3+1 formulation of light modes in nonlinear electrodynamics: Minkowski spacetime

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Abstract. We study the light modes in nonlinear electrodynamics of Plebanski class, such as Post-Maxwellian Lagrangian, Born-Infeld Lagrangian, ModMax Lagrangian, and Heisenberg-Euler-Schwinger QED Lagrangian. Nonlinear electrodynamics provides light, a probe photon, with not only the permittivity and permeability but also the magneto-electric effect, which result in vacuum birefringence in the studied models but multirefringence in the most general Plebanski-class Lagrangians. In the Minkowski spacetime, we advance a 3+1 formulation for polarization vectors and propagation of light, and find a factorization of the Fresnel equation into two quadratures, practically allowing to find refractive indices. The polarization vectors form a basis of reciprocal vectors in the non-degenerate case. We apply the formulation to find the refractive indices and polarization vectors for the Post-Maxwellian Lagrangian, Born-Infeld Lagrangian, ModMax Lagrangian, and HES Lagrangian in an arbitrarily strong magnetic field.

Keywords: nonlinear electrodynamics, nonlinear vacuum, light propagation, polarization vectors, vacuum birefringence

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1 Introduction

Light propagation has been an interesting theoretical issue in curved spacetimes [1]. Recently Event Horizon Telescope (EHT) observed the shadow of supermassive black holes from light propagating around the event horizon and provided a way to understand the geometry of black holes [2, 3]. Nonlinear electrodynamics has also provided nontrivial backgrounds for light propagation. Born and Infeld introduced a nonlinear Lagrangian of the Maxwell scalar and its pseudo-scalar, which recovers the Maxwell theory in the weak field limit but significantly modifies the physics of light-matter and light-light interactions in strong field limit [4]. The causality of Born-Infeld Lagrangian was studied in [5]. Heisenberg and Euler obtained the exact one-loop effective Lagrangian for electrons in a constant electromagnetic field [6]. Later Schwinger introduced the proper-time method for quantum field theory and obtained the one-loop effective Lagrangian for spinless charged bosons and spin-1/2 fermions in a constant electromagnetic field [7].

The prominent feature of Heisenberg-Euler-Schwinger (HES) QED Lagrangian is the vacuum polarization [8, 9] and the so-called (Sauter-)Schwinger pair production of charged particles and anti-particles [7, 10]. Interestingly, the vacuum becomes unstable in strong electric fields due to pair production, which is a consequence of the imaginary part of the effective Lagrangian [11, 12]. The polarized vacua in strong electromagnetic fields provide a probe photon (light) with nontrivial backgrounds. In recent years, nonlinear electrodynamics (NED) has been intensively studied partly because ultra-intense lasers using the chirped pulse amplification (CPA) technology have been developed [13] and partly because highly magnetized neutron stars and magnetars have been observed; these astrophysical bodies have
magnetic fields comparable to or stronger than the critical field $B_c = m^2 c^2 / e\hbar = 4.4 \times 10^{13}$ G [14]. An electromagnetic field makes the vacuum a polarized medium, and the polarized vacuum behaves similarly as the dielectric or ferromagnetic medium [15]. Nonlinear electrodynamics in general exhibits the magneto-electric effect, in which a magnetic field induces electric polarization whereas an electric field induces magnetization [16–18]. A strong electric field, on the other hand, create pairs of charged particles and anti-particles, known as the (Sauter-)Schwinger pair production.

Nonlinear electrodynamics exhibits a rich structure of vacuum polarization, such as vacuum birefringence, photon propagation, and magneto-electric effect. In a strong magnetic field, the polarized vacuum acquires non-trivial refractive indices, which give birefringence of photon propagation either along the magnetic field or in the perpendicular plane [19, 20]. Light propagation has been intensively studied in NED of Plebanski Lagrangians [21], in which a probe photon of low energy obeys a wave equation in nonlinear backgrounds. Most literature has employed the covariant four-vector formulation of the light propagation (for instance, see [19, 22–26]), while the 3+1 formulation of photon propagation was used for the post-Maxwellian theory in weak electromagnetic fields [27].

We studied the light propagation in a supercritical magnetic field and a weak electric field, especially the case of a non-zero electric field along the magnetic field (electromagnetic wrench) [28, 29]. In contrast to the pure magnetic field case, the electromagnetic wrench introduces magneto-electric effects and thus significantly modifies the light modes. In [28, 29], we focused on the configuration of parallel electric and magnetic fields to study the effect in the simplest setting. However, considering complicated field configurations around neutron stars, we need to extend the previous formulation to an arbitrary field configuration. Furthermore, it is worthy to make the formulation applicable to other Lagrangians considered in the context of general NED.

The purpose of this paper is to introduce the 3+1 formulation of light modes in general NED of the Plebanski Lagrangians $L(F,G)$ [21], where $F \equiv F_{\mu\nu} F^{\mu\nu}/4$ is the Maxwell scalar, and $G \equiv F_{\mu\nu}^* F^{\mu\nu}/4$ is the Maxwell pseudo-scalar. $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the field-strength tensor, and $F^{*\mu\nu} = \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}/2 (\varepsilon_{0123} = 1)$ is the dual field-strength tensor [30]. In the Minkowski spacetime with the metric $(+,−,−,−)$, the timelike Killing vector $\partial_t$ foliates the spacetime manifold into one-parameter spacelike hypersurface $\Sigma_t$. On each hypersurface $\Sigma_t$, the energy-momentum of a photon is given by $k^\mu = (\omega, k)$ and the field by $F^{\mu\nu}$. In curved spacetimes, the Maxwell theory also has the 3+1 formulation [31]. The advantage of the 3+1 formulation is that the propagation and polarization of probe photons (light) are expressed in terms of directly measurable electromagnetic fields $(E,B)$ and the direction of photon $k$ on each $\Sigma_t$.

