Sporadic SICs and Exceptional Lie Algebras

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Abstract

Sometimes, mathematical oddities crowd in upon one another, and the exceptions to one classification scheme reveal themselves as fellow-travelers with the exceptions to a quite different taxonomy.

1 Preliminaries

A set of equiangular lines is a set of unit vectors in a \(d\)-dimensional vector space such that the magnitude of the inner product of any pair is constant:

\[
|⟨v_j, v_k⟩| = \begin{cases} 1, & j = k; \\ α, & j \neq k. \end{cases}
\] (1)

The maximum number of equiangular lines in a space of dimension \(d\) (the so-called Gerzon bound) is \(d(d+1)/2\) for real vector spaces and \(d^2\) for complex. In the real case, the Gerzon bound is only known to be attained in dimensions 2, 3, 7 and 23, and we know it can’t be attained in general. If you like peculiar alignments of mathematical topics, the appearance of 7 and 23 might make your ears prick up here. If you made the wild guess that the octonions and the Leech lattice are just around the corner... you’d be absolutely right. Meanwhile, the complex case is of interest for quantum information theory, because a set of \(d^2\) equiangular lines in \(\mathbb{C}^d\) specifies a measurement that can be performed upon a quantum-mechanical system. These measurements are highly symmetric, in that the lines which specify them are equiangular, and they are “informationally complete” in a sense that quantum theory makes precise. Thus, they are known as SICs [1–4]. Unlike the real case, where we can only attain the Gerzon bound in a few sparse instances, it appears that a SIC exists for each dimension \(d\), but nobody knows for sure yet.

Before SICs became a physics problem, constructions of \(d^2\) complex equiangular lines were known for dimensions \(d = 2, 3\) and 8. These arose from topics like higher-dimensional polytopes and generalizations thereof [5–8]. Now, we have exact solutions for SICs in the following dimensions [9–12]:

\[
d = 2–28, 30, 31, 35, 37–39, 42, 43, 48, 49, 52, 53, 57, 61–63, 67, 73, 74, 78, 79, 84, 91, 93, \\
95, 97–99, 103, 109, 111, 120, 124, 127, 129, 134, 143, 146, 147, 168, 172, 195, 199, \\
228, 259, 292, 323, 327, 399, 849, 844, 1299.
\] (2)

Moreover, numerical solutions to high precision are known for the following cases:

\[
d = 2–189, 191, 192, 204, 224, 255, 288, 528, 725, 1155, 2208.
\] (3)

These lists have grown irregularly in the years since the quantum-information community first recognized the significance of SICs. (Many entries are due to A. J. Scott and M. Grassl [3, 13, 14]. Other pioneers include M. Appleby, I. Bengtsson, T.-Y. Chien, S. T. Flammia, G. S. Kopp and S. Waldron.) It is fair to say that researchers feel that SICs should exist for all integers \(d \geq 2\), but we have no proof one way or the other. The attempts to resolve this question have extended into algebraic number theory [11, 15–18], an
intensely theoretical avenue of research with the surprisingly practical application of converting numerical solutions into exact ones [9]. For additional (extensive) discussion, we refer to the review article [4] and the textbooks [19,20].

In what follows, we will focus our attention mostly on the sporadic SICs, which comprise the SICs in dimensions 2 and 3, as well as one set of them in dimension 8 [21]. These SICs have been designated “sporadic” because they stand out in several ways, chiefly by residing outside the number-theoretic patterns observed for the rest of the known SICs [18]. After laying down some preliminaries, we will establish a connection between the sporadic SICs and the exceptional Lie algebras $E_6$, $E_7$ and $E_8$ by way of their root systems.

2 Quantum Measurements and Systems of Lines

A *positive-operator-valued measure* (POVM) is a set of “effects” (positive semidefinite operators satisfying $0 < E < I$) that furnish a resolution of the identity:

$$\sum_i E_i = \sum_i w_i \rho_i = I,$$

for some density operators $\{\rho_i\}$ and weights $\{w_i\}$. Note that taking the trace of both sides gives a normalization constraint for the weights in terms of the dimension of the Hilbert space. In this context, the Born Rule says that when we perform the measurement described by this POVM, we obtain the $i$-th outcome with probability

$$p(i) = \text{tr} (\rho E_i),$$

where $\rho$ without a subscript denotes our quantum state for the system. The weighting $w_i$ is, up to a constant, the probability we would assign to the $i$-th outcome if our state $\rho$ were the maximally mixed state $\frac{1}{d}I$, the “state of maximal ignorance.”

SICs are a special type of POVM. Given a set of $d^2$ equiangular unit vectors $\{|\pi_i\rangle\} \subset \mathbb{C}^d$, we can construct the operators which project onto them, and in turn we can rescale those projectors to form a set of effects:

$$E_i = \frac{1}{d} \Pi_i,$$

where $\Pi_i = |\pi_i\rangle\langle\pi_i|$. The equiangularity condition on the $\{|\pi_i\rangle\}$ turns out to imply that the $\{\Pi_i\}$ are linearly independent and thus they span the space of Hermitian operators on $\mathbb{C}^d$. Because the SIC projectors $\{\Pi_i\}$ form a basis for the space of Hermitian operators, we can express any quantum state $\rho$ in terms of its (Hilbert–Schmidt) inner products with them. But, by the Born Rule, the inner product $\text{tr} (\rho \Pi_i)$ is, apart from a factor $1/d$, just the probability of obtaining the $i$-th outcome of the SIC measurement $\{E_i\}$. The formula for reconstructing $\rho$ given these probabilities is quite simple, thanks to the symmetry of the projectors:

$$\rho = \sum_i \left[ (d+1)p(i) - \frac{1}{d} \right] \Pi_i,$$

where $p(i) = \text{tr} (\rho E_i)$ by the Born Rule. This furnishes us with a map from quantum state space into the probability simplex, a map that is one-to-one but not onto. In other words, we can fix a SIC as a “reference measurement” and then transform between density matrices and probability distributions without ambiguity, but the set of valid probability distributions for our reference measurement is a proper subset of the probability simplex.

We don’t need equiangularity for informational completeness, just that the $d^2$ operators which form the reference measurement are linearly independent and thus span the operator space. But equiangularity implies the linear independence of those operators, and it makes the formula for reconstructing $\rho$ from the overlaps particularly clean [22, 23].

Because we can treat quantum states as probability distributions, we can apply the concepts and methods of probability theory to them, including Shannon’s theory of information. The structures that I will discuss in the following sections came to my attention thanks to Shannon theory. In particular, the question of recurring interest is, “Out of all the extremal states of quantum state space — i.e., the ‘pure’ states $\rho = |\psi\rangle\langle\psi|$ —
which minimize the Shannon entropy of their probabilistic representation?" I will focus on the cases of dimensions 2, 3 and 8, where the so-called sporadic SICs occur. In these cases, the information-theoretic question of minimizing Shannon entropy leads to intricate geometrical structures.

Any time we have a vector in $\mathbb{R}^3$ of length 1 or less, we can map it to a $2 \times 2$ Hermitian matrix by the formula

$$ \rho = \frac{1}{2} \left( I + x\sigma_x + y\sigma_y + z\sigma_z \right), $$

where $(x, y, z)$ are the Cartesian components of the vector and $(\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. This yields a positive semidefinite matrix $\rho$ with trace equal to 1; when the vector has length 1, we have $\rho^2 = \rho$, and the density matrix is a rank-1 projector that can be written as $\rho = |\psi\rangle \langle \psi|$ for some vector $|\psi\rangle$.

