Dimensionless Fluctuation Balance Principle: New Statistical Perspectives Applied to Boltzmann, Planck, Fermi-Dirac, Bose-Einstein and Schrödinger Distributions

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ABSTRACT
In this work we propose a completely new way to obtain statistics distributions from fluctuations balance. By dimensionless fluctuation analysis we obtain Boltzmann, Planck, Fermi-Dirac, Bose-Einstein and Schrödinger Distributions using the same fundamental principle. Our result point to a general foundation that was successful verified to principal Physics Distributions. We name it as Dimensionless Fluctuation Balance Principle. This is a great achievement which enable us to discuss exchange between different physical quantities, like we do when treat energy conservation when some type of energy is converted to another, but with more generality, because we can exchange one physical quantity to any other. All physics model which needs distribution can take advantage of methodology presented in this paper including: Statistical Physics, Schrödinger’s Quantum Mechanics, Nanomaterials, Thin Films and New Materials Modeling. Keywords: Fluctuations, PDEs, Boltzmann, Planck, Entropy, Fermi-Dirac, Bose-Einstein, Schrödinger, Distributions.

1 Introduction
Many times, in Physics we need to extend some fundamental law and their application [1], or just produce new models relating experimental facts slightly different from original [2], like Perturbation Theory [3], Energy Conservation [1], Gauss Divergence Theorem [5], Stokes Curl Theorem [6], Reynolds Theorem [7], each one with their better applicability to singular context.

Observing some new materials applications [8], we can see that distributions plays a central role to modeling this system [9]. Sometimes some modification in theoretical model due to experimental requirement is needed as theoretical proposition [10]. We feel the urgency to some principle to obtain distributions quickly, as the main goal of our work. We name it as Dimensionless Fluctuation Balance Principle.

Our “Ansatz” starts when we observe Boltzmann distribution [11], which was applied to Plank’s hypothesis resulting their Radiation Law [12]. In the ultraviolet catastrophe case as in [13], with \( E = kT \) we can see a pair of quantities energy and temperature \((E,T)\), and this one takes a form of Boltzmann distribution like \( u(E,T) \propto e^{-E/kT} \) [14].

Our research starts from Dimensionless Fluctuations of \((E,T)\) pair and we verify how to obtain a Partial Differential Equation (PDE), which the solution is desired distribution. In this case, Boltzmann distribution \( u(E,T) \propto e^{-E/kT} \), after this we try to apply the same procedures to other distributions searching for a fundamental principle, our tests results was verified with success to principal Physics Distributions.

¹ where \( k \) is the Boltzmann’s constant
This work shows in completely new way, how to obtain statics distributions from fluctuations balance, theses fluctuations relate fractions from domain dimensionless terms. A special point about our analysis is that with this one we can obtain all fundamentals distributions (including Boltzmann Entropy Law) without probability as a required initial concept. This achievement doesn’t neglect statistical theory in measurement of states, because we can think any mean value as inner product between quantity to be measured as distribution on Hilbert Spaces.

2 Small Fluctuations Balance

In many cases, when we study a control volume conservation by Reynolds Transport Theorem 7, we just observe net rate exchange of property per unit time. As a classical example, the continuity equation case when balancing charges in some region of space which can decrease when some current leaves the volume across boundary surface 15.

Our effort here is to study localized phenomenon, as simple case, gas inside a closed box or some portion of fluid inside a thermal-mechanical set. To reach this goal, instead of work with classical balance like continuity equation, we will study small portions of a property inside a fraction of material.

As illustrative case study, being a given group of molecules with the same characteristics, the summation over \( N \) gives the total number of molecules, then

\[
\sum_{i=1}^{n} \left( \frac{N_i}{N} \right) = 1. \tag{1}
\]

For simplicity we can see that a portion instead of sum of all contributions, can be denoted as,

\[
\left( \frac{\delta N}{N} \right) = c_N. \tag{2}
\]

From equation (1) we can see that the sum of all portions of molecules number reach the maximum value which is the total of molecules amount. Otherwise, a small portion of molecules

\(^N\)N also represent a group of molecules of gas.
number presented in (2) gives us the fluctuation as small fraction of molecules amounts much less than one.

Can we extend this concept of fluctuations to more than one property? Does this extension give us some new aspects? To reach a concise response to this question, let us relate more than one property in some balance around frontier of a surrounded control volume.

