GLOBAL EXISTENCE AND LARGE TIME BEHAVIOR FOR THE CHEMOTAXIS–SHALLOW WATER SYSTEM IN A BOUNDED DOMAIN

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ABSTRACT. In this paper, we consider the chemotaxis–shallow water system in a bounded domain $\Omega \subset \mathbb{R}^2$. By energy method, we establish the global existence of strong solution with small initial perturbation and obtain the exponential decaying rate of the solution. We divide the bounded domain into interior domain and the domain up to the boundary. In the interior domain, the problem is treated like the Cauchy problem. In the domain up to the boundary, the tangential and normal directions are treated differently. We use different method to get the estimates for the tangential and normal directions.

1. Introduction. Chemotaxis–shallow water system was first proposed in [3] to describe the movement of bacteria cells under the influence of the signals of chemical and the fluid transportation. In this paper, we consider the chemotaxis–shallow water system

$$
\begin{align*}
  n_t + \text{div}(nu) &= D_n \Delta n - \nabla \cdot (n\chi(c)\nabla c), \\
  c_t + \text{div}(cu) &= D_c \Delta c - nf(c), \\
  h_t + \text{div}(hu) &= 0, \\
  hu_t + hu \cdot \nabla u + h^2 \nabla h + \frac{1}{2}(1 + n)\nabla h^2 &= \mu \Delta u + (\mu + \lambda)\nabla(\text{div} u)
\end{align*}
$$

in a bounded domain. Here, the unknown functions $n, c, h, u$ represent density of bacterial, substrate concentration, the fluid height and the fluid velocity field respectively. Two constants $D_n$ and $D_c$ are diffusion coefficients of cells and substrate. Given functions $\chi(c)$ and $f(c)$ denote the chemotactic sensitivity and the rate of substrate consumed by cells respectively. $\mu$ is the constant representing the shear viscosity and $\mu > 0$. $\lambda$ is the bulk viscosity coefficients satisfying $\mu + \lambda \geq 0$.  

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Chemotaxis models describe the response of bacteria cells and microorganisms to signal chemicals. The classical chemotaxis model is Keller–Segel system
\[
\begin{aligned}
    n_t &= \Delta n - \nabla \cdot (n \chi(c) \nabla c), \\
    c_t &= \Delta c - c + n.
\end{aligned}
\]
Here \( n \) is the density of cells and \( c \) is the concentration of chemical. It was proposed in [14] and [15]. The initial boundary value problem of the classical parabolic–parabolic Keller–Segel model is
\[
\begin{aligned}
    n_t &= \Delta n - \nabla \cdot (n \nabla c), & x \in \Omega, t > 0, \\
    c_t &= \Delta c - c + n, & x \in \Omega, t > 0, \\
    \frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
    (n(x,0), c(x,0)) &= (n_0, c_0), & x \in \Omega.
\end{aligned}
\]
where \( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary. \( \frac{\partial}{\partial \nu} := \nu \cdot \nabla \) and \( \nu = (\nu^1, \nu^2) \) is the outward unit normal vector on \( \partial \Omega \). For 1D case, the global solution exists and is bounded (see [20]). For 2D case, it was proved in [7] and [19] that the global solution exists and is bounded if \( \|n_0\|_{L^1} < C^* \). If \( \|n_0\|_{L^1} > C^* \), blow-up of the solution was presented in [9], [10] and [24]. Here \( C^* = 8 \pi \) if the domain \( \Omega \) is radially symmetric and \( C^* = 4 \pi \) if the domain \( \Omega \) is non-radially symmetric. For higher dimensions, it was proved in [1] and [27] that the global solution exists and tends to a constant state with an exponential decaying rate. For the radially symmetric initial data \((n_0, c_0)\), blow-up of the solution in finite time was obtained in [29].

Recently it becomes popular to study the chemotaxis model coupled with fluid equations. It seems more reasonable and also with practical interest since the chemotaxis processes should always happen in a fluid setting instead of vacuum. For chemotaxis processes in an incompressible fluid setting, a well-studied model is the chemotaxis–(Navier–)Stokes system
\[
\begin{aligned}
    n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \chi(c) \nabla c), & x \in \Omega, t > 0, \\
    c_t + u \cdot \nabla c &= \Delta c - n f(c), & x \in \Omega, t > 0, \\
    \nabla \cdot u &= 0, & x \in \Omega, t > 0, \\
    \nabla \cdot u &= 0, & x \in \partial \Omega, t > 0, \\
    (n, c, u)(x,0) &= (n_0, c_0, u_0)(x), & x \in \Omega.
\end{aligned}
\]
In the case \( \Omega \subset \mathbb{R}^2 \), for \( k \neq 0 \), the global existence and steady state in \( L^\infty \) norm for initial boundary value problem were obtained in [28] and [30]. It was further proved in [34] that the solution tends to equilibrium with an exponential decaying rate. Later, under weaker conditions, the global existence was established in [5]. For \( k = 0 \), under rater general assumptions on \( \chi \), the global existence and asymptotic theories were achieved in [32] and [33]. In the case \( \Omega \subset \mathbb{R}^3 \) and \( k \neq 0 \), the global weak solution was obtained in [31] and the global stability was given in [2].

For the chemotaxis in a compressible fluid, as we know, only the chemotaxis–shallow water system has been considered. There are many results for the Cauchy problem of this system. One could refer to [3] for the local existence of strong solution and a blow-up criterion. If the bacterial density does not allowed to vanish, the global existence of strong solution and the \( L^p \) decay estimates for the global
solution were studied in [22]. If the bacterial density is allowed to vanish, the global existence of classical solution and the $L^p$ decay estimates for the global solution were obtained in [25]. In order to overcome the difficulties of vacuum, in [25], we use the low and high frequency decomposition method. For the initial boundary value problem, only the local existence of strong solution and a blow-up criterion were obtained in [3].

There are also many other chemotaxis coupled with fluid systems, such as two-species chemotaxis–Navier–Stokes system [13], chemotaxis–non-newtonian system [17], attraction-repulsion coupled Navier–Stokes system [6] and some general chemotaxis Navier–Stokes system [11, 12, 26].

The main purpose of this paper is to establish the global existence of strong solution and study the asymptotic behavior of the global solution in a bounded domain of $\mathbb{R}^2$. In the study of the initial boundary value problem, for the usual Keller–Segel system, we can solve the problem by energy estimates. But for this coupled-system, as the boundary conditions are different, it is difficult to get estimates for high order derivatives since it is always highly nontrivial to obtain boundary information for high order derivatives. It becomes the main trouble to apply energy method. However, it is also difficult to apply Green’s function method or other spectral method to solve the initial boundary value problem. Recently, there are successful applications of Green’s function in a bounded domain for chemotaxis model alone (without fluid equations) (see [27]). The difference is that chemotaxis model alone is a pure parabolic system and the Green’s function in fact is just heat kernel while the coupled model is parabolic–hyperbolic and the corresponding Green’s function is far more complicated. In fact, to the best of our knowledge, for the initial boundary value problem in a bounded domain of pure parabolic system (instead of parabolic–hyperbolic system), the Green’s function method is well developed (see [16] and [27]). In this paper, motivated by [18] and [21], we divide the bounded domain into interior domain and the domain up to the boundary. For the previous one, we do not need to deal with the boundary information and it is like a Cauchy problem. To get the estimates of the latter one, we make full use of the structure of the system to convert a high order normal derivative into a high order tangential derivative.

For simplicity, let

$$D_n = D_c = 1, \chi(c) = 1, f(c) = c, \mu + \lambda = 0.$$  

In this paper, we consider the system

$$\begin{align*}
n_t + \text{div} (nu) &= \Delta n - \nabla \cdot (n\nabla c), \quad x \in \Omega, t > 0, \\
c_t + \text{div} (cu) &= \Delta c - nc, \quad x \in \Omega, t > 0, \\
h_t + \text{div} (hu) &= 0, \quad x \in \Omega, t > 0, \\
hu_t + hu \cdot \nabla u + h^2 \nabla n + \frac{1}{2}(1 + n)\Delta h^2 &= \mu \Delta u, \quad x \in \Omega, t > 0,
\end{align*}$$

(2)

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary. We give the initial conditions

$$(n, c, h, u)(x, 0) = (n_0, c_0, h_0, u_0)(x), \quad x \in \Omega$$

(3)

and boundary conditions

$$\begin{align*}
\left(\frac{\partial n}{\partial \nu}, \frac{\partial c}{\partial \nu}, u\right) &= (0, 0, 0), \quad \text{for } x \in \partial \Omega \text{ and } t > 0.
\end{align*}$$

(4)

Before we list the main result, we introduce some notations. Throughout this paper, $\partial_t$ stands for the derivative with respect to time variable and $f_t = \partial_t f$. The
symbol $\partial_i f (i = 1, 2)$ means partial derivative with respect to $x_i$

$$\partial_i f = \frac{\partial f}{\partial x_i}.$$  

We employ the notation $D^l f$ to mean the partial derivative of order $l$. That is if $l$ is a nonnegative integer, then

$$D^l f := \{D^\alpha f | \alpha = (\alpha_1, \alpha_2), |\alpha| = l\}$$

is a set of all partial derivatives of order $l$, endowed with the norm

$$\| D^l f \|_{L^2}^2 = \sum_{|\alpha|=l} \| D^\alpha f \|_{L^2}^2,$$

where

$$D^\alpha f := \partial_{\alpha_1}^\alpha \partial_{\alpha_2}^\alpha f.$$  

As usual, we use $D$ to stand for $D^1$. In the following, we define $\xi = (\tau, \eta)$ is a new coordinate transformed from the original coordinate $x = (x_1, x_2)$. The symbols $\partial_x^\alpha$ and $\partial^\alpha$ stand for the tangential and normal derivatives of order $l$. Here $l$ is a nonnegative integer. When $l = 1$, we omit the superscript $l$. For simplicity, we denote $\int f dx = \int_{\Omega_1} f dx$. The notation $\bar{\int} = \frac{1}{|\Omega|} \int f dx.$

Now the main result can be stated as follows.

**Theorem 1.1.** Assume that the initial data (3) satisfies

$$\|(n_0 - \bar{n}_0, c_0, h_0 - \bar{h}_0, u_0)\|_{H^3} \leq \delta_0$$

and $n_0 \geq 0, c_0 \geq 0, h_0 > 0$. Then the solution of (2)–(4) has a global strong solution

\[
\begin{cases}
(n - \bar{n}, c, h - \bar{h}, u) \in L^\infty(0, +\infty; H^3), \\
h - \bar{h} \in L^2(0, +\infty; H^3), \\
(n - \bar{n}, c, u) \in L^2(0, +\infty; H^4), \\
h_t \in L^\infty(0, +\infty; H^2) \cap L^2(0, +\infty; H^2), \\
(n_t, c_t, u_t) \in L^\infty(0, +\infty; H^1) \cap L^2(0, +\infty; H^2),
\end{cases}
\]

which satisfies

\[
\begin{align*}
\|(n - \bar{n}, c, h - \bar{h}, u)\|_{H^3}^2 &+ \|(n_t, c_t, u_t)\|_{H^1}^2 + \|h_t\|_{H^2}^2 \\
+ \int_0^t (\|(n - \bar{n}, c, u)\|_{H^4}^2 + \|h - \bar{h}\|_{H^3}^2 + \|(n_t, c_t, h_t, u_t)\|_{H^2}^2) \, ds \\
&\leq C\|(n_0 - \bar{n}_0, c_0, h_0 - \bar{h}_0, u_0)\|_{H^3}^2.
\end{align*}
\]

Furthermore, the solution decays exponentially

$$\|(n, c, h, u) - (\bar{n}, 0, \bar{h}, 0)\|_{H^3}^2 \leq Ce^{-kt}.$$  

Here $\delta_0$ is a sufficiently small positive constant, $k$ is a suitably small constant and $C$ is a positive constant.

**Remark 1.** From (2)$_1$ and (2)$_3$, we can easily obtain

$$\int ndx = \int n_0 dx, \int hdx = \int h_0 dx.$$  

That is $\bar{n} = \bar{n}_0, \bar{h} = \bar{h}_0.$ Thus by Poincaré inequality, (5) and $u|_{\partial\Omega} = 0$, we have

$$\|n - \bar{n}\|_{L^2}^2 \leq C\|\nabla n\|_{L^2}^2, \|h - \bar{h}\|_{L^2}^2 \leq C\|\nabla h\|_{L^2}^2, \|u\|_{L^2}^2 \leq C\|\nabla u\|_{L^2}^2.$$
The rest of this paper is organized as follows. In Section 2, we give the preliminary knowledge of this paper. In Section 3, we get the basic estimates. In Section 4, estimates are established in the interior domain. In Section 5, we investigate the estimates in the domain up to the boundary. In Section 6, we complete the proof of the main theorem.

2. The preliminary knowledge. In this section, we establish the local existence of strong solution and prove the nonnegativity of $n$ and $c$. The proof of the local existence is analogous to the discussions in [3] and [4]. We only sketch the proof for completeness.

Lemma 2.1. (Local well-posedness) Assume that the initial data (3) satisfies

$$(n_0 - \bar{n}_0, c_0, h_0 - \bar{h}_0, u_0) \in H^3$$

and $n_0 \geq 0, c_0 \geq 0, h_0 > 0$. Then there exists $T^* > 0$ such that the system (2)–(4) has a unique strong solution satisfying

$$\begin{aligned}
(n - \bar{n}, c, h - \bar{h}, u) &\in L^\infty(0, T^*; H^3),
\h - \bar{h} \in L^2(0, T^*; H^3),
(n - \bar{n}, c, u) &\in L^\infty(0, T^*; H^4) \cap L^2(0, T^*; H^2),
(h_t, c_t, u_t) &\in L^\infty(0, T^*; H^1) \cap L^2(0, T^*; H^2).
\end{aligned}$$

Proof. First we define a space

$$\mathcal{M} = \{f \mid \sup_{0 \leq t \leq T^*} \|f_t\|^2_{H^1} + \|f\|^2_{H^3} + \int_0^{T^*} (\|f_t\|^2_{H^2} + \|f\|^2_{H^4}) \, ds \leq R\},$$

where $T^*$ and $R$ will be decided later. Then we give two known functions $\tilde{n}, \tilde{u} \in \mathcal{M}$ satisfying the initial conditions

$$\begin{aligned}
(\tilde{n}, \tilde{u})(x, 0) &= (n_0, u_0)(x), \ x \in \Omega \ \text{and boundary conditions}
\end{aligned}$$

and boundary conditions

$$\begin{aligned}
(\frac{\partial \tilde{n}}{\partial \nu}, \tilde{u}) &= (0, 0), \ \text{for } x \in \partial \Omega \ \text{and } t > 0.
\end{aligned}$$

In the following, we consider the system

$$\begin{aligned}
n_t + \text{div}(n \tilde{u}) &= \Delta n - \nabla \cdot (n \nabla c),
c_t + \text{div}(c \tilde{u}) &= \Delta c - \tilde{n}c, 
\h_t + \text{div}(\h \tilde{u}) &= 0, 
\h \tilde{u} + h \tilde{u} \cdot \nabla u + h^2 \nabla n + \frac{1}{2}(1 + n)\nabla h^2 &= \mu \Delta u
\end{aligned}$$

with the initial conditions

$$\begin{aligned}
(n, c, h, u)(x, 0) &= (n_0, c_0, h_0, u_0)(x), \ x \in \Omega \ \text{and boundary conditions}
\end{aligned}$$

and boundary conditions

$$\begin{aligned}
(\frac{\partial n}{\partial \nu}, \frac{\partial c}{\partial \nu}, \tilde{u}) &= (0, 0, 0), \ \text{for } x \in \partial \Omega \ \text{and } t > 0.
\end{aligned}$$

By the characteristic method, the existence and regularity of (6) can be obtained. The solution can be represented as

$$h(x, t) = h_0 \exp(-\int_0^t \text{div} \tilde{u} \, ds).$$
The equation (6)\textsubscript{2} is a linear parabolic equation. Thus the existence, uniqueness and regularity of $c$ can be achieved by the standard energy method for the linear parabolic equation. Then we can get the estimates for $n$ by the same way. As the equation (6)\textsubscript{4} is also a linear parabolic equation, utilizing the estimates for $(h,n)$ and the energy method, we obtain the existence, uniqueness and regularity of $u$. The above argument is that if $(\bar{n}, \bar{u}) \in \mathcal{M}$, then we can define a map mapping $\mathcal{M}$ to itself

$$\Phi : (\bar{n}, \bar{u}) \rightarrow (n, u).$$

Finally, by fixed point theorem, we get the local well-posedness for (2)–(4). \qed

The proof of the following lemma can be found in [5].

