What Maxwell Theory in $d \neq 4$ teaches us about scale and conformal invariance

Sheer El-Showk\textsuperscript{a}, Yu Nakayama\textsuperscript{b}, Slava Rychkov\textsuperscript{c}

\textsuperscript{a} Institut de Physique Théorique, CEA Saclay, and CNRS URA 2306, F-91191 Gif-sur-Yvette, France

\textsuperscript{b} California Institute of Technology, 452-48, Pasadena, CA 91125, USA

\textsuperscript{c} Laboratoire de Physique Théorique, École Normale Supérieure, and Faculté de Physique, Université Pierre et Marie Curie, France

Abstract

The free Maxwell theory in $d \neq 4$ dimensions provides a physical example of a unitary, scale invariant theory which is NOT conformally invariant. The easiest way to see this is that the field strength operator $F_{\mu\nu}$ is neither a primary nor a descendant. We show how conformal multiplets can be completed, and conformality restored, by adding new local operators to the theory. In $d \geq 5$, this can only be done by sacrificing unitarity of the extended Hilbert space. We analyze the full symmetry structure of the extended theory, which turns out to be related to the $OSp(d,2|2)$ superalgebra.

January 2011
1 Introduction

Conformal invariance plays a central role in quantum field theory (QFT). It is often assumed to be an almost inevitable consequence of scale invariance. For instance, in the world-sheet formulation of string theory, the target space equations of motion can be obtained by imposing the vanishing of the beta function of the underlying two-dimensional world-sheet field theory. A vanishing beta function only indicates scale invariance, and does not automatically imply conformal invariance. Yet, it is the conformal invariance that is necessary for the consistency of the world-sheet string theory. Can we find a string background that satisfies the equations of motion but is inconsistent just because of the lack of conformal invariance? Another example arises in condensed matter physics. In order to classify the possible critical phenomena, we use the conformal field theory (CFT) techniques. Is it really the case that critical phenomena must always enjoy conformal invariance while their definition only requires scale invariance? Is there any deep physics behind the distinction between scale invariance and conformal invariance?

That scale invariance should generically imply conformal invariance can be intuitively understood as follows. If a theory is scale invariant, the trace of its energy-momentum tensor should be a total divergence \[ T_{\mu}^{\mu} = \partial_{\mu}k^{\mu}. \] (1.1) Here \( k_{\mu} \) is the ‘virial current’ (internal part of the scale current). Since \( T_{\mu\nu} \) has dimension \( d \), \( k_{\mu} \) must have dimension exactly \( d - 1 \) at the fixed point. However, barring coincidences, only conserved currents do not acquire anomalous dimensions. Thus, generically, \( k_{\mu} \) will be conserved, so that (1.1) implies

\[ T_{\mu}^{\mu} = 0, \]

which is the condition for conformal invariance.

The word “generically” in the above paragraph is important, since scale but not conformally invariant theories do exist. One class of examples is formed by non-unitary theories, such as the theory of elasticity (i.e. the free vector field without gauge invariance) \[3\], or theories based on higher-derivative Lagrangians. A second class includes theories which are unitary but do not have an energy-momentum tensor operator, such as the linearized gravity, or a Gaussian vector field with a non-conformally invariant two-point function \[4\]. A third class is based on non-compact models like the deformed Liouville theory \[5,6\], which typically break unitarity as well.
It turns out that in $d = 2$ spacetime dimensions, this classification of counterexamples is complete: a fundamental theorem due to Zamolodchikov and Polchinski \cite{7,8} says that any scale invariant 2D QFT which is unitary, has a well-defined energy-momentum tensor, and has a discrete spectrum, is conformally invariant.

For $d \geq 3$, the situation remains unclear. Polchinski \cite{8} in 1987 undertook a detailed review of the pre-existing literature and found no counterexamples. Systematic searches among theories having candidates for a nonconserved virial current \cite{8,4} have not turned up any counterexamples either. Consequently, a general conjecture seems to be that the Zamolodchikov-Polchinski theorem is true in $d \geq 3$, even though a proof is so far elusive.

In this note we explain that this conjecture is false, at least in $d = 3$ and in $d \geq 5$. The counterexample is astonishingly simple: it is the free Maxwell theory! The theory is unitary, has a well-defined energy-momentum tensor and has a discrete spectrum, and it is legitimate to call it a physical scale invariant but non-conformal field theory. Moreover, in $d \geq 5$, we will see a far richer structure of the theory by embedding it (in the BRST sense) into a non-unitary CFT, whose physical subsector coincides with the unitary Maxwell theory.

The note is organized as follows. In Section 2 we briefly remind the reader the axioms of conformal field theory, and demonstrate them on the $d = 4$ Maxwell theory, which is a bona fide CFT. In Sections 3,4,5 we discuss the Maxwell theory in $d = 3$ and $d \geq 5$. These theories are not conformal since $F_{\mu\nu}$, the lowest dimension gauge invariant operator, is neither a primary nor a descendant. Interestingly, in both cases conformality can be saved but only at the price of changing the theory by extending the space of local operators. In the extended Hilbert space $F_{\mu\nu}$ is a descendant. The difference between $d = 3$ and $d \geq 5$ is that in the second case the extended Hilbert space is, necessarily, not unitary. This poses further conceptual puzzles, which we attempt to resolve. We summarize and conclude in Section 6.

**Note added.** When our paper was being prepared for publication, paper \cite{27} appeared which also discusses scale and conformal (non)invariance of the $d \neq 4$ Maxwell theory, at the classical level.

\footnote{See also \cite{9} for an independent work on the theorem.}
2 Maxwell theory in \( d = 4 \) as Conformal Field Theory

In this paper we will consistently emphasize the arguments based on correlation functions over the conventional Lagrangian methods. On the most basic level, QFT is simply a set of correlators of local operators. These correlators must satisfy well-known and natural axioms (the Wightman axioms), such as Lorentz invariance, microcausality, and unitarity, to which we add the existence of the energy-momentum tensor operator \( \mathbf{T} \). A Lagrangian is useful inasmuch as it allows one to compute the correlators; having a good Lagrangian also guarantees that the correlators will satisfy the axioms. Once the correlators are computed, however, and the theory thus defined, we can safely discard the Lagrangian and never recall it.

