On convergence properties for generalized Schrödinger operators along tangential curves

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Abstract: In this paper, we consider convergence properties for generalized Schrödinger operators along tangential curves in \( \mathbb{R}^n \times \mathbb{R} \) with less smoothness comparing with Lipschitz condition. Firstly, we obtain sharp convergence rate for generalized Schrödinger operators with polynomial growth along tangential curves in \( \mathbb{R}^n \times \mathbb{R} \), \( n \geq 1 \). Secondly, it was open until now on pointwise convergence of solutions to the Schrödinger equation along non-\( C^1 \) curves in \( \mathbb{R}^n \times \mathbb{R} \), \( n \geq 2 \), we obtain the corresponding results along some tangential curves when \( n = 2 \) by the broad-narrow argument and polynomial partitioning. Moreover, the corresponding convergence rate will follow. Thirdly, we get the convergence result along a family of restricted tangential curves in \( \mathbb{R} \times \mathbb{R} \). As a consequence, we obtain the sharp \( L^p \)-Schrödinger maximal estimates along tangential curves in \( \mathbb{R} \times \mathbb{R} \).

Keywords: Schrödinger operator; Pointwise convergence; Convergence rate; Tangential curve.

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1 Introduction

Let $P(\xi)$ be a real continuous function defined on $\mathbb{R}^n$, $D = \frac{1}{i}(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})$. $P(D)$ is defined via its real symbol

$$P(D)f(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} P(\xi) \hat{f}(\xi) d\xi,$$

where $\hat{f}(\xi)$ denotes the Fourier transform of $f$.

The solution to the generalized Schrödinger equation

$$\begin{cases}
\partial_t u(x, t) - iP(D)u(x, t) = 0 & x \in \mathbb{R}^n, t \in \mathbb{R}^+, \\
u(x, 0) = f
\end{cases}$$  \hspace{1cm} (1.1)

can be formally written as

$$e^{itP(D)}f(x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi + itP(\xi)} \hat{f}(\xi) d\xi.$$  \hspace{1cm} (1.2)

For example, when $P(\xi) = |\xi|^2$, the operator $e^{itP(D)}$ is the celebrated Schrödinger operator $e^{it\Delta}$.

The related pointwise convergence problem is to determine the optimal $s$ for which

$$\lim_{t \to 0^+} e^{itP(D)}f(x) = f(x)$$  \hspace{1cm} (1.3)
almost everywhere whenever \( f \in H^s(\mathbb{R}^n) \). For various \( P(\xi) \), many experts have made many valuable contributions on the development of this subject, see the related articles \[1, 2, 3, 4, 5, 8, 12, 20, 22, 30, 32, 34, 44\].

There are various interesting generalizations of the pointwise convergence problem. The readers can see \[24\] and references therein. One of such generalizations is to consider convergence properties for generalized Schrödinger operators along curves \((\gamma(x,t), t)\) instead of the vertical line \((x,t)\). Here \(\gamma(x,t)\) maps \(\mathbb{R}^n \times \mathbb{R}\) to \(\mathbb{R}^n\), \(\gamma(x,0) = x\). When \(\gamma(x,t)\) is a \(C^1\) function in \(t\), the convergence properties are very similar with that in the vertical case, see \[23\]. However, much less is known when \(\gamma(x,t)\) is just \(\alpha\)-Hölder continuous in \(t\), \(0 < \alpha < 1\). Such curves \((\gamma(x,t), t)\) are called tangential curves since as \(t \to 0\), \((\gamma(x,t), t)\) approaches \((x,0)\) tangentially to the hyperplane \(\{(y,t) \in \mathbb{R}^n \times \mathbb{R} : t = 0\}\). One can see figure 3 in Subsection 1.2 below for the geometry.

In this paper, we mainly consider four kinds of convergence properties for (generalized) Schrödinger operators along tangential curves:

(A) a.e. convergence rate for generalized Schrödinger operators along tangential curves in \(\mathbb{R}^n \times \mathbb{R}, n \geq 1\);

(B) a.e. pointwise convergence for Schrödinger operator along tangential curves in \(\mathbb{R}^2 \times \mathbb{R}\);

(C) a.e. convergence for Schrödinger operator along a family of restricted tangential curves in \(\mathbb{R} \times \mathbb{R}\);

(D) sharp \(L^p\)-Schrödinger maximal estimates along tangential curves in \(\mathbb{R} \times \mathbb{R}\).

In the rest of this introduction, we will introduce (A), (B), (C), (D) in Subsection 1.1, Subsection 1.2, Subsection 1.3, Subsection 1.4 respectively. Our results obtained in Subsection 1.3 and Subsection 1.4 can be extended to more general operators, such as elliptic operators and fractional operators. It will appear in our subsequent articles.

1.1 Convergence rate for generalized Schrödinger operators and applications

The problems on a.e. convergence rate of important operators (such as Fourier multipliers, certain integral means and summability means of Fourier integrals) were investigated in a lot of works \[4, 6, 24, 31, 38\] etc. In \[24\], the authors studied the relationship between smoothness of the functions and the convergence rate for generalized Schrödinger operators with polynomial growth along the curves in \(\mathbb{R}^n \times \mathbb{R}\).
Denote by \( B(x_0, r) \) the ball with center \( x_0 \in \mathbb{R}^n \) and radius \( r \leq 1 \). Suppose that \( \gamma(x, t) \) satisfies

\[
|\gamma(x, t) - \gamma(x, t')| \leq C|t - t'|^{\alpha}, \quad 0 < \alpha \leq 1
\]

uniformly for \( x \in B(x_0, r) \) and \( t, t' \in [0, 1] \), \( \gamma(x, 0) = x \).

**Theorem 1.1.** \([24]\) If there exist real numbers \( m \geq 1 \) and \( s_0 \geq 0 \) such that

\[
|P(\xi)| \lesssim |\xi|^m, \quad |\xi| \to +\infty,
\]

and for each \( s > s_0 \),

\[
\left\| \sup_{0 < t < 1} |e^{itP(D)}(f)(\gamma(x, t))| \right\|_{L^p(B(x_0, r))} \lesssim \|f\|_{H^s(\mathbb{R}^n)}, \quad p \geq 1,
\]

then for all \( f \in H^{s+\delta}(\mathbb{R}^n), \, 0 \leq \delta < m \),

\[
e^{itP(D)}(f)(\gamma(x, t)) - f(x) = o(t^{\alpha\delta/m}), \quad \text{a.e.} \quad x \in B(x_0, r) \quad \text{as} \quad t \to 0^+.
\]

The notation "\( o \)" means infinitesimal of high order.

Li and Wang \([24]\) showed that in the case of vertical line (\( \alpha = 1 \)), \( t^{\delta/m} \) in the inequality (1.7) cannot be replaced by \( t^{\delta/m'} \) for some \( 0 < m' < m \). However, for general case, there is no counterexample in \([24]\) to show \( t^{\alpha\delta/m} \) in the inequality (1.7) is optimal.

Moreover, from the red line in Figure 1 and Figure 2, we observed that when smoothness \( \alpha \) of the curve is fixed, the convergence rate will become close to \( t^\alpha \) as long as smoothness \( \delta \) of the function \( f \) tends to \( m \). The problem is whether it can also guarantee the convergence rate will be close to \( t^\alpha \), if smoothness \( \delta \) of the function is far away from \( m \). We will give a confirmed answer in this article.

**Theorem 1.2.** If there exist real numbers \( m \geq 1 \) and \( s_0 \geq 0 \) such that \( P(\xi) \) satisfies (1.5) and for each \( s > s_0 \),

\[
\left\| \sup_{0 < t < 1} |e^{itP(D)}(f)(\gamma(x, t))| \right\|_{L^p(B(x_0, r))} \lesssim \|f\|_{H^s(\mathbb{R}^n)}, \quad p \geq 1.
\]

Then we have,

1. when \( 1/m \leq \alpha < 1 \), for each \( s > s_0 \), \( f \in H^{s+\delta}(\mathbb{R}^n), \, 0 \leq \delta < \alpha m, \) it holds

\[
\left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f)(\gamma(x, t)) - f(x)|}{t^{\delta/m}} \right\|_{L^q(B(x_0, r))} \lesssim \|f\|_{H^{s+\delta}(\mathbb{R}^n)}, \quad q = \min\{p, 2\};
\]

2. when \( 0 < \alpha < 1/m \), for each \( s > s_0 \), \( f \in H^{s+\delta}(\mathbb{R}^n), \, 0 \leq \delta < 1, \) it holds

\[
\left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f)(\gamma(x, t)) - f(x)|}{t^{\alpha\delta}} \right\|_{L^q(B(x_0, r))} \lesssim \|f\|_{H^{s+\delta}(\mathbb{R}^n)}, \quad q = \min\{p, 2\}.
\]
By standard arguments, we obtain the following convergence rate result.

**Theorem 1.3.** Under the assumption of Theorem 1.2, we have

1. If $1/m \leq \alpha < 1$, then for all $f \in H^{s+\delta}(\mathbb{R}^n)$,
   $$e^{itP(D)}(f)(\gamma(x,t)) - f(x) = o(t^h), \text{ a.e. } x \in B(x_0,r) \text{ as } t \to 0^+,$$
   whenever $(\delta, h) \in D_1 := \{(x, y) : x \geq 0, y \geq 0, y \leq x/m, y < \alpha\}$;

2. If $0 < \alpha < 1/m$, then for all $f \in H^{s+\delta}(\mathbb{R}^n)$,
   $$e^{itP(D)}(f)(\gamma(x,t)) - f(x) = o(t^h), \text{ a.e. } x \in B(x_0,r) \text{ as } t \to 0^+,$$
   whenever $(\delta, h) \in D_2 := \{(x, y) : x \geq 0, y \geq 0, y \leq \alpha x, y < \alpha\}$.

In order to compare the results of Theorem 1.1 (Red line) and Theorem 1.3 (Green line), we will give two figures as follows.

**Figure 1.** $1/m \leq \alpha < 1$.

$x$ means smoothness $\delta$ of the function $f$, $y$ means the convergence rate
Figure 2. $0 < \alpha < 1/m$.

$x$ means smoothness $\delta$ of the function $f$, $y$ means the convergence rate

We notice that it is difficult for us to extend the regions obtained in Theorem 1.3 under such a general assumption on $\gamma$, $P$ and $s_0$. On one hand, for non-zero Schwartz functions, the convergence rate seems no faster than $t^\alpha$ as $t$ tends to 0 along some curves $(\gamma(x,t), t)$, where functions $\gamma(x,t)$ are chosen as in Theorem 1.4 below.

**Theorem 1.4.** There exists

$$\gamma(x,t) = x - e_1 t^\alpha, \; e_1 = (1,0,...,0),$$

such that for each Schwartz function $f$ and $0 < \alpha < 1$, if

$$\lim_{t \to 0^+} \frac{e^{itP(D)}(f)(\gamma(x,t)) - f(x)}{t^\alpha} = 0, \; \text{a.e.} \; x \in \mathbb{R}^n,$$

then $f \equiv 0$, where $P(\xi)$ satisfies (1.5).

On the other hand, a stronger version of Theorem 1.2 can be obtained from its proof.

**Theorem 1.5.** For any $\varepsilon > 0$, if a function $f$ defined on $\mathbb{R}^n$ with $\text{supp } \hat{f} \subset \{ \xi : |\xi| \sim R \}$ for some $R \gg 1$ satisfies

$$\left\| \sup_{0 < t < 1} \left| e^{itP(D)}(f)(\gamma(x,t)) - f(x) \right| \right\|_{L^p(B(x_0,r))} \leq C \varepsilon R^\varepsilon \|f\|_{L^2(\mathbb{R}^n)}, \; p \geq 1,$$

where $C \varepsilon > 0$, $P(\xi)$ satisfies (1.5) for some $m \geq 1$, then we have

(1) when $1/m \leq \alpha < 1$,

$$\left\| \sup_{0 < t < 1} \left| e^{itP(D)}(f)(\gamma(x,t)) - f(x) \right| \right\|_{L^q(B(x_0,r))} \leq C \varepsilon R^{\varepsilon + \delta} \|f\|_{L^2(\mathbb{R}^n)}, \; q = \min\{p,2\}; \tag{1.14}$$

(2) when $0 < \alpha < 1/m$,

$$\left\| \sup_{0 < t < 1} \left| e^{itP(D)}(f)(\gamma(x,t)) - f(x) \right| \right\|_{L^q(B(x_0,r))} \leq C \varepsilon R^{\varepsilon + \delta} \|f\|_{L^2(\mathbb{R}^n)}, \; q = \min\{p,2\}. \tag{1.15}$$

Under the conditions of Theorem 1.5, $t^\delta/m$ on the left hand side of inequality (1.14) can not be replaced by $t^{\delta/m'}$ for some $0 < m' < m$. Also, $t^{\alpha \delta}$ on the left hand side of inequality (1.15) can not be replaced by $t^{\alpha' \delta}$ for some $\alpha' > \alpha$. Two counterexamples will be given in Subsection 2.2 below.
Here we will give some applications. Let’s first recall some results on convergence problem for (fractional) Schrödinger operators along tangential curves.

In $\mathbb{R} \times \mathbb{R}$, [10] established convergence result for Schrödinger operator along the curve $(\gamma(x,t), t)$, where $\gamma(x,t)$ is given in Theorem 1.6 below. The authors obtained the following maximal estimate from which the pointwise convergence result follows.

**Theorem 1.6.** [10] Let $n = 1$ and $0 < \alpha \leq 1$. Denote by $B_r(x_0)$ the interval which has center at $x_0$ with length $2r$, and by $I_T(t_0)$ the interval which has center at $t_0$ with length $2T$. Suppose that a function $\gamma$ satisfies Hölder condition of order $\alpha$, $0 < \alpha \leq 1$ in $t$

$$|\gamma(x,t) - \gamma(x,t')| \leq C|t - t'|^\alpha,$$

and is bilipschitz in $x$

$$C_1|x - y| \leq |\gamma(x,t) - \gamma(y,t)| \leq C_2|x - y|,$$

for arbitrary $x, y \in B_r(x_0)$ and $t, t' \in I_T(t_0)$, $\gamma(x,0) = x$. Then

$$\left\| \sup_{t \in I_T(t_0)} |e^{it\Delta} f(\gamma(x,t))| \right\|_{L^2(B_r(x_0))} \leq C\|f\|_{H^s(\mathbb{R})},$$

if $s > \max\{1/2 - \alpha, 1/4\}$.

Based on the result from Theorem 1.6, we will give an application for Theorem 1.3.

**Theorem 1.7.** Under the assumption of Theorem 1.6 we have,

(1) if $1/2 \leq \alpha < 1$, then for each $s > 1/4$ and all $f \in H^{s+\delta}(\mathbb{R})$,

$$e^{it\Delta}(f)(\gamma(x,t)) - f(x) = o(t^h), \quad \text{a.e. } x \in B(x_0, r) \quad \text{as } t \to t_0^+, \quad (1.16)$$

whenever $(\delta, h) \in D_1 := \{(x,y) : x \geq 0, y \geq 0, y \leq x/2, y < \alpha\}$.

(2) if $0 < \alpha < 1/2$, then for each $s > \max\{1/2 - \alpha, 1/4\}$ and all $f \in H^{s+\delta}(\mathbb{R})$,

$$e^{it\Delta}(f)(\gamma(x,t)) - f(x) = o(t^h), \quad \text{a.e. } x \in B(x_0, r) \quad \text{as } t \to 0^+, \quad (1.17)$$

whenever $(\delta, h) \in D_2 := \{(x,y) : x \geq 0, y \geq 0, y \leq \alpha x, y < \alpha\}$.

In [10], the authors adopted the $TT^*$ method, the time localizing lemma and the van der Corput’s lemma to establish Theorem 1.6. Cho-Lee extended this result in [9] where they obtained the capacity dimension of the divergence set. Recently, getting around of using the time localizing lemma, the corresponding result was obtained for fractional Schrödinger operators by Cho-Shiraki [11]. Combing with Theorem 1.3, we can get the convergence rate results. But we omit the results here.
1.2 Convergence results along tangential curves in $\mathbb{R}^2 \times \mathbb{R}$

Comparing with the case in $\mathbb{R} \times \mathbb{R}$, much less is known about the convergence problem for Schrödinger operator along tangential curves in higher dimensional case $\mathbb{R}^n \times \mathbb{R}$, $n \geq 2$. It follows from [10, Proposition 4.3] that if $P(\xi)$ satisfies

$$|D_\beta^\beta P(\xi)| \lesssim |\xi|^{m-|\beta|-1}, \quad |P(\xi)| \sim |\xi|^{m-1}$$

for $|\xi| \gg 1$ and $m \geq 2$, $\gamma(x,t)$ is bi-lipschitz in $x$ and satisfies Hölder condition of order $\frac{1}{m-1}$ in $t$. Then the convergence of $e^{itP(D)}f$ along the curve $(\gamma(x,t), t)$ follows from the convergence of $e^{itP(D)}f$ along the vertical line $(x,t)$. It is clear that some convergence result along tangential curves for generalized Schrödinger operators can be obtained when $m > 2$.

For $m = 2$ and $n = 1$, Cho-Lee-Vargas [10] showed the pointwise convergence along the curve $(\gamma(x,t), t)$ as required by Theorem 1.6. Ding and Niu [14] improved the above theorem, i.e. $f \in H^s(\mathbb{R})$ for $s \geq 1/4$, if $1/2 \leq \alpha \leq 1$.

However, the convergence problem along tangential curve in $\mathbb{R}^n \times \mathbb{R}$ ($n \geq 2$) was open until now when $m = 2$.

It is now well-known that the method of restriction estimates for the paraboloid adopted by the references [19, 20] can be applied to get the sharp convergence result for $e^{it\Delta}f$ along the vertical line $(x,t)$, one can see the articles [16] for $n = 2$ and [18] for $n \geq 3$. So it is interesting to seek if these methods can be applied to get convergence result for $e^{it\Delta}f$ along tangential curves. In this Subsection, we give a convergence result for Schrödinger operator along tangential curves in $\mathbb{R}^2 \times \mathbb{R}$ by the broad-narrow argument and polynomial partitioning. Moreover, we obtain the corresponding convergence rate.

Let $\Gamma_\alpha := \{ \gamma : [0, 1] \to \mathbb{R}^2 : \text{for each } t, t' \in [0, 1], |\gamma(t) - \gamma(t')| \leq C_\alpha |t - t'|^{\alpha} \}$, $C_\alpha \geq 1$. We consider the convergence problem of the Schrödinger operator along the curves $(x + \gamma(t), t)$ with $\gamma \in \Gamma_\alpha$ for some $\alpha \in [1/2, 1)$. For example, $\gamma(t) = t^\alpha \mu, \alpha \in [1/2, 1)$ and $\mu$ is a bounded vector in $\mathbb{R}^2$.

We have the maximal estimate below.

**Theorem 1.8.** Let $p = 3.2$. For all $\varepsilon > 0$ and $f \in H^{3/8+\varepsilon}(\mathbb{R}^2)$, it holds

$$\left\| \sup_{t \in (0,1)} |e^{it\Delta}f(x + \gamma(t))| \right\|_{L^p(B(0,1))} \leq C\|f\|_{H^s(\mathbb{R}^2)}.$$  \hspace{1cm} (1.18)

Here, the constant $C$ depends only on $\varepsilon$ and $C_\alpha$, but does not depend on the choice of $\gamma$. 

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By Theorem 1.8 if $\gamma \in \Gamma_\alpha$ and $\gamma(0) = (0, 0)$, then the convergence result

$$\lim_{t \to 0^+} e^{it\Delta} f (x + \gamma(t)) = f(x) \text{ a.e. } x \in \mathbb{R}^2$$

(1.19)

holds whenever $f \in H^s(\mathbb{R}^2)$, $s > 3/8$. Hence, the corresponding convergence rate follows from Theorem 1.8 and Theorem 1.3.

**Theorem 1.9.** For each $s > 3/8$ and all $f \in H^{s+\delta}(\mathbb{R}^2)$. If $\gamma \in \Gamma_\alpha$, $\gamma(0) = (0, 0)$, then

$$e^{it\Delta} f (x + \gamma(t)) - f(x) = o(t^h), \text{ a.e. } x \in \mathbb{R}^2 \text{ as } t \to 0^+,$$

(1.20)

whenever $(\delta, h) \in D_1 := \{(x, y) : x \geq 0, y \geq 0, y \leq x/2, y < \alpha\}$.

Before the proof of Theorem 1.8 we give the following remark. If Theorem 1.8 holds true, then for each $(x_0, t_0) \in \mathbb{R}^2 \times \mathbb{R}$, and $\gamma$ which maps $[t_0, t_0 + 1]$ to $\mathbb{R}^2$ such that for each $t, t' \in [t_0, t_0 + 1],

$$|\gamma(t) - \gamma(t')| \leq C_\alpha |t - t'|^{\alpha},$$

it holds

$$\left\| \sup_{t \in (t_0, t_0+1)} |e^{it\Delta} f (x + \gamma(t))| \right\|_{L^p(B(x_0,1))} \leq C\|f\|_{H^s(\mathbb{R}^2)}, \ s > 3/8,$$

(1.21)
where the constant $C$ depends only on $\varepsilon$ and $C_\alpha$. Indeed, changing variables implies that
\[
\left\| \sup_{t \in (t_0, t_0 + 1]} |e^{it\Delta} f(x + \gamma(t))| \right\|_{L^p(B(x_0, 1))} = \left\| \sup_{t \in (0, 1)} |e^{it\Delta} f(x + \gamma(t + t_0))| \right\|_{L^p(B(0, 1))},
\]
in which $\hat{g}(\xi) = e^{ix_0 \cdot \xi + it_0 |\xi|^2} \hat{f}(\xi)$. It is obvious that $\gamma(t + t_0) \in \Gamma_\alpha$, then by Theorem 1.8, we have
\[
\left\| \sup_{t \in (0, 1)} |e^{it\Delta} g(x + \gamma(t + t_0))| \right\|_{L^p(B(0, 1))} \leq C \|g\|_{H^s(\mathbb{R}^2)}, \quad s > 3/8.
\]
Here the constant $C$ depends only on $\varepsilon$ and $C_\alpha$, but does not depend on the choice of $\gamma$. Since $\|f\|_{H^s} = \|g\|_{H^s}$, inequality (1.21) follows. Moreover, it follows from inequality (1.21) that
\[
\lim_{t \to t_0^+} e^{it\Delta} f(x + \gamma(t)) = e^{it_0\Delta} f(x + \gamma(t_0)) \quad \text{a.e. } x \in \mathbb{R}^2.
\]

Now let’s turn to reduce the proof of Theorem 1.8. By Littlewood-Paley decomposition, it suffices to show that
\[
\left\| \sup_{t \in (0, 1)} |e^{it\Delta} f(x + \gamma(t))| \right\|_{L^p(B(0, 1))} \leq CR^{3/8+\varepsilon} \|f\|_{L^2}, \quad (1.22)
\]
whenever supp $\hat{f} \subset \{\xi \in \mathbb{R}^2 : |\xi| \sim R\}$, $R \gg 1$, the constant $C$ depends on $\varepsilon$ and $C_\alpha$, but does not depend on the choice of $\gamma$. For each $\gamma \in \Gamma_\alpha$, according to the time localizing lemma, the global estimate
\[
\left\| \sup_{t \in (0, 1)} |e^{it\Delta} f(x + \gamma(t))| \right\|_{L^p(B(0, 1))} \leq C'R^{3/8+\varepsilon} \|f\|_{L^2} \quad (1.23)
\]
follows from the local estimate on each interval $(t_0, t_0 + R^{-1}) \subset (0, 1),$
\[
\left\| \sup_{t \in (t_0, t_0 + R^{-1})} |e^{it\Delta} f(x + \gamma(t))| \right\|_{L^p(B(0, 1))} \leq C''R^{3/8+\varepsilon} \|f\|_{L^2}. \quad (1.24)
\]
Notice that the constant $C'$ in the global estimate (1.23) depends on $C_\alpha$ and the constant $C''$ in the local estimate (1.24). Therefore, in order to show inequality (1.22), we just need to prove the theorem below.

