As Time Goes By:
Reflections on Treewidth for Temporal Graphs*

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Abstract

Treewidth is arguably the most important structural graph parameter leading to algorithmically beneficial graph decompositions. Triggered by a strongly growing interest in temporal networks (graphs where edge sets change over time), we discuss fresh algorithmic views on temporal tree decompositions and temporal treewidth. We review and explain some of the recent work together with some encountered pitfalls, and we point out challenges for future research.

Keywords: Network science, time-evolving network, link stream, NP-hardness, parameterized complexity, tree decomposition, monadic second-order logic (MSO).

1 Introduction

You must remember this: treewidth is one of the most important structural graph parameters [9], being extremely popular in parameterized algorithms. Without the contributions of Hans Bodlaender, this would be much less so.

Intuitively, the fundamental observation behind treewidth is that many NP-hard graph problems turn easy when restricted to trees. Indeed, typically a simple bottom-up greedy algorithm from the leaves to the (arbitrarily chosen) root of the tree suffices to solve many fundamental graph problems (including Vertex Cover and Dominating Set) efficiently on trees. This naturally leads to the investigation on how “tree-likeness” of graphs helps to solve problems efficiently. Fruitful results on this are provided by the concept of a tree decomposition and, correspondingly, the treewidth of a graph: if the treewidth is small, then otherwise NP-hard problems can be solved “fast”. Notably, the concepts of tree decomposition and treewidth are tightly connected to the existence of small graph separators (that is, vertex sets whose deletion partitions the graph into at least two connected components) that are arranged in a tree-like structure (see Section 2 for formalities and an example).

It is fair to say that tree decompositions currently are the most popular structural graph decompositions used in (parameterized) algorithms for (NP-hard) problems on (static) graphs. More specifically, these algorithmic results typically are “fixed-parameter tractability” results with respect to the parameter, that is, the studied problems then can be solved by an exponential-time algorithm whose exponential part exclusively depends on the treewidth of the input graph.

Computing the treewidth of a graph is NP-hard [4], even on graphs of maximum degree nine [17], but linear-time solvable if the treewidth is some fixed constant [11, 14] (more specifically, the running time is $c^{c^k} n$ [11]) and 5-approximable in $(c')^k n$ time [21] on n-vertex graphs of treewidth $k$, for some constants $c, c' > 1$. Polynomial-time algorithms are known for several restricted graph classes [16, 19, 20, 22]. From an algorithmic point of view, a tree decomposition of a graph typically allows for a dynamic programming approach. The twist is that these dynamic programs for many NP-hard problems run in polynomial time when the width of the tree decomposition is constant [12]. Indeed, many NP-hard problems are known to be fixed-parameter tractable when parameterized by treewidth [27, 33], underpinning the reputation of treewidth as one of the most fundamental algorithmically exploitable graph parameters, as confirmed in experimental studies [13, 15] (practically useful implementations for computing tree decompositions are also available [30, 31]). In this work, our goal is to discuss the role treewidth currently plays in the strongly emerging field of temporal graphs; these are graphs where the edge set may change over time. Further, we reflect on possible definitions of a temporal version of treewidth: temporal treewidth.

Temporal graphs model networks where adjacencies of vertices change over discrete time steps. In fact, many natural time-dependent networks can be modeled by temporal graphs, for instance interaction/contact networks, connection/availability networks, or bio-physical networks (see, e.g., [52, Section II]). Applications range from epidemiology over sociology to transportation. In the last decade, problems on temporal graphs gained increased attention in theoretical computer science [2, 3]

*Also known as time-varying graphs, evolving graphs, link streams, or dynamic graphs where no changes on the vertex set are allowed.

**Dedicated to Hans L. Bodlaender on the occasion of his 60th birthday.

The inclined reader, besides hopefully discovering interesting science, is also invited to enjoy a few quotes from a famous movie scattered around our text; the paper title is partially taken from the theme song of this movie.
This problem captures no information about time and occurrences of temporal edges. As we will see in Section 3, many temporal graph problems remain NP-hard even when the underlying treewidth is constant, indicating that the underlying treewidth is most probably missing useful “time-structural” information. This lack of information seems to be even larger for the layer treewidth, defined as $\text{tw}_{\infty}(G) := \max_{i \in \{1, \ldots, \tau\}} \text{tw}(G_i)$ for a temporal graph $G$ with layers $G_1, \ldots, G_\tau$. This is also expressed by the fact that $\text{tw}_{\infty}(G) \leq \text{tw}_{\downarrow}(G)$ for every temporal graph $G$.

Nevertheless, there are problems that are polynomial-time solvable on temporal graphs of constant underlying treewidth. As we will see in Section 4, several of these problems are actually fixed-parameter tractable when parameterized by the combination of the underlying treewidth and the lifetime. Indeed, in Section 5, we present a (temporal) adaption of the well-known technique of employing treewidth together with monadic second-order logic. Again, we can obtain several fixed-parameter tractability results when parameterized by $\text{tw}_{\downarrow} + \tau$.

In search of a more useful definition of temporal treewidth, we derive two requirements from observations in Sections 3 and 4: it should be at least as large as the underlying treewidth and upper-bounded by some function in the combination of the underlying treewidth and the lifetime. In Section 6, we will elaborate on this while reflecting on some possibly useful definitions for temporal treewidth. All temporal graph problems encountered in this work are summarized in the appendix.

## 2 Preliminaries

> But what about us? In this section, we provide some basic definitions and facts. By $\mathbb{N}$ and $\mathbb{N}_0$ we denote the natural numbers excluding and including zero, respectively. For any set $A$, we write $\binom{A}{k}$ for the set of all size-$k$ subsets from $A$.

### 2.1 Static Graphs and Treewidth

Let $G = (V, E)$ be a (static) graph with vertex set $V$ and edge set $E \subseteq \binom{V}{2}$. Alternatively, $V(G)$ and $E(G)$ also denote the vertex set and edge set of $G$, respectively. We write $G[W]$ for the subgraph induced by a set of vertices $W \subseteq V$ and use $G - W$ as a shorthand for $G[V \setminus W]$.

**Tree Decompositions and Treewidth.** In the following, we define (rooted and nice) tree decompositions and treewidth of static graphs, and we explain the connection to a cops-and-robber game.

**Definition 1** (Tree Decomposition, Treewidth). Let $G = (V, E)$ be an undirected graph. Then a tuple $T = (T, \{B_u \mid u \in V(T)\})$ consisting of a tree $T$ and

[Diagram of temporal graphs and tree decompositions shown here.]
a set of so-called bags \( B_u \subseteq V \) is a tree decomposition \((tdc)\) of \( G \) if

(i) \( \bigcup_{u \in V(T)} B_u = V \),

(ii) for every \( e \in E \) there is a node \( u \in V(T) \) such that \( e \subseteq B_u \), and

(iii) for every \( v \in V \), the graph \( T[\{u \in V(T) \mid v \in B_u\}] \) is a tree.

The width of \( \mathcal{T} \) is \( \text{width}(\mathcal{T}) := \max_{u \in V(T)} |B_u| - 1 \). The treewidth of \( G \) is the minimum width over all tree decompositions of \( G \), that is,

\[
\text{tw}(G) = \min_{\mathcal{T} \text{ is tdc of } G} \text{width}(\mathcal{T}).
\]

It follows from the definition, that for any edge \( \{u, u'\} \) of \( T \), the intersection of the corresponding bags \( B_u \cap B_{u'} \) is a separator of \( G \) of size at most width(\( \mathcal{T} \)) (as long as \( \mathcal{T} \) does not contain redundant bags). Refer to Figure 2 for an illustrative example.

