Abstract

We study a special combination of the compact directions of the spacetime (compactified on tori) and the worldsheet parity transformed of these directions. The transformations that change the compact part of the spacetime to this combination, are more general than the T-duality transformations. Many properties of this combination and also of the corresponding worldsheet fermions are studied. By using the boundary state formalism, we study the effects of the above transformations to D-branes. For the special cases the resulted branes reduce to the known mixed branes, or reduce to the modified mixed branes.
1 Introduction

T-duality transformation is an exact symmetry of the closed string theory \[1\]. The problems that might at first sight seem different, can be related by T-duality. In type II superstring theory, the action of the T-duality on a compact direction of a D-brane, changes this direction to a transverse direction of a new brane. When T-duality acts on a transverse compact direction of a D-brane, this direction becomes along the new brane. Therefore the action of T-duality on the odd number of the compact directions changes the type IIA theory to the type IIB theory and vice-versa \[2, 3, 4\]. This means, the type IIA theory compactified on a circle of radius R is equivalent to the type IIB theory compactified on a circle of radius \(\alpha'/R\) \[1, 2, 5, 6, 7\].

We concentrate on the compact part of the spacetime (compactified on tori) and its worldsheet parity transformed. In fact, by special combination of these parts (Y-space), the target-space duality is generalized. In this combined space the mass operator of the closed superstring is the same as that in the spacetime. Application of the transformations that transform spacetime to the above combination to a D-brane, produces a brane that is more general than a mixed brane. For the special cases of these transformations the super virasoro operators and consequently the Hamiltonian of the closed superstring are invariant. Note that the boundary conditions of the closed superstring, emitted by a mixed brane, are combinations of Dirichlet and Neumann boundary conditions. This is due to the NS\(\otimes\)NS background B-field \[2, 8, 9, 10, 11\].

This paper is organized as the follows. In section 2, we give a brief review of the T-duality and its effects on D-branes. In section 3, some properties of the Y-space will be studied. This space appears similar to a compact space, therefore under the duality transformations, some interesting effects appear. In section 4, the corresponding fermions of the Y-space will be obtained. In section 5, we apply the transformations that change spacetime to the Y-space, to the D-branes. We obtain the transformed boundary state of a closed superstring, emitted from a brane. In section 6, by neglecting the worldsheet parity part of the Y-space we obtain the usual mixed branes. Putting away the spacetime part of it, leads to the modified mixed branes. The field strengths on these branes, are some combinations of the transformation matrices that change spacetime to the Y-space.

A useful tool for describing branes (specially in non-zero background fields) is boundary state formalism \[12, 13, 14, 15\]. In studying the branes, we will use this formalism.
2 T-duality

Since we will use the T-duality in the next sections, we discuss the T-duality in type II superstring theory in this section. The mode expansion of the closed string coordinate $X^\mu$, is

$$X^\mu(\sigma, \tau) = X^\mu_L(\tau + \sigma) + X^\mu_R(\tau - \sigma) ,$$  
(1)

$$X^\mu_L(\tau + \sigma) = x^\mu_L + 2\alpha' p^\mu_L(\tau + \sigma) + \frac{i}{2} \sqrt{2\alpha'} \sum_{n \neq 0} \left( \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \right) ,$$  
(2)

$$X^\mu_R(\tau - \sigma) = x^\mu_R + 2\alpha' p^\mu_R(\tau - \sigma) + \frac{i}{2} \sqrt{2\alpha'} \sum_{n \neq 0} \left( \frac{1}{n} \alpha_n^\mu e^{-2in(\tau-\sigma)} \right) .$$  
(3)

For non-compact direction $X^\mu$, since $X^\mu$ must be single valued at $\sigma$ and $\sigma + \pi$, there is $p^\mu_L = p^\mu_R = \frac{1}{2} p^\mu$.

If the direction $X^\bar{\mu}$ is compactified on a circle of radius $R_{\bar{\mu}}$, we have the identification

$$x^\bar{\mu} \equiv x^\bar{\mu} + 2\pi L^\bar{\mu} ,$$  
(4)

where

$$x^\bar{\mu} = x^\bar{\mu}_L + x^\bar{\mu}_R ,$$  
(5)

$$L^\bar{\mu} = N^\bar{\mu} R_{\bar{\mu}} , \quad \text{(no sum on \bar{\mu}).}$$  
(6)

Therefore the compacted coordinates at $\sigma + \pi$ and $\sigma$ satisfy

$$X^\bar{\mu}(\sigma + \pi, \tau) - X^\bar{\mu}(\sigma, \tau) = 2\pi L^\bar{\mu},$$  
(7)

which gives

$$L^\bar{\mu} = \alpha'(p^\bar{\mu}_L - p^\bar{\mu}_R) = N^\bar{\mu} R_{\bar{\mu}} ,$$  
(8)

the integer $N^\bar{\mu}$ is the winding number of closed string around the direction $X^\bar{\mu}$. According to the identification (4) the momentum component $p^\bar{\mu}$ of the closed string is quantized,

$$p^\bar{\mu} = p^\bar{\mu}_L + p^\bar{\mu}_R = \frac{M^\bar{\mu}}{R_{\bar{\mu}}} ,$$  
(9)

where $M^\bar{\mu}$, the closed string momentum number, is an integer.
Under the T-duality transformations,

\[ T_\bar{\mu} : \frac{R_{\bar{\mu}}}{\sqrt{\alpha'}} \leftrightarrow \sqrt{\alpha'} \hat{R}_{\bar{\mu}}, \quad M^{\bar{\mu}} \leftrightarrow N^{\bar{\mu}}, \]  

(10)

the closed string mass operator and the Virasoro operators are invariant. According to the eqs. (8)-(10) there are \( p^\bar{\mu}_L \rightarrow p^\bar{\mu}_L \) and \( p^\bar{\mu}_R \rightarrow -p^\bar{\mu}_R \). Generalizing this for all oscillators i.e.

\[
\begin{align*}
\bar{\alpha}^\mu &\rightarrow \bar{\alpha}^\mu, \\
\alpha^\mu &\rightarrow -\alpha^\mu,
\end{align*}
\]  

(11)

and also considering

\[
\begin{align*}
x^\bar{\mu}_L &\rightarrow x^\mu_L, \\
x^\bar{\mu}_R &\rightarrow -x^\mu_R,
\end{align*}
\]  

(12)

we obtain

\[ T_\bar{\mu} : X^{\bar{\mu}}(\sigma, \tau) \rightarrow X'^{\bar{\mu}}(\sigma, \tau) = X^\mu_L - X^\mu_R, \]  

(13)

which is spacetime parity transformation acting only on the right moving degree of freedom.

According to the eqs. (2) and (3), the dual coordinate \( X'^{\bar{\mu}}(\sigma, \tau) \) at \( \sigma \) and \( \sigma + \pi \) is not single valued,

\[ X'^{\bar{\mu}}(\sigma + \pi, \tau) - X'^{\bar{\mu}}(\sigma, \tau) = 2\pi \alpha' p^\bar{\mu} = 2\pi M^{\bar{\mu}} \frac{\alpha'}{R_{\bar{\mu}}}, \]  

(14)

which says the dual coordinate \( X'^{\bar{\mu}} \) is compact on a circle with radius \( R_{\bar{\mu}}' = \alpha' / R_{\bar{\mu}} \). Therefore we have the following identification

\[ x'^{\bar{\mu}} \equiv x^{\mu} + 2\pi \alpha' p^\bar{\mu}, \]  

(15)

where

\[ x^{\mu} = x^\mu_L - x^\mu_R. \]  

(16)

Substitution of eqs. (2) and (3) in the transformation (13) gives the exchange

\[ \partial_\tau X^{\bar{\mu}} \leftrightarrow \partial_\sigma X^{\bar{\mu}}, \]  

(17)

i.e. under the T-duality, the Dirichlet and Neumann boundary conditions of the closed string exchange. Worldsheet supersymmetry requires the following T-duality transformations on the worldsheet fermions

\[ T_\bar{\mu} : \left\{ \begin{array}{l}
\psi_+^{\bar{\mu}} \rightarrow \psi_+^{\bar{\mu}} \\
\psi_-^{\bar{\mu}} \rightarrow -\psi_-^{\bar{\mu}}
\end{array} \right. \]  

(18)
Now we apply the T-duality transformations on a D$_p$-brane, and briefly study their effects on branes.

