Quantum thermodynamics in adiabatic open systems and its trapped-ion experimental realization

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Quantum thermodynamics aims at investigating both the emergence and the limits of the laws of thermodynamics from a quantum mechanical microscopic approach. In this scenario, thermodynamic processes with no heat exchange, namely, adiabatic transformations, can be implemented through quantum evolutions in closed systems, even though the notion of a closed system is always an idealization and approximation. Here, we first theoretically discuss thermodynamic adiabatic processes in open quantum systems, which evolve non-unitarily under decoherence due to its interaction with its surrounding environment. From a general approach for adiabatic non-unitary evolution, we establish heat and work in terms of the underlying Liouville superoperator governing the quantum dynamics. As a consequence, we derive the conditions that an adiabatic open-system quantum dynamics implies in the absence of heat exchange, providing a connection between quantum and thermal adiabaticity. Moreover, we determine families of decohering systems exhibiting the same maximal heat exchange, which imply in classes of thermodynamic adiabaticity in open systems. We then approach the problem experimentally using a hyperfine energy-level quantum bit of an Ytterbium \(^{171}\)Yb\(^+\) trapped ion, which provides a work substance for thermodynamic processes, allowing for the analysis of heat and internal energy throughout a controllable engineered dynamics.

The notion of adiabaticity is a fundamental concept in a number of different areas in physics, including quantum information processing\(^{1-4}\) and quantum thermodynamics\(^{5-7}\). In the context of closed quantum systems, adiabaticity is understood as the phenomenon in which the Hilbert space of the system can be (quasi-)perfectly decomposed into uncoupled Schrodinger-eigenspaces, composed by the eigenvectors of the Hamiltonian with distinct non-crossing instantaneous energies\(^{8-10}\). Then, by initially preparing a quantum system in an energy eigenstate, the system undergoes a decoupled evolution to the corresponding energy eigenstate at later times. However, the concept of a closed system is always an idealization and approximation. Indeed, real quantum systems are always coupled to a surrounding environment. In open quantum systems described by convolutionless master equations, the definition of adiabaticity can be naturally extended to the decomposition of the Hilbert-Schmidt space into Lindblad-Jordan eigenspaces associated with distinct eigenvalues of the generator of the dynamics\(^{11-14}\).

In thermodynamics, adiabaticity is associated to a process with no heat exchange between the system and its reservoir. In general, it is not possible to associate an observable for the thermodynamic definition of heat and of work\(^{15}\). Then, the starting point widely used to define such physical quantities in quantum systems is from the definition of internal energy given as \(U(t) = \langle H(t) \rangle\)\(^{5,16}\). From this definition, we obtain the work (\(\delta W\)) and exchanged heat (\(\delta Q\)) between the reservoir and system as

\[
\delta W = \text{Tr}[\rho(t)\dot{H}(t)]dt \quad \text{and} \quad \delta Q = \text{Tr}[\dot{\rho}(t)H(t)]dt , \quad \text{(1)}
\]

respectively. These quantities are well-defined when at least one of them vanishes, thus the non-vanishing quantity can be identified with the internal energy variation \(\Delta U(t)\) during the entire process. For example, for a unitary transformation associated with a closed quantum system, we necessarily have \(\delta Q_d = 0\), so that any variation \(\Delta U(t)\) is due some work performed on/by the system\(^{5,17}\). Eq. (1) can be directly employed to analyze quantum thermodynamical cycles, as an efficient way of assuring that no heat is exchanged in intermediate steps\(^{18-20}\) or to minimize quantum friction in a nonequilibrium setup\(^{21-23}\).

Here, we theoretically and experimentally discuss thermodynamical adiabatic processes in real (open) quantum systems evolving under decoherence. To this end, we address the problem from a general approach for adiabatic dynamics in decohering systems. In contrast with closed systems, some amount of heat may be exchanged in the case of non-unitary evolution. In particular, we will establish a sufficient condition to ensure that an adiabatic open-system dynamics (associated with Lindblad-Jordan decoupled eigenspaces) leads to an adiabatic thermodynamical process (associated with no heat exchange). Moreover, for thermodynamically non-adiabatic processes, we discuss how to minimize the entropy production as a function of their total evolution time. Our results are then experimentally implemented by using a hyperfine energy-level quantum bit (qubit) of an Ytterbium \(^{171}\)Yb\(^+\) trapped ion, where reservoir engineering is performed to achieve a controllable adiabatic dynamics. Due to requirements of the usual definitions of heat and work, the investigation of thermodynamic quantities in adiabatic dynamics is achieved with time-dependent decoherence effects. To this end, we introduce an efficient control to a Gaussian noise with time-dependent amplitude, which is then used to simulate a dephasing channel with a time-dependent decoherence rate \(\gamma(t)\).

Results
We start by introducing heat and work in a general formalism for adiabaticity in open quantum systems, namely, the superoperator formalism. In this work, we will consider a discrete quantum system $S$ defined over a $d$-dimensional Hilbert space. The system $S$ interacts with its surrounding environment $A$. The dynamics is assumed to be described by a time-local master equation $\dot{\rho}(t) = L(t)\rho(t)$, where $\rho(t)$ is the density operator associated with $S$ and $L(t)$ is a time-dependent Liouville operator. The Liouville operator takes the form $L(t)\rho(t) = \mathcal{H}[\rho(t)] + \mathcal{R}[\rho(t)]$, where $\mathcal{H}[\bullet] = (1/i\hbar)[H(t), \bullet]$ is the unitary part of the dynamics and $\mathcal{R}[\bullet]$ describes the decohering effects of $A$ over $S$.

In the superoperator formalism, the open-system dynamics can be provided from a Schrödinger-like equation $\dot{\rho}(t) = \mathbb{L}(t)\rho(t))$, where the density operator $\rho(t))$ is represented by a $D^2$-dimensional vector (hence the double ket notation), whose components $\varrho_i(t)$ can be suitably expanded in terms of tensor products of the Pauli basis $\{|\sigma_i\rangle\rangle_i, i = 1, 2, \ldots, D\rangle$. For instance, for the case of a single qubit ($D = 2$), we have $\rho(t) = \sum_{i,j} \varrho_{ij}(t) |\sigma_i\rangle\langle\sigma_j|$, with $\varrho_{ij}$ denoting an element of the Pauli basis. Moreover, $\mathbb{L}(t) = \mathbb{H}(t) + \mathbb{R}(t)$, where $\mathbb{H}(t)$ and $\mathbb{R}(t)$ are $(D^2 \times D^2)$-dimensional supermatrices, whose elements are $\mathbb{H}_{ik}(t) = (1/D)\text{Tr}[\sigma_i^k\mathcal{H}[\sigma_i]]$ and $\mathbb{R}_{ik}(t) = (1/D)\text{Tr}[\sigma_i^k\mathcal{R}[\sigma_i]]$, respectively. The thermodynamic quantities defined in Eq. (1) are then rewritten as (see Supplemental Material)

$$\delta W_{\text{ad}} = \frac{1}{D} \langle \langle \hat{h}(t)| \rho(t) | \rho(t) \rangle \rangle dt,$$
$$\delta Q_{\text{ad}} = \frac{1}{D} \langle \langle \hat{h}(t)| \mathbb{L}(t)| \rho(t) \rangle \rangle dt,$$
$$d(\mathcal{S}) = \langle \langle \hat{h}(t)| \mathbb{L}(t)| \rho(t) \rangle \rangle dt,$$ (2)