A tree decomposition \( \mathcal{T} = (T, \{B_u \mid u \in V(T)\}) \) is rooted if there is a designated node \( r \in V(T) \) being the root of \( T \) (this allows to talk about children, parents, ancestors, descendants, etc. of the nodes of \( T \)). A rooted tree decomposition \( \mathcal{T} = (T, \{B_u \mid u \in V(T)\}) \) is nice if each node \( u \in V(T) \) is either (i) a leaf node (\( u \) has no children), (ii) an introduce node (\( u \) has one child \( v \) with \( B_v \subseteq B_u \) and \( |B_v \setminus B_u| = 1 \)), (iii) a forget node (\( u \) has one child \( v \) with \( B_v \supseteq B_u \) and \( |B_v \setminus B_u| = 1 \)), or (iv) a join node (\( u \) has two children \( v, w \) with \( B_v = B_w = B_u \)). Given a tree decomposition, one can compute a corresponding nice tree decomposition in linear time [53].

Alternatively, treewidth can be defined through a cops-and-robber game [63] as follows. Let \( G = (V, E) \) be an undirected graph, and \( k \in \mathbb{N} \).

- At the start, the \( k \) cops choose a set \( C_0 \subseteq \binom{V}{k} \) of vertices, and then the robber chooses a vertex \( r_0 \in V \setminus C_0 \).

- In round \( i \in \mathbb{N} \), first the cops choose \( C_i \subseteq \binom{V}{k} \), and then the robber chooses \( r_i \in V \setminus C_i \) such that \( r_i \) and \( r_{i-1} \) are connected in \( G \setminus (C_i \cap C_{i-1}) \).

The cops win if, after finitely many rounds, the robber is caught, that is, the robber has no vertex left to choose. The connection to treewidth is the following (which also implies an alternative definition for treewidth).

**Lemma 1** ([63]). Graph \( G \) has treewidth at most \( k \) if and only if at most \( k + 1 \) cops win the cops-and-robber game.

The **pathwidth** of a graph \( G \) is the minimum width over all tree decompositions \( \mathcal{T} = (T, \{B_u \mid u \in V(T)\}) \) with \( T \) being a path. Note that for every graph its treewidth is at most its pathwidth. For the graph in Figure 2, the treewidth and the pathwidth are equal (take the union of the bags \( A \) and \( B \)).

### 2.2 Temporal Graphs

Let \( G = (V, E, \tau) \) be a temporal graph (see Section 1). We also denote by \( V(G) \) the vertex set of \( G \). For any vertex subset \( W \subseteq V \), the temporal graph \( G[W] \) induced by \( W \) is defined as \( (W, \{e \in E \mid e \subseteq W\}, \tau) \). Further, we define \( G - W := G[V \setminus W] \). For a subset of temporal edges \( E' \subseteq E \), the temporal graph \( G - E' \) is defined as \((V, E \setminus E', \tau)\).

A **temporal walk** is defined as a sequence of temporal edges \( \{(v_1, v_2), (v_2, v_3), \ldots, (v_p, v_{p+1})\}, t_p \), each contained in \( E \) and \( t_1 \leq t_2 \leq \cdots \leq t_p \) (also called a temporal walk when the terminals are specified). A temporal walk is called **strict** if \( t_1 < t_2 < \cdots < t_p \). A **(strict) temporal path** is a (strict) temporal walk where additionally \( \alpha \leq t_{i+1} - t_i \leq \beta \) holds.

When analyzing problems on temporal graphs, the following concept often comes in handy.

**Definition 2** **((Strict) Static Expansion).** The static expansion of a temporal graph \( G = (V, E, \tau) \) is a directed graph \( H := (V', A) \), with vertices \( V' = \{v \in V \mid t_j \leq \tau\} \) and arcs \( A = A' \cup A_{col} \) where the first set \( A' := \{(u_{i,j}, v_{i,j}) \mid (v_i, v_j, t) \in E\} \) contains the arcs within the layers, and the second set \( A_{col} := \{(u_{t,j}, u_{t+1,j}) \mid v_j \in V, t \in \{1, \ldots, \tau - 1\}\} \) contains the arcs connecting different layers. We refer to \( A_{col} \) as column-edges of \( H \).

A static expansion is called **strict** if its vertex set \( V' \) additionally contains the vertex set \( \{u_{t+1,j} \mid v_j \in V\} \) and its arc set \( A' \) is replaced by the set \( A'' := \{(u_{t,j}, u_{t+1,j}), (u_{t,j}, u_{t+1, i}) \mid (v_i, v_j, t) \in E\} \).

Note that (strict) temporal walks correspond exactly to walks within the (strict) static expansion. Moreover, note that strict static expansions are always directed acyclic graphs.

### 2.3 Parameterized Complexity

We use standard notation and terminology from parameterized complexity theory [27, 33, 34, 43, 61]. A parameterized problem with parameter \( k \) is a language \( L \subseteq \{x, k\} \in \Sigma^* \times \mathbb{N} \) for some finite alphabet \( \Sigma \). A parameterized problem \( L \) is called **fixed-parameter tractable** if every instance \( (x, k) \) can be decided for \( L \) in \( f(k) \cdot |x|^{O(1)} \) time, where \( f \) is some computable function only depending on \( k \). The tool for proving that a parameterized problem is presumably not fixed-parameter tractable is to show that it is hard for the parameterized complexity class \( W[1] \). A parameterized problem \( L \) is contained in the complexity class \( \text{XP} \) if every instance \( (x, k) \) can be decided for \( L \) in \( |x|^{|\alpha|k} \) time, where \( g \) is some computable function only depending on \( k \). A parameterized problem \( L \) is **para-NP-hard** if the problem is NP-hard for some constant value of the parameter.
Graph Exploration (RTB-TGE) to decide whether there is a strict temporal walk start- 
poral graph Return-To-Base Temporal
3.1 Temporal Exploration

In the remainder of this section we present some concrete
example reduction in moderate detail. Our selected NP-
hardness reductions cover problems from the fields of
temporal exploration (Section 3.1), temporal reachability
(Section 3.2), and temporal matching (Section 3.3).

3.1 Temporal Exploration

In the problem called RETURN-TO-BASE TEMPORAL
GRAPH EXPLORATION (RTB-TGE), one is given a tem-
poral graph $G$ and a designated vertex $s$, and the task is
to decide whether there is a strict temporal walk start-
ning and ending at $s$ that visits all vertices in $V(G)$. The
NP-hardness of RTB-TGE follows by a simple reduc-
tion from HAMILTONIAN CYCLE (HC): given a directed
graph $G$, decide whether there is a (simple) cycle in $G$
that contains all vertices from $G$. However, from a pa-
rameterized view regarding the (underlying) treewidth,

RTB-TGE is much harder than HC; while HC param-
eterized by treewidth is fixed-parameter tractable, for
RTB-TGE we have the following.

Theorem 1 ([2, 18]). RETURN-TO-BASE TEMPORAL
GRAPH EXPLORATION is NP-hard even if
(i) the underlying graph is a star or
(ii) each layer is a tree and the underlying graph has
pathwidth at most two.

Akrida et al. [2] proved Theorem 1(i) via a reduction from
3-SAT(3), a special case of 3-SAT where each vari-
able appears in at most three clauses. See Figure 3(a) for
an illustration. In the reduction from 3-SAT(3), a star is
constructed where for each variable and clause in the
input 3-SAT formula there is a leaf in the star. Moreover,
each variable $x_i$ has two unique entry time steps $f_t$, $t_i$
and two unique exit time steps $f'_t$, $t'_i$, corresponding to set-
ing $x_i$ to false (entering at $f_t$ and leaving at $f'_t$) or true
(entering at $t_i$ and leaving at $t'_i$) with $f_t < f'_t < t_i < t'_i$.
Clearly, it is never beneficial for the explorer to linger in
a leaf longer than necessary. Now assume clause $C_j$ to
contain variable $x_i$ unnegated. Then we add to $C_j$ an
entry time step $f_t - \varepsilon$ and an exit time step $f'_t - \varepsilon$. Since
$f_t - \varepsilon < f_t < f'_t - \varepsilon < f'_t$, clause $C_j$ can be visited at
time $f_t - \varepsilon$ if and only if $x_i$ is set to true.