**Theorem 1.10.** Let $p = 3.2$, $R \gg 1$ and
\[
\Gamma_{\alpha,R^{-1}} := \{ \gamma : [0, R^{-1}] \to \mathbb{R}^2 : \text{for each } t, t' \in [0, R^{-1}], |\gamma(t) - \gamma(t')| \leq C_\alpha |t - t'|^\alpha \}.
\]
For any $\varepsilon > 0$, we have
\[
\left\| \sup_{t \in (0,R^{-1})} |e^{it\Delta} f(x + \gamma(t))| \right\|_{L^p(B(0, 1))} \leq CR^{3/8+\varepsilon} \|f\|_{L^2}, \quad (1.25)
\]
for all $f$ with supp $\hat{f} \subset \{\xi \in \mathbb{R}^2 : |\xi| \sim R\}$, where the constant $C$ depends only on $\varepsilon$ and $C_\alpha$. 
If Theorem 1.10 holds true, then for any interval \((t_0, t_0 + R^{-1}) \subset (0, 1)\) and arbitrary \(\gamma \in \Gamma_\alpha\),

\[
\left\| \sup_{t \in (t_0, t_0 + R^{-1})} |e^{it\Delta} f(x + \gamma(t))| \right\|_{L^p(B(0,1))} = \left\| \sup_{t \in (0,R^{-1})} |e^{it\Delta} g(x + \gamma(t + t_0))| \right\|_{L^p(B(0,1))},
\]

where \(g(\xi) = e^{it|\xi|^2} \hat{f}(\xi)\). It is clear that \(\gamma(t + t_0) \in \Gamma_{\alpha,R^{-1}}\) and Theorem 1.10 implies that

\[
\left\| \sup_{t \in (0,R^{-1})} |e^{it\Delta} g(x + \gamma(t + t_0))| \right\|_{L^p(B(0,1))} \leq CR^{3/8+\varepsilon} \|g\|_{L^2},
\]

where the constant \(C\) depends only on \(\varepsilon\) and \(C_\alpha\). Then we have the local estimate (1.24) with the constant \(C''\) depends only on \(\varepsilon\) and \(C_\alpha\), so does the constant \(C'\) in (1.23). Finally we arrive at inequality (1.22).

By parabolic rescaling, Theorem 1.10 can be reduced to show that for each \(\gamma \in \Gamma_{\alpha,R^{-1}}\),

\[
\left\| \sup_{t \in (0,R)} |e^{it\Delta} f(x + R\gamma(\frac{t}{R^2}))| \right\|_{L^p(B(0,R))} \leq CR^{2/p-5/8+\varepsilon} \|f\|_{L^2},
\]

for all \(f\) with \(\text{supp} \hat{f} \subset B(0,1)\). But in order to apply the induction argument in the frequency space of \(f\), we will prove the following theorem.

**Theorem 1.11.** Let \(p = 3.2\). For arbitrary \(\gamma \in \Gamma_{\alpha,R^{-1}}\), any \(\varepsilon > 0\), all balls \(B(\xi_0,M^{-1}) \subset B(0,1)\) and \(f\) with \(\text{supp} \hat{f} \subset B(\xi_0,M^{-1})\), it holds

\[
\left\| \sup_{t \in (0,R)} |e^{it\Delta} f(x + R\gamma(\frac{t}{R^2}))| \right\|_{L^p(B(0,R))} \leq CM^{-\varepsilon^2} R^{2/p-5/8+\varepsilon} \|f\|_{L^2}. \tag{1.26}
\]

Here the constant \(C\) depends only on \(\varepsilon\) and \(C_\alpha\), but does not depend on the choice of \(\gamma\).

We notice that the following feature plays key role in the proof of Theorem 1.11 \(\gamma \in \Gamma_{\alpha,R^{-1}}\) implies that \(R\gamma(\frac{t}{R^2})\) is \(\alpha\)-Hölder continuous for \(t \in [0,R]\) when \(\alpha \in [1/2, 1)\). However this feature is no longer true for \(\alpha \in (0, 1/2]\), then the method in this paper does not work. We leave the case \(\alpha \in (0, 1/2]\) for further consideration.

We put the detailed proof of Theorem 1.11 into Section 3 and briefly discuss the method here. We would like to prove Theorem 1.11 by the broad-narrow argument and polynomial partitioning. Such methods work very well in the study of Fourier restriction operators, see 19 and 20 for instance. Du-Li 15 firstly applied the method in 20 to study convergence problem for Schrödinger operator along vertical lines. Further, with the help of decoupling methods, the result of 15 was improved to sharp convergence in 16. The main difference between our problem and the vertical case is that the support for the Fourier transform of \(e^{it\Delta} f(x + R\gamma(\frac{t}{R^2}))\)
is not clear, since \( \gamma(t) \) is not smooth. This leads to the failure of many nice properties which are very important in the research of the works \[15,16,19,20\]. In order to overcome these difficulties, we prove Lemma 3.1 below. Lemma 3.1 works as a substitution for the locally constant property. However, we still do not know if the decoupling method can help to improve the result obtained by Theorem 1.11, also by Theorem 1.8.

The convergence result obtained by Theorem 1.8 may not be sharp. In fact, by Lemma 4.1 below and Bourgain’s counterexample in \[2\], we can get the following necessary condition.

**Theorem 1.12.** Let \( \gamma(x,t) = x - e_1 t^\alpha \), \( e_1 = (1,0) \), \( \alpha \in [1/2, 1) \). Then
\[
\left\| \sup_{0<t<1} |e^{it\Delta} f(\gamma(x,t))| \right\|_{L^1(B(0,1))} \leq C \|f\|_{H^s(\mathbb{R}^2)}
\]  
(1.27)
holds for all \( f \in H^s(\mathbb{R}^2) \) only if \( s \geq 1/3 \).

It is clear that there is a gap between 1/3 and 3/8. The next theorem implies that if one wants to improve the convergence result obtained by Theorem 1.8 then the range of \( p \) should be chosen very carefully. More concretely, if \( s \to 1/3^+ \), then \( p \) should not be larger than 3.

**Theorem 1.13.** Let \( \gamma(x,t) = x - e_1 t^\alpha \), \( e_1 = (1,0) \), \( \alpha \in [1/2, 1) \). If \( 0 < s < 1/2 \) and
\[
\left\| \sup_{0<t<1} |e^{it\Delta} f(\gamma(x,t))| \right\|_{L^p(B(0,1))} \leq C \|f\|_{H^s(\mathbb{R}^2)}
\]  
(1.28)
holds for all \( f \in H^s(\mathbb{R}^2) \), then \( p \leq \frac{1}{1-2s} \).

Proofs of Theorem 1.12 and Theorem 1.13 will appear in Section 4.

### 1.3 Convergence results along a family of restricted tangential curves in \( \mathbb{R} \times \mathbb{R} \)

In this section, we consider the convergence problem along a class of restricted tangential curves in \( \mathbb{R} \times \mathbb{R} \) given by \( \{(y,t) : y \in \Gamma_{x,t}\} \) for each \( t \in [0,1] \), where
\[
\Gamma_{x,t} = \{ \gamma(x,t,\theta) : \theta \in \Theta \}
\]
for a given compact set \( \Theta \) in \( \mathbb{R} \). \( \gamma \) is a map from \( \mathbb{R} \times [0,1] \times \Theta \) to \( \mathbb{R} \), which satisfies \( \gamma(x,0,\theta) = x \) for all \( x \in \mathbb{R}, \theta \in \Theta \), and the following conditions (C1)-(C3) hold:

(C1) for fixed \( t \in [0,1], \theta \in \Theta \), \( \gamma \) has at least \( C^1 \)-regularity in \( x \), and there exists a constant \( C_1 \geq 1 \) such that for each \( x,x' \in \mathbb{R}, \theta \in \Theta, t \in [0,1] \),
\[
C_1^{-1}|x-x'| \leq |\gamma(x,t,\theta) - \gamma(x',t,\theta)| \leq C_1|x-x'|; 
\]  
(1.29)
(C2) there exists a constant $C_2 > 0$ and $\alpha \in (0, 1)$ such that for each $x \in \mathbb{R}$, $\theta \in \Theta$, $t, t' \in [0, 1]$,

$$|\gamma(x, t, \theta) - \gamma(x, t', \theta)| \leq C_2 |t - t'|^\alpha; \quad (1.30)$$

(C3) there exists a constant $C_3 > 0$ such that for each $x \in \mathbb{R}$, $t \in [0, 1]$, $\theta, \theta' \in \Theta$,

$$|\gamma(x, t, \theta) - \gamma(x, t, \theta')| \leq C_3 |\theta - \theta'|. \quad (1.31)$$

Let’s study the relationship between the dimension of $\Theta$ and the optimal $s$ for which

$$\lim_{(y,t) \to (x,0)} e^{it\Delta} f(y) = f(x) \quad a.e. \ x \in \mathbb{R} \quad (1.32)$$

whenever $f \in H^s(\mathbb{R})$. In order to understand such convergence problem better, we give the exact example below, see Figure 4 and Figure 5. Here we introduce the so-called logarithmic density or upper Minkowski dimension of $\Theta$ to characterize its size, which is defined by

$$\beta(\Theta) = \limsup_{\delta \to 0^+} \frac{\log N(\delta)}{-\log \delta},$$

where $N(\delta)$ is the minimum number of closed balls of diameter $\delta$ to cover $\Theta$. Apparently, when $\Theta$ is a single point, $\beta(\Theta) = 0$; when $\Theta$ is a compact subset of $\mathbb{R}^n$ with positive Lebesgue measure, $\beta(\Theta) = n$.

![Figure 4](image-url)

Figure 4. $\Theta = \{1, \frac{5}{6}, \cdots, \frac{1}{2} + \frac{1}{k+1}, \cdots, \frac{1}{2} : k \in \mathbb{N}_+\}$. 
Figure 5. The convergence path (green curve) is determined by the set
\[ \bigcup_{t \in [0, 1/2]} \{(y, t) : y \in \Gamma_{x,t}\} \]
whose element consists of all black points where
\[ \Gamma_{x,t} = \{x + \theta \sqrt{t}, \theta \in \Theta\}. \]

In [10], this question is considered for a family of restricted straight lines in \( \mathbb{R} \times \mathbb{R} \). Exactly, for \( t \in [-1, 1] \), let \( \Gamma_{x,t} = \{x + t\theta : \theta \in \Theta\} \), where \( \Theta \) is a given compact set in \( \mathbb{R} \). In [10], they proved that the corresponding non-tangential convergence result holds for \( s > \frac{\beta(\Theta)+1}{4} \).

Then Shiraki [33] generalized this result to a wider class of equations which includes the fractional Schrödinger equation. Very recently, Li-Wang-Yan [25] obtained the corresponding non-tangential convergence result in any dimensions and extend the straight lines to more general curves with Lipschitz regularity in time variable. The problem is what will happen if the curves satisfy just \( \alpha \)-Hölder regularity \( (0 < \alpha < 1) \). Next, we will give an answer about it.

By standard argument, the convergence result follows from the maximal estimate below.

**Theorem 1.14.** When \( \gamma \) satisfies (C1)-(C3), considering the maximal estimate
\[
\left\| \sup_{t \in (0,1), \theta \in \Theta} |e^{it\Delta} f(\gamma(x,t,\theta))| \right\|_{L^p(B(x_0,r))} \leq C \|f\|_{H^s(\mathbb{R})}, \quad f \in H^s(\mathbb{R}),
\] (1.33)
where \( B(x_0, r) \subset \mathbb{R} \). We have,

1. for each \( \alpha \in [1/2, 1) \), inequality (1.33) holds if \( s > s_0 = \frac{\beta(\Theta)+1}{4} \) and \( p = 4 \);
2. for each \( \alpha \in (1/4, 1/2) \), inequality (1.33) holds if \( s > s_0 = \frac{\beta(\Theta)+1}{4} \) and \( p = 8\alpha \);
3. for each \( \alpha \in (0, 1/4] \), inequality (1.33) holds if \( s > s_0 = \alpha \beta(\Theta) + \frac{1}{2} - \alpha \) and \( p = 2 \).

Moreover, we notice that the constant on the right hand side of inequality (1.33) depends only on \( C_1, C_2, C_3, \Theta \) and the choice of \( B(x_0, r) \), but does not depend on \( f \).

Then we have the following convergence result for Schrödinger operator along a family of restricted tangential curves.

**Theorem 1.15.** Let \( \gamma \) satisfy (C1)-(C3). The convergence result (1.32) holds almost everywhere if

1. \( \alpha \in [1/2, 1) \), \( f \in H^s(\mathbb{R}) \) and \( s > s_0 = \frac{\beta(\Theta)+1}{4} \);
2. \( \alpha \in (1/4, 1/2) \), \( f \in H^s(\mathbb{R}) \) and \( s > s_0 = \frac{\beta(\Theta)+1}{4} \);
3. \( \alpha \in (0, 1/4] \), \( f \in H^s(\mathbb{R}) \) and \( s > s_0 = \alpha \beta(\Theta) + \frac{1}{2} - \alpha \).

Theorem 1.15 is sharp when \( \beta(\Theta) = 0 \) (see [10] or Theorem 1.6 in this paper) and \( \beta(\Theta) = 1 \) (see [34]). We leave necessity of the case \( 0 < \beta(\Theta) < 1 \) for further consideration.
Here we briefly sketch the main idea for the proof of Theorem 1.14 and leave the details to Section 5. By Littlewood-Paley decomposition, we only need to consider $f$ whose Fourier transform is supported in $\{\xi \in \mathbb{R} : |\xi| \sim \lambda\}$, $\lambda \gg 1$. Next we decompose $\Theta$ into small subsets $\{\Theta_k\}$ such that $\Theta = \bigcup_k \Theta_k$ with bounded overlap, where each $\Theta_k$ is contained in a closed ball with diameter $\lambda^{-\mu}$, $\mu > 0$. Then the number of $\Theta_k$ is dominated by $\lambda^{\mu\beta(\Theta) + \varepsilon}$, for any $\varepsilon > 0$.

Theorem 1.14 is reduced to consider the estimate
\[
\sup_{t \in (0,1), \theta \in \Theta_k} |e^{i \Delta f(\gamma(x,t,\theta))}|_{L^p(B(x_0,r))} \leq C \lambda^\nu \|f\|_{L^2},
\]
where the constant $C$ does not depend on $k$. By [10] or Theorem 1.6 in this paper, $\nu$ cannot be smaller than $\max\{1/2 - \alpha, 1/4\}$. Then we have to solve two problems:

(P1) what is the smallest possible value for $\mu$ such that (1.34) holds for $\nu = \max\{1/2 - \alpha, 1/4\}$;

(P2) what is the largest possible value for $p$ such that (1.34) holds for $\nu = \max\{1/2 - \alpha, 1/4\}$.

These two problems are solved by Lemma 5.1 below. What’s more, we will further discuss the problem (P2) in the next Subsection.

1.4 Sharp $L^p$–Schrödinger maximal estimates along tangential curves in $\mathbb{R} \times \mathbb{R}$

We notice that the problem (P2) is of independent interest in the study of Schrödinger maximal function. For celebrated $L^p$-Schrödinger maximal estimate, one would find the optimal $p$ and $s$ such that the maximal estimate holds. When spatial dimension $n = 2$, Du-Guth-Li [16] proved the sharp $L^p$-estimates for all $p \leq 3$ and $s > 1/3$. When spatial dimension $n \geq 3$, Du-Zhang [18] proved the sharp $L^2$-estimate with $s > n/2(n + 1)$, but the sharp $L^p$-estimate of Schrödinger maximal function is still unknown for $p > 2$. Partial results on this problem are obtained by using polynomial partitioning and refined Strichartz estimates in [7,17,43].

As a consequence of Theorem 1.14 we achieve the sharp $L^p$-Schrödinger maximal estimates along tangential curves in $\mathbb{R} \times \mathbb{R}$. In fact, we take $\Theta$ to be the set only consisting of a single point $\theta_0$, and rewrite the conditions (C1)-(C3) as follows. Here we abuse the notation a bit and replace $\gamma(x,t,\theta_0)$ by $\gamma(x,t)$. Let $\gamma$ be a map from $\mathbb{R} \times [0,1]$ to $\mathbb{R}$, which satisfies $\gamma(x,0) = x$ for all $x \in \mathbb{R}$ and the following conditions (C1)',(C2)' hold:

(C1)' for fixed $t \in [0,1]$, $\gamma$ has at least $C^1$-regularity in $x$, and there exists a constant $C_1 \geq 1$ such that for each $x, x' \in \mathbb{R}$, $t \in [0,1],
\[
C_1^{-1}|x - x'| \leq |\gamma(x,t) - \gamma(x',t)| \leq C_1|x - x'|; \tag{1.35}
\]
there exists a constant $C_2 > 0$ and $\alpha \in (0, 1)$ such that for each $x \in \mathbb{R}$, $t, t' \in [0, 1]$,

$$|\gamma(x, t) - \gamma(x, t')| \leq C_2|t - t'|^{\alpha}. \quad (1.36)$$

Specially, Sjölin [36] has studied $L^p$-Schrödinger maximal estimates for the case $\gamma(x, t) = x$.

**Theorem 1.16.** Let $n = 1$ and $0 < \alpha < 1$. Suppose that a function $\gamma$ satisfies conditions (C1)', (C2)' for arbitrary $x, y \in B(x_0, r) \subset \mathbb{R}$ and $t, t' \in [0, 1]$, $\gamma(x, 0) = x$. Considering the $L^p$-maximal estimate,

$$\left\| \sup_{t \in (0, 1)} |e^{it\Delta} f(\gamma(x, t))| \right\|_{L^p(B(x_0, r))} \leq C \| f \|_{H^s(\mathbb{R})}, \quad (1.37)$$

we have,

1. for each $s > 1/4$ and $\alpha \in [1/2, 1)$, inequality (1.37) holds if $p \leq 4$;
2. for each $s > 1/4$ and $\alpha \in (1/4, 1/2)$, inequality (1.37) holds if $p \leq 8\alpha$;
3. for each $s > 1/2 - \alpha$ and $\alpha \in (0, 1/4]$, inequality (1.37) holds if $p \leq 2$.

Moreover, the constant $C$ on the right hand side of inequality (1.37) depends only on $C_1, C_2$ and the choice of $B(x_0, r)$.

Figure 6. Relationship between $p$ and $\alpha$ when $s$ is fixed.

Figure 7. Relationship between $s$ and $\alpha$ when $p$ is fixed.
It is clear that Theorem 1.16 improves the previous $L^2$-Schrödinger maximal estimates of [10], see also Theorem 1.6 in this paper. Finally, we will show that the upper bound for $p$ obtained by Theorem 1.16 cannot be improved when $\gamma(x, t)$ are chosen as in Theorem 1.17 below.

**Theorem 1.17.** Taking $\gamma(x, t) = x + t^\alpha$, we consider the $L^p$-maximal estimate

$$
\left\| \sup_{t \in (0,1)} |e^{it\Delta}f(\gamma(x, t))| \right\|_{L^p(B(0,1))} \leq C\|f\|_{H^s(\mathbb{R})},
$$

(1.38)

we have,

(1) inequality (1.38) holds for each $s > 1/4$ and $\alpha \in [1/2, 1)$ only if $p \leq 4$;

(2) inequality (1.38) holds for each $s > 1/4$ and $\alpha \in (1/4, 1/2)$ only if $p \leq 8\alpha$;

(3) inequality (1.38) holds for each $s > 1/2 - \alpha$ and $\alpha \in (0, 1/4]$ only if $p \leq 2$.

We will show the proof for Theorem 1.17 in Section 6.

**Conventions:** Throughout this article, we shall use the notation $A \ll B$, which means that there is a sufficiently large constant $G$, which does not depend on the relevant parameters arising in the context in which the quantities $A$ and $B$ appear, such that $GA \leq B$. We write $A \sim B$, and mean that $A$ and $B$ are comparable. By $A \lesssim B$ we mean that $A \leq CB$ for some constant $C$ independent of the parameters related to $A$ and $B$. We write $A \wedge B = \min\{A, B\}$. Given $\mathbb{R}^n$, we write $B(0,1)$ instead of the unit ball $B^n(0,1)$ in $\mathbb{R}^n$ centered at the origin for short, and the same notation is valid for $B(x_0, r)$. We denote $\text{RapDec}(R)$ for terms rapidly decaying in $R$ which can be ignored in our estimate.

2 Proof of Theorem 1.2

We will prove the sufficiency and necessity of Theorem 1.2 in Subsection 2.1 and Subsection 2.2 respectively. We notice that some of the details during the proofs have been appeared in [24], but we write most of them for the reader’s convenience.

2.1 Sufficiency

**Lemma 2.1.** ([24]) Assume that $g$ is a Schwartz function whose Fourier transform is supported in the annulus $A(\lambda) = \{\xi \in \mathbb{R}^n : |\xi| \sim \lambda\}$. $\gamma(x, t)$ satisfies

$$
|\gamma(x, t) - x| \lesssim t^\alpha, \quad \gamma(x, 0) = x
$$

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for all $x \in B(x_0, r)$ and $t \in (0, \lambda^{-\frac{1}{\alpha}})$. Then for each $x \in B(x_0, r)$ and $t \in (0, \lambda^{-\frac{1}{\alpha}})$,

$$|e^{itP(D)}g(\gamma(x,t))| \leq \sum_{l \in \mathbb{Z}^n} C_n \left| \int_{\mathbb{R}^n} e^{i(x+\frac{1}{\lambda})\xi + itP(\xi)} \hat{g}(\xi) d\xi \right|. \quad (2.1)$$

**Proof of Theorem 1.2.** (1) When $\frac{1}{m} \leq \alpha < 1$, it is sufficient to show that for arbitrary $\varepsilon > 0$, $s_1 = s_0 + \varepsilon$,

$$\left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f)(\gamma(x,t)) - f(x)|}{t^{\delta/m}} \right\|_{L^q(B(x_0,r))} \lesssim \|f\|_{H^{1+\delta}(\mathbb{R}^n)}.$$

(2.2)

where $0 < \delta < \alpha m$.

