The form factors in the finite volume

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Abstract

The form factors of integrable models in finite volume are studied. We construct the explicit representations for the form factors in terms of determinants.

1 Introduction

In this paper we study form factors of quantum integrable models in finite volume. The possibility to obtain exact non-perturbative results in this field is of great importance. The Quantum Inverse Scattering Method created by L.D. Faddeev and his school [1,2,3] allows one to formulate general methods of solving this problem. Namely, in order to obtain an explicit representation for a form factor of an operator $O$, it is sufficient to embed this operator into the algebra

$$R(;) \ T() \ T() = T() \ T() \ R(;):$$

(1.1)

After this is done, the problem is reduced to the standard Algebraic Bethe Ansatz procedures.

We would like to mention that our approach is drastically different from the ones, used in relativistic models [3][14]. We consider form factors between the states constructed on the bare vacuum, instead of the physical one. We also do not use the axioms, describing analytical properties of form factors. All our calculations are based on the algebra [14] only. In spite of this approach meets serious combinatorial difficulties, nevertheless it provides us with explicit determinant representations for form factors.

In the present paper we consider Quantum Nonlinear Schrodinger equation (Q-NLS) and XXZ (XXZ) Heisenberg chains of spin $1/2$. The Q-NLS model in a finite volume $L$ with periodically boundary conditions is described by the Hamiltonian

$$H_{Q-NLS} = \sum_{L=2}^{Z} \int_{L=2}^{Z} dx \ \partial_x \ \partial_x \ + c \ \partial_x \ \partial_x ;$$

(1.2)
where \((x; t)\) and \(\gamma(x; t)\) are canonical Bose fields
\[
[ (x; t); \gamma(y; t) ] = \{ x \ y \}; \quad \gamma_0 = 0; \quad \hbar \gamma = 0; \quad (1.3)
\]
and \(\gamma\) is the coupling constant.

The Hamiltonian of the XXZ Heisenberg chain with \(M\) sites is
\[
H_{XXZ} = \sum_{m=1}^{M} (\sigma^x_m (\sigma^x_{m+1}) + \sigma^z_m (\sigma^z_{m+1}) + C^z_m (C^z_{m+1})); \quad (1.4)
\]
Here \(\sigma^a_m\) are local spin operators (Pauli matrices), associated to the \(m\)th site, \(\sigma^a_{M+1} = (1)\). Parameter \(\delta\) defines the anisotropy of the model. Particular choice \(\delta = 1\) corresponds to the XXZ chain.

Let us describe now the content of the paper. In the section 2 we recall the basic definitions of the Algebraic Bethe Ansatz. In the section 3 the determinant representation for the form factor of the local field of QNLS is obtained. In the section 4 we consider different representations for form factors of QNLS and prove their equivalency. The last section is devoted to the form factors of Heisenberg chains.

2 Algebraic Bethe ansatz

We consider integrable models, which can be solved via the Algebraic Bethe Ansatz. Let the monodromy matrix is
\[
T() = \begin{array}{cccc}
A() & B() & ! \\
C() & D()
\end{array} ; \quad (2.1)
\]
and the highest vector \(\gamma_0\) (pseudovacuum) of this matrix has the following properties:
\[
A() \gamma_0 = a() \gamma_0; \quad D() \gamma_0 = d() \gamma_0; \quad C() \gamma_0 = 0; \quad (2.2)
\]
The dual vector \(\hbar \gamma_0\) is defined similarly
\[
\hbar \gamma_0 A() = a() \hbar \gamma_0; \quad \hbar \gamma_0 D() = d() \hbar \gamma_0; \quad \hbar \gamma_0 B() = 0; \quad (2.3)
\]
The explicit form of functions \(a()\) and \(d(),\) entering Eqs. (2.2), (2.3), depends on specific model.

The \(R\) matrix, defining the commutation relations of the monodromy matrix entries (1.1), can be written in the form:
\[
\begin{pmatrix}
0 & f(); & 0 & 0 & 0 & 0 & 1 \\
0 & g(); & 1 & 0 & 0 & C^0 & 0 \\
0 & 1 & g(); & 0 & C^0 & 0 & 0 \\
0 & 0 & 0 & f(); & 0 & 0 & 0
\end{pmatrix} = (2.4)
\]

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For the models QNLs and XX X Heisenberg chain the R-matrix is rational:

$$f(\; ; ) = \frac{1}{\; ; } + \frac{i c}{\; ; } ; \quad g(\; ; ) = \frac{i c}{\; ; } ; \quad (2.5)$$

Here \( c \) is a constant. Below we shall use also two new functions

$$h(\; ; ) = \frac{f(\; ; )}{g(\; ; )} = \frac{1}{\; ; } + \frac{i c}{\; ; } ;$$

$$t(\; ; ) = \frac{g(\; ; )}{h(\; ; )} = \frac{c^2}{(\; ; ) (\; ; ) + ic} ; \quad (2.6)$$

