_\mu : J^{-1}(\mu) \to (T^*Q)_\mu \) is the projection. The pair \((T^*Q)_\mu, \omega_\mu\) is called Marsden-Weinstein reduced space of \((T^*Q, \omega)\) at \( \mu \). Let \( H : T^*Q \to \mathbb{R} \) be a \( G \)-invariant Hamiltonian, the flow \( F_t \) of the Hamiltonian vector field \( X_H \) leaves the connected components of \( J^{-1}(\mu) \) invariant and commutes with the \( G \)-action, so it induces a flow \( f^\mu_t \) on \((T^*Q)_\mu\), defined by \( f^\mu_t \cdot \pi_\mu = \pi_\mu \cdot F_t \cdot i_\mu \), and the vector field \( X_{h_\mu} \) generated by the flow \( f^\mu_t \) on \((T^*Q)_\mu, \omega_\mu\) is Hamiltonian with the associated regular point reduced Hamiltonian function \( h_\mu : (T^*Q)_\mu \to \mathbb{R} \) defined by \( h_\mu \cdot \pi_\mu = H \cdot i_\mu \), and the Hamiltonian vector fields \( X_H \) and \( X_{h_\mu} \) are \( \pi_\mu \)-related. On the other hand, from Marsden et al \([18]\), we know that the regular point reduced space \((T^*Q)_\mu, \omega_\mu\) is symplectic diffeomorphic to a symplectic fiber bundle. Thus, we can introduce a regular point reducible RCH systems as follows.

**Definition 3.1 (Regular Point Reducible RCH System)** A 6-tuple \((T^*Q, G, \omega, H, F, W)\), where the Hamiltonian \( H : T^*Q \to \mathbb{R} \), the fiber-preserving map \( F : T^*Q \to T^*Q \) and the fiber submanifold \( W \) of \( T^*Q \) are all \( G \)-invariant, is called a regular point reducible RCH system, if there exists a point \( \mu \in g^* \), which is a regular value of the momentum map \( J \), such that the regular point reduced system, that is, the 5-tuple \((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, W_\mu \), where \((T^*Q)_\mu = J^{-1}(\mu)/G_\mu \), \( \pi^*_\mu \omega = i^*_\mu \omega \), \( h_\mu \cdot \pi_\mu = H \cdot i_\mu \), \( f_\mu \cdot \pi_\mu = \pi_\mu \cdot F \cdot i_\mu \), \( W_\mu = \pi_\mu(W) \), is a RCH system, which is simply written as \( R_P \)-reduced RCH system. Where \((T^*Q)_\mu, \omega_\mu\) is the \( R_P \)-reduced space, which is also called Marsden-Weinstein reduced space, the function \( h_\mu : (T^*Q)_\mu \to \mathbb{R} \) is called the reduced Hamiltonian, the fiber-preserving map \( f_\mu : (T^*Q)_\mu \to (T^*Q)_\mu \) is called the reduced (external) force map, \( W_\mu \) is a fiber submanifold of \((T^*Q)_\mu \) and is called the reduced control subset.

Denote by \( X_{(T^*Q, G, \omega, H, F, W)} \) the vector field of regular point reducible RCH system \((T^*Q, G, \omega, H, F, W)\) with a control law \( u \), then

\[
X_{(T^*Q, G, \omega, H, F, W)} = (dH)^{\phi} + \text{vlift}(F) + \text{vlift}(u).
\] (3.2)

Moreover, for the regular point reducible RCH system we can also introduce the regular point reduced controlled Hamiltonian equivalence (RpCH-equivalence) as follows.

**Definition 3.2 (RpCH-equivalence)** Suppose that we have two regular point reducible RCH systems \((T^*Q_1, G_1, \omega_1, H_1, F_1, W_1)\), \( i = 1, 2 \), we say them to be RpCH-equivalent, or simply, \((T^*Q_1, G_1, \omega_1, H_1, F_1, W_1) \sim_{RpCH} (T^*Q_2, G_2, \omega_2, H_2, F_2, W_2)\), if there exists a diffeomorphism \( \varphi : Q_1 \to Q_2 \) such that the following Hamiltonian matching conditions hold:

**RpHM-1:** The cotangent lift map \( \varphi^* : T^*Q_2 \to T^*Q_1 \) is symplectic.

**RpHM-2:** For \( \mu_i \in g^*_1 \), the regular points of RpCH systems \((T^*Q_i, G_i, \omega_i, H_i, F_i, W_i)\), \( i = 1, 2 \), the map \( \varphi^*_\mu = i^{-1}_{\mu_1} \cdot \varphi^* \cdot i_{\mu_2} : J^{-1}_{\mu_2}(\mu_2) \to J^{-1}_{\mu_1}(\mu_1) \) is \((G_2, G_1)-equi\)-equivariant and \( W_1 = \varphi^*_\mu(W_2) \), where \( \mu = (\mu_1, \mu_2) \), and denote by \( i^{-1}_{\mu_1}(S) \) the preimage of a subset \( S \subset T^*Q_1 \) for the map \( i_{\mu_1} : J^{-1}_{\mu_1}(\mu_1) \to T^*Q_1 \).

**RpHM-3:** \( \text{Im}[(dH_1)^{\phi} + \text{vlift}(F_1) - ((\varphi^*)dH_2)^{\phi} - \text{vlift}(\varphi^*F_2\varphi_*)] \subset \text{vlift}(W_1) \).

It is worthy of note that for the regular point reducible RCH system, the induced equivalent map \( \varphi^* \) not only keeps the symplectic structure, but also keeps the equivariance of \( G \)-action at the regular point. If a feedback control law \( u_\mu : (T^*Q)_\mu \to W_\mu \) is chosen, the \( R_P \)-reduced RCH
system \(((T^*Q)_{\mu}, \omega_{\mu}, h_{\mu}, f_{\mu}, u_{\mu})\) is a closed-loop regular dynamic system with a control law \(u_{\mu}\). Assume that its vector field \(X_{(T^*Q)_{\mu}, \omega_{\mu}, h_{\mu}, f_{\mu}, u_{\mu}}\) can be expressed by

\[
X_{(T^*Q)_{\mu}, \omega_{\mu}, h_{\mu}, f_{\mu}, u_{\mu}} = (dh_{\mu})^2 + \text{vlift}(f_{\mu}) + \text{vlift}(u_{\mu}),
\]

where \((dh_{\mu})^2 = X_{h_{\mu}}, \text{vlift}(f_{\mu}) = \text{vlift}(f_{\mu})X_{h_{\mu}}, \text{vlift}(u_{\mu}) = \text{vlift}(u_{\mu})X_{h_{\mu}}\), and satisfies the condition

\[
X_{(T^*Q)_{\mu}, \omega_{\mu}, h_{\mu}, f_{\mu}, u_{\mu}} \cdot \pi_{\mu} = T\pi_{\mu} \cdot X_{(T^*Q, G, \omega, H, F, u)} \cdot i_{\mu}.
\]

Then we can obtain the following regular point reduction theorem for RCH system, which explains the relationship between the RpCH-equivalence for regular point reducible RCH systems with symmetry and the RCH-equivalence for associated \(R_P\)-reduced RCH systems, its proof is given in Marsden et al [18]. This theorem can be regarded as an extension of regular point reduction theorem of Hamiltonian systems under regular controlled Hamiltonian equivalence conditions.

**Theorem 3.3** Two regular point reducible RCH systems \( (T^*Q_i, G_i, \omega_i, H_i, F_i, W_i) \), \( i = 1, 2 \), are RpCH-equivalent if and only if the associated \(R_P\)-reduced RCH systems \( ((T^*Q_i)_{\mu_i}, \omega_{i\mu_i}, h_{i\mu_i}, f_{i\mu_i}, W_{i\mu_i}) \), \( i = 1, 2 \), are RCH-equivalent.

### 3.2 Hamilton-Jacobi Theorem of \(R_P\)-reduced RCH System

In the following we first prove the Hamilton-Jacobi theorem of regular point reduced RCH system by using reduced symplectic form and Lemma 2.5. Then we give a theorem to state the relationship between solutions of Hamilton-Jacobi equations and regular point reduction. Moreover, we prove that the solution of Hamilton-Jacobi equation for regular point reducible RCH system with symmetry leaves invariant under the conditions of RpCH-equivalence.

At first, for the regular point reducible RCH system \( (T^*Q, G, \omega, H, F, W) \), we can prove the following Hamilton-Jacobi theorem for \(R_P\)-reduced RCH system \( ((T^*Q)_{\mu}, \omega_{\mu}, h_{\mu}, f_{\mu}, u_{\mu}) \). For convenience, the maps involved in the following theorem and its proof are shown in Diagram-3.

\[
\begin{align*}
\mathbf{J}^{-1}_1(\mu) & \xrightarrow{i_{\mu}} T^*Q \\
T^*Q & \xrightarrow{T^*T^*Q} T^*Q \\
T^*(T^*Q)_{\mu} & \xrightarrow{\pi_{\mu}} (T^*Q)_{\mu}
\end{align*}
\]

**Diagram-3**

**Theorem 3.4** (Hamilton-Jacobi Theorem of \(R_P\)-reduced RCH System) For a regular point reducible RCH system \( (T^*Q, G, \omega, H, F, W) \), assume that \( \gamma : Q \rightarrow T^*Q \) is an one-form on \( Q \), and \( \gamma \) is closed with respect to \( T\pi_Q : TT^*Q \rightarrow TQ \), and \( \bar{X} = X_{(T^*Q, G, \omega, H, F, u)} \) is the dynamical vector field of the regular point reducible RCH system \( (T^*Q, G, \omega, H, F, W) \) with a control law \( u \). Moreover, assume that \( \mu \in g^* \) is the regular reducible point of the RCH system, and \( \text{Im}(\gamma) \subset \mathbf{J}^{-1}(\mu) \), and it is \( G_{\mu} \)-invariant, and \( \bar{\gamma} = \pi_{\mu}(\gamma) : Q \rightarrow (T^*Q)_{\mu} \). Then the following two assertions are equivalent:
(i) $\dot{X}^\gamma$ and $\dot{X}_\mu$ are $\gamma$-related, that is, $T \gamma \cdot \dot{X}^\gamma = \dot{X}_\mu \cdot \dot{\gamma}$, where $\dot{X}_\mu = X_{((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, u_\mu)}$ is the dynamical vector field of $R_P$-reduced RCH system $((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, u_\mu)$;

(ii) $X_{h_\mu \cdot \gamma} + \text{vlift}(f_\mu \cdot \gamma) + \text{vlift}(u_\mu \cdot \gamma) = 0$, or $\dot{X}^\gamma$.

Where the equation that $X_{h_\mu \cdot \gamma} + \text{vlift}(f_\mu \cdot \gamma) + \text{vlift}(u_\mu \cdot \gamma) = 0$, is the Hamilton-Jacobi equation for $R_P$-reduced RCH system $((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, u_\mu)$, and $\dot{\gamma}$ is called a solution of the Hamilton-Jacobi equation.

**Proof:** We first prove that (i) implies (ii). By using the $R_P$-reduced symplectic form $\omega_\mu$, note that $\text{Im}(\gamma) \subset J^{-1}(\mu)$, and it is $G_\mu$-invariant, in this case $\pi^*_\mu \omega_\mu = \pi^*_Q \omega = \pi^*_Q \omega$, along $\text{Im}(\gamma)$. Thus, we take that $v = \dot{X} \cdot \gamma \in TT^*Q$, and for any $w \in TT^*Q$, and $T \pi_\mu \cdot w \neq 0$, from Lemma 2.5(i) we have that

$$
\omega_\mu(T \gamma \cdot \dot{X}^\gamma, T \pi_\mu \cdot w) = \omega_\mu(T(\pi_\mu \cdot \gamma) \cdot \dot{X}^\gamma, T \pi_\mu \cdot w) = \pi^*_\mu \omega_\mu(T \gamma \cdot \dot{X}^\gamma, w)
$$

Thus, for any $w \in TT^*Q$, and $T \pi_\mu \cdot w \neq 0$, from Lemma 2.5(i) we have that

$$
\omega_\mu(T \gamma \cdot \dot{X}^\gamma, T \pi_\mu \cdot w) = \omega_\mu(T(\pi_\mu \cdot \gamma) \cdot \dot{X}^\gamma, T \pi_\mu \cdot w) = \pi^*_\mu \omega_\mu(T \gamma \cdot \dot{X}^\gamma, w)
$$

where we used that $T \pi_\mu \cdot \dot{X} = T \pi_\mu \cdot X_{((T^*Q)_\mu, \omega_\mu, h_\mu, f_\mu, u_\mu)} = \dot{X}_\mu$. By assuming (i), $T \gamma \cdot \dot{X}^\gamma = \dot{X}_\mu \cdot \dot{\gamma}$, we can obtain that $\omega_\mu(\dot{X}_\mu \cdot \dot{\gamma}, T \gamma \cdot T \pi_\mu \cdot w) = 0$. On the other hand, note that $\dot{X}_\mu \cdot \dot{\gamma} = X_{h_\mu \cdot \gamma} + \text{vlift}(f_\mu \cdot \gamma) + \text{vlift}(u_\mu \cdot \gamma) = T \gamma \cdot X_{h_\mu \cdot \gamma} + T \gamma \cdot \text{vlift}(f_\mu \cdot \gamma) + T \gamma \cdot \text{vlift}(u_\mu \cdot \gamma)$, then we have that

$$
0 = \omega_\mu(\dot{X}_\mu \cdot \dot{\gamma}, T \gamma \cdot T \pi_\mu \cdot w)
$$

$$
= \omega_\mu(T \gamma \cdot (X_{h_\mu \cdot \gamma} + \text{vlift}(f_\mu \cdot \gamma) + \text{vlift}(u_\mu \cdot \gamma)), T \gamma \cdot T \pi_\mu \cdot w)
$$

$$
= \gamma^* \omega_\mu(X_{h_\mu \cdot \gamma} + \text{vlift}(f_\mu \cdot \gamma) + \text{vlift}(u_\mu \cdot \gamma), T \pi_\mu \cdot w)
$$

$$
= \gamma^* \pi^*_\mu \omega_\mu(X_{h_\mu \cdot \gamma} + \text{vlift}(f_\mu \cdot \gamma) + \text{vlift}(u_\mu \cdot \gamma), T \pi_\mu \cdot w)
$$

$$
= \gamma^* \pi^*_Q \omega(X_{h_\mu \cdot \gamma} + \text{vlift}(f_\mu \cdot \gamma) + \text{vlift}(u_\mu \cdot \gamma), T \pi_\mu \cdot w)
$$

$$
= -d\gamma(X_{h_\mu \cdot \gamma} + \text{vlift}(f_\mu \cdot \gamma) + \text{vlift}(u_\mu \cdot \gamma), T \pi_\mu \cdot w).
$$

It follows that either $X_{h_\mu \cdot \gamma} + \text{vlift}(f_\mu \cdot \gamma) + \text{vlift}(u_\mu \cdot \gamma) = 0$, or there is someone $\tilde{v} \in TT^*Q$, such that $X_{h_\mu \cdot \gamma} + \text{vlift}(f_\mu \cdot \gamma) + \text{vlift}(u_\mu \cdot \gamma) = T \pi_\mu \cdot \tilde{v}$. But from assuming (i), we have that

$$
T \gamma \cdot \dot{X}^\gamma = \dot{X}_\mu \cdot \dot{\gamma} = T \gamma \cdot (X_{h_\mu \cdot \gamma} + \text{vlift}(f_\mu \cdot \gamma) + \text{vlift}(u_\mu \cdot \gamma)),
$$

and from Lemma 2.5(ii), $T \gamma : TQ \to TT^*Q$ is injective with respect to $T \pi_\mu : TT^*Q \to TQ$, and hence, $T \gamma = T \pi_\mu \cdot T \gamma$ is injective with respect to $T \pi_\mu : TT^*Q \to TQ$, and $X_{h_\mu \cdot \gamma} + \text{vlift}(f_\mu \cdot \gamma) + \text{vlift}(u_\mu \cdot \gamma) = \dot{X}^\gamma$. Thus, (i) implies (ii).

Conversely, from the above arguments we have that for any $w \in TT^*Q$, and $T \pi_\mu \cdot w \neq 0$, then

$$
\omega_\mu(T \gamma \cdot \dot{X}^\gamma, T \pi_\mu \cdot w) = \omega_\mu(\dot{X}_\mu \cdot \dot{\gamma}, T \pi_\mu \cdot w)
$$

$$
= d\gamma(X_{h_\mu \cdot \gamma} + \text{vlift}(f_\mu \cdot \gamma) + \text{vlift}(u_\mu \cdot \gamma), T \pi_\mu \cdot w).
$$
Thus, since \( \omega_{\mu} \) is nondegenerate, the proof that (ii) implies (i) follows in the same way. ■

In particular, if both the external force and control of the regular point reducible RCH system \((T^*Q, G, \omega, H, F, u)\) are zero, in this case the RCH system is just a regular point reducible Hamiltonian system \((T^*Q, G, \omega, H)\). By using the same way of the proof of the above Theorem 3.4, we can also get the Hamilton-Jacobi theorem of Marsden-Weinstein reduced Hamiltonian system, which is given in Wang [25]. Thus, Theorem 3.4 can be regarded as an extension of Hamilton-Jacobi theorem for regular point reduced Hamiltonian system. Moreover, for a regular point reducible RCH system \((T^*Q, G, \omega, H, F, W)\) with a control law \( u \), we know that the dynamical vector fields \( X_{(T^*Q, G, \omega, H, F, u)} \) and \( \bar{X}_{((T^*Q)_{\mu}, \omega_{\mu}, f_{\mu}, u_{\mu})} \) are \( \pi_{\mu} \)-related, that is, \( X_{((T^*Q)_{\mu}, \omega_{\mu}, h_{\mu}, f_{\mu}, u_{\mu})} = T\pi_{\mu} \cdot X_{(T^*Q, G, \omega, H, F, u)} \cdot i_{\mu} \). Then we can prove the following Theorem 3.5, which states the relationship between the solutions of Hamilton-Jacobi equations and regular point reduction.

