Bayesian Nonparametric Space Partitions: A Survey

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Abstract
Bayesian nonparametric space partition (BNSP) models provide a variety of strategies for partitioning a \(D\)-dimensional space into a set of blocks. In this way, the data points lie in the same block would share certain kinds of homogeneity. BNSP models can be applied to various areas, such as regression/classification trees, random feature construction, relational modeling, etc. In this survey, we investigate the current progress of BNSP research through the following three perspectives: models, which review various strategies for generating the partitions in the space and discuss their theoretical foundation ‘self-consistency’; applications, which cover the current mainstream usages of BNSP models and their potential future practises; and challenges, which identify the current unsolved problems and valuable future research topics. As there are no comprehensive reviews of BNSP literature before, we hope that this survey can induce further exploration and exploitation on this topic.

1 Introduction
Many machine learning algorithms can be boiled down into exploring implicit and complex relationships between different features (dimensions) of data points. Among various approaches, Bayesian Nonparametric Space Partition (BNSP) models provide an interesting and geometrically interpretable way to describe these highly expressive relationships. In general, the main idea of BNSP is to use various partition strategies to tailor a \(D\)-dimensional \((D \geq 2)\) space into a number of ‘blocks’. On one hand, the partition model can thus fit the data points using these blocks such that the data within each block exhibit certain types of label homogeneity; on the other hand, we can choose an arbitrarily fine resolution of partition (i.e. arbitrary number of blocks in the space) according to the number of data points and their relationships, such that the data points can be fitted reasonably well.

Those classical Bayesian nonparametric priors, such as the Dirichlet process [Ferguson, 1973] and its explicit form, the stick-breaking process [Sethuraman, 1994], are often applied only in one-dimensional-space problem settings. For BNSP models, with various definitions of the spaces and modeling objectives, they have found many successful examples in real-world applications. For example, when the space is spanned by data features and the modeling objective is the labels attached to the data points, BNSP models can be applied in regression/classification trees [Chipman et al., 2010; Lakshminarayanan et al., 2014; Linero, 2018; Fan et al., 2019b] and online learning [Lakshminarayanan et al., 2014; Lakshminarayanan et al., 2016; Mourtada et al., 2017; Fan et al., 2020]; when the space is spanned by the communities in a social network setting and the modeling objective is the linkages between the nodes in the network, BNSP models can be involved in relational modeling [Kemp et al., 2006; Airoldi et al., 2009; Roy and Teh, 2009; Miller et al., 2009; Nakano et al., 2014a; Wang et al., 2015; Fan et al., 2016a; Fan et al., 2018a] and community detection [Nowicki and Snijders, 2001; Karrer and Newman, 2011]. Other examples of applications include random feature construction [Balog et al., 2016], voice recognition [Nakano et al., 2014b], co-clustering [Wang et al., 2011], and matrix permutation approximation [Kuck et al., 2019]. Also, BNSP models can be potentially used in the areas of spatial-temporal modeling, image detection and segmentation.

In this survey, we will cover the following aspects of BNSP: (1) models, in which we will first introduce an important property in defining the model – self-consistency, and then review various strategies for generating the partitions in the space, including grid-style partitions, hierarchical partitions, rectangular tiling partitions, rectangular bounding partitions, and other partition strategies. We will also summarize and compare the characteristics of all these approaches as no single strategy will dominate the others in all cases; (2) applications, in which we will show the approaches of applying BNSP models in the real-world studies of classification/regression trees (including online learning), random feature construction and relational modeling; (3) challenges & future work, in which we will provide insightful discussions of the current challenges for BNSP, including scalable inference methods, flexibility of partitions and posterior concentration analysis.

