Solution curve for linear control systems on Lie groups

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Abstract
The purpose of this paper is to describe explicitly the solution for linear control system on Lie groups. Furthermore, we present some applications when a linear control systems has an inner derivation and we study some aspects of controllability.

Keywords
Linear control system · Solutions · Controllability

Mathematics Subject Classification
93B05 · 93C25 · 34H05

1 Introduction
Linear control systems on $\mathbb{R}^n$ are given by the differential equations

$$\dot{x} = Ax + Bu, \ x \in \mathbb{R}^n,$$

(1)

where $A$ is a $n \times n$-matrix, $B$ is a $n \times m$, and $u = (u_1, u_2, \ldots, u_m)$ are the admissible control functions. Markus in [17] introduced the linear control to matrix groups aiming to study controllability of this control system. Later, Ayala and Tirao [2] extended the research of the linear control system for general Lie groups. Recall that a linear control system on a
connected Lie group $G$ is a control system given by the differential equations

$$
\Sigma_L: \dot{g} = \frac{dg}{dt} = \mathcal{X}(g) + \sum_{j=1}^{m} u_j Y_j(g), \quad g \in G,
$$

(2)

where $\mathcal{X}$ is a linear vector field (we note that its flow $\varphi_t$ is a family of automorphisms of $G$), $Y_1, \ldots, Y_m$ are right-invariant vector fields and the admissible control functions $u: \mathbb{R} \to U \subset \mathbb{R}^m$ belong to a subset $\mathcal{U} \subset L^1_{loc}(\mathbb{R}; \mathbb{R}^m)$ of the space of the locally integrable functions.

Despite linear control systems (1) have good descriptions of their solutions (see for instance [1]), this is not true in case of systems on Lie groups (2). The first result in this context are found in [2]. Later, Ayala, da Silva and Kizil [3] improve this result. In both cases the solution is described as a series of functions. In consequence, the solutions presented in [2, 3] depends of an interval convergence, which may be bounded. Our purpose is to describe the solutions of (2) using a technique of Cardetti and Mittenhuber [9], that is, considering an invariant control system $\Sigma_I$ that is the lift of (2) on the semi-direct product $G \times_{\varphi} \mathbb{R}$. In fact, in Theorem 9 we present the solution of the invariant control system $\Sigma_I$ and, consequently, we obtain the solution of the linear system $\Sigma_L$. When derivation $\mathcal{D}$ associated to the linear vector field $\mathcal{X}$ is inner then the solution of linear control system has a simpler description as showed in Theorem 12.

From this study it is natural to think that results of the controllability of the invariant system $\Sigma_I$ imply ones of linear control system (2) because $\Sigma_I$ is projected in the linear control system (2). A little more, as $\Sigma_I$ is an invariant control system there are many results in literature about controllability. However, we show that $\Sigma_I$ is not controllable and that there is not a control set associated to $\Sigma_I$ as one can see in Proposition 14 and Theorem 15. Despite this, we use the invariant control system $\Sigma_I$ to show that linear control system (2) is controllable on $G$ if (2) is controllable at origin and if $\mathcal{X}$ has periodic orbits. For more results we refer the reader to [4, 7, 9, 12, 13, 15, 20] for controllability and to [5, 6] for control sets.

The paper is organized as follow, in the second section we establish some basic facts about linear control systems. In the third section we construct the solution of these systems. In fourth section we study solutions of linear control system (2) when the derivation is inner, and, as application of this section, we can construct solutions on matrix groups $GL(n, \mathbb{R})^+$ and on all 3-dimensional, semisimple Lie groups: $SL(2, \mathbb{R}), SU(2), SO(3, \mathbb{R})$, and $SO(2, 1)$. Finally, in fifth section we show that $\Sigma_I$ is not controllable and that there is not a control set associated to $\Sigma_I$.

## 2 Linear vector fields

In this section we recall some necessary facts about linear vector fields and linear control systems (as usual we call just as linear systems) on Lie groups (see e.g. [15] for more details). Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. Throughout this paper, $\mathfrak{g}$ is the set of right invariant vector fields. For every $g \in G$, the maps $R_g, L_g : G \to G$ are, respectively, the right and left translations on $G$.

A vector field $\mathcal{X}$ on $G$ is called linear if its flow, which is denoted by $\varphi_t$, is an one-parameter group of automorphisms. In [15] it is showed that a linear vector field $\mathcal{X}$ can be characterized by one of the following conditions:

(i) for all $t \in \mathbb{R}$, $\varphi_t$ is an automorphism of $G$;

(ii) for all $Y \in \mathfrak{g}$, $[\mathcal{X}, Y] \in \mathfrak{g}$ and $\mathcal{X}(e) = 0$, where $e$ is identity of $G$; and,

(iii) for all $g, h \in G$, $\mathcal{X}(gh) = d(R_h)_g \mathcal{X}(g) + d(L_g)_h \mathcal{X}(h)$.
For any linear vector field $\mathcal{X}$ it is possible to associate a derivation $\mathcal{D}: g \to g$ defined by $\mathcal{D}(Y) = -[\mathcal{X}, Y]$. From the derivation $\mathcal{D}$ and a linear flow $\varphi$ given by $\mathcal{X}$ we can see that

$$d(\varphi_t)_e = e^{t\mathcal{D}}$$

and $\varphi_t(\exp(Y)) = \exp(e^{t\mathcal{D}}Y)$.

A particular case of derivations are the inner derivations, that is, the derivation is written as $\mathcal{D} = -ad(X)$ where $X \in g$ comes from the decomposition of linear vector field $\mathcal{X} = X + dI X$. Here, $dI X$ is the left-invariant vector field induced by the inverse map $I(g) = g^{-1}$.

For $u \in U$ and $g \in G$, we denote the solution of the linear control system $\Sigma_L$ given by (2) starting at $g$ associated to $u$ by $\phi_L(t, g, u)$ with $t \in \mathbb{R}$. For $g, h \in G$, we say that $h$ is reachable from $g$ in time $t$ if there is $u$ such that $\phi_L(t, g, u) = h$. Let us denote by $A_t(g)$ the set of all points in $G$ reachable from $g$ in time $t$. The reachable set from $g$ is defined as follows

$$A(g) = \bigcup_{t \geq 0} A_t(g).$$

The system $\Sigma_L$ is said to be controllable if any $h$ is reachable for any $g$. Equivalently, $A(g) = G$ for any $g \in G$.

### 3 Solution for linear systems

In this section we come up with a construction presented in [9] which associates to a linear system $\Sigma_L$ an invariant one, denoted by $\Sigma_I$. We begin by remembering that the flow $\varphi_t$ of the linear vector field $\mathcal{X}$ yields a representation $\varphi: \mathbb{R} \to Aut(G)$. This allows us to define the semi-direct product $G \times_{\sigma} \mathbb{R}$, that is, the set $G \times \mathbb{R}$ endowed with the product $(g, t)(h, s) = (g\varphi_t(h), t + s)$ (see e.g. [19]). It is well-known that $G \times_{\sigma} \mathbb{R}$ is a Lie group. Furthermore, its correspondent Lie algebra is the semi-direct product of Lie algebras $g \times_{\sigma} \mathbb{R}$, where $\sigma: \mathbb{R} \to Der(g)$ is defined by

$$\sigma(t)(Y) = ad_{\mathcal{X}}(Y) = [t\mathcal{X}, Y].$$

The relation between $\varphi$ and $\sigma$ is given by $d\varphi_0 = \sigma$.

