PERMANENCE IN POLYMATRIX REPLICATORS

TELMO PEIXE

ISEG - Lisbon School of Economics & Management, Universidade de Lisboa
REM - Research in Economics and Mathematics
CEMAPRE - Centro de Matemática Aplicada à Previsão e Decisão Económica
(Communicated by Elvio Accinelli)

Abstract. Generally a biological system is said to be permanent if under small perturbations none of the species goes to extinction. In 1979 P. Schuster, K. Sigmund, and R. Wolff [15] introduced the concept of permanence as a stability notion for systems that models the self-organization of biological macromolecules. After, in 1987 W. Jansen [9], and J. Hofbauer and K. Sigmund [6] give sufficient conditions for permanence in the replicator equations. In this paper we extend these results for polymatrix replicators.

1. Introduction. In the 1970’s J. Maynard Smith and G. Price [11] applied the theory of strategic games developed by J. von Neumann and O. Morgenstern [18] in the 1940’s to investigate the dynamical processes of biological populations, giving rise to the field of Evolutionary Game Theory (EGT).

Some classes of ordinary differential equations which play a central role in EGT are the Lotka-Volterra (LV) system, the replicator equation, the bimatrix replicator and the polymatrix replicator.

The Lotka-Volterra (LV) system, independently introduced in 1920s by A. J. Lotka [10] and V. Volterra [17], are perhaps the most widely known systems used in scientific areas as diverse as physics, chemistry, biology, and economics.

Given $n$ species competing against each other, we call Lotka-Volterra equation to the following system of differential equations

$$\frac{dx_i}{dt}(t) = x_i(t) \left( r_i + \sum_{j=1}^{n} a_{ij} x_j(t) \right), \quad i = 1, \ldots, n, \quad (1)$$

where $x_i(t) \geq 0$ represents the density of species $i$ in time $t$, $r_i$ its intrinsic rate of decay or growth, and each coefficient $a_{ij}$ represents the effect of species $j$ over population $i$.

Another classical model widely used is the replicator equation which J. Hofbauer [5] proved is equivalent to the LV system. The replicator equation was introduced by P. Taylor and L. Jonker [16]. It models the time evolution of the probability distribution of behavioural strategies in a biological population. Given

2020 Mathematics Subject Classification. 34D05, 34D20, 37B25, 37C75, 37N25, 37N40, 91A22.

Key words and phrases. Permanence, replicator equation, Lotka-Volterra, polymatrix replicator, evolutionary game theory.

The author was supported by FCT - Fundação para a Ciência e a Tecnologia, under the project CEMAPRE - UID/MULTI/00491/2013 through national funds.
a payoff matrix $A \in \mathbb{M}_{n \times n}(\mathbb{R})$, the replicator equation refers to the ordinary differential equations system

$$\frac{dx_i}{dt}(t) = x_i(t) \left((Ax(t))_i - x(t)^T A x(t)\right), \quad i = 1, \ldots, n,$$

defined on the simplex $\Delta^{n-1} = \{x \in \mathbb{R}_+^n : \sum_{j=1}^n x_j = 1\}$.

In the case we want to model the interaction between two populations (or a population divided in two groups, for example, males and females), where each group have a different set of strategies (asymmetric games), and all interactions involve individuals of different groups, the common used model is the bimatrix replicator equation, that first appeared in [12] and [14]. Given two payoff matrices $A \in \mathbb{M}_{n \times m}(\mathbb{R})$ and $B \in \mathbb{M}_{m \times n}(\mathbb{R})$, the bimatrix replicator equation refers to the ordinary differential equations system

$$\begin{align*}
\frac{dx_i}{dt}(t) &= x_i(t) \left((Ay(t))_i - x(t)^T A y(t)\right), \quad i = 1, \ldots, n \\
\frac{dy_j}{dt}(t) &= y_j(t) \left((Bx(t))_j - y(t)^T B x(t)\right), \quad j = 1, \ldots, m
\end{align*}$$

(3)

defined on the product of simplices $\Delta^{n-1} \times \Delta^{m-1}$. Each state in this case is a pair of frequency vectors, which represents the frequencies of use of the behavioural strategies of each group, and the bimatrix replicator equation describes the temporal evolution of these frequencies.

Suppose now that we want to study a population divided in a finite number of groups, each one with a finite number of behavioural strategies. Interactions between individuals from any two groups (including the same group) are allowed, but competition takes place within each group, that is, the relative success of each strategy is assessed within the corresponding group.

H. Alishah and P. Duarte [1] introduced the model that they designated as polymatrix games to study this kind of population. In [2] H. Alishah, P. Duarte and T. Peixe study particular classes of polymatrix games. The system of ordinary differential equations, designated as the polymatrix replicator, that model this game, will be presented in detail in section 3. The phase space of these systems are prisms, products of simplices $\Delta^{n_1-1} \times \ldots \times \Delta^{n_p-1}$, where $p$ is the number of groups and $n_j$ the number of behavioural strategies inside the $j$-th group, for $j = 1, \ldots, p$. This class of evolutionary systems includes the replicator equation (the case of only one group of individuals) and the bimatrix replicator equation (the case of two groups of individuals).

In 1979 P. Schuster, K. Sigmund, and R. Wolff introduced in [15] the concept of permanence as a stability notion for systems that models the self-organization of biological macromolecules. Generally, we say that a biological system is permanent if, for small perturbations, none of the species goes to extinction.

In 1987 J. Hofbauer and K. Sigmund [6] and W. Jansen [9] give sufficient conditions for permanence in the replicator equations. In addition to introducing the permanence concept for polymatrix replicators, in this paper we extend those results from J. Hofbauer, K. Sigmund and W. Jasen to these replicators.

This paper is organized as follows. In section 2, we recall the replicator equation, its relation with the LV system, some properties of these systems, and the concept of permanence. In section 3 we present the definition of the polymatrix replicator. In section 4 we extend the concept of permanence to polymatrix replicators and the results given by J. Hofbauer and K. Sigmund [6] and W. Jansen [9]. Finally,
in section 5 we illustrate our main results of permanence in polymatrix replicators with two examples.

