A Kolmogorov proof of the Clauser, Horne, Shimony and Holt inequalities

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Boolean logic is used to prove the CHSH inequalities. The proof elucidates the connection between Einstein elements of reality and quantum non locality. The violation of the CHSH inequality by quantum theory is discussed and the two stage view of quantum measurement relevance to incompatible observables is outlined.

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I. INTRODUCTION

A convenient specification of Einstein, Podolsky, Rosen and Bohm experimental set up that we adopt is: An experiment is made of numerous runs. In each run two photons (1,2) in the state \( |\psi(1, 2)\rangle = \frac{1}{\sqrt{2}} (|+1 +2 \rangle + |+1 -2 \rangle) \) are involved. (Here +(-) means positive (negative) polarization along a common z axis.) The photons propagates to two separated ports (\( \alpha, \beta \)) therein their polarizations are measured by one of two polarizers: A or A’ at \( \alpha \), and B or B’ at \( \beta \). The experiments are assumed flawless: In each run both photons reach the counters at \( \alpha \) and \( \beta \) and record a reading at both: +1 if the photon pass the polarizer and -1 if it doesn’t.

The issue under study is: can the experimental results of the EPRB set up be accounted for in terms of canonical (Kolmogorov’s) probability theory based on hidden variable, \( \lambda \)?

Two point correlation of the outcomes with the polarizers set at A and B, \( < AB > \), is defined by:

\[
< AB > = \frac{1}{N} \sum_{i=1}^{N} a_i b_i, \tag{1}
\]

Here i enumerates the run and a,b the outcome at the ports \( \alpha, \beta \) respectively.

An account by probability theory based on hidden variable implies that two point correlations may be expressed by

\[
< AB > = \int d\lambda \rho(\lambda) A(\lambda) B(\lambda), \tag{2}
\]

with a distribution, \( \rho(\lambda) \), common to all correlations, \( \lambda \) is the "hidden variable".

We note that within such an account the correlations \( < AA' > \) and \( < BB' > \) are defined though are not measured (perhaps even not measureable e.g. due to technical difficulties). Thus, e.g.,

\[
< AA' > = \int d\lambda \rho(\lambda) A(\lambda) A'(\lambda). \tag{3}
\]

The canonical (Kolmogorov’s) probability theory involves definition of a sample space wherein each of the points accounts for possible experimental outcomes. The total sample space, \( \Omega \), is a set of points that cover all possible experimental outcomes. The measure assigned to these points is their probability. Observable such as e.g. \( A(\lambda) \), are termed events. Our analysis does not require the specification of this space - suffice it to know that such exists for a canonical probability theory. This allows us to analyse interrelations among events via the algebra of sets aided by intuitively appealing Venn diagrams.

In the next section we obtain, assuming the existence of a probability theory and using simple Boolean logic \([1 \ 2 \ 3]\) implied consistency relations among the two points correlations it allows. These interrelations will then be shown to be the so called Bell’s inequalities.
II. INTERRELATIONS AMONG CORRELATIONS IN PROBABILITY THEORY

In classical Kolmogorov theory interrelations among probabilities may be analysed in terms of Venn diagrams pertaining to the corresponding events [5],[6]. Thus, e.g., we may consider the "area" in a Venn diagram "occupied" by the event \((AB) \). This event relates to the probability of observing equal readings, \(A=B\), i.e. polarizer A (at \(\alpha\)) and B (at \(\beta\)) both read +1 or both -1. In our notation (using A and B as an example) this means: \((AB) \) is the set of all the sample space points with \(A=B\). Its measure is the probability of this event, \(P((AB) \)\). The complementary event \(\overline{(AB)}\), containing the sample points with unequal values for A and B, is designated by \((AB)\). Its probability is denoted by \(P((AB)\).\)

Thus,

\[
\overline{(AB)} = (AB) \Rightarrow P(\overline{(AB)}) = P((AB)\).
\]

and

\[
P[(AB) \cup (AB')] + P[(\overline{(AB)}) \cup (\overline{(AB')})] = 1.
\]

Similar relations hold for the other two points correlations of interest, e.g., \((AB')\), \((A'B)\), \((A'B')\), etc.

Boolean logic, i.e. set algebra, dictates \(1,3,4,6\),

\[
((AB) \cup (AB')) = (AB) \cap (\overline{(AB')}) = (AB) \times (\overline{(AB')})\]

(6)

Where we used Eq.(4) in the last step.

We have trivially that \((AB) \times (AB') \Rightarrow (BB')\) i.e. :

\[
((AB) \times (AB')) \subseteq (BB')\]

(7)

implying [7],

\[
P((AB)\) + P((AB')\) + P((BB')\) \geq 1.
\]

Eq.(8) is a consistency requirement stemming from pairing the probabilities of \((AB)\) with \((AB')\).