By adding analogous entry and exit time steps to $C_j$ for
all its contained variables (negated or unnegated), it
follows that $C_j$ can be visited if and only if at least one
of its literals is set to true.

Bodlaender and van der Zanden [18] proved Theo-
rem 1(ii) via a reduction from RTB-TGE with the un-
derlying graph being a star to (RTB-)TGE by adding a
long path to each layer, which is connected to some of
the star’s leaves in such a way that each layer is a tree
and the underlying graph has pathwidth at most two. See
Figure 3(b) for an illustration. In the reduction from RTB-TGE, let $G$ be the temporal graph with life-
time $\tau$ and the underlying graph being a star on ver-
ces $s$ and $v_1, \ldots, v_n$. Then a temporal graph $G'$ with
lifetime $\tau' = Q + \tau + 1$ is constructed from $G$ by adding
a path on vertices $p_0, \ldots, p_Q$ (highlighted in gray and
present in all layers), where $Q = \tau \cdot (n + 4)$, and app-
pending $Q + 1$ layers, in which each vertex $v_i$ is adjacent
only to $s$. Furthermore, in each of the first $\tau$ layers, each
vertex $v_i$ is connected to some vertex on the path $P$ if
and only if $v_i$ is the lowest numbered vertex in a con-
ected component. This guarantees that every layer is
connected. Equivalence holds since in the first $\tau$ time
steps, $G$ must be explored, and in the remaining $Q + 1$
Table 1: Overview on treewidth-related results for NP-hard temporal graph problems. See Appendix A for respective problem definitions. “FPT”, “p-NP-h”, and “?” abbreviate “fixed-parameter tractable”, “para-NP-hard”, and “open”, respectively.

| Problem | Parameter | $tw_\infty$ | $tw_\bot$ | $\tau$ | $|V|$ | $tw_\bot + \tau$ | Ref. |
|---------|-----------|-------------|-----------|-------|------|----------------|-----|
| **Temporal Graph Exploration** | | | | | | | |
| | | | | | | | |
|返回基时间图探索 | | | | | | | |
| | | | | | | | |
| $(\alpha, \beta)$-可达性时间边删除 | | | | | | | |
| | | | | | | | |
| $\text{Min-Max}$-可达性时间排序 | | | | | | | |
| | | | | | | | |
| $\text{Min}$-可达性时间合并 | | | | | | | |
| | | | | | | | |
| Temporal Matching | | | | | | | |
| | | | | | | | |
| Temporal Separation | | | | | | | |
| | | | | | | | |
| Minimum Single-Source Temporal Connectivity | | | | | | | |

| | | | | | | | |

| Figures: Illustration of reductions behind Theorem 1 (i) and (ii). (a) A star with leaves corresponding to clauses and variables [2]. (b) A star with center $s$ and leaves $v_1, \ldots, v_n$; a path (highlighted in gray) on $Q+1$ vertices is attached to the star [18].

Theorem 2 ([37]). $(\alpha, \beta)$-可达性时间边删除是NP-hard even if the underlying graph consists of two stars with adjacent centers.

The proof of Theorem 2 employs a reduction from CLIQUE: given an undirected graph $G$ and an integer $r$, decide whether $G$ contains a clique (a graph where each pair of vertices is adjacent) with $r$ vertices. In the corresponding construction, adjacencies among the vertices in the CLIQUE instance are encoded by time stamps. See Figure 4(a) for an illustration. So suppose that a CLIQUE instance with the vertex set $\{v_1, \ldots, v_m\}$, edge set $\{e_1, \ldots, e_m\}$, and solution size $r$ is given. The constructed underlying graph consists of two stars with $m$ leaves each and adjacent centers $x$ and $y$. The leaves of the first star are only connected to $x$ at time step 1, thus $x$ is the source vertex that reaches the most other vertices. For each $e_i$, the edge $(x, y)$ is present at time step $i\beta + 2$. For each edge $e_i = \{e_i, v_j\}$, the edge connecting $y$ and the vertex $e_i$ (the vertex corresponding to edge $e_i$) is present at time steps $i\beta + \alpha + 2$ and $j\beta + \alpha + 2$. Observe that reaching $e_i$ from $x$ by a strict $(\alpha, \beta)$-temporal path requires the edge $(x, y)$ to be present at time $i\beta + 2$ or at time $j\beta + 2$. Thus, if $v_{i_1}, \ldots, v_{i_r}$ form a clique on $r$ vertices, then deleting...
the temporal edge set \( \{(x,y), (\ell \beta + 2) \mid \ell \in \{i_1, \ldots, i_r\}\} \) reduces the \((\alpha, \beta)\)-reachability of \(x\) by \(\binom{r}{2}\). Conversely, reducing the \((\alpha, \beta)\)-reachability of \(x\) by \(\binom{r}{2}\) with only \(r\) deletions is impossible unless a clique of size \(r\) exists in the input.

**Reordering of Layers.** Enright et al. [36] proved the following hardness result for the Min-Max Reachability Temporal Ordering (Min-Max RTO) problem: given a temporal graph \(G = (V, E, \tau)\) and an integer \(k \in \mathbb{N}\), decide whether there is a bijection \(\phi : \{1, \ldots, \tau\} \rightarrow \{1, \ldots, \tau\}\) such that the maximum reachability (that is, the maximum number of vertices any vertex can reach via a strict temporal path) in \(G' = (V, \{(e, \phi(t)) \mid (e, t) \in E\}, \tau)\) is at most \(k\). Correspondingly, Max-Min RTO is defined by exchanging “maximum” by “minimum” and “at most” by “at least”.

**Theorem 3 ([36]).** Min-Max Reachability Temporal Ordering and Max-Min Reachability Temporal Ordering are NP-hard even when the underlying graph is a tree obtained by connecting two stars using a path.

Enright et al. proved Theorem 3 (similarly to the previously presented reduction by Enright et al. [37]) via a reduction from CLIQUE. See Figure 4(b) for an illustration. In their reduction, the input consists of vertex set \(\{v_1, \ldots, v_n\}\), edge set \(\{e_1, \ldots, e_m\}\), and solution size \(r\). Each layer \(G_i\) corresponds to a vertex \(v_i\): the edge \(\{y, e_\ell\}\) is present in \(G_i\) if and only if \(v_i \in e_\ell\). That is, the incidence of an edge with vertex \(v_i\) is represented by the presence of that edge in layer \(G_i\). Hence, if \(v_{i_1}, \ldots, v_{i_r}\) form a clique on \(r\) vertices, then mapping \(i_1, \ldots, i_r\) to the first \(r\) layers disallows \(s\) to reach \(\binom{r}{2}\) leaves adjacent to \(y\) (since the \(s-y\) path contains \(r + 1\) vertices).

**Merging of Layers.** Deligkas and Potapov [29] studied reachability minimization/ maximization under certain layer-merging operations and showed hardness results on trees and paths. In this context, merging an interval of time stamps \(M \subseteq \{1, \ldots, \tau\}\) in \(G\) means replacing each temporal edge \((e, \ell)\) with \(\ell \in M\) by a new temporal edge \((e, \max(M))\). Thus, the appearance of all temporal edges within this interval \(M\) is shifted to the end of \(M\). More precisely, Deligkas and Potapov considered the Min Reachability Temporal Merging (MRTM) problem: given a temporal graph \(G = (V, E, \tau)\), a set of sources \(S \subseteq V\), and three integers \(\lambda, \mu, k \in \mathbb{N}\), decide whether there are \(\mu\) disjoint intervals \(M_1, \ldots, M_\mu\), each of size \(|M_i \cap \{1, \ldots, \tau\}| = \lambda\), such that, after merging each of them in \(G\), the number of vertices reachable from \(S\) is at most \(k\).

