ON CATEGORIES OF O-MINIMAL STRUCTURES

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Abstract. Our aim in this paper is to look at some transfer results in model theory (mainly in the context of o-minimal structures) from the category theory viewpoint.

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1. INTRODUCTION

Our aim in this paper is to look at some transfer results in the context of o-minimal structures from the category theory viewpoint. Recall that an o-minimal structure $\mathcal{M}$ is an expansion of an ordered set $(|\mathcal{M}|, \leq)$ such that every unary set definable in $\mathcal{M}$ (with parameters in $|\mathcal{M}|$) is a finite union of open intervals and points. For a detailed exposition of this topic, see [2].

In [1] A. Berarducci and M. Otero point out some transfer results with respect to topological properties from one o-minimal structure to another. Specifically, if $\mathcal{M}$ is an o-minimal expansion of an ordered field and $\varphi$ is a first order formula in the language of the ordered rings, then the following statements concerning the definable subsets $\varphi^\mathcal{M}$ and $\varphi^\mathbb{R}$ hold: (1) $\varphi^\mathcal{M}$ is definably connected if and only if $\varphi^\mathbb{R}$ is connected; (2) $\varphi^\mathcal{M}$ is definably compact if and only if $\varphi^\mathbb{R}$ is compact; (3) there is a natural isomorphism between the homology groups $H^\text{def}_*(\varphi^\mathcal{M})$ and $H^\text{def}_*(\varphi^\mathbb{R})$.

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there is a natural isomorphism between the fundamental groups \( \pi_*(\varphi^R) \); (4) there is a natural isomorphism between the fundamental groups \( \pi^\text{def} (\varphi^M, x_0) \cong \pi(\varphi^R, x_0) \); and assuming that \( \varphi^R \) is compact it follows that (5) if \( \varphi^M \) is a definable manifold, then \( \varphi^R \) is a (topological) manifold; and (6) if moreover \( \varphi^M \) is definably orientable, then \( \varphi^R \) is an orientable manifold.

In [4], C. Miller and S. Starchenko prove a dichotomy theorem on o-minimal expansions of ordered groups:

**Fact 1** (Theorem A, [4]). Suppose that \( \mathcal{R} \) is an o-minimal expansion of an ordered group \((R, <, +)\). Then exactly one of the following holds: (a) \( \mathcal{R} \) is linearly bounded (that is, for each definable function \( f: R \to R \) there exists a definable endomorphism \( \lambda: R \to R \) such that \(|f(x)| \leq \lambda(x)\) for all sufficiently large positive arguments \( x \)); (b) \( \mathcal{R} \) defines a binary operation \( \cdot \) such that \((R, <, +, \cdot)\) is an ordered real closed field. If \( \mathcal{R} \) is linearly bounded, then for every definable \( f: R \to R \) there exists \( c \in R \) and a definable \( \lambda \in \{0\} \cup \text{Aut}(R, +) \) with \( \lim_{x \to +\infty} [f(x) - \lambda(x)] = c \).

Such a dichotomy on o-minimal expansions of ordered groups is the analogue of the subsequent dichotomy for o-minimal expansions of the real field \( \mathbb{R} \), due to C. Miller:

**Fact 2** (Theorem and Proposition, [3]). Let \( \mathcal{R} \) be an o-minimal expansion of the ordered field of real numbers \((\mathbb{R}, <, +, \cdot, 0, 1)\). If \( \mathcal{R} \) is not polynomially bounded (that is, for every definable function \( f: \mathbb{R} \to \mathbb{R} \) there exists \( N \in \mathbb{N} \) such that \(|f(x)| \leq x^N \) for all sufficiently large positive \( x \)), then the exponential function is definable (without parameters) in \( \mathcal{R} \). If \( \mathcal{R} \) is polynomially bounded, then for every definable function \( f: \mathbb{R} \to \mathbb{R} \), with \( f \) not identically zero for all sufficiently large positive arguments, there exist \( c, r \in \mathbb{R} \) with \( c \neq 0 \) such that \( x \mapsto x^r: (0, +\infty) \to \mathbb{R} \) is definable in \( \mathcal{R} \) and \( \lim_{x \to +\infty} f(x)/x^r = c \).

