Stability, uniqueness and existence of solutions to McKean-Vlasov SDEs: a multidimensional Yamada-Watanabe approach

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Abstract

We establish stability and pathwise uniqueness of solutions to Wiener noise driven McKean-Vlasov equations with random non-Lipschitz continuous coefficients. In the deterministic case, we also obtain the existence of unique strong solutions. By using our approach, which is based on an extension of the Yamada-Watanabe ansatz to the multidimensional setting and which does not rely on the construction of Lyapunov functions, we prove first moment and pathwise exponential stability. Furthermore, Lyapunov exponents are computed explicitly.

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1 Introduction

In this work, let $d, m \in \mathbb{N}$ and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space that satisfies the usual conditions and carries a standard $d$-dimensional $(\mathcal{F}_t)_{t \geq 0}$-Brownian motion $W$. McKean-Vlasov stochastic differential equations (McKean-Vlasov SDEs), or alternatively mean-field SDEs, are integral equations of the form

$$X_t = X_0 + \int_0^t b(s, X_s, \mathcal{L}(X_s)) \, ds + \int_0^t \sigma(s, X_s, \mathcal{L}(X_s)) \, dW_s \quad \text{for } t \geq 0 \text{ a.s.} \quad (1.1)$$

with the drift and diffusion coefficients $b$ and $\sigma$ defined on $\mathbb{R}_+ \times \mathbb{R}^m \times \mathcal{P}(\mathbb{R}^m)$ and taking their respective values in $\mathbb{R}^m$ and $\mathbb{R}^{m \times d}$, where $\mathcal{P}(\mathbb{R}^m)$ denotes the convex space of all Borel probability measures on $\mathbb{R}^m$.

Originally motivated by Boltzmann’s equation in kinetic gas theory, McKean-Vlasov equations were first studied by Kac [28], McKean [34] and Vlasov [36] and utilised to describe the stochastic dynamics of large interacting particle systems. In particular, Vlasov analysed the propagation of chaos of charged particles with long-range interaction in a plasma, where the interaction of particles is modelled by means of a system of SDEs. It turns out that the convergence of such an $N$-particle system, as the positive integer $N$ tends to infinity, which is referred to as propagation of chaos, can be described by an equation of the form (1.1).

Since the seminal papers [28], [34] and [36], mean-field SDEs attracted much interest and sparked off new ground-breaking developments in both theory and applications in
a variety of other research areas. For example, Lasry and Lions \cite{31} employ mean-field techniques to study mean-field games in economic applications. There, the interaction of strategies of ‘rational’ players in an $N$-player differential game is described by a system of SDEs. Then, in the limiting case $N \to \infty$, the authors derive a system of partial differential equations that consists of a Hamilton-Jacobi and a Kolmogorov equation, which they use to examine the behaviour of agents in a vast network.

As for other works in this direction, but based on different methods, which rely on a probabilistic analysis, we refer to \cite{13}, \cite{14}, \cite{15}, \cite{16} and \cite{17}. Recent applications of mean-field methods include e.g. the modelling of systemic risk in financial networks, see \cite{18}, \cite{19}, \cite{21}, \cite{22}, \cite{23}, \cite{24} and \cite{30}. Further, extensions of (1.1) to the setting of mean-field SDEs driven by Lévy processes or backward mean-field SDEs were studied in \cite{10}, \cite{11}, \cite{12} and \cite{27}. More recently, a robust solution theory of mean-field SDEs from the perspective of rough path theory was developed in \cite{4}.

Mean-field equations of type (1.1) were also examined for non-Lipschitz vector fields $b$ and $\sigma$. See e.g. \cite{26}, where the author invokes martingale techniques to construct unique weak solutions in the case of a bounded drift $b$, which is Lipschitz continuous in the law variable. We also refer to \cite{20}, \cite{35} in the case of weak solutions. As for path-dependent coefficients, see \cite{32}. More recently, by using techniques based on Malliavin calculus, the existence of unique strong solutions to mean-field SDEs with additive Brownian noise and singular drift $b$ were obtained in \cite{8}, \cite{9} and \cite{17} and a Bismut-Elworthy-Li formula for such equations was derived there. See also \cite{6} for the case of a certain non-Markovian and ‘rough’ Gaussian driving noise in a Hilbert space setting.

Now let $t_0 \geq 0$ and $\mathcal{P}$ be a separable metrisable space in $\mathcal{P}(\mathbb{R}^m)$. In this article, we consider a McKean-Vlasov SDE with random drift and diffusion coefficients of the form

$$dX_t = B_t(X_t, \mathcal{L}(X_t)) \, dt + \Sigma_t(X_t) \, dW_t \quad \text{for } t \geq t_0,$$

(1.2)

where $B : [t_0, \infty) \times \Omega \times \mathbb{R}^m \times \mathcal{P} \to \mathbb{R}^m$ and $\Sigma : [t_0, \infty) \times \Omega \times \mathbb{R}^m \to \mathbb{R}^{m \times d}$ are measurable in an appropriate meaning. This includes controlled SDEs, as Example 2.8 shows. While the diffusion $\Sigma$ does not depend on the measure variable $\mu \in \mathcal{P}$, both $B$ and $\Sigma$ are allowed to be non-Lipschitz, as stated more precisely below.

The measure state space $\mathcal{P}$, whose topology may be finer than the topology of weak convergence, is required to be admissible in a suitable measurable sense, as introduced in Section 2. For instance, $\mathcal{P}$ may stand for $\mathcal{P}(\mathbb{R}^m)$, endowed with the Prokhorov metric, or the Polish space $\mathcal{P}_1(\mathbb{R}^m)$ of all measures in $\mathcal{P}(\mathbb{R}^m)$ with a finite first absolute moment, equipped with the Wasserstein metric.

In this general framework, the main objective of our work is to establish Lyapunov stability, uniqueness and existence of solutions to McKean-Vlasov SDEs of type (1.2). Our methodology is based on a multidimensional Yamada-Watanabe approach and allows for irregular drifts. More precisely, under an Osgood condition on compact sets on $\Sigma$, our paper offers the following novel contributions compared to the existing literature:

1. **Pathwise uniqueness** for (1.2) follows from Corollary 3.7 under a partial Osgood condition on $B$, and if the drift is independent of $\mu \in \mathcal{P}$, in which case (1.2) reduces to a SDE, then this condition is imposed on compact sets only.

2. **(Asymptotic) moment stability** for (1.2) is inferred from Proposition 3.11, which yields a general $L^1$-comparison estimate, and stated in Corollary 3.13 under a partial mixed Hölder continuity condition on $B$ and verifiable integrability conditions on the random partial Hölder coefficients with respect to $(x, \mu) \in \mathbb{R}^m \times \mathcal{P}$.

3. **Exponential moment stability** is asserted by Corollary 3.14 if $B$ satisfies a partial Lipschitz condition and the partial Lipschitz coefficients are bounded by a sum of power functions that determines the moment Lyapunov exponent explicitly.
(4) **Pathwise exponential stability** is implied by Corollary 3.17 if the preceding conditions on $B$ hold, the random Lipschitz coefficient of $B$ relative to $\mu \in \mathcal{P}$ is of suitable growth and the Osgood condition on $\Sigma$ is replaced by an $\frac{1}{2}$-Hölder condition. In particular, the pathwise Lyapunov exponent is half the moment Lyapunov exponent.

(5) Thereby, we show that all these stability results can be obtained under verifiable assumptions, **without resorting to the existence of Lyapunov functions**.

(6) **Existence of unique strong solutions** is established in Theorem 3.25 when $B$ and $\Sigma$ are deterministic, $B$ satisfies a partial affine growth and a partial Lipschitz condition and $B_s(\cdot,\cdot)$ is continuous and $\Sigma_s(0) = 0$ for all $s \geq t_0$. In particular, $B$ and $\Sigma$ may fail to be of affine growth.

Concisely, our methods for proving these main results rest on the pathwise uniqueness approach of Yamada and Watanabe [37], which we extend to the multidimensional setting. In this context, let us consider two articles, which employ the Yamada-Watanabe ansatz to show pathwise uniqueness and stability of solutions:

In [1] the authors verify pathwise uniqueness of solutions to SDEs with deterministic coefficients. However, the Yamada-Watanabe condition given in the multidimensional case is rather restrictive and essentially reduces to a Lipschitz condition on the diffusion. The article [3] pertains to the study of one-dimensional mean-field SDEs with bounded drift and diffusion coefficients by means of the Yamada-Watanabe approach. There, it is essentially assumed that the drift coefficient is Lipschitz continuous in the spatial and the law variable, while the diffusion satisfies a global Osgood condition.

In our paper, however, the conditions on the coefficients, even in the one-dimensional deterministic case, are weaker than in [3], since no growth conditions are imposed. Further, we only require that $B$ and $\Sigma$ satisfy a partial Osgood condition and an Osgood condition on compact sets, respectively. Finally, in the context of stability results for SDEs with irregular coefficients, we also mention the work [2], where the authors prove moment exponential stability of solutions to SDEs driven by a drift vector field with discontinuities on a hyperplane.

Our paper is organised as follows. In Section 2 we prove auxiliary results on measure state spaces and introduce the probabilistic setting. In Section 3 our main results are presented, whose proofs are given in Section 5. Section 4 is devoted to a priori estimates and a pathwise asymptotic analysis for random Itô processes. The results of this section are of independent interest and serve as basis for the derivation of the main results.

## 2 Preliminaries

In the following, $|\cdot|$ is used as absolute value function, Euclidean norm or Hilbert-Schmidt norm, $|\cdot|_1$ stands for the 1-norm and $A'$ is the transpose of a matrix $A \in \mathbb{R}^{m \times d}$. Further, for any interval $I$ with infimum $a$ and supremum $b$ and each monotone function $f : I \to \mathbb{R}$, we set $f(a) := \lim_{v \downarrow a} f(v)$, if $a \notin I$, and $f(b) := \lim_{v \uparrow b} f(v)$, if $b \notin I$.

### 2.1 Admissible Polish spaces of Borel probability measures

We consider a tractable class of spaces of Borel probability measures on $\mathbb{R}^m$, which serve as part of the domain of the random drift and diffusion coefficients of the McKean-Vlasov equation (1.2).

**Definition 2.1.** A metrisable space $\mathcal{P}$ in $\mathcal{P}(\mathbb{R}^m)$ is called **admissible** if for any metrisable space $S$, each probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and every process $X : S \times \Omega \to \mathbb{R}^m$ with continuous paths satisfying

$$\mathcal{L}(X_s) \in \mathcal{P} \quad \text{for all } s \in S,$$
the map $S \to \mathcal{P}$, $s \mapsto \mathcal{L}(X_s)$ is Borel measurable.

To give sufficient conditions for the admissibility of a metrisable space $\mathcal{P}$ in $\mathcal{P}(\mathbb{R}^m)$, we introduce stochastic convergence of probability measures. Namely, a sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\mathcal{P}(\mathbb{R}^m)$ converges stochastically to some $\mu \in \mathcal{P}(\mathbb{R}^m)$ if there is a sequence $(\theta_n)_{n \in \mathbb{N}}$ of Borel measures on $\mathbb{R}^m \times \mathbb{R}^m$ such that $\theta_n \in \mathcal{P}(\mu_n, \mu)$ for each $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} \theta_n \left( \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m | |x - y| \geq \delta \} \right) = 0 \quad \text{for all } \delta > 0. \quad (2.1)$$

Thereby, $\mathcal{P}(\mu, \nu)$ denotes the convex space of all Borel probability measures $\theta$ on $\mathbb{R}^m \times \mathbb{R}^m$ with first and second marginal distributions $\mu$ and $\nu$, respectively, for all $\mu, \nu \in \mathcal{P}(\mathbb{R}^m)$.

**Proposition 2.2.** Let $(\varphi_n)_{n \in \mathbb{N}}$ be a sequence of $\mathbb{R}^m$-valued bounded uniformly continuous maps on $\mathbb{R}^m$ such that $|\varphi_k| \leq |\varphi_{k+1}|$ and $\lim_{n \to \infty} \varphi_n(x) = x$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}^m$.

Then $\mathcal{P}$ is admissible under the following two conditions:

(i) Each $\mu \in \mathcal{P}$ satisfies $\mu \circ \varphi_n^{-1} \in \mathcal{P}$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \mu \circ \varphi_n^{-1} = \mu$ in $\mathcal{P}$.

(ii) For any sequence $(\mu_k)_{k \in \mathbb{N}}$ in $\mathcal{P}$ that converges stochastically to some $\mu \in \mathcal{P}$ we have $\lim_{k \to \infty} \mu_k \circ \varphi_n^{-1} = \mu \circ \varphi_n^{-1}$ in $\mathcal{P}$ for any $n \in \mathbb{N}$.

**Example 2.3.** We may let $\varphi_n$ be the radial retraction of the closed ball with center 0 and radius $n$ for each $n \in \mathbb{N}$. That is, $\varphi_n(x) = x$, if $|x| \leq n$, and $\varphi_n(x) = \frac{n}{|x|} x$, if $|x| > n$, for any $x \in \mathbb{R}^m$. Indeed, $\varphi_n$ is Lipschitz continuous, $|\varphi_n| \leq n \wedge |\varphi_{n+1}|$ and $|\varphi_n(x) - x| = (|x| - n)^+$.

To give more concrete conditions, let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing and $\mathcal{P}_\rho(\mathbb{R}^m)$ denote the convex space of all $\mu \in \mathcal{P}(\mathbb{R}^m)$ for which $\int_{\mathbb{R}^m} \rho(|x|) \mu(dx)$ is finite.

**Corollary 2.4.** Assume that $\mathcal{P} \subseteq \mathcal{P}_\rho(\mathbb{R}^m)$, $\rho$ is continuous and $\rho(0) = 0$. Then the admissibility of $\mathcal{P}$ holds under the following two conditions:

(i) If $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ is bounded and uniformly continuous and $|\varphi(x)| \leq |x|$ for all $x \in \mathbb{R}^m$, then $\mu \circ \varphi^{-1} \in \mathcal{P}$ for any $\mu \in \mathcal{P}$.

(ii) A sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\mathcal{P}$ converges to some $\mu \in \mathcal{P}$ if there is a sequence $(\theta_n)_{n \in \mathbb{N}}$ of Borel measures on $\mathbb{R}^m \times \mathbb{R}^m$ such that $\theta_n \in \mathcal{P}(\mu_n, \mu)$ for all $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} \int_{\mathbb{R}^m \times \mathbb{R}^m} \rho(|x - y|) d\theta_n(x, y) = 0.$$

**Remark 2.5.** By the measure transformation formula, any measurable map $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ satisfies $\int_{\mathbb{R}^m} \rho(|x|) (\mu \circ \varphi^{-1})(dx) = \int_{\mathbb{R}^m} \rho(|\varphi(x)|) \mu(dx)$. Thus, the first condition of the corollary holds for $\mathcal{P}_\rho(\mathbb{R}^m)$.

**Examples 2.6.** (i) The Prokhorov metric $\vartheta_\rho$ turns $\mathcal{P}(\mathbb{R}^m)$ into a Polish space and can be represented as follows: For $\varepsilon > 0$ let $N_\varepsilon(B)$ be the $\varepsilon$-neighbourhood of a set $B$ in $\mathbb{R}^m$.

Then the $[0, 1]$-valued functional on $\mathcal{P}(\mathbb{R}^m) \times \mathcal{P}(\mathbb{R}^m)$ given by

$$\vartheta_0(\mu, \nu) := \inf \{ \varepsilon > 0 | \forall B \in \mathcal{B}(\mathbb{R}^m) : \mu(B) \leq \nu(N_\varepsilon(B)) + \varepsilon \}$$

satisfies the triangle inequality and $\vartheta_\rho(\mu, \nu) \vee \vartheta_0(\nu, \mu)$ for all $\mu, \nu \in \mathcal{P}(\mathbb{R}^m)$. As convergence with respect to $\vartheta_\rho$ is equivalent to weak convergence, all requirements of Proposition 2.2 are met by $\mathcal{P}(\mathbb{R}^m)$ if endowed with $\vartheta_\rho$.

(ii) For $p \geq 1$ consider the Polish space $\mathcal{P}_p(\mathbb{R}^m)$ of all $\mu \in \mathcal{P}(\mathbb{R}^m)$ with finite $p$-th absolute moment $\int_{\mathbb{R}^m} |x|^p \mu(dx)$, equipped with the $p$-th Wasserstein metric given by

$$\vartheta_p(\mu, \nu) := \inf \left\{ \frac{1}{\theta \in \mathcal{P}(\mu, \nu)} \left( \int_{\mathbb{R}^m \times \mathbb{R}^m} |x - y|^p d\theta(x, y) \right)^{\frac{1}{p}} \right\}.$$

As the definition of $\vartheta_p$ entails that the second condition of Corollary 2.4 is valid for the choice $\rho(v) = v^p$ for all $v \geq 0$, we see that $\mathcal{P}_p(\mathbb{R}^m)$ is admissible.
2.2 Notions of solutions, pathwise uniqueness and stability

In what follows, let $\mathcal{P}$ be an admissible separable metrisable space in $\mathcal{P}(\mathbb{R}^m)$ and $\mathcal{A}$ denote the progressive $\sigma$-field on $[t_0, \infty) \times \Omega$. Thus, a set $A$ in $[t_0, \infty) \times \Omega$ lies in $\mathcal{A}$ if and only if $\mathbb{1}_A$ is progressively measurable. We shall call a map

$$F : [t_0, \infty) \times \Omega \times \mathbb{R}^m \times \mathcal{P} \to \mathbb{R}^{m \times d}, \quad (s, \omega, x, \mu) \mapsto F_s(x, \mu)(\omega)$$

admissible if it is measurable relative to the product $\sigma$-field $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathcal{P})$. In this case, for any $\mathbb{R}^m$-valued progressively measurable process $X$ and each Borel measurable map $\mu : [t_0, \infty) \to \mathcal{P}$, the process

$$[t_0, \infty) \times \Omega \to \mathbb{R}^{m \times d}, \quad (s, \omega) \mapsto F_s(X_s(\omega), \mu(s))(\omega)$$

is progressively measurable. In particular, if $X$ is continuous and $\mathcal{L}(X_t) \in \mathcal{P}$ for all $t \geq t_0$, then the law map $[t_0, \infty) \to \mathcal{P}$, $t \mapsto \mathcal{L}(X_t)$ is a feasible choice for $\mu$. In this sense, we require the drift $B$ and the diffusion $\Sigma$ of the McKean-Vlasov equation \[[\ref{12}]\] to be admissible.

**Definition 2.7.** A solution to \[[\ref{12}]\] is an $\mathbb{R}^m$-valued adapted continuous process $X$ such that $\mathcal{L}(X_s) \in \mathcal{P}$ for all $s \geq t_0$, $\int_{t_0}^{t_1}|B_s(X_s, \mathcal{L}(X_s))| + |\Sigma_s(X_s)|^2 \, ds < \infty$ and

$$X = X_{t_0} + \int_{t_0}^{t} B_s(X_s, \mathcal{L}(X_s)) \, ds + \int_{t_0}^{t} \Sigma_s(X_s) \, dW_s \quad \text{a.s.}$$

**Example 2.8.** For $l \in \mathbb{N}$ let $Y$ be an $\mathbb{R}^l$-valued progressively measurable process and $b : [t_0, \infty) \times \mathbb{R}^m \times \mathcal{P} \times \mathbb{R}^l \to \mathbb{R}^l$ and $\sigma : [t_0, \infty) \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^{m \times d}$ be Borel measurable such that

$$B_s(x, \mu) = b(s, x, \mu, Y_s) \quad \text{and} \quad \Sigma_s(x) = \sigma(s, x, Y_s)$$

for any $(s, x, \mu) \in [t_0, \infty) \times \mathbb{R}^m \times \mathcal{P}$. Then $B$ and $\Sigma$ are indeed admissible and \[[\ref{12}]\] turns into a McKean-Vlasov SDE whose drift and diffusion coefficients are controlled by $Y$.

We observe that $B$ and $\Sigma$ are independent of $\omega \in \Omega$ if and only if there are two Borel measurable maps $b : [t_0, \infty) \times \mathbb{R}^m \times \mathcal{P} \to \mathbb{R}^l$ and $\sigma : [t_0, \infty) \times \mathbb{R}^m \to \mathbb{R}^{m \times d}$ such that

$$B_s(x, \mu) = b(s, x, \mu) \quad \text{and} \quad \Sigma_s(x) = \sigma(s, x) \quad \text{for all} \quad (s, x, \mu) \in [t_0, \infty) \times \mathbb{R}^m \times \mathcal{P}. \quad \text{\cite{2.3}}$$

For the deterministic coefficients $b$ and $\sigma$, we may also consider solutions in the strong and weak sense and write \[[\ref{12}]\] formally as

$$dX_t = b(t, X_t, \mathcal{L}(X_t)) \, dt + \sigma(t, X_t) \, dW_t \quad \text{for} \quad t \geq t_0. \quad \text{\cite{2.4}}$$

Namely, for a fixed $\mathbb{R}^m$-valued $\mathcal{F}_{t_0}$-measurable random vector $\xi$ we let $(\mathcal{E}_t^\xi)_{t \geq t_0}$ denote the process of the natural filtration of the process $[t_0, \infty) \times \Omega \to \mathbb{R}^m \times \mathbb{R}^d$, $(t, \omega) \mapsto (\xi, W_t - W_{t_0})(\omega)$. That means,

$$\mathcal{E}_t^\xi = \sigma(\xi) \vee \sigma(W_s - W_{t_0} : s \in [t_0, t]) \quad \text{for all} \quad t \geq t_0.$$ 

Then a solution $X$ to \[[\ref{2.4}]\] satisfying $X_{t_0} = \xi$ a.s. is called strong if it is adapted to the right-continuous augmented filtration of $(\mathcal{E}_t^\xi)_{t \geq t_0}$. A weak solution to \[[\ref{2.4}]\] is a solution $X$ defined on some filtered probability space

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq t_0}, \tilde{\mathbb{P}})$$

that satisfies the usual conditions and on which there exists a standard $d$-dimensional $(\tilde{\mathcal{F}}_t)_{t \geq t_0}$-Brownian motion $W$. That is, $X$ is an $\mathbb{R}^m$-valued $(\tilde{\mathcal{F}}_t)_{t \geq t_0}$-adapted continuous process satisfying $\mathcal{L}(X_s) \in \mathcal{P}$ for all $s \geq t_0$,

$$\int_{t_0}^{\infty} |b(s, X_s, \mathcal{L}(X_s))| + |\sigma(s, X_s)|^2 \, ds < \infty$$
and $X = X_{t_0} + \int_{t_0}^{t_1} b(s, X_s, \mathcal{L}(X_s)) \, ds + \int_{t_0}^{t_1} \sigma(s, X_s) \, d\tilde{W}_s$ a.s. In this case, we will say that $X$ solves (2.4) weakly relative to $\tilde{W}$. Let us now return to stochastic coefficients.