**Theorem 3.5** For a regular point reducible RCH system \((T^*Q, G, \omega, H, F, W)\) with a control law \( u \), assume that \( \gamma : Q \to T^*Q \) is an one-form on \( Q \), and \( \gamma \) is closed with respect to \( T\pi_Q : TT^*Q \to TQ \). Moreover, assume that \( \mu \in g^*_\gamma \) is the regular reducible point of the RCH system, and \( \text{Im}(\gamma) \subset J^{-1}_{\gamma}(\mu) \), and it is \( G_{\mu} \)-invariant, and \( \bar{\gamma} = \pi_{\mu}(\gamma) : Q \to (T^*Q)_{\mu} \). Then \( \gamma \) is a solution of Hamilton-Jacobi equation for the regular point reducible RCH system \((T^*Q, G, \omega, H, F, u)\) with a control law \( u \), if and only if \( \gamma = \pi_{\mu}(\gamma) : Q \to (T^*Q)_{\mu} \) is a solution of Hamilton-Jacobi equation for \( Rp \)-reduced RCH system \(((T^*Q)_{\mu}, \omega_{\mu}, h_{\mu}, f_{\mu}, u_{\mu})\).

**Proof:** In fact, from the proof of Theorem 2.6, we know that

\[
T_{\gamma} \cdot (X_{H_{\gamma}} + \text{vlift}(F \cdot \gamma) + \text{vlift}(u \cdot \gamma)) = \bar{X} \cdot \bar{\gamma} = T_{\gamma} \cdot \bar{X} \bar{\gamma},
\]
and from the proof of Theorem 3.4, we have that

\[
T_{\gamma} \cdot \bar{X} \bar{\gamma} = \bar{X}_{h_{\mu_{\gamma}}} \cdot \bar{\gamma} = T_{\gamma} \cdot (X_{h_{\mu_{\gamma}}} + \text{vlift}(f_{\mu_{\gamma}} + \text{vlift}(u_{\mu_{\gamma}})) + vlift(u_{\mu_{\gamma}} - \bar{\gamma})).
\]

Note that both maps \( T_{\gamma} : TQ \to TT^*Q \) and \( T\gamma = T\pi_{\mu} \cdot T_{\gamma} : TQ \to T(T^*Q)_{\mu} \) are injective with respect to \( T\pi_Q : TT^*Q \to TQ \). Thus,

\[
X_{H_{\gamma}} + \text{vlift}(F \cdot \gamma) + \text{vlift}(u \cdot \gamma) = 0, \quad \Leftrightarrow \quad \bar{X} \bar{\gamma} = 0, \quad \Leftrightarrow \quad X_{h_{\mu_{\gamma}}} + \text{vlift}(f_{\mu_{\gamma}} + \text{vlift}(u_{\mu_{\gamma}})) = 0.
\]
It follows that the conclusion of Theorem 3.5 holds. ■

Moreover, if considering the \( RpCH \)-equivalence of regular point reducible RCH systems, we can obtain the following Theorem 3.6, which states that the solution of Hamilton-Jacobi equation for regular point reducible RCH system with symmetry leaves invariant under the conditions of \( RpCH \)-equivalence.

**Theorem 3.6** Suppose that two regular point reducible RCH systems \((T^*Q_i, G_i, \omega_i, H_i, F_i, W_i)\), \( i = 1, 2 \), are \( RpCH \)-equivalent with an equivalent map \( \varphi : Q_1 \to Q_2 \).

(i) If \( \gamma_2 : Q_2 \to T^*Q_2 \) is a solution of the Hamilton-Jacobi equation for RCH system \((T^*Q_2, G_2, \omega_2, H_2, F_2, W_2)\), and \( \gamma_2 \) is closed with respect to \( T\pi_{Q_2} : TT^*Q_2 \to TQ_2 \), \( \gamma_1 = \varphi^* \cdot \gamma_2 \cdot \varphi : Q_1 \to T^*Q_1 \). Moreover, assume that \( \mu_i \in g^*_\gamma \), \( i = 1, 2 \), are the regular reducible points of the two RCH systems, and \( \text{Im}(\gamma_i) \subset J^{-1}_{\gamma_i}(\mu_i) \), and it is \( G_{\mu_i} \)-invariant, \( i = 1, 2 \). Then \( \gamma_1 = \varphi^* \cdot \gamma_2 \cdot \varphi \) is a solution of the Hamilton-Jacobi equation for RCH system \((T^*Q_1, G_1, \omega_1, H_1, F_1, W_1)\);
(ii) If $\gamma_1 : Q_1 \rightarrow T^*Q_1$ is a solution of the Hamilton-Jacobi equation for RCH system $(T^*Q_1, G_1, \omega_1, H_1, F_1, W_1)$, and $\gamma_1$ is closed with respect to $T\pi_{Q_1} : T^*Q_1 \rightarrow TQ_1$, $\gamma_2 = (\varphi^{-1})^* \cdot \gamma_1 : Q_2 \rightarrow T^*Q_2$. Moreover, assume that $\mu_i \in g_i^*$, $i = 1, 2$, are the regular reducible points of the two RCH systems, and $\text{Im}(\gamma_i) \subset J_i^{-1}(\mu_i)$, and it is $G_\mu$-invariant, $i = 1, 2$. Then $\gamma_2 = (\varphi^{-1})^* \cdot \gamma_1 : Q_2 \rightarrow T^*Q_2$ is a solution of the Hamilton-Jacobi equation for RCH system $(T^*Q_2, G_2, \omega_2, H_2, F_2, W_2)$.

**Proof:** We first prove the conclusion (i). If $(T^*Q_1, G_1, \omega_1, H_1, F_1, W_1) \sim_{RCH} (T^*Q_2, G_2, \omega_2, H_2, F_2, W_2)$, then from Definition 3.2 there exists a diffeomorphism $\varphi : Q_1 \rightarrow Q_2$, such that $\varphi^* : T^*Q_2 \rightarrow T^*Q_1$ is symplectic, and for $\mu_i \in g_i^*$, $i = 1, 2$, $\varphi^*_\mu = i_{\mu_1}^{-1} \cdot \varphi^* \cdot i_{\mu_2} : J_2^{-1}(\mu_2) \rightarrow J_1^{-1}(\mu_1)$ is $(G_{2\mu_2}, G_{1\mu_1})$-equivariant. From the following commutative Diagram-4:

![Diagram-4](image)

we have a well-defined symplectic map $\varphi^*_\mu / G : (T^*Q_2)_{\mu_2} \rightarrow (T^*Q_1)_{\mu_1}$, such that $\varphi^*_\mu / G \cdot \pi_{\mu_2} = \pi_{\mu_1} \cdot \varphi^*_\mu$, see Marsden et al [18]. Then from Theorem 3.3 we know that the associated $R_P$-reduced RCH systems $((T^*_Q)_{\mu_1}, \omega_{\mu_1}, h_{\mu_1}, f_{\mu_1}, W_{\mu_1})$, $i = 1, 2$, are RCH-equivalent with an equivalent map $\varphi^*_{\mu/G} : (T^*Q_2)_{\mu_2} \rightarrow (T^*Q_1)_{\mu_1}$. If $\gamma_2 : Q_2 \rightarrow T^*Q_2$ is a solution of the Hamilton-Jacobi equation for RCH system $(T^*Q_2, G_2, \omega_2, H_2, F_2, W_2)$, from Theorem 3.5 we know that $\gamma_2 = \pi_{\mu_2}(\gamma_2) : Q_2 \rightarrow (T^*Q_2)_{\mu_2}$ is a solution of Hamilton-Jacobi equation for $R_P$-reduced RCH system $((T^*_Q)_{\mu_2}, \omega_{\mu_2}, h_{\mu_2}, f_{\mu_2}, u_{\mu_2})$. Note that $\gamma_1 = \varphi^* \cdot \varphi^*_\mu / G : Q_1 \rightarrow T^*Q_1$, then $\gamma_1 = \pi_{\mu_1}(\gamma_1) = \varphi^* \cdot \varphi^*_\mu \cdot \gamma_2 : \varphi = \varphi^*_{\mu/G} \cdot \pi_{\mu_2} \cdot \gamma_2 : \varphi = \varphi^*_\mu / G \cdot \varphi$. From Theorem 2.7(i) we know that $\gamma_1 = \varphi^*_{\mu/G} \cdot \varphi$ is a solution of Hamilton-Jacobi equation for RCH-equivalent system $((T^*_Q)_{\mu_1}, \omega_{\mu_1}, h_{\mu_1}, f_{\mu_1}, u_{\mu_1})$, and hence from Theorem 3.5 we know that $\gamma_1 = \varphi^* \cdot \varphi$ is a solution of the Hamilton-Jacobi equation for RCH system $(T^*Q_1, G_1, \omega_1, H_1, F_1, W_1)$. It follows that the conclusion (i) of Theorem 3.6 holds.

Conversely, note that $\varphi : Q_1 \rightarrow Q_2$ is a diffeomorphism, by using the same way, we can also prove the conclusion (ii) of Theorem 3.6. □

**Remark 3.7** If $(T^*Q, \omega)$ is a connected symplectic manifold, and $J : T^*Q \rightarrow g^*$ is a non-equivariant momentum map with a non-equivariance group one-cocycle $\sigma : G \rightarrow g^*$, which is defined by $\sigma(g) := J(g \cdot z) - Ad^*_{g^{-1}} J(z)$, where $g \in G$ and $z \in T^*Q$. Then we know that $\sigma$ produces a new affine action $\Theta : G \times g^* \rightarrow g^*$ defined by $\Theta(g, \mu) := Ad^*_{g^{-1}} \mu + \sigma(g)$, where $\mu \in g^*$, with respect to which the given momentum map $J$ is equivariant. Assume that $G$ acts freely and properly on $T^*Q$, and $\tilde{G}_\mu$ denotes the isotropy subgroup of $\mu \in g^*$ relative to this affine action $\Theta$ and $\mu$ is a regular value of $J$. Then the quotient space $(T^*Q)_{\mu} = J^{-1}(\mu) / \tilde{G}_\mu$ is also a symplectic manifold with symplectic form $\omega_\mu$ uniquely characterized by (3.1), see Ortega and Ratiu [21] and Marsden et al [15]. In this case, we can also define the regular point reducible RCH system $(T^*Q, G, \omega, H, F, W)$ and RpCH-equivalence, and prove the Hamilton-Jacobi theorem for $R_P$-reduced RCH system $((T^*_Q)_{\mu}, \omega_{\mu}, h_{\mu}, f_{\mu}, u_{\mu})$ by using the above same way, and state that the solution of Hamilton-Jacobi equation for regular point reducible RCH system with symmetry leaves invariant under the conditions of RpCH-equivalence, where the $R_P$-reduced space $((T^*_Q)_{\mu}, \omega_{\mu})$ is determined by the affine action.
4 Hamilton-Jacobi Theorem of Regular Orbit Reduced RCH System

In this section, we first review some relevant definitions and basic facts about regular orbit reducible RCH systems and RoCH-equivalence, then we prove the Hamilton-Jacobi theorem of regular orbit reduced RCH system, by using reduced symplectic form and Lemma 2.5, and state the relationship between the solutions of Hamilton-Jacobi equations and regular orbit reduction, as well as that the solution of Hamilton-Jacobi equation for RCH system with symmetry leaves invariant under the conditions of RoCH-equivalence. We shall follow the notations and conventions introduced in Marsden et al [18], Wang [25].

4.1 Regular Orbit Reducible RCH System

Let $\Phi : G \times Q \to Q$ be a smooth left action of the Lie group $G$ on $Q$, which is free and proper. Then the cotangent lifted left action $\Phi^{T^*} : G \times T^*Q \to T^*Q$ is symplectic, free and proper, and admits a $\text{Ad}^*$-equivariant momentum map $J : T^*Q \to \mathfrak{g}^*$. Assume that $\mu \in \mathfrak{g}^*$ is a regular value of the momentum map $J$ and $\mathcal{O}_\mu = G \cdot \mu \subset \mathfrak{g}^*$ is the $G$-orbit of the coadjoint $G$-action through the point $\mu$. Since $G$ acts freely, properly and symplectically on $T^*Q$, then the quotient space $(T^*Q)_{\mathcal{O}_\mu} = J^{-1}(\mathcal{O}_\mu)/G$ is a regular quotient symplectic manifold with the symplectic form $\omega_{\mathcal{O}_\mu}$ uniquely characterized by the relation

$$i_{\mathcal{O}_\mu} \omega = \pi^*_{\mathcal{O}_\mu} \omega_{\mathcal{O}_\mu} + J^*_{\mathcal{O}_\mu} \omega^+_{\mathcal{O}_\mu}, \quad (4.1)$$

where $J_{\mathcal{O}_\mu}$ is the restriction of the momentum map $J$ to $J^{-1}(\mathcal{O}_\mu)$, that is, $J_{\mathcal{O}_\mu} = J \circ i_{\mathcal{O}_\mu}$ and $\omega^+_{\mathcal{O}_\mu}$ is the $+$-symplectic structure on the orbit $\mathcal{O}_\mu$ given by

$$\omega^+_{\mathcal{O}_\mu}(\nu)(\xi g^*(\nu), \eta g^*(\nu)) = \langle \nu, [\xi, \eta] \rangle, \quad \forall \nu \in \mathcal{O}_\mu, \xi, \eta \in \mathfrak{g}. \quad (4.2)$$

The maps $i_{\mathcal{O}_\mu} : J^{-1}(\mathcal{O}_\mu) \to T^*Q$ and $\pi_{\mathcal{O}_\mu} : J^{-1}(\mathcal{O}_\mu) \to (T^*Q)_{\mathcal{O}_\mu}$ are natural injection and the projection, respectively. The pair $((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is called the symplectic orbit reduced space of $(T^*Q, \omega)$. Let $H : T^*Q \to \mathbb{R}$ be a $G$-invariant Hamiltonian, the flow $F_t$ of the Hamiltonian vector field $X_H$ leaves the connected components of $J^{-1}(\mathcal{O}_\mu)$ invariant and commutes with the $G$-action, so it induces a flow $f^H_{t_{\mathcal{O}_\mu}}$ on $(T^*Q)_{\mathcal{O}_\mu}$, defined by $f^H_{t_{\mathcal{O}_\mu}} \circ \pi_{\mathcal{O}_\mu} = \pi_{\mathcal{O}_\mu} \circ F_t \circ i_{\mathcal{O}_\mu}$, and the vector field $X_{h_{\mathcal{O}_\mu}}$ generated by the flow $f^H_{t_{\mathcal{O}_\mu}}$ on $((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is Hamiltonian with the associated regular orbit reduced Hamiltonian function $h_{\mathcal{O}_\mu} : (T^*Q)_{\mathcal{O}_\mu} \to \mathbb{R}$ defined by $h_{\mathcal{O}_\mu} \circ \pi_{\mathcal{O}_\mu} = H \circ i_{\mathcal{O}_\mu}$ and the Hamiltonian vector fields $X_H$ and $X_{h_{\mathcal{O}_\mu}}$ are $\pi_{\mathcal{O}_\mu}$-related.

When $Q = G$ is a Lie group with Lie algebra $\mathfrak{g}$, and the $G$-action is the cotangent lift of left translation, then the associated momentum map $J_L : T^*G \to \mathfrak{g}^*$ is right invariant. In the same way, the momentum map $J_R : T^*G \to \mathfrak{g}^*$ for the cotangent lift of right translation is left invariant. For regular value $\mu \in \mathfrak{g}^*$, $\mathcal{O}_\mu = G \cdot \mu = \{ \text{Ad}_{g^{-1}}^\mu | g \in G \}$ and the Kostant-Kirillov-Souriau(KKS) symplectic forms on coadjoint orbit $\mathcal{O}_\mu(\subset \mathfrak{g}^*)$ are given by

$$\omega_{\mathcal{O}_\mu}(\nu)(\text{ad}^*_\xi(\nu), \text{ad}^*_\eta(\nu)) = - \langle \nu, [\xi, \eta] \rangle, \quad \forall \nu \in \mathcal{O}_\mu, \xi, \eta \in \mathfrak{g}. \quad (4.3)$$

From Ortega and Ratiu [21], we know that by using the momentum map $J_R$ can induce a symplectic diffeomorphism from the symplectic point reduced space $((T^*G)_\mu, \omega_\mu)$ to the symplectic orbit space $(\mathcal{O}_\mu, \omega^-_{\mathcal{O}_\mu})$. In general case, we maybe thought that the structure of the symplectic orbit reduced space $((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is more complex than that of the symplectic point reduced
space \((T^*Q)_\mu, \omega_\mu\), but, from the regular reduction diagram, we know that the regular orbit reduced space \(((T^*Q)_{O_\mu}, \omega_{O_\mu})\) is symplectic diffeomorphic to the regular point reduced space \(((T^*Q)_\mu, \omega_\mu)\), and hence is also symplectic diffeomorphic to a symplectic fiber bundle, see Marsden et al \cite{Marsden18}. Thus, we can introduce a kind of the regular orbit reducible RCH systems as follows.

**Definition 4.1** (Regular Orbit Reducible RCH System) A 6-tuple \((T^*Q, G, \omega, H, F, W)\), where the Hamiltonian \(H : T^*Q \to \mathbb{R}\), the fiber-preserving map \(F : T^*Q \to T^*Q\) and the fiber submanifold \(W\) of \(T^*Q\) are all \(G\)-invariant, is called a regular orbit reducible RCH system, if there exists an orbit \(O_\mu, \mu \in \mathfrak{g}^*\), where \(\mu\) is a regular value of the momentum map \(\mathfrak{J}\), such that the regular orbit reduced system, that is, the 5-tuple \(((T^*Q)_{O_\mu}, \omega_{O_\mu}, h_{O_\mu}, f_{O_\mu}, W_{O_\mu})\), where \((T^*Q)_{O_\mu} = J^{-1}(O_\mu)/G, \pi_{O_\mu}^\omega = i^*_\mu \omega - J^*_\mu \omega_{O_\mu}, h_{O_\mu} = H \cdot i^*_\mu, f_{O_\mu} = \pi_{O_\mu} = \pi_{O_\mu} \cdot F \cdot i^*_\mu, W \subset J^{-1}(O_\mu), W_{O_\mu} = \pi_{O_\mu}(W)\), is a RCH system. Where \((T^*Q)_{O_\mu}, \omega_{O_\mu}\) is the \(R_0\)-reduced CH system. Moreover, for the regular orbit reducible RCH system we can also introduce the regular orbit reducible controlled Hamiltonian equivalence (RoCH-equivalence) as follows.

