Kroll-Lee-Zumino quantum field theory of pionic interactions: rho-meson propagator at two-loop level and electromagnetic pion form factor

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The Kroll-Lee-Zumino renormalizable Abelian quantum field theory of strong interactions is used to compute the rho-meson propagator at the two-loop level. As an application, the result is used to determine the electromagnetic form factor of the pion in the time-like region. Unlike the known and reasonably successful one-loop expression, the pion form factor at the two-loop level is not in better agreement with data in the low momentum region below the rho-meson peak. This is in spite of the effective strong coupling constant being relatively small, i.e. $g_{\rho\pi\pi}/(4\pi) \approx 0.4$ at the one-loop level, and $g_{\rho\pi\pi}/(4\pi) \approx 0.2$ at the two-loop level.

I. INTRODUCTION

A renormalizable, Abelian, quantum field theory of strong interactions among pions and a massive neutral rho-meson was proposed long ago by Kroll, Lee, and Zumino (KLZ) [4]. In spite of the presence of a massive gauge boson this KLZ theory is renormalizable, due to this boson coupling to a conserved current [2]. An attractive feature of the theory is that it provides the quantum field theory justification for the Vector Meson Dominance (VMD) model [3]. In addition, it is a potential candidate to fill the energy gap between chiral perturbation theory at threshold and QCD above 1 GeV. A successful application was made some time later with the calculation of the rho-meson self energy at the one-loop level [4]. In fact, after using this result in the VMD expression of the electromagnetic pion form factor, a good agreement was found with data in the time-like region. In particular, this form factor agrees with the well known Gounaris-Sakurai formula [5]-[6] in the vicinity of the rho-meson peak. This agreement is intriguing, given the fact that this formula is purely empiric. Another successful application is that of the pion form factor in the space-like region, determined from the triangle diagram \[7\]. There is excellent agreement with data up to $q^2 \approx -10$ GeV$^2$, with a chi-squared per degree of freedom of $\chi^2_{KLZ} = 1.1$, in contrast with the VMD value of $\chi^2_{VMD} = 5.0$. Furthermore, the pion mean-squared radius is predicted to be $\langle r_{\pi}^2 \rangle_{KLZ} = 0.46$ fm$^2$, compared with the experimental value $\langle r_{\pi}^2 \rangle_{\text{EXP}} = 0.439 \pm 0.008$ fm$^2$, and the VMD prediction $\langle r_{\pi}^2 \rangle_{\text{VMD}} = 0.39$ fm$^2$. Finally, two equally successful applications of KLZ are the scalar radius of the pion [10], and the scalar form factor of the pion in the space-like region [11], both in good agreement with Lattice QCD [12]-[13].

Motivated by these results we calculate in this paper the KLZ rho-meson propagator at the two-loop level in perturbation theory. The expectation being that given the size of the one-loop effective strong coupling, $g_{\rho\pi\pi}/(4\pi) \approx 0.4$, the two-loop result for the pion form factor, in the time-like region, could improve the agreement with data.

The KLZ Lagrangian is given by

$$\mathcal{L}_{KLZ} = \partial_\mu \phi \partial^\mu \phi - m^2 \phi \phi - \frac{i}{4} \rho_{\mu\nu} \rho^{\mu\nu} + \frac{1}{2} M^2 \rho_\mu \rho^\mu + g_{\rho\pi\pi} \rho_\mu J^\mu_{\pi\pi} + g^2_{\rho\pi\pi} \rho_\mu \rho_\nu \phi \phi^*,$$

(1)

with $m$ and $M$ the pion and rho-meson masses, respectively, $\rho_\mu$ a vector field of the $\rho^0$ meson ($\partial_\mu \rho^\mu = 0$), $\phi$ a complex pseudo-scalar field describing the $\pi^\pm$ mesons, $\rho_{\mu\nu}$ is the field strength tensor, $\rho_\mu = \partial_\mu \rho_\nu - \partial_\nu \rho_\mu$, and $J^\mu_{\pi\pi}$ is the $\pi^\pm$ current, $J^\pm_\mu = i \phi^* \partial_\nu \rho^\nu$. In spite of the explicit presence of the $\rho^0$ mass term above, the theory is renormalizable. This is due to the neutral vector meson being coupled to a conserved current [1]-[2].

The $\rho$-meson self energy at the one-loop level, Fig.1, is given by

$$i \Pi^{\mu\nu}(p^2) = g^2 \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{(2k + p)^\mu (2k + p)^\nu}{(k^2 - m^2)((k + p)^2 - m^2)} - 2 \frac{g_{\mu\nu}}{k^2 - m^2} \right\} i F_{\text{vac}}(p^2),$$

(2)

where the vacuum polarization, $F_{\text{vac}}(p^2)$, in the time-like region, after dimensional regularization and renormalization-
tion, is given by [4]

\[ F_{\text{vac}}(p^2) = \frac{g^2 p^2}{48 \pi^2} \left(1 - \frac{4m^2}{p^2}\right)^{3/2} \left[\ln \left|1 + \sqrt{1 - \frac{4m^2}{p^2}}\right| - \ln \left|1 - \sqrt{1 - \frac{4m^2}{p^2}}\right| \right] \]

- \left[i \pi \theta (p^2 - 4m^2) \right] + \frac{8m^2}{p^2} + \left(\frac{4m^2}{p^2} - 1\right)^{1/2} \times \cos^{-1} \left(1 - \frac{p^2}{2m^2}\right) \left[\theta(p^2 - 4m^2) - \theta(p^2) \right] + C \right), \tag{3}

where

\[ C = - \frac{8m^2}{M^2} + \left(1 - \frac{4m^2}{M^2}\right)^{3/2} \ln \left|1 - \sqrt{1 - \frac{4m^2}{M^2}}\right| - \ln \left|1 + \sqrt{1 - \frac{4m^2}{M^2}}\right| \right]. \tag{4}

The gauge invariance of the KLZ theory can be revealed using St"uckelberg’s procedure [2, 14], to wit. Quantizing KLZ as a gauge theory in the path-integral formalism introduces a gauge-fixing parameter, \( \xi \), so that the Lagrangian to be used in perturbative calculations is

\[ \mathcal{L}_{\text{KLZ}} = \partial_{\mu} \phi_0^* \partial^{\mu} \phi_0 - m_0^2 \phi_0^* \phi_0 - \frac{1}{4} \rho_0^{\mu\nu} \rho_0_{\mu\nu} \]

\[ + \frac{1}{2} M_0^2 \rho_0^{\mu\nu} \rho_0_{\mu\nu} + ig_0 \rho_0^{\mu} \rho_0^{\nu} \left( \phi_0^* \partial_{\mu} \phi_0 \right) \]

\[ + g_0^2 \rho_0^{\mu\nu} \rho_0_{\mu\nu} \phi_0^* \phi_0 - \frac{1}{2 \xi_0} \left( \partial_{\mu} \rho_0^{\mu} \right)^2. \tag{9} \]

Notice the appearance of the renormalization scale \( \mu \), due to the Lagrangian being written in \( n = 4 - 2\epsilon \) dimensions. Each of the unrenormalized fields, \( \phi_0 \) and \( \rho_0^\mu \), has a zero-subscript to distinguish it from their renormalized counterparts, \( \phi \) and \( \rho^{\mu} \), respectively. Similarly, the bare parameters \( m_0, M_0, g_0 \) and \( \xi_0 \) are written with a zero-subscript whereas their physical/renormalized counterparts, \( m, M, g, \) and \( \xi \), are written without subscripts. To relate bare quantities to their renormalized partners we introduce renormalization constants as follows

\[ \phi_0 = \sqrt{Z_2} \phi, \quad \rho_0^{\mu} = \sqrt{Z_3} \rho^{\mu}, \quad Z_1 g = g_0 Z_2 \sqrt{Z_3}, \]
Each counter-term is expressed as an expansion in the coupling constant, \( g \), i.e.

\[
\delta Z_2 = \sum_{k=1}^{\infty} \delta Z_2^{(k)} \left( \frac{g^2}{16\pi^2} \right)^k ,
\]

\[
\delta Z_3 = \sum_{k=1}^{\infty} \delta Z_3^{(k)} \left( \frac{g^2}{16\pi^2} \right)^k ,
\]

\[
\delta Z_m = \sum_{k=1}^{\infty} \delta Z_m^{(k)} \left( \frac{g^2}{16\pi^2} \right)^k ,
\]

where each coefficient of the expansion is to be determined order-by-order in perturbation theory.