Given any polyhedron of unit radius or less in $\mathbb{R}^3$, we can feed its vertices into the Bloch representation and obtain a set of density operators (which are pure states if they lie on the surface of the Bloch sphere). For a simple example, we can do a regular tetrahedron. Let $s$ and $s'$ take the values $\pm 1$, and define

$$ \rho_{s,s'} = \frac{1}{2} \left( I + \frac{1}{\sqrt{3}}(s\sigma_x + s'\sigma_y + ss'\sigma_z) \right). $$

To make these density matrices into a POVM, scale them down by the dimension. That is, take

$$ E_{s,s'} = \frac{1}{2} \rho_{s,s'}. $$

Then, the four operators $E_{s,s'}$ will sum to the identity. In fact, they comprise a SIC.

By introducing a sign change, we can define another SIC,

$$ \tilde{\rho}_{s,s'} = \frac{1}{2} \left( I + \frac{1}{\sqrt{3}}(s\sigma_x + s'\sigma_y - ss'\sigma_z) \right). $$

Each state in the original SIC is orthogonal to exactly one state in the second. In the Bloch sphere representation, orthogonal states correspond to antipodal points, so taking the four points that are antipodal to the vertices of our original tetrahedron forms a second tetrahedron. Together, the states of the two SICs form a cube inscribed in the Bloch sphere.

Here we have our first appearance of Shannon theory entering the story. With respect to the original SIC, the states $\{\tilde{\pi}_i\}$ of the antipodal SIC all minimize the Shannon entropy. The two interlocking tetrahedra are, entropically speaking, dual structures.

3 E8

In what follows, I will refer to H. S. M. Coxeter’s Regular Complex Polytopes [7]. Coxeter devotes a goodly portion of chapter 12 to the Hessian polyhedron, which lives in $\mathbb{C}^3$ and has 27 vertices. These 27 vertices lie on nine diameters in sets of three apiece. (In a real vector space, only two vertices of a convex polyhedron can lie on a diameter. But in a complex vector space, where a diameter is a complex line through the center of the polyhedron, we can have more [24].) He calls the polyhedron “Hessian” because its nine diameters and twelve planes of symmetry interlock in a particular way. Their incidences reproduce the Hesse configuration, a set of nine points on twelve lines such that four lines pass through each point and three points lie on each line.

Coxeter writes the 27 vertices of the Hessian polyhedron explicitly, in the following way. First, let $\omega$ be a cube root of unity, $\omega = e^{2\pi i/3}$. Then, construct the complex vectors

$$ (0, \omega^\mu, -\omega^\nu), (-\omega^\nu, 0, \omega^\mu), (\omega^\mu, -\omega^\nu, 0), $$

where $\mu$ and $\nu$ range over the values 0, 1 and 2. As Coxeter notes, we could just as well let $\mu$ and $\nu$ range over 1, 2 and 3. He prefers this latter choice, because it invites a nice notation: We can write the vectors above as

$$ 0\mu\nu, \nu0\mu, \mu\nu0. $$
For example,
\[ 230 = (\omega^2, -1, 0), \]  
and
\[ 103 = (-\omega, 0, 1). \]  
Coxeter then points out that this notation was first introduced by Beniamino Segre, “as a notation for the 27 lines on a general cubic surface in complex projective 3-space. In that notation, two of the lines intersect if their symbols agree in just one place, but two of the lines are skew if their symbols agree in two places or nowhere.” Consequently, the 27 vertices of the Hessian polyhedron correspond to the 27 lines on a cubic surface “in such a way that two of the lines are intersecting or skew according as the corresponding vertices are non-adjacent or adjacent.”

Every smooth cubic surface in the complex projective space \( \mathbb{CP}^3 \) has exactly 27 lines that can be drawn on it. For a special case, the Clebsch surface, we can actually get a real surface (that is, in \( \mathbb{RP}^3 \)) that we can mold in plaster and contemplate in the peaceful stillness of a good library. Intriguingly, the coefficients for the lines on the Clebsch surface live in the “golden field” \( \mathbb{Q}(\sqrt{5}) \), which we will meet again later in this article.

Casting the Hessian polyhedron into the real space \( \mathbb{R}^6 \), we obtain the polytope known as \( 2_{21} \), which is related to \( E_6 \), since the Coxeter group of \( 2_{21} \) is the Weyl group of \( E_6 \). The Weyl group of \( E_6 \) can also be thought of as the Galois group of the 27 lines on a cubic surface.

We make the connection to symmetric quantum measurements by following the trick that Coxeter uses in his Eq. (12.39). We transition from the space \( \mathbb{C}^3 \) to the complex projective plane by collecting the 27 vertices into equivalence classes, which we can write in homogeneous coordinates as follows:

\[
\begin{align*}
(0, 1, -1), & \quad (-1, 0, 1), \quad (1, -1, 0) \\
(0, 1, -\omega), & \quad (-\omega, 0, 1), \quad (1, -\omega, 0) \\
(0, 1, -\omega^2), & \quad (-\omega^2, 0, 1), \quad (1, -\omega^2, 0) 
\end{align*}
\]  

(16)

Let \( u \) and \( v \) be any two of these vectors. We find that

\[ |\langle u, u \rangle|^2 = 4 \]  
when the vectors coincide, and

\[ |\langle u, v \rangle|^2 = 1 \]  
when \( u \) and \( v \) are distinct. We can normalize these vectors to be quantum states on a three-dimensional Hilbert space by dividing each vector by \( \sqrt{2} \).

We have found a SIC for \( d = 3 \). When properly normalized, Coxeter’s vectors furnish a set of \( d^2 = 9 \) pure quantum states, such that the magnitude squared of the inner product between any two distinct states is \( 1/(d + 1) = 1/4 \).

Every known SIC has a group covariance property. Talking in terms of projectors, a SIC is a set of \( d^2 \) rank-1 projectors \( \{\Pi_j\} \) on a \( d \)-dimensional Hilbert space that satisfy the Hilbert–Schmidt inner product condition

\[ \text{tr} (\Pi_j \Pi_k) = \frac{d\delta_{jk} + 1}{d+1}. \]  

(19)

These form a POVM if we rescale them by \( 1/d \). In every known case, we can compute all the projectors \( \{\Pi_j\} \) by starting with one projector, call it \( \Pi_0 \), and then taking the orbit of \( \Pi_0 \) under the action of some group. The projector \( \Pi_0 \) is known as the fiducial state. (I don’t know who picked the word “fiducial”; I think it was something Carl Caves decided on, way back.)

In all known cases but one, the group is the Weyl–Heisenberg group in dimension \( d \). To define this group, fix an orthonormal basis \( \{\{|n\rangle\}\} \) and define the operators \( X \) and \( Z \) such that

\[ X|n\rangle = |n + 1\rangle, \]  
interpreting addition modulo \( d \), and

\[ Z|n\rangle = e^{2\pi in/d}|n\rangle. \]  

(21)
The Weyl–Heisenberg displacement operators are

\[ D_{l\alpha} = (e^{i\pi/d})^{l\alpha} X^l Z^\alpha. \]  

(22)

Because the product of two displacement operators is another displacement operator, up to a phase factor, we can make them into a group by inventing group elements that are displacement operators multiplied by phase factors. This group has Weyl's name attached to it, because he invented \( X \) and \( Z \) back in 1925, while trying to figure out what the analogue of the canonical commutation relation would be for quantum mechanics on finite-dimensional Hilbert spaces \([25–27]\). It is also called the generalized Pauli group, because \( X \) and \( Z \) generalize the Pauli matrices \( \sigma_x \) and \( \sigma_z \) to higher dimensions (at the expense of no longer being Hermitian).

