A NOTE ON THE QUANTUM FAMILY OF MAPS

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Abstract. The notion and theory of the quantum space of all maps from a quantum space pioneered by Sołtan have been mainly focused on finite-dimensional C*-algebras which are matrix algebra bundles over a finite set \( S \). We propose a modification of this notion to cover the case of \( C(X) \) for general compact Hausdorff spaces \( X \) instead of finite sets \( S \) while taking into account of the topology of \( X \). A notion of free product of copies of a unital C*-algebra topologically indexed by a compact Hausdorff space arises naturally, and satisfies some desired functoriality.

1. Introduction. In a series of papers \([5,6,7]\), Sołtan pioneered in the development and study of a theory of quantum spaces of maps between quantum spaces. Following a natural categorical viewpoint, he proposes a conceptual definition of the quantum space \( \mathbb{A} \) of all maps from a quantum space \( \mathbb{X} \) to a quantum space \( \mathbb{Y} \), via the universality that there is a C*-algebra homomorphism \( \eta: C(\mathbb{Y}) \to C(\mathbb{X}) \otimes C(\mathbb{A}) \) such that for any quantum space \( \mathbb{M} \) with a C*-algebra homomorphism \( \psi: C(\mathbb{Y}) \to C(\mathbb{X}) \otimes C(\mathbb{M}) \) (viewed as a quantum family of maps from \( \mathbb{X} \) to \( \mathbb{Y} \) parametrized by \( \mathbb{M} \)), there is a unique C*-algebra homomorphism...
$\bar{\psi} : C(\mathbb{A}) \to C(\mathbb{M})$ such that $\psi = (\text{id} \otimes \bar{\psi}) \circ \eta$. Here we adopt the view that all C*-algebras $\mathcal{B}$ are the function algebra $C(\mathbb{Y})$ of some virtual quantum space $\mathbb{Y}$, and as Sołtan does, we limit our consideration to unital C*-algebras or equivalently compact quantum spaces only. Most of Sołtan’s results are focused on finite-dimensional C*-algebras $C(\mathbb{X})$ and finitely generated C*-algebras $C(\mathbb{Y})$, since the existence of $\mathbb{A}$ is generally established only for such cases.

An attempt to extend the study of quantum spaces of all maps to cover infinite-dimensional C*-algebras $C(\mathbb{X})$ was initiated in Kang’s thesis [2], which considers the commutative C*-algebra $C(\mathbb{X}) \equiv C(\mathbb{N}^+)$ for the one-point compactification $\mathbb{N}^+ = \mathbb{N} \cup \{\infty\}$ of the discrete infinite space $\mathbb{N}$ of all natural numbers. It is noted that the quantum space $\mathbb{A}$ satisfying the above definition fails to exist since the operator-norm continuity required by the C*-algebra tensor product $C(\mathbb{N}^+) \otimes \mathcal{B}(\mathcal{H}) \cong C(\mathbb{N}^+; \mathcal{B}(\mathcal{H}))$ does not survive under taking an infinite direct sum $\oplus$ of operators.

In this short note, we report some observations made in extending the approach taken in Kang’s thesis to propose a version of the quantum space of all maps from a classical compact Hausdorff space $\mathbb{X}$, i.e. the quantum space underlying a unital commutative C*-algebra $C(\mathbb{X}) \equiv C(\mathbb{X})$, to a compact quantum space $\mathbb{Y}$.

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2. Relaxed C*-algebras. In order to cover the case of $C(\mathbb{X}) = C(\mathbb{N}^+)$ for the compact infinite topological space $\mathbb{N}^+ \equiv \mathbb{N} \cup \{\infty\}$, the ordinary notion of (minimal) C*-algebra tensor product $C(\mathbb{N}^+) \otimes \mathcal{B}$ was modified in Kang’s thesis [2]. A key idea in the approach taken was to replace norm continuity in the notion $C(\mathbb{N}^+) \otimes \mathcal{B}$ by strong continuity in a representation theoretic context.

In this paper, to handle compact Hausdorff spaces in full generality, we will start with replacing the minimal tensor product functor $C(\mathbb{X}) \otimes -$ by a “relaxed” variant $C(\mathbb{X}) \boxtimes -$ whose definition involves an additional locally convex topology on the second factor.

In the following discussion, unless otherwise stated, *-algebras, *-homomorphisms, and *-representations are all assumed to be unital and all compact spaces are assumed to be Hausdorff.

**Definition 1.** A relaxed C*-algebra is a C*-algebra $\mathcal{A}$ with an additional locally convex topology $\Sigma$, called the “relaxed topology” of $\mathcal{A}$, which is determined by a separating family $S$ of norm-continuous seminorms on $\mathcal{A}$, making each of the multiplication, involution and addition operations of $\mathcal{A}$ continuous on the open unit ball $(\mathcal{A})_1$ of $\mathcal{A}$ (and hence on any bounded subset of $\mathcal{A}$), e.g. the multiplication binary operation $(\mathcal{A})_1 \times (\mathcal{A})_1 \ni (a, b) \mapsto ab \in (\mathcal{A})_1$ is (jointly) continuous when each copy of $(\mathcal{A})_1$ involved is equipped with the inherited topology $\Sigma$. 
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Definition 2. A morphism of unital relaxed C*-algebras, also called a relaxed *-homomorphism, is a unital *-homomorphism that is continuous with respect to the relaxed topologies.

