Phase Transitions of Charged Kerr-AdS Black Holes from Large-\(N\) Gauge Theories

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We study \(\mathcal{N} = 4\) super Yang-Mills theories on a three-sphere with two types of chemical potential. One is associated with the R-symmetry and the other with the rotational symmetry of \(S^3\) (\(SO(4)\) symmetry). These correspond to charged Kerr-AdS black holes via AdS/CFT. The exact partition functions at zero coupling are computed and the thermodynamical properties are studied. We find a nontrivial gap between the confinement/deconfinement transition line and the boundary of the phase diagram when we include more than four chemical potentials. In dual gravity, we find such a gap in the phase diagram by studying the thermodynamics of the charged Kerr-AdS black hole. This shows that the qualitative phase structures agree between both theories. We also find that the ratio between the thermodynamical quantities is close to well-known factor of \(3/4\) even at low temperatures.

§1. Introduction and summary

The AdS/CFT correspondence\(^1\)–\(^3\) has played a central role for about ten years in the study of the strongly coupled region of \(\mathcal{N} = 4\) super Yang-Mills (SYM) theory with the \(SU(N)\) gauge group because it is simply described by type IIB supergravity on \(AdS_5 \times S^5\) (see Ref. 4) for a review). It is well-known that the thermodynamical quantities in free gauge theory agree with those in dual gravity up to a factor of \(3/4\).\(^5\),\(^6\) This factor does not change if we consider other SYM theories such as the one with R-symmetry chemical potentials (dual to R-charged black holes)\(^7\)–\(^9\) or the others with \(SO(4)\) symmetry chemical potentials associated with the angular momenta of fields on a three-sphere (dual to Kerr-AdS black holes).\(^10\),\(^11\) Also it has been shown quantitatively\(^12\),\(^13\) that this discrepancy is always nearly \(3/4\) for infinitely many \(\mathcal{N} = 1\) SCFTs, which can be constructed systematically\(^14\),\(^15\) from dual \(AdS_5 \times Y_5\) geometries, where \(Y_5\) is a toric Sasaki-Einstein manifold.\(^16\),\(^17\) This agreement suggests that the free approximation of gauge theory captures the significant properties in the strongly coupled theory if the AdS/CFT correspondence holds.

In the AdS space, there is a phase transition between the thermal AdS space and the AdS-Schwarzschild black hole, the so-called Hawking-Page transition.\(^18\) It has been pointed out in Ref. 6) that this corresponds to the confinement/deconfinement

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transition in the strongly coupled gauge theory. Although naively there seems to
be no phase transition in the gauge theory defined on a compact space $S^3$ since
we have only a finite degree of freedom, we know of such a example in the large-$N$
limit: the Gross-Witten-Wadia transition.\(^{19),20}\) The infinite degree of freedom in
the large-$N$ limit causes the phase transition even in a finite-volume system. It was
shown in Refs. 21) and 22) that there exists such a phase transition even at zero
coupling by studying large-$N$ gauge theories with constituent states in the adjoint
representation of the gauge group. In the absence of any interaction, but with a
singlet (Gauss’ law) constraint considered, the gauge theory becomes an exactly
solvable unitary matrix model of the Polyakov loop. This model exhibits a phase
transition such that the expectation value of the Polyakov loop is zero below some
critical temperature and becomes nonzero above it. Also the free energy scales as
$O(1)$ in the low-temperature phase and $O(N^2)$ in the high-temperature phase. This
is precisely the confinement/deconfinement phase transition. This model has been
extensively studied in Refs. 23)–28) including weak coupling, the finite-$N$ effect and
the orbifold.

Similar analysis has been performed regarding the presence of the global R-
symmetry chemical potentials in Refs. 29)–32), and a weak coupling region with near-
critical chemical potentials has recently been investigated.\(^{33}\) The resulting phase
diagram of the zero-coupling limit is very similar to that of the gravitational solutions
of five-dimensional $\mathcal{N} = 2$ gauged supergravity.

In this paper, we study $\mathcal{N} = 4$ SYM theory on a three-sphere with general
(R- and $SO(4)$ symmetry) chemical potentials following the method in Refs. 21)
and 22).\(^{\ast}\) The dual theory to this gauge theory is the five-dimensional maximal
$SO(6)$-gauged $\mathcal{N} = 8$ supergravity. The $SO(6)$ gauge symmetry, which originates
from the $S^5$ compactification of ten-dimensional type IIB supergravity, incorporates
$U(1)^3$ symmetry. This symmetry corresponds to the R-symmetry of SYM theory on
$S^3$. Thus, we should compare the solutions in this five-dimensional $U(1)^3$-gauged
$\mathcal{N} = 2$ supergravity theory with the fields in SYM theory with R-charge chemical
potentials. Despite considerable effort devoted toward finding the exact black hole
solution within this theory,\(^{35)–44}\) the most general solution with three charges and
two angular momenta has not been yet found. However, we can construct the most
general dual gauge theory with three R-symmetry chemical potentials and two $SO(4)$
chemical potentials. Therefore, we may expect that the various properties of the
undiscovered black hole solution can be induced from the analysis of dual gauge
theory. In this paper we focus on a solution of the five-dimensional charged Kerr-AdS
black hole constructed in Refs. 36) and 37), which has two equal angular momenta
and three independent charges.

Setting the R-symmetry chemical potential to zero, we obtain $\mathcal{N} = 4$ SYM with
$SO(4)$ chemical potentials. This has already been considered in Ref. 45), and the
AdS/CFT correspondence about the Kerr-AdS black hole was investigated. How-
ever, the constraint of Gauss’ law was not taken into account, which plays a crucial
role in gauge theory on a compact space. Therefore the analysis is only valid at

\(^{\ast}\) Similar setup has been studied in the decoupling limit in Ref. 34).
the high-temperature limit where the compact space can be approximated to a flat space. It is necessary to maintain Gauss' law even at zero coupling to obtain valid results in the limit of the interacting theory as pointed out in Ref. 30). We will consider the Gauss' law constraint in the analysis of gauge theory, and show that the confinement/deconfinement transition occurs at zero coupling. This transition was not observed in the analysis by Ref. 45).

The organization of this paper is as follows. In §2, we study $\mathcal{N} = 4$ SYM theory on a three-sphere with chemical potentials associated with the R-symmetry of $\mathcal{N} = 4$ supersymmetry and the $SO(4)$ symmetry of $S^3$. The $SU(4)$ R-symmetry has a $U(1)^3$ Cartan subalgebra; thus, we can introduce the three chemical potentials ($\mu_1, \mu_2, \mu_3$) discussed in Ref. 30), while the $SO(4)$ symmetry has a $U(1)^2$ Cartan subalgebra, so we have two associated chemical potentials ($\Omega_1, \Omega_2$). Then we construct a partition function for free $\mathcal{N} = 4$ SYM with these chemical potentials following Refs. 21) and 22), and we determine the phase diagram in the phase space ($\Omega_1, \Omega_2, \mu_1, \mu_2, \mu_3$) (Fig. 1(a)). In this diagram, we have found the maximal chemical potential $\mu_{\text{max}}$ as a function of the other chemical potentials, below which the entire body of the transition line is enclosed in the phase diagram. The appearance of $\mu_{\text{max}}$ is related to the divergence of the fermion partition function, but the theory is still valid because of Pauli exclusion principle. We also determine upper bounds for the chemical potentials above which some field becomes tachyonic. We call this boundary line the unitarity line since the unitarity of the theory breaks down above it. We show that a gap appears between the confinement/deconfinement transition line and the unitarity line when there are more than four chemical potentials in the gauge theory. This is a new phenomenon discovered in this paper. The theories with only R-symmetry or $SO(4)$ symmetry are included in the above general theory and we also study these specific theories.

In §3, the dual gravity theory is analyzed. We study the Hawking-Page transitions and the thermodynamical instability of charged Kerr-AdS black holes, and reveal the phase structures for these black holes. A schematic of the resulting phase diagram is shown in Fig. 1(b) and we found a gap between the Hawking-Page line and the instability line. We compare these phase diagrams with those of the gauge theory, and find remarkable agreement between them (Fig. 1). Furthermore, we calculate the ratio of the effective actions between these two theories and show that the ratio takes a value close to 3/4 even at a low temperature. This quantitative result shows that the deconfinement phase of free $\mathcal{N} = 4$ SYM with chemical potentials well describes the dual black hole. Section 4 is devoted to discussion.

§2. Large-$N$ gauge theory

In this section, we study the thermodynamics of $\mathcal{N} = 4$ SYM theory with the $U(N)$ gauge group on $S^3$. First, we summarize the symmetry of this theory and the spectrum of its fields. The $SU(4)$ and $SO(4)$ groups arise as the R-symmetry of the $\mathcal{N} = 4$ supersymmetry and the rotational symmetry of $S^3$, respectively. The symmetry group has a $U(1)^3$ Cartan subgroup, and we can consider a grand canonical ensemble with five chemical potentials.
Then, we derive a partition function with chemical potentials. We see that the partition function is reduced to a matrix model of the Polyakov loop by summing over gauge invariant states or by integrating all the massive modes. The distribution of the eigenvalues of the matrix model exhibits a phase transition from the uniform phase to the nonuniform phase at some critical temperature. The low-temperature phase has thermodynamical quantities of order one, while the high-temperature phase has those of order $N^2$. The phase transition line is depicted in the phase diagram.

We will find that interesting phenomena occur in the case when more than four chemical potentials are turned on. In this case, the maximal chemical potential $\mu_{\text{max}}$ appears, below which the entire body of the transition line is contained in the phase diagram. We will also study the bounds of the chemical potentials above which the unitarity of the theory breaks down.

We also consider the theory with only R-symmetry chemical potentials or only $SO(4)$ chemical potentials as specific cases. The R-symmetry case has already been studied in Ref. 30 and we obtain the same result here.

2.1. Symmetry of $\mathcal{N} = 4$ super Yang-Mills theory and chemical potentials

The AdS boundary of a charged Kerr-AdS black hole has $S^3$ topology in global coordinates. Therefore, we need to study $\mathcal{N} = 4$ SYM on $S^3$. The action is given by:

$$S = -\int d^4 x \sqrt{-g} \text{tr} \left[ \frac{1}{2} (F_{\mu\nu})^2 + (D_\mu \phi_m)^2 + l^{-2} \phi_m^2 + i \bar{\lambda}^A \Gamma^\mu D_\mu \lambda_A \right]$$

*1) We take the normalization of the generator $T^a$ of the gauge group as $\text{tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$. 

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Fig. 1. Phase diagrams of $\mathcal{N} = 4$ large-$N$ SYM theory (a) and the charged Kerr-AdS black hole (b). We take $(\Omega_1, \Omega_2, \mu_1, \mu_2, \mu_3) = (0.9, 0.9, \mu, \mu, 0)$, so that we can see the typical features of general phase diagrams.
The grand canonical partition function is given by
\[ -\frac{g^2}{2}[\phi_m, \phi_n]^2 - g\lambda^A \Gamma^m[\phi_m, \lambda_A], \] (2.1)
and the background metric is
\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + l^2 d\Omega_3^2, \] (2.2)
where \( \mu = 0, 1, 2, 3, m = 1, 2, \ldots, 6, A = 1, \ldots, 4, F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu], \)
\( A_\mu \) is the gauge field of \( U(N) \), \( \phi_m \) is a scalar field and \( \lambda_A \), which is originally a gaugino in the \( 16 \) representation of ten-dimensional type IIB supergravity, is a four-dimensional spinor in the \( (2,4) + (2,4) \) representation under \( SO(1,3) \times SU(4) \). All fields are adjoint representation of \( U(N) \). The gauge covariant derivative is defined by \( D_\mu = \nabla_\mu + i g[A_\mu, \cdot] \). \( l \) is the radius of \( S^3 \) and we set \( l = 1 \) for simplicity. The mass term of the scalar field \( (R/6)\phi_m^2 = l^{-2}\phi_m^2 \) is needed to make the theory conformal invariant, where \( R \) is the Ricci scalar in (2.2). This action has two types of global symmetry. One of them is \( R_t \times SO(4) \), which arises from the symmetry of the background spacetime (2.2), where \( R_t \) represents the time translation invariance. The other one is \( SO(6) \simeq SU(4) \), which originates from the R-symmetry of \( N = 4 \) supersymmetry.

The conserved charges are associated with commutative (Cartan) subgroups of global symmetry \( R_t \times SO(4) \times SU(4) \). Due to the time translation symmetry \( R_t \), the Hamiltonian \( \hat{H} \) is conserved. The \( SO(4) \) group contains a \( U(1)^2 \) Cartan subgroup and we denote the associated charges as \( \hat{J}_1 \) and \( \hat{J}_2 \). These charges represent angular momenta on \( S^3 \). The \( SU(4) \) group also contains a \( U(1)^3 \) Cartan subgroup and we will denote the associated charges as \( \hat{Q}_a \) \((a = 1, 2, 3)\). Therefore, we can consider a grand canonical ensemble with five chemical potentials in SYM at a finite temperature. The grand canonical partition function is given by
\[ Z(\beta) = \text{Tr} \left[ e^{-\beta(\hat{H} - \sum_{a=1}^3 \mu_a \hat{Q}_a - \Omega_1 \hat{J}_1 - \Omega_2 \hat{J}_2)} \right], \] (2.3)
where \( \mu_a, \Omega_1 \) and \( \Omega_2 \) are the chemical potentials conjugate to \( \hat{Q}_a, \hat{J}_1 \) and \( \hat{J}_2 \), respectively. To calculate this partition function, we need to know the eigenvalues of the conserved charges, \( \hat{H}, \hat{Q}_a, \hat{J}_1 \) and \( \hat{J}_2 \).