The regularity of the drift $B$ relative to the variable $\mu \in \mathcal{P}$ will be stated in terms of an $\mathbb{R}_+^*$-valued Borel measurable functional $\vartheta$ on $\mathcal{P} \times \mathcal{P}$ satisfying

$$\vartheta(\mathcal{L}(X), \mathcal{L}(\tilde{X})) \leq E[|X - \tilde{X}|]$$

(2.5)

for any two $\mathbb{R}^m$-valued random vectors $X, \tilde{X}$ with $\mathcal{L}(X), \mathcal{L}(\tilde{X}) \in \mathcal{P}$. For instance, this estimate holds if $\vartheta$ is dominated by the Wasserstein metric $\vartheta_1$, introduced in Examples 2.6 in the sense that

$$\vartheta(\mu, \nu) \leq \vartheta_1(\mu, \nu) \quad \text{for all } \mu, \nu \in \mathcal{P},$$

(2.6)

where the definition of $\vartheta_1$ in (2.2) is extended for any $\mu, \nu \in \mathcal{P}(\mathbb{R}^m)$ by allowing infinite values. If in fact $\mathcal{P} \subseteq \mathcal{P}_1(\mathbb{R}^m)$, then this extension is not used. In this case, $\mathcal{P}$ is admissible as soon as $\vartheta$ is a metric inducing its topology and condition (i) of Corollary 2.4 holds.

**Example 2.9.** Let $\phi : [-\infty, \infty] \to \mathbb{R}_+$ and $\varphi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be measurable, $\rho \in C(\mathbb{R}_+)$ be increasing and vanish at 0 and $c > 0$ be such that

$$|\varphi(x, y)| \leq \rho(|x - y|) \quad \text{and} \quad \rho(v + w)/c \leq \rho(v) + \rho(w)$$

for all $x, y \in \mathbb{R}^m$ and $v, w \geq 0$. For instance, we may take $\rho(v) = \lambda v^p$ for each $v \geq 0$ and some $\lambda, p > 0$. Suppose that $\mathcal{P} \subseteq \mathcal{P}_\rho(\mathbb{R}^m)$ and

$$\vartheta(\mu, \nu) = \phi \left( \inf_{\vartheta \in \mathcal{P}(\mu, \nu)} \int_{\mathbb{R}^m \times \mathbb{R}^m} \varphi(x, y) \, d\vartheta(x, y) \right) \quad \text{for any } \mu, \nu \in \mathcal{P}.$$ 

Then $\vartheta$ is well-defined and the following three assertions are readily checked:

1. If $\varphi(x, y) = \psi(x) - \psi(y)$ for all $x, y \in \mathbb{R}^m$ and some uniformly continuous function $\psi : \mathbb{R}^m \to \mathbb{R}$ that admits $\rho$ as modulus of continuity, then $\vartheta$ is of the form

$$\vartheta(\mu, \nu) = \phi \left( \int_{\mathbb{R}^m} \psi(x) \, \mu(dx) - \int_{\mathbb{R}^m} \psi(y) \, \nu(dy) \right) \quad \text{for any } \mu, \nu \in \mathcal{P}.$$

2. In the case $\varphi(v) = v$ for all $v \geq 0$, $\varphi(x, y) = |x - y|$ for any $x, y \in \mathbb{R}^m$ and $\rho(v) = v$ for each $v \geq 0$, we obtain that $\vartheta(\mu, \nu) = \vartheta_1(\mu, \nu)$ for any $\mu, \nu \in \mathcal{P}$.

3. Assume that $\phi(v) \leq v^+$ for all $v \in [-\infty, \infty]$ and $\rho(v) = v$ for each $v \geq 0$. Then the general domination estimate (2.6) holds.

Let for the moment $\mathcal{P} = \mathcal{P}_1(\mathbb{R}^m)$, $\vartheta = \vartheta_1$ and the following affine growth condition be valid: $|B(\cdot, \mu)| \leq c_0 + c_1 \vartheta_1(\mu, \delta_0)$ for all $\mu \in \mathcal{P}_1(\mathbb{R}^m)$ and some $c_0, c_1 \geq 0$, where $\delta_0$ is the Dirac measure in 0. Then any $\mathbb{R}^m$-valued continuous process $X$ satisfies

$$\int_{t_0}^{t_f} |B_s(X_s, \mathcal{L}(X_s))| \, ds \leq \int_{t_0}^{t_f} c_0 + c_1 E[|X_s|] \, ds \quad \text{for all } t \geq t_0,$$

by using that $\vartheta_1(\mu, \delta_0) = \int_{\mathbb{R}^m} |x| \, \mu(dx)$ for any $\mu \in \mathcal{P}_1(\mathbb{R}^m)$. While the left-hand integral is finite if $X$ solves (1.2), the right-hand integral may be infinite. Indeed, the absolute moment function

$$[t_0, \infty] \to \mathbb{R}_+, \quad s \mapsto E[|X_s|]$$

of $X$ is lower semicontinuous and hence, measurable, by Fatou’s lemma, but it is not necessarily locally integrable. If $c_1 = 0$, then $E[|X|]$ would even be locally bounded for any solution $X$ to (1.2), by Lemmas 3.20 and 3.21.

However, to allow for growth in the measure variable, we study uniqueness, stability and existence under a local integrability condition, which leads to more generality. To this end, let $\Theta$ be an $[0, \infty]$-valued functional on $[t_0, \infty] \times \mathcal{P} \times \mathcal{P} \times \mathcal{P}(\mathbb{R}^m)$.
Definition 2.10. We say that pathwise uniqueness holds for (1.2) (relative to Θ) if any two solutions $X$ and $\tilde{X}$ with $X_{t_0} = \tilde{X}_{t_0}$ a.s. (and for which the function

$$\Theta(s, L(X_s), L(\tilde{X}_s), L(X_s - \tilde{X}_s))$$

(2.7)

is measurable and locally integrable) are indistinguishable.

Example 2.11. Suppose that $\rho, g \in C(\mathbb{R}_+)$ are positive on $[0, \infty]$ and vanish at 0, $\rho$ is concave and $\eta, \lambda : [t_0, \infty] \rightarrow \mathbb{R}_+$ are measurable and locally integrable such that

$$\Theta(s, \mu, \hat{\mu}, \nu) = \lambda(s) \rho(\hat{\vartheta}(\mu, \hat{\mu})) + \eta(s) \int_{\mathbb{R}^m} \rho(|x|) \nu(dx)$$

for all $s \geq t_0$, $\mu, \hat{\mu} \in \mathcal{P}$ and $\nu \in \mathcal{P}(\mathbb{R}^m)$. Then $\Theta(s, \mu, \hat{\mu}, \nu)$ is finite if $\nu \in \mathcal{P}_1(\mathbb{R}^m)$, by Jensen’s inequality, and for any two continuous processes $X$ and $\tilde{X}$ with $L(X_s), L(\tilde{X}_s) \in \mathcal{P}$ we have

$$\Theta(s, L(X_s), L(\tilde{X}_s), L(X_s - \tilde{X}_s)) = \eta(s) \rho(\hat{\vartheta}(L(X_s), L(\tilde{X}_s))) + \lambda(s) E[\rho(|X - \tilde{X}|)].$$

Thus, the measurable function (2.7) is locally integrable as soon as $E[|X - \tilde{X}|]$ is locally bounded.

Now we present generalised notions of stability for (1.2) in a global sense, which apply directly without shifting the stochastic drift and diffusion. Namely, in the literature for stability of SDEs, it is a convenient assumption that drift and diffusion vanish at all times at the origin of $\mathbb{R}^m$, ensuring that the constant zero process is a solution.

If a reader seeks to use stability results for McKean-Vlasov SDEs and the required normalisations $B(0, \delta_0) = 0$ and $\Sigma(0) = 0$ fail, then there should exist at least one solution $\tilde{X}$ to (1.2). In this case, the maps $\hat{B}$ and $\hat{\Sigma}$ on $[t_0, \infty] \times \Omega \times \mathbb{R}^m \times \mathcal{P}$ and $[t_0, \infty] \times \Omega \times \mathbb{R}^m$ with values in $\mathbb{R}^m$ and $\mathbb{R}^{m \times d}$, respectively, given by

$$\hat{B}_t(x, \mu) := B_t(\tilde{X}_t + x, \hat{l}(t, \mu)) - B_t(\tilde{X}_t, \mathcal{L}(\tilde{X}_t)) \quad \text{and} \quad \hat{\Sigma}_t(x) := \Sigma_t(\tilde{X}_t + x) - \Sigma_t(\tilde{X}_t)$$

are admissible and satisfy $\hat{B}(0, \delta_0) = 0$ and $\hat{\Sigma}(0) = 0$ for any fixed Borel measurable map $\hat{l} : [t_0, \infty] \times \mathcal{P} \rightarrow \mathcal{P}$ such that $\hat{l}(t, \delta_0) = \mathcal{L}(\tilde{X}_t)$ for all $t \geq t_0$. In effect, the reader is forced to replace the drift $B$ and the diffusion $\Sigma$ by $\hat{B}$ and $\hat{\Sigma}$, respectively, and use stability concepts that are stated in terms of the particular solution $\tilde{X}$.

Further, even if $B$ and $\Sigma$ were deterministic as in (2.3), the coefficients $\hat{B}$ and $\hat{\Sigma}$ would in general become random, unless $\tilde{X}_t$ is constant for each $t \geq t_0$. This translation procedure may certainly have its justification for a local stability analysis, but, as it is not necessary for global comparisons of solutions, we do not apply it.

Definition 2.12. Let $\alpha > 0$.

(i) We say that (1.2) is stable in moment (with respect to $\Theta$) if any two solutions $X$ and $\tilde{X}$ (for which the function (2.7) is measurable and locally integrable) satisfy

$$\sup_{t \geq t_0} E[|X_t - \tilde{X}_t|] < \infty$$

under the condition that $E[|X_{t_0} - \tilde{X}_{t_0}|] < \infty$. If in addition $\lim_{t \uparrow \infty} E[|X_t - \tilde{X}_t|] = 0$, then we speak about asymptotic stability in moment.

(ii) Equation (1.2) is said to be $\alpha$-exponentially stable in moment (relative to $\Theta$) if there are $\lambda < 0$ and $c \geq 0$ such that for any two solutions $X$ and $\tilde{X}$ to (1.2),

$$E[|X_t - \tilde{X}_t|] \leq c e^{\lambda(t-t_0)^{\alpha}} E[|X_{t_0} - \tilde{X}_{t_0}|] \quad \text{for all } t \geq t_0$$

(2.8)

(whenever (2.7) is measurable and locally integrable). In this case, $\lambda$ is said to be a moment $\alpha$-Lyapunov exponent for (1.2).
values in $[0, \infty)$, on which there is a standard $d$-dimensional Brownian motion. These concepts carry over to the case (2.3) of deterministic Osgood continuity condition.

We seek to compare solutions to (1.2) with varying drifts and thereby show pathwise uniqueness. For this purpose, let the map $\tilde{W} : [t_0, \infty] \times \Omega \times \mathbb{R}^m \times \mathcal{P} \to \mathbb{R}^m$ be admissible. For $p \geq 1$ we define two sublinear functionals $[\cdot]_p$ and $[\cdot]_\infty$ with respective values in $[0, \infty]$ and $[0, \infty]$ on the linear space of all random variables by

$$[X]_p := E[(X^+)^p]^\frac{1}{p} \quad \text{and} \quad [X]_\infty := \text{ess sup} X.$$

Note that if $\alpha, \beta \in [0, 1]$ satisfy $\alpha + \beta \leq 1$ and $X$ and $Y$ are two random variables such that $Y \geq 0$, $[X]_\frac{1}{1+\alpha} < \infty$ and $E[Y] < \infty$, then the inequalities of Hölder and Young entail that $XY^\alpha$ is quasi-integrable and

$$E[XY^\alpha]E[Y]^\beta \leq [X]_\frac{1}{1+\alpha} (1 - (\alpha + \beta) + (\alpha + \beta)E[Y]).$$

This bound leads to the quantitative $L^1$-estimates of Theorems 4.5 and 4.6, on which our main results are based. As first requirement, we introduce an Osgood continuity condition on compact sets for the rows of $\Sigma$.

3 Main results

3.1 A quantitative first moment estimate and pathwise uniqueness

We seek to compare solutions to (1.2) with varying drifts and thereby show pathwise uniqueness. For this purpose, let the map

$$\tilde{B} : [t_0, \infty] \times \Omega \times \mathbb{R}^m \times \mathcal{P} \to \mathbb{R}^m$$

be admissible. For $p \geq 1$ we define two sublinear functionals $[\cdot]_p$ and $[\cdot]_\infty$ with respective values in $[0, \infty]$ and $[0, \infty]$ on the linear space of all random variables by

$$[X]_p := E[(X^+)^p]^\frac{1}{p} \quad \text{and} \quad [X]_\infty := \text{ess sup} X.$$

Note that if $\alpha, \beta \in [0, 1]$ satisfy $\alpha + \beta \leq 1$ and $X$ and $Y$ are two random variables such that $Y \geq 0$, $[X]_\frac{1}{1+\alpha} < \infty$ and $E[Y] < \infty$, then the inequalities of Hölder and Young entail that $XY^\alpha$ is quasi-integrable and

$$E[XY^\alpha]E[Y]^\beta \leq [X]_\frac{1}{1+\alpha} (1 - (\alpha + \beta) + (\alpha + \beta)E[Y]).$$

This bound leads to the quantitative $L^1$-estimates of Theorems 4.5 and 4.6, on which our main results are based. As first requirement, we introduce an Osgood continuity condition on compact sets for the rows of $\Sigma$. 

Remark 2.13. If $\tilde{W}, u : [t_0, \infty] \to \mathbb{R}_+$ are locally bounded, $\psi > 0$ on $[0, \infty]$ and $\lambda \in \mathbb{R}$ satisfies $\limsup_{t \to \infty} \frac{1}{t} \log(u(t)) \leq \lambda$, then for any $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that

$$u(t) \leq c_\varepsilon e^{\psi(t)(\lambda + \varepsilon)} \quad \text{for all } t \geq t_0.$$
Remark 3.3. If $B = \tilde{B}$ and for all $(C.1)$ $F$ or any

Remark 3.1. The condition forces the

Example 3.2. In the one-dimensional setting $m = 1$ let $l \in \mathbb{N}$, $\varphi : \mathbb{R} \to \mathbb{R}^{d \times l}$ be locally \( \frac{1}{2} \)-Hölder continuous and $\zeta$ and $\eta$ be progressively measurable processes with values in $\mathbb{R}^d$ and $\mathbb{R}^{d \times l}$, respectively, such that

\[
\Sigma(x) = \begin{pmatrix}
\hat{\Sigma}(1,1)(x_1) & \ldots & \hat{\Sigma}(1,d)(x_1) \\
\vdots & \ddots & \vdots \\
\hat{\Sigma}(m,1)(x_m) & \ldots & \hat{\Sigma}(m,d)(x_m)
\end{pmatrix}
\]

for some admissible map $\hat{\Sigma} : [t_0, \infty[ \times \Omega \times \mathbb{R} \to \mathbb{R}^{m \times d}$. Further, for every $n \in \mathbb{N}$ and $\alpha_n \in [\frac{1}{4}, 1]$, we may take $\hat{\rho}_n(v) = v^{\alpha_n}$ for all $v \geq 0$ as appearing modulus of continuity.

for any $x \in \mathbb{R}^m$ a.s. for any $i \in \{1, \ldots, m\}$, then \( (C.2) \) reduces to a partial uniform continuity condition on $B$. In the particular case that there are $\alpha_0, \beta_0 \in [0, 1]$ such that

\[
\alpha_0 \leq \alpha, \quad \rho(v) = v^{\alpha_0} \quad \text{and} \quad \varrho(v) = v^{\beta_0} \quad \text{for all } v \geq 0,
\]

we obtain a partial Hölder condition. Following this reasoning, the term $\varepsilon$ provides an error estimate. That is, if $\varepsilon(i)$ holds for $B = B$ and $\varepsilon = 0$, then it is valid in general as soon as $|B^{(i)} - B^{(i)}| \leq \varepsilon(i)$ for any $i \in \{1, \ldots, m\}$.

As our first estimation result is based on Bihari’s inequality, we recall that for any $\rho \in C(\mathbb{R}_+)$ that is positive on $[0, \infty[$ and vanishes at 0, the function $\Phi_\rho \in C^1([0, \infty[)$ defined via

\[
\Phi_\rho(w) := \int_1^w \frac{1}{\rho(v)} \, dv \tag{3.2}
\]
is a strictly increasing $C^1$-diffeomorphism onto the interval $[\Phi_\rho(0), \Phi_\rho(\infty)]$. Let $D_\rho$ denote the set of all $(v, w) \in \mathbb{R}^2_+$ with $\Phi_\rho(v) + w < \Phi_\rho(\infty)$. Then $\Psi_\rho : D_\rho \to \mathbb{R}^+$ given by

$$\Psi_\rho(v, w) := \Phi_\rho^{-1}(\Phi_\rho(v) + w) \quad (3.3)$$

is a continuous extension of a locally Lipschitz continuous function and it is increasing in each variable.

Based on these considerations, we obtain a quantitative $L^1$-bound under (C.2) by introducing for fixed $\beta \in [0, 1]$ the two measurable locally integrable functions

$$\gamma := \alpha[|\eta|_1] \frac{1}{1 - \alpha} + \beta E[|\lambda|_1] \quad \text{and} \quad \delta := (1 - \alpha)[|\eta|_1] \frac{1}{1 - \alpha} + (1 - \beta)E[|\lambda|_1].$$

**Proposition 3.4.** Let (C.1) and (C.2) hold, $X$ and $\tilde{X}$ be two solutions to (1.2) with respective drifts $B$ and $\tilde{B}$ such that

$$E[|Y_{t_0}|_1] < \infty \quad \text{for } Y := X - \tilde{X}$$

and $E[|\lambda|_1] \rho(\partial(\mathcal{L}(X), \mathcal{L}(\tilde{X})))$ is locally integrable. Define $g_0 \in C(\mathbb{R}^+)$ by

$$g_0(v) := \rho(v) \frac{1}{1 - \alpha} + \varrho(v) \frac{1}{1 - \beta}$$

and suppose that $\Phi_{\rho^\alpha}(\infty) = \infty$ or $E[|\eta|_1\rho(|Y|_1)]$ is locally integrable. Then $E[|Y|_1]$ is locally bounded and

$$\sup_{s \in [t_0, t]} E[|Y_s|_1] \leq \Psi_{\rho^\alpha} \left( E[|Y_{t_0}|_1] + \int_{t_0}^{t} E[|\varepsilon_s|_1] + \delta(s) ds, \int_{t_0}^{t} \gamma(s) ds \right)$$

for any $t \in [t_0, t_0^+]$, where $t_0^+ > t_0$ stands for the supremum over all $t \geq t_0$ for which

$$\left( E[|Y_{t_0}|_1] + \int_{t_0}^{t} E[|\varepsilon_s|_1] + \delta(s) ds, \int_{t_0}^{t} \gamma(s) ds \right) \in D_{g_0}.$$

**Remark 3.5.** If in fact $\Phi_{\rho^\alpha}(\infty) = \infty$, then $\Phi_{\rho^\alpha}(\infty) = \infty$ and $D_{g_0} = \mathbb{R}_+^2$. In this case, $Y$ is bounded in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ as soon as $E[|\varepsilon|_1]$, $\gamma$ and $\delta$ are integrable. Further, if

$$\Phi_{\rho^\alpha}(0) = -\infty, \quad Y_{t_0} = 0 \quad \text{a.s.} \quad \text{and} \quad E[|\varepsilon|_1] = \delta = 0 \quad \text{a.e.,}$$

then $t_0^+ = \infty$ and $Y = 0$ a.s. This implication will be used to deduce pathwise uniqueness.

**Example 3.6.** Let $\alpha_0 \in [0, \alpha]$ be such that $\rho(v) = \varrho(v) = v^{\alpha_0}$ for all $v \geq 0$ and $\alpha = \beta$. Then $\Phi_{\rho^\alpha}(\infty) = \infty$ and in the case $\alpha_0 < \alpha$ we get that

$$\Psi_{\rho^\alpha}(v, w) = \left( v^{1 - \hat{\alpha}} + (1 - \hat{\alpha})w \right)^{1 - \alpha} \text{ for all } v, w \geq 0$$

with $\hat{\alpha} := \frac{\alpha_0}{\alpha}$. If instead $\alpha_0 = \alpha$, then $\Psi_{\rho^\alpha}(v, w) = v \exp(w)$ for any $v, w \geq 0$. Thus, Proposition 3.2 provides an estimate for any choice of $\hat{\alpha} \in [0, 1]$.

To infer pathwise uniqueness for (1.2) from the comparison, we specify (C.2) for $B = \tilde{B}$, $\varepsilon = 0$, $\alpha = 1$ and a deterministic choice of $\eta$. Further, if $B$ does not depend on $\mu \in \mathcal{P}$, then it suffices to pose this condition on compact sets only.

(C.3) There are $\rho, \varrho \in C(\mathbb{R}^+_\infty)$ that are positive on $[0, \infty]$ and vanish at 0, a measurable locally integrable function $\eta : [t_0, \infty[ \to \mathbb{R}^+_\infty$ and an $\mathbb{R}^m_\infty$-valued progressively measurable process $\lambda$ with locally integrable paths such that

$$\sgn(x_i - \tilde{x}_i)(B^{(i)}(x, \mu) - B^{(i)}(\tilde{x}, \tilde{\mu})) \leq \eta \rho(|x - \tilde{x}|_1) + \lambda^{(i)} \varrho(\vartheta(\mu, \tilde{\mu}))$$

for any $x, \tilde{x} \in \mathbb{R}^m$ and $\mu, \tilde{\mu} \in \mathcal{P}$ a.s. for all $i \in \{1, \ldots, m\}$. In addition, $\rho$ is concave, $\varrho$ is increasing and $E[|\lambda|_1]$ is locally integrable.
(C.4) B is independent of $\mu \in \mathcal{P}$ and for any $n \in \mathbb{N}$ there are a concave $\rho_n \in C(\mathbb{R}_+)$ that is positive on $[0, \infty]$ and vanishes at 0 and a measurable locally integrable function $\eta_n : [t_0, \infty] \to \mathbb{R}_+$ satisfying
\[
\text{sgn}(x - \tilde{x}_i)(\hat{B}^{(i)}(x) - \hat{\hat{B}}^{(i)}(\tilde{x})) \leq \eta_n \rho_n(|x - \tilde{x}|_1)
\]
for all $x, \tilde{x} \in \mathbb{R}^m$ with $|x| \vee |\tilde{x}| \leq n$ a.s. for any $i \in \{1, \ldots, m\}$, where $\hat{B} := B(\cdot, \hat{\mu})$ for some $\hat{\mu} \in \mathcal{P}$.

