The Varchenko Determinant of Arrangement Collages

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Abstract
A collage is another way to consider a set of hyperplane arrangements in separate Euclidean spaces. We define a distance function on the chambers in a collage which gives rise to a determinant whose entry in position \((C, D)\) is the distance between \(C\) and \(D\): it is the Varchenko determinant of a collage. The purpose of this article is to show that the Varchenko determinant of a collage has a nice factorization. As example of application, we prove the realizability of a model in infinite statistics based on a collage.

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1 Introduction
Varchenko introduced in 1993 a distance function on the chambers of a hyperplane arrangement that he called quantum bilinear form [9 § 1]. That gave rise to a determinant indexed by the chambers whose entry in position \((C, D)\) is the quantum bilinear form of \(C\) and \(D\): that is the original Varchenko determinant. He proved that the latter has a nice factorization [9, Theorem 1.1]. Over time, several generalizations of that determinant have appeared. In 2013, Gente computed the Varchenko determinant indexed by the chambers lying in a cone of a hyperplane arrangement [3, Theorem 4.5]. In 2017, Aguiar and Mahajan defined a generalization of the quantum bilinear form [1 § 8.1.1], and proved that, for a central hyperplane arrangement and its cones, the determinant given rise by their distance function has a nice factorization [1, Theorem 8.11, 8.12]. In 2018, Hochstättler and Welker even computed the Varchenko determinant of oriented matroids [5, Theorem 1] which are abstractions for central hyperplane arrangements. And in 2019, Randriamaro proved that, for an arbitrary hyperplane arrangement and its apartments, the determinant given rise by the generalized quantum bilinear form of Aguiar and Mahajan has a nice factorization [7, Theorem 1.8, 1.12]. In this article, we propose a slight generalization of their distance function over a collage. Besides, the matrix given rise by the quantum bilinear form, called Varchenko matrix, has also been investigated. For instances, Hanlon and Stanley computed the nullspace of the Varchenko matrix of braid arrangements [4, Theorem 3.3]. Then Denham and Hanlon studied

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the connection between the Smith normal form of the Varchenko matrix for special matrix coefficients and the Betti numbers of its complexified hyperplane arrangement [2, Theorem 3.3]. Furthermore, the Varchenko determinant plays a key role to prove the realizability of variant models of quon algebras in quantum mechanics like a deformed quon algebra [6, Theorem 4.2], or a multiparametric quon algebra [8, Proposition 2.1]. The quon algebra is an approach to particle statistics in order to provide a theory in which the Pauli exclusion principle and Bose statistics are violated. We use the Varchenko determinant of a collage to prove the realizability of a model based on a direct sum of quon algebras.

**Definition 1.1.** Let \( a_1, \ldots, a_n, b \in \mathbb{R} \) such that \((a_1, \ldots, a_n) \neq (0, \ldots, 0)\). A hyperplane in \( \mathbb{R}^n \) is a \((n-1)\)-dimensional affine subspace \( \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \cdots + a_nx_n = 0\} \).

A hyperplane arrangement is a finite set of hyperplanes in \( \mathbb{R}^n \).

To every hyperplane \( H \) in \( \mathbb{R}^n \) can be associated two connected open half-spaces \( H^+ \) and \( H^- \) such that \( H^+ \cup H^0 \cup H^- = \mathbb{R}^n \) and \( \overline{H^+} \cap \overline{H^-} = H^0 \), letting \( H^0 := H \). A face of a hyperplane arrangement \( A \) is a nonempty subset of \( \mathbb{R}^n \) having the form

\[
F := \bigcap_{H \in A} H^\epsilon H(F) \quad \text{with} \quad \epsilon_H(F) \in \{+, 0, -\}.
\]

Denote the set formed by the faces of \( A \) by \( F_A \). The sign sequence of a face \( F \in F_A \) is \( \epsilon_A(F) := (\epsilon_H(F))_{H \in A} \). A chamber of a hyperplane arrangement \( A \) is a face in \( F_A \) whose sign sequence contains no 0. Denote the set formed by the chambers of \( A \) by \( C_A \).

**Definition 1.2.** A pointed arrangement is a pair \((A, C)\), where \( A \) is a hyperplane arrangement and \( C \subseteq C_A \) is the pointing chamber. A collage is a finite set of pointed arrangements in Euclidean spaces not necessarily having the same dimension.

If \( \Gamma = \{(A_i, C_i)\}_{i \in [r]} \) is a collage, and \( i, j \in [r] \) such that \( i \neq j \), then we assume that \( A_i \) and \( A_j \) do not lie in the same euclidean space.

