Analysis Method of Subjective Uncertainty

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Abstract: Understanding the uncertainty of objective things in actual engineering is inevitable, and it has a direct impact on engineering decision-making. In view of this, there are many theories—fuzzy mathematics, evidence theory and other related theories have been put forward, and which have been widely applied in practical engineering at present. However, these theories are based on their specific uncertainties, which are large different from each other and do not completely reveal the common attributes of such uncertainties. In this paper, a basic analysis method of subjective uncertainty is established by based on the concepts of degree of belief and unascertained quantity. The paper will introduce the axiomatic operation rules of degree of belief, and further elaborate the subjective uncertainty analysis method. Based on the basic analysis methods of fuzzy mathematics and evidence theory, a unified method of subjective uncertainty analysis is reviewed and improved. It can help to establish a unified understanding of subjective uncertainty, and can help to comprehensively analyze the influence of various subjective uncertainties on engineering decision-making.

1. Introduction

In engineering decision-making, people's understanding of objective things is often uncertain. According to the origin, this kind of cognitive uncertainty at a certain level can be called the subjective uncertainty, including the fuzziness and unascertainness proposed at present [1,2]. It originates from the subject (person) of cognition and is determined by people's cognitive level which is restricted by cognitive means, information resources, knowledge level, natural and social conditions, etc. The subjective uncertainty is unavoidable in engineering practice, and has a direct impact on engineering decision [3,4]. In this respect, basic theories such as fuzzy mathematics and evidence theory have been put forward [5-8], and Bayesian school and belief school in statistics also involve this kind of uncertainty [9,10]. These theories are mainly put forward for their own specific uncertainty, among which fuzzy mathematics and evidence theory focus on the uncertainty of concept and proposition understanding respectively, Bayesian school and belief school focus on the uncertainty of random things understanding, and there are great differences in their basic analysis methods. In fact, the uncertainty of understanding objective things should have a common attribute. It is necessary to establish a unified analysis method, which not only helps to establish a unified understanding of the subjective uncertainty, but also helps to comprehensively analyze the impact of various subjective uncertainties on project decision-making [11-13].

Literature [1,2] established the basic analysis method of subjective uncertainty using concepts such as reliability and unascertained quantity, as well as calculation rules of proposition sets and reliability. The axiomatic operation rules of reliability are introduced, and the analysis methods of subjective uncertainty are explained in more depth. Based on this, the axiomatic operation rules of degree of belief are introduced, and the analysis methods of subjective uncertainty are explained in more
depth. The basic analysis methods in fuzzy mathematics and evidence theory are reviewed and improved, and a unified analysis method of subjective uncertainty is established. Analysis method of subjective uncertainty

2. Analysis method of subjective uncertainty

2.1. Proposition domain and proposition set

Similar to the definition of random events in probability theory [14], the most basic and indivisible results of people’s understanding of objective things can be called basic propositions \( \omega \); The set consisting of all basic propositions is the domain \( \Omega \), and any proposition composed of basic propositions can be expressed as a subset of the domain \( \Omega \), in which the empty \( \emptyset \) set represents a completely unproven proposition, and the complete set \( \Omega \) represents a proposition that must be established, and they are not compatible; The set composed of all subsets in the domain \( \Omega \) is called the proposition domain \( \Gamma \) [1,2].

To ensure the integrity of proposition field \( \Gamma \), we can set [1,2]:

1. \( \Omega \in \Gamma \). The inevitable proposition \( \Omega \) (complete set \( \Omega \)) belongs to the domain of proposition \( \Gamma \).
2. If \( A \in \Gamma \), \( A^c = \Omega - A \), then \( A^c \in \Gamma \). If the proposition \( A \) belongs to the proposition domain \( \Gamma \), then the propositions \( A^c \) (complements of sets \( A^c \) ) outside the propositions \( A \) in the domain also belong to the proposition domain \( \Omega \).
3. If \( A_i \in \Gamma (i = 1,2,\cdots) \), then \( \bigcup_{i=1}^{n} A_i \in \Gamma \). If all propositions \( A_1, A_2, \cdots \) belong to propositional domain \( \Gamma \), then their or propositions \( \bigcup_{i=1}^{n} A_i \) also belong to propositional domain \( \Gamma \).