Under (C.1) and (C.3), pathwise uniqueness can be shown with respect to the Borel measurable functional $\Theta : [t_0, \infty] \times \mathcal{P} \times \mathcal{P}(\mathbb{R}^m) \to [0, \infty]$ defined by
\[
\Theta(s, \mu, \tilde{\mu}, v) := E[|\lambda_\nu|] \phi(\vartheta(\mu, \tilde{\mu})) + 1_{[0, \infty]}(\Phi_{\rho}(\infty))\eta(s)\int_{\mathbb{R}_m^m} \rho(|y|) v(dy).
\]

**Corollary 3.7.** Suppose that (C.1) is satisfied.

(i) Let (C.3) be valid and $\int_0^1 \frac{1}{\rho_n(v)} \, dv = \infty$. Then pathwise uniqueness holds for (1.2) relative to $\Theta$.

(ii) If (C.4) is satisfied and $\int_0^1 \frac{1}{\rho_n(v)} \, dv = \infty$ for each $n \in \mathbb{N}$, then pathwise uniqueness for the SDE (1.2) follows.

**Remark 3.8.** Let $B$ and $\Sigma$ be deterministic, that is, (2.3) holds. Then the corollary yields pathwise uniqueness for (2.4) in the standard sense if the conditions are specified as follows:

1. The Osgood condition (C.1) on compact sets is formulated when $\tilde{\eta}^{(n)}$ is independent of $\omega \in \Omega$ for all $n \in \mathbb{N}$.

2. The partial uniform continuity condition (C.3) is stated when $\lambda$ is deterministic and the required estimate (2.5) for $\vartheta$ is replaced by the domination condition (2.6).

Let us consider a class of drift maps to which these uniqueness results apply.

**Example 3.9.** Let $F$ and $G$ be two $\mathbb{R}^m$-valued admissible maps on $[t_0, \infty] \times \Omega \times \mathbb{R}$ and $[t_0, \infty] \times \Omega \times \mathbb{R}^m \times \mathcal{P}$, respectively, such that
\[
B^{(i)}(x, \mu) = F^{(i)}(x_i) + G^{(i)}(x, \mu)
\]
for all $(x, \mu) \in \mathbb{R}^m \times \mathcal{P}$ and $i \in \{1, \ldots, m\}$. Then (C.3) and (C.4) are implied by the following respective conditions:

1. There is $\vartheta \in C(\mathbb{R}_+)$ that is positive on $[0, \infty]$ and vanishes at 0, a measurable locally integrable function $\eta : [t_0, \infty] \to \mathbb{R}_+$ and an $\mathbb{R}^m_+$-valued progressively measurable process $\lambda$ such that
\[
\text{sgn}(v - \tilde{v})(F^{(i)}(v) - F^{(i)}(\tilde{v})) \leq \eta|v - \tilde{v}|
\]
and
\[
|G^{(i)}(x, \mu) - G^{(i)}(\tilde{x}, \tilde{\mu})| \leq \eta|x - \tilde{x}| + \lambda^{(i)}(\vartheta(\mu, \tilde{\mu}))
\]
for any $v, \tilde{v} \in \mathbb{R}, x, \tilde{x} \in \mathbb{R}^m, \mu, \tilde{\mu} \in \mathcal{P}$ and $i \in \{1, \ldots, m\}$. Moreover, $\vartheta$ is increasing, $\lambda$ has locally integrable paths and $E[|\lambda_\nu|]$ is locally integrable.

2. $G$ is independent of $\mu \in \mathcal{P}$ and for each $n \in \mathbb{N}$ there are $\rho_n \in C(\mathbb{R}_+)$ that is positive on $[0, \infty]$ and vanishes at 0 and a measurable locally integrable function $\eta_n : [t_0, \infty] \to \mathbb{R}_+$ such that
\[
\text{sgn}(v - \tilde{v})(F^{(i)}(v) - F^{(i)}(\tilde{v})) \leq \eta_n \rho_n(|v - \tilde{v}|), \quad |\hat{G}(x) - \hat{\hat{G}}(\tilde{x})| \leq \eta_n \rho_n(|x - \tilde{x}|)
\]
for all $v, \tilde{v} \in [-n, n], x, \tilde{x} \in \mathbb{R}^m$ with $|x| \vee |\tilde{x}| \leq n$ and $i \in \{1, \ldots, m\}$, where $\hat{G} := G(\cdot, \hat{\mu})$ for fixed $\hat{\mu} \in \mathcal{P}$. Further, $\rho_n$ is concave and increasing.
For instance, for $l \in \mathbb{N}$, an $\mathbb{R}^{m \times l}_+$-valued progressively measurable process $\eta$ and $\alpha \in [0, \infty]$, we could take

$$F^{(i)}(v) = -\eta^{(i,1)}(v^+)\alpha_1 - \cdots - \eta^{(i,l)}(v^+)\alpha_l$$

(3.4)

for all $v \in \mathbb{R}$ and $i \in \{1, \ldots, m\}$, in which case both conditions (1) and (2) are met by $F$. In the general case, if (C.1) is imposed on $\Sigma$, then Corollary 3.7 entails two assertions:

(3) Let condition (1) hold. Then we have pathwise uniqueness for (1.2) relative to the Borel measurable functional $\Theta : [t_0, \infty[ \times \mathcal{P} \times \mathcal{P} \to [0, \infty]$ given by

$$\Theta(s, \mu, \tilde{\mu}) := E[|\lambda_n|]g(\vartheta(\mu, \tilde{\mu})),$$

provided $\int_0^1 \frac{1}{v} \sqrt{\varrho(v)} \, dv = \infty$.

In particular, $\varrho(v) = v$ for all $v > 0$ and $\varrho(v) = \alpha v(\log(v)) + 1$ for any $v > 0$ with $\alpha > 0$ are feasible choices.

(4) There is pathwise uniqueness for the SDE (1.2) under condition (2) as soon as $\int_0^1 \frac{1}{\varrho(v)} \, dv = \infty$ for each $n \in \mathbb{N}$.

3.2 An explicit moment estimate and stability in first moment

Now we provide a comparison bound, from which stability results in first moment can be inferred. In this regard, we require a partial uniform error and mixed Hölder continuity condition on $B$ and $\tilde{B}$:

(C.5) There are $i \in \mathbb{N}$, $\alpha, \beta \in [0, 1]^l$ and progressively measurable processes $\varepsilon$, $\eta$ and $\lambda$ with values in $\mathbb{R}^n_+$, $\mathbb{R}^{m \times m \times l}$ and $\mathbb{R}^{m \times l}$, respectively, such that

$$\text{sgn}(x_i - \tilde{x}_i)(B^{(i)}(x, \mu) - B^{(i)}(\tilde{x}, \tilde{\mu})) \leq \varepsilon^{(i)}$$

$$+ \sum_{k=1}^l \left( \sum_{j=1}^m \eta^{(i,j,k)} |x_j - \tilde{x}_j|^{\alpha_k} \right) + \lambda^{(i,k)} \varrho(\mu, \tilde{\mu})^{\beta_k}$$

for all $x, \tilde{x} \in \mathbb{R}^m$ and $\mu, \tilde{\mu} \in \mathcal{P}$ a.s. for any $i \in \{1, \ldots, m\}$. In addition, $\varepsilon$, $\eta$ and $\lambda$ have locally integrable paths and we have $\eta^{(i,j,k)} \geq 0$, if $i \neq j$, and

$$E[\varepsilon^{(i)}], \quad [\eta^{(i,j,k)}]_{1=\alpha_k}, \quad E[\lambda^{(i,k)}]$$

are locally integrable for all $i, j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, l\}$.

Remark 3.10. If (C.4) is satisfied, $\alpha_1 < \cdots < \alpha_l$ and $\beta_1 < \cdots < \beta_l$, then (3.2) follows for $\rho, \varrho \in C(\mathbb{R}_+)$ given by

$$\rho(v) := v^{\alpha_1} \mathbb{1}_{[0,1]}(v) + v^{\alpha_2} \mathbb{1}_{[1,\infty)}(v) \quad \text{and} \quad \varrho(v) := v^{\beta_1} \mathbb{1}_{[0,1]}(v) + v^{\beta_2} \mathbb{1}_{[1,\infty)}(v).$$

Under (C.5), the two measurable locally integrable functions $\gamma_1 : [t_0, \infty[ \to [-\infty, \infty]$ and $\hat{\gamma}_1 : [t_0, \infty[ \to [0, \infty]$ defined by

$$\gamma_1(s) := \max_{j=1,\ldots,m} \sum_{k=1}^l \alpha_k \left( \sum_{i=1}^m \eta^{(i,j,k)} \right) + \beta_k \sum_{i=1}^m E[\lambda^{(i,k)}]$$

(3.5)

and

$$\hat{\gamma}_1(s) := \sum_{k=1}^l (1 - \alpha_k) \left( \sum_{j=1}^m \sum_{i=1}^m \eta^{(i,j,k)} \right) + (1 - \beta_k) \sum_{i=1}^m E[\lambda^{(i,k)}]$$

(3.6)
soley depend on the regularity of $B$ and $\tilde{B}$. By means of these coefficients we get an \textit{explicit} $L^1$-\textit{comparison estimate} relative to the $[0, \infty]$-valued Borel measurable functional $\Theta$ on $[t_0, \infty] \times \mathcal{P} \times \mathcal{P}$ given by

\[
\Theta(s, \mu, \tilde{\mu}) := \sum_{k=1}^{l} \sum_{i=1}^{m} E[\lambda_s^{(i,k)}] \vartheta(\mu, \tilde{\mu})^\delta_k.
\]  

(3.7)

\begin{proposition}
Let (C.1) and (C.5) be valid, $X$ be a solution to (1.2) and $\tilde{X}$ solve (1.2) with $\tilde{B}$ instead of $B$ such that

\[
E[|Y_t|_1] < \infty \quad \text{for } Y := X - \tilde{X}
\]

and $\Theta(\cdot, \mathcal{L}(X), \mathcal{L}(\tilde{X}))$ is locally integrable. Then

\[
E[|Y_t|_1] \leq e^{\int_{t_0}^{t} \gamma_1(s) \, ds} E[|Y_{t_0}|_1] + \int_{t_0}^{t} e^{\int_{s}^{t} \gamma_1(\tilde{x}) \, ds} (E[|\varepsilon_s|_1] + \hat{\delta}_1(s)) \, ds
\]  

(3.8)

for any $t \geq t_0$. In particular, if $\gamma^+_1$, $E[|\varepsilon|_1]$ and $\hat{\delta}_1$ are integrable, then $E[|Y|_1]$ is bounded. If additionally $\int_{t_0}^{\infty} \gamma^-_1(s) \, ds = \infty$, then

\[
\lim_{t \uparrow \infty} E[|Y_t|_1] = 0.
\]

\end{proposition}

\begin{remark}
While $\varepsilon$ serves as error estimate for $B - \tilde{B}$, the coefficient $\hat{\delta}_1$ arises from all the partial Hölder exponents in $[0, 1]$ that appear in (C.5). Namely, $\hat{\delta}_1(s)$ vanishes for given $s \geq t_0$ if and only if each $k \in \{1, \ldots, l\}$ satisfies

\[
1_{[0,1]}(\alpha_k) \max_{j=1, \ldots, m} \sum_{i=1}^{m} \eta_s^{(i,j,k)} \leq 0 \text{ a.s. and } 1_{[0,1]}(\beta_k) \max_{i=1, \ldots, m} \lambda_s^{(i,k)} = 0 \text{ a.s.}
\]

Although the bound in Proposition 3.11 applies to different types of moduli of continuity that are specified in (C.2), the estimate (3.8) is generally sharper, as Example 3.6 and Remark 3.10 show, bearing in mind that $\gamma_1$ may take negative values.

Based on the \textit{partial mixed Hölder continuity condition} (C.5) for $B$, assuming that $B = \tilde{B}$ and $\varepsilon = 0$ there, we get (asymptotic) stability in moment as direct consequence.

\begin{corollary}
Let (C.1) and (C.5) be satisfied for $B = \tilde{B}$ and $\varepsilon = 0$. Then (1.2) is (asymptotically) stable in moment relative to $\Theta$ defined by (3.7) if $\gamma^+_1$ and $\hat{\delta}_1$ are integrable (and $\int_{t_0}^{\infty} \gamma^-_1(s) \, ds = \infty$).

For a description of the $L^1$-boundedness and the rate of $L^1$-convergence for solutions in Corollary 3.11 below, let us restrict (C.5) to a partial Lipschitz continuity condition:

(C.6) There are a measurable locally integrable map $\eta : [t_0, \infty] \to \mathbb{R}^{m \times m}$ and an $\mathbb{R}^m_+$-valued progressively measurable process $\lambda$ with locally integrable paths such that

\[
\text{sgn}(x_i - \tilde{x}_i)(B^{(i)}(x, \mu) - B^{(i)}(\tilde{x}, \tilde{\mu})) \leq \left( \sum_{j=1}^{m} \eta_{i,j}|x_j - \tilde{x}_j| \right) + \lambda^{(i)}(\vartheta(\mu, \tilde{\mu})
\]

for any $x, \tilde{x} \in \mathbb{R}^m$ and $\mu, \tilde{\mu} \in \mathcal{P}$ a.s. for all $i \in \{1, \ldots, m\}$. Moreover, $\eta_{i,j} \geq 0$ for all $i, j \in \{1, \ldots, m\}$ with $i \neq j$ and $E[|\lambda|_1]$ is locally integrable.

If the preceding condition holds, then the coefficient $\hat{\delta}_1$ in (3.5) vanishes and the stability coefficient $\gamma_1$ in (3.5) becomes

\[
\gamma_1 = \max_{j=1, \ldots, m} \eta_{1,j} + \cdots + \eta_{m,j} + E[|\lambda|_1].
\]  

(3.9)

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Remarkably, $\gamma_1$ is merely influenced by the regularity of $B$. Further, under (C.6), the functional in (3.7) is of the form

$$
\Theta(\cdot, \mu, \tilde{\mu}) = E[|\lambda_1|]\vartheta(\mu, \tilde{\mu})
$$

(3.10)

for any $\mu, \tilde{\mu} \in \mathcal{P}$. To deduce exponential moment stability, we impose an upper bound on $\gamma_1$ that involves sums of power functions.

(C.7) Condition (C.6) is valid and there are $l \in \mathbb{N}, \alpha \in [0, \infty[^l$ and $\tilde{\lambda}, s \in \mathbb{R}^l$ such that $\alpha_1 < \cdots < \alpha_l, \tilde{\lambda}_l < 0$ and

$$
\gamma_1(s) \leq \tilde{\lambda}_1 \alpha_1(s - s_1)^{\alpha_1 - 1} + \cdots + \tilde{\lambda}_l \alpha_l(s - s_l)^{\alpha_l - 1}
$$

for a.e. $s \geq t_1$ for some $t_1 \geq t_0$ satisfying $\max_{k=1, \ldots, l} s_k \leq t_1$.

By using the negativity of the constant $\tilde{\lambda}_l$ associated to the greatest power $\alpha_l$ in the preceding condition, the following stability properties hold.

**Corollary 3.14.** Under (C.1), the following two assertions hold:

(i) If (C.6) is valid and $\gamma_1^+$ is integrable, then the difference $Y$ of any two solutions $X$ and $\tilde{X}$ to (1.2) satisfies

$$
\sup_{t \geq t_0} e^{\int_0^t \gamma_1^-(s) \, ds} E[|Y_t|] < \infty,
$$

provided $E[|Y_{t_0}|] < \infty$ and $\Theta(\cdot, \mathcal{L}(X), \mathcal{L}(\tilde{X}))$ is locally integrable. If in addition $\gamma_1^-$ fails to be integrable, then

$$
\lim_{t \to \infty} e^{\int_0^t \gamma_1^-(s) \, ds} E[|Y_t|] = 0 \quad \text{for any } \alpha \in [0, 1].
$$

(ii) Let (C.7) be valid. Then (1.2) is $\alpha_l$-exponentially stable in moment relative to $\Theta$ with any moment $\alpha_l$-Lyapunov exponent in $]\lambda_1, 0[$. Further, $\tilde{\lambda}_l$ serves as Lyapunov exponent as soon as

$$
\max_{k=1, \ldots, l} \tilde{\lambda}_k \leq 0 \quad \text{and} \quad s_l \leq t_0.
$$

Let us conclude with a specification of Example 3.9, which for $m = 1$ plays a major role in the volatility modelling in Section 3.1, as the representation (5.3) suggests.

**Example 3.15.** Let $l \in \mathbb{N}, \zeta : [t_0, \infty[ \times \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ be measurable and locally integrable, $\eta$ be an $\mathbb{R}^m_{+ \times l}$-valued progressively measurable process with locally integrable paths,

$$
f : \mathbb{R} \rightarrow \mathbb{R}^m_{+ \times l}
$$

be measurable and $G$ be an $\mathbb{R}^m$-valued admissible map on $[t_0, \infty[ \times \Omega \times \mathbb{R}^m \times \mathcal{P}$ such that

$$
B^{(i)}(x, \mu) = \zeta_i x_i + \eta^{(i,1)} f_i(x_i) + \cdots + \eta^{(i,l)} f_i(x_i) + G^{(i)}(x, \mu)
$$

(3.11)

for all $(x, \mu) \in \mathbb{R}^m \times \mathcal{P}$ and $i \in \{1, \ldots, m\}$. Then (C.5) follows for $B = \hat{B}$ and $\epsilon = 0$ from the following condition:

1. There are $\alpha_0, \beta_0 \in [0, 1], \alpha \in [0, 1]^l$, $\tilde{\eta} \in \mathbb{R}^m_{+ \times l}$ and $\mathbb{R}^m_{+ \times l}$-valued progressively measurable processes $\tilde{\gamma}$ and $\tilde{\lambda}$ with locally integrable paths such that

$$
\text{sgn}(v - \hat{v}) (f_{i,k} (v) - f_{i,k} (\hat{v})) \leq \tilde{\eta}_{i,k} |v - \hat{v}|^{\alpha_k}
$$

and

$$
|G^{(i)}(x, \mu) - G^{(i)}(\hat{x}, \hat{\mu})| \leq \tilde{\gamma}^{(i)} |x - \hat{x}|^{\alpha_0} + \lambda^{(i)} \vartheta(\mu, \hat{\mu})^{\beta_0}
$$

for all $v, \hat{v} \in \mathbb{R}, x, \hat{x} \in \mathbb{R}^m, \mu, \hat{\mu} \in \mathcal{P}, i \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, l\}$. Further, $\sum_{k=1}^l \sum_{j=1}^m [\eta^{(j,k)} \tilde{\eta}_{j,k}]^{1/\alpha_k}, E[|\tilde{\eta}|_1]$ and $E[|\lambda|_1]$ are locally integrable.
For instance, if $f_{i,k}$ is decreasing for all $i \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, l\}$, then $\hat{\eta} = 0$ is possible in (1). In general, if (C.1) is met by $\Sigma$ and (1) holds, then Proposition 3.11 and Corollaries 3.13 and 3.14 yield the following statements:

(2) The difference $Y$ of any two solutions $X$ and $\tilde{X}$ to (1.2) for which $E[|Y_0|] < \infty$ and $E[|\lambda_1|] \vartheta(L(X), L(\tilde{X}))^{\beta_0}$ is locally integrable satisfies the estimate (3.8) with

$$\gamma_1 = \max_{j=1, \ldots, m} \frac{\eta_j}{\alpha_0} + \alpha_0 \frac{|\eta_j|}{1-\alpha_0} + \beta_0 E[|\lambda_1|],$$

$$\tilde{\gamma}_1 = (\sum_{k=1}^{L}(1-\alpha_k) \sum_{j=1}^{m} |\eta^{(j,k)} \theta_{j,k}^{(k)}| \frac{1}{1-\alpha_k}) + (1-\alpha_0)m |\eta_j| \frac{1}{1-\alpha_0} + (1-\beta_0)E[|\lambda_1|]$$

$\varepsilon = 0$, where the measurable map $\hat{\eta} : [t_0, \infty[ \to [0, \infty]$ is given by

$$\hat{\eta}_j(s) = \zeta_j(s) + \sum_{k=1}^{L} \alpha_k [\eta^{(j,k)} \theta_{j,k}^{(k)}] \frac{1}{1-\alpha_k} \tag{3.12}$$

(3) For equation (1.2) to be (asymptotically) stable in moment with respect to the Borel measurable functional $\Theta : [t_0, \infty[ \times P \times P \to [0, \infty]$ given by

$$\Theta(s, \mu, \hat{\mu}) := E[|\lambda_s|] \vartheta(\mu, \hat{\mu})^{\beta_0},$$

it is sufficient that the coefficients $\eta_1^+, \ldots, \eta_m^+, [\eta_1^+] \frac{1}{1-\alpha_0}$ and $E[|\lambda|]$ are integrable (and $\int_{t_0}^{\infty} \min_{j=1, \ldots, m} \eta_j^{-}(s) ds = \infty$).

(4) Assume that $\alpha_0 = \cdots = \alpha_L = 1$. This ensures the validity of (C.6). Hence, if there are $\lambda < 0$ and $\hat{\lambda} > 0$ such that

$$\max_{j=1, \ldots, m} \eta_j(s) + [\eta_s^+] \leq \lambda \hat{\lambda}(s-t_0)^{\hat{\lambda}^{-1}}$$

for a.e. $s \geq t_0$,

then $\hat{\lambda}$-exponential stability in moment relative to $\Theta$ with Lyapunov exponent $\hat{\lambda}$ holds.

### 3.3 Pathwise stability and moment growth estimates

In this section, our first aim is to derive pathwise exponential stability for (1.2). For this purpose, we replace the Osgood condition (C.1) on compact sets by the following stronger $\frac{1}{2}$-H"older continuity condition for the diffusion:

(C.8) There is a measurable locally square-integrable map $\hat{\eta} : [t_0, \infty[ \to \mathbb{R}^m$ such that

$$|e^t(\Sigma(x) - \Sigma(\hat{x}))| \leq \hat{\eta}_i|x_i - \hat{x}_i|^2$$

for any $x, \hat{x} \in \mathbb{R}^m$ a.s. for all $i \in \{1, \ldots, m\}$.

Moreover, we require that the partial Lipschitz condition (C.0) and the regularity coefficients $\lambda$ and $\hat{\eta}$ satisfy a suitable growth condition, which follows if $E[|\lambda_1|]$ and $\hat{\eta}$ are locally bounded, for instance.

