HIGHER ORDER DEGREE IN SIMPLICIAL COMPLEXES, MULTI COMBINATORIAL LAPLACIAN AND APPLICATIONS OF TDA TO COMPLEX NETWORKS.

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ABSTRACT. Many real networks in social, biological or computer sciences have an inherent structure of a simplicial complex, which reflects the multi interactions among agents (and groups of agents) and constitutes the basics of Topological Data Analysis. Normally, the relevance of an agent in a network of graphs is given in terms of the number of edges incident to it, its degree, and in a simplicial network there are already notions of adjacency and degree for simplices that, as far as we know, are not valid for comparing simplices in different dimensions. We propose new notions of higher order lower, upper and generalised adjacency degrees for simplices in a simplicial complex, allowing any dimensional comparison among them and their faces. New multi parameter boundary and coboundary operators in an oriented simplicial complex are also given and a novel multi combinatorial Laplacian is defined. These operators generalise the known ones and are proved to be an effective tool for calculating the higher order degrees here presented. Thus, this mathematical framework allows us to elucidate the relevance not only of an agent, but of a bunch of them as a simplicial community, and also to study the degree of collaboration between different communities in a simplicial complex. In addition, they are effective and programmable computational techniques. Some potential applications to simplicial Network Science are also proposed.

INTRODUCTION.

The simplest way to mathematically describe real world networks is to use graphs, where nodes represent the agents of the network and edges are thought to be the interactions between these agents. Real world networks present a complex and highly
irregular behaviour, in the sense that local small changes might address to significant changes in the global network, but this random and disorganised phenomena displays a meaningful structure. Essentially, this is the reason why such systems are called complex networks. From this point of view, Network Science and Statistical Mechanics of complex networks ([3] [2]) provide a universal language which allows to classify networks, elucidate patterns of interactions and make predictions about the structure and evolution of such systems.

Despite of the success of complex networks analysis and Network Science, there is a major drawback in this approach due to the fact that an implicit assumption is made: the complex system is described by combinations of pairwise interactions, that is, binary relations. Nonetheless, many complex systems and datasets in real world networks come with a richer inherent structure, since there are higher-order interactions involving group of agents. Nowadays there are many important examples of these richer complex systems; see for instance [10] [6] [11] [13] [16] [5] for social systems, [14] for infrastructural systems or [17] [18] [4] [7] for brain and biological networks.

All of the above results are based in the use of a powerful algebraic topology tool: simplicial complexes. This notion originally comes from a discretisation of manifolds that allows to perform computational calculus. It have been also widely used in modern physics (gauge theories, quantum gravity, elasticity, ...) and is the keystone of topological data analysis (TDA). Simplicial complexes generalise the standard graph tools by allowing many-body interactions, providing robust results under continuous deformations of the system or dataset. The basic idea is simple, in graph theory we can not distinguish from three agents which are pairwise linked (for example, if they have written a paper pairwise, so that it is represented as a triangle), from the situation where the three of them have published a joint paper (and thus, in particular they have written it also pairwise, again a triangle). In simplicial theory, the triangle is a 2-simplex which, by definition, contains all of its faces (1-simplices are pairwise connections and 0-simplices are the agents), and this simplicial point of view naturally allows to keep track of the multi-interactions among the agents or group of agents. Given the finite set of vertices \( \{v_0, v_1, \ldots, v_n\} \) in a network, a \( q \)-simplex is a subset \( \sigma^{(q)} = \{v_0, v_1, \ldots, v_q\} \) such that \( v_i \neq v_j \) for all \( i \neq j \), and a \( p \)-face (for \( p < q \)) of \( \sigma^{(q)} \) is just a subset \( \tau^{(p)} = \{v_{i_1}, \ldots, v_{i_p}\} \) of \( \sigma^{(q)} \). A simplicial complex \( K \) is a collection of simplices such that if \( \sigma \) is a simplex in \( K \), then all the faces of \( \sigma \) are also in \( K \).

If one attempts to define centrality measures based on a degree notion for simplices (which would allow to characterise certain relevance of an agent, or group of agents, in a simplicial network), the definition of adjacency between simplices is required. In
definitions of lower, upper and general adjacency for $q$-simplices (where $q$ is called the dimension of the simplex) are proposed: two simplices $\sigma^{(q)}$ and $\sigma'^{(q)}$ are lower adjacent if there exists a $(q - 1)$-simplex $\tau^{(q-1)}$ which is a common $(q - 1)$-face of both of them; they are said to be upper adjacent if there exists $\tau^{(q+1)}$ having both as $q$-faces; and they are considered adjacent if they are strictly lower adjacent but not upper adjacent. Thus, it is possible to upper compare two $q$-simplices if they are faces of the same (one more dimensional) $(q + 1)$-simplex, and lower compare them if they share a common (one less dimensional) $(q - 1)$-simplex. Moreover, in those references it is proved that the degrees, associated with those notions of adjacencies, in a simplicial complex can be effectively computed by the entries of a matrix associated with the $q$-combinatorial Laplacian operator. All this implies, in particular, that two triangles can be lower compared if they share a common edge, but cannot be compared if they only share a common vertex. Similarly, one could compute with the notions at hand the number of upper adjacent triangles (upper degree) to an edge, but not the ones which are upper adjacent to a single vertex.

Having this problem in mind, in [13] the notion of upper degree for a $q$-simplex is further expanded. The expanded definition follows the idea that a vertex-to-triangle degree can be computed by counting the number of triangles incident to each edge which is incident to the vertex, and then dividing by 2, and the counting procedure proposed can be also stated in terms of certain entries of several Laplacian matrices. Numerical analysis is done in [13] to study the associated degree distributions for the co-authorship network finding that, other than the usual degree, there are not clear models for the “$q$-simplex to $(q + h)$-simplex degree of a $q$-simplex”. Thus, an alternative extension of this higher order notion of degree is introduced: the “vertex to facets degree”. In social networks, a facet represents the different groups within which the social individual interact, thus the “vertex to facets degree of a vertex” is the number of distinct maximal collaborative groups which the vertex belongs to. This nice approach also have a drawback: the authors affirm that, given a facet list, that degree can be “computed with relatively straightforward searching and counting procedure”, but no explicit computational method is given, nor a lower or a general adjacency degree notions are stated there.

This disadvantage, and the fact that there is not, as far as we know, a higher dimensional notion of all lower, upper and general degree for simplices in the literature, are the reasons why we are introducing in this paper a mathematical framework to generalise the notions of lower, upper and general adjacency and their associated degrees, that are valid for any simplicial dimension comparison. We define new notions of higher
order upper and lower adjacency which generalise the notions commented above and that, in particular, allow us to define new notions of lower and upper degrees and to redefine the “q-simplex to \((q + h)\)-simplex degree of a \(q\)-simplex”. Then, we will show how to explicitly compute the strict lower and upper degrees by giving a closed formula, thus stating an explicit mechanism to compute the “\(q\)-simplex to facets degree” of \([13]\). We will also give closed formulas for our general adjacency degree. Moreover, since the new degree notions need to be effectively computed, we define a higher order boundary operator which allows us to introduce a novel higher-order combinatorial Laplacian (generalising the already known \(q\)-combinatorial Laplacian and the graph Laplacian), so that certain entries of the associated higher order Laplacian matrix compute some of the higher dimensional degrees of a simplex. In addition, we use the boundary and coboundary operators to state closed and computational effective formulas for all the higher order degrees of simplices in a simplicial network. The importance of these notions for general real world networks rely in the fact that, we can now study the relevance of a (simplicial) community in terms of other communities collaborating with some of its (smaller dimensional) subcommunities, and also in terms of the (higher dimensional) collaborative communities on which our community is nested in. We also illustrate these ideas with some examples.

The paper is organised as follows. Section 1 starts by recalling well-known definitions of simplices, simplicial complexes, adjacency for simplices and the \(q\)-combinatorial Laplacian operator in a simplicial complex. In Section 2 we introduce the new notions of lower, upper and general adjacency for simplices of any dimension: two simplices \(\sigma^{(q)}\) and \(\sigma^{(q')}\) are \(p\)-lower adjacent if there exists a \(p\)-simplex \(\tau^{(p)}\) which is a common \(p\)-face of both of them; they are said to be \(p\)-upper adjacent if there exists \(\tau^{(p)}\) having both as faces; and they are considered \(p\)-adjacent if they are strictly \(p\)-lower adjacent (meaning \(p\)-lower adjacent and not \((p + 1)\)-lower adjacent) but not \(p'\)-upper adjacent for a certain (explicitly given) dimension \(p'\). The associated degrees to these notions of adjacencies are defined, they are proved to generalise the usual notions (and thus to recover the known definitions for particular values of \(q\), \(q'\) and \(p\)) and closed formulas to explicitly compute these higher dimensional degrees are also tackled in this section. In Section 3 a new higher order multi combinatorial Laplacian operator in an oriented simplicial complex is defined by using a novel multi-parameter boundary operator. We propose a definition for two simplices of being similarly (or dissimilarly) oriented and state a notion of an (upper and lower) oriented degree of two simplices with respect to other one. This permits to explicitly define the multi-parameter boundary operator (and its adjoint operator, the coboundary operator) only in terms of the oriented degree,
and to explicitly compute the multi combinatorial Laplacian matrices. In addition, the boundary and coboundary operators are proved to be computationally effective tools for calculating all the higher order degrees of the previous section and closed formulas are given in this section. We finish by adding in Section 4 conclusions, potential applications and future lines of research.

1. Simplicial complexes, adjacency and combinatorial Laplacian.

Simplicial complexes have been very much studied in the literature and during the last decade they have been proved to be a powerful tool in topological data analysis (TDA). Very recently, the simplicial techniques of TDA are being also applied in the contest of Complex Networks and Network Science. We shall start with some well-known definitions and properties on the category of simplicial complexes. We refer to ([15, 8]) for a wide exposition and details.

Roughly speaking, given a finite set of points \( \{v_0, v_1, \ldots, v_n\} \), which we call vertices, a \( q \)-simplex is a subset of vertices \( \{v_0, v_1, \ldots, v_q\} \) such that \( v_i \neq v_j \) for all \( i \neq j \). A \( p \)-face (for \( p < q \)) of a \( q \)-simplex is just a subset \( \{v_i, \ldots, v_p\} \) of the \( q \)-simplex. A simplicial complex \( K \) is a collection of simplices such that if \( \sigma \) is a simplex in \( K \), then all the faces of \( \sigma \) are also in \( K \).

Formally, a set \( \{v_0, \ldots, v_q\} \) of points of \( \mathbb{R}^n \) is said to be geometrically independent if the vectors \( \{v_0 - v_1, \ldots, v_0 - v_q\} \) are linearly independent.

The \( q \)-simplex spanned by these points is its the convex envelope, that is, the set of all points of \( \mathbb{R}^n \) such that

\[
\sigma = \left\{ \sum_{i=0}^{q} \lambda_i v_i : \sum_{i=0}^{q} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \text{ for all } i \right\}
\]

The points \( \{v_0, \ldots, v_q\} \) that span \( \sigma \) are called vertices of \( \sigma \) and the number \( q \) is the dimension of \( \sigma \). The simplex spanned by a proper nonempty subset \( \{v_i, \ldots, v_p\} \) of \( \{v_0, \ldots, v_q\} \) is called a \( p \)-face of \( \sigma \). If a simplex is not a face of any other simplex, then it is called a facet.

A (finite) simplicial complex in \( \mathbb{R}^n \) is a (finite) collection \( K \) of simplices in \( \mathbb{R}^n \) satisfying the following conditions:

1. If \( \sigma \in K \) and \( \tau \) is a face of \( \sigma \), then \( \tau \in K \).
2. The non-empty intersection of any two simplices of \( K \) is a face of each of them.

Each element \( \sigma \in K \) is called a \( q \)-simplex of \( K \), being \( q + 1 \) the cardinality of \( \sigma \). The union of 0-simplices of \( K \) is called the vertex set of \( K \). The dimension of \( K \) is defined
as $\dim K = \max \{ \dim \sigma : \sigma \in K \}$. We shall use the notation $\sigma^{(q)}$ to denote a simplex $\sigma$ of dimension $q$.

Hence, simplices can be understood as higher dimensional generalisations of a point, line, triangle, tetrahedron, and so on. Since simplices can codify multi interaction relations in classical networks (co-authorship network, social networks, protein interaction network, biological networks, ...), they are starting to be introduced in Network Science.

**Remark 1.** Even if simplicial complexes and many of its associated properties can be defined over a commutative ring with unity, for the sake of clarity we shall restrict ourselves to the base field $\mathbb{R}$.

Recall that the degree of a vertex is the number of its incident edges. It has local relevance in determining the centrality of a vertex and global importance in modelling the network in virtue of its degree distribution. This notion can be generalized to $q$-simplices.

Notice that as a 0-simplex, a vertex has degree $d$ if there are $d$ edges, 1-simplices, incident to it, but a 1-simplex have two 0-simplices adjacent to it (the two vertices the edge has) but it also might be adjacent to a 2-simplex (triangle). That is, we need a notion of upper and lower adjacency in order to define the degree for $q$-simplices when $q > 0$ (see [7, 8] for details).

