Contact symmetries and conservation laws of the first order of the equation of one-dimensional shallow water over a rough bottom in Lagrange’s variables

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Abstract. The systems of equations of one-dimensional shallow water over a rough bottom in Euler’s and Lagrange’s variables are considered. The problem of group classification of contact symmetries of the shallow water equation in Lagrange’s variables is solved. First order conservation laws of the shallow water equation in Lagrange’s variables obtained using the Noether’s theorem. Bottom profiles, providing additional conservation laws, are given.

1. Introduction
There are different approaches to the construction of conservation laws for equations of mathematical physics [1, 2, 3, 4]. The most well-known method of constructing conservation laws is based on the use of Noether’s theorem.

Hydrodynamic conservation laws of one-dimensional shallow water equations over a rough bottom in Euler’s variables were obtained in [5, 6].

In present work the problem of group classification of contact symmetries for the equation of shallow water in Lagrangian variables is solved and the first order conservation laws using the Noether’s theorem are obtained.

2. Basic equations
In dimensionless variables, the system of one-dimensional shallow-water equations over uneven bottom has the following form [7]

\begin{align}
    u_t + uu_x + \eta_x &= 0, \\
    \eta_t + (\eta + h(x))u_x &= 0.
\end{align}

Here \( h(x) \) is the thickness of the unperturbed layer of the liquid, \( u = u(x, t) \) is the depth-average horizontal velocity and \( \eta = \eta(x, t) \) is the deviation of the free surface \( (\eta + h(x) \geq 0) \). The bottom profile is given by the relation \( z = -h(x) \) (\( z \) is the vertical coordinate).

Using the second equation of the system (1) we introduce a new variable \( m = m(x, t) \) and consider the following system of equations

\begin{align}
    u_t + uu_x + \rho_x &= h'(x), \\
    m_x &= \rho, \\
    m_t &= -u\rho,
\end{align}
where \( \rho = \eta + h(x) \). From the second and third equations of the system of equations (2) it follows that due to the relation
\[
\frac{dm}{dt} = m_t + um_x = 0
\]
the variable \( m \) is Lagrange’s variable.

One can get the equation of one-dimensional shallow water in Lagrange’s variables by choosing \( m \) and \( t \) as independent variables \([8]\)
\[
x_{tt} - \frac{x_{mm}}{x_m^3} = h'(x).
\]

3. Group classification of contact symmetries of the equation of one-dimensional shallow water in Lagrange’s variables

A contact symmetry operator of the equation (3) has canonical form \([9, 10]\)
\[
Y = W \frac{\partial}{\partial x}.
\]
Here \( W = W(m, t, x, x_m, x_t) \) is characteristic of the symmetry
\[
X = \xi^1 \frac{\partial}{\partial m} + \xi^2 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x_m} + \zeta^1 \frac{\partial}{\partial x_t},
\]
where
\[
\xi^1 = -\frac{\partial W}{\partial x_m}, \quad \xi^2 = -\frac{\partial W}{\partial x_t}, \quad \eta = W + \xi^1 x_m + \xi^2 x_t,
\]
\[
\zeta^1 = \frac{\partial W}{\partial m} + x_m \frac{\partial W}{\partial x}, \quad \zeta^2 = \frac{\partial W}{\partial t} + x_t \frac{\partial W}{\partial x}.
\]