Now we decompose $f$ as

$$f = \sum_{k=0}^{\infty} f_k,$$

where supp $\hat{f}_0 \subset B(0, 1)$, supp $\hat{f}_k \subset \{\xi : |\xi| \sim 2^k\}$, $k \geq 1$. It follows that

$$\left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)|}{t^{\delta/m}} \right\|_{L^q(B(x_0,r))} \leq \sum_{k=0}^{\infty} \left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)|}{t^{\delta/m}} \right\|_{L^q(B(x_0,r))}. \quad (2.3)$$

For $k \lesssim 1$, since $P(\xi)$ is continuous,

$$\left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(\gamma(x,t))|}{t^{\delta/m}} \right\|_{L^q(B(x_0,r))} \leq \left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)|}{t^{\delta/m}} \right\|_{L^q(B(x_0,r))} + \left\| \sup_{0 < t < 1} \frac{|f_k(\gamma(x,t)) - f_k(x)|}{t^{\delta/m}} \right\|_{L^q(B(x_0,r))} \lesssim \|f\|_{H^{1+\delta}(\mathbb{R}^n)}, \quad (2.4)$$

where we used

$$\left| e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(\gamma(x,t)) \right| \leq t^{1-\delta/m} \int_{\mathbb{R}^n} |P(\xi)||\hat{f}_k(\xi)| d\xi,$$

and

$$\left| f_k(\gamma(x,t)) - f_k(x) \right| \leq t^{\alpha-\delta/m} \int_{\mathbb{R}^n} |\xi||\hat{f}_k(\xi)| d\xi.$$
For $k \gg 1$, 

\[
\left\| \sup_{0 < t < 1} \left| e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x) \right| \right\|_{L^q(B(x_0,r))} \\
\leq \left\| \sup_{2^{-m_0} \leq t < 1} \left| e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x) \right| \right\|_{L^q(B(x_0,r))} \\
+ \left\| \sup_{0 < t < 2^{-m_0}} \left| e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x) \right| \right\|_{L^q(B(x_0,r))} \\
:= I + II.
\]  

(2.5)

We first estimate $I$, inequality (1.8) implies 

\[
\left\| \sup_{2^{-m_0} \leq t < 1} \left| e^{itP(D)}(f_k)(\gamma(x,t)) \right| \right\|_{L^p(B(x_0,r))} \lesssim 2^{(s_0 + \frac{\ell}{2})k} \|f_k\|_{L^2(\mathbb{R}^n)},
\]  

(2.6)

therefore, 

\[
I \leq 2^{\ell k} \left\| \sup_{2^{-m_0} \leq t < 1} \left| e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x) \right| \right\|_{L^q(B(x_0,r))} \\
\leq 2^{\ell k} \left\{ \left\| \sup_{2^{-m_0} \leq t < 1} \left| e^{itP(D)}(f_k)(\gamma(x,t)) \right| \right\|_{L^q(B(x_0,r))} + \|f_k\|_{L^q(B(x_0,r))} \right\} \\
\lesssim 2^{\ell k} \left\{ \left\| \sup_{2^{-m_0} \leq t < 1} \left| e^{itP(D)}(f_k)(\gamma(x,t)) \right| \right\|_{L^p(B(x_0,r))} + \|f_k\|_{L^2(B(x_0,r))} \right\} \\
\lesssim 2^{\ell k} 2^{(s_0 + \frac{\ell}{2})k} \|f_k\|_{L^2(\mathbb{R}^n)} \\
\lesssim 2^{\ell k} \|f\|_{H^{s_0 + \frac{\ell}{2}}(\mathbb{R}^n)}.
\]  

(2.7)

For $II$, by the triangle inequality, 

\[
II \leq \left\| \sup_{0 < t < 2^{-m_0}} \left| \frac{e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)}{t^{\ell/m}} \right| \right\|_{L^q(B(x_0,r))} \\
+ \left\| \sup_{0 < t < 2^{-m_0}} \left| \frac{f_k(x) - f_k(x)}{t^{\ell/m}} \right| \right\|_{L^q(B(x_0,r))}.
\]  

(2.8)

Using Taylor’s formula, we have 

\[
\lesssim \sum_{j \geq 1} \frac{1}{j!} \sum_{h_1, h_2, \ldots, h_j \in \{1, 2, \ldots, n\}} \left\| \sup_{0 < t < 2^{-m_0}} \left| \frac{\Pi_{d=1}^{j} |h_d(x,t) - x_{h_d}|}{t^{\ell/m}} \int_{\mathbb{R}^n} e^{i\xi \cdot \xi} \Pi_{d=1}^{j} \hat{h}_d(\xi) d\xi \right| \right\|_{L^2(B(x_0,r))} \\
\lesssim \sum_{j \geq 1} \frac{2^{-\alpha m j + \ell k}}{j!} \sum_{h_1, h_2, \ldots, h_j \in \{1, 2, \ldots, n\}} \left\| \int_{\mathbb{R}^n} e^{i\xi \cdot \xi} \Pi_{d=1}^{j} \hat{h}_d(\xi) d\xi \right\|_{L^2(B(x_0,r))}
\]
\begin{align*}
\lesssim \sum_{j \geq 1} \frac{2^{-\alpha mkj + \delta k} 2^j}{j!} \|f_k\|_{L^2(\mathbb{R}^n)} \\
\lesssim 2^\delta \|f\|_{L^2(\mathbb{R}^n)} \\
\lesssim 2^{-s_1 k} \|f\|_{H^{s_1+\delta}(\mathbb{R}^n)}.
\end{align*}

(2.9)

By Taylor’s formula and Lemma 2.1 (notice that $2^{-mk} \leq 2^{-k/\alpha}$, since $m \geq 1/\alpha$), we get

\begin{align*}
\left\| \sup_{0 < t < 2^{-mk}} \left| e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(\gamma(x,t)) \right| \right\|_{L^q(B(x_0,r))} &
\leq \sum_{j=1}^\infty \frac{2^{-mkj + \delta k}}{j!} \left\| \sup_{0 < t < 2^{-mk}} \int_{\mathbb{R}^n} e^{i\gamma(x,t) \cdot \xi} P(\xi) \hat{f}_k(\xi) d\xi \right\|_{L^q(B(x_0,r))} \\
&\leq \sum_{j=1}^\infty \frac{2^{-mkj + \delta k}}{j!} \sum_{l \in \mathbb{Z}^n} \frac{C_n}{(1 + |l|)^{n+1}} \left\| \int_{\mathbb{R}^n} e^{i(x + l \cdot \xi) \cdot \xi} P(\xi) \hat{f}_k(\xi) d\xi \right\|_{L^q(B(x_0,r))} \\
&\lesssim \sum_{j=1}^\infty \frac{2^{-mkj + \delta k}}{j!} \|P(\xi) \hat{f}_k(\xi)\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \sum_{j=1}^\infty \frac{2^{-mkj + \delta} \cdot 2^j}{j!} \|\hat{f}_k(\xi)\|_{L^2(\mathbb{R}^n)} \\
&\lesssim 2^{-s_1 k} \|f\|_{H^{s_1+\delta}(\mathbb{R}^n)}.
\end{align*}

(2.10)

Inequalities (2.7), (2.9) and (2.10) yield for $k \gg 1$,

\begin{align*}
\left\| \sup_{0 < t < 1} \left| e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x) \right| \right\|_{L^q(B(x_0,r))} &\lesssim 2^{-\delta} \|f\|_{H^{s_1+\delta}(\mathbb{R}^n)}.
\end{align*}

(2.11)

Obviously, inequality (2.2) follows from inequalities (2.3), (2.4) and (2.11).

(2) When $0 < \alpha < 1/m$, we just need to show that for arbitrary $\varepsilon > 0$, $s_1 = s_0 + \varepsilon$,

\begin{align*}
\left\| \sup_{0 < t < 1} \left| e^{itP(D)}(f)(\gamma(x,t)) - f(x) \right| \right\|_{L^q(B(x_0,r))} &\lesssim \|f\|_{H^{s_1+\delta}(\mathbb{R}^n)}
\end{align*}

(2.12)

where $0 < \delta < 1$. The proof is very similar with that of part (1), we write the details only for completeness.

For this goal, we decompose $f$ as

$$f = \sum_{k=0}^\infty f_k,$$

where $\text{supp} \hat{f}_0 \subset B(0,1)$, $\text{supp} \hat{f}_k \subset \{\xi : |\xi| \sim 2^k\}$, $k \geq 1$. 20
For $k \lesssim 1$, just as the similar argument in the part (1), we have
\[
\left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)|}{t^{\alpha/\delta}} \right\|_{L^\alpha(B(x_0,r))} \lesssim \|f\|_{H^{s_0 + \frac{s}{2}}(\mathbb{R}^n)},
\] (2.13)
which follows from two inequalities below
\[
\left| \frac{e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(\gamma(x,t))}{t^{\alpha/\delta}} \right| \leq t^{1 - \alpha\delta} \int_{\mathbb{R}^n} |P(\xi)||\hat{f}_k(\xi)|\,d\xi,
\]
and
\[
\left| f_k(\gamma(x,t)) - f_k(x) \right| \leq t^{\alpha - \alpha\delta} \int_{\mathbb{R}^n} |\xi||\hat{f}_k(\xi)|\,d\xi.
\]
For $k \gg 1$,
\[
\left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)|}{t^{\alpha/\delta}} \right\|_{L^\alpha(B(x_0,r))}
\leq \left\| \sup_{2^{-k/\alpha} \leq t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)|}{t^{\alpha/\delta}} \right\|_{L^\alpha(B(x_0,r))}
\]
\[
+ \left\| \sup_{0 < t < 2^{-k/\alpha}} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)|}{t^{\alpha/\delta}} \right\|_{L^\alpha(B(x_0,r))}
\]
:= I' + II'.
\] (2.14)
From inequality (1.8) we obtain
\[
\left\| \sup_{2^{-k/\alpha} \leq t < 1} |e^{itP(D)}(f_k)(\gamma(x,t))| \right\|_{L^p(B(x_0,r))} \lesssim 2^{(s_0 + \frac{s}{2})k} \|f_k\|_{L^2(\mathbb{R}^n)}.
\] (2.15)
Thus,
\[
I' \leq 2^k \left\| \sup_{2^{-k/\alpha} \leq t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)|}{t^{\alpha/\delta}} \right\|_{L^\alpha(B(x_0,r))}
\leq 2^k \left\{ \left\| \sup_{2^{-k/\alpha} \leq t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t))|}{t^{\alpha/\delta}} \right\|_{L^\alpha(B(x_0,r))} + \|f_k\|_{L^\alpha(B(x_0,r))} \right\}
\leq 2^k \left\{ \left\| \sup_{2^{-k/\alpha} \leq t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t))|}{t^{\alpha/\delta}} \right\|_{L^\alpha(B(x_0,r))} + \|f_k\|_{L^2(B(x_0,r))} \right\}
\leq 2^k \|f_k\|_{H^{s_0 + \frac{s}{2}}(\mathbb{R}^n)}
\leq 2^{-\frac{sk}{2}} \|f\|_{H^{s_0 + \frac{s}{2}}(\mathbb{R}^n)}.
\] (2.16)
For $II'$, by the triangle inequality,
\[
II' \leq \left\| \sup_{0 < t < 2^{-k/\alpha}} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(\gamma(x,t))|}{t^{\alpha/\delta}} \right\|_{L^\alpha(B(x_0,r))}
\]
\[
+ \left\| \sup_{0 < t < 2^{-k/\alpha}} \frac{|f_k(\gamma(x,t)) - f_k(x)|}{t^{\alpha/\delta}} \right\|_{L^\alpha(B(x_0,r))}.
\] (2.17)
Applying Taylor’s formula, we have
\[
\left\| \sup_{0 < t < 2^{-k/\alpha}} \frac{|f_k(\gamma(x,t)) - f_k(x)|}{t^{\alpha\delta}} \right\|_{L^q(B(x_0,r))} \lesssim \sum_{j \geq 1} \frac{1}{j!} \sum_{h_1, h_2, \ldots, h_j \in \{1, 2, \ldots, n\}} \left\| \sup_{0 < t < 2^{-k/\alpha}} \Pi_{d=1}^{j} |\gamma_{h_d}(x,t) - x_{h_d}| \right\|_{L^q(B(x_0,r))}
\]
\[
\lesssim \sum_{j \geq 1} \frac{\delta^j}{j!} \sum_{h_1, h_2, \ldots, h_j \in \{1, 2, \ldots, n\}} \left\| \int e^{ix \cdot \xi} \Pi_{d=1}^{j} \xi_{h_d} \hat{f}_k(\xi) d\xi \right\|_{L^2(B(x_0,r))}
\]
\[
\lesssim \sum_{j \geq 1} \frac{\delta^j k^j}{j!} \|f_k\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-s_1 k} \|f\|_{H^{s_1+\delta}(\mathbb{R}^n)}. \quad (2.18)
\]
Using Taylor’s formula and Lemma 2.1, we get
\[
\left\| \sup_{0 < t < 2^{-k/\alpha}} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)|}{t^{\alpha\delta}} \right\|_{L^q(B(x_0,r))} \lesssim \sum_{j=1}^{\infty} \frac{2^{-k/j + \alpha + \delta}}{j!} \left\| \sup_{0 < t < 2^{-k/\alpha}} \int e^{ix \cdot \xi} P(\xi)^j \hat{f}_k(\xi) d\xi \right\|_{L^q(B(x_0,r))}
\]
\[
\lesssim \sum_{j=1}^{\infty} \frac{2^{-k/j + \alpha + \delta}}{j!} \sum_{l \in \mathbb{Z}^n} C_n \frac{1}{(1 + |l|)^{n+1}} \left\| \int e^{ix \cdot \xi + \frac{lx}{2^j}} P(\xi)^j \hat{f}_k(\xi) d\xi \right\|_{L^q(B(x_0,r))}
\]
\[
\lesssim \sum_{j=1}^{\infty} \frac{2^{-k/j + \alpha + \delta}}{j!} \sum_{l \in \mathbb{Z}^n} C_n \frac{1}{(1 + |l|)^{n+1}} \|P(\xi)^j \hat{f}_k(\xi)\|_{L^2(\mathbb{R}^n)} \lesssim \sum_{j=1}^{\infty} \frac{2^{-k/j + \alpha + \delta} C_n}{j!} \|\hat{f}_k(\xi)\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-s_1 k} \|f\|_{H^{s_1+\delta}(\mathbb{R}^n)}. \quad (2.19)
\]
So we arrive at
\[
\left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f_k)(\gamma(x,t)) - f_k(x)|}{t^{\alpha\delta}} \right\|_{L^q(B(x_0,r))} \lesssim 2^{-\frac{2k}{\alpha}} \|f\|_{H^{s_1+\delta}(\mathbb{R}^n)} \quad (2.20)
\]
for \(k \gg 1\), then finish the proof of part (2). \ \Box

### 2.2 Necessity

In this subsection, we give two counterexamples to show the sharpness of the exponent of the variable \(t\) obtained by Theorem 1.5.
Set \( f_R \) with

\[ \vec{f}_R(\xi) = \chi_{Q_R}(\xi), \]

where \( \chi_{Q_R} \) is the characteristic function on the cube \( Q_R := \{ \xi : R \leq \xi_i \leq R + 1, 1 \leq i \leq n \} \), \( R \geq 1 \). Let

\[ P(\xi) = \xi^m. \]

Choose \( \gamma(x,t) = x + \mu t^\alpha \), \( 0 < \alpha < 1 \), \( \mu = (1/1000, 0, \ldots, 0) \). Hölder’s inequality implies

\[ \left\| \sup_{0 < t < 1} |e^{itP(D)}f_R(\gamma(x,t))| \right\|_{L^p(B(0,1))} \lesssim \|f_R\|_{L^2(\mathbb{R}^n)}, \quad p \geq 1, \]

obviously,

\[ \left\| \sup_{0 < t < 1} |e^{itP(D)}f_R(\gamma(x,t))| \right\|_{L^p(B(0,1))} \lesssim R^\varepsilon \|f_R\|_{L^2(\mathbb{R}^n)}, \quad p \geq 1, \tag{2.21} \]

for arbitrary \( \varepsilon > 0 \). The convergent rate result corresponding to inequality (2.21) is

\[ \left\| \sup_{0 < t < 1} \left| \frac{e^{itP(D)}(f_R(\gamma(x,t)) - f_R(x))}{t^{\delta_1}} \right| \right\|_{L^1(B(0,1))} \lesssim R^{\varepsilon + \delta_2} \|f_R\|_{L^2(\mathbb{R}^n)}. \tag{2.22} \]

We observed that the implied constant in inequality (2.22) does not depend on \( R \). It is not hard to see that

\[ R^{\varepsilon + \delta_2} \|f_R\|_{L^2(\mathbb{R}^n)} \sim R^{\varepsilon + \delta_2}. \tag{2.23} \]

By Taylor’s formula, for arbitrary \( t \in (0, 1) \), we get

\[
t^{-\delta_1} |e^{itP(D)}(f_R(\gamma(x,t)) - f_R(x))|
\]

\[
= t^{-\delta_1} \left| \int_{Q_R} e^{ix \cdot \xi}(e^{it^{\alpha} \xi_1^m + itP(\xi)} - 1) d\xi \right|
\]

\[
= t^{-\delta_1} \left| \int_{Q_R} e^{ix \cdot \xi}(i \frac{t^{\alpha}}{1000} \xi_1 + it \xi_1^m) d\xi + \sum_{j \geq 2} \frac{1}{j!} \int_{Q_R} e^{ix \cdot \xi}(i \frac{t^{\alpha}}{1000} \xi_1 + it \xi_1^m)^j d\xi \right|
\]

\[
\geq t^{-\delta_1} \left| \int_{Q_R} e^{ix \cdot \xi}(i \frac{t^{\alpha}}{1000} \xi_1 + it \xi_1^m) d\xi - \sum_{j \geq 2} \frac{1}{j!} \int_{Q_R} e^{ix \cdot \xi}(i \frac{t^{\alpha}}{1000} \xi_1 + it \xi_1^m)^j d\xi \right|. \tag{2.24} \]

When \( 1/m \leq \alpha < 1 \), we take \( t_0 = \frac{R^{-m}}{1000} \), for each \( \xi \in Q_R \),

\[ \frac{1}{100} \leq \frac{t_0^{\alpha}}{1000} \xi_1 + t_0 \xi_1^m \leq \frac{1}{50} \frac{(R + 1)^m}{R^m}. \]

Then for any \( x \in B(0, \frac{1}{1000}) \),

\[ \left| \int_{Q_R} e^{ix \cdot \xi}(i \frac{t_0^{\alpha}}{1000} \xi_1 + it \xi_1^m) d\xi \right| = \left| \int_{\{\eta : \eta_n \leq 1, 1 \leq i \leq n\}} e^{ix \cdot \eta}(i \frac{t_0^{\alpha}}{1000} (\eta_1 + R) + t_0(\eta_1 + R)^m) d\eta \right|. \]
where the phase function satisfies
\[ |x \cdot \eta| \leq \frac{1}{100}. \]
So we have
\[
\left| \int_{Q_R} e^{ix \cdot \xi} \left( i \frac{t_0^\alpha}{1000} \xi_1 + it_0 \xi_1^m \right) d\xi \right| \geq \frac{1}{200} \tag{2.25}
\]
for each \( x \in B(0, \frac{1}{1000}) \). When \( R \) is sufficiently large so that \((R + 1)^m/R^m \leq 2\), we obtain
\[
\left| \sum_{j \geq 2} \frac{1}{j!} \int_{Q_R} e^{ix \cdot \xi} \left( i \frac{t_0^\alpha}{1000} \xi_1 + t_0 \xi_1^m \right)^j d\xi \right|
\leq \sum_{j \geq 2} \frac{1}{j!} \int_{Q_R} \left| \frac{t_0^\alpha}{1000} \xi_1 + t_0 \xi_1^m \right|^j d\xi
\leq \sum_{j \geq 2} \frac{1}{j!} \left( \frac{1}{50} \right)^j \left( \frac{(R + 1)^m}{R^m} \right)^j
\leq \frac{e - \frac{2}{625}}{\epsilon} \tag{2.26}
\]
for any \( x \in B(0, \frac{1}{1000}) \). Therefore, for each \( R \) which satisfies \((R + 1)^m/R^m \leq 2\), inequalities (2.24), (2.25) and (2.26) imply
\[
\left\| \sup_{0 < t < 1} \frac{|e^{itP(D)}(f_R)(\gamma(x,t)) - f_R(x)|}{t^{\delta_1}} \right\|_{L^1(B(0,1))}
\geq \left\| \frac{|e^{it_0P(D)}(f_R)(\gamma(x,t_0)) - f_R(x)|}{t_0^{\delta_1}} \right\|_{L^1(B(0,\frac{1}{1000}))}
\geq R^{\delta_1 - m \delta_2 - \epsilon}. \tag{2.27}
\]
If inequality (2.22) holds true for each \( \epsilon > 0 \), inequalities (2.23) and (2.27) yield
\[ R^{\delta_1 m - \delta_2 - \epsilon} \lesssim 1. \]
When \( R \) tends to infinity, this is possible only if
\[ \delta_1 \leq \delta_2/m, \]
since \( \epsilon \) is arbitrarily small.

When \( 0 < \alpha < 1/m \), we choose \( t_0 = R^{-1/\alpha} \), then for each \( \xi \in Q_R \),
\[
\frac{1}{1000} \leq \frac{t_0^\alpha}{1000} \xi_1 + t_0 \xi_1^m \leq \frac{1}{1000} \frac{(R + 1)}{R} + \frac{(R + 1)^m}{R^{1/\alpha}}.
\]
Therefore, we have
\[
\left| \int_{Q_R} e^{ix\xi(i \frac{t_0^\alpha}{1000} \xi_1 + it_0 \xi_1^m)} d\xi \right| \geq \frac{1}{2000} \tag{2.28}
\]
for any \( x \in B(0, \frac{1}{1000}) \). When \( R \) is sufficiently large so that \((R+1)/R \leq 3/2 \) and \((R+1)^m/R^{1/\alpha} < 1/2000\), we obtain
\[
\left| \sum_{j \geq 2} \frac{1}{j!} \int_{Q_R} e^{ix\xi(i \frac{t_0^\alpha}{1000} \xi_1 + t_0 \xi_1^m)^j} d\xi \right|
\leq \sum_{j \geq 2} \frac{1}{j!} \left( \frac{1}{500} \right)^j
\leq \frac{e - 2}{250000} \tag{2.29}
\]
for each \( x \in B(0, \frac{1}{1000}) \). Therefore, for large enough \( R \), inequalities (2.24), (2.28) and (2.29) imply
\[
\left\| \sup_{0<\tau<1} \left| e^{itP(D)}(f_R(\gamma(x, t)) - f_R(x)) \right| \right\|_{L^1(B(0,1))}
\geq \left\| \left| e^{it_0P(D)}(f_R(\gamma(x, t_0)) - f_R(x)) \right| \right\|_{L^1(B(0, \frac{1}{1000}))}
\geq R^{\delta_1/\alpha} \tag{2.30}
\]
If inequality (2.22) holds true for each \( \varepsilon > 0 \), inequality (2.23) and (2.30) yield
\[
R^{\delta_1/\alpha - \delta_2 - \varepsilon} \preceq 1.
\]
When \( R \) tends to infinity, since \( \varepsilon \) is arbitrarily small, this is possible only if
\[
\delta_1 \leq \alpha \delta_2.
\]

3 Proof of Theorem 1.11

Before the proof of Theorem 1.11 we first prove the following lemma.

