The Korteweg-de Vries (KdV) equation with step boundary conditions is considered, with an emphasis on soliton dynamics. When one or more initial solitons are of sufficient size they can propagate through the step; in this case the phase shift is calculated via the inverse scattering transform. On the other hand, when the amplitude is too small they become trapped. In the trapped case the transmission coefficient of the associated linear Schrödinger equation can become large at a point exponentially close to the continuous spectrum. This point is referred to as a pseudo-embedded eigenvalue. Employing the inverse problem it is shown that the continuous spectrum associated with a branch cut in the neighborhood of the pseudo-embedded eigenvalue plays the role of discrete spectra, which in turn leads to a trapped soliton in the KdV equation.

Most research associated with the KdV equation has been posed on spatial domains with either decaying or periodic boundary values. There is, however, an important related problem, which is eq. (1) subject to step BCs

$$\lim_{x \to -\infty} u = 0, \quad \lim_{x \to +\infty} u = \pm c^2,$$

(3)

where $c > 0$ is constant and $u$ goes to these limits sufficiently fast; we require that

$$\int_{-\infty}^{\infty} |u(x,t) + c^2 H(x)(1 + x^2)| dx < \infty,$$

(4)

where $H(x)$ is the Heaviside function. We refer to the increasing boundary condition $+c^2$ case as “step up” and the decreasing boundary condition $-c^2$ case as “step down”. Since the KdV equation is Galilean invariant, it suffices to consider $u \to 0$ as $x \to -\infty$; i.e. any nonzero boundary condition $u \to u_0 \neq 0$ as $x \to -\infty$ can be made zero through the transformation $u(x,t) = u_0 + \tilde{u}(x - 6u_0t,t)$.

The step problem has been studied by a number of authors. With step down boundary values, the basic direct/inverse scattering theory of the Schrödinger equation was developed over 50 years ago by Buslaev and Fomin [11]. Their results were later used to discuss the asymptotic behavior of certain solutions to the KdV equation in [12]. Subsequently the problem was studied by a number of authors in [13] and later by [14] with the main aim of developing a rigorous understanding of the direct/inverse scattering of this problem.

In terms of the KdV wave dynamics, pure step down data for $t > 0$ leads to the development of collisionless or dispersive shock waves (DSWs) [15], whereas pure step up data for $t > 0$ leads to the development of a linear ramp from $u = 0$ to $u = c^2$ with small associated oscillations [16]. The asymptotic development of dispersive shock waves due to multi-step down initial data was considered in [20].

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### I. INTRODUCTION

The realization that solitary waves are important dynamical entities goes back to seminal observations and experiments of shallow water waves by Russell [1]. Motivated by these observations Korteweg and deVries [2] discovered an important equation in shallow water waves. Indeed the KdV equation is a universal nonlinear system which arises whenever there is a balance of weak dispersion and quadratic nonlinearity [3].

In particular, they showed that solitons were related to time-independent eigenvalues/bound states of eq. (4) and obtained pure soliton solutions explicitly. The linearization is in terms of a Gel’fand-Levitan-Marchenko (GLM) integral equation which provides the inverse scattering/reconstruction of the solution $u(x,t)$ to eq. (1). This method of solution is now called the Inverse Scattering Transform (IST) and considerable research using these techniques has ensued and continues today cf. [8, 9, 7]. The analytical underpinnings of the direct/inverse scattering problems associated with eq. (2) for decaying boundary data can be found in [4, 10].

### Solitons, the Korteweg-de Vries equation with step boundary values and pseudo-embedded eigenvalues

M.J. Ablowitz,¹ X-D. Luo,² and J.T. Cole¹

¹Department of Applied Mathematics, University of Colorado, Boulder, Colorado 80309
²Department of Mathematics, State University of New York at Buffalo, Buffalo, New York 14260-2900

The Korteweg-deVries (KdV) equation with step boundary conditions is considered, with an emphasis on soliton dynamics. When one or more initial solitons are of sufficient size they can propagate through the step; in this case the phase shift is calculated via the inverse scattering transform. On the other hand, when the amplitude is too small they become trapped. In the trapped case the transmission coefficient of the associated linear Schrödinger equation can become large at a point exponentially close to the continuous spectrum. This point is referred to as a pseudo-embedded eigenvalue. Employing the inverse problem it is shown that the continuous spectrum associated with a branch cut in the neighborhood of the pseudo-embedded eigenvalue plays the role of discrete spectra, which in turn leads to a trapped soliton in the KdV equation.

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The step problem has been studied by a number of authors. With step down boundary values, the basic direct/inverse scattering theory of the Schrödinger equation was developed over 50 years ago by Buslaev and Fomin [11]. Their results were later used to discuss the asymptotic behavior of certain solutions to the KdV equation in [12]. Subsequently the problem was studied by a number of authors in [13] and later by [14] with the main aim of developing a rigorous understanding of the direct/inverse scattering of this problem.