For any vector fields $(Y, t), (W, s) \in g \times_{\sigma} \mathbb{R}$, the Lie bracket is given by the formula

$$[(Y, t)(W, s)] = ([Y + t\mathcal{X}, W + s\mathcal{X}], 0).$$

For $(g, r) \in G \times_{\sigma} \mathbb{R}$, $R_{(g, r)}$ denotes the right translation and, when not specified, $dR_{(g, r)}$ means that the differential is evaluated at the group identity.

Now we determine the value of a vector field $(W, s)$ on an arbitrary point $(g, r)$.

**Proposition 1** If $(W, s) \in g \times_{\sigma} \mathbb{R}$ and $(g, r) \in G \times_{\sigma} \mathbb{R}$, then

$$(W, s)(g, r) = (W(g) + s\mathcal{X}(g), s).$$

**Proof** We first write $(W, s)(g, r) = dR_{(g, r)}(W, s)$. Thus, in matrix notation, the differential $dR_{(g, r)}$ gives

$$dR_{(g, r)}(W, s) = \begin{pmatrix} dR_g & \mathcal{X}(g) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} W \\ s \end{pmatrix} = (W(g) + s\mathcal{X}(g), s).$$

From this Proposition we have
Lemma 2 If \((W, 0), (0, s) \in g \times_{\sigma} \mathbb{R}\), then their exponentials are smooth curves \((\exp(tW), 0)\) and \((e, st)\), respectively, with \(t \in \mathbb{R}\).

Proof We first compute the exponential for \((W, 0)\). First from Proposition 1 we note that \((W, 0)(g, r) = (W(g), 0)\) for all \((g, r) \in G \times_{\varphi} \mathbb{R}\). By definition of exponential,

\[
\frac{d}{dt} \exp(tW), 0 \right) = \left( \frac{d}{dt} \exp(tW), 0 \right) = (W(\exp(tW)), 0) = (W, 0)(\exp(tW), 0).
\]

The result follows by uniqueness of solution. Analogously, we can see that the curve \((e, st)\) is the exponential of \((0, s)\).

In the following, we use the previous Lemma to determine the exponential of an invariant vector field \((W, s) \in g \times_{\sigma} \mathbb{R}\).

Proposition 3 Let \((W, s)\) be an invariant vector field on \(G \times_{\sigma} \mathbb{R}\). It follows that

\[
\exp(t(W, s)) = \left( \lim_{n \to \infty} \prod_{i=0}^{n-1} \varphi \circ \exp(t/n \cdot W), st \right) = \left( \lim_{n \to \infty} \prod_{i=0}^{n-1} \varphi \circ \exp(t/n \cdot W), st \right).
\]

Proof We first write \((W, s) = (W, 0) + (0, s)\). Applying the Lie product formula we have

\[
\exp(t(W, s)) = \lim_{n \to \infty} \exp((t/n(W, 0)) \cdot \exp(t/n(0, s)))^n.
\]

Using the above Lemma and the semi-direct product we see that

\[
\exp(t(W, s)) = \lim_{n \to \infty} ((\exp(t/n \cdot W), 0)(e, st/n))^n
\]

\[
= \lim_{n \to \infty} (\exp(t/n \cdot W), st/n)^n
\]

\[
= \left( \lim_{n \to \infty} \prod_{i=0}^{n-1} \varphi \circ \exp(t/n \cdot W), st \right),
\]

and the proof is complete.

Using the relation \(d(\varphi_t)_e = e^{tD}\) we can rewrite Formula (3) as

\[
\exp(t(W, s)) = \left( \lim_{n \to \infty} \prod_{i=0}^{n-1} \exp\left( t/n \cdot e^{\bar{D}t} \right), st \right),
\]

where, for simplification, we denote \(\bar{D}_t = istD\).

Consider the vector fields \(\bar{X} = (0, 1), \bar{Y}_j = (Y_j, 0) \in g \times_{\sigma} \mathbb{R}\), for each \(j = 1, \ldots, m\). It follows from Proposition 1 that these vector fields can be expressed as \(\bar{X}(g, r) = (\bar{X}(g), 1)\) and \(\bar{Y}_j(g, r) = (\bar{Y}_j(g), 0)\). We define the following invariant control system on \(G \times_{\varphi} \mathbb{R}\):

\[
\Sigma_I: \frac{d(g, r)}{dt} = \bar{X}(g, r) + \sum_{j=1}^{m} u_j \bar{Y}_j(g, r).
\]

Equivalently, we have

\[
\begin{pmatrix}
\frac{dg/dt} \\
\frac{dr/dt}
\end{pmatrix} = \begin{pmatrix}
\bar{X}(g) + \sum_{j=1}^{m} u_j Y_j(g) \\
1
\end{pmatrix}.
\]
It shows that \( \phi_I(t, (e, 0), u) = (\phi_L(t, e, u), t) \) and, consequently, we have

\[
\pi(\phi_I(t, (e, 0), u)) = \phi_L(t, e, u),
\]

where \( \pi : G \times \mathbb{R} \rightarrow G \) is the projection on the first coordinate.

**Remark 1** The lift of linear control system (4) is different of lift present by Cardetti and Mittenhuber in [9] because they choose to work with left invariant vector fields on \( G \) and we choose to work with right invariant vector fields on \( G \). It is the key of the work because this choice allow us project the solution \( \phi_I(t, (e, 0), u) \) of \( \Sigma_I \) directly on solution \( \phi_L(t, e, u) \) of \( \Sigma_L \). Consequently, this projection allows us to describe a solution of \( \phi_L(t, e, u) \) below.

**Proposition 4** For any piece-wise constant control \( u \) and any time \( t \), there exist a \( N \in \mathbb{N} \) and \( t_1, t_2, \ldots, t_N \) such that \( t = t_1 + t_2 + \cdots + t_N \) and

\[
\phi_L(t, e, u) = \phi_L(t_1, e, u_1) \phi_I(t_1, (e, 0), u_1) \cdots \phi_I(t_{N-1}, (e, 0), u_{N-1}) \phi_I(t_N, (e, 0), u_N).
\]

where \( u(t) = u_i \) when \( t \in [\sum_{j=0}^{i-1} t_j, \sum_{j=0}^{i} t_j] \).