Consider the free Maxwell theory in \( d = 4 \) from this point of view. The Lagrangian

\[
\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2
\]

defines the two-point function of the gauge potential \( A_\mu \). In the position space (we do not keep track of the overall field normalization)

\[
\langle A_\mu(x)A_\nu(0) \rangle = \frac{\eta_{\mu\nu}}{x^2} + \text{gauge terms} \quad (d = 4).
\]

This two-point function is gauge-variant and not physical, but it serves to define the correlators of the gauge invariant field strength \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). For example, the two-point function of \( F_{\mu\nu} \) comes out to be

\[
\langle F_{\mu\nu}(x)F_{\lambda\sigma}(0) \rangle = \frac{I_{\mu\lambda}I_{\nu\sigma} - \mu \leftrightarrow \nu}{(x^2)^2}, \quad I_{\mu\nu} \equiv \eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2} \quad (d = 4). \tag{2.1}
\]

The latter set of gauge invariant correlators defines the theory. It is not yet the full theory because other local and gauge invariant fields can be constructed by taking the OPE of \( F_{\mu\nu} \) with itself. Since we are dealing with a free theory, these fields are nothing but normal ordered powers of \( F_{\mu\nu} \). Among them, two notable fields are the scalar

\[
\Phi = :((F_{\mu\nu})^2):
\]

and the symmetric tensor

\[
T_{\mu\nu} = :F_{\rho\mu}F_{\nu}^\rho - \frac{\eta_{\mu\nu}}{4}(F_{\rho\sigma})^2:
\] \tag{2.2}

\(^2\)The Wightman axioms assume the existence of the total energy-momentum but not necessarily of its density.
The latter field is in fact the energy-momentum tensor of the theory. In textbook treatments, its expression is obtained from the Lagrangian, e.g. by varying the action with respect to the external metric. Notice, however, that this is not strictly speaking necessary. Instead, one could have simply checked that the correlators of $T_{\mu\nu}$ defined by (2.2) are conserved and satisfy the Ward identities.

The $d = 4$ Maxwell theory is obviously scale invariant (all correlators scale with distance). As is well known, it is also conformally invariant. The fastest way to see this is to observe that the energy-momentum tensor (2.2) is traceless. What if we did not know the explicit expression of the energy-momentum tensor? How would we then decide if the theory is conformal or not?

The answer is: look at the correlators and see if they are consistent with conformal symmetry. In a CFT, local fields are classified into primaries and their derivatives (descendants). Correlators of primaries are strongly constrained by conformal invariance. For example, the correlator of three scalar primaries $\phi_i$ of equal dimensions $\Delta_i = \Delta$ must take the form

$$\langle \phi_i(x_1) \phi_j(x_2) \phi_k(x_3) \rangle = \frac{c_{ijk}}{[(x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2]^{\Delta/2}}.$$

Following this logic, let us compute the three-point function of the $\Phi = : (F_{\mu\nu})^2 :$. We find that it vanishes:

$$\langle \Phi(x) \Phi(y) \Phi(z) \rangle = 0 \quad (d = 4).$$

This equation is consistent with $\Phi$ being a scalar primary, with an (accidentally) vanishing three-point function coefficient $c_{\Phi \Phi \Phi}$.

A further constraint is that the two-point functions of tensor primaries have to be constructed out of the $I_{\mu\nu}$ tensor (see e.g. [11]). Eq. (2.1) is thus consistent with $F_{\mu\nu}$ being an antisymmetric tensor primary.

It is good news for the conformal invariance of the theory that both $\Phi$ and $F_{\mu\nu}$ can be interpreted as primaries. Had we found a contradiction at this point, the only remaining possibility (consistent with conformality of the theory) would be to try to interpret them as descendants. However, there are no candidate gauge invariant local fields in the theory of which $\Phi$ and $F_{\mu\nu}$ could be descendants.

Let us finally discuss unitarity in some detail. Unitarity is an important part of the Wightman axioms, and means positivity of the spectral density for Minkowski-space...
non time-ordered correlators. Applied to two-point functions $\langle 0|O^\dagger(x)O(y)|0\rangle$, unitarity implies lower bounds on the scaling dimensions of local operators. For example, scalars have dimensions $\Delta \geq 1$. In conformal theories, unitarity bounds for each primary type (vector, symmetric traceless and antisymmetric two-tensors, etc.) can be obtained [12]. The dimension of an antisymmetric tensor primary is always $\Delta \geq 2$, saturated by $F_{\mu\nu}$.

The vectors have dimensions $\Delta \geq 3$. There is no contradiction that the Maxwell gauge potential violates this bound, since it does not even belong to the Hilbert space of the theory (let alone being a primary).

3 Maxwell theory in $d \neq 4$. Conformality lost.

Let us now consider the Maxwell theory in $d \geq 3, d \neq 4$. By the Maxwell theory we mean the same as in the previous section, i.e. the set of correlation functions of the gauge invariant fields, such as $F_{\mu\nu}$ and its operator products. The gauge potential is not a physical field of the theory: its gauge dependent two-point function

$$\langle A_\mu(x)A_\nu(0)\rangle = \frac{\eta_{\mu\nu}}{(x^2)^{(d-2)/2}} + \text{gauge terms}$$

is only used to define the correlators of $F_{\mu\nu}$.

It is obvious that the $d \neq 4$ Maxwell is scale invariant; it is also unitary, just like in $d = 4$. Is it conformally invariant? If we are to follow the textbook approach, we have to compute the trace of the energy-momentum tensor (2.2). We find

$$T^\mu_\mu = \frac{4 - d}{4}(F_{\mu\nu})^2,$$

nonzero in $d \neq 4$. However, by itself this does not automatically imply the absence of conformal invariance. For example, the free scalar energy-momentum tensor

$$T^\phi_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2$$

suffers from the same problem if $d \neq 2$, yet it is known that it can be ‘improved’, so that the free scalar theory is in fact conformally invariant in any dimension. This is because the virial current happens to be a total divergence (see below).

---

4In the Euclidean signature unitarity is replaced by the (Osterwalder-Schaeder) reflection positivity.

5The $d = 2$ Maxwell theory has no propagating degrees of freedom.
In a short while, we will discuss the impossibility of improving (2.2), unlike in the scalar case. But first, let us look at the two-point function of $F_{\mu\nu}$ and the three-point function of $\Phi = (F_{\mu\nu})^2$. For a general $d$ they turn out to have the form:

$$
\langle F_{\mu\nu}(x) F_{\lambda\sigma}(0) \rangle = \frac{2d - 4}{(x^2)^{d/2}} \left[ \left( \eta_{\mu\lambda} - \frac{d}{2} \frac{x_{\mu}x_{\lambda}}{x^2} \right) \left( \eta_{\nu\sigma} - \frac{d}{2} \frac{x_{\nu}x_{\sigma}}{x^2} \right) - \mu \leftrightarrow \nu \right], \quad (3.2)
$$

$$
\langle \Phi(x_1) \Phi(x_2) \Phi(x_3) \rangle = \frac{-8(d - 2)^3(d - 4)d}{(x_{12}^2)^{d/2}(x_{13}^2)^{d/2}(x_{23}^2)^{d/2}} \times \left[ 2 + d^2 (x_{12} \cdot x_{13})(x_{12} \cdot x_{23})(x_{13} \cdot x_{23}) - d(x_{12} \cdot x_{23}) \frac{x_{13}^2 + 2 \text{ perms}}{x_{12}^2 x_{13}^2 x_{23}^2} \right], \quad (3.3)
$$

(here $x_{ij} \equiv x_i - x_j$).