To relate this with the Coxeter construction we discussed earlier, turn the first of Coxeter’s vectors into a column vector:

\[
\begin{pmatrix}
0
1
-1
\end{pmatrix}
\]  

(23)

Apply the \( X \) operator twice in succession to get the other two vectors in Coxeter’s table (converted to column-vector format). Then, apply \( Z \) twice in succession to recover the right-hand column of Coxeter’s table. Finally, apply \( X \) to these vectors again to effect cyclic shifts and fill out the table. This set of nine states is known as the Hesse SIC.

Each of the 27 lines corresponds to a weight in the minimal representation of \( E_6 \) \([28]\), and so each element in the Hesse SIC corresponds to three weights of \( E_6 \).

In dimension \( d = 3 \), we encounter a veritable cat’s cradle of vectors \([29]\). First, there’s the Hesse SIC. Like all informationally complete POVMs, it defines a probabilistic representation of quantum state space, in this case mapping from \( 3 \times 3 \) density matrices to the probability simplex for 9-outcome experiments. As suggested earlier, we can look for the pure states whose probabilistic representations minimize the Shannon entropy. The result is a set of twelve states, which sort themselves into four orthonormal bases of three states apiece. What’s more, these bases are mutually unbiased: The Hilbert–Schmidt inner product of a state from one basis with any state from another is always constant. In a sense, the Hesse SIC has a “dual” structure, and that dual is a set of Mutually Unbiased Bases (MUB). This duality relation is rather intricate: Each of the 9 SIC states is orthogonal to exactly 4 of the MUB states, and each of the MUB states is orthogonal to exactly 3 SIC states \([29]\).

An easy way to remember these relationships is to consider the finite affine plane on nine points. This configuration is also known as the discrete affine plane on nine points, and as the Steiner triple system of order 3. That’s a lot of different names for something which is pretty easy to put together! To construct it, first draw a \( 3 \times 3 \) grid of points, and label them consecutively:

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{array}
\]  

(24)

These will be the points of our discrete geometry. To obtain the lines, we read along the horizontals, the verticals and the leftward and rightward diagonals:

\[
\begin{array}{ccc}
(123) & (456) & (789) \\
(147) & (258) & (369) \\
(159) & (267) & (348) \\
(168) & (249) & (357) \\
\end{array}
\]  

(25)

Each point lies on four lines, and every two lines intersect in exactly one point. For our purposes today, each of the points corresponds to a SIC vector, and each of the lines corresponds to a MUB vector, with point-line incidence implying orthogonality. The four bases are the four ways of carving up the plane into parallel lines (horizontals, verticals, diagonals and other diagonals).

To construct a MUB vector, pick one of the 12 lines we constructed above, and insert zeroes into those slots of a 9-entry probability distribution, filling in the rest uniformly. For example, picking the line (123),
we construct the probability distribution
\[
\left(0, 0, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right).
\]
(26)

This represents a pure quantum state that is orthogonal to the quantum state
\[
\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, 0, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)
\]
and to
\[
\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, 0, 0, \frac{1}{6}, \frac{1}{6}\right),
\]
while all three of these have the same Hilbert–Schmidt inner product with the quantum state represented by
\[
\left(0, \frac{1}{6}, \frac{1}{6}, 0, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right),
\]
for example.

Considering all the lines in the original structure that are orthogonal to a given line in the dual yields a maximal set of real equiangular lines in one fewer dimensions. (Oddly, I noticed this happening up in dimension 8 before I thought to check in dimension 3 [30], but we’ll get to that soon.) To visualize the step from $\mathbb{C}^3$ to $\mathbb{R}^2$, we can use the Bloch sphere representation for two-dimensional quantum state space. Pick a state in the dual structure, i.e., one of the twelve MUB vectors. All the SIC vectors that are orthogonal to it must crowd into a 2-dimensional subspace. In other words, they all fit into a qubit-sized state space, and we can draw them on the Bloch sphere. When we do so, they are coplanar and lie at equal intervals around a great circle, a configuration sometimes called a trine [31]. This configuration is a maximal equiangular set of lines in the plane $\mathbb{R}^2$.

What happens if, starting with the Hesse SIC, you instead consider all the lines in the dual structure that are orthogonal to a given vector in the original? This yields a SIC in dimension 2. I don’t know where in the literature that is written, but it feels like something Coxeter would have known.

Another path from the sporadic SICs to $E_8$ starts with the qubit SICs, i.e., regular tetrahedra inscribed in the Bloch sphere. Shrinking a tetrahedron, pulling its vertices closer to the origin, yields a type of quantum measurement (sometimes designated a SIM [32]) that has more intrinsic noise. Apparently, $E_8$ is part of the story of what happens when the noise level becomes maximal and the four outcomes of the measurement merge into a single degenerate case. This corresponds to a singularity in the space of all rotated and scaled tetrahedra centered at the origin. Resolving this singularity turns out to involve the Dynkin diagram of $E_8$: We invent a smooth manifold that maps to the space of tetrahedra, by a mapping that is one-to-one and onto everywhere except the origin. The pre-image of the origin in this smooth manifold is a set of six spheres, and two spheres intersect if and only if the corresponding vertices in the Dynkin diagram are connected [33].

4 $E_8$

On the fourteenth of March, 2016, Maryna Viazovska published a proof that the $E_8$ lattice is the best way to pack hyperspheres in eight dimensions [34]. I celebrated the third anniversary of this event by writing a guest post at the $n$-Category Café, explaining how this relates to another packing problem that seems quite different: how to fit as many equiangular lines as possible into the complex space $\mathbb{C}^8$. The answer to this puzzle is another example of a SIC.

In the previous section, we saw how to build SICs by starting with a fiducial vector and taking the orbit of that vector under the action of a group, turning one line into $d^2$. We said that the Weyl–Heisenberg group was the group we use in all cases but one. Now, we take on that exception. It will lead us to the exceptional root systems $E_7$ and $E_8$. Actually, it will be a bit easier to tackle the latter first. Whence $E_8$ in the world of SICs?
Table 1: Four of the states from the \(\{\Pi^-\}\) Hoggar-type SIC, written in the probabilistic representation of three-qubit state space provided by the \(\{\Pi^+\}\) SIC. Up to an overall normalization by 1/36, these states are all binary sequences, i.e., they are uniform over their supports.

We saw how to generate the Hesse SIC by taking the orbit of a fiducial state under the action of the \(d = 3\) Weyl–Heisenberg group. Next, we will do something similar in \(d = 8\). We start by defining the two states

\[ |\psi^+_0\rangle \propto (-1 \pm 2i, 1, 1, 1, 1, 1, 1, 1)^T. \]  

(30)

Here, we are taking the transpose to make our states column vectors, and we are leaving out the dull part, in which we normalize the states to satisfy

\[ \langle \psi^+_0 | \psi^+_0 \rangle = \langle \psi^-_0 | \psi^-_0 \rangle = 1. \]  

(31)

First, we focus on \( |\psi^+_0\rangle \). To create a SIC from the fiducial vector \( |\psi^+_0\rangle \), we take the set of Pauli matrices, including the identity as an honorary member: \(\{I, \sigma_x, \sigma_y, \sigma_z\}\). We turn this set of four elements into a set of sixty-four elements by taking all tensor products of three elements. This creates the Pauli operators on three qubits. By computing the orbit of \( |\psi^+_0\rangle \) under multiplication (equivalently, the orbit of \( \Pi^+_0 = |\psi^+_0\rangle \langle \psi^+_0| \) under conjugation), we find a set of 64 states that together form a SIC set.