**Proof** We know that \( \pi(\phi_I(t, (e, 0), u)) = \phi_L(t, e, u) \) for any real \( t \) and any piece-wise constant control \( u \). From Lemma 3.2 in [18] there exists a \( N \in \mathbb{N} \) and \( t_0, t_1, t_2, \ldots, t_N \) such that \( t_1 + t_2 + \cdots + t_N = t \) and that

\[
\phi_I(t, (e, 0), u) = \exp(t_1 \tilde{W}_1) \cdots \exp(t_N \tilde{W}_N),
\]

where \( \tilde{W}_i = \tilde{X} + \sum_{j=1}^m u_i^j \tilde{X}_j = (\sum_{j=1}^m u_i^j \tilde{X}_j, 1) \) with \( u_i \in \mathbb{R}^m \) and \( u(t) = u_i \) when \( t \in [\sum_{j=0}^{i-1} t_j, \sum_{j=0}^{i} t_j] \). It follows that

\[
\phi_I(t, (e, 0), u) = (\phi_L(t_1, e, u_1), t_1) \cdots (\phi_L(t_{N-1}, e, u_{N-1}), t_N) = (\phi_L(t_1, e, u_1) \phi_I(t_1, (e, 0), u_1) \cdots \phi_I(t_N, (e, 0), u_N), t)
\]

It implies that

\[
\phi_L(t, e, u) = \phi_L(t_1, e, u_1) \phi_I(t_1, (e, 0), u_1) \cdots \phi_I(t_N, (e, 0), u_N).
\]

Based on Proposition above our next step is to describe the solution of linear control flow \( \phi_L \) for a constant control.

**Theorem 5** For \( u = (u_1, \ldots, u_m) \in \mathbb{R}^m \) the curve

\[
\phi_L(t, e, u) = \lim_{n \rightarrow \infty} \prod_{i=0}^{n-1} \varphi_{it/n} \circ \exp \left( \frac{t}{n} \sum_{j=1}^{m} u_j Y_j \right)
\]

is the solution, with initial condition \( \phi_L(0, e, u) = e \), of the system

\[
\Sigma_L: \frac{dg}{dt} = \mathcal{X}(g) + \sum_{j=1}^{m} u_j Y_j(g).
\]
Proposition 3 to give a description of 

\[ (W, 1)(\phi_L(t, e, u), t) = (W(\phi_L(t, e, u)) + \mathcal{X}(\phi_L(t, e, u)), 1) \]

On the other side, the curve \((\phi_L(t, e, u), t)\) is the integral curve of \((W, 1)\). Therefore

\[ (W, 1) \exp(t(W, 1)) = (W, 1)(\phi_L(t, e, u), t) = \left( \frac{d\phi_L(t, e, u)}{dt}, 1 \right). \]

We thus get

\[
\begin{pmatrix}
\frac{d\phi_L(t, e, u)}{dt} \\
1
\end{pmatrix} = \begin{pmatrix}
\mathcal{X}(\phi_L(t, e, u)) + \sum_{j=1}^{m} u_j Y_j(\phi_L(t, e, u)) \\
1
\end{pmatrix}.
\]

Taking the projection on the first coordinate we conclude that the curve \(\phi_L(t, e, u)\) satisfies the system. As \(\phi_L(0, e, u) = e\) we have that this is the solution of the system at the identity. Being \((W, 1)\) a time-invariant, right invariant vector field on \(G \times \phi \mathbb{R}\), it is possible to use the Proposition 3 to give a description of \(\phi_L(t, e, u)\). In fact, as \(\exp(t(W, 1)) = (\phi_L(t, e, u), t)\) we have

\[ \phi_L(t, e, u) = \lim_{n \to \infty} \prod_{i=0}^{n-1} \phi_{i/n} \circ \exp \left( \frac{t}{n} \sum_{j=1}^{m} u_j Y_j \right). \]

Following we present an example about the method developed above.

**Example 1** (Linear Systems on Heisenberg Group 1) Let \(G\) be the Heisenberg group, that is, the set of all real matrix of the form

\[
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}.
\]

As usual, we identify this group with \(\mathbb{R}^3\) with the group product defined as

\[(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2).\]

In this case, the Lie algebra of the Heisenberg group is the vector space \(\mathbb{R}^3\) with the Lie bracket defined as \([\langle x_1, y_1, z_1 \rangle, \langle x_2, y_2, z_2 \rangle] = (0, 0, x_1 y_2 - x_2 y_1)\) and the exponential map \(\exp: \mathbb{R}^3 \to \mathbb{R}^3\) as

\[ \exp(x, y, z) = \left( x, y, \frac{xy}{2} + z \right). \]
The right invariant vector fields $Y = (m, n, p)$ in $G$ have the form

$$Y(x, y, z) = m \frac{\partial}{\partial x} + n \frac{\partial}{\partial y} + (my + p) \frac{\partial}{\partial z},$$

while the matrix of a derivation $\mathcal{D}$ associated to a linear vector field $\mathcal{X}$ is written as

$$\mathcal{D} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & a + d \end{pmatrix}.$$ 

By functional calculus, we know that

$$e^{t\mathcal{D}} = \alpha(t)\mathcal{D}^2 + \beta(t)\mathcal{D} + \gamma(t)I_3,$$

where $I_3$ is the identity matrix and $a_1(t), b_1(t)$ and $c_1(t)$ appropriates functions. Furthermore, it is possible to compute that

$$e^{t\mathcal{D}} = \begin{pmatrix} \alpha(t)a^2 + \beta(t)a + \gamma(t) & \alpha(t)b^2 + \beta(t)b & 0 \\ \alpha(t)c^2 + \beta(t)c & \alpha(t)d^2 + \beta(t)d + \gamma(t) & 0 \\ \alpha(t)e^2 + \beta(t)e & \alpha(t)f^2 + \beta(t)f & e^{(a+d)t} \end{pmatrix}. \tag{7}$$

We now consider the following linear system on $G$

$$\frac{dg}{dt} = \mathcal{X}(g) + uY(g),$$

where $Y = (0, 0, p)$ and $\mathcal{X}$ is linear vector field associated to derivation $\mathcal{D}$.

We first express the solution of the system on an interval $[0, T]$ in which the control function $u$ is constant. In formula (6), we note, by a direct calculation, that

$$\frac{t}{n}u e^{i a d / n} Y = \left( 0, 0, \int_0^t upe^{(a+d)s} ds \right).$$

Taking the exponential we obtain

$$\phi_L(t, e, u) = \left( 0, 0, \lim_{n \to \infty} \sum_{i=0}^{n-1} up e^{(a+d)i} \right) = \left( 0, 0, \int_0^t upe^{(a+d)s} ds \right),$$

where $u(t) = u$, for $0 \leq t \leq T$. Now, we define a function $f$ by $f(t) = t$ if $a + d = 0$ and $f(t) = \frac{e^{(a+d)t}}{a+d}$ if contrary. Therefore

$$\phi_L(t, e, u) = (0, 0, up f(t)).$$

Now from Proposition 4 we have for a piecewise control $u$ that there exist a $N \in \mathbb{N}$ and $t_1, t_2, \ldots, t_N$ such that $t = t_1 + t_2 + \cdots + t_N$ and $\phi_L(t, e, u)$ is equal to

$$(0, 0, pu_1 f(t_1)) \varphi_1 ((0, 0, pu_2 f(t_2)) \cdots \varphi_{\sum_{i=1}^{N-1} t_i} ((0, 0, pu_n f(t_n))
$$

$$(0, 0, pu_1 f(t_1)) \varphi_1 (\text{exp}(0, 0, pu_2 f(t_2)) \cdots \varphi_{\sum_{i=1}^{N-1} t_i} ((0, 0, pu_n f(t_n))$$

$$(0, 0, pu_1 f(t_1))\text{exp}(e^{t_i \mathcal{D}} (0, 0, pu_2 f(t_2)) \cdots \text{exp}(e^{(\sum_{i=1}^{N-1} t_i) \mathcal{D}} (0, 0, pu_n f(t_n)).$$