with the components of $\rho_{ij}(t)$ defined by $h_{ij}(t) = \text{Tr}[H(t)\sigma_i\sigma_j]$. In this notation, the inner product of vectors $|u\rangle$ and $|v\rangle$ associated with operators $u$ and $v$, respectively, is defined as $\langle u|v\rangle = (1/D)^2\text{Tr}(u^\dagger v)$. Since we are analyzing far-from-equilibrium thermodynamical processes, we also consider the entropy production during the dynamics. The entropy production $\mathcal{S}$ during a non-equilibrium process is defined as $\mathcal{S} = \text{Tr}[\rho(t)\log \rho(t) - \rho_{\text{eq}} \log \rho_{\text{eq}}]$, where $\rho_{\text{eq}}$ describes the equilibrium state at inverse temperature $\beta_{\text{eq}}$. In the superoperator formalism it is possible to show that (see Supplemental Material)

$$d(\mathcal{S}) = (1/D) \left[ \langle \langle \rho_{\text{log}}(t)| \mathbb{L}(t)|\rho(t) \rangle \rangle - \langle \langle \rho_{\text{log}}^\text{th}(t)| \rho(t) \rangle \rangle \right] dt,$$ (3)

with the components of $\langle \langle \rho_{\text{log}}(t)|$ and $\langle \langle \rho_{\text{log}}^\text{th}(t)|$ given by $\rho_{\text{log}}(t) = \text{Tr}[\sigma_i \log \rho(t)]$ and $\rho_{\text{log}}^\text{th} = \text{Tr}[\sigma_i \log \rho_{\text{eq}}]$, respectively. For a general process, Eq. (3) may be hard to be computed. However, as it will be shown, it can be analytically derived for a general adiabatic quantum dynamics.

Because $\mathbb{L}(t)$ is non-Hermitian, it cannot always be diagonalized. Then, the definition of adiabaticity in this scenario is subtler than in the case of closed systems. For open systems, the adiabatic dynamics can be defined in terms of the Jordan decomposition of $\mathbb{L}(t)^{11}$. More specifically, adiabaticity is associated with a completely positive trace-preserving dynamics that can be decomposed into decoupled Lindblad-Jordan eigenspaces associated with distinct non-crossing instantaneous eigenvalues $\lambda_i(t)$ of $\mathbb{L}(t)$. We notice here that some care is required in order to find a basis for describing the density operator. The standard technique is to start from the instantaneous right and left eigenstates of $\mathbb{L}(t)$, completing these eigensets in order to compose right $|\mathcal{D}_i^{(k)}(t)\rangle$ and left $|\mathcal{E}_i^{(k)}(t)\rangle$ vector bases, where $|\mathcal{D}_i^{(k)}(t)\rangle$ and $|\mathcal{E}_i^{(k)}(t)\rangle$ are the $k$-th right and left vector, respectively, associated with the eigenspace with eigenvalue $\lambda_i(t)$ in the Jordan decomposition of $\mathbb{L}(t)$. These Jordan-pressing left and right bases can always be built such that they satisfy a bi-orthonormal relationship $\langle \mathcal{E}_i^{(k)}(t)|\mathcal{D}_j^{(l)}(t)\rangle = \delta_{ij}\delta_{kl}$. Assuming an open-system adiabatic dynamics, we can analytically derive work, heat, and entropy production. Indeed, by taking the initial density operator as $\rho(0) = \sum_i \varrho_i |\mathcal{D}_i^{(k)}(0)\rangle\langle\mathcal{D}_i^{(k)}(0)|$, we obtain that work, heat, and entropy production are provided by

$$\delta W_{\text{ad}} = \frac{1}{D} \sum_i \varrho_i \langle \langle \hat{h}(t)| \mathcal{L}_i^{(k)}(t) | \mathcal{D}_i^{(k)}(t) \rangle \rangle dt,$$
$$\delta Q_{\text{ad}} = \frac{1}{D} \sum_i \varrho_i \langle \langle \hat{h}(t)| \mathbb{L}(t)| \mathcal{D}_i^{(k)}(t) \rangle \rangle dt,$$
$$d(\mathcal{S}) = \frac{1}{D} \sum_i \varrho_i \langle \langle \hat{h}(t)| \mathbb{L}(t)| \mathcal{D}_i^{(k)}(t) \rangle \rangle dt,$$ (4)

where $\mathbb{L}_i^{(k)}(t) = \langle (\langle \rho_{\text{log}}(t)| | \mathcal{D}_i^{(k)}(t) \rangle \rangle - \langle \langle \rho_{\text{log}}^\text{th}(t)| | \mathcal{D}_i^{(k)}(t) \rangle \rangle$, $\mathbb{L}_i^{(k)}(t)$, and $\mathbb{L}_i^{(k)}(t)$ stand for the adiabatic evolved state associated with $\langle \langle \rho_{\text{log}}(t)| ||\mathcal{D}_i^{(k)}(t)\rangle \rangle$ and $\langle \langle \rho_{\text{log}}^\text{th}(t)| ||\mathcal{D}_i^{(k)}(t)\rangle \rangle$, with $\langle \langle \rho_{\text{log}}(t)|$ and $\delta W_{\text{ad}}$ being identified to the amount of work (heat) performed on/by the system when $\delta Q_{\text{ad}} = 0$ ($\delta W_{\text{ad}} = 0$).

The validity of Eqs. (4), (5), and (6) is shown in the Supplemental Material. They provide general expressions for work, heat, and entropy production during an open-system adiabatic dynamics. In particular, as a consequence we can obtain a sufficient condition for avoiding heat exchange in a quantum mechanical adiabatic evolution. More specifically, if the initial state $\rho(0)$ of the system can be written as a superposition of the eigenstate set $|\mathcal{D}_i^{(k)}(0)\rangle\rangle$, with eigenvalue $\lambda_i(t)$, then $t \in [0, \tau]$, the adiabatic dynamics implies in no heat exchange. Therefore, we can establish that an adiabatic dynamics in quantum mechanics is not in general associated with an adiabatic process in quantum thermodynamics, with a sufficient condition for thermal adiabaticity being the evolution within an eigenstate set with vanishing eigenvalue of $\mathbb{L}(t)$.

**Heat exchange for a qubit adiabatic dynamics.** As an illustration, let us begin by considering a two-level system initialized in a thermal equilibrium state $\rho_{\text{eq}}(0)$ for the Hamiltonian $H(0)$ at inverse temperature $\beta = 1/k_B T$, where $k_B$ and $T$ are the Boltzmann’s constant and the absolute temperature, respectively. Let the system be governed by a Lindblad equation, where the environment acts as a dephasing channel in the energy eigenstate basis $|\mathcal{E}_n(t)\rangle$ of $H(t)$. Thus, we describe the coupling between the system and its reservoir through $R_{\mathcal{S}(k)} = (\gamma(t)|\mathcal{E}_n(t)\rangle\langle\mathcal{E}_n(t)| - \epsilon_n)$, where $\gamma(t) = |\mathcal{E}_n(t)|\langle\mathcal{E}_n(t)| = |\mathcal{E}_n(t)(1)/\mathcal{E}_n(t)|\langle\mathcal{E}_n(t)|$. In this case, the set of eigenvectors of $\mathbb{L}(t)$ can be obtained from set of operators $P_{nm} = |\mathcal{E}_n(t)\rangle\langle\mathcal{E}_m(t)|$, where the components $\mathcal{D}_n^{(k)}(t)$, $\mathcal{E}_n^{(k)}(t)$, are given by $\mathcal{D}_n^{(k)}(t) = \text{Tr}[P_{nm}(\sigma_i)]$. Moreover, the eigenvalue equation for $\mathbb{L}(t)$ can be written as $\mathbb{L}(t)|\mathcal{D}_n^{(k)}(t)\rangle = \mathbb{L}(t)|\mathcal{D}_n^{(k)}(t)\rangle = \lambda_i(t)|\mathcal{D}_n^{(k)}(t)\rangle$.
\(\lambda_{nm}(t)\mathcal{D}_{nm}(t))\), where \(\lambda_{nm}(t) = E_n(t) - E_m(t) - 2(1 - \delta_{nm})\gamma(t)\).

