HADWIGER NUMBERS OF SELF-COMPLEMENTARY GRAPHS

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Abstract. The Hadwiger number of a graph $G$, denoted by $h(G)$, is the order of the largest complete minor of $G$. A graph is said to be self-complementary if it is isomorphic to its complement. We prove that for all $n \equiv 0, 1(\mod 4)$ and for all $\left\lfloor \frac{n+1}{2} \right\rfloor \leq h \leq \left\lfloor \frac{3n}{5} \right\rfloor$, there exists a self-complementary graph $G$ with $n$ vertices whose Hadwiger number is $h$.

1. Introduction

A minor of a simple undirected graph $G$ is a graph $H$ that can be obtained from $G$ through a series of vertex deletions, edge deletions, and edge contractions. The Hadwiger number of a simple undirected graph $G$, denoted by $h(G)$, is the order of the largest complete minor of $G$. The Hadwiger conjecture, one of the famous problems in graph theory, states that for any graph $G$, $\chi(G) \leq h(G)$, where $\chi(G)$ is the chromatic number of $G$. While the validity of the conjecture is still unknown in general, Girse and Gillman [1] and Rao and Sahoo [5] proved the conjecture to be true for self-complementary graphs. A graph is said to be self-complementary if it is isomorphic to its complement. Nordhaus and Gaddum [3] proved that for a self-complementary graph $G$ on $n$ vertices, $\chi(G) \leq \left\lfloor \frac{n+1}{2} \right\rfloor$. Rao and Sahoo [5] showed that $\left\lfloor \frac{n+1}{2} \right\rfloor \leq h(G)$. The same lower bound for $h(G)$ was independently found by the authors in [4]. In [5] and [4], different classes of examples show that this lower bound is attained.

Motivated by a conjecture of Kostochka [2], Stiebitz [6] proved that if $G$ is a graph on $n$ vertices, then $h(G) + h(cG) \leq \left\lfloor \frac{6n}{5} \right\rfloor$, where $cG$ is the complement of $G$. This implies that for a self-complementary graph with $n$ vertices $G$, $h(G) \leq \left\lfloor \frac{3n}{5} \right\rfloor$. Given these upper and lower bounds for the Hadwiger numbers of self-complementary graphs, Rao and Sahoo [5] asked whether each integer within this allowable range is realized as a Hadwiger number of a self-complementary graph. In what follows, we answer this question in the positive. We first show that the $\left\lfloor \frac{3n}{5} \right\rfloor$ upper bound is realized for all $n \equiv 0, 1(\mod 4)$, thus proving the upper bound for the Hadwiger number is sharp. Then we use induction to prove the following:

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Theorem 1. For all \( n \equiv 0, 1 \pmod{4} \) and for all \( \left\lfloor \frac{n+1}{2} \right\rfloor \leq h \leq \left\lfloor \frac{3n}{5} \right\rfloor \), there exists a self-complementary graph \( G \) with \( n \) vertices whose Hadwiger number is \( h \).

We prove Theorem 1 in the next two sections.

2. CONSTRUCTIONS FOR THE UPPER BOUND

Self-complementary graphs with \( n \) vertices exist only for \( n \equiv 0, 1 \pmod{4} \). One is prompted by the upper bound of \( \left\lfloor \frac{3n}{5} \right\rfloor \) to consider the remainders of \( n \) modulo 5. The Chinese Remainder Theorem yields ten cases that need to be considered. The following construction, introduced in \([5]\), provides examples of self-complementary graphs with maximum Hadwiger number in eight of the cases.

Consider \( X_r \) a self-complementary graph on \( r \) vertices, \( K_q \) the complete graph on \( q \) vertices, and \( E_q \) the graph with \( q \) vertices and no edges. The graph \( G \) with \( n = 4q + r \) vertices described in Figure 1 is self-complementary. In this figure, the triple lines mark that all edges between the respective subgraphs are present (i.e. a complete bipartite graph).

![Figure 1. A self-complementary graph with \( n = 4q + r \) vertices.](image)

By assigning different values to \( r \) and \( q \) in the graph described in Figure 1 we obtain self-complementary graphs of maximal Hadwiger number for all \( n \equiv 0, 1 \pmod{4} \), except \( n \equiv 12, 17 \pmod{20} \). These graphs are presented in Table 1. We explain why each of these eight graphs attain the maximum Hadwiger number of \( \left\lfloor \frac{3n}{5} \right\rfloor \).

Let \( p \) denote the minimum between \( q \) and \( r \). For each copy of \( E_q \), contract \( p \) disjoint edges between \( p \) of its vertices and the same \( p \) vertices of \( X_r \). This way, we obtain a \( K_{2q+p} \) minor of \( G \). Notice that for all eight graphs in Table 1 \( 2q + p \) equals the upper bound \( \left\lfloor \frac{3n}{5} \right\rfloor \), thus \( h(G) = \left\lfloor \frac{3n}{5} \right\rfloor \).

For \( n = 20s + 12 \) and \( n = 20s + 17 \), we found that no values of \( r \) and \( q \) yield graphs whose Hadwiger number is \( \left\lfloor \frac{3n}{5} \right\rfloor \) (\( 12s + 7 \) and \( 12s + 10 \), respectively). For these two cases, we provide different classes of examples.
Table 1.

| n = 4q + r | r   | q   | h = ⌊3n/5⌋ |
|------------|-----|-----|-------------|
| 20s        | 4s  | 4s  | 12s         |
| 20s + 1    | 4s + 1 | 4s  | 12s         |
| 20s + 4    | 4s  | 4s + 1 | 12s + 2   |
| 20s + 5    | 4s + 1 | 4s + 1 | 12s + 3   |
| 20s + 8    | 4s  | 4s + 2 | 12s + 4   |
| 20s + 9    | 4s + 1 | 4s + 2 | 12s + 5   |
| 20s + 13   | 4s + 1 | 4s + 3 | 12s + 7   |
| 20s + 16   | 4s + 4 | 4s + 3 | 12s + 9   |

For \( n = 12 \), \( \lfloor \frac{3n}{5} \rfloor = 7 \). The self-complementary graph with 12 vertices presented in Figure 2 contains a \( K_7 \) minor obtained by contracting the five marked edges.