Lemma 3.1. Let \( \gamma \in \Gamma_{\alpha, R^{-1}}, \alpha \in [1/2, 1) \). Suppose that \( \tilde{\gamma} \) is a measurable function from \([0, R]\) to \( \mathbb{R}^2 \).
(1) If \( \text{supp} \hat{f} \subset B(\xi_0, \rho^{-\alpha}) \) for some \( 1 \leq \rho \leq R \), then for each \( (x, t) \in B(x_0, \rho) \times (t_0, t_0 + \rho) \subset B(0, R) \times [0, R] \), we have
\[
\left| e^{it\Delta} f(x + \tilde{\gamma}(t) + R\gamma(t) \left( \frac{t}{R^2} \right)) \right| \lesssim \sum_{l \in \mathbb{Z}^2} \frac{1}{(1 + |l|)^{100}} \left| e^{it\Delta} f_l(x + \tilde{\gamma}(t) + R\gamma(t) \left( \frac{t_0}{R^2} \right)) \right|,
\]
(3.1)
where \( \hat{f}_l(\xi) = e^{i\rho^\alpha \xi} \hat{f}(\xi) \), the implied constant does not depend on the choice of \( \gamma \) and \( \tilde{\gamma} \).

(2) If \( \hat{f} \) is supported in a ball of radius \( \rho^{-1} \), then for each \( (x, t) \in B(x_0, \rho) \times (t_0, t_0 + \rho) \subset B(0, R) \times [0, R] \), we have
\[
\left| e^{it\Delta} f(x + R\gamma(t) \left( \frac{t}{R^2} \right)) \right| \lesssim \int_{B(x_0, \rho)} \int_{t_0}^{t_0 + \rho} \left| e^{it\Delta} f_l(m + R\gamma(t) \left( \frac{t}{R^2} \right)) \right| dt dy,
\]
(3.2)
where \( \hat{f}_l(m) = e^{i\rho^\alpha \xi + i\rho^\alpha m} \hat{f}(\xi) \), the implied constant does not depend on the choice of \( \gamma \).

**Proof.** (1) We choose a cutoff function \( \phi \) which equals 1 on \( B(0, 1) \) and is supported in \([-\pi, \pi]^2\), then
\[
\left| e^{it\Delta} f(x + \tilde{\gamma}(t) + R\gamma(t) \left( \frac{t}{R^2} \right)) \right|
= \left| \int_{\mathbb{R}^2} e^{ix\xi + i\tilde{\gamma}(t) \cdot \xi + iR\gamma(t) \left( \frac{t}{R^2} \right) \cdot \xi + ut \xi^2} \phi \left( \frac{\xi - \xi_0}{\rho^{-\alpha}} \right) \hat{f}(\xi) d\xi \right|
= \rho^{-2\alpha} \left| \int_{\mathbb{R}^2} e^{ix\xi + i\tilde{\gamma}(t) \cdot \xi + iR\gamma(t) \left( \frac{t}{R^2} \right) \cdot \xi + ut |\xi|^{2\alpha} + R\gamma(t) \left( \frac{t}{R^2} \right) \cdot \xi_0 + it |\xi_0|^{2\alpha}} \times \phi(\eta) \hat{f}(\rho^{-\alpha} \eta + \xi_0) d\eta \right|
= \rho^{-2\alpha} \left| \int_{\mathbb{R}^2} e^{ix\xi - R\gamma(t) \left( \frac{t}{R^2} \right) \cdot \xi_0} \times \phi(\eta) \hat{f}(\rho^{-\alpha} \eta + \xi_0) d\eta \right|.
\]
By Fourier expansion,
\[
e^{R\gamma(t) \left( \frac{t}{R^2} \right) \cdot \xi_0} \phi(\eta) = \sum_{t \in \mathbb{Z}^2} c_t(t) e^{it \cdot \eta}.
\]
Since for each \( t \in (t_0, t_0 + \rho) \), we get
\[
|R\gamma(t) \left( \frac{t}{R^2} \right) - R\gamma(t_0) \left( \frac{t_0}{R^2} \right)| \rho^{-\alpha} \leq C_\alpha R^{1-2\alpha} \rho^\alpha \rho^{-\alpha} \leq C_\alpha.
\]
Integration by parts implies that,
\[
|c_t(t)| \lesssim \frac{1}{(1 + |l|)^{100}},
\]
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where the implied constant does not depend on the choice of \( \gamma \). Therefore, we obtain
\[
|e^{it\Delta} f(x + \tilde{\gamma}(t) + R\gamma(t_0 R^2))| \\
\leq \sum_{l \in \mathbb{Z}^2} \rho^{-2\alpha} |c_l(t)| \left| \int_{\mathbb{R}^2} e^{ix \cdot \psi(t, \tilde{\gamma}(t) + R\gamma(t_0 R^2))} \right| d\eta \\
\times \hat{f}(\rho^{-\alpha} \eta + \xi_0) d\eta \\
= \sum_{l \in \mathbb{Z}^2} |c_l(t)| \left| \int_{\mathbb{R}^2} e^{ix \cdot \psi(t, \tilde{\gamma}(t) + R\gamma(t_0 R^2))} \hat{f}(\rho^{-\alpha} \eta + \xi_0) d\eta \right| \\
\lesssim \sum_{l \in \mathbb{Z}^2} \frac{1}{(1 + |l|)^{100}} \left| e^{it\Delta} f_l(x + \tilde{\gamma}(t) + R\gamma(t_0 R^2)) \right|.
\] (3.3)

(2) By inequality (3.1), taking \( \tilde{\gamma} \equiv 0 \), we have
\[
|e^{it\Delta} f(x + R\gamma(t_0 R^2))| \lesssim \sum_{l \in \mathbb{Z}^2} \frac{1}{(1 + |l|)^{100}} \left| e^{it\Delta} f_l(x + R\gamma(t_0 R^2)) \right|.
\] (3.4)

Since the Fourier transform of \( e^{it\Delta} f_l(x + R\gamma(t_0 R^2)) \) is supported in a ball of radius \( \rho^{-1} \), by locally constant property, for any \((x, t) \in B(x_0, \rho) \times (t_0, t_0 + \rho)\), we get
\[
|e^{it\Delta} f_l(x + R\gamma(t_0 R^2))| \lesssim \rho^{-3} \int_{B(x_0, \rho)} \left| e^{it\Delta} f_l(y + R\gamma(t_0 R^2)) \right| dt dy,
\]
where the implied constant does not depend on \( t \). Taking \( \tilde{\gamma}(t) = R\gamma(t R^2) \), and applying inequality (3.1) with \( -\gamma(t) + \gamma(t_0 R^2) \) instead of \( \gamma(t) \), then for each \((y, t) \in B(x_0, \rho) \times (t_0, t_0 + \rho)\),
\[
|e^{it\Delta} f_l(y + R\gamma(t_0 R^2))| \\
= \left| e^{it\Delta} f_l(y + R\gamma(t R^2) - R\gamma(t_0 R^2)) \right| \\
\lesssim \sum_{m \in \mathbb{Z}^2} \frac{1}{(1 + |m|)^{100}} \left| e^{it\Delta} f_{l,m}(y + R\gamma(t R^2)) \right|
\]
and inequality (3.2) is proved. \( \square \)

Now we are ready to prove Theorem 1.11 by induction on \( M \). Let’s first check the base of the induction. When \( M^{-1} \leq R^{-1/2} \) and \( \text{supp } \hat{f} \subset B(\xi_0, R^{-1/2}) \), we can apply the same method as described in the proof of Lemma 3.1 (1) to obtain that for each \((x, t) \in B(0, R) \times (0, R)\),
\[
|e^{it\Delta} f(x + R\gamma(t R^2))| \lesssim \sum_{l \in \mathbb{Z}^2} \frac{1}{(1 + |l|)^{100}} \left| e^{it\Delta} f_l(x + R\gamma(0)) \right|.
\] (3.5)
where \( \hat{f}(\xi) = e^{iR^{1/2} \xi \hat{f}(\xi)} \), then
\[
\left\| \sup_{t \in (0, R)} \left| e^{it\Delta} f(x + R\gamma(t \frac{t}{R^2})) \right| \right\|_{L^p(B(0, R))} 
\lesssim \sum_{l \in \mathbb{Z}^2} \frac{1}{(1 + |l|)^{100}} \left\| \sup_{t \in (0, R)} \left| e^{it\Delta} f_\tau(x + R\gamma(0)) \right| \right\|_{L^p(B(0, R))}
\]
\[
\leq \sum_{l \in \mathbb{Z}^2} \frac{1}{(1 + |l|)^{100}} CM^{-2} R^2/|p| \left\| e^{iR^{1/2} \xi + iR\gamma(0) \cdot \xi} \hat{f}(\xi) \right\|_{L^2}
\]
\[
\leq CM^{-2} R^2/|p| \left\| f \right\|_{L^2}, \tag{3.6}
\]
and the constant \( C \) does not depend on the choice of \( \gamma \). Hence inequality (1.26) holds when \( M^{-1} \leq R^{-1/2} \). Note that we have used the result obtained by [15] for Schrödinger operator along vertical lines here.

Now we assume that inequality (1.26) holds for \( f \) whose Fourier transform is supported in a ball of radius \((KM)^{-1}, K > 1\).

We decompose \( B(0, R) \) into balls \( B_K \) of radius \( K \), and interval \((0, R)\) into intervals \( I^j_K \) of length \( K \), then
\[
\left\| \sup_{t \in (0, R)} \left| e^{it\Delta} f(x + R\gamma(t \frac{t}{R^2})) \right| \right\|_{L^p(B(0, R))} = \left( \sum_{B_K \subset B(0, R)} \int_{B_K} \sup_{l \in I^j_K} \left| e^{it\Delta} f_\tau(x + R\gamma(t \frac{t}{R^2})) \right|^p \, dx \right)^{1/p}. \tag{3.7}
\]
We divide \( B(\xi_0, M^{-1}) \) into balls \( (KM)^{-1}, f = \sum_\tau f_\tau, \hat{f}_\tau = \hat{f} \mid_\tau \). By Lemma 3.1 (2), for each \((x, t) \in B_K \times I^j_K\),
\[
\left| e^{it\Delta} f(x + R\gamma(t \frac{t}{R^2})) \right| \leq \sum_\tau \left| e^{it\Delta} f_\tau(x + R\gamma(t \frac{t}{R^2})) \right| \lesssim \sum_{l, m \in \mathbb{Z}^2 \times \mathbb{Z}} \frac{K^{-3}}{(1 + |l|)(1 + |m|)^{100}} \sum_\tau \int_{B_K} \int_{I^j_K} \left| e^{it\Delta} f_{\tau, l, m}(y + R\gamma(t \frac{t}{R^2})) \right| dtdy. \tag{3.8}
\]
Therefore,
\[
\left\| \sup_{t \in (0, R)} \left| e^{it\Delta} f(x + R\gamma(t \frac{t}{R^2})) \right| \right\|_{L^p(B(0, R))} \lesssim \sum_{l, m \in \mathbb{Z}^2 \times \mathbb{Z}} \frac{1}{(1 + |l|)(1 + |m|)^{100}}
\]
\[
\times \left\{ \sum_{B_K \subset B(0, R)} \int_{B_K} \sup_{I^j_K \subset (0, R)} \left( \sum_\tau K^{-3} \int_{B_K} \int_{I^j_K} \left| e^{it\Delta} f_{\tau, l, m}(y + R\gamma(t \frac{t}{R^2})) \right| dtdy \right)^p \, dx \right\}^{1/p}
\]
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\[
\sum_{l,m \in \mathbb{Z}^2 \times \mathbb{Z}^2} \frac{1}{(1 + |l|)^{100}(1 + |m|)^{100}} \times \left\{ \sum_{B_K \subset B(0,R)} \sup_{I_K^1 \subset (0,R)} \left[ \sum_{\tau} K^{-1/p} \left( \int_{B_K} \int_{I_K^1} |e^{it\Delta} f_{\tau,l,m}(y + R\gamma(\frac{t}{R^2})^p) dt \right)^{1/p} dy \right] \right\}^{1/p}.
\] (3.9)

Fix \(I, m\), for each \(B_K \times I_K^1\) and a parameter \(A \in \mathbb{Z}^+\), we choose \((k - 1)\)-dimensional subspaces \(V_1^0, V_2^0, \ldots, V_A^0\) such that
\[
\mu e^{it\Delta f_{l,m}(x + R\gamma(\frac{t}{R^2}))} (B_K \times I_K^1) := \min_{V_1, V_2, \ldots, V_A} \left\{ \max_{\tau \in \mathbb{Z}^+} \left( \int_{B_K} \int_{I_K^1} |e^{it\Delta} f_{\tau,l,m}(y + R\gamma(\frac{t}{R^2})^p) dt \right)^{1/p} dy \right\}.
\] (3.10)

achieves the minimum, in our case \(k = 2\). We say that \(\tau \in V_{\alpha}\) if
\[
\inf_{\xi \in \tau} \text{Angle} \left( \frac{(-2\xi, 1)}{|(-2\xi, 1)|}, V_{\alpha} \right) \leq (KM)^{-1}.
\]

Then from inequality (3.9),
\[
\left\| \sup_{t \in (0,R)} |e^{it\Delta} f(x + R\gamma(\frac{t}{R^2}))| \right\|_{L^p(B(0,R))} \lesssim \sum_{l,m \in \mathbb{Z}^2 \times \mathbb{Z}^2} \frac{1}{(1 + |l|)^{100}(1 + |m|)^{100}} \times \left\{ \sum_{B_K \subset B(0,R)} \sup_{I_K^1 \subset (0,R)} \left[ \sum_{\tau \in \mathbb{Z}^+} \frac{1}{K^{1/p}} \left( \int_{B_K} \int_{I_K^1} |e^{it\Delta} f_{\tau,l,m}(y + R\gamma(\frac{t}{R^2})^p) dt \right)^{1/p} dy \right] \right\}^{1/p}.
\]

\[
\leq K^{10} \sum_{l,m \in \mathbb{Z}^2 \times \mathbb{Z}^2} \frac{1}{(1 + |l|)^{100}(1 + |m|)^{100}} \left( \sum_{B_K \subset B(0,R)} \sup_{I_K^1 \subset (0,R)} \mu e^{it\Delta f_{l,m}(x + R\gamma(\frac{t}{R^2}))} (B_K \times I_K^1) \right)^{1/p}.
\]

\[
+ A \sum_{l,m \in \mathbb{Z}^2 \times \mathbb{Z}^2} \frac{1}{(1 + |l|)^{100}(1 + |m|)^{100}} \times \left( \sum_{\tau \in \mathbb{Z}^+} \sup_{B_K \subset B(0,R)} \frac{1}{K} \int_{B_K} \int_{I_K^1} |e^{it\Delta} f_{\tau,l,m}(y + R\gamma(\frac{t}{R^2}) |^p dt dy \right)^{1/p}.
\]

\[
:= I_1 + I_2.
\] (3.11)

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If $I_1$ dominates, we define
\[
\left\| e^{it\Delta} f_{l,m}(x + R\gamma(t - \frac{t}{R^2})) \right\|_{BL_{c,\phi}^p L^\infty(B(0,R) \times [0,R])} := \left( \sum_{B_K \subset B(0,R)} \sup_{I_K \subset (0,R)} \mu_{e^{it\Delta} f_{l,m}(x + R\gamma(t - \frac{t}{R^2}))}(B_K \times I_K) \right)^{1/p}.
\]
(3.12)

If Theorem 3.2 below holds true, we have
\[
I_1 \leq K^{10} \sum_{l,m \in \mathbb{Z}^2 \times \mathbb{Z}^2} \frac{1}{(1 + |l|)^{100}(1 + |m|)^{100}} C \left( K, \frac{\varepsilon}{2}, C_\alpha \right) R^{2\gamma} M^{-\varepsilon^2} R^2 \frac{2}{\gamma^2} \frac{5}{8} + \frac{5}{4} \|f_{l,m}\|_{L^2}.
\]

Theorem 1.11 follows from the fact that $R$ is sufficiently large and $\|f_{l,m}\|_{L^2} = \|f\|_{L^2}$.

If $I_2$ dominates, then we will prove (1.26) by the induction on the frequency radius $1/M$. Notice that (1.26) holds for frequency radius $1/(KM)$ by previous assumption. We get
\[
I_2 \leq A \sum_{l,m \in \mathbb{Z}^2 \times \mathbb{Z}^2} \frac{1}{(1 + |l|)^{100}(1 + |m|)^{100}}
\]
\[
\times \left( \sum_{\tau : \tau \in \text{some} V_0} \sum_{B_K \subset B(0,R)} \int_{B_K} \sup_{I_K} \frac{1}{K} \int_{I_K} \left| e^{it\Delta} f_{\tau,l,m}(y + R\gamma(t - \frac{t}{R^2})) \right|^p dt \right)^{1/p}
\]
\[
\leq A \sum_{l,m \in \mathbb{Z}^2 \times \mathbb{Z}^2} \frac{1}{(1 + |l|)^{100}(1 + |m|)^{100}}
\]
\[
\times \left( \sum_{\tau : \tau \in \text{some} V_0} \int_{B(0,R)} \sup_{t \in (0,R)} \left| e^{it\Delta} f_{\tau,l,m}(y + R\gamma(t - \frac{t}{R^2})) \right|^p dt \right)^{1/p}
\]
\[
= A \sum_{l,m \in \mathbb{Z}^2 \times \mathbb{Z}^2} \frac{1}{(1 + |l|)^{100}(1 + |m|)^{100}}
\]
\[
\times \left( \sum_{\tau : \tau \in \text{some} V_0} \int_{B(0,R)} \sup_{t \in (0,R)} \left| e^{it\Delta} f_{\tau,l,m}(y + R\gamma(t - \frac{t}{R^2})) \right|^p dt \right)^{1/p}
\]
\[
\leq A \sum_{l,m \in \mathbb{Z}^2 \times \mathbb{Z}^2} \frac{1}{(1 + |l|)^{100}(1 + |m|)^{100}} \left( \sum_{\tau} \left( KM \right)^{-\varepsilon^2} R^{2\gamma} \frac{2}{\gamma^2} + \frac{5}{4} \|f_{\tau,l,m}\|_{L^2} \right)^{1/p}
\]
\[
\leq C A K^{-\varepsilon^2} C_\varepsilon M^{-\varepsilon^2} R^{2\gamma} \frac{2}{\gamma^2} + \frac{5}{4} \|f\|_{L^2}.
\]

Inequality (1.26) follows if we choose $K$ sufficiently large such that $AK^{-\varepsilon^2} \ll 1$. Notice the constants through our proof are all independent of the choice of $\gamma$.

In the proof of Theorem 1.11 we used the following theorem.
This means,\[ R \text{ very important for the induction on the scale constants here are all independent of the choice of } \gamma \text{ for all } 1 \] which can be approximated by Theorem 3.2.

For 2 \leq p \leq 3.2 and k = 2, for each \( \gamma \in \Gamma_{a,R^{-1}} \) and any \( \varepsilon > 0 \), there exist positive constants \( A = A(\varepsilon) \) and \( C(K,\varepsilon,C_\alpha) \) such that

\[
\left\| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \right\|_{BL^{p}_{k,a}L^\infty(B(0,R) \times [0,R])} \leq C(K,\varepsilon,C_\alpha) R^{2-\frac{3}{8} + \varepsilon} \| f \|_{L^2}, \tag{3.13}
\]

for all \( R \geq 1, \xi_0 \in B(0,1), M^{-1} \geq R^{-1/2} \), all \( f \) with supp \( \hat{f} \subset B(\xi_0, M^{-1}) \). The constant \( C(K,\varepsilon,C_\alpha) \) does not depend on the choice of \( \gamma \).

As the similar argument in [15], for any subset \( U \subset B(0,R) \times [0,R] \), we define

\[
\left\| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \right\|_{BL^{p}_{k,a}L^\infty(U)} := \left( \sum_{B_K \subset B(0,R)} \sup_{I_K' \subset [0,R]} \left( \left\| U \cap \left( B_K \times I_K' \right) \right\|_{BL^{p}_{k,a}L^\infty(U)} \right) \right)^{1/p},
\]

which can be approximated by

\[
\left\| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \right\|_{BL^{p}_{k,a}L^q(U)} := \left( \sum_{B_K \subset B(0,R)} \left( \left\| U \cap \left( B_K \times I_K' \right) \right\|_{BL^{p}_{k,a}L^q(U)} \right) \right)^{1/p}.
\]

This means,

\[
\left\| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \right\|_{BL^{p}_{k,a}L^\infty(U)} = \lim_{q \to +\infty} \left\| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \right\|_{BL^{p}_{k,a}L^q(U)},
\]

hence Theorem 3.2 can be deduced to prove Theorem 3.3 below.

**Theorem 3.3.** Let 2 \leq p \leq 3.2 and k = 2. For each \( \gamma \in \Gamma_{a,R^{-1}} \) and any \( \varepsilon > 0 \), 1 \leq q < +\infty, there exist positive constants \( \bar{A} = \bar{A}(\varepsilon) \) and \( C(K,\varepsilon,C_\alpha) \) such that

\[
\left\| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \right\|_{BL^{p}_{k,a}L^q(B(0,R) \times [0,R])} \leq C(K,\varepsilon,C_\alpha) R^{6(\text{Log} \bar{A} - \text{Log} A)} R^{2-\frac{3}{8} + \varepsilon} \| f \|_{L^2}, \tag{3.14}
\]

for all 1 \leq A \leq \bar{A}, R \geq 1, \xi_0 \in B(0,1), all \( f \) with supp \( \hat{f} \subset B(\xi_0, M^{-1}) \), \( M^{-1} \geq R^{-1/2} \). The constants here are all independent of the choice of \( \gamma \).

We notice that the result in Theorem 3.3 is translation invariance in both \( x \) and \( t \). This is very important for the induction on the scale \( R \). The translation invariance property also plays a key role in the proof of the corresponding result in the vertical case. We leave the proof of Theorem 3.3 to Appendix, because it is quite long and technical.
4 Proofs of Theorem 1.12 and Theorem 1.13

In order to prove Theorem 1.12 and Theorem 1.13, we first show the following lemma.

Lemma 4.1. Let $\gamma(x,t) = x - e_1 t^\alpha$, $e_1 = (1,0)$, $\alpha \in [1/2, 1)$. Then for $p \geq 1$ and $s \geq 0$,

$$\left\| \sup_{0 < t < R^{-1}} |e^{it\Delta} f(\gamma(x,t))| \right\|_{L^p(B(0,1))} \leq C \|f\|_{H^s},$$  \hspace{1cm} (4.1)

holds whenever $\hat{f}$ has compact support and supp $\hat{f} \subset \{ (\xi_1, \xi_2) : |\xi_1 - \xi_1^0| \lesssim R^{1/2} \}$ if and only if

$$\left\| \sup_{0 < t < R^{-1}} |e^{it\Delta} f(x)| \right\|_{L^p(B(0,1))} \leq C \|f\|_{H^s},$$  \hspace{1cm} (4.2)

holds no matter when $\hat{f}$ has compact support and supp $\hat{f} \subset \{ (\xi_1, \xi_2) : |\xi_1 - \xi_1^0| \lesssim R^{1/2} \}$.