In the superoperator formalism, the initial state \(\rho_0(0)\) is written as \(\rho_0(0) = \mathcal{Z}^{-1}(0) \sum_n e^{-\beta E_n(0)\mathcal{D}_{nm}(0)})\), where \(\mathcal{Z}(t) = \text{Tr}[e^{-\beta \mathcal{H}t}]\) is the partition function of the system. Therefore, since \(\rho_0(0))\) is given by a superposition of eigenvectors of \(\mathcal{L}(t)\) with eigenvalue \(\lambda_{nm}(t) = 0\), we obtain from Eq. (5) that \(\delta Q^{\text{ad}} = 0\). Therefore, thermal adiabaticity is achieved for an arbitrary open-system adiabatic dynamics subject to dephasing in the energy eigenbasis. Hence, any internal energy variation for this situation should be identified as work.

In contrast, we can use a similar qubit system to find a process in which heat can be exchanged, i.e., \(\delta Q^{\text{ad}} \neq 0\). To this end, let us consider dephasing in the computational basis, with the coupling between the system and its reservoir through \(\mathcal{R}_{2}^{\text{q}[\bullet]} = \gamma(t) [\sigma_x \cdot \sigma_x - \bullet]\). In order to guarantee that any internal energy variation is associated to heat exchange, we consider a constant Hamiltonian during the entire non-unitary evolution (so that \(\delta W^{\text{ad}} = 0\)). Since \(\mathcal{R}_{2}^{\text{q}[\bullet]}\) must not be written in the eigenbasis of the Hamiltonian, we assume a Hamiltonian \(\hat{H}_S = \hbar \omega \sigma_y\), where the system is initialized in the typical initial state of a thermal machine, namely, the thermal state of the Hamiltonian \(\hat{H}_S\) at inverse temperature \(\beta\). At this stage, the system is unitarily driven by the time-dependent Hamiltonian \(\hat{H}(t)\), which varies from \(\hat{H}_S\) to \(\hat{H}_R = \hbar \omega \sigma_y\), with \(\bar{\omega} \neq \omega\), so that no heat is exchanged. After this stage we have the contact with the reservoir, which is then governed by \(\hat{H}_R\). By letting the system undergo a non-unitary adiabatic dynamics, the evolved state is

\[
\rho^{\text{ad}}(t) = \frac{1}{2} \left[ 1 - e^{-\tilde{\gamma}(\xi) \xi d\xi} \tanh[\beta \hbar \omega] \sigma_y \right],
\]

so that we can compute the exchanged heat during an infinitesimal time interval \(dt\) as

\[
\delta Q^{\text{ad}}(t) = 2h \tanh[\beta \hbar \omega] \tilde{\gamma}(t)e^{-\tilde{\gamma}(t) / \beta \hbar \omega} dt.
\]

The negative argument in the exponential of Eq. (8) shows that the higher the mean-value of \(\gamma(t)\) the faster the heat exchange ends. In fact, we can use the well-known mean-value theorem for real functions to write \(e^{-\tilde{\gamma}(\xi) \xi d\xi} = e^{-\tilde{\gamma} (1/\Delta t) \int_{0}^{\Delta t} \gamma(\xi) \xi d\xi}\) where \(\tilde{\gamma} = (1/\Delta t) \int_{0}^{\Delta t} \gamma(\xi) \xi d\xi\) is the mean-value of \(\gamma(t)\) within the interval \(\Delta t\). Thus, if we define the amount of exchanged heat during the entire evolution as \(\Delta Q(\tau_{\text{dec}}) = \int_{0}^{\Delta t} \delta Q^{\text{ad}}(t)\), where \(\tau_{\text{dec}}\) is the total evolution time of the nonunitary dynamics, we get

\[
\Delta Q(\tau_{\text{dec}}) = \hbar \bar{\omega} \tanh[\beta \hbar \omega] \left( 1 - e^{-2\gamma \tau_{\text{dec}}} \right).
\]

Notice that \(\Delta Q(\tau_{\text{dec}}) > 0\) for any value of the average dephasing rate \(\bar{\gamma}\). Therefore, the dephasing channel considered here works like an artificial thermal reservoir at inverse temperature \(\beta_{\text{deph}} < \beta\). We can further compute the maximum exchanged heat from Eq. (9) as a quantity independent on the environment parameters and given by \(\Delta Q_{\text{max}} = \hbar \bar{\omega} \tanh[\beta \hbar \omega]\). It would be worth to highlight that, for quantum thermal machines weakly coupled to thermal reservoirs at different temperatures\(^{16}\), the maximum heat \(\Delta Q_{\text{max}}\) is obtained with high-temperature hot reservoirs\(^{18,25,28}\).

![FIG. 1: Experimental scheme to investigate the thermodynamics of adiabaticity in open quantum systems. (a) Schematic diagram of the six-needle Paul trap and relevant levels of the \(^{171}\text{Yb}^+\) ion. (b) Experimental microwave instrument for generating the field to drive the two level system. The AWG is programmed to implement the target Hamiltonian and control the amplitude of the Gaussian noise which is used as a dephasing channel.](image)

Despite we have illustrated two specific models of thermal adiabaticity and heat exchange in open-system adiabatic evolutions, we can determine infinite classes of systems exhibiting the same amount of heat exchange \(\Delta Q\). This is provided in Theorem 1 below.

\[\text{Theorem 1:} \ \text{Let} \ S \ \text{be an open quantum system governed by a time-local master equation in the form} \ \dot{\rho}(t) = \mathcal{H}[\rho(t)] + \mathcal{R}_\text{dec}\gamma(t), \ \text{where} \ \mathcal{H}[\bullet] = (1/\hbar)(\hat{H}[\bullet] \bullet) \ \text{and} \ \mathcal{R}_\text{dec}\gamma(t) = \sum_n \gamma_n(t) |n\rangle \langle n| \text{with} \ |n\rangle \langle n| \sim |n\rangle \langle n| \sim \Gamma_n(t) \langle n| \langle n| \sim \Gamma_n(t) \Gamma_n(t), \bullet|]. \ \text{The Hamiltonian} \ \hat{H} \ \text{is taken as a constant operator so that no work is realized by/on the system. Assume that the heat exchange between} \ S \ \text{and its reservoir during the quantum evolution is given by} \ \Delta Q. \ \text{Then, any unitarily related adiabatic dynamics driven by} \ \dot{\rho}(t) = \mathcal{H}[\rho(t)] + \mathcal{R}_\text{dec}\gamma(t), \ \text{where} \ \dot{\rho}(t) = U(t) \rho(t) U^\dagger \Gamma_n(t) \Gamma_n(t), \ \Gamma_n(t) = U(t) \hat{H}(t) U^\dagger \text{for some constant unitary} \ U, \ \text{implies in an equivalent heat exchange} \ \Delta Q' = \Delta Q.\]

A proof of Theorem 1 can be found in the Supplemental Material\(^{24}\). It guarantees that there is an infinite class of models with a maximum heat exchanged given by \(\Delta Q_{\text{max}}\), providing a procedure to inversely engineer environments to extract \(\Delta Q_{\text{max}}\). As an example of application of Theorem 1, let us consider a system-reservoir interaction governed by \(\mathcal{R}_\text{dec}\gamma(t) = \gamma(t) [\sigma_x \cdot \sigma_x - \bullet] \text{ (bit-flip channel)}\). We can then use Theorem 1 to show that the results previously obtained for dephasing can be reproduced if the quantum system is initially prepared in thermal state of \(\hat{H}^\text{th}\) with \(\tilde{\gamma} = \omega/2\). Such result is clear if we choose \(U = R_{\text{th}}(\pi/2)R_{\text{th}}(\sigma_z/2)\). Then, it follows that \(\mathcal{R}_\text{dec}\gamma(t) = \hat{H}_\text{th}\) and \(\hat{H}_\text{th}\) is rotation matrices with angle \(\theta\) around \(z\)-axes for the case of a single qubit. Thus, Theorem 1 assures that the maximum exchanged heat will be \(\Delta Q_{\text{max}} = \hbar \bar{\omega} \tanh[\beta \hbar \omega]\).

