THE GROUP $\Gamma(2)$ AND THE FRACTIONAL QUANTUM HALL EFFECT

Yvon Georgelin, Jean-Christophe Wallet

$^a$ Division de Physique Théorique†, Institut de Physique Nucléaire
F-91406 ORSAY Cedex, France

Abstract:
We analyze the action of the inhomogeneous modular group $\Gamma(2)$ on the three cusps of its principal fundamental domain in the Poincaré half plane. From this, we obtain an exhaustive classification of the fractional quantum Hall numbers. This classification is somehow similar to the one given by Jain. We also present some resulting remarks concerning direct phase transitions between the different quantum Hall states.

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I) Introduction

Since the discovery of the two dimensional quantized Hall conductivity by [1] for the integer plateaus and by [2] for the fractional ones, the quantum Hall effect has been a constant and intensive field of theoretical and experimental investigations.

As far as the theoretical viewpoint is concerned, pioneering contributions by [3a-c] have related the basic features of the hierarchy of the Hall plateaus with the properties of a two dimensional incompressible fluid with fractional charges collective excitations [3a]. This has been followed by a large number of related works, dealing with condensed matter theory [4], quantum field theory [5], and mathematical physics [6].

The purpose of this paper is to set up a link between the similarity transformations as introduced by Jain [7] in the hierarchical model of [3a-c] and some basic operations of the group Γ(2), which is known in the mathematical literature as the level 2–principal congruence subgroup of the inhomogeneous modular group Γ(1) [8]. It will be shown that the similarity transformations of Jain [7] (see also [10]) may well be described as a family of specific transformations of Γ(2) acting on the three cusps \([i\infty], [0]\) and \([1 \approx -1]\) (that is, the well known vertices of one of its fundamental domains [9]). In particular, the whole structure of the integer and fractional plateaus can be entirely recovered from this family of transformations, independantly of any microscopic models. Furthermore, the results obtained from this arithmetical construction may provide a new insight on recently proposed Hall phases diagrams [10-12], as it will be briefly discussed.

II) Γ(2) and the quantum Hall effect.
IIa – Basic properties

Let \(P\) be the upper complex plane and \(z\) a complex coordinate \((Im \, z > 0)\) on \(P\). We recall that the transformations \(G\) acting on \(P\) and pertaining to the inhomogeneous modular group \(\Gamma(1)\) are defined by

\[
G(z) = \frac{az + b}{cz + d}; \quad a, b, c, d \in \mathbb{Z}; \quad ad - bc = 1 \text{(unimodularity condition)}
\]  

(1) 

The group \(\Gamma(1)\) leaves also invariant the real numbers and the set of rational numbers.

Up to now, two subgroups of \(\Gamma(1)\) underlying the main models for the Hall effect have essentially been considered. The first one, \(\Gamma_\theta\) \((\equiv \Gamma_S(2))\), appears to be the natural symmetry group of the Landau problem on a torus; it is generated by the following transformations on \(P\):

\[
T^2(\tau) = \tau + 2; \quad S(\tau) = -\frac{1}{\tau}
\]  

(2a; b).
Here, the complex parameter \( \tau \) characterizes the considered torus. As clearly demonstrated in [13], the restriction to the even translations is required in order to recover the exact periodic conditions on the wave functions, up to an allowed gauge transformation. This subgroup appears also in the construction of the many body Landau states on the torus [14] in terms of the Coulomb gas vertex operators [15]. One of the reasons why \( \Gamma_\theta \) can be hardly retained as a good classifying symmetry for the physical Hall system is that it has only two equivalence classes among the real fractions \( p/q \) (namely \( pq \in 2\mathbb{Z} \) and \( pq \in 2\mathbb{Z} + 1 \)), which apparently do not correspond to any observed feature in the quantum Hall hierarchy.