The KLZ theory, being gauge invariant, has associated Ward identities which can be used to relate the counter-terms as

\[
\delta Z_1 = \delta Z_2 = \delta Z'_1 ,
\]

\[
\delta Z_M = 0 = \delta Z'_3 .
\]

This greatly simplifies the lagrangian and therefore all calculations. As a further simplification we make the gauge choice \( \xi = 1 \), the so-called Feynman gauge, which leads to

\[
\mathcal{L}_{KLZ} = \partial_\mu \phi^* \partial^\mu \phi - Z_m m^2 \phi^* \phi - \frac{1}{4} Z_3 \rho^\mu \rho_\mu - \frac{1}{2} Z_M M^2 \rho^\mu \rho_\mu + i Z_1 g \rho^\mu \left( \phi^* \partial_\mu \phi \right)
\]

\[
+ Z'_1 g^2 M^2 \rho^\mu \rho_\mu \phi^* \phi - \frac{1}{2} Z_3 (\partial_\mu \rho_\mu)^2 .
\]  

Each renormalization constant can be written as the unity plus a counter-term

\[
Z_1 = 1 + \delta Z_1 , \quad Z'_1 = 1 + \delta Z'_1 , \quad Z_2 = 1 + \delta Z_2 , \quad Z_3 = 1 + \delta Z_3 , \quad Z_M = 1 + \delta Z_M , \quad Z_3 = 1 + \delta Z_3
\]

\[
Z_m = 1 + \delta Z_m 
\]

Each counter-term is expressed as an expansion in the coupling constant, \( g \), i.e.

\[
\delta Z_2 = \sum_{k=1}^{\infty} \delta Z_2^{(k)} \left( \frac{g^2}{16\pi^2} \right)^k ,
\]

\[
\delta Z_3 = \sum_{k=1}^{\infty} \delta Z_3^{(k)} \left( \frac{g^2}{16\pi^2} \right)^k ,
\]

\[
\delta Z_m = \sum_{k=1}^{\infty} \delta Z_m^{(k)} \left( \frac{g^2}{16\pi^2} \right)^k ,
\]

FIG. 2: Full set of KLZ Feynman rules in the Feynman gauge.

III. SELF-ENERGY OF THE RHO-MESON

At two-loop order, i.e. \( \mathcal{O}(g^4) \), the \( \rho^0 \) self-energy, denoted by \( \Pi^{\rho\rho} \), is obtained by summing the diagrams shown in
Figure 3 as follows

\[
\Pi^{\mu\nu}(p^2) = I^{\mu\nu} + J^{\mu\nu} + \frac{1}{2}\xi_{1\mu} + \xi_{2\mu}
+ \Omega^{\mu\nu} + 2Z^{\mu\nu} + 4X^{\mu\nu} + W^{\mu\nu} + A^{\mu\nu}
+ 2C^{\mu\nu}_1 + 2C^{\mu\nu}_2 + 2C^{\mu\nu}_3 + C^{\mu\nu}_4 + C^{\mu\nu}_5 .
\] (21)

where \( p \) is the external momentum to each diagram. Yet another consequence of the theory’s Ward identities is that \( \Pi^{\mu\nu} \) is transverse \[ 2 \Rightarrow 15 \]

\[
\Pi^{\mu\nu} = \left( g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) F_{\text{vac}}(p^2) .
\] (22)

This means that \( F_{\text{vac}} \) can be extracted by contracting \( \Pi^{\mu\nu} \) with \( g_{\mu\nu} \). Using Eq. (21) and Eq. (22) gives

\[
F_{\text{vac}}(p^2) = \frac{1}{n-1} \left\{ g_{\mu\nu}I^{\mu\nu} + g_{\mu\nu}J^{\mu\nu} + \frac{1}{2}g_{\mu\nu}\xi_{1\mu}^{\nu} + g_{\mu\nu}\xi_{2\mu}^{\nu} + 2g_{\mu\nu}Z^{\mu\nu} + 4g_{\mu\nu}X^{\mu\nu} + g_{\mu\nu}W^{\mu\nu} + g_{\mu\nu}A^{\mu\nu} + g_{\mu\nu}\Omega^{\mu\nu} + 2g_{\mu\nu}C^{\mu\nu}_1 + g_{\mu\nu}C^{\mu\nu}_2 + 2g_{\mu\nu}C^{\mu\nu}_3 + g_{\mu\nu}C^{\mu\nu}_4 + g_{\mu\nu}C^{\mu\nu}_5 \right\} .
\] (23)

It should be clear from Eq. (23) that the quantities of interest are the contractions of \( g_{\mu\nu} \) with each of the diagrams. Before examining the integrals arising from each of the Feynman diagrams we define

\[
\alpha = \frac{g^2}{16\pi^2} ,
\]

\[
D^n k = \frac{d^n k}{i\pi^2 (2\pi)^{n-1}} .
\] (25)

It is to be understood that \( k \) and \( q \) represent loop momenta, while \( p \) stands for the external momentum. The Feynman rules for the one-loop diagrams \( I^{\mu\nu} \) and \( J^{\mu\nu} \) yield

\[
I^{\mu\nu} = 2ig^2 \mu^2 g^{\mu\nu} \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m^2}
= -2\alpha g^{\mu\nu} \int D^n k \frac{1}{k^2 - m^2} ,
\] (26)

and

\[
J^{\mu\nu} = -ig^2 \mu^2 \int \frac{d^n k}{(2\pi)^n} \frac{(2k + p)^\mu (2k + p)^\nu}{(k^2 - m^2)((k + p)^2 - m^2)}
= \alpha \int D^n k \frac{(2k + p)^\mu (2k + p)^\nu}{(k^2 - m^2)((k + p)^2 - m^2)} .
\] (27)

Similarly, the two-loop diagrams involve

\[
\zeta_1^{\mu\nu} = 4\alpha^2 g^{\mu\nu} \int D^n k D^n q \frac{1}{(k^2 - m^2)^2 (q^2 - M^2)} ,
\] (28)

\[
\zeta_2^{\mu\nu} = 2\alpha^2 g^{\mu\nu} \int D^n k D^n q \frac{(2k + p)^\mu (2k + p)^\nu}{D_1^2 D_2 D_3 D_4} ,
\] (29)

\[
\Omega^{\mu\nu} = -2\alpha^2 g^{\mu\nu} \int D^n k D^n q \frac{(k + q)^2}{D_1^3 D_2 D_3} ,
\] (30)

\[
Z^{\mu\nu} = \alpha^2 g^{\mu\nu} \int D^n k D^n q \frac{1}{D_2^3 D_3 D_4} ,
\] (31)

\[
X^{\mu\nu} = -2\alpha^2 g^{\mu\nu} \int D^n k D^n q \frac{(k + q)^\mu (2k + p)^\nu}{D_1 D_2 D_3 D_4} ,
\] (32)

\[
W^{\mu\nu} = 4\alpha^2 g^{\mu\nu} \int D^n k D^n q \frac{1}{D_2 D_3 D_4} ,
\] (33)

\[
A^{\mu\nu} = \alpha^2 \int D^n k D^n q \left\{ \frac{(k + q) \cdot (k + q + p)}{D_1 D_2 D_3 D_4 D_5} \times (2k + p)^\mu (2q + p)^\nu \right\} ,
\] (34)

where

\[
D_1 = k^2 - m^2 , \quad D_2 = (k + p)^2 - m^2 ,
D_3 = q^2 - m^2 , \quad D_4 = (k + q)^2 - M^2 ,
D_5 = (q + p)^2 - m^2 .
\] (35)

Finally, we consider diagrams with counter-term insertions. Applying the Feynman rules to \( C_1^{\mu\nu} \) gives

\[
C_1^{\mu\nu} = \alpha \int D^n k D^n q \left\{ \frac{(2k + p)^\mu (2k + p)^\nu}{(k^2 - m^2)((k + p)^2 - m^2)} \times (k^2 \delta Z_2 - m^2 \delta Z_3) \right\}
= \alpha \int D^n k D^n q \left\{ \frac{(2k + p)^\mu (2k + p)^\nu}{(k^2 - m^2)((k + p)^2 - m^2)} \times \left( k^2 \sum_{j=1}^{\infty} \delta Z_2^{(j)} \alpha^j - m^2 \sum_{j=1}^{\infty} \delta Z_3^{(j)} \alpha^j \right) \right\} ,
\] (36)

where we invoked Eqs. (15 and 17). To order \( \alpha^2 \), each of the above series is truncated at first order in \( \alpha \), so that
Eq. (36) becomes

\[ C_1^{\mu\nu} = \alpha^2 \int D^n k D^n q \left\{ \frac{(2k+p)^\mu(2k+p)^\nu}{(k^2-m^2)((k+p)^2-m^2)} \times \left( k^2 \delta Z_2^{(1)} - m^2 \delta Z_m^{(1)} \right) \right\}. \]

