On relaxed averaged alternating reflections (RAAR) algorithm for phase retrieval with structured illumination

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Abstract

In this paper, we consider the phase retrieval problem with structured illumination, which leads to a pixel-dependent deterministic phase shift term in the physical model. The existence of the phase shift term can ease the numerical algorithm for phase retrieval. The relaxed averaged alternating reflections (RAAR) algorithm is modified to adapt to two or more diffraction patterns, and the modified RAAR algorithm operates in the Fourier domain rather than the space domain. The RAAR is motivated by a linear combination of the reflection projection and projection onto the measurement space, with parameter $\beta$ trading off the two projections. Although the local convergence of the RAAR algorithm with an initialization within the basin of attraction is proved for $0 < \beta \leq 1$, the numerical performance with a random initialization varies with different values of $\beta$. Numerical simulations are presented to demonstrate the effectiveness and stability of the algorithm with $0.5 < \beta < 1$, compared to the ER ($\beta = 0.5$) method and Douglas–Rachford ($\beta = 1$) method. The numerical global convergence of the RAAR with $0.5 < \beta < 1$ is also illustrated in our tests.

Keywords: phase retrieval, iterative projective method, RAAR, local convergence

(Some figures may appear in colour only in the online journal)

1. Introduction

The phase retrieval problem arises in many engineering and science applications, such as x-ray crystallography [29], electron microscopy [30], x-ray coherent diffraction imaging [34], optics [22], astronomy [14] and antenna characterization [10], just to name a few. In these applications, one often has recorded the Fourier transform intensity of a complex signal, while
the phase information is lost. The recovery of the signal from the intensity of its Fourier transform is called phase retrieval. We refer the reader to the recent survey papers [19, 34] for the recent progress on this problem.

There are two fundamental issues that accompany the phase retrieval problem. The first one is the non-uniqueness of the solution. Clearly, the solution to phase retrieval has the following three ‘trivial associates’: the solutions up to a unit magnitude complex coefficient, a shift in space-domain and a conjugate reflection through the origin. Fortunately, these solutions do not change the structure of the solution, and they are accepted in practice. References on theoretical results in terms of uniqueness can be found in [1–3, 35] for the continuous model and in [4, 17, 32, 33] for the discrete model. These references point out that the solution is almost relatively unique in multidimensional cases, up to the above three ‘trivial associates’. In literature [26], it is pointed out that, these uniqueness results are of fundamental importance, but they are not applied to numerical algorithms, particularly for noisy data. The second issue is how to design efficient numerical algorithms. The most widely used methods are perhaps error reduction (ER) and its variants, such as hybrid input output (HIO) [14], hybrid projection reflection (HPR) [25], relaxed averaged alternating reflections (RAAR) [24] and the difference map. Since these methods involve sequential projection onto the constraint sets, they are called iterative projection methods. Although these iterative projection methods sometimes work provided sufficiently oversampled measurements and a priori information, they easily stagnate at a local minimizer or take many iterations to achieve a satisfactory solution. A priori information, such as real-valuedness and nonnegativity, does not always increase the probability of finding a trivial associate solution. A unified evaluation of the iterative projection algorithms for the phase retrieval can be found in literature [27]. The above iteration schemes are the counterparts of the corresponding iterations in the framework of convex set feasible problems [5]. For example, the HIO method with relaxation parameter \( \beta = 1 \) is the Douglas–Rachford algorithm. Since the phase retrieval involves the intensity constraint set in Fourier space, which is nonconvex, there is no theory to guarantee the convergence, unlike the convex setting.

The stagnation may be due to the non-uniqueness [13] in the absolute sense, i.e. except for the trivial associates. The structured illumination approach is proposed to restore the uniqueness only up to a complex constant with unit magnitude, referred as absolute uniqueness in this paper. Incorporating structured illumination, i.e. collecting the diffraction patterns of the modulated object, where the modulating waveforms or patterns are known, can obtain more measurement data or encode more phase information into the diffraction patterns. Researchers have investigated this measurement setup and found that structured illumination eases the numerical algorithms. The optimization-based algorithms are studied in [6, 8, 12]. The phase retrieval from structured illumination was formulated as a matrix completion problem, whose convex relaxation is a convex trace-norm minimization problem [6, 7]. However, due to the lifting from vector to matrix, the approach is prohibitive for two-dimensional problems. To overcome the memory consumption issue, a common least-squares optimization was proposed and solved by gradient descent method from a special spectral initialization with local convergence guarantee [8]. Besides, the ‘absolute uniqueness’ holds with high probability for a simplified Gaussian model [9]. The iterative projection schemes can also be applied to the phase retrieval problem with structured illumination. An important difference between the optimization approaches in [6–8] and the standard iterative projection methods is that their coded diffraction patterns are not oversampled. When provided oversampled data, the ‘absolute uniqueness’ holds for continuous model [28] and for random phase masks with high probability [13]. Incorporating random phase masks, even the ER algorithm behaves well for non-negative images. And recently the local geometrical convergence to the solution for Douglas–Rachford algorithm has been proved [11]. Without the oversampled data, to ensure
the effectiveness of the lifting method and gradient descent method, a large number of patterns are needed. We emphasize that reducing the number of coded diffraction patterns is crucial for the diffract-before-destruct approach and oversampling is a small price to pay with current sensor technology [11]. So we consider the iterative projection method under the oversampling assumption. To the best of our knowledge, the RAAR algorithm is here applied to phase retrieval with structured illumination for the first time.

