Half-skyrmion and meron pair in spinor condensates

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We propose a simple experimental scheme to generate spin textures in the ground state of interacting ultracold bosonic atoms loaded in a two-dimensional harmonic trap. Our scheme is based on two co-propagating Laguerre-Gauss laser beams illuminating the atoms and coupling two of their internal ground state Zeeman sublevels. Using a Gross-Pitaevskii description, we show that the ground state of the atomic system has different topological properties depending on the interaction strength and the laser beam intensity. A half-skyrmion state develops at low interactions while a meron pair develops at large interactions.

I. INTRODUCTION

Because of their ability to materialize abstract theoretical models into carefully designed and controlled experiments, ultracold quantum gases have successfully pervaded many diverse fields of physics ranging from lattice and spin systems, quantum information, quantum simulators, to gauge fields and Anderson localization to cite a few [1]. This is particularly true in the condensed-matter realm where they became a key player in many-body physics as exemplified by the first observation of the Mott-superfluid transition [2, 4].

In recent years, the physics of the quantum Hall effects has become one important focus of the ultracold atoms community. Because atoms are neutral, one needed effective schemes to mimic the action of a magnetic field. A first idea was to set quantum gases into rapid rotation [5] but it faded away because more promising alternatives using light-atom coupling were quickly proposed and experimentally studied [6, 12]. A large variety of Hamiltonians, including non-Abelian ones, either in lattices or in the bulk [13, 19], have been now proposed to mimic magnetic field configurations like artificial Dirac monopoles [17, 18], spin-orbit (SO) coupling [19, 21] or topological phases [22, 23]. For instance, for atoms loaded in a square optical lattice, SO coupling leads to highly nontrivial properties like ground states breaking time reversal invariance and/or magnetic textures with topological properties, like a skyrmion crystal [24, 26]. Such skyrmionic structures have been experimentally observed in excitations of cold atomic gases [27], but not yet in the ground state. From a theoretical point of view, some papers have proposed to generate these topological configurations with cold atomic gases either in transient excitations, which decay eventually to a non-topological configuration [28], or directly in the ground state [29, 30]. However the actual experimental implementation of the latter proposal remains quite challenging.

In the present paper, we provide a rather simple experimental set-up to generate spin textures in the ground state of interacting ultracold bosonic atoms loaded in a two-dimensional harmonic trap. Our scheme is based on two co-propagating Laguerre-Gauss laser beams illuminating the atoms and coupling two of their internal ground state Zeeman sublevels. At the mean field level, i.e. using a Gross-Pitaevskii description, we show that the ground state of the atomic system has different topological properties depending on the interaction strength and the laser beam intensity. A half-skyrmion state, also known as a Mermin-Ho vortex [31], develops at low interactions while a meron pair develops at large interactions.

In the following, we first introduce our model and its effective Hamiltonian, then we briefly present the essential properties of the single particle states. Next, we analyse the topological properties of the ground state in the weak interaction limit. Finally, we show that at large interaction there is a transition to a ground state made of a vortex-antivortex pair separated by a finite distance. The separation between the two opposite vortices vanishes at the transition and increases with the interaction.

II. MODEL HAMILTONIAN

A. Experimental Setup

We consider here bosonic ultracold atoms with 3 internal ground state levels, for instance the F = 1 states of 87Rb. We assume the atoms are harmonically-trapped in the two-dimensional plane (Ox,Oy) and tightly-confined in the third direction Oz (chosen as the quantization axis) so that the atomic dynamics is effectively two-dimensional. The atoms are further subjected to a static magnetic field along Oz splitting the Zeeman degeneracy and are illuminated by two far-detuned laser beams (blue detuning) co-propagating along Oz with opposite circular polarizations. These two laser beams create a resonant Raman coupling between the Zeeman sublevels...
state manifold reads \[32\]:

\[
\text{dynamics in the (positive (blue-detuning). The atoms are tightly confined in}
\]

is thus barely populated. The detuning \(\Delta\) is here assumed

two-photon transitions through an intermediate excited state
\(|e\rangle\) which is non-resonantly coupled to the ground state and

is thus barely populated. The detuning \(\Delta\) is here assumed

positive (blue-detuning). The atoms are tightly confined in

the \(Oz\) direction and harmonically trapped in the transverse

plane \((Ox, Oy)\).