**Definition 1.** We say that two $q$-simplices $\sigma_i^{(q)}$ and $\sigma_j^{(q)}$ of a simplicial complex $K$ are lower adjacent if they share a common $(q-1)$-face, which is called their common lower simplex. Lower adjacency is denoted as $\sigma_i^{(q)} \sim_L \sigma_j^{(q)}$.

We say that two $q$-simplices $\sigma_i^{(q)}$ and $\sigma_j^{(q)}$ of a simplicial complex $K$ are upper adjacent if they are both faces of the same common $(q+1)$-simplex, called their common upper simplex. Upper adjacency is denoted as $\sigma_i^{(q)} \sim_U \sigma_j^{(q)}$.

Notice that if two $q$-simplices are upper adjacent, then they are also lower adjacent. Moreover, if $\sigma_i^{(q)}$ and $\sigma_j^{(q)}$ are upper adjacent (resp. lower adjacent), then their common upper $(q+1)$-simplex (resp. their common lower simplex) is unique.

**Definition 2.** We define the lower degree of a $q$-simplex $\sigma^{(q)}$, $\deg_L(\sigma^{(q)})$, as the number of $(q-1)$-simplices in $K$ which are contained in $\sigma^{(q)}$, which is always $\binom{q+1}{q} = q+1$. We define the upper degree of a $q$-simplex $\sigma^{(q)}$, $\deg_U(\sigma)$, as the number of $(q+1)$-simplices in $K$ of which $\sigma^{(q)}$ is a $q$-face.

The degree of a $q$-simplex is defined as:

$$\deg(\sigma^{(q)}) := \deg_L(\sigma^{(q)}) + \deg_U(\sigma^{(q)}) = \deg_U(\sigma^{(q)}) + q + 1.$$
In network theory, the degree of a vertex also appears as a diagonal entry of the graph Laplacian matrix, defined as \( D - A \), where \( D \) is a diagonal matrix with the degree of the vertices as diagonal entries, and \( A \) is the usual vertex adjacency matrix. We will recall here the definition of the \( q \)-combinatorial Laplacian for \( q \)-simplices, and that of its associated matrix (which takes control of the degrees of \( q \)-simplices in a simplicial complex \( K \) and their adjacency relations). As we shall see, the \( q \)-Laplacian operator makes use of the \( q \)-boundary operator, so that, an orientation is needed in the simplicial complex.

Let \( \sigma \) be a simplex, we define two orderings of its vertex set to be equivalent if they differ from one another by an even permutation. If \( \dim \sigma > 0 \), this relation provides two equivalence classes and each of them is called an orientation of \( \sigma \). An oriented simplex is a simplex \( \sigma \) together with an orientation of \( \sigma \). For a geometrically independent set of points \( \{v_0, v_1, \ldots, v_q\} \) we denote by \([v_0, \ldots, v_q]\) and \([-v_0, \ldots, v_q]\) the opposite oriented simplices spanned by \( \{v_0, v_1, \ldots, v_q\} \). We say that a finite simplicial complex \( \mathcal{K} \) is oriented if all of its simplices are oriented. We shall denote by \( \tilde{S}_p(\mathcal{K}) \) the set of oriented \( p \)-simplices of the simplicial complex \( \mathcal{K} \), and by \( S_p(\mathcal{K}) \) the set of non-oriented \( p \)-simplices.

Given and oriented simplicial complex \( \mathcal{K} \), we define the group of \( q \)-chains as the free abelian group \( C_q(\mathcal{K}) \) with basis the set of oriented \( q \)-simplices of \( \mathcal{K} \). The dimension \( f_q \) of \( C_q(\mathcal{K}) \) is the number of \( q \)-dimensional simplices of the simplicial complex \( \mathcal{K} \), and it is codified in a topological invariant called the \( f \)-vector \( f = (f_0, f_1, \ldots, f_q, \ldots, f_{\dim \mathcal{K}}) \).

By assumption \( C_q(\mathcal{K}) \) is trivial if \( q \notin [0, \dim \mathcal{K}] \).

**Definition 3.** The \( q \)-boundary operator \( \partial_q : C_q(\mathcal{K}) \to C_{q-1}(\mathcal{K}) \) is the homomorphism given as the linear extension of

\[
\partial_q([v_0, \ldots, v_q]) = \sum_{i=0}^{q} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_q]
\]

where \([v_0, \ldots, \hat{v}_i, \ldots, v_q]\) denotes the oriented \((q-1)\)-simplex obtained from removing the vertex \( v_i \) in \([v_0, \ldots, v_q]\).

Let \( C^q(\mathcal{K}) = \text{Hom}_k(C_q(\mathcal{K}), k) \) the dual vector space of \( C_q(\mathcal{K}) \) (the field \( k \) is allowed to be \( \mathbb{Z}_p \), \( \mathbb{Q} \), \( \mathbb{R} \) or \( \mathbb{C} \)). Its elements, called cochains, are completely determined by specifying its value on each simplex (since chains are linear combinations of simplices). Fixing an auxiliary inner product (with respect to which the basis of \( C_q(\mathcal{K}) \) can be chosen to be orthonormal), we can identify (via the associated polarity): \( C_q(\mathcal{K}) \simeq \)
$C^q(K)$. Then, we can define the co-boundary operator:

$$\delta_{q-1}: C^{q-1}(K) \to C^q(K),$$

which is nothing but the adjoint operator:

$$\partial^*: C_{q-1}(K) \to C_q(K)$$

of the boundary map $\partial_q$. Given a linear form $\omega \in C^{q-1}(K)$ and an oriented simplex $\sigma = [v_0, \ldots, v_q] \in C_q(K)$, the map $\delta_{q-1}$ is defined as follows:

$$\delta_{q-1}(\omega)(\sigma) = \omega \left( \sum_{i=0}^q (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_q] \right),$$

which represents the evaluation of $\omega$ on all the faces of $\sigma$.

**Definition 4.** Given an oriented simplicial complex $K$, for $q \geq 0$ the $q$-combinatorial Laplacian is the linear operator $\Delta_q: C_q(K) \to C_q(K)$ defined by:

$$\Delta_q := \partial_{q+1} \circ \partial^*_q + \partial^*_q \circ \partial_q.$$

We define the upper $q$-combinatorial Laplacian $\Delta^U_q := \partial_{q+1} \circ \partial^*_q + \partial^*_q \circ \partial_q$ and the lower $q$-combinatorial Laplacian $\Delta^L_q := \partial^*_q \circ \partial_q$.

**Definition 5.** Let $K$ be an oriented simplicial complex and $\sigma_i^{(q)}, \sigma_j^{(q)} \in C_q(K)$ two $q$-simplices which are upper adjacent with common upper $(q + 1)$-simplex $\tau^{(q+1)}$. We say that $\sigma_i^{(q)}$ and $\sigma_j^{(q)}$ are similarly oriented with respect to $\tau^{(q+1)}$ if both have the same sign in $\partial_{q+1}(\tau^{(q+1)})$. If the signs are opposite we say that they are upper dissimilarly oriented.
Definition 6. Let $K$ be an oriented simplicial complex and $\sigma_i^{(q)}, \sigma_j^{(q)} \in C_q(K)$ two $q$-simplices which are lower adjacent with common lower $(q-1)$-simplex $\tau^{(q-1)}$. We say that $\sigma_i^{(q)}$ and $\sigma_j^{(q)}$ are similarly oriented with respect to $\tau^{(q-1)}$ if $\tau^{(q-1)}$ has the same sign in both $\partial_q(\sigma_i^{(q)})$ and $\partial_q(\sigma_j^{(q)})$. If the sign is different we say that they are lower dissimilarly oriented.

Let $B_q$ be the associated matrix to the boundary operator $\partial_q$ with respect to the orthonormal basis of elementary chains with some ordering for $C_q(K)$ and $C_{q-1}(K)$. Then, the associated matrix to its adjoint operator $\partial_q^* = \delta_q - 1$ with respect to the same ordered basis is the transpose matrix $B_q^t$. We call the $q$-Laplacian matrix of $K$ the associated matrix of $\Delta_q$:

$$L_q = B_{q+1}B_{q+1}^t + B_q^tB_q,$$

where we shall also denote $L_q^U = B_{q+1}B_{q+1}^t$ the upper $q$-Laplacian matrix and $L_q^L = B_q^tB_q$ the lower $q$-Laplacian matrix.

Then, the $q$-Laplacian matrices are given by (8):

$$(L_q^U)_{ij} = \begin{cases} 
\deg_U(\sigma_i^{(q)}) & \text{if } i = j, \\
1 & \text{if } i \neq j, \sigma_i^{(q)} \sim_U \sigma_j^{(q)} \text{ with similar orientation}, \\
-1 & \text{if } i \neq j, \sigma_i^{(q)} \sim_U \sigma_j^{(q)} \text{ with dissimilar orientation}, \\
0 & \text{otherwise}. 
\end{cases}$$

$$(L_q^L)_{ij} = \begin{cases} 
\deg_L(\sigma_i^{(q)}) = q + 1 & \text{if } i = j, \\
1 & \text{if } i \neq j, \sigma_i^{(q)} \sim_L \sigma_j^{(q)} \text{ with similar orientation}, \\
-1 & \text{if } i \neq j, \sigma_i^{(q)} \sim_L \sigma_j^{(q)} \text{ with dissimilar orientation}, \\
0 & \text{otherwise}. 
\end{cases}$$

$$(L_q)_{ij} = \begin{cases} 
\deg(\sigma_i^{(q)}) & \text{if } i = j, \\
1 & \text{if } i \neq j, \sigma_i^{(q)} \not\sim_U \sigma_j^{(q)} \text{ and } \sigma_i^{(q)} \sim_L \sigma_j^{(q)} \text{ with similar orientation}, \\
-1 & \text{if } i \neq j, \sigma_i^{(q)} \not\sim_U \sigma_j^{(q)} \text{ and } \sigma_i^{(q)} \sim_L \sigma_j^{(q)} \text{ with dissimilar orientation}, \\
0 & \text{if } i \neq j \text{ and either } \sigma_i^{(q)} \sim_U \sigma_j^{(q)} \text{ or } \sigma_i^{(q)} \not\sim_L \sigma_j^{(q)}. 
\end{cases}$$

Remark 2. The off diagonal entries of the $q$-adjacency matrix given in [7] are the absolute value of the off diagonal entries of the $q$-Laplacian matrix $L_q$ just given following [8,10].
Remark 3. Since a network is the 1-skeleton of a simplicial complex and $\delta_0$ and $\partial^*_1$ are zero maps, then $L_0 = \partial_1 \circ \partial^*_1$ and $L_0 = \partial_1 B^*_1$ is given by:

$$
(L_0)_{ij} = \begin{cases} 
\deg(v_i) = \deg_U(v_i) & \text{if } i = j, \\
-1 & \text{if } v_i \text{ is upper adjacent to } v_j, \\
0 & \text{otherwise.}
\end{cases}
$$

Thus, $L_0 = D - A$ is the traditional Laplacian matrix of a network.

2. Higher order adjacency and simplicial degree.

In [13] the notion of upper degree for a simplex is further expanded. Their definition follows the idea that a vertex-to-triangle degree (that is the number of triangles connected to a vertex) can be computed by counting the number of triangles incident to each edge which is incident to the vertex, and then dividing by 2 (since each triangle incident to the vertex has to be incident to a pair of edges connected to the vertex). Thus, they show that the “vertex to triangle degree” is given by:

$$
\frac{1}{2} \sum_{j_1,j_2} |(B_1)_{i,j_1}| |(B_2)_{j_1,j_2}|,
$$

(2)

where $B_1$ is the vertex-edge incidence matrix of equation (1). By an inductive procedure they propose the following formula for the “vertex to $h$-simplex degree”:

$$
\frac{1}{h!} \sum_{j_1,\ldots,j_h} |(B_1)_{i,j_1}| |(B_2)_{j_1,j_2}| \cdots |(B_h)_{j_{h-1},j_h}|
$$

(3)

And, for the “$q$-simplex to $(q + h)$-simplex degree”:

$$
\frac{1}{h!} \sum_{j_1,\ldots,j_h} |(B_{q+1})_{i,j_1}| |(B_{q+2})_{j_1,j_2}| \cdots |(B_{q+h})_{j_{h-1},j_h}|
$$

(4)

They do numerical analysis to study the associated degree distributions for the co-authorship network finding that, a part from the usual degree, there are not clear models for the “$q$-simplex to $(q + h)$-simplex degree of a $q$-simplex”. Thus, they introduce an alternative extension of this higher order notion of degree: the “vertex to facets degree”. In social networks, a facet represents the number of different groups within which the social individual interacts, thus the “vertex to facets degree of a vertex” is the number of distinct maximal collaborative groups which the vertex belongs to. That is, for each $h$ is the number of $h$-simplices incident to the vertex $v$, and such
that they are not incident to any other \((h + 1)\)-simplex incident to the vertex \(v\):

\[
\sum_{h \geq 1} \#\{\sigma^{(h)} \mid v \in \sigma^{(h)} \cap \sigma^{(h+1)} \text{ and } \sigma^{(h)} \not\subset \sigma^{(h+1)}\}
\]

(5)

A generalization for “\(q\)-simplex to facets degree” is also given in [13]:

\[
\sum_{h \geq 1} \#\{\sigma^{(q+h)} \mid \sigma^{(q)} \subseteq \sigma^{(q+h)} \text{ and } \sigma^{(q+h)} \not\subset \sigma^{(q+h+1)} \text{ if } \sigma^{(q)} \subseteq \sigma^{(q+h+1)}\}
\]

(6)

They affirm that, given a facet list, this degree can be “computed with relatively straightforward searching and counting procedure”, but no explicit formula is given.