The same construction works for the other choice of sign, \( |\psi^-_0\rangle \), creating another SIC with the same symmetry group. We can call both of them SICs of Hoggar type, in honor of Stuart Hoggar.

To make this connection, we consider the stabilizer of the fiducial vector, i.e., the group of unitaries that map the SIC set to itself, leaving the fiducial where it is and permuting the other \(d^2 - 1\) vectors. Huangjun Zhu observed that the stabilizer of any fiducial for a Hoggar-type SIC is isomorphic to the group of 3 \(\times\) 3 unitary matrices over the finite field of order 9 [35,36]. This group is sometimes written \(U_3(3)\) or \(PSU(3,3)\).

In turn, this group is up to a factor \(Z_2\) isomorphic to \(G_2(2)\), the automorphism group of the Cayley integers, a subset of the octonions also known as the octavians [37]. Up to an overall scaling, the lattice of octavians is also the lattice known as \(E_8\).

The octavian lattice contains a great deal of arithmetic structure. Of particular note is that it contains 240 elements of norm 1. In addition to the familiar +1 and −1, which have order 1 and 2 respectively, there are 56 units of order 3, 56 units of order 6 and 126 units of order 4. The odd-order units generate subrings of the octavians that are isomorphic to the Eisenstein integers and the Hurwitz integers, lattices in the complex numbers and the quaternions [37]. From the symmetries of these lattices, we can in fact read off the stabilizer groups for fiducials of the qubit and Hesse SICs [21]. It is as if the sporadic SICs are drawing their strength from the octonions.

Before moving on, we pause to note how peculiar it is that by trying to find a nice packing of complex unit vectors, we ended up talking about an optimal packing of Euclidean hyperspheres [34].

Now that we’ve met \(E_8\), it’s time to visit the root system we skipped: Where does \(E_7\) fit in?

5 \(E_7\)

With respect to the probabilistic representation furnished by the \(\Pi^+_0\) SIC, the states of the \(\Pi^-_0\) SIC minimize the Shannon entropy, and vice versa [30,38].

Recall that when we invented SICs for a single qubit, they were tetrahedra in the Bloch ball, and we could fit together two tetrahedral SICs such that each vector in one SIC was orthogonal (in the Bloch picture, antipodal) to exactly one vector in the other. The two Hoggar-type SICs made from the fiducial states \(\Pi^+_0\) and \(\Pi^-_0\) satisfy the grown-up version of this relation: Each state in one is orthogonal to exactly twenty-eight states of the other.
We can understand these orthogonalities as corresponding to the antisymmetric elements of the three-qubit Pauli group. It is simplest to see why when we look for those elements of the $\Pi^+_0$ SIC that are orthogonal to the projector $\Pi^+_0$. These satisfy
\[
\text{tr} (\Pi^+_0 D \Pi^+_0 D^\dagger) = 0
\]
for some operator $D$ that is the tensor product of three Pauli matrices. For which such tensor-product operators will this expression vanish? Intuitively speaking, the product $\Pi^+_0 \Pi^+_0$ is a symmetric matrix, so if we want the trace to vanish, we ought to try introducing an asymmetry, but if we introduce too much, it will cancel out, on the "minus times a minus is a plus" principle. Recall the Pauli matrices:
\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
(32)

Note that of these three matrices, only $\sigma_y$ is antisymmetric, and also note that we have
\[
\sigma_z \sigma_x = -\sigma_x \sigma_z = i\sigma_y.
\]
(33)

This much is familiar, though that minus sign gets around. For example, it is the fuel that makes the GHZ thought-experiment go [39, 40], because it means that
\[
(\sigma_z \sigma_x \sigma_y) (\sigma_x \sigma_z \sigma_y) (\sigma_y \sigma_z \sigma_x) (\sigma_z \sigma_y \sigma_x).
\]
(34)

Let’s consider the finite-dimensional Hilbert space made by composing three qubits. This state space is eight-dimensional, and we build the *three-qubit Pauli group* by taking tensor products of the Pauli matrices, considering the $2 \times 2$ identity matrix to be the zeroth Pauli operator. There are 64 matrices in the three-qubit Pauli group, and we can label them by six bits. The notation
\[
\begin{pmatrix} m_1 & m_3 & m_5 \\ m_2 & m_4 & m_6 \end{pmatrix}
\]
(35)

means to take the tensor product
\[
\sigma_x^{m_1} \sigma_z^{m_2} \sigma_z^{m_3} \sigma_y^{m_3} \sigma_z^{m_4} \sigma_x^{m_4} \sigma_y^{m_5} \sigma_z^{m_6}.
\]
(36)

Now, we ask: Of these 64 matrices, how many are symmetric and how many are antisymmetric? We can only get antisymmetry from $\sigma_y$, and (speaking heuristically) if we include too much antisymmetry, it will cancel out. More carefully put: We need an odd number of factors of $\sigma_y$ in the tensor product to have the result be an antisymmetric matrix. Otherwise, it will come out symmetric. Consider the case where the first factor in the triple tensor product is $\sigma_y$. Then we have $(4 - 1)^2 = 9$ possibilities for the other two slots. The same holds true if we put the $\sigma_y$ in the second or the third position. Finally, $\sigma_y \otimes \sigma_y \otimes \sigma_y$ is antisymmetric, meaning that we have $9 \cdot 3 + 1 = 28$ antisymmetric matrices in the three-qubit Pauli group. In the notation established above, they are the elements for which
\[
m_1 m_2 + m_3 m_4 + m_5 m_6 = 1 \mod 2.
\]
(37)

Moreover, these 28 antisymmetric matrices correspond exactly to the 28 bitangents of a quartic curve, and to pairs of opposite vertices of the Gosset polytope $3_{21}$. In order to make this connection, we need to dig into the octonions.

To recap: Each of the 64 vectors (or, equivalently, projectors) in the Hoggar SIC is naturally labeled by a displacement operator, which up to an overall phase is the tensor product of three Pauli operators. Recall that we can write the Pauli operator $\sigma_y$ as the product of $\sigma_x$ and $\sigma_z$, up to a phase. Therefore, we can label each Hoggar-SIC vector by a pair of binary strings, each three bits in length. The bits indicate the power to which we raise the $\sigma_x$ and $\sigma_z$ generators on the respective qubits. The pair $(010, 101)$, for example, means that on the three qubits, we act with $\sigma_x$ on the second, and we act with $\sigma_z$ on the first and third. Likewise, $(000, 111)$ stands for the displacement operator which has a factor of $\sigma_z$ on each qubit and no factors of $\sigma_x$ at all.
There is a natural mapping from pairs of this form to pairs of unit octonions. Simply turn each triplet of bits into an integer and pick the corresponding unit from the set \( \{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\} \), where each of the \( e_j \) square to \(-1\).

We can choose the labeling of the unit imaginary octonions so that the following nice property holds. Up to a sign, the product of two imaginary unit octonions is a third, whose index is the \text{xor} of the indices of the units being multiplied. For example, in binary, \( 1 = 001 \) and \( 4 = 100 \); the \text{xor} of these is \( 101 = 5 \), and \( e_1 \) times \( e_4 \) is \( e_5 \).