For the XXZ chain the R-matrix has the same form (2.4), however the functions \( g(\; ; ) \) and \( f(\; ; ) \) become trigonometric ones

$$f(\; ; ) = \frac{\sinh(\; ; ) + i}{\sinh(\; ; )} ; \quad g(\; ; ) = \frac{i \sin(\; ; )}{\sinh(\; ; )} ; \quad (2.7)$$

Respectively the functions \( h(\; ; ) \) and \( t(\; ; ) \) should be replaced by their trigonometric analogs according to the definition (2.6). The constant \( c \) is related to the anisotropy of the model by \( \cos = \). The entries of the monodromy matrix (2.1) act in a space consisting of states

$$j_N (f g)i = \sum_{j=1}^{N} B(j) j_i ; \quad N = 0; 1; \ldots ; \quad (2.8)$$

Here \( f \) and \( g \) are arbitrary complex parameters. Similarly dual states can be constructed by the operators \( C(\; ; ) \)

$$h_N (f g) j = \sum_{j=1}^{N} C(j) j ; \quad N = 0; 1; \ldots ; \quad (2.9)$$

In what follows we shall refer to the states (2.8), (2.9) with arbitrary \( f \)-s as \( BA \)-vectors' (Bethe Ansatz vectors).

The transfer matrix \( T(\; ; ) = \text{tr}T(\; ; ) = A(\; ; ) + D(\; ; ) \) generates the complete set of the conservation laws of the model. The eigenstates of the transfer matrix have the form (2.2), (2.3), however the parameters \( f \) and \( g \) are not arbitrary, but satisfy the system of Bethe equations

$$r(j) \frac{f(j; k)}{f(k; j)} = 1 ; \quad \text{where} \quad r(\; ; ) = \frac{a(\; ; )}{d(\; ; )} ; \quad (2.10)$$

The eigenvalues of the transfer matrix are

$$(f g) j_N (f g)i = (f g) j_N (f g)i ; \quad (2.11)$$

$$(f g) = a(\; ; m-1) f(\; ; m) + d(\; ; m-1) f(\; ; m) ;$$
Let us focus now at the models, which will be considered in the next sections. For the Q NLS the pseudovacuum (2.2) coincides with the Fock vacuum (1.3). The coupling constant c is equal to the constant in (2.5), the functions \(a()\) and \(d()\) are

\[
a() = \exp(iL=2); \quad d() = \exp(iL=2): \tag{2.12}
\]

In the \(XXZ\) chain with \(M\) sites the pseudovacuum vectors \(\Omega_i\) and \(\Omega_j\) are

\[
\Omega_i = m^{-1}_{= 1} m_i; \quad \Omega_j = (\Omega_i)^T: \tag{2.13}
\]

For intermediate calculations the inhomogeneous \(XXZ\) chain is suitable. For this case the monodromy matrix is given by

\[
\mathbf{T}() = L_M (\ M \ i (L \ i)); \tag{2.14}
\]

where \(m\) are arbitrary complex numbers, and local \(L\)-operators are

\[
L_m() = \begin{bmatrix}
0 & \cosh - i_2 \frac{e^{i \frac{\pi}{2}}}{m} & \sin \frac{\pi}{2} & \frac{1}{m} \\
\sin \frac{\pi}{2} & \cosh & - i_2 \frac{e^{i \frac{\pi}{2}}}{m} & \frac{1}{m} \\
\end{bmatrix} \tag{2.15}
\]

The functions \(a()\) and \(d()\) are equal to

\[
a() = \sum_{m=1}^{\Psi} \cosh (m \ i=2); \quad d() = \sum_{m=1}^{\Psi} \cosh (m + i = 2): \tag{2.16}
\]

In the limit \(m! 0\) we turn back to the homogeneous \(XXZ\) chain.

Particular case \(= 1\) corresponds to \(XXX\) Heisenberg magnet. This model is described by rational \(R\)-matrix (2.4) with \(c = 1\). The vacuum eigenvalues are

\[
a() = \sum_{m=1}^{\Psi} (m \ i=2); \quad d() = \sum_{m=1}^{\Psi} (m + i = 2): \tag{2.17}
\]

In the conclusion of this section we would like to mention that the Hamiltonians, presented in the Introduction, are not of the most general form. One can consider, for instance, Heisenberg magnets in external magnetic field. Similarly it is possible to consider Q NLS with a chemical potential. In order to do this, one need to add to the Hamiltonians (1.2), (1.4) the corresponding terms. However these additional terms do not play an essential role in the models in finite volume. They become significant only in the thermodynamic limit, in particular they define the structure of the ground state and the thermodynamic properties of the models. Since the thermodynamic limit is not considered in this paper, therefore we restrict our selves with the Hamiltonians (1.2) and (1.4) only.
3 Form factor of the local field in the QNLS

