Inverse Kinematic Problem of Movement of Six Degrees of Freedom Robotic Arm (Solved by a Numerical Method)

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Abstract. Robot manipulators are widely used in engineering for solving various tasks in accordance with their purpose. Manipulators usually consist of sequentially connected rigid links and connections between them – joints. One of the most complex types of robot manipulators is the articulated robot, also called the Robotic Arm. It has a structure similar to a human arm with three rigid links and only rotational joints. Such a structure provides greater flexibility, manageability and versatility. At the same time, the accuracy is reduced and the kinematic calculation is complicated. The direct kinematic problem is fairly well known and is uniquely solved using Euler angles or quaternion matrices. At the same time, the inverse kinematic problem has a multivariate solution and, therefore, it must be an optimization. The article proposes a new approach to solving the inverse kinematic problem based on numerical methods and parametric optimization. Formulas of transformations of Cartesian coordinates for a direct kinematic problem act as a mathematical model. However, in the formulas, the angles of rotation of the joints are unknown. Their values are determined during the minimization of the objective function in the form of a sum of squares of the difference between the specified and obtained Cartesian coordinates of the end-effector. The article describes a robot with 6 Degrees of Freedom. Based on the above approach, modeling was performed in the MATLAB software environment. It is proposed to use the sum of the angles of the links of the robot as an optimization criterion. Analysis of the results showed that the optimal solution allows reducing the rotation angles of all links of the robot approximately two times.

1. Introduction

Robot manipulator is a type of industrial robots [1-5]. Manipulators are composed of an assembly of links and joints. Links are defined as the rigid sections that make up the mechanism and joints are defined as the connection between two links. The device attached to the manipulator which interacts with its environment to perform tasks is called the end-effector.

The combination and mutual arrangement of links and joints determines the number of Degrees of Freedom (DOF), as well as the scope of the manipulation system of the robot. Following types of robot mechanisms are available: Cartesian coordinate robot, Cylindrical coordinate robot, Spherical coordinate robot, SCARA robot (Selective compliance automatic robot arm), articulated robot [6-25]. There are industrial robots with closed-loop kinematic chains – parallel linkages robots. Examples include the Stewart Platform and delta robots. An articulated robot is a robot with all rotary joints. Articulated robots can range from simple two-jointed structures to systems with 10 or more interacting
joints. It has great amount of flexibility, manageability, and versatility. Such robots are also called Robotic Arm.

Robot manipulators are extensively used in the industrial manufacturing sector and also have many other specialized applications (for example, the Canadarm was used on space shuttles to manipulate payloads). The study of robot manipulators involves dealing with the positions and orientations of the several segments that make up the manipulators. First of all, it is necessary to determine the Cartesian coordinates of the end-effector.

2. The formulation of the problem
The direct problem is to calculate the Cartesian coordinates of an end-effector from its kinematic scheme and given angles of links. The well-known methods for solving the direct problem are constructed on coordinate transformations using the matrix of direction cosines, Euler angles, quaternion apparatus, and linear fractional transformations with Kayleigh-Klein parameters. The direct problem has an unambiguous solution and is described in sufficient detail in [1-5, 27-31].

The inverse problem is to calculate the angles of the links for a given position of the end-effector with a known scheme of its kinematics. The inverse problem, in contrast to the direct one, has many solutions, so it is advisable to solve it on the basis of parametric optimization. To do this, you need to find a set of solutions, and then choose one of them by some criterion. Thus, the inverse problem is more complex than the direct one. The actual question is the choice of criteria for optimization.

Currently, existing methods for obtaining solutions to the inverse problem can be divided into geometric, analytical, and numerical. Finding generic coordinates explicitly is a difficult task, since equations are non-linear. Therefore, an analytical solution exists only for robots with a specific design. In the Euler angle method, it is proposed to successively multiply both sides of the equation by inverse transformation matrices and determines the required angles from the matrix equations thus obtained.