(C.9) Conditions (C.6) and (C.8) are satisfied and there is some $\hat{\delta} > 0$ such that

$$\sup_{t \geq t_0} \int_{t}^{t+\hat{\delta}} f(s) ds < \infty$$

for $f \in \{E[|\lambda_1|], |\hat{\eta}|^2\}

Under the following abstract condition on the stability coefficient $\gamma_1$, we obtain a general pathwise stability estimate from Theorem 1.13 a pathwise result for random Itô processes based on the Borel-Cantelli Lemma.

(C.10) Condition (C.9) holds and there are $\varepsilon \in [0, 1[$ and a strictly increasing sequence $(t_n)_{n \in \mathbb{N}}$ in $[t_0, \infty[$ such that $\gamma_1 \leq 0$ a.e. on $[t_1, \infty[$,

$$\sup_{n \in \mathbb{N}} (t_{n+1} - t_n) < \hat{\delta}, \quad \lim_{n \to \infty} t_n = \infty$$

and

$$\sum_{n=1}^{\infty} \exp(\frac{\varepsilon}{2} \int_{t_n}^{t_{n+1}} \gamma_1(s) ds) < \infty$$

for all $\varepsilon \in [0, 1[$.
Proposition 3.16. Assume \((\text{C.10})\) and let \(X\) and \(\tilde{X}\) be two solutions to \((1.2)\) such that 
\[E[|\lambda|_1] \theta(\mathcal{L}(X), \mathcal{L}(\tilde{X}))\] 
is locally integrable. Then \(Y := X - \tilde{X}\) satisfies
\[\limsup_{t \uparrow \infty} \frac{1}{\varphi(t)} \log(|Y_t|_1) \leq \frac{1}{2} \limsup_{n \uparrow \infty} \frac{1}{\varphi(t_n)} \int_{t_1}^{t_n} \gamma_1(s) \, ds \quad \text{a.s.}\]
for any increasing continuous function \(\varphi : [t_1, \infty] \to \mathbb{R}_+\) that is positive on \([t_1, \infty]\), provided \(E[|Y_{t_0}|_1] < \infty\) or \(\lambda = 0\).

Now we employ the upper bound \((\text{C.7})\) on \(\gamma_1\) to derive pathwise exponential stability, since \((\text{C.10})\) already follows from \((\text{C.9})\) in this case, as will be shown.

Corollary 3.17. Let \((\text{C.7})\) and \((\text{C.9})\) hold and define \(\Theta : [t_0, \infty] \times \mathcal{P} \times \mathcal{P} \to [0, \infty]\) by \((\text{3.10})\).

(i) Then the McKean-Vlasov SDE \((1.2)\) is pathwise \(\alpha_1\)-exponentially stable with Lyapunov exponent \(\frac{\lambda}{\alpha} \) with respect to an initial absolute moment and \(\Theta\).

(ii) If in fact \(B\) is independent of \(\mu \in \mathcal{P}\), then the SDE \((1.2)\) is pathwise \(\alpha_1\)-exponentially stable with Lyapunov exponent \(\frac{\lambda}{\alpha} \).

The corollary directly applies to the class of drift coefficients in Example 3.15.

Example 3.18. Suppose that \(B\) is of the form \((3.11)\) and let \((\text{C.8})\) hold for \(\Sigma\) when \(\tilde{\eta}\) is locally bounded. Then the following sharpened version of condition (1) in Example 3.15 implies \((\text{C.9})\):

(2) There are \(\tilde{\eta} \in \mathbb{R}^{m \times l}\), a measurable locally bounded map \(\tilde{\eta} : [t_0, \infty] \to \mathbb{R}^m_n\) and an \(\mathbb{R}^n\)-valued progressively measurable process \(\lambda\) with locally integrable paths so that
\[\text{sgn}(v - \tilde{v})(f_{i,k}(v) - f_{i,k}(\tilde{v})) \leq \tilde{\eta}_{i,k}|v - \tilde{v}| \quad \text{and} \quad |G^{(i)}(x, \mu) - G^{(i)}(\tilde{x}, \tilde{\mu})| \leq \tilde{\eta}_i|x - \tilde{x}| + \lambda^{(i)} \vartheta(\mu, \tilde{\mu})\]
for any \(v, \tilde{v} \in \mathbb{R}, x, \tilde{x} \in \mathbb{R}^m, \mu, \tilde{\mu} \in \mathcal{P}, i \in \{1, \ldots, m\}\) and \(k \in \{1, \ldots, l\}\). In addition, \(\sum_{k=1}^{m} \sum_{j=1}^{m} \tilde{\eta}_{i,k} \tilde{\eta}_{i,k,\infty}\) and \(E[|\lambda|_1]\) are locally bounded.

Under this condition, the formula \((3.12)\) reduces to \(\tilde{\eta}_j = \zeta_j + \sum_{k=1}^{m} |\eta^{(j,k)}_{i,k,\infty}|\) for all \(j \in \{1, \ldots, m\}\), and we suppose in addition that
\[\max_{j=1, \ldots, m} \eta_j(s) + |\tilde{\eta}(s)|_1 + E[|\lambda_s|_1] \leq \lambda \hat{\alpha}(s - t_0)^{\hat{\alpha} - 1} \quad \text{for a.e.} \ s \geq t_0\]
for some \(\lambda < 0\) and \(\hat{\alpha} > 0\). Then Corollary 3.17 entails the subsequent two assertions:

(3) Equation \((1.2)\) is pathwise \(\hat{\alpha}\)-exponentially stable with Lyapunov exponent \(\frac{\lambda}{\hat{\alpha}}\) relative to an initial absolute moment and \(\Theta : [t_0, \infty] \times \mathcal{P} \times \mathcal{P} \to [0, \infty]\) given by \((3.10)\).

(4) If \(\lambda = 0\), in which case \(B\) is independent of \(\mu \in \mathcal{P}\), then the SDE \((1.2)\) is pathwise \(\hat{\alpha}\)-exponentially stable with Lyapunov exponent \(\frac{\lambda}{\hat{\alpha}}\).

Next, we give two first moment bounds for solutions to \((1.2)\), each showing that their absolute moment functions are locally bounded. Hence, local integrability relative to the functional \(\Theta\) in each of the Corollaries 3.15, 3.16, 3.17 and 3.18 holds in these cases. First, we require an \textit{Osgood growth condition on compact sets} for \(\Sigma\).
(C.11) For any \( n \in \mathbb{N} \) there are an increasing \( \phi_n \in C(\mathbb{R}_+) \) and an \( \mathbb{R}_+ \)-valued progressively measurable process \( \hat{\phi}^{(n)} \) with locally square-integrable paths such that

\[
\phi_n > 0 \text{ on } [0, \infty[ , \quad \int_0^1 \frac{1}{\phi_n(v)^2} dv = \infty \quad \text{and} \quad |\epsilon'_l \Sigma(x)| \leq \hat{\phi}^{(n)}(\hat{\phi}(x))
\]

for any \( x \in \mathbb{R}^m \) with \(|x| \leq n \) a.s. for every \( i \in \{1, \ldots, m\} \).

**Remark 3.19.** If the Osgood condition (C.11) on compact sets holds and \( \Sigma(0) = 0 \) a.s., then (C.11) follows. In particular, this implication applies to Example 3.2 when \( \varphi(0) = 0 \) and \( \zeta = 0 \).

Let us also consider two partial growth conditions on \( B \). While the first allows for various kinds of growth behaviour, the second is of affine type and explicitly measures the growth components:

(C.12) There are \( \phi, \varphi \in C(\mathbb{R}_+) \) that are positive on \([0, \infty[\) and vanish at 0 and \( \mathbb{R}^m \)-valued progressively measurable processes \( \kappa, \nu, \chi \) with locally integrable paths so that

\[
\text{sgn}(x_i) B^{(i)}(x, \mu) \leq \kappa^{(i)} + \nu^{(i)} \varphi(|x_i|) + \chi^{(i)} \varphi(\vartheta(\mu, \delta_0))
\]

for any \((x, \mu) \in \mathbb{R}^m \times \mathcal{P}\) a.s. for each \( i \in \{1, \ldots, m\} \). Moreover, \( \phi^{1/\alpha} \) is concave for some \( \alpha \in [0, 1] \), \( \varphi \) is increasing and \( E[|\kappa|_1], E[|\nu|_1] \), \( E[|\chi|_1] \) are locally integrable.

(C.13) There are \( l \in \mathbb{N}, \alpha, \beta \in [0, 1]^l \) and progressively measurable processes, \( \kappa, \nu \) and \( \chi \) with values in \( \mathbb{R}^m_+, \mathbb{R}^{m \times m \times l}_+ \) and \( \mathbb{R}^{m \times l}_+ \), respectively, such that

\[
\text{sgn}(x_i) B^{(i)}(x, \mu) \leq \kappa^{(i)} + \sum_{k=1}^l \left( \sum_{j=1}^m \nu^{(i,j,k)} |x_j|^{a_k} \right) + \chi^{(i,k)} \vartheta(\mu, \delta_0)^{\beta_k}
\]

for all \((x, \mu) \in \mathbb{R}^m \times \mathcal{P}\) a.s. for any \( i \in \{1, \ldots, m\} \). Further, \( \kappa, \nu \) and \( \chi \) have locally integrable paths and it holds that \( \nu^{(i,j,k)} \geq 0 \) if \( i \neq j \), and

\[
E[\kappa^{(i)}], \quad [\nu^{(i,j,k)}]^{1/1-\alpha_k}, \quad E[\chi^{(i,k)}]
\]

are locally integrable for any \( i, j \in \{1, \ldots, m\} \) and \( k \in \{1, \ldots, l\} \).

If (C.12) holds, we introduce for given \( \beta \in [0, 1] \) two measurable locally integrable functions by

\[
g := \alpha \left[ |v|_1 \right]^{1-\alpha} + \beta E[|\chi|_1] \quad \text{and} \quad h := (1-\alpha) \left[ |v|_1 \right]^{1-\alpha} + (1-\beta)E[|\chi|_1]
\]

and infer a quantitative first moment estimate from Theorem 4.5, which reduces to an explicit bound in the framework of Example 3.6.

**Lemma 3.20.** Let (C.11) and (C.12) be valid and \( X \) be a solution to (1.2) for which \( E[|X_{t_0}|] < \infty \) and \( E[|\chi|_1] \vartheta(\mathcal{L}(X), \delta_0) \) is locally integrable. Define \( \varphi_0 \in C(\mathbb{R}_+) \) by

\[
\varphi_0(v) := \varphi(v)^{\frac{1}{\beta}} \vee \varphi(v)^{\frac{1}{\alpha}}
\]

and suppose that \( \Phi_{\varphi_0}(\infty) = \infty \). Then \( E[|X|_1] \) is locally bounded and

\[
\sup_{s \in [t_0, t]} E[|X_s|_1] \leq \Psi_{\varphi_0} \left( E[|X_{t_0}|_1] + \int_{t_0}^t E[|\kappa_s|_1] + h(s) ds, \int_{t_0}^t g(s) ds \right)
\]

for all \( t \geq t_0 \). In particular, if \( E[|\kappa|_1] \), \( g \) and \( h \) are integrable, then \( E[|X|_1] \) is bounded.
Under (C.13), two measurable locally integrable functions \( g_1 : [t_0, \infty] \to [0, \infty] \) and \( h_1 : [t_0, \infty] \to [0, \infty] \) can be defined by

\[
g_1(s) := \max_{j=1, \ldots, m} \sum_{k=1}^l \alpha_k \left[ \sum_{i=1}^m v_s^{(i,j,k)} \right] + \beta_k \sum_{i=1}^m E[\chi_s^{(i,k)}] \quad (3.13)
\]

and

\[
h_1(s) := \sum_{k=1}^l (1-\alpha_k) \left[ \sum_{i=1}^m v_s^{(i,j,k)} \right] + (1-\beta_k) \sum_{i=1}^m E[\chi_s^{(i,k)}]. \quad (3.14)
\]

These formulas are in spirit the same as those for the stability coefficients in (3.5) and (3.6), since the subsequent *explicit* \( L^1 \)-growth bound follows, just as the \( L^1 \)-comparison estimate in Proposition 3.11 from Theorem 4.6.

**Lemma 3.21.** Let (C.11) and (C.13) be satisfied and \( X \) be a solution to (1.2) such that \( E[|X_{t_0}|] < \infty \) and \( \sum_{k=1}^l \sum_{i=1}^m E[\chi_s^{(i,k)}] \vartheta(L(X), \delta_0)^{\beta_k} \) is locally integrable. Then

\[
E[|X_t|] \leq \int_{t_0}^t g_1(s) \, ds E[|X_{t_0}|] + \int_{t_0}^t E[|X_s|] \, ds (E[|\kappa_s|] + h_1(s)) \, ds \quad (3.15)
\]

for each \( t \geq t_0 \). In particular, suppose that \( g_1^+, E[|\kappa|] \) and \( h_1 \) are integrable. Then \( E[|X_t|] \) is bounded, and \( \lim_{t \to \infty} E[|X_t|] = 0 \) as soon as \( \int_{t_0}^\infty g_1^+(s) \, ds = \infty \).

**Example 3.22.** Let \( B \) be the representation (3.11) of Example 3.15. Then (C.13) is implied by the following conditions:

1. There are \( \alpha_0, \beta_0 \in [0, 1], \alpha \in [0, 1]^l \) and \( \nu \in \mathbb{R}^{m \times l} \) and \( \nu^* \)-valued progressively measurable processes \( \kappa, \nu \) and \( \chi \) with locally integrable paths such that

\[
\text{sgn}(v) f_{i,k}(v) \leq \nu_{i,k}|v|^{\alpha_k} \quad \text{and} \quad |G^{(i)}(x,\mu)| \leq \kappa^{(i)} + \nu^{(i)}|x|^{\alpha_0} + \chi^{(i)} \vartheta(\mu, \delta_0)^{\beta_0}
\]

for all \( v \in \mathbb{R}, (x,\mu) \in \mathbb{R}^m \times \mathcal{P}, i \in \{1, \ldots, m\} \) and \( k \in \{1, \ldots, l\} \). Moreover, \( E[|\kappa_t|], \sum_{k=1}^l \sum_{j=1}^m |\eta_{i,k}^{(j,k)} \nu_{i,k}| \leq \frac{E[|\nu_t|]}{1-\alpha_0}, \frac{E[|\nu_t|]}{1-\alpha_0}, \frac{E[|\chi_t|]}{1-\beta_0} \) and \( E[|\chi_t|] \) are locally integrable.

Certainly, we may choose \( \nu = 0 \) in (1) whenever \( f_{i,k} \geq 0 \) on \( [0, \infty) \) and \( f_{i,k} \leq 0 \) on \( [0, \infty) \) for all \( i \in \{1, \ldots, m\} \) and \( k \in \{1, \ldots, l\} \). Moreover, if (C.11) holds for \( \Sigma \) and (1) is satisfied, then two facts follow from Lemma 3.21.

2. For any solution \( X \) to (1.2) for which \( E[|X_{t_0}|] < \infty \) and \( E[|\kappa_t|] \vartheta(L(X), \delta_0)^{\beta_0} \) is locally integrable the bound (3.15) holds for

\[
g_1 = \max_{j=1, \ldots, m} \nu_j + \alpha_0 \frac{|\nu_t|}{1-\alpha_0} + \beta_0 E[|\chi_t|]
\]

and

\[
h_1 = \left( \sum_{k=1}^l (1-\alpha_k) \sum_{i=1}^m \eta_{i,k}^{(j,k)} \frac{|\nu_{i,k}|}{1-\alpha_k} \right) + \left( 1 - \alpha_0 \right) m \frac{|\nu_t|}{1-\alpha_0} + \left( 1 - \beta_0 \right) E[|\chi_t|],
\]

where the measurable map \( \nu : [t_0, \infty] \to [0, \infty] \) is defined by

\[

\]

(3) In particular, \( E[|X_t|] \) is bounded whenever \( \nu_1^+, \ldots, \nu_m^+, \frac{|\nu_t|}{1-\alpha_0} \) and \( E[|\chi_t|] \) are integrable. If in addition

\[

\int_{t_0}^{\infty} \min_{j=1, \ldots, m} \nu_j^{-}(s) \, ds = \infty, \quad \text{then} \quad \lim_{t \to \infty} E[|X_t|] = 0.
\]

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3.4 Strong solutions with locally bounded absolute moment functions

In what follows, let $b : [t_0, \infty[\times \mathbb{R}^m \times \mathcal{P} \to \mathbb{R}^m$ and $\sigma : [t_0, \infty[\times \mathbb{R}^m \to \mathbb{R}^{m \times d}$ be Borel measurable and $\xi : \Omega \to \mathbb{R}^m$ be $\mathcal{F}_{t_0}$-measurable. We derive a strong solution $X$ to (2.4) such that $X_{t_0} = \xi$ a.s. and the measurable absolute moment function

$$[t_0, \infty[ \to [0, \infty[, \quad t \mapsto E(|X_t|)]$$

is finite and locally bounded but not necessarily continuous, by combining the preceding results with a fixed-point approach. Thereby, neither $b(s, \cdot, \mu)$ nor $\sigma(s, \cdot)$ need to be locally Lipschitz continuous for any $(s, \mu) \in [t_0, \infty[\times \mathcal{P}$.

For a Borel measurable map $\mu : [t_0, \infty[ \to \mathcal{P}$ we define an $\mathbb{R}^m$-valued measurable map $b_\mu$ on $[t_0, \infty[\times \mathbb{R}^m$ by $b_\mu(s, x) := b(s, x, \mu(s))$ and show that the induced SDE

$$dX_t = b_\mu(t, X_t) \, dt + \sigma(t, X_t) \, dW_t \quad \text{for } t \geq t_0$$

admits a solution. To this end, $\sigma$ should vanish at the origin of $\mathbb{R}^m$ at any time and satisfy an Osgood continuity condition on compact sets:

(D.1) $\sigma(\cdot, 0) = 0$ and for any $n \in \mathbb{N}$ there are an increasing $\hat{\rho}_n \in C(\mathbb{R}_+)$ and a measurable locally bounded function $\hat{\eta}_n : [t_0, \infty[ \to \mathbb{R}_+$ such that

$$\hat{\rho}_n > 0 \text{ on } ]0, \infty[, \quad \int_0^1 \frac{1}{\hat{\rho}_n(v)^2} \, dv = \infty \quad \text{and} \quad |\epsilon_i(\sigma(\cdot, x) - \sigma(\cdot, \tilde{x}))| \leq \hat{\eta}_n \hat{\rho}_n(|x_i - \tilde{x}_i|)$$

for all $x, \tilde{x} \in \mathbb{R}^m$ with $|x| \lor |\tilde{x}| \leq n$ for all $i \in \{1, \ldots, m\}$.

Remark 3.23. This condition still allows for Example 3.2 when $\Sigma = \sigma$, $\varphi(0) = 0$, $\zeta = 0$ and $\eta = \hat{\eta}$ for some measurable locally bounded map $\hat{\eta} : [t_0, \infty[ \to \mathbb{R}^{d \times l}$.

We introduce a partial growth condition and a continuity and boundedness condition as well as a partial Osgood condition on compact sets on $b$:

(D.2) There are $\phi, \varphi \in C(\mathbb{R}_+)$ that are positive on $]0, \infty[$ and vanish at 0 and measurable locally integrable maps $\kappa, v, \chi : [t_0, \infty[ \to \mathbb{R}_+^m$ such that

$$\text{sgn}(x_i) b_i(\cdot, x, \mu) \leq \kappa_i + v_i \phi(|x|_1) + \chi_i \varphi(\vartheta(\mu, \delta_0))$$

for all $(x, \mu) \in \mathbb{R}^m \times \mathcal{P}$ and $i \in \{1, \ldots, m\}$. Further, $\phi$ is concave, $\varphi$ is increasing and $\int_0^1 \frac{1}{\varphi(v)} \, dv = \infty$.

(D.3) $b(s, \cdot, \mu)$ is continuous for all $(s, \mu) \in [t_0, \infty[ \times \mathcal{P}$ and for any $n \in \mathbb{N}$ there is $c_n \geq 0$ such that

$$|b(s, x, \mu)| \leq c_n$$

for each $(s, x, \mu) \in [t_0, t_0 + n] \times \mathbb{R}^m \times \mathcal{P}$ with $|x| \leq n$ and $\vartheta(\mu, \delta_0) \leq n$.

(D.4) For each $n \in \mathbb{N}$ there are a concave $\rho_n \in C(\mathbb{R}_+)$ that is positive on $]0, \infty[$ and a measurable locally integrable function $\eta_n : [t_0, \infty[ \to \mathbb{R}_+$ satisfying

$$\int_0^1 \frac{1}{\rho_n(v)} \, dv = \infty$$

and

$$\text{sgn}(x_i - \tilde{x}_i)(b_i(\cdot, x, \mu) - b_i(\cdot, \tilde{x}, \mu)) \leq \eta_n \rho_n(|x - \tilde{x}|_1)$$

for all $x, \tilde{x} \in \mathbb{R}^m$ with $|x| \lor |\tilde{x}| \leq n$, $\mu \in \mathcal{P}$ and $i \in \{1, \ldots, m\}$.

Let $B_{b,loc}(\mathcal{P})$ be the set of all Borel measurable maps $\mu : [t_0, \infty[ \to \mathcal{P}$ for which $\vartheta(\mu, \delta_0)$ is locally bounded. By means of a local weak existence result from [25] and Corollary 3.7 we settle questions of uniqueness and existence of solutions to (3.16).
Proposition 3.24. For $\mu \in B_{b,loc}(\mathcal{P})$ the following three assertions hold:

(i) Let $\{D.1\}$ and $\{D.4\}$ be valid. Then pathwise uniqueness for the SDE $3.16$ follows.

(ii) Let $\{D.1\} \land \{D.3\}$ hold. Then $3.16$ admits a weak solution $X$ with $\mathcal{L}(X_{t_0}) = \mathcal{L}(\xi)$. Further, if $\xi$ is integrable, then $\vartheta_1(\mathcal{L}(X), \delta_0)$ is locally bounded.

(iii) If $\{D.1\} \land \{D.4\}$ are satisfied, then there is a unique strong solution $X^{\xi,\mu}$ to $3.16$ such that $X^{\xi,\mu}_{t_0} = \xi$ a.s.

In the sequel, let $B_{b,loc}(\mathcal{P}_1(\mathbb{R}^m))$ denote the convex space of all Borel measurable maps $\mu : [t_0, \infty[ \to \mathcal{P}_1(\mathbb{R}^m)$ for which $\int_{\mathbb{R}^m} |x| \mu(dx)$ is locally bounded, equipped with the topology of local uniform convergence.

