Cantilever self-excited with a higher mode by a piezoelectric actuator

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Abstract The sensitivity of vibration-type sensors can be improved using a higher resonating frequency of a cantilever resonator. Resonance in such systems can be achieved using a self-excited oscillation, which overcomes the difficulty of using an external excitation in viscous environments. To enhance the sensitivity of cantilever resonators, several groups have proposed ways to increase the natural frequency of the first mode by changing the cantilever geometry. However, the sensitivity can be further improved by using a self-excited oscillation with a higher mode in addition to the geometry-changing technique. In this study, we present a method to realize this goal. We perform a nonlinear analysis of the governing equation of a cantilever excited by a piezoelectric actuator. For each mode, we also clarify the dependence of the critical feedback gain on the location of a displacement sensor, the output of which is used for feedback control. With the aid of filters, we then devise a way to generate a self-excited oscillation with a higher mode associated with a desired higher natural frequency. Finally, we carry out experiments using a macro-cantilever and report the observation of a self-excited oscillation with the second natural frequency, which is higher than the first natural frequency.

Keywords Cantilever sensor · Sensitivity · Self-excited oscillation · Higher modes

1 Introduction

Resonating cantilever beams have been widely used in vibration-type sensors to measure properties such as viscosity [1,2], pressure [3], mass [4–6], and surface topography [7–9]. A higher resonating frequency of resonators can improve the sensitivity of vibration-type sensors, and such a frequency has been extensively used in microcantilevers and nanocantilevers [10–18]. Moreover, previous studies have considered using a mode higher than the first mode and increasing the first natural frequency by changing the geometry of the cantilevers [19–21].

Most of the measurement techniques mentioned above involve detecting the peak frequency of a frequency response curve under external excitation [22–25]. Resonance in a higher mode under the external excitation is easily achieved because the frequency of the external excitation equals to the response frequency of the resonator. However, these sensing methods cannot be used in viscous environments because the peak of the frequency response curve is ambiguous or non-existent [26]. Fortunately, a self-excited oscillation based on feedback control can be applied even in viscous environments to eliminate the effect of viscous dissipation on the resonator. Therefore, such an oscillation
can overcome the aforementioned limitation of external excitation (e.g., [27,28]).

To boost the sensitivity of cantilever sensors that use a self-excited oscillation, a previous paper suggested applying such an oscillation in microcantilever measurements through feedback-control methods [29]. In that study, the microcantilever was thickened to increase the first natural frequency, and a piezoelectric film was also slightly thickened. However, changes to the geometric properties were limited by the fabrication process. In such a scenario, because the cantilever is a system of infinitely many degrees of freedom, one can consider a self-excited oscillation with a higher natural frequency associated with a higher mode. In the sensitivity of cantilever sensors under external excitation, the enhancement of the sensitivity has been carried out [19,20]. However, unlike for the aforementioned external excitation, it is difficult to produce such an oscillation because the excitation is applied through the feedback of the velocity of the resonator itself, and one cannot directly excite the resonator with the desired frequency. No method exists to generate a self-excited oscillation with a higher mode to the best of our knowledge. Thus, we propose a method to produce cantilever sensors self-excited with a desired higher mode.

In this study, we propose a technique based on linear and nonlinear feedback to produce a self-excited oscillation with a higher natural frequency of a cantilever, which is a practical model for the resonators used in nano- and micro-sensors. In particular, the nonlinear feedback causes self-excited oscillation with a finite constant amplitude and changes its magnitude. By analyzing a nonlinear partial differential equation that governs the dynamics under the feedback, we obtain an averaged equation that expresses the linear and nonlinear characteristics of the self-excited oscillation for each mode.

Using the averaged equation, we derive a bifurcation set that shows the relationship between the critical feedback gain and the location of a sensor that produces the self-excited oscillation. By examining the bifurcation set, we determine the sensor location that minimizes the linear-feedback gain. With the aid of a low-pass filter, this process enables us to generate a self-excited oscillation in the higher mode associated with the desired higher natural frequency. Cantilevers have more or less nonlinear components in the elasticity, and the dependence of the nonlinear natural frequency on the oscillation amplitude is described by a backbone curve. Using the averaged equation, we show that the relationship between the response frequency and the oscillation amplitude is equivalent to the backbone curve that characterizes the nonlinearity of the cantilever itself. This finding indicates that the reduction in the response amplitude caused by the high-gain nonlinear feedback enables the generation of a self-excited oscillation with a linear natural frequency.

To confirm the validity of our method, we perform experiments using a macro-cantilever. We use the approach to produce a self-excited oscillation with the second mode. Moreover, we experimentally verify that the relationship between the response frequency and amplitude is equivalent to the backbone curve. Finally, under high-gain nonlinear feedback, we achieve self-excitation of the cantilever with the linear second natural frequency.