(37)

Proceeding similarly for the remaining counter-term diagrams in Figure 3 gives

\[ C_2^{\mu\nu} = 2\alpha^2 g^{\mu\nu} \int D^n k \frac{k^2 \delta Z_2^{(1)} - m^2 \delta Z_m^{(1)}}{(k^2-m^2)^2}, \]

(38)

\[ C_3^{\mu\nu} = \alpha^2 \delta Z_1^{(1)} \int D^n k \frac{(2k+p)^\mu(2k+p)^\nu}{(k^2-m^2)((k+p)^2-m^2)}, \]

(39)

\[ C_4^{\mu\nu} = -2\alpha^2 \delta Z_1^{(1)} g^{\mu\nu} \int D^n k \frac{1}{k^2-m^2}, \]

(40)

\[ C_5^{\mu\nu} = \left( g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \left( -\alpha^2 p^2 \delta Z_3^{(2)} \right) \]

(41)

A. Reduction of Feynman Integrals

The strategy for evaluating \( F_{\text{vac}} \) is to express each function in Eq. (23) in terms of well known scalar integrals. The one-loop integrals \( g^{\mu\nu} I^{\mu\nu} \) and \( g^{\mu\nu} J^{\mu\nu} \) are reduced, starting from Eqs. (26)-(27), as follows

\[ g^{\mu\nu} I^{\mu\nu} = -2\alpha \int D^n k \frac{g^{\mu\nu}}{k^2-m^2} \]

(42)

\[ = -2\alpha n \int D^n k \frac{1}{k^2-m^2} \]

\[ = -2\alpha n A_0 \left( m^2 \right), \]

and

\[ g^{\mu\nu} J^{\mu\nu} = \alpha \int D^n k \frac{(2k+p)^2}{(k^2-m^2)((k+p)^2-m^2)} \]

\[ = \alpha \int D^n k \frac{2D_1 + 2D_2 + 4m^2 - p^2}{D_1 D_2} \]

\[ = \alpha \left\{ 4A_0(m^2) + \left( 4m^2 - p^2 \right) B_0 \left( p^2; m_1^2, m_2^2 \right) \right\}, \]

(43)

where we have introduced the one-loop basic integrals

\[ A_0(m_1^2) = \int D^n k \frac{1}{k^2-m_1^2}, \]

(44)

\[ B_0 \left( p^2; m_1^2, m_2^2 \right) = \int D^n k \frac{1}{(k^2-m_1^2)((k+p)^2-m_2^2)}. \]

(45)
Both these integrals are known analytically [16]. Two-loop self-energy-type integrals may be similarly reduced in terms of a scalar basis, the so-called T-integrals [17]. To define the T-integrals we first label the propagator momenta as

\[ k_1 = k , \quad k_2 = k + p , \quad k_3 = k - q , \]
\[ k_4 = q , \quad k_5 = q + p . \]  

(46)

A general T-integral is then defined as

\[ T_{i_1 i_2 \ldots i_r} (p^2; m^2_{j_1}, m^2_{j_2}, \ldots m^2_{j_r}) \]
\[ = \int \frac{\mathcal{D}^n k \mathcal{D}^n q}{(k_i - m^2_{j_1})(k_i - m^2_{j_2}) \ldots (k_i - m^2_{j_r})} , \]  

(47)

where each \( i \) \( \in \{1, 2, 3, 4, 5\} \) labels an internal momentum, and the \( j \) \( \in \) are arbitrary indices labelling masses. For instance

\[ T_{234}(p^2; m^2_w, m^2_b, m^2_j) \]
\[ = \int \frac{\mathcal{D}^n k \mathcal{D}^n q}{(k^2 - m^2_w)(k^2 - m^2_b)(k^2 - m^2_j)} \]
\[ = \int \frac{((k + p)^2 - m^2_w)((k - q)^2 - m^2_b)(q^2 - m^2_j)}{\mathcal{D}^n k \mathcal{D}^n q} . \]  

(48)

Whenever the only factors entering the denominator are those in Eq. (35), we use the notation

\[ T_{i_1 i_2 \ldots i_r} = \int \frac{1}{\mathcal{D}^n k \mathcal{D}^n q D_{i_1} D_{i_2} \ldots D_{i_r}} . \]  

(49)

Additional integrals will be of the form

\[ Y_{i_1 \ldots i_r} = \int \frac{k_{i_1}^2 \ldots k_{i_r}^2}{\mathcal{D}^n k \mathcal{D}^n q D_{i_1} \ldots D_{i_r}} . \]  

(50)

These Y-integrals can be reduced to T-integrals as shown in [17]. T-integrals are well-known, and there are various methods to evaluate them, both analytically and numerically [18][21].

We can now reduce the two-loop integrals entering Eq. (23). From Eq. (33) we find

\[ g_{\mu \nu} W^{\mu \nu} = 4 \alpha^2 \int \mathcal{D}^n k \mathcal{D}^n q \frac{g_{\mu \nu} g^{\mu \nu}}{D_2 D_3 D_4} , \]
\[ = 4 \alpha^2 T_{234} , \]  

(51)

and from Eq. (30)

\[ g_{\mu \nu} \Omega^{\mu \nu} = -2 \alpha^2 \int \mathcal{D}^n k \mathcal{D}^n q \frac{(k + q)^2}{D_1^2 D_3 D_4} \]
\[ = -2 \alpha^2 \int \mathcal{D}^n k \mathcal{D}^n q \frac{2 D_1 + 2 D_4 - D_3 + 4 m^2 - M^2}{D_1^2 D_3 D_4} \]
\[ = \alpha^2 \{ 2 T_{114} - 4 T_{134} - 4 T_{113} - 2(4 m^2 - M^2) T_{1134} \} . \]  

(52)

Similarly, the remaining integrals in Eq. (23) are

\[ g_{\mu \nu} C_1^{\mu \nu} = 4 \alpha^2 \alpha^2 T_{113} , \]  

(53)

\[ g_{\mu \nu} C_2^{\mu \nu} = -2 \alpha^2 \{ (n - 1) T_{123} + T_{113} \} , \]  

(54)

\[ g_{\mu \nu} X^{\mu \nu} = \alpha^2 \{ -5 T_{234} - T_{134} + 2 T_{124} - T_{123} \}
\[ - T_{1234} - (7 m^2 - 2 M^2 - 2 p^2) T_{1134} \} , \]  

(55)

\[ g_{\mu \nu} Z^{\mu \nu} = \alpha^2 \{ 4 T_{234} + 4 T_{134} - (n - 1) T_{124} - T_{114} + 2(n - 1) T_{123} + 2 T_{113} + 2(8 m^2 - M^2 - p^2) T_{1234} \]
\[ + 2(4 m^2 - M^2) T_{1134} + (4 m^2 - p^2)(4 m^2 - M^2) T_{1134} \} . \]  

(56)

\[ g_{\mu \nu} A^{\mu \nu} = \alpha^2 \{ 4 T_{2345} + 4 T_{134} + 4 T_{134} - 8 T_{124} + 4 T_{123} - (8 m^2 - 2 M^2 - 4 p^2) T_{1245} + 4(7 m^2 - 3 M^2 - 3 p^2) \]
\[ \times T_{1234} + (4 m^2 - 2 M^2 - p^2)(4 m^2 - M^2 - 2 p^2) T_{1245} \} . \]  

(57)

For the counter-term integrals we obtain

\[ g_{\mu \nu} C^{\mu \nu}_2 = \]
\[ - \alpha^2 \left\{ C_2^{(1)} \left[ (n - 1) B_0(p^2; m^2, m^2) + \frac{m - 2}{2 m^2} A_0(m^2) \right] \right. \]
\[ + \delta Z_2^{(1)} \left[ 4 A_0(m^2) + (4 m^2 - p^2) B_0(p^2; m^2, m^2) \right] \right\} , \]  

(58)

\[ g_{\mu \nu} C^{\mu \nu}_3 = 2 \alpha^2 \left( \delta Z_2^{(1)} + C_2^{(1)} \frac{n - 2}{2 m^2} A_0(m^2) \right) , \]  

(59)

\[ g_{\mu \nu} C^{\mu \nu}_4 = \alpha^2 \delta Z_2^{(1)} \left[ (4 m^2 - p^2) B_0(p^2; m^2, m^2) + 4 A_0(m^2) \right] \} , \]  