That means, a sequence $(\mu_n)_{n \in \mathbb{N}}$ in this space converges locally uniformly to some $\mu \in B_{b,loc}(\mathcal{P}_1(\mathbb{R}^m))$ if $\lim_{n \to \infty} \sup_{s \in [t_0, t]} \vartheta_1(\mu_n, \mu)(s) = 0$ for every $t \geq t_0$. Then, as the Wasserstein metric $\vartheta_1$ is complete, $B_{b,loc}(\mathcal{P}_1(\mathbb{R}^m))$ is completely metrisable.

To construct a solution to $2.4$ from a local uniform limit of a Picard iteration in $B_{b,loc}(\mathcal{P}_1(\mathbb{R}^m))$, we strengthen the partial Osgood condition $\{D.4\}$ on compact sets by a partial Lipschitz condition on $b$:

$\{D.5\}$ There are measurable locally integrable maps $\eta$ and $\lambda$ on $[t_0, \infty[$ with values in $\mathbb{R}^{m \times m}$ and $\mathbb{R}_+^m$, respectively, such that

$$\text{sgn}(x_i - \tilde{x}_i)(b_i(\cdot, x, \mu) - b_i(\cdot, \tilde{x}, \tilde{\mu})) \leq \left( \sum_{j=1}^m \eta_{i,j} |x_j - \tilde{x}_j| \right) + \lambda_i \vartheta(\mu, \tilde{\mu})$$

for any $x, \tilde{x} \in \mathbb{R}^m$, $\mu, \tilde{\mu} \in \mathcal{P}$ and $i \in \{1, \ldots, m\}$. Moreover, we have $\eta_{i,j} \geq 0$ for all $i, j \in \{1, \ldots, m\}$ with $i \neq j$.

If $\{D.5\}$ is satisfied, which implies $\{C.6\}$ for $B = b$, then we use the formula in $8.9$ when $\lambda = 0$ for the definition of $\gamma_{1,0} : [t_0, \infty[ \to \mathbb{R}$. Namely, we set

$$\gamma_{1,0} := \max_{j=1, \ldots, m} \eta_{1,j} + \cdots + \eta_{m,j}.$$

Further, if the following partial affine growth condition for $b$ holds, which is stronger than $\{D.2\}$, then we can deduce an estimate for the Picard iteration.

$\{D.6\}$ There are $l \in \mathbb{N}$, $\alpha, \beta \in ]0, 1[^l$ and measurable locally integrable maps $\kappa, v$ and $\chi$ on $[t_0, \infty[$ with values in $\mathbb{R}_+^l$, $\mathbb{R}^{m \times m \times l}$ and $\mathbb{R}^{m \times l}$, respectively, such that

$$\text{sgn}(x_i) b_i(\cdot, x, \mu) \leq \kappa_i + \sum_{k=1}^l \left( \sum_{j=1}^m v_{i,j,k} |x_j|^{\alpha_k} \right) \chi_{i,k} \vartheta(\mu, \delta_0)^{\beta_k}$$

for any $(x, \mu) \in \mathbb{R}^m \times \mathcal{P}$ and $i \in \{1, \ldots, m\}$. In addition, it holds that $v_{i,j,k} \geq 0$ for all $i, j \in \{1, \ldots, m\}$ with $i \neq j$ and $k \in \{1, \ldots, l\}$.

Since $\{D.6\}$ entails $\{C.13\}$ for $B = b$, we may use the formulas $8.13$ and $8.14$ for the functions $g_1$ and $h_1$ when all appearing coefficients are deterministic. Namely,

$$g_1 = \max_{j=1, \ldots, m} \sum_{k=1}^l \alpha_k \left( \left( \sum_{i=1}^m v_{i,j,k} \right)^+ - \left( \sum_{i=1}^m v_{i,j,k} \right)^- \right) \chi_{i,k} + \beta_k \sum_{i=1}^m \chi_{i,k}$$

and

$$h_1 = \sum_{k=1}^l (1 - \alpha_k) \left( \sum_{j=1}^m \left( \sum_{i=1}^m v_{i,j,k} \right)^+ \right) + (1 - \beta_k) \sum_{i=1}^m \chi_{i,k}.$$

Based on these considerations, we obtain a strong existence result with explicit error and growth estimates.
Theorem 3.25. Let (D.1), (D.3) and (D.5) hold, \( \mathcal{P}_t(\mathbb{R}^m) \subseteq \mathcal{P} \), \( \mu_0 \in B_{b,loc}(\mathcal{P}) \) and \( E[|\xi_1|] < \infty \). Further, define \( \Theta : [t_0, \infty) \times \mathcal{P} \times \mathbb{R} \to \mathbb{R}_+ \) by \( \Theta(t, \cdot, \mu, \tilde{\mu}) := |\lambda|_1 \vartheta(\mu, \tilde{\mu}) \).

(i) Then pathwise uniqueness for (2.1) relative to \( \Theta \) holds and there exists a unique strong solution \( X^\xi \) to (2.1) such that \( X^\xi_0 = \xi \) a.s. and \( E[|X^\xi(t)|] \) is locally bounded.

(ii) The map \( [t_0, \infty[ \to \mathcal{P}_t(\mathbb{R}^m) \), \( t \mapsto \mathcal{L}(X^\xi_0) \) is the local uniform limit of the sequence \((\mu_n)_{n \in \mathbb{N}} \) in \( B_{b,loc}(\mathcal{P}_t(\mathbb{R}^m)) \) recursively given by \( \mu_n := \mathcal{L}(X^\xi_{\mu_{n-1}}) \)

\[
\sup_{s \in [t_0, t]} \vartheta(\mu_n(s), \mathcal{L}(X^\xi_s)) \leq \Delta(t) \sum_{i=0}^{n-1} \frac{1}{i!} \left( \int_{t_0}^{t} e^{\int_{s}^{t} \gamma_i(s) ds} |\lambda(s)|_1 ds \right)^i
\]

for all \( t \geq t_0 \) with \( \Delta(t) := \sup_{s \in [t_0, t]} \vartheta(\mathcal{L}(X^\xi_{\mu_0}), \mu_0)(s) \).

(iii) If in fact (D.6) holds, then \((\mu_n)_{n \in \mathbb{N}} \) is a sequence in the closed convex space \( M \) of all \( \mu \in B_{b,loc}(\mathcal{P}_t(\mathbb{R}^m)) \) satisfying

\[
\vartheta(\mu(t), \delta_0) \leq e^{\int_{t_0}^{t} g_i(s) ds} E[|\xi|_1] + \int_{t_0}^{t} e^{\int_{s}^{t} g_i(s) ds} (|\kappa_1 + h_1|)(s) ds
\]

for each \( t \geq t_0 \) as soon as \( \mu_0 \in M \).

Remark 3.26. For \( \mu_0 = \delta_0 \) it holds that \( \Delta(t) \leq \sup_{s \in [t_0, t]} E[|X^\xi_{\delta_0}|] \) for any \( t \geq t_0 \). If instead \( \mu_0 = \mathcal{L}(X^\xi) \), then \( \Delta = 0 \) and \( \mu_n = \mu_0 \) for all \( n \in \mathbb{N} \).

Let us conclude with a deterministic variant of Example 3.15.

Example 3.27. Let \( l \in \mathbb{N} \), \( \eta : [t_0, \infty[ \to \mathbb{R}_+^{m \times l} \) be measurable and locally integrable, \( f : \mathbb{R} \to \mathbb{R}^{m \times l} \) be continuous and \( \hat{g} : [t_0, \infty[ \times \mathbb{R}^{m \times l} \to \mathbb{R}^m \) be measurable such that

\[
b_i(s, x, \mu) = \eta_{i,1}(s)f_{i,1}(x_i) + \ldots + \eta_{i,l}(s)f_{i,l}(x_i) + \hat{g}_i(s, x, \mu)
\]

for any \((s, x, \mu) \in [t_0, \infty[ \times \mathbb{R}^{m} \times \mathcal{P} \) and \( i \in \{1, \ldots, m\} \). Then the following three assertions hold:

(1) Suppose that \( \tilde{\eta} \in \mathbb{R}^{m \times l} \) and \( \tilde{\eta}, \lambda : [t_0, \infty[ \to \mathbb{R}_+^m \) are measurable locally integrable maps such that \( \text{sgn}(v - \hat{v})(f_{i,k}(v) - f_{i,k}(\hat{v})) \leq \tilde{\eta}_{i,k}|v - \hat{v}| \) and

\[
|\hat{g}_i(\cdot, x, \mu) - \hat{g}_i(\cdot, \hat{x}, \hat{\mu})| \leq \tilde{\eta}_i|x - \hat{x}| + \lambda_i \vartheta(\mu, \hat{\mu})
\]

for all \( v, \hat{v} \in \mathbb{R}, x, \hat{x} \in \mathbb{R}^m, \mu, \hat{\mu} \in \mathcal{P}, i \in \{1, \ldots, m\} \) and \( k \in \{1, \ldots, l\} \). Then the partial Lipschitz condition (D.5) for \( b \) is valid.

(2) Let \( \hat{v} \in \mathbb{R}^{m \times l} \) and \( \kappa, \bar{\tau}, \chi : [t_0, \infty[ \to \mathbb{R}_+^m \) be measurable locally integrable maps satisfying \( \text{sgn}(v)f_{i,k}(v) \leq \bar{v}_{i,k}|v| \) and

\[
|\hat{g}_i(\cdot, x, \mu)| \leq \kappa_i + \bar{\tau}_i|x|_1 + \chi_i \vartheta(\mu, \delta_0)
\]

for any \( v \in \mathbb{R}, (x, \mu) \in \mathbb{R}^{m} \times \mathcal{P}, i \in \{1, \ldots, m\} \) and \( k \in \{1, \ldots, l\} \). Then the partial affine growth condition (D.6) for \( b \) is satisfied.

(3) Let the conditions in (1) and (2) hold and the maps \( \eta, \kappa, \bar{\tau}, \chi \) be actually locally bounded. Then the continuity and boundedness condition (D.3) for \( b \) follows.
For example, if \( f_{i,k} \) is decreasing, \( f_{i,k} \geq 0 \) on \([-\infty, 0[\) and \( f_{i,k} \leq 0 \) on \](0, \infty[\) for all \( i \in \{1, \ldots, m\} \) and \( k \in \{1, \ldots, l\} \), then \( \eta = \tilde{\nu} = 0 \) is feasible in (1) and (2). More specifically, we may take
\[
f_{i,1}(v) = -v^{n_i,1}, \ldots, f_{i,l}(v) = -v^{n_i,l}
\]
for all \( v \in \mathbb{R} \) and \( i \in \{1, \ldots, m\} \) and some \( n \in \mathbb{N}^{m \times l} \) with odd coordinates. In general, if \( \sigma \) satisfies (12), \( \mathcal{P}_1(\mathbb{R}^m) \subseteq \mathcal{P} \), \( E[|\xi|_1] < \infty \) and the conditions in (1)-(3) hold, then all assertions of Theorem 3.25 apply and the coefficients reduce to
\[
\gamma_{1,0} = \max_j \eta_j + |\eta_1|, \quad g_1 = \max_j \nu_j + |\nu_1| + |\chi_1| \quad \text{and} \quad h_1 = 0
\]
with the measurable locally bounded maps \( \eta_j, \nu_j : [t_0, \infty[ \rightarrow \mathbb{R}^m \) coordinatewise given by
\[
\eta_j := \sum_{k=1}^l \eta_{i,k} \tilde{\nu}_{j,k} \quad \text{and} \quad \nu_j := \sum_{k=1}^l \eta_{j,k} \tilde{\nu}_{j,k}.
\]

4 Moment and pathwise asymptotic estimations for random Itô processes

4.1 Auxiliary moment bounds

In the sequel, let \( B \) and \( \Sigma \) be two progressively measurable processes with values in \( \mathbb{R}^m \) and \( \mathbb{R}^{m \times d} \), respectively, such that \( \int_{t_0} \|B_s\| + \|\Sigma_s\|^2 \, ds < \infty \). We will derive quantitative \( L^1 \)-estimates for an \( \mathbb{R}^m \)-valued adapted continuous process \( Y \) satisfying
\[
Y = Y_{t_0} + \int_{t_0}^T B_s \, ds + \int_{t_0}^T \Sigma_s \, dW_s \quad \text{a.s.,}
\]
which we call a random Itô process with drift \( B \) and diffusion \( \Sigma \). First, let us recall an approximation of the identity function on \( \mathbb{R}_+ \), used by Yamada and Watanabe [37] to prove pathwise uniqueness for SDEs.

For any \( i \in \{1, \ldots, m\} \) and each increasing \( \hat{\rho}_i \in C(\mathbb{R}_+) \) that is positive on \([0, \infty[\) and satisfies \( \int_0^1 \hat{\rho}_i(v)^{-2} \, dv = \infty \), there are a strictly decreasing zero sequence \( (a_{i,n})_{n \in \mathbb{N}_0} \) in \([0,1] \) and an increasing sequence \( (\psi_{i,n})_{n \in \mathbb{N}} \) of non-negative functions in \( C^2(\mathbb{R}_+) \) such that
\[
\psi_{i,n} \in [0,1], \quad \psi'_{i,n} = \psi'_{i,n} 1_{[0,a_{i,n}-1]} + 1_{[a_{i,n}-1,\infty[} \quad \text{and} \quad 0 \leq \psi''_{i,n} \leq \frac{2}{n} \hat{\rho}_i^{-2} 1_{[a_{i,n},a_{i,n-1}]} \]
for any \( n \in \mathbb{N} \) and hence, \( \psi_{i,n}(0) = \psi'_{i,n}(0) = \psi''_{i,n}(0) = 0 \). These conditions ensure that \( \sup_{n \in \mathbb{N}} \psi_{i,n}(x) = x \) and \( \lim_{n \to \infty} \psi_{i,n}(x) = 1 \) for all \( x > 0 \), which we combine with an application of Itô’s formula.

**Lemma 4.1.** Let \( \psi \in C^2(\mathbb{R}_+) \) satisfy \( \psi'(0) = \psi''(0) = 0 \) and \( U \) be an \( \mathbb{R}^m \)-valued adapted locally absolutely continuous process. Then
\[
u \psi(|U'| Y) = u(t_0) \psi(u_{t_0}^U Y_{t_0}) + \int_{t_0}^T \psi(|U'_s Y_s|) \, du(s)
\]
\[
+ \int_{t_0}^T u(s) \left( \psi'(|U'_s Y_s|) \text{sgn}(U'_s Y_s) (U'_s Y_s + U'_s B_s) + \frac{1}{2} \psi''(|U'_s Y_s|) |U'_s Y_s|^2 \right) \, ds
\]
\[
+ \int_{t_0}^T u(s) \psi'(|U'_s Y_s|) \text{sgn}(U'_s Y_s) U'_s \Sigma_s \, dW_s \quad \text{a.s.}
\]
for any continuous function \( u : [t_0, \infty[ \rightarrow \mathbb{R} \) that is locally of bounded variation.
Proof. We define \( \varphi \in C^{2,2}(\mathbb{R}^m \times \mathbb{R}^m) \) by \( \varphi(x, y) := \psi(|x' y|) \) with first- and second-order derivatives with respect to the first coordinate

\[
D_x \varphi(x, y) = \psi'(|x' y|) \text{sgn}(x' y)yy' \quad \text{and} \quad D_x^2 \varphi(x, y) = \psi''(|x' y|)yy'
\]

for any \( x, y \in \mathbb{R}^m \). Its first- and second-order derivatives relative to the second coordinate satisfy \( D_y \varphi(x, y) = D_x \varphi(y, x) \) and \( D_y^2 \varphi(x, y) = D_x^2 \varphi(y, x) \), as \( \varphi(x, y) = \varphi(y, x) \). Thus,

\[
\varphi(U, Y) = \varphi(U_0, Y_0) + \int_{t_0}^t D_x \varphi(Y_s, U_s) \Sigma_s dW_s \\
+ \int_{t_0}^t D_x \varphi(U_s, Y_s) \hat{U}_s + D_x \varphi(Y_s, U_s)B_s + \frac{1}{2} \text{tr}(D_x^2 \varphi(Y_s, U_s) \Sigma_s \Sigma_s') \quad \text{a.s.,}
\]

by Itô’s formula. As \( u \) is locally of bounded variation, the asserted identity follows from Itô’s product rule. \( \square \)

Now we can state an auxiliary moment estimate.

**Proposition 4.2.** Suppose that \( Z \) and \( \hat{\eta} \) are two progressively measurable processes with values in \( \mathbb{R}^m \) and \( \mathbb{R}_+ \), respectively, and \( \tau \) is a stopping time satisfying \( \int_{t_0}^t |Z_s| + \hat{\eta}_s^2 \) for all \( s \in [t_0, \tau] \) a.s. for any \( \eta \in \{1, \ldots, m\} \). If \( u : [t_0, \infty] \rightarrow \mathbb{R}_+ \) is locally absolutely continuous, then

\[
E[u(t \wedge \tau)|Y_t|] \leq u(t_0)E[|Y_{t_0}|] \\
+ E\left[ \int_{t_0}^{t \wedge \tau} \hat{u}(s)|Y_s| + u(s) \sum_{i=1}^m Z_s^{(i)} \mathbb{1}_{\{Y_s^{(i)} \neq 0\}} ds \right]
\]

(4.1)

for each \( t \geq t_0 \) for which \( \int_{t_0}^{t \wedge \tau} \hat{u}(s)|Y_s| + u(s) \sum_{i=1}^m Z_s^{(i)} \mathbb{1}_{\{Y_s^{(i)} \neq 0\}} ds \) is integrable.

**Proof.** For fixed \( n \in \mathbb{N} \) we infer from the preceding lemma that the \( \mathbb{R}_+ \)-valued adapted continuous process \( X^{(n)} := \sum_{i=1}^m \psi_{i,n}(|Y^{(i)}|) \) is a semimartingale satisfying

\[
u(\cdot \wedge \tau) X_{t \wedge \tau}^{(n)} \leq u(t_0)X_{t_0}^{(n)} + \int_{t_0}^{t \wedge \tau} \hat{u}(s)X_s^{(n)} + u(s) \left( \sum_{i=1}^m \psi_{i,n}(|Y_s^{(i)}|)Z_s^{(i)} + \frac{m}{n} \hat{\eta}_s^2 \right) ds \\
+ \int_{t_0}^{t \wedge \tau} u(s) \sum_{i=1}^m \psi_{i,n}(|Y_s^{(i)}|) \text{sgn}(Y_s^{(i)})e_s^{(i)} \Sigma_s dW_s \quad \text{a.s.}
\]

Let us now assume that \( E[|Y_{t_0}|] < \infty \), as otherwise the right-hand terms in (4.1) are infinite. In this context, we readily notice that \( E[X_{t_0}^{(n)}] \leq E[|Y_{t_0}|] \) and

\[
\sum_{i=1}^m \psi'_{i,n}(|Y_s^{(i)}|) \text{sgn}(Y_s^{(i)})e_s^{(i)} \Sigma_s \leq \hat{\eta}_s \sum_{i=1}^m \psi_{i,n}(|Y_s^{(i)}|) \mathbb{1}_{\{Y_s^{(i)} \neq 0\}} \quad \text{for all } s \in [t_0, \tau] \ \text{a.s.}
\]

So, we define a stopping time by \( \tau_k := \inf\{t \geq t_0 \mid |Y_t| \geq k \text{ or } \int_{t_0}^t |Z_s| + \hat{\eta}_s^2 \) for given \( k \in \mathbb{N} \) to get that

\[
E[u(t \wedge \tau_k)X_{t \wedge \tau_k}^{(n)}] \leq u(t_0)E[X_{t_0}^{(n)}] \\
+ E\left[ \int_{t_0}^{t \wedge \tau_k} \hat{u}(s)X_s^{(n)} + u(s) \left( \sum_{i=1}^m \psi_{i,n}(|Y_s^{(i)}|)Z_s^{(i)} + \frac{m}{n} \hat{\eta}_s^2 \right) ds \right]
\]

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for fixed $t \geq t_0$. By monotone and dominated convergence, we may take the limit $n \uparrow \infty$ to deduce (4.1) when $\tau$ is replaced by $\tau_k$, as $(X^{(n)})_{n \in \mathbb{N}}$ is an increasing sequence converging pointwise to $|Y|_1$.

Finally, Fatou’s lemma, another application of the dominated convergence theorem and the fact that $\sup_{k \in \mathbb{N}} \tau_k = \tau$ give the claimed bound, under the stated integrability condition.

**Remark 4.3.** If there are $i \in \{1, \ldots, m\}$, a measurable function $\rho : \mathbb{R}_+ \to \mathbb{R}$ and a progressively measurable process $\alpha$ such that $\rho(0) = 0$ and $Z^{(i)} = \alpha + \rho(|Y^{(i)}|)$, then

$$Z_s^{(i)} \mathbb{1}_{\{Y_s^{(i)} \neq 0\}} \leq \alpha_s^+ + \rho(|Y_s^{(i)}|) \quad \text{for all } s \geq t_0.$$ 

We recall a Burkholder-Davis-Gundy inequality for stochastic integrals driven by $W$ from [23][Theorem 7.3]. Namely, for $p \geq 2$ let $\overline{\mathbb{m}}_p := (p^{p+1}/(2(p-1)^{p-1}))^{p/2}$, if $p > 2$, and $\overline{\mathbb{m}}_p := 4$, if $p = 2$. Then

$$E \left[ \sup_{\hat{a} \in [t_0, t]} \left| \int_{t_0}^{\hat{a}} X_s \, dW_s \right|^p \right] \leq \overline{\mathbb{m}}_p E \left[ \left( \int_{t_0}^{t} |X_s|^2 \, ds \right)^{\frac{p}{2}} \right]$$

(4.2)

for each $\mathbb{R}^{m \times d}$-valued progressively measurable process $X$ and any $t \geq t_0$ for which $\int_{t_0}^{t} |X_s|^2 \, ds < \infty$. Now we conclude with an auxiliary moment estimate in the supremum norm.