However, the random phase mask is difficult to implement in practice [13]. To reduce the randomness of the random phase mask, we replace the random phase mask by pixel-dependent deterministic phase shift. The physically realizable setup can be implemented in two ways. If the diffraction is in the regime of Fraunhofer, the phase shift can result from optical instruments, such as optical grating, ptychography and oblique illumination (see [20, 23, 37, 38]). In the regime of Fresnel, the phase shift is automatically introduced in the physical model. In this paper, we consider the more stable RAAR algorithm instead of the Douglas–Rachford method (corresponding RAAR method with \( \beta \) being unity) used in [11]. A common and vexing problem with the Douglas–Rachford method is that iterations will oscillate around the solution in the noiseless case and even wander away from the neighborhood of the solution in the presence of noise (see our numerical simulations in section 6). It is commonly argued that two measurements are not only sufficient but also necessary for ‘absolute uniqueness’ for phase retrieval with deterministic phase shift [21]. While the theory of uniqueness is improved, stating that only one pattern is needed, two patterns are required to reconstruct a complex-valued image in numerical algorithms. We modify the RAAR algorithm (still called RAAR) to adapt to two or more diffraction patterns, which operates in Fourier domain rather than space domain.

The rest of the paper is organized as follows. In section 2, we state the physical model of the phase retrieval with structured illumination from the continuous setting. In section 3, we give the discrete model that we will study in this paper and clarify the oversampling measurement scheme in the discrete setting and provide some preliminaries. In section 4, we describe the RAAR algorithm for two or more diffraction patterns. The local convergence of RAAR is proved under some assumptions in section 4.2 and the proof details in section 5. In section 6, we present numerical examples and demonstrate numerical global convergence of the RAAR algorithm. Section 7 concludes the paper.

2. Physical model

As we have claimed in the introduction, the applications of phase retrieval arise in many imaging engineering sciences. For example, we use the setup of x-ray coherent diffractive imaging to describe the physical model [28]. Other applications will conclude with the same physical model. We consider the physical model of image formation in propagation-based coherent diffractive imaging. For concreteness, the simplified and idealized setup is depicted in figure 1. Since the setup is three-dimensional, we denote the lateral dimensions as \( x = (x, y) \) and the optical axial dimension as \( z \). The incident plane waves (e.g. electromagnetic waves) of wave-number \( k \) are scattered on the compactly supported sample of thickness \( L \). We describe the sample by its complex refractive index \( n = 1 - \delta(x) + i\beta(x) \), where \( \delta \) and \( \beta \) govern refraction and absorption, encoding the phase shifts and intensity attenuation respectively. The interaction between the incident wave and the sample leads to a slightly perturbed wavefield distribution at the exit-surface \( \Psi_0 \). If we record the intensity difference between the before- and after-interaction with the sample, this is the common attenuation-based CT (computerized tomography) setup, which is to image the \( \beta \). If we are interested in imaging the more sensitive
parameter $\delta$, this information should be encoded in the diffractive patterns, whose intensity can be recorded by the CCD or other photonic films at distance $z$ along the exit-surface $\Psi_0$.

There are two stages to describing the overall process of the plane wave field distribution along the $z$-axis: interaction with sample and free-space propagation. From the well-known Helmholtz equation

$$(\nabla^2 + n^2k^2)\Psi = 0,$$

which models the complex-valued time-independent field $\Psi(x, z)$ traveled in a medium of refractive index $n$, and the assumption that the sample size is smaller than the distance $z$, we exploit the common paraxial approximation of $\Psi(x, z)$. It corresponds to an ansatz of the form

$$\Psi(x, z) = e^{ikz}\widetilde{\Psi}(x, z),$$

where the envelope $\widetilde{\Psi}$ is assumed to be slowly varying on axial length scales $1/k$. Neglecting the contribution $\partial_z^2 \Psi$ against higher orders in $k$, leading to the paraxial Helmholtz equation

$$(2ik\partial_z + k^2(n^2 - 1) + \nabla_x^2)\widetilde{\Psi} = 0,$$

where $\nabla_x^2$ denotes the Laplacian in the lateral coordinates $x$.

The phase retrieval problem arises at the stage of free space propagation, which only involves the wave field relation between $\Psi_0$ and $\Psi_z$ at distance $z$ far from the exit-surface $\Psi_0$. At this stage of interaction between the x-ray and the sample, the complex refractive index $n$ information is encoded into the exit-surface $\Psi_0$. We refer the interested reader to [28].

The scattered wave propagates in a vacuum ($n = 1$) between the exit-surface $E_0$ and the recording plane $E_z$, in which the free-space propagation is governed by equation

$$(2ik\partial_z + \nabla_x^2)\widetilde{\Psi} = 0.$$

Applying Fourier transformation to the above equation and the relation

$$\mathcal{F}(\nabla_x^2\Psi)(\xi, z) = -(2\pi \xi)^2 \hat{\Psi}(\xi, z),$$

where the $\hat{\Psi}(\xi, z)$ is the Fourier transform of $\Psi(x, z)$, we have the ordinary differential equation

$$(2ik\partial_z - (2\pi \xi)^2)\hat{\Psi}(\xi, z) = 0.$$
By inverse Fourier transform, we have an expression for the propagation of general paraxial wave fields

$$\tilde{\Psi}_z = \mathcal{F}_z^{-1}(\tilde{\Psi}_0) = \mathcal{F}^{-1} \left( \exp \left( -\frac{i(2\pi \xi^2 z)}{2k} \right) \mathcal{F}(\tilde{\Psi}_0) \right), \tag{3}$$

where we have set \( \tilde{\Psi}_z = \tilde{\Psi}(\cdot, z) \) for notational convenience. We denote \( \mathcal{F}_z^{-1} \) as the Fresnel propagator. Moreover, it defines a unitary operator as a composition of the unitary maps \( \mathcal{F} \), \( \mathcal{F}^{-1} \) and a multiplication with a function of modulus 1. Physically, this property corresponds to energy conservation of the wave field as it is propagated between lateral planes over a distance \( z \).