2 Problem Statement
The aim of BNSP models is to generate partitions in a \(D\)-dimensional \((D \geq 2)\) space. The partition result (denoted
as \( \mathbb{D} \) is defined as a set of blocks, \( \mathbb{D} = \{ \square^{(k)} \}_{k \in \mathbb{N}^+} \). For \( N \) data points \( \{(x_n, y_n)\}_{n=1}^{N} \), where \( x_n \) and \( y_n \) refer to the feature and label of the \( n \)-th data point respectively, we have \( x_n \in \square_k \) if the \( n \)-th data point belongs to the \( k \)-th block \( \square_k \). It is expected that the labels for the data points in the same block, i.e. \( \{y_n : x_n \in \square_k\} \), would exhibit some kinds of homogeneity. Here are two formulations of representative BNSP applications:

**Example 1: Classification tree for credit risk modeling.** Credit risk modeling task involves the risk assessment for \( N \) people. Specifically, \( x_n \) and \( y_n \) respectively refer to the attribute and credit risk level of the \( n \)-th person. The block \( \square_k \) may specify a rectangular box in the feature space as: \([20, 25]\) years old \( \times \)$2000, $3000] monthly expense \( \times \)$4000, $4500] monthly income. Individuals located in this \( \square_k \) are assumed to follow the same categorical distribution in the risk levels and get the same risk label.

**Example 2: Relational modeling for link prediction.** Each data point of the relational modeling task refers to the linkage information of entities, with the value of the link (e.g. \( R_{ij} \), between person \( i \) and person \( j \)) as label. Each entity is associated with one latent covariate (e.g. uniformly distributed variable \( u_i \) for person \( i \)). The feature information for \( R_{ij} \) is thus a pair of random variables \( [u_i, u_j] \), which concatenates the latent covariates of entities \( i \) and \( j \). The block is defined by community-by-community interactions and the links locating in the same block would follow the same Bernoulli distribution.

### 3 Partition Strategies

We summarize the partition strategies for BNSP into 5 categories and illustrate how they work in the 2-dimensional space using 7 strategies as examples in Figure 2. Before reviewing these existing partition strategies, we would like to introduce an important property ‘self-consistency’ which underpins these BNSP processes.

**Self-consistency** The main idea of self-consistency is to ensure that the probability of partition keeps invariant under the expansion or shrinkage of the space.

More formally, assume we have a partition \( \mathbb{D}_Y \) of a space \( Y \) sampled from a BNSP. When restricting the BNSP process to a sub-space \( X \), where \( X \subset Y \in \mathcal{F}(\mathbb{R}^D) \) and \( \mathcal{F}(\mathbb{R}^D) \) denotes the collection of all finite boxes in \( \mathbb{R}^D \), the resulting partition of \( \mathbb{D}_Y \) restricted to \( X \) should distribute as if it is generated through the BNSP process working directly on \( X \). Figure 1 illustrates the self-consistency of the Rectangular Boundaries Process (RBP) [Fan et al., 2018b], which will be reviewed in detail later. A BNSP process is defined to be self-consistent if and only if the probability of a partition on \( X \) equals to the sum of probabilities of all possible partitions extended to space \( Y \), which can be denoted by \( p_X(x) = \sum_{Y \supseteq X} p_Y(y) \).

![Figure 1: Illustration of self-consistency of the Rectangular Boundary Process (RBP).](image)

**Space expansion**

- \( p^X( ) = p^Y( ) + p^Y( ) + \ldots \)

**Space restriction**

\( X \subset \mathbb{D} \)

\( Y \supseteq \mathbb{D} \)

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3.1 Grid-Style Partition

Grid-style BNSP models use the crossover of each dimension’s partition to divide the space such that the generated blocks of the partition exhibit grid patterns (Figure 2 (a–c)).

(a) **Regular-grid partition** originates from the one-dimensional partition (e.g. the stick-breaking process) and it is an earlier development of the BNSP models. For a \( D \)-dimensional space, the model is composed of \( D \) independent stick-breaking processes, each of which corresponds to the partition of one dimension. The orthogonal crossover of the partition on these dimensions produces regular-grid blocks in the space. Due to the self-consistency of the stick-breaking process in each dimension, the regular-grid partition is self-consistent.

The regular-grid partition is an over-simplified strategy, as it ignores the dependency among dimensions and generates partitions of each dimension independently. This strategy is very likely to generate trivial and undesired blocks in sparse regions. Currently, this approach is mainly used in the application of relational modeling in 2-dimensional space, including the Infinite Relational Model (IRM) [Kemp et al., 2006], which is an infinite-state variant of the Stochastic Block Model (SBM) [Nowicki and Snijders, 2001]. Other models in this category include dynamic IRMs [Ishiguro et al., 2010] for modeling dynamic relational data and multi-membership relational modeling [Schmidt and Mørup, 2013].