The second relevant subgroup of \( \Gamma(1) \) is related to the similarity transformations of Jain used in the hierarchical model. In this framework, two fundamental exchange symmetries acting on the fractional Hall numbers \( \nu = p/q \) have been singled out [7]. They are defined by

\[
\nu \leftrightarrow \nu + 1; \quad \nu \leftrightarrow \frac{\nu}{2\nu + 1} \quad (3a; b).
\]

Physically, (3a) corresponds to the Landau level addition transformations whereas (3b) represents flux attachment transformations (see [7] and [10]). Another exchange symmetry is also present at least for the first Landau level; it is given by \( \nu \leftrightarrow 1 - \nu \) and represents a particle-hole symmetry. It is an important remark that (3a) and (3b) generate the subgroup \( \Gamma_0(2) \equiv \Gamma_T(2) \) of \( \Gamma(1) \). Notice also that \( \Gamma_0(2) \) has been proposed in [10-12] as the best candidate for a dynamical flow symmetry group generated in renormalization group equations for the complex Hall conductivity. Its nice feature is that it respects the parity of the denominators when acting on the fractions \( p/q \). However, the way how it could take into account finer structures as well as the particle-hole similarity is not so clear.

One of the reasons leading us to look for other subgroups of \( \Gamma(1) \) sufficient to generate the Hall fractional numbers hierarchy comes from a comparison between the experimental situation for the Hall system and the (many body) Landau problem on the torus. Indeed, it has been underlined in [16] that in the true macroscopic Hall experiment, the sample with the threads attached to it can be viewed as a topological subsystem of the (many body) Landau problem on the torus (see figure 1). This is to be contrasted with the mathematical property that \( \Gamma_0(2) \) and \( \Gamma_\theta \) bear no inclusion relation the one with respect to the other but instead are conjugate groups in \( \Gamma(1) \). Therefore, if one considers roughly the Hall system as a kind of subsystem of the Landau problem on the torus, it is quite natural to look for a subgroup of \( \Gamma(1) \), included both in \( \Gamma_\theta \) and \( \Gamma_0(2) \), and which can generate the quantum Hall number hierarchy.

We find that the largest group sharing the above properties is the inhomogeneous prin-
principal congruence group at level 2, usually denoted by $\Gamma(2)$ in the mathematical literature. Its action on $\mathcal{P}$ is given by

$$G(z) = \frac{az + b}{cz + d}; \quad ad - bc = 1 \quad \text{(unimodularity condition); } a, d \text{ odd; } b, c \text{ even} \quad (4).$$

It is a free group generated by

$$T^2(z) = z + 2; \quad \Sigma(z) \equiv ST^{-2}S(z) = \frac{z}{2z + 1} \quad (5),$$

and the following inclusion relation holds:

$$\Gamma(2) \subset (\Gamma_0(2), \Gamma_\theta) \subset \Gamma(1) \quad (6).$$

The main properties of $\Gamma(2)$ can be found e.g in [8], [9]. For the present purpose, it is crucial to recall that its principal fundamental domains $D$, is obtained by identifying the frontiers as indicated on figure (2). Moreover, $D$ has three cusps denoted by $[i\infty]$, $[0]$ and $[1 \simeq -1]$ which are identified with the three integers $\infty$, 0 and 1.

Corresponding to this, the action of $\Gamma(2)$ on real rational fractions generates also three well separated orbits on the rational numbers

$$\left\{ \frac{2k}{2m + 1} \right\}; \left\{ \frac{2k + 1}{2m + 1} \right\}; \left\{ \frac{2k + 1}{2m} \right\} \cup \{\infty\}; \quad k, m \in \mathbb{Z} \quad (7),$$

suggesting that it could be of some relevance for the quantum Hall hierarchy [12]. Actually, we claim that the action of $\Gamma(2)$ on the cusps allows one to describe a more detailed structure in this hierarchy. Our proposal is the following: The action of $\Gamma(2)$ on the three cusps $[i\infty]$, $[0]$ and $[1 \simeq -1]$ generates natural families of transformations in one-to-one correspondence with families of the Jain hierarchy, which permits one to obtain an exhaustive classification of all the fractional Hall numbers observed up to now. It could give also some new light on the structure of the phase diagram for the quantum Hall effect.

IIIb – The construction.

