Long-range order and pinning of charge-density waves in competition with superconductivity

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We consider the possibility of long-range charge-density wave order in a non-linear sigma model of a clean layered system where such order competes with superconductivity. Using a large-\(N\) approximation, and Monte-Carlo simulations, we show that weak inter-layer coupling can stabilize long-range order only in the presence of a magnetic field, which suppresses the superconducting order parameter inside vortex cores. This fact is related to the low temperature behavior of the charge-density wave structure factor, which vanishes linearly with decreasing temperature in the absence of a magnetic field and diverges in its presence. Such behavior is inconsistent with recent x-ray scattering measurements of the structure factor in cuprate high-temperature superconductors.

On the other hand, qualitative agreement with experiments is obtained when the effect of a random pinning potential is taken into account in our simulations and large-\(N\) analysis. In particular, we find that at low temperatures the structure factor attains a non-zero finite value, which grows linearly with magnetic field.

The pseudogap state of the cuprate high-temperature superconductors (HTSCs) harbors various fluctuating electronic orders \cite{1}. In particular, recent nuclear magnetic resonance \cite{2–4} and x-ray scattering \cite{5–15} measurements have found evidence of charge-density wave (CDW) fluctuations in underdoped YBa$_2$Cu$_3$O$_{6+x}$. The observed strength of the CDW fluctuations is anti-correlated with superconductivity (SC) in the sense that the intensity of the CDW scattering peak grows as the temperature is reduced towards the superconducting transition temperature, \(T_c\), and then decreases upon entering the SC phase. In addition, the CDW signal is enhanced when a magnetic field is used to quench SC, while the effect of a magnetic field above \(T_c\) is negligible.

Motivated by these findings, Hayward et al. \cite{16,17} have proposed a phenomenological non-linear sigma model (NLSM), which formulates the competition between fluctuating SC and CDW order parameters. Similar models emerge also from more microscopic considerations \cite{15,21}. Using Monte-Carlo simulations, Hayward et al. calculated the temperature dependence of the x-ray structure factor in the absence of a magnetic field and showed that it exhibits a maximum slightly above the Berezinskii-Kosterlitz-Thouless temperature, \(T_{BKT}\), of their two-dimensional model and that it remains finite at all temperatures. However, concerns were raised about whether the latter feature, which is essential in order to obtain agreement with the experimental results, is robust to the effects of weak interlayer couplings \cite{11}.

The purpose of the present letter is to evaluate the effects of such interlayer couplings on the CDW ordering tendencies in the presence of two additional ingredients that are important for a comparison with experiments, namely, a magnetic field and random pinning potentials. To this end we employ a large-\(N\) approximation, which we have previously used \cite{22} to calculate the signatures of thermally excited vortices in the NLSM, and calculate averages over disorder using the replica method. We concentrate on the low temperature SC phase where the effects of the additional factors are significant. Finally, we present complementary results of Monte-Carlo simulations, which we used to study the model beyond the limits of our analytical approach.

We show that in a clean system, without a magnetic field, the competition with SC establishes a threshold interlayer coupling for the stabilization of long-range CDW order. On the other hand, in the presence of a magnetic field, any small interlayer coupling suffices to induce long-range order between the CDW regions which nucleate around the cores of the Abrikosov vortices. These results are also reflected in the low-temperature CDW structure factor of weakly coupled layers. While it vanishes linearly with decreasing temperature in the field-free system it diverges when a field is present. Both behaviors are inconsistent with the x-ray data on YBa$_2$Cu$_3$O$_{6+x}$.

In contrast, qualitative agreement with the experimental phenomenology is obtained when the effects of CDW pinning by a random potential are taken into account. Specifically, since some regions of the system overcome the competition with SC and assume a CDW configuration even as the temperature is reduced to zero, the structure factor attains a non-zero finite value in this limit. This value grows linearly with magnetic field, but does not diverge, as required from general considerations \cite{23}. At higher temperatures the structure factor exhibits a maximum close to \(T_c\), which is washed away by a magnetic field, while at even higher temperatures it becomes magnetic field-independent.

The clean, single layer NLSM. We begin with the model considered by Hayward et al. \cite{10}, for a real 6-dimensional order parameter, equivalent to a complex SC field \(\Psi = n_1 + in_2\) and two complex CDW fields,
\( \Phi_x = n_3 + i n_4 \) and \( \Phi_y = n_5 + i n_6 \). Here, for the sake of simplicity, we disregard quartic and anisotropic CDW terms, which appear in Ref. [16] and follow our previous strategy [22] of using a saddle-point approximation for the CDW fields, which is formally justified when their number is large. Thus, we analyze a system described by a complex SC field \( \{ \psi_1, \psi_2 \} \), and \( N - 2 \) real CDW fields \( \{ n_a \} \), where \( \alpha = 1 \ldots N - 2 \), whose Hamiltonian is

\[
H_0[\psi, n_a] = \frac{1}{2} \rho_s \int d^2r \left\{ \left| (\nabla + 2i e A) \psi \right|^2 + \sum_{\alpha=1}^{N-2} \left[ \lambda (\nabla n_\alpha)^2 + g n_\alpha^2 \right] \right\},
\]

where \( \rho_s \) is the stiffness of the SC order, \( \lambda \rho_s \) is the corresponding quantity for the CDW components, and \( g \rho_s \) is the energy density penalty for CDW ordering. We assume that some type of order (SC or CDW) is always locally present, in the sense of its amplitude, but that the different order parameters compete, as expressed by the constraint

\[
|\psi|^2 + \sum_{\alpha=1}^{N-2} n_\alpha^2 = 1.
\]