Compared with the event domain in probability theory, proposition domain has two different characteristics. First, its basic propositions are not necessarily incompatible. For example, for basic propositions \( A_1 = \{ \text{The 62 year old is an old man} \} \) and \( A_2 = \{ \text{The 62 year old is not an old man} \} \), although their meanings are completely opposite, they may be affirmed to some extent at the same time, these two basic propositions are not incompatible. Second, the propositional domain is not completely composed of valid propositions. The so-called effective proposition refers to the!proposition that makes a meaningful judgment on things, such as {Air Pollution Index will not be lower than 150}, {High-speed rail running speed exceeds 300 km/h} and other topics. There are three cases in which the propositional domain is not completely composed of valid propositions:

1. General concept. After listing the N factors \( \{ A_1, A_2, \cdots, A_n \} \) that affect the flow of people in the malls, people may still worry about missing other factors, which means that the complement of union \( \bigcup_{i=1}^{n} A_i \) is not an empty set \( \emptyset \). If you remember it as \( \Delta \), then the complete propositional domain \( \Gamma \) should consist of all subsets in the domain \( \Omega = \{ A_1, A_2, \cdots, A_n, \Delta \} \). The complement \( \Delta \) here refers to proposition \( \{ \text{it is impossible to determine whether a 62 year old is an old person} \} \), it is impossible to determine if \( \bigcup_{i=1}^{n} A_i \) is a complete set, it is not a valid proposition.
2. Fuzzy concept. The two basic propositions which are totally opposite to the practical significance \( A_1 = \{ \text{The 62 year old is an old man} \} \) and \( A_2 = \{ \text{The 62 year old is not an old man} \} \). It is impossible to make a completely positive judgment on their or propositions. This means that the complements of \( A_1 \cup A_2 \) is not empty \( \emptyset \). Also remember it as \( \Delta \), then the complete propositional domain \( \Gamma \) should consist of all subsets in the domain \( \Omega = \{ A_1, A_2, \Delta \} \). At this time, the complement \( \Delta \) refers to the proposition \( \{ \text{it is impossible to determine whether a 62 year old is an old person} \} \), and it is not an effective proposition.
(3) Clear concept and insufficient criteria. Let $A_1 = \{\text{there is life outside the star}\}$, $A_2 = \{\text{there is no life outside the star}\}$. It has no practical significance and is not an effective proposition. Due to the lack of evidence, the degree of affirmation of these two basic propositions is very low, which obviously means that the complement of $A_1 \cup A_2$ is not an empty set $\phi$, and the complement $\Delta$ of $A_1 \cup A_2$ need to be added to the domain $\Omega$, so that $\Omega = \{A_1, A_2, \Delta\}$. At this time, the complement $\Delta$ refers to the proposition \{can't determine whether there is life outside the star\}, which has no practical significance and is not an effective proposition.

In order to ensure the integrity of proposition domain $\Gamma$, in addition to all valid basic propositions $A_1, A_2, \cdots$, a special proposition should be introduced into the domain $\Omega$, that is, \{Or proposition $\bigcup_{i=1}^{\infty} A_i$ cannot be completely determined\}, which is equivalent to the complement of $\bigcup_{i=1}^{\infty} A_i$ here, it is collectively called complement $\Delta$, and can be expressed as

$$\Delta = \Omega - \bigcup_{i=1}^{\infty} A_i$$  \hspace{1cm} (1)

The introduction of complements $\Delta$ is determined by the characteristics of people's cognitive activities. In fact, for any valid proposition, in addition to "affirmative" and "negative", people may also have a third attitude, that is, "can't be completely determined", and the introduction of complement $\Delta$ is exactly to reflect this attitude of people.