For $d \neq 4$, these equations imply that neither $F_{\mu\nu}$ nor $\Phi$ can be conformal primaries. If the $d \neq 4$ Maxwell is to be conformal, these fields have to be descendants of some other local gauge invariant fields. But of which ones? There are no other gauge invariant fields of sufficiently low dimension which could serve this purpose. In fact, $F_{\mu\nu}$ is the lowest-dimensional gauge invariant field, so it cannot be anyone else’s descendant. The $\Phi$ and $T_{\mu\nu}$ are the next-to-lowest dimensional fields after $F_{\mu\nu}$. Thus $\Phi$ could only be a descendant of $F_{\mu\nu}$. But $F_{\mu\nu}$ has no scalar descendants, since $\partial^\mu F_{\mu\nu} = 0$.

We are led to conclude that the $d \neq 4$ Maxwell is not conformally invariant.

For completeness, let us discuss how the same conclusion could be reached using the textbook approach, by showing that the energy-momentum tensor cannot be improved. To begin with, note that the energy-momentum tensor trace can be represented as a total derivative

$$
T^\mu_\mu = \frac{4 - d}{8} \partial_\mu k^\mu, \quad k^\mu = A_\mu F^{\rho\mu}. \quad (3.4)
$$

This had to happen since the theory is scale invariant. Notice that it is somewhat peculiar that the virial current $k_\mu$ is not gauge invariant. As a consequence, the conserved dilatation current $J^D_\mu = x^\nu T_{\nu\mu} - k_\mu$ is not gauge invariant. However, one can check that the dilatation charge $D = \int d^3x J^D_0$ is gauge invariant. Using the equations of motion,

$$
\delta_\Lambda D = \int d^3x \partial^i \Lambda F_{i0} = - \int d^3x \Lambda \partial^i F_{i0} = 0. \quad (3.4)
$$

\footnote{The assumption that $k_\mu$ must be a physical, gauge invariant (or BRST invariant), operator was implicit in \cite{8}. The $d \neq 4$ Maxwell theory apparently violates this assumption.}
The condition for the improvement of the energy-momentum tensor (so that it can be traceless) is that the $k_\mu$ be a total derivative \cite{2, 8}:

$$k^\mu = \partial^\nu L_{\nu \mu}, \quad (3.5)$$

where $L_{\mu \nu}$ must be a dimension $d - 2$ symmetric tensor constructed out of the local fields of the theory (for $d > 2$). On dimensional grounds the only possibility is

$$L_{\mu \nu} = a_1 A_\mu A_\nu + a_2 \eta_{\mu \nu} (A_\rho)^2,$$

but it is easy to check that this cannot generate the above $k_\mu$ no matter how we choose $a_1$ and $a_2$.

4 Recovering conformality in \(d = 3\)

As we have seen in the previous section, the \(d \neq 4\) Maxwell theory is scale invariant but not conformal, since there are gauge invariant local fields which are neither primaries nor descendants.

It turns out that a partial fix is possible. Namely, we can try to add new local fields to the theory. The new extended theory will contain all the correlation functions of the original Maxwell theory and the correlators of the new fields. The new fields will be primaries, and the old fields like $F_{\mu \nu}$ or $\Phi$ will now be descendants of the new fields. Thus the extended theory will be conformal.

This procedure turns out to work differently in \(d = 3\) and \(d \geq 5\); in this section we focus on \(d = 3\).

Let us add to the theory a free scalar field $B$. We will postulate that $F_{\mu \nu}$ is a descendant of $B$, according to the formula

$$F_{\mu \nu} = \epsilon_{\mu \nu \rho} \partial^\rho B. \quad (4.1)$$

Physically, $B$ is the non-local magnetic dual of the gauge potential.

\footnote{Another way to turn \(d \neq 4\) Maxwell theory into a classically conformal theory would be to couple it to a conformal compensator scalar field, i.e. by considering actions of the form $F_{\mu \nu}^2 \phi^\alpha + (\partial \phi)^2$ for an appropriate $\alpha$. We are grateful to A. Tseytlin for this remark. Notice however that at the quantum level, such an action could describe only an effective field theory in a phase of spontaneously broken conformal invariance (since one would have to give $\phi$ a vev). There would be severe difficulties in defining such a theory in the UV. In this paper we deal with much simpler theories, but with the virtue that they make sense quantum-mechanically at all energies.}
It is not difficult to check that, in $d = 3$, prescription (4.1) gives the $F_{\mu\nu}$ two-point function that is identical to Eq. (3.2). Furthermore, the extended theory will also contain new composite fields constructed out of $B$. Their existence is essential to complete all conformal multiplets. For example, $\Phi$ will now be a descendant of $:B^2:.$.

The outlined construction can be rephrased as follows: In $d = 3$, free scalar theory contains a subsector which is isomorphic to the Maxwell theory. By a subsector we mean here a set of correlators closed with respect to the OPE.

In other words, we have saved conformality by extending the $d = 3$ Maxwell into the free scalar theory, which is conformal for any $d$. It has to be stressed that by passing from Maxwell to free scalar we really changed the theory (changed the set of local operators). The extended theory is unitary, so we can successfully embed the non-conformal Maxwell theory in $d = 3$ into a unitary CFT. It would be interesting to understand if there is another reason (except for the need to recover conformal invariance) which would justify such an extension.

One idea would be to see if modular invariance of the torus partition function requires this. This would then be analogous to the Ising model in $d = 2$, where the OPE closes in the $\epsilon$ sector, yet modular invariance makes the presence of the $\sigma$ field mandatory [13].

In $d = 3$, another example which comes to mind is the M2-brane gauge theory (i.e. the ABJM model [14]). At levels $k = 1, 2$, $N = 8$ supersymmetry is not manifest in the original variables appearing in the action of this theory. We have to introduce magnetic monopole operators (essentially the same field $B$) to construct $N = 8$ SUSY multiplets (see e.g. [15]). In this example, the manifestation of the maximal supersymmetry required the existence of the magnetic dual operators.