Using the convolution theorem and the definition of the Fourier transform, we can rewrite (3) in convolution form. In the distributional sense, it holds that

$$\mathcal{F}^{-1} \left( \exp \left( -\frac{i(2\pi \xi^2 z)}{2k} \right) \right)(x) = -i \frac{\lambda_z}{\lambda} \exp \left( \frac{ikx^2}{2z} \right).$$

This yields that

$$\tilde{\Psi}_0(x') = \left( \tilde{\Psi}_0 \ast -i \frac{\lambda_z}{\lambda} \exp \left( \frac{ikx^2}{2z} \right) \right)(x')$$

$$= -i \frac{\lambda_z}{\lambda} \exp \left( \frac{ikx^2}{2z} \right) \int_{\mathbb{R}^2} \tilde{\Psi}_0(x) \exp \left( \frac{ikx^2}{2z} \right) \exp \left( -\frac{i2\pi x \cdot x'}{\lambda_z} \right) dx.$$

It is assumed that the wave field \( \tilde{\Psi}_0 \) to be propagated is non-negligible only within a region of lengthscale \( b \), e.g. due to confinement by a suitable aperture, such that the dimensionless Fresnel number

$$N_F = \frac{b^2}{2\lambda_z}$$

is \( \ll 1 \). Then for all relevant contributions to the Fourier integral, the unitary factor is close to unity and thus may be suppressed. Accordingly, free-space propagation in this limit reduces to a Fourier transform of the original wave field (up to a coordination scaling). This approximation is denoted as the far-field or Fraunhofer diffraction.

To this end, the relation between the exit-surface wave field \( \tilde{\Psi}_0 \) and recorded plane wave field \( \Psi_0(x') \) (\( x' \) are the lateral coordinates in the recorded plane) is described by

$$\Psi_0(x') = \exp(ikz)\tilde{\Psi}_0(x') = -i \frac{\lambda_z}{\lambda} \exp \left( \frac{ikx'^2}{2z} \right) \cdot \mathcal{F} \left( \Psi_0 \exp \left( \frac{ikx^2}{2z} \right) \right) \bigg|_{x'}. \tag{4}$$

Due to the limitation of physical recorder, we only detect the intensity of wave field \( \Psi_z \). And after scaling, we have the forward physical imaging model

$$I_z = \left| \mathcal{F} \left( \Psi_0 \exp \left( \frac{ikx^2}{2z} \right) \right) \right|^2. \tag{5}$$

If the Fresnel number \( N_F \approx 0 \), this diffraction pattern is given by the Fourier transform, which is called far-field regime. We consider the near-field regime diffraction imaging, in which the distance-dependent phase shift term eases the numerical difficulty encountered in solving the
far-field phase retrieval. In terms of the uniqueness, the far-field phase retrieval is unique up to three kinds of 'trivial ambiguities', while the near-field phase retrieval is unique only up to a constant phase factor. It has been shown that the absolute uniqueness holds from only one intensity measurement with the proviso that the sample is supported. To obtain more than one structured illuminations, we can record the diffraction patterns at different distances. This kind of multiple measurement approach is referred to as phase diversity in astronomy [15, 16].

The difference between far-field and near-field phase retrieval is whether the phase shift occurs or not. Only relative uniqueness of the classical phase retrieval holds for the near-field phase retrieval. Introducing structured illumination, absolute uniqueness holds for near-field phase retrieval. This will ease the numerical algorithms. On the other hand, the diffraction may occur in the far-field regime; we can implement some mask or spread spectrum phase modulation to simulate the artificial phase shift as the near-field phase retrieval. No matter where the phase shift comes from, either from the near-field recording or from the phase modulation, we refer to this two model theory as phase retrieval with structured illumination.

3. Mathematical model and some preliminaries

We deal with the discrete version of the phase retrieval with structured illumination. We denote the two-dimensional signal (sequences/image) using symbol \( x(n) \in \mathbb{C}^{n_1 \times n_2} \), where \( n = (n_1, n_2) \in \mathbb{N}^2 \) denotes the discrete grid. The forward propagation matrix with one phase shift diffraction pattern is given by the matrix (operator)

\[
A(x(n)) = \Phi(x(n) \exp(id \cdot n)),
\]

where \( \Phi \) is the oversampled two-dimensional discrete Fourier transform (DFT) and \( d \) depends on the illumination wavenumber \( k \) and the recorded distance \( z \).

For the two diffraction pattern case, the propagation matrix (operator) is the stacked DFTs, i.e.

\[
A(x(n)) = \begin{bmatrix}
\Phi(x(n) \exp(id \cdot n)) \\
\Phi(x(n) \exp(id \cdot n))
\end{bmatrix}.
\]

To be more concrete, we will denote a two-dimensional signal (sequences/image) \( x \in \mathbb{C}^{m \times n} \), as \( x = (x_1, n) \in \mathbb{N} \) by stacking the matrix into a one-dimensional vector. Generally the required oversampling ratio is 4 for a two-dimensional signal. For a given signal \( x(n) \) with size \( n = n_1 \times n_2 \), we apply the two dimensional Fourier transform to the extended signal \( 2n_1 \times 2n_2 \) with zero-padding [24, 25]. The propagation operator can be written as a matrix \( A \in \mathbb{C}^{m \times n} \) with slight abuse of notation.

The discrete model is described as follows, given the propagation matrix \( A \in \mathbb{C}^{m \times n} \), which is isometric, i.e. \( A^*A = I \). We are to recover the signal/image \( x \in \mathcal{X}(\mathbb{C}^n \text{ or } \mathbb{R}^n) \) from the non-linear measurement data

\[
|Ax| = b \in \mathbb{R}^m.
\]

We focus on two cases in this paper:

(a) One-pattern case: \( A \) is given by (6), \( X = \mathbb{R}^n \), and \( m = 4n \).
(b) Two-pattern case: \( A \) is given by (7), \( X = \mathbb{C}^n \), and \( m = 8n \).

We review some preliminaries on the relative uniqueness for the far-field phase retrieval. These materials are adapted from the literature [17, 31].
Given a vector with nonnegative integer elements \( n = (n_1, \ldots, n_d) \in \mathbb{N}^d \) and a vector with complex components \( z = (z_1, \ldots, z_d) \in \mathbb{C}^d \), we define the multi-index notation \( z^n = z_1^{n_1}z_2^{n_2} \cdots z_d^{n_d} \).