**Proposition 4.4.** Let $p \geq 1$, $Z$ be an $\mathbb{R}^m$-valued progressively measurable process with locally integrable paths, $\hat{\eta} : [t_0, \infty[ \to \mathbb{R}^m_+$ be measurable and locally square-integrable and $\tau$ be a stopping time such that

$$\text{sgn}(Y_s^{(i)})B_s \leq Z_s^{(i)} \quad \text{on } \{Y_s^{(i)} \neq 0\} \quad \text{and} \quad |e_i^T \Sigma_s| \leq \hat{\eta}_i(s)\hat{\rho}_i(|Y_s^{(i)}|)$$

for any $s \in [t_0, \tau]$ a.s. for all $i \in \{1, \ldots, m\}$. Then any locally absolutely continuous function $u : [t_0, \infty[ \to \mathbb{R}_+$ satisfies

$$E \left[ \left( \sup_{s \in [t_0, t]} (u(s \wedge \tau)|Y_s^{(i)}|_1 - u(t_0)|Y_{t_0}|_1) \right)^{\frac{1}{p}} \right]$$

$$\leq E \left[ \left( \int_{t_0}^{t \wedge \tau} (\hat{u}(s)|Y_s|_1 + u(s) \sum_{i=1}^{m} Z_s^{(i)} \mathbb{1}_{\{Y_s^{(i)} \neq 0\}}) \, ds \right)^{\frac{1}{p}} \right]$$

$$+ \left( \int_{t_0}^{t} |\hat{\eta}_i(s)|^2 \, ds \right)^{\frac{1}{p}} \overline{\mathbb{m}}_p \int_{t_0}^{t} |u(s)|^{p_0} E \left[ \hat{\rho}(|Y_s|_1)^{p_0} \mathbb{1}_{\{\tau > s\}} \right] \, ds$$

(4.3)

for all $t \geq t_0$ with $p_0 := p \vee 2$ and $\hat{\rho} := \max_{i=1, \ldots, m} \hat{\rho}_i$.

**Proof.** For given $k, n \in \mathbb{N}$ we set $\tau_k := \inf \{t \geq t_0 \mid |Y_t|_1 \geq k \text{ or } \int_{t_0}^{t} |Z_s| \, ds \geq k \} \wedge \tau$ and $X^{(n)} := \sum_{i=1}^{m} \psi_{i,n}(|Y_i^{(i)}|)$. Then Lemma [4,1] and Minkowski’s inequality show that

$$E \left[ \left( \sup_{s \in [t_0, t]} (u(s \wedge \tau_k)X_s^{(n)}(s \wedge \tau_k) - u(t_0)X_{t_0}^{(n)}) \right)^{\frac{1}{p}} \right]$$

$$\leq E \left[ \left( \int_{t_0}^{t \wedge \tau_k} (\hat{u}(s)X_s^{(n)} + u(s) \sum_{i=1}^{m} \psi_{i,n}(|Y_s^{(i)}|)Z_s^{(i)}) \, ds \right)^{\frac{1}{p}} \right]$$

$$+ \frac{1}{n} \int_{t_0}^{t} u(s)\sum_{i=1}^{m} |\hat{\eta}_i(s)|^2 \, ds + E \left[ \left( \sup_{s \in [t_0, t]} I_{s \wedge \tau_k}^{(n)} \right)^{\frac{1}{p}} \right]$$

(4.4)
for fixed $t \geq t_0$, where $I^{(n)}$ denotes a continuous local martingale with $I^{(n)}_{t_0} = 0$ that is indistinguishable from the stochastic integral

$$
\int_{t_0}^{t} u(s) \sum_{i=1}^{m} \psi_{i,n}(Y^{(i)}_s) \text{sgn}(Y^{(i)}_s) e_i^t \Sigma_s \, dW_s.
$$

Moreover, as $\sum_{i=1}^{m} \psi_{i,n}(Y^{(i)}_s) |e_i^t \Sigma_s| \leq |\hat{\eta}(s)|_1 \hat{\rho}(|Y_s|_1)$ for all $s \in [t_0, t]$ with $s < \tau_k$ a.s., it follows from Hölder’s inequality, (4.2) and Jensen’s inequality that

$$
\frac{1}{mp} E \left[ \sup_{s \in [t_0, t]} |I^{(n)}_{s \wedge \tau_k}|^p \right]^{\frac{1}{p}} \leq E \left[ \int_{t_0}^{t \wedge \tau_k} |\hat{\eta}(s)|_1^2 u(s)^2 \hat{\rho}(|Y_s|_1) \, ds \right]^{\frac{1}{2p}} \leq \left( \int_{t_0}^{t} |\hat{\eta}(s)|_1^2 \, ds \right)^{\frac{1}{p} - 1} \int_{t_0}^{t} |\hat{\eta}(s)|_1^2 u(s)^p \hat{\rho}(|Y_s|_1)^p \, ds \right].
$$

(4.5)

Now recall that any sequence $(x_n)_{n \in \mathbb{N}}$ of real-valued functions on $[t_0, t]$ and each function $x : [t_0, t] \to \mathbb{R}$ such that $x(s) \leq \liminf_{n \uparrow \infty} x_n(s)$ for all $s \in [t_0, t]$ satisfies

$$
\sup_{s \in [t_0, t]} x(s) \leq \liminf_{n \uparrow \infty} \sup_{s \in [t_0, t]} x_n(s).
$$

In combination with Fatou’s lemma, this shows that (4.3) follows when $\tau$ is replaced by $\tau_k$ from (4.4), (4.5) and dominated convergence. As $\sup_{k \in \mathbb{N}} \tau_k = \tau$, monotone convergence yields the asserted estimate. \hfill \square

### 4.2 Quantitative first moment estimates

To deduce an $L^1$-estimate based on Bihari’s inequality from the results of Section 4.1, we fix $l \in \mathbb{N}$ and $\alpha, \beta \in [0, 1]^l$ and introduce two assumptions on the random Itô process $Y$:

(A.1) For any $n \in \mathbb{N}$ there are increasing $\hat{\rho}_{i,n}, \ldots, \hat{\rho}_{m,n} \in C(\mathbb{R}_+)$ and an $\mathbb{R}_+$-valued progressively measurable process $\hat{\eta}^{(n)}$ with locally square-integrable paths so that

$$
\hat{\rho}_{i,n} > 0 \text{ on } [0, \infty[, \quad \frac{1}{\hat{\rho}_{i,n}(v)} \, dv = \infty \quad \text{and} \quad \sqrt{e_i^t \Sigma_s} \leq \hat{\eta}^{(n)}_s \hat{\rho}_{i,n}(Y^{(i)}_s)
$$

for all $s \geq t_0$ with $|Y_s|_1 \leq n$ a.s. for every $i \in \{1, \ldots, m\}$.

(A.2) There exist $\rho_1, \ldots, \rho_l, \varrho_1, \ldots, \varrho_l \in C(\mathbb{R}_+)$, a measurable map $\theta : [0, \infty[ \to \mathbb{R}_+^l$ and an $\mathbb{R}_+^{m \times l}$-valued process $\kappa$ and two $\mathbb{R}_+^{m \times l}$-valued processes $\eta, \lambda$ that are all progressively measurable and have locally integrable paths such that

$$
\text{sgn}(Y^{(i)}) B^{(i)} \leq \kappa^{(i)} + \sum_{k=1}^{l} \eta^{(i,k)} \rho_k(|Y|_1) + \lambda^{(i,k)} \varrho_k \circ \theta_k \text{ a.s.}
$$

for each $i \in \{1, \ldots, m\}$. Further, $\rho_k, \varrho_k$ are positive on $[0, \infty[$ and vanish at $0$, $\hat{\rho}_{i,n}^{-1}$ is concave, $\varrho_k$ is increasing and

$$
E[|\kappa|_1], \quad \left[ \sum_{i=1}^{m} \eta^{(i,k)} \right]_{1-a_k}, \quad \sum_{i=1}^{m} E[\lambda^{(i,k)}]
$$

are locally integrable for all $k \in \{1, \ldots, l\}$.
For a measurable map $\theta : [t_0, \infty] \to \mathbb{R}_+$ and an $\mathbb{R}_+^m$-valued progressively measurable process $\lambda$, as in (A.2), we rely on a domination condition:

(A.3) We have $\theta_k(s) \leq E[|Y_t|_{1}]$ for all $s \geq t_0$ with $\sum_{i=1}^{m} E[\lambda_s^{(i,k)}] > 0$ for any $k \in \{1, \ldots, l\}$.

If (A.2) is satisfied, then we may define two measurable locally integrable functions $\gamma, \delta : [t_0, \infty] \to [0, \infty]$ via

$$
\gamma(s) := \sum_{k=1}^{l} \alpha_k \left[ \sum_{i=1}^{m} \eta_s^{(i,k)} \right] + \beta_k \sum_{i=1}^{m} E[\lambda_s^{(i,k)}] \\
\delta(s) := \sum_{k=1}^{l} (1 - \alpha_k) \left[ \sum_{i=1}^{m} \eta_s^{(i,k)} \right] + (1 - \beta_k) \sum_{i=1}^{m} E[\lambda_s^{(i,k)}].
$$

We also recall the definitions (3.2) and (3.3). This leads to a general estimation result.

**Theorem 4.5.** Let (A.1), (A.3) hold, $E[|Y_{t_0}|_{1}] < \infty$, $\sum_{k=1}^{l} \sum_{i=1}^{m} E[\lambda_s^{(i,k)}] \theta_k \circ \theta_k$ be locally integrable and $\rho_0, \varrho_0 \in C(\mathbb{R}_+)$ be defined by

$$
\rho_0(v) := \max_{k=1, \ldots, l} \rho_k(v)^{\frac{1}{\alpha_k}} \quad \text{and} \quad \varrho_0(v) := \rho_0(v) \vee \max_{k=1, \ldots, l} \rho_k(v)^{\frac{1}{\alpha_k}}.
$$

If $\Phi_{\rho_0}(\infty) = \infty$ or $\sum_{k=1}^{l} \sum_{i=1}^{m} E[\eta_s^{(i,k)} \rho_k(|Y_t|_{1})]$ is locally integrable, then $E[|Y|_{1}]$ is locally bounded and

$$
\sup_{s \in [t_0, t]} E[|Y_s|_{1}] \leq \Psi_{\varrho_0} \left( E[|Y_{t_0}|_{1}] + \int_{t_0}^{t} E[|\kappa_s|_{1}] + \delta(s) \, ds, \int_{t_0}^{t} \gamma(s) \, ds \right)
$$

for any $t \in [t_0, t_0^+]$, where $t_0^{+} > t_0$ denotes the supremum over all $t \geq t_0$ for which

$$
\left( E[|Y_{t_0}|_{1}] + \int_{t_0}^{t} E[|\kappa_s|_{1}] + \delta(s) \, ds, \int_{t_0}^{t} \gamma(s) \, ds \right) \in D_{\varrho_0}.
$$

**Proof.** We introduce the stopping time $\tau_n := \inf\{t \geq t_0 \mid |Y_t|_{1} \geq n\}$ for fixed $n \in \mathbb{N}$ and set $\kappa := E[|\kappa_s|_{1}] + \sum_{k=1}^{l} \sum_{i=1}^{m} E[\lambda_s^{(i,k)}] \theta_k \circ \theta_k$. Then Proposition 4.2 and Remark 4.3 imply that

$$
E[|Y_{t_0}|_{1}] \leq E[|Y_{t_0}|_{1}] + \int_{t_0}^{t} \kappa(s) + \sum_{k=1}^{l} \sum_{i=1}^{m} E[\eta_s^{(i,k)} \rho_k(|Y_{t_{0}}|_{1}) 1_{\{\tau_n > s\}}] \, ds
$$

(4.6)

given $t \geq t_0$. Thereby, we notice that the local integrability of the measurable function $[t_0, \infty] \to [0, \infty], s \mapsto \sum_{i=1}^{m} E[\eta_s^{(i,k)} \rho_k(|Y_{t_{0}}|_{1}) 1_{\{\tau_n > s\}}]$ follows from (3.1), which yields

$$
\sum_{i=1}^{m} E[\eta_s^{(i,k)} \rho_k(|Y_{t_{0}}|_{1}) 1_{\{\tau_n > s\}}] \leq \left[ \sum_{i=1}^{m} \eta_s^{(i,k)} \right] \left( 1 - \alpha_k + \alpha_k \rho_k(E[|Y_{t_{0}}|_{1}]) \right)^{\frac{1}{\alpha_k}}
$$

(4.7)

for all $s \in [t_0, t]$, each $k \in \{1, \ldots, l\}$ and every stopping time $\tau$ for which $E[|Y_{\tau}|_{1}]$ is finite, because $\rho_k^{1/\alpha_k}$ is concave, by assumption.

Thus, let us set $\delta := \sum_{k=1}^{l} \{1 - \alpha_k\}|\sum_{i=1}^{m} \eta_s^{(i,k)}|^{1-\alpha_k}$. If $\Phi_{\rho_0}(\infty) = \infty$ holds, then we apply Bihari’s inequality to (4.6) and infer from Fatou’s lemma that

$$
E[|Y_1|_{1}] \leq \liminf_{n \to \infty} E[|Y_{t_0}|_{1}] \leq \Psi_{\rho_0} \left( E[|Y_{t_0}|_{1}] + \int_{t_0}^{t} (\kappa + \delta)(s) \, ds, \int_{t_0}^{t} \sum_{k=1}^{l} \alpha_k \left[ \sum_{i=1}^{m} \eta_s^{(i,k)} \right] \, ds \right),
$$

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as the domain of $\Psi_{\rho_0}$ satisfies $D_\rho = \mathbb{R}^2_+$. For this reason, $E[|Y_1|]$ is locally bounded in this case. By choosing $\tau = \infty$ in (4.7), we see that it suffices to consider the case when $\sum_{k=1}^m \sum_{i=1}^m E[\eta^{(i,k)}|\rho_k(Y_1)]$ is locally integrable. Then

$$E[|Y_1|] \leq E[|Y_{t_0}|] + \int_{t_0}^{t} \left( \tilde{k} + \delta \right)(s) + \sum_{k=1}^{l} \alpha_k \left( \sum_{i=1}^{m} \eta^{(i,k)} \right) \rho_k(E[|Y_s|])^{l/\alpha_k} \, ds$$

follows from (4.5). Fatou’s lemma and (4.7). Thereby, we readily checked that $E[|Y_1|]$ is actually locally bounded. Finally, Young’s inequality gives us that

$$\sum_{i=1}^{m} E[\lambda^{(i,k)}] \theta_k \leq \sum_{i=1}^{m} E[\lambda^{(i,k)}] (1 - \beta_k + \beta_k \rho_k(E[|Y_1|])^{\gamma_k})$$

on $[t_0, t]$ for all $k \in \{1, \ldots, l\}$ and we conclude the proof with another application of Bihari’s inequality, since $\delta + \sum_{k=1}^{l} (1 - \beta_k) \sum_{i=1}^{m} E[\lambda^{(i,k)}] = \delta$. \hfill \qed

For a stability analysis we consider another condition, which explicitly measures the dependence on each coordinate and implies (A.2):

(A.4) There are a measurable map $\theta : [t_0, \infty] \to \mathbb{R}_+^l$ and progressively measurable processes $\kappa, \eta$ and $\lambda$ with values in $\mathbb{R}_+^m, \mathbb{R}^{m \times m \times l}$ and $\mathbb{R}^{m \times l}$, respectively, such that

$$\text{sgn}(Y^{(i)})B^{(i)} \leq \kappa^{(i)} + \sum_{k=1}^{l} \left( \sum_{j=1}^{m} \eta^{(i,j,k)}|Y^{(j)}|^{\alpha_k} \right) + \lambda^{(i,k)} \theta_k a.s.$$ 

for any $i \in \{1, \ldots, m\}$. Moreover, the paths of $\kappa, \eta, \lambda$ are locally integrable and we have $\eta^{(i,j,k)} \geq 0$, if $i \neq j$, and

$$E[\kappa^{(i)}], \quad E[\eta^{(i,j,k)} \theta_k^{\gamma_k}], \quad E[\lambda^{(i,k)}]$$

are locally integrable for any $i, j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, l\}$.

If (A.4) and (A.3) hold, then we may utilise the two functions $\gamma_1$ and $\tilde{\delta}_1$ given by (3.5) and (3.6) to get an explicit moment estimate.

**Theorem 4.6.** Let (A.1), (A.4), (A.3) be valid, $E[|Y_{t_0}|] < \infty$ and $\sum_{k=1}^{l} \sum_{i=1}^{m} E[\lambda^{(i,k)}] \theta_k^{\gamma_k}$ be locally integrable. Then

$$E[|Y_t|] \leq e^{\int_{t_0}^{t} \gamma_1(s) \, ds} E[|Y_{t_0}|] + \int_{t_0}^{t} e^{\int_{s}^{t} \gamma_1(\hat{s}) \, d\hat{s}} \left( E[|\kappa_s|] + \tilde{\delta}_1(s) \right) \, ds$$

for all $t \geq t_0$. In particular if $\gamma_1^+, E[|\kappa_s|]$ and $\tilde{\delta}_1$ are integrable, then $E[|Y_1|]$ is bounded. If in addition $\int_{t_0}^{\infty} \gamma_1^+(s) \, ds = \infty$, then $\lim_{t \to \infty} E[|Y_t|] = 0$.

**Proof.** First, we observe that (A.2) holds when the appearing process $\eta$ there is replaced by the $\mathbb{R}^{m \times l}$-valued process $\tilde{\eta}$ defined coordinatewise by $\tilde{\eta}^{(i,k)} := \sum_{j=1}^{m} (\eta^{(i,j,k)})^+$ and it holds that

$$\rho_k(v) = v^{\alpha_k} \quad \text{and} \quad \theta_k(v) = v^{\beta_k}$$

for all $v \geq 0$. As $\rho_0 \in C(\mathbb{R}_+)$ given by $\rho_0(v) = v$ satisfies $\Phi_{\rho_0}(\infty) = \infty$, Theorem 4.5 shows us that $E[|Y_1|]$ is locally bounded.

Thus, we define an $\mathbb{R}^{m \times l}$-valued process $\hat{\eta}$ coordinatewise by $\hat{\eta}^{(i,j,k)} := \sum_{i=1}^{m} \eta^{(i,j,k)}$. Then Proposition 4.2 and Remark 4.3 imply that

$$u(t)E[|Y_t|] \leq u(t_0)E[|Y_{t_0}|] + \int_{t_0}^{t} u(s)E[|\kappa_s|] + \hat{u}(s)E[|Y_s|] \, ds$$

$$+ \int_{t_0}^{t} u(s) \left( \sum_{k=1}^{l} \left( \sum_{j=1}^{m} E[\hat{\eta}^{(j,k)}|Y^{(j)}|^{\alpha_k}] \right) + \sum_{i=1}^{m} E[\lambda^{(i,k)}] \theta_k(s)^{\beta_k} \right) \, ds$$

(4.8)

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for fixed $t \geq t_0$ and any locally absolutely continuous function $u : [t_0, \infty[ \rightarrow \mathbb{R}_+$. From (3.1) we directly obtain that

$$E[\tilde{\eta}^{(j,k)}|Y^{(j)}|^{\alpha_k}] \leq [\tilde{\eta}^{(j,k)}] \frac{1}{1-\alpha_k} (1 - \alpha_k + \alpha_k E[|Y^{(j)}|])$$

and $\sum_{i=1}^m E[\lambda^{(i,k)}] \phi_k \leq \sum_{i=1}^m E[\lambda^{(i,k)}](1 - \beta_k + \beta_k E[|Y^{(j)}|])$ on $[t_0, t]$ for all $j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, l\}$. As a consequence,

$$\sum_{k=1}^l \left( \sum_{j=1}^m E[\tilde{\eta}^{(j,k)}|Y^{(j)}|^{\alpha_k}] \right) + \sum_{i=1}^m E[\lambda^{(i,k)}] \phi_k \leq \delta_1 + \gamma_1 E[|Y^{(j)}|]$$

a.e. on $[t_0, t]$. Thus, by choosing $u(\tilde{s}) = \exp(-\int_0^{\tilde{s}} \gamma_1(s) \, ds)$ for any $\tilde{s} \geq t_0$ in (4.8), we get the asserted estimate after dividing by $u(t)$.

Next, let us assume that $\gamma_1^+ = E[|\kappa_s|_1]$ and $\delta_1$ are integrable. Then the second assertion follows from the bound

$$\sup_{t \geq t_0} E[|Y_t|_1] \leq e^{\int_{t_0}^{\infty} \gamma_1^+(s) \, ds} \left( E[|Y_{t_0}|_1] + \int_{t_0}^{\infty} E[|\kappa_s|_1] + \delta_1(s) \right).$$

For the last claim, let additionally $\int_{t_0}^{\infty} \gamma_1^-(s) \, ds = \infty$. Then $\lim_{t \uparrow \infty} \exp(\int_{t_0}^{t} \gamma_1(s) \, ds) = 0$ for every $s \geq t_0$, by monotone convergence. Thus,

$$\lim_{t \uparrow \infty} \int_{t_0}^{t} e^{\int_{t_0}^{s} \gamma_1^-(t) \, dt} (E[|\kappa_s|_1] + \delta_1(s)) \, ds = 0$$

follows from dominated convergence, which completes the proof. \qed

**Remark 4.7.** Suppose that $\kappa = 0$ a.s., and let $\delta_1$ vanish a.e., which holds if $\alpha_k = \beta_k = 1$ for any $k \in \{1, \ldots, l\}$. If $\gamma_1^+$ is integrable, then

$$\sup_{t \geq t_0} e^{\int_{t_0}^{t} \gamma_1^-(s) \, ds} E[|Y_t|_1] \leq e^{\int_{t_0}^{\infty} \gamma_1^+(s) \, ds} E[|Y_{t_0}|_1] < \infty.$$

If additionally $\int_{t_0}^{\infty} \gamma_1^-(s) \, ds = \infty$, then from $a \gamma_1^- + \gamma_1 = \gamma_1^+ - (1-a) \gamma_1^-$ we infer that

$$\lim_{t \uparrow \infty} e^{a \int_{t_0}^{t} \gamma_1^-(s) \, ds} E[|Y_t|_1] = 0 \quad \text{for all} \quad a \in [0,1].$$

These two facts give more insight into the rate of convergence.

### 4.3 Moment estimates in the supremum norm

In this section we deduce absolute $p$-th moment estimates in the supremum norm for $p \in [1,2]$. To this end, we require a Hölder condition instead of the weaker Osgood condition (A.1) on compact sets and restrict (A.2) and (A.4) as follows:

(A.5) There exists a measurable locally square-integrable function $\hat{\eta} : [t_0, \infty[ \rightarrow \mathbb{R}_+^m$ such that $|e^i \Sigma| \leq \hat{\eta}_i |Y^{(i)}|^{\frac{1}{2}}$ a.s. for all $i \in \{1, \ldots, m\}$.

(A.6) Assumption (A.2) holds when $\rho_k(v) = v^\alpha_k$ for all $v \geq 0$ and $k \in \{1, \ldots, l\}$ and $\eta = \overline{\eta}$ for some measurable locally integrable map $\overline{\eta} : [t_0, \infty[ \rightarrow \mathbb{R}_{+}^{m \times m \times l}$.

(A.7) Assumption (A.4) is valid and there exists a measurable locally integrable map $\overline{\eta} : [t_0, \infty[ \rightarrow \mathbb{R}_{+}^{m \times m \times l}$ such that $\eta = \overline{\eta}$. 