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Unlike previous research on the step BC problem, we focus on the dynamical situation when solitons/sech$^2$ profiles in addition to a step are initially given. To be concrete we use as initial data a delta function, a box,
or a soliton/$\text{sech}^2$ profile located well to the left of a step/Heaviside function. The step problem is different from the decaying to zero problem since we do not have ‘pure solitons’, i.e. along with solitons there is always additional continuous spectrum (no reflectionless potential).

In the context of equation 11 there are two types of pulses associated with localized initial data positioned well to the left of a step. The first case is that of a pulse with a large enough amplitude that allows it to pass all the way thorough the step. The amplitude is related to the discrete spectra/eigenvalues. These are ‘proper’ eigenvalues and correspond to zeros of the inverse of the transmission coefficient, i.e. poles of the transmission coefficient. Proper eigenvalues are associated with standard solitons that, as mentioned above, propagate through the step with only a suitable Galilean shift (velocity increase) and phase shift. From the IST we derive the phase shift of the soliton as it passes through the rarefaction ramp that evolves from an initial step up or the DSW in the step down case, with or without other solitons. The phase shift formulae are similar to those derived for the decaying problem with continuous spectrum 22. As an example, we find the phase shift for a single soliton passing through a step. The J-soliton phase shift can be calculated by similar methods (see Appendix C).

However, if the initial localized profile is not large enough we find that the inverse of the transmission coefficient has no zeros i.e. no proper eigenvalues. What we do find in this case are spectral values that are exponentially close to continuous spectra. We term such a point a pseudo-embedded eigenvalue. Such pseudo-embedded eigenvalues are associated with soliton-like pulses which propagate as though they were true solitons for a while, but eventually become trapped inside the rarefaction ramp for the step up initial condition, or the DSW in the step down case; hence they have interesting physical manifestations. In this paper we discuss the step up case; the step down case is similar.

Below we show that in the trapped case the ‘branch cut’ term associated with the continuous spectrum in the the inverse scattering problem leads to a contribution that plays the role of discrete spectra. Said differently, the pseudo-embedded eigenvalue leads to a dominant contribution from the branch cut associated with the continuous spectrum that has exactly the form as discrete spectra from a proper eigenvalue. This term gives rise to a pulse which travels uniformly like a soliton until it encounters the rarefaction ramp/DSW where it eventually becomes trapped. The pseudo-embedded eigenvalue provides a spectral interpretation of the trapped soliton.

We point out that in recent experiments 22 solitons have been transmitted through rarefaction waves (step up) and dispersive shock waves (step down), moreover, it has been shown that small amplitude solitons can become trapped in rarefaction ramp/DSW that develops from step initial data.

In this paper we concentrate on step up BCs; i.e. eq. (3) with the positive sign. The analogous theory can be developed for step down BCs; see also additional remarks in the conclusion of this article. These studies were motivated by lectures by M. Hoefner discussing analytical/experimental research summarized in 22.

II. SCATTERING/INVERSE SCATTERING THEORY AND KDV SOLITONS

The KdV eq. (11) is the compatibility condition (Lax pair) for the following two linear equations

\[ v_{xx} + \left( u(x,t) + k^2 \right) v = 0, \]

\[ v_t = (ux(x,t) + \gamma) v + \left( 4k^2 - 2u(x,t) \right) v_x, \]

where \( k \) is the spectral parameter, \( \gamma \) is a constant; the potential \( u(x,t) \) satisfies the BCs 3 with a plus sign.

A. Eigenfunctions

Eigenfunctions of (15) are defined by the following BCs

\[ \phi(x,k) \sim e^{-ikx}, \quad \bar{\phi}(x,k) \sim e^{ikx} \text{ as } x \to -\infty, \]

\[ \psi(x,\lambda) \sim e^{i\lambda x}, \quad \bar{\psi}(x,\lambda) \sim e^{-i\lambda x} \text{ as } x \to +\infty, \]

where \( k, \lambda \) are real and

\[ \lambda(k) = (k^2 + c^2)^{1/2}. \]

We take the branch cut of \( \lambda(k) \) to be \( k \in [-ic,ic] \), and the branch cut of \( k(\lambda) \) to be \( \lambda \in [-c,c] \); then \( \Im \lambda \geq 0 \) when \( \Re \lambda \geq 0 \) and \( \Im k \leq 0 \) when \( \Im \lambda \leq 0 \). From the governing integral equations, the eigenfunctions \( \phi, \psi \) can be analytically continued into the upper half plane (UHP) of \( k, \lambda \), while \( \bar{\phi}, \bar{\psi} \) can be analytically continued into the corresponding lower half plane (LHP). From eq. (5) and the BCs (7)-(8) we see that the eigenfunctions are related by:

\[ \phi(x,k) = \bar{\phi}(x,-k) = \phi^*(x,k), \]

\[ \psi(x,\lambda) = \bar{\psi}(x,-\lambda) = \psi^*(x,\lambda), \]

for \( \lambda, k \) real and where asterisk represents complex conjugate. When \( k = ik, \kappa \in [-c,c] \)

\[ \phi(x,k) = \phi^*(x,k). \]

