The Proton Gluon Distribution from the Color Dipole Picture

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Employing the representation of the experimental data on deep inelastic electron-proton scattering (DIS) in the color-dipole picture (CDP), we determine the gluon distribution of the proton at small Bjorken $x$. At sufficiently large momentum transfer, $Q^2$, the extracted gluon distribution fulfills the standard evolution equation for the proton structure function. For low values of $Q^2$, e.g. for $Q^2 = 1.9\,\text{GeV}^2$, the evolution equation for the proton structure function is violated. The standard procedure of adopting a low-$Q^2$ starting scale for the extraction of the gluon density is questionable and requires further investigations.

I. INTRODUCTION

The extraction of the gluon-distribution function [1–4] of the proton from deep-inelastic electron-proton scattering (DIS) [5] rests on the determination of the fit parameters in an ad-hoc parametrization of the gluon $x$-distribution at a low-$Q^2$ input scale [6]. It is well-known, and was recently emphasized in ref. [6] that the results on the gluon distribution function from different collaborations [1–4] show a significant spread [2] in the low-$x$, low-$Q^2$ domain. The quality of the fits is considered [6] not to be satisfactory. This led to the introduction of phenomenological power corrections [7, 8] to the structure functions, and to modifications [6] of the DGLAP evolution equations [9] for the gluon distribution by a non-linear term [10, 11].

In the color-dipole picture (CDP) [12, 13, 14, 15] of DIS, the photoabsorption cross section at low $x$ is represented in terms of the colour-gluon exchange. The corresponding forward scattering amplitude is given by an ansatz for the color-gluon invariant interaction of $q\bar{q}$ states with the proton via two-gluon exchange. The $q\bar{q}$ states form a massive continuum as a function of the $q\bar{q}$ masses (as observed in $e^+e^- \rightarrow q\bar{q}$ annihilation), including a smooth interpolation of the low-lying vector meson peaks. Accordingly, the massive $q\bar{q}$ continuum starts at a mass squared of $m_0^2 \lesssim m_{p^2}$, where $m_{p^2}$ denotes the square of the $\rho^0$ meson mass. The general structure (at leading order of $\alpha_s(Q^2)$) of the two-gluon exchange interaction of $\gamma^* g \rightarrow q\bar{q}$ (where $\gamma^*$ denotes the photon of virtuality $Q^2$) from the pQCD improved parton model is assumed to remain valid when proceeding from large values of $Q^2$ (the genuine region of pQCD) to small values of $Q^2$, for $Q^2$ towards $Q^2 = 0$.

In the CDP, accordingly, the representation of DIS is continued to include the $Q^2$ towards $Q^2 = 0$ limit, and the photoproduction cross section $\sigma_{\gamma p}(W^2)$ may be used as a normalization of the $\gamma^* p$ interaction cross section, $\sigma_{\gamma p}(W^2, Q^2)$.

The basic quantity of the CDP ansatz, the cross section $\sigma_{(q\bar{q})p}(W^2)$, is dependent on $W^2$ via $\Lambda_{sat}(W^2) \equiv C_1 \left( \frac{W^2}{1 \,\text{GeV}^2} \right)^{C_2}$, where $C_1$ and $C_2$ are adjustable parameters.

The dominance of the $\gamma^* g \rightarrow q\bar{q}$ interaction is common to both, the description of DIS at low $x$ in the CDP and in the pQCD-based parton model, where instead of $\sigma_{(q\bar{q})p}(W^2)$ the gluon distribution function $G(x, Q^2)$ enters as the basic quantity. The basic mechanism of $\gamma^* g \rightarrow q\bar{q}$ being the same, the underlying gluon distribution may be deduced from the CDP representation.

In the present paper, at leading order in the strong coupling $\alpha_s(Q^2)$, we accordingly determine the gluon distribution, $G(x, Q^2)$, from the CDP representation of the DIS experimental data.

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1 In standard notation, the virtuality of the photon is denoted by $Q^2$, and $x$ is the Bjorken scaling variable, $x \equiv Q^2/W^2$, where $W$ denotes the virtual-photon-proton center of mass energy.

2 See Fig.2 in ref.[6]
Adopting an implicit validity of the $\gamma^* g \to q\bar{q}$ interaction, a gluon distribution may also be extracted from a fit to the experimental data not manifestly being based on the $\gamma^* g \to q\bar{q}$ interaction of the CDP. Assuming implicit validity of the $\gamma^* g \to q\bar{q}$ interaction, we in addition accordingly deduce $G(x, Q^2)$ from the Froissart-bounded representation of the DIS data. We find results consistent with the ones based on the CDP. Our results on the gluon distribution do not exclusively depend on the CDP representation of the DIS experimental data.

In the present paper, accordingly, the leading order gluon distribution of the perturbative-QCD-improved (pQCD) parton model is extracted from the representation of the proton structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ in the CDP. We find that at low $x$, for $Q^2$ sufficiently large ($Q^2 \gtrsim 10 \text{ GeV}^2$ to 20 GeV$^2$), the experimental results on DIS parametrised by the CDP fulfill evolution in distinction from low values of $Q^2$ (e.g. for $Q^2 = 1.9 \text{ GeV}^2$), where due to hadronlike behaviour of DIS, the standard evolution equation is strongly violated. This result suggests that the unsatisfactory fits mentioned in Ref.[6] at low $Q^2$ are due to an improper use of a low-$Q^2$ input scale, such as $Q^2 = 1.9 \text{ GeV}^2$, in the widely employed analysis of the DIS experimental results.

In Section II, we briefly review the determination of the gluon distribution at leading order of $\alpha_s(Q^2)$ from the pQCD improved parton model. In Section III, we review the relevant parts of the CDP. In Section IV, as an alternative to the CDP, we consider and summarise the Froissart-bounded representation of $F_2(x, Q^2)$. In Section V, the results obtained for the gluon distribution are presented for both the CDP and the Froissart-bounded parametrisation of the DIS data. At low $Q^2$ ($Q^2 = 1.9\text{ GeV}^2$), we observe drastic differences between our results on the gluon distribution and the results published by several different collaborations. Note that for the extraction of the gluon distribution in Sections II to V, no use is being made of the pQCD evolution equations [9].

In Sections VI and VII, we examine the consistency between the low-$Q^2$ dependence at low-$x$ of our results for the gluon distribution and the validity of the evolution equation for the structure function $F_2(x, Q^2)$. At large $Q^2$, specifically for approximately 20 GeV$^2 \lesssim Q^2 \lesssim 100\text{ GeV}^2$, the CDP gluon distribution is consistent with the validity of evolution, leading to the important constraint of $C_2 = 0.29$ for the exponent in the dependence of $F_2(x, Q^2)$ on $(W^2)^{C_2} = (Q^2/x)^{C_2}$. The predicted value of $C_2 \cong 0.29$ agrees with experiment. At low-$Q^2$, the logarithmic derivative of the structure function $F_2(x, Q^2)$ deviates from the prediction of the evolution equation by a factor of magnitude up to 3. Final Conclusions are presented in Section VIII.

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See Fig.2 in ref.[6]
TABLE I. The table shows the ratio of the proton structure function of the CDP in (5.1) upon rescaling to the CDP input for the gluon distribution from (2.8) by inserting the explicit CDP expression for the gluon distribution from (2.8) by inserting the explicit analytic expression derived from the structure function (2.7) of the proton structure function at a rescaled value of $x \to \xi_L x$. In a brief summary, the photon in (2.8) has the preferred value $\xi_L \approx 0.40$.

The table shows the ratio of $F_L (W^2, Q)$ from (2.7) upon insertion of the gluon distribution from the proportionality (2.8), to the CDP input for $F_L (x, Q^2)$ from (5.1). The ratio is indeed close to unity.

| $W^2 = 10^3$ [GeV$^2$] | $W^2 = 10^4$ [GeV$^2$] | $W^2 = 10^5$ [GeV$^2$] |
|-------------------------|-------------------------|-------------------------|
| $Q^2 = 100$ [GeV$^2$]  | 1.043                   | 1.037                   | 0.981                   |
| $Q^2 = 50$ [GeV$^2$]   | 1.041                   | 1.037                   | 1.012                   |
| $Q^2 = 10$ [GeV$^2$]   | 1.035                   | 1.035                   | 1.030                   |
| $Q^2 = 2$ [GeV$^2$]    | 1.021                   | 1.026                   | 1.029                   |
| $Q^2 = 1$ [GeV$^2$]    | 1.015                   | 1.021                   | 1.026                   |

The simple proportionality (2.8) indeed consistently provides a gluon distribution that fulfills the pQCD relationship (2.7). The $Q^2$ dependence of the gluon distribution in (2.7) is indeed determined by the $Q^2$ dependence of the proton structure function of the CDP in (5.1) upon insertion into (2.8).

III. THE COLOR-DIPOLE REPRESENTATION.

For a detailed representation of the CDP, we refer to Refs.[14] and [15]. In a brief summary, the photon interaction (2.5) with the gluon field in the proton at low values of $x$ is interpreted as fluctuation of the photon into $q\bar{q}$-color-dipole states, see (2.6), that interact with the gluon field in the proton via color-gauge-invariant gluon couplings to quark and antiquark states.