(b) **Copula regular-grid partition** (cRG) [Fan et al., 2016b] introduces copula functions [Nelsen, 2007] to fully describe the dimensional dependency. We can use various forms of copula functions to capture different dependencies. From the BNSP perspective, the copula function can be integrated out to provide some kind of ‘curved’ crossover partition. Since the copula function keeps the marginal distribution of each dimension’s partition invariant, the marginal distribution of
cRG in any dimension still follows the stick-breaking process. As a result, the self-consistency of cRG is guaranteed. However, the appropriate choice of copula function remains a challenge of cRG. Moreover, although the main benefit of copula function is to describe the heavy-tail dependency, the cRG has not taken advantage of this heavy-tail description capability.

(c) **Deep regular-grid partition** [Fan et al., 2019a] introduces a layer-wise deep architecture for the partition of each dimension. In a deep regular-grid partition, the Dirichlet distribution is placed for the partitions of each layer, and the partitions of current layer are further used as concentration parameters in the partition of the next layer. Further, it engages the graph convolutional networks in the layer-wise connection, such that the nodes would only propagate its information to its connected nodes in the next layer. In order to obtain the self-consistency for deep regular-grid partitions, we can easily extend the Dirichlet distribution to its infinite number of states case (i.e. stick-breaking process).

### 3.2 Hierarchical Partition

Hierarchical partitions follow a top-down strategy to recursively cut an existing block into two new blocks. In this way, the blocks are organized in a binary tree, which displays the hierarchical partition structure (Figure 2(d–e)).

(d) **Hierarchical partition with axis-aligned cuts** The Mondrian process (MP) [Roy et al., 2007; Roy and Teh, 2009; Roy, 2011] is a representative work of hierarchical partition. In general, an MP recursively generates axis-aligned cuts on a unit hypercube and divides the space in a hierarchical fashion known as kd-tree. The kd-tree construction process is regulated by attaching exponentially distributed cost to each axis-aligned cut, and the process is terminated when the accumulated sum of cost exceeds a provided budget value. Further, since the MP can partially consider inter-dimensional dependency, it can produce fewer trivial blocks. MP keeps the self-consistent property by carefully modeling the relationship between the cut cost and the generation of axis-aligned cuts.

(e) **Hierarchical partition with sloped cuts** Another group of tree-structured hierarchical partition models, which considers sloped cuts for cutting the space, includes the Ostomachion process [Fan et al., 2016a], the Binary Space Partition-Tree (BSP-Tree) process [Fan et al., 2018a], and Random Tessellation Forests (RTF) [Ge et al., 2019]. In contrast to the MP producing axis-aligned partitions, the Ostomachion process and BSP-Tree process consider inter-dimensional dependency to generate sloped cuts and form convex polygon-shaped blocks in the 2-dimensional space. The Binary Space Partition Forests [Fan et al., 2019b] extend the sloped cuts from 2-dimensional spaces to D-dimensional spaces ($D \geq 2$) and produce convex-polyhedron blocks, with the restriction that the cutting hyperplane is parallel to $D - 2$ dimensions. Random Tessellation Forests are proposed to generate arbitrary sloped cutting hyperplanes in D-dimensional spaces. These sloped cuts concentrate more on describing the inter-dimensional dependency and the model claims to produce partitions more efficiently in the space. Recent work generalizes cut directions efficiently using the technique of iteration stable (STIT) tessellations [O’Reilly and Tran, 2020]. Except the Ostomachion process, all the above hierarchical partitions with sloped cuts possess the self-consistency property, which can be proved by following the proof procedure in [Roy and Teh, 2009].