5 Recovering conformality in $d \geq 5$

5.1 No unitary extension

We will now discuss whether the $d \geq 5$ Maxwell can also be extended into a conformal theory, similarly to $d = 3$. To answer this question, we have to try to add a field that satisfies the following property: it has the two-point function of a primary field, and we can construct $F_{\mu\nu}$ as its descendant. However, this runs into the following difficulty. Taking into account that all bosonic primaries are symmetric traceless or antisymmetric tensors,
there are only two possible descendant relations:
\[ \begin{align*}
F_{\mu\nu} &= \partial_{\mu} Y_{\nu} - \partial_{\nu} Y_{\mu}, \\
F_{\mu\nu} &= \epsilon_{\mu\nu\lambda...} \partial^{\lambda} Z^{...},
\end{align*} \]

where \( Y \) is a hypothetical primary vector (not to be confused with the vector potential \( A_{\mu} \)), while \( Z \) would be a totally antisymmetric tensor of rank \( d - 3 \). To be consistent with the scaling dimension of \( F_{\mu\nu} \), \( \Delta_{F} = d/2 \), these fields would have to have dimension \( d/2 - 1 \). The trouble is, such low dimensions are inconsistent with unitarity. Unitarity bounds for higher dimensional conformal fields theories were derived in \[16\],\[17\]; in the cases of interest for us they read:
\[ \begin{align*}
\Delta &\geq d - 1 \quad \text{(primary vector)} \quad (5.1) \\
\Delta &\geq \max(d - k, k) \quad \text{(primary rank } k \text{ antisymmetric tensor)} \quad (5.2)
\end{align*} \]

We are thus led to the conclusion that it is impossible to extend the \( d \geq 5 \) Maxwell into a unitary conformal theory.

### 5.2 A non-unitary extension

Despite the difficulty we have mentioned, it turns out that an extension into a non-unitary conformal theory is possible. Namely, let us add to the theory a local field \( Y_{\mu} \) having a two-point function of a dimension \( d/2 - 1 \) vector primary:
\[ \langle Y_{\mu}(x) Y_{\nu}(0) \rangle = \frac{I_{\mu\nu}}{(x^2)^{d/2-1}}. \]

Assume that this field is Gaussian, i.e. all higher order correlators are computed via Wick’s theorem. This, then, is a conformal field theory, which is non-unitary, since the unitarity bound \((5.1)\) is violated (see Appendix \[13\] for an elementary derivation of this bound).

Assume further that \( F_{\mu\nu} \) is a descendant of \( Y_{\mu} \) via
\[ F_{\mu\nu} = \partial_{\mu} Y_{\nu} - \partial_{\nu} Y_{\mu}. \quad (5.3) \]

It is not difficult to check that this ansatz reproduces the two-point function of \( F_{\mu\nu} \) given by \((3.2)\), up to a normalization factor of \((d - 4)/(d - 2)\). This is all we need to demonstrate the extension.

This result suggests several interesting questions, to be discussed below.

---

\(^8\)Metsaev \[16\] used a field-theoretic AdS realization which gives results equivalent to a purely group-theoretic derivation given later by Minwalla \[17\].
5.3 The origin of $Y_\mu$

In Sections 2,3 we have not specified the gauge in which we compute the gauge field propagator (since, of course, all gauges give the same $F_{\mu\nu}$ two-point functions). Let us now focus on the $\xi$-gauge,

$$\mathcal{L}_\xi = -\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2,$$

so that the momentum space propagator takes the well-known form

$$\langle A_\mu(-p)A_\nu(p) \rangle = \frac{\eta_{\mu\nu}}{p^2} \left(1 - \frac{1 - \xi}{2} \frac{p_\mu p_\nu}{p^2} \right),$$

from which the coordinate-space propagator is found to be

$$\langle A_\mu(x)A_\nu(0) \rangle = \frac{1}{(x^2)^{(d-1)/2}} \left(\eta_{\mu\nu} + (d-2)\frac{1 - \xi}{1 + \xi} \frac{x_\mu x_\nu}{x^2} \right)$$

(up to an overall constant factor).

Now notice a curious thing. For a particular choice of the gauge parameter

$$\xi^* = \frac{d}{d-4},$$

the two-point function of $A_\mu$ becomes proportional to the $I_{\mu\nu}$ tensor, and thus takes the form consistent with $A_\mu$ being a conformal primary. For this value of $\xi$, the gauge-fixed theory (5.4) is conformal. (In Appendix C we also show that its energy-momentum tensor can be improved to become traceless.)

This, then, is the underlying reason which made it possible to extend the Maxwell theory into a CFT in which the field strength is a descendant of a primary vector field. The primary vector $Y_\mu$ is nothing but the vector potential $A_\mu$ in a particular $\xi$-gauge. This of course guarantees that the identification (5.3) will produce the correct $F_{\mu\nu}$ two-point function.

5.4 BRST-like interpretation

From now on we will consider the theory (5.4) with $\xi = \xi^*$ and will rename $A_\mu$ to $Y_\mu$, in order to avoid any possible confusion. As discussed above, this is a CFT, however of an unusual sort. While the full theory is non-unitary, there is a subsector of the theory (the Maxwell theory) which is unitary. The conformal multiplet of which $Y_\mu$ is the primary contains both non-unitary ($Y_\mu$ itself) and unitary ($F_{\mu\nu}$ and its derivatives) fields. This
is clearly not a bona fide unitary CFT, where all local fields must belong to the unitary Hilbert space.

One would like to come up with a procedure to distinguish the unitary sector from non-unitary sectors of the theory. As usual in gauge-fixed theories, BRST is the way to implement the distinction. Thus, let us add to the theory the ghost sector

$$\mathcal{L}_{gh} = -\frac{1}{2} \epsilon_{ab} c_a \partial^2 c_b.$$  \hspace{1cm} (5.5)

On the one hand, the new local fields $c_{1,2}$, anticommuting scalars, also violate unitarity. On the other hand, the full theory now has a BRST-like symmetry with two fermionic generators (‘BRST’ and ‘anti-BRST’)

$$[Q_a, Y_\mu] = -i \partial_\mu c_a, \quad \{Q_a, c_b\} = -i \xi^{-1} \epsilon_{ab} \partial_\mu Y_\mu.$$ \hspace{1cm} (5.6)

The unitary subsector of the theory can now be picked out by saying that it consists of all $Q$-invariant fields.\footnote{One of the $Q_a$ or both of them would equally work.}

There is one curious thing about the above construction which merits further attention: the BRST transformations do not commute with the conformal algebra. This is already obvious from the fact that a BRST-variant primary field $Y_\mu$ has a BRST-invariant descendant $F_{\mu\nu}$. This looks pretty unusual: for example in string-theoretic 2D CFTs the BRST operator always commutes with the conformal algebra. This also provokes us to try to identify the full symmetry (super)algebra of the theory, containing both the BRST and conformal generators.