The organization of this paper is as follows. In section 2, we introduce the 3+1 formulation of light modes in general NED of Plebanski Lagrangians. On each $\Sigma_t$ whose observers measure the electromagnetic fields $(E,B)$ and the photon 4-vector $(\omega,k)$, the constitutive equations $\{D(E,B), H(E,B)\}$ for electric induction and magnetic field strength are derived. In the linear response theory for weak low-frequency photons, a probe photon (light) experiences vacuum media with not only electric permittivity and magnetic permeability tensors but also magneto-electric responses. Then, expressing the constitutive equation for the probe photon in terms of dyadics, we obtain the complete information about the light modes, i.e., refractive indices and polarization vectors. Also as found in section 2.4, we compare the results from the 3+1 formulation with those from the covariant formulation in the literature. These two formulations agree with each other, while the 3+1 formulation is more conve-
nient for observations or measurements. In section 3, we apply the 3+1 formulation to the Post-Maxwellian Lagrangian, the Born-Infeld Lagrangian, the ModMax Lagrangian, and the HES QED Lagrangian. Finally in section 4, summarizing the results, we conclude to discuss the implications to the vacuum birefringence research, the terrestrial experiments using ultra-intense lasers in the weak field regime and the astrophysical observations for highly magnetized neutron stars in the strong field regime.

2 3+1 formulation of the light modes in a nonlinear vacuum

In this section, we study the propagation of light in NED and find the polarization vectors in the 3+1 formulation in the Minkowski spacetime. To do so, we consider the Plebanski class of Lagrangians for NED, which includes the Post-Maxwellian Lagrangian [27], the Born-Infeld Lagrangian [4], the ModMax Lagrangian [32], and the HES QED Lagrangian in constant electromagnetic fields [6, 7].

2.1 Polarization and magnetization of a nonlinear vacuum

The Plebanski class of Lorentz- and gauge-invariant Lagrangians takes the form [21]

\[ \mathcal{L} = \mathcal{L}(F, G), \]  

(2.1)

where \( F \) and \( G \) are the Maxwell scalar and the Maxwell pseudo-scalar, respectively:

\[ F \equiv \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (B^2 - E^2), \quad G \equiv \frac{1}{4} F_{\mu\nu} F^*_{\mu\nu} = -E \cdot B. \]  

(2.2)

Here and hereafter, we use the Lorentz-Heaviside units with \( \hbar = c = 1 \) and \( \alpha = e^2/4\pi \), where \( e \) is the elementary charge. Note that \( \mathcal{L}(F, -G) \neq \mathcal{L}(F, G) \) gives a parity violating Lagrangian while the parity conserving Lagrangian \( \mathcal{L}(F, G) \) is an even function of \( G \).

The nonlinear Lagrangian (2.1) gives the constitutive relations [6, 33]:

\[ D = E + \mathcal{P} = \frac{\partial \mathcal{L}}{\partial E}, \quad H = B - \mathcal{M} = -\frac{\partial \mathcal{L}}{\partial B}, \]  

(2.3)

where \( \mathcal{P} \) is the polarization, and \( \mathcal{M} \) is the magnetization. Substituting \( \mathcal{L} = \mathcal{L}(F, G) \) into (2.3), we obtain \( D \) and \( H \) as

\[ D = -\mathcal{L}_F E - \mathcal{L}_G B, \quad H = -\mathcal{L}_F B + \mathcal{L}_G E, \]  

(2.4)

where the subscripts \( F \) and \( G \) denote the partial differentiation of \( \mathcal{L} \) with respect to them: e.g., \( \mathcal{L}_{FG} \equiv \partial^2 \mathcal{L}/(\partial F \partial G) \). In general, \( D \) and \( H \) depend on \( E \) and \( B \) non-linearly because \( \mathcal{L} \) can take any analytical function and generically \( \mathcal{L}_G \neq 0 \). However, the Maxwell theory \( \mathcal{L} = -F \) has the zero polarization and magnetization as expected.

2.2 Permittivity and permeability for a weak low-frequency light

Now we find the permittivity and permeability for a weak low-frequency (\( \omega \ll m = 0.5 \text{MeV} \)) light (\( \delta E, \delta B \)) under a strong constant background field (\( E_0, B_0 \)). The light can be treated as a perturbation added to the background field [20, 34]. Furthermore, when the background field varies slowly over the magnetic or electric length (\( \lambda_B = 1/\sqrt{\varepsilon B_0}, \lambda_E = 1/\sqrt{\varepsilon E_0} \)) of the background field, the locally constant field approximation holds.
Then, we can decompose \( \mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}, \mathbf{P}, \) and \( \mathcal{M} \) in the form of
\[
\mathbf{A} = \mathbf{A}_0 + \delta \mathbf{A},
\]
where \( \mathbf{A}_0 \) (\( \delta \mathbf{A} \)) refers to the background (light) field. Furthermore, as \( \mathbf{A}_0 \) is uniform (locally constant) while \( \delta \mathbf{A} \) is varying, the relations (2.3) should be satisfied separately between the uniform and varying quantities:
\[
\begin{align*}
\mathbf{D}_0 &= \mathbf{E}_0 + \mathbf{P}_0, & \mathbf{H}_0 &= \mathbf{B}_0 - \mathcal{M}_0, \\
\delta \mathbf{D} &= \delta \mathbf{E} + \delta \mathbf{P}, & \delta \mathbf{H} &= \delta \mathbf{B} - \delta \mathcal{M}.
\end{align*}
\]
Because the light is weak, we consider only the linear response of the field-modified-vacuum to the light and thus define the permittivity and permeability tensors for the light as
\[
\begin{align*}
\epsilon_E &= -L_F \mathbf{I} + \left[ L_{FF} \mathbf{E}_0 \mathbf{E}_0 + L_{FG} \left( \mathbf{E}_0 \mathbf{B}_0 + \mathbf{B}_0 \mathbf{E}_0 \right) + L_{GG} \mathbf{B}_0 \mathbf{B}_0 \right], \\
\epsilon_B &= -L_G \mathbf{I} + \left[ -L_{FF} \mathbf{E}_0 \mathbf{B}_0 + L_{FG} \left( \mathbf{E}_0 \mathbf{E}_0 - \mathbf{B}_0 \mathbf{B}_0 \right) + L_{GG} \mathbf{E}_0 \mathbf{E}_0 \right], \\
\mu_B &= -L_F \mathbf{I} + \left[ -L_{FF} \mathbf{B}_0 \mathbf{B}_0 + L_{FG} \left( \mathbf{B}_0 \mathbf{E}_0 + \mathbf{E}_0 \mathbf{B}_0 \right) - L_{GG} \mathbf{E}_0 \mathbf{E}_0 \right], \\
\mu_E &= L_G \mathbf{I} + \left[ L_{FF} \mathbf{B}_0 \mathbf{E}_0 + L_{FG} \left( \mathbf{B}_0 \mathbf{B}_0 - \mathbf{E}_0 \mathbf{E}_0 \right) - L_{GG} \mathbf{E}_0 \mathbf{E}_0 \right],
\end{align*}
\]
where the products of two vectors are dyadics\(^1\), and \( \mathbf{I} \) is the unit tensor \([37]\). Dyadics will prove useful in finding the polarization vectors later on. The symbols \( L_F, L_G, L_{FF}, L_{FG}, \) and \( L_{GG} \) refer to \( \mathcal{L}_F, \mathcal{L}_G, \mathcal{L}_{FF}, \mathcal{L}_{FG}, \) and \( \mathcal{L}_{GG} \) evaluated at \( \mathbf{E} = \mathbf{E}_0 \) and \( \mathbf{B} = \mathbf{B}_0 \), respectively. From now on, we omit the subscript \( 0 \) in representing the background fields for notational simplicity.