\[
m_F = \pm 1, \text{ see Fig. 1 (\(\Lambda\) scheme). In the rotating-wave}
\]

approximation, and after adiabatic elimination of the excited

state, the effective 2 \(\times\) 2 Hamiltonian describing the
dynamics in the \((Ox, Oy)\) plane for the \(m_F = \pm 1\) ground

state manifold reads \[32\]:

\[
H_e + \left( \frac{p^2}{2m} + \frac{1}{2} m \omega^2 r^2 \right) 1 + \frac{1}{4\Delta} \left( \frac{\Omega_1^2}{\Omega_1 \Omega_2^*} |\Omega_2|^2 \right)
\]

(1)

where \(m\) is the mass of the atoms, \(\omega\) the harmonic trapping

frequency and \(r = \sqrt{x^2 + y^2}\) the radial distance in

the plane and where we have used the pseudo-spin representation \(|\downarrow\rangle \equiv |m_F = -1\rangle\) and \(|\uparrow\rangle \equiv |m_F = 1\rangle\). In the

specific case of Laguerre-Gauss beams \[27, 33\] with equal

(real) strength \(\Omega_0\) and carrying opposite orbital angular

momentum \(\pm \hbar\), the respective Rabi frequencies read:

\[
\Omega_1 = \Omega_0 \frac{r}{R} e^{i(kz+\varphi)} \quad \Omega_2 = \Omega_0 \frac{r}{R} e^{i(kz-\varphi)}
\]

(2)

where \(R\) is the size of the "doughnut" core, \(k\) the laser

wave number and \(\varphi\) the polar angle of vector \(r = (x, y)\).

We assume here that the transverse size of the laser

beams is much larger than the atomic cloud.

In the following, we use the harmonic oscillator quantum

of energy \(\hbar \omega\), the harmonic length \(a_0 = \sqrt{\hbar/m \omega}\)

and \(\hbar/a_0\) as energy, space and momentum units. We

also denote the usual Pauli spin matrices by \(\sigma_x, \sigma_y\) and

\(\sigma_z\). The dimensionless single-particle effective Hamiltonian

then reads:

\[
H_0 = \left( \frac{1}{2} p^2 + \frac{1}{2} \mu^2 \right) \mathbb{1} + \frac{1}{2} \Omega^2 \sigma^2 \left( \frac{1}{2} - e^{2i \varphi} \right)
\]

(3)

with \(\Omega^2 = \Omega_0^2/(2m \omega^2 R^2 \Delta)\) and where \(\mathbb{1} = -i \mathbf{\nabla}\).

As easily checked, this Hamiltonian is invariant under

a combined space and spin rotation, namely \(H_0 = R(\varphi_0) H_0 R^\dagger(\varphi_0)\) where \(R(\varphi_0) = e^{i \varphi_0 (L_x + \sigma_x)}\) is the operator

associated to a rotation by an angle \(\varphi_0\) around \(Oz\)

both in coordinate and spin space. Here \(L_x = -i \partial/\partial \varphi\) is

the orbital angular momentum operator around \(Oz\).

Applying the unitary transformation \(U(\varphi) = e^{i \varphi \sigma_z}\), one gets

the unitary-equivalent Hamiltonian \(\tilde{H}_0 = U H_0 U^\dagger\) with:

\[
\tilde{H}_0 = \frac{1}{2} (p^2 + \frac{\hbar^2}{r} \sigma_z)^2 + \frac{1}{2} (1 + \Omega^2) \mathbb{1}^2 + \frac{1}{2} \Omega^2 r^2 \sigma_x.
\]

(4)

In this new gauge, \(\tilde{H}_0\) can be viewed as the Hamiltonian

of a particle subjected to the artificial gauge potential \(A = -\frac{1}{2} \sigma_z \mathbb{1}\) \[34\] associated to two infinite strings carrying

opposite magnetic fluxes \(\Phi = \pm 2\pi\), one along the

positive \(Oz\) axis, the other one along the negative \(Oz\) axis.

The corresponding magnetic field is simply given by \(B = 2\pi \delta(r) e_z \sigma_z\).