The calculation of form factors in the framework of the Algebraic Bethe Ansatz form ally can be reduced to the calculation of scalar products of a special type:

\[ S_N (\psi_0; \psi_0) = \hbar^j C (j) \hat{A} \otimes B (j) \hat{A} \hat{d}_i: \]

In the last equation at least one of the states, say \( \hbar^j C (j) \), is an eigenfunction of the transfer matrix. In other words, the parameters \( \omega_j \) satisfy the system of the Bethe equations \( \omega_j = 2 \pi m_j \lambda_j \), while the parameters \( \omega_j \) remain free, i.e., \( Q_N (j) \) is an arbitrary BA-vector.

Let us demonstrate the general idea of the method, which will be used. Consider a form factor of an operator \( 0 \)

\[ F_0 = \hbar^j C (j) \hat{A} \otimes B (j) \hat{A} \hat{d}_i: \]

Here both states \( \hbar^j C (j) \) and \( Q_N (j) \) are eigenstates of the transfer matrix. The operator \( 0 \) acting on the vector \( Q_N (j) \) gives a new state, which generally speaking is not an eigenstate of \( (\) \). However, if this new state can be presented as a linear combination of BA-vectors with numerical coefficients \( c_k \)

\[ F_0 = \hbar^j C (j) \hat{A} \otimes B (j) \hat{A} \hat{d}_i: \]

then evidently the form factor \( F_0 \) is equal

\[ F_0 = \hbar^j C (j) \hat{A} \otimes B (j) \hat{A} \hat{d}_i: \]

Anyways the state \( \hbar^j C (j) \) remains the eigenstate of the transfer matrix. Thus, in order to find explicit expression for the form factor \( 0 \), one need to calculate the scalar products \( <\psi|\psi_0> \). In order to do this, it is enough to use the algebra \( [1,2] \) only. Moreover, the specific form of the vacuum eigenvalues \( a(\) \) and \( d(\) \), as well as the structure of the pseudovacuum, are not essential. However, the straightforward calculation of the scalar products appears to be a complicated combinatorial problem. These difficulties have been partly overcome in [6, 17]. In the paper [18] the determinant representation for the scalar product of two arbitrary BA-vectors was obtained in terms of auxiliary quantum operators\| dual fields. The explicit determinant representation for the scalar products \( (3.1) \) was found in [19] (see also [20]):

\[ S_N(f,g) = \sum_{a=1}^{8} d(a)d(a) h(a; b); \]

\[ g < \Psi^0 \Psi^0 \hat{h}^a = \]

\[ \sum_{a,b=1}^{8} g(a; b)g(b; a); \]

\[ \det M_{jk}; \]

\[ (3.5) \]
where

$$M_{jk} = t(k; j) t(j; k) t_{m-1}^{\gamma} f(j; m) f(m; j).$$

(3.6)

Recall that here the parameters $f, g$ satisfy the Bethe equations (2.10), while the parameters $f, g$ are free. Before applying this formula to the calculations of form factors we would like to mention several features of the representation (3.3).

This representation is valid for arbitrary model, possessing rational $R$-matrix (2.4) or trigonometric one. The representation (3.3) also allows one to prove the orthogonality of the transfer matrix eigenstates directly. Indeed, the particular case of an $BA$-vector $Q_{m}^{m}$ $j_{1} \cdots B(j_{1}) \cdots$ is an eigenstate of ( ). Therefore, one can demand the parameters $f, g$ to satisfy the Bethe equations (2.10), but to be different from the parameters $f, g$. In this case we obtain the scalar product of two different eigenstates, which must be equal to zero. It is easy to check, that for the such choice of $f, g$ the determinant $\text{det}M$ in (3.3) does vanishes. Indeed, the matrix (3.4) turns into

$$M_{jk} = t(k; j) + V_{j} t(j; k)$$

(3.7)

where

$$V_{j} = \gamma \frac{h(j; m) h(m; j)}{h(m; j) h(j; m)}.$$  

(3.8)

This matrix has the eigenvector with zero eigenvalue:

$$\gamma^{-1} \quad M_{jk} \quad k = 0; \quad k = \frac{g(k; m)}{\gamma \gamma_{m-1} \gamma_{m} \gamma_{m+1}}.$$  

(3.9)

The proof of the (3.4) is given in Appendix 1. Hence $\text{det}M = 0$.