2 Equation for the cantilever under a self-excited oscillation

In this section, we present the equation of motion for a cantilever beam, the supporting point of which is subject to lateral displacement excitation by a stack-type piezoelectric actuator. In recent years, composite material like carbon nano tube (CNT) has become to be widely utilized [30]. It is also used as resonators in Atomic Force Microscope (AFM) [31,32]. However, because this work is a proposition for the fundamental method to produce the self-excited oscillation with a higher mode, we used the conventional non-composite resonator.

We consider the analytical cantilever shown in Fig. 1a, where $s$ is the arch-length coordinate, $l$ is the cantilever length, $\Psi$ is the rotation angle, and $u$ and $v$ are the displacements in the $x$ and $y$ directions, respectively. The displacement excitation $\eta$ is applied at the supporting end ($s = 0$). Figure 1b shows the positions of an element of the deflected cantilever beam.

Our formulation takes into account geometric nonlinearity. We assume that the cantilever behaves like an Euler–Bernoulli beam and impose the following inextensible condition, which assumes that the neutral surface does not change in length during deformation [35].

$$f = 1 - \left(1 + \frac{\partial u}{\partial s}\right)^2 - \left(\frac{\partial v}{\partial s}\right)^2 = 0. \quad (1)$$

This assumption is widely used in the nonlinear dynamics of a cantilever beam [33]. The limitation is as
follows. The assumption can be used in the condition when the displacement in the axial direction is small enough to be ignored. For example, the condition is satisfied in the cases when one end of the beam is not constrained such as a cantilever beam and a simply supported beam, i.e., a hinged-hinged beam whose one end is on a roller support in the axial direction. On the other hand, in a hinge-hinged beam in which both ends are axially immovable, the neutral surface can be stretched and the assumption Eq. (1) is not applicable. The theoretical and experimental discussions can be found in Ref. [34].

Using the Lagrange multiplier $\lambda$, the Lagrangian $L$ to an accuracy of $O(v^4)$ can be formulated as

$$
L = \int_0^L \left\{ \frac{1}{2} \rho A cs \left[ \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial v}{\partial t} + \frac{\partial \eta}{\partial t} \right)^2 \right] + \frac{1}{2} EI \left( \frac{\partial \Psi}{\partial s} \right)^2 + f \lambda \right\} ds,
$$

(2)

where $\rho A cs$ is the mass per unit length and $EI$ is the bending stiffness of the cantilever. By applying Hamilton’s principle [35], we derive the following nondimensional equation and boundary conditions [36]:

$$
\dddot{v} + \dddot{\eta} + \mu \dddot{v} + v^{1/3} + v^{1/2} + 4v^2 \dot{v} + v^{1/2} + v^{1/2} = 0,
$$

(3)

where the viscous damping effect is taken into account by the third term on the left-hand side of Eq. (3). All lengths and the time are nondimensionalized by the representative length $L = l$ and the representative time $T = \sqrt{\frac{\rho A cs}{EI}}$, respectively. We use the dimensionless parameters $t^* = \frac{t}{T}$, $v^* = \frac{v}{L}$, $\eta^* = \frac{\eta}{L}$, $s^* = \frac{s}{L}$, and $\mu = \frac{\mu}{L^2}$. Finally, $\left[ \cdot \right]$ and $\left( \cdot \right)$ denote $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$, respectively.

3 Method to produce a self-excited oscillation with a higher natural frequency

3.1 Setup for the control of the linear and nonlinear feedback

We use the feedback-control method described in Reference [37] to generate a self-excited oscillation. When applying only linear feedback, the amplitude of such an oscillation grows over time and eventually becomes infinite. In actual experiments, this growth is saturated, and high-frequency noise is included in the response, owing to the limited power supplied to the actuator. Therefore, in addition to linear feedback, we apply the following nonlinear feedback (similar to our previous study [38] in which we produced a self-excited oscillation with the first mode):

$$
\eta = k_{\text{lin}} \frac{\partial v}{\partial t} \bigg|_{s=s_{\text{sr}}} - k_{\text{nl}} \left( \frac{\partial v}{\partial t} \right)^3 \bigg|_{s=s_{\text{sr}}},
$$