In this section we shall define a new notion of higher order lower, upper and general adjacency for simplices and their associated degrees, which in particular allows us to redefine the “\(q\)-simplex to \((q + h)\)-simplex degree of a \(q\)-simplex” of [13]. Then we will present properties and closed formulae for these degrees, and we shall also illustrate how to explicitly compute the “\(q\)-simplex to facets degree” by using our upper degree definition.

2.1. Generalised adjacency for simplices. Let \(\sigma_i^{(q)}\) be a \(q\)-simplex and \(\sigma_j^{(q')}\) be a \(q'\)-simplex of a simplicial complex \(K\). For simplicity we shall omit the subscripts \(i\) and \(j\), unless confusion can arise.

**Definition 7.** We say that \(\sigma^{(q)}\) and \(\sigma^{(q')}\) are \(p\)-lower adjacent if there exists a \(p\)-simplex \(\tau^{(p)}\) which is a common face of both \(\sigma^{(q)}\) and \(\sigma^{(q')}\):

\[
\sigma^{(q)} \sim_{L_p} \sigma^{(q')} \iff \exists \tau^{(p)} : \tau^{(p)} \subseteq \sigma^{(q)} \& \tau^{(p)} \subseteq \sigma^{(q')}.
\]

Note that if \(\sigma^{(q)} \sim_{L_{p'}} \sigma^{(q')}\), then \(\sigma^{(q)} \sim_{L_{p'}} \sigma^{(q')}\) for all \(0 \leq p' \leq p\). Therefore, we say that \(\sigma^{(q)}\) and \(\sigma^{(q')}\) are strictly \(p\)-lower adjacent, referred as \(p^*\)-lower adjacent, if \(\sigma^{(q)} \sim_{L_p} \sigma^{(q')}\) and \(\sigma^{(q)} \not\sim_{L_{p+1}} \sigma^{(q')}\). We shall write \(\sigma^{(q)} \sim_{L_{p^*}} \sigma^{(q')}\) for the strict lower adjacency.

**Definition 8.** We say that \(\sigma^{(q)}\) and \(\sigma^{(q')}\) are \(p\)-upper adjacent if there exists a \(p\)-simplex \(\tau^{(p)}\) having both \(\sigma^{(q)}\) and \(\sigma^{(q')}\) as faces:

\[
\sigma^{(q)} \sim_{U_p} \sigma^{(q')} \iff \exists \tau^{(p)} : \sigma^{(q)} \subseteq \tau^{(p)} \& \sigma^{(q')} \subseteq \tau^{(p)}.
\]

We say that \(\sigma^{(q)}\) and \(\sigma^{(q')}\) are strictly \(p\)-upper adjacent, referred as \(p^*\)-upper adjacent and denoted as \(\sigma^{(q)} \sim_{U_{p^*}} \sigma^{(q')}\), if \(\sigma^{(q)} \sim_{U_p} \sigma^{(q')}\) and \(\sigma^{(q)} \not\sim_{U_{p+1}} \sigma^{(q')}\).

**Remark 4.** Let us point out some comments.
(1) If \( q = q' \) and \( p = q - 1 \) (resp. \( p = q + 1 \)), then the notion \((q-1)\)-lower (resp. \((q+1)\)-upper) adjacency recovers the ordinary lower (resp. upper) adjacency for \( q \)-simplices of Definition 1. Thus, \((q+1)\)-upper adjacency implies \((q-1)\)-lower adjacency for \( q \)-simplices. However, in contrast to the ordinary case, the uniqueness of the common lower (resp. upper) simplex is no longer true. 

(2) If \( h \geq 0 \) and \( \sigma^{(q)} \sim_{U_{q+h}} \sigma^{(q+h)} \), then \( \sigma^{(q)} \) is a face of \( \sigma^{(q+h)} \) and thus \( \sigma^{(q)} \sim_{L_q} \sigma^{(q+h)} \).

(3) Although, in general is no longer true that \( p \)-upper adjacency implies \( p' \)-lower adjacency, one has that for \( q \geq q' \geq h \) if \( \sigma^{(q)}_i \sim_{U_{q+h}} \sigma^{(q')}_{j} \), then \( \sigma^{(q)}_i \sim_{L_{q-h}} \sigma^{(q')}_{j} \).

**Proposition 1.** Assume \( \sigma^{(q)}_i \sim_{L_{p'}} \sigma^{(q')}_{j} \) for some \( p \) and put \( p' = q + q' - p \). If \( \sigma^{(q)}_i \not\sim_{U_{p'}} \sigma^{(q')}_{j} \), then \( \sigma^{(q)}_i \not\sim_{U_{p'+h}} \sigma^{(q')}_{j} \) for all \( h \geq 1 \).

**Proof.** Assume \( \sigma^{(q)}_i \sim_{U_{p'+h}} \sigma^{(q')}_{j} \) for some \( h \geq 1 \). Then, there exists a \((p'+h)\)-simplex \( \tau^{(p'+h)} \) such that \( \sigma^{(q)}_i \cup \sigma^{(q')}_{j} \subseteq \tau^{(p'+h)} \). In particular, \( \tau^{(p'+h)} \) has a \( p' \)-face containing both \( \sigma^{(q)}_i \) and \( \sigma^{(q')}_{j} \) as faces, that is, \( \sigma^{(q+1)}_i \sim_{U_{p'}} \sigma^{(q')}_{j} \). \( \square \)

**Remark 5.** Notice that if \( \sigma^{(q)}_i \sim_{L_{p'}} \sigma^{(q')}_{j} \), then \( \sigma^{(q)}_i \) and \( \sigma^{(q')}_{j} \) share \( p+1 \)-vertices. Thus, the smallest simplex which might contain both as faces (and therefore all of their vertices) has to have \( q+1+q'+1-(p+1) = q+q'-p+1 \) vertices, and thus it should be a \( p' = q + q' - p \)-simplex.

This justifies the following definition.

**Definition 9.** We say that \( \sigma^{(q)} \) and \( \sigma^{(q')} \) are \( p \)-adjacent if they are \( p^* \)-lower adjacent and not \( p' \)-upper adjacent, for \( p' = q + q' - p \):

\[
\sigma^{(q)} \sim_{A_p} \sigma^{(q')} \iff \sigma^{(q)} \sim_{L_{p^*}} \sigma^{(q')} \& \sigma^{(q)} \not\sim_{U_{p'}} \sigma^{(q')}.
\]

In order to agree with graph theory, for \( q = 0 \) we say that two vertices \( v_i \) and \( v_j \) are adjacent if \( v_i \sim_{U_0} v_j \).

We say that \( \sigma^{(q')} \) is maximal \( p \)-adjacent to \( \sigma^{(q)} \) if:

\[
\sigma^{(q')} \sim_{A_{p^*}} \sigma^{(q)} \iff \sigma^{(q')} \sim_{A_p} \sigma^{(q)} \& \sigma^{(q')} \not\subset \sigma^{(q'')} \forall \sigma^{(q'')} \mid \sigma^{(q'')} \sim_{A_p} \sigma^{(q)}.
\]

**Remark 6.** With the maximal \( p \)-adjacency we are saying that \( \sigma^{(q')} \) and \( \sigma^{(q)} \) are maximal collaborative simplicial communities in the sense that, even if some faces (subcommunities) of \( \sigma^{(q')} \) might be \( p \)-adjacent to \( \sigma^{(q)} \), they are not taken into account since the biggest one \( p \)-adjacent to \( \sigma^{(q)} \) is \( \sigma^{(q')} \).
2.2. Generalised lower degree for simplices.

**Definition 10.** We define the $p$-lower degree of a $q$-simplex $\sigma^{(q)}$ as the number of $q'$-simplices which are $p$-lower adjacents to $\sigma^{(q)}$:

$$\deg_p^{L}(\sigma^{(q)}) := \#\{\sigma^{(q')} : \sigma^{(q')} \sim_{L_p} \sigma^{(q)}\}.$$ 

The strictly $p$-lower degree of a $q$-simplex $\sigma^{(q)}$ is the number of $q'$-simplices which are $p^*$-lower adjacents to $\sigma^{(q)}$:

$$\deg_p^{*L}(\sigma^{(q)}) := \#\{\sigma^{(q')} : \sigma^{(q')} \sim_{L_p} \sigma^{(q)} \& \sigma^{(q')} \not\sim_{L_{p+1}} \sigma^{(q)}\}.$$ 

We define the $(h,p)$-lower degree of a $q$-simplex $\sigma^{(q)}$ as the number of $(q-h)$-simplices which are $p$-lower adjacents to $\sigma^{(q)}$:

$$\deg_{L}^{h,p}(\sigma^{(q)}) := \#\{\sigma^{(q-h)} : \sigma^{(q-h)} \sim_{L_p} \sigma^{(q)}\}.$$ 

The strictly $(h,p)$-lower degree of a $q$-simplex $\sigma^{(q)}$ is the number of $(q-h)$-simplices which are $p^*$-lower adjacent to $\sigma^{(q)}$:

$$\deg_{L}^{h,p^{*}}(\sigma^{(q)}) := \#\{\sigma^{(q-h)} : \sigma^{(q-h)} \sim_{L_p} \sigma^{(q)} \& \sigma^{(q-h)} \not\sim_{L_{p+1}} \sigma^{(q)}\}.$$ 

**Example 1.** In Figure 2 (a) we have that the blue edge $\sigma^{(1)}$ is lower adjacent to the yellow triangle $\sigma^{(2)}$ (in the vertex $v$), but there does not exists $p'$ such that $\sigma^{(1)}$ and $\sigma^{(2)}$ were $p'$-upper adjacent. That is, $\sigma^{(1)} \sim_{L_0} \sigma^{(2)}$ and $\sigma^{(1)} \not\sim_{U_p} \sigma^{(2)}$, and thus $\sigma^{(1)} \sim_{A_0} \sigma^{(2)}$.

In this same picture, if we consider $q = 1$, $h = 0$ and $p = 0$, then the $(0,0)$-lower degree of $\sigma^{(1)}$:

$$\deg_{L}^{0,0}(\sigma^{(1)}) = \#\{\tau^{(1)} : \tau^{(1)} \sim_{L_0} \sigma^{(1)}\}$$ 

is the number of adjacent edges distinct to $\sigma^{(1)}$ in one of its vertices, so that it is 3. If we choose $q = 1$, $h = 1$ and $p = 0 = q - h$, then the $(1,0)$-lower degree of $\sigma^{(1)}$:

$$\deg_{L}^{1,0}(\sigma^{(1)}) = \#\{\tau^{(0)} : \tau^{(0)} \sim_{L_0} \sigma^{(1)}\}$$ 

is the number of vertices of $\sigma^{(1)}$, which is 2.
Figure 2 (b) shows that the green vertex $\sigma^{(0)}$ is not lower adjacent to the blue edge $\sigma^{(1)}$ but they are 2-upper adjacent, since there exists a triangle $\tau^{(2)}$ (in pink) containing both as faces. This gives an example that with our notion of adjacency, upper adjacency does not imply lower adjacency in general.

In Figure 2 (c) we have that $\sigma^{(1)} \sim_{L_0} \sigma^{(3)}$ (they intersects in the vertex $v$) but they are not upper adjacent, so that $\sigma^{(1)} \sim_{A_0} \sigma^{(3)}$. Setting $q = 3$, $h = 2$ and $p = 0$ then $\deg^L_{L_0}(\sigma^{(3)})$ is the number of edges lower incident to the blue tetrahedron $\sigma^{(3)}$ in a vertex, and thus it is 7 (six coming from the edges of $\sigma^{(3)}$ plus the edge $\sigma^{(1)}$).

We have the following properties:

- If $h = 1$ and $p = q - h = q - 1$ then the $(1, q - 1)$-lower degree of a $q$-simplex is the lower degree of the $q$-simplex of Definition 2:
  \[
  \deg^{L,q-1}_L(\sigma^{(q)}) := \#\{\tau^{(q-1)} \mid \tau^{(q-1)} \sim_{L_{q-1}} \sigma^{(q)}\} = \#\{(q - 1) - \text{faces of } \sigma^{(q)}\} = q + 1
  \]

- If $p = q - h$ we have that:
  \[
  \deg^{h,q-h}_L(\sigma^{(q)}) := \#\{\tau^{(q-h)} \mid \tau^{(q-h)} \sim_{L_{q-h}} \sigma^{(q)}\} = \#\{(q - h) - \text{simplices of } \sigma^{(q)}\} = \binom{q + 1}{q - h + 1}
  \]

- From the very definition we have that:
  \[
  \deg^{h,p}\ast_L(\sigma^{(q)}) = \deg^{h,p}_L(\sigma^{(q)}) - \deg^{h,p+1}_L(\sigma^{(q)}).
  \]

- $\deg^{p}_L(\sigma^{(q)}) = \sum_{h=q-\dim K}^{q-p} \deg^{h,p}_L(\sigma^{(q)}).

