Universal finite-size corrections of the entanglement entropy of quantum ladders and the entropic area law

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Abstract. We investigate the finite-size corrections of the entanglement entropy of critical ladders and propose a conjecture for its scaling behaviour. The conjecture is verified for free fermions, for Heisenberg ladders, and for quantum Ising ladders. Our results support that the prefactor of the logarithmic correction of the entanglement entropy of critical ladder models is universal and is associated with the central charge of the one-dimensional version of the models and with the number of branches associated with gapless excitations. Our results suggest that it is possible to infer whether there is a violation of the entropic area law in two-dimensional critical systems by analyzing the scaling behaviour of the entanglement entropy of ladder systems, which are easier to deal with.

Keywords: conformal field theory (theory), entanglement in extended quantum systems (theory)
1. Introduction

Entanglement is a very peculiar property of composite systems, which has intrigued physicists since the beginning of quantum mechanics. Entanglement is a fundamental ingredient of teleport quantum states and it is also an important key in quantum computation and quantum information [1]. Among the various quantifiers of entanglement, entanglement entropy (EE) is one of the most used, since it is sensitive to the long-distance quantum correlations of critical systems.

In the last few years, physicists working in distinct areas (such as quantum information, quantum field theory, and condensed matter) have made a great effort to understand the scaling behaviour of the EE of bipartite systems. In particular, the violation of the entropic area law has been a highly debated issue in recent years [2–16]. The EE of two composite subsystems $A$ and $B$ is defined as von Neumann entropy $S_A = -\text{Tr} \rho_A \ln \rho_A$, and is associated with the reduced density matrix $\rho_A = \text{Tr}_B \rho$. Since $S_A = S_B$, the information is shared only among the degrees of freedom localized around the surface (‘area’) separating both systems, due to this fact it is expected that the EE of a cube $A$ with side $N$ behaves as $S_A \sim N^{d-1}$, where $d$ is the dimension and $N^{d-1}$ is the boundary ‘area’ separating the regions $A$ and $B$. Indeed, this scaling behaviour is expected for gapped systems [17] and was also observed for some critical systems (see [13] and references therein). On the other hand, some models such as the one-dimensional critical systems [18], the free fermion systems with a finite Fermi surface in any dimension [7, 8], the two-dimensional (2D) Heisenberg model [19, 20], and the 2D conformal critical systems [21, 22] present beyond the $N^{d-1}$ correction a logarithmic term.

It is well known that the prefactor of the logarithmic correction of critical 1D systems of size $L$ is universal, and is associated with the central charge $c$ by the following
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Figure 1. Illustration of six-leg ladders divided into two entangled blocks. In (a) the subsystem \( \mathcal{A} \) is immersed in the middle of the system, while in (b) the subsystem \( \mathcal{A} \) is in the corner of the ladder. We also present the labels of the sites.

Equation [18]:

\[
S(L, \ell) = \frac{c}{3\eta} \ln \left[ \frac{\eta L}{\pi} \sin \left( \frac{\pi \ell}{L} \right) \right] + a, \quad (1)
\]

where \( \ell \) is the size of the subsystem \( \mathcal{A} \), \( a \) is a non-universal constant, and \( \eta = 1(2) \) is for the systems under periodic (open/fixed) boundary conditions. Note that other subleading corrections exist and are related to the scaling dimensions [23].

For any dimension \( d \), the following general behaviour is expected for the EE of a cube \( \mathcal{A} \) with side \( \mathcal{N} \) (see figure 1):

\[
S(\ell) = A \mathcal{N}^{d-1} + C(\mathcal{N}) \ln(\mathcal{N}) + B. \quad (2)
\]

In this work, we determine numerically \( C(\mathcal{N}) \) for some quantum ladders, finding that it is universal. The \( \mathcal{N} \)-leg ladders are characterized by \( \mathcal{N} \) parallel chains of size \( L \) coupled one to each other [24]. We denote the size of the ladders by \( \mathcal{N} \times L \). The \( \mathcal{N} \)-leg ladders are easier to deal with than two-dimensional systems, and can be used as a simple route to study the EE of two-dimensional systems. Here, we consider ladders composed of the following critical chains: free fermion chains, Heisenberg chains, and quantum Ising chains.