Let us choose the following parametrization for $G \in \Gamma(2)$

$$G(z) = \frac{(2s + 1)z + 2n}{2rz + (2k + 1)} \quad k, n, r, s \in \mathbb{Z} \quad (8).$$

The unimodularity condition imposes

$$(2s + 1)(2k + 1) - 4rn = 1 \quad (9).$$
Let us now select \( \{G^\lambda\} \), a family of transformations in \( \Gamma(2) \) parametrized by \( \lambda \), such that each \( G^\lambda \) sends the cusp \( [i\infty] \) onto a given irreducible fraction with even denominator, \( \lambda = \frac{2s + 1}{2r} \) (so that \( \lim_{|z| \to \infty} G^\lambda(z) = \lambda \)). Then, it can be easily realized that the image by the \( G^\lambda \)'s of the two other cusps \([0]\) and \([1 \simeq -1]\) of \( D \) generates precisely a Jain hierarchy.

Indeed, let us pick for example \( \lambda = 1/2 \); then any transformation \( G^{1/2} \) which sends the cusp \([i\infty]\) onto the irreducible fraction \( \lambda = \frac{1}{2} \) (implying that \( s = 0 \) and \( r = 1 \)) could be parametrized, using (8) as

\[
G^{1/2}(z) = \frac{z + 2n}{2z + 2k + 1}, \quad k, n \in \mathbb{Z}
\]

(10),

together with the unimodularity condition which implies \( 2k = 4n \); then, (10) becomes

\[
G^{1/2}_n(z) = \frac{z + 2n}{2z + 4n + 1}
\]

(11).

Now, the images by \( G^{1/2}_n \) of the cusps \([0]\) and \([1 \simeq -1]\) are given by

\[
G^{1/2}_n(0) = \frac{2n}{4n + 1}; \quad G^{1/2}_n(1) = \frac{2n + 1}{4n + 3}
\]

(12),

corresponding to the families of fractional levels collected in table 1. There, one easily identifies the Hall fractions located on each side of the \( \lambda = 1/2 \) fraction; this includes the numbers 0 and 1, the integer Hall numbers which border this family.

Starting from \( \lambda = 3/4 \), a similar construction gives rise to the Hall numbers family located on each side of the \( \lambda = 3/4 \) fraction, including the border numbers 1 and 2/3. Indeed, one must have

\[
G^{3/4}(i\infty) = \frac{3}{4}
\]

(13).

This, combined with (8) and (9) (implying \( r = 2 \) and \( s = 1 \)) yields

\[
G^{3/4}_n(z) = \frac{3z + 2n}{4z + (2k + 1)}, \quad n, k \in \mathbb{Z}; \quad 3k = 4n - 1 \text{ (unimodularity)}
\]

(14a; b).

From this later relations, the images of \([0]\) and \([1 \simeq -1]\) under the \( G^{3/4}_n \)'s are easily computed. The results, collected in table 2, are nothing but the Hall numbers family located on each side of the \( \lambda = 3/4 \) fraction, including the borders 1 and 2/3.

\[\text{† The fraction must be irreducible because of the unimodularity condition}\]
The construction proceeds in the same way, starting also from any irreducible even denominator fraction larger than 1. Pick for example $\lambda = 3/2$. One has

$$G^{3/2}_n(z) = \frac{3z + 2n}{2z + (2k + 1)}, \quad n, k \in \mathbb{Z}; \quad 3k = 2n - 1 \quad \text{(unimodularity)}$$

The images of $[0]$ and $[1 \simeq -1]$ by $G^{3/2}$ are collected in table 3, which reproduces the Jain hierarchy symmetrical about $\lambda = 3/2$, including the borders 1 and 2.

The results for $\lambda = 1/4$ are also collected in table 4.

**III) Discussion and conclusion.**

Summarizing, we claim that any Jain hierarchy symmetrical about any fraction $\lambda$ with even denominator is the image of the cusps $[0]$ and $[1 \simeq -1]$ of $\mathcal{D}$ by the family of transformations $G^\lambda \in \Gamma(2)$ sending the cusp $[i\infty]$ to $\lambda$. The fraction $\lambda$ belongs to the orbit $\{(2m + 1)/2r\} \cup \{\infty\}$.

