CONTROLLABILITY PROPERTIES OF DEGENERATE PSEUDO-PARABOLIC BOUNDARY CONTROL PROBLEMS

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Abstract. This paper concerns with the boundary control of a degenerate pseudo-parabolic equation. Compare to the results those for degenerate parabolic equations, we discover that the null controllability property for the degenerate pseudo-parabolic equation is false, but the approximate controllability in some proper state space holds.

1. Introduction and main results. Theoretical research of pseudo-parabolic equations was started due to their numerous physical applications, such as unidirectional propagation of long waves of small amplitude [1], capillary imbibition [9], and the well-known Benjamin-Bona-Mahony (BBM) equations [28, 29]. Mathematical study of pseudo-parabolic equations goes back to [22] in early 1970s, since then, linear and nonlinear pseudo-parabolic equations are investigated in the mathematical literatures for decades (see [2, 16] and so on).

Control issues for parabolic equations, including deterministic and stochastic parabolic equations, have received a lot of attention in the past few decades (see, for instance, [11], [12], [18], [19], [27] and the rich references therein). Pseudo-parabolic equations, as an important nonclassical parabolic equations, have been studied extensively on its control problems. The optimal control and the controllability property for this type of equations were first established in [25] and [26], respectively. Later in [15], the authors proposed the control laws that stabilize systems governed by the linear and the nonlinear pseudo-parabolic equations and illustrated the application of their results by numerical simulations. Further the internal controllability with the moving control for the pseudo-parabolic equations was considered in [23].

In the last decade, there are a large number of works attributed to the mathematical analysis of the degenerate pseudo-parabolic equations. Several topics on the degenerate pseudo-parabolic model such as traveling wave solutions and existence of weak solutions have been addressed, see [3, 20, 24] and other literatures. However,
very few results, to the best of our knowledge, are obtainable on the control problems of the degenerate pseudo-parabolic equations. Recently, the controllability of a class of linear stochastic degenerate pseudo-parabolic equations was considered in [17]. Therefore, it is natural to investigate the controllability of the deterministic degenerate pseudo-parabolic equations in this paper.

Let $T > 0$, $\Omega = (0, 1)$, $Q = \Omega \times (0, T)$. We consider the following controlled system governed by a linear degenerate pseudo-parabolic equation:

\[
\begin{align*}
\left\{ \begin{array}{l}
y_t - (a(x)y_x)_x - k(a(x)y_{xt})_x = 0 \quad \text{in } Q, \\
y(0, t) = 0 \quad (0 < s < 1) \\
(ay_x)(0, t) = 0 \quad (s \geq 1) \quad \text{in } (0, T), \\
y(1, t) = u(t) \quad \text{in } (0, T), \\
y(x, 0) = y_0(x) \quad \text{in } \Omega,
\end{array} \right. \\
\end{align*}
\]

where $y$ is the state variable, $u$ denotes the control function, $y_0$ is a given initial value, $a(x) = x^s$ with $s > 0$, and $k > 0$ is a constant.

When $s = 0$, the equation (1) is non-degenerate. For the non-degenerate pseudo-parabolic equations, the controllability properties have been investigated by many authors. For instance, the approximate controllability of the pseudo-parabolic equation was analysed in [26]. The authors found that the approximate controllability result of the pseudo-parabolic equation on finite interval can be obtained, but the equation is not approximately controllable on semi-infinite interval. In [23], the authors discussed the null controllability of a pseudo-parabolic equation with moving control on the one-dimensional torus by moment method. On the other hand, when the pseudo-parabolic term dropped (i.e. $k = 0$), (1) is reduced to the degenerate parabolic equation which is closely related to the pseudo-parabolic equation in this paper. The controllability of this type of degenerate parabolic equations has been studied widely and is quite well understood. For example, in [4] the authors deduced the null controllability results for degenerate parabolic equations, where the control acts on a nonempty subinterval of $\Omega$. The null controllability for degenerate parabolic equations with the boundary control located at the degenerate point was considered in [14]. The authors, in [5], obtained the controllability with control acting on the non-degenerate boundary. In [7], the authors considered the cost of controlling weakly degenerate parabolic equations by a boundary control acting at the degeneracy point based on spectral decomposition of the solution in terms of Bessel’s functions. Another relevant work concerning to the cost of the strongly degenerate parabolic equations by a boundary control acting at non-degeneracy point, or by a locally distributed control is [6]. More precisely, the null controllability holds when $s \in (0, 2)$ and the null controllability fails when $s \geq 2$ in this case. However, the controllability property of the system (1) is completely different from the degenerate parabolic equations (we will explain why the different was arised in Remark 1).

The purpose of our paper is to prove the non-controllability and approximate controllability of the system (1). Inspired by [13], we construct a terminal value such that observability estimate for the adjoint system does not hold and obtain the non-controllability of the system (1). Next, the approximate controllability for the system (1) is obtained by means of the eigenvector expansion of the solution.
Let
\[
H^1_s(\Omega) := \{ f \in L^2(\Omega) | x^s f \in L^2(\Omega), f(0) = f(1) = 0 \}, \quad \text{for } s \in (0, 1),
\]
\[
H^1_k(\Omega) := \{ f \in L^2(\Omega) | x^s f \in L^2(\Omega), f(1) = 0 \}, \quad \text{for } s \geq 1,
\]
\[
H^2_k(\Omega) = \{ f \in H^1_k(\Omega) | x^s f \in H^1(\Omega) \}.
\]
All these spaces are equipped with their graph norms, and \( H^{-1}_s(\Omega) \) stands for the dual space of \( H^1_s(\Omega) \).