Although most of the works in the literature consider the subsystem \( \mathcal{A} \) immersed in a ‘reservoir’, as illustrated in figure 1(a), for ladder systems it is convenient to consider the subsystem \( \mathcal{A} \) in the corner of the ladders (see figure 1(b)). Our main aim is to present a conjecture to the scaling behaviour of the EE of critical ladders. Surprisingly, we verify that the finite-size corrections of the EE of quantum ladders are very similar to those of critical chains (equation (1)). Consider a ladder system composed of \( \mathcal{N} \) quantum chains of size \( L \), and let \( \ell \) be the number of sites of the block \( \mathcal{A} \) labelled as shown in figure 1(b). We propose that the scaling behaviour of the EE of critical ladders is given by

\[
S(\ell) = AN + \frac{c}{3\eta x} N_{gl} \ln \left[ \frac{\sin \left( \frac{\pi \ell}{N} \right)}{\sin \left( \frac{\pi}{L} \right)} \right] + B + \sum_{j=1}^{\left\lfloor \frac{\mathcal{N}}{2} \right\rfloor} a_j \cos \left( 2\pi \ell j / N \right), \quad (3)
\]

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where $c$ is the central charge (of the quantum chain used to build the ladders), $N_{gl}$ is the number of dispersion branches associated with the gapless excitations for a given energy, $\eta_x = 1 \, (2)$ for ladders under a periodic (open/fixed) boundary in the $x$ direction, and $A$, $B$, and $\alpha_j$ are non-universal constants. The last term in the above equation is an ansatz, which has been shown to be efficient for describing the oscillations of EE. The importance of the number of gapless modes in EE have been discussed in spin systems [25] and boson systems [26]. The above conjecture indicates that the prefactor of the logarithmic correction of the EE of critical ladders is universal, and it is related to the universality class of the critical behaviour of the chains that are used to build quantum ladders. Note that for gapped systems $N_{gl} = 0$ and the equation (3) suggests that the entropic area law holds in this case, as expected. Below, we present the results for the critical ladders that support our conjecture.

2. Free fermion ladders

Let us first consider a free-fermion ladder whose Hamiltonian is given by

$$H = \sum_{k_x, k_y} E(k_x, k_y) c_{k_x, k_y}^\dagger c_{k_x, k_y},$$

(4)

where the dispersion is $E(k_x, k_y) = -2 [\cos(k_x) + \cos(k_y)]$ and the sum is taken for all the wave numbers in the Brillouin zone. The momenta are given by $k_x = j_x 2\pi \frac{L}{L+1}$ and $k_y = j_y \frac{N}{L} \frac{\pi}{N+1}$ for periodic [open] boundary condition in $x$ and $y$ directions, respectively. The variables $j_x$ and $j_y$ are integers and their values depend on the boundary conditions.

In the case of free fermion systems it is possible to determine the EE for very large systems by using the correlation matrix method [27]. Note that in principle it is possible to use the Widom conjecture [6,28] to determine the prefactor that appears in the logarithmic correction (see for example [12]). However, we observe that this prefactor is easier to understand in terms of the number of gapless modes $N_{gl}$ that cross the Fermi level. For the sake of clarification, we show in figure 2 the band dispersions for the four-leg ladder as well as the values of $N_{gl}$ for some densities $\rho$. For the half-filling case with a periodic boundary condition (PBC) in the $x$ direction and an open boundary condition (OBC) in the $y$ direction, the number of gapless modes that cross the Fermi level is equal to the number of legs, i.e. $N_{gl} = N$ (for the other boundary conditions $N_{gl} \approx N$ for large values of $N$). So, based on our conjecture we expect that the EE for large values of $N$ and $L$ behaves as $S(\ell = NL/2) = AN + \frac{1}{6} N \ln(\frac{L}{\pi}) + B$, which suggests that the entropic area law is broken for the half-filling case. Indeed, this was observed in free fermion systems in two dimensions [5–7, 11, 29].

In figure 3(a), we present $S(\ell)$ as a function of $\ell$ for a cluster of size $4 \times 750$ with PBC [OBC] in the $x$ [$y$] direction and three values of densities. As we observe, the data obtained by the correlation matrix method agree perfectly with the conjecture proposed (equation (3)). In the fitting procedure, we used $c = 1$ (which corresponds to the central charge of the one-dimensional chain) and the values of $N_{gl}$ used were obtained counting the number of gapless modes that cross the Fermi level, as illustrated in figure 2. Similar agreements are found for several other ladders, as shown in figure 3(b).
Figure 2. The band dispersions of the four-leg free fermion ladders for different boundary conditions. The horizontal dashed lines indicate the positions of the Fermi levels for three values of densities $\rho$. We also indicate the values of $N_{gl}$ associated with each density. Note that some branches are degenerate.