### 2.3 Polarization vectors and refractive indices

In a source-free space, the light fields obey the following Maxwell equations
\[
- \frac{\partial \delta \mathbf{B}}{\partial t} = \nabla \times \delta \mathbf{E}, \quad \frac{\partial \delta \mathbf{D}}{\partial t} = \nabla \times \delta \mathbf{H}.
\]
As both the constitutive relation (2.7) and the Maxwell equations (2.9) are linear in the light fields (linear response theory), we use a plane wave with an angular frequency of \( \omega \) and a propagation vector of \( \mathbf{k} = \omega \mathbf{n} = \omega \mathbf{n} \)\(^2\) for the light \((n > 0 \text{ without loss of generality})\): \( \delta \mathbf{E}, \delta \mathbf{B}, \delta \mathbf{D}, \) and \( \delta \mathbf{H} \) are all proportional to \( e^{-i k \cdot r - \omega t} = e^{i \omega (\mathbf{n} \cdot \mathbf{r} - t)} \). Then eq. (2.9) reduces to
\[
\begin{align*}
\delta \mathbf{B} &= \mathbf{n} \times \delta \mathbf{E}, \\
\delta \mathbf{D} &= -\mathbf{n} \times \delta \mathbf{H}.
\end{align*}
\]
\(^1\)Two vectors \( \mathbf{A} \) and \( \mathbf{B} \) form a dyadicic \( \mathbf{AB} \), a rank-2 tensor in 3D space, with components \((\mathbf{AB})_{ij} = A_i B_j\).
\(^2\)For a direction vector \( \mathbf{n} \) and an arbitrary vector \( \mathbf{A} \), we use the following notations: \( A_n := \mathbf{A} \cdot \mathbf{n}, A_\perp := \mathbf{A} \times \mathbf{n}, A = (\mathbf{n} \times \mathbf{A}) \times \mathbf{n}, A^2 := \mathbf{A} \cdot \mathbf{A}, \) and \( A^2_\perp := A^2 - A^2_n \).
where \( \Lambda \) is a 3-by-3 matrix and can be explicitly found:

\[
\Lambda = \epsilon_E \tilde{n} \cdot \tilde{\mu}_E + \epsilon_B \cdot \tilde{n} + \tilde{n} \cdot \tilde{\mu}_B \cdot \tilde{n},
\]

(2.12)

where \( \tilde{n} \) is a 3-by-3 matrix with the components \( \tilde{n}_{ij} = -\epsilon_{ijk}n_k \).

Substituting (2.8) into (2.12), we obtain \( \Lambda \) as

\[
\Lambda = -L_F [(1 - n^2) I + nn] + L_{FF}PP + L_{FG} (PQ + QP) + L_{GG}QQ,
\]

(2.13)

where

\[
P = E + n \times B, \quad Q = B - n \times E.
\]

Furthermore, in trying to form a complete square of \( P \) and \( Q \), we find \( \Lambda \) as a sum of self-conjugate dyadics (dyadics of two identical vectors) \cite{37} and a constant multiple of the unit tensor:

\[
\Lambda = XX + YY + ZZ + U,
\]

(2.15)

where

\[
X = aP + bQ, \quad Y = cQ, \quad Z = d\tilde{n}, \quad U = eI
\]

(2.16)

and

\[
a^2 = L_{FF}, \quad ab = L_{FG}, \quad b^2 = \frac{L_{FG}^2}{L_{FF}}, \quad c^2 = \frac{L_{FF}L_{GG} - L_{FG}^2}{L_{FF}}, \quad d^2 = -L_F, \quad e = -L_F (1 - n^2).
\]

The form (2.15) suggests that we may use \( X, Y, \) and \( Z \) or their reciprocal vectors as basis vectors to represent \( \delta E \), as in crystallography. However, it is possible only when the volume \( V := X \times Y \cdot Z \) is not zero, called non-degenerate. The degenerate condition of \( V = 0 \) is brought by either \( abc = 0 \) or when \( P \times Q \cdot \tilde{n} = 0 \), as can be found from (2.14) and (2.16).