### 3.3 Rectangular Tiling Partition

(f) **Rectangular tiling partition** (RTP) [Nakano et al., 2014a] produces a flat partition structure on a 2-dimensional array. In general, it uses the geometric distribution to generate the length of block, with the constraint that the length does not violate the rectangular restriction of the other existing blocks. In order to avoid generating small and trivial blocks, RTP further incorporates the MP [Roy and Teh, 2009] to generate blocks of moderate size. By relaxing the restriction of regular-grid or hierarchical structures, RTP desires to provide more flexible partitions. The RTP keeps self-consistency due to the memoryless property of the geometric distribution. However, there are two weaknesses of RTP. First, the posterior inference of RTP is complicated, which makes it difficult to be applied in real-world settings. Second, RTP is restricted to discrete space (i.e., multi-dimensional arrays) and can only be used in the relational modeling application, while the partition strategies discussed above can be applied to both continuous space and multi-dimensional arrays (with trivial modifications).

### 3.4 Rectangular Bounding Partition

(g) **Rectangular bounding partition** In contrast to the cutting-based strategies (including grid-style partitions and hierarchical partitions), the Rectangular Bounding Process (RBP) [Fan et al., 2018b] uses a bounding strategy to par-
Bayesian plaid models \cite{Miller2016}. In particular, RBP requires a budget parameter $\tau$ to control the number of bounding boxes $K_\tau$, which follows a Poisson distribution, and a length parameter $\lambda$, which follows an Exponential distribution, to mainly control the size and location of the bounding boxes (the occupation of a box in each dimension is closely related to the Exponential distribution $\text{Exp}(\lambda)$). The RBP also keeps self-consistency, and can generate more bounding boxes given a larger budget or a larger space.

The most important advantage of RBP is the parsimonious partition of space. Data points often do not distributed evenly in the entire space, but cluster in local regions. The cutting-based strategies such as grid-style and hierarchical partitions would inevitably produce too many cuts for the sparse regions with few data points while trying to fit data in the dense regions. Recall the credit risk modeling problem in Example 1, where the feature space is composed of ‘age’ and ‘salary’. The traditional cutting-based models may inevitably cut the regions of young age and very high salary, even if there are very few people in those regions. On the contrary, RBP can place more bounding boxes to those important regions and fewer to those sparse and noisy regions, as the bounding boxes are generated independently. As a result, RBP can well balance fitness and parsimony of the partition.

### 3.5 Other Un-self-consistent Partitions

In terms of the space-partition models without the self-consistency property, we would like to discuss three of them. Bayesian plaid models \cite{Miller2009, Ishiguro2016, Caldas2008} generate ‘plaid’-like partitions on a 2-dimensional array. Usually, the plaid is generated through the Beta-Bernoulli process or the Indian Buffet Process (IBP) \cite{Griffiths2011} for an infinite number of plaid. Regardless of the difference between the continuous space and discrete arrays, the plaid models can produce boxes similar to the RBP. However, each box is formed through individual row/column permutations. As these permutations are different, it is usually impossible to form all the boxes through a single permutation of rows and columns. Plaid models do not possess the important self-consistency property.

Similar to Bayesian plaid models, the Matrix Tile Analysis (MTA) \cite{Givoni2006} works on a 2-dimensional array, but it is a non-Bayesian MTA. MTA can generate rectangular boxes on the discrete arrays, with a constraint that the boxes cannot be overlapped. Due to its non-Bayesian nature, the self-consistency is not applicable to it.

Bayesian Additive Regression Trees (BART) \cite{Chipman2010} is a space-partition model using the hierarchical partition strategy without the self-consistency property. In general, the BART assigns uniform distributions to the cutting positions and uses the Bernoulli distribution to regulate the tree depth. As the parameter in the Bernoulli distribution is set as inverse to the depth of the node in the tree, deeper nodes would have lower probability to be split.

### 3.6 Comparison

Table 1 compares the reviewed space-partition strategies comprehensively through several important aspects, including the self-consistency property, whether applicable to continuous spaces, number of dimensions, inference methods, inter-dimensional dependency, and applications. As can be seen, there is not a single model that can dominate the others in all these aspects, so it is critical to select the most appropriate technique considering the case.