2.2. Spectrum of conserved charges

First let us determine the R-charges of the fields using the method in Ref. 30. The vector field is invariant under the \( SO(6) \) group, thus has no R-charge. When we write the six scalar fields as three complex fields
\[ \Phi_1 \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \quad \Phi_2 \equiv \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4), \quad \Phi_3 \equiv \frac{1}{\sqrt{2}}(\phi_5 + i\phi_6), \] (2.4)
\( ^* \) One can see from Table I that the spinor \( \lambda_A \) satisfies the unitarity condition of the superconformal algebra\(^{46,47}\)
\[ 2\{S, Q\} = E - J_1 - J_2 - Q_1 - Q_2 - Q_3 \geq 0, \]
\[ 2\{\bar{S}, \bar{Q}\} = E - J_1 + J_2 - Q_1 - Q_2 + Q_3 \geq 0, \]
only when we choose the \( 16 \) representation for \( \lambda_A \). This choice is appropriate: since the \( 16 \) representation does not satisfy this condition.
the generators $Q_i$ ($i = 1, 2, 3$) of the Cartan subalgebra $U(1)^3$ of the R-symmetry $SO(6)$ act on the complex vector
\[ \vec{\Phi} = (\Phi_1, \Phi_1^*, \Phi_2, \Phi_2^*, \Phi_3, \Phi_3^*)^T \]  
(2.5)
as rotations
\[
Q_1^6 = \text{diag}(1, -1, 0, 0, 0, 0), \\
Q_2^6 = \text{diag}(0, 0, 1, -1, 0, 0), \\
Q_3^6 = \text{diag}(0, 0, 0, 0, 1, -1). 
\]
(2.6)

Four Weyl fermions $\lambda^A$ transform as fundamental representation $4$ under $SU(4)_R$. We choose to represent the generators of the Cartan subalgebra, in the fundamental representation $4$, following Ref. 30, as
\[
Q_1^4 = \frac{1}{2} \text{diag}(1, 1, -1, -1), \\
Q_2^4 = \frac{1}{2} \text{diag}(1, -1, 1, -1), \\
Q_3^4 = \frac{1}{2} \text{diag}(1, -1, -1, 1). 
(2.7)
\]

This choice is consistent with the assignment of the R-charges on the scalar fields (2.6), so that the antisymmetric representation $6$ can be constructed from the tensor representation $4 \otimes 4$. Similarly, four conjugate Weyl fermions $\bar{\lambda}_A$ with the representation $\bar{4}$ have the R-charges
\[
Q_1^\bar{4} = -\frac{1}{2} \text{diag}(1, 1, -1, -1), \\
Q_2^\bar{4} = -\frac{1}{2} \text{diag}(1, -1, 1, -1), \\
Q_3^\bar{4} = -\frac{1}{2} \text{diag}(1, -1, -1, 1). 
(2.8)
\]

We now move on to the charges associated with the rotational group $SO(4)$. We denote the generators of $SO(4)$ as $\hat{J}_1$ and $\hat{J}_2$ ($(\hat{J}_1)_3 \equiv \hat{J}_1, (\hat{J}_2)_3 \equiv \hat{J}_2$), which satisfy the following commutation relation:
\[
\{(\hat{J}_1)_i, (\hat{J}_1)_j\} = i\epsilon_{ijk}(\hat{J}_1)_k, \\
\{(\hat{J}_1)_i, (\hat{J}_2)_j\} = i\epsilon_{ijk}(\hat{J}_2)_k, \\
\{(\hat{J}_2)_i, (\hat{J}_2)_j\} = i\epsilon_{ijk}(\hat{J}_1)_k. \quad (i, j, k = 1, 2, 3) 
(2.9)
\]

$SO(4)$ can be represented as two independent $SU(2)$ spins as
\[
\hat{J}_1 = \hat{J}_L + \hat{J}_R, \quad \hat{J}_2 = \hat{J}_L - \hat{J}_R, \quad (2.10)
\]
where $\hat{J}_L$ and $\hat{J}_R$ represent the generators of two $SU(2)$ groups respectively, which satisfy
\[
\{(\hat{J}_L)_i, (\hat{J}_L)_j\} = i\epsilon_{ijk}(\hat{J}_L)_k, 
\]
\[
\{(\hat{J}_R)_i, (\hat{J}_R)_j\} = i\epsilon_{ijk}(\hat{J}_R)_k, \\
\{(\hat{J}_L)_i, (\hat{J}_R)_j\} = 0.
\] (2.11)

All the fields on \(S^3\) are characterized by the eigenvalues of their spins \((j_L, j_R)\) under the two \(SU(2)\) groups. The representations of the form \((j, j \pm s)\) \((j = 0, 1/2, 1, \ldots)\) describe particles of spin \(s\).\(^{45}\) The Laplacian on \(S^3\) and the Casimir operator are related to each other. The relations for scalar fields \(\phi\), spinor fields \(\psi\) and divergenceless vector fields \(A_i\) are given by

\[
2(\hat{J}_L^2 + \hat{J}_R^2)\phi = -\nabla^2_{S^3}\phi, \\
2(\hat{J}_L^2 + \hat{J}_R^2)\psi = \left( -\nabla^2_{S^3} + \frac{R}{8} \right)\psi, \\
2(\hat{J}_L^2 + \hat{J}_R^2)A_i = \left( -\nabla^2_{S^3} + \frac{R}{3} \right)A_i,
\] (2.12, 2.13, 2.14)

where \(R = 6/l^2 = 6\) is the Ricci scalar of the three-sphere and \(\nabla^2_{S^3}\) is the Laplacian on \(S^3\). The operations of \(\hat{J}_L\) and \(\hat{J}_R\) are defined by the Lie derivative along the \(SU(2)\) generators. The proof of these relations is given in Appendix A.

We now evaluate the spectrum of the conformally coupled scalar with the representation \((j, j)\). The equation of motion is

\[
\left[ \partial^2_t - \nabla^2_{S^3} + \frac{R}{6} \right] \phi = 0.
\] (2.15)

From (2.12) and (2.15), the energy spectrum for a scalar field is given by

\[
E^2_s \equiv -\partial^2_t = -\nabla^2_{S^3} + 1 \\
= 2(\hat{J}_L^2 + \hat{J}_R^2) + 1 = (2j + 1)^2, \quad j = 0, \frac{1}{2}, 1, \ldots.
\] (2.16)

The degeneracy of the state with \((j, j)\) is \((2j + 1)^2\).

Next we move on to the analogous calculation for Dirac fermions represented as two Majorana fermions \((j, j + 1/2) + (j + 1/2, j)\). The equation of motion for fermions is

\[
\left[ \partial^2_t - \nabla^2 \right] \psi = 0,
\] (2.17)

where \(\nabla \equiv \Gamma^i \nabla_i\) and \(x^i\) are coordinates on \(S^3\). The spinor Laplacian is obtained from the square of the Dirac operator\(^{48}\)

\[
-\nabla^2 = -\nabla^2_{S^3} + \frac{R}{4}.
\] (2.18)

This is conformally covariant and we do not need the extra coupling to the Ricci scalar in (2.15). From (2.13), (2.17) and (2.18), the energy of the fermion \((j, j + 1/2)\) (or \((j + 1/2, j)\)) is

\[
E^2_f = -\nabla^2 = 2(\hat{J}_L^2 + \hat{J}_R^2) + \frac{3}{4} = \left( 2j + \frac{3}{2} \right)^2.
\] (2.19)
Table I. Spectrum and R-charges of free fields. The angular momentum \( j \) has half-integer values \( 0, \frac{1}{2}, 1, \ldots \).

| Field         | \( E \) | Degeneracy  | Representation | \((Q_1, Q_2, Q_3)\) |
|---------------|---------|-------------|----------------|----------------------|
| scalar        | \( 2j + 1 \) | \((2j + 1)^2\) | \((j, j)\)       | \((\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\) |
| fermion \((2, \tilde{4})\) | \( 2j + \frac{3}{2} \) | \((2j + 1)(2j + 2)\) | \((j + \frac{1}{2}, j)\) | \((\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}), (-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}), (-\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2})\) |
| fermion \((\tilde{2}, 4)\) | \( 2j + \frac{3}{2} \) | \((2j + 1)(2j + 2)\) | \((j, j + \frac{1}{2})\) | \((\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\) |
| vector        | \( 2j + 2 \) | \((2j + 1)(2j + 3)\) | \((j, j + 1) + (j + 1, j)\) | \((0, 0, 0)\) |

and its degeneracy is \((2j + 1)(2j + 2)\).

Finally, we consider the divergenceless vector field with the representation \((j, j + 1) + (j + 1, j)\). The equation of motion for the divergenceless vector is

\[
\left[ \partial_t^2 - \nabla^2 + \frac{R}{3} \right] A_i = 0.
\] (2.20)

From (2.14) and (2.20), the energy spectrum for the vector field becomes

\[
E_v^2 = -\nabla^2 + 2 = 2(\hat{\mathbf{j}}_L^2 + \hat{\mathbf{j}}_R^2) = (2j + 2)^2,
\] (2.21)

and the degeneracy is \((2j + 1)(2j + 3)\). We summarize these spectra and the R-charges of free fields in Table I.

2.3. Thermodynamics of \( \mathcal{N} = 4 \) super Yang-Mills theory and phase transition

Consider free Yang-Mills theory with an arbitrary gauge group and matter on any compact space with any chemical potential at a finite temperature. In a compact space, all modes of the matter fields in the gauge theory are massive and only the zero modes of the temporal gauge field remain. In this case, an exact expression for the partition function is given as follows: \( ^{21,22} \)

\[
Z(x) = \text{Tr} \left[ e^{-\beta(\hat{H} - \sum \hat{\mu}_i \hat{N}_i)} \right],
\] (2.22)

\[
= \int_G [dU] \exp \left\{ \sum_R \sum_{n=1}^{\infty} \frac{1}{n} [z_R^n(x^n, \mu_i) + (-)^{n+1} z_R^n(x^n, \mu_i)] \chi_R(U^n) \right\},
\]

where we denote the gauge group as \( G \), its element as \( U \), the character \( \chi_R \) for the representation \( R \). The \( \hat{N}_i \) are conserved charges and \( \mu_i \) are chemical potentials. We define single-particle partition functions of the boson and fermion for each representation \( R \) as

\[
z_B^R(x, \mu_i) \equiv \text{Tr}_R x^{\hat{H} - \sum \mu_i \hat{N}_i}, \quad z_F^R(x, \mu_i) \equiv \text{Tr}_R x^{\hat{H} - \sum \mu_i \hat{N}_i},
\] (2.23)

where \( x = e^{-\beta} \).

We focus on \( \mathcal{N} = 4 \) SYM on \( S^1 \times S^3 \) with the gauge group \( U(N) \). In this case, all matters are in the adjoint representation of the gauge group and we only
The single-particle partition function becomes
\[ z_S(x, \Omega_1, \Omega_2, \mu_1, \mu_2, \mu_3) = \text{Tr}_{\text{scalar}} \left[ x^{\hat{A} + \sum_{a=1}^{3} \mu_a \hat{Q}_a - \sum_{i=1}^{3} \Omega_i \hat{J}_i} \right] \]
\[ = \sum_{\text{scalar}} \sum_{j=0,1/2,...} \sum_{m_L=-j}^{j} \sum_{m_R=-j}^{j} x^{2j+1-\Omega_1(m_L+m_R)-\Omega_2(m_L-m_R)} \]
\[ = \frac{x(1-x^2)(x^{\mu_1} + x^{-\mu_1} + x^{\mu_2} + x^{-\mu_2} + x^{\mu_3} + x^{-\mu_3})}{(1-x^{1+\Omega_1})(1-x^{1+\Omega_2})(1-x^{1-\Omega_1})(1-x^{1-\Omega_2})}. \] (2.24)

Here we use the relations \( \hat{J}_1 = (\hat{J}_L + \hat{J}_R)_3 \) and \( \hat{J}_2 = (\hat{J}_L - \hat{J}_R)_3 \). The Majorana fermion modes form the representation \((j, j + 1/2) + (j + 1/2, j)\) with energy \( E = 2j + 3/2 \).