In order to understand the contribution of the first term of equation (3), we present in figure 3(c) the EE for the $20 \times 60$ and the $40 \times 120$ clusters with a PBC [OBC] in the $x$ [$y$] direction at half-filling. As we can note, $S(\ell)$ grows linearly for $\ell \leqslant N$ and the logarithmic scaling is present only for $\ell \geqslant N$ (see the inset of figure 3(c)). If we impose an ansatz for $S(\ell)$ similar to the equation (1) and use the fact that $S(\ell)$ is continuous at $\ell = N$ (i.e. $AN + B = \frac{1}{\sin} N_{gl} \ln \left[ \frac{N}{\pi} \sin \left( \frac{\pi}{2} \right) \right] + a$) we realize that the EE must behave like equation (3). This is very interesting, since in principle we can obtain the prefactor $A$ by studying the behaviour of $S(\ell)$ for $\ell < N$, which is easier to obtain.

3. Heisenberg ladders

Now, let us consider the $N$-leg spin-$s$ Heisenberg ladders whose Hamiltonian is given by

$$H = J \sum_{i=1}^{N} \sum_{j=1}^{L-1} S_{i,j} \cdot S_{i,j+1} + J \sum_{i=1}^{N-1} \sum_{j=1}^{L} S_{i,j} \cdot S_{i+1,j},$$

where $S_{i,j}$ is the spin-$s$ operator at the $i$-th leg and $j$-th rung. We have set $J = 1$ to fix the energy scale. It is well known that the $N$-leg spin-$s$ Heisenberg ladders is gapless (gapped) if $sN$ is a semi-integer (integer) [24,30], see also [31] and the references therein.
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Figure 3. $S(\ell)$ versus $\ell$ for the free fermion ladders. (a) Results for a cluster $4 \times 750$ and three values of $\rho$. Inset: $S(\ell)$ for a few sites. In order to show all the data in the figure we added some constants with the values of $S$. (b) The data of the EE for several ladders at half-filling. From these fits we get $A = 0.56$, $B = 0.37$. The non-universal constants $a_j$ are small and vary from $-0.04$ to $-0.01$. (c) The results for the twenty- and forty-leg ladders at half-filling. In (a) and (b) the symbols are the data obtained by the correlation matrix method (see text) and the solid lines connect the fitted points by using our conjecture (equation (3)).

Here, we focus on the case of critical ladders, i.e. $sN$ is a semi-integer. For the Heisenberg ladders case, we obtained the EE numerically by using the DMRG [32]. For simplicity we consider only OBC in both directions. The spin-$s$ Heisenberg chains with semi-integer spins have a central charge $c = 1$ [33]. Moreover, based on the spin wave approximation it is expected that the dispersion of the 2D Heisenberg model has one Goldstone mode.
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Figure 4. (a) $S(\ell)$ for the Heisenberg ladders with spins $s = 1/2$ and $s = 3/2$. The symbols are the data obtained by using the density-matrix renormalization group (DMRG) and the solid lines connect the fitted points by using our conjecture (equation (3)) with $c = 1$ and $N_{gl} = 1$. From these fits we get $A = 0.27$ and $B = 0.16$ for $s = 1/2$. The inset shows $S(\ell)$ for few sites. (b) $S(\ell = NL/2) - AN - B$ versus $-1/6 \ln \left( \sin \left( \frac{\pi}{\ell} \right) \right)$ for several cluster sizes with $s = 1/2$.

$E(k) \sim \sqrt{k_x^2 + k_y^2}$. Since the number of legs $N$ is finite, the values of $k_y$ are discrete. Due to this fact, analogous to the free fermion case, there is just one dispersion branch ($E(k_x, 0) \sim |k_x|$) associated with gapless excitations, which crosses the energy of the ground state, i.e. $N_{gl} = 1$. In figure 4(a), we show the $S(\ell)$ as function of $\ell$ for the Heisenberg ladders with spins $s = 1/2$ and $s = 3/2$. Similar to the free fermion case, the equation (3) reproduces the scaling behaviour of $S(\ell)$ quite well if we use $c = 1$ and $N_{gl} = 1$. Note that in this case, our results suggest that a violation of the entropic area law is not expected in 2D systems. The EE for large values of $N$ and $L$ should behave as $S(\ell = NL/2) = AN + \frac{1}{6} \ln \left( \frac{\ell}{\pi} \right) + B$. In order to verify this, we present in figure 4(b) $S(\ell = NL/2) - AN - B$ as a function of $-1/6 \ln \left( \sin \left( \frac{\pi}{\ell} \right) \right)$. As we can see, the data strongly indicate that the prefactor of the logarithmic term is $1/6$ for the Heisenberg ladders when the subsystem is in the corner. Note that this result is intriguing, at least from the point of view of $N$ uncoupled chains under an OBC, which could suggest that the prefactor is $N/6$. Note that Monte Carlo simulations [19], as well as the DMRG results [20], show a similar behaviour for the scaling of the EE for another aspect ratio.