Proof. We first show that if inequality (4.2) holds whenever $\hat{f}$ has compact support and supp $\hat{f} \subset \{ (\xi_1, \xi_2) : |\xi_1 - \xi_1^0| \lesssim R^{1/2} \}$, then inequality (4.1) holds for such $f$. As the similar argument in [23], we introduce a cut-off function $\phi$ which is real-valued, smooth and equal to 1 on $(-2,2)$ and supported on $(-\pi, \pi)$. Making a change of variables,

$$\xi_1 \rightarrow R^{1/2} \eta_1 + \xi_1^0, \quad \xi_2 \rightarrow \eta_2,$$

one gets

$$e^{it\Delta} f(\gamma(x,t)) = \int_{\mathbb{R}^2} e^{ix \cdot \xi - it^\alpha \xi_1 + it |\xi|^2} \hat{f}(\xi) d\xi = R^{1/2} e^{-it^\alpha \xi_1^0} \times \int_{\mathbb{R}^2} e^{ix \cdot (R^{1/2} \eta_1 + \xi_1^0, \eta_2) - iR^{1/2} t^\alpha \eta_1 + it(R^{1/2} \eta_1 + \xi_1^0, \eta_2)^2} \phi(\eta_1) \hat{f}(R^{1/2} \eta_1 + \xi_1^0, \eta_2) d\eta_1 d\eta_2. \quad (4.3)$$

Since for each $t \in (0, R^{-1})$ and $\alpha \in [1/2, 1)$,

$$R^{1/2} t^\alpha \leq 1,$$

then by Fourier expansion,

$$\phi(\eta_1) e^{iR^{1/2} t^\alpha \eta_1} = \sum_{l \in \mathbb{Z}} c_l(t) e^{il\eta_1}.$$ 

Integration by parts shows that

$$|c_l(t)| \leq \frac{C}{(1 + |l|)^2}.$$
uniformly for each $l \in \mathbb{Z}$ and $t \in (0, R^{-1})$. Then we have

$$|e^{it \Delta} f(\gamma(x, t))|$$

$$= \sum_{l \in \mathbb{Z}} c_l(t) e^{-it \xi^0_l} \int_{\mathbb{R}^2} e^{i x \cdot (R^{1/2} \eta_1 + \xi^0_1, n_2) - il \eta_1 + it(R^{1/2} \eta_1 + \xi^0_1, n_2)^2} \hat{f}(R^{1/2} \eta_1 + \xi^0_1, \eta_2) d\eta_1 d\eta_2$$

$$= \sum_{l \in \mathbb{Z}} c_l(t) e^{-it \xi^0_l} \int_{\mathbb{R}^2} e^{i x \cdot \xi - il(\xi_1 - \xi^0_1) / R^{1/2} + it |\xi|^2} \hat{f}(\xi_1, \xi_2) d\xi_1 d\xi_2$$

$$= \sum_{l \in \mathbb{Z}} c_l(t) e^{-it \xi^0_l + il \xi^0_l / R^{1/2}} \int_{\mathbb{R}^2} e^{i x \cdot \xi + il \xi_1 / R^{1/2}} \hat{f}(\xi_1, \xi_2) d\xi_1 d\xi_2$$

$$\leq \sum_{l \in \mathbb{Z}} \frac{C}{(1 + |l|)^2} |e^{it \Delta} f_R(x)|,$$

where $\hat{f}_R(\xi_1, \xi_2) = e^{-il \xi_1 / R^{1/2}} \hat{f}(\xi_1, \xi_2)$ and supp $\hat{f}_R \subset \{ (\xi_1, \xi_2) : |\xi_1 - \xi^0_1| \lesssim R^{1/2} \}$. Therefore, applying inequality (4.2) to get

$$\left\| \sup_{0 < |l| < R^{-1}} |e^{it \Delta} f(\gamma(x, t))| \right\|_{L^p(B(0, 1))} \leq \sum_{l \in \mathbb{Z}} \frac{C}{(1 + |l|)^2} \left\| \sup_{0 < |l| < R^{-1}} |e^{it \Delta} f_R(x)| \right\|_{L^p(B(0, 1))}$$

$$\leq \sum_{l \in \mathbb{Z}} \frac{C}{(1 + |l|)^2} \| f_R \|_{H^s}$$

$$\leq C \| f \|_{H^s}, \quad (4.4)$$

then we arrive at inequality (4.1). By the same method, we can also prove if inequality (4.1) holds whenever $\hat{f}$ has compact support and supp $\hat{f} \subset \{ (\xi_1, \xi_2) : |\xi_1 - \xi^0_1| \lesssim R^{1/2} \}$ for some $\xi^0_1$, then inequality (4.2) holds also for such $f$. We omit its proof here. 

**Proof of Theorem 1.12.** If inequality (1.27) holds for all $f \in H^s(\mathbb{R}^2)$, by Lemma 4.1, inequality (4.2) holds whenever $\hat{f}$ has compact support and supp $\hat{f} \subset \{ (\xi_1, \xi_2) : |\xi_1 - \xi^0_1| \lesssim R^{1/2} \}$. Bourgain [3] actually showed that there exists $f_R$,

$$\hat{f}_R(\xi) = \chi_{A_R}(\xi),$$

where $A_R$ is the subset of $\{ \xi \in \mathbb{R}^2 : |\xi| \sim R \}$ defined by

$$A_R = \bigcup_{l \in \mathbb{Z}^+} A_{R,l},$$

$$A_{R,l} = [R - R^{1/2}, R + R^{1/2}] \times [R^{2/3} l, R^{2/3} l + 1].$$
Here $\chi_{A_R}(\xi)$ is the characteristic function of the set $A_R$. One can find a set $S$ with positive measure such that for each $x \in S$, there exists $t$ with $0 < t < R^{-1}$,

$$
\left| e^{it\Delta} f_R(x) \right| \geq R^{3/4}.
$$

(4.5)

Hence,

$$
\sup_{0 < t < R^{-1}} \left| e^{it\Delta} f(x) \right| \geq R^{3/4}.
$$

(4.6)

It is obvious that $\text{supp} \hat{f}_R \subset \{ (\xi_1, \xi_2) : |\xi_1 - R| \lesssim R^{1/2} \}$, applying inequality (4.2) to $f_R$, and we obtain that

$$
R^{3/4} \lesssim R^{8 R^{5/12}}.
$$

Finally we get $s \geq 1/3$ since $R$ can be sufficiently large.

Proof of Theorem 1.13. If inequality (1.28) holds for some $0 < s < 1/2$, we get inequality (4.2) whenever $\hat{f}$ has compact support and $\text{supp} \hat{f} \subset \{ (\xi_1, \xi_2) : |\xi_1 - \xi_0| \lesssim R^{1/2} \}$. According to the proof of [36, Theorem 5], there is $f_R$ with $\text{supp} \hat{f}_R \subset (\gamma_R(0), R_1, \gamma_R(R_1)) \times (\gamma_R(0), R_1, \gamma_R(R_1))$ and

$$
\|f_R\|_{L^p(B(x_0, r))} \leq C R^{s-1/2}.
$$

(5.1)

For each $x \in \{ (x_1, x_2) : x_1 \in I, |x_1 - x_2| \leq \delta R^{-1/2} \}$, there exists $t \in (0, R^{-1})$ such that

$$
|e^{it\Delta} f(x)| \geq c,
$$

here $I$ is a small interval around the origin and $\delta, c$ are small positive numbers. Combining this argument with inequality (4.2), we have

$$
R^{-1/2p} \leq C R^{s-1/2},
$$

which implies

$$
p \leq \frac{1}{1 - 2s},
$$

since $R$ can be sufficiently large.

5 Proof of Theorem 1.14

Proof. Using Littlewood-Paley decomposition, we only need to show that for $f$ with $\text{supp} \hat{f} \subset \{ \xi \in \mathbb{R} : |\xi| \sim \lambda \}$, $\lambda \gg 1$,

$$
\sup_{(x, \theta) \in (0, 1) \times \Theta} \left| e^{i(x, \theta)} (\gamma(x, t, \theta)) \right|_{L^p(B(x_0, r))} \leq C \lambda^{s_0 + \varepsilon} \|f\|_{L^2}, \quad \forall \varepsilon > 0,
$$

(5.1)
where \( s_0 \) and \( p \) are chosen as in Theorem 1.14.

We decompose \( \Theta \) into small subsets \( \{ \Theta_k \} \) such that \( \Theta = \bigcup_k \Theta_k \) with bounded overlap, where each \( \Theta_k \) is contained in a closed ball with diameter \( \lambda^{-\mu}, \mu = \min\{1, 2\alpha\} \). Due to the definition of \( \beta(\Theta) \), we have

\[
1 \leq k \leq \chi^{\mu \beta(\Theta) + \epsilon}. \tag{5.2}
\]

We claim that

\[
\sup_k \left\| \sup_{(t, \theta) \in (0, 1) \times \Theta} |e^{it\Delta} f(\gamma(x, t, \theta))| \right\|_{L^p(B(x_0, r))} \leq C \chi^{\nu + (p-1)\epsilon} \|f\|_{L^2}, \tag{5.3}
\]

where \( \nu = \max\left\{ \frac{1}{2} - \alpha, \frac{1}{4}\right\} \). Then

\[
\left\| \sup_{(t, \theta) \in (0, 1) \times \Theta} |e^{it\Delta} f(\gamma(x, t, \theta))| \right\|_{L^p(B(x_0, r))} \leq \left( \sum_k \left\| \sup_{(t, \theta) \in (0, 1) \times \Theta} |e^{it\Delta} f(\gamma(x, t, \theta))| \right\|_{L^p(B(x_0, r))}^p \right)^{1/p}
\leq C \left( \sum_k \chi^{\nu + (p-1)\epsilon} \|f\|_{L^2}^p \right)^{1/p}
\leq C \chi^{\nu + (\nu + \epsilon) \|f\|_{L^2}}
\]

which implies inequality (5.1).

Now let’s turn to prove inequality (5.3). In fact, inequality (5.3) comes from the following lemma.

**Lemma 5.1.** Under the assumption of Theorem 1.14 and \( f \) is a Schwartz function whose Fourier transform is supported in the annulus \( A_\lambda = \{ \xi \in \mathbb{R} : |\xi| \sim \lambda \} \). Then for each \( k \),

\[
\left\| \sup_{t \in (0, 1), \theta \in \Theta_k} |e^{it\Delta} f(\gamma(x, t, \theta))| \right\|_{L^p(B(x_0, r))} \leq C \chi^{\nu} \|f\|_{L^2}, \tag{5.4}
\]

where \( p \) is chosen as in Theorem 1.14 and \( \nu = \max\{1/2 - \alpha, 1/4\} \). Moreover, the constant \( C \) in inequality (5.4) depends on \( C_1, C_2, C_3, \Theta \) and the choice of \( B(x_0, r) \), but does not depend on \( f \) and \( k \).

Next we will prove Lemma 5.1. By the Kolmogorov-Seliverstov-Plessner linearization, we choose \( t(x) \in (0, 1), \theta(x) \in \Theta_k \) to be measurable functions defined on \( B(x_0, r) \), such that

\[
\sup_{t \in (0, 1), \theta \in \Theta_k} \left| e^{it\Delta} f(\gamma(x, t, \theta)) \right| \leq 2 \left| e^{it(x)\Delta} f(\gamma(x, t(x), \theta(x))) \right|.
\]

Set

\[
Tf(x) := \int_{A_\lambda} e^{i\gamma(x,t(x),\theta(x))\xi + it(x)\xi^2} f(\xi) d\xi.
\]
It is sufficient to show that
\[ \|Tf\|_{L^p(B(x_0,r))} \leq C\lambda^r\|f\|_{L^2(A_\lambda)} \]
holds for all \( f \) with \( \text{supp } f \subset A_\lambda \). Notice that we used the Plancherel’s theorem here to replace \( \hat{f} \) by \( f \). By duality, this is equivalent to show
\[ \|T^*g\|_{L^2(A_\lambda)} \leq C\lambda^r\|g\|_{L^{p'}(B(x_0,r))}, \quad 1/p + 1/p' = 1, \]
holds for all \( g \in L^{p'}(B(x_0,r)) \), where
\[ T^*g(x) := \int_{B(x_0,r)} e^{-i\gamma(x,t(x),\theta(x))\xi - it(x)d}\phi(x)g(x)dx. \]
We choose a real-valued cutoff function \( \phi \) which is equal to 1 on \( A_1 \) and rapidly decay outside.

Then
\[
\|T^*g\|^2_{L^2(A_\lambda)}
= \int_{B(x_0,r)}\int_{B(x_0,r)} g(x)\bar{g}(y)\int_{A_\lambda} e^{i\gamma(y,t(y),\theta(y))\beta - i\gamma(x,t(x),\theta(x))\xi - it(x)d}\xi d\beta d\gamma
\]
\[
= \int_{B(x_0,r)}\int_{B(x_0,r)} g(x)\bar{g}(y)K(x,y)d\beta d\gamma.
\]
Here
\[ K(x,y) := \int_{A_\lambda} e^{i\gamma(y,t(y),\theta(y))\beta - i\gamma(x,t(x),\theta(x))\xi - it(x)d}\phi(x)\lambda d\xi. \]

We have the following kernel estimates:

(E1) for each \( x,y \in B(x_0,r) \), \( |K(x,y)| \leq \lambda \).

(E2) for each \( x,y \in B(x_0,r) \) and \( x \neq y \), if \( |t(x) - t(y)| \geq 5(C_1r + C_2 + C_3\text{diam}(\Theta))\lambda^{-1} \), then
\[ |K(x,y)| \leq C\lambda^{-100}. \]

(E3) for each \( x,y \in B(x_0,r) \) and \( x \neq y \), if \( |t(x) - t(y)| < 5(C_1r + C_2 + C_3\text{diam}(\Theta))\lambda^{-1} \) and \( |x - y| \geq 2C_1C_3\text{diam}(\Theta_k) \), then
\[ |K(x,y)| \leq C \max\left\{ \frac{\lambda^{1/2}}{|x - y|^{1/2}} , |x - y|^{-1/2} \right\}. \]

We also remark that the constant \( C \) in (E2) and (E3) depends only on \( C_1, C_2, C_3, r \) and \( \Theta \).

(E1) is trivial so we will only prove (E2) and (E3) by stationary phase method. By rescaling,
\[ K(x,y) = \lambda \int_{A_1} e^{i\lambda\gamma(y,t(y),\theta(y))\eta - i\gamma(x,t(x),\theta(x))\eta + i\lambda(y)\eta^2 - i\lambda(x)\eta^2}\phi(\eta)d\eta. \]
Denote
\[ \psi(x, y, \eta) := \gamma(y, t(y), \theta(y))\eta - \gamma(x, t(x), \theta(x))\eta + \lambda t(y)\eta^2 - \lambda t(x)\eta^2. \]

Then
\[ \frac{\partial}{\partial \eta} \psi(x, y, \eta) = \gamma(y, t(y), \theta(y)) - \gamma(x, t(x), \theta(x)) + 2\lambda t(y)\eta - 2\lambda t(x)\eta, \]
\[ \frac{\partial^2}{\partial \eta^2} \psi(x, y, \eta) = 2\lambda t(y) - 2\lambda t(x). \]

We first prove (E2). Note that
\[
\left| \gamma(y, t(y), \theta(y)) - \gamma(x, t(x), \theta(x)) \right|
= \left| \gamma(y, t(y), \theta(y)) - \gamma(x, t(y), \theta(y)) + \gamma(x, t(y), \theta(y)) - \gamma(x, t(x), \theta(y)) + \gamma(x, t(x), \theta(x)) - \gamma(x, t(x), \theta(x)) \right|
\leq C_1|x - y| + C_2|t(x) - t(y)| + C_3|\theta(x) - \theta(y)|
\leq 2C_1 r + C_2 + C_3 \text{diam}(\Theta). \tag{5.5}
\]
Therefore, if \(|t(x) - t(y)| \geq 5(C_1 r + C_2 + C_3 \text{diam}(\Theta))\lambda^{-1}\), then integration by parts implies (E2), and the constant \(C\) depends only on \(C_1, C_2, C_3, r\) and \(\Theta\).

Next we prove (E3). By the triangle inequality,
\[
\left| \gamma(y, t(y), \theta(y)) - \gamma(x, t(x), \theta(x)) \right|
\geq \left| \gamma(y, t(y), \theta(y)) - \gamma(x, t(y), \theta(y)) + \gamma(x, t(x), \theta(y)) - \gamma(x, t(x), \theta(x)) \right|
- \left| \gamma(x, t(y), \theta(y)) - \gamma(x, t(x), \theta(y)) \right|.
\]
According to the assumption of (E3), \(|x - y| \geq 2C_1 C_3 \text{diam}(\Theta_k)\), then
\[
\left| \gamma(y, t(y), \theta(y)) - \gamma(x, t(x), \theta(y)) + \gamma(x, t(x), \theta(y)) - \gamma(x, t(x), \theta(x)) \right|
\leq C_1|x - y| + C_3 \text{diam}(\Theta_k)
\leq (C_1 + \frac{1}{2C_1})|x - y|,
\]

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and

$$\left| \gamma(y, t(y), \theta(y)) - \gamma(x, t(y), \theta(y)) + \gamma(x, t(x), \theta(y)) - \gamma(x, t(x), \theta(x)) \right|$$

$$\geq C^{-1}_1|x - y| - C_3 \text{diam}(\Theta_k)$$

$$\geq (2C_1)^{-1}|x - y|.$$ 

If $|x - y| \leq 100C_1C_2|t(x) - t(y)|^\alpha$,

$$\left| \frac{\partial^2}{\partial \eta^2} \psi(x, y, \eta) \right| = 2\lambda|t(x) - t(y)| \geq 2\lambda(100C_1C_2)^{-1/\alpha}|x - y|^{1/\alpha}.$$ 

Using Van der Corput’s lemma, we have

$$|K(x, y)| \leq 2^{-1/2}(100C_1C_2)^{1/2\alpha}|x - y|^{-1/2\alpha}.$$ 

If $|x - y| > 100C_1C_2|t(x) - t(y)|^\alpha$ and $|x - y| > 100C_1\lambda|t(x) - t(y)|$,

$$\left| \frac{\partial}{\partial \eta} \psi(x, y, \eta) \right| \geq (10C_1)^{-1}|x - y|,$$

and

$$\left| \frac{\partial^2}{\partial \eta^2} \psi(x, y, \eta) \right| = 2\lambda|t(x) - t(y)| \leq 10(C_1^r + C_2 + C_3 \text{diam}(\Theta)).$$ 

Integration by parts implies

$$|K(x, y)| \leq \frac{C_N\lambda}{(1 + \lambda(10C_1)^{-1}|x - y|)^N}.$$ 

If $|x - y| > 100C_1C_2|t(x) - t(y)|^\alpha$ and $|x - y| \leq 100C_1\lambda|t(x) - t(y)|$,

$$\left| \frac{\partial^2}{\partial \eta^2} \psi(x, y, \eta) \right| = 2\lambda|t(x) - t(y)| \geq (50C_1)^{-1}|x - y|.$$ 

Applying Van der Corput’s lemma to get

$$|K(x, y)| \leq (50C_1)^{1/2} \frac{\lambda^{1/2}}{|x - y|^{1/2}}.$$ 

We finish the estimate of (E3).

According to the kernel estimate, we break $B(x_0, r) \times B(x_0, r)$ into $\Omega_1, \Omega_2$, where

$$\Omega_1 := \{(x, y) \in B(x_0, r) \times B(x_0, r) : |t(x) - t(y)| \leq 5(C_1^r + C_2 + C_3 \text{diam}(\Theta))\lambda^{-1}\},$$

$$\Omega_2 := \{(x, y) \in B(x_0, r) \times B(x_0, r) : |t(x) - t(y)| > 5(C_1^r + C_2 + C_3 \text{diam}(\Theta))\lambda^{-1}\}.$$
By (E2), we have
\[
\left| \int \int_{\Omega_2} g(x)\bar{g}(y)K(x, y)dxdy \right| \leq C\lambda^{-100}\|g\|_{L^p(B(x_0, r))}^2,
\]
here the constant \(C\) depends on \(C_1, C_2, C_3, B(x_0, r)\) and \(\Theta\). To establish the estimate on \(\Omega_1\), we will consider the following three cases, \(\alpha \in [1/2, 1)\), \(\alpha \in (1/4, 1/2)\) and \(\alpha \in (0, 1/4)\), respectively.

**Case 1.** When \(\alpha \in [1/2, 1)\). Note that we have \(\text{diam}(\Theta_k) = \lambda^{-1}\). If \((x, y) \in \Omega_1\) and \(|x - y| \geq 2C_1C_3\lambda^{-1}\),
\[
\lambda^{1/2}|x - y|^{1/2\alpha - 1/2} \geq \lambda^{1 - 1/2\alpha} \geq 1.
\]
Hence,
\[
|K(x, y)| \leq \frac{C\lambda^{1/2}}{|x - y|^{1/2}}.
\]
Then we have
\[
\left| \int \int_{\Omega_1} g(x)\bar{g}(y)K(x, y)dxdy \right| \\
\leq \int \int_{\{x, y\in \Omega_1\, |x - y| < 2C_1C_3\lambda^{-1}\}} |g(x)\bar{g}(y)K(x, y)|dxdy \\
+ \int \int_{\{x, y\in \Omega_1\, |x - y| \geq 2C_1C_3\lambda^{-1}\}} |g(x)\bar{g}(y)K(x, y)|dxdy \\
\leq \lambda \int \int_{\{x, y\in \Omega_1\, |x - y| < 2C_1C_3\lambda^{-1}\}} |x - y|^{1/2}|x - y|^{-1/2}|g(x)\bar{g}(y)|dxdy \\
+ C\lambda^{1/2} \int \int_{\{x, y\in \Omega_1\, |x - y| \geq 2C_1C_3\lambda^{-1}\}} |g(x)\bar{g}(y)||x - y|^{-1/2}dxdy \\
\leq C\lambda^{1/2} \int \int_{\mathbb{R}} |g(x)|\chi_{B(x_0, r)}(x)|\bar{g}(y)||\chi_{B(x_0, r)}(y)||x - y|^{-1/2}dxdy \\
\leq C\lambda^{1/2} \|g\|_{L^{1/3}(B(x_0, r))} \|g\chi_{B(x_0, r)} * |\cdot|^{-1/2}\|_{L^4(\mathbb{R})} \\
\leq C\lambda^{1/2} \|g\|_{L^{1/3}(B(x_0, r))}^2.
\]
Here we applied the HLS inequality
\[
\|g\chi_{B(x_0, r)} * |\cdot|^{-\gamma}\|_{L^{p_1}} \leq A_{p_1, p_2} \|g\chi_{B(x_0, r)}\|_{L^{p_2}}, \quad (5.8)
\]
\(1 < p_1, p_2 < \infty\), \(1/p_1 = 1/p_2 - 1 + \gamma\), \(0 < \gamma < 1\).