By matrix (7),

$$e^{(\sum_{i=1}^{m-1} t_i) \mathcal{D}} (0, 0, pu_m f(t_m)) = (0, 0, pu_m f(t_m) e^{(a+d)(\sum_{i=1}^{m-1} t_i)}),$$
which implies that
\[ \exp(e^{\sum_{i=1}^{n-1} t_i} D(0, 0, p u_n f(t_n))) = (0, 0, p u_n f(t_n)) e^{(a + d)(\sum_{i=1}^{n-1} t_i)}. \]

Therefore,
\[
\phi_L(t, e, u) = (0, 0, p u_1 f(t_1))(0, 0, p u_2 f(t_2)e^{(a + d)t_1}) \ldots (0, 0, p u_n f(t_n)e^{(a + d)(\sum_{i=1}^{n-1} t_i)})
\]
\[
= \left(0, 0, p(u_1 f(t_1) + u_2 f(t_2)e^{(a + d)t_1} + \ldots + u_n f(t_n)e^{(a + d)(\sum_{i=1}^{n-1} t_i)})\right)
\]

In the case that the right invariant vector field \( Y \) is given by \( Y = (m, n, p) \) the accounts increase in quantity and complexity, but the way to compute is the same.

Our next step is to improve the way of writing the solution of linear control system \( \Sigma_1 \).

To do this, we need the concept of Lie Wedge. The Lie Wedge of the invariant system \( \Sigma_1 \) is given by
\[
L(\tilde{A}) = \{ \tilde{A} \in g \times_\sigma \mathbb{R} : \exp(t \tilde{A}) \in \text{cl}(\tilde{A}), \forall t \geq 0 \},
\]
where \( \tilde{A} \) is the reachable set from \((e, 0)\). We now prove a necessary result.

**Proposition 6** If \((W, 0)\) is a right invariant vector field on \(g \times_\sigma \mathbb{R} \), then \((W, 0)\) belongs to \(L(\tilde{A})\) if and only if \(W = 0\).

**Proof** The sufficiency is trivial. We work on the necessary part. To do this, take an arbitrary positive \(t\). Assume that \(\exp(t(W, 0)) \in \text{cl}(\tilde{A})\). Then there exists a sequence \(a_n \in \tilde{A}\) such that \(\exp(t(W, 0)) = \lim_{n \to \infty} a_n\). Now, for each \(n\) there exists a piecewise constant control \(u\) and a time \(n\) such that \(a_n = \phi_t(t_n, e, u_n)\). Hence, for each \(t_n\) and \(u_n\) there exists a natural \(N_n\) and non-negative times \(t_1^n, \ldots, t_{N_n}^n\) such that \(t_1^n + \cdots + t_{N_n}^n = t_n\) and
\[
a_n = \exp(t_1^n \tilde{A}_1^n) \cdots \exp(t_{N_n}^n \tilde{A}_{N_n}^n).
\]

Observe that \(\tilde{A}_i^n = \tilde{X} + \sum_{j=1}^{m} u_{i,n}^j \tilde{X}_j = (\sum_{j=1}^{m} u_{i,n}^j X_j, 1)\) for \(i = 1, \ldots, n\). Thus, by Proposition 3,
\[
\exp(t_1^n \tilde{A}_1^n) = (\lim_{k \to \infty} \prod_{l=0}^{n-1} \phi_{t_{i,l}/k} \circ \exp(t_{i,l}/k \cdot (\sum_{j=1}^{m} u_{i,l,n}^j X_j)), t_{i,l}) = (B_i^n, t_i^n).
\]

Thus, by the product rule of semidirect product we have
\[
a_n = (B_1^n, t_1^n) \cdots (B_{N_n}^n, t_{N_n}^n) = (C_n, t_1^n + \cdots + t_{N_n}^n) = (C_n, t_n).
\]
It implies that \(\lim_{n \to \infty} t_n = 0\). Thus, given any \(\epsilon > 0\) there exists a \(n_0 \in \mathbb{N}\) such that \(t_n < \epsilon\) for \(n > n_0\). Since \(t_1^n + \cdots + t_{N_n}^n = t_n\), for \(n > n_0\) we see that \(t_i^n < \epsilon\) for \(i = 1, \ldots, n\). We claim that \(\lim a_n \to (e, 0)\). In fact, let \(d\) be a right invariant Riemannian distance on \(G \times_\sigma \mathbb{R}\). Then, for \(n > n_0\) we see that
\[
d(a_n, (e, 0)) = d(\exp(t_1^n \tilde{A}_1^n) \cdots \exp(t_{N_n}^n \tilde{A}_{N_n}^n), (e, 0))
\]
\[
\leq d(\exp(t_1^n \tilde{A}_1^n) \cdots \exp(t_{N_n}^n \tilde{A}_{N_n}^n), \exp(t_2^n \tilde{A}_2^n) \cdots \exp(t_{N_n}^n \tilde{A}_{N_n}^n)) + \cdots
\]
\[
+ d(\exp(t_{N_n-1} \tilde{A}_{N_n-1}^n) \exp(t_{N_n}^n \tilde{A}_{N_n}^n), \exp(t_{N_n}^n \tilde{A}_{N_n}^n))
\]
\[
+ d(\exp(t_{N_n}^n \tilde{A}_{N_n}^n), (e, 0))
\]
\[
= d(\exp(t_1^n \tilde{A}_1^n), (e, 0)) + \cdots + d(\exp(t_{N_n-1} \tilde{A}_{N_n-1}^n), (e, 0))
\]
\[ +d(\exp(t^n_{N_0} \bar{A}^n_{N_0}), (e, 0)). \]

Now using the Gauss Lemma (see Lemma 3.70 in [14]), it is possible to show that
\[ d(\exp(t^n_i \bar{A}^n_i), (e, 0)) \leq t^n_i \| \bar{A}^n_i \|, \quad i = 1, \ldots, n. \]

It implies that
\[ d(a_n, (e, 0)) \leq t_n \max_i \| \bar{A}^n_i \|. \]

In consequence,
\[ \lim a_n = (e, 0). \]

We conclude that
\[ (\exp(tW), 0) = \lim a_n = (e, 0), \]

hence \( \exp(tW) = e \), and finally \( W = 0 \), which completes the proof. \( \square \)

As first consequence we see that \( \text{cl}(\bar{A}) \) is a proper subset of \( G \times \mathbb{R}^+ \). A second one is that the Lie Bracket of two invariant vector fields that belongs to Lie wedge is null.