Let \( \mathcal{C}(n) \) denote the set of complex-valued sequences on \( \mathbb{N}^d \) vanishing outside
\[
\mathcal{N} = \{ 0 \leq n \leq N \}, \quad N = (N_1, \ldots, N_d),
\]
where \( n \geq 0 \) if \( n_j \geq 0, \forall j \), and the cardinality of \( \mathcal{N} \) is \( |\mathcal{N}| = \prod_j N_j \). Then the \( d \)-dimensional \( z \)-transform of a sequence \( x(n) \in \mathcal{C}(n) \) may be written compactly as
\[
X(z) = \sum_n x(n)z^{-n}.
\]

All sequences \( x(n) \) are assumed to have \( z \)-transforms with a region of convergence that includes the unit ball \( |z_k| = 1, k = 1, \ldots, d \), so that the Fourier transform may be obtained as
\[
X(\omega) = X(z)|_{z=e^{i\omega n}} = \sum_n x(n)e^{-i2\pi \omega n}, \quad \omega \in \mathbb{R}^d.
\]

Written in polar form, \( X(\omega) \) is represented with respect to its magnitude and phase as
\[
X(\omega) = |X(\omega)|\exp(\imath\phi(\omega)).
\]

**Definition 1 (Equivalence).** We say that \( y(n) \) is equivalent to \( x(n) \) if
\[
y(n) = \begin{cases} 
e^{\imath \theta}x(k + n), \\ e^{\imath \theta}x(k - n). \end{cases}
\]

for some real scalar \( \theta \) and some vector \( k \) with integer components. We denote this by \( y \sim x \).

**Theorem 1 (Uniqueness of phase retrieval).** Let \( x(n) \in \mathcal{C}(n) \) have a \( z \)-transform with at most one irreducible nonconjugate symmetric factor, i.e.
\[
X(z) = P(z) \prod_{k=1}^p X_k(z),
\]
where \( P(z) \) is irreducible and \( X_k(z) \) are irreducible and conjugate symmetric. If \( y(n) \in \mathcal{C}(n) \) with \( |X(\omega)| = |Y(\omega)| \), then \( y \sim x \).

Though we can obtain at most \( 2^{(p-1)} \) different signals with the same Fourier transform magnitude in one-dimensional phase retrieval, the solutions for phase retrieval in two or more dimensions are almost unique in the equivalent sense. This is due to the fact that ‘almost all’ polynomials in two or more variables are irreducible [18]. It has also been shown for the case of polynomials with real coefficients that the geometric character of the set of reducible polynomials provides a stable framework for the retrieval of the phase magnitude [33].

**Theorem 2 (Uniqueness of magnitude retrieval).** Let \( x(n), y(n) \in \mathcal{C}(n) \). If \( X(z) \) and \( Y(z) \) have no nontrivial symmetric factors, i.e. trivial linear phase factors are excluded. Furthermore, \( \tan(\phi(x)(\omega)) = \tan(\phi(y)(\omega)) \) holds for all \( \omega \), then \( x(n) = \beta y(n) \) for some real number \( \beta \).

**Proof.** Consider the sequence \( g(n) \)
\[
g(n) = x(n) \ast y(n),
\]
whose \( z \)-transform is given by
Since the phase of the Fourier transform of \( g(n) \) satisfies 
\[
\tan(\phi_g(\omega)) = 0.
\]
So it follows that \( G(\omega) \) is real-valued. By analytic continuation, we have 
\[
G(z) = G^*(1/z^*).
\]
and so 
\[
X(z)Y^*(1/z^*) = X^*(1/z^*)Y(z).
\]
Multiplying both sides of above equality by \( z^{-N} \) results in the following polynomial equation in \( z^{-1} \),
\[
X(z)Y(z)z^{-m} = \tilde{X}(z)Y(z)z^{-n},
\]
where \( m \) and \( n \) are integer-valued vectors with \( m \geq 0 \) and \( n \geq 0 \). Now consider an arbitrary nontrivial irreducible factor \( X_i(z) \) of \( X(z) \). If \( X_i(z) \) is associated with a factor of \( \tilde{X}(z) \), then 
\[
X_i(z) = \alpha \tilde{X}_i(z)
\]
for some \( i \). If \( i = k \), then \( X_i(z) = \alpha^2 X_k(z) \). Therefore, \( \alpha = \pm 1 \) and \( X_i(z) \) is symmetric. If \( i \neq k \), then 
\[
X_i(z)X_i(z) = \alpha \tilde{X}_i(z)X_i(z)
\]
and \( \tilde{X}_i(z)X_i(z) \) is a symmetric factor of \( X(z) \). Consequently, each nontrivial irreducible factor of \( X(z) \) must be associated with a factor of \( Y(z) \). By the same argument, each nontrivial irreducible factor of \( Y(z) \) must be associated with a factor of \( X(z) \). Therefore, \( X(z) \) and \( Y(z) \) may differ by at most a trivial factor, i.e.
\[
Y(z) = \beta e^k X(z).
\]
However, if the tangent of the phase of \( x(n) \) and \( y(n) \) are equal, then \( k = 0 \). This completes the proof. \( \square \)

The advantage of structured illumination is that the uniqueness up to a unit complex constant is generally guaranteed. For example, assume that the signal is real-valued and the \( z \)-transform of the product signal \( \tilde{x}(n) = x(n) \exp(i d n \cdot n) \) has at most one irreducible non-conjugate symmetric factor, then its equivalences should be of the form:
\[
y(n) = \begin{cases} 
  e^{i\theta} x(m + n) \exp(d(m + n) \cdot (m + n))/\exp(dn \cdot n), \\
  e^{i\theta} x^*(m - n) \exp(-d(m - n) \cdot (m - n))/\exp(dn \cdot n). 
\end{cases}
\]
Since the signal is real-valued, we have that \( y(n) = \pm x(n) \). If the signal is complex, then two phase-shift oversampled diffraction patterns along two distances \( d_1, d_2 \) determine the signal \( y(n) = e^{i\theta} x(n) \) for some real constant \( \theta \).
4. Relaxed averaged alternating reflections (RAAR) algorithm