Denote by \(X_{(T^*Q, G, \omega, H, F, W,u)}\) the vector field of the regular orbit reducible RCH system \((T^*Q, G, \omega, H, F, W)\) with a control law \(u\), then

\[
X_{(T^*Q, G, \omega, H, F, W,u)} = (dH)^\sharp + vlift(F) + vlift(u). \tag{4.3}
\]

Moreover, for the regular orbit reducible RCH system we can also introduce the regular orbit reducible controlled Hamiltonian equivalence (RoCH-equivalence) as follows.

**Definition 4.2** (RoCH-equivalence) Suppose that we have two regular orbit reducible RCH systems \((T^*Q_1, G_1, \omega_1, H_1, F_1, W_1)\), \(i = 1, 2\), we say them to be RoCH-equivalent, or simply \((T^*Q_1, G_1, \omega_1, H_1, F_1, W_1) \sim \text{RoCH} (T^*Q_2, G_2, \omega_2, H_2, F_2, W_2)\), if there exists a diffeomorphism \(\varphi : Q_1 \to Q_2\) such that the following Hamiltonian matching conditions hold:

**RoHM-1:** The cotangent lift map \(\varphi^* : T^*Q_2 \to T^*Q_1\) is symplectic.

**RoHM-2:** For \(O_{\mu_i}, \mu_i \in \mathfrak{g}^*_i\), the regular reducible orbits of RCH systems \((T^*Q_i, G_i, \omega_i, H_i, F_i, W_i)\), \(i = 1, 2\), the map \(\varphi_{O_{\mu_2}}^* = \varphi_{O_{\mu_1}}^* \cdot \varphi^* : J^*_2(O_{\mu_2}) \to J^*_1(O_{\mu_1})\) is \((G_2, G_1)\)-equivariant, \(W_1 = \varphi_{O_{\mu_2}}^*(W_2)\), and \(J^*_{2\omega_{\mu_2}} \omega_{\mu_2} = (\varphi_{O_{\mu_1}}^* \cdot J^*_{1\omega_{\mu_1}}) \omega_{\mu_1}\), where \(\mu = (\mu_1, \mu_2)\), and denote by \(i^*_{O_{\mu_1}}(S)\) the preimage of a subset \(S \subset T^*Q_1\) for the map \(i^*_{O_{\mu_1}} : J^*_1(O_{\mu_1}) \to T^*Q_1\).

**RoHM-3:** \(\text{Im}[(dH_1)^\sharp + vlift(F_1) - ((\varphi^*)^* dH_2)^\sharp - vlift(\varphi^* F_2 \varphi^*)] \subset vlift(W_1)\).

It is worthy of note that for the regular orbit reducible RCH system, the induced equivalence map \(\varphi^*\) not only keeps the symplectic structure and the restriction of the \(+\)-symplectic structure on the regular orbit to \(J^{-1}(O_\mu)\), but also keeps the equivariance of \(G\)-action on the regular orbit. If a feedback control law \(u_{O_\mu} : (T^*Q)_{O_\mu} \to W_{O_\mu}\) is chosen, the \(R_0\)-reduced RCH system \(((T^*Q)_{O_\mu}, \omega_{O_\mu}, h_{O_\mu}, f_{O_\mu}, u_{O_\mu})\) is a closed-loop regular dynamic system with a control law \(u_{O_\mu}\). Assume that its vector field \(X_{((T^*Q)_{O_\mu}, \omega_{O_\mu}, h_{O_\mu}, f_{O_\mu}, u_{O_\mu})}\) can be expressed by

\[
X_{((T^*Q)_{O_\mu}, \omega_{O_\mu}, h_{O_\mu}, f_{O_\mu}, u_{O_\mu})} = (dh_{O_\mu})^\sharp + vlift(f_{O_\mu}) + vlift(u_{O_\mu}), \tag{4.4}
\]

where \((dh_{O_\mu})^\sharp = X_{h_{O_\mu}}, vlift(f_{O_\mu}) = vlift(f_{O_\mu})X_{h_{O_\mu}}, vlift(u_{O_\mu}) = vlift(u_{O_\mu})X_{h_{O_\mu}}, \) and satisfies the condition

\[
X_{((T^*Q)_{O_\mu}, \omega_{O_\mu}, h_{O_\mu}, f_{O_\mu}, u_{O_\mu})} \cdot \pi_{O_\mu} = T \pi_{O_\mu} \cdot X_{(T^*Q, G, \omega, H, F, W,u)} \cdot i_{O_\mu}. \tag{4.5}
\]
Then we can obtain the following regular orbit reduction theorem for RCH system, which explains the relationship between the RoCH-equivalence for the regular orbit reducible RCH systems with symmetry and the RCH-equivalence for associated $R_O$-reduced RCH systems, its proof is given in Marsden et al [18]. This theorem can be regarded as an extension of regular orbit reduction theorem of Hamiltonian systems under regular controlled Hamiltonian equivalence conditions.

**Theorem 4.3** If two regular orbit reducible RCH systems $(T^*Q_i, G_i, \omega_i, H_i, F_i, W_i), i = 1, 2$, are RoCH-equivalent, then their associated $R_O$-reduced RCH systems $((T^*Q)_{\mu_i}, \omega_i, \nu_i, h_i, f_i, \omega_i_{\mu_i}), i = 1, 2$, must be RCH-equivalent. Conversely, if $R_O$-reduced RCH systems $((T^*Q)_{\mu_i}, j_{i \mu}, \omega_i_{\mu_i}, \nu_i_{\mu_i}), i = 1, 2$, are RCH-equivalent and the induced map $\varphi^*_{\mu_i} : J_2^{-1}(\mathcal{O}_{\mu_2}) \to J_1^{-1}(\mathcal{O}_{\mu_1})$, such that $J_{20}^{\omega_{\mu_2}} \omega_{\mu_2} = (\varphi^*_{\mu_i})^* \cdot J_1^{\omega_{\mu_1}} \omega_{\mu_1}$, then the regular orbit reducible RCH systems $(T^*Q_i, G_i, \omega_i, H_i, F_i, W_i), i = 1, 2$, are RoCH-equivalent.

**4.2 Hamilton-Jacobi Theorem of $R_O$-reduced RCH System**

In the following we first prove the Hamilton-Jacobi theorem of regular orbit reduced RCH system by using reduced symplectic form and Lemma 2.5. Then we give a theorem to state the relationship between solutions of Hamilton-Jacobi equations and regular orbit reduction. Moreover, we prove that the solution of Hamilton-Jacobi equation for regular orbit reducible RCH system with symmetry leaves invariant under the conditions of RoCH-equivalence.

At first, for the regular orbit reducible RCH system $(T^*Q, G, \omega, H, F, W)$, we can prove the following Hamilton-Jacobi theorem for $R_O$-reduced RCH system $((T^*Q)_{\mu_i}, \omega_{\mu_i}, h_{\mu_i}, f_{\mu_i}, u_{\mu_i})$. For convenience, the maps involved in the following theorem and its proof are shown in Diagram 5.

![Diagram 5](image)

**Theorem 4.4** (Hamilton-Jacobi Theorem of $R_O$-reduced RCH System) For a regular orbit reducible RCH system $(T^*Q, G, \omega, H, F, W)$, assume that $\gamma : Q \to T^*Q$ is an one-form on $Q$, and $\gamma$ is closed with respect to $T\pi_Q : TT^*Q \to TQ$, and $\tilde{X}^\gamma = T\pi_Q \tilde{X} \cdot \gamma$, where $\tilde{X} = X_{(T^*Q, G, \omega, H, F, W)}$ is the dynamical vector field of the regular orbit reducible RCH system $(T^*Q, G, \omega, H, F, W)$ with a control law $u$. Moreover, assume that $O_{\mu_i}, \mu_i \in g^*$ is the regular reducible orbit of the RCH system, and $\text{Im}(\gamma) \subset J^{-1}(\mu_i)$, and it is $G$-invariant, and $\tilde{\gamma} = \pi_{O_{\mu_i}}(\gamma) : Q \to (T^*Q)_{\mu_i}$. Then the following two assertions are equivalent:

(i) $\tilde{X}^\gamma$ and $\tilde{X}_{O_{\mu_i}}$ are $\tilde{\gamma}$-related, that is, $T\tilde{\gamma} \cdot \tilde{X}^\gamma = \tilde{X}_{O_{\mu_i}} \cdot \tilde{\gamma}$, where $\tilde{X}_{O_{\mu_i}} = X_{((T^*Q)_{\mu_i}, \omega_{\mu_i}, h_{\mu_i}, f_{\mu_i}, u_{\mu_i})}$ is the dynamical vector field of $R_O$-reduced RCH system $((T^*Q)_{\mu_i}, \omega_{\mu_i}, h_{\mu_i}, f_{\mu_i}, u_{\mu_i})$;

(ii) $X_{\mu_i} \cdot \tilde{\gamma} + \text{vlift}(f_{\mu_i} \cdot \tilde{\gamma}) + \text{vlift}(u_{\mu_i} \cdot \tilde{\gamma}) = 0$, or $\tilde{X}^\gamma$. 

1
Where the equation that \( X_{h_{O_\mu}} \cdot \bar{\gamma} + \text{vlift}(f_{O_\mu} \cdot \bar{\gamma}) + \text{vlift}(u_{O_\mu} \cdot \bar{\gamma}) = 0 \), is the Hamilton-Jacobi equation for the \( R_0 \)-reduced RCH system \(((T^*Q)_{O_\mu}, \omega_{O_\mu}, h_{O_\mu}, f_{O_\mu}, u_{O_\mu})\), and \( \bar{\gamma} \) is called a solution of the Hamilton-Jacobi equation.

**Proof:** Note that the \( R_0 \)-reduced space \((T^*Q)_{O_\mu} = J^{-1}(O_\mu)/G \cong J^{-1}(\mu)/G \times O_\mu\), with the reduced symplectic form \( \omega_{O_\mu} \) uniquely characterized by the relation \( i_{\mu}^* \omega = \pi^*_{O_\mu} \omega + J_{O_\mu}^* \bar{\gamma} \). Since \( \text{Im}(\gamma) \subset J^{-1}(\mu) \), and it is \( G \)-invariant, in this case for any \( V \in TQ \), and \( w \in TT^*Q \), we have that \( J_{O_\mu}^* \omega_{O_\mu} = 0 \), and hence \( i_{\mu}^* \omega_{O_\mu} = i_{\mu}^* \omega = \pi^*_{O_\mu} \omega \), along \( \text{Im}(\gamma) \). In the following we first prove that (i) implies (iii). We take that \( v = \bar{X} \cdot \gamma \in TT^*Q \), and for any \( w \in TT^*Q \), and \( T\pi_{O_\mu} \cdot w \neq 0 \), from Lemma 2.5(i) we have that

\[
\omega_{O_\mu}(T \bar{\gamma} \cdot \bar{X}, T \pi_{O_\mu} \cdot w) = \omega_{O_\mu}(T(\pi_{O_\mu} \cdot \gamma) \cdot \bar{X}, T \pi_{O_\mu} \cdot w) = \pi^*_{O_\mu} \omega_{O_\mu}(T \gamma \cdot \bar{X}, T \pi_{O_\mu} \cdot w) = \pi^*_{O_\mu} \omega_{O_\mu}(T(\gamma \cdot \bar{X}, T \pi_{O_\mu} \cdot w) = \pi^*_{O_\mu} \omega_{O_\mu}(T(\gamma \cdot \bar{X}, T \pi_{O_\mu} \cdot w) = \pi^*_{O_\mu} \omega_{O_\mu}(T(\gamma \cdot \bar{X}, T \pi_{O_\mu} \cdot w) = \pi^*_{O_\mu} \omega_{O_\mu}(T(\gamma \cdot \bar{X}, T \pi_{O_\mu} \cdot w) = \pi^*_{O_\mu} \omega_{O_\mu}(T(\gamma \cdot \bar{X}, T \pi_{O_\mu} \cdot w) = \pi^*_{O_\mu} \omega_{O_\mu}(T(\gamma \cdot \bar{X}, T \pi_{O_\mu} \cdot w)
\]

where we used that \( T\pi_{O_\mu}(\bar{X}) = T\pi_{O_\mu} \cdot X(T^*Q,G,w,H,F,w) = X((T^*Q)_{O_\mu},\omega_{O_\mu},h_{O_\mu},f_{O_\mu},u_{O_\mu}) = \tilde{X}_{O_\mu}. \) By assuming (i), \( T \tilde{\gamma} \cdot \tilde{X} \gamma = \tilde{X}_{O_\mu} \cdot \gamma \), we can obtain that \( \omega_{O_\mu}(\tilde{X}_{O_\mu} \cdot \gamma, \tilde{T} \cdot \pi_{O_\mu} \cdot w) = 0 \). On the other hand, note that \( \tilde{X}_{O_\mu} \cdot \gamma = X_{h_{O_\mu}} \cdot \gamma + \text{vlift}(f_{O_\mu} \cdot \bar{\gamma}) + \text{vlift}(u_{O_\mu} \cdot \bar{\gamma}) = \tilde{T} \gamma \cdot X_{h_{O_\mu}} \cdot \gamma + \tilde{T} \tilde{\gamma} \cdot \text{vlift}(f_{O_\mu} \cdot \bar{\gamma}) + \tilde{T} \tilde{\gamma} \cdot \text{vlift}(u_{O_\mu} \cdot \bar{\gamma}) \), then we have that

\[
0 = \omega_{O_\mu}(\tilde{X}_{O_\mu} \cdot \gamma, T \tilde{T} \cdot \pi_{O_\mu} \cdot w)
= \omega_{O_\mu}(T \tilde{T} \gamma \cdot (X_{h_{O_\mu}} \cdot \gamma + \text{vlift}(f_{O_\mu} \cdot \bar{\gamma}) + \text{vlift}(u_{O_\mu} \cdot \bar{\gamma})), T \tilde{T} \cdot \pi_{O_\mu} \cdot w)
= \tilde{\gamma}^* \omega_{O_\mu}(X_{h_{O_\mu}} \cdot \gamma + \text{vlift}(f_{O_\mu} \cdot \bar{\gamma}) + \text{vlift}(u_{O_\mu} \cdot \bar{\gamma}), T \pi_{O_\mu} \cdot w)
= \gamma^* \cdot \pi^*_{O_\mu} \omega_{O_\mu}(X_{h_{O_\mu}} \cdot \gamma + \text{vlift}(f_{O_\mu} \cdot \bar{\gamma}) + \text{vlift}(u_{O_\mu} \cdot \bar{\gamma}), T \pi_{O_\mu} \cdot w)
= \gamma^* \cdot \pi^*_{O_\mu} \omega_{O_\mu}(X_{h_{O_\mu}} \cdot \gamma + \text{vlift}(f_{O_\mu} \cdot \bar{\gamma}) + \text{vlift}(u_{O_\mu} \cdot \bar{\gamma}), T \pi_{O_\mu} \cdot w)
= -d\gamma(X_{h_{O_\mu}} \cdot \gamma + \text{vlift}(f_{O_\mu} \cdot \bar{\gamma}) + \text{vlift}(u_{O_\mu} \cdot \bar{\gamma}), T \pi_{O_\mu} \cdot w)
\]

It follows that either \( X_{h_{O_\mu}} \cdot \gamma + \text{vlift}(f_{O_\mu} \cdot \bar{\gamma}) + \text{vlift}(u_{O_\mu} \cdot \bar{\gamma}) = 0 \), or there is someone \( \tilde{\tilde{v}} \in TT^*Q \), such that \( X_{h_{O_\mu}} \cdot \gamma + \text{vlift}(f_{O_\mu} \cdot \bar{\gamma}) + \text{vlift}(u_{O_\mu} \cdot \bar{\gamma}) = T \pi_{O_\mu} \cdot \tilde{\tilde{v}} \). But from assuming (i), we have that

\[
T \tilde{\gamma} \cdot \tilde{X} \gamma = \tilde{X}_{O_\mu} \cdot \gamma = T \tilde{\gamma} \cdot (X_{h_{O_\mu}} \cdot \gamma + \text{vlift}(f_{O_\mu} \cdot \bar{\gamma}) + \text{vlift}(u_{O_\mu} \cdot \bar{\gamma})),
\]

and from Lemma 2.5(ii), \( T \gamma : TQ \rightarrow TT^*Q \) is injective with respect to \( T\pi_Q : TT^*Q \rightarrow TQ \), and hence, \( T \tilde{\gamma} = T\pi_{O_\mu} \cdot \tilde{\gamma} \) is injective with respect to \( T\pi_Q : TT^*Q \rightarrow TQ \), and \( X_{O_\mu} \cdot \gamma + \text{vlift}(f_{O_\mu} \cdot \bar{\gamma}) + \text{vlift}(u_{O_\mu} \cdot \bar{\gamma}) = \tilde{X} \gamma \). Thus, (i) implies (ii).

Conversely, from the above arguments we have that for any \( w \in TT^*Q \), and \( T\pi_{O_\mu} \cdot w \neq 0 \), then

\[
\omega_{O_\mu}(T \tilde{T} \gamma \cdot \tilde{X} \gamma, T \pi_{O_\mu} \cdot w) = \omega_{O_\mu}(\tilde{X}_{O_\mu} \cdot \gamma, T \pi_{O_\mu} \cdot w)
= d\gamma(X_{h_{O_\mu}} \cdot \gamma + \text{vlift}(f_{O_\mu} \cdot \bar{\gamma}) + \text{vlift}(u_{O_\mu} \cdot \bar{\gamma}), T \pi_{O_\mu} \cdot w).
\]
Thus, since $\omega_{O_\mu}$ is nondegenerate, the proof that (ii) implies (i) follows in the same way.

In particular, if both the external force and control of the regular orbit reducible RCH system $(T^*Q, G, \omega, H, F, u)$ are zero, in this case the RCH system is just a regular orbit reducible Hamiltonian system $(T^*Q, G, \omega, H)$. By using the same way of the proof of the above Theorem 4.4, we can also get the Hamilton-Jacobi theorem of $R_O$-reduced Hamiltonian system, which is given in Wang [25]. Thus, Theorem 4.4 can be regarded as an extension of Hamilton-Jacobi theorem for regular orbit reduced Hamiltonian system. Moreover, for a regular orbit reducible RCH system $(T^*Q, G, \omega, H, F, W)$ with a control law $u$, we know that the dynamical vector fields $X_{(T^*Q,G,\omega,H,F,u)}$ and $X_{((T^*Q)_{O_\mu},\omega_{O_\mu},h_{O_\mu},f_{O_\mu},u_{O_\mu})}$ are $\pi_{O_\mu}$-related, that is,

$$X_{((T^*Q)_{O_\mu},\omega_{O_\mu},h_{O_\mu},f_{O_\mu},u_{O_\mu})} \cdot \pi_{O_\mu} = T\pi_{O_\mu} \cdot X_{(T^*Q,G,\omega,H,F,u)} \cdot i_{O_\mu}. $$

Then we can prove the following Theorem 4.5, which states the relationship between the solutions of Hamilton-Jacobi equations and regular orbit reduction.