The method of transformations consists in transferring first one of the unknown quantities from the right to the left side of the equation, finding it and transferring it to the left side of the next unknown and repeating this procedure until all variables are found. It is also known to solve the inverse problem of kinematics in the dual Rodrig – Hamilton parameters, namely, using biquaternionic matrices [32].

The geometric approach is associated with the use of an analytical solution, taking into account the peculiarities of the kinematic scheme, which makes it possible to reduce the number of independent equations.

The disadvantages of the analytical solution include the ambiguity of the result obtained, due to the trigonometric functions used. This requires additional analysis of the choice of the correct solution. Therefore, the article proposes a new approach to solving the inverse kinematic problem based on numerical methods and parametric optimization. The mathematical model is a formula for the transformations of the Cartesian coordinates of the links of the robot, obtained for the direct kinematic problem. However, in the formulas, the angles of rotation of the links are unknown and are determined on the basis of parametric optimization. The objective function is minimized as a sum of squares of the difference between the specified and obtained Cartesian coordinates of the end-effector. The article deals with a robot manipulator with 6 DOF, which has only rotational joints. This robot scheme is more general than the well-known PUMA robots (Programmable Universal Machine for Assembly or Programmable Universal Manipulation Arm).

3. Mathematical Model
We introduce the following coordinate systems (Fig. 1):

1) Conditionally fixed system associated with the first link of the robot – the base – \((x_0, y_0, z_0)\);
2) Moving coordinate systems associated with the first and second links, respectively \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\);
3) Moving coordinate system associated with point 01 – the end-effector – \((x_3, y_3, z_3)\).

To determine the coordinates of the position of the manipulator it is necessary to consistently find the coordinates of the position of its entire links. The expressions for the Cartesian coordinates of
the position of the first link will be: 

\[
\begin{align*}
    x_1 &= x_0 + l_1 \cos \alpha_1 \sin \alpha_2; \\
    y_1 &= y_0 + l_1 \sin \alpha_1; \\
    z_1 &= z_0 + l_1 \cos \alpha_1 \cos \alpha_2,
\end{align*}
\]  

(1)

where \( l_1 \) is the length of the first link of the robot; \( x_0, y_0, z_0 \) – coordinates of the base of the robot; \( \alpha_1, \alpha_2 \) – the angles of rotation of the first link around the \( Y \) and \( X \) axes respectively.

**Figure 1.** Coordinate scheme of the 6 DOF robot.

Formulas for the Cartesian coordinates of the second link:

\[
\begin{align*}
    x_2 &= x_1 + l_2 \cos (\alpha_1 + \alpha_3) \sin (\alpha_2 + \alpha_4); \\
    y_2 &= y_1 + l_2 \sin (\alpha_1 + \alpha_3); \\
    z_2 &= z_1 + l_2 \cos (\alpha_1 + \alpha_3) \cos (\alpha_2 + \alpha_4),
\end{align*}
\]  

(2)

where \( l_2 \) is the length of the second link of the robot; \( x_1, y_1, z_1 \) – coordinates of the first joint of the robot; \( \alpha_3, \alpha_4 \) – the angles of rotation of the second link around the \( Y \) and \( X \) axes respectively.

Formulas for the Cartesian coordinates of the third link:

\[
\begin{align*}
    x_3 &= x_2 + l_3 \cos (\alpha_1 + \alpha_3 + \alpha_5) \sin (\alpha_2 + \alpha_4 + \alpha_6); \\
    y_3 &= y_2 + l_3 \sin (\alpha_1 + \alpha_3 + \alpha_5); \\
    z_3 &= z_2 + l_3 \cos (\alpha_1 + \alpha_3 + \alpha_5) \cos (\alpha_2 + \alpha_4 + \alpha_6),
\end{align*}
\]  

(3)

where \( l_3 \) is the length of the third link of the robot; \( x_2, y_2, z_2 \) – coordinates of the second joint of the robot; \( \alpha_5, \alpha_6 \) – the angles of rotation of the third link around the \( Y \) and \( X \) axes respectively.

The system of equations (3) gives a solution to the direct kinematic problem – the determination of the \( x, y, z \) coordinates for known angles \( \alpha \). If in these equations to set the values of \( x, y, z \) and the angles \( \alpha \) are considered unknown, then we get the inverse kinematic problem.