The first condition \( abc = 0 \) is independent of \( \tilde{n} \) but depends only on the functional form of \( \mathcal{L}(F, G) \) and the background field. In fact, the condition reduces to \( c = 0 \) because \( a^2 = L_{FF} \) and \( d^2 = -L_F \) do not vanish in general for non-trivial Lagrangians. An example of having \( c = 0 \) is the ModMax Lagrangian, as will be shown in section 3.3. In contrast, the second condition \( P \times Q \cdot \tilde{n} = 0 \) depends also on \( \tilde{n} \). It is satisfied in a very specific configuration, we call the free propagation configuration (FPC), in which \( B_t = \tilde{n} \times E_t \). In such a case, \( n = 1, S := E \times B \) heads along \( \tilde{n} \), and any vector perpendicular to \( \tilde{n} \) is a polarization vector: the light with \( \tilde{n} \) propagates as if it were in a free vacuum. If we assume \( n = 1 \) first, the only possible configuration is the FPC in the non-degenerate case. However, the degenerate case \( (c = 0) \) has another configuration in addition to the FPC, as shown in section 2.3.2. A special case of the FPC is the colinear configuration: \( E \parallel B \parallel \tilde{n} \). Below we deal with the non-degenerate case of \( c \neq 0 \) first and then the degenerate case of \( c = 0 \), disregarding the FPC due to its triviality.

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3 The cross product of two vectors can be rewritten as a matrix-vector product. For \( n = (n_1, n_2, n_3) \), we construct an associated matrix \( \tilde{n} \) with \( \tilde{n}_{ij} = -\epsilon_{ijk}n_k \). Then, the following identities hold: \( n \times A = \tilde{n} \cdot A \), \( (n \times A)B = (\tilde{n} \cdot A)B \), and \( (AB) \cdot \tilde{n} = -A(\tilde{n} \cdot B) \).
In the non-degenerate case, $V \neq 0$, and thus we use the reciprocal vectors $X'$, $Y'$, and $Z'$ as basis vectors:

$$\delta E = pX' + qY' + rZ', \quad (2.18)$$

where

$$X' = \frac{Y \times Z}{V}, \quad Y' = \frac{Z \times X}{V}, \quad Z' = \frac{X \times Y}{V}. \quad (2.19)$$

Assuming $d^2 = -L_F \neq 0$, we can obtain the linear equations of $p$, $q$, and $r$ by multiplying (inner product) $X$, $Y$, and $Z$, respectively, from the left to the equation $\Lambda \cdot \delta E = 0$ with $\Lambda$ in (2.15):

$$Mp + Hq = 0, \quad Hp + Nq = 0, \quad r = -\frac{Z \cdot X}{d^2} p - \frac{Z \cdot Y}{d^2} q, \quad (2.20)$$

where

$$H = \left[ X \cdot Y - \frac{(X \cdot Z)(Z \cdot Y)}{d^2} \right], \quad M = \left[ X^2 + e - \frac{(X \cdot Z)^2}{d^2} \right], \quad N = \left[ Y^2 + e - \frac{(Y \cdot Z)^2}{d^2} \right]. \quad (2.21)$$

The linear equations (2.20) have non-trivial solutions when $MN - H^2 = 0$, from which $n$ is determined. Then $(p, q, r)$ and thus the polarization vectors are determined as shown in Table 1. In the particular case of $H = M = N = 0$, $(p, q)$ can have arbitrary values, and the resulting polarization vector spans a plane:

$$\delta E = p \left( X' - \frac{X \cdot Z}{d^2} Z' \right) + q \left( Y' - \frac{Y \cdot Z}{d^2} Z' \right). \quad (2.22)$$
In contrast to a photon in a free vacuum, a photon in a nonlinear vacuum acquires a longitudinal component when \( r \neq 0 \): \( \mathbf{\delta E} \cdot \mathbf{n} = r/d \) from (2.16), (2.18), and (2.19). Such a longitudinal component is familiar from anisotropic dielectric medium [35].

So far we have proceeded without determining the refractive index \( n \), which is obtained from the null determinant condition for eqs. (2.20): \( MN - H^2 = 0 \). Factorizing \( MN - H^2 \) by using (2.14), (2.16), (2.17), and (2.21), we obtain the Fresnel equation:

\[
\frac{\tilde{P}^2}{n^2 - 1} - \lambda^{(+)}, \quad \frac{\tilde{P}^2}{n^2 - 1} - \lambda^{(-)} = 0, \tag{2.23}
\]

where

\[
\tilde{P}^2 := P^2 - n^2 p_n^2 = E^2 - 2nS_n + n^2(B_t^2 - E_n^2), \quad \lambda^{(\pm)} = \frac{-\beta \pm \sqrt{\beta^2 - 4\gamma}}{2\gamma}. \tag{2.24}
\]

The parameters \( \alpha, \beta, \) and \( \gamma \) are determined solely by the background field, regardless of \( \mathbf{n} \), and thus so is \( \lambda^{(i)} \):

\[
\begin{align*}
\alpha &= L_F^2 + 2L_F (GL_{FG} - FL_{GG}) - (L_{FG}L_{GG} - L_{FG}^2) G^2 \\
d &= d^2 - 2d [abG - (b^2 + c^2)F] - a^2 c^2 G^2, \\
\beta &= L_F (L_{FG} + L_{GG}) - 2F(L_{FG}L_{GG} - L_{FG}^2) = -d^2(a^2 + b^2 + c^2) - 2a^2 c^2 F, \\
\gamma &= L_{FG}L_{GG} - L_{FG}^2 = a^2 c^2.
\end{align*}
\tag{2.25}
\]

