Noncommutative Solitons and the $W_{1+\infty}$ Algebras in Quantum Hall Theory

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Abstract

We show that $U(\infty)$ symmetry transformations of the noncommutative field theory in the Moyal space are generated by a combination of two $W_{1+\infty}$ algebras in the Landau problem. Geometrical meaning of this infinite symmetry is illustrated by examining the transformations of an invariant subgroup on the noncommutative solitons, which generate deformations and boosts of solitons. In particular, we find that boosts of solitons, which was first introduced in Ref[22], are valid only in the infinite $\theta$ limit.

Keywords: $W_{1+\infty}$ algebra; Noncommutative Solitons.

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# 1 Introduction

Noncommutative field theory has been an intense topic of research among string/field theory community for the last few years. One reason is that such a theory naturally appears in the low-energy description of D-brane dynamics in the presence of a constant Neveu-Schwarz B field background [1] [2] [3] [4] [5]. More recently, soliton solutions of noncommutative scalar [6] and scalar-gauge [7] [8] [9] theories were constructed. These topological objects were then applied to the construction of D-branes as solitons of the tachyon field in noncommutative open string theory [10] [11] [12].

The notion of noncommutative coordinates is not a new idea in physics community. In fact, it has been known for a long time in condense matter physics that two-dimensional quantum Hall system realizes the simplest non-commutative geometry [13], whose appearance can be traced back to the quantum realization of projective representation for the magnetic translation group. Consequently, it is not surprising that quantum Hall system offers some kind of toy model in the study of noncommutative string/field theories. Recent development of exact realization of the first quantized Laughlin wave function by an abelian noncommutative Chern-Simons theory was along this line of thought [14].

In this paper, we shall unfold another interesting connection between noncommutative field theory and the quantum Hall theory. Namely, we show that $U(\infty)$ symmetry transformations of the non-commutative field theory in the Moyal space are generated by a combination of two $W_{1+\infty}$ algebras in the Landau problem [15] [16] [17] [18] [19] [20].

As an application, one can use this algebra as a soliton generating algebra to generate non-radially symmetric solitons. To get some geometrical meaning of our new solitons, we discuss and explicitly calculate an invariant subalgebra of this soliton generating algebra and, in particular, use it to generate deformations and boosts of solitons. We find that boosts of solitons, which was first introduced in Ref[22], are valid only in the infinite $\theta$ limit. The argument of eqs (12) and (13) used in Ref[22] was misleading for the finite $\theta$ case. This can be seen from our calculation that area-preserving, or $W_{1+\infty}$ symmetry, is violated for the finite $\theta$ case.
2 \( U(\infty) \) Symmetry in Noncommutative Field theory

The two-dimensional noncommutative plane is defined by a commutation rule between its coordinates, \( \vec{x} = (\hat{x}, \hat{y}) \),

\[
[\hat{x}, \hat{y}] = i\theta. \tag{1}
\]

The operator algebra \( \mathcal{A} \) defined on this noncommutative plane is generated by the formal power series in the noncommutative coordinates,

\[
\mathcal{A} \equiv \{ \hat{A}(\vec{x}) | \hat{A}(\vec{x}) = \sum_{mn} a_{mn} \hat{x}^m \hat{y}^n \}. \tag{2}
\]

The Weyl-ordered operator on the noncommutative plane can be generated from a function defined on the Moyal space,

\[
\hat{\phi}(\vec{x}) = \int d^2 \vec{x} \int \frac{d^2 \vec{k}}{(2\pi)^2} e^{i\vec{k}(\vec{x} - \vec{x})} \phi(\vec{x}), \tag{3}
\]

and we have the following isomorphism between operator product and star-product,

\[
(\hat{\phi} \cdot \hat{\psi})(\vec{x}) = \int d^2 \vec{x} \int \frac{d^2 \vec{k}}{(2\pi)^2} e^{i\vec{k}(\vec{x} - \vec{x})} (\phi * \psi)(\vec{x}), \tag{4}
\]

where the star-product is defined as

\[
\phi(\vec{x}) * \psi(\vec{x}) \equiv \exp \left[ i\frac{\theta_{ij}}{2} \frac{\partial}{\partial y_i} \frac{\partial}{\partial z_j} \right] \phi(y) \psi(z) \big|_{y = z = x}, \quad i, j = 1, 2. \tag{5}
\]