**Proposition 7** If \( (Y, r), (W, s) \in L(\bar{A}) \), then \( [(Y, r), (W, s)] = 0. \)

**Proof** We first see that
\[ [(Y, r), (W, s)] = [(Y + rX, W + sX'), 0). \]

Then for any \( t > 0 \) it follows that
\[ \exp(t [(Y, r), (W, s)]) = (\exp(t[Y + rX, W + sX']), 0). \]

On the other side, it is known that if \( (Y, r), (W, s) \in L(\bar{A}) \), then \( [(Y, r)(W, s)] \in L(\bar{A}) \). It implies that \( [Y + rX, W + sX'] = 0 \). \( \square \)

From Propositions above we present a simpler way of writing the trajectories of the invariant control system \( \Sigma_1 \).

**Proposition 8** For any \( t \geq 0 \) and any piecewise constant control \( u \), there are \( N \in \mathbb{N} \) and \( t_1, t_2, \ldots, t_N \) such that \( t = t_1 + t_2 + \cdots + t_N \) such that
\[ \phi_I(t, (e, 0), u) = \exp \left( t \bar{X} + \sum_{i=1}^m (t_1 u_1^i + \cdots + t_N u_N^i) \bar{X}_i \right). \]

where \( u(t) = u_i \) when \( t \in [\sum_{j=0}^{i-1} t_j, \sum_{j=0}^i t_j] \). In consequence,
\[ \phi_L(t, e, u) = \pi \left( \exp \left( t \bar{X} + \sum_{i=1}^m (t_1 u_1^i + \cdots + t_N u_N^i) \bar{X}_i \right) \right). \]

**Proof** Let \( t > 0 \) and a piecewise constant control \( u \). Since \( \phi_I(t, (e, 0), u) \) is an invariant system, there exist \( N \in \mathbb{N} \) and \( t_1, t_2, \ldots, t_N \) such that \( t = t_1 + t_2 + \cdots + t_N \) and
\[ \phi_I(t, (e, 0), u) = \exp(t_1 \bar{A}_1) \exp(t_2 \bar{A}_2) \cdots \exp(t_N \bar{A}_N) \]
with \( \tilde{A}_i = \vec{X} + \sum_{j=1}^m u_i^j \vec{X}_j \) from Lemma 3.2 in [18]. By construction of orbits of a control system \( \Sigma_I \), it follows that \( \tilde{A}_i \in L(\tilde{A}) \). From Proposition 7 we see that
\[
\phi_I(t, (e, 0), u) = \exp \left( t_1 \tilde{A}_1 + t_2 \tilde{A}_2 + \cdots + t_N \tilde{A}_N \right).
\]

Therefore, a simple calculus shows that
\[
\phi_I(t, (e, 0), u) = \exp \left( t \vec{X} + \sum_{i=1}^m \left( t_1 u^i_1 + \cdots + t_N u^i_N \right) \vec{X}_i \right).
\]

\[ \Box \]

From the above Theorem we describe the general solution of control systems \( \Sigma_I \) and \( \Sigma_L \).

**Theorem 9** If \( u \) is a locally integrable control on \([0, t]\), then
\[
\phi_I(t, (e, 0), u) = \exp \left( t \vec{X} + \sum_{i=1}^m \left( \int_0^t u^i(\tau) d\tau \right) \vec{X}_i \right).
\]

In consequence,
\[
\phi_L(t, e, u) = \pi \left( \exp \left( t \vec{X} + \sum_{i=1}^m \left( \int_0^t u^i(\tau) d\tau \right) \vec{X}_i \right) \right).
\]

**Proof** We begin by recalling that if \( u \) is locally integrable control then there is a sequence of piecewise constant controls \( (u_n) \) converging almost everywhere to \( u \). Since \( L^\infty[0, t] \) is the dual space of \( L^1[0, t] \), then it is possible to show that \( (u_n) \) converges weakly to \( u \) (see for instance section 7 in [16, ch.4]). According to Theorem 10 in [16, ch.4]), we have
\[
\lim_{n \to \infty} \phi_I(t, (e, 0), u_n) = \phi_I(t, (e, 0), u).
\]

Now for any \( n \in \mathbb{N} \) there are \( N(n) \in \mathbb{N} \) and \( t_{n,1}, t_{n,2}, \ldots, t_{n,N(n)} \) such that \( t = t_{n,1} + t_{n,2} + \cdots + t_{n,N(n)} \) with \( u_n(t) = u^i_n \) when \( t \in [\sum_{j=0}^{i-1} t_{n,j}, \sum_{j=0}^{i} t_{n,j}] \). From the above Theorem we see that
\[
\phi_I(t, (e, 0), u_n) = \exp \left( t \vec{X} + \sum_{i=1}^m \left( t_{n,1} u^i_{n,1} + \cdots + t_{n,N(n)} u^i_{n,N(n)} \right) \vec{X}_i \right).
\]

Observe that \( G \times_\varphi \mathbb{R} \) and \( g \times_\sigma \mathbb{R} \) are metric spaces and that \( \exp : g \times_\sigma \mathbb{R} \to G \times_\varphi \mathbb{R} \) is a continuous map. We thus get
\[
\lim_{n \to \infty} \exp \left( t \vec{X} + \sum_{i=1}^m \left( t_{n,1} u^i_{n,1} + \cdots + t_{n,N(n)} u^i_{n,N(n)} \right) \vec{X}_i \right)
= \exp \left( t \vec{X} + \sum_{i=1}^m \left( \lim_{n \to \infty} \left( t_{n,1} u^i_{n,1} + \cdots + t_{n,N(n)} u^i_{n,N(n)} \right) \vec{X}_i \right) \right)
= \exp \left( t \vec{X} + \sum_{i=1}^m \left( \int_0^t u^i(\tau) d\tau \right) \vec{X}_i \right),
\]
where \( \int_0^t u^i(\tau) d\tau \) is a Riemannian integral for any \( i = 1, 2, \ldots, n \). \[ \Box \]
**Example 2** (Linear Systems on Heisenberg Group II) We go back to Example 1 and consider a locally integrable control $u$ on $[0, t]$ instead a piecewise constant control. From the above Theorem the solution of invariant control system on $G \times \mathbb{R}$ is given by

$$
\phi_I(t, (e, 0), u) = \exp \left( t\bar{X} + \int_0^t u(\tau)d\tau\bar{Y} \right) = \exp \left( \left( 0, 0, p \int_0^t u(\tau)d\tau \right), t \right).
$$

From Proposition 3 we see that

$$
\phi_I(t, (e, 0), u) = \left( \lim_{n \to \infty} \prod_{i=0}^{n-1} \exp \left( \frac{1}{n} e^{\frac{1}{n} \mathcal{D}} \left( 0, 0, p \int_0^t u(\tau)d\tau \right) \right), t \right).
$$

By matrix (7),

$$
\frac{1}{n} e^{\frac{1}{n} \mathcal{D}} \left( 0, 0, p \int_0^t u(\tau)d\tau \right) = \left( 0, 0, \frac{1}{n} e^{(a+d)\frac{s}{n}} \int_0^t u(\tau)d\tau \right)
$$

Taking the exponential we obtain

$$
\phi_I(t, (e, 0), u) = \left( \left( 0, 0, \lim_{n \to \infty} \sum_{i=0}^{n-1} \frac{1}{n} e^{(a+d)i\frac{s}{n}} \int_0^t u(\tau)d\tau \right), t \right).
$$

This gives

$$
\phi_I(t, (e, 0), u) = \left( \left( 0, 0, \frac{p}{t} \int_0^t e^{(a+d)s} ds \int_0^t u(\tau)d\tau \right), t \right).
$$

We thus obtain

$$
\phi_L(t, e, u) = \left( 0, 0, \frac{p}{t} \int_0^t e^{(a+d)s} ds \int_0^t u(\tau)d\tau \right).
$$

### 4 Some applications to the inner derivation case

We now remember that a derivation $\mathcal{D}$ is inner if there exists a right invariant vector field $X$ such that $\mathcal{D} = -\text{ad}(X)$. In this case, it is simple to see that the linear flow $\phi_t$ can be written as $\phi_t(g) = \exp(tX) \cdot g \cdot \exp(-tX)$.