We denote by \( \mathcal{R} \) the category of all unital relaxed C*-algebras.

Example. Let \( \mathcal{A} \) be a unital C*-algebra with a faithful representation \( \pi \) on some Hilbert space \( \mathcal{H} \), and denote by \( T_\pi \) the topology on \( \mathcal{A} \) pulled back via \( \pi \) from the *-strong topology on \( \mathcal{B}(\mathcal{H}) \). Then \( (\mathcal{A}, T_\pi) \) is a unital relaxed C*-algebra.

In particular, given a Hilbert space \( \mathcal{H} \), all unital C*-algebras \( \mathcal{A} \subseteq \mathcal{B}(\mathcal{H}) \) can be regarded as relaxed C*-algebras in a natural way. Thus, all unital C*-subalgebras of \( \mathcal{B}(\mathcal{H}) \) yield a full subcategory \( \mathcal{R}^H \) of \( \mathcal{R} \). Here we remark that it might be of interest to focus on specific full subcategories like this one, for example, the construction of a topological free product \( \mathcal{B}_\diamond \mathcal{X} \) discussed later in this paper could be carried out in this subcategory \( \mathcal{R}^H \), which would potentially give us another C*-algebra.

Denote by \( \mathcal{C} \) the category formed by all unital C*-algebras and all unital *-homomorphisms.

3. The relaxed tensor product. Let \( X \) be a compact Hausdorff space and \( (\mathcal{A}, T) \) be a unital relaxed C*-algebra. We denote by \( \mathcal{C}(X) \boxotimes (\mathcal{A}, T) \) the set of all functions \( f: X \to \mathcal{A} \) that are norm-bounded and continuous with respect to the relaxed topology \( T \) on \( \mathcal{A} \).

Lemma 1. The set \( \mathcal{C}(X) \boxotimes (\mathcal{A}, T) \) is a C*-algebra with respect to the point-wise operations and the supremum norm.

Proof. By the assumption on \( T \), the set \( \mathcal{C}(X) \boxotimes (\mathcal{A}, T) \) is closed under point-wise addition, multiplication and involution of functions. Let \( (f_n)_n \) be a sequence in \( \mathcal{C}(X) \boxotimes (\mathcal{A}, T) \) converging to some function \( f: X \to \mathcal{A} \) with respect to the supremum norm. We need to show that \( f \) is continuous with respect to the relaxed topology on \( \mathcal{A} \). Recall that \( T \) is determined by some separating family \( S \) of norm-continuous seminorms on \( \mathcal{A} \). Given any semi-norm \( \rho \in S \), a point \( x \in X \) and an \( \epsilon > 0 \), we can choose

1. \( C > 0 \) such that \( \rho \leq C \| \cdot \| \),
2. \( N \) such that \( \| f_n - f \|_\infty \leq \epsilon/(3C) \) for all \( n > N \), and
3. a neighbourhood \( U \) of \( x \) such that \( \rho(f_n(y) - f_n(x)) < \epsilon/3 \) for all \( y \in U \).

Then for all \( y \in U \),

\[
\rho(f(y) - f(x)) \leq \rho(f(y) - f_n(y)) + \rho(f_n(y) - f_n(x)) + \rho(f_n(x) - f(x)) \\
\leq C \cdot \epsilon/(3C) + \epsilon/3 + C \cdot \epsilon/(3C) = \epsilon.
\]

So \( f \) is continuous with respect to the topology determined by any \( \rho \in S \), and hence continuous with respect to the topology \( T \) determined by \( S \).

Definition 3. Let \( X \) be a compact Hausdorff space and let \( (\mathcal{A}, T) \) be a unital relaxed C*-algebra. Then the relaxed tensor product of \( \mathcal{C}(X) \) and \( (\mathcal{A}, T) \) is the C*-algebra \( \mathcal{C}(X) \boxotimes (\mathcal{A}, T) \).
By construction, the relaxed tensor product is functorial in the following sense:

**Lemma.** Given a continuous map \( F : X \to Y \) of compact Hausdorff spaces and a morphism \( \pi \) of relaxed C*-algebras \( (A, \mathcal{T}_A) \) and \( (B, \mathcal{T}_B) \), there exists a unital *-homomorphism

\[
F^* \boxtimes \pi : C(Y) \boxtimes (A, \mathcal{T}_A) \ni f \mapsto \pi \circ f \circ F \in C(X) \boxtimes (B, \mathcal{T}_B).
\]

The proof is trivial.

### 4. Quantum family of maps.

Now we give a modified definition of the quantum space \( A \) of all maps from a compact Hausdorff space \( X \) to a compact quantum space \( Y \).

Conceptually we will replace the notion of a quantum space \( M \) of maps on the category of all maps from a compact Hausdorff space \( X \) to the category of compact Hausdorff spaces and a unital C*-algebra \( (C(M), \mathcal{T}_M) \), and replace the ordinary (minimal) tensor product \( \boxtimes \) by the relaxed tensor product \( \boxtimes \).