Equation (2.23) can be rewritten as

\[
(B_t^2 - E_n^2 - \lambda^{(\pm)})n^2 - 2S_n n + E_n^2 + \lambda^{(\pm)} = 0, \tag{2.26}
\]

solving which, we obtain the refraction indices as

\[
n^{(i)}_{\pm} = \frac{-S_n \pm \sqrt{S_n^2 + (\lambda^{(i)} - B_t^2 + E_n^2)(\lambda^{(i)} + E_n^2)}}{\lambda^{(i)} - B_t^2 + E_n^2}, \tag{2.27}
\]

where \( i = \pm \). Without a loss of generality, we take only positive values of \( n \) and solve (2.20) to obtain the corresponding polarization vectors (see Table 1). It should be mentioned that we may have multirefringence, even beyond birefringence, because (2.23) is a quartic equation of \( n \). In a pure magnetic field background, \((n^{(i)})^2\) has a simpler form of

\[
(n^{(i)})^2 = \left| \frac{\lambda^{(i)}}{\lambda^{(i)} - B_t^2} \right|. \tag{2.28}
\]

So far, we have implicitly assumed that \( n \neq 1 \). When \( n = 1 \), the dyadic \( \mathbf{A} \) in (2.15) becomes \( \mathbf{A} = \mathbf{XX} + \mathbf{YY} + \mathbf{ZZ} \) as \( e \to 0 \). Then, the condition of \( MN - H^2 = 0 \) reduces to \( P_t = Q_t = 0 \), which is satisfied only for the FPC. The light sees no effect of the nonlinear vacuum in such a case.

### 2.3.2 Degenerate case \((c = 0)\)

When \( c = 0 \), \( \mathbf{Y} = 0 \) from (2.16), and \( \mathbf{A} \) becomes simpler. Unless \( \mathbf{Z} \parallel \mathbf{X} \), which turns out to be the FPC, we can proceed similarly as in the non-degenerate case by introducing two-dimensional reciprocal vectors as

\[
\begin{align*}
\mathbf{X}' &= \frac{Z^2 \mathbf{X} - (\mathbf{X} \cdot \mathbf{Z}) \mathbf{Z}}{Z^2 X^2 - (\mathbf{X} \cdot \mathbf{Z})^2}, \\
\mathbf{Z}' &= \frac{X^2 \mathbf{Z} - (\mathbf{Z} \cdot \mathbf{X}) \mathbf{X}}{X^2 Z^2 - (\mathbf{Z} \cdot \mathbf{X})^2}.
\end{align*}
\tag{2.29}
\]
where $X'$ and $Z'$ span the same plane spanned by $X$ and $Z$; $\delta E$ has no component perpendicular to the plane. The counterpart to the equations (2.18), (2.20), and (2.21) becomes as follows:

$$\delta E = pX' + rZ', \quad \tilde{M}p + \tilde{H}r = 0, \quad \tilde{H}p + \tilde{N}r = 0,$$

(2.30)

The null determinant condition $\tilde{M}\tilde{N} - \tilde{H}^2 = 0$ is given as a quadratic equation of $n$:

$$[a^2(E_n^2 - B_n^2) + b^2(B_n^2 - E_n^2) - L_F - 2abG] n^2 + 2S_n(a^2 + b^2)n - a^2E_n^2 + 2abG - b^2B_n^2 + L_F = 0.$$

(2.31)

Once $n$ is available by solving this equation, the polarization vectors can be found from $(p, r)$ in table 1.

In contrast to the non-degenerate case, the degenerate case always has a non-trivial mode with $n = 1$. With $n = 1$, $\Lambda = XX + ZZ$. When $Z \parallel X$ (non-FPC), $Z \times X$ becomes the polarization vector because $\Lambda \cdot \delta E = 0$ is satisfied by construction. The case of $Z \parallel X$ corresponds to the FPC.

2.4 Equivalence with the covariant formulation

Most literature employs the covariant four-vector formulation for light propagation in NED [19, 23, 24, 38–40]. The presented 3+1 formulation is equivalent to the covariant formulation and complements it with a complete determination of the polarization vectors.

In the 3+1 formulation in the Minkowski spacetime, we use the Killing vector $\partial_t$ for a decomposition of the spacetime into a one-parameter spacelike hypersurface $\Sigma_t$. Each observer on $\Sigma_t$ measures the energy-momentum of photons as $k^\mu = \omega(1, n)$ and the electromagnetic field $E, B$. Thus, the covariant vectors used for polarization tensors take the form

$$F_{\mu
u}k_\nu = \omega(E \cdot n, P), \quad F^*_{\mu\nu}k_\nu = \omega(B \cdot n, Q).$$

(2.32)

The Fresnel equation in the covariant formulation, for instance eq. (12) of $f^2 := F_{\alpha\mu}F^{\alpha\nu}k_\mu k_\nu$ in [40], recovers eq. (2.23) with the same $\{\alpha, \beta, \gamma\}$. The advantage of the 3+1 formulation is that the refractive indices and polarization vectors are given in terms of $E, B$ and the propagation vector $n$ on each $\Sigma_t$.

3 Application to nonlinear electrodynamics models

In this section, we apply the 3+1 formulation to several NED models with a background magnetic field: post-Maxwellian (PM) Lagrangian [27], Born-Infeld (BI) Lagrangian [4], Mod-Max (MM) Lagrangian [32, 41], and the Heisenberg-Euler-Schwinger (HES) Lagrangian for an arbitrarily strong magnetic field [42]. Without loss of generality, we assume a magnetic field along the $z$-axis and the light’s propagation direction vector on the $xz$-plane: $B = B_0\hat{z}$ and $\hat{n} = (\sin \theta, 0, \cos \theta)$. The light modes for a given Lagrangian, i.e., $n$ and $\delta E$, is found by following the procedure in table 2 and appendix A.
Table 2. Procedure to find the light modes in a nonlinear non-degenerate vacuum. The steps from 7 to 9 should be implemented for each value of \( n \) found in the step 6. In appendix A, we present the essential formulas used in the procedure in order. A similar procedure can be set up the degenerate case.