**Lemma 2.2.** Suppose that the assumptions of Theorem 1.1 hold. Then the global strong solution $(n, c, h, u)$ to the initial boundary value problem of system (2)–(4) satisfies

$$n(x,t) \geq 0, \ c(x,t) \geq 0 \ a.e. \ in \ [0, +\infty) \times \Omega.$$ 

3. **The basic estimates.** In this section, we establish the basic estimates which can be obtained in the domain $\Omega$ itself without dividing it into interior domain and the domain up to the boundary.

Denoting $v = n - \bar{n}$ and $\rho = h - \bar{h}$, we linearize the equation (2) near $(\bar{n}, 0, \bar{h}, 0)$

$$\begin{cases}
v_t + \bar{n} \text{div } u - \Delta v + \bar{n} \Delta c = F_1, \\
c_t - \Delta c + \bar{nc} = F_2, \\
\rho_t + \bar{h} \text{div } u + u \nabla \rho = F_3, \\
u_t + \bar{h} \nabla v + (1 + \bar{n})\nabla \rho + v \nabla \rho = (\mu/\bar{h})\Delta u - (\mu/\bar{h}^2)\rho \Delta u.
\end{cases}$$

(7)

In the following, we estimate $(v, c, \rho, u)$ under a *priori* assumption

$$N^2(0,t) = \sup_{0 \leq s \leq \tau} \left\{ \| (v, c, \rho, u) \|^2_{H^1} + \| (v_t, c_t, u_t) \|^2_{H^1} + \| \rho_t \|^2_{H^2} \right\}
+ \int_0^t \left\{ \| (v, c, \rho, u) \|^2_{H^3} + \| (v_t, c_t, \rho_t, u_t) \|^2_{H^2} \right\} ds \leq \delta^2,$$

where $(v, c, \rho, u)$ is the solution of (2)–(4) and $\delta$ is a small constant with $0 < \delta \ll 1$.

We rewrite (7) as

$$\begin{cases}
v_t + \bar{n} \text{div } u - \Delta v + \bar{n} \Delta c = F_1, \\
c_t - \Delta c + \bar{nc} = F_2, \\
\rho_t + \bar{h} \text{div } u + u \nabla \rho = F_3, \\
u_t + \bar{h} \nabla v + (1 + \bar{n})\nabla \rho - (\mu/\bar{h})\Delta u = F_4,
\end{cases}$$

(8)

where

$$\begin{align*}
F_1 &= - \text{div } (vu) - \nabla \cdot (v \nabla c), \\
F_2 &= - \text{div } (cu) - vc, \\
F_3 &= - \rho \text{div } u, \\
F_4 &= -u \cdot \nabla u - \nabla (\rho v) - (\mu/\bar{h}^2)\rho \Delta u.
\end{align*}$$

(9)
Lemma 3.1. For $2 \leq l \leq 4$, it holds that
\[
\|D^l v\|_{L^2}^2 \leq C(\|D^{l-2} v\|_{L^2}^2 + \|D^{l-2} (\text{div } u)\|_{L^2}^2 + \|D^l c\|_{L^2}^2 + \|D^{l-2} F_1\|_{L^2}^2),
\]
\[
\|D^l c\|_{L^2}^2 \leq C(\|D^{l-2} c\|_{L^2}^2 + \|D^{l-2} c\|_{L^2}^2 + \|D^{l-2} F_2\|_{L^2}^2),
\]
\[
\|u\|_{H^l}^2 \leq C(\|u\|_{H^{l-2}}^2 + \|v\|_{H^{l-1}}^2 + \|\rho\|_{H^{l-1}}^2 + \|F_4\|_{L^2}^2).
\]

Proof. Since the equations of $v, c, u$ have the strongly elliptic operator, we can get
\[
\|D^l v\|_{L^2}^2 \leq C(\|D^{l-2} v\|_{L^2}^2 + \|D^{l-2} (\text{div } u)\|_{L^2}^2 + \|D^l c\|_{L^2}^2 + \|D^{l-2} F_1\|_{L^2}^2),
\]
\[
\|D^l c\|_{L^2}^2 \leq C(\|D^{l-2} c\|_{L^2}^2 + \|D^{l-2} c\|_{L^2}^2 + \|D^{l-2} F_2\|_{L^2}^2),
\]
\[
\|D^l u\|_{L^2}^2 \leq C(\|D^{l-2} u\|_{L^2}^2 + \|D^{l-1} v\|_{L^2}^2 + \|D^{l-2} F_4\|_{L^2}^2).
\]

For $u$, according to the boundary condition, it satisfies the Poincaré inequality and Gagliardo-Nirenberg inequality for whole space. We can also get estimates for low order derivatives. \hfill \square

Lemma 3.2. For the system (8), we have
\[
\|(v, c, u)\|_{L^2}^2 + \int_0^t \|(Dv, c, Dc, Du)\|_{L^2}^2 ds \
\leq C\|(v, c, u)(0)\|_{L^2}^2 + \epsilon \int_0^t \|\rho\|_{L^2}^2 ds \
+ C\epsilon \int_0^t (\|F_1\|_{L^2}^2 + \|F_2\|_{L^2}^2 + \|F_3\|_{L^2}^2 + \|u\nabla \rho\|_{L^2}^2 + \|F_4\|_{L^2}^2) ds,
\]
where $\epsilon$ is suitably small.

Proof. Multiplying the system (8) by $\tilde{h}^2 v, (\tilde{n}\tilde{h})^2 c, \tilde{n}(1+\tilde{n}) \rho, (\tilde{n}\tilde{h}) u$ respectively, integrating over $\Omega$ and summing up, according to the boundary conditions and integration by parts, we have
\[
\tilde{h}^2 \frac{d}{dt} \int v^2 dx + \frac{(\tilde{n}\tilde{h})^2}{2} \frac{d}{dt} \int c^2 dx + \frac{\tilde{n}(1+\tilde{n})}{2} \frac{d}{dt} \int \rho^2 dx + \frac{(\tilde{n}\tilde{h})}{2} \frac{d}{dt} \int u^2 dx \
+ \tilde{h}^2 \int (Dv)^2 dx + (\tilde{n}\tilde{h})^2 \int (Dc)^2 dx + \tilde{n}(\tilde{n}\tilde{h})^2 \int c^2 dx \
+ \mu \tilde{n} \int (Du)^2 dx + \tilde{n}\tilde{h}^2 \int \Delta c v dx \
= \tilde{h}^2 \int F_1 v dx + (\tilde{n}\tilde{h})^2 \int F_2 c dx + \tilde{n}(1+\tilde{n}) \int F_3 \rho dx \
+ (\tilde{n}\tilde{h}) \int F_4 u dx - \tilde{n}(1+\tilde{n}) \int (u\nabla \rho) \rho dx,
\]
which implies
\[
\tilde{h}^2 \frac{d}{dt} \int v^2 dx + \frac{(\tilde{n}\tilde{h})^2}{2} \frac{d}{dt} \int c^2 dx + \frac{\tilde{n}(1+\tilde{n})}{2} \frac{d}{dt} \int \rho^2 dx + \frac{(\tilde{n}\tilde{h})}{2} \frac{d}{dt} \int u^2 dx \
+ \frac{\tilde{h}^2}{2} \int (Dv)^2 dx + \frac{(\tilde{n}\tilde{h})^2}{2} \int (Dc)^2 dx + \tilde{n}(\tilde{n}\tilde{h})^2 \int c^2 dx + \mu \tilde{n} \int (Du)^2 dx \tag{11}
\]
\[
\leq \epsilon \left( v^2 + c^2 + \rho^2 + u^2 \right) dx + C\epsilon \left( F_1^2 + F_2^2 + F_3^2 + (u\nabla \rho)^2 + F_4^2 \right) dx.
\]
Applying Poincaré inequality and integrating (11) with respect to time variable, it is easy to obtain the inequality (10). \hfill \square
Lemma 3.3. For the system (8), we get the estimates

\[ \|(v_t, c_t, \rho_t, u_t)\|_{L^2}^2 + \int_0^t \|(Dv_t, c_t, Dc_t, Du_t)\|_{L^2}^2 ds \leq C \|(v_t, c_t, \rho_t, u_t)(0)\|_{L^2}^2 + \epsilon \int_0^t \|\rho_t\|_{L^2}^2 ds \]

\[ + C \epsilon \int_0^t (\|F_1\|_{L^2}^2 + \|F_2\|_{L^2}^2 + \|F_3\|_{L^2}^2 + \|(u \nabla \rho)\|_{L^2}^2 + \|F_4\|_{L^2}^2)ds, \]

where \((\bar{F}_1)\) = \(-(\nu u + v \nabla c)_t\).

Proof. We use \(\partial_t\) to (8), multiply the resulting system by \(\bar{h}^2 v_t, (\bar{n} \bar{h})^2 c_t, \bar{n}(1 + \bar{n})\rho_t, (\bar{n} \bar{h}) u_t\) respectively, integrate over \(\Omega\) and sum up to get

\[ \frac{\bar{h}^2}{2} \frac{d}{dt} \int v_t^2 dx + \frac{(\bar{n} \bar{h})^2}{2} \frac{d}{dt} \int c_t^2 dx + \frac{\bar{n}(1 + \bar{n})}{2} \frac{d}{dt} \int \rho_t^2 dx + \frac{(\bar{n} \bar{h})}{2} \frac{d}{dt} \int u_t^2 dx \]

\[ + \bar{h}^2 \int (Dv_t)^2 dx + (\bar{n} \bar{h})^2 \int (Dc_t)^2 dx + \bar{n}(\bar{h})^2 \int c_t^2 dx \]

\[ + \mu \bar{n} \int (Du_t)^2 dx + \bar{n} \bar{h}^2 \int \Delta c_t v_t dx \]

\[ = \bar{h}^2 \int (F_1)_t v_t dx + (\bar{n} \bar{h})^2 \int (F_2)_t c_t dx + \bar{n}(1 + \bar{n}) \int (F_3)_t \rho_t dx \]

\[ + (\bar{n} \bar{h}) \int (F_4)_t u_t dx - \bar{n}(1 + \bar{n}) \int (u \nabla \rho)_t \rho_t dx. \]

One has

\[ \int (F_1)_t v_t dx = - \int (\bar{F}_1)_t Dx_t dx. \]

Substituting (14) into (13) and then integrating the resulting equation with respect to time variable, we obtain the inequality (12).

Lemma 3.4. For \(Dv\) and \(Dc\), we have

\[ \|Dv\|_{L^2}^2 + \int_0^t \|v_t\|_{L^2}^2 ds \leq C \|Dv(0)\|_{L^2}^2 + C \int_0^t (\|Dv\|_{L^2}^2 + \|Dc\|_{L^2}^2 + \|\text{div} u\|_{L^2}^2 + \|F_1\|_{L^2}^2) ds \]

and

\[ \|Dc\|_{L^2}^2 + \int_0^t \|c_t\|_{L^2}^2 ds \leq C \|Dc(0)\|_{L^2}^2 + C \int_0^t (\|c\|_{L^2}^2 + \|F_2\|_{L^2}^2) ds. \]

Proof. Multiplying (8)_1, (8)_2 by \(v_t, c_t\) respectively and integrating over \(\Omega\), we can easily get the estimates.

Lemma 3.5. For \(0 \leq l \leq 2\), we have the estimates for \(\rho_t\)

\[ \|D^l \rho_t\|_{L^2}^2 + \int_0^t \|D^l \rho_t\|_{L^2}^2 ds \leq C (\|D^l (\text{div} u)\|_{L^2}^2 + \|D^l (u \nabla \rho)\|_{L^2}^2 + \|D^l F_3\|_{L^2}^2) \]

\[ + C \int_0^t (\|D^l (\text{div} u)\|_{L^2}^2 + \|D^l (u \nabla \rho)\|_{L^2}^2 + \|D^l F_3\|_{L^2}^2) ds. \]
Proof. Applying $D^t$ to equation (8)$_3$, the construction of the resulting equation asserts
\[
\|D^t \rho_t \|_{L^2}^2 \leq C(\|D^t(\text{div } u)\|_{L^2}^2 + \|D^t(u \nabla \rho)\|_{L^2}^2 + \|D^tF_3\|_{L^2}^2).
\]
Combining (15) and (15) integrating with respect to time variable, we derive
\[
\|D^t \rho_t \|_{L^2}^2 + \int_0^t \|D^t \rho_t \|_{L^2}^2 ds \leq C(\|D^t(\text{div } u)\|_{L^2}^2 + \|D^t(u \nabla \rho)\|_{L^2}^2 + \|D^tF_3\|_{L^2}^2) + C \int_0^t (\|D^t(\text{div } u)\|_{L^2}^2 + \|D^t(u \nabla \rho)\|_{L^2}^2 + \|D^tF_3\|_{L^2}^2) ds.
\]
Therefore, we complete the proof of this lemma. \[\square\]

Lemma 3.6. For $v_t, c_t, u_t$, we obtain the estimates for high order derivatives with respect to spatial variable
\[
\|D(v_t, c_t, u_t)\|_{L^2}^2 + \int_0^t \|D^2(v_t, c_t, u_t)\|_{L^2}^2 ds \leq C \|D(v_t, c_t, u_t)(0)\|_{L^2}^2 + C \int_0^t \|D(v_t, c_t, u_t)\|_{L^2}^2 ds + C \int_0^t (\|D_1\|_{L^2}^2 + \|D_2\|_{L^2}^2 + \|D_3\|_{L^2}^2) ds.
\]

Proof. Now consider the system
\[
\begin{cases}
    v_{tt} + \bar{n} \text{div } u_t - \Delta v_t + \bar{n} \Delta c_t = (F_1)_t, \\
    c_{tt} - \Delta c_t + \bar{n} c_t = (F_2)_t, \\
    u_{tt} + \bar{h} \nabla v_t + (1 + \bar{n}) \nabla \rho_t - (\mu/\bar{h}) \Delta u_t = (F_3)_t.
\end{cases}
\]
Multiplying the above system by $-\bar{h}^2 \Delta v_t, -(\bar{n}\bar{h})^2 \Delta c_t, -(\bar{n}\bar{h}) \Delta u_t$ respectively, integrating with respect to space and time variable gives the inequality (16). \[\square\]

In order to get the high order derivatives with respect to spatial variable, we need to divide the bounded domain into finite subdomains. Assume that $\{\Omega_j\}_{j=0}^n$ is a cover of $\Omega$ a.e. $\Omega \subset \bigcup_{j=0}^n \Omega_j$ and satisfies
- $\bar{\Omega}_0 \subset \Omega$ and $\Omega_j \cap \partial \Omega \neq \emptyset (j = 1, \ldots, n)$;
- In each $\Omega \cap \Omega_j (j = 1, \ldots, n)$, it is given locally by a smooth function $\psi_j$
  \[
  \Omega \cap \Omega_j = \{(y_1, y_2) | y_2 > \psi_j(y_1)\} \cap \Omega_j \text{ and }
  \\partial \Omega \cap \Omega_j = \{(y_1, y_2) | y_2 = \psi_j(y_1)\} \cap \Omega_j.
\]
  Here $\psi_j \in C^2$ and $(y_1, y_2)$ is a new coordinate introduced in Section 5;
- $\{\Omega_{j\varepsilon}\}_{j=0}^n$ is also a cover of $\Omega$. Here
  $\Omega_{j\varepsilon} = \{x \in \Omega_j | \text{dist}(x, \partial \Omega_j) > \varepsilon\} \subset \Omega_j$;
- In each $\Omega_j (j = 0, 1, \ldots, n)$, we define a cut-off function $\chi_j \in C_0^\infty$ and
  \[
  \chi_j = \begin{cases}
    1, x \in \Omega_{j\varepsilon}; \\
    0, x \in \Omega_j^c.
  \end{cases}
\]
In this paper, the domain $\Omega_0$ is called the interior domain and $\{ \Omega_1 \}_{j=1}^n$ is called the domain up to the boundary. Below, we assume $\text{diam}(\Omega_j)$ ($j = 1, \cdots , n$) is small such that $\delta \psi = \max_{1 \leq j \leq n} \| \psi_j \|_{L^\infty(\Omega_j)}$ is suitably small.

