FINITE ELEMENT APPROXIMATION OF POWER MEAN CURVATURE FLOW

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Abstract. In [21] the evolution of hypersurfaces in $\mathbb{R}^{n+1}$ with normal speed equal to a power $\kappa > 1$ of the mean curvature is considered and the levelset solution $u$ of the flow is obtained as the $C^0$-limit of a sequence $u^\varepsilon$ of smooth functions solving the regularized levelset equations.

We prove a rate for this convergence.

Then we triangulate the domain by using a tetraeder mesh and consider continuous finite elements, which are polynomials of degree $\leq 2$ on each tetraeder of the triangulation. We show in the case $n = 1$ (i.e. the evolving hypersurfaces are curves), that there are solutions $u^\varepsilon_h$ of the above regularized equations in the finite element sense, which satisfy for every $0 < \Theta < \frac{1}{2}$ an error estimate of the form

\begin{align}
\|u - u^\varepsilon_h\|_{C^0,\Theta} \leq ce^\lambda + ce^{-\gamma}h^\delta,
\end{align}

where values for $\lambda, \gamma, \delta > 0$ can be obtained explicitly.

Our method can be extended to the case $n > 1$, if one uses higher order finite elements.

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1. Introduction and main results

The famous mean curvature flow, cf. e.g. [9] and [14], evolves hypersurfaces in the direction of their normal with normal speed equal to the mean curvature. This flow has—apart from being of great interest by itself—important applications in image processing. During the last thirty years many variants of extrinsic curvature flows have been analyzed, which differ mainly in the prescribed normal velocity and

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the ambient space, in which the evolution takes place, cf. e.g. the inverse mean curvature flow [15], the Gauss curvature flow [1] and the inverse mean curvature flow in a Lorentzian manifold [11].

Concerning the numerical analysis for these flows there exist results in the case of mean curvature flow, cf. e.g. [7], [8] and the references therein, and [18] in the case of anisotropic mean curvature flow in higher codimension. In [6] K. Deckelnick proves a rate of convergence for the approximation of the levelset solution of mean curvature flow by using a finite difference scheme; for the approximation he uses the solution of the regularized levelset equation as an intermediate step and divides the error estimate correspondently into the approximation error between the levelset solution and the solution of the regularized levelset equation and the error for the finite difference approximation of the regularized levelset equation. See also [16] for the former error estimate.

Recently F. Schulze [21] considered the evolution of hypersurfaces in $\mathbb{R}^{n+1}$ in the direction of their normal, for which the normal speed is given by a power $k > 1$ of the mean curvature.

To the author’s knowledge there do not exist any numerical results for Schulze’s flow [21] so far. Our aim is to approximate the levelset solution of this flow using the method of finite elements and to prove a convergence rate. This is done similarly to [6] by using the solution of the regularized levelset equation as an intermediate step. It will come out that a ‘polynomial coupling’ between the regularization parameter $\epsilon$ and the numerical parameter $h$ will ensure a polynomial convergence rate, cf. Theorem 1.2 and Corollary 1.3. In contrast to [6] we use a levelset formulation as suggested in [21], for which the levelset function does not depend on the time, cf. (1.3) and [6, equation (1.1)]. This ensures that the nonlinearity coming from the exponent $k$ affects only lower order (spatial) derivatives of the levelset function.

We introduce our setting more precisely.

Let $M$ be a smooth $n$-dimensional compact manifold without boundary, $k > 1$ and $x_0 : M \to \mathbb{R}^{n+1}$ a smooth embedding such that $x_0(M)$ has positive mean curvature, then there exist a small $T > 0$ and a smooth mapping

$$x : M \times [0, T) \to \mathbb{R}^n$$

with

$$x(0, \cdot) = x_0$$

$$\dot{x}(t, \xi) = -H^k \nu.$$  \hspace{1cm} (1.2)

Here, $H$ and $\nu$ denote the mean curvature and the outer normal of $x(t, \cdot)(M)$ at $x(t, \xi)$ respectively, cf. [21, Section 1].

We call this a power mean curvature flow (PMCF).

We give a level set formulation of PMCF. Let $\Omega \subset \mathbb{R}^{n+1}$ be open, connected and bounded having smooth boundary $\partial \Omega$ with positive mean curvature. We call the level sets $\Gamma_t = \partial \{ x \in \Omega : u(x) > t \}$ of the continuous function $0 \leq u \in C^0(\bar{\Omega})$ a level set PMCF, if $u$ is a viscosity solution of

$$\text{div} \left( \frac{Du}{|Du|^k} \right) = -\frac{1}{|Du|^{\frac{k}{2}}}$$

$$u|_{\partial \Omega} = 0.$$  \hspace{1cm} (1.3)

If $u$ is smooth in a neighborhood of $x \in \Omega$ with non vanishing gradient and satisfies there (1.3), then the level set $\{ u = u(x) \}$ moves locally at $x$ according to (1.2).
Using elliptic regularization of level set PMCF we obtain the equation

\[ \text{div} \left( \frac{Du^\varepsilon}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}} \right) = - \left( \varepsilon^2 + |Du^\varepsilon|^2 \right)^{-\frac{1}{2}} \frac{1}{n} k \text{ in } \Omega \]

\[ u^\varepsilon = 0 \text{ on } \partial \Omega, \]

which has unique smooth solutions \( u^\varepsilon \) for sufficiently small \( \varepsilon > 0 \); moreover, there is \( c_0 > 0 \) such that

\[ \|u^\varepsilon\|_{C^1(\bar{\Omega})} \leq c_0 \]

and (for a subsequence)

\[ u^\varepsilon \rightarrow u \in C^{0,1}(\bar{\Omega}) \]

in \( C^0(\bar{\Omega}) \). We call \( u \) a weak solution of (1.3), which is unique for \( n \leq 6 \).

All the above facts are proved in [21, Section 4].

A weak solution of (1.3) satisfies (1.3) in the viscosity sense, cf. Section 2. We formulate our first main result.

**Theorem 1.1.** For every \( 0 < \Theta < 1 \), there is \( 0 < \lambda = \lambda(\Theta, k) \) so that

\[ \|u - u^\varepsilon\|_{C^0(\bar{\Omega})} \leq c \varepsilon^\lambda, \]

where \( c = c(\Theta, k, \Omega) > 0 \) is a constant.

We need some notations before we formulate our second main result in Theorem 1.2. Let \( \{T_h : 0 < h < h_0\} \) be a family of regular triangulations of \( \Omega \), \( h \) the mesh size of \( T_h \) and \( h_0 = h_0(\Omega) > 0 \) small, so that for each boundary tetraeder \( T \in T_h \) \( n + 1 \) vertexes lie on \( \partial \Omega \). We define

\[ \Omega^h = \bigcup_{T \in T_h} T; \]

since \( \Omega \) might lack convexity, there will not hold in general \( \Omega^h \subset \bar{\Omega} \). Let

\[ \Omega^h = \{w \in C^0(\bar{\Omega}^h) : \forall_{T \in T_h} w|_T \text{ polynom of degree } \leq 2, w|_{\partial \Omega^h} = 0\}. \]

Let \( d : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \), be the signed distance function of \( \partial \Omega \) (sign convention so that \( d|_{\Omega} < 0 \)) and \( \delta_0 = \delta_0(\Omega) > 0 \) small. For \( 0 < \delta < \delta_0 \) we define

\[ \Omega^\delta = \{d < \delta\} \]

and have \( \partial \Omega^\delta \subset C^\infty, \partial \Omega^\delta \|_C \leq c(\Omega)\|\partial \Omega\|_C \). Furthermore, there is a constant \( 0 < c = c(\Omega) \) so that

\[ \|u^\delta\|_C \leq c \|u^\varepsilon\|_C, \]

cf. [12, Lemma 1.2.16 in PDE II].