3.1 Post-Maxwellian Lagrangian

The PM Lagrangian [27] is given as

\[
\mathcal{L}_{\text{PM}} = - F + \eta_1 F^2 + \eta_2 G^2, \tag{3.1}
\]

where \( \eta_1 \) and \( \eta_2 \) are parameters. It is useful as it represents the weak-field limit of other Lagrangians. When \( \eta_1/4 = \eta_2/7 = e^4/(360\pi^2m^4) \), it is the HES Lagrangian in the weak field limit [6, 7, 43], and, when \( \eta_1 = \eta_2 = 1/(2T) \), it is the BI Lagrangian in the weak field limit (see section 3.2). It has been widely used to design vacuum birefringence experiments as only subcritical fields are available in laboratory. One may extend \( \mathcal{L}_{\text{PM}} \) by adding \( \eta_3 FG \) (parity violating) [27] or further by adding \( \eta_4 F\sqrt{G^2} \) (parity conserving).

The procedure in table 2 yields the coefficients (2.17) and (2.25), regardless of the specific configuration:

\[
\begin{align*}
    a^2 &= 2\eta_1, \quad ab = 0, \quad b^2 = 0, \quad c^2 = 2\eta_2, \quad d^2 = 1 - 2F\eta_1 \\
    \alpha &= 4F^2\eta_1^2 - 4(2F^2 + G^2)\eta_1\eta_2 - 4F(\eta_1 - \eta_2) + 1 \\
    \beta &= 4F\eta_1^2 - 4F\eta_1\eta_2 - 2(\eta_1 + \eta_2), \quad \gamma = 4\eta_1\eta_2. 
\end{align*} \tag{3.2}
\]

For the configuration of a pure magnetic field, the light modes are given as shown in table 3. Expanding the expressions in \( B_0 \), we obtain the well-known result for a weak magnetic field: \( n = 1 + \eta_1 B_0^2 \sin^2 \theta \) with \((0, 1, 0)\) and \( n = 1 + \eta_2 B_0^2 \sin^2 \theta \) with \((\cos \theta, 0, -\sin \theta)\) [20, 39]. The former mode is called the perpendicular mode as \( \delta \mathbf{E} \) is perpendicular to the plane formed by \( \hat{\mathbf{n}} \) and \( \mathbf{B} \). The latter is called the parallel mode and has a longitudinal component: \( \delta \mathbf{E} \cdot \hat{\mathbf{n}} = \eta_2 B_0^2 \sin 2\theta \).

3.2 Born-Infeld Lagrangian

The BI Lagrangian, introduced to remove the self-energy divergence in classical electrodynamics, is given as [4]

\[
\mathcal{L}_{\text{BI}} = T - \sqrt{T^2 + 2FT - G^2}, \tag{3.3}
\]

where \( T \) is a parameter. In the weak-field limit \((T \gg F, G)\), the BI Lagrangian becomes the PM Lagrangian with \( \eta_1 = \eta_2 = 1/(2T) \).
The coefficients (2.17), (2.25), (2.24) are given in a configuration-independent form as

\[
\begin{align*}
    a^2 &= \frac{T^2}{D^{3/2}}, \quad ab = -\frac{GT}{D^{3/2}}, \quad b^2 = \frac{G^2}{D^{3/2}}, \quad c^2 = \frac{1}{D^{1/2}}, \quad d^2 = \frac{T}{D^{1/2}} \\
    \alpha &= \frac{T^2(2F + T)}{D^2}, \quad \beta = -\frac{2T^2(2F + T)}{D^2}, \quad \gamma = \frac{T^2}{D^2}, \quad \lambda^{(\pm)} = 2F + T
\end{align*}
\]

(3.4)

where \( D = T^2 + 2FT - G^2 \).

As shown in table 3, the BI Lagrangian does not have birefringence in a pure magnetic field [41, 44]. Furthermore, \( H = M = N = 0 \) holds (see table 1), and thus the polarization vector space is two-dimensional as in a free vacuum. However, \( \delta \mathbf{E} \) can have a longitudinal component here: \( \delta \mathbf{E} \cdot \hat{n} = qB_0^2 \sin 2\theta/(2T) \).

### 3.3 ModMax Lagrangian

The MM Lagrangian was introduced as a modification of the Maxwell theory to preserve duality invariance and conformal invariance [32]. It has the form of

\[
\mathcal{L}_{\text{MM}} = -F \cosh g + \sqrt{F^2 + G^2} \sinh g
\]

(3.5)

with \( g \geq 0 \) for a lightlike or subluminal propagation of light. It reduces to the Maxwell Lagrangian when \( g = 0 \). Regardless of the configuration, the coefficients (2.17) given as

\[
\begin{align*}
    a^2 &= \frac{G^2}{R^3} \sinh g, \quad ab = -\frac{FG}{R^3} \sinh g, \quad b^2 = \frac{F^2}{R^3} \sinh g, \\
    c^2 &= 0, \quad d^2 = \cosh g - \frac{F}{R} \sinh g
\end{align*}
\]

(3.6)

where \( R = \sqrt{F^2 + G^2} \). As \( c^2 = 0 \), the MM Lagrangian is degenerate, and the formulas in section 2.3.2 are used for analysis.

The light modes in a pure magnetic field are shown in table 3. What is peculiar is that the refractive indices and polarization vectors are independent of the strength of the background magnetic field unlike other Lagrangians, which is a consequence of conformal invariance [41]. The mode with \( n = 1 \) is always present regardless of the configuration, as shown in section 2.3.2. In this case in a pure magnetic field, a photon in the mode propagates as if it sees no background field.