2. The replicator equation and permanence. In this section we present the definition of the replicator equation, some of its properties, and the concept or permanence. For more details on the subject see [7] for instance.

Consider a population where individuals interact with each other according to a set of \( n \) possible strategies. The state of the population concerning this interaction is fully described by a vector \( x = (x_1, \ldots, x_n)^T \), where \( x_i \) represents the frequency of individuals using strategy \( i \), for \( i = 1, \ldots, n \). The set of all population states is the simplex \( \Delta^{n-1} = \{ x \in \mathbb{R}^n_+ : \sum_{j=1}^n x_j = 1 \} \).

If an individual using strategy \( i \) interacts with an individual using strategy \( j \), the coefficient \( a_{ij} \) represents the average payoff for the individual using strategy \( i \). Let \( A = (a_{ij}) \in \mathbb{M}_{n \times n}(\mathbb{R}) \) be the matrix consisting of these \( a_{ij} \)'s. Assuming random encounters between individuals of that population, the average payoff for strategy \( i \) is given by

\[
(Ax)_i = \sum_{k=1}^n a_{ik}x_k,
\]

and the global average payoff of all population strategies is given by

\[
x^T A x = \sum_{i=1}^n \sum_{k=1}^n a_{ik}x_i x_k.
\]

The growth rate \( \frac{dx_i}{dt} / x_i \) of the frequency of strategy \( i \) is equal to the payoff difference \( (Ax)_i - x^T A x \), which yields the replicator equation (2) defined on the simplex \( \Delta^{n-1} \), that is invariant under (2) (see for example [7, Section 7.1]).

The replicator equation models the frequency evolution of certain strategical behaviours within a biological population. In fact, the equation says that the logarithmic growth of the usage frequency of each behavioural strategy is directly proportional to how well that strategy fares within the population.

This system of ordinary differential equations was introduced in 1978 by P. Taylor and L. Jonker [16] and was designated as the replicator equation by P. Schuster and K. Sigmund [13] in 1983.

In 1981 J. Hofbauer [5] stated an important relation between the LV systems and the replicator equation. The replicator equation is a cubic equation on the compact set \( \Delta^{n-1} \) while the LV equation is quadratic on \( \mathbb{R}^n_+ \). However, Hofbauer proved that the replicator equation in \( n + 1 \) variables is equivalent to the LV equation in \( n \) variables (see also [7]).

In LV systems the existence of an equilibrium point in \( \mathbb{R}^n_+ \) is related with the orbit’s behaviour. Namely, a LV system admits an interior equilibrium point if and only if \( \text{int}(\mathbb{R}^n_+) \) contains \( \alpha \) or \( \omega \)-limit points. Moreover, if there exists a unique interior equilibrium point and if the solution does not converge to the boundary nor to infinity, then its time average converges to the equilibrium point, as stated in the following result.

**Theorem 2.1.** Suppose that \( x(t) \) is a solution of a \( n \)-dimensional LV system such that \( 0 < m \leq x_i(t) \leq L \), for all \( t \geq 0 \) and \( i \in \{1, \ldots, n\} \). Then, there exists a sequence \( (T_k)_{k \in \mathbb{N}} \) such that \( T_k \to +\infty \) and an equilibrium point \( q \in \text{int}(\mathbb{R}^n_+) \) such
that
\[
\lim_{k \to +\infty} \frac{1}{T_k} \int_0^{T_k} x(t) \, dt = q.
\]
Moreover, if the LV system has only one equilibrium point \( q \in \text{int}(\mathbb{R}^+_n) \), then
\[
\lim_{T \to +\infty} \frac{1}{T} \int_0^T x(t) \, dt = q.
\]

**Proof.** A proof of this theorem can be seen in [4].

By the equivalence between the LV and the replicator equation, together with Theorem 2.1, we have the following known result from J. Hofbauer [8].

**Proposition 1.** If the replicator equation (2) has no equilibrium point in \( \text{int}(\Delta^{n-1}) \), then every solution converges to the boundary of \( \Delta^{n-1} \).

J. Hofbauer in [8] also proved a natural extension of Theorem 2.1 in LV systems to the replicator equation.

**Theorem 2.2.** If the replicator equation (2) admits a unique equilibrium point \( q \in \text{int}(\Delta^{n-1}) \), and if the \( \omega \)-limit of the orbit of \( x(t) \) is in \( \text{int}(\Delta^{n-1}) \), then
\[
\lim_{t \to \infty} \frac{1}{T} \int_0^T x(t) \, dt = q.
\]

We recall now the concept of **permanence** in the replicator equation, that is a stability notion introduced by Schuster et al. in [15].

**Definition 2.3.** The vector field associated to the replicator equation (2) defined on \( \Delta^{n-1} \) is said to be **permanent** if there exists \( \delta > 0 \) such that, for all \( x \in \text{int}(\Delta^{n-1}) \),
\[
\liminf_{t \to \infty} d(\varphi^t(x), \partial\Delta^{n-1}) > \delta,
\]
where \( d \) is the Euclidean distance, \( \varphi^t \) denotes the flow determined by system (2) and \( \partial\Delta^{n-1} \) is the boundary of \( \Delta^{n-1} \).

In the context of biology, a system being permanent means that sufficiently small perturbations cannot lead any species to extinction.

The following theorem due to Jansen [9] is also valid for LV systems.

**Theorem 2.4.** Let \( X \) be the replicator vector field defined by (2). If there is a point \( p \in \text{int}(\Delta^{n-1}) \) such that for all boundary equilibria \( x \in \partial\Delta^{n-1} \),
\[
p^T A x > x^T A x,
\]
then the vector field \( X \) is permanent.

This Theorem 2.4 is a corollary of the following theorem which gives sufficient conditions for a system to be permanent. This result is stated and proved by Hofbauer and Sigmund in [6, Theorem 1] or [7, Theorem 12.2.1].

