Normal Forms of Conditional Belief Bases Respecting Inductive Inference

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Abstract
Normal forms of syntactic entities play an important role in many different areas in computer science. In this paper, we address the question of how to obtain normal forms and minimal normal forms of conditional belief bases in order to, e.g., ease reasoning with them or to simplify their comparison. We introduce notions of equivalence of belief bases taking nonmonotonic inductive inference operators into account. Furthermore, we also consider renamings of belief bases induced by renamings of the underlying signatures. We show how renamings constitute another dimension of normal forms. Based on these different dimensions, we introduce and illustrate various useful normal forms and show their properties, advantages, and interrelationships.

1 Introduction
Conditional belief bases consisting of conditionals of the form "If A then usually B" are commonly used to represent and reason with beliefs. Various semantics have been proposed for conditionals, e.g., (Benferhat, Dubois, and Prade 1999; Spohn 2012; Kern-Isberner 2001; Beierle and Kern-Isberner 2012). Generally, the inference properties of the semantics have been in the focus of the research e.g. (Adams 1965; Kraus, Lehmann, and Magidor 1990; Lehmann and Magidor 1992), less attention has been paid do the study of normal form for conditional belief bases, e.g. (Beierle and Kutsch 2019a; Beierle 2019; Beierle and Haldimann 2020). In this paper, we investigate normal forms of belief bases in particular from the viewpoint obtained by respecting inference methods satisfying corresponding properties. We introduce notions of equivalence of belief bases taking inductive inference operators into account, leading to various normal forms and to unique minimal normal forms. Orthogonal to this dimension, we employ signature renamings and show how they can be combined systematically with other normal forms. We investigate the properties of the introduced normal forms and their interrelationships and present observations from our empirical evaluation of normal forms that support our formal investigations.

2 Background: Conditional Logic
Let \( \mathcal{L}(\Sigma) \), or just \( \mathcal{L} \), be the propositional language over a finite signature \( \Sigma \). We call a signature \( \Sigma \) with a linear ordering \( \prec \) an ordered signature and denote it by \((\Sigma, \prec)\). For \( A, B \in \mathcal{L} \), we write \( AB \) for \( A \land B \) and \( \overline{A} \) for \( \neg A \). We identify the set of all complete conjunctions over \( \Sigma \) with the set \( \Omega \) of possible worlds over \( \mathcal{L} \). For \( \omega \in \Omega \) and \( A \in \mathcal{L}, \omega \models A \) means that \( A \) holds in \( \omega \). Two formulas \( A, B \) are equivalent, denoted as \( A \equiv B \), if \( \Omega_A = \Omega_B \), with \( \Omega_A = \{ \omega \mid \omega \models A \} \).

We define the set \((\mathcal{L} \mid \mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}\) of conditionals over \( \mathcal{L} \). The intuition of a conditional \((B|A)\) is that if \( A \) holds then usually \( B \) holds, too. As semantics for conditionals, we use functions \( \kappa : \Omega \to \mathbb{N} \) such that \( \kappa(\omega) = 0 \) for at least one \( \omega \in \Omega \), called ordinal conditional functions (OCF), introduced (in a more general form) by Spohn. They express degrees of plausibility where a lower degree denotes “less surprising”. Each \( \kappa \) uniquely extends to a function \( \kappa : \mathcal{L} \to \mathbb{N} \cup \{\infty\} \) with \( \kappa(A) = \min\{\kappa(\omega) \mid \omega \models A\} \) where \( \min \emptyset = \infty \). An OCF \( \kappa \) accepts a conditional \((B|A)\), written \( \kappa \models (B|A) \), if \( \kappa(AB) < \kappa(\overline{A}\overline{B}) \). A conditional \((B|A)\) is trivial if it is self-fulfilling \((A \models B)\) or contradictory \((A \models \overline{B})\). We say that \((B|A)\) and \((B'|A')\) are conditionally equivalent, denoted by \((B|A) \equiv_{ce} (B'|A')\), if \( A \equiv A' \) and \( AB \equiv A'B' \). A finite set \( \mathcal{R} \subseteq (\mathcal{L} \mid \mathcal{L}) \) is a belief base. An OCF \( \kappa \) accepts \( \mathcal{R} \) if \( \kappa \) accepts all conditionals in \( \mathcal{R} \), and \( \mathcal{R} \) is consistent if an OCF accepting \( \mathcal{R} \) exists.

