Elliptic curve arithmetic and superintegrable systems

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Abstract

Harmonic oscillator and the Kepler problem are superintegrable systems which admit more integrals of motion than degrees of freedom and all these integrals are polynomials in momenta. We present superintegrable deformations of the oscillator and the Kepler problem with algebraic and rational first integrals. Also, we discuss a family of superintegrable metrics on the two-dimensional sphere, which have similar first integrals.

1 Introduction

In 1757-1759 Euler created the theory of elliptic integrals which in turn gave birth to the Abel theory of Abelian integrals, to the Jacobi theory of elliptic functions, to the Riemann theory of algebraic functions, etc. This paper considers two themes in algebraic geometry and elliptic curve cryptography descended from Eulers work: elliptic curve arithmetic and algebraic integrals of Abel’s equations, see Problems 81-84 in Euler’s textbook [12].

In 1760-1767 Euler applied this mathematical theory to searching of algebraic trajectories in the two fixed centers problem. In [9, 10, 11] he reduced equations of motion to one equation defining trajectory

\[ \frac{dr}{\sqrt{R}} + \frac{ds}{\sqrt{S}} = 0, \]

identified algebraic integral of this equation with a partial first integral in the phase space and separated partial algebraic trajectories from transcendental trajectories. In particular, Euler obtained an additional first integral for the superintegrable Kepler problem, which is a partial case of two fixed centers problem, in terms of elliptic coordinates on the plane. In [18] Lagrange proved that equations of motion for the two fixed centers problem with three degrees of freedom are separable in prolate spheroidal coordinates, considered generalized two fixed centers problem and then used algebraic integral of Abel’s equation for searching algebraic trajectories in this generalized two fixed centers problem, see [18, 19] and comments by Serret [23] and Darboux [7]. Modern description of algebraic trajectories in the two centers problem may be found in [8].

Thus, if generic or partial equations of motion are reduced to Abel’s equation on the elliptic curve, then we have additional partial or complete first integral obtained by Euler in his solution of Problems 81-84 in [12]. In [13, 24, 26, 27] we used Euler’s construction in order to classify known superintegrable systems with additional first integrals which are polynomials in momenta. This paper considers superintegrable deformations of the Kepler problem, harmonic oscillators on the plane and geodesics on the sphere which have algebraic and rational additional integrals of motion.

1.1 Arithmetic of elliptic curves

Let us consider smooth nonsingular elliptic curve \( X \) on the projective plane defined by an equation of the form

\[ X : \quad y^2 = f(x), \quad f(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0. \] (1.1)

The prime divisors are points on \( X \), denoted \( P_i = (x_i, y_i) \), including point at infinity \( P_\infty \), which plays the role of neutral element 0 in arithmetic of elliptic curves.
In 1757 Euler proved an addition formulae for elliptic integrals, in modern terms he proved that
by adding two points on $X$

$$(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$$

one gets the third point with the following abscissa and ordinate

$$x_3 = -x_1 - x_2 - \frac{2b_0b_2 + b_1^2 - a_2}{2b_1b_2 - a_3}, \quad \text{and} \quad y_3 = -P(x_3), \quad (1.2)$$

where

$$P(x) = b_2x^2 + b_1x + b_0 = \sqrt{a_4(x - x_1)(x - x_2)} + \frac{(x - x_2)y_1}{x_1 - x_2} + \frac{(x - x_1)y_2}{x_2 - x_1}.$$ 

Then Euler explicitly defined doubling of divisor

$$[2]P_1 = (x_1, y_1) + (x_1, y_1) = ([2]x_1, [2]y_1),$$

i.e. point of $X$ with coordinates

$$[2]x_1 = -2x_1 - \frac{2b_0b_2 + b_1^2 - a_2}{2b_1b_2 - a_3}, \quad [2]y_1 = -P([2]x_1), \quad (1.3)$$

and tripling of divisor

$$[3]P_1 = ([2]x_1, [2]y_1) + (x_1, y_1) = ([3]x_1, [3]y_1),$$

i.e. point of $X$ with coordinates

$$[3]x_1 = -3x_1 - \frac{a_3 - 2b_1b_2}{a_4 - b_2^2}, \quad [3]y_1 = -P([3]x_1),$$

$$P(x) = b_2x^2 + b_1x + b_0 = \frac{(x - x_1)^2(4a_4x_1^3 + 3a_3x_1^2 + 2a_2x_1 + a_1)^2}{8y_1^3} + \frac{(x - x_1)(x(6a_4x_1^2 + 3a_3x_1 + a_2) - 2a_4x_1^3 + a_2x_1 + a_1)}{2y_1} + y_1, \quad (1.4)$$

and described an algorithm for multiplication on any integer $m$, see Problem 83 in [12]. Later Abel
used elliptic curve point multiplication in proving his theorem on $m$-division points of the lemniscate
when he introduced some pre-image of the division polynomials. Modern computer algorithms for
performing addition and multiplication on elliptic curve are discussed in [3, 15, 34].

Lagrange proves Euler’s addition equation introducing time $t$ and motion of two points $P_1(t)$ and
$P_2(t)$ on $X$ governed by the following Newton equations

$$\frac{d^2x_1}{dt^2} = 2y_1^2, \quad \frac{d^2x_2}{dt^2} = 2y_2^2,$$

see details in [14], p.144. In this terms Eulers solutions of Problems 81-84 [12] can be reformulated in
the following form: If two points $P_1 = (x_1, y_1)$ and $P_2(x_2, y_2)$ move along a curve $y^2 = f(x)$, there is
an algebraic constraint on their motion with the property that

$$\int \omega(x_1, y_1)dx_1 + \int \omega(x_2, y_2)dx_2$$

can be expressed, for any differential $\omega(x, y)$, in terms of elementary functions in coordinates $x_1, y_1$
and $x_2, y_2$ when these coordinates satisfy the algebraic constraint. In his famous Paris memoir [11],
Abel states Euler’s conclusion in almost exactly this form as a preamble to his famous theorem.
In Problem 81 Euler calculates algebraic constraint associated with addition of points and proves that
\[ C = \left( \frac{y_1 - y_2}{x_1 - x_2} \right)^2 - a_4(x_1 + x_2)^2 - a_3(x_1 + x_2) \] (1.5)
is the general integral of the differential relation
\[ \frac{dx_1}{y_1} + \frac{dx_2}{y_2} = 0 \] (1.6)
when \( C \) is a constant and particular integral of
\[ \frac{dx_1}{y_1} + \frac{dx_2}{y_2} + \frac{dx_3}{y_3} = 0 \]
when
\[ C = 2a_4x_3^2 + a_3x_4 + a_2 - 2\sqrt{a_4}y_1. \] (1.7)
In 1863 Clebsch proposed geometric approach to construction of algebraic constraints, closely interwoven with the intersection theory, which was continued by Brill and Noether in 1857 and formalized by Poincaré in 1901 and Severy in 1914, see classical textbooks [14, 16], review [4] and modern discussion in [6].