5.5 Extended conformal algebra

Conformal algebra commutation relations and the generator action on the primary fields are summarized in Appendix A. In particular, the action on the ghosts $c_a$, which are scalars, is given by Eq. (A.2) with $\Delta = (d - 2)/2$ and $\Sigma \equiv 0$. The action on the primary vector $Y_\mu$ is given by the same equations but now $\Sigma$ is nontrivial: $(\Sigma_{\mu\nu} Y)_\rho = \eta_{\mu\rho} Y_\nu - \eta_{\nu\rho} Y_\mu$.

In order to understand the full symmetry algebra, we have to include Eq. (5.6), which defines the action of fermionic generators $Q_a$.\footnote{In the section $\sim$ means equality up to an irrelevant c-number factor.}

As already mentioned above, $Q$’s cannot commute with the conformal algebra: the relations

$$[Q_a, F_{\mu\nu}] = 0, \quad [Q_a, Y_\mu] \neq 0, \quad Y_\mu \sim [K_\nu, F_{\mu\nu}]$$
can only be consistent if
\[ Q_{a\mu} \equiv i[Q_a, K_\mu] \neq 0. \] (5.7)

An explicit computation shows that, indeed, these new ‘fermionic rotation’ generators \( Q_{a\mu} \) act nontrivially on the primary fields:
\[
\{Q_{a\mu}, c_b(x)\} = -i\xi^* \epsilon_{ab}[d \eta_{\mu\nu} - 2x_{\mu}\partial_{\nu}]Y_\nu(x),
\]
\[
[Q_{a\mu}, Y_\nu(x)] = i[(d - 2)\eta_{\mu\nu} + 2x_{\mu}\partial_{\nu}]c_a(x).
\]

What is, then, the full symmetry (super)algebra of the theory we are dealing with?

A natural conjecture could be that this is the orthosymplectic superalgebra \( OSp(d, 2|2) \). Indeed, this superalgebra has the bosonic subgroup \( SO(d, 2) \times Sp(2) \), and is the smallest (simple) superalgebra that contains this symmetry \([18]\). The first factor could then be identified with the conformal algebra, while the second factor acts on the ghosts which form the fundamental multiplet of \( Sp(2) \), so that the ghost action (5.5) is invariant. The full (anti)commutation relations of \( OSp(d, 2|2) \) are \([19]\)
\[
[J_{AB}, J_{CD}] = i(\eta_{\alpha\beta} J_{a\beta} - \epsilon_{ab} J_{a\beta}) ,
\]
\[
\eta_{AB} = (\eta_{\alpha\beta}, \epsilon_{ab}), \quad \alpha, \beta = 0, \ldots, d + 1, \quad a, b = 1, 2; \quad \text{gradings} \quad [\alpha] = 0, [a] = 1.
\]

Here \( J_{[\alpha\beta]} \) are the \( SO(d, 2) \) generators (see Appendix [A]), \( J_{ab}(ab = 11, 12, 22) \) are the \( Sp(2) \) generators. The fermionic generators are \( J_{a\pm} (x^\pm = x^{d+1} \pm x^d) \) and \( J_{a\mu} \); they satisfy the commutation relations
\[
[J_{a+}, J_{\mu-}] = -\frac{i}{2} J_{a\mu} ,
\]
\[
[J_{a\mu}, J_{\mu-}] = i\eta_{\mu\nu} J_{a\nu} ,
\]
\[
\{J_{a\alpha}, J_{b\beta}\} = i(\eta_{\alpha\beta} J_{ab} - \epsilon_{ab} J_{a\beta}) .
\]

Recall that \( K_\mu \sim J_{\mu-}, P_\mu \sim J_{\mu+} \). Let us try to identify \( Q_a \sim J_{a+} \) so that the BRST operator becomes a sort of fermionic \( P_\mu \). This is consistent with \( \{Q_a, Q_b\} = 0 \). Then Eq. \([5.8]\) identifies \( J_{a\mu} \) with the fermionic rotation \( Q_{a\mu} \) as defined above.

However, unfortunately, this \( OSp(d, 2|2) \) conjecture does not hold up since the algebra with the above identifications does not close. To see this, consider the partial case of Eq. \([5.10]\),
\[
\{Q_a, Q_{b\mu}\} \sim i\epsilon_{ab} P_\mu \quad (?)
\]

12
On the other hand, an explicit computation using the known action of $Q_a$ and $Q_{b\mu}$ on the fields shows:

$$\{Q_a, Q_{b\mu}\} = i\epsilon_{ab}(d - 4)(P_{\mu} + P'_{\mu}),$$  \hspace{1cm} \text{(5.11)}

where $P'_{\mu}$ is a new bosonic generator which acts only on $Y_\mu$ according to

$$[P'_{\mu}, Y_\nu(x)] = -i[\partial_\nu Y_\mu - \partial_\mu Y_\nu + \xi^{-1}\eta_{\mu\nu}(\partial Y)].$$

These new generators do not commute:

$$P'_{[\mu\nu]} \equiv i[P'_{\mu}, P'_{\nu}] \neq 0,$$  \hspace{1cm} \text{(5.12)}

where $P'_{[\mu\nu]}$ generate higher spin symmetries of the theory of the form

$$\delta Y_\mu = a_{[\mu\nu]}\partial_\nu(\partial^\rho Y_\rho) + \xi_{a[\rho\sigma]}\partial_\sigma \partial^\rho Y_\rho.$$  \hspace{1cm} \text{(5.13)}

By iterating this procedure $[P'_{\mu_1}, [P'_{\mu_2}, [P'_{\mu_3}, \cdots]]]$, one can construct an infinite number of antisymmetric tensor conserved currents. One can also construct their fermionic counterparts by commuting with the (anti-)BRST charges $Q_a$.

Of course, since the theory we are dealing with is free, we should not be surprised by the presence of infinite-dimensional higher spin symmetry algebras. It was not obvious that these currents must be generated by repeated commutators of $Q_a$ and $K_\mu$ (because $OSp(d, 2|2)$ could be a minimal closure of the algebra), yet the above discussion shows that the generation of the higher spin symmetry does happen in our theory.