Both Facts 1 and 2 can be viewed as implied transfer results of o-minimality property from one structure to another (see Section 4) and served as our main motivation for this work.

2. Preliminaries

Recall that a *signature* is a triple \( L := (\mathcal{F}, \mathcal{R}, \text{ar}) \), where \( \mathcal{F} \) and \( \mathcal{R} \) are disjoint sets whose members are called respectively function symbols and predicative symbols and \( \text{ar}: \mathcal{F} \cup \mathcal{R} \to \mathbb{N} \) is a function which assigns a nonnegative integer, called *arity*, to every function or predicative
symbol. A function or a predicative symbol is said to be \(n\)-ary if its arity is \(n\). A 0-ary function symbol is called a constant symbol.

The cardinality \(\text{card}(L)\) of a signature \(L = (\mathcal{F}, \mathcal{R}, \text{ar})\) is defined to be \(\text{card}((\mathcal{F}) + \text{card}(\mathcal{R})\).

The first-order language of a signature \(L\) is the set of all (well formed) terms and formulas arising from \(L\), and is denoted by \(\mathcal{L}\). If we denote by \(\text{Term}(L)\) the set of all \(L\)-terms, and by \(\text{Form}(L)\) the set of all \(L\)-formulas then \(\mathcal{L} = \text{Term}(L) \sqcup \text{Form}(L)\).

Let \(\mathcal{L}\) and \(\mathcal{L}'\) be two first-order languages. A language morphism from \(\mathcal{L}\) to \(\mathcal{L}'\) is a (set-theoretic) map \(H: \mathcal{L} \to \mathcal{L}'\) such that \(h\) maps terms from \(\mathcal{L}\) to terms from \(\mathcal{L}'\), and formulas from \(\mathcal{L}\) to formulas from \(\mathcal{L}'\).

3. A CATEGORY OF THE FIRST-ORDER LANGUAGES

Fix a countable set of variable symbols \(\text{Var} = \{x_i : i \in \mathbb{N}\}\).

In what follows we make a brief description of the category \(\text{FOL}\) of the first-order languages.

Let \(\text{Ob}(\text{FOL})\) denote the set of all first-order languages.

Given two languages \(\mathcal{L}, \mathcal{L}' \in \text{Ob}(\text{FOL})\), with underlying signatures \(L = (\cup_{n \geq 0} F_n, \cup_{n \geq 0} R_n)\) and \(L' = (\cup_{n \geq 0} F'_n, \cup_{n \geq 0} R'_n)\) respectively, the correspondence for each \(n \geq 0\)

(i) \(f \mapsto h(f)\), an \(\mathcal{L}'\)-term whose variable symbols occurring in it are precisely \(x_0, \ldots, x_{n-1}\), \(f \in F_n\);

(ii) \(R \mapsto h(R)\), an \(\mathcal{L}'\)-atomic formula whose variable symbols occurring in it are precisely \(x_0, \ldots, x_{n-1}\), \(R \in R_n\).

gives rise to a language morphism \(H: \mathcal{L} \to \mathcal{L}'\), where the restriction \(H(t)\) to \(\text{Term}(L)\) is given by

(iii) \(H(t) := x_i\), if \(t = x_i \in \text{Var}\);

(iv) \(H(t) = h(f)[H(t_0)/x_0, \ldots, H(t_{n-1})/x_{n-1}],\) if \(t = f(t_0, \ldots, t_{n-1})\) with \(f \in F_n\) and \(t_0, \ldots, t_{n-1} \in \text{Term}(L)\),

and the restriction \(H(\varphi)\) to \(\text{Form}(L)\) is defined to be

(v) \(H(\varphi) := (H(t) = H(s))\), if \(\varphi\) is the \(\mathcal{L}\)-atomic formula \((t = s)\) with \(s, t \in \text{Term}(L)\);