(5)

where the displacement of the beam $v$ at $s = s_{\text{sr}}$ is measured by a displacement sensor that generates the feedback-control signal and $s_{\text{sr}}$ denotes the sensor location on the arch-length coordinate. The variables $k_{\text{lin}}$ and $k_{\text{nl}}$ are the linear- and nonlinear-feedback gains, respectively. The first term on the right-hand side of Equation (5) provides positive or negative damping to the cantilever beam, and $k_{\text{lin}}$ can control whether or
not a self-excited oscillation occurs. Using this nonlinear feedback, we make the cantilever behave as a Rayleigh-type self-excited oscillator [39], which has a nontrivial steady-state amplitude that is both finite and stable. Then, we realize a self-excited oscillation with a steady-state amplitude. By tuning the nonlinear-feedback gain \( k_{nl} \), this amplitude is reduced (a steady-state self-excited oscillation with a desired linear higher natural frequency is described in Sect. 3.3). The dimensionless form of Eq. (5) is

\[
\eta^* = k_{lin} \frac{\partial v^*}{\partial \tau^*} |_{\tau^* = s_{nl}^*} - k_{nl} \left( \frac{\partial v^*}{\partial t^*} \right)^3 |_{t^* = s_{nl}^*},
\]  

(6)

where the dimensionless linear- and nonlinear-feedback gains are given by \( k_{lin} = \frac{L_{lin}^3}{t^3} \) and \( k_{nl} = \frac{L_{nl}^3}{t^3} \), respectively. Hereafter, we omit the \([\cdot]^*\) notation for simplicity.

3.2 Nonlinear analysis of a self-excited oscillation with the nth mode

The magnitude of the damping coefficient \( \mu \) is assumed to be

\[
\mu = \epsilon^2 \hat{\mu},
\]  

(7)

where \( 0 < \epsilon < 1 \), and \( \hat{\mu} = O(1) \) [37]. To derive the amplitude equation for the nth mode, we use the method of multiple scales [40] and introduce the following time scales:

\[
t_0 = t, \quad t_2 = \epsilon^2 t.
\]  

(8)

Then, we express \( v \) in a uniform expansion described in the following orders:

\[
v = \epsilon \hat{v}_1 + \epsilon^3 \hat{v}_3 + O(\epsilon^5),
\]  

(9)

where \( \hat{v}_1 = O(1) \) and \( \hat{v}_3 = O(1) \). We set the orders of the linear feedback gain \( k_{lin} \) and nonlinear feedback gain \( k_{nl} \) so that these effects are included in the following solvability condition [37]:

\[
k_{lin} = \epsilon^2 \hat{k}_{lin}, \quad k_{nl} = \hat{k}_{nl},
\]  

(10)

where \( \hat{k}_{lin} = O(1) \) and \( \hat{k}_{nl} = O(1) \) (see Appendix B for the base of Eqs. (7) and (10)). By combining Eqs. (7–10) with the equation of motion [Eq. (3)], the boundary conditions [Eq. (4)], and the equation of feedback control [Eq. (6)], we obtain the following equations of motion and associated boundary conditions at the orders of \( O(\epsilon) \) and \( O(\epsilon^3) \):

\[
O(\epsilon):
\]

\[
D_0^2 \hat{v}_1 + \hat{\theta}''' = 0,
\]  

(11)

\[
\hat{v}_1 |_{\tau = 0} = \hat{\theta}' |_{\tau = 0} = \hat{\theta}'' |_{\tau = 1} = \hat{\theta}''' |_{\tau = 1} = 0,
\]  

(12)

\[
O(\epsilon^3):
\]

\[
D_0^2 \hat{v}_3 + \hat{\theta}''' = -2D_0 D_2 \hat{v}_1 - \hat{\mu} D_0 \hat{v}_1 - 4 \hat{\theta}' |_{\tau = 1}'''' - \hat{\theta}''' |_{\tau = 1} = 0,
\]  

(13)

where \( \hat{v}_1 \) and \( \hat{v}_3 \) are the amplitude and mode function of the nth mode, respectively. By substituting Eq. (15) into Eqs. (11) and (12), we obtain the following solution for \( \varphi_n \):

\[
\begin{align*}
\varphi_n &= \left( \sin \lambda_n s - \sinh \lambda_n s \right) + \left( \cos \lambda_n s + \cosh \lambda_n s \right) \\
\end{align*}
\]  

(14)

where \( \lambda_n = \sqrt{\omega_n} \) and \( \Gamma \) is a constant that satisfies

\[
\int_0^1 \varphi_n'^2 ds = 1.
\]  

(15)

For the nth mode, we numerically obtain \( \lambda_n \) from Eq. (16) and use this result to derive \( \Phi_n \) from Eq. (17). We assume the solutions to Eqs. (13) and (14), and take the following form (e.g., [37]):

\[
\hat{v}_3 = \left( e^{i \omega_0 t_0} + e^{-i \omega_0 t_0} \right) \varphi_n (s, t_2).
\]  

(18)

