Large-Time Asymptotics of Solutions to the Kramers-Fokker-Planck Equation with a Short-Range Potential

Xue Ping Wang

Laboratoire de Mathématiques Jean Leray, UMR CNRS 6629, Université de Nantes, 44322 Nantes Cedex 3, France. E-mail: xue-ping.wang@univ-nantes.fr

Received: 15 April 2014 / Accepted: 24 September 2014
Published online: 15 January 2015 – © Springer-Verlag Berlin Heidelberg 2015

Abstract: In this work, we use a scattering method to study the Kramers-Fokker-Planck equation with a potential whose gradient tends to zero at the infinity. For short-range potentials in dimension three, we show that complex eigenvalues do not accumulate at low-energies and obtain the low-energy resolvent asymptotics. This, combined with high energy pseudospectral estimates valid in more general situations, gives the large-time asymptotics of solutions in weighted \( L^2 \) spaces.

1. Introduction

The Kramers equation [23], also called the Fokker-Planck equation [7,10,11] or the Kramers-Fokker-Planck equation [12,13], is the evolution equation for the distribution functions describing the Brownian motion of particles in an external field:

\[
\frac{\partial W}{\partial t} = \left( -v \cdot \nabla_x + \nabla_v \cdot (\gamma v - \frac{F(x)}{m}) + \frac{\gamma kT}{m} \Delta_v \right) W, \quad (1.1)
\]

where \( W = W(t; x, v) \) for \( x, v \in \mathbb{R}^n \) and \( t > 0 \), and \( F(x) = -m \nabla V(x) \) is the external force. In this equation, \( x \) and \( v \) represent the position and velocity of particles, \( m \) the mass, \( k \) the Boltzmann constant, \( \gamma \) the friction coefficient and \( T \) the temperature of the media. This equation is a special case of the more general Fokker-Planck equation [23] or the Kolmogorov forward equation for continuous-time diffusion processes [16].

After a change of unknowns and for appropriate values of physical constants, the Kramers-Fokker-Planck equation (1.1) can be written into the form [10,23]

\[
\partial_t u(t; x, v) + Pu(t; x, v) = 0, \quad (x, v) \in \mathbb{R}^n \times \mathbb{R}^n, \quad n \geq 1, \ t > 0, \quad (1.2)
\]

Research supported in part by French ANR Project NOSEVOL BS01019 01 and by Chinese Qian Ren programme at Nanjing University.
with some initial data

\[ u(0; x, v) = u_0(x, v). \]  

Here \( P \) is the Kramers-Fokker-Planck (KFP, in short) operator:

\[ P = -\Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2} + v \cdot \nabla_x - \nabla V(x) \cdot \nabla_v, \]  

where the potential \( V(x) \) is supposed to be a real-valued \( C^1 \) function. Define \( m \) by

\[ m(x, v) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}\left(\frac{|v|^2}{2} + V(x)\right)}. \]  

Then one has \( Pm = 0 \).

The large-time asymptotics of the solution is motivated by mathematical analysis of the trend to equilibrium in statistical physics and is studied by several authors in the case where \( |\nabla V(x)| \to +\infty \) as \( |x| \to +\infty \). See for example [7, 9–12, 24] and also [2, 8, 18, 19, 22] for related equations. Note that in the works [11, 12, 24], the case where \( |\nabla V(x)| \geq C > 0 \) outside some compact set is also analyzed. In all these situations, the spectrum of \( P \) is discrete in a neighbourhood of zero and nonzero eigenvalues of \( P \) are of strictly positive real parts. If in addition \( V(x) > 0 \) outside some compact set, \( m^2 \) is the Maxwellian if \( V(x) \) is normalized by

\[ \int_{\mathbb{R}^n} e^{-V(x)} dx = 1, \]  

and \( m \) is an eigenfunction of \( P \) associated with the eigenvalue zero. It is proven in these situations that the solution \( u(t) \) to (1.2) with initial data \( u_0 \) converges exponentially rapidly to \( m \) in the sense that there exists some \( \sigma > 0 \) such that for any nice initial data \( u_0 \), one has in appropriate spaces

\[ u(t) = \langle m, u_0 \rangle m + O(e^{-\sigma t}), \quad t \to +\infty. \]  

The rate \( \sigma \) can be evaluated in some regimes in terms of the spectral gap between zero and the real parts of other nonzero eigenvalues. Since the change of unknowns from (1.1) to (1.2) is essentially given by \( W(t) = mu(t) \) with appropriate choice of physical constants, this result shows that the distribution functions governed by (1.1) always tend, up to some multiplicative constant depending only on the initial data, to the Maxwellian \( m^2 \), as \( t \to +\infty \) and justifies the phenomenon of the return to equilibrium in statistical physics. Since the KFP operator is neither elliptic nor selfadjoint, the proof of such a result is highly nontrivial and is first realized by the entropy method in [7] (see also [24]) and later on by microlocal and spectral methods in [10–12]. The existence of a spectral gap between the eigenvalue zero and the other part of the spectrum of \( P \) is crucial in these works.

If \( V(x) \) is slowly increasing or decreasing so that \( |\nabla V(x)| \to 0 \) as \( |x| \to \infty \), zero may be or may not be an eigenvalue of \( P \). But now one can check that the essential spectrum of \( P \) is equal to \([0, +\infty] \) and there is no spectral gap between zero and the other part of the spectrum of \( P \). A natural question at this connection is whether there still exists some phenomenon of the trend to equilibrium in such cases.

The goal of this work is to study spectral properties of the KFP operator and large-time asymptotics of the solutions to the KFP equation for potentials \( V(x) \) such that
\(|\nabla V(x)| \to 0\) as \(|x| \to +\infty\). Although our final result concerns only short-range potentials in dimension three, some preliminary results hold for slowly increasing potentials in any space dimension. Throughout this work, we assume that \(V(x)\) is real-valued and \(C^1\) on \(\mathbb{R}^n\) and

\[
|V(x)| + \langle x|\nabla V(x)\rangle |\leq C(\langle x\rangle)^{-\rho}, \quad x \in \mathbb{R}^n, \tag{1.8}
\]

for some \(\rho \geq -1\), where \(\langle x\rangle = (1 + |x|^2)^{1/2}\). The potential is said to be of short-range if \(\rho > 1\). Remark that Eq. (1.2) depends only on \(\nabla V(x)\) and there is a freedom in the choice of an additive constant in \(V(x)\). When \(\rho > 0\), the condition (1.8) means that this “normalizing constant” is chosen such that \(|V(x)| \to 0\) as \(|x| \to \infty\) in contrast to (1.6) for increasing potentials. The KFP operator \(P\) with the maximal domain in \(L^2\) is a closed, accretive (\(\Re P \geq 0\)) and hypoelliptic operator [10, 11]. Denote

\[
P = P_0 + W, \tag{1.9}
\]

with \(P_0 = v \cdot \nabla - \Delta_v + \frac{1}{2}|v|^2 - \frac{n}{2}\) and \(W = -\nabla V(x) \cdot \nabla v\). If \(\rho > -1\), \(W\) is a relatively compact perturbation of the free KFP operator \(P_0\); \(W(P_0 + 1)^{-1}\) is a compact operator in \(L^2\). The general spectral theorem says that the essential spectrum of \(P\) is equal to \(\sigma(P_0) = [0, +\infty]\) and that all possible accumulation points of complex eigenvalues are in the essential spectrum \([0, +\infty]\). For non selfadjoint Schrödinger operators, it is known that complex eigenvalues do accumulate to some number \(\lambda > 0\) in some cases.

For KFP operators with potentials whose gradient tend to zero, scattering phenomena arise due to the transport part of \(P\). For this reason, we introduce the weighted \(L^2\) spaces, \(L^{2,r}\), \(r \in \mathbb{R}\), using only weight in \(x\)-variables:

\[
L^{2,r} = L^2(\mathbb{R}^{2n}; \langle x \rangle^{2r} \, dx \, dv). \tag{1.10}
\]

Let \(S(t) = e^{-tP}\) be the strongly continuous semigroup of contractions generated by \(-P\) in \(L^2(\mathbb{R}^{2n}; \, dx \, dv)\). If \(s \geq 0\), \(L^{2,s} \subset L^2 \subset L^{2,-s}\) and \(S(t)\) maps naturally \(L^{2,s}\) into \(L^{2,-s}\). The main results of this paper can be summarized as follows.

**Theorem 1.1.** (a) Assume \(n \geq 1\) and the condition (1.8) with \(\rho \geq -1\). Then there exists \(C > 0\) such that \(\sigma(P) \cap \{z; |\Im z| > C, \Re z \leq \frac{1}{C} |\Im z|^2\} = \emptyset\) and the resolvent \(R(z) = (P - z)^{-1}\) satisfies the estimates

\[
\|R(z)\| \leq \frac{C}{|z|^\frac{1}{2}}, \tag{1.11}
\]

and

\[
\|(1 - \Delta_v + v^2)^{\frac{1}{2}} R(z)\| \leq \frac{C}{|z|^\frac{1}{2}}, \tag{1.12}
\]

for \(|\Im z| > C\) and \(|\Re z| \leq \frac{1}{C} |\Im z|^\frac{1}{2}\).

(b) Assume \(n = 3\) and \(\rho > 1\). Then \(P\) has no eigenvalues in a neighborhood of zero. One has for any \(s > \frac{3}{2}\),

\[
\|S(t)\|_{L^{2,s} \to L^{2,-s}} \leq C_s t^{-\frac{3}{2}}, \tag{1.13}
\]

for \(t > 0\).
(c) Assume \( n = 3 \) and \( \rho > 2 \). Then for any \( s > \frac{3}{2} \), there exists \( \epsilon > 0 \) such that the following asymptotics holds
\[
S(t) = \frac{1}{(4\pi t)^\frac{3}{2}} \langle m, \cdot \rangle m + O(t^{-\frac{3}{2} - \epsilon}), \quad t \to +\infty, \tag{1.14}
\]
as operators from \( L^{2,s} \) to \( L^{2,-s} \).

**Remark.** Let us begin with some comments about the sharpness of the results. The sharpness of the order of the resolvent bound in (1.12) has been discussed in Sect. 3 of the work of Lebeau [19] for geometric KFP operators on cotangent bundles over compact manifolds. In the Euclidean case studied here, the sharpness of this resolvent bound can be shown by the similar argument. For the free KFP operator whose spectrum is equal to \([0, +\infty[\), a related open question is raised in Sect. 2, after Corollary 2.8. The sharpness of the upper bound in (1.13) can be checked for the free KFP operator (see Remark after Theorem 2.4). As to (1.14), the condition \( \rho > 2 \) is necessary by our method, because it is needed to give a meaning to some intermediate terms appearing in our calculation. But we do not think that it is optimal and feel that it may be improved by means of other methods and with some new ideas. See the remark at the end of Sect. 4. The quantity \( \epsilon > 0 \) appeared in the remainder estimate in (1.14) may be made more precise in terms of the values of \( \rho > 2 \) and \( s > \frac{3}{2} \), but we shall not enter into such technical details.

It may be interesting to compare (1.14) with the known results of the form (1.7). The space distributions of solutions to (1.1) in both cases are governed by the Maxwellian \( m^2 \), but for decreasing potentials, the density of distribution decays in time like \( t^{-\frac{3}{2}} \).

Recall that it is well-known for Schrödinger operator \( H = -\Delta_x + U(x) \) with a real-valued potential \( U(x) \) that the space-decay rates of solutions to \( Hu = 0 \) determine the low-energy asymptotics of the resolvent \((H - z)^{-1}\) near the threshold zero, which in turn determine the large-time asymptotics of solutions to the evolution equation (see, for example, [25]). From this point of view, the difference between (1.7) and (1.14) may be explained by the lack of decay in \( x \)-variables in the Maxwellian \( m^2 \) in our case. One may even expect that the solution to the KFP equation behaves like \( t^{-a} \) with \( a \) depending on \( a \) for critical potentials \( V(x) \sim a \ln |x|, \ a \in ]\frac{2}{3} - \frac{2}{3}\), \( \frac{2}{3} \]. Although the KFP operator is a differential operator of the first order in \( x \)-variables, the large-time behavior of solutions looks like those that the heat equation described by \( e^{t\Delta_x} \) as \( t \to +\infty \). This is due to the interplay between the diffusion part and the transport part of the KFP operator \( P \) and will become clear from the spectral decomposition formula of \( e^{-tP_0} \). Under stronger assumption on \( \rho \), one can calculate in appropriate spaces the second term in large-time asymptotics of solutions, which is of the order \( O(t^{-\frac{5}{2}}) \) in dimension three.

The method used in this work is scattering in nature: we regard the full KFP operator \( P \) as a perturbation of the free KFP operator \( P_0 \) without potential. A large part of this work is devoted to a detailed analysis of the free KFP operator. Several basic questions remain open for \( P_0 \), such as the high energy estimates for the free resolvent near the positive real axis. A key step in the proof of the large-time asymptotics of solutions to the full KFP equation is to show that the complex eigenvalues of \( P \) do not accumulate towards the threshold zero. So far as the author knows, such a statement is not yet proven for non-selfadjoint Schrödinger operators \(-\Delta + U(x)\) with a general complex-valued potential satisfying \( |U(x)| = O(|x|^{-\rho}), \ \rho > 2 \). (See, however, [17, 26] for dissipative potentials \( \Im U(x) \leq 0 \)). Results like (1.13) and (1.14) may fail if there is a sequence of complex eigenvalues tending to zero. For the KFP operator \( P \) with a short-range potential
in dimension three, we prove that there is no complex eigenvalue in the neighborhood of zero by making use of the method of threshold spectral analysis. In particular, we use the internal symmetric structure of the KFP operator to show that zero is neither an eigenvalue nor a resonance of $P$.

The organisation of this work is as follows. In Sect. 2, we study in detail the spectral properties of the model operator $P_0$ and establish some dispersive estimate for the semigroup generated by $-P_0$. We also prove the limiting absorption principles and the low-energy asymptotics of the resolvent $R_0(z) = (P_0 - z)^{-1}$, as well as some high energy resolvent estimates. These estimates are pseudo-spectral in nature, because the numerical range of the free KFP operator is equal to the right half complex plane. The threshold spectral properties of the full KFP operator with a short-range potential is analyzed in Sect. 3. We prove that under the condition (1.8) with $\rho > 1$, zero is neither an eigenvalue nor a resonance of $P$, and we calculate the lower-energy resolvent asymptotics. For technical reasons, we only prove these results when the dimension $n$ is equal to three, but we believe that they remain true when $n \geq 4$. Finally, in Sect. 4, we prove the time-decay and the large-time asymptotics of solutions. In “Appendix A”, we study a family of nonselfadjoint harmonic oscillators that may be regarded as complex translation in variables of selfadjoint harmonic oscillators. We prove some quantitative estimates with respect to the parameters of translation, establish a spectral decomposition formula and prove some uniform time-decay estimates of the semigroup. These results are used in Sect. 2 to analyze the free KFP operator.

2. The Free Kramers-Fokker-Planck Operator

Denote by $P_0$ the free KFP operator (with $\nabla V = 0$):

$$P_0 = v \cdot \nabla_x - \Delta_v + \frac{1}{4} |v|^2 - \frac{n}{2}. \quad (2.1)$$

$P_0$ is a non-selfadjoint and hypoelliptic operator with loss of $\frac{1}{3}$ derivative in $x$ variables, whose numerical range fills the whole right half complex plane. The following result is known (see [10,21]). In particular, the essential maximal accretivity is discussed in Proposition 2.4 of [21].

**Proposition 2.1.** One has

$$\|\Delta_v u\| + \||v|^2 u\| + \|\langle D_x \rangle^{\frac{2}{3}} u\| \leq C(\|P_0 u\| + \|u\|), \quad u \in S(\mathbb{R}^{2n}_{x,v}) \quad (2.2)$$

$P_0$ defined on $S(\mathbb{R}^{2n}_{x,v})$ is essentially maximally accretive, i.e., the closure of $P_0$ in $L^2(\mathbb{R}^{2n}_{x,v})$ with core $S(\mathbb{R}^{2n}_{x,v})$ is of maximal domain $D(P_0) = \{ u \in L^2(\mathbb{R}^{2n}_{x,v}); P_0 u \in L^2(\mathbb{R}^{2n}_{x,v}) \}$ and $\Re \langle P_0 u, u \rangle \geq 0$ for $u \in D(P_0)$.

Henceforth we still denote by $P_0$ its closed extension in $L^2$ with maximal domain $D(P_0) = \{ u \in L^2(\mathbb{R}^{2n}_{x,v}); P_0 u \in L^2(\mathbb{R}^{2n}_{x,v}) \}$.

In terms of Fourier transform in $x$-variables, we have for $u \in D(P_0)$

$$P_0 u(x, v) = \mathcal{F}^{-1}_{x \rightarrow \xi} \hat{P}_0(\xi) \hat{u}(\xi, v), \quad \text{where} \quad (2.3)$$

$$\hat{P}_0(\xi) = -\Delta_v + \frac{1}{4} \sum_{j=1}^n (v_j + 2i\xi_j)^2 - \frac{n}{2} + |\xi|^2 \quad (2.4)$$

$$\hat{u}(\xi, v) = \langle \mathcal{F}_{x \rightarrow \xi} u \rangle(\xi, v) \equiv \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x, v) \, dx. \quad (2.5)$$
Denote
\[ D(\hat{P}_0) = \{ f \in L^2(\mathbb{R}^{2n} \setminus \mathbb{R}^{2n}_0); \hat{P}_0(\xi) f \in L^2(\mathbb{R}^{2n}_0) \}. \]  

Then \( \hat{P}_0 \triangleq F_{x \to \xi} P_0 F_{\xi \to x}^{-1} \) is a direct integral of the family of complex harmonic operators \( \{ \hat{P}_0(\xi); \xi \in \mathbb{R}^n \} \) which is studied in Appendix A.

