Weight hierarchies of a family of linear codes associated with degenerate quadratic forms

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Abstract We restrict a degenerate quadratic form $f$ over a finite field of odd characteristic to subspaces. Thus, a quotient space related to $f$ is introduced. Then we get a non-degenerate quadratic form induced by $f$ over the quotient space. Some related results on the subspaces and quotient space are obtained. Based on this, we solve the weight hierarchies of a family of linear codes related to $f$.

Keywords Weight hierarchy · Generalized Hamming weight · Linear code · Quadratic form · Quotient space

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1 Introduction

Weight hierarchies of linear codes have been an interesting topic for their important value in theory and applications to cryptography. In 1991, Wei in

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the paper [25] presented his wonderful results about weight hierarchies. It has been shown that the weight hierarchy of a linear code completely characterizes the performance of the code on the type II wire-tap channel. Readers can refer to [22] for a detailed survey on the results up to 1995 about weight hierarchies. The interest towards the knowledge of the weight hierarchy of a linear code has been continually increasing. Many authors devoted themselves to weight hierarchies of particular classes of codes [1,2,3,7,10,13,14,26,27]. In general, it is hard to settle the weight hierarchy of a linear code.

Let \( p \) be an odd prime and \( F_p^m \) be the finite field with \( p^m \) elements. Denote by \( C \subset F_p^n \) an \([n,k,d]\) \( p \)-ary linear code with minimum Hamming distance \( d \) [12]. Let \([C,r]_p \) be the set of all \( r \)-dimensional subspaces of \( C \). For \( V \in [C,r]_p \), define

\[
\text{Supp}(V) = \{ i : x_i \neq 0 \text{ for some } x = (x_1, x_2, \ldots, x_n) \in V \}.
\]

Then we define the \( r \)-th (\( 1 \leq r \leq k \)) generalized Hamming weight \( d_r(C) \) of linear code \( C \) by

\[
d_r(C) = \min\{|\text{Supp}(V)| : V \in [C,r]_p \}.
\]

In particular, \( d_1(C) = d \). The weight hierarchy of \( C \) is defined as the set \( \{d_i(C) : 1 \leq i \leq k \} \) (see [11,12,15]).

Denote by \( F_p^* \) the set of nonzero elements in the finite field \( F_p^m \). A generic construction of linear code was proposed by Ding et al. ([4,6]). It is as follows. Let \( Tr \) denote the trace function from \( F_p^m \) to \( F_p \) and \( D = \{d_1, d_2, \ldots, d_n\} \subset F_p^* \). Define a linear code \( C_D \) with length \( n \) as follows:

\[
C_D = \{(\text{Tr}(xd_1), \text{Tr}(xd_2), \ldots, \text{Tr}(xd_n)) : x \in F_p^m \}, \tag{1}
\]

and \( D \) is called the defining set. Many classes of linear codes with a few weights were obtained by choosing properly defining sets [9,18,25,30,21].

In this paper, we discuss the generalized Hamming weights of a class of linear codes \( C_D \), whose defining set is chosen to be

\[
D = D_f^a = \{ x \in F_p^m : f(x) = a \}, \quad a \in F_p^*, \tag{2}
\]

Here \( f \) is a degenerate quadratic form over \( F_p^m \) with values in \( F_p \).
In the paper, we settle the weight hierarchy of $C_D^a$, $a \in \mathbb{F}_p^*$. In our previous work \[20\], the weight hierarchy of $C_D^a$ ($a \in \mathbb{F}_p^*$) relating to non-degenerate quadratic forms was solved. In \[23,24\], Z. Wan and X. Wu calculated the weight hierarchies of the projective codes from quadrics by the theory of finite geometry. In the case $a = 0$, the weight hierarchy of $C_D^a$ can be deduced from Theorem 18 in \[24\].

The weight distributions of $C_D^a$ have been settled. In reference \[5\], K. Ding and C. Ding constructed the linear codes $C_D^a$ in the case $a = 0$ relating to the special quadratic form $\text{Tr}(X^2)$ and determined their weight distributions. In \[8,30,29\], the authors calculated the weight distributions of $C_D^a$ for general quadratic forms. In these articles, it was shown that the linear codes $C_D^a$ have a few weights and can be used to get association schemes, authentication codes, secret sharing schemes with interesting access structures.

Also, by these results, we know that $C_D^a$ is an $m$-dimensional linear code. So we can employ a general formula for calculating the generalized Hamming weights of linear codes defined in (1). It is given as follows.