It is convenient to introduce the annihilation and creation operators,

\[
c \equiv \frac{\hat{x} + i\hat{y}}{\sqrt{2\theta}}, \quad c^\dagger \equiv \frac{\hat{x} - i\hat{y}}{\sqrt{2\theta}}, \quad [c, c^\dagger] = 1. \tag{6}
\]

Any operator in \( \mathcal{A} \) can be expanded in terms of the occupation-number operator basis,

\[
\hat{A}(\vec{x}) = \sum_{m,n=0}^{\infty} a_{mn} \hat{\phi}_{mn}, \tag{7}
\]

where the operator basis \( \hat{\phi}_{mn} \) is defined as

\[
\hat{\phi}_{mn} \equiv |m > < n| = \frac{(c^\dagger)^m}{\sqrt{m!}} |0 > < 0| \frac{c^n}{\sqrt{n!}}. \tag{8}
\]

We also take the following definitions of commutative variables,

\[
z \equiv \sqrt{\frac{2}{\theta}}(x + iy), \quad \bar{z} = \sqrt{\frac{2}{\theta}}(x - iy); \quad k_c \equiv \sqrt{\frac{\theta}{2}}(k_x + ik_y), \quad \bar{k}_c = \sqrt{\frac{\theta}{2}}(k_x - ik_y). \tag{9}
\]
The action of the noncommutative scalar-gauge theory can be written as

$$L = Tr[-\frac{1}{4}\hat{F}_{ij}\hat{F}^{ij} + (D_i\hat{\phi})^\dagger(D^i\hat{\phi}) + V(\hat{\phi})],$$

where the field strength tensor is

$$F_{ij} \equiv \frac{\epsilon_{ki}}{i\theta}[\hat{x}_k, \hat{A}_j] - \frac{\epsilon_{kj}}{i\theta}[\hat{x}_k, \hat{A}_i] + ig[\hat{A}_i, \hat{A}_j]$$

and the covariant derivatives are

$$D_i\hat{\phi} \equiv \frac{\epsilon_{ki}}{i\theta}[\hat{x}_k, \hat{\phi}] - ig[\hat{A}_i, \hat{\phi}],$$

$$D_i\hat{\phi} \equiv \frac{\epsilon_{ki}}{i\theta}[\hat{x}_k, \hat{\phi}] - ig\hat{A}_i\hat{\phi},$$

for real and complex scalar fields, respectively. In the complex case eq.(13), for our purpose, we only introduce one piece of gauge field (see eq.(16)).

The action in eq.(10) obviously contains the $U(\infty)$ symmetry which will be discussed in the next section. One interesting discovery recently was that the theory possesses various soliton solutions which turns out to be important in many applications [21]. The original GMS solitons, for example, are radially symmetric solutions of real scalar theory in the infinite $\theta$ limit,

$$\phi_n(z, \bar{z}) = 2(-1)^n L_n(|z|^2) \exp^{-|z|^2}, \quad n = 0, 1, ...$$

which can be constructed from the corresponding projection operators in the operator algebra $\mathcal{A}$. The $U(\infty)$ symmetry mentioned above can then be used to generate more general non-radially symmetric solitons through the following transformation

$$\hat{\phi} \to U\hat{\phi}U^\dagger.$$ 

Similar technique and its generalization [21] can be applied to the case of complex scalar-gauge theory. The transformation for the scalar is

$$\hat{\phi} \to \hat{\phi}U.$$ 

In this case, for simplicity, we consider only right-handed $U(1)$ gauge symmetry or half of the $U(\infty)$ symmetry. In the next section, we will give an explicit form of these transformations on the corresponding Moyal function space.
3 \ W_{1+\infty} \ Algebras \ in \ Quantum \ Hall \ Theory \ from \ U(\infty) \ Symmetry

To calculate eq.(15) in the corresponding Moyal function space, we first choose the occupation-number operators as a basis of the operator algebra \( A \),

\[
\hat{\phi}_{mn} = |m > < n| = \frac{(c^\dagger)^m}{\sqrt{m!}} : \exp[-c^\dagger c] : \frac{c^n}{\sqrt{n!}},
\]

(17)

where the normal-ordering has been introduced in the second equality. On the other hand, the corresponding function basis \( \varphi_{mn} \) on the Moyal space can be defined through eq.(3)

\[
\hat{\varphi}_{mn} \equiv |m > < n| = \int d^2\vec{x} \int \frac{d^2\vec{k}}{(2\pi)^2} e^{i\vec{k}(\vec{x} - \vec{x})} \varphi_{mn}(\vec{x}).
\]