B. Single-particle eigenstates

Since \(H_0\) is invariant under a combined spin and space

rotation, its spinor eigenstates in the pseudo-spin basis

\((|\downarrow\rangle, |\uparrow\rangle)\) have the general structure:

\[
\phi_m(r) = \left( \phi_m^\dagger(r) \right) = \left( \begin{array}{c} f_m(r) e^{-i \varphi} \\ g_m(r) e^{i \varphi} \end{array} \right) e^{im \varphi} \frac{\sqrt{2\pi}}{\sqrt{\mathcal{E}_m}}
\]

(5)

where \(m\) is an integer. Inspection of the coupled

Schrödinger equations for the eigenstates shows that both

radial functions \(f_m(r)\) and \(g_m(r)\) can be chosen real. \(H_0\) is also invariant under the operator \(T = \sigma_x C, \quad H_0 = T H_0 T^{-1}\), where \(C\) represent complex conjugation.

This implies that both \(\phi_m(r)\) and \(T \phi_m(r)\) are eigenstates

of \(H_0\) with the same eigenenergy \(\mathcal{E}_m(\Omega)\).

Since \(T \phi_m(r) = \pm \phi_{-m}(r)\), we have \(g_m = \pm f_{-m}\) and we can

restrict the analysis to the \(m \geq 0\) sector. Noting that

\(\phi_m(r)\) and \(T \phi_m(r)\) are orthogonal spinors when \(m \neq 0\),

we conclude that their corresponding eigenenergy is dou-

bly degenerate when \(\Omega > 0\).

Fig. 2 displays the two lowest eigenenergies of \(H_0\) as a

function of the dimensionless Rabi frequency \(\Omega\). Below

\(\Omega \approx 3.35\), the ground state manifold is doubly de-

generate and is spanned by the two spinor states \(\phi_1(r)\) and

\(T \phi_1(r)\). We find that \(f_1(r)\) reaches a finite value at

the origin \(r = 0\) while \(g_1(r)\) vanishes. This means that

the spin-down component of \(\phi_1(r)\), \(g_1(r)\) \(\exp(2i \varphi)\),

depicts a vortex with vorticity equal to \(2\) while the spin-up

component of \(T \phi_1(r)\), \(g_1(r)\) \(\exp(-2i \varphi)\), depicts the

opposite vortex. A convenient parametrization of the spinor

proves to be \(\phi_1(r) = \sqrt{n_1(r) \chi_1(r)}\) where

\[
\chi_1(r) = \left( \begin{array}{c} x_{\uparrow 1}(r) \\ x_{\downarrow 1}(r) \end{array} \right) = \left( \begin{array}{c} -\cos \frac{\beta(r)}{2} \\ \frac{\beta(r)}{2} \end{array} \right) e^{i2 \varphi \sin \frac{\beta(r)}{2}}
\]

(6)
and \( n_{1}(r) = f_{1}^{2}(r) + g_{1}^{2}(r) \) is the total density. As \( n_{1}(r) \) is finite at the origin and \( g_{1}(r) \) vanishes, we must have \( \beta(0) = 0 \). We also find \( \beta(\infty) = \pi/2 \) corresponding to \( n_{r} = n_{\perp} \) at large distances. A configuration satisfying such boundary conditions is known as a Mermin-Ho vortex \cite{31}, also called a half-skyrmion since one has \( \beta(\infty) = \pi \) for a "full" skyrmion. The local spin texture is defined by

\[
S(r) = \chi_{1}(r) \sigma \chi_{1}(r) = -\sin \beta(r)(\cos 2\varphi \hat{e}_{x} + \sin 2\varphi \hat{e}_{y}) + \cos \beta(r) \hat{e}_{z}
\]

with modulus \( |S(r)| = 1 \). It characterizes a 2D Skyrmion with topological charge \cite{37}

\[
Q = \int d^{2}r q(r) = \int d^{2}r \epsilon^{ij} S \cdot (\partial_{i} S \times \partial_{j} S) / 8\pi
\]

where \( i, j = x, y \) and where \( \epsilon^{ij} \) is the antisymmetric tensor. Using the parametrization given by Eq. (7), the topological charge density is simply

\[
q(r) = \epsilon^{ij} S \cdot (\partial_{i} S \times \partial_{j} S) / 8\pi = -\frac{1}{2\pi r} \frac{d \cos \beta(r)}{dr}.
\]

We thus find that, for \( \Omega < \Omega_{c} \), the topological properties of the ground state spin texture of our non-interacting system are described by a Mermin-Ho vortex with unit topological charge \( Q = \cos \beta(0) - \cos \beta(\infty) = 1 \).