The following abstract result may be useful to determine the spectrum of operators which are direct integral of a family of unbounded nonselfadjoint operators. Since we are unable to find a reference for this seemingly standard result, we give a short proof for it. See also Theorem 8.3 of [5] for families of bounded nonselfadjoint operators.

**Theorem 2.2.** Let \( H \) be a separable Hilbert space, \( X \) a non empty open set of \( \mathbb{R}^n \) and \( \mathcal{H} = \{ f : X \to H; \| f \| = \left( \int_X \| f(x) \|^2_H dx \right)^{1/2} < +\infty \} \). Here \( dx \) is the Lebesgue measure of \( \mathbb{R}^n \). Suppose that both \( \{ Q(x); x \in X \} \) and the adjoints \( \{ Q(x)^*; x \in X \} \) are strongly continuous families of closed, densely defined operators with constant domains \( D, D^* \subset H \), respectively. Suppose in addition that for each \( z \in \mathbb{C} \setminus \bigcup_{x \in X} \sigma(Q(x)) \), one has
\[ \sup_{x \in X} \| (Q(x) - z)^{-1} \| < +\infty. \]  

Let \( Q \) be the closed, densely defined operator in \( \mathcal{H} \) such that for any \( f \) in the domain of \( Q \), one has \( Qf = Q(x)f(x) \) in \( \mathcal{H} \).

Then one has
\[ \sigma(Q) = \overline{\bigcup_{x \in X} \sigma(Q(x))} \]  

**Proof.** If \( z \notin \bigcup_{x \in X} \sigma(Q(x)) \), define \( R_z : \mathcal{H} \to \mathcal{H} \) by \( (R_z f)(x) = (Q(x) - z)^{-1} f(x), f \in \mathcal{H} \). Then (2.7) shows that \( R_z \) is bounded on \( \mathcal{H} \). One can check that \( (Q - z)R_z = 1 \) on \( \mathcal{H} \) and \( R_z(Q(x) - z) = 1 \) on \( D(Q) \). Thus \( z \) is in resolvent set of \( Q \). This shows that \( \sigma(Q) \subseteq \bigcup_{x \in X} \sigma(Q(x)) \).

Conversely, if \( z \in \sigma(Q(x_0)) \) for some \( x_0 \in X \), then either \( \| (Q(x_0) - z)u_n \| \to 0 \) or \( \| (Q(x_0) - z)^*v_n \| \to 0 \) for some sequences \( \{ u_n \}, \{ v_n \} \) in \( H \) with \( \| u_n \| = \| v_n \| = 1 \). In fact if one had both \( \| (Q(x_0) - z)u \| \geq c \| u \| \) and \( \| (Q(x_0) - z)^*v \| \geq c \| v \| \) for some \( c > 0 \) and for all \( u \in D \) and \( v \in D^* \), \( z \) would belong to the resolvent set of \( Q(x_0) \). Since \( x \to Q(x) \) is strongly continuous on \( D \), for any \( \epsilon > 0 \), we can find \( f \in \mathcal{H} \) in the form \( f = \chi(x)u_{n_0} \) or \( g = \chi(x)v_{n_0} \in \mathcal{H} \) in the second case \( \| (Q(x_0) - z)^*v_n \| \to 0 \) for some \( n_0 \) and for some numerical function \( \chi \) supported in a sufficiently small neighborhood of \( x_0 \) such that \( \| f \| = 1 \) and \( \| (Q - z)f \| < \epsilon \) (or \( \| g \| = 1 \) and \( \| (Q - z)^*g \| < \epsilon \)). This shows that \( z \in \sigma(Q) \). Consequently, \( \sigma(Q) \supseteq \bigcup_{x \in X} \sigma(Q(x)) \).

Remark that the condition (2.7) is always satisfied for a family of selfadjoint operators. In non selfadjoint case, if this condition is not satisfied, the equality (2.9) may fail. See Theorem 8.3 of [5].

**Proposition 2.3.** Let \( P_0 \) denote the free KFP operator with the maximal domain. Then one has: \( \sigma(P_0) = [0, +\infty] \).
Proof. We want to apply Theorem 2.2. One sees that \( P_0 \) is unitarily equivalent with \( \hat{P}_0 \) which is a direct integral of a family of operators \( \{ \hat{P}_0(\xi); \xi \in \mathbb{R}^n \} \) with constant domain \( D \). Lemma A.1 shows that

\[
\bigcup_{\xi \in \mathbb{R}^n} \sigma(\hat{P}_0(\xi)) = \bigcup_{\xi \in \mathbb{R}^n} \{ k + |\xi|^2; k \in \mathbb{N} \} = [0, +\infty[.
\]

It is clear that \( D(\hat{P}_0(\xi)^n) = D \) and \( \xi \to \hat{P}_0(\xi) \) and \( \xi \to \hat{P}_0(\xi)^n \) are strongly continuous on \( D \). To apply Theorem 2.2 to show that \( \sigma(P_0) = [0, +\infty[ \), it remains to check the condition (2.7): for each \( z \not\in [0, +\infty[ \), there exists some constant \( C_z \) such that

\[
\| (\hat{P}_0(\xi) - z)^{-1} \| \leq C_z \tag{2.10}
\]

uniformly in \( \xi \in \mathbb{R}^n \). For \( \xi \) in a compact, this follows from the fact that since \( \hat{P}_0(\xi) \) forms a holomorphic family of type (A) in sense of Kato, the resolvent \( (\hat{P}_0(\xi) - z)^{-1} \) is locally bounded in \( \xi \in \mathbb{R}^n \) for each \( z \not\in [0, +\infty[ \) \cite{15}. For \( |\xi| \) large \((|\xi|^2 > |\Re z| + 1)\), using the representation

\[
(\hat{P}_0(\xi) - z)^{-1} = -\int_0^T e^{-t(\hat{P}_0(\xi) - z)} \, dt - \int_T^\infty e^{-t(\hat{P}_0(\xi) - z)} \, dt
\]

with \( T \geq 3 \) fixed, one deduces from Corollary A.4 that there exists \( C = C(|\Re z|) \) independent of \( \xi \) such that

\[
\| (\hat{P}_0(\xi) - z)^{-1} \| \leq C + \frac{C}{\xi^2 - |\Re z|} \tag{2.11}
\]

for \( \xi^2 > |\Re z| + 1 \). This proves (2.10) which finishes the proof of Proposition 2.3. \( \square \)

Remark. The proof of Proposition 2.3 gives an idea of the spectral structure of \( P_0 \). The spectrum of \( P_0 \) is the union of an infinite number of half-lines: \( \sigma(P_0) = \bigcup_{k \in \mathbb{N}} l_k \), where \( l_k = [k, +\infty[ \) can roughly be interpreted as the spectra of \( P_0 \) restricted on the range of the operator \( \Pi_k^D \) defined by (2.25), although \( \Pi_k^D \) is not a bounded operator in \( L^2 \). The numbers \( k \in \mathbb{N} \), which are extreme points of some parts of the essential spectrum, are called thresholds of the free KFP operator \( P_0 \). This situation is analogous to the selfadjoint Schrödinger operator with a constant magnetic field in odd dimensions where Landau levels are called thresholds in spectral analysis.

From Proposition A.3 in Appendix A on a family of non-selfadjoint harmonic oscillators, one can deduce some time-decay estimates for \( e^{-tP_0} \) in appropriate spaces. Denote

\[
\mathcal{L}^p = L^p(\mathbb{R}^n; L^2(\mathbb{R}^n_1)), \quad p \geq 1,
\]

equipped with their natural norms.

Theorem 2.4. (a). One has the following dispersive type estimate: \( \exists C > 0 \) such that

\[
\| e^{-tP_0} u \|_{\mathcal{L}^\infty} \leq \frac{C}{t^{\frac{n}{2}}} \| u \|_{\mathcal{L}^1}, \quad t \geq 3, \tag{2.13}
\]

for \( u \in \mathcal{L}^1 \).
(b). For \( s > \frac{n}{2} \), one has for some \( C_s > 0 \)

\[
\|e^{-tP_0}u\|_{L^{2,-s}} \leq \frac{C_s}{t^{\frac n 2}} \|u\|_{L^{2,s}},
\]  

(2.14) for \( t > 0 \) and \( u \in L^{2,s} \).

**Proof.** For \( u \in \mathcal{S}(\mathbb{R}^{2n}_{x,v}) \), we denote by \( \hat{u} \) the Fourier transform of \( u \) in \( x \)-variables. Then one has

\[
\|e^{-tP_0}u\|_{L^\infty} \leq \frac{1}{(2\pi)^n} \|e^{-t\hat{P}_0(\xi)}\hat{u}\|_{L^1_\xi}.
\]  

(2.15)

Here \( L^1_\xi \) denotes the space \( L^1(\mathbb{R}^n_\xi; L^2(\mathbb{R}^n_v)) \). Corollary A.4 gives for any \( \xi \in \mathbb{R}^n \)

\[
\|e^{-t\hat{P}_0(\xi)}\hat{u}\|_{L^2_v} \leq \frac{e^{-\xi^2(2-\frac{4}{\epsilon^2-1})}}{(1-e^{-t})^n} \|\hat{u}(\xi, \cdot)\|_{L^2_v}. 
\]  

(2.16)

Since \( t - 2 - \frac{4}{\epsilon^2-1} \geq c_0 t > 0 \) for some \( c_0 > 0 \) when \( t \geq 3 \), one obtains that

\[
\|e^{-tP_0}u\|_{L^1_\xi} \leq \int_{\mathbb{R}^n_\xi} \frac{e^{-\xi^2(2-\frac{4}{\epsilon^2-1})}}{(1-e^{-t})^n} d\xi \|u\|_{L^1} 
\]  

\[
\leq Ct^{-\frac{n}{2}} \|u\|_{L^1} 
\]  

(2.17) for \( t \geq 3 \) and for any \( u \in \mathcal{S} \). An argument of density proves (a) for any \( u \in L^1 \). Theorem 2.4 (b) is a consequence of Theorem 2.4 (a). \( \square \)

**Remark.** The upper bounds given in Theorem 2.4 are sharp. In fact, let \( u \in \mathcal{S}(\mathbb{R}^{2n}) \) such that \( u \geq 0, u \neq 0 \) and \( e^{2\Delta_s^x}u \in L^1(\mathbb{R}^{2n}) \). Then one has

\[
\|e^{-tP_0}u - e^{t\Delta_s^x}u_0\|_{L^\infty} = O(e^{-t}), \quad t \to +\infty, 
\]  

(2.18)

where \( u_0 = \Pi_0 D_s u = e^{-2\Delta_s^x} p_0^w(v, D_x, D_v) u \). See Proposition 2.6 below. Making use of the explicit formula for the integral kernel of the heat semigroup \( e^{t\Delta_s^x} \), one can compute the large-time asymptotics of \( e^{t\Delta_s^x}u_0 \) and obtain for any \( s > \frac{n}{2} \), there exists some \( \epsilon > 0 \) such that

\[
e^{t\Delta_s^x} u_0 = \frac{1}{(4\pi t)^\frac{n}{2}} \left( \int_{\mathbb{R}^n_y} (p_0^w u)(y, v) \, dy + O(t^{-\epsilon}) \right), \quad t \to +\infty.
\]  

(2.19)

Notice that

\[
\int_{\mathbb{R}^n_y} (p_0^w u)(y, v) \, dy = c_0 \psi_0(v)
\]  

where \( \psi_0 \) is the first eigenfunction of the selfadjoint harmonic oscillator \(-\Delta_v + \frac{v^2}{4}\) and

\[
c_0 = \int_{\mathbb{R}^{2n}} u(y, v') \psi_0(v') \, dy \, dv' \neq 0,
\]

because \( u \geq 0 \) and \( u \neq 0 \). Therefore, one has a lower bound of the form \( ct^{-\frac{n}{2}}, c > 0 \), for the norms of \( e^{-tP_0}u \) in \( L^\infty \) and \( L^{2,-s}, s > \frac{n}{2} \), which is of the same order as the upper bounds given in Theorem 2.4.
For a symbol \( a(x, v; \xi, \eta) \), denote by \( a^w(x, v, D_x, D_v) \) the associated Weyl pseudo-differential operator defined by
\[
a^w(x, v, D_x, D_v)u(x, v) = \frac{1}{(2\pi)^{2n}} \int \int e^{i(x - x') \cdot \xi + i(v - v') \cdot \eta} a \left( \frac{x + x'}{2}, \frac{v + v'}{2}, \xi, \eta \right) u(x', v') dx' dv' d\xi d\eta
\] (2.20)
for \( u \in S(\mathbb{R}^{2n}) \).

The following Proposition can be regarded as a spectral decomposition for the semi-group \( e^{-tP_0} \) which follows from Proposition A.3 given in Appendix A and some elementary symbolic calculus for Weyl pseudo-differential operators (see Chap. 18, [14]).

**Proposition 2.5.** For any \( t > 3 \), one has
\[
e^{-tP_0} = \sum_{l=0}^{\infty} e^{-lt+(t-2)\Delta_x} p_l^w(v, D_x, D_v),
\] (2.21)
where the series is norm-convergent as operators in \( L^2(\mathbb{R}^{2n}) \) and \( p_l(v, \xi, \eta) \) is given by
\[
p_l(v, \xi, \eta) = \int_{\mathbb{R}^n} e^{-iv' \cdot \eta} \left( \sum_{|\alpha|=l} e^{-2|\xi|^2} \psi_{\alpha}(v + \frac{v'}{2} + 2i\xi) \psi_{\alpha}(v - \frac{v'}{2} + 2i\xi) \right) dv', \quad l \in \mathbb{N}.
\] (2.22)

In particular,
\[
p_0(v, \xi, \eta) = 2^n e^{-\frac{1}{2}v^2 - 2(\eta + \xi)^2}.
\] (2.23)

We just indicate that for \( t > 3 \), one has \( t - 2 - \frac{4}{t^2 - 1} > 0 \) and
\[
e^{t(t-2)\Delta_x} p_l^w(v, D_x, D_v) = e^{t\Delta_x} \Pi_l^{D_x}
\] (2.24)
is a bounded operator, where
\[
\Pi_l^{D_x} = \mathcal{F}^{-1}_{x \rightarrow \xi} \Pi_l^{x} \mathcal{F}_{x \rightarrow \xi}
\] (2.25)
and \( \Pi_l^{x} \) is the Riesz projection of \( \hat{P}_0(\xi) \) associated with the eigenvalue \( E_l = l + |\xi|^2 \) (see Lemma A.1 in Appendix A). The norm-convergence as operators in \( L^2(\mathbb{R}^{2n}) \) of the right-hand side of (2.21) follows from (A.10) which gives:
\[
\sum_{l=0}^{\infty} \| e^{-lt+(t-2)\Delta_x} p_l^w(v, D_x, D_v) \| \leq \frac{1}{(1 - e^{-t})^n}, \quad t > 3.
\]
The detail of the proof is omitted.

As an application of this spectral decomposition (2.21), we can establish the following result on large-time approximation of solutions to the free KFP equation.
Proposition 2.6. There exists $C > 0$ such that
\[ \| e^{-tP_0} u - e^{(t-2)\Delta} p_0^w u \|_{L^\infty} \leq C \frac{e^{-t}}{t^2} \| u - e^{-2\Delta} p_0^w u \|_{L^1}, \]  
(2.26)
for $t \geq 3$ and for any $u \in L^1$ with $e^{-2\Delta} u \in L^1$.

Proof. By a direction calculation, one can check that
\[ \mathcal{F}^{-1}_{x \to \xi} \Pi_0^\xi \mathcal{F}_{x \to \xi} = e^{-2\Delta} p_0^w, \]  
(2.27)
\[ \mathcal{F}^{-1}_{x \to \xi} e^{-tP_0(\xi)} \Pi_0^\xi \mathcal{F}_{x \to \xi} = e^{(t-2)\Delta} p_0^w. \]  
(2.28)
$p_0^w$ is continuous on $L^1$. In fact, let $\tau : u(x, v) \to u(x + 2v, v)$. Then one has
\[ p_0^w u(x, v) = \langle \psi_0, \tau u \rangle_{L^2(\mathbb{R}^n)}(x) \psi_0(v), \]
where $\psi_0 = \frac{1}{(2\pi)^{\frac{n}{4}}} e^{-\xi^2/4}$ is the first eigenfunction of $-\Delta_v + \frac{v^2}{4}$. It follows that
\[ \| p_0^w u \|_{L^1} \leq \int_{\mathbb{R}^{2n}} \psi_0(v)|u(x + 2v, v)|dxdv = \int_{\mathbb{R}^{2n}} \psi_0(v)|u(y, v)|dydv \leq \| u \|_{L^1}. \]

Denote $u_0 = e^{-2\Delta} p_0^w u$. For $u \in L^1$ with $e^{-2\Delta} u \in L^1$, one has $u_0 = p_0^w (e^{-2\Delta} u) \in L^1$. The proof of Lemma A.2 shows that if $n = 1$, one has
\[ \sum_{k=1}^\infty e^{-t(k+\xi^2)} \| \Pi_k^\xi \| \]
\[ = \sum_{k=1}^\infty e^{-t(k+\xi^2)+2\xi^2} \sum_{j=0}^k \frac{C_k^j}{j!} (4\xi^2)^j \]
\[ = e^{\xi^2(t-2)-t} \left( 1 - e^{-t} \right)^{-1} \sum_{j=1}^{\infty} \frac{1}{j!} \left( \frac{4\xi^2}{e^t-1} \right)^j \]
\[ \leq \frac{e^{\xi^2(t-2)-t}}{1 - e^{-t}} \left( 1 + \frac{4\xi^2}{e^t-1} \right), \quad t > 0. \]  
(2.29)

When $n \geq 1$, we deduce from the explicit formula for the Riesz projections $\Pi_k^\xi$ and the above one-dimensional bound that there exist some constants $C, a > 0$ such that
\[ \sum_{k=1}^\infty e^{-t(k+\xi^2)} \| \Pi_k^\xi \| \leq C (1 + \xi^2) e^{-at\xi^2-t}, \quad t > 3. \]  
(2.30)

Making use of the above estimate and following the proof of Theorem 2.4 with $u$ replaced by $u - u_0$, one obtains
\[ \| e^{-tP_0} (u - u_0) \|_{L^\infty} \leq C \sum_{k=1}^\infty \| e^{-t(k+\xi^2)} \Pi_k^\xi (\hat{u} - \hat{u}_0) \|_{L^1} \]
\[ \leq C' \frac{e^{-t}}{t^2} \| u - u_0 \|_{L^1}, \quad t > 3. \]  
(2.31)

This proves Proposition 2.6. \[\square\]
Proposition 2.6 says that \( e^{-tP_0} \) decay exponentially in some sense and that the time-decay of \( e^{-tP_0} \) is governed by the first eigenvalue of the harmonic oscillator in \( v \)-variables and propagation of energy due to the transport term \( v \cdot \nabla_x \). Although this term is of the first order in \( \xi \), time-decay of the solutions to the free KFP equation is the same as those to the heat equation in space variables.