(18)

The momentum space function basis can be calculated to be

\[
\tilde{\varphi}_{mn}(\vec{k}) = \frac{1}{\sqrt{m!n!}} e^{\frac{k_{ckc}}{2}} \left(i \frac{\partial}{\partial k_c}\right)^m \left(i \frac{\partial}{\partial k_c}\right)^n \tilde{\chi}_{00}(\vec{k}),
\]

(19)

where

\[
\tilde{\chi}_{00}(\vec{k}) \equiv (2\pi) e^{-\frac{\vec{k}^2}{4\theta}}.
\]

(20)

Taking Fourier transform and after a series of integration by parts, we get

\[
\varphi_{mn}(z, \bar{z}) = \frac{1}{\sqrt{m!n!}} \int d^2\vec{k} \left(\frac{-i}{\partial k_c}\right)^m e^{-\frac{i}{2}(k_c k_c + i k_c \bar{z} + i k_c z)} \left(\frac{i}{\partial k_c}\right)^n \tilde{\chi}_{00}(\vec{k})
\]

\[
= \frac{1}{\sqrt{m!n!}} \left(\frac{-\partial}{\partial z} + \frac{\bar{z}}{2}\right)^m \left(\frac{-\partial}{\partial \bar{z}} + \frac{z}{2}\right)^n \tilde{\chi}_{00}(\vec{k}).
\]

(21)

Similar procedure can be used to simplify the \( k_c \) partial derivatives, and we have

\[
\varphi_{mn}(z, \bar{z}) = \frac{1}{\sqrt{m!n!}} \left[\left(-\frac{\partial}{\partial z} + \frac{\bar{z}}{2}\right)^m \left(-\frac{\partial}{\partial \bar{z}} + \frac{z}{2}\right)^n \varphi_{00}(z, \bar{z})\right],
\]

\[
= \frac{\sqrt{2\pi\theta}}{\sqrt{m!n!}} (a^\dagger)^m (b^\dagger)^n \Phi_{00}(z, \bar{z}) = \sqrt{2\pi\theta} \Phi_{mn}(z, \bar{z}),
\]

(21)

where \( a^\dagger, b^\dagger \) and \( \Phi_{00} \) are given by

\[
a^\dagger \equiv -\frac{\partial}{\partial z} + \frac{\bar{z}}{2}, \quad b^\dagger \equiv -\frac{\partial}{\partial \bar{z}} + \frac{z}{2},
\]

(22)

\[
\Phi_{00}(\vec{x}) = \sqrt{\frac{2}{\pi\theta}} \exp[-\frac{\vec{x}^2}{\theta}].
\]

(23)
It is important that $a^\dagger$ and $b^\dagger$ are commuting operators, so that their ordering is immaterial.

The physical meaning of these operators in the quantum Hall theory \cite{13,15} will be discussed at the end of this section.

We are now ready to calculate the corresponding Moyal functional form of general non-radially symmetric solitons in eq.(15). To be more specific, we examine the unitary transformation, $U = \exp[i\hat{F}]$, on the noncommutative scalar function

$$\delta \hat{\phi} = U \hat{\phi} U^\dagger - \hat{\phi} \approx i[\hat{F}, \hat{\phi}],$$

(24)

where $\hat{F}$ is the generating function of the $U(\infty)$ transformation, and can be expressed in terms of the occupation number basis

$$\hat{F} = \sum_{r,s=0}^{\infty} \xi_{rs} J_{rs}, \quad J_{rs} \equiv (c^\dagger_r c^s).$$

(25)

For a Hermitian generator, $\hat{F} = \hat{F}^\dagger$, the group parameters satisfy $\xi_{rs}^* = \xi_{sr}$. One can show that

$$(-i)\delta_{rs} \phi_{mn} \equiv [J_{rs}, \phi_{mn}] = (c^\dagger_r c^s)|m \rangle \langle n| - \langle m \rangle \langle n|(c^\dagger_r c^s)$$

$$= \theta(m - s) \sqrt{m!} (m + r - s)! (m - s)! \phi_{m+r-s,n}$$

$$- \theta(n - r) \sqrt{n!} (n + r - s)! (n - r)! \phi_{m,n-r+s},$$

(26)

where $\theta(n) = 1$ if $n \geq 0$, and $\theta(n) = 0$ otherwise.

