ON THE EQUATION \( N_{p_1}(E)N_{p_2}(E) \cdots N_{p_k}(E) = n \)

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Abstract. For a given elliptic curve \( E/\mathbb{Q} \), set \( N_p(E) \) to be the number of points on \( E \) modulo \( p \) for a prime of good reduction for \( E \). Given integer \( n \), let \( G_k(E, n) \) be the number of \( k \)-tuples of pairwise distinct primes \( p_1, \ldots, p_k \) of good reduction for \( E \), for which equation in the title holds, then on assuming the Generalized Riemann Hypothesis for elliptic curves without CM (and unconditionally if the curves have complex multiplication), I show that \( \lim_{n \to \infty} G_k(E, n) = \infty \) for any integer \( k \geq 3 \). I also conjecture that this result also holds for \( k = 1 \) and \( k = 2 \). In particular for \( k = 1 \) this conjecture says that there are “elliptic progressions of primes” i.e. sequences of primes \( p_1 < p_2 \cdots < p_m \) of arbitrary lengths \( m \) such that \( N_{p_1}(E) = N_{p_2}(E) = \cdots = N_{p_m}(E) \).

\[ N \text{ was a net} \]
\[ \text{Which was thrown in the sea} \]
\[ \text{To catch fish for dinner} \]
\[ \text{For you and for me.} \]

Edward Lear ([5])

1. Introduction

Let me begin with an example which explains the problem I consider in this paper. Consider the elliptic curve \( E : y^2 + y = x^3 - x/\mathbb{Q} \). This is an elliptic curve with conductor 37. Let \( p \) be a prime of good reduction, here and at all relevant places I will always assume \( p \) is a prime of good reduction and call such a prime a good prime (for \( E \)). Let \( N_p(E) \) be the number of points on this elliptic curve modulo \( p \). It is a well-known theorem of Hasse and Weil [13] that

\[ p + 1 - 2\sqrt{p} \leq N_p(E) \leq p + 1 + 2\sqrt{p}. \]

A simple computation with [11] and [17] reveals that following equalities hold:

\[ N_2(E) \cdot N_{13}(E) \cdot N_{43}(E) = 3360 = N_3(E) \cdot N_5(E) \cdot N_{67}(E) \]
\[ N_5(E) \cdot N_{43}(E) \cdot N_{73}(E) = 25200 = N_{17}(E) \cdot N_{19}(E) \cdot N_{61}(E) \]
\[ N_{101}(E) \cdot N_{107}(E) \cdot N_{251}(E) = 3107520 = N_{113}(E) \cdot N_{127}(E) \cdot N_{167}(E) \]
\[ 99 \cdot 120 \cdot 254 = 3107520 = 132 \cdot 127 \cdot 180. \]
\[ N_{1009}(E) \cdot N_{1181}(E) \cdot N_{1601}(E) = 1988217000 = N_{1063}(E) \cdot N_{1283}(E) \cdot N_{1399}(E) \]
\[ 1057 \cdot 1125 \cdot 1648 = 1988217000 = 1057 \cdot 1320 \cdot 1425. \]

All the triples of primes are all pairwise distinct; moreover the six numbers which enter these equalities are typically distinct but occasionally not.

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More generally, for a fixed \( k \geq 3 \), any elliptic curve \( E/\mathbb{Q} \), and a given natural number \( n \geq 1 \), I consider the equation (in pairwise distinct primes \( p_1, \ldots, p_k \))

\[
N_{p_1}(E) \cdots N_{p_k}(E) = n.
\]

Let \( G_k(E, n) \) (for \( k \geq 3 \)) be the number of solutions to (1.8) in pairwise distinct primes \( p_1, \ldots, p_k \). The number \( G_k(E, n) \) is always finite. Indeed for any prime \( p \) one has

\[
\frac{p + 1 - 2\sqrt{p}}{p} \geq \begin{cases} 
1 - \frac{2}{\sqrt{p}} & \text{if } p \geq 5 \\
1 + \frac{1}{3} - \frac{2}{\sqrt{3}} & \text{if } p = 3 \\
1 + \frac{1}{2} - \frac{2}{\sqrt{2}} & \text{if } p = 2
\end{cases}
\]

Hence \( N_p(E) \geq p/100 \) for all primes \( p \) and so

\[ n \geq \frac{p_1 \cdots p_k}{100^k}. \]

Thus the primes which contribute to \( G_k(E, n) \) are finite in number and hence \( G_k(E, n) \) is finite for any \( n \). The purpose of this paper is to prove the following theorem which shows that these equalities (1.2) are not small numerical accidents.

**Theorem 1.10.** Let \( E/\mathbb{Q} \) be an elliptic curve. If \( E \) does not have complex multiplication assume GRH holds for \( E \). Then for any \( k \geq 3 \)

\[
\lim_{n \to \infty} G_k(E, n) = \infty.
\]

More precisely, for every \( k \geq 3 \) and for all sufficiently large \( x \) there exist integers \( n \leq x \) with

\[ G_k(E, n) \geq (\log x)^\delta \]

for some \( \delta = \delta(k) > 0 \).

It is unlikely that the growth of \( G_k(E, n) \) given by the proof of Theorem 1.10 is optimal. Let me point of that in [1], Paul Erdos considered the question of solutions (in triples of pairwise distinct primes \( p_1, p_2, p_3 \)) to the equation

\[
(p_1 - 1)(p_2 - 1)(p_3 - 1) = n,
\]

and proved the corresponding result in this case and that *loc. cit.* served as an inspiration to this note. Let me also point out that one can also replace the assumption that GRH holds for \( E \) by other types of hypothesis such as \( \theta \)-quasi-GRH etc., however since I make no claim of providing sharpest error terms nor best possible constants, the choice of stronger or weaker hypothesis is insignificant (to me). While [1] served as inspiration for this paper, my approach differs from that of *loc. cit.* in several important points. Notably I use the fourth moment to prove Lemma 3.5 instead of the Hardy-Ramanujan Theorem. I also prove Theorem 2.26, which was proved in [1] for \( k = 3 \), a bit differently. Note that Theorem 1.10 assumes \( k \geq 3 \). But in fact I expect that the result also holds for \( 1 \leq k \leq 2 \). I hope to pursue \( k = 2 \) in a separate paper.

