NON-TRIVIAL DARBOUX SOLUTIONS OF CLASSICAL PAINLEVÉ SECOND EQUATION

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Abstract. In this article an other equivalent linear representation of classical Painlevé second equation is derived by introducing a gauge transformation to old Lax pair. The new linear system of that equation carries similar structure as other integrable systems possess in AKNS scheme. That system yields non-trivial Darboux solutions of classical Painlevé second equation which are further generalized to the $N$-th form in terms of Wranskian. Finally we present the exact solutions of that equation through its associated Riccati system.

1. Introduction

The six classical Painlevé equations first were introduced by Paul Painlevé and his colleagues while classifying nonlinear second-order ordinary differential equations with respect to their solutions [1]. In beginning these were well entertained in mathematics because of their connection to nonlinear partial differential equation, where they appear as as ordinary differential equation reduction of higher dimensional integrable that further pursued to establish the Painlevé test for these partial differential equations [2, 3]. for example the Painlevé second (P-II) equation arises as ODE reduction of the Korteweg-de Vries (KdV) equation [4, 5]. Although various properties of these equations have been studied from mathematical point of such as their integrability through the Painlevé test, their zero-curvature representation as compatibility condition of associated set of linear systems [6] - [8] as well as in the framework of hamiltonian formalism [9]. Later on these equation got considerable attention from physical point of view because number of problems in applied mathematics and in physics are made be integrable in terms of their solutions, Painlevé transcendent, The Painlevé second equation is one of these six equation

\[ v_{xx} = v^3 + xv - \alpha. \]  

applied as a model to study electric field in semiconductor [10], where \( v = \frac{d^2 u}{dx^2} \) and \( \alpha \) is parameter, whose rational solutions have been constructed in [11] by means of Yablonski-Vorobév polynomials and further the determinantal generalization of its rational solutions are presented by [12]. In this article we construct the non-trivial Darboux solutions in determinantal form, the Wronskians, by introducing gauged Lax pair. This has been show that the Painlevé second equation (1.1) arises as the compatibility condition of linear systems

\[ \Psi_\lambda = A(x; \lambda)\Psi, \quad \Psi_x = B(x; \lambda)\Psi \]  

with Lax pair

\[
\begin{cases}
A = (4\lambda v + \alpha \lambda^{-1})\sigma_1 - 2v'\sigma_2 - (4i\lambda^2 + ix + 2iv^2)\sigma_3 \\
B = v\sigma_1 - i\lambda\sigma_3
\end{cases}
\]
yields Painlevé second equation (1.1) where $\sigma_1$, $\sigma_2$, and $\sigma_3$ are Pauli spin matrices

Remark 1.1

Under the standard Darboux transformation [13, 14] on components of column vector $\Psi = \begin{pmatrix} X \\ Y \end{pmatrix}$ as

\[
\begin{aligned}
X \rightarrow X[1] &= \lambda Y - \lambda_1 \left( \frac{Y_1}{X_1} \right) X \\
Y \rightarrow Y[1] &= \lambda X - \lambda_1 \left( \frac{X_1}{Y_1} \right) Y
\end{aligned}
\] (1.4)

one can construct the one-fold Darboux transformation to the solution of equation (1.1) by using its associated linear system (1.2) in following form

\[v[1] = v \left( X X_{\lambda}^{-1} \right)^2 \]

that yields trivial Darboux solutions to equation (1.1) taking $v = 0$ as seed solution at $\alpha = 0$, which are meaningless. In next section, we will present non-trivial Darboux solutions of that equation introducing gauge transformation that brings its Lax pair (1.3) to new form as similar as we have for many integrable systems in AKNS scheme. Further, by iteration the $N$-fold Darboux transformation will be generalized to determinantal form in terms of Wronskian.