Translate the Cayley–Graves table here into binary if enlightenment has not yet struck:

\[
\begin{array}{c|cccccccc}
  e_i e_j & 1 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
  \hline
  1 & 1 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
  e_1 & e_1 & -1 & e_3 & -e_2 & e_5 & -e_4 & -e_7 & e_6 \\
  e_2 & e_2 & -e_3 & -1 & e_1 & e_6 & e_7 & -e_4 & -e_5 \\
  e_3 & e_3 & e_2 & -e_1 & -1 & e_7 & -e_6 & e_5 & -e_4 \\
  e_4 & e_4 & -e_5 & -e_6 & -e_7 & -1 & e_1 & e_2 & e_3 \\
  e_5 & e_5 & e_4 & -e_7 & e_6 & -e_1 & -1 & -e_3 & e_2 \\
  e_6 & e_6 & e_7 & e_4 & -e_5 & -e_2 & e_3 & -1 & -e_1 \\
  e_7 & e_7 & -e_6 & e_5 & e_4 & -e_3 & -e_2 & e_1 & -1 \\
\end{array}
\] (39)

So, each projector in the Hoggar SIC is labeled by a pair of octonions, and the group structure of the displacement operators is, almost, octonion multiplication. There are sign factors all over the place, but for this purpose, we can neglect them. They will crop up again soon, in a rather pretty way.

Another way to express the Cayley–Graves multiplication table is with the Fano plane, a set of seven points grouped into seven lines that has been called “the combinatorialist’s coat of arms”. We can label the seven points with the imaginary octonions \( e_1 \) through \( e_7 \). When drawn on the page, a useful presentation of the Fano plane has the point \( e_4 \) in the middle and, reading clockwise, the points \( e_1, e_7, e_2, e_5, e_3 \) and \( e_6 \) around it in a regular triangle. The three sides and three altitudes of this triangle, along with the inscribed circle, provide the seven lines: \( (e_1, e_2, e_3), (e_1, e_4, e_5), (e_1, e_7, e_6), (e_2, e_4, e_6), (e_2, e_5, e_7), (e_3, e_4, e_7), (e_3, e_6, e_5) \). It is apparent that each line contains three points, and it is easy to check that each point lies within three distinct lines, and that each pair of lines intersect at a single point. One consequence of this is
that if we take the incidence matrix of the Fano plane, writing a 1 in the $ij$-th entry if line $i$ contains point $j$, then every two rows of the matrix have exactly the same overlap:

$$M = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 
\end{pmatrix}.$$ \hspace{1cm} (40)

The rows of the incidence matrix furnish us with seven equiangular lines in $\mathbb{R}^7$. We can build upon this by considering the signs in the Cayley–Graves multiplication table, which we can represent by adding orientations to the lines of the Fano plane. Start by taking the first row of the incidence matrix $M$, which corresponds to the line $(e_1, e_2, e_3)$, and give it all possible choices of sign by multiplying by the elements not on that line. Multiplying by $e_4, e_5, e_6$ and $e_7$ respectively, we get

$$\begin{pmatrix}
+ & + & + & 0 & 0 & 0 & 0 \\
- & + & - & 0 & 0 & 0 & 0 \\
- & - & + & 0 & 0 & 0 & 0 \\
+ & - & - & 0 & 0 & 0 & 0 
\end{pmatrix}.$$ \hspace{1cm} (41)

The sign we record here is simply the sign we find in the corresponding entry of the Cayley–Graves table. Doing this with all seven lines of the Fano plane, we obtain a set of 28 vectors, each one given by a choice of a line and a point not on that line. Moreover, all of these vectors are equiangular. This is easily checked: For any two vectors derived from the same Fano line, two of the terms in the inner product will cancel, leaving an overlap of magnitude 1. And for any two vectors derived from different Fano lines, the overlap always has magnitude 1 because each pair of lines always meets at exactly one point. Van Lint and Seidel noted that the incidence matrix of the Fano plane could be augmented into a full set of 28 equiangular lines [41, 42], but to my knowledge, extracting the necessary choices of sign from octonion multiplication is not reported in the literature.

So, the 28 lines in a maximal equiangular set in $\mathbb{R}^7$ correspond to point-line pairs in the Fano plane, where the point and the line are not coincident. In discrete geometry, the combination of a line and a point not on that line is known as an anti-flag. It is straightforward to show from Eq. (38) that the antisymmetric matrices in the three-qubit Pauli group also correspond to the anti-flags of the Fano plane: Simply take the powers of the $\sigma_x$ operators to specify a point and the powers of the $\sigma_z$ operators to specify a line [30].

Two fun things have happened here: First, we started with complex equiangular lines. By carefully considering the orthogonalities between two sets of complex equiangular lines, we arrived at a maximal set of real equiangular lines in $\mathbb{R}^7$. And since one cannot actually fit more equiangular lines into $\mathbb{R}^8$ than into $\mathbb{R}^7$, we have a connection between a maximal set of equiangular lines in $\mathbb{C}^8$ and a maximal set of them in $\mathbb{R}^8$.

Second, our equiangular lines in $\mathbb{R}^7$ are the diameters of the Gosset polytope $3_{21}$. And because we have made our way to the polytope $3_{21}$, we have arrived at $E_7$. To quote a fascinating paper by Manivel [28],

Gosset seems to have been the first, at the very beginning of the 20th century, to understand that the lines on the cubic surface can be interpreted as the vertices of a polytope, whose symmetry group is precisely the automorphism group of the configuration. Coxeter extended this observation to the 28 bitangents, and Todd to the 120 tritangent planes. Du Val and Coxeter provided systematic ways to construct the polytopes, which are denoted $n_{21}$ for $n = 2, 3, 4$ and live in $n + 4$ dimensions. They have the characteristic property of being semiregular, which means that the automorphism group acts transitively on the vertices, and the faces are regular polytopes. In terms of Lie theory they are best understood as the polytopes in the weight lattices of the exceptional simple Lie algebras $\mathfrak{e}_{n+4}$, whose vertices are the weights of the minimal representations.

When we studied the Hesse SIC, we met the case $n = 2$ and $\mathfrak{e}_6$. The intricate orthogonalities between two conjugate SICs of Hoggar type have led us to the case $n = 3$ and $\mathfrak{e}_7$. 

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\textbf{Line} & \textbf{Point} & \textbf{Overlap} \\
\hline
$(e_1, e_2, e_3)$ & & \\
$(e_4, e_5, e_6)$ & & \\
$(e_7)$ & & \\
\hline
\end{tabular}
\caption{Equiangular Lines in $\mathbb{R}^7$}
\end{table}
6 The Regular Icosahedron and Real-Vector-Space Quantum Theory

In the previous sections, we uncovered correspondences between equiangular lines in $\mathbb{C}^3$ and $\mathbb{R}^2$, and between $\mathbb{C}^8$ and $\mathbb{R}^7$. It would be nice to have a connection like that between $\mathbb{C}^4$ and $\mathbb{R}^3$, but I have not found one yet. Instead, there is a slightly different relationship that brings $\mathbb{R}^3$ into the picture.

Suppose that, unaccountably, we wished to build the Hesse SIC, but in real vector space. What might this even mean? It would entail finding a fiducial vector and an appropriate group, closely analogous to the qutrit Weyl–Heisenberg group, such that the orbit of said fiducial is a maximal set of equiangular lines. How big would such a set of lines be? Recall that the Gerzon bound is $d^2$ for $\mathbb{C}^d$, but only $d(d + 1)/2$ in $\mathbb{R}^d$. In both cases, this is essentially because those values are the dimensions of the appropriate operator spaces (symmetric for operators on $\mathbb{R}^d$, self-adjoint for operators on $\mathbb{C}^d$). It is not difficult to show that, if the Gerzon bound is attained, the magnitude of the inner product between the vectors is $1/\sqrt{d+1}$ in $\mathbb{C}^d$ and $1/\sqrt{d+2}$ in $\mathbb{R}^d$.

