Gravity and the Tenacious Scalar Field
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Abstract
Scalar fields have had a long and controversial life in gravity theories, having progressed through many deaths and resurrections. The first scientific gravity theory, Newton’s, was that of a scalar potential field, so it was natural for Einstein and others to consider the possibility of incorporating gravity into special relativity as a scalar theory. This effort, though fruitless in its original intent, nevertheless was useful in leading the way to Einstein’s general relativity, a purely two-tensor field theory. However, a universally coupled scalar field again appeared, both in the context of Dirac’s large number hypothesis and in five dimensional unified field theories as studied by Fierz, Jordan, and others. While later experimentation seems to indicate that if such a scalar exists its influence on solar system size interactions is negligible, other reincarnations have been proposed under the guise of dilatons in string theory and inflatons in cosmology. This paper presents a brief overview of this history.

1 Scalar Gravity?
After the conceptual foundations of special relativity had been laid by Einstein and the natural four-dimensional formalism for space-time had been clarified by him, Minkowski and others, it was natural to consider how field theories should fit into the new framework. Of course, since it lay at the foundations of special relativity, Maxwell’s electromagnetic theory translated beautifully using the four-vector potential and two-form field formalism. The other classical field theory was gravity, so the question of incorporating gravity into the new relativity arose next. The standard textbook introductions to the subject naturally emphasize the logical path to Einstein’s resolution of this, his general theory of relativity. Such pedagogical treatments can even seduce the reader into believing that Einstein’s general relativity is the logically necessary consequence of special relativity and the gravitational principle of equivalence. The actual history was of course not so linear, and the logical path not so obvious to the participants. The first, apparently most simple, approach is to generalize Newton’s scalar gravitational theory to a special relativistic scalar one. In fact, this was precisely what was tried. In the following we will sketch a brief overview of the physics of this period, 1907 to 1915. This a is fascinating and highly instructive story. Fortunately, John Norton has provided an excellent, very readable, review of the history of this subject, [1].

Let us start with the natural four-force generalization of Newtonian mechanics,

\[
\frac{d}{d\tau} (m \frac{dx^\mu}{d\tau}) = F^\mu,
\]  

1Contribution to Festschrift volume for Englebert Schücking
where the right side is the four-force vector. Implicit in the physics of this equation is that the path parameter, $\tau$, must proper time, so the auxiliary condition

$$\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^{\nu}}{d\tau} = -1,$$

must be satisfied. In modern terms this fixing of the path parameter would be described as a “conformal gauge fixing condition.” At any rate, (1), together with the assumption that $m$ is constant in the (2) gauge, implies that

$$\eta_{\mu\nu} F^\mu \frac{dx^{\nu}}{d\tau} = 0.$$

As mentioned earlier, Maxwell’s electromagnetism fits into special relativity quite naturally, since electromagnetism was the theory whose consistency triggered the re-investigation of space-time that ultimately led to it. For electromagnetism the force in (1) is

$$F^\mu_{\text{em}} = \eta^{\nu\rho} \frac{dx^\rho}{d\tau},$$

and (3) follows neatly from the antisymmetry of $F^{\mu\nu}$.

Now, what of gravity? In Newtonian-Galilean theory, the field can be described by a scalar potential, $\phi$, with

$$\nabla^2 \phi = \frac{\kappa}{2} \rho,$$

where $\rho$ is mass density, $\kappa \equiv 8\pi G$, and $G$ the usual Newton constant. Using Galilean three-vector notation,

$$E_g = -\nabla \phi,$$

the equations of motion become

$$\frac{d}{dt} \left( m \frac{dx}{dt} \right) = m E_g.$$

Note that this potential has units of velocity squared, so that in the standard relativistic choice used in this paper, $c = 1$, $\phi$ is dimensionless.

The natural, perhaps logically minimal, approach to including gravity in special relativity might thus seem to be “relativizing” (1),(2), and (3), simply extending them from three to four dimensions,

$$\square \phi = \frac{\kappa}{2} \rho,$$

$$F^\mu_g = -m \phi^\nu,$$
and (1). However, (3), applied to (9) results in
\[
\frac{dx^\mu}{d\tau} \frac{\partial \phi}{\partial x^\mu} = \frac{d\phi}{d\tau} = 0,
\]
along the particle’s path. That is, the potential is constant along the path of every particle, so the gravitational force must necessarily be zero on every particle! Although the historical details are not entirely clear it seems likely that this is the problem to which Einstein referred in 1907 in discounting the appropriateness of a scalar special relativistic theory of gravitation without allowing “...the inert mass of a body to depend on the gravitational potential.” [2] In the context of the time, this seemed unacceptable, and so the problem of incorporating gravity into special relativity was the root of a great deal of concern on Einstein’s part. It also provided fuel for criticism of the entire structure of special relativity by others, notably Abraham, [1].

For our purposes, the next significant contribution was by Nordström, [3], who attacked the question of the possible dependence of mass on gravitational field directly, using,
\[
m = m_0 e^\phi.
\]
This is a direct precursor to the idea that the time component of the metric, or perhaps the proper time itself (gauge dependence of mass) depends on the gravitational potential.