2. GAUGE TRANSFORMATION AND NONTRIVIAL DARBOUX SOLUTIONS

**Proposition 2.0.1.** Let consider a matrix $G = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ that transforms old Lax pair (1.3) $(A, B)$ to new set as $\tilde{A} = GAG^{-1}$, $\tilde{B} = GBG^{-1}$, where new Lax pair $(\tilde{A}, \tilde{B})$ reproduces Painlevé second equation (1.1) as subjected to zero-curvature condition $\tilde{A}_x - \tilde{B}_\lambda = [\tilde{B}, \tilde{A}]$

Proof:

Under above gauge transformations the new Lax will take the following form

\[
\begin{aligned}
\tilde{A} &= -2v'\sigma_1 - (4i\lambda^2 + ix + 2iv^2)\sigma_2 + (4\lambda v + \alpha\lambda^{-1})\sigma_3 \\
\tilde{B} &= v\sigma_3 - i\lambda\sigma_2
\end{aligned}
\] (2.1)

with linear system

\[\Psi_\lambda = \tilde{A}(x; \lambda)\Psi, \quad \Psi_x = \tilde{B}(x; \lambda)\Psi\] (2.2)

whose compatibility condition gives zero-curvature form in terms of Lax matrices $(\tilde{A}, \tilde{B})$ as $\tilde{A}_x - \tilde{B}_\lambda = [\tilde{B}, \tilde{A}]$. Now having values for $\tilde{A}_x$, $\tilde{B}_\lambda$ and for commutator $[\tilde{B}, \tilde{A}]$ from gauged Lax pair (2.1) and substituting in corresponding zero-curvature condition, then after some computations system reproduces the classical Painlevé second equation (1.1).

**Proposition 2.0.2.** The linear system (2.2) under the Darboux transformation (1.4) connects the next new solution, say $v[1]$ of (1.1) to its old one $v$ as

\[v[1] = -v + \frac{d}{dx} \ln \frac{X_1}{Y_1}\]

where $X_1$ and $Y_1$ are particular solutions of linear system (2.2) at $\lambda = \lambda_1$.

Proof:
Let us start with standard Darboux transformations (1.4) on components of column vector consider a column matrix \( \Psi = \begin{pmatrix} X \\ Y \end{pmatrix} \) with linear system (2.2), we have following transformed equation

\[
X'[1] = v[1]X[1] - \lambda Y[1], \tag{2.3}
\]

and now from (1.4)

\[
X'[1] = \lambda_o Y'_o - \lambda_1 (\frac{Y'_1}{X'_1})X_o + \lambda (\frac{Y_1}{X_1})X'_o, \tag{2.4}
\]

now after combining (2.3) and (2.4) and then comparing the co-efficients of \( X_o \) and \( Y_o \), at \( \lambda_o = \lambda_1 = \lambda \): we obtain

\[
v(\frac{Y_1}{X_1}) - \frac{Y_1^2}{X_1^2} \lambda = -v[1](\frac{Y_1}{X_1})\lambda - \lambda,
\]

above expression yields after simplification, the one fold Darboux transformation as a follow

\[
v[1] = -v + (\frac{Y_1}{X_1} - \frac{X_1}{Y_1})\lambda_1, \tag{2.5}
\]

\[
v[1] = -v + (Y_1 X_1^{-1} - X_1 Y_1^{-1})\lambda_1. \tag{2.6}
\]

where in above expression (2.6) \( v \) is the old solution of classical Painlevé II equation that generates its new solution \( v[1] \) with having \( X_1 \) and \( Y_1 \) particular solutions of system (2.2) at \( \lambda = \lambda_1 \). Now this can bee shown that after substituting the values of \( \lambda_1 X_1 \) and \( \lambda_1 X_1 \) in (2.6) from linear system (2.2), the one fold Darboux transformation can be expressed as

\[
v[1] = -v + \frac{d}{dx} \ln \frac{X_1}{Y_1}. \tag{2.7}
\]

3. N-fold Darboux solutions in terms of Wronskians

Here we genralize Darboux transformation in terms of Wroński., the iterated Darbous solutions can be describe in a more compact way as Nth wroński of the components the eigenfunctions.