The complete sequences of Hall fractionals is of course not still experimentally determined but the hierarchical families obtained from our construction are in complete agreement with what is up to now experimentally confirmed [17].

It has been recently argued that it can be associated to the Hall system a two-dimensional (i.e magnetic field – disorder variable) phase diagram [10-12] where the even denominator fractions label Hall metallic states whereas the odd denominator fractions as well as the non zero integers label Hall liquid states, the number zero corresponding to a unique Hall insulator state. In that framework, each Hall metallic state appears to be surrounded by a defined family of Hall liquid states, the Hall insulator state being ”in contact” with a restricted number of other states [10].

Our construction agrees globally with this picture. Here too, a central role is played by the (metallic) even denominator fractions, each of them generating two symmetric families of odd denominator and integer Hall liquid states (corresponding to the action of the $G^\lambda$’s on the cusps $[0]$ and $[1]$). In each of these families labeled by $\lambda$, it could eventually appears (see tables) the Hall insulator $\nu = 0$ (for example, this happens for $\lambda = 1/2, 1/4$ but not for $\lambda = 3/4, 3/2$). We suspect in that case the occurrence of a direct insulator/metal phase transition. Furthermore, according to our description ($\nu = 0$ appears only in $G^\lambda(0)$), a direct transition between the insulator and the nearest liquid state appearing in $G^\lambda(1)$ is also probably allowed (see tables). It is amazing to observe that, if we accept such a ”neighboring principle” for possible direct Hall phase transitions between the different entries of our tables, we actually predict a direct phase transition between the insulator
and the liquid states $2/3, 2/5, 2/7, 2/9$ (see table 1 and 4). In this respect, our classification is closer to the one proposed in [18] rather than the one given by [10-11].

The complete description of the Hall phase diagram is still subject to some controversy. Nevertheless, our present mathematical (arithmetical) construction fits quite well with the existing datas [17,18].

Let us add a comment concerning the interpretation of the particle-hole similarity [7] $\nu' + \nu = 1$ within our construction. Clearly, this last relation can be satisfied only for $0 \leq \nu(\nu') \leq 1$ (i.e. inside the first Landau level). The way we constructed the hierarchies from the $G^\lambda$'s indicates we should have $0 < \lambda < 1$. Now, it is known in arithmetics that given a positive odd number $2m+1$, the set of fractions with that given denominator obtained from our present construction is the same as the set of fractions with denominators equal to $2m+1$ belonging to the order $2m+1$ Farey sequence $F_{2m+1}$ [19]. It is a nice property of Farey sequences that the sum of two symmetric fractions about $\lambda = 1/2$ is equal to 1. This is identical to $\nu' + \nu = 1$. We can translate this in terms of our $G^\lambda$'s: the particle-hole similarity inside the first Landau level corresponds to the relation

$$G^\lambda(0) + G^{\lambda'}(1) = 1$$

which holds only when

$$\lambda + \lambda' = 1 \quad \lambda, \lambda' > 0, \quad \lambda, \lambda' < 1$$

The solution of these equations gives any pairs of fractions related by a particle-hole similarity. For example, pick $\lambda = \lambda' = 1/2$, then $G_{n}^{1/2}(0) + G_{n'}^{1/2}(1) = 1$ is verified provided $n + n' = -1$.

One mathematical remark is now in order. Among all the inhomogenous modular subgroups of $\Gamma(1)$, the group $\Gamma(2)$ is the only one possessing the principal fundamental domain $D$ with only three cusps $[0], [1 \simeq -1]$ and $[i\infty]$. In this sense, our construction is unique.

The physical interpretation (if any) of the cusps is obscure to us at the present time. The same remark could be made concerning the "shift operators" $G^\lambda$'s. Nevertheless, as a final conclusion, we recall that microscopic models proposed recently [20] to describe the quantum Hall dynamics exhibit an approximate $\Gamma(1)$ invariance from which some physical usefull observables (for instance the height of each longitudinal conductivity peaks) could in principle be determined. We suggest here that $\Gamma(2)$ could in fact be an alternative candidate to $\Gamma(1)$ as a dynamical group to be exploited as a starting point to built future microscopic models.
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FIGURE CAPTIONS

**Figure 1:** A Hall sample “sitting” on a torus with the thickness of the threads largely exaggerated; the magnetic field is constant and perpendicular to the sample.