**Theorem 1.2.** Let \( n = 1 \) and \( n + 1 < \mu < 4 \) and \( 1 < \delta < \frac{1}{2} + \frac{2}{\mu} \), then there exist \( \beta, \gamma, c > 0 \) depending on \( \mu, \delta, k, \Omega \), so that for every \( 0 < \epsilon < \epsilon_0 \), where \( \epsilon_0 > 0 \) small, and \( h \leq c \epsilon^\beta \) the equation

\[ \int_{\Omega^h} \frac{\langle Du^\varepsilon_h, D\varphi_h \rangle}{\sqrt{\varepsilon^2 + |Du^\varepsilon_h|^2}} = \int_{\Omega^h} \left( \varepsilon^2 + |Du^\varepsilon_h|^2 \right)^{-\frac{1}{2}} \varphi_h \]

\[ \forall \varphi_h \in V_h, \]
has a unique solution \( u^*_h \) in

\[
\tilde{B}^h_\rho := \{ w_h \in V_h : \| w_h - u^\epsilon \|_{H^{1,\rho}(\Omega^h)} \leq \rho \},
\]

where

\[
\rho = c\epsilon^{-\gamma} h^\delta.
\]

**Corollary 1.3.** In the situation of Theorem 1.1 and Theorem 1.2 holds for \( 0 < \Theta < \frac{1}{2} \)

\[
\| u - u^*_h \|_{C^0,\Theta(\Omega^h)} \leq c\epsilon^{\lambda} + c\epsilon^{-\gamma} h^\delta.
\]

**Remark 1.4.** (i) Our proofs of Theorems 1.1 and 1.2 are constructive in the sense, that possible values for \( \Theta, \beta, \gamma \) can be calculated explicitly.

(ii) An analogous result to Theorem 1.2 can be obtained in case \( n > 1 \) by using higher order finite elements.

The remaining part of the paper deals with the proof of the above Theorems. In Section 2 we give the definition of a viscosity solution of (1.3), that accounts for the fact, that \( Du \) might vanish. Therefore we adapt the definitions in [9] for the mean curvature flow (where a time dependent levelset function \( u \) is used) to our situation. Furthermore, we show that \( u \) in (1.6) is a viscosity solution of (1.3).

In Section 3 we prove Theorem 1.1, for what we modify an argument, which is used in [6] to prove a corresponding result for the levelset formulation of mean curvature flow (in [6] a time dependent levelset function is considered too).

In Section 4 we derive higher order estimates for \( u^\epsilon \) and Section 5 provides explicit constants in some estimates concerning linear equations, which are applied in Section 6 in order to prove Theorem 1.2. In Section 6 we proceed similarly to [10], where the regularized levelset equation for the inverse mean curvature flow is approximated by finite elements; in contrast to our paper, cf. especially Corollary 1.3, [10] does not provide any quantitative information about the approximation error between the finite element solution and the original geometric problem, i.e. the solution of the not regularized levelset equation.

### 2. The viscosity solution \( u \)

We give the definition of a viscosity solution of (1.3) and prove that the limit \( u \) in (1.6) is a viscosity solution of (1.3). Both seems to be standard, but since these things are not carried out in [21], we present them for reasons of completeness here.

To define a viscosity solution of (1.3) we adapt the corresponding definitions in [9, Sections 2.2 and 2.3] and [4, Section 2].

By formal differentiation we get from (1.3) that

\[
F(u) := -|Du|^{-\frac{1}{2}}(\delta_{ij} - u_i u_j |Du|^{-1})u_{ij} = 1.
\]

and from (1.4) that

\[
F_\epsilon(u^\epsilon) := -(|Du^\epsilon|^2 + \epsilon^2)^{\frac{1}{2}}(\delta_{ij} - u^\epsilon_i u^\epsilon_j |Du^\epsilon|^2 + \epsilon^2)u_{ij}^\epsilon = 1.
\]

We need the following definitions.
Definition 2.1. Let \( u \in C^0(\Omega) \) and \( \hat{x} \in \Omega \), then we define
\[
J^2_{\Omega} u(\hat{x}) = \{(p, X) \in \mathbb{R}^{n+1} \times S(n + 1) : u(x) \leq u(\hat{x}) + \langle p, x - \hat{x} \rangle + \frac{1}{2} \langle X(x - \hat{x}), x - \hat{x} \rangle + o(|x - \hat{x}|^2) \text{ as } x \to \hat{x}\}
\]
(2.3)
and for \( x \in \Omega \)
\[
J^2_{\Omega} u(x) = \{(p, X) \in \mathbb{R}^{n+1} \times S(n + 1) : \text{there are } x_k \in \Omega, \text{ and } \langle p_k, X_k \rangle \in J^2_{\Omega} u(x_k), \text{ so that } (x_k, p_k, X_k) \to (x, p, X) \},
\]
(2.4)
where \( S(l), l \in \mathbb{N} \), denotes the set of symmetric \( n \times n \) matrices.

Definition 2.2. (i) A continuous function \( u : \Omega \to \mathbb{R} \) is a viscosity subsolution of (1.3), if for all \((\eta, X) \in J^2_{\Omega} u(x), x \in \Omega\), there holds
\[
-|\eta|^{1/2} \langle \delta_{ij} - \frac{\eta_i \eta_j}{|\eta|^2} \rangle X_{ij} \leq 1,
\]
(2.5)
if \( \eta \neq 0 \) and
\[
- \langle \delta_{ij} - \tilde{\eta}_i \tilde{\eta}_j \rangle X_{ij} \leq 0
\]
(2.6)
for some \( \tilde{\eta} \) with \( |\tilde{\eta}| \leq 1 \), if \( \eta = 0 \).

(ii) A continuous function \( u : \Omega \to \mathbb{R} \) is a viscosity supersolution of (1.3), if for all \((\eta, X) \in J^2_{\Omega} u(x), x \in \Omega\), there holds
\[
-|\eta|^{1/2} \langle \delta_{ij} - \frac{\eta_i \eta_j}{|\eta|^2} \rangle X_{ij} \geq 1,
\]
(2.7)
if \( \eta \neq 0 \) and
\[
- \langle \delta_{ij} - \tilde{\eta}_i \tilde{\eta}_j \rangle X_{ij} \geq 0
\]
(2.8)
for some \( \tilde{\eta} \) with \( |\tilde{\eta}| \leq 1 \), if \( \eta = 0 \).

(iii) A function \( u \), which is supersolution and subsolution of (1.3) is a viscosity solution of (1.3).

Remark 2.3. A simple inspection shows that we could have replaced \( J^2_{\Omega} u(x) \) in the preceding definition by \( J^2_{\Omega} u(x) \) and \( J^2_{\Omega} u(x) \) by \( J^2_{\Omega} u(x) \).

Sometimes it is useful to have another definition available.

Definition 2.4. (i) A function \( u \in C^0(\Omega) \) is a viscosity subsolution of (1.3), provided that if
\[
u - \varphi \text{ has a local maximum at a point } x_0 \in \Omega
\]
(2.9)
for each \( \varphi \in C^\infty(\Omega) \), then
\[
\begin{cases}
- |D\varphi|^{1/2} \langle \delta_{ij} - \frac{\varphi_i \varphi_j}{|D\varphi|^2} \rangle \varphi_{ij} \leq 1 \text{ at } x_0 \\
\text{if } D\varphi(x_0) \neq 0,
\end{cases}
\]
(2.10)
and
\[
\begin{cases}
- \langle \delta_{ij} - \eta_i \eta_j \rangle X_{ij} \leq 0 \text{ at } x_0 \\
\text{for some } \eta \in \mathbb{R}^n \text{ with } |\eta| \leq 1, \text{ if } D\varphi(x_0) = 0.
\end{cases}
\]
(2.11)
(ii) A function \( u \in C^0(\Omega) \) is a viscosity supersolution of (1.3), provided that if
\[
u - \varphi \text{ has a local maximum at a point } x_0 \in \Omega
\]
(2.12)
for each $\varphi \in C^\infty(\Omega)$, then

$$\begin{align*}
(2.13) & \quad -|D\varphi|^2^{-1}(\delta_{ij} - \frac{\varphi_{ij}}{|D\varphi|^2})\varphi_{ij} \geq 1 \text{ at } x_0 \\
& \quad \text{if } D\varphi(x_0) \neq 0,
\end{align*}$$

and

$$\begin{align*}
(2.14) & \quad -(\delta_{ij} - \eta_i \eta_j)X_{ij} \geq 0 \text{ at } x_0 \\
& \quad \text{for some } \eta \in \mathbb{R}^n \text{ with } |\eta| \leq 1, \text{ if } D\varphi(x_0) = 0.
\end{align*}$$

**Theorem 2.5.** Definitions 2.2 and 2.4 are equivalent.

**Proof.** We only consider the case of viscosity subsolutions.

(i) We assume that $u$ is a viscosity subsolution according to Definition 2.2. Assume that $u - \varphi$ has a local maximum at a point $x_0 \in \Omega$ for a fixed $\varphi \in C^\infty(\Omega)$. Hence for $x \in \Omega$ close to $x_0$ we get

$$\begin{align*}
(2.15) & \quad u(x) \leq \varphi(x) + u(x_0) - \varphi(x_0) \\
& \quad = u(x_0) + D\varphi(x_0)(x - x_0) + \frac{1}{2}D^2\varphi(x_0)(x - x_0)(x - x_0) + o(|x - x_0|^2),
\end{align*}$$

which implies

$$\begin{align*}
(2.16) & \quad (D\varphi(x_0), D^2\varphi(x_0)) \in J^2_{\Omega}(u)(x_0)
\end{align*}$$

and the claim follows.