3.1. One fold Darboux solution. With following settings

\[
Y^{(1)}_i = \lambda_i Y_i, \tag{3.1}
\]

\[
X^{(1)}_i = \lambda_i X_i, \tag{3.2}
\]

the one fold Darboux transformation \( X[1], Y[1] \) and \( v[1] \) can be written as below

\[
X[1] = \frac{W_1(X_1, Y_1, X_o, Y_o)[2]}{W_1(X_1, Y_1)[1]}, \tag{3.3}
\]

\[
Y[1] = \frac{W_2(X_1, Y_1, X_o, Y_o)[2]}{W_2(X_1, Y_1)[1]}, \tag{3.4}
\]

and
\[ v[1] = -v + \partial_x \ln \left( \frac{W_1(X_1, Y_1)[1]}{W_2(X_1, Y_1)[1]} \right). \]  

(3.5)

where \( W_1 \) and \( W_2 \) are Wronskians defined as

\[ W_1(X_1, Y_1, X_o, Y_o)[2] = \begin{vmatrix} X_1 & X_o \\ Y_1^{(1)} & Y_o^{(1)} \end{vmatrix}, \]

\[ W_2(X_1, Y_1, X_o, Y_o)[2] = \begin{vmatrix} Y_1 & Y_o \\ X_1^{(1)} & X_o^{(1)} \end{vmatrix}, \]

\[ W_1(X_1, Y_1)[1] = X_1, \]

\[ W_2(X_1, Y_1)[1] = Y_1, \]

3.2. Wronskians for two fold Darboux transformation. The two fold Darboux transformation for the \( X \) component can be written as

\[ X[2] = \lambda_o Y_o[1] - \lambda_1 \left( \frac{Y_1^{(1)}[1]}{X_1^{(1)}[1]} \right) X_o[1], \]  

(3.6)

or

\[ X[2] = Y_o^{(1)}[1] - \left( \frac{Y_1^{(1)}[1]}{X_1^{(1)}[1]} \right) X_o[1], \]

and now in terms of Wronskian can be expressed as

\[ X[2] = \frac{W_1(X_k, Y_k, X_o, Y_o)[3]}{W_1(X_k, Y_k)[2]}. \]  

(3.7)

Similarly for the \( Y \) component we have

\[ Y[2] = \lambda_o X_o[1] - \lambda_1 \left( \frac{X_1^{(1)}[1]}{Y_1^{(1)}[1]} \right) Y_o[1], \]  

(3.8)

equivalently can be expressed as

\[ Y[2] = X_o^{(1)}[1] - \left( \frac{X_1^{(1)}[1]}{Y_1^{(1)}[1]} \right) Y_o[1], \]

or

\[ Y[2] = \frac{W_2(X_k, Y_k, X_o, Y_o)[3]}{W_2(X_k, Y_k)[2]}, \]  

(3.9)

where \( k = 1, 2 \). Now the second iteration on (2.7) yields the next new solution to equation (1.1)

\[ v[2] = -v + \partial_x \ln \left( \frac{W_1(X_1, Y_1)[2]}{W_2(X_1, Y_1)[2]} \right). \]  

(3.10)
where the Wronskians are defined as follow

\[
W_1(X_k, Y_k, X_o, Y_o)[3] = \begin{vmatrix} Y_2 & Y_1 & Y_o \\ X_2^{(1)} & X_1^{(1)} & X_o^{(1)} \\ Y_2^{(2)} & Y_1^{(2)} & Y_o^{(2)} \end{vmatrix},
\]

\[
W_2(X_k, Y_k, X_o, Y_o)[3] = \begin{vmatrix} X_2 & X_1 & X_o \\ Y_2^{(1)} & Y_1^{(1)} & Y_o^{(1)} \\ X_2^{(2)} & X_1^{(2)} & X_o^{(2)} \end{vmatrix},
\]

\[
W_1(X_1, Y_1)[2] = \begin{vmatrix} X_2 & X_1 \\ Y_2^{(1)} & Y_1^{(1)} \end{vmatrix},
\]

\[
W_2(X_1, Y_1)[2] = \begin{vmatrix} Y_2 & Y_1 \\ X_2^{(1)} & X_1^{(1)} \end{vmatrix},
\]