For orderings like \( \leq \) or \( \preceq \), the strict variants are denoted by \( < \) or \( \prec \), respectively, i.e., \( a < b \) iff \( a \leq b \) and \( b \not< a \).

3 Inductive Inference Operators
The notion of inductive inference operator formalizes how an inference relation \( \models \subseteq \mathcal{L} \times \mathcal{L} \) is obtained by inductive completion of a given belief base.

Definition 1 (inductive inference operator (Kern-Isberner, Beierle, and Brewka 2020)). An inductive inference operator is a mapping \( C : \Delta \mapsto \models_{\Delta} \) that maps a belief base to an inference relation such that direct inference (DI) and trivial vacuity (TV) are fulfilled:

(DI) if \((B|A) \in \Delta\) then \( A \models_{\Delta} B \)

(TV) if \( \Delta = \emptyset \) and \( A \models_{\Delta} B \) then \( A \models B \)

If no confusion arises, we will often simply use \( \models \) to denote the inductive inference operator mapping \( \Delta \) to \( \models_{\Delta} \). Examples of inductive inference operators are:

\( p\)-entailment \( \models^p \) (Goldszmidt and Pearl 1996) considers all ranking models and coincides with system P-inference
system Z \models (And) (Goldszmidt and Pearl 1996) uses the inclusion maximal tolerance partition of \( \Delta \) and it coincides with rational closure (Lehmann and Magidor 1992).

e-inference \models (c) (Beierle et al. 2018; 2021) considers all e-representations (Kern-Isberner 2004).

system W \models w (Komo and Beierle 2022) captures both e-inference and system Z and thus rational closure.

In the following, we formalize some properties an inductive inference operator can have: (AND) and right weakening (RW) from system P, self-fulfilling (SF), semi-monotony (SM) (Reiter 1980; Goldszmidt and Pearl 1996), syntax in-
(Kutsch and Beierle 2021) supports the conjecture that ANF is also \( \vdash \)-complete for system \( Z \), \( c \)-inference, and system \( W \). A systematic generation of belief bases over \( \Sigma_{ab} = \{a, b\} \) using the approach given in (Beierle and Haldimann 2020) and a comparison with respect to \( \equiv_{\vdash} \) suggests that for \( \vdash \in \{\vdash^2, \vdash^c, \vdash^w\} \) and all \( R \) over \( \Sigma_{ab} \) the inference relation \( \vdash_{\mathcal{R}} \) can already be obtained from a belief base in ANF.

6 Reduced Antecedent Normal Form

A belief base in ANF may still contain redundancies in form of conditionals that can be inferred form the other conditionals in \( R \). For instance, in \( R = \{(ab)(a), (ab)(b), (ab(a \lor b))\} \), the third conditional can be derived from the first two conditionals with system \( P \) axiom (\( OR \)); omitting it does not change the induced inference relation of \( R \) with respect to system \( P \) inference. The reduced ANF (Beierle and Haldimann 2020) avoids such redundancies with respect to system \( P \) inference. Here, we generalize this concept by taking any inductive inference operator into account.

Definition 14 (\( \vdash \)-reduced, RANF\(_{\vdash} \)). A belief base \( R \) is \( \vdash \)-reduced if there is no conditional \( (B|A) \in R \) such that \( A \vdash (B|A)B \). \( R \) in \( \vdash \)-reduced antecedent normal form (in RANF\(_{\vdash} \)) if \( R \) is \( \vdash \)-reduced and in ANF.