Now let us apply the representation (3.3) to the calculation of specific form factors. Namely, consider the form factor of the local field $(k; t)$ in the QNLS

$$F(k; t) = h_{j}^{0} \cdots C(j)^{A} (k; t)^{B(j)^{A} \cdots B(j)^{A} \cdots F(0; 0):}$$

(3.10)

The dependency on time $t$ and distance $x$ evidently separated

$$F(k; t) = \exp \sum_{k=1}^{\gamma} \frac{it}{\gamma} k x_k \quad \sum_{k=1}^{\gamma} \frac{it}{\gamma} j x_j; \quad F(0; 0):$$

(3.11)

The action of the operator $(0; 0)$ on the $BA$-vector had been found in [15]:

$$\sum_{j=1}^{\gamma} B(j)^{A} \cdots B(j)^{A} \cdots F(0; 0):$$

(3.12)
In spite of the parameters $1; \ldots; N+1$ satisfy the Bethe equations
\begin{equation}
\mathbf{r}(j) = \frac{N^j \mathbf{h}(p; j)}{\mathbf{h}(j; p)};
\end{equation}
the states entering the r.h.s. of (3.12) are not eigenstates of the transfer matrix, since each of them depends only on $N$ parameters. Nevertheless all these states are $BA$-vectors, therefore we can apply (3.3) for calculation of the scalar products
\begin{equation}
S_N^{(')}(g; f; g) = h_0 \prod_{j=1}^{N} C(j)^{\frac{N^j}{p-1}} B(j)^{\frac{N^{j+1}}{j}} \mathbf{d}_j;
\end{equation}
since the state $h_0 \prod_{j=1}^{N} C(j)$ remains an eigenstate. After simple algebra [21] we obtain for the form factor $F(0; 0)$:
\begin{equation}
F(0; 0) = \mathbf{P} \mathbf{C} \prod_{a > b}^{y} g(a; b) g(b; a) h(a; b) \prod_{a > b}^{y} d(a) d(b)
\end{equation}
Here we introduce new matrix $S$:
\begin{equation}
S_{jk} = \prod_{a = 1}^{N} d(a) \prod_{b = 1}^{N} d(b) (1)^{p-1} \mathbf{d} S_{jk}^{(')};
\end{equation}
The matrix $S^{(')}$ can be obtained from the matrix $S$ via the replacement of the $i$-th row by $S_{N+1}^{pk}$:
\begin{equation}
S_{jk}^{(')} = \begin{pmatrix}
S_{jk} \\
S_{N+1,k} \\
\end{pmatrix} j \neq i;
\end{equation}
The sum with respect to $i$ in (3.16) gives a single determinant:
\begin{equation}
(1)^{p-1} \mathbf{d} S_{jk}^{(')} = \mathbf{d} (S_{jk} S_{N+1,k});
\end{equation}
In order to prove (3.16) one can use the Laplace formula for determinant of sum of two matrices, and to take into account that $S_{N+1,k}$ is the first-rank matrix. Thus we arrive at the determinant representation for the form factor of the local field in QNLS [21]:
\begin{equation}
F(0; 0) = \mathbf{P} \mathbf{C} \prod_{a > b}^{y} g(a; b) g(b; a) h(a; b) \prod_{a > b}^{y} d(a) d(b) \mathbf{d} (S_{jk} S_{N+1,k});
\end{equation}
Here $S_{jk}$ is given by (3.16), and $S_{N+1,k}$ can be obtained from $S_{jk}$ via the replacement $j \rightarrow N+1$.

In the conclusion of this section we would like to emphasize once more that the most serious problem of the described method is to embed the operator $O$, whose form factors wanted to be found, into the algebra [1.2]. However, after the commutation relations between $O$ and the monodromy matrix are defined, then the problem can be solved via standard methods of the Algebraic Bethe Ansatz.

4 Other representations for form factors

As we have mentioned already, the method used for calculations of form factors of relativistic models is essentially different from the method, presented in the previous section. Nevertheless a series of the results, obtained in this domain for the quantum field theory models, also can be applied for the models in finite volume. Thus, for instant, the form factors of $O(3)$-invariant model unsuspectingly had been found closely related to the form factors of the QNLS [22]. In the present section we consider several examples.

The approach, developed in papers [13, 14], allows to express form factors of the QNLS in terms of some rational functions (originally they were denoted as $\phi_k$), which depend on $N$ arguments $f_a(x_i)$ and $N$ arguments $g_i(x_j)$. The only singularities of these functions are simple poles at $j = k$, and the residues in these points can be expressed in terms of $\phi_k$, depending on the argument $f_a(x_i)$ and $g_i(x_j)$. This property allows one to construct via a recurrence, however during a long time this recurrence have not been solved in general form.