B. Scattering data

The two eigenfunctions \( \psi(x,\lambda), \bar{\psi}(x,\lambda) \) are linearly independent for \( k \neq 0 \). Hence, we can write \( \phi(x,k), \bar{\phi}(x,k) \) as a linear combinations of \( \psi(x,\lambda) \) and \( \bar{\psi}(x,\lambda) \). Thus, we have the relations, formulated on the left, termed the left scattering problem

\[ \phi(x,k) = a(k)\bar{\psi}(x,\lambda) + b(k)\psi(x,\lambda), \]
where and are real. The scattering data is given by

\[ a(k) = \frac{1}{2i\lambda} W(\phi, \psi), \quad b(k) = \frac{1}{2i\lambda} W(\bar{\psi}, \phi), \]

where \( W(u, v) = uv_x - vu_x \) is the Wronskian. We remark that from the relation \( \lambda^2 = k^2 + c^2 \) the scattering data \( a, b \) can be written in terms of either \( k \) or \( \lambda \); i.e. \( a = a(k) \) or \( a = a(\lambda) \). Similar relations hold for \( \bar{b}(k), \bar{a}(k) \), and we can show \( \bar{a}(k) = b^*(k), \bar{b}(k) = a^*(k) \) for \( k \) real and \( a(-\lambda) = b(\lambda) \) for real \( \lambda \) such that \( |\lambda| \leq c \). We note that \( k = i\kappa, |\kappa| < c \) corresponds to real \( \lambda, |\lambda| < c \). This forms part of the continuous spectrum and plays an important role below.

For the left scattering problem \( (\mathbb{I}) \) the usual transmission and reflection coefficients of quantum mechanics are, respectively,

\[ \tau(k) = \frac{1}{a(k)}, \quad \rho(k) = \frac{b(k)}{a(k)}. \]

These correspond to a unit wave denoted by \( e^{-ikx} \) propagating into the potential \( u(x) \) from \( x = -\infty \).

In the decaying problem, a soliton solution of the KdV equation is given by

\[ u(x, t) = 2\kappa^2 \text{sech}^2 \left[ \kappa (x - 4\kappa^2 t - x_0) \right], \quad x_0 \in \mathbb{R}. \] (17)

Solitons are associated with zeros of \( a(k) \): \( k = i\kappa, \kappa > 0 \) such that \( a(k) = 0 \). We also call \( k = i\kappa \) an eigenvalue; it is related to a bound state of the Schrödinger eq. (5).

The situation is quite different in the step problem. Consider a soliton initially positioned far to the left \( (x_0 \ll -1) \) of a localized step centered at \( x = 0 \). A soliton with amplitude parameter: \( \kappa_1 > 0 \), suggests it has a corresponding eigenvalue: \( k = i\kappa_1 \) with \( \lambda_1 \) given by

\[ \lambda_1 = \sqrt{c^2 - \kappa_1^2}, \quad \exists \lambda_1 > 0. \] (18)

When the ‘incoming’ soliton has sufficient size, i.e. the corresponding eigenvalue \( k = k_1 = i\kappa_1 \) satisfies \( \kappa_1 > c \), and \( a(k_1) = 0, \lambda = \lambda_1 = i\eta_1, \eta_1 > 0 \), then this is an instance of a proper eigenvalue and corresponds to a soliton tunneling through the rarefaction ramp (as an example, see Fig. 1). An analogous equation was obtained via Whitham theory in the weakly dispersive regime and was termed the ‘transmission condition’ \( (\mathbb{I}) \). As in the decaying problem, the eigenvalue corresponds to a bound state; i.e. the eigenmodes in eq. (4) which are square integrable. In this case we find the phase shift of the soliton.

If, however, the initial soliton/\text{sech}^2(x) profile is not large enough, then the ‘soliton’ becomes trapped \( (\mathbb{II}) \) \( (\mathbb{III}) \). We term this a trapped soliton since it does look and travel like the \text{sech}^2 solution in \( (\mathbb{I}) \) to the left of the step. In fact we find this mode eventually becomes trapped in the rarefaction ramp that evolves from the step (see Fig. 2), never reaching the top of the ramp. This is unlike a normal, or proper, soliton. Below we show that these trapped solitons correspond to \textit{pseudo-embedded} eigenvalues which are points that are exponentially close to the continuous spectrum.