By using eqs.(21),(26), one can calculate the corresponding transformation on the function basis of the Moyal space

$$(-i)\delta_{rs} \varphi_{mn}(z, \bar{z}) = \sqrt{2\pi \theta} [ \theta(m - s) \sqrt{m!} (m - s)! \sqrt{n!} (a^\dagger)^{m+r-s} (b^\dagger)^n$$

$$- \theta(n - r) \sqrt{n!} (n - r)! \sqrt{m!} (a^\dagger)^m (b^\dagger)^{n-r+s} \Phi_{00}(z, \bar{z})$$

$$= [(a^\dagger)^r a^s - (b^\dagger)^s b^r] \varphi_{mn}(z, \bar{z}).$$

(27)

One can thus easily identify the corresponding symmetry generators in the Moyal space to be

$$J_{rs} \leftrightarrow L_{rs}^a - L_{sr}^b \equiv (a^\dagger)^r a^s - (b^\dagger)^s b^r.$$
Similar consideration for complex scalar function in eq (16) gives us
\[ \mathcal{J}_{rs} \leftrightarrow \mathcal{L}_{rs}^{b} \equiv (b^\dagger)^{r}b^{s}. \] (29)

If we have used the left-handed gauge transformation in eq.(16), the corresponding generators will be \((a^\dagger)^{r}a^{s}\).

It is interesting to note that either \(\mathcal{L}^{b}\) or \(\mathcal{L}^{a}\) in eq.(28) generates a \(W_{1+\infty}\) algebra \[15\] \[16\] \[17\] \[18\] \[19\] \[20\], respectively,

\[ [\mathcal{L}_{mn}, \mathcal{L}_{kl}] = \sum_{s} \frac{(n + 1)!(k + 1)!}{(n - s)!(s + 1)!(k - s)!} \mathcal{L}_{m+k-s,n+l-s} - (m \leftrightarrow k, n \leftrightarrow l). \] (30)

This infinite symmetry was discovered some time ago in the study of quantum Hall theory, where the underlying physics is described by the Landau problem \[13\].

The Hamiltonian of the Landau problem in the symmetry gauge is
\[ H = -\frac{\hbar^2}{2\mu}(\vec{p} - e\vec{A})^2 = \hbar\omega(a^\dagger a + 1/2), \quad \vec{A} = (-\frac{1}{2}By, \frac{1}{2}Bx), \] (31)

where the cyclotron frequency is \(\omega = \frac{eB}{\mu}\), and the noncommutative parameter \(\theta\) is related to the magnetic field by
\[ \theta = \frac{4\hbar}{eB}. \] (32)

In eq.(31), \(a\) and \(a^\dagger\) as defined in eq.(22) are the mechanical momenta of the electron, and \(W_{1+\infty}^{a}\) can be interpreted as the energy generating algebra. On the other hand, the physical meaning of \(b\) and \(b^\dagger\) defined in eq.(22) are the so-called magnetic translations of the Hall system. They both commute with the Hamiltonian due to the trivial translational symmetry of the Hall system. The projective representation of these two noncommutative translations is given by \[13\]
\[ T_{\xi,\bar{\xi}} \equiv \exp[\xi b^\dagger - \bar{\xi}b] \Rightarrow T_{\xi,\bar{\xi}} T_{\eta,\bar{\eta}} = \exp[\frac{\xi\bar{\eta} - \bar{\xi}\eta}{2}] T_{\xi+\eta,\bar{\xi}+\bar{\eta}}. \] (33)

It is remarkable that one can extend this symmetry to an infinite symmetry due to the noncommutativity of these two magnetic translations. Defining an infinite number of conserved charges as in eq.(29), it is not difficult to see that they form the \(W_{1+\infty}\) algebra in eq.(30).

One geometric implication of these two \(W_{1+\infty}\) algebras is the manifestation of the well-known area preserving property of each Landau states as will be discussed in the next section.
We thus have shown that the star-product representation of $U(\infty)$ symmetry of eq.(15) or eq.(16) in the non-commutative field theory is generated by two $W_{1+\infty}^a$ algebras, $W_{1+\infty}^a$ and $W_{1+\infty}^b$, of the Landau problem. One obvious application is to use this algebra to explicitly construct the deformed GMS solitons.