Let us consider two different properties \( A \) and \( B \), and a region in the space like a control volume or membrane, at which this two properties can exchange one to another inside this boundaries region. So considering this,

\[
\left( \frac{\delta A}{A} \right) = c_A, \quad (3)
\]

and

\[
\left( \frac{\delta B}{B} \right) = c_B. \quad (4)
\]

Here \( c_A \) and \( c_B \) are fractions of a total amount, their total summation reaches one. Then, summing all portions over complete region from (3) and (4) we obtain,

\[
\sum_n \left( \frac{\delta A}{A} \right) = 1, \quad (5)
\]

and

\[
\sum_n \left( \frac{\delta B}{B} \right) = 1. \quad (6)
\]

Observing (5) and (6) we can see as first impression a balance condition as,

\[
\sum_n \left( \frac{\delta A}{A} \right) = 1 = \sum_n \left( \frac{\delta B}{B} \right). \quad (7)
\]

But the question doesn’t close so easily, because in physical cases we have some law relating \( A \) and \( B \). As a good choice we point here to relate fraction of quantities as their appear in (3) and (4) which gives,

\[
\left( \frac{\delta A}{A} \right) = \frac{c_A}{c_B} \left( \frac{\delta B}{B} \right), \quad (8)
\]

that can be simplified to initial form as,

\[
\left( \frac{\delta A}{A} \right) \propto \left( \frac{\delta B}{B} \right). \quad (9)
\]

As example, our first case study starts from temperature, considering it as important parameter of a gas, and for this reason let’s analyze the fluctuation of temperature, searching for new useful relations for balance of properties in a gas.

### 3 Boltzmann Distribution PDE

#### 3.1 A Closer View to First Law of Thermodynamics

Let our thermodynamics system with internal energy \( U \), \( Q \) as energy that external agent delivery to the system and \(-W\) the work done by system. The portion \( dQ \) is delivered by heating system at each cycle and some portion of energy that we deliver to the system will be spent with mechanical work \(-dW\). Therefore, the internal energy of system, which is a balance of energy delivered to the system \( dQ \) with discount of the mechanical work done by the system \(-dW\), is given by,
\[ dU = dQ - dW, \] (10)

which is the First Law of Thermodynamics [16].

For simple analysis, with the aim of verify temperature fluctuations, we can put the equation (10) in another terms: changing the heating energy of system \( dQ \) to material components temperature increase as \( CdT \), where \( C \) is thermal capacity and \( T \) is temperature, and thinking as a work done \(-dW\) like expansion of system. We get

\[ dU = CdT - pdV. \] (11)

Isolating small fluctuation of temperature, we can see that,

\[ \frac{dT}{T} = \frac{dU + pdV}{CT}. \] (12)

Equation (12) shows that temperature fluctuation is related to internal energy modification and work done by system.

In next steps we will consider how these laws can contribute to understand gas dynamics.

### 3.2 Small Fluctuation Balance to Gas System

As we can see at Right Hand Side (RHS) of equation (12), the internal energy term and work term are divided by thermal energy amount, expressing energy fluctuations and this fact explains the fundamental relation between temperature fluctuation and energy fluctuation. As we discuss at equation (9), a first appointment to this energy-temperature balance of fluctuations can be done as,

\[ \frac{\delta E}{E} \propto \frac{\delta T}{T}. \] (13)

This relation is based on thermal system, which has a balance between: input energy, work done and internal energy of system. The continuation of the analysis that improve equation (13), will be made for a gas inside a box as our object of study and the most important thing to study this gas dynamic system. The analysis consists of a small virtual work considering the box frontiers slightly moving from initial state to most outside or inside position. This analysis will give us how dependency between energy and temperature fluctuations increase or decrease.

We start this analysis first considering a gas inside a box with \( T \) as gas temperature and \( E \) as energy, as two fundamental quantities. If we think this gas can do small displacement pushing box walls to outside, like a small virtual work, we need to balance where the energy fluctuation will be related to work done by gas, which will cost a loss of thermal energy of the gas and as effect temperature will decrease. In mathematical terms equation (13) became,

\[ \frac{\delta E}{E} \propto -\frac{\delta T}{T}. \] (14)

Another change that must be done is related to dimensional analysis, because Left Hand Side (LHS) is a fluctuation of Energy, and RHS is related to temperature.

At first glance the thermal energy \( kT \) could be a good proposition, considering that a gas inside a box is well represented by temperature of gas in insulated system. Then,

\[ \frac{\delta E}{E} = -\frac{\delta kT}{kT}, \] (15)

as we can see, a natural simplification of constant \( k \) occurs. This is a strong aspect of fluctuation analysis, because all fluctuation terms are dimensionless in essence.