The single-particle partition function becomes
\[ z_F(x, \Omega_1, \Omega_2, \mu_1, \mu_2, \mu_3) = \sum_{\text{chiral}} \sum_{j=0,1/2,...} \sum_{m_L=-j}^{j} \sum_{m_R=-j}^{j} x^{2j+\frac{3}{2}-\Omega_1(m_L+m_R)-\Omega_2(m_L-m_R)} \]
\[ + \left[ \text{anti-chiral} \right] \]
\[ = \frac{x^{\frac{3}{2}}(x^{\frac{\mu_1}{2}} + x^{-\frac{\mu_1}{2}} - x(x^{\frac{\mu_2}{2}} + x^{-\frac{\mu_2}{2}}))}{(1-x^{1+\Omega_1})(1-x^{1+\Omega_2})(1-x^{1-\Omega_1})(1-x^{1-\Omega_2})} \]
\[ + \left[ (\mu_1, \mu_2, \mu_3, \Omega_2) \rightarrow -(\mu_1, \mu_2, \mu_3, \Omega_2) \right], \] (2.25)

where we denote \( \Omega_+ \equiv \Omega_1 + \Omega_2 \) and \( \Omega_- \equiv \Omega_1 - \Omega_2 \). The vector modes form the representation \((j, j + 1) + (j + 1, j)\) with energy \( E = 2j + 2 \) and no R-charge. The single-particle partition function becomes
\[ z_V(x, \Omega_1, \Omega_2, \mu_1, \mu_2, \mu_3) = \sum_{j=0,1/2,...} \sum_{m_L=-j}^{j} \sum_{m_R=-j}^{j} x^{2j+2-\Omega_1(m_L+m_R)-\Omega_2(m_L-m_R)} + (\Omega_2 \rightarrow -\Omega_2) \]
\[ = \frac{x^2(1 + x^2 - x^{1+\Omega_1} - x^{1-\Omega_1} - x^{1+\Omega_2} - x^{1-\Omega_2} + x^{\Omega_1+\Omega_2} + x^{\Omega_1-\Omega_2})}{(1-x^{1+\Omega_1})(1-x^{1+\Omega_2})(1-x^{1-\Omega_1})(1-x^{1-\Omega_2})} \]
\[ + (\Omega_2 \rightarrow -\Omega_2). \] (2.26)

If we set \( \Omega_i \) or all the chemical potentials to zero, expressions (2.24)–(2.26) precisely reduce to the single-particle partition functions given in Refs. 30 and 22.)
respectively. We obtain the partition function as a unitary matrix model:

\[
Z(x) = \int [dU] \exp \left( \sum_{m=1}^{\infty} \frac{1}{m} (z_B(x^m) + (-1)^{m+1}z_F(x^m)) \text{tr}(U^m)\text{tr}(U^m) \right),
\]

(2.27)

where \(z_B(x) = z_S(x) + z_V(x)\). This expression is also derived by a path integral in Appendix C.

The partition function (2.27) can be expressed by the eigenvalues \(\{e^{i\alpha_i}\} (-\pi < \alpha_i < \pi, i = 1, \ldots, N)\) of \(U\) after rewriting the Haar measure given in (B.4). The final expression becomes

\[
Z(x) = \int N \prod_{i=1}^{N} d\alpha_i \exp \left( -\sum_{i \neq j} V(\alpha_i - \alpha_j) \right),
\]

(2.28)

where

\[
V(\theta) = \log 2 + \sum_{n=1}^{\infty} \frac{1}{n} (1 - z_B(x^n) - (-1)^{n+1}z_F(x^n)) \cos(n\theta).
\]

(2.29)

In the large-\(N\) limit, the density of the eigenvalues becomes a continuous function \(\rho(\theta)\). It must be nonnegative everywhere on \(-\pi < \theta < \pi\) and can be normalized as \(\int_{-\pi}^{\pi} d\theta \rho(\theta) = 1\). The effective action \(I_{\text{gauge}} \equiv -\ln Z\) of (2.28) for \(\rho(\theta)\) becomes

\[
I_{\text{gauge}}[\rho(\theta)] = N^2 \int d\theta_1 \int d\theta_2 \rho(\theta_1)\rho(\theta_2)V(\theta_1 - \theta_2) = N^2 \sum_{n=1}^{\infty} \rho_n^2 V_n,
\]

(2.30)

where we define \(\rho_n \equiv \int_{-\pi}^{\pi} d\theta \rho(\theta) \cos(n\theta)\) and

\[
V_n \equiv \frac{1}{\pi} \int_{-\pi}^{\pi} d\theta V(\theta) \cos(n\theta) = \frac{1}{n} (1 - z_B(x^n) - (-)^{n+1}z_F(x^n)).
\]

(2.31)

This shows that the uniform eigenvalue distribution \(\rho_n = 0\) is an absolute minimum when the inequality

\[
V_n > 0 \iff z_B(x^n) + (-)^{n+1}z_F(x^n) < 1 \quad \text{for all } n
\]

(2.32)

is satisfied. Since the single-particle partition functions increase monotonically with \(x\), and \(x\) takes values \(0 < x < 1\), the condition of \(n = 1\) gives the lowest upper bound of \(x\) above which the uniform distribution does not give a minimum of \(I_{\text{gauge}}\). Therefore, the critical temperature \(T_H\), which separates the uniform phase and the non-uniform phase, is determined by

\[
z_B(x_H) + z_F(x_H) = 1,
\]

(2.33)

\footnote{In the following calculation, we will omit the arguments \(\Omega_1, \Omega_2, \mu_1, \mu_2\) and \(\mu_3\) to simplify the expressions.}
where $x_H \equiv e^{-1/T_H}$. That is to say, the sign of the coefficient $V_1$ determines whether or not the density of the eigenvalues is uniform.

Near the above critical line, the coefficients $V_{n \geq 2}$ are positive, whereas $V_1$ is negative. Hence, the configuration that gives minimal $I_{\text{gauge}}$ is realized at $\rho_1 = 1/2$ and $\rho_{n \geq 2} = 0$, and thus $I_{\text{gauge}}$ in (2.30) becomes $O(N^2)$. Below the critical line, on the other hand, the configuration $\rho_{n \geq 1} = 0$ minimizes $I_{\text{gauge}}$. In this case $I_{\text{gauge}}$ becomes $O(1)$. Therefore, this phase transition is a confinement/deconfinement transition of gauge theory: the phase of $T > T_H$ is the deconfinement phase and that of $T < T_H$ is the confinement phase. We will solve this equation and reveal the phase structure in §2.4.

In Refs. 21) and 22), the exact solution for $T > T_H$ is obtained in the large-$N$ limit, while one can approximate this solution as follows if $z_n(x) \equiv z_B(x^n) + (-)^{n+1} z_F(x^n)$ decreases exponentially with $n$ for $n > 1$:

$$\rho(\theta) = \begin{cases} \sqrt{\sin^2 \left(\frac{\theta_0}{2}\right) - \sin^2 \left(\frac{\theta}{2}\right)} \cos \frac{\theta}{2} / \pi \sin^2 \left(\frac{\theta_0}{2}\right) & (|\theta| < \theta_0) \\ 0 & (\text{elsewhere}) \end{cases}$$

$$\sin^2 \left(\frac{\theta_0}{2}\right) = 1 - \sqrt{1 - \frac{1}{z_1(x)}}. \quad (2.35)$$

The factor $z_n$ does in fact decrease exponentially, and thus we can use (2.34) as a good approximation. Substituting (2.34) into (2.30), we obtain the effective action in a very simple form:

$$I_{\text{gauge}} = -N^2 \left( \frac{1}{2 \sin^2 \left(\frac{\theta_0}{2}\right)} + \frac{1}{2} \log \left( \sin^2 \left(\frac{\theta_0}{2}\right) \right) - \frac{1}{2} \right). \quad (2.36)$$

For $T > T_H$ ($z_1 > 1$), this action is well-defined and exhibits a first-order transition of $O(N^2)$, while the action is zero for $T < T_H$ since all $\rho_n$ must be zero. We can calculate this effective action once the single-particle partition functions (2.23) are given. It will be compared quantitatively with that of dual gravity in §3.6.

2.4. Phase structure

In the previous subsection, we derived Eq. (2.33) for the critical temperature of the phase transition. In this section, we solve this equation numerically and depict the transition lines on the phase space. The phase space is the six-dimensional space of $(T, \Omega_1, \Omega_2, \mu_1, \mu_2, \mu_3)$ and we cannot cover the whole phase space. We thus focus on several slices, which are $(\mu_1, \mu_2, \mu_3) = (\mu, 0, 0), (\mu, \mu, 0), (\mu, \mu, \mu)$ and $(\mu, \Omega_1, \Omega_2)$, while the lines for $(\mu, 0, 0)$ with large $\Omega$ end at some maximal chemical potential $\mu_{\text{max}}(\Omega)$ (Fig. 2(b)). These behaviors can be understood
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by specifying where the partition function diverges at zero temperature. In the limit of \( x \to 0 \), the bosonic partition functions (2.24) and (2.26) diverge only when one of the chemical potentials approaches one. On the other hand, the fermionic partition function (2.25) can also diverge at some maximal chemical potential less than one if there are more than four chemical potentials. The general conditions for the convergence of a partition function for \( x \to 0 \) can be written as

\[
\begin{align*}
|\Omega_1|, |\Omega_2|, |\mu_1|, |\mu_2|, |\mu_3| &< 1, \\
3 - |\mu_1 + \mu_2| + \mu_3 - |\Omega_1 + \Omega_2| &> 0, \\
3 - \mu_1 - |\mu_2 + \mu_3| - |\omega_1 - \omega_2| &> 0,
\end{align*}
\]

where we have assumed \( \mu_1 \geq \mu_2 \geq \mu_3 \) without loss of generality. In the cases of \((\mu_1, \mu_2, \mu_3) = (\mu, 0, 0), (\mu, \mu, 0)\) and \((\mu, \mu, \mu)\), there exist maximum values of \( \mu \) above which the inequalities (2.37) are not satisfied. These maximum \( \mu \), which we denote as \( \mu_{\text{max}}^{(1)}, \mu_{\text{max}}^{(2)} \) and \( \mu_{\text{max}}^{(3)} \), respectively, are given by

\[
\mu_{\text{max}}^{(1)} = 1, \quad \mu_{\text{max}}^{(2)} = \min \left( \frac{3 - \Omega_1 - \Omega_2}{2}, 1 \right), \quad \mu_{\text{max}}^{(3)} = 1 - \frac{|\Omega_1 - \Omega_2|}{3}.
\]
These maximum values coincide with the end points of the transition lines at $T = 0$ in Fig. 2.

2.5. Unitarity line

In this section, we determine the unitarity line where the phase diagram is bounded. In the presence of the chemical potentials, the time derivative in the Lagrangian shifts as

$$\partial_0 \rightarrow \partial_0 - i \left( \sum_{a=1}^{3} \mu_a Q_a + \sum_{i=1}^{2} \Omega_i J_i \right).$$

Then, the Hamiltonian is shifted as $H \rightarrow H - \sum_{a=1}^{3} \mu_a Q_a - \sum_{i=1}^{2} \Omega_i J_i$, and the chemical potentials are introduced into the path integral as explained in Appendix C. By the replacement of the time derivative in (2.39), the mass of the scalar with the representation $(E_s = 2j + 1, m_L, m_R, 1, 0, 0), \ |m_L| \leq j, |m_R| \leq j$ shifts as

$$m_{scalar}^2 = E_s^2 = (2j+1)^2 \quad \rightarrow \quad m_{scalar}^2 = E_s^2 - (\mu_1 + (\Omega_1 + \Omega_2)m_L + (\Omega_1 - \Omega_2)m_R)^2.$$ (2.40)

The $j = 0$ mode first becomes tachyonic as the chemical potentials increase, and this gives the bound $\mu_1 = 1$ above which the theory breaks down. The scalar modes with $(\mu_1, \mu_2, \mu_3) = (0, 1, 0)$ and $(0, 0, 1)$ also give the bounds $\mu_2 = 1$ and $\mu_3 = 1$, respectively. Similarly, the $j = \infty$ mode also requires the upper bound $\Omega_i = 1$. The $j = \infty$ mode of the vector field also becomes tachyonic for $\Omega_i > 1$, and thus it imposes the same upper bound on $\Omega_i$. Note that although the fermionic single-particle partition function diverges for $\mu > \mu_{max}$, the theory does not break down owing to Pauli’s exclusion principle, while the tachyonic boson causes the theory to breakdown above the unitarity line.*

It is noteworthy that the $j = \infty$ mode first becomes tachyonic as each $\Omega_i$ increases for the following reason. In dual gravity theory, a similar phenomenon occurs: the $j = \infty$ mode on the Kerr-AdS black hole background first becomes unstable as we increase $\Omega_i$. This instability is called a superradiant instability, which is caused by wave amplification via a mechanism similar to the Penrose process and by wave reflection due to the potential barrier of the AdS spacetime.49)-51) This similarity with the $j = \infty$ mode suggests that the bound $\Omega_i < 1$ in gauge theory may correspond to the bound for the superradiant instability of Kerr-AdS black holes in dual gravity theory.45)

This unitarity line meets to the transition line at $T = 0$ for many cases as shown in Fig. 2, while in general a gap appears between these two lines when there are more than four chemical potentials, as shown in (2.37) and (2.38).** In §3.5 we will provide a dual description of this unitarity line, which we think is the line representing the black hole instability, and we find remarkable agreement between their behaviors.

* We thank H. Kawai for providing us with this interpretation.

** Note that in Fig. 2(c), we set $\Omega_1 = \Omega_2$, thus there is no gap.
§3. Comparison with dual gravity

In this section, we briefly review the properties of the five-dimensional asymptotically AdS black hole, which is dual to the gauge theory we have considered in the previous section. We study the Hawking-Page transition and the thermodynamical instability of charged Kerr-AdS black holes, and reveal phase structures for these black holes. We compare the phase structure of the charged Kerr-AdS black hole with that of the dual gauge theory. We also compute the ratio of the effective actions between the gauge theory and its gravity dual, and show that the ratio is close to the universal value of $3/4$ over a wide range of temperatures.