Above \( \Omega_{c} \), the ground state manifold is non degenerate and the eigenstate is now the spinor \( \phi_{0}(r) \) where \( g_{0}(r) = -f_{0}(r) \). Since \( f_{0}(r) \) vanishes at the origin, we see that the two spin components, \( \phi_{0\uparrow}(r) = f_{0}(r)e^{-i\varphi} \) and \( \phi_{0\downarrow}(r) = -f_{0}(r)e^{i\varphi} \), describe opposite vortices with unit vorticity.

### III. INTERACTING BOSONS

We assume here that the atoms in the \( m_{F} = \pm 1 \) Zeeman states interact through a fully \( SU(2) \)-symmetric interaction and are not coupled to the \( m_{F} = 0 \) state. The corresponding second-quantized Hamiltonian reads

\[
H_{\text{int}} = \frac{g}{2} \int d^{2}r \Psi_{b}^{\dagger} \Psi_{b} \Psi_{a} \Psi_{a}^{\dagger}
\]

where \( g \) is the dimensionless interaction strength and where summation over the dummy pseudo-spin indices \( a \) and \( b \) is understood. Here \( \Psi_{a} \) and \( \Psi_{b} \) stand for the creation and destruction operators of a particle at point \( r \) in spin component \( a = \uparrow, \downarrow \). They satisfy the usual bosonic commutation relations \( [\Psi_{a}, \Psi_{b}^{\dagger}] = \delta_{ab} \). We next assume that, in the zero temperature limit, all the bosons condense into a single spinor coherent state \( \Phi(r) \) with spin components \( \Phi_{\uparrow}(r) \) and \( \Phi_{\downarrow}(r) \) and we describe the interacting system within a mean-field approach. The Gross-Pitaevskii (GP) energy functional reads

\[
E[g, \Phi(r)] = \int d^{2}r \left[ \Phi^{\dagger} H_{0} \Phi + \frac{g}{2} (\Phi^{\dagger} \Phi)^{2} \right]
\]

where \( \Phi^{\dagger} \Phi = n(r) = n_{\uparrow}(r) + n_{\downarrow}(r) = |\Phi_{\uparrow}(r)|^{2} + |\Phi_{\downarrow}(r)|^{2} \) is subjected to the normalization condition \( \int d^{2}r n(r) = 1 \).

#### A. Weak interaction regime

1. **Case \( \Omega < \Omega_{c} \)**

   In the limit \( g \to 0 \), only states with an energy separation \( |\beta E| \lesssim g \) or lower, are efficiently coupled. Therefore, in first approximation, we expect the ground state \( \phi_{g}(r) \) to belong to the single-particle ground state manifold. We thus look for the simple ansatz \( \phi_{g}(r) = \alpha \phi_{\uparrow}(r) + \beta \phi_{\downarrow}(r) \), where the minimization parameters \( \alpha \) and \( \beta \) are two constant complex numbers satisfying \( |\alpha|^{2} + |\beta|^{2} = 1 \). It is easy to check that the corresponding GP energy functional is always larger than the one computed with \( \phi_{\uparrow}(r) \) alone (which is also equal to that computed with \( \Phi_{\uparrow}(r) \) alone). This means that, when \( g \to 0 \), the \( T \)-symmetry is spontaneously broken: \( \phi_{g}(r) = \phi_{\uparrow}(r) \) (\( \alpha = 1 \)) or \( \phi_{g}(r) = \Phi_{\uparrow}(r) \) (\( \beta = 1 \)). The spin texture associated to this weakly-interacting GP ground state is a Mermin-Ho vortex with unit topological charge. We have numerically computed the exact GP ground state and checked that the previous ansatz provides a qualitatively correct picture at small values of the interaction strength \( g \). For instance, the density profiles \( n(r) \), \( n_{\uparrow}(r) \), \( n_{\downarrow}(r) \) and the topological charge density \( q(r) \) of the exact GP spinor ground state are displayed in Fig. 3 for \( g = 0.1 \) and \( \Omega = 2 \). One can clearly see that \( n_{\uparrow} \) remains finite whereas \( n_{\downarrow} \) vanishes at the center of the trap; in addition, the ground state depicts a non-trivial topological charge density, with a total topological charge \( Q = \int d^{2}r q(r) = 1 \).
This emphasizes that the GP ground state has the same topology as a Mermin-Ho vortex with unit topological charge.