Then in Problem 83 Euler proves that
\[ C_{mn} = \left( \frac{[m]y_1 - [n]y_2}{[m]x_1 - [n]x_2} \right)^2 - a_4([m]x_1 + [n]x_2)^2 - a_3([m]x_1 + [n]x_2) \] (1.8)
is the general integral of the differential relation
\[ \frac{mx_1}{y_1} + \frac{nx_2}{y_2} = 0, \] (1.9)
associated with scalar multiplication of points on integer numbers \( m, n \) and addition of the obtained results. Here and below we write the coordinates of \([m](x, y)\) as \(([m]x, [m]y)\).

In fact Euler proposed only an algorithm of computations, because explicit expression for \( C_{mn} \) is a cumbersome formula. For instance, we have
\[ C_{21} = \frac{A}{B^2} = \frac{16A_6y_1^6 + 32A_5y_1^5 + 16A_4y_1^4 + 16f'_1A_3y_1^3 + 4f_1^2A_2y_1^2 + 4f_1^3A_1y_1 + f_1^4A_0}{(B_3y_1^3 + B_2y_1^2 + B_1y_1 + B_0)^2} \] (1.10)
where \( f_1' = df(x_1)/dx_1 \) is a derivative of the polynomial \( f(x) \) from (1.1) at point \( x = x_1 \),
\[ B = 8\sqrt{a_4}y_1^3 - 4\left(2a_4x_1(x_1 + 2x_2) + a_3(2x_1 + x_2) + a_2y_1^2 \right) - 4\sqrt{a_4}(x_1 - x_2)f_1'y_1 + f_1'^2 \]
and
\[ A_0 = 2a_4x_2^2 + a_3x_2 + a_2 - 2\sqrt{a_4}y_2, \]
\[ A_1 = -4a_2^{3/2}x_1x_2^2 - a_2^{3/2}(2a_3x_1x_2 + a_3x_2^2 + 2a_2x_1 + a_1) + (4a_4x_1 + a_3)y_2, \]
\[ A_2 = -4(x_1^2 + x_2^2)a_2^2 - 4\left(2a_3(x_1^2 + x_2^2) + a_2(x_1^2 + x_2^2) - a_1x_1 \right)a_4 + a_3 + a_2 \]
\[ -2a_2^{3/2} - 2a_3(2x_1x_2)a_4 - 2a_2(4x_1 - x_2)a_3 + 4a_4y_2^2, \]
\[ A_3 = 8x_1(x_1^2 + x_2^2)a_2^2/2 + (2a_3(x_1^2 + x_2^2)(5x_1^2 + x_2 + x_2^2) + 8a_2x_1(2x_1^2 + x_2^2) + 3a_2(2x_1^2 + x_2^2))a_1^{3/2} \]
\[ + (a_3(2x_1^2 + x_2 + x_1^2 + x_2^2) + 2a_2x_1 + a_1a_3(3x_1 + x_2) + 2a_3(9x_1^2 + 4x_1 + x_2 + x_2^2) + 2a_1a_2)a_4^{1/2} \]
\[ - (16a_2^{3/2}x_1 + 2(6a_3x_1^2 - a_0a_4 + a_3(3a_3x_1 + a_2))y_2 - 2a_2^{3/2}(4a_4x_1 + a_3)y_2^2, \]
\[
A_4 = -8a_1x_1^2(3x_1^2+2x_2^2)a_1^4 - 4x_1(3a_1(2x_1^2+x_2^2)+a_2x_1(3x_1^2+2x_2^2)+a_3x_2(3x_1^2+6x_1x_2+4x_1x_2^2+2x_2^2))a_1^4
- (a_1^2+8a_1x_1+a_1a_3(18x_2^2+4x_1x_2+2x_2^2)-2a_2x_1^2+4a_2x_1x_2^2+2a_2x_2^2(4x_1+8x_2^2+8x_1x_2^2+2x_2^2))a_4
- a_1(3a_1^2x_1+a_1a_2x_1+a_2x_1^2(4x_1+x_2)+a_2a_3(3x_1^2+4x_1x_2-x_2^2)-a_2x_1^2+x_2^2)(x_1^2-3x_1x_2-x_2^2)
+ 2a_4^{1/2}(20a_2x_1^2+4x_1(5a_3x_1^2+a_2x_1-a_1)a_4+6a_2x_1^2+4a_2a_3x_1-a_1a_3)\)y_2
+ (16a_2x_1^2+8a_4a_3x_1+a_3)\)y_2,
\]
\[
A_5 = -8a_1^{5/2}x_1^2(2x_1^2+x_2^2)+4a_1^{3/2}(4a_1x_1^2+a_2x_1^2+a_3x_1^2+2x_1^2+3a_2x_2^2+2a_2)
+ a_1^{1/2}(3x_1^2x_2+a_3x_2^2+6a_2a_3x_1+2a_2a_3x_2+2a_2^2)- (8a_1^2x_1^2+4a_4(a_3x_1+a_2)-a_2^2)\)y_2,
\]
\[
A_6 = 8(2x_1^2+x_2^2)a_2^4+(2a_3x_1+a_3x_2+a_2)a_1^2+a_3^2+8a_4^{1/2}y_2.
\]

If \(m = 3\) and \(n = 1\), then one gets an algebraic integral with similar structure

\[
C_{31} = \frac{A}{B^2} = \frac{A_{14}y_1^{14} + A_{13}y_1^{13} + \ldots + A_0}{(B_6y_1^6 + B_5y_1^5 + B_2y_1^2 + B_0)^2},
\]

where \(f_1'' = d^2f(x_1)/dx_1^2\) and

\[
B = 64(3a_4x_1 + a_4x_2 + a_3)y_1^6 - 8f_1''(2a_4x_1^2(x_1 + 3x_2) + 3a_4x_1(x_1 + x_2) + a_2(3x_1 + x_2) + 2a_1)y_1^4
+ 8f_1''(2a_4x_1^2(3x_2 + x_1) + 3a_4x_1x_2 + a_2(x_1 + x_2) + a_1)y_1^2 + f_1''(x_1 - x_2).
\]

For brevity we omit expressions for functions \(A_k\), which are polynomials in coordinates \(x_{1,2}\) and \(y_2\). They can be obtained using any computer algebra system.

In the next Section we apply these Euler’s results to construction of rational and algebraic functions commuting with Hamilton functions of superintegrable systems with two degrees of freedom.

2 Superintegrable systems with two degrees of freedom

The traditional way of writing an elliptic curve equation of is to use its short or long Weierstrass form. In elliptic curve cryptography we can find also other forms of the elliptic curve such as Edwards curves, Jacobi intersections and Jacobi quartics, Hessian curves, Huff curves, etc.

Following [20] we begin with nonsingular elliptic curve \(X\) defined by a short Weierstrass equation

\[
y^2 = f(x), \quad f(x) = x^3 + ax + b,
\]

and arithmetic equation on \(X\)

\[
P_1 + P_2 + P_3 = 0.
\]

Here \(P_1, P_2\) and \(P_3\) are intersection points of \(X\) with a straight line, see standard picture in Figure 1 and in textbooks [3, 15, 34].

![Figure 1: Addition of points \(P_1 + P_2 + P_3 = 0\) on the elliptic curve.](image)
Using coordinates of points $P_1 = (x_1, y_1)$ and $P_2(x_2, y_2)$ we can easily define coordinates of the third point
$$x_3 = \lambda^2 - (x_1 + x_2), \quad y_3 = y_1 + \lambda(x_3 - x_1), \quad \lambda = \frac{y_2 - y_1}{x_2 - x_1}.$$ 
In this case Euler’s integral (1.5) coincides with abscissa of $P_3$, i.e. $C = x_3$ and it is the required algebraic constraint for motion of two points $P_1(t)$ and $P_2(t)$ along the elliptic curve $X$.