First, the solution in the sense of transposition of (1) is defined as follows.

**Definition 1.1.** For any \( y_0 \in L^2(\Omega) \), \( u \in H^1(0, T) \), a function \( y \in C([0, T]; H^{-1}_s(\Omega)) \) is said to be a solution to (1) in the sense of transposition if, for any \( f \in L^1(0, T; H^1_s(\Omega)) \), the following identity holds:
\[
\langle y, f \rangle_{L^\infty(0, T; H^{-1}_s(\Omega)), L^1(0, T; H^1_s(\Omega))} = \int_\Omega y_0(x) (\varphi(x, 0) - k(a\varphi_x)_x(x, 0)) \, dx + \int_0^T \varphi_x(1, t) (u + ku_t) \, dt,
\]
where \( \varphi \) is the weak solution of
\[
\begin{cases}
\varphi_t + (a\varphi_x)_x - k(a\varphi_xt)_x = f & \text{in } Q, \\
\varphi(0, t) = 0 & (0 < s < 1) \\
(a\varphi_x)(0, t) = 0 & (s \geq 1) \\
\varphi(1, t) = 0 & \text{in } (0, T), \\
\varphi(x, T) = 0 & \text{in } \Omega.
\end{cases}
\]

The following theorem states the well-posedness of (1).

**Theorem 1.2.** Let \( y_0 \in L^2(\Omega) \) and \( u \in H^1(0, T) \), then the system (1) admits a unique solution \( y \in C([0, T]; H^{-1}_s(\Omega)) \) in the sense of Definition 1.1 satisfying
\[
\|y\|_{L^\infty(0, T; H^{-1}_s(\Omega))} \leq C \left( \|y_0\|_{L^2(\Omega)} + \|u\|_{H^1(0, T)} \right).
\]

System (1) is said to be null controllable at time \( T \) if for any \( y_0 \in L^2(\Omega) \), there exists a control \( u \in H^1(0, T) \) such that the solution \( y \) of (1) corresponding to \( u \) satisfies \( y(x, T) = 0 \) (in the sense of transposition) for almost all \( x \in \Omega \).

We show our main result on non-controllability for the system (1).

**Theorem 1.3.** Let \( T > 0 \). Then for any \( s > 0 \), the system (1) is not null controllable at time \( T \).

**Remark 1.** The controllability of degenerate pseudo-parabolic equations is completely different from that of degenerate parabolic equations due to the third-order term. Now, we explain it in details. When \( s \in (0, 2) \), the eigenvalues of degenerate parabolic equations accumulate at \( \infty \), but the eigenvalues of system (1) have an accumulation point \(-\frac{1}{k}\) (see Lemma 3.2). Compared with the observability inequality for the degenerate parabolic equation, there is one more item \((a\varphi_x)_x(\cdot, 0)\) (see (13)) in the initial information. Therefore, the initial information cannot be estimated from above by the right hand of the observability inequality.
Another main result is concerned with the approximate controllability for the system (1). To this end, we introduce some notations. First, consider the following eigenvalue problem:

\[
\begin{aligned}
- \left( a w_x \right)_x &= \lambda w & \text{in } \Omega, \\
\left( a w_x \right)(0) &= 0 & (0 < s < 1), \\
\left( a w_x \right)(0) &= 0 & (1 \leq s < 2), \\
w(1) &= 0.
\end{aligned}
\]

Then, for any \( n \geq 1 \), the eigenvalues and the corresponding eigenfunctions of (5) are given by (see [14])

\[
\begin{aligned}
w_n(x) &= \left( \frac{(2s)^{\frac{3}{2}}}{\Gamma(\frac{3}{2})} x^{1-s} \right) J_{\alpha,n}(x), & x \in \Omega, \\
\lambda_n &= (\beta j_{\alpha,n})^2,
\end{aligned}
\]

where \( \alpha = \left\{ \begin{array}{ll}
0 < s < 1, \\
1 \leq s < 2, 
\end{array} \right. \beta = \frac{2-s}{2} \), \( J_{\alpha,n}(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{\Gamma(l+\alpha+1)} \left( \frac{x}{2} \right)^{2l+\alpha} \) for \( x \geq 0 \)

is the Bessel function, \( \Gamma(\cdot) \) is the Gamma function, and \( \{j_{\alpha,n}\}_{n \geq 1} \) are the positive zeros of \( J_{\alpha} \).
Moreover, \( \|w_n\|_{L^2(\Omega)} = 1 \), \( \forall \ n \in \mathbb{N}^* \), and \( \{w_n\}_{n \in \mathbb{N}^*} \) is an orthonormal basis of \( L^2(\Omega) \) when \( \alpha \geq -\frac{1}{2} \) and \( \beta > 0 \).