**Definition 1.3.** Let \( \Gamma = \{(A_i, C_i)\}_{i \in [r]} \) be a collage. The set formed by the hyperplanes resp. faces resp. chambers of \( \Gamma \) is

\[
A_\Gamma := \bigsqcup_{i \in [r]} A_i \quad \text{resp.} \quad F_\Gamma := \bigsqcup_{i \in [r]} F_{A_i} \quad \text{resp.} \quad C_\Gamma := \bigsqcup_{i \in [r]} C_{A_i}.
\]

For two chambers \( C, D \in C_{A_i} \), denote the set of half-spaces containing \( C \) but not \( D \) by

\[
H_i(C, D) := \{H^\epsilon H(C) \mid H \in A_i, \epsilon_H(C) = -\epsilon_H(D)\}.
\]

Assign a variable \( h_H^\epsilon, \epsilon \in \{+, -\} \), to every half-space \( H^\epsilon \), \( H \in A_i \), and define the polynomial ring \( R_{A_i} := \mathbb{Z}[h_H^\epsilon \mid \epsilon \in \{+, -\}, H \in A_i] \). The Aguiar-Mahajan distance function \( v_i : C_{A_i} \times C_{A_i} \to R_{A_i} \) is defined by

\[
v_i(C, C) = 1 \quad \text{and} \quad v_i(C, D) = \prod_{H^\epsilon \in H_i(C, D)} h_H^\epsilon \text{ if } C \neq D.
\]

Consider a variable \( q \), and the polynomial ring \( R_\Gamma := \mathbb{Z}[q, h_H^\epsilon \mid \epsilon \in \{+, -\}, H \in A_\Gamma] \).
Definition 1.4. Let $\Gamma = \{(A_i, C_i)\}_{i \in [r]}$ be a collage. We define the distance function $v : C_\Gamma \times C_\Gamma \to R_\Gamma$ on the chambers of $\Gamma$ by

$$v(C, D) = \begin{cases} 1 & \text{if } C = D, \\ v_i(C, D) & \text{if } C, D \in C_{A_i} \text{ and } C \neq D, \\ q v_i(C, C_i) v_j(C_j, D) & \text{if } C \in C_{A_i}, D \in C_{A_j}, \text{ and } i \neq j. \end{cases}$$

Definition 1.5. The Varchenko determinant of a collage $\Gamma$ is $|v(D, C)|_{C, D \in C_\Gamma}$.

Definition 1.6. The centralization of a collage $\Gamma$ to a face $F \in F_\Gamma \setminus C_\Gamma$ is the hyperplane arrangement $A_F$ defined by $A_F := \{H \in A_\Gamma \mid F \subseteq H\}$. The weight of a face $F \in F_\Gamma \setminus C_\Gamma$ is the monomial $b_F := \prod_{H \in A_F} h_H^+ h_H^-$, and, for $H \in A_F$, its multiplicity is the integer independent of $H$ [7, Theorem 5.6]

$$\beta_F := \frac{\#\{C \in C_\Gamma \mid C \cap H = F\}}{2}$$

Theorem 1.7. Let $\Gamma = \{(A_i, C_i)\}_{i \in [r]}$ be a collage. Then,

$$|v(D, C)|_{C, D \in C_\Gamma} = (1 + (r - 1)q)(1 - q)^{r-1} \prod_{F \in F_\Gamma \setminus C_\Gamma} (1 - b_F)^{\beta_F}.$$ 

Note that the Varchenko determinant is independent of the chosen pointing chambers.

Consider, for instance, the collage $\{(A_1, C_1), (A_2, C_2)\}$ in Figure [1] such that, for $i \in [3]$, $h_{A_1}^+ = h_{A_1}^- = h_1$, and $h_{A_2}^+ = h_{A_2}^- = h_2$. Its Varchenko determinant is

$$\begin{vmatrix} 1 & h_1^2 & h_1 & h_1 & h_1 & h_1 & q & qh_2 \\ h_1^2 & 1 & h_1 & h_1 & h_1 & h_1 & q & qh_2 \\ h_1 & h_1 & 1 & h_1 & h_1 & h_1 & q & qh_2 \\ h_1 & h_1 & h_1 & 1 & h_1 & h_1 & q & qh_2 \\ h_1 & h_1 & h_1 & h_1 & 1 & h_1 & q & qh_2 \\ h_1 & h_1 & h_1 & h_1 & h_1 & 1 & q & qh_2 \\ h_1 & h_1 & h_1 & h_1 & h_1 & h_1 & 1 & qh_2 \\ q & qh_1 & q & q & h_1 & q & qh_2 \\ qh_2 & h_1 & q & q & h_1 & q & qh_2 \\ h_1 & q & q & h_1 & q & q & q & qh_2 \end{vmatrix} = (1-q^2)(1-h_1^2)^2 (1-h_2^2)^2.$$ 

This article is structured as follows: We establish the combinatorial properties of a collage requisite to compute its Varchenko determinant in Section 2. Then, we prove Theorem 1.7 in Section 3. We conclude in Section 4 with an example of application of our main result in infinite statistics.

## 2 Combinatorial Properties of the Faces of a Collage

Like for the set formed by the faces of a hyperplane arrangement, we will see in this section that one can also construct a monoid on the set formed by the faces of a collage. Moreover, we provide an extension of a Witt identity [1] Proposition 7.30] to collages.
Definition 2.1. Let \( \Gamma = \{(A_i, C_i)\}_{i \in [r]} \) be a collage, \( j \in [r] \), and \( F \in F_{A_j} \). The **collage sign sequence** of \( F \) is

\[
\epsilon_{\Gamma}(F) := (u_H)_{H \in A_{\Gamma}} \quad \text{with} \quad u_H = \begin{cases} 
\epsilon_H(F) & \text{if } H \in A_j, \\
\epsilon_H(C_i) & \text{if } H \in A_i \text{ and } i \neq j.
\end{cases}
\]