(60)

\[ g_{\mu \nu} C^{\mu \nu}_5 = -2 \alpha^2 \delta Z_2^{(1)} A_0(m^2) , \]  

(61)

\[ g_{\mu \nu} C^{\mu \nu}_5 = -(n - 1) p^2 \left\{ \alpha \delta Z_2^{(1)} + \alpha^2 \delta Z_2^{(2)} \right\} , \]  

(62)

where

\[ C_2^{(1)} = m^2 \left( \delta Z_2^{(1)} - \delta Z_2^{(1)} \right) . \]  

(63)

The combination of these counter-terms entering Eq. (23) reduces to

\[ 2 g_{\mu \nu} C_1^{\mu \nu} + g_{\mu \nu} C_2^{\mu \nu} + 2 g_{\mu \nu} C_3^{\mu \nu} + g_{\mu \nu} C_4^{\mu \nu} + g_{\mu \nu} C_5^{\mu \nu} \]
\[ = - \alpha(n - 1) p^2 \delta Z_2^{(1)} + \alpha^2(n - 1) \left\{ - p^2 \delta Z_2^{(2)} \right\} + 2 C_2^{(1)} \left[ \frac{n - 2}{2 m^2} A_0(m^2) - B_0(p^2; m^2, m^2) \right] \} . \]  

(64)
implying
\[ C^{(1)}_\pi = (4m^2 - M^2)B_0(m^2; M^2, m^2) \]
\[ - A_0(m^2) - (n - 2)A_0(M^2) \] (75)

Substituting Eq. (75) into Eq. (64) gives
\[ 2g_{\mu\nu}C^{\mu\nu}_1 + g_{\mu\nu}C^{\mu\nu}_2 + 2g_{\mu\nu}C^{\mu\nu}_3 + g_{\mu\nu}C^{\mu\nu}_4 + g_{\mu\nu}C^{\mu\nu}_5 \]
\[ = -\alpha(n - 1)p^2\delta Z_3^{(2)} + \alpha^2(n - 1)\left\{ - p^2\delta Z_3^{(2)} \right\} \]
\[ + 2\left\{ (4m^2 - M^2)B_0(m^2; M^2, m^2) - (n - 2)A_0(M^2) \right\} \]
\[ - A_0(m^2) \times \left[ \frac{n - 2}{2m^2}A_0(m^2) - B_0(p^2; m^2, m^2) \right]. \] (76)

In Eq. (76) there are some products of one-loop basis integrals which can be expressed in terms of T-integrals to obtain
\[ 2g_{\mu\nu}C^{\mu\nu}_1 + g_{\mu\nu}C^{\mu\nu}_2 + 2g_{\mu\nu}C^{\mu\nu}_3 + g_{\mu\nu}C^{\mu\nu}_4 + g_{\mu\nu}C^{\mu\nu}_5 \]
\[ = -\alpha(n - 1)p^2\delta Z_3^{(2)} + \alpha^2(n - 1)\left\{ - 2[T_{114} - T_{124}] \right\} \]
\[ + 2(4m^2 - M^2)B_0(m^2; M^2, m^2) \left\{ \frac{n - 2}{2m^2}A_0(m^2) \right\} \]
\[ - B_0(p^2; m^2, m^2) \right\} - 2(n - 2)[T_{113} - T_{123}] \]
\[ - p^2\delta Z_3^{(2)} \right\}. \] (77)

We now consider the remaining contributions to \( F_{\text{vac}} \) as expressed in Eq. (23). Starting with the one-loop contributions, Eqs. (42)-(43), leads to
\[ g_{\mu\nu}F^{\mu\nu} = \alpha\left\{ 2(2 - n) + (4m^2 - p^2)B_0(p^2; m^2, m^2) \right\}, \] (78)

and from Eqs. (55)-(54) one finds
\[ \frac{1}{2}g_{\mu\nu}\xi^{\mu\nu}_1 + g_{\mu\nu}\xi^{\mu\nu}_2 = 2\alpha(n - 1)\alpha^2[T_{113} - T_{123}] \]. (79)

Using Eqs. (51)-(52), and Eqs. (55)-(57) leads to
\[ 2g_{\mu\nu}Z^{\mu\nu} + 4g_{\mu\nu}X^{\mu\nu} + g_{\mu\nu}W^{\mu\nu} + g_{\mu\nu}A^{\mu\nu} + g_{\mu\nu}\Omega^{\mu\nu} \]
\[ = \alpha^2\left\{ - 4(n - 1)[T_{113} - T_{123}] + 2(n - 1)[T_{114} - T_{124}] + 4(n - 2)[T_{234} - T_{134}] - 2(n - 2)(4m^2 - M^2)T_{1134} \right\} \]
\[ + 4(8m^2 - 2M^2 - 2p^2)T_{1234} - (8m^2 - 2M^2 - 4p^2)T_{1245} + 2(4m^2 - 2M^2 - p^2)(4m^2 - M^2 - 2p^2)T_{12345} \]. (80)
This completes all the information needed for $F_{\text{vac}}$, Eq. (23). Using Eqs. (77)-(80) gives

$$F_{\text{vac}} = \left[ (4m^2 - p^2)B_0(p^2; m^2, m^2) - 2(n - 2)A_0(m^2) \right]$$

$$\times \frac{\alpha}{n - 1} - \frac{\alpha^2}{n - 1} \left[ 4(n - 2)T_{234} - 4(n - 2)T_{134} - 2(n - 2)(4m^2 - M^2)T_{1134} - (8m^2 - 2M^2 - 4p^2)T_{1245} + 4(8m^2 - 2M^2 - 2p^2)T_{1234} + (4m^2 - 2M^2 - p^2)(4m^2 - M^2)T_{12345} \right] + \alpha^2 T_{\odot} - \alpha^2 p^2 \delta Z_{\odot}^2,$$  

with

$$r_1 = \frac{1}{2} \left\{ \frac{2 - p^2}{m^2} + \sqrt{\left( \frac{2 - p^2}{m^2} \right)^2 - 4} \right\},$$  

$$r_2 = \frac{1}{2} \left\{ \frac{2 - p^2}{m^2} - \sqrt{\left( \frac{2 - p^2}{m^2} \right)^2 - 4} \right\},$$  

$$\tilde{r}_1 = \frac{1}{2} \left\{ 1 + y - x + \sqrt{(x - y - 1)^2 - 4y} \right\},$$  

$$\tilde{r}_2 = \frac{1}{2} \left\{ 1 + y - x - \sqrt{(x - y - 1)^2 - 4y} \right\}.$$  

Finally,

$$w = \frac{1 - \tilde{r}_1}{\tilde{r}_2 - \tilde{r}_1}, \quad \tilde{w} = \frac{1 - \tilde{r}_2}{\tilde{r}_1 - \tilde{r}_2},$$  

$$z = \frac{\tilde{r}_1(1 - \tilde{r}_2)}{\tilde{r}_1 - \tilde{r}_2}, \quad \tilde{z} = \frac{\tilde{r}_2(1 - \tilde{r}_1)}{\tilde{r}_2 - \tilde{r}_1},$$  

$$w_0 = \frac{1 - r_1}{r_2 - r_1}, \quad \tilde{w}_0 = \frac{1 - r_2}{r_1 - r_2},$$  

$$z_0 = \frac{r_1(1 - r_2)}{r_1 - r_2}, \quad \tilde{z}_0 = \frac{r_2(1 - r_1)}{r_2 - r_1}.$$  

The $\eta$-function, needed for the addition of logarithms with complex arguments, is given by

$$\eta(a, b) = \ln(ab) - \ln a - \ln b$$

$$= 2\pi i \left\{ \theta(-Im a)\theta(-Im b)\theta(Im(ab)) - \theta(Im(a))\theta(Im(b))\theta(-Im(ab))\right\}.$$  

The expressions for the one-loop scalar integrals are [16] [20] [23] [24]

$$A_0(m^2) = \int \mathcal{D}^n k \frac{1}{k^2 - m^2}$$

$$= \frac{m^2}{\epsilon} + m^2(1 - L_m)$$

$$+ cm^2 \left\{ \frac{1}{2} \zeta(2) + \frac{1}{2} \frac{L_m^2 - L_m + 1}{2} \right\},$$

$$B_0(p^2; m^2, M^2) = \int \mathcal{D}^n k \frac{1}{(k^2 - m^2)((k + p) - M^2)}$$

$$= \frac{1}{\epsilon} - \frac{1}{2} (L_m + L_M) + 2 - \frac{y - \frac{1}{2} \ln y + \tilde{R}}{2x}$$

$$+ \frac{\epsilon}{2} \left\{ \zeta(2) + 8 + \frac{1}{4} (L_m + L_M)^2 + \frac{1}{4} \ln^2 y + 4\tilde{R} \right\}$$

$$+ \left( L_m + L_M \right) \left( -2 + \frac{y - \frac{1}{2} \ln y - \tilde{R}}{2x} \right)$$

$$+ \frac{2(y - 1)}{x} \ln y + \frac{\tilde{r}_1 - \tilde{r}_2}{x} \left\{ \ln w \ln z - \ln \tilde{w} \ln \tilde{z} \right\}$$

$$+ \text{Li}_2(z) - \text{Li}_2(\tilde{z}) + \text{Li}_2(w) - \text{Li}_2(\tilde{w}) \right\}.$$  