In current formalism we guarantee consistency by starting from an action approach. For a single point particle of mass \( m \), the field variable can be regarded as the path functions, \( z^\mu(\tau) \), and the single particle action is
\[
A_p = - \int m \sqrt{-\dot{z}^\mu \dot{z}_\mu} d\tau,
\]
where \( \dot{z}^\mu \equiv d\dot{z}^\mu / d\tau \). To fit into the volume integration needed for field theories, this can be written
\[
A_p = - \int \left( \int m \sqrt{-\dot{z}^\mu \dot{z}_\mu} \delta^4(x^\mu - z^\mu(\tau)) d\tau \right) d^4x.
\]
As a model for particle-field interaction consider electromagnetism, with total field plus interaction actions,
\[
A_{em} + A_I = \frac{1}{16\pi} \int (A_{\mu,\nu} - A_{\nu,\mu})(A^{\mu,\nu} - A^{\nu,\mu}) d^4x + q \int \left( \int \dot{z}^\mu(\tau) A_\mu \delta^4(x^\nu - z^\nu(\tau)) d\tau \right) d^4x.
\]
Using \( A_p + A_{em} + A_I \) as the total action results in the Maxwell field equations with current density source \( J^\mu(x^\nu) = q \int \dot{z}^\mu(\tau) \delta^4(x^\nu - z^\nu(\tau)) d\tau \), together with the correct equation of motion for the particles. Note that the inertial mass, \( m \), in \( A_p \) is independent of the coupling constant, \( q \) in \( A_I \).
By analogy, the total action for scalar gravity would be

\[
A_p + A_I + A_\phi = -\int \left( \int m \sqrt{-\dot{z}^\mu \dot{z}_\mu} \delta^4(x^\mu - z^\mu(\tau)) d\tau \right) d^4x - \int \phi \left( \int m \sqrt{-\dot{z}^\mu \dot{z}_\mu} \delta^4(x^\mu - z^\mu(\tau)) d\tau \right) d^4x - \frac{1}{\kappa} \int \phi,_{\mu} \phi^{\mu} d^4x. \tag{15}
\]

Clearly the field variation results in (8) with \( \rho(x^\mu) = m \int \delta^4(x^\mu - z^\mu(\tau)) d\tau \) in the conformal gauge, (2). However, the variation over the particle’s variables, \( z^\mu(\tau), \dot{z}^\mu(\tau) \) results in something quite different from (8) and (9), namely,

\[
\frac{d}{d\tau} (m(1 + \phi) \dot{z}^\mu(\tau)) = -m \phi^{\mu}. \tag{16}
\]

This can of course be identified with Nordström’s (11), with \( \exp(\phi) \approx 1 + \phi \), to first order in \( \phi \).

While this provides a self-consistent and thus potentially viable scalar gravitational theory, it was not an attractive approach to Einstein, who seemed to have some strong intuitive drive to incorporate the principle of equivalence at the basis for the correct relativistic gravitational theory, even though he apparently was not yet aware of the Eötvös experimental work on this matter.

Einstein proceeded to criticize scalar special relativistic gravitational theories as being specifically inconsistent with the equivalence principle. When the four-vector equations of motion, say (8) or (16) are expressed in terms of local coordinate time, it is clear that local coordinate acceleration of a particle will depend on the particle’s full kinetic energy, thus violating the equal acceleration principle. For example, replacing the two \( d\tau \)’s in the denominator of the left side of (16) with their coordinate expressions gives

\[
\frac{d^2 r}{d\tau^2} \sim -(1 - v^2) \nabla \phi. \tag{17}
\]

Thus, for example, spinning bodies would have smaller accelerations in a gravitational field than non-spinning identical ones, hot bodies than cold, etc. Nordström pointed out that this effect would be too small to be measured by contemporary technology, but nevertheless it was regarded by Einstein as a serious obstacle to the consideration of such theories.

By a coincidence, von Laue was investigating the influence of special relativity on the theory of stress in bodies, and in so doing discovered what we now call the four dimensional stress-energy tensor, with \( T^{00} \) identified with energy density, and \( T^{ij} = p^{ij} \) the components of the spatial stresses on the body, \( p^{ij} = p \delta^{ij} \), for an isotropic fluid. Clearly a Lorentz velocity transformation would then mix the purely spatial stress components into the energy density, so that the energy density of a moving body would depend on its stress and quadratically on speed.
At this point it is appropriate to take a close look at the concepts of active and passive gravitational mass, as well as inertial mass. A complete study of these questions is beyond our scope here. In fact, the extent to which Newton’s laws are laws as opposed to simply definitions of force and mass is not an easy one to settle. For a review of the ideas of Mach and others on these questions, see the book by Ray [4]. Here we content ourselves with the following somewhat superficial review. First assume some adequate operational definition has been given for Newtonian force. Then gravitational force turn out to be proportional to a single scalar parameter for a given particle at a particular spacetime point. This parameter is called “passive gravitational mass, \( m_{gp} \), so

\[
F = m_{gp} E_g. \tag{18}
\]

On the other hand, the gravitational field is determined by field equations with source density proportional to “active gravitational mass, \( m_{ga} \).” It is now easy to see however, that these two parameters cannot be independent of other without violation of Newton’s third law and thus conservation of momentum. Thus, for a pair of particles, the force of particle 1 on particle 2 is proportional to \( m_{ga}^1 m_{gp}^2 \) while the reaction force is proportional to \( m_{ga}^2 m_{gp}^1 \), so that Newton’s third law requires

\[
\frac{m_{ga}}{m_{gp}} = \text{universal constant}. \tag{19}
\]