**Lemma 1. (Theorem 1, \[19\])** For each $r$ ($1 \leq r \leq m$), if the dimension of $C_D$ is $m$, then $d_r(C_D) = n - \max\{|D \cap H| : H \in [\mathbb{F}_p^m, m - r]/_p\}$.

The rest of this paper is organized as follows: in Sect. 2, we present some basic definitions and results of quadratic forms restricted to subspaces and of induced quadratic forms over quotient spaces of finite fields; in Sect. 3, using the results in Sect. 2, we give all the generalized Hamming weights of linear codes defined in (2).

**2 Quadratic Form, Dual Space and Quotient Space**

2.1 Restricting Quadratic Forms to Subspaces

The finite field $\mathbb{F}_{p^m}$ can be viewed as an $m$-dimensional vector space over $\mathbb{F}_p$. Fix a basis $v_1, v_2, \ldots, v_m \in \mathbb{F}_{p^m}$ and express each vector $X \in \mathbb{F}_{p^m}$ in the unique form $X = x_1v_1 + x_2v_2 + x_m v_m$, with $x_1, x_2, \ldots, x_m \in \mathbb{F}_p$. We can write $X = (x_1, x_2, \ldots, x_m)^T$, where $T$ represents the transpose of a matrix.
Let $f : \mathbb{F}_{p^m} \to \mathbb{F}_p$ be a quadratic form over $\mathbb{F}_{p^m}$ with values in $\mathbb{F}_p$. Set

$$F(X, Y) = \frac{1}{2}[f(X + Y) - f(X) - f(Y)].$$

We can write $f(X) = X^TAX$, where $A$ is the symmetric matrix $(a_{ij})_{1 \leq i, j \leq m}$ and $a_{ii} = f(v_i), a_{ij} = F(v_i, v_j)$.

The rank $R_f$ of quadratic form $f$ is defined to be the rank of matrix $A$. We say that $f$ is non-degenerate if $R_f = m$ and degenerate, otherwise.

We can find an invertible matrix $M$ such that $M^TAM$ is a diagonal matrix $A = diag(\lambda_1, \lambda_2, \ldots, \lambda_{R_f}, 0, \ldots, 0)$. Let $\Delta_f = \lambda_1 \cdot \lambda_2 \cdots \lambda_{R_f}$. When $R_f = 0$, we define $\Delta_f = 1$. Let $\eta$ be the quadratic character of $\mathbb{F}_p$, i.e., $\eta(a) = a^{\frac{p-1}{2}}$ for $a \in \mathbb{F}_p^\times$. In the paper, $\eta(0)$ is defined to be zero. Under the congruent transformation of $A \to M^TAM$, $\eta(\Delta_f)$ is an invariant. We called $\eta(\Delta_f)$, denoted by $\epsilon_f$, the sign of the quadratic form $f$.

For a subspace $H \subseteq \mathbb{F}_{p^m}$, define

$$H^\perp = \{x \in \mathbb{F}_{p^m} : F(x, y) = 0 \text{ for each } y \in H\}.$$ 

Then $H^\perp$ is called the dual space of $H$. And $R_f$ can also be defined as the codimension of $\mathbb{F}_{p^m}^\perp$. Namely, $R_f + \dim(\mathbb{F}_{p^m}^\perp) = m$.

Let $H$ be a $d$-dimensional subspace of $\mathbb{F}_{p^m}$. Restricting the quadratic form $f$ to $H$, we get a quadratic form over $H$ in $d$ variables. It is denoted by $f|_H$. Similarly, we define the dual space $H^\perp|_H$ of $H$ under $f|_H$ in itself by

$$H^\perp|_H = \{x \in H : f(x + y) = f(x) + f(y) \text{ for each } y \in H\}.$$ 

Let $R_H, \epsilon_H$ be the rank and sign of $f|_H$ over $H$, respectively. Obviously, $H^\perp|_H = H \cap H^\perp$ and $R_H = d - \dim(H^\perp|_H)$.

**Example 1** Let $f(X) = x_1^2 - 2x_1x_2 + x_2^2$ with $X = (x_1, x_2)$. It is a degenerate quadratic form over $\mathbb{F}_p^2$. After simple calculation, we have $F(X, Y) = x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2$, where $Y = (y_1, y_2)$. Let $H = \{(x_1, x_2) \in \mathbb{F}_p^2 : x_1 = x_2\}$. It is not hard to get $\mathbb{F}_p^m = H, R_f = 1, f|_H = 0, H^\perp|_H = H, R_H = 0$ and $\epsilon_H = 1$.

For $a \in \mathbb{F}_p$, the following lemma tells us the number of solutions in $H$ of the equation $f(x) = a$. 