Notice that for the above conclusion it was important that we included both BRST and anti-BRST generators into the symmetry algebra. Were we to artificially leave out one of them, the closure would be a finite-dimensional super-algebra which is a subalgebra of $OSp(d, 2|2)$. This subalgebra would include the conformal generators $J_{a\beta}$, the BRST charge $Q_B = J_{1-}$, the fermionic vector charges $J_{1\mu}$ and the fermionic scalar charge $J_{1+}$ (here 1 denotes a component of the $Sp(2)$ index $a$). The point is that the $OSp(d, 2|2)$ has a grading with respect to the ghost number generator $J_{12}$ within $Sp(2)$. We can thus construct a subalgebra by restricting to the non-negative ghost number sector. In particular, the problematic anti-commutation (5.11) is avoided, and $P'$ would not appear in the algebra. This subalgebra is not simple (and thus does not appear in the Kac classification), because it contains a non-trivial ideal consisting of the strictly positive ghost number generators.

13
6 Summary, discussion and lessons

In this paper, we started by showing the following three technical results:

- The Maxwell theory in \( d \neq 4 \) is an example of a unitary, scale invariant theory which is not conformally invariant. We demonstrated this in two ways: a) by showing that the physical, gauge invariant fields \( F_{\mu\nu} \) and \( (F_{\mu\nu})^2 \) are neither primaries nor descendants; b) in the textbook way - by showing that the energy-momentum tensor cannot be improved to be traceless.

- The \( d = 3 \) Maxwell theory can be extended to a unitary CFT by introducing a new local field: a primary scalar \( B \). In the extended theory, \( F_{\mu\nu} \) is a descendant of \( B \).

- The \( d \geq 5 \) Maxwell theory cannot be extended to a unitary CFT, since this would contradict unitarity bounds. However, it can be extended to a non-unitary CFT, by introducing a primary vector \( Y_\mu \) of which \( F_{\mu\nu} \) is a descendant.

This group of results is interesting for two reasons. First, it provides a clearcut counterexample to the often assumed conjecture that any unitary and scale invariant theory is conformal. It is pretty amazing that this counterexample escaped the attention of the recent literature on the subject.\textsuperscript{11} Second, our results point a way to generalize the conjecture so that it has a chance to remain true: Could it be that in fact any unitary and scale invariant theory can be made conformal by extending the set of local fields?

At present we do not have any evidence in favor of this generalized conjecture except for the fact that it does not contradict our examples. A most puzzling thing is that non-unitary conformal extensions need to be allowed, as the \( d \geq 5 \) Maxwell theory case shows. We thus proceeded to take a closer look at this case. Our findings here can be summarized as follows:

- We showed how to separate the unitary part of the Hilbert space by using the usual BRST language. In fact, the non-unitary field \( Y_\mu \) can be interpreted as the vector potential in a particular \( \xi \)-gauge in which its propagator is conformally invariant.

\textsuperscript{11}We have noticed that Birrell and Davis \textsuperscript{20}, Section 3.8, do mention in passing that the Maxwell theory is conformally invariant only in \( d = 4 \). Another precursor result is a classification, for any \( d \), of all primary operators \( \phi \) which can consistently satisfy the free field equation \( \partial^2 \phi = 0 \) \textsuperscript{21,22,17}. An antisymmetric two-tensor is not in the list for \( d \neq 4 \).
• The (anti-)BRST operators do not commute with the conformal algebra.

• An attempt to close the algebra starting from BRST, anti-BRST and conformal generators leads to an infinite-dimensional superalgebra which includes higher-spin symmetry generators (present in the Maxwell theory since it is free). On the other hand, starting from just the BRST and conformal generators (i.e. omitting anti-BRST) we can generate a finite-dimensional algebra, which is a subalgebra of the orthosymplectic superalgebra $OSp(d, 2|2)$.

Based on this study, we can speculate that if other, and interacting, examples of theories following the above generalized conjecture with a non-unitary CFT extension do exist, they should also perhaps realize $OSp(d, 2|2)$ or its nonnegative ghost-number subalgebra in their Hilbert space.

Unfortunately, it seems quite difficult to search for such counterexamples, since it is impossible to marginally deform the $d \geq 5$ Maxwell theory by coupling it to matter. Indeed, the dimension of $A_\mu$ is different from that of $\partial_\mu$ so that the covariant derivative $\partial_\mu + ieA_\mu$ with a dimensionless coupling $e$ cannot be introduced while preserving scale invariance. The higher derivative couplings like the Pauli interactions are always irrelevant, so it is impossible to classically deform the $d \geq 5$ Maxwell theory while preserving even the scale invariance. In this sense, the $d \geq 5$ Maxwell theory looks like an isolated IR fixed point.

This would not exclude the possibility that the UV fixed point may be asymptotically safe, and we might expect a scale invariant but non-conformal field theory with a strongly coupled fixed point. Such a theory may or may not possess the hidden non-unitarity realized conformal invariance of our free Maxwell theory. In such a case, the anti-BRST transformation and its commutation relation with the conformal charge must be modified compared with the free Maxwell theory because we would not expect an infinite number of conserved charges in interacting field theories.

Another obstruction to constructing such theories comes from their dual holographic descriptions (when they are expected to exist). It was proven that as long as the effective supergravity description is valid with matter satisfying the strict null energy condition, the solutions of the equation of motion (even with higher derivative corrections) automatically possess the AdS isometry if we impose the Poincaré isometry and the scaling isometry [23]. One could argue that a free theory is not expected to have a dual that is in any sense accurately described by supergravity, as the dual of a free theory is necessarily very
strongly coupled. Nevertheless these holographic arguments do suggest that it may be difficult to construct interacting examples of scale invariant but non-conformal theories.

Finally, we should stress that our examples are strictly in $d \geq 3$, $d \neq 4$. The $d = 4$ case of the Zamolodchikov-Polchinski theorem remains open; it could still be that in this dimension, like in $d = 2$, scale invariance + unitarity $\implies$ conformal invariance.

Acknowledgements

S. R. is grateful to Jan Troost, Manuela Kulaxizi and Andrei Parnachev for useful discussions. Y. N. thanks John Cardy for the encouraging and stimulating correspondence on the subject. S. E. would like to thank the CEA Saclay for hospitality during the completion of part of this work. The research of S.E. is partially supported by the Netherlands Organisation for Scientific Research (NWO) under a Rubicon grant. The work of S. R. was supported in part by the European Programme “Unification in the LHC Era”, contract PITN-GA-2009-237920 (UNILHC). The work of Y. N. is supported by Sherman Fairchild Fellowship at California Institute of Technology.