One of the goals of the remaining part of this work is to prove a result similar to Theorem 2.4 (b) for the full KFP operators through method of resolvent estimates. In order to study the resolvent of \( P \), we establish here some limiting absorption principles for the resolvent of \( P_0 \) and its low-energy asymptotics. Different from the limiting absorption principle for selfadjoint operators, the problem we want to study here is pseudospectral in nature, because \( \Re P \) is located in the interior of the numerical range of \( P_0 \). Set \( R_0(z) = (P_0 - z)^{-1} \), \( \hat{R}_0(z) = (\hat{P}_0 - z)^{-1} \) and \( \hat{R}_0(z, \xi) = (\hat{P}_0(\xi) - z)^{-1} \) for \( z \not\in \mathbb{R}_+ \). Then \( R_0(z) = \mathcal{F}_{x \to \xi}^{-1} \hat{R}_0(z) \mathcal{F}_{x \to \xi} \). Note that \( \hat{R}_0(z) \) is multiplication in \( \xi \)-variables by \( \hat{R}_0(z, \xi) \).

**Proposition 2.7.** Let \( l \in \mathbb{N} \) and \( l < a < l + 1 \) be fixed. Take \( \chi \geq 0 \) and \( \chi \in C_0^\infty (\mathbb{R}_n^\xi) \) with \( \text{supp} \chi \subset \{ \xi, |\xi|^2 \leq a + 4 \} \), \( \chi(\xi) = 1 \) when \( |\xi|^2 \leq a + 3 \) and \( 0 \leq \chi(\xi) \leq 1 \). Then one has

\[
\hat{R}_0(z, \xi) = \sum_{k=0}^l \chi(\xi) \frac{\Pi_k^\xi}{\xi^2 + k - z} + r_l(z, \xi),
\]

for any \( \xi \in \mathbb{R}^n \) and \( z \in \mathbb{C} \) with \( \Re z < a \) and \( \Im z \neq 0 \). Here \( r_l(z, \xi) \) is holomorphic in \( z \) with \( \Re z < a \) verifying the estimate

\[
\sup_{\Re z < a, \xi \in \mathbb{R}^n} \| r_l(z, \xi) \|_{\mathcal{L}(L^2(\mathbb{R}_n^\xi))} < \infty.
\]

**Proof.** Let \( \chi_1 = 1 - \chi \). For \( \Re z < 0 \), one has

\[
\hat{R}_0(z, \xi) = \int_0^\infty e^{-t(\hat{P}_0(\xi) - z)} \, dt = \int_0^\infty \chi_1(\xi) e^{-t(\hat{P}_0(\xi) - z)} \, dt + \int_0^\infty \chi(\xi) e^{-t(\hat{P}_0(\xi) - z)} \, dt
\]

\[
\triangleq I_1(z, \xi) + I_2(z, \xi).
\]

Since \( \Re P_0(\xi) \geq 0 \), it is clear that for each fixed \( T \), \( \int_0^T \chi_1(\xi) e^{-t(\hat{P}_0(\xi) - z)} \, dt \) is uniformly bounded in \( \xi \) and \( z \) with \( \Re z \leq a \):

\[
\left\| \int_0^T \chi_1(\xi) e^{-t(\hat{P}_0(\xi) - z)} \, dt \right\|_{\mathcal{B}(L^2(\mathbb{R}_n^\xi))} \leq Te^{aT},
\]

for all \( \xi \in \mathbb{R}^n \). Corollary A.4 with \( T = 3 \) shows that

\[
\| e^{-t(\hat{P}_0(\xi) - z)} \|_{\mathcal{B}(L^2(\mathbb{R}_n^\xi))} \leq Ce^{-t(\xi^2 - 2) - e^{-1} + \Re z}
\]

for \( t \geq 3 \). Since \( \chi_1 \) is supported in \( \xi^2 \geq a + 3 \), \( I_1(z, \xi) \) is holomorphic in \( z \) with \( \Re z < a \) and verifies the estimate (2.33). To study \( I_2(z, \xi) \), we decompose \( e^{-t\hat{P}_0(\xi)} \) as

\[
e^{-t\hat{P}_0(\xi)} = J_1(t, \xi) + J_2(t, \xi),
\]
where \( J_j(t, \xi) = e^{-t \hat{\rho}_j(\xi)} S_j^\xi \) with \( S_1^\xi = \sum_{k=0}^l \Pi_k^\xi \) and \( S_2^\xi = 1 - S_1^\xi \). For \( \Im z < 0 \), the contribution of \( J_1(t, \xi) \) to \( \hat{R}_0(z, \xi) \) is
\[
\int_0^\infty e^{\imath z} J_1(t, \xi) dt = \sum_{k=0}^l \frac{\Pi_k^\xi}{\xi^2 + k - z}.
\]
By (A.4), one has for \( t \geq T > 0 \)
\[
\|J_2(t, \xi)\|_{B(L^2(\mathbb{R}^n_v))} \leq \sum_{k=l+1}^\infty e^{-(k+\xi^2) + 2\xi^2} \sum_{j=0}^k \frac{C_k^j (4\xi^2)^j}{j!} \leq J_{21}(t, \xi) + J_{22}(t, \xi)
\]
(2.36)
where
\[
J_{21}(t, \xi) = e^{-\xi^2(t-2)} \sum_{j=0}^{l+1} \frac{(4\xi^2)^j}{j!} \sum_{k=l+1}^\infty C_k^j e^{-tk}
\]
\[
J_{22}(t, \xi) = e^{-\xi^2(t-2)} \sum_{j=l+2}^\infty \frac{(4\xi^2)^j}{j!} \sum_{k=j}^\infty C_k^j e^{-tk}.
\]
\( J_{21}(t, \xi) \) and \( J_{22}(t, \xi) \) can be evaluated as in the proof of Proposition A.3 (see also the proof of (2.30) in the case \( l = 0 \)) and we omit the details here. One has for some \( C, a > 0 \)
\[
J_{21}(t, \xi) \leq e^{-\xi^2(t-2) - (l+1)t} \sum_{j=0}^{l+1} \frac{(4\xi^2)^j}{j!} \sum_{k=l+1}^\infty C_k^j e^{-tk}
\]
(2.37)
\[
J_{22}(t, \xi) \leq C e^{-a\xi^2 t - (l+1)t} (2.38)
\]
for \( t \geq T \). Since \(|\xi|\) is bounded on the support of \( \chi \), this implies that there exists some constant \( C \) such that
\[
\|J_2(t, \xi)\|_{B(L^2(\mathbb{R}^n_v))} \leq C e^{-(l+1)t} (2.39)
\]
uniformly in \( \xi \in \text{supp} \chi \) and \( t \geq T \). We obtain a decomposition for \( \hat{R}_0(z, \xi) \) when \( \Im z < 0 \):
\[
\hat{R}_0(z, \xi) = \sum_{k=0}^l \chi(\xi) \frac{\Pi_k^\xi}{\xi^2 + k - z} + r_1(z, \xi),
\]
(2.40)
where
\[
r_1(z, \xi) = I_1(z, \xi) + \int_0^\infty e^{\imath z} J_2(t, \xi) dt.
\]
By the estimates (2.35) and (2.38), \( r_1(z, \xi) \) is holomorphic in \( z \) with \( \Im z < a \) and verifies the estimate (2.33). Since the both sides of (2.40) are holomorphic in \( z \in \mathbb{C} \setminus \mathbb{R}_+ \) with \( \Im z < a \), this representation formula remains valid for \( z \) in this region. \( \square \)
For \( r, s \in \mathbb{R} \), we introduce the weighted Sobolev space according to hypoellipticity of \( P_0 \):

\[
\mathcal{H}^{r,s} = \{ u \in \mathcal{S}'(\mathbb{R}^{2n}) ; (1 + \langle D_v \rangle^2 + |v|^2 + \langle D_x \rangle^2)^{\frac{s}{2}} \chi(x)^su \in L^2 \}.
\]

Denote \( \mathcal{B}(r, s; r', s') \) the space of continuous linear operators from \( \mathcal{H}^{r,s} \) to \( \mathcal{H}^{r',s'} \). The hypoellipticity of \( P_0 \) (Proposition 2.1) shows that \((P_0 + 1)^{-1} \in \mathcal{B}(0, 0; 2, 0)\). An argument of successive commutators shows that \((P_0 + 1)^{-1} \in \mathcal{B}(0, s; 2, s)\) for any \( s \in \mathbb{R} \). Similarly, one can show that \((P_0 + 1)^{-1} \in \mathcal{B}(s, 0; -2, s)\) for any \( s \in \mathbb{R} \). A complex interpolation gives \((P_0 + 1)^{-1} \in \mathcal{B}(-1, s; 1, s)\) for any \( s \in \mathbb{R} \).

**Corollary 2.8.** Set \( R_0(z) = (P_0 - z)^{-1} \), \( z \notin \mathbb{R}_+ \).

(a). Assume \( n \geq 1 \). Let \( I \) be a compact interval of \( \mathbb{R} \) which does not contain any non negative integer. Then for any \( s > \frac{1}{2} \), one has

\[
\sup_{\lambda \in I; \varepsilon \in [0, 1]} \| R_0(\lambda \pm i\varepsilon) \|_{\mathcal{B}(0,s;2,-s)} < \infty \tag{2.41}
\]

The boundary values of the resolvent \( R_0(\lambda \pm i0) = \lim_{\varepsilon \to 0^+} R_0(\lambda \pm i\varepsilon) \) exists in \( \mathcal{B}(0, s; 2, -s) \) for \( \lambda \in I \) and is continuous in \( \lambda \).

(b). Assume \( n \geq 3 \). Let \( I \) be a compact interval containing some non negative integer. Then for any \( s > 1 \), one has

\[
\sup_{\lambda \in I; \varepsilon \in [0, 1]} \| R_0(\lambda \pm i\varepsilon) \|_{\mathcal{B}(0,s;2,-s)} < \infty \tag{2.42}
\]

for any \( k \in \mathcal{N} \), the limits \( R_0(k \pm i0) = \lim_{\varepsilon \to 0^+} R_0(k \pm i\varepsilon) \) exist in \( \mathcal{B}(0, s; 2, -s) \) for any \( s > 1 \). One has \( R_0(0 + i0) = R_0(0 - i0) \) and \( R_0(k + i0) - R_0(k - i0) \in \mathcal{B}(0, s; 2, -s) \) for any \( s > \frac{1}{2} \) if \( k \geq 1 \).

**Proof.** Proposition 2.7 shows that

\[
R_0(z) = \sum_{k=0}^{\infty} \chi(D_x)\Pi_k^{D_x}(-\Delta_x + k - z)^{-1} + r_1(z), \tag{2.43}
\]

for \( z \in \mathbb{C} \) with \( \Re z < a \) and \( \Im z \neq 0 \) and that \( r_1(z) \) is bounded on \( L^2 \) and holomorphic in \( z \) with \( \Re z < a \). \( \chi(D_x)\Pi_k^{D_x}, k = 0, 1, \ldots \), are Weyl pseudodifferential operators with nice symbols \( b_k \) independent of \( x \):

\[
\chi(D_x)\Pi_k^{D_x} = b_k(v, D_x, D_v) \tag{2.44}
\]

with \( b_k(v, \xi, \eta) \) given by

\[
b_k(v, \xi, \eta) = \int_{\mathbb{R}^n} e^{-i\nu \cdot \eta/2} \left( \sum_{|\alpha|=k} \chi(\xi)\psi_\alpha(v + \nu' + 2i\xi)\psi_\alpha(v - \nu' + 2i\xi) \right) dv'.
\]

\[
(2.45)
\]

In particular,

\[
b_0(v, \xi, \eta) = 2^\alpha \chi(\xi) e^{-v^2-\eta^2+2iv\xi+2\xi^2}. \tag{2.46}
\]
These Weyl pseudodifferential operators with exponentially decaying symbols belong to $\mathcal{B}(r; s; r', s)$ for any $r, r', s \in \mathbb{R}$ (Chap. 18, [14]).

It is well-known that for any compact interval $I' \subset \mathbb{R}$, one has

$$\sup_{\lambda \in I'; \epsilon \in [0,1]} \| (x)^{-s} (-\Delta_x - (\lambda \pm i \epsilon))^{-1} (x)^{-s} \|_{\mathcal{B}(L^2(\mathbb{R}^d_n))} < \infty \quad (2.47)$$

for any $s > 1/2$ if $I'$ does not contain 0 (see [1]) and for any $s > 1$ and $n \geq 3$ if $I'$ contains 0 (which follows immediately from the explicit formula for the integral kernel of $(-\Delta_x - (\lambda \pm i \epsilon))^{-1}$). It follows from (2.43) that for $I \subset ] - \infty, a[$

$$\sup_{\lambda \in I; \epsilon \in [0,1]} \| R_0(\lambda \pm i \epsilon) \|_{\mathcal{B}(0,s;0,-s)} < \infty \quad (2.48)$$

for $s > \frac{1}{2}$ if $I \cap \mathbb{N} = \emptyset$ or $s > 1$ and $n \geq 3$ if $I \cap \mathbb{N} \neq \emptyset$. Estimates (2.41) and (2.42) follow from (2.48) and the resolvent equation

$$R_0(z) = R_0(-1) + (1 + z)R_0(-1)R_0(z)$$

by noticing that $R_0(-1) \in \mathcal{B}(0, s; 2, s)$ for any $s \in \mathbb{R}$. The other assertions of Corollary 2.8 can be proven by making use of the properties of $(-\Delta_x - (\lambda \pm i 0))^{-1}$. □

Remark. Below we shall give low-energy analysis for the boundary values of the resolvent $R_0(\lambda \pm i 0)$. A related non-trivial open question is whether one can obtain some high energy pseudospectral estimates for $R_0(\lambda \pm i 0)$ when $n \geq 3$.

The formula (2.43) can also be used to study the threshold asymptotics of the resolvent $R_0(z)$ as $z \to k$, $\Im z \neq 0$, $k \in \mathbb{N}$. To simplify calculations, we only consider the threshold zero in the case $n = 3$.

**Proposition 2.9.** Let $n = 3$. One has the following low-energy resolvent asymptotics for $R_0(z)$: for $s, s' > \frac{1}{2}$ and $s + s' > 2$, there exists $\epsilon > 0$ such that

$$R_0(z) = G_0 + O(|z|^\epsilon), \quad \text{as } z \to 0, z \notin \mathbb{R}_+,$$

as operators in $\mathcal{B}(-1, s; 1, -s')$. More generally, for any integer $N \geq 1$ and $s > N + \frac{1}{2}$, there exists $\epsilon > 0$

$$R_0(z) = \sum_{j=0}^{N} z^{\frac{j}{2}} G_j + O(|z|^{N+\epsilon}), \quad \text{as } z \to 0, z \notin \mathbb{R}_+,$$

as operators in $\mathcal{B}(-1, s; 1, -s)$. Here the branch of $z^{\frac{1}{2}}$ is chosen such that its imaginary part is positive when $z \notin \mathbb{R}_+$ and $G_j \in \mathcal{B}(-1, s; 1, -s)$ for $s > j + \frac{1}{2}$, $j \geq 1$. In particular,

$$G_0 = F_0 + F_1, \quad (2.51)$$

where $F_0$ is the operator with integral kernel

$$F_0(x, v; x', v') = \frac{\psi_0(v)\psi_0(v')}{4\pi |x - x'|} \quad (2.52)$$
and $F_1 \in \mathcal{B}(-1, s; 1, -s')$ for any $s, s' \geq 0$ and $s + s' > \frac{3}{2}$. $G_1 : \mathcal{H}^{-1,s} \to \mathcal{H}^{1,-s}$, $s > \frac{3}{2}$, is an operator of rank one with integral kernel given by

$$K_1(x, x'; v, v') = \frac{i}{4\pi} \psi_0(v) \psi_0(v').$$

(2.53)

Here $\psi_0 = (2\pi)^{-\frac{3}{4}} e^{-\frac{r^2}{4}}$ is the first eigenfunction of the harmonic oscillator $-\Delta_v + \frac{v^2}{4}$.