Using the invariance criterion \([9, 10]\), we get the following overdetermined system of defining equations
\[
\frac{\partial^2 W}{\partial x_t^2} - x_m \frac{\partial^2 W}{\partial x_m^2} = 0,
\]
\[
\frac{\partial^2 W}{\partial m \partial x_t} - x_m^3 \frac{\partial^2 W}{\partial m \partial x_m} - x_m^2 x_t \frac{\partial^2 W}{\partial x \partial x_m} + x_m \frac{\partial^2 W}{\partial x \partial x_t} - h'(x) x_m^3 \frac{\partial^2 W}{\partial x_m \partial x_t} = 0,
\]
\[
2x_m \frac{\partial^2 W}{\partial m \partial x_m} - 2x_m \frac{\partial^2 W}{\partial t \partial x_m} + 2x_m^2 \frac{\partial^2 W}{\partial x \partial x_m} - 2x_m x_t \frac{\partial^2 W}{\partial x \partial x_t} - 2h'(x) x_m^3 \frac{\partial^2 W}{\partial x^2} - 3 \frac{\partial W}{\partial m} - 3x_m \frac{\partial W}{\partial x} = 0,
\]
\[
\frac{\partial^2 W}{\partial m^2} + 2x_m \frac{\partial^2 W}{\partial m \partial x} - 2h'(x) x_m^3 \frac{\partial^2 W}{\partial m \partial x_m} - x_m^3 \frac{\partial^2 W}{\partial t^2} - 2x_m x_t \frac{\partial^2 W}{\partial t \partial x} + x_m^2 (1 - x_m x_t) \frac{\partial^2 W}{\partial x^2} - 2h'(x) x_m^4 \frac{\partial^2 W}{\partial x \partial x_m} + (h'(x))^2 x_m^6 \frac{\partial^2 W}{\partial x^2} + 3h'(x) x_m^2 \frac{\partial W}{\partial m} + 2h'(x) x_m^3 \frac{\partial W}{\partial x} - h''(x) x_m^3 \left( x_m \frac{\partial W}{\partial x} + \frac{\partial W}{\partial x_t} - W \right).
\]

This system was tested for compatibility. The following are the results of this investigation for all possible bottom profiles \( h = h(x) \).

1. \( h = h(x) \) is an arbitrary function. For any bottom profile \( h(x) \), the equation (3) has contact symmetries with characteristics
\[
W_1 = x_m, \quad W_2 = 2x_t.
\]
Corresponding symmetry operators are
\[ X_1 = -\frac{\partial}{\partial m}, \quad X_2 = -2\frac{\partial}{\partial t}. \]

2. \( h = a_1 x + a_2. \) Denote \( r = x_t - a_1 t, \ s = 1/x_m, \) then characteristics of additional contact symmetries of the equation (3) are
\[ W_3 = 2t, \quad W_4 = 48tr^2 + 48ts + 2(9a_1 t^2 - 18x)r - \frac{12mr}{s}, \]
\[ W_5 = 20tr + 6a_1 t^2 - 12x - \frac{4m}{s}, \quad W_6 = 2x - 3tr - a_1 t^2, \quad W_\infty = \frac{w(r, s)}{s}, \]
where \( w = w(r, s) \) is an arbitrary solution of the linear equation
\[ w_{rr} = sw_{ss}. \]

Corresponding symmetry operators are
\[ X_3 = 2t \frac{\partial}{\partial x} + 2 \frac{\partial}{\partial x_t}, \]
\[ X_4 = \left( 48ts^2 + 12mr \right) \frac{\partial}{\partial m} + \left( \frac{12m}{s} - 96tr - 18a_1 t^2 + 36x \right) \frac{\partial}{\partial t} + \]
\[ + \left( 96ts + 48r^2 - 2a_1 t(9a_1 t^2 - 18x) - 96t(r + a_1 t)r + \frac{12m}{s} (r + a_1 t) \right) \frac{\partial}{\partial x}, \]
\[ + \left( 48s + 12r^2 - 96a_1 tr - 2a_1 (9a_1 t^2 - 18x) + \frac{12a_1 m}{s} \right) \frac{\partial}{\partial x_m}, \]
\[ X_5 = 4m \frac{\partial}{\partial m} - 20t \frac{\partial}{\partial t} - (14a_1 t^2 + 12x) \frac{\partial}{\partial x_m} - \frac{16m}{s} \frac{\partial}{\partial x_t} + \frac{8}{s} (8r - 20a_1 t) \frac{\partial}{\partial x_t}, \]
\[ X_6 = 3t \frac{\partial}{\partial t} + 2(x + a_1 t^2) \frac{\partial}{\partial x} + 2 \frac{\partial}{\partial x_m} - (3a_1 t - r) \frac{\partial}{\partial x_t}, \]
\[ X_\infty = (sw_s - w) \frac{\partial}{\partial m} - w_t \frac{\partial}{\partial t} + (w_s - (r + a_1 t) \frac{w_r}{s}) \frac{\partial}{\partial x} - a_1 w_r \frac{\partial}{\partial x_t}. \]