Substituting Eqs. (15) and (18) into Eq. (13) and retaining the terms proportional to \( e^{i \omega_0 t_0} \), we obtain the following equation for \( \varphi_n \):

\[
\varphi_n'''' - \omega_n^2 \varphi_n = |A|^2 AC(s) - 2i \omega_n D_2 A \Phi_n
\]

\[
- i \hat{\mu} \omega_n A \Phi_n
\]

\[
+ i k_{lin} A^3 \Phi_n |_{s = s_{nl}}
\]

\[
- 3i \hat{k}_{nl} |\omega_n|^5 A^2 \Phi_n^3 |_{s = s_{nl}},
\]  

(19)
where
\[
C(s) = -3 \Phi_n'' - 12 \Phi_n'' + 3 \Phi_n''' - 3 \Phi_n'''
+ 2 \omega_n^2 \Phi_n' \int_1^s \Phi_n' ds + 2 \omega_n^2 \Phi_n' \int_1^s \Phi_n' ds.
\]

The associated boundary conditions for \( \varphi_n \) are as follows:
\[
\varphi_n|_{x=0} = \varphi_n'|_{x=0} = \varphi_n''|_{x=1} = \varphi_n'''|_{x=1} = 0. \tag{21}
\]

Multiplying both sides of Eq. (19) by \( \Phi_n \) and integrating from 0 to 1 with respect to \( s \) yields
\[
D_2 A + \left( \frac{\hat{\mu}}{2} - \beta_1 \omega_n^2 \nu \right) A
+ \beta_2 \omega_n^2 k_n |A|^2 A + i \beta_3 \omega_n |A|^2 A = 0, \tag{22}
\]

where
\[
\beta_1 = \frac{\Phi_n|_{s=sa} - \int_0^1 \Phi_n ds}{2}, \tag{23}
\]
\[
\beta_2 = \frac{3 \Phi_n|_{s=sa} - \int_0^1 \Phi_n ds}{2}, \tag{24}
\]
\[
\beta_3 = \int_0^1 \Phi_n C(s) ds \tag{25}
\]

The variable \( A \) is the complex amplitude and is expressed in polar form. Substituting \( A = \frac{1}{2} \hat{\alpha}(t_2) e^{-i \gamma(t_2)} \) into Eq. (22), we obtain the following equations that govern the time evolution of the amplitude \( a \) and phase \( \gamma \) of the \( n \)th mode:
\[
\frac{da}{dt} + \left( \frac{\mu}{2} - \beta_1 \omega_n^2 k_n + \beta_2 \omega_n^2 k_n |a|^2 \right) a = 0, \tag{26}
\]
\[
\frac{d\gamma}{dt} - \beta_3 \omega_n a^2 = 0, \tag{27}
\]
where \( \alpha = e^{-i \hat{\alpha}}. \) Because \( \beta_1 \) and \( \beta_2 \) include \( \Phi_n \) and \( \Phi_n|_{s=sa}, \) the effect of the linear feedback depends on the mode and the sensor location.

3.3 Relationship between the response frequency and the steady-state amplitude

To investigate the magnitude of the steady-state amplitude, we substitute \( \frac{da}{dt} = 0 \) into Eq. (26). Then, the solutions of the steady-state amplitude \( a_{st} \) in Eq. (26) are given by
\[
a_{st} = 0, \quad a_{st} = \sqrt{\frac{4 \beta_1 \omega_n^2 k_n - 2 \mu}{\beta_2 \omega_n^2 k_n}}. \tag{28}
\]

where the first and second solutions are trivial steady-state amplitude and nontrivial steady-state amplitude, respectively. And the phase associated with the nontrivial steady-state amplitude is expressed as
\[
\gamma = \frac{\beta_3 a_{st}^2}{4 \omega_n} t + \gamma_0. \tag{29}
\]

where the integration constant \( \gamma_0 \) can be neglected without loss of generality. The second equation in Equation (28) indicates that the oscillation of the cantilever has a nontrivial steady-state amplitude that depends on the linear- and nonlinear-feedback gains, \( k_{lin} \) and \( k_{nl}. \)

Using the second equation in Eq. (28), the relationship between the steady-state amplitude \( a_{st} \) and the linear-feedback gain \( k_{lin} \) can be represented as in Fig. 2, where the solid and dashed lines denote stable and unstable steady-state amplitudes, respectively. In Fig. 2a, for \( k_{nl} > 0, \) the nontrivial steady-state amplitude exists when \( k_{lin} > \frac{\mu}{2 \omega_n^2 \beta_1} \) owing to supercritical Hopf bifurcation and is stable. In Fig. 2b, for \( k_{nl} < 0, \) the amplitude exists when \( k_{lin} < \frac{\mu}{2 \omega_n^2 \beta_1} \) owing to subcritical Hopf bifurcation but is unstable. Therefore, to realize a self-excited oscillation with a stable nontrivial steady-state amplitude, the feedback gains should satisfy \( k_{nl} > 0 \) and \( k_{lin} > \frac{\mu}{2 \omega_n^2 \beta_1}. \)