**Definition.** The quantum space \( A \) of all maps from a compact Hausdorff space \( X \) to a compact quantum space \( Y \) is the relaxed C*-algebra \( (C(A), \mathcal{T}_A) \) (if exists) with a unital C*-algebra homomorphism \( \eta : C(Y) \to C(X) \boxtimes (C(A), \mathcal{T}_A) \) such that for any unital relaxed C*-algebra \( (C(M), \mathcal{T}_M) \) with a unital *-homomorphism \( \psi : C(Y) \to C(X) \boxtimes (C(M), \mathcal{T}_M) \), there is a unique unital relaxed *-homomorphism \( \psi : (C(A), \mathcal{T}_A) \to (C(M), \mathcal{T}_M) \) such that \( \psi = (\text{id} \boxtimes \psi) \circ \eta \). We call a unital *-homomorphism \( \phi : C(Y) \to C(X) \boxtimes (C(M), \mathcal{T}_M) \) a quantum family of maps from \( X \) to \( Y \) parametrized by \( M \).

Our aim is to show that such a quantum space \( A \) exists in general. In fact, a suitably defined free product \( B^{\otimes X} \) of copies of \( B \equiv C(Y) \) topologically parametrized by \( X \) is shown in the next section to be \( C(A) \), generalizing Soltan’s and Kang’s results on the existence of \( A \) for \( X \) finite or \( X = \mathbb{N}^+ \).

### 5. The universal quantum family of maps.

Given a compact Hausdorff space \( X \) and a unital C*-algebra \( B \), we denote by \( B^{*X} \) the unital free product of \( |X| \) copies of \( B \) and by \( \iota_{x,B} : B \to B^{*X} \) the canonical C*-algebra embedding associated to each point \( x \in X \), where \( |X| \) is the cardinality of \( X \). The assignment \( B \mapsto B^{*X} \) extends to a functor as follows. Given a map \( F : X \to Y \) of compact Hausdorff spaces and a unital *-homomorphism \( \pi : B \to C \), we obtain a unital *-homomorphism

\[
\pi^{*F} : B^{*X} \ni b \mapsto \iota_{F(x),C}(\pi(b)) \in C^Y \quad \text{for } x \in X \text{ and } b \in B.
\]

Now, the assignments \( (X,B) \mapsto B^{*X} \) and \( (F,\pi) \mapsto \pi^{*F} \) form a functor from \( \text{Comp} \times \mathcal{C} \) to \( \mathcal{C}^* \), where \( \text{Comp} \) denotes the category of compact Hausdorff spaces with continuous maps, and fixing \( X \), we obtain a functor \( (-)^{*X} \) on \( \mathcal{C}^* \).

Clearly this notion of free product \( B^{*X} \) works for any set \( X \), and existing studies of free products (e.g. \([1]\) for a reduced version and \([8]\) mainly focused on a finite set \( X \) while showing important connections to quantum theory. Now bringing the topology of \( X \) into consideration, we define a smaller but more topologically sensitive free product C*-algebra \( B^{\otimes X} \).
Let $\mathcal{B}$ be a unital $C^*$-algebra again. Given a unital relaxed $C^*$-algebra $(\mathcal{A}, \mathcal{T})$, we call a unital $*$-homomorphism $\pi : \mathcal{B}^{*X} \to \mathcal{A}$ admissible if for every $b \in \mathcal{B}$, the map

$$X \ni x \mapsto \pi(\iota_x(b)) \in \mathcal{A}$$

is continuous with respect to the relaxed topology $\mathcal{T}$ on $\mathcal{A}$. Let

$$J_{X,\mathcal{B}} := \bigcap_{\pi} \ker \pi \subseteq \mathcal{B}^{*X},$$

where the intersection is taken over all unital relaxed $C^*$-algebras $(\mathcal{A}, \mathcal{T})$ and all admissible $*$-homomorphisms $\pi$ from $\mathcal{B}^{*X}$ to the underlying $C^*$-algebra $\mathcal{A}$. Denote by

$$\mathcal{B}^{\otimes_X} := \mathcal{B}^{*X} / J_{X,\mathcal{B}}$$

the quotient $C^*$-algebra, by

$$\eta_{X,\mathcal{B}} : \mathcal{B}^{*X} \to \mathcal{B}^{\otimes_X}$$

the quotient map and let

$$\bar{\iota}_{x,\mathcal{B}} := \eta_{X,\mathcal{B}} \circ \iota_{x,\mathcal{B}} : \mathcal{B} \to \mathcal{B}^{\otimes_X}$$

for every $x \in X$.