The case \( k = 1 \) seems to be interesting in its own right: I conjecture that the result holds for \( k = 1 \). In other words I conjecture that there exists a sequence of integers \( n \to \infty \) and a sequence of primes \( p_1 < p_2 < \cdots < p_m \) with \( m \to \infty \) (with \( n \)) such that

\[ N_{p_1}(E) = N_{p_2}(E) = \cdots = N_{p_m}(E) = n. \]

I call this an elliptic progression of primes and thus the assertion for \( k = 1 \) is equivalent to existence of arbitrarily long sequences of primes in elliptic progressions (for any given
elliptic curve \( E/\mathbb{Q} \). This seems difficult at the moment. Even the weakest assertion: that for any \( E/\mathbb{Q} \) there exists infinitely many pairs of primes \( N_p(E) = N_q(E) \) seems difficult at the moment. See Section 4 for a detailed discussion of what I expect to be true for \( k = 1 \) and numerical examples. Let me also point out that in [14] it has been conjectured that there exists infinitely many pairs of primes \( p, q \) such that \( N_p(E) = q \) and \( N_q(E) = p \) (I am grateful to Joseph Silverman for providing this reference).

I thank M. Ram Murty for his comments which have improved the readability of this paper and for pointing out that in [10] and [9] estimation of higher moments (used here and calculated by the sieving method of [3]) is carried using a different method.

2. Basic Estimates

Throughout this paper \( \varepsilon \) will be a positive number which will be sufficiently small. Symbols \( c_1, c_2, \ldots \) will be positive constants as will \( c_1(\varepsilon), c_2(\varepsilon), \ldots \) which will depend on \( \varepsilon \). In either of the cases the numerical values of these constants will be immaterial and I caution the reader that in some situations the same constant may be denoted by different symbols. The letter \( x \geq 1 \) will denote a real number and symbols \( x_0, x_0(\varepsilon), x_1, x_1(\varepsilon), \ldots \) will also be real numbers whose values will not be important to us except for the fact that such numbers will exist (in the contexts where they appear) and again such symbols may occur in multiple contexts (and may differ from the ones which appearing in other contexts). Hopefully there will be no confusion caused by this notational conflation which I will indulge in throughout this paper.

For an integer \( n \) let \( \omega(n) \) be the number of distinct prime factors of \( n \) and let \( \Omega(n) \) be the number of prime factors of \( n \), counted with multiplicity. Let \( E \) be an elliptic curve over \( \mathbb{Q} \). Let \( E[m] \) be the \( m \)-torsion of the elliptic curve \( E \) for an integer \( m \geq 1 \). I will say that \( \text{GRH} \) holds for \( E/\mathbb{Q} \) if the Generalized Riemann Hypothesis holds for the Dedekind zeta function of the finite Galois extension \( \mathbb{Q}(E[m])/\mathbb{Q} \) for every integer \( m \geq 1 \). A prime \( p \) will be called a good prime for \( E \) if \( E \) has good reduction at \( p \).

Let \( d \) be a square-free integer. Let \( \pi_E(x, d) \) be the number of good primes \( p \leq x \) such that \( d|N_p(E) \). Write

\[
\pi_E(x, d) = \delta(d)\text{Li}(x) + r_d,
\]

where \( \text{Li}(x) = \int_{2}^{x} \frac{dt}{\log(t)} \), \( \delta(d) \) is a positive fraction (whose explicit form is not needed for the moment) and \( r_d \) is the "error term." If we assume that GRH holds for \( E \) then the precise form of error term can be made explicit (this the only point where one needs to assume GRH). This is done by means of the explicit Chebotarev density theorem of [8, 12] and hence one has the following estimate for \( r_d \) due to [16, 15]:

\[
|r_d| = O \left( d^{3/2}x^{1/2}\log(dx) \right),
\]

and the implied constant is dependent on \( E \) (specifically the conductor of \( E \)).

The following result of [6, 7] will be crucial in the proof of Theorem 1.10.

**Theorem 2.3.** Let \( E/\mathbb{Q} \) be an elliptic curve. If \( E \) does not have complex multiplication assume GRH holds for \( E \). Then there exists a positive constant \( c_i \) such that for all \( x \) sufficiently large one has

\[
\sum_{p \leq x} (\omega(N_p(E)) - \log \log x)^2 = c_1x \log \log x(1 + o(1)).
\]
Theorem 2.3 says that the numbers $\omega(N_p(E))$ follow a normal distribution with mean $\log \log x$ and variance $(\log \log x)^{1/2}$. The next lemma computes the fourth moment of this distribution.

**Lemma 2.4.** Let $E/\mathbb{Q}$ be an elliptic curve. If $E$ does not have complex multiplication assume GRH holds for $E$. There is a constant $c_i$ such that for all $x$ sufficiently large, one has

$$
\sum_{p \leq x} (\omega(N_p(E)) - \log \log x)^4 = c_2 \pi(x)(\log \log x)^2 + o(\pi(x)(\log \log x)^2).
$$

**Proof.** This proved by the sieving argument of [4]. An alternative method to this using [9] was suggested to me by M. Ram Murty. Let me write $z = x^\beta$ where $\beta < 1$ is a constant. At the end of this proof one is led to the choice $\beta = \frac{1}{29}$. Let $P = \prod_{q < z} q$. Define

$$
\omega_z(n) = \sum_{q | n, q < z} 1,
$$

so $\omega_z(n)$ counts the number of distinct prime divisors $q | n$ such that $q < z$. As $z = x^\beta$, any $n \leq x$ has at most a bounded number of prime divisors $q \geq z$. So one has

(2.5)  \quad \omega(N_p(E)) - \log \log x = (\omega_z(N_p(E)) - \log \log x) + (\omega(N_p(E)) - \omega_z(N_p(E)))

(2.6)  \quad = (\omega_z(N_p(E)) - \log \log x) + O(1).