3. \( h = a_1 x^2/2 + a_2 x + a_3. \)

3.1. \( h = a_1 x^2/2 + a_2 x + a_3, \ a_1 > 0. \) In this case characteristics of additional contact symmetries of the equation (3) are
\[ W_3 = \sqrt{a_1} e^{-t\sqrt{a_1}}, \quad W_4 = -\sqrt{a_1} e^{t\sqrt{a_1}}, \quad W_5 = 3a_1 mx_m - a_1 x - a_2. \]

Corresponding symmetry operators are
\[ X_3 = \sqrt{a_1} e^{-t\sqrt{a_1}} \frac{\partial}{\partial x} - a_1 e^{-t\sqrt{a_1}} \frac{\partial}{\partial x_t}, \quad X_4 = -\sqrt{a_1} e^{t\sqrt{a_1}} \frac{\partial}{\partial x} - a_1 e^{t\sqrt{a_1}} \frac{\partial}{\partial x_t}, \]
\[ X_5 = -3a_1 m \frac{\partial}{\partial m} - (a_1 x + a_2) \frac{\partial}{\partial x} - a_1 x_t \frac{\partial}{\partial x_t} + 2a_1 x_m \frac{\partial}{\partial x_m}. \]

3.2. \( h = a_1 x^2/2 + a_2 x + a_3, \ a_1 < 0. \) In this case characteristics of additional contact symmetries of the equation (3) are
\[ W_3 = \sqrt{-a_1} \sin(t\sqrt{-a_1}), \quad W_4 = -\sqrt{-a_1} \cos(t\sqrt{-a_1}), \quad W_5 = 3a_1 mx_m - a_1 x - a_2. \]

Corresponding symmetry operators are
\[ X_3 = \sqrt{-a_1} \sin(t\sqrt{-a_1}) \frac{\partial}{\partial x} - a_1 \cos(t\sqrt{-a_1}) \frac{\partial}{\partial x_t}, \]
\[ X_4 = -\sqrt{-a_1} \cos(t\sqrt{-a_1}) \frac{\partial}{\partial x} - a_1 \sin(t\sqrt{-a_1}) \frac{\partial}{\partial x_t}, \]
\[ X_5 = -3a_1 m \frac{\partial}{\partial m} - (a_1 x + a_2) \frac{\partial}{\partial x} - a_1 x_t \frac{\partial}{\partial x_t} + 2a_1 x_m \frac{\partial}{\partial x_m}. \]
4. \( h = a_1(x + a_2)^{a_4} + a_3, \quad a_1 \neq 0, \quad a_4 \neq 0, 1, 2 \). In this case characteristic of additional contact symmetry of the equation (3) is
\[
W_3 = 3(a_4 - 2)tx + 6(a_4 + 1)m + 6(x + a_2).
\]
Corresponding symmetry operator is
\[
X_3 = -3(a_4 - 2)t \frac{\partial}{\partial t} + 6(a_4 + 1)m \frac{\partial}{\partial m} + 6(x + a_2) \frac{\partial}{\partial x} - 6a_4x_m \frac{\partial}{\partial x_m} + 3a_4x_t \frac{\partial}{\partial x_t}.
\]
5. \( h = a_1 \ln(x + a_2) + a_3, \quad a_1 \neq 0, \quad x > -a_2 \). In this case characteristic of additional contact symmetry of the equation (3) is
\[
W_3 = tx + mx - (x + a_2).
\]
Corresponding symmetry operator is
\[
X_3 = -t \frac{\partial}{\partial t} - m \frac{\partial}{\partial m} - (x + a_2) \frac{\partial}{\partial x}.
\]
6. \( h = a_1 e^{a_2x} + a_3, \quad a_1 \neq 0, \quad a_2 \neq 0 \). In this case characteristic of additional contact symmetry of the equation (3) is
\[
W_3 = a_2tx - 2a_2mx + 2.
\]
Corresponding symmetry operator is
\[
X_3 = -a_2t \frac{\partial}{\partial t} + 2a_2m \frac{\partial}{\partial m} + 2 \frac{\partial}{\partial x} - 2a_2x_m \frac{\partial}{\partial x_m} + a_2x_t \frac{\partial}{\partial x_t}.
\]