Lemma 2. (Proposition 1, [20]) Let \( f \) be a quadratic form over \( \mathbb{F}_p^m \), \( a \in \mathbb{F}_p \) and \( H \) be a \( d \)-dimensional \((d > 0)\) subspace of \( \mathbb{F}_p^m \), then the number of solutions of \( f(X) = a \) in \( H \) is

\[
|H \cap D^2_a| = \begin{cases} 
 p^{d-1} + v(a)\eta((-1)^{\frac{p-1}{2}})\epsilon_H p^{d-\frac{p+1}{2}} & , \text{if } R_H \equiv 0 \mod 2, \\
 p^{d-1} + \eta((-1)^{\frac{p-1}{2}})\epsilon_H p^{d-\frac{p+1}{2}} & , \text{if } R_H \equiv 1 \mod 2,
\end{cases}
\]

where \( v(a) = p - 1 \) if \( a = 0 \), otherwise \( v(a) = -1 \).

Remark In Proposition 1 of [20], \( f \) is set to be non-degenerate. In fact, we know from the proof that this condition is unnecessary. Namely, \( f \) can be any quadratic form.

For later use, we need two results in the case that \( f \) is a non-degenerate quadratic form. They are listed as follows.

Lemma 3. (Proposition 2, [20]) Let \( f \) be a non-degenerate quadratic form over \( \mathbb{F}_p^m \). For each \( r \) with \( 0 < 2r < m \), there exist an \( r \)-dimensional subspace \( H \subseteq \mathbb{F}_p^m \) \((m > 2)\) such that \( H \subseteq H^\perp \).

For \( k \) elements \( \beta_1, \beta_2, \ldots, \beta_k \in \mathbb{F}_p^m \), the matrix \( M(\beta_1, \beta_2, \ldots, \beta_k) \) of them is defined as the \( k \times k \) square matrix \((F(\beta_i, \beta_j))_{1 \leq i, j \leq k}\). The discriminant \( \Delta(\beta_1, \beta_2, \ldots, \beta_k) \) of them is defined to be \( \det(M(\beta_1, \beta_2, \ldots, \beta_k)) \). We denote by \( \langle \beta_1, \beta_2, \ldots, \beta_k \rangle \) the subspace spanned by \( \beta_1, \beta_2, \ldots, \beta_k \).

Proposition 1. Let \( f \) be a non-degenerate quadratic form over \( \mathbb{F}_p^m \) and \( H \subseteq \mathbb{F}_p^m \) a subspace with \( \dim(H \cap H^\perp) = e \). Then, \( \epsilon_H \epsilon_{H^\perp} = (-1)^{\frac{d(d-1)}{2}} \epsilon_f \).

Proof. Suppose \( \dim(H) = r \). By hypothesis, we can set

\[
H = \langle \alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \beta_2, \ldots, \beta_e \rangle,
H^\perp = \langle \gamma_1, \gamma_2, \ldots, \gamma_m-r, \beta_1, \beta_2, \ldots, \beta_e \rangle,
\langle \alpha_1, \alpha_2, \ldots, \alpha_r, \gamma_1, \gamma_2, \ldots, \gamma_m-r \rangle^\perp = \langle \eta_1, \eta_2, \ldots, \eta_e, \beta_1, \beta_2, \ldots, \beta_e \rangle.
\]

We have \( \epsilon_H = \eta(\Delta(\alpha_1, \alpha_2, \ldots, \alpha_r)) \), \( \epsilon_{H^\perp} = \eta(\Delta(\gamma_1, \gamma_2, \ldots, \gamma_m-r)) \). And

\[
\epsilon_f = \eta(\Delta(\alpha_1, \alpha_2, \ldots, \alpha_r, \gamma_1, \gamma_2, \ldots, \gamma_m-r, \beta_1, \beta_2, \ldots, \beta_e, \eta_1, \eta_2, \ldots, \eta_e))
= \eta(\Delta(\alpha_1, \ldots, \alpha_r, \gamma_1, \ldots, \gamma_m-r, \beta_1, \beta_2, \ldots, \beta_e, \eta_1, \eta_2, \ldots, \eta_e))
= \eta(\Delta(\alpha_1, \ldots, \alpha_r)) \eta(\Delta(\beta_1, \ldots, \beta_e, \eta_1, \ldots, \eta_e))
= \epsilon_H \epsilon_{H^\perp} \eta((-1)^{e}) \eta(\det(M^2_e)).
\]
Here $M_e$ is the square matrix $(F(\beta_i,\eta_j))_{1 \leq i,j \leq e}$. Then the desired result follows and we complete the proof.