Concerning the entropy production rate, it can be obtained from Eq. (6), yielding

\[
\langle \Sigma \rangle = 2g(t)\gamma(t)[\beta \hbar \bar{\omega} - \tanh^{-1} g(t)],
\]
with \( g(t) = e^{-2 \int_0^t \gamma(t) \omega(t) \, dt} \cdot \tanh(\beta \hbar \omega) \). The coefficient \( g(t) \) behaves such that, in limit \( \gamma \Delta t \to \infty \), we get \( \langle \Sigma \rangle \to 0 \). Therefore \( \langle \Sigma \rangle \) achieves its steady value \( \langle \Sigma \rangle_s \). It is important to highlight that the sign of \( \langle \Sigma \rangle \) can change during the adiabatic evolution. In fact, if the initial thermal state is such that \( \omega(t) \) changes during the adiabatic evolution. In particular, it can be optimized to be vanishing by for a suitable time \( \tau_{\text{dec}} \). The solid curves in Fig. 2a are computed from Eq. (9), while the experimental points are directly computed through the variation of internal energy as \( \Delta Q(t_{\text{dec}}) = U_{\text{fin}}(t_{\text{dec}}) - U_{\text{ini}}(t_{\text{dec}}) \), where \( U_{\text{fin}}(t_{\text{dec}}) = \text{Tr}(\rho_{\text{fin}}(t_{\text{dec}}) H(t_{\text{dec}})) \). The computation of \( U_{\text{fin}}(t_{\text{dec}}) \) is directly obtained from quantum state tomography of \( \rho_{\text{fin}}(t_{\text{dec}}) \) for each value of \( \tau_{\text{dec}} \). Although the maximum exchanged heat is independent of \( \gamma_0 \), the initial dephasing rate \( \gamma_0 \) affects the power for which the system exchanges heat with the reservoir for a given evolution time \( \tau_{\text{dec}} \). By defining \( \bar{P}(t_{\text{dec}}) = |\Delta Q(t_{\text{dec}})|/\tau_{\text{dec}} \), we can quantify the average power for extracting/introducing the amount \( |\Delta Q(t_{\text{dec}})| \) in the time interval \( \tau_{\text{dec}} \). We then obtain \( \bar{P}(t_{\text{dec}}) = \Delta Q_{\text{max}} \gamma(t_{\text{dec}}) \), where \( \eta(t_{\text{dec}}) = (1 - e^{-2\gamma(t_{\text{dec}})})/\tau_{\text{dec}} \). This result is illustrated in Fig. 2b, where we have plotted \( \bar{P}(t_{\text{dec}}) \) during the entire heat exchange (within the interval \( \tau_{\text{dec}} \) as a function of \( \tau_{\text{dec}} \)).

To begin with, we need to guarantee that the dynamics of the system is really adiabatic. Then, we compute the fidelity \( F(t) \) of finding the system in a path given by Eq. (7), where \( F(t) = \text{Tr}[\rho_{\text{exp}}(t)(\rho_{\text{ini}}(t))^{1/2}]^{1/2} \), with \( \rho_{\text{ini}}(t) \) the density matrix provided Eq. (7) and \( \rho_{\text{exp}}(t) \) the experimental density matrix obtained from quantum tomography. In Table I we show the minimum experimental fidelity \( F_{\text{min}} = \min_{\tau_{\text{dec}}} F(t_{\text{dec}}) \) for several choices of the parameter \( \gamma_0 \). This result shows that the system indeed evolves as predicted by the adiabatic solution for every \( \gamma_0 \) and \( \tau_{\text{dec}} \), with excellent experimental agreement. As a further development, we analyze in Fig. 2a the experimental results for the heat exchange \( \Delta Q(t_{\text{dec}}) \) as a function of \( \tau_{\text{dec}} \), where we have chosen \( \omega(t) = \omega_0, \omega(t) = 2\omega_0 \) and \( \gamma(t) = \gamma_0 (t + t_{\text{dec}}) \). The solid curves in Fig. 2a are computed from Eq. (9), while the experimental points are directly computed through the variation of internal energy as \( \Delta Q(t_{\text{dec}}) = U_{\text{fin}}(t_{\text{dec}}) - U_{\text{ini}}(t_{\text{dec}}) \), where \( U_{\text{fin}}(t_{\text{dec}}) = \text{Tr}(\rho_{\text{fin}}(t_{\text{dec}}) H(t_{\text{dec}})) \). The computation of \( U_{\text{fin}}(t_{\text{dec}}) \) is directly obtained from quantum state tomography of \( \rho_{\text{fin}}(t_{\text{dec}}) \) for each value of \( \tau_{\text{dec}} \). Although the maximum exchanged heat is independent of \( \gamma_0 \), the initial dephasing rate \( \gamma_0 \) affects the power for which the system exchanges heat with the reservoir for a given evolution time \( \tau_{\text{dec}} \). By defining \( \bar{P}(t_{\text{dec}}) = |\Delta Q(t_{\text{dec}})|/\tau_{\text{dec}} \), we can quantify the average power for extracting/introducing the amount \( |\Delta Q(t_{\text{dec}})| \) in the time interval \( \tau_{\text{dec}} \). We then obtain \( \bar{P}(t_{\text{dec}}) = \Delta Q_{\text{max}} \gamma(t_{\text{dec}}) \), where \( \eta(t_{\text{dec}}) = (1 - e^{-2\gamma(t_{\text{dec}})})/\tau_{\text{dec}} \). This result is illustrated in Fig. 2b, where we have plotted \( \bar{P}(t_{\text{dec}}) \) during the entire heat exchange (within the interval \( \tau_{\text{dec}} \) as a function of \( \tau_{\text{dec}} \)). Notice that, in the case of \( \Delta Q(t_{\text{dec}}) \), the asymptotic behavior of the average power is independent of \( \gamma_0 \). Moreover, as predicted by Eq. (10), we can control the entropy production along an adiabatic quantum evolution. As shown in Fig. 2c, the entropy production varies as a function of \( \tau_{\text{dec}} \). In particular, it can be optimized to be vanishing by for a suitable time \( \tau_{\text{dec}} \) for which the system is kept in touch with its surrounding environment.

**Conclusions**

From a general approach for adiabaticity in open quantum systems, it is possible to design realistic experimental implementations of quantum systems that can be used for various applications in quantum information and quantum computation.
systems, we provided a relationship between adiabaticity in quantum mechanics and in quantum thermodynamics. In particular, we derived a sufficient condition for which the adiabatic dynamics in open quantum systems leads to adiabatic processes in thermodynamics. By using a particular example of a single qubit undergoing an open-system adiabatic evolution path, we have illustrated the existence of both adiabatic and diabatic regimes in quantum thermodynamics, computing the associated heat fluxes in the processes. As a further result, we also proved the existence of an infinite family of decohering systems exhibiting the same maximum heat exchange. From the experimental side, we have realized adiabatic open-system evolutions using an Ytterbium trapped ion, with its hyperfine energy level encoding a qubit (work substance). In particular, heat exchange and entropy production can be optimized along the adiabatic path as a function of the total evolution time. Our implementation exhibits high controllability, opening perspectives for analyzing thermal machines (or refrigerators) in open quantum systems under adiabatic evolutions. The associated effects of the engineered reservoirs on the thermal efficiencies are left for future research.

