Custodial $SU(2)$ Violation and the Origin of Fermion Masses

A. Blumhofer and M. Lindner

Institut für Theoretische Physik
der Universität Heidelberg
Philosophenweg 16, D–W–6900 Heidelberg

Abstract

Custodial $SU(2)$ breaking due to dynamical fermion masses is studied in a rather general context and it is shown how some well known limiting cases are correctly described. The type of “gap equation” which can systematically lead to extra negative contributions to the so–called $\rho$–parameter is emphasized. Furthermore general model independent features are discussed and it is shown how electro–weak precision measurements can be sensitive to the fermion content and/or dynamical features of a given theory.

*Heisenberg Fellow
Email: T36 (A.B.) or Y29 (M.L.) at VM.URZ.UNI-HEIDELBERG.DE
I. Introduction

The Standard Model of electro–weak interactions is today in very good shape even though the Higgs mechanism is for a number of reasons unsatisfactory. The model agrees however with all known experimental facts and there is even evidence for quantum corrections. On the other hand a Higgs particle has not yet been found and the symmetry breaking mechanism is untested. Besides the vacuum expectation value (given by the Fermi constant) other essential experimental information is expressed by model independent parametrizations of radiative corrections in terms of the so–called $S$, $T$, $U$ variables [1], where $T$ is related to the old $\rho$–parameter [2] by 
$$
\alpha(T - T_0) = \rho - 1 = \Delta \rho \quad (\text{where } \alpha = e^2/4\pi).
$$
This $\rho$–parameter (which is experimentally very close to unity) is actually defined in terms of the charged and neutral electro–weak Goldstone Boson decay constants $F_\pm(0)$ and $F_3(0)$ as
$$
\rho := \frac{F_\pm^2(0)}{F_3^2(0)} = 1 + \Delta \rho \quad ; \quad 0 \lesssim \Delta \rho \lesssim 0.01 ,
$$
and $T \equiv T_0$, i.e. $\rho \equiv 1$, can be understood in terms of an extra global “custodial” symmetry transforming charged and neutral Goldstone Bosons into each other such that $F_\pm$ and $F_3$ must be identical. Small deviations from $\rho = 1$ are perturbations of this symmetry and this article deals with such deviations due to a dynamical origin of fermion masses.

In the Standard Model the four real components of the Higgs doublet $\Phi$ correspond to a global $SO(4) \simeq SU(2)_L \times SU(2)_R$ invariance of the pure scalar Lagrangian with an extra custodial $SU(2)$ symmetry. This can be made explicit by defining the matrix field $\Omega := (\tilde{\Phi}, \Phi)$ which transforms as $\Omega \rightarrow U_L \Omega U_R^\dagger$, where $\tilde{\Phi} = -i\sigma_2 \Phi^*$ and $U_{L/R} := \exp(i\tau_a \lambda_a^{L/R})$. Due to $\Phi^+\Phi = 1/2 \, Tr(\Omega^+\Omega)$ those parts of the Lagrangian which depend only on $\Phi^+\Phi$ possess an extra $SU(2)$ symmetry. If this were an exact symmetry of the full Lagrangian then it would guarantee exactly (i.e. to all orders) $\rho \equiv 1$. The Standard Model contains however two sources of custodial $SU(2)$ violations outside of the pure Higgs sector, namely the $U(1)$ hypercharges and the asymmetries of Yukawa couplings. In terms of $\Omega$ these custodial $SU(2)$ violating pieces can be written as

$$
\delta \mathcal{L}_{\text{custodial}} = -g_1 B_\mu R \text{e} \left\{ T_R \left[ \tau_3 \Omega^+(D_2^a \Omega) \right] \right\} - \left( \frac{g_t - g_b}{2} \right) \left( \overline{T} \Omega \tau_3 R + \overline{R} \tau_3 \Omega^+ L \right) ,
$$

where $D_2^a = \partial^a - ig_a W_a^\mu \tau_\mu$ and $L = (t_L, b_L)$, $R = (t_R, b_R)$. Due to their smallness we have ignored all tiny Yukawa couplings and we will even drop the bottom Yukawa coupling from now on. It is easily verified that $\delta \mathcal{L}_{\text{custodial}}$ does not spoil $\rho_{\text{tree}} \equiv 1$ upon symmetry breaking (i.e. $\Omega = v \, 1 + \delta \Omega$) even though the $SU(2)_R$ symmetry is destroyed. Consequently custodial $SU(2)$ violating vertices enter only via loops into the renormalization of the Higgs sector.
and guarantee an expansion of the form

$$\rho = 1 + G(g_1^2, g_t^2, \ldots) = 1 + c_1 g_1^2 + c_t g_t^2,$$  \hspace{1cm} (1.3)

where \( G \) is a homogeneous function in \( g_1^2 \) and \( g_t^2 \). Note that \( c_1 \) and \( c_t \) depend in general on all other couplings, but eq. (1.3) guarantees in a perturbative expansion Veltman screening, i.e. the dependence on \( \lambda \) (i.e. \( m_H^2 \)) is reduced compared to naive expectations [3].

The coefficient \( c_t \) arises in the Standard Model at the one loop level from the diagrams shown in Fig. 1. Numerically\(^1\) \( c_t \) is typically about four times bigger than \( c_1 \) and the current direct lower limit on the top quark mass [4] of \( m_t > 91 \text{ GeV} \simeq m_Z \) leads to \( g_t^2 / g_1^2 = m_Z^2 / m_t^2 > 2 \sin^2 \theta_W < 2 \sin^2 \theta_W \simeq 0.46 \), i.e. \( g_t^2 \gtrsim 2.2 g_1^2 \). The biggest correction to \( \rho = 1 \) comes therefore from the top quark and the experimental data for \( \Delta \rho_{\text{exp}} \) can be translated into a prediction of \( g_t \), i.e. the top mass:

$$\Delta \rho_{\text{exp}} = \Delta \rho_{\text{theo}} \simeq \frac{N_c}{32 \pi^2} \frac{g_t^2}{g_1^2} \frac{m_t^2}{v^2} = \frac{N_c \alpha_em_t^2}{16 \pi^2 \sin^2 \theta_W \cos^2 \theta_W M_Z^2}.$$  \hspace{1cm} (1.4)

This is actually the dominating effect in top mass predictions based on the analysis of radiative corrections of the Standard Model. This leads today for \( m_H = 300 \text{ GeV} \) to \( m_t = 152 \pm 18 \) GeV [5].