2.2 Induced Quadratic Form over a Quotient Space

From now on, we suppose $f$ is a degenerate quadratic form. Let $\mathbb{F}_{p^m}$ be the quotient space $\mathbb{F}_{p^m}/\mathbb{F}_{p^m}^\perp$. For $\overline{\alpha} \in \mathbb{F}_{p^m}$, define $\overline{f}(\overline{\alpha}) = f(\alpha)$. It is well-defined. We obtain a non-degenerate quadratic form $\overline{f}$, induced by $f$, over $\mathbb{F}_{p^m}$. Without confusion, we still use $f$ to denote the rank and sign of $f$ over $\mathbb{F}_{p^m}$, respectively. It is easy to see $R_{\overline{f}} = R_f$ and $\epsilon_{\overline{f}} = \epsilon_f$.

Since $f$ is a non-degenerate quadratic form over $\mathbb{F}_{p^m}$, the results in Lemmas 2, 3 and Proposition 1 can be applied to $\mathbb{F}_{p^m}$.

For $a \in \mathbb{F}_p$, set
$$D_f^a = \{ \overline{\alpha} \in \mathbb{F}_{p^m} : f(\alpha) = a \}.$$ Obviously, $|D_f^a| = |D_f^a|$. Applying Lemma 2 to $\mathbb{F}_{p^m}$, we have
$$|D_f^a| = \begin{cases} \frac{p^{m-1} + v(a)\eta((-1)^{m_f}a)}{2}, & \text{if } R_f \equiv 0 \text{ (mod 2)}, \\ \frac{p^{m-1} + \eta((-1)^{m_f-a})\epsilon_f p^{m_f-a}}{2}, & \text{if } R_f \equiv 1 \text{ (mod 2)}, \end{cases}$$
where $m_f = \dim(\mathbb{F}_{p^m}) = m - \dim(\mathbb{F}_{p^m}^\perp)$.

Example 2 Just like Example 1, let $f(X) = x_1^2 - 2x_1x_2 + x_2^2$ with $X = (x_1, x_2)$, a degenerate quadratic form over $\mathbb{F}_p^2 \cong \mathbb{F}_p^2$. Because $\mathbb{F}_p^2 = \{(x_1, x_2) \in \mathbb{F}_p^2 : x_1 = x_2\}$, $\mathbb{F}_p^2$ is isomorphic to $\{(x_1, x_2) \in \mathbb{F}_p^2 : x_1 = -x_2\}$. So, $\overline{f} = 4x_1^2$ is a non-degenerate quadratic form over $\mathbb{F}_p^2$. It is not hard to get $R_{\overline{f}} = 1$, $\epsilon_{\overline{f}} = \epsilon_f = 1$, and
$$|D_f^{a}| = \begin{cases} 1, & \text{if } a = 0, \\ 2, & \text{if } \eta(a) = 1, \\ 0, & \text{if } \eta(a) = -1. \end{cases}$$

Let $\varphi$ be the canonical map from $\mathbb{F}_{p^m}$ to $\mathbb{F}_{p^m}$. For a subspace $H \subset \mathbb{F}_{p^m}$, denote by $\overline{H}$ the image of $H$ under $\varphi$, i.e., $\overline{H} = \varphi(H)$. In the absence of confusion, also we use $\overline{H}$ to represent a subspace of $\mathbb{F}_{p^m}$. Let $R_{\overline{H}}$ and $\epsilon_{\overline{H}}$ denote the rank and sign of $f$ over $\overline{H}$, respectively.
**Proposition 2.** Let $H$ be a subspace of $\mathbb{F}_p^m$ and $\overline{H} = \varphi(H) \subseteq \mathbb{F}_p^m$, then $R_{\overline{H}} = R_H$, $\epsilon_{\overline{H}} = \epsilon_H$.

**Proof.** Suppose $\dim(H) = r$, $\dim(H \cap H^\perp) = t$. Then we set $H \cap H^\perp = \langle \beta_1, \beta_2, \ldots, \beta_t \rangle$, $H = \langle \alpha_1, \alpha_2, \ldots, \alpha_{r-t}, \beta_1, \beta_2, \ldots, \beta_t \rangle$. So we have $H = \langle \alpha_1, \alpha_2, \ldots, \alpha_{r-t}, \beta_1, \beta_2, \ldots, \beta_t \rangle$. The matrix $M(\alpha_1, \alpha_2, \ldots, \alpha_{r-t}, \beta_1, \beta_2, \ldots, \beta_t)$ is the block matrix
\[
\begin{pmatrix}
M_1 & O \\
O & O
\end{pmatrix},
\]
where $M_1 = M(\alpha_1, \alpha_2, \ldots, \alpha_{r-t})$. Then $R_H = \text{Rank}(M_1)$, $\epsilon_H = \eta(\det(M_1))$.