Most of the inference algorithms for the BNSP models are based on Markov chain Monte Carlo (MCMC) methods. In particular, Gibbs samplers are used for the grid-style partition models, Bayesian plaid models and BARTs, as the conjugacy property between the prior and posterior distributions is satisfied by each latent variable. The Reversible-Jump MCMC (RJ-MCMC) \cite{Pitman2006} and Particle Gibbs (PG) \cite{Andrieu2010} algorithms are developed for the hierarchical partition models, due to the tree structures of their latent variables. The Metropolis-Hastings (M-H) algorithm is used for RBP and RTP, due to the specially designed distributions of their latent variables. The Iterative Condition Modes (ICM) method \cite{Givoni2006} is used for the non-Bayesian MTA.

### 4 Applications

In this section, we will review three typical applications of BNSP, including regression trees, random feature construction and relational modeling.

#### 4.1 Regression Trees

To construct regression trees based on BNSP models, the corresponding space is spanned by the feature data $\{x_n\}_{n=1}^N$ and the blocks $\{\square_k\}_k$. We assume that the blocks are generated in this feature space. For the data points locating in the same block, their labels are expected to be similar. In addition, an intensity variable $\omega_k$ is usually associated to each block $\square_k$, such that $\omega_k$ can impose an impact with intensity $\omega_k$ to the labels of all the data points belonging to it. With different settings of the likelihood function for the label data, regression trees can be built for either regression or classification tasks. For example, we can set the Bernoulli distribution as the likelihood function, where the probability can be the logistic transform of the sum of intensities. For example, if we are interested in the regression task and use a Gaussian distribution to generate the label data, the generative process can be developed as:

\[
\begin{align*}
(1) & \quad \{\square_k\}_k \sim \text{BNSP}([0, 1]^2, -) \\
(2) & \quad \{\omega_k\}_k \sim \mathcal{N}(0, \delta^2) \\
(3) & \quad y_n \sim \mathcal{N}\left(\sum_k \omega_k \cdot 1_{x_n \in \square_k}(x_n), \sigma^2\right)
\end{align*}
\]

where $1_{x_n \in \square_k}(x_n)$ equals to 1 if $x_n$ belongs to $\square_k$; otherwise 0. Step (1) generates the blocks from a BNSP model on the space spanned by the feature data $\{x_n\}_{n=1}^N$. Step (2) generates the intensity values $\{\omega_k\}_k$ for all the blocks from a Gaussian distribution, with mean and variance being 0 and $\delta^2$; Step (3) generates the label data from a Gaussian distribution, with
the mean being the sum of intensities of all covering blocks, and variance being the error variance $\sigma^2$.

**Online Learning** BNSP can also be applied in the online learning setting of regression trees [Lakshminarayanan et al., 2014; Lakshminarayanan et al., 2016; Mourtada et al., 2017; Fan et al., 2020]. Suppose we observe a set of $N$ labeled data points $(x_n, y_n)_{n=1}^N \in \mathbb{R}^D \times \mathbb{R}$ arriving over time, with $y_n$ as the corresponding label of $x_n$. When a new data point arrives, we incorporate it to update the BNSP structure accordingly, which can enhance the model’s prediction ability.

The online learning application of BNSP shares the same spirit as online random forest-type algorithms [Breiman, 2000; Genuer, 2012; Arlot and Genuer, 2014], which assume that the tree-structured model is generated independently with the data labels. The Mondrian Forest (MF) [Lakshminarayanan et al., 2014; Lakshminarayanan et al., 2016] is the first model to apply BNSP to online learning settings, which uses the Mondrian process [Roy and Teh, 2009] to place a probability distribution over all kd-tree partitions of the space. To regularize the MF to be universal consistent (which ensures the prediction error converge to Bayes error), the budget parameter can be increased with the amount of data [Mourtada et al., 2017], and the model can achieve the minimax rate in a multi-dimensional space for single decision trees. [Mourtada et al., 2018] shows the advantage of forest settings through improved convergence results. The Online Binary Space Partition Forest [Fan et al., 2020] extends the BSP-Tree to the online learning setting by randomly generating sloped hyperplanes to cut the feature space.