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3. As default in Classical Mechanics to obtain motion equations.
4. In this context, temperature \( T \) is a macroscopic value and energy \( E \) is a microscopic value related to the first one.
5. The work done by gas costs internal energy decreasing the temperature of a gas.
In the next steps let’s consider this fractions $\delta E$ and $\delta T$ so small that we can assume them as differentials quantities $dE$ and $dT$. So,

$$\frac{dE}{E} = -\frac{dT}{T}. \quad (16)$$

Considering that we can have some distribution function like $u(E, T)$ and other dependencies of interest, a partial differential approach is most affordable to this purpose, then we can change (16) to,

$$\frac{\partial E}{\partial T} = -\frac{E}{T}. \quad (17)$$

Using chain rule below, with the goal to include distribution function $u(E, T)$, given by

$$\frac{\partial E}{\partial T} = \frac{\partial u}{\partial T} \frac{\partial E}{\partial u}. \quad (18)$$

Replacing (18) in (17), we get a partial differential equation in the form,

$$\frac{\partial u}{\partial T} = -\frac{E}{T} \frac{\partial u}{\partial E}. \quad (19)$$

This is a new form to study distribution, from fluctuation analysis. Here, we present the Boltzmann Distribution PDE. Now, to obtain the Boltzmann distribution, we only need to solve Partial Differential Equation (19).

We point here a function that depends on two properties which we made fluctuation analysis. For simplicity, we use a fashion of exponents $a$ and $b$ like we do in dimensional analysis,

$$u = e^{-(kT)^a \cdot (E)^b}. \quad (20)$$

Replacing (20) in (19), we get,

$$\frac{\partial u}{\partial T} = -k^a E^b a T^{a-1} e^{-(kT)^a \cdot (E)^b}, \quad (21)$$

$$\frac{\partial u}{\partial E} = -k^a T^b b E^{b-1} e^{-(kT)^a \cdot (E)^b}. \quad (22)$$

To relate (21) and (22), we just divide booth terms and simplify in order to analyse our general solution. Then,

$$\frac{\partial u}{\partial T} = \frac{a}{b} \frac{E}{T} \frac{\partial u}{\partial E}. \quad (23)$$

The way between (20) to (23) show us that $a$ and $b$ are parameters that we can choose to satisfy the PDE. Finally, comparing (23) to (19) we can see that choosing $a = -b$ set PDE in (23) as the same as Boltzmann PDE in (19), with the same sort this choice implies in tentative solution of equation (20) to the form,

$$u = e^{-(E/kT)^b}. \quad (24)$$

At last step, if we choose $b = 1$, we find the Boltzmann Distribution function,\textsuperscript{6}

$$u \propto e^{-E/kT}. \quad (25)$$

Next step we study application of this distribution in Planck hypothesis context, which refers to using Boltzmann Distribution function to get Planck’s Radiation Law.

\textsuperscript{6}Note that we start solving a PDE using a general solution method inspired by exponentials combinations, like when we try find some law of Physics using Dimensional Analysis. As expected, this method gives us free to choose some parameters to get result, but it’s a strong point of this method, not a weakness.
4 Planck’s Law as Boltzmann Distribution PDE Solution

4.1 Planck hypothesis

Planck when solving ultraviolet catastrophe explains two points. First one is their achievement foundations related to Boltzmann Distribution and second that he finds their formula heuristically searching for a curve that satisfy experimental facts.

In our Appendix A session, we treat aspects related to Planck Ansatz which consist proposing that number of states like a photon gas is related to Boltzmann distribution, and we point additionally to particular regime $h\nu \ll kT$ as an experimental foundation too. Boltzmann distribution is given by

$$u \propto e^{+E/kT}.$$  \hfill (26)

Replacing $E = h\nu$ in (26), the initial form of distribution stay

$$u = e^{h\nu/kT}.$$ \hfill (27)

Planck solves breaking region in ultraviolet zone, which occurs because $kT$ term is much greater than $h\nu$, or just, $kT \gg h\nu$.

At same fashion as Einstein explains easily mass-energy equivalence, expanding kinetic energy because of velocity $v$ is much less than $c$ \cite{17}, let’s expand Boltzmann distribution using Planck hypothesis, with Taylor series at variable $\nu$ around ultraviolet zone limit. So,

$$e^{h\nu/kT} = \lim_{h\nu \ll kT} e^{h\nu/kT} \left[ 1 + \frac{h\nu}{kT} + \frac{1}{2} \left( \frac{h\nu}{kT} \right)^2 + \sum_{n=3}^{\infty} \frac{1}{n!} \left( \frac{h\nu}{kT} \right)^n \right].$$ \hfill (28)

Considering $h\nu \ll kT$ as in (28), let’s truncate terms to get an approximation, for all terms with $n$ equal two or higher, resulting,

$$e^{h\nu/kT} \simeq 1 + \frac{h\nu}{kT}.$$ \hfill (29)

Isolating $kT$ term,

$$kT \simeq \frac{h\nu}{e^{h\nu/kT} - 1},$$ \hfill (30)

that is a thermal energy related to a photon in the gas at state $\mu$. Or just $E_\mu$,

$$E_\mu = \frac{h\nu}{e^{h\nu/kT_\mu} - 1}.$$ \hfill (31)

This equation is the Planck’s Law of Radiation, given energy per photon state $\mu$ in the gas, valid across ultraviolet zone. Planck shows by his law a mathematical relation that attends phenomenology not solved before by the classical formulation.