Furthermore, if \( v \in L^2(\Omega) \), then \( v \) can be written the following Fourier expansion form:

\[
v(x) = \sum_{n \in \mathbb{N}^*} \tilde{v}_n w_n(x), \quad \text{where} \quad \tilde{v}_n = \int_{\Omega} v(x) w_n(x) dx.
\]

For any \( \gamma \geq 0 \), we define \( \mathbb{H}_s^\gamma := \left\{ v = \sum_{n \in \mathbb{N}^*} \tilde{v}_n w_n(x) : \|v\|_s^\gamma = \sum_{n \in \mathbb{N}^*} |\tilde{v}_n|^2 \lambda_n^\gamma < +\infty \right\} \), and \( \mathbb{H}_s^\gamma \) is a Hilbert space with the inner product \( (u,v)_{\mathbb{H}_s^\gamma} = \sum_{n \in \mathbb{N}^*} \tilde{u}_n \tilde{v}_n \lambda_n^\gamma \), \( \forall \ u = \sum_{n \in \mathbb{N}^*} \tilde{u}_n w_n(x), \ v = \sum_{n \in \mathbb{N}^*} \tilde{v}_n w_n(x) \). Moreover, \( \mathbb{H}_s^{-\gamma} := [\mathbb{H}_s^\gamma]' \), \( \langle \cdot, \cdot \rangle_{\mathbb{H}_s^\gamma, \mathbb{H}_s^{-\gamma}} \) denotes the duality product between \( \mathbb{H}_s^\gamma \) and \( \mathbb{H}_s^{-\gamma} \). Obviously, \( \mathbb{H}_s^1 = H_s^1(\Omega) \) and \( \mathbb{H}_s^{-1} = H_s^{-1}(\Omega) \).

**Theorem 1.4.** If \( \nu > \frac{3}{2} \), then for any \( s \in (0, \frac{4}{3}] \), the system (1) is approximately controllable in \( \mathbb{H}_s^{-\nu} \) at time \( T \).

**Remark 2.** The proof of Theorem 1.4 depends on the property for the zeros of Bessel function \( J_{\alpha} \), and this property holds only when \( s \in (0, \frac{4}{3}] \). Therefore, new techniques should be developed to investigate the approximate controllability for the case of \( s > \frac{4}{3} \) in our future works.

**Remark 3.** In this paper, the control acts on the non-degenerate boundary. If \( s \in (0, 1) \), we can also obtain the same results (Theorem 1.3 and Theorem 1.4) even if the control acts on the whole boundary or the degenerate boundary in a similar way.

This paper is organized as follows. In Section 2, we prove the well-posedness of system (1). Section 3 is devoted to the proofs of non-null-controllability and approximate controllability for system (1) (i.e. Theorem 1.3 and Theorem 1.4).
2. Well-posedness. In this section, we investigate the well-posedness for the system (1). First, we are concerned with the following degenerate pseudo-parabolic equation with homogeneous boundary conditions:

$$
\begin{aligned}
z_t - (a z_x)_x - k(a z_{xt})_x &= g & \text{in } Q, \\
(az)_x(0, t) &= 0 & (0 < s < 1) \\
(az_x)_x(0, t) &= 0 & (s \geq 1) \\
z(1, t) &= 0 & \text{in } (0, T), \\
z(x, 0) &= z_0(x) & \text{in } \Omega,
\end{aligned}
$$

where $g \in L^1(0, T; L^2(\Omega))$, $z_0 \in H^1_0(\Omega)$.

Similar to pseudo-parabolic equations, it is easy to establish the well-posedness of the system (8) as follows (see [22]).

**Proposition 2.1.** Let $z_0 \in H^1(\Omega)$ and $g \in L^1(0, T; L^2(\Omega))$. Then (8) admits a unique solution $z \in C([0, T]: H^1(\Omega))$. Furthermore, if $z_0 \in H^2(\Omega)$, $g \in L^1(0, T; H^2(\Omega))$ then there exists a unique solution $z \in C([0, T]; H^2(\Omega))$ satisfying

$$
\|z\|_{L^\infty(0,T;H^2(\Omega))} \leq C (\|g\|_{L^1(0,T;H^1(\Omega))} + \|z_0\|_{H^2(\Omega)}).
$$

Now, let us prove Theorem 1.2.

**Proof of Theorem 1.2.** Multiply the first equation of (1) by $\varphi$ ($\varphi$ is the solution of (3)) and integrate over $Q$, by (1), (3), and integrations by parts, one gets

$$
\int_Q fy dxdt = -\int_\Omega y_0(x)(\varphi(x, 0) - k(a \varphi_x)_x(x, 0))dx + \int_0^T \varphi_x(1, t)(u + ku_t)dt.
$$

Notice that $f \in L^1(0, T; H^1(\Omega))$, by Proposition 2.1, we get $\varphi \in C([0, T]; H^2(\Omega))$. Moreover,

$$
\|\varphi\|_{L^\infty(0,T;H^2(\Omega))} \leq C\|f\|_{L^1(0,T;H^1(\Omega))}.
$$

Using the classical trace theorem yields

$$
\int_0^T \varphi_x^2(1, t)dt \leq C\|f\|_{L^1(0,T;H^1(\Omega))}^2.
$$

We introduce a linear functional on $L^1(0, T; H^1_0(\Omega))$:

$$
\mathcal{L}(f) = -\langle y_0, \varphi(\cdot, 0) - k(a \varphi_x)_x(\cdot, 0) \rangle_{L^2(\Omega)} + \int_0^T \varphi_x(1, t)(u + ku_t)dt.
$$

we use (11) and (12) to drive that

$$
|\mathcal{L}(f)| \leq \|y_0\|_{L^2(\Omega)} \cdot (\|\varphi(\cdot, 0)\|_{L^2(\Omega)} + k\|a \varphi_x\|_{L^2(\Omega)}) + \|\varphi_x(1, \cdot)\|_{L^2([0, T])} \cdot (\|u\|_{L^2(0, T)} + k\|u_t\|_{L^2(0, T)})
$$

$$
\leq C (\|y_0\|_{L^2(\Omega)} + \|u\|_{L^2(0, T)} + \|u_t\|_{L^2(0, T)}) \cdot \|f\|_{L^1(0,T;H^1(\Omega))}.
$$

Then, $\mathcal{L}$ is a bounded linear functional on $L^1(0, T; H^1_0(\Omega))$. Therefore, we can find a function $y \in L^\infty(0,T; H^{-1}_s(\Omega))$ such that (2) holds. By the smoothing technique, we have $y \in C([0, T]; H^{-1}_s(\Omega))$ and the estimate (4) holds. Thus, we complete the proof of Theorem 1.2. \qed
3. Proof of main results. Let us now turn to prove the controllability results of (1). First, by the duality argument between controllability and observability, one has the following lemma.