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Supplemental material for:
Quantum thermodynamics in adiabatic open systems and its trapped-ion experimental realization

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Appendix A: Theoretical results

1. Heat and work in superoperator formalism

Let us consider the heat exchange as
\[ dQ_{\text{op}} = \text{Tr}(\dot{\rho}(t)H(t))dt = \text{Tr}(\mathcal{L}[\rho(t)]H(t))dt . \] (A1)
where we have used the equation \( \dot{\rho}(t) = \mathcal{L}[\rho(t)] \). To derive the corresponding expression in the superoperator formalism we first define the basis of operators given by \( \{\sigma_i\}, i = 0, \cdots, D^2 - 1 \), where \( \text{Tr}[\sigma_i \sigma_j] = d \delta_{ij} \). In this basis, we can write \( \rho(t) \) and \( H(t) \) generically as
\[ H(t) = \frac{1}{D} \sum_{n=0}^{D^2-1} h_n(t)\sigma_n \quad \text{and} \quad \rho(t) = \frac{1}{D} \sum_{n=0}^{D^2-1} \rho_n(t)\sigma_n , \] (A2)
where we have \( h_n(t) = \text{Tr}[H(t)\sigma_n] \) and \( \rho_n(t) = \text{Tr}[\rho(t)\sigma_n^\dagger] \). Then, we get
\[ dQ_{\text{op}} = \frac{1}{D} \left( \sum_{n=0}^{D^2-1} \text{Tr}[\mathcal{L}[\rho_n(t)]h_n(t)\sigma_n^\dagger] \right) dt , \]
\[ = \frac{1}{D} \left( \sum_{n=0}^{D^2-1} \rho_n(t)h_n(t)\text{Tr}[\sigma_n^\dagger] \right) dt . \] (A3)
Now, we use the definition of the matrix elements of the superoperator \( \mathcal{L}(t) \), associated with \( \mathcal{L}[^\bullet] \), which reads \( \mathcal{E}_{mn} = (1/\mathcal{D})\text{Tr}[\sigma_m^\dagger\mathcal{L}[^\bullet] \sigma_n] \), so that we write
\[ dQ_{\text{op}} = \frac{1}{D} \left( \sum_{n=0}^{D^2-1} h_n(t)\mathcal{E}_{nm}\rho_m(t) \right) dt . \] (A4)
In conclusion, by defining the vector elements
\[ \langle \hat{h}(t) \rangle = \left[ h_0(t) \quad h_1(t) \quad \cdots \quad h_{D^2-1}(t) \right]^\dagger , \] (A5)
\[ \rho(t)) = \left[ \rho_0(t) \quad \rho_1(t) \quad \cdots \quad \rho_{D^2-1}(t) \right] , \] (A6)
we can rewrite Eq. (A4), yielding
\[ dQ_{\text{op}} = \frac{1}{D} \langle \langle \hat{h}(t)\mathcal{E}(t)\rho(t) \rangle \rangle dt . \] (A7)

Equivalently,
\[ dW_{\text{op}} = \text{Tr}[\rho(t)H(t)]dt , \] (A8)
where we have used Eq. (A2) to write \( \dot{H}(t) = (1/D)\sum_{n=0}^{D^2-1} \dot{h}_n(t)\sigma_n^\dagger \) and, consequently,
\[ dW_{\text{op}} = \frac{1}{D} \sum_{n=0}^{D^2-1} \dot{h}_n(t)\text{Tr}[\rho(t)\sigma_n^\dagger]dt , \] (A9)
so that we use the definition of the coefficients \( \dot{\rho}_n(t) \) to get
\[ dW_{\text{op}} = \frac{1}{D} \sum_{n=0}^{D^2-1} \dot{\rho}_n(t)\dot{\rho}_n(t)dt . \] (A10)
By using Eqs. (A5) and (A6) into Eq. (A10), we conclude that
\[ \delta W_{\text{op}} = \frac{1}{D} \langle \langle \dot{h}(t)\rho(t) \rangle \rangle dt \] (A11)

2. Entropy production in superoperator formalism

Our starting point is the definition
\[ \langle \Sigma(t) \rangle = \text{Tr}[\rho(t)\log[\rho(t)]] - \text{Tr}[\rho(t)\log[\rho_{\text{th}}]] \] , \] (A12)
where \( \rho_{\text{th}} \) is a thermal reference state. Now, let us study the dynamics of \( \langle \Sigma(t) \rangle \) by taking its time derivative
\[ \frac{d}{dt} \langle \Sigma(t) \rangle = \frac{d}{dt} \left[ \text{Tr}[\rho(t)\log[\rho(t)]] \right] - \frac{d}{dt} \left[ \text{Tr}[\rho(t)\log[\rho_{\text{th}}]] \right] \] (A13)
Then, we find
\[ \langle \Sigma(t) \rangle = \text{Tr}[\dot{\rho}(t)\log[\rho(t)]] + \text{Tr}[\dot{\rho}_{\text{th}}] - \text{Tr}[\dot{\rho}(t)\log[\rho_{\text{th}}]] \] (A14)
By using that \( \text{Tr}[\dot{\rho}(t)] = 1 \), we get \( \text{Tr}[\dot{\rho}_{\text{th}}] = 0 \). Therefore
\[ \langle \Sigma(t) \rangle = \text{Tr}[\dot{\rho}(t)\log[\rho(t)]] - \text{Tr}[\dot{\rho}(t)\log[\rho_{\text{th}}]] , \] (A15)
or equivalently (by using \( \dot{\rho}(t) = \mathcal{L}[\rho(t)] \))
\[ \langle \Sigma(t) \rangle = \text{Tr}[\mathcal{L}[\rho(t)]\log[\rho(t)]] - \text{Tr}[\mathcal{L}[\rho(t)]\log[\rho_{\text{th}}]] . \] (A16)
Now, let us to write
\[ \log \rho(t) = \frac{1}{D} \sum_{n=0}^{D^2-1} g_n^{\log}(t) \sigma_n^1, \tag{A17} \]
\[ \log \rho_{th} = \frac{1}{D} \sum_{n=0}^{D^2-1} g_n^{\log,th} \sigma_n^1, \tag{A18} \]
so that we can define the vectors \( \langle \rho_{\log}(t) \rangle \) and \( \langle \rho_{\log}^{th}(t) \rangle \) associated to \( \log \rho(t) \) and \( \rho_{th} \), where their components \( g_n^{\log}(t) \) and \( g_n^{\log,th} \) are obtained as \( g_n^{\log}(t) = \text{Tr} [\sigma_n \log \rho(t)] \) and \( g_n^{\log,th} = \text{Tr} [\sigma_n \log \rho_{th}] \). Thus, we get
\[ \langle \dot{\Sigma}(t) \rangle = \frac{1}{D^2} \sum_{m=0}^{D^2-1} \sum_{n=0}^{D^2-1} g_m(t) g_n^{\log}(t) \text{Tr} [\mathcal{L}_m \sigma_n^1] - \frac{1}{D^2} \sum_{m=0}^{D^2-1} \sum_{n=0}^{D^2-1} g_m(t) g_n^{\log,th} \text{Tr} [\mathcal{L}_m \sigma_n^1], \tag{A19} \]
so that we get the equation for \( \langle \dot{\Sigma}(t) \rangle \) in superoperator formalism as
\[ \langle \dot{\Sigma}(t) \rangle = \frac{1}{D} \langle \langle \rho_{\log}(t) \rangle \partial_t \rho(t) \rangle - \frac{1}{D} \langle \langle \rho_{\log}^{th}(t) \rangle \partial_t \rho(t) \rangle. \tag{A20} \]