The essence of Planck proposition is that his Ansatz consists of relating radiation as gas of photons and applying Boltzmann distribution as hypothesis to study this gas.

4.2 Planck’s Law as Boltzmann’s Differential Equation Solution

When Planck starting from Boltzmann distribution and obtain their radiation law, he defines the case study as a photon gas system which has compatibility with Boltzmann gas. In this perspective that Planck’s law was originated from Boltzmann Distribution, let’s investigate if Planck’s law satisfies Boltzmann PDE equation \cite{19}.

\footnote{Another solution to Boltzmann Distribution PDE \cite{19} have positive signal.}
\[ \frac{\partial u}{\partial T} = -\frac{E}{T} \frac{\partial u}{\partial E}, \]  

(32)

or just,

\[ \frac{\partial u}{\partial T} \left( \frac{\partial u}{\partial E} \right)^{-1} = -\frac{E}{T}. \]  

(33)

Starting from equation (31),

\[ E_\mu \equiv u = \frac{1}{e^{\hbar \nu / kT_\mu} - 1}. \]  

(34)

Without loss of generality let’s introduce two free parameters \( a \) and \( b \) in (34). Then,

\[ u = \left( e^{\hbar \nu / kT_\mu} + a \right)^b. \]  

(35)

Applying (35) in (33), we get

\[ \frac{\partial u}{\partial T} = -b \left( e^{\hbar \nu / kT_\mu} + a \right)^{b-1} \frac{\hbar \nu}{kT_\mu} e^{\hbar \nu / kT_\mu}, \]  

(36)

and,

\[ \frac{\partial u}{\partial E} = b \left( e^{\hbar \nu / kT} + a \right)^{b-1} \frac{1}{kT_\mu} e^{\hbar \nu / kT_\mu}. \]  

(37)

Relating (36) and (37) at same fashion we relate (21) and (22), gives

\[ \frac{\partial u}{\partial T} = \frac{\hbar \nu}{T_\mu} \frac{\partial u}{\partial E}. \]  

(38)

So, we can see that Planck’s Radiation law satisfies Boltzmann equation with \( \hbar \nu \) as energy \( E \).

Another question that we can investigate, is related to modeling thermodynamic systems. Macroscopic variables like temperature, pressure and volume can be measured by experimentation. Otherwise, we know that these ones are manifestation of micro states configuration, as the case of temperature which is the average molecular kinetic energy. The entropy is a quantity that turns possible to relate micro states to macro states [18].

Planck interpreted radiation as a photon gas and obtain their law with success. In next session let’s investigate how to relate entropy to distribution of microstates using Boltzmann PDE.

## 5 Boltzmann Entropy Law

In previous sections we have success when obtaining some known distributions. Considering their close relationship with Statistical Physics, let we show how to obtain Boltzmann’s Entropy Law using a distribution.

Supposing some gas volume and their free expansion, when volume grow entropy increases, but growing entropy by increasing volume costs thermal energy of gas which will decay. So,

\[ \frac{\delta (E_S)}{E_S} = -\frac{\delta (E_T)}{E_T}. \]  

(39)

At proper limit we can adjust this equation to,

\[ \frac{dE_S}{E_S} = -\frac{dE_T}{E_T}. \]  

(40)
Let us set a distribution \( \Omega \) which will depend on energy displaced to entropy \( E_S \) when volume increases and the thermal energy costs to volume increase \( E_T \). Given \( \Omega \equiv \Omega (E_S, E_T) \), we must change from total to partial differential equation for this purpose,

\[
\frac{\partial E_S}{\partial E_T} = -\frac{E_S}{E_T}.
\]  

(41)

Applying the chain rule like below,

\[
\frac{\partial E_S}{\partial E_T} = \frac{\partial \Omega}{\partial E_T} \frac{\partial E_S}{\partial \Omega}.
\]  

(42)

Replacing (42) in (41) we get a final form of partial differential equation which we name as Boltzmann’s Entropy Distribution PDE,

\[
\frac{\partial \Omega}{\partial E_T} = -\frac{E_S}{E_T} \frac{\partial \Omega}{\partial E_S}.
\]  