**Lemma 3.1.** The control system (1) is null controllable at time $T$ if and only if the following property holds: there exists a constant $C > 0$ such that

$$\|\phi(\cdot, 0) - k(a\phi_x)_x(\cdot, 0)\|_{L^2(\Omega)}^2 \leq C \int_0^T \phi^2_k(1, t) dt, \ \forall \ \phi_t \in H^1_0(\Omega),$$

where $\phi$ is the solution of the following adjoint equation

$$\begin{cases}
\phi_t + (a\phi_x)_x - k(a\phi_{xt})_x = 0 & \text{in } Q, \\
\phi(0, t) = 0 & (0 < s < 1) \\
(a\phi_x)(0, t) = 0 & (s \geq 1) \\
\phi(1, t) = 0 & (0, T), \\
\phi(x, T) = \phi_T(x) & \text{in } \Omega.
\end{cases}$$

**Proof.** $(\Rightarrow)$ We suppose that the system (1) is null controllable, i.e., for any $y_0 \in L^2(\Omega)$, there exists $u \in H^1(0, T)$ such that the solution of (1) satisfies $y(x, T) = 0$, $\forall \ x \in \Omega$. We need to prove the estimate (13). To do so, let $\mathcal{P} : H^1_0(\Omega) \to L^2(\Omega)$ be defined by

$$\mathcal{P}(\phi_T) := \phi(\cdot, 0) - k(a\phi_x)_x(\cdot, 0).$$

Using the contradiction argument, we suppose that (13) is not true. Then, we can find a sequence of functions $\{\phi^n_T\}_{n=1}^\infty \subset H^1_0(\Omega)$ such that the solution $\{\phi^n\}$ of (14) corresponding to $\{\phi^n_T\}_{n=1}^\infty$ satisfy:

$$0 \leq \int_0^T \left(\phi^n_T(1, t)\right)^2 dt \leq \frac{1}{n^2} \|\phi^n(\cdot, 0) - k(a\phi^n_{x})_x(\cdot, 0)\|^2_{L^2(\Omega)}.$$  \hspace{1cm} (16)

We take $\tilde{\phi}^n_T = \frac{\sqrt{n}\phi^n_T}{\|\phi^n(\cdot, 0) - k(a\phi^n_{x})_x(\cdot, 0)\|_{L^2(\Omega)}}$, let $\tilde{\phi}^n$ be the solution of (14) associated to $\tilde{\phi}^n_T$. Then, $\tilde{\phi}^n = \frac{\sqrt{n}\phi^n}{\|\phi(\cdot, 0) - k(a\phi_x)_x(\cdot, 0)\|_{L^2(\Omega)}}$. We use (16) to obtain

$$\int_0^T |\tilde{\phi}^n_T(1, t)|^2 dt \leq \frac{1}{n} \text{ and } \|\mathcal{P}(\tilde{\phi}^n_T)\|_{L^2(\Omega)} = \sqrt{n}.$$ \hspace{1cm} (17)

On the other hand, multiply both sides of (1) by $\tilde{\phi}^n$ and perform integration over $Q$, by (1), (14), and notice that the system (1) is null controllable, one gets

$$\int y_0(x)(\tilde{\phi}^n(x, 0) - k(a\tilde{\phi}^n_x)_x(x, 0)) dx = \int_0^T \tilde{\phi}^n_T(1, t)(u + ku) dt.$$ \hspace{1cm} (18)

By (15), it follows that

$$\langle y_0, \mathcal{P}(\tilde{\phi}^n_T) \rangle_{L^2(\Omega)} = \int_0^T \tilde{\phi}^n_T(1, t)(u + ku) dt.$$

From the first inequality in (17), we can deduce that $\mathcal{P}(\tilde{\phi}^n_T) \to 0$ in $L^2(\Omega)$, which implies that $\mathcal{P}(\tilde{\phi}^n_T)$ is uniformly bounded in $L^2(\Omega)$. This contradicts to the second equality in (17).
(⇐) We suppose that (13) holds. Introduce the following subspace of $L^2(0,T)$:
\[ \mathcal{H} := \left\{ \phi_x(1,\cdot) \bigg| \phi \text{ is the solution of (14)} \right\}. \]
For any $\phi_x(1,\cdot) \in \mathcal{H}$, define a linear functional: $\mathcal{A}(\phi_x(1,\cdot)) := \int_{\Omega} y_0(x)(\phi(x,0) - k(a\phi_x)_x(x,0))dx$. From (13), one gets $|\mathcal{A}(\phi_x(1,\cdot))| \leq C\|y_0\|_{L_2(\Omega)} \left(\int_0^T \phi_x^2(1,t)dt\right)^{\frac{1}{2}}$.