For an initial soliton/\text{sech}^2 profile, delta, or box function we find the spectral coefficient \( a(k) \) takes the form

\[ a(k) = a_1(k)(k - i\kappa_0) + a_2(k), \quad \epsilon = e^{2\kappa_0 x_0} \ll 1. \] (19)

with \( 0 < \kappa_0 < c \) and \( a_2(i\kappa_0) \neq 0 \); where \( x_0 \) is the initial position of a soliton or delta function or box. The calculations leading to equation (19) are discussed in Appendix \( \mathbb{A} \). Since we take \( x_0 \ll -1 \), \( \epsilon \) is exponentially small. Although \( a(k) \) is not found to be exactly zero \textit{anywhere} in the upper half plane, it does possess values of \( k, 0 \leq \Im k \leq c \), that are exponentially close to the imaginary \( k \) axis and thus have properties analogous to those of discrete spectra/eigenvalues. The value \( \kappa_0 \approx i\kappa_0 \) provides the spectral meaning of a trapped soliton.

In Sec. (\mathbb{III}) we discuss the inverse scattering problem in the case of proper eigenvalues and proper solitons. In Sec. (\mathbb{IV}) we show how the form of \( a(k) \) given in (19) leads to a spectral contribution that plays the role of a discrete eigenvalue even though, by assumption, there are no proper eigenvalues/discrete spectra in the GLM equation. The scattering data for three prototypical examples (delta function, box function, soliton) are given in Appendix \( \mathbb{A} \).

**III. IST VIA GLM AND SOLITON PHASE SHIFTS**

The left scattering problem in eq. (13) can be transformed into a GLM equation from \( x \) to \( x \rightarrow \infty \). To do this we assume \( \psi \) has the following triangular form

\[ \psi(x, \lambda; t) = e^{i\lambda x} + \int_{x}^{\infty} G(x, s; t)e^{i\lambda s} ds. \] (20)
Substituting the above representation into eq. (13), using (10), then dividing by $a(k)$, integrating over $\mathbb{R}$ with $\int d\lambda \exp(i\lambda(y-x))/(2\pi)$ for $y > x$ and carrying out requisite calculations yields the following GLM equation,

$$G(x, y; t) + \Omega(x + y; t) + \int_{-\infty}^{\infty} \Omega(y + s; t)G(x, s; t)ds = 0,$$

(21)

where the kernel is given by

$$\Omega(z; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\lambda; t)e^{i\lambda z}d\lambda - \sum_{j=1}^{J} \rho^{j}(x; t),$$

(22)

with $\rho^{j}(x; t) = \frac{\rho(x; t)}{a(x; t)}$, $c_j(t) = c_j(0)e^{i\lambda_j t}$, $\rho(\lambda; t) = \frac{b(\lambda, t)}{a(\lambda, t)}$,

where $\rho(\lambda; t) = \rho(\lambda; 0)e^{i\lambda t}$, $c_j(t) = c_j(0)e^{i\lambda_j t}$, $\rho^{j}(x; t) = \frac{\rho(x; t)}{a(x; t)}$,

(24)

where $\rho(\lambda; t) = \rho(\lambda; 0)e^{i\lambda t}$, $c_j(t) = c_j(0)e^{i\lambda_j t}$, $\rho^{j}(x; t) = \frac{\rho(x; t)}{a(x; t)}$.

When the eigenvalues are proper ($\kappa_j > 0$), solitons will move through the rarefaction ramp and the phase shift can be calculated. For $J = 1$ and as $t \to \infty$, while neglecting the contribution from the continuous spectrum, the following one soliton solution is obtained from eqs. (21)-(23)

$$u(x, t) \sim \frac{2}{\pi} \rho(0) \exp(-i\kappa z)$$

for $x > \kappa_0$, $t \to \infty$.

The time evolution of the data is given by

$$\rho(0) \exp(-i\kappa z)$$

for $x > \kappa_0$, $t \to \infty$.

The soliton phase shift of proper solitons as $t \to -\infty$ can be obtained from the above GLM equation. Assuming no pseudo-embedded eigenvalues, and neglecting the contribution from the continuous spectrum, the following one soliton solution is obtained from eqs. (25)-(30)

$$u(x, t) \sim \frac{2}{\pi} \rho(0) \exp(-i\kappa z)$$

for $x > \kappa_0$, $t \to \infty$.

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for $x > \kappa_0$, $t \to \infty$.  

IV. TRAPPED SOLITONS AND PSEUDO-EMBEBBED EIGENVALUES

In this section we show how an initial soliton/sech^2 profile with amplitude parameter $0 < \kappa_0 < c$ can be described using the GLM approach. To this end we consider the GLM equation from $-\infty$ to $x$ given in (28).
We assume no proper eigenvalues; so we only have two terms, both from the continuous spectrum: the first and last terms in (29). The last term arises due to the branch cut.

We assume a localized initial condition: a soliton/sech$^2$ form with corresponding amplitude parameter $\kappa_0$, or a box function or a delta function, located well to the left of the step centered at $x = 0$; we call $\kappa_0 = i\kappa_0$ a pseudo-embedded eigenvalue. Additional solitons/sech$^2$ profiles can be added in a similar manner. The first term in the kernel (29) is small in the neighborhood of this pulse, so we only need to consider the branch cut contribution, i.e. the third term.