There exists a well-known solution generating technique to generate new solitons with different topological charges. In particular, instead of using the unitary $U$ in eq.(24), one introduces the non-unitary isometry or shift operator,

$$S_k \equiv \sum_{p=0}^{\infty} \langle p+k | < p |,$$

and the new solitons generated in the Moyal-space will be

$$S_k \hat{\phi}_{mn} S_k^\dagger \Rightarrow \sqrt{\frac{2\pi \theta}{(m+k)!(n+k)!}} (a^\dagger)^k (b^\dagger)^k \Phi_{mn}.$$

### 4 Geometrical Meaning of the $U(\infty)$ Symmetry in Non-commutative Field Theory

Having established the connections between $U(\infty)$ symmetry in the noncommutative field theory and $W_{1+\infty}$ algebras in the quantum Hall theory, it is instructive to illustrate the concrete geometrical meaning of this infinite symmetry. For this purpose, it is sufficient to examine the finite subgroup transformation of the $U(\infty)$ symmetry on the noncommutative solitons in eq.(14). This finite group is defined by a truncation of the generating function of the symmetry

$$U = \exp[i\hat{F}(c, c^\dagger)], \quad \hat{F}^\dagger = \hat{F},$$

up to second order in $c$ and $c^\dagger$,

$$\hat{F}(c, c^\dagger) = \alpha_0 + \xi c^\dagger + \eta c + \beta_1 (c^\dagger c) + \beta_2 (c^\dagger - c)^2 + \beta_3 [(c^\dagger)^2 + c^2].$$

Due to the simple commutation relation, $[c, c^\dagger] = 1$, one can verify that this finite set of generators is closed under group multiplication. We shall evaluate the induced transformations of this subgroup on the noncommutative scalar function. To solve for the induced transformations, we first decompose any elements in the subgroup into four independent transformations,
1. translation: \( U_T(\xi) \equiv \exp[\xi c^\dagger - \bar{\xi} c], \quad U_T \vec{x} U_T^\dagger = \vec{x} - \bar{\xi}. \)
2. rotation: \( U_R(\theta) \equiv \exp[i\theta c^\dagger c], \quad U_R \vec{x} U_R^\dagger = R\vec{x}. \)
3. squeeze: \( U_S(\alpha) \equiv \exp\left[\frac{\alpha}{4}(c^2 - (c^\dagger)^2)\right], \quad U_S \vec{x} U_S^\dagger = (e^\alpha \hat{x}, e^{-\alpha} \hat{y}). \)
4. shearing: \( U_H(\beta) \equiv \exp[-i\beta(c^\dagger - c)^2], \quad U_H \vec{x} U_H^\dagger = (\hat{x} + \beta \hat{y}, \hat{y}). \)

It is easy to check that
\[
U e^{i\vec{k} \cdot \vec{x}} U^\dagger = e^{i\vec{k} \cdot (V \vec{x})},
\] where \( V \) represents the corresponding transformations, translation, rotation, etc. on the plane.

The induced transformations of the scalar function is then calculated to be
\[
\hat{\phi}_U(\vec{x}) \equiv U \hat{\phi}(\vec{x}) U^\dagger \equiv \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k} \cdot \vec{x}} \tilde{\phi}_U(\vec{k}) \Rightarrow \varphi_U(\vec{x}) = \varphi(V \vec{x}).
\]

The induced transformations above generate a finite subgroup of \( w_{1+\infty} \), namely, the two-dimensional area-preserving diffeomorphism (APD). This finite subgroup is defined as the direct product of translation group (parameterized by a vector \( \vec{\xi} \)) and the \( SL(2, \mathbb{R}) \) group (parameterized by a \( 2 \times 2 \) matrix \( M \) with unit determinant), and the action of group element on the two-dimensional coordinates is given by
\[
V(x, y) = M \vec{x} + \vec{\xi} = (ax + by + \xi_x, cx + dy + \xi_y), \quad \det M = ad - bc = 1.
\]

The subgroup has an identity element \((M = 1, \vec{\xi} = 0)\), and one can show that the composition rule defines a closed algebra under group multiplication.
\[
V_1 \circ V_2 \equiv (M_2, \vec{\xi}_2) \circ (M_1, \vec{\xi}_1) = (M_2 \cdot M_1, M_2 \vec{\xi}_1 + \vec{\xi}_2).
\]

Finally, the associativity follows naturally from the definition of group multiplication.