Therefore, $\mathcal{A}$ is a bounded linear functional on $\mathcal{H}$. We use the Hahn-Banach Theorem to obtain that $\mathcal{A}$ can be extended to a bounded linear functional on $L^2(0,T)$ (which is still denoted by itself). It then follows from Reisz Representation Theorem that there exists a function $v \in L^2(0,T)$ such that $\mathcal{A}(\eta) = \int_0^T \eta v dt$, $\forall \eta \in L^2(0,T)$. Set $\eta = \phi_x(1,\cdot)$ one has
\[ \int_{\Omega} y_0(x)(\phi(x,0) - k(a\phi_x)_x(x,0))dx = \int_0^T \phi_x(1,t)v(t)dt. \quad (19) \]

On the other hand, we multiply (1) by $\phi$ and integral the obtained equality on $Q$. Combining (1) and (14), we have
\[ \int_{\Omega} y_0(x)(\phi(x,0) - k(a\phi_x)_x(x,0))dx - \int_{\Omega} y(x,T)(\phi(x,T) - k(a\phi_x)_x(x,T))dx \]
\[ = \int_0^T \phi_x(1,t)(u + ku_t)dt. \quad (20) \]
We can deduce that there exists a function $u \in H^1(0,T)$ such that
\[ \int_0^T \phi_x(1,t)v(t)dt = \int_0^T \phi_x(1,t)(u + ku_t)dt. \]
The preceding identity, together with (19) and (20), indicate, $y(x,T) = 0$ in $\Omega$, the proof of Lemma 3.1 is completed. \hfill \box

Next, for simplicity, we take $k = 1$. Let us consider the following degenerate pseudo-parabolic equation:
\[
\begin{aligned}
\psi_t - (a\psi_x)_x - (a\psi_{xt})_x &= 0 \quad \text{in } Q, \\
\psi(0,t) &= 0 \quad (0 < s < 1) \\
(a\psi_x)(0,t) &= 0 \quad (s \geq 1) \quad \text{in } (0,T), \\
\psi(1,t) &= 0 \quad \text{in } (0,T), \\
\psi(x,0) &= \psi_0(x) \quad \text{in } \Omega.
\end{aligned}
\]
An explicit solution of (21) is given by the following lemma.

\textbf{Lemma 3.2.} If $\psi_0(x) = \sum_{n \in \mathbb{N}^*} \psi_0^n w_n(x)$, then the solution of (21) can be written as follows
\[ \psi(x,t) = \sum_{n \in \mathbb{N}^*} \psi_0^n e^{\tilde{\lambda}_n t} w_n(x), \quad (22) \]
where $\tilde{\lambda}_n = -\frac{\lambda_n}{1 + \lambda_n} = -\frac{(\beta_j \alpha_{\bar{n},n})^2}{1 + \beta_j \alpha_{\bar{n},n}^2}$ and $w_n(x) = \frac{\beta_j \alpha_{\bar{n},n}}{\beta_j \alpha_{\bar{n},n}^2} x \int_0^1 J_\lambda(j_{\alpha,n}x^\beta).$ Moreover, $\tilde{\lambda}_n < 0$, the sequence $\{\tilde{\lambda}_n\}$ is decreasing with respect to $n$, and $\tilde{\lambda}_n \to -1$, as $n \to \infty.$
Proof. Using the method of separation of variables, set \( \psi(x,t) = T(t)X(x) \neq 0 \). By the first equation of (21) one gets

\[
\frac{T'(t)}{T(t) + T'(t)} = \frac{(aX_x(x))_x}{X(x)} := \mu. \tag{23}
\]

Combining this and the second and third equalities of (21), we obtain

\[
\begin{cases}
(aX_x(x))_x = \mu X(x) \quad \text{in } \Omega, \\
X(0) = X(1) = 0 \quad (0 < s < 1), \\
(aX_x(0)) = X(1) = 0 \quad (s \geq 1).
\end{cases}
\]

Therefore, by (5) and (6), we have

\[
\mu = -\lambda_n = -\left(\beta j_{\alpha,n}\right)^2, \quad X_n(x) = C_n w_n(x) = C_n \left(\frac{(2\beta)^{\frac{1}{2}}}{|J'_\alpha(j_{\alpha,n})|}\right)^{\frac{1}{2}} \int_{0}^{s} J_\alpha(j_{\alpha,n}x^\beta), \tag{24}
\]

where \( C_n \) is an arbitrary constant.

On the other hand, by (23) and (24), one gets

\[
T'(t) + \lambda_n T'(t) + \lambda_n T(t) = 0.
\]

Then \( T_n(t) = C_t e^{\hat{\lambda}_n t} \), where \( \hat{\lambda}_n = \frac{-(\beta j_{\alpha,n})^2}{1 + (\beta j_{\alpha,n})^2} \). As \( \psi_0(x) = \sum_{n \in \mathbb{N}^*} \psi_n^2 w_n(x) \), we get the solution of (21) as follows

\[
\psi(x,t) = \sum_{n \in \mathbb{N}^*} \psi_n^2 e^{\hat{\lambda}_n t} w_n(x).
\]

Moreover, by the expression of \( \hat{\lambda}_n \), it is obvious that \( \hat{\lambda}_n < 0 \). Notice that \( j_{\alpha,1} < j_{\alpha,2} < \ldots \), converging to infinity (see [14, Lemma 1]), then the sequence \( \{\hat{\lambda}_n\} \) is decreasing with respect to \( n \), and \( \hat{\lambda}_n \to -1 \), as \( n \to \infty \).