The free energy \( F_0 \) is given by

\[
e^{-\beta F_0} = \int \mathcal{D}\psi^* \mathcal{D}\psi \mathcal{D}n \delta \left( |\psi|^2 + \sum_{\alpha=1}^{N-2} n_\alpha^2 - 1 \right) e^{-\beta H_0}
= \int \mathcal{D}\psi^* \mathcal{D}\psi \mathcal{D}n \mathcal{D}\sigma e^{-\beta H_0}
\times e^{\frac{\beta}{2} \rho_s \int d^2r \left( |\psi|^2 + \sum_{\alpha=1}^{N-2} n_\alpha^2 - 1 \right)},
\]

where \( \beta = 1/T \) is the inverse temperature. In the limit \( N \to \infty \) we integrate over \( n_a \) while assuming that the Lagrange multiplier field, \( \sigma \), attains its saddle-point configuration, \( \sigma = \sigma_0 \). Since we are focusing on signatures of the CDW deep inside the SC phase, \( T \ll T_{BRKT} \), we also assume that the SC fields \( \psi, \psi^* \) take their saddle-point configurations. Within this approximation [24], the free energy of the clean layer is given by

\[
\beta F_0 = \frac{N - 2}{2} \text{Tr} \ln \left[ \frac{1}{2} \rho_s \left( -\lambda \nabla^2 + g + \sigma \right) \right]
+ \frac{1}{2} \rho_s \int d^2r \left[ (\nabla + 2i e A) \psi (|\psi|^2 - 1) \right],
\]

where the fields \( \psi, \psi^* \) and \( \sigma \) are determined by the coupled saddle-point equations

\[
\frac{\delta \beta F_0}{\delta \psi^*(r)} = \frac{1}{2} \rho_s \left[ - (\nabla + 2i e A)^2 + \sigma \right] \psi = 0,
\]

and

\[
\frac{\delta \beta F_0}{\delta \sigma(r)} = \frac{N - 2}{2} \text{Tr} \left[ (\nabla^2 + g + \sigma)^{-1} \right] \psi
+ \frac{1}{2} \rho_s \left( |\psi|^2 - 1 \right) = 0,
\]

with \( \delta(r', r'') = \delta(r - r') \delta(r'' - r) \).

We first consider the case of zero magnetic field, where the SC field assumes a uniform configuration, \( \psi(r) = \psi_0 \), and \( \sigma = 0 \). By substituting this solution in Eq. (6), we find that

\[
|\psi_0|^2 = 1 - \frac{T}{T_{MF}},
\]

where the mean-field transition temperature is given by

\[
\frac{\rho_s}{T_{MF}} = \frac{N - 2}{2} \text{Tr} \left[ (\nabla^2 + g/\lambda)^{-1} \right] \delta(r)
\approx \frac{N - 2}{4\pi \lambda \rho_s} \ln \left( \frac{32 \lambda}{g a^2} \right).
\]

To obtain the last expression we have regularized the theory by putting it on a square lattice with lattice constant \( a \). Here, and in the following, we assume that \( ga^2/\lambda \ll 1 \). The CDW structure factor, is defined by

\[
S_{CDW} = \frac{1}{L^2} \int d^2r d^2r' \langle n_a(r) n_a(r') \rangle_0,
\]

where \( L^2 \) is the layer’s area, and \( \langle \cdots \rangle_0 \) denotes thermal averaging with respect to \( H_0 \). In the present case, due to the uniformity of \( \psi \) and \( \sigma \), it is readily evaluated with the result

\[
S_{CDW} = \frac{T}{g \rho_s},
\]

which vanishes as \( T \to 0 \).

This situation changes upon applying a magnetic field, \( B \). The solution of the saddle-point equations (5) is expected to take the form of an Abrikosov lattice of vortices, whose density is determined by the magnetic field. Far away from the vortex cores, \( \sigma = 0 \) and \( \psi = \psi_0 \), just as in the zero field case. However, close to the center of each vortex, \( \psi \) vanishes linearly with the distance from the vortex center, and \( \sigma \) becomes negative. As a result, there is a trapped CDW mode inside each core, in addition to a continuum of scattering modes, which exists also without a magnetic field. Using a tight-binding approximation [23] for these trapped modes in Eq. (6), we can estimate their contribution to \( S_{CDW} \).

The result depends on the order of limits. At a low but non-zero temperature, and \( B \to 0 \) (more precisely, when \( t \sim g \phi_0 \sqrt{g \phi_0} / B \ll gT/\rho_s \)), we find that

\[
S_{CDW} = \frac{T}{g \rho_s} + A_1 \left[ 1 - \left( \frac{A_2}{\rho_s} + \frac{1}{T_{MF}} \right) \right] \frac{B}{g^2 \phi_0},
\]

where \( \phi_0 = \pi \hbar c/e \) is the flux quantum, and \( c_1, A_{1,2} \) are numerical constants. In the other limit, of a finite magnetic field and \( T \to 0 \) (when \( t \gg gT/\rho_s \)), we obtain

\[
S_{CDW} = A_3 \frac{T}{T_{MF}} \frac{B}{g \phi_0} e^{-c(t/g)\rho_s(1/T_{MF})},
\]

where \( \phi_0 = \pi \hbar c/e \) is the flux quantum, and \( c_1, A_{1,2} \) are numerical constants. In the other limit, of a finite magnetic field and \( T \to 0 \) (when \( t \gg gT/\rho_s \)), we obtain

\[
S_{CDW} = A_3 \frac{T}{T_{MF}} \frac{B}{g \phi_0} e^{-c(t/g)\rho_s(1/T_{MF})},
\]
where $c_2, A_3$ are additional numerical constants. Therefore, the structure factor diverges at low temperatures in the presence of a magnetic field.