Proposition 2.2. Let \( \Gamma = \{(A_i, C_i)\}_{i \in [r]} \) be a collage. The face set \( F_{\Gamma} \) is a monoid with the following multiplication: If \( F, G \in F_{\Gamma} \), then \( FG \) is the face in \( F_{\Gamma} \) such that

\[
\forall H \in A_{\Gamma} : u_H(FG) = \begin{cases} 
\epsilon_H(F) & \text{if } u_H(F) \neq 0, \\
u_H(G) & \text{otherwise.}
\end{cases}
\]

Proof. Recall that \( F_{A_i} \) is a monoid with the following multiplication [7, Proposition 2.1]: If \( F, G \in F_{A_i} \), then \( FG \in F_{A_i} \) such that, for every \( H \in A_i \), \( \epsilon_H(FG) := \begin{cases} 
\epsilon_H(F) & \text{if } \epsilon_H(F) \neq 0, \\
\epsilon_H(G) & \text{otherwise.}
\end{cases} \)

Now, suppose that \( F, G \in F_{\Gamma} \):

- If \( F, G \in F_{A_j} \), then \( u_H(FG) = \begin{cases} 
\epsilon_H(FG) & \text{if } H \in A_j, \\
\epsilon_H(C_i) & \text{if } H \in A_i \text{ and } i \neq j
\end{cases} \)

which corresponds to the face \( FG \in F_{A_j} \).

- If \( F \in F_{A_j} \) and \( G \in F_{A_k} \) with \( j \neq k \), then \( u_H(FG) = \begin{cases} 
\epsilon_H(FC_j) & \text{if } H \in A_j, \\
\epsilon_H(C_i) & \text{if } H \in A_i \text{ and } i \neq j
\end{cases} \)

which corresponds to the face \( FC_j \in F_{A_j} \).

\[\square\]

Corollary 2.3. The face set \( F_{\Gamma} \) of a collage \( \Gamma \) is a monoid whose binary operation is the product \( FG \), for \( F, G \in F_{\Gamma} \), defined in Proposition 2.2.
The face set of a collage $\Gamma$ is a poset with partial order defined, for $F, G \in F_\Gamma$, by

$$F \preceq G \iff \forall H \in A_\Gamma : \epsilon_H(F) \neq 0 \Rightarrow \epsilon_H(F) = \epsilon_H(G).$$

**Definition 2.4.** Let $\Gamma = \{(A_i, C_i)\}_{i \in [r]}$ be a collage. A *nested face* is a pair $(F, G)$ of faces in $F_\Gamma$ such that $F \prec G$.

For a nested face $(F, G)$, define the set of faces $F^{(F,G)}_\Gamma := \{K \in F_\Gamma \mid F \preceq K \preceq G\}$.

**Definition 2.5.** Let $\Gamma$ be a collage, $D \in C_\Gamma$, and $(A, D)$ a nested face. The *chamber opposite* of $D$ with respect to $A$ is the chamber $\tilde{D}_A \in C_\Gamma$ whose sign sequence is defined, for every $H \in A_\Gamma$, by [7, Lemma 4.2]

$$u_H(\tilde{D}_A) = \begin{cases} -\epsilon_H(D) & \text{if } \epsilon_H(A) = 0, \\ \epsilon_H(A) & \text{otherwise.} \end{cases}$$

**Definition 2.6.** Let $\Gamma = \{(A_i, C_i)\}_{i \in [r]}$ be a collage, and $c_i := \min\{\dim F \mid F \in F_{A_i}\}$. The *rank* of a face $F \in F_{A_i} \subseteq F_\Gamma$ is $\text{rk } F := \dim F - c_i$.

Assign a variable $x_C$ to each chamber $C \in C_\Gamma$.

**Lemma 2.7.** Let $\Gamma$ be a collage, $D \in C_\Gamma$, and $(A, D)$ a nested face. Then,

$$\sum_{F \in F^{(A,D)}_\Gamma} (-1)^{\text{rk } F} \sum_{C \in C_\Gamma \mid FC = D} x_C = (-1)^{\text{rk } D} \sum_{C \in C_\Gamma} x_C.$$ 

**Proof.** We have

$$\sum_{F \in F^{(A,D)}_\Gamma} (-1)^{\text{rk } F} \sum_{C \in C_\Gamma \mid FC = D} x_C = \sum_{C \in C_\Gamma} \left( \sum_{F \in F^{(A,D)}_\Gamma} (-1)^{\text{rk } F} \right) x_C$$

$$= (-1)^{\text{rk } D} \sum_{C \in C_\Gamma} x_C \text{ using the proof of [7 Proposition 4.4]}$$

$$= (-1)^{\text{rk } D} \sum_{C \in C_\Gamma} x_C.$$  

□

Let $\tilde{C}_\Gamma$ be the set formed by the bounded chambers of a collage $\Gamma$. The *m-ball* $B^m$ consists of a point if $m = 0$, otherwise $B^m := \{x \in \mathbb{R}^m \mid \|x\| < 1\}$. We distinguish three types of unbounded chambers $C \in C_\Gamma \setminus \tilde{C}_\Gamma$ according to their frontiers. For some $m \in \mathbb{N}$:

1. if $\partial C$ is homeomorphic to $B^m$, we say that $C$ is of type 1, and write $C \in C^{(1)}_\Gamma$,
2. if $\partial C$ is homeomorphic to $\partial B^m \times \mathbb{R}$, we say that $C$ is of type 2, and write $C \in C^{(2)}_\Gamma$,
3. if $\partial C$ is homeomorphic to $\partial B^1 \times \mathbb{R}^m$, we say that $C$ is of type 3, and write $C \in C^{(3)}_\Gamma$.  