The dominant contribution to this integral, comes from values of $\kappa$ near $\kappa_0$ where $a(ik)$ is nearly zero. We substitute the form of $a(k)$ given in eq. (19) (see Appendix A for more details) into the branch cut integral in (29), expand around the point $k = i\kappa_0$, and insert the time dependence of $a(k,t)$ from equation (32) into this term, which we call $\Omega_3(z,t)$. The dominant contribution is given by the integral

$$\Omega_3(z,t) \sim \frac{\kappa_0 e^{\kappa_0 z - s_{\kappa_0 z}^2 t}}{2\pi \lambda_0} \int_{-\infty}^{\infty} \frac{1}{\Delta(k')} dk', \quad (34)$$

where $\lambda_0 = \sqrt{\epsilon^2 - \kappa_0^2}$.

$$\Delta(k') = |a_0|^2 \epsilon^2 - 2|a_0|a_2|\sin(\varphi_1 - \varphi_2)|\kappa' + |a_2|^2_0, \quad (35)$$

and $\kappa - \kappa_0 = \epsilon \kappa', \quad a_j(ik_0) = |a_j|e^{i\varphi_j}, \quad j = 1,2$. We note that $|a(ik,0)|^2$ given in the third term of the kernel (29) is approximated by $\Delta(\kappa)$ in the neighborhood of $\kappa_0$. Evaluating the above integral we find

$$\Omega_3(z,t) \sim \frac{\kappa_0}{2\lambda_0 \alpha \epsilon} e^{-\kappa_0 z - s_{\kappa_0 z}^2 t}. \quad (36)$$

where $\alpha = |a_0|a_2|0| \cos(\varphi_1 - \varphi_2)| > 0$. Remarkably, this has exactly the same form as that from the discrete spectra in the GLM equation given in (29). The corresponding solution is given by

$$u(x,t) \sim 2\kappa_0^2 \text{sech}^2 \left[ \kappa_0(x - 4\kappa_0^2 t - x_0^-) \right], \quad (37)$$

where

$$x_0^- = \frac{\ln(4\lambda_0 \alpha \epsilon)}{2\kappa_0}, \quad (38)$$

and valid when $x_0 \ll -1$ and the soliton position $x - x_0$ is well to the left of the ramp.

Thus, far to the left of the step ($x_0 \ll -1$), a soliton-like pulse travels with pseudo-eigenvalue $\kappa_0$. This soliton/sech$^2$ mode travels unimpeded until it comes into contact with the ramp that emanates from the step up initial condition. This soliton/sech$^2$ becomes trapped by the ramp (see Fig. 2). We refer to this as a trapped soliton. To carry out the details of this long time asymptotic analysis of the trapping from the inverse problem is outside the scope of this paper. The weakly dispersive case is discussed in (10) where analysis and numerical calculations further show how the soliton becomes trapped in the ramp and never makes it to the top of the ramp.

### A. Conclusion

The scattering/inverse scattering theory associated with the time-independent Schrödinger equation and its relationship to soliton solutions of the KdV equation for step potentials was analyzed.

The first case we considered was that of “proper” eigenvalues: $a(k) = 0, k = ik; \kappa > c$ where we find “proper” solitons. In this case the inverse scattering theory and linearization of the KdV equation can be carried out via a GLM equation with the solitons calculated from the discrete spectrum of the GLM kernel. Here a soliton that is initially well separated from the step propagates all the way through a ramp; doing so it acquires a phase shift which can be calculated exactly. This phase shift has encoded in it the continuous spectra which arises from the step. Numerical calculations confirm these formulae (19).

The second case was that of spectral data which had no proper eigenvalues, yet behaved as though it did. In terms of the soliton pulse, the amplitude is not large enough to pass though the rarefaction ramp that develops from the step up initial condition. This becomes a trapped soliton. In spectral terms there is a point, $k = ik, 0 \leq \kappa \leq c$, where the inverse of the transmission coefficient: $a(k)$, is exponentially close to, but not zero. In this case the continuous spectrum associated with the branch cut $0 \leq \kappa \leq c$ gives rise to a contribution that approximates a discrete eigenvalue located at $k = ik_0$. We call such $\kappa_0$ a pseudo-embedded eigenvalue.

Although the analysis here is developed for step up boundary conditions, the step down case is similar. In (23) the correspondence between the step up and step down case is referred to as ‘hydrodynamic reciprocity’. In the step down case a localized initial profile is inserted to the right of the step. Upon evolution it gets trapped by a DSW that emanates out of the initial data. From a mathematical viewpoint, we have the relationship $\lambda^2 = k^2 - c^2$ for step down boundary data (compare this with eq. (29)). Here the scattering/inverse scattering theory corresponds to interchanging the roles of $\lambda$ and $k$.