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Figure 5. (a) Finite-size estimates of the critical point, $\lambda_c(N, L)$, as a function of $1/L$ for the two- and three-leg Ising ladders. Inset: $\lambda_c^N$ versus $1/N$. (b) $S(\ell)$ versus $\ell$ for three values of $N$ at the critical points. The symbols are the DMRG results and the solid lines connect the fitted points by using our conjecture (equation (3)) with $c = 1/2$ and $N_{gl} = 1$. In order to show all the data in this figure we added some constants in the values of $S$. The inset shows $S(\ell)$ for few sites.

4. Quantum Ising ladders

Finally, let us consider the $N$-leg quantum Ising ladders whose Hamiltonian is given by:

$$H = \sum_{i=1}^{N} \sum_{j=1}^{L-1} \sigma^x_{i,j} \sigma^x_{i,j+1} + \sum_{i=1}^{N-1} \sum_{j=1}^{L} \sigma^x_{i,j} \sigma^x_{i+1,j} + \lambda \sum_{i=1}^{N} \sum_{j=1}^{L} \sigma^z_{i,j},$$

where $\sigma^x_{i,j}$ and $\sigma^z_{i,j}$ are Pauli matrices at the $i$-th leg and $j$-th rung. The one-dimensional case, i.e. $N = 1$, has a critical point at $\lambda_c = 1$ and its critical behaviour is described by a conformal field theory with central charge $c = 1/2$. In order to test the validity of equation (3) for the Ising ladders, we have first to determine the critical values of $\lambda_c^N$ for each value of $N$. First, we get the finite-size estimates of $\lambda_c(N, L)$ using the EE, as reported in [34]. Then, we assume that $\lambda_c(N, L)$ behaves as $\lambda_c(N, L) = \lambda_c(N) + a/L + b/L^2$, and finally we fit the data to obtain $\lambda_c(N)$. As an illustration of this, we present in figure 5(a) $\lambda_c(N, L)$ as a function of $1/L$ for the two and three-leg Ising ladders. By fitting our data we obtained $\lambda_c(N) = 1.838, 2.219, 2.443, 2.578$, and $2.670$ for $N = 2, 3, 4, 5$ and $6$, respectively. It is

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interesting to note that if we extrapolate these estimates to obtain $\lambda_c(\infty)$, as reported in the inset of figure 5(a), we obtain $\lambda_c^{2D} = \lambda_c(\infty) = 3.1$, which is close to the estimates of the critical point of the two-dimensional quantum Ising model obtained by Monte Carlo [35] ($\lambda_c^{2D} = 3.044$) and by the multiscale entanglement renormalization ansatz [36] ($\lambda_c^{2D} = 3.07$). The small discrepancy between our estimate and the last ones is very probably associated with the small lattice sizes considered to extrapolate our data.

As in the Heisenberg model, it is expected that $N_{gl} = 1$ is for the critical Ising ladders, and we do not anticipate a violation of the entropic area law for the two-dimensional quantum Ising model. The EE should behave, at the critical point, as $S(\ell = NL/2) = AN + \frac{1}{12} \ln\left(\frac{L}{\pi}\right) + B$, for an OBC in both directions. In figure 5(b), we present the EE of the Ising ladders at the critical points acquired by the DMRG for $N = 2, 3$ and $N = 4$. As shown in this figure, the conjecture proposed (equation (3)) also reproduces the scaling behaviour of the EE of the critical Ising ladders quite well.

5. Conclusions

We present an ansatz (equation (3)) for the finite-size corrections of the entanglement entropy of critical ladders. We verify that the ansatz is able to reproduce the scaling behaviour of the entanglement entropy of some critical ladders quite well, namely: Free fermion ladders, Heisenberg ladders, and Ising ladders. The preliminary results of the quantum $q = 3$ Potts ladders (not shown) also corroborate the scaling behaviour of the entanglement entropy proposed. All these results support that the prefactor of the logarithmic correction of the critical ladders is universal and is related to the central charge of the one-dimensional version of the model, as well as the number of branches associated with gapless excitations. Note that equation (3) is valid for $L \gg N$ and only when the subsystem $\mathcal{A}$ is considered in the corner of the ladder. One unsolved puzzle is to find the exact value of the prefactor of the logarithmic term, when the subsystem $\mathcal{A}$ is immersed in the middle of the ladders.

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