**Proof.** Note that by a complex interpolation, the results on the resolvent $R_0(z)$ in Corollary 2.8 also hold in $\mathcal{B}(-1, s; 1, -s)$.

For $z \notin \mathbb{R}_+$, (2.43) with $l = 0$ shows that

$$R_0(z) = b_0^w(v, D_v, D_v)(-\Delta_v - z)^{-1} + r_0(z),$$

(2.54)

with $r_0(z) \in \mathcal{B}(-1, 0; 1, 0)$ holomorphic in $z$ when $\Im z < a$ for some $a \in [0, 1]$. Here the cut-off $\chi(\xi)$ is chosen such that $\chi \in C^\infty_0$ and $\chi(\xi) = 1$ in a neighbourhood of $||\xi||^2 \leq a$. Therefore $r_0(z)$ admits a convergent expansion in powers of $z$ for $z$ near $0$

$$r_0(z) = r_0(0) + z r_0'(0) + \cdots$$

in $\mathcal{B}(-1, 0; 1, 0)$. It is sufficient to analyze the lower-energy expansion of $b_0^w(v, D_v, D_v)(-\Delta_v - z)^{-1}$.

The integral kernel of $b_0^w(v, D_v, D_v)(-\Delta_v - z)^{-1}$, $z \notin \mathbb{R}_+$, is given by

$$K(x, x'; v, v'; z) = \int_{\mathbb{R}^3} e^{i\sqrt{|z|}y - (x - x')|} \frac{1}{4\pi |y - (x - x')|} \Phi(v, v', y) \, dy$$

(2.55)

with

$$\Phi(v, v', y) = (2\pi)^{-\frac{9}{2}} e^{-\frac{1}{4}(v^2 + v'^2)} \int_{\mathbb{R}^3} e^{i(y - v - v') \cdot \xi + 2\xi^2} \chi(\xi) \, d\xi.$$  

Since $\chi \in C^\infty_0$, one has the following asymptotic expansion for $K(x, x'; v, v'; z)$: for any $\epsilon \in [0, 1]$ and $N \geq 0$

$$|K(x, x'; v, v'; z) - \sum_{j=0}^N z^j K_j(x, x', v, v')| \leq C_{N,\epsilon} |z|^\frac{N+\epsilon}{2} |x - x'|^{N-1+\epsilon} e^{-\frac{1}{2}(v^2+v'^2)}$$

(2.56)

where

$$K_j(x, x'; v, v') = \frac{i^j}{4\pi} \int_{\mathbb{R}^3} |y - (x - x')|^{j-1} \Phi(v, v', y) \, dy.$$  

(2.57)

Remark that for $N \geq 1$, $s'$, $s > N + \frac{1}{2}$ and $0 < \epsilon < \min\{s, s'\} - N - \frac{1}{2}$,

$$\langle x \rangle^{-s} \langle x' \rangle^{-s'} |x - x'|^{N-1+\epsilon} e^{-\frac{1}{4}(v^2+v'^2)} \in L^2(\mathbb{R}^3)$$

and the same is true if $N = 0$ and $s, s' > \frac{1}{2}$ with $s + s' > 2$. We obtain the asymptotic expansion for $b_0^w(v, D_v, D_v)(-\Delta_v - z)^{-1}$ in powers of $z^\frac{1}{2}$ for $z$ near $0$ and $z \notin \mathbb{R}_+$.

$$b_0^w(v, D_v, D_v)(-\Delta_v - z)^{-1} = \sum_{j=0}^N z^j K_j + O(|z|^\frac{N+\epsilon}{4}),$$

(2.58)
as operators in $B(0, s'; 0, -s), s', s > N + \frac{1}{2}$ (and $s + s' > 2$ if $N = 0$). This proves that the expansion (2.50) holds in $B(0, s'; 0, -s)$ with $G_k$ given by

$$G_{2j} = K_{2j} + \frac{r_{j}^{(0)}}{j!}, \quad G_{2j+1} = K_{2j+1}, \quad j \geq 0. \quad (2.59)$$

To see that this expansion still holds in $B(-1, s'; 1, -s)$, we use the hypoelliticity of $P_0$ and the argument used in Corollary 2.8: writing the resolvent equation $R_0(z) = R_0(-1) + (1 + z)R_0(-1)R_0(z)$ and using the hypoelliticity of $P_0$ which implies that $R_0(-1) \in B(0, r; 2, r)$ for any $r$, one deduces from the expansion (2.50) holds in $B(0, s'; 0, s)$ that this expansion still holds $B(0, s'; 2, s)$. Similarly, one can show that it also holds in $B(-2, s'; 0, s)$. A complex interpolation gives the expansion (2.50) in $B(-1, s'; 1, s)$.

To show (2.51) and (2.53), note that since $\chi(0) = 1$, one has

$$\int_{\mathbb{R}^3} \Phi(v, v', y) dy = \chi(0)\psi_0(v)\psi_0(v') = \psi_0(v)\psi_0(v').$$

One can then calculate that

$$K_0(x, x', v, v') = \frac{1}{4\pi} \int_{\mathbb{R}^3} \Phi(v, v', y) \frac{1}{|y - (x - x')|} dy
= \frac{1}{4\pi|x - x'|} \psi_0(v)\psi_0(v')$$

$$+ \frac{1}{4\pi} \int_{\mathbb{R}^3} \Phi(v, v', y) \left( \frac{1}{|y - (x - x')|} - \frac{1}{|x - x'|} \right) dy \quad (2.60)$$

and

$$K_1(x, x', v, v') = \frac{i}{4\pi} \int_{\mathbb{R}^3} \Phi(v, v', y) dy = \frac{i}{4\pi} \psi_0(v)\psi_0(v'). \quad (2.61)$$

This shows (2.53) and that $G_0 = F_0 + F_1$ with $F_1 = K_{0,1} + r_0(0)$, $K_{0,1}$ being the operator with the integral kernel

$$K_{0,1}(x, x', v, v') = \frac{1}{4\pi} \int_{\mathbb{R}^3} \Phi(v, v', y) \left( \frac{1}{|y - (x - x')|} - \frac{1}{|x - x'|} \right) dy,$$

which is a smooth function and

$$K_{0,1}(x, x', v, v') = O(\psi_0(v)\psi_0(v')|x - x'|^{-2})$$

for $|x - x'|$ large. Therefore $K_{0,1}$ is bounded in $B(-1, s; 1, -s')$ for any $s, s' \geq 0$ and $s + s' > \frac{3}{2}$. This shows that $F_1 = K_{0,1} + r_0(0)$ has the same continuity property, which proves the decomposition (2.51) for $G_0$. □

The following high-energy pseudo-spectral estimate is used in the proof of Theorem 1.1 (a). In the earlier version of this work, we obtained an upper bound of the form $O(|z|^{-\frac{1}{2}})$ for $R_0(z)$ which is also sufficient to study the large-time asymptotics of solutions. We learned recently from F. Nier that the optimal bound is $O(|z|^{-\frac{1}{2}})$. See Proposition A.2 of [22].
Proposition 2.10. Let \( n \geq 1 \). Then for every \( \delta > 0 \), there exists \( M > 0 \) such that

\[
\| R_0(z) \| \leq \frac{M}{|z|^\frac{3}{2}},
\]

(2.62)

and

\[
\|(1 - \Delta_v + v^2)^\frac{1}{2} R_0(z)\| \leq \frac{M}{|z|},
\]

(2.63)

for \( |\Im z| > \delta \) and \( \Re z \leq \frac{1}{M} |\Im z|^\frac{1}{2} \).

Proof. We first prove that for some constant \( C > 0 \)

\[
\| R_0(z) \| \leq \frac{C}{|z|^\frac{1}{2}},
\]

(2.64)

for \( z = -\frac{n}{2} + i\mu \) with \( \mu \in \mathbb{R} \). It suffices to show that

\[
\| \hat{R}_0(-\frac{n}{2} + i\mu, \xi) \|_{B(L^2(\mathbb{R}^n))} \leq \frac{C}{|\xi|^\frac{1}{2}},
\]

(2.65)

uniformly in \( \xi \in \mathbb{R}^n \). Notice that \( \hat{P}_0(0) \) is selfadjoint and that one has

\[
\| \hat{R}_0(z, 0) \| \leq \frac{1}{|\Im z|} \quad \text{and} \quad \| v \cdot \xi \hat{R}_0(z, 0) \| \leq \frac{C|\xi|}{\sqrt{|\Im z|}},
\]

for \( \Im z \neq 0 \). Making use of the resolvent equation

\[
\hat{R}_0(z, \xi) = \hat{R}_0(z, 0) - \hat{R}_0(z, \xi)i v \cdot \xi \hat{R}_0(z, 0),
\]

one obtains

\[
\| \hat{R}_0(z, \xi) \|_{B(L^2(\mathbb{R}^n))} \leq \frac{C}{|\Im z|},
\]

(2.66)

if \( |\xi| \leq c_1 \sqrt{|z|} \) for some \( c_1 > 0 \). For \( c|z|^{\frac{1}{2}} \leq |\xi| \) with \( c > 0 \) small enough, since we are concerned with estimates for \( |z| \) large, we can use a rotation and a rescaling to reduce to a semiclassical problem. Set \( A(h) = -h^2 \Delta_v + \frac{v^2}{4} + i v_1 \). Then

\[
\| \hat{R}_0(z, \xi) \| = |\xi|^{-2} \|(A(h) - z')^{-1}\|
\]

where \( h = |\xi|^{-2} \) and \( z' = |\xi|^{-2}(\frac{n}{2} + z) = i\lambda \) with \( \lambda = |\xi|^{-2}\mu \in \mathbb{R} \). In the earlier versions of this work, we used Theorem 1.4 of [6] to conclude

\[
\|(A(h) - i\lambda)^{-1}\| \leq C h^{-\frac{2}{3}},
\]

(2.67)

if \( 0 < h \leq 1, |\lambda| \leq C \), which allowed to obtain an upper bound of the resolvent \( R_0(z) \) of the form \( O(|z|^{-\frac{1}{2}}) \). In fact, one can show for this model operator the following better estimate:

\[
\|(A(h) - i\lambda)^{-1}\| \leq C (h^\frac{2}{3} + |h\lambda|^\frac{1}{2})^{-1}
\]

(2.68)
uniformly in $h \in ]0, 1]$ and $\lambda \in \mathbb{R}$. See Proposition A.2 of [22] in the case $h = 1$. The proof for $0 < h \leq 1$ follows from the same arguments and is omitted here.

It follows from (2.68) that for $z = -\frac{n}{2} + i \mu$ with $\mu$ real, one has

$$\| \hat{R}_0(z, \xi) \| \leq C \langle |\xi|^{\frac{3}{2}} + |\mu|^{\frac{1}{2}} \rangle^{-1}, \quad (2.69)$$

for $c|z|^{\frac{1}{2}} \leq |\xi|$. This proves (2.64). Now to prove (2.62), set $z = \lambda + i \mu$ with $\lambda, \mu \in \mathbb{R}$ and write

$$R_0(\lambda + i \mu) = R_0(-\frac{n}{2} + i \mu) - (\lambda + \frac{n}{2})R_0(-\frac{n}{2} + i \mu)R_0(\lambda + i \mu).$$

According to (2.64),

$$\| (\lambda + \frac{n}{2})R_0(-\frac{n}{2} + i \mu) \| \leq \frac{C|\lambda + \frac{n}{2}|}{|\xi|^{\frac{1}{2}}} \leq \frac{1}{2}$$

if $|\lambda| \leq \frac{1}{M}|\mu|^{\frac{1}{2}}$ and $|\mu| \geq M$ for some $M > 1$ large enough (2.62) follows from (2.64) and the equation $R_0(\lambda + i \mu) = (1+(\lambda + \frac{n}{2})R_0(-\frac{n}{2} + i \mu))^{-1}R_0(-\frac{n}{2} + i \mu)$ when $|\lambda| \frac{1}{M}|\Im z|^{\frac{1}{2}}$ with $M > 0$ sufficiently large. The estimate (2.62) for $\Re z = \lambda < -\frac{1}{M}|\Im z|^{\frac{1}{2}}$ follows from the accretivity of $P_0$.

To show (2.63), notice that for $z = \lambda + i \mu$ with $\lambda, \mu \in \mathbb{R}$, one has the identity

$$\| \nabla v u \|^2 + \frac{1}{4} \| v u \|^2 = (\lambda + \frac{n}{2})\| u \|^2 + \Im (P_0 - z)u, u)$$

for $u \in D$. One obtains from (2.62) that

$$\| \nabla v R_0(z)w \|^2 + \frac{1}{4} \| v R_0(z)w \|^2$$

$$\leq |\lambda + \frac{n}{2}| \| R_0(z)w \|^2 + \| w \| \| R_0(z)w \|$$

$$\leq C \left( \frac{|\lambda|}{|\xi|} + \frac{1}{2} \right) \| w \|, \quad \forall w \in L^2,$$

for $|\Im z| > M$ and $|\Re z| \leq \frac{1}{M}|\Im z|^{\frac{1}{2}}$ with $M > 1$ sufficiently large. (2.63) is proven. $\square$

Remark that when $n \geq 3$, Corollary 2.8 shows the existence of the boundary values of the free resolvent $R_0(\lambda \pm i 0)$ for any $\lambda > 0$. But so far it is still unclear what kind of upper bound one can expect for $R_0(\lambda \pm i 0)$ as $\lambda \rightarrow +\infty$. From Propositions 2.9 and 2.10, one can deduce the following

**Corollary 2.11.** Let $n = 3$. Let $S_0(t)$, $t \geq 0$, denote the semigroup generated by $-P_0$. Then for any integer $N \geq 1$ and $s > N + \frac{1}{2}$, the following asymptotic expansion holds for some $\epsilon > 0$

$$e^{-tP_0} = \sum_{k \in \mathbb{N}, 2k+1 \leq N} t^{-\frac{2k+3}{2}} \beta_k G_{2k+1} + O(t^{-\frac{N+2}{2}-\epsilon}), \quad t \rightarrow +\infty, \quad (2.70)$$
Lemma 2.12. Let $n$ themselves. We establish some formulae on the evolution of observables which may be of interest in 2.13 below to compute the leading term. As a preparation for the proof of this proposition, the operator $P$ is a rank-one operator given by

$$
\frac{1}{8\pi^2} \langle \mathbf{m}_0, \cdot \rangle \mathbf{m}_0 : \mathcal{L}^{2,s} \to \mathcal{L}^{2,-s}
$$

(2.71) for any $s > \frac{3}{2}$. Here $\mathbf{m}_0(x, v) = 1 \otimes \psi_0(v)$.

The proof of Corollary 2.11 uses a representation of the free semigroup $e^{-tP_0}$ as contour integral of the resolvent $R_0(z)$ in the right half complex plane. See Sect. 4 for more details in the case $V \neq 0$ where we shall prove a similar result for the full KFP operator $P$ [see (1.14)]. In the final step of the proof of (1.14, we shall apply Proposition 2.13 below to compute the leading term. As a preparation for the proof of this proposition, we establish some formulae on the evolution of observables which may be of interest in themselves.