3.1. Dual gravity theory

The most general dual gravity solution (black hole) that is asymptotically AdS spacetime is expected to be constructed within five-dimensional maximal $SO(6)$-gauged $\mathcal{N} = 8$ supergravity, because this theory arises from the reduction of type IIB supergravity on $S^5$. $SO(6)$ has three $U(1)$ Cartan subgroups, therefore the black hole solution can have three independent charges. Hence, we may concentrate on the $U(1)^3$ parts of $SO(6)$ and consider $U(1)^3$-gauged $\mathcal{N} = 2$ five-dimensional supergravity. These $U(1)^3$ charges correspond to the R-charges in dual gauge theory.

The Lagrangian for the relevant bosonic sector of the maximal gauged supergravity in five dimensions is given by

$$\mathcal{L} = \mathcal{R} \star 1 - \frac{1}{2} \sum_{i=1}^{2} * d\varphi_i \wedge d\varphi_i - \frac{1}{2} \sum_{a=1}^{3} X_a^{-2} \star F^a \wedge F^a + 4 \sum_{a=1}^{3} X_a^{-1} \star 1 + F^1 \wedge F^2 \wedge A^3,$$

(3.1)

where

$$X_1 = e^{-\frac{1}{\sqrt{6}} \varphi_1 - \frac{1}{\sqrt{2}} \varphi_2}, \quad X_2 = e^{-\frac{1}{\sqrt{6}} \varphi_1 + \frac{1}{\sqrt{2}} \varphi_2}, \quad X_3 = e^{\frac{2}{\sqrt{6}} \varphi_1}. \quad (3.2)$$

The charges are given by the Gaussian integrals

$$Q_a = \frac{1}{16\pi G_5} \int_{S^3} \left( X_a^{-2} \star F^a - \frac{1}{2} \epsilon_{abc} A^b \wedge A^c \right), \quad (3.3)$$

and the angular momenta are calculated from the Komar integral

$$J = \frac{1}{16\pi G_5} \int_{S^3} * dK, \quad (3.4)$$

where $K$ is the Killing vector, that generates the rotational symmetry $U(1)$ of spacetime. $G_5$ is Newton’s constant in five dimensions and it can be written as $1/G_5 = \pi^3/G_{10} = 2N^2/\pi$ by setting the AdS space radius $l$ to one. In the following subsections, we will sketch some black hole solutions within this theory (3.1).

3.2. Five-dimensional charged Kerr-AdS black hole

The most general solution of (3.1) can have two independent rotations and three independent charges. These are five degrees of freedom excluding the mass parameter. We denote these charges as $(J_1, J_2, Q_1, Q_2, Q_3)$, where $J_1$ and $J_2$ are angular...
momenta and \(Q_1, Q_2\) and \(Q_3\) are \(U(1)\) charges. Unfortunately such a general solution has not yet been discovered. The currently known charged Kerr-AdS black holes have three or four degrees of freedom. The solutions with three degree of freedom are \((J_1, J_2, Q_1, Q_1, Q_1)\), \((J_1, J_2, Q_1, 0, 0)\) \(^{40,41}\) and \((J_1, J_2, Q_1, Q_1, Q_3(J_1, J_2, Q_1))\). \(^{39}\) The solutions with four degrees of freedom are \((J_1, J_2, Q_1, Q_3)\), \((J_1, J_1, Q_1, Q_2, Q_3)\) \(^{36,37}\) and \((J_1, J_2, Q_1, Q_2, Q_3)\) with one constraint and supersymmetry. \(^{42}\)

In this paper, we focus on the solution \((J_1, J_1, Q_1, Q_2, Q_3)\). \(^{36,37}\) The metric is given by

\[
ds^2 = -\frac{Y - f_3}{r^4 H^{1/3}} dt^2 + \frac{r^4 H^{1/3}}{Y} dr^2 + r^2 H^{1/3} d\Omega_3^2 + \frac{f_1 - r^6 H}{r^4 H^{2/3}} (\sin^2 \theta d\phi_1 + \cos^2 \theta d\phi_2)^2 - \frac{2f_2}{r^4 H^{2/3}} dt (\sin^2 \theta d\phi_1 + \cos^2 \theta d\phi_2),
\]

\[
A^a = \frac{2}{r^2 H_a} \left\{ s_a c_a dt + a (c_a s_b s_c - s_a c_b c_c) (\sin^2 \theta d\phi_1 + \cos^2 \theta d\phi_2) \right\},
\]

\[
X_a = H_a^{-1} H^{1/3},
\]

where the indices \(a, b\) and \(c\) run through 1, 2, 3, where \(a \neq b \neq c \neq a\), and

\[
H = H_1 H_2 H_3, \quad H_a = 1 + \frac{2m s_a^2}{r^2},
\]

\[
d\Omega_3^2 = d\theta^2 + \sin^2 \theta d\phi_1^2 + \cos^2 \theta d\phi_2^2,
\]

\[
s_a = \sinh \delta_a, \quad c_a = \cosh \delta_a,
\]

and the functions \(f_1, f_2, f_3\) and \(Y\) are given by

\[
f_1 = r^6 H + 2m a^2 r^2 + 4m^2 a^2 \left[ 2 \left( \prod_a c_a - \prod_a s_a \right) \prod_b s_b - \sum_{a<b} s_a^2 s_b^2 \right],
\]

\[
f_2 = 2ma \left( \prod_a c_a - \prod_a s_a \right) r^2 + 4m^2 a \prod_a s_a,
\]

\[
f_3 = 2ma^2 (1 + r^2) + 4m^2 a^2 \left[ 2 \left( \prod_a c_a - \prod_a s_a \right) \prod_b s_b - \sum_{a<b} s_a^2 s_b^2 \right],
\]

\[
Y = f_3 + r^6 H + r^4 - 2mr^2.
\]

The inverse temperature, entropy, angular velocity and electric potentials are given as

\[
\beta = \frac{2\pi \sqrt{f_1(r_+)}}{3r_+^4 + 2(1 + 2m \sum_a s_a^2) r_+^2 + 4m^2 \sum_{a<b} s_a^2 s_b^2 - 2m(1 - a^2)},
\]

\[
S = N^2 \pi \sqrt{f_1(r_+)},
\]

\[
\Omega = \frac{f_2(r_+)}{f_1(r_+)},
\]

\(^{4)}\) The thermodynamics of the solution \((J_1, J_1, Q_1, Q_1, Q_1)\) constructed in Ref. 35) were studied and the field theory dual was discussed in Ref. 38).
where the outer horizon $r_+$ is defined as the largest root of the function $Y(r)$. The conserved charges are

$$M = N^2 \frac{m(3 + a^2 + 2 \sum_i s_i^2)}{2}, \quad J = N^2 ma(\prod_a c_a - \prod_a s_a), \quad Q_a = N^2 ms_ac_a.$$  \hfill (3.12)

The effective action is given by

$$I_{\text{gravity}} = (M - TS - 2\Omega J - \mu_1 Q_1 - \mu_2 Q_2 - \mu_3 Q_3) / T.$$  \hfill (3.13)

The value of the effective action and the free energy $F \equiv TI_{\text{gravity}}$ of the black hole are measured relative to the thermal AdS space without a black hole. Therefore, when the sign of the effective action or the free energy is negative (or positive), the black hole phase is stable (or unstable) against the thermal AdS phase. This phase transition is well known to be the Hawking-Page transition, and the transition line is characterized as $I_{\text{gravity}} = 0$.

### 3.3. Five-dimensional Kerr-AdS black hole

The charged Kerr-AdS black hole (3.5) considered in the previous section contains the Kerr-AdS black hole with equal two rotations but does not contain the one with two independent rotations. Hence, here we separately treat the Kerr-AdS black hole with two independent rotations.

The five-dimensional Kerr-AdS black hole is defined by the following metric:

$$ds^2 = -\frac{\Delta_r}{\rho^2} \left( dt - \frac{a_1 \sin^2 \theta}{\Xi_1} d\phi_1 - \frac{a_2 \cos^2 \theta}{\Xi_2} d\phi_2 \right)^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left( a_1 dt - \frac{(r^2 + a_1^2)}{\Xi_1} d\phi_1 \right)^2$$

$$+ \frac{\Delta_\theta \cos^2 \theta}{\rho^2} \left( a_2 dt - \frac{(r^2 + a_2^2)}{\Xi_2} d\phi_2 \right)^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2$$

$$+ \frac{(1 + r^2)}{r^2 \rho^2} \left( a_1 a_2 dt - \frac{a_2 (r^2 + a_1^2) \sin^2 \theta}{\Xi_1} d\phi_1 - \frac{a_1 (r^2 + a_2^2) \cos^2 \theta}{\Xi_2} d\phi_2 \right)^2,$$  \hfill (3.14)

where

$$\Delta_r = \frac{1}{r^2} (r^2 + a_1^2)(r^2 + a_2^2)(1 + r^2) - 2m,$$

$$\Delta_\theta = 1 - a_1^2 \cos^2 \theta - a_2^2 \sin^2 \theta,$$

$$\rho^2 = r^2 + a_1^2 \cos^2 \theta + a_2^2 \sin^2 \theta,$$

$$\Xi_i = 1 - a_i^2.$$  \hfill (3.15)

The scalar and gauge fields are given by $X_a = 1$ and $A^a = 0$, respectively. This metric is nonsingular outside the horizon at $r = r_+$ defined by the larger root of
the equation $\Delta_+(r_+) = 0$ provided $a_i^2 < 1$ ($i = 1, 2$). To obtain the appropriate conformal boundary, we use the following coordinates:

$$T = t,$$
$$\Xi_1 y^2 \sin^2 \Theta = (r^2 + a_1^2) \sin^2 \theta,$$
$$\Xi_2 y^2 \cos^2 \Theta = (r^2 + a_2^2) \cos^2 \theta,$$
$$\phi_i = \phi_i + a_i t,$$

(3.16)

which are nonrotating at infinity. Using these coordinates, the angular velocities become

$$\Omega_i = \frac{a_i(1 + \frac{r^2}{a_i^2})}{r_+^2 + a_i^2},$$

(3.17)

and the conformal boundary becomes $R_t \times S^3$:

$$ds^2 = -dT^2 + d\Theta^2 + \sin^2 \Theta d\phi_1^2 + \cos^2 \Theta d\phi_2^2,$$

(3.18)

as expected. The inverse Hawking temperature is determined to avoid a conical singularity of the metric as

$$\beta = \frac{2\pi r_+(r_+^2 + a_1^2)(r_+^2 + a_2^2)}{2r_+^6 + (1 + a_1^2 + a_2^2)r_+^4 - a_1^2 a_2^2}.$$

(3.19)

The action relative to pure AdS space is

$$I_{\text{gravity}} = -\frac{N^2 \beta(r_+^2 + a_1^2)(r_+^2 + a_2^2)(r_+^2 - 1)}{4r_+^2(1 - a_1^2)(1 - a_2^2)}.$$

(3.20)

This action is only negative for $r_+ > 1$, and then the Hawking-Page transition takes place at $r_+ = 1$. The entropy of this black hole is given by$^{52,*}$

$$S = \frac{N^2 \pi(r_+^2 + a_1^2)(r_+^2 + a_2^2)}{r_+(1 - a_1^2)(1 - a_2^2)},$$

(3.21)

and the mass and angular momenta are

$$M = \frac{N^2 m(2\Xi_1 + 2\Xi_2 - \Xi_1 \Xi_2)}{2\Xi_1^2 \Xi_2^2}, \quad J_1 = \frac{N^2 a_1 m}{\Xi_1^2 \Xi_2}, \quad J_2 = \frac{N^2 a_2 m}{\Xi_1 \Xi_2^2}.$$

(3.22)

3.4. Phase structure

Now let us consider the phase structure of the charged Kerr-AdS black hole. The transition temperature is determined by the condition $I_{\text{gravity}}(T, \mu_i, \Omega_i) = 0$ for the action (3.13). This equation is too complicated to obtain $T$ analytically in terms of $\mu_a$ and $\Omega_i$, except for the limiting case of vanishing electric charges (Kerr-AdS black hole case) or vanishing rotations (R-charged black hole case). We therefore plot the diagrams numerically, which are shown in Fig. 3.

$^{*}$ The black hole entropy given in Ref. 11) is different from that in Ref. 52) up to $\frac{\pi}{2}$. 
First, we consider the phase diagrams of the charged Kerr-AdS black hole (Figs. 3(a)–(c)). These are similar to the phase diagrams for the dual gauge theory (Figs. 2(a)–(c)). In particular, we can see the strong agreement between Fig. 2(b) and Fig. 3(b). In this case, for $\Omega \gtrsim 0.9$, the transition line ends at $(\mu, T) = (\mu_{\text{max}}, 0)$, where $\mu_{\text{max}} < 1$. This appearance of $\mu_{\text{max}}$ also occurs in the gauge theory for $\Omega > 0.5$ (see Fig. 2(b)). This similarity may be evidence for the AdS/CFT correspondence.