And the matrix $M(\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_{r-t}, \overline{\beta}_1, \overline{\beta}_2, \ldots, \overline{\beta}_t)$ is the block matrix
\[
\begin{pmatrix}
\overline{M}_1 & O \\
O & O
\end{pmatrix},
\]
where $\overline{M}_1 = M(\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_{r-t})$. Then $R_{\overline{H}} = \text{Rank}(\overline{M}_1)$, $\epsilon_{\overline{H}} = \eta(\det(\overline{M}_1))$. In fact, $\overline{M}_1 = M_1$, since $f(x) = f(\overline{x})$ for each $x \in \mathbb{F}_p^m$. Hence the desired results follow directly and we complete the proof.

Define the dual space $\overline{H}^\perp$ of $\overline{H}$ by
\[
\overline{H}^\perp = \{ \overline{x} \in \mathbb{F}_p^m : f(\overline{x} + \overline{y}) = f(\overline{x}) + f(\overline{y}) \text{ for each } \overline{y} \in \overline{H} \}.
\]
For the dual spaces, we have an interesting conclusion as below.

**Proposition 3.** Let $H$ be a subspace of $\mathbb{F}_p^m$, then $\overline{H}^\perp = \overline{H^\perp}$.

**Proof.** Let $\overline{x}$ be an element of $\overline{H}^\perp$ with $x \in H^\perp$. We have $f(x + y) = f(x) + f(y)$ for each $y \in H$. So $f(x + \overline{y}) = f(\overline{x}) + f(\overline{y})$. Since $x + \overline{y} = \overline{x} + \overline{y}$, $f(x + \overline{y}) = f(\overline{x}) + f(\overline{y})$. By definition, $\overline{x} \in \overline{H}^\perp$, which means $\overline{H^\perp} \subseteq \overline{H}^\perp$.

On the other hand, let $\overline{x}$ be an element of $\overline{H}^\perp$. For each $\overline{y} \in \overline{H}$, we have $f(\overline{x} + \overline{y}) = f(\overline{x}) + f(\overline{y})$. So $f(\overline{x} + \overline{y}) = f(\overline{x}) + f(\overline{y})$ and $f(x + y) = f(x) + f(y)$. Thus $x \in H^\perp$ and $\overline{x} \in \overline{H}^\perp$. Therefore $\overline{H^\perp} \supseteq \overline{H}^\perp$. In a word, $\overline{H}^\perp = \overline{H^\perp}$. The proof is finished.
3 Weight Hierarchies of Linear Codes Defined in (2)

By our method, we have successfully settled the weight hierarchies of \( C_{D_f^s} \). In this case \( a = 0 \), the weight hierarchies can be derived from Theorem 18 in [24]. In this section, we will just present the weight hierarchies of \( C_{D_f^s} \) in the case \( a \in \mathbb{F}_p^* \).

**Theorem 1.** Let \( f \) be a degenerate quadratic form over \( \mathbb{F}_{p^m} \) with rank \( R_f = 2s \) and \( a \) a non-zero element in \( \mathbb{F}_p^* \). Suppose \( m = 2s + l, \) \( l = \dim(\mathbb{F}_p^\perp) \), then for the linear codes defined in (2), we have

\[
d_r(C_{D_f^s}) = \begin{cases} 
    p^{m-1} - p^{m-r-1} - ((-1)\frac{d_{m-1}}{2} \epsilon_f + 1)p^{s+l-1}, & \text{if } 1 \leq r \leq s, \\
    p^{m-1} - 2p^{m-r-1} - (-1)\frac{d_{m-1}}{2} \epsilon_f p^{s+1}, & \text{if } s \leq r < m, \\
    p^{m-1} - (-1)\frac{d_{m-1}}{2} \epsilon_f p^{s+l-1}, & \text{if } r = m.
\end{cases}
\]

**Proof.** We will use Lemma 1 to compute \( d_r(C_{D_f^s}) \). To do so, we need to know the value of \( \max\{|D_f^s \cap H| : H \in [\mathbb{F}_{p^m}, m - r]_p\} \).