3. Validity of Eqs. (4), (5), and (6)

Let \( \rho(0) = \sum_{i,k} c_{i,k}^{(0)} |D_i^{(k)}(0)\rangle \rangle \) be the initial state of the system associated with the initial matrix density \( \rho(0) \). Under adiabatic evolution, the state at a later time \( t \) will be given by
\[ |\rho^{ad}(t)\rangle = \sum_{i,k} c_{i,k}^{(0)} e^{\int t \lambda_{i,k}(t)dt} |D_i^{(k)}(t)\rangle \tag{A21} \]
with \( \lambda_{i,k}(t) = \lambda_i(t) - \langle \langle E^{(k)}_i(t) \rangle \rangle \), where \( \langle \langle E^{(k)}_i(t) \rangle \rangle \) and \( \langle \langle D^{(k)}_i(t) \rangle \rangle \) denote the instantaneous Jordan-preserving left and right bases of \( \mathbb{L}(t) \), respectively. Therefore, from Eq. (A11), we can write the work \( \delta W_{op} \) for an adiabatic dynamics as
\[ \delta W_{op} = \frac{1}{D} \sum_{i,k} c_{i,k}^{(0)} \sum_{r} e^{\int t \lambda_{i,k}(t)dt} \langle \langle \rho(t) \rangle \rangle \delta W_{i,k}(r) \tag{A22} \]
On the other hand, when no work is realized, we can obtain the heat \( dQ_{op} \) for an adiabatic dynamics as
\[ dQ^{ad} = \frac{1}{D} \sum_{i,k} c_{i,k}^{(0)} \sum_{r} e^{\int t \lambda_{i,k}(t)dt} \langle \langle \rho(t) \rangle \rangle \delta W_{i,k}(r) \tag{A23} \]
so that \( dQ^{ad} \) represents the exchanged heat if no work is performed during such dynamics. Moreover, from Eq. (A20), we can write the entropy production variation as
\[ \langle \dot{\Sigma}(t) \rangle = \frac{1}{D} \langle \langle \rho_{\log}^{ad}(t) \rangle \partial_t \rho(t) \rangle - \frac{1}{D} \langle \langle \rho_{\log}^{th}(t) \rangle \partial_t \rho(t) \rangle = \frac{1}{D} \sum_{i,k} c_{i,k}^{(0)} \sum_{r} e^{\int t \lambda_{i,k}(t)dt} \langle \langle \rho_{\log}^{ad}(t) \rangle \rangle \delta W_{i,k}(r) \tag{A24} \]
so that we conclude that
\[ d\langle \Sigma \rangle = \frac{1}{D} \sum_{i,k} c_{i,k}^{(0)} e^{\int t \lambda_{i,k}(t)dt} \Gamma_{i,k}(t) \tag{A25} \]
where \( \Gamma_{i,k}(t) = \langle \langle \rho_{\log}^{ad}(t) \rangle \rangle - \langle \langle \rho_{\log}^{th}(t) \rangle \rangle \), with \( \langle \langle \rho_{\log}^{ad}(t) \rangle \rangle \) standing for the adiabatic evolved state associated with \( \langle \rho_{\log}(t) \rangle \).

4. Proof of Theorem 1

**Theorem 1**: Let \( \mathcal{S} \) be an open quantum system governed by a time-local master equation in the form \( \dot{\rho}(t) = \mathcal{H} \rho(t) + \mathcal{R} \rho(t) \), where \( \mathcal{H} = \frac{1}{\hbar} \{ H, \rho \} \) and \( \mathcal{R} = \sum_n \gamma_n(t) \Gamma_n(t) \), \( \Gamma_n(t) = \frac{1}{2} \{ \Gamma_n^\dagger, \Gamma_n \} \). The Hamiltonian \( H \) is taken as a constant operator so that no work is realized by/on the system. Assume that the heat exchange between \( \mathcal{S} \) and its reservoir during the quantum evolution is given by \( \Delta Q \). Then, any unitarily related adiabatic dynamics driven by \( \rho'(t) = U \rho(t) U^\dagger \), \( \mathcal{H}' = U \mathcal{H} U^\dagger \) and \( \mathcal{R}' = U \mathcal{R} U^\dagger \), for some constant unitary \( U \), implies an equivalent heat exchange \( \Delta Q' = \Delta Q \).

**Proof.** Let us consider that \( \rho(t) \) is solution of
\[ \dot{\rho}(t) = \mathcal{H} \rho(t) + \mathcal{R} \rho(t), \tag{A26} \]
so, by multiplying both sides of the above equation by \( U \) (on the left-hand-side) and \( U^\dagger \) (on the right-hand-side), we get
\[ \dot{\rho}'(t) = \frac{1}{i\hbar} [H, \rho'] + \mathcal{R} \rho', \tag{A27} \]
with \( \rho'(t) = U \rho(t) U^\dagger \), \( \mathcal{H}' = U \mathcal{H} U^\dagger \) and \( \mathcal{R}' = U \mathcal{R} U^\dagger \). We find
\[ \dot{\rho}'(t) = \frac{1}{i\hbar} [H, \rho'] + \sum_n \gamma_n(t) \Gamma_n^\dagger(t) \rho'(t) \Gamma_n(t), \tag{A28} \]
where \( \Gamma(t) = U \Gamma_n(t) U^\dagger \). In conclusion, we get that \( \rho'(t) = U \rho(t) U^\dagger \) is a solution of
\[ \dot{\rho}(t) = \mathcal{H}' \rho(t) + \mathcal{R}' \rho(t), \tag{A29} \]
where
\[ \mathcal{H}' = \frac{1}{i\hbar} [H, \rho'] = \mathcal{H} \rho(t) + \mathcal{R} \rho(t), \tag{A30} \]
\[ \mathcal{R}' = \sum_n \gamma_n(t) \Gamma_n^\dagger(t) \rho'(t) \Gamma_n(t) - \frac{1}{2} [\Gamma_n^\dagger(t) \Gamma_n(t), \rho'(t)] = U \mathcal{R} U^\dagger. \tag{A31} \]
Now, by taking into account that the Hamiltonian \( H \) is a constant operator, we have that no work is realized by/on the system. Then, by computing the amount of heat extracted from...
the system in the prime dynamics during an interval $t \in [0, \tau]$, we obtain
\[ \Delta Q' = \text{Tr}[H' \rho'(\tau)] - \text{Tr}[H' \rho'(0)], \] (A32)
where, by definition, we can use $\rho'(t) = U \rho(t) U^\dagger$, $\forall t \in [0, \tau]$. Hence
\[
\Delta Q' = \text{Tr}[H' U \rho(\tau)] - \text{Tr}[H' U \rho(0)] = \text{Tr}[H \rho(\tau)] - \text{Tr}[H \rho(0)] = \Delta Q
\] (A33)
where we have used the cyclical property of the trace and that $\Delta Q = \text{Tr}[H \rho(\tau)] - \text{Tr}[H \rho(0)]$.