4. Conservation laws of the first order
One can show that the equation (3) is an Euler-Lagrange equation for a variational problem with Lagrange function
\[
L = \frac{x^2}{2} - \frac{1}{2x_m} + h(x).
\]
One can derive conservation laws of the equation (3) with Noether’s theorem [1, 2]. By conservation laws of the equation (3) we understand relations in divergent form
\[
D_m(P) + D_t(Q) = 0
\]
that hold modulo (3). Let us recall, that symmetry with operator (4), is variational symmetry if there is a vector field \((T_1, T_2)\), that
\[
X(L) + L(D_m(\xi^1) + D_t(\xi^2)) = D_m(T_1) + D_t(T_2),
\]
where \(D_m, D_t\) are total derivatives.
Using the results of group classification of the equation (3) and the relation (5), we find the variational symmetries and corresponding conservation laws.
1. \( h = h(x) \) is an arbitrary function. Symmetries of the equation (3) with operators
\[
X_1 = -\frac{\partial}{\partial m}, \quad X_2 = -2\frac{\partial}{\partial t}
\]
are variational symmetries. Characteristics of these symmetries are characteristics of conservation laws with functions
\[
P_1 = -\frac{x^2}{2} + \frac{1}{x_m} - h(x), \quad Q_1 = x_t x_m,
\]
\[
P_2 = x_t \left( \frac{1}{x_m} - h^2(x) \right), \quad Q_2 = x_t^2 + x_m \left( \frac{1}{x_m} - h(x) \right)^2.
\]
2. \( h = a_1x + a_2 \). In this case additional variational symmetries of the equation (3) are
symmetries with operators
\[ X_3 = 2t \frac{\partial}{\partial x} + 2 \frac{\partial}{\partial x_t}, \]
\[ X_4 = (48ts^2 + 12mr) \frac{\partial}{\partial m} + \left( \frac{12m}{s} - 96t - 18a_1 t^2 + 36x \right) \frac{\partial}{\partial t} + \]
\[ + \left( 96ts + 48t^2 - 2a_1 t(9a_1 t^2 - 18x) - 96t(r + a_1 t) + \frac{12m}{s} r (r + a_1 t) \right) \frac{\partial}{\partial x} + \]
\[ + \left( 48s + 12r^2 - 96a_1 t r - 2a_1 (9a_1 t^2 - 18x) + \frac{12a_1 m}{s} \right) \frac{\partial}{\partial x_t} - \frac{48r}{s} \frac{\partial}{\partial x_m}, \]
\[ X_5 = 4m \frac{\partial}{\partial m} - 20t \frac{\partial}{\partial t} - (14a_1 t^2 + 12x) \frac{\partial}{\partial x} - \frac{16}{s} \frac{\partial}{\partial x_m} + (8r - 20a_1 t) \frac{\partial}{\partial x_t}, \]
\[ X_\infty = (sq_{rs} - q_r) \frac{\partial}{\partial m} - \frac{q_{rs}}{s} \frac{\partial}{\partial t} + \left( q_{rs} - (r + a_1 t) \frac{q_{rr}}{s} \right) \frac{\partial}{\partial x} - a_1 \frac{q_{rr}}{s} \frac{\partial}{\partial x_t}, \]
where \( p = p(r, s), q = q(r, s) \) is an arbitrary solution of the linear system of equations
\[ p_r = sq_r + q_r, \quad p_s = rq_s + q_r. \] (6)
Characteristics of these symmetries are characteristics of conservation laws with functions
\[ P_3 = (a_1 a_2 t^2 - 2a_2 x) r + ts^2 - a_2^2 t, \quad Q_3 = 2t r + \frac{2a_2 x - a_1 a_2 t^2}{s} + a_1 t^2 - 2x, \]
\[ P_4 = 2mr^3 + 24mr^2 s - (18x - 9a_1 t^2) rs^2 - 12m rs + 16ts^3, \]
\[ Q_4 = 16tr(r^2 + 3s) + (9a_1 t^2 - 18x)(r^2 + s) - \frac{6m r^2}{s} - 12m \ln s, \]
\[ P_5 = 10tr^2 s^2 + (3a_1 t^2 - 6x)s^2 - 4ms, \quad Q_5 = 10r^2 t + 10ts + (6a_1 t^2 - 12x)r - \frac{4mr}{s}, \]
\[ P_\infty = p(r, s) - rq(r, s), \quad Q_\infty = \frac{q(r, s)}{s}, \]
where \( r = x_t - a_1 t, s = 1/x_m; \) pair \( p(r, s), q(r, s) \) is an arbitrary solution of the system of equations (6).