Since the complement $\Delta$ is not a valid proposition, it can be assumed that it is incompatible with any valid proposition, that is

$$A_i \cap \Delta = \phi \hspace{1cm} i = 1, 2, \cdots$$  \hspace{1cm} (2)

$$A_i \cup \Delta = A_i + \Delta \hspace{1cm} i = 1, 2, \cdots$$  \hspace{1cm} (3)

It can be proved that the proposition set after the introduction of complement set $\Delta$ satisfies the idempotent law, the interchange law, the combination law, the absorption law, the distribution law, the bipolar law, the restoration law and the duality law of set operations. In order to show whether it satisfies the complementary law, the complement of proposition set needs to be redefined. For the set $A_i$, after the delta complement $\Delta$ is introduced, its complement $A_i^c$ should be

$$A_i^c = \Omega - A_i = \bigcup_{i=1}^{\infty} A_i + \Delta - A_i = A_i^{\infty} + \Delta$$  \hspace{1cm} (4)

Among

$$A_i^{\infty} = \bigcup_{i=1}^{\infty} A_i - A_i = \Omega - \Delta - A_i$$  \hspace{1cm} (5)

In the formula: $A_i^{\infty}$ is the complement of the set $A_i$ when the complement $\Delta$ is not considered at present, and it can be called an effective complement. In this case, the actual complement set $A_i^c$ of the set $A_i$ contains the complement $\Delta$, which has the following relationship with the set $A_i$:

$$A_i \cup A_i^c = A_i \cup \left( \bigcup_{i=1}^{\infty} A_i + \Delta - A_i \right) = \bigcup_{i=1}^{\infty} A_i + \Delta = \Omega$$  \hspace{1cm} (6)

$$A_i \cap A_i^c = A_i \cap \left( \bigcup_{i=1}^{\infty} A_i + \Delta - A_i \right) = A_i \cap \left( \bigcup_{i=1}^{\infty} A_i - A_i \right) \neq \phi$$  \hspace{1cm} (7)

They satisfy the union operation in the complementary law, but because the basic propositions in the propositional domain are not necessarily incompatible with each other, they do not always satisfy the intersection operation. The set of propositions does not completely satisfy the law of complementarity, which is different from the event domain in probability theory.
2.2. Reliability and unascertained quantity

Let Bel be a set of real-valued functions defined in the proposition field $\Gamma$, that is, according to the set function, any proposition in the proposition field $\Gamma$ corresponds to a specific real number. If the following conditions are met, Bel is called the trust measure on the propositional domain $\Gamma$, and $Bel(A)$ is the degree of belief of the proposition $A$.

1) Non-negativity. For any proposition $A \in \Gamma$, there is 
$$0 \leq Bel(A) \leq 1$$

(8)

2) Regularity. For the inevitable propositions $\Omega$, there is 
$$Bel(\Omega) = 1$$

(9)

3) Additivity. For any propositions, there is 
$$Bel\bigg(\bigcup_{i} A_i\bigg) = \sum_i Bel(A_i) - \sum_i Bel(A_i \cap A_j) + \sum_{i,j,k} Bel(A_i \cap A_j \cap A_k) - \cdots$$

(10)

The additivity of degree of belief is different from the current viewpoint in evidence theory, which will be explained in detail later.

Similar to the definition of random variable [14], let $\xi = \xi(\omega)$ be a single-valued real function defined on the domain $\Omega$. If there is $\{\omega: \xi(\omega) \leq x\} \in \Gamma$ for any real number $x$, that is, a proposition composed of all basic propositions satisfying the $\xi(\omega) \leq x$ condition belongs to the proposition domain $\Gamma$, then $\xi = \xi(\omega)$ is called an unknown variable, and calling the function 
$$Bel_i(x) = Bel(\{\omega: \xi(\omega) \leq x\})$$

(11)

It is the belief function of the unknown quantity $\xi$, and it can also be recorded as $Bel(\xi \leq x)$, which indicates the degree of trust to proposition $\{\xi \leq x\}$. When the degree of belief function $Bel_i(x)$ is absolutely continuous, the trust density function $bel_i(x)$ can be introduced, which satisfies 
$$Bel_i(x) = \int_{-\infty}^{x} bel_i(t) \, dt$$

(12)

2.3. Reliability calculation rules

For the degree of belief of proposition, it can be set according to the basic logical relationship:

(1) For the inevitable propositions $\Omega$, there are 
$$Bel(\Omega) = 1$$

(13)