5. Proof of Eqs. (7)-(10)

Consider the Hamiltonian $H_s = \hbar \omega \sigma_z$, where the system is initialized in the thermal of $H_s$ at inverse temperature $\beta$. In this case, the initial state can be written as
\[ \rho(0) = \frac{1}{2} \left( \mathbb{1} + \tanh[\beta \hbar \omega] \sigma_s \right). \] (A34)
If we rewrite the above state in superoperator formalism as the state $|\varphi(0)\rangle$, we can compute the components $\rho^i_\alpha(0)$ of $|\varphi(0)\rangle$ from $\rho^i_\alpha(0) = \text{Tr}[\rho(0) \sigma_n]$, where $\sigma_n = [\mathbb{1}, \sigma_x, \sigma_y, \sigma_z]$. Thus we get
\[ |\varphi(0)\rangle = |1\rangle - \tanh[\beta \hbar \omega]|x\rangle, \] (A35)
where we define the basis $|k\rangle = [\delta_{k1} \delta_{kx} \delta_{ky} \delta_{kz}]^T$. Moreover, it is possible to show that the set $\{|1\rangle, |x\rangle\}$ satisfies the eigenvalue equation for $\mathbb{L}(t)$ as
\[ \mathbb{L}(t)|1\rangle = 0, \quad \mathbb{L}(t)|x\rangle = -2\gamma(t)|x\rangle. \] (A36)
It can be shown that these eigenstates are nondegenerate. Therefore, if the dynamics is adiabatic, we can write the evolved state as $|\varphi(\tau)\rangle = c_1(t)|1\rangle + c_x(t)|x\rangle$, where $c_1(t) = c_1(0) = 0$ and $c_x(t) = c_x(0) = 0$ because the coefficients evolve independently form each other. Thus, from the adiabatic solution in open quantum system given in Eq. (A21), we obtain $c_1(t) = 1$ and $c_x(t) = -\tanh[\beta \hbar \omega]$, so that we can use $\lambda_1 = 0$ and $\lambda_x = -2\gamma(t)$ to obtain
\[ |\varphi(\tau)\rangle = |\mathbb{D}(t)| - \tanh[\beta \hbar \omega]|\mathbb{D}(t)| |x\rangle. \] (A37)
Notice that Eq. (7) in the main text directly follows by rewriting Eq. (A37) in the standard operator formalism. Moreover, heat can be computed from Eq. (A23) as
\[ dQ^{ad} = \frac{1}{D} \sum_{k \in \mathcal{L}} e^{\lambda_k t} e^{\lambda_k t} |\langle \mathbb{h}(t)|\mathbb{L}(t)|1\rangle\rangle | \mathbb{D}(k)(t) | d\tau = \frac{1}{2} \left[ c_1(\langle \mathbb{h}(t)|\mathbb{L}(t)|1\rangle) + c_x e^{-\frac{\tau}{\hbar} \gamma(t)} (\langle \mathbb{h}(t)|\mathbb{L}(t)|x\rangle) \right] dt, \] (A38)
where we already used $c_1 = 0$, for $i = y, z$. Now, we can use that the vector $\langle \mathbb{h}(t)\rangle$ has components $\mathbb{h}_n(t)$ given by $\mathbb{h}_n(t) = \text{Tr}[\rho(0)H(t)]$, in which $H(t)$ is the Hamiltonian that acts on the system during the non-unitary dynamics. Therefore, $H(t) = \tilde{H} = \hbar \omega \sigma_x$, so that we write $\langle \mathbb{h}(t)\rangle = \hbar \omega \langle x \rangle$. In conclusion, by using this result and Eq. (A36), we get
\[ \delta Q^{ad}(t) = 2\hbar \tanh[\beta \hbar \omega] \gamma(t) e^{-\frac{\tau}{\hbar} \gamma(t)} dt. \] (A39)
Now, by integrating the above result
\[ \Delta Q(\tau_{dec}) = \int_0^{\tau_{dec}} \delta Q^{ad}(t) \] (A40)
To solve the above equation, we need to solve
\[ F(t) = \int_0^\tau \gamma(t) e^{-\frac{\tau}{\hbar} \gamma(t)} dt, \] (A41)
where we can note that
\[ \frac{d}{dt} \left( e^{-\frac{\tau}{\hbar} \gamma(t)} \right) = -2\int_0^\tau \gamma(t) e^{-\frac{\tau}{\hbar} \gamma(t)} dt \] (A42)
Therefore, we can write the Eq. (A41) as
\[ F(t) = -\frac{1}{2} \int_0^\tau \frac{d}{dt} \left[ e^{-\frac{\tau}{\hbar} \gamma(t)} \right] dt, \] (A43)
so that
\[ F(t) = -\frac{1}{2} \left[ e^{-\frac{\tau}{\hbar} \gamma(t)} - 1 \right]. \] (A44)
As a last step, we can use the mean-value theorem to write
\[ F(t) = -\frac{1}{2} \left[ e^{-2\tau \gamma} - 1 \right], \] (A45)
where $\bar{\gamma} = [1/(\tau - t_0)] \int_0^\tau \gamma(t) dt$. Therefore, by using this result in Eq. (A40), we find
\[ \Delta Q(\tau_{dec}) = \hbar \tanh[\beta \hbar \omega] \beta \gamma \left( 1 - e^{-\gamma(t_{dec})} \right). \] (A46)
By following the same procedure, in our particular dynamics the Eq. (A25) becomes
\[ d \langle \mathcal{S} \rangle = \frac{1}{2} \left[ c_1(\Gamma_1(t) + c_x e^{-\frac{\tau}{\hbar} \gamma(t)} \delta(t) \Gamma_x(t)) \right] dt, \] (A47)
where
\[ \Gamma_1(t) = \langle \rho_{log}^{ad}(t)|\mathbb{L}(t)|1\rangle - \langle \rho_{log}^{th} |\mathbb{L}(t)|1\rangle, \] (A48)
\[ \Gamma_x(t) = \langle \rho_{log}^{ad}(t)|\mathbb{L}(t)|x\rangle - \langle \rho_{log}^{th} |\mathbb{L}(t)|x\rangle. \] (A49)
By computing $\langle \rho_{log}^{ad}(t) \rangle$ and $\langle \rho_{log}^{th} \rangle$, we find
\[ \langle \rho_{log}^{ad}(t) \rangle = \left\{ \log \left[ 1 + \frac{\tau}{4} \right] \right\} - 2\tanh^{-1}(g(t) 0 0) \] (A50)
\[ \langle \rho_{log}^{th} \rangle = \left\{ \log \left[ \frac{1}{2} \tanh^2(h \hbar \omega) \right] \right\} - 2h \hbar \omega \beta \beta_0 0 \] (A51)
where $g(t) = e^{-\frac{\tau}{\hbar} \gamma(t)} \tanh[\beta \hbar \omega]$. Therefore, from above equations and using the eigenvalues relations in Eq. (A36), we find
\[ \langle \mathcal{S} \rangle = 2g(t) \beta \beta_0 \hbar \omega - \tanh^{-1}(g(t)) \] (A52)
system Hamiltonian reads
\[ \mu \]
with space \(| \psi \rangle \).

In particular, in our experiment to implement the Hamiltonian \( \mu \) we can efficiently control the parameters \( \omega_\text{rf} \) and \( \Omega_R \).

### Appendix B: Trapped-ion experimental setup

1. **Manipulation and readout of the hyperfine qubit**

   We encode a qubit into hyperfine energy levels of a trapped Ytterbium ion \({}^{171}\text{Yb}^+\), denoting its associated states by \(|0\rangle \equiv \left| \frac{1}{2}S_{1/2}; F = 0, m = 0 \right\rangle \) and \(|1\rangle \equiv \left| \frac{1}{2}S_{1/2}; F = 1, m = 0 \right\rangle \). By using an arbitrary waveform generator (AWG) we can drive the qubit through either a unitary or a non-unitary dynamics (via a frequency mixing scheme). The detection of the ion state is obtained from use of a “readout” laser with wavelength 369.526 nm.