Denote by $\mathcal{T}_{\mathcal{B}^{\otimes_X}}$ the weakest topology on $\mathcal{B}^{\otimes_X}$ that makes $\pi' : \mathcal{B}^{\otimes_X} \to (\mathcal{A}, \mathcal{T})$ continuous with respect to the relaxed topology $\mathcal{T}$ on $\mathcal{A}$ for any admissible $*$-homomorphism $\pi : \mathcal{B}^{*X} \to (\mathcal{A}, \mathcal{T})$, where $\pi'$ is the unique $*$-homomorphism determined by $\pi' \circ \eta_{X,\mathcal{B}} = \pi$. Then $(\mathcal{B}^{\otimes_X}, \mathcal{T}_{\mathcal{B}^{\otimes_X}})$ evidently is a relaxed $C^*$-algebra with $\mathcal{T}_{\mathcal{B}^{\otimes_X}}$ consisting of seminorms $\bar{\rho}$ on $\mathcal{B}^{\otimes_X}$ such that $\bar{\rho} \circ \eta_{X,\mathcal{B}} = \rho \circ \pi$ for some admissible $*$-homomorphism $\pi : \mathcal{B}^{*X} \to (\mathcal{A}, \mathcal{T})$ and some $\rho$ in the set $S$ of seminorms on $\mathcal{A}$ that determines the topology $\mathcal{T}$.

**Proposition 1.** Let $\mathcal{B}$ be a unital $C^*$-algebra. Then there exists a unital $*$-homomorphism

$$\alpha_{\mathcal{B}} : \mathcal{B} \to C(X) \boxtimes (\mathcal{B}^{\otimes_X}, \mathcal{T}_{\mathcal{B}^{\otimes_X}})$$

such that for all $b \in \mathcal{B}$ and $x \in X$,

$$\alpha_{\mathcal{B}}(b)(x) = \bar{\iota}_{x,\mathcal{B}}(b) = (\eta_{X,\mathcal{B}} \circ \iota_{x,\mathcal{B}})(b),$$

or equivalently, $\eta_{X,\mathcal{B}} : \mathcal{B}^{*X} \to \mathcal{B}^{\otimes_X}$ is an admissible $*$-homomorphism.

**Proof.** We only need to check that for every $b \in \mathcal{B}$, the map $\alpha_{\mathcal{B}}(b) : X \to \mathcal{B}^{\otimes_X}$ is continuous with respect to the relaxed topology. So, we take a semi-norm $\bar{\rho}$ that arises by factorizing a composition $\rho \circ \pi$ as above, that is, $\bar{\rho} \circ \eta_{X,\mathcal{B}} = \rho \circ \pi$. Then the map

$$x \mapsto \bar{\rho}(\bar{\iota}_{x,\mathcal{B}}(b)) = (\rho \circ \pi \circ \iota_{x,\mathcal{B}})(b)$$

is continuous because $\pi$ is admissible. $\blacksquare$

Clearly, the assignment

$$(X, \mathcal{B}) \mapsto (\mathcal{B}^{\otimes_X}, \mathcal{T}_{\mathcal{B}^{\otimes_X}})$$

extends to a functor

$$(-) \otimes (-) : \text{Comp} \times C^* \to \mathcal{rC}^*.$$

**Theorem 1.** Let $X$ be a compact Hausdorff space. Then the functor $(-) \otimes X : C^* \to \mathcal{rC}^*$ is left adjoint to the functor $C(X) \boxtimes - : \mathcal{rC}^* \to C^*$. 
Proof. Let $\mathcal{B}$ be a unital C*-algebra and $\mathcal{A}$ a unital relaxed C*-algebra with $\Sigma$ determined by a separating family $S$ of norm-continuous seminorms on $\mathcal{A}$.

We claim that the map
\[\text{hom}_{\mathbb{E}^*}(\mathcal{B}^\otimes X, \Sigma_{\mathcal{B}^\otimes X}), (\mathcal{A}, \Sigma) \ni \phi \mapsto \bar{\phi} := (\text{id} \otimes \phi) \circ \alpha_{\mathcal{B}} \in \text{hom}_{\mathbb{E}^*}(\mathcal{B}, C(X) \boxtimes (\mathcal{A}, \Sigma))\]
is bijective. Indeed, given a unital *-homomorphism $\psi: \mathcal{B} \to C(X) \boxtimes (\mathcal{A}, \Sigma)$, the unital *-homomorphism $\Psi: \mathcal{B}^*X \to \mathcal{A}$ defined by $\Psi(\iota_{x, \mathcal{B}}(b)) := \psi(b)(x)$ is admissible and therefore factorizes to a *-homomorphism $\tilde{\psi}: \mathcal{B}^\otimes X \to \mathcal{A}$ such that $\tilde{\psi} \circ \eta_{X, \mathcal{B}} = \Psi$. This *-homomorphism $\tilde{\psi}$ is relaxed because $\Sigma_{\mathcal{B}^\otimes X}$ is defined as the weakest topology on $\mathcal{B}^\otimes X$ that makes $\pi'$ continuous for all admissible *-homomorphisms $\pi$, and $\pi' = \tilde{\psi}$ when $\pi = \Psi$. Clearly, the assignment $\psi \mapsto \tilde{\psi}$ is inverse to the assignment $\phi \mapsto \bar{\phi}$.

Moreover, it is straightforward to check that both assignments $\psi \mapsto \tilde{\psi}$ and $\phi \mapsto \bar{\phi}$ are natural in $\mathcal{B}$ and $(\mathcal{A}, \Sigma)$.