- $\deg^{p}\ast_L(\sigma^{(q)}) = \sum_{h=q-\dim K}^{q-p} \deg^{h,p}\ast_L(\sigma^{(q)}).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{simplicial_complexes.png}
\caption{Simplicial complexes. Computing lower and upper strict degrees.}
\end{figure}
Example 2. Let us use Figure 3 (a) to perform a few computations for the lower degree of equation (8).

- Set $q = 3$, $h = 1$ and $p = 1$. Let us compute the $(1,1)$-lower degree of the blue tetrahedron $\sigma^{(3)}$:

$$
\deg^{1,1}_L(\sigma^{(3)}) = \# \{ \text{incident triangles to } \sigma^{(3)} \text{ in an edge} \} = \\
= \# \{ \text{incident triangles to } \sigma^{(3)} \text{ in an edge and not in a triangle} \} + \\
+ \# \{ \text{triangles of } \sigma^{(3)} \} = \\
= \deg^{1,1\ast}_L(\sigma^{(3)}) + \left( \frac{q+1}{q-h+1} \right) = \deg^{1,1\ast}_L(\sigma^{(3)}) + \deg^{1,2}_L(\sigma^{(3)}) = \\
= (2 + 3) + 4 = 9.
$$

Where the 2 comes from the two triangles of the green tetrahedron $\tau^{(3)}$ incident to $\sigma^{(3)}$ in an edge and not in a triangle, the number 3 comes from the three triangles of the white tetrahedron $\tau^{(3)}$ incident to $\sigma^{(3)}$ in an edge and not in a triangle, and finally the number 4 is the number of triangles in a tetrahedron.

- Set $q = 3$, $h = 2$ and $p = 0$. Let us compute the $(2,0)$-lower degree of the blue tetrahedron $\sigma^{(3)}$:

$$
\deg^{2,0}_L(\sigma^{(3)}) = \# \{ \text{incident edges to } \sigma^{(3)} \text{ in a vertex} \} = \\
= \# \{ \text{incident edges to } \sigma^{(3)} \text{ in a vertex and not in an edge} \} + \\
+ \# \{ \text{edges of } \sigma^{(3)} \} = \\
= \deg^{2,0\ast}_L(\sigma^{(3)}) + \left( \frac{q+1}{q-h+1} \right) = \deg^{2,0\ast}_L(\sigma^{(3)}) + \deg^{2,1}_L(\sigma^{(3)}) = \\
= (3 + 4 + 3 + 2) + 6 = 12 + 6 = 18.
$$

Where the first 3 comes from the three edges of the white tetrahedron $\tau^{(3)}$ incident to $\sigma^{(3)}$ in a vertex and not in a edge, the number 4 comes from the four edges of the green tetrahedron $\tau^{(3)}$ incident to $\sigma^{(3)}$ in a vertex and not in a edge, the second 3 comes from the three edges of the pink tetrahedron $\tau^{\prime\prime}(3)$ incident to $\sigma^{(3)}$ in a vertex and not in a edge, the number 2 comes from the two edges of the yellow triangle $\tau^{(2)}$ incident to $\sigma^{(3)}$ in a vertex, and 6 is the number of edges in a tetrahedron.

2.3. Generalised upper degree for simplices.
Definition 11. We define the $p$-upper degree of a $q$-simplex $\sigma^{(q)}$ as the number of $q'$-simplices which are $p$-upper adjacent to $\sigma^{(q)}$:

$$\deg^p_U(\sigma^{(q)}) := \#\{\sigma^{(q')} : \sigma^{(q)} \sim_U \sigma^{(q')}\}.$$ 

The strictly $p$-upper degree of a $q$-simplex $\sigma^{(q)}$ as the number of $q'$-simplices which are $p^*$-upper adjacent to $\sigma^{(q)}$:

$$\deg^{p^*}_U(\sigma^{(q)}) := \#\{\sigma^{(q')} : \sigma^{(q)} \sim_U \sigma^{(q')} & \sigma^{(q)} \not\sim_{U_{p+1}} \sigma^{(q')}\}.$$ 

We define the $(h,p)$-upper degree of a $q$-simplex $\sigma^{(q)}$ as the number of $(q+h)$-simplices which are $(h,p)$-upper adjacent to $\sigma^{(q)}$:

$$\deg^{h,p}_U(\sigma^{(q)}) := \#\{\sigma^{(q+h)} : \sigma^{(q)} \sim_U \sigma^{(q+h)}\}.$$ 

The strictly $(h,p)$-upper degree of a $q$-simplex $\sigma^{(q)}$ as the number of $(q+h)$-simplices which are $p^*$-upper adjacent to $\sigma^{(q)}$:

$$\deg^{h,p^*}_U(\sigma^{(q)}) := \#\{\sigma^{(q+h)} : \sigma^{(q)} \sim_U \sigma^{(q+h)} & \sigma^{(q)} \not\sim_{U_{p+1}} \sigma^{(q+h)}\}.$$ 

Notice that:

- For $q = 0$, $h = 1$ and $p = q + h = 1$ then the $(1,1)$-upper degree of a vertex $v$ is the usual degree:

$$\deg_{U}^{1,1}(v) = \#\{\text{edges incident to } v\} = \deg(v).$$

- $\deg_{U}^{p}(\sigma^{(q)}) = \sum_{h=-q}^{p-q} \deg_{U}^{h,p}(\sigma^{(q)}).$

- $\deg_{U}^{h,p^*}(\sigma^{(q)}) = \sum_{h=-q}^{p-q} \deg_{U}^{h,p^*}(\sigma^{(q)}).$

If $h > 0$ and $p = q + h$ we recover the following known notions of degrees given in [13]:

- For $q = 0$, $h = 2$ and $p = q + h = 2$ then the $(2,2)$-upper degree of a vertex $v$ is the “vertex to triangle degree” of equation (2):

$$\deg_{U}^{2,2}(v) = \#\{\text{triangles incident to } v\} = \frac{1}{2} \sum_{j_1,j_2} |\langle B_1 \rangle_{i,j_1}||\langle B_2 \rangle_{j_1,j_2}|.$$ 

- For $q = 0$ and $p = q + h$ then the $(h,h)$-upper degree of a vertex $v$ is the “vertex to $h$-simplex degree” of equation (3):
\[ \deg_{U}^{h,h}(v) = \# \{ (h)\text{-simplices incident to } v \} = \]
\[ = \frac{1}{h!} \sum_{j_1,\ldots,j_h} |(B_1)_{i,j_1}|(B_2)_{j_1,j_2}|\cdots|(B_h)_{j_{h-1},j_h}|. \]

- In general, for \( p = q + h \), the \((h,q + h)\)-upper degree of a \( q\)-simplex \( \sigma^{(q)} \) is the “\( q \)-simplex to \((q + h)\)-simplex degree” of equation (4):

\[ \deg_{U}^{h,q+h}(\sigma^{(q)}) = \# \{ (q + h)\text{-simplices incident to } \sigma^{(q)} \} = \]
\[ = \frac{1}{h!} \sum_{j_1,\ldots,j_h} |(B_{q+1})_{i,j_1}|(B_{q+2})_{j_1,j_2}|\cdots|(B_{q+h})_{j_{h-1},j_h}|. \]

In the following section, we will show a different way of computing these degrees by introducing a single new combinatorial Laplacian matrix.

- The “\( q \)-simplex to facets degree” of equation (6) can be given by the formula:

\[ \sum_{h=1}^{\dim K-q} \deg_{U}^{h,(q+h)^*}(\sigma^{(q)}). \]

Now, let us show how to compute the \((h,p^*)\)-upper degree of a \( q \)-simplex, \( \deg_{U}^{h,(p^*)}(\sigma^{(q)}) \), and thus the facets degree.

Let us consider the simplicial complex of Figure 3 (b), having a total of 11 vertices (0-simplices), 20 edges (1-simplices), 15 triangles (2-simplices), 6 tetrahedrons (3-simplices) and 1 pentahedron (4-simplices). Its maximum dimension is therefore 4.

- Let us count the \((1,1^*)\)-upper degree of the vertex \( v \). We are in the case \( q = 0 \), \( h = 1 \) and \( p = q + h = 1 \). It is clear from the figure that the number of edges incident to \( v \) which are not contained in a higher dimensional simplex is 1. The count can be also performed in the following way: we start by counting the number of edges incident to \( v \) (its usual degree, which is 10), then many of these edges are contained in higher dimensional simplices, so we start by subtracting the ones belonging to the same triangle still containing the vertex \( v \), and we have to multiply this number by the number of triangles incident to \( v \). That is, there are 10 triangles incident to \( v \) (1 alone in yellow, 3 coming from the green tetrahedron and 6 coming from the blue pentahedron) and there are always 2 edges incident to a vertex in a triangle (thus we are subtracting 20). At this step we have over subtracted edges incident to \( v \): the ones that are also contained in a tetrahedron. Then we sum the number of incident tetrahedrons to \( v \) (which are 1 from the green tetrahedron and 3 coming from the blue pentahedron) times the number of edges containing \( v \) in a tetrahedron (which is always 3). Again,
we have over counted edges incident to $v$ which are contained in a pentahedron, thus we have to subtract the number of incident pentahedrons to $v$ (which is 1, the blue one) times the number of edges containing $v$ in a pentahedron (which is 4). We have now finished since the maximum dimension of the simplicial complex is 4. Thus, we have:

$$
\deg_{U}^{1,1}(v) = \#\{\text{incident edges to } v\} - \#\{\text{incident triangles to } v\} \cdot \#\{\text{incident edges to } v \text{ in a triangle}\} + \#\{\text{incident tetrahedrons to } v\} \cdot \#\{\text{incident edges to } v \text{ in a tetrahedron}\} - \#\{\text{incident pentahedrons to } v\} \cdot \#\{\text{incident edges to } v \text{ in a pentahedron}\} = 10 - (1 + 3 + 6) \cdot 2 + (1 + 4) \cdot 3 - 1 \cdot 4 = 10 - 20 + 15 - 4 = 1.
$$

- Let us perform the computation for the $(2,2^*)$-upper degree of $v$, which is the number of incident triangles to $v$ which are not contained in a higher dimensional simplex (again the from the figure we read that this number is 1). We are in the case $q = 0$, $h = 2$ and $p = q + h = 2$.

$$
\deg_{U}^{2,2^*}(v) = \#\{\text{incident triangles to } v\} - \#\{\text{incident tetrahedrons to } v\} \cdot \#\{\text{incident triangles to } v \text{ in a tetrahedron}\} + \#\{\text{incident pentahedrons to } v\} \cdot \#\{\text{incident triangles to } v \text{ in a pentahedron}\} = (1 + 3 + 6) - (1 + 4) \cdot 3 - 1 \cdot 6 = 10 - 15 + 6 = 1.
$$

A straightforward generalisation of this strategy produces the following formula for the strict upper degree in terms of the upper degree (see definition 11):

$$
\deg_{U}^{h,(q+h)^*}(\sigma^{(q)}) = \sum_{i=0}^{\dim K-(q+h)} (-1)^i \deg_{U}^{h+i,q+h+i}(\sigma^{(q)}) \cdot \binom{h+i}{h}. \tag{10}
$$

**Remark 7.** The combinatorial number $\binom{h+i}{h} = \binom{h+i}{i}$ comes from counting the number of $(q+h)$-simplices having $\sigma^{(q)}$ as a $q$-face in a single $(q+h+i)$-simplex. Since we have a total of $q+h+i+1$ points, and there are $q+1$ in $\sigma^{(q)}$, we have $(q+h+i+1)-(q+1) = h+i$ to form, joint with the $q+1$ points of $\sigma^{(q)}$, a $(q+h)$-simplex $(q+h+1 \text{ points})$. Thus we need to group the $h+i$ points in subsets of $i$ points.

Formula (10) allows to compute the the “$q$-simplex to facets degree” (a sum of the strict $(q,q + h)$-upper degrees) of equation 9 in terms of the $(h+i,q+h+i)$-upper degrees. We will show in the following section how to compute these generalised upper degrees as the diagonal entries of a single (multi parameter) combinatorial Laplacian.
matrix, instead of as a product of several entries of different matrices associated with distinct $q$-boundary operators, as stated in [13].

2.4. Generalised adjacency degree for simplices and simplicial degree.

Let us finish this section by defining the generalised adjacency degrees associated with Definition 9 and a general simplicial degree of a simplex.

Definition 12.