Starting from one of the single-particle ground states selected in the limit \( g \to 0 \), we now increase the interaction strength \( g \). Spinor \( \phi_0 \) can no longer be ignored now, especially when \( \Omega \) is close to \( \Omega_c \) and \( \phi_1 \), \( T\phi_1(r) \) and \( \phi_0 \) are almost degenerate. An updated ansatz simply reads \( \phi_2(r) = \alpha \phi_1(r) + \beta T\phi_1(r) + \gamma \phi_0(r) \) with constant complex parameters satisfying \( |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1 \). We find that there exists a critical interaction strength \( g_c(\Omega) \) such that \( \phi_2 = \phi_1 \) (or \( T\phi_1 \)) when \( g < g_c \) and \( \phi_2 = \phi_0 \) when \( g > g_c \). The reason for this phase transition is that the spinor \( \phi_0 \) carries less interaction energy than \( \phi_1 \) and \( T\phi_1 \). Indeed its total density \( n_0(r) \) is vanishing at the trap center whereas the total density \( n_1(r) \) is maximum there. At the mean-field level, this transition is first order since the states have different vorticities.

A rough estimate of the critical interaction strength \( g_c \) is obtained for each \( \Omega \) by equating the GP energy functionals computed with these single-particle ground states, namely \( (\epsilon + g_c V_1) \) and \( (\epsilon_0 + g_c V_0) \), where \( V_m = \frac{1}{4\pi} \int r dr n_m^2(r) \) \( (m = 0, 1) \). The result is shown in Fig. 4 (dashed line). One may notice that this predicted \( g_c \) is not really weak unless \( \Omega \) is very close to \( \Omega_c \). This means that approximating the true GP ground state by one of the single-particle states becomes questionable. A more accurate estimate is obtained as follows. For each value of \( \Omega \), we compute, in each sector \( (i.e. m = 0 \) or \( m = 1) \), the GP ground states for the interaction \( g_c \) computed above. In practice, this is done by running the imaginary time evolution algorithm, starting from either the single-particle states \( \phi_0 \) or \( \phi_1 \). The invariance of the GP equation under a combined spin and space rotation ensures that the imaginary time evolved state always remains inside the chosen symmetry sector. We then compute the GP functionals at interaction strength \( g \) in each sector with these improved ground states and we find the new improved critical interaction strength by equating them. The result is shown in Fig. 4 (continuous line) and is in very good agreement with the exact value for \( g_c \) obtained by monitoring the symmetry and the topological properties of the ground state (obtained by globally minimizing the GP energy functional) as a function of \( g \) (star symbols).

In this case, the single-particle ground state is \( \phi_0 \) and it qualitatively describes the properties of the GP ground state in the weakly-interacting regime. This is confirmed by our exact numerical results which show that the GP ground state indeed hosts a vortex in each of its components, but with opposite vorticity.

### B. Strong interaction regime

In the strong interacting regime, higher single-particle states are coupled and no simple ansatz can be made. In this case we obtain the interacting ground state \( \phi_q \) by direct minimization of the GP energy functional \( Eq. \( 11 \). This is achieved by imaginary-time evolution of the corresponding GP equation. Fig. 5 shows the ground state density of the up and down components and their relative phase \( (\theta_1 - \theta_2) \) when \( g = 100 \) and \( \Omega = 4 \). As one can see, each component density vanishes at an off-centered location, at which the other component reaches its maximum,
Two segments \[ \int_{-\infty}^{x_m} \] two components exhibits two clear 2\( \pi \) jumps along the two segments \[ | -\infty, -x_m \rangle \] and \[ | x_m, +\infty \rangle \] on axis \( Ox \).