**Clean, weakly coupled layers.** Next, we would like to ask whether a weak interlayer CDW coupling is sufficient to stabilize long-range CDW order. First, let us note that the diverging SC susceptibility of each layer at $T_{BKT}$ implies that any weak interlayer Josephson coupling of the form $-\rho_s J_{SC} \int d^2 r \sum_i (\psi_i^\dagger \psi_{i+1} + H.c.)$, where $i$ is the layer index, induces long-range SC order. However, for weak $J_{SC}$ the SC transition at $T_{BKT} = T_{BKT} \left[ 1 + b_1 \ln^2(b_2 g/J_{SC}) \right]$, with $b_{1,2}$ constants, has only a small effect on $|\psi|$ and thus on the amplitude and ordering tendencies of $n$. Consequently, we concentrate on the following multi-layer Hamiltonian

$$H = \sum_i H_0[\psi_i, n_{\alpha,i}] - \rho_s J_\perp \int d^2 r \sum_{\alpha,i} n_{\alpha,i} n_{\alpha,i+1}. \quad (13)$$

To estimate the effect of $J_\perp$ we use the interlayer mean-field approximation [25, 27] which results in the following condition for the putative CDW ordering transition

$$\langle n_\alpha(r) \rangle = 2 \rho_s J_\perp \int d^2 r' \sum_\beta \chi_{\alpha\beta}(r, r') \langle n_\beta(r') \rangle, \quad (14)$$

where $\chi_{\alpha\beta}(r, r') = \langle n_\alpha(r) n_\beta(r') \rangle / T$ is the response function of a single layer.

In the absence of a magnetic field $\int d^2 r' \chi_{\alpha\beta}(r, r') = \delta_{\alpha\beta} S_{CDW}/T$, and Eq. (16) implies that condition (14) can be fulfilled only if $J_\perp \geq g/2$. When this happens uniform CDW order is established, and the interlayer coupling term in Eq. (13) leads to the effective modification $g \rightarrow g - 2 J_\perp \langle n_\alpha \rangle^2 / \langle n_\alpha^2 \rangle$. Hence, for $J_\perp \geq g/2$ and $T \rightarrow 0$ the effective $g$ turns negative, SC disappears and the system becomes purely CDW ordered.

In the presence of a magnetic field the response function is still related to $S_{CDW}$, and the condition for long range CDW order is $2 \rho_s J_\perp S_{CDW}/T = 1$. However, the divergence of $S_{CDW}$, Eq. (12), means that even for $J_\perp \rightarrow 0$, long-range order between the CDW regions around the vortex cores does set in at

$$T_{CDW} = \frac{c_2 \rho_s}{g \ln(g \phi_0 / 2A_3 J_\perp B)}, \quad (15)$$

and coexists with the long-range SC order.

**Disordered single layer.** Finally, we consider the effects of a random potential, which pins the CDW. Because a continuous symmetry can not be spontaneously broken in a disordered system below four dimensions [23], no long-range CDW order can be established, even in the presence of interlayer coupling. Thus, we neglect any small residual effects of $J_\perp$ and focus on a single disordered layer described by the Hamiltonian

$$H = H_0 - \rho_s \int d^2 r \sum_\alpha V_\alpha n_\alpha, \quad (16)$$

where $V_\alpha$ are independent random Gaussian fields satisfying $V_\alpha = 0$ and $V_\alpha(r)V_\beta(r') = V^2 \delta_\alpha \delta_\beta(r - r')$, with the overline signifying disorder averaging.

Applying the replica method to the $N \rightarrow \infty$ limit [28, 29] we adapt the saddle-point equations (5,6) to the disordered case, and calculate the structure factor $S_{CDW}$, averaged over realizations of the pinning field. In the zero field case we find [28]

$$\overline{S_{CDW}} = \frac{T}{g \rho_s} + \frac{V^2}{g^2}, \quad (17)$$

which decreases linearly to a finite value as the temperature is reduced to zero. Such behavior reflects the fact that due to the random field certain regions assume local CDW order even at $T = 0$. When the system is subject to a magnetic field, superconductivity is suppressed inside the vortex cores, around which CDW halos are formed. As a result, a larger fraction of the system’s area supports pinned local CDW order, and $S_{CDW}$ increases. As long as $T \ll \max(gT/\rho_s, \sqrt{gV^2})$ we find that [28]

$$\overline{S_{CDW}} = \frac{T}{g \rho_s} + \frac{V^2}{g^2} + A_1 \left[ 1 - \left( \frac{A_2}{\rho_s} + \frac{1}{T_{MF}} \right) \right] T - A_2 \frac{V^2}{g^2 \phi_0} \frac{B}{g^2 \phi_0}, \quad (18)$$

with the additional numerical constant $A_4$. A similar functional form characterizes the $T \rightarrow 0$ spatially averaged Edwards-Anderson order parameter

$$q_{EA} = \frac{1}{L^2} \int d^2 r \overline{n_\alpha(r)^2} = \frac{V^2}{4\pi \lambda g} + A_5 \frac{B}{g \phi_0}, \quad (19)$$

with $A_5$ a constant. Random-field models in the $N \rightarrow \infty$ limit do not exhibit a glass transition [28], and $q_{EA} > 0$ for all $T$.