(1) We define the $p$-adjacency degree of a $q$-simplex $\sigma^{(q)}$ by:

$$\deg^p_A(\sigma^{(q)}) := \# \{ \sigma^{(q)}' \mid \sigma^{(q)} \sim_A \sigma^{(q)}' \}.$$  

(2) We define the maximal $p$-adjacency degree of a $q$-simplex $\sigma^{(q)}$ by:

$$\deg^{p*}_A(\sigma^{(q)}) := \# \{ \sigma^{(q)}' \mid \sigma^{(q)}' \sim_A \sigma^{(q)} \}.$$  

Remark 8. Let us remark that with the $p$-adjacency degree we might be over counting certain simplices in the following sense: imaging we a triangle $\sigma^{(2)}$ to which another triangle $\sigma'^{(2)}$ is 0-adjacent in a vertex $v$, then, since there are two edges (1-faces) of $\sigma'^{(2)}$ that are 0-adjacent to $\sigma^{(2)}$, they are also being counted with the $p$-adjacency degree. That is, we are counting the community $\sigma'^{(2)}$ and two of its 1-faces. This suggests that we should use in certain applications the maximal $p$-adjacency degree for a $q$-simplex, which only counts the maximal collaborative communities $p$-adjacent to a given simplex.

Example 3. In Figure 3 (a) we have that there are two triangles of $\tau'^{(3)}$ which are 1-adjacent to $\sigma^{(3)}$, but which are not maximal 1-adjacent to $\sigma^{(3)}$; The tetrahedron $\tau'^{(3)}$ is maximal 1-adjacent to $\sigma^{(3)}$; the tetrahedron $\sigma^{(3)}$ is maximal 2-adjacent to $\tau^{(3)}$.

If one would like to count all the collaborations of a simplex with different simplicial communities, both the ones collaborating with its faces and also the bigger simplicial communities on which the simplex is nested in, we can define a two parameter simplicial degree using both the adjacency degree and the upper degree as follows.

Definition 13. Given $p_1 > q$ and $p_2 < q$, we define the $(p_1, p_2^*)$-degree of a $q$-simplex $\sigma^{(q)}$ by:

$$\deg^{(p_1, p_2^*)}(\sigma^{(q)}) := \deg^{p_1}_{U}(\sigma^{(q)}) + \deg^{p_2}_{A}(\sigma^{(q)}).$$

Similarly, for strict upper degree, we define the $(p_1^*, p_2^*)$-degree of a $q$-simplex $\sigma^{(q)}$ by:

$$\deg^{(p_1^*, p_2^*)}(\sigma^{(q)}) := \deg^{p_1^*}_{U}(\sigma^{(q)}) + \deg^{p_2^*}_{A}(\sigma^{(q)}).$$
Finally, let us propose a definition of maximal simplicial degree of a \(q\)-simplex, which counts all the maximal communities collaborating with the faces of the \(q\)-simplex (the ones that are maximal \(p\)-adjacent) and also the maximal communities to which the \(q\)-simplex belongs to (these last being strictly upper adjacent).

**Definition 14.** We define the maximal simplicial degree of \(\sigma^{(q)}\) by:

\[
\deg^*(\sigma^{(q)}) = \deg^*_A(\sigma^{(q)}) + \deg^*_U(\sigma^{(q)}),
\]

where:

\[
\deg^*_A(\sigma^{(q)}) := \sum_{p=0}^{q-1} \deg'^*(\sigma^{(q)}) ; \quad \deg^*_U(\sigma^{(q)}) := \sum_{h=1}^{\dim K-q} \deg'^{h,(q+h)}(\sigma^{(q)}).
\]

3. The Multi Combinatorial Laplacian

We will define in this section generalised multi parameter boundary and coboundary operators in an oriented simplicial complex, and a new higher order multi combinatorial Laplacian will be introduced. They will give us a way to effectively compute all the higher order degrees of the previous section.

3.1. The generalized boundary operator. Let \(\sigma^{(q)}\) be a \(q\)-simplex spanned by the set of points \(\{v_0, \ldots, v_q\}\). Given the \((q-h)\)-face \(\sigma^{(q-h)}\) of \(\sigma^{(q)}\) spanned by the set of vertices \(\{v_0, \ldots, \widehat{v}_{j_1}, \ldots, \widehat{v}_{j_h}, \ldots, v_q\}\) let us denote by \(\epsilon_{j_1\ldots j_h}\) the permutation

\[
\begin{pmatrix}
0 & \cdots & h - 1 & h & \cdots & q \\
\cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
 j_1 & \cdots & j_h & 0 & \cdots & q
\end{pmatrix}
\]

As oriented \(q\)-simplex, \(\sigma^{(q)}\) is represented by \([v_{\eta(0)}, \ldots, v_{\eta(q)}]\), for some permutation \(\eta\) in the set of its vertices.

**Definition 15.** We define the \((q, h)\)-boundary operator

\[
\partial_{q,h} : C_q(K) \to C_{q-h}(K)
\]

as the homomorphism given as the linear extension of:

\[
\partial_{q,h}([v_{\eta(0)}, \ldots, v_{\eta(q)}]) = \sum_{j_1, \ldots, j_h} \text{sign}(\eta) \text{sign}(\epsilon_{j_1\ldots j_h})[v_0, \ldots, \widehat{v}_{j_1}, \ldots, \widehat{v}_{j_h}, \ldots, v_q]
\]

where \([v_0, \ldots, \widehat{v}_{j_1}, \ldots, \widehat{v}_{j_h}, \ldots, v_q]\) denotes the oriented \(q\)-simplex obtained from removing the vertices \(v_{j_1}, \ldots, v_{j_h}\) in \([v_0, \ldots, v_q]\).
Note that \([v_0, \ldots, v_{q}] = [v_0', \ldots, v_{q'}]\) if and only if \(\text{sign}(\eta) = \text{sign}(\eta')\), so that \([v_0, \ldots, v_{q}] = \text{sign}(\eta) [v_0, \ldots, v_q]\). Then, this operator is well defined and \(\partial_{q,h}(-\sigma^{(q)}) = -\partial_{q,h}(\sigma^{(q)})\). Moreover, for \(h = 1\) the operator \(\partial_{q,h}\) is the ordinary \(q\)-boundary operator \(\partial_q\).

**Figure 4.** Examples of \((q, h)\)-boundary operators where we denote \([v_0, \ldots, v_p] = v_{0\cdots p}\).

Given \(\tau^{(p)}\) a \(p\)-simplex and \(\sigma^{(q)}\) a \(q\)-face in \(K\), with \(q < p\), we denote by \(\text{sign} (\tau^{(p)}, \sigma^{(q)})\) the coefficient of \(\sigma^{(q)}\) in the sum \(\partial_{p,q}(\tau^{(p)})\).

**Definition 16.** Let \(\sigma_i^{(q)}\) and \(\sigma_j^{(q')}\) be two simplices which are \(p\)-upper adjacent. Let \(\tau^{(p)}\) be a common upper \(p\)-simplex. We say that \(\sigma_i^{(q)}\) and \(\sigma_j^{(q')}\) are upper similarly oriented with respect to \(\tau^{(p)}\) if \(\text{sign} (\tau^{(p)}, \sigma^{(q)}) = \text{sign} (\tau^{(p)}, \sigma^{(q')})\).

We shall denote it by \(\sigma_i^{(q)} \sim_{U^+(p)} \sigma_j^{(q')}\). If the signs are different, we say that they are dissimilarly oriented with respect to \(\tau^{(p)}\). We shall denote it by \(\sigma_i^{(q)} \sim_{U^-(p)} \sigma_j^{(q')}\).

**Remark 9.** The equality or inequality of sign \((\tau^{(p)}, \sigma^{(q)})\) and \((\tau^{(p)}, \sigma^{(q')})\) does not depend on the orientation of \(\tau^{(p)}\) but only on the orientations of \(\sigma^{(q)}\) and \(\sigma^{(q')}\).

**Remark 10.** For \(h = 1\) this definition recovers Definition [3].

Let \(\sigma_i^{(q)}, \sigma_j^{(q')}\) and \(\tau^{(p)}\) oriented simplices.

**Definition 17.** We define the upper sign of \(\sigma_i^{(q)}\) and \(\sigma_j^{(q')}\) with respect to \(\tau^{(p)}\) as the following function:

\[
\text{sign}_{U^{+}}(\sigma_i^{(q)}, \sigma_j^{(q')}; \tau^{(p)}) := \begin{cases} 
0 & \text{if } \sigma_i^{(q)} \cup \sigma_j^{(q')} \nsubseteq \tau^{(p)} \\
1 & \text{if } \sigma_i^{(q)} \sim_{U^{+(p)}} \sigma_j^{(q')} \\
-1 & \text{if } \sigma_i^{(q)} \sim_{U^{-}(p)} \sigma_j^{(q')} 
\end{cases}
\]
Note that if $\tau^{(p)}$ is a common upper $p$-simplex to $\sigma_i^{(q)}$ and $\sigma_j^{(q')}$, then
\[ \text{sig}_{U}(\sigma_i^{(q)}, \sigma_j^{(q')}; \tau^{(p)}) = \text{sign}(\tau^{(p)}, \sigma^{(q)}) \cdot \text{sign}(\tau^{(p)}, \sigma^{(q')}) , \]
which does not depend on the orientation of $\tau^{(p)}$. This justifies the following definition.

**Definition 18.** Let $\sigma_i^{(q)}$ and $\sigma_j^{(q')}$ two simplices.
We define the $p$-upper oriented degree of $\sigma_i^{(q)}$ and $\sigma_j^{(q')}$ as the following sum:
\[ \text{odeg}^p_{U}(\sigma_i^{(q)}, \sigma_j^{(q')}) := \frac{1}{2} \sum_{\tau^{(p)} \in \tilde{S}_p(K)} \text{sig}_{U}(\sigma_i^{(q)}, \sigma_j^{(q')}; \tau^{(p)}) . \]
where by $\tilde{S}_p(K)$ we denote the set of oriented $p$-simplices of the simplicial complex $K$.

**Remark 11.** Note that we are dividing by 2 since, as mentioned above, we have
\[ \text{sig}_{U}(\sigma_i^{(q)}, \sigma_j^{(q')}; \tau^{(p)}) = \text{sig}_{U}(\sigma_i^{(q)}, \sigma_j^{(q')}; -\tau^{(p)}). \]
Let us point out that if $p = q + 1$ and $\sigma_i^{(q)} \sim_{U^{q+1}} \sigma_j^{(q)}$, then there exists a unique common $(q + 1)$-simplex $\tau^{(q+1)}$, and thus:
\[ \text{odeg}^{q+1}_{U}(\sigma_i^{(q)}, \sigma_j^{(q)}) = \text{sig}_{U}(\sigma_i^{(q)}, \sigma_j^{(q)}; \tau^{(q+1)}) = \]
\[ = \begin{cases} 1 & \text{if } \sigma_i^{(q)} \sim_{U^{q+1}} \sigma_j^{(q)} \text{ and similarly oriented w.r.t. } \tau^{(q+1)} , \\ -1 & \text{if } \sigma_i^{(q)} \sim_{U^{q+1}} \sigma_j^{(q)} \text{ and dissimilarly oriented w.r.t. } \tau^{(q+1)} , \end{cases} \]
which therefore recovers the notion of similarly and dissimilarly oriented.

Let us now give the analogous definition for the lower adjacency.

**Definition 19.** Let $\sigma_i^{(q)}$ and $\sigma_j^{(q')}$ two simplices which are $p$-lower adjacent and $\tau^{(p)}$ be a common lower $p$-face. We say that $\sigma_i^{(q)}$ and $\sigma_j^{(q')}$ are lower similarly oriented with respect to $\tau^{(p)}$ if $\text{sign}(\sigma^{(q)}, \tau^{(p)}) = \text{sign}(\sigma^{(q')}, \tau^{(p)})$. If the signs are different, we say that they are dissimilarly oriented with respect to $\tau^{(p)}$. As before, we shall denote it by $\sigma_i^{(q)} \sim_{L^+(p)} \sigma_j^{(q')}$ and $\sigma_i^{(q)} \sim_{L^-(p)} \sigma_j^{(q')}$, respectively.

**Remark 12.** The equality or inequality of $\text{sign}(\sigma^{(q)}, \tau^{(p)})$ and $\text{sign}(\sigma^{(q')}, \tau^{(p)})$ does not depend on the orientation of $\tau^{(p)}$ but only on the orientations of $\sigma^{(q)}$ and $\sigma^{(q')}$. 

**Remark 13.** For $p = q - 1$ this definition recovers Definition 6.

**Definition 20.** Let $\sigma_i^{(q)}$ and $\sigma_j^{(q')}$ two simplices. We define the lower sign of $\sigma_i^{(q)}$ and $\sigma_j^{(q')}$ with respect to a $p$-simplex $\tau^{(p)}$ as the following function:
\[ \text{sig}_{L}(\sigma_i^{(q)}, \sigma_j^{(q')}; \tau^{(p)}) := \begin{cases} 0 & \text{if } \tau^{(p)} \not\subseteq \sigma_i^{(q)} \cap \sigma_j^{(q')} , \\ 1 & \text{if } \sigma_i^{(q)} \sim_{L^+(p)} \sigma_j^{(q')}, \\ -1 & \text{if } \sigma_i^{(q)} \sim_{L^-(p)} \sigma_j^{(q')} . \end{cases} \]
As we pointed out for the upper sign, notice that if \( \tau^{(p)} \) is a common lower \( p \)-face of \( \sigma_i^{(q)} \) and \( \sigma_j^{(q')} \), then
\[
\text{sig}_L(\sigma_i^{(q)}, \sigma_j^{(q')}; \tau^{(p)}) = \text{sign} \left( \sigma^{(q)}, \tau^{(p)} \right) \cdot \text{sign} \left( \sigma^{(q')}, \tau^{(p)} \right),
\]
which does not depend on the orientation of \( \tau^{(p)} \). This justifies the following definition.