**Proposition 10** Let $(W, s)$ be an invariant vector field on $G \times \mathbb{R}$. If $\mathcal{D} = -\text{ad}(X)$ is an inner derivation on $\mathfrak{g}$ then

$$
\exp(t(W, s)) = \exp \left( tsX + \sum_{j=1}^m tu_j Y_j \right) \exp(-tX), ts \right).
$$

**Proof** We first simplify the notation by denoting $Y = \sum_{j=1}^m u_j Y_j$. Replacing

$$
\varphi_{it\frac{s}{n}} \left( \exp \frac{t}{n} Y \right) = \exp \left( \frac{its}{n} X \right) \exp \left( \frac{t}{n} Y \right) \exp \left( -\frac{its}{n} X \right)
$$

\[ \mathcal{D} \] Springer
in formula (3) we obtain
\[
\lim_{n \to \infty} \prod_{i=0}^{n-1} \phi_{\frac{t}{n}} (\exp - \frac{t}{n} Y) = \lim_{n \to \infty} \prod_{i=0}^{n-1} \exp \left( \frac{its}{n} X \right) \exp \left( - \frac{its}{n} X \right) \\
= \lim_{n \to \infty} \left( \exp \left( \frac{t}{n} Y \exp \frac{ts}{n} X \right) \exp \left( - \frac{t}{n} Y \right) \exp \left( \frac{1 - n)ts}{n} X \right) \right).
\]

Inserting \( \exp \left( \frac{t}{n} Y \right) \exp \left( \frac{ts}{n} X \right) \exp \left( - \frac{t}{n} Y \right) \) in the above expression yields
\[
\lim_{n \to \infty} \prod_{i=0}^{n-1} \phi_{\frac{t}{n}} (\exp - \frac{t}{n} Y) = \lim_{n \to \infty} \left( \exp \left( \frac{t}{n} Y \exp \frac{ts}{n} X \right) \right) \exp (-tX).
\]

Using the Lie product formula we conclude that
\[
\exp(t(W, s)) = \left( \exp \left( tsX + \sum_{j=1}^{m} tu_j Y_j \right) \right) \exp(-tX),
\]
and proof is complete. \(\square\)

**Corollary 11** Let \( \mathcal{D} \) be a derivation such that \( \mathcal{D} = -\text{ad}(X) \) for a right invariant vector field \( X \). For any \( t \geq 0 \) and any piecewise constant control, there exist a \( N \in \mathbb{N} \) and \( t_1, t_2, \ldots, t_N > 0 \) such that \( t = t_1 + t_2 + \cdots + t_N \) and
\[
\phi_L(t, e, u) = \exp \left( tX + \sum_{i=1}^{m} \left( \int_{0}^{t} u_i(t)dt \right) X_i \right) \cdot \exp(-tX).
\]

**Proof** It is direct from Proposition 10 and Theorem 8. \(\square\)

**Theorem 12** Let \( \mathcal{D} \) be a derivation such that \( \mathcal{D} = -\text{ad}(X) \) for a right invariant vector field \( X \). If \( u \) is an locally integrable control \([0, t]\), then
\[
\phi_L(t, e, u) = \exp \left( tX + \sum_{i=1}^{m} \left( \int_{0}^{t} u^i(t)dt \right) X_i \right) \cdot \exp(-tX).
\]

**Proof** The proof is similar to the proof of Theorem 9, where one must use the above Corollary instead of Proposition 8. \(\square\)

Following, we present some applications of the results above. Initially, we consider a system defined in \( \text{Gl}(n; \mathbb{R})^+ \) in the same way as in [17], and describe the solution curve for this case. According to [8], there are four three-dimensional semisimple Lie groups, which are \( \text{Sl}(2) \), \( \text{SU}(2) \), \( \text{SO}(3) \) and \( \text{SO}(2, 1)_0 \), so we construct the solutions of linear systems in these Lie groups.

**4.1 Lie group in \( \text{Gl}(n; \mathbb{R})^+ \)**

Let \( G = \text{Gl}(n; \mathbb{R})^+ \) be the set of all \( n \times n \) real matrices with positive determinant and \( \mathfrak{g} = \text{gl}(n; \mathbb{R}) \) its Lie algebra. For \( A \in \mathfrak{g} \), the vector field \( \mathcal{X}_A(g) = Ag - gA \) is linear and

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its associated flow is \( \varphi_t(g) = e^{tA}ge^{-tA} \). Given \( B_1, \ldots, B_m \in g \), consider the right-invariant fields \( B_j(g) = B_j g \) and the linear system

\[
\frac{dg}{dt} = Ag - gA + \sum_{j=1}^{m} u_j B_j(g).
\]

Applying Corollary (11) we can write the solution at the identity for a piecewise constant control \( u \) as

\[
\phi_L(t, e, u) = e^{tA + \sum_{i=1}^{m} (t_{i1}u_1 + \cdots + t_{iN}u_N^N)B_i} e^{-tA}.
\]

where \( t = t_1 + t_2 + \cdots + t_N \) for some \( N \in \mathbb{N} \) and \( t_1, t_2, \ldots, t_N > 0 \). When \( u \) is a locally integrable control on \([0, t]\) we obtain

\[
\phi_L(t, e, u) = e^{tA + \sum_{i=1}^{m} \left( \int_{t_{i1}}^{t_{i+1}} u^i(t) dt \right) B_i} e^{-tA}
\]

by Theorem 12.

### 4.2 Semisimple Lie groups of dimension 3

We study linear system only on \( SL(2, \mathbb{R}) \) because on others the computations are similar. In this case, the Lie algebra \( sl(2, \mathbb{R}) \) is the set of \( 2 \times 2 \) real matrices with null trace. Then the eigenvalues \( \alpha_1, \alpha_2 \) of an element \( Z \in g \) satisfy \( \alpha_1 + \alpha_2 = 0 \).