The value \( k_{lin} = \frac{\mu}{2 \omega_n^2 \beta_1} \) in Fig. 2 is the critical feedback gain, which determines the smallest feedback gain that can produce a self-excited oscillation of the \( n \)th mode with a nontrivial stable steady-state amplitude. Moreover, as the critical gain \( \frac{\mu}{2 \omega_n^2 \beta_1} \) consists of the viscous damping effect \( \mu, \) the linear feedback can eliminate the effect of viscosity. There is large damping effect on vibration-type sensors under microscale environment. In our previous research [41], the coupled microcantilevers in water are self-excited with the first mode by applying the same method as the present research and compensating the viscosity effect. In that research, the self-excited mode is the first mode, but the principle based on the feedback control for compensating for the viscosity is the same as that in the present study. Therefore, the proposed method is also applicable to the self-excitation in microscale environment.
Fig. 2 Relationship between the steady-state amplitude \( a_{st} \) and the linear-feedback gain \( k_{\text{lin}} \). The solid and dashed lines indicate stable and unstable amplitudes, respectively. (a) shows the Hopf bifurcation when \( k_{nl} > 0 \). (b) shows the Hopf bifurcation when \( k_{nl} < 0 \). A stable nontrivial steady-state amplitude exists when \( k_{nl} > 0 \) and \( k_{\text{lin}} > \frac{\mu}{2\omega_n^2 \beta_1} \).

The critical feedback gain is a function of \( \beta_1 \) and therefore depends on the sensor location \( s_{sr} \). We determine the sensor location that minimizes the critical linear-feedback gain \( \frac{\mu}{2\omega_n^2 \beta_1} \). We then rewrite this gain as follows:

\[
\frac{\mu}{2\omega_n^2 \beta_1} = \frac{\mu}{\omega_n^2 \Phi_n |_{s=s_{sr}} \int_0^1 \Phi_n ds} = k_{\text{lin-\text{cr}(n)}}(n) \mu, \quad (30)
\]

where \( k_{\text{lin-\text{cr}(n)}}(n) = \frac{1}{\omega_n^2 \Phi_n |_{s=s_{sr}} \int_0^1 \Phi_n ds} \) shows the dependence of the mode on the critical feedback gain and is hereafter called the critical coefficient. Using Eq. (30), this coefficient with respect to the sensor location can be depicted as in Fig. 3a, where the black, red, blue, and green lines correspond to the first, second, third, and fourth modes, respectively. The markers in Fig. 3 are not data points, they are only used to distinguish different lines. Figure 3b depicts the expansion of Fig. 3a. Figure 3 can be regarded as a Hopf-bifurcation set with respect to the linear-feedback gain and the sensor location.

Given Eq. (30), we propose a method to generate a self-excited oscillation with a higher mode. In the following discussion, as an example, we consider the case in which the second mode is desired. We select the sensor location that minimizes the critical coefficient \( k_{\text{lin-\text{cr}(n)}}(n) \) of the second mode and compare this coefficient with those of other modes.

As seen in Fig. 3b, the critical coefficient of the second mode is smallest when the nondimensional sensor location \( s_{sr} \) is set at \( s = 1 \), i.e., the tip of the cantilever. Moreover, Fig. 3 shows that the critical gain of a mode lower than the desired second mode, i.e., the first mode,
is larger than that of the second mode. Because the critical gain of the modes higher than the second mode can be smaller than that of the second mode, we use a low-pass filter to eliminate the influence of these higher modes. As a result, a self-excited oscillation with the desired second mode can be produced. This strategy is applicable independent of the mode that needs to be self-excited.

Substituting Eqs. (15), (29) and  
\[ A = \frac{1}{2} \hat{a}(t_2)e^{-i\gamma(t_2)} \] into Equation (9), the displacement in the steady state is given by

\[ v = a_{st} \cos(\omega_n t - \gamma) \Phi_n(s) + O(\epsilon^3) \]

\[ = a_{st} \cos \left( \omega_n - \frac{\beta_3 a_{st}^2}{4\omega_n} t - \gamma_0 \right) \Phi_n(s) + O(\epsilon^3). \]

(31)

The steady-state amplitude \( a_{st} \) depends on the nonlinear-feedback gain \( k_{nl} \), as seen in Equation (28) and depicted in Fig. 4. In particular, a high nonlinear-feedback gain produces a small steady-state amplitude. The response frequency \( \omega_{n-nl} \) is expressed as

\[ \omega_{n-nl} = \omega_n - \frac{\beta_3 a_{st}^2}{4\omega_n}. \]

(32)

Therefore, the response frequency depends on the steady-state amplitude, which can be adjusted by tuning the nonlinear-feedback gain.