Thus

(2.7)  \quad (\omega(N_p(E)) - \log \log x)^4 = ((\omega_z(N_p(E)) - \log \log x) + (\omega(N_p(E)) - \omega_z(N_p(E))))^4

(2.8)  \quad = ((\omega_z(N_p(E)) - \log \log x) + O(1))^4.

So one has

(2.9)  \quad \sum_{p \leq x} (\omega(N_p(E)) - \log \log x)^4 = \sum_{p \leq x} (\omega_z(N_p(E)) - \log \log x)^4 +

(2.10)  \quad + O \left( \sum_{p \leq x} (\omega_z(N_p(E)) - \log \log x)^3 \right)

(2.11)  \quad + O \left( \sum_{p \leq x} (\omega_z(N_p(E)) - \log \log x)^2 \right)

(2.12)  \quad + O \left( \sum_{p \leq x} (\omega_z(N_p(E)) - \log \log x)^x \right)

(2.13)  \quad + O (\pi(x)).

(2.14)

By [4, Proposition 3] one has

(2.15)  \quad \sum_{p \leq x} (\omega_z(N_p(E)) - \log \log x)^4 = c_3 \pi(x)(\log \log x)^2 \left( 1 + O \left( \frac{1}{\log \log x} \right) \right)

(2.16)  \quad + O \left( (\log \log x)^4 \sum_{d \in D_4} |r_d| \right),
where in the last sum “$d \in D_4$” is short for sum over sum over all square-free products $d = q_1 \cdots q_4$ of at most four primes $q_i$, each $q_i < z$. Similarly

\begin{equation}
\sum_{p \leq x} (\omega_z(N_p(E)) - \log \log x)^3 \leq c_4 \pi(x) \log \log x \left( 1 + O \left( \frac{1}{\log \log x} \right) \right)
\end{equation}

\begin{equation}
+ O \left( (\log \log x)^3 \sum_{d \in D_3} |r_d| \right),
\end{equation}

where in the last sum “$d \in D_4$” is short for sum over sum over all square-free products $d = q_1 \cdots q_4$ of at most four primes $q_i$, each $q_i < z$.

The sum $\sum_{p \leq x} (\omega_z(N_p(E)) - \log \log x)^2$ is estimated by Theorem 2.3 and is $O(\pi(x) \log \log x)$ and so by Cauchy-Schwartz inequality one has

\[ | \sum_{p \leq x} (\omega_z(N_p(E)) - \log \log x) | \ll \pi(x) \log \log x. \]

Let me indicate how to estimate the term (the estimation of other terms is similar to this one)

\begin{equation}
O \left( (\log \log x)^4 \sum_d |r_d| \right)
\end{equation}

where the sum over $d$ runs over all square-free integers which are products of at most four primes $q < z$. Each $d$ which contributes to the sum is at most a product of four primes $q < z$ so

\begin{equation}
d = q_1 \cdots q_4 < z^4,
\end{equation}

and hence $d^{3/2} \log(d) \leq d^{3/2} < (z^4)^{5/2} = z^{10}$. Further since there are $\pi(z)$ primes less that $z$, so the number of such $d$ is $O(\pi(z)^4)$. Thus one has

\begin{equation}
\sum |r_d| = O \left( x^{1/2} \log x z^{10} \pi(z)^4 \right) = O \left( x^{1/2} \log x \frac{z^{14}}{(\log(z))^4} \right)
\end{equation}

which is

\begin{equation}
O \left( \frac{x^{1/2} x^{14\beta}}{(\log x)^3} \right)
\end{equation}

Now choose $\beta$ so that $14\beta + \frac{1}{2} < 1$ which gives $\beta < \frac{1}{28}$. So if we choose $\beta = \frac{1}{29}$ one gets

\begin{equation}
\sum_d |r_d| = O \left( \frac{x^{57/58}}{(\log x)^3} \right),
\end{equation}

and finally with this estimate the last term in (2.19) is

\begin{equation}
\leq c_5 \frac{x^{57/58}}{(\log x)^4} = o(\pi(x)(\log \log x)^2).
\end{equation}

The sum

\begin{equation}
O \left( (\log \log x)^3 \sum_{d \in D_3} |r_d| \right)
\end{equation}
can also be estimated similarly and is \( o(\pi(x)(\log \log x)^2) \). The sum \( O(\log \log x \sum_{d \in D_1} |r_d|) \) can also be estimated similarly and is certainly \( o(\pi(x)(\log \log x)^2) \). This completes the proof of Lemma 2.4.

□

The following theorem was proved in \([1]\) for \( k = 3 \).

**Theorem 2.26.** Let \( k \geq 3 \) be an integer. Let \( 0 < \varepsilon < \frac{2}{(20(k^2+k))] \) be a real number. Let

\[
D(x) = \{n = n_1 \cdots n_k \leq x : \omega(n_i) > (1 - \varepsilon) \log \log x \text{ for } 1 \leq i \leq k\}.
\]

Then

\[
|D(x)| \leq c_6(\varepsilon) \frac{x}{(\log x)^{1+\delta}},
\]

with \( \delta = \log(2)(k - \varepsilon \frac{k^2+k}{2}) - 2 > \log(2)(k - \frac{1}{10}) - 2 > 0 \).

**Proof.** Let \( n \in D(x) \). Let us write \( n = m_1 \cdots m_2 \) where \( m_1 \) is square-free and for every prime \( q|m_2 \) one has \( q^2|m_2 \). Let \( D_1(x) = \{n \in D(x) : \omega(m_2) > \varepsilon \log \log x\} \) and let \( D_2(x) = \{n \in D(x) : \omega(m_2) \leq \varepsilon \log \log x\} \). Then clearly \( D(x) = D_1(x) \cup D_2(x) \) and \( D_1(x) \cap D_2(x) = \emptyset \). The cardinalities of each of \( D_1(x) \) and \( D_2(x) \) will be estimated separately.