**Theorem 4 ([29]).** Min Reachability Temporal Merging is NP-hard even when the underlying graph is a path.

The proof of Theorem 4 employs a reduction from MAX2SAT(3), a variant of the MAX2SAT problem where each variable occurs in at most three clauses. (In the MAX2SAT problem, the goal is to find a truth assignment maximizing the number of satisfied clauses of a given 2-SAT formula.)

For each clause in a given input instance, a separate subpath containing nine vertices of the underlying path is used and labeled as shown in Figure 5(a). Here, \(c\) is the index of the clause \((x_i \lor \neg x_j)\) and we may assume \(c\) to always be much smaller than \(i\) and \(j\). The middle vertex \(s\) of this subpath is added to the set \(S\) of sources and we take the merge size as \(\lambda = 2\). Then it is possible to either merge \(\{4c, 4c + 1\}\), thus preventing \(s\) from reaching the three bottom left vertices, or to merge \(\{4c + 1, 4c + 2\}\), thus blocking the three bottom right vertices, but not both (due to the disjointness condition). Hence, given a large enough number of merges, each source \(s\) can reach at most five other vertices. If we want to reduce this number to four, then one must additionally merge \(\{4i, 4i + 1\}\) (thus setting \(x_i\) to true) or merge \(\{4j + 1, 4j + 2\}\) (thus setting \(x_j\) to false), i.e., give an assignment satisfying clause \(c\).

If the underlying graph is allowed to be a ternary tree, then this construction can be modified to only require a single source vertex [29].

### 3.3 Temporal Matching

Mertzios et al. [58] proved hardness for the Temporal Matching (TM) problem: given a temporal graph \(G = (V, E, \tau)\) and integers \(k, \Delta \geq 0\), decide whether there is a \(\Delta\)-temporal matching of cardinality at least \(k\) in \(G\). A \(\Delta\)-temporal matching is a set \(E' \subseteq E\) of temporal edges such
that for every two temporal edges \((e, t), (e', t')\) ∈ \(\mathcal{E}'\), we have that \(e \cap e' = \emptyset\) or \(|t - t'| \geq \Delta\).

**Theorem 5** ([58]). **Temporal Matching** is NP-hard even when the underlying graph is a path.

The crucial observation is that solving TM on a temporal graph \(G\) is equivalent to solving INDEPENDENT SET on the so-called \(\Delta\)-temporal line graph of \(G\), which contains a vertex for each temporal edge of \(G\) and two vertices adjacent if the corresponding temporal edges cannot be both contained in a \(\Delta\)-temporal matching \([58, \text{Definition 2}]\). For an illustration see Figure 5(b). Moreover, if the underlying graph \(G'_1\) is a path with \(m\) edges, then its 2-temporal line graph is an induced subgraph of a diagonal grid graph of size \(m \times \tau\), and conversely each such grid can be obtained as a 2-temporal line graph. Here, a diagonal grid graph is simply a grid that additionally contains the two diagonal edges of every grid cell. Subsequently, Mertzios et al. proved that INDEPENDENT SET is NP-complete on induced subgraphs of diagonal grid graphs, thus also showing NP-hardness of TM.

4 Dynamic Programming Based on an Underlying Tree Decomposition

»It’s still the same old story...« For many graph problems, algorithms exploiting small treewidth are dynamic programs over a corresponding tree decomposition. For temporal graph problems, few such dynamic programs are known. Yet, we present four dynamic programs known from the literature: Two XP-algorithms for two NP-hard problems, and two polynomial-time algorithms. For the former two algorithms, the running time depends exponentially on the lifetime \(\tau\) (hence proving fixed-parameter tractability regarding \(\text{tw}_1 + \tau\) in both cases). This supports our intuition that while capturing the structure of the graph, the underlying treewidth is missing relevant time aspects.

4.1 Two XP-Algorithms

The two XP-algorithms \([6, 46]\) we sketch indeed both are FPT-algorithms regarding the combination \(\text{tw}_1 + \tau\) of the underlying treewidth and the lifetime.

**An XP-Algorithm for Temporal Separation.** Fluschnik et al. \([46]\) studied TEMPORAL SEPARATION, which is the problem of deciding whether all (strict) temporal paths connecting two given terminal vertices \(s\) and \(z\) in a temporal graph \(G\) can be destroyed by removing a set \(S \subseteq V\setminus \{s, z\}\) of at most \(k\) vertices. Such a set \(S\) is called a (strict) \(s-z\) separator. Fluschnik et al. \([46]\) employed dynamic programming on a given tree decomposition to prove that this problem is fixed-parameter tractable when parameterized by \(\text{tw}_1 + \tau\), and is in XP when parameterized by \(\text{tw}_1\).

**Theorem 6** ([46]). **Temporal Separation with given tree decomposition of the underlying graph is solvable in** \(O((\tau + 2)^{\text{tw}_1+2} \cdot \text{tw}_1 \cdot |V| \cdot |E|)\) **time.**

The dynamic program behind Theorem 6 is based on the fact that for each vertex \(v \in V\setminus \{s\}\) in a temporal graph \(G = (V, \mathcal{E}, \tau)\) there is a time step \(t \in \{1, \ldots, \tau\}\) such that \(v\) cannot be reached from \(s\) before \(t\). In particular, one guesses a partition \(V = A_1 \uplus A_2 \uplus \ldots \uplus A_r \uplus S \uplus Z\) such that (i) \(S\) is a temporal \(s-z\) separator, (ii) \(G - S\) no vertex contained in \(Z\) is reachable from \(s\), and (iii) no vertex \(v \in A_t\) can be reached from \(s\) before time step \(t\), where \(t \in \{1, \ldots, \tau\}\). See Figure 6 for an illustrative example.

**An XP-Algorithm for Temporally Connected Subgraphs.** Given a temporal graph \(G = (V, \mathcal{E}, \tau)\) and a designated vertex \(r \in V\), a temporally \(r\)-connected spanning subgraph of \(G\) is a temporal graph \(G' = (V, \mathcal{E}', \tau)\) with \(\mathcal{E}' \subseteq \mathcal{E}\) such that \(G'\) contains a temporal path from \(r\) to any other vertex \(v \in V\) in \(G'\). The task in MINIMUM SINGLE-SOURCE TEMPORAL CONNECTIVITY (\(r\)-MTC) is to find a temporally \(r\)-connected spanning subgraph for a given vertex \(r\) such that the total weight \(\sum_{(e,t) \in \mathcal{E}} w(e, t)\) is minimized, where \(w\) is an arbitrary nonnegative weight function. Axiotis and Fotakis \([6]\) employed dynamic programming on a nice tree decomposition \([53]\) to prove that MINIMUM SINGLE-SOURCE TEMPORAL CONNECTIVITY (\(r\)-MTC) is fixed-parameter tractable when parameterized by \(\text{tw}_1 + \tau\), and is contained in XP when parameterized by \(\text{tw}_1\) alone.