The nonlinear characteristic of cantilevers is described by a backbone curve. Because Eq. (32) is equivalent to the equation that represents such a curve, a self-excited oscillation produced by the proposed method has a frequency response that corresponds to the nonlinearity of the cantilever. As seen in Fig. 4, a self-excited oscillation with a linear natural frequency, which is used in most sensors, is realized by decreasing the response amplitude of high nonlinear-feedback gain \( k_{nl} \). The reason is that the response frequency for an infinitesimally small steady-state amplitude is equal to the linear natural frequency.

4 Experiments

4.1 Experimental setup

We perform experiments to validate our proposed method. We use a laser displacement sensor (LK-G35A, Keyence Corp.; resolution: 0.001 mm, sampling period: 0.1 ms) to measure the response displacement of a cantilever beam. The sensor location is set at 45% of the cantilever length (see Appendix A for a discussion of this location). The displacement excitation \( \eta (m) \) in Eq. (5) is given by

\[ \eta = d_{33} G_a \left( k_{lin-v} \frac{\partial v}{\partial t} \bigg|_{x=x_{st}} - k_{nl-v} \left( \frac{\partial v}{\partial t} \bigg|_{x=x_{st}} \right)^3 \right). \]

(33)

where \( d_{33} (m/V) \) is the piezoelectric constant and \( G_a \) is the gain of the actuator driver, which is always set to 8.4 in our experiments. The variables \( k_{lin-v} (Vs/m) \) and \( k_{nl-v} (Vs^3/m^3) \) are the linear- and nonlinear-feedback gains, respectively, and are adjusted by a digital signal processing (DSP) board (DS1104, dSPACE GmbH).

Figure 5 depicts the experimental setup, and Fig. 6 shows the signal flow in the experiments. A piezoelectric actuator (AHB151C362ND0LF, Tokin Corp.; piezoelectric constant \( d_{33}: 0.97 \times 10^{-6} m/V \)) applies
the displacement excitation to a phosphor-bronze cantilever (dimensions: $0.209 \times 0.02 \times 0.0002$ m, mass: $1.44 \times 10^{-3}$ kg) through an actuator driver (ENP-152, Echo Corp.). The response amplitude is analyzed by a fast Fourier transform (FFT). The natural frequencies of the first and second modes of the cantilever are 2.52 Hz (FFT resolution: 0.02 Hz) and 15.8 Hz (FFT resolution: 0.1 Hz), respectively, and are measured by free vibrations. Finally, a low-pass filter (dual-channel programmable filter 3624, NF Corp.; cut-off frequency: 30 Hz) eliminates the influence of the $n$th ($n > 2$) modes.

4.2 Identification of the backbone curve and linear natural frequency of the cantilever under external excitation

For preliminary experiments, we apply the following external excitation to obtain frequency response curves for different values of the excitation amplitude $a_e$:

$$\eta = d_{33}a_e \cos \nu t.$$  \hfill (34)

Figure 7 shows the dependence of the response frequency on $a_e$, where the symbols ▼, □, △, and ▲ correspond to cases in which $a_e$ is 9 V, 13 V, 19 V, and 24 V, respectively. The curve connecting the peaks of the frequency response curves for different excitation amplitudes is equivalent to a backbone curve [40]. Given Eq. (32), we assume the backbone curve has the following form:

$$\omega_n - \omega_{nl} = \omega_n + \alpha a_e^2,$$  \hfill (35)

where $\omega_n$ is the linear natural frequency. Then, using the least-squares method, we determine $\omega_n$ and $\alpha$ to be 99.36 rad/s and $8.501 \times 10^5$ rad/(m$^2$ \cdot s), respectively. Therefore, the linear natural frequency of the cantilever is 15.815 Hz. The backbone curve is depicted by the solid line in Fig. 7.

4.3 Vibration state and mode shape

Figure 8a shows the time evolution of the response displacement at 45% of the cantilever length (the linear- and nonlinear-feedback gains $k_{lin-v}$ and $k_{nl-v}$ are set to 0.5 V/s/m and 0.001 V s$^3$/m$^3$, respectively). Initially, the amplitude increases over time. Then, after this transient state, the system enters a stable steady state with a constant amplitude. Figure 8b presents a few periods of the steady state in Fig. 8a. Figure 8c shows the results of a fast Fourier transform ( FFT) for 320 s of the steady state. Because the response frequency is 15.888 Hz, the cantilever is self-excited with the second mode. However, this frequency deviates from the linear second natural frequency, which is identified as 15.815 Hz in Sect. 4.2. The deviation is caused by the inherent nonlinearity of the cantilever (see Sect. 4.4). Finally, Fig. 8d illustrates the mode shape of the self-excited cantilever. (d-1) is the shape at a particular time, and (d-2), (d-3), and (d-4) are the shapes after one-quarter, one-half, and three-quarters of a period, respectively.
4.4 Response amplitude and frequency

Figures 9a and b show the dependence of the response amplitude in the case of a self-excited oscillation on the nonlinear-feedback gain $k_{nl-v}$ for $k_{lin-v} = 0.5$ Vs/m and $k_{lin-v} = 1.7$ Vs/m, respectively. The amplitude is inversely proportional to the nonlinear-feedback gain, as theoretically shown by the second equation in Eq. (28) and Fig. 4.