(ii) We assume that $u$ is a viscosity subsolution according to Definition 2.4. Let $(\eta, X) \in J^2_{\Omega}(u)(x), x \in \Omega$. Define for $\delta > 0$

$$\begin{align*}
(2.17) & \quad \varphi_\delta(y) = u(x) + \eta(y - x) + (y - x)^tX(y - x) + \delta|y - x|^2
\end{align*}$$

then $u - \varphi_\delta$ has a local maximum in $x$. Hence (2.10), (2.11) hold with

$$\begin{align*}
(2.18) & \quad D\varphi_\delta(x) = \eta, \quad D^2\varphi_\delta(x) = X + \delta E.
\end{align*}$$

Letting $\delta \to 0$ proves the claim.

**Lemma 2.6.** The function $u$ in (1.6) is a viscosity solution of (1.3).

**Proof.** We adapt [9, Section 4.3]. Let $\varphi \in C^\infty(\Omega)$ and suppose $u - \varphi$ has a strict local maximum at a point $x_0 \in \Omega$. As $u^\epsilon \to u$ uniformly, $u^\epsilon - \varphi$ has a local maximum at a point $x_\epsilon \in \Omega$ with

$$\begin{align*}
(2.19) & \quad x_\epsilon \to x_0 \quad \text{as } \epsilon \to 0.
\end{align*}$$

Since $u^\epsilon$ and $\varphi$ are smooth, we have

$$\begin{align*}
(2.20) & \quad Du^\epsilon = D\varphi, \quad D^2u^\epsilon \leq D^2\varphi \quad \text{at } x_\epsilon.
\end{align*}$$

Thus (2.2) implies

$$\begin{align*}
(2.21) & \quad -|D\varphi|^2 + \epsilon^2)\varphi_{ij} \leq 1 \text{ at } x_\epsilon.
\end{align*}$$

Suppose first $D\varphi(x_0) \neq 0$. Then $D\varphi(x_\epsilon) \neq 0$ for small $\epsilon > 0$. We consequently may pass to limits in (2.21), recalling (2.19) to deduce

$$\begin{align*}
(2.22) & \quad -|D\varphi|^2 \frac{1}{2} - \frac{\varphi_{ij}}{|D\varphi|^2} \varphi_{ij} \leq 1 \text{ at } x_0.
\end{align*}$$
Next, assume instead $D\varphi(x_0) = 0$. Set

$$\eta^\epsilon := \frac{D\varphi(x_\epsilon)}{(|D\varphi(x_\epsilon)|^2 + \epsilon^2)^{\frac{1}{2}}}$$

so that (2.21) becomes

$$-(\delta_{ij} - \eta^\epsilon_i \eta^\epsilon_j)\varphi_{ij} \leq (|D\varphi|^2 + \epsilon^2)^{\frac{1}{2}} - \frac{1}{2\epsilon^4} \text{ at } x_\epsilon.$$

Since $|\eta^\epsilon| \leq 1$, we may assume, upon passing to a subsequence and relabeling if necessary, that $\eta^\epsilon \to \eta$ in $\mathbb{R}^n$ for some $|\eta| \leq 1$. Sending $\epsilon$ to 0 in (2.24) we discover

$$-(\delta_{ij} - \eta_i \eta_j)\varphi_{ij} \leq 0 \text{ at } x_0.$$

If $u - \varphi$ has a local maximum, but not necessarily a strict maximum at $x_0$, we repeat the argument above with $\varphi(x)$ replaced by

$$\tilde{\varphi}(x) = \varphi(x) + |x - x_0|^4,$$

again to obtain (2.22) or (2.25).

Consequently, $u$ is a weak subsolution. That $u$ is a weak supersolution follows analogously. \hfill \Box

3. Estimate of $u - u^\epsilon$

We first define some constants, which will determine an error estimate for $u^\epsilon - u$, as will become clear in the succeeding Theorem 3.1.

Let

$$\gamma > 1 + k$$

and $\alpha, s > 0$ be small so that

$$\beta_1(\alpha, s) > \beta_2(\alpha, s),$$

where

$$\beta_1(\alpha, s) := \frac{2 - s + \alpha(2 - \frac{1}{k})}{\gamma(2 - \frac{1}{k}) + \frac{1}{k} - 1}, \quad \beta_2(\alpha, s) := \frac{\alpha + ks}{\gamma - k - 1}$$

and choose

$$0 < r < \frac{\alpha}{\gamma}.$$

**Theorem 3.1.** There is $c = c(k, \Omega) > 0$ such that

$$\|u^\epsilon - u\|_{C^\alpha(\Omega)} \leq c\epsilon^{\min(r,s)}$$

for all $\epsilon > 0$.

**Corollary 3.2.** By interpolation we get in the situation of Theorem 3.1 for $0 < \theta < 1$ that

$$[u^\epsilon - u]_{\theta, \Omega} \leq c(\theta, k, \Omega)\epsilon^{\min(r,s)(1-\theta)} \forall \epsilon > 0,$$

where the bracket denotes the Hölder semi-norm.
Remark 3.3. We explain how we can deduce rates of convergence explicitly. Since inequality (3.2) ‘improves’ for decreasing $s > 0$ we choose $s = \frac{\alpha}{\gamma}$ in view of (3.4). Then we maximize $\frac{\alpha}{\gamma}$ with respect to $\alpha, \gamma$ under the constraints $\alpha > 0, (3.1)$ and (3.2). This can be done by assuming equality in (3.2) and solving this equation for $\alpha$, which is possible; then it suffices to maximize a nonlinear expression for $\gamma$ under the constraint that the weak inequality $\geq$ holds in (3.1). Small perturbations if necessary of maximizers of the latter optimization problem lead to feasible values for $r, s$.

In the remaining part of this section we prove Theorem 3.1 by adapting the proof of [6, Theorem 1.2].

For $\epsilon > 0$ we define $w_{\epsilon} : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ by

$$w_{\epsilon}(x, y) := \mu u(x) - u'(y) - \frac{\epsilon - \alpha}{\gamma} |x - y|^{\gamma}, \quad x, y \in \bar{\Omega},$$

where

$$\mu = \mu(\epsilon) = (1 - \epsilon^{s})^{k}. \quad \text{We use the abbreviation}$$

$$\varphi(x, y) := \frac{\epsilon - \alpha}{\gamma} |x - y|^{\gamma}. \quad \text{Let} \ \hat{x}, \hat{y} \in \Omega \text{ such that}$$

$$w_{\epsilon}(\hat{x}, \hat{y}) = \sup_{\bar{\Omega} \times \bar{\Omega}} w.$$  

Lemma 3.4. There holds $\hat{x} \in \partial\Omega$ or $\hat{y} \in \partial\Omega$.