**Theorem 7** ([6]). **Minimum Single-Source Temporal Connectivity with given nice tree decomposition of the underlying graph is solvable in** \(O(3^w \cdot (\tau + \text{tw}_1)^{\text{tw}_1+1} \cdot |V|)\) **time.**
The idea of the dynamic program behind Theorem 7 using a nice tree decomposition rooted at the source $r$ is as follows (see Figure 7 for an illustration): for each bag, the vertices contained in the bag are (bi-)partitioned into vertices connected to $r$ (we call them local sources) and vertices not (yet) connected to $r$. Vertices in the subgraph induced by the vertices in the bag and all its descendant bags must be reachable from the local sources by a temporal path starting “late enough” (i.e., after the local source has been reached from source $r$). For each node $x$ in the nice tree decomposition, a table entry $f(x | a_1, t_1, \ldots, a_{tw_j}, t_{tw_j})$ stores the minimum cost of a temporal subgraph such that in the graph induced by all the vertices in the bags of $x$ and all its descendants, every vertex is reachable from some vertex $v^j$ in $B_x$ with $a_j = 1$ (the local sources) by some temporal path starting at time step $t_j$ or later. Hence, each such node $x$ has a table entry for each of the $2^{tw_j}$ possible bipartitions, and each of the $\tau^{tw_j}$ possible starting times for the temporal paths starting at local sources.

For both of the two presented problems, it appears to be crucial to guess the time steps in which a solution “touches” the corresponding bag. However, in both it is open whether the dependencies on $\tau$ (being the base of exponent $tw_j$; see Theorems 6 and 7) can be avoided: Is Temporal Separation (TS) or Minimum Single-Source Temporal Connectivity (r-MTC) fixed-parameter tractable when parameterized by $tw_j$?

### 4.2 Two Fixed-Parameter Polynomial-Time Algorithms

The underlying tree decomposition, when part of the input, can also be used for tasks solvable in polynomial time. In this section, we present two algorithms making use of the underlying tree decomposition, one for temporal exploration, and one for computing foremost temporal walks.

Temporal Exploration. In Section 3, we discussed the NP-hardness of determining the exact time required to fully explore a temporal graph. However, as long as each layer of the input temporal graph is connected and the underlying treewidth is low, it can be shown that a subquadratic number of steps is always sufficient. More precisely, Erlebach et al. [40] proved the following by giving an algorithm that utilizes a given tree decomposition.

**Theorem 8 ([40]).** If every layer is connected, then a temporal graph $G$ can be explored in $O(|V|^{3/2} \cdot tw_\hat{V}^{3/2} \cdot \log |V|)$ steps.

The proof of Theorem 8 builds upon the observation that an agent needs at most $n - 1$ steps to move from any vertex to any other vertex if both of these vertices are connected in every layer. The idea is then to divide up $G_i(G)$ into sufficiently small subgraphs to which this observation can then be applied.

To this end, select a vertex set $S$ as the union of $O(\sqrt{|V|/tw_\hat{V}})$ bags of a nice tree decomposition of $G_i(G)$ in such a way that every connected component of $G_i(G) - S$ has size at most $O(\sqrt{|V|/tw_\hat{V}})$. If we consider a time window of $\Theta(tw_\hat{V} \cdot \sqrt{|V|/tw_\hat{V}})$ layers, then, by the pigeonhole principle, for any vertex $v$ in any of these connected components, there is a vertex $w \in S$ that is in the same connected component in at least $\Theta(\sqrt{|V|/tw_\hat{V}})$ layers. Thus, by the above observation, an agent at $w$ can reach $v$ and return to $w$ within $O(tw_\hat{V} \cdot \sqrt{|V|/tw_\hat{V}})$ time steps.

Hence, if we use $\Theta(tw_\hat{V} \cdot \sqrt{|V|/tw_\hat{V}})$ agents, then each starting at a vertex of $S$, we can explore $G$ in at most $O(tw_\hat{V} \cdot \sqrt{|V|/tw_\hat{V}} \cdot \sqrt{|V|/tw_\hat{V}}) = O(|V|)$ steps. From this, one can derive an upper bound of $O(|V|^{3/2} \cdot tw_\hat{V}^{3/2} \cdot \log(|V|))$ steps if only a single agent is used to perform these explorations sequentially.

Computing Foremost Walks. A (strict) foremost $s$-$z$ walk is a temporal walk that arrives earliest among all $s$-$z$ temporal walks. Himmel [50] proved that foremost walk queries can be answered quickly using a specific data structure that relies on a (given) underlying tree decomposition.

**Theorem 9 ([50]).** There exists a data structure of size $O(tw_\hat{V}^2 \cdot \tau \cdot |V|)$ computable in $O(tw_\hat{V}^2 \cdot \tau^2 \cdot |V|)$ time such that one can find a foremost walk between two vertices on temporal graphs with underlying treewidth $tw_\hat{V}$ in $O(tw_\hat{V}^2 \cdot \tau \cdot \log |V| \cdot \log (tw_\hat{V} + \tau \cdot \log |V|))$ time.

The data structure behind Theorem 9 was originally introduced by Abraham et al. [1] for computing shortest path queries in static graphs. It exploits binary tree decompositions of depth $O(\log |V|)$. Basically, the preprocessing for the data structure computes the earliest
α function Courcelle’s famous theorem, connects

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\[\text{are constructed from atomic formulas using boolean}\]

paths connecting \(v\) to any vertex \(w\) at any possible starting time, where \(v\) and \(w\) are contained in the same bag of the tree decomposition.

5 Monadic Second-Order Logic for Temporal Graphs

»Honest as the day is long!« Courcelle’s famous theorem states that every graph property definable in monadic second-order logic is fixed-parameter tractable when simultaneously parameterized by the treewidth of the graph and the length of the formula [26]. In this section, we review how one could lift this powerful classification tool to temporal graphs and spot some pitfalls having led to flaws in the literature.

For monadic second-order (MSO) logic on a graph \(G\) we need a structure consisting of a universe \(U = V(G) \cup E(G)\) and a vocabulary consisting of two unary relations \(V \subseteq U\) and \(E \subseteq U\) containing the vertices and the edges, respectively, and two binary relations \(\text{adj} \subseteq U \times U\) and \(\text{inc} \subseteq U \times U\), where \((v, w) \in \text{adj}\) if and only if \(\{v, w\} \in E(G)\), and \((v, e) \in \text{inc}\) if and only if \(v \in e\).

For a fixed finite set of (monadic) variables, an atomic formula over vocabulary \(v\) is of the form \(x_1 = x_2\) or \(R(x_1, x_2)\) or \(R'(x_1)\), where \(R \in \{\text{adj}, \text{inc}\}, R' \in \{V, E\}\) and \(x_1, x_2 \in U\). Here, \(R(x_1, x_2)\) (\(R'(x_1)\)) evaluates to true if and only if \((x_1, x_2) \in R(x_1, R')\).

MSO formulas are constructed from atomic formulas using boolean operations \(\neg, \lor, \land\) and existential and universal quantifiers \(\exists, \forall\) over variables and set variables. For further details, refer to Courcelle and Engelfriet [26].

Example 1. The well-known CLIQUE problem can be expressed by the following MSO formula: \(\exists X. (\forall x, y \in X. (V(x) \land V(y) \land \text{adj}(x, y)))\).

The following, known as an optimization variant of Courcelle’s theorem, connects MSO and treewidth.

Theorem 10 ([5, 26]). There exists an algorithm that, given (i) an MSO formula \(\rho\) with free monadic variables \(X_1, \ldots, X_r\), (ii) an \(n\)-vertex graph \(G\), and (iii) an affine function \(\alpha(x_1, \ldots, x_r)\), finds the minimum (maximum)

\[\text{of } \alpha(|X_1|, \ldots, |X_r|)\text{ over evaluations of }X_1, \ldots, X_r\text{ for which formula }\rho\text{ is satisfied on }G.\]

The running time of this algorithm is \(f(|\rho|, tw(G)) \cdot n\), where \(f\) is a computable function, \(|\rho|\) is the length of \(\rho\), and \(tw(G)\) is the treewidth of \(G\).