**Theorem 4.5** For a regular orbit reducible RCH system $(T^*Q, G, \omega, H, F, W)$ with a control law $u$, assume that $\gamma: Q \to T^*Q$ is an one-form on $Q$, and $\gamma$ is closed with respect to $T\pi_Q: TT^*Q \to TQ$. Moreover, assume that $O_{\mu}, \mu \in g^*$ is the regular reducible orbit of the RCH system, and $\text{Im}(\gamma) \subset J^{-1}(\mu)$, and it is $G$-invariant, and $\tilde{\gamma} = \pi_{O_\mu}(\gamma): Q \to (T^*Q)_{O_\mu}$. Then $\gamma$ is a solution of Hamilton-Jacobi equation for the regular orbit reducible RCH system $(T^*Q, G, \omega, H, F, u)$ with a control law $u$, if and only if $\tilde{\gamma} = \pi_{O_\mu}(\gamma): Q \to (T^*Q)_{O_\mu}$ is a solution of Hamilton-Jacobi equation for $R_O$-reduced RCH system $((T^*Q)_{O_\mu}, \omega_{O_\mu}, h_{O_\mu}, f_{O_\mu}, u_{O_\mu})$.

**Proof:** In fact, from the proof of Theorem 2.6, we know that

$$T\gamma \cdot (X_{H,\gamma} + \text{vlift}(F \cdot \gamma) + \text{vlift}(u \cdot \gamma)) = \tilde{X} \cdot \gamma = T\gamma \cdot \tilde{X} \gamma,$$

and from the proof of Theorem 4.4, we have that

$$T\tilde{\gamma} \cdot \tilde{X} \gamma = \tilde{X}_{\tilde{O}_\mu} \cdot \tilde{\gamma} = T\tilde{\gamma} \cdot (X_{h_{O_\mu}, \gamma} + \text{vlift}(f_{O_\mu} \cdot \gamma) + \text{vlift}(u_{O_\mu} \cdot \gamma)).$$

Note that both maps $T\gamma: TQ \to TT^*Q$ and $T\tilde{\gamma} = T\pi_{O_\mu} \cdot T\gamma: TQ \to T(T^*Q)_{O_\mu}$ are injective with respect to $T\pi_Q: TT^*Q \to TQ$. Thus,

$$X_{H, \gamma} + \text{vlift}(F \cdot \gamma) + \text{vlift}(u \cdot \gamma) = 0 \iff \tilde{X} \gamma = 0 \iff X_{h_{O_\mu}, \gamma} + \text{vlift}(f_{O_\mu} \cdot \gamma) + \text{vlift}(u_{O_\mu} \cdot \gamma) = 0.$$ It follows that the conclusion of Theorem 4.5 holds.

Moreover, if considering the RoCH-equivalence of regular orbit reducible RCH systems, we can obtain the following Theorem 4.6, which states that the solution of Hamilton-Jacobi equation for regular orbit reducible RCH system with symmetry leaves invariant under the conditions of RoCH-equivalence.

**Theorem 4.6** Suppose that two regular orbit reducible RCH systems $(T^*Q_i, G_i, \omega_i, H_i, F_i, W_i)$, $i = 1, 2$, are RoCH-equivalent with an equivalent map $\varphi: Q_1 \to Q_2$.

(i) If $\gamma_2: Q_2 \to T^*Q_2$ is a solution of the Hamilton-Jacobi equation for RCH system $(T^*Q_2, G_2, \omega_2, H_2, F_2, W_2)$, and $\gamma_2$ is closed with respect to $T\pi_{Q_2}: TT^*Q_2 \to TQ_2$, $\gamma_1 = \varphi^* \gamma_2 \cdot \varphi: Q_1 \to T^*Q_1$.

Moreover, assume that $O_{\mu_i}, \mu_i \in g_i^*; i = 1, 2$, are the regular reducible orbits of the two RCH systems, and $\text{Im}(\gamma_i) \subset J_i^{-1}(\mu_i)$, and it is $G_i$-invariant, $i = 1, 2$. Then $\gamma_1 = \varphi^* \gamma_2 \cdot \varphi$ is a solution of the Hamilton-Jacobi equation for RCH system $(T^*Q_1, G_1, \omega_1, H_1, F_1, W_1)$;
(ii) If $\gamma_1 : Q_1 \to T^*Q_1$ is a solution of the Hamilton-Jacobi equation for RCH system $(T^*Q_1, G_1, \omega_1, H_1, F_1, W_1)$, and $\gamma_1$ is closed with respect to $T\pi_{Q_1} : T^*Q_1 \to TQ_1$, $\gamma_2 = (\varphi^{-1})^* \cdot \gamma_1 \cdot \varphi^{-1} : Q_2 \to T^*Q_2$. Moreover, assume that $\mathcal{O}_{\mu_i}$, $\mu_i \in g^*_i$, $i = 1, 2$, are the regular reducible orbits of the two RCH systems, and $\text{Im}(\gamma_i) \subset J_i^{-1}(\mu_i)$, and it is $G_i$-invariant, $i = 1, 2$. Then $\gamma_2 = (\varphi^{-1})^* \cdot \gamma_1 \cdot \varphi^{-1}$ is a solution of the Hamilton-Jacobi equation for RCH system $(T^*Q_2, G_2, \omega_2, H_2, F_2, W_2)$.

**Proof:** We first prove the conclusion (i). If $(T^*Q_1, G_1, \omega_1, H_1, F_1, W_1) \overset{\text{RCH}}{\sim} (T^*Q_2, G_2, \omega_2, H_2, F_2, W_2)$, then from Definition 4.2 there exists a diffeomorphism $\varphi : Q_1 \to Q_2$, such that $\varphi^* : T^*Q_2 \to T^*Q_1$ is symplectic, and for $\mu_i \in g^*_i$, $i = 1, 2$, $\varphi^* \mathcal{O}_{\mu_i} = i_{\mathcal{O}_{\mu_i}}^* \mathbf{J}_2^{-1}(\mathcal{O}_{\mu_2}) \to \mathcal{J}_1^{-1}(\mathcal{O}_{\mu_1})$ is $(G_2, G_1)$-equivariant, $\mathcal{J}_2^* \mathbf{J}_2^{-1}(\mathcal{O}_{\mu_2}) = (\varphi^*_\mathcal{O}_{\mu_i})^* \mathbf{J}_1^* \mathbf{J}_1^{-1}(\mathcal{O}_{\mu_1}) \mathbf{J}_1^* \mathbf{J}_1^* \mathcal{O}_{\mu_1} \mathbf{J}_2^* \mathcal{O}_{\mu_2}$. From the following commutative Diagram 6,

$$
\begin{array}{ccc}
Q_2 & \xrightarrow{\gamma_2} & T^*Q_2 \\
\varphi \downarrow & & \varphi \downarrow \\
Q_1 & \xrightarrow{\gamma_1} & T^*Q_1
\end{array}
$$

Diagram 6

we have a well-defined symplectic map $\varphi^*_{\mathcal{O}_{\mu_i}/G} : (T^*Q_2) \mathcal{O}_{\mu_2} \to (T^*Q_1) \mathcal{O}_{\mu_1}$, such that $\varphi^*_{\mathcal{O}_{\mu_i}/G} \cdot \pi_{\mathcal{O}_{\mu_i}} = \pi_{\mathcal{O}_{\mu_1}} \cdot \varphi^*_{\mathcal{O}_{\mu_i}}$, see Marsden et al [18]. Then from Theorem 3.3 we know that the associated $RO$-reduced RCH systems $((T^*Q) \mathcal{O}_{\mu_i}, \omega_{\mathcal{O}_{\mu_i}}, h_{\mathcal{O}_{\mu_i}}, f_{\mathcal{O}_{\mu_i}}, W_{\mathcal{O}_{\mu_i}})$, $i = 1, 2$, are RCH-equivalent with an equivalent map $\varphi^*_{\mathcal{O}_{\mu_i}/G} : (T^*Q_2) \mathcal{O}_{\mu_2} \to (T^*Q_1) \mathcal{O}_{\mu_1}$. If $\gamma_2 : Q_2 \to T^*Q_2$ is a solution of the Hamilton-Jacobi equation for RCH system $(T^*Q_2, G_2, \omega_2, H_2, F_2, W_2)$, from Theorem 4.5 we know that $\gamma_2 = \pi_{\mathcal{O}_{\mu_2}}(\gamma_2) : Q_2 \to (T^*Q_2) \mathcal{O}_{\mu_2}$ is a solution of Hamilton-Jacobi equation for $RO$-reduced RCH system $((T^*Q_2) \mathcal{O}_{\mu_2}, \omega_{\mathcal{O}_{\mu_2}}, h_{\mathcal{O}_{\mu_2}}, f_{\mathcal{O}_{\mu_2}}, u_{\mathcal{O}_{\mu_2}})$. Note that $\gamma_1 = \varphi^* \cdot \gamma_2 \cdot \varphi : Q_1 \to T^*Q_1$, then $\gamma_1 = \pi_{\mathcal{O}_{\mu_1}}(\gamma_1) = \pi_{\mathcal{O}_{\mu_1}} \cdot \varphi^* \cdot \gamma_2 \cdot \varphi = \varphi^*_{\mathcal{O}_{\mu_i}/G} \cdot \pi_{\mathcal{O}_{\mu_2}} \cdot \gamma_2 \cdot \varphi = \varphi^*_{\mathcal{O}_{\mu_i}/G} \cdot \gamma_2 \cdot \varphi$. From Theorem 2.7(i) we know that $\gamma_1 = \gamma^*_{\mathcal{O}_{\mu_i}/G} \cdot \gamma_2 \cdot \varphi$ is a solution of Hamilton-Jacobi equation for RCH-equivalent system $($(T^*Q_1) \mathcal{O}_{\mu_1}, \omega_{\mathcal{O}_{\mu_1}}, h_{\mathcal{O}_{\mu_1}}, f_{\mathcal{O}_{\mu_1}}, u_{\mathcal{O}_{\mu_1}})$, and hence from Theorem 4.5 we know that $\gamma_1 = \varphi^* \cdot \gamma_2 \cdot \varphi$ is a solution of the Hamilton-Jacobi equation for RCH system $(T^*Q_1, G_1, \omega_1, H_1, F_1, W_1)$. It follows that the conclusion (i) of Theorem 4.6 holds.

Conversely, note that $\varphi : Q_1 \to Q_2$ is a diffeomorphism, by using the same way, we can also prove the conclusion (ii) of Theorem 4.6.

**Remark 4.7** If $(T^*Q, \omega)$ is a connected symplectic manifold, and $\mathbf{J} : T^*Q \to g^*$ is a non-equivariant momentum map with a non-equivariance group one-cocycle $\sigma : G \to g^*$, which is defined by $\sigma(g) := \mathbf{J}(g \cdot z) - \text{Ad}_{g^{-1}} \mathbf{J}(z)$, where $g \in G$ and $z \in T^*Q$. Then we know that $\sigma$ produces a new affine action $\Theta : G \times g^* \to g^*$ defined by $\Theta(g, \mu) := \text{Ad}_{g^{-1}} \mu + \sigma(g)$, where $\mu \in g^*$, with respect to which the given momentum map $\mathbf{J}$ is equivariant. Assume that $G$ acts freely and properly on $T^*Q$, and $\mathcal{O}_\mu = G \cdot \mu \subset g^*$ denotes the $G$-orbit of the point $\mu \in g^*$ with respect to this affine action $\Theta$, and $\mu$ is a regular value of $\mathbf{J}$. Then the quotient space $(T^*Q) \mathcal{O}_\mu = \mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ is also a symplectic manifold with symplectic form $\omega_{\mathcal{O}_\mu}$ uniquely characterized by (4.1), see Ortega and Ratiu [21] and Marsden et al [15]. In this case, we can also define the regular orbit reducible RCH system $(T^*Q, G, \omega, H, F, W)$ and RoCH-equivalence, and prove the Hamilton-Jacobi theorem for $RO$-reduced Hamiltonian system $((T^*Q) \mathcal{O}_\mu, \omega_{\mathcal{O}_\mu}, h_{\mathcal{O}_\mu}, f_{\mathcal{O}_\mu}, u_{\mathcal{O}_\mu})$ by using the above same way, and state that the solution of Hamilton-Jacobi equation for regular orbit reducible RCH system with symmetry leaves invariant under the conditions of RoCH-equivalence, where the $RO$-reduced space $((-T^*Q) \mathcal{O}_\mu, \omega_{\mathcal{O}_\mu})$ is determined by the affine action.
5 Applications

In this section, as an application of the theoretical results, we consider the regular point reducible RCH system on the generalization of a Lie group, and give Hamilton-Jacobi equation of the reduced system. In particular, we show the Hamilton-Jacobi theorems of rigid body and heavy top with internal rotors on the generalization of rotation group SO(3) and on the generalization of Euclidean group SE(3) by calculation in detail, respectively. We shall follow the notations and conventions introduced in Marsden et al [16], Marsden and Ratiu [17], Marsden et al [18], Wang [23].

5.1 Hamilton-Jacobi Theorem on the Generalization of a Lie Group

In order to describe the Hamilton-Jacobi theorems of rigid body and heavy top with internal rotors, we need to first consider the regular point reducible RCH system on the generalization of a Lie group \( G = G \times V \), where \( G \) is a Lie group with Lie algebra \( \mathfrak{g} \) and \( V \) is a \( k \)-dimensional vector space. Defined the left \( G \)-action \( \Phi : G \times Q \rightarrow Q \), \( \Phi(g, (h, \theta)) := (gh, \theta) \), for any \( g, h \in G \), \( \theta \in V \), that is, the \( G \)-action on \( Q \) is the left translation on the first factor \( G \), and \( G \) acts trivially on the second factor \( V \). Because \( T^*Q = T^*G \times T^*V \), and \( T^*V = V \times \mathfrak{g}^* \), by using the left trivialization of \( T^*G \), that is, \( T^*G = G \times \mathfrak{g}^* \), where \( \mathfrak{g}^* \) is the dual of \( \mathfrak{g} \), and hence we have that \( T^*Q = G \times \mathfrak{g}^* \times V \times V^* \). If the left \( G \)-action \( \Phi : G \times Q \rightarrow Q \) is free and proper, then the cotangent lift of the action to its cotangent bundle \( T^*Q \), given by \( \Phi^* : G \times T^*Q \rightarrow T^*Q \), \( \Phi^*(g, (h, \mu, \theta, \lambda)) := (gh, \mu, \theta, \lambda) \), for any \( g, h \in G \), \( \mu, \theta \in \mathfrak{g}^* \), \( \lambda \in V \), \( \mu \in \mathfrak{g}^* \), \( \theta \in V \), \( \lambda \in V^* \), is also a free and proper action, and the orbit space \( (T^*Q)/G \) is a smooth manifold and \( \pi : T^*Q \rightarrow (T^*Q)/G \) is a smooth submersion. Since \( G \) acts trivially on \( \mathfrak{g}^* \), \( V \) and \( V^* \), it follows that \( (T^*Q)/G \) is diffeomorphic to \( \mathfrak{g}^* \times V \times V^* \).

We know that \( \mathfrak{g}^* \) is a Poisson manifold with respect to the \((\pm)\)-Lie-Poisson bracket \( \{\cdot, \cdot\}_{\pm} \) defined by

\[
\{f, g\}_{\pm}(\mu) := \pm < \mu, [\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}] >, \quad \forall f, g \in C^\infty(\mathfrak{g}^*), \quad \mu \in \mathfrak{g}^*, \tag{5.1}
\]

where the element \( \frac{\delta f}{\delta \mu} \in \mathfrak{g} \) is defined by the equality \( v, \frac{\delta f}{\delta \mu} := Df(\mu) \cdot v \), for any \( v \in \mathfrak{g}^* \), see Marsden and Ratiu [17]. For \( \mu \in \mathfrak{g}^* \), the coadjoint orbit \( O_\mu \subset \mathfrak{g}^* \) has the induced orbit symplectic forms \( \omega_\mu^\pm \) given by

\[
\omega_\mu^\pm(\nu)(\text{ad}_\xi^\mu(\nu), \text{ad}_\eta^\mu(\nu)) = \pm <\nu, [\xi, \eta]>, \quad \forall \xi, \eta \in \mathfrak{g}, \quad \nu \in O_\mu \subset \mathfrak{g}^*, \tag{5.2}
\]

which are coincide with the restriction of the \( (\pm) \)-Lie-Poisson brackets on \( \mathfrak{g}^* \) to the coadjoint orbit \( O_\mu \). From the Symplectic Stratification theorem we know that the coadjoint orbits \( (O_\mu, \omega_\mu^\pm) \), \( \mu \in \mathfrak{g}^* \), form the symplectic leaves of the Poisson manifolds \( (\mathfrak{g}^*, \{\cdot, \cdot\}_\pm) \). Let \( \omega_V \) be the canonical symplectic form on \( T^*V \cong V \times V^* \) given by

\[
\omega_V((\theta_1, \lambda_1), (\theta_2, \lambda_2)) = \lambda_2, \theta_1 > - \lambda_1, \theta_2 >,
\]

where \( (\theta_1, \lambda_1) \in V \times V^* \), \( i = 1, 2, < \cdot, \cdot > \) is the natural pairing between \( V^* \) and \( V \). Thus, we can induce a symplectic forms \( \omega_\mu^\pm \otimes V \times V^* \), \( \mu \in \mathfrak{g}^* \), \( \pi_\mu : O_\mu \times V \times V^* \rightarrow O_\mu \), \( \pi_\mu : O_\mu \times V \times V^* \rightarrow O_\mu \) are canonical projections. on the other hand, note that for \( F, K : T^*V \cong V \times V^* \rightarrow \mathbb{R} \), by using the canonical symplectic form \( \omega_V \) on \( T^*V \cong V \times V^* \), we can define the Poisson bracket \( \{\cdot, \cdot\}_V \) on \( T^*V \) as follows

\[
\{F, K\}_V(\theta, \lambda) = \frac{\delta F}{\delta \theta} \cdot \frac{\delta K}{\delta \lambda} - \frac{\delta K}{\delta \theta} \cdot \frac{\delta F}{\delta \lambda}.
\]
If \( \theta_i, i = 1, \ldots, k \), is a base of \( V \), and \( \lambda_i, i = 1, \ldots, k \), a base of \( V^* \), then we have that

\[
\{F, K\}_V(\theta, \lambda) = \sum_{i=1}^k \left( \frac{\partial F}{\partial \theta_i} \frac{\partial K}{\partial \lambda_i} - \frac{\partial K}{\partial \theta_i} \frac{\partial F}{\partial \lambda_i} \right).
\]