4. The interior estimates. In this section, we aim to establish the estimates in the interior domain. In the interior domain, we do not need to consider the boundary conditions. We first estimate the high order derivatives with respect to $\rho$. We have

Lemma 4.1. For $1 \leq l \leq 3$, we have

$$
\| \chi_0 D^l \rho \|_{L^2}^2 + \int_0^t \| \chi_0 D^l \rho \|_{L^2}^2 ds \\
\leq C(\| D^l \rho(0) \|_{L^2}^2 + \| D^{l-1} u(0) \|_{L^2}^2 + \| D^{l-1} u \|_{L^2}^2 )
\quad + C(\int_0^t (\| D^l v \|_{L^2}^2 + \| u \|_{L^2}^2 ) ds + C(\int_0^t (\| D^l F_3 \|_{L^2}^2 + \| D^{l-1} F_4 \|_{L^2}^2 ) ds.
$$

Proof. First we consider $l = 1$. Applying $D$ to (8), we have

$$
D\rho_1 + \bar{h} D(\text{div} u) + D(\text{u} \nabla \rho) = DF_3.
$$

Multiplying (19), (8)$_{4}$ by $\mu \chi_0^2 D \rho$ and $\bar{h} \chi_0^2 D \rho$ respectively, integrating over $\Omega_0$ and summing up, one gets

$$
\frac{\mu}{2} \frac{d}{dt} \int (\chi_0 D \rho)^2 dx + (1 + \bar{h}) \bar{h} \int (\chi_0 D \rho)^2 dx
\quad = -\mu \bar{h} \int \chi_0^2 D(\text{div} u) D \rho dx - \mu \int \chi_0^2 D(u \nabla \rho) D \rho dx - \bar{h} \int \chi_0^2 u D \rho dx
\quad - \bar{h} \int \chi_0^2 u D \rho dx + \mu \bar{h} \int \chi_0^2 D u D \rho dx + \mu \int \chi_0^2 D F_3 D \rho dx
\quad + \bar{h} \int \chi_0^2 g D \rho dx.
$$

It follows from integration by parts that

$$
\int \chi_0^2 D(\text{div} u) D \rho dx - \int \chi_0^2 D u D \rho dx
\quad = -2 \int \chi_0 \partial_2 \chi_0 (\partial_1 u_2 - \partial_2 u_1) \partial_1 \rho dx - 2 \int \chi_0 \partial_1 \chi_0 (\partial_2 u_1 - \partial_1 u_2) \partial_2 \rho dx
\quad \leq \| u \|_{H^1} \| \chi_0 D \rho \|_{L^2}.
$$

Now we turn to the nonlinear term

$$
\int \chi_0^2 D(u \nabla \rho) D \rho dx = \frac{1}{2} \int \chi_0^2 \text{div} (u D(\rho))^2 dx - \int \chi_0 D \chi_0 (D \rho)^2 u dx
\quad \leq \| \text{div} u \|_{L^\infty} \| \chi_0 D \rho \|_{L^2}^2 + \| \chi_0 D \rho \|_{L^2} \| D \rho \|_{L^\infty} \| u \|_{L^2}.
$$

In order to deal with $\int \chi_0^2 u D(\rho) dx$, following the idea in [8], making use of equation (8)$_{3}$, we can replace time derivative by the spatial derivative, that is

$$
\int \chi_0^2 u D \rho_1 dx = \frac{d}{dt} \int \chi_0^2 u D \rho dx - \int \chi_0^2 u D \rho_1 dx
\quad = \frac{d}{dt} \int \chi_0^2 u D \rho dx + \bar{h} \int \chi_0^2 u D(\text{div} u) dx + \int \chi_0^2 u D(u \nabla \rho) dx - \int \chi_0^2 u D F_3 dx
\quad \leq \frac{d}{dt} \int \chi_0^2 u D \rho dx + C \| u \|_{H^1}^2 + C(\| D F_3 \|_{L^2}^2 + \| u \nabla \rho \|_{L^2}^2).
$$
Above all, we have
\[
\|\chi_0 D\rho\|_{L^2}^2 + \int_0^t \|\chi_0 D\rho\|_{L^2}^2 ds \\
\leq C(\|D\rho(0)\|_{L^2}^2 + \|\mathbf{u}(0)\|_{H^1}^2 + \|\mathbf{u}\|_{L^2}^2) \\
+ C \int_0^t (\|D\mathbf{u}\|_{L^2}^2 + \|\mathbf{u}\|_{H^1}^2) ds + C \int_0^t (\|DF_3\|_{L^2}^2 + \|\mathbf{u}\nabla \rho\|_{L^2}^2 + \|F_4\|_{L^2}^2) ds.
\]

Then we consider \(2 \leq l \leq 3\). Operating \(D^l\) and \(D^{l-1}\) on the equations (8)\(_3\) and (8)\(_4\) respectively, it has that
\[
D^l \rho_1 + \bar{h} D^l (\text{div} \mathbf{u}) + D^l (\mathbf{u} \nabla \rho) = D^l F_3
\]
and
\[
D^{l-1} \mathbf{u}_1 + \tilde{h} D^{l-1} v + (1 + \bar{n}) D^l \rho - (\mu/\bar{h}) D^{l-1} (\Delta \mathbf{u}) = D^{l-1} F_4.
\]
Multiplying the equations (22), (23) by \(\mu_0^2 D^l \rho\) and \(\bar{h}^2 \chi_0^2 D^l \rho\) respectively and integrating over \(\Omega_0\) and summing up gives
\[
\frac{\mu}{2} \frac{d}{dt} \int (\chi_0 D^l \rho)^2 dx + (1 + \bar{n})\tilde{h}^2 \int (\chi_0 D^l \rho)^2 dx \\
= -\mu \bar{h} \int \chi_0^2 D^l (\text{div} \mathbf{u}) D^l \rho dx - \mu \int \chi_0^2 D^l (\mathbf{u} \nabla \rho) D^l \rho dx - \tilde{h}^2 \int \chi_0^2 D^{l-1} \mathbf{u}_1 D^l \rho dx \\
- \bar{h}^3 \int \chi_0^2 D^l v D^l \rho dx + \mu \bar{h} \int \chi_0^2 D^l (\Delta \mathbf{u}) D^l \rho dx + \mu \int \chi_0^2 D^l F_3 D^l \rho dx \\
+ \tilde{h}^2 \int \chi_0^2 D^{l-1} F_4 D^l \rho dx.
\]
From (20), we deduce that each term of
\[
\int \chi_0^2 D^l (\Delta \mathbf{u}) D^l \rho dx - \int \chi_0^2 D^l ((\text{div} \mathbf{u}) D^l \rho dx
\]
can be controlled by \(\|\chi_0 D^l \rho\|_{L^2}^2\) and \(\|\mathbf{u}\|_{H^1}^2\). Let us deal with \(\int \chi_0^2 D^{l-1} \mathbf{u}_1 D^l \rho dx\) as follows
\[
\int \chi_0^2 D^{l-1} \mathbf{u}_1 D^l \rho dx \\
= \frac{d}{dt} \int \chi_0^2 D^{l-1} \mathbf{u} D^l \rho dx - \int \chi_0^2 D^{l-1} \mathbf{u} D^l \rho dx \\
= \frac{d}{dt} \int \chi_0^2 D^{l-1} \mathbf{u} D^l \rho dx + \bar{h} \int \chi_0^2 D^{l-1} \mathbf{u} (\text{div} \mathbf{u}) dx \\
+ \int \chi_0^2 D^{l-1} \mathbf{u} (\mathbf{u} \nabla \rho) dx - \int \chi_0^2 D^{l-1} \mathbf{u} F_3 dx \\
\leq \frac{d}{dt} \int \chi_0^2 D^{l-1} \mathbf{u} D^l \rho dx + C\|\mathbf{u}\|_{H^1}^2 + C(\|DF_3\|_{L^2}^2 + \|D^{l-1}(\mathbf{u} \nabla \rho)\|_{L^2}^2).
\]
The nonlinear term can be estimated as
\[
\int \chi_0^2 D^2 (\mathbf{u} \nabla \rho) D^2 \rho dx \\
= \frac{3}{2} \int \chi_0^2 (D^2 \rho)^2 D \mathbf{u} dx + \int \chi_0^2 D \rho D^2 \mathbf{u} D^2 \rho dx - \int \chi_0 D \chi_0 \mathbf{u} (D^2 \rho)^2 dx \\
\leq \|D \mathbf{u}\|_{L^\infty} \|\chi_0 D^2 \rho\|_{L^2}^2 + \|D \rho\|_{L^\infty} \|D^2 \mathbf{u}\|_{L^2} \|\chi_0 D^2 \rho\|_{L^2} \\
+ \|\chi_0 D^2 \rho\|_{L^2} \|D^2 \rho\|_{L^2} \|\mathbf{u}\|_{L^\infty}.
and
\[
\int \chi_0^2 D^3 (u \nabla \rho) D^3 \rho dx
\]
\[=
\frac{5}{2} \int \chi_0^2 (D^3 \rho)^2 D^3 u dx + 3 \int \chi_0^2 D^2 \rho D^2 u D^3 \rho dx
\]
\[- \int \chi_0 D^2 \rho \rho D^2 u D^3 \rho dx + \int \chi_0^2 D^3 \rho D^3 \rho dx
\]
\[\leq \|Du\|_{L^\infty} \|\chi_0 D^3 \rho\|_2^2 + \|D^2 \rho\|_{L^1} \|D^2 u\|_{L^4} \|\chi_0 D^3 \rho\|_{L^2}
\]
\[+ \|\chi_0 D^3 \rho\|_{L^2} \|D^3 \rho\|_{L^2} \|\rho\|_{L^\infty} + \|D^3 \rho\|_{L^\infty} \|D^3 u\|_{L^2} \|\chi_0 D^3 \rho\|_{L^2}.
\]
Combining all the above estimates, yields the inequality (18). 

**Lemma 4.2.** For \(1 \leq l \leq 3\), it holds
\[
\|\chi_0 D^l (v, c, \rho, u)\|_{L^2}^2 + \int_0^t \|\chi_0 D^{l+1} (v, c, u)\|_{L^2}^2 ds
\]
\[\leq C \|D^l (v, c, \rho, u)(0)\|_{L^2}^2 + C \int_0^t (\|D^l (v, c)\|_{L^2}^2 + \|u\|_{H^1}^2 + \|\chi_0 D^l \rho\|_{L^2}^2) ds
\]
\[+ C \int_0^t (\|D^{l-1} F_1\|_{L^2}^2 + \|D^l F_2\|_{L^2}^2 + \|D^l F_3\|_{L^2}^2 + \|D^{l-1} F_4\|_{L^2}^2) ds.
\]

**Proof.** First, we consider \(l = 1\). Applying \(D\) to (8) and multiplying by \(\tilde{h}^2 \chi_0^2 D_v\), \((\tilde{h}^2)^2 \chi_0^2 D_c\), \((\tilde{h}^2)^2 \chi_0^2 D_\rho\), \((\tilde{h}^2)^2 \chi_0^2 D_u\) respectively, it gives that
\[
\frac{\tilde{h}^2}{2} \frac{d}{dt} \int (\chi_0 Dv)^2 dx + \frac{(\tilde{h}^2)^2}{2} \frac{d}{dt} \int (\chi_0 Dc)^2 dx + \frac{(\tilde{h}^2)^2}{2} \frac{d}{dt} \int (\chi_0 D^2 \rho)^2 dx
\]
\[+ \mu \tilde{h} \int (\chi_0 Dv)^2 dx + \frac{(\tilde{h}^2)^2}{2} \int (\chi_0 D^2 c)^2 dx + \tilde{h} \int (\chi_0 D^2 \rho)^2 dx
\]
\[+ K_1 + K_2 + K_3
\]
\[= \tilde{h}^2 \int \chi_0^2 D^3 F_1 Dv dx + (\tilde{h}^2)^2 \int \chi_0^2 D^3 F_2 Dc dx + (\tilde{h}^2)^2 \int \chi_0^2 D^3 F_3 D\rho dx
\]
\[- \tilde{h} \int \chi_0^2 D(u \nabla \rho) Dv dx + \tilde{h} \int \chi_0^2 D^3 F_4 Dv dx,
\]
where
\[
K_1 = \tilde{h} \int \chi_0^2 D(\text{div } u) Dx v dx + \tilde{h} \int \chi_0 D^2 u D^2 v dx,
\]
\[
K_2 = \tilde{h} (1 + \tilde{h}) \int \chi_0^2 D(\text{div } u) D\rho dx + \tilde{h} \int \chi_0^2 D^2 u D^2 \rho dx
\]
and
\[
K_3 = 2 \tilde{h} \int \chi_0 D^2 \Delta v Dx v dx + 2 (\tilde{h}^2)^2 \int \chi_0 D^2 \Delta c Dx c dx + 2 \mu \tilde{h} \int \chi_0 D^2 \Delta u Dx u dx.
\]
We use integration by parts to get
\[
\int \chi_0^2 D(\text{div } u) Dx v dx
\]
\[= - \int \chi_0^2 D^2 u Dx v dx - 2 \int \chi_0 \partial_1 \chi_0 D u_1 Dx v dx - 2 \int \chi_0 \partial_2 \chi_0 D u_2 Dx v dx,
\]
\[
\int \chi_0^2 D(div u) D\rho dx \\
= -\int \chi_0^2 D\rho u D\rho dx - 2 \int \chi_0 \partial_1 \chi_0 D\rho u_1 D\rho dx - 2 \int \chi_0 \partial_2 \chi_0 D\rho u_2 D\rho dx
\]

and
\[
\int \chi_0^2 D(\Delta c) D\rho dx = -2 \int \chi_0 D\chi_0 \Delta c D\rho dx - \int \chi_0^2 \Delta c \Delta \rho dx.
\]

For the nonlinear term \( \int \chi_0^2 D(u\nabla \rho) D\rho dx \), it has been dealt with in Lemma 4.1. Hence, it has that
\[
\|\chi_0 D(v, c, \rho, u)\|^2_{L^2} + \int_0^t \|\chi_0 D^2(v, c, u)\|^2_{L^2} dt \\
\leq C\|D(v, c, \rho, u)(0)\|^2_{L^2} + C \int_0^t (\|D(v, c)\|^2_{L^2} + \|u\|^2_{H^1} + \|\chi_0 D\rho\|^2_{L^2}) dt \\
+ C \int_0^t (\|F_1\|^2_{L^2} + \|DF_2\|^2_{L^2} + \|DF_3\|^2_{L^2} + \|F_4\|^2_{L^2}) dt.
\]

Let
\[
R_1 = 2\bar{h}^2 (\int \chi_0 D\chi_0 \Delta c D\rho dx + \bar{n}^2 \int \chi_0 D\chi_0 \Delta c D\rho dx) + 2\mu \bar{n} \int \chi_0 D\chi_0 D\rho D\rho dx \\
- 2\bar{n} \bar{h} (\int \chi_0 \partial_1 \chi_0 D\rho u_1 D\rho dx + \int \chi_0 \partial_2 \chi_0 D\rho u_2 D\rho dx) \\
- \bar{n} (1 + \bar{n}) \int \chi_0 D\chi_0 (D\rho)^2 u dx + 2\bar{h}^2 \int \chi_0 D\chi_0 F_1 D\rho dx \\
+ 2\bar{n} \bar{h} \int \chi_0 D\chi_0 F_4 D\rho dx.
\]

We notice that each term of \( R_1 \) can be controlled by \( \|\chi_0 D^2(v, c, u)\|^2_{L^2} \), \( \|\chi_0 D\rho\|^2_{L^2} \), \( \|D(v, c, u)\|^2_{L^2} \) and \( \|(F_1, F_4)\|^2_{L^2} \). In the following, we use \( R_i \) to conclude these terms that the derivatives loss on the cut-off function. We do not concern about the specific form of \( R_i \). We know that it is easy to be estimated.