**Case 2.** When \(\alpha \in (1/4, 1/2)\). Notice that we now have \(\text{diam}(\Theta_k) = \lambda^{-2\alpha}\). For each \((x, y) \in \Omega_1\),
\[
|x - y| \leq 2r,
\]
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where the constant $C$ depends on $r$. So for each $x, y \in B(x_0, r)$ satisfying $|x - y| \geq 2C_1C_3\lambda^{-2\alpha}$,

$$|K(x, y)| \leq C \max \left\{ \frac{\lambda^{1/2}}{|x - y|^{1/2}}, |x - y|^{-1/2\alpha} \right\} \leq C \max \left\{ \frac{\lambda^{1/2}}{|x - y|^{1/4\alpha}}, |x - y|^{-1/2\alpha} \right\}, \tag{5.9}$$

where the constant $C \geq 1$ depends on $C_1$, $C_2$ and $r$. When $|x - y| \geq 2C_1C_3\lambda^{-2\alpha}$,

$$\frac{C \lambda^{1/2}|x - y|^{-1/2\alpha}}{|x - y|^{1/4\alpha}} = C \lambda^{1/2}|x - y|^{1/4\alpha} \geq 1.$$ 

Thus

$$|K(x, y)| \leq \frac{C \lambda^{1/2}}{|x - y|^{1/4\alpha}}. \tag{5.10}$$

Then we have

$$\left| \int \int_{\Omega_1} g(x)\bar{g}(y)K(x, y)dxdy \right| \leq \int \int \{ (x,y)\in\Omega_1:|x-y|<2C_1C_3\lambda^{-2\alpha} \} |g(x)\bar{g}(y)K(x, y)|dxdy + \int \int \{ (x,y)\in\Omega_1:|x-y|\geq2C_1C_3\lambda^{-2\alpha} \} |g(x)\bar{g}(y)K(x, y)|dxdy \leq \lambda \int \int \{ (x,y)\in\Omega_1:|x-y|<2C_1C_3\lambda^{-2\alpha} \} |x - y|^{1/4\alpha}|x - y|^{-1/4\alpha}|g(x)\bar{g}(y)|dxdy + C \lambda^{1/2} \int \int \{ (x,y)\in\Omega_1:|x-y|\geq2C_1C_3\lambda^{-2\alpha} \} |g(x)\bar{g}(y)||x - y|^{-1/4\alpha}dxdy \leq C \lambda^{1/2} \int \int_{\Omega_1} |g|_{L^{8\alpha/(8\alpha-1)}(B(x_0, r))}^2 ||g|\chi_{B(x_0, r)} * |·|^{-1/4\alpha} ||_{L^{8\alpha/(8\alpha-1)}(B(x_0, r))} \leq C \lambda^{1/2} ||g||_{L^{8\alpha/(8\alpha-1)}(B(x_0, r))}^2 \tag{5.11}$$

Here we applied the HLS inequality \((5.8)\).

**Case 3.** When $\alpha \in (0, 1/4)$, similar proof with the previous discussion in Case 1 and Case 2, we divide

$$\left| \int \int_{\Omega_1} g(x)\bar{g}(y)K(x, y)dxdy \right| \leq \int \int \{ (x,y)\in\Omega_1:|x-y|<2C_1C_3\lambda^{-2\alpha} \} |g(x)\bar{g}(y)K(x, y)|dxdy + \int \int \{ (x,y)\in\Omega_1:2C_1C_3\lambda^{-2\alpha}\leq|x-y|<\lambda^{-\alpha/(1-\alpha)} \} |g(x)\bar{g}(y)K(x, y)|dxdy + \int \int \{ (x,y)\in\Omega_1:|x-y|\geq\lambda^{-\alpha/(1-\alpha)} \} |g(x)\bar{g}(y)K(x, y)|dxdy. \tag{5.12}$$
Let’s estimate these three terms respectively. For the first term, by Hölder’s inequality and $L^2$-estimate for the Hardy-Littlewood maximal function, we have
\[
\int \int \{ (x,y) \in \Omega_1 : |x-y|<2C_1C_3 \lambda^{-2\alpha} \} \quad |g(x)\bar{g}(y)K(x,y)|\,dxdy \\
\leq C\lambda^{1-2\alpha} \int M(\|g\chi_{B(x_0,r)}\|)\|\bar{g}(y)\chi_{B(x_0,r)}\|\,dy \\
\leq C\lambda^{1-2\alpha} \|M(\|g\chi_{B(x_0,r)}\|)\|L^2\|g\|L^2(B(x_0,r)) \\
\leq C\lambda^{1-2\alpha} \|g\|^2_{L^2(B(x_0,r))}. \tag{5.13}
\]

Let’s turn to evaluate the second term. By Hölder’s inequality and Schur’s lemma,
\[
\int \int \{ (x,y) \in \Omega_1 : 2C_1C_3 \lambda^{-2\alpha} \leq |x-y|<\lambda^{-\alpha/(1-\alpha)} \} \quad |g(x)\bar{g}(y)K(x,y)|\,dxdy \\
\leq C \int \left\{ \{ x,y \in \mathbb{R} : 2C_1C_3 \lambda^{-2\alpha} \leq |x-y|<\lambda^{-\alpha/(1-\alpha)} \} \right\} \quad |x-y|^{-1/2} \chi_{B(x_0,r)}(x)|g(x)|\,dx \|g\|L^2(B(x_0,r)) \\
\leq C\lambda^{1-2\alpha} \|g\|^2_{L^2(B(x_0,r))}. \tag{5.14}
\]

For the last term, we can apply the HLS inequality and Hölder’s inequality to obtain
\[
\int \int \{ (x,y) \in \Omega_1 : |x-y|\geq\lambda^{-\alpha/(1-\alpha)} \} \quad |g(x)\bar{g}(y)K(x,y)|\,dxdy \\
\leq C\lambda^{1/2} \int \int |g(x)\chi_{B(x_0,r)}(x)||\bar{g}(y)\chi_{B(x_0,r)}(y)||x-y|^{-1/2}\,dxdy \\
\leq C\lambda^{1/2} \|g\|_{L^{4/3}(B(x_0,r))} \left\| g\chi_{B(x_0,r)} * |\cdot|^{-1/2} \right\|_{L^4(\mathbb{R})} \\
\leq C\lambda^{1/2} \|g\|^2_{L^{4/3}(B(x_0,r))} \\
\leq C\lambda^{1/2} \|g\|^2_{L^2(B(x_0,r))}. \tag{5.15}
\]

Notice that $g \in L^2(B(x_0,r))$ here, and $1-2\alpha \geq 1/2$ since $\alpha \in (0,1/4]$.

\section{Proof of Theorem 1.17}

\textbf{Proof of Theorem 1.17.} The original idea of this proof comes from [10] Proposition 1.5.

Put
\[
\hat{f}(\xi) = \chi_{B(0,\lambda^{1/2})}(\xi).
\]

Then
\[
\|f\|_{H^s(\mathbb{R})} \leq \lambda^{1/4} \lambda^{s/2}.
\]

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By rescaling, 
\[ |e^{it\Delta}f(\gamma(x,t))| = \lambda^{1/2} \left| \int_{B(0,1)} e^{i\lambda^{1/2}(x-t^\alpha)\eta + i\lambda t\eta^2} d\eta \right|. \]
If \( t \in (0, \frac{1}{100}\lambda^{-1}) \) and \( x \in S := \{ y : |y - t^\alpha| \leq \frac{1}{100}\lambda^{-1/2} \text{ for some } t \in (0, \frac{1}{100}\lambda^{-1}) \} \), then
\[ |\lambda^{1/2}(x - t^\alpha)\eta + \lambda t\eta^2| \leq 1/50 \]
and
\[ |e^{it\Delta}f(\gamma(x,t))| \geq \lambda^{1/2}. \]

When \( \alpha \in [1/2, 1) \), we have \(|S| \sim \lambda^{-1/2}\) and it follows from inequality (1.38) that
\[ \lambda^{1/2-1/2p} \lesssim \lambda^{1/4+s/2}. \]
Apparently, \( p \) can not be larger than 4 when \( s \) is sufficiently close to 1/4, since \( \lambda \) can be sufficiently large.

When \( \alpha \in (0, 1/2) \), we get \(|S| \sim \lambda^{-\alpha}\) and it follows from inequality (1.38) that
\[ \lambda^{1/2-\alpha/p} \lesssim \lambda^{1/4+s/2}. \]
It is easy to see that if \( \alpha \in (1/4, 1/2) \), \( p \) can not be larger than 8\( \alpha \) when \( s \) is sufficiently close to 1/4, since \( \lambda \) can be sufficiently large. By the same reason, if \( \alpha \in (0, 1/4] \), \( p \) can not be larger than 2 when \( s \) is sufficiently close to 1/2 - \( \alpha \). \( \square \)

7 Appendix

In this appendix, we will prove Theorem 3.3. The original idea comes from [20] and [15]. The proof here looks long and technical, but we write most of the details for completeness.

7.1 Wave packets decomposition

We first introduce the wave packets decomposition for \( f \). Let \( \varphi \) be a Schwartz function from \( \mathbb{R} \) to \( \mathbb{R} \), \( \hat{\varphi} \) is non-negative and supported in a small neighborhood of the origin, and identically equal to 1 in another smaller interval. Denote by \( \theta = \prod_{j=1}^2 \theta_j \) the rectangle in the frequency space with center \((c(\theta_1), c(\theta_2))\) and
\[ \hat{\varphi}_{\theta}(\xi_1, \xi_2) = \prod_{j=1}^2 \frac{1}{|\theta_j|^{1/2}} \hat{\varphi} \left( \frac{\xi_j - c(\theta_j)}{|\theta_j|} \right). \]
A rectangle $\nu$ in the physical space is said to be dual to $\theta$ if $|\theta_j|\nu_j| = 1$, $j = 1, 2$, and $(\theta, \nu)$ is said to be a tile. Let $T$ be a collection of all tiles with fixed dimensions and coordinate axes.

Define

$$
\tilde{\varphi}_{\theta, \nu}(\xi) = e^{-ic(\nu)\cdot\xi}\tilde{\varphi}_\theta(\xi).
$$

It is well-known that a Schwartz function $f$ can be decomposed by

$$
f = \sum_{(\theta, \nu) \in T} \langle f, \varphi_{\theta, \nu}\rangle \varphi_{\theta, \nu},
$$

and

$$
\|f\|_2^2 \sim \sum_{(\theta, \nu) \in T} |\langle f, \varphi_{\theta, \nu}\rangle|^2.
$$

Define $\tilde{\varphi}_{\theta, \nu}$ whose Fourier transform is given by

$$
\tilde{\varphi}_{\theta, \nu}(\xi) = e^{-iR\gamma(0)\cdot\xi}\tilde{\varphi}_{\theta, \nu}(\xi).
$$

We claim that

$$
f = \sum_{(\theta, \nu) \in T} f_{\theta, \nu} = \sum_{(\theta, \nu) \in T} \langle f, \tilde{\varphi}_{\theta, \nu}\rangle \tilde{\varphi}_{\theta, \nu}.
$$

(7.1)

Indeed, by Plancherel’s theorem, for each Schwartz function $f$,

$$
\hat{f}(\xi) = \sum_{(\theta, \nu) \in T} \langle \hat{f}, \tilde{\varphi}_{\theta, \nu}\rangle \tilde{\varphi}_{\theta, \nu}(\xi).
$$

(7.2)

Then

$$
\hat{f}(\xi) = \sum_{(\theta, \nu) \in T} \langle \hat{f}, e^{-iR\gamma(0)\cdot\xi}\tilde{\varphi}_{\theta, \nu}\rangle e^{-iR\gamma(0)\cdot\xi}\tilde{\varphi}_{\theta, \nu}.\tag{7.3}
$$

The claim follows by applying the Plancherel’s theorem again. And it is not hard to check that

$$
\|f\|_2^2 \sim \sum_{(\theta, \nu) \in T} |\langle f, \tilde{\varphi}_{\theta, \nu}\rangle|^2.
$$

Next, we consider the localization of $e^{it\Delta}\tilde{\varphi}_{\theta, \nu}(x + R\gamma(\frac{t}{R^2}))$ in $B(0, R) \times [0, R]$. In fact,

$$
e^{it\Delta}\tilde{\varphi}_{\theta, \nu}(x + R\gamma(\frac{t}{R^2}))
$$

$$
= \frac{1}{\sqrt{R}} \int_{\mathbb{R}^2} e^{i\left(\xi - c(\nu)\cdot\xi\right)\cdot\left(R^{-1/2}\eta + c(\theta)\right) + i\gamma(R\gamma(\frac{t}{R^2}) - R\gamma(0))\cdot\left(R^{-1/2}\eta + c(\theta)\right) + tin\left|R^{-1/2}\eta + c(\theta)\right|^2} \prod_{j=1}^2 \tilde{\varphi}(\eta_j) d\eta.
$$

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After simple calculation,

\[
\left| R\gamma \left( \frac{t}{R^2} \right) - R\gamma (0) \right| \leq C_\alpha R^{1/2}.
\]

Put

\[
T_{\theta, \nu} := \left\{ (x, t) : 0 \leq t \leq R, |x - c(\nu) + 2tc(\theta)| \leq R^{1+\delta} \right\}, \delta \ll \varepsilon, \tag{7.4}
\]

which is a tube with direction \( G(\theta) = (-2c(\theta), 1) \). Integration by parts implies that in \( B(0, R) \times [0, R] \),

\[
\left| e^{it\Delta} \tilde{\varphi}_{\theta, \nu}(x + R\gamma(\frac{t}{R^2})) \right| \leq \frac{1}{\sqrt{R}} \chi_{T_{\theta, \nu}}^*(x, t), \tag{7.5}
\]

where \( \chi_{T_{\theta, \nu}}^* \) denotes a bump function satisfying \( \chi_{T_{\theta, \nu}}^* = 1 \) on the tube \( T_{\theta, \nu} \), and \( \chi_{T_{\theta, \nu}}^* = O(R^{-1000}) \) outside \( T_{\theta, \nu} \). So we can essentially treat \( \chi_{T_{\theta, \nu}}^* \) by \( \chi_{T_{\theta, \nu}} \) which is the indicator function on the tube \( T_{\theta, \nu} \).

We do not know if the Fourier transform of \( e^{it\Delta} \tilde{\varphi}_{\theta, \nu}(x + R\gamma(\frac{t}{R^2})) \) is concentrated near a paraboloid. This brings the main difficulty for us to apply the decoupling method to improve our result in Theorem 3.3.

### 7.2 Proof of Theorem 3.3

To prove Theorem 3.3, we just consider the case \( p = 3 \), other cases will follow from Hölder’s inequality. Next we will show inequality (3.14) via induction on both \( R \) and \( A \).

Fix \( \gamma \in \Gamma_{\alpha, R^{-1}} \). We say that we are in the algebraic case if there is a transverse complete intersection \( Z(P) \) of dimension 2, where \( \deg P(z) \leq D = D(\varepsilon) \) (we will give the definition about \( D(\varepsilon) \) later), so that

\[
\left\| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \right\|_{BL^p_{k, A}L^q(B(0, R) \times [0, R])} \leq C D^3 \left\| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \right\|_{BL^p_{k, A}L^q(B(0, R) \times [0, R])},
\]

where \( N_{R^{1/2+\delta}}(Z(P)) \) denotes the \( R^{1/2+\delta} \)-neighborhood of \( Z(P) \). Otherwise we are in the cellular case.

We will use polynomial partitioning as in [15]. There exist a non-zero polynomial \( P(z) = \prod Q_l(z) \) of degree at most \( D \) and 2-dimensional transverse complete intersection \( Z(P) \), such that \( (\mathbb{R}^2 \times \mathbb{R}) \setminus Z(P) \) is a union of \( \sim D^3 \) disjoint cells \( O_i \), and for each \( i \), we have

\[
\left\| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \right\|_{BL^p_{k, A}L^q(B(0, R) \times [0, R])}^p \leq C D^3 \left\| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \right\|_{BL^p_{k, A}L^q(B(0, R) \times [0, R])}^p,
\]

otherwise we are in the cellular case.
Set
\[ W := N_{R^{1/2+\delta}}(Z(P)), \quad O'_i := O_i \setminus W. \]

**Cellular case.** In this case, \( W \subset \cup_l N_{R^{1/2+\delta}}(Z(Q_l)) \), the contribution from \( W \) is negligible. Hence for each \( i \),
\[
\left\| e^{it\Delta} f(x + R\gamma(t/R^2)) \right\|_{BL^p_{k,A}L^q(B(0,R)\times[0,R])}^p 
\leq CD^3 \left\| e^{it\Delta} f_i(x + R\gamma(t/R^2)) \right\|_{BL^p_{k,A}L^q((B(0,R)\times[0,R])\cap O'_i)}^p. \tag{7.7}
\]

We do wave packets decomposition for \( f \) as inequality \([7.1]\) in \( B(0,R) \times [0,R] \). For each cell \( O'_i \), we put
\[
T_i := \left\{ (\theta,\nu) \in T : T_{\theta,\nu} \cap O'_i \neq \emptyset \right\}.
\]
For the function \( f \), we define
\[
f_i := \sum_{(\theta,\nu) \in T_i} f_{\theta,\nu}.
\]
It follows that on \( O'_i \),
\[
\left\| e^{it\Delta} f(x + R\gamma(t/R^2)) \right\| \sim \left\| e^{it\Delta} f_i(x + R\gamma(t/R^2)) \right\|.
\]
By Fundamental Theorem of Algebra, see [20], for each \( (\theta,\nu) \in T \), we have
\[
\text{Card} \{ i : (\theta,\nu) \in T_i \} \leq D + 1.
\]
Hence
\[
\sum_i \| f_i \|_{L^2}^2 \leq CD \| f \|_{L^2}^2,
\]
by pigeonhole principle, there exists \( O'_i \) such that
\[
\| f_i \|_{L^2}^2 \leq CD^{-2} \| f \|_{L^2}^2. \tag{7.8}
\]
So for such \( i \), by inequality \([7.7]\) we get
\[
\left\| e^{it\Delta} f(x + R\gamma(t/R^2)) \right\|_{BL^p_{k,A}L^q(B(0,R)\times[0,R])}^p 
\leq CD^3 \left\| e^{it\Delta} f_i(x + R\gamma(t/R^2)) \right\|_{BL^p_{k,A}L^q((B(0,R)\times[0,R])\cap O'_i)}^p 
\leq CD^3 \sum_{B_{R/2} \text{ cover } B(0,R)\times[0,R]} \left\| e^{it\Delta} f_i(x + R\gamma(t/R^2)) \right\|_{BL^p_{k,A}L^q(B_{R/2})}^p. \tag{7.9}
\]

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In order to apply the induction on $R$, we need the following observation. Suppose that the projection of $B_{\frac{R}{2}}$ on $t$-direction is contained in an interval $[t_0, t_0 + R/2] \subset [0, R]$. We write $R\gamma(t) = \frac{R}{2} 2\gamma(t/(R/2)^2)^{1/4} := \frac{R}{2} \gamma(t/(R/2)^2)$, then check that the function $\tilde{\gamma}(t) := 2\gamma(t)$ satisfies

$$|\tilde{\gamma}(t) - \tilde{\gamma}(t')| \leq C_\alpha |t - t'|^\alpha,$$

(7.10)

for each $t, t' \in [t_0/(R/2)^2, (t_0 + R/2)/(R/2)^2]$. Indeed, for such $t, t'$, it is easy to see that $t/4, t'/4 \in [t_0/R^2, (t_0 + R/2)/R^2] \subset [0, R^{-1}]$. Since $\gamma \in \Gamma_{\alpha, R^{-1}}$ and $\alpha \in [1/2, 1)$, we obtain

$$|\tilde{\gamma}(t) - \tilde{\gamma}(t')| = \left| 2\gamma(t) - 2\gamma(t') \right| \leq 2^{1-2\alpha} C_\alpha |t - t'| \leq C_\alpha |t - t'|^\alpha,$$

and inequality (7.10) is established. So we can apply the induction for $R/2$ and translation invariance to get

$$\left\| e^{it\Delta} f(x + R\gamma(t/(R^2))) \right\|_{BL^{p}_{k,A}(L^q(B_{R/2}))}^p \leq C(K, \varepsilon, C_\alpha) (R/2)^{(\log \bar{A} - \log A)(\delta_1 \varepsilon)} \left\| f \right\|_{L^2}^p.\tag{7.11}$$

Then it follows from inequalities (7.9), (7.11), (7.8) that

$$\left\| e^{it\Delta} f(x + R\gamma(t/(R^2))) \right\|_{BL^{p}_{k,A}(L^q(B(0,R) \times [0, R]))}^p \leq CD^{\delta - p} \left( C(K, \varepsilon, C_\alpha) R^{(\log \bar{A} - \log A) R^\varepsilon} \right)^p.$$

We choose $D = D(\varepsilon)$ sufficiently large such that $CD^{\delta - p} \ll 1$, this completes the induction in the cellular case.

**Algebraic case.** We decompose $B(0, R) \times [0, R]$ into balls $B_j$ of radius $\rho_j$, $\rho_j^{1/2 + \delta_2} = R^{1/2 + \delta}$. Choose $\delta_2 \gg \delta$, so that $\rho_j \sim R^{1-O(\delta_2)}$. For each $j$ we define

$$T_j := \{(\theta, \nu) \in T : T_{\theta, \nu} \cap N_{R^{1/2 + \delta}}(Z(P)) \cap B_j \neq \emptyset\},$$

and

$$f_j := \sum_{(\theta, \nu) \in T_j} f_{\theta, \nu}.$$

On each $B_j \cap N_{R^{1/2 + \delta}}(Z(P))$, we have

$$\left| e^{it\Delta} f(x + R\gamma(t/(R^2))) \right| \sim \left| e^{it\Delta} f_j(x + R\gamma(t/(R^2))) \right|.$$

Therefore,

$$\left\| e^{it\Delta} f(x + R\gamma(t/(R^2))) \right\|_{BL^{p}_{k,A}(L^q(B(0,R) \times [0, R]))}^p \leq \sum_j \left\| e^{it\Delta} f_j(x + R\gamma(t/(R^2))) \right\|_{BL^{p}_{k,A}(L^q(B_j \cap N_{R^{1/2 + \delta}}(Z(P))))}^p.$$
We further divide $T_j$ into tubes which are tangential to $Z(P)$ and tubes which are transverse to $Z(P)$. We say that $T_{\theta,\nu}$ is tangential to $Z(P)$ in $B_j$ if the following two conditions hold:

**Distance condition:**

$$T_{\theta,\nu} \cap 2B_j \subset N_{\rho^{1/2+\delta}}(Z(P)) \cap 2B_j = N_{\rho^{1/2+\delta}}(Z(P)) \cap 2B_j.$$  

**Angle condition:** If $z \in Z(P) \cap N_O(R^{1/2+\delta})(T_{\theta,\nu}) \cap 2B_j = Z(P) \cap N_O(R^{1/2+\delta})(T_{\theta,\nu}) \cap 2B_j,$ then

$$\text{Angle}(G(\theta), T_z Z(P)) \leq C \rho^{-1/2+\delta}. $$

Here $T_z Z(P)$ denotes the tangential plane of $Z(P)$ at point $z$. The tangential wave packets are defined by

$$T_{j,\text{tang}} := \{(\theta, \nu) \in T_j : T_{\theta,\nu} \text{ is tangential to } Z(P) \text{ in } B_j\},$$

and the transverse wave packets

$$T_{j,\text{trans}} := T_j \setminus T_{j,\text{tang}}.$$ 

Setting

$$f_{j,\text{tang}} := \sum_{(\theta, \nu) \in T_{j,\text{tang}}} f_{\theta,\nu}, \quad f_{j,\text{trans}} := \sum_{(\theta, \nu) \in T_{j,\text{trans}}} f_{\theta,\nu},$$

so

$$f_j = f_{j,\text{tang}} + f_{j,\text{trans}}.$$ 

Hence, we have

$$\left\| e^{it\Delta} f(x + R\gamma(t R^2)) \right\|_{BL^p_k L^q(B(0,R) \times [0,R])} \leq \sum_j \left\| e^{it\Delta} f_j(x + R\gamma(t R^2)) \right\|_{BL^p_k L^q(B_j)} \leq \sum_j \left\| e^{it\Delta} f_{j,\text{tang}}(x + R\gamma(t R^2)) \right\|_{BL^p_k L^q(B_j)} + \sum_j \left\| e^{it\Delta} f_{j,\text{trans}}(x + R\gamma(t R^2)) \right\|_{BL^p_k L^q(B_j)}.$$

We will treat the tangential term and the transverse term respectively.