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Under (A.5) and (A.6), we define two \([0, \infty]\)-valued measurable functions \(f_p\) and \(g_p\) on the set of all \((t_1, t) \in [t_0, \infty]^2\) with \(t_1 \leq t\) by

\[
f_p(t_1, t) := E\left[\left(\int_{t_1}^t |\kappa_s| \, ds + \sum_{k=1}^m \sum_{i=1}^l \lambda^{(i,k)}_s \varrho_k(\theta_k(s)) \, ds\right)^{p}\right] + 2\left(\int_{t_1}^t |\eta(s)| \, ds \right)^{\frac{p}{2}} E[|Y_{t_1}|^1] \left(\int_{t_1}^t |\eta(s)| \, ds \right)^{\frac{1}{2}}
\]

and

\[
g_p(t_1, t) := f_p(t_1, t) + \sum_{k=1}^l E[|Y_{t_1}|^{\alpha_k p}] \frac{1}{p} \int_{t_1}^t |\eta_{i,k}(s)| \, ds.
\]

In addition, let us set \(\alpha := \min_{k=1,\ldots,l} \alpha_k\) and \(\overline{\alpha} := \max_{k=1,\ldots,l} \alpha_k\).

**Proposition 4.8.** Let (A.5), (A.6) and (A.3) be valid, \(\sum_{k=1}^l \sum_{i=1}^m E[|\lambda^{(i,k)}| \varrho_k \circ \theta_k] \) be locally integrable and \(\rho_0 \in C(\mathbb{R}_+)\) be given by \(\rho_0(v) := v^{\alpha} \mathbb{1}_{[0,1]}(v) + v^\overline{\alpha} \mathbb{1}_{[1,\infty)}(v)\). If

\[
E[|Y_{t_0}|^p], \quad E\left[\left(\int_{t_0}^t |\kappa_s| \, ds\right)^{p}\right] \quad \text{and} \quad E\left[\left(\int_{t_0}^t \sum_{k=1}^m \lambda^{(i,k)}_s \, ds\right)^{p}\right]
\]

are finite, then \(\sup_{s \in [t_0, t]} |Y_{s}|^1\) is \(p\)-fold integrable and

\[
E\left[\left(\sup_{s \in [t_1, t]} |Y_{s}| - |Y_{t_1}|\right)^p\right] \leq \Psi_{\rho_0}(l+1)^{p-1} g_p(t_1, t)^p, (l+1)^{p-1} \sum_{k=1}^l \left(\int_{t_1}^t |\eta_{i,k}(s)| \, ds\right)^{p}\]

for any \(t_1, t \geq t_0\) with \(t_1 \leq t\). In particular, \(E[|Y|^p]\) is continuous.

**Proof.** If the integrability assertion is true, then \(\lim_{n \to \infty} E[|Y_{t_n}|^p] = E[|Y_{t_0}|^p]\) for every sequence \((t_n)_{n \in \mathbb{N}}\) in \([t_0, \infty]\) that converges to some \(t \geq t_0\), by dominated convergence. For this reason, it suffices to show the first two claims.

We set \(\tau_n := \inf\{t \geq t_1 : |Y_t|^n \geq n\}\) for given \(t_1 \geq t_0\) and \(n \in \mathbb{N}\). Then Proposition 4.4 and the inequalities of Minkowski and Jensen give

\[
E\left[\left(\sup_{s \in [t_1, t]} |Y_{s}| - |Y_{t_1}|\right)^p\right] \leq E\left[\left(\int_{t_1}^t |\kappa_s| \, ds\right)^{p}\right] + E\left[\left(\int_{t_1}^t \sum_{k=1}^m \lambda^{(i,k)}_s \varrho_k(\theta_k(s)) \, ds\right)^{p}\right]
\]

for fixed \(t \geq t_1\). We recall that \(\Phi_{\rho_0}(\infty) = \infty\) and \(D_{\rho_0} = \mathbb{R}_+^2\). Thus, if \(E[|Y_{t_1}|^p] < \infty\), then another application of Minkowski’s inequality together with Bihari’s inequality yield the asserted bound for

\[
E\left[\left(\sup_{s \in [t_1, t]} |Y_{s}| - |Y_{t_1}|\right)^p\right]
\]

and from Fatou’s lemma we readily infer the claimed result. In this context, we may use the fact that \(E[|Y_{t_1}|]\) is locally bounded, by Theorem 4.5. This ensures that \(g_p(t_1, t)\) is finite in this case.

Further, by choosing \(t_1 = t_0\) the \(p\)-fold integrability of \(\sup_{s \in [t_0, t]} |Y_{s}|\) follows from that of \(|Y_{t_0}|\) and the finiteness of \(g_p(t_0, t)\), which completes the proof.
If \((A.7)\) holds, then the definitions \((3.5)\) and \((3.6)\) for the choice \(\lambda = 0\) lead us to two measurable locally integrable functions

\[
\gamma_{1,0} := \max_{j=1,\ldots,m} \sum_{k=1}^l \alpha_k \left( \left( \sum_{i=1}^m \eta_{i,j,k} \right)^+ + \left( \sum_{i=1}^m \eta_{i,j,k} \right)^- \right) \mathbb{1}_{\{1\}}(\alpha_k)
\]

and

\[
\delta_{1,0} := \sum_{k=1}^l (1 - \alpha_k) \sum_{j=1}^m \left( \sum_{i=1}^m \eta_{i,j,k} \right)^+ .
\]

For a measurable locally integrable function \(\gamma : [t_0, \infty[ \rightarrow \mathbb{R}\) we introduce an \([0, \infty]\)-valued measurable function \(h_{\gamma,p}\) on the set of all \((t_1, t) \in [0, \infty]^2\) with \(t_1 \leq t\) by

\[
h_{\gamma,p}(t_1, t) := E \left[ \left( \int_{t_1}^t e^{-\int_s^{t_1} \gamma(s) \, ds} \left( |\kappa_s|_1 + \delta_{1,0}(s) + \sum_{k=1}^l \sum_{i=1}^m \lambda_{s,k} \theta_k(s) \right) \, ds \right)^p \right]^{\frac{1}{p}}
\]

\[
+ 2 \left( \int_{t_1}^t |\hat{\gamma}(s)|^2 e^{-2\int_s^{t_1} \gamma(s) \, ds} E[|Y_s|_1] \, ds \right)^{\frac{1}{2}}
\]

and state the analogue of Proposition 4.8 when \((A.7)\) instead of \((A.6)\) holds.

**Lemma 4.9.** Let \((A.5)\), \((A.7)\) and \((A.3)\) be satisfied and \(\sum_{k=1}^l \sum_{i=1}^m E[|\lambda^{(i,k)}| \theta_k^p] \) be locally integrable. If the expectations in \((4.9)\) are finite, then

\[
E \left[ \left( \sup_{s \in [t_1, t]} e^{-\int_s^{t_1} \gamma(s) \, ds} |Y_s|_1 - |Y_{t_1}|_1 \right)^p \right]^{\frac{1}{p}} \leq h_{\gamma,p}(t_1, t)
\]

\[
+ \left( \int_{t_1}^t (\gamma_{1,0} - \gamma)^+ (s) \, ds \right)^{\frac{1}{p}} \left( \int_{t_1}^t (\gamma_{1,0} - \gamma)^+ (s) e^{-p \int_s^{t_1} \gamma(s) \, ds} E[|Y_s|_1^p] \, ds \right)^{\frac{1}{p}}
\]

for each measurable locally integrable function \(\gamma : [t_0, \infty[ \rightarrow \mathbb{R}\) and all \(t_1, t \geq 0\) with \(t_1 \leq t\).

**Proof.** As \((A.7)\) is a special case of \((A.6)\), Proposition 4.8 entails that \(\sup_{s \in [t_0, t]} |Y_s|_1\) is \(p\)-fold integrable. Moreover, we readily see that

\[
\sum_{k=1}^l \sum_{i=1}^m \eta_{i,j,k} |Y^{(j)}| \chi^0_k \leq \delta_{1,0} + \gamma_{1,0} |Y|_1,
\]

by Young’s inequality. Hence, the claim follows immediately from Proposition 4.4 and the inequalities of Minkowski and Jensen.

We consider a last restriction that still allows for the mixed Hölder condition in \((A.3)\) on a finite time interval:

\[(A.8)\] Assumptions \((A.5)\) and \((A.7)\) hold and there are \(t_1 \geq t_0, \delta > 0\) and \(\theta_0 \geq 0\) such that

\[
\kappa^{(j)}(s) = (1 - \beta_k) \lambda^{(j,k)}(s) = 0 \quad \text{and} \quad (1 - \alpha_k) \sum_{i=1}^m \eta_{i,j,k} \leq 0 \quad \text{on} \ [t_1, \infty[\]

for any \(j \in \{1, \ldots, m\}\) and \(k \in \{1, \ldots, l\}\) and

\[
\sup_{t \geq t_1} \int_{t}^{t+\delta} \sum_{i=1}^m E[|\lambda^{(i,k)}|] \, ds \vee \int_{t}^{t+\delta} |\hat{\gamma}(s)|^2 \, ds \leq \theta_0
\]

for every \(k \in \{1, \ldots, l\}\) with \(\beta_k = 1\).
Then the following first moment estimate in the supremum norm will contribute to the pathwise asymptotic analysis of $Y$ in the next section.

**Proposition 4.10.** Let (A.8) and (A.3) hold, $E[|Y_{t_0}|] < \infty$ and $\sum_{k=1}^{l} \sum_{i=1}^{m} E[\lambda^{(i,k)}] \phi_k$ be locally integrable. Further, suppose that there is a measurable locally integrable function $\gamma : [t_0, \infty] \to \mathbb{R}$ and $\tau_{\gamma, -1}, \ldots, \tau_{\gamma, 3} \geq 0$ such that

\[
\int_{t_2}^{t} (\gamma_{t_1, 0} - \gamma) (s) ds \leq \tau_{\gamma, -1}, \quad \int_{t_2}^{t} (\gamma_{t_1, 0} - q \gamma) (s) ds \leq \tau_{\gamma, q}, \quad \int_{t_2}^{t} \gamma (s) ds \leq \tau_{\gamma, 3}
\]

for all $t_2, t \geq t_1$ with $t_2 \leq t < \hat{\delta}$ and $q \in \{0, 1, 2\}$. Then there is $\bar{c} > 0$ such that

\[
E \left[ \sup_{s \in [t_2, t]} |Y_s| \right] \leq \bar{c} \varphi \left( E[|Y_{t_0}|] + t_0 \right) + \frac{1}{2} \left( \int_{t_2}^{t} |\gamma (s)|^2 ds + \int_{t_2}^{t} \gamma (s) ds \right)^{\frac{1}{2}}
\]

(4.10)

for the function $\gamma := (\gamma_{t_1, 0} - \gamma) + \sum_{k=1, \beta_k=1}^{l} \sum_{i=1}^{m} E[\lambda^{(i,k)}]$. Further, because $\hat{\delta}_1 = \hat{\delta}_{1,0} + \sum_{k=1, \beta_k=1}^{l} \sum_{i=1}^{m} E[\lambda^{(i,k)}]$, we infer from the moment stability estimate of Theorem 4.6 that

\[
e^{-q \int_{t_2}^{t} \gamma (s) ds - c_{\gamma, q}} E[|Y_{t_0}|] \leq e^{f_{t_0}^{\tau_{\gamma, 1}} (s) ds} E[|Y_{t_0}|] + \frac{1}{2} \int_{t_0}^{t_1} e^{f_{t_0}^{\tau_{\gamma, 1}} (s) ds} (E[|\kappa_{s_0}|] + \hat{\delta}_1 (s)) ds 0
\]

for any $s \in [t_2, t]$ and each $q \in \{0, 1, 2\}$, where $c_{\gamma, q} := \tau_{\gamma, q} + \tau_0 \sum_{k=1, \beta_k=1}^{l} 1$. Hence, the first two terms on the right-hand side in (4.10) do not exceed

\[
\overline{\tau}_1 \left( E[|Y_{t_0}|] + \int_{t_0}^{t_1} E[|\kappa_{s}|] + \hat{\delta}_1 (s) ds \right) e^{\frac{1}{2} f_{t_1}^{\tau_{\gamma, 1}} (s) ds}
\]

with $\overline{\tau}_1 := \exp(f_{t_0}^{\tau_{\gamma, 1}} (s) ds + c_{\gamma, 0}/2) (1 + e^{c_{\gamma, 1}} (\tau_{\gamma, 1} + \tau_0 \sum_{k=1, \beta_k=1}^{l} 1))$. Moreover, the third expression in (4.10) is bounded by

\[
\overline{\tau}_2 \left( E[|Y_{t_0}|] + \int_{t_0}^{t_1} E[|\kappa_{s}|] + \hat{\delta}_1 (s) ds \right) e^{\frac{1}{2} f_{t_1}^{\tau_{\gamma, 1}} (s) ds}
\]

for $\overline{\tau}_2 := 2^{\frac{1}{2}} \exp(\overline{\tau}_{\gamma, 3})$. Since $\exp(-f_{t_0}^{\tau_{\gamma, 1}} (s) ds) \geq \exp(-\overline{\tau}_{\gamma, 3})$ for all $s \in [t_2, t]$, the assertion follows for $\overline{\tau} := \exp(\overline{\tau}_{\gamma, 3}) (\overline{\tau}_1 \vee \overline{\tau}_2)$.

**4.4 Pathwise asymptotic behaviour**

To aim of this section is to deduce the limiting behaviour of $Y$ from the moment estimate of Proposition 4.10 by using the following application of the Borel-Cantelli Lemma.
Lemma 4.11. Let $A \in \mathcal{F}$ and $X$ be an $\mathbb{R}_+^+$-valued right-continuous process for which there are a strictly increasing sequence $(t_n)_{n \in \mathbb{N}}$ in $[t_0, \infty]$ with $\lim_{n \uparrow \infty} t_n = \infty$ and a sequence $(c_n)_{n \in \mathbb{N}}$ in $[0, \infty]$ such that

$$\sum_{n=1}^{\infty} P \left( \sup_{s \in [t_n, t_{n+1}]} X_s I_A > c_n \right) < \infty. \tag{4.11}$$

Then for any lower semicontinuous function $\varphi : A, \infty \mapsto [0, \infty]$ it holds that

$$\limsup_{t \uparrow \infty} \frac{\log(X_t)}{\varphi(t)} \leq \limsup_{n \uparrow \infty} \frac{\log(c_n)}{\inf_{s \in [t_n, t_{n+1}]} \varphi(s)} \quad \text{a.s. on } A. \tag{4.12}$$

Proof. By the Borel-Cantelli Lemma, there is a null set $N \in \mathcal{F}$ such that for any fixed $\omega \in N^c \cap A$ there is $n_0 \in \mathbb{N}$ so that $s \in [t_n, t_{n+1}] X_s(\omega) \leq c_n$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Hence, for every $n_1 \in \mathbb{N}$ with $n_1 \geq n_0$. This in turn shows us that

$$\limsup_{t \uparrow \infty} \frac{\log(X_t(\omega))}{\varphi(t)} = \inf_{n \in \mathbb{N}; n \geq n_0} \sup_{s \geq t_n} \frac{\log(X_s(\omega))}{\varphi(s)} \leq \limsup_{n \uparrow \infty} \frac{\log(c_n)}{\inf_{s \in [t_n, t_{n+1}]} \varphi(s)}.$$ 

Remark 4.12. Suppose that instead of (4.11) there are $\hat{c} > 0$ and $\hat{\varepsilon} \in (0, 1]$ such that $E[\sup_{s \in [t_n, t_{n+1}]} X_s I_A] \leq \hat{c} n$ for every $n \in \mathbb{N}$ and

$$\sum_{n=1}^{\infty} c_n^\varepsilon < \infty \quad \text{for all } \varepsilon \in (0, \hat{\varepsilon}]. \tag{4.13}$$

Then (4.12) follows as well. Indeed, for $\varepsilon \in (0, \hat{\varepsilon}]$ Chebyshev’s inequality and Lemma 4.11 yield a null set $N_\varepsilon \in \mathcal{F}$ such that

$$\limsup_{t \uparrow \infty} \frac{\log(X_t(\omega))}{\varphi(t)} \leq (1 - \varepsilon) \limsup_{n \uparrow \infty} \frac{\log(c_n)}{\inf_{s \in [t_n, t_{n+1}]} \varphi(s)} \tag{4.14}$$

for all $\omega \in N_\varepsilon^c \cap A$. So, any $\omega \in A$ that lies in the complement of $N := \bigcup_{\varepsilon \in \mathbb{Q} \cap [0, \hat{\varepsilon}]} N_\varepsilon$ satisfies (4.14) for $\varepsilon = 0$, which is the sharpest bound.

More specifically, one may derive the conditions in Remark 4.12 in the case that there are $n_0 \in \mathbb{N}$ and a decreasing function $\psi : [n_0, \infty] \mapsto \mathbb{R}_+$ such that $c_n = \psi(n)$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Then (4.13) holds for some $\hat{\varepsilon} \in (0, 1]$ if and only if

$$\int_{n_0}^{\infty} \psi(v)\varepsilon \, dv < \infty \quad \text{for all } \varepsilon \in (0, \hat{\varepsilon}], \tag{4.15}$$

as the integral test for the convergence of series shows. We conclude with a pathwise estimate and stress the fact that the fraction $\frac{1}{2}$ comes from the Hölder condition (A.3).

Theorem 4.13. Let (A.8) and (A.3) hold and $\sum_{k=0}^{n-1} \sum_{i=1}^{m_k} E[|\lambda(i,k)|^{\frac{\tilde{\beta}_k}{2}}] < \infty$ be locally integrable. Suppose that $\gamma_1 \leq 0$ a.e. on $[t_1, \infty]$ and there is a strictly increasing sequence $(t_n)_{n \in \mathbb{N}\setminus\{1\}}$ in $[t_1, \infty]$ such that

$$\sup_{n \in \mathbb{N}} (t_{n+1} - t_n) < \hat{\delta}, \quad \lim_{n \uparrow \infty} t_n = \infty$$

and $\sum_{n=1}^{\infty} \exp((\varepsilon/2) \int_{t_1}^{t_n} \gamma_1(s) \, ds) < \infty$ for every $\varepsilon \in (0, \hat{\varepsilon}]$ and some $\hat{\varepsilon} \in (0, 1]$. If $E[|Y_{t_0}|]$ is $\infty$ or $\lambda = 0$, then

$$\limsup_{t \uparrow \infty} \frac{1}{\varphi(t)} \log(|Y_{t_1}|) \leq \frac{1}{2} \limsup_{n \uparrow \infty} \frac{1}{\varphi(t_n)} \int_{t_1}^{t_n} \gamma_1(s) \, ds \quad \text{a.s.}$$

for any increasing continuous function $\varphi : [t_1, \infty] \mapsto \mathbb{R}_+$ that is positive on $]t_1, \infty[.$
Remark 4.12. By noting that

\[ \bigcup_{n} \] holds, Proposition 4.10 gives \( \hat{c} > 0 \) a.e. on \([t_0, t_1]\) for some \d\ > 0. Thus, if \( \gamma \leq -\gamma \) a.e. on \([t_1, t_1 + \delta]\) for some \( \delta > 0 \), then

\[
\limsup_{t \uparrow \infty} \frac{\log(|Y_t|)}{\int_{t_0}^{t} \gamma(s) \, ds} \leq -\frac{1}{2} (1 - \gamma) \quad \text{a.s.}
\]

Remark 4.14. Let \( \gamma : [t_0, \infty[ \to \mathbb{R}_+ \) be a measurable locally integrable function satisfying \( \gamma > 0 \) a.e. on \([t_1, t_1 + \delta]\) for some \( \delta > 0 \). If \( \gamma \leq -\gamma \) a.e. on \([t_1, \infty[\), then

\[
\limsup_{t \uparrow \infty} \frac{\log(|Y_t|)}{\int_{t_0}^{t} \gamma(s) \, ds} \leq -\frac{1}{2} (1 - \gamma) \quad \text{a.s.}
\]

with \( \hat{\gamma} := \int_{t_0}^{t} \gamma(s) \, ds/ \int_{t_0}^{\infty} \gamma(s) \, ds \). In this case, the sharpest bound is attained for \( \hat{\gamma} = 0 \), which occurs if and only if \( \gamma = 0 \) a.e. on \([t_0, t_1]\) or \( \gamma \) fails to be integrable.

5 Proofs of the main results

5.1 Proofs for admissible spaces of probability measures

Proof of Proposition 2.2. Let \( S \) be a metrisable space, \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}) \) be a probability space and \( X : S \times \tilde{\Omega} \to \mathbb{R}^m \) be a continuous process so that \( \mathcal{L}(X_s) \in \mathcal{P} \) for all \( s \in S \). By condition (i), the sequence \( (F_n)_{n \in \mathbb{N}} \) of \( \mathcal{P}\)-valued maps on \( S \) defined via \( F_n(s) := \mathcal{L}(\varphi_n \circ X_s) \) satisfies \( \lim_{n \uparrow \infty} F_n(s) = \mathcal{L}(X_s) \) in \( \mathcal{P} \) for any given \( s \in S \).

Since \( (X_{s_k})_{k \in \mathbb{N}} \) converges pointwise to \( X_s \) for any sequence \( (s_k)_{k \in \mathbb{N}} \) in \( S \) converging to \( s \), it follows from (ii) that \( \lim_{k \uparrow \infty} F_n(s_k) = F_n(s) \) in \( \mathcal{P} \) for each \( n \in \mathbb{N} \). Thus, the map \( S \to \mathcal{P}, \ s \mapsto \mathcal{L}(X_s) \) is Borel measurable as pointwise limit of a sequence of continuous maps.

Proof of Corollary 2.4. We show that the two conditions of Proposition 2.2 are met by the sequence \( (\varphi_n)_{n \in \mathbb{N}} \) of radial retractions, introduced in Example 2.3.

Let \( \mu \in \mathcal{P} \) and note that (i) directly yields \( \mu \circ \varphi_n^{-1} \in \mathcal{P} \) for fixed \( n \in \mathbb{N} \). If we define \( \phi_n : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m \) by \( \phi_n(x) := (\varphi_n(x), x) \), then \( \theta_n := \mu \circ \phi_n^{-1} \) belongs to \( \mathcal{P}(\mu \circ \varphi_n^{-1}, \mu) \) and

\[
\int_{\mathbb{R}^m \times \mathbb{R}^m} \rho(|x - y|) \, d\theta_n(x,y) = \int_{\mathbb{R}^m} \rho(|\varphi_n(x) - x|) \, d\mu(dx),
\]

by the measure transformation formula. Since \( \rho(|\varphi_n(x) - x|) \leq \rho(|x|) \) for all \( x \in \mathbb{R}^m \), the integral on the right-hand side converges to zero as \( n \uparrow \infty \), by dominated convergence. Thus, from (ii) we infer that \( \lim_{n \uparrow \infty} \mu \circ \varphi_n^{-1} = \mu \) in \( \mathcal{P} \).