On the other hand the same functions appeared to be useful for the analysis of the form factors of $O(3)$-invariant model. The series of the explicit determinant representations for $\phi_k$, depending on elementary symmetric polynomials $\phi_k^{(n)}$, were found in [23]:

$$\phi_k^{(n)}(x_1, \ldots, x_n) = \frac{1}{(n)_k} \frac{\prod_{m=1}^{n} (\kappa + x_m)}{\prod_{m=1}^{n} (\kappa + x_m)} : (4.1)$$

The explicit solution for the functions has the form:

$$\phi_k^{(n)}(x_1, \ldots, x_n) = \frac{\det_{a,b}(x_a + x_b)}{\prod_{m=1}^{n} (\kappa + x_m)} : (4.2)$$

Here $(N+N)$ $(N+N)$ matrix has the following entries:

$$M_{jk} = (N+N) \begin{pmatrix} f_a + \frac{ic}{2} g_f a + \frac{ic}{2} g_b \end{pmatrix} \begin{pmatrix} f_a + \frac{ic}{2} g_f a + \frac{ic}{2} g_b \end{pmatrix}$$

$$e \begin{pmatrix} f_a + \frac{ic}{2} g_f a + \frac{ic}{2} g_b \end{pmatrix} = \begin{pmatrix} f_a + \frac{ic}{2} g_f a + \frac{ic}{2} g_b \end{pmatrix}$$

$$k = 1; \ldots; N : (4.3)$$
Index \(j\) runs through \(0;1;\ldots;N + N - 1\). By (4.1) we denote the W andermone detem inants.

Form factors of the QNLS can be expressed in terms of the functions \(J\). In particular the form factor of the local field \(F(0;0)\), considered in the previous section, is proportional to the function \(0\) with \(N = N + 1\) :

\[
F(0;0) = i^{2N} C d_{(a)}^{N+1} d_{(b)} (f g_N ; f g_{N+1}) _{a=0} 
\]

(4.5)

It is worth mentioning, that in this case \((2N + 1) (2N + 1)\) matrix. On the other hand, as we have seen above, the form factor of the local field is proportional to the determinant of the \(N \times N\) matrix (4.3). There exists a nice method to reduce the determinant of the matrix (4.3), (4.4) to the determinant of (4.2). We shall demonstrate this method for more simple form factor.

Consider an operator \(Q_1\) of the number of particles in the interval \([0;x]\). In the QNLS this operator is equal to

\[
Q_1 = \int_0^x \psi(y) \psi(y) dy 
\]

(4.6)

The form factor of this operator was calculated in [17] in terms of the function with \(N = N + 1\):

\[
F_{Q_1} = i^{2N} C \text{ det } (f g_N ; f g_{N+1}) _{a=0} 
\]

(4.7)

Let us show, how one can reduce the determinant of \(2N \times 2N\) matrix (4.2) to a determinant of \(N \times N\) matrix.

Theorem 4.1 The function \(f g_N ; f g_{N+1}\) is proportional to the determinant of \(N \times N\) matrix

\[
(f g_N ; f g_{N+1}) = \prod_{a > b} g(j; k) g(k; j) \prod_{a, b=1} h(a; b) 
\]

(4.8)

where

\[
V_j = \prod_{m=1}^{\psi} h(j; m) h(m; j) 
\]

(4.9)

(compare with (3.3)).
Proof. The determinant representation for the function \((f g N ; f g N)\) is given by

\[ (4.2) \]

with \(N = N = N\). Let us multiply the matrix from the left by matrix \(U\)

\[ U_{jk} \quad P_{jk} = P_j^{2N} k^1; \quad j = 1; \ldots; N ; \]

\[ (4.10) \]

\[ U_{j+N k} \quad Q_{jk} = Q_j^{2N} k^1; \quad j = 1; \ldots; N ; \]

\[ (4.11) \]

Index \(k\) runs through the set \(0; 1; \ldots; 2N\), parameters \(p_j\) and \(q_j\) are some complex numbers, which will be fixed later. Obviously

\[ \det = \frac{\det(U)}{\det(U)}; \]

\[ (4.12) \]

and

\[ \det(U) = (p_j \quad q_j); \]

\[ (4.13) \]

The product \(U\) can be written as a block matrix

\[ U = \begin{pmatrix} P M & P \cr Q M & Q \end{pmatrix}; \]

\[ (4.14) \]

where each of these blocks is \(N \times N\) matrix. Using the representation for elementary symmetric polynomials \((4.2)\), one can easily find the entries of these blocks, for example:

\[ (P M)_{jk} (f g; f g) = \prod_{a=1}^{N} p_j + a + \frac{ic}{2} \prod_{a=1}^{N} p_j + a + \frac{ic}{2} \]

\[ e^{\frac{ic}{2}} p_j + a + \frac{ic}{2} e^{\frac{ic}{2}} p_j + a + \frac{ic}{2}; \]

\[ (4.15) \]

Other blocks have similar form. Namely, replacing \(p_j\) by \(q_j\) in \((4.15)\) we obtain the block \(Q M\). As for the right blocks, they can be obtained from the left ones, in which one should replace \(f g\) by \(f g\) and put \(0\). Thus we obtain the representation for the function \((f g N ; f g N)\), containing \(2N\) arbitrary complex parameters \(p_j\) and \(q_j\), but does not depending on their specific choice.