It is possible that the top quark is not precisely found where required by the Standard Model and therefore corrections to \( \Delta \rho \) from new physics should be studied. We discuss here modifications of custodial \( SU(2) \) violation due to a possible dynamical origin of fermion masses. If e.g. the top mass has dynamical origin then \( m_t \) is replaced by a dynamical top mass function \( \Sigma_t(p^2) \) while the physical top mass is given by one point only, namely the solution of the on–shell–condition \( m_t = \Sigma(m_t^2) \). In Section II we calculate \( \Delta \rho \) for an arbitrary fermionic weak isospin doublet with momentum dependent mass functions \( \Sigma_i(p^2) \). In Section III we present some limiting cases and illustrate magnitude and sign of typical modifications. We show that relative to the Standard Model positive and negative corrections to \( \Delta \rho \) can occur and we will point out that it is in principle possible to keep \( \Delta \rho \) fixed while the physical top mass can essentially take any value. In Section IV we relate these results to the type of gap equation and show that this may provide in a certain class of models a natural compensation mechanism which makes \( \Delta \rho \) systematically smaller than expected. The implications for electro–weak precision measurements on general Dynamical Symmetry Breaking scenarios are discussed in Section V.

\(^1\)Note that \( c_1 \) and \( c_t \) are defined without powers of coupling constants.
II. $\Delta \rho$ for Dynamical Fermion Masses

Suppose the Higgs sector is replaced by some dynamical scenario which is responsible for the breaking of the electro–weak symmetry and for quark and lepton masses. Consequently the underlying Lagrangian would be the Standard Model without the Higgs sector amended by a new (presumably strongly coupled) sector triggering dynamical symmetry breaking. This new sector may contain new fundamental fermions and/or bosons, but may also stand for an effective description of non–perturbative effects of the known fermions and gauge fields. In any case there must be a scalar operator which develops a condensate (or VEV) such that the broken global symmetries give rise to those Goldstone Bosons which can give mass to $W$ and $Z$. Well known examples are Technicolor \[6, 7\], top condensation \[8\] and even the Standard Model Higgs mechanism can be phrased in this way.

Besides breaking the $SU(2)_L$ gauge symmetry, Dynamical Symmetry Breaking (DSB) should also explain fermion masses like those which arise via Yukawa interactions in the Standard Model. When fundamental scalars are absent this is achieved by connecting the fermions in a suitable way to some electro–weak symmetry breaking fermionic condensate. A given fermion is therefore either condensing itself such that its mass is the result of a “critical” Schwinger–Dyson (or gap) equation or alternatively the fermion is coupled indirectly via some (e.g. “see–saw” or “horizontal”) interaction to the condensation mechanism. In both cases the fermion masses become therefore momentum dependent functions $\Sigma(p^2)$ related directly or indirectly to some gap equation. For an asymptotically free condensing force $\Sigma(p^2)$ approaches zero at high momenta $p^2 \to \infty$. This asymptotic behaviour starts typically around some generic DSB scale, which – if the underlying condensation mechanism is to solve the old hierarchy problem – should not be many TeV. Note that this should imply structure in $\Sigma(p^2)$ at a few TeV.

We assume now that symmetry breaking is the result of unspecified new strong forces acting on some fermion doublet(s) and that – like in the Standard Model – custodial $SU(2)$ violation does not change significantly if the weak $U(1)_Y$ coupling $g_1$ is set to zero. In that limit custodial $SU(2)$ breaking must stem entirely from the new sector which is coupled to the $W_3$ and $W_{\pm}$ propagators only via those fermions which are representations under both $SU(2)_L$ and the new strong force. All custodial $SU(2)$ violations arise then from the contributions of fermionic vacuum polarizations to the $W$ propagator. In an expansion in powers of $g_2^2$ the leading contribution is given by fermion loop corrections to the $W$

\[\text{indirectly (via vacuum alignment) a small custodial } SU(2) \text{ violating } U(1)_Y \text{ coupling can however become very important.}\]
propagator which do not contain any electro–weak gauge boson propagation inside the loop. Insertions of fermionic vacuum polarizations into electro–weak loop diagrams are suppressed by corresponding powers of \( g_2^2 \). In leading order \( g_2^2 \), but exact in the new strong coupling, the custodial \( SU(2) \) violating contributions to the \( W \) propagator are graphically represented in Fig. 2. The first contribution is the generalization of the type of diagram shown in Fig. 1 with hard masses replaced by \( \Sigma \)‘s, i.e. all diagrams which contribute to the dynamically generated fermion masses. The second contribution contains the exact Kernel \( K \) of the strong forces responsible for condensation and it is useless to expand this Kernel perturbatively in powers of the coupling constants of the new strong force. The Goldstone theorem tells us however that the Kernel must contain poles of massless Goldstone Bosons due to the global symmetries broken by the fermionic condensates. This is symbolically expressed by the second line of Fig. 2, where \( \tilde{K} \) does not contain any further poles of massless particles. But \( \tilde{K} \) may (and typically will) contain all sort of massive bound states like vectors, Higgs–like scalars etc. in all possible channels.