The idea of proving the non-null-controllability of the system (1) is partially motivated by [13]. Now, we give the proof of Theorem 1.3.

Proof of Theorem 1.3. Step 1. The case of \( 0 < s < 2 \). By Lemma 3.1, it suffices to prove the estimate (13) fails for (14). Set \( \phi_T(x) = \sum_{n \in \mathbb{N}^*} \gamma_n w_n(x), \quad x \in \Omega \), where \( \{\gamma_n\}_{n \in \mathbb{N}^*} \in l^2 \equiv \{\{a_i\}_{i \in \mathbb{N}^*} : \sum_{i \in \mathbb{N}^*} a_i^2 < +\infty\} \). Then, by Lemma 3.2, we obtain that the solution of (14) is given by:

\[
\phi(x,t) = \sum_{n \in \mathbb{N}^*} \xi_n(t) w_n(x), \quad \forall (x,t) \in Q, \tag{25}
\]

where \( \xi_n(t) = \gamma_n e^{\hat{\lambda}_n(T-t)}, \quad \hat{\lambda}_n = \frac{-(\beta j_{\alpha,n})^2}{1 + (\beta j_{\alpha,n})^2}, \quad w_n(x) = \left(\frac{(2\beta)^{\frac{1}{2}}}{|J'_\alpha(j_{\alpha,n})|}\right)^{\frac{1}{2}} x^{\frac{1+\alpha}{2}} J_\alpha(j_{\alpha,n}x^\beta), \quad \alpha = \begin{cases} 1 - \frac{1}{s} & 0 < s < 1, \\ \frac{1}{2} - \frac{1}{s} & 1 \leq s < 2. \end{cases} \]

By (5), (6) and notice that \( \{w_n\}_{n \in \mathbb{N}^*} \) is an orthonormal basis of \( L^2(\Omega) \) when \( 0 < s < 2 \), we get

\[
\begin{align*}
\|\phi(\cdot,0) - (a\phi_x)_x(\cdot,0)\|^2_{L^2(\Omega)} &= \int_{\Omega} \left| \sum_{n \in \mathbb{N}^*} \gamma_n e^{\hat{\lambda}_n T} w_n(x) - \sum_{n \in \mathbb{N}^*} \gamma_n e^{\hat{\lambda}_n T} (aw_n)_x(x) \right|^2 dx \\
&= \int_{\Omega} \left| \sum_{n \in \mathbb{N}^*} \gamma_n e^{\hat{\lambda}_n T} \left(1 + (\beta j_{\alpha,n})^2\right) w_n(x) \right|^2 dx
\end{align*}
\]
Then, we have an equation and two unknowns
\[
C \text{equation (31), we can choose nontrivial solutions } \gamma_s \text{ sufficiently large entire number, we choose } N \leq K. \text{ Therefore, by (29)-(31) and (32), we have }
\]

On the other hand, by (6), we have
\[
\phi_n'(x) = \frac{(2\beta)^{\frac{1}{2}}}{[J_\alpha(J_{\alpha,n})]} \left[ \frac{1 - 2x^{1-\beta}J_{\alpha}(J_{\alpha,n}x^\beta)}{2} + \beta j_{\alpha,n}x^{1-2\beta}J'_{\alpha}(J_{\alpha,n}x^\beta) \right].
\]
Then,
\[
\phi_n'(1) = \frac{(2\beta)^{\frac{1}{2}}}{[J_\alpha(J_{\alpha,n})]} \beta j_{\alpha,n}J'_{\alpha}(j_{\alpha,n}).
\]
Therefore,
\[
\int_0^T \phi_n^2(1,t)dt = \int_0^T \left[ \sum_{n \in \mathbb{N}^*} \gamma_n e^{\lambda_n(T-t)} \frac{(2\beta)^{\frac{1}{2}}}{[J_\alpha(J_{\alpha,n})]} \beta j_{\alpha,n}J'_{\alpha}(j_{\alpha,n}) \right]^2 dt. \tag{28}
\]
We first estimate the right hand of (13): \(\int_0^T \phi_n^2(1,t)dt\). Suppose that \(K > 0\) is a sufficiently large entire number, we choose \(\gamma_n = 0\), if \(n \notin X_{K,2} := \{ K + m : m \in \mathbb{N}, 1 \leq m \leq 2 \}\). Then, by (28), we have
\[
\int_0^T \phi_n^2(1,t)dt
\]
\[
= \int_0^T \left[ \sum_{m=1}^2 \gamma_{K+m} e^{\lambda_{K+m}(T-t)} \frac{(2\beta)^{\frac{1}{2}}}{J'_\alpha(j_{\alpha,K+m})} \beta j_{\alpha,K+m}J'_{\alpha}(j_{\alpha,K+m}) \right]^2 dt. \tag{29}
\]
By \(\lambda_n = (\beta j_{\alpha,n})^2\), we choose \(n\) large enough, then
\[
e^{\lambda_n(T-t)} = e^{-\frac{1}{\lambda_n}x^{\lambda_n}(T-t)} = e^{-(T-t)} \left( 1 + \frac{T-t}{\lambda_n} + o(\lambda_n^{-2}) \right). \tag{30}
\]
We suppose that \(\gamma_n\) satisfy the following equation:
\[
\sum_{m=1}^2 \gamma_{K+m} \frac{(2\beta)^{\frac{1}{2}}}{J'_\alpha(j_{\alpha,K+m})} \beta j_{\alpha,K+m}J'_{\alpha}(j_{\alpha,K+m}) = 0. \tag{31}
\]
Then, we have an equation and two unknowns \(\{\gamma_n\}_{n \in X_{K,2}}\). By the linearity of equation (31), we can choose nontrivial solutions \(\{\gamma_n\}_{n \in X_{K,2}}\) satisfy \(|\gamma_n| \leq C\) with \(C\) independent of \(K\). Therefore, by (29)-(31) and \(\lambda_n = (\beta j_{\alpha,n})^2\), one gets
\[
\int_0^T \phi_n'^2(1,t)dt
\]
\[
= \int_0^T \left[ \sum_{m=1}^2 \gamma_{K+m} \frac{(2\beta)^{\frac{1}{2}}}{J'_\alpha(j_{\alpha,K+m})} \beta j_{\alpha,K+m}J'_{\alpha}(j_{\alpha,K+m})e^{-(T-t)} \right]^2 dt
\]
\[
= \int_0^T e^{-2(T-t)} \left[ \sum_{m=1}^2 \gamma_{K+m} \frac{(2\beta)^{\frac{1}{2}}}{J'_\alpha(j_{\alpha,K+m})} \beta j_{\alpha,K+m}J'_{\alpha}(j_{\alpha,K+m}) \right]^2 dt
\]
\[
\left( \frac{T - t}{\lambda_{K+m}} + o(\lambda_{K+m}^{-2}) \right)^2 dt \leq \frac{C}{(j_{\alpha,K})^2},
\]
where \( C > 0 \) is independent of \( K \).