**T-duality and D-branes**

A D$_p$-brane can be described by the boundary state $|B\rangle$, [12, 13, 14, 15]. This state satisfies the following boundary conditions of the closed string emitted by a D$_p$-brane

$$(\partial_\tau X^\alpha)_{\tau=0}|B\rangle = 0 ,$$

$$(\partial_\sigma X^i)_{\tau=0}|B\rangle = 0 ,$$

where the set $\{X^\alpha\} = \{X^0, X^{\alpha_1}, ..., X^{\alpha_p}\}$ shows the directions along the world volume of the brane and the set $\{X^i\}$ shows the directions perpendicular to the D$_p$-brane. The fermionic part of the closed string boundary conditions are

$$(\psi_\alpha - i\eta\psi_\alpha^\dagger)_{\tau=0}|B\rangle = 0 ,$$

$$(\psi^i - i\eta\psi^i_\dagger)_{\tau=0}|B\rangle = 0 ,$$

where $\eta = \pm 1$ is introduced to simplify the “GSO” projection.

Now assume that the $X^{\alpha}$-direction of the above D$_p$-brane is compact on a circle. The action of the T-duality (i.e. transformations (17) and (18)) on this direction changes the boundary conditions (19) and (21) for $\alpha = \bar{\alpha}$, i.e.

$$(\partial_\sigma X^{\bar{\alpha}})_{\tau=0}|B'\rangle = 0 ,$$

$$(\psi^{\bar{\alpha}} - i\eta\psi^{\bar{\alpha}}_\dagger)_{\tau=0}|B'\rangle = 0 ,$$

This means that the direction $X^{\bar{\alpha}}$ is perpendicular to the new brane, which $|B'\rangle$ describes it. Hence one unit of the dimension of the brane is reduced, which means for even (odd) values of $p$ type IIA (type IIB) theory changes to type IIB (type IIA) theory.

The action of the T-duality on the transverse compact direction $X^{\bar{i}}$, changes the equations (20) and (22) for $i = \bar{i}$, as

$$(\partial_\tau X^{\bar{i}})_{\tau=0}|B''\rangle = 0 ,$$

$$(\psi^{\bar{i}} - i\eta\psi^{\bar{i}}_\dagger)_{\tau=0}|B''\rangle = 0 ,$$

5
therefore the direction $X^i$ is along the brane. The corresponding boundary state of this brane is $|B''\rangle$.

Application of T-duality on arbitrary number of brane directions and transverse directions to it, changes the brane to another brane, that can be obtained by the above method. One can obtain a brane with background fields from a D$p$-brane by appropriate actions of T-duality. For example the action of the T-duality along the directions that make an angle with the brane, or along the boosted directions of the brane produces a brane with magnetic and electric fields \cite{10, 11}.

3 The Combined Space $\{Y^{\bar{\mu}}\}$

The Subspace $\{\tilde{X}^{\bar{\mu}}\}$

Now we define the tilde coordinate $\tilde{X}^{\bar{\mu}} = \tilde{X}^{\bar{\mu}}_L + \tilde{X}^{\bar{\mu}}_R$ as follows

$$\tilde{X}^{\bar{\mu}}_L(\tau + \sigma) = \tilde{x}^{\bar{\mu}}_L + 2\alpha' \tilde{p}^{\bar{\mu}}_L(\tau + \sigma) + \frac{i}{2}\sqrt{2}\alpha' \sum_{n \neq 0} \left( \frac{1}{n} \tilde{\alpha}^{\bar{\mu}}_n e^{-2in(\tau+\sigma)} \right),$$

$$\tilde{X}^{\bar{\mu}}_R(\tau - \sigma) = \tilde{x}^{\bar{\mu}}_R + 2\alpha' \tilde{p}^{\bar{\mu}}_R(\tau - \sigma) + \frac{i}{2}\sqrt{2}\alpha' \sum_{n \neq 0} \left( \frac{1}{n} \tilde{\alpha}^{\bar{\mu}}_n e^{-2in(\tau-\sigma)} \right),$$

where

$$\begin{align*}
\tilde{p}^{\bar{\mu}}_L &= \frac{1}{\sqrt{2}\alpha'} \tilde{\alpha}^{\bar{\mu}}_0 = \tilde{p}^{\bar{\mu}}_R, \\
\tilde{p}^{\bar{\mu}}_R &= \frac{1}{\sqrt{2}\alpha'} \tilde{\alpha}^{\bar{\mu}}_0 = \tilde{p}^{\bar{\mu}}_L ,
\end{align*}$$

which gives $\tilde{p}^{\bar{\mu}} = \tilde{p}^{\bar{\mu}}$. For appropriate $\tilde{x}^{\bar{\mu}}_R$ and $\tilde{x}^{\bar{\mu}}_L$, for example

$$\begin{align*}
\tilde{x}^{\bar{\mu}}_L &= x^{\bar{\mu}}_L , \\
\tilde{x}^{\bar{\mu}}_R &= x^{\bar{\mu}}_R
\end{align*}$$

the coordinate $\tilde{X}^{\bar{\mu}}(\sigma, \tau)$ is the worldsheet parity transformed $(\sigma \rightarrow -\sigma)$ of $X^{\bar{\mu}}(\sigma, \tau)$, that is

$$\tilde{X}^{\bar{\mu}}(\sigma, \tau) = X^{\bar{\mu}}(-\sigma, \tau) .$$

If the subspace $\{X^{\bar{\mu}}\}$ is compact, according to the equation (29), the tilde subspace will be also compact,

$$\tilde{L}^{\bar{\mu}} = \alpha'(\tilde{p}^{\bar{\mu}}_L - \tilde{p}^{\bar{\mu}}_R) = -L^{\bar{\mu}} ,$$

which means, worldsheet parity changes the windings of the closed string. There is therefore an identification,

$$\tilde{x}^{\bar{\mu}} \equiv \tilde{x}^{\bar{\mu}} + 2\pi \tilde{L}^{\bar{\mu}} = \tilde{x}^{\bar{\mu}} - 2\pi L^{\bar{\mu}} .$$
The dual coordinate of the $\tilde{X}^\mu$ is

$$\tilde{X}^\mu(\sigma, \tau) = \tilde{X}^\mu_L(\tau + \sigma) - \tilde{X}^\mu_R(\tau - \sigma).$$

(34)

Since $\tilde{X}^\mu$ is a compact coordinate, $\tilde{x}^\mu = \tilde{x}^\mu_L - \tilde{x}^\mu_R$ is identified with $\tilde{x}^\mu + 2\pi \alpha' \tilde{p}^\mu$,

$$\tilde{x}^\mu \equiv \tilde{x}^\mu + 2\pi \alpha' \tilde{p}^\mu = \tilde{x}^\mu + 2\pi \alpha' \tilde{p}^\mu.$$ 

(35)

The Y-space

This space is the following combination of the subspaces $\{X^\mu\}$ and $\{\tilde{X}^\mu\}$. The left and the right moving parts of the coordinates of this combined space are defined by

$$\begin{cases} 
Y^\mu_L = \mathcal{A}^\mu_\nu X^\nu_R(\tau - \sigma) + \tilde{\mathcal{A}}^\mu_\nu \tilde{X}^\nu_R(\tau - \sigma), \\
Y^\mu_R = \mathcal{B}^\mu_\nu \tilde{X}^\nu_L(\tau + \sigma) + \tilde{\mathcal{B}}^\mu_\nu X^\nu_L(\tau + \sigma), 
\end{cases}$$

(36)

therefore

$$Y^\mu(\sigma, \tau) = Y^\mu_L(\tau + \sigma) + Y^\mu_R(\tau - \sigma).$$

(37)