**Case :** \( s \leq r < m \). If \( H_{m-r} \) is an \((m-r)\)-dimensional subspace of \( \mathbb{F}_{p^m} \), then, by Lemma 2, we have

\[
|H_{m-r} \cap D_f^s| \leq 2p^{m-r-1},
\]

and \( |H_{m-r} \cap D_f^s| \) may reach the upper bound \( 2p^{m-r-1} \) if \( R_{H_{m-r}} = 1 \) or 0. We assert that there exists an \((m-r)\)-dimensional subspace \( H_{m-r} \subset \mathbb{F}_{p^m} \) satisfying \( R_{H_{m-r}} = 1 \) and \( \epsilon_{H_{m-r}} \) may take values \(-1\) or \(1\). Applying Lemma 3 to \( \mathbb{F}_{p^m} \), there is an \((s-1)\)-dimensional subspace \( \overline{H}_{s-1} \subset \mathbb{F}_{p^m} \) with \( \overline{H}_{s-1} \subset \overline{H}_{s-1} \). So \( \dim(\overline{H}_{s-1}) = s + 1, \) \( R_{\overline{H}_{s-1}} = 2 \). Applying Lemma 2 to \( \mathbb{F}_{p^m} \), for each \( b \in \mathbb{F}_p^* \), \( |D_f^s \cap \overline{H}_{s-1}| > p^{s-1} \). We choose an element \( \alpha \in (D_f^s \cap \overline{H}_{s-1}) \setminus \overline{H}_{s-1} \) and let \( \overline{H}_s = \langle \alpha \rangle \oplus \overline{H}_{s-1} \). Then \( \dim(\overline{H}_s) = s, \) \( R_{\overline{H}_s} = 1 \) and the values of \( \epsilon_{\overline{H}_s} = \eta(b) \) may take \(-1\) or \(1\). Note that the hypothesis \( l = \dim(\mathbb{F}_p^\perp) \). Therefore, there exists an \((s+l)\)-dimensional subspace \( H_{s+l} \subset \mathbb{F}_{p^m} \) with \( \overline{H}_{s+l} = \varphi(H_{s+l}) = \overline{H}_s \). Thus the assertion is true since \( 1 \leq m-r \leq s+l \). By Lemma 2, for \( s \leq r < m \), we have that \( \max\{|D_f^s \cap H| : H \in [\mathbb{F}_{p^m}, m - r]_p\} = 2p^{m-r-1} \).

**Case :** \( 1 \leq r < s \). For an \((m-r)\)-dimensional subspace \( H_{m-r} \subset \mathbb{F}_{p^m} \), we have

\[
\dim(\overline{H}_{m-r}) = \dim(H_{m-r}/(H_{m-r} \cap \mathbb{F}_p^\perp)) \geq m - r - l = 2s - r.
\]
So, we have $\dim(\overline{H}_{m-r} \cap \overline{H}_{m-r}^\perp) \leq r$, since $\dim(\overline{H}_{m-r}) + \dim(\overline{H}_{m-r}^\perp) = 2s$. Noting that $R_{\overline{H}_{m-r}} = \dim(\overline{H}_{m-r}) - \dim(\overline{H}_{m-r} \cap \overline{H}_{m-r}^\perp)$. By Proposition 2, we have $R_{H_{m-r}} = R_{\overline{H}_{m-r}} \geq 2s - 2r$. By Lemma 2, we have

$$|H_{m-r} \cap D_j^p| \leq p^{m-r-1} + p^{s+l-1},$$

and $|H_{m-r} \cap D_j^p|$ may reach the upper bound $p^{m-r-1}+p^{s+l-1}$ if $R_{H_{m-r}} = 2s-2r+1$ or $2s-2r$. We assert that there is such an $(m-r)$-dimensional subspace $H_{m-r} \subset \mathbb{F}_p^m$ with $|H_{m-r} \cap D_j^p| = p^{m-r-1} + p^{s+l-1}$. By the construction of $\Pi_s$ as above, we have an $r$-dimensional subspace $\overline{H}_r \subset \mathbb{F}_p^m$ satisfying $\epsilon_{\overline{H}_r} = 1$ and $\epsilon_{\overline{H}_r}$ may take values $-1$ or $1$. And $\dim(\overline{H}_r) = 2s-r, R_{\overline{H}_r} = 2s-2r+1$. By Proposition 1, the values of $\epsilon_{\overline{H}_r}$ may take $-1$ or $1$, too. Note that $l = \dim(\mathbb{F}_p^m)$ and $m-r = 2s-r+l$. Thus we can construct an $(m-r)$-dimensional subspace $H_{m-r} \subset \mathbb{F}_p^m$ satisfying $H_{m-r} = \overline{H}_r^\perp$. Notice that $\epsilon_{H_{m-r}} = \epsilon_{\overline{H}_r}$. Therefore, $|D_j^p \cap H_{m-r}| = p^{m-r-1} + p^{s+l-1}$. By Lemma 2, we have that $\max(|D_j^p \cap H| : H \in [\mathbb{F}_p^m, m-r]_p) = p^{m-r-1} + p^{s+l-1}$.