Clearly this constant can be chosen to be one and active and passive masses assumed to be equal, unless Newton’s third law is to be relaxed. The remaining mass is “inertial mass, \( m_i \),” with

\[
F = m_i a. \tag{20}
\]

One form of the equivalence principle, the universality of gravitational acceleration at a fixed spacetime point, is then simply the statement that

\[
\frac{m_{ga}}{m_i} = \text{universal constant}. \tag{21}
\]

Returning to the historical matter of a scalar gravitational mass, the discovery by von Laue and others that special relativity would imply that stresses contribute to mass, the question was how to fit this into a gravitational theory. Nordström had already taken into account the contribution of the gravitational potential to mass, in [14], or in more modern formalism, [16], but still needed to account for the questions raised by von Laue’s study of \( T^{\mu\nu} \). Specifically, what should be the \( \rho \) on the right side of (8)? Nordström first considered \( T^{00} \), evaluated in the rest frame of the particle, or invariantly \( T^{\mu\nu} u_\mu u_\nu \), where \( u_\mu \) are the components of the body’s four-velocity. Einstein suggested that the trace of the tensor, \( T = T^{\mu\nu} \) explicitly including stresses, would be more appropriate. However, what of stress energy tensors defined by null fields, such as electromagnetism, for which \( T = 0 \)? Einstein pointed out that if such fields are to be
treated as localized bodies, they must be contained and their contribution to the gravitational field might be accounted for in terms of the stresses the confined fields exert on the container.

At this point, the prospects for a scalar special relativistic gravitational theory looked good. However, a critical thought experiment remained. Consider a closed cycle in which a stressed body (a rod) is lowered in a gravitational field, then unstressed and raised again. Clearly energy is not conserved in this cycle. The net gain in energy is associated with the lack of energy associated with a pure stress (no strain). Nordström and Einstein were then able to show that energy conservation could be restored if they assumed that movement through the gravitational field was also associated with a change in length of the rod along the direction of the stress. Thus work is done and energy released. In other words,

Gravity and Geometry? A full working through of the implications of a special relativistic scalar gravitational theory leads to the suggestions that the lengths of rods, and rates of clocks, might depend on their location in a gravitational field.

In modern notation Nordström’s theory would be expressed by saying that the metric is conformally flat,

$$ds^2 = e^{2\phi}(dx^2 + dy^2 + dz^2 - dt^2), \quad (22)$$

with the gravitational field, $\phi$ determined by

$$\Box \phi = \frac{\kappa}{2} T, \quad (23)$$

where the exact form of the source field equations was developed through several trial and error stages. The crucial step of associating gravity with a distortion of spacetime measurements, thus with empirical spacetime geometry had been taken. Einstein and Fokker built on this work, exploring the full geometric implications of Nordström’s ideas. Einstein noted that Nordström’s geometric form, (22), conformally flat, was too specialized and eventually generalized to an arbitrary metric form, leading to the full Einstein equations, with metric tensor playing the role of gravitational potential. Apparently this finally put an end to the search for a purely scalar special relativistic gravitational theory. Nevertheless, a universally coupled scalar field pops up in yet another context!

2 Kaluza-Klein theories

Very early in the development of relativity searches began for a unification of gravity with electromagnetism. One of the most enduring of such attempts is that associated with the names of Kaluza and Klein. The collection of papers on this subject edited by Appelquist et al provides a valuable resource for this subject, both classically and in more modern contexts.
Briefly, the Kaluza-Klein idea is to embed the electromagnetic four-potential into the metric by enlarging the dimension of the space to five, and inserting the four-potential as
\[
\gamma_{AB} = \left( \frac{V^2}{V^2 A_\alpha} - \frac{V^2 A_\beta}{g_{\alpha\beta} + V^2 A_\alpha A_\beta} \right).
\]
(24)
The vector tangent to the extra dimension is assumed to be a Killing vector, so all variables depend only on the spacetime coordinates, \(x^\alpha\). The existence of this Killing vector gives rise to a natural foliation of the five-space of codimension 1, with local expressions as in (24). Maintaining this foliation and simply counting variables, we see the usual 10 components of the spacetime metric, \(g_{\alpha\beta}\), the four \(A_\alpha\), a four-vector, and a fifteenth, surprise quantity, a four-scalar, \(V\). In the 1921 paper of Kaluza, [7], this field is described as "noch ungedeutet," and left at that. In 1948 Thiry, [8], wrote out the full field equations five-Ricci, \(R_{AB}\), equals zero, explicitly, in effect taking this new scalar seriously as a possible universally coupled scalar field. The five-dimensional vacuum Einstein tensor reduces to a scalar part,
\[
S_{55} = \frac{3V^2 F_{\alpha\beta} F^{\alpha\beta}}{8} - \frac{R}{2},
\]
(25)
a four-vector set,
\[
S_{5\alpha} = \frac{3}{2} V_\alpha F^\alpha \beta / 2 + \frac{V}{2} F_\beta \alpha :\alpha,
\]
(26)
and the four-tensor part,
\[
S_{\alpha\beta} = S_{\alpha\beta} + \frac{V^2}{2} \left( F_{\alpha\mu} F^\mu_\beta + \frac{\eta_{\alpha\beta}}{4} F_{\mu\nu} F^{\mu\nu} \right) - \left( \frac{V}{V} \right) \eta_{\alpha\beta} \frac{4V}{V}.
\]
(27)
Here
\[
dA = \frac{1}{2} (A_\alpha|\beta - A_\beta|\alpha) \sigma^\beta \wedge \sigma^\alpha = \frac{1}{2} F_\beta \alpha \sigma^\beta \wedge \sigma^\alpha.
\]
(28)
The full five dimensional analog of the Einstein equations are
\[
S_{AB} = 0,
\]
(29)
derived from
\[
\delta \int d^5x R \sqrt{|g(5)|}.
\]
(30)
However, the resulting (25),(26),and (27) contain the new field, \(V\), apparently unrelated to either gravity or electromagnetism. With the \textit{ad hoc} choice,
\[
V^2 = \frac{k}{2\pi} = \text{const},
\]
(31)
the result is standard vacuum Einstein-Maxwell,
\[
F_{[\alpha\beta,\mu]} = 0,
\]
(32)
from (28),
\[ F^\alpha_{\beta \alpha \beta \alpha} = 0, \]  
(33)
from (29), and
\[ S_{\alpha \beta} = \frac{\kappa}{4\pi} (F_{\alpha \mu} F^\mu_{\beta} - \frac{g_{\alpha \beta}}{4} F_{\mu \nu} F^{\mu \nu}), \]  
(34)
from (27) plus one additional condition,
\[ F_{\mu \nu} F^{\mu \nu} = 0, \]  
(35)
which follows from \( S_{55} = 0 \). Of course, this last condition is not a part of Maxwell’s theory. To eliminate it and reproduce the full Maxwell theory, the variational principle can be replaced by
\[ \delta \int d^5 x \sqrt{|g(5)|} [\mathcal{R} + \lambda (g_{55} - \frac{\kappa}{2\pi})] = 0. \]  
(36)
From this the combined vacuum Maxwell-Einstein field equations are recovered. However, there are some residual difficulties in the interpretation of the five-geodesic as the path of a particle subject to both gravity and electromagnetic forces, so this approach has never been widely accepted as fully satisfactory in its original intent. Nevertheless, the basic ideas of the Kaluza-Klein approach continue to be used in the wider sense of modern gauge theories, [6].