Figure 2. A self-complementary graph with \( n = 12 \) vertices containing a \( K_7 \) minor. This minor can be obtained by contracting the five marked edges.

For \( n = 20s + 12 \) with \( s \geq 1 \), let \( G \) be the self-complementary graph obtained from the graph in Figure 3, together with all edges between the vertices of \( T_1, T_2, T_3, T_4 \) and the vertices of the two copies of \( E_s \), and all edges between the vertices of \( T_1, T_2, T_3, T_4 \) and the vertices of the two copies of \( K_s \). For a natural number \( m \), \( K_m \) denotes the complete graph with \( m \) vertices and \( E_m \) the graph on \( m \) vertices with an empty edge set. This graph admits a \( K_{12s+7} \) minor obtained by performing the following sequence of edge contractions:

- For \( i \in \{1, 3\} \), choose one vertex \( t_i \) of \( T_i \) and contract \( 2s \) disjoint edges between the remaining vertices of \( T_i \) and the \( 2s \) vertices of the two copies of \( K_s \);
For \( i \in \{2, 4\} \), choose one vertex \( t_i \) of \( T_i \) and contract \( 2s \) disjoint edges between the remaining vertices of \( T_i \) and the \( 2s \) vertices of the two copies of \( E_s \);

• the edge \( a_3a_4 \) contracts to form a single vertex \( a_3 = a_4 \);
• the edges \( a_1t_1 \) and \( a_1t_3 \) contract to form the vertex \( a_1 = t_1 = t_3 \).
• the edges \( a_2t_2 \) and \( a_2t_4 \) contract to form the vertex \( a_2 = t_2 = t_4 \).

Note that the four copies of \( K_{2s+1} \) induce a \( K_{8s+4} \) subgraph of \( G \). The contractions of edges in between the vertices of the \( T_i \)'s and those of the \( K_s \)’ and those of \( E_s \), respectively, take place within bipartite graphs. As such, there are many possible choices of the set of contracted edges. This sequence of edge contractions produces a \((8s + 4) + 4s + 3 = 12s + 7\) complete minor.

Figure 3. A self-complementary graph on \( n = 20s + 12 \) vertices containing a \( K_{12s+7} \) minor. All the drawn edges represent complete bipartite graphs, and \( a_1, a_2, a_3, \) and \( a_4 \) are single vertices. All edges between \( T_1, T_2, T_3, T_4 \) and the two copies of \( E_s \) and the two copies of \( K_s \) are also included in the graph. These are not drawn to preserve readability.

For \( n = 20s + 17, s \geq 0 \), we build a self-complementary graph \( G \) on \( n \) vertices by adding a vertex \( a \) to the graph in Figure 1 with \( r = 4s + 4 \) and \( q = 4s + 3 \). See Figure 1.

Consider the subgraph of \( G \) induced by the vertices of \( X_{4s+4} \). Since the average degree of the vertices of \( X_{4s+4} \) is \( \frac{4s+3}{2} \), exactly half of them have degree less than \( \frac{4s+3}{2} \) (see [4]). Let \( b \) be one of these vertices. The vertex \( a \) neighbors all the vertices of the two copies of \( K_{4s+3} \), and half of the vertices of \( X_{4s+4} \), namely those of small degree, \( b \) included. To obtain the \( K_{12s+10} \) complete minor, contract \( 4s + 3 \) disjoint edges between one copy of \( E_{4s+3} \) and the \( 4s + 3 \) vertices of \( X_{4s+4} \) except \( b \), another \( 4s + 3 \) disjoint edges between the other copy of \( E_{4s+3} \) and the same \( 4s + 3 \) vertices of \( X_{4s+4} \), and the edge \( ab \).
3. **Inductive Step**

Let $n \geq 12$, $n \equiv 0, 1 (\text{mod } 4)$. Assume that for each integer $k$ between $\lfloor \frac{n+1}{2} \rfloor$ and $\lfloor \frac{3n}{5} \rfloor$, there exists a self-complementary graph $G$ on $n$ vertices whose Hadwiger number is $k$. Using the construction in Figure 1 with $X_r = G$ and $q = 1$, we obtain a self-complementary graph on $n + 4$ vertices and Hadwiger number $k + 2$. As

$$\left\lfloor \frac{n + 4 + 1}{2} \right\rfloor - \left\lfloor \frac{n + 1}{2} \right\rfloor = 2 \quad \text{and} \quad \left\lfloor \frac{3(n + 4)}{5} \right\rfloor - \left\lfloor \frac{3n}{5} \right\rfloor \leq 3,$$

it follows that, for every $k$ in between $\lfloor \frac{n+4+1}{2} \rfloor$ and $\lfloor \frac{3(n+4)}{5} \rfloor$, one inductively builds an example of a self-complementary graph on $n + 4$ vertices and Hadwiger number $k$, with the exception of $k = \lfloor \frac{3(n+4)}{5} \rfloor$ when $\left\lfloor \frac{3(n+4)}{5} \right\rfloor - \left\lfloor \frac{3n}{5} \right\rfloor = 3$. Since the case of the upper bound $k = \lfloor \frac{3(n+4)}{5} \rfloor$ was already covered in the previous section, the proof is complete by induction on $n$.

**References**

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