**Theorem 2.5.** Let \( P : \Delta^{n-1} \to \mathbb{R} \) be a smooth function such that \( P = 0 \) on \( \partial\Delta^{n-1} \) and \( P > 0 \) on \( \text{int}(\Delta^{n-1}) \). Assume there is a continuous function \( \Psi : \Delta^{n-1} \to \mathbb{R} \) such that
\begin{enumerate}
  \item for any orbit \( x(t) \) in \( \text{int}(\Delta^{n-1}) \), \( \frac{d}{dt} \log P(x(t)) = \Psi(x(t)) \),
  \item for any orbit \( x(t) \) in \( \partial\Delta^{n-1} \), exists \( T > 0 \) such that \( \int_0^T \Psi(x(t)) \, dt > 0 \).
\end{enumerate}
Then the vector field \( X \) is permanent.
3. The polymatrix replicator. In this section we present the definition of the polymatrix replicator. For more details on the subject, namely some of its properties or special classes, see [1] and [2].

Consider a population divided in $p$ groups, labelled by an integer $\alpha$ ranging from 1 to $p$. Individuals of each group $\alpha \in \{1, \ldots, p\}$ have exactly $n_{\alpha}$ strategies to interact with other members of the population, including the same group. For each $\alpha, \beta \in \{1, \ldots, p\}$, consider a real $n_{\alpha} \times n_{\beta}$ matrix, say $A^{\alpha,\beta}$, whose entries $a_{ij}^{\alpha,\beta}$, with $i \in \{1, \ldots, n_{\alpha}\}$ and $j \in \{1, \ldots, n_{\beta}\}$, represent the average payoff of an individual of the group $\alpha$ using the $i^{th}$ strategy (of the group $\alpha$) when interacting with an individual of the group $\beta$ using the $j^{th}$ strategy (of the group $\beta$). Thus, considering a matrix, say $A$, consisting of all of these entries $a_{ij}^{\alpha,\beta}$ for all $\alpha, \beta \in \{1, \ldots, p\}$ and $i \in \{1, \ldots, n_{\alpha}\}$ and $j \in \{1, \ldots, n_{\beta}\}$, we have that $A$ is a square block matrix of order $n$, made up of these block matrices $A^{\alpha,\beta}$, where $n = n_1 + \ldots + n_p$.

Let $\mathfrak{u}_\alpha = (n_1, \ldots, n_p)$. The state of the population in time $t$ is described by a point $x(t) = (x^\alpha(t))_{1 \leq \alpha \leq p}$ in the prism

$$
\Gamma_\mathfrak{u} := \Delta^{n_1-1} \times \cdots \times \Delta^{n_p-1} \subset \mathbb{R}^n,
$$

where $\Delta^{n-1} = \{ x \in \mathbb{R}_{+}^{n_\alpha} : \sum_{i=1}^{n_\alpha} x_i^\alpha = 1 \}$, $x^\alpha(t) = (x_1^\alpha(t), \ldots, x_{n_\alpha}^\alpha(t))$ and the entry $x_i^\alpha(t)$ represents the usage frequency of the $i^{th}$ strategy within the group $\alpha$ at time $t$. The prism $\Gamma_\mathfrak{u}$ is a $(n-p)$-dimensional simple polytope whose affine support is the $(n-p)$-dimensional subspace of $\mathbb{R}^n$ defined by the $p$ equations

$$
\sum_{i=1}^{n_\alpha} x_i^\alpha = 1, \quad \alpha \in \{1, \ldots, p\}.
$$

We denote by $\partial \Gamma_\mathfrak{u}$ the boundary of $\Gamma_\mathfrak{u}$.

Assuming random encounters between individuals, for each group $\alpha \in \{1, \ldots, p\}$, the average payoff for a strategy $i \in \{1, \ldots, n_\alpha\}$, is given by

$$
(Ax(t))_i' = \sum_{\beta=1}^{p} A^{\alpha,\beta}_i x^\beta(t) = \sum_{\beta=1}^{p} \sum_{k=1}^{n_\beta} a_{ik}^{\alpha,\beta} x_k^\beta(t),
$$

where $i' := n_1 + \cdots + n_{\alpha-1} + i$, and the average payoff of all strategies in $\alpha$ is given by

$$
\sum_{i=1}^{n_\alpha} x_i^\alpha(t) (Ax(t))_i',
$$

which can also be written as

$$
\sum_{\beta=1}^{p} (x^\alpha(t))^T A^{\alpha,\beta} x^\beta(t).
$$

The growth rate $\frac{dx_i^\alpha(t)}{dt}$ of the frequency of the strategy $i \in \{1, \ldots, n_\alpha\}$, for each $\alpha \in \{1, \ldots, p\}$, is equal to the payoff difference $(Ax(t))_i' - \sum_{\beta=1}^{p} (x^\alpha(t))^T A^{\alpha,\beta} x^\beta(t)$, which yields the following system of ordinary differential equations defined on the prism $\Gamma_\mathfrak{u}$:

$$
\frac{dx_i^\alpha(t)}{dt} = x_i^\alpha(t) \left( (Ax(t))_i' - \sum_{\beta=1}^{p} (x^\alpha(t))^T A^{\alpha,\beta} x^\beta(t) \right), \quad \alpha \in \{1, \ldots, p\}, i \in \{1, \ldots, n_\alpha\},
$$

(5)

called the polymatrix replicator.
Notice that interactions between individuals of any two groups (including the same) are allowed. Notice also that this equation implies that competition takes place inside the groups, i.e., the relative success of each strategy is evaluated within the corresponding group.

The flow \( \phi^t_{\mathbb{A}_n} \) of this equation leaves the prism \( \Gamma_n \) invariant. (The proof of this result follows by the same argument presented in the proof that the Cartesian product \( \Delta^{n-1} \times \Delta^{m-1} \) is invariant for the bimatrix replicator equation, see [7, Section 10.3]). Hence, by compactness of \( \Gamma_n \), the flow \( \phi^t_{\mathbb{A}_n} \) is complete. The underlying vector field on \( \Gamma_n \) will be denoted by \( X_{\mathbb{A}_n} \).

