Separability Criterion for all bipartite Gaussian States

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(Dated: March 31, 2022)

We provide a necessary and sufficient condition for separability of Gaussian states of bipartite systems of arbitrarily many modes. The condition provides an operational criterion since it can be checked by simple computation. Moreover, it allows us to find a pure product–state decomposition of any given separable Gaussian state. Our criterion is independent of the one based on partial transposition, and is strictly stronger.

PACS numbers: 03.67.-a, 03.65.Bz, 03.65.Ca, 03.67.Hk

Entanglement is the basic ingredient in the philosophical implications of Quantum Theory. It also plays a crucial role in some fundamental issues of this theory, such as decoherence or the measurement process. Furthermore, it is the basis of most applications in the field of Quantum Information. However, despite of its importance, the entanglement properties of systems are far from being understood. In particular, we do not even know how to solve the following question [1]: given two systems A and B in a state described by a density operator \( \rho \), is \( \rho \) separable? This question constitutes the so-called separability problem, and it represents one of the most important theoretical challenges of the emerging theory of quantum information.

During the last few years a significant amount of work in the field of quantum information has been devoted to the separability problem [2]. For the moment, the basic tool to study this problem is a linear map called partial transposition operation. Introduced in this context by Peres [3], it provides us with a necessary condition for a density operator to be separable (equivalently, not entangled). This condition turns out to be sufficient as well for two particular cases: (a) A and B are two qubits or one qubit and one qutrit [4]; (b) A and B are two modes (continuous variable systems) in a Gaussian state [5]. Thus, in these two cases the separability problem can be fully solved. However, for higher dimensional systems as well as in the case in which A and B consist of several modes in a joint Gaussian state, partial transposition alone does not provide a general criterion for separability. In both cases, examples of states which despite of being entangled satisfy the partial transposition criterion have been provided [6, 7].

In this Letter we solve the problem of separability for Gaussian states of an arbitrary number of modes per site. Our method does not rely in any sense on the concept of partial transposition, and therefore is diametrically different from the ones that have been introduced so far to study the separability problem [2]. It is based on a non-linear map \( f:\gamma_N \mapsto \gamma_{N+1} \) between matrices \( \gamma_N \) which reveals whether a state \( \rho \) is entangled state or not. Furthermore, once \( \rho \) is shown to be separable, our method allows to find an explicit decomposition of \( \rho \) as a convex combination of product states.

Let us start by fixing the notation and recalling some properties of correlation matrices (CMs). A Gaussian state of \( n \) modes is completely characterized by a matrix \( \gamma \in M_{2n,2n} \) (the set of \( 2n \times 2n \) matrices), called correlation matrix [3], whose elements are directly measurable quantities. A matrix \( \gamma \in M_{2n,2n} \) is a CM if it is real, symmetric, and \( \gamma - iJ_n \geq 0 \). Here \( J_n \) is a matrix of the form

\[
J_n \equiv \bigoplus_{k=1}^{n} J_1, \quad J_1 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]  

(1)

In the following we will consider two systems A and B, composed of \( n \) and \( m \) modes, respectively, in a Gaussian state. The corresponding CM will be written as

\[
\gamma_0 = \begin{pmatrix} A_0 & C_0 \\ C_0^T & B_0 \end{pmatrix} \geq iJ_{n,m}
\]  

(2)

where \( A_0 \in M_{2n,2n} \) and \( B_0 \in M_{2m,2m} \) are CM themselves, \( C_0 \in M_{2n,2m} \) and \( J_{n,m} \equiv J_n \oplus J_m \). In order to simplify the notation, when it is clear from the context we will not write the subscripts to the matrices \( J \) and we will not specify the dimensions of the matrices involved in our derivations. In [3] it was shown that a CM of the form \( \gamma_0 \) is separable (i.e., it corresponds to a separable state) iff there exist two CMs, \( \gamma_{A,B} \), such that

\[
\gamma_0 \geq \gamma_A \oplus \gamma_B.
\]  

(3)

This condition, even though it can be very useful to show that some particular states are entangled [3, 8], cannot be directly used in practice to determine whether an arbitrary state is entangled or not, since there is no way of determining \( \gamma_{A,B} \) in general. If one can determine them, however, then one can automatically construct an explicit decomposition of the corresponding density operator as a convex combination of product states [8].