Proof. We assume $\hat{x}, \hat{y} \in \Omega$. From [4, Theorem 3.2] we deduce that for every $\rho > 0$ there are $X, Y \in S(n + 1)$ such that

$$(D_{x}(\varphi(\hat{x}, \hat{y}), X) \in \mathcal{J}_{\Omega}^{+}(\mu u)(\hat{x}) \quad \land \quad (D_{y}(\varphi(\hat{x}, \hat{y}), Y) \in \mathcal{J}_{\Omega}^{+}(-u')(\hat{y})$$

and

$$-(\frac{1}{\rho} + \|A\|)I \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq A + \rho A^{2},$$

where $A := D^{2}\varphi(\hat{x}, \hat{y})$. We calculate

$$D_{x}(\varphi(\hat{x}, \hat{y}) = \epsilon^{-\alpha} |\xi|^{\gamma - 2}\xi = -D_{y}(\varphi(\hat{x}, \hat{y}), \quad \xi = \hat{x} - \hat{y},$$

and

$$A = \begin{pmatrix} B & -B \\ -B & B \end{pmatrix}, \quad B = \epsilon^{-\alpha} |\xi|^{\gamma - 4}((\gamma - 2)\xi \otimes \xi + |\xi|^{2}I).$$

Using

$$F(\mu u) = \mu^{\frac{k}{4}}, \quad F_{\epsilon}(-u') = -1,$$

we conclude from (3.11) that

$$- (\delta_{ij} - \frac{D_{x}(\varphi D_{x}\varphi)}{|D_{x}\varphi|^{2}})X_{ij} \leq \mu^{\frac{k}{4}}|D_{x}\varphi|^{1 - \frac{k}{4}} \text{ at } \hat{x}$$

if $D_{x}(\varphi) \neq 0$ and

$$-(\delta_{ij} - \eta_{i}\eta_{j})X_{ij} \leq 0 \text{ at } \hat{x}$$

$$\delta_{ij} - \eta_{i}\eta_{j}$$

$$r, s$$

$$\mu = \mu(\epsilon) = (1 - \epsilon^{s})^{k}. \quad \text{We use the abbreviation}$$

$$\varphi(x, y) := \frac{\epsilon - \alpha}{\gamma} |x - y|^{\gamma}. \quad \text{Let} \ \hat{x}, \hat{y} \in \Omega \text{ such that}$$

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and

$$-(\frac{1}{\rho} + \|A\|)I \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq A + \rho A^{2},$$

where $A := D^{2}\varphi(\hat{x}, \hat{y})$. We calculate

$$D_{x}(\varphi(\hat{x}, \hat{y}) = \epsilon^{-\alpha} |\xi|^{\gamma - 2}\xi = -D_{y}(\varphi(\hat{x}, \hat{y}), \quad \xi = \hat{x} - \hat{y},$$

and

$$A = \begin{pmatrix} B & -B \\ -B & B \end{pmatrix}, \quad B = \epsilon^{-\alpha} |\xi|^{\gamma - 4}((\gamma - 2)\xi \otimes \xi + |\xi|^{2}I).$$

Using

$$F(\mu u) = \mu^{\frac{k}{4}}, \quad F_{\epsilon}(-u') = -1,$$

we conclude from (3.11) that

$$- (\delta_{ij} - \frac{D_{x}(\varphi D_{x}\varphi)}{|D_{x}\varphi|^{2}})X_{ij} \leq \mu^{\frac{k}{4}}|D_{x}\varphi|^{1 - \frac{k}{4}} \text{ at } \hat{x}$$

if $D_{x}(\varphi) \neq 0$ and

$$-(\delta_{ij} - \eta_{i}\eta_{j})X_{ij} \leq 0 \text{ at } \hat{x}$$

$$\delta_{ij} - \eta_{i}\eta_{j}$$

$$r, s$$
for some \( \eta \in \mathbb{R}^n \) with \( |\eta| \leq 1 \) if \( D\varphi(\hat{x}) = 0 \); furthermore, there holds

\[
(3.18) \quad - (\delta_{ij} - \frac{D_{ij}\varphi D_{ij}\varphi}{|D\varphi|^2 + \epsilon^2}) Y_{ij} \leq -(|D\varphi|^2 + \epsilon^2)^{\frac{1}{2}} - \frac{1}{\varphi}.
\]

From (3.12) we get for all \( \zeta \in \mathbb{R}^n \)

\[
(3.19) \quad \zeta^t (X + Y) \zeta = (\zeta^t, \zeta^t) \left( \begin{array}{cc} X & 0 \\ 0 & Y \end{array} \right) \left( \begin{array}{c} \zeta \\ \zeta \end{array} \right)
\]

\[
\leq (\zeta^t, \zeta^t) \left\{ \left( \begin{array}{cc} B & -B \\ -B & B \end{array} \right) + 2\rho \left( \begin{array}{cc} B^2 & B^2 \\ B^2 & B^2 \end{array} \right) \right\} \left( \begin{array}{c} \zeta \\ \zeta \end{array} \right)
\]

\[
= 0,
\]

i.e.

\[
(3.20) \quad X + Y \leq 0,
\]

and

\[
(3.21) \quad \xi^t Y \xi = (0, \xi^t) \left( \begin{array}{cc} X & 0 \\ 0 & Y \end{array} \right) \left( \begin{array}{c} 0 \\ \xi \end{array} \right)
\]

\[
\leq \xi^t B \xi + 2\rho \xi^t B^2 \xi
\]

\[
\leq (\gamma - 1)e^{-\alpha} |\xi|^\gamma + 2\rho \xi^t B^2 \xi.
\]

**Case \( \hat{x} \neq \hat{y} \):** We add the inequalities (3.16) and (3.18) and get

\[
(3.22) \quad LHS := - (\delta_{ij} - \frac{D_{ij}\varphi D_{ij}\varphi}{|D\varphi|^2 + \epsilon^2}) X_{ij} - (\delta_{ij} - \frac{D_{ij}\varphi D_{ij}\varphi}{|D\varphi|^2 + \epsilon^2}) Y_{ij}
\]

\[
\leq (\mu + 1)|D\varphi|^{1 - \frac{1}{\varphi}}.
\]

We estimate \( LHS \) from below

\[
LHS = - (\delta_{ij} - \frac{D_{ij}\varphi D_{ij}\varphi}{|D\varphi|^2})(X_{ij} + Y_{ij}) - \epsilon^2 \frac{D_{ij}\varphi D_{ij}\varphi}{|D\varphi|^2 + \epsilon^2} Y_{ij}
\]

\[
\geq - \frac{\epsilon^2 \xi^t Y \xi}{|\xi|^2 (|\xi|^2 - 2e^{-2\alpha} + \epsilon^2)}
\]

\[
\geq - (\gamma - 1)\epsilon^2 - \alpha |\xi|^\gamma - 2\epsilon^2 \rho \xi^t B^2 \xi
\]

\[
\geq - (\gamma - 1)\epsilon^2 - \alpha |\xi|^{2\gamma - 2} e^{-2\alpha} + \epsilon^2
\]

where we used (3.20) and (3.21). Combining (3.22) with (3.23), letting \( \rho \to 0 \) and applying the relations (3.8) and (3.13) yield

\[
(3.24) \quad - (\gamma - 1)\epsilon^2 - \alpha |\xi|^{\gamma - 2} - \epsilon^2 \leq - \epsilon^{s - \alpha (1 - \frac{1}{\varphi})} |\xi|^{(\gamma - 1)(1 - \frac{1}{\varphi})}.
\]

We multiply this inequality by the denominator of the left-hand side and deduce two inequalities

\[
(3.25) \quad -(\gamma - 1)\epsilon^2 - \alpha |\xi|^{\gamma - 2} \leq - \epsilon^{s - \alpha (3 - \frac{1}{\varphi})} |\xi|^{(\gamma - 1)(1 - \frac{1}{\varphi}) + 2\gamma - 2}
\]

\[
-(\gamma - 1)\epsilon^2 - \alpha |\xi|^{\gamma - 2} \leq - \epsilon^{s + 2 - \alpha (1 - \frac{1}{\varphi})} |\xi|^{(\gamma - 1)(1 - \frac{1}{\varphi})},
\]
which lead to

\[(3.26)\]

\[ (\gamma - 1) \epsilon^{2 - s + \alpha(2 - \frac{k}{s})} \geq |\xi|^{\gamma(2 - \frac{k}{s}) - 1 + \frac{k}{s}} \]

\[ (\gamma - 1) k \epsilon^{\alpha - ks} \geq |\xi|^{-\gamma + k + 1}. \]

Accounting for (3.1) we have

\[(3.27)\]

\[ |\xi| \leq (\gamma - 1) \epsilon^{\frac{1}{(2 - \frac{k}{s}) - 1 + \frac{k}{s}}} \epsilon^{\frac{2 - s + \alpha(2 - \frac{k}{s})}{2 - \frac{k}{s}}} =: c_1 \epsilon^{\beta_1(\alpha, s)} \]

\[ |\xi| \geq (\gamma - 1) \epsilon^{\frac{k}{(2 - \frac{k}{s}) - 1 + \frac{k}{s}}} \epsilon^{\frac{\alpha + ks}{(2 - \frac{k}{s}) - 1 + \frac{k}{s}}} =: c_2 \epsilon^{\beta_2(\alpha, s)}. \]

In view of (3.2) we get a contradiction for small \( \epsilon > 0 \).