To solve it, we use parametric optimization. The objective function is the sum of the squares of the difference between the specified and obtained Cartesian coordinates of the end-effector (Fig. 2):

\[
F(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = \sum (x_i - X_i)^2 + (y_i - Y_i)^2 + (z_i - Z_i)^2,
\]  

(4)

where \( X_1, Y_1, Z_1 \) – the initial coordinates of the end-effector; \( X_2, Y_2, Z_2 \) – the required coordinates of the end-effector; \( \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \) – the unknown angles of rotation of the links of the robot.

The software implementation is performed in the MATLAB environment (Fig. 3). The optimization method is the Nelder-Mead simplex method.
Figure 2. Estimated movement scheme of the robot.

Figure 3. Visualization of the calculation of the positions of the robot in the MATLAB environment.

4. Simulation results
Perform the calculation of the inverse kinematic problem for the previously considered 6 DOF robot manipulator with link lengths $l_1 = 200$ mm, $l_2 = 150$ mm, $l_3 = 50$ mm. Baseline data are presented in Table 1. It shows the initial values of the angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ of rotation of the links, the initial and final coordinates of the end-effector $X, Y, Z$.

| Position of the end-effector | Angles of rotation of links, degree | Coordinates, mm |
|-----------------------------|------------------------------------|----------------|
|                             | $\alpha_1$  $\alpha_2$  $\alpha_3$  $\alpha_4$  $\alpha_5$  $\alpha_6$ | $X$  $Y$  $Z$ |
| Initial                     | 50.0  20.0  120.0  20.0  5.0  40.0 | -100.04  183.61  -1.00 |
| Final                       | - - - - - - | 120.0  -190.0  250.0 |
A numerical experiment showed that the inverse kinematic problem has many solutions. All these solutions are equivalent – the objective function has a minimum close to zero (about $10^{-8}$). Achievement of one or another local minimum depends on the choice of initial values of start points during optimization. The physical justification for multivariance is the excess number of degrees of freedom of the robot. Four calculation options are illustrated in Fig. 4, the numerical values are given in Table 2.

### Table 2. The estimated coordinates of the links of the robot.

| Calculation option | Final angles of rotation of links, degree | β₁ | β₂ | β₃ | β₄ | β₅ | β₆ |
|--------------------|------------------------------------------|-----|-----|-----|-----|-----|-----|
| 1                  |                                          | 148.97 | 155.14 | -101.34 | 41.32 | -0.74 | -0.17 |
| 2                  |                                          | -65.75 | 16.92 | 141.17 | 98.19 | 110.37 | 202.65 |
| 3                  |                                          | -61.52 | 9.90 | -160.48 | -9.67 | 7.46 | 204.23 |
| 4                  |                                          | -53.98 | -0.18 | 186.77 | 0.14 | -37.56 | 0.19 |

Data analysis of Table 2 showed that different sums of corners of the links of the robot correspond to different versions of the solution of the inverse kinematic problem. The results in the form of the angles of rotation of the first $|α₁-β₁|+|α₂-β₂|$, second $|α₃-β₃|+|α₄-β₄|$, third $|α₅-β₅|+|α₆-β₆|$ and all links of the robot are given in Fig. 5. It can be seen from it that option 4 provides the sum of the angles of rotation of the links about two times less than option 3.

![Figure 4](image-url)

**Figure 4.** Options for solving the inverse kinematic problem: a – option 1, b – option 2, c – option 3, d – option 4.
5. Analysis of the results and conclusions
The conducted numerical experiment showed that the solution of the inverse kinematic problem for a robot with 6 degrees of freedom is multivariate. There are many equivalent solutions with different angles of joints. Therefore, it is advisable to set an optimization problem. As a criterion, the sum of the angles of all the joints of the robot is proposed. Using the example of calculation, it is shown that the optimal solution makes it possible to reduce the sum of the angles by about 2 times and thereby to reduce move the robot links.

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