(2) For any proposition $A_i$ and $A_j$, $i,j = 1,2,\ldots$, if $A_i \subseteq A_j$, that is, any element of proposition set $A_i$ belongs to proposition set $A_j$, and proposition $A_i$ is contained in the proposition $A_j$, then 
$$Bel(A_i \cap A_j) = Bel(A_i) \leq Bel(A_j) \quad A_i \subseteq A_j$$

(14)

(3) For any proposition $A_i$ and $A_j$, $i,j = 1,2,\ldots$, if $A_i \cap A_j = \phi$, that is, the intersection of proposition sets $A_i$ and $A_j$ is empty set, and propositions $A_i$ and $A_j$ are incompatible, then 
$$Bel(A_i \cup A_j) = Bel(A_i) + Bel(A_j) \leq 1 \quad A_i \cap A_j = \phi$$

(15)

These three basic operation rules of degree of belief are axiomatic. Combined with the operation rules of proposition set, the non-negativity, regularity and additivity of the degree of belief can be verified. Only additivity of the degree of belief is verified here.

After considering the complement $\Delta$, for any proposition $A_i$ and $A_j$, its or proposition, and the degree of belief of proposition is decomposed into the sum of the following incompatible propositions, that is 
$$Bel(A_i \cup A_j) = Bel(A_i : A_i \cap A_j = \phi) + Bel(A_i : A_i \cap A_j \neq \phi, A_j \subseteq A_i)$$

(16)

$$+ Bel(A_j : A_i \cap A_j \neq \phi, A_i \subseteq A_j) + Bel(A_i : A_i \cap A_j = \phi) \quad i = 1,2,\ldots$$

(17)

In the formula, “$A_i \cup A_j$” means that the B proposition is the condition of degree of belief value of proposition $a$. Add the two formulas and pay attention to
\( \text{Bel}(A) = \text{Bel}(A : A \cap A_i = \phi) + \text{Bel}(A : A \cap A_i \neq \phi, A_i \subseteq A_j) + \text{Bel}(A : A \cap A_j \neq \phi, A_j \subseteq A_i) \) \hfill (18)

\( \text{Bel}(A) = \text{Bel}(A : A \cap A_i = \phi) + \text{Bel}(A : A \cap A_j \neq \phi, A_j \subseteq A_i) + \text{Bel}(A : A \cap A_j = \phi) \) \hfill (19)

Finally available

\( \text{Bel}(A \cup A_i) = \text{Bel}(A) + \text{Bel}(A_i) - \text{Bel}(A \cap A_i) \) \hfill (20)

For the general case of multiple propositions, according to the calculation rules of propositional set and using equation (20), the additivity of degree of belief in general case can also be proved, that is, the general formula shown in formula (10).

For any proposition \( A_i \) and \( A_j \), according to the relationship between propositions, the degree of belief can be

\[
\text{Bel}(A_i | A_j) = \begin{cases} 
0 & A_i \cap A_j = \phi \\
\text{Bel}(A_i) & A_i \cap A_j = \phi, A_i \subseteq A_j \\
\text{Bel}(A_i) \text{Bel}(A_j) & A_i \cap A_j = \phi \text{ and they are independent} \\
\text{Bel}(A_i | A_j) \text{Bel}(A_j) & A_i \cap A_j = \phi \text{ and they are related}
\end{cases}
\]

(21)

Where:

\( \text{Bel}(A_i | A_j) \) refers to the degree of belief of proposition \( A_i \) under the condition of affirming proposition \( A_j \), that is, the conditional degree of belief of proposition \( A_j \).