The matrices $\mathcal{A}$, $\tilde{\mathcal{A}}$, $\mathcal{B}$ and $\tilde{\mathcal{B}}$ will be limited by some conditions such as the invariance of the mass operator of closed string, under changing $\{X^\mu\}$-subspace to the Y-space. As next we will discuss about their properties. Introducing mode expansions (2), (3), (27) and (28) in the definition (36), shows the oscillating modes of the $Y^\mu$

$$Y^\mu_L = y^\mu_L + 2\alpha' \Pi^\mu_L(\tau + \sigma) + \frac{i}{2} 2\alpha' \sum_{n \neq 0} \left( \frac{1}{n} a^\mu_n e^{-2in(\tau + \sigma)} \right),$$

(38)

$$Y^\mu_R = y^\mu_R + 2\alpha' \Pi^\mu_R(\tau - \sigma) + \frac{i}{2} 2\alpha' \sum_{n \neq 0} \left( \frac{1}{n} a^\mu_n e^{-2in(\tau - \sigma)} \right),$$

(39)

where the parameters are

$$y^\mu_R = \mathcal{A}^\mu_\nu x^\nu_R + \tilde{\mathcal{A}}^\mu_\nu \tilde{x}^\nu_R,$$

(40)

$$y^\mu_L = \mathcal{B}^\mu_\nu \tilde{x}^\nu_L + \tilde{\mathcal{B}}^\mu_\nu x^\nu_L,$$

(41)

$$\Pi^\mu_L = \mathcal{A}^\mu_\nu \tilde{p}_R^\nu + \tilde{\mathcal{A}}^\mu_\nu p_R^\nu,$$

(42)

$$\Pi^\mu_R = \mathcal{B}^\mu_\nu \tilde{p}_L^\nu + \tilde{\mathcal{B}}^\mu_\nu p_L^\nu,$$

(43)
\[ a_n^\mu = A_\bar{\mu} \alpha_n^\bar{\mu} + \tilde{A}_\bar{\mu} \tilde{\alpha}_n^\bar{\mu} , \quad (44) \]
\[ \tilde{a}_n^\mu = B_\bar{\mu} \alpha_n^\bar{\mu} + \tilde{B}_\bar{\mu} \tilde{\alpha}_n^\bar{\mu} . \quad (45) \]

To find eqs. (42) and (43) we have used the eq. (29).

Now look at the mass operator of the closed string. The bosonic part of it, is
\[ \alpha' M_b^2 = \alpha_0^\mu \alpha_{0\bar{\mu}} + \tilde{\alpha}_0^\mu \tilde{\alpha}_{0\bar{\mu}} + 2 \sum_{n=1}^{\infty} (\alpha_{-n}^\mu \alpha_{n\bar{\mu}} + \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_{n\bar{\mu}}) \]
\[ + 2 \sum_{n=1}^{\infty} (\tilde{\alpha}_{-n}^{\mu'} \alpha_{n\mu'} + \tilde{\alpha}_{n}^{\mu'} \tilde{\alpha}_{n\mu'}) + \text{ghost part} , \quad (46) \]

where \( \mu' \notin \{ \bar{\mu} \} \). Consider the transformations
\[ \begin{cases} 
X_L^\bar{\mu} \rightarrow Y_L^\bar{\mu} , \\
X_R^\mu \rightarrow Y_R^\mu ,
\end{cases} \quad (47) \]

which give \( \alpha_n^\mu \rightarrow a_n^\mu \) and \( \tilde{\alpha}_n^\mu \rightarrow \tilde{a}_n^\mu \). The invariance of the mass operator under the transformations (47) gives the following conditions on the matrices \( A, \tilde{A}, B \) and \( \tilde{B} \),
\[ A^T A + B^T B = 1 , \quad (48) \]
\[ \tilde{A}^T \tilde{A} + \tilde{B}^T \tilde{B} = 1 , \quad (49) \]
\[ A^T \tilde{A} + B^T \tilde{B} = 0 . \quad (50) \]

One can show that these matrices satisfy the following relations
\[ AA^T + \tilde{A} \tilde{A}^T = 1 , \quad (51) \]
\[ BB^T + \tilde{B} \tilde{B}^T = 1 , \quad (52) \]
\[ AB^T + \tilde{A} \tilde{B}^T = 0 . \quad (53) \]

These equations are not independent of eqs. (48)-(50). We will use them for later purposes.

For some special choices of these matrices, the \( Y \)-space reduces to the known cases. For example the \( Y \)-space is,
- \( \{ X^\mu \} \)-subspace for, \( \tilde{A} = B = 0, \ A = \tilde{B} = 1 \),
- T-dual of \( \{ X^{\bar{\mu}} \} \)-subspace for, \( \tilde{A} = B = 0, \ A = -\tilde{B} = -1 \),
\{ \bar{X}^\mu \} \text{-subspace for}, \quad \mathcal{A} = \hat{\mathcal{B}} = 0, \quad \tilde{\mathcal{A}} = \mathcal{B} = 1, \\
T\text{-dual of } \{ \bar{X}^\mu \} \text{-subspace for}, \quad \mathcal{A} = \hat{\mathcal{B}} = 0, \quad \tilde{\mathcal{A}} = -\mathcal{B} = -1.

These choices of the matrices satisfy all conditions (48)-(53).

Oscillators of the \( Y \)-space, \( \bar{a}^\mu_n \) and \( \tilde{\bar{a}}^\mu_n \) satisfy the same commutation relations as \( \alpha^\mu_n \) and \( \tilde{\alpha}^\mu_n \),

\[
\begin{cases}
[a^\mu_m, a^\nu_n] = [\tilde{a}^\mu_m, \tilde{a}^\nu_n] = m \delta_{m+n,0} \eta^\mu\nu, \\
[a^\mu_m, \tilde{a}^\nu_n] = 0.
\end{cases}
\]  

Also \( y^\mu \) and \( \Pi^\mu \) are

\[
y^\mu = y^\mu_R + y^\mu_L = \mathcal{A}^\mu_\nu \bar{x}^\nu_R + \tilde{\mathcal{A}}^\mu_\nu \tilde{x}^\nu_R + \mathcal{B}^\mu_\nu \bar{x}^\nu_L + \tilde{\mathcal{B}}^\mu_\nu \tilde{x}^\nu_L,
\]

\[
\Pi^\mu = \Pi^\mu_L + \Pi^\mu_R = \frac{1}{2}(\mathcal{A} + \mathcal{B} + \tilde{\mathcal{A}} + \tilde{\mathcal{B}})^\mu_\nu p^\nu + \frac{1}{2\alpha'}(-\mathcal{A} - \mathcal{B} + \tilde{\mathcal{A}} + \tilde{\mathcal{B}})^\mu_\nu L^\nu.
\]

They satisfy the following commutation relation,

\[
[y^\mu, \Pi^\nu] = i \eta^\mu\nu,
\]

where we have used the relation (30).

Define \( \frac{1}{\alpha'} \Lambda^\mu \) as the difference of \( \Pi^\mu_L \) and \( \Pi^\mu_R \)

\[
\frac{1}{\alpha'} \Lambda^\mu = \Pi^\mu_L - \Pi^\mu_R = \frac{1}{2}(-\mathcal{A} + \mathcal{B} - \tilde{\mathcal{A}} + \tilde{\mathcal{B}})^\mu_\nu p^\nu + \frac{1}{2\alpha'}(-\mathcal{A} - \mathcal{B} - \tilde{\mathcal{A}} + \tilde{\mathcal{B}})^\mu_\nu L^\nu.
\]

We see that

\[
Y^\mu(\sigma + \pi, \tau) - Y^\mu(\sigma, \tau) = 2 \pi \Lambda^\mu,
\]

which means \( Y^\mu(\sigma, \tau) \) is not single valued at \( \sigma \) and \( \sigma + \pi \), therefore it appears like a compact coordinate.