Then we turn to \( 2 \leq l \leq 3 \). Applying \( D^l \) to (8) and multiplying by \( \bar{h}^2 \chi_0^2 D^l v \), \( (\bar{n} \bar{h})^2 \chi_0^2 D^l c \), \( \bar{n} (1 + \bar{n}) \chi_0^2 D^l \rho \), \( (\bar{n} \bar{h}) \chi_0^2 D^l u \) respectively, we have
\[
\frac{\bar{h}^2}{2} \frac{d}{dt} \int (\chi_0 D^l v)^2 dx + \frac{(\bar{n} \bar{h})^2}{2} \frac{d}{dt} \int (\chi_0 D^l c)^2 dx + \frac{\bar{n} (1 + \bar{n})}{2} \frac{d}{dt} \int (\chi_0 D^l \rho)^2 dx \\
+ \frac{(\bar{n} \bar{h})}{2} \frac{d}{dt} \int (\chi_0 D^l u)^2 dx + \bar{h}^2 \int (\chi_0 D^{l+1} v)^2 dx + (\bar{n} \bar{h})^2 \int (\chi_0 D^{l+1} c)^2 dx \\
+ \mu \bar{n} \int (\chi_0 D^{l+1} u)^2 dx + \bar{n} (\bar{n} \bar{h})^2 \int (\chi_0 D^{l+1} c)^2 dx \\
+ \bar{n} (1 + \bar{n}) \int (\chi_0 D^{l+1} \rho)^2 dx \\
= \bar{h}^2 \int \chi_0^2 D^l F_1 D^l v dx + (\bar{n} \bar{h})^2 \int \chi_0^2 D^l D^l \rho dx \\
- \bar{n} (1 + \bar{n}) \int \chi_0^2 D^l (u \nabla \rho) D^l \rho dx + (\bar{n} \bar{h}) \int \chi_0^2 D^l F_4 D^l u dx + R_2.
\]

Repeating all the above procedures, we obtain inequality (24).
5. The estimates in the domain up to the boundary. In this section, we establish the estimates in the domain up to the boundary. We use the different method to get estimates in the tangential and normal directions respectively. For any point \( x_0 \in \partial \Omega \), we can establish a new coordinate system \( y = (y_1, y_2) \) by shifting and rotating the coordinate \( x = (x_1, x_2) \). The point \( x_0 \) is the origin of \( y \)-coordinate, the direction of \( y_1 \)-axis is parallel with the tangent of \( \partial \Omega \) at the point \( x_0 \) and the positive direction of \( y_2 \)-axis is inward pointing into \( \Omega \). The coordinate \((y_1, y_2)\) can be expressed by \((x_1, x_2)\)

\[
\begin{pmatrix}
  y_1 \\
y_2
\end{pmatrix}
= 
\begin{pmatrix}
  \cos \theta_j & -\sin \theta_j \\
  \sin \theta_j & \cos \theta_j
\end{pmatrix}
\begin{pmatrix}
x_1 + b_{1j} \\
x_2 + b_{2j}
\end{pmatrix},
\]

where

\[
\begin{pmatrix}
  \cos \theta_j & -\sin \theta_j \\
  \sin \theta_j & \cos \theta_j
\end{pmatrix}
\]

is a rotation matrix and

\[
\begin{pmatrix}
b_{1j} \\
b_{2j}
\end{pmatrix}
\]

is a translation vector. Then using \( \tau = y_1, \eta = y_2 - \psi_j(y_1) \) to express the boundary \( \Omega \cap \Omega_j \)

\[
\begin{pmatrix}
\tau \\
\eta
\end{pmatrix}
= 
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}
- 
\begin{pmatrix}
0 \\
\psi_j(y_1)
\end{pmatrix}
= 
\begin{pmatrix}
\cos \theta_j(x_1 + b_{1j}) - \sin \theta_j(x_2 + b_{2j}) \\
\sin \theta_j(x_1 + b_{1j}) + \cos \theta_j(x_2 + b_{2j}) - \psi_j(y_1)
\end{pmatrix}.
\]

Through the new coordinate \( \xi = (\tau, \eta) \), we transform the boundary of \( \Omega \cap \Omega_j \) into \( \eta = 0 \). So the tangential direction always satisfies the boundary conditions. For convenience, we only consider in one \( \Omega_j \). We omit the subscripts \( j \). According to the transformation, we have

\[
\begin{align*}
\partial_1 &= \cos \theta \partial_{\tau} + (\sin \theta - \cos \theta \psi') \partial_{\eta}, \\
\partial_2 &= -\sin \theta \partial_{\tau} + (\cos \theta + \sin \theta \psi') \partial_{\eta}, \\
\partial_1^2 &= \cos^2 \theta \partial_1^2 + 2 \cos \theta (\sin \theta - \cos \theta \psi') \partial_1 \partial_2 + (\sin \theta - \cos \theta \psi')^2 \partial_2^2 - \cos^2 \theta \psi'' \partial_\eta, \\
\partial_2^2 &= \sin^2 \theta \partial_2^2 - 2 \sin \theta (\cos \theta + \sin \theta \psi') \partial_2 \partial_\eta + (\cos \theta + \sin \theta \psi')^2 \partial_1^2 - \sin^2 \theta \psi'' \partial_\eta, \\
\Delta &= \partial_1^2 + \partial_2^2 + (1 + (\psi')^2) \partial_\eta^2 - \psi'' \partial_\eta - 2 \psi' \partial_\eta.
\end{align*}
\]

In the new \( \xi \)-coordinate, for simplicity, we introduce the following notations

\[
\begin{align*}
\mathcal{D}_1 &\triangleq \cos \theta \partial_{\tau} + (\sin \theta - \cos \theta \psi') \partial_{\eta}, \\
\mathcal{D}_2 &\triangleq -\sin \theta \partial_{\tau} + (\cos \theta + \sin \theta \psi') \partial_{\eta}, \\
\mathcal{D}_1^2 &+ \mathcal{D}_2^2 \triangleq \partial_\xi^2 + (1 + (\psi')^2) \partial_\eta^2 - \psi'' \partial_\eta - 2 \psi' \partial_\eta.
\end{align*}
\]

Then we can rewrite the equation (8) as

\[
\begin{align*}
v_t + \bar{n}(\mathcal{D}_1 u_1 + \mathcal{D}_2 u_2) - (\mathcal{D}_1^2 + \mathcal{D}_2^2) v + \bar{n}([\mathcal{D}_1^2 + \mathcal{D}_2^2] c] &= F_1, \\
c_t - ([\mathcal{D}_1^2 + \mathcal{D}_2^2] c] + \bar{n}c &= F_2, \\
u_1 + \bar{h} (\mathcal{D}_1 u_1 + \mathcal{D}_2 u_2) + u_1 \mathcal{D}_1 \rho + u_2 \mathcal{D}_2 \rho &= F_3, \\
u_{11} + \bar{h} \mathcal{D}_1 v + (1 + \bar{n}) \mathcal{D}_1 \rho - (\mu / \bar{h}) ([\mathcal{D}_1^2 + \mathcal{D}_2^2] u_1) &= F_{41}, \\
u_{21} + \bar{h} \mathcal{D}_2 v + (1 + \bar{n}) \mathcal{D}_2 \rho - (\mu / \bar{h}) ([\mathcal{D}_1^2 + \mathcal{D}_2^2] u_2) &= F_{42}.
\end{align*}
\]

(25)
Lemma 5.1. For $1 \leq l \leq 3$, we have the estimates for tangential derivatives

$$
\|\partial^l_x (v, c, u)\|_{L^2}^2 + \int_0^t (\|\partial^{l+1}_x (v, c, u)\|_{L^2}^2 + \|\partial^l_x \partial_v (v, c, u)\|_{L^2}^2) \, ds
$$

$$
\leq C \|D^l (v, c, u)(0)\|_{L^2}^2 + \epsilon \int_0^t (\|\partial^l_x \rho\|_{L^2}^2 + \|\partial^l_x \partial_v \rho\|_{L^2}^2) \, ds
$$

$$
+ C \epsilon \int_0^t (\|\partial^l_x \partial_v c\|_{L^2}^2 + \|\partial^l_x \partial_v u\|_{L^2}^2) \, ds
$$

$$
+ C \int_0^t (\|D_1^{l-1} F_1\|_{L^2}^2 + \|D^l F_2\|_{L^2}^2 + \|D_1^{l-1} F_4\|_{L^2}^2) \, ds.
$$

Proof. First we consider $l = 1$. Applying $\partial_v$ to the equation (25) and multiplying each equation by $\bar{h}^2 \partial_v, (\bar{n}h)^2 \partial_v c, \bar{n}(1 + \bar{n}) \partial_v \rho, (\bar{n}h) \partial_v u_1, (\bar{n}h) \partial_v u_2$ respectively, then integrating and summing up, we have

$$
\frac{\bar{h}^2}{2} \frac{d}{dt} \int (\partial_v v)^2 d\xi + \frac{(\bar{n}h)^2}{2} \frac{d}{dt} \int (\partial_v c)^2 d\xi + \frac{\bar{n}(1 + \bar{n})}{2} \frac{d}{dt} \int (\partial_v \rho)^2 d\xi
$$

$$
+ \frac{(\bar{n}h)^2}{2} \frac{d}{dt} \int (\partial_v u)^2 d\xi + \bar{n}(\bar{n}h)^2 \int (\partial_v v)^2 d\xi + \frac{\bar{h}^2}{2} A_1 (v) + \frac{(\bar{n}h)^2}{2} A_1 (c)
$$

$$
+ \mu \bar{n} A_1 (u) + \bar{h}^2 B_1 (v, v) - \bar{n} (\bar{n}h)^2 B_1 (c, v) + (\bar{n}h)^2 B_1 (c, c) + \mu \bar{n} B_1 (u, u)
$$

$$
= \bar{h}^2 \int \chi^2 \partial_v F_1 \partial_v u \, d\xi + (\bar{n}h)^2 \int \chi^2 \partial_v F_2 \partial_v c \, d\xi + \bar{n}(1 + \bar{n}) \int \chi^2 \partial_v F_3 \partial_v \rho \, d\xi
$$

$$
+ (\bar{n}h) \int \chi^2 \partial_v F_4 \partial_v u_1 \, d\xi + (\bar{n}h) \int \chi^2 \partial_v F_4 \partial_v u_2 \, d\xi
$$

$$
- \bar{n}(1 + \bar{n}) \int \chi^2 \partial_v (u \nabla \rho) \partial_v \rho \, d\xi + R_3,
$$

where

$$
A_l (f) = \int \{ (\chi \partial_x^{l+1} f)^2 + (\chi \sqrt{1 + (\psi')^2}) \partial_x \partial_n f \} \, d\xi,
$$

$$
B_l (f, g) = \int \chi^2 \psi'' \partial_x \partial_n f \partial_x g \, d\xi - 2 \int \chi^2 \psi' \partial_x \partial_n f \partial_x^{l+1} g \, d\xi,
$$

$$
R_3 = R_{31} + R_{32} + R_{33} + R_{34} + R_{35} + R_{36} + R_{37}
$$

$$
= \bar{n}h^2 C(v) + \bar{h}^2 D(v, v) - \bar{n} (\bar{n}h)^2 D(c, v) + (\bar{n}h)^2 D(c, c)
$$

$$
+ \bar{n}(1 + \bar{n}) C(\rho) + \mu \bar{n} D(u_1, u_1) + \mu \bar{n} D(u_2, u_2)
$$

The main purpose of this subsection is to get the high order tangential derivatives with respect to $v, c, u$. The estimates for tangential derivatives are provided.
and
\[
C(f) = \cos \theta \int \partial_r \chi^2 \partial_r u_1 \partial_r f d\xi + \cos \theta \int \chi^2 \partial_r \psi' \partial_r u_1 \partial_r f d\xi \\
+ \frac{C}{\tau} \left( \int \partial_r \chi^2 \partial_r u_1 \partial_r f d\xi + \int \partial_r \chi^2 \partial_r \psi' \partial_r u_1 \partial_r f d\xi \right) \\
- \sin \theta \int \partial_r \chi^2 \partial_r u_2 \partial_r f d\xi - \sin \theta \int \chi^2 \partial_r \psi' \partial_r u_2 \partial_r f d\xi \\
- \sin \theta \int \chi^2 \partial_r \psi' \partial_r u_2 \partial_r f d\xi + \int \partial_r \chi^2 \partial_r \psi' \partial_r u_2 \partial_r f d\xi.
\]

\[
D(f, g) = -\int \partial_r \chi^2 \partial_r f \partial_r g d\xi - \int \chi^2 \partial_r \psi' \partial_r f \partial_r g d\xi - 2 \int \chi^2 \partial_r \psi' \partial_r h \partial_r f \partial_r g d\xi \\
+ 2 \int \partial_r \chi^2 (\psi') \partial_r \psi' \partial_r f \partial_r g d\xi - \int \partial_r \chi^2 (\psi') \partial_r f \partial_r g d\xi \\
- \int \chi^2 \partial_r (\psi')^2 \partial_r f \partial_r g d\xi - \int \partial_r \chi^2 (1 + (\psi')) \partial_r f \partial_r g d\xi.
\]

We observe that each term in \( R_3 \) can be easily estimated. In the following, we use \( R_i \) to represent all these terms that the derivatives loss on the cut-off function and \( \psi \). \( R_i \) is not involved in our estimates and thus its expression is not needed.