The Goldstone Boson contributions shown in Fig. 2 were used by Pagels and Stokar\cite{Pagels1981} to obtain a relation between the \( \Sigma \)‘s and the Goldstone Boson decay constant. Their derivation uses the fact that only the Goldstone Bosons contribute a term proportional \( p_\mu p_\nu / p^2 \) to the \( W \) polarization at vanishing external momentum, but this method ignores possible contributions from \( \tilde{K} \) which enter indirectly via the use of Ward identities. The \( p_\mu p_\nu / p^2 \) contributions to \( \Pi_{\mu\nu} \) are balanced (up to small corrections from \( \tilde{K} \)) by \( g_{\mu\nu} \) terms created by the first diagram on the rhs of Fig. 2. We derive now a relation between the \( \Sigma \)’s and the Goldstone Boson decay constants from these \( g_{\mu\nu} \) terms and compare the result later with the Pagels Stoker relation. We will further argue that contributions from \( \tilde{K} \) are significantly suppressed. Let us therefore work with rescaled fields such that gauge couplings appear in the kinetic terms of the gauge boson Lagrangian like \(( -1/4g^2 ) ( W_{\mu\nu} )^2 \). Since we do not include any propagating \( W \) bosons we need not gauge fix at this stage and the inverse \( W \) propagator can be written as

\[
\frac{1}{g_2^2} D_{W,\mu\nu}(p^2) = \frac{1}{g_2^2} \left( -g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} \right) p^2 - \Pi_{\mu\nu}(p^2)
\]

with the polarization tensor \( \Pi_{\mu\nu}(p^2) = ( -g_{\mu\nu} p^2 + p_\mu p_\nu ) \Pi(p^2) \). At vanishing external momentum the first fermion loop on the rhs of Fig. 2 contributes to \( \Pi_{\mu\nu} \)

\[
\Pi_{\mu\nu} = -i Z^2 N_c \int \frac{d^4k}{(2\pi)^4} \frac{\text{Tr} \left[ \Gamma_{\mu}(k) + \Sigma_1(k) \right] \Gamma_{\nu}(k) + \Sigma_2(k) \right] }{(k^2 - \Sigma_1(k)^2)(k^2 - \Sigma_2(k)^2)} ,
\]

(2.2)

where \( Z^{-1} = \sqrt{2} \), in the charged and neutral channel, respectively, \( \Gamma_{\alpha} = (1 - \gamma_5)^{\gamma_\alpha} \), and \(+\iota\) is generally implied in the denominator. By naive power counting eq. (2.2) has\footnote{Which are very important for a gauge invariant dynamical Higgs mechanism.}
quadratic and logarithmic divergences, but assuming\(^5\) \(\Sigma_i(p^2) \xrightarrow{p^2 \to \infty} 0\) we find that the divergences of \(\Pi_{\mu \nu}(p^2)\) are identical to those calculated for \(\Sigma_i \equiv 0\). It makes therefore sense to split \(\Pi_{\mu \nu}(p^2) = \Pi^0_{\mu \nu}(p^2) + \Delta \Pi_{\mu \nu}(p^2)\) where \(\Pi^0_{\mu \nu}\) is defined as \(\Pi_{\mu \nu}\) for \(\Sigma_i \equiv 0\). \(\Pi^0_{\mu \nu}\) is then an uninteresting \(\Sigma_i\) independent constant which contains all divergences and needs renormalization. Contrary the interesting \(\Sigma_i\) dependent piece \(\Delta \Pi_{\mu \nu} = \Pi_{\mu \nu} - \Pi^0_{\mu \nu}\) is finite, even when the external momentum is sent to zero. Thus

\[
\Delta \Pi_{\mu \nu} = -i Z^2 N_c \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{\text{Tr} \left[ \Gamma_\mu(k + \Sigma_1)\Gamma_\nu(k + \Sigma_2) \right]}{(k^2 - \Sigma_1^2)(k^2 - \Sigma_2^2)} - \frac{\text{Tr} \left[ \Gamma_\mu k \Gamma_\nu k \right]}{k^4} \right\}, \tag{2.3}
\]

\[
= -i Z^2 N_c \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ \Gamma_\mu k \Gamma_\nu k \right] \left\{ \frac{1}{(k^2 - \Sigma_1^2)(k^2 - \Sigma_2^2)} - \frac{1}{k^4} \right\}
- i Z^2 N_c \int \frac{d^4k}{(2\pi)^4} \Sigma_1 \Sigma_2 \text{Tr} \left[ \Gamma_\mu \Gamma_\nu \right] \left\{ \frac{1}{(k^2 - \Sigma_1^2)(k^2 - \Sigma_2^2)} \right\}, \tag{2.4}
\]

where \(N_c\) is the number of colors and \(\Gamma_i = (1 - \gamma_5)\gamma_i\). Note that our separation procedure for \(\Delta \Pi_{\mu \nu}\) will not spoil gauge invariance. The first trace in eq. (2.4) gives under the integral \(-\frac{1}{2} g_{\mu \nu} k^2\) while the second trace vanishes. Angular integration is trivially performed in Euclidean space and continued back to Minkowski space:

\[
\Delta \Pi_{\mu \nu} = -g_{\mu \nu} \frac{Z^2 N_c}{(4\pi)^2} \int_0^\infty dk^2 \frac{k^2 (\Sigma_1^2 + \Sigma_2^2) - \Sigma_1^2 \Sigma_2^2}{k^2 - \Sigma_1^2 (k^2 - \Sigma_2^2)} . \tag{2.5}
\]

As anticipated this result is homogenous in \(\Sigma_i\) and finite with the assumptions made on \(\Sigma_i\). For neutral channels eq. (2.5) must be summed over all fermion anti–fermion pairs with \(\Sigma_1 = \Sigma_2\) and for charged channels one must sum over all doublets, where \(\Sigma_1\) and \(\Sigma_2\) represent then the fermion masses of the isospin doublet. We can for example neglect the bottom quark mass for the contribution of the \(t - b\) doublet and set \(\Sigma_1 = \Sigma_2 = \Sigma_t\) in the neutral channel and \(\Sigma_1 = \Sigma_t\), \(\Sigma_2 = \Sigma_b \equiv 0\) in the charged channel, respectively. The contributions of any other fermion doublet are given by the same formula provided \(N_c\) is suitably replaced.