A Conformal algebra

Recall the conformal algebra\textsuperscript{12}.

\[
[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} \pm \text{perms})
\]

\[
[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu)
\]

\[
[D, P_\mu] = -iP_\mu
\]

\[
[D, K_\mu] = +iK_\mu
\]

\[
[P_\mu, K_\nu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu}) .
\]

\textsuperscript{12}To avoid any possible confusion, let us note that the special conformal generator $K_\mu$ has nothing to do with the virial current $k_\mu$. 

16
The generators act on the primary fields (not necessarily scalars) by

\[ [P_\mu, \phi(x)] = -i \partial_\mu \phi(x) \]
\[ [D, \phi(x)] = -i (\Delta + x^\mu \partial_\mu) \phi(x) \]  
\[ [M_{\mu\nu}, \phi(x)] = \{ \Sigma_{\mu\nu} - i (x_\mu \partial_\nu - x_\nu \partial_\mu) \} \phi(x) \]
\[ [K_\mu, \phi(x)] = (-i 2x_\mu \Delta - 2x^\lambda \Sigma_{\lambda\mu} - i 2x_\mu x^\rho \partial_\rho + ix^2 \partial_\mu) \phi(x), \]  

where the finite-dimensional matrices \( \Sigma \) act in the space of \( \phi \)'s Lorentz indices; they have to satisfy the commutation relation (notice the sign difference from the first equation in (A.1))

\[ [\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = +i (\eta_{\mu\rho} \Sigma_{\nu\sigma} \pm \text{perms}). \]

The algebra (A.1) corresponds to the mostly minus Minkowski signature. Beware that the literature uses inconsistent sign conventions for various generators, in particular \( M_{\mu\nu} \) and \( D \) (our conventions are those of [24]). Also, the generator action is usually given with relative sign errors among various terms; this is not surprising because in practice these expressions are actually rarely used. However, we will need them, so we re-checked from scratch by using the original method of Mack and Salam [25].

As is well known, the algebra (A.1) is isomorphic to \( SO(d,2) \). The isomorphism is exhibited by identifying

\[ J_{\mu\nu} = M_{\mu\nu}, \quad J_{d,d+1} = D \]
\[ J_{d,\mu} = \frac{1}{2} (P_\mu - K_\mu), \quad J_{d+1,\mu} = \frac{1}{2} (P_\mu + K_\mu), \]

and then \( J_{\alpha\beta} \) \( (\alpha, \beta = 0 \ldots d + 1) \) satisfy the \( SO(d,2) \) commutation relations

\[ [J_{\alpha\beta}, J_{\gamma\delta}] = -i (\eta_{\alpha\gamma} J_{\beta\delta} \pm \text{perms}), \quad \eta_{\alpha\beta} = \text{diag} (+, -, \ldots, -; -, +). \]

**B Unitarity bound for vectors**

Metsaev [16] and Minwalla [17] have shown that in any number of spacetime dimensions \( d \) a unitary primary vector must have dimension \( \Delta \geq d - 1 \). We will give here an independent derivation of this result (see [26], [1] for similar arguments) based on the fact that conformal invariance fixes the primary vector two-point function to have the (Euclidean) form:

\[ \langle Y_\mu(x) Y_\nu(0) \rangle = \frac{1}{(x^2)^\Delta} \left( \delta_{\mu\nu} - 2 x_\mu x_\nu x^2 \right). \]
Write this as a sum of derivatives:

\[
\langle Y_\mu(x)Y_\nu(0) \rangle = \left(1 - \frac{1}{\Delta}\right) \frac{1}{(x^2)^\Delta} \delta_{\mu\nu} - \frac{1}{2(\Delta - 1)\Delta} \partial_\mu \partial_\nu \frac{1}{(x^2)^{\Delta - 1}}
\]

and pass to momentum space by using:

\[
\frac{1}{(x^2)^\Delta} \rightarrow \text{const} \frac{\Gamma(d/2 - \Delta)}{4^\Delta \Gamma(\Delta)} (p^2)^{\Delta - d/2},
\]

where const does not depend on \(\Delta\). Keeping track only of the relative factor among the two tensor structures, we have:

\[
\langle Y_\mu(p)Y_\nu(-p) \rangle \propto \left( B_1 \delta_{\mu\nu} + B_2 \frac{p_\mu p_\nu}{p^2} \right) (p^2)^{\Delta - d/2}, \quad B_1 = 1, \ B_2 = -2 \frac{\Delta - d/2}{\Delta - 1}.
\]

The spectral density of the Wightman function in the forward Minkowski cone can be extracted from the Euclidean spectral density via the Wick rotation. We have:

\[
\langle 0|Y_\mu(p)Y_\nu(-p)|0 \rangle \propto -\theta(p^0)\theta(p^2) \left( B_1 \eta_{\mu\nu} + B_2 \frac{p_\mu p_\nu}{p^2} \right) (p^2)^{\Delta - d/2}
\]

(where we fixed the sign by matching the spatial components). Unitarity implies that for any complex constant four-vector \(\chi_{\mu}\), the contraction \(\chi.Y\) must have a positive spectral density, i.e.

\[
-B_1 \chi.\chi^\dagger - B_2 \frac{|\chi.p|^2}{p^2} \geq 0,
\]

for any \(\chi\) and any \(p\) in forward cone. Taking \(\vec{p} = 0\), \(p_0 = 1\) we get:

\[
-B_1 (|\chi_0|^2 - |\vec{\chi}|^2) - B_2 |\chi_0|^2 \geq 0 \iff B_1 \geq 0, B_1 + B_2 \leq 0 \iff \Delta \geq d - 1.
\]

Notice that for \(\Delta = d - 1\) the spectral density of \(\partial_\mu Y^\mu\) vanishes. Thus a dimension \(d - 1\) vector primary is a conserved current, just like in \(d = 4\).

### C Energy-momentum tensor of the \(\xi\)-gauge Maxwell theory

Here we show that the energy-momentum tensor of the gauge fixed Maxwell theory (5.4) can be improved to be traceless, for \(\xi = \xi^*\). This gives an alternative proof of the conformal invariance at this particular value of the gauge fixing parameter.