As a consequence of this categorical statement, we get the following existence theorem for the quantum space of all maps from a compact Hausdorff space $X$ to a compact quantum space.

**Theorem 2.** The quantum space $\mathcal{A}$ of all maps from a compact Hausdorff space $X$ to a compact quantum space $Y$ (or equivalently, a unital C*-algebra $\mathcal{B} \equiv C(Y)$) exists and $(C(\mathcal{A}), \Sigma_{\mathcal{A}}) = (\mathcal{B}^\otimes X, \Sigma_{\mathcal{B}^\otimes X})$ the topological free product of copies of $\mathcal{B}$ parametrized by $X$. More precisely, the *-homomorphism $\alpha_{\mathcal{B}}: \mathcal{B} \to C(X) \boxtimes (\mathcal{B}^\otimes X, \Sigma_{\mathcal{B}^\otimes X})$ satisfies the universality that for any unital C*-algebra homomorphism $\psi: \mathcal{B} \to C(X) \boxtimes (C(\mathcal{M}), \Sigma_{\mathcal{M}})$, there is a unique unital relaxed *-homomorphism $\tilde{\psi}: (\mathcal{B}^\otimes X, \Sigma_{\mathcal{B}^\otimes X}) \to (C(\mathcal{M}), \Sigma_{\mathcal{M}})$ such that $\psi = (\text{id} \otimes \tilde{\psi}) \circ \alpha_{\mathcal{B}}$.

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\alpha_{\mathcal{B}}} & C(X) \boxtimes (\mathcal{B}^\otimes X, \Sigma_{\mathcal{B}^\otimes X}) \\
\text{id} \downarrow & & \text{id} \otimes \tilde{\psi} \\
\mathcal{B} & \xrightarrow{\psi} & C(X) \boxtimes (C(\mathcal{M}), \Sigma_{\mathcal{M}}).
\end{array}
\]

**Proof.** The assignment $\psi \mapsto \tilde{\psi}$ constructed in the proof of Theorem 1 satisfies the stated universality exactly.

Since for a finite set $N$, the free product $\mathcal{B}^{|N|} \equiv \mathcal{B}^N$ of $|N|$ copies of a C*-algebra $\mathcal{B}$ is characterized by the universality condition in Sołtan’s definition of quantum space of all maps from $N$ to $\mathcal{B}$, the above theorem motivates our definition of the topological free product $\mathcal{B}^\otimes X$ of a family of copies of $\mathcal{B}$ parametrized by a topological space $X$. As a special case, the theorem says that given any collection of *-representations $\psi_x: \mathcal{B} \to \mathcal{B}(\mathcal{H})$ parametrized by $x \in X$ such that for all $b \in \mathcal{B}$, $x \mapsto \psi_x(b)$ is strongly continuous (and hence *-strongly continuous since $\mathcal{B}$ is involutive) on $X$, there is a unique *-representation $\tilde{\psi}: \mathcal{B}^\otimes X \to \mathcal{B}(\mathcal{H})$ such that $\tilde{\psi} \circ \iota_{x, \mathcal{B}} = \psi_x$ for all $x \in X$.

It is easy to see that $\mathcal{B}^N = \mathcal{B}^\otimes N$ when $N$ is a finite (discrete) space. But in general, it is not clear whether $\mathcal{B}^N \equiv \mathcal{B}^\otimes N$ is embedded in $\mathcal{B}^\otimes X$ for any finite subset $N$ of a compact Hausdorff space $X$. It seems that a non-discrete topology on $X$ can seriously limit the existence of enough many admissible *-representations of $\mathcal{B}^X$. However $\mathcal{B}^N$ is embedded
in $\mathcal{B}^\otimes X$ for any finite open subset $N$ of $X$ since any *-representation $\rho : \mathcal{B}^* N \to \mathcal{B}(\mathcal{H})$ can be extended to a *-representation $\pi$ of $\mathcal{B}^* X$ with $x \mapsto \pi(\iota_{x,\mathcal{B}}(b))$ *-strongly continuous on $X$ by setting $\pi(\iota_{x,\mathcal{B}}(b)) := \rho(\iota_{x,\mathcal{B}}(b)) \otimes I_{\mathcal{H}_0}$ for $x \in N$ and $\pi(\iota_{x,\mathcal{B}}(b)) := I_{\mathcal{H}} \otimes \pi_0(b)$ for all $x \in X \setminus N$, where $\pi_0 : \mathcal{B} \to \mathcal{B}(\mathcal{H}_0)$ can be any fixed unital *-representation of $\mathcal{B}$.

6. Some functorial properties. In this section, we discuss some functorial properties about the notion of topological free product $\mathcal{B}^\otimes X$.

We recall that in the category of unital C*-algebras, the pushout for a pair of C*-algebra homomorphisms $\mathcal{C} \xrightarrow{h} \mathcal{A}$ and $\mathcal{C} \xrightarrow{k} \mathcal{B}$ is given by the amalgamated product $\mathcal{A} *_{\mathcal{C}} \mathcal{B}$. For the theory of amalgamated product of C*-algebras or pushout in the category of C*-algebras, we refer to the systematic study given by G. Pedersen in [4].