Meanwhile, by the selection of \( \gamma_n \) and (26), note that \( \hat{\lambda}_n \geq -1 \), we get
\[
\|\phi(-,0) - (a\phi_x)_x(-,0)\|^2_{L^2(\Omega)} = \sum_{n \in X_{K+2}} |\gamma_n|^2 e^{2\hat{\lambda}_n T} \left( 1 + (\beta j_{\alpha,n})^2 \right) \geq C(j_{\alpha,K})^4, \tag{33}
\]
where \( C > 0 \) is independent of \( K \). If we suppose that (13) holds, then by (32) and (33), we have
\[
C(j_{\alpha,K})^4 \leq \|\phi(-,0) - (a\phi_x)_x(-,0)\|^2_{L^2(\Omega)} \leq C \int_0^T \phi_x^2(1,t) dt \leq \frac{C}{(j_{\alpha,K})^2}.
\]
This contradicts to \((j_{\alpha,K})^4 > \frac{C}{(j_{\alpha,K})^2}\) by taking \( K \) being an entire number and large enough, where \( C > 0 \) is independent of \( K \). Therefore, we find a terminal value \( \phi_T(x) \) such that (13) does not hold. By Lemma 3.1, this completes the proof of Theorem 1.3 when \( 0 < s < 2 \).

**Step 2.** The case of \( s \geq 2 \). The method in Step 1 can not applied to the case of \( s \geq 2 \), since \( \{w_n\}_{n \in \mathbb{N}} \) can not form an orthonormal basis in \( L^2(\Omega) \). Therefore, we use a standard change of variables (see \([8]\)):
\[
X = \int_x^1 \theta^{-\frac{1}{2}} d\theta \quad \text{and} \quad Y(X,t) = x^2 y(x,t).
\]
If \( s > 2 \), then the system (1) transforms into a control problem on the semi-infinite interval \( \mathbb{R}^+ \):
\[
\begin{cases}
(1 + F(X))Y_t(X,t) - Y_{XX}(X,t) - Y_{Xt}(X,t) + F(X)Y(X,t) = 0 & \text{in } \mathbb{R}^+ \times (0,T),
Y(0,t) = u(t) & \text{in } (0,T),
Y(X,0) = Y_0(X)
:= \left[ 1 - (1 - \frac{s}{2})X \right] \frac{1}{\sqrt{(4-2s)X}} y_0 \left( \left[ 1 - (1 - \frac{s}{2})X \right] \frac{x}{\sqrt{2}} \right) & \text{in } \mathbb{R}^+,
\end{cases}
\tag{34}
\]
where \( F(X) = \frac{s(3s-4)}{[4s-2s^2]X^2} \). The null controllability of (34) with the boundary control is equivalent to the existence of a constant \( C > 0 \) such that
\[
\|(1 + F(X))\Phi(X,0) - \Phi_{XX}(X,0)\|_{L^2(\mathbb{R}^+)}^2 \leq C \int_0^T \Phi_X^2(0,t) dt, \tag{35}
\]
for the adjoint system
\[
\begin{cases}
(1 + F(X))\Phi_t(X,t) + \Phi_{XX}(X,t) - \Phi_{Xt}(X,t) - F(X)\Phi(X,t) = 0 & \text{in } \mathbb{R}^+ \times (0,T),
\Phi(0,t) = 0 & \text{in } (0,T),
\Phi(X,T) = \Phi_T(X) & \text{in } \mathbb{R}^+.
\end{cases}
\tag{36}
\]
Let \( \Phi \in C_0^\infty(\mathbb{R}^+) \) and \( \Phi^m_m(X) = \Phi(X - m) \) with \( m > 0 \) large enough. \( \Phi^m_m \) denotes the solution of (36) associated to the terminal data \( \Phi^m_T \). Then, \( \int_0^T (\Phi^m_X)^2(0,t) dt \rightarrow \)
Proof. Let \( \Phi \) denote the solution of system (14) associated to \( \Phi_T \). Then, for any \( 0 < s \leq \frac{4}{3} \), \( \Phi(1, \cdot) \in C[0, T] \).