**The dual space \{ Y'^\mu \}**

Now we define the dual of the \( Y \)-space, i.e. \( Y' \)-space as the difference between the left and the right moving parts of \( Y^\mu \)

\[
Y'^\mu(\sigma, \tau) = Y^\mu_L - Y^\mu_R = \mathcal{B}^\mu_\nu \bar{X}^\nu_L + \tilde{\mathcal{B}}^\mu_\nu \bar{X}^\nu_L - \mathcal{A}^\mu_\nu X^\nu_R - \tilde{\mathcal{A}}^\mu_\nu \tilde{X}^\nu_R.
\]

Similar to the \( Y \)-space, the \( Y' \)-space for some special choices of the matrices reduces to the \{ \bar{X}^\mu \} -subspace, \{ \tilde{\bar{X}}^\mu \} -subspace or to the duals of them. According to the eq. (60) for dualizing, it is sufficient to change \( \mathcal{A} \) and \( \tilde{\mathcal{A}} \) as

\[
\begin{cases}
\mathcal{A} \rightarrow -\mathcal{A}, \\
\tilde{\mathcal{A}} \rightarrow -\tilde{\mathcal{A}}.
\end{cases}
\]
According to the eq. (30), \( y^{\mu} \) of the dual space is
\[ y^{\mu} = y^{\mu}_L - y^{\mu}_R = -\mathcal{A}^{\mu}_{\nu} x^{\nu}_R - \tilde{\mathcal{A}}^{\mu}_{\nu} \tilde{x}^{\nu}_R + \mathcal{B}^{\mu}_{\nu} x^{\nu}_L + \tilde{\mathcal{B}}^{\mu}_{\nu} \tilde{x}^{\nu}_L . \] (62)
This with \( \frac{1}{\alpha'} \Lambda^{\mu} \) satisfy the relation
\[ [y^{\mu}, \frac{1}{\alpha'} \Lambda^{\nu}] = i \eta^{\mu\nu} , \] (63)
as expected. Under the dualizing of the \( Y \)-space, i.e. transformations (61), there is the following exchange,
\[ \Pi^{\mu} \leftrightarrow \frac{1}{\alpha'} \Lambda^{\mu} . \] (64)
In studying the branes, we will see other effects of this dualizing.

Now return to the equations (4), (5), (15) and (16). We find the following identifications for \( x_L^{\mu} \) and \( x_R^{\mu} \),
\[
\begin{cases}
  x_L^{\mu} \equiv x_L^{\mu} + \pi(L^{\mu} + \alpha' p^{\mu}) , \\
  x_R^{\mu} \equiv x_R^{\mu} + \pi(L^{\mu} - \alpha' p^{\mu}) .
\end{cases}
\] (65)
Similarly the eqs. (33) and (35) give the identifications
\[
\begin{cases}
  \tilde{x}_L^{\mu} \equiv \tilde{x}_L^{\mu} + \pi(-L^{\mu} + \alpha' p^{\mu}) , \\
  \tilde{x}_R^{\mu} \equiv \tilde{x}_R^{\mu} - \pi(L^{\mu} + \alpha' p^{\mu}) .
\end{cases}
\] (66)
Therefore, there are following identifications for \( y^\mu \) and \( y'^\mu \),
\[ y^{\mu} \equiv y^{\mu} + 2\pi \Lambda^{\mu} , \] (67)
\[ y'^{\mu} \equiv y'^{\mu} + 2\pi \alpha' \Pi^{\mu} . \] (68)
Note that under dualizing, \( y^\mu \) and \( y'^{\mu} \) exchange, therefore \( \Pi^{\mu} \) and \( \frac{1}{\alpha'} \Lambda^{\mu} \) must also be exchanged. This takes place as we saw in (64). Also the dual coordinate \( Y'^{\mu}(\sigma, \tau) \) satisfies the equation,
\[ Y'^{\mu}(\sigma + \pi, \tau) - Y'^{\mu}(\sigma, \tau) = 2\pi \alpha' \Pi^{\mu} , \] (69)
which is expected and is consistent with the identification (68).

Consider the transformations (47) from \( \{X^{\mu}\} \)-subspace to \( \{Y'^{\mu}\} \)-space. These transformations form a group that is isomorphic to the group \( O(2k) \), where \( k \) is the dimension of the compact part of the spacetime \( \{X^{\mu}\} \). This orthogonality implies that the squared mass of the closed string to be invariant.
4 The fermions of Y-space

Consider the R⊗R and the NS⊗NS sectors of type II superstrings. In terms of the oscillating modes the worldsheet fermions of these sectors are

\[
\begin{align*}
\psi^\mu_- &= \sum_{n \in \mathbb{Z}} (d_n^\mu e^{-2in(\tau-\sigma)}) , \\
\psi^\mu_+ &= \sum_{n \in \mathbb{Z}} (\bar{d}_n^\mu e^{-2in(\tau+\sigma)}) ,
\end{align*}
\]

(70)

for the R⊗R sector, and

\[
\begin{align*}
\psi^\mu_- &= \sum_{r \in \mathbb{Z} + \frac{1}{2}} (\tilde{b}_r^\mu e^{-2ir(\tau-\sigma)}) , \\
\psi^\mu_+ &= \sum_{r \in \mathbb{Z} + \frac{1}{2}} (\tilde{d}_r^\mu e^{-2ir(\tau+\sigma)}) ,
\end{align*}
\]

(71)

for the NS⊗NS sector. According to the worldsheet supersymmetry, the tilde fermions \( \tilde{\psi}^\mu_\pm \) corresponding to the \( \{\tilde{X}^\mu\}\)-subspace, are

\[
\begin{align*}
\tilde{\psi}^\mu_- &= \sum_{n \in \mathbb{Z}} (\tilde{d}_n^\mu e^{-2in(\tau-\sigma)}) , \\
\tilde{\psi}^\mu_+ &= \sum_{n \in \mathbb{Z}} (\tilde{d}_n^\mu e^{-2in(\tau+\sigma)}) ,
\end{align*}
\]

(72)

for the R⊗R sector, and

\[
\begin{align*}
\tilde{\psi}^\mu_- &= \sum_{r \in \mathbb{Z} + \frac{1}{2}} (\tilde{b}_r^\mu e^{-2ir(\tau-\sigma)}) , \\
\tilde{\psi}^\mu_+ &= \sum_{r \in \mathbb{Z} + \frac{1}{2}} (\tilde{b}_r^\mu e^{-2ir(\tau+\sigma)}) ,
\end{align*}
\]

(73)

for the NS⊗NS sector. The fermions (72) and (73), are worldsheet parity transformed \((\sigma \rightarrow -\sigma)\) of the worldsheet fermions (70) and (71).

Let \( \chi_- (\sigma, \tau) \) and \( \chi_+ (\sigma, \tau) \) be the right and the left moving components of the fermions of the Y-space, therefore

\[
\begin{align*}
\chi^\mu_- (\tau - \sigma) &= A^\mu_\rho \psi^\rho_- + \tilde{A}^\mu_\rho \tilde{\psi}^\rho_- , \\
\chi^\mu_+ (\tau + \sigma) &= B^\mu_\rho \psi^\rho_+ + \tilde{B}^\mu_\rho \tilde{\psi}^\rho_+ .
\end{align*}
\]

(74)

For the special choices of the matrices that mentioned for the Y-space, these fermions reduce to the fermions of \( \{X^\mu\}\)-subspace, \( \{\tilde{X}^\mu\}\)-subspace, or to the fermions of their dual spaces. Oscillators of the Y-space fermions are,

\[
\begin{align*}
D^\mu_n &= A^\mu_\rho d^\rho_n + \tilde{A}^\mu_\rho \tilde{d}^\rho_n , \\
\tilde{D}^\mu_n &= B^\mu_\rho d^\rho_n + \tilde{B}^\mu_\rho \tilde{d}^\rho_n ,
\end{align*}
\]

(76)
for the R⊗R sector. For the NS⊗NS sector, \( d_n^\mu \) and \( \tilde{d}_n^\mu \) should be replaced with \( b_r^\mu \) and \( \tilde{b}_r^\mu \), respectively. Oscillators \( D_n^\mu \) and \( \tilde{D}_n^\mu \) satisfy the same anti-commutation relations as \( d_n^\mu \) and \( \tilde{d}_n^\mu \),

\[
\begin{cases}
\{ D_m^\mu, D_n^\nu \} = \{ \tilde{D}_m^\mu, \tilde{D}_n^\nu \} = \delta_{m+n,0} \eta^{\mu\nu}, \\
\{ D_n^\mu, \tilde{D}_n^\nu \} = 0.
\end{cases}
\]  

(77)

Similar relations hold for the NS⊗NS sector.