5
The set of faces composing the closure of a chamber $D \in C_\Gamma$ is $F_\Gamma := \{ F \in F_\Gamma \mid F \preceq D \}$.

Besides, denote by $C_\Gamma^* := C_\Gamma \setminus \{ C_i \}_{i \in [r]}$ the set of nonpointing chambers.

**Lemma 2.8.** Let $\Gamma = \{ (A_i, C_i) \}_{i \in [r]}$ be a collage, and $D \in C_\Gamma^* \setminus C_\Gamma^{(1)}$. Then,

$$\sum_{F \in F_\Gamma} (-1)^{rkF} \sum_{\substack{C \in C_\Gamma \\text{at } FC = D}} x_C = \begin{cases} (-1)^{c_i} x_D & \text{if } D \in \hat{C}_A, \\ (-1)^{1+c_i} x_D & \text{if } D \in C_A^{(2)}, \\ (-1)^{c_i+m} x_D & \text{if } D \in C_A^{(3)} \text{ and dim } D = m + 1. \end{cases}$$

**Proof.** Suppose that dim $D = m + 1$. We have

$$\sum_{F \in F_\Gamma} (-1)^{rkF} \sum_{\substack{C \in C_\Gamma \\text{at } FC = D}} x_C = \sum_{\substack{C \in C_\Gamma \\text{at } FC = D}} \left( \sum_{F \in F_\Gamma} (-1)^{rkF} \right) x_C \begin{cases} (-1)^{c_i} x_D & \text{if } D \in \hat{C}_A, \\ (-1)^{1+c_i} x_D & \text{if } D \in C_A^{(2)}, \\ (-1)^{c_i+m} x_D & \text{if } D \in C_A^{(3)}, \end{cases}$$

using the proof of [7, Proposition 4.5].

\[ \square \]

### 3 The Varchenko Matrix of a Collage

We establish the main result in this section.

The module of $R_\Gamma$-linear combinations of chambers in $C_\Gamma$ is $M_\Gamma := \left\{ \sum_{C \in C_\Gamma} x_C C \mid x_C \in R_\Gamma \right\}$.

Let $\{ C^* \}_{C \in C_\Gamma}$ be the dual basis of the basis $C_\Gamma$ of $M_\Gamma$. Define the linear map $\gamma_\Gamma : M_\Gamma \to M_\Gamma^*$, for $D \in C_\Gamma$, by

$$\gamma_\Gamma(D) := \sum_{C \in C_\Gamma} v(D, C) C^*.$$ 

For a nested face $(A, D)$, with $D \in C_\Gamma$, define $m(A, D) := \sum_{\substack{C \in C_\Gamma \\text{at } AC = D}} v(D, C) C^*$.

Consider the extension ring $B_\Gamma$ of $R_\Gamma$ defined by

$$B_\Gamma := \left\{ \frac{p}{\left( (1 + (r-1)q)(1-q)^{r-1} \right)^k \prod_{F \in F_\Gamma \setminus C_\Gamma} (1-b_F)^{k_F}} \bigg| p \in R_\Gamma, k, k_F \in \mathbb{N} \right\}.$$ 

**Proposition 3.1.** Let $\Gamma$ be a collage, $D \in C_\Gamma$, and $(A, D)$ a nested face. Then,

$$m(A, D) = \sum_{C \in C_\Gamma} x_C \gamma_\Gamma(C) \text{ with } x_C \in B_\Gamma.$$ 

**Proof.** The backward induction proof is inspired by a part of the proof of [1, Proposition 8.13]. Clearly, $m(D, D) = \gamma_\Gamma(D)$. Applying $x_C = v(D, C) C^*$ to Lemma 2.7 we get

$$\sum_{F \in F_\Gamma^{(A,D)}} (-1)^{rkF} m(F, D) = (-1)^{rkD} \sum_{\substack{C \in C_\Gamma \\text{at } AC = D_A}} v(D, C) C^*.$$ 

6
Using Definition 1.4 and [7] Lemma 5.1, we can prove that \( v(D, C) = v(D, AC) v(AC, C) \). Then,

\[
\sum_{F \in F_{\Gamma}^{(A,D)}} (-1)^{rk F} m(F, D) = (-1)^{rk F} v(D, \tilde{D}_A) \sum_{C \in C_{\Gamma}} v(\tilde{D}_A, C) C^* \\
= (-1)^{rk F} v(D, \tilde{D}_A) m(A, \tilde{D}_A). 
\]

Hence, \( m(A, D) - (-1)^{rk F} v(D, \tilde{D}_A) m(A, \tilde{D}_A) = \sum_{F \in F_{\Gamma}^{(A,D)} \setminus \{A\}} (-1)^{rk F} \sum_{C \in C_{\Gamma}} v(F, D) e_C \gamma_T(C) \) with \( a_C, e_C \in B_{\Gamma} \).