In the case \( p = 1 \), we have \( \Gamma_n = \Delta^{n-1} \) and (5) is the replicator equation associated to the payoff matrix \( \mathbb{A} \).

When \( p = 2 \), and \( A^{11} = A^{22} = 0 \), \( \Gamma_n = \Delta^{n_1-1} \times \Delta^{n_2-1} \) and (5) becomes the bimatrix replicator equation associated to the pair of payoff matrices \( (A^{22}, A^{21}) \).

More generally, it also includes the replicator equation for n-person games (when \( A^{\alpha,\alpha} = 0 \) for all \( \alpha \in \{1, \ldots, p\} \)).

4. Permanence in the polymatrix replicator. In this section we extend to polymatrix replicators the definition and some properties of permanence stated in the context of LV and the replicator equation.

If an orbit in the interior of the state space converges to the boundary, this corresponds to extinction. Despite we give a formal definition of permanence in polymatrix replicators (see Definition 4.1), as we saw in the context of the LV systems and the replicator equation, we say that a system is permanent if there exists a compact set \( K \) in the interior of the state space such that all orbits starting in the interior of the state space end up in \( K \). This means that the boundary of the state space is a repellor.

Consider a polymatrix replicator (5) and \( X := X_{\mathbb{A}_n} \) its associated vector field defined on the \((n-p)\)-dimensional prism \( \Gamma_n \). For each \( \alpha \in \{1, \ldots, p\} \), we denote by \( \pi_\alpha : \mathbb{R}^n \to \mathbb{R}^n \) the projection \( x \mapsto y \) defined by

\[
y^\beta_i := \begin{cases} x^\alpha_i & \text{if } \beta = \alpha \\ 0 & \text{if } \beta \neq \alpha \end{cases}, \text{ for all } \beta \in \{1, \ldots, p\}, i \in \{1, \ldots, n_\beta\}.
\]

The following result is an extension of the average principle in LV systems (see Theorem 2.1) and replicator equation (see Theorem 2.2) to the framework of the polymatrix replicator systems.

**Proposition 2** (Average Principle). Let \( x(t) \in \text{int}(\Gamma_n) \) be an interior orbit of the vector field \( X \) such that for some \( \varepsilon > 0 \) and some time sequence \( T_k \to +\infty \), as \( k \to +\infty \), one has

1. \( \frac{d}{dt} (x(T_k), \partial \Gamma_n) \geq \varepsilon \) for all \( k \geq 0 \), where \( d \) is the Euclidean distance,

2. \( \lim_{k \to +\infty} \frac{1}{T_k} \int_0^{T_k} x(t) \, dt = q \),

3. \( \lim_{k \to +\infty} \frac{1}{T_k} \int_0^{T_k} \pi_\alpha (x(t))^T A x(t) \, dt = a_\alpha \), for all \( \alpha \in \{1, \ldots, p\} \).

Then \( q \) is an equilibrium of \( X \) and \( a_\alpha = \pi_\alpha(q)^T A q \), for all \( \alpha \in \{1, \ldots, p\} \). Moreover,

\[
\lim_{k \to +\infty} \frac{1}{T_k} \int_0^{T_k} x(t)^T A x(t) \, dt = q^T A q.
\]
Let $\alpha \in \{1, \ldots, p\}$ and $i, j \in \{1, \ldots, n_\alpha\}$. Let $i' := n_1 + \cdots + n_{\alpha-1} + i$ and $j' := n_1 + \cdots + n_{\alpha-1} + j$. Notice that from (2) we obtain
\[
\lim_{k \to +\infty} \frac{1}{T_k} \int_0^{T_k} (Ax(t))_i - (Ax(t))_j \, dt = 0.
\]
By (1) we have that $\varepsilon < x_i^0(T_k) < 1 - \varepsilon$ for all $k$. Hence, considering $e_k$ the $k^{th}$-vector of the canonical basis of $\mathbb{R}^n$,
\[
(Aq)_i - (Aq)_j = e_i^T Aq - e_j^T Aq
\]
\[
= \lim_{k \to +\infty} \frac{1}{T_k} \int_0^{T_k} (e_i^T A x(t) - e_j^T A x(t)) \, dt
\]
\[
= \lim_{k \to +\infty} \frac{1}{T_k} \left( \log x_i^0(T_k) - \log x_j^0(T_k) \right) = 0.
\]

It follows that $q$ is an equilibrium of $X$, and for all $i, j \in \{1, \ldots, n_\alpha\}$, $(Aq)_i - (Aq)_j = \pi_\alpha(q)^T A q$.

Finally, using (1)-(3),
\[
0 = \lim_{k \to +\infty} \frac{1}{T_k} \left( \log x_i^0(T_k) - \log x_i^0(0) \right)
\]
\[
= \lim_{k \to +\infty} \frac{1}{T_k} \int_0^{T_k} \frac{dx_i^0(t)}{x_i^0(t)} \, dt
\]
\[
= \lim_{k \to +\infty} \frac{1}{T_k} \int_0^{T_k} \left( (Ax(t))_i - \sum_{\beta=1}^p (x^\alpha(t))^T A^{\alpha\beta} x^\beta(t) \right) \, dt
\]
\[
= \lim_{k \to +\infty} \frac{1}{T_k} \int_0^{T_k} \left( (Ax(t))_i - \pi_\alpha(x(t))^T A x(t) \right) \, dt
\]
\[
= (Aq)_i - \lim_{k \to +\infty} \frac{1}{T_k} \int_0^{T_k} \pi_\alpha(x(t))^T A x(t) \, dt = (Aq)_i - a_\alpha,
\]
which implies that $a_\alpha = \pi_\alpha(q)^T A q$, and hence
\[
\lim_{k \to +\infty} \frac{1}{T_k} \int_0^{T_k} x(t)^T A x(t) \, dt = q^T A q.
\]

The definition of permanence in the replicator equation (see Definition 2.3) can be naturally extended to the polymatrix replicators, as follows.