Having Theorem 10 at hand, we can prove that CLIQUE (see Example 1) is fixed-parameter tractable when parameterized by the treewidth of the input graph. The formula given in Example 1 has one free monadic variable and constant length \(c\), hence, with \(\alpha\) being the identity function, we can decide CLIQUE in \(f(c, tw(G)) \cdot |V|\) time.

We are aware of two successful approaches and one flawed approach to lift Theorem 10 to the temporal setting. We will survey in Sections 5.1 and 5.2 the two successful approaches, and discuss in Section 5.3 the flawed approach.

5.1 Using Labels

Arnborg et al. [5] showed that it is possible to apply Theorem 10 to graphs in which edges have labels from a fixed finite set, either by augmenting the graph logic to incorporate predicates describing the labels, or by representing the labels by unquantified edge set variables. Zschoche et al. [65] exploited this for temporal graphs as follows (see Figure 8 for an example): For a given temporal graph \(G\) of lifetime \(\tau\), define the edge-
labeled graph $L(\mathcal{G})$ as the underlying graph $G_1$ with the added edge-labeling $\omega: E(G_1) \to \{1, \ldots, 2^t - 1\}$ such that $\omega((v, w)) = \sum_{i=1}^{2^t-1} I_{(v, w) \in E_i} \cdot 2^{-i}$, where $I_{(v, w) \in E_i} = 1$ if and only if $\{(v, w), i\} \in \mathcal{E}$, and 0 otherwise. Observe that the $i$-th bit of a label now expresses whether the edge is present in the $i$-th layer of the temporal graph. Hence, we can check whether an edge $e$ is present in layer $t$ using the MSO formula $\text{layer}(e, t) := \bigvee_{j=1}^{t} \bigwedge_{\sigma(i, 2^j-1)} (t = i \land \omega(e) = j)$ of length $2^{O(t)}$, where $\sigma(i, z) := \{x \in \{1, \ldots, z\} | \text{$i$-th bit of $x$ is 1}\}$. Furthermore, we can determine whether two vertices $v$ and $w$ are adjacent in layer $t$ using the MSO formula $\text{adj}(v, w, t) := 3e \in E. (\text{inc}(e, v) \land \text{inc}(e, w) \land \text{layer}(e, t))$ of length $2^{O(t)}$. Altogether, in a nutshell we get the following: If a temporal graph problem $\Pi$ can be formulated by an MSO-formula which uses $\text{layer}(e, t)$ and $\text{adj}(v, w, t)$ as black boxes, then $\Pi$ is fixed-parameter tractable when parameterized by the combination of the length of the formula, the underlying treewidth, and the lifetime $\tau$. Zschoche et al. [65] derived an MSO-formula for TEMPORAL SEPARATION (TS), where the length of the formula is upper-bounded by some function in $\tau$. Hence, TS is fixed-parameter tractable when parameterized by the combination of the underlying treewidth and the lifetime.

5.2 Enriching the Vocabulary

Another approach, used by Enright et al. [37], can be applied to exchange the dependency on $\tau$ with a dependency on the maximum temporal total degree $\Delta_\tau$, which is the maximum number of temporal edges incident to the same vertex in temporal graph $\mathcal{G}$. Observe that the maximum temporal total degree is at least the maximum degree of the underlying graph. Moreover, the parameters lifetime and maximum temporal total degree are unrelated to each other, meaning that the maximum temporal total degree can be large while the lifetime is small and vice versa (see Figure 9 for two examples).

In a nutshell, we alter the universe and the vocabulary of the structure (we refer to this structure as enriched) in order to express a temporal graph problem. We add all temporal edges $(e, t)$ of the temporal graph $\mathcal{G}$ to the universe and equip the vocabulary with two binary relation symbols $\mathcal{L}$ and $\mathcal{R}$, where

- $(e, (e, t)) \in \mathcal{L}$ if and only if $e$ is an edge in the underlying graph and $(e, t)$ is a temporal edge of $\mathcal{G}$, and
- $(e_1, t_1), (e_2, t_2) \in \mathcal{R}$ if and only if $(e_1, t_1)$ and $(e_2, t_2)$ are temporal edges where $e_1$ and $e_2$ have a vertex in common and $t_1 < t_2$.

It is easy to see that the treewidth of the Gaifman graph\footnote{In the Gaifman graph of a structure, there is one vertex for each element in the universe and two vertices have an edge if and only if the corresponding elements occur together in the same relation.} for the enriched structure is upper-bounded by a function of the treewidth of the underlying graph of the temporal graph and the maximum temporal total degree. Hence, due to Courcelle and Engelfriet [26], if a temporal graph problem $\Pi$ can be formulated by an MSO-formula in the enriched structure, then $\Pi$ is fixed-parameter tractable when parameterized by the combination of the underlying treewidth, the maximum temporal total degree, and the length of that formula. Enright et al. [37] derived an MSO-formula in the enriched structure for TEMPORAL REACHABILITY EDGE DELETION, where the length of the formula depends on $h$ (the size of the set of reachable vertices), hence proving fixed-parameter tractability for the problem when parameterized by the combination of $h$, the underlying treewidth, and the maximum temporal total degree.

5.3 Pitfalls in the Literature

Mans and Mathiason [54] also explored the direction of enriching the vocabulary in the context of dynamic graphs. In their model, vertices can (dis)appear over time as well. Furthermore, the layers are not necessarily arranged in a linear (time) ordering. Hence, their model of dynamic graphs is more general than temporal graphs. However, some of their results seem flawed. In the remainder of this section we discuss these flaws in the special case of temporal graphs.

Mans and Mathiason construct a so-called treewidth-preserving structure. Here, the universe has for each vertex $v$ of the temporal graph $\tau$ many copies $v^1, \ldots, v^\tau$, one element $t_i$ for each $i \in \{1, \ldots, \tau\}$, and an additional element $s$. Note, that there is a unary relation symbol $L_v$ which contains an element $x$ if and only if the element $x$ is generated from the vertex $v$ ($x \equiv v^t$, for some $t \in \{1, \ldots, \tau\}$). The Gaifman graph of a treewidth-preserving structure of a temporal graph is the disjoint union of the layers. Additionally, there is one long path starting at some special vertex $s$ and then “visits” all layers in the time induced order, see Figure 10 for an illustration.

On the good side, this keeps the treewidth of the Gaifman graph upper-bounded by a function in the maximum treewidth over all layers. Furthermore, one can still express (in MSO) time relations between elements of different layers, for example by measuring the distance to $s$.

On the problematic side, having two elements $v$ and $w$ at hand which represent vertices in some layer of the temporal graph, it seems difficult to get an MSO-formula which evaluates to true if and only if $v$ and $w$ are generated from the same vertex. To do so, Mans and Mathiason [54] used an expression $f_V(v) = f_V(w)$. It is unclear whether $f_V$ is in fact part of the treewidth-preserving structure or not. Note that the length of such an expression in terms of the unary relation symbols $L_v$ depends on the number of vertices in the temporal graph. If the expression $f_V(v) = f_V(w)$ is an short cut for an expression of size at least the number of vertices in the temporal graph, then Lemmata 13 and 17 and hence Corollaries 14–16 and 18 of Mans and Mathiason [54] break. We believe that it is rather unlikely that one can provide arguments to repair the idea of Mans and Mathiason [54] because of the following example.

**Example 2.** The following is a polynomial-time algorithm for the NP-complete 3-COLORING problem.
of the following two straightforward ways:

1. Take the maximum over all layer treewidths, resulting in $\text{tw}_{\infty}$.
2. Take the treewidth of the underlying graph, resulting in $\text{tw}_{\downarrow}$.