For the dual space \( Y' \), also there are corresponding fermions. The right and the left moving components of these fermions are

\[
\begin{cases}
\chi_{-n}^{\prime \mu} = -\chi_{-n}^{\mu}, \\
\chi_{+n}^{\prime \mu} = \chi_{+n}^{\mu},
\end{cases}
\]  

(78)

where, we have used the transformations (61).

Again return to the mass operator of the closed string. In the R⊗R sector, for instance its fermionic part has the form,

\[
\alpha' M_f^2 = 2 \sum_{n=1}^{\infty} n (d_{-n}^\mu d_{n\bar{\mu}} + \tilde{d}_{-n}^{\bar{\mu}} \tilde{d}_{n\mu}) + 2 \sum_{n=1}^{\infty} n (d_{-n}^{\prime \mu} d_{n\mu'} + \tilde{d}_{-n}^{\prime \bar{\mu}} \tilde{d}_{n\mu'})
+ \text{superghost part},
\]  

(79)

where \( \mu' \notin \{ \bar{\mu} \} \). Under the transformations

\[
\begin{cases}
\psi_{-n}^\bar{\mu} \rightarrow \chi_{-n}^{\bar{\mu}}, \\
\psi_{+n}^\bar{\mu} \rightarrow \chi_{+n}^{\bar{\mu}},
\end{cases}
\]  

(80)

or equivalently

\[
\begin{cases}
d_{n}^{\mu} \rightarrow D_{n}^{\mu}, \\
\tilde{d}_{n}^{\bar{\mu}} \rightarrow \tilde{D}_{n}^{\bar{\mu}},
\end{cases}
\]  

(81)

the fermionic part of the mass operator is also invariant. This invariance does not impose new conditions on the matrices \( A, \tilde{A}, B \) and \( \tilde{B} \). This also holds for the NS⊗NS sector.

5 Transformed branes

To study the effects of the transformations (47) and (80) on a \( D_p \)-brane, we discuss the transformations for the brane directions and the transformations for the transverse directions of the brane. For simplicity we neglect the transformations that mix coordinates \( \{ X_R^\alpha \} \) with
\( \{X_R^i\} \cup \{\bar{X}_R^i\} \) and also \( \{X_L^i\} \cup \{\bar{X}_L^i\} \). Similarly for the left moving coordinates.

Consider the following transformations for the compact directions along the brane i.e. \( \{X^\alpha\} \),

\[
\begin{align*}
X_R^\alpha & \rightarrow Y_R^\alpha = A_\alpha^\beta X_R^\beta + \tilde{A}_\alpha^\beta \bar{X}_R^\beta , \\
X_L^\alpha & \rightarrow Y_L^\alpha = B_\alpha^\beta \bar{X}_L^\beta + \tilde{B}_\alpha^\beta X_L^\beta ,
\end{align*}
\]

(82)

for the bosonic part, and

\[
\begin{align*}
\psi_-^\alpha & \rightarrow \chi_-^\alpha = A_\alpha^\beta \psi_-^\beta + \tilde{A}_\alpha^\beta \bar{\psi}_-^\beta , \\
\psi_+^\alpha & \rightarrow \chi_+^\alpha = B_\alpha^\beta \bar{\psi}_+^\beta + \tilde{B}_\alpha^\beta \psi_+^\beta ,
\end{align*}
\]

(83)

for the fermionic part. The four matrices \( A, \tilde{A}, B \) and \( \tilde{B} \) satisfy the conditions (48)-(53).

For some of the compact transverse directions of the brane i.e. \( \{X^i\} \), we introduce the transformations

\[
\begin{align*}
X_R^i & \rightarrow Y_R^i = A_i^\alpha X_R^\alpha + \tilde{A}_i^\alpha \bar{X}_R^\alpha , \\
X_L^i & \rightarrow Y_L^i = B_i^\alpha \bar{X}_L^\alpha + \tilde{B}_i^\alpha X_L^\alpha ,
\end{align*}
\]

(84)

for the bosonic part, and

\[
\begin{align*}
\psi_-^i & \rightarrow \chi_-^i = A_i^\alpha \psi_-^\alpha + \tilde{A}_i^\alpha \bar{\psi}_-^\alpha , \\
\psi_+^i & \rightarrow \chi_+^i = B_i^\alpha \bar{\psi}_+^\alpha + \tilde{B}_i^\alpha \psi_+^\alpha ,
\end{align*}
\]

(85)

for the fermionic part. Again four matrices \( A', \tilde{A}', B' \) and \( \tilde{B}' \) satisfy the conditions (48)-(53).

Now we apply the transformations (82)-(85) to the boundary state equations corresponding to a Dp-brane, i.e. equations (19)-(22). For the bosonic part there are,

\[
\begin{align*}
\left\{(A + \tilde{B})_\alpha^\beta \partial_\tau X^\beta - (A - \tilde{B})_\alpha^\beta \partial_\sigma X^\beta \\
+ (\tilde{A} + B)_\alpha^\beta \partial_\tau \bar{X}^\beta - (\tilde{A} - B)_\alpha^\beta \partial_\sigma \bar{X}^\beta \right\}_{\tau = 0} |B_t\rangle = 0 , \\
(\partial_\tau X^\alpha)_{\tau = 0} |B_t\rangle = 0 ,
\end{align*}
\]

(86)

where the set \( \{X^\alpha'\} \) shows the non-compact directions along the brane. The state \( |B_t\rangle \) is transformed boundary state that describes the new brane. Also the transformed form of the equation (20) is given by

\[
\begin{align*}
\left\{(A' - \tilde{B}')_i^j \partial_\tau X^j - (A' + \tilde{B}')_i^j \partial_\sigma X^j \\
+ (\tilde{A}' - B')_i^j \partial_\tau \bar{X}^j - (\tilde{A}' + B')_i^j \partial_\sigma \bar{X}^j \right\}_{\tau = 0} |B_t\rangle = 0 , \\
(\partial_\sigma X^i')_{\tau = 0} |B_t\rangle = 0 ,
\end{align*}
\]

(87)

where \( X^i' \) is a non-compact direction transverse to the brane.
Transformations on the fermionic equations (21) and (22), give the following boundary state equations,

\[
\begin{align*}
\left( A^{\dot{\alpha}}_{\dot{\beta}} \psi_{-}^- - i \eta B^{\dot{\alpha}}_{\dot{\beta}} \psi_{+}^+ + \tilde{A}^{\dot{\alpha}}_{\dot{\beta}} \tilde{\psi}_{-}^- - i \eta B^{\dot{\alpha}}_{\dot{\beta}} \tilde{\psi}_{+}^+ \right) | B_t \rangle = 0 , \\
(\psi_{-}^- - i \eta \psi_{+}^+ )_{\tau=0} | B_t \rangle = 0 , \\
\end{align*}
\]

(88)

For some special matrices, the eqs. (86)-(89) reduce to the known cases. For example for \( \tilde{A} = B = A' = B' = 0 \) and \( A = -\tilde{B} = -A' = -\tilde{B}' = 1 \), the directions \( \{ X^{\dot{\alpha}} \} \) become perpendicular to the brane. This is usual T-duality along the compact directions of the brane. The case \( \tilde{A} = B = \tilde{A}' = B' = 0 \) and \( A = \tilde{B} = A' = -\tilde{B}' = 1 \), is T-duality along the directions \( \{ X^{\tilde{\alpha}} \} \). In the next section other special cases will be discussed.