**Proposition 2.** The functor $(-)^\otimes X$ preserves pushout in the sense that for any pushout $\mathcal{A} *_{\mathcal{C}} \mathcal{B}$ given by morphisms $\mathcal{C} \xrightarrow{h} \mathcal{A}$ and $\mathcal{C} \xrightarrow{k} \mathcal{B}$ in the category $\mathcal{C}^*$, if $(\mathcal{A}^\otimes X, \mathcal{I}_{\mathcal{A}^\otimes X}) \xrightarrow{H} (\mathcal{D}, \mathcal{I}_D)$ and $(\mathcal{B}^\otimes X, \mathcal{I}_{\mathcal{B}^\otimes X}) \xrightarrow{K} (\mathcal{D}, \mathcal{I}_D)$ are relaxed *-homomorphisms such that $H \circ h^\otimes X = K \circ k^\otimes X$, then there is a unique relaxed *-homomorphism $\Psi : ((\mathcal{A}^\otimes \mathcal{B})^\otimes X, \mathcal{I}_{(\mathcal{A}^\otimes \mathcal{B})^\otimes X}) \to (\mathcal{D}, \mathcal{I}_D)$ such that $H = \Psi \circ (\iota_{\mathcal{A}})^\otimes X$ and $K = \Psi \circ (\iota_{\mathcal{B}})^\otimes X$, where $\iota_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} *_{\mathcal{C}} \mathcal{B}$ and $\iota_{\mathcal{B}} : \mathcal{B} \to \mathcal{A} *_{\mathcal{C}} \mathcal{B}$ are the canonical *-homomorphisms.

**Proof.** Since left adjoint functors preserve colimits in the category theory ([3, section V.5, p. 115]) and the functor $(-)^\otimes X : \mathcal{C}^* \to \mathcal{C}^*$ is shown in Theorem 1 to be left adjoint to the functor $C(X) \boxtimes - : \mathcal{C}^* \to \mathcal{C}^*$, the assignment $\mathcal{B} \mapsto (\mathcal{B}^\otimes X, \mathcal{I}_{\mathcal{B}^\otimes X})$ preserves the colimits and in particular, the pushouts. ■

Now we consider another functor associated with the construction of topological free product $\mathcal{B}^\otimes X$.

**Proposition 3.** Given a C*-algebra $\mathcal{B}$ and a continuous map $f : X \to Y$ between compact Hausdorff spaces $X$ and $Y$, there is a unique well-defined relaxed *-homomorphism $\mathcal{B}^\otimes f : (\mathcal{B}^\otimes X, \mathcal{I}_{\mathcal{B}^\otimes X}) \to (\mathcal{B}^\otimes Y, \mathcal{I}_{\mathcal{B}^\otimes Y})$ such that $\mathcal{B}^\otimes f(\iota_{x,\mathcal{B}}(b)) = \iota_{f(x),\mathcal{B}}(b)$ for all $x \in X$ and $b \in \mathcal{B}$.

**Proof.** Clearly since $\iota_{x,\mathcal{B}}(b)$ for $(x, b) \in X \times \mathcal{B}$ generate the C*-algebra $\mathcal{B}^\otimes X$, the uniqueness of $\mathcal{B}^\otimes f$ is clear. It remains to show that $\mathcal{B}^\otimes f$ can be well-defined as a relaxed *-homomorphism.

Note that the map $f$ induces a well-defined *-homomorphism $f^\otimes \boxtimes \text{id} : C(Y) \boxtimes (\mathcal{B}^\otimes Y, \mathcal{I}_{\mathcal{B}^\otimes Y}) \ni g \mapsto g \circ f \in C(X) \boxtimes (\mathcal{B}^\otimes Y, \mathcal{I}_{\mathcal{B}^\otimes Y})$ because $g \circ f : X \to (\mathcal{B}^\otimes Y, \mathcal{I}_{\mathcal{B}^\otimes Y})$ is continuous if $g : Y \to (\mathcal{B}^\otimes Y, \mathcal{I}_{\mathcal{B}^\otimes Y})$ is, and clearly $\|g \circ f\|_\infty \leq \|g\|_\infty$.

Let $\alpha_{Y,\mathcal{B}} : \mathcal{B} \to C(Y) \boxtimes (\mathcal{B}^\otimes Y, \mathcal{I}_{\mathcal{B}^\otimes Y})$ be the canonical *-homomorphism $\alpha_{\mathcal{B}}$ associated with the space $Y$ instead of $X$. By applying Theorem 2 to the *-homomorphism $\psi := (f^\otimes \boxtimes \text{id}) \circ \alpha_{Y,\mathcal{B}} : \mathcal{B} \to C(X) \boxtimes (\mathcal{B}^\otimes Y, \mathcal{I}_{\mathcal{B}^\otimes Y})$
shown in the diagram