### 4.2 Random Feature Construction

The generative process of BNSP models can also support constructing random features to approximate kernels [Rahimi and Recht, 2008]. Given $N$ data points and $K$ generated blocks, we can use a binary random feature matrix $\Phi \in \{0, 1\}^{N \times K}$ to record the box coverage status for the data points, where the $(n,k)$th entry represents whether the $n$th data point is covered by the $k$th block. The random feature $\Phi_n$ for each data point can replace their true features $x_n$ and be used to approximate certain kinds of kernels.

The Mondrian Kernel (MK) [Balog et al., 2016] is the first model to introduce the random feature construction application of BNSP. In particular, the MK generates a set of $M$ Mondrian process partitions on the feature space and incorporates all $M$ partitions into one random feature matrix $\Phi \in \{0, 1\}^{N \times (M \cdot K)}$. As a result, the product of $\Phi_{n_1}^\top \Phi_{n_2}$ represents the count of blocks covering both the $n_1$th and $n_2$th data points. Moreover, they have shown that both the expectation and the $M \to \infty$ case of $\Phi \Phi^\top$ is the Laplace kernel. [O’Reilly and Tran, 2020] further extends the cuts into arbitrary directions and characterizes all possible kernels (including the radial basis function kernel and the Laplace kernel) that the hierarchical partitions can approximate.

| Models              | Self-consistency | Continuous | $D$ | Inference | IDD | Applications                       |
|---------------------|------------------|------------|-----|-----------|-----|-----------------------------------|
| Copula regular-grid | ✓                | ✓          | 2   | Gibbs     | ✓   | RM & RT                          |
| Deep regular-grid   | ✓                | ✓          | 2   | Gibbs     | √   | RM                               |
| MP                  | ✓                | ✓          | $\geq 2$ | Gibbs & PG | ✓   | RM & RT                          |
| BSP                 | ✓                | ✓          | $\geq 2$ | Gibbs & PG | ✓   | RM & RT                          |
| RTF                 | ✓                | ✓          | $\geq 2$ | SMC       | ✓   | RT                               |
| RBF                 | ✓                | ✓          | $\geq 2$ | M-H       | ×   | RM & RT                          |
| RFT                 | ✓                | ×          | 2   | M-H       | ✓   | RM                               |
| Plaid               | ×                | ×          | 2   | Gibbs     | ×   | RM                               |
| MTA                 | ×                | ×          | 2   | ICM       | ×   | RM                               |
| BART                | ×                | ✓          | $\geq 2$ | Gibbs     | ×   | RT                               |

4.3 Relational Modeling

For the relational modeling, the observed linkage data $R$ is regarded as the label data, which is usually presented as a symmetric (undirected) or asymmetric (directed) matrix $R \in \{0, 1\}^{N \times N}$, with $R_{ij} = 1$ indicating that entity $i$ interacts with entity $j$, otherwise $R_{ij} = 0$. Since $R$ represents pairwise linkage relations, the space of this application is spanned by two community distributions, which can be encoded as a unit square $[0, 1]^2$. For the feature data of each linkage $R_{ij}$, the model generates a pseudo attribute $u_i$ for each entity $i$, and concatenates attributes (i.e. $[u_i, u_j]')$ for entities $i$ and $j$ as the pseudo feature.

We can summarize the generative process of relational modeling based on BNSP as follows:

1. $\square_k \sim \text{BNSP}([0, 1]^2, -)$
2. $\omega_k \sim \text{Beta}(-)$
3. $u_i \sim \text{Uniform}[0, 1]$
4. $R_{ij} \sim \text{Bernoulli}(\omega_k(u_i, u_j) \in \square_k)$

where the block $\square_k$ represents meaningful community-by-community interaction groups and the intensity $\omega_k$ denotes the influence to the links belonging to $\square_k$.