(43)

Our Ansatz to solve this equation consists in using the tentative solution below,

\[
\Omega (E_S, E_T) \equiv e^{E_S/E_T}.
\]  

(44)

To obtain the PDE solution let us derive our tentative from (44). First in terms of \( E_T \) given \( \frac{\partial \Omega}{\partial E_T} = bE_T^{a-1} \Omega \) and after in terms of \( E_S \), \( \frac{\partial \Omega}{\partial E_S} = aE_S^{a-1}E_T^b \Omega \). Relating both derivatives we can equate,

\[
\frac{\partial \Omega}{\partial E_T} \left( \frac{\partial \Omega}{\partial E_S} \right)^{-1} = \frac{bE_T^{a-1} \Omega}{aE_S^{a-1}E_T^b \Omega} = \frac{bE_S}{aE_T}.
\]  

(45)

To satisfy Boltzmann’s Entropy Distribution PDE we can set \( b/a = -1 \), where our particular choice is \( b = -a \). So, finally we obtain the solution of PDE in (43),

\[
\Omega (E_S, E_T) = e^{E_S/E_T},
\]  

(46)

At next steps let us do a connection between this distribution to Boltzmann’s Entropy law. Using as first principles dimensional analysis and thermodynamics laws. From entropy fundamental concept defined in Thermodynamics \( dQ = TdS \) from who we can see that \([dQ] = dE_S \) establishing relation between heat energy and entropy energy, and as macroscopic version we can set dimensional dependence of entropy energy as \( E_S \propto [T]^a[S]^b \) which dimensional solution points to \( E_S = TS \). Another demand is to thermal energy \( E_T \) which we get as \( E_T = kT \). So,

\[
\Omega (E_S, E_T) = e^{TS/kT} = e^{S/k},
\]  

(47)

where, to isolate entropy \( S \) in the final form, let us assume the inverse function from (47). Which gives

\[
S = k \ln |\Omega|.
\]  

(48)

### 6 Fermi-Dirac and Bose-Einstein Distributions

Now, we will verify how to Fermi-Dirac and Bose-Einstein are related to a similar format as that obtained in equation of Planck’s law (35). So,

\[
u = \left(e^{E/kT} + a\right)^b.
\]  

(49)

Here, our emphasis consists in using the Boltzmann’s partial differential equation to inspect how close Fermi-Dirac and Bose-Einstein are to Plank’s law format. With a little modification in energy from (49), we get
\[ u = \left( e^{(\varepsilon - \mu)/kT} + a \right)^b. \]  
\[ (50) \]

And the analogous form to \( (33) \), is

\[ \frac{\partial u}{\partial T} \left( \frac{\partial u}{\partial \varepsilon} \right)^{-1} = -\frac{(\varepsilon - \mu)}{T}. \]  
\[ (51) \]

Next steps let’s apply \( (50) \) in \( (51) \) to study both Fermi-Dirac and Bose-Einstein particular cases.

### 6.1 Fermi-Dirac and Bose-Einstein Distributions Analysis

Comparing general format from \( (50) \) with Fermi-Dirac distribution equation in \[19\] we see that \( a \) and \( b \) must assume,

\[ (a, b) \equiv (+1, -1). \]  
\[ (52) \]

This choice gives us,

\[ u = \frac{1}{e^{(\varepsilon - \mu)/kT} + 1}. \]  
\[ (53) \]

Derivating in temperature and energy we have their respective equations,

\[ \frac{\partial u}{\partial T} = \left[ e^{(\varepsilon - \mu)/kT} + 1 \right]^{-2} e^{(\varepsilon - \mu)/kT} \frac{(\varepsilon - \mu)}{kT^2}, \]  
\[ (54) \]

and

\[ \frac{\partial u}{\partial \varepsilon} = -\left[ e^{(\varepsilon - \mu)/kT} + 1 \right]^{-2} e^{(\varepsilon - \mu)/kT} \frac{1}{kT}. \]  
\[ (55) \]

Now relating both \( (54) \) and \( (55) \), we finally see that,

\[ \frac{\partial u}{\partial T} \left( \frac{\partial u}{\partial \varepsilon} \right)^{-1} = -\frac{(\varepsilon - \mu)}{T}. \]  
\[ (56) \]

Fermi-Dirac’s distribution satisfy general form of Boltzmann’s distribution PDE. At same fashion, comparing general format from \( (50) \) with Bose-Einstein distribution equation in \[20\] we see that \( a \) and \( b \) must assume,

\[ (a, b) \equiv (-1, -1). \]  
\[ (57) \]

This choice gives us,

\[ u = \frac{1}{e^{(\varepsilon - \mu)/kT} - 1}. \]  
\[ (58) \]

Which verification results will led us to, at same way that obtained in \( (56) \), concluding that Bose-Einstein’s distribution satisfy general form of Boltzmann’s distribution PDE too.