Let us also present a well-known expression for the elliptic curve point multiplication on any positive integer $m$:
$$[m](x, y) \equiv [m]x, [m]y = \left( x - \frac{\psi_{m-1} \psi_{m+1}}{\psi_m^2}, \frac{\psi_{2m}}{2\psi_m^4} \right),$$
where $\psi_m$ are the so-called division polynomials in $\mathbb{Z}[x, y, a, b]$, which are the ratio of two Weierstrass $\sigma$-functions, see [3, 20, 34]. It is easy to see that abscissa of $[m]P$ is a rational function strictly in terms of $x$ whereas ordinate has the form $yR(x)$, where $R(x)$ is a rational function.

If we identify abscissas and ordinates of points $P_1$ and $P_2$ with canonical coordinates on the phase space
$$x_{1,2} = u_{1,2}, \quad y_{1,2} = p_{u_{1,2}},$$
where
$$\{u_1, u_2\} = 0, \quad \{p_{u_1}, p_{u_2}\} = 0, \quad \{u_i, p_j\} = \delta_{ij},$$
and solve a pair of equations $y_i^2 = f(x_i), \ i = 1, 2$, with respect to $a, b$, we obtain two functions on the phase space
$$a = \frac{p_{u_1}^2}{u_1 - u_2} + \frac{p_{u_2}^2}{u_2 - u_1} - u_1^2 - u_1u_2 - u_2^2, \quad b = \frac{u_2p_{u_1}^2}{u_2 - u_1} + \frac{u_1p_{u_2}^2}{u_1 - u_2} + (u_1 + u_2)u_1u_2, \quad (2.13)$$
which are in involution with respect to canonical Poisson brackets.

Taking $H = a$ as a Hamiltonian, one gets integrable system on the phase space $T^*\mathbb{R}^2$ with quadratures
$$\omega_1 = \int \frac{u_3 du_1}{\sqrt{u_1^3 + au_1 + b}} + \int \frac{u_2 du_2}{\sqrt{u_2^3 + au_2 + b}} = -2t$$
and
$$\omega_2 = \int \frac{du_1}{\sqrt{u_1^3 + au_1 + b}} + \int \frac{du_2}{\sqrt{u_2^3 + au_2 + b}} = \text{const}.$$ 
In physical terms $a, b$ and $\omega_1, \omega_2$ are action-angle variables associated with this motion, whereas Euler’s algebraic constraint (1.5) is an additional first integral [33].

Second quadrature in the differential form coincides with (1.6) and, therefore, we have superintegrable system with additional first integrals which are abscissa and ordinate of the third point $P_3$ on a projective plane
$$x_3 = \left( \frac{p_{u_1} - p_{u_2}}{u_1 - u_2} \right)^2 - (u_1 + u_2), \quad y_3 = p_{u_1} + \left( \frac{p_{u_1} - p_{u_2}}{u_1 - u_2} \right)(x_3 - u_1).$$
Functions $a, b$ from (2.13) and functions $x_3, y_3$ on the phase space $T^*\mathbb{R}^2$ form an algebra of integrals
$$\{a, b\} = 0, \quad \{a, x_3\} = 0, \quad \{a, y_3\} = 0, \quad \{b, x_3\} = 2y_3, \quad \{b, y_3\} = 3x_3^2 + a, \quad \{x_3, y_3\} = -1,$$
in which Weierstrass equation (2.12) plays the role of syzygy
$$y^2_3 = x_3^3 + ax_3 + b.$$ 
In this case two points $P_1(t)$ and $P_2(t)$ move along a curve $y^2 = f(x)$ with fixed third point $P_3$ because its abscissa $x_3$ and ordinate $y_3$ are additional first integrals, see the picture in Figure 2.
Periodic motion of the points $P_1(t)$ and $P_2(t)$ on two bound pieces of the curve $X$ on the projective plane generates motion by algebraic curves in the phase space, similar to algebraic trajectories in the two fixed centers problem [9, 10, 11].

After canonical transformation of variables

$$u_1 = q_1 - \sqrt{q_2}, \quad u_2 = q_1 + \sqrt{q_2}, \quad p_{u_1} = \frac{p_1}{2} + \frac{(u_1 - u_2)p_2}{2}, \quad p_{u_2} = \frac{p_1}{2} - \frac{(u_1 - u_2)p_2}{2}$$

these first integrals look like

$$a = p_1p_2 - 3q_1^2 - q_2, \quad b = \frac{p_1^2}{4} - q_1p_1p_2 + q_2p_2^2 + 2q_1(q_1^2 - q_2),$$

$$x_3 = p_2^2 - 2q_1, \quad y_3 = \frac{p_1}{2} + p_3 - 3p_2q_1.$$

Superintegrable Hamiltonian $H = a$ (2.13) belongs to a family of superintegrable Hamiltonians on the plane depending on two integer numbers $k_{1,2}$. Indeed, let us consider Hamiltonian

$$H = A_{k_{1,k_2}} = \frac{(k_1^{-1}p_{u_1})^2}{u_1 - u_2} + \frac{(k_2^{-1}p_{u_2})^2}{u_2 - u_1} - u_1^2 - u_1u_2 - u_2^2, \quad (2.14)$$

commuting with the following integral of motion

$$B_{k_{1,k_2}} = \frac{u_2(k_1^{-1}p_{u_1})^2}{u_2 - u_1} + \frac{u_1(k_2^{-1}p_{u_2})^2}{u_1 - u_2} + (u_1 + u_2)u_1u_2.$$

These functions can be obtained from (2.13) by using non-canonical transformation $p_{u_i} \rightarrow k_i^{-1}p_{u_i}$, see discussion in [33]. The corresponding quadratures

$$k_1 \int \frac{du_1}{\sqrt{u_1^3 + au_1 + b}} + k_2 \int \frac{u_2du_2}{\sqrt{u_2^3 + au_2 + b}} = -2t$$

and

$$k_1 \int \frac{du_1}{\sqrt{u_1^3 + au_1 + b}} + k_2 \int \frac{du_2}{\sqrt{u_2^3 + au_2 + b}} = \text{const},$$

are related to arithmetic equation on the elliptic curve $X$

$$[k_1]P_1 + [k_2]P_2 + P_3 = 0.$$
According to Euler Hamiltonians $H = A_{k_1,k_2}$ are in involution with the additional first integral of the form \cite{18,19}
\[ x_3 = C_{k_1,k_2} = \left( \frac{|k_1|y_1 - |k_2|y_2}{|k_1|x_1 - |k_2|x_2} \right)^2 - \left( |k_1|x_1 + |k_2|x_2 \right), \]
which is abscissa $x_3$ of fixed point $P_3$ that is a well-defined rational function on $x_1, y_1, x_2, y_2$ on the projective plane.