According to the degree of belief operation rules of or proposition , and proposition, the reliability rules for complements are available, that is

\[
\text{Bel}(A_i \cup A_j) = \text{Bel}(A_i) + \text{Bel}(A_j) - \text{Bel}(A_i \cap A_j) = 1
\]

(22)

\[
\text{Bel}(A_i \cup A_j) = \text{Bel}(A_i \cup (\Omega - A_j)) = \text{Bel}(\Omega - A_j) = 1 - \text{Bel}(A_j) \leq 1
\]

(23)

\[
\text{Bel}(A_i \cup A_j) = \text{Bel}(A_i \cup (\Omega - A_j)) = \text{Bel}(A_i \cap (\Omega - A_j)) \geq 0
\]

(24)

\[
\text{Bel}(A_i \cup A_j) = \text{Bel}(A_i \cup (\Omega - A_j)) = 1
\]

(25)

\[
\text{Bel}(A_i \cup A_j) = \text{Bel}(A_i \cup (\Omega - A_j)) \geq 0
\]

(26)

\[
\text{Bel}(A_i \cap A_j) = \text{Bel}(A_i \cap (\Omega - A_j))
\]

(27)

\[
\text{Bel}(A_i \cap A_j) = \text{Bel}(A_i \cap (\Omega - A_j)) \geq 0
\]

(28)

Obviously, the degree of belief of proposition set and its effective complement does not satisfy the complementarity law, and the reliability of its actual complement satisfies the union operation in the complementarity law, but does not always satisfy the intersection operation.

3. Review and improvement of relevant theories

3.1. evidence theory

Evidence theory does not negate the complementary law of the set of propositions, but believes that the degree of belief of propositions is not additive, it only satisfies the following inequalities, that is

\[
\text{Bel}(A_i) \geq \sum_{A_{i,j}} \text{Bel}(A_{i,j}) - \sum_{A_{i,j}} \text{Bel}(A_i \cap A_{i,j}) + \sum_{A_{i,j}} \text{Bel}(A_i \cap A_j \cap A_i) - \ldots
\]

(29)

Its typical example is the proposition \( A = \{ \text{life outside the star} \} \) and its converse proposition \( \overline{A} = \{ \text{extraterrestrial life} \} \), there should be

\[
\text{Bel}(A \cup \overline{A}) = \text{Bel}(\Omega) > \text{Bel}(A) + \text{Bel}(\overline{A}) - \text{Bel}(A \cap \overline{A})
\]

(30)

The root of this difference is that the complement \( \Delta \) is not considered in the construction of proposition domain \( \Gamma \).

In order to reflect the characteristics of cognitive activities, complement \( \Delta \) should also be introduced into evidence theory. It not only make the degree of belief of proposition have the additivity shown in formula (10), but also make a clearer explanation for the degree of belief of proposition \( A \cup \overline{A} \) in the current evidence theory according to formula (25): it only represents the reliability of all valid
propositions, and does not reflect the third attitude of valid propositions determination, that is, “it can not be completely judged”. The degree of belief of \( A \cup \tilde{A} \) does not represent the degree of belief of complete set \( \Omega \), but the degree of belief of proposition \( A \cup A' \) or after the introduction of complement \( \Delta \).

In addition to the degree of belief function, the likelihood function is also proposed in evidence theory, and its expression is\[7\]

\[
\text{Pl}(A) = 1 - \text{Bel}(\tilde{A})
\]

(31)

It indicates that the degree of belief of no negative proposition \( A \). In this case, \([0, \text{Bel}(A)]\) represents the supporting evidence interval of proposition \( A \), \([\text{Bel}(A), \text{Pl}(A)]\) represents the uncertainty interval of proposition \( A \), \([0, \text{Pl}(A)]\) represents the rejection evidence interval of proposition \( A \). After introducing complement \( \Delta \), the likelihood function can be expressed as

\[
\text{Pl}(A) = 1 - \text{Bel}(\tilde{A}) = 1 - \text{Bel}(\omega - \Delta - A) = \text{Bel}(A) + \text{Bel}(\Delta)
\]

(32)

It consists of two parts: the degree of belief \( \text{Bel}(A) \) of proposition \( A \) and the degree of belief \( \text{Bel}(\tilde{A}) \) which cannot be completely determined for all valid propositions. According to this structure, the significance of the above-mentioned intervals can be revealed more clearly. In this case, the larger the value of \( \text{Bel}(A) \), that is, the higher the degree of "unable to determine completely", the larger the width of uncertainty interval and pseudo-belt interval.