(vi) \(H(\varphi) := h(R)[H(t_0)/x_0, \ldots, H(t_{n-1})/x_{n-1}]\), if \(\varphi\) denotes the \(\mathcal{L}\)-atomic formula \(R(t_0, \ldots, t_{n-1})\) with \(R \in R_n\) and \(t_0, \ldots, t_{n-1} \in \text{Term}(L)\);

(vii) \(H(\varphi) := \neg H(\phi)\), if \(\varphi\) is the \(\mathcal{L}\)-formula \(\neg \phi\) with \(\phi \in \text{Form}(L)\);

(viii) \(H(\varphi) := H(\phi) \lor H(\psi)\), if \(\varphi\) is the \(\mathcal{L}\)-formula \(\phi \lor \psi\) with \(\phi, \psi \in \text{Form}(L)\);
(ix) $H(\varphi) := \exists x H(\phi)$, if $\varphi$ is the $\mathcal{L}$-formula $\exists x \phi$ with $\phi \in \text{Form}(\mathcal{L})$ and $x$ a variable symbol in Var.

Observe that $\text{FV}(\varphi) = \text{FV}(H(\varphi))$, where $\text{FV}(\varphi)$ denotes the set of all free variables occurring in $\varphi$.

The composition rule in FOL is given in the most natural way. Indeed, for any language morphisms $H: \mathcal{L} \to \mathcal{L}'$ and $H': \mathcal{L}' \to \mathcal{L}''$, the map $H' \circ H: \mathcal{L} \to \mathcal{L}''$ is the language morphism obtained by extending to $\mathcal{L}$, as above, the following associations: for all $n \geq 0$

- $f \mapsto H'(h(f))$, $f \in F_n$;
- $R \mapsto H'(h(R))$, $R \in R_n$,

where $H'$ is the extension to $\mathcal{L}'$ of $h$. The identity element with respect to $\circ$ is the language morphism $1: \mathcal{L} \to \mathcal{L}$ obtained from the extension of the rules: for all $n \geq 0$

- $f \mapsto f(x_0, \ldots, x_{n-1})$, $f \in F_n$;
- $R \mapsto R(x_0, \ldots, x_{n-1})$, $R \in R_n$.

In other words, $1: \mathcal{L} \to \mathcal{L}$ is the map which associates each $\mathcal{L}$-term to itself, and each $\mathcal{L}$-formula to itself. It is not hard to see that $\circ$ and $1$ satisfy the associativity and identity laws. Therefore, FOL is indeed a category.

Note that FOL has a subcategory of “simple morphisms” given by

- $f \mapsto f'(x_0, \ldots, x_{n-1})$, $f' \in F'_n$ and $R \in R_n \mapsto R'(x_0, \ldots, x_{n-1})$, $R' \in R'_n$.

Here and throughout “language morphism” will mean “a morphism constructed in (i)-(ix)”, unless otherwise stated.

4. Categories of o-minimal structures

Throughout this section we fix an order relation symbol $<$. For each language $\mathcal{L}$, $\mathcal{L}_<$ stands for its extension $\text{Term}(\mathcal{L} \cup \{<\}) \sqcup \text{Form}(\mathcal{L} \cup \{<\})$, which is an object in FOL. Similarly, any morphism $H: \mathcal{L} \to \mathcal{L}'$ in FOL can be extended to a morphism $H_<: \mathcal{L}_< \to \mathcal{L}'_<$ in FOL as defined in the previous section. Such a morphism $H_<$ is the unique language morphism from $\mathcal{L}_<$ to $\mathcal{L}'_<$ satisfying the equality $H_<> \circ i = i' \circ H$, where $i: \mathcal{L} \to \mathcal{L}_<$ and $i': \mathcal{L}' \to \mathcal{L}'_<$ indicate the inclusion maps.