The Goldstone Boson decay constants \(F^2_i\) are the poles of \(\Pi(p^2)\) at vanishing external momentum. For our definition of \(\Pi_{\mu \nu}\) we find that \(F^2_i\) is identical to the \(g_{\mu \nu}\) piece eq. (2.3) without the factor \(-g_{\mu \nu}\). Taking into account \(Z = 1/\sqrt{2}\) in the charged channel and \(Z = 1/2\) in the neutral channel and allowing for further arbitrary custodial SU(2) symmetric contributions \(F^2_o\) one finds

\[
F^2_\pm = F^2_0 + \frac{N_c}{32\pi^2} \int_0^\infty dk^2 \frac{k^2 (\Sigma_1^2 + \Sigma_2^2) - \Sigma_1^2 \Sigma_2^2}{k^2 - \Sigma_1^2 (k^2 - \Sigma_2^2)} .
\]

\(^5\)This is justified for asymptotically free theories where chiral symmetry breaking disappears as \(p^2 \to \infty\).
\[ F_3^2 = F_0^2 + \frac{N_c}{32\pi^2} \int_0^\infty dk^2 \frac{\Sigma_i^2}{k^2 - \Sigma_i^2}, \]  
\[ F_3^2 = F_0^2 + \frac{N_c}{32\pi^2} \int_0^\infty dk^2 \left\{ \frac{k^2\Sigma_1^2 - \frac{1}{2}\Sigma_1^4}{(k^2 - \Sigma_1^2)^2} + \frac{k^2\Sigma_2^2 - \frac{1}{2}\Sigma_2^4}{(k^2 - \Sigma_2^2)^2} \right\}, \]  
\[ F_3^2 = F_0^2 + \frac{N_c}{32\pi^2} \int_0^\infty dk^2 \frac{k^2\Sigma_i^2 - \frac{1}{2}\Sigma_i^4}{(k^2 - \Sigma_i^2)^2}, \]  
\[ F_3^2 - F_\pm^2 = \frac{N_c}{64\pi^2} \int_0^\infty dk^2 \frac{k^4(\Sigma_1^2 - \Sigma_2^2)^2}{(k^2 - \Sigma_1^2)^2(k^2 - \Sigma_2^2)^2} \overset{\Sigma_1 = \Sigma_2 = 0}{\longrightarrow} \frac{N_c}{64\pi^2} \int_0^\infty dk^2 \frac{\Sigma_i^4}{(k^2 - \Sigma_i^2)^2}. \]  

Eq. (2.6) for $F_\pm^2$ is equivalent to the result obtained by Pagels and Stokar from the $q_\mu q_\nu/q^2$ contributions of Goldstone Bosons to $\Pi_{\mu\nu}$. The result for the neutral channel, eq. (2.7), looks however somewhat different. By using the integral identity
\[ \int_0^\infty dx \frac{x^2f(x)' - f(x)^2}{(x - f(x))^2} = f(\infty), \]
for $x = k^2$ and $f = \Sigma_i^2$ we can rewrite eq. (2.7) for example in the case $\Sigma_1 = \Sigma_1, \Sigma_2 = 0$
\[ F_3^2 = F_0^2 + \frac{N_c}{32\pi^2} \int_0^\infty dk^2 k^2 \frac{\Sigma_i^2 - k^2\Sigma_i\Sigma_i'}{(k^2 - \Sigma_i^2)^2}, \]  
where $\Sigma_i' = d\Sigma_i/dk^2$. Even though this looks now formally similar to the Pagels Stokar result it differs by a factor 2 in front of the derivative term in the nominator of eq. (2.11). This difference may appear less important, but we will see in Section III that in the limit of a hard top mass our method produces the correct $\rho$-parameter, while the Pagels Stokar result produces 3/2 times the correct answer. In addition to the correct $\rho$-parameter limit our expression leads also to a better numerical estimate of $f_\pi$ if we follow the methods of ref. 4. The difference between our result and the Pagels Stokar result must be resolved by $g_{\mu\nu}$ and $q_\mu q_\nu/q^2$ contributions to $\Pi_{\mu\nu}$ from $\tilde{K}$ in the second line of Fig. 4 such that the full result is transverse.

The $\rho$-parameter can be rewritten as
\[ \rho = 1 + \Delta \rho = \frac{F_3^2}{F_3^2} = \left(1 + \frac{(F_3^2 - F_\pm^2)}{F_\pm^2}\right)^{-1} \simeq 1 - 2 \frac{(F_3^2 - F_\pm^2)}{v^2}, \]  

(2.11)
and from eq. (2.8) we find the contribution of any fermion doublet to the $\rho$-parameter

$$
\Delta \rho = \frac{-N_c}{32\pi^2 v^2} \int_0^\infty dk^2 \frac{k^4 (\Sigma_1^2 - \Sigma_2^2)^2}{(k^2 - \Sigma_1^2)^2 (k^2 - \Sigma_2^2)^2} \xrightarrow{\Sigma_1 = \Sigma_2 \to 0} \frac{-N_c}{32\pi^2 v^2} \int_0^\infty dk^2 \frac{\Sigma_4^4}{(k^2 - \Sigma_2^2)^2},
$$

(2.12)

where we used $F_{\pm} = v^2/2$ with $v \simeq 175 \text{ GeV}$ in the denominator and the custodial $SU(2)$ symmetric contributions $F_0^2$ have disappeared as they should.