**Definition 4.1.** We say that a vector field $X$ defined in $\Gamma_n$ is **permanent** if there exists $\delta > 0$ such that $x \in \text{int}(\Gamma_n)$ implies
\[
\liminf_{t \to +\infty} d \left( \varphi^t_X(x), \partial \Gamma_n \right) \geq \delta,
\]
where $d$ is the Euclidean distance and $\varphi^t_X$ the associated flow to the vector field $X$.

The following theorem extends Theorem 2.5 for polymatrix replicators.

**Theorem 4.2.** Let $\Phi : \Gamma_n \to \mathbb{R}$ be a smooth function such that $\Phi = 0$ on $\partial \Gamma_n$ and $\Phi > 0$ on $\text{int}(\Gamma_n)$. Assume there is a continuous function $\Psi : \Gamma_n \to \mathbb{R}$ such that
\[
(1) \text{ for any orbit } x(t) \text{ in } \text{int}(\Gamma_n), \quad \frac{d}{dt} \log \Phi(x(t)) = \Psi(x(t)),
\]
(2) for any orbit \(x(t)\) in \(\partial \Gamma_n\), \(\exists T > 0\) s.t. \(\int_0^T \Psi(x(t)) \, dt > 0\).

Then the vector field \(X\) is permanent.

J. Hofbauer and K. Sigmund in [7, Theorem 12.2.1] state and prove a result that is abstract and applicable to a much wider class of systems, including polymatrix replicator systems. In fact, where they refer to \(S^n\) we can consider any \(d\)-dimensional simple polytope, since the proof is the same replacing \(S^n\) by this polytope. Hence, the result stated in Theorem 4.2, as the one stated in the following Remark 1, are an adaptation for polymatrix replicator systems of the result in [7, Theorem 12.2.1] and [7, Theorem 12.2.2], respectively, whose proofs are the same as the ones made by J. Hofbauer and K. Sigmund, just replacing \(S^n\) by \(\Gamma_n\).

**Remark 1.** If we consider \(\Gamma_n\) instead of \(S^n\) in the result stated and proved by J. Hofbauer and K. Sigmund in [7, Theorem 12.2.2], we obtain an analogous result for polymatrix replicator systems saying that for the conclusion in Theorem 4.2 it is enough to check (2) for all \(\omega\)-limit orbits in \(\partial \Gamma_n\). Thus, defining

\[
(2') \text{ for any } \omega\text{-limit orbit } x(t) \text{ in } \partial \Gamma_n, \quad \int_0^T \Psi(x(t)) \, dt > 0 \text{ for some } T > 0,
\]

we have that condition \(2'\) implies (2).

Let \(k \in \mathbb{N}_0\) with \(0 \leq k \leq n - p - 1\), where \(p\) is the number of groups in some population and \(n = n_1 + \ldots + n_p\), where \(n_\alpha\) is the number of strategies in each group \(\alpha \in \{1, \ldots, p\}\), as defined at the beginning of section 3. Now let the \(k\)-dimensional face skeleton of \(\Gamma_n\), denoted by \(\partial_k \Gamma_n\), be the union of all \(j\)-dimensional faces of \(\Gamma_n\), with \(0 \leq j \leq k\). In particular, the vertex skeleton of \(\Gamma_n\) is the union \(\partial_0 \Gamma_n\) of all vertices of \(\Gamma_n\), and the edge skeleton of \(\Gamma_n\) is the union \(\partial_1 \Gamma_n\) of all vertices and edges of \(\Gamma_n\). We will use these sets in the proof of the following theorem, which is an extension of Theorem 2.4 to polymatrix replicator systems.

**Theorem 4.3.** If there is a point \(q \in \text{int}(\Gamma_n)\) such that for all boundary equilibria \(x \in \partial \Gamma_n\),

\[
q^T Ax > x^T Ax, \quad \text{(6)}
\]

then \(X\) is permanent.

**Proof.** The proof we present here is an adaptation of the argument used in the proof of Theorem 13.6.1 in [7].

Take the given point \(q \in \text{int}(\Gamma_n)\) and consider \(\Phi : \Gamma_n \to \mathbb{R}\),

\[
\Phi(x) := \prod_{\alpha=1}^p \prod_{i=1}^{n_\alpha} (x_i^\alpha)^{q_i^\alpha}.
\]

We have that \(\Phi = 0\) on \(\partial \Gamma_n\) and \(\Phi > 0\) on \(\text{int}(\Gamma_n)\).

Consider now the continuous function \(\Psi : \Gamma_n \to \mathbb{R}\) defined by

\[
\Psi(x) := q^T Ax - x^T Ax.
\]

Then

\[
\frac{d}{dt} \log \Phi(x(t)) = \Psi(x(t)).
\]

It remains to show that for any orbit \(x(t)\) in \(\partial \Gamma_n\), there exists a \(T > 0\) such that

\[
\int_0^T \Psi(x(t)) \, dt > 0. \quad \text{(7)}
\]
We prove by induction in \( k \in \mathbb{N}_0 \) that if \( x(t) \in \partial_k \Gamma_n \), then (7) holds for some \( T > 0 \).

If \( x(t) \in \partial_0 \Gamma_n \) then \( x(t) \equiv q' \) for some vertex \( q' \) of \( \Gamma_n \). Since by (6) we have \( \Psi(q') > 0 \), then (7) follows. Hence the induction step is true for \( k = 0 \).

Assume now that conclusion (7) holds for every orbit \( x(t) \in \partial_{n-1} \Gamma_n \), and consider an orbit \( x(t) \in \partial_n \Gamma_n \). Then there is an \( m \)-dimensional face \( \sigma \subset \partial_n \Gamma_n \) that contains \( x(t) \). We consider two cases:

(i) If \( x(t) \) converges to \( \partial \sigma \) (the boundary of \( \sigma \)), i.e., \( \lim_{t \to +\infty} d(x(t), \partial \sigma) = 0 \), then the \( \omega \)-limit of \( x(t) \), \( \omega(x) \), is contained in \( \partial \sigma \). By the induction hypothesis, (7) holds for all orbits inside \( \omega(x) \), and consequently, by Remark 1 the same is true about \( x(t) \).

