KMS States, Entropy and a Variational Principle for Pressure

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Abstract

We want to introduce the concept of entropy and pressure for $C^*$ Algebras. Several different definitions of entropy are known in our days. The one presented here is quite natural and extends the usual one for Dynamical Systems in Thermodynamic Formalism Theory. It has the advantage of been very easy to be introduced. It is basically obtained from transfer operators. Later we introduce the concept of pressure as a min-max principle.

Finally, we consider the concept of a KMS state been an equilibrium state for a potential (in the context of $C^*$-Algebras).

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1 Introduction and Main Result

We want to introduce the concept of entropy and pressure for $C^*$ Algebras. Several different definitions of entropy are known in our days [11]. The one presented here is quite natural and extends the usual one for Dynamical Systems in Thermodynamic Formalism Theory [9]. It has the advantage of been very easy to be introduced. It is basically obtained from transfer operators.

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Finally, we consider the concept of a KMS state been an equilibrium state for a potential (in the context of $C^*$-Algebras).

The KMS states are quite important in Quantum Statistical Mechanics [1]. We refer the reader for general references on $C^*$-Algebras to [13] [1] [8] [3]

Suppose $A$ is a commutative $C^*$-Algebra with unity and $\alpha : A \rightarrow A$ is an injective endomorphisms which preserves unity. Considering $A = C(X)$, via Gelfand-Naimark theorem, we get that $\alpha$ is of the form $\alpha(a) = a \circ T$, where $T : X \rightarrow X$ is a continuous transformation.

**Definition 1** A transference operator for $\alpha$ is linear transformation $L : A \rightarrow A$, such that $L(\alpha(a)b) = aL(b)$, $\forall a, b \in A$. Moreover, if $L(1) = 1$, then we say that the transfer operator is normalized.

In the case $A$ is a commutative algebra, the transfer operator takes the form of a Ruelle operator [12] [5] [6] [8] [2].

**Proposition 2** [7] If $L$ is a transference operator for $\alpha$ then:

1. $L(\alpha(a)b) = L(a)b$, $\forall a, b \in A$;
2. $L(1)$ is a central positive element of $A$.

**Proposition 3** [7] If $T$ is a local homeomorphism, Then all transference operator for $\alpha$ is of the form:

$$L(a)|_x = \sum_{T(y) = x} \rho(y)a(y),$$

where $\rho : X \rightarrow [0, \infty)$ is a continuous function. Moreover, any continuous function $\rho : X \rightarrow [0, \infty)$, defines a transfer operator by the above expression.
From now on, we suppose $T$ is a local homeomorphism and we denote by $L_\rho$ transfer operator defined by $\rho$. We want to introduce a notion of entropy for a state $\phi : A \to \mathbb{C}$, using the transfer operator. This generalizes the point of view described in [9] [10]. Denote $A_+ := \{ a \in A \mid \sigma(a) \in (0, \infty) \}$

**Definition 4** Given a state $\phi$ in $A$, we define the entropy of $\phi$ by:

$$h(\phi) = \inf_{a \in A_+} \left\{ \phi \left( \ln \left( \frac{L_\rho(a)}{\rho a} \right) \right) \right\},$$

where $L_\rho$ is the transfer operator for $\rho : X \to (0, \infty)$.

Note that the above definition is independent of the choice of $\rho$, indeed, if $\rho' : X \to (0, \infty)$ is another continuous function, then taking $a' = a\rho(\rho')^{-1}$, for some arbitrary $a \in A$, we get that $a' \in A_+$, and

$$\frac{L_\rho'(a')}{\rho'a'} = \frac{1}{\rho a} \sum_{y = T(x)} \rho'(x)a(x)\rho(x)\rho'(x)^{-1} = \frac{L_\rho(a)}{\rho a}.$$ 

Therefore we are considering the infimum over the same set.

**Definition 5** We say that a state $\phi$ is $\alpha$-invariant, if, $\phi \circ \alpha = \phi$.

**Definition 6** Given an element $b \in A_+$, we define the topological pressure of $b$ by

$$p(b) = \sup_{\phi \text{ inv}} \{ h(\phi) + \phi(\ln b) \}.$$ 

If $\phi$ is an $\alpha$-invariant state, such that, $h(\phi) + \phi(\ln b) = p(b)$, Then, we say that $\phi$ is an $C^*$-equilibrium state for $b$.

**Proposition 7** If $L_\rho(1) = 1$, then, there exists an state $\phi$ such that $\phi \circ L_\rho = \phi$.

**Proof.** As $L_\rho(1) = 1$, we have that $L_\rho^*(S) \subset S$, where $S$ is the set of all states of $A$. Using Tychonoff-Schauder theorem, we get that $L_\rho^*|S$ has a fixed point.