B. Evaluation of the Basis Scalar Integrals

The scalar integrals entering Eq. (81) are well known, either analytically or numerically [18] [19] [21] [22]. Before writing their expressions, the following definitions are needed

$$L_M = \gamma_E + \ln \left( \frac{M^2}{4\pi \mu^2} \right),$$

$$L_m = \gamma_E + \ln \left( \frac{m^2}{4\pi \mu^2} \right),$$

$$L_{|p|} = \gamma_E + \ln \left( \frac{|p|^2}{4\pi \mu^2} \right),$$

where $\gamma_E$ is Euler’s constant. Next, we define

$$x = \frac{p^2}{M^2},$$

$$y = \frac{m^2}{M^2},$$

$$R = \frac{m^2(r_1 - r_2)}{p^2} \ln r_1,$$

$$\tilde{R} = \frac{1}{2} \frac{\tilde{r}_1 - \tilde{r}_2}{x} \left( \ln \tilde{r}_1 - \ln \tilde{r}_2 \right),$$

$$\tilde{R} = \frac{1}{2} \frac{\tilde{r}_1 - \tilde{r}_2}{x} \left( \ln \tilde{r}_1 - \ln \tilde{r}_2 \right).$$
\[ B_0(p^2; m^2, m^2) = \frac{1}{\epsilon} + R + 2 - L_m + \frac{\epsilon}{2} \left\{ \zeta(2) \right\} \]
\[ + 8 + L_m^2 - 2(R + 2) + 4R + \frac{m^2 (r_1 - r_2)}{p^2} \]
\[ \times \left[ \ln w_0 \ln z_0 - \ln \tilde{w}_0 \ln \tilde{z}_0 + \text{Li}_2(z_0) - \text{Li}_2(\tilde{z}_0) \right] \]  
\[ + \text{Li}_2(w_0) - \text{Li}_2(\tilde{w}_0) \} \right\} . \quad \text{(101)} \]

The following products of one-loop integrals enter in Eq. [11]
\[ T_{1245} = \left( B_0(p^2; m^2, m^2) \right)^2 \]
\[ T_\otimes = 2 \left( \frac{n - 2}{2m^2} A_0(m^2) - B_0(p^2; m^2, m^2) \right) \]
\[ \times \left( 4m^2 - M^2 \right) B_0(m^2; M^2, m^2) , \quad \text{(102)} \]

leading to
\[ T_{1245} = \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left( 2R + 4 - 2L_m \right) + 2L_m^2 \]
\[ - 4(R + 2)L_m + R^2 + 8R + 12 + \zeta(2) \]
\[ + \frac{m^2}{p^2} (r_1 - r_2) \left\{ \ln w_0 \ln z_0 - \ln \tilde{w}_0 \ln \tilde{z}_0 \right. \]
\[ + \text{Li}_2(z_0) - \text{Li}_2(\tilde{z}_0) + \text{Li}_2(w_0) - \text{Li}_2(\tilde{w}_0) \} \right\} , \quad \text{(103)} \]

\[ T_\otimes = \left( 4m^2 - M^2 \right) (R + 2) \left\{ \frac{-2}{\epsilon} + 3L_m + L_M \right\} \]
\[ - \frac{m^2}{p^2} \left( 4m^2 - M^2 \right) (r_1 - r_2) \left\{ \ln w_0 \ln z_0 + \text{Li}_2(z_0) \right. \]
\[ - \ln \tilde{w}_0 \ln \tilde{z}_0 - \text{Li}_2(\tilde{z}_0) + \text{Li}_2(w_0) - \text{Li}_2(\tilde{w}_0) \} \right\} \]
\[ + \left\{ \frac{1}{m^2} \ln \left( \frac{m^2}{M^2} \right) - 8 - \left. \tilde{R} \right|_{p^2 = m^2} \right\} \]
\[ \times \left( 4m^2 - M^2 \right) (R + 2) . \quad \text{(104)} \]

The vacuum integrals \( T_{134} \) and \( T_{1134} \) are: (see [19][22][25][27])
\[ T_{134} = \frac{1}{2\epsilon^2} \left( 2m^2 + M^2 \right) + \frac{1}{\epsilon} \left\{ \frac{3}{2} \left( 2m^2 + M^2 \right) - 2m^2 L_m \right\} \]
\[ - M^2 L_m \right\} + 2m^2 (L_m^2 - 3L_m) + M^2 (L_M^2 - 3L_M) \]
\[ + \left\{ \frac{7}{2} + \frac{\zeta(2)}{2} \right\} \left( 2m^2 + M^2 \right) - \frac{M^2}{2} \ln \left( \frac{m^2}{M^2} \right) \]
\[ + \left\{ \frac{1}{2} \left( - 4\text{Li}_2 \left( \frac{1 - \lambda}{2} \right) + 2 \ln^2 \left( \frac{1 - \lambda}{2} \right) \right) \right. \]
\[ - \ln^2 \left( \frac{m^2}{M^2} \right) + \frac{\pi^2}{3} \right\} , \quad \text{(105)} \]
\[ T_{1134} = \frac{1}{2\epsilon^2} + \frac{1}{\epsilon} \left\{ \frac{1}{2} - L_m \right\} + \frac{1}{2} + \frac{\zeta(2)}{2} \left( L_m^2 \right) \]
\[ - L_m - \frac{1}{2} \left\{ - 4\text{Li}_2 \left( \frac{1 - \lambda}{2} \right) + 2 \ln^2 \left( \frac{1 - \lambda}{2} \right) \right\} \right. \]
\[ - \ln^2 \left( \frac{m^2}{M^2} \right) + \frac{\pi^2}{3} \right\} , \quad \text{(106)} \]

where
\[ \lambda = \sqrt{1 - 4m^2} / M^2 . \quad \text{(107)} \]

The master integral \( T_{12345} \) is the only one that remains finite as \( n \to 4 \). It will be evaluated numerically using the following integral representation, first obtained in [18]
\[ T_{12345}(p^2; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) \]
\[ = - \frac{1}{\pi^4} \int \left( \frac{d^4 k d^4 q}{(k^2 - m_1^2) ((k + p)^2 - m_2^2) ((q - k)^2 - m_3^2) \right) \]
\[ \times \left( q^2 - m_4^2 \right) \left( (q + p)^2 - m_5^2 \right) \}
\[ \times \ln \left( \frac{w_1 + w_2}{w_2 + w_3 + w_5} \right) \frac{(w_1 + w_3 + w_4)(w_2 + w_3 + w_5)}{(w_1 + w_3 + w_5)(w_2 + w_3 + w_4)} \right\} , \quad \text{(108)} \]

where
\[ w_1 = \sqrt{x^2 - \frac{m_2^2}{p^2} + i\epsilon} , \quad \text{(109)} \]
\[ w_2 = \sqrt{(x + 1)^2 - \frac{m_2^2}{p^2} + i\epsilon} , \quad \text{(110)} \]
\[ w_3 = \sqrt{(x + y)^2 - \frac{m_2^2}{p^2} + i\epsilon} , \quad \text{(111)} \]
\[ w_4 = \sqrt{y^2 - \frac{m_2^2}{p^2} + i\epsilon} , \quad \text{(112)} \]
\[ w_5 = \sqrt{(y - 1)^2 - \frac{m_2^2}{p^2} + i\epsilon} . \quad \text{(113)} \]

Notice that this representation holds for arbitrary masses, so that to recover \( T_{12345} \) in \( n = 4 \) dimensions one sets \( m_1 = m_2 = m_4 = m_5 = m \) and \( m_3 = M \).

The Semi-Numerical Algorithm

The remainder of the \( p^2 \)-dependent integrals, \( T_{234}, T_{1245}, T_{11234} \), will be evaluated using the semi-numerical algorithm described in [22], in conjunction
with some analytically calculated integrals from [20].