Thus, by the \((\pm)\)-Lie-Poisson brackets on \( \mathfrak{g}^* \) and the Poisson bracket \( \{\cdot, \cdot\}_V \) on \( T^*V \), for \( F, K : \mathfrak{g}^* \times V \times V^* \to \mathbb{R} \), we can define the Poisson bracket on \( \mathfrak{g}^* \times V \times V^* \) as follows

\[
\{F, K\}_\pm(\mu, \theta, \lambda) = \{F, K\}_\pm(\mu) + \{F, K\}_V(\theta, \lambda) = \pm \mu \left( \frac{\delta F}{\delta \mu} \frac{\delta K}{\delta \mu} \right) + \sum_{i=1}^k \left( \frac{\partial F}{\partial \theta_i} \frac{\partial K}{\partial \lambda_i} - \frac{\partial K}{\partial \theta_i} \frac{\partial F}{\partial \lambda_i} \right).
\]

(5.3)

See Krishnaprasad and Marsden [9]. In particular, for \( F_\mu, K_\mu : \mathcal{O}_\mu \times V \times V^* \to \mathbb{R} \), we have that

\[
\tilde{\omega}|_{\mathcal{O}_\mu \times V \times V^*}(X_{F_\mu}, X_{K_\mu}) = \{F_\mu, K_\mu\}_\pm|_{\mathcal{O}_\mu \times V \times V^*}.
\]

On the other hand, from \( T^*Q = T^*G \times T^*V \) we know that there is a canonical symplectic form \( \omega_Q = \pi^*_2 \omega_0 + \pi^*_1 \omega_V \) on \( T^*Q \), where \( \omega_0 \) is the canonical symplectic form on \( T^*G \) and the maps \( \pi_1 : Q = G \times V \to G \) and \( \pi_2 : Q = G \times V \to V \) are canonical projections. Then the cotangent lift of the left \( G \)-action \( \Phi^{T^*} : G \times T^*Q \to T^*Q \) is also symplectic, and admits an associated \( \text{Ad}^* \)-equivariant momentum map \( J_Q : T^*Q \to \mathfrak{g}^* \) such that \( J_Q \cdot \pi^*_1 = J_L \), where \( J_L : T^*G \to \mathfrak{g}^* \) is a momentum map of left \( G \)-action on \( T^*G \), and \( \pi^*_1 : T^*G \to T^*Q \). If \( \mu \in \mathfrak{g}^* \) is a regular value of \( J_Q \), then \( \mu \in \mathfrak{g}^* \) is also a regular value of \( J_L \) and \( J_Q^{-1}(\mu) \cong J_L^{-1}(\mu) \times V \times V^* \). Denote by \( G_\mu = \{ g \in G \mid \text{Ad}^*_g \mu = \mu \} \) the isotropy subgroup of coadjoint \( G \)-action at the point \( \mu \in \mathfrak{g}^* \). It follows that \( G_\mu \) acts also freely and properly on \( J_Q^{-1}(\mu) \), the regular point reduced space \( (T^*Q)_\mu = J_Q^{-1}(\mu)/G_\mu \cong (T^*G)_\mu \times V \times V^* \) of \( (T^*Q, \omega_Q) \) at \( \mu \), is a symplectic manifold with symplectic form \( \omega_\mu \) uniquely characterized by the relation \( \pi^*_\mu \omega_\mu = i^*_\mu \omega_Q = i^*_\mu \pi^*_1 \omega_0 + i^*_\mu \pi^*_2 \omega_V \), where the map \( i_\mu : J_Q^{-1}(\mu) \to T^*Q \) is the inclusion and \( \pi_\mu : J_Q^{-1}(\mu) \to (T^*Q)_\mu \) is the projection. Because from Abraham and Marsden [1], we know that \( (T^*G)_\mu, \omega_\mu \) is symplectically diffeomorphic to \( (\mathcal{O}_\mu, \omega_{\mathcal{O}_\mu}) \), and hence we have that \( (T^*Q)_\mu, \omega_\mu \) is symplectically diffeomorphic to \( (\mathcal{O}_\mu \times V \times V^*, \tilde{\omega}|_{\mathcal{O}_\mu \times V \times V^*}) \), which is a symplectic leaf of the Poisson manifold \( (\mathfrak{g}^* \times V \times V^*, \{\cdot, \cdot\}_-) \).

We now consider the Lagrangian \( L(g, \xi, \theta, \dot{\theta}) : TQ \cong G \times \mathfrak{g} \times TV \to \mathbb{R} \), which is usual the total kinetic minus potential energy of the system, where \( (g, \xi) \in G \times \mathfrak{g} \), and \( \theta \in V \), \( \xi^i \) and \( \dot{\theta}^j = \frac{\partial L}{\partial \theta^j} \), \( (i = 1, \ldots, n, j = 1, \ldots, k, n = \text{dim} G, k = \text{dim} V) \), regarded as the velocity of system. If we introduce the conjugate momentum \( p_i = \frac{\partial L}{\partial \dot{\theta}^i}, l_j = \frac{\partial L}{\partial \theta^j}, i = 1, \ldots, n, j = 1, \ldots, k \), and by the Legendre transformation \( FL : TQ \cong G \times \mathfrak{g} \times V \times V \to T^*Q \cong G \times \mathfrak{g}^* \times V \times V^* \), \( (g^i, \xi^i, \theta^j, \dot{\theta}^j) \to (g^i, p_i, \theta^j, l_j) \), we have the Hamiltonian \( H(g, p, \theta, l) : T^*Q \cong G \times \mathfrak{g}^* \times V \times V^* \to \mathbb{R} \) given by

\[
H(g^i, p_i, \theta^j, l_j) = \sum_{i=1}^n p_i \xi^i + \sum_{j=1}^k l_j \dot{\theta}^j - L(g^i, \xi^i, \theta^j, \dot{\theta}^j).
\]

(5.4)

If the Hamiltonian \( H(g, p, \theta, l) : T^*Q \cong G \times \mathfrak{g}^* \times V \times V^* \to \mathbb{R} \) is left cotangent lifted \( G \)-action \( \Phi^{T^*} \) invariant, for \( \mu \in \mathfrak{g}^* \) we have the associated reduced Hamiltonian \( h_\mu(v, \theta, l) : (T^*Q)_\mu \cong \mathcal{O}_\mu \times V \times V^* \to \mathbb{R} \), defined by \( h_\mu \cdot \pi_\mu = H \cdot i_\mu \), and the reduced Hamiltonian vector field \( X_{h_\mu} \) given by \( X_{h_\mu}(K_\mu) = \{K_\mu, h_\mu\}_{\mathcal{O}_\mu \times V \times V^*} \). Thus, if the fiber-preserving map \( F : T^*Q \to T^*Q \) and the fiber submanifold \( \mathcal{W} \) of \( T^*Q \) are all left cotangent lifted \( G \)-action \( \Phi^{T^*} \) invariant, then the 6-tuple \( (T^*Q, G, \omega_Q, H, F, \mathcal{W}) \) is a regular point reducible RCH system. For a point \( \mu \in \mathfrak{g}^* \), the regular value of the momentum map \( J_Q : T^*Q \to \mathfrak{g}^* \), the \( R_P \)-reduced system is the 5-tuple
acting on the rotor. The configuration space is (orthonormal) body axes. Assume that the rotor and the body coordinate axes are aligned. The corresponding phase space is the cotangent bundle \( T^*Q \). Denote the system center of mass by \( \mathcal{O}_Q \). SO(3) acts freely and properly on \( T^*Q \) given by \( \Phi : SO(3) \times T^*Q \rightarrow T^*Q \). \( T^*Q \) is an one-form on \( Q \), and \( \gamma \) is closed with respect to \( T\pi_Q : T^*Q \rightarrow TQ \), and \( \Im(\gamma) \subseteq J_Q^{-1}(\mu) \), and it is \( G_{\mu} \)-invariant, and \( \gamma = \pi_\mu(\gamma) : Q \rightarrow \mathcal{O}_Q \times V \times V^* \). By using the same way in the proof of Hamilton-Jacobi theorem for the \( R_P \)-reduced RCH system, we can get the following theorem.

**Theorem 5.1** For the regular point reducible RCH system \((T^*Q, G, \omega_Q, H, F, W)\) on the generalization of a Lie group \( Q = G \times V \), where \( G \) is a Lie group and \( V \) is a 3-dimensional vector space, assume that \( \gamma : Q \rightarrow T^*Q \) is an one-form on \( Q \), and \( \gamma \) is closed with respect to \( T\pi_Q : T^*Q \rightarrow TQ \), and \( \bar{X} = X(T^*Q, G, \omega_Q, H, F, W) \) is the dynamical vector field of the regular point reducible RCH system \((T^*Q, G, \omega_Q, H, F, W)\) with a control law \( u \). Moreover, assume that \( \mu \in \mathfrak{g}^* \) is the regular reducible point of the RCH system, and \( \Im(\mu) \subseteq J_Q^{-1}(\mu) \), and it is \( G_{\mu} \)-invariant, and \( \bar{\mu} = \pi_\mu(\gamma) : Q \rightarrow \mathcal{O}_Q \times V \times V^* \). Then the following two assertions are equivalent: (i) \( \bar{X} \) and \( \bar{X}_\mu \) are \( \bar{\mu} \)-related, and \( \bar{X}_\mu = X_{(O_Q \times V \times V^*, \omega_Q, \mu, f_\mu, u_\mu)} \) is the dynamical vector field of \( R_P \)-reduced RCH system \((O_Q \times V \times V^*, \omega_Q, \mu, f_\mu, u_\mu)\); (ii) \( X_{h_{\mu, \bar{\gamma}}} + \text{vlift}(f_\mu, \bar{\gamma}) + \text{vlift}(u_\mu, \bar{\gamma}) = 0 \), or \( X_{h_{\mu, \bar{\gamma}}} + \text{vlift}(f_\mu, \bar{\gamma}) + \text{vlift}(u_\mu, \bar{\gamma}) = \bar{X} \). Moreover, \( \gamma \) is a solution of the Hamilton-Jacobi equation \( X_{h_{\mu, \bar{\gamma}}} + \text{vlift}(f_\mu, \bar{\gamma}) + \text{vlift}(u_\mu, \bar{\gamma}) = 0 \).

In particular, when \( Q = G \), we can obtain the Hamilton-Jacobi theorem for the \( R_P \)-reduced RCH system on Lie group \( G \). In this case, note that the symplectic structure on the coadjoint orbit \( O_\mu \) is induced by the (-)-Lie-Poisson brackets on \( \mathfrak{g}^* \), then the Hamilton-Jacobi equation \( X_{h_{\mu, \bar{\gamma}}} + \text{vlift}(f_\mu, \bar{\gamma}) + \text{vlift}(u_\mu, \bar{\gamma}) = 0 \) for \( R_P \)-reduced RCH system \((O_\mu, \omega_Q, h_\mu, f_\mu, u_\mu)\) is also called Lie-Poisson Hamilton-Jacobi equation. See Wang [25], Marsden and Ratiu [17], and Ge and Marsden [7].

### 5.2 Hamilton-Jacobi Equation of Rigid Body with Internal Rotors

In the following we regard the rigid body with three symmetric internal rotors as a regular point reducible RCH system on the generalization of rotation group \( SO(3) \times \mathbb{R}^3 \), and give the Hamilton-Jacobi equation of its reduced RCH system by calculation in detail. Note that our description of the motion and the equations of rigid body with internal rotors in this subsection follows some of the notations and conventions in Marsden and Ratiu [17], Marsden [14], Marsden et al [18].

We consider a rigid body (to be called the carrier body) carrying three symmetric rotors. Denote the system center of mass by \( O \) in the body frame and at \( O \) place a set of (orthonormal) body axes. Assume that the rotor and the body coordinate axes are aligned with principal axes of the carrier body. The rotor spins under the influence of a torque \( u \) acting on the rotor. The configuration space is \( Q = \text{SO}(3) \times V \), where \( V = S^1 \times S^1 \times S^1 \), with the first factor being rigid body attitude and the second factor being the angles of rotors. The corresponding phase space is the cotangent bundle \( T^*Q = T^*\text{SO}(3) \times T^*V \), where \( T^*V = T^*(S^1 \times S^1 \times S^1) \cong T^*\mathbb{R}^3 \), with the canonical symplectic form. Assume that Lie group \( G = \text{SO}(3) \) acts freely and properly on \( Q \) by the left translations on \( \text{SO}(3) \), then the action of \( \text{SO}(3) \) on the phase space \( T^*Q \) is by cotangent lift of left translations on \( \text{SO}(3) \) at the identity, that is, \( \Phi : \text{SO}(3) \times T^*\text{SO}(3) \times T^*V \cong \text{SO}(3) \times \text{SO}(3) \times \text{so}^*(3) \times \mathbb{R}^3 \rightarrow \text{SO}(3) \times \text{so}^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \), given by \( \Phi(B, (A, \Pi, \alpha, l)) = (BA, \Pi, \alpha, l) \), for any \( A, B \in \text{SO}(3) \), \( \Pi \in \text{so}^*(3) \), \( \alpha, l \in \mathbb{R}^3 \), which
is also free and proper, and admits a associated $\text{Ad}^*$-equivariant momentum map $J_Q : T^*Q \cong \text{SO}(3) \times \mathfrak{so}^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathfrak{so}^*(3)$ for the left $\text{SO}(3)$ action. If $\Pi \in \mathfrak{so}^*(3)$ is a regular value of $J_Q$, then the regular point reduced space $(T^*Q)_\Pi = J_Q^{-1}(\Pi)/\text{SO}(3)_\Pi$ is symplectically diffeomorphic to the coadjoint orbit $O_\Pi \times \mathbb{R}^3 \times \mathbb{R}^3 \subset \mathfrak{so}^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3$.

Let $I = \text{diag}(I_1, I_2, I_3)$ be the moment of inertia of the carrier body in the principal body-fixed frame, and $J_i$, $i = 1, 2, 3$ be the moments of inertia of rotors around their rotation axes. Let $J_{ik}$, $i = 1, 2, 3$, $k = 1, 2, 3$, be the moments of inertia of the $i$th rotor with $i = 1, 2, 3$, around the $k$th principal axis with $k = 1, 2, 3$, respectively, and denote by $\bar{I}_i = I_i + J_{1i} + J_{2i} + J_{3i} - J_{ii}$, $i = 1, 2, 3$. Let $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ be the vector of body angular velocities computed with respect to the axes fixed in the body and $(\Omega_1, \Omega_2, \Omega_3) \in \mathfrak{so}(3)$. Let $\alpha_i$, $i = 1, 2, 3$, be the relative angles of rotors and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ the vector of rotor relative angular velocities about the principal axes with respect to a carrier body fixed frame.

Consider the Lagrangian of the system $L(A, \Omega, \alpha, \dot{\alpha}) : TQ \cong \text{SO}(3) \times \mathfrak{so}(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, which is the total kinetic energy of the rigid body plus the total kinetic energy of the rotor, given by

$$L(A, \Omega, \alpha, \dot{\alpha}) = \frac{1}{2} \left[ \bar{I}_1 \Omega_1^2 + \bar{I}_2 \Omega_2^2 + \bar{I}_3 \Omega_3^2 + J_1 (\Omega_1 + \dot{\alpha}_1)^2 + J_2 (\Omega_2 + \dot{\alpha}_2)^2 + J_3 (\Omega_3 + \dot{\alpha}_3)^2 \right],$$

where $A \in \text{SO}(3)$, $\Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathfrak{so}(3)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, $\dot{\alpha} = (\dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}_3) \in \mathbb{R}^3$. If we introduce the conjugate angular momentum, which is given by

$$\Pi_i = \frac{\partial L}{\partial \Omega_i} = \bar{I}_i \Omega_i + J_i (\Omega_i + \dot{\alpha}_i), \quad l_i = \frac{\partial L}{\partial \dot{\alpha}_i} = J_i (\dot{\alpha}_i), \quad i = 1, 2, 3,$$

and by the Legendre transformation $FL : TQ \cong \text{SO}(3) \times \mathfrak{so}(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to T^*Q \cong \text{SO}(3) \times \mathfrak{so}^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3$, $(A, \Omega, \alpha, \dot{\alpha}) \to (A, \Pi, \alpha, l)$, where $\Pi = (\Pi_1, \Pi_2, \Pi_3) \in \mathfrak{so}^*(3)$, $l = (l_1, l_2, l_3) \in \mathbb{R}^3$, we have the Hamiltonian $H(A, \Pi, \alpha, l) : T^*Q \cong \text{SO}(3) \times \mathfrak{so}^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$H(A, \Pi, \alpha, l) = \Omega \cdot \Pi + \dot{\alpha} \cdot l - L(A, \Omega, \alpha, \dot{\alpha})$$

$$= \bar{I}_1 \Omega_1^2 + J_1 (\Omega_1 + \dot{\alpha}_1) + \bar{I}_2 \Omega_2^2 + J_2 (\Omega_2 + \dot{\alpha}_2) + \bar{I}_3 \Omega_3^2 + J_3 (\Omega_3 + \dot{\alpha}_3)$$

$$+ \Omega_3 \dot{\alpha}_3 + J_1 (\dot{\alpha}_1 \Omega_1 + \dot{\alpha}_1^2) + J_2 (\dot{\alpha}_2 \Omega_2 + \dot{\alpha}_2^2) + J_3 (\dot{\alpha}_3 \Omega_3 + \dot{\alpha}_3^2)$$

$$- \frac{1}{2} \left[ \bar{I}_1 \Omega_1^2 + l_1^2 + \bar{I}_2 \Omega_2^2 + l_2^2 + \bar{I}_3 \Omega_3^2 + l_3^2 + J_1 (\Omega_1 + \dot{\alpha}_1)^2 + J_2 (\Omega_2 + \dot{\alpha}_2)^2 + J_3 (\Omega_3 + \dot{\alpha}_3)^2 \right]$$

$$= \frac{1}{2} \left[ (\Pi_1 - l_1)^2 + (\Pi_2 - l_2)^2 + (\Pi_3 - l_3)^2 \right] + \frac{l_1^2}{I_1} + \frac{l_2^2}{I_2} + \frac{l_3^2}{I_3}.$$

From the above expression of the Hamiltonian, we know that $H(A, \Pi, \alpha, l)$ is invariant under the left $\text{SO}(3)$-action $\Phi : \text{SO}(3) \times T^*Q \to T^*Q$. For the case $\Pi_0 = \mu \in \mathfrak{so}^*(3)$ is the regular value of $J_Q$, we have the reduced Hamiltonian $h_\mu(\Pi, \alpha, l) : \mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3 \subset \mathfrak{so}^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by $h_\mu(\Pi, \alpha, l) = H(A, \Pi, \alpha, l)_{|\mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}$. From the rigid body Poisson bracket on $\mathfrak{so}^*(3)$ and the Poisson bracket on $T^*\mathbb{R}^3$, we can get the Poisson bracket on $T^*Q$, that is, for $F, K : \mathfrak{so}^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, we have that

$$\{F, K\}_\Pi(\alpha, l) = -\Pi \cdot (\nabla_\Pi F \times \nabla_\Pi K) + \{F, K\}_\Pi \Pi(\alpha, l). \quad (5.5)$$

In particular, for $F_\mu, K_\mu : \mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, we have that $\omega_{\mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}^{-1}(XF_\mu, XK_\mu) = \{F_\mu, K_\mu\}_{|\mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}$. Moreover, for reduced Hamiltonian $h_\mu(\Pi, \alpha, l) : \mathcal{O}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, we
have the Hamiltonian vector field $X_{h_{\mu}}(K_{\mu}) = \{K_{\mu}, h_{\mu}\}|_{\mathcal{O}_{\mu} \times \mathbb{R}^3 \times \mathbb{R}^3}$.