Using well-known formula for a determinant of a block matrix

\[ \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A \quad \det(D \quad CA^{-1}B)); \]

\[ (4.16) \]

one can reduce \(\det(U)\) to the determinants of \(N \times N\) matrices. However, in order to simplify the calculations, we shall fix the parameters \(p_j\) and \(q_j\). Let, for example, \(p_j = \frac{j}{2} + \frac{ic}{2}\) and \(q_j = j + \frac{ic}{2}\). Then we have for two left blocks:

\[ (P M)_{jk} = e^{\frac{ic}{2}} (a \quad j) \quad (a \quad j); \]

\[ (4.17) \]
\[ (QM)_{jk} = \sum_{a=1}^{N} \left( \sum_{j \neq a}^{N} \left( \sum_{i=1}^{\gamma} (a_j + ic) \right) \right) \]  
\[ \text{Here } a_j = a_j \text{. Observe that} \]

\[ (QM)_{jk} = u_j (PM)_{jk} ; \text{ where } u_j = e^{\sum_{a=1}^{N} \frac{a_j + ic}{a_j}} \]  
\[ \text{Hence, due to (4.18) we have} \]

\[ \det(U) = \det(P)_{jk} \det(Q)_{jk} + u_j (P)_{jk} \]  

After simple algebra we arrive at

\[ \det(U) = W \det(G) \]  
\[ \text{where} \]

\[ W = e^{\sum_{a,b=1}^{N} (ab + ic)(a + ib)} [e^{\sum_{a>b}^{N} (ab + ic)}] \]  
\[ \text{and} \]

\[ G_{jk} = \sum_{a=1}^{N} \left( \sum_{j \neq a}^{N} \left( \sum_{i=1}^{\gamma} (a_j + ic) \right) \right) \]  
\[ \text{In order to reduce } \det(G) \text{ to the determinant (4.13), one need to make one more step:} \]

\[ \det(G) = \frac{\det(G)}{\det} \]  
\[ \text{where} \]

\[ jk = \frac{1}{\sum_{a=1}^{N} (j a)} ; \text{ and } \det = e^{\sum_{a=1}^{N} (ab + ic)} \]  

The product \( G \) can be computed explicitly via the method, describing in the Appendix 1. The resulting representation for the form factor \( F_{Q1} \) exactly coincides with (4.2) (4.3).

One can use the same idea in order to reduce the determinant of \((2N + 1) \times (2N + 1)\) matrix (4.5) to the determinant of the \(N \times N\) matrix (3.6). A small complication is caused by the fact that in this case the blocks in the matrix (4.14) have different dimension, for example, the block \( PM \) is \(N \times N\) matrix, while the block \( QM \) is \((N + 1) \times (N + 1)\) matrix etc. This leads to the appearance of the term \( S_{N+1,jk} \) in (3.13).
Thus we have demonstrated different representations for form factors and proved their equivalence. The representations in terms of elementary symmetric polynomials are quite characteristic for the form factors of the relativistic models (see, for instance, [10]). At the same time the representations (3.3), (4.2) look more natural, since they depend only on the functions, entering the R-matrix of the model. These representations also are suitable for the computation of the correlation functions.

5 Form factors of Heisenberg chains

As we have seen already, the calculation of the form factors is based on the explicit determinant representation (3.3), (3.4) for the scalar products of the special type (3.1). However the calculation itself of the scalar products in the framework of the Algebraic Bethe Ansatz deals with rather complicated combinatorial problems. The method based on the explicit representations for the functions in terms of elementary symmetric polynomials also makes use of solving of complicated recurrence. In the paper [24] a new approach was proposed, which allows to simplify significantly the calculation of scalar products and form factors. There a factorizing Drinfeld twist F was constructed. The new basis generated by the twist F is extremely suitable to study the structure of BA-vectors, and for representation of local operators in terms of the entries of the monodromy matrix. In the paper [25] this method was used for calculation of the form factors in Heisenberg chains. We would like to present here the main results of the paper [25].