From the viewpoint of this paper, however, Kaluza-Klein is important for introducing the new scalar \( V \). In fact, from the identification required in (31), this new scalar can be associated with the Newtonian gravitational constant.

### 3 Dirac’s Numbers

Meanwhile Dirac[8], building on the work of Eddington and Milne, pointed out some remarkable clustering of dimensionless numbers composed from observed values of certain fundamental constants. Dirac’s starting point was the value of present cosmological age of the universe, \( T_u \), as defined by the best available value of the Hubble constant in 1938. Of course, the value of this number depends on arbitrary choice of units and so, of itself, cannot have any particular physical significance. Thus, another natural time unit is needed. A natural choice is provided by any of a number of “atomic” time scales, such as \( e^2/m \), or \( h/m \), where \( e \) is the electronic charge and \( m \) is some natural mass, such as that of the electron, or of a nucleon. Clearly, the range of numbers available for this choice is \( 10^3 \), and this arbitrariness will not affect the agreement to orders of magnitude that follow. Dirac then noticed that this fundamental time ratio results in
\[ t \equiv \frac{T_n}{T_a} \approx 10^{40}, \]  
(37)
where $T_a$ is one of the natural atomic time units. Next Dirac decided to look at the dimensionless ratio of two of the fundamental forces, electrical and gravitational, on some standard atomic particle,

$$\gamma \equiv \frac{e^2}{\kappa m^2} \approx 10^{40}.$$  

(38)

Another number to consider is the ratio of the present observed mass of the universe to the standard atomic mass,

$$\mu \equiv \frac{M_u}{m} \approx 10^{80}.$$  

(39)

These empirical numbers were first discussed by Eddington, and are generally known as Eddington numbers. From Dirac’s perspective, the clustering of these natural, dimensionless constants into groups of widely varying magnitude $10^{40}, 10^{80}$ is remarkable indeed and led Dirac to speculate that this clustering might well have some causal basis in some physical theory. He proceeded to develop a cosmological model in which

$$\mu \approx t^2,$$  

(40)

$$\gamma \approx t,$$  

(41)

so that the quantities $\mu, \gamma$ change with the age of the universe, the “epoch” in Dirac’s terms. Dirac’s cosmology was later reconsidered in more detail by Canuto and others[9]. At this point, Dirac’s clustering leads to a number

$$\frac{\mu}{\gamma} \approx 10^0,$$  

(42)

or,

$$\kappa \frac{M_u}{R} \approx 10^9.$$  

(43)

Later, as a preface to inflationary cosmological models, these “large number coincidences” will play an important role. This will be discussed in the section on “inflatons,” below. However, at this point they raise the question of whether $\kappa$ is a truly universal constant, or would change in circumstances with different values for $M/R$. In other words, (43) raises the possibility that $\kappa$ is determined by the mass distribution in the universe.