In general, for an inductive inference operator \( \vdash \) and a belief base \( R \) there may be several \( R' \), \( R'' \in \text{RANF}_{\vdash} \) with \( R \equiv_{\vdash} R' \) and \( R \equiv_{\vdash} R'' \). But in contrast to the CDNF, CNF, and ANF normal forms, there is not a unique RANF\(_{\vdash} \) for every belief base.

Definition 15 (\( \mathcal{R} \), \( \mathcal{A} \), \( \mathcal{F}_{\vdash} \), \( \mathcal{R} \)). The set of RANF\(_{\vdash} \) representations of \( R \), denoted by \( \mathcal{R} \), \( \mathcal{A} \), \( \mathcal{F}_{\vdash} \), \( \mathcal{R} \), is given by \( \mathcal{R} = \{ R' | R' \equiv_{\vdash} R, R' \in \text{RANF}_{\vdash} \} \).

For instance, the non-deterministic transformation system \( \Theta^a \) provided in (Beierle and Haldimann 2020) takes system \( P \) inference into account and ensures that every \( R' \in \Theta^a \) (in RANF\(_{\vdash} \)) and \( R \equiv_{\vdash} R' \). But not every belief base in \( \mathcal{R} \) is in \( \Theta^a \).

Example 16. For a shorter and more concise notation of formulas in CDNF we use \( \nu(F) \) to denote the CDNF of a formula \( F \) in this example; e.g., for \( \Sigma = \{a, b, c, d\} \), we have \( \nu(abc) = CDNF(abc) = \{abcd, abc\} \). Consider the belief bases \( R = \{\nu(ab)|\nu(a), \nu(ab)|\nu(b), \nu((a \lor c)\lor d)|\nu(a \lor c)\} \) and \( \mathcal{R} = \{\nu(ab)|\nu(a), \nu(ab)|\nu(b), \nu((b \lor c)\lor d)|\nu(b \lor c)\} \). We have \( R \equiv_{\vdash} \mathcal{R} \) and \( \mathcal{R} \) is in \( \text{RANF}_{\vdash} \), and thus \( \mathcal{R} \in \mathcal{R} \).

The completeness property about ANF in Proposition 13 can be generalized to RANF\(_{\vdash} \).

Proposition 17. RANF\(_{\vdash} \) is \( \vdash \)-complete if \( \vdash \) satisfies (SF), (CE), (AND), (RW), and (SM).

Thus, RANF\(_{\vdash} \) is \( \vdash \)-complete.

Observation 2. An extension of the empirical evaluation discussed in Observation 1 showed that for \( \vdash \in \{\vdash^2, \vdash^c, \vdash^w\} \) and all \( R \) in ANF over \( \Sigma_{ab} \), the inference relation \( \vdash_{\mathcal{R}} \) can already be obtained from a belief base in RANF\(_{\vdash} \), suggesting that RANF\(_{\vdash} \) is \( \vdash \)-complete for system \( Z \), \( c \)-inference, and for system \( W \).

7 Minimal Normal Form

Here, we employ a linear ordering on the set of belief bases over \( NFC(\Sigma) \) as it is developed in (Beierle and Haldimann 2020). This ordering uses signature renamings, where a function \( \rho : \Sigma \rightarrow \Sigma \) is a renaming if \( \rho \) is a bijection. E.g., the function \( \rho_{ab} \) with \( \rho_{ab}(a) = b \) and \( \rho_{ab}(b) = a \) is a renaming for \( \Sigma_{ab} \). As usual, \( \rho \) is extended canonically to worlds, formulas, conditionals, belief bases, and to sets thereof.

Definition 18 (\( \preceq \)). Let \( X, X' \) be two signatures, worlds, formulas, belief bases, sets, or relations over one of these items. We say that \( X \) and \( X' \) are isomorphic with respect to signature renamings, denoted by \( X \approx X' \), if there exists a renaming \( \rho \) such that \( \rho(X) = X' \).