As usual we denote the category of all locally small categories by CAT. The category $\text{Str}(\mathcal{L})$ of all $\mathcal{L}$-structures whose morphisms are the homomorphisms between $\mathcal{L}$-structures is an object from CAT. A (non full) subcategory of $\text{Str}(\mathcal{L})$ is the category $\text{Str}_e(\mathcal{L})$ of all $\mathcal{L}$-structures whose morphisms are the elementary homomorphisms (hence
embeddings) between $L$-structures. We denote by $\text{Str}_{\text{omin}}(L_<)$ the full (small) subcategory of $\text{Str}(L_<)$ whose objects are the o-minimal $(L \cup \{<\})$-structures.

**Definition 1 (Induced functor).** In view of this discussion, we can form the following contravariant functor $\mathcal{E}: \text{FOL} \rightarrow \text{CAT}$:

$$\mathcal{L} \mapsto \text{Str}(\mathcal{L})$$

and

$$\mathcal{L} \xrightarrow{H} \mathcal{L}' \mapsto \text{Str}(\mathcal{L}) \xleftarrow{\mathcal{E}(H)} \text{Str}(\mathcal{L}')$$

where $\mathcal{E}(H)$ is the functor given by:

- $\mathcal{M}' \in \text{Ob}(\text{Str}(\mathcal{L}')) \mapsto \mathcal{M} := \mathcal{E}(H)(\mathcal{M}')$, with $|\mathcal{M}| := |\mathcal{M}'| := M'$, and for each $f \in F_n$ and each $R \in R_n$ we have $f^\mathcal{M} := H(f)^\mathcal{M}' : M^m \rightarrow M'$ (that is, $f^\mathcal{M}$ is the interpretation of the $L'$-term $H(f)$ in $\mathcal{M}'$) and $R^\mathcal{M} := H(R)^\mathcal{M}' \subseteq M^m$ (that is, $R^\mathcal{M}$ is the interpretation of the atomic $L'$-formula $H(R)$ in $\mathcal{M}'$). Thus, for any $L$-formula $\varphi(x_0, \ldots, x_{n-1})$ and any valuation $\nu: \{x_0, \ldots, x_{n-1}\} \rightarrow M'$ we obtain

$$\left(\ast\right) \mathcal{M} \models_{\nu} \varphi(x_0, \ldots, x_{n-1}) \text{ if and only if } \mathcal{M}' \models_{\nu} H(\varphi)(x_0, \ldots, x_{n-1}),$$

by induction on the complexity of $\varphi$.

- $\alpha' \in \text{Hom}_{\text{Str}(\mathcal{L}')}((\mathcal{M}_1, \mathcal{M}_2)) \mapsto \mathcal{E}(H)(\alpha') := \alpha' \in \text{Hom}_{\text{Str}(\mathcal{L})}(\mathcal{M}_1, \mathcal{M}_2)$.

**Remark 1.** There are some variants of the functor $\mathcal{E}$, namely:

(a) the contravariant functor $\mathcal{E}_e: \text{FOL} \rightarrow \text{CAT}$ given by

$$\mathcal{L} \mapsto \text{Str}_e(\mathcal{L})$$

and

$$\mathcal{L} \xrightarrow{H} \mathcal{L}' \mapsto \text{Str}_e(\mathcal{L}) \xleftarrow{\mathcal{E}_e(H)} \text{Str}_e(\mathcal{L}')$$

where $\mathcal{E}_e(H)$ is defined the same way as above for the category $\text{Str}(\mathcal{L}')$. It is worth noticing that $\alpha' \in \text{Hom}_{\text{Str}_e(\mathcal{L}')}((\mathcal{M}_1, \mathcal{M}_2)) \mapsto \mathcal{E}_e(H)(\alpha') := \alpha' \in \text{Hom}_{\text{Str}_e(\mathcal{L})}(\mathcal{M}_1, \mathcal{M}_2)$ is well defined by virtue of $\left(\ast\right)$.