Lemma 2.12. Let $n \geq 1$. For $t > 0$ and $0 \leq s \leq t$, one has the following equalities as operators from $\mathcal{S}(\mathbb{R}^{2n}_{x,v})$ to $L^2(\mathbb{R}^{2n}_{x,v})$:

$$
e^{-(t-s)P_0}v_je^{-sP_0} = e^{-tP_0}(v_j \cosh s - 2\partial_{v_j} \sinh s + 2(\cosh s - 1)\partial_{x_j})
$$

(2.72)

$$
e^{-(t-s)P_0}\partial_{v_j}e^{-sP_0} = -\frac{1}{2}e^{-tP_0}((v_j \sinh s - 2\partial_{v_j} \cosh s + 2\partial_{x_j} \sinh s))
$$

(2.73)

$$
e^{-(t-s)P_0}x_je^{-sP_0} = e^{-tP_0}(x_j + v_j \sinh s - 2(\cosh s - 1)\partial_{v_j} + 2(\sinh s - s)\partial_{x_j})
$$

(2.74)

Proof. For fixed $t > 0$, set $f(s) = e^{-(t-s)P_0}v_j e^{-sP_0}$, $0 \leq s \leq t$. Proposition 2.1 shows that for $u \in \mathcal{S}$, $A^k e^{-tP_0}u \in L^2$ for any $k \in \mathbb{N}$, where $A$ may be one of the operators $v_j, \partial_{v_j}, \partial_{x_j}$, $j = 1, \ldots, n$. As operators from $\mathcal{S}$ to $L^2$, one has:

$$
f'(s) = e^{-(t-s)P_0}[P_0, v_j]e^{-sP_0} = -2e^{-(t-s)P_0}\partial_{v_j}e^{-sP_0} = e^{-(t-s)P_0}
$$

(2.75)

$$
f''(s) = -2e^{-(t-s)P_0} \left[ \frac{v_j^2}{4} + v \cdot \partial_x, \partial_{v_j} \right] e^{-sP_0} = 2(\sinh s - s)\partial_{x_j}
$$

(2.76)

This shows that $f(s) = C_1 e^s + C_2 e^{-s} - 2\partial_{x_j} e^{-tP_0}$. $C_1, C_2$ can be determined by the initial data $f(0) = e^{-tP_0}v_j$ and $f'(0) = -2e^{-tP_0}\partial_{v_j}$:

$$
C_1 = e^{-tP_0} \left( \frac{1}{2} v_j - \partial_{v_j} + \partial_{x_j} \right), \quad C_2 = e^{-tP_0} \left( \frac{1}{2} v_j + \partial_{v_j} + \partial_{x_j} \right).
$$

This proves (2.72), (2.73) follows from (2.72) and the equality

$$
e^{-(t-s)P_0}\partial_{v_j}e^{-sP_0} = -\frac{1}{2} f'(s).
$$

To prove (2.74), one can check the following commutator relation:

$$
[e^{-tP_0}, x_j] = -\int_0^t e^{(t-s)P_0}v_j e^{-sP_0} ds
$$

$$
= -e^{-tP_0} \left( v_j \sinh t - 2(\cosh t - 1)\partial_{v_j} + 2(\sinh t - t)\partial_{x_j} \right),
$$

(2.77)
which means that the commutator initially defined as forms on $S \times S$ extends to operators from $S$ to $L^2$ and the equality (2.77) holds. A successive application of this commutator relation shows that if $u \in S$, then $(x)^r e^{-tP_0}u \in L^2$ for any $r \in \mathbb{R}$. It follows from (2.72) that

$$e^{-(t-s)P_0}x_j e^{-sP_0} = e^{-tP_0}x_j + \int_0^s e^{-(t-\tau)P_0}v_j e^{-\tau P_0} \, d\tau$$

$$= e^{-tP_0}(x_j + v_j \sinh s - 2(\cosh s - 1)\partial v_j + 2(\sinh s - s)\partial x_j).$$

This proves (2.74).  \[\square\]

**Proposition 2.13.** Let $n = 3$. Assume that $u \in \mathcal{L}^{2, -s}$ for some $\frac{3}{2} < s < 2$ such that there exists some constant $c_0$ and $\psi \in L^2_v$ with $(-\Delta_v + v^2)\psi \in L^2_v$ such that

$$u(x, v) - c_0(1 \otimes \psi) \in \mathcal{L}^{2, \delta}([\mathbb{R}^6_x, v])$$

for some $\delta > 0$. Then on has

$$\lim_{\lambda \to 0_-} \lambda R_0(\lambda)u = -c_0(\psi_0, \psi)_{L^2_v}m_0$$

in $\mathcal{L}^{2, -s}$ for any $s > \frac{3}{2}$, where $m_0 = 1 \otimes \psi_0(v)$.

**Proof.** For $\lambda < 0$, $R_0(\lambda)$ maps $\mathcal{L}^{2, -s}$ to $\mathcal{L}^{2, -s}$ for any $s$ and one has

$$\|R_0(\lambda)\|_{\mathcal{B}(0, 0; 0, 0)} \leq \frac{C}{|\lambda|}, \quad \|R_0(\lambda)\|_{\mathcal{B}(0, s'; 0, -s)} \leq C_{s,s'}$$

if $s, s' > \frac{1}{2}$ with $s + s' > 2$, uniformly in $\lambda \in ]-1, 0[$. An argument of complex interpolation shows that for any $s, s' > 0$, there exists $\epsilon > 0$ such that

$$\|R_0(\lambda)\|_{\mathcal{B}(0, s'; 0, -s)} \leq C|\lambda|^{-1+\epsilon}. \quad (2.79)$$

This shows that

$$\lambda R_0(\lambda)(u - c_0(1 \otimes \psi)) = o(1), \quad \text{as } \lambda \to 0_-$$

in $\mathcal{L}^{2, -s}$ for any $s > 0$. To prove Proposition 2.13, it suffices to study the limit $\lim_{\lambda \to 0_-} \lambda R_0(\lambda)(1 \otimes \psi)$. By Proposition 2.7, the resolvent $R_0(\lambda)$ can be decomposed as

$$R_0(\lambda) = b_0^{wx}(v, D_x, D_v)(-\Delta_v - \lambda)^{-1} + r_0(\lambda) \quad (2.81)$$

where

$$b_0(v, \xi, \eta) = 2^3 \chi(\xi) e^{-\omega^2 - \eta^2 + 2i\omega \xi + 2\xi^2}$$

with $\chi$ a smooth cut-off around 0 with compact support, and $r_0(\lambda)$ is uniformly bounded as operators in $L^2$ for $\lambda < a$ for some $a \in ]0, 1[$.

We claim that the following estimate holds uniformly in $\lambda < a$:

$$\|\langle x \rangle^{-2} r_0(\lambda) \langle x \rangle^2 f \| \leq C(\|f\| + \|H_0 f\|) \quad (2.82)$$
for any \( f \in D(H_0) \), where \( H_0 = -\Delta_v + v^2 - \Delta_x \). Remark that \( r_0(\lambda) \) is a pseudodifferential operator in \( x \)-variables: \( r_0(\lambda) = r_0(\lambda, D_x) \), with operator-valued symbol \( r_0(\lambda, \xi) \in B(L_v^2) \) [see (2.40)]. The proof of Proposition 2.7 shows that \( r_0(\lambda, \xi) \) can be decomposed as

\[
r_0(\lambda, \xi) = r_{0,1}(\lambda, \xi) + r_{0,2}(\lambda, \xi)
\]

with

\[
r_{0,1}(\lambda, \xi) = \int_0^T \chi_1(\xi)e^{-t(\hat{P}_0(\xi) - \lambda)}dt
\]

for some fixed \( T \geq 3 \) and \( r_{0,2}(\lambda, \xi) \) is smooth and rapidly decreasing in \( \xi \), uniformly for \( \lambda < a \) [see (2.35), (2.37) and (2.38)]. Since \( r_{0,2}(\lambda, D_x) \) is a convolution in \( x \)-variables with a smooth and rapidly decreasing kernel, one has \( \langle x \rangle^{-s}r_{0,2}(\lambda, D_x)(x)^s \) is uniformly bounded for any \( s \). To study \( r_{0,1}(\lambda, D_x) \), we use commutator techniques. One writes

\[
\langle x \rangle^{-2}r_{0,1}(\lambda, D_x)x^2 = \langle x \rangle^{-2} \left( x^2r_{0,1}(\lambda, D_x) + \sum_{j=1}^n [r_{0,1}(\lambda, D_x), x_j], x_j \right).
\]

Making use of (2.77), one can calculate

\[
[r_{0,1}(\lambda, D_x), x_j] = -i(\partial_{\xi_j} \chi_1)(D_x) \int_0^T e^{-t(\hat{P}_0 - \lambda)}dt + \chi_1(D_x) \int_0^T [e^{-t(\hat{P}_0 - \lambda)}, x_j]dt
\]

\[
= -i(\partial_{\xi_j} \chi_1)(D_x) \int_0^T e^{-t(\hat{P}_0 - \lambda)}dt + \chi_1(D_x) \int_0^T e^{-t(\hat{P}_0 - \lambda)}dt
\]

\[
\quad + \chi_1(D_x) \int_0^T e^{-t(\hat{P}_0 - \lambda)}(v_j \sinh t - 2(cosh t - 1)\partial v_j + 2(sinh t - t)\partial x_j)dt
\]

Since \( \hat{P}_0 \) is accretive, one obtains

\[
|[r_{0,1}(\lambda, D_x), x_j], f|| \leq C(||f|| + ||v_j f|| + ||\partial_{v_j} f|| + ||\partial_{x_j} f||), \quad f \in S.
\]

Similarly, one can check that the second commutator \([r_{0,1}(\lambda, D_x), x_j], x_j\) verifies

\[
||[r_{0,1}(\lambda, D_x), x_j], x_j], f|| \leq C(||f|| + ||H_0 f||).
\]

This proves that

\[
\langle x \rangle^{-2}r_{0,1}(\lambda)(x)^2 f \| \leq C(||f|| + ||H_0 f||)
\]

for any \( f \in D(H_0) \), which gives (2.82). It follows from (2.82) and the uniform boundedness of \( \|r_0(\lambda)\| \) for \( \lambda < a, a > 0 \), that for any \( s \in [0, 2] \)

\[
\langle x \rangle^{-s}r_0(\lambda)\langle x \rangle^s f \| \leq C(||f|| + ||H_0 f||),
\]

Since \( 1 \otimes \psi = \langle x \rangle^s f_s \) with \( f_s = \langle x \rangle^{-s} \otimes \psi \in D(H_0) \) for any \( s > \frac{3}{2} \), it follows that

\[
\lambda r_0(\lambda)(1 \otimes \psi) = O(||\lambda||), \quad \lambda \to 0,
\]

(2.86)
in $L^{2,-s}$ for any $s > \frac{3}{2}$, which together with (2.80) implies that
\[
\lambda R_0(\lambda)u - c_0\lambda b_0^w(v, D_x, D_v)(-\Delta_x - \lambda)^{-1}(1 \otimes \psi) = o(1)
\]
in $L^{2,-s}$, $s < \frac{3}{2}$, as $\lambda \to 0_+$. This estimate is valid in all dimensions.

Finally we calculate $\lambda b_0^w(v, D_x, D_v)(-\Delta_x - \lambda)^{-1}(1 \otimes \psi)$ which is independent of $\lambda < 0$ in dimension $n = 3$. In fact, according to (2.55) one has for $\lambda < 0$ and $n = 3$
\[
\lambda b_0^w(v, D_x, D_v)(-\Delta_x - \lambda)^{-1}(1 \otimes \psi)
= \int_{\mathbb{R}^3} \frac{\lambda e^{-\sqrt{|\lambda|}|y - (x - x')|}}{4\pi |x - x'|} \Phi(v(v'), y)\psi(v') dy dx' dv'
= \int_{\mathbb{R}^3} \frac{\lambda e^{-\sqrt{|\lambda|}|x'|}}{4\pi |x'|} dx' \int_{\mathbb{R}^6} \Phi(v(v'), y)\psi(v') dy dv'
= -\chi(0)(\psi_0, \psi_{L^2})\psi_0(v) = -(\psi_0, \psi_{L^2})\psi_0(v).
\]
This finishes the proof of Proposition 2.13. \qed

3. Threshold Spectral Properties of the Kramers-Fokker-Planck Operator

Consider the Kramers-Fokker-Planck operator $P = v \cdot \nabla_x - \nabla_v V(x) \cdot V_v - \Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2}$ with a $C^1$ potential $V(x)$ satisfying (1.8) for some $\rho \geq -1$. In fact, it is sufficient to suppose that $V(x)$ is a Lipschitz function satisfying (1.8) almost everywhere and one can even include some mild local singularities in $\nabla V(x)$, by using hypoellipticity of the operator. But we will not care about such generalization. Operator $P$ defined on $D(P) = D(P_0)$ is maximally dissipative. By Proposition 2.1, if $\rho > -1$, $\nabla_x V(x) \cdot V_v$ is relatively compact with respect to $P_0$. Consequently, the spectrum of $P$ is discrete outside $\mathbb{R}_+$ and the complex eigenvalues of $P$ may only accumulate towards points in $\mathbb{R}_+$. The set $\mathbb{N}$ is called thresholds of $P$. Notice that in spectral analysis of selfadjoint operators, thresholds often refer to the extreme points of the essential spectrum of some model operator. Our terminology is consistent from this point of view (see the remark after Proposition 2.3). Large-time asymptotics of solutions are closely related to the low-energy spectral properties of the operator. The notion of threshold resonance is important in this context, because threshold resonances give rise to non-trivial singularities in the resolvent asymptotics. The analysis made in Sect. 2 shows that near thresholds, the free resolvent $R_0(z)$ bears many similarities with the resolvent of the Laplacian $-\Delta_x$. This leads us to introduce the following definition of threshold resonance for the full KFP operator $P$.

**Definition.** Let $k \in \mathbb{N}$. We call $u$ a resonant state of the KFP operator $P$ at the threshold $k$ if $u$ satisfies the equation $Pu = ku$ and $u \in L^{2,-s} \backslash L^2$ for any $s > 1$. The number $k$ is then called a threshold resonance of $P$.

**Remarks.** (a). For non selfadjoint Schrödinger operators $Q = -\Delta + U(x)$ with a complex-valued short-range potential $U(x)$, one call $\lambda \geq 0$ a real resonance of $Q$ if the equation $Qu = \lambda u$ admits a non-trivial solution $u$ verifying some Sommerfeld radiation condition. If the potential $U(x)$ is real, it is known that under fairly
general assumptions, positive real resonances are absent (see [1]). For complex-valued potentials, there may exist positive real resonances. See [27] for an example in the case of dissipative Schrödinger operators. The possible existence of real resonances is one of the difficulties for understanding the spectral properties of non-selfadjoint Schrödinger operators.

(b). For the KFP operator $P$, if $V(x)$ is $C^1$ on $\mathbb{R}^n$ and $V(x) = a \ln |x|$ for $|x|$ large with $a \in [\frac{k}{2}, \frac{2}{k}]$, $n \geq 2$, then $0$ is a threshold resonance of $P$ and $m$ a resonant state and while if $V(x) \to 0$ as $|x| \to \infty$, $m$ is not a resonant state although it still verifies the equation $Pm = 0$. This is because in the former case $m \in L^{2,-s} \setminus L^2$ for any $s > 1$ while in the later case, $m \not\in L^{2,-s}$ for some $s > 1$.

In the following, we only study the first threshold 0 in dimension $n = 3$ under the short-range assumption $\rho > 1$. Denote $W = -\nabla_x V(x) \cdot \nabla_v$. One has $W^* = -W$ and

$$(P - \varepsilon)R_0(z) = 1 + R_0(z)W = 1 + G_0W + O(|z|^\delta), \quad \varepsilon > 0,
$$
in $B(0, -s; 0, -s)$ for $1 < s < (1 + \rho)/2$ and $z$ near 0 and $z \not\in \mathbb{R}_+$. 

**Lemma 3.1.** Assume $n = 3$. Let (1.8) be satisfied with $\rho > 1$ (i.e., the potential is of short-range). Then, $G_0W$ is a compact operator in $L^{2,-s}$ for $1 < s < (1 + \rho)/2$ and

$$\ker_{L^{2,-s}}(1 + G_0W) = \{0\}. \quad (3.1)$$

**Proof.** For $1 < s < (1 + \rho)/2$, take $1 < s' < s$. Proposition 2.9 (and the arguments used in the proof of Corollary 2.8) shows that $G_0W \in B(0, -s; 1, -s')$. The injection $\mathcal{H}^{1,-s'} \hookrightarrow L^{2,-s}$ is compact when $s' < s$. Therefore $G_0W$ is a compact operator in $L^{2,-s}$.

Let $u \in L^{2,-s}$ with $u + G_0Wu = 0$. Then by the hypoellipticity of $P_0$, one can check that $u \in \mathcal{H}^{2,-s}$ for any $s > 1$. In addition, one has $Pu = 0$. According to (2.51), $u$ can be decomposed as

$$u = -F_0Wu - F_1Wu.$$ 

Since $Wu \in \mathcal{H}^{-1,\rho+1-s}$ and $F_0 \in B(-1, s; 1, -s')$ for any $s, s' > \frac{1}{2}$ and $s + s' > 2$, it follows that $u$ is in fact in $\mathcal{H}^{2,-s}$ for any $s > \frac{1}{2}$. Using the condition $\rho > 1$ and repeating the above argument, we deduce that $F_1Wu \in L^2$. By (2.52), one can calculate the asymptotic behavior of $F_0Wu$ for $|x|$ large and obtains that

$$u(x, v) = w(x, v) + r(x, v), \quad (3.2)$$

where $\langle v \rangle^2 r \in L^2(\mathbb{R}^6_{x,v})$ and $w(x, v) = C(u) e^{-\frac{v^2}{|x|^2}}$ with

$$C(u) = \int_{\mathbb{R}^6} \int_{\mathbb{R}^6} \frac{e^{-\frac{v^2}{|x|^2}}}{2(2\pi)^{3\frac{1}{2}}} \nabla_x V(x) \cdot \nabla_v u(x, v) \, dx \, dv. \quad (3.3)$$

Let $\chi \in C_0^\infty(\mathbb{R})$ be a cut-off with $\chi(\tau) = 1$ for $|\tau| \leq 1$ and $\chi(\tau) = 0$ for $|\tau| \geq 2$ and $0 \leq \chi(\tau) \leq 1$. Set $\chi_R(x) = \chi(\frac{|x|}{R})$, $R > 1$ and $x \in \mathbb{R}^3$ and $u_R(x) = \chi_R(x)u(x, v)$. Then one has

$$Pu_R = \frac{v \cdot \hat{x}}{R} \chi'(\frac{|x|}{R})u.$$
Theorem 3.2. Assume $n = 3$ and $\rho > 1$. Then zero is not an accumulation point of the eigenvalues of $P$ and one has for any $s, s' > 1/2$ with $s + s' > 2$, $\exists \epsilon > 0$ such that
\[
R(z) = A_0 + O(|z|^\epsilon), \quad z \to 0, \quad z \notin \mathbb{R}_+,
\]
in $\mathcal{B}(-1, s; 1, -s')$, where
\[
A_0 = (1 + G_0 W)^{-1} G_0. \tag{3.9}
\]
There exists $\delta > 0$ such that boundary values of the resolvent
\[
R(\lambda \pm i0) = \lim_{\epsilon \to 0^+} R(\lambda \pm i\epsilon), \quad \lambda \in [0, \delta]
\]
exist in $\mathcal{B}(-1, s; 1, -s)$ for any $s > 1/2$ and are continuous in $\lambda$. 