Case \( \hat{x} = \hat{y} \): Due to \( \gamma > 2 \) and (3.14) we have \( B = 0 \), so that a calculation as in (3.21) (now with \( \eta \) instead of \( \xi \)) shows

\[(3.28)\]

\[ \eta^T Y \eta \leq 0. \]

Hence, adding (3.17) to (3.18) and having (3.20) in mind we get

\[(3.29)\]

\[ \epsilon^{1 - \frac{k}{s}} (\delta_{ij} - \eta_i \eta_j) X_{ij} + \delta_{ij} Y_{ij} \leq (\delta_{ij} - \eta_i \eta_j) (X_{ij} + Y_{ij}) + \eta^T Y \eta \leq 0, \]

which is a contradiction.

\[ \square \]

**Lemma 3.5.** There is \( c_4 > 0 \) such that

\[(3.30)\]

\[ w_\epsilon(\hat{x}, \hat{y}) \leq c_4 \epsilon^r. \]

**Proof.** In view of Lemma 3.4 we can assume in equation (3.9) w.l.o.g. that \( \hat{y} \in \partial \Omega \). Hence we can write

\[(3.31)\]

\[ w_\epsilon(\hat{x}, \hat{y}) = \mu u(\hat{x}) - \mu u(\hat{y}) - \frac{\epsilon^{-\alpha}}{\gamma} |\hat{x} - \hat{y}|^\gamma. \]

In case \( |\hat{x} - \hat{y}| \leq \epsilon^r \) we get using the lipschitz continuity of \( u \)

\[(3.32)\]

\[ w_\epsilon(\hat{x}, \hat{y}) \leq \mu c_0 |\hat{x} - \hat{y}| \leq \mu c_0 \epsilon^r, \]

which proves the lemma.

The remaining case \( |\hat{x} - \hat{y}| > \epsilon^r \) is not available for sufficiently small \( \epsilon > 0 \), for we estimate

\[(3.33)\]

\[ w_\epsilon(\hat{x}, \hat{y}) \leq 2 \mu c_0 - \frac{\epsilon^{\gamma - \alpha}}{\gamma} \rightarrow -\infty, \quad \epsilon \rightarrow 0. \]

\[ \square \]

Now, collecting facts we finish the estimate for \( u - u^r \). Let \( x \in \Omega \) arbitrary. Then

\[(3.34)\]

\[ u(x) - u^r(x) = \mu u(x) - u^r(x) + (1 - \mu) u(x) \]

\[ = w_\epsilon(x, x) + (1 - \mu) u(x) \]

\[ \leq c_4 \epsilon^r + c_0 \epsilon^a \]

\[ \leq c_5 \epsilon^{\min(r, s)}, \]

with a positive constant \( c_5 \). Interchanging the roles of \( u \) and \( u^r \) we see, that there is a positive constant \( c_0 \) with

\[(3.35)\]

\[ |u(x) - u^r(x)| \leq c_6 \epsilon^{\min(r, s)}. \]
4. Higher order estimates of $u^\epsilon$

In this section we make the $\epsilon$-dependence of a bound for higher order derivatives of $u^\epsilon$ explicit.

We recall that the $u^\epsilon$ are $C^\infty$, bounded $\|u^\epsilon\|_{C^1(\bar{\Omega})} \leq c_0$ and satisfy the quasilinear equations in divergence form

$$-D_i a^i(Du^\epsilon) = f, \quad u^\epsilon|_{\partial\Omega} = 0,$$

where

$$a^i(p) = \frac{p^i}{\sqrt{\epsilon^2 + |p|^2}}, \quad p \in \mathbb{R}^d,$$

and

$$f = -(\epsilon^2 + |Du^\epsilon|^2)^{\frac{-1}{2}}.$$

Let us denote

$$a^{ij}(p) := \frac{\partial a^i}{\partial p_j}(p) = \frac{\epsilon^2 \delta^{ij} + |p|^2 \delta^{ij} - p^i p^j}{(\epsilon^2 + |p|^2)^{\frac{3}{2}}},$$

the largest and smallest eigenvalue of $a^{ij}(p)$ by $\Lambda(p)$ and $\lambda(p)$, respectively, and

$$\Lambda = \sup_{\bar{B}_c(0)} \Lambda(p), \quad \lambda = \inf_{\bar{B}_c(0)} \lambda(p).$$

In $\bar{B}_c(0) \subset \mathbb{R}^{n+1}$ we have

$$0 < c\epsilon^2 \delta^{ij} \leq a^{ij} \leq \frac{c}{\epsilon} \delta^{ij}, \quad \frac{\Lambda(p)}{\lambda(p)} \leq \frac{c}{\epsilon^2}, \quad \frac{\Lambda}{\lambda} \leq \frac{c}{\epsilon^3}.$$

From standard $L^2$-regularity theory of quasilinear equations in divergence form we get, see for example the proof of [12, Theorem 1.5.1 in PDE II], that all second derivatives of $u^\epsilon$ except for the second derivative in normal direction at the boundary are bounded in the $L^2$-norm by

$$\frac{c}{\epsilon^2} \|f\|_{L^2(\Omega)} + \frac{c}{\epsilon^2} c_0 \leq \frac{c}{\epsilon^2 + \epsilon^2}.$$

Hence

$$\|u^\epsilon\|_{H^2(\Omega)} \leq \frac{c}{\epsilon^4 + \epsilon^2}$$

and bounds for higher order derivatives of $u^\epsilon$ are obtained iteratively.

5. Tracking constants in linear equations

We consider linear equations of the form

$$Lu = D_i(a^{ij} D_j u) + c^i D_i u = g + D_i f^i,$$

in $\tilde{\Omega}$, where we assume that

$$\lambda > 0, \quad a^{ij} \geq \lambda \delta^{ij}, \quad \sum |a^{ij}|^2 \leq \Lambda^2, \quad \lambda^{-1} \sum |c^i|^2 \leq \nu^2$$

and $\tilde{\Omega} = \Omega^h, \quad 0 < h < h_0$, or $\tilde{\Omega} = \Omega^\delta, \quad 0 < \delta < \delta_0$. In the following results constants are uniform with respect to $h, \delta$.

Our aim in the present section is to provide Corollary 5.6, which will be needed in Section 6. We assume in this section $\lambda < 1 < \nu$. 

Theorem 5.1. Let $f^i \in L^q(\tilde{\Omega})$, $g \in L^2(\tilde{\Omega})$, $q > n + 1$. Then if $u \in H^{1,2}(\tilde{\Omega})$ is a subsolution (supersolution) of

\begin{equation}
Lu = g + D_i f^i
\end{equation}

in $\tilde{\Omega}$ satisfying $u \leq 0 \geq 0$ on $\partial \tilde{\Omega}$, we have

\begin{equation}
\sup_{\tilde{\Omega}} u(-u) \leq C(||u^+(u^-)||_{L^2(\tilde{\Omega})} + R),
\end{equation}

where $R = \lambda^{-1}(\|f\|_{L^q(\tilde{\Omega})} + \|g\|_{L^4(\tilde{\Omega})})$ and

\begin{equation}
C = C(n, q, |\tilde{\Omega}|)\nu^{-\frac{3\lambda}{2}}.
\end{equation}

Proof. A careful view of the proof of [13, Theorem 8.15] shows the claim. □

Theorem 5.2. In the situation of Theorem 5.1 holds

\begin{equation}
\sup_{\tilde{\Omega}} u(-u) \leq \sup_{\partial \tilde{\Omega}} u^+(u^-) + CR,
\end{equation}

where $R = \lambda^{-1}(\|f\|_{L^q(\tilde{\Omega})} + \|g\|_{L^4(\tilde{\Omega})})$ and

\begin{equation}
C = C(n, q, |\tilde{\Omega}|)(1 + \nu^{-\frac{3\lambda}{2}}).
\end{equation}

Proof. Use Theorem 5.1 and the proof of [13, Theorem 8.16]. □

Let $q > n + 1$.