When studying the uncertainty of the cognition of random phenomena, evidence theory holds that the degree of belief of future change in random things is controlled by probability. The greater the probability of a certain result, the higher the certainty that the result will appear, and the degree of belief of a proposition is composed of the basic probability number of the evidence it contains. The latter refers to the degree of belief given to each evidence according to the basic degree of belief distribution \[7\]. In this case, the degree of belief of proposition \( A_i \) and \( A_j \) can be expressed as

\[
\text{Bel}(A_i) = \sum_{B_i \subseteq A_i} m(B_i)
\]

(33)

\[
\text{Bel}(A_j) = \sum_{B_j \subseteq A_j} m(B_j)
\]

(34)

Where, \( m(B_i) \) and \( m(B_j) \) are the basic probability values of evidence \( B_i \) and \( B_j \) contained in proposition \( A_i \) and \( A_j \) respectively. Then

\[
\text{Bel}(A) = \text{Bel}(A \cap A_i) = \sum_{B_i \subseteq A \cup A_i} m(B_i)m(B_j)
\]

(35)

The formula is consistent with the degree of belief operation rules of the proposition shown in formula (21). However, there may be a phenomenon of \( B_i \cap B_j = \emptyset \) between the evidences of proposition \( A \). It is unreasonable to assign the basic probability value to this part of evidences, and it should be discarded, but the total reliability of domain \( \Omega \) will be less than 1. In order to ensure the regularity of degree of belief, the Dempster synthesis rule of evidence is proposed by the method of normalization in evidence theory, that is \[7\]

\[
\text{Bel}(A) = \frac{\sum_{B_i \subseteq A \cup A_i} m(B_i)m(B_j)}{1 - \sum_{A \cup A_i \neq \emptyset} m(B_i)m(B_j)}
\]

(36)

This is just a methodological treatment, and the reason for it is not explored in the theory of evidence.

The Brownian motion of particles on a circular water surface with a diameter of 1 shown in Fig. 1 is investigated. When observed on the x-axis, proposition \( A_{11} \) = {particles appear in the x-axis interval [0,0.5]}, \( A_{12} \) = {particles appear in the x-axis interval (0.5,1]}; when observed on the y-axis, the
proposition $A_1 = \{\text{particles appear in the y-axis interval [0,0.5]}\}$, $A_2 = \{\text{particles appear in the y-axis interval (0.5,1]}\}$. The reliability of the four propositions is 0.5. Whether it is observed on the x-axis or the y-axis, the total reliability of each domain is 1.

![Random Brownian motion of particles on a circular water surface](image)

When inferring the area where particles appear on a circular water surface based on observations, it is unreasonable to assign basic probability values to parts outside the water. The intersection of evidences $B_x$ and $B_y$ supporting propositions $A_x$ and $A_y$ outside the circular water surface and inside the square should be empty set $\emptyset$, but its actual basic probability value is not 0, but $1 - \pi/4$, so it should be discarded. Order

$$A_y = A_x \cap A_y \quad i, j = 1,2$$  \hspace{1cm} (37)

After discarding the degree of belief of $B_x \cap B_y = \emptyset$, According to formula (36), there should be

$$Bel(A_y) = \frac{\sum_{i \in [0,\pi/\angle \Omega]} m(B_i) m(B_j)}{\sum_{i \in [0,\pi/\angle \Omega]} m(B_i) m(B_j)} = \frac{4 \pi}{16} = \frac{1}{4} \quad i, j = 1,2$$  \hspace{1cm} (38)

After applying this correction according to the theory of evidence, the total reliability of particles appearing on the actual circular water surface is 1.

No matter in the x-axis or y-axis observation, the proposition’s domain $\Omega$ is [0,1], but when inferring the area where particles appear on the circular water surface, the domain $\Omega$ will evolve into a circular plane with a diameter of 1, while the evidence theory defaults it to a square plane with a side length of 1, expanding the scope of domain $\Omega$, making the total reliability of the actual domain $\Omega$ less than 1. The failure to determine the scope of domain $\Omega$ reasonably is the fundamental reason why the total reliability of evidence theory is less than 1.