**Figure 2:** The principal fundamental domain of $\Gamma(2)$ in the Poincaré plane showing the cusps $[0]$, $[1 \simeq -1]$ and $[i\infty]$. The identification operations $T^2$ and $\Sigma$ are indicated.
**TABLE 1:** $\lambda = 1/2$; $(k = 2n)$

| $n$  | ... | 3  | 2  | 1  | 0  | 1  | 2  | 3  | ... |
|------|-----|----|----|----|----|----|----|----|-----|
| $k$  | ... | -6 | -4 | -2 | 0  | 2  | 4  | 6  | ... |
| $G_{n/2}^{1/2}(0)$ | ... | $\frac{6}{11}$ | $\frac{4}{7}$ | $\frac{2}{3}$ | 0 | $\frac{2}{5}$ | $\frac{4}{9}$ | $\frac{6}{13}$ | ... |
| $G_{n/2}^{1/2}(1)$ | ... | $\frac{5}{9}$ | $\frac{3}{5}$ | 1 | $\frac{1}{3}$ | $\frac{3}{5}$ | $\frac{5}{7}$ | $\frac{7}{9}$ | ... |

**TABLE 2:** $\lambda = 3/4$; $(3k = 4n - 1)$

| $n$  | ... | -8  | -5  | -2  | 1  | 4  | 7  | ... | ... |
|------|-----|-----|-----|-----|----|----|----|-----|-----|
| $k$  | ... | -11 | -7  | -3  | 1  | 5  | 9  | ... | ... |
| $G_{n}^{3/4}(0)$ | ... | $\frac{16}{21}$ | $\frac{10}{13}$ | $\frac{4}{5}$ | $\frac{2}{3}$ | $\frac{8}{11}$ | $\frac{14}{19}$ | ... | ... |
| $G_{n}^{3/4}(1)$ | ... | $\frac{13}{17}$ | $\frac{7}{9}$ | 1 | $\frac{5}{7}$ | $\frac{11}{15}$ | $\frac{17}{23}$ | ... | ... |

**TABLE 3:** $\lambda = 3/2$; $(3k = 2n - 1)$

| $n$  | ... | -7  | -4  | -1  | 2  | 5  | 8  | ... | ... |
|------|-----|-----|-----|-----|----|----|----|-----|-----|
| $k$  | ... | -5  | -3  | -1  | 1  | 3  | 5  | ... | ... |
| $G_{n/2}^{3/2}(0)$ | ... | $\frac{14}{17}$ | $\frac{8}{11}$ | 2 | $\frac{4}{5}$ | $\frac{10}{13}$ | $\frac{16}{19}$ | ... | ... |
| $G_{n/2}^{3/2}(1)$ | ... | $\frac{11}{14}$ | $\frac{7}{9}$ | 1 | $\frac{5}{7}$ | $\frac{13}{15}$ | $\frac{19}{23}$ | ... | ... |

**TABLE 4:** $\lambda = 1/4$; $(k = 4n)$

| $n$  | ... | -3  | -2  | -1  | 0  | 1  | 2  | ... | ... |
|------|-----|-----|-----|-----|----|----|----|-----|-----|
| $k$  | ... | -12 | -8  | -4  | 0  | 4  | 8  | ... | ... |
| $G_{n}^{1/4}(0)$ | ... | $\frac{6}{23}$ | $\frac{4}{15}$ | $\frac{2}{7}$ | 0 | $\frac{2}{5}$ | $\frac{4}{7}$ | ... | ... |
| $G_{n}^{1/4}(1)$ | ... | $\frac{5}{19}$ | $\frac{3}{11}$ | $\frac{1}{3}$ | $\frac{1}{5}$ | $\frac{3}{13}$ | $\frac{5}{21}$ | ... | ... |