If \( n \in D_1(x) \) then there is a divisor \( d|n \) such that \( \omega(d) = j \) where \( j = [\varepsilon \log \log x] \) and for each prime \( q|d \) one has \( q^2|n \). Hence \( |D_1(x)| \) is less than the number of such numbers less than \( x \) i.e.,

\[
|D_1(x)| < c_7x\left(\sum_{q, \alpha \geq 2} \frac{1}{q^\alpha}\right)^j \cdot \frac{1}{j!},
\]

where the sum is over all primes \( q \) and all integers \( \alpha \geq 2 \) and clearly the sum converges so let \( K = \sum_{q, \alpha \geq 2} \frac{1}{q^\alpha} \). Then

\[
|D_1(x)| < c_8 \frac{xK^j}{j!} < c_9x(\frac{K}{j})^j.
\]

Now as \( j = [\varepsilon \log \log x] \) note that one has the trivial bounds (for all \( x \) sufficiently large):

\[
(2.31) \quad j \leq \varepsilon \log \log x
\]

\[
(2.32) \quad j \geq \frac{1}{2}\varepsilon \log \log x.
\]

Hence one has

\[
K^j = e^{j \log(K)} \leq e^{\varepsilon \log \log x \log(K)},
\]

\[
(2.34) \quad = e^{\log((\log x)^{\varepsilon \log(K)})}
\]

\[
(2.35) \quad = (\log x)^{\varepsilon \log(K)},
\]

and

\[
(2.36) \quad \frac{1}{j} \leq \frac{2}{\varepsilon \log \log x},
\]

so

\[
(2.37) \quad \frac{1}{j^j} \leq \left(\frac{2}{\varepsilon}\right)^j \frac{1}{(\log \log x)^j},
\]

The following theorem was proved in \([1]\) for \( k = 3 \).
and this gives

\[(2.38) \quad \frac{1}{j^j} \leq \frac{(\log x)^{\varepsilon \log(2/\varepsilon)}}{(\log \log x)^j}.
\]

Hence

\[(2.39) \quad \frac{K^j}{j^j} < \frac{(\log x)^{\varepsilon \log(K) + \varepsilon \log(2/\varepsilon)}}{(\log \log x)^j}.
\]

Now as \(j \geq \frac{1}{2} \varepsilon \log \log x\) one has

\[(2.40) \quad \frac{1}{(\log \log x)^j} \leq \frac{1}{(\log \log x)^{(\varepsilon/2) \log \log \log x}}
\]

\[(2.41) \quad \frac{1}{(\log \log x)^j} \leq \frac{1}{(\log \log x)^{(\varepsilon/2) \log \log \log x}}.
\]

So finally one sees that this gives

\[(2.42) \quad \frac{K^j}{j^j} \leq \frac{(\log x)^{\varepsilon \log(K) + \varepsilon \log(2/\varepsilon)}}{(\log x)^{(\varepsilon/2) \log \log \log x}},
\]

which is certainly \(\leq \frac{1}{(\log x)^{1+\delta}}\) for any \(\delta > 0\) for sufficiently large \(x\). Hence one deduces that for all \(x\) sufficiently large one has

\[(2.43) \quad |\mathcal{D}_1(x)| \leq c_{10} x K^j \leq c_{11} \frac{x}{(\log x)^{1+\delta}}
\]

for any positive \(\delta\).

Now let me estimate \(|\mathcal{D}_2(x)|\). Each \(n \in \mathcal{D}_2(x)\) is of the form \(n = n_1 \cdots n_k\) with \(\omega(n_i) \geq (1 - \varepsilon) \log \log x\) for all \(1 \leq i \leq k\). Hence one has

\[(2.44) \quad \Omega(n) \geq (k - k\varepsilon) \log \log x.
\]

As \(n \in \mathcal{D}_2(x)\), one has \(n = m_1 \cdot m_2\) with \(\omega(m_2) \leq \varepsilon \log \log x\). Therefore each pair \(n_i, n_j\) can have at most \(\varepsilon \log \log x\) common prime factors (as such factors contribute to \(m_2\)). Therefore it follows that each \(n \in \mathcal{D}_2(x)\) has at least

\[(2.45) \quad (k - k\varepsilon - \varepsilon k(k - 1)/2) \log \log x = (k - \varepsilon k(k + 1)/2) \log \log x
\]

distinct prime factors. In other words

\[(2.46) \quad \omega(n) \geq (k - \varepsilon k(k + 1)/2) \log \log x.
\]

If \(\varepsilon < \frac{2}{20(k + k)}\) then the number on the right is positive. Thus each such \(n\) has at least \(2^{(k - \varepsilon k(k + 1)/2) \log \log x}\) distinct divisors.

On the other hand as

\[(2.47) \quad \sum_{n \leq x} d(n) \leq c_{12} x \log x
\]

one has

\[(2.48) \quad 2^{(k - \varepsilon k(k + 1)/2) \log \log x} \sum_{n \in \mathcal{D}_2(x)} 1 \leq \sum_{n \leq x} d(n) \leq c_{13} x \log x
\]
which gives

\begin{align}
|D_2(x)| & \leq c_{14} \frac{x \log x}{2^{(k-\varepsilon k(k+1)/2)} \log \log x} \\
& \leq c_{15} \frac{x \log x}{\varepsilon \log(2) \log(2)^{(k-\varepsilon k(k+1)/2)}} \\
& \leq c_{16} \frac{x \log x}{(\log x) \log(2)^{(k-\varepsilon k(k+1)/2)}} \\
& \leq c_{17} \frac{x}{(\log x) \log(2)^{(k-\varepsilon k(k+1)/2)-1}}.
\end{align}