It follows from a priori assumption and Sobolev inequality that
\[
\int \chi^2 \partial_r (u \nabla \rho) \partial_r \rho d\xi = \cos \theta \int \chi^2 \partial_r u_1(\partial_r \rho)^2 d\xi - \sin \theta \int \chi^2 \partial_r u_2(\partial_r \rho)^2 d\xi \\
- \cos \theta \int \chi^2 u_1 \partial_r \psi' \partial_r \rho \partial_r \rho d\xi \sin \theta \int \chi^2 u_2 \partial_r \psi' \partial_r \rho \partial_r \rho d\xi \\
- \frac{1}{2} \int \partial_r \chi^2 (\sin \theta - \cos \theta \psi' u_1)(\partial_r \rho)^2 d\xi - \frac{1}{2} \cos \theta \int \partial_r \chi^2 (u_1)(\partial_r \rho)^2 d\xi \\
- \frac{1}{2} \int \partial_r \chi^2 (\cos \theta + \sin \theta \psi' u_2)(\partial_r \rho)^2 d\xi + \frac{1}{2} \sin \theta \int \partial_r \chi^2 (u_2)(\partial_r \rho)^2 d\xi \\
+ \int \chi^2 (\sin \theta - \cos \theta \psi' \partial_r u_1 \partial_r \rho \partial_r \rho d\xi + \int \chi^2 (\cos \theta + \sin \theta \psi' \partial_r u_2 \partial_r \rho \partial_r \rho d\xi \\
\leq \delta (\| \partial_r \rho \|^2_{L^2} + \| \chi \partial_r \rho \|^2_{L^2} + \| u \|^2_{L^2}).
\]

(28)

Together with (27) and (28), yields that
\[
\| \chi \partial_r (v, c, \rho, u) \|^2_{L^2} + \int_0^t (\| \chi \partial_r^2 (v, c, u) \|^2_{L^2} + \| \chi \partial_r \eta (v, c, u) \|^2_{L^2}) dt \\
\leq C \| D (v, c, \rho, u) (0) \|^2_{L^2} + C \int_0^t (\| \chi \partial_r \rho \|^2_{L^2} + \| \chi \partial_r \rho \|^2_{L^2}) dt \\
+ C \int_0^t (\| v, c, u \|^2_{H^1} + \| DF_3 \|^2_{L^2}) dt + C \int_0^t (\| F_1 \|^2_{L^2} + \| DF_2 \|^2_{L^2} + \| F_4 \|^2_{L^2}) dt.
\]

Then we get the estimates for \( 2 \leq l \leq 3 \). Applying \( \partial_r^l \) to equation (25) and multiplying by \( \tilde{n} \partial_r^l v, (\tilde{n})^2 \partial_r^l c, \tilde{n}(1 + \tilde{n}) \partial_r^l \rho, (\tilde{n}) \partial_r^l u_1, (\tilde{n}) \partial_r^l u_2 \) respectively and integrating, one gets
\[
\frac{\tilde{n}^2}{2} \frac{d}{dt} \int (\chi \partial_r^l v)^2 d\xi + \frac{\tilde{n}^2}{2} \frac{d}{dt} \int (\chi \partial_r^l c)^2 d\xi + \frac{\tilde{n}(1 + \tilde{n})}{2} \frac{d}{dt} \int (\chi \partial_r^l \rho)^2 d\xi
\]
In this section, we will get the estimates for normal derivatives. As the equation of \( \rho \) does not have strongly elliptic operator \( \Delta \), we must get the high order normal derivatives with respect to \( \rho \). Using \( \partial_t \) to (25)3 and multiplying the resulting equation by \((\mu/\bar{h})(1 + (\psi')^2)\), we have
\[
\frac{d\rho}{dt} = \mu(\psi')^2 \partial_{\eta}(\frac{d\rho}{dt}) + \mu(1 + (\psi')^2)(\cos \theta \partial_{\varphi} \partial_{\eta} u_1 - \cos \theta \partial_{\eta} \psi \partial_{\eta} u_1 + (\sin \theta - \cos \theta \psi') \partial_{\eta}^2 u_1) + \mu(1 + (\psi')^2)(\sin \theta \partial_{\varphi} \partial_{\eta} u_2 - \sin \theta \partial_{\eta} \psi \partial_{\eta} u_2 - (\cos \theta + \sin \theta \psi') \partial_{\eta}^2 u_2) 
\]
\[
= (\mu/\bar{h})(1 + (\psi')^2) \partial_{\eta} F_3, 
\]
where
\[
\frac{d\rho}{dt} = \rho_t + u \nabla \rho. 
\]
Then we multiply (25)4 and (25)5 by \( \bar{h}(\sin \theta - \cos \psi') \), \( \bar{h}(\cos \theta + \sin \psi') \) respectively and add (29), we have
\[
(\mu/\bar{h}) \partial_{\eta}(\frac{d\rho}{dt}) + \bar{h} A_1(\psi) u_{11} + \bar{h} A_2(\psi) u_{21} - \mu A_1(\psi) \partial_{\varphi}^2 u_1 + \mu A_3(\psi) \partial_{\varphi} \partial_{\eta} u_1 + \mu A_4(\psi) \partial_{\eta} u_1 - \mu A_2(\psi) \partial_{\varphi}^2 u_2 + \mu A_5(\psi) \partial_{\varphi} \partial_{\eta} u_2 + \mu A_6(\psi) \partial_{\eta}^2 u_2 
\]
\[
+ \bar{h}^2(\epsilon_2(\psi) \partial_{\varphi} v + \partial_{\eta} v) + (1 + \bar{n}) \bar{h}(\epsilon_2(\psi) \partial_{\varphi} \rho + \partial_{\eta} \rho) 
\]
\[
= (\mu/\bar{h}) \partial_{\eta} F_3 + \bar{h} A_1(\psi) F_{41} + \bar{h} A_2(\psi) F_{42}, 
\]
where
\[
A_1(\psi) = \frac{\sin \theta - \cos \psi'}{1 + (\psi')^2}, \quad A_2(\psi) = \frac{\cos \theta + \sin \psi'}{1 + (\psi')^2}, 
A_3(\psi) = \frac{\cos \theta(1 - (\psi')^2) + 2\sin \theta \psi'}{1 + (\psi')^2}, 
A_4(\psi) = \frac{(\sin \theta - \cos \psi') \psi'' - (1 + (\psi')^2)\cos \theta \partial_{\eta} \psi'}{1 + (\psi')^2}, 
A_5(\psi) = \frac{\sin \theta(\psi')^2 - 1 + 2\cos \theta \psi'}{1 + (\psi')^2}, 
A_6(\psi) = \frac{(\cos \theta + \sin \psi') \psi'' + (1 + (\psi')^2)\sin \theta \partial_{\eta} \psi'}{1 + (\psi')^2}, 
A_7(\psi) = \frac{\psi'}{1 + (\psi')^2}. 
\]
Lemma 5.2. For $0 \leq m + k \leq 2$, it holds that

$$\|\chi \partial_r^m \partial_n^{k+1} \rho\|_{L^2}^2 + \int_0^t (\|\chi \partial_r^m \partial_n^{k+1} \rho\|_{L^2}^2 + \|\chi \partial_r^m \partial_n^{k+1} \left(\frac{d\rho}{dt}\right)\|_{L^2}^2) \, ds \leq C\|D^{m+k+1} \rho(0)\|_{L^2}^2 + \delta_\psi \int_0^t \|\chi \partial_r^{m+1} \partial_n^k \rho\|_{L^2}^2 \, ds$$

$$+ C \int_0^t (\|\chi \partial_r^{m+2} \partial_n^k u\|_{L^2}^2 + \|\chi \partial_r^{m+1} \partial_n^{k+1} u\|_{L^2}^2) \, ds$$

$$+ C \int_0^t (\|u\|_{H^{m+k}}^2 + \|u\|_{H^{m+k+1}}^2 + \|v\|_{H^{m+k}}^2 + \|D\rho\|_{H^{m+k-1}}^2 + \delta_{m+k} \|D\rho\|_{H^{m+k+1}}^2) \, ds$$

$$+ C \int_0^t (\|F_1\|_{H^{m+k+1}}^2 + \|F_4\|_{H^{m+k}}^2) \, ds,$$

where $\delta_\psi$ is given in Section 3, $\delta_{m+k} = 1$ if $m+k \geq 1$, else $\delta_{m+k} = 0$.

Proof. First we take $m = k = 0$. Multiplying (30) by $\chi^2 \partial_\eta \rho$ and $\chi^2 \partial_\eta (\frac{d\eta}{dt})$ respectively, integrating with respect to time and spatial variable, we have

$$\|\chi \partial_\eta \rho\|_{L^2}^2 + \int_0^t \|\chi \partial_\eta \rho\|_{L^2}^2 \, ds$$

$$\leq C\|D\rho(0)\|_{L^2}^2 + \delta_\psi \int_0^t \|\chi \partial_\eta \rho\|_{L^2}^2 \, ds + C \int_0^t (\|\chi \partial_\eta^2 u\|_{L^2}^2 + \|\chi \partial_\eta \partial_\eta u\|_{L^2}^2) \, ds$$

$$+ C \int_0^t (\|u\|_{L^2}^2 + \|u\|_{H^1}^2 + \|v\|_{H^1}^2 + \|F_3\|_{H^1}^2 + \|F_4\|_{L^2}^2) \, ds$$

and

$$\|\chi \partial_\eta \rho\|_{L^2}^2 + \int_0^t \|\chi \partial_\eta (\frac{d\rho}{dt})\|_{L^2}^2 \, ds$$

$$\leq C\|D\rho(0)\|_{L^2}^2 + \delta_\psi \int_0^t \|\chi \partial_\eta \rho\|_{L^2}^2 \, ds + C \int_0^t (\|\chi \partial_\eta^2 u\|_{L^2}^2 + \|\chi \partial_\eta \partial_\eta u\|_{L^2}^2) \, ds$$

$$+ C \int_0^t (\|u\|_{L^2}^2 + \|u\|_{H^1}^2 + \|v\|_{H^1}^2 + \|F_3\|_{H^1}^2 + \|F_4\|_{L^2}^2) \, ds.$$
Multiplying the equation (32) by $\chi^2 \partial_t \partial_t \rho$ and $\chi^2 \partial_t (\frac{d\rho}{dt})$ respectively and then integrating with respect to time and spatial variable, one gets

$$\|\chi \partial_t \partial_t \rho\|_{L^2_t}^2 + \int_0^t \|\chi \partial_t \partial_t \rho\|_{L^2_t}^2 ds \leq C\|D^2 \rho(0)\|_{L^2_t}^2 + \delta_0 \int_0^t \|\chi \partial_t^2 \rho\|_{L^2_t}^2 ds + C\int_0^t (\|\chi \partial_t^2 \mathbf{u}\|_{L^2_t}^2 + \|\chi \partial_t^2 \partial_t \mathbf{u}\|_{L^2_t}^2) ds$$

and

$$\|\chi \partial_t \partial_t \rho\|_{L^2_t}^2 + \int_0^t \|\chi \partial_t \partial_t (\frac{d\rho}{dt})\|_{L^2_t}^2 ds \leq C\|D^2 \rho(0)\|_{L^2_t}^2 + \delta_0 \int_0^t \|\chi \partial_t^2 \rho\|_{L^2_t}^2 ds + C\int_0^t (\|\chi \partial_t^2 \mathbf{u}\|_{L^2_t}^2 + \|\chi \partial_t^2 \partial_t \mathbf{u}\|_{L^2_t}^2) ds$$

Therefore, for $m = 1, k = 0$, it has that

$$\|\chi \partial_t \partial_t \rho\|_{L^2_t}^2 + \int_0^t (\|\chi \partial_t \partial_t \rho\|_{L^2_t}^2 + \|\chi \partial_t \partial_t (\frac{d\rho}{dt})\|_{L^2_t}^2) ds \leq C\|D^2 \rho(0)\|_{L^2_t}^2 + \delta_0 \int_0^t \|\chi \partial_t^2 \rho\|_{L^2_t}^2 ds + C\int_0^t (\|\chi \partial_t^2 \mathbf{u}\|_{L^2_t}^2 + \|\chi \partial_t^2 \partial_t \mathbf{u}\|_{L^2_t}^2) ds$$

Now we take $m = 0, k = 1$. Using $\partial_t$ to (30), we have

$$\frac{(\mu/\bar{h}) \partial_t^2 (\frac{d\rho}{dt})}{\partial_t} + \bar{h} A_1(\psi) \partial_t \rho_{14} + \bar{h} A_2(\psi) \partial_t \rho_{15} - \mu A_1(\psi) \partial_t^2 \partial_t \rho_{14} + \mu A_3(\psi) \partial_t \partial_t \rho_{14} + \mu A_4(\psi) \partial_t^2 \partial_t \rho_{14} + \mu A_5(\psi) \partial_t \partial_t \rho_{14} + \mu A_6(\psi) \partial_t^2 \partial_t \rho_{14}$$

$$= \frac{\mu}{\bar{h}} \partial_{t}^2 F_3 + \bar{h} A_1(\psi) \partial_t F_{41} + \bar{h} A_2(\psi) \partial_t F_{42} + \mathbf{R}_6.$$
and
\[
\|\chi \partial^2_{\eta} \rho\|_{L^2}^2 + \int_0^t \|\chi \partial^2_{\eta} \frac{d\rho}{dt}\|_{L^2}^2 ds
\]
\[
\leq C \|D^2 \rho(0)\|_{L^2}^2 + 2\delta \int_0^t \|\chi \partial_\tau \partial_\eta \rho\|_{L^2}^2 ds + C \int_0^t (\|\chi \partial^2_\tau \partial_\eta u\|_{L^2}^2 + \|\chi \partial_\tau \partial^2_\eta u\|_{L^2}^2) ds
\]
\[
+ C \int_0^t (\|u\|_{H^1}^2 + \|u\|_{H^2}^2 + \|v\|_{H^2}^2 + \|D\rho\|_{L^2}^2 + \|F_3\|_{H^2}^2 + \|F_4\|_{H^2}^2) ds.
\]
Combining the two inequalities, if \(m = 0, k = 1\), yields that
\[
\|\chi \partial^2_{\eta} \rho\|_{L^2}^2 + \int_0^t (\|\chi \partial^2_{\eta} \rho\|_{L^2}^2 + \|\chi \partial^2_{\eta} \frac{d\rho}{dt}\|_{L^2}^2) ds
\]
\[
\leq C \|D^2 \rho(0)\|_{L^2}^2 + 2\delta \int_0^t \|\chi \partial_\tau \partial_\eta \rho\|_{L^2}^2 ds + C \int_0^t (\|\chi \partial^2_\tau \partial_\eta u\|_{L^2}^2 + \|\chi \partial_\tau \partial^2_\eta u\|_{L^2}^2) ds
\]
\[
+ C \int_0^t (\|u\|_{H^1}^2 + \|u\|_{H^2}^2 + \|v\|_{H^2}^2 + \|D\rho\|_{L^2}^2 + \|F_3\|_{H^2}^2 + \|F_4\|_{H^2}^2) ds.
\]
Finally, we consider \(m + k = 2\). First we prove \(m = 2, k = 0\). Applying \(\partial_\tau\) to (32), we have
\[
(\mu/\bar{h}) \partial^2_{\eta} \partial_\eta (\frac{d\rho}{dt}) + \bar{h} A_1(\psi) \partial^2_{\tau} u_{14} + \bar{h} A_2(\psi) \partial^2_{\tau} u_{24} - \mu A_1(\psi) \partial^2_{\tau} u_1
\]
\[
+ \mu A_3(\psi) \partial^2_{\tau} \partial_\eta u_1 + \mu A_4(\psi) \partial^2_{\tau} \partial_\eta u_1 - \mu A_2(\psi) \partial^2_{\tau} u_2
\]
\[
+ \mu A_5(\psi) \partial^2_{\tau} \partial_\eta u_2 + \mu A_6(\psi) \partial^2_{\tau} \partial_\eta u_2
\]
\[
+ \bar{h}^2 (A_7(\psi) \partial^2_{\tau} v + \partial^2_{\tau} \partial_\eta v) + (1 + \bar{n}) \bar{h} (A_7(\psi) \partial^2_{\tau} \rho + \partial^2_{\tau} \partial_\eta \rho)
\]
\[
= (\mu/\bar{h}) \partial^2_{\eta} \partial_\eta F_3 + \bar{h} A_1(\psi) \partial^2_{\tau} F_{41} + \bar{h} A_2(\psi) \partial^2_{\tau} F_{42} + R_7.
\]
Multiplying (34) by \(\chi^2 \partial^2_{\eta} \partial_\eta \rho\) and \(\chi^2 \partial^2_{\eta} \partial_\eta (\frac{d\rho}{dt})\) respectively and then integrating with respect to time and spatial variable, one has
\[
\|\chi \partial^2_{\eta} \rho\|_{L^2}^2 + \int_0^t \|\chi \partial^2_{\eta} \partial_\eta \rho\|_{L^2}^2 ds
\]
\[
\leq C \|D^2 \rho(0)\|_{L^2}^2 + 2\delta \int_0^t \|\chi \partial^2_{\eta} \partial_\eta \rho\|_{L^2}^2 ds + C \int_0^t (\|\chi \partial^2_{\eta} \partial_\eta u\|_{L^2}^2 + \|\chi \partial^2_{\eta} \partial_\eta u\|_{L^2}^2) ds
\]
\[
+ C \int_0^t (\|u\|_{H^1}^2 + \|u\|_{H^2}^2 + \|v\|_{H^2}^2 + \|D\rho\|_{L^2}^2 + \|F_3\|_{H^2}^2 + \|F_4\|_{H^2}^2) ds.
\]
So for \( m = 2, k = 0 \), we obtain
\[
\| \chi \partial_{\tau}^2 \partial_n \|_{L^2}^2 + \int_0^t \left( \| \chi \partial_{\tau}^2 \partial_n \|_{L^2}^2 + \| \chi \partial_{\tau}^2 \partial_n \left( \frac{d \rho}{dt} \right) \|_{L^2}^2 \right) ds \\
\leq C \| D^3 \rho(0) \|_{L^2}^2 + \delta_\nu \int_0^t \| \chi \partial_{\tau}^2 \partial_n | \|_{L^2}^2 ds + C \int_0^t \left( \| \chi \partial_{\tau}^2 \partial_n \|_{L^2}^2 + \| \chi \partial_{\tau}^2 \partial_n u \|_{L^2}^2 \right) ds \\
+ C \int_0^t \left( \| u \|_{H^2}^2 + \| v \|_{H^3}^2 + \| D \rho \|_{H^1}^2 + \| F_3 \|_{H^2}^2 + \| F_4 \|_{H^2}^2 \right) ds.
\]