Assume that $\gamma : SO(3) \times \mathbb{R}^3 \rightarrow T^*SO(3) \times \mathbb{R}^3$ is an one-form on $SO(3) \times \mathbb{R}^3$, and $\gamma$ is closed with respect to $T^*\pi_{SO(3) \times \mathbb{R}^3} : TT^*(SO(3) \times \mathbb{R}^3) \rightarrow T(SO(3) \times \mathbb{R}^3)$, and $\text{Im}(\gamma) \subset J^{-1}_Q(\mu)$, and it is $SO(3)_{\mu}$-invariant, and $\bar{\gamma} = \pi_{\mu}(\gamma) : SO(3) \times \mathbb{R}^3 \rightarrow \mathcal{O}_{\mu} \times \mathbb{R}^3 \times \mathbb{R}^3$. Denote by $\bar{\gamma}(A, \alpha) = (\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3, \bar{\gamma}_4, \bar{\gamma}_5, \bar{\gamma}_6, \bar{\gamma}_7, \bar{\gamma}_8, \bar{\gamma}_9)(A, \alpha) \in \mathcal{O}_{\mu} \times \mathbb{R}^3 \times \mathbb{R}^3(\subset \mathfrak{so}^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3)$, and $(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)(A, \alpha) \in \mathcal{O}_{\mu}$, then $h_{\mu} \cdot \bar{\gamma} : SO(3) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is given by

$$h_{\mu} \cdot \bar{\gamma}(A, \alpha) = H \cdot \bar{\gamma}(A, \alpha)|_{\mathcal{O}_{\mu} \times \mathbb{R}^3 \times \mathbb{R}^3} = \frac{1}{2} \left( \frac{\tilde{\gamma}_1 - \tilde{\gamma}_7}{I_1} + \frac{\tilde{\gamma}_2 - \tilde{\gamma}_8}{I_2} + \frac{\tilde{\gamma}_3 - \tilde{\gamma}_9}{I_3} + \frac{\tilde{\gamma}_7^2}{J_1} + \frac{\tilde{\gamma}_8^2}{J_2} + \frac{\tilde{\gamma}_9^2}{J_3} \right),$$

and the vector field

$$X_{h_{\mu} \cdot \bar{\gamma}}(\Pi) = \{\Pi, h_{\mu} \cdot \bar{\gamma}(A, \alpha)\}|_{\mathcal{O}_{\mu} \times \mathbb{R}^3 \times \mathbb{R}^3} = -\Pi \cdot (\nabla_{\Pi} \Pi \times \nabla_{\Pi} (h_{\mu} \cdot \bar{\gamma})) + \{\Pi, h_{\mu} \cdot \bar{\gamma}(A, \alpha)\}|_{\mathcal{O}_{\mu} \times \mathbb{R}^3 \times \mathbb{R}^3} = -\nabla_{\Pi} \Pi \cdot (\nabla_{\Pi} (h_{\mu} \cdot \bar{\gamma}) \times \Pi) + \sum_{i=1}^{3} \left( \frac{\partial \Pi}{\partial \alpha_i} \frac{\partial (h_{\mu} \cdot \bar{\gamma})}{\partial l_i} - \frac{\partial (h_{\mu} \cdot \bar{\gamma})}{\partial \alpha_i} \frac{\partial \Pi}{\partial l_i} \right)$$

$$= \left( \frac{I_2 I_3 (\tilde{\gamma}_3 - \tilde{\gamma}_9) - I_1 I_3 (\tilde{\gamma}_2 - \tilde{\gamma}_8)}{I_1 I_2 I_3} - \frac{I_2 I_3 (\tilde{\gamma}_1 - \tilde{\gamma}_7) - I_1 I_3 (\tilde{\gamma}_2 - \tilde{\gamma}_8)}{I_1 I_2 I_3} \right)$$

$$= \left( \frac{I_1 I_2 (\tilde{\gamma}_2 - \tilde{\gamma}_8) - I_2 I_3 (\tilde{\gamma}_1 - \tilde{\gamma}_7)}{I_1 I_2} \right),$$

since $\nabla_{\Pi} \Pi = 1$, and $\nabla_{\Pi j}(h_{\mu} \cdot \bar{\gamma}) = (\tilde{\gamma}_j - \tilde{\gamma}_{j+6})/I_j$, $j = 1, 2, 3$, and $\frac{\partial \Pi}{\partial \alpha_i} = \frac{\partial (h_{\mu} \cdot \bar{\gamma})}{\partial \alpha_i} = 0$, $i = 1, 2, 3$.

$$X_{h_{\mu} \cdot \bar{\gamma}}(\alpha) = \{\alpha, h_{\mu} \cdot \bar{\gamma}(A, \alpha)\}|_{\mathcal{O}_{\mu} \times \mathbb{R}^3 \times \mathbb{R}^3} = -\Pi \cdot (\nabla_{\Pi} \alpha \times \nabla_{\Pi} (h_{\mu} \cdot \bar{\gamma})) + \{\alpha, h_{\mu} \cdot \bar{\gamma}(A, \alpha)\}|_{\mathcal{O}_{\mu} \times \mathbb{R}^3 \times \mathbb{R}^3} = -\nabla_{\Pi} \alpha \cdot (\nabla_{\Pi} (h_{\mu} \cdot \bar{\gamma}) \times \Pi) + \sum_{i=1}^{3} \left( \frac{\partial \alpha}{\partial \alpha_i} \frac{\partial (h_{\mu} \cdot \bar{\gamma})}{\partial l_i} - \frac{\partial (h_{\mu} \cdot \bar{\gamma})}{\partial \alpha_i} \frac{\partial \alpha}{\partial l_i} \right)$$

$$= \left( \frac{\tilde{\gamma}_1 - \tilde{\gamma}_7}{I_1} + \frac{\tilde{\gamma}_7}{I_1} - \frac{\tilde{\gamma}_2 - \tilde{\gamma}_8}{I_2} - \frac{\tilde{\gamma}_8}{I_2} - \frac{\tilde{\gamma}_3 - \tilde{\gamma}_9}{I_3} + \frac{\tilde{\gamma}_9}{I_3} \right),$$

since $\nabla_{\Pi} \alpha = 0$, $\frac{\partial \alpha_i}{\partial \alpha_i} = 1$, $j = i$, and $\frac{\partial \alpha_i}{\partial \alpha_j} = 0$, $i \neq j$, $\frac{\partial (h_{\mu} \cdot \bar{\gamma})}{\partial \alpha_i} = 0$, $i = 1, 2, 3$.

$$X_{h_{\mu} \cdot \bar{\gamma}}(l) = \{l, h_{\mu} \cdot \bar{\gamma}(A, \alpha)\}|_{\mathcal{O}_{\mu} \times \mathbb{R}^3 \times \mathbb{R}^3} = -\Pi \cdot (\nabla_{\Pi} l \times \nabla_{\Pi} (h_{\mu} \cdot \bar{\gamma})) + \{l, h_{\mu} \cdot \bar{\gamma}(A, \alpha)\}|_{\mathcal{O}_{\mu} \times \mathbb{R}^3 \times \mathbb{R}^3} = -\nabla_{\Pi} l \cdot (\nabla_{\Pi} (h_{\mu} \cdot \bar{\gamma}) \times \Pi) + \sum_{i=1}^{3} \left( \frac{\partial l}{\partial \alpha_i} \frac{\partial (h_{\mu} \cdot \bar{\gamma})}{\partial l_i} - \frac{\partial (h_{\mu} \cdot \bar{\gamma})}{\partial \alpha_i} \frac{\partial l}{\partial l_i} \right) = (0, 0, 0),$$

since $\nabla_{\Pi} l = 0$, and $\frac{\partial l}{\partial \alpha_i} = \frac{\partial (h_{\mu} \cdot \bar{\gamma})}{\partial \alpha_i} = 0$, $i = 1, 2, 3$. From Theorem 5.1, if we consider the rigid body-rotor system with a control torque $u : T^*Q \rightarrow T^*Q$ acting on the rotors, and $u \in$
$W \subset J_Q^{-1}(\mu)$ is invariant under the left SO(3)-action, and its reduced control torque $u_{\mu} : O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3 \to O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3$ is given by $u_{\mu}(\Pi, \alpha, l) = \pi_\mu(u(A, \Pi, \alpha, l)) = u(A, \Pi, \alpha, l)|_{\omega_{O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}}$, where $\pi_\mu : J_Q^{-1}(\mu) \to O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3$. The dynamical vector field of $R_P$-reduced RCH system $(O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3, \omega_{O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}, h_\mu, u_{\mu})$ is given by

$$X_{(O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3, \omega_{O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}, h_\mu, u_{\mu})} = X_{h_\mu} + \text{vlift}(u_{\mu}),$$

where $\text{vlift}(u_{\mu}) = \text{vlift}(u_{\mu})X_{h_\mu} \in T(O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3)$. Assume that $\text{vlift}(u_{\mu} \cdot \gamma)(A, \alpha) = (U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8, U_9)(A, \alpha) \in T(O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3)$, then the Hamilton-Jacobi equations for rigid body-rotor system with the control torque $u$ acting on the rotors are given by

$$\begin{align*}
\dot{\gamma} &= \frac{\gamma - \gamma_0}{\gamma - \gamma_0} - \dot{\gamma}_l \Pi_3(\gamma_2 - \gamma_9) + \dot{\gamma}_l \Pi_3 U_1 = 0, \\
\dot{\gamma}_l \Pi_1(\gamma_2 - \gamma_8) - \dot{\gamma}_l \Pi_2(\gamma_7 - \gamma_9) + \dot{\gamma}_l \Pi_1 U_2 = 0, \\
- J_1(\gamma_1 - \gamma_7) + \dot{\gamma} \gamma_7 + \dot{\gamma}_l J_1 U_4 = 0, \\
- J_2(\gamma_2 - \gamma_8) + \dot{\gamma}_l \gamma_8 + \dot{\gamma}_l J_2 U_5 = 0, \\
- J_3(\gamma_3 - \gamma_9) + \dot{\gamma}_l \gamma_9 + \dot{\gamma}_l J_3 U_6 = 0, \\
U_7 = U_8 = U_9 = 0.
\end{align*}$$

(5.6)

To sum up the above discussion, we have the following proposition.

**Proposition 5.2** The 5-tuple $(T^*Q, SO(3), \omega_Q, H, u)$, where $Q = SO(3) \times \mathbb{R}^3$, is a regular point reducible RCH system. For a point $\mu \in so^*(3)$, the regular value of the momentum map $J_Q : SO(3) \times so^*(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \to so^*(3)$, the $R_P$-reduced system is the 4-tuple $(O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3, \omega_{O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}, h_\mu, u_{\mu})$, where $O_\mu \subset so^*(3)$ is the coadjoint orbit, $\omega_{O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}$ is orbit symplectic form on $O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3$, $h_\mu(\Pi, \alpha, l) = H(A, \Pi, \alpha, l)|_{O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}$, $u_{\mu}(\Pi, \alpha, l) = u(A, \Pi, \alpha, l)|_{O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}$. Assume that $\gamma : SO(3) \times \mathbb{R}^3 \to T^*(SO(3) \times \mathbb{R}^3)$ is an one-form on $SO(3) \times \mathbb{R}^3$, and $\gamma$ is closed with respect to $T\pi_{SO(3) \times \mathbb{R}^3} : TT^*(SO(3) \times \mathbb{R}^3) \to T(SO(3) \times \mathbb{R}^3)$, and $\text{Im}(\gamma) \subset J_Q^{-1}(\mu)$, and it is $SO(3)_{\mu}$-invariant, and $\tilde{\gamma} = \pi_\mu(\gamma) : SO(3) \times \mathbb{R}^3 \to O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3$. Then $\tilde{\gamma}$ is a solution of either Hamilton-Jacobi equation of rigid body with three symmetric internal rotors given by (5.6), or the equation $X_{h_{\mu} \cdot \gamma} + \text{vlift}(u_{\mu} \cdot \gamma) = \tilde{X}_\gamma$, if and only if $\tilde{X}_\gamma$ and $\tilde{X}_\mu$ are $\gamma$-related, where $\tilde{X}_\gamma = T\pi_{SO(3) \times \mathbb{R}^3} \cdot \tilde{X}_\gamma$, $\tilde{X} = X_{(T^*Q, SO(3), \omega_Q, H, u)}$, and $\tilde{X}_\mu = X_{(O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3, \omega_{O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}, h_\mu, u_{\mu})}$. 

**Remark 5.3** If we consider that Lie group $G = SO(3) \times \mathbb{R}^3$ acts freely and properly on the cotangent bundle $T^*Q = T^*SO(3) \times T^*\mathbb{R}^3$, by cotangent lift at the identity $\Phi : SO(3) \times \mathbb{R}^3 \to \Phi((B, \phi)(A, \Pi, \alpha, l)) = (BA, \Pi, \alpha + \phi, l)$, and it admits a associated $Ad^*$-equivariant momentum map $J_Q : T^*Q \cong SO(3) \times so^*(3) \times \mathbb{R}^3 \to so^*(3) \times \mathbb{R}^3$ for the left $SO(3) \times \mathbb{R}^3$ action. For $\mu \in g^* = so^*(3) \times \mathbb{R}^3$, a regular value of $J_Q$, then the regular point reduced space $(T^*Q)_{\mu} = J_Q^{-1}(\mu)/G_{\mu}$ is symplectically diffeomorphic to the coadjoint orbit $O_\mu \times \mathbb{R}^3 \subset so^*(3) \times \mathbb{R}^3$. By using the above same way, we can state that the 5-tuple $(T^*(SO(3) \times \mathbb{R}^3), SO(3) \times \mathbb{R}^3, \omega_{SO(3) \times \mathbb{R}^3}, H, u)$, is a regular point reducible RCH system, and the $R_P$-reduced system is the 4-tuple $(O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3, \omega_{O_\mu \times \mathbb{R}^3 \times \mathbb{R}^3}, h_\mu, u_{\mu})$. Moreover, assume that $\gamma : SO(3) \times \mathbb{R}^3 \to T^*(SO(3) \times \mathbb{R}^3)$ is an one-form on $SO(3) \times \mathbb{R}^3$, and $\gamma$ is closed with respect to $T\pi_{SO(3) \times \mathbb{R}^3} : TT^*(SO(3) \times \mathbb{R}^3) \to T(SO(3) \times \mathbb{R}^3)$, and $\text{Im}(\gamma) \subset J_Q^{-1}(\mu)$, and it is $G_{\mu}$-invariant, and $\tilde{\gamma} = \pi_\mu(\gamma) : SO(3) \times \mathbb{R}^3 \to O_\mu \times \mathbb{R}^3$. Then $\tilde{\gamma}$ is a solution of either Hamilton-Jacobi equation of rigid body with three symmetric internal rotors given by the
following (5.7), or the equation \( X_{h^i} \gamma + \text{lift}(u_\mu \cdot \gamma) = \bar{X}^\gamma \), if and only if \( \bar{X}^\gamma \) and \( \bar{X}_\mu \) are \( \tilde{\gamma} \)-related, where \( \bar{X}^\gamma = T\pi_{SO(3) \times \mathbb{R}^3} \cdot \tilde{X} \cdot \gamma \), \( \tilde{X} = X((T^*Q)_{SO(3) \times \mathbb{R}^3}, \omega Q, H, u) \), and \( \bar{X}_\mu = X((O_\mu \times \mathbb{R}^3, \tilde{\omega}^\cdot_{O_\mu \times \mathbb{R}^3}, h_\mu, u_\mu) \).

\[
\begin{align*}
\bar{I}_2 \bar{I}_2 (\bar{\gamma}_3 - \bar{\gamma}_6) - \bar{I}_3 \bar{I}_3 (\bar{\gamma}_2 - \bar{\gamma}_5) + \bar{I}_3 \bar{I}_3 U_1 &= 0, \\
\bar{I}_3 \bar{I}_3 (\bar{\gamma}_1 - \bar{\gamma}_4) - \bar{I}_1 \bar{I}_1 (\bar{\gamma}_3 - \bar{\gamma}_6) + \bar{I}_3 \bar{I}_3 U_2 &= 0, \\
\bar{I}_1 \bar{I}_1 (\bar{\gamma}_2 - \bar{\gamma}_5) - \bar{I}_2 \bar{I}_2 (\bar{\gamma}_1 - \bar{\gamma}_4) + \bar{I}_1 \bar{I}_2 U_3 &= 0, \\
U_4 = U_5 = U_6 &= 0.
\end{align*}
\] (5.7)

5.3 Hamilton-Jacobi Equation of Heavy Top with Internal Rotors

In the following we regard the heavy top with two pairs of symmetric internal rotors as a regular point reducible RCH system on the generalization of Euclidean group \( SE(3) \times \mathbb{R}^2 \), and give the Hamilton-Jacobi equation of its reduced RCH system by calculation in detail. Note that our description of the motion and the equations of heavy top with internal rotors in this subsection follows some of the notations and conventions in Marsden and Ratiu [17], Marsden [14], Marsden et al [18].

