A PROBLEM OF BOUNDARY CONTROLLABILITY FOR A PLATE

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Abstract. The boundary controllability problem, here discussed, might be described by a two-dimensional space equation modeling, at the same time \( t \), different physical phenomena in a composite solid made of different materials. These phenomena may be governed, at the same time \( t \), for example, by the heat equation and by the Schrödinger equation in two separate regions. Interface transmission conditions are imposed.

1. Introduction. Recently, in the joint paper [1] with Walter Littman we have investigated a null boundary controllability problem of two partial differential equations, modeling a composite solid with different physical properties in each layer. That problem was described by a one-dimensional space equation, in divergence form, for functions \( u = u(x,t) \), of the type

\[
    u_t = (c(x)u_x)_x, \quad x \in \mathbb{R}, \quad t > 0,
\]

being \( c(x) = a \) for \( x > 0 \) and \( c(x) = b \) for \( x < 0 \), with \( a \) and \( b \) real or complex numbers different from zero. The function \( u \) was subjected to initial condition and proper interface conditions were assumed. Here in this paper, I would like to extend such an investigation to a two-dimensional space equation for functions \( u = u(x,y,t) \), \( (x,y) \in \mathbb{R}^2, \ t > 0 \).

Actually, we will deal with a null boundary controllability problem related to the following equation:

\[
    u_t = c(x,y)\left[u_{xx} + u_{yy}\right], \quad \text{in} \quad \mathbb{R}^2 \times (0, +\infty),
\]

where

\[
    c(x,y) = \begin{cases} 
    a & \text{in the half-plane } x < 0, \text{ for any } y \in \mathbb{R} \\
    b & \text{in the half-plane } x > 0, \text{ for any } y \in \mathbb{R}
    \end{cases}
\]

being \( a \) and \( b \) real or complex numbers different from zero and different each other, in general. For instance, if \( a = 1 \) and \( b = i \), equation (1) may be modeling phenomena in a composite solid comprising two parts governed, at same time \( t \), one of them by the heat equation and the other by the Schrödinger equation. The function \( u = u(x,y,t) \) is required to satisfy the initial condition:

\[
    u(x,y,0) = f(x,y),
\]

with \( f(x,y) \) a function with compact support in \( \mathbb{R}^2 \).

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The following interface conditions are required to be satisfied for \( t > 0 \):

\[
u(0^-, y, t) = \alpha u(0^+, y, t), \quad \text{for any} \ y \in \mathbb{R}, \tag{3}\]

with \( \alpha \in \mathbb{R} \) (possibly \( \alpha \neq 1 \)) and

\[
k_1 u_x(0^-, y, t) = k_2 u_x(0^+, y, t), \quad \text{for any} \ y \in \mathbb{R}, \tag{4}\]

with \( k_1, k_2 \) physical constants \( \neq 1 \), different from each other, in general.

Further, for \( t > 0 \):

\[
u(x, y, t) = h(x, y, t), \quad (x, y) \in \partial \Omega, \tag{5}\]

where \( \Omega \) will be properly chosen later. For the above described problem (1) – (5), we are interested in the null boundary controllability of the system, that is, given \( T > 0 \), we look for proper control functions \( h \) such that, given initial data \( f \) in the appropriate space, the solution \( u(x, y, t) \) of the system vanishes for \( t \geq T \).

As was pointed out in [1], since the 70’s a large number of authors have studied boundary controllability problems and applied different methods. In this context, it is worthy to recall results of meaningful interest as those, among the others, by Avalos and Lasiecka [3] on null-controllability and by Lasiecka and Triggiani [5], [6] where the method of multipliers was employed to obtain boundary controllability results for the Schrödinger equation. Similar null controllability problems for one-dimensional models where also analyzed in [14] and [15].

Nevertheless, in [7] Littman presented a general direct method for boundary control for beams and plates. In the framework of [7], “direct methods” were used subsequently by Littman and Taylor [8], [9], [10].

Now, in this paper, I will follow Littman’s approach [7], paying attention to the four steps of the procedure. Accordingly, section 2 will be devoted to the study of the pure initial value problem for (1) (step 1). Then, in section 3 the second step will be described; precisely, the obtained solution \( u = u(x, y, t) \) will be multiplied by a cut-off function. Section 4 will deal with the third step solving related side-ways Cauchy problems. In section 5 the control functions \( h_i \) will be found.

## 2. The pure initial value problem (1)-(4). To start let us recall the assumptions and the notations we are going to use.

We denote \( X_- \) and \( X_+ \), respectively, the half-plane \( x < 0 \) and the half-plane \( x > 0 \).