In Fig. 1, reproduced from refs. [14, 15], we show the results for the photoabsorption cross section, $\sigma_{\gamma p}(W^2, Q^2)$, in the CDP. The results are obtained from the explicit analytic expression

$$\sigma_{\gamma p}(\eta, \xi) = \frac{\alpha R_{e+e^-}}{3\pi} \sigma^{(\infty)}(W^2) \times \left( I_T^{(1)} (\eta, \mu) G_T(u) + I_L^{(1)} (\eta, \mu) G_L(u) \right),$$

for $\sigma_{\gamma p}(W^2, Q^2)$,

$$\sigma_{\gamma p}(W^2, Q^2) = \sigma_{\gamma_{\perp} p}(W^2, Q^2) + \sigma_{\gamma_{\parallel} p}(W^2, Q^2)$$

derived from an ansatz [13] for the W-dependent dipole cross section that essentially, via the structure

$$I_q(x, Q^2) \text{ from } (2.3) \text{ in (2.7) yields the well-known approximate result [19, 14] for the longitudinal structure function (2.7)}$$

$$F_L (\xi_L x, Q^2) = \frac{\alpha_s(Q^2)}{3\pi} \sum_q Q_q^2 G(x, Q^2).$$

A given gluon distribution determines the longitudinal proton structure function at a rescaled value of $x \to \xi_L x$. The rescaling factor $\xi_L$ in (2.8) has the preferred value of $\xi_L \approx 0.40$.

We have explicitly tested the validity of the replacement of the pQCD integral representation (2.7) between the longitudinal structure function of the CDP and the gluon distribution by the simple proportionality (2.8) for the determination of the gluon distribution. Determining the gluon distribution from (2.8) by inserting the explicit CDP expression for $F_L (\xi_L x, Q^2)$ given by (5.1) below, and inserting the result into (2.7), we indeed reproduce the CDP input for $F_L (x, Q^2)$ within a deviation of up to at most 3.5%. Compare the results in Table 1.

FIG. 1. The theoretical results for the photoabsorption cross section $\sigma_{\gamma p}(W^2, Q^2), \xi)$ in the CDP as a function of the low-$x$ scaling variable $\eta(W^2, Q^2) = (Q^2 + m_0^2)/\Lambda^2_{\text{sat}}(W^2)$ for different values of the parameter $\xi$ that determines the (squared) mass range $M_{\eta q}^2 \leq m_0^2(W^2) = \xi\Lambda^2_{\text{sat}}(W^2)$ of the $\gamma^* \to q\bar{q}$ fluctuations that are taken into account. The experimental results for $\sigma_{\gamma p}(\eta(W^2, Q^2), \xi)$ lie on the full line corresponding to $\xi = \xi_0 = 130$, compare refs. [14, 15].
of the coupling of the quark-antiquark state to two gluons, represents the color-gauge-invariant interaction of the $q\bar{q}$ dipole with the gluon-field in the nucleon. In (3.1), $R_{c+e^{-}} = 3 \sum \sigma Q_q^2$, where $q$ runs over the active quark flavors, and $Q_q$ denotes the quark charge. The smooth transition to $Q^2 = 0$ photoproduction in (3.1) allows one [15] to replace (see (2.37) in ref. [13]) $\sigma^{(\infty)}(W^2)$, which stems from the normalization of the $q\bar{q}$-dipole-proton cross section [7] by the photoproduction cross section, and (3.1) becomes

$$\sigma_{\gamma p}(W^2, Q^2) = \text{lim}_{\eta \to \mu(W^2)} \frac{\sigma_{\gamma p}(W^2)}{I_T^{(1)} \left( \frac{\eta}{\mu(W^2)} \right)} G_T(u) \times \left( I_T^{(1)} \left( \frac{\eta}{\mu(W^2)} \right) G_T(u) + I_L^{(1)}(\eta, \mu) G_L(u) \right)$$

(3.2)

We note that $I_T^{(1)}(\eta, \mu)$ vanishes in the photoproduction $Q^2 = 0$ limit of $\eta(W^2, Q^2) = 0 = m_0^2/\Lambda_{\text{sat}}(W^2) \equiv \mu(W^2)$, and $G_T(u = \xi/\eta) \approx 1$, and for later reference we also note

$$\text{lim}_{\eta \to \mu(W^2)} I_T^{(1)} \left( \frac{\eta}{\mu(W^2)} \right) = \ln \frac{\mu}{\mu(W^2)}$$

(3.3)

For the general explicit analytic expressions for the functions $I_T^{(1)}(\rho, \mu(W^2))$ and $I_L^{(1)}(\eta, \mu)$ we refer to Appendix A. The functions $G_T \left( u = \frac{\xi}{\eta} \right)$ and $G_L \left( u = \frac{\xi}{\eta} \right)$ are given by

$$G_T(u) = \frac{2u^3 + 3u^2 + 3u}{2(1 + u)^3} \approx \left\{ \begin{array}{ll} \frac{3}{2}\frac{\xi}{\eta} & , \quad (\eta \gg \xi), \\ 1 - \frac{3}{2}\frac{\xi}{\eta} & , \quad (\eta \ll \xi), \end{array} \right.$$

(3.4)

and

$$G_L(u) = \frac{2u^3 + 6u^2}{2(1 + u)^3} \approx \left\{ \begin{array}{ll} \frac{3}{2} \left( \frac{\xi}{\eta} \right)^2 & , \quad (\eta \gg \xi), \\ 1 - 3 \left( \frac{\xi}{\eta} \right)^2 & , \quad (\eta \ll \xi), \end{array} \right.$$  

(3.5)

where $u \equiv \xi/\eta$, and the constant parameter $\xi$ restricts the masses of the contributing mass $q\bar{q}$ states via

$$M_{q\bar{q}}^2 \leq m_1^2(W^2) = \xi \Lambda_{\text{sat}}^2(W^2).$$

(3.6)

The numerical results for the photoabsorption cross section in Fig. 1 are obtained by numerical evaluation of (3.2) upon insertion of a $(\ln W^2)^2$ fit to the experimental results for the photoproduction cross section $\sigma_{\gamma p}(W^2)$ from the Particle Data Group [21]. The results in Fig. 1 were obtained for $W = 275$ GeV from

$$\sigma_{\gamma p}(W^2) = 0.003056 \left( 34.71 + \frac{0.38947}{M^2} \ln^2 \frac{W^2}{M_{\text{cutoff}}^2} \right) + 0.0128 \left( \frac{M_{\text{cutoff}}^2}{W^2} \right)^{0.462}.$$  

(3.7)

In (3.7), $M_p$ denotes the proton mass, $M = 2.15$ GeV and $\sigma_{\gamma p}(W^2)$ is given in units of millibarn.

Before going into more detail, we note that the full curve in Fig. 1, which for the parameter $\xi$ corresponds to the choice of $\xi = \xi_0 = 130$, provides a representation of the full set of experimental data on $\sigma_{\gamma p}(W^2, Q^2)$ at low $x \equiv Q^2/W^2$, compare Fig. 8 in ref. [14].

In (3.1) and (3.2), the low-x scaling variable $\eta(W^2, Q^2)$ is given [13] by

$$\eta \equiv \eta(W^2, Q^2) = \frac{Q^2 + m_0^2}{\Lambda_{\text{sat}}^2(W^2)}.$$  

(3.8)

with

$$\mu \equiv \mu(W^2) = \eta(W^2, Q^2) = 0 = \frac{m_0^2}{\Lambda_{\text{sat}}^2(W^2)}.$$  

(3.9)

the "saturation scale" $\Lambda_{\text{sat}}^2(W^2)$, being parametrized by

$$\Lambda_{\text{sat}}^2(W^2) = C_1 \left( \frac{W^2}{1\text{GeV}^2} \right)^{C_2},$$  

(3.10)

and numerically the results in Fig. 1 are based on

$$m_0^2 = 0.15 \text{ GeV}^2, \quad C_1 = 0.31 \text{ GeV}^2; \quad C_2 = 0.27.$$  

(3.11)

The parameter $\rho = \text{const}$ in (3.1) and (3.2) is related to the longitudinal-to-transverse ratio $R(W^2, Q^2)$ of the photoabsorption cross section, and approximately we have $R(W^2, Q^2) \approx 1/2\rho$ for $\eta(W^2, Q^2) \gg \mu(W^2)$, while $R(W^2, Q^2) = 0$ for $Q^2 = 0$. The total cross section $\sigma_{\gamma p}(W^2, Q^2)$ is fairly insensitive to the value of $\rho$ for realistic values of $\rho$ around $\rho \approx 1$, and the evaluation presented in Fig. 1 is based on $\rho = 0.4$.

In terms of the photoabsorption cross sections $\sigma_{\gamma_{L,T}p}(W^2, Q^2)$, in (3.1) and (3.2), the proton structure functions, relevant for the extraction of the gluon-density distribution, see (2.8), are given by

$$F_{L,T}(W^2, Q^2) = \frac{Q^2}{4\pi^2\alpha} \sigma_{\gamma_{L,T}p}(W^2, Q^2).$$  

(3.12)

Note that the photon fluctuates [13,16] into on-mean-shell massive $q\bar{q}$ states (see e.g. Ref.[18] for an explicit proof of this point) implying a dipole-proton cross section that depends on $W^2$ (besides the transverse dipole size) and not on $x$, a dependence on $x$ nevertheless being adopted frequently without justification.

5 The value of $\Lambda_{\text{sat}}^2(W^2)$ via $\eta(W^2, Q^2) \approx 5$ determines the transition from color transparency of $\eta(W^2, Q^2) \gtrless 5$ to hadronlike saturation of $\eta \lesssim 5$. 