At $k_1 = 2$ and $k_2 = 1$ this additional first integral is a rational function of the form
\[ C_{21} = 4u_1 - 3u_2 - \frac{8(u_1 - u_2) \left( 6u_1^2 - 12u_1^2u_2 + 6u_1u_2^2 + p_u, p_{u_2} - 2p_{u_2}^2 \right)}{(8u_1^2 - 12u_1^2u_2 + 4u_2^2 - p_{u_1}^2 + 4p_{u_1}p_{u_2} - 4p_{u_2}^2)}, \]
\[ + \frac{64^2(u_1 - u_2)^2 \left( 2u_1^3 - 3u_1^2u_2 + u_2^3 + p_u, p_{u_2} - p_{u_2}^2 \right)}{(8u_1^2 - 12u_1^2u_2 + 4u_2^2 - p_{u_1}^2 + 4p_{u_1}p_{u_2} - 4p_{u_2}^2)^2}. \]

At $k_1 = 3$ and $k_2 = 1$ additional first integral is equal to
\[ C_{31} = -u_1 + u_2 + \frac{(p_{u_1}^2 - 9p_{u_2}^2)^2}{81(p_{u_1} + p_{u_2})^2(u_1 - u_2)^2} + \frac{8p_{u_1}A_1}{9(p_{u_1} + p_{u_2})^2B} - \frac{64p_{u_2}^3A_2}{9(p_{u_1} + p_{u_2})^2B^2}, \]
where
\[ B = p_{u_1}^4 - 8p_{u_1}^3p_{u_2} + 18 \left( p_{u_2}^2 - u_1^2u_2 + 2u_1u_2^2 - u_2^3 \right) p_{u_1}^2 - 27 \left( p_{u_2}^2 - 2u_1^3 + 3u_1^2u_2 - u_2^3 \right)^2, \]
and
\[ A_1 = (15u_1 + 19u_2)p_{u_1}^4 - 6(13u_1 + 5u_2)p_{u_1}^3p_{u_2} - 3(11u_1 + 19u_2)(2u_1 + u_2)(u_1 - u_2)^2p_{u_1}^2 \]
\[ + 54((u_1 + u_2)p_{u_2}^2 + (2u_1 + u_2)(u_1 - u_2)^3)p_{u_1}p_{u_2} \]
\[ -27p_{u_2}^2(5u_1 + 2u_2)(p_{u_2}^2 - (2u_1 + u_2)(u_1 - u_2)^2), \]
\[ A_2 = 2(u_1 + u_2)p_{u_1}^4 - 4(7u_1 + 5u_2)p_{u_1}^3p_{u_2} + 54p_{u_2}^2(5u_1 + u_2)(p_{u_2}^2 - (2u_1 + u_2)(u_1 - u_2)^2) \]
\[ + 3(42u_1 + u_2)p_{u_2}^2 - (7u_1^2 + 16u_1u_2 + 13u_2^2)(u_1 - u_2)^2)p_{u_1}^2 \]
\[ -9(12(3u_1 + u_2)p_{u_2}^2 - (23u_1^2 + 38u_1u_2 + 11u_2^2)(u_1 - u_2)^2)p_{u_1}p_{u_2}. \]

Algebraic trajectories for these superintegrable systems are generated by rotation of a parabola and a cubic around fixed point $P_3$ on the projective plane instead of rotation of the straight line, see the picture in Figure 2.

Similarly we can take superintegrable systems on the plane with Hamiltonians
\[ H = p_1p_2 + V(q_1, q_2) \]
listed in \cite{22,25,26} and obtain families of the superintegrable Hamiltonians $H_{k_1,k_2}$ depending on integer numbers $k_{1,2}$, see discussion in \cite{33}.

### 2.1 Elliptic coordinates on the plane

Let us come back to physical systems and introduce elliptic coordinates on the plane following Euler \cite{9} and Lagrange \cite{18,19}. If $r$ and $r' = r(x_1,y_1)$ are distances from a point on the plane to the two fixed centers, then elliptic coordinates $u_{1,2}$ are
\[ r + r' = 2u_1, \quad r - r' = 2u_2. \]
If two centres are taken to be fixed at $-\kappa$ and $\kappa$ on the first axis of the Cartesian coordinate system, then we have standard Euler’s definition of the elliptic coordinates on the plane

$$ q_1 = \frac{u_1 u_2}{\kappa}, \quad \text{and} \quad q_2 = \frac{\sqrt{(u_1^2 - \kappa^2)(\kappa^2 - u_2^2)}}{\kappa}. $$

Coordinates $u_{1,2}$ are curvilinear orthogonal coordinates, which take values only on in the intervals

$$ u_2 < \kappa < u_1, $$

i.e. they are locally defined coordinates.

In terms of elliptic coordinates $u_{1,2}$ and the corresponding momenta $p_{u_{1,2}}$ kinetic energy has the following form

$$ 2T = p_1^2 + p_2^2 = \frac{u_1^2 - \kappa^2}{u_1^2 - u_2^2} p_{u_1}^2 + \frac{u_2^2 - \kappa^2}{u_2^2 - u_1^2} p_{u_2}^2, $$

see [7, 13, 23]. By adding separable in elliptic coordinates potentials one gets a Hamiltonian and a second integral of motion

$$ 2H = I_1 = \frac{(u_1^2 - \kappa^2)(p_{u_1}^2 + V_1(u_1))}{u_1^2 I_1 - V_1(u_1)(u_1^2 - \kappa^2) + I_2} + \frac{(u_2^2 - \kappa^2)(p_{u_2}^2 + V_2(u_2))}{u_2^2 I_1 - V_2(u_2)(u_2^2 - \kappa^2) + I_2}, $$

(2.15)

According to Euler and Lagrange there are an equation defining time

$$ \frac{u_1^2 du_1}{\sqrt{(u_1^2 - \kappa^2)(u_1^2 I_1 - V_1(u_1)(u_1^2 - \kappa^2) + I_2)}}, \quad \frac{u_2^2 du_2}{\sqrt{(u_2^2 - \kappa^2)(u_2^2 I_1 - V_2(u_2)(u_2^2 - \kappa^2) + I_2)}}, $$

and an equation defining trajectories of motion

$$ \frac{du_1}{\sqrt{(u_1^2 - \kappa^2)(u_1^2 I_1 - V_1(u_1)(u_1^2 - \kappa^2) + I_2)}}, \quad \frac{du_2}{\sqrt{(u_2^2 - \kappa^2)(u_2^2 I_1 - V_2(u_2)(u_2^2 - \kappa^2) + I_2)}} = 0. $$

In these equations $I_{1,2}$ are the values of integrals of motion, see terminology and discussion in the Lagrange textbook [13] and comments by Darboux and Serret [7, 23].