If \( s = 2 \), the derivation can be made in a similar way with \( F(X) = \frac{1}{2} \). The proof of Theorem 1.3 is now completed.

To prove Theorem 1.4, we need the following lemma.

**Lemma 3.3.** Assume that \( \phi_T \in \mathbb{H}_s^\nu \) with \( \nu > \frac{3}{2} \). Let \( \phi \) denote the solution of system (14) associated to \( \phi_T \). Then, for any \( 0 < s \leq \frac{4}{3} \), \( \phi(x, \cdot) \in C[0, T] \).

Proof. Let \( \phi_T(x) = \sum_{n \in \mathbb{N}^*} \hat{\gamma}_n w_n(x) \), \( \nu = \Omega \). By \( \phi_T \in \mathbb{H}_s^\nu \), we have \( \sum_{n \in \mathbb{N}^*} |\hat{\gamma}_n|^2 \lambda_n^\nu < +\infty \). Then \( \phi(x, t) = \sum_{n \in \mathbb{N}^*} \hat{\gamma}_n e^{\lambda_n(t-t)}w_n(x) \), \( \forall (x, t) \in Q \). By (27) and Lemma 3.2, we obtain

\[
|\phi_x(1, \cdot)| = \left| \sum_{n \in \mathbb{N}^*} \hat{\gamma}_n e^{\lambda_n(t-T)} \left| J_\alpha^j(j_{\alpha, n}) \right| \beta_{j_{\alpha, n}} J_\alpha^j(j_{\alpha, n}) \right|
\]

\[
\leq C \left( \sum_{n \in \mathbb{N}^*} \hat{\gamma}_n j_{\alpha, n} \right) \leq C \left( \sum_{n \in \mathbb{N}^*} \hat{\gamma}_n^2 \lambda_n^\nu \right)^{\frac{1}{2}} \left( \sum_{n \in \mathbb{N}^*} j_{\alpha, n}^{2-2\nu} \right)^{\frac{1}{2}}
\]

We know that \( j_{\alpha, n} \) satisfies (see [10]) \( \nu s + \frac{\pi}{2} (\alpha - \frac{1}{2}) < j_{\alpha, n} < n \pi, \forall n \in \mathbb{N}^*, \) if \( -\frac{1}{2} \leq \alpha \leq \frac{1}{2} \). i.e., the preceding inequality holds when \( 0 < s \leq \frac{4}{3} \). Then \( \left( \sum_{n \in \mathbb{N}^*} j_{\alpha, n}^{2-2\nu} \right)^{\frac{1}{2}} < +\infty \), if \( \nu > \frac{3}{2} \) and \( 0 < s \leq \frac{4}{3} \). Therefore, \( \phi_x(1, \cdot) \in C[0, T] \).

Proof of Theorem 1.4. There is no loss of generality in assuming \( y_0 = 0 \). Similar to (20), we can deduce that

\[
\langle y(\cdot, T), \phi_T - k(\alpha \phi_T, x) \rangle_{\mathbb{H}_s^\nu, \mathbb{H}_s^\nu} = -\int_0^T \phi_x(1, t)(u + u_t) dt.
\]

(37)

Suppose that the system (1) is not approximately controllable in \( \mathbb{H}_s^\nu \). It means that \( R(T) \) is not dense in \( \mathbb{H}_s^\nu \). We use Hahn-Banach Theorem to obtain that there exists a function \( h = \sum_{n \in \mathbb{N}^*} b_n w_n \in \mathbb{H}_s^\nu \), \( h \neq 0 \) such that

\[
\langle h, y(\cdot, T) \rangle_{\mathbb{H}_s^\nu, \mathbb{H}_s^\nu} = 0, \forall y(\cdot, T) \in R(T).
\]

(38)

By the definition of \( w_n(x) \), it follows that there exists \( \phi_T = \sum_{n \in \mathbb{N}^*} \hat{\gamma}_n w_n \in \mathbb{H}_s^\nu \), \( \phi_T \neq 0 \) such that \( \langle y(\cdot, T), \phi_T - k(\alpha \phi_T, x) \rangle_{\mathbb{H}_s^\nu, \mathbb{H}_s^\nu} = 0, \forall y(\cdot, T) \in R(T) \). This and (37) give that

\[
\int_0^T \phi_x(1, t)(u + u_t) dt = 0, \forall u \in H^1(0, T).
\]

That is,

\[
\int_0^T \phi_x(1, t)v(t) dt = 0, \forall v \in L^2(0, T).
\]

Therefore, \( \phi_x(1, t) = 0 \) a.e. \( t \in (0, T) \). Furthermore,

\[
\phi_x(1, t) = \sum_{n \in \mathbb{N}^*} \hat{\gamma}_n e^{\lambda_n(t-T)} \left( \frac{2\beta}{J_\alpha^j(j_{\alpha, n})} \right) \beta_{j_{\alpha, n}} J_\alpha^j(j_{\alpha, n}).
\]
Thus, by Lemma 3.3 and [21, Lemma 1], we have $\hat{\gamma}_n = 0$, $\forall n \in \mathbb{N}^*$, which contradicts to the fact that $\phi_T \neq 0$. The proof of Theorem 1.4 is completed.

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