**Definition 21.** Let \( \sigma_i^{(q)} \) and \( \sigma_j^{(q')} \) two simplices. We define the \( p \)-lower oriented degree of \( \sigma_i^{(q)} \) and \( \sigma_j^{(q')} \) as the following sum:
\[
\text{odeg}_L^{(p)}(\sigma_i^{(q)}, \sigma_j^{(q')}) := \frac{1}{2} \sum_{\tau^{(p)} \in \tilde{S}_p(K)} \text{sig}_L(\sigma_i^{(q)}, \sigma_j^{(q')}, \tau^{(p)}),
\]
where by \( \tilde{S}_p(K) \) we denote the set of oriented \( p \)-simplices of the simplicial complex \( K \).

**Remark 14.** Note that we are dividing by 2 since, as mentioned above, we have
\[
\text{sig}_L(\sigma_i^{(q)}, \sigma_j^{(q')}, \tau^{(p)}) = \text{sig}_L(\sigma_i^{(q)}, \sigma_j^{(q')}, -\tau^{(p)}).
\]

**Example 4.** Let \( K \) be the simplicial complex given by Figure 5. Let \( C_0(K) \) be the set of vertices \( \{v_0, v_1, \ldots, v_5\} \) and \( C_2(K) = \langle v_{012}, v_{024}, v_{034}, v_{035}, v_{045}, v_{345} \rangle \) the free abelian group generated by triangles (choosing one of its orientations) where we are writing \( v_{ijk} \) for the oriented triangle \( [v_i, v_j, v_k] \). Let
\[
\partial_{2,2} : C_2(K) \to C_0(K)
\]
\[
v_{ijk} \mapsto \text{sign}(\bar{\epsilon}_{j,k})v_i + \text{sign}(\bar{\epsilon}_{i,k})v_j + \text{sign}(\bar{\epsilon}_{i,j})v_k
\]
be the \((2,2)\)-boundary operator, with
\[
\bar{\epsilon}_{j,k} = \begin{pmatrix} i & j & k \\ j & k & i \end{pmatrix}, \quad \bar{\epsilon}_{i,k} = \begin{pmatrix} i & j & k \\ k & i & j \end{pmatrix}, \quad \bar{\epsilon}_{i,j} = \begin{pmatrix} i & j & k \\ i & j & k \end{pmatrix}
\]
Thus:
\[
\partial_{2,2}(v_{ijk}) = v_i - v_j + v_k
\]
We have that:
odeg\_2^U(v_0, v_2) := \sum_{\tau^{(2)} \in C_2(K)} \text{sig}_U(v_0, v_2; \tau^{(2)}) = \\
= \text{sig}_U(v_0, v_2; v_{012}) + \text{sig}_U(v_0, v_2; v_{024}) = 1 - 1 = 0

odeg\_0^L(v_{012}, v_{024}) := \sum_{\tau^{(0)} \in C_0(K)} \text{sig}_L(v_{012}, v_{024}; \tau^{(0)}) = \\
= \text{sig}_L(v_{012}, v_{024}; v_0) + \text{sig}_L(v_{012}, v_{024}; v_2) = 1 - 1 = 0

odeg\_2^U(v_0, v_3) := \sum_{\tau^{(2)} \in C_2(K)} \text{sig}_U(v_0, v_3; \tau^{(2)}) = \\
= \text{sig}_U(v_0, v_3; v_{034}) + \text{sig}_U(v_0, v_3; v_{035}) = -1 - 1 = -2.

Assume that \(\tau^{(q)}\) is a \(q\)-simplex and \(\sigma^{(p)}\) is a \(p\)-face of \(\tau^{(q)}\). By the above definitions we have:

1. \(\text{odeg}_q^U(\sigma^{(p)}, \tau^{(q)}) = \text{sig}_U(\sigma^{(p)}, \tau^{(q)}, \tau^{(q)}).\)

2. \(\text{odeg}_p^L(\sigma^{(p)}, \tau^{(q)}) = \text{sig}_L(\sigma^{(p)}, \tau^{(q)}, \sigma^{(p)}).\)

3. \(\text{sig}_U(\sigma^{(p)}, \tau^{(q)}; \tau^{(q)}) = \text{sign}(\tau^{(q)}, \sigma^{(p)})) = \text{sig}_L(\sigma^{(p)}, \tau^{(q)}; \sigma^{(p)}).\)

Then one obtains the following result.

Proposition 2.

\[ \partial_{q,h}(\tau^{(q)}) = \sum_{\sigma^{(q-h)} \in S_{q-h}(K)} \text{odeg}_{q-h}^L(\tau^{(q)}, \sigma^{(q-h)}; \sigma^{(q-h)}). \tag{11} \]

Let us now show that there exists a coboundary operator in the oriented simplicial complex \(K\), and that it can be written down in terms of the oriented degree.

Proposition 3. Given an oriented simplicial complex \(K\), let \(\partial_{q,h}\) be the \((q, h)\)-boundary operator. There exists a unique homomorphism:

\[ \partial^*_{q,h} : C_{q-h}(K) \rightarrow C_q(K) \]

declared as:

\[ \partial^*_{q,h}(\sigma^{(q-h)}) = \sum_{\tau^{(q)} \in S_q(K)} \text{odeg}^q_{q-h}(\sigma^{(q-h)}, \tau^{(q)}; \tau^{(q)}; \tau^{(q)}). \tag{12} \]

and such that \(\partial_{q,h}\) and \(\partial^*_{q,h}\) are adjoint operators.
Proof. Consider \( \{\tau_1, \ldots, \tau_n\} \) and \( \{\sigma_1, \ldots, \sigma_m\} \) basis of \( C_q(K) \) and \( C_{q-h}(K) \) respectively. For all \( \sigma_i \subseteq \tau_j \) we have

\[
\langle \text{sig}_U(\sigma_i, \tau_j; \tau_j, \tau_k) \rangle = \langle \text{sig}_L(\sigma_i, \tau_j; \sigma_i) \tau_j, \tau_k \rangle = \langle \tau_j, \text{sig}_L(\sigma_i, \tau_j; \sigma_i) \tau_k \rangle
\]

where \( \langle \cdot, \cdot \rangle \) denote the standard inner product \( \langle \sum_i \alpha_i \tau_i, \sum_j \beta_j \tau_j \rangle = \sum_k \alpha_k \beta_k \). Therefore,

\[
\langle \sum_{\tau_j} \text{odeg}^q_U(\sigma_i, \tau_j) \tau_j, \tau_k \rangle = \text{sig}_U(\sigma_i, \tau_k; \tau_k) = \text{sig}_L(\sigma_i, \tau_k; \sigma_i) =
\]

\[
= \langle \sigma_i, \sum_{\tau_j} \text{odeg}^q_{-h}(\sigma_j, \tau_k) \sigma_j \rangle = \langle \sigma_i, \partial_q, h(\tau_k) \rangle
\]

and the result follows. \( \square \)

3.2. The multi combinatorial Laplacian. Once we have the boundary and coboundary operators in an oriented simplicial complex \( K \), we can define a multi combinatorial Laplacian and show how it computes some higher order degrees of simplices.

**Definition 22.** Let \( q, h, h' \) non negative integers. We define the \((q, h, h')\)-Laplacian operator

\[
\Delta_{q,h,h'} : C_q(K) \to C_q(K)
\]

as the following operator:

\[
\Delta_{q,h,h'} := \partial_{q+h,h} \circ \partial_{q+h,h}^* + \partial_{q,h'}^* \circ \partial_{q,h'}.
\]

\( \Delta_{q,h}^{U} = \partial_{q+h,h} \circ \partial_{q+h,h}^* : C_q(K) \to C_q(K) \) is named the upper \((q, h)\)-Laplacian operator and \( \Delta_{q,h'}^{L} = \partial_{q,h'} \circ \partial_{q,h'}^* : C_q(K) \to C_q(K) \) is called the \((q, h')\)-Laplacian operator.

Let us fix basis of ordered simplices for \( C_{q+h}(K) \), \( C_{q}(K) \) and \( C_{q-h'}(K) \) and denote by \( B_{q+h,h} \) and \( B_{q,h'} \) the corresponding matrix representation of \( \partial_{q+h,h} : C_{q+h}(K) \to C_{q}(K) \) and \( \partial_{q,h'} : C_{q}(K) \to C_{q-h'}(K) \), respectively. Then the associated matrix of the \((q, h, h')\)-Laplacian operator is

\[
L_{q,h,h'} = B_{q+h,h} B_{q+h,h}^t + B_{q,h'}^t B_{q,h'}.
\]

We shall call it the \((q, h, h')\)-Laplacian matrix and as before we use the notation \( L_{q,h}^{U} = B_{q+h,h} B_{q+h,h}^t \) and \( L_{q,h'}^{L} = B_{q,h'}^t B_{q,h'} \).

**Theorem 1.** Let \( K \) be an oriented simplicial complex and fix oriented basis on the \( q \)-chains \( C_q(K) \) of \( K \). With respect to this basis, the \((i, j)\)-th entry of the associated
matrices of the \((q, h)\)-Laplacian operators is given by:

\[
\begin{align*}
(L^U_{q, h})_{i,j} &= \begin{cases} 
\deg_{U}^{h, q+h}(\sigma^{(q)}_i) & \text{if } i = j \\
\text{odeg}_{U}^{q+h}(\sigma^{(q)}_i, \sigma^{(q)}_j) & \text{if } i \neq j
\end{cases} \\
(L^L_{q, h'})_{i,j} &= \begin{cases} 
\deg_{L}^{h', q-h'}(\sigma^{(q)}_i) = \binom{q+1}{q-h'+1} & \text{if } i = j \\
\text{odeg}_{L}^{q-h'}(\sigma^{(q)}_i, \sigma^{(q)}_j) & \text{if } i \neq j
\end{cases}
\end{align*}
\]

Proof. Fix \(\{\tau^{(q+h)}_1, \ldots, \tau^{(q+h)}_r\}, \{\sigma^{(q)}_1, \ldots, \sigma^{(q)}_n\}\) and \(\{\gamma^{(q-h')}_1, \ldots, \gamma^{(q-h')}_r\}\) basis of \(C_{q+h}(K), C_q(K)\) and \(C_{q-h'}(K)\) respectively. By using the above notations one has that the \((i, j)\)-th entry of \(B_{q, h'}\) is

\[
b^{(q, h')}_{ij} = \langle \partial_{q, h'}(\sigma^{(q)}_j), \gamma^{(q-h')}_i \rangle = \text{sig}_L(\sigma^{(q)}_j, \gamma^{(q-h')}_i, \gamma^{(q-h')}_i).
\]

Then

\[
(L^L_{q, h'})_{i,j} = \sum_{k=1}^{r} b^{(q, h')}_{ki} b^{(q, h')}_{kj}
\]

\[
= \sum_{k=1}^{r} \text{sig}_L(\sigma^{(q)}_i, \gamma^{(q-h')}_k, \gamma^{(q-h')}_k) \text{sig}_L(\sigma^{(q)}_j, \gamma^{(q-h')}_k, \gamma^{(q-h')}_k)
\]

\[
= \begin{cases} 
\sum_{k=1}^{r} \text{sig}_L(\sigma^{(q)}_i, \sigma^{(q)}_j, \gamma^{(q-h')}_k) & \text{for } i \neq j \\
\deg_{L}^{h', q-h'}(\sigma^{(q)}_i) & \text{if } i = j
\end{cases}
\]

The explicit description of the adjoint operator of \(\partial_{q+h, h}\) previously given in Equation 12 shows that the \((i, j)\)-th entry of \(B^L_{q+h, h}\) is

\[
b^{(q+h, h)}_{ij} = \langle \partial^*_{q+h, h}(\sigma^{(q)}_j), \tau^{(q+h)}_i \rangle = \text{sig}_U(\sigma^{(q)}_j, \tau^{(q+h)}_i, \tau^{(q+h)}_i)
\]

so, we get \((L^U_{q, h})_{i,j} = \)

\[
\begin{cases} 
\sum_{l=1}^{m} \text{sig}_U(\sigma^{(q)}_i, \sigma^{(q)}_j, \gamma^{(q+h)}_l) & \text{for } i \neq j \\
\deg_{U}^{h, q+h}(\sigma^{(q)}_i) & \text{if } i = j
\end{cases}
\]

Notice that for \(h = h' = 1\) we recover the \(q\)-combinatorial Laplacian (see \(\S\ 10\)).
Remark 15. As opposite with the \( q \)-combinatorial Laplacian matrix and the graph Laplacian matrix, there might be 0 entries in the multi combinatorial Laplacian matrix coming not only from non adjacent simplices, but also from simplices which being, for example, lower adjacent, the orientation of a common face is opposite from one another, and thus it cancels the corresponding oriented degree. See for instance the second computation of Example 4.

Example 5. With the notations of example 4, let us compute an upper and a lower multi combinatorial Laplacian of the simplicial complex \( K \) of Figure 5. Recall that we have the basis \( \{v_{012}, v_{024}, v_{035}, v_{045}, v_{345}\} \) of \( C_2(K) \) and \( \{v_0, \ldots, v_5\} \) of \( C_0(K) \).