**Theorem 3.2.** Let \( \Gamma = \{ (A_i, C_i) \}_{i \in [r]} \) be a collage, and \( D \in C_{\Gamma} \). Then,

\[
D^* = \sum_{C \in C_{\Gamma}} x_C \gamma_T(C) \quad \text{with} \quad x_C \in B_{\Gamma}. 
\]

**Proof.** If \( D \in C_{\Gamma}^{(1)} \setminus C_{\Gamma}^{(2)} \), applying \( x_C = v(D, C) C^* \) to Lemma 2.8, we obtain

\[
\sum_{F \in F_{\Gamma}} (-1)^{rk F} m(F, D) = \lambda D^* \quad \text{with} \quad \lambda \in \{-1, 1\}. 
\]

From Proposition 3.1, we conclude that \( D^* = \sum_{C \in C_{\Gamma}} x_C \gamma_T(C) \) with \( x_C \in B_{\Gamma} \).

If \( D \in C_{\Gamma}^{(1)} \setminus \{ C_i \} \), consider the hyperplane arrangement \( A'_i = A_i \cup \{ H' \} \) such that

- \( H' \) divides \( D \) into two chambers \( D_b \in \tilde{C}_{A'_i} \cup C_{A'_i}^{(2)} \cup C_{A'_i}^{(3)} \), and \( D_u \in C_{A'_i}^{(1)} \),

- if \( C_{A'_i} \) is the set of chambers in \( C_{A_i} \) cut by \( H' \), then \( C_{A'_i} \cap (\tilde{C}_{A_i} \cup C_{A_i}^{(2)} \cup C_{A_i}^{(3)}) = \emptyset \) and, for every \( C \in C_{A_i} \setminus C_{A'_i}^{(1)} \), \( C \subset H'^{+} \),

- if we denote by \( C_b \) and \( C_u \) the two chambers obtained from the cut of \( C \in C_{A_i}^{(1)} \) by \( H' \), then assume that \( C_b \subset H'^{+} \).

Let \( \Gamma' \) be the collage obtained from \( \Gamma \) by replacing \( (A_i, C_i) \) with \( (A'_i, C'_i) \) or \( (A'_i, (C_i)_b) \) if \( C_i \in C_{A_i}^{(1)} \). With an argument similar to that in the proof of [7] Theorem 5.3, we prove that setting \( h_{H'}^+ = h_{H'}^- = 0 \), we get

\[
D_b^* - \sum_{C \in C_{\Gamma} \setminus C_{A_i}} x_C \gamma_T(C) - \sum_{C' \in C_{A_i}'} x_{C_b'} \gamma_T(C_b') = \sum_{C' \in C_{A_i}'} x_{C_u'} \gamma_T(C_u') = 0. 
\]

Replacing \( C_b' \) by \( C' \), we obtain \( D^* = \sum_{C \in C_{\Gamma} \setminus C_{A_i}} x_C \gamma_T(C) + \sum_{C' \in C_{A_i}'} x_{C'} \gamma_T(C') \) with \( x_C, x_{C'} \in B_{\Gamma} \).
Finally, for a pointing chamber $C_i$, we have $\gamma_\Gamma(C_i) = C_i^* + q \sum_{j \in [r] \setminus \{i\}} C_j^* + \sum_{C \in C^*_{\Gamma}} v(C, C) C^*$. Then,

$$
\begin{pmatrix}
\gamma_\Gamma(C_1) - \sum_{C \in C^*_{\Gamma}} v(C, C) C^*
\gamma_\Gamma(C_2) - \sum_{C \in C^*_{\Gamma}} v(C, C) C^*
\vdots
\gamma_\Gamma(C_{r-1}) - \sum_{C \in C^*_{\Gamma}} v(C, C) C^*
\gamma_\Gamma(C_r) - \sum_{C \in C^*_{\Gamma}} v(C, C) C^*
\end{pmatrix}
= \begin{pmatrix}
1 & q & q & \cdots & q \\
q & 1 & q & \cdots & q \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
q & \cdots & q & 1 & q \\
q & \cdots & q & q & 1 \\
\end{pmatrix}
\begin{pmatrix}
C_1^*
C_2^*
\vdots
C_{r-1}^*
C_r^*
\end{pmatrix}.
$$

Since $\det \gamma_\Gamma = (1 + (r-1)q)(1-q)^{r-1}$, then $C_i^* = \sum_{C \in C^*_{\Gamma}} x_C \gamma_\Gamma(C)$ with $x_C \in B_{\Gamma}$.

**Definition 3.3.** The Varchenko matrix of a collage $\Gamma$ is $V_\Gamma := (v(D, C))_{C,D \in C^*_{\Gamma}}$.

**Theorem 3.4.** Let $\Gamma = \{(A_i, C_i)\}_{i \in [r]}$ be a collage. Then,

$$
\det V_\Gamma = (1 + (r-1)q)(1-q)^{r-1} \prod_{F \in \mathcal{F}_{\Gamma}(C)} (1 - b_F)^{\beta_F}.
$$

**Proof.** It is clear that $V_\Gamma$ is the matrix representation of $\gamma_\Gamma$. As $\det V_\Gamma$ is a polynomial in $R_\Gamma$ with constant term 1, $V_\Gamma$ is then invertible. From Theorem 3.2, we know that, for every $D \in C^*_{\Gamma}$, there exist $x_C \in B_{\Gamma}$ such that $D^* = \sum_{C \in C^*_{\Gamma}} x_C \gamma_\Gamma(C)$. Hence, $\gamma_\Gamma^{-1}(D^*) = \sum_{C \in C^*_{\Gamma}} x_C C^*$.