We are familiar with the complex case, in which we define a shift operator $X$ and a phase operator $Z$ that both have order $d$. A cyclic shift is nice and simple, so we’d like to keep that idea, but the only “phase” we have to work with is the choice of positive or negative sign. So, let us consider the operators

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ (42)

The shift operator $X$ still satisfies $X^3 = I$, while for the phase operator $Z$, we now have $Z^2 = I$.

What group can we make from these operators? Note that

$$(ZX)^3 = -I,$$ (43)

and so

$$(-Z)^2 = X^3 = (-ZX)^3 = I,$$ (44)

meaning that the operators $X$ and $-Z$ generate the tetrahedral group, so designated because it is isomorphic to the rotational symmetry group of a regular tetrahedron. Equivalently, we can use $Z$ as a generator, since $-Z = (ZX)^3Z$ by the above.

Now, we want to take the orbit of a vector under this group! But what vector? It should not be an eigenvector of $X$ or of $Z$, for then we know we could never get a full set. Therefore, we don’t want a flat vector, nor do we want any of the basis vectors, so we go for the next simplest thing:

$$v = \begin{pmatrix} 0 \\ 1 \\ y \end{pmatrix},$$ (45)

where $y$ is a real number. We now have

$$Zv = \begin{pmatrix} 0 \\ -1 \\ y \end{pmatrix}$$ (46)

and also

$$X^2v = \begin{pmatrix} 1 \\ y \\ 0 \end{pmatrix},$$ (47)

so if we want equality between the inner products,

$$\langle Zv, v \rangle = \langle X^2v, v \rangle,$$ (48)

then we need to have

$$-1 + y^2 = y.$$ (49)
The positive solution to this quadratic equation is

\[ y = \frac{1 + \sqrt{5}}{2}, \]  

so we can in fact take our \( y \) to be \( \phi \), the golden ratio.

In the group we defined above, \( X \) performs cyclic shifts, \( Z \) changes the relative phase of the components, and we have the freedom to flip all the signs. Therefore, the orbit of the fiducial \( v \) is the set of twelve vectors

\[
\begin{pmatrix}
0 \\
\pm 1 \\
\pm \phi
\end{pmatrix},
\begin{pmatrix}
\pm 1 \\
\pm \phi \\
0
\end{pmatrix},
\begin{pmatrix}
\pm \phi \\
0 \\
\pm 1
\end{pmatrix}.
\]

These are the vertices of a regular icosahedron, and the diagonals of that icosahedron are six equiangular lines. The inner products between these vectors are always \( \pm \phi \). Since \( d(d + 1)/2 = 6 \), there cannot be any larger set of equiangular lines in \( \mathbb{R}^3 \).

Recall that the reciprocal of the golden ratio \( \phi \) is

\[
\phi^{-1} = \frac{2}{1 + \sqrt{5}} \frac{1 - \sqrt{5}}{1 - \sqrt{5}} = \frac{2 - 2\sqrt{5}}{1 - 5} = \frac{-1 + \sqrt{5}}{2} = \phi - 1. \tag{52}
\]

The golden ratio \( \phi \) is a root of the monic polynomial \( y^2 - y - 1 \), and being a root of a monic polynomial with integer coefficients, it is consequently an algebraic integer. The same holds for its reciprocal, so \( \phi^{-1} \) is also an algebraic integer, making the two of them units in the number field \( \mathbb{Q}(\sqrt{5}) \).

To summarize: For the diagonals of the regular icosahedron, the vector components are given by the units of the “golden field” \( \mathbb{Q}(\sqrt{5}) \). But it has been discovered [17] that the vector components for the Weyl–Heisenberg SICs in dimension four are derived from a unit in the ray class field over \( \mathbb{Q}(\sqrt{5}) \). Therefore, in a suitably perplexing way, the icosahedron is the Euclidean version of the Hesse SIC, and the SICs in \( d = 4 \) are the number-theoretic extension of the icosahedron.

Futhermore, it has been observed empirically that in dimensions

\[ d_k = \phi^{2k} + \phi^{-2k} + 1, \tag{53} \]

there exist Weyl–Heisenberg SICs with additional group-theoretic properties that make their exact expressions easier to find. These are known as Fibonacci–Lucas SICs [14].

There are exactly four known cases where the Gerzon bound can be attained in \( \mathbb{R}^d \): when \( d = 2, 3, 7 \) and 23. Three out of these four examples relate to SICs, specifically to the sporadic SICs. We can obtain the maximal equiangular sets in \( \mathbb{R}^2 \) and \( \mathbb{R}^7 \) from SICs in \( \mathbb{C}^3 \) and \( \mathbb{C}^8 \) respectively, while the set in \( \mathbb{R}^3 \) turns out to be the real analogue of our example in \( \mathbb{C}^3 \). All of this raises a natural question: What about \( \mathbb{R}^{23} \)? Does the equiangular set there descend from a SIC in \( \mathbb{C}^{24} \)? That, nobody knows.

We do know that the maximal equiangular line set in \( \mathbb{R}^{23} \) can be extracted from the Leech lattice [43]. It contains 276 lines, and its automorphism group is Conway’s group \( \text{Co}_3 \) [44]. Further study of this structure connects back with our use of SICs to give a probabilistic representation of quantum state space. When we fix a SIC in \( \mathbb{C}^d \) as a reference measurement, the condition \( \text{tr} \rho^2 = 1 \), which is satisfied when \( \rho \) is a pure state, becomes

\[ \sum_i p(i)^2 = \frac{2}{d(d + 1)}. \tag{54} \]

Now, flipping this equation upside down makes both sides look like counting! The right-hand side becomes combinatorics: It’s just the binomial coefficient for choosing two things out of \( d + 1 \). Meanwhile, the left-hand side becomes the effective number of outcomes, which we are familiar with because it is a biodiversity index [45]:

\[ N_{\text{eff}} = \left( \sum_i p(i)^2 \right)^{-1}. \tag{55} \]
So, when we ascribe a pure state to a quantum system of Hilbert-space dimensionality $d$, we are saying that the effective number of possible outcomes for a reference measurement is

$$N_{\text{eff}} = \binom{d+1}{2}. \quad (56)$$

Consequently, ascribing a pure state means that we are effectively ruling out a number of outcomes equal to

$$d^2 - N_{\text{eff}} = \frac{d(d-1)}{2} = \binom{d}{2}. \quad (57)$$

This motivates the following question: What is the upper bound on the number of entries in $\vec{p}$ that can equal zero? A brief calculation with the Cauchy–Schwarz inequality [29] reveals that the answer is, in fact, exactly $d^2 - N_{\text{eff}}$. In other words, no vector $\vec{p}$ in the image of quantum state space can contain more than $d(d-1)/2$ zeros. One reason why the sporadic SICs are distinguished from all the others is that they provide the examples where this bound is known to be saturated. We can attain it for qubit SICs (where it equals 1), the Hesse SIC (where it equals 3) and the Hoggar-type SICs (where it equals 28). The states which saturate this bound are also those that minimize the Shannon entropy, as we discussed above.