- Let us set \( q = 0 \) and \( h = 2 \). The associated matrix of the \((q + h, h) = (2, 2)\)-boundary operator \( \partial_{2,2} \) is:

\[
B_{2,2} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 1 \\
0 & 1 & 1 & 0 & -1 & -1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

and, since the associated matrix (with respect to the corresponding dual basis) of its adjoint operator \( \partial_{2,2}^* \) is its transpose, the matrix of the \((q, h) = (0, 2)\)-upper Laplacian operator \( L_{0,2}^U = \partial_{2,2} \circ \partial_{2,2}^* : C_0(K) \to C_0(K) \) is:

\[
B_{2,2} \cdot B_{2,2}^t = \begin{pmatrix}
5 & -1 & 0 & -2 & 1 & 2 \\
-1 & 1 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 & -1 & 0 \\
-2 & 0 & 0 & 3 & -2 & 0 \\
1 & 0 & -1 & -2 & 4 & -2 \\
2 & 0 & 0 & 0 & -2 & 3
\end{pmatrix}
\]

whose diagonal entries are the upper degrees of the vertices \( \deg_{h, q + h}^U(v_i) = \deg_{2,2}^U(v_i) \) (for \( i = 0, 1, \ldots, 5 \)), and the off diagonal entries are the upper oriented degrees \( o\deg_{2,2}^U(v_i, v_j) \) for \( i \neq j \) (see example 4).

- Similarly, let us set \( q = 2 \) and \( h = 2 \), then the matrix of the \((q, h) = (2, 2)\)-boundary operator \( \partial_{2,2} \) is again \( B_{2,2} \), and the matrix of the \((q, h) = (2, 2)\)-lower
Laplacian operator \( L_{2,2}^L = \partial_{2,2}^* \circ \partial_{2,2} : C_2(K) \rightarrow C_2(K) \) is:

\[
B_{2,2}^t \cdot B_{2,2} = \begin{pmatrix}
3 & 0 & 1 & 1 & 1 & 0 \\
0 & 3 & 2 & 1 & 0 & -1 \\
1 & 2 & 3 & 2 & 0 & -2 \\
1 & 1 & 2 & 3 & 2 & 0 \\
1 & 0 & 0 & 2 & 3 & 2 \\
0 & -1 & -2 & 0 & 2 & 3
\end{pmatrix}
\]

whose diagonal entries are the lower degrees of the vertices \( \deg^{h}_{L}^{q-h}(v_{ijk}) = \deg^{2,0}_{L}(v_{ijk}) \), and the off diagonal entries are \( \deg^{0}_{L}(v_{ijk}, v_{ij'k'}) \), the lower oriented degrees (see example 4).

3.3. The boundary and coboundary operators compute the higher order degrees. Notice that the multi combinatorial Laplacian of Theorem 1 does not compute all the higher order degrees of simplices, for instance the general p-lower degree of a simplex \( \sigma^{(q)} \) is not computed, only its lower degree \( \deg^{h,q-h}_{L}(\sigma^{(q)}) \) is contained in the multi combinatorial Laplacian, and we already knew that it were equal to \( \binom{q+1}{q-h+1} \). Let us finish this section by giving an explicit description of all the higher order degrees in terms of the generalised boundary and coboundary operators. The key point to perform these computations is to use Propositions 2 and 3.

We start with the p-lower degree. Recall that the p-lower degree of a q-simplex \( \sigma^{(q)} \) is the number of \( q' \)-simplices \( \tau^{(q')} \) which are p-lower adjacent to \( \sigma^{(q)} \), that is, those \( q' \)-simplices which contain a p-face \( \gamma^{(p)} \) of \( \sigma^{(q)} \). Hence \( \tau^{(q')} \) contributes to \( \deg^{p}_{L}(\sigma^{(q)}) \) as long as \( \text{sig}_{L} (\sigma^{(q)}, \tau^{(q')}; \gamma^{(p)}) \) does not vanish for some \( \gamma^{(p)} \). Then \( \text{sig}_{L} (\sigma^{(q)}, \tau^{(q')}; \gamma^{(p)}) \) should be related to \( \deg^{p}_{L}(\sigma^{(q)}) \).

Following the proof of Theorem 1 is a straightforward computation to show that \( \text{sig}_{L} (\sigma^{(q)}, \tau^{(q')}; \gamma^{(p)}) \) can be given in terms of the entries of the matrices corresponding to \( \partial \) and \( \partial^* \) operators. However, as the common lower simplex might not be unique, the sum \( \sum_{\gamma^{(p)}} | \text{sig}_{L} (\sigma^{(q)}, \tau^{(q')}; \gamma^{(p)}) | \) could be bigger than 1 and we would be counting the \( q' \)-simplex \( \tau^{(q')} \) more than once when computing \( \deg^{p}_{L}(\sigma^{(q)}) \).

**Definition 23.** Let \( \sigma^{(q)} \) and \( \tau^{(q')} \) be two simplices. We define the p-lower order of \( \sigma^{(q)} \) and \( \tau^{(q')} \) as the number \( \text{ord}^{p}_{L}(\sigma^{(q)}, \tau^{(q')}) \) of \( p \)-simplices of \( K \) which are \( p \)-faces of both \( \sigma^{(q)} \) and \( \tau^{(q')} \). That is,

\[
\text{ord}^{p}_{L}(\sigma^{(q)}, \tau^{(q')}) = \sum_{\gamma^{(p)} \in S_{p}(K)} | \text{sig}_{L} (\sigma^{(q)}, \tau^{(q')}; \gamma^{(p)}) |.
\]

In that case, we shall say that \( \sigma^{(q)} \) and \( \tau^{(q')} \) are p-lower adjacent in order \( \text{ord}^{p}_{L}(\sigma^{(q)}, \tau^{(q')}) \).
Let \( p, h \) and \( h' \) be non-negative integers, put \( q = p + h, q' = p + h' \) and fix \( \{ \tau_1^{(q')}, \ldots, \tau_m^{(q')} \}, \{ \sigma_1^{(q)}, \ldots, \sigma_n^{(q)} \} \) and \( \{ \gamma_1^{(p)}, \ldots, \gamma_r^{(p)} \} \) basis of \( C_q'(K), C_q(K) \) and \( C_p(K) \) respectively. Given the boundary operators \( \partial_{q,h} : C_q(K) \to C_p(K) \) and \( \partial_{q',h'} : C_q'(K) \to C_p(K) \), denote by \( B_{q,h} \) and \( B_{q',h'} \) their corresponding matrices with respect to those bases. For the composition

\[
C_q(K) \xrightarrow{\partial_{q,h}} C_p(K) \xrightarrow{\partial_{q',h'}} C_q'(K)
\]

one has:

\[
(\partial_{q',h'} \circ \partial_{q,h})(\sigma_j^{(q)}) = \sum_{i,k} b_{ij}^{(q,h)} b_{ik}^{(q',h')} \tau_k^{(q')}
\]

\[
= \sum_{i,k} \text{sig}_L \left( \sigma_j^{(q)}, \gamma_i^{(p)}; \gamma_i^{(p)} \right) \text{sig}_L \left( \tau_k^{(q')}, \gamma_i^{(p)}; \gamma_i^{(p)} \right) \tau_k^{(q')}
\]

\[
= \sum_{i,k} \text{sig}_L \left( \sigma_j^{(q)}, \tau_k^{(q')}; \gamma_i^{(p)} \right) \tau_k^{(q')}
\]

\( \gamma_i^{(p)} \) is a \( p \)-face of both \( \sigma_j^{(q)} \) and \( \tau_k^{(q')} \) if and only if \( |\text{sig}_L \left( \sigma_j^{(q)}, \tau_k^{(q')}; \gamma_i^{(p)} \right)| = 1 \), so we obtain that:

- the number of \( p \)-faces of both \( \sigma_j^{(q)} \) and \( \tau_k^{(q')} \) is

\[
\text{ord}_L^{p} \left( \sigma_j^{(q)}, \tau_k^{(q')} \right) = \sum_i |\text{sig}_L \left( \sigma_j^{(q)}, \tau_k^{(q')}; \gamma_i^{(p)} \right)| = \sum_i |b_{ij}^{(q,h)}||b_{ik}^{(q',h')}|,
\]

- the number of \( q' \)-simplices which are \( p \)-lower adjacent to \( \sigma_j^{(q)} \) in \( \gamma_i^{(p)} \) is

\[
\sum_k |\text{sig}_L \left( \sigma_j^{(q)}, \tau_k^{(q')}; \gamma_i^{(p)} \right)| = \sum_k |b_{ij}^{(q,h)}||b_{ik}^{(q',h')}|.
\]

Under these assumptions and notations we have the following statement.

**Theorem 2.** Let \( p \) and \( q \) be non-negative integers. The \( p \)-lower degree of a \( q \)-simplex \( \sigma_j^{(q)} \) is:

\[
\deg_L^p(\sigma_j^{(q)}) = -1 + \sum_{q'=p}^{\dim K} \sum_k \min \left( 1, \sum_i |b_{ij}^{(q,h)}||b_{ik}^{(q',h')}| \right)
\]

with \( h = q - p \) and \( h' = q' - p \).

**Proof.** Fixed basis as above, we define the following sign matrix:

\[
S_{q,h,h'}(j) := \begin{pmatrix}
|\text{sig}_L \left( \sigma_j^{(q)}, \tau_1^{(q')}; \gamma_1^{(p)} \right)| & \cdots & |\text{sig}_L \left( \sigma_j^{(q)}, \tau_1^{(q')}; \gamma_r^{(p)} \right)| \\
\vdots & \ddots & \vdots \\
|\text{sig}_L \left( \sigma_j^{(q)}, \tau_m^{(q')}; \gamma_1^{(p)} \right)| & \cdots & |\text{sig}_L \left( \sigma_j^{(q)}, \tau_m^{(q')}; \gamma_r^{(p)} \right)|
\end{pmatrix}
\]  

(13)
whose \((k, i)\)-th entry is:

\[
s_{\sigma_{kh}}^{(q,h',h')}(j) = |b_{ij}^{(q,h)}b_{ik}^{(q',h')}| = \begin{cases} 1 & \text{if } \sigma_{i}^{(p)} \subseteq \sigma_{j}^{(q)} \cap \tau_{k}^{(q')} \\ 0 & \text{otherwise} \end{cases}
\]

Note that if \(s_{\sigma_{kh}}^{(q,h',h')}(j) \neq 0\) for some \(i\), then \(\sigma_{j}^{(q)}\) and \(\tau_{k}^{(q')}\) are \(p\)-lower adjacent in order \(\text{ord}_{p}^{q}(\sigma_{j}^{(q)}; \tau_{k}^{(q')}\) = \(\sum_{i} |b_{ij}^{(q,h)}||b_{ik}^{(q',h')}|\). Hence the number of \(q'\)-simplices \(p\)-lower adjacent to \(\sigma_{j}^{(q)}\), counted each one with its order, is \(\sum_{k} |b_{ij}^{(q,h)}||b_{ik}^{(q',h')}|\). To avoid that \(\tau_{k}^{(q')}\) to be counted more than once in \(\text{deg}_{p}^{q}(\sigma_{j}^{(q)}\), we consider the minimum between 1 and its order, so that (assuming \(\tau_{k}^{(q')} \neq \sigma_{j}^{(q)}\)):

\[
\tau_{k}^{(q')} \sim_{p} \sigma_{j}^{(q)} \iff \min\left(1, \sum_{i} |b_{ij}^{(q,h)}||b_{ik}^{(q',h')}|\right) = 1.
\]

Therefore, the number of \(q'\)-simplices \(p\)-lower adjacent to \(\sigma_{j}^{(q)}\) is:

\[
\text{deg}_{p}^{q,q'}^{q,q'}(\sigma_{j}^{(q)}) = \begin{cases} \sum_{k} \min\left(1, \sum_{i} |b_{ij}^{(q,h)}||b_{ik}^{(q',h')}|\right) & \text{for } q' \neq q \\ \sum_{k} \left( \min\left(1, \sum_{i} |b_{ij}^{(q,h)}||b_{ik}^{(q,h)}|\right) - 1 \right) & \text{for } q' = q \end{cases}
\]

and the result follows. Notice that, by definition, \(\sigma_{j}^{(q)}\) is not \(p\)-lower adjacent to itself, so that if \(q = q'\), then the degree \(\text{deg}_{p}^{q,q}(\sigma_{j}^{(q)}) = \sum_{k} \min\left(1, \sum_{i} |b_{ij}^{(q,h)}||b_{ik}^{(q,h)}|\right)\) minus 1.

A similar argument yields an analogous formula to compute the \(p\)-upper degree. Let us recall that the \(p\)-upper degree of a \(q\)-simplex \(\sigma_{j}^{(q)}\) is the number of \(q'\)-simplices \(\tau^{(q')}\) which are \(p\)-upper adjacent to \(\sigma_{j}^{(q)}\). That is, those \(\tau^{(q')}\) such that \(|\text{sig}_{U}(\sigma_{j}^{(q)}; \tau^{(q')}; \gamma^{(p)})| = 1\), for some \(p\)-simplex \(\gamma^{(p)}\). As in the lower degree setting, the common upper simplex \(\gamma^{(p)}\) could be not unique, which motives the following definition.