Finally, we compare the relationship between the steady-state amplitude and the response frequency under the proposed self-excited method with the cantilever backbone curve obtained in the preliminary experiment described in Sect. 4.2. We set the linear feedback gain $k_{lin-v}$ at 0.5 Vs/m and change the nonlinear feedback gain $k_{nl-v}$ from 0.01 Vs$^3$/m$^3$ to 0.3 Vs$^3$/m$^3$ to get different steady-state amplitudes. The dependence of the response amplitude on the response frequency and the backbone curve depicted in Sect. 4.2 are shown in Fig. 10.

As shown in Fig. 10, the amplitude-frequency relationship follows the backbone curve, and as the amplitude decreases, the response frequency approaches the linear natural frequency. Cantilever sensors are typically operated near the linear natural frequency. On the basis of the above discussion, such a situation can be realized by reducing the response amplitude under high nonlinear feedback. Moreover, if the inherent nonlinearity of the resonator is used in the future, this nonlinear characteristic can be provided by lowering the nonlinear-feedback gain.

![Fig. 8](image-url)
Fig. 9 Steady-state amplitude versus nonlinear-feedback gain

(a) $k_{nl-v} = 0.5 \text{Vs/m}$

(b) $k_{nl-v} = 1.7 \text{Vs/m}$

Fig. 10 Dependence of the response amplitude on the response frequency, and the backbone curve. The ● symbols shows the amplitude-frequency relationship and indicates that the variation of the response frequency depends on the amplitude settled to various values by changing the nonlinear feedback gain $k_{nl-v}$. The linear feedback gain $k_{lin-v}$ is set at 0.5 Vs/m and the nonlinear feedback gain $k_{nl-v}$ is tuning from 0.01 Vs/m$^3$ to 0.3 Vs/m$^3$. The solid line denotes the backbone curve obtained from the frequency response curves in Fig. 7.

4.5 Validation of the experimental results

For observing the change of the sensitivity, we add mass $m$ [kg] to the free end of the cantilever (mass: $1.44 \times 10^{-3}$ kg) and detect the frequency shifts with second mode produced by the proposed method in this paper and the frequency shifts with the first mode produced by the method in the previous study [5]. As we discuss in Sect. 4.4, the response frequency approaches the linear natural frequency with the decrease of the amplitude. Therefore, we detect the frequency with small amplitude by tuning the nonlinear feedback gain to be large.

In Table 1, we show the natural frequencies of the cantilever of first mode $f_{1-0}$ and second mode $f_{2-0}$ without measured mass. Then, we add different masses $m$ [kg] to the free end of the cantilever. The frequency shifts of first mode $f_{1} - f_{1-0}$ and the second mode $f_{2} - f_{2-0}$ are shown in Table 2. As the frequency shift of second mode is larger than first mode for the same measured mass, it is experimentally verified that the sensitivity of cantilever sensor in the second mode is higher than that in first mode.

5 Conclusions

To increase the sensitivity of resonating sensors, we proposed a way to use linear and nonlinear feedback to produce a self-excited oscillation of a cantilever with a higher mode. Self-excitation of resonating sensors has attracted much attention because, unlike for external excitation, measurements can be performed even in viscous environments.

Using an averaged equation analytically obtained from a nonlinear equation governing the dynamics of a cantilever, we designed linear- and nonlinear-feedback gains. We then determined a bifurcation set that shows the relationship between the sensor location and the linear-feedback gain. Using this result and a low-pass filter, we devised a method to generate a self-excited oscillation with a higher mode associated with a desired higher natural frequency. We also found that the relationship between the response frequency of the self-excited oscillation and the response amplitude follows a backbone curve that reflects the inherent nonlinearity of the cantilever. A reduction in the amplitude by high-gain nonlinear feedback leads to a self-excited oscillation with a linear higher mode.
Finally, we carried out experiments on a self-excited oscillation of a cantilever with the second mode that verified the validity of our approach. We experimentally confirmed that the nonlinear amplitude-frequency relationship for the self-excited oscillation is equivalent to the backbone curve. Moreover, we experimentally verified that high-gain nonlinear feedback realizes a self-excited oscillation with a linear second natural frequency.

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**Data Availability Statement** The datasets of the current study are available from the corresponding author on reasonable request.