The similarities of the phase diagrams for the gravity and the gauge theory can also be seen for Kerr-AdS black holes (Fig. 3(d)) and R-charged black holes (Fig. 3(e)), which can be obtained as the nonrotating limit of the charged Kerr-AdS black holes. For R-charged black holes, we can reproduce the phase diagram already obtained in Refs. 8), 30) and 53).

These similarities show that a global phase structure such as a confinement/deconfinement transition does not depend on the coupling constant if we regard the gravity theory to be the strongly coupled gauge theory via AdS/CFT. Instead of these marked similarities, there are some differences between the phase diagrams for these two theories. The transition temperatures for the gravity theory are higher than those for the gauge theory in all cases. Furthermore, the transition lines in the gravity theory can end at $\mu = 1, T > 0$, but those in the gauge theory always end at $T = 0$. This discrepancy may be due to the strong-coupling effect; the classical gravity theory is considered to be dual to the gauge theory in the strong-coupling regime, whereas we used the free gauge theory to calculate the effective action and other quantities in §2.

3.5. Instability of charged Kerr-AdS black hole

In §2.5, we studied the unitarity line for gauge theory. On the basis of the analysis of R-charged black holes, 8), 30), 54) it has been suggested that this unitarity line in the gauge theory corresponds to the thermodynamical instability line on the phase diagram in dual gravity theory. It will be interesting to study the thermodynamical stability line of charged Kerr-AdS black holes and compare it with the unitarity line in the gauge theory.*)

The thermodynamical stability of a system can be analyzed as follows. Suppose that we have a system in thermal equilibrium, and we consider a small deviation from the equilibrium state. The second law of thermodynamics is then written as

$$\delta M - T\delta S - 2\Omega \delta J - \mu_a \delta Q_a = \delta M - x_i \delta X_i \leq 0, \quad (3.23)$$

where we define $x_i = (T, 2\Omega, \mu_a)$ and $X_i = (S, J, Q_a)$. If there is a deviation that satisfies this second law (3.23), it implies that the system is unstable thermodynamically. Therefore, the stability condition is stated as

$$0 \leq \delta M - x_i \delta X_i = \left( \frac{\partial M(X)}{\partial X_i} - x_i \right) \delta X_i + \frac{1}{2} \frac{\partial^2 M(X)}{\partial X_i \partial X_j} \delta X_i \delta X_j, \quad (3.24)$$

where we have neglected $O((\delta X_i)^3)$ terms. The $O(\delta X_i)$ terms in (3.24) vanish owing to Maxwell’s relations if the system is in thermal equilibrium. Thus, if

*) We are grateful to D. Yamada for giving us advice on the computation.
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Fig. 3. Phase diagrams for charged Kerr-AdS black holes with two equal rotations and three independent R-charges: \((J, J, Q_1, Q_2, Q_3)\). We plot the transition lines for the R-symmetry chemical potentials (a) \((\mu, 0, 0)\), (b) \((\mu, \mu, 0)\) and (c) \((\mu, \mu, \mu)\), varying the angular velocities \(\Omega = \Omega_1 = \Omega_2\). We also depict the transition lines of Kerr-AdS and R-charged black holes. The lines represent the temperature of the Hawking-Page transition between the thermal AdS space and the Kerr-AdS black hole. The thermal AdS space is preferentially realized below the lines, and the black hole is preferentially formed above the lines. These figures are drawn in the same scale as the phase diagrams for the gauge theory in Fig. 2.

\[
M_{ij} \equiv \frac{\partial^2 M(X)}{\partial X_i \partial X_j} \text{ is positive definite, the thermal equilibrium system is stable. However, the explicit expression of } M_{ij} \text{ becomes complicated due to the derivatives of } M \text{ with respect to } X_i. \text{ To simplify the analysis of } M_{ij}, \text{ it is convenient to use parameters } (r_+, a, m, s_1, s_2, s_3) \text{ instead of } X_i, \text{ because the derivatives of } M \text{ with respect to these parameters are much simpler. Not all these parameters are independent of each other because of the equation } Y(r_+) = 0. \text{ Thus we may eliminate the rotation parameter } a \text{ using the equation } Y(r_+) = 0, \text{ and use the parameters } y_i \equiv (r_+, m, s_1, s_2, s_3). \text{ To use these parameters, it is convenient to define}
\]

\[
\tilde{F}(X, \tilde{x}) \equiv M(X) - \tilde{x}_i X_i, \quad (3.25)
\]

where \(\tilde{x}_i\) are free parameters independent of \(X_i\). This function \(\tilde{F}\) is equal to the
Then the Hessian of \( \tilde{F} \) becomes

\[
H_{ij} = \frac{\partial^2 \tilde{F}(X(y), \tilde{x})}{\partial y_i \partial y_j} \bigg|_{\tilde{x} = x} = \frac{\partial X_i(y) \partial^2 \tilde{F}(X, \tilde{x}) \partial X_k(y)}{\partial y_i \partial X_i \partial X_k} \bigg|_{\tilde{x} = x} + \frac{\partial^2 X_k(y) \partial \tilde{F}(X, \tilde{x})}{\partial y_j \partial X_k} \bigg|_{\tilde{x} = x}.
\]

In the final equality we used (3.26). Therefore, the positivity of \( \det(M_{ij}) \) is equivalent to that of \( \det(H_{ij}) \) as long as \( \partial X_i(y)/\partial y \) is nondegenerate, and thus the thermodynamical instability occurs when \( \det(H_{ij}) = 0 \) and \( |\partial X_i(y)/\partial y| \neq 0 \). This \( H_{ij} \) is defined by the derivatives with respect to the convenient parameters \( y_k \) and it is easy to evaluate.

We evaluate \( H_{ij} \) in the \((\mu, T)\) space for the cases of \((\mu_1, \mu_2, \mu_3) = (\mu, 0, 0), (\mu, \mu, 0)\) and \((\mu, \mu, \mu)\), fixing \( \Omega = 0.9 \). Figure 4 shows the resultant instability lines on which \( \det(H_{ij}) \) becomes zero, along with the Hawking-Page transition lines. We also depict the confinement/deconfinement transition lines and the unitarity lines of the dual gauge theory.

In Fig. 4, we can see qualitative similarities between the instability line of the gravity theory and the unitarity line of the dual gauge theory. In particular, in Fig. 4(b) for the case \((\mu_1, \mu_2, \mu_3) = (\mu, \mu, 0)\), a gap appeared between the Hawking-Page line and the instability line. We found such a gap in the dual gauge theory in §2.5. At this point we can see a strong agreement between both theories.

It seems that the instability line corresponds to the unitarity line in the gauge theory, whereas there is a discrepancy between the gradients of the lines. The instability line leans toward the large-\( \mu \) region, while the unitarity line is vertical at \( \mu = 1 \). This discrepancy can be resolved if we take a quantum correction (nonzero gauge coupling) into account in the gauge theory. Actually, in the case of \( \Omega = 0 \), it has been shown in Ref. 30) that the unitarity line is inclined and approaches the instability line of the dual gravity as ’t Hooft coupling increases in a high-temperature regime. The same behavior has also been found at a small finite temperature in Ref. 33). Hence, we may expect that the unitarity line for \( \Omega \neq 0 \) also begins to incline as ’t Hooft coupling increases and finally coincides with the instability line in the strong-coupling limit. Further investigations with nonzero gauge coupling are required to confirm our prediction.

\footnote{We require a careful treatment for the evaluation of \( H_{ij} \). For example, in the case of \((\mu, \mu, \mu)\), we have to evaluate \( H_{ij} \) assuming that \( s_1, s_2 \) and \( s_3 \) are independent of each other, and then substitute \( s_1 = s_2 = s_3 \) to obtain the final result. Otherwise we cannot find the instability in the cases \((\mu, \mu, 0)\) and \((\mu, \mu, \mu)\).}
3.6. Ratio of free energy at finite temperature

To quantitatively study the discrepancy between the free gauge theory and its dual gravity, we evaluate the ratio of the effective actions as

\[ f(T, \Omega_1, \Omega_2, \mu_1, \mu_2, \mu_3) \equiv \frac{I_{\text{gravity}}}{I_{\text{gauge}}}, \tag{3.28} \]

where the effective action of the free gauge theory is given by (2.36) and \( I_{\text{gravity}} \) is given by (3.13) or (3.20). The ratio is plotted as a function of \( T \) while fixing the chemical potentials.

We depict the ratios for the charged Kerr-AdS black holes for several values of \( \Omega_1 = \Omega_2 \equiv \Omega \) and \( \mu_1 = \mu_2 = \mu_3 \equiv \mu \) in Fig. 5(a). In Fig. 5(b), we depict the ratios for the Kerr-AdS black holes with unequal rotation: \( \Omega_1 \neq 0 \) and \( \Omega_2 = 0 \). We obtained the ratios for other chemical potentials, such as a purely R-charged case, which we do not show here because their behaviors are similar to those of the cases above. We find that the ratios approach \( 3/4 \) as the temperature increases for any value of \( \Omega_i \) or \( \mu_a \). Note that in the high-temperature limit, we cannot use the expression of the effective action for the gauge theory (2.36), because (2.36) is valid only when \( z_1 > 1 \) and \( z_{n \geq 2} < 1 \). However, we can show analytically that the ratio of the effective actions approaches \( 3/4 \) as \( T \to \infty \) using the Poincaré patch in the limit where \( S^3 \) radius goes to infinity, as done in leading order.\(^{45}\) The subleading order was computed in Ref. 55).

Surprisingly, the ratio remains approximately \( 3/4 \) even at low temperatures. This fact shows that the gauge theory corresponds fairly well to the dual charged Kerr-AdS black hole even at low temperatures for any value of \( \Omega_i \) and \( \mu_a \). However, the ratio becomes zero at some temperature in all cases. This disagreement in the effective actions does not imply the breakdown of the duality; it is merely due to the fact that the temperature of the Hawking-Page transition is always higher than that of the confinement/deconfinement transition.

Fig. 4. Phase diagrams for \( \Omega = 0.9 \) including the unitarity lines. The solid lines are the Hawking-Page transition line and the instability line of a charged Kerr-AdS black hole. The dashed lines are the confinement/deconfinement transition line and the unitarity line of the dual gauge theory.
§4. Discussion

We have studied the free $\mathcal{N}=4$ SYM theory dual to the charged Kerr-AdS black hole using the unitary matrix model. We have seen that five chemical potentials can be introduced into the thermodynamics of this theory; these chemical potentials are associated with the R-charges and the angular momenta. We found the confinement/deconfinement transition and specified the unitarity bound for this theory. In the dual gravity theory, the Hawking-Page transition and the thermodynamical instability of charged Kerr-AdS black holes have been investigated. The resulting phase diagrams for gauge theory and charged Kerr-AdS black holes resemble each other, and, in particular, we have found that the confinement/deconfinement transition line and the unitarity line of gauge theory correspond to the Hawking-Page transition line and the instability line in dual gravity theory, respectively. We have also found that the ratio of the effective actions of the two theories is always $3/4$ at high temperatures, and close to $3/4$ even at low temperatures around the Hawking-Page transition point, for all values of the chemical potentials. This result implies that the deconfinement phase of free $\mathcal{N}=4$ SYM with chemical potentials describes the dual black hole well for all cases.

We have found interesting phenomena in gauge theory and dual gravity theory when more than four chemical potentials are turned on. In gauge theory, the transition line touches the $T=0$ line at $\mu = \mu_{\text{max}} < 1$ when the chemical potentials are set to $(\mu_1, \mu_2, \mu_3, \Omega_1, \Omega_2) = (\mu, \mu, 0, \Omega, \Omega)$ (Fig. 4(b)). In other words, a gap appears between the transition line and the unitarity line only for this case, while the two lines touch at $T=0$ and $\mu = 1$ for the other cases (Figs. 4(a) and (c)). The appearance of $\mu_{\text{max}}$ is caused by the divergence of the fermion partition function in the region $\mu > \mu_{\text{max}}$. In dual gravity theory, on the other hand, a gap appears between the transition line and the instability line only in the case $(\mu_1, \mu_2, \mu_3, \Omega_1, \Omega_2) = (\mu, \mu, 0, \Omega, \Omega)$ (Fig. 4(b)). At this point the correspondence between these two theories is perfect. However, the physical origin of this gap in the gravity theory is not yet clear. It will
be interesting to investigate the origin and the reason for the correspondence.