The goal of the algorithm is to write a T integral as a sum $T_A + T_N$, where $T_A$ involves massless propagators, thus expressed analytically, and $T_N$ is finite in $n = 4$ dimensions, thus evaluated numerically. We consider the case of $T_{234}$ with arbitrary masses

\[
T_{234}(p^2; m_2^2, m_3^2, m_4^2) = \int \frac{D^n k D^n q}{((k + q)^2 - m_2^2)((k - q)^2 - m_3^2)(q^2 - m_4^2)} .
\]  

(114)

The algorithm entails a substitution using the following simple identities

\[
\frac{1}{(k - q)^2 - m_3^2} = \frac{1}{(k - q)^2} + \frac{m_3^2}{(k - q)^2 - m_3^2} 
\]  

(115)

\[
\frac{1}{q^2 - m_4^2} = \frac{1}{q^2} + \frac{m_4^2}{q^2 - m_4^2} .
\]  

(116)

On the right-hand side of each of the above equations the first term replaces a massive propagator with a massless one and the second term decreases the degree of divergence of the integral. Equation (114) then simplifies to

\[
T_{234}(p^2; m_2^2, m_3^2, m_4^2) = T_{234}(p^2; m_2^2, m_3^2, 0) + T_{234}(p^2; m_2^2, 0, m_3^2) - T_{234}(m_2^2, 0, 0) + m_3^2 m_4^2 T_{23344}(p^2; m_2^2, m_3^2, 0, 0, m_4^2, 0) .
\]  

(117)

The first three terms on the right-hand side of Eq. (117) have analytic expressions to be found in the literature [20] [22], and the last term is finite for $n \to 4$. Hence, the goal of the algorithm is achieved in the case of $T_{234}$, leading to

\[
T_{2344}(p^2; m_2^2, m_3^2, m_4^2, m_5^2) = T_{234}(p^2; m_2^2, m_3^2, 0) + T_{234}(p^2; m_2^2, 0, m_3^2) - T_{234}(m_2^2, 0, 0) + m_3^2 m_4^2 T_{23344}(p^2; m_2^2, m_3^2, 0, 0, m_4^2, 0) .
\]  

(118)

and

\[
T_{234N}(p^2; m_2^2, m_3^2, m_4^2) = m_3^2 m_4^2 T_{23344}(p^2; m_2^2, m_3^2, 0, 0, m_4^2, 0) .
\]  

(119)

The remaining two integrals

\[
T_{1234}(p^2; m_1^2, m_2^2, m_3^2, m_4^2) = \int D^n k D^n q \frac{1}{(k - p)^2 - m_1^2} \times \frac{1}{((k - q)^2 - m_2^2)(q^2 - m_4^2)} ,
\]  

(120)

\[
T_{11234}(p^2; m_1^2, m_2^2, m_3^2, m_4^2) = \int D^n k D^n q \frac{1}{(k - p)^2 - m_1^2} \times \frac{1}{((k - q)^2 - m_2^2)(q^2 - m_4^2)} ,
\]  

(121)

can be treated similarly using Eqs. (115)-(116) to give

\[
T_{1234A}(p^2; m_1^2, m_2^2, m_3^2, m_4^2) = T_{1234}(p^2; m_1^2, m_2^2, 0, 0) ,
\]  

(122)

\[
T_{1234N}(p^2; m_1^2, m_2^2, m_3^2, m_4^2) = m_1^2 T_{12334}(p^2; m_1^2, m_2^2, m_3^2, 0, 0) + m_2^2 T_{12344}(p^2; m_1^2, m_2^2, 0, m_4^2, 0) + m_3^2 m_4^2 T_{13344}(p^2; m_1^2, m_2^2, 0, 0, m_4^2, 0) ,
\]  

(123)

as well as

\[
T_{11234A}(p^2; m_1^2, m_2^2, m_3^2, m_4^2) = T_{1234}(p^2; m_1^2, m_2^2, 0, 0) ,
\]  

(124)

\[
T_{11234N}(p^2; m_1^2, m_2^2, m_3^2, m_4^2) = m_1^2 T_{11334}(p^2; m_1^2, m_2^2, m_3^2, 0, 0) + m_2^2 T_{11344}(p^2; m_1^2, m_2^2, 0, m_4^2, 0) + m_3^2 m_4^2 T_{113344}(p^2; m_1^2, 0, 0, 0, m_4^2, 0) .
\]  

(125)

Analytical expressions for Eq. (122) and Eq. (124) are derived in [20] using a combination of Cutkosky’s cutting rules to extract the imaginary part of the integrals, and dispersion relations to recover their respective real parts. Equations (123) and (125) only involve finite integrals in the limit $n \to 4$ and so can be treated numerically. Berends and Tausk [23] show how to evaluate these integrals by adapting Kreimer’s method [15] to obtain analogous double-integral representations
\[ T_{11234N}(p^2; m_1^2, m_2^2, m_3^2, m_4^2) = \frac{4}{p^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \]

\[
\frac{1}{(w_1^2 - w_2^2)^2} \left\{ \ln \left[ \frac{(w_1 + \bar{w}_3 + \bar{w}_4)(w_2 + \bar{w}_3 + \bar{w}_4)}{(w_1 + w_3 + w_4)(w_2 + w_3 + w_4)} \right] - \frac{2}{w_1(w_1 + w_3 + w_4)(w_1 + \bar{w}_3 + \bar{w}_4)} \right\},
\]

where the new terms \( \bar{w}_3 \) and \( \bar{w}_4 \) are

\[
\bar{w}_3 = \sqrt{(x + y)^2 + \epsilon},
\]

\[
\bar{w}_4 = \sqrt{y^2 + \epsilon}.
\]

The analytical parts of the T-integrals are given by \([22, 20]\)

\[
T_{234A} = \frac{1}{2\epsilon^2}(2m^2 + M^2) + \frac{1}{\epsilon} \left\{ 3m^2 + \frac{3}{2}M^2 \right\}
\]

\[+ 2m^2 L_m - M^2 L_M - \frac{1}{4}\left( \frac{p}{2} \right)^2 + M^2 \left( L_M^2 - 3L_M \right) \]

\[+ 2m^2 (L_m^2 - 3L_m) + \frac{1}{2} L|p| + 3(2m^2 + M^2) \]

\[+ \frac{1}{2}M^2 \zeta(2) - \frac{1}{4}(m^2 + M^2) \ln \left( \frac{m^2}{M^2} \right) \]

\[+ \left\{ \text{Li}_2 \left( \frac{m^2 - M^2}{m^2} \right) - \text{Li}_2 \left( \frac{M^2 - m^2}{M^2} \right) \right\} \]

\[\times \frac{1}{2}(m^2 - M^2) - \frac{p^2}{4} \left\{ \ln \left( \frac{p^2}{m^2} \right) + \ln \left( \frac{p^2}{M^2} \right) + \frac{13}{2} \right\} \]

\[+ \frac{1}{4}p^2 \left\{ \left( \frac{m^2}{p^2} \right)^2 - \left( \frac{M^2}{p^2} \right)^2 \right\} \ln \left( \frac{m^2}{M^2} \right) \]

\[+ \frac{1}{2}m^2 \left( \frac{m^2}{p^2} - \frac{p^2}{m^2} \right) \ln \left( 1 - \frac{p^2}{m^2} \right) - m^2 \text{Li}_2 \left( \frac{p^2}{m^2} \right) \]

\[+ \frac{1}{2}(p^2 + 2m^2)R - \frac{1}{2}(p^2 + m^2 + M^2) \tilde{R} \]

\[+ 2m^2 \left( 1 - \frac{m^2}{p^2} \right) \left\{ \text{Li}_2 \left( 1 - \tilde{r}_1 \right) + \text{Li}_2 \left( 1 - \tilde{r}_2 \right) \right\} \]

\[+ m^2 \left( 1 - \frac{M^2}{p^2} \right) \left\{ \text{Li}_2 \left( 1 - \tilde{r}_1 \right) + \text{Li}_2 \left( 1 - \tilde{r}_2 \right) \right\} \]

\[= \text{Li}_2 \left( \frac{m^2 - M^2}{m^2} \right) \]

\[+ M^2 \left( 1 - \frac{m^2}{p^2} \right) \left\{ \text{Li}_2(1 - \tilde{r}_1) + \text{Li}_2(1 - \tilde{r}_2) \right\} \]

\[= \text{Li}_2 \left( \frac{M^2 - m^2}{M^2} \right) \}

\[T_{11234A} = \frac{1}{2\epsilon^2} + \frac{1}{\epsilon} \left\{ \frac{5}{2} - L_m + R \right\} + \frac{19}{2} + \frac{3}{2}\zeta(2) \]

\[+ L_M^2 - (5 + 2R)L_M + \left( \frac{m^2}{p^2} - 1 \right) \ln \left( 1 - \frac{p^2}{m^2} \right) \]

\[+ \text{Li}_2 \left( \frac{p^2}{m^2} \right) + 4R + \frac{m^2(r_1 - r_2)}{2p^2} \left\{ \ln^2(1 + r_2) \right. \]

\[\left. - \ln^2(1 + r_1) + 2\text{Li}_2 \left( \frac{1}{1 + r_2} \right) \right\} \]

\[- 2\text{Li}_2 \left( \frac{1}{1 + r_1} \right) - \text{Li}_2(1 - r_1) + \text{Li}_2(1 - r_2) \]

\[- \text{Li}_2(r_2(1 - r_2)) - \eta\left( 1 - \frac{p^2}{m^2}, r_2 \right) \ln(r_2(1 - r_2)) \]

\[+ \text{Li}_2(r_1(1 - r_1)) + \eta\left( 1 - \frac{p^2}{m^2}, r_1 \right) \ln(r_1(1 - r_1)) \right\}, \]

\[C. Renormalization \]