(ii) If \( x(t) \) does not converge to \( \partial \sigma \), there exists \( \varepsilon > 0 \) and a sequence \( T_k \to +\infty \) such that \( d(x(T_k), \partial \sigma) \geq \varepsilon \) for all \( k \geq 0 \). Let us write

\[
\bar{x}(T) = \frac{1}{T} \int_0^T x(t) \, dt \quad \text{and} \quad a_\alpha(T) = \frac{1}{T} \int_0^T \pi_\alpha(x(t))^T Ax(t) \, dt
\]

for all \( \alpha \in \{1, \ldots, p\} \). Since the sequences \( \bar{x}(T_k) \) and \( a_\alpha(T_k) \) are bounded, there is a subsequence of \( T_k \), that we will keep denoting by \( T_k \), such that \( \bar{x}(T_k) \) and \( a_\alpha(T_k) \) converge, say to \( q' \) and \( a_\alpha \), respectively, for all \( \alpha \in \{1, \ldots, p\} \). Thus, since \( x(t) \) is contained in the \( m \)-dimensional face \( \sigma \) and we are in the case where \( x(t) \) does not converge to \( \partial \sigma \), considering the system restricted to \( \sigma \), we can apply Proposition 2 and deduce that \( q' \) is an equilibrium point in \( \sigma \) and \( a_\alpha = \pi_\alpha(q')^T Aq' \). Therefore

\[
\frac{1}{T_k} \int_0^{T_k} \Psi(x(t)) \, dt
\]

converges to \( q^T Aq' - q'^T Aq' \), which by (6) is positive. This implies (7) and hence proves the permanence of \( X \).

A particular class of interest of polymatrix replicators is the **dissipative polymatrix replicator**. For formal definitions and properties of conservative and dissipative polymatrix replicators see [2]. If a dissipative polymatrix replicator only has one globally attractive interior equilibrium, then it is permanent. Given this, and by the definition of permanence, an interesting question is whether dissipativity is a necessary condition for permanence. This is not true, as illustrated by the first example in the following section.

5. **Examples.** We present here two examples of polymatrix replicators that are permanent. We prove that the first example is permanent because it satisfies condition (6) of Theorem 4.3. Moreover, we show that this system is not dissipative, illustrating that dissipativity is not necessary for permanence.

In the second example we prove that the system is permanent since it is dissipative and has a unique globally attractive interior equilibrium. However, it does not satisfy condition (6) (of Theorem 4.3), thus illustrating that this condition (6) is not necessary for permanence.

There is much more to analyse in the structure/dynamics of these two examples. Namely, by the method presented in [3] we can analyse the asymptotic dynamics of this examples along the heteroclinic network formed out of the polytope’s vertices and edges, which will be done in future work to appear.

All computations and pictures presented in this section were carried out with Wolfram Mathematica and Geogebra software.
5.1. Example 1. Consider a population divided in 4 groups where individuals of each group have exactly 2 strategies to interact with other members of the population, where associated payoff matrix is

\[
A = \begin{pmatrix}
1 & -1 & -1 & 1 & -100 & 100 & -100 & 100 \\
-1 & 1 & 1 & -1 & 100 & -100 & 100 & -100 \\
101 & -101 & 10 & -10 & 1 & -100 & 100 & -100 \\
-101 & 101 & 10 & -10 & 1 & -100 & 100 & -100 \\
1 & -1 & 100 & -100 & -190 & 190 & -101 & 101 \\
-1 & 1 & -100 & 100 & 190 & -190 & 101 & -101 \\
1 & -1 & 5 & -5 & 100 & -100 & -100 & 100 \\
-1 & 1 & -5 & 5 & -100 & 100 & -100 & 100
\end{pmatrix}.
\]

The phase space of the associated polymatrix replicator defined by the payoff matrix \(A\) is the prism \(\Gamma_{(2,2,2,2)} := \Delta^1 \times \Delta^1 \times \Delta^1 \times \Delta^1 \equiv [0,1]^4\).

Besides the 16 vertices of \(\Gamma_{(2,2,2,2)}\) (see Table 1), this system has 2 equilibria on 3d-faces of \(\Gamma_{(2,2,2,2)}\) (see Table 2), 6 equilibria on 2d-faces of \(\Gamma_{(2,2,2,2)}\) (see Table 3), and 12 equilibria on 1d-faces (the edges) of \(\Gamma_{(2,2,2,2)}\) (see Table 4).

| Vertices of \(\Gamma_{(2,2,2,2)}\) | \(f(v_i)\) |
|----------------------------------|----------|
| \(v_1 = (1,0,1,0,1,0,1,0)\)      | -394     |
| \(v_2 = (0,1,0,1,0,1,0,1)\)      | -4       |
| \(v_3 = (1,0,0,0,1,1,0)\)        | -392     |
| \(v_4 = (1,0,1,0,0,1,0,1)\)      | -6       |
| \(v_5 = (1,0,0,1,1,0,1,0)\)      | -602     |
| \(v_6 = (1,0,0,1,1,0,0,1)\)      | -592     |
| \(v_7 = (1,0,0,1,0,1,1,0)\)      | -204     |
| \(v_8 = (1,0,0,1,0,1,0,1)\)      | -198     |
| \(v_9 = (0,1,1,0,1,0,1,0)\)      | -198     |
| \(v_{10} = (0,1,1,0,1,0,0,1)\)   | -204     |
| \(v_{11} = (0,1,1,0,0,1,1,0)\)   | -592     |
| \(v_{12} = (0,1,1,0,0,1,0,1)\)   | -602     |
| \(v_{13} = (0,1,0,1,1,0,1,0)\)   | -6       |
| \(v_{14} = (0,1,0,1,1,0,0,1)\)   | -392     |
| \(v_{15} = (0,1,0,1,0,1,1,0)\)   | -4       |
| \(v_{16} = (0,1,0,1,0,1,0,1)\)   | -394     |