So, when we treat Plank’s law of radiation, we see that same PDE which gives us Boltzmann’s Distribution is satisfied to Planck’s law. We extend this analysis using a general form of distribution and conclude that also Fermi-Dirac’s and Bose-Einstein’s distributions emerges as Planck’s law from the same Boltzmann’s PDE.
7 Schrödinger’s Distribution

The uncertainty principle emerges from experimental and Gaussian widths relations [21]. We use uncertainty as a strong fundamental to point our dimensionless fluctuation and obtain a fundamental distribution that we can use to recover Schrödinger equation.

To obtain Schrödinger distribution, we must recover the most fundamental aspect, which lies over uncertainty relations [22],

\[ \delta x \delta p \geq \frac{\hbar}{2}, \tag{59} \]

and

\[ \delta E \delta t \geq \frac{\hbar}{2}. \tag{60} \]

Another association that we can make consist in show interrelation between both (59) and (60), starting from energy relation \( \delta E = m\delta(v)^2/2 = mv\delta v \). Replacing it in (60) we get \( \delta E \delta t \geq \hbar/2 \), and remember that \( \delta x = v\delta t \) and \( \delta p = m\delta v \) we recover the other uncertainty relation \( \delta x \delta p \geq \hbar/2 \) as in reference [22].

Now, we are supposing a system with some initial energy amount \( E \) and a free particle with momentum \( p \) and an isolated system. In terms of dimensionless fluctuation analysis, if the particle momentum \( p \) increase a quantity \( \delta p \), it will cost to system a decrease amount of energy \(-\delta E\). So, we can write

\[ \frac{\delta p}{p} \propto \frac{\delta E}{E}. \tag{61} \]

As next steps, we start replacing uncertainty relations from (59) and (60), in the limit that we equate both. So,

\[ \delta E \delta t = \delta x \delta p. \tag{62} \]

Multiplying (62) by inverse of \( E \) and, after a little algebra, we get

\[ \frac{\delta E}{E} = 2 \frac{\delta p}{p}. \tag{63} \]

Now comparing this format of uncertainty relation presented in (63) to dimensionless fluctuation relation of (61), we can see that instead of proportionality factor assume value 1 as in many physical cases [23], a factor 2 will be needed, given

\[ \frac{\delta E}{E} = -2 \frac{\delta p}{p}. \tag{64} \]

After a little algebra, and considering the proper limit when \( \delta E \) and \( \delta p \) can be assumed as differentials, we get

\[ \frac{dE}{dp} = -2 \frac{E}{p}. \tag{65} \]

Assuming that we have some distribution function, \( \Psi \equiv \Psi(p,E) \), the partial derivatives turn (65) to a most interesting format,

\[ \frac{\partial E}{\partial p} = -v. \tag{66} \]

Assuming a chain rule using \( \Psi \), relating \( E \) and \( p \) we get,

\[ \frac{\partial E}{\partial p} = \frac{\partial \Psi}{\partial p} \frac{\partial E}{\partial \Psi}. \tag{67} \]
Replacing (67) in (66), we finally get the Schrödinger distribution equation PDE,

$$\frac{\partial \Psi}{\partial p} = -v \frac{\partial \Psi}{\partial E}.$$  \hspace{1cm} (68)

We can easily see that a simple solution is given by,

$$\Psi = e^{i p x / \hbar} e^{-i E t / \hbar}.$$  \hspace{1cm} (69)

This distribution equation is very important, because from this one we can obtain all properties of the system, including schrödinger Equation. From expected value as statistical concept $\langle p \rangle = \int_{-\infty}^{+\infty} p \Psi(x, t) dx$, we can see that distribution play a central role. Because of complex variable format at (69) we extend this to square integrate as $\langle p \rangle = \int_{-\infty}^{+\infty} \Psi^* (x, t) \hat{p} \Psi (x, t) dx$. At same fashion for $\langle H \rangle$ and $\langle V \rangle$ and after putting it all together in Hamiltonian energy conservative form of square integrable expected values, we easily recover Schrödinger equation.