Below we will present a criterion which allows us to determine whether a given CM, \( \gamma_0 \), is separable or not, and which allows us to determine a product–state decomposition if this is the case. To this aim, we define a sequence of matrices \( \{ \gamma_N \}_{N=0}^{\infty} \) of the form \( \gamma_0 \). The matrix \( \gamma_{N+1} \)
can be determined through the discrete map defined as follows: (i) if $\gamma_N$ is not a CM then $\gamma_{N+1} = 0$; (ii) if $\gamma_N$ is a CM then

\[
A_{N+1} \equiv B_{N+1} \equiv A_N - \text{Re}(X_N), \quad (4a)
\]
\[
C_{N+1} \equiv -\text{Im}(X_N), \quad (4b)
\]

where $X_N \equiv C_N(B_N - iJ)^{-1}C_N^T$. Note that for $N \geq 1$ we have that $A_N = A_N^T = B_N$ and $C_N = -C_N^T$ are real matrices. The importance of this sequence is that, as we will show below, $\gamma_0$ is separable if $\gamma_N$ is a valid separable CM. In particular, for some finite number of iterations $\gamma_N$ will acquire a form in which it is simple to check that it is separable. Furthermore, starting from that CM we will be able to construct the CMs $\gamma_{A,B}$ of Eq. (5) for the original $\gamma_0$. Now we will present several propositions from which the above results will follow. Two lemmas are presented in an appendix.

First we show that if $\gamma_N$ is separable, so is $\gamma_{N+1}$. Moreover, the CMs $\gamma_{A,B}$ associated to $\gamma_N$ [cf. Eq. (5)] allow us to construct the corresponding CMs for $\gamma_{N+1}$.

**Proposition 1** If for some CMs, $\gamma_{A,B}$, we have $\gamma_N \geq \gamma_{A+B}$ then $\gamma_{N+1} \geq \gamma_A + \gamma_B$.

**Proof:** We use the equivalence (i)–(iii) of Lemma 1 to obtain that $B_N - C_N^T(A_N - \gamma_A)^{-1}C_N \geq \gamma_B \geq iJ$, where the last inequality follows from the fact that $A_B$ is a CM. Using the equivalence (ii)–(iii) of Lemma 1 we obtain $\gamma_A \leq A_N - C_N(B_N - iJ)^{-1}C_N^T = A_{N+1} + iC_{N+1}$, where we have also used the map (4). According to Lemma 2, this immediately proves the proposition.

Now, we show that the converse of Prop. 1 is true. That is, if $\gamma_{N+1}$ is separable, so is $\gamma_N$. Apart from that, the following proposition exhibits how to construct the matrices $\gamma_{A,B}$ [cf. Eq. (5)] related to $\gamma_N$ starting from the ones corresponding to $\gamma_{N+1}$.

**Proposition 2** If for some CM, $\gamma_A$, we have that $\gamma_{N+1} \geq \gamma_{A+B}$ then $\gamma_N \geq \gamma_A + \gamma_B$, where

\[
\gamma_B \equiv B_N - C_N(A_N - \gamma_A)^{-1}C_N^T, \quad (5)
\]

is a CM.

**Proof:** We use Lemma 2 and the map (4) to transform the inequality $\gamma_{N+1} \geq \gamma_{A+B}$ into $A_N - C_N(B_N - iJ)^{-1}C_N \geq \gamma_A$. According to the equivalence (ii)–(iii) of Lemma 1 this implies that $\gamma_B \geq iJ$. Since it is clear from its definition (5), $\gamma_B$ is also real and symmetric, it is a CM. On the other hand, using the equivalence (i)–(iii) of Lemma 1 we immediately obtain that $\gamma_N \geq \gamma_A + \gamma_B$.

Using the fact that for $N \geq 1$, $A_N = B_N$ and the symmetry of the corresponding matrix $\gamma_N$ we have

**Corollary 1** Under the conditions of Prop. 1, if $N \geq 1$ we have that $\gamma_N \geq \gamma_A + \gamma_B$, where $\gamma_A \equiv (\gamma_A + \gamma_B)/2 \geq iJ$ is a CM.