### 4 Scalar-Tensor Theories

Pascual Jordan[10] was intrigued by the occurrence of the new scalar field in the Kaluza-Klein type of theories, and especially its possible role as a generalized gravitational constant in the spirit of Dirac’s hypothesis, (43). Building on this idea Jordan and his colleagues, including Englebert Schücking, began an
investigation of Kaluza-Klein theories with special concern for the idea that the new five-dimensional metric component, a spacetime scalar, might play the role of a varying gravitational “constant.” The resulting four-dimensional form of the field equations can be so interpreted. However, Jordan and his colleagues took the next step of separating the scalar field from the original five-dimensional metric context unified gravitational-electromagnetic context. Later Brans and Dicke \textsuperscript{11} independently arrived at a similar proposal. Brans and Dicke were especially motivated by Mach’s ideas on inertial induction. Sciama\textsuperscript{12} had proposed a model theory of inertial induction, that is, a theoretical mechanism for generating the inertial forces felt during acceleration of a reference frame. These forces were hypothesized to be of gravitational origin, occurring only during acceleration relative to the “fixed stars.” In this model the ratio of inertial to gravitational mass will depend on the average distribution of mass in the universe, in effect making $\kappa$ a function of the mass distribution in the universe.

In commonly used notation such theories introduce a scalar field, $\phi$, which will (locally and approximately) play the role of reciprocal Newtonian gravitational constant, $\kappa$. One obvious motivation for this choice of field quantity (rather than $\kappa$ itself) is Dirac’s large number hypothesis in the form

$$\frac{1}{\kappa} \approx \frac{M}{R}. \quad (44)$$

From this, rather than (43), we see the possibility that $\frac{1}{\kappa}$ itself might be a field variable and satisfy a field equation with mass as a source, something like

$$\Box \frac{1}{\kappa} = \rho. \quad (45)$$

Of course, the usual Lagrangian for Einstein theory including matter has $\kappa$ directly multiplying the matter contributions. In this form, changes would be made to the local behavior of matter, the local equations of motion, as a result of variations in $\phi$. Consequently, in order to incorporate a Mach’s principle by way of a variable gravitational “constant,” some modifications must be made to standard general relativity. Start by writing the standard Einstein action as

$$\delta \int d^4x \sqrt{-g} (R + \kappa L_m) = 0, \quad (46)$$

where $L_m$ is the “usual” matter Lagrangian, presumably derived from some classical or quantum model.

At this point it is useful to consider several aspects of the famous “principle of equivalence.” First is the statement that all bodies at the same spacetime point in a given gravitational field will undergo the same acceleration. We will refer to this as the “weak” equivalence principle, WEP. A stronger statement, on which standard Einstein’s general relativity is built, is that the only influence of gravity is through the metric, and can thus (apart from tidal effects)
be locally, approximately transformed away, by going to an appropriately accelerated reference frame. This is the “strong” principle, SEP. If we start from an action of the form in (46) with variable $\kappa$, we will be risking the geodesic equation for test particles, thus, possibly the WEP, and even mass conservation. However, we are allowing for a possible violation of the SEP, since gravity, the universal interaction of mass, will influence local physics by changing the local $\kappa$. As Dicke noted, the Eötvös experiment verifies only the WEP and not the SEP, so, in the 1960’s, it was reasonable to consider such alternatives.

Returning to the form of the action, let us then isolate $\kappa$ from matter in the original (46) by dividing by it,

$$\delta \int d^4x \sqrt{-g} (\phi R + L_m) = 0,$$

(47)

where $\phi \equiv 1/\kappa$. While we seem to have thus saved the geodesic equations for test particles, it is now known, of course, that the motion of composite bodies is more complex. It turns out that with refined observation techniques, even the coupling of $\phi$ directly to the gravitational field gives rise to observable effects for matter configurations to which gravitational energy contributes significantly. This is now known as the “Dicke-Nordtvedt” effect and has been investigated in the earth-moon system with the lunar laser reflector.

Nevertheless, let us proceed to see what follows from (46). We are anticipating field equations for $\phi$ so some action for this new field must be supplied,

$$\delta \int d^4x \sqrt{-g} (\phi R + L_m + L_\phi) = 0.$$

(48)

The usual requirement that the field equations be second order leads to

$$L_\phi = L(\phi, g_{\mu\nu}).$$

(49)

Apart from this, there seem to be few $a$ priori restrictions on $L_\phi$. At first glance, the standard choice for a scalar field,

$$L_\phi = -\omega \phi,_{\mu} \phi,^{\mu} g^{\mu\nu},$$

(50)

leading to a wave equation for $\phi$ with $R$ as source would seem to be natural. However, the coupling constant $\omega$ would itself then need to have the same dimensions as the gravitational $\kappa$ that the new field is to replace! It would at least seem reasonable to require that any new coupling constant be dimensionless for various reasons, so a natural minimal choice is

$$L_\phi = -\omega \phi,_{\mu} \phi,^{\mu} g^{\mu\nu} / \phi,$$

(51)

in which $\phi$ has dimensions of inverse gravitational constant,

$$[\phi] = [\kappa^{-1}].$$

(52)
The form (51) leads to an action which is often referred to as the “Jordan-Brans-Dicke,” JBD, action,

$$\delta \int d^4x \sqrt{-g}(\phi R + L_m - \frac{\omega}{\phi} \phi_{,\mu} \phi_{,\nu} g^{\mu\nu}) = 0. \quad (53)$$