Since the matrix representation of $\gamma_\Gamma^{-1}$ is $V_\Gamma^{-1}$, each entry of $V_\Gamma^{-1}$ is then an element of $B_{\Gamma}$. Besides, $V_\Gamma^{-1} = \frac{\text{adj}(V_\Gamma)}{\det V_\Gamma}$, where each entry of $\text{adj}(V_\Gamma)$ is a polynomial in $R_\Gamma$. Thus, $\det V_\Gamma$ has the form $m \left(1 + (r-1)q\right)(1-q)^{r-1} \prod_{F \in \mathcal{F}_{\Gamma}(C)} (1 - b_F)^{\beta_F}$, with $m \in \mathbb{Z}$, $k, k_F \in \mathbb{N}$.

As the constant term of $\det V_\Gamma$ is 1, we deduce that $m = 1$.

Setting $q = 0$, we get $V_\Gamma = \bigoplus_{i \in [r]} V_{A_i}$. As $\det V_{A_i} = \prod_{F \in \mathcal{F}_{\Gamma}(C_{A_i})} (1 - b_F)^{\beta_F}$ [Theorem 1.8], we conclude that, for every $F \in \mathcal{F}_{\Gamma} \setminus C_{\Gamma}$, $k_F = \beta_F$.

Now, setting $h_H = h_H^+ = 0$ for every $H \in A_{\Gamma}$, we get $V_\Gamma = (v(C_j, C_i))_{i,j \in [r]} \bigoplus 1_{\mathbb{I}}$, where $1_{\mathbb{I}}$ is the identity matrix of order $\#C_{\Gamma} - r$. Since $\det (v(C_j, C_i))_{i,j \in [r]} = (1 + (r-1)q)(1-q)^{r-1}$, we obtain $k = 1$. \qed
4 Direct Sum of Multiparametric Quon Algebras

We prove the realizability of a model of infinite statistics based on the collage $\Gamma_n = \{(A_i, C_i)\}_{i \in [r]}$, where $A_i$ is the hyperplane arrangement in $\mathbb{R}^n$ associated the permutation group of $n$ elements, and $C_i$ is the chamber associated to the identity permutation.

For $i \in \mathbb{N}^*$, denote by $\mathbb{C}[q_{awi}]$ the polynomial ring $\mathbb{C}[q_{awi} | u, v \in \mathbb{N}^*]$. The $q_{awi}$-conjugate of a monomial $P = \mu \prod_{k \in [n]} q_{uki} \in \mathbb{C}[q_{awi}]$ is the monomial

$$\tilde{P} := \tilde{\mu} \prod_{k \in [n]} \tilde{q}_{uki} \quad \text{with} \quad \tilde{\mu} = \mu \quad \text{and} \quad \tilde{q}_{uki} = q_{uki}.$$

A Hilbert module is a module $V_i$ over $\mathbb{C}[q_{awi}]$ endowed with an indefinite inner product $(.,.)_i : V_i \times V_i \to \mathbb{C}[q_{awi}]$ such that, for $\mu, v, w \in V_i$, we have

- $(\mu u, v)_i = \tilde{\mu}(u, v)_i$ and $(u + v, w)_i = (u, w)_i + (v, w)_i$,
- $(u, v)_i = (v, u)_i$,
- and, if $u \neq 0$, $(u, u)_i \in \mathbb{C}[q_{awi}] \setminus \{0\}$.

By multiparametric quon algebra is meant a free algebra $A_i = \mathbb{C}[q_{awi}] [a_{ui} | u \in \mathbb{N}^*]$ subject to the anti-involution $\dagger$ exchanging $a_{ui}$ with $a^\dagger_{ui}$, and to the commutation relations

$$a_{ui}a^\dagger_{ui} = q_{uvi} a_{ui} a^\dagger_{ui} + \delta_{uv},$$

where $\delta_{uv}$ is the Kronecker delta. In a Fock representation, the generators of $A_i$ are linear operators $a_{ui}, a^\dagger_{ui} : V_i \to V_i$ satisfying $a_{ui}|i\rangle = 0$, where $a^\dagger_{ui}$ is the adjoint of $a_{ui}$, and $|i\rangle$ is a nonzero distinguished vector of $V_i$.

The submodule $H_i := \{a|i\} | a \in A_i\}$ is also a Hilbert module for the indefinite inner product $[,]_i : H_i \times H_i \to \mathbb{C}[q_{awi}]$ defined, for $\mu, \nu \in \mathbb{C}[q_{awi}]$, and $a, b \in A_i$, by $[\mathbb{S} \text{ Theorem 1.3}]$

$$[\mu a|i\rangle, \nu b|i\rangle]_i := \tilde{\mu} \nu \langle i| a^\dagger b|i\rangle \quad \text{with} \quad \langle i|i\rangle = 1.$$

Besides, $B_i := \{a^\dagger_{ui_1} \ldots a^\dagger_{ui_n}|i\} | (u_1, \ldots, u_n) \in (\mathbb{N}^*)^n\}$ is a basis of $H_i$, $[\mathbb{S} \text{ Lemma 3.1}]$.