The final expression for the $\Delta \rho$ implies that for given $\Sigma_t(p^2) \xrightarrow{p^2 \to \infty} 0$ we can calculate three observable quantities which are one of the Goldstone Boson decay constants $F_i^2$, $\Delta \rho$ and furthermore the physical top mass defined via $\Sigma_t(m_t^2) = m_t$. These three quantities are dominated by different momenta and therefore $\Sigma \neq \text{constant}$ leads to a different answer than a constant, i.e. hard mass. In this context it is instructive to look at the degree of convergence of the above integrals. The Goldstone Boson decay constants $F_i^2$ are formally log. divergent, but are finite with our assumption on $\Sigma_t(p^2)$. In that case renormalization is not needed, but due to the formal log. divergence $\Sigma$ contributes with equal weight at all momentum scales. In other words, the magnitude of $F_i^2$ depends crucially on the high energy tail of $\Sigma_i$. The difference $F_+^2 - F_0^2$ has better convergence properties and is always finite, even for $\Sigma_t(p^2) = \text{constant}$. This implies that $\rho$ is finite, as it should be, and it is most sensitive to infrared scales somewhat above $m_t$. We will illustrate now effects of structure in $\Sigma$ and postpone a discussion how certain $\Sigma$ emerge from a gap equation of the underlying dynamics in Section IV.

III. Magnitude and Sign of Effects

The result eq. (2.12) for $\Delta \rho$ has several interesting limiting cases. First we would like to see if the correct Standard Model result emerges for a $t - b$ doublet. Therefore we set

$$
\Sigma_t(p^2) = m_t \Theta(\Lambda^2 - p^2),
$$

(3.1)

and ignore the $b$ quark mass. From eq. (2.12) we obtain

$$
\Delta \rho = -\frac{N_c}{32\pi^2 v^2} \int_0^\Lambda dk^2 \frac{m_t^4}{(k^2 - m_t^2)^2} = \frac{N_c m_t^2}{32\pi^2 v^2} \left( \frac{1}{1 - m_t^2/\Lambda^2} \right),
$$

(3.2)

which becomes in the limit $\Lambda \to \infty$ (i.e. a hard, constant top mass)

$$
\Delta \rho = \frac{N_c}{32\pi^2} \frac{m_t^2}{v^2} = \frac{N_c \alpha_{\text{em}}}{16\pi \sin^2 \theta_W \cos^2 \theta_W} \frac{m_t^2}{M_Z^2},
$$

(3.3)

6I.e. this formula applies to many cases such as for example for Technicolor.

7They are related to the $W$- and $Z$- masses via $M_W^2 = g_2^2 F_Z^2$ and $M_Z^2 = g_2^2 F_Z^2 = (g_1^2 + g_2^2) F_Z^2$. 

7
which is correctly the leading Standard Model result. Note that the Pagels Stokar relation produces in this limit incorrectly \(3/2\) times the Standard Model result while our expression gives the correct answer. For finite \(\Lambda\) eq. (3.2) describes furthermore the modification of the Standard Model result due to a high energy momentum cutoff
\[
\Delta \rho_\Lambda = \Delta \rho_{SM} \left( \frac{1}{1 - m_t^2/\Lambda^2} \right) \simeq \Delta \rho_{SM} \left( 1 + m_t^2/\Lambda^2 \right), \tag{3.4}
\]
where the last simplification is valid for \(m_t \gg \Lambda\). The cutoff makes \(\Delta \rho_\Lambda\) more positive than in the Standard Model which implies for a fixed experimental value of \(\Delta \rho\) a lower top mass prediction. An ansatz like eq. (3.1) can be viewed as the result of the Nambu–Jona-Lasinio gap equation of top condensation \[10\] and exhibits the leading correction to \(\Delta \rho_{SM}\) for such models.

The corrections to \(\Delta \rho\) can in principle also go into the opposite direction. Consider for example a modification of the above ansatz
\[
\Sigma_t(p^2) = \begin{cases} 
  m_t & \text{for } p^2 < \Lambda_1^2; \Lambda_1^2 > m_t^2; \\
  \sqrt{r} \cdot m_t & \text{for } \Lambda_1^2 \leq p^2 \leq \Lambda^2; \\
  0 & \text{for } p^2 > \Lambda^2; 
\end{cases} \tag{3.5}
\]
where \(\Sigma\) is enhanced \(r\)-fold above \(\Lambda_1 < \Lambda\) before it vanishes at \(\Lambda\) as before. The modified result is
\[
\Delta \rho = \frac{N_c m_t^2}{32\pi^2 v^2} \left( \frac{1}{1 - m_t^2/\Lambda^2} - \frac{r^2 m_t^2(\Lambda^2 - \Lambda_1^2)}{(\Lambda^2 - r m_t^2)(\Lambda_1^2 - r m_t^2)} - \frac{m_t^2(\Lambda^2 - \Lambda_1^2)}{(\Lambda^2 - m_t^2)(\Lambda_1^2 - m_t^2)} \right), \tag{3.6}
\]
\[
\Lambda^2, \Lambda_1^2 \gg m_t^2, r m_t^2 \simeq \frac{N_c m_t^2}{32\pi^2 v^2} \left( 1 + m_t^2/\Lambda^2 - \frac{m_t^2(\Lambda^2 - \Lambda_1^2)}{\Lambda^2 \Lambda_1^2} (r^2 - 1) \right), \tag{3.7}
\]
where extra contributions due to \(r \neq 1\) and \(\Lambda_1 \neq \Lambda\) are isolated in square brackets. Compared to eq. (3.1) the ansatz eq. (3.5) has for \(r > 1\) an extra “bump” between \(\Lambda_1\) and \(\Lambda\). This bump counteracts the effect of the cutoff and makes \(\Delta \rho\) less positive and it is easy to see that the bump can even become more important than the cutoff. This illustrates that scales somewhat above \(m_t\) are very important for the magnitude and sign of \(\Delta \rho\) and it is natural to ask if \(\Sigma\) can be chosen such that \(\Delta \rho\) vanishes for an arbitrary value of \(m_t\). This can indeed be done by choosing for example by hand
\[
\Sigma_t(p^2) = \frac{241 (118 m_t^4 + p^4)}{7 (4096 m_t^6 + p^6)} m_t^3, \tag{3.8}
\]
which is shown graphically in Fig. 3 to have only very moderate structure.