Consider inhomogeneous X X Z (X X X) model (2.14):

\[ T ( ) = L_1 ( ) 1 (L - 1); \]

Then the local spin operators \( \alpha \) can be presented in terms of the monodromy matrix entries as

\[
\begin{align*}
\alpha_-^m &= \frac{\det}{\det} (a) \ B (m) \ A \ ; \\
\alpha^m &= \frac{\det}{\det} (a) \ C (m) \ A \ ; \\
\alpha^z &= \frac{\det}{\det} (a) \ (A (m) \ D (m)) \ A \ ;
\end{align*}
\]

Here \( A ( ) + D ( ) \) is the transform matrix.

The representation (5.2) allows to find the form factors of local spin operators of X X Z (X X X) chains. Consider, for example, the form factor \( F^m \) of the operator \( \alpha_-^m \):

\[
F^m = hD^{j \theta} \ C (j) \ A \ B (j) \ A \ D_i;
\]

(5.3)
Both states in (5.3) are eigenstates of the transfer matrix, hence substituting here (5.2) we obtain

\[ F^{(m)} = \sum_{a=1}^{m} (a \neq g) \sum_{a=m+1}^{N} (a \neq g) \prod_{j=1}^{N+m-1} \frac{C(j)^{A} B \theta_{j}^{1}}{B(j)^{A} \theta_{j}} \]  

Here is eigenvalue of the transfer matrix (2.11). Thus we again have reduced the form factor to the scalar product of eigenstate by BA-vector. It is enough now to use the representation (3.5), which gives us the representation for the form factor \( F^{(m)} \) in terms of a determinant of \((N+1) \times (N+1)\) matrix.

Let us present the result in the form given in [25]. Introduce \((N+1) \times (N+1)\) matrix

\[ T_{jk}(f \ g_{N+1} \ g_{N+1}) = \frac{\theta_{j}}{\theta_{k}} (j \neq g_{N+1}) \]  

Then the form factor \( F^{(m)} \) is proportional to the ratio of two determinants

\[ F^{(m)} = \prod_{a=1}^{Q} \prod_{b=1}^{Q} \frac{f(a;b)}{f(a;b)} \frac{\det T_{jk}(f \ g_{m} \ g_{N} \ g)}{\det K_{jk}(f \ g_{m} \ g \ g \ g \ g)} \]  

Here \( K_{jk} \) is the Cauchy matrix

\[ K_{jk}(f \ g \ g) = \theta^{1}(j \ k) \]  

where \( \theta^{1}(\ ) = \text{for XX chain and } \theta^{1}(\ ) = \sinh \text{ for XXZ chain}. Actually the function \( \theta^{1}(\ ) \) coincides with the function \( g(\ ) \) up to constant factor. The representation similar to (5.6) also exists for the scalar products [3.3].

6 Conclusion

We considered form factors of exactly solvable (completely integrable) models in the finite volume. We demonstrated that the form factors can be represented as determinants of matrices. The dimension of the matrix is related to the number of particles in the corresponding state. In our earlier publications we showed that this representation is useful for the theory of correlation functions. The contributions of all form factors can be taken into account and the correlation function also can be represented as a determinant.

A Zero eigenvector

Consider for simplicity the case of the rational \( R \)-matrix. Let \( N \times N \) matrix (3.7) is given

\[ M_{jk} = t(k; j) + V_{j} t(j; k) \]  

\[ (A.1) \]
Here

\[ V_j = \chi^\text{m} \frac{\left(\begin{array}{cc} j & m + \text{i}c \\ m - 1 & j + \text{i}c \end{array}\right)}{\left(\begin{array}{cc} m & j + \text{i}c \\ m - 1 & j + \text{i}c \end{array}\right)}: \tag{A.2} \]

Recall also that \( \tau(\ : \ ) = (\text{i}c)^2 \left(\begin{array}{cc} \text{i}c & \text{i}c \\ m & m \end{array}\right) \left(\begin{array}{cc} m & m \\ j & j \end{array}\right) \). Let us prove that

\[ \chi^\text{m} \sum_{k=1}^{\text{N}} M_{jk}^m k = 0; \quad \text{where} \quad k = \frac{m - 1}{\chi^\text{m}} \left(\begin{array}{cc} k & m \\ m & m \end{array}\right): \tag{A.3} \]

We have

\[ \chi^\text{m} \sum_{k=1}^{\text{N}} M_{jk}^m k = (\text{i}c)^2 \ G + V_j G \ ; \tag{A.4} \]

where

\[ G = \chi^\text{m} \frac{1}{\chi^\text{m}} \left(\begin{array}{cc} k & j \\ k & j \end{array}\right) \left(\begin{array}{cc} k & m \\ m & m \end{array}\right): \tag{A.5} \]

In order to find \( G \) consider auxiliary integral

\[ I = \frac{1}{2} \int_{|z|=R} \frac{dz}{(z-j)^m (z-j+\text{i}c)^m} \chi^\text{m} \frac{(z-m)}{(z-m)}: \tag{A.6} \]