Since there are only three independent parameters in \( SL(2, \mathbb{R}) \), we can decompose any group element in the \( SL(2, \mathbb{R}) \) by three independent transformations,
\[
M = M_R \cdot M_H \cdot M_S,
\]
where
\[
M_R \equiv \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad M_H \equiv \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \quad M_S \equiv \begin{pmatrix} e^\alpha & 0 \\ 0 & e^{-\alpha} \end{pmatrix}.
\]
These three transformations represent rotation, shearing, and squeeze, respectively. Given any $SL(2, R)$ matrix, one can solve the three independent group parameters as follows,

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \rightarrow \begin{cases} 
\alpha = \frac{1}{2} \ln (m_{11}^2 + m_{21}^2) \\
\beta = m_{11} m_{12} + m_{21} m_{22} \\
\theta = \arctan \left( -\frac{m_{21}}{m_{11}} \right)
\end{cases}.$$  \hspace{1cm} (44)

Finally, we list the generating functions associated with these transformations:

1. **translation**

$$\delta x = \xi_x = \frac{\partial F_T}{\partial y}, \quad \delta y = \xi_y = -\frac{\partial F_T}{\partial x}, \quad \Rightarrow \quad F_T = -\xi_y x + \xi_x y.$$ \hspace{1cm} (45)

2. **rotation**

$$\delta x = \theta y = \frac{\partial F_R}{\partial y}, \quad \delta y = -\theta x = -\frac{\partial F_R}{\partial x}, \quad \Rightarrow \quad F_R = \frac{\theta}{2} (x^2 + y^2).$$ \hspace{1cm} (46)

3. **squeeze**

$$\delta x = \alpha x = \frac{\partial F_S}{\partial y}, \quad \delta y = -\alpha y = -\frac{\partial F_S}{\partial x}, \quad \Rightarrow \quad F_S = \alpha xy.$$ \hspace{1cm} (47)

4. **shearing**

$$\delta x = \beta y = \frac{\partial F_H}{\partial y}, \quad \delta y = 0 = -\frac{\partial F_H}{\partial x}, \quad \Rightarrow \quad F_H = \frac{\beta}{2} y^2.$$ \hspace{1cm} (48)

One can now use these transformations to explicitly generate new solitons. For example, we can use squeeze and shearing to generate nonradially symmetric solitons. One other interesting case is the boosted solitons [22], which has been discussed in the literature. It has been known that Lorentz boost accompanied by a rescaling of $\theta$ remains a symmetry of noncommutative field theory. The boosted soliton is easily calculated to be

$$\phi^v_n(x, y, t; \theta) = \phi_n(\gamma(x - vt), \gamma y; \gamma \theta) = \phi_n(\sqrt{\gamma}(x - vt), \frac{y}{\sqrt{\gamma}}; \theta)$$ \hspace{1cm} (49)

where $\gamma$ is the Lorentz boost parameter and $\phi_n$ are the radially symmetrical GMS solitons, eq.(14). The first equality of eq.(49) defines the Lorentz boost of noncommutative field theory. It is now clear from the second equality that the boosted soliton in fact results from a combined transformations of translation and squeeze (without $\theta$ rescaling) discussed in this section. The key mechanism that is functioning behind these transformations is the
principle of area-preserving, or $W_{1+\infty}$ algebra, discussed in this paper. It is worth noting that the boosted solution of finite $\theta$ scalar field theory considered in [22] is area-preserving only approximately since the $U(\infty)$ symmetry is violated by the kinetic term.

5 Summary and Discussion

In this paper, we have identified the $U(\infty)$ symmetries of noncommutative field theory and $W_{1+\infty}$ algebras of quantum Hall theory. Many important properties of the $W_{1+\infty}$ algebra, such as area-preserving, Moyal noncommutativity and magnetic translations etc. can then help us to understand the dynamics of D-branes and noncommutative field theory. In particular, we have applied this result to explicitly construct various deformed noncommutative solitons including boosted solitons. The result presented here applies to both pure scalar theory with infinite $\theta$ parameter and scalar-gauge theories with finite or infinite $\theta$ parameter. It also justifies the close relation between quantum Hall theory and noncommutative field theory as have been suggested recently [14]. Many interplays between these two fast developing fields remain to be uncovered.

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