There are various extensions to the above basic generative process. For the case of overlapped blocks [Fan et al., 2018b], the intensity is assumed to follow a Normal distribution. Positive (negative) intensity would promote (suppress) links locating in the corresponding block. For the case of categorical or real-valued links, different types of likelihood functions can be adopted. For example, nonnegative real-valued links can be modeled by setting the Gamma distribution as the likelihood function.
Connection to graphons The relational modeling applications are closely related to the graphon (graph function) literature [Orbanz and Roy, 2015]. Given the exchangeable relational data for relational modeling, the Aldous–Hoover theorem [Hoover, 1979; Aldous, 1981] provides the theoretical foundation to model exchangeable multi-dimensional arrays conditioned on a stochastic partition model. A random 2-dimensional array is separately exchangeable if its distribution is invariant under separate permutations of rows and columns. Specifically, the theorem states that:

**Theorem 1.** [Orbanz and Roy, 2015; Lloyd et al., 2012] A random array \((R_{i,j})\) is separately exchangeable if and only if it can be represented as follows: there exists a random measurable function \(F : [0,1]^2 \mapsto \mathcal{X}\) such that \((R_{i,j}) \overset{d}{=} F(u_i^{(1)}, u_j^{(2)}), \nu_{ij})\), where \(\{u_i^{(1)}\}_{i}, \{u_j^{(2)}\}_{j}\) and \(\{\nu_{ij}\}_{i,j}\) are two sequences and an array of i.i.d. uniform random variables in \([0,1]\), respectively.

Many of the existing BNSP models comply with this theorem, with specific forms of mapping function \(F\). For instance, as illustrated in Figure 3, given the uniformly distributed node coordinates \((u_i^{(1)}, u_j^{(2)})\), the regular-grid partition is related to a regular-grid graphon; the MP is related to a kd-tree structured graphon; the RTP is related to a guillotine partition graphon; and the RBP is related to a box-wise constant graphon. All these graphons are piece-wise constant in \([0,1]^2\).

5 Challenges & Future Work

With such fruitful modeling strategies and interesting applications, there is no doubt that the research of BNSP would continue inspiring the machine learning communities in its own way. However, there are still some challenges and interesting work left for further explorations.

5.1 Scalable Inference Methods

As far as we concern, most of the current inference methods for these BNSP models rely on the MCMC methods (see details in Table 1). Since MCMC methods have often been criticized due to its long computational time and the difficulty of convergence diagnosis, alternative scalable inference methods are necessary to deal with the current large-scale data problems. Variational methods, in particular the popular variational auto-encoder methods [Kingma and Welling, 2013], are promising solutions as they are optimization-based and can also provide posterior approximations to the ground-truth at the same time. However, we have not seen much progress of developing variational inference methods for BNSP models, except for the regular-grid partition. One of the possible reasons might be the complications of the partition structure, such as the tree structure in a hierarchical partition.

Specific to hierarchical partitions with sloped cuts: In the hierarchical partition models, the strategy of using sloped cuts for BNSP seems to be more effective and flexible than the axis-aligned cutting strategy. However, the improvement of modeling capability comes with a price of computational cost. The cost of sloped-cut models scales at least quadratically with the number of dimensions of the space, whereas the cost of axis-aligned models often scales linearly with the number of dimensions. Efficient ways of circumventing computational complexity and keeping self-consistency at the same time would be interesting topics to work on.

5.2 Partition Flexibility

Dependent BNSP partitions Currently, the BNSP models are assumed to independently generate space partitions. We may consider the dependence between different partitions and extend the application to deal with various data formats, such as dynamic, cross-domain or multiview data. However, these extensions would be nontrivial, as the important self-consistency property would be easily violated. We may need to propose innovative ways to incorporate dependence and keep the self-consistency at the same time.

Convex-polygon Bounding Blocks In the aforementioned bounding based strategy, we assume that the box occupations in all dimensions are independent and thus produce rectangular blocks. In practice, the shape of a block might be convex or even irregular polygons. For example, in credit risk modeling (with expense and salary), one high risk block might be formed by a sloped cutting line of expense ≥ salary. In this problem, the way of defining the convex polygon and proving the self-consistency would be challenging and important.
5.3 Posterior Concentration Analysis

The importance of posterior concentration behavior in the Bayesian generative process has been repeatedly emphasized in the literature. Currently, [van der Pas and Ročková, 2017; Rocková and van der Pas, 2017; Ročková and Saha, 2019] has developed a series of posterior concentration analysis for the BART, based on the work of [Ghosal et al., 2000; Ghosal et al., 2007]. It would be quite interesting to see how the posterior concentration analysis can be integrated into the hierarchical partition or even bounding based partition BNSP models.

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