Since the treewidth of a graph does not increase when edges are removed, we naturally get that for any temporal graph $G$ it holds that $\text{tw}_{\infty}(G) \leq \text{tw}_{\downarrow}(G)$. We can also observe that these two variants of temporal treewidth are invariant under reordering of the layers and hence might not be considered truly temporal since they also apply to the unordered “multilayer setting”.

There is a further generic way to transfer a structural graph parameter to the temporal setting. This one is particularly interesting in the context of problems that make use of $\Delta$-time windows\(^3\), as done in recent work on Restless Temporal Paths\([24]\), Temporal Clique\([7, 51, 60, 64]\), Temporal Coloring\([57]\), Temporal Matching\([58]\), and Temporal Vertex Cover\([3]\). In the case of treewidth we call this parameter $\Delta$-slice treewidth\(^4\), and as the name suggests, it depends on an additional natural number $\Delta$ that is typically part of the input or the problem specification. The $\Delta$-slice treewidth is the maximum of the treewidths of the union graphs of all $\Delta$-time windows, formally defined as follows:

**Definition 3 ($\Delta$-Slice Treewidth).** For a temporal graph $G = (V, E_1, \ldots, E_\tau)$ and a natural number $\Delta \leq \tau$, the $\Delta$-slice treewidth $\text{tw}_\Delta(G)$ of $G$ is defined as

$$\text{tw}_\Delta(G) := \max_{i \in \{1, \ldots, \tau - \Delta + 1\}} \text{tw}(G_i^{(\Delta)}),$$

where $G_i^{(\Delta)} = (V, \bigcup_{j \in \{i, \ldots, i+\Delta-1\}} E_j)$.

\(^3\)A $\Delta$-time window is a set of $\Delta$ consecutive time steps.

\(^4\)To the best of our knowledge, the concept of a “$\Delta$-slice parameter” was introduced by Himmel et al.\([51]\) to define a temporal version of degeneracy. It was later also used by Bentert et al.\([7]\).
It is easy to see the \( \Delta \)-slice treewidth interpolates between layer treewidth and underlying treewidth, hence we have that \( \tw_{\infty}(G) \leq \tw_{\Delta}(G) \leq \tw_{1}(G) \), for all temporal graphs \( G \) and all \( \Delta \leq \tau \).

In light of the known results of temporal graph problems where treewidth is used as a parameter (see Table 1 in Section 3), we observe that even for the largest of our established concepts of treewidth of a temporal graph, namely the underlying treewidth, we already obtain para-NP-hardness results for many temporal graph problems. Hence, temporal treewidth versions such as \( \Delta \)-slice treewidth, which are upper-bounded by the underlying treewidth, are not desirable since on their own they presumably do not offer new ways to obtain tractability results.

As to islands of tractability (see Table 1 or apply Theorem 10), we find many FPT-algorithms for the combined parameter \( \tw_{1} + \tau \) and for the parameter number \( |V| \) of vertices for a variety of temporal graph problems. Further examples of FPT results that are not included in Table 1 are Multistage Vertex Cover [45], Restless Temporal Paths [24], and Temporal Coloring [57]. This means that if we »round up the usual suspects« of which (combination of) parameters should lower- and upper-bound the temporal treewidth in our endeavor of finding useful definitions, then we might want to be on the look-out for something between \( \tw_{1} \) and \( \tw_{1} + \tau \) or something between \( \tw_{1} \) and \( |V| \), or something that is incomparable to the aforementioned parameters.

In the following, we are going to discuss three canonical ways to approach defining treewidth for temporal graphs:

1. Adapting tree decompositions to temporal graphs (Section 6.1).
2. Deriving static graphs from a temporal graph in a natural way and using the treewidth of those graphs (Section 6.2).
3. Looking at ways to play cops-and-robber games on temporal graphs (Section 6.3).

### 6.1 Adapts of the Tree Decomposition

One beacon of treewidth applications has always been the tree decomposition. Hence, it is only logical to begin our quest for temporal treewidth in the decomposition territory. First, we have to ask ourselves which general properties we want a temporal tree decomposition to have. Should it be temporal as well? We could try to take inspiration from Bodlaender [10] who showed how to maintain a tree decomposition under edge additions and deletions (when the treewidth is at most two). However, it seems difficult to perform dynamic programming (which is the standard way to design FPT-algorithms for problems parameterized by treewidth) on tree decompositions that keep changing over time. Hence, we focus on static tree decompositions for temporal graphs, even though the idea of a temporal tree decomposition that itself is temporal as well probably deserves further consideration.

Second, we have to seek for something to put into our bags. There are two canonical choices: the vertices \( V \) of a temporal graph \( G = (V,E,\tau) \), or its vertex appearances, that is, \( V \times \{1, \ldots, \tau\} \). If we put the vertices into our bags, then it seems difficult to end up with something that is significantly different to the treewidth of the underlying graph and captures the temporal nature of the setting. So let us see what we can end up with if we put vertex appearances into the bags. We probably would want to require that for each temporal edge there is a bag that contains both endpoints of the edge, which in terms of vertex appearances would be the endpoints of the edge labeled with the time stamp of the temporal edge. However, if we stop here and add the straightforward adaptation of the third condition of tree decompositions, namely that for every vertex appearance, all bags that contain this vertex appearance should form a connected subtree, then we end up with the layer treewidth, which is something we do not want. To fix this, we may want to consider requiring every two vertex appearances with the same vertex and adjacent time stamps to be contained in at least one bag. This would surely give us something that is at least as large as the underlying treewidth and at most as large as the underlying treewidth times the lifetime. The following definition formalizes this idea.

**Definition 4** (Temporal Tree Decomposition). Let \( G = (V,E,\tau) \) be a temporal graph. A tuple \( T = (T,\{B_u \mid u \in V(T)\}) \) consisting of a tree \( T \) and a set of bags \( B_u \subseteq V \times \{1, \ldots, \tau\} \) is a temporal tree decomposition (ttdc) of \( G \) if

(i) \( \bigcup_{u \in V(T)} B_u = V \times \{1, \ldots, \tau\} \),

(ii) for every \( (\{v,w\},t) \in E \) there is a node \( u \in V(T) \) such that \( (v,t) \in B_u \) and \( (w,t) \in B_u \),

(iii) for every \( v \in V \) and \( t \in \{1, \ldots, \tau - 1\} \) there is a node \( u \in V(T) \) such that \( (v,t) \in B_u \) and \( (v,t+1) \in B_u \), and

(iv) for every \( (v,t) \in V \times \{1, \ldots, \tau\} \), the graph \( T[\{u \in V(T) \mid (v,t) \in B_u\}] \) is a tree.

The width of \( T \) is \( \operatorname{width}(T) := \max_{u \in V(T)} |B_u| - 1 \).

As with static treewidth, this definition of a graph decomposition would give a canonical definition of a temporal treewidth: The temporal treewidth of a temporal graph \( G \) is the minimum width over all temporal tree decompositions of \( G \), that is,

\[
\ttw(G) = \min_{T \text{ is ttdc of } G} \operatorname{width}(T).
\]

As we will see in the next subsection, this definition is equivalent to using the treewidth of a certain type of static expansion of the temporal graph. Then it also will become clearer why the proposed definition gives a temporal treewidth that is at least as large as the underlying treewidth and at most as large as (roughly) the underlying treewidth times the lifetime.
6.2 Treewidth of the Static Expansion

Another direction to define a temporal version of treewidth would be to use the treewidth of static graphs as we know it, and apply it to a graph that can be naturally derived from a given temporal graph. The most canonical graph of this type is the static expansion (see Definition 2) of a temporal graph which, however, is typically directed. One possibility would be to apply treewidth adaptations for directed graphs [34, Chapter 16]. Another possibility is to compute the treewidth of the undirected version of the static expansion of a temporal graph. Observe that in this case, we end up with the same temporal treewidth as in Definition 4.