The second equation is reduced to Euler’s differential relation (1.6) for the Kepler problem

$$ 2H = I_1 = p_1^2 + p_2^2 + \frac{\alpha}{r}, \quad V_1(u_1) = \frac{\alpha u_1}{u_1^2 - \kappa^2} $$

(2.16)

when

$$ \int \frac{du_1}{\sqrt{(u_1^2 - \kappa^2)(I_1 u_1^2 + \alpha u_1 + I_2)}} + \int \frac{du_2}{\sqrt{(u_2^2 - \kappa^2)(I_1 u_2^2 + \alpha u_2 + I_2)}} = \text{const}. $$

and for the harmonic oscillator

$$ 2H = I_1 = p_1^2 + p_2^2 - \alpha^2 (q_1^2 + q_2^2), \quad V_1(u_1) = -\alpha^2 u_1^2 $$

(2.17)

when

$$ \int \frac{du_1}{\sqrt{(u_1^2 - \kappa^2)(I_1 u_1^2 + \alpha^2 u_1^2 (u_1^2 - \kappa^2) + I_2)}} + \int \frac{du_2}{\sqrt{(u_2^2 - \kappa^2)(I_1 u_2^2 + \alpha^2 u_2^2 (u_2^2 - \kappa^2) + I_2)}} = \text{const}. $$

The equation for the harmonic oscillator coincides with equation

$$ \frac{dp}{\sqrt{a_6 p^6 + a_4 p^4 + a_2 p^2 + a_0}} \pm \frac{dq}{\sqrt{a_6 q^6 + a_4 q^4 + a_2 q^2 + a_0}} = 0, $$

8
which Euler studied in Problem 82 in [12] using reduction of this equation to (1.6).

Elliptic curve \( C \) associated with the Kepler problem is defined by the following polynomial of fourth order in \( x \)

\[
 f(x) = I_1 x^4 - \alpha x^3 + (I_2 - I_1 \kappa^2)x^2 + \kappa^2 \alpha x - I_2 \kappa^2, \quad x_{1,2} = u_{1,2}.
\]  

(2.18)

For the harmonic oscillator the corresponding quartic polynomial looks like

\[
 f(x) = \alpha^2 x^4 + (I_1 - 2\alpha \kappa^2)x^3 + (\alpha^2 \kappa^4 - I_1 \kappa^2 + I_2)x^2 - I_2 \kappa^2 x, \quad x_{1,2} = u_{1,2}^2.
\]

(2.19)

when abscissas of divisors \( x_{1,2} = u_{1,2}^2 \) are equal to the squared elliptic coordinates.

Substituting elliptic coordinates and first integrals \( I_{1,2} \) into \( y^2 = f(x) \) one gets expressions for ordinates \( y_{1,2} \) of points \( P_1 = (x_1, y_1) \) and \( P_2 = (x_2, y_2) \). For the Kepler problem we have

\[
y_1 = (u_1^2 - \kappa^2)p_{u_1}, \quad \text{and} \quad y_2 = (u_2^2 - \kappa^2)p_{u_2},
\]

whereas for the harmonic oscillator we obtain

\[
y_1 = u_1(u_1^2 - \kappa^2)p_{u_1}, \quad \text{and} \quad y_2 = u_2(u_2^2 - \kappa^2)p_{u_2}.
\]

Substituting coefficients \( a_4 = I_1, \ a_3 = -\alpha \) and coordinates of divisors \( x_{1,2} \) and \( y_{1,2} \) into the Euler algebraic relation (1.5) one gets an additional first integral for the Kepler problem

\[
 C = \frac{(u_1^2 - \kappa^2)(u_2^2 - \kappa^2)(p_{u_1} - p_{u_2})^2}{(u_1 - u_2)^2},
\]

(2.20)

which is independent on first integrals \( I_{1,2} \). This partial integral of motion in the two fixed centers problem and the corresponding algebraic trajectories were studied by Euler and Lagrange [9][10][11][18][19].

Substituting coefficients \( a_4 = \alpha^2, \ a_3 = I_1 - 2\alpha \kappa^2 \) and coordinates of divisors \( x_{1,2} \) and \( y_{1,2} \) into the Euler algebraic relation (1.5) one gets an additional first integral for the harmonic oscillator

\[
 C = \frac{(u_1^2 - \kappa^2)(u_2^2 - \kappa^2)(p_{u_1} - p_{u_2})^2}{(u_1 - u_2)^2} - 2u_1u_2(u_1^2 - \kappa^2)(u_2^2 - \kappa^2)p_{u_1}p_{u_2} + (u_2^2 - \kappa^2)(u_1^2 - \kappa^2 u_2^2)p_{u_2}^2
\]

\[
 + \alpha^2 \kappa^2(u_1^2 + u_2^2).
\]

(2.21)

First integrals (2.20) and (2.21) have two different forms in elliptic and especially in Cartesian coordinates, but they have a common simple form in terms of coordinates of divisors (1.5).

Other superintegrable systems separable in elliptic coordinates on the plane with first integral of the form (1.5) are discussed in [5][13][20][27][28].

2.2 Elliptic coordinates on the sphere

Two-dimensional metrics which geodesic flows admit three functionally independent first integrals are called superintegrable metrics. Superintegrable metrics with first integrals which are second order polynomials in momenta were described by Koenigs [17]. In this Section we consider a well-known superintegrable metric on the two-dimensional sphere with quadratic first integrals, one of which has the Euler form (1.5). In the next Section we present superintegrable metrics on the sphere with algebraic and rational first integrals which are easily constructed using multiplication of points on elliptic curve. We have to underline that our main aim is a search of algebraic trajectories for dynamical systems following to Euler [9][10][11] and Lagrange [18][19]. Construction of the first integrals is only a tool for the searching of such trajectories.

Let us introduce elliptic coordinates on the two-dimensional sphere \( S^2 \subset \mathbb{R}^3 \) embedded into three-dimensional Euclidean space with Cartesian coordinates \( q_1, q_2 \) and \( q_3 \). The elliptic coordinate system \( u_{1,2} \) on the sphere \( S^2 \) with parameters \( \alpha_1 < \alpha_2 < \alpha_3 \) is defined through equation

\[
 \frac{q_1^2}{z-\alpha_1} + \frac{q_2^2}{z-\alpha_2} + \frac{q_3^2}{z-\alpha_3} = \frac{(z-u_1)(z-u_2)}{\phi(z)}, \quad \phi(z) = (z-\alpha_1)(z-\alpha_2)(z-\alpha_3),
\]

\[9\]
which should be interpreted as an identity with respect to \( z \). It implies

\[ q_1^2 + q_2^2 + q_3^2 = 1, \]

which is a standard description of the sphere in \( \mathbb{R}^3 \). Similar to elliptic coordinates on the plane, elliptic coordinates on the sphere are also orthogonal and only locally defined. They take values in the intervals

\[ \alpha_1 < u_1 < \alpha_2 < u_2 < \alpha_3. \]

The coordinates and the parameters can be subjected to a simultaneous linear transformation \( u_i \rightarrow au_i + b \) and \( \alpha_i \rightarrow a\alpha_i + b \), so it is always possible to choose \( \alpha_1 = 0 \) and \( \alpha_3 = 1 \).

Let us consider free motion on the sphere \( S^2 \) defined by the Hamiltonian

\[ H = p_1^2 + p_2^2 + p_3^2. \]

In elliptic coordinates this Hamiltonian and the corresponding second integral of motion have the following form

\[ 2H = I_1 = \frac{\phi(u_1)p_{u_1}^2}{u_1 - u_2} + \frac{\phi(u_2)p_{u_2}^2}{u_2 - u_1}, \quad I_2 = \frac{u_2\phi(u_1)p_{u_1}^2}{u_2 - u_1} + \frac{u_1\phi(u_2)p_{u_2}^2}{u_1 - u_2}. \]

As above, there are two Abel’s equations

\[ \frac{u_1du_1}{\sqrt{\phi(u_1)(u_1I_1 + I_2)}} + \frac{u_2du_2}{\sqrt{\phi(u_2)(u_2I_1 + I_2)}} = 2dt, \]

\[ \frac{du_1}{\sqrt{\phi(u_1)(u_1I_1 + I_2)}} + \frac{du_2}{\sqrt{\phi(u_2)(u_2I_1 + I_2)}} = 0. \]

defining time and trajectories according to Euler and Lagrange terminology [19].