### V. Acknowledgements

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[1] J.S., Russell, Report on Waves 14, 311 (1844).
[2] D. Korteweg and G. de Vries, Phil. Mag. 5, 422 (1895).
where $c > 0$, $\kappa_0 > 0$. If we consider the decaying problem ($c = 0$) corresponding to the above sech$^2$ potential we can calculate $\phi(x, k), \psi(x, k)$ exactly. The reason for this is that when $b(k) = 0$ in eq. (19), or when $b(k) = 0$ in eq. (20), there are significant simplifications. In eqs. (19), (20) we divide by $a(k)$, subtract the pole contributions, use the symmetries in (11) and take a minus projector. Evaluation at $k = i\kappa_0$ yields the bound state and then using the bound state for general $k$ we can calculate the eigenfunction (cf. 5). In this way we find the eigenfunction $\phi(x, k)$ which is valid for $x \leq 0$ in the step problem; similarly we can get $\psi(x, k)$. However for $\psi(x, k)$ in the step problem $k$ is replaced by $\lambda$ since the spectral parameter satisfies $\lambda^2 = k^2 + 2c$. The results are

\[
\phi(x, k) = e^{-ikx} \left( 1 - \frac{2i\kappa_0}{k + i\kappa_0} \frac{1}{1 + e^{-2\kappa_0(x-x_0)}} \right), \quad x \leq 0
\]

(A4)

\[
\psi(x, \lambda) = e^{\lambda x} \left( 1 - \frac{2i\kappa_0}{\lambda + i\kappa_0} \frac{1}{1 + e^{2\kappa_0(x-x_0)}} \right), \quad x \geq 0.
\]

(A5)

From these results we can calculate the scattering data using the Wronskian. Using eq. (19) $a(k)$ is found to be

\[
a(k) = \frac{e^2 k + (k + \lambda)(k^2 + \kappa_0^2) + ic^2 \kappa_0 \tanh(\kappa_0 x_0)}{2\lambda(k + i\kappa_0)(\lambda + i\kappa_0)}.
\]

(A6)

From the above relation we can calculate the first two terms of the asymptotic approximation of the form given by eq. (19) which for convenience we give again below

\[
a(k) = a_1(k)(k - i\kappa_0) + a_2(k), \quad \epsilon = e^{-2\kappa_0 x_0} \ll 1.
\]

The values $a_1(k), a_2(k)$ for the soliton plus step (A8) are given by

\[
a_1(k) = \frac{\lambda + k}{2\lambda(k + i\kappa_0)}, \quad a_2(k) = \frac{ic^2 \kappa_0}{\lambda(k + i\kappa_0)(\lambda + i\kappa_0)}.
\]

(A7)

To approximate the branch cut integral in (20) in the case of pseudo-embedded eigenvalues we focus on values of $k$ near $i\kappa_0$ since that is where $a(k)$ is at a minimum. As such, to get the asymptotic integral in eq. (14) we expand $a(k)$ around $k = i\kappa_0$ which results in evaluating $a_1(k)$ and $a_2(k)$ in eq. (19) at $k = i\kappa_0$.

We remark that a similar example can also be calculated exactly, namely that of a soliton truncated at zero at $x = 0$:

\[
u(x) = 2\kappa_0^2 \text{sech}^2[\kappa_0(x - x_0)] \left[ 1 - H(x) + c^2 H(x) \right].
\]

(A8)
In this case \( \phi(x,k) \) is still given by eq. (A4) and \( \psi(x,\lambda) = e^{ikx} \) for \( x \geq 0 \). Hence \( a(k) \) can be calculated from the Wronskian formula
\[
a(k) = \frac{2\kappa_0^2 + (k + \lambda) |k + k \cosh(2\kappa_0 x_0) + ik_0 \sinh(2\kappa_0 x_0)|}{4\lambda(k + i\kappa_0) \cosh^2(\kappa_0 x_0)},
\]
which has the same \( a_1(k) \) as (A7), but different \( a_2(k) \).

We also note that while there are solutions \( a(k) = 0 \) for any \( \kappa > 0 \) we do not find any solutions to \( a(k) = 0 \) when \( 0 < \kappa < c \). An asymptotic expansion suggests that the zeros of \( a(k) \) are complex i.e. \( \kappa_0 = \xi_0 + i\kappa_0 \) where \( \kappa_0 > 0, \xi_0 \neq 0 \). This is, in fact, a contradiction since any eigenvalue corresponding to a bound state must be purely imaginary (see Appendix B).