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and
\[ F_2(W^2, Q^2) = \frac{Q^2}{4\pi^2} \sigma_{\gamma p}(W^2, Q^2) \]
\[ = \frac{Q^2}{4\pi^2} \left( \sigma_{\gamma p}(W^2, Q^2) + \sigma_{\gamma_{Lp}}(W^2, Q^2) \right). \]  
(3.13)

Upon introducing the longitudinal-to-transverse ratio \( R(W^2, Q^2) \), the longitudinal structure function becomes
\[ F_L(W^2, Q^2) = \frac{R}{1 + R} F_2(W^2, Q^2). \]  
(3.14)

Explicitly, according to (3.2), we have
\[ R(W^2, Q^2) = \frac{I_L^{(1)}(\eta, \mu) G_L(u)}{I_T^{(1)}(\frac{2}{\rho}, \frac{u}{\rho}) G_T(u)}. \]  
(3.15)

For \( Q^2 = 0 \), as a consequence of electromagnetic gauge invariance,
\[ R(W^2, Q^2 = 0) = 0, \]  
(3.16)
while for \( \eta(W^2, Q^2) \), restricted by the interval of \( 1 \ll \eta(W^2, Q^2) \ll \xi = \xi_0 = 130 \) that will be of relevance subsequently, we have
\[ R(W^2, Q^2) \approx \frac{1}{2\rho}, \]  
(3.17)
with \( \rho = \text{const} \) in the vicinity of \( \rho \approx 1 \), and, accordingly, (3.14) becomes
\[ F_L(W^2, Q^2) = \frac{1}{2\rho} + 1 \]
\[ (1 \ll \eta(W^2, Q^2) \ll \xi_0). \]  
(3.18)

While (3.18) holds strictly for sufficiently large \( Q^2 \), even for low values of \( Q^2 \gtrapprox 1 \text{GeV}^2 \) it may be used as an approximation of (3.14). The accuracy of (3.18) as approximation for low \( Q^2 \) is seen by comparing the results in Fig. 3 and Fig. 4 in Section V below.

For large \( Q^2 \), specifically for \( 10 \text{GeV}^2 \lesssim Q^2 \lesssim 100 \text{GeV}^2 \) the range of \( Q^2 \) of particular relevance in connection with pQCD, the experimental data on the proton structure function \( F_2(W^2, Q^2) \) in (3.13) depend on the single variable \( W^2 \), compare Fig. 2 quoted from ref. [14].

A simple two-parameter eye-ball fit to the experimental data in Fig. 2 yields [14]
\[ F_2(W^2) = f_2 \left( \frac{W^2}{1 \text{GeV}^2} \right)^{C_2}, \]  
(3.19)
where
\[ f_2 = 0.063, \quad C_2 = 0.29. \]  
(3.20)

The fit result is understood as a consequence of the CDP in the large-\( \eta(W^2, Q^2) \) approximation. The structure function \( F_2(W^2, Q^2) \) for \( Q^2 \gg \Lambda_{\text{sat}}^2(W^2) \) according to (3.13) upon substitution of (3.1) takes the simple form
\[ F_2(W^2, Q^2) = \frac{R_{e+e^-}\sigma^{(\infty)}(W^2)}{24\pi^3} \left( 1 + \rho \right) \frac{1}{3} \Lambda_{\text{sat}}^2(W^2) \]
\[ \times \left( 1 + 0 \left( \frac{1}{\eta} \right) \right). \]  
(3.21)

Upon specifying \( \Lambda_{\text{sat}}^2(W^2) = C_1 \left( \frac{W^2}{1 \text{GeV}^2} \right)^{C_2} \) according to (3.10), the structure function (3.21) can directly be compared with (3.19),
\[ F_2(W^2, Q^2) = \frac{R_{e+e^-}\sigma^{(\infty)}(W^2)}{24\pi^3} \left( 1 + \rho \right) C_1 \left( \frac{W^2}{1 \text{GeV}^2} \right)^{C_2} \]
\[ \times \left( 1 + 0 \left( \frac{1}{\eta} \right) \right) \]  
(3.22)
\[ \equiv \hat{f}_2(W^2) \left( \frac{W^2}{1 \text{GeV}^2} \right)^{C_2} \left( 1 + 0 \left( \frac{1}{\eta} \right) \right). \]

The product \( R_{e+e^-}\sigma^{(\infty)}(W^2) \) is determined by the photoproduction cross section (3.7) according to
\[ R_{e+e^-}\sigma^{(\infty)}(W^2) = \frac{3\pi}{\alpha} \frac{\sigma_{\gamma p}(W^2)}{\ln \rho \Lambda_{\text{sat}}^2(W^2) \eta^2}. \]  
(3.23)

\[ \hat{f}_2(W^2) = \begin{cases} 0.067, & \text{for } W^2 = 10^4 \text{GeV}^2, \\ 0.068, & \text{for } W^2 = 10^5 \text{GeV}^2. \end{cases} \]  
(3.24)

These values of \( \hat{f}_2(W^2) \) are approximately 6\% to 7\% larger than the values from the eye-ball fit (3.19) in
TABLE II. The evaluation of \( f_2(W^2) \) defined by (3.22). The parameters \( \rho \) and \( m_0^2 \) in (3.22) and (3.23) are given by \( \rho = 4/3 \) and \( m_0^2 = 0.15 \text{GeV}^2 \).

| \( W^2 \text{ [GeV}^2] \) | 10^4 | 10^5 |
|----------------------|------|------|
| \( \sigma_{\text{np}} \text{ [mb]} \) | 0.146 | 0.175 |
| \( \Lambda_{\text{sat}}^2(W^2) = C_1 \left( \frac{W^2}{1 \text{GeV}^2} \right)^{C_2} \) | 4.48 | 8.74 |
| \( C_1 = 0.31 \text{ GeV}^2, \) | | |
| \( C_2 = 0.29 \) | | |
| \( R_{\gamma \gamma - \gamma}(\infty)(W^2) \text{ [mb]} \) | 50.9 | 51.9 |
| \( f_2(W^2) \) | 0.067 | 0.068 |

For the determination of the gluon distribution according to (2.8), the structure functions in (3.12) and (3.13) have to be evaluated at a rescaled or shifted value of \( x \to \xi_L x \), corresponding to a shift of \( W^2 \to \xi_L^{-2} W^2 \). In terms of the low-\( x \) scaling variable \( \eta(W^2, Q^2) \), the shift becomes

\[
\eta(W^2, Q^2) = \frac{Q^2 + m_0^2}{\Lambda_{\text{sat}}^2(W^2)} \rightarrow \frac{Q^2 + m_0^2}{\Lambda_{\text{sat}}^2(\xi_L^{-2} W^2)}. \tag{3.26}
\]

For

\[
\Lambda_{\text{sat}}^2(W^2) = C_1 \left( \frac{W^2}{1 \text{GeV}^2} \right)^{C_2}, \tag{3.27}
\]

to be employed subsequently, the shift becomes

\[
\eta(W^2, Q^2) \rightarrow \xi_L^{-2} \eta(W^2, Q^2). \tag{3.28}
\]

The photoabsorption cross section (3.2) essentially only depends on \( \eta(W^2, Q) \),

\[
\sigma_{\gamma_L \gamma p}(W^2, Q^2) \equiv \sigma_{\gamma_L \gamma p}(\eta(W^2, Q^2)), \tag{3.29}
\]

since the \( Q^2 = 0 \) photoproduction factor in (3.2) depends only weakly on \( W^2 \). The rescaling shift (3.28) applied to the cross section (3.2), accordingly, amounts to

\[
\sigma_{\gamma_L \gamma p}(\eta(W^2, Q^2)) \rightarrow \sigma_{\gamma_L \gamma p}(\xi_L^{-2} \eta(W^2, Q^2)). \tag{3.30}
\]

Numerically, for \( C_2 = 0.29 \) and \( \xi_L = 0.4 \), we have

\[
\xi_L^{-2} \eta = 0.4^0.29 \eta \equiv 0.77 \eta(W^2, Q^2). \tag{3.31}
\]

Explicitly, the gluon distribution (2.8) becomes

\[
\alpha_s(Q^2) G(x, Q^2) = \frac{3\pi}{\sum_q Q_q^2 4\pi^2 \alpha_s} \sigma_{\gamma_L \gamma p}(\xi_L^{-2} \eta(W^2, Q^2)). \tag{3.32}
\]

We note that (3.32) provides the connection between the gluon distribution (the basic quantity of the pQCD-improved parton model) and the dipole cross section (the basic quantity of the CDP), since \( \sigma_{\gamma_L \gamma p} \) in (3.32) may be replaced by the dipole cross section \( \sigma_{(q\bar{q})p}(r_\perp, z(1 - z), W^2 = Q^2) \), according to

\[
\sigma_{\gamma_L \gamma p}(W^2 = \frac{Q^2}{x}, Q^2) = \int dz \int d^2 r_\perp |\Psi_L(r_\perp, z(1 - z), Q^2)|^2 \times \sigma_{(q\bar{q})p}(r_\perp, z(1 - z), W^2 = \frac{Q^2}{x}), \tag{3.33}
\]

The empirically verified low-\( x \) scaling of the photoabsorption cross section in the variable \( \eta(W^2, Q^2) \) translates from the photoabsorption cross section to the gluon distribution. For sufficiently large \( Q^2 \), with \( \sigma_{\gamma_L \gamma p} \propto \Lambda_{\text{sat}}^2(W^2)/Q^2 \) \cite{15}, according to (3.32) we obtain the asymptotic behavior of

\[
\alpha_s(Q^2) G(x, Q^2) \propto \Lambda_{\text{sat}}^2(W^2). \tag{3.34}
\]

The quantity \( \Lambda_{\text{sat}}^2(W^2) \), also known as saturation scale, determines the gluon distribution function in the pQCD-large-\( Q^2 \) limit.