The variational principle, with standard topological and surface term assumptions, results in

$$\delta_m \int dx^4 \sqrt{-gL_m} = 0, \quad (54)$$

$$\phi S_{\alpha\beta} = T_{(m)\alpha\beta} + \phi_{,\alpha;\beta} - g_{\alpha\beta} \phi + \omega (\phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} g_{\alpha\beta} \phi_{,\lambda} \phi_{,\lambda}), \quad (55)$$

$$\omega (\phi_{,\phi} - \phi_{,\lambda} \phi_{,\lambda}) = -R. \quad (56)$$

The first of these, (54), is the standard variational principle for matter, which follows the same equations as in Einstein theory, thus (apparently) satisfying the weak equivalence principle. For test particles, (54), results in the geodesic equations. However, for extended, or composite, particles, this is may no longer be true, even in standard general relativity. The second order interaction of matter by way of the scalar-metric coupling gives rise to violations of the weak equivalence principle, so that bodies of different mass may have different gravitational accelerations in identical gravitational fields. Of course, because of the free standing $L_m$ in (53), the energy tensor for matter is still conserved,

$$T_{(m)\alpha\beta} = 0. \quad (57)$$

Taking the trace of (55), solving for $R$, leads to another form for (56),

$$\Box \phi = \frac{1}{(2\omega + 3)} T_{(m)}, \quad (58)$$

in which $T_{(m)}$ is the trace of the ordinary matter tensor. In a weak field model situation, within a static spherical shell of mass $M$, radius $R$ and otherwise empty universe this equation produces

$$\phi \approx \phi_{\infty} + \frac{1}{4\pi(2\omega + 3)} \frac{M}{R}. \quad (59)$$

If $\phi$ can be identified with the local reciprocal gravitational constant, and $\phi_{\infty}$ is set zero as a default asymptotic condition, then this equation is seen to be consistent with the Dirac coincidence, (44). Another natural approximation to (58) is to consider the effect of local matter over some background $\phi_0$ equal to the present observed value,

$$\phi \approx \phi_0 + \frac{1}{4\pi} \sum_{\text{local } m} \frac{m}{r}. \quad (60)$$
In equation (55) \( T_{(m)\alpha\beta} \) are the components of the stress-energy tensor for matter derived from the matter Lagrangian \( L_m \) in the standard fashion. This equation, which describes the sources of the gravitational field, can be re-written

\[
S_{\alpha\beta} = \frac{1}{\phi} (T_{(m)\alpha\beta} + T_{(\phi)\alpha\beta}). \tag{61}
\]

This form clearly suggests that \( (1/\phi) \) does indeed act as a generalized gravitational “constant”, with both ordinary matter and the field \( \phi \) itself serving as sources for the metric. However, it turns out that the presence of the \( \phi \) term on the right hand side of (55), together with (58) results in two occurrences of the matter tensor as a source, effectively producing a constant renormalization of \( \phi \) as “gravitational constant.”

The earliest serious investigations of this theory were by Jordan and his group, prominent among whom was Englebert Schücking. Heckman gave the first non-trivial exact vacuum solution, the generalization of the Schwarzschild solution of standard Einstein theory. Later, Schücking[13] investigated the natural question of a Birkhoff type theorem for scalar-tensor equations. He was able to show that again the most general spherically symmetric solution must be static, if the scalar field is assumed to be static, or to have a light-like gradient. However, if \( \phi \) is allowed to be a function of time, more general solutions can exist, of course. A class of such solutions was also presented in Schücking’s paper, and opened the way for studies of spherically symmetric, non-static, phenomena occurring in scalar-tensor but not standard Einstein theory.

Early on questions of the choice of “conformal gauge” for the metric were considered. In other words, replacing the metric, \( g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} = \psi g_{\mu\nu} \) leads to a replacement of the action (53) discarding the surface (topological) part, by

\[
\delta \int d^4x \sqrt{-\bar{g}} \left( \frac{\phi}{\psi} \bar{R} + \frac{3\phi}{2} \frac{|\nabla\psi|^2}{\psi^3} - 3\nabla\phi \cdot \nabla\phi/\psi^2 + L_m/\psi^2 - \frac{\omega}{\phi\psi} |\nabla\phi|^2 \right) = 0. \tag{62}
\]

In particular, if \( \psi \) is chosen to be \( \phi \), (62) becomes

\[
\delta \int d^4x \sqrt{-\bar{g}} (\bar{R} - (\omega + \frac{3}{2})|\nabla\alpha|^2 + e^{-2\alpha} L_m(\bar{g})) = 0, \tag{63}
\]

where \( \phi = e^{\alpha} \). This variational principle is of course just the Einstein one for a massless scalar field (dimensionless), \( \alpha \), but universally coupled to all other matter through the \( e^{-2\alpha} \) factor. Regarding conformal rescalings of the metric as a “gauge,” (63) is an expression of the theory in the “Einstein gauge,” as opposed to the original (53), the “Jordan” gauge. However, it should be clear that there is more to the conformal scaling than merely the formal expression of the equations. In fact, the universal coupling of \( \alpha \) to all matter in (63) means that in this metric test particles will not follow geodesics, nor have conserved inertial mass, etc., in the Einstein gauge. In effect, the identification of the Einstein metric used in the formulation (63) as the “physical” metric leads
to significant and observable violations of mass conservation and the WEP. Nevertheless, the choice of various conformal gauges continues to be studied.