For a general \(\xi\), the energy-momentum tensor and its trace are given by

\[
T_{\mu\nu} = \frac{\eta_{\mu\nu}}{4} (F_{\rho\sigma})^2 - F_\mu^\rho F_{\rho\nu} + \xi^{-1} \left( A_\mu \partial_\nu (\partial A) + A_\nu \partial_\mu (\partial A) - \eta_{\mu\nu} \left( A^\rho \partial_\rho (\partial A) + (\partial A)^2 / 2 \right) \right),
\]

\[
T^\mu_\mu = \left( \frac{d}{2} - 2 \right) (\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial_\nu A^\mu) + \xi^{-1} \left( (2 - d) A_\mu \partial^\rho \partial^\nu A^\nu - \frac{d}{2} (\partial A)^2 \right).
\]
The condition under which an improvement is possible was discussed in Section 3. In the present case, the trace of the energy-momentum tensor must take the form

\[ T^\mu_{\mu} = \alpha \partial^2 (A_\mu A^\mu) + \beta \partial_\mu \partial_\nu (A^\mu A^\nu), \]

up to the equations of motion:

\[ \partial^2 A_\mu + \frac{1 - \xi}{\xi} \partial_\mu (\partial A) = 0. \]

By collecting coefficients of four independent terms \((\partial A)^2, A_\mu \partial^\mu (\partial A), \partial_\nu A_\mu \partial^\nu A^\mu,\) and \(\partial_\nu A^\mu \partial_\mu A^\nu,\) we obtain

\[ \beta = - \frac{d}{2\xi}, \quad \frac{d}{2} - 2 = 2\alpha = -\beta, \quad \frac{2 - d}{\xi} = 2\alpha \left(1 - \frac{1}{\xi}\right) + 2\beta. \]

These equations have one and only solution \(\xi = \xi^*.\)

References

[1] J. Wess, “The Conformal Invariance in Quantum Field Theory”, Nuovo Cim. 18 (1960) 1086.
[2] S. R. Coleman and R. Jackiw, “Why dilatation generators do not generate dilatations?,” Annals Phys. 67 (1971) 552.
[3] V. Riva and J. L. Cardy, “Scale and conformal invariance in field theory: A physical counterexample,” Phys. Lett. B 622, (2005) 339, arXiv:hep-th/0504197.
[4] D. Dorigoni and V. S. Rychkov, “Scale Invariance + Unitarity \(\Rightarrow\) Conformal Invariance?,” arXiv:0910.1087 [hep-th].
[5] A. Iorio, L. O’Raifeartaigh, I. Sachs and C. Wiesendanger, “Weyl gauging and conformal invariance,” Nucl. Phys. B 495, 433 (1997) arXiv:hep-th/9607110.
[6] C. M. Ho and Y. Nakayama, “Dangerous Liouville Wave – exactly marginal but non-conformal deformation,” JHEP 0807, 109 (2008) arXiv:0804.3635 [hep-th].
[7] A. B. Zamolodchikov, “Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory,” JETP Lett. 43, (1986) 730 [Pisma Zh. Eksp. Teor. Fiz. 43, (1986) 565].
[8] J. Polchinski, “Scale And Conformal Invariance In Quantum Field Theory,” Nucl. Phys. B 303 (1988) 226.
[9] M. Lüscher and G. Mack, “The energy momentum tensor of a critical quantum field theory,” 1976, unpublished; G. Mack, “Introduction to conformal invariant quantum field theory in two or more dimensions,” in Nonperturbative Quantum Field Theory. Proceedings, NATO Advanced Study Institute, Cargese, France, July 16-30, 1987.

[10] F. A. Dolan, H. Osborn, “Conformal partial waves and the operator product expansion,” Nucl. Phys. B678, 491-507 (2004). arXiv:hep-th/0309180

[11] H. Osborn, A. C. Petkou, “Implications of conformal invariance in field theories for general dimensions,” Annals Phys. 231, 311-362 (1994). [hep-th/9307010].

[12] S. Ferrara, R. Gatto and A. F. Grillo, “Positivity Restrictions On Anomalous Dimensions,” Phys. Rev. D 9, 3564 (1974); G. Mack, “All Unitary Ray Representations Of The Conformal Group SU(2,2) With Positive Energy,” Commun. Math. Phys. 55, 1 (1977).

[13] J. L. Cardy, “Operator Content of Two-Dimensional Conformally Invariant Theories,” Nucl. Phys. B 270, 186-204 (1986).

[14] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, “N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” JHEP 0810, 091 (2008) arXiv:0806.1218 [hep-th].

[15] A. Gustavsson and S. J. Rey, “Enhanced N=8 Supersymmetry of ABJM Theory on R(8) and R(8)/Z(2),” arXiv:0906.3568 [hep-th].

[16] R.R. Metsaev, “Massless mixed symmetry bosonic free fields in d-dimensional anti-de Sitter space-time,” Phys. Lett B354, 78-84 (1995)

[17] S. Minwalla, “Restrictions imposed by superconformal invariance on quantum field theories,” Adv. Theor. Math. Phys. 2, 781-846 (1998). arXiv:hep-th/9712074.

[18] V. G. Kac, “Lie superalgebras,” Advances in Math. 26 (1977), no. 1, 8–96.

[19] A. Barducci, R. Casalbuoni, D. Dominici et al., “The IOSp(d,2/2) Symmetry From The BRST Quantization Of The Relativistic Spinning Particle,” Phys. Lett. B187, 135 (1987).

[20] N. D. Birrell, P. C. W. Davies, “Quantum Fields In Curved Space,” Cambridge, Uk: Univ. Pr. ( 1982) 340p.

[21] W. Siegel, “All Free Conformal Representations In All Dimensions,” Int. J. Mod. Phys. A4:2015, 1989.

[22] R.R. Metsaev, “All conformal invariant representations of d-dimensional anti-de Sitter group,” Mod. Phys. Lett A10, 1719-1731 (1995).
[23] Y. Nakayama, “No Forbidden Landscape in String/M-theory,” JHEP 1001, 030 (2010) arXiv:0909.4297; Y. Nakayama, “Higher derivative corrections in holographic Zamolodchikov-Polchinski theorem,” arXiv:1009.0491 [hep-th].

[24] S. Ferrara, R. Gatto, A. F. Grillo, “Conformal algebra in space-time and operator product expansion,” Springer Tracts Mod. Phys. 67, 1-64 (1973).

[25] G. Mack, A. Salam, “Finite component field representations of the conformal group,” Annals Phys. 53, 174-202 (1969).

[26] B. Grinstein, K. A. Intriligator and I. Z. Rothstein, “Comments on Unparticles,” Phys. Lett. B 662, 367 (2008) arXiv:0801.1140 [hep-ph].

[27] R. Jackiw and S. Y. Pi, “Tutorial on Scale and Conformal Symmetries in Diverse Dimensions,” arXiv:1101.4886 [math-ph].