**Monte-Carlo results.** In order to go beyond our mean-field results, we performed Monte-Carlo (MC) simulations of the NLSM, incorporating the effects of a finite magnetic field, interlayer coupling, and disorder. We used standard Metropolis updating to study systems on a square $L \times L$ lattice, with $L$ ranging from 32 to 200 sites, and cylindrical boundary conditions. We present here MC results for the experimentally relevant case, $N = 6$, and additionally set $\lambda = 1$ and $g_0^2 = 0.3$. For these parameters, $T_{BKT}/\rho_s = 0.345$. To facilitate comparison with the results of Ref. [10], we present in Fig. 1 the structure factor $S_{\Phi_2}/a^2 = 2S_{CDW}/a^2$ as a function of $T$ in a clean layer with and without a magnetic field. For $B = 0$, $S_{\Phi_2}$ vanishes linearly as $T$ approaches zero in agreement with Eq. (16), but diverges at low $T$ for finite $B > 0$, as in Eq. (12). Our results are generally independent of the system size, except for the $B > 0$, $T \rightarrow 0$ limit, where the diverging $S_{\Phi_2}$ increases with $L$. The inset of Fig. 1 depicts the CDW ordering temperature, $T_{CDW}$, estimated from a clean $32 \times 32 \times 8$ layered system, as a function
of the interlayer CDW coupling $J_L a^2$. As expected from our interlayer mean-field considerations, we find a transition for $J_L > g/2$ and $B = 0$, and down to the lowest accessible values of $J_L$ when $B > 0$. In addition, however, for $J_L$ slightly below $g/2$ and $B = 0$, we observe a transition to an ordered phase which vanishes at a lower critical temperature. This behavior can be traced to a maximum, $\chi_{max}$, in $\chi = S_{CDW}/T$, which gives two solutions to Eq. (14) in the range $1/2\rho_s \chi_{max} < J_L < g/2$. Finally, we present results for a disordered layer, with $V^2 a^2 = 0.15$. The structure factor, $S_{\Phi_x}$, averaged over 60 disorder realizations, is shown in Fig. 2 with the inset depicting its low-temperature $B$ dependence. In accordance with our analytical result, Eq. (15), $S_{\Phi_x}$ assumes a finite value for $T, B \to 0$, and grows linearly with both $T$ and $B$. The error bars in our MC results, which grow with increasing $B$ and decreasing $T$, reflect the low convergence rates and sensitivity to initial conditions which arise in this limit. However, our MC simulations clearly show that $S_{\Phi_x}$ does not diverge when $B > 0$, even for temperatures below the range presented in Fig. 2.

Discussion. Including the effects of a random pinning potential leads to a qualitative agreement between our theory and the x-ray scattering data of Chang et al. [6], in the following aspects: The nucleation of CDW at regions of strong attractive disorder makes $S_{CDW}$ attain a finite value at $T = 0$, even for $B = 0$. This value increases linearly with $B$, due to pinned CDW around vortex cores, similar to the checkerboard halos observed in scanning-tunneling experiments [30]. We note, however, that while the calculated low-$T$ behavior of $S_{CDW}$ resembles the experimental results of Chang at non-zero $B$, the predicted linear $T$ dependence for $B = 0$, see Eq. (17) and Fig. 2 is not reflected in the scattering data. At higher temperatures both our simulations and the experiments exhibit a maximum in $S_{CDW}$ close to $T_c$, which disappears with increasing $B$, while there is practically no $B$ dependence beyond this temperature.

A commonly advocated scenario for explaining recent quantum oscillations experiments, invokes long-range CDW order as a cause for Fermi-surface reconstruction. As such experiments are carried out at relatively high magnetic fields, it is unclear to what extent they reflect the nature of cuprate HTSCs at $B = 0$. Indeed, our reasoning indicates a qualitative difference between the $B = 0$ and $B > 0$ cases within the classical model which we have studied. Namely, finite temperature long range CDW order forms in clean weakly-coupled layers for $B > 0$, but not at $B = 0$. However, this sharp distinction is blurred in the presence of disorder, where no long-range CDW order is possible [22], and applying a magnetic field leads only to an increase in the density of pinned CDW regions, as expressed by the Edwards-Anderson order parameter, Eq. (19). The degree of local CDW ordering needed to explain the quantum oscillations experiments remains to be resolved.

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CDW SPECTRUM IN THE ABRIKOSOV VORTEX LATTICE STATE

Consider the NLSM for a clean layer with $B > 0$ and $T < T_{BKT}$, where an Abrikosov lattice of vortices is expected to develop. In terms of the orthonormal eigenfunctions, $\phi_s(r)$, and eigenvalues, $\varepsilon_s$, of the operator

$$\hat{L} = -\lambda \nabla^2 + g + \sigma(r),$$

the saddle-point equations take the form

$$[-(\nabla + 2ie\mathbf{A}(r))^2 + \varepsilon_s] \psi(r) = 0,$$

and

$$\sum_s |\phi_s(r)|^2 = \frac{\beta\rho_s}{N-2} [1 - |\psi(r)|^2].$$

We have previously derived an effective Ginzburg-Landau theory for the NLSM, and showed that the vortex core radius scales at low temperatures as $r_0 \sim \rho^{-1/2}$. From Eq. \ref{eq:21} it then follows that inside the core $\varepsilon_s \sim -\lambda r_0^2 \sim -\lambda g$, while $\varepsilon_s = 0$ for $r \gg r_0$, where $|\psi| = |\psi_0|$. Consequently, one expects that in the presence of a vortex, the spectrum of $L$ consists of a continuum of scattering states with $\varepsilon_s > g$, and a discrete set of bound states with $\varepsilon_s < g$. Our numerical solution of the saddle-point equations confirms these expectations and shows that for $\lambda = 1$ there is a single bound state, $\phi_0(r)$, with eigenvalue $\varepsilon_0 \ll g$, which decays at large distances as $\phi_0(r) \sim e^{-r/r_0}$.