**Declaration**

**Conflict of interest** Conflict of interest: The authors declare that they have no conflict of interest.

### Appendix A: Discussion for the experimental location of sensor

In Sect. 3.1, the nondimensional symbol [ * ] is omitted for simplicity. In this appendix, we express nondimensional terms with [ * ] and dimensional terms without [ * ].

We consider the setting of the sensor. In the theory of our proposed method, we assume that the sensor detects the displacement at the location \( s = s_{sr} \), which is fixed on the beam, moves not only in the \( y \)-direction but also in the \( x \)-direction, and it moves with \( v \) and \( u \) in Fig. 1b. The following relationship [33] implies that \( u^* \) is of the order \( O(v^* 2) \):

\[
u^* = -\frac{1}{2} \int_0^{s^*} \left( \frac{\partial v^*}{\partial s^*} \right)^2 ds^*.
\]

Therefore, if the sensor is set at \( x^* = s_{sr}^* \) on the fixed \( x - y \) plane, the beam deflection can be only approximately measured at \( s = s_{sr} \). Consequently, we use the result for the sensor at \( x^* = s_{sr}^* \) as the deflection at \( s = s_{sr} \). In the production of the self-excitation with the second mode (discussed in Sects. 3 and 4), the sensor location at the tip of the beam \( (s^* = 1) \) leads to the smallest possible linear-feedback gain \( k_{lin-cr} \). However, if the sensor is fixed at \( x^* = 1 \), because \( u^* \) is negative, the laser beam does not strike the cantilever tip when the cantilever is deflected. Therefore, in the experiment described in Sect. 4, we set the sensor at \( x^* = 0.45 \) because the bifurcation set with respect to the second mode in Fig. 3 is a local minimum at \( s^* \approx 0.45 \).

### Appendix B: Setting the orders of linear and nonlinear feedback gains

The basis is to use the limit cycle in Rayleigh oscillator to realize the steady-state amplitude. The order estimation of Eqs. (7) and (10) is in accordance with the order balance of the governing equation in the steady state of the oscillator, which is well known (for example, [42]), as follows.

The Rayleigh equation is generally expressed as:

\[
\ddot{x} - \epsilon(\dot{x} - \frac{1}{3} x^3) + x = 0,
\]

where \( x \) is the deflection, \( \epsilon \) is a small parameter, and \( \dot{x} \) and \( \ddot{x} \) are the first and second derivatives of \( x \) with respect to time, respectively. The steady-state solution of this equation is a limit cycle, and the order balance is given by:

\[
\frac{\partial v^*}{\partial s^*} = \frac{\partial u^*}{\partial s^*} = \frac{\partial \dot{v}^*}{\partial s^*} = \frac{\partial \ddot{v}^*}{\partial s^*} = 0.
\]

This implies that the nonlinear terms are of higher order than the linear terms, and the steady-state solution is a limit cycle.
where $0 < \epsilon \ll 1$. The linear and nonlinear damping terms $-\epsilon \dot{x}$ and $\epsilon \frac{1}{2} \dot{x}^2$ are the same order of $O(\epsilon)$, which are smaller than the inertia and linear stiffness terms, i.e., $\ddot{x}$ and $x$. We realize this order balance in our system, in which $x$ corresponds to the dimensionless deflection of the cantilever in our system $\dot{u}^*$. The parameter $\epsilon$ in Eq. (7) is a book keeping devise [43].

The order of damping effect expressed by the third term in Eq. (3) is $O(\epsilon^2 \dot{u}^*)$ because of $\mu = O(\epsilon^2)$. First, to compensate for the viscous damping, we need to set the order of the linear feedback to be $O(\epsilon^2 \dot{u}^*)$. Then, because the order of $k_{\text{lin}}^* \cdot \frac{\partial u^*}{\partial \dot{u}^*} |_{\dot{u}^*=u^*}$ should be the same as $O(\epsilon^2 \dot{u}^*)$ and the order of the feedback gain should be $O(k_{\text{lin}}) = \epsilon^2$, we set $k_{\text{lin}}$ as Eq. (10).

Furthermore, to realize the steady state as in Rayleigh oscillator, we artificially add the nonlinear effect in Rayleigh oscillator by the nonlinear feedback control, i.e., the second term in Eq. (6). Then, we set the order as $O(\epsilon^2 \dot{u}^*) = O \left( k_{\text{nn}}^* \left( \frac{\partial u^*}{\partial \dot{u}^*} \right)^3 |_{\dot{u}^*=u^*} \right)$ so that the nonlinear feedback is balanced to the viscous damping and the linear feedback terms as Rayleigh oscillator mentioned before. As a result, the nonlinear feedback gain $k_{\text{nn}}^*$ should be $O(1)$ and is set to be the second equation in Eq. (10).

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