To gain further insight, we introduce again the pseudospin representation and decompose the GP spinor components as \( \phi_{g,a} = \sqrt{n} \chi_a \) with \( \chi_a = |\chi_a| e^{i\theta_a} \)\( \alpha = \uparrow, \downarrow \). The total density is \( n = |\phi_{\uparrow}|^2 + |\phi_{\downarrow}|^2 \) and the spinor \( \chi \) thus satisfies \( |\chi_{\uparrow}|^2 + |\chi_{\downarrow}|^2 = 1 \). The corresponding local spin texture \( S = \chi^\dagger \sigma \chi \) reads
\[
S_{x} = 2|\chi_{\uparrow}| |\chi_{\downarrow}| \cos(\theta_{\uparrow} - \theta_{\downarrow}) \\
S_{y} = 2|\chi_{\uparrow}| |\chi_{\downarrow}| \sin(\theta_{\uparrow} - \theta_{\downarrow}) \\
S_{z} = |\chi_{\uparrow}|^2 - |\chi_{\downarrow}|^2.
\]
and has unit modulus \( |S| = 1 \). This local spin is parallel to axis \( Oz \), namely \( S = e_z \) (resp. \( S = -e_z \)) at space points where \( n_{\uparrow} \) (resp. \( n_{\downarrow} \)) vanishes. The relative phase \( (\theta_{\uparrow} - \theta_{\downarrow}) \) is undefined at these two points and they correspond to a vortex-antivortex pair. These properties appear clearly in Fig. 4, where the spin components \( (S_x, S_y) \) are plotted in the plane \( (Ox, Oy) \). Writing \( r = (-x_m + \delta x, \delta y) \), we find \( (S_x, S_y) \propto (\delta x, \delta y) \) around the left vortex \( S = -e_z \). By the same token, writing \( r = (x_m + \delta x, \delta y) \), we find \( (S_x, S_y) \propto (\delta x, -\delta y) \) around the right vortex \( S = +e_z \). The topological charge density, computed with Eq. (8), is displayed in Fig. 6. It emphasizes that the GP ground state spinor depicts two vortices with opposite vorticity (with respect to \( S_z \)), such that the total topological charge vanishes.

In the pseudospin representation \[ 30,38 \], the GP energy functional reads
\[
E = \int d^2 r \left[ \frac{1}{2} (\nabla \sqrt{n})^2 + \frac{n}{8} (\nabla S)^2 + \frac{n}{2} v_e^2 + \frac{\Omega^2}{2} n^2 + \frac{1}{2} \Omega^2 r^2 (I + S_x \cos 2\varphi + S_y \sin 2\varphi) + \frac{g}{2} n^2 \right],
\]
where
\[
(\nabla S)^2 \overset{\text{def}}{=} (\nabla S_x)^2 + (\nabla S_y)^2 + (\nabla S_z)^2.
\]
The effective velocity field is given by \[ 30,38 \]
\[
v_e = \frac{1}{2} \left[ \nabla \Theta + \frac{S_x (S_y \nabla S_x - S_x \nabla S_y)}{S_x^2 + S_y^2} \right].
\]
and depends on the gradient of the total phase $\Theta = \theta_1 + \theta_2$ and of the pseudo-spin. In analogy with the meron pair solution discussed in [28, 30, 31], we parameterize the spin texture as follows:

$$\begin{align*}
S_x &= -r^2 \cos 2\varphi + \lambda^2 e^{-\alpha r^2} \\
S_y &= -r^2 \sin 2\varphi \\
S_z &= -2 \lambda e^{-\alpha r^2} r \cos \varphi.
\end{align*} \tag{16}$$

The usual meron pair parametrization is obtained for $\alpha = 0$. The corresponding topological charge density is

$$q(r) = -\frac{\mu x}{\pi (r^2 + \mu^2)^2} - \alpha \frac{\mu x r^2}{\pi (r^2 + \mu^2)^2}, \tag{17}$$

where $\mu = \lambda e^{-\alpha r^2/2}$. The vortex-antivortex nature of the meron pair results in a topological density $q(r)$ which is an odd function of coordinate $x$, see Fig. [6]. As a consequence, the total topological charge is $Q = \int dr q(r) = 0$.