**Table 1.** The vertices of \(\Gamma_{(2,2,2,2)}\) and the value of \(f(v_i)\), where \(f(x) = (x-q)^TAx\) and \(q = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in \text{int}(\Gamma_{(2,2,2,2)})\).

| Equilibrium on 3d-faces of \(\Gamma_{(2,2,2,2)}\) | \(f(q_i)\) |
|-----------------------------------------------|----------|
| \(q_1 = (0.05266, 0.9473, 0.93275, 0.0672483, 0.991199, \frac{9049}{1028189}, 0.1)\) | -201.7 |
| \(q_2 = (0.9473, 0.05266, 0.0672483, 0.93275, \frac{9049}{1028189}, 0.991199, 1.0)\) | -201.7 |

**Table 2.** The equilibria on 3d-faces of \(\Gamma_{(2,2,2,2)}\) and the value of \(f(q_i)\), where \(f(x) = (x-q)^TAx\) and \(q = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in \text{int}(\Gamma_{(2,2,2,2)})\).

All these equilibria belong to \(\partial \Gamma_{(2,2,2,2)}\) and satisfy
(1) $(Ax)_1 = (Ax)_2 = (Ax)_3 = (Ax)_4 = (Ax)_5 = (Ax)_6 = (Ax)_7 = (Ax)_8,$

(2) $x_1^\alpha + x_2^\alpha = 1,$ for all $\alpha \in \{1, \ldots, 4\},$

where $x = (x_1^1, x_1^2, x_1^3, x_1^4, x_2^1, x_2^2, x_2^3, x_2^4) \in \mathbb{R}^8.$

We have that all equilibria on $\partial \Gamma(2,2,2,2)$ satisfy (6). In fact, we have that for all equilibria $x \in \partial \Gamma(2,2,2,2),$ $(x - q)^T A x < 0,$

with

$$q = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \in \text{int} \left(\Gamma(2,2,2,2)\right),$$

as we can see in Table 1, Table 2, Table 3, and Table 4.
Hence, by Theorem 4.3, we can conclude that the system defined by the payoff matrix $A$ is permanent.

We prove now that this system is not dissipative. By definition (see [2, Definition 5.1]) this system is dissipative if there exists a positive diagonal matrix

$$D = \begin{bmatrix}
    d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & d_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & d_2 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & d_3 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & d_4 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & d_4 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & d_4 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & d_4
\end{bmatrix},$$

such that the quadratic form $Q_{AD} : H_{(2,2,2,2)} \rightarrow \mathbb{R}$ defined by $Q_{AD}(x) = x^TADx$ is negative semidefinite, where $H_{(2,2,2,2)} = \{x \in \mathbb{R}^8 : x_1^\alpha + x_2^\alpha = 0, \text{ for all } \alpha \in \{1, \ldots, 4\}\}$. By the definition of $H_{(2,2,2,2)}$ we have that the symmetric matrix associated to the quadratic form $Q_{AD}(x)$ is the four dimensional square matrix

$$S = \begin{bmatrix}
    2d_1 & 101d_1 - d_2 & d_1 - 100d_3 & d_1 - 100d_4 \\
    101d_1 - d_2 & -200d_2 - 100d_3 & 5d_2 - 100d_4 \\
    d_1 - 100d_3 & 100d_2 - d_3 & -380d_3 & 100d_3 - 101d_4 \\
    d_1 - 100d_4 & 5d_2 - 100d_4 & 100d_3 - 101d_4 & -200d_4
\end{bmatrix}.$$

We can see, for example by the criterion of the principal minors, that this symmetric matrix $S$ is not negative semidefinite (notice that $d_1, d_2, d_3$ and $d_4$ must be positive). Hence, we conclude that the system defined by the payoff matrix $A$ is not dissipative. Since we have already seen that this system is permanent, this example illustrates that dissipativity is not necessary for permanence.

5.2. Example 2. Consider a population divided in 3 groups where individuals of each group have exactly 2 strategies to interact with other members of the population, where associated payoff matrix is

$$A = \begin{bmatrix}
    0 & -102 & 0 & 79 & 0 & 18 \\
    102 & 0 & 0 & -79 & -18 & 9 \\
    0 & 0 & 0 & 0 & 9 & -18 \\
    -51 & 51 & 0 & 0 & 0 & 0 \\
    0 & 102 & -79 & 0 & -18 & -9 \\
    -102 & -51 & 158 & 0 & 9 & 0
\end{bmatrix}.$$

The phase space of the associated polymatrix replicator defined by the payoff matrix $A$ is the prism $\Gamma_{(2,2,2)} := \Delta^1 \times \Delta^1 \times \Delta^1 \equiv [0,1]^3$, represented in Figure 1.

This system only has one interior equilibrium,

$$q = \left(\begin{array}{ccccc}
    1 & 1 & 71 & 87 & 2 \\
    2 & 2 & 158 & 158 & 3 \\
    3 & 3 & 3 & 3 & 3
\end{array}\right) \in \text{int} (\Gamma_{(2,2,2,2)}).$$

Moreover, besides the 8 vertices of $\Gamma_{(2,2,2,2)},$

$$v_1 = (1, 0, 1, 0, 1, 0), \quad v_2 = (1, 0, 1, 0, 0, 1), \quad v_3 = (1, 0, 0, 1, 1, 0),$$
$$v_4 = (1, 0, 0, 1, 0, 1), \quad v_5 = (0, 1, 1, 0, 1, 0), \quad v_6 = (0, 1, 1, 0, 0, 1),$$
$$v_7 = (0, 1, 0, 1, 1, 0), \quad v_8 = (0, 1, 0, 1, 0, 1).$$
it has 2 equilibria on two opposite 2d-faces of \( \Gamma_{(2,2,2)} \),

\[
q_1 = \left( \frac{7}{17}, 10, \frac{37}{79}, \frac{42}{79}, 1, 0 \right), \quad \text{and} \quad q_2 = \left( \frac{23}{34}, \frac{11}{34}, \frac{65}{158}, \frac{93}{158}, 0, 1 \right),
\]

as represented in Figure 1. In fact, all these equilibria satisfy

1. \((Ax)_1 = (Ax)_2, (Ax)_3 = (Ax)_4, (Ax)_5 = (Ax)_6,\)
2. \(x_1^\alpha + x_2^\alpha = 1, \text{ for all } \alpha \in \{1, \ldots, 3\},\)

where \(x = (x_1^1, x_2^1, x_1^2, x_2^2, x_1^3, x_2^3) \in \mathbb{R}^6.\)