This will be \( \ll \frac{x}{(\log x)^{1+\delta}} \) with \( \delta > 0 \) if

\begin{align}
\log(2)^{(k-\varepsilon k(k+1)/2)} - 1 > 1.
\end{align}

Now one choose \( \varepsilon \) as follows:

\begin{align}
\varepsilon < \frac{2}{20(k^2 + k)}
\end{align}

then it follows that

\begin{align}
\log(2)^{(k-\varepsilon k(k+1)/2)} - 1 > \log(2)^{(k - \frac{1}{20})} - 1 > 1
\end{align}

holds for all \( k \geq 3 \). Hence in this case one has

\begin{align}
|D_2(x)| & \leq c_{18}(\varepsilon) \frac{x}{\log x)^{1+\delta}}
\end{align}

with \( \delta(k) = \log(2)^{(k-\varepsilon k(k+1)/2)} - 2 > 0 \) for all \( k \geq 3 \). This is the only place where \( k \geq 3 \) is used in this proof. Unfortunately as \( 2 \log(2) = 1.38 \cdots \) no choice of \( \varepsilon > 0 \) however small can give \( \delta(k) > 0 \). Now one has \( |D(x)| = |D_1(x)| + |D_2(x)| \). Putting the estimates (2.43) and (2.56) together one obtains that there is a \( \delta > 0 \) (for \( k \geq 3 \)) such that for all sufficiently large \( x \) one has

\begin{align}
|D(x)| & \leq c_{19}(\varepsilon) \frac{x}{(\log x)^{1+\delta}}.
\end{align}

This proves the theorem. \( \Box \)

3. Proof of Theorem 1.10

**Lemma 3.1.** Let \( E/\mathbb{Q} \) be an elliptic curve. If \( E \) does not have complex multiplication assume GRH holds for \( E \). Let \( \varepsilon \) be a fixed, sufficiently small positive real number. Let \( A(x) \) denote the number of primes \( p \leq x \) such that

\begin{align}
\omega(N_p(E)) < (1 - \varepsilon) \log \log x
\end{align}

Then there is a positive constant \( c_4(\varepsilon) \) such that for all \( x \geq x_0 \) one has

\begin{align}
|A(x)| \leq c_{20}(\varepsilon) \frac{\pi(x)}{(\log \log x)^2}.
\end{align}

**Proof.** It is sufficient to prove that the number of \( p \leq x \) such that

\begin{align}
\omega(N_p(E)) < (1 - \varepsilon) \log \log x \quad \text{or} \quad \omega(N_p(E)) > (1 + \varepsilon) \log \log x
\end{align}

part. 1
is \( \ll \frac{\pi(x)}{(\log \log x)^2} \) as the number of \( p \leq x \) counted in the assertion of this Lemma is certainly less than the number of primes \( p \leq x \) with this property. For \( p \leq x \) satisfying the above property one has either
\[
\omega(N_p(E)) - \log \log x < -\varepsilon \log \log x
\]
i.e.
\[
-(\omega(N_p(E)) - \log \log x) > \varepsilon \log \log x
\]
or
\[
\omega(N_p(E)) - \log \log x > \varepsilon \log \log x
\]
and hence at any rate
\[
(\omega(N_p(E)) - \log \log x)^4 > \varepsilon (\log \log x)^4.
\]
Hence
\[
\sum_{p \in A(x)} \varepsilon^4 (\log \log x)^4 < \sum_{p \leq x} (\omega(N_p(E)) - \log \log x)^4 \leq c_{21} \pi(x) (\log \log x)^2,
\]
by Lemma 2.4. Thus the assertion follows. \( \square \)

**Lemma 3.5.** Let \( E/\mathbb{Q} \) be an elliptic curve. If \( E \) does not have complex multiplication assume GRH holds for \( E \). The series \( \sum_{p \in A(x)} \frac{1}{p} \) converges.

**Proof.** This follows from the previous lemma by Abel summation formula and the tautological bound \( \mathcal{A}(t) \leq \pi(t) \) for all \( t \geq 1 \) as the integral
\[
\int_{1}^{x} \frac{dt}{t \log(t)(\log \log(t))^2}
\]
converges as \( x \to \infty \). \( \square \)

**Lemma 3.6.** Let \( E/\mathbb{Q} \) be an elliptic curve. If \( E \) does not have complex multiplication assume GRH holds for \( E \). Let \( \varepsilon \) be a fixed, sufficiently small positive real number. Let \( \mathcal{C}(x) \) denote the number of primes \( p \leq x \) such that
\[
(3.7) \quad \omega(N_p(E)) \geq (1 - \varepsilon) \log \log x.
\]
Then there is a positive constant \( c_5(\varepsilon) \) such that for all \( x \geq x_0 \) one has
\[
|\mathcal{C}(x)| \geq \pi(x) - c_{22}(\varepsilon) \frac{\pi(x)}{(\log \log x)^2} > c_{23}(\varepsilon) \pi(x).
\]

**Proof.** By Lemma 3.1 the number of primes \( p \leq x \) such that \( \omega(N_p(E)) \geq (1 - \varepsilon) \log \log x \) does not hold is \( o(\pi(x)) \) hence the assertion follows. \( \square \)

**Lemma 3.8.** Let \( E/\mathbb{Q} \) be an elliptic curve. If \( E \) does not have complex multiplication assume GRH holds for \( E \). Suppose \( \varepsilon \) is sufficiently small positive real number. Let \( k \geq 1 \) be an integer and \( 0 < a < b < 1 \) be fixed positive real numbers such that \( \log(b) - \log(a) > 1 \). Let
\[
\sum_1 = \sum_{x^a \leq p_1, \ldots, p_k < x^b} \frac{1}{p_1 \cdots p_k},
\]
where the sum is over all the pairwise distinct primes \( p_1, \ldots, p_k \) such that \( x^a < p_i < x^b \) and \( \omega(N_{p_i}(E)) > (1 - \varepsilon) \log \log x \) for all \( 1 \leq i \leq k \). Then there is a positive constant (depending on \( a, b \)) such that
\[
\sum_1 = c_{24}(a, b, k) - \varepsilon(x) > 1,
\]
where the right hand side is greater than one for all sufficiently large $x$ and $\epsilon(x) \to 0$ as $x \to \infty$.

Proof. For the purpose of this paper, I say that a prime $p$ is admissible if $\omega(N_p(E)) \geq (1 - \epsilon) \log \log x$, otherwise I say $p$ is inadmissible. Let me first remark that the proof shows that one may take $a, b$ satisfying $0 < b < \frac{1}{e^{k-1}}$ and $a < \frac{1}{e^{k-1}}$ then one has $0 < a < b < 1$ and $\log(b/a) > 1$. For $k = 3$, in [1] $b = \frac{1}{4} < \frac{1}{2}$ and $a = \frac{1}{8} < \frac{1}{2e}$ so that $\log(b/a) = \log(4) > 1$.