In the presence of a dilute Abrikosov lattice of vortices, i.e., one for which the inter-vortex distance, $R$, obeys $R \gg r_0$, the small overlap between bound states in neighboring cores leads to the formation of a tight-binding band

$$\phi_{0,k}(r) = \frac{1}{\sqrt{N_V}} e^{ik \cdot r} \sum_R \phi_0(r - R),$$

where the sum extends over the positions of the $N_V$ vortices, and $\mathbf{k}$ lies within the magnetic Brillouin zone. For a square vortex lattice its dispersion takes the form

$$\varepsilon_0(\mathbf{k}) = \tilde{\varepsilon}_0 - 2t [\cos(k_x R) + \cos(k_y R)],$$

with $|k_{x,y}| < \pi/R$, $\tilde{\varepsilon}_0 = \varepsilon_0 - \Delta \varepsilon_0$, and

$$\Delta \varepsilon_0 = -\int d^2r \Delta \sigma_{\mathbf{R} = 0}(r) \phi_0^2(r),$$

$$t = -\int d^2r \phi_0^2(r) \Delta \sigma_{\mathbf{R} = R\hat{x}}(r) \phi_0(r - R\hat{x}),$$

$$\Delta \varepsilon_0 \int d^2r \phi_0^2(r) \phi_0(r - R\hat{x}).$$

Here,

$$\Delta \sigma_{\mathbf{R}}(r) = \sum_{\mathbf{R}' \neq \mathbf{R}} \sigma_{\mathbf{R}}(r - \mathbf{R}'),$$

where $\sigma_{\mathbf{R}}(r)$ is the configuration assumed by $\sigma$ in the presence of a single vortex. Consequently, using $R = \sqrt{\varepsilon_0/B}$, both $\Delta \varepsilon_0$ and $t$ scale as $g \exp(-c_1 \sqrt{\varepsilon_0/B})$, where $c_1$ is a constant that depends on $\lambda$.

Under the specified conditions the scattering states still form a continuum with $\varepsilon_s \geq g$. Since $\phi_{0,k}(r)$ vanish rapidly between vortices it follows from Eq. \ref{eq:22} that

$$\sum_{s \in \text{scattering}} \frac{|\phi_s(r)|^2}{\varepsilon_s} = \frac{\beta\rho_s}{N-2} [1 - |\psi_0|^2 - |\delta\psi(r)|^2],$$

where $|\delta\psi(r)|$ is appreciable only within the cores. Therefore,

$$\sum_{\mathbf{k}} \frac{|\phi_{0,k}(\mathbf{r})|^2}{\varepsilon_0(\mathbf{k})} = \frac{\beta\rho_s}{N-2} \left( |\psi_0|^2 - |\psi(r)|^2 + |\delta\psi(r)|^2 \right),$$

whose integral over $\mathbf{r}$ gives

$$\int_{BZ} d^2k \frac{1}{\varepsilon_0(\mathbf{k})} = C \beta\rho_s |\psi_0|^2 \left( \frac{r_0}{R} \right)^2,$$

with a constant $C$. Evaluating the integral and using $|\psi_0|^2 = 1 - T/T_{MF}$, gives

$$\tilde{\varepsilon}_0 - 4t \approx \left\{ \begin{array}{ll} \frac{C}{4\pi} \rho_s \left( \frac{1}{T} - \frac{1}{T_{MF}} \right) r_0^2 & : t \ll T/\rho_s r_0^2 \\ \frac{1}{4\pi r_0^2} \exp \left( \frac{C}{4\pi} \rho_s \left( \frac{1}{T} - \frac{1}{T_{MF}} \right) r_0^2 \right) & : t \gg T/\rho_s r_0^2 \end{array} \right.$$

The effective action for the CDW fields, $n_\alpha$, is of the form $(\beta \rho_s/2) \int d^2r \sum_\alpha n_\alpha \tilde{L}_n_\alpha$, with the result that

$$S_{CDW} = \frac{1}{\beta \rho_s} \frac{1}{L^2} \sum_\alpha \frac{1}{\varepsilon_0} \int d^2r \tilde{n}_\alpha(r)^2.$$

Due to the spatial integration, the scattering states contribution to $S_{CDW}$ is dominated by the state with $\varepsilon_s = g$, which is the descendent of the $\mathbf{k} = 0$ scattering state of the system with $B = 0$. Therefore, its integral satisfies $(1/L^2) \int d^2r \tilde{n}_\alpha(r)^2 = 1 - O(r_0^2/R^2)$, where the correction is due to its deviations from uniformity in the vicinity of the cores. States with $\varepsilon_s > g$ provide further contributions of order $O(r_0^2/R^2)$. For $\phi_{0,k}$ we find using Eq. \ref{eq:23}

$$\int d^2r \phi_{0,k}(r) = \frac{1}{\sqrt{N_V}} \sum_\mathbf{R} e^{ik \cdot \mathbf{R}} \int d^2r e^{ik \cdot r} \phi_0(r) = \sqrt{N_V} \phi_{k,0} \int d^2r \phi_0(r).$$
Since $\varphi_0(r)$ is normalized and appreciable within $r \lesssim r_0$ the last integral scales as $r_0$, and the contribution of the band of core states to $S_{CDW}$ is of order $(r_0/R)^2T/\rho_s\varepsilon_0(k = 0)$. Using Eq. (31) and combing the two contributions, we finally arrive at Eqs. (11) and (12) of the main text.