\[8\] Which may not only stand for the falloff of \(\Sigma\) but also for some other cancellation mechanism.
At this point it is necessary to say a few words on the integration over the pole of eq. (2.12). Instead of performing an analytic continuation for any ansatz individually one can rewrite eq. (2.12) exactly into

\[
\Delta \rho = \frac{N_c m_t^2}{32 \pi^2 v^2} \left( 1 + \frac{4 m_t \Sigma_U^2(m_t^2) - 4 m_t^2 \Sigma_U^2(m_t^2)^2}{(1 - 2 m_t \Sigma_U^2(m_t^2))^2} \right) \nonumber 
- \int_0^\infty \frac{dk^2}{m_t^2} \left[ \frac{\Sigma_t(k^2)^4}{(k^2 - \Sigma_t(k^2)^2)^2} - \frac{m_t^4}{(k^2 - m_t^2)^2} \right] \nonumber 
- \frac{4 m_t^5}{k^4 - m_t^4} \left( \frac{2 \Sigma_t^3(m_t^2)}{(1 - 2 m_t \Sigma_t^2(m_t^2))^2} + \frac{m_t \Sigma_t^4(m_t^2)^2 + m_t^2 \Sigma_t^6(m_t^2)^2}{(1 - 2 m_t \Sigma_t^2(m_t^2))^3} \right) \right),
\]

which has the advantage that the integrand in square brackets does not have an explicit pole for any arbitrary given \(\Sigma_t(p^2)\).

In order to illustrate that our result is not just limited to the contributions of a \(t - b\) doublet we can look for example at Technicolor \[6\] where an extra doublet of Techni–fermions \(U - D\) condenses and breaks the electro–weak symmetry. Ordinary quark and lepton masses (like the top mass) must be generated by so–called Extended Technicolor \[7\] interactions\[9\]. The coupled system of gap equations leads in a rough approximation \[11\] to the relation \(\Sigma_U - \Sigma_D = \Sigma_t\). Assuming this relation and \(\Sigma_i = m_i \Theta(\Lambda^2 - p^2)\) we obtain from eq. (2.12)

\[
\Delta \rho = \frac{N_c m_t^2}{32 \pi^2 v^2} \left( 1 + \frac{4 m_t \Sigma_t^2(m_t^2) - 4 m_t^2 \Sigma_t^2(m_t^2)^2}{(1 - 2 m_t \Sigma_t^2(m_t^2))^2} \right) \left( 1 + \frac{4}{9} N_{TC} \frac{m_U^2}{\Lambda^2} + \frac{4}{3} N_{TC} \frac{m_U^2}{\Lambda^2} + O(m^4/\Lambda^4) \right),
\]

which becomes for \(\Lambda \rightarrow \infty\) the result which is quoted in the literature \[11\]. For finite \(\Lambda\) we find the \(m_t^2/\Lambda^2\) correction of eq. (3.4) and additionally a term proportional to \(m_U^2/\Lambda^2\). These \(1/\Lambda^2\) terms are small and are usually omitted. This Technicolor example illustrates that our result works generally for cases where our assumptions are fulfilled. \(\Delta \rho\) is given as soon as all fermionic doublets, their color factors and their \(\Sigma\)’s are known. One might think that this does not contain much information without specifying a detailed theory, but we will see that there are interesting model independent consequences.

**IV. Reduced \(\Delta \rho\) and the Type of Gap Equation**

In the discussion of the previous Section we showed that a \(\Sigma_t\) with a “bump” leads to a \(\Delta \rho\) which is considerably smaller than expected from the pole mass. For a fixed experimental

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\[9\]Which must be settled at very high scales in order to be compatible with experimental limits on Flavour Changing Neutral Currents (FCNC).
value of $\Delta \rho^{\text{exp}}$ this would imply systematically a higher top mass prediction from radiative corrections if such a bump arises naturally. Such a bump can be phrased as a negative contribution to $\Delta \rho$ compared to the Standard Model and since there are only a few known ways to get such a negative contribution to $\Delta \rho$ we would like to show what kind of gap equation could lead to such a scenario. We consider therefore a situation where $\Sigma_t$ arises from the exchange of a boson with mass $M_X$ as indicated in Fig. 4. The full gap equation Fig. 4 is too complicated and therefore one uses the so-called ladder approximation Fig. 5 which can be written as

$$S^{-1}(p) - \not{p} = \int \frac{d^4k}{(2\pi)^4} \Gamma^a S^{-1}(k) \Gamma_a \frac{-i}{(p-k)^2 - M_X^2},$$

(4.1)

where

$$S(p) = \frac{i}{p - \Sigma(p^2)},$$

(4.2)

is the fermion propagator and $\Gamma^a$ the vertex. The index $a$ runs over all Minkowski and internal group indices corresponding to the interaction structure. Angular integration leads to

$$\Sigma(p^2) = C \int_0^\infty dk^2 \frac{\Sigma(k^2)k^2}{k^2 - \Sigma(k^2)^2} K(p^2, k^2, M_X^2), \quad C = -\frac{\Gamma^a \Gamma_a}{(4\pi)^2},$$

(4.3)

with the Kernel

$$K(p^2, k^2, M_X^2) = \frac{2}{(k^2 + p^2 - M_X^2) \left(1 + \sqrt{1 - \frac{k^2p^2}{(k^2 + p^2 - M_X^2)^2}}\right)},$$

(4.4)

where $C$ is a constant which depends only on the strength and group structure of the new interaction.

In this approximation exist a number of simple arguments why $\Sigma$ should have a bump when $M_X \neq 0$:

1. The self–energy graph of the ladder approximation has a resonance like structure at $p^2 = (M_X + m_t)^2$ due to the generation of real particles above that scale. This explains also why there is no bump in QCD for momenta higher than the constituent quark masses.