Consider (the sum over all primes)

$$
\sum_{x^a \leq p \leq x^b} \frac{1}{p} = \log \log(x^b) - \log \log(x^a) + o(1) = \log(b/a) - \log(a) + o(1)
$$

by [2, Mertens Theorem] and in particular, by the choice of $a, b$, this sum is greater than a positive constant for all $x$ sufficiently large. Now write this sum as

$$(3.9) \sum_{x^a \leq p \leq x^b} \frac{1}{p} = \sum_2 \frac{1}{p} + \sum_3 \frac{1}{p}$$

where $\sum_2$ is over primes $p$ for which $\omega(N_p(E)) > (1 - \epsilon) \log \log x$, in other words it is a sum over admissible primes, and

$$(3.10) \sum_3 = \sum_{x^a \leq p \leq x^b, \omega(N_p(E)) \leq (1 - \epsilon) \log \log x} \frac{1}{p}$$

is a sum over inadmissible primes. By the convergence of the sum over inadmissible primes given by Lemma 3.5 one sees that if we write

$$(3.11) \sum_3 = \sum_{x^a \leq p \leq x^b, \omega(N_p(E)) \leq (1 - \epsilon) \log \log x} \frac{1}{p} = \epsilon(x)$$

then $\epsilon(x) = o(1)$ as $x \to \infty$ and in particular for all sufficiently large $x$ one has $|\epsilon(x)| < \frac{1}{100}$. The choice of $\frac{1}{100}$ is arbitrary, the point being that $\epsilon(x)$ can be made smaller than any given positive real number by choosing $x$ sufficiently large. So one gets that the sum of the reciprocals of admissible primes (on the left)

$$(3.12) \sum_2 = \sum_{x^a \leq p \leq x^b} \frac{1}{p} - \epsilon(x) = \log(b/a) - \epsilon(x)$$

and hence for sufficiently large $x$ the right hand side is greater than one as $\epsilon(x)$ can be made arbitrarily small by choosing $x$ sufficiently large. Thus the result is true for $k = 1$.

Now before I give the general case let me illustrate the method of proof with the case $k = 2$ as this also will allow me to set up notational conventions needed to do the general case. The sums over primes which occur below are sums over admissible primes. Since the result is true for $k = 1$, one gets by squaring

$$(3.13) \left( \sum_2 \right)^2 = (\log(b/a) - \epsilon(x))^2$$

on the other hand by squaring the sum over admissible primes on the left one gets a sum over admissible primes

$$(3.14) \left( \sum_2 \right)^2 = \sum \frac{1}{p_1 p_2} + 2 \sum \frac{1}{p_1^2} = (\log(b/a))^2 - 2 \log(b/a)\epsilon(x) + \epsilon^2(x).$$
Now the sum $\sum_p \frac{1}{p^x}$, taken over all primes (and not just admissible ones) is convergent and hence the sum $\sum_{x^a<p_1<x^b} \frac{1}{p_1}$ can be made arbitrarily small by choosing $x$ sufficiently large, in other words the sum is $\epsilon(x)$ for some function $\epsilon(x)$. Since $\log(b/a)$ is fixed one may also write middle term in the above expression as $\epsilon(x)$ for yet another $\epsilon(x)$. Since there are finitely many such $\epsilon(x)$ it is possible to arrange $x$ so large that all of them be less than a given positive real number, similarly $\epsilon^2(x)$ is also $\epsilon(x)$. Note that I am systematically conflating all the $\epsilon(x)$—a diligent reader may easily workout the full notational genuflections required to separate them. Thus one gets from all this that

$$\sum_{x^a<p_1, p_2<x^b} \frac{1}{p_1 p_2} = c_{25} - \epsilon(x) > 1$$

for all sufficiently large $x$.

Now let me return to the general case which is proved by induction on $k$. Suppose the result is true for such a sum in the statement of the lemma and all natural numbers $\leq k$. Then I show that the result is also true for $k + 1$. Since the result is true for $k$ one has

$$\sum_1 = \sum_{x^a \leq p_1, \ldots, p_k < x^b} \frac{1}{p_1 \cdots p_k} = c_{26}(a, b, k) - \epsilon(x),$$

Then consider the product

$$\left( \sum_{x^a \leq p_1, \ldots, p_k < x^b} \frac{1}{p_1 \cdots p_k} \right) \left( \sum_{x^a \leq p_1, \ldots, p_k < x^b} \frac{1}{p} \right)$$

where both the sums are over all (pairwise) distinct admissible primes in the asserted range. Then by induction hypothesis (applied to both the sums) one has

$$\left( \sum_{x^a \leq p_1, \ldots, p_k < x^b} \frac{1}{p_1 \cdots p_k} \right) \left( \sum_{x^a \leq p_1, \ldots, p_k < x^b} \frac{1}{p} \right) = (c_{27} - \epsilon(x))(c_{28} - \epsilon(x)) = c_{29} - \epsilon(x),$$

for some $\epsilon(x)$ and some constant depending on $a, b$ and in particular the right hand side is greater than one.

Now multiplying out the product one gets one term of the form

$$\sum_1 = \left( \sum_{x^a \leq p_1, \ldots, p_{k+1} < x^b} \frac{1}{p_1 \cdots p_{k+1}} \right)$$

a sum over pairwise distinct admissible primes in the asserted range and the remaining terms are of the form

$$\sum_1 = \left( \sum_{x^a \leq p_1, \ldots, p_{k+1} < x^b} \frac{1}{p_{\ell_1} \cdots p_{\ell_j}} \right),$$

where the primes are admissible and $\ell_1 + \cdots + \ell_j = k + 1$ and $j \leq k$ and at least one of the exponents $\ell_1, \ldots, \ell_j$ is $\geq 2$, and which arise out of coincidences amongst the primes when one takes the product in (3.17). So one has to deal with these sort of sums.