3. \( h = a_1 x^2/2 + a_2 x + a_3. \)

3.1. \( h = a_1 x^2/2 + a_2 x + a_3, \quad a_1 > 0. \) In this case additional variational symmetries of the equation (3) are symmetries with operators
\[ X_3 = \sqrt{a_1} e^{-t \sqrt{a_1}} \frac{\partial}{\partial x} - a_1 e^{-t \sqrt{a_1}} \frac{\partial}{\partial x_t}, \quad X_4 = -\sqrt{a_1} e^{t \sqrt{a_1}} \frac{\partial}{\partial x} - a_1 e^{t \sqrt{a_1}} \frac{\partial}{\partial x_t}, \]
Characteristics of these symmetries are characteristics of conservation laws with functions
\[ P_3 = e^{-t \sqrt{a_1}} \left( x_t h + \sqrt{a_1} \left( \frac{1}{x_m^2} - h^2 \right) \right), \quad Q_3 = e^{-t \sqrt{a_1}} (\sqrt{a_1} x_t + (1 - x_m h) h'), \]
\[ P_4 = e^{t \sqrt{a_1}} \left( x_t h' - \sqrt{a_1} \left( \frac{1}{x_m^2} - h^2 \right) \right), \quad Q_4 = e^{t \sqrt{a_1}} (-\sqrt{a_1} x_t + (1 - x_m h) h'). \]