Taking the real part of the equality $\langle Pu_R, u_R \rangle = \langle \frac{v^*}{R} \chi' \left( \frac{|x|}{R} \right) u, u_R \rangle$, one obtains
\[
\int \int_{\mathbb{R}^6} |(\partial_v + \frac{v}{2})u(x, v)|^2 \chi \left( \frac{|x|}{R} \right)^2 \, dx \, dv = \langle \frac{v^*}{R} \chi' \left( \frac{|x|}{R} \right) u, u_R \rangle \tag{3.4}
\]
Since $w$ is even in $v$, one has
\[
\langle \frac{v^*}{R} \chi' \left( \frac{|x|}{R} \right) w, \chi R w \rangle = 0.
\]
The right hand side of the equality (3.4) satisfies
\[
|\langle \frac{v^*}{R} \chi' \left( \frac{|x|}{R} \right) u, u_R \rangle|
\leq \frac{2}{R} \langle \frac{v^*}{R} \chi' \left( \frac{|x|}{R} \right) w, \chi R r \rangle + \langle \frac{v^*}{R} \chi' \left( \frac{|x|}{R} \right) r, \chi R r \rangle
\leq CR^{-(1-s)}(||w||_{L^{2,-s}} + ||r||_{L^2})||\langle v \rangle r||_{L^2} \tag{3.5}
\]
for some $\frac{1}{2} < s < 1$. Taking the limit $R \to +\infty$ in (3.4), one obtains that
\[
\int \int_{\mathbb{R}^6} |(\partial_v + \frac{v}{2})u(x, v)|^2 \, dx \, dv = 0. \tag{3.6}
\]
This shows that $(\partial_v + \frac{v}{2})u(x, v) = 0$, a.e. in $x, v$. Since $u \in L^{2,-s}$ for any $s > \frac{1}{2}$ and $Pu = 0$, one sees that $u$ is of the form $u(x, v) = C(x)e^{-\frac{v^2}{2}}$ for some $C \in L^{2,-s}(\mathbb{R}_3^3)$ verifying the equation
\[
v \cdot \nabla_x C(x) + \frac{1}{2} v \cdot \nabla V(x) C(x) = 0 \tag{3.7}
\]
a.e. in $x$ for all $v \in \mathbb{R}_3^3$. This proves that $C(x) = c_0 e^{-\frac{v^2}{2}}$ a.e. for some constant $c_0$ and
\[
u(x, v) = c_0 m.
\]
Since $u \in L^{2,-s}$ for any $s > \frac{1}{2}$ and $m \notin L^{2,-s}$ if $\frac{1}{2} < s < \frac{3}{2}$ when $V(x)$ is bounded, one concludes that $c_0 = 0$, therefore $u = 0$. This proves that $\text{ker} L^{2,-s}(1 + G_0 W) = \{0\}$. 

Lemma 3.1 says that zero is neither an eigenvalue nor a resonance of the KFP operator $P$ if the potential is of short-range. This is in sharp contrast to Schrödinger operators for which zero resonance may exist even for smooth and compactly supported potentials. Lemma 3.1 makes easier the threshold spectral analysis for the KFP operator $P$. 

Theorem 3.2. Assume $n = 3$ and $\rho > 1$. Then zero is not an accumulation point of the eigenvalues of $P$ and one has for any $s, s' > 1/2$ with $s + s' > 2$, $\exists \epsilon > 0$ such that
\[
R(z) = A_0 + O(|z|^\epsilon), \quad z \to 0, \quad z \notin \mathbb{R}_+,
\]
in $\mathcal{B}(-1, s; 1, -s')$, where
\[
A_0 = (1 + G_0 W)^{-1} G_0. \tag{3.9}
\]
There exists $\delta > 0$ such that boundary values of the resolvent
\[
R(\lambda \pm i0) = \lim_{\epsilon \to 0^+} R(\lambda \pm i\epsilon), \quad \lambda \in [0, \delta]
\]
exist in $\mathcal{B}(-1, s; 1, -s)$ for any $s > 1/2$ and are continuous in $\lambda$. 

X. P. Wang
**Proof.** Remark that $G_0 \in \mathcal{B}(-1, s; 1, -s')$ for any $s, s' > \frac{1}{2}$ and $s + s' > 2$. Lemma 3.1 implies that $\ker_{\mathcal{H}^{1,-s}}(1 + G_0 W) = \{0\}$ and $G_0 W$ is compact on $\mathcal{H}^{1,-s}$ for $1 < s < (1 + \rho)/2$. Therefore $1 + G_0 W$ is invertible on $\mathcal{H}^{1,-s}$ with bounded inverse. Since

$$1 + R_0(z) W = 1 + G_0 W + O(|z|^\varepsilon), \quad \text{in } \mathcal{B}(1, -s; 1, -s), \quad \varepsilon > 0,$$

for $|z|$ small and $z \not\in \mathbb{R}_+$, it follows that $1 + R_0(z) W$ is invertible on $\mathcal{H}^{1,-s}$ for $|z|$ small and $z \not\in \mathbb{R}_+$ and that

$$(1 + R_0(z) W)^{-1} = (1 + G_0 W)^{-1} + O(|z|^\varepsilon), \quad \text{as } |z| \to 0, z \not\in \mathbb{R}_+. \quad (3.10)$$

In particular, $1 + R_0(z) W$ is injective in $\mathcal{L}^{2,-s}$ for $z$ near 0 and $z \not\in \mathbb{R}_+$. Since $R_0(z)(P - z) = 1 + R_0(z) W$, this shows that $P$ has no eigenvalues in a small disk $\{z; |z| < \delta\}$ for some $\delta > 0$ and that

$$R(z) = (1 + R_0(z) W)^{-1} R_0(z) = A_0 + O(|z|^\varepsilon), \quad z \not\in \mathbb{R}_+, \quad (3.11)$$

with $A_0 = (1 + G_0 W)^{-1} G_0$. The existence of the boundary values $R(\lambda \pm i0)$ for $0 < \lambda < \delta$ follows from the first equality in (3.11) and the corresponding results for the free resolvent $R_0(z)$. \( \square \)

**Theorem 3.3.** Assume $n = 3$ and $\rho > 2$. Then for any $s > \frac{3}{2}$, there exists $\varepsilon > 0$ such that

$$R(z) = A_0 + z^{\frac{1}{2}} A_1 + O(|z|^{\frac{1}{2} + \varepsilon}), \quad (3.12)$$

in $\mathcal{B}(-1, s; 1, -s)$ for $|z|$ small and $z \not\in \mathbb{R}_+$, where $A_1$ is an operator of rank one given by

$$A_1 = (1 + G_0 W)^{-1} G_1 (1 - W A_0). \quad (3.13)$$

**Proof.** For $s > \frac{3}{2}$, one has

$$R_0(z) = G_0 + z^{\frac{1}{2}} G_1 + O(|z|^{\frac{1}{2} + \varepsilon})$$

in $\mathcal{B}(-1, s; 1, -s)$. If $\rho > 2$, one has $W \in \mathcal{B}(0, -r; 1, \rho + 1 - r)$. Therefore for $\frac{3}{2} < s < (\rho + 1)/2$, it

$$R_0(z) W - (G_0 + z^{\frac{1}{2}} G_1) W = O(|z|^{\frac{1}{2} + \varepsilon})$$

in $\mathcal{B}(0, -s; 0, -s)$. Since $B_0 \triangleq (1 + G_0 W)^{-1} \in \mathcal{B}(0, -s; 0, -s)$, it follows that

$$(1 + R_0(z) W)^{-1} = B_0 - z^{\frac{1}{2}} B_0 G_1 W B_0 + O(|z|^{\frac{1}{2} + \varepsilon}).$$

From the resolvent equation $R(z) = (1 + R_0(z) W)^{-1} R_0(z)$, we obtain that

$$R(z) = B_0 G_0 + z^{\frac{1}{2}} B_0 G_1 (1 - W B_0 G_0) + O(|z|^{\frac{1}{2} + \varepsilon})$$

in $\mathcal{B}(0, s; 0, -s)$. An argument of hypoellipticity shows that the same asymptotics holds in $\mathcal{B}(-1, s; 1, -s)$. Remark that $A_1 = B_0 G_1 (1 - W B_0 G_0)$ is a rank one operator, because $G_1$ is of rank one. See Proposition 2.9. \( \square \)
4. Large-Time Behaviors of Solutions

The large-time behaviors of solutions to the KFP equation with a potential will be deduced from resolvent asymptotics and a representation formula of the semigroup $S(t) = e^{-tP}$ in terms of the resolvent. To this purpose, we need the following high energy pseudospectral estimate.

**Theorem 4.1.** Let $n \geq 1$ and assume (1.8) with $\rho \geq -1$. Then there exists $C > 0$ such that $\sigma(P) \cap \{z; |\Re z| > C, \Im z \leq \frac{1}{C}|\Re z|^{\frac{1}{2}}\} = \emptyset$ and

$$
\|R(z)\| \leq \frac{C}{|z|^{\frac{1}{2}}}, \tag{4.1}
$$

and

$$
\|(1 - \Delta_v + v^2)\frac{1}{2} R(z)\| \leq \frac{C}{|z|^{\frac{1}{4}}}, \tag{4.2}
$$

for $|\Re z| > C$ and $\Re z \leq \frac{1}{C}|\Re z|^{\frac{1}{2}}$.

**Proof.** Let $W = -\nabla_x V(x) \cdot \nabla_v$. (2.63) shows that $\|WR_0(z)\| + \|R_0(z)W\| = O(|z|^{-\frac{1}{2}})$ for $z$ in the region $\{z; |\Re z| > M, \Re z \leq \frac{1}{M}|\Re z|^{\frac{1}{2}}\}$. Therefore $(1 + R_0(z)W)^{-1}$ exists and is uniformly bounded if $\{z; |\Re z| > M, \Re z \leq \frac{1}{M}|\Re z|^{\frac{1}{2}}\}$ with $M$ sufficiently large. Theorem 4.1 follows from Proposition 2.10 and the resolvent equation $R(z) = (1 + R_0(z)W)^{-1}R_0(z)$ for $|\Re z| > C$ and $\Re z \leq \frac{1}{C}|\Re z|^{\frac{1}{2}}$ with $C \geq M$ sufficiently large. $\square$

**Lemma 4.2.** Let $n \geq 1$ and assume (1.8) with $\rho \geq -1$. Then

$$
S(t)f = \frac{1}{2\pi i} \int_{\gamma} e^{-t\zeta} R(\zeta) f d\zeta \tag{4.3}
$$

for $f \in L^2$ and $t > 0$, where the contour $\gamma$ is chosen such that

$$
\gamma = \gamma_- \cup \gamma_0 \cup \gamma_+
$$

with $\gamma_{\pm} = \{z; z = \pm iC + \lambda \pm iC\lambda^3, \lambda \geq 0\}$ and $\gamma_0$ is a curve in the left-half complexe plane joining $-iC$ and $iC$ for some $C > 0$ sufficiently large, $\gamma$ being oriented from $-i\infty$ to $+i\infty$.

**Proof.** The spectrum of $P$ is void in the left side of $\gamma$. By Theorem 4.1, for $C > 0$ sufficiently large, $\gamma$ is contained in the resolvent set of $P$ and

$$
\|R(z)\| \leq \frac{C}{|z|^{\frac{1}{2}}}, \quad z \in \gamma.
$$

Therefore, the integral $\tilde{S}(t) = \frac{1}{2\pi i} \int_{\gamma} e^{-t\zeta} R(\zeta) d\zeta$ is norm convergent. In addition, one can check as in the standard case (see, for example, [15]) that $\tilde{S}'(t)f = -P\tilde{S}(t)f$ for $f \in D(P_0)$ and that $\lim_{t \to 0^+} \tilde{S}(t) = I$ strongly. The uniqueness of solution to the evolution equation $u'(t) + Pu(t) = 0$ for $t > 0$ and $u(0) = u_0$ implies that $\tilde{S}(t) = S(t)$, $t > 0$. $\square$
Corollary 4.3. Assume that $n = 3$ and $\rho > 1$. One has

$$
(S(t)f, g) = \frac{1}{2\pi i} \int_{\Gamma} e^{-itz} \langle R(z)f, g \rangle dz, \quad t > 0,
$$

for any $f, g \in L^{2,s}, s > 1$. Here

$$
\Gamma = \Gamma_- \cup \Gamma_0 \cup \Gamma_+
$$

with $\Gamma_{\pm} = \{ z; z = \delta + \lambda \pm i\delta^{-1}\lambda^3, \lambda \geq 0 \}$ for $\delta > 0$ small enough and $\Gamma_0 = \{ z = \lambda \pm i0; \lambda \in [0, \delta] \}$. $\Gamma$ is oriented from $-i\infty$ to $+i\infty$.

Proof. $P$ has no eigenvalues with real part equal to zero (see the arguments used in the proof of Lemma 3.1) and their only possible accumulation points are in $\mathbb{R}_+$. If $n = 3$ and $\rho > 1$, Theorem 3.2 and Theorem 4.1 imply that there exists some $\delta_0 > 0$ such that

$$
\sigma_P(P) \cap \{ z; \Re z \leq \delta_0 \} = \emptyset.
$$

Consequently if $0 < \delta < \delta_0$ is small enough, there is no spectrum of $P$ in the interior of the region between $\gamma$ and $\Gamma$ and the resolvent is holomorphic there. In addition, the limiting absorption principles at low energies ensure that the integral

$$
\int_{\Gamma} e^{-itz} \langle R(z)f, g \rangle dz, \quad t > 0,
$$

is convergent for any $f, g \in L^{2,s}$ with $s > 1$. By analytic deformation, we conclude from (4.1) that

$$
(S(t)f, g) = \frac{1}{2\pi i} \int_{\gamma} e^{-itz} \langle R(z)f, g \rangle dz = \frac{1}{2\pi i} \int_{\Gamma} e^{-itz} \langle R(z)f, g \rangle dz, \quad t > 0,
$$

for any $f, g \in L^{2,s}$ with $s > 1$. \qedsymbol

The formula (4.4) is useful for studying the time-decay of solutions to the KFP equation with a short-range potential (satisfying (1.8) with $\rho > 1$).

Theorem 4.4. Assume $n = 3$.

(a) If $\rho > 1$, one has for any $s > \frac{3}{2}$

$$
\|S(t)\|_{B(0,s;0,−s)} \leq C_s t^{-\frac{3}{2}}, \quad t > 0.
$$

(b) If $\rho > 2$, then for any $s > \frac{3}{2}$, there exists some $\epsilon > 0$ such that

$$
S(t) = t^{-\frac{3}{2}} B_1 + O(t^{-\frac{3}{2}−\epsilon})
$$

in $B(0,s;0,−s)$ as $t \to +\infty$, where

$$
B_1 = \frac{1}{2i\sqrt{\pi}} A_1
$$

is an operator of rank one.
Proof of Theorem 4.4. (a). Assume that \( n = 3 \) and \( \rho > 1 \). By Corollary 4.3, one has for \( f, g \in L^{2,s} \) with \( s > 1 \) and for \( t > 0 \)

\[
\langle S(t), f, g \rangle = \frac{1}{2\pi i} \int_0^\delta e^{-t\lambda}((R(\lambda + i0) - R(\lambda - i0)) f, g) d\lambda
\]

\[
+ \frac{e^{-t\delta}}{2\pi i} \int_0^\infty e^{-t(\lambda + i\delta^{-1}\lambda^3)}(R(\delta + \lambda + i\delta^{-1}\lambda^3) f, g)(1 + 3i\delta^{-1}\lambda^2) d\lambda
\]

\[
- \frac{e^{-t\delta}}{2\pi i} \int_0^\infty e^{-t(\lambda - i\delta^{-1}\lambda^3)}(R(\delta + \lambda - i\delta^{-1}\lambda^3) f, g)(1 - 3i\delta^{-1}\lambda^2) d\lambda
\]

\[
\triangleq I_1 + I_2 + I_3. \tag{4.9}
\]

For \( I_2 \) and \( I_3 \), one can apply Theorem 4.1 to estimate

\[
|\langle R(\delta + \lambda \pm i\delta^{-1}\lambda^3) f, g \rangle| \leq C_M \|f\|_{\mathcal{H}^{-1,s}} \|g\|_{\mathcal{H}^{-1,s}}
\]

for \( s > \frac{1}{2} \) and \( \lambda \in [0, M] \) for each fixed \( M > 0 \) and

\[
|\langle R(\delta + \lambda \pm i\delta^{-1}\lambda^3) f, g \rangle| \leq C_M \lambda^{-\frac{1}{2}} \|f\|_{L^2} \|g\|_{L^2}
\]

for \( \lambda > M \) with \( M > 1 \) sufficiently large. Therefore, if \( \rho > 1 \)

\[
|I_k| \leq C e^{-t\delta} \|f\|_{\mathcal{H}^{0,s}} \|g\|_{\mathcal{H}^{0,s}} \tag{4.10}
\]

for \( k = 2, 3 \) and for any \( s > \frac{1}{2} \).