The arguments in Section II and the present Section III may be summarized as follows: In the CDP, at leading order of pQCD, DIS at low \( x \) is (successfully) described as interaction of the photon with the \( q\bar{q} \) sea according to (2.6). This implies that the pQCD equation (2.1) for the longitudinal proton structure function can be approximated by the gluon contribution, see (2.7), and, finally, the gluon distribution can be determined from (2.8). On the left-hand side, the proton structure function from the CDP has to be substituted.

Independently from the CDP, the approximation of (2.1) by (2.8) as leading term at large \( Q^2 \) was justified \cite{19} by evaluating suitable models for the gluon distribution.

Note that the pQCD equation (2.8) for its validity requires sufficiently large values of \( Q^2 \). Since the CDP description by the proton structure functions to be substituted into (2.8) includes the low-\( Q^2 \) limit, the use of (2.8) contains a smooth extrapolation of the gluon density to low values of \( Q^2 \).

IV. THE FROISSART-BOUNDED REPRESENTATION OF \( F_2(x, Q) \).

A very accurate fit to the measured proton structure function of DIS, due to Block et al. \cite{22}, is provided by a representation of the DIS data satisfying the (\( \log W^2 \))^2 bound derived from general-field-theory principles by Froissart \cite{23}. Even though the determination of the gluon distribution (2.8) contains the CDP representation for the proton structure function, one may expect that different precise fits to the DIS data will
at least approximately lead to the same results for the gluon distribution. In this spirit, we shall employ the Froissart-bounded representation in addition to the CDP representation for the proton structure function.

The results from DIS at low $x \leq 0.1$ and for a large range of $Q^2$ from 0.15GeV$^2 \leq Q^2 \leq 3000$GeV$^2$ are represented by the structure function

$$F_2(x, Q^2) = D(Q^2)(1 - x)^n \left[ C(Q^2) + A(Q^2) \ln \left( \frac{1}{x Q^2 + \mu^2} \right) + B(Q^2) \ln^2 \left( \frac{1}{x Q^2 + \mu^2} \right) \right]. \quad (4.1)$$

The power $n$ and the scale $\mu^2$ are given by $n = 11.49 \pm 0.99$ and $\mu^2 = 2.82 \pm 0.290$ GeV$^2$. The logarithmic dependence on $Q^2$ of the functions $A(Q^2), B(Q^2), C(Q^2)$ and $D(Q^2)$ is given in Appendix B and Table B1 reproducing Table II from ref. [22].

In distinction from the CDP, the Froissart-bounded fit does not explicitly provide a representation of the longitudinal-to-transverse ratio $R$. When deducing the gluon distribution function, the value of $R$ from the CDP will have to be adopted.

V. THE RESULTS FOR THE GLUON DISTRIBUTION FUNCTION.

We turn to the explicit results for the gluon distribution which follow from the evaluation of (2.8) upon substitution of the representation of the DIS experimental results in the CDP and the Froissart-bounded representation, as given in Sections III and IV, respectively.

Substitution into (2.8) of the CDP results with $\xi$ for $F_L(W^2, Q^2)$ yields

$$\alpha_s(Q^2)G(x, Q^2) = \frac{3\pi}{\sum_q Q_q^2} F_L(\xi_x, Q^2) = \frac{9Q^2}{4\pi\alpha_{R_{e+e^-}}} \times \left( \frac{\sigma_{pp}(W^2)}{\left( \ln \frac{Q^2}{\mu^2} \right) G_T(\frac{\xi}{\eta}) I_L^{1(1)}(\eta, \mu) G_L(\frac{\xi}{\eta})} \right)_{W^2 \rightarrow \xi_L^{-1} W^2} \quad (5.1)$$

The variables $\eta = \eta(W^2, Q^2)$ and $\mu(W^2)$ are given by (see (3.8), (3.9)),

$$\eta(W^2, Q^2) = \frac{Q^2 + m_0^2}{\Lambda_{sat}^2(W^2)},$$
$$\mu(W^2) = \frac{m_0^2}{\Lambda_{sat}^2(W^2)}, \quad (5.2)$$

where

$$\Lambda_{sat}^2(W^2) = C_1(W^2)C_2 = 0.31 \left( \frac{W^2}{1\text{GeV}^2} \right)^{0.29} \text{GeV}^2,$$

with $C_2 = 0.29$ from (3.20) and

$$m_0 = 0.15 \text{GeV}^2. \quad (5.4)$$

For the assumed number of four flavors,

$$R_{e+e^-} = \sum_q Q_q^2 = \frac{10}{3}. \quad (5.5)$$

The parameter $\rho$, denoting the size enhancement of transversely relative to longitudinally polarized $q\bar{q}$ fluctuations, is given by [24]

$$\rho = \frac{4}{3}. \quad (5.6)$$

The parameter $\xi$ that stands for the upper limit, $\xi_{sat}(W^2)$, of masses of $q\bar{q}$ fluctuations actively contributing, is given by $\xi = 130$. At the relevant (low) values of $\eta(W^2, Q^2)$, the limit of $\xi \rightarrow \infty$ and $G_T(\xi) = G_L(\xi) = 1$ (see (3.4) and (3.5)) may be taken without loss of generality, compare Fig. 1. The $\log(W^2)$ fit to the photoproduction experimental results is given by (3.7), and, finally, the shift factor $\xi_L$ according to (2.8) is given by

$$\xi_L = 0.4. \quad (5.7)$$

The essential effect of the shift $W^2 \rightarrow \xi_L^{-1} W^2$, in (5.1) amounts to

$$\eta(W^2, Q^2) \rightarrow \xi_L^{-1} \eta(W^2, Q^2), \quad (5.8)$$

compare (3.30), the $Q^2 = 0$ photoproduction term in (5.1), including its normalization by the denominator in (5.1), being hardly affected due to its weak dependence on $W^2$.

According to (2.8), the gluon distribution may equivalently be expressed in terms of the structure function $F_2(\xi_x, Q^2)$ instead of $F_L(\xi_x, Q^2)$, together with the longitudinal-to-transverse ratio $R$. For sufficiently large $Q^2$, according to (2.8) and (3.18), with $R = 1/2\rho$,

$$\alpha_s(Q^2)G(x, Q^2) = \frac{9\pi}{R_{e+e^-} - 2\rho + 1} F_2(\xi_x, Q^2). \quad (5.9)$$

The representation (5.9), even at small $Q^2$ may be used as an approximation of the gluon distribution.

For $Q^2$ sufficiently large, specifically for $20 \text{GeV}^2 \lesssim Q^2 \lesssim 100 \text{GeV}^2$, the large-$Q^2$ limit of (5.1) given by (3.19) becomes relevant. It implies the simple large-$Q^2$ representation of

$$\alpha_s(Q^2)G(x, Q^2) = \frac{9\pi}{R_{e+e^-} - 2\rho + 1} F_2(\xi_x, Q^2) = \frac{9\pi}{R_{e+e^-} - 2\rho + 1} F_2\left( \xi_x, Q^2 \right) \left( \frac{W^2}{1\text{GeV}^2} \right)^{0.29}, \quad (5.10)$$
FIG. 3. The gluon distribution $\alpha_s(Q^2) x_g(x,Q^2) \equiv \alpha_s(Q^2) G(x,Q^2)$ of the CDP, compare (5.1), as a function of $W^2$ for various values of $Q^2$. The solid line shows the asymptotic limit (5.10) that is reached at $Q^2 \gtrsim 30$ GeV$^2$.

where $f_2 = 0.063$, and $\rho = \frac{4}{3}$.

In Fig. 3, we show the gluon distribution (2.8) deduced from $F_L(W^2, Q^2)$ according to (5.1), as a function of $W^2$ for various values of $Q^2$, where $1$ GeV$^2 \leq Q^2 \leq 100$ GeV$^2$. The results in Fig. 3, for $Q^2$ sufficiently above $Q^2 \approx 10$ GeV$^2$, indeed converge towards the asymptotic representation (5.10).

FIG. 4. As Fig. 3, but based on (5.9), employing $R = \text{const} = 1/2p = 3/8$. Compare text for details.

The results in Fig. 4, for low $Q^2$, $1$ GeV$^2 \leq Q^2 \leq 10$ GeV$^2$, compared with the results in Fig. 3, show the enhanced gluon distribution resulting from employing the large-$Q^2$ approximation (5.9). For sufficiently large $Q^2$, the results in Fig. 4, based on $F_L(\xi_L, x, Q^2)$ according to (3.18) and (5.9), coincide with the ones in Fig. 3 based on $F_L(\xi_L, x, Q^2)$ according to (5.1).

For Figs. 5 and 6, the energy variable, $W$, of the CDP is replaced by $x \simeq Q^2 / W^2$. The increase of $x$ for $Q^2 = 1.9$ GeV$^2$ in the range of $10^{-4} \leq x \leq 10^{-1}$ corresponds to an increase of $\eta(W^2, Q^2 = 1.9$ GeV$^2$) according to $0.35 \leq \eta \leq 2.6$, with $W^2$ decreasing according to $1.9 \times 10^4$ GeV$^2 \geq W^2 \geq 19$ GeV$^2$. Taking into account the proportionality of the gluon distribution to the longitudinal photon cross section (3.32) and consulting the results in Fig. 1, we expect a (hadronlike) increase of the gluon distribution by a factor of approximately about 3.5 with decreasing $x$ in the interval $10^{-1} \geq x \geq 10^{-4}$. This factor of 3.5 for $x = 10^{-4}$ is seen in Figure 5.