**Algebraic transverse case.** In this case, the transverse term dominates, by induction on
the radius $R$,
\[ \left\| e^{it\Delta} f_{j,\text{trans}}(x + R\gamma(\frac{t}{R^2})) \right\|_{BL^p_{k, \frac{1}{2} L^6(B_j)}} = \left\| e^{it\Delta} f_{j,\text{trans}}(x + \rho\bar{\gamma}(\frac{t}{\rho^2})) \right\|_{BL^p_{k, \frac{1}{2} L^6(B_j)}} \]
\[ \leq C(K, \varepsilon, C_\alpha) \rho^\delta (\log A - \log \frac{A}{2}) \rho^\varepsilon \| f_{j,\text{trans}} \|_{L^2} \]
\[ \leq R^{O(\delta) - \varepsilon O(\delta_2)} C(K, \varepsilon, C_\alpha) R^\delta (\log \bar{A} - \log A) R^\varepsilon \| f_{j,\text{trans}} \|_{L^2}. \]

Here we used the induction on $R$, and the similar observation for the function $\bar{\gamma}(t) = R \rho^{\gamma}(t \rho^2)$ as we did to establish inequality (7.11). By Subsection 8.4 in [20] we have
\[ \sum_j \| f_{j,\text{trans}} \|_{L^2}^2 \leq C(D) \| f \|_{L^2}^2. \quad (7.12) \]

Then
\[ \sum_j \left\| e^{it\Delta} f_{j,\text{trans}}(x + R\gamma(\frac{t}{R^2})) \right\|_{BL^p_{k, \frac{1}{2} L^6(B_j)}}^p \]
\[ \leq R^{O(\delta) - \varepsilon O(\delta_2)} [C(K, \varepsilon, C_\alpha) R^\delta (\log \bar{A} - \log A) R^\varepsilon] \sum_j \| f_{j,\text{trans}} \|_{L^2}^p \]
\[ \leq R^{O(\delta) - \varepsilon O(\delta_2)} C(D) [C(K, \varepsilon, C_\alpha) R^\delta (\log \bar{A} - \log A) R^\varepsilon] \| f \|_{L^2}^p. \]

The induction follows by choosing $\varepsilon \delta_2 \gg \delta$ and the fact that $R$ is sufficiently large.

**Algebraic tangential case.** In this case, the tangential term dominates, we need to do wave packets decomposition in $B_j$ at scale $\rho$.

**Wave packets decomposition in $B_j$.** Choose tiles $(\vartheta, \nu)$ as in Subsection 7.1, where $\vartheta$ is a $\rho^{-1/2}$-cube in frequency space and $\nu$ is a $\rho^{1/2}$-cube in physical space. Let $T$ be a collection of all such tiles with fixed dimensions and coordinate axes. Put
\[ \varphi_{\vartheta, \nu}(\xi) = e^{-ix_j \cdot \xi - it_j \cdot \xi} e^{-ic(\nu) \cdot \xi} \varphi_{\vartheta}(\xi), \quad (7.13) \]
\[ \varphi_{\vartheta}(\xi_1, \xi_2) = \frac{1}{\rho^{-1/2}} \prod_{j=1}^2 \varphi \left( \frac{\xi_j - c(\theta_j)}{\rho^{-1/2}} \right). \]

Assume that $(x_j, t_j)$ is the center of the ball $B_j$. We set
\[ \varphi_{\vartheta, \nu}(\xi) = e^{-iR\gamma(\frac{t}{R^2}) \cdot \xi} \varphi_{\vartheta, \nu}(\xi), \quad (7.14) \]

\[ f = \sum_{(\vartheta, \nu) \in T} \left\langle f, \varphi_{\vartheta, \nu} \right\rangle \varphi_{\vartheta, \nu}. \]
Therefore, 
\[ e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) = \sum_{(\bar{p}, \bar{\nu}) \in T} \left\langle f, \tilde{\varphi}_{\bar{p}, \bar{\nu}} \right\rangle \tilde{\psi}_{\bar{p}, \bar{\nu}} \]
where
\[ \tilde{\psi}_{\bar{p}, \bar{\nu}}(x, t) = e^{it\Delta} \tilde{\varphi}_{\bar{p}, \bar{\nu}}(x + R\gamma(\frac{t}{R^2})). \]

By similar argument as in Subsection 7.1, we restrict \((x, t) \in B_j\), then
\[ \left| \tilde{\psi}_{\bar{p}, \bar{\nu}} (x, t) \right| \leq \rho^{-1/2} \chi_{T_{\bar{p}, \bar{\nu}}}(x, t), \]
where the tube \(T_{\bar{p}, \bar{\nu}}\) is defined by
\[ T_{\bar{p}, \bar{\nu}} := \left\{ (x, t) \in B_j : \left| x - x_{\nu} - c(\bar{\nu}) + 2c(\bar{\theta}) (t - t_j) \right| \leq \rho^{1/2+\delta}, |t - t_j| \leq \rho \}. \]

For each \((\theta, \nu) \in T_{j, \text{tang}}\), we consider the decomposition of \(f_{\theta, \nu}\) in \(B_j\),
\[ f_{\theta, \nu} = \sum_{(\bar{p}, \bar{\nu}) \in T} \left\langle f_{\theta, \nu}, \tilde{\varphi}_{\bar{p}, \bar{\nu}} \right\rangle \tilde{\varphi}_{\bar{p}, \bar{\nu}}, \]
wave packets \((\bar{\theta}, \bar{\nu})\) which contribute to \(f_{\theta, \nu}\) satisfy
\[ |c(\theta) - c(\bar{\theta})| \leq 2\rho^{-1/2}, \]
and
\[ |c(\nu) - c(\bar{\nu}) - x_{\nu} - 2t_j c(\theta)| \leq R^{1/2+\delta}. \]

From inequality (7.17) we know that
\[ \text{Angle} \left( G(\theta), G(\bar{\theta}) \right) \leq 2\rho^{-1/2}, \]
and inequality (7.18) implies that if \((x, t) \in T_{\bar{p}, \bar{\nu}}\), then
\[ |x - c(\nu) + 2c(\theta) t| \leq CR^{1/2+\delta}, \]
i.e., \(T_{\bar{p}, \bar{\nu}} \subset N_{R^{1/2+\delta}}(T_{\theta, \nu} \cap B_j)\).

We introduce the definition of \(\left( R' \right)^{-1/2+\delta_m}\)-tangent to \(Z\) in \(B\) with radius \(R'\), \(\delta_m\) is a small positive constant, \(m = 1, 2\). Suppose that \(Z = Z(P_1, ..., P_{3-m})\) is a transverse complete intersection in \(\mathbb{R}^2 \times \mathbb{R}\). We say that \(T_{\bar{p}, \bar{\nu}}\) (with scale \(R'\)) is \(\left( R' \right)^{-1/2+\delta_m}\)-tangent to \(Z\) in \(B\) if the following two conditions hold:
(1) Distance condition: 
\[ T_{\theta, R} \subset N_{R^{'}} \cap B. \]

(2) Angle condition: If \( z \in Z \cap \bigcup \mathcal{F}_{\theta, R} \cap B \), then 
\[ \text{Angle} \left( G \left( \theta \right), T_z Z \right) \leq C \left( R^{'} \right)^{-1/2+\delta_m}. \]

Moreover, set 
\[ T_z := \{ (\theta, \nu) : T_{\theta, R} \text{ is } R^{'} \text{-tangent to } Z \text{ in } B \}, \]
we say that \( f \) is concentrated in wave packets from \( T_z \) in \( B \) if there is some \( \gamma \in \Gamma_{\alpha, R-1} \), such that for each \( (\theta, \nu) \in T_z \), \( e^{it\Delta} f_{\theta, \nu}(x + R\gamma(\frac{t}{R^2})) \) is essentially supported in \( T_{\theta, R} \), and
\[
\sum_{(\theta, \nu) \notin T_z} \| f_{\theta, \nu} \|_{L^2} \leq \text{RapDec} \left( R^{'} \right) \| f \|_{L^2}.
\]

We claim that new wave packets of \( f_{j,t} \) are \( \rho^{-1/2+\delta_2} \)-tangent to \( Z \cap B \) (note that we do not make a separate notation for convenience). In fact, if \( z \in Z \cap \bigcup \mathcal{F}_{\theta, R} \cap B \), 
\[ z \in Z \cap \bigcup \mathcal{F}_{\theta, R} \cap B \]
and
\[ \text{Angle} \left( G \left( \theta \right), T_z Z \right) \leq \text{Angle} \left( G \left( \theta \right), G \left( \theta \right) \right) + \text{Angle} \left( G \left( \theta \right), T_z Z \right) \leq C \rho^{-1/2+\delta_2}. \]

Also,
\[ T_{\theta, R} \subset N_{R^{'}} \cap B \cap B_j = N_{\rho^{1/2+\delta_2}} \cap B_j \subset \bigcup \mathcal{F}_{\theta, R} \cap B_j \cap B_j \subset N_{\rho^{1/2+\delta_2}} \cap B_j. \]

Moreover, for each \( t \in \{ t : (x, t) \in B_j \} \),
\[
\sum_{(\theta, \nu) \notin T_z} \| f_{\theta, \nu} \|_{L^2} \leq \sum_{(\theta, \nu) \in T_{j,tang}} \sum_{(\theta, \nu) \notin T_z} \left| \langle f_{\theta, \nu}, \varphi_{\theta, \nu} \rangle \right|
\]
\[ \leq \left( \sum_{(\theta, \nu) \in T_{j,tang}} \left| \langle f_{\theta, \nu}, \varphi_{\theta, \nu} \rangle \right| \right)^2 \left( \sum_{(\theta, \nu) \in T_{j,tang}} \left| \langle \varphi_{\theta, \nu}, \varphi_{\theta, \nu} \rangle \right| \right)^{1/2}
\]
\[ \leq \left( \sum_{(\theta, \nu) \in T_{j,tang}} \left| \langle f_{\theta, \nu}, \varphi_{\theta, \nu} \rangle \right| \right)^2 \left( \sum_{(\theta, \nu) \in T_{j,tang}} \left| \langle \varphi_{\theta, \nu}, \varphi_{\theta, \nu} \rangle \right| \right)^{1/2}
\]
\[ \leq \text{RapDec}(R) \| f_{j,t} \|_{L^2}. \]  

(7.21)
Here we used inequalities (7.5) and (7.15). Therefore, $f_{j,\text{tang}}$ is concentrated in wave packets from $T_{Z}$ in $B_{j}$. If Theorem 7.1 below holds true, using the similar observation for the function $R_{p}^{-1/2} \gamma \left( t_{R}^{2} \right)$ as we did to establish (7.11), we have

$$
\| e^{it\Delta} f_{j,\text{tang}} (x + R \gamma \left( \frac{t}{R} \right)) \|_{BL_{p}^{2} L^{q}(B_{j})} 
\leq \rho^{(2+1/q)(1/p-1/(4+\delta))} \| e^{it\Delta} f_{j,\text{tang}} (x + R \gamma \left( \frac{t}{R} \right)) \|_{BL_{p}^{2} L^{q}(B_{j})}
\leq \rho^{(2+1/q)(1/p-1/(4+\delta))} C \left( K, D, C_{\alpha} \right) R^{\delta/(\log A - \log A') \log A / 2} \| f_{j,\text{tang}} \|_{L^{2}}
\leq R^{O(\delta)-\varepsilon/2} C \left( K, D, C_{\alpha} \right) R^{\delta/(\log A - \log A') R^{\varepsilon}} \| f \|_{L^{2}},
$$

where we choose $C \left( K, D, C_{\alpha} \right) \geq C \left( K, D, C_{\alpha} \right)$. Thus,

$$
\sum_{j} \left\| e^{it\Delta} f_{j,\text{tang}} (x + R \gamma \left( \frac{t}{R} \right)) \right\|_{BL_{p}^{2} L^{q}(B_{j})}^{p} \leq R^{O(\delta)} R^{O(\delta)-\varepsilon/2} \left[ C \left( K, D, C_{\alpha} \right) R^{\delta/(\log A - \log A') R^{\varepsilon}} \| f \|_{L^{2}} \right]^{p}.
$$

The induction closes for the fact that $\delta \ll \delta_{2} \ll \varepsilon$ and $R$ is sufficiently large. Note that the constants throughout the proof are all independent of the choice of $\gamma$.

**Theorem 7.1.** Let $\gamma \in \Gamma_{\alpha,R^{-1}}$. Suppose that $Z = Z \left( P \right) \subset \mathbb{R}^{2} \times \mathbb{R}$ is a transverse complete intersection determined by some $P \left( \varepsilon \right)$ with $\text{deg} P \left( \varepsilon \right) \leq D_{Z}$. For all $f$ with $\text{supp} \hat{f} \subset B \left( 0, 1 \right)$, $f$ can be decomposed by

$$
f = \sum_{(\theta,\nu) \in T} f_{\theta,\nu} := \sum_{(\theta,\nu) \in T} \langle f, \hat{\varphi}_{\theta,\nu} \rangle \hat{\varphi}_{\theta,\nu}.
$$

In $B \left( 0, R \right) \times [0, R]$, $e^{it\Delta} f_{\theta,\nu} (x + R \gamma \left( \frac{t}{R} \right))$ is essentially supported in $T_{\theta,\nu}$, $(\theta, \nu) \in T$ (given by equality (7.4)), and satisfies

$$
\sum_{(\theta,\nu) \notin T_{Z}} \| f_{\theta,\nu} \|_{L^{2}} \leq \text{RapDec} \left( R \right) \| f \|_{L^{2}},
$$

where

$$
T_{Z} := \left\{ (\theta, \nu) : T_{\theta,\nu} \text{ is } R^{-1/2+\delta_{2}} - \text{tangent to } Z \left( P \right) \text{ in } B \left( 0, R \right) \times [0, R] \right\}.
$$

Then for any $\varepsilon > 0$ and $p > 4$, there exist positive constants $\alpha = \alpha \left( \varepsilon \right)$ and $C \left( K, D_{Z}, \varepsilon, C_{\alpha} \right)$ such that

$$
\left\| e^{it\Delta} f (x + R \gamma \left( \frac{t}{R} \right)) \right\|_{BL_{p}^{2} L^{q}(B \left( 0, R \right) \times [0, R])} \leq C \left( K, D_{Z}, \varepsilon, C_{\alpha} \right) R^{\delta/(\log A - \log A') R^{\varepsilon}} \| f \|_{L^{2}}
$$

(7.22)
holds for all $1 \leq A \leq \overline{A}$. The constant here does not depend on the choice of $\gamma$.

We first show that Theorem 7.1 is translation invariance both in $x$ and $t$. Suppose that $\gamma$ satisfies $\alpha$-Hölder condition with constant $C_\alpha$ on an interval $[t_0/R^2, t_0/R^2 + R^{-1}]$, $t_0 \in \mathbb{R}$. $Z(P)$ is an algebraic surface required by Theorem 7.1. If $f$ with $\text{supp} \hat{f} \subset B(0, 1)$ can be decomposed by

$$f = \sum_{(\theta, \nu) \in T} f_{\theta, \nu}^{x_0, t_0} = \sum_{(\theta, \nu) \in T} \langle f, \hat{\varphi}_{\theta, \nu}^{x_0, t_0, \gamma} \rangle \hat{\varphi}_{\theta, \nu}^{x_0, t_0, \gamma},$$

where

$$\hat{\varphi}_{\theta, \nu}^{x_0, t_0, \gamma} (\xi) = e^{-ix_0 \xi - it_0 |\xi|^2 / 2 - iR \xi \cdot \gamma (t_0 / R^2)} \hat{\varphi}_{\theta, \nu} (\xi),$$

and $\varphi_{\theta, \nu}$ was given in Subsection 7.1. In $B(x_0, R) \times [t_0, t_0 + R]$, by previous analysis in Subsection 7.1, $e^{it\Delta} f_{\theta, \nu}^{x_0, t_0} (x + R \gamma (t_0 / R^2))$ is essentially supported in $T_{\theta, \nu}^{x_0, t_0} := \{(x, t) : t_0 \leq t \leq t_0 + R, |x - x_0 - c(\nu) + 2(t - t_0) c(\theta)| \leq R^{1/2 + \delta}, \delta \ll \epsilon \}$. Denote

$$T_Z := \{ (\theta, \nu) : T_{\theta, \nu}^{x_0, t_0} \text{ is } R^{-1/2 + \delta_2}, \text{ tangent to } Z(P) \text{ in } B(x_0, R) \times [t_0, t_0 + R] \}.$$

Assume that $f$ satisfies

$$\sum_{(\theta, \nu) \notin T_Z} \left\| f_{\theta, \nu}^{x_0, t_0} \right\|_{L^2} \leq \text{RapDec} (R) \left\| f \right\|_{L^2}.$$

Changing variables implies,

$$\left\| e^{it\Delta} f (x + R \gamma (t / R^2)) \right\|_{BL^p_{k,A} L^q (B(x_0, R) \times [t_0, t_0 + R])} = \left\| e^{it\Delta} g (x + R \gamma (t + t_0 / R^2) - R \gamma (t_0 / R^2)) \right\|_{BL^p_{k,A} L^q (B(0, R) \times [0, R])},$$

(7.23)

where $\hat{g}(\xi) = e^{ix_0 \xi + t_0 \xi^2 / 2 + iR \xi \cdot \gamma (t_0 / R^2)} \hat{f}(\xi)$. From the decomposition of $f$, we get

$$g = \sum_{(\theta, \nu) \in T} g_{\theta, \nu} = \sum_{(\theta, \nu) \in T} \langle g, \varphi_{\theta, \nu} \rangle \varphi_{\theta, \nu}.$$

The function $e^{it\Delta} g_{\theta, \nu} (x + R \gamma (t + t_0 / R^2) - R \gamma (t_0 / R^2))$ is essentially supported in $T_{\theta, \nu}^{x_0, t_0}$ defined by (7.4). It is easy to see that $T_{\theta, \nu} = T_{\theta, \nu}^{x_0, t_0} - (x_0, t_0)$. Therefore, $(\theta, \nu) \in T_Z$ if and only if $(\theta, \nu) \in T_{Z - (x_0, t_0)}$ which is defined by

$$T_{Z - (x_0, t_0)} := \{ (\theta, \nu) : T_{\theta, \nu} \text{ is } R^{-1/2 + \delta_2} \text{ tangent to } Z(P) - (x_0, t_0) \text{ in } B(0, R) \times [0, R] \}.$$
Moreover,
\[ \langle f, \hat{\varphi}_{x_0, t_0, \gamma} \rangle = \langle g, \varphi_{\theta, \nu} \rangle, \]
then
\[ \sum_{(\theta, \nu) \notin T_{Z - (x_0, t_0)}} \| g_{\theta, \nu} \|_{L^2} = \sum_{(\theta, \nu) \notin T_{Z}} \| f_{x_0, t_0} \|_{L^2} \leq \text{RapDec} (R) \| f \|_{L^2}. \]
So it is proved that \( g \) is concentrated in wave packets which are tangential to \( Z(P) - (x_0, t_0) \) in \( B(0, R) \times [0, R] \). Also notice that \( \gamma(t + t_0/R^2) - \gamma(t_0/R^2) \in \Gamma_{\alpha, R^-1} \). Thus we can apply Theorem 7.1 to \( g \) to get
\[ \| e^{it\Delta} g(x + R\gamma(t + t_0/R^2)) \|_{BL^p_{k, A} L^q(B(0, R) \times [0, R])} \leq C (K, D_Z, \varepsilon, C_\alpha) R^{\frac{1}{2} + \varepsilon} \| g \|_{L^2}. \] (7.24)
Because of \( \| g \|_{L^2} = \| f \|_{L^2} \), we have
\[ \| e^{it\Delta} f(x + R\gamma(t/R^2)) \|_{BL^p_{k, A} L^q(B(x_0, R) \times [x_0, t_0 + R])} \leq C (K, D_Z, \varepsilon, C_\alpha) R^{\frac{1}{2} + \frac{1}{2p} + \frac{1}{4} + \varepsilon} \| f \|_{L^2}. \] (7.25)
Theorem 7.1 remains valid under translation.

7.3 Proof of Theorem 7.1

We will again use the induction on \( R \) and \( A \) to prove Theorem 7.1. The base of the induction is the fact that Theorem 7.1 is trivial when \( R \lesssim 1 \) or \( \bar{A} \) large enough and \( A = 1 \).

Let \( D = D(\varepsilon, D_Z) \), we will define it later. We say we are in algebraic case if there is transverse complete intersection \( Y \subset Z \) of dimension 1 defined by polynomials of degree no more than \( D \), such that
\[ \| e^{it\Delta} f(x + R\gamma(t/R^2)) \|_{BL^p_{k, A} L^q((B(0, R) \times [0, R]) \cap N_{r/2 + \delta_2}(Y))} \]
Otherwise we are in the cellular case.

**Cellular case.** For fixed \( \omega \in \Lambda^2 \mathbb{R}^3 \), let \( Z_{\omega} := \{ z \in Z(P) : \nabla P(z) \wedge \omega = 0 \} \) be a transverse complete intersection of dimension 1. As the similar argument in [20], we can choose a finite set
of \( \omega \in \Lambda^2 \mathbb{R}^3 \) such that the angle of \( T_z(Z(P)) \) changes smaller than \( 1/1000 \) on each connected exponent of \( Z(P) \setminus \cup \omega Z_\omega \). By pigeonhole principle, we can identify a significant piece \( N_1 \) of \( (B(0,R) \times [0,R]) \cap N_{R^{1/2+\delta}}(Z(P)) \), where locally \( Z(P) \) behaves like a 2-plane \( V \). Notice that for each \( \omega, Z_\omega \) is a transverse complete intersection determined by polynomial with degree less than \( D \) and we are in the cellular case, so we obtain that

\[
\| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \|_{BL^p_{k,A} L^q(B(0,R) \times [0,R])} \\
\leq C \| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \|_{BL^p_{k,A} L^q((B(0,R) \times [0,R]) \cap N_{R^{1/2+\delta}}(Z(P)))} \\
\leq C \| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \|_{BL^p_{k,A} L^q(N_1)} . \tag{7.26}
\]

By Theorem 5.5 in [20], there exists a polynomial \( Q(z) := \prod_{l=1}^s Q_l \) with \( \deg Q \leq D \), where polynomials

\[ Q_l(z) = Q_{V,l}(\pi(z)), \quad l = 1, 2, \ldots, s, \]

\( \pi \) is the orthogonal projection from \( \mathbb{R}^2 \times \mathbb{R} \) to \( V \), then \( \mathbb{R}^2 \times \mathbb{R} \setminus Z(Q) \) is divided into \( \sim D^2 \) cells \( O_i \) such that

\[
\| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \|_{BL^p_{k,A} L^q(N_1)} \leq CD^2 \| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \|_{BL^p_{k,A} L^q(N_1 \cap O_i)} . \tag{7.27}
\]

For each \( l \), the variety \( Y_l := Z(P, Q_l) \) is a transverse complete intersection of dimension 1. Define \( W := N_{R^{1/2+\delta}}(Z(Q)) \), \( O'_i := O_i \setminus W \). From the analysis in [20], we have

\[ W \cap N_1 \subset \cup_i N_{O(R^{1/2+\delta})}(Y_l), \]

since we are in the cellular case, the contribution from \( W \cap N_1 \) is negligible. So we have

\[
\| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \|_{BL^p_{k,A} L^q(N_1 \cap O'_i)} \leq CD^2 \| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \|_{BL^p_{k,A} L^q(N_1 \cap O'_i)} . \tag{7.28}
\]

From inequalities (7.26)-(7.28) we actually obtain

\[
\| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \|_{BL^p_{k,A} L^q(B(0,R) \times [0,R])} \\
\leq CD^2 \| e^{it\Delta} f(x + R\gamma(\frac{t}{R^2})) \|_{BL^p_{k,A} L^q((B(0,R) \times [0,R]) \cap O'_i)} . \tag{7.29}
\]

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For each cell $O'_i$, we set

$$T_i := \left\{ (\theta, \nu) \in T : T_{\theta, \nu} \cap O'_i \neq \emptyset \right\}.$$  

For the function $f$, we define

$$f_i := \sum_{(\theta, \nu) \in T_i} f_{\theta, \nu}.$$  

It follows that on $O'_i$,

$$\left| e^{it\Delta} f(x + R\gamma \left( \frac{t}{R^2} \right)) \right| \sim \left| e^{it\Delta} f_i(x + R\gamma \left( \frac{t}{R^2} \right)) \right|. \quad (7.30)$$  

By the Fundamental Theorem of Algebra, for each $(\theta, \nu) \in T$, we have

$$\text{Card} \{ i : (\theta, \nu) \in T_i \} \leq D + 1.$$  

Hence

$$\sum_i \| f_i \|_{L^2}^2 \leq CD \| f \|_{L^2}^2,$$  

by pigeonhole principle, there exists $O'_i$ such that

$$\| f_i \|_{L^2}^2 \leq CD^{-1} \| f \|_{L^2}^2. \quad (7.31)$$  

So by inequalities (7.29), (7.30), the induction on $R$, and inequality (7.31), we have

$$\left\| e^{it\Delta} f(x + R\gamma \left( \frac{t}{R^2} \right)) \right\|_{L^p(B(0,R) \times [0,R])}^p \leq CD^2 \left\| e^{it\Delta} f_i(x + R\gamma \left( \frac{t}{R^2} \right)) \right\|_{L^p(B(0,R) \times [0,R])}^p \leq CD^2 \sum_{R/2 \text{ cover } B(0,R) \times [0,R]} \left\| e^{it\Delta} f_i(x + R\gamma \left( \frac{t}{R^2} \right)) \right\|_{L^p(B(0,R/2))}^p \leq CD^2 \left( C(K, D_Z, \varepsilon, C_\alpha) \left( \frac{R}{2} \right)^{\delta (\log A - \log A \left( \frac{R}{2} \right)^{\frac{1}{p}} - \frac{1}{4} + \varepsilon} \| f \|_{L^2} \right)^p \right.$$  

Choosing $D := D(\varepsilon, D_Z)$ sufficiently large such that $CD^{2 - \frac{p}{2}} \ll 1$, this completes the induction.