To show (4.6), it remains to prove that if \( \rho > 1 \) and \( n = 3 \), one has

\[
|\int_{\Gamma_0} e^{-t\lambda z} (R(z) f, g) dz| \leq C t^{-\frac{3}{2}} \|f\|_{L^{2,\infty}} \|g\|_{L^{2,\infty}} \tag{4.11}
\]

for any \( f, g \in L^{2,\infty}, s > \frac{3}{2} \). For \( 1 < s < (\rho + 1)/2 \), one has for some \( \epsilon_0 > 0 \)

\[
W(R_0(z) - G_0) = O(|z|^{\epsilon_0}), \quad \text{in} \ B(0, s; 0, s)
\]

for \( z \) near 0 and \( z \notin \mathbb{R}_+ \). By Lemma 3.1, \( 1 + W G_0 \) is invertible in \( B(0, s; 0, s) \) for any \( 1 < s < (\rho + 1)/2 \). One obtains

\[
R(z) = R_0(z)(1 + W R_0(z))^{-1} = \sum_{j=0}^N R_0(z) T(z)^j (1 + W G_0)^{-1} + O(|z|^{(N+1)\epsilon_0}) \tag{4.12}
\]

in \( B(0, s; 0, s) \) with \( s > 1 \), where \( N \) is taken such that \( (N + 1)\epsilon_0 > \frac{1}{2} \) and

\[
T(z) = (1 + W G_0)^{-1} W (R_0(z) - G_0). \tag{4.13}
\]

Consequently

\[
|\int_{\Gamma_0} e^{-t\lambda z} (\langle R(z) - R_0(z) \rangle \sum_{j=0}^N T(z)^j (1 + W G_0)^{-1}, f, g) dz| \leq C t^{-\frac{3}{2}-\epsilon} \|f\|_{L^{2,\infty}} \|g\|_{L^{2,\infty}}, \quad \epsilon > 0, \tag{4.14}
\]

if \( s > 1 \). (4.11) follows from the following lemma which achieves the proof of (4.6). 

\[\square\]
Lemma 4.5. For each \( j \geq 0 \) and \( \frac{3}{2} < s < \rho + \frac{1}{2} \), there exists some \( C > 0 \) such that
\[
| \int_{\Gamma_0} e^{-tz} \langle R_0(z)T(z)^j (1 + WG_0)^{-1} f, g \rangle dz | \leq Ct^{-\frac{3}{2}} \| f \|_{L^{2,s}} \| g \|_{L^{2,s}}
\] (4.15)
for any \( f, g \in L^{2,s} \) and \( t > 0 \).

Proof. We want to show that
\[
|(R_0(z)T(z)^j - R_0(\bar{z})T(\bar{z})^j)(1 + WG_0)^{-1} f, g)| \leq C \sqrt{\lambda} \| f \|_{L^{2,s}} \| g \|_{L^{2,s}}
\] (4.16)
for \( z = \lambda + i0 \) with \( \lambda \in [0, \delta] \) and for \( f, g \in L^{2,s} \) with \( \frac{3}{2} < s < \rho + \frac{1}{2} \). Recall that
\[
R_0(z) = b_0^w(v, D_x, D_x)((-\Delta_x - z)^{-1} + r_0(z)) \text{ with } r_0(z) \text{ a bounded operator-valued function holomorphic in } z \text{ when } \Re z < a \text{ for some } 0 < a < 1 \text{ [see (2.54)]}. Without loss, we can choose \( \delta > 0 \) small such that \( 0 < \delta < a \) which gives
\[
R_0(z) - R_0(\bar{z}) = b_0^w(v, D_x, D_x)((-\Delta_x - z)^{-1} - (-\Delta_x - \bar{z})^{-1})
\] (4.17)
for \( z = \lambda + i0 \) with \( \lambda \in [0, \delta] \), \( 0 < \delta < 1 \). Making use of the explicit formula for the integral kernel of \((-\Delta_x - z)^{-1}\), one obtains
\[
R_0(z) - R_0(\bar{z}) = O(\sqrt{\lambda}), \quad \lambda \in [0, \delta]
\] (4.18)
in \( B(-1, s; 1, -s) \) for any \( s > \frac{3}{2} \). By Proposition 2.9, \( R_0(z) \) is uniformly bounded in \( B(-1, s; 1, -s') \) for \( s, s' > \frac{1}{2} \) and \( s + s' > 2 \). We deduce that for any \( \frac{1}{2} < s < \rho + \frac{1}{2} \), \( WG_0 \) and \((1 + WG_0)^{-1}\) belongs to \( B(0, s; 0, 0) \) and
\[
T(z) = O(|z|^{c_0}) \quad \text{in } B(0, s; 0, 0).
\] (4.19)
Seeing (4.18), one obtains that for any \( \frac{3}{2} < s < \rho + \frac{1}{2} \) and \( s' = 1 + \rho - s > \frac{1}{2} \),
\[
T(z) - T(\bar{z}) = (1 + WG_0)^{-1} W(R_0(z) - R_0(\bar{z})) = O(\sqrt{\lambda})
\] (4.20)
in \( B(0, s; 0, 0) \) for \( z = \lambda + i0 \). Since for \( j \geq 1 \),
\[
R_0(z)T(z)^j - R_0(\bar{z})T(\bar{z})^j
\]
\[
= (R_0(z) - R_0(\bar{z}))T(z)^j + R_0(\bar{z}) \sum_{k=0}^{j-1} T(z)^k (T(z) - T(\bar{z})) T(\bar{z})^{j-k-1}
\] (4.21)
one can estimate \(|\langle (R_0(z)T(z)^j - R_0(\bar{z})T(\bar{z})^j)(1 + WG_0)^{-1} f, g \rangle|\) by
\[
|\langle (R_0(z)T(z)^j - R_0(\bar{z})T(\bar{z})^j)(1 + WG_0)^{-1} f, g \rangle| \leq C \| (R_0(z) - R_0(\bar{z})) \|_{B(0, s; 0, -s)} \| T(z) \|_{B(0, s; 0, s)}^j
\]
\[
+ \sum_{k=0}^{j-1} \| R_0(\bar{z}) \|_{B(0, s'; 0, -s)} \| (T(z) - T(\bar{z})) \|_{B(0, s; 0, s')} \| T(z) \|_{B(0, s; 0, s')}^k \times \| T(\bar{z}) \|_{B(0, s; 0, s')}^{j-k-1} \| f \|_{L^{2,s}} \| g \|_{L^{2,s}}
\]
\[
\leq C' \sqrt{\lambda} \| f \|_{L^{2,s}} \| g \|_{L^{2,s}}
\] (4.22)
for $\frac{3}{2} < s < \rho + \frac{1}{2}$ and $s' = 1 + \rho - s > \frac{1}{2}$. The above estimate is clearly also true when $j = 0$. This proves (4.16) which implies (4.15) and consequently (4.6).

**Proof of Theorem 4.4.** (b) Assume now $\rho > 2$. Theorem 3.3 gives that if $\rho > 2$, 
\[ R(z) = A_0 + \sqrt{z}A_1 + O(|z|^{\frac{1}{2} + \varepsilon}), \quad z \to 0, z \not\in \mathbb{R}_+, \]

in $B(-1, s; 1, -s)$, $s > \frac{3}{2}$. According to (4.23), $I_1$ can be evaluated by

\[ I_1 = \frac{1}{\pi i} \int_0^\delta e^{-t\lambda}((\sqrt{\lambda}A_1 + O(\lambda^{\frac{1}{2} + \varepsilon}))f, g) d\lambda \]

and

\[ |I_1 - \frac{1}{t^2} \langle B_1 f, g \rangle| \leq Ct^{-\left(\frac{3}{2} + \varepsilon\right)}\|f\|_{\mathcal{H}^{-1, s}}\|g\|_{\mathcal{H}^{-1, s}} \]

with

\[ B_1 = \frac{1}{\pi i} A_1 \int_0^\infty e^{-s} \sqrt{s} d s = \frac{1}{2i\sqrt{\pi}} A_1. \]

This proves (4.7). Since operator $A_1$ is of rank one, so is operator $B_1$.

**Proof of Theorem 1.1.** Theorem 1.1 (a) is just Theorem 4.1, Theorem 1.1 (b) is contained in Theorem 3.2 and Theorem 4.4 (a). Seeing Theorem 4.4 (b), to prove Theorem 1.1 (c), it remains to calculate the operator $B_1$ given in Theorem 4.4.

For $u \in L^{2, s}$ with $s > \frac{3}{2}$, one can write

\[ B_1 u = \frac{1}{8\pi\frac{3}{2}} \langle m_0, (1 - WA_0)u \rangle (1 + G_0 W)^{-1} m_0 \]

\[ = \frac{1}{8\pi\frac{3}{2}} \langle (1 - WA_0)^* m_0, u \rangle (1 + G_0 W)^{-1} m_0 \]

\[ = \frac{1}{8\pi\frac{3}{2}} \langle v_0, u \rangle \mu_0 \]

where

\[ \mu_0 = (1 + G_0 W)^{-1} m_0. \quad v_0 = (1 - WA_0)^* m_0. \]

Since $m_0 \in L^{2, -s}$ for any $s > \frac{3}{2}$ and $(1 + G_0 W)^{-1} \in \mathcal{B}(0, -s; 0, -s)$, $G_0 W$ and $(WA_0)^* \in \mathcal{B}(0, -s; 0, -s')$ for any $s' > \frac{1}{2}$ and $\frac{3}{2} < s < \frac{\rho + 1}{2}$ (if $\rho > 2$), $v_0, \mu_0$ belong to $L^{2, -s}$ for any $s > \frac{3}{2}$. In addition, $\mu_0$ satisfies the equation

\[ \lim_{z \to 0, z \not\in \mathbb{R}_+} (1 + R_0(z) W) \mu_0 = (1 + G_0 W) \mu_0 = m_0 \]

in $L^{2, -s}$. It follows that

\[ P \mu_0 = P_0 m_0 = 0. \]
Similarly, since $A_0 = \lim_{z \to 0, z \neq \mathbb{R}_+} R(z)$ in $B(-1, s; 1, -s')$ for any $s, s' > \frac{1}{2}$ with $s + s' > 2$ and $(1 - WA_0)^* = 1 + A_0^* W$ ($W$ being skew-adjoint), one can check that
\[ P^* v_0 = (P_0 + W)^* + W)m_0 = P_0^* m_0 = 0. \] (4.29)

To prove that $\mu_0 = m$, we remark that the solution of the equation $(1 + G_0 W)\mu = m_0$ is unique in $L^{2,-s}$ for any $s > \frac{3}{2}$ because of the invertibility of $1 + G_0 W$. Therefore since $\mu_0$ verifies the equation $(1 + G_0 W)\mu_0 = m_0$, it suffices to check that $m$ also verifies the equation
\[ (1 + G_0 W)m = m_0. \] (4.30)

To show this, we notice that since $(P_0 + W)m = 0$, one has for $\lambda < 0$

\[ (1 + R_0(\lambda) W)m = -\lambda R_0(\lambda)m \] (4.31)

For $\rho > 2$, one has $m - m_0 \in L^{2,s}$ for any $0 < s < \rho - \frac{3}{2}$. Proposition 2.13 with $\psi = \psi_0$ shows that
\[ \lim_{\lambda \to 0_-} \lambda R_0(\lambda)m = -m_0. \] (4.32)

Taking the limit $\lambda \to 0_-$ in (4.31), one obtains $(1 + G_0 W)m = m_0 = (1 + G_0 W)\mu_0$. According to Lemma 3.1 with $\rho > 2$, the operator $1 + G_0 W$ is invertible in $L^{2,-s}$ for any $\frac{3}{2} < s < \frac{1+\rho}{2}$, which gives $\mu_0 = m = (1 + G_0 W)^{-1} m_0$.

To show that $v_0 = m$, we notice that $1 + W G_0 \in B(0, s; 0, s)$ for any $\frac{3}{2} < s < \frac{\rho+1}{2}$ and is invertible and its inverse is given by:
\[ (1 + W G_0)^{-1} = 1 - W(1 + G_0 W)^{-1} G_0 = 1 - W A_0. \] (4.33)

Therefore $v_0 = (1 - W A_0)^* m_0 = (1 - G_0^* W)^{-1} m_0$. Since $m$ verifies also the equation $P^* m = (P_0^* - W)m = 0$, the same arguments as those used above allow to conclude that $(1 - G_0^* W)m = m_0$ which shows $v_0 = (1 - G_0^* W)^{-1} m_0 = m$. This shows
\[ B_1 u = \frac{1}{8\pi^\frac{3}{2}} \langle m, u \rangle m \text{ for } u \in L^{2,s} \text{ with } s > \frac{3}{2}, \] (4.34)

which proves (1.14) of Theorem 1.1. □

In the proof of Theorem 1.1 (c), we showed that solutions to the equation $Pu = 0$ with $u \in L^{2,-s}$ for any $s > \frac{3}{2}$, are given by $u = cm$ for some constant $c$. If $V$ is smooth and coercive ($|\nabla V(x)| \to \infty$ and $V(x) > 0$ outside some compact), the hypoelliptic estimate for $P$ allows to conclude that if $Pu = 0$ and $u \in S'$, then $u \in S$ and $u = cm$ for some constant $c$. See [10,13]. This kind of uniqueness result seems to be unknown for potentials whose gradient tends to zero. Our proof not only shows that $\mu_0 = cm$, but also compute the constant $c$ which allows to give the universal constant in the leading term of (1.14).

Open questions. We make some comments about open questions related to this work.

(a) In Theorem 2.4, we proved some $L^1 - L^\infty$ estimate for the free semigroup $e^{-tP_0}$, while results for $e^{-tP}$ in Theorem 1.1 are only proven in weighted $L^2$ spaces. Can one prove that results like (1.13) and (1.14) hold for $e^{-tP}$ as operators from $L^1$ to $L^\infty$?
(b) The assumptions on dimension and on the decay rate of the potential are only used in low-energy resolvent asymptotics. While we believe that the condition \( n = 3 \) is only technical, the condition on the decay rate \( \rho \) is more essential to our approach which consists in regarding the free KFP operator \( P_0 \) as model operator for the full KFP operator with a potential. What can one say about the large-time behaviors of solutions to (1.2) if \( V(x) \) satisfies (1.8) with \(-1 < \rho \leq 1\)? More precisely, can one prove that if \( V(x) = c(x)^\mu, \ c > 0 \) and \( \mu \in ]0, 1[ \), for \( |x| \) large, then one has

\[
e^{-tP} = \langle \cdot, m \rangle m + O(e^{-at^\beta})?
\]

Here \( V \) is normalized by \( \int_{\mathbb{R}^n} e^{-V(x)} \, dx = 1, \ a > 0 \) and \( \beta = \frac{\mu}{2 - \mu} \). To answer these questions, one may try to use the Witten Laplacian as the model for the KFP operator. See \([9, 10, 12]\) for the relationship between the KFP operator and the Witten Laplacian in eigenvalue problems. In the case where \( V(x) = c(x)^\mu, \ c > 0 \), for \( |x| \) large, the Witten Laplacian

\[
-\Delta_V = (\nabla_x + \nabla V(x)) \cdot (\nabla_x + \nabla V(x))
\]

is a Schrödinger operator \(-\Delta + U(x)\) with the potential \( U(x) = |\nabla V(x)|^2 - \Delta V(x)\) positive outside some compact and lowly decreasing at the infinity. For the case when the potential \( U(x) \) is, in addition, globally positive, the threshold spectral analysis for this class of Schrödinger operators has been done in \([20, 29]\). In fact for a class of more general non selfadjoint Schrödinger operators, one can prove some Gevrey type resolvent estimates at the threshold zero \([28]\). An approach to the above open questions for the KFP operators could be first to make a precise threshold spectral analysis for some non selfadjoint model operator that is a generalization of the Witten Laplacian and then to see to that extent the approximation of the KFP operator by such model operator is valid in a scattering framework.

Acknowledgements. The question studied here came to my mind during a talk given by F. Nier about the work \([11]\) in January 2004 at Analysis Seminar in Nantes. I take the opportunity to thank F. Hérau and F. Nier for useful discussions related to the Kramers-Fokker-Planck equation. I also thank the anonymous referees for their constructive comments and suggestions, which improved this paper.

Appendix A. A Family of Complex Harmonic Oscillators

In this appendix, we study spectral properties of a family of non-selfadjoint harmonic oscillators

\[
\hat{P}_0(\xi) = -\Delta_v + \frac{v^2}{4} - \frac{n}{2} + iv \cdot \xi,
\]

where \( \xi \in \mathbb{R}^n \) are regarded as parameters. By Fourier transform in \( x \)-variables, the free KFP \( P_0 \) is a direct integral of the family \( \{\hat{P}_0(\xi); \ \xi \in \mathbb{R}^n\} \). We give here some quantitative results with explicit bounds in \( \xi \) which are used in Sect. 2. Note that non-selfadjoint harmonic operators with complex frequency \(-\Delta + \omega x^2, \ \omega \in \mathbb{C}, \) are studied by several authors. See for example \([3, 4]\) and references quoted therein. To compare with, our family of harmonic oscillators is only weakly non-selfadjoint, because the term \( iv \cdot \xi \) is a relatively compact perturbation of the selfadjoint part. This may explain why we can obtain better results in our case.
The operator $\hat{P}_0(\xi)$ can be written as

$$\hat{P}_0(\xi) = -\Delta_v + \frac{1}{4} \sum_{j=1}^{n} (v_j + 2i\xi_j)^2 - \frac{n}{2} + |\xi|^2.$$ 

$\{\hat{P}_0(\xi), \xi \in \mathbb{R}^n\}$ is a holomorphic family of type (A) in sense of Kato with constant domain $D = D(-\Delta_v + \frac{v^2}{4})$ in $L^2(\mathbb{R}^n)$. The numerical range of $\hat{P}_0(\xi)$ is contained in the region $\{z \in \mathbb{C}; \Im z \geq 0, \Re z \leq 2\Im z + n + \frac{|\xi|^2}{2}\}$.