The above propositions imply that $\gamma_0$ is separable iff $\gamma_N$ is separable for all $N > 0$. Thus, if we find some $\gamma_N$ fulfilling (3) then $\gamma_0$ is separable. Thus, we can establish now the main result of this work.

**Theorem 1** (Separability criterion)

(1) If for some $N \geq 1$ we have $A_N \geq iJ$ then $\gamma_0$ is not separable.

(2) If for some $N \geq 1$ we have

\[
L_N \equiv A_N - ||C_N||_{\text{op}} \geq iJ
\]

then $\gamma_0$ is separable (3).

**Proof:** (1) It follows directly from Prop. 1; (2) We will show that $\gamma_N \geq L_N \oplus L_N$, so that according to Prop. 2 $\gamma_0$ is separable. We have

\[
\gamma_N = L_N \oplus L_N + \left( ||C_N||_{\text{op}} \frac{C_N}{C_N^T} \right), \quad (7)
\]

so that we just have to prove that the last matrix is positive. But using Lemma 2 this is equivalent to $||C_N||_{\text{op}} \geq C_N^T C_N$, which is always the case. ■

This theorem tells us how to proceed in order to determine if a CM is separable or not. We just have to iterate the map (4) until we find that either $A_N$ is no longer a CM or $L_N$ is a CM. In the first case, we have that $\gamma_0$ is not separable, whereas in the second one it is separable. If we wish to find a decomposition of the corresponding density operator as a convex set of product vectors we simply use the construction given in Corollary 1 until $N = 1$ and then the one of Prop. 2. This will give us the CMs, $\gamma_{A,B}$, such that $\gamma_0 \geq \gamma_A + \gamma_B$, from which the decomposition can be easily found.

In order to check how fast our method converges we have taken families of CMs and applied to them our criterion. We find that typically with less than 5 iterations we are able to decide whether a given CM is entangled or not. The most demanding states for the criterion are those which lie very close to the border of the set of separable states (see Corollary 2 below). We challenged the criterion by applying it to states close to the border of the set of separable states and still the convergence was very fast (always below 30 steps). Figure 1 illustrates this behavior. We have taken $n = m = 2$ modes, an entangled CM $\gamma_a$ of the GHZ form (4) (Fig. 1a) and an entangled CM $\gamma_b$ with positive partial transposition (4) (Fig. 1b). We produced two families of CMs as $\gamma_{a,b}(\epsilon) = \gamma_{a,b} + \epsilon I$.

We have determined $\epsilon_{a,b}$ such that the CMs become separable. In the figure we see that in both cases, as we approach exponentially fast $\epsilon_{a,b}$ the number of steps only increases linearly. We have also added, instead of $I$ other positive projectors with all possible ranks and found the same behavior. By taking other initial CMs we also find the same results.
Even though we have tested numerically the rapid convergence of our method, we still have to prove that, except for a zero measure set, it can decide whether a CM is entangled or not after a finite number of steps \([14]\). Since for them is entangled or not after a finite number of steps \([14]\).

By using our procedure, we will be able to detect it in a finite number of steps. We will start out by considering the set of separable states. Since for them \(\gamma_0 \geq \gamma_A \oplus \gamma_B \) with \(\gamma_A, \gamma_B \geq iJ\), if we just consider those with \(\gamma_A > iJ\), we will leave out a zero measure set. In this case we can show that after a finite number of steps these separable states will be detected using our procedure.

**Proposition 3** If \(\gamma_0 \geq \gamma_A \oplus \gamma_B \) with \(\gamma_A \geq iJ + \epsilon \mathbf{1}\), then there exists some

\[
N < N_0 \equiv \frac{1}{\epsilon}(|A_0|_{tr} - 2n) + 1,
\]

for which condition (2) is fulfilled.