3.2. \( h = a_1 x^2/2 + a_2 x + a_3, \quad a_1 < 0. \) In this case additional variational symmetries of the equation (3) are symmetries with operators
\[ X_3 = -\sqrt{-a_1} \sin(t \sqrt{-a_1}) \frac{\partial}{\partial x} - a_1 \cos(t \sqrt{-a_1}) \frac{\partial}{\partial x_t}, \quad X_4 = -\sqrt{-a_1} \cos(t \sqrt{-a_1}) \frac{\partial}{\partial x} - a_1 \sin(t \sqrt{-a_1}) \frac{\partial}{\partial x_t}. \]
Characteristics of these symmetries are characteristics of conservation laws with functions
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References
[1] Noether E 1918 Invariante Variationsprobleme Nachr. D. König. Gesellsch. D. Wissen. Zu Göttingen, Math.-Phys. Klasse. 235–257 (English translation: Transport Theory and Stat. Phys. 1971 1 (3) 186–207)
[2] Olver P J 1993 Applications of Lie Groups to Differential Equations (New York: Springer-Verlag)
[3] Symmetries and Conservation Laws for Differential Equations of Mathematical Physics Editors: A M Vinogradov and I S Krasil’schik 1999 (American Mathematical Society)
[4] Bluman G W Cheviakov A F and Anco S C Applications of Symmetry Methods to Partial Differential Equations 2009 (New York: Springer-Verlag)
[5] Aksenov A V and Druzhkov K P 2016 Zakony sokhranenija, simmetrii i tochnye reshenija sistemy uravnenij melkoj vody nad nerovnym dnom [ Conservation Laws, Symmetries, and Exact Solutions of the System of Equations of Shallow Water over an Irregular Bottom], Vestnik Nacionalnogo issledovatelskogo yadernogo universiteta MIIF 5 (1) 38–46
[6] Aksenov A V and Druzhkov K P 2016 Conservation laws and symmetries of the shallow water system above rough bottom J. Phys.: Conf. Ser. 2016 722 (012001) 1–7
[7] Stoker J J 1957 Water Waves. The Mathematical Theory With Applications (New York: Interscience Publishers)
[8] Cherny G G 1988 Gas dynamics (Moscow: Nauka) (in Russian).
[9] Ovsyannikov L V 1982 Group Analysis of Differential Equations (New York: Academic Press)
[10] Ibragimov N H 1985 Transformation Groups Applied to Mathematical Physics (New York: Springer-Verlag)

\[
P_3 = \cos(t\sqrt{-a_1})x_t x_h + \frac{\sqrt{-a_1}}{2} \sin(t\sqrt{-a_1}) \left( \frac{1}{x_m} - h^2 \right),
\]

\[
Q_3 = \sin(t\sqrt{-a_1}) \sqrt{-a_1} x_t + \cos(t\sqrt{-a_1})(1 - x_m h)h',
\]

\[
P_4 = \sin(t\sqrt{-a_1})x_t h' - \frac{\sqrt{-a_1}}{2} \cos(t\sqrt{-a_1}) \left( \frac{1}{x_m} - h^2 \right),
\]

\[
Q_4 = -\cos(t\sqrt{-a_1}) \sqrt{-a_1} x_t + \sin(t\sqrt{-a_1})(1 - x_m h)h'.
\]

4. \( h = a_1(x + a_2)^{-4/3} + a_3, \ a_1 \neq 0, \ x > -a_2. \) In this case additional variational symmetry of the equation (3) is symmetry with operator

\[
X_3 = 10t \frac{\partial}{\partial t} - 2m \frac{\partial}{\partial m} + 6(x + a_2) \frac{\partial}{\partial x} + 8x_m \frac{\partial}{\partial x_m} - 4x_t \frac{\partial}{\partial x_t}.
\]

Characteristic of this symmetry is characteristic of conservation law with functions

\[
P_3 = -5 \left( \frac{x_t}{x_m^2} - m x_t^2 + \frac{3(x + a_2)}{x_m^2} + 2m \left( \frac{1}{x_m} - h \right) \right),
\]

\[
Q_3 = -5 \left( x_t^2 + \frac{1}{x_m^2} \right) + 6(x + a_2)x_t + (10h - 8a_3)t + 2m x_t x_m.
\]

One can show that other symmetries of the equation (3) are non-variational.

5. Conclusion
The main results of the work are the solution of a group classification problem for the equation of one-dimensional shallow water in Lagrangian variables and the derivation of all it’s first order conservation laws. The paper presents bottom profiles with extensions of contact symmetry operators kernel and conservation laws kernel. These results can be used to build new exact solutions.