### 3.2. Fuzzy mathematics

In fuzzy mathematics, membership function $\mu_A(x)$, membership degree “$\cup$” and “$\cap$” operators are used to describe and analyze fuzzy sets [5], which are quite different from the analysis methods of proposition sets, but the difference is only formal. By using unascertained quantity and the degree of belief operation rules, the same degree of belief analysis can be carried out on Fuzzy sets, and the same results can be obtained as membership degree analysis.

From the point of view of fuzzy mathematics, if any element $x$ in domain $\Omega$, there is

$$\mu_A(x) \leq \mu_A(x)$$  \hspace{1cm} (39)

The fuzzy set $A_1$ is included in the fuzzy set $A$, which is recorded as $A_1 \subseteq A$. For any two fuzzy sets $A_1$ and $A$, the membership degrees of Union and intersection are respectively

$$\mu_{A \cup B}(x) = \mu_A(x) \cup \mu_B(x)$$  \hspace{1cm} (40)

$$\mu_{A \cap B}(x) = \mu_A(x) \cap \mu_B(x)$$  \hspace{1cm} (41)

If for any element $x$ in domain $\Omega$, there is
\[ \mu_{\text{r}}(x) = 1 - \mu_{\text{i}}(x) \] (42)

Then \( B \) is called the complement of fuzzy set \( A \), and
\[ \mu_{(\text{r} \cup \text{i})}(x) = \mu_{\text{r}}(x) \lor \mu_{\text{i}}(x) = \mu_{\text{i}}(x) \lor [1 - \mu_{\text{r}}(x)] \leq 1 \] (43)
\[ \mu_{(\text{r} \cap \text{i})}(x) = \mu_{\text{r}}(x) \land \mu_{\text{i}}(x) = \mu_{\text{i}}(x) \land [1 - \mu_{\text{r}}(x)] \geq 0 \] (44)

Obviously, the operations of fuzzy sets do not satisfy the complementary law, including the Union operation and intersection operation in the complementary law.

The uncertainty of fuzzy set \( A \) mainly lies in its left and right boundaries. They are not two exact values, but two variables in different intervals. They describe the unascertained boundaries of fuzzy sets, which can be regarded as unascertained quantities \([1, 2]\), and here they are respectively recorded as \( a \) and \( b \). Let the proposition \( A = \{ x \in A \} \), then its reliability should be
\[ \text{Bel}(A) = \mu_{\text{i}}(a \leq x \land x \leq b) \] (45)

Generally, the left bound \( a \) of fuzzy set \( A \) monotonically increases within its value range \([a_{1}, a_{2}]\), and the right bound \( b \) decreases monotonically in its value range \([b_{1}, b_{2}]\). Let
\[ \text{bel}_{\text{i}}(x) = \begin{cases} \frac{d\mu_{\text{i}}(x)}{dx} & x \in (a_{1}, a_{2}) \\ 0 & x \notin (a_{1}, a_{2}) \end{cases} \] (46)
\[ \text{bel}_{\text{r}}(x) = \begin{cases} \frac{d\mu_{\text{r}}(x)}{dx} & x \in (b_{1}, b_{2}) \\ 0 & x \notin (b_{1}, b_{2}) \end{cases} \] (47)

According to the degree of belief operation rules when the situation is included in equation (21), it can prove that \([1,2]\)

\[ \text{Bel}(A) = \text{Bel}\{ x \in A \} = \mu_{\text{i}}(x) \] (48)

At present, complement \( \Delta \) is not introduced in fuzzy mathematics, so the complement of fuzzy set \( A \) is actually its effective complement \( A^c \). Using equation (48), it can be proved that
\[ \text{Bel}(A^c) = \text{Bel}\{ x \in A^c \} = 1 - \mu_{\text{i}}(x) \] (49)
\[ \text{Bel}(A \cap A^c) = \text{Bel}\{ x \in A \cap x \in A^c \} = \mu_{\text{i}}(x) \land [1 - \mu_{\text{i}}(x)] = \mu_{\text{g}(A^c)}(x) \] (50)
\[ \text{Bel}(A \cup A^c) = \text{Bel}\{ x \in A \cup x \in A^c \} = \mu_{\text{i}}(x) \lor [1 - \mu_{\text{i}}(x)] = \mu_{\text{g}(A)}(x) \] (51)

They are the same as the results of membership calculation of fuzzy sets, and they also do not satisfy the complementary law.