Let $r \in \mathbb{N}^*$, $q$ a real variable, and $\mathbb{C}[q, q_{awi}] = \mathbb{C}[q, q_{awi} | u, v \in \mathbb{N}^*, i \in [r]]$. We construct an indefinite inner product on the direct sum $H_{\otimes} := \bigoplus_{i \in [r]} H_i$.

Let $\binom{(\mathbb{N}^*)}{n}$ be the set of multisets of $n$ elements in $\mathbb{N}^*$. For $I \in \binom{(\mathbb{N}^*)}{n}$, denote by $G_I$ the permutation set of $I$, and $\iota_I$ the permutation in $G_I$ such that $\iota_I(1) \leq \cdots \leq \iota_I(n)$. Moreover, if $I \in \binom{(\mathbb{N}^*)}{n}$ and $I' \in \binom{(\mathbb{N}^*)}{n'}$, then extend the Kronecker delta by $\delta_{II'} := \begin{cases} 1 & \text{if } I = I', \\ 0 & \text{otherwise}. \end{cases}$

**Theorem 4.1.** Let $I \in \binom{(\mathbb{N}^*)}{n}$, $I' \in \binom{(\mathbb{N}^*)}{n'}$, $\sigma \in G_I$, and $\tau \in G_{I'}$. Moreover, define

$$\beta_I := \begin{cases} 1 & \text{if } I \in \binom{(\mathbb{N}^*)}{n}, \\ 0 & \text{otherwise}. \end{cases}$$

Under the conditions $\frac{1}{1-r} < q < 1$ and $|q_{awi}| < 1$, the direct sum...
\( \mathbf{H}_n = \bigoplus_{i \in [r]} \mathbf{H}_i \) is an indefinite Hilbert module for the sesquilinear form \([\cdot, \cdot]_\mathbb{B}: \mathbf{H}_n \times \mathbf{H}_n \to \mathbb{C}[q, q^\ast] \) defined, for \( \mathbf{a} = \mathbf{a}_{(\sigma(1)i)} \cdots \mathbf{a}_{(\sigma(n)i)} \in \mathbf{A}_i \) and \( \mathbf{a}' = \mathbf{a}_{(\tau(1)i)} \cdots \mathbf{a}_{(\tau(n')j)} \in \mathbf{A}_j \), by
\[
[\mathbf{a}|i\rangle, \mathbf{a}'|j\rangle]_\mathbb{B} := \begin{cases} 
\langle i|\mathbf{a}^\dag \mathbf{a}'|i\rangle \\
q^{|j|_I - |i|_I} \beta_I \langle i|\mathbf{a}^\dag_{(\tau(1)i)} \cdots \mathbf{a}^\dag_{(\tau(n')j)}|i\rangle \langle j|\mathbf{a}_{(\tau(n')j)} \cdots \mathbf{a}_{(\tau(1)i)}|a'|_j\rangle 
\end{cases}
\text{ if } i = j,
\text{ otherwise.}
\]

**Proof.** The infinite matrix associated to the sesquilinear form of Theorem 4.1 is \( \mathbf{M}_n := \left( [\mathbf{v}|i\rangle, \mathbf{v}'|j\rangle]_\mathbb{B} \right)_{\mathbf{v}, \mathbf{v}' \in \mathbf{B}_n} \). From its definition and [8, Lemma 3.2], we know that
\[
\forall i, j \in [r], \forall \sigma, \tau \in \mathcal{S}_I: I \neq I' \Rightarrow \left[ \mathbf{a}_{(\tau(1)i)} \cdots \mathbf{a}_{(\tau(n')j)}|j\rangle, \mathbf{a}_{(\sigma(1)i)} \cdots \mathbf{a}_{(\sigma(n')j)}|i\rangle \right]_\mathbb{B} = 0.
\]
Therefore,
\[
\mathbf{M}_n = \bigoplus_{n \in \mathbb{N}^r} \bigoplus_{\mathbf{I} \in \left( \left( \mathbb{N}^r \right)^n \right)} \mathbf{M}_I \text{ with } \mathbf{M}_I := \left( \left[ \mathbf{a}_{(\tau(1)i)} \cdots \mathbf{a}_{(\tau(n')j)}|j\rangle, \mathbf{a}_{(\sigma(1)i)} \cdots \mathbf{a}_{(\sigma(n')j)}|i\rangle \right]_\mathbb{B} \right)_{i, j \in [r], \sigma, \tau \in \mathcal{S}_I}.
\]

Let \( \Gamma_n = \{ (\mathbf{A}_i, C_i) \}_{i \in [r]} \) be the collage such that, for every \( i \in [r] \),
- \( \mathbf{A}_i = \{ H_{uv} | u, v \in [n], u < v \} \) with \( H_{uv} := \{ (x_1, \ldots, x_n) \in \mathbb{R}^n | x_u = x_v \} \),
- \( C_{A_i} = \{ C_{\sigma} | \sigma \in \mathcal{S}_n \} \), and \( C_i = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 < x_2 < \cdots < x_n \} \).