When applying the above considerations to a triangular Abrikosov lattice, one needs to take into account the facts that now $BR^2 = 2\phi_0/\sqrt{3}$ and $\varepsilon_0(k) = -4t\left[\cos^2(k_xR/2) + \cos(k_yR/2)\cos(\sqrt{3}k_yR/2) - 1/2\right]$. However, these changes do not affect the functional dependence of $S_{CDW}$, but only the various numerical constants, which appear in the solution.

**THE INTERLAYER MEAN-FIELD APPROXIMATION**

In the presence of interlayer Josephson coupling the Hamiltonian reads

$$H = \sum_i H_0[\psi_i, n_{\alpha,i}] - \rho_s J_{SC} \int d^2r \mid \psi_{i+1}^* \psi_i + H.c.\mid, \tag{34}$$

where $i$ is the layer index. The interlayer mean-field approximation amounts to replacing $H$ by an effective single-layer Hamiltonian of the form

$$H_{MF} = H_0 - z\rho_s J_{SC} \int d^2r \mid \psi^* \langle \psi \rangle_{MF} + H.c.\mid, \tag{35}$$

where $z = 2$ is the coordination number of the layer, and $\langle \cdot \cdot \cdot \rangle_{MF}$ denotes averaging with respect to $H_{MF}$.

We are interested in using this approximation to estimate $T_c$ in the multilayer system. To this end, we calculate $\langle \psi \rangle_{MF}$. Since it is small in the vicinity of $T_c$, we may carry out the averaging over $H_{MF}$ perturbatively in the Josephson coupling term. As a result, in the absence of a magnetic field and using the fact that $\langle \psi(r)\psi(r')\rangle_0 = 0$, we obtain the following condition for the SC transition

$$T_c = 2\rho_s J_{SC} \int d^2r \langle \psi^*(r')\psi(r) \rangle_0. \tag{36}$$

For weak $J_{SC}$, $T_c$ lies close to $T_{BKT}$ of a single layer, where $\langle \psi^*(r')\psi(r) \rangle_0 \approx |\psi_0(T_c)|^2|\langle \psi(r\theta-\delta(r'\theta) \rangle_0$. Since for $T > T_{BKT}$ the SC phase correlations decay exponentially over the BKT correlation length $\xi$, we obtain

$$T_c = 2\pi \rho_s J_{SC}|\psi_0(T_c)|^2\xi^2(T_c). \tag{37}$$

On a square lattice $T_{BKT} \approx 0.9\rho_s |\psi_0(T_{BKT})|^2$, thereby establishing, for $T_c \approx T_{BKT}$, a relation between $|\psi_0(T_c)|^2$ and $T_{BKT}$. Finally, using BKT critical behavior of $\xi(T_c) = r_0 \exp[\sqrt{b}T\xi/\xi(T_{BKT})]$, where $b$ is a constant, we arrive at

$$T_c = T_{BKT} \left[1 + \frac{4b^2}{\ln^2(0.9/4\pi J_{SC}r_0^2)}\right]. \tag{38}$$

A similar type of interlayer mean-field analysis is carried out in the main text for the estimation of the CDW ordering temperature.

**THE NLSM WITH A RANDOM PINNING POTENTIAL**

We next consider the NLSM of a single layer with independent Gaussian random potentials, $V_\alpha$. The system is described by the action

$$S = \beta H_0 - \beta \rho_s \int d^2r \sum_\alpha V_\alpha(r)n_\alpha(r) + \frac{1}{2}\int d^2r \sum_\alpha J_\alpha(r)n_\alpha(r)n_\alpha(r') - \int d^2r \sum_{\alpha\beta} K_{\alpha\beta}(r,r')n_\alpha(r)n_\beta(r'), \tag{39}$$

where $\alpha$ and $\beta$ denote two types of interlayer hopping, and $J_{\alpha\beta}$ is a Josephson coupling term. As a result, we have introduced sources in order to calculate correlation and response functions in terms of the free energy,

$$e^{-\beta F} = \int D\psi^* D\psi Dn \delta \left(|\psi|^2 + \sum_\alpha n_\alpha^2 - 1\right) e^{-S}. \tag{40}$$

Consequently,

$$G_{\alpha\beta}(r,r') \equiv \langle n_\alpha(r)n_\beta(r') \rangle = -\frac{\delta \beta F}{\delta K_{\alpha\beta}(r,r')} \bigg|_{K=0}. \tag{41}$$

and

$$T_{\chi_{\alpha\beta}}(r,r') \equiv \frac{\langle n_\alpha(r)n_\beta(r') \rangle - \langle n_\alpha(r)\rangle\langle n_\beta(r') \rangle}{\delta J_\alpha(r)\delta J_\beta(r')} \bigg|_{J=0} \tag{42}$$