Observation 1. Let \( G = (V, E, \tau) \) be a temporal graph and let \( H = (V', A) \) be its static expansion. Let \( G = (V', E) \) with \( E = \{ \{v, w\} \mid (v, w) \in A \} \) be the undirected static expansion of \( G \). Then

\[
\text{ttw}(G) = \text{tw}(G).
\]

We can check that Observation 1 is true by realizing that the bags in a temporal tree decomposition contain the vertex appearances, which are also the vertices of a static expansion. Furthermore, the edges of a static expansion connect all vertex appearances that we want to be together in at least one bag.

Using this observation, we can also check easily that the claim we made earlier holds. The precise bounds that we can show are

\[
\text{tw}_1(G) \leq \text{ttw}(G) \leq (\text{tw}_1(G) + 1) \cdot \tau - 1.
\]

The lower bound for the temporal treewidth follows from the fact that the underlying graph is a minor of the undirected static expansion. The upper bound follows from the observation that the following is a tree decomposition for the undirected static expansion: take a tree decomposition of the underlying graph and replace every vertex in every bag by all its appearances. This increases the size of all bags by a factor of \( \tau \).

Now we can also more easily understand how temporal graphs with very small temporal treewidth look like. A temporal graph whose temporal treewidth is one necessarily needs to have a forest as underlying graph. However, even in this case, the temporal treewidth can still be as large as \( \min\{ |V|, \tau \} \) if every edge appears at every time step. Take a path as underlying graph as an example where every edge appears at every time step. Then the undirected static expansion is a \( |V| \times \tau \)-grid. In fact, as soon as an edge appears at more than one time step, the undirected static expansion contains a cycle. Hence, a temporal graph with temporal treewidth one has a forest as underlying graph and every edge appears in exactly one time step. This seems to be a good property of temporal treewidth since many problems are indeed easy to solve on temporal graphs of this form.

6.3 Playing Cops-and-Robber Games on Temporal Graphs

Since the treewidth of a static graph can be defined via a cops-and-robber game on static graphs (see Section 2.1), we can also try to transfer these games to temporal graphs in a meaningful way.

Recently, Erlebach and Spooner [39] investigated a cops-and-robber game on temporal graphs with infinite lifetime and periodic edge appearances. Here, whenever the cops and the robber have taken their turn, time moves forward one step, and when making their moves, the cops and the robber can only use edges that are present at the current time.

The first obvious issue with this approach is that the temporal graphs we want to investigate have neither infinite lifetime nor periodic edge appearances. If the game would just stop when the lifetime finished and the robber wins if he or she does not get caught, then we would need more cops on temporal graphs with shorter lifetime. Deriving a temporal treewidth concept from this would lead to the probably undesirable property that temporal graphs with short lifetime have higher treewidth than temporal graphs with a long lifetime. To circumvent this, we could repeat the temporal graph ad infinitum. This would also make edge appearances periodic. However, then we will also get the property that the moves a robber can make in this temporal graph is a subset of the moves a robber could make in the underlying graph (or, equivalently, when all edges are always present). This means the number of cops necessary to catch a robber in this scenario is upper-bounded by the treewidth of the underlying graph—a property that we do not want to have.

Summarizing, we can say that designing cops-and-robber games on temporal graphs that lead to useful treewidth definitions seems to be a challenging task. However, since cops-and-robber games already inherently have a temporal character, maybe they are the best-suited way to define temporal treewidth.

7 Conclusion

*Here’s looking at you*, temporal treewidth. Indeed, it is a worthwhile endeavor to explore the prospects and limitations of parameters such as treewidth transformed to the context of temporal graphs. A lot of exploration and clarification is yet to do. So let us agree, in temporal treewidth future we see. Hans, can you?

Acknowledgments. TF acknowledges support by DFG, project TORE (NI 369/18). HM and MR acknowledge support by DFG, project MATE (NI 369/17).

We thank Mark de Berg, Anne-Sophie Himmel, Frank Kammer, Sándor Kisfaludi-Bak, Erik Jan van Leeuwen, and George B. Mertzios for their constructive feedback which helped us to improve the presentation of the paper.

We further thank Bernard Mans and Luke Mathieson for helpful discussions concerning the issues presented in Section 5.3.
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A Temporal Graph Problem Zoo

((α, β)-Temporal Reachability Edge Deletion ($(α, β)$-TRED))

**Input:** A temporal graph $G = (V, E, \tau)$ and two integers $k, h \in \mathbb{N}_0$.

**Question:** Is there a subset $E' \subseteq E(G_s)$ of the underlying graph's edges with $|E'| \leq k$ such that in $G - (E' \times \{1, \ldots, \tau \})$, the size of the set of vertices reachable from every vertex $s \in V$ via a strict $(\alpha, \beta)$-temporal path is at most $h$?

((α, β)-Temporal Reachability Time-Edge Deletion ($(α, β)$-TRED))

**Input:** A temporal graph $G = (V, E, \tau)$ and two integers $k, h \in \mathbb{N}_0$.

**Question:** Is there a subset $E' \subseteq E$ of temporal edges with $|E'| \leq k$ such that in $G - E'$, the size of the set of vertices reachable from every vertex $s \in V$ via a strict $(\alpha, \beta)$-temporal path is at most $h$?

Minimum Single-Source Temporal Connectivity ($r$-MTC)

**Input:** A temporal graph $G = (V, E, \tau)$ with edge weights $w : E \to \mathbb{Q}$, a designated vertex $r \in V$, and a number $k \in \mathbb{Q}$.

**Question:** Is there a temporally $r$-connected spanning subgraph of $G$ of weight at most $k$?
Min-Max Reachability Temporal Ordering (MIN-MAX RTO)

**Input:** A temporal graph $G = (V,E,\tau)$, and an integer $k \in \mathbb{N}$.

**Question:** Is there a bijection $\phi : \{1, \ldots, \tau\} \rightarrow \{1, \ldots, \tau\}$ such that the maximum reachability in $G' = (V,\{(s,\phi(t)) | (e,t) \in E\},\tau)$ is at most $k$?

Min Reachability Temporal Merging (MRTM)

**Input:** A temporal graph $G = (V,E,\tau)$, a set of sources $S \subseteq V$, and three integers $\lambda, \mu, k \in \mathbb{N}$.

**Question:** Are there $\mu$ disjoint intervals $M_1, \ldots, M_\mu \subseteq \{1, \ldots, \tau\}$, each of size $\lambda$, such that, after merging each of them in $G$, the number of vertices reachable from $S$ is at most $k$?

Return-To-Base Temporal Graph Exploration (RTB-TGE)

**Input:** A temporal graph $G = (V,E,\tau)$ and a designated vertex $s \in V$.

**Question:** Is there a strict temporal walk starting and ending at $s$ that visits all vertices in $V$?

Temporal Graph Exploration (TGE)

**Input:** A temporal graph $G = (V,E,\tau)$ and a designated vertex $s \in V$.

**Question:** Is there a strict temporal walk starting at $s$ that visits all vertices in $V$?

Temporal Matching (TM)

**Input:** A temporal graph $G = (V,E,\tau)$ and integers $k, \Delta \in \mathbb{N}_0$.

**Question:** Is there a set of $k$ temporal edges $E' \subseteq E$ such that any pair $\{(e, t), (e', t')\} \subseteq E'$ has $e \cap e' = \emptyset$ or $|t - t'| \geq \Delta$?

Temporal Separation (TS)

**Input:** A temporal graph $G = (V,E,\tau)$, two designated vertices $s, z \in V$, and an integer $k \in \mathbb{N}$.

**Question:** Is there a temporal $s$-$z$ separator of size at most $k$ in $G$?