Consider the positive diagonal matrix

\[
D = \begin{bmatrix}
\frac{1}{51} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{51} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{79} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{79} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{9} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{9}
\end{bmatrix},
\]

and the affine subspace \(H_{(2,2,2)} = \{x \in \mathbb{R}^6 : x_1^\alpha + x_2^\alpha = 0, \text{ for all } \alpha \in \{1, \ldots, 3\}\}.\)

The quadratic form \(Q_{AD} : H_{(2,2,2)} \longrightarrow \mathbb{R} \) defined by \(Q_{AD}(x) = x^T ADx,\) is \(Q_{AD}(x) = -2(x_1^3)^2 \leq 0.\) Hence, by definition (see [2, Definition 5.1]), this system is dissipative.

In fact this system has a strict global Lyapunov function \(h : \text{int}(\Gamma_{(2,2,2,2)}) \rightarrow \mathbb{R} \) for \(X_A,\) defined by

\[
h(x) = -\sum_{\alpha=1}^{3} \sum_{i=1}^{2} \frac{q_i^\alpha}{d_\alpha} \log x_i^\alpha,
\]

where \(X_A\) is the associated vector field, \(q\) is the interior equilibrium, and \(d_1 = \frac{1}{51},\)
\(d_2 = \frac{1}{79},\)
\(d_3 = \frac{1}{9}\) \(\) (the elements on the main diagonal of matrix \(D\)). This function \(h\) has an absolute minimum at \(q\) and satisfies \(\frac{dh}{dt} = Dh(x_A) < 0\) for all \(x \in \text{int}(\Gamma_{(2,2,2,2)})\) with \(x \neq q.\) Hence, by Proposition 13 and Proposition 17 in [2], the
\( \omega \)-limit of any interior point \( x \in \text{int} \left( \Gamma_{(2,2,2,2)} \right) \) is the equilibrium \( q \). The equilibria \( q_1 \) and \( q_2 \) in the faces of \( \Gamma_{(2,2,2,2)} \) are centres in each corresponding face, i.e., for any initial condition in one of these faces its orbit will be periodic around the equilibrium point in that corresponding face.

Since the \( \omega \)-limit of any interior point \( x \in \text{int} \left( \Gamma_{(2,2,2,2)} \right) \) is the equilibrium \( q \), and this is the only interior equilibrium, it follows that the system is permanent. However, this second example does not satisfies condition (6) of Theorem 4.3. In fact there is no \( q \in \text{int} \left( \Gamma_{(2,2,2,2)} \right) \) such that

\[
x^T A x - q^T A x < 0,
\]

for all equilibria \( x \in \partial \Gamma_{(2,2,2,2)} \). Hence, this example illustrates that condition (6) is not necessary for permanence.

Acknowledgments. The author wishes to express his gratitude to Pedro Duarte for stimulating conversations and many significant suggestions.

REFERENCES

[1] H. N. Alishah and P. Duarte, Hamiltonian evolutionary games, Journal of Dynamics and Games, 2 (2015), 33–49.
[2] H. N. Alishah, P. Duarte and T. Peixe, Conservative and dissipative polymatrix replicators, Journal of Dynamics and Games, 2 (2015), 157–185.
[3] H. N. Alishah, P. Duarte and T. Peixe, Asymptotic Poincaré maps along the edges of polytopes, Nonlinearity, 33 (2020), 469–510.
[4] P. Duarte, R. L. Fernandes and W. M. Oliva, Dynamics of the attractor in the Lotka-Volterra equations, J. Differential Equations, 149 (1998), 143–189.
[5] J. Hofbauer, On the occurrence of limit cycles in the Volterra-Lotka equation, Nonlinear Anal., 5 (1981), 1003–1007.
[6] J. Hofbauer and K. Sigmund, Permanence for Replicator Equations, Springer Berlin Heidelberg, 1987.
[7] J. Hofbauer and K. Sigmund, Evolutionary Games and Population Dynamics, Cambridge University Press, Cambridge, 1998.
[8] J. Hofbauer, A general cooperation theorem for hypercycles, Monatsh. Math., 91 (1981), 233–240.
[9] W. Jansen, A permanence theorem for replicator and Lotka-Volterra systems, J. Math. Biol., 25 (1987), 411–422.
[10] A. J. Lotka, Elements of mathematical biology, (formerly published under the title Elements of Physical Biology), Dover Publications, Inc., New York, 1958.
[11] J. M. Smith, The logic of animal conflict, Nature, 246 (1973), 15–18.
[12] P. Schuster and K. Sigmund, Coyness, philandering and stable strategies, Animal Behaviour, 29 (1981), 186–192.
[13] P. Schuster and K. Sigmund, Replicator dynamics, J. Theoret. Biol., 100 (1983), 533–538.
[14] P. Schuster, K. Sigmund, J. Hofbauer and R. Wolff, Self-regulation of behaviour in animal societies, Biol. Cybernet., 40 (1981), 9–15.
[15] P. Schuster, K. Sigmund and R. Wolff, Dynamical systems under constant organization. iii. cooperative and competitive behavior of hypercycles, Journal of Differential Equations, 32 (1979), 357–368.
[16] P. D. Taylor and L. B. Jonker, Evolutionarily stable strategies and game dynamics, Math. Biosci., 40 (1978), 145–156.
[17] V. Volterra, Leçons sur la Théorie Mathématique de la Lutte pour la Vie (Reprint of the 1931 original), Editions Jacques Gabay, Sceaux, 1990.
[18] J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, Princeton, New Jersey, 1944.

Received for publication May 2020.

E-mail address: telmop@iseg.ulisboa.pt