We introduce the following subscript notation for the T-integrals

\[ T_{i_1...i_j} = T_{i_1...i,j,D} + T_{i_1...i,F}, \]

where \( T_{i_1...i,j,D} \) contains (i) the divergent part of \( T_{i_1...i,j} \), and (ii) terms dependent on the renormalization scale. \( T_{i_1...i,F} \) contains the remainder of the finite \( (O(\epsilon^0)) \) terms of \( T_{i_1...i,j} \). For some of the T-integrals, \( T_{i_1...i,F} \) will be at least partially evaluated numerically using Kreimer’s double integral representation \([13]\). Turning to each of
the integrals entering Eq. (81), we have
\[
T_{134D} = \frac{2m^2 + M^2}{2\epsilon^2} + \frac{1}{\epsilon} \left\{ 3m^2 + \frac{3}{2} M^2 - 2m^2 L_m ight\} - M^2 L_M \bigg| + 2m^2(L_m^2 - 3L_m) + M^2(L_M^2 - 3L_M),
\]
\[
T_{1134D} = \frac{1}{2\epsilon^2} + \frac{1}{\epsilon} \left\{ \frac{1}{2} - L_m \right\} + L_m^2 - L_m,
\]
\[
T_{1245D} = \frac{1}{\epsilon^2} + \frac{2}{\epsilon} (R + 2 - L_m) + 2L_m^2 - 4(R + 2)L_m,
\]
\[
T_{0D} = (4m^2 - M^2)(R + 2) \left\{ -\frac{2}{\epsilon} + 3L_m + L_M \right\},
\]
\[
T_{234D} = \frac{1}{2\epsilon^2} (2m^2 + M^2) + \frac{1}{\epsilon} \left\{ 3m^2 + \frac{3}{2} M^2 - 2m^2 L_m - M^2 L_M - \frac{1}{4} R^2 \right\} + 2m^2(L_m^2 - 3L_m) + M^2(L_M^2 - 3L_M) + \frac{1}{2} p^2 L_{[p]},
\]
\[
T_{1234D} = \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left\{ \frac{5}{2} - L_m + R \right\} + L_m^2 - (5 + 2R)L_m,
\]
\[
T_{11234D} = \frac{R}{4m^2 - p^2} \left\{ \frac{1}{\epsilon} - 2L_m \right\},
\]
Returning to the rho-meson self-energy, Eq. (81) can be split into a one-loop and a two-loop contribution
\[
F_{\text{vac}} = \alpha F_{\text{vac}}^{(1)} + \alpha^2 F_{\text{vac}}^{(2)},
\]
with
\[
F_{\text{vac}}^{(1)} = \frac{1}{3 - 2\epsilon} \left\{ (4m^2 - p^2)B_0(p^2; m^2, m^2) - 4(1 - \epsilon)A_0(m^2) \right\} - p^2 \delta Z_3^{(1)},
\]
\[
F_{\text{vac}}^{(2)} = \frac{1}{3 - 2\epsilon} \left\{ -8(1 - \epsilon)T_{134D} + 8(1 - \epsilon)T_{234} - 4(1 - \epsilon)(4m^2 - M^2)T_{1134D} + 4(8m^2 - 2M^2 - 2p^2) \times T_{1234D} - (8m^2 - 2M^2 - 4p^2)T_{1245D} + 2(4m^2 - p^2) \times (4m^2 - M^2)T_{11234D} + 4m^2 - 2M^2 - p^2 \times (4m^2 - M^2 - 2p^2)T_{12345D} + T_{0} - p^2 \delta Z_3^{(2)},
\]
where \( n = 4 - 2\epsilon \) has been used, and \( \alpha \) is defined in Eq. (24). Imposing the on-shell renormalization condition
\[
\text{Re} \left[ F_{\text{vac}}|p^2=M^2\right] = 0,
\]
ensures that \( M \) is the physical mass of the \( \rho^0 \)-meson, \( M = 775.5 \text{ MeV} \) [8], and
\[
\text{Re} \left[ F_{\text{vac}}^{(1)}|p^2=M^2\right] = 0,
\]
\[
\text{Re} \left[ F_{\text{vac}}^{(2)}|p^2=M^2\right] = 0.
\]

**One-loop Contribution**

The one-loop contribution to the self energy is
\[
F_{\text{vac}}^{(1)} = \frac{1}{3} \left\{ 1 + \frac{2\epsilon}{3} \right\} \left\{ (4m^2 - p^2)B_0(p^2; m^2, m^2) - 4(1 - \epsilon)A_0(m^2) \right\} - p^2 \delta Z_3^{(1)} + O(\epsilon).
\]
Substituting this expression in Eqs. (89) and (101), \( F_{\text{vac}} \) becomes
\[
F_{\text{vac}}^{(1)} = \frac{1}{3} \left\{ -\frac{p^2}{\epsilon} + p^2 L_m + 8m^2 - \frac{8p^2}{3} + (4m^2 - p^2)R \right\} - p^2 \delta Z_3^{(1)} + O(\epsilon).
\]
Notice that \( R, \) Eq. (88), can be written as
\[
R = \left( 1 - \frac{4m^2}{p^2} \right)^\frac{1}{2} \left\{ \ln \frac{1 - \sqrt{1 - \frac{4m^2}{p^2}}}{1 + \sqrt{1 - \frac{4m^2}{p^2}}} + \frac{4m^2}{p^2} - 1 \right\}^{\frac{1}{2}} \cos^{-1} \left( 1 - \frac{p^2}{2m^2} \right) \times \left[ \theta(p^2 - 4m^2) - \theta(p^2) \right],
\]
so that Eq. (149) becomes
\[
F_{\text{vac}}^{(1)} = \frac{1}{3} \left\{ -\frac{p^2}{\epsilon} + p^2 L_m + 8m^2 - \frac{8p^2}{3} + p^2 \left( 1 - \frac{4m^2}{p^2} \right) \right\}^{\frac{1}{2}} \left\{ \ln \frac{1 + \sqrt{1 - \frac{4m^2}{p^2}}}{1 - \sqrt{1 - \frac{4m^2}{p^2}}} - \frac{4m^2}{p^2} - 1 \right\}^{\frac{1}{2}} \cos^{-1} \left( 1 - \frac{p^2}{2m^2} \right) \times \left[ \theta(p^2 - 4m^2) - \theta(p^2) \right] - p^2 \delta Z_3^{(1)} + O(\epsilon).
\]
The on-shell renormalization condition, Eq. (145), leads to the one-loop contribution to the \( \rho^0 \) wave-function renor-
malization constant
\[
\delta Z_3^{(1)} = \frac{1}{3} \left\{ -\frac{1}{\epsilon} + L_\text{m} + \frac{8m^2}{M^2} - \frac{8}{3} \left[ \left( 1 - \frac{4m^2}{M^2} \right)^{3/2} \left( 1 + \frac{1}{\sqrt{1 - \frac{4m^2}{M^2}}} \right) - O(\epsilon) \right] \right\},
\]

as well as to the one loop contribution to the rho-meson self-energy in the limit \(\epsilon \to 0\)
\[
P_{\text{vac}}^{(1)} = \frac{\mu^2}{3} \left\{ -\frac{1}{\epsilon} + L_\text{m} + \frac{8m^2}{M^2} - \frac{8}{3} \left[ \left( 1 - \frac{4m^2}{M^2} \right)^{3/2} \left( 1 + \frac{1}{\sqrt{1 - \frac{4m^2}{M^2}}} \right) - O(\epsilon) \right] \right\}.
\]