2. Because of this resonance structure in the complex plane there is a cut on the real axis for momenta $p^2$ higher than $(M_X + m_t)^2$. We demand that $\Sigma$ is analytic at all other points, which is plausible in ladder approximation. If $\Sigma$ does not have zeros in the complex plane, we know from the theory of analytic functions that the maximum of $|\Sigma|$ must be at the boundary. Therefore there must be a bump at the cut since $|\Sigma| \to 0$ for $|p^2| \to \infty$. 

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3. Demanding maximal analyticity one can also use the gap equation in Euclidean space

\[\Sigma(-p^2) = C \int_0^\infty dk^2 \frac{\Sigma(-k^2)k^2}{k^2 + \Sigma(-k^2)^2} K(-p^2, -k^2, M_X^2), \quad (4.5)\]

and one finds

\[\Sigma'(0) = C \int_0^\infty dk^2 \frac{\Sigma(-k^2)k^2}{k^2 + \Sigma(-k^2)^2} \frac{3k^2 + 4M_X^2}{4(k^2 + M_X^2)^3}, \quad (4.6)\]

which is positive, even if \(\Sigma\) has zeros at large \(k^2\). Furthermore \(\Sigma'\) is positive for small \(p^2\) which shows also that there must be a bump.

In solving the gap equation numerically one runs easily into problems because of the slow decrease of the \(\Sigma\)-function(s). The integral equation is best transformed into a discrete eigenvalue problem by dividing the \(k^2\)-axis up to a cutoff into \(n\) intervals or by using a special discrete function space, e.g. a Taylor expansion on the Möbius-transformed \(k^2\)-axis. With this methods we found a critical value for \(C\) which is in good agreement with the bound \(C_{\text{crit}} > 1/4\) derived by T. Maskawa and H. Nakajima [12]. The effects on \(\Delta \rho\) are for reasonable parameters typically 10-20% corrections to the Standard Model value and become biggest when \(m_t\) is of the same magnitude as \(M_X\).

Clearly such a calculation is not exact but gives only a qualitative impression of the magnitude of the effects. In principle one can also calculate the Goldstone Boson decay constants and the \(W\) mass directly. This leads typically to a result which is to small by a factor 2. But this can easily be due to the uncertainty in the asymptotic high energy behaviour of the solutions of such gap equations. In contrast \(\Delta \rho\) does not get big contributions from the asymptotic part because of the strong convergence of the integral in eq. (2.12). Therefore \(\rho\) is not sensitive to the ultra high energy details of \(\Sigma\).

The ladder approximation omits a lot of graphs which could in principle be relevant in the exact gap equation. Important effects could arise for example for the following reasons:

1. The analyticity properties are not obvious such that \(\Delta \rho\) might even be negative.

2. The feedback of a composite Higgs resonance is ignored in this ladder approximation which could even be dominating the gap equation if the top mass (i.e. the Yukawa coupling) is very big. Due to this feedback there could be a bump at \(M_H \approx 2m_t\) allowing a drastically smaller value of \(\Delta \rho\) and therefore a rather high top mass. Such effects could be relevant in a realization of Nambu’s bootstrap idea of electro–weak symmetry breaking. In this case there is further amplification since \(g_t\) at the condensation scale is considerably higher than the on–shell value \(g_t(m_t)\).
quark might therefore condense for a top quark mass which is even 1.5 to 2 times smaller than naive values.

Despite of all the technical uncertainties we believe that a massive strongly coupled gap equation should lead to a “bump” scenario which might e.g. play a role in proposed gauge models of top condensation where a strongly interacting broken gauge group triggers condensation [13, 14].

V. Discussion

We studied effects of dynamical chiral symmetry breaking of the Standard Model on custodial $SU(2)$ violation in the limit where the $U(1)_Y$ coupling $g_1$ vanishes and where only fermion doublets contribute. Under the assumption $\Sigma_i(p^2) \xrightarrow{p^2 \to \infty} 0$ we were able to derive very general results to leading order in $g_2^2$ and – in principle – to arbitrary order in the new strong dynamics by calculating the finite, $\Sigma_i$ dependent $g_{\mu\nu}$ pieces of the vacuum polarization tensor. However, since we do not know the spectrum of the theory under consideration we have to restrict ourself to the leading contribution of a dynamical fermion loop and we ignore possible $g_{\mu\nu}$ contributions from massive bound states. For a given scenario one might assume to know the masses $M_j$ and couplings of bound states and estimate their contributions, but if these states are heavy then their contributions would typically be suppressed by factors of $\Sigma_i^2/M_j^2$. Our results, which apply for an arbitrary $SU(2)_L$ doublet of fermions, are formally similar to the old Pagels Stokar expressions. It turns however out that the difference cannot be explained by the integral identity eq. (2.9) and the difference must find an explanation in the remaining contributions of $\tilde{K}$. Since our result reproduces in the limit of a hard top mass correctly the well known Standard Model $\rho$–parameter we believe that it should be better suited for phenomenological studies.

We emphasized that the three observables $m_t = \Sigma_t(m_t^2)$, $\Delta \rho$ and one of the Goldstone Boson decay constants have different sensitivities to details of $\Sigma_t$. This implies that the uncertainties which are introduced via truncations made to obtain approximate solutions of $\Sigma_t$ enter in different ways. For example in numerical simulations of the problem the asymptotic high energy tail of $\Sigma_t$ turns out to be very unstable. This implies that $m_W/m_t$ is very unstable due to the logarithmic sensitivity of this ratio to the high energy details (for a Technicolor example of this statement see for example [13]). Contrary $\Delta \rho$ is very insensitive to the high energy tail.

We showed that in general it is possible to obtain negative and positive corrections to $\Delta \rho$ compared to the result of a hard, constant top mass. Negative contributions to $\Delta \rho$ are
usually hard to obtain and we discussed therefore somewhat the type of gap equation that could systematically lead to such negative corrections. This lead to what we called “bump” solutions for $\Sigma_t$ which might be relevant in gauge models of top condensation or some sort of electro–weak bootstrap.