Then we estimate \( m = k = 1 \). Utilizing \( \partial_\tau \) to (33), we have
\[
(\mu / \overline{h}) \partial_{\tau} \partial_{\tau}^2 \frac{d \rho}{dt} + \overline{h} A_1(\psi) \partial_{\tau} \partial_n u_{1t} + \overline{h} A_2(\psi) \partial_{\tau} \partial_n u_{2t} - \mu A_1(\psi) \partial_{\tau}^2 \partial_n u_1 \\
+ \mu A_3(\psi) \partial_{\tau}^2 \partial_n u_1 + \mu A_4(\psi) \partial_{\tau} \partial_n \partial_n u_1 - \mu A_2(\psi) \partial_{\tau}^2 \partial_n u_2 \\
+ \mu A_5(\psi) \partial_{\tau}^2 \partial_n u_2 + \mu A_6(\psi) \partial_{\tau} \partial_n \partial_n u_2 \\
+ \overline{h}^2 (A_1(\psi) \partial_{\tau} \partial_n v + \partial_{\tau} \partial_n \partial_n v) + (1 + \overline{h}) A_7(\psi) \partial_{\tau}^2 \partial_n \rho + \partial_{\tau} \partial_n \partial_n \rho \\
= (\mu / \overline{h}) \partial_{\tau} \partial_{\tau}^2 F_3 + \overline{h} A_1(\psi) \partial_{\tau} \partial_n F_{41} + \overline{h} A_2(\psi) \partial_{\tau} \partial_n F_{42} + \mathbf{R}_8.
\]

Multiplying (35) by \( \chi^2 \partial_{\tau} \partial_{\tau}^2 \rho \) and \( \chi^2 \partial_{\tau} \partial_{\tau} \left( \frac{d \rho}{dt} \right) \) respectively and then integrating with respect to time and spatial variable, we have
\[
\| \chi \partial_{\tau} \partial_{\tau}^2 \|_{L^2}^2 + \int_0^t \| \chi \partial_{\tau} \partial_{\tau} \|_{L^2}^2 ds \\
\leq C \| D^3 \rho(0) \|_{L^2}^2 + \delta_\nu \int_0^t \| \chi \partial_{\tau} \partial_{\tau} \|_{L^2}^2 ds + C \int_0^t \left( \| \chi \partial_{\tau} \partial_{\tau} \|_{L^2}^2 + \| \chi \partial_{\tau} \partial_{\tau}^2 u \|_{L^2}^2 \right) ds \\
+ C \int_0^t \left( \| u \|_{H^2}^2 + \| u \|_{H^3}^2 + \| v \|_{H^3}^2 + \| D \rho \|_{H^1}^2 + \| F_3 \|_{H^2}^2 + \| F_4 \|_{H^2}^2 \right) ds.
\]

and
\[
\| \chi \partial_{\tau} \partial_{\tau} \|_{L^2}^2 + \int_0^t \| \chi \partial_{\tau} \partial_{\tau} \left( \frac{d \rho}{dt} \right) \|_{L^2}^2 ds \\
\leq C \| D^3 \rho(0) \|_{L^2}^2 + \delta_\nu \int_0^t \| \chi \partial_{\tau} \partial_{\tau} \|_{L^2}^2 ds + C \int_0^t \left( \| \chi \partial_{\tau} \partial_{\tau} \|_{L^2}^2 + \| \chi \partial_{\tau} \partial_{\tau}^2 u \|_{L^2}^2 \right) ds \\
+ C \int_0^t \left( \| u \|_{H^2}^2 + \| u \|_{H^3}^2 + \| v \|_{H^3}^2 + \| D \rho \|_{H^1}^2 + \| F_3 \|_{H^2}^2 + \| F_4 \|_{H^2}^2 \right) ds.
\]

Combining the above two inequalities, for \( m = k = 1 \), we get
\[
\| \chi \partial_{\tau} \partial_{\tau}^2 \|_{L^2}^2 + \int_0^t \left( \| \chi \partial_{\tau} \partial_{\tau} \|_{L^2}^2 + \| \chi \partial_{\tau} \partial_{\tau} \left( \frac{d \rho}{dt} \right) \|_{L^2}^2 \right) ds \\
\leq C \| D^3 \rho(0) \|_{L^2}^2 + \delta_\nu \int_0^t \| \chi \partial_{\tau} \partial_{\tau} \|_{L^2}^2 ds + C \int_0^t \left( \| \chi \partial_{\tau} \partial_{\tau} \|_{L^2}^2 + \| \chi \partial_{\tau} \partial_{\tau}^2 u \|_{L^2}^2 \right) ds \\
+ C \int_0^t \left( \| u \|_{H^2}^2 + \| u \|_{H^3}^2 + \| v \|_{H^3}^2 + \| D \rho \|_{H^1}^2 + \| F_3 \|_{H^2}^2 + \| F_4 \|_{H^2}^2 \right) ds.
\]
Finally we take $m = 0, k = 2$. We apply $\partial_\eta$ to (33)

\[
(\mu/\bar{h})\partial_\eta^3 (\frac{d\rho}{dt}) + \bar{h}A_1(\psi)\partial^2_\eta u_{1t} + \bar{h}A_2(\psi)\partial^2_\eta u_{2t} - \mu A_1(\psi)\partial^2_\eta u_1 + \mu A_2(\psi)\partial^2_\eta u_2
\]

\[
+ \mu A_3(\psi)\partial_\eta^3 u_1 + \mu A_4(\psi)\partial_\eta^3 u_2 - \mu A_2(\psi)\partial^2_\eta u_2
\]

\[
+ \mu A_5(\psi)\partial_\eta^3 u_2 + \mu A_6(\psi)\partial^2_\eta u_2
\]

\[
+ \bar{h}^2(A_7(\psi)\partial_\eta^3 \rho + \partial^3_\eta \rho) + (1 + \bar{n})\bar{h}(A_7(\psi)\partial_\eta^2 \rho + \partial^2_\eta \rho)
\]

\[
= (\mu/\bar{h})\partial_\eta^3 F_3 + \bar{h}A_1(\psi)\partial_\eta^2 F_{41} + \bar{h}A_2(\psi)\partial^2_\eta F_{42} + R_9.
\]

Multiplying (36) by $\chi^2 \partial^2_\eta \rho$ and $\chi^2 \partial^2_\eta (\frac{d\rho}{dt})$ respectively and then integrating with respect to time and spatial variable, we obtain

\[
\left\| \chi \partial^3_\eta \rho \right\|^2_{L^2} + \int_0^t \left\| \chi \partial^3_\eta \rho \right\|^2_{L^2} ds
\]

\[
\leq C\left\| D^3 \rho(0) \right\|^2_{L^2} + \delta_0 \int_0^t \left\| \chi \partial_\eta \partial^2_\eta \rho \right\|^2_{L^2} ds + C \int_0^t \left( \left\| \chi \partial^2_\eta \partial_\eta^2 \rho \right\|_{L^2}^2 + \left\| \chi \partial_\eta \partial^2_\eta \rho \right\|_{L^2}^2 \right) ds
\]

\[
+ C \int_0^t \left( \left\| u \right\|_{H^2}^2 + \left\| u \right\|_{H^3}^2 + \left\| v \right\|_{H^2}^2 + \left\| D\rho \right\|_{H^2}^2 + \left\| F_3 \right\|_{H^3}^2 + \left\| F_4 \right\|_{H^2}^2 \right) ds
\]

and

\[
\left\| \chi \partial^3_\eta \rho \right\|^2_{L^2} + \int_0^t \left\| \chi \partial^3_\eta (\frac{d\rho}{dt}) \right\|^2_{L^2} ds
\]

\[
\leq C\left\| D^3 \rho(0) \right\|^2_{L^2} + \delta_0 \int_0^t \left\| \chi \partial_\eta \partial^2_\eta \rho \right\|^2_{L^2} ds + C \int_0^t \left( \left\| \chi \partial^2_\eta \partial_\eta^2 \rho \right\|_{L^2}^2 + \left\| \chi \partial_\eta \partial^2_\eta \rho \right\|_{L^2}^2 \right) ds
\]

\[
+ C \int_0^t \left( \left\| u \right\|_{H^2}^2 + \left\| u \right\|_{H^3}^2 + \left\| v \right\|_{H^2}^2 + \left\| D\rho \right\|_{H^2}^2 + \left\| F_3 \right\|_{H^3}^2 + \left\| F_4 \right\|_{H^2}^2 \right) ds.
\]

Through the two inequalities, for $m = 0, k = 2$, it is easy to obtain

\[
\left\| \chi \partial^3_\eta \rho \right\|^2_{L^2} + \int_0^t \left( \left\| \chi \partial^3_\eta \rho \right\|_{L^2}^2 + \left\| \chi \partial^3_\eta (\frac{d\rho}{dt}) \right\|_{L^2}^2 \right) ds
\]

\[
\leq C\left\| D^3 \rho(0) \right\|^2_{L^2} + \delta_0 \int_0^t \left\| \chi \partial_\eta \partial^2_\eta \rho \right\|^2_{L^2} ds + C \int_0^t \left( \left\| \chi \partial^2_\eta \partial_\eta^2 \rho \right\|_{L^2}^2 + \left\| \chi \partial_\eta \partial^2_\eta \rho \right\|_{L^2}^2 \right) ds
\]

\[
+ C \int_0^t \left( \left\| u \right\|_{H^2}^2 + \left\| u \right\|_{H^3}^2 + \left\| v \right\|_{H^2}^2 + \left\| D\rho \right\|_{H^2}^2 + \left\| F_3 \right\|_{H^3}^2 + \left\| F_4 \right\|_{H^2}^2 \right) ds.
\]

Combining all the cases of $m$ and $k$ completes the proof of Lemma 5.2.

In the following, we give the Stokes lemma. It is very important because we can get some derivatives with respect to $\mathbf{u}$ and $\rho$ at the same time by this lemma.

**Lemma 5.3.** [23] For the system

\[
\begin{align*}
\tilde{h} \text{div} \mathbf{u} &= g^0, \\
-\Delta \mathbf{u} + \nabla \rho &= g^1, \\
\mathbf{u}|_{\partial \Omega} &= 0,
\end{align*}
\]

where $g^0 \in H^{m+1}$ and $g^1 \in H^m$. Then the system (37) has a solution $(\rho, \mathbf{u})$ satisfying

\[
\left\| \mathbf{u} \right\|_{H^{m+2}}^2 + \left\| D\rho \right\|_{H^m}^2 \leq C\left( \left\| g^0 \right\|_{H^{m+1}}^2 + \left\| g^1 \right\|_{H^m}^2 \right).
\]
If we take $\partial_t^k$ to (7) and use the cut-off function $\chi$ on the resulting equation, through Lemma 5.3, we have

$$
\| \chi D^{2+m}\partial_t^k u \|_{L^2_x}^2 + \| \chi D^{1+m} \partial_t^{k+1} \|_{L^2_x}^2 \\
\leq C(\| D^{1+m} \partial_t^{k} (\frac{d\rho}{dt}) \|_{L^2_x}^2 + \| F_3 \|_{H^{m+1}}^2 + \| F_4 \|_{H^{m+1}}^2 ) \\
+ C(\| u \|_{H^{m+1}}^2 + \| \rho \|_{H^{m+1}}^2 + \| v \|_{H^{m+1}}^2 + \| u_t \|_{H^{m+1}}^2 ) .
$$

6. The proof of Theorem 1.1. Now we combine the Lemmas 3.1–5.3 to close a priori assumption.

**Lemma 6.1.** For the system (2)–(4), we have

$$
\| (v, c, \rho, u) \|_{H^3}^2 + \| (v_t, c_t, u_t) \|_{H^1}^2 + \| \rho_t \|_{H^2}^2 \\
+ \int_0^t \| (v, c, u) \|_{H^4}^2 + \| \rho \|_{H^3}^2 + \| (v_t, c_t, \rho_t, u_t) \|_{L^2_x}^2 ) ds \\
\leq C \| (v, c, \rho, u(0)) \|_{H^3}^2 + C \int_0^t (\| (F_1) \|_{L^2_x}^2 + \| (u\nabla \rho)t \|_{L^2_x}^2 ) ds \\
+ C \sup_{0 \leq s \leq t} \{ \| F_1 \|_{H^1}^2 + \| F_2 \|_{H^1}^2 + \| u\nabla \rho \|_{H^2}^2 + \| F_3 \|_{H^2}^2 + \| F_4 \|_{H^1}^2 \} \\
+ C \int_0^t (\| F_1 \|_{H^2}^2 + \| F_2 \|_{H^2}^2 + \| u\nabla \rho \|_{H^2}^2 + \| F_3 \|_{H^2}^2 + \| F_4 \|_{H^2}^2 ) ds \\
+ C \int_0^t (\| (F_1)t \|_{L^2_x}^2 + \| (F_2)t \|_{L^2_x}^2 + \| (F_3)t \|_{L^2_x}^2 + \| (F_4)t \|_{L^2_x}^2 ) ds .
$$