3.3. **N fold Darboux transformation.** For N-times iterated Darboux transformation, we have the following results for \(X\) and \(Y\) components respectively

\[
X[N] = \frac{W_1(X_k, Y_k, X_o, Y_o)[N + 1]}{W_1(X_k, Y_k)[N]},
\]

\[
Y[N] = \frac{W_2(X_k, Y_k, X_o, Y_o)[N + 1]}{W_2(X_k, Y_k)[N]},
\]

where \(k = 1, 2, \ldots, N\) and \(X_k\) and \(Y_k\) are particular solutions of Lax pair at \(\lambda = \lambda_k\).

For odd values of \(N\), determinants \(W_1\) and \(W_2\) are

\[
W_1(X_k, Y_k, X_o, Y_o)[N + 1] = \begin{vmatrix} X_N & X^{(N-1)} & \ldots & X_1 & X_o \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Y^{(N-1)}_N & Y^{(N-1)}_{(N-1)} & \ldots & Y^{(N-1)}_1 & Y^{(N-1)}_o \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X^{(N)}_N & X^{(N)}_{(N-1)} & \ldots & X^{(N)}_1 & X^{(N)}_o \end{vmatrix},
\]

\[
W_2(X_k, Y_k, X_o, Y_o)[N + 1] = \begin{vmatrix} Y_N & Y^{(N-1)} & \ldots & Y_1 & Y_o \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X^{(N-1)}_N & X^{(N-1)}_{(N-1)} & \ldots & X^{(N-1)}_1 & X^{(N-1)}_o \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Y^{(N)}_N & Y^{(N)}_{(N-1)} & \ldots & Y^{(N)}_1 & Y^{(N)}_o \end{vmatrix}.
\]
For even values of $N$ we have following determinants.

$$W_1(X_k, Y_k, X_o, Y_o)[N + 1] = \begin{vmatrix} Y_N & Y_{N-1} & \ldots & Y_1 & Y_0 \\ \vdots & X_N^{(N-1)} & X_{N-1}^{(N-1)} & \ldots & X_1^{(N-1)} & X_o^{(N-1)} \\ \vdots & Y_N^{(N)} & Y_{N-1}^{(N)} & \ldots & Y_1^{(N)} & Y_o^{(N)} \end{vmatrix},$$

$$W_2(X_k, Y_k, X_o, Y_o)[N + 1] = \begin{vmatrix} X_N & X_{N-1} & \ldots & X_1 & X_0 \\ \vdots & Y_N^{(N-1)} & Y_{N-1}^{(N-1)} & \ldots & Y_1^{(N-1)} & Y_o^{(N-1)} \\ \vdots & X_N^{(N)} & X_{N-1}^{(N)} & \ldots & X_1^{(N)} & X_o^{(N)} \end{vmatrix}.$$  

Finally the all possible Darboux solutions of (1.1) can generalized to the $N$-th form in terms of Wronskian as below

$$v[N] = -v + \partial_x \ln \left( \frac{W_1(X_k, Y_k)[N]}{W_2(X_k, Y_k)[N]} \right).$$  (3.11)

where $v$ is the initial or "seed" solution that can be taken as trivial at $\alpha = 0$ in (1.1), with this simplest choice one can construct all possible non-trivial solutions to that equation.

4. Conclusion:

Above we have presented a technique to construct an other Lax pair to the classical Painlevé second equation through the gauge like transformations with the help of this new linear systems we have derived the all possible non-trivial Darboux solutions of that equation taking seed solution as zero in that transformation. Still work is in progress to construct the exact solution through its Darboux transformation involving the associated Riccati forms of classical Painlevé second.

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