**Boundary state**

To know the properties of the state \( | B_t \rangle \), we obtain this state from the eqs. (86)-(89). In terms of oscillators, the first equations in (86) and (88) have the form

\[
\left( A^{\dot{\alpha}}_{\dot{\beta}} \alpha_{n}^+ + B^{\dot{\alpha}}_{\dot{\beta}} \beta_{-n}^+ + \tilde{A}^{\dot{\alpha}}_{\dot{\beta}} \tilde{\alpha}_{n}^+ + \tilde{B}^{\dot{\alpha}}_{\dot{\beta}} \tilde{\beta}_{-n}^+ \right) | B_t \rangle = 0 ,
\]

(90)

for the bosonic part, and

\[
\left( A^{\dot{\alpha}}_{\dot{\beta}} d_{n}^+ + \tilde{A}^{\dot{\alpha}}_{\dot{\beta}} \tilde{d}_{n}^+ - i \eta (B^{\alpha}_{\beta} d_{-n}^- + \tilde{B}^{\alpha}_{\beta} \tilde{d}_{-n}^-) \right) | B_t \rangle = 0 ,
\]

(91)

for the \( R \otimes R \) sector of the fermionic part. For the \( NS \otimes NS \) sector there is

\[
\left( A^{\dot{\alpha}}_{\dot{\beta}} b_{r}^+ + \tilde{A}^{\dot{\alpha}}_{\dot{\beta}} \tilde{b}_{r}^+ - i \eta (B^{\alpha}_{\beta} b_{-r}^- + \tilde{B}^{\alpha}_{\beta} \tilde{b}_{-r}^-) \right) | B_t \rangle = 0 .
\]

(92)

The first equations in (87) and (89), in terms of oscillators, can be obtained from the eqs. (90)-(92) by the changes,

\[
\begin{align*}
\tilde{\alpha} \rightarrow \tilde{\gamma} , & \quad \tilde{\beta} \rightarrow \tilde{\jmath} , \\
A \rightarrow A' , & \quad \tilde{A} \rightarrow \tilde{A}' , \quad B \rightarrow -B' , \quad \tilde{B} \rightarrow -\tilde{B}'.
\end{align*}
\]

(93)

Equation (90) for \( n = 0 \), gives

\[
\Pi^{\dot{\alpha}} = 0 ,
\]

(94)

this means that in the \( Y \)-space closed string has no momentum along the \( Y^{\dot{\alpha}} \)-direction. According to the changes (93), we also have

\[
\Lambda^{\tilde{\gamma}} = 0 .
\]

(95)
In other words we obtain a relation between momentum (momentum numbers) and winding numbers of closed string, around the compact directions \( \{X^\alpha\} \) and \( \{\overline{X}^\alpha\} \),

\[
p^\alpha = -\frac{1}{\alpha'} f^{\alpha}_\beta L^\beta , \tag{96}
\]

\[
f = (A + B + \tilde{A} + \tilde{B})^{-1}(-A - B + \tilde{A} + \tilde{B}) , \tag{97}
\]

\[
p^\beta = -\frac{1}{\alpha'} f^{\beta}_\gamma L^\gamma , \tag{98}
\]

\[
f' = (A' - B' + \tilde{A}' - \tilde{B}')^{-1}(-A' + B' + \tilde{A}' - \tilde{B}') . \tag{99}
\]

In the special cases such as \( \tilde{A} = B = \tilde{A}' = B' = 0 \), or \( A = \tilde{B} = A' = \tilde{B}' = 0 \), the matrices \( f \) and \( f' \) are antisymmetric, therefore they can be interpreted as background fields on the brane. For more detail of the relation between momentum and winding numbers of closed string see Ref. [8].

As next we will concentrate on the solutions of the R⊗R sector. The solutions of the NS⊗NS sector can be obtained by similar method. First we separate the eqs. (90) and (91) for positive, negative and zero values of the integer “\( n \)”. After using of the relations (48)-(53) for the matrices \( A, B, \tilde{A}, \) and \( \tilde{B} \) we obtain

\[
\left( \alpha_n^\alpha + (A^T \tilde{B} + B^T \tilde{A})^\alpha_\beta \tilde{\alpha}_n^\beta + (A^T B + B^T A)^\alpha_\beta \tilde{\alpha}_n^\beta \right) |B_t\rangle = 0 , \tag{100}
\]

\[
\left( \tilde{\alpha}_n^\alpha + (A^T B + B^T A)^\alpha_\beta \tilde{\alpha}_n^\beta + (\tilde{A}^T \tilde{B} + \tilde{B}^T \tilde{A})^\alpha_\beta \tilde{\alpha}_n^\beta \right) |B_t\rangle = 0 , \tag{101}
\]

\[
\left( (A + B)^\alpha_\beta \tilde{\alpha}_n^\beta + (\tilde{A} + \tilde{B})^\alpha_\beta \tilde{\alpha}_n^\beta \right) |B_t\rangle = 0 , \tag{102}
\]

for the bosonic part, and

\[
\left( d_n^\alpha - i\eta [(A^T B - B^T A)^\alpha_\beta d_n^\beta + (A^T \tilde{B} - B^T \tilde{A})^\alpha_\beta \tilde{d}_n^\beta] \right) |B_t\rangle = 0 , \tag{103}
\]

\[
\left( \tilde{d}_n^\alpha - i\eta [(\tilde{A}^T \tilde{B} - \tilde{B}^T \tilde{A})^\alpha_\beta \tilde{d}_n^\beta + (\tilde{A}^T B - B^T A)^\alpha_\beta d_n^\beta] \right) |B_t\rangle = 0 , \tag{104}
\]

\[
\left( d_0^\alpha + [(A - i\eta B)^{-1} (\tilde{A} - i\eta \tilde{B})]^\alpha_\beta \tilde{d}_0^\beta \right) |B_t\rangle = 0 , \tag{105}
\]
for the fermionic part. For the directions \( \{X^i\} \), apply the changes (93) to the eqs. (100)-(105).

The complete boundary state in each sector is the following product

\[
|B_t\rangle_{R,NS} = |B_t\rangle_b|B_{gh}\rangle|B_t^{(f)}\rangle_{R,NS}|B_{sgh}\rangle_{R,NS},
\]

where \( |B_{gh}\rangle \) and \( |B_{sgh}\rangle \) are the ghost and the superghost parts of the boundary states respectively [16, 17]. These states do not change under the spacetim e transformations (47) and (80). The ghost part of the boundary state is

\[
|B_{gh}\rangle = \exp \left\{ \sum_{n=1}^{\infty} (c_{-n} \bar{b}_{-n} - b_{-n} \bar{c}_{-n}) \right\} \frac{c_0 + \bar{c}_0}{2} |P = 1\rangle |\bar{P} = 1\rangle.
\]

(107)

For the superghost parts there are

\[
|B_{sgh}\rangle_R = \exp \left\{ i\eta \sum_{n=1}^{\infty} (\gamma_{-n} \bar{\beta}_{-n} - \beta_{-n} \bar{\gamma}_{-n}) + i\eta \gamma_0 \bar{\beta}_0 \right\} |P = -1/2 , \bar{P} = -3/2\rangle,
\]

(108)

for the \( R \otimes R \) sector, and

\[
|B_{sgh}\rangle_{NS} = \exp \left\{ i\eta \sum_{r=1/2}^{\infty} (\gamma_{r} \bar{\beta}_{r} - \beta_{r} \bar{\gamma}_{r}) \right\} |P = -1 , \bar{P} = -1\rangle,
\]

(109)

for the \( NS \otimes NS \) sector. The ghost and the superghost vacuums are in the \((1, 1), (-1/2, -3/2)\) and \((-1, -1)\) pictures respectively [14].

The bosonic part of the boundary state is,

\[
|B_t\rangle_b = \frac{T_p}{2} N_b \exp \left\{ - \sum_{n=1}^{\infty} \frac{1}{n} \left[ 2 \alpha_n^{\alpha} (A^T B)_{\bar{\alpha} \bar{\beta}} \alpha_n^{\bar{\beta}} - 2 \bar{\alpha}_n^{\bar{\alpha}} (\bar{A}^T \bar{B})_{\alpha \bar{\beta}} \bar{\alpha}_n^{\alpha} + \alpha_n^{' \alpha} \bar{\alpha}_n^{\bar{\alpha}} \right] \right\} |0\rangle,
\]

(110)

where \( N_b \) is an appropriate normalizing factor and \( T_p \) is the tension of the initial \( D_p \)-brane.

The two terms containing the indices \( \alpha' \) and \( i' \), are the solutions of the second equations of (86) and (87), for the untransformed directions \( \{X^{\alpha'}\} \) and \( \{X^{i'}\} \).