**Proof:** Using Prop. 3 we have that for all \(N\),

\[
A_N - iJ \geq \epsilon \mathbf{1}.
\]

Thus, \(0 \leq \text{Re}(X_N) = A_N - A_{N+1}\). Since all the matrices in this expression are positive, taking the trace norm we have \(||A_N||_{tr} - ||A_{N+1}||_{tr} = ||\text{Re}(X_N)||_{tr}\). Adding both sides of this equation from \(N = 0\) to \(N_0\), taking into account that \(||\ldots||_{tr} \geq ||\ldots||_{op}\) and \(||\text{Re}(X_N)||_{op} \geq ||C_{N+1}||_{op}\) [since \(\text{Re}(X_N) \geq \pm i \text{Im}(X_N)\)], we have

\[
\sum_{N=0}^{N_0-1} ||C_{N+1}||_{op} \leq ||A_0||_{tr} - ||A_N||_{tr} \leq ||A_0||_{tr} - 2n,
\]

where the last inequality is a consequence of the fact that \(A_N \geq iJ\) for all \(N\). Thus, among \(\{C_N\}_{N=1}^{N_0}\) there must be at least one for which \(||C_N||_{op} \leq \epsilon\). Thus, \(A_N - ||C_N||_{op} \mathbf{1} \geq A_N - \epsilon \mathbf{1} \geq 0\) where for the last inequality we have used Eq. (5), and therefore, for that particular value of \(N\), condition (6) must be fulfilled.

It is worth stressing that from the proof of Prop. 3 it follows directly that if \(\gamma_0\) is separable, then the sequence \(N_{\gamma}\) converges to a fixed point \(\gamma_{\infty} = A_{\infty} \oplus B_{\infty}\), where \(A_{\infty} = B_{\infty} \geq iJ\) are CMs. On the other hand, for the sake of completeness, we will now show that if \(\gamma_0\) is not separable, then we can always detect it in a finite number of steps. We will use the fact that the CMs of inseparable Gaussian states form an open set, a fact that can be directly inferred from condition (6). This means that if \(\gamma_0\) is inseparable, there always exist some \(\epsilon_0 > 0\) such that if \(\epsilon < \epsilon_0\) then \(\gamma_0 + \epsilon \mathbf{1}\) is still inseparable and therefore condition (6) is never fulfilled. However, if \(\gamma_0\) was separable, then, according to Prop. 3, \(\gamma_0 + \epsilon \mathbf{1}\) should fulfill that condition before reaching \(N = N_0\). This can be summarized as follows.

**Corollary 2** If \(\gamma\) is inseparable then there exists some \(\epsilon > 0\) such that starting out from \(\gamma_0 = \gamma + \epsilon \mathbf{1}\), condition (6) is not fulfilled for any \(N \leq N_0 \equiv ||A_0||_{tr} - 2n/\epsilon\).

Together, Prop. 3 and Corollary 2 indicate that whether \(\gamma_0\) is separable or not, and except for a set of zero measure, we will be able to detect it in a finite number of steps. However, as mentioned above, according to our numerical calculations we see that the process always converges very fast and in practice one can directly use the method sketched after Theorem 1.

In conclusion, we have obtained a necessary and sufficient condition for Gaussian states to be separable. The condition provides an operational criterion in that it can be easily checked by direct computation. It is also worth mentioning that our criterion can be used to study the separability properties with respect to bipartite splitting of multipartite systems in Gaussian states \([15]\). Our criterion is based on a non-linear map, and is more powerful than partial transposition. This fact indicates that in other situations, like the one in which the systems A and B are \(n\) and \(m\)-level systems with \(n \times m > 6\), there might exist a more powerful criterion than partial transposition to determine whether states are separable or not. This problem still remains open. However, the results presented here represent a significant step in understanding the separability problem, which is one of the most challenging problems in the field of quantum information.

G.G. acknowledges financial support by the Friedrich-Naumann-Stiftung. This work was supported by the Austrian Science Foundation (SFB “Control and Measurement of Coherent Quantum Systems”, Project 11), the EU (EQUIP, contract IST-1999-11053), the ESF, the Institute for Quantum Information GmbH in Innsbruck, and the DFG (SFB 407 and Schwerpunkt “Quanteninformationsverarbeitung”).