4.1. RAAR algorithm in the Fourier domain

Let $\mathcal{X}$ be a nonempty closed convex set in $\mathbb{C}^n$ and $[x]_{\mathcal{X}} = \arg\min_{x' \in \mathcal{X}} \|x' - x\|$ be its projection onto $\mathcal{X}$. We consider applying the iterative projection method to the phase retrieval from two or more illuminations. This problem can be formulated as the set feasibility problem in signal/space domain as follows, to find $x \in \mathcal{X} \cap \mathcal{Y}$,

where $\mathcal{X}$ is the set with a priori information constraints and $\mathcal{Y} = \{x \in \mathbb{C}^n | Ax = b\}$. The difficulty is that the projection onto set $\mathcal{Y}$ is not easy to compute. Generally, the quantity $x' = A'(b \circ Ax)/|Ax| \in \mathbb{C}^n$ is not a projection of $x$ onto $\mathcal{Y}$, since $|A(x')| = b$ due to $AA^* = I$. In contrast, we consider the set feasible problem in the Fourier domain, to find $y \in A\mathcal{X} \cap \mathcal{Y}$, (9)

where $\mathcal{Y}$ is the set which satisfies the Fourier domain constraint, i.e. $\mathcal{Y} = \{y \in \mathbb{C}^m | |y| = b\}$. Let $P_1$ be the projection onto $A\mathcal{X}$ and $P_2$ the projection onto $\mathcal{Y}$. For the phase retrieval problem, the projections are of the following form:

$P_1 y = A[A^*y]_\mathcal{X}, \quad P_2 y = b \circ y/|y|, \quad y \in \mathbb{C}^m,$

(10)

where $\circ$ is the Hadamard product, which operates element-wise product of the vectors. When $|y| = 0$, the phase can be assigned arbitrarily and we set $y/|y| = 0$.

The RAAR method in the Fourier domain is given by the following iteration:

$y_{k+1} = \beta T(y_k) + (1 - \beta)P_2(y_k),$

(11)

where

$T(y_k) = \frac{1}{2}(R_1R_2 + I)(y_k) = [I + P_1(2P_2 - I)](y_k)$

(12)

is the Douglas–Rachford iteration and $R_1 = 2P_1 - I$ are the reflection operators.

The RAAR iteration can be written in the following form:

$y_{k+1} = \frac{1}{\beta}(\beta I + P_1(2P_2 - I) - (2\beta - 1)P_2)(y_k)$

(13)

$= \left(\beta y_k + A\left(A^*\left(2b \circ \frac{y_k}{|y_k|} - y_k\right)\right)|_{\mathcal{X}} - \frac{2\beta - 1}{\beta} b \circ \frac{y_k}{|y_k|}\right).$

(14)

Note that the RAAR algorithm with $\beta = 0.5$ becomes the ER (Error Reduction) algorithm in the Fourier domain, since $P_1$ is linear for $\mathcal{X} = \mathbb{R}^n$. When $\beta = 1$, the unrelaxed RAAR algorithm becomes the Douglas–Rachford (or HIO) algorithm in the Fourier domain [11]. Since each of these algorithms involves the same basic operations at each iteration, the rate
of convergence of one algorithm relative to another boils down to iteration counts. It is well-known that, for far-field phase retrieval, the ER algorithm stagnates at a local minimizer, while the Douglas–Rachford (RAAR with \( \beta = 1 \), or HIO) has the tendency to escape local minima [27]. As we have demonstrated in numerical simulations, the near-field phase retrieval (with structured illumination) setup eases the ER algorithm. Although it performs worse than the RAAR algorithm with \( 0.5 < \beta \leq 1 \), it will avoid the local minimizer and can converge to the true solution for real cases. Stability is of importance in numerical algorithms, referring to the property that the algorithm reliably approaches a neighborhood of a solution and remains there, especially in the presence of noise. A common and vexing problem for the Douglas–Rachford algorithm is that iterations will wander away from the neighborhood of a solution. The RAAR algorithm is a trade-off between the ER algorithm and Douglas–Rachford algorithm, whose performance depends on the relaxation parameter \( \beta \). Generally, the RAAR algorithm with \( 0.5 < \beta < 1 \) avoids the oscillation and instability of the Douglas–Rachford algorithm.

4.2. Local convergence

In the following, we assume that the data \( b \neq 0 \) and \( |Ax| \neq 0 \) at the neighborhood of the solution to phase retrieval. We consider the operator

\[
K(y) = y + AA^\epsilon \left( \frac{2b \circ[y]}{|y|} - y \right) - \gamma b \circ \frac{y}{|y|}.
\]

According to the definition of Gâteaux derivative, it follows that

\[
K(y + \epsilon \eta) - K(y) = \epsilon (I - AA^\epsilon) \eta + (2AA^\epsilon - \gamma I) \left( \frac{y + \epsilon \eta}{y + \epsilon \eta} - \frac{y}{|y|} \right) \circ b
\]

\[
= \epsilon (I - AA^\epsilon) \eta + i \epsilon (2AA^\epsilon - \gamma I) \text{Im} \left( \frac{y \circ \eta}{|y|^2} \right) \circ b,
\]

where we use the equality \(|z| \neq 0\),

\[
D \left( \frac{z}{|z|^2} \right)(h) = \frac{h}{|z|^2} - \frac{z \text{Re}(\bar{z}h)}{|z|^3}.
\]

We denote

\[
\Omega = \text{diag} \left( \frac{y}{|y|} \right), \quad B = \Omega^* A,
\]

then it yields

\[
K(y + \epsilon \eta) - K(y) = \epsilon \Omega (I - BB^*) \Omega^* \eta + i \epsilon \Omega (2BB^* - \gamma I) \text{Im} \left( \frac{b \circ \eta}{|y|^2} \right) \text{Im} (\Omega^* \eta) + o(\epsilon)
\]

\[
= \epsilon \Omega J(\nu) + o(\epsilon),
\]

where

\[
J(\nu) = (I - BB^*)\nu + i(2BB^* - \gamma I) \text{Im} \left( \frac{b}{|y|^2} \right) \text{Im}(\nu), \quad \nu = \Omega^* \eta.
\]
When $|y| = b$, we have

$$J(y) = (I - BB^*)y + i(2BB^* - \gamma I)\text{Im}(y).$$

The main result is local, geometric convergence of the RAAR algorithm. We have adapted the proof technique from the literature [11] in which the local convergence of the Douglas–Rachford algorithm is given. Besides, the model setup with random phase mask considered in [24] is different from our model with deterministic Fresnel phase shift. We shall clarify that our new theorem holds for the RAAR algorithm with all $\beta$ within $(0, 1]$. The result for convergence of the Douglas–Rachford algorithm is a special case with $\beta = 1$.