Multiplication of the results in Fig. 5 by $\alpha_s(Q^2 = 1.9$ GeV$^2)^{-1} = 0.480^{-1} = 2.083$ yields the

FIG. 5. The gluon-distribution function $\alpha_s(Q^2) x_g(x,Q^2)$ of the CDP as a function of $x \equiv Q^2 / W^2$ at the (low) value of $Q^2 = 1.9$ GeV$^2$, the scale frequently used as input scale [6]. The solid curve is due to $C_1 = 0.31$ and $C_2 = 0.29$ and the uncertainties are due to $C_1 = 0.34\pm0.05$ and $C_2 = 0.27\pm0.01$ [19] (G. Cvetic, D. Schildknecht, B. Surrow, M. Tentyukov, Eur. Phys. J. C 20, 77 (2001)).

FIG. 6. As Fig. 5, but for $x_g(x, Q^2)$ instead of $\alpha_s(Q^2) x_g(x,Q^2)$. The CDP results are compared with the results from ref.[6] and the parametrization methods at the NLO approximation, CJ12 [25], NNPDF3.0 [26] and MMHT14 [27] as accompanied with total errors.
results for \( xg(x, Q^2) \equiv G(x, Q^2) \) that are shown in Fig. 6.

The fairly small error bars of our results for the gluon distribution in Fig. 5 are a reflection of the small errors of the original fit to the experimental data of DIS.

The comparison of our CDP results for the gluon distribution in Fig. 6 and Fig. 10 below with published distributions reveals a significantly different shape of the \( x \) distribution at fixed \( Q^2 = 1.9 \text{GeV}^2 \) and a large deviation in absolute normalization outside the total errors of these distributions. Compare also the comment in the Conclusions on the interpretation of the observed strong deviations.

In Fig. 6, we compare with the results obtained in ref. [6] upon introducing absorptive corrections to the measured proton structure functions, and a modification of \( \alpha_s(Q^2) \) by \( \alpha_s(Q^2 + \mu_0^2) \). The gluon distributions from ref. [6] in Fig. 6 are given by [6]

\[
\begin{align*}
xg(x) &= 5.63x^{0.103}(1 - x)^{10.261} \text{ for } \mu_0 = 0 \text{ GeV}, \\
xg(x) &= 4.21x^{0.021}(1 - x)^{9.427} \text{ for } \mu_0 = 1 \text{ GeV}.
\end{align*}
\]

As seen in Fig. 6, the results from ref. [6] are drastically different from ours. The results obtained by exploiting the smooth low-\( Q^2 \) transition for \( F_L(x, Q^2) \) of the CDP by substitution into the pQCD-improved parton model do not support the modifications of evolution suggested and applied to DIS in Ref.[6]. For further discussions on the consequences of our approach of incorporating the CDP representation for \( F_L(x, Q^2) \) into pQCD, see sections VI and VII.

For completeness, in Fig.7, we show the CDP gluon distribution for \( Q^2 \) between \( Q^2 = 2 \text{ GeV}^2 \) and \( Q^2 = 100 \text{ GeV}^2 \).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{Fig7.png}
\caption{The CDP gluon distribution \( xg(x, Q^2) \) in a wide range of \( Q^2 \) values, \( 2 \text{ GeV}^2 \leq Q^2 \leq 100 \text{ GeV}^2 \).}
\end{figure}

The analysis of the gluon distribution function that led to the results in Figs. 3 to 7 is based on the representation of the proton structure functions in the CDP. We expect that other precise and theoretically well-founded representations (not manifestly including \( q \bar{q} \) two-gluon couplings) of the measured structure functions will lead to (at least approximately) identical results for the gluon distribution. Explicitly this equivalence is tested by the example of employing the fit to the proton structure function in the Froissart-bounded representation from ref. [22], described in Section IV.

Since the theoretical representation in this approach does not isolate the longitudinal structure function, the gluon distribution must be deduced from \( F_2(\xi_L x, Q^2) \) with \( \rho = \frac{3}{2} \) in (5.9).

---

8 The value of \( \alpha_s(Q^2 = 1.9 \text{ GeV}^2) = 0.480 \) is obtained for \( \Lambda_{QCD} = 437\text{MeV} \) corresponding to \( \alpha_s(M_Z^2) = 0.118 \).

9 Employing (2.8), we find that \( F_L(x, Q^2) \) resulting by inserting the gluon distribution (5.11) differs from the CDP \( F_L(x, Q^2) \) by the same factor as observed for the gluon distribution, see Fig.6.

10 We evaluate the gluon distribution from the pQCD approximation (2.8) as well as (6.5) below, in distinction from the approach in ref. [22] that is based on an exact determination of \( G(x, Q^2) \) upon converting (6.1) below into an inhomogeneous second order differential equation for \( G(x, Q^2) \).
The results in Fig. 8 for large \( Q^2 \), asymptotically, agree with the CDP result. For \( Q^2 \lesssim 10 \text{ GeV}^2 \), there are acceptable deviations, seen upon comparing the results in Fig. 8 with the ones in Figs. 3 and 4.

In Figs. 9 and 10, respectively, we present the comparison of the gluon distribution from the Froissart-bounded representation of \( F_2(x, Q^2) \) with the CDP, and from Froissart-bounded representations in refs. [22, 29] and [24, 26].

![FIG. 9. The gluon-distribution function \( e_s(Q^2)xg(x, Q^2) \) as a function of \( x \equiv Q^2/W^2 \) at \( Q^2 = 1.9 \text{ GeV}^2 \) from the CDP, and from Froissart-bounded representations in refs. [22, 29] and [24, 26].](image)

VI. THE CONSISTENCY OF PQCD AND CDP EVOLUTION AT LARGE \( Q^2 \)

In this Section, we examine the consistency of the \( Q^2 \) evolution thus obtained for the gluon distribution from the CDP, with the \( Q^2 \) dependence predicted from pQCD. To order \( \alpha_s(Q^2) \), only process (2.5) contributes, and accordingly we can restrict ourselves to investigating the consistency of the \( Q^2 \) dependence from the CDP with the \( Q^2 \) dependence from the first DGLAP evolution equation [9].

The change of \( F_2(x, Q^2) \) with \( Q^2 \), its evolution at large \( Q^2 \), is given by [9]

\[
\frac{\partial}{\partial \ln Q^2} F_2(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 dz P_{qg}(z) F_2 \left( \frac{x}{z}, Q^2 \right) + \frac{R_{e+e^-}}{3\pi} \alpha_s(Q^2) \int_0^{1-x} dz P_{qg}(z) G \left( \frac{x}{1-z}, Q^2 \right),
\]
where
\[ P_{qg}(z) = \frac{4}{3} \left( 1 + x^2 \right) \frac{1}{(1-x)^2} + \frac{3}{2} \delta(1-x), \tag{6.2} \]
and
\[ P_{qg}(z) = \frac{1}{2} \left( z^2 + (1-z)^2 \right). \tag{6.3} \]
According to our basic assumption (2.7), saying that the photon exclusively interacts with $q\bar{q}$ pairs originating from a gluon, $g \to q\bar{q}$ see (2.6), we exclude the first term on the right-hand side in (6.1) to obtain
\[ \frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} = \frac{R_{e^+e^-}}{3\pi} \int_0^{1-x} dz P_{qg}(z) G \left( x, \frac{Q^2}{1-z} \right). \tag{6.4} \]
Exploiting the symmetry of $P_{qg}(z)$ in (6.4) around $z = 1/2$, the first derivative of $G(x/(1-z), Q^2)$ in a Taylor expansion of $G(x/(1-z), Q^2)$ around $z = 1/2$ yields a vanishing contribution to the integral in (6.4), and approximately
\[ \frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} = \frac{R_{e^+e^-}}{9\pi} \alpha_s(Q^2) G \left( \frac{x}{\xi_2}, Q^2 \right). \tag{6.5} \]
with $x/\xi_2 = 2x$, or $\xi_2 = 1/2$. Replacing the gluon distribution in (6.5) by its proportionality (2.8) to the structure function $F_L$, (6.5) becomes
\[ \frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} = F_L \left( \frac{\xi_L}{\xi_2}, Q^2 \right) = \frac{1}{2\rho + 1} F_2 \left( \frac{\xi_L}{\xi_2} x, Q^2 \right), \tag{6.6} \]
where the second equality is due to (3.18).
So far no specific ansatz for the proton structure function was used. Introducing the large-$Q^2$ power-law of the CDP (3.21) with (3.10),
\[ F_2(x, Q^2) \sim (W^2)^{C_2} = \left( \frac{Q^2}{x} \right)^{C_2}, \tag{6.7} \]
from (6.6) we obtain the important constraint
\[ C_2 \left( \frac{Q^2}{x} \right)^{C_2} = \frac{1}{2\rho + 1} \left( \frac{Q^2}{x} \right)^{C_2} \left( \frac{\xi_L}{\xi_2} \right)^{C_2}, \tag{6.8} \]
or
\[ C_2(2\rho + 1) \left( \frac{\xi_L}{\xi_2} \right)^{C_2} = 1. \tag{6.9} \]
Expansion of the exponential to first order in $C_2$ yields the convenient expression for $C_2$,
\[ C_2 \approx \frac{1}{2\rho + 1} \left( 1 - \frac{1}{2\rho + 1} \ln \frac{\xi_L}{\xi_2} \right). \tag{6.10} \]
For $\xi_2/\xi_L = 0.5/0.4 = 1.25$, with $\rho = 4/3$, one finds $C_2 \approx 0.29$ for the exponent $C_2$ in (6.9).