It is easy to see that second equation

\[ \frac{du_1}{\sqrt{f(u_1)}} + \frac{du_2}{\sqrt{f(u_2)}} = 0, \]

coincides with Euler equation \((1.6)\) on the elliptic curve \( X \) defined by

\[ y^2 = f(x), \quad f(x) = \phi(x)(xI_1 + I_2) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0. \]

It allows us to directly obtain an additional integral of motion \((1.5)\) which has the following form in elliptic coordinates

\[ C = \frac{(a_1\alpha_2\alpha_3 - (a_1\alpha_2 + a_1\alpha_3 + a_2\alpha_3)u_1 + (a_1 + a_2 + \alpha_3)u_2^2 - u_1u_2^2)\phi_1 p_{u_1}^2}{(u_1 - u_2)^2}, \]

\[ -\frac{(a_1\alpha_2\alpha_3 - (a_1\alpha_2 + a_1\alpha_3 + a_2\alpha_3)u_2 + (a_1 + a_2 + \alpha_3)u_1^2 - u_1^2u_2)\phi_2 p_{u_2}^2}{(u_1 - u_2)^2}, \]

\[ -\frac{2\phi_1\phi_2 p_{u_1} p_{u_2}}{(u_1 - u_2)^2}. \]

In modern terms, two-dimensional metric \( g(u_1, u_2) \) in \((2.22)\)

\[ H = \sum g_{ij}p_i p_j, \quad g = \begin{pmatrix} \frac{(u_1 - a_1)(u_1 - a_2)(u_1 - a_3)}{u_1 - u_2} & 0 \\ 0 & \frac{(u_2 - a_1)(u_2 - a_2)(u_2 - a_3)}{u_2 - u_1} \end{pmatrix} \]

is a superintegrable metric on the sphere, thus, we have global superintegrable Hamiltonian system on a compact manifold with closed trajectories. Recall, that problem of finding and describing global
integrable Hamiltonian systems on a compact manifold is one of the central topics in the classical mechanics, see discussion in [2].

Summing up, equations of motion for the Kepler system, for the harmonic oscillator on the plane and for the geodesic motion on the sphere are reduced to equation which coincide with Abel’s equation defining rotation of the parabola around a fixed point on the elliptic curve, see Figure 3.

Figure 3: Rotation of the parabola with points $P_1(t)$ and $P_2(t)$ around fixed point $P_3$.

For all these superintegrable systems motion of two points $P_1(t)$ and $P_2(t)$ on bound pieces of the curve $X$ generates motion by algebraic trajectories in the phase space $T^*\mathbb{R}^2$ or $T^*S^2$.

For all these superintegrable systems additional first integrals (2.20, 2.21) and (2.23) are defined by coordinates of this fixed point $P_3$ by equation (1.7). Of course, abscissa $x_3$ and ordinate $y_3$ are also first integrals depending on $I_1, I_2$ and $C$.

3 Superintegrable systems with algebraic and rational first integrals

In this Section we consider Abel’s equations (1.9)

$$k_1 \frac{dx_1}{y_1} + k_2 \frac{dx_2}{y_2} = 0,$$

defining motion of the straight line, quadric, cubic, quartic and so on around a fixed point on the elliptic curve when two movable points $P_1(t)$ and $P_2(t)$ of degree $k_1, k_2$ and one fixed point $P_3$ form an intersection divisor of elliptic curve $X$ with the straight line, quadric, cubic, quartic, etc.

In order to construct superintegrable systems associated with this motion of points on the elliptic curve we just have to identify Abel’s equations (1.9) on a projective plane with Abel’s equations on some phase space.

3.1 New superintegrable systems on the plane

Let us consider integrable systems with the following Hamiltonian and second integral of motion

$$2H = I_1 = \frac{(u_1^2 - \kappa^2) \left( \frac{p_{u_1}}{k_1} \right)^2 + V_1}{u_1^2 - u_2^2} + \frac{(u_2^2 - \kappa^2) \left( \frac{p_{u_2}}{k_1} \right)^2 + V_2}{u_2^2 - u_1^2},$$

$$I_2 = - \frac{u_2^2(u_2^2 - \kappa^2) \left( \frac{p_{u_2}}{k_1} \right)^2 + V_1}{u_1 - u_2} - \frac{u_1^2(u_1^2 - \kappa^2) \left( \frac{p_{u_1}}{k_1} \right)^2 + V_2}{u_2 - u_1},$$

where $u_{1,2}$ and $p_{u_{1,2}}$ are elliptic coordinates and the corresponding momenta. At $k_1 = k_2$ these integrals of motion coincide with the integral of motion (2.16) up to the scalar factor.
If potentials $V_{1,2}$ are given by \( (2.16) \) or \( (2.17) \), and $k_{1,2}$ are integer positive numbers, then all the trajectories of motion are algebraic trajectories because equation defined trajectories coincides with the Abel’s equation

$$k_1 \frac{dx_1}{y_1} + k_2 \frac{dx_2}{y_2} = 0, \quad k_{1,2} \in \mathbb{Z}_+, $$
onumber

on the elliptic curve

$$y^2 = f(x), \quad f(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0. $$

For the Kepler potential and the harmonic oscillator potential quartic polynomials \( f(x) \) are given by \( (2.18) \) and \( (2.19) \), respectively. Consequently, complete integral \( C_{k_1,k_2} \) \( (1.8) \) of the Abel’s equation gives rise to a complete first integral for the corresponding Hamiltonian system.

**Proposition 1** Hamiltonian $H = 2I_1$ \( (3.24) \) commutes with integral of motion $I_2$

$$\{H, I_2\} = 0. $$

for arbitrary potentials $V_{1,2}$ and parameters $k_{1,2}$.

If potentials are equal to

$$V_i(u_i) = \frac{\alpha u_i}{u_i^2 - \kappa^2} \quad \text{or} \quad V_i(u_i) = -\alpha^2 u_i^2, $$

and $k_{1,2}$ are positive integers, then Hamiltonian $H = 2I_1$ \( (3.24) \) commutes

$$\{H, C_{k_1,k_2}\} = 0. $$

with additional first integral $C_{k_1,k_2}$ \( (1.8) \) which is independent on first integrals $I_{1,2}$.

The proof of this proposition is completely based on the Euler solution of Problem 83 in \[12\].