   Applying a static magnetic field with intensity 6.40 G, we get a frequency transition between the qubit states given by \( \omega_\text{rf} = 2\pi \times 12.642825 \text{ GHz} \). Therefore, by denoting the states \(|0\rangle \) and \(|1\rangle \) as ground and excited states, respectively, the inner system Hamiltonian is given by

   \[ H_0 = \frac{\hbar \omega_\text{rf}}{2} \sigma_z \tag{B1} \]

where \( \sigma_z = |1\rangle \langle 1| - |0\rangle \langle 0| \). Therefore, to unitarily drive the system through coherent population inversions within the subspace \(|0\rangle, |1\rangle \), we use a microwave whose magnetic field

   \[ \vec{B}_\text{mw}(t) = \vec{B}_0 \cos \omega t \tag{B2} \]

interacts with the electron magnetic dipole moment \( \vec{\mu} = \mu_M \vec{S} \), with \( \mu_M \) a constant and \( \vec{S} \) is the electronic spin. Then, the system Hamiltonian reads

   \[ H(t) = H_0 - \vec{\mu} \cdot \vec{B}_\text{mw}(t) \tag{B3} \]

Thus, by defining the *Rabi frequency* \( \hbar \Omega_R \equiv -\mu_M| \vec{B}_0 |/4 \), we obtain that the effective Hamiltonian that drives the qubit is (in interaction picture)

   \[ H_I(t) = \frac{\hbar \omega_0}{2} \sigma_z + \frac{\hbar \Omega_R}{2} \sigma_x \tag{B4} \]

where \( \omega_0 = \omega_\text{rf} - \omega \) and \( \sigma_x = |1\rangle \langle 0| + |0\rangle \langle 1| \). By using the AWG we can efficiently control the parameters \( \omega_\text{rf} \) and \( \Omega_R \).

In particular, in our experiment to implement the Hamiltonian \( \hat{H}_s \), we have used a resonant \( \left( \omega_\text{rf} = \omega \right) \) microwave with Rabi frequency \( \Omega_R = \tilde{\omega} \), while the frequency \( \omega_\text{rf} \) has been adjusted around \( 2\pi \times 12.642 \text{ GHz} \), with \( \tilde{\omega} \) modulated by using the channel 1 (CH1) of the AWG.

After the experimental qubit operation, we use the state-dependent florescence detection method to implement the quantum state binary measurement. We can observe on average 13 photons for the bright state \(|1\rangle \) and zero photon for the dark state \(|0\rangle \) in the 500 \( \mu \text{s} \) detection time interval, as shown in Fig. 3. These scattered photons at 396.526 nm are collected by an objective lens with numerical aperture \( NA = 0.4 \). After the capture of these photons, they go through an optical bandpass filter and a pinhole, after which they are finally detected by a photomultiplier tube (PMT) with 20% quantum efficiency. By using this procedure, the measurement fidelity is measured to be 99.4%.

2. **The dephasing channel**

   Due to the long coherence time of the hyperfine qubit, the decoherence effects can be neglected in our experimental timescale. However, since we are interested in a nontrivial non-unitary evolution, we need to perform environment engineering. This task can be achieved by using a Gaussian noise source to mix the carrier microwave \( \vec{B}_\text{mw}(t) \) by a frequency modulation (FM) method. Thus, by considering the noise source encoded in the function \( \eta(t) = A g(t) \), where \( A \) is average amplitude of the noise and \( g(t) \) is a random analog voltage signal, the driving magnetic field will be in form

   \[ \vec{B}_\text{mw}(t) = \vec{B}_0 \cos(\omega t + C \eta(t) t) \tag{B5} \]

where \( \vec{B}_0 \) is field intensity and \( C \) is the modulation depth supported by the commercial microwave generator E8257D. If \( C \) is a fixed parameter (for example, \( C = 96.00 \text{ KHz/V} \)), the dephasing rate \( \gamma(t) \) associated with Lindblad equation

   \[ \dot{\rho}(t) = \frac{1}{i\hbar} [\hat{H}_s, \rho(t)] + \gamma(t) \left[ \sigma_z \rho(t) \sigma_z - \rho(t) \right] \tag{B6} \]

is controlled from the average amplitude of the Gaussian noise function \( \eta(t) \). To see that \( \eta(t) \) is a Gaussian function in the frequency domain, we show its spectrum in Fig. 4.

![FIG. 3: Spectrum of the noise source. The noise source is provided by the commercial microwave generator E8257D. Dots are measured data and the solid curve is a Gaussian fit to the data.](image1)

![FIG. 4: Histograms of detected photons after the ion is prepared in |0⟩ and |1⟩. All data is obtained under 100 000 measurement repetitions.](image2)
In order to certify that the decoherence channel is indeed a $\sigma_z$ channel (dephasing channel) in our experiment, we employed quantum process tomography. A general quantum evolution can be typically described by the operator-sum representation associated to a trace-preserving map $\epsilon$. For an arbitrary input state $\rho$, the output state $\epsilon(\rho)$ can be written as

$$
\epsilon(\rho) = \sum_{m,n} \chi_{mn} A_m \rho A_n^\dagger,
$$

where $A_m$ are basis elements (usually a fixed reference basis) that span the state space associated with $\rho$ and $\chi_{mn}$ is the matrix element of the so-called process matrix $\chi$, which can be measured by quantum state tomography. In a single qubit system, we take $A_0 = I, A_1 = \sigma_x, A_2 = \sigma_y, A_3 = \sigma_z$. The quantum process tomography is carried out for the quantum process described by the Lindblad equation given by Eq. (B6), where $H(t) = \omega \sigma_x$, with $\omega = 5.0 \times 2\pi$ KHz and $\gamma = 2.5$ KHz. We fixed the total evolution time as 0.24 ms (here, the noise amplitude is 1.62 V and the modulation depth is 96.00 KHz). The resulting estimated process matrix is shown in Fig. 5. We can calculate the fidelity between the experimental process matrix $\chi_{\text{exp}}$ and the theoretical process matrix $\chi_{\text{id}}$

$$
\mathcal{F}(\chi_{\text{exp}}, \chi_{\text{id}}) = \left| \text{Tr} \sqrt{\chi_{\text{exp}} \chi_{\text{id}} \chi_{\text{exp}}^\dagger} \right|^2
$$

We measured several process with different evolution times. For example, when the amplitude of the noise is set to 1.54V, the process fidelities are measured as $\mathcal{F}_{t_1} = 99.27\%$, $\mathcal{F}_{t_2} = 99.50\%$, $\mathcal{F}_{t_3} = 99.72\%$, $\mathcal{F}_{t_4} = 99.86\%$ and $\mathcal{F}_{t_5} = 99.87\%$, at times $t_1 = 0.08$ ms, $t_2 = 0.16$ ms, $t_3 = 0.24$ ms, $t_4 = 0.32$ ms and $t_5 = 0.40$ ms, respectively. Thus, the dephasing channel can be precisely controlled as desired and it can support the scheme to implement the time-dependent dephasing in experiment.

The function $\eta(t)$ depends on an amplitude parameter $A$, which is used to control $\gamma(t)$. As shown in Fig. 6, we experimentally measured the relation between $A$ and $\gamma(t)$ for a situation where $\gamma(t)$ is a time-independent value $\gamma_0$. As result, we find a linear relation between $\sqrt{\gamma_0}$ and $A$, which reads

$$
\sqrt{\gamma_0} = 29.81 A + 1.74.
$$

For the case $A = 0$, we get the natural dephasing rate $\gamma_{\text{id}} = 1.74$ Hz of the physical system. Thus, we can see that, if we change the parameter $A$, which we can do with high controllability, the quantity $\sqrt{\gamma_0}$ can be efficiently controlled. On the other hand, if we need a time-dependent rate $\gamma(t)$, we just need to consider a way to vary $A$ as a function $A(t)$. To this end, we use a second channel (CH2) of the AWG to perform amplitude modulation (AM) of the Gaussian noise. The temporal dependence of $A(t)$ is achieved by programming the channel (CH2) to change during the evolution time.