**Proof.** First we get the estimates for low order derivatives.

I. By Lemma 3.2, we have the basic estimate

$$
\| (v, c, \rho, u) \|_{L^2_x}^2 + \int_0^t \| (Dv, c, Dc, Du) \|_{L^2_x}^2 ds \\
\leq C \| (v, c, \rho, u)(0) \|_{L^2_x}^2 + \epsilon \int_0^t \| \rho \|_{L^2_x}^2 ds \\
+ C_\epsilon \int_0^t (\| F_1 \|_{L^2_x}^2 + \| F_2 \|_{L^2_x}^2 + \| F_3 \|_{L^2_x}^2 + \| u\nabla \rho \|_{L^2_x}^2 ) ds .
$$

II. By Lemma 3.5, for $l = 0$, we have the estimate

$$
\int_0^t \| \rho_t \|_{L^2_x}^2 ds \leq C \int_0^t (\| \text{div } u \|_{L^2_x}^2 + \| u\nabla \rho \|_{L^2_x}^2 + \| F_3 \|_{L^2_x}^2 ) ds
$$

and for $l = 1$, we have

$$
\| D\rho_t \|_{L^2_x}^2 + \int_0^t \| D\rho_t \|_{L^2_x}^2 ds \\
\leq C(\| D(\text{div } u) \|_{L^2_x}^2 + \| D(u\nabla \rho) \|_{L^2_x}^2 + \| DF_3 \|_{L^2_x}^2 ) \\
+ C \int_0^t (\| D(\text{div } u) \|_{L^2_x}^2 + \| D(u\nabla \rho) \|_{L^2_x}^2 + \| DF_3 \|_{L^2_x}^2 ) ds .
$$
III. By Lemma 3.3, we have the estimate

$$\|v(t, c, \rho, u_t)\|_{L^2}^2 + \int_0^t \|(Dv_t, c_t, Dc_t, Du_t)\|_{L^2}^2 ds$$

$$\leq C\|v(0, c_0)\|_{L^2}^2 + \epsilon \int_0^t \|\rho_t\|_{L^2}^2 ds$$

$$+ C\epsilon \int_0^t (\|\bar{F}_1\|_{L^2}^2 + \|\bar{F}_2\|_{L^2}^2 + \|\bar{F}_3\|_{L^2}^2 + \|\bar{F}_4\|_{L^2}^2) ds.$$ 

IV. By Lemma 4.1, for $l = 1$, we have

$$\|\chi_0 D\rho\|_{L^2}^2 + \int_0^t \|\chi_0 D\rho\|_{L^2}^2 ds$$

$$\leq C(\|D\rho(0)\|_{L^2}^2 + \|\bar{u}(0)\|_{L^2}^2 + \|\bar{u}\|_{L^2}^2)$$

$$+ C\int_0^t (\|Dv\|_{L^2}^2 + \|\bar{u}\|_{H^1}^2) ds + C\int_0^t (\|DF_3\|_{L^2}^2 + \|F_4\|_{L^2}^2) ds$$

and for $l = 2$, we have

$$\|\chi_0 D^2\rho\|_{L^2}^2 + \int_0^t \|\chi_0 D^2\rho\|_{L^2}^2 ds$$

$$\leq C(\|D^2\rho(0)\|_{L^2}^2 + \|D\bar{u}(0)\|_{L^2}^2 + \|D\bar{u}\|_{L^2}^2)$$

$$+ C\int_0^t (\|D^2v\|_{L^2}^2 + \|\bar{u}\|_{H^2}^2) ds + C\int_0^t (\|D^2F_3\|_{L^2}^2 + \|DF_4\|_{L^2}^2) ds.$$ 

V. By Lemma 4.2, for $l = 1$, we have

$$\|\chi_0 D(v, c, \rho, u)\|_{L^2}^2 + \int_0^t \|\chi_0 D^2(v, c, u)\|_{L^2}^2 ds$$

$$\leq C\|D(v, c, \rho, u)(0)\|_{L^2}^2 + C\int_0^t (\|D(v, c)\|_{L^2}^2 + \|\bar{u}\|_{H^1}^2 + \|\chi_0 D\rho\|_{L^2}^2) ds$$

$$+ C\int_0^t (\|F_1\|_{L^2}^2 + \|DF_2\|_{L^2}^2 + \|DF_3\|_{L^2}^2 + \|F_4\|_{L^2}^2) ds$$

and for $l = 2$, we have

$$\|\chi_0 D^2(v, c, \rho, u)\|_{L^2}^2 + \int_0^t \|\chi_0 D^3(v, c, u)\|_{L^2}^2 ds$$

$$\leq C\|D^2(v, c, \rho, u)(0)\|_{L^2}^2 + C\int_0^t (\|D^2(v, c)\|_{L^2}^2 + \|\bar{u}\|_{H^2}^2 + \|\chi_0 D^2\rho\|_{L^2}^2) ds$$

$$+ C\int_0^t (\|DF_1\|_{L^2}^2 + \|D^2F_2\|_{L^2}^2 + \|D^2F_3\|_{L^2}^2 + \|DF_4\|_{L^2}^2) ds.$$ 

VI. By Lemma 5.1, for $l = 1$, we have

$$\|\chi\partial_x (v, c, \rho, u)\|_{L^2}^2 + \int_0^t (\|\chi\partial_x^2 (v, c, u)\|_{L^2}^2 + \|\chi\partial_x\partial_y (v, c, u)\|_{L^2}^2) ds$$

$$\leq C\|D(v, c, \rho, u)(0)\|_{L^2}^2 + \epsilon \int_0^t (\|\chi\partial_x\rho\|_{L^2}^2 + \|\chi\partial_y\rho\|_{L^2}^2) ds$$
\[ C \int_0^t (\| (v, c, u) \|_{L^2}^2 + \|DF_3\|_{L^2}^2) \, ds \]
and for \( l = 2 \), we have
\[ \| \chi \partial_t^2 (v, c, \rho, u) \|_{L^2}^2 + \int_0^t (\| \chi \partial_t^2 (v, c, u) \|_{L^2}^2 + \| \chi \partial_t^2 \partial_n (v, c, u) \|_{L^2}^2) \, ds \]
\[ \leq C \| D^2 (v, c, \rho, u) (0) \|_{L^2}^2 + \delta \int_0^t (\| \chi \partial_t \rho \|_{L^2}^2 + \| \chi \partial_t \partial_n \rho \|_{L^2}^2) \, ds \]
\[ + \int_0^t \left( \| (v, c, u) \|_{H^2}^2 + \| D \rho \|_{L^2}^2 + \| D^2 F_3 \|_{L^2}^2 \right) \, ds \]
\[ + C \int_0^t \left( \| D F_1 \|_{L^2}^2 + \| D^2 F_2 \|_{L^2}^2 + \| D F_4 \|_{L^2}^2 \right) \, ds. \]

VII. By Lemma 5.2, for \( m = k = 0 \), we have
\[ \| \chi \partial_t \rho \|_{L^2}^2 + \int_0^t (\| \chi \partial_t \rho \|_{L^2}^2 + \| \partial_t \partial_n (\frac{d \rho}{dt}) \|_{L^2}^2) \, ds \]
\[ \leq C \| D \rho (0) \|_{L^2}^2 + \delta \int_0^t (\| \chi \partial_t \rho \|_{L^2}^2 + \| \chi \partial_t \partial_n \rho \|_{L^2}^2) \, ds \]
\[ + \int_0^t \left( \| \rho \|_{H^2}^2 + \| D \rho \|_{L^2}^2 + \| F_3 \|_{H^2}^2 + \| F_4 \|_{L^2}^2 \right) \, ds, \]
for \( m = 1, k = 0 \), we have
\[ \| \chi \partial_t \partial_n \rho \|_{L^2}^2 + \int_0^t (\| \chi \partial_t \partial_n \rho \|_{L^2}^2 + \| \partial_t \partial_n (\frac{d \rho}{dt}) \|_{L^2}^2) \, ds \]
\[ \leq C \| D^2 \rho (0) \|_{L^2}^2 + \delta \int_0^t (\| \chi \partial_t \partial_n \rho \|_{L^2}^2 + \| \chi \partial_t \partial_n \partial_n (\frac{d \rho}{dt}) \|_{L^2}^2) \, ds \]
\[ + \int_0^t \left( \| \rho \|_{H^2}^2 + \| D \rho \|_{L^2}^2 + \| F_3 \|_{H^2}^2 + \| F_4 \|_{H^2}^2 \right) \, ds, \]
and for \( m = 0, k = 1 \), we have
\[ \| \chi \partial_t^2 \rho \|_{L^2}^2 + \int_0^t (\| \chi \partial_t \rho \|_{L^2}^2 + \| \partial_t \partial_n (\frac{d \rho}{dt}) \|_{L^2}^2) \, ds \]
\[ \leq C \| D^2 \rho (0) \|_{L^2}^2 + \delta \int_0^t (\| \chi \partial_t \partial_n \rho \|_{L^2}^2 + \| \partial_t \partial_n \partial_n (\frac{d \rho}{dt}) \|_{L^2}^2) \, ds \]
\[ + \int_0^t \left( \| \rho \|_{H^2}^2 + \| D \rho \|_{L^2}^2 + \| F_3 \|_{H^2}^2 + \| F_4 \|_{H^2}^2 \right) \, ds. \]

VIII. By Lemma 5.3, we have
\[ \int_0^t (\| \rho \|_{H^2}^2 + \| D \rho \|_{L^2}^2) \, ds \leq C \int_0^t (\| \frac{d \rho}{dt} \|_{L^2}^2 + \| F_3 \|_{H^2}^2 + \| \rho \|_{L^2}^2 + \| D \rho \|_{L^2}^2) \, ds, \]
Now combining all the above inequalities, we have
\[
\int_0^t (\| x D^2 \partial_t u \|_{L^2}^2 + \| x D \partial_x \rho \|_{L^2}^2) \, ds \\
\leq C \int_0^t (\| x D \partial_t (\frac{d \rho}{dt}) \|_{L^2}^2 + C \int_0^t (\| u \|_{H^2}^2 + \| \rho \|_{H^1}^2 + \| v \|_{H^2}^2 + \| u_t \|_{H^1}^2) \, ds \\
+ C \int_0^t (\| F_3 \|_{H^2}^2 + \| F_4 \|_{H^1}^2) \, ds
\]
and
\[
\int_0^t (\| u \|_{H^3}^2 + \| D \rho \|_{H^1}^2) \, ds \leq C \int_0^t (\| \frac{d \rho}{dt} \|_{H^2}^2 + \| F_3 \|_{H^2}^2 + \| u_t \|_{H^1}^2 + \| D v \|_{H^2}^2 + \| F_4 \|_{H^1}^2) \, ds.
\]

IX. By Lemma 3.4, we have
\[
\| D v \|_{L^2}^2 + \int_0^t \| v_t \|_{L^2}^2 \, ds \\
\leq C (\| D v(0) \|_{L^2}^2 + C \int_0^t (\| D v_t \|_{L^2}^2 + \| D c \|_{L^2}^2 + \| D c \|_{L^2}^2 + \| \text{div} \ u \|_{L^2}^2 + \| F_1 \|_{L^2}^2) \, ds,
\]
\[
\| D c \|_{L^2}^2 + \int_0^t \| c_t \|_{L^2}^2 \, ds \leq C (\| D c(0) \|_{L^2}^2 + C \int_0^t (\| c_t \|_{L^2}^2 + \| F_2 \|_{L^2}^2) \, ds.
\]

X. By Lemma 3.1, we have
\[
\int_0^t \| D^2 c \|_{L^2}^2 \, ds \leq C \int_0^t (\| c_t \|_{L^2}^2 + \| c_t \|_{L^2}^2 + \| 2 \|_{L^2}^2) \, ds,
\]
\[
\int_0^t \| D^3 c \|_{L^2}^2 \, ds \leq C \int_0^t (\| D c_t \|_{L^2}^2 + \| D c \|_{L^2}^2 + \| D F_2 \|_{L^2}^2) \, ds,
\]
\[
\int_0^t \| D^2 v \|_{L^2}^2 \, ds \leq C \int_0^t (\| v_t \|_{L^2}^2 + \| \text{div} \ u \|_{L^2}^2 + \| D^2 c \|_{L^2}^2 + \| F_1 \|_{L^2}^2) \, ds,
\]
\[
\int_0^t \| D^3 v \|_{L^2}^2 \, ds \leq C \int_0^t (\| D v_t \|_{L^2}^2 + \| D (\text{div} \ u) \|_{L^2}^2 + \| D^3 c \|_{L^2}^2 + \| D F_1 \|_{L^2}^2) \, ds,
\]
\[
\| u \|_{H^2}^2 \leq C (\| u \|_{H^2}^2 + \| v \|_{H^2}^2 + \| \rho \|_{H^2}^2 + \| F_4 \|_{L^2}^2),
\]
\[
\| D^2 c \|_{L^2}^2 \leq C (\| c_t \|_{L^2}^2 + \| c_t \|_{L^2}^2 + \| F_2 \|_{L^2}^2),
\]
\[
\| D^2 v \|_{L^2}^2 \leq C (\| v_t \|_{L^2}^2 + \| \text{div} \ u \|_{L^2}^2 + \| D^2 c \|_{L^2}^2 + \| F_1 \|_{L^2}^2).
\]

Now combining all the above inequalities, we have
\[
\| (v, c, \rho, u) \|_{H^2}^2 + \| (v, c, \rho, u_t) \|_{H^2}^2 + \| \rho_t \|_{H^1}^2
\]
\[
+ \int_0^t (\| (v, c, u) \|_{H^3}^2 + \| \rho \|_{H^2}^2 + \| (v_t, c_t, \rho_t, u_t) \|_{H^1}^2) \, ds
\]
\[
\leq C (\| (v, c, \rho, u)(0) \|_{H^2}^2 + C \int_0^t (\| (\tilde{F}_1) \|_{L^2}^2 + \| (\text{div} \ u) \|_{L^2}^2) \, ds
\]
\[
+ C \sup_{0 \leq s \leq t} \{ \| F_3 \|_{L^2}^2 + \| F_2 \|_{H^2}^2 + \| F_3 \|_{H^1}^2 + \| \text{div} \ u \|_{H^1}^2 + \| F_4 \|_{L^2}^2 \}
\]
\[
+ C \int_0^t (\| F_1 \|_{H^1}^2 + \| F_2 \|_{H^2}^2 + \| F_3 \|_{H^2}^2 + \| \text{div} \ u \|_{H^1}^2 + \| F_4 \|_{H^1}^2) \, ds
\]
\[
+ C \int_0^t (\| (F_2) \|_{L^2}^2 + \| (F_3) \|_{L^2}^2 + \| (F_4) \|_{L^2}^2) \, ds.
\]

Repeating the above procedures, we can get the estimates for high order derivatives. 
\[\square\]
Lemma 6.2. For the nonlinear terms, we have

\[
\sup_{0 \leq s \leq t} \{ \| F_1 \|_{H^1}^2 + \| F_2 \|_{H^1}^2 + \| F_3 \|_{L^2}^2 + \| \mathbf{u} \nabla \rho \|_{H^2}^2 + \| F_4 \|_{H^1}^2 \} \\
+ \int_0^t \left( \| F_1 \|_{H^1}^2 + \| F_2 \|_{H^1}^2 + \| F_3 \|_{H^1}^2 + \| \mathbf{u} \nabla \rho \|_{H^2}^2 + \| F_4 \|_{H^1}^2 \right) ds \\
+ \int_0^t \left( \| (F_1)_t \|_{L^2}^2 + \| (F_2)_t \|_{L^2}^2 + \| (F_3)_t \|_{L^2}^2 + \| (F_4)_t \|_{L^2}^2 \right) ds \\
+ \int_0^t \left( \| (F_1)_t \|_{L^2}^2 + \| (\mathbf{u} \nabla \rho)_t \|_{L^2}^2 \right) ds \\
\leq C(N^2(0, t))^2.
\]