The integral is taken with respect to the closed contour around in infinity (in other words all the poles of the integrand lie within the contour). Obviously the integral \( I \) is equal to the residue at infinity, which in turn is equal to zero, hence \( I = 0 \). On the other hand the value of the integral \( I \) is equal to the sum of the residues in the poles \( z = k; \ k = 1; \ldots; \text{N} \) and \( z = j + \text{i}c \). Thus, we obtain

\[ I = G \ \frac{1}{\chi^\text{m}} \left(\begin{array}{cc} m & j + \text{i}c \\ m & j + \text{i}c \end{array}\right)^! : \tag{A.7} \]

The first term in the r.h.s. of (A.7) corresponds to the residues at the points \( z = k; \ k = 1; \ldots; \text{N} \), the second term corresponds to the residue at the point \( z = j + \text{i}c \). Hence

\[ G = \frac{1}{\chi^\text{m}} \left(\begin{array}{cc} m & j + \text{i}c \\ m & j + \text{i}c \end{array}\right)^! : \tag{A.8} \]

Substituting \( G \) into (A.4) and using explicit expression (A.2) for \( V_j \), we arrive at

\[ \chi^\text{m} \sum_{k=1}^{\text{N}} M_{jk}^m k = 0; \tag{A.9} \]

In the case of trigonometric R-matrix, after the replacement \( \exp f2g \) and \( \exp f2g \) the proof reduces to the rational case.
References

[1] L.D. Faddeev and E.K. Sklyanin, Dokl. Acad. Nauk SSSR 243 (1978) 1430.
[2] E.K. Sklyanin, Dokl. Acad. Nauk SSSR 244 (1978) 1337.
[3] L.D. Faddeev, E.K. Sklyanin, and L.A. Takhtajan, Theor. Math. Phys. 40 (1979) 194.
[4] F.A. Smirnov, Form factors in completely integrable models of quantum field theory (World Scientific, Singapore, 1992).
[5] A.B. Zamolodchikov and A.B. Zamolodchikov, Ann. Phys. 120 (1979) 253; A.B. Zamolodchikov, Advanced Studies in Pure Math. 19 (1989) 641; Int. J. Mod. Phys. A 3 (1988) 743.
[6] A.E. Arinshtein, V.A. Fateev, and A.B. Zamolodchikov, Phys. Lett. B 87 (1979) 3389.
[7] B.Berg, M. Karowski and P. Weisz, Phys. Rev. D 19 (1979) 2477; M. Karowski and P. Weisz, Nucl. Phys. B 139 (1978) 445; M. Karowski, Phys. Rep. 49 (1979) 229.
[8] A.B. Zamolodchikov, Nucl. Phys. B 348 (1991) 619.
[9] J.L. Cardy, G. Mussardo, Nucl. Phys. B 340 (1990) 387.
[10] A. Koubek and G. Mussardo, Phys. Lett. B 311 (1993) 193.
[11] A. Fring, G. Mussardo and P. Simonetti, Nucl. Phys. B 393 (1990) 413.
[12] C. Ahn, G. Del no and G. Mussardo, Phys. Lett. B 317 (1993) 573.
[13] S. Lukyanov, Phys. Lett. B 408 (1997) 192; V. Brazhnikov and S. Lukyanov, preprints RU-97-58, CLNS 97/1488, hep-th/9707091.
[14] J. Balog, T. Hauer and M. R. Niedermayer, Phys. Lett. B 386 (1996) 224; Nucl. Phys. B 440 (1995) 603.
[15] A.G. Izergin, V.E. Korepin and N.Yu. Reshetikhin, J. Phys. A 20 (1987) 4799.
[16] V.E. Korepin, Commun. Math. Phys. 86 (1982) 391.
[17] A.G. Izergin and V.E. Korepin, Commun. Math. Phys. 99 (1985) 271.
[18] V.E. Korepin, Commun. Math. Phys. 113 (1987) 177.
[19] N.A. Slavnov, Theor. Math. Phys. 79 (1989) 502.
[20] N.A. Slavnov, Zap. Nauchn. Sem. in POM I 245 (1997) 270.
[21] T. Kojma, V.E. Korepin and N.A. Slavnov, Commun. Math. Phys. 188 (1997) 657.
[22] A. N. Kirillov and F. A. Smirnov, Int. Journ. Mod. Phys. A 3 (1988) 731

[23] A. N. Kirillov and F. A. Smirnov, J. Sov. Math. 47 (1989) 2413.

[24] J. M. Maillet and J. Sanchez de Santos, Drinfeld twists and Algebraic Bethe Ansatz, [algebra/9612012].

[25] N. A. Kitani, J. M. Maillet and V. Terras, Form factors of the XXZ Heisenberg spin-1=2 finite chain, [math-ph/9807020].