Quite generally this approach allows derivation of consistency relations among probabilities. These are equivalent to Bell’s inequalities which are formulated in terms of correlations. We now proceed to reformulate our consistency relations in term of correlations.

\[
P((AB)\) + P((AB)\) = 1,
P((AB)\) - P((AB)\) = < AB >, \Rightarrow
2P((AB)\) = 1 + < AB >,
2P((AB)\) = 1 - < AB >.
\]

(9)
Utilizing Eq.\((\text{9})\), Eq.\((\text{8})\) may be written in terms of two point correlations as

\[
\langle AB \rangle + \langle AB' \rangle + \langle BB' \rangle \geq -1 \tag{10}
\]

Going through similar reasoning with the pair \((AB)_x \cup (AB')_x\) gives,

\[
\langle AB \rangle - \langle AB' \rangle - \langle BB' \rangle \geq -1 \tag{11}
\]

The inequality for \((A'B)_x\) and \((A'B')_x\), is of course identical to Eq.\((\text{10})\) with \(A'\) replacing \(A\):

\[
\langle A'B \rangle + \langle A'B' \rangle + \langle BB' \rangle \geq -1 \tag{12}
\]

Combining the inequality, Eq.\((\text{11})\), with that of Eq.\((\text{12})\) yields a Bell’s inequality:

\[
\langle AB \rangle - \langle AB' \rangle + \langle A'B \rangle + \langle A'B' \rangle \geq -2 \tag{13}
\]

There are 4 possible pairings: 1 \((AB), (AB')\); 2 \((A'B), (A'B')\); 3 \((AB), (A'B)\); 4 \((AB'), (A'B')\).

There are 4 combinations for each, e.g.: 1. \((AB)_x, (AB')_x\); 2. \((AB)_x, (AB')_x\); 3. \((AB)_x, (A'B')_x\); 4. \((AB)_x, (A'B')_x\).

Each implies consistency inequality among the relevant correlations. This gives 16 consistency inequalities.

Eliminating the unmeasured correlations \(\langle AA' \rangle, \langle BB' \rangle\) gives the following inequalities

\[
|\langle AB \rangle \mp \langle A'B \rangle | + |\langle AB \rangle + \langle A'B' \rangle | \leq 2 \tag{14}
\]

identified as Bell’s inequalities. It is accepted that these inequalities are violated experimentally \([8]\). Thence an account of the EPRB set up by a canonical probability theory fails. (No violation has been observed of the quantum mechanical (QM) predictions \([9]\).)

### III. THE QUANTUM MECHANICAL STATE

The culprit in the failure of an account of the EPRB set up by canonical probability theory appears to be the presence therein of correlations among observables that do not have values simultaneously i.e. that can be revealed by the same experiment. (These are \(A\) with \(A'\) and \(B\) with \(B'\) in the EPRB set up. E.g. Eq.\((\text{3})\) assigns instantaneous relation between values for \(A\) and \(A'\), within the same distribution.) Such instantaneous relation are, we contend, disallowed by nature. Thence the distribution function for values of, say, \(A\) must be different from that of \(A'\) (though both may pertain to the same quantum state) and their measurements must involve distinct experiments.

The observed correlations violate the inequalities yet abide by QM. This may seem unexpected in view of the analysis above since the evaluation of the correlations within QM does involves the same quantum mechanical state, \(\rho\), for all the correlations. This, seemingly, is equivalent to using the same sample space and measure (i.e probability) in the canonical probability formulation. Yet the later led to consistency relations which are violated experimentally indicating thereby its inadequacy. However closer look at the QM calculations show that mathematical attributes of the Hilbert space formalism in effect assigns distinct (probability) distributions to non commuting observables. Thus in evaluating the expectation of an arbitrary operator, \(\hat{A} = \sum_a |a > a < a|\) for a state specified by an arbitrary \(\rho\) the effective \(\rho\) is \([10, 11]\) in effect (indicated by an arrow) diagonal,

\[
\hat{\rho} \rightarrow \rho_A \equiv \sum_a |a > < a|\rho|a > < a|. \tag{15}
\]

Likewise, when evaluating \(\hat{A}'\), it is \(\rho_{A'} \equiv \sum_{a'} |a' > < a'|\rho|a' > < a'| \neq \rho_A\), in general. I.e. in effect we evaluate two non commuting observables with two different distribution functions though the QM state is the same.
These considerations were raised elsewhere [11] and led to viewing QM measurement as made of two stages. The first, termed unrecorded measurement (URM), involves an unrecorded von Neumann measurement. This is associated with, in general, an actual change in the QM state. The second stage view the resultant state as amendable for a classical distribution account (for compatible observables) and the uncovering of the final outcome is handled much like within classical probability theory. Translated to the case at hand, the QM state is viewed as an information code encoding the possible distributions for the system [12]. The relevant distribution for a particular measurement is attained, by a measurement (this is illustrated below). Thus, whereas the classical state may be viewed as a set of values for attributes - the quantum state may be viewed as an encoded set of possible distributions for these attributes. It "is (the symbolic representation of) the ensemble to which it belongs" [12]. ρ within a quantum measurement (i.e. Hilbert space formalism) is "projected" to classical like distribution for compatible observables. Once the distribution is determined the measurement’s second stage is purely classical: gaining the numerical value for, say, A within the distribution ρ_A. Measurement selects the distribution [13], e.g.,

\[ \hat{\rho} \rightarrow \rho_A \text{ (via Unrecorded measurement) } \rightarrow a \text{ (via classical measurement.)} \]  

\[ (16) \]