For a set \( M, m \in M \), and an equivalence relation \( \equiv \) on \( M \), the set of equivalence classes induced by \( \equiv \) is denoted by \( [M]_{\equiv} \), and the unique equivalence class containing \( m \) is denoted by \( [m]_{\equiv} \). E.g., \( [\Omega_{\Sigma_{ab}}]_{\equiv} = \{[ab], [ab], [\overline{ab}], [\overline{\overline{ab}}] \} \) are the three equivalence classes of worlds over \( \Sigma_{ab} = \{a, b\} \) and we have \( [ab] \lor [ab] = [ab] \lor [\overline{ab}] \).

Based on the equivalence classes with respect to \( \equiv \), the linear ordering \( \preceq \) on \( NFC(\Sigma) \) is defined in (Beierle and Haldimann 2020) for each ordered signature \( \Sigma \). We will omit the formal definition \( \preceq \) in this paper as it is not of importance here. The \( \preceq \)-minimal conditional in each equivalence class in \( [NFC(\Sigma_{ab})]_{\equiv} \) is the canonical representative of that class, called canonical normal form conditional.

We can extend \( \preceq \) to an ordering on belief bases.

Definition 19 (\( \mathcal{R} \preceq \mathcal{R}' \)). The lexicographic extension of the ordering on \( NFC(\Sigma) \) to strings over \( NFC(\Sigma) \) is denoted by \( \preceq_{\text{lex}} \). For belief bases \( R = \{r_1, \ldots, r_n\} \) and \( R' = \{r'_1, \ldots, r'_n\} \) over \( NFC(\Sigma) \) with \( r_1 \preceq r_{i+1} \) and \( r'_1 \preceq r'_{j+1} \) the ordering \( \preceq_{\text{lex}} \) is given by: \( R \preceq_{\text{lex}} R' \) if \( u < v \), or if \( u = v \) and \( r_1 \prec r_{i+1} \). Furthermore, \( R \preceq_{\text{lex}} R' \) stands for \( R \preceq_{\text{set}} R' \).

Note that \( \preceq \) is a linear ordering on belief bases.

Definition 20 (MNF\(_{\vdash} \)). A belief base \( R \) is in minimal normal form with respect to \( \vdash \) (in MNF\(_{\vdash} \)), if \( \vdash \) is in CNF and for every \( R' \) in CNF with \( R \equiv_{\vdash} R' \) it holds that \( R \preceq_{\text{set}} R' \).

As immediate consequence, we get the following:

Proposition 21 (MNF\(_{\vdash} \)). For every inductive inference operator \( \vdash \) and every consistent belief base \( R \) in CNF there is a uniquely determined belief base in MNF\(_{\vdash} \), denoted by MNF\(_{\vdash} \)(\( R \)), with \( R \equiv_{\vdash} \) MNF\(_{\vdash} \)(\( R \)).

Completeness for CNF (Prop. 9) also holds for MNF\(_{\vdash} \).

Proposition 22. MNF\(_{\vdash} \) is \( \vdash \)-complete if \( \vdash \) satisfies (SF) and (CE).

If \( \vdash \) also satisfies (AND), (RW), and (SM) then MNF\(_{\vdash} \) is among the RANF\(_{\vdash} \) representations of \( R \).

Proposition 23. If \( \vdash \) is in MNF\(_{\vdash} \) and \( \vdash \) satisfies (SF), (CE), (AND), (RW), and (SM) then \( R \in \mathcal{R} \) and MNF\(_{\vdash} \)(\( R \)) is in RANF\(_{\vdash} \)(\( R \)).

Thus, for \( \vdash \) satisfying (SF), (CE), (AND), (RW), and (SM), MNF\(_{\vdash} \) is a refinement of RANF\(_{\vdash} \) in the sense that \( \Delta(\text{MNF}_{\vdash}) \subseteq \Delta(\text{RANF}_{\vdash}) \); for instance, \( \Delta(\text{MNF}_{\vdash}) \subseteq \Delta(\text{RANF}_{\vdash}) \) holds. Furthermore, according to the study of
\[ \Delta(\text{RANF}_{\sim p}) \leftrightarrow \Delta(\text{ANF}) \]
\[ \Delta(\text{MNF}_{\sim p}) \leftrightarrow \Delta(\text{CNF}) \leftrightarrow \Delta(\text{CDNF}) \]

Figure 1: Overview of normal forms for conditional belief bases. Arrows indicate subset relationships. The dashed arrow holds if \( \sim \) satisfies (SF), (CE), (AND), (RW), and (SM), cf. Proposition 23.