We are able to prove the following theorem.

**Theorem 2.1.** Let \( a, b \in \mathbb{C}, \ Re \ a > 0, \ Re \ b > 0, \alpha, k_1, k_2 \in \mathbb{R}, \alpha k_1 a^{-1/2} + k_2 b^{-1/2} \neq 0; \) let \( Y = (0, y) \in \mathbb{R}^2, \ f \in C^0(\mathbb{R}^2) \) with compact support, \( f|_Y = 0, \int_{\mathbb{R}} f(x, y)dy = 0, \) for every \( x \in \mathbb{R}. \) The problem

\[
\frac{\partial u}{\partial t} = a \Delta u \quad \text{in} \quad x < 0, \tag{6}
\]

\[
\frac{\partial u}{\partial t} = b \Delta u \quad \text{in} \quad x > 0, \tag{7}
\]

\[
u(x, y, 0) = f(x, y), \quad (x, y) \in \mathbb{R}^2 \tag{8}
\]

\[
u(0^-, y, t) = \alpha u(0^+, y, t), \tag{9}
\]

\[
k_1 \frac{\partial u}{\partial x}(0^-, y, t) = k_2 \frac{\partial u}{\partial x}(0^+, y, t), \quad y \in \mathbb{R}, t > 0. \tag{10}
\]

has a solution

\[
u(x, y, t) = \mathcal{F}^{-1} [v(x, (\cdot), t)](y)
\]
(here $F^{-1}$ is the inverse Fourier transform) where
\[ v(x, \xi, t) = \mathcal{L}^{-1}[U(x, \xi, \cdot)](t) \]
(\text{here $\mathcal{L}^{-1}$ is the inverse Laplace transform}) and
\[
U(x, \xi, s) = \alpha \frac{a^{-1}k_1 \int_{0}^{\infty} \phi(x_3, \xi) e^{(x_3 + x_3) \sqrt{s^2 + s/b}} dx_3}{\alpha k_1 \sqrt{\xi^2 + s/a} + k_2 \sqrt{\xi^2 + s/b}} + \\
+ \frac{b^{-1}k_2 \int_{0}^{\infty} \phi(x_4, \xi) e^{-x \sqrt{s^2/a - x^2 + s/b}} dx_4}{\alpha k_1 \sqrt{\xi^2 + s/a} + k_2 \sqrt{\xi^2 + s/b}} + \\
- \alpha \int_{0}^{x} dx_1 \int_{-\infty}^{x_1} \phi(x_3, \xi) e^{(x_3 + x_3 - 2x_1) \sqrt{s^2 + s/b}} dx_3 \quad \text{in} \quad x < 0
\]
and
\[
U(x, \xi, s) = \alpha \frac{a^{-1}k_1 \int_{0}^{\infty} \phi(x_3, \xi) e^{x^2 \sqrt{s^2 + s/b}} dx_3}{\alpha k_1 \sqrt{\xi^2 + s/a} + k_2 \sqrt{\xi^2 + s/b}} + \\
+ \frac{b^{-1}k_2 \int_{0}^{\infty} \phi(x_4, \xi) e^{-x \sqrt{s^2/a - x^2 + s/b}} dx_4}{\alpha k_1 \sqrt{\xi^2 + s/a} + k_2 \sqrt{\xi^2 + s/b}} + \\
+ \frac{b^{-1} \int_{0}^{x} dx_2 \int_{x_2}^{x_3} \phi(x_4, \xi) e^{-x \sqrt{s^2/a - x^2 + s/b}} dx_4}{\alpha k_1 \sqrt{\xi^2 + s/a} + k_2 \sqrt{\xi^2 + s/b}} \quad \text{in} \quad x > 0;
\]
here $\phi(x, \xi)$ is the Fourier transform of $f(x, \cdot)$.

\textbf{Proof.} Let $u(x, y, t)$ be a solution of (6) - (9). Applying the Fourier transform on the second variable and the Laplace transform on the third variable, $u$ is changed to a solution $U(x, \xi, s)$ of the problem
\[
\frac{d^2 U}{dx^2} - \left(\xi^2 + s/a\right) U = -\phi/a \quad \text{in} \quad x < 0, \quad (11)
\]
\[
\frac{d^2 U}{dx^2} - \left(\xi^2 + s/b\right) U = -\phi/b \quad \text{in} \quad x > 0, \quad (12)
\]
\[
U(0^-, \xi, s) = \alpha U(0^+, \xi, s), \quad k_1 \frac{\partial U}{\partial x}(0^-, \xi, s) = k_2 \frac{\partial U}{\partial x}(0^+, \xi, s). \quad (13)
\]
With elementary computations, it can be proved that the above problem, with the condition $\lim_{x \to \pm \infty} U(x, \xi, s) = 0$ has solution the function $U(x, \xi, s)$ introduced above and that the transforms indicated above can be performed.