We first describe a heavy top with two pairs of symmetric rotors. We mount two pairs of rotors within the top so that each pair’s rotation axis is parallel to the first and the second principal axes of the top. The rotor spins under the influence of a torque acting on the rotor. The configuration space is \( Q = SE(3) \times V \), where \( V = S^1 \times S^1 \), with the first factor being the position of the heavy top and the second factor being the angles of rotors. The corresponding phase space is the cotangent bundle \( T^*Q = T^*SE(3) \times T^*V \), where \( T^*V = T^*(S^1 \times S^1) \cong T^*\mathbb{R}^2 \), with the canonical symplectic form. Let Lie group \( G = SE(3) \) acts freely and properly on \( Q \) by the left translations on \( SE(3) \), then the action of \( SE(3) \) on the phase space \( T^*Q \) is by cotangent lift of left translations on \( SE(3) \) at the identity, that is, \( \Phi : SE(3) \times T^*SE(3) \times T^*V \cong SE(3) \times SE(3) \times \mathfrak{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2 \to SE(3) \times \mathfrak{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2 \), given by \( \Phi((B, u)((A, v), (\Pi, w), (\alpha, l)) = ((BA, v), (\Pi, w), (\alpha, l), for any A, B \in SO(3), \Pi \in \mathfrak{so}^*(3), u, v, w \in \mathbb{R}^3, \alpha, l \in \mathbb{R}^2 \), which is also free and proper, and admits a associated \( \mathfrak{se}^* \)-equivariant momentum map \( J_Q : T^*Q \cong SE(3) \times \mathfrak{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathfrak{se}^*(3) \) for the left \( SE(3) \) action. If \( (\Pi, w) \in \mathfrak{se}^*(3) \) is a regular value of \( J_Q \), then the regular point reduced space \( (T^*Q)_{(\Pi, w)} = J_Q^{-1}(\Pi, w)/SE(3)_{(\Pi, w)} \) is symplectically diffeomorphic to the coadjoint orbit \( O_{(\Pi, w)} \times \mathbb{R}^2 \times \mathbb{R}^2 \subset \mathfrak{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2 \).

Let \( I = \text{diag}(I_1, I_2, I_3) \) be the moment of inertia of the heavy top in the body-fixed frame. Let \( J_i, i = 1, 2 \) be the moments of inertia of rotors around their rotation axes. Let \( J_{ik}, i = 1, 2, k = 1, 2, 3 \), be the moments of inertia of the \( i \)-th rotor with \( i = 1, 2 \) around the \( k \)-th principal axis with \( k = 1, 2, 3 \), respectively, and denote by \( \bar{I}_i = I_i + J_{1i} + J_{2i} - J_{ii}, i = 1, 2, \) and \( \bar{I}_3 = I_3 + J_{13} + J_{23} \). Let \( \Omega = (\Omega_1, \Omega_2, \Omega_3) \) be the vector of heavy top angular velocities computed with respect to the axes fixed in the body and \( (\Omega_1, \Omega_2, \Omega_3) \in \mathfrak{so}(3) \). Let \( \theta_i, i = 1, 2 \), be the relative angles of rotors and \( \bar{\theta} = (\bar{\theta}_1, \bar{\theta}_2) \) the vector of rotor relative angular velocities about the principal axes with respect to the body fixed frame of heavy top. Let \( m \) be that total mass of the system, \( g \) be the magnitude of the gravitational acceleration and \( h \) be the distance from the origin \( O \) to the center of mass of the system.

Consider the Lagrangian \( L(A, v, \Omega, \Gamma, \theta, \bar{\theta}) : TQ \cong SE(3) \times \mathfrak{se}(3) \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \), which is the total kinetic energy of the heavy top plus the total kinetic energy of the rotor minus the
potential energy of the system, given by

\[ L(A, v, \Omega, \theta, \dot{\theta}) = \frac{1}{2}[I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2 + J_1(\Omega_1 + \dot{\theta}_1)^2 + J_2(\Omega_2 + \dot{\theta}_2)^2] - mgh\Gamma \cdot \chi, \]

where \((A, v) \in SE(3), (\Omega, \Gamma) \in \mathfrak{se}(3)\) and \(\Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathfrak{so}(3), \Gamma \in \mathbb{R}^3, \theta = (\theta_1, \theta_2) \in \mathbb{R}^2, \dot{\theta} = (\dot{\theta}_1, \dot{\theta}_2) \in \mathbb{R}^2\). If we introduce the conjugate angular momentum, which is given by

\[ \Pi_i = \frac{\partial L}{\partial \dot{\Omega}_i} = I_i\dot{\Omega}_i + J_i(\Omega_i + \dot{\theta}_i), \quad i = 1, 2, \]
\[ \Pi_3 = \frac{\partial L}{\partial \dot{\theta}_3} = I_3\dot{\theta}_3, \quad l_i = \frac{\partial L}{\partial \theta_i} = J_i(\Omega_i + \dot{\theta}_i), \quad i = 1, 2, \]

and by the Legendre transformation \(FL : TQ \cong SE(3) \times \mathfrak{se}(3) \times \mathbb{R}^2 \times \mathbb{R}^2 \to T^*Q \cong SE(3) \times \mathfrak{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2\), \((A, v, \Omega, \Gamma, \theta, \dot{\theta}) \to (A, v, \Pi, \Gamma, \theta, l)\), where \(\Pi = (\Pi_1, \Pi_2, \Pi_3) \in \mathfrak{so}^*(3), l = (l_1, l_2) \in \mathbb{R}^2\), we have the Hamiltonian \(H(A, v, \Pi, \Gamma, \theta, l) : T^*Q \cong SE(3) \times \mathfrak{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}\) given by

\[ H(A, v, \Pi, \Gamma, \theta, l) = \Omega \cdot \Pi + \theta \cdot l - L(A, v, \Omega, \Gamma, \theta) \]
\[ = \frac{1}{2}(\Pi_1 - l_1)^2 + \frac{1}{2}(\Pi_2 - l_2)^2 + \frac{\Pi_3^2}{I_3} + \frac{l_1^2}{J_1} + \frac{l_2^2}{J_2} + mgh\Gamma \cdot \chi. \]

From the above expression of the Hamiltonian, we know that \(H(A, v, \Pi, \Gamma, \theta, l)\) is invariant under the left \(SE(3)\)-action \(\Phi : SE(3) \times T^*Q \to T^*Q\). For the case \((\Pi_0, \Gamma_0) = (\mu, a) \in \mathfrak{se}^*(3)\) is the regular value of \(J_Q\), we have the reduced Hamiltonian \(h_{(\mu, a)}(\Pi, \Gamma, \theta, l) : O_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2(\subset \mathfrak{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2) \to \mathbb{R}\) given by \(h_{(\mu, a)}(\Pi, \Gamma, \theta, l) = H(A, v, \Pi, \Gamma, \theta, l)|_{O_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2}\). From the heavy top Poisson bracket on \(\mathfrak{se}^*(3)\) and the Poisson bracket on \(T^*\mathbb{R}^2\), we can get the Poisson bracket on \(T^*\mathbb{R}^2\), that is, for \(F, K : \mathfrak{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}\), we have that

\[ \{F, K\}_-(\Pi, \Gamma, \theta, l) = -\Pi \cdot (\nabla_\Pi F \times \nabla_\Pi K) - \Gamma \cdot (\nabla_\Pi F \times \nabla_\Gamma K - \nabla_\Pi K \times \nabla_\Gamma F) + \{F, K\}_+(\theta, l). \]

In particular, for \(F_{(\mu, a)}, K_{(\mu, a)} : O_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}\), we have that

\[ \tilde{\omega}_{O_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2}(X_{F_{(\mu, a)}}, X_{K_{(\mu, a)}}) = \{F_{(\mu, a)}, K_{(\mu, a)}\}_-|_{O_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2}. \]

Moreover, for reduced Hamiltonian \(h_{(\mu, a)}(\Pi, \Gamma) : O_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}\), we have the Hamiltonian vector field \(X_{h_{(\mu, a)}}(K_{(\mu, a)}) = \{K_{(\mu, a)}, h_{(\mu, a)}\}_-|_{O_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2}\).

Assume that \(\gamma : SE(3) \times \mathbb{R}^2 \to T^*(SE(3) \times \mathbb{R}^2)\) is an one-form on \(SE(3) \times \mathbb{R}^2\), and \(\gamma\) is closed with respect to \(T\pi_{SE(3) \times \mathbb{R}^2} : TT^*(SE(3) \times \mathbb{R}^2) \to T(SE(3) \times \mathbb{R}^2)\), and \(\text{Im}(\gamma) \subset J^{-1}(\mu, a)\), and it is \(SE(3)_{(\mu, a)}\)-invariant, and \(\tilde{\gamma} = \pi_{(\mu, a)}(\gamma) : SE(3) \times \mathbb{R}^2 \to O_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2\). Denote by \(\tilde{\gamma}(A, v, \theta) = (\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3, \tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\Gamma}_3, \tilde{\gamma}_4, \tilde{\gamma}_5, \tilde{\gamma}_6, \tilde{\gamma}_7)(A, v, \theta) \in O_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2(\subset \mathfrak{se}^*(3) \times \mathbb{R}^2 \times \mathbb{R}^2)\), then \(h_{(\mu, a)} \cdot \tilde{\gamma} : SE(3) \times \mathbb{R}^2 \to \mathbb{R}\) is given by

\[ h_{(\mu, a)} \cdot \tilde{\gamma}(A, v, \theta) = H \cdot \tilde{\gamma}(A, v, \theta)|_{O_{(\mu, a)} \times \mathbb{R}^2 \times \mathbb{R}^2} \]
\[ = \frac{1}{2} \left( \frac{(\tilde{\gamma}_1 - \tilde{\gamma}_6)^2}{I_1} + \frac{(\tilde{\gamma}_2 - \tilde{\gamma}_7)^2}{I_2} + \frac{\tilde{\gamma}_3^2}{I_3} + \frac{\tilde{\gamma}_4^2}{J_1} + \frac{\tilde{\gamma}_5^2}{J_2} \right) + mgh\Gamma \cdot \chi. \]
and the vector field

\[ X_{\Pi(A, v)} \cdot \gamma(\Pi) = \{ \Pi, h_{(\mu, a)} \cdot \gamma(A, v) \} - O(\mu, a) \times \mathbb{R}^2 \times \mathbb{R}^2 \]

\[ = - \Pi \cdot (\nabla_{\Pi} \times \nabla_{\Pi}(h_{(\mu, a)} \cdot \gamma)) - \Gamma \cdot (\nabla_{\Pi} \times \nabla_{\Gamma}(h_{(\mu, a)} \cdot \gamma)) - \nabla_{\Pi}(h_{(\mu, a)} \cdot \gamma) \times \nabla_{\Gamma} \Pi \\
+ \{ \Gamma, h_{(\mu, a)} \cdot \gamma(A, v) \} \Pi^2 \cdot O(\mu, a) \times \mathbb{R}^2 \times \mathbb{R}^2 \]

\[ = - \nabla_{\Pi} \Pi \cdot (\nabla_{\Pi}(h_{(\mu, a)} \cdot \gamma) \times \Pi) - \nabla_{\Pi} \Pi \cdot (\nabla_{\Gamma}(h_{(\mu, a)} \cdot \gamma) \times \Gamma) \\
+ \frac{2}{\sum_{i=1}^{2} \left( \frac{\partial \Pi}{\partial \theta_i} \frac{\partial (h_{(\mu, a)} \cdot \gamma)}{\partial l_i} - \frac{\partial (h_{(\mu, a)} \cdot \gamma)}{\partial \theta_i} \frac{\partial \Pi}{\partial l_i} \right)} \]

\[ = (\Pi_1, \Pi_2, \Pi_3) \times \left( \frac{\gamma_1 - \gamma_6}{I_1}, \frac{\gamma_2 - \gamma_7}{I_2}, \frac{\gamma_3}{I_3} \right) + mgh(\Gamma_1, \Gamma_2, \Gamma_3) \times (\chi_1, \chi_2, \chi_3) \]

\[ = \left( \frac{I_2 \Pi_2 \gamma_3 - I_3 \Pi_3 (\gamma_2 - \gamma_7)}{I_2 I_3}, \frac{I_3 \Pi_3 \gamma_1 (\gamma_2 - \gamma_6) - I_1 \Pi_1 \gamma_3}{I_3 I_1}, \frac{I_1 \Pi_1 (\gamma_2 - \gamma_7) - I_2 \Pi_2 (\gamma_1 - \gamma_6)}{I_1 I_2} \right) + mgh(\Gamma_2 \chi_3 - \Gamma_3 \chi_2), \]

since \( \nabla_{\Pi} \Pi = 1, \ nabla_{\Gamma} \Pi = 0, \ \frac{\partial \Pi}{\partial \theta_i} = \frac{\partial \Pi}{\partial l_i} = 0, \ i = 1, 2, \ Gamma = (\Gamma_1, \Gamma_2, \Gamma_3), \chi = (\chi_1, \chi_2, \chi_3), \) and \( \nabla_{\Pi}(h_{(\mu, a)} \cdot \gamma) = (\gamma_j - \gamma_{j+5})/I_j, \ j = 1, 2, \nabla_{\Pi}(h_{(\mu, a)} \cdot \gamma) = \gamma_3/I_3. \)

\[ X_{h(\mu, a)} \cdot \gamma(\Gamma) = \{ \Gamma, h_{(\mu, a)} \cdot \gamma(A, v) \} - O(\mu, a) \times \mathbb{R}^2 \times \mathbb{R}^2 \]

\[ = - \Pi \cdot (\nabla_{\Pi} \Gamma \times \nabla_{\Pi}(h_{(\mu, a)} \cdot \gamma)) - \Gamma \cdot (\nabla_{\Pi} \Gamma \times \nabla_{\Gamma}(h_{(\mu, a)} \cdot \gamma)) - \nabla_{\Pi}(h_{(\mu, a)} \cdot \gamma) \times \nabla_{\Gamma} \Gamma \\
+ \{ \Gamma, h_{(\mu, a)} \cdot \gamma(A, v) \} \Pi^2 \cdot O(\mu, a) \times \mathbb{R}^2 \times \mathbb{R}^2 \]

\[ = \nabla_{\Gamma} \Gamma \cdot (\Gamma \times \nabla_{\Pi}(h_{(\mu, a)} \cdot \gamma)) + \frac{2}{\sum_{i=1}^{2} \left( \frac{\partial \Gamma}{\partial \theta_i} \frac{\partial (h_{(\mu, a)} \cdot \gamma)}{\partial l_i} - \frac{\partial (h_{(\mu, a)} \cdot \gamma)}{\partial \theta_i} \frac{\partial \Gamma}{\partial l_i} \right)} \]

\[ = (\Gamma_1, \Gamma_2, \Gamma_3) \times \left( \frac{\gamma_1 - \gamma_6}{I_1}, \frac{\gamma_2 - \gamma_7}{I_2}, \frac{\gamma_3}{I_3} \right) \]

\[ = \left( \frac{I_2 \Gamma_2 \gamma_3 - I_3 \Gamma_3 (\gamma_2 - \gamma_7)}{I_2 I_3}, \frac{I_3 \Gamma_3 \gamma_1 (\gamma_2 - \gamma_6) - I_1 \Gamma_1 \gamma_3}{I_3 I_1}, \frac{I_1 \Gamma_1 (\gamma_2 - \gamma_7) - I_2 \Gamma_2 (\gamma_1 - \gamma_6)}{I_1 I_2} \right) \]

since \( \nabla_{\Gamma} \Gamma = 1, \ \nabla_{\Pi} \Gamma = 0, \ \frac{\partial \Gamma}{\partial \theta_i} = \frac{\partial \Gamma}{\partial l_i} = 0, \ i = 1, 2. \)

\[ X_{h(\mu, a)} \cdot \gamma(\theta) = \{ \theta, h_{(\mu, a)} \cdot \gamma(A, v) \} - O(\mu, a) \times \mathbb{R}^2 \times \mathbb{R}^2 \]

\[ = - \Pi \cdot (\nabla_{\Pi} \theta \times \nabla_{\Pi}(h_{(\mu, a)} \cdot \gamma)) - \Gamma \cdot (\nabla_{\Pi} \theta \times \nabla_{\Gamma}(h_{(\mu, a)} \cdot \gamma)) - \nabla_{\Pi}(h_{(\mu, a)} \cdot \gamma) \times \nabla_{\Gamma} \theta \\
+ \{ \theta, h_{(\mu, a)} \cdot \gamma(A, v) \} \Pi^2 \cdot O(\mu, a) \times \mathbb{R}^2 \times \mathbb{R}^2 \]

\[ = \frac{2}{\sum_{i=1}^{2} \left( \frac{\partial \theta}{\partial l_i} \frac{\partial (h_{(\mu, a)} \cdot \gamma)}{\partial l_i} - \frac{\partial (h_{(\mu, a)} \cdot \gamma)}{\partial \theta_i} \frac{\partial \theta}{\partial l_i} \right)} \]

\[ = (-\frac{\gamma_1 - \gamma_6}{I_1} + \frac{\gamma_6}{J_1}, -\frac{\gamma_2 - \gamma_7}{I_2} + \frac{\gamma_7}{J_2}) \]

since \( \nabla_{\Pi} \theta = 0, \ \frac{\partial \theta}{\partial \theta_i} = 1, \ \frac{\partial \theta}{\partial l_i} = 0, \ j \neq i, \ \frac{\partial \theta}{\partial l_i} = 0, \ i = 1, 2. \)
\[ X_{h(\mu,a)} \gamma(l) = \{ l, h_\omega(\mu,a) \cdot \gamma(A,v) \} - |\omega(\mu,a) \times \mathbb{R}^2| \]