We can also deduce the corresponding bound for the case of real equiangular lines. In $\mathbb{R}^d$, the Gerzon bound is $d(d+1)/2$, and in those cases where we have a complete set of equiangular lines, we can play the game of doing “real-vector-space quantum mechanics”, using our $d(d+1)/2$ equiangular lines to define a reference measurement. As in $\mathbb{C}^d$, the Cauchy–Schwarz inequality gives an upper bound on the number of zero-valued entries in a probability distribution $p$, which works out to be

$$N_Z = \frac{d^2 - 1}{3}. \quad (58)$$

Thinking about it for a moment, we realize that this is telling us about the maximum number of equiangular lines that can all simultaneously be orthogonal to a common vector. In particular, if we fix $d = 23$, we have a “real-vector-space SIC” that we can derive from the Leech lattice, and we know that

$$N_Z = \frac{23^2 - 1}{3} = 176 \quad (59)$$

of the elements of that set can be orthogonal to a “pure quantum state”, i.e., a vector in $\mathbb{R}^{23}$. All 176 such lines have to crowd together into a 22-dimensional subspace, while still being equiangular. They comprise a maximal set of equiangular lines in $\mathbb{R}^{22}$, whose symmetries form the Higman–Sims finite simple group [46].

And that’s what the classification of finite simple groups has to do with biodiversity!

7 Open Puzzles Concerning Exceptional Objects

While we’re thinking about equiangular lines in spaces other than $\mathbb{C}^d$, here is a puzzle: What about the octonionic space $\mathbb{O}^3$, which figures largely in the study of exceptional objects? The Gerzon bound for this space works out to be 27. Cohn, Kumar and Minton give a nonconstructive proof that a set saturating the Gerzon bound in $\mathbb{O}^3$ exists, along with a numerical solution [47], but that numerical solution doesn’t look like an approximation of a really pretty exact solution in any obvious way. (Their set of mutually unbiased bases in $\mathbb{O}^4$ does look like a generalization of a familiar set thereof in $\mathbb{C}^3$, which might raise our hopes.) Both in $\mathbb{R}^3$ and in $\mathbb{C}^3$, we can construct a maximal set of equiangular lines by starting with a fairly nice fiducial vector and applying a straightforward set of transformations. Is the analogous statement true in $\mathbb{O}^3$?

An equiangular set of 27 lines would provide a map from the set of density matrices for an “octonionic qutrit” to the probability simplex in $\mathbb{R}^{27}$, yielding a convex body that would be a higher-dimensional analogue of the Bloch ball. The extreme points of this Bloch body, the images of the “pure states”, might form an interesting variety.

The 27 we have quoted here is related to a 27 that we encountered above. The algebra of self-adjoint operators on $\mathbb{O}^3$ — the “observables” for an octonionic qutrit — is known as the octonionic Albert algebra, and it is 27-dimensional. The group of linear isomorphisms of $\mathbb{O}^3$ that preserve the determinant in the
octonionic Albert algebra is a noncompact real form of $E_6$ [48]. As we saw earlier, the weights of the minimal representation of the Lie algebra $e_6$ yield the polytope $2_{21}$, from which we can derive the Hesse SIC in $\mathbb{C}^3$. An exact solution for an “octonionic qutrit SIC” might close this circuit of ideas.

It is possible to fit the Leech lattice into the traceless part of the octonionic Albert algebra [49]. This means that each point in the Leech lattice is a “Hamiltonian” for a three-level octonionic quantum system. This sounds a bit like a toy version of a vertex operator algebra construction.

Having reached a point where the tone has taken a rather speculative turn, we now embrace that attitude, just for the fun of it.

Another “28” that appears in the study of exceptional or unusual mathematical objects is the size of the 28-element Dedekind lattice. This is a lattice in the sense of order theory, a partially ordered set with the property that we can trace subsets of elements upward through the ordering to where they join and downward to where they meet. It is the free modular lattice on three generators with the top and bottom elements removed, and Dedekind showed how to construct it as a sublattice within the lattice of subspaces of $\mathbb{R}^8$. Baez has suggested that its size is therefore related to the Lie group $SO(8)$, which is 28-dimensional [50]. Without resolving this conjecture, we note that the structure does sound a bit like something one would see in quantum theory, or a close relative of it. The “quantum logic” people have argued for a good long while that the lattice of closed subspaces of a Hilbert space, ordered by inclusion, can be thought of as a lattice of propositions pertaining to a quantum system. If the system in question is a set of three qubits, then we’d be talking about the lattice of subspaces of $\mathbb{C}^8$. To make this look exactly like the setting of Dedekind’s lattice, we would have to do quantum mechanics over real vector space (“rebits” instead of qubits), but that’s not so bad as far as pure math is concerned [51–53].

There’s an idea, going back to Birkhoff and von Neumann in the 1930s, that in quantum physics, we should relax the distributive law of logic. The argument goes that we can measure the position of a particle, say, or we can measure its momentum, but per the uncertainty principle, we cannot precisely measure its position and its momentum at once. Thus, we should reconsider how the logical connectives

$$\land = \text{“and”}, \lor = \text{“or”}$$

interact. I’m not convinced this is really the way to dig deep into the quantum mysteries: We can always impose an “uncertainty principle” on top of a classical theory [54,55]. Still, the idea is good enough to wring some mathematics out of. In Boolean logic, we can distribute “or” over “and”:

$$a \lor (b \land c) = (a \lor b) \land (a \lor c).$$

But if $a$, $b$ and $c$ are propositions about the outcomes of experiments upon a quantum system, then we cannot combine them arbitrarily and still have the result be physically meaningful. “Complementary” actions are mutually exclusive. We should only require that the distribution trick above works in restricted circumstances. When we organize all the propositions pertaining to a quantum system into a lattice, we say that $a \leq b$ when $a$ implies $b$. Then, unlike Boolean logic, we say that the distributive trick works when $a \leq b$ or $a \leq c$. This makes our lattice modular instead of distributive.

The free modular lattice on 3 generators is the structure we get when we introduce a set of elements $\{a, b, c\}$ and build up by combining them, assuming that the only restrictions are those due to requiring that the lattice be modular. In other words, it is the logic we build by starting with three propositions and saying nothing about them except that they should be “quantum” propositions, in this very stripped-down way of being “quantum”.

We can also try approaching from the $SO(8)$ direction. Manogue and Schray point out that we can label a set of 28 generators for $SO(8)$ in the following way [56]. 7 of them correspond to the unit imaginary octonions $e_1$ through $e_7$. Then, for each of the $e_i$, there are three pairs of unit imaginary octonions $(e_j, e_k)$ that multiply to $e_i$. These pairs give the other 21 generators. Considering the Fano plane, we have four generators for each point: one for the point itself, and one for each line through that point.

We can associate each generator with a line in $\mathbb{R}^7$ in the following way. First, we label each point in the Fano plane by the lines which meet there. For example,

$$\{1,1,0,0,0,0\}$$

stands for the point at which the first three lines coincide. There are seven such vectors, any two of which coincide at a single entry (because any two points in the Fano plane have exactly one line between them),
and so any two of these vectors have the same inner product. Next, we pick one of the three lines through our chosen point, and we mark it by flipping the sign of that entry. For example,

$$ (1, -1, 1, 0, 0, 0, 0, 0) $$

(63)

There are three ways to do this for each point in the Fano plane, and so we get 28 vectors in all. Note that the magnitude of the inner product is constant for all pairs. If the vectors correspond to different Fano points, then their supports overlap at only a single entry, and so the inner product is ±1. If they correspond to the same Fano point, then the magnitude of the inner product is $1 - 1 + 1 = 1$ again.

So, the 28 of $SO(8)$ seems to be tied in with the other 28’s: The same procedure counts generators of the group and equiangular lines in $\mathbb{R}^7$. (Counting — a kind of math I understand, some of the time.)