Lemma 5.3. Let $g \in L^2(\tilde{\Omega})$ and $f^i \in L^q(\tilde{\Omega})$ then there exists a unique solution $u \in H^{1,2}(\tilde{\Omega})$ of (5.1) and there holds

\begin{equation}
\|Du\|_{L^2(\tilde{\Omega})} \leq c_2(\|f\|_{L^q(\tilde{\Omega})} + \|g\|_{L^4(\tilde{\Omega})}),
\end{equation}

where

\begin{equation}
c_2 := c_1(g)\left(\nu + \frac{1}{\lambda}\right)^{\nu^{-\frac{3\lambda}{2}}}.\end{equation}

Proof. Use $u$ as test function, apply standard estimates and Theorem 5.2. □

Let $\tilde{\Omega} = \Omega_{\delta}$, $0 < \delta < \delta_0$ arbitrary but fixed, then there holds the following Lemma with constants being uniform in $\delta$.

Lemma 5.4. Let $u \in H^{1,2}_{0}(\tilde{\Omega})$ be the solution of (5.1) with $f^i = 0$, $g \in L^2(\tilde{\Omega})$ and

\begin{equation}
a^{ij}, c^i \in C^1(\tilde{\Omega}), \quad \|Da^{ij}\|_{C^0(\tilde{\Omega})} + \|Dc^i\|_{C^0(\tilde{\Omega})} \leq a_1.
\end{equation}

Then there holds

\begin{equation}
\|u\|_{H^{2,2}(\tilde{\Omega})} \leq c_3\|g\|_{L^2(\tilde{\Omega})},
\end{equation}

where

\begin{equation}
c_3 := cc_1(g)c_2a_1\left(\frac{a_1}{\lambda^2} + \frac{\nu}{\lambda}\right).
\end{equation}

Proof. The proof is a straight forward calculation. □

We have the following theorem.
Theorem 5.5. We assume the situation of Lemma 5.3 with \( \tilde{\Omega} = \Omega^h \) and

\[
5.13 \quad a^{ij}, c^i \in C^1(\tilde{\Omega}_{\delta_h}), \quad \|Da^{ij}\|_{C^0(\tilde{\Omega}_{\delta_h})} + \|Dc^i\|_{C^0(\tilde{\Omega}_{\delta_h})} \leq a_1.
\]

Let \( u \) be the unique solution of (5.1) in \( \Omega^h \). Then for

\[
5.14 \quad 0 < h \leq h_0 := \min(\frac{\nu^2 c^2_2}{4}, (\frac{\delta_0}{4})^2),
\]

cf. (5.44), (5.38) and (5.40), there exists a unique FE solution \( u_h \in V_h \) of (5.1) in \( \Omega^h \), we have

\[
5.15 \quad \|u - u_h\|_{H^{1,2}(\Omega^h)} \leq c_4 \inf_{v_h \in V_h} \|u - v_h\|_{H^{1,2}(\Omega^h)},
\]

and

\[
5.16 \quad \|u_h\|_{H^{1,2}(\Omega^h)} \leq c_5 \|u\|_{H^{1,2}(\Omega^h)},
\]

where

\[
5.17 \quad c_4 := c(\frac{\Lambda}{\lambda} + \nu), \quad c_5 := c_4 + 1.
\]

Corollary 5.6. In the situation of Theorem 5.5 holds

\[
5.18 \quad \|u_h\|_{H^{1,2}(\Omega^h)} \leq c_8 \left( \|f\|_{L^2(\Omega^h)} + \|g\|_{L^2(\Omega^h)} \right)
\]

with \( c_8 := c_2 c_5 \).

To prove Theorem 5.5 we would like to apply the Schatz argument, cf. [3, Theorem 5.7.6] or [19], which uses the adjoint operator \( L^* \) given by

\[
5.19 \quad L^* : H^{1,2}_{0}(\tilde{\Omega}) \rightarrow H^{-1,2}(\tilde{\Omega}), \quad L^* u = D_i(a^{ij} D_j u - c^i u),
\]

i.e.

\[
5.20 \quad \langle Lu, v\rangle_{H^{-1,1},H^1} = \langle u, L^* v\rangle_{H^{1,2},H^{-1,2}} \quad \forall u, v \in H^{1,2}_{0}(\tilde{\Omega}),
\]

and needs that in the situation \( \tilde{\Omega} = \Omega^h \) the space \( H^{1,2}_{0}(\Omega^h) \) lies in the image of \( L^* \) and—for our case—that \( L^2 \)-estimates for \( L^* \) with explicit constants are available. But both is not ensured, because \( D_i c^i \) does not have the right sign necessarily and \( \partial \Omega^h \) might lack the needed regularity (e.g. \( \partial \Omega^h \in C^{0,1} \) and \( \Omega^h \) convex).

In the remaining part of this section we prove Theorem 5.5 using a modified Schatz argument.

In view of [13, Theorem 8.6] there exists a countable set \( \Sigma \subset \mathbb{R} \) so that for all \( \sigma \notin \Sigma \) and all \( g \in L^2(\tilde{\Omega}) \) there exists a unique solution \( u \in H^{1,2}_{0}(\tilde{\Omega}) \) of the equation

\[
5.21 \quad (L^* + \sigma)u = g.
\]

\( \Sigma \) depends on \( h \) and \( \delta \), and in the following we will only use, that for \( h \) and \( \delta \) fixed the corresponding \( \mathbb{R}\setminus \Sigma \) has 0 as accumulation point.

Proof of Theorem 5.5. (i) Let \( u \) be the unique solution of (5.1) in \( \Omega^h \). We assume that \( u_h \) is a FE solution of (5.1) in \( V_h \) and extend \( u, u_h \) by 0 to \( \mathbb{R}^{n+1} \). Set \( \delta = h^\frac{\tilde{s}}{2} \), then

\[
5.22 \quad \forall 0 < h < h_0 \quad \partial \Omega^h \subset \Omega_{\frac{h}{2}} \backslash \Omega_{\frac{h}{4}},
\]

cf. (1.11).

Choose a positive

\[
5.23 \quad \frac{1}{4c_3} > \sigma \notin \Sigma
\]
and let \( w \in H_0^{1,2}(\Omega_h) \) be the unique solution of
\[
(5.24) \quad (L^* + \sigma)w = u - u_h
\]
in \( \Omega_h \). Then for all \( w_h \in V_h \) we have
\[
\| u - u_h \|_{L^2(\Omega_h)}^2 = \langle (L^* + \sigma)w, u - u_h \rangle_{H^{-1}(\Omega_h), H^1(\Omega_h)}
\]
\[
= \int_{\Omega_h} \sigma w (u - u_h) + \langle w, L(u - u_h) \rangle_{H^1(\Omega_h), H^{-1}(\Omega_h)}
\]
\[
- \langle w_h, L(u - u_h) \rangle_{H^1(\Omega_h), H^{-1}(\Omega_h)}
\]
\[
\leq \sigma \| w \|_{L^2(\Omega_h)} \| u - u_h \|_{L^2(\Omega_h)}
\]
\[
+ \left( \Lambda + \nu \lambda \right) \| u - u_h \|_{H^{1,2}(\Omega_h)} \| w - w_h \|_{H^{1,2}(\Omega_h)}
\]
For \( w_h \in V_h \) we have
\[
\| w - w_h \|_{H^1(\Omega^h)} \leq \| w - \tilde{w} \|_{H^1(\Omega^h)} + \| \tilde{w} \|_{H^1(\Omega^h)}.
\]
(5.36)

Since \( w - \tilde{w} \in H^{1,2}(\Omega^h) \) there holds
\[
\inf_{w_h \in \Omega^h} \| w - w_h \|_{H^{1,2}(\Omega^h)} \leq c c_6 h \| u - u_h \|_{L^2(\Omega^h)}
\]
and, furthermore, choosing
\[
p \left\{ \begin{array}{ll}
\frac{1}{2} & \text{if } n = 0, \\
\frac{n}{p} & \text{if } n > 0,
\end{array} \right.
\]
we have
\[
\| \tilde{w} \|_{H^{1,2}(\Omega^h)} \leq \| \tilde{w} \|_{H^{2,2}(\Omega^h)}
\]
\[
\leq c \delta \| \tilde{w} \|_{H^{1,2}(\Omega^h)}
\]
\[
\leq c \| \tilde{w} \|_{H^{2,2}(\Omega^h)}
\]
\[
\leq c c_6 \delta \| u - u_h \|_{L^2(\Omega^h)},
\]
(5.39)

where
\[
q = \frac{1}{2(p/2)^2},
\]
and hence
\[
\inf_{w_h \in V_h} \| w - w_h \|_{H^{1,2}(\Omega^h)} \leq c c_6 (\delta + \delta^2) \| u - u_h \|_{L^2(\Omega^h)}.
\]
(5.41)