### Table 3

Refractive indices and polarization vectors for various Lagrangians in a magnetic background field: \( B = B_0\hat{z} \) and \( \hat{n} = (\sin \theta, 0, \cos \theta) \). The acronyms of the Lagrangians are as follows: PM for post-Maxwellian, BI for Born-Infeld, and MM for ModMax. In BI, \( p \) and \( q \) are arbitrary. The FPC is not included here.

| Lagrangian | (refractive index)^2 (\( n^2 \)) | polarization vector (\( \delta \mathbf{E} \)) |
|-----------|---------------------------------|------------------------------------------|
| PM        | \( \frac{1-\eta_1 B_0^2}{1-\eta_1 B_0^2 - 2\eta_2 B_0^2 \sin^2 \theta} \) | \( (0, 1, 0) \) \( (1 - \eta_1 B_0^2 + 2\eta_2 B_0^2) \cos \theta, 0, - (1 - \eta_1 B_0^2) \sin \theta \) |
| BI        | \( \frac{B_0^2 + T}{B_0^2 \cos^2 \theta + T^2} \) | \( p (0, 1, 0) + q (1 + B_0^2/T) \cos \theta, 0, - \sin \theta \) |
| MM        | \( \frac{1}{\cos^2 \theta + e^{-2g} \sin^2 \theta} \) | \( (0, 1, 0) \) \( (e^{2g} \cos \theta, 0, - \sin \theta) \) |
3.4 Heisenberg-Euler-Schwinger Lagrangian

Heisenberg, Euler, and Schwinger obtained the one-loop correction to the Maxwell Lagrangian due to spin-1/2 fermions of mass \( m \) in a constant electric and magnetic fields of arbitrary strengths \([6, 7]\). The exact one-loop action is given as the proper-time integral:

\[
\mathcal{L}^{(1)}(h, g) = -\frac{1}{8\pi^2} \int_0^\infty ds \frac{e^{-m^2 s}}{s^3} \left\{ (ehs) \coth(ehs)(egs) \cot(egs) - \left[ 1 + \left( \frac{ehs}{3} \right) \left( \frac{egs}{3} \right) \right] \right\},
\]

where

\[
h = \sqrt{F^2 + G^2 + F}, \quad g = \sqrt{F^2 + G^2 - F}.\]

The HES Lagrangian is obtained by adding the correction to the Maxwell Lagrangian:

\[
\mathcal{L}_{\text{HES}}(h, g) = g^2 - \frac{h^2}{2} + \mathcal{L}^{(1)}(h, g).
\]

The closed analytical form of \( \mathcal{L}_{\text{HES}} \) for arbitrary constant electromagnetic fields is not known yet. However, a closed form for the case of \( G = 0 \) was obtained by using the dimensional regularization \([42]\), and it was confirmed by the gamma-function regularization in the in-out formalism \([45]\):

\[
\mathcal{L}_{\text{HES}}^{G=0}(\bar{h}) = m^4 \frac{8\pi^2}{8\pi^2 h^2} \left[ -\frac{\pi}{4\alpha_e} + \zeta'(-1, \bar{h}) - \frac{1}{12} + \frac{\bar{h}^2}{4} - \left( \frac{1}{12} - \frac{\bar{h}^2}{2} + \frac{\bar{h}^2}{2} \right) \ln \bar{h} \right],
\]

where \( \bar{h} = m^2/(2eh) \), \( \alpha_e = e^2/(4\pi) \) is the fine structure constant, \( \zeta(s, \tilde{h}) \) is the Hurwitz zeta function, and \( \zeta'(-1, \bar{h}) = d\zeta(s, \bar{h})/ds|_{s=-1} \). For the case with \( G \neq 0 \), the correction was obtained up to the order of \( G^4 \) \([46]\). Recently, the correction was obtained as a power series in a small parameter \((g/h)\) \([29]\).

In finding the light modes in a pure magnetic field, one may try to start with the Lagrangian in a pure magnetic field, \( \mathcal{L}_{\text{HES}}^{G=0}(\bar{h})|_{h=B_0} \). However, the Lagrangian is not sufficient, and the Lagrangian with a first-order correction in \( G \) is necessary because the combined fields of the background field and the light field can have \( G \neq 0 \) in general \((\delta\mathbf{E}, \mathbf{B}) \) is not necessarily zero). The first-order correction is given as

\[
\mathcal{L}_{\text{cor}}(\tilde{h}, \tilde{g}) = m^4 \frac{8\pi^2}{8\pi^2 \tilde{h}^2} \frac{1}{\tilde{g}^2} \left( -\frac{1}{24h} + \frac{\ln \tilde{h} - \psi^{(0)}(\bar{h})}{12} \right),
\]

where \( \tilde{g} = m^2/(2eg) \), and \( \psi^{(0)}(\bar{h}) \) is the 0th-order polygamma function \([29]\). The condition of a pure magnetic field, \( h = B_0 \) and \( g = 0 \), should be imposed on \( \{L_F, L_{FF}, L_{FG}, L_{GG}\} \) after they are calculated without specifying a configuration. If the correction is not included, \( L_{GG} \) will miss an amount of

\[
\Delta L_{GG} = -\frac{e^4 \bar{h}^2}{6\pi^2 m^4} \left[ 2\psi^{(0)}(\bar{h}) + \frac{1}{\bar{h}} - 2\ln \bar{h} \right],
\]
whereas the other derivatives remain the same. The parameters \( \{a^2, ab, b^2, c^2, d^2\} \), essential in finding light modes (see appendix A), are given as follows:

\[
  a^2 = \frac{16\alpha^2\tilde{h}^2}{m^4} \left[ 2\tilde{h}^2(\psi^{(0)}(\tilde{h}) - 1) + \tilde{h} \left( 1 - \ln \frac{\tilde{h}\Gamma(\tilde{h})}{2\pi} \right) + 1 \right], \quad ab = b^2 = 0,
\]

\[
  c^2 = -\frac{8\alpha^2\tilde{h}}{3m^4} \left[ 6\tilde{h}^3 - 6\tilde{h}^2 \ln \frac{2\pi\tilde{h}}{\Gamma(\tilde{h})^2} + \tilde{h} \left( 1 + 2\psi^{(0)}(\tilde{h}) - 24\zeta'(1, \tilde{h}) \right) + 1 \right],
\]

\[
  d^2 = \frac{\alpha_e}{6\pi} \left[ 6\tilde{h}^2 - 6\tilde{h} \ln \frac{2\pi\tilde{h}}{\Gamma(\tilde{h})^2} - 24\zeta'(1, \tilde{h}) + 2 \ln \tilde{h} + 1 \right] + 1.
\]

With \( \{a^2, ab, b^2, c^2, d^2\} \) and \( \{B, \hat{n}\} \) available, we can find the light modes in the QED vacuum in an arbitrarily strong magnetic field, following the procedure in table 2 and appendix A. However, the formulas of \( \lambda^{(i)} \) are formidable, and thus one should resort to numerical evaluation of \( \lambda^{(i)} \), albeit the formulas of the refractive indices are available when \( \hat{n} \perp B \) [28]. We show a detailed calculation in a recent paper [29], in which the effect of a weak electric field along the magnetic field is studied.