Proof. First we estimate \( F_1, (F_1)_t \) and \( (F_1)_t \).

\[
\sup_{0 \leq s \leq t} \| F_1 \|_{H^1}^2 \leq \sup_{0 \leq s \leq t} \{ \| \mathbf{u} \|_{H^2}^2 + \| \nabla c \|_{H^2}^2 \} \\
\leq \sup_{0 \leq s \leq t} \{ \| \nabla \|_{H^2}^2 \| \mathbf{u} \|_{H^3}^2 + \| \nabla \|_{H^2}^2 \| c \|_{H^3}^2 \} \leq C(N^2(0, t))^2,
\]

\[
\int_0^t \| F_1 \|_{H^1}^2 ds \leq \int_0^t \left( \| \mathbf{u} \|_{H^3}^2 + \| \nabla c \|_{H^2}^2 \right) ds \\
\leq \int_0^t \left( \| \mathbf{u} \|_{H^3}^2 \| \mathbf{u} \|_{H^3}^2 + \| \nabla \mathbf{u} \|_{H^2}^2 \| c \|_{H^3}^2 \right) ds \\
\leq \sup_{0 \leq s \leq t} \| \mathbf{u} \|_{H^3}^2 \int_0^t \| \mathbf{u} \|_{H^3}^2 + \| c \|_{H^3}^2 \right) ds \leq C(N^2(0, t))^2,
\]

\[
\int_0^t \| (F_1)_t \|_{L^2}^2 ds \leq \int_0^t \left( \| v_t \|_{L^2}^2 \| \mathbf{u} \|_{L^\infty}^2 + \| v_t \|_{L^\infty}^2 \| c_t \|_{L^2}^2 \right) ds \\
+ \int_0^t \left( \| v_t \|_{L^2}^2 \| \nabla c \|_{L^\infty}^2 + \| v_t \|_{L^\infty}^2 \| \nabla c_t \|_{L^2}^2 \right) ds \\
\leq \sup_{0 \leq s \leq t} \| v_t \|_{L^2}^2 \int_0^t \| \mathbf{u} \|_{H^2}^2 + \| c \|_{H^3}^2 \right) ds \\
+ \sup_{0 \leq s \leq t} \| v_t \|_{H^3}^2 \int_0^t \| \mathbf{u} \|_{L^2}^2 + \| c \|_{H^3}^2 \right) ds \leq C(N^2(0, t))^2,
\]

\[
\int_0^t \| (F_1)_t \|_{L^2}^2 ds \leq \int_0^t \left( \| \nabla v_t \|_{L^2}^2 \| \mathbf{u} \|_{L^\infty}^2 + \| v_t \|_{L^2}^2 \| \nabla \mathbf{u} \|_{L^\infty}^2 \right) ds \\
+ \int_0^t \left( \| \nabla v_t \|_{L^2}^2 \| \mathbf{u} \|_{L^\infty}^2 + \| v_t \|_{L^2}^2 \| \nabla \mathbf{u} \|_{L^\infty}^2 \right) ds \\
+ \int_0^t \left( \| \nabla v_t \|_{L^2}^2 \| \mathbf{u} \|_{L^\infty}^2 + \| v_t \|_{L^2}^2 \| \nabla \mathbf{u} \|_{L^\infty}^2 \| D^2 c_t \|_{L^2}^2 \right) ds \\
+ \int_0^t \left( \| \nabla v_t \|_{L^2}^2 \| \mathbf{u} \|_{L^2}^2 + \| v_t \|_{L^2}^2 \| D^3 c \|_{L^2}^2 \| D^2 c \|_{L^2}^2 + \| D^2 c \|_{L^2}^2 \right) ds \\
\leq C \sup_{0 \leq s \leq t} \| v_t \|_{H^3}^2 \int_0^t \left( \| \mathbf{u} \|_{H^3}^2 + \| c \|_{H^3}^2 \right) ds \\
+ C \sup_{0 \leq s \leq t} \| v_t \|_{H^3}^2 \int_0^t \left( \| \mathbf{u} \|_{H^3}^2 + \| c \|_{H^3}^2 \right) ds \leq C(N^2(0, t))^2.
\]
Then we consider the estimates for $F_2$ and $(F_2)_t$.

$$\sup_{0 \leq s \leq t} \|F_2\|_{H^2}^2 \leq \sup_{0 \leq s \leq t} \{ \|c u\|_{H^2}^2 + \|v c\|_{H^1}^2 \} \leq \sup_{0 \leq s \leq t} \{ \|c\|_{H^2}^2 \|u\|_{H^2}^2 + \|v\|_{H^2}^2 \|c\|_{H^3}^2 \} \leq C(N^2(0, t))^2,$$

$$\int_0^t \|F_2\|_{H^3}^2 ds \leq \int_0^t (\|c u\|_{H^4}^2 + \|v c\|_{H^3}^2) ds$$

$$\leq \int_0^t (\|c\|_{H^2}^2 \|u\|_{H^2}^2 + \|c\|_{H^3}^2 \|u\|_{H^3}^2) ds$$

$$+ \int_0^t (\|c\|_{H^2}^2 \|u\|_{H^2}^2 + \|v\|_{H^2}^2 \|c\|_{H^3}^2) ds$$

$$\leq C \sup_{0 \leq s \leq t} \|c\|_{H^2}^2 \int_0^t (\|u\|_{H^2}^2 + \|v\|_{H^2}^2) ds$$

$$+ C \sup_{0 \leq s \leq t} \|u\|_{H^2}^2 \int_0^t \|c\|_{H^2}^2 ds \leq C(N^2(0, t))^2,$$

$$\int_0^t \|(F_2)_t\|_{H^2}^2 ds \leq \int_0^t ((\|\nabla c\|_{L^2}^2 \|u\|_{L^\infty}^2 + \|c_t\|_{L^2}^2 \|\text{div} u\|_{L^\infty}^2) ds$$

$$+ \int_0^t ((\|\nabla c\|_{L^2}^2 \|u\|_{L^2}^2 + \|c\|_{L^\infty}^2 \|u\|_{L^2}^2) ds$$

$$+ \int_0^t ((\|v_t\|_{L^2}^2 \|c\|_{L^\infty}^2 + \|v\|_{L^\infty}^2 \|c\|_{L^2}^2) ds$$

$$\leq C \sup_{0 \leq s \leq t} \|c\|_{H^1}^2 \int_0^t (\|u\|_{H^2}^2 + \|v\|_{H^2}^2) ds$$

$$+ C(\sup_{0 \leq s \leq t} \|u_t\|_{H^2}^2 \int_0^t \|c\|_{H^2}^2 ds + \sup_{0 \leq s \leq t} \|v_t\|_{H^2}^2 \int_0^t \|c\|_{H^2}^2 ds)$$

$$\leq C(N^2(0, t))^2.$$  

Let us deal with the nonlinear term $u \nabla \rho$ and $(u \nabla \rho)_t$ as follows.

$$\sup_{0 \leq s \leq t} \|u \nabla \rho\|_{H^2}^2 \leq \sup_{0 \leq s \leq t} \|u\|_{H^2}^2 \|ho\|_{H^3}^2 \leq C(N^2(0, t))^2,$$

$$\int_0^t \|u \nabla \rho\|_{H^3}^2 ds \leq \int_0^t \|u\|_{H^2}^2 \|ho\|_{H^3}^2 ds$$

$$\leq \sup_{0 \leq s \leq t} \|u\|_{H^2}^2 \int_0^t \|ho\|_{H^3}^2 ds \leq C(N^2(0, t))^2,$$

$$\int_0^t \|(u \nabla \rho)_t\|_{L^2}^2 ds \leq \int_0^t \|u \nabla \rho\|_{L^2}^2 ds + \int_0^t \|u \nabla \rho_t\|_{L^2}^2 ds$$

$$\leq \sup_{0 \leq s \leq t} \|ho\|_{H^3}^2 \int_0^t \|u_t\|_{L^2}^2 ds + \sup_{0 \leq s \leq t} \|u\|_{H^2}^2 \int_0^t \|\rho_t\|_{H^1}^2 ds$$

$$\leq C(N^2(0, t))^2.$$  

Now we turn to $F_3$ and $(F_3)_t$.

$$\sup_{0 \leq s \leq t} \|F_3\|_{H^2}^2 \leq \sup_{0 \leq s \leq t} \|ho\|_{H^2}^2 \|u\|_{H^3}^2 \leq C(N^2(0, t))^2,$$
\[ \int_0^t \| F_3 \|^2_{L^2} ds \leq \int_0^t \| \rho \|^2_{H^1} \| u \|^2_{H^1} ds \]
\[ \leq \sup_{0 \leq s \leq t} \| \rho \|^2_{H^1} \int_0^t \| u \|^2_{H^1} ds \leq C(N^2(0, t))^2, \]
\[ \int_0^t \| (F_3)_t \|^2_{L^2} ds \leq \int_0^t (\| \rho_t \|^2_{H^1} \| \nabla u \|^2_{L^\infty} + \| \rho \|^2_{H^1} \| \nabla u \|^2_{L^2}) ds \]
\[ \leq \int_0^t (\| \rho_t \|^2_{H^1} \| u \|^2_{H^1} + \| \rho \|^2_{H^1} \| u \|^2_{H^1}) ds \]
\[ \leq \sup_{0 \leq s \leq t} \| \rho_t \|^2_{L^2} \int_0^t \| u \|^2_{H^1} ds + \sup_{0 \leq s \leq t} \| u \|^2_{H^1} \int_0^t \| \rho \|^2_{H^1} ds \]
\[ \leq C(N^2(0, t))^2. \]

Finally, we consider \( F_4 \) and \((F_4)_t\).
\[ \sup_{0 \leq s \leq t} \| F_4 \|^2_{H^1} \leq \sup_{0 \leq s \leq t} \{ \| u \cdot \nabla u \|^2_{H^1} + \| \rho v \|^2_{H^1} + \| \rho \Delta u \|^2_{H^1} \} \]
\[ \leq \sup_{0 \leq s \leq t} \{ \| u \|^2_{H^2} \| u \|^2_{H^3} + \| \rho \|^2_{H^1} \| v \|^2_{H^2} + \| \rho \|^2_{H^1} \| u \|^2_{H^1} \} \]
\[ \leq C(N^2(0, t))^2, \]
\[ \int_0^t \| F_4 \|^2_{H^2} ds \leq \int_0^t (\| u \cdot \nabla u \|^2_{H^1} + \| \rho v \|^2_{H^1} + \| \rho \Delta u \|^2_{H^1}) ds \]
\[ \leq \int_0^t (\| u \|^2_{H^2} \| u \|^2_{H^3} + \| \rho \|^2_{H^1} \| v \|^2_{H^2} + \| \rho \|^2_{H^1} \| u \|^2_{H^1}) ds \]
\[ \leq \sup_{0 \leq s \leq t} \| u \|^2_{H^2} \int_0^t \| u \|^2_{H^3} ds + \sup_{0 \leq s \leq t} \| v \|^2_{H^2} \int_0^t \| \rho \|^2_{H^2} ds \]
\[ + \sup_{0 \leq s \leq t} \| \rho \|^2_{H^2} \int_0^t \| u \|^2_{H^1} ds \]
\[ \leq C(N^2(0, t))^2, \]
\[ \int_0^t \| (F_4)_t \|^2_{L^2} ds \leq \int_0^t (\| u_t \|^2_{L^2} \| \nabla u \|^2_{L^\infty} + \| u_t \|^2_{L^\infty} \| \nabla u \|^2_{L^2} + \| \nabla u \|^2_{L^2} \| v_t \|^2_{L^2} + \| v_t \|^2_{L^2} \| \nabla v_t \|^2_{L^2}) ds \]
\[ + \int_0^t (\| \rho_t \|^2_{L^2} \| \nabla v \|^2_{L^\infty} + \| \nabla \rho \|^2_{L^\infty} \| v_t \|^2_{L^2} + \| v_t \|^2_{L^\infty} \| \nabla v_t \|^2_{L^2}) ds \]
\[ + \int_0^t (\| \rho_t \|^2_{L^\infty} \| \Delta u \|^2_{L^2} + \| \Delta \rho \|^2_{L^\infty} \| u_t \|^2_{L^2}) ds \]
\[ \leq C \sup_{0 \leq s \leq t} \| u_t \|^2_{H^1} \int_0^t \| u \|^2_{H^3} ds + C \sup_{0 \leq s \leq t} \| v_t \|^2_{H^1} \int_0^t \| \rho \|^2_{H^3} ds \]
\[ + C \sup_{0 \leq s \leq t} \| \rho_t \|^2_{H^2} \int_0^t (\| v \|^2_{H^3} + \| u \|^2_{H^3}) ds \]
\[ + C \sup_{0 \leq s \leq t} \| \rho \|^2_{H^2} \int_0^t \| u \|^2_{H^2} ds \leq C(N^2(0, t))^2. \]

Together all the estimates, we obtain (38). \( \square \)

Now we complete the proof of Theorem 1.1.
Proof. By Lemma 6.1 and Lemma 6.2, we establish the global existence

\[
\| (v, c, \rho, u) \|_{H^3}^2 + \| (v_t, c_t, u_t) \|_{H^1}^2 + \| \rho_t \|_{H^2}^2 \\
+ \int_0^t \left( \| (v, c, u) \|_{H^1}^2 + \| \rho \|_{H^3}^2 + \| (v_t, c_t, u_t) \|_{H^2}^2 \right) ds \leq C \| (v, c, \rho, u)(0) \|_{H^3}^2.
\]

(39)

For the decay of the solution, we multiply the equation (8) by \( e^{kt} \) and get

\[
\begin{cases}
\partial_t (e^{kt} v) + n \text{div} \ (e^{kt} u) - \Delta (e^{kt} v) + \tilde{n} \Delta (e^{kt} c) = e^{kt} F_1 + ke^{kt} v, \\
\partial_t (e^{kt} c) - \Delta (e^{kt} c) + \tilde{n} (e^{kt} c) = e^{kt} F_2 + ke^{kt} c, \\
\partial_t (e^{kt} \rho) + h \text{div} \ (e^{kt} u) + e^{kt} (u \nabla \rho) = e^{kt} F_3 + ke^{kt} \rho, \\
\partial_t (e^{kt} u) + h \nabla (e^{kt} v) + (1 + \tilde{n}) \nabla (e^{kt} \rho) - (\mu / \tilde{h}) \Delta (e^{kt} u) = e^{kt} F_4 + ke^{kt} u.
\end{cases}
\]

(40)

Repeating the whole process, if \( k \) is suitably small, similar to (39) we get estimates for \( e^{kt} (v, c, \rho, u) \)

\[
\| e^{kt} (v, c, \rho, u) \|_{H^3}^2 + \| \partial_t (e^{kt} (v, c, u)) \|_{H^1}^2 + \| \partial_t (e^{kt} \rho) \|_{H^2}^2 \\
+ \int_0^t \left( \| e^{kt} (v, c, u) \|_{H^1}^2 + \| e^{kt} \rho \|_{H^3}^2 + \| \partial_t (e^{kt} (v, c, \rho, u)) \|_{H^2}^2 \right) ds \\
\leq C \| (e^{kt} (v, c, \rho, u)(0)) \|_{H^3}^2.
\]

(41)

From (41), we can get the exponential decaying rate

\[
\| (v, c, \rho, u) \|_{H^3}^2 \leq Ce^{-kt}.
\]

(42)

\[ \square \]

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