The interpretation has special significance when dealing with two (or more) particles state that allows separate measurements for each: measurement of one particle discloses partial distribution, in general:

\[ \rho \rightarrow \sum_{a,b,b'} |a > |b > < b'| < a| \rho |a > |b' > < b'| < a|, \]

\[ (17) \]

where only the partial distribution of the first particle pertaining to the attribute \(\hat{A}\) is revealed. In the special case wherein \(\rho\) relates to a (maximally) entangled state viewing the state as symbolic representation of an ensemble has conceptual advantage. Thus, e.g., consider measuring the polarity along some axis, \(\theta\), on particle 1 in a two photon maximally entangled state,

\[ \rho = \frac{1}{2} [(|+ \theta >_1 |+_\theta >_2 + |_\theta >_1 |-_\theta >_2 + |-_\theta >_1 |+_\theta >_2)] \]

\[ (18) \]

The form of this state is independent of the direction of the polarizers. Thus rotating the polarizers to the \(\theta\) direction leaves the form of state invariant:

\[ \rho \Rightarrow \rho_{\theta} = \frac{1}{2} [(|+ \theta >_1 |+_\theta >_2 + |_\theta >_1 |-_\theta >_2 + |-_\theta >_1 |+_\theta >_2)] \]

\[ (19) \]

Undertaking a measurement along \(\theta\) of one particle, gives:

\[ \rho \Rightarrow \rho_{u} = \frac{1}{2} [(|+ \theta >_1 |+_\theta >_2 + |_\theta >_1 |-_\theta >_2 + |-_\theta >_1 |+_\theta >_2)] \]

\[ (20) \]

I.e., in the case of a (maximally) entangled state, the one particle measurement (to uncover its distribution) revealed the complete distribution.

IV. CONCLUDING REMARKS

A hidden variables account for the Einstein, Podolsky, Rosen and Bohm (EPRB) set up considered in the literature [8] is an account in terms of classical (Kolmogorov’s) probability theory. Such an account, necessarily, assigns values to all the EPRB two point correlations both measured and unmeasured.

A mathematical attribute of standard quantum mechanics (QM) (i.e. within its Hilbert space formulation) is an involvement of non commuting observables. This may be viewed as reflecting nature’s disallowance for values of non-commuting observables to be simultaneously (within the same experimental run) revealed. I.e. their having a
defined instantaneous correlations within the same distribution is disallowed - they are incompatible.

QM accommodates the above mentioned natures' ruling, classical probability theory in terms of hidden variables does not. This precludes an account via a canonical probability theory of the EPRB set up: such an account implies, in principle, simultaneous values for some of these (un measured) incompatible observables. Canonical probability formulation entails consistency requirements expressible as interrelation among correlations evaluated within the theory. They were identified as Bell’s inequalities. We argued that their (experimental) violation reflects classical probability theory inadequacy in handling the incompatible observables.

Phase space formulation of QM aspire to provide it with classical like view. The formulation assigns the Wigner function the role of distribution [14]( The role of hidden variables is played by phase space coordinates.) A consistency condition for viewing the Wigner function as a classical like distribution is it be non negative. Thence the quantal demonstration of "violation" of positivity of the Wigner function is similar to the quantal violation Bell’s inequalities. In either case the direct classical like formalism fails to uphold nature’s disallowance of prescribed values for instantaneous correlations of incompatible observables. Both are violated within QM that does uphold the disallowance.

An appropriate classical like hidden variable account for the EPRB set up will, thus, require intricate information on ensemble of distributions (which is incorporated within the QM formalism). This is beyond the classical Kolmogorov probability theory. The issue of entanglement enters since for such states the full distribution is revealed via measurement of one of the constituents.

To summarize:

1. A novel concise derivation of Bell’s inequalities is presented. The derivation allow better isolation of the reason for their violation which support viewing quantum measurement as two stage process and quantum states as encoding information on distributions in addition to their probabilistic attributes.

2. The experimentally observed Bell’s inequality violation does not relate in any obvious way to locality attribute of hidden variable account for Einstein, Podolsky, Rosen and Bohm (EPRB) set up. The violation does not rule out EPR contention that QM is an incomplete theory of real physical entities.

3. Bell’s inequalities are consistency conditions for classical probability theory. Their violation indicate inadequacy of Kolmogorov’s classical probability theory to account for the physics involved in EPRB set up. A proper account should require an extended theory, one that could deal with states encoding distributions of classical like distributions.

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