Given a linear system \( \Sigma_L \) on \( SL(2, \mathbb{R}) \), for a constant control function \( u = (u_1, \ldots, u_m) \), we denote

\[
\Sigma_L = X + \sum_{i=1}^{m} u_i X_i.
\]

With these notations, by Theorem 12, the solution of \( \Sigma_L \) is

\[
\phi_L(t, e, u) = \exp(t \Sigma_L) \exp(-tX), 0 \leq t \leq T.
\]

We denote by \( \lambda_1, \lambda_2 \) the eigenvalues of \( \Sigma_L \) and by \( \delta_1, \delta_2 \) the eigenvalues of \( X \). Applying methods of functional calculus we have \( \exp(t \Sigma_L) = p(\Sigma_L) \) and \( \exp(tX) = q(X) \), where \( p(z) = az + b \) and \( q(z) = cz + d \) are polynomials satisfying the following conditions:

\[
\begin{align*}
 p(\lambda_i) &= a\lambda_i + b = e^{\lambda_i}, \text{ for } i = 1, 2. \\
 q(\delta_j) &= c\delta_j + d = e^{\delta_j}, \text{ for } j = 1, 2.
\end{align*}
\]

Using the fact that \( \lambda_1 + \lambda_2 = \delta_1 + \delta_2 = 0 \) we obtain

\[
\begin{align*}
 p(z) &= \frac{\text{senh}(t\lambda)}{\lambda} z + \cosh(t\lambda) \quad \text{and} \quad q(z) = \frac{\text{senh}(t\delta)}{\delta} z + \cosh(t\delta),
\end{align*}
\]

where \( \lambda \) and \( \delta \) may be any of the correspondent eigenvalues. Then the solution of the linear system is written as

\[
\phi_L(t, e, u) = \left( \frac{\text{senh}(t\lambda)}{\lambda} \Sigma_L + \cosh(t\lambda) I d \right) \left( \frac{\text{senh}(t\delta)}{\delta} X + \cosh(t\delta) I d \right),
\]

for \( 0 \leq t \leq T \).

As mentioned above, on the other 3-dimensional, semisimple Lie groups \( SU(2), SO(3) \) and \( SO(2, 1) \), the argument to construct the solutions of linear systems are the same. Summarizing:
Applying the product rules and (9) we obtain

\[ \text{\lambda controllable set for } I_{\Sigma_1} \times G \text{ for any } \]

Lemma 13

Let \( \bar{\Sigma}_1 \) origin of \( \Sigma_1 \)

It implies that \( \bar{\Sigma}_1 \text{ cannot be reached from } I \).

Proof

We begin by recalling that the control system \( \Sigma_I \) given in (4) is a right invariant control system. It is an important observation because there are a large literature about invariant control system on Lie groups (see for instance [8, 18] and references given there). Another observation from (5) is that \( \pi(\bar{A}(e, 0)) = A(e, 0) \) is the reachable set of \( \Sigma_I \) at \((0, e)\) and \( A(e, 0) \) is the reachable set of \( \Sigma_L \) at \( e \). Both observations together suggest that the controllability of \( \Sigma_I \) at \((e, 0)\) implies the controllability of \( \Sigma_L \) at \( e \).

In this section, we show that \( \Sigma_I \) is not a controllable system, furthermore, there is not a controllable set for \( \Sigma_I \) as we show in Theorem 15. We see these facts as an important contribution because no other researcher needs to try to show some aspect of controllability of \( \Sigma_I \). Despite this, we use the invariant control system \( \Sigma_I \) to show that controllability at origin of \( \Sigma_L \) implies controllability on \( G \) of \( \Sigma_L \) since \( \lambda \) has periodic orbits.

To show the first purpose in this section we begin remembering that

\[ \phi_L(t, e, u) = \phi_L(t, e, u)\phi_i(g) \] (9)

for any \( x \in G \) and any control \( u \). For simplicity of notation, we write \( \tilde{G} = G \times_\varphi \mathbb{R} \).

Lemma 13

Let \( \bar{g} = (g, s) \in \tilde{G} \), then

\[ \phi_I(t, \bar{g}, u) = (\phi_L(t, g, u), t + s). \]

Proof

We first observe that invariance gives

\[ \phi_I(t, \bar{g}, u) = \phi_I(t, \bar{e}, u) \cdot \bar{g} = (\phi_L(t, e, u), t) \cdot (g, s). \]

Applying the product rules and (9) we obtain

\[ \phi_I(t, \bar{g}, u) = (\phi_L(t, e, u)\phi_i(g), t + s) = (\phi_L(t, g, u), t + s). \]

\[ \square \]

Proposition 14

Let \( \bar{x} = (x, r) \) and \( \bar{y} = (y, s) \) be points in \( \tilde{G} \). Suppose that \( s < r \). Then \( \bar{y} \) cannot be reached from \( \bar{x} \).

Proof

We begin by supposing that \( \bar{y} \in \bar{A}(\bar{x}) \). Then there exist a control \( u \) and a time \( t > 0 \) such that \( \bar{y} = \phi_I(t, \bar{x}, u) \). From Lemma above it follows that \( (y, s) = (\phi_L(t, x, u), t + r) \).

It implies that \( s = t + r \). Since \( t > 0 \), it follows that \( s > r \), which is a contradiction. \[ \square \]
The meaning of Proposition above is that \( \phi_I(t, e, u) \) is not controllable in \( \tilde{G} \). Despite not controllability of \( \phi_I(t, e, u) \), it is possible to study control sets of \( \Sigma_I \) and then project their\n
on control sets of \( \Sigma_L \). Thus, we begin by remembering the definition of control set.

**Definition 1** A set \( D \subset \tilde{G} \) is called a control set of system \( \Sigma_I \) if

i. for all \( x \in D \) there is a control \( u \in U \) such that \( \varphi(t, u, x) \in D \) for all \( t \geq 0 \);

ii. for all \( x \in D \) one has \( D \subset cl A(x) \);

iii. \( D \) is maximal with this properties.

The basic idea about a control set is that this is a set where the control system is controllable. The standard work on control set is the book due to Colonius and Kliemann [10].

**Theorem 15** There are no control sets associated to control system \( \Sigma_I \).

**Proof** Suppose that there exist a control set \( D \) associated to \( \Sigma_I \). Take \( \bar{x} = (x, t), \bar{y} = (y, s) \in D \) arbitrary points. Furthermore, from item ii, we see that \( \bar{x} \in cl \bar{A}(\bar{y}) \) and \( \bar{y} \in cl \bar{A}(\bar{x}) \). It implies that \( \bar{x} = \lim \bar{y}_n \) and \( \bar{y} = \lim \bar{x}_n \) with \( \bar{x}_n \in \bar{A}(\bar{x}) \) and \( \bar{y}_n \in \bar{A}(\bar{y}) \). From Lemma 13 it follows that

\[
\bar{x} = (x, t) = \lim(\varphi_L(s_n, y, u_n), s + s_n) = (\lim \varphi_L(s_n, y, u_n), \lim s + s_n).
\]

Therefore \( \lim s + s_n = t \). We also have

\[
\bar{y} = (y, s) = \lim(\varphi_L(t_n, x, v_n), t + t_n) = (\lim \varphi_L(t_n, x, v_n), \lim t + t_n).
\]

Hence \( \lim t + t_n = s \). We now analyze three cases: \( s < t, s > t \) and \( s = t \). If \( s < t \) then there exists a real number \( c \) such that \( s < c < t \). Thus for large \( n \) we have that \( t + t_n < c < t \). Hence \( t_n < 0 \) for large \( n \), which is an absurd. An analogous situation occur when \( s > t \).