4 Conclusion

In this paper, we have provided a general formulation of the light modes in nonlinear vacua, which are described by Plebanski-type Lagrangians, i.e., \( \mathcal{L} = \mathcal{L}(F, G) \). Instead of Lorentz-covariant quantities, we used spatial vectors and dyadics (3+1 formulation) so that the formulation can be conveniently applicable to the situations of laboratory experiments or astrophysical observations. We found that nonlinear vacua can be classified into two cases, non-degenerate and degenerate, in which the latter always possesses a mode with \( n = 1 \). For each case, we derived general explicit formulas of refractive indices and polarization vectors. For illustrations, we applied the formulation to a magnetic background field described by the well-known NED Lagrangians, such as the post-Maxwellian Lagrangian, Born-Infeld Lagrangian, ModMax Lagrangian and Heisenberg-Euler-Schwinger QED action. Finally, we provided a streamlined procedure of determining light modes for practical applications (see appendix A).

The presented formulation is essential in detailed investigations of NED. The NED effects, i.e., the modification of the vacuum, may appear in various phenomena, but among them the vacuum birefringence is the most promising to be observed because it relies on high-precision optical technologies. The terrestrial experiments employ strong permanent magnets [47] or ultra-intense lasers/x-ray free electron lasers [48, 49] to prove the existence of vacuum birefringence. Though in the current or near future experiments, the strength of the background magnetic fields is far below the critical field, the configuration can be designed as simple and convenient as possible for optimal observations.

In contrast, the proposed astrophysical observations of the x-rays from highly magnetized neutron stars, such as the enhanced X-ray Timing and Polarimetry (eXTP) [50] and the Compton Telescope project [51], will have an access to supercritical magnetic fields [52, 53], albeit the configuration of background electromagnetic field is complicated and uncontrollable. As the combined electric and magnetic fields, for instance, the Goldreich-Julian dipole model, vary over macroscopic scale lengths, our 3+1 formulation can provide the local values of the refractive indices and polarization vectors. Then, from this local information, we should find the propagation path of light by solving the vector wave equation incorporating birefringence.
A Procedure to find the light modes in a nonlinear non-degenerate vacuum and essential formulas

The procedure to find the light modes in a nonlinear non-degenerate vacuum is outlined in table 2. Here, we present the essential formulas used for the procedure in order. It is assumed that the formulas for \{L_F, L_{FF}, L_{FG}, L_{GG}\}, and thus \{a^2, ab, b^2, c^2, d^2\} in (2.17) are known regardless of the field configuration.

Given a configuration specified by \(\{E, B, \hat{n}\}\) in the 3+1 formulation, we then specialize \{a^2, ab, b^2, c^2, d^2\} to the configuration and calculate \{\alpha, \beta, \gamma\} in (2.25) to obtain the refractive index \(n\) from (2.27). The refractive index may be numerically evaluated if the Lagrangian has a complicated functional form. With the refractive index, we calculate \(P\) and \(Q\) in (2.14) and the following auxiliary parameters:

\[
P_c = E_c + nB_t, \quad Q_c = B_c - nE_t, \\
P \times Q = S - n(E^2 + B^2)\hat{n} + n(E_nE + B_nB) + n^2S_n\hat{n}, \\
\frac{PQ}{\ell} := P \cdot Q - n^2P_nQ_n = -(1 - n^2)G, \\
\tilde{P}^2 := P^2 - n^2P_n^2 = E^2 - 2nS_n + n^2(B_b^2 - E_n), \\
\tilde{Q}^2 := Q^2 - n^2Q_n^2 = B^2 - 2nS_n + n^2(E_t^2 - B_n^2),
\]

where the subscripts \(n\) or \(t\) denote components parallel or transverse to \(n\), respectively.

Then \(acH, M,\) and \(N\) in (2.21) are expressed as

\[
a^2c^2PQ + abc^2\tilde{Q}^2, \\
M = a^2\tilde{P}^2 + 2ab\tilde{P}Q + b^2\tilde{Q}^2 + d^2(1 - n^2), \\
N = c^2\tilde{Q}^2 + d^2(1 - n^2).
\]

Unless \(acH = M = N = 0\), we form the following two vectors proportional to the polarization vectors in table 1:

\[
\delta E_1 = a^2MP_c + (acH + abM)Q_c - \frac{(a^2P_n + abQ_n)acH - a^2c^2MQ_n^2P_c}{d^2} \times Q, \\
\delta E_2 = -\frac{acHa^2}{c^2}P_c - \left(a^2N + \frac{acHab}{c^2}\right)Q_c + \frac{a^2[(a^2P_n + abQ_n)N - acHQ_n]}{d^2} \times Q \\
\]

Between \(\delta E_1\) and \(\delta E_2\), we choose any non-zero vector as the polarization vector for the refractive index \(n\). When both are non-zero, they differ from each other only by a multiplicative factor. When \(acH = M = N = 0\), the polarization vector spans a plane:

\[
\delta E = \left[d^2Q_c - (a^2P_n + abQ_n)P \times Q\right] p + \left(a^2d^2P_c + abd^2Q_c + a^2c^2Q_nP \times Q\right) q,
\]

where \(p\) and \(q\) are arbitrary real numbers.

In general, light modes are completely determined by the formulas (2.25), (2.27), (A.2), (A.3), and (A.4) once \{\(a^2, ab, b^2, c^2, d^2\)\} (2.17) and the configurational parameters (A.1) are known.
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