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\alpha_B} & C(X) \boxtimes (\mathcal{B}^\otimes X, \Sigma_{B^\otimes X}) \\
\beta & \downarrow \alpha_{Y, B} & \downarrow \text{id} \boxtimes \mathcal{B}^\otimes f \\
C(Y) \boxtimes (\mathcal{B}^\otimes Y, \Sigma_{B^\otimes Y}) & \xrightarrow{f^* \boxtimes \text{id}} & C(X) \boxtimes (\mathcal{B}^\otimes Y, \Sigma_{B^\otimes Y})
\end{array}
\]

we get a unique relaxed *-homomorphism denoted as \( \mathcal{B}^\otimes f : (\mathcal{B}^\otimes X, \Sigma_{B^\otimes X}) \rightarrow (\mathcal{B}^\otimes Y, \Sigma_{B^\otimes Y}) \) making this diagram commute. Now for all \((x, b) \in X \times \mathcal{B}\),

\[
\mathcal{B}^\otimes f(\tilde{i}_{x,B}(b)) = \mathcal{B}^\otimes f(\alpha_B(b)(x)) = (((f^* \boxtimes \text{id}) \circ \alpha_{Y, B})(b))(x) = ((f^* \boxtimes \text{id})(\alpha_{Y, B}(b)))(x) = \alpha_{Y, B}(b)(f(x)) = \tilde{i}_{f(x),B}(b)
\]

where it is understood that \( \tilde{i}_{x,B} : \mathcal{B} \rightarrow \mathcal{B}^\otimes X \) and \( \tilde{i}_{f(x),B} : \mathcal{B} \rightarrow \mathcal{B}^\otimes Y \) are embeddings of \( \mathcal{B} \) into different free products.

**Corollary 1.** The assignment \( X \mapsto (\mathcal{B}^\otimes X, \Sigma_{B^\otimes X}) \) and \( f \mapsto \mathcal{B}^\otimes f \) for continuous maps \( f : X \rightarrow Y \) constitute a covariant functor \( \mathcal{B}^\otimes - : \text{Comp} \rightarrow \text{rc}^* \) from the category of compact Hausdorff spaces to the category of unital relaxed C*-algebras, for any given unital C*-algebra \( \mathcal{B} \).

**Proof.** For any \( X \xrightarrow{f} Y \xrightarrow{g} Z \), \( x \in X \), and \( b \in \mathcal{B} \),

\[
\mathcal{B}^\otimes (g \circ f)(\tilde{i}_{x,B}(b)) = \tilde{i}_{(g \circ f)(x),B}(b) = \tilde{i}_{g(f(x)),B}(b) = \mathcal{B}^\otimes g(\tilde{i}_{f(x),B}(b)) = \mathcal{B}^\otimes g(\mathcal{B}^\otimes f(\tilde{i}_{x,B}(b)))
\]

implies that \( \mathcal{B}^\otimes (g \circ f) = \mathcal{B}^\otimes g \circ \mathcal{B}^\otimes f \).

Given continuous maps \( Z \xrightarrow{f} X \) and \( Z \xrightarrow{g} Y \) between compact Hausdorff spaces, the pushout \( X \cup_Z Y \) in the category of compact Hausdorff spaces for the pair \((f, g)\) is well defined and can be constructed as follows.

Let \( \mathcal{R} \) be the collection of all equivalence relations \( R \subset (X \cup Y) \times (X \cup Y) \) on \( X \cup Y \) such that \((f(z), g(z)) \in R \) for all \( z \in Z \) and the (automatically compact) quotient topological space \((X \cup Y)/R \) is Hausdorff. Clearly \( \mathcal{R} \) is not empty since it contains the equivalence relation \((X \cup Y) \times (X \cup Y) \). It is clearly that the intersection \( \sim \) of all \( R \in \mathcal{R} \) is an equivalence relation on \( X \cup Y \) containing \((f(z), g(z)) \) for all \( z \in Z \). Furthermore the quotient space \((X \cup Y)/\sim \) is still Hausdorff. In fact, if \([p] \neq [q] \) in \((X \cup Y)/\sim \) for some \( p, q \in X \cup Y \) then \((p, q) \notin \sim \) and hence \((p, q) \notin R \) for some \( R \in \mathcal{R} \), which implies that the two distinct points \([p]_R \) and \([q]_R \) in the Hausdorff space \((X \cup Y)/R \) are separated by disjoint open sets \( U \) and \( V \) of \((X \cup Y)/R \). Now the inverse images of \( U \) and \( V \) under the canonical quotient map \((X \cup Y)/\sim \rightarrow (X \cup Y)/R \) are disjoint open sets separating \([p] \) and \([q] \) in \((X \cup Y)/\sim \). So \((X \cup Y)/\sim \) is Hausdorff.

It is routine to check that the compact Hausdorff space \( X \cup_Z Y := (X \cup Y)/\sim \) is a pushout for the pair \((f, g)\), i.e. for any continuous maps \( f' : X \rightarrow W \) and \( g' : Y \rightarrow W \) with \( f' \circ f = g' \circ g \), there is a unique continuous map \( h : X \cup_Z Y \rightarrow W \) such that \( h \circ \varepsilon_X = f' \) and \( h \circ \varepsilon_Y = g' \) where \( \varepsilon_X : X \rightarrow X \cup_Z Y \) and \( \varepsilon_Y : Y \rightarrow X \cup_Z Y \) are the canonical quotient map \( X \cup Y \rightarrow X \cup_Z Y \) restricted to \( X \) and \( Y \) respectively.
We remark that more abstractly, one can get the pushout $X \cup Z Y$ from the commutative $C^*$-algebra $C(X) \oplus C(Z) C(Y)$ which is the pullback of the pair $f^* : C(X) \to C(Z)$ and $g^* : C(Y) \to C(Z)$.