I claim that these second sort of sums are always $\epsilon(x)$ for some $\epsilon(x) \to 0$ as $x \to \infty$. Suppose this assertion is true for the moment. Then one gets
\[
(3.2) \left( \sum_{x^a \leq p_1 \cdots p_k < x^b} \frac{1}{p_1 \cdots p_k} \right) \left( \sum_{x^a \leq p_1 \cdots p_k < x^b} \frac{1}{p} \right) = c_{30} - \epsilon(x)
\]

\[
(3.22) = \left( \sum_{x^a \leq p_1 \cdots p_{k+1} < x^b} \frac{1}{p_1 \cdots p_{k+1}} \right) + \sum_{j \ell_1, \ldots, \ell_j} \left( \sum_{x^a \leq p_1 \cdots p_{j+1} < x^b} \frac{1}{p_{\ell_1} \cdots p_{\ell_j}} \right) \]

\[
(3.23) = \left( \sum_{x^a \leq p_1 \cdots p_{k+1} < x^b} \frac{1}{p_1 \cdots p_{k+1}} \right) + \epsilon(x),
\]

where the second sum is \( o(1) \) as \( \sum_{x^a < p \leq x^b} \frac{1}{p^2} \) converges.

and hence rearranging terms

\[
(3.25) \left( \sum_{x^a \leq p_1 \cdots p_{k+1} < x^b} \frac{1}{p_1 \cdots p_{k+1}} \right) = c_{31} - \epsilon(x) > 1.
\]

Now let me return to prove that the sums where one has coincidences between some of the primes are small. I claim that for any \( k \) and \( \ell_1 + \cdots + \ell_j = k \) and at least one \( \ell_i \geq 2 \),

\[
\sum_1 = \left( \sum_{x^a \leq p_1 \cdots p_{j+1} < x^b} \frac{1}{p_1 \cdots p_{j+1}} \right) = \epsilon(x),
\]

This is again proved by induction on \( j \). Suppose \( j = 1 \). Then one has \( \ell_1 \geq 2 \) as at least one of the exponents is required to be greater than one. Then the assertion follows as \( \sum \frac{1}{p^2} \) converges. Hence assume the result is true when the number of primes factors is less than or equal to some \( j \) and any \( \ell_1, \ldots, \ell_j \) with one of them being greater than that two. Then I prove the result is true for \( j + 1 \) and any \( \ell_1, \ldots, \ell_{j+1} \) with at least one of them being greater than two.

If at least one of \( \ell_i \geq 2 \) for some \( 1 \leq i \leq j \) then the assertion follows by induction as \( \sum_{x^a < p \leq x^b} \frac{1}{p^{\ell+1}} \) is bounded if \( \ell_{j+1} = 1 \) and \( o(1) \) if \( \ell_{j+1} \geq 2 \).

So now assume that \( \ell_i = 1 \) for all \( 1 \leq i \leq j \) and \( \ell_{j+1} \geq 2 \). Then the sum in consideration occurs in the product

\[
(3.27) \left( \sum_{x^a < p_i \leq x^b} \frac{1}{p_1 \cdots p_i} \right) \left( \sum_{p} \frac{1}{p^{\ell_{j+1}}} \right)
\]

and the first is a summand of \( \left( \sum_{x^a < p \leq x^b} \frac{1}{p} \right)^j \) and hence is bounded (again one uses induction hypothesis to deal with series which coincidences as the number of factors will be \( < j + 1 \) and the second sum is \( o(1) \) as \( \sum_{p} \frac{1}{p^{\ell_{j+1}}} \) converges as \( \ell_{j+1} \geq 2 \). Thus in all cases the claim 3.26 is proved and hence Lemma 3.8 is proved.  \( \square \)
Proof of Theorem 1.10. Let $E/\mathbb{Q}$ be an elliptic curve. If $E$ does not have complex multiplication assume GRH holds for $E$. Let $\varepsilon$ be a fixed, sufficiently small positive real number. Let $k \geq 1$ be an integer. Let

\begin{equation}
B_k(x) = \{ p_1 \cdot p_2 \cdots p_k \leq x : p_i \neq p_j \text{ if } i \neq j \text{ and } p_i \text{ admissible for all } 1 \leq i \leq k \}.
\end{equation}

Then I claim that it is sufficient to prove that for any $k \geq 1$ one has

\begin{equation}
|B_k(x)| \geq c_{32}(\varepsilon) \frac{x}{\log x},
\end{equation}

Note that this claim does not use the assumption in Theorem 1.10 that $k \geq 3$. This assumption enters the proof only through the use of Theorem 2.26 which needs $k \geq 3$.

Suppose for the moment that (3.29) has been established. Then one can complete the proof of the theorem as follows. On one hand one has $|B_k(x)| \geq c_{33}(\varepsilon) \frac{x}{\log x}$ on the other hand by Theorem 2.26 one has $|D(x)| \leq c_{34}(\varepsilon) \frac{x}{(\log x)^{1+r}}$. So if one considers the mapping $B_k(x) \to D(x)$ given by $p_1 \cdots p_k \in B_k(x) \mapsto N_{p_1}(E) \cdots N_{p_k}(E)$, then this map cannot have all fibers of bounded cardinality and the number $G_k(n)$ is precisely the cardinality of the fiber over $n$ under this mapping.