**Definition 24.** Let \(\sigma_{j}^{(q)}\) and \(\tau^{(q')}\) be two simplices. The \(p\)-upper order of \(\sigma_{j}^{(q)}\) and \(\tau^{(q')}\), written \(\text{ord}_{p}^{U}(\sigma_{j}^{(q)}; \tau^{(q')}\), is the number of \(p\)-simplices of which \(\sigma_{j}^{(q)}\) and \(\tau^{(q')}\) are faces. That is,

\[
\text{ord}_{p}^{U}(\sigma_{j}^{(q)}; \tau^{(q')}\) = \sum_{\gamma^{(p)} \in \text{sig}_{U}(\sigma_{j}^{(q)}; \tau^{(q')}; \gamma^{(p)})} |\text{sig}_{U}(\sigma_{j}^{(q)}; \tau^{(q')}; \gamma^{(p)})|.
\]

In that case, we shall say that \(\sigma_{j}^{(q)}\) and \(\tau^{(q')}\) are \(p\)-upper adjacent in order \(\text{ord}_{p}^{U}(\sigma_{j}^{(q)}; \tau^{(q')}\).

Let \(p, h\) and \(h'\) be non negative integers, denote \(q = p - h, q' = p - h'\) and fix \(\{\tau_{1}^{(q')}, \ldots, \tau_{m}^{(q')}\}, \{\sigma_{1}^{(q)}, \ldots, \sigma_{n}^{(q)}\}\) and \(\{\gamma_{1}^{(p)}, \ldots, \gamma_{r}^{(p)}\}\) basis of \(C_{q'}(K), C_{q}(K)\) and \(C_{p}(K)\)
respectively. We denote by $B_{q+h,h'}$ and $B_{q'+h',h'}$ the corresponding matrices (with respect to those basis) to the boundary operators:

$$C_p(K) \xrightarrow{\partial_{q+h,h}} C_q(K) \text{ and } C_p(K) \xrightarrow{\partial_{q'+h',h'}} C_{q'}(K).$$

For the composition

$$C_q(K) \xrightarrow{\partial_{q+h,h}} C_p(K) \xrightarrow{\partial_{q'+h',h'}} C_{q'}(K)$$

one has:

$$(\partial_{q'+h',h'} \circ \partial_{q+h,h})(\sigma_j^q) = \sum_{i,k} b_{j_i}^{(q+h,h)} b_{k_i}^{(q'+h',h')} \gamma_i^q.$$ 

$$\gamma_i^q$$ is a $p$-simplex containing $\sigma_j^q$ and $\tau_k^q$ as faces if and only if $|\text{sig}_U(\sigma_j^q, \tau_k^q; \gamma_i^q)| = 1$, so that, we obtain that:

- the number of $p$-simplices which contain both $\sigma_j^q$ and $\tau_k^q$ as faces is

$$\text{ord}_U^p(\sigma_j^q, \tau_k^q) = \sum_i |\text{sig}_U(\sigma_j^q, \tau_k^q; \gamma_i^q)| = \sum_i |b_{j_i}^{(q+h,h)}||b_{k_i}^{(q'+h',h')}|,$$

- the number of $q'$-simplices $\tau_k^q$ such that $\sigma_j^q$ and $\tau_k^q$ are faces of $\gamma_i^q$ is

$$\sum_k |\text{sig}_U(\sigma_j^q, \tau_k^q; \gamma_i^q)| = \sum_k |b_{j_i}^{(q+h,h)}||b_{k_i}^{(q'+h',h')}|. $$

Hence, the following holds.

**Theorem 3.** Let $p$ and $q$ be non negative integers. The $p$-upper degree of a $q$-simplex $\sigma_j^q$ is:

$$\text{deg}_U^p(\sigma_j^q) = -1 + \sum_{q'=0}^p \sum_{k} \min \left(1, \sum_i |b_{j_i}^{(q+h,h)}||b_{k_i}^{(q'+h',h')}|\right)$$

where $h = p - q$ and $h' = p - q'$.

**Proof.** It is entirely analogous to $p$-lower degree case, so we omit it. \qed

We finish this section by giving explicit formulas to compute de $p$-adjacency degree for a simplex and its maximal $p$-adjacency degree.

Assume $\sigma^{(q)}$ is a $q$-simplex, the $p$-adjacency degree of $\sigma^{(q)}$ has been defined as the number of $q'$-simplices $\sigma^{(q')}$ such that $\sigma^{(q)} \sim_{U_p}$ $\sigma^{(q')}$ and $\sigma^{(q)} \sim_{U_{p'}} \sigma^{(q')}$, with $p' = q + q' - p$, and its maximal $p$-adjacent degree is the number of $q'$-simplices $\sigma^{(q')} such that $\sigma^{(q')} \sim_{A_p} \sigma^{(q)}$ and $\sigma^{(q')}$ is not a face of a $q''$-simplex $\sigma^{(q'')}$ which is also $p$-adjacent to $\sigma^{(q)}$ (see Definitions 9 and 12). As we have already remarked, the fact that a $q'$-simplex
σ(q′) to be p-lower adjacent to σ(q) can be encoded in terms of the lower sign of both simplices. In other words, if one sets a base \{γ_1^{(p)}, \ldots, γ_{i_i}^{(p)}\} of C_p(K), then a q′-simplex σ(q′) is p-lower adjacent to σ(q) if and only if \(\text{sig}_L(σ(q), σ(q′); γ_i^{(p)}) \neq 0\) for some \(γ_i^{(p)}\), so that, one has:

\[
σ(q′) \sim_{L_p} σ(q) \iff \sum_i |\text{sig}_L(σ(q), σ(q′); γ_i^{(p)})| \geq 1 \\
\iff \text{min}\left(1, \sum_i |\text{sig}_L(σ(q), σ(q′); γ_i^{(p)})|\right) = 1.
\]

In a similar way, one has:

\[
σ(q′) \sim_{U_p} σ(q) \iff \sum_i |\text{sig}_U(σ(q), σ(q′); γ_i^{(p)})| \geq 1 \\
\iff \text{min}\left(1, \sum_i |\text{sig}_U(σ(q), σ(q′); γ_i^{(p)})|\right) = 1.
\]

For simplicity we shall denote:

\[
m^p_L(σ(q), σ(q′)) = \text{min}\left(1, \sum_i |\text{sig}_L(σ(q), σ(q′); γ_i^{(p)})|\right) \\
m^p_U(σ(q), σ(q′)) = \text{min}\left(1, \sum_i |\text{sig}_U(σ(q), σ(q′); γ_i^{(p)})|\right) \\
adj^p(σ(q), σ(q′)) = m^p_L(σ(q), σ(q′))(1 - m^{p+1}_L(σ(q), σ(q′)))(1 - m^{p′}_U(σ(q), σ(q′))).
\]

Hence,

\[
σ(q′) \sim_{A_p} σ(q) \iff \text{adj}^p(σ(q), σ(q′)) = 1. \quad (14)
\]

Therefore, if we take q, q′ and p non negative integers, put \(p′ = q + q′ - p\) and fix basis of \(C_q(K), C_q(K), C_p(K), C_{p+1}(K)\) and \(C_{p′}(K)\), namely, \(\{σ_1^{(q′)}, \ldots, σ_n^{(q′)}\}, \{γ_1^{(p)}, \ldots, γ_{i_i}^{(p)}\}, \{γ_1^{(p+1)}, \ldots, γ_{i_i}^{(p+1)}\} \) and \(\{τ_1^{(p′)}, \ldots, τ_{i_i}^{(p′)}\}\), respectively, then the following formulas compute the adjacency degrees for simplices.

**Theorem 4.** Let q and p be non negative integers. Then:

\[
\deg_A^p(σ_j^{(q)}) = \sum_{q′=p}^{\dim K} \sum_{k=1}^{f_{q′}} \text{adj}^p(σ_j^{(q)}, σ_k^{(q′)})
\]

with \(p′ = q + q′ - p\) and \(f_{q′} = \dim C_{q′}(K)\).
(2) \[
\deg^p_A(\sigma_j^{(q)}) = \deg_A(\sigma_j^{(q)}) - \sum_{q'=p}^{\dim K} \sum_{k=1}^{f_{q'}} \Delta_{q',k}
\]
with
\[
\Delta_{q',k} = \min \left( 1, \sum_{q'' \leq q'} |\text{sig}_L(\sigma_k^{(q')}, \sigma_k^{(q''); \sigma_j^{(q)'})| \cdot \text{adj}^p(\sigma_j^{(q')}, \sigma_k^{(q'')}) \right)
\]
where \( p \leq q' \leq \dim K, 1 \leq k \leq f_{q'}, q' + 1 \leq q'' \leq \dim K, 1 \leq \ell \leq f_{q''} \) and \( \{\sigma_\ell^{(q''})\}_\ell \) is a basis of \( C_{q''}(K) \).

Proof. (1) follows from formula \[14\]. To prove (2) we need only to check the number of those \( q'\)-simplices, which being \( p \)-adjacent to \( \sigma_j^{(q)} \), are also faces of \( q''\)-simplices adjacent to \( \sigma_j^{(q)} \). Assume that \( \sigma_k^{(q')} \) is a \( q'\)-simplex such that the following hold:

(a) \( \sigma_k^{(q')} \sim_{Ap} \sigma_j^{(q)} \)

(b) There exists a (likely not uniquely determined) \( q''\)-simplex \( \sigma^{(q'')} \) such that \( \sigma_k^{(q')} \subset \sigma^{(q'')} \) and \( \sigma^{(q'')} \sim_{Ap} \sigma_j^{(q)} \).

By assumption (a), we have that \( \text{adj}^p(\sigma_j^{(q')}, \sigma_k^{(q)}) = 1 \), and assumption (b) is equivalent to say that \( |\text{sig}_L(\sigma_k^{(q')}, \sigma_k^{(q'')}; \sigma_j^{(q)'})| \cdot \text{adj}^p(\sigma_j^{(q')}, \sigma_k^{(q''}) = 1 \). Therefore, under these assumptions, \( \sigma_k^{(q')} \) is not a maximal \( p \)-adjacent to \( \sigma_j^{(q)} \) simplex and we don’t have to take this simplex into account to compute \( \deg^p_A(\sigma_j^{(q)}) \). Thus, for every \( \sigma_k^{(q')} \), the expression:

\[
\sum_{q'=q'+1}^{\dim K} \sum_{\ell=1}^{\dim C_{q''}(K)} |\text{sig}_L(\sigma_k^{(q')}, \sigma_k^{(q'')}; \sigma_j^{(q)'})| \cdot \text{adj}^p(\sigma_j^{(q')}, \sigma_k^{(q'')})
\]

gives the number of \( q''\)-simplices (where \( q'' \) runs over all dimensions from \( q' + 1 \) to \( \dim K \)) which are \( p \)-adjacent to \( \sigma_j^{(q)} \) and contain \( \sigma_k^{(q')} \). Then \( \sigma_k^{(q')} \) is \( p \)-adjacent but not maximal to \( \sigma_j^{(q)} \) if and only if:

\[
\min \left( 1, \sum_{q''} \sum_{\ell} |\text{sig}_L(\sigma_k^{(q')}, \sigma_k^{(q''); \sigma_j^{(q)'})| \cdot \text{adj}^p(\sigma_j^{(q')}, \sigma_k^{(q'')}) \right) \cdot \text{adj}^p(\sigma_j^{(q')}, \sigma_k^{(q''}) = 1 ,
\]

and we get the statement. \( \square \)

4. Conclusions: potential applications and future research.

Many real networks in social sciences, biological and biomedical sciences or computer science have an inherent structure of simplicial complexes, which reflect the multi interactions among agents and groups of agents. As far as we know, higher order notions of adjacency and degree for simplices valid for any dimensional simplicial comparison are lacked in the literature. We propose these notions and give explicit methods for
computing them, showing its potential use in simplicial network science. These results can be applied to study topological and dynamical properties of simplicial networks; we will propose in [9] new centrality measures based on the higher degree definitions here presented, which should contain meaningful information about the relevance of an agent, or a simplicial community of agents, in terms of other collaborative simplicial communities. In particular, using the maximal $p$-adjacency we can define $p$-walks in a simplicial complex (and thus a $p$-distance), so that closeness and betweenness centralities can be now defined. These measures represent a starting point in studying the geometry and robustness of the simplicial networks, together with the importance of certain simplicial communities in how the information travels throughout a simplicial network.

If one would attempt to classify simplicial networks or to know the dynamics of a simplicial network, further research should be conducted; it may start with the study of higher order simplicial degree distributions, trying to generalise to the simplicial case some of the results of [3, 2]. New measures, such as a simplicial clustering coefficient, will be defined using the higher order maximal adjacency degree and might be needed to study the relationships among simplicial communities. A preferential attachment algorithm and a configuration model should be stated in order to generalise (beyond the $d$-pure simplicial networks) some of the results of [5]. Hopefully these notes contribute in developing new applications of topological data analysis in complex networks, and thus to expand the basis of an emergent Simplicial Network Science.

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