**Theorem 3.** Let $x_0 \in \mathbb{C}^n$ be the solution of the phase retrieval problem and the propagation matrix $A \in \mathbb{C}^{m \times n}$ be isometric. Suppose $m \geq 2n$ and

$$\max_{z \in \mathbb{C}^n, z \perp i x_0} \|z\|^{-1} \|\text{Im}(Bz)\| < 1, \quad B = \text{diag}(\frac{Ax_0}{|Ax_0|}) A.$$  \hfill (15)

Let $y_k$ be a RAAR iteration sequence with values of $0 < \beta \leq 1$ and $x_0 = A'y_k$, $k = 1, \ldots$. If $x_1$ is sufficiently close to $x_0$, then for some constant $\eta < 1$,

$$\text{dist}(x_k, x_0) \leq \eta^{k-1} \text{dist}(x_1, x_0),$$  \hfill (16)

where $\text{dist}(x_k, x_0)$ is the distance

$$\text{dist}(x_k, x_0) = \min_{c \in \mathbb{C}, |c| = 1} \|cx_k - x_0\|.$$  \hfill (17)

**5. Spectral gap and the proof of theorem 3**

In this section, we provide the result that (15) is satisfied for almost all two-dimensional phase retrieval. We introduce the following matrix $B$ and map $G$, which maps a complex vector to its real and imaginary parts:

$$B = [\text{Re}(B) - \text{Im}(B)] \in \mathbb{R}^{m \times 2n}, \quad G(z) = (\text{Re}(z), \text{Im}(z))^T,$$

where $B = \Omega^*_A$. We have

$$G(Bz) = \begin{bmatrix} BG(z) \\ BG(-iz) \end{bmatrix} \in \mathbb{R}^{2m}, \quad z \in \mathbb{C}^n$$

and

$$B^T = \begin{bmatrix} \text{Re}(B^*) \\ \text{Im}(B^*) \end{bmatrix} \in \mathbb{R}^{2n \times m}, \quad \|B^T \text{Im}(y)\| = \|B^T \text{Im}(y)\|.$$  

We first derive the singular values of the corresponding matrix $B$. Since $Bx_0 = |y_0| = |b|$, we have

$$BG(x_0) = |y_0|, \quad BG(-ix_0) = 0.$$  

Let $B = U\Sigma V^*$, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{2n}$ be the singular values of matrix $B$, then $\sigma_1 = 1$, $\nu_1 = G(x_0), \mu_1 = |y_0|$ and $\sigma_{2n} = 0$, $\nu_{2n} = G(-ix_0)$.

**Lemma 1.**
\[ \sigma_2 = \max_{w \in \mathbb{R}^n, w \perp v_i} \frac{\|Bw\|}{\|w\|} = \max_{z \in \mathbb{C}^n, z \perp x_0} \frac{\|\text{Im}(Bz)\|}{\|z\|}. \]

**Proof.** Let \( z_0 \perp ix_0 \) and \( \sigma_2 = \frac{\|\text{Im}(Bz_0)\|}{\|z_0\|} \), then \( 0 = \text{Re}(z_0, ix_0) = \langle G(-iz_0), G(x_0) \rangle = 0 \). Take \( w_0 = G(-iz_0) \), we have \( \max_{z \in \mathbb{C}^n, z \perp ix_0} \|z\|^{-1} \|\text{Im}(Bz)\| \leq \max_{w \in \mathbb{R}^n, w \perp v_i} \|w\|^{-1} \|Bw\| \). The inverse inequality can be obtained by the fact that \( \text{Im}(Bz) = BG(-iz) \) and \( \|G(-iz)\| = \|z\| \). This completes the proof. \( \square \)

To this end, we have to show the assumption of theorem 3 is satisfied for our measurement setup for the two-dimensional phase retrieval problem. Furthermore, we show that this condition (15) is satisfied for almost all two-dimensional signals with at least one oversampled coded diffraction pattern. This excludes some two-dimensional signals whose \( z \)-transforms have nontrivial symmetric factors—see theorem 2. However, it should hold for the signals we consider in practice.

The fact that the second singular value \( \sigma_2 \) is strictly less than the leading one is the immediate consequence of the following result.

**Proposition 1.** Let \( A \) be isometric and \( B = \Omega_0^* A \). Then \( \|\text{Im}(Bz)\| = 1 \) holds for some unit vector \( z \) if and only if

\[ \text{Re}(a_j^*z)\text{Re}(a_jx_0) + \text{Im}(a_j^*z)\text{Im}(a_jx_0) = 0, \forall j = 1, \ldots, m, \]

where \( a_j \) are the rows of \( A \), or equivalently

\[ \omega = \pm i\omega_0, \quad \omega = \frac{Az}{|Az|}, \quad \omega_0 = \frac{Au_0}{|Au_0|}, \]

where the \( \pm \) sign may be element-wise-dependent.