So let me now prove the bound asserted in (3.29). If $k = 1$ then this follows from Lemma 3.1 and Lemma 3.6. So assume $k \geq 2$. If $m = p_1 \cdots p_k \leq x$ then $p_k = \frac{m}{p_1 \cdots p_{k-1}} \leq \frac{x}{p_1 \cdots p_{k-1}}$ and hence $p_k \in B_1(x/(p_1 \cdots p_{k-1}))$. Conversely if $p_1, \ldots, p_{k-1}$ are admissible primes and $p_1 \cdots p_{k-1} \leq x$ and $p_k \leq \frac{x}{p_1 \cdots p_{k-1}}$ is an admissible prime then $p_1 \cdots p_k \leq x$ and is a product of admissible primes and hence is in $B_k(x)$. Moreover any of the $k!$ permutations of the factors $p_1, \ldots, p_k$ give the same $m = p_1 \cdots p_k$. So one has

\begin{equation}
k!|B_k(x)| \geq \sum_{x^a \leq p_1, \ldots, p_{k-1} < x^b} |B_1(x/(p_1 \cdots p_{k-1}))|,
\end{equation}

where the sum is over all the pairwise distinct $p_1, \ldots, p_{k-1}$ and excludes the terms corresponding to the coincidences $p_k = p_i$ for any of the $1 \leq i \leq k-1$. Since one wants a lower bound, it is in fact enough to replace this sum by one of the same sort except where the primes $p_i$ are in some fixed range $x^a \leq p_i < x^b$ for $0 < a < b < 1$, and which satisfy the requirements of Lemma 3.8. Thus

\begin{equation}
k!|B_k(x)| \geq \sum_{x^a \leq p_1, \ldots, p_{k-1} < x^b} |B_1(x/(p_1 \cdots p_{k-1}))|,
\end{equation}

If one chooses $b$ so that

\begin{equation}
1 - b(k - 1) > 0
\end{equation}

then $x/(p_1 \cdots p_{k-1}) \geq x/x^{b(k-1)} \gg 1$ and one can apply Lemma 3.6 to get

\begin{equation}
k!|B_k(x)| \geq \sum_{x^a \leq p_1, \ldots, p_{k-1} < x^b} |B_1(x/(p_1 \cdots p_{k-1}))| \geq c_{35}(\varepsilon) \sum_{x^a \leq p_1, \ldots, p_{k-1} < x^b} \pi(x/(p_1 \cdots p_{k-1})).
\end{equation}

Hence

\begin{equation}
k!|B_k(x)| \geq c_{36}(\varepsilon) \sum_{x^a \leq p_1, \ldots, p_{k-1} < x^b} \frac{x/(p_1 \cdots p_{k-1})}{\log(x/(p_1 \cdots p_{k-1}))}.
\end{equation}
Since \( \frac{1}{\log(x/(p_1 \cdots p_{k-1}))} \geq \frac{1}{\log x} \) this gives

\[
(3.35) \quad k! |B_k(x)| \geq c_{37}(\varepsilon) \frac{x}{\log x} \left( \sum_{x^{a} \leq p_1 \cdots p_{k-1} < x^{b}} \frac{1}{p_1 \cdots p_{k-1}} \right).
\]

Now the proposition follows from Lemma 3.8 provided that in addition to the condition (3.32) one choose \( a \) so that the conditions of Lemma 3.8 are met. Thus on choosing \( 0 < b < \frac{1}{k-1} \) and \( 0 < a < \frac{1}{\alpha_{k-1}} \) one gets the assertion of Theorem 1.10.

4. The case \( k = 1 \)

Now let us consider the case \( k = 1 \). This is a case which does not occur in the classical context considered in [1]. Let \( G_1(E, n) \) be the number of solutions (in primes \( p \)) to the equation \( N_p(E) = n \). For example for the curve \( E : y^2 + y = x^3 - x \) one has

\[ N_{1009}(E) = N_{1063}(E) = 1057. \]

So \( G_1(E, 1057) \geq 2 \) and a simple calculation using (1.1) and a computer search shows that equality holds so \( G_1(1057) = 2 \).

Here is a sample computation for another elliptic curve (chosen for no particular reason)

\[ E : y^2 + 3y = x^3 - x + 2 \]

which tabulates a small list of solutions to the equation \( G_1(n) = 3 \) (in other words \( n \) for which there are three primes \( p_1, p_2, p_3 \) with \( N_p(E) = n \)):

| \( n \) | \( p_1 \) | \( p_2 \) | \( p_3 \) |
|-------|-------|-------|-------|
| 624   | 593   | 661   | 619   |
| 6495  | 6337  | 6449  | 6389  |
| 7440  | 7369  | 7523  | 7487  |
| 8568  | 8563  | 8423  | 8527  |
| 11422 | 11299 | 11617 | 11519 |
| 12312 | 12161 | 12421 | 12391 |
| 12672 | 12721 | 12791 | 12619 |
| 32022 | 31873 | 31699 | 32213 |
| 34240 | 34603 | 34217 | 34327 |
| 37464 | 37693 | 37571 | 37517 |

For the same curve \( E : y^2 + 3y = x^3 - x + 2 \) here is an extract from the table of values of \( G_1(n) \geq 2 \):

...
I conjecture that for any elliptic curve \( E \), one has \( \lim_{n \to \infty} G_1(E, n) = \infty \). In other words, I conjecture that for any elliptic curve \( E/\mathbb{Q} \), there exists integers \( n \) (going to infinity) and arbitrarily large strings of primes \( p_1 < p_2 < \cdots < p_m \) such that

\[
N_{p_1}(E) = N_{p_2}(E) = \cdots = N_{p_m}(E) = n.
\]

But this seems substantially more difficult at this point (even on GRH), and of course it would even more astounding if this turns out to be false. It seems natural to call such primes an *elliptic progression*—progression of primes given by \( E \) or simply *primes in an elliptic progression* if the curve is unambiguously defined. At the very least I expect that \( G_1(E, n) \geq 2 \) infinitely often, i.e., there are infinitely many pairs of primes in an elliptic progression. Let me summarize these conjectures:

**Conjecture 4.1.** Let \( E/\mathbb{Q} \) be an elliptic curve and for any integer \( n \geq 1 \) let \( G_1(n) \) be the number of good primes \( p \) such that \( N_p(E) = n \). Then

1. \( \lim_{n \to \infty} G_1(n) = \infty \).
2. \( \lim_{n \to \infty} G_1(n) \geq 2 \)
3. There exists infinitely many \( n \) such that \( G_1(n) = 2 \).

Clearly 4.1(1) \( \implies \) 4.1(2) and 4.1(3) \( \implies \) 4.1(2). Note that 4.1(3) says that for any elliptic curve \( E/\mathbb{Q} \), there are infinitely many distinct good primes \( p, q \) such that \( N_p(E) = N_q(E) \).

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