Two-loop Contribution

Next, we consider the two-loop contribution to the self-energy. The cancellation of divergences is now less obvious. To analyse \(P_{\text{vac}}^{(2)}\) we write it as
\[
P_{\text{vac}}^{(2)} = \frac{\mu^2}{3} \left\{ -\frac{1}{\epsilon} + L_\text{m} + \frac{8m^2}{M^2} - \frac{8}{3} \left[ \left( 1 - \frac{4m^2}{M^2} \right)^{3/2} \left( 1 + \frac{1}{\sqrt{1 - \frac{4m^2}{M^2}}} \right) - O(\epsilon) \right] \right\}.
\]

where
\[
F_D^{(2)} = \frac{1}{3} \left\{ -\frac{8}{3} \left( 1 - \frac{4m^2}{M^2} \right)^{3/2} \left( 1 + \frac{1}{\sqrt{1 - \frac{4m^2}{M^2}}} \right) - O(\epsilon) \right\},
\]
and
\[
F_F^{(2)} = \frac{1}{3} \left\{ -\frac{8}{3} \left( 1 - \frac{4m^2}{M^2} \right)^{3/2} \left( 1 + \frac{1}{\sqrt{1 - \frac{4m^2}{M^2}}} \right) - O(\epsilon) \right\}.
\]

Next, we consider the divergent terms in \(F_{\text{vac}}^{(2)}\), i.e. only \(F_D^{(2)}\) and \(p^2 \delta Z_3^{(2)}\), as \(F_F^{(2)}\) is entirely finite. Substituting Eqs. (135)-(141) into Eq.(156), and after some cancellations one finds
\[
F_D^{(2)} = -\frac{2p^2}{\epsilon} + \frac{8p^2}{3} L_\text{m} + \frac{4p^2}{3} L_{|p|} + \left[ \ln \left( \frac{M^2}{m^2} \right) + \frac{4}{3} \right] R + 2 \left[ \ln \left( \frac{M^2}{m^2} \right) + \frac{5}{3} \right] - \frac{2p^2}{3}.
\]

The divergent part of this result is cancelled by \(p^2 \delta Z_3^{(2)}\), thus rendering the self energy finite. Notice that all terms proportional to \(\frac{1}{\epsilon^2}\) vanish. Next, we determine \(\delta Z_3^{(2)}\) explicitly using the renormalization condition Eq.(147). Equation (155) can be written as
\[
P_{\text{vac}}^{(2)} = -\frac{2p^2}{\epsilon} + \frac{8p^2}{3} L_\text{m} + \frac{4p^2}{3} L_{|p|} + \left( 4m^2 - M^2 \right)
\times \left\{ \left( \ln \left( \frac{M^2}{m^2} \right) + \frac{4}{3} \right) R + 2 \left( \ln \left( \frac{M^2}{m^2} \right) + \frac{5}{3} \right) - \frac{2p^2}{3} \right\},
\]

Imposing the on-mass-shell renormalization condition, Eq. (145), leads to
\[
\delta Z_3^{(2)} = -\frac{2p^2}{\epsilon} + \frac{8p^2}{3} L_\text{m} + \frac{4p^2}{3} L_{|p|} - \frac{2p^2}{3} + 2 \left( \ln \left( \frac{M^2}{m^2} \right) + \frac{5}{3} \right) + \frac{1}{M^2} \left( F_F^{(2)} \right|_{p^2 = M^2}.
\]
Using

\[ \text{Re} \left[ R|_{p^2=M^2} \right] = \sqrt{1- \frac{4m^2}{M^2}} \ln \left| 1 - \sqrt{1 - \frac{4m^2}{M^2}} \right| , \quad (161) \]

one obtains

\[ \delta Z_3^{(2)} = -\frac{2}{\alpha} + \frac{8}{3} L_m + \frac{4}{3} L_M = -\frac{2}{\alpha} - \left(1 - \frac{4m^2}{M^2}\right)^{3/2} \times \left( \ln \left( \frac{M^2}{m^2} \right) + \frac{4}{3} \right) \ln \left| 1 - \sqrt{1 - \frac{4m^2}{M^2}} \right| + 2 \left( \frac{4m^2}{M^2} - 1 \right) \left( \ln \left( \frac{M^2}{m^2} \right) + \frac{5}{3} \right) + \frac{1}{M^2} \text{Re} \left[ F_F^{(2)} \right] |_{p^2=M^2} , \quad (162) \]

where \( F_F^{(2)} \) is defined in Eq.\((157)\). Substituting Eq.\((162)\) into Eq.\((159)\) gives

\[ F_F^{(2)} = f(p^2) + F_F^{(2)} , \quad (163) \]

where

\[ f(p^2) = \frac{4p^2}{3} \ln \left| \frac{p^2}{M^2} \right| + \left( \frac{4m^2 - M^2}{M^2} \right) \left( \ln \left( \frac{M^2}{m^2} \right) + \frac{4}{3} \right) \times \left\{ R - \frac{p^2}{M^2} \sqrt{1 - \frac{4m^2}{M^2}} \ln \left| 1 - \sqrt{1 - \frac{4m^2}{M^2}} \right| \right\} + 2 \left( \frac{4m^2}{M^2} - 1 \right) \left( \ln \left( \frac{M^2}{m^2} \right) + \frac{5}{3} \right) \left( \frac{1 - p^2}{M^2} \right) - \frac{p^2}{M^2} \text{Re} \left[ F_F^{(2)} \right] |_{p^2=M^2} . \quad (164) \]

IV. ELECTROMAGNETIC PION FORM FACTOR

The rho-meson self energy, \( F_{vac} \), to two-loop order, Eq.\((142)\), is determined from Eqs.\((153)\) and \((163)\), up to the value of the coupling \( \alpha \), Eq.\((24)\). This is to be fixed by fitting the pion form factor, Eq.\((8)\), to experimental data. Using the BESIII data \([28]\) we find \( g_{\rho\pi\pi} = 5.37 \) in good agreement with VMD, \( g_{\rho\pi\pi} \approx f_{\rho} \), and \( f_{\rho} = 4.97 \pm 0.07 \) \([8]\). It is also in agreement with a recent Lattice QCD result \([29]\)

\[ g_{\rho\pi\pi} = 5.69(13)(16) . \quad (165) \]

The effective coupling, absorbing the dimensional regularization factor, at one-loop level, is \( (g_{\rho\pi\pi}/4\pi)^2 = 0.18 \). At the two-loop level we find, instead, \( g_{\rho\pi\pi} = 2.24 \), and

\[ \text{FIG. 5: One and two-loop KLZ results, together with the BESIII data \([28]\) for the squared electromagnetic form factor of the pion in the time-like region.} \]

\[ \text{FIG. 6: Imaginary part of } \alpha F_{vac}^{(1)} \text{ (broken line), and of } \alpha^2 F_{vac}^{(2)} \text{ (solid line), as a function of } s \equiv p^2 \geq 0. \]
\[(g_{\rho\pi\pi}/4\pi)^2 = 0.03.\] At this level the pion form factor is given by
\[
F_\pi(s) = \frac{M^2}{M^2 - s + \alpha F_{\text{vac}}^{(1)}(s) + \alpha^2 F_{\text{vac}}^{(2)}(s)},
\]
where \(F_{\text{vac}}^{(1)}(s)\) is given in Eq. (153), and \(F_{\text{vac}}^{(2)}(s)\) in Eq. (163). This form factor is shown in Fig. 5 together with the BESIII data [28], and the one-loop result. Clearly, the two-loop result does not improve the agreement with data. This is in spite of the substantial reduction of the numerical value of the strong coupling at the two-loop level.

V. CONCLUSION

As mentioned in the Introduction, at the one-loop level, the KLZ pion form factor in the space-like region, from the triangle diagram [7], provides an excellent fit to the data. In particular, the pion radius agrees quite well with experiment. However, in the time-like region one-loop results from the vacuum polarization, while good, do not quite match such an agreement, as shown Fig. 5. Nevertheless, the magnitude of this disagreement would suggest that a two-loop determination could improve the outcome, given that the expansion parameter at the one-loop level is \(g^2/(4\pi)^2 \approx 0.18\), and at two-loops it becomes \(g^2/(4\pi)^2 = 0.03\). Unfortunately, this is not the case, as the strong coupling quenching is overwhelmed by the size of the two-loop imaginary part of the vacuum polarization, as shown in Fig. 6. Hence, the KLZ theory would have to remain an effective strong interaction theory at the one-loop level.

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