Now let \( (\mu_k)_{k \in \mathbb{N}} \) be a sequence in \( \mathcal{P} \) that converges stochastically to some \( \mu \in \mathcal{P} \). That is, there is a sequence \( (\theta_k)_{k \in \mathbb{N}} \) of Borel measures on \( \mathbb{R}^m \times \mathbb{R}^m \) such that \( \theta_k \in \mathcal{P}(\mu_k, \mu) \) for any \( k \in \mathbb{N} \) and (2.1) holds.
For fixed $n \in \mathbb{N}$ and $\psi_n : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$ given by $\psi_n(x, y) := (\varphi_n(x), \varphi_n(y))$, the measure $\hat{\theta}_k := \theta_k \circ \psi_n^{-1}$ lies in $\mathcal{P}(\mu_k \circ \varphi_n^{-1}, \mu \circ \varphi_n^{-1})$ and the measure transformation formula ensures that

$$\int_{\mathbb{R}^m \times \mathbb{R}^m} \rho(|x-y|) \, d\hat{\theta}_k(x, y) = \int_{\mathbb{R}^m \times \mathbb{R}^m} \rho(|\varphi_n(x) - \varphi_n(y)|) \, d\theta_k(x, y)$$

for all $k \in \mathbb{N}$. For given $\varepsilon > 0$ we choose $\delta > 0$ and $k_0 \in \mathbb{N}$ such that $\rho(2\delta) < \varepsilon/2$ and $\theta_k:\{(x,y) \in \mathbb{R}^m \times \mathbb{R}^m \mid |x-y| \geq \delta\} \subset (\varepsilon/2)(1+\rho(2n))^{-1}$ for any $k \in \mathbb{N}$ with $k \geq k_0$. Then the integral on the right-hand side cannot exceed $\varepsilon$ for every such $k \in \mathbb{N}$. This shows $\lim_{k \uparrow \infty} \mu_k \circ \varphi_n^{-1} = \mu \circ \varphi_n^{-1}$ in $\mathcal{P}$ and the assertion follows. \hfill $\square$

5.2 Proofs of the moment estimates, uniqueness and moment stability

Proof of Proposition 3.4. We define two progressively measurable processes $\hat{B}$ and $\hat{\Sigma}$ with values in $\mathbb{R}^m$ and $\mathbb{R}^{m \times d}$, respectively, by

$$\hat{B}_s := B_s(X_s, \mathcal{L}(X_s)) - \tilde{B}_s(\tilde{X}_s, \mathcal{L}(\tilde{X}_s)) \quad \text{and} \quad \hat{\Sigma}_s := \Sigma_s(X_s) - \Sigma_s(\tilde{X}_s). \quad (5.1)$$

Then the difference $Y$ of $X$ and $\tilde{X}$ is a random Itô process with drift $\hat{B}$ and diffusion $\hat{\Sigma}$ such that

$$\text{sgn}(Y^{(i)})\hat{B}^{(i)} \leq \varepsilon^{(i)} + \eta^{(i)}\rho(|Y|_1) + \lambda^{(i)}\rho \circ \theta \quad \text{a.s.}$$

for all $i \in \{1, \ldots, m\}$ with the measurable function $\theta := \hat{\theta}(\mathcal{L}(X), \mathcal{L}(\tilde{X}))$. Hence, the assertion follows from an application of Theorem 4.6 \hfill $\square$

Proof of Corollary 3.7. To show uniqueness in both cases, we suppose that $X$ and $\tilde{X}$ are two solutions to (1.2) such that $X_{t_0} = \tilde{X}_{t_0}$ a.s. In case (i) we require, as stated, the local integrability of the function $\Theta(\cdot, \mathcal{L}(X), \mathcal{L}(\tilde{X}), \mathcal{L}(X - \tilde{X}))$, which equals

$$E[|\lambda|_1|\phi(\hat{\theta}(\mathcal{L}(X), \mathcal{L}(\tilde{X})))] + \mathbb{I}_{|0,\infty|}(\Phi_{\rho}(\infty))\eta E[|X - \tilde{X}|_1].$$

Then $E[|X_t - \tilde{X}_t|_1]$ vanishes for all $t \geq t_0$, due to Proposition 3.4. Indeed, $\phi_0 := \rho \vee \varphi$ satisfies $(0, w) \in D_{\varphi_0}$ and $\Psi_{\varphi_0}(0, w) = 0$ for all $w \geq 0$, as $\Phi_{\varphi_0}(0) = -\infty$. So, $X = \tilde{X}$ a.s., by the continuity of paths.

In case (ii) we set $\tau_n := \inf\{t \geq t_0 \mid |X_t| \geq n \text{ or } |\tilde{X}_t| \geq n\}$ for fixed $n \in \mathbb{N}$ and observe that the difference $Y$ of $X^{\tau_n}$ and $\tilde{X}^{\tau_n}$ is a random Itô process with drift $\hat{B}$ and diffusion $\hat{\Sigma}$ given by

$$\hat{B}_s := (\hat{B}_s(X_s) - \check{B}_s(\check{X}_s))\mathbb{I}_{\{\tau_n>s\}} \quad \text{and} \quad \hat{\Sigma}_s := (\Sigma_s(X_s) - \Sigma_s(\check{X}_s))\mathbb{I}_{\{\tau_n>s\}}.$$ 

Then $\text{sgn}(Y^{(i)})\hat{B}^{(i)} \leq \eta \rho \rho(|Y|_1)$ a.s. for each $i \in \{1, \ldots, m\}$. Thus, Theorem 4.6 gives $Y = 0$ a.s. From $\sup_{n \in \mathbb{N}} \tau_n = \infty$ we conclude that $X = \tilde{X}$ a.s. \hfill $\square$

Proof of Proposition 3.11. As in the proof of Proposition 3.4, we let the two processes $\hat{B}$ and $\hat{\Sigma}$ with values in $\mathbb{R}^m$ and $\mathbb{R}^{m \times d}$, respectively, be given by (5.1). Then

$$\text{sgn}(Y^{(i)})Y^{(i)} \leq \varepsilon^{(i)} + \sum_{k=1}^l \sum_{j=1}^m \eta^{(i,j,k)}|Y^{(j)}|_{\alpha_k} + \lambda^{(i)}\rho \circ \theta(\mathcal{L}(X), \mathcal{L}(\tilde{X})) \check{B}^{(i)}$$

a.s. for any $i \in \{1, \ldots, m\}$. For this reason, all assertions are implied by Theorem 4.6 \hfill $\square$

Proof of Corollary 3.13. By Definition 2.12, the stability assertions directly follow from Proposition 3.11 in the case that $B = \hat{B}$ and $\varepsilon = 0$. \hfill $\square$
Proof of Corollary 5.14. (i) Based on Proposition 3.11 we immediately infer both claims from the reasoning in Remark 4.7.

(ii) Let $X$ and $\hat{X}$ be two solutions to (1.2) for which $E[|X_{t_0} - \hat{X}_{t_0}|]$ is finite and $E[|\lambda_1|^0(L(X), L(\hat{X}))]$ is locally integrable. Then Proposition 3.11 gives

$$E[|X_t - \hat{X}_t|] \leq \sqrt{m} e^{\int_0^t \gamma_1(s) \, ds} E[|X_{t_0} - \hat{X}_{t_0}|]$$

for each $t \geq t_0$. Thus, for the first assertion let $P(X_{t_0} \neq \hat{X}_{t_0}) > 0$, as otherwise $E[|X - \hat{X}|]$ vanishes. Then from (C.7) we get that

$$\limsup_{t \to \infty} \frac{1}{t} \log (E[|X_t - \hat{X}_t|]) \leq \limsup_{t \to \infty} \sum_{k=1}^l \hat{\lambda}_k \frac{(t - s_k)^{\alpha_k} - (t_1 - s_k)^{\alpha_k}}{t^{\alpha_l}} = \hat{\lambda}_l,$$

since $\lim_{t \to \infty} (t - s_k)^{\alpha_k} / t^{\alpha_l} = 1_{\{l\}}(k)$ for any $k \in \{1, \ldots, l\}$. Now Remark 2.13 yields the correct result.

For the second assertion it is sufficient to consider the case $l = 1$. First, we set

$$\hat{\gamma}_0 := \max_{t \in [t_0, t_1]} \exp(\int_{t_0}^t \gamma_1(s) \, ds - \hat{\lambda}_1(t - t_0)^{\alpha_1})$$

and get that

$$E[|X_t - \hat{X}_t|] \leq \sqrt{m} \hat{\gamma}_0 e^{\hat{\lambda}_1(t - t_0)^{\alpha_1}} E[|X_{t_0} - \hat{X}_{t_0}|]$$

for each $t \in [t_0, t_1]$. Since $\int_{t_0}^t \gamma_1(s) \, ds < \hat{\lambda}_1((t - s_1)^{\alpha_1} - (t_1 - s_1)^{\alpha_1})$ for all fixed $t \geq t_1$, we see that (5.2) holds if we replace $\hat{\gamma}_0$ by $\hat{\gamma} := \hat{\gamma}_0 \vee \exp(\int_{t_0}^t \gamma_1(s) \, ds - \hat{\lambda}_1(t - s_1)^{\alpha_1})$.

\[ \square \]

5.3 Proofs for pathwise stability and the moment growth estimates

Proof of Corollary 3.14. As we have seen, $Y$ is a random Itô process with drift $\hat{B}$ and diffusion $\hat{\Sigma}$ given by (5.1) when $\hat{B} = B$. Therefore, Theorem 4.13 entails the claim. \[ \square \]

For the proof of Corollary 3.17 we need to check whether the series in (C.10) converges when the upper bound for $\gamma_1$ in (C.7) is used.

Lemma 5.1. Let $l \in \mathbb{N}$, $\alpha \in [0, \infty]^l$ and $\beta, s \in \mathbb{R}^l$ satisfy $\alpha_1 < \cdots < \alpha_l$, $\beta_l < 0$ and $\max_{k=1, \ldots, l} s_k \leq t_1$ for some $t_1 \geq 0$. Then

$$\int_0^\infty \exp \left( \varepsilon \sum_{k=1}^l \beta_k \int_{t_1}^{t_1 + \delta t} \alpha_k(s - s_k)^{\alpha_k - 1} \, ds \right) \, dt < \infty \quad \text{for all } \delta, \varepsilon > 0.$$ 

Proof. It suffices to consider the case $\varepsilon = 1$, since $\beta$ may be replaced by $\varepsilon \beta$. We set

$$c := - \sum_{k=1}^l \beta_k (t_1 - s_k)^{\alpha_k}$$

and readily note that there is $t_2 > 0$ such that

$$\sum_{k=1}^l \beta_k \int_{t_1}^{t+\delta t} \alpha_k(s - s_k)^{\alpha_k - 1} \, ds = c + \sum_{k=1}^l \beta_k (t + \delta t - s_k)^{\alpha_k} \leq \frac{\beta_l}{2} (\delta t)^{\alpha_l}$$

for all $t \geq t_2$. Further, a substitution shows us that $\int_0^\infty e^{-c(\delta t)^{\alpha_l}} \, dt = \frac{1}{\alpha_l} e^{-\frac{1}{\alpha_l} \Gamma(\frac{1}{\alpha_l})}$ for any $c > 0$, where $\Gamma$ is the gamma function. So, the claim follows. \[ \square \]

Proof of Corollary 3.17. To apply Proposition 3.16 we verify (C.10). For this purpose, we choose $\hat{t}_1 \geq t_1$, such that $\gamma_1 \leq 0$ a.e. on $[\hat{t}_1, \infty]$ and define a sequence $(t_n)_{n \in \mathbb{N} \setminus \{1\}}$ in $[t_0, \infty]$ by $t_n := \hat{t}_1 + \delta(n - 1)$ for some $\delta > 0$. The integral test for the convergence of series, recalled in [4.15], shows that

$$\sum_{n=1}^\infty e^{\varepsilon \int_{t_n}^{t_1} \gamma_1(s) \, ds} < \infty \quad \Leftrightarrow \quad \int_0^\infty \exp \left( \varepsilon \int_{t_n}^{t_1 + \delta t} \gamma_1(s) \, ds \right) \, dt < \infty$$

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for any given \( \varepsilon > 0 \) and the latter condition is always satisfied, due to the imposed upper bound on \( \gamma_1 \) in (C.7) and Lemma 5.1.

Thus, for (C.10) to be valid, it suffices to take \( \delta < \delta \). Then, by Proposition 3.16 the difference \( Y \) of any two solutions \( X \) and \( \tilde{X} \) to (1.2) for which \( E[|\lambda|] \theta(L(X), L(\tilde{X})) \) is locally integrable satisfies

\[
\limsup_{t \uparrow \infty} \frac{\log (|Y_t|)}{m_t} \leq \frac{1}{2} \limsup_{n \uparrow \infty} \sum_{k=1}^{\hat{t}} \hat{\lambda}_k \frac{(t_n - s_k)\alpha_k - (\hat{t}_{k-1} - s_k)\alpha_k}{m_t} = \frac{\hat{\lambda}_1}{2} \text{ a.s.}
\]
as soon as \( E[|Y_0|] < \infty \) or \( B \) is independent of \( \mu \) in \( P \).

**Proof of Lemma 3.20.** Because \( X \) is a random Itô process with drift \( \hat{B} \) and \( \hat{\Sigma} \) defined by \( \hat{B} := B(X, \mathcal{L}(X)) \) and \( \hat{\Sigma} := \Sigma(X) \), the claim is a direct consequence of Theorem 4.3.

**Proof of Lemma 3.27.** By the same reasoning as in Lemma 3.20 the assertions follow from an application of Theorem 4.6.

### 5.4 Derivation of strong solutions

**Proof of Proposition 3.24.** (i) As the partial uniform continuity condition (C.3) holds in the case that \( B = b_{\mu} \), pathwise uniqueness for (3.16) is implied by Corollary 3.7.

(ii) We shall first suppose that \( \xi \) is essentially bounded. Then the support of \( \mathcal{L}(\xi) \) is compact and it essentially follows from Theorem 2.3 in [25][Chapter IV] that there is a local weak solution \( \tilde{X} \) to (3.16).

By using the one-point compactification of \( \mathbb{R}^m \), we can view \( \tilde{X} \) as a \( \mathbb{R}^m \cup \{ \infty \} \)-valued adapted continuous process on a filtered probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{P}) \) that allows for an \( (\tilde{\mathcal{F}}_t)_{t \geq 0} \)-Brownian motion \( \tilde{W} \) such that the usual and the following conditions are satisfied:

1. If \( \tilde{X}_s(\omega) = \infty \) for some \( (s, \omega) \in [t_0, \infty] \times \tilde{\Omega} \), then \( \tilde{X}_t(\omega) = \infty \) for all \( t \geq s \).
2. \( \mathcal{L}(\tilde{X}_{t_0}) = \mathcal{L}(\xi) \) and the supremum \( \tau \) of the sequence \( (\tau_n)_{n \in \mathbb{N}} \) of stopping times given by \( \tau_n := \inf \{ t \geq t_0 | |\tilde{X}_t| \geq n \} \) satisfies \( \tau > t_0 \) a.s.
3. For any \( n \in \mathbb{N} \) the process \( \tilde{X}^{\tau_n} \) is a solution to (1.2) relative to \( \tilde{W} \) when \( B \) and \( \Sigma \) are replaced by the admissible maps

\[
[t_0, \infty] \times \tilde{\Omega} \times \mathbb{R}^m \to \mathbb{R}^m, \quad (s, \tilde{\omega}, x) \mapsto b_{\mu}(s, x) 1_{\{\tau_n > s\}}(\tilde{\omega})
\]

and \( [t_0, \infty] \times \tilde{\Omega} \times \mathbb{R}^m \to \mathbb{R}^{m \times d}, \quad (s, \tilde{\omega}, x) \mapsto \sigma(s, x) 1_{\{\tau_n > s\}}(\tilde{\omega}) \), respectively.

We readily observe that (D.2) implies the partial growth condition (C.4) for \( b_{\mu} \) instead of \( B \). In fact, we may define \( \kappa_{\mu} : [t_0, \infty] \to \mathbb{R}^m \) coordinatewise by \( (\kappa_{\mu})_i := \kappa_i + \chi_i(\varphi(\mu, \delta_0)) \) and get that

\[
\text{sgn}(x_i)(b_{\mu})_i(s, x) 1_{\{\tau_n > s\}} \leq (\kappa_{\mu})_i(s) + \nu_i(s) \phi(|x|)
\]

for any \( (s, x) \in [t_0, \infty] \times \mathbb{R}^m \), all \( i \in \{1, \ldots, m\} \) and each stopping time \( \hat{\tau} \). Further, \( |e'_i(\sigma(s \cdot)) 1_{\{\tau_n > s\}}| \leq |e'_i(\sigma(s \cdot))| \). Hence, we infer from Fatou’s lemma that

\[
E[|\tilde{X}^{\tau_n}_t|] \leq \liminf_{n \uparrow \infty} E[|\tilde{X}^{\tau_n}_t|] \leq \Psi(\hat{E}[|\xi|]) + \int_{t_0}^{t} |\kappa_{\mu}(s)|_1 ds, \int_{t_0}^{t} |\nu(s)|_1 ds \tag{5.3}
\]

for all \( t \geq t_0 \), by the virtue of Lemma 3.20. In particular, \( \tau = \infty \) and \( \tilde{X} \in \mathbb{R}^m \tilde{P}\)-a.s. So, \( X : [t_0, \infty] \times \Omega \to \mathbb{R}^m \) given by \( X_t(\tilde{\omega}) := \tilde{X}_t(\tilde{\omega}) \), if \( \tau(\tilde{\omega}) = \infty \), and \( X_t(\tilde{\omega}) := 0 \), otherwise, is a weak solution to (3.16) and \( E[|X|] \) is locally bounded.
Now we remove the boundedness hypothesis on $\xi$. As we have shown that to each $x \in \mathbb{R}^m$ there is a weak solution to (3.16) with initial value condition $x$, it follows from Remark 2.1 in [24] (Chapter IV) that there exists a weak solution $X$ to (3.16) satisfying $X_{t_0} = \xi$ a.s. Further, its absolute moment function is locally bounded if $\xi$ is integrable, as it cannot exceed the right-hand estimate in (3.3) in this case, due to Lemma 3.20.

(iii) By the first two assertions, we have pathwise uniqueness for (3.16) and there is a weak solution for any $\mathbb{R}^m$-valued $\mathcal{F}_{t_0}$-measurable random vector serving as initial condition. As postulated by Theorem 1.1 in [25] (Chapter IV), there is a unique strong solution $X^{\xi,\mu}$ with initial condition $\xi$. Therefore, (3.17) follows from (5.5) by taking the limit

\[ \lim_{m \to \infty} \vartheta_m(t) = \vartheta(t), \]

for each $t \geq t_0$. We notice that the function $[s, \infty[ \to \mathbb{R}_+, t \mapsto \exp(\int_s^t \gamma_{1,0}(\lambda(s)) \, ds) |\lambda(s)|_1$ is increasing for any $s \geq t_0$. Consequently, Gronwall’s inequality entails that $\Psi$ admits at most a unique fixed-point of the operator

\[ \Psi : B_{b,loc}(\mathcal{P}) \to B_{b,loc}(\mathcal{P}_1(\mathbb{R}^m)), \quad \Psi(\nu)(t) := \mathcal{L}(X_t^{\xi,\nu}). \]

In this case, $X^{\xi,\mu}$ must be a strong solution. For any $\xi, \eta \in B_{b,loc}(\mathcal{P})$ we readily see that condition (C.5) holds for $(b, b)$ instead of $(B, B)$. For this reason, Proposition 3.11 yields that

\[ \vartheta_1(\Psi(\mu), \Psi(\eta))(t) \leq \mathbb{E}[|X_t^{\xi,\mu} - X_t^{\xi,\eta}|] \leq \int_{t_0}^t e^{\int_s^t \gamma_{1,0}(\lambda(s)) \, ds} |\lambda(s)|_1 \vartheta(\mu, \eta)(s) \, ds \]  

(5.4)

for each $t \geq t_0$. We notice that the function $[s, \infty[ \to \mathbb{R}_+, t \mapsto \exp(\int_s^t \gamma_{1,0}(\lambda(s)) \, ds) |\lambda(s)|_1$ is increasing for any $s \geq t_0$. Consequently, Gronwall’s inequality entails that $\Psi$ admits at most a unique fixed-point.

As $B_{b,loc}(\mathcal{P}_1(\mathbb{R}^m))$ is completely metrisable, existence and the error estimate (3.17), which implies the local uniform convergence assertion, follow from an application of the fixed-point theorem for time evolution operators in [29]. In fact,

\[ \vartheta_1(\mu_m, \mu_n)(t) \leq \Delta(t) \sum_{i=0}^{m-1} \frac{1}{i} \left( \int_{t_0}^t e^{\int_s^t \gamma_{1,0}(\lambda(s)) \, ds} |\lambda(s)|_1 \, ds \right)^i \]  

(5.5)

for any $m, n \in \mathbb{N}$ with $m > n$ and $t \geq t_0$, by induction and the triangle inequality. So, $(\mu_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $B_{b,loc}(\mathcal{P}_1(\mathbb{R}^m))$ and, due to (5.4), its limit $\mu$ is a fixed-point of $\Psi$. Hence, (3.11) follows from (3.17) by taking the limit $m \uparrow \infty$.

(iii) Since the bound in (3.15) is independent of $\mu \in B_{b,loc}(\mathcal{P}_1(\mathbb{R}^m))$, the set $M$ is closed and convex. We set $g_{1,1} := \sum_{k=1}^m \beta_k \sum_{i=1}^n \chi_{i,k}$ and $g_{1,0} = g_1 - g_{1,1}$. Then

\[ \vartheta_1(\Psi(\mu)(t), h_0) \leq e^{\int_{t_0}^t g_{1,0}(s) \, ds} \mathbb{E}[|\xi_1|] + \int_{t_0}^t e^{\int_s^t g_{1,0}(\lambda(s)) \, ds} (|\xi_1| + h_1 + g_{1,1} \vartheta_1(\mu, h_0))(s) \, ds \]

for all $\mu \in B_{b,loc}(\mathcal{P})$ and $t \geq t_0$, by Lemma 3.21 and Young’s inequality. Then it follows from the Fundamental Theorem of Calculus for Lebesgue-Stieltjes integrals and Fubini’s theorem that $\Psi$ maps $M$ into itself. Hence, the claim holds.

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