8 Concluding Remarks

After our tests results was verified with success to principal Physics Distributions, we find a collection of procedures as registered in this paper, which we can summarize in a single principle: “Given some fundamental relations between two or more quantities, the Dimensionless Fluctuation Analysis point to a Partial Differential Equation which solution gives desired distribution”. Which we name as Dimensionless Fluctuation Balance Principle. At least this work shows a new way to relate two quantities using dimensionless number that enable us to obtain distributions as differential equations solutions. This is not only a practical tool, but also turns possible to understand some meaning beyond distributions. As example, we can see that when some particle or energy or another system entity was confined submitted to some compact region like localized phenomena or when required to interact with some other entities not in large space field, distributions emerges as natural imposition of space region to a group of entities when the system size is so small as needed to impose statistical reality to every entity within it.

9 Appendix A

Observing black line in the figure (2), winch is a curve for classical theory to 5000K, this curve diverges when compared with Planckian blue curve to 5000K.

In classical curve, a breaking point appears in classical curve near $\lambda = 1 \times 10^{-6} m$ but we can observe this problem starts at the ultraviolet region, because other curves like red 3000K, green 4000K and blue 5000K, which is expected by experiments, cross ultraviolet region without blow up.

In other words we need to investigate $\lambda = 0.375 \times 10^{-6} m$, which is the ultraviolet region, where the classical curve problem begin.

Considering $c = \lambda \nu$, we easily get $\nu = 8 \times 10^{14} Hz$ as frequency in the point that wavelength is $\lambda = 0.375 \times 10^{-6} m$.

To understand this limit when classical curve blow up at ultraviolet region, let’s investigate the relation $(h \nu / kT)$, which appears in equation (28), using ultraviolet frequency $\nu = 8 \times 10^{14} Hz$ and $T = 5 \times 10^3 K$ as in figure (2). Then,

$$\frac{h \nu}{kT} = \frac{(6.63 \times 10^{-34} m^2 \cdot Kg \cdot s^{-1}) (8 \times 10^{14} s^{-1})}{(1.38 \times 10^{-23} m^2 \cdot Kg \cdot s^{-2} \cdot K^{-1}) (5 \times 10^3 K)} = 0.77 \times 10^{-43}.$$  \hspace{1cm} (70)
Figure 2: Classical radiation distribution, evidencing temperature of 5000$K$ and a breaking point near wavelength 1$\mu m$. Graphic units: wavelength $\mu m$ and Spectral Radiance $kW \cdot sr^{-1} \cdot m^{-2} \cdot nm^{-1}$.

Equation (41) tells us that the breaking region in the ultraviolet zone occurs because $kT$ term is much greater than $h\nu$. Or just,

$$h\nu \ll kT.$$  (71)

10 Appendix B

In this session, we just recover Schrödinger equation in a quick way, starting from Schrödinger distribution obtained as PDE solution (69) as a fundamental to discover operators from mean value concept, the aim of this session is show a close relation between a distribution that became from experimental measurable parameters like uncertainty relations and schrödinger quantum mechanics.

Starting from Hamiltonian of expected values $\langle H \rangle = \langle p \rangle^2 / 2m + \langle V \rangle$ in square integrable system, we easily can see that,

$$\int_{-\infty}^{+\infty} \Psi^* (x,t) \hat{H} \Psi (x,t) \, dx = \frac{1}{2m} \int_{-\infty}^{+\infty} \Psi^* (x,t) \hat{p}^2 \Psi (x,t) \, dx + \int_{-\infty}^{+\infty} \Psi^* (x,t) \hat{V} \Psi (x,t) \, dx.$$  (72)

Another useful concept is that $\hat{p}$ operator must extract eigenvalue from distribution, as in the form $\hat{p}^2 \Psi (x,t) = p^2 \Psi (x,t)$. Because of format of distribution we get in (69) only a specific $\hat{p}$ operator can extract $p$ eigenvalue. So,

$$\Psi (x,t) = e^{ipx/\hbar} e^{-iEt/\hbar} \Leftrightarrow \hat{p} \equiv -i\hbar \frac{\partial}{\partial x}.$$  (73)

At same fashion, we get $\hat{H}$ operator (that extract energy $E$) and $\hat{V}$ operator (that impose potential energy $V$) in (72) to get the Schrödinger equation from expected values and Schrödinger distribution. Which gives,
\[
\int_{-\infty}^{+\infty} \Psi^* (x,t) \left[ i \hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - V(x,t) \right] \Psi (x,t) \, dx = 0,
\]
(74)
or just,
\[
 i \hbar \frac{\partial}{\partial t} \Psi (x,t) = - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi (x,t) + V(x,t) \Psi (x,t).
\]
(75)

Here, we present the Schrödinger equation as Heisenberg uncertainty relations Mechanics. The strong aspect relies in distribution, because from this one operators assume their form. We show by unique way with fluctuations analysis that Heisenberg uncertainty principle is determinant to obtain distribution that will lead us to find Schrödinger equation as a simple consequence.