The prediction of $C_2 \approx 0.29$ is based on the assumption that the evolution in $Q^2$ is entirely due to $q\bar{q}$ pairs originating from gluons $g \to q\bar{q}$, see (6.4) and (6.5). The agreement of the prediction of $C_2 \approx 0.29$ from (6.9) with the experimentally determined value, given in (3.20), provides empirical evidence for, or confirms the approximation of reducing the complete evolution equation (6.1) to the reduced form (6.5) employed to arrive at (6.9). The incoming photon dominantly interacts with $q\bar{q}$ pairs originating from the $g \to q\bar{q}$ transition, see (2.6).

We conclude: the CDP with the power-law ansatz of $F_2(x, Q^2) \sim (Q^2/x)^{C_2}$ for large $Q^2$ fulfills evolution according to (6.4) with $Q^2$ at large $Q^2$, and it implies the empirically successful prediction of $C_2 \approx 0.29$.

**VII. EVOLUTION AT LOW $Q^2 \gtrsim 1.9GeV^2$.**

In considering the validity of the pQCD evolution equation (6.5) for $F_2(x, Q^2)$, we so far restricted ourselves to large values of $Q^2$, sufficiently large with respect to $\Lambda_{sat}(W^2)$, effectively, $Q^2 \gtrsim 10GeV^2$ to $20GeV^2$. Employing the large-$Q^2$ representation for $F_2(x, Q^2) \sim (Q^2/x)^{C_2}$, the validity of the evolution equation (6.6) implied the restriction (6.9) that predicts the exponent $C_2$ in $\Lambda_{sat}^2(W^2) \sim (W^2/x)^{1/2}$ in terms of the transverse-$q\bar{q}$-size parameter $\rho$, where $\rho$ has the preferred value [24] of $\rho = 4/3$.

In the present Section, we lift the restriction on the magnitude of $Q^2$ by allowing for values of $Q^2$ as low as $Q^2 \gtrsim 1.9GeV^2$, the value frequently adopted [6] when applying pQCD to electron-proton DIS.

The CDP proton structure functions entering (6.6) according to (3.1), (3.2) and (3.12), without restriction to large $Q^2$ are given by
\[ F_2(x, Q^2) \equiv F_2 \left( \eta \left( W^2 = \frac{Q^2}{x}, Q^2 \right), \mu \left( W^2 = \frac{Q^2}{x} \right) \right) = \frac{Q^2}{4\pi \alpha_s(\mu)} \left( \frac{\eta}{\mu} \right) G_T(u) + I^{(1)}_L(\eta, \mu) G_L(u), \tag{7.1} \]
and
\[ F_L(x, Q^2) \equiv F_L \left( \eta \left( W^2 = \frac{Q^2}{x}, Q^2 \right), \mu \left( W^2 = \frac{Q^2}{x} \right) \right) = \frac{Q^2}{4\pi \alpha_s(\mu)} I^{(1)}_L G_L(u). \tag{7.2} \]

The fit to the experimental data of $\sigma_{pp}(W^2)$ is given by (3.7). For the explicit expressions for $I^{(1)}_T$ and $I^{(1)}_L$ in (7.1) and (7.2), we refer to Appendix A. In the present context we can restrict ourselves to the range of $\eta \ll \xi = 130,$
and accordingly $G_{L,T}(u) = 1$ in (7.1) and (7.2), see (3.4) and (3.5).

Adopting (3.10),

$$\Lambda_{sat}^2(W^2) = C_1 \left( \frac{W^2}{1 \text{GeV}^2} \right)^{C_2}, \quad W^2 = \frac{Q^2}{x}, \quad (7.3)$$

the low-$x$ scaling variable $\eta(W^2, Q^2)$ from (3.8), expressed in terms of the parton variables $x$ and $Q^2$, becomes

$$\eta \equiv \eta(W^2) = \frac{Q^2}{x}, \quad Q^2 = \frac{x^2 \gamma}{C_1} \left( \frac{Q^2 + m_0^2}{(2Q^2)^{C_2}} \right), \quad (7.4)$$

and $\mu(W^2)$ from (3.9) is given by

$$\mu \equiv \mu(W^2) = \frac{Q^2}{x} = \frac{x^{C_2}}{C_1} \left( \frac{Q^2 + m_0^2}{(2Q^2)^{C_2}} \right). \quad (7.5)$$

The numerical value of the exponent $C_2$, for a given value of $\rho$, is fixed by the large-$Q^2$ constraint (6.9).

Turning to the examination of the validity of the evolution equation for $F_2(x, Q^2)$ in the form of the first equality in (6.6) relating the logarithmic derivative of $F_2(x, Q^2)$ to the longitudinal structure function $F_L(x, Q^2)$, we introduce the ratio

$$\text{Ratio} = \frac{\partial F_2(x, Q^2)}{F_L \left( \frac{\xi_L}{\xi_2}, x, Q^2 \right)} = 1 + \Delta \left( \eta \left( W^2 = \frac{Q^2}{x}, Q^2 \right), \mu \left( W^2 = \frac{Q^2}{x} \right) \right), \quad (7.6)$$

where, according to the validity of (6.6) under constraint (6.9), we have $\Delta(\eta, \mu) \equiv 0$ for sufficiently large $Q^2 \gtrsim 10 \text{GeV}^2$. Deviations from the validity of the evolution equation (6.5) are accordingly parametrized by $\Delta(\eta, \mu)$ according to (7.6).

Note that $\Delta(\eta, \mu) \neq 0$ in (7.6) may be due to either a consequence of a violation of the evolution equation (6.5), under validity of the relation between the gluon distribution and the longitudinal structure function (2.8), or else to a violation of (2.8), not excluding both cases simultaneously. In any case, $\Delta(\eta, \mu) \neq 0$ implies violation of the underlying parton model. At low $\eta$, the interaction is dominantly due to the interaction of low-lying $q\bar{q}$ vector states with the gluon field in the proton, or, equivalently (see (2.6)) to the interaction of the photon with low-lying $q\bar{q}$ vector states and not with freely moving quarks and gluons. Nevertheless, one may formally define a gluon distribution function according to (2.9) and (3.32).

From (7.1), the derivative of $F_2(x, Q^2)$ is obtained as

$$\frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} = \frac{Q^2 \sigma_{\gamma p}(W^2)}{4\pi^2} \ln \left( \frac{\rho}{\mu} \right) \left[ \left( I_T^{(1)} \left( \eta, \frac{\mu}{\rho} \right) + I_L^{(1)}(\eta, \mu) \right) \cdot \left( I_T^{(1)} \left( \eta, \frac{\mu}{\rho} \right) + I_L^{(1)}(\eta, \mu) \right) \right], \quad (7.7)$$

and the denominator of (7.6) is given by

$$F_L \left( \frac{\xi_L}{\xi_2}, x, Q^2 \right) = \frac{Q^2}{4\pi^2} \ln \left( \frac{\rho}{\mu} \right) \left( I_T^{(1)} \left( \eta, \frac{\rho}{\mu} \right) + I_L^{(1)}(\eta, \mu) \right). \quad (7.8)$$

In connection with the derivatives appearing in (7.7), we mention the equality $\frac{\partial}{\partial \ln Q^2} = \frac{\partial}{\partial \ln \rho}$, and moreover, we note the derivatives

$$Q^2 \frac{\partial\eta}{\partial Q^2} = (1 - C_2) \eta - \mu, \quad \quad Q^2 \frac{\partial\mu}{\partial Q^2} = -C_2 \mu. \quad (7.9)$$

Since $\sigma_{\gamma p}(W^2)$ only weakly depends on $W^2$, one may approximate “Ratio” defined in (7.6) by ignoring the derivative of $\sigma_{\gamma p}(W^2)$ in (7.7). Explicitly, Ratio from (7.6) becomes

$$\text{Ratio} = \frac{1}{I_T^{(1)} \left( \frac{\xi_L}{\xi_2}, \eta, \frac{\rho}{\mu} \right)} \times \left[ I_T^{(1)} \left( \eta, \frac{\rho}{\mu} \right) + I_L^{(1)}(\eta, \mu) \right] \times \left[ I_T^{(1)} \left( \eta, \frac{\rho}{\mu} \right) + I_L^{(1)}(\eta, \mu) \right] \cdot [1 - C_2 \eta - \mu] \frac{\partial}{\partial \eta} I_T^{(1)} \left( \eta, \frac{\rho}{\mu} \right) + I_L^{(1)}(\eta, \mu)]. \quad (7.10)$$

Moreover, since $\mu(W^2) = m_0^2/A_{sat}^2(W^2)$, with $m_0^2 = 0.15 \text{GeV}^2$, for sufficiently large $W^2$, $\mu(W^2)$ is small compared to unity, $\mu(W^2) \ll 1$, and the approximation of $I_T^{(1)}$ and $I_L^{(1)}$ in terms of $I_0(\eta)$ in (7.3) given by (A3) to (A5) in Appendix A may be inserted when numerically evaluating Ratio in (7.6) and (7.10).