In order to reflect the characteristics of cognitive activities, complement \( \Delta \) should also be introduced into fuzzy mathematics. This should be due
\[ \text{Bel}(\Delta) = 1 - \text{Bel}(A \cup A^c) = \mu_{\text{i}}(x) \land [1 - \mu_{\text{i}}(x)] \] (52)

Equation (51) can be rewritten as
\[ \text{Bel}(A \cup A^c) = 1 - \text{Bel}(\Delta) = \mu_{\text{i}}(x) \lor [1 - \mu_{\text{i}}(x)] \] (53)

It also only represents the degree of belief of all valid propositions \( A \cup A^c \), and does not represent the degree of belief of the whole set. For the actual complement \( A^c \) of proposition set \( A \), there should be
\[ \text{Bel}(A^c) = \text{Bel}(A^c + \Delta) = \text{Bel}(A^c) + \text{Bel}(\Delta) = 1 - \mu_{\text{i}}(x) + \mu_{\text{i}}(x) \land [1 - \mu_{\text{i}}(x)] \] (54)
\[ \text{Bel}(A \cap A^c) = \text{Bel}(A \cap (A^c + \Delta)) = \text{Bel}(A \cap A^c) = \mu_{\text{i}}(x) \land [1 - \mu_{\text{i}}(x)] \geq 0 \] (55)
\[ \text{Bel}(A \cup A^c) = \text{Bel}(A) + \text{Bel}(A^c) - \text{Bel}(A \cap A^c) = 1 \] (56)

They satisfy the union operation in the complementary law and the intersection operation indefinitely, which is the same as the result of the degree of belief operation of the general proposition set.

After introducing the complement, for the general fuzzy sets \( \tilde{A}, \tilde{A}^c \), it can be proved by the operation rules of the degree of belief
\[
Bel\{x \in A \cap A_j\} = \begin{cases} 
\min\{Bel(A_i), Bel(A_j)\} & A_i \cap A_j \neq \emptyset \\
0 & A_i \cap A_j = \emptyset 
\end{cases} 
\] (57)

\[
Bel\{x \in A \cup A_j\} = Bel(A_i) + Bel(A_j) - Bel(A_i \cap A_j) = \begin{cases} 
\max\{Bel(A_i), Bel(A_j)\} & A_i \cap A_j \neq \emptyset \\
Bel(A_i) + Bel(A_j) & A_i \cap A_j = \emptyset 
\end{cases} 
\] (58)

They are the same as the membership operation results of fuzzy sets, but considering the characteristics of complement, the intersection of them and any valid proposition should be empty set \(\emptyset\). Formula (57) and formula (58) show that the operation rules of proposition reliability are also applicable to the degree of belief analysis of fuzzy sets, and the same results as membership analysis can be obtained. Using these two formulas, fuzzy relation synthesis, fuzzy cluster analysis, and fuzzy comprehensive evaluation can also be explained and performed from the perspective of the degree of belief.

4. conclusion

(1) Compared with the current event domain and proposition domain, the proposition domain set up in this paper has two outstanding characteristics: basic propositions are not necessarily incompatible; in addition to the effective proposition, it may also include the basic proposition that cannot be completely determined for all the effective propositions, that is complement. According to this, a more complete proposition domain can be constructed, and the task set meets all set operation rules except the complementary law intersection operation.

(2) According to the three axiomatic calculation rules of degree of belief and the calculation rules of propositions and proposition sets, a complete degree of belief calculation rule can be established, which satisfies the non-negativity, regularity and additivity of degree of belief.

(3) In order to reflect the characteristics of cognitive activities, complements should be introduced into current fuzzy mathematics and evidence theory, which can make its membership and reliability additivity, and meet all set operation rules except complementary law intersection operations, and make a clearer explanation of the operation results.

(4) As long as the actual domain of multiple propositions can be determined, and the relationship between the propositions is reasonably considered, the degree of belief calculation rules in the article can be used for the synthesis of multiple evidences in evidence theory.

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