The main difficulty is in calculating $\beta F$, the free energy averaged over realizations of disorder. This can be done by employing the replica method in which we consider $m$ replicas of the original model. Analytically continuing $m \rightarrow 0$ we have $\beta F = \lim_{m \rightarrow 0} F(m)/m$, where $F(m)$ is defined by
$e^{-\beta F(m)} = \int D\psi^a D\bar{\psi}^a Dn_a^a D\sigma_a D\hat{\sigma}_a \delta \left( |\psi|^2 + \sum_{a=1}^{N-2} (n_a^a)^2 - 1 \right) e^{-\sum_a S[\psi^a, n_a^a, J_a, K_a, \hat{\sigma}_a]} \frac{1}{\sqrt{2\pi}} \int d^2r \sum_a V_a^a$ \(\int D\psi^a D\bar{\psi}^a Dn_a^a D\sigma_a D\hat{\sigma}_a e^{-\sum_a S[\psi^a, n_a^a, J_a, K_a, \hat{\sigma}_a]} \frac{1}{\sqrt{2\pi}} \int d^2r \sum_a V_a^a + \frac{1}{2} \beta \rho_s \int d^2r i\sigma_a [\psi|^2 + \sum_a (n_a^a)^2 - 1]. \) (43)

Integration over $V_a$ and analytically continuing to $\bar{\sigma} = i\sigma$ gives 

$$
S(m) = \frac{1}{2} \beta \rho_s \int d^2r \left\{ \sum_a \left[ (\nabla + 2ieA)\psi^a|^2 + \sigma^a (|\psi|^2 - 1) \right] + \sum_{ab} \sum_a n_a^a \left[ \delta_{ab} \hat{L}^a - \beta \rho_s V^2 \right] n_b^a \right\} 
- \int d^2r \sum_a \sum_{\alpha} J_a(r) n_a^a(r) - \int d^2r d^2r' \sum_{ab} \sum_{\alpha\beta} K_{\alpha\beta}(r, r') n_a^a(r) n_b^b(r'),
$$

and $\hat{L}^a = -\Lambda \nabla^2 + g + \sigma^a(r)$. Integrating over the CDW fields, $n_a^a$, gives 

$$e^{-\beta F(m)} = \int D\psi^a D\bar{\psi}^a Dn_a^a D\sigma_a e^{-S(m)}, \quad S(m) \text{ is defined by}$$

$$S(m) = \frac{1}{2} \text{Tr} \ln(G^{-1} - 2K) + \frac{1}{2} \beta \rho_s \int d^2r \sum_a \left[ (\nabla + 2ieA)\psi^a|^2 + \sigma^a (|\psi|^2 - 1) \right] 
- \frac{1}{2} \int d^2r d^2r' \sum_{ab} \sum_{\alpha\beta} J_{ab}(r, r') \left[ (G^{-1} - 2K)^{-1}\right]_{\alpha\beta}(r, r')J_{\alpha\beta}(r'), \quad (45)$$

and similarly,

$$T_{\alpha\beta}(r, r') = \delta_{\alpha\beta} \lim_{m \to 0} \frac{1}{m} \sum_{ab} G_{ab}(r, r'). \quad (51)$$

We will calculate $G^{ab}$ by assuming a replica-symmetric solution of the saddle-point equations, $ \psi^a = \psi$ and $\sigma^a = \sigma$. Under this assumption the operator $\hat{L}^a = \hat{L}$ is also replica symmetric, and $G^{ab}$ must obey

$$\sum_c \beta \rho_s (\delta_{ac} \hat{L} - \beta \rho_s V^2) G^{cb}(r, r') = \delta_{ab} \delta(r - r'). \quad (52)$$

Expanding $G^{aa}$ in the eigen-basis of $\hat{L}$

$$G^{ab}(r, r') = \sum_{st} G_{st}^a \phi_s(r) \phi_t^*(r'), \quad (53)$$

we find the solution

$$G_{st}^a = \delta_{st} \left[ \frac{\delta_{ab}}{\beta \rho_s \varepsilon_s} + \frac{V^2}{\varepsilon_s (\varepsilon_s - m\beta \rho_s V^2)} \right], \quad (54)$$

expressed in terms of the eigenvalues, $\varepsilon_s$, of $\hat{L}$.

In the absence of a magnetic field, the saddle-point equation for $\psi$, Eq. (49), assumes a uniform solution $\psi = \psi_0$ with $\sigma = 0$. For this case the spectrum of $\hat{L}$ is spanned by plane waves $s \equiv k$, $\phi_k(r) = e^{ikr}/L$ and