(b) the contravariant functor $\mathcal{E}_<: \text{FOL} \rightarrow \text{CAT}$ given by

$$\mathcal{L}_< \mapsto \text{Str}(\mathcal{L}_<)$$

and

$$\mathcal{L}_< \xrightarrow{H_<} \mathcal{L}_<' \mapsto \text{Str}(\mathcal{L}_<) \xleftarrow{\mathcal{E}_<(H_<)} \text{Str}(\mathcal{L}_<')$$

where $\mathcal{E}_<(H_<)$ is defined analogously to $\mathcal{E}(H)$.

\[1\] Clearly, other similar contravariant functors can be defined, corresponding to other kinds of morphisms between structures.
**Theorem 1.** The functor $E_<(H_<) : \text{Str}(\mathcal{L}'_<) \to \text{Str}(\mathcal{L}<)$ (see Remark 1(b)) maps o-minimal structures in the language $\mathcal{L}'_<$ to o-minimal structures in the language $\mathcal{L}<$, in other words, the following diagram commutes

$$
\begin{array}{ccc}
\text{Str}(\mathcal{L}'_<) & \xrightarrow{E_<(H_<)} & \text{Str}(\mathcal{L}<) \\
\uparrow & & \uparrow \\
\text{Str}_{omin}(\mathcal{L}'_<) & \xrightarrow{E_<(H_<)|} & \text{Str}_{omin}(\mathcal{L}<)
\end{array}
$$

where $E_<(H_<)|$ denotes the restriction of $E_<(H_<)$ to the subcategory $\text{Str}_{omin}(\mathcal{L}<)$.

**Proof.** It follows immediately from ($\ast$) and the fact $FV(\varphi) = FV(H_<(\varphi))$, for any first order formula $\varphi$ in $\mathcal{L}<$. □

The dichotomy result stated in Fact 1 (see Section 1) can be translated in this section into diagrams of categories of o-minimal structures and functors induced by language morphisms:

![Diagram 1](image1.png)

**Figure 1.** Diagram in FOL

where $\mathcal{L}_{or}$ is the language generated by the signature of the ordered rings $L_{or}$, $\mathcal{L}_{og}$ is the language generated by the signature of the ordered groups $L_{og}$ and $\tilde{\mathcal{L}}$ expands $\mathcal{L}_{og}$ arbitrarily. Applying the functor $E_<$, we get

![Diagram 2](image2.png)

**Figure 2.** Diagram in CAT

where $\tilde{R}$ is an o-minimal expansion of ordered group $(R, <, +)$.

Similarly, the dichotomy stated in Fact 2 in Section 1 can be read out of the following diagrams:
where $L_{\text{exp}}$ is the language generated by the signature $L_{or} \cup \{\text{exp}\}$ and $\tilde{L}$ expands $L_{or}$ arbitrarily, and

\[
\begin{array}{c}
\tilde{R} \\
\text{Exp} \\
\rightarrow \text{CAT}
\end{array}
\]

where $\mathbb{R}$ stands for the ordered field of real numbers, $\mathbb{R}_{\text{exp}}$ is the exponential real field $(\mathbb{R}, \text{exp})$ and $\tilde{R}$ is an o-minimal expansion of $\mathbb{R}$.

Observe that the dichotomy theorems in Facts 1 and 2 characterize the images of the induced functors as considered above (Definition 1).

The above remarks suggest that may be useful to consider the following notion:

**Definition 2.** We define the category $\text{STR}$ of all structures by means of the Grothendieck construction as follows.