Making use of

$$\frac{d}{d\eta} I_0(\eta) = \frac{-2}{1 + 4\eta} \left( I_0(\eta) - \frac{1}{2\eta} \right). \quad (7.11)$$

together with (7.9), Ratio in (7.10) becomes a function that depends on $\eta$, $\mu$, and $I_0(\eta)$. The different dependence of $I_0(\eta)$ on $\eta$ at large $\eta \gg 1$, where

$$I_0(\eta) \approx \frac{1}{2\eta} \left( 1 - \frac{1}{6\eta^2} \right) + \frac{1}{\eta^3}, \quad (7.12)$$

and at small $\eta$ of $\mu \leq \eta \ll 1$, where

$$I_0(\eta) \approx \frac{1}{\eta} \left( 1 - 2\eta \left( 1 - \frac{1}{\ln \frac{\rho}{\mu}} \right) \right) + ..., \quad (7.13)$$

11 In (7.10), the shift $\xi_L/\xi_2 W^2$ in $\sigma_{\gamma p}(\xi_L/\xi_2 W^2)/\ln \left( \frac{\rho}{\mu} \right)$ from (7.8) is ignored.
determines the different behavior of the numerical results for Ratio to be presented next. 

For the numerical evaluation of Ratio from (7.6) with (7.7) and (7.8) inserted, and of Ratio explicitly given in (7.10), we use the parameters\footnote{\footnote{We add the comment that only the value of \(C_1 = 0.31\text{GeV}^2\) is an entirely free fit parameter, whereas \(\rho = \frac{4}{3}\) is a consequence of the enhanced transverse size of \(q\bar{q}\) states originating from transversely relative to longitudinally polarized photons, and the exponent \(C_2 = 0.29\) is accordingly fixed by (6.9). The value of \(m_0^2 = 0.15\text{GeV}^2 < m^2\) is the starting point of the effective \(q\bar{q}\) continuum, including the \(\rho^1, \omega, \phi\) peaks. We add the remark that \(\sqrt{0.31\text{GeV}^2\approx 0.57\text{GeV}}\) may be associated with the level spacing between the vector mesons \(\rho^0\) and \(\rho^1\).}}

\[
\rho = \frac{4}{3}, \quad C_2 = 0.29, \quad C_1 = 0.31 \text{ GeV}^2, \quad m_0^2 = 0.15 \text{ GeV}^2.
\]  

(7.14)

We also note \(\xi_L = 0.4\) (see (2.8)) and \(\xi_2 = 0.5\) (see (6.5)).

In Fig. 11, we show the derivative of the structure function as well as the longitudinal structure function according to (7.7) and (7.8) respectively, and their ratio defined by (7.8). For \(W^2\), the value of \(W^2 = 10^5 \text{ GeV}^2\) was used\footnote{\footnote{As a consequence of the dominant dependence on \(\eta(W^2, Q^2)\), the absolute value of \(W^2\) in only relevant with respect to \(\sigma_{\gamma p}(W^2)\) and \(\mu(W^2)\) in (7.7) and (7.8).}}.

Since \(\sigma_{\gamma p}(W^2)\) depends weakly on \(W^2\), we expect the numerical results of Fig. 11 not being drastically affected, if in (7.7) the term proportional to the derivative of \(\sigma_{\gamma p}(W^2)\) is put to zero. The corresponding value of Ratio is explicitly given by (7.10). Comparison of the results in Fig. 12 with the ones in Fig. 11 indeed confirms that Ratio can reliably be evaluated according to the simple expression \(\sigma_{\gamma p}(W^2)\) upon insertion of \(I_L^{(1)}\) and \(I_L^{(1)}\) from (A.3) to (A.5).

In Fig. 13, we show the deviation \(\Delta \left(\eta \left(W^2 = \frac{Q^2}{x}\right), \mu(W^2)\right)\) of Ratio in (7.6) from unity on a linear scale. Remember, a value of unity means strict validity of the first DGLAP evolution equation (6.1) in the approximation (6.4) and (6.5). A deviation smaller than \(\Delta \left(\eta \left(W^2 = \frac{Q^2}{x}\right), \mu(W^2)\right) \lesssim 0.1\) to 0.2 according to Fig. 13 requires \(\eta(W^2, Q^2) \gtrsim 5\). The weaker increase, and finally the decrease of \(F_L(\eta(x, Q^2), Q^2)\) relative to the derivative of \(F_2(\eta(x, Q^2), Q^2)\) with decreasing \(\eta(x, Q^2)\), or decreasing \(x\) at fixed \(Q^2\), see (7.4), implies the substantial violation of the evolu-
tion equation reaching \( \Delta(\eta(W^2, Q^2), \mu(W^2)) \approx 2.7 \) at \( \eta(W^2, Q^2) \approx 0.1 \).

Above \( \eta \approx 5 \), we have pQCD-evolution or CDP-color-transparency, while below \( \eta \approx 5 \), we have a violation of conventional evolution that is equivalent to hadronlike saturation.

In terms of \( x \) and \( Q^2 \), according to Fig.15, at fixed value \( Q^2 \) with decreasing \( x \), or at fixed value of \( x \) with decreasing \( Q^2 \), the region of hadronlike behavior is reached that in connected with a violation of standard evolution.

In Fig. 16, we address the question of the dependence of the above conclusion on the value of \( \rho \) that specifies the asymptotic behavior of the longitudinal-to-transverse ratio in photoabsorption via \( R = 1/2 \rho \). As long as precision experimental data on \( R_t \) and accordingly on \( \rho \) are lacking, it is interesting to vary \( \rho \) around its preferred value of \( \rho = 4/3 \), implying different values of the exponent \( C_2 \) according to (6.9). From Fig. 16, we infer that a change of \( \rho \) does qualitatively not significantly change our general conclusion obtained for \( \Delta(\eta(W^2 = Q^2/x), Q^2) \).

In Fig.17, we show Ratio according to (7.10) as a function of \( Q^2 \) at fixed \( x \), upon replacing \( \eta \) and \( \mu \) in terms of \( x \) and \( Q^2 \) according to (7.4) and (7.5). The figure shows for decreasing fixed values of \( x \) the increasingly stronger deviation of Ratio from Ratio=1 with decreasing \( Q^2 \).

In Figs. 11 to 14, we showed Ratio from (7.6) as a function of \( \eta(W^2, Q^2) \). In connection with the extraction of the gluon distribution from DIS experimental data, one frequently uses a fixed value of \( Q^2 \), chosen as low as \( Q^2 = 1.9 \text{ GeV}^2 \approx 2 \text{ GeV}^2 \), as starting scale for the evolution of the gluon distribution with increasing \( Q^2 \) at fixed \( x \).

In Table III, we show the results for \((1 + \Delta)^{-1}\) according to (7.6) with (7.7) and (7.8) for fixed \( Q^2 = 2 \text{ GeV}^2 \) and the typical range of \( 10^{-2} \leq x \leq 10^{-5} \), corresponding to an interval of \( \eta \) approximately given by \( 1.5 \geq \eta \geq 0.2 \). According to Table III, we observe a dramatic correction factor to standard evolution of magnitude \( 0.6 \geq (1 + \Delta)^{-1} \geq 0.4 \). This correction to evolution
is due to a strong violation of the impulse approximation of the pQCD improved parton model at low $Q^2$. The associated transition to hadronlike $(q\bar{q})p$ interactions is outside the range of validity of the standard evolution equations. It comes without surprise that global DIS data fits based on imposing evolution from a low-$Q^2$-starting input scale “do not describe the deep inelastic scattering data in the low-$x$, low-$Q^2$ region very well”, see ref. [3]. We add a comment on the interpretation of the huge discrepancy between the CDP gluon distributions in Figs. 6 and 10 and the published results from several collaborations. The CDP gluon distribution is deduced from color gauge-invariant CDP structure functions explicitly incorporating the $Q^2\rightarrow0$ limit. The standard determinations contain a relatively ad hoc continuation to low values of $Q^2$.

In Table III, in addition to the choice of $Q^2 = 2$ GeV$^2$, we also give the results for the correction factor $(1+\Delta)^{-1}$ at $Q^2 = 20$ GeV$^2$. For $x = 10^{-2}$, with $(1+\Delta)^{-1} = 0.96 \cong 1$, we have consistency with perturbative evolution, as expected. For sufficiently large $W^2$, or $10^{-3} > x > 10^{-5}$, we observe the expected transition from unmodified pQCD evolution to hadronlike saturation.

We turn to the discussion of the gluon distribution function. Rewriting (7.6), and substituting the gluon distribution from (2.8), we find

$$\frac{\alpha_s(Q^2)}{\alpha_s(Q)} \frac{dF_3(x, Q^2)}{dlog_2(x, Q^2)} = F_L \left( \frac{\xi^2 x, Q^2}{\xi^2} \right) = \frac{\alpha_s(Q^2)}{\alpha_s(Q)} R_{e^+e^-} G \left( \frac{x}{\xi^2}, Q^2 \right),$$  (7.15)

where $R_{e^+e^-} = 10/3$ for four flavors of quarks. The factor $(1+\Delta)^{-1}$ in (7.15), for $\Delta \neq 0$ represents a violation of, or alternatively, a correction to the evolution equation for the gluon distribution with decreasing $\eta$ for $\eta(W^2 = Q^2) \lesssim 5$.