**Appendix**

In this Appendix we present the lemmas which are needed in order to prove Props. 3 and 4.
Let us consider three real matrices $0 \leq A = A^T \in M_{n,n}$, $0 \leq B = B^T \in M_{m,m}$, and $C \in M_{n,m}$, and
\[
M = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} = M^T \in M_{n+m,n+m}.
\] (11)

**Lemma 1** The following statements are equivalent:
(i) $M \geq 0$.
(ii) $\ker(B) \subseteq \ker(C)$ and $A - CB^{-1}C^T \geq 0$.
(iii) $\ker(A) \subseteq \ker(C^T)$ and $B - C^T A^{-1} C \geq 0$ \footnote{1}.

**Proof:** We will just prove the first equivalence since the other one is analogous. We use that $M \geq 0$ iff for any two real vectors $a, b \in \mathbb{R}^n$ we have
\[
a^T A a + b^T B b + a^T C b + b^T C^T a \geq 0.
\] (12)

On the other hand, $A - CB^{-1}C^T \geq 0$ iff for any $a \in \mathbb{R}^n$ we have
\[
a^T A a - a^T C B^{-1} C^T a \geq 0.
\] (13)

(i) $\Rightarrow$ (ii): We assume \footnote{2}. First, $\ker(B) \subseteq \ker(C)$ since otherwise we could always choose a $b \in \ker(B)$ so that $-a^T C b > a^T A a$. Second, if we choose $b = -B^{-1} C^T a$ then we obtain \footnote{3}. Then, $A = CB^{-1} C^T + P$, where $P \geq 0$. Defining $\tilde{a} = B^{-1} C^T a$, we have that $C^T a = B \tilde{a}$ (since $\ker(B) \subseteq \ker(C)$), and thus the lhs of \footnote{4} can be expressed as $a^T P a + (\tilde{a} + b) C B (\tilde{a} + b)$, which is positive.

In the derivations of Props. \footnote{5} and \footnote{6} we have not included explicitly the conditions imposed by the present lemma on the kernels of $B$ and $C$. However, one can easily verify that all the problems that may arise from these kernels are eliminated by using pseudoinverses instead of inverses of matrices \footnote{7}.

Let us consider two real matrices $A = A^T \in M_{n,n}$ and $C = -C^T \in M_{n,n}$, and
\[
M = \begin{pmatrix} A & C \\ C^T & A \end{pmatrix} = M^T \in M_{2n,2n}.
\] (14)

**Lemma 2** $M \geq 0$ iff $A + iC \geq 0$.

**Proof:** This follows from the observation that $M$ is real, and that for any pair of real vectors $a, b \in \mathbb{R}^N$ we have $(a - ib)^T (A + iC) (a - ib) = (a \oplus b)^T M (a \oplus b)$.

\[\text{References:}\]

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[8] If $X_k, P_k$ are position- and momentum-like operators in each mode fulfilling canonical commutation relation $[X_k, P_k] = i$, we define
\[
\gamma_{\alpha,\beta} = 2 \text{Re}[(R_{\alpha} - d_{\alpha})(R_{\beta} - d_{\beta})],
\]
where $d_{\alpha} = \langle R_{\alpha} \rangle$ and $R_{2k-1} = X_k$ and $R_{2k} = P_k$ ($k = 1, 2, \ldots, n$).
[9] For convenience we use direct sum notation for matrices and vectors. That is, if $A \in M_{n,n}$ and $B \in M_{m,m}$, $A \oplus B \in M_{n+m,n+m}$ is a block diagonal matrix of blocks $A$ and $B$. Similarly, if $f_1 \in \mathbb{R}^n$ and $f_2 \in \mathbb{R}^m$ are two vectors, then $f_1 \oplus f_2 \in \mathbb{R}^{n+m}$ is a vector whose first $n$ components are given by the entries of $f_1$ and the last $m$ by those of $f_2$.
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[11] Throughout this work we will denote by $B^{-1}$ the pseudoinverse of $B$, that is, $BB^{-1} = B^{-1}B$ is the projector on the range of $B$. If $B$ is invertible, $B^{-1}$ coincides with the inverse of $B$.
[12] $\|A\|_{\text{tr}} \equiv \text{tr}(A^T A)^{1/2}$ denotes the trace norm of $A$. The operator norm of $A$, $\|A\|_{\text{op}}$ is the maximum eigenvalue of $(A^T A)^{1/2}$.
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