\textbf{Remark.} On analyticity of $u = u(x, y, t)$ in the spatial variables and thus Gevrey $\gamma^2$ in $t$.

Assuming $Re a > 0$ and $Re b > 0$, the analyticity question is a local fact to be studied in any cylinder. Actually, the question was considered in J. L. Doob’s book “Classical potential theory and its probabilistic counterpart” [16] (see pages 262-272), where for the equation $u_t = \frac{\sigma^2}{2} \Delta u$ with $\sigma^2 > 0$, the analyticity follows from formula (7.2) there.

However, it is not difficult to see that formula (7.2) holds for $Re \sigma^2 > 0$. The key point is that
\[
\int_{-\infty}^{+\infty} \frac{1}{(\sigma^2)^{3/2}} \frac{1}{(2\pi t)^{1/2}} \exp\left(-\frac{x^2}{2\sigma^2 t}\right) dx = 1,
\]
if $Re \sigma^2 > 0$, and $(\sigma^2)^{3/2}$ is the root with positive real part, as can be checked through a detailed computation.
Finally, note that \( u = u(x, y, t) \) is \( C^\infty \) in the variable \( t \).

Anyway, it is worthwhile to stress that we expect this result to hold for Schrödinger equation as well. To this aim, recall our previous paper [1] in which the result holds in one space dimension. Nevertheless, now, more analysis would be needed to show this fact for the present problem.

3. The 2nd step of our procedure. Let \( \psi = \psi(t) \) be a cut-off function:

\[
\psi(t) = \begin{cases} 
  1 & \text{for } \ t \leq \frac{T}{2} \\
  0 & \text{for } \ t \geq T
\end{cases}
\]

belonging to a Gevrey class \( \gamma^\delta \); choose \( \psi \) so that \( \psi \in \gamma^{3/2} \) (see [1]). Such function can be constructed explicitly (see [4]). Recall that a function \( F(x, t) \) is said to belong to the space of Gevrey class \( \gamma^\delta \), with respect to the positive \( t \) variable, uniformly for \((x, t)\) in the compact set \( K \), if for every \((x, t)\) \( \in K \) and for every \( \theta > 0 \):

\[
\left| \frac{\partial^n}{\partial t^n} F(x, t) \right| \leq C_{K, \theta} \theta^n (n!)^\delta
\]

for all \( n = 1, 2, \ldots \), with \( C_{K, \theta} \) a positive constant.

Let us now call:

\[ u_1(x, y, t) := u(x, y, t) \quad \text{in } X_-, y \in \mathbb{R}, t \geq 0 \]
\[ u_2(x, y, t) := u(x, y, t) \quad \text{in } X_+, y \in \mathbb{R}, t > 0. \]

Multiply \( u_1 \) and \( u_2 \) by \( \psi(t) \) and set, for \( x \neq 0 \) and \( y \in \mathbb{R}, t > 0 \):

\[ g_i(x, y, t) = \left[ c(x, y) \Delta - \frac{\partial}{\partial t} \right] (u_i \psi) = -u_i(x, y, t) \psi'(t), \quad i = 1, 2. \]

**Define:** \( \tilde{u}_i(x, y, t) := u_i(x, y, t) \psi(t) \). Remark that: \( \tilde{u}_i \) has the advantage that it vanishes for \( t \geq T \), but at a price; it generates the “garbage terms” \( g_i \).

4. The 3rd step of our procedure. For every \( i \), \( g_i \) vanishes for \( t \) outside \([\frac{T}{2}, T]\) and is analytic function of variables \( x \) and \( y \) and belongs to a Gevrey class, as function of \( t \) (because of the choice of \( \psi \)).

To get rid of these “garbage terms”, we have to find solution to the non-homogeneous equations:

\[
\begin{cases} 
  a \Delta W^- - \frac{\partial W^-}{\partial t} = g_1 & \text{in } X_-, y \in \mathbb{R}, t > 0 \\
  b \Delta W^+ - \frac{\partial W^+}{\partial t} = g_2 & \text{in } X_+, y \in \mathbb{R}, t > 0
\end{cases}
\]

with zero initial conditions for \( W^- \) and \( W^+ \) on \( x = 0^\pm \).

The solution \( W^- \) and \( W^+ \) vanish outside \([\frac{T}{2}, T]\).