\[ = -\Pi \cdot (\nabla \Pi \times \nabla (h_\omega(\mu,a) \cdot \gamma)) - \Gamma \cdot (\nabla \Pi \times \nabla (h_\omega(\mu,a) \cdot \gamma)) - \nabla (h_\omega(\mu,a) \cdot \gamma) \times \nabla \Pi \]

\[ + \{ l, h_\omega(\mu,a) \cdot \gamma(A,v) \} \}_{\mathbb{R}^2} |\omega(\mu,a) \times \mathbb{R}^2| \]

\[ = \sum_{i=1}^2 \left( \frac{\partial}{\partial \theta_i} \frac{\partial (h_\omega(\mu,a) \cdot \gamma)}{\partial l} - \frac{\partial (h_\omega(\mu,a) \cdot \gamma)}{\partial \theta_i} \frac{\partial l}{\partial l} \right) = (0,0), \]

since \( \nabla \Pi l = \nabla \Gamma l = 0 \), \( \frac{\partial l}{\partial \theta_i} = 0 \), and \( \frac{\partial (h_\omega(\mu,a) \cdot \gamma)}{\partial l} = 0 \), \( i = 1, 2 \). From Theorem 5.1, if we consider the heavy top-rotor system with a control torque \( u : T^*Q \rightarrow T^*Q \) acting on the rotors, and \( u \in W \subset J_Q^{-1}(\mu,a) \) is invariant under the left \( SE(3) \)-action, and its reduced control torque \( u_\omega(\mu,a) : O(\mu,a) \times \mathbb{R}^2 \rightarrow O(\mu,a) \times \mathbb{R}^2 \) is given by \( u_\omega(\mu,a)(\Pi, \Gamma, \theta, l) = \pi(\mu,a)(u(A,v, \Pi, \Gamma, \theta, l)) = u(A,v, \Pi, \Gamma, \theta, l)(O(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2) \), where \( \pi(\mu,a) : J_Q^{-1}(\mu,a) \rightarrow O(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2 \). The dynamical vector field of \( R \)-reduced \( R \)CH system \( (O(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2, \tilde{\omega}^{-}_O(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2, h_\omega(\mu,a), u_\omega(\mu,a)) \) is given by

\[ X_{(O(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2, \tilde{\omega}^{-}_O(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2, h_\omega(\mu,a), u_\omega(\mu,a))} = X_{h(\mu,a)} + \text{vlift}(u(\mu,a)), \]

where \( \text{vlift}(u(\mu,a)) = \text{vlift}(u(\mu,a))X_{h(\mu,a)} \in T(O(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2) \). Assume that \( \text{vlift}(u(\mu,a) \cdot \gamma)(A,v, \theta) = (U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8, U_9, U_{10})(A,v, \theta) \in T(O(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2) \), then the Hamilton-Jacobi equations for heavy top-rotor system with the control torque \( u \) acting on the rotors are given by

\[
\begin{aligned}
\hat{I}_2 \Pi_2 \gamma_3 - \hat{I}_3 \Pi_3 (\gamma_2 - \gamma_7) + mgh \hat{I}_2 \hat{I}_3 (\Gamma_2 \chi_3 - \Gamma_3 \chi_2) + \hat{I}_2 \hat{I}_3 U_1 &= 0, \\
\hat{I}_3 \Pi_3 (\gamma_1 - \gamma_6) - \hat{I}_1 \Pi_1 \gamma_3 + mgh \hat{I}_3 \hat{I}_1 (\Gamma_3 \chi_1 - \Gamma_1 \chi_3) + \hat{I}_3 \hat{I}_1 U_2 &= 0, \\
\hat{I}_1 \Pi_1 (\gamma_2 - \gamma_7) - \hat{I}_2 \Pi_2 (\gamma_1 - \gamma_6) + mgh \hat{I}_1 \hat{I}_2 (\Gamma_1 \chi_2 - \Gamma_2 \chi_1) + \hat{I}_1 \hat{I}_2 U_3 &= 0, \\
\hat{I}_1 \hat{I}_2 \gamma_3 - \hat{I}_2 \hat{I}_3 (\gamma_2 - \gamma_7) + \hat{I}_2 \hat{I}_3 U_4 &= 0, \\
\hat{I}_2 \hat{I}_3 (\gamma_1 - \gamma_6) - \hat{I}_1 \hat{I}_1 \gamma_3 + \hat{I}_3 \hat{I}_1 U_5 &= 0, \\
\hat{I}_1 \gamma_1 (\gamma_2 - \gamma_7) - \hat{I}_2 \gamma_2 (\gamma_1 - \gamma_6) + \hat{I}_1 \hat{I}_2 U_6 &= 0, \\
- J_1 (\gamma_1 - \gamma_6) + \hat{I}_1 \hat{I}_1 U_7 &= 0, \\
J_2 (\gamma_2 - \gamma_7) + \hat{I}_2 \hat{I}_2 U_8 &= 0, \\
U_9 &= U_{10} = 0.
\end{aligned}
\]

To sum up the above discussion, we have the following proposition.

**Proposition 5.4** The 5-tuple \((T^*Q, SE(3), \omega_Q, H, u)\), where \( Q = SE(3) \times \mathbb{R}^2 \), is a regular point reducible \( R \)CH system. For a point \((\mu,a) \in se^*(3)\), the regular value of the momentum map \( J_Q : SE(3) \times se^*(3) \times \mathbb{R}^2 \rightarrow se^*(3) \), the \( R \)-reduced system is the 4-tuple \((O(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2, \tilde{\omega}^{-}_O(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2, h_\omega(\mu,a), u_\omega(\mu,a))\), where \( O(\mu,a) \subset se^*(3) \) is the coadjoint orbit, \( \tilde{\omega}^{-}_O(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2 \) is orbit symplectic form on \( O(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2, h_\omega(\mu,a), u_\omega(\mu,a) \), and \( u_\omega(\mu,a) : O(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow T(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2 \) is orbit symplectic form on \( O(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2, h_\omega(\mu,a), u_\omega(\mu,a) \). Assume that \( \gamma : SE(3) \times \mathbb{R}^2 \rightarrow T^*(SE(3) \times \mathbb{R}^2) \) is an one-form on \( SE(3) \times \mathbb{R}^2 \), and \( \gamma \) is closed with respect to \( T \pi_{SE(3) \times \mathbb{R}^2} : T^*(SE(3) \times \mathbb{R}^2) \rightarrow T(SE(3) \times \mathbb{R}^2), \) and \( Im(\gamma) \subset J^{-1}(\mu,a) \), and it is \( SE(3)_{(\mu,a)} \)-invariant, and \( \tilde{\gamma} = \pi(\mu,a)(\gamma) : SE(3) \times \mathbb{R}^2 \rightarrow O(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2 \). Then \( \gamma \) is a solution of either the Hamilton-Jacobi equation of heavy top with two pairs of symmetric internal rotors given by (5.9), or the equation \( X_{h_\omega(\mu,a)} \gamma + \text{vlift}(u(\mu,a) \cdot \gamma) = X\gamma, \) if and only if \( \hat{X} \gamma \) and \( \hat{X}_{(\mu,a)} \) are \( \gamma \)-related, where \( \hat{X} = X(T^*Q, SE(3), \omega_Q, H, u) \), and \( \hat{X}_{(\mu,a)} = X(O(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2, \tilde{\omega}^{-}_O(\mu,a) \times \mathbb{R}^2 \times \mathbb{R}^2, h_\omega(\mu,a), u_\omega(\mu,a)) \).
When the heavy top does not carry any internal rotor, in this case $Q = G = SE(3)$, and the heavy top is a regular point reducible Hamiltonian system $(T^*SE(3), SE(3), \omega, H)$, and hence it is also a regular point reducible RCH system without the external force and control. For a point $(\mu, a) \in \mathfrak{se}^*(3)$, the regular value of the momentum map $J : T^*SE(3) \rightarrow \mathfrak{se}^*(3)$, the Marsden-Weinstein reduced system is 3-tuple $(\mathcal{O}(\mu,a), \omega_{\mathcal{O}(\mu,a)}, h_{\mathcal{O}(\mu,a)})$, where $\mathcal{O}(\mu,a) \subset \mathfrak{se}^*(3)$ is the coadjoint orbit, $\omega_{\mathcal{O}(\mu,a)}$ is orbit symplectic form on $\mathcal{O}(\mu,a)$, which is induced by the heavy top Poisson bracket on $\mathfrak{se}^*(3)$, $h_{\mathcal{O}(\mu,a)}(\Pi, \Gamma) = H(A, v, \Pi, \Gamma)|_{\mathcal{O}(\mu,a)}$. From Wang [25] we know that the Lie-Poisson Hamilton-Jacobi equation of heavy top is given by

\[
\begin{align*}
I_2 \Pi_2 \gamma_3 - I_3 \Pi_3 \gamma_2 + mgh I_2 I_3 (\Gamma_2 \chi_3 - \Gamma_3 \chi_2) &= 0, \\
I_3 \Pi_3 \gamma_1 - I_1 \Pi_1 \gamma_3 + mgh I_3 I_1 (\Gamma_3 \chi_1 - \Gamma_1 \chi_3) &= 0, \\
I_1 \Pi_1 \gamma_2 - I_2 \Pi_2 \gamma_1 + mgh I_1 I_2 (\Gamma_1 \chi_2 - \Gamma_2 \chi_1) &= 0, \\
I_2 \Gamma_2 \gamma_3 - I_3 \Gamma_3 \gamma_2 &= 0, \\
I_3 \Gamma_3 \gamma_1 - I_1 \Gamma_1 \gamma_3 &= 0, \\
I_1 \Gamma_1 \gamma_2 - I_2 \Gamma_2 \gamma_1 &= 0.
\end{align*}
\tag{5.10}
\]

On the other hand, from Marsden et al [18] we know that as two $R_P$-reduced RCH systems, the rigid body with internal rotors and the heavy top are RCH-equivalent. If $\varphi : Q_1 = SO(3) \times \mathbb{R}^3 \rightarrow Q_2 = SE(3)$ is a diffeomorphism, and from Remark 5.3 and Theorem 3.6 there is a induced RCH-equivalent map $\varphi^{*}_{\mu/G} : \mathcal{O}_{(\mu,a)} \rightarrow \mathcal{O}_{\mu} \times \mathbb{R}^3$, $(\Pi, \Gamma) \rightarrow (\Pi, I)$, where $\mathcal{O}_{\mu} \times \mathbb{R}^3 \subset \mathfrak{so}^*(3) \times \mathbb{R}^3$ and $\mathcal{O}_{(\mu,a)} \subset \mathfrak{se}^*(3)$ are coadjoint orbits. Moreover, from RCH-equivalence and Theorem 2.4, we know that there exists a control law $u : T^*(SO(3) \times \mathbb{R}) \rightarrow W$, and its reduced control law $u_{\mu : \mathcal{O}_{\mu} \times \mathbb{R}^3 \rightarrow W_{\mu}$. Such that $\text{vlift}(u_{\mu}) = -X_{h_{\mu}} + T \varphi^{*}_{\mu/G} : X_{h_{\mathcal{O}_{(\mu,a)}}}$. In consequence, if $\gamma_b : SO(3) \times \mathbb{R}^3 \rightarrow T^*(SO(3) \times \mathbb{R}^3)$ is an one-form on $SO(3) \times \mathbb{R}^3$, and $\gamma_b$ is closed with respect to $T \pi_{SO(3) \times \mathbb{R}^3} : TT^*(SO(3) \times \mathbb{R}^3) \rightarrow T(SO(3) \times \mathbb{R}^3)$, and $\text{Im}(\gamma_b) \subset J_{Q}^{-1}(\mu)$, and it is $G_{\mu}$-invariant, and $\gamma_b = \pi_{\mu}(\gamma_b) : SO(3) \times \mathbb{R}^3 \rightarrow \mathcal{O}_{\mu} \times \mathbb{R}^3$, is a solution of Hamilton-Jacobi equation of rigid body with three symmetric internal rotors given by (5.7), then from Theorem 2.7, we know that $\tilde{\gamma}_t = (\varphi^{-1})^{*}_{\mu/G} \cdot \gamma_b \cdot (\varphi)^{-1} : SE(3) \rightarrow \mathcal{O}_{(\mu,a)}$, is a solution of Lie-Poisson Hamilton-Jacobi equation of heavy top given by (5.10).

The theory of mechanical control system is a very important subject, following the theoretical development of geometric mechanics, a lot of important problems about this subject are being explored and studied. In this paper, we study the Hamilton-Jacobi theory of regular controlled Hamiltonian systems with the symplectic structure and symmetry. But if we define a controlled Hamiltonian system on the cotangent bundle $T^*Q$ by using a Poisson structure, see Wang and Zhang in [26] and Ratiu and Wang in [22], and the way given in this paper cannot be used, what and how we could do? This is a problem worthy to be considered in detail. On the other hand, we also note that there have been a lot of results in recent years about reduction and Hamilton-Jacobi theory of (nonholonomic) mechanical systems on Lie algebroids and Lie groupoids, see Balseiro et al [3], León et al [12] and Barbero-Liñán et al [4]. Thus, it is an important topic to study the reduction and Hamilton-Jacobi theory of controlled mechanical systems on Lie algebroids and Lie groupoids, and it needs the deeper understanding for the structures of Lie algebroid and Lie groupoid and controlled mechanical systems, and for the reduction and Hamilton-Jacobi theory. This is our goal in future research.
References

[1] R. Abraham, J.E. Marsden, Foundations of Mechanics, second ed., Addison-Wesley, Reading, MA, 1978.

[2] V.I. Arnold, Mathematical Methods of Classical Mechanics, second ed., in: Graduate Texts in Mathematics, vol. 60, Springer-Verlag, 1989.

[3] P. Balseiro, M. de León, J.C. Marrero and D. Martín de Diego, The ubiquity of the symplectic Hamiltonian equations in mechanics, J. Geom. Mech. 1(2009), No.1, 1-34.

[4] M. Barbero-Liñán, M. de León, J.C. Marrero and D. Martín de Diego and M.C. Muñoz-Lecanda, Kinematic reduction and the Hamilton-Jacobi equation, J. Geom. Mech. 4(2012), No.3, 207-237.

[5] J.F. Cariñena, X. Gràcia, G. Marmo, E. Martínez, M. Muñoz-Lecanda and N. Román-Roy: Geometric Hamilton-Jacobi theory, Int. J. Geom. Methods Mod. Phys. 3(2006), 1417-1458.

[6] J.F. Cariñena, X. Gràcia, G. Marmo, E. Martínez, M. Muñoz-Lecanda and N. Román-Roy: Geometric Hamilton-Jacobi theory for nonholonomic dynamical systems, Int. J. Geom. Methods Mod. Phys. 7(2010), 431-454.

[7] Z. Ge and J.E. Marsden, Lie-Poisson integrators and Lie-Poisson Hamilton-Jacobi theory, Phys. Lett. A, 133(1988), 134-139.

[8] D. Iglesias-Ponte, M. de León and D. Martín de Diego: Towards a Hamilton-Jacobi theory for nonholonomic mechanical systems, J. Phys. A: Math. Theor. 41(2008), 1-14.

[9] P.S. Krishnaprasad and J.E. Marsden, Hamiltonian structure and stability for rigid bodies with flexible attachments, Arch. Rat. Mech. An. 98(1987), 137–158.

[10] J-A Lázaro-Camí and J-P Ortega, The stochastic Hamilton-Jacobi equation, J. Geom. Mech. 1(2009), 295-315.

[11] M. de León, J.C. Marrero and D. Martín de Diego, A geometric Hamilton-Jacobi theory for classical field theories, In: Variations, Geometry and Physics, Nova Sci. Publ., New York, 2009, 129-140.

[12] M. de León, J.C. Marrero and D. Martín de Diego, Linear almost Poisson structures and Hamilton-Jacobi equation, Applications to nonholonomic mechanics, J. Geom. Mech. 2(2010), 159-198.

[13] P. Libermann, C.M. Marle, Symplectic Geometry and Analytical Mechanics, Kluwer Academic Publishers, 1987.

[14] J.E. Marsden, Lectures on Mechanics, in: London Mathematical Society Lecture Notes Series, vol. 174, Cambridge University Press, 1992.

[15] J.E. Marsden, G. Misiolek, J.P. Ortega, M. Perlmutter, T.S. Ratiu, Hamiltonian Reduction by Stages, in: Lecture Notes in Mathematics, vol. 1913, Springer, 2007.

[16] J.E. Marsden, R. Montgomery, T.S. Ratiu, Reduction, Symmetry and Phases in Mechanics, in: Memoirs of the American Mathematical Society, vol. 88, American Mathematical Society, Providence, Rhode Island, 1990.
[17] J.E. Marsden, T.S. Ratiu, Introduction to Mechanics and Symmetry, second ed., in: Texts in Applied Mathematics, vol. 17, Springer-Verlag, New York, 1999.

[18] J.E. Marsden, H. Wang, Z.X. Zhang, Regular reduction of controlled Hamiltonian system with symplectic structure and symmetry, (2010, arXiv: 1202.3564, To submit to Diff. Geom. Appl.).

[19] J.E. Marsden and M. West, Discrete mechanics and variational integrators, Acta Numerica, (2001) 357-514.

[20] T. Ohsawa and A.M. Bloch, Nonholonomic Hamilton-Jacobi equation and integrability, J. Geom. Mech., 1 (2009), 461-481.

[21] J.P. Ortega, T.S. Ratiu, Momentum Maps and Hamiltonian Reduction, in: Progress in Mathematics, vol. 222, Birkhäuser, 2004.

[22] T.S.Ratiu and H.Wang, Poisson reduction of controlled Hamiltonian system by controllability distribution, (2012).

[23] L. Vitagliano, The Hamilton-Jacobi formalism for higher order field theories, Int. J. Geom. Methods Mod. Phys., 7(2010) 8, 1413-1436.

[24] H. Wang, Singular reduction of regular controlled Hamiltonian system with symmetry, (2012).

[25] H. Wang, Hamilton-Jacobi theorem for regular reducible Hamiltonian system on a cotangent bundle, (2013a, arXiv: 1303.5840, To submit to Jour. Geom. Phys.).

[26] H.Wang and Z.X.Zhang, Optimal reduction of controlled Hamiltonian system with Poisson structure and symmetry, Jour. Geom. Phys., 62 (5)(2012), 953-975.