2. Delta potential

Consider a delta function of height \( Q \) positioned well to the left (\( -x_0 \gg 1 \)) of a step function:
\[
u(x) = Q\delta(x-x_0) + e^2H(x), \quad Q > 0. \tag{A10}
\]

The time-independent Schrödinger equation can be explicitly calculated in this case. The solution takes the form
\[
\phi(x,k) = \begin{cases} 
  e^{-ikx}, & x < x_0 \\
  \alpha_1(k)e^{-ikx} + \beta_1(k)e^{ikx}, & x_0 < x < 0 \\
  a(k)e^{-i\lambda x} + b(k)e^{i\lambda x}, & x > x_0
\end{cases}
\]
\[
\text{for } \lambda = \sqrt{c^2 + k^2}. \text{ At } x = x_0 \text{ the eigenfunction } \phi(x,k) \text{ satisfies the jump condition } [\partial_x \phi(x,k)]_{x_0}^+ + Qe^{-ikx_0} = 0 \text{ and continuity. Using continuity of } \phi(x,k) \text{ and its derivative at } x = 0 \text{ yields the remaining coefficients. We only give } a(k) \text{ for the delta function plus step below}
\]
\[
a(k) = \frac{\lambda+k}{2\lambda} \left( 1 + \frac{Q}{2ik} \right) + \frac{\lambda-k}{2\lambda} \left( -\frac{Q}{2ik} e^{-2ikx_0} \right). \tag{A12}
\]

In the case of proper eigenvalues there exists \( k_1 = i\kappa_1, \kappa_1 > c \) such that \( a(k_1) = 0 \). For \( -x_0 \gg 1, \kappa_1 \approx Q/2 \) which is the same as the decaying (non-step) problem. When \( 0 < \kappa < c \) the unperturbed term, i.e. without the exponential term in equation (A12), suggests that \( \kappa_0 \) should be approximated by \( Q/2 \). But keeping the exponential term and carrying out an asymptotic expansion for \( -x_0 \gg 1 \) leads to the zeros of \( a(k) \) being complex i.e. \( \kappa_0 = \xi_0 + i\kappa_0 \) where \( \kappa_0 > 0 \) and \( \xi_0 \neq 0 \). This is a contradiction since any true bound state eigenvalue is purely imaginary (see Appendix B). In fact trying to solve \( a(k) = 0 \) numerically does not lead to a convergent iteration for this or any of the pseudo-embedded/trapped soliton examples discussed in this appendix.

From formula (A12) we can calculate the pseudo-eigenvalue and the first two terms of the asymptotic approximation given in eq. (A19): they are
\[
\kappa_0 = \frac{Q}{2}, \quad a_1(k) = \frac{\lambda+k}{2\lambda} \quad \text{and} \quad a_2(k) = \frac{ik_0(\lambda-k)}{2\lambda}. \tag{A13}
\]

3. Box potential

Consider a box function positioned well to the left (\( x_0 \ll -1 \)) of a Heaviside function (A2):
\[
u(x) = h^2B(x-x_0) + \frac{c^2}{2}H(x), \quad (A14)
\]
\[
B(x-x_0) = \begin{cases} 
  0, & \text{if } |x-x_0| > L/2 \\
  1, & \text{if } |x-x_0| \leq L/2.
\end{cases}
\]

with height \( h^2, h > 0 \) and width \( L > 0 \). The solution takes the form
\[
\phi(x,k) = \begin{cases} 
  e^{-ikx}, & x < x_0 - L/2 \\
  \alpha_1(k)e^{-ikx} + \beta_1(k)e^{ikx}, & \text{if } |x-x_0| < L/2 \\
  \alpha_2(k)e^{-ikx} + \beta_2(k)e^{ikx}, & x_0 + L/2 < x < 0 \\
  a(k)e^{-i\lambda x} + b(k)e^{i\lambda x}, & x > 0
\end{cases}
\]
\[
\text{where } \eta = \sqrt{h^2 + k^2}. \text{ We enforce continuity of the solution and its derivative at } x = x_0 \text{ and } x = 0. \text{ All coefficients can be calculated. We only give } a(k) \text{ for this case below}
\]
\[
a(k) = -\frac{(\lambda+k)}{8\lambda\eta k} e^{ikL} \left[ (\eta - k)^2 e^{i\eta L} - (\eta + k)^2 e^{-i\eta L} \right] + \frac{(\eta^2 - k^2)(\lambda-k)}{8\lambda\eta k} \left[ e^{i\eta L} - e^{-i\eta L} \right] e^{-2ikx_0}. \tag{A15}
\]

If we neglect the term multiplied by \( e^{-2ikx_0} \) i.e. take \( x_0 \ll -1 \), the eigenvalues satisfying \( a(k) = 0 \) yield solutions for \( \eta \in \mathbb{R} \) satisfying
\[
\tan(\eta L) = -\frac{2ik\eta}{k^2 + \eta^2}. \tag{A17}
\]

Solutions to the above equation are the same as those obtained in the decaying no-step problem (B). Graphical analysis shows that there can be one or more solutions depending on the size of \( h \) and \( L \). For example, when \( h < \pi/L \) there is one solution, which we denote as \( \kappa_0 \) for \( 0 < \kappa < h \).