The violation of the evolution with decreasing $Q^2$ may be seen directly in terms of the gluon distribution by showing the gluon distribution as a function of $x$ for e.g. $Q^2 = 100$ GeV$^2$ and $Q^2 = 2$ GeV$^2$ i.e. rewriting (7.15) as

$$G \left( x, Q^2 \right) = \frac{\partial F_3(x, Q^2)}{\partial \ln Q^2} \frac{9\pi}{1 + \Delta(\eta, \mu)} \frac{\alpha_s(Q^2) R_{e^+e^-}}{\alpha_s(Q^2)},$$  (7.16)

where $\Delta(\eta, \mu)$ is inserted from (7.6). In Fig.18 we show the gluon distribution according to (7.16). In addition, we show the gluon distribution under the ad hoc assumption of $\Delta(\eta, \mu) = 0$ corresponding to validity of evolution in violation of the CDP result where $\Delta(\eta, \mu) \neq 0$.
at low $Q^2$, i.e., specifically for $Q^2=1.9$ GeV$^2$. Since the experimental data, described by the $\gamma^* g \rightarrow q\bar{q}$ interaction of the CDP, violate the evolution equation at low $Q^2$, the assumption of a universal validity of the evolution equation at low $Q^2$ is excluded. Relying on the universal absolute validity of the evolution equation at a low-$Q^2$ input scale in e.g. global fits to the experimental data is highly questionable.

The Ratio may be explicitly interpreted as a modification of the evolution equation (6.6). Upon substitution of (7.10) equation (6.6) becomes

$$
\frac{\partial F_2(x,Q^2)}{\partial \ln Q^2} = F_L \left( \frac{\xi_L}{\xi_T} x, Q^2 \right) \times \text{Ratio}
$$

$$
= F_L \left( \frac{\xi_L}{\xi_T} x, Q^2 \right) \times \left[ I_L^{(1)} \left( \frac{\eta}{\mu}, \frac{\xi_T}{\xi_L} \right) + I_L^{(1)} (\eta, \mu) \right] + \frac{2}{\partial \ln Q^2} \left( I_T^{(1)} \left( \frac{\eta}{\mu}, \frac{\xi_T}{\xi_L} \right) + I_L^{(1)} (\eta, \mu) \right),
$$

(7.17)

where the simple equations for $I_{L,T}^{(1)}$ in Appendix (A3) to (A5) are to be inserted. At small $\mu$, the CDP evolution equation (7.17) becomes

$$
\frac{\partial F_2(x,Q^2)}{\partial \ln Q^2} \simeq \frac{F_L \left( \frac{\xi_L}{\xi_T} x, Q^2 \right)}{I_0(\eta)} \times \left\{ \int_0^{\eta} \frac{(1 - C_2)^{\alpha_s(Q^2)}}{1 - 2\xi_T I_0(\eta)} \right\}
$$

(7.18)

where $I_0$ is defined in (A.5). The factor multiplying the longitudinal structure function continuously converges towards unity at large $Q^2$. We note that the CDP evolution equation (7.17), differs from an ad hoc modification of conventional evolution, since it is an analytically determined smooth extrapolation to low $Q^2$ consistent with DIS measurements.

VIII. CONCLUSIONS.

Employing the pQCD relation between the longitudinal proton structure function and the gluon distribution function allows one to define a gluon distribution associated with the proton structure functions of the CDP at low $x$ and any $Q^2 \geq 0$. The CDP gluon distribution depends on $W^2 = Q^2/x$ and $Q^2$. For $Q^2$ sufficient large ($10$ GeV$^2 \lesssim Q^2 \lesssim 100$ GeV$^2$ at presently available energies) the CDP gluon distribution, multiplied by $\alpha_s(Q^2)$, converges towards the asymptotic limit proportional to the saturation scale $\Lambda_{sat}^2(W^2) \sim (\frac{W^2}{Q^2})^{C_2=0.29}$.

The deviation between the CDP gluon distribution and several published distributions, at low $Q^2$ in particular, is a consequence of the representation of the structure functions of DIS in the CDP. The CDP structure functions based on two-gluon exchange explicitly include large as well as small values of $Q^2$, including the $Q^2 \rightarrow 0$ limit. The associated CDP gluon distribution is valid for large as well as small $Q^2$. The usual determination of the gluon distribution however is not based on a very detailed justification of deducing the gluon distribution from the experimental data at a low-$Q^2$ input scale. The CDP explicitly includes the $Q^2 \rightarrow 0$ structure functions, in distinction from the standard approach based on a low $Q^2$ input scale that lacks a very detailed justification for a low $Q^2$ input scale.

Concerning evolution, the dependence of the logarithmic derivative of the proton structure function on $x$ and $Q^2$, we find a CDP evolution equation that agrees with the standard model evolution equation in the limit of large $Q^2$ at any fixed low $x$.

At low $Q^2$, where photoabsorption cross section becomes hadronlike, the CDP evolution differs significantly from the standard one by an explicitly analytically given factor. Relying on a universality of the parton model and standard evolution at a low-$Q^2$ starting scale, such as $Q^2 \approx 1.9$ GeV$^2$, as frequently employed in global fits, is questionable and requires further investigations.

APPENDIX A

The explicit expressions for the functions $I_{L,T}^{(1)}(\eta(W^2, Q^2), \mu(W^2))$ in (3.2) are given by:

$$
I_{L,T}^{(1)}(\eta(W^2, Q^2), \mu(W^2)) = I_{L,T}^{(1)}(\eta(W^2, Q^2), \mu(W^2)) \times (1 + \mu(W^2)),
$$

(A.1)

where $I_{L}^{(1)}(\eta(W^2, Q^2), \mu(W^2))$ and $I_{T}^{(1)}(\eta(W^2, Q^2), \mu(W^2))$ are given by

$$
I_{L}^{(1)}(\eta, \mu) = \frac{\eta - \mu}{\eta} \times \left( 1 + \frac{\eta}{\sqrt{1 + 4(\eta - \mu)}} \right)
$$

$$
\times \ln \left( \frac{\eta(1 + \sqrt{1 + 4(\eta - \mu)})}{4\mu - 1 - 3\eta + \sqrt{(1 + 4(\eta - \mu))(1 + \eta)^2 - 4\mu}} \right),
$$

(A.2)

$$
I_{T}^{(1)}(\eta, \mu) = \frac{1}{2} \ln \frac{\eta - 1 + \sqrt{(1 + \eta)^2 - 4\mu}}{2\eta} - \frac{\eta - \mu}{2\sqrt{1 + 4(\eta - \mu)}} \times \ln \frac{\eta(1 + \sqrt{1 + 4(\eta - \mu)})}{4\mu - 1 - 3\eta + \sqrt{(1 + 4(\eta - \mu))(1 + \eta)^2 - 4\mu}}.
$$

For $\mu(W^2) \ll 1$ (and $\eta(W^2, Q^2) \geq \mu(W^2)$) the functions $I_{L}^{(1)}(\eta, \mu)$ and $I_{T}^{(1)}(\eta, \mu)$ can be simplified to become

$$
I_{L}^{(1)}(\eta, \mu) = \frac{\eta - \mu}{\eta} \times (1 - 2\eta I_{0}(\eta))
$$

(A.3)
where
\[ I_0(\eta) = \frac{1}{\sqrt{1 + 4\eta}} \ln \frac{\sqrt{1 + 4\eta} + 1}{\sqrt{1 + 4\eta} - 1} \]

(A.5)

We also note that for the relevant range of \( \eta \gg \mu \), or \( Q^2 \gg M^2 \), we may put \( \mu = 0 \) in (A.3) and (A.4).

**APPENDIX B**

In this Appendix we show the coefficients of the structure functions \( F_2(x, Q^2) \) given in (4.1), as taken from ref. [22]. The coefficients in (4.1),
\[
A(Q^2) = a_0 + a_1 \ln \left( 1 + \frac{Q^2}{\mu^2} \right) + a_2 \ln^2 \left( 1 + \frac{Q^2}{\mu^2} \right),
\]
\[
B(Q^2) = b_0 + b_1 \ln \left( 1 + \frac{Q^2}{\mu^2} \right) + b_2 \ln^2 \left( 1 + \frac{Q^2}{\mu^2} \right),
\]
\[
C(Q^2) = c_0 + c_1 \ln \left( 1 + \frac{Q^2}{\mu^2} \right),
\]
\[
D(Q^2) = \frac{Q^2(Q^2 + \lambda M^2)}{(Q^2 + M^2)^2},
\]
depend on the fit parameters listed in Table II.

**TABLE B.1.** The effective parameters at low \( x \) for \( 0.15 GeV^2 < Q^2 < 3000 GeV^2 \) provided by the following values. The fixed parameters are defined by the Block-Halzen fit to the real photon-proton cross sections as \( M^2 = 0.753 \pm 0.068 \text{ GeV}^2, \mu^2 = 2.82 \pm 0.290 \text{ GeV}^2 \) and \( c_0 = 0.255 \pm 0.016 \).

| parameters | value |
|------------|-------|
| \( a_0 \)  | \( 8.205 \times 10^{-4} \pm 4.62 \times 10^{-4} \) |
| \( a_1 \)  | \( -5.148 \times 10^{-2} \pm 8.19 \times 10^{-3} \) |
| \( a_2 \)  | \( -4.725 \times 10^{-3} \pm 1.01 \times 10^{-3} \) |
| \( b_0 \)  | \( 2.217 \times 10^{-3} \pm 1.42 \times 10^{-4} \) |
| \( b_1 \)  | \( 1.244 \times 10^{-2} \pm 8.56 \times 10^{-4} \) |
| \( b_2 \)  | \( 5.958 \times 10^{-4} \pm 2.32 \times 10^{-4} \) |
| \( c_1 \)  | \( 1.475 \times 10^{-3} \pm 3.025 \times 10^{-4} \) |
| \( n \)    | \( 11.49 \pm 0.99 \) |
| \( \lambda \) | \( 2.430 \pm 0.153 \) |
| \( \chi^2 \) (goodness of fit) | 0.95 |
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