\( \sim p \)-relations in (Beierle, Haldimann, and Kutsch 2021), we have \( |\Delta(\text{MNF}_{\sim p})| = 485 \) and \( |\Delta(\text{RANF}_{\sim p})| = 4.168 \).

An overview over the relations between the sets of all belief bases in a certain normal form is given in Figure 1.

Observation 3. Our empirical evaluations suggest that \( \text{MNF}_{\sim} \) is \( \sim \)-complete for system Z, for \( c \)-inference, and for system W although (SM) does not hold in these cases. Furthermore, they revealed that for \( \sim \in \{ \sim^a, \sim^c, \sim^w \} \), the \( \sim \)-relations of \( \Sigma_{ab} \) can be obtained from \( \Delta(\text{MNF}_{\sim p}) \) and that \( \Delta(\text{MNF}_{\sim p}) \subseteq \Delta(\text{MNF}_{\sim p}) \).

8 Normal Forms Respecting Renamings

The linear ordering \( \equiv \) ensures that there is a unique renaming normal form (Beierle and Haldimann 2020).

Definition 24 (\( \rho \text{NF}, \rho \text{NF}(R) \)). A belief base \( R \) in \( \text{CNF} \) is in renaming normal form (\( \rho \text{NF} \)) if for every \( R' \) with \( R \sim R' \) it holds that \( R \not\sim R' \). For every consistent \( R \) in \( \text{CNF} \), the renaming normal form \( \rho \text{NF}(R) \) of \( R \) is the uniquely determined belief base in \( \rho \text{NF} \) such that \( R \sim \rho \text{NF}(R) \).

If \( \langle NF \rangle \) is one of the other normal forms, we say that a belief base \( R \) is in renaming \( \langle NF \rangle \), abbreviated by \( \rho \langle NF \rangle \), if \( R \) is in \( \rho \text{NF} \) and also in \( \langle NF \rangle \).

Proposition 25 (\( \rho \langle NF \rangle(R) \)). Let \( \sim \) be an inductive inference operator and \( R \) be in \( \text{CNF} \). For \( \langle NF \rangle \in \{ \text{CNF}, \text{ANF}, \text{MNF}_{\sim p} \} \), the \( \rho \langle NF \rangle \) of \( R \), denoted by \( \rho \langle NF \rangle(R) \), is uniquely determined by \( \rho \langle NF \rangle(R) = \rho \text{NF}(\langle NF \rangle(R)) \). The set of \( \rho \text{RANF}_{\sim p} \)-representations of \( R \), denoted by \( \rho \text{RANF}_{\sim p}(R) \), is given by \( \rho \text{RANF}_{\sim p}(R) = \{ \rho \text{NF}(R') \mid R' \in \rho \text{RANF}_{\sim p}(R) \} \).

When generalizing the notions of \( \equiv_{\sim} \) and of \( \sim \)-complete (Definitions 3 and 4) by taking renamings into account, the results of Propositions 9, 13, 17, and 22 carry over to the corresponding renaming normal forms.

Observation 4. Over the signature \( \Sigma_{ab} \), there are 4.168 belief bases in \( \text{RANF}_{\sim p} \). For \( p \)-entailment, we have \( |\Delta(\text{MNF}_{\sim p})| = 484 \) and \( |\Delta(\rho \text{MNF}_{\sim p})| = 262 \) using renamings. For system Z, we have \( |\Delta(\text{MNF}_{\sim p})| = 75 \) and \( |\Delta(\rho \text{MNF}_{\sim p})| = 44 \).

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