To this end, it is enough to apply separately in \( X_- \) and \( X_+ \) the Hörmander’s result [4] on Cauchy problems for differential operators with constant coefficients having Cauchy date belonging to a Gevrey class \( \gamma^\delta (1 < \delta \leq 2) \). Precisely, see Theorem 5.7.3 by L. Hörmander and the example on page 150 [4].
The boundary control functions. At this stage, we may accomplish our procedure with the last step.

In this direction, let \( \Omega \) be a smooth open set of \( \mathbb{R}^2 \), containing \((0, 0)\), and let us denote \( C \) the boundary set
\[
C = \partial \Omega \times [0, +\infty).
\]

We are able to prove the following result.

**Theorem 5.1.** Let \( T > 0. \) There exist a function \( h \), defined on \( C \), and a function \( u \) solution to the problem (1)-(4), satisfying
\[
|u|_C = h
\]
vanishing for \( t \geq T \).

The function \( h \) is a boundary control of the studied system.

**Proof.** Let \( u \) be the solution to (1)-(4). We summarize our procedure.

Multiply \( u \) by the cut-function \( \psi = \psi(t) \), introduced in section 3,
\[
\psi(t) = \begin{cases} 
1, & 0 \leq t < \frac{T}{2} \\
0, & t \geq T
\end{cases}
\]
belonging to the Gevrey class \( \gamma^{3/2} \), i.e. satisfying
\[
\left| \frac{\partial^n}{\partial t^n} \psi(t) \right| \leq c \theta^n (n!)^{3/2}.
\]

Next, for \( t > 0 \) and any \( y \in \mathbb{R} \) evaluate:
\[
\begin{align*}
\begin{cases}
(a \triangle - \frac{\partial}{\partial t}) (u_1 \psi) = -u_1(x, y, t) \psi'(t), & \text{in } X_- \\
(b \triangle - \frac{\partial}{\partial t}) (u_2 \psi) = -u_2(x, y, t) \psi'(t), & \text{in } X_+
\end{cases}
\end{align*}
\]
where \( u_1 \) and \( u_2 \) denote \( u \), respectively, in \( X_- \times [0, \infty) \) and \( X_+ \times [0, \infty) \).

Call \( g_1 \) and \( g_2 \) the right-hand side in the above equalities. Such functions \( g_1 \) and \( g_2 \), as functions of \( t \), belong to the Gevrey class \( \gamma^2 \). Actually, for our purposes, we need to get rid of these terms. To this end, we solve - via Hörmander's result [4] - the non-homogeneous problems:
\[
\begin{align*}
\begin{cases}
(a \triangle - \frac{\partial}{\partial t}) W^- = g_1 \\
W^-(0, y, t) = 0 \\
W^- x(0, y, t) = 0
\end{cases} & \text{ in } X_- \times [0, +\infty), \\
\begin{cases}
(b \triangle - \frac{\partial}{\partial t}) W^+ = g_2 \\
W^+(0, y, t) = 0 \\
W^+ x(0, y, t) = 0
\end{cases} & \text{ in } X_+ \times [0, +\infty).
\end{align*}
\]

Thus
\[
\begin{align*}
\omega^-(x, y, t) &= u_1(x, y, t) \psi(t) - W^-(x, y, t) & \text{ in } X_- \times [0, +\infty) \\
\omega^+(x, y, t) &= u_2(x, y, t) \psi(t) - W^+(x, y, t) & \text{ in } X_+ \times [0, +\infty)
\end{align*}
\]
satisfy (1)-(4) and vanish for \( t \geq T \).

Then
\[
h(x, y, t) = \begin{cases} 
\omega^-, & \text{on } \partial \Omega \cap X_-, \quad t \geq 0 \\
\omega^+, & \text{on } \partial \Omega \cap X_+, \quad t \geq 0
\end{cases}
\]
together with (1)-(4), has a solution of the system that becomes zero in a finite time.

The proof is so completed. \( \square \)
6. Some final remarks. It is worthy to point out that the result obtained would enable us to solve the boundary control problem even for different geometric regions. As a matter of fact, the problem we have discussed here might be describing physical phenomena in a composite solid composed by four parts governed, at the same time $t$, by different equations. In this context, the coefficient $c(x, y)$ in the equation (1) should assume different values, real or complex, $A_i$ in the quadrant $Q_i$ of the plane $\mathbb{R}^2$, for $i = 1, 2, 3, 4$. Of course, initial condition and proper interface conditions should be required. The study of this boundary control problem, under this more general frame, is in progress and will appear later.

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