**Proposition 8** If $L_\rho(1) = 1$, then $p(\rho) = 0$. Moreover, the states $\phi$ which satisfies $\phi \circ L_\rho = \phi$ are equilibrium states for $\rho$. 

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Proof. Note that using $L_\rho$ in the definition of entropy we have that

$$p(\rho) = \sup_{\phi} \left\{ \inf_{a \in A^+} \left\{ \phi \left( \ln \left( \frac{L_\rho(a)}{a} \right) \right) \right\} \right\}.$$ 

Choosing $a = 1$, inside the infimum, it follows that $p(\rho) \leq 0$.

By the other hand, as $L_\rho$ is normalized, we have that $L_\rho \circ \alpha = Id$, and, if $\phi \circ L_\rho = \phi$, then $\phi \circ \alpha = \phi \circ L_\rho \circ \alpha = \phi$. As $\ln$ is a concave funtion, then

$$\ln(L_\rho(a)) \geq L_\rho(\ln a).$$

Therefore,

$$\phi \left( \ln \left( \frac{L_\rho(a)}{a} \right) \right) = \phi \left( \ln(L_\rho(a)) - \ln a \right) \geq \phi \left( L_\rho(\ln a) - \ln a \right).$$

If $\phi \circ L_\rho = \phi$, the right hand side of the inequality above is equal to zero, and therefore, $\inf_{a \in A^+} \left\{ \phi \left( \ln \left( \frac{L_\rho(a)}{a} \right) \right) \right\} = 0$. It follows that $p(\rho) \geq 0$, and in the case of an eigen-state, we have $h(\phi) + \phi(\rho) = \inf_{a \in A^+} \left\{ \phi \left( \ln \left( \frac{L_\rho(a)}{a} \right) \right) \right\} = 0 = p(\rho).$ 

If we consider the context of an algebra $A$, an injective endomorphism preserving unity $\alpha$, and a normalized transfer operator $L$, we can consider (among others) two different C*álgebras: the cross-product endomorphism $A \rtimes_{\alpha,L} \mathbb{N}$ (see [2]) and the C*-álgebra given by approximately proper equivalence relations $C^*(\mathcal{R},\mathcal{E})$ (see [4]). The second algebra is related with the equivalence relation $x \sim y \iff \exists n \in \mathbb{N}$, such that, $T^n(x) = T^m(y)$. The first one considers a more broad equivalence relation $x \sim y \iff \exists n, m \in \mathbb{N}$, such that, $T^n(x) = T^m(y)$.

We want to relate KMS states of $A \rtimes_{\alpha,L} \mathbb{N}$ and $C^*(\mathcal{R},\mathcal{E})$, with the equilibrium states (in $A$) of the potential $h^{-\beta}$. Here $\beta$ suppose to represent the inverse of temperature.

In the case the algebra $A$ is commutative, we have unique conditional expectations $F : A \rtimes_{\alpha,L} \mathbb{N} \to A$ and $G : C^*(\mathcal{R},\mathcal{E}) \to A$. Moreover, if $E := \alpha \circ L : A \to \alpha(A)$, for a conditional expectation with finite index, then the KMS states $\psi$ of $A \rtimes_{\alpha,L} \mathbb{N}$ can be decomposed as $\psi = \phi \circ F$, where $\phi$ is a state on $A$ which satisfies

$$\phi(a) = \phi(L(\Lambda a)), \forall a \in A,$$

and $\Lambda = h^{-\beta}\text{ind}(E)$. The KMS state $\psi$ of $C^*(\mathcal{R},\mathcal{E})$ can be decomposed as $\psi = \phi \circ G$, where $\phi$ is a state in $A$ which satisfies

$$\phi(a) = \phi(\Lambda^{[n]} E_n(\Lambda^{[n]} a)), \forall a \in A, \forall n \in \mathbb{N},$$

where $E_n = \alpha^n \circ L^n$, and $\Lambda^{[n]} = \prod_{i=0}^{n-1} \alpha^i(h^{-\beta}\text{ind}(E))$. 

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Proposition 9 If \( \psi = \phi \circ F \) is a \((h, \beta)\)-KMS state for \( A \rtimes_{\alpha, L} \mathbb{N} \), and \( L(A_1) = 1 \), then \( \phi \) is an equilibrium state (in \( A \)) for the potential \( h^{-\beta} \).