- $\text{Ob}(\text{STR})$: $(\mathcal{L}, \mathcal{M})$, where $\mathcal{L}$ is a language and $\mathcal{M} \in \text{Str}(\mathcal{L})$;
- For any pair $(\mathcal{L}, \mathcal{M})$ and $(\mathcal{L}', \mathcal{N'})$, $\text{Hom}_{\text{STR}}((\mathcal{L}, \mathcal{M}), (\mathcal{L}', \mathcal{N'}))$ is the set of pairs $(H, \alpha)$ where $H: \mathcal{L} \to \mathcal{L}'$ is a language morphism and $\alpha: E(H)(\mathcal{N'}) \to \mathcal{M}$ is a morphism in $\text{Str}(\mathcal{L}')$;
- Composition: $(H', \alpha') * (H, \alpha) := (H' \circ H, \alpha \circ E(H)(\alpha'))$;
- Identities: $\text{id}_{(\mathcal{L}, \mathcal{M})} := (\text{id}_\mathcal{L}, \text{id}_\mathcal{M})$.

We have some variants of $\text{STR}$ such as:

(a) $\text{STR}_{e_1}$, where $\alpha$ as in $\text{STR}$ are taken to be elementary homomorphims;
(b) $\text{STR}_{e_1}$, where $\alpha$ as in $\text{STR}$ preserve only the validity of first order unary formulas;
(c) $\text{STR}_{<}$ constructed analogously to $\text{STR}$ for all language expansions $\mathcal{L}_{<}$;
(d) \( \text{STR}_{\text{o-min}} \) as \( \text{STR}_{<} \), with \( \mathcal{M} \in \text{Str}_{\text{o-min}}(\mathcal{L}_{<}) \).

Note that the dichotomy results expressed in Facts 1 and 2 can also be read in this global context, since the morphism from \((\mathcal{L}_{<}, \mathcal{M})\) to \((\mathcal{L}'_{<}, \mathcal{N}')\) is the pair \( H_{<}: \mathcal{L}_{<} \rightarrow \mathcal{L}'_{<} \) and \( \alpha: \mathcal{E}(H_{<})(\mathcal{N}') \rightarrow \mathcal{M} \) is the identity homomorphism, that is, \( \mathcal{E}(H_{<})(\mathcal{N}') = \mathcal{M} \) and \( \alpha = \text{id}_{\mathcal{M}} \).

On the other hand, a more general case in which the map \( \alpha \) is not necessarily the identity also occurs in the literature. For instance,

**Fact 3** ([5]). If \( \mathcal{M} \) is any nonstandard model of PA, with \((\text{HF}^\mathcal{M}, \in^\mathcal{M})\) the corresponding nonstandard hereditary finite sets of \( \mathcal{M} \) (by Ackerman coding: the natural numbers of \( \text{HF}^\mathcal{M} \) are isomorphic to \( \mathcal{M} \)), then for any consistent computably axiomatized theory \( T \) extending ZF in the language of set theory, there is a submodel \( \mathcal{N}' \subseteq (\text{HF}^\mathcal{M}, \in^\mathcal{M}) \) such that \( \mathcal{N}' \models T \).

5. **Final remarks**

- It is natural to consider even more general forms of induced functors by changing of languages as in [6]: for instance, something in this direction already occurred in Facts 1 (and 2) since \( \cdot \) is \( \tilde{L} \) definable in \( \tilde{R} \). This would complete the picture of Facts 1, 2 (that is, it would name the dot arrows in the diagrams shown in Figures 1, 2, 3 and 4).

- Are there natural examples of the phenomenon appeared in Fact 3 in the setting of o-minimal structures? That is, a situation involving o-minimal structures and a morphism from \((\mathcal{L}_{<}, \mathcal{M})\) to \((\mathcal{L}'_{<}, \mathcal{N}')\), which is the \( H_{<}: \mathcal{L}_{<} \rightarrow \mathcal{L}'_{<} \) and \( \alpha: \mathcal{E}(H_{<})(\mathcal{N}') \rightarrow \mathcal{M} \), where \( \mathcal{E}(H_{<})(\mathcal{N}') \neq \mathcal{M} \) and/or \( \alpha \neq \text{id}_{\mathcal{M}} \). What about with \( \alpha \) being an embedding? Or an elementary embedding? Or an \( e_1 \)-elementary embedding, that is, an embedding which preserves formulas with one free variable?

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