By Lemma 2, we have $|D_j^p| = p^{m-1} - \epsilon_f(-1)^{(m-1)/2}p^{s+l-1}$. Then the desired results follow directly from Lemma 1. And we complete the proof.

**Example 3** Let $(p, m) = (3, 4)$ and $f(x) = \text{Tr}(x^{12}) = \text{Tr}(x^{3^2+3})$. Then $s = 1, l = 2, \epsilon_f = 1$ and the weight hierarchy of $C_{D_j}$ is $d_1 = 18, d_2 = 30, d_3 = 34, d_4 = 36$.

**Theorem 2.** Let $f$ be a degenerate quadratic form over $\mathbb{F}_p^m$ with rank $R_f = 2s+1$ and $a$ a non-zero element in $\mathbb{F}_p^*$. Suppose $m = 2s+1+l$, $l = \dim(\mathbb{F}_p^m)$. If $\eta(a) = (-1)^{(m-1)/2} \epsilon_f$, then for the linear codes defined in (2) we have

$$d_r(C_{D_j}) = \begin{cases} 
 p^{m-1} - p^{m-r-1}, & \text{if } 1 \leq r \leq s, \\
 p^{m-1} + p^{s+l-1} - 2p^{m-r-1}, & \text{if } s < r < m, \\
 p^{m-1} + p^{s+l}, & \text{if } r = m.
\end{cases}$$

**Proof.** Case 1: $1 \leq r \leq s$. For an $(m-r)$-dimensional subspace $H_{m-r} \subset \mathbb{F}_p^m$, we have

$$\dim(\overline{H}_{m-r}) = \dim(H_{m-r}/(H_{m-r} \cap \mathbb{F}_p^m)) \geq m - r - l = 2s+1 - r.$$
We have \( \dim(\overline{H}_{m-r} \cap \overline{H}_{m-r}) \leq r \), since \( \dim(\overline{H}_{m-r}) + \dim(\overline{H}_{m-r}) = 2s + 1 \). Noting that \( R_{\overline{H}_{m-r}} = \dim(\overline{H}_{m-r}) - \dim(\overline{H}_{m-r} \cap \overline{H}_{m-r}) \). By Proposition 2, we have \( R_{\overline{H}_{m-r}} = R_{\overline{H}_{m-r}} \geq 2s + 1 - 2r \). By Lemma 2, we have

\[
|H_{m-r} \cap D_j^j| \leq p^{m-r-1} + p^{s+l},
\]

and \( |H_{m-r} \cap D_j^j| \) may reach the upper bound \( p^{m-r-1} + p^{s+l} \) if \( R_{\overline{H}_{m-r}} = 2s + 1 - 2r \). We assert that there is such an \((m-r)\)-dimensional subspace \( H_{m-r} \subset \mathbb{F}_{p^m} \) with \( |H_{m-r} \cap D_j^j| = p^{m-r-1} + p^{s+l} \), which is constructed as follows. Applying Lemma 3 to \( \mathbb{F}_{p^m} \), we know there is an \( r \)-dimensional subspace \( \overline{H}_r \subset \mathbb{F}_{p^m} \) with \( \dim(\overline{H}_r) = 2s + 1 - r \), \( R_{\overline{H}_r} = 2s - 2r + 1 \). Note that \( l = \dim(\mathbb{F}_{p^m}) \) and \( m - r = 2s - r + 1 + l \). Thus we have an \((m-r)\)-dimensional subspace \( H_{m-r} \subset \mathbb{F}_{p^m} \) satisfying \( \dim(\overline{H}_{m-r}) = \dim(\overline{H}_r) \). By Proposition 1, we have \( \epsilon_{H_{m-r}} = \eta(-1)^r \epsilon_r \), since \( \epsilon_{H_{m-r}} = 1 \), \( \epsilon_r = \epsilon_r \) and \( \epsilon_{H_{m-r}} = \epsilon_{H_{m-r}} = \epsilon_{H_{r}} \). Therefore, by hypothesis and Lemma 2, \( |D_j^j \cap H_{m-r}| = p^{m-r-1} + p^{s+l} \). By Lemma 2, we have that \( \max\{|D_j^j \cap H| : H \in [\mathbb{F}_{p^m}, m-r]_p\} = p^{m-r-1} + p^{s+l} \).