Let $F_j(s) = (-1)^j e^{\frac{s^2}{2}} \frac{d^j}{ds^j} e^{-\frac{s^2}{2}}, j \in \mathbb{N}$, be the Hermite polynomials and 

$$\varphi_j(s) = (j! \sqrt{2\pi})^{-\frac{1}{2}} e^{-\frac{s^2}{2}} F_j(s)$$

the normalized Hermite functions. For $\xi \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, define 

$$\psi_\alpha(v) = \prod_{j=1}^{n} \varphi_{\alpha_j} (v_j)$$

and 

$$\psi_\alpha^\xi(v) = \psi_\alpha(v + 2i\xi). \quad (A.2)$$

For $\xi \in \mathbb{R}^n \setminus 0$, the eigenfunctions $\{\psi_\alpha^\xi, \alpha \in \mathbb{N}^n\}$ are no longer orthonormal. For $\alpha, \beta \in \mathbb{N}^n, \xi \to \langle \psi_\alpha^\xi, \psi_\beta^\xi \rangle$ extends to an entire function for $\xi \in \mathbb{C}$ and is constant on $i\mathbb{R}$.

Therefore $\langle \psi_\alpha^\xi, \psi_\beta^{-\xi} \rangle$ is constant for $\xi \in \mathbb{C}$ and one has

$$\langle \psi_\alpha^\xi, \psi_\beta^{-\xi} \rangle = \delta_{\alpha\beta} \begin{cases} \frac{1}{2^{n/2}} \delta_{\alpha\beta}, & \alpha = \beta, \\ 0, & \alpha \neq \beta, \end{cases} \quad \forall \alpha, \beta \in \mathbb{N}^n, \xi \in \mathbb{R}^n. \quad (A.3)$$

Using the definition of Hermite functions, one can check that for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$

$$\|\psi_\alpha^\xi\|^2 = e^{2\xi^2} \prod_{m=1}^{n} \left( \sum_{j=1}^{\alpha_m} \frac{C^j_m}{j!} (2\xi_m)^{2j} \right). \quad (A.4)$$

In fact, when $n = 1, \xi = \xi_1$ and $k \in \mathbb{N}$, one has

$$\|\psi_k^\xi\|^2 = \int_{\mathbb{R}} \varphi_k(y + 2i\xi) \varphi_k(y - 2i\xi) dy$$

$$= (k! \sqrt{2\pi})^{-1} e^{2\xi^2} \int_{\mathbb{R}} F_k(y + 2i\xi) F_k(y - 2i\xi) e^{-\frac{y^2}{2}} dy$$

$$= (k! \sqrt{2\pi})^{-1} e^{2\xi^2} \int_{\mathbb{R}} \left( \sum_{j=0}^{k} C_j^l (2i\xi)^{k-j} F_j(y) \right) \sum_{l=0}^{k} C_k^l (-2i\xi)^{k-l} F_l(y) e^{-\frac{y^2}{2}} dy$$

$$= e^{2\xi^2} \sum_{j,l=0}^{k} (k! \sqrt{2\pi})^{-1} C_j^l C_k^l (j! \sqrt{2\pi})^{\frac{1}{2}} (l! \sqrt{2\pi})^{\frac{1}{2}} (2i\xi)^{k-j} (-2i\xi)^{k-l}$$

$$\times \int_{\mathbb{R}} \psi_j(y) \psi_l(y) dy$$

$$= e^{2\xi^2} \sum_{j=0}^{k} \frac{j!(C_j^l)^2}{k!} (4\xi^2)^{k-j}$$
\[ e^{2\xi^2} \sum_{j=0}^{k} \frac{C_j}{j!} (4\xi^2)^j. \]

The general case \( n \geq 1 \) follows from the product formula:

\[ \| \psi_\alpha^\xi \|^2 = \prod_{j=1}^n \| \varphi_{\alpha,j} (\cdot + 2i\xi_j) \|^2. \]

(A.4) shows that if \( \xi \neq 0 \), \( \| \psi_\alpha^\xi \| \) grows exponentially as \( |\alpha| \to +\infty \).

**Lemma A.1.** The spectrum of \( \hat{P}_0(\xi) \) is purely discrete:

\[ \sigma(\hat{P}_0(\xi)) = \{ E_l \triangleq l + \xi^2 ; l \in \mathbb{N} \}. \]  

Each eigenvalue \( E_l \) is semi-simple (i.e., its algebraic multiplicity and geometric multiplicity are equal) with multiplicity \( m_l = \# \{ \alpha \in \mathbb{N}^n ; |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n = l \} \). The Riesz projection associated with the eigenvalue \( l + \xi^2 \) is given by

\[ \Pi_l^\xi \phi = \sum_{\alpha,|\alpha|=l} \langle \psi_\alpha^\xi, \phi \rangle \psi_\alpha^\xi, \quad \phi \in L^2. \]  

**Proof.** It is clear that the spectrum of \( \hat{P}_0(\xi) \) is purely discrete and \( \psi_\alpha^\xi(\cdot) \) is an eigenfunction associated with the eigenvalue \( E_l \). This means that \( \sigma(\hat{P}_0(\xi)) \supset \{ l + \xi^2 ; l \in \mathbb{N} \} \). Since \( \hat{P}_0(\xi)^* = \hat{P}_0(-\xi) \) and that the linear span of \( \{ \psi_\alpha^\xi(\cdot) ; \alpha \in \mathbb{N}^n \} \) is dense in \( L^2 \), \( \hat{P}_0(-\xi) \) can not have other eigenvalues than \( E_l, l = 0, 1, \ldots \). This proves that \( \sigma(\hat{P}_0(\xi)) = \{ E_l = l + \xi^2 ; l \in \mathbb{N} \} \).

To show that \( E_l \) is semisimple, assume by contradiction that \( \exists \phi \in D \) such that \( (\hat{P}_0(\xi) - E_l)\phi = c\psi_\alpha^\xi, |\alpha| = l \) and \( c \in \mathbb{C}^* \). Then \( \psi_\alpha^\xi \) is in the range of \( \hat{P}_0(\xi) - E_l \) and hence is orthogonal to the kernel of \( (\hat{P}_0(\xi) - E_l)^* = \hat{P}_0(-\xi) - E_l \). In particular, one has

\[ \langle \psi_\alpha^\xi, \psi_\alpha^{-\xi} \rangle = 0. \]

This is impossible due to (A.3). This contradiction shows that the eigenvalue \( E_l = l + \xi^2 \) is semisimple. Since the pole of the resolvent \( (\hat{P}_0(\xi) - z)^{-1} \) is simple at \( z = E_l \), the range of the associated Riesz projection defined by

\[ \Pi_l^\xi \phi = \frac{i}{2\pi} \int_{|z-E_l|=\frac{1}{2}} (\hat{P}_0(\xi) - z)^{-1} \phi dz \]

is equal to \( \ker(\hat{P}_0(\xi) - E_l) \) and the kernel of \( \Pi_l^\xi \) is equal to the range of \( \hat{P}_0(\xi) - E_l \).

The latter is the orthogonal complement of \( \ker(\hat{P}_0(-\xi) - E_l) \) which is spanned by \( \{ \psi_\alpha^{-\xi} ; |\alpha| = l \} \). The representation formula (A.6) then follows from (A.3). \( \square \)

We emphasize here that the Riesz projections \( \Pi_j^\xi, j \in \mathbb{N} \), are not orthogonal and the series \( \sum_{j=0}^{+\infty} \Pi_j^\xi \) does not convergent in the strong topology when \( \xi \neq 0 \), because the sequence \( \{ \| \Pi_j^\xi \| ; j \in \mathbb{N} \} \) is not bounded when \( \xi \neq 0 \). However one can establish a spectral decomposition formula for \( e^{-t\hat{P}_0(\xi)} \) for any \( t > 0 \) (Proposition A.3). Let us begin with the following remarkable identity.
Lemma A.2. Let \( n = 1 \). Then one has for \( t > 0 \)

\[
\sum_{k=0}^{\infty} e^{-t(k+\xi^2)} \| \Pi_k \| = \frac{e^{-\xi^2(t-2)}}{1 - e^{-t}} e^{\frac{4\xi^2}{t-1}}, \quad \xi \in \mathbb{R}. \tag{A.7}
\]

**Proof.** In the case \( n = 1 \), one has for any \( k \in \mathbb{N} \)

\[
\| \Pi_k \| = \| \psi_k \| \| \psi_k^{-1} \| = \| \psi_k \|^2 = e^{2\xi^2} \sum_{j=0}^{k} \frac{C_j}{j!} (4\xi^2)^j. \tag{A.8}
\]

The left-hand side of (A.7) is norm convergent when \( t > 0 \). In fact, one can calculate the sum of the series as follows

\[
\sum_{k=0}^{\infty} e^{-t(k+\xi^2)} \| \Pi_k \| = \sum_{k=0}^{\infty} e^{-t(k+\xi^2)} + 2\xi^2 \sum_{j=0}^{\infty} \frac{C_j}{j!} (4\xi^2)^j
\]

The following result is a kind of spectral decomposition for the non-selfadjoint harmonic oscillator \( \hat{P}_0(\xi) \).

**Proposition A.3.** Let \( n \geq 1 \). For any \( \xi \in \mathbb{R}^n \) and \( t > 0 \), one has the following formula of spectral decomposition:
\[ e^{-t\hat{P}_0(\xi)} = \sum_{l=0}^{\infty} e^{-t(l+\xi^2)} \Pi_l^\xi, \quad (A.9) \]

where \( \Pi_l^\xi \) is the Riesz projection associated with the eigenvalue \( l \) of \( \hat{P}_0(\xi) \) and the series is norm convergent as operators on \( L^2(\mathbb{R}^n) \).

**Proof.** For \( n \geq 1 \), one has
\[
\| \Pi_l^\xi \| \leq \sum_{\alpha=(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n; |\alpha|=l} \prod_{j=1}^n \| \varphi_{\alpha_j}(\cdot + 2i\xi_j) \|^2.
\]

By Lemma A.2, the right-hand side of (A.9) is norm convergent for every \( t > 0 \) and can be evaluated by
\[
\sum_{l=0}^{\infty} e^{-t(l+\xi^2)} \| \Pi_l^\xi \| \leq \prod_{j=1}^n \left( \sum_{\alpha_j=0}^{\infty} e^{-t(\alpha_j+\xi_j^2)} \| \Pi_{\alpha_j}^\xi \| \right)
= \frac{e^{-\xi^2(t-2-\frac{4}{t-1})}}{(1-e^{-t})^n}.
\]

Since the both sides of (A.9) are equal on the subspace spanned by \( \{ \psi_\alpha^\xi; \alpha \in \mathbb{N} \} \) which is dense in \( L^2 \), an argument of density shows that they are equal on the whole space \( L^2 \). \( \square \)

For nonselfadjoint harmonic oscillators with complex frequency, it is proven in [4] that there exists some critical value \( \omega_0 > 0 \) such that an expansion like (A.9) holds when \( t > \omega_0 \) and does not hold when \( 0 \leq t < \omega_0 \).

As a consequence of the proof of Proposition A.3 (A.10), we obtain the following estimate on the semigroup \( e^{-t\hat{P}_0(\xi)} \) which is useful in the analysis of the free KFP operator \( P_0 \).

**Corollary A.4.** The following estimate holds for \( t > 0 \) and \( \xi \in \mathbb{R}^n \)
\[
\| e^{-t\hat{P}_0(\xi)} \|_{B(L^2(\mathbb{R}^n))} \leq \frac{e^{-\xi^2(t-2-\frac{4}{t-1})}}{(1-e^{-t})^n} \quad (A.11)
\]

**Remarks.** Let \( \hat{R}_0(z, \xi) = (\hat{P}_0(\xi) - z)^{-1} \) for \( z \not\in \mathbb{R}_+ \). By a rescale argument, one can show the following pseudospectral estimate
\[
\| \hat{R}_0(-(\frac{n}{2} + i\mu, \xi)) \| \leq C(\langle \xi \rangle^\frac{i}{2} + \langle \mu \rangle^\frac{1}{2})^{-1} \quad (A.12)
\]
for \( \mu \in \mathbb{R} \) and \( \xi \in \mathbb{R}^n \). See Proposition A.2 in [22]. By an argument of perturbation, one has the same upper bound for the resolvent \( \hat{R}_0(z, \xi) \) when \( \Re z \leq \delta(\|z\|^\frac{1}{2} - 1) \) for some \( \delta > 0 \). Can one obtain some upper bound on \( \| \hat{R}_0(z, \xi) \| \) for \( \xi \in \mathbb{R}^n \) and for \( \xi \) in a larger domain like \( \{ z; |\arg z| \geq \delta \} \) for some \( \delta \in \]1, \( \pi/2[? \]

**References**
1. Agmon S.: Spectral properties of Schrödinger operators and scattering theory. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)2 , no. 2, 151–218 (1975)
2. Arnold A., Gamba I.M., Gualdani M.P., Mischler S., Mouhot C., Sparber C.: The Wigner-Fokker-Planck equation: stationary states and large-time behavior. Math. Models Methods Appl. Sci. 22, no. 11, 1250034, 31 pp. (2012)
3. Boulton, L.S.: Non-self-adjoint harmonic oscillator, compact semigroups and pseudospectra. J. Operator Theory 47(2), 413–429 (2002)
4. Davies, E.B., Kuijlaars, A.B.J.: Spectral asymptotics of the non-self-adjoint harmonic oscillator. J. Lond. Math. Soc. 2(70), 420–426 (2004)
5. Davies E.B.: Linear operators and their spectra, Web Supplement. Preprint. http://www.mth.kcl.ac.uk/staff/eb_davies/LOTSwebsupp32W
6. Dencker, N., Sjöstrand, J., Zworski, M.: Pseudospectra of semiclassical (pseudo-)differential operators. Commun. Pure Appl. Math. LIV, 1, 184–415 (2004)
7. Desvillettes, L., Villani, C.: On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation. Commun. Pure Appl. Math. LIV, 1–42 (2001)
8. Eckmann, J.P., Pillet, C.A., Rey-Bellet, L.: Non-equilibrium statistical mechanics of anharmonic chains coupled to two heat baths at different temperatures. Commun. Math. Phys. 201(3), 657–697 (1999)
9. Helffer, B., Klein, M., Nier, F.: Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach. Math. Contemp. 26, 41–85 (2004)
10. Helffer B., Nier F.: Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians. Lecture Notes in Mathematics, 1862. Springer, Berlin, x+209 pp (2005). ISBN: 3-540-24200-7
11. Hérau, F., Nier, F.: Isotropic hypoellipticity and trend to equilibrium for the Fokker-Planck equation with a high-degree potential. Arch. Ration. Mech. Anal. 171(2), 151–218 (2004)
12. Hérau, F., Hitrik, M., Sjöstrand, J.: Tunnel effect for Kramers-Fokker-Planck type operators: return to equilibrium and applications. Int. Math. Res. Not. IMRN, no. 15, Art. ID rnm057, 48 pp (2008)
13. Hérau, F., Sjöstrand, J., Stolk, C.: Semiclassical analysis for the Kramers-Fokker-Planck equation. Commun. Partial Differ. Equus. 30(4–6), 689–760 (2005)
14. Hörmander, L.: The Analysis of Linear Partial Differential Operators. III. Pseudo-Differential Operators. Classics in Mathematics. Springer, Berlin (2007)
15. Kolmogorov, A.: Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung (German). Math. Ann. 104(1), 415–458 (1931)
16. Laptev, A., Safronov, O.: Eigenvalues estimates for Schrödinger operators with complex potentials. Commun. Math. Phys. 292(1), 29–54 (2009)
17. Lebeau, G.: Geometric Fokker-Planck equations. Port. Math. 62(4), 469–530 (2005)
18. Lebeau, G.: Équations de Fokker-Planck géométriques. II. Estimations hypoelliptiques maximales. Ann. Inst. Fourier (Grenoble) 57(4), 1285–1314 (2007)
19. Nakamura, S.: Low energy asymptotics for Schrödinger operators with slowly decreasing potentials. Commun. Math. Phys. 161(1), 63–76 (1994)
20. Nier, F.: Hypoellipticity for Fokker-Planck operators and Witten Laplacians. Lectures on the analysis of nonlinear partial differential equations. pp. 31–84. In: Lin, F.H., Wang, X.P., Zhang P. (eds.) Morningside Lect. in Math., Vol. 1, Int. Press, Somerville, MA, iv+317 pp (2012). ISBN: 978-1-57146-235-0
21. Nier, F.: Boundary conditions and subelliptic estimates for geometric Kramers-Fokker-Planck operators on manifolds with boundaries, preprint. arXiv:1309.5070
22. Risken, H.: The Fokker-Planck Equation, Methods of Solutions and Applications. Springer, Berlin (1989)
23. Villani, C.: Hypocoercivity. Mem. Amer. Math. Soc. 202, no. 950, iv+141 pp (2009). ISBN: 978-0-8218-4498-4
24. Wang, X.P.: Asymptotic expansion in time of the Schrödinger group on conical manifolds. Ann. Inst. Fourier 56(6), 1903–1945 (2006)
25. Wang, X.P.: Number of eigenvalues for dissipative Schrödinger operators under perturbation. J. Math. Pures Appl. 96, 409–422 (2011)
26. Wang X.P.: Time-decay of semigroups generated by dissipative Schrödinger operators. J. Differ. Equus. 253(12), 3523–3542 (2012)
27. Wang X.P.: Gevrey type resolvent estimates for a class of non selfadjoint Schrödinger operators. In: Talk given at analysis seminar, University of Bologna, Dec 2014 (2014)
28. Yafaev, D.: The low energy scattering for slowly decreasing potentials. Commun. Math Phys. 85, 117–196 (1982)

Communicated by C. Mouhot