We studied custodial $SU(2)$ violations in terms of $\Delta \rho = \alpha (T - T_0)$ which is less sensitive to model details than other electro–weak observables like $S$, $U$ or the $Zb\bar{b}$ vertex. Note however, that the $m_t$ dependence of all of these quantities is dominated by infrared loop momenta. For given $\Sigma_t$ all these observables should therefore initially be consistent with one single, constant top mass very close to the pole mass. Only when the precision is increased it may be possible to measure the contributions of structure in $\Sigma_t$ to these observables. In this context it should also be mentioned that structure in $\Sigma_i$ at some scale $\Lambda$ can also be understood as a synonym for contributions due to new particle states above the threshold $\Lambda$.

Remarkably there are some completely model independent conclusions. First we remark that for analytical functions $\Sigma_i$ and the absence of poles in the first quadrant $\Delta \rho$ can receive only positive contributions. This can be seen by rewriting eq. (2.12) in Euclidean space with a positive integrand:

$$\Delta \rho = \frac{N_c}{32\pi^2 v^2} \int_0^\infty dk^2 \left( \frac{k^2 (\Sigma_1^2 - \Sigma_2^2)}{(k^2 + \Sigma_1^2)(k^2 + \Sigma_2^2)} \right)^2. \quad (5.1)$$

This positivity may in principle be arbitrarily weak and does especially not forbid that $\Delta \rho$ is smaller than in the Standard Model. An example which illustrates this point was given by the “bump” solution. In terms of the variable $T$ this implies that fermionic contributions can only lead to $T > T_0 \simeq -0.7$ whatever the details of the model are.

Next there are further general features of the corrections to $\Delta \rho$ even without a specific theory. These are – like in the case of the Technicolor example – multiplicative corrections to the Standard Model value of $\Delta \rho$ which are either counting with appropriate weights the number of involved fermions and/or terms $m_t^2/\Lambda^2$ which are sensitive to structure in $\Sigma_t$. These days it is often said that Technicolor is phenomenologically in trouble due to the $S$ parameter. We would like to emphasize that the $T$ parameter will soon become much more important due to the $4/9 N_{TC}$ correction in eq. (3.10). This term which counts extra fermions will essentially be forbidden if the lower top mass limits increase further. In a more general context this fermion counting depends of course on the way how the gap equations are coupled. Typically there are corrections which count the fermionic degrees of freedom and the weight should not be very tiny. It is however possible to build models where this counting is completely absent. In that case there are only $m_t^2/\Lambda^2$ corrections
due to structure in $\Sigma_t$. But even then one can make interesting conclusions by precision comparisons of theory and experiment. If the top mass were discovered somewhat outside the actual Standard Model window then this could be due to structure in $\Sigma_t$ which is (besides other possibilities like Higgs triplets, $W'$ and/or $Z'$ etc.) another important way to bring experimental and theoretical values of $\Delta \rho$ in agreement. On the other side the absence of any mismatch can be used to limit contributions of new physics – including fermion counting and the scale where structure could show up. If one assumes for example the absence of fermion counting and for $\Sigma_t$ the ansatz eq. (3.1) then the mismatch between the Standard Model expectation $m_{t,SM}$ from radiative corrections and the physical top mass $m_t$ translates into a bound for $\Lambda$

$$\Lambda^2 \lesssim \frac{m_{t,phys}^4}{m_{t,SM}^2 - m_{t,phys}^2}. \quad (5.2)$$

This bound is shown in Fig. 5 as dashed line and a comparison of the theoretical predicted top mass with its experimental value to 5 GeV would imply sensitivity to scales $\Lambda \simeq 0.5$ TeV. For arbitrary shapes of $\Sigma_t$ such a bound does of course strictly speaking not exist, but without fine–tuning of the shape one will always find a similar bound. If we take for example the numerical solutions of the gap eq. (4.1) and identify $\Lambda \simeq M_X$ then we obtain the even more interesting solid line of Fig. 6. But it will be hard to find such a deviation as long as the Higgs and top mass are not known precise enough. For the future it is however conceivable that the top mass is known very precisely from the $tt$ threshold, that the Higgs mass is roughly known (or at least stronger bounded) and that the theoretical precision of radiative corrections is a small part of a percent. In that case it is possible to come to $\Delta m_t$ values below 1 GeV or even a few hundred MeV which would probe extremely interesting $\Lambda$ values.

Acknowledgements: We would like to thank H.G. Dosch, D. Gromes, P. Haberl, N. Krasnikov, D. Ross and B. Stech for useful discussions.
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## Figure Captions

1. The graphs which are responsible for the leading $m_t$ dependence of $\Delta \rho$ in the Standard Model.  

2. The $W$ propagator in leading order $g_2^2$ and exact in the new non-perturbative interactions. Fermionic self-energies are represented as fat dots and the four-fermion Kernel $K$ is represented by a fat circle. In the second line the Kernel is split into Goldstone Boson contributions (which arise due to the broken global symmetries with some non-trivial vertex function) and $\tilde{K}$ (which has no further massless poles).  

3. Example for a $\Sigma_t(p^2)$ with $\Delta \rho \equiv 0$. Note that a rather mild “bump” can already result in big corrections. The on-shell mass is given by the intersection with the dashed line $\Sigma(p^2) = p$.  

4. The massive gap (Schwinger Dyson) equation involves the exact contribution from the new interactions carried by a heavy $X$ boson (with unspecified quantum numbers). The full one particle irreducible fermionic self-energy is given in terms of the full propagators and the vertex function.  

5. Ladder approximation of Fig. 4.  

6. Limit on $\Lambda$ from precision measurements of $m_t$. Shown are the scales $\Lambda$ which are tested by a precision comparison of $m_t$ as predicted from radiative corrections with the physical top mass. The dashed line represents eq. (5.2) while the solid line was obtained from our numerical simulations of eq. (4.1) where we identify $\Lambda = M_X$.  


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