Keeping the exponential term modifies the above result. For \( \eta \in \mathbb{R}, k = i\kappa, \kappa > c \) define \( \lambda = i\bar{\kappa}, \bar{\lambda} \in \mathbb{R} \). Solutions of \( a(k) = 0 \) must satisfy
\[
\left( \eta + i\kappa \right) \left( \eta - i\kappa \right)^2 \left( 1 - \frac{(\eta^2 + \kappa^2)^2(\lambda - \kappa)^2}{(\lambda + \kappa)^2(\lambda - \kappa)^2} \right) e^{2ikx_0 + \kappa L} = e^{2\eta L}. \tag{A18}
\]

Note that both left and right sides of the above formula have unit modulus and hence a perturbative solution for \( -x_0 \gg 1 \) is expected; numerical solutions have been found.

For \( \eta \in \mathbb{R}, k = i\kappa, 0 < \kappa < c, \lambda \in \mathbb{R} \) solutions of \( a(k) = 0 \) now satisfy
\[
\left( \eta + i\kappa \right) \left( \eta - i\kappa \right)^2 \left( 1 - \frac{(\eta^2 + \kappa^2)^2(\lambda - \kappa)^2}{(\lambda + \kappa)^2(\lambda - \kappa)^2} \right) e^{2ikx_0 + \kappa L} = e^{2\eta L}. \tag{A19}
\]
In this case the left-hand side is not of unit magnitude; no solution is expected; a numerical solution has not been found. The values of \(a_1(k)\) and \(a_2(k)\) can be found from equation (A18).

**Appendix B: Bound states**

In this appendix we establish that all solutions of \(a(k) = 0\) associated with bound states, i.e.

The values are proper. Moreover, we point out that \(\phi\) and \(a\) and \(\phi\) then \(\phi\psi\psi\) respectively, with 

Thus, \(\phi(x, k) \sim e^{-ikx} \phi_0 e^{ikx} \phi\) as \(x \to -\infty\), as \(x \to +\infty\), (B1)

and 

where \(\lambda_0 := \sqrt{k_0^2 + c^2}\). Assuming that 

then necessarily \(\Re \lambda_0 > 0\) and hence \(\kappa_0 > c\). Such eigenvalues are proper. Moreover, we point out that \(\phi(x, k)\) and its complex conjugate \(\phi^*(x, k)\) satisfy the equations 

\[ \phi_{xx} + (u(x) + k^2) \phi = 0, \]

\[ \phi_{xx}^* + (u(x) + (k^*)^2) \phi^* = 0, \]

respectively, with \(u(x)\) real. Hence, 

\[ \frac{\partial}{\partial x} W(\phi, \phi^*) + ((k^*)^2 - k^2) \phi \phi^* = 0. \]

(B3)

Since \(\phi \to 0\), \(\phi_x \to 0\) as \(x \to \pm \infty\), we have

\[ ((k^*)^2 - k^2) \int_{-\infty}^{\infty} |\phi(x, k)|^2 dx = 0. \]

(B4)

If \(a(k_0) = 0\), where \(k_0 = \xi_0 + i\kappa_0\), then 

\[ \xi_0 \kappa_0 \int_{-\infty}^{\infty} |\phi(x, k)|^2 dx = 0, \]

for \(\phi(x, k) \in L^2(\mathbb{R})\) so \(\int_{-\infty}^{\infty} |\phi(x, k)|^2 dx > 0\), and \(\xi_0 \kappa_0 = 0\). For decay as \(|x| \to \infty\) we require \(\kappa_0 > 0\); thus \(\xi_0 = 0\).

**Appendix C: Phase shift for \(J\) solitons**

1. **\(J\)-soliton solution from the GLM equation (21)**

As above, we take \(J\) eigenvalues, i.e.

\[ \Omega(z; t) = -\sum_{j=1}^{J} c_j(0)e^{(\lambda_j^2 + 12c^2)t - \eta_j z}, \]

(C1)

where \(\lambda_j = i\eta_j, \eta_j > 0\) and \(\eta_1 < \eta_2 < \ldots < \eta_J\), and \(c_j(0)\) is defined below eq. (22). Then as \(t \to \infty, x \sim 4\kappa_j^2 t\) we find that the fastest, or \(J\)th soliton, is asymptotically given by (22)

\[ u_J(x; t) \sim c^2 + 2\eta_J^2 \sech^2 \left[ \eta_J \left( x - (6c^2 + 4\eta_J^2)t - x_J^0 \right) \right], \]

(C2)

where

\[ \eta_J x_J^0 = \frac{1}{2} \log \left[ \frac{c_j(0)}{2\eta_J} \right] + \sum_{j=1}^{J-1} \log \left| \frac{\eta_J - \eta_j}{\eta_J + \eta_j} \right|, \]

(C3)

defines the phase of \(J\)th soliton when \(t \to +\infty\).