The spin texture Eq. (16) corresponds to the GP spinor condensate:

$$\phi_\uparrow = \sqrt{\frac{n}{2}} \frac{\mu - re^{-i\varphi}}{\sqrt{r^2 + \mu^2}} \quad \phi_\downarrow = \sqrt{\frac{n}{2}} \frac{\mu + re^{i\varphi}}{\sqrt{r^2 + \mu^2}}. \tag{18}$$

The meron pair is polarized along axis $Ox$ due to the $\sigma_x$-term in Eq. (4) which describes an effective magnetic field along $Ox$. The locations of the two vortex cores are determined by the two extremas of $S_z$. They are found at $((\pm x_m, 0))$ where [29]

$$x_m^2 = \lambda^2 e^{-\alpha x_m^2}. \tag{19}$$

The relative phase is given by

$$e^{i(\phi_\downarrow - \phi_\uparrow)} = \frac{\mu^2 - r^2 e^{2i\varphi}}{\sqrt{(r^2 + \mu^2)^2 - 4\mu^2 r^2 \cos^2 \varphi}}, \tag{20}$$

and is singular at the two vortex cores $(\pm x_m, 0)$. Writing $(x, y) = ((\pm x_m + \delta x, \delta y))$, a first-order expansion gives $\theta_1 - \theta_1 = \delta \varphi + \pi$ around $(x_m, 0)$ and $\theta_2 - \theta_1 = \delta \varphi$ around $(-x_m, 0)$, where $\delta \varphi = \arctan(\delta y/\delta x)$ is the local polar angle. When circling around each vortex core, the accumulated relative phase is $2\pi$. Similarly, in the large distance limit $r \gg x_m$, the relative phase is $\theta_2 - \theta_1 = 2\varphi + \pi$ and a full loop around the two vortices generates a total phase change of $4\pi$. This is slightly different from the usual meron pair situation [29], where the relative phase reaches a constant value at large distance, which corresponds to a spin texture pointing in a fixed direction. In the present case, $(S_x, S_y) \approx (\cos 2\varphi, -\sin 2\varphi)$. This difference explains why, in the present situation, the phase jumps happen on each outer side of the meron pair and not in between the two vortices, see Fig. [5]. Apart from this, the GP ground state properties are similar to those of the meron pair already studied in a double-layer quantum Hall system [29, 32].

By fitting our numerical data with ansatz Eq. (16), we have determined the parameters $\lambda$ and $\alpha$ as a function of $\Omega$ and $g$. The results are shown in Fig. 7 for $\Omega = 4$. One can clearly see a phase transition happening at $g \approx 20$. Below, the GP ground state exhibits topological properties similar to the $m = 0$ single-particle state. Above, the GP ground state describes a meron pair with two off-centered and opposite vortices. It means that the energy cost to separate and shift away the vortex cores is less than the interaction energy. Above the transition point, the value of $\lambda$ increases with $g$, which means that the size $2x_m$ of the meron pair increases. Finally, from the pseudo-spin point of view, the transition occurs between a uniformly vanishing $S_z(r)$ component and a well-defined structure $S_z(r)$. Therefore we expect the spin susceptibility along axis $Oz$ to diverge at the transition and the phase transition is second order.

![Figure 7](image_url)

**Figure 7.** The size $2x_m$ of the meron pair as a function of $g$ for $\Omega = 4$. The system exhibits a second-order phase transition at $g \approx 20$ between a ground state with topological properties similar to the $m = 0$ single-particle state and a meron pair with two opposite and off-centered vortices.

**IV. CONCLUSION**

In this paper we have proposed an experimental scheme leading to non-trivial spin textures in the interacting ground state of a two-component spinor condensate. More precisely, we have shown that a second-order phase transition occurs between a Mermin-Ho vortex and a meron pair when the interaction strength increases. A possible extension of the work is to study the excitations of the system and their topological properties. Finally, from an experimental point of view, $F = 1$ spinor condensates have also an effective spin-spin interaction $g_2 S_z^2/2$. This interaction term breaks the $SU(2)$ invariance and converts a pair of bosons in the $|m_F = -1\rangle$ and $|m_F = +1\rangle$ spin states into a pair of bosons in the $|m_F = 0\rangle$ spin state. In the present situation, these collision processes correspond to losses. In the case of $^{87}$Rb, fortunately $g_2 < g$ and one should be able to observe the...
Mermin-Ho vortex and the meron pair before the effect of spin-spin interaction sets in. An alternative would be to lift the energy of the $|n_F = 0\rangle$ spin state and suppress the detrimental pair conversion processes by rendering them energetically less favorable.

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