Although the qualitative coincidence of gauge theory with chemical potentials and charged Kerr-AdS black holes is good, we have also found some discrepancies between these theories. First, the Hawking-Page transition temperature in the gravity theory is higher than the confinement/deconfinement transition temperature in dual gauge theory (Figs. 2 and 3). Second, the instability lines incline toward the large-$\mu$ direction for charged Kerr-AdS black holes, while the unitarity lines in dual gauge theory are vertical at $\mu = 1$ (Fig. 4). Finally, the ratio of effective actions is not one but almost $3/4$ (Fig. 5). These discrepancies may be resolved if we consider strong coupling gauge theories, not the free theory we investigated in the literature. For the case of zero chemical potentials, there have been some works on finite gauge coupling effects. It is known that the Hawking-Page transition temperature $T_{HP}$ decreases due to a string correction ($O(\alpha'^3)$ correction). The $\alpha'$ correction corresponds to the $1/\lambda$ correction in the strongly coupled gauge theory with gauge coupling $\lambda = \infty$; thus, it is expected that the transition temperature monotonically increases from $T_H$ at $\lambda = 0$ to $T_{HP}$ at $\lambda = \infty$. Weak coupling analysis of the transition temperature $T_H$ in the gauge side has not yet been studied, but the coupling dependence of $T_H$ in pure Yang-Mills theory was studied in Refs. 58 and 59, and it was shown that the temperature increases as the coupling becomes larger. From these results we may expect that the phase transition line rises as the coupling $\lambda$ becomes large (see Fig. 6). In addition, in the case when the chemical potentials are zero, the coupling constant dependence of the ratio of the effective actions is computed as

$$f(\lambda) = \frac{I_{\text{gravity}}}{I_{\text{gauge}}(\lambda)} = \begin{cases} \frac{3}{4} + \frac{9}{8\pi^2} \frac{\lambda}{15} \left(\frac{c(3)}{(2\lambda)^{3/2}}\right) \lambda \sim 0, \\ 1 - \frac{15}{8} \frac{\zeta(3)}{(2\lambda)^{3/2}} \lambda \sim \infty, \end{cases}$$

(4.1)

where we used a stringy corrected gravity theory as a strongly coupled gauge theory. This coupling dependence may not change even in the presence of chemical potentials since the effect of chemical potentials is negligible at high temperatures. Recently, nonperturbative approaches to SYM theory have been developed in Refs. 62–65, which are expected to clarify whether or not these discrepancies can be resolved beyond the zero coupling approximation.

It is interesting to compare our work with Ref. 66, in which the gauge theory is analyzed by a hydrodynamic approach. In this work the thermodynamical quantities of black holes were successfully reproduced from the gauge theory using a hydrodynamic approximation. The results in Ref. 66 are in agreement with ours, at least in the case when this approximation is valid, i.e. in the case that the temperature is sufficiently high. Note that they treat the gauge theory in the strongly coupled region as
a perfect fluid, while we treat the gauge theory in the weakly coupled region as a free gas. It is notable that in Ref. 66) the thermodynamical quantities exhibit the same behavior as ours, although the regions of the gauge coupling studied are very different.

In this paper, we have used the black hole solutions with one degree of freedom in the angular momentum and three degrees of freedom in the R-charges. To check the correspondence for the most general case, we have to use the solution with two independent rotations and three independent charges, although such a solution has not been constructed yet. We have found, however, that the parameter dependence of the black hole solutions is regular and smooth. In addition, the behavior of the gauge theory is reasonably smooth for the chemical potential values even in the most general case. Thus we expect that the properties of the most general black hole are similar to those we used in the literature. It is of course desirable to check this directly using the most general exact solution, which is expected to be found in the future. One prediction for this most general black hole solution is that a gap will appear between the transition line and the instability line for the case \((\mu, \mu, \mu, \Omega_1, \Omega_2)\), where \(\Omega_1 \neq \Omega_2\), for which a gap appears in the dual gauge theory (see Eq. (2.38)). It will be another nontrivial test of the correspondence to verify that such a gap really appears in this case.

Another further investigation in the weak coupling analysis of field theory is to search for the phase transition dual to the Gregory-Laflamme instability\(^{67)-69}\) in the gravity theory (see Ref. 68 for a comprehensive review). It has been shown that the black hole/black string transition corresponds to the phase transition in 1 + 1 dimensional SYM on a circle in Ref. 71), and its generalization to a higher-dimensional case was carried out in Refs. 72) and 73). Such a phase transition of \(\mathcal{N} = 4\) SYM on \(S^3\) was studied in Ref. 74) by computing an effective action at a finite temperature and weak 't Hooft coupling. It was shown that the effective potential has a new saddle point that preserves only an \(SO(5)\) subgroup of the \(SO(6)\) R-symmetry above some critical temperature, which was identified as the Gregory-Laflamme instability of the small AdS black hole predicted in Ref. 75). It would be interesting to search for such a phase transition in the presence of chemical potentials.

Our results suggest that the dynamical instability of AdS black holes can be understood in terms of the unitarity violation in the gauge theory. It is known that AdS black holes become unstable due to the superradiant instability when their rotation is too fast. Although the stability analysis of rotating black holes is difficult because of the difficulty in separating the variables, there are some works on this subject.\(^{49)-51), 76), 77}\) In these works, it was found that a Kerr-AdS black hole with \(\Omega_i > 1\) suffers from the superradiant instability, and the modes with a higher wave number first becomes unstable as \(\Omega_i\) increases. In §2.5, we found that this behavior also appears in dual gauge theory, that is, higher modes of scalar and vector fields become tachyonic when \(\Omega_i > 1\). Because of these tachyonic fields, the path integral diverges and the thermodynamical quantities cannot be defined. However, if we take into account the nonzero gauge coupling, the path integral will converge owing to the \(\phi^4\) term and \(A^4_\mu\) terms, and the thermodynamics will become well-defined. In
this case the vector field $A_\mu$ will acquire a nonzero vacuum expectation value and will break the $SO(4)$ symmetry, which is the rotational symmetry of $S^3$. In gravity theory, on the other hand, the $U(1)^2$ rotational symmetry of spacetime is broken by the superradiant instability. Thus, there is a possibility that the symmetry breaking vacuum in dual gauge theory, which emerges due to the nonzero gauge coupling effect, is the AdS/CFT counterpart of the final state of the black hole spacetime after the superradiant instability occurs. We do not know much about such a spacetime, thus it is interesting that we may be able to shed new light on this issue by analysis of dual gauge theory. This needs further investigation with nonzero gauge coupling and also some analysis in gravity theory to confirm our expectation.

The generalization to lower supersymmetry is an ambitious issue. The $\mathcal{N} = 2$ case has already been done in Ref. 78) by considering the supersymmetric orbifold gauge theory dual to $AdS_5 \times S^5/\mathbb{Z}_M$. If we change $\mathcal{N} = 4$ SYM to $\mathcal{N} = 1$ SCFTs, the five-dimensional Newton constant will be modified in dual gravity reduced to five dimensions. It will be interesting to study the phase structure of $\mathcal{N} = 1$ SCFTs at zero coupling and verify the ratio of the thermodynamical quantities. At high temperatures these ratios are known to always be approximately $3/4$, which gives quantitative evidence for the AdS/CFT correspondence for $\mathcal{N} = 1$ SUSY. Investigation in this direction would give further evidence for the correspondence at low temperatures.

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Appendix A

--- Laplacian and Casimir Operator on $S^3$ ---

In this appendix, we will prove of the relations (2.12)–(2.14). Since these relations can be proved in similar ways, we will only show the relation for fermions (2.13). The left hand side of (2.13) is interpreted as the spinor Lie derivative acting on fermions. The spinor Lie derivative is defined as

$$\mathcal{L}_X \psi = X^i \nabla_i \psi - \frac{1}{4} \nabla_i X^j \Gamma^{ij} \psi.$$  \hspace{1cm} (A.1)

We have to take the Casimir operators $\hat{j}_L$ and $\hat{j}_R$ of a three-sphere for $X^i$ in this expression, so let us start with the construction of their explicit forms. The three-
sphere is parameterized as
\[ ds^2 = g_{ij}dx^i dx^j = \frac{1}{4}(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2, \]  
where we have used the invariant forms \( \sigma^a \) \( (a = 1, 2, 3) \) of \( SU(2) \) satisfying the relation \( d\sigma^a = \frac{1}{2} \epsilon^{abc} \sigma^b \wedge \sigma^c \) with
\[ \sigma^1 = -\sin \chi d\theta + \cos \chi \sin \theta d\phi, \quad \sigma^2 = \cos \chi d\theta + \sin \chi \sin \theta d\phi, \quad \sigma^3 = d\chi + \cos \theta d\phi. \]
The coordinate ranges are \( 0 \leq \theta < \pi, \quad 0 \leq \phi < 2\pi \) and \( 0 \leq \chi < 4\pi \). We define the dual vectors \( e_a \) of \( \sigma^a \) by \( \sigma^a e_a = \delta^a_b \). Using this relation and (A.2), we obtain the following identity for the dual vectors:
\[ \sum_{a=1}^{3} e_a^i e_a^j = \frac{g^{ij}}{4}. \]  
The explicit forms of the dual vectors are
\[ e_1 = -\sin \chi \partial_\theta + \cos \chi \frac{\sin \theta}{\sin \theta} \partial_\phi - \cot \theta \cos \chi \partial_\chi, \]
\[ e_2 = \cos \chi \partial_\theta + \sin \chi \frac{\sin \theta}{\sin \theta} \partial_\phi - \cot \theta \sin \chi \partial_\chi, \quad e_3 = \partial_\chi, \]
which are in fact Killing vectors. There is another set of Killing vectors \( \xi_a \) \( (a = 1, 2, 3) \),
\[ \xi_1 = \cos \phi \partial_\theta + \sin \phi \frac{\sin \theta}{\sin \theta} \partial_\phi - \cot \theta \sin \phi \partial_\chi, \]
\[ \xi_2 = -\sin \phi \partial_\theta + \cos \phi \frac{\sin \theta}{\sin \theta} \partial_\phi - \cot \theta \cos \phi \partial_\chi, \quad \xi_3 = \partial_\phi, \]
which satisfy a similar identity to (A.4):
\[ \sum_{a=1}^{3} \xi_a^i \xi_a^j = \frac{g^{ij}}{4}. \]  
By an explicit calculation, we obtain the useful relation
\[ \sum_{a=1,2,3} \left( e_a^i \nabla_j e_a^k + \xi_a^i \nabla_j \xi_a^k \right) = 0. \]  
We now introduce the following notation:
\[ (E_p^i) \equiv (e_1^i, e_2^i, e_3^i, \xi_1^i, \xi_2^i, \xi_3^i). \]  
In this notation, we can express (A.4), (A.7) and (A.8) as
\[ E_p^i E_p^j = \frac{g^{ij}}{2}, \quad E_p^i \nabla_j E_p^k = 0, \]  
\(*\) We use the parameterization in Ref. 77).
where $\sum_{p=1}^{6}$ is omitted.

The Casimir operators are related to the two sets of Killing vectors as

$$\langle \hat{j}_L \rangle_a = i \xi_a, \quad \langle \hat{j}_R \rangle_a = -i e_a, \quad (A.11)$$

and the left hand side of (2.13) becomes

$$2(\hat{j}_L^2 + \hat{j}_R^2) = -2 {\mathcal L}_{E_p} {\mathcal L}_{E_p}. \quad (A.12)$$

Using the definition of the Lie derivative for spinor (A.1), we can calculate the right hand side of (A.12) as

$${\mathcal L}_{E_p} {\mathcal L}_{E_p} \psi = E_p^i \nabla_i \nabla_k E_p^k \nabla_k \psi + E_p^i (\nabla_i E_p^k) \nabla_k \psi - \frac{1}{4} E_p^i (\nabla_i \nabla_k E_p^k) \Gamma^{kl} \psi$$

$$- \frac{1}{2} E_p^i (\nabla_k E_p^i) \Gamma^{kl} \nabla_i \psi + \frac{1}{16} (\nabla_i E_p^j) (\nabla_k E_p^j) \Gamma^{ij} \Gamma^{kl} \psi$$

$$= \frac{1}{2} \nabla^2 \psi - \frac{1}{4} E_p^i (\nabla_i \nabla_k E_p^k) \Gamma^{kl} \psi + \frac{1}{16} (\nabla_i E_p^j) (\nabla_k E_p^j) \Gamma^{ij} \Gamma^{kl} \psi. \quad (A.13)$$

In the last equality, (A.10) has been used. Here we need the following formulae to proceed further:

$$\nabla_i \nabla_k E_p^k = -R_{klij} E_p^j, \quad (A.14)$$

$$\Gamma^{ij} \Gamma^{kl} = \Gamma^{ijkl} - 2(g^{k[i} g^j]l - g^{l[i} g^j]k) - (g^{k[i} g^j]l - g^{l[i} g^j]k), \quad (A.15)$$

where $R_{ijkl}$ is the Riemann tensor of $S^3$. For three-dimensional space, the $\Gamma^{ijkl}$ must vanish in (A.15). From (A.14), we obtain

$$E_p^i \nabla_i \nabla_k E_p^k = -R_{klij} E_p^i E_p^j = 0. \quad (A.16)$$

Thus, the second term in (A.13) vanishes. From (A.15) and (A.10),

$$(\nabla_i E_p^j) (\nabla_k E_p^k) \Gamma^{ij} \Gamma^{kl} = -4(\nabla_i E_p^j) (\nabla_i E_p^j) \Gamma^{jl} - 2(\nabla_i E_p^j) (\nabla_i E_p^j)$$

$$= -2 \nabla_i (E_p^j \nabla^i E_p^j) + 2 E_p^j \nabla^2 E_p^j$$

$$= -2 R_{jk} E_p^j E_p^k = -R. \quad (A.17)$$

Therefore, the Casimir operator can be written as

$$2(\hat{j}_L^2 + \hat{j}_R^2) \psi = \left( -\nabla_{S^3}^2 + \frac{R}{8} \right) \psi. \quad (A.18)$$