Then, we must have \( s = t \). From item i. of Definition there exists a control \( u \) such that \( \phi_I(r, \bar{x}, u) \in D \) for all \( r \geq 0 \). From Lemma 13 it follows that

\[
\phi_I(r, \bar{x}, u) = (\phi_L(r, x, u), t + r).
\]

Since \( \phi_I(r, \bar{x}, u) \in cl \bar{A}(\bar{x}) \), it follows that \( t = t + r \). We thus get \( r = 0 \), which is a contradiction. \( \square \)

Despite of the above theorem to show that there are not control sets of the control system \( \Sigma_I \), there are results about control sets of linear control system \( \Sigma_L \) as the reader can find in the references [5, 6]. Our contribution is given an answer to question due to Jouan in [15]: Is it true that controllability of the linear control system \( \Sigma_L \) at origin implies controllability of one on \( G \)?

Firstly, we show that controllability of the linear control system \( \Sigma_L \) is equivalent an geometric property about reachable set of the invariant control system \( \Sigma_I \). We recall that \( \Sigma_L \) is controllable at point \( g \in G \) if for any \( h \in G \) there exist a time \( t > 0 \) and an admissible control \( u \) such that \( \phi_L(t, e, u) = h \).

**Proposition 16** The linear control system \( \Sigma_L \) is controllable at point \( g \) if and only if \( \bar{A}(g, s) \cap \pi^{-1}(h) \neq \emptyset \) for some \( (g, s) \in \tilde{G} \) and any \( h \in G \).

**Proof** We first see that for any time \( t > 0 \) and an admissible control \( u \) we have that

\[
\phi_I(t, (g, s), u) = R_{[g,s]} \phi_I(t, (e, 0), u) = R_{[g,s]} (\phi_L(t, e, u), t) = (\phi_L(t, e, u) \varphi_I(g), t + s) = (\phi_L(t, g, u), t + s).
\]
Thus, if $\phi_L$ is controllable at point $g$ then for any $h \in G$ there exists a time $t > 0$ and an admissible control $u$ such that $\phi_L(t, g, u) = h$. It implies that

$$(h, t + s) = (\phi_L(t, g, u), t + s) = \phi_I(t, (g, s), u),$$

which show that $\bar{A}(g, s) \cap \pi^{-1}(h) \neq \emptyset$.

On contrary, suppose that $\bar{A}(g, s) \cap \pi^{-1}(h) \neq \emptyset$ for any $h \in G$. Take a $(x, r) \in \bar{A}(g, s) \cap \pi^{-1}(h)$. Then there exist a time $t > 0$ and an admissible control $u$ such that $(x, r) = \phi_I(t, (g, s), u)$. Since $(x, r) \in \pi^{-1}(h)$, then $(x, r) = (h, r)$. We thus get

$$(h, r) = (\phi_L(t, g, u), t + s),$$

and consequently $h = \phi_L(t, g, u)$, which show that $\phi_L$ is controllable at point $g$. \qed

**Corollary 17** The linear control system $\Sigma_L$ is controllable if and only if $\bar{A}(g, s) \cap \pi^{-1}(h) \neq \emptyset$ for any $(g, s) \in \tilde{G}$ and any $h \in G$.

**Proof** The proof follows from the above Proposition. \qed

A difficult to apply the above results is about the nature of the fibers of projection $\pi$ over the action to right of $\tilde{G}$ into $\tilde{G}$. In fact, it is clear that $\pi^{-1}(h) = \{(h, s) : s \in \mathbb{R}\}$ for any $h \in G$. It follows that $R_{(g, s)} \pi^{-1}(h) = \{h\phi_I(g, t + s) : s \in \mathbb{R}\}$ for $(g, t) \in \tilde{G}$. It shows that the fiber $\pi^{-1}(h)$ twists itself over the action to right. For other side, we want to use the following property of invariant control system $\phi_I$: $\phi_I(t, (g, s), u) = R_{(g, s)} \phi_I(t, (e, 0), u)$ for any $(g, s) \in \tilde{G}$. Thus our idea is to solve this problem is to ask that the flow $\varphi_t$ has periodic orbits.

**Theorem 18** If $\Sigma_L$ is controllable at origin and $g$ is a periodic point of the linear flow $\varphi_t$, then $\Sigma_L$ is controllable at point $g$.

**Proof** Suppose that $\Sigma_L$ is controllable at origin, then for any $h \in G$ there exist a time $t > 0$ and an admissible control $u$ such that $\phi_L(t, e, u) = h$. By construction, the invariant control $\phi_I$ satisfies $\phi_I(t, (e, 0), u) = (h, t)$. Now,

$$\phi_I(t, (g, 0), u) = R_{(g, 0)} \phi_I(t, (e, 0), u) = R_{(g, 0)}(h, t) = (h \varphi_I(g), t).$$

Since $g$ is a periodic point of the linear flow $\varphi_t$, it follows that there exists a time $T > 0$ such that $\varphi_T(g) = g$. We thus get

$$\phi_I(T, (g, 0), u) = (hg, t).$$

It implies that $\bar{A}(g, 0) \cap \pi^{-1}(hg) \neq \emptyset$. From Proposition 16 it follows that $\phi_I$ is controllable at point $g$. \qed

A direct application of the above Theorem is presented below with hypothesis that assures periodic orbits of the linear flow $\varphi_t$.

**Corollary 19** Let $G$ be a connected Lie group. Suppose that $X$ is a linear vector field on $G$, and denote by $D$ and $\varphi_t$ their derivation and flow, respectively. If the eigenvalues of the derivation $D$ are semisimple and they are null or $\pm \alpha_1, \ldots, \pm \alpha_r$ with rational quotient $\alpha_i/\alpha_j$ for $i, j = 1, \ldots, r$, and if $\phi_L$ is controllable at origin, then $\phi_L$ is controllable.

**Proof** We begin by seeing that the linear flow has periodic orbits from Theorem 3.5 in [22]. From the above Theorem it follows that $\Sigma_L$ is controllable on $G$ since $\Sigma_L$ is controllable at origin. \qed
**Remark 2** Even if the invariant control system $\Sigma_I$ is not controllable, there are still open questions about the controllability of the linear control system $\Sigma_L$ that may use the lift presented here. A first open question is if the orbit of invariant control system $\Sigma_I$ has codimension 1 or if the orbit is transverse with respect to lift vector field $\bar{X}$, it implies in the controllability of the linear control system $\Sigma_L$? Other problem, suggested by referee, is if the linear control system $\Sigma_L$ is driftless ($X = 0$) is possible to use our results to improve the classical Rashevski-Chow’s Theorem? For Rashevski–Chow’s Theorem we refer the reader to [11, ch.3].

**Data availability** No datasets were generated or analyzed during the current study.

**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

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