Combining (5.41) and (5.25) yields
\[
\| u - u_h \|_{L^2(\Omega^h)} \leq c c_6 (\Lambda + \nu \lambda) \| u - u_h \|_{H^{1,2}(\Omega^h)}
\]
\[
\| u - u_h \|_{L^2(\Omega^h)} \leq c (\delta + \delta^2) \| u - u_h \|_{H^{1,2}(\Omega^h)}
\]
with
\[
c_7 := c c_6 (\Lambda + \nu \lambda).
\]
(5.43)

(iv) We have for any \( v_h \in V_h \)
\[
\frac{\lambda}{2} \| u - u_h \|_{H^{1,2}(\Omega^h)}^2 \leq \langle L(u - u_h), u - u_h \rangle + \nu^2 \lambda \| u - u_h \|_{L^2(\Omega^h)}^2
\]
\[
= \langle L(u - u_h), u - v_h \rangle + \nu^2 \lambda \| u - u_h \|_{L^2(\Omega^h)}^2
\]
\[
\leq (\Lambda + \nu \lambda) \| u - u_h \|_{H^{1,2}(\Omega^h)} \| u - v_h \|_{H^{1,2}(\Omega^h)} + \nu^2 \lambda c_7^2 (\delta + \delta^2) \| u - u_h \|_{H^{1,2}(\Omega^h)}^2
\]
(5.45)

and hence
\[
\| u - u_h \|_{H^{1,2}(\Omega^h)} \leq c \frac{\lambda}{\lambda + \nu} \| u - v_h \|_{H^{1,2}(\Omega^h)}.
\]
(5.46)

(v) Existence of a FE solution \( u_h \) of (5.1) follows in the usual way. Due to the quadratic structure of the corresponding system of linear equations, which determines \( u_h \), we deduce existence from uniqueness, at which the latter is given in view of (5.8) and (5.16).
6. Proof of Theorem 1.2

In this section we will prove Theorem 1.2, for it we obtain the solution \( u_h^\epsilon \) of (1.13) as the unique fixed point of a map \( T : V_h \to V_h \) in \( \bar{B}^h \), cf. (1.14), which will be defined in (6.10). We show that in the situation of Theorem 1.2 we can choose \( \beta, \gamma, \eta > 0 \) (and these values can be calculated explicitly) so that
\[
\bar{B}^h \neq \emptyset,
\]
and
\[
(Tw_h - Tv_h)_{H^{1,\nu}(\bar{\Omega}^h)} \leq \chi h^n\|w_h - v_h\|_{H^{1,\nu}(\bar{\Omega}^h)} \quad \forall w_h, v_h \in \bar{B}^h
\]
and
\[
T(\bar{B}^h) \subset \bar{B}^h,
\]
i.e. Theorem 1.2 follows from Banach’s fixed point theorem.

(i) We define the map \( T \).

We define for \( \epsilon > 0 \) and \( z \in \mathbb{R}^n \)
\[
|z|_\epsilon := f_\epsilon(z) := \sqrt{|z|^2 + \epsilon^2}
\]
and denote derivatives of \( f_\epsilon \) with respect to \( z^i \) by \( D_{z^i} f_\epsilon \). There holds
\[
D_{z^i} f_\epsilon(z) = \frac{z_i}{|z|_\epsilon}, \quad D_{z^i} D_{z^j} f_\epsilon(z) = \delta_{ij} \frac{z_i z_j}{|z|_\epsilon} - \frac{z_i z_j}{|z|_\epsilon^2}.
\]
We define the operator \( \Phi_\epsilon \) by
\[
\Phi_\epsilon : H^{1,2}_0(\Omega) \to H^{-1,2}_0(\Omega), \quad \Phi_\epsilon(v) = -D_i \left( \frac{D_i u}{|Dv|_\epsilon} \right) - \frac{1}{|Dv|_\epsilon^2},
\]
so that (1.4) can be written as
\[
\Phi_\epsilon(u^\epsilon) = 0.
\]
We denote the derivative of \( \Phi_\epsilon \) in \( u^\epsilon \) by
\[
L_\epsilon := D\Phi_\epsilon(u^\epsilon)
\]
and have for all \( \varphi \in H^{1,2}_0(\Omega) \)
\[
L_\epsilon \varphi = -D_i \left( D_{z^i} D_{z^j} f_\epsilon(D u^\epsilon) D_j \varphi \right) + \frac{1}{k} f_\epsilon(D u^\epsilon)^{-1 - \frac{1}{2}} D_{z^i} f_\epsilon(D u^\epsilon) D_j \varphi.
\]
We will apply the results of Section 5 to the linear differential operator \( L = L_\epsilon \), where we consider—having (1.12) in mind—\( L_\epsilon \) (and also \( M_\epsilon \)) to be defined in \( H^{1,2}_0(\Omega^h) \), \( h > 0 \) small; one observes that \( L_\epsilon \) has the structure (5.1) and explicit values for the constants \( \lambda, \Lambda, \nu, a_1, h_0 \) in (5.2), (5.14) and (5.13) can be obtained in terms of \( \epsilon \) (this dependence is polynomial in \( \epsilon \) and \( \frac{1}{\epsilon} \)) using the results of Section 4 and (1.12). Let us denote the constant \( c_8 \) in Corollary 5.6 adapted to the case of \( L = L_\epsilon \) (as an operator defined in \( H^{1,2}_0(\Omega^h) \)) by \( \bar{c}_8 \).

We define \( T : V_h \to V_h \) by
\[
L_\epsilon(w_h - Tw_h) = \Phi_\epsilon(w_h), \quad w_h \in V_h.
\]
(ii) We check condition (6.1).

Let
\[
I_h : C^0(\bar{\Omega}^h) \to \bar{V}_h,
\]
be the unique interpolation operator with
\[
I_h u(p) = u(p)
\]
for all \( u \in C^0(\Omega^h) \) and \( p \in N_h \), where

\[
\bar{V}_h := \{ w \in C^0(\Omega^h) : \forall T \in T_h, w|_T \text{ polynomial of degree } \leq 2 \}
\]

and

\[
N_h := \{ p \in \Omega^h : p \text{ vertex or midpoint of an edge of a tetraeder } T \in T_h \}. 
\]

We have

\[
\| u - I_h u \|_{L^\infty(\Omega^h)} \leq ch^2 \| u \|_{C^3(\Omega^h)} \quad \forall u \in C^3(\Omega^h),
\]

define \( z_h \in \bar{V}_h \) by

\[
z_h(p) = \begin{cases} 
I_h u^\varepsilon(p), & \text{if } p \in N_h \cap \partial \Omega^h, \\
0, & \text{if } p \in N_h \setminus \partial \Omega^h
\end{cases}
\]

and set

\[
\tilde{u}^\varepsilon := I_h u^\varepsilon - z_h.
\]

Then \( \tilde{u}^\varepsilon \in V_h \) and for all \( 1 \leq q \leq \infty \)

\[
\| \tilde{u}^\varepsilon - u^\varepsilon \|_{H^1,q(\Omega^h)} \leq ch^{1 + \frac{1}{q}} \| u^\varepsilon \|_{C^3(\bar{\Omega}^h)},
\]

which follows from

\[
\| z_h \|_{C^0(\bar{\Omega}^h)} \leq ch^2, \quad \| Dz_h \|_{L^\infty(\Omega^h)} \leq ch
\]

and that the support of \( z_h \) lies in a boundary strip of measure \( \leq ch \).

We conclude \( \tilde{u}^\varepsilon \in \bar{B}_h^\varepsilon \) provided \( \beta, \gamma > 0 \) are sufficiently large.

(iii) We check condition \((6.2)\).