Similarly, we can obtain the relation of the Casimir operator and the Laplacian for scalar and vector fields as

$$2\langle \hat{j}_L \rangle \phi = -\nabla_{S^3}^2 \phi, \quad 2\langle \hat{j}_L \rangle A_i = \left( -\nabla_{S^3}^2 + \frac{R}{3} \right) A_i. \quad (A.19)$$
Appendix B

--- Haar Measure of $U(N)$ ---

We define the metric of a unitary matrix as

$$||dU||^2 = \text{tr}(dUdU^\dagger),$$

(B.1)

where $U$ is an element of the unitary group $U(N)$ and we suppose that $U$ depends on one parameter $t$. It is clear that this metric is invariant under constant unitary matrix rotations $U(t) \to VU(t)V^\dagger$. The unitary matrix $U(t)$ can be diagonalized by some unitary matrix $\Omega(t)$, where

$$U(t) = \Omega(t)M(t)\Omega(t)^\dagger, \quad M = \text{diag}(e^{i\alpha_1}, \ldots, e^{i\alpha_N}), \quad \Omega(t) \in U(N).$$

(B.2)

We express the unitary matrix $\Omega(t)$ as $\Omega(t) = e^{iT(t)}$ using the Hermite matrix $T(t)$. Using the above decomposition, the metric becomes a separated form

$$||dU||^2 = \text{tr}(||dM||^2 + ||M, \Omega^\dagger d\Omega||^2) = \sum_{i=1}^{N} (d\alpha_i)^2 + \sum_{i,j=1}^{N} |e^{i\alpha_i} - e^{i\alpha_j}|^2 |dT_{ij}|^2,$$

(B.3)

where we use $\text{tr}(dM^\dagger[M, \Omega^\dagger d\Omega]) = \text{tr}([dM^\dagger, M]\Omega^\dagger d\Omega) = 0$. The independent variables are $\alpha_i$, $\text{Re}T_{ij}$ and $\text{Im}T_{ij}$ for $i < j$. Taking a basis of $(\alpha_i, \text{Re}T_{ij}, \text{Im}T_{ij})$ ($i < j$) and denoting $\lambda_{ij} = |e^{i\alpha_i} - e^{i\alpha_j}|$, the metric becomes

$$G = \begin{pmatrix}
1 & \cdots & 1 \\
& \ddots & \lambda_{12} \\
& & \ddots & \lambda_{N-1,N} \\
& \lambda_{12} & \cdots & \lambda_{N-1,N}
\end{pmatrix}. \quad \text{(B.4)}$$

The Haar measure can be read from (B.3):

$$[dU] = \prod_{i=1}^{N} [d\alpha_i] \prod_{j<k} d\text{Re}T_{jk}d\text{Im}T_{jk} \cdot \sqrt{\det G}$$

$$= \prod_{i=1}^{N} [d\alpha_i] \prod_{j<k} |e^{i\alpha_j} - e^{i\alpha_k}|^2 d\text{Re}T_{jk}d\text{Im}T_{jk}$$

$$= \prod_{i=1}^{N} [d\alpha_i] \prod_{j<k} 4\sin^2 \left(\frac{\alpha_j - \alpha_k}{2}\right) [d\Omega]. \quad \text{(B.5)}$$

The term $[d\Omega]$ is the gauge volume and should be divided when we consider a gauge invariant action.
Appendix C

Derivation of Partition Function with Chemical Potentials

In this appendix, we calculate the partition function (2.3) and derive (2.27). This partition function has been derived by a group theoretical method in Refs. 21 and 22. However, the path integral derivation is also important because it will be a first step to take into account the gauge coupling.\(^{30}\)

Using (2.10), we rewrite the partition function (2.3) as

\[
Z(\beta) = \text{Tr} \left[ e^{-\beta (H - \mu_a Q_a - \Omega_+ \hat{m}_L - \Omega_- \hat{m}_R)} \right], \tag{C.1}
\]

where we define \((\hat{J})_3 = \hat{m}_L, (\hat{J})_3 = \hat{m}_R, \Omega_+ \equiv \Omega_1 + \Omega_2\) and \(\Omega_- \equiv \Omega_1 - \Omega_2\). It is well known that the partition function without chemical potentials is given by the Euclidean path integral. To introduce nonzero chemical potentials, we need some tricks.\(^{30}\) We replace \(D_0\) by \(D_0 - i \mu_a Q_a - i \Omega_+ \hat{m}_L - i \Omega_- \hat{m}_R\) in the Lorentzian action (C.4). Then the Hamiltonian is replaced by \(\hat{H} - \mu_a \hat{Q}_a - \Omega_+ \hat{m}_L - \Omega_- \hat{m}_R\). For the Euclidean signature case, we thus obtain

\[
D_0 \rightarrow D_0 - \mu_a \hat{Q}_a - \Omega_+ \hat{m}_L - \Omega_- \hat{m}_R \equiv D'_0. \tag{C.2}
\]

It is difficult to evaluate the partition function (C.1) for finite gauge coupling; thus, we consider the free theory taking the limit of \(g \rightarrow 0\). We should take this limit carefully. Since we are considering field theory on a compact space, the Gauss' law constraint becomes important. That is, the total \(U(N)\) charge on the compact space should be neutral. To take this into account, we decompose the gauge field as

\[
A_0(x^\mu) = \tilde{A}_0(x^\mu) + \frac{1}{g} a(t), \quad \frac{1}{g} a(t) \equiv \frac{1}{\omega_3} \int_{S^3} A_0(x^\mu), \tag{C.3}
\]

where \(a(t)\) is the zero mode of \(A_0\) and \(\omega_3\) is the area of the unit \(S^3\). Then, the zero mode of \(A_0\) becomes \(\mathcal{O}(g^{-1})\) and \(a(t)\) couples with other fields even in the limit of \(g \rightarrow 0\). Using this device, in the limit of \(g \rightarrow 0\), the SYM action (2.1) of the Euclidean signature becomes

\[
S = \int d^4x \sqrt{g} \text{tr} \left[ \frac{1}{2} F_{\mu \nu} F^{\mu \nu} + (D'_\mu \phi_m)^2 + l^{-2} (\phi_m)^2 + i \tilde{\lambda}_A \mathcal{D}' \lambda_A \right], \tag{C.4}
\]

where the background metric is

\[
ds^2 = g_{\mu \nu} dx^\mu dx^\nu = d\tau^2 + l^2 d\Omega_3^2 \tag{C.5}
\]

and the differential operators and field strength are defined by

\[
D'_\mu = (D'_0, D_i), \quad D_0 = \partial_0 + i [a, \cdot], \quad D_i = \nabla_i, \quad D_i = \partial_i \tilde{A}_0, \quad F_{ij} = \partial_i A_j - \partial_j A_i, \tag{C.6}
\]

where \(\nabla_i\) is the covariant derivative on \(S^3\) and \(D'_0\) is defined in (C.2). The finite temperature partition function (C.1) can be written as the Euclidean path integral,

\[
Z(\beta) = \int D\lambda D\tilde{A}_0 D\phi D\lambda \exp(-S[\tilde{A}_0, A_i, \phi, \lambda, a]). \tag{C.7}
\]
The Euclidean time $\tau$ is periodic under $\tau \sim \tau + \beta$. The boson and fermion fields in the Euclidean action are periodic and antiperiodic under $\tau \sim \tau + \beta$, respectively.

Now we take the Coulomb gauge for gauge fixing:

$$\nabla_i A^i = 0. \quad (C.8)$$

This leaves one remaining degree of gauge freedom $a(\tau) \to a(\tau) + D'_0 u_0(\tau)$, where $u_0(\tau)$ is an arbitrary function consistent with the periodicity. We fix this as$^*)$

$$\partial_\tau a(\tau) = 0. \quad (C.9)$$

Then the partition function for $\tau \sim \tau + \beta$ can be written as

$$Z = \int DA \mathcal{D} A_0 \mathcal{D} A \mathcal{D} \phi \mathcal{D} \lambda \delta(\partial_\tau a) \delta(\nabla_i A^i) \Delta_1[a] \Delta_2[A_i] \exp(-S[A_0, A_i, \phi, \lambda, a]),$$

$$= \int da_0 \mathcal{D} A_0 \mathcal{D} A \mathcal{D} \phi \mathcal{D} \lambda \delta(\nabla_i A^i) \Delta_1[a_0] \Delta_2[A_i] \exp(-S[A_0, A_i, \phi, \lambda, a_0]), \quad (C.10)$$

where $a_0$ is the zero mode of $a(t)$, which is defined by $a_0 = \beta^{-1} \int_0^\beta dt \, a(t)$. The Faddeev-Popov determinant can be written as

$$\Delta_1[a] = \text{Det}'(\partial_\tau D_0), \quad \Delta_2 = \text{Det}(\nabla^2), \quad (C.11)$$

where the domain of the functional determinant $\Delta_1$ is the zero mode of $S^3$ and the nonzero modes of $S^1$, which is the time direction. Because the zero mode of $S^3$ has eigenvalues $m_L = 0$ and $m_R = 0$ and the gauge field does not have an R-charge, we can substitute $D'_0 = D_0$ in the expression for $\Delta_1[a]$. Hence, $\Delta_1[a]$ can be calculated as

$$\Delta_1[a] = \text{Det}'(\partial_\tau) \cdot \text{Det}'(\partial_\tau + i[a_0, \cdot]) = \prod_{m \neq 0} \frac{2\pi im}{\beta} \cdot \prod_{n \neq 0} \prod_{i,j} \left[ \frac{2\pi in}{\beta} + i(\alpha_i - \alpha_j) \right]$$

$$= \left( \prod_{m \neq 0} \frac{2\pi im}{\beta} \right)^{N^2+1} \prod_{i<j} \prod_{n=1}^\infty \left[ 1 - \frac{\beta^2(\alpha_i - \alpha_j)^2}{4\pi^2 n^2} \right]$$

$$= \left( \prod_{m \neq 0} \frac{2\pi im}{\beta} \right)^{N^2+1} \prod_{i<j} \frac{4}{\beta^2(\alpha_i - \alpha_j)^2} \sin^2 \left( \frac{\beta(\alpha_i - \alpha_j)}{2} \right), \quad (C.12)$$

where $\alpha_i (i = 1, \ldots, N)$ are eigenvalues of $a_0$. In the last equality, we have used the infinite product formula

$$\prod_{n=1}^\infty \left( 1 - \frac{x^2}{n^2} \right) = \frac{1}{\pi x} \sin(\pi x). \quad (C.13)$$

$^*)$ We cannot take the gauge condition $a(\tau) = 0$, because it does not make $u_0(\tau)$ periodic under $\tau \sim \tau + \beta$. 

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The left-right invariant integration measure over Hermitian matrices $a_0$ is

$$da_0 = \prod_i d\alpha_i \prod_{i<j} (\alpha_i - \alpha_j)^{2}[d\Omega] ,$$  \hspace{1cm} (C.14)

where $[d\Omega]$ is the gauge volume arising from the diagonalization of $a_0$. Neglecting this volume, we obtain

$$da_0 \Delta_1[a_0] = \prod_i d\alpha_i \left[ \left( \prod_{m \neq 0} \frac{2\pi im}{\beta} \right)^{N^2+1} \left( \prod_{i<j} \frac{1}{\beta^2} \right) \right] \prod_{i<j} 4 \sin^2 \left( \frac{\beta(\alpha_i - \alpha_j)}{2} \right) .$$  \hspace{1cm} (C.15)

The contents of the square bracket in (C.15) do not depend on $\alpha_i$ and thus we can neglect this factor. Then, from Appendix B, $da_0 \Delta_1[a]$ is equivalent to the Haar measure of $U(N)$. Hence, we denote $da_0 \Delta_1[a]$ as $dU$. Then, the partition function can be written as

$$Z = \int dU \int \mathcal{D}\tilde{A}_0 \mathcal{D}A_i \delta(\nabla_i A^i) \Delta_2[A_i] e^{-S_{\text{gauge}}[A_0,A_i,a_0]}$$

$$\times \int \mathcal{D}\phi e^{-S_{\text{scalar}}[\phi,a_0]} \int \mathcal{D}\lambda e^{-S_{\text{fermion}}[\lambda,a_0]} ,$$  \hspace{1cm} (C.16)

where $S_{\text{gauge}}$, $S_{\text{scalar}}$ and $S_{\text{fermion}}$ are the gauge, scalar and spinor field sectors of action (C.4), respectively.