**Proof.** By the isometry of \( B \), we have

\[ \|\text{Im}(Bz)\|^2 \leq \|Bz\|^2 = \|z\|^2. \]

And the inequality becomes an equality if and only if

\[ \text{Re}(Bz) = \text{Re} \left( \frac{Ax_0}{|Ax_0|} \circ Az \right) = 0. \]

So the arguments of \( \omega_0 \) and \( \omega \) differ by \( \frac{\pi}{2} \). This completes the proof. \( \square \)

Since we have \( \angle \omega_0 = \angle \pm i\omega \), we have \( \tan(\angle \omega_0) = \tan(\angle \pm i\omega) \), so that \( z = icx_0 \) for a real constant \( c \) by the uniqueness of magnitude retrieval. So

\[ \|\text{Im}(Bz)\| = 1, \|z\| = 1 \iff z = \pm ix_0/\|x_0\|, \]

and hence

\[ \sigma_2 = \max_{z \in \mathbb{C}^n, z \perp x_0} \|z\|^{-1} \|\text{Im}(Bz)\| < 1. \]
Proof of theorem 3. Let the solution to the phase retrieval be \( \mathbf{x}_0 \), then we have \( y_0 = A \mathbf{x}_0 \). Then, letting \( v_k = \Omega_0^* (c_k y_k - y_0) \), where \( c_k \) is the minima phase such that \( \| c_k y_k - y_0 \| \) is the minimum, we have

\[
\Omega_0^* (c_k y_{k+1} - y_0) = \beta \Omega_0^* (K(y_k) - K(y_0)) \\
= \beta (J(v_k) + o(\|v_k\|)).
\]

Moreover, multiplying \( B^* A^* \Omega_0 \), it follows that

\[
c_k x_{k+1} - x_0 = B^* \Omega_0^* (c_k y_{k+1} - y_0) \\
= \beta (B^* J(v_k) + o(\|c_k x_k - x_0\|)) \\
= \beta (B^* (I - BB^*) v_k + i (2B^* B B^* + \gamma B^*) \text{Im}(v_k)) \\
= iB^* \text{Im}(v_k)
\]

by the isometric property \( B^* B = I \) and \( \gamma = (2\beta - 1)/\beta \).

By the optimal phase \( c_k \), we have

\[
\text{Re}(v_k, i|y_0>) = \text{Re}(c_k y_k - y_0, y_0) = 0,
\]

so \( \text{Im}(v_k) \) is orthogonal to the leading right singular vector \( |y_0\rangle \) of \( \mathcal{B} \).

So,

\[
\| c_k x_{k+1} - x_0 \| = \| B^* \text{Im}(v_k) \| + o(\|c_k x_k - x_0\|) \\
= \| B^* \text{Im}(v_k) \| + o(\|c_k x_k - x_0\|) \\
\leq \sigma_2 \| \text{Im}(v_k) \| + o(\|c_k x_k - x_0\|) \\
\leq \eta \| v_k \|,
\]

where we use the fact that \( \sigma_2 < 1 \), so we have \( \eta < 1 \). \( \square \)

Remark 1. It can be seen that for the ER algorithm, i.e. \( \beta = 0.5 \), the local convergence holds. For the Douglas–Rachford algorithm, i.e. \( \beta = 1 \), the local convergence has been proved in [11].

From the proof, we find that one oversampled Fourier diffraction pattern can ensure the local convergence, and the local convergence holds for ER and Douglas–Rachford algorithm, which are two special cases of the RAAR algorithm corresponding to two specific values of \( \beta \). Note that the spectral gap requirement (i.e. \( \sigma_2 < 1 \)) holds only in the two-dimensional setting.

6. Numerical simulations

We explore the performance of the RAAR algorithm in two cases, i.e. one-pattern and two-pattern cases, which differ by the number of the diffraction patterns recorded. We test three real images and one complex image with different sizes: (a) cameraman with size \( 128 \times 128 \), (b) mandrill with size \( 512 \times 512 \), (c) barbara with size \( 512 \times 512 \), and (d) golden balls with size \( 512 \times 512 \). The complex image is obtained by adding a random phase to the original magnitude image. Figure 2 displays the three real images and the complex golden balls image. For the complex image, we plot the magnitude instead of its real and imaginary parts. For the first two images, one structured illumination diffraction pattern with \( d = 3 \) is recorded for each image. For image (c), the structured illumination diffraction data are obtained with \( d = 5 \). For complex image (d), two structured illumination diffraction patterns \( (d_1 = 3, d_2 = -3) \) are recorded.
To compare the rate of convergence, we denote the relative error of the iteration $x_k$ as
c\[ \text{rel err} = \frac{\|c_k x_k - x_0\|}{\|x_0\|}, \]
c\[ c_k = \arg \min_{c \in \mathbb{C}} \|c x_k - x_0\|, \]
where $x_0$ is the true solution. We stop the iteration when either of the two following criteria is satisfied: the maximum number of iterations is reached or the relative error is below $10^{-5}$. The iteration maximum is set to 150 for the first two images and to 300 for the last two images. Two initialization schemes are tested to explore whether the performance varies with the initialization. Constant initialization sets each pixel value to unity, while random initialization is generated from complex numbers with unit amplitude and random phase.

6.1. Comparison of different values of $\beta$

As we have said, the performance of the general RAAR algorithm depends on the relaxed parameter $\beta$. If $\beta = 0.5$, then it becomes the ER algorithm, and $\beta = 1$, it becomes the Douglas–Rachford algorithm or HIO algorithm. We test ten different values of $\beta$, from 0.1 to 1 with stepsize 0.1, with constant initialization. The performance of the RAAR algorithm with the ten values of $\beta$ for the four test images is depicted in figure 3 for the noiseless case. Although the performance of the RAAR algorithm varies with different problems and different values of $\beta$, the instability and oscillation occur in all cases for RAAR algorithm with $\beta = 1$. It is shown that the RAAR algorithm with $0 < \beta \leq 0.5$ performs worse than that with $0.5 < \beta \leq 1$. For the ER algorithm ($\beta = 0.5$), the relative error decreases very slowly, so as the algorithms with values of $\beta$ below 0.5. It will converge to the true solution with sufficiently large iteration maximum, as compared to applying ER for far-field phase retrieval. For the three real images, the oscillation phenomenon with the Douglas–Rachford ($\beta = 1$) algorithm can be observed (RAAR algorithm with $\beta$ value 0.9 shows little oscillation for image (b) mandrill) when the iteration is in the neighborhood of the solution. Except for the unwanted oscillation, the performance comparison of $\beta = 0.8$ and $\beta = 0.9$ does not show a uniform result for different problems, and it depends on the problem we consider. For the small size image (a), the case $\beta = 0.8$ is superior to 0.9. For image (b), $\beta = 0.8$ is better than other values of $\beta$. For image (c) and complex image (d), the performance behavior is different: the RAAR algorithm with $\beta = 0.9$ is superior to Douglas–Rachford algorithm (i.e. $\beta = 1$).