**Case:** \( s < r < m \). The proof is similar to that of Theorem 1.

By Lemma 2, we have \( |D_j^j| = p^{m-1} + p^{s+l} \). Then the desired conclusions follow from Lemma 1. And the proof is completed.

**Example 4** Let \((p, m) = (3, 4)\) and \( f(x) = \text{Tr}(x^2 + x^{3+1}) \). Then \( s = 1 \), \( l = 1 \), \( \epsilon_f = -1 \) and the weight hierarchy of \( C_{D_1^j} \) is \( d_1 = 18, d_2 = 30, d_3 = 34, d_4 = 36 \).

**Theorem 3.** Let \( f \) be a degenerate quadratic form over \( \mathbb{F}_{p^m} \) with rank \( R_f = 2s + 1 \) and \( a \) a non-zero element in \( \mathbb{F}_{p}^* \). Suppose \( m = 2s + 1 + l \), \( l = \dim(\mathbb{F}_{p^m}) \). If \( \eta(a) = (-1)^{\frac{(p-1)}{2}} \epsilon_f \), then for the linear codes defined in (2) we have

\[
d_r(C_{D_j^j}) = \begin{cases} 
p^{m-1} - p^{m-r-1} - p^{s+l} - p^{s+l-1}, & \text{if } 1 \leq r \leq s, 
p^{m-1} - p^{s+l} - 2p^{m-r-1}, & \text{if } s < r < m, 
p^{m-1} - p^{s+l}, & \text{if } r = m.
\end{cases}
\]

**Proof.** **Case:** \( 1 \leq r \leq s \), for an \((m-r)\)-dimensional subspace \( H_{m-r} \subset \mathbb{F}_{p^m} \), we have \( R_{H_{m-r}} \geq 2s - 2r + 1 \). By the corresponding proof of Theorem 2, we know that \( \epsilon_{H_{m-r}} = \eta(-1)^r \epsilon_r \) if \( R_{H_{m-r}} = 2s - 2r + 1 \). By hypothesis and Lemma 2, we have \( |D_j^j \cap H_{m-r}| = p^{m-r-1} - p^{s+l} \).
Next we will construct an \((m-r)\)-dimensional subspace \(H_{m-r} \subset \mathbb{F}_p^m\) with \(R_{H_{m-r}} = 2s - 2r + 2\) and discuss the value of \(|D_f^a \cap H_{m-r}|\). Applying Lemma 3 to \(F_p^m\), there is an \((r - 1)\)-dimensional subspace \(H_{r-1} \subset \mathbb{F}_p^m\) with \(H_{r-1} \subseteq H_{r-1}^\perp\). So \(\dim(H_{r-1}) = 2s - r + 2\), \(R_{H_{r-1}} = 2s - 2r + 3\). Applying Lemma 2 to \(\mathbb{F}_p^m\), we have, for each \(b \in \mathbb{F}_p^m\), \(|D_b^f \cap H_{r-1}^\perp| > 1\). We choose an element \(\alpha \in (D_f^a \cap H_{r-1}^\perp)\) and let \(H_r = \langle \alpha \rangle \oplus H_{r-1}^\perp\). Then \(\dim(H_r) = r\), \(R_{H_r} = 1\) and the values of \(\epsilon_{H_r} = \eta(b)\) may take \(-1\) or 1. Therefore, there exists an \((m-r)\)-dimensional subspace \(H_{m-r} \subset \mathbb{F}_p^m\) with \(H_{m-r} = H_r^\perp\). By Lemma 2, we have that \(|D_f^a \cap H| = p^{m-r-1} \pm p^{s+l-1}\). Therefore, also by Lemma 2, we have \(\max\{|D_f^a \cap H| : H \in \mathbb{F}_p^m, m-r\} = p^{m-r-1} + p^{s+l-1}\).

**Case:** \(s < r < m\). The proof is similar to that of Theorem 1. We omit the details.

By Lemma 2, we have \(|D_f^a| = p^{m-1} - p^{s+l}\). Then the desired conclusions follow from Lemma 1. And the proof is completed.

**Example 5** Let \((p, m) = (3, 4)\) and \(f(x) = \text{Tr}(x^2 - x^{3+1})\). Then \(s = 1, l = 1, \epsilon_f = 1\) and the weight hierarchy of \(C_{D_f^a}\) is \(d_1 = 6, d_2 = 12, d_3 = 16, d_4 = 18\).

Examples 3-5 have been verified by Magma.

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