First, we focus on the gauge field in the Lagrangian (C.4). We can write it as follows after integration by parts:

$$S_{\text{gauge}} = \int d^4x \sqrt{g} \text{tr} \left[ - \tilde{A}_0 \nabla^2 \tilde{A}_0 - A^i((D_0^2 + \nabla^2)g_{ij} - \mathcal{R}_{ij})A^j \right.$$  

$$+ 2\tilde{A}_0 D_0^i \nabla_i A^j - A^j \nabla_j \nabla_i A^i \].$$  \hspace{1cm} (C.17)

The last two terms vanish because of the Coulomb gauge (C.8). The path integral for the gauge field becomes

$$\int \mathcal{D}\tilde{A}_0 \mathcal{D}A_i \delta(\nabla_i A^i) \Delta_2[A_i] e^{-S_{\text{gauge}}[A_0,A_i,a_0]}$$

$$= \int \mathcal{D}\tilde{A}_0 \mathcal{D}A_i \delta(\nabla_i A^i) \Delta_2[A_i] \exp \left( \int d^4x \sqrt{g} \text{tr} \left[ \tilde{A}_0 \nabla^2 \tilde{A}_0 + A^i(D_0^2 + \nabla^2 - 2)A_i \right] \right)$$

$$= \text{Det}(\nabla^2) \cdot \text{Det}(\nabla^2)^{-1/2} \int \mathcal{D}A_i \delta(\nabla_i A^i) \exp \left( \int d^4x \sqrt{g} \text{tr} \left[ A^i(D_0^2 + \nabla^2 - 2)A_i \right] \right) ,$$  \hspace{1cm} (C.18)

where we use $\mathcal{R}_{ij} = \mathcal{R}_{gij} / 3 = 2g_{ij}$ for $S^3$. We decompose $A_i$ into a divergenceless vector and a scalar part as

$$A_i = B_i + \partial_i \varphi ,$$  \hspace{1cm} (C.19)
where $\nabla_i B^i = 0$. By this field redefinition, the measure is replaced by $DA_i = DB_i D\varphi \operatorname{Det}(\nabla^2)^{1/2}$, and Eq. (C.18) becomes

$$
\operatorname{Det}(\nabla^2) \int DB_i D\varphi \delta(\nabla^2 \varphi) \exp \left( \int d^4x \sqrt{g} \operatorname{tr} \left[ B^i \left( D_0^2 + \nabla^2 - 2 \right) B_i \right] \right)
= \int DB_i \exp \left( \int d^4x \sqrt{g} \operatorname{tr} \left[ B^i \left( D_0^2 + \nabla^2 - 2 \right) B_i \right] \right) = \operatorname{Det}(D_0^2 + \nabla^2 - 2)^{-1/2},
$$

where the functional determinant $\operatorname{Det}(\nabla^2)$ has been canceled completely. In the final expression of (C.20), the domain of the functional determinant is a divergenceless vector. The eigenvalues of $D_0'$ and $\nabla^2$ for the gauge field are given by

$$
D_0' = \frac{2\pi in}{\beta} + i\alpha_{ij} - \Omega_+ m_L - \Omega_- m_R,
$$

$$
-\nabla^2 + 2 = E_v^2 = (2j + 2)^2,
$$

where $\alpha_{ij} \equiv \alpha_i - \alpha_j$ and $E_v$ is the energy of the divergenceless vector derived in §2.2. Since the gauge field does not have an R-charge, we can set $\hat{Q}_a = 0$ in (C.2). From §2.2, $m_L$ and $m_R$ satisfy $|m_L| \leq j$, $|m_R| \leq j + 1$ for the $(j, j + 1)$ representation or $|m_L| \leq j + 1$, $|m_R| \leq j$ for the $(j + 1, j)$ representation. Hence, the functional determinant can be calculated as

$$
\ln \operatorname{Det}(-D_0'^2 - \nabla^2 + 2)^{-1/2} = -\frac{1}{2} \operatorname{Tr} \ln(-D_0'^2 - \nabla^2 + 2)
= -\frac{1}{2} \sum_{(j, m_L, m_R)} \sum_{i,j} \sum_{n=-\infty}^{\infty} \ln \left[ \left( \frac{2\pi n}{\beta} + \alpha_{ij} + i\Omega_+ m_L + i\Omega_- m_R \right)^2 + E_v^2 \right]
= -\frac{1}{2} \sum_{(j, m_L, m_R)} \sum_{i,j} \left\{ \sum_{n \neq 0} \ln \left( 1 + \frac{\beta(\alpha + iE_v)}{2\pi n} \right) + \ln \left( 1 + \frac{\beta(\alpha - iE_v)}{2\pi n} \right) \right\}
+ \ln \left( \beta^2(\alpha + iE_v)(\alpha - iE_v) \right) + \sum_{n \neq 0} \ln \left( \frac{2\pi n}{\beta} \right)^2 - \ln(\beta^2) \right\},
$$

where we have defined $\alpha \equiv \alpha_{ij} + i\Omega_+ m_L + i\Omega_- m_R$. Because the last two terms in (C.22) are constant and independent of $\alpha$ and $E_v$, we will neglect them. Then, the curly brackets in (C.22) become

$$
\sum_{n=1}^{\infty} \left[ \ln \left( 1 - \frac{\beta^2(\alpha + iE_v)^2}{4\pi^2n^2} \right) + \ln \left( 1 - \frac{\beta^2(\alpha - iE_v)^2}{4\pi^2n^2} \right) \right] + \ln \left( \beta^2(\alpha + iE_v)(\alpha - iE_v) \right)
= \ln \left[ 4 \sin \left( \frac{\beta(\alpha + iE_v)}{2} \right) \sin \left( \frac{\beta(\alpha - iE_v)}{2} \right) \right]
= \ln \left[ e^{\beta E_v} (1 - e^{-\beta E_v + i\alpha})(1 - e^{-\beta E_v - i\alpha}) \right]
= \beta E_v + \ln \left[ (1 - e^{-\beta E_v + i\alpha})(1 - e^{-\beta E_v - i\alpha}) \right].
$$

(C.23)
In the first equality, we have used (C.13). The first term in the final line of (C.23) represents the Casimir energy. However, we should not take this term into account since, in gravity theory, we have measured the thermodynamical quantities relative to AdS spacetime, and the contribution from the Casimir energy has been subtracted. Then, (C.22) becomes

\[
-\frac{1}{2} \sum_{(j,m_L,m_R)} \sum_{i,j} \left[ \ln(1 - e^{-\beta E_v + i\beta \alpha}) + \ln(1 - e^{-\beta E_v - i\beta \alpha}) \right]
\]

\[
= \frac{1}{2} \sum_{(j,m_L,m_R)} \sum_{i,j} \sum_{n=1}^{\infty} \frac{1}{n} e^{-n\beta(E_v - i\alpha)} + (\alpha \to -\alpha)
\]

\[
= \frac{1}{2} \sum_{(j,m_L,m_R)} \sum_{n=1}^{\infty} \frac{1}{n} e^{-n\beta(E_v + \Omega_L + \Omega_L + \Omega_R - \Omega_R)} \sum_{i=1}^{N} e^{-i\beta \alpha_i} \sum_{j=1}^{N} e^{-i\beta \alpha_j} + (\alpha \to -\alpha)
\]

\[
= \sum_{n=1}^{\infty} \frac{z_V(x^n, \mu_a, \Omega_i)}{n} \text{tr}(U^n)\text{tr}(U^{-n}),
\]

where \( x = e^{-\beta} \) and \( U = e^{i\beta_0} \). \( z_V(x, \mu_a, \Omega_i) \) is the single-particle partition function of the gauge field defined by (2.26). Therefore, the path integral for the gauge field is given by

\[
\int D\tilde{A}_0 D\tilde{A}_i \delta(\nabla_i A^i) \Delta_2[A_i] e^{-S_{\text{gauge}}[\tilde{A}_0, A_i, a_0]}
\]

\[
= \exp \left( \sum_{n=1}^{\infty} \frac{z_V(x^n, \mu_a, \Omega_i)}{n} \text{tr}(U^n)\text{tr}(U^{-n}) \right). \tag{C.25}
\]

Next, we consider the scalar fields in the action (C.4). The path integral for scalar fields is

\[
\int D\phi e^{-S_{\text{scalar}}[\phi, a_0]} = \text{Det}(-D_0^2 - \nabla^2 + 1)^{-1/2} . \tag{C.26}
\]

The eigenvalues of \( D_0' \) and \( \nabla^2 \) for the scalar fields are given by

\[
D_0' = \frac{2\pi in}{\beta} + i\alpha_{ij} - \sum_{p=1}^{3} \mu_a Q_a - \Omega_L \Omega_L - \Omega_R \Omega_R ,
\]

\[
\nabla^2 + 1 = E_s^2 = (2j + 1)^2 , \tag{C.27}
\]

where \( Q_a = \pm 1 \) and \( |m_L|, |m_R| \leq j \). By a similar calculation to that in the gauge field case, the path integral for the scalar fields can be written as

\[
\int D\phi e^{-S_{\text{scalar}}[\phi, a_0]} = \exp \left( \sum_{n=1}^{\infty} \frac{z_S(x^n, \mu_a, \Omega_i)}{n} \text{tr}(U^n)\text{tr}(U^{-n}) \right), \tag{C.28}
\]

where \( z_S(x, \mu_a, \Omega_i) \) is the single-particle partition function for the scalar field defined by (2.24).
Finally, we consider the fermions. The path integral for fermions is
\[
\int D\lambda e^{-S_{\text{fermion}}[\lambda,a]} = \text{Det}(iD') = \text{Det}(-D'^2)^{1/2},
\]
\[
= \text{Det} \left( -D'^2_0 - \nabla^2 + \frac{3}{2} \right)^{1/2}. \tag{C.29}
\]

The eigenvalues for \(D'_0\) and \(\nabla^2\) are given by
\[
D'_0 = \frac{(2n+1)\pi i}{\beta} + i\alpha_{ij} - \sum_{p=1}^{3} \mu_a Q_a - \Omega_m L - \Omega_{-m} R,
\]
\[-\nabla^2 + \frac{3}{2} = E_f^2 = \left(2j + \frac{3}{2}\right)^2, \tag{C.30}
\]
where \(Q_a = \pm 1/2\). From §2.2, \(m_L\) and \(m_R\) satisfy \(|m_L| \leq j, |m_R| \leq j + 1/2\) for the \((j, j+1/2)\) representation or \(|m_L| \leq j + 1/2, |m_R| \leq j\) for the \((j+1/2, j)\) representation. Since the fermions are antiperiodic on \(S^1\), the eigenvalue of \(\partial_0\) becomes \((2n+1)\pi i/\beta\) and the calculation of the functional determinant is slightly different from that of boson fields.

\[
\ln \text{Det}(-D'^2_0 - \nabla^2 + \frac{3}{2})^{1/2}
\]
\[= \frac{1}{2} \text{Tr} \ln(-D'^2_0 - \nabla^2 + \frac{3}{2})
\]
\[= \frac{1}{2} \sum_{(j,m_L,m_R,Q_a)} \sum_{i,j} \left\{ \sum_{n=-\infty}^{\infty} \left[ \ln \left( 1 + \frac{\beta(\alpha + iE_f)}{2\pi(n+1/2)} \right) + \ln \left( 1 + \frac{\beta(\alpha - iE_f)}{2\pi(n+1/2)} \right) \right] + \ln(4) + \sum_{n=-\infty}^{\infty} \ln \left( \frac{2\pi(n+1/2)}{\beta} \right)^2 - \ln(4) \right\}. \tag{C.31}
\]

Because the last two terms in \(C.31\) are constant, we will neglect them. Then, the curly brackets in the above equation become
\[
\sum_{n=1}^{\infty} \left[ \ln \left( 1 - \frac{\beta^2(\alpha + iE_f)^2}{4\pi^2(n-1/2)^2} \right) + \ln \left( 1 - \frac{\beta^2(\alpha - iE_f)^2}{4\pi^2(n-1/2)^2} \right) \right] + \ln(4)
\]
\[= \ln \left[ 4 \cos \left( \frac{\beta(\alpha + iE_f)}{2} \right) \cos \left( \frac{\beta(\alpha - iE_f)}{2} \right) \right]
\[= \ln \left[ e^{\beta E_f}(1 + e^{-\beta E_f+i\beta\alpha})(1 + e^{-\beta E_f-i\beta\alpha}) \right]
\[= \beta E_f + \ln \left[ (1 + e^{-\beta E_f+i\beta\alpha})(1 + e^{-\beta E_f-i\beta\alpha}) \right]. \tag{C.32}
\]

In the first equality, we have used the infinite product formula
\[
\prod_{n=1}^{\infty} \left( 1 - \frac{x^2}{(n-1/2)^2} \right) = \cos(\pi x). \tag{C.33}
\]
The first term in (C.32) represents the Casimir energy, which we neglect. In the second term, we can replace $-i\beta\alpha$ by $+i\beta\alpha$ in the second parentheses because of the summations over $(i,j)$ and $(j,m_{L,R},Q_{a})$. Then, (C.31) becomes

$$\sum_{(j,m_{L,R},Q_{a})} \sum_{i,j} \ln(1 + e^{-\beta E_{f} + i\beta\alpha}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}z_{F}(x^{n},\mu_{a},\Omega_{i})}{n} \text{tr}(U^{n}) \text{tr}(U^{-n}),$$

where $z_{F}(x,\mu_{a},\Omega_{i})$ is the single-particle partition function for fermions defined by (2.25).

Therefore, the partition function taking into account the gauge, scalar and Majorana fields is given by

$$Z = \int dU \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} \left\{ z_{B}(x^{n},\mu_{a},\Omega_{i}) + (-1)^{n+1}z_{F}(x^{n},\mu_{p},\Omega_{i}) \right\} \text{tr}(U^{n}) \text{tr}(U^{-n}) \right],$$

where $z_{B}(x,\mu_{a},\Omega_{i}) \equiv z_{V}(x,\mu_{a},\Omega_{i}) + z_{S}(x,\mu_{a},\Omega_{i})$. This expression is equal to (2.27), which was derived by a group theoretical approach.

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