The fermionic part of the \( R \otimes R \) sector has the solution,

\[
|B_t^{(f)}\rangle_R = N_f \exp \left\{ i\eta \sum_{n=1}^{\infty} \left[ d_{-n}^{\bar{\alpha}} (A^T B)_{\bar{\alpha} \bar{\beta}} d_{-n}^{\bar{\beta}} + d_{-n}^{\bar{\beta}} (\bar{A}^T \bar{B})_{\alpha \bar{\beta}} \bar{d}_{-n}^{\alpha} \right] + d_{-n}^{\alpha} (A^T \bar{B} - B^T \bar{A})_{\bar{\alpha} \bar{\beta}} \bar{d}_{-n}^{\bar{\beta}} - d_{-n}^{\bar{\beta}} (\bar{A}^T \bar{B}^T)_{\bar{\alpha} \bar{\beta}} \bar{d}_{-n}^{\alpha} - d_{-n}^{\alpha} (\bar{A}^T B')_{\bar{\alpha} \bar{\beta}} \bar{d}_{-n}^{\bar{\beta}} \right\} |B_t\rangle_R^{(0)},
\]

(111)
where $N_f$ is a normalizing factor. The state $|B_t\rangle_R^{(0)}$ is the solution of the zero mode part. The boundary state equation of it, is

$$
(d^\mu_0 - i\eta S^\mu_\nu \bar{d}^\nu_0)|B_t\rangle_R^{(0)} = 0,
$$

(112)

where the matrix $S^\mu_\nu$ has the components,

$$
S^\bar{\alpha}_{\bar{\beta}} \equiv Q^\bar{\alpha}_{\bar{\beta}} = [(A - i\eta B)^{-1}(\bar{B} + i\eta A)]^\bar{\alpha}_{\bar{\beta}},
$$

(113)

$$
S^\bar{i}_{\bar{j}} \equiv Q^\bar{i}_{\bar{j}} = [(A' + i\eta B')^{-1}(-\bar{B}' + i\eta A')]^\bar{i}_{\bar{j}},
$$

(114)

$$
S^{\alpha'_{\beta'}} = \delta^{\alpha'_{\beta'}}, \quad S^{\bar{i'}_{\bar{j}'}} = -\delta^{\bar{i'}_{\bar{j}'}} ,
$$

(115)

and the other components of the matrix $S$ are zero. According to the conditions (51)-(53), both $Q$ and $Q'$ are orthogonal.

The solution of the eq. (112) can be written as

$$
|B_t\rangle_R^{(0)} = \mathcal{M}^{(n)}_{CD} |C\rangle |\bar{D}\rangle ,
$$

(116)

where the vacuum for the fermionic zero modes $d^\mu_0$ and $\bar{d}^\mu_0$ is 

$$
|C\rangle |\bar{D}\rangle = \lim_{z,\bar{z} \to 0} S^C(z)\bar{S}^D(\bar{z})|0\rangle ,
$$

(117)

in which $S^C$ and $\bar{S}^D$ are the spin fields in the 32-dimensional Majorana representation. By substituting (116) in the eq. (112) and using the Ref. [12] for the action of $d^\mu_0$ and $\bar{d}^\mu_0$ on the vacuum $|C\rangle |\bar{D}\rangle$, we obtain an equation for the $32 \times 32$-matrix $\mathcal{M}^{(n)}$,

$$
(\Gamma^\mu)^T \mathcal{M}^{(n)} - i\eta S^{\mu_\nu} \Gamma_{11} \mathcal{M}^{(n)} \Gamma^{\nu} = 0 .
$$

(118)

The matrix $\mathcal{M}^{(n)}$ can be written as

$$
\mathcal{M}^{(n)} = C' \Gamma^0 \Gamma^{a_1} ... \Gamma^{a_n} \left(\frac{1 + i\eta \Gamma_{11}}{1 + i\eta}\right)G ,
$$

(119)

where $C'$ is the charge conjugation matrix [12], and the indices $\{a_1, a_2, ..., a_n\}$ are

$$
\{\bar{\alpha}\} \cup \{\alpha'\} \cup \{\bar{i}\} = \{0, a_1, ..., a_n\} .
$$

(120)

By separating the eq. (118) for various indices $\{\mu\}$, the matrix $G$ must satisfy the equations,

$$
\begin{cases}
\Gamma^\mu G = GT^\mu, \\
\Gamma^{\alpha'} G = GT^{\alpha'}, \\
\Gamma_i G = Q^\bar{i}_{\bar{j}} G T^{\bar{j}}, \\
\Gamma^\alpha G = Q^{\alpha}_{\bar{\beta}} G T^{\bar{\beta}} .
\end{cases}
$$

(121)
Therefore, the matrix $G$ has the form

$$G = \left( e^{\frac{i}{2} F_{\bar{\alpha} \bar{\beta}} \Gamma^\alpha \Gamma^\beta} \right) \left( e^{\frac{1}{2} F'_{ij} \Gamma^i \Gamma^j} \right),$$

with the convention that all gamma matrices anticommute. Hence there are a finite number of terms in each factor. The matrices $F_{\bar{\alpha} \bar{\beta}}$ and $F'_{ij}$ are

$$F = (Q + 1)^{-1}(Q - 1) = \left( A + \bar{B} + i\eta(\bar{A} - B) \right)^{-1} \left( -A + \bar{B} + i\eta(\bar{A} + B) \right),$$

$$F' = (Q' + 1)^{-1}(Q' - 1) = \left( A' - \bar{B}' + i\eta(\bar{A}' + B') \right)^{-1} \left( -A' - \bar{B}' + i\eta(\bar{A}' - B') \right),$$

since $Q$ and $Q'$ are orthogonal, $F$ and $F'$ are antisymmetric.

The boundary state of the NS\$\otimes$NS sector fermions is

$$|B_{i}^{(f)}\rangle_{NS} = \exp \left\{ i\eta \sum_{r=1/2}^{\infty} \left[ b^\alpha_{-r}(A^T B)_{\bar{\alpha} \bar{\beta}} b^\beta_{-r} + \bar{b}_{-r}^\alpha(\bar{A}^T \bar{B})_{\bar{\alpha} \bar{\beta}} \bar{b}^\beta_{-r} \right. \right.$$  

$$\left. + b^\alpha_{-r}(A^T \bar{B} - B^T \bar{A})_{\bar{\alpha} \bar{\beta}} \bar{b}^\beta_{-r} - \bar{b}_{-r}^i(A^T B')_{ij} \bar{b}^j_{-r} - \bar{b}_{-r}^i(\bar{A}'^T \bar{B}')_{ij} \bar{b}^j_{-r} \right.$$  

$$\left. - b_{-r}^i(A^T \bar{B}' - B^T \bar{A}')_{ij} \bar{b}^j_{-r} + b_{-r}^\alpha \bar{b}_{-ra'} - b_{-r}^i \bar{b}_{-ra'} \right\} |0\rangle,$$

The boundary state corresponding to a D-brane [12, 13, 14, 15] or corresponding to a brane with internal background B-field [2, 3, 11, 12], contains those closed superstring states that the left moving and the right moving oscillator modes couple. But the boundary states (110), (111) and (125) have three kinds of the closed superstring states, which contain left-right, left-left and right-right coupling of oscillator modes. In the interaction of the two transformed branes, these three kinds of the superstring states contribute. Therefore these brane interactions would be more general than the mixed branes interactions [8, 9].

6 Special cases

6.1. The first special case

Now let us study the special case

$$\begin{align*}
\bar{A} &= B = 0, \\
\bar{A}' &= B' = 0.
\end{align*}$$

According to the conditions (48)-(53), the matrices $A$, $A'$, $\bar{B}$ and $\bar{B}'$ are orthogonal, and the matrices $Q$, $Q'$, $F$ and $F'$ are real, with $F = f$ and $F' = f'$. Also the boundary state
equations (86)-(89) reduce to the usual forms of the mixed brane equations. For this case, under the transformations (47) and (80) the virasoro operators, the super virasoro operators and BRST charge are invariant. For example the zero mode of the super virasoro operator of the $R \otimes R$ sector
\[
L_0^{(\alpha,d)} = \frac{1}{4} \alpha' p^\mu p_\mu - \frac{1}{2} p^\beta L_\beta + \frac{1}{4} \alpha' L^\mu L_\mu + \sum_{n=1}^{\infty} \left( \alpha^\mu_{-n} \alpha_n \mu + nd_{-n} d_n \right),
\]
under the transformations (47) and (80), is invariant. Similarly $\tilde{L}_0^{(\tilde{\alpha},\tilde{d})}$ is invariant. Therefore, the Hamiltonian of the closed superstring is an invariant operator.