For the Kepler potential we have to substitute into $C_{k_1,k_2}$ \( (1.8) \) the following coordinates of divisors

$$x_i = u_i, \quad y_i = (u_i^2 - \kappa^2) \frac{p_{u_i}}{k_i}, \quad i = 1, 2, $$

and coefficients

$$a_4 = I_1, \quad a_3 = \alpha, \quad a_2 = I_2 - \kappa^2 I_1, \quad a_1 = \alpha \kappa^2, \quad a_0 = -\kappa^2 I_2, $$

whereas for the harmonic oscillator potential these coordinates and coefficients are equal to

$$x_i = u_i^2, \quad y_i = u_i(u_i^2 - \kappa^2) \frac{p_{u_i}}{k_i}, \quad i = 1, 2, $$

and

$$a_4 = \alpha^2, \quad a_3 = I_1 - 2\alpha^2 \kappa^2, \quad a_2 = \alpha^2 \kappa^4 - \kappa^2 I_1 + I_2, \quad a_1 = -\kappa^2 I_2, \quad a_0 = 0. $$

We calculated integrals $C_{21}$ and $C_{31}$ in two computer algebra systems Mathematica and Maple and directly verified that these integrals are in involution with the corresponding Hamiltonians $H = 2I_1$ \( (3.24) \).

In case of the Kepler potential \( (2.16) \) additional first integrals $C_{21}$ \( (1.10) \) depends on $\sqrt{a_4} = \sqrt{I_1}$, i.e. it is the algebraic function on momenta $p_{u_{1,2}}$. Additional first integral $C_{31}$ \( (1.11) \) is the rational function on elliptic coordinates $u_{1,2}$ and momenta $p_{u_{1,2}}$.

In case of the harmonic oscillator potential \( (2.17) \) both the additional integrals of motion $C_{21}$ and $C_{31}$ are rational functions on elliptic coordinates $u_{1,2}$ and momenta $p_{u_{1,2}}$. For instance, we explicitly present additional first integral $C_{21}$, which is in involution with the Hamiltonian

$$2H = I_1 = \frac{(u_1^2 - \kappa^2)p_{u_1^2}}{4(u_1^2 - u_2^2)} + \frac{(\kappa^2 - u_2^2)p_{u_2^2}}{u_1^2 - u_2^2} + \alpha^2(\kappa^2 - u_1^2 - u_2^2),$$

\[12\]
and does not commute with the second polynomial integral of motion

\[ I_2 = \frac{u_2^2(\kappa^2 - u_1^2)p_{u_1}^2}{4(u_1^2 - u_2^2)} - \frac{u_1^2(\kappa^2 - u_2^2)p_{u_2}^2}{u_1^2 - u_2^2} + \alpha^2 u_1^2 u_2^2. \]

Using various tools in Mathematica and Maple one gets the following observable expression

\[ C_{21} = -\kappa^2 H + (u_2^2 - \kappa^2)(2\alpha u_1 + p_{u_2})p_{u_2} + \alpha^2(\kappa^4 - \kappa^2u_2^4 + u_2^4) + \frac{p_{u_1}C_1}{D} + \frac{p_{u_2}^2 C_2}{D^2}, \]

where

\[ D = \frac{(\kappa^2 u_2^2 + u_1^4 - 2u_1 u_2^2) p_{u_1}^2}{\kappa^2 - u_2^2} + \frac{4\alpha u_1(u_1^2 - u_2^2)^2(\alpha u_1 + p_{u_1})}{\kappa^2 - u_2^2} - 4u_1 u_2 p_{u_1} p_{u_2} + 4u_1^2 p_{u_2}^2, \]

and

\[ C_1 = u_2^2(2\kappa^2 - u_1^2 - 2u_2^2) p_{u_1} + 2\alpha u_1 (\alpha u_1(\kappa^2 - u_1^2))(\alpha u_1 + 4u_2^2 - 3\kappa^2) p_{u_2} + \alpha u_2 (\kappa^2 u_2^2 + 2u_2^2 - 2u_1^2 - u_2^2) + 2\alpha(\kappa^2 u_2^2 + 2u_2^2 - u_1^2) p_{u_2} + \left(\kappa^4 - 8u_2^2\kappa^2 + u_2^2(2u_2^2 - 7u_2^2)\right) p_{u_2} \]

\[ + \frac{8u_1(\alpha u_2 + p_{u_2})}{\alpha(u_1^2 - u_2^2) p_{u_2} + u_2^2(p_{u_1} + (\kappa^2 - u_2^2) p_{u_2})} \]

\[ C_2 = u_2^2(4u_1^2 - \kappa^2(u_1^2 + 2u_2^2)) p_{u_1} - 4u_1 u_2(\kappa^2 u_1^2 - 3\alpha^2 u_1^2 + 2u_1^2) - (\kappa^2(u_1^2 + 4u_2^2) - 5u_1^2 u_2^2 - 2u_1^2) p_{u_2} + 4u_1 u_2\left(\kappa^2 u_2^2 + 2u_2^2 - 2u_1^2 - u_2^2\right) p_{u_2} + \left(\kappa^4 - \kappa^2(12u_2^2 - 7u_2^2) + 24u_1^2 u_2^2\right) p_{u_2} \]

\[ - 16u_1 u_2(\alpha^2(u_1^2 - u_2^2) + \alpha u_2 (\kappa^2 - 5u_2^2) p_{u_2} + \alpha u_2 (\kappa^2(6u_1^2 - 5u_2^2) - 12u_1^2 u_2^2 + 11u_2^2) p_{u_2} + \left(\kappa^4 - 2\kappa^2(u_1^2 - 4u_2^2) + 4u_1^2 u_2^2 + 7u_2^2\right) p_{u_2} \]

\[ + 16u_1^2(4\alpha u_2(\kappa^2 - 2u_2^2) + (\kappa^4 - 8\kappa^2 u_2^2 + 8u_2^4) p_{u_2}) \]

In our opinion, it is practically impossible to use such sophisticated expressions in the direct search of additional integrals of motion or for investigations of algebras of integrals of motion, see also discussion in [5]. Nevertheless, because additional first integrals of motion are easily expressed via coordinates of third point \(P_3 = [k_1]p_1 + [k_2]p_2\) at any \(k_{1,2}\), for instance

\[ C_{k_1 k_2} = 2a_4 x_3^2 + a_3 x_3 + a_2 - 2\sqrt{a_4} y_3, \]

we can derive the algebra of integrals using the well-known syzygies on elliptic curve [14][16].

### 3.2 New superintegrable metrics on the sphere

Let us consider geodesic motion on the two-dimensional sphere \(S^2\) defined by Hamiltonian and second integral of motion

\[ H = I_1 = \frac{\phi(u_1)p_{u_1}^2}{k_1^2(u_1 - u_2)} + \frac{\phi(u_2)p_{u_2}^2}{k_2^2(u_2 - u_1)}, \quad I_2 = \frac{u_2 \phi(u_1)p_{u_1}^2}{k_1^2(u_2 - u_1)} + \frac{u_1 \phi(u_2)p_{u_2}^2}{k_2^2(u_1 - u_2)}. \]  

(3.25)

At \(k_1 = k_2\) these first integrals coincide with (2.22) up to the scalar factor \(k_i^2\). Trajectories of motion are defined as solutions of equation

\[ k_1 \frac{du_1}{\sqrt{\phi(u_1)}(u_1 I_1 + I_2)} + k_2 \frac{du_2}{\sqrt{\phi(u_2)}(u_2 I_1 + I_2)} = 0, \]

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which is the Abel type equation \(1.9\)

\[ k_1 \frac{dx_1}{y_1} + k_2 \frac{dx_2}{y_2} = 0, \quad x_{1,2} = u_{1,2}, \]

on the elliptic curve defined by an equation of the form

\[ y^2 = f(x), \quad f(x) = \phi(x)(x I_1 + I_2) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0. \]

Consequently, we have superintegrable systems with additional first integral \(C_{k_1 k_2}\) \(1.8\).