**Algebraic case.** In the algebraic case, there exists a transverse complete intersection $Y \subset Z(P)$ of dimension 1, determined by polynomial with degree no more than $D = D(\varepsilon, D_Z)$, so
that
\[
\|e^{it\Delta}f(x + R\gamma(\frac{t}{R^2}))\|_{B^{p}_{k,A}L^{q}(B(0,R)\times[0,R])} \leq C \|e^{it\Delta}f(x + R\gamma(\frac{t}{R^2}))\|_{B^{p}_{k,A}L^{q}((B(0,R)\times[0,R])\cap N_{R^{1/2+\delta_2}(Y)}^\circ)}.
\]
(7.32)

We decompose \( B(0,R) \times [0,R] \) into balls \( B_j \) of radius \( \rho, \rho^{1/2+\delta_1} = R^{1/2+\delta_2}, \delta_2 \ll \delta_1 \), in fact \( \rho \sim R^{1-O(\delta_1)} \).

For each \( j \), we define
\[
T_j := \{ (\theta, \nu) \in T : T_{\theta,\nu} \cap N_{R^{1/2+\delta_2}}(Y) \cap B_j \neq \emptyset \},
\]
and
\[
f_j := \sum_{(\theta, \nu) \in T_j} f_{\theta,\nu}.
\]
On each \( B_j : B_j \cap N_{R^{1/2+\delta_2}}(Y) \neq \emptyset \), we have
\[
\left| e^{it\Delta}f(x + R\gamma(\frac{t}{R^2})) \right| \sim \left| e^{it\Delta}f_j(x + R\gamma(\frac{t}{R^2})) \right|.
\]
Therefore,
\[
\left\| e^{it\Delta}f(x + R\gamma(\frac{t}{R^2})) \right\|_{B^{p}_{k,A}L^{q}(B(0,R)\times[0,R])} \leq \sum_j \left\| e^{it\Delta}f_j(x + R\gamma(\frac{t}{R^2})) \right\|_{B^{p}_{k,A}L^{q}(B_j)}.
\]
We further divide \( T_j \) into tubes that are tangential to \( Y \) and tubes that are transverse to \( Y \).

We say that \( T_{\theta,\nu} \) is tangential to \( Y \) in \( B_j \) if the following two conditions hold:

**Distance condition:**
\[
T_{\theta,\nu} \cap 2B_j \subset N_{R^{1/2+\delta_2}}(Y) \cap 2B_j = N_{\rho^{1/2+\delta_1}}(Y) \cap 2B_j.
\]
(7.33)

**Angle condition:** If \( z \in Y \cap N_{O(R^{1/2+\delta_2})}(T_{\theta,\nu}) \cap 2B_j = Y \cap N_{O(\rho^{1/2+\delta_1})}(T_{\theta,\nu}) \cap 2B_j \), then
\[
\text{Angle } (G(\theta), T_z Y) \leq C \rho^{-1/2+\delta_1}.
\]
(7.34)

The tangential wave packets are defined by
\[
T_{j,tang} := \{ (\theta, \nu) \in T_j : T_{\theta,\nu} \text{ is tangent to } Y \text{ in } B_j \},
\]
and the transverse wave packets
\[
T_{j,trans} := T_j \setminus T_{j,tang}.
\]
Set

\[ f_{j,\text{tang}} := \sum_{(\theta, \nu) \in T_{j,\text{tang}}} f_{\theta, \nu}, \quad f_{j,\text{trans}} := \sum_{(\theta, \nu) \in T_{j,\text{trans}}} f_{\theta, \nu}, \]

so

\[ f_j = f_{j,\text{tang}} + f_{j,\text{trans}}. \]

Therefore, we have

\[
\left\| e^{it\Delta} f(x + R\gamma\left(\frac{t}{R^2}\right) \right\|_{BL^p_{k,A}L^q(B(0,R) \times [0,R])}^p \leq \sum_j \left\| e^{it\Delta} f_j(x + R\gamma\left(\frac{t}{R^2}\right) \right\|_{BL^p_{k,A}L^q(B_j)}^p \\
\leq \sum_j \left\| e^{it\Delta} f_{j,\text{tang}}(x + R\gamma\left(\frac{t}{R^2}\right) \right\|_{BL^p_{k,A}L^q(B_j)}^p \\
+ \sum_j \left\| e^{it\Delta} f_{j,\text{trans}}(x + R\gamma\left(\frac{t}{R^2}\right) \right\|_{BL^p_{k,A}L^q(B_j)}^p.
\]

We will treat the tangential term and the transverse term respectively. Again, we need to use wave packets decomposition in \( B_j \).

**Algebraic tangential case.** In this case, the tangential term dominates. We claim that the new wave packets \( T_{\tilde{\theta}, \nu} \) of \( f_{j,\text{tang}} \) are \( \rho^{-1/2+\delta_1} \)-tangent to \( Y \) in \( B_j \). In fact, by (7.19) and (7.20), if \( z \in Y \cap N_{O(\rho^{1/2+\delta_1})}(T_{\tilde{\theta}, \nu}) \cap B_j \), then \( z \in Y \cap N_{O(\rho^{1/2+\delta_2})}(T_{\theta, \nu}) \cap B_j \), we have

\[
\text{Angle} \left( G(\bar{\theta}), T_z Y \right) \leq \text{Angle} \left( G(\bar{\theta}), G(\theta) \right) + \text{Angle} \left( G(\theta), T_z Y \right) \leq C\rho^{-1/2+\delta_1}.
\] (7.35)

Also,

\[
T_{\tilde{\theta}, \nu} \subset N_{R^{1/2+\delta_2}}(T_{\theta, \nu} \cap B_j) \cap B_j = N_{\rho^{1/2+\delta_1}}(T_{\theta, \nu} \cap B_j) \cap B_j \subset N_{O(\rho^{1/2+\delta_1})}(Y) \cap B_j.
\] (7.36)

Consider \( B_K \times I^j_K \) such that

\[
\left[ N_{O(\rho^{1/2+\delta_1})}(Y) \cap B_j \right] \cap \left( B_K \times I^j_K \right) \neq \emptyset,
\]

there exists \( z_0 \in Y \cap B_j \cap N_{O(\rho^{1/2+\delta_1})}(B_K \times I^j_K) \), for each \( T_{\tilde{\theta}, \nu} \) such that \( T_{\tilde{\theta}, \nu} \cap \left( B_K \times I^j_K \right) \neq \emptyset \), we have that \( z_0 \in Y \cap B_j \cap N_{O(\rho^{1/2+\delta_1})}(T_{\tilde{\theta}, \nu}) \), it holds

\[
\text{Angle} \left( G(\bar{\theta}), T_{z_0} Y \right) \leq C\rho^{-1/2+\delta_1}.
\]

Then for each \( \tau \) with such a \( \bar{\theta} \) in it, it follows

\[
\text{Angle} \left( G(\tau), T_{z_0} Y \right) \leq (KM)^{-1}.
\]
By the angle condition, it turns out that each $T\theta,\nu$ can be negligible. So we only need to consider the transverse case.

Consider the new wave packets decomposition of $f_{j,\text{trans}}$ in $B_j$, 

$$\left\| e^{it\Delta} f_{j,\text{trans}}(x + R\gamma\left(\frac{t}{R^2}\right)) \right\|_{BL^p_k \frac{\partial}{\partial t} L^q(B_j)}^p \leq \text{RapDec}(\rho) \|f\|_{L^2}^p,$$

which can be negligible. So we only need to consider the transverse case.

**Algebraic transverse case.** In this case, the transverse term dominates. So we need to estimate

$$\sum_j \left\| e^{it\Delta} f_{j,\text{trans}}(x + R\gamma\left(\frac{t}{R^2}\right)) \right\|_{BL^p_k \frac{\partial}{\partial t} L^q(B_j)}^p.$$

Consider the new wave packets decomposition of $f_{j,\text{trans}}$ in $B_j$, by (7.19) and (7.20), the new wave packets $T_{\overline{\theta},\overline{\nu}}$ satisfy 

$$T_{\overline{\theta},\overline{\nu}} \subset N_{R^{1/2+\delta_2}}(T_{\theta,\nu} \cap B_j) \cap B_j \subset N_{R^{1/2+\delta_2}}(Z) \cap B_j. \quad (7.37)$$

And if $z \in Z \cap N_{O(\rho^{1/2+\delta_2})}(T_{\overline{\theta},\overline{\nu}}) \cap B_j \subset Z \cap N_{O(R^{1/2+\delta_2})}(T_{\theta,\nu}) \cap B_j$, then

$$\text{Angle} \left( G(\overline{\theta}), T_z Z \right) \leq \text{Angle} \left( G(\theta), T_z Z \right) + \text{Angle} \left( G(\theta), G(\overline{\theta}) \right) \leq C \rho^{-1/2+\delta_2}. \quad (7.38)$$

$T_{\overline{\theta},\overline{\nu}}$ is no longer $\rho^{-1/2+\delta_2}$-tangent to $Z$ in $B_j$ because the distance condition is not satisfied.

For each vector $b$ with $|b| \leq R^{1/2+\delta_2}$, define

$$\overline{T}_{Z+b} := \left\{ (\overline{\theta}, \overline{\nu}) : T_{\overline{\theta},\overline{\nu}} \text{ is } \rho^{-1/2+\delta_2} \text{-tangent to } Z + b \text{ in } B_j \right\}.$$

By the angle condition, it turns out that each $T_{\overline{\theta},\overline{\nu}} \in \overline{T}_{Z+b}$ for some $b$. We set

$$f_{j,\text{trans},b} := \sum_{(\overline{\theta},\overline{\nu}) \in \overline{T}_{Z+b}} f_{\overline{\theta},\overline{\nu}}.$$

Then on $B_j$, it holds

$$\left| e^{it\Delta} f_{j,\text{trans},b}(x + R\gamma\left(\frac{t}{R^2}\right)) \right| \sim c \chi_{N_{\rho^{1/2+\delta_2}}(Z+b)}(x,t) \left| e^{it\Delta} f_{j,\text{trans}}(x + R\gamma\left(\frac{t}{R^2}\right)) \right|. \quad (7.39)$$

Next we choose a family of vectors $b \in B_{R^{1/2+\delta_2}}$. We cover $N_{R^{1/2+\delta_2}}(Z) \cap B_j$ with disjoint balls of radius $R^{1/2+\delta_2}$. In each ball $B$, we will dyadically pigeonhole the volume of $N_{\rho^{1/2+\delta_2}}(Z) \cap B$. 

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For each \( s \in \mathbb{Z} \), denote
\[
B_s := \left\{ B \left( \omega, R^{1/2 + \delta_2} \right) \subset N_{R^{1/2 + \delta_2}} (Z) \cap B_j : \left| B \left( \omega, R^{1/2 + \delta_2} \right) \cap N_{\rho_1^1/2 + \xi_2} (Z) \right| \sim 2^s \right\}.
\]

We take a value of \( s \) so that
\[
\left\| e^{it\Delta} f_{j,\text{trans}} (x + R \gamma \left( t \frac{R^2}{R^2} \right) ) \right\|^p_{BL^p_{k,\frac{4}{5}} L^q(B_j)} \leq \left( \log R \right) \left\| e^{it\Delta} f_{j,\text{trans}} (x + R \gamma \left( t \frac{R^2}{R^2} \right) ) \right\|^p_{BL^p_{k,\frac{4}{5}} L^q((\cup B \in B_s) B)}.
\]

Therefore, we only consider \( (\theta, \nu) \) such that \( T_{\theta, \nu} \) meets at least one of the balls in \( B_s \). We choose a random set of \( |B_{R^{1/2 + \delta_2}}| / 2^s \) vectors \( b \in B_{R^{1/2 + \delta_2}} \). For a typical ball \( B \left( \omega, R^{1/2 + \delta_2} \right) \in B_s \), the union \( \cup b N_{\rho_1^1/2 + \delta_2} (Z + b) \cap B_j \) covers a definite fraction of the ball with high probability. It follows
\[
\left\| e^{it\Delta} f_{j,\text{trans,b}} (x + R \gamma \left( t \frac{R^2}{R^2} \right) ) \right\|^p_{BL^p_{k,\frac{4}{5}} L^q(B_j)} \leq \left( \log R \right) \sum_b \left\| e^{it\Delta} f_{j,\text{trans,b}} (x + R \gamma \left( t \frac{R^2}{R^2} \right) ) \right\|^p_{BL^p_{k,\frac{4}{5}} L^q(N_{\rho_1^1/2 + \delta_2} (Z + b) \cap B_j)}.
\]

By the induction on \( R \), we have
\[
\left\| e^{it\Delta} f_{j,\text{trans,b}} (x + R \gamma \left( t \frac{R^2}{R^2} \right) ) \right\|^p_{BL^p_{k,\frac{4}{5}} L^q(N_{\rho_1^1/2 + \delta_2} (Z + b) \cap B_j)} \leq \left[ C \left( K, D \gamma, 1, |\alpha|, R^\|a\| \right) \right] \left( \log R \right) \left( 1 + \frac{1}{\rho} \right) R^\frac{1}{2} \| f_{j,\text{trans,b}} \|^2_{L^2} \leq \left[ C \left( K, D \gamma, 1, |\alpha|, R^\|a\| \right) \right] \left( \log R \right) \left( 1 + \frac{1}{\rho} \right) R^\frac{1}{2} \| f_{j,\text{trans,b}} \|^2_{L^2}.
\]

Here we used the induction on the radius of \( B_j \) and the observation for \( R^\gamma (t \frac{R^2}{R^2}) \) as we did in order to get inequality (7.11). If
\[
\sum_j \sum_b \| f_{j,\text{trans,b}} \|^2_{L^2} \leq \sum_j \| f_{j,\text{trans}} \|^2_{L^2} \leq D \| f \|^2_{L^2}, \tag{7.41}
\]
and
\[
\max_b \| f_{j,\text{trans,b}} \|^2_{L^2} \leq R^\|a\| \left( \frac{R^2}{\rho} \right)^{-1/2} \| f_{j,\text{trans}} \|^2_{L^2}, \tag{7.42}
\]

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then we have
\[
\sum_{j} \sum_{b} \left\| e^{it\Delta} f_{j,\text{trans},b}(x + R\gamma(\frac{t}{R^2})) \right\|_{L^p_{k,\frac{1}{2}}(B_j)}^p \\
\leq \left[ C(K, DZ, \varepsilon, C_\alpha) R^{O(\delta)} R^{O(\delta_1)} R^{O((\log \delta - \log A) R^{\frac{1}{4} + \varepsilon})} \right] \sum_{j} \sum_{b} \left\| f_{j,\text{trans},b} \right\|_{L^2}^p \\
\leq \left[ C(K, DZ, \varepsilon, C_\alpha) R^{O(\delta)} R^{O(\delta_1)} R^{O((\log \delta - \log A) R^{\frac{1}{4} + \varepsilon})} \right] \sum_{j} \sum_{b} \left\| f_{j,\text{trans},b} \right\|_{L^2}^2 \max_{b} \left\| f_{j,\text{trans},b} \right\|_{L^2}^{p-2} \\
\leq \left[ C(K, DZ, \varepsilon, C_\alpha) R^{O(\delta)} R^{O(\delta_1)} R^{O((\log \delta - \log A) R^{\frac{1}{4} + \varepsilon})} \right] \sum_{j} \sum_{b} \left\| f_{j,\text{trans},b} \right\|_{L^2}^p \max_{b} \left\| f_{j,\text{trans},b} \right\|_{L^2}^{p-2} \\
\leq R^{O(\delta_2)} R^{O(\delta_1)} \left[ C(K, DZ, \varepsilon, C_\alpha) R^{O((\log \delta - \log A) R^{\frac{1}{4} + \varepsilon})} \right] \max_{b} \left\| f_{j,\text{trans},b} \right\|_{L^2}^p,
\]
so the induction closes by taking $\delta_2 \ll \varepsilon \delta_1$ and the fact that $R$ is sufficiently large. This completes the proof of Theorem 7.1.

Next we will prove inequalities (7.41) and (7.42). It is easy to check that $f_{j,\text{trans},b}$ and $f_{j,\text{trans}}$ are concentrated in wave packets from $(\hat{\theta}, \hat{\nu}) \in \mathcal{T}_{Z+b}$ and $(\theta, \nu) \in \mathcal{T}_Z$ respectively, and the tubes $T_{\hat{\theta},\hat{\nu}}$’s and $T_{\theta,\nu}$’s are distributed as required by [20] Lemma 7.4 and Lemma 7.5. But Guth’s results were built for the extension operator for the paraboloid, here we identify it as the free Schrödinger operator $e^{it\Delta}$ by Plancherel’s theorem. In order to apply [20] Lemma 7.4 and Lemma 7.5 to Schrödinger operator along tangential curves, we construct a new function $\tilde{f}$ whose Fourier transform is given by $e^{iR\gamma(\frac{b}{R^2})} \tilde{f}$. We recall that $(x_j, t_j)$ is the center of $B_j$.

We first do wave packets decomposition for $\tilde{f}$ in $B(0, R) \times [0, R]$. Choosing $\tilde{\varphi}_{\theta,\nu}$ whose Fourier transform is given by $e^{iR\gamma(\frac{b}{R^2})-i\gamma(0)} \widehat{\tilde{\varphi}_{\theta,\nu}}$, and $\widehat{\varphi}_{\theta,\nu}$ was given in Subsection 7.1. Decompose
\[
\tilde{f} = \sum_{(\theta, \nu) \in \mathcal{T}} \tilde{f}_{\theta,\nu} = \sum_{(\theta, \nu) \in \mathcal{T}} \langle \tilde{f}, \tilde{\varphi}_{\theta,\nu} \rangle \tilde{\varphi}_{\theta,\nu}.
\]
Also we will make the wave packet decomposition for $\tilde{f}$ in the ball $B_j$ with radius $\rho$. We decompose
\[
\tilde{f} = \sum_{(\tilde{\theta}, \tilde{\nu}) \in \mathcal{T}} \tilde{f}_{\tilde{\theta},\tilde{\nu}} = \sum_{(\tilde{\theta}, \tilde{\nu}) \in \mathcal{T}} \langle \tilde{f}, \varphi_{\tilde{\theta},\tilde{\nu}} \rangle \varphi_{\tilde{\theta},\tilde{\nu}},
\]
where $\varphi_{\tilde{\theta},\tilde{\nu}}$ was defined by inequality (7.13).

Now we check the following facts.

(a) By the same arguments as in Subsection 7.1, it is not hard to see that $e^{it\Delta} \tilde{f}_{\theta,\nu}$ is essentially supported in $T_{\theta,\nu}$ which is defined by inequality (7.4).
(b) For each \((\bar{\theta}, \bar{\nu}) \in \bar{T}\), \(e^{it\Delta} \tilde{f}_{\bar{\theta}, \bar{\nu}}\) is essentially supported in \(T_{\bar{\theta}, \bar{\nu}}\) (see (7.16)).

(c) For each \((\theta, \nu) \in T\), we have \(\|f_{\theta, \nu}\|_{L^2} = \|\tilde{f}_{\theta, \nu}\|_{L^2}\).

(d) For each \((\bar{\theta}, \bar{\nu}) \in \bar{T}\), it holds \(\|f_{\bar{\theta}, \bar{\nu}}\|_{L^2} = \|\tilde{f}_{\bar{\theta}, \bar{\nu}}\|_{L^2}\).

Denote
\[
\hat{f}_{j,\text{trans}} = \sum_{(\theta, \nu) \in T_{j,\text{trans}}} \hat{f}_{\theta, \nu}; \quad \hat{f}_{j,\text{trans},b} = \sum_{(\bar{\theta}, \bar{\nu}) \in \bar{T}_{Z+b}} \hat{f}_{\bar{\theta}, \bar{\nu}}.
\]

By facts (a) and (b), we can apply the result of [20, Lemma 7.4 and Lemma 7.5] to obtain that
\[
\sum_{b} \|\hat{f}_{j,\text{trans},b}\|_{L^2}^2 \leq \|\hat{f}_{j,\text{trans}}\|_{L^2}^2, \quad (7.43)
\]
and
\[
\max_{b} \|\hat{f}_{j,\text{trans},b}\|_{L^2}^2 \leq R^{O(\delta)} \left(\frac{R}{\rho}\right)^{-1/2} \|\hat{f}_{j,\text{trans}}\|_{L^2}^2. \quad (7.44)
\]

Inequalities (7.41) and (7.42) can be implied by combining inequalities (7.43), (7.44) and (7.12) with
\[
\|f_{j,\text{trans}}\|_{L^2} = \|\tilde{f}_{j,\text{trans}}\|_{L^2}, \quad \|f_{j,\text{trans},b}\|_{L^2} = \|\tilde{f}_{j,\text{trans},b}\|_{L^2},
\]
which follow from facts (c) and (d).

**Conflict of interest**

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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