Let \( q > n + 1 \) and \( v_h, v_h \in \bar{B}_h^\varepsilon \), \( \xi_h = v_h - w_h \), \( \alpha(t) = w_h + t \xi_h, 0 \leq t \leq 1 \), then using \((6.10)\) we conclude

\[
L^\varepsilon(Tv_h - Tw_h) = L^\varepsilon \xi_h + \Phi^\varepsilon(w_h) - \Phi^\varepsilon(v_h).
\]

The right-hand side of \((6.20)\) is of the form \( D_1 f^\varepsilon + g \) with

\[
f^\varepsilon = D_2 f_\varepsilon(Dv_h) - D_2 f_\varepsilon(Dw_h) - D_2 D_2 f_\varepsilon D_2 \xi_h
\]

\[
= \int_0^1 (D_2 D_2 f_\varepsilon(D\alpha(t)) - D_2 D_2 f_\varepsilon(D\alpha(t)) D_2 \xi_h)
\]

and

\[
g = \frac{1}{k} f_\varepsilon^{-\frac{1}{k}} D_2 f_\varepsilon D_2 \xi_h + f_\varepsilon(Dv_h)^{-\frac{1}{k}} - f_\varepsilon(Dw_h)^{-\frac{1}{k}}
\]

\[
= \frac{1}{k} \int_0^1 \left( f_\varepsilon^{-\frac{1}{k}} D_2 f_\varepsilon f_\varepsilon(D\alpha(t)) D_2 f_\varepsilon(D\alpha(t)) D_2 \xi_h \right)
\]

We have

\[
\| Dw_h - D u^\varepsilon \|_{L^\infty(\Omega^h)} \leq \| Dw_h - DI_h u^\varepsilon \|_{L^\infty(\Omega^h)} + \| DI_h u^\varepsilon - D u^\varepsilon \|_{L^\infty(\Omega^h)}
\]

\[
\leq ch^{-\frac{1}{2}} \left( \| Dw_h - D u^\varepsilon \|_{L^\infty(\Omega^h)} + \| D u^\varepsilon - DI_h u^\varepsilon \|_{L^\infty(\Omega^h)} \right)
\]

\[
\leq ch^{-\frac{1}{2}} (\rho + h^2 \| u^\varepsilon \|_{C^3(\bar{\Omega}^h)}),
\]

\[
(6.23)
\]
where we used an inverse estimate and (6.15). We estimate the integrals in (6.22) and (6.21) by mean value theorem and get with a constant $c_9 := c_9(\epsilon, k)$

$$\|f^i\|_{L^q(\Omega^h)} + \|g\|_{L^2(\Omega^h)} \leq c_{10}h^{\frac{n+1}{2}} (\rho + \frac{h^2}{\mu})\rho,$$

Therefore we get

$$\|Tv_h - T\eta_{\alpha}\|_{H^1(\Omega^h)} \leq h^{\frac{n+1}{2}} \|Tv_h - T\eta_{\alpha}\|_{H^2(\Omega^h)}$$

$$\leq c_{10}h^{\frac{n+1}{2}} (\rho + \frac{h^2}{\mu})\rho,$$

in view of Corollary 5.6.

Assuming $n = 1$ and $2 < q < \mu$ we have

$$1 + \frac{2}{\mu} - \frac{2}{q} < 1 < \delta$$

and hence (6.2) holds provided $\beta > 0$ is sufficiently large.

(iv) We check condition (6.3).

Let $q > n + 1$ and $w_h \in V_h$. We have

$$\|Tw_h - u^\epsilon\|_{H^1(\Omega^h)} \leq \|Tw_h - T\eta_{\alpha}\|_{H^1(\Omega^h)} + \|T\eta_{\alpha} - \eta_{\alpha}\|_{H^1(\Omega^h)} + \|\eta_{\alpha} - u^\epsilon\|_{H^1(\Omega^h)}$$

We estimate the three terms on the right-hand side of this inequality separately and get

$$\|\eta_{\alpha} - u^\epsilon\|_{H^1(\Omega^h)} \leq c_{11}h^{\frac{n+1}{2}}\|u^\epsilon\|_{C^3(\Omega^h)}$$

and

$$\|Tw_h - T\eta_{\alpha}\|_{H^1(\Omega^h)} \leq ch^\frac{n}{2} \|w_h - \eta_{\alpha}\|_{H^1(\Omega^h)}$$

$$\leq ch^\frac{n}{2} \|w_h - u^\epsilon\|_{H^1(\Omega^h)} + ch^\frac{n}{2} \|u^\epsilon - \eta_{\alpha}\|_{H^1(\Omega^h)}$$

$$\leq ch^\frac{n}{2} \rho + c_{11}h^{\frac{n+1}{2}}\|u^\epsilon\|_{C^3(\Omega^h)}.$$
with $c_{10} := c_{10}(\epsilon)$ suitable and $c_{11} := \sup_{\Omega_h} |D(\Phi_{\epsilon}(u^\epsilon))|$. Finally, we get

$$
\| \tilde{u}^\epsilon - T(\tilde{u}^\epsilon) \|_{H^{1,\mu}(\Omega^h)} \leq c_8 h^{\frac{n+1}{2}} \left( c_{10} h^{1+\frac{1}{q}} \|u^\epsilon\|_{C^3(\bar{\Omega}^h)} + c_{11} h^{3} \right)
$$

(6.34)

with $c_{12} = c_{12}(\epsilon)$.

To allow for (6.3) in case $n = 1$ it is sufficient to have

$$
\delta < \frac{2}{\mu} + \frac{1}{q},
$$

(6.35)

which holds for $q > 2$ close to 2, and $\beta > 0$ sufficiently large.

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References

[1] B. Andrews: Gauss curvature flow: The fate of the rolling stones, Invent. Math. 138, (1999), 151-161.
[2] J. W. Barrett, H. Garcke, R. Nürnberg: Parametric approximation of isotropic and anisotropic elastic flow for closed and open curves, Numer. Math. 120, (2012), 489-542.
[3] S. C. Brenner, L. R. Scott: The Mathematical Theory of Finite Element Methods, Texts in Applied Mathematics 15, Springer 1996.
[4] M. G. Crandall, H. Ishii, P.-L. Lions: User’s guide to viscosity solutions of second order partial differential equations, Bulletin (new Series) of the American Mathematical Society 27, no 1, (1992), 1-67.
[5] M. G. Crandall, P.-L. Lions: Convergent difference schemes for nonlinear parabolic equations and mean curvature flow, Numer. Math. 75, (1996), 17-41.
[6] K. Deckelnick: Error bounds for a difference scheme approximating viscosity solutions of mean curvature flow, Interfaces Free Bound. 2, (2000), 117-142.
[7] K. Deckelnick, G. Dziuk: Convergence of a finite element method for non–parametric mean curvature flow, Numer. Math. 72, (1995), 197-222.
[8] K. Deckelnick, G. Dziuk, C. M. Elliott: Computation of geometric partial differential equations and mean curvature flow, Acta Numer. 14, (2005), 139-232.
[9] L. C. Evans, J. Spruck: Motion of level sets by mean curvature, J. Differ. Geom. 33, (1991), 635-681.
[10] X. Feng, M. Neilan, A. Prohl: Error analysis of finite element approximations of the inverse mean curvature flow arising from the general relativity, Numer. Math. 108, no 1, (2007), 93-119.
[11] C. Gerhardt: The inverse mean curvature flow in cosmological spacetimes, Adv. Theor. Math. Phys. 12, (2008), 1183-1207.
[12] C. Gerhardt: Partial differential equations I & II , Lecture Notes, University of Heidelberg, http://www.math.uni-heidelberg.de/studinfo/gerhardt/lecture-notes.
[13] D. Gilbarg, N. S. Trudinger: Elliptic Partial Differential Equations of Second Order, (Springer, Berlin, Heidelberg, New York etc., 2001).
[14] G. Huisken: Flow by mean curvature of convex surfaces into spheres, J Differ. Geom. 20, (1984), 117-138.
[15] G. Huisken, T. Ilmanen: The inverse mean curvature flow and the Riemannian Penrose inequality, J. Differ. Geom. 59, no 3, (2001), 353-437.
[16] H. Mitake: On convergence rates for solutions of approximate mean curvature equations, Proceedings of the American Mathematical Society 139, no. 10, (2011), 3691-3696.
[17] R. Rannacher: Numerische Mathematik 2, Lecture Notes, University of Heidelberg, http://numerik.iwr.uni-heidelberg.de/~lehre/notes/num2/numerik2.pdf.
[18] P. Pozzi: Anisotropic mean curvature flow for two dimensional surfaces in higher codimension: a numerical scheme, Interfaces Free Bound. 10, (2008), no. 4, 539-576.
[19] A. Schatz: An observation concerning Ritz-Galerkin methods with indefinite bilinear forms, Math. Comp. 28, (1974), 959-962.
[20] F. Schulze: Evolution of convex hypersurfaces by powers of the mean curvature, Math. Z. 251, no. 4, (2005), 721-733.
[21] F. Schulze: Nonlinear evolution by mean curvature and isoperimetric inequalities, J. Diff. Geom. 79 (2008), 197-241.