$$G_{\alpha\beta}(r, r') = \delta_{\alpha\beta} \lim_{m \to 0} \frac{1}{m} \sum_{a} G^{aa}(r, r'), \quad (50)$$
\[ \varepsilon_k = \lambda k^2 + g. \] Thus, we find for the correlation function
\[ G_{\alpha\beta}(\mathbf{r}, \mathbf{r}') = \delta_{\alpha\beta} \lim_{m \to 0} \frac{1}{m} \sum_a \sum_k C_{ka}^a \frac{1}{L^2} e^{ik \cdot (\mathbf{r} - \mathbf{r}')} \]
\[ = \delta_{\alpha\beta} \int \frac{d^2 k}{(2\pi)^2} \left[ \frac{1}{\beta \rho_s (\lambda k^2 + g)} + \frac{V^2}{(\lambda k^2 + g)^2} \right] e^{ik \cdot (\mathbf{r} - \mathbf{r}')}, \]
from which we can determine the averaged structure factor
\[ \overline{S}_{CDW} = \frac{1}{L^2} \int d^2 r \, d^2 r' \, G_{\alpha\alpha}(\mathbf{r}, \mathbf{r}') = \frac{T}{\rho_s g} + \frac{V^2}{g^2}. \] (56)
Similarly, we find that the response function is given by
\[ T \chi_{\alpha\beta}(\mathbf{r}, \mathbf{r}') = \delta_{\alpha\beta} \lim_{m \to 0} \frac{1}{m} \sum_a \sum_k C_{ka}^a \frac{1}{L^2} e^{ik \cdot (\mathbf{r} - \mathbf{r}')} \]
\[ = \delta_{\alpha\beta} \frac{1}{\beta \rho_s} \int \frac{d^2 k}{(2\pi)^2 \lambda k^2 + g} e^{ik \cdot (\mathbf{r} - \mathbf{r}')}. \] (57)
such that the \( k = 0 \) susceptibility is
\[ \chi = \frac{1}{L^2} \int d^2 r \, d^2 r' \, \chi_{\alpha\alpha}(\mathbf{r}, \mathbf{r}') = \frac{1}{\rho_s g}. \] (58)
Using Eqs. (53) and (54) we obtain that in the \( m \to 0 \) limit the saddle-point equation for \( \sigma \), Eq. (48), takes the form
\[ \int \frac{d^2 k}{(2\pi)^2} \left[ \frac{1}{\beta \rho_s (\lambda k^2 + g)} + \frac{V^2}{(\lambda k^2 + g)^2} \right] = 1 - \frac{|\psi_0|^2}{N - 2}, \]
which, for \( \lambda \gg ga^2 \), gives
\[ |\psi_0|^2 \approx 1 - \frac{(N - 2)V^2}{4\pi \lambda g} - \frac{T}{T_{MF}^0}, \] (59)
where \( T_{MF}^0 \) is the value of \( T_{MF} \) in the clean system. We therefore find that disorder reduces \( |\psi_0|^2 \), as well as \( T_{MF} \).

We conclude with the calculation of \( \overline{S}_{CDW} \) in the disordered system, subject to a magnetic field. Just as for the clean system, we expect that the saddle-point equations of the replicated action possess a solution in the form of an Abrikosov lattice. Hence, we assume that the spectrum of \( \hat{L} \) consists of a continuum of scattering states, similar to those of the magnetic-field-free system, and a band originating from bound states inside vortex cores. The reasoning that was used for the derivation of Eq. (29) is then applicable here. If, in addition, we assume that \( t \ll \varepsilon_0 \), we can ignore the dispersion of the tight-binding band and approximate it by a flat band with a constant eigenvalue \( \varepsilon_0 \). As a result we find that the saddle-point equation for \( \sigma \) becomes, in the \( m \to 0 \) limit,
\[ \sum_{\mathbf{R}} \left( \frac{1}{\beta \rho_s \varepsilon_0} + \frac{V^2}{\varepsilon_0^2} \right) |\varphi_0(\mathbf{r} - \mathbf{R})|^2 \]
\[ \approx \frac{||\psi_0||^2 - |\psi(\mathbf{r})|^2 + |\delta\psi(\mathbf{r})|^2}{N - 2}. \] (60)
Integrating over \( \mathbf{r} \) and dividing by the system area gives
\[ \frac{1}{R^2} \left( \frac{1}{\beta \rho_s \varepsilon_0} + \frac{V^2}{\varepsilon_0^2} \right) = C |\psi_0|^2 \left( \frac{r_0}{R} \right)^2, \] (61)
where \( C \) is a numerical constant. From Eq. (62) we find that the assumption \( t \ll \varepsilon_0 \) is indeed satisfied under reasonable conditions, i.e., \( t \ll \max(gT/\rho_s, \sqrt{gV^2}) \).

In the presence of disorder the expansion of the structure factor in terms of the eigenstates and eigenvalues of \( \hat{L} \) takes the form
\[ \overline{S}_{CDW} = \frac{1}{L^2} \sum_a \left( \frac{1}{\beta \rho_s \varepsilon_0} + \frac{V^2}{\varepsilon_0^2} \right) \int d^2 r |\phi_a(\mathbf{r})|^2. \] (63)
Due to the same consideration used for the clean systems, we find that the main contribution of the scattering states comes from the state with \( \varepsilon_s = g \). An additional contribution comes from the states \( \varphi_0(\mathbf{r} - \mathbf{R}) \) bound to the vortex cores at positions \( \mathbf{R} \). Noting that \( L^{-2} \sum_{\mathbf{R}} |\int d^2 r \varphi_0(\mathbf{r} - \mathbf{R})|^2 \approx 1/(gR^2) \), we obtain Eq. (18) of the main text. In addition, the \( T \to 0 \) limit of the spatially averaged Edwards-Anderson order parameter can be easily calculated from the saddle-point equation, Eq. (61),
\[ q_{EA}(T \to 0) = \frac{1}{L^2} \int d^2 r \, G_{\alpha\alpha}(\mathbf{r}, \mathbf{r}) \]
\[ = \frac{1 - |\psi_0|^2 + C|\psi_0|^2 (\frac{r_0}{R})^2}{N - 2}, \] (64)
which yields Eq. (19) of the main text.