The investigation of such scalar-tensor generalizations of Einstein theory was strongly influenced by the work of Dicke. In fact, the 1960’s and 1970’s saw an explosion of interest in relativity and gravitational theories prompted at least in part by the presence of theoretically viable alternatives to standard Einstein theory, and Dicke’s energetic promotion of them. By fortuitous coincidence this was also the time when NASA was coming of age and searching for space related experiments of fundamental importance. Simultaneously, Nordtvedt, Will and others [14] were led to provide rigorous underpinnings to the operational significance of various theories, especially in solar system context, developing the parameterized post Newtonian (PPN) formalism as a theoretical standard for expressing the predictions of relativistic gravitational theories in terms which could be directly related to experimental observations. The equations of scalar-tensor theory approach those of standard Einstein theory as $\phi$ approaches a constant. From (59) this would seem to occur in the limit of large $\omega$. In fact, it is generally true that the predictions of scalar-tensor approach those of Einstein for large $\omega$, although there are interesting questions to be considered in general, [15].

The ultimate outcome of these efforts was to set limits on the value of the parameter $\omega$ so large as to make the predictions of this theory essentially equivalent to those of standard Einstein theory. In other words, solar system experimentation led to the conclusion that scalar-tensor modifications of standard Einstein theory would necessarily differ insignificantly from the standard, leading many workers to regard such theories as irrelevant. Nevertheless, as the next two sections show, universally coupled, thus gravitational, scalar fields continue to play important roles in contemporary physics.

5 Dilatons

In the preceding discussion the scalar field was universally coupled to all matter and played a role determining the locally measured Newtonian gravitational “constant.” Of course, scalar fields occur throughout physics, especially as quantum fields. Investigations of internal spaces for particle symmetries directly involve gauge theories with internal symmetry spaces occupied by families of fields which while having interesting transformation properties from the internal gauge group viewpoint are nonetheless spacetime scalars. Some of the earliest are the $SO(N)$ bosons of the dual model, the Nambu-Goldstone bosons and the famous Higgs fields. Certainly, these scalars, as quantum fields, are based on different motivations than those leading to the scalar field in scalar-tensor theories. Nevertheless, the formalism, and perhaps macroscopic manifestations may turn out to be not too different.

Historically, quantum dual models led to string theory and later superstring
theory. In this process, a scalar field referred to as a “dilaton” appears quite naturally. This field couples to the trace of the two-dimensional string stress tensor. It thus manifestly breaks the Weyl conformal (dilation) symmetry of the string. Nevertheless it is precisely what is needed to balance the quantum anomalies of this tensor by way of beta functionals of this tensor. Along the way, the Einstein equations can be derived as the beta functions related to some external spacetime metric. The two volumes of Green, Schwartz and Witten provide useful description of the origin and role of dilatons.

This is clearly a long and complicated subject, which we only summarize here. Consider a string action as a natural generalization of a point particle action. For a background metric, \( g_{\alpha\beta} \), an obvious action choice is

\[
S_1 = - \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{|h|} h^{ab} \partial_a X^\alpha \partial_b X^\beta g_{\alpha\beta}(X^c),
\]

(64)

with internal coordinate area \( d^2\sigma \), internal string metric, \( h_{ab} \), \( a, b... = 1, 2 \), and \( \alpha' \) a tension related coupling parameter. If \( S_1 \) is compared to the relativistic point particle action, the role of point particle parameter has been replaced by the intrinsic surface metric, \( h_{ab} \). As in the point particle case, the derived physics should be independent of the internal parameterization, and in particular, the choice of string metric. Thus, \( \delta S_1 / \delta h_{ab} = 0 \), or

\[
T_{ab} = \partial_a X \cdot \partial_b X = 0.
\]

(65)

Trivially, this implies that the trace of \( T \) vanishes, but this would also be predicted by the conformal invariance of the string metric in (64). Actually, an even stronger result obtains: any two-dimensional metric is conformally flat (but only locally, in general!),

\[
h_{ab} = \phi \eta_{ab},
\]

(66)

with constant \( \eta_{ab} \). Thus, the surface element appearing in (64) reduces to the flat one,

\[
d^2\sigma \sqrt{|h|} h^{ab} = d^2\sigma \eta^{ab}.
\]

(67)

The independence of the classical action from the choice of string metric is thus equivalent to invariance under Weyl (conformal) transformation internal to the string surface. In addition to \( S_1 \), it is natural to consider two additional terms, with associated fields derived from the string quantities. The first contains a spacetime two-form field, \( B_{\alpha\beta} \), derived from the intrinsic volume two-form in the string,

\[
S_2 = - \frac{1}{4\pi\alpha'} \int d^2\sigma \varepsilon_{ab} \partial_a X^\alpha \partial_b X^\beta B_{\alpha\beta}(X^c).
\]

(68)

The second introduces the geometry of the string,

\[
\chi = \frac{1}{4\pi} \int d^2\sigma \sqrt{|h|} R^{(2)}.
\]

(69)
However, one of the first discoveries relating geometry and topology was that this integral depends only on the topology of the string surface, and not the particular geometry. This is in fact the first Chern class for two dimensions. The value for $\chi$ is the Euler number of the surface, and cannot be a dynamical variable. The two-geometry of the string can be introduced non-trivially by adding to (67) a scalar field, the “dilaton,” $\Phi$, giving

$$S_3 = \frac{1}{4\pi} \int d^2 \sigma \sqrt{|h|} \Phi(X^c) R^{(2)}.$$  \hspace{1cm} (70)