The normalizing factors reduce to the normalizations of the mixed branes
\[
N_b = 1/N_f = \sqrt{\det(1 - F) \det(1 - F')} = \left( \frac{2^{k_\alpha + k_i} \det A \det A'}{\det(A + B) \det(A' - B')} \right)^{1/2},
\]
where $k_\alpha (k_i)$ is the number of the compact directions along (transverse to) the initial $D_p$-brane.

The boundary states (110), (111) and (125) also take simple forms
\[
|B_t\rangle_R = \frac{T_p}{2} N_b \exp \left\{ \frac{\sum_{n=1}^{\infty} \left( - \frac{1}{n} \alpha^\mu_{-n} S_{\mu \nu} \tilde{\alpha}^\nu_{-n} + i \eta d_{-n} \alpha \nu_{-n} \right) }{ \det(A + B) \det(A' - B')} \right\} |B_t\rangle_R^{(0)} ,
\]
\[
|B_t\rangle_{NS} = \frac{T_p}{2} N_b \exp \left\{ \sum_{n=1}^{\infty} \left( - \frac{1}{n} \alpha^\mu_{-n} S_{\mu \nu} \tilde{\alpha}^\nu_{-n} \right) + i \eta \sum_{r=1/2}^{\infty} \left( b^\mu_{-r} S_{\mu \nu} b^\nu_{-r} \right) \right\} |0\rangle ,
\]
where $S_{\mu \nu}$ is given by the eqs. (113)-(115), in which the eq. (126) must insert in to them. For $|B_t\rangle_R^{(0)}$, insert (126) to (123) and (124) to obtain the matrix $G$ for this case. For appropriate matrices $A, A', B$ and $\tilde{B}'$, the boundary states (129) and (130) describe a mixed brane, along the directions $\{X^\alpha\} \cup \{X^i\}$, with the field strength $(F^\alpha, F^n)$, whose components are on the subspaces $\{X^\alpha\}$ and $\{X^i\}$. Note that one can consider more special $A, A', B$ and $B'$, such that some of the directions $\{X^\alpha\}$ and $\{X^i\}$ be perpendicular to (or along) the brane.
For example for $A = \text{diag}(1, -1, -1, \ldots, 1)$, two directions of $\{X^\alpha\}$ become perpendicular to the brane. For $A = -1$ and $B = 1$, we have usual T-duality on the directions $\{X^\alpha\}$. Similarly $A' = -1$ and $\tilde{B}' = 1$ give T-duality on the directions $\{X^i\}$.

6.2. The second special case

The other interesting case is,
\[
\begin{cases}
A = \tilde{B} = 0,
A' = \tilde{B}' = 0.
\end{cases}
\]
Now the remaining matrices $\tilde{A}, B, \tilde{A}'$ and $B'$ are orthogonal. Also the matrices $Q, Q', \mathcal{F}$ and $\mathcal{F}'$ are real. If all directions transform, i.e. the indices $\alpha'$ and $i'$ disappear, the super virasoro operators $L^{(\alpha,d)}_m$ and $\tilde{L}^{(\tilde{\alpha},\tilde{d})}_m$ change to each other. Also the exchange takes place between $F^{(\alpha,d)}_m$ and $\tilde{F}^{(\tilde{\alpha},\tilde{d})}_m$, where

$$F^{(\alpha,d)}_m = \sum_{n=-\infty}^{\infty} \alpha_{-n} d_{m+n},$$
$$\tilde{F}^{(\tilde{\alpha},\tilde{d})}_m = \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{-n} \tilde{d}_{m+n}. \quad (132)$$

The boundary states (110), (111) and (125) take the forms,

$$|B_t\rangle_b = \frac{T_b}{2} N_b \exp \left\{ \sum_{n=1}^{\infty} \left( -\frac{1}{n} \alpha_{-n} \mu S^{(b)}_{\mu \nu} \tilde{\alpha}_{-n} \nu \right) \right\} |0\rangle, \quad (133)$$
$$|B_t^{(f)}\rangle_R = N_f \exp \left\{ i \eta \sum_{n=1}^{\infty} \left( d_{-n}^{\mu} S^{(f)}_{\mu \nu} \tilde{d}_{-n}^{\nu} \right) \right\} |B_t\rangle_{R}^{(0)}, \quad (134)$$
$$|B_t^{(f)}\rangle_{NS} = \exp \left\{ i \eta \sum_{r=1/2}^{\infty} \left( b_{-r}^{\mu} S^{(f)}_{\mu \nu} \tilde{b}_{-r}^{\nu} \right) \right\} |0\rangle, \quad (135)$$

where the normalizing factors are

$$N_b = 1/N_f = \left( \frac{2^{k_{\alpha}+k_{i}} \det \tilde{A} \det \tilde{A}'}{\det(\tilde{A} + B) \det(\tilde{A}' - B')} \right)^{1/2}. \quad (136)$$

Also the matrices $S^{(b)}_{\mu \nu}$ and $S^{(f)}_{\mu \nu}$ are

$$S^{(b)}_{\alpha \beta} = (B^T \tilde{A})^{\tilde{\alpha}}_{\tilde{\beta}},$$
$$S^{(b)}_{\alpha' \beta'} = \delta^{\alpha'}_{\beta'},$$
$$S^{(f)}_{\bar{i} j} = -(B'^T \tilde{A}')^{\bar{i}}_{\bar{j}},$$
$$S^{(f)}_{i' j'} = -\delta_{i' j'}. \quad (137)$$

$$S^{(f)}_{\alpha \beta} = -S^{(b)}_{\bar{i} j},$$
$$S^{(f)}_{\alpha' \beta'} = S^{(b)}_{\alpha' \beta'},$$
$$S^{(f)}_{i j} = -S^{(b)}_{\bar{i} j},$$
$$S^{(f)}_{i' j'} = S^{(b)}_{i' j'}. \quad (138)$$

the other components of $S^{(b)}$ and $S^{(f)}$ are zero. For zero modes state in (134), insert the eq. (131) to the eqs. (123) and (124) to obtain the matrix $G$ in (122). Two matrices that
appear in the bosonic part (i.e. $S_{(b)}$) and in the fermionic part (i.e. $S_{(f)}$), are different. For this reason we call the corresponding brane to the states (133)-(135) as a “modified mixed brane”.

According to the eqs. (131) and (81), the opposite signs of $(S_{(f)}^{(f)}, S_{(f)}^{(f)})$ of the fermionic part with respect to the bosonic part i.e. $(S_{(b)}^{(b)}, S_{(b)}^{(b)})$, have originated from the exchange of $\{d^{a}_{n}\}$ with $\{d^{a}_{n}\}$, and also $\{\tilde{d}^{a}_{n}\}$ with $\{\tilde{d}^{a}_{n}\}$. Similarly these exchanges take place for the fermions of the NS$\otimes$NS sector.

7 Conclusion

We found that the $Y$-space (which contains the compact part of the spacetime and its worldsheet parity transformed) is an interesting space for studying the superstring theory and its target space dualities. For the special cases this space reduces to the T-dual of the compact part of the spacetime, or reduces to the T-dual of the worldsheet parity transformed of the compact part of the spacetime. The $Y$-space appears similar to a compact space, therefore its dualities give some expected results corresponding to a compact space.

By going from the spacetime to the $Y$-space, we obtained the boundary state of a closed superstring, emitted from a brane that is more general than the boundary state corresponding to a mixed brane. In other words, a brane with background NS$\otimes$NS B-field in spacetime appears as a D-brane in $Y$-space. For example in the $Y$-space, closed string has no momentum along the brane, which is similar to the closed string emitted from a wrapped mixed brane in the spacetime.

By neglecting the worldsheet parity part of the $Y$-space, the transformed brane reduces to a mixed brane. In this case the super virasoro operators and the BRST charge are invariant. Also by putting away the compact part of the spacetime, we obtained the boundary state corresponding to a modified mixed brane. The background fields of the transformed branes, are special combinations of the transformation matrices.

Acknowledgment

The author would like to thank H. Arfaei for useful discussions and N. Sadooghi for reading the manuscript.

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