We also run the RAAR algorithm with the same setting from a random initialization. To avoid the randomness effect, tens of tests are conducted. The numerical performance is similar.
Although it varies with different random initialization, the performance of RAAR algorithm with different values of $\beta$ is uniform when the random initialization is fixed. The global convergence of RAAR algorithm in convex setting is well-known. Since we deal with nonconvex set in phase retrieval problem, only local convergence is guaranteed for $0 < \beta \leq 1$. From our numerical tests, the RAAR algorithm with values of $0.5 \leq \beta \leq 1$ shows numerical global convergence from an initialization which is not in the basin of attraction. All the reconstructed images are artifact-free and no visible difference occurs. The improved performance of the RAAR algorithm may result from the ‘absolute uniqueness’ of the near-field phase retrieval.

6.2. Noisy measurements

In the next set of experiments we consider the same test images but with noisy measurements. Since, in practice, the main source of noise has Poisson distribution, resulting from the photon counting, we add random Poisson noise to the measurements for five different SNR levels, ranging from 30 dB to 50 dB with step 5 dB. The performance of the RAAR algorithm with different values of $\beta$ is shown in figure 4 in the case of SNR level 50 dB. It is observed that in the presence of noise, the common instability of the Douglas–Rachford algorithm is obvious. If it runs longer, the iteration remains further from the solution. This wandering of the iterations near a local solution has been reported in [24]. The relaxation parameter $\beta$ in the RAAR

Figure 3. Relative error versus parameter $\beta$ in the noiseless case. Data for $\beta$ below 0.5 are not shown. (a) Cameraman. (b) Mandrill. (c) Barbara. (d) Golden balls.
algorithm can dampen the iterations near a solution. The performance with $0.5 \leq \beta < 1$ is similar to the noiseless case. From the noiseless and noisy cases, it is difficult to say with which $\beta$ the RAAR algorithm performs best, since it varies with the problem considered. We suggest that $\beta = 0.8$ and $\beta = 0.9$ should be the starting points in applying the RAAR algorithm for real and complex images respectively. The Douglas–Rachford algorithm should be avoided due to its severe oscillation in the noiseless case and much poorer performance in the noisy case.

Figure 4. Relative error versus parameter $\beta$ from Poisson noisy data with SNR being 50 dB. Legends for $\beta$ below 0.5 are not shown. (a) Cameraman. (b) Mandrill. (c) Barbara. (d) Golden balls.

Figure 5. Relative Error in dB versus SNR, the values of $\beta$ are 0.8 and 0.9 for real images and complex images respectively. (a) Three real images. (b) Complex image.
We present the reconstructions of RAAR algorithm with $\beta = 0.8$ and $\beta = 0.9$ for real and complex images respectively. Figure 5 shows the average relative error in dB versus the SNR. The error curve clearly shows the linear behavior between SNR and relative error for complex images (see figure 5(b)). For real images, the curves imply the instability of the phase retrieval from structured illumination: the SNR becomes lower, the reconstruction becomes more worse. To be clear, we show the reconstructions from two SNR level data. Figures 6 and 7 depict the resulting reconstructions for SNR = 50 dB and SNR = 40 dB, respectively.

6.3. The effect of the phase shift

We explore the effect of the phase shift parameter $d$ on the performance of the phase retrieval in the noiseless setting. Figure 8 depicts the performance with varying phase shift $d$. We apply the RAAR algorithm with $\beta$ being 0.8 for images (a) and (b) and with $\beta$ being 0.9 for images (c) and (d) respectively. The smaller the $d$ is, the more iterations are needed for reaching the same relative error level. It is difficult to explain why larger values of $d$ ease convergence. This can be partly explained by the diffraction patterns in the propagated holograms. For smaller propagation distances, corresponding to larger $d$, more fringes appear by diffraction of the propagated wave field, encoding phase information. For larger distances, corresponding to the far-field regime, the structure of the original image can no longer be identified in the diffraction patterns. In some sense, the smaller the distance $d$ is, the more nonlinear the problem. To be clear, we have now no strict mathematical theory to explain this phenomenon.

Figure 6. Reconstructions from Poisson noisy data with SNR being 50 dB. The ‘golden balls’ data (d) are courtesy of Stefano Marchesini [36]. © IOP Publishing Ltd. All rights reserved. (a) Cameraman. (b) Mandril. (c) Barbara. (d) Golden balls.

Figure 7. Reconstructions from Poisson noisy data with SNR being 40 dB. The ‘golden balls’ data (d) are courtesy of Stefano Marchesini [36]. © IOP Publishing Ltd. All rights reserved. (a) Cameraman. (b) Mandril. (c) Barbara. (d) Golden balls.
The knowledge of the Fourier transform intensity from structured illumination with random phase masks specifies the signal up to a single complex constant [13]. However, the random phase mask is difficult to realize in practice; we consider the structured illumination with a pixel-dependent deterministic phase shift, which can be implemented by recording the diffraction pattern in the Fresnel regime. We extend the RAAR algorithm for two and more diffraction patterns, which operates in the Fourier domain rather than in the space domain. We prove the local convergence of the RAAR algorithm. The ER and Douglas–Rachford algorithms are then two special cases, with the relaxation parameter $\beta$ being 0.5 and 1 respectively. We find that the iterations of the Douglas–Rachford algorithm oscillate in the neighborhood of the solution for the noiseless case and wander away from the neighborhood of the solution for noisy data. From the numerical simulations, it is difficult to specify which value of $\beta$ is the best choice, since the performance varies with the problem. The effectiveness and stability of the algorithm with $0.5 < \beta < 1$, compared to the ER ($\beta = 0.5$) method and Douglas–Rachford ($\beta = 1$) method are demonstrated. The linear relation between the relative error and the noise level shows that phase retrieval with two patterns is numerically stable.

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