This term apparently breaks with the conformal invariance classically, thus violating the desired invariance at the classical level. However, paradoxically, it is precisely this term which can restore this invariance after quantization. Thus, when the action $S = S_1 + S_2 + S_3$ is quantized, conformal invariance is broken (an anomaly) unless the external fields satisfy three equations, as described in detail in GSW, volume 1, page 180. For brevity, we drop the antisymmetric field, setting $B_{\alpha\beta} = 0$, and get (in the magical string dimension 26!) Einstein-like equation,

$$0 = R_{\alpha\beta} - 2\Phi_{;\alpha;\beta},$$  \hspace{1cm} (71)  

$$0 = 4\Phi_{,\alpha} \Phi^{,\alpha} - 4\Phi^{,\alpha} + R.$$  \hspace{1cm} (72)

Equivalently, these background field conditions can be derived from an “effective action,”

$$\delta \int d^D X e^{-2\Phi} (R - 4\Phi_{,\alpha} \Phi^{,\alpha}).$$  \hspace{1cm} (73)

It is easy to verify then that this action is a special case of the vacuum scalar-tensor one, \[53\], with $-2\Phi = \ln \phi$, and $\omega = 1$. Nevertheless, it is difficult not to notice the close parallel between the universally coupled scalar of the old scalar-tensor theories and the new dilaton.

6 Inflatons

Cosmological models in standard general relativity have long been known to contain serious conceptual difficulties. In particular, using standard general relativistic models, initial conditions must be fantastically fine-tuned in order to result in the universe as we now see it some $10^{10}$ years later. See for example Peebles \[17\], Linde \[18\]. Consider the standard Robertson-Walker isotropic homogeneous metric model,

$$ds^2 = -dt^2 + R(t)^2 d\sigma^2,$$  \hspace{1cm} (74)

where the three-space metric, $d\sigma^2$, is hyperbolic, flat, or spherical depending on whether $\epsilon$ is -1, 0 or +1. The Einstein equations result in

$$\left(\frac{\dot{R}}{R}\right)^2 = \kappa \rho / 3 + \left(\frac{\epsilon}{R(t)^2}\right)^2 \Lambda / 3.$$  \hspace{1cm} (75)
Defining the Hubble variable as usual, this can be rewritten,

\[ 1 = \Omega + \epsilon \Omega_R + \Omega_\Lambda, \]  

(76)

where

\[ \Omega \equiv \frac{\kappa \rho}{3H^2}, \]  

(77)

\[ \Omega_R \equiv \frac{1}{(RH)^2}, \]  

(78)

and

\[ \Omega_\Lambda \equiv \frac{\Lambda}{3H^2}. \]  

(79)

Present data certainly gives values for these three quantities each in the ballpark of one. In fact,

\[ \Omega(\text{now}) \approx \frac{\kappa M}{R} \approx 10^9, \]  

(80)

which is one of Dirac’s large number coincidences which was so instrumental in leading to the scalar-tensor theories. Now, however, we note it in the context of a universe evolving from earlier (“initial”) data drastically different from that at present. For example, in the present matter dominated era the equation of state leads to

\[ \rho R^3 = M \approx \text{const}, \]  

(81)

whereas in an earlier radiation dominated state

\[ \rho R^4 \approx \text{const}. \]  

(82)

An analysis of the time evolution of these quantities under drastically different regimes show that an extremely small variation of the values of the \( \Omega \)’s at early times would result in drastically different values now. But this is not the only conceptual problem. For example, there are questions of how the universe could have homogenized itself from random early data (the “horizon” problem), and others, \[17, 18\].

Guth [19] pointed out that this myriad of difficulties could be at least partially resolved if the early stages of evolution were “inflationary,” that is

\[ R(t) = R(0)e^{Ht}, \]  

(83)

with constant \( H \). Such a model is consistent with (75) for \( \rho = \epsilon = 0, \Lambda \neq 0 \). Of course, this is not consistent with present data, so something other than a cosmological constant is needed. One way to achieve it is to introduce a new massless scalar field was the “inflaton,” \( \phi \), with Lagrangian density,

\[ \mathcal{L} = g^{\alpha\beta} \phi_\alpha \phi_\beta - V(\phi). \]  

(84)
The resulting stress tensor produces an effective mass density and pressure given by
\[ \rho_\phi = \frac{\dot{\phi}^2}{2} + V, \quad p_\phi = \frac{\dot{\phi}^2}{2} - V. \] (85)

By “fine-tuning” the potential, \( V \), at least some, but certainly not all, of the problems discussed above can be resolved. In some versions, the inflaton has a dilaton-like nature, in others it is reminiscent of the \( \phi \) in the old scalar-tensor theories, with \( \omega \) so large as to make the deviations from general relativity insignificant in contemporary solar system physics, but very significant in earlier cosmological contexts. At present, it seems likely that more than one scalar field will be required. The entire field of inflationary models is a very active one at present with many competing models. However, the role of scalar fields such as \( \phi \), is prominent in many of them.

7 Conclusion

Universally coupled, thus gravitational, scalar fields are still active players in contemporary theoretical physics. So, what is the relationship between the scalar of scalar-tensor theories, the dilaton and the inflaton? Clearly this is an unanswered and important question. The scalar field is still alive and active, if not always well, in current gravity research.

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John Norton was of great help teaching me some of the interesting early history of scalar gravity and its role in the development of General Relativity.
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