We now recall that the stabilizer subgroup for each vector in a Hoggar-type SIC is isomorphic to the projective special unitary group of $3 \times 3$ matrices over the finite field of order 9, known for short as PSU(3,3). Finite-group theorists also refer to this structure as $U_3(3)$, and as $G_2(2)^\prime$, since it is isomorphic to the commutator subgroup of the automorphism group of the octavians, the integer octonions. The symbol $G_2(2)^\prime$ arises because the automorphism group of the octonions is called $G_2$, when we focus on the octavians we add a 2 in parentheses, and when we form the subgroup of all the commutators, we affix a prime.

We also recall that, given one SIC of Hoggar type, we can construct another by antiunitary conjugation, and each vector in the first SIC will be orthogonal to exactly 28 vectors out of the 64 in the other SIC. Furthermore, the Hilbert–Schmidt inner products that are not zero are all equal. Said another way, if we use the first SIC to define a probabilistic representation of three-qubit state space, then each vector in the second SIC is a probability distribution that is uniform across its support. Up to normalization, such a probability distribution is a binary string composed of 28 zeros and 36 ones.

Let $\{\Pi_j^+\}$ be a SIC of Hoggar type, and let $\{\Pi_j^-\}$ be its conjugate SIC. Suppose that $U$ is a unitary that permutes the $\{\Pi_j^+\}$. Then a linear combination of 36 equally weighted projectors drawn from $\{\Pi_j^+\}$ will be sent to a linear combination of 36 equally weighted projectors from the set $\{\Pi_j^-\}$, possibly a different combination. But the only sequences of 36 ones and 28 zeros that correspond to valid quantum states are the representations of $\{\Pi_j^-\}$. Therefore, a unitary that shuffles the + SIC will also shuffle the − SIC.

Furthermore, a unitary that stabilizes a projector, say $\Pi_0^-$, must permute the $\{\Pi_j^+\}$ in such a way that 1’s go to 1’s and 0’s go to 0’s.

To repeat: Because each SIC provides a basis, we can uniquely specify a vector in one SIC by listing the vectors in the other SIC with which it has nonzero overlap.

A unitary symmetry of one SIC set corresponds to a permutation of the other. Using the second SIC to define a representation of the state space, each vector in the first SIC is essentially a binary string, and sending one vector to another permutes the 1’s and 0’s. In particular, a unitary that stabilizes a vector in one SIC must permute the vectors of the second SIC in such a way that the list of 1’s and 0’s remains the same. The 1’s can be permuted among themselves, and so can the 0’s, but the binary sequence as a whole does not change.

It would be nice to have a way of visualizing this with a more tangible structure than eight-dimensional complex Hilbert space — something like a graph. Thinking about the permutations of 36 vectors, we imagine a graph on 36 vertices, and we try to draw it in such a way that its group of symmetries is isomorphic to the stabilizer group of a Hoggar fiducial. Can this be done? Well, almost — that is, up to a “factor of two”:

It’s called the $U_3(3)$ graph, and its automorphism group has the stabilizer group of a Hoggar fiducial as an index-2 subgroup.

Now, is there a way to illustrate the structure of both SICs together as a graph? We want to record the fact that each vector in one SIC is nonorthogonal to exactly 36 vectors of the other, that the stabilizer of each vector is PSU(3,3), and that a stabilizer unitary shuffles the nonorthogonal set within itself. So, we start with one vertex to represent a fiducial vector, then we add 63 more vertices to stand for the other vectors in the first SIC, and then we add 36 vertices to represent the vectors in the second SIC that are nonorthogonal to the fiducial of the first. We’d like to connect the vertices in such a way that the stabilizer of any vertex is isomorphic to PSU(3,3). In fact, because any vector in either SIC can be identified by the list of the 36 nonorthogonal vectors in the other, the graph should look locally like the $U_3(3)$ graph everywhere! Is this possible? Yes! The result is the Hall–Janko graph, whose automorphism group has the Hall–Janko finite simple group as an index-2 subgroup.
The Hall–Janko group can also be constructed as the symmetries of the Leech lattice cast into a quaternionic form \([57,58]\). Speculating only a little wildly, we can contemplate a possible path from the Hoggar-type SICs in \(\mathbb{C}^8\) to the Hall–Janko group and thence to the Leech lattice and the “real SIC” in \(\mathbb{R}^{23}\).

That’s what we get when we think about the permutations of the 1’s. What about \(\text{PSU}(3, 3)\) acting to permute the 0’s?

This seems to lead us in the direction of the Rudvalis group. Wilson’s textbook *The Finite Simple Groups* has this to say (§5.9.3):

The Rudvalis group has order 145 926 144 000 = \(2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29\), and its smallest representations have degree 28. These are actually representations of the double cover \(2 \cdot \text{Ru}\) over the complex numbers, or over any field of odd characteristic containing a square root of \(-1\), but they also give rise to representations of the simple group Ru over the field \(\mathbb{F}_2\) of order 2.

Wilson then describes in some detail the 28-dimensional complex representations of \(2 \cdot \text{Ru}\), using a basis in which \(2^6 \cdot \text{G}_2(2)\) appears as the monomial subgroup \([59]\). And the 28 appears to be the same 28 we saw before; that is, it’s the 28 pairs of cube roots of the identity in the octavians mod 2, so it’s a set of 28 that is naturally shuffled by \(\text{G}_2(2)\)\(^\prime\) \([60]\).

The Rudvalis group can be constructed as a rank-3 permutation group acting on 4060 points, where the stabilizer of a point is the Ree group \(2 \cdot F_4(2)\). This group is in turn given by the symmetries of a “generalized octagon” \([61]\). (Generalized \(n\)-gons abstract the properties of the more familiar \(n\)-gons that, as graphs, they have diameter \(n\) and their shortest cycles have length \(2n\). The Fano plane is a generalized triangle.) Compare this situation to the Hall–Janko group, which has a rank-3 permutation representation on 100 points where the point stabilizer is \(\text{U}_3(3)\), and \(\text{U}_3(3)\) is furnished by the symmetries of a generalized hexagon \([62]\).

Ru has a maximal subgroup given by a semidirect product \(2^6 : \text{G}_2(2)\)\(^\prime\) : 2. This is what first caught my eye. Neglecting the issue of the group actions required to define the semidirect products, consider the factors: we have \(\text{G}_2(2)\)\(^\prime\) from the Hoggar stabilizer, 2 from conjugation and \(2^6\) from the three-qubit Pauli group.

This is reminiscent of a theorem proved by O’Nan \([63]\):

Let \(G\) be a finite simple group having an elementary abelian subgroup \(E\) of order 64 such that \(E\) is a Sylow 2-subgroup of the centralizer of \(E\) in \(G\) and the quotient of the normalizer of \(E\) in \(G\) by the centralizer is isomorphic to the group \(\text{G}_2(2)\) or its commutator subgroup \(\text{G}_2(2)\)\(^\prime\). Then \(G\) is isomorphic to the Rudvalis group.

The peculiar thing is that the Hall–Janko group is part of the “Happy Family”, i.e., it is a subgroup of the Monster, while the Rudvalis group is a “pariah”, floating off to the side. The Hoggar SIC almost seems to be acting as an intermediary between the two finite simple groups, one of which fits within the Monster while the other does not.

Finally, what connects the largest of sporadic simple groups with the second-smallest among quantum systems?

I was greatly amused to find the finite affine plane on nine points also appearing in the theory of the Monster group and the Moonshine module \([64, 65]\). In that case, the 9 points and 12 lines correspond to involution automorphisms. All the point-involutions commute with one another, and all the line-involutions commute with each other as well. The order of the product of a line-involution and a point-involution depends on whether the line and the point are incident or not.

This is probably of no great consequence — just an accident of the same small structures appearing in different places, because there are only so many small structures to go around. But it’s a cute accident all the same.

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