**Proposition 2** The following Hamiltonian \(H = \sum g_{ij} p_i p_j\) on the two-dimensional sphere with diagonal metric

\[
g = \frac{1}{u_1 - u_2} \begin{pmatrix} (u_1 - \alpha_1)(u_1 - \alpha_2)(u_1 - \alpha_3) & 0 \\ k_1^2 & (\alpha_1 - u_2)(\alpha_2 - u_2)(\alpha_3 - u_2) \\ 0 & k_2^2 \end{pmatrix}, \quad k_1, k_2 \in \mathbb{Z}_+ \]

is in involution with two independent first integrals

\[ \{H, I_2\} = 0, \quad \{H, C_{k_1 k_2}\} = 0. \]

Thus, this metric is a superintegrable metric.

The proof of this proposition is completely based on the Euler solution of Problem 83 in \[12\].

In Cartesian variables \(q_1, q_2, q_3\) and momenta \(p_1, p_2, p_3\) in \(T^* \mathbb{R}^3\), so that

\[ q_1^2 + q_2^2 + q_3^2 = 1, \quad q_1 p_1 + q_2 p_2 + q_3 p_3 = 0, \]

this Hamiltonian has the following form

\[
H = \frac{k_1^2 + k_2^2}{8k_1 k_2} (p_1^2 + p_2^2 + p_3^2) \\
+ \frac{k_1^2 - k_2^2}{2k_1^2 k_2^2 (u_2 - u_1)} (\alpha_1 q_2 q_3 p_2 p_3 + \alpha_2 q_1 q_3 p_1 p_3 + \alpha_3 q_1 q_2 p_1 p_2 + \beta_1 p_1^2 + \beta_2 p_2^2 + \beta_3 p_3^2),
\]

where

\[
\beta_1 = \frac{1}{4} \left( q_2^2 + q_3^2 \right) \alpha_1 + \left( q_1^2 - q_3^2 \right) \alpha_2 + \left( q_1^2 - q_2^2 \right) \alpha_3, \\
\beta_2 = \frac{1}{4} \left( q_2^2 - q_3^2 \right) \alpha_1 + \left( q_1^2 + q_3^2 \right) \alpha_2 + \left( -q_1^2 + q_3^2 \right) \alpha_3, \\
\beta_3 = \frac{1}{4} \left( -q_2^2 + q_3^2 \right) \alpha_1 + \left( q_1^2 + q_3^2 \right) \alpha_2 + \left( q_1^2 + q_2^2 \right) \alpha_3,
\]

and

\[
u_2 - u_1 = \left( (\alpha_3 - \alpha_2)^2 q_1^2 + (\alpha_1 - \alpha_3)^2 q_2^2 + (\alpha_2 - \alpha_1)^2 q_3^2 + 2(\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1)q_1 q_2 q_3 \right)^{1/2} \\
+ 2(\alpha_3 - \alpha_2)(\alpha_1 - \alpha_2)q_1 q_3^2 + 2(\alpha_3 - \alpha_1)(\alpha_2 - \alpha_3)q_2 q_3^2
\]

is a difference of the elliptic coordinates, which is the globally defined strictly positive function on the sphere.

At \(k_1 = 2, 3\) and \(k_2 = 1\) explicit expression for the first integral \(C_{k_1 k_2}\) can be obtained substituting

\[ x_{1,2} = u_{1,2}, \quad y_{1,2} = (u_{1,2} - \alpha_1)(u_{1,2} - \alpha_2)(u_{1,2} - \alpha_3) p_{u_{1,2}} \]

and

\[
a_4 = I_1, \quad a_3 = I_2 - (\alpha_1 + \alpha_2 + \alpha_3) I_1 \quad a_2 = (\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3) I_1 - (\alpha_1 + \alpha_2 + \alpha_3) I_2 \\
a_1 = (\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3) I_2 - \alpha_1 \alpha_2 \alpha_3 I_1, \quad a_0 = -\alpha_1 \alpha_2 \alpha_3 I_2
\]
into the divisors doubling and tripling operations \( (1.3, 1.4) \) and then into the definition of Euler’s integral \( (1.8) \).

Because \( \sqrt{a^4} = \sqrt{I_1} \), first integral \( C_{21} \) \( (1.10) \) is an algebraic function on the elliptic coordinates \( u_{1,2} \) and the corresponding momenta \( p_{u_{1,2}} \). Additional first integral \( C_{31} \) is a rational function on variables of separation of the form \( (1.11) \).

As above at \( k_1 = 2, 3 \) and \( k_2 = 1 \) we directly verified that Hamiltonians \( H = \sum g_{ij} p_{u_i} p_{u_j} \) and first integrals \( C_{k_1,k_2} \) are in involution by using computer algebra systems Mathematica and Maple.

### 4 Conclusion

Consider motion of \( k \) points \( P_1, \ldots, P_k \) around \( m \) fixed points \( P_{k+1}, \ldots, P_{k+m} \) along a plane curve \( X \), which is governed by Abel’s equations generated by the addition of points on \( X \)

\[
(P_1 + \cdots + P_k) + (P_{k+1} + \cdots + P_{k+m}) = 0.
\]

If the same Abel’s equations arise when studying motion of an integrable Hamiltonian or non-Hamiltonian system, then this dynamical system is a superintegrable system with additional partial or complete integrals of motion which are given by the coordinates of fixed points \( P_{k+1}, \ldots, P_{k+m} \). According to Abel’s theorem these integrals are algebraic functions in coordinates of movable points \( P_1, \ldots, P_m \) and, therefore, they are well-defined algebraic functions on original physical variables. Evolution of movable points around the fixed points gives rise to algebraic trajectories of this superintegrable system similar to algebraic trajectories in the Euler two centers problem.

In this note we study motion of two points of degree \( k_1 \) and \( k_2 \) around one fixed point on the elliptic curve. The corresponding Abel’s equations also arise when studying the following superintegrable systems with two degrees of freedom: the Kepler problem, the harmonic oscillator, the geodesic motion on the sphere and their superintegrable deformations. Here we only present the corresponding first integrals, which are polynomial functions in momenta at \( k_1 = k_2 \) and algebraic/rational functions at \( k_1 \neq k_2 \). We plan to discuss the corresponding algebraic trajectories in forthcoming publication.

Arithmetic on elliptic curves has been an object of study in mathematics for well over a century. Recently arithmetic on elliptic curves has proven useful in applications such as factoring \[21\], elliptic curve cryptography \[3, 15, 34\], and in the proof of Fermat’s last theorem \[35\]. In real world elliptic curve point multiplication is one of the most widely used methods for digital signature schemes in cryptocurrencies, which is applied in both Bitcoin and Ethereum for signing transactions. In \[29, 30, 32\] we apply elliptic and hyperelliptic curve point multiplication to discretization of some known integrable systems in Hamiltonian and non-Hamiltonian mechanics and to construction of new integrable Hamiltonian systems in \[50, 31\]. In this note we use this universal mathematical tool to construct new superintegrable systems with algebraic and rational integrals of motion. It will be interesting to discuss other possible applications of arithmetic on elliptic and hyperelliptic curves in classical and quantum mechanics.

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