**Proposition 4.** The functor $B^\otimes$ preserves pushout in the sense that for any pushout $X \cup Z Y$ given by $Z \xrightarrow{f} X$ and $Z \xrightarrow{g} Y$, the relaxed $C^*$-algebra $(B^\otimes(X \cup Z Y), \xi_{B^\otimes(X \cup Z Y)})$ satisfies: if $(B^\otimes X, \xi_{B^\otimes X}) \overset{F}{\to} (C, \xi_C)$ and $(B^\otimes Y, \xi_{B^\otimes Y}) \overset{G}{\to} (C, \xi_C)$ are relaxed *-homomorphisms such that $F \circ B^\otimes f = G \circ B^\otimes g$, then there is a unique relaxed *-homomorphism

$$H : (B^\otimes(X \cup Z Y), \xi_{B^\otimes(X \cup Z Y)}) \to (C, \xi_C)$$

such that $F = H \circ B^\otimes \epsilon_X$ and $G = H \circ B^\otimes \epsilon_Y$ for the canonical maps $\epsilon_X : X \to X \cup Z Y$ and $\epsilon_Y : Y \to X \cup Z Y$.

**Proof.** Note that $F$ and $G$ give rise to two well-defined *-homomorphisms

$$\text{id} \otimes F : C(X) \otimes (B^\otimes X, \xi_{B^\otimes X}) \to C(X) \otimes (C, \xi_C)$$

and

$$\text{id} \otimes G : C(Y) \otimes (B^\otimes Y, \xi_{B^\otimes Y}) \to C(Y) \otimes (C, \xi_C)$$

respectively. Composing them with the canonical *-homomorphisms $\alpha_X : B \to C(X) \otimes (B^\otimes X, \xi_{B^\otimes X})$ and $\alpha_Y : B \to C(Y) \otimes (B^\otimes Y, \xi_{B^\otimes Y})$ respectively, we get two *-homomorphisms $\phi : B \to C(X) \otimes (C, \xi_C)$ and $\gamma : B \to C(Y) \otimes (C, \xi_C)$. With

$$(\phi(b))(x) = F(\alpha_X(b)(x)) = F(\bar{i}_x, B(b))$$

and

$$(\gamma(b))(y) = G(\alpha_Y(b)(y)) = G(\bar{i}_y, B(b))$$

for all $(x, y) \in X \times Y$, we get

$$(\phi(b))(f(z)) = F(\bar{i}_{f(z)}, B(b)) = F(B^\otimes f(\bar{i}_z, B(b))) = G(B^\otimes f(\bar{i}_z, B(b))) = G(\bar{i}_{g(z)}, B(b)) = (\gamma(b))(g(z))$$

for all $z \in Z$. Thus $\phi(b) \in C(X) \otimes (C, \xi_C)$ and $\gamma(b) \in C(Y) \otimes (C, \xi_C)$ can be merged together to well define an element $\psi(b) \in C(X \cup Z Y) \otimes (C, \xi_C)$ such that $(\psi(b))(\epsilon_X(x)) = (\phi(b))(x)$ and $(\psi(b))(\epsilon_Y(y)) = (\gamma(b))(y)$ for all $(x, y) \in X \times Y$, and $\psi : B \to C(X \cup Z Y) \otimes (C, \xi_C)$ is a well-defined *-homomorphism.

Then by Theorem 2 there exists a unique relaxed *-homomorphism

$$\bar{\psi} : (B^\otimes(X \cup Z Y), \xi_{B^\otimes(X \cup Z Y)}) \to (C, \xi_C)$$

such that

$$(\ast) \quad \bar{\psi}(\bar{i}_p, B(b)) = (\psi(b))(p) \equiv \begin{cases} (\phi(b))(x) = F(\bar{i}_x, B(b)), & \text{if } p = \epsilon_X(x) \\ (\gamma(b))(y) = G(\bar{i}_y, B(b)), & \text{if } p = \epsilon_Y(y) \end{cases}$$

for all $p \in X \cup Z Y$ and $b \in B$. Now with

$$\bar{i}_p, B(b) \equiv \begin{cases} \bar{i}_{\epsilon_X(x)}, B(b) \equiv B^\otimes \epsilon_X(\bar{i}_x, B(b)), & \text{if } p = \epsilon_X(x) \\ \bar{i}_{\epsilon_Y(y)}, B(b) \equiv B^\otimes \epsilon_Y(\bar{i}_y, B(b)), & \text{if } p = \epsilon_Y(y) \end{cases}$$

the equality (\ast) translates exactly to the equalities $F = \bar{\psi} \circ B^\otimes \epsilon_X$ and $G = \bar{\psi} \circ B^\otimes \epsilon_Y$, and so we simply take $H := \bar{\psi}$. ■
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