11 Authors Declaration Interests

The authors declare that they have no know competing financial interests or personal relationship that could have appeared to influence the work reported in this paper.

References

[1] J. Clerk Maxwell, A Dynamical Theory of the Electromagnetic Field, Royal Society Publishing, (1864).
https://doi.org/10.1098/rstl.1865.0008

[2] T. Whestphal, H. Hepach, J. Pfaff and M. Aspelmeyer, “Measurement of Gravitational Coupling Between Millimetre-Sized Masses”, Nature Physics, (2021).
https://doi.org/10.1038/s41586-021-03250-7

[3] R. Merlin, Rabi oscillations, Floquet states, Fermis Golden Rule, and all that Insights From an Exactly Solvable Two-Level Model, American Journal of Physics, 89, 26, (2021).
https://doi.org/10.1119/10.0001897

[4] I D Landau, E M Lifshitz, The Classical Theory of Fields,Elsevier,(1980).

[5] Basil S. Davis and Lev Kaplan, Poynting Vector Flow in a Circular Circuit, American Journal of Physics 79, 1155 (2011).
https://doi.org/10.1119/1.3630927

[6] George B. Arfken , Hans J. Weber, et al, Mathematical Methods for Physicists, 7th edition, Elsevier, (2012).

[7] Reynolds Osborne, On the sub-mechanics of the Universe, Proc. R. Soc. Lond. 69425–433, (1902).
http://doi.org/10.1098/rspl.1901.0127

[8] Ilana Abayev, Properties of the Electronic Density of States in TiO2 Nanoparticles Surrounded with Aqueous Electrolyte, J Solid State Electrochem, (2006).
https://doi.org/10.1007/s10008-006-0220-1

[9] Luca Pierantoni et all, Dirac Equation Based Formulation for the Quantum Conductivity in 2D Nanomaterials, Applied Sciences MDPI, (2021).
https://doi.org/10.3390/app11052398

[10] Dieter Vollath, Energy Distribution in an Ensemble of Nanoparticles and its Consequences, Beilstein Journal of Nanotechnology, (2019).
https://doi.org/10.3762/bjnano.10.143
[11] Rowlinson, J. S. The Maxwell-Boltzmann distribution, Molecular Physics, 103.21-23: 2821-2828, (2005).
http://dx.doi.org/10.1080/00268970500044749

[12] Agudelo, Andrés; Cortés, Cristóbal. Thermal radiation and the second law. Energy, 35.2: 679-691, (2010).

[13] Rayleigh, Lord. Remarks Upon the Law of Complete Radiation”. Philosophical Magazine, 49: 539-540, (1900).
http://dx.doi.org/10.1080/14786440009463878

[14] D.H. Trevena, The Boltzmann Distribution and Related Topics, Statistical Mechanics, Woodhead Publishing, Pages 18-26, (2010).
https://doi.org/10.1533/9780857099662.18

[15] Jerry B. Marion, Mark A. Heald, Classical Electromagnetic Radiation, 2nd Edition, Academic Press, Pages 104-130,(1980).
https://doi.org/10.1016/B978-0-12-472257-6.50008-2

[16] Mortimer, R. G., Work, Heat, and Energy. Physical Chemistry, 45–93,(2000).
https://doi.org/10.1016/b978-012508345-4/50006-x

[17] Einstein, Relativity, the Special and the General Theory, General Press, New Delhi , 61-63 (2012).

[18] Keith Endrew, Entropy, American Journal of Physics 52, 492, (1984).
https://doi.org/10.1119/1.13892

[19] Frank S. Crawford, Using Einstein’s Method to Derive Both the Planck and Fermi-Dirac Distributions, American Journal of Physics 56, 883, (1988).
https://doi.org/10.1119/1.15402

[20] W. C. Barford, Derivation of Classical and Quantum Statistical Distributions, American Journal of Physics 44, 940, (1976).
https://doi.org/10.1119/1.10236

[21] Nairz, Olaf and Arndt, Markus and Zeilinger, Anton, Experimental verification of the Heisenberg uncertainty principle for fullerene molecules, Phys. Rev. A, 65, 3, pages 032109, (2002)
https://doi.org/10.1103/PhysRevA.65.032109

[22] Heisenberg, W. Über den Anschaulichen Inhalt der Quantentheoretischen Kinematik und Mechanik. Z. Physik 43, 172–198 (1927).
https://doi.org/10.1007/BF01397280

[23] Barenblat, G. I.,Dimensional Analysis. USSR Academy of Science, Moscow, Gordon and Breach Science Publishers, (1987).