**Proof.** The condition \( L(A_1) = 1 \) implies that \( L_{h^{-\beta}} \) is a normalized transfer operator, and, therefore \( p(h^{-\beta}) = 0 \). Moreover, the KMS condition says that \( \phi(a) = \phi(L_{h^{-\beta}}(a)) \), which implies that \( \phi(\alpha(a)) = \phi(a) \). It follows from Proposition 8 that \( \phi \) is an equilibrium state for \( h^{-\beta} \).

In the construction of the algebras we are interested, the choice of two normalized transfer operator define isomorphic algebras, in such way that, the choice of the operator can be arbitrary.

Suppose that \( L_\rho(k) = \lambda k \), for some \( \lambda > 0 \) and \( k \in A_+ \). Defining \( \tilde{\rho} = \frac{\rho k}{\lambda \alpha(k)} \), we have that \( L_{\tilde{\rho}} \) is a normalized transfer operator for \( \alpha \), and, therefore, we can use it to obtain \( C^*(\mathbb{R}, \mathcal{E}) \).

Proposition 10 Suppose that \( \psi = \phi \circ G \) is a \((h, \beta)\)-KMS state of \( C^*(\mathbb{R}, \mathcal{E}) \).

Consider \( \rho = h^{-\beta} \) and suppose that \( L_\rho(k) = \lambda k \), for some \( \lambda > 0 \) e \( k \in A_+ \).

Denote \( \tilde{\rho} = \frac{\rho k}{\lambda \alpha(k)} \) and consider \( \tilde{\phi} \) the state of \( A \) given by \( \tilde{\phi}(a) = \phi(ka) \). If

\[
\lim_{n \to \infty} \left\| L^n_{\tilde{\rho}}(a) - \tilde{\phi}(a) \right\| = 0, \forall a \in A, \text{ then } \tilde{\phi} \text{ is an equilibrium state for } \tilde{\rho}.
\]

**Proof.** We can suppose, w.l.g., that \( C^*(\mathbb{R}, \mathcal{E}) \) was obtained from \( L_{\tilde{\rho}} \), in such way that \( \text{ind}(E) = \tilde{\rho}^{-1} \). Therefore,

\[
\Lambda^{[1]} = (\rho \tilde{\rho}^{-1}) = \rho \frac{\lambda \alpha(k)}{\rho k} = \frac{\lambda \alpha(k)}{k},
\]

and, more generally

\[
\Lambda^{[n]} = \prod_{i=0}^{n-1} \alpha^i (\rho \tilde{\rho}^{-1}) = \lambda^n \prod_{i=0}^{n-1} \frac{\alpha^{i+1}(k)}{\alpha^i(k)} = \frac{\lambda^n \alpha^n(k)}{k}
\]

The KMS condition implies

\[
\phi(a) = \phi \left( k \frac{\lambda^n \alpha^n(k)}{\lambda^n \alpha^n(k)} L^n_{\tilde{\rho}} \left( \frac{\lambda^n \alpha^n(k)}{k} a \right) \right) = \\
= \phi \left( k \alpha^n L^n_{\tilde{\rho}} \left( \frac{a}{k} \right) \right),
\]

for all \( n \in \mathbb{N} \). It follows that

\[
\tilde{\phi}(a) = \phi(ak) = \phi \left( k \alpha^n L^n_{\tilde{\rho}} \left( \frac{a}{k} \right) \right) = \tilde{\phi} \left( \alpha^n L^n_{\tilde{\rho}} (a) \right).
\]
Now,\[\|\hat{\phi}(L_{\bar{\rho}}(a) - a)\| = \|\hat{\phi}(\alpha^n(L_{\bar{\rho}}^{n+1}(a) - L_{\bar{\rho}}^n(a))\| \leq \hat{\phi}(\|\alpha^n(L_{\bar{\rho}}^{n+1}(a) - L_{\bar{\rho}}^n(a))\|) \leq \hat{\phi}(\|L_{\bar{\rho}}^{n+1}(a) - L_{\bar{\rho}}^n(a)\|) \leq \hat{\phi}(\|L_{\bar{\rho}}^{n+1}(a) - \tilde{\phi}(a)\|) - \hat{\phi}(\|\tilde{\phi}(a) - L_{\bar{\rho}}^n(a)\|) \rightarrow_{n \to \infty} 0,\]

and, therefore \(\hat{\phi} \circ L_{\bar{\rho}} = \hat{\phi}\). From proposition 8, the claim follows. \(\blacksquare\)

Note that the hypothesis of the convergence of \(L_{\bar{\rho}}^n\) is one of the conclusions of Ruelle-Perron-Frobenius theorem (see [12], [4], [6]) so that the classical setting satisfies the hypothesis of the previous proposition.

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