Assign the variable \( q_{uv} \) to the half-space \( \{ (x_1, \ldots, x_n) \in \mathbb{R}^n | x_u < x_v \} \). The module generated by the chambers of \( \Gamma_n \) is \( M_{\Gamma_n} := \{ \sum_{i \in [r]} x_{\sigma} C_{\sigma} | x_{\sigma} \in \mathbb{C}[q, q^\ast] \} \). Define the distance function \( v_n: M_{\Gamma_n} \times M_{\Gamma_n} \to \mathbb{C}[q, q^\ast] \), for \( C_{\sigma}, C_{\tau} \in C_{\Gamma_n} \), by
\[
v_n(C_{\sigma}, C_{\tau}) = \begin{cases} 
1 & \text{if } C_{\sigma} = C_{\tau}, \\
\prod_{\{u, v\} \in \binom{\left( \mathbb{N}^r \right)}{2}} q_{\sigma(u)\sigma(v)} & \text{if } i = j \text{ and } C_{\sigma} \neq C_{\tau},
\end{cases}
\]
\[
\text{if } i \neq j.
\]

Using [8] Lemma 3.3, we get \( \left[ \mathbf{a}_{(\tau(1)i)} \cdots \mathbf{a}_{(\tau(n')j)}|j\rangle, \mathbf{a}_{(\sigma(1)i)} \cdots \mathbf{a}_{(\sigma(n')j)}|i\rangle \right]_\mathbb{B} = v_n(C_{\tau}, C_{\sigma}). \) Define the linear map \( \gamma_n: M_{\Gamma_n} \to M_{\Gamma_n} \), for \( C_{\tau} \in C_{\Gamma_n} \), by
\[
\gamma_n(C_{\tau}) := \sum_{i \in [r]} v_n(C_{\tau}, C_{\sigma}) C_{\sigma}.
\]

\( M_{[n]} \) is obviously the matrix representation of \( \gamma_n \). Using Theorem 1.7 and [8] Proposition 2.1, we obtain
\[
\det M_{[n]} = (1 + (r - 1)q)(1 - q)^{r - 1} \prod_{i \in [r]} \prod_{\# K \geq 2} \left( 1 - \prod_{\{u, v\} \in \binom{K}{2}} q_{uv}(q_{uv})^{-1} \right)^{(\# K - 2)!} (n - \# K + 1)!
\]
Take a partition \( \Lambda = (p_1, \ldots, p_k) \) of \( n \). Denote by \( \mathfrak{S}_\Lambda \) the subgroup \( \prod_{u \in [k]} \mathfrak{S}_{\Lambda_u} \) of \( \mathfrak{S}_n \), where \( \mathfrak{S}_{\Lambda_u} \) is the permutation group of the set \( \Lambda_u = \{ p_u + p_u - 1 + \cdots + p_1 \} \). Consider the multiset \( I = \{ p_1 \text{ times } 1, \ldots, p_k \text{ times } k \} \) \( \in \binom{\mathbb{N}^*}{n} \). For \( s \in [n] \), let \( s := u \) if \( s \in \Lambda_u \). Let \( p : \mathfrak{S}_n \to \mathfrak{S}_I \) be the projection \( p(\sigma) := \dot{\sigma}(1) \dot{\sigma}(2) \cdots \dot{\sigma}(n) \). For \( \dot{\sigma} \in \mathfrak{S}_I \), define the element \( C_{\dot{\sigma}i} := \sum_{\sigma \in p^{-1} (\dot{\sigma})} C_{\sigma i} \in M_{\Gamma_n} \). Consider the submodule of chambers \( M_\Lambda \Gamma_n := \{ \sum_{i \in [r]} \dot{\sigma} \in \mathfrak{S}_I \} \). For \( s, t \in [n] \) with \( s \neq t \), assign the variable \( q_{\dot{\sigma}i} \) to the half-space \( \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_s < x_t \} \), and suppose that the generators of the quon algebra \( A_i \) satisfy the commutation relations \( a_{\dot{\sigma}i} a_{\dot{\tau}i}^\dagger = q_{\dot{\sigma}i} a_{\dot{\tau}i}^\dagger a_{\dot{\sigma}i} + \delta_{\dot{\sigma} \dot{\tau}} \). From [8, Proposition 2.2], we deduce that \( \gamma_n (M_\Lambda \Gamma_n) = M_\Lambda \Gamma_n \). And since, for every \( \sigma \in p^{-1}(\dot{\sigma}) \), \( \langle i | a_{\dot{\tau}(n)i} \cdots a_{\dot{\tau}(1)i} a_{\dot{\sigma}(1)i} \cdots a_{\dot{\sigma}(n)i} | i \rangle = \nu_n (C_{\dot{\tau}i}, C_{\sigma i}) \) [8, Lemma 3.4], then \( M_I \) is the matrix representation of \( \gamma_n \) on the basis \( \{ C_{\sigma i} \}_{\sigma \in \mathfrak{S}_I} \).

We can finally conclude that, for every \( I \in \binom{\mathbb{N}^*}{n} \), \( M_I \) is nonsingular for \( \frac{1}{1 - r} < q < 1 \) and \( |q_{\dot{\alpha}i}| < 1 \). \( \square \)
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