Degree vs. Approximate Degree and Quantum Implications of Huang’s Sensitivity Theorem

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Abstract

Based on the recent breakthrough of Huang (2019), we show that for any total Boolean function \( f \),

1. \( \deg(f) = O(\widetilde{\deg}(f)^2) \): The degree of \( f \) is at most quadratic in the approximate degree of \( f \). This is optimal as witnessed by the OR function.

2. \( D(f) = O(Q(f)^4) \): The deterministic query complexity of \( f \) is at most quartic in the quantum query complexity of \( f \). This matches the known separation (up to log factors) due to Ambainis, Balodis, Belovs, Lee, Santha, and Smotrovs (2017).

We apply these results to resolve the quantum analogue of the Aanderaa–Karp–Rosenberg conjecture. We show that if \( f \) is a nontrivial monotone graph property of an \( n \)-vertex graph specified by its adjacency matrix, then \( Q(f) = \Omega(n) \), which is also optimal. We also show that the approximate degree of any read-once formula on \( n \) variables is \( \Theta(\sqrt{n}) \).

1 Introduction

Last year, Huang resolved a major open problem in the analysis of Boolean functions called the sensitivity conjecture [Hua19], which was open for nearly 30 years [NS94]. Surprisingly, Huang’s elegant proof takes less than 2 pages—truly a “proof from the book.” Specifically, Huang showed that for any total Boolean function, which is a function \( f : \{0,1\}^n \rightarrow \{0,1\} \), we have

\[
\deg(f) \leq s(f)^2,
\]

where \( \deg(f) \) is the real degree of \( f \) and \( s(f) \) is the (maximum) sensitivity of \( f \). These measures and other standard measures appearing in this introduction are defined in Section 2.

In this paper, we describe some implications of Huang’s resolution of the sensitivity conjecture to polynomial-based complexity measures of Boolean functions and quantum query complexity. Our first observation is that Huang actually proves a stronger claim than Eq. (1), in which \( s(f) \) can be replaced by \( \lambda(f) \), a spectral relaxation of sensitivity we define in Definition 7.

**Theorem 1.** For all Boolean functions \( f : \{0,1\}^n \rightarrow \{0,1\} \), we have \( \deg(f) \leq \lambda(f)^2 \).
In short, while \( s(f) \) can be viewed as the maximum number of 1s in any row or column of a certain Boolean matrix, \( \lambda(f) \) is the largest eigenvalue of that matrix, which could potentially be smaller. This observation has several implications because, as we show, \( \lambda(f) \) lower bounds many other complexity measures. One of the messages of this work is that \( \lambda(f) \) is an interesting complexity measure and can be used to establish relationships between other complexity measures.

We use this observation to prove two main results: Our first result is an optimal relationship between deterministic and quantum query complexity for total functions, and our second result is an optimal relationship between degree and approximate degree for total functions. We then apply the first result to prove the quantum analogue of the Aanderaa–Karp–Rosenberg conjecture and apply the second result to show that the approximate degree of any read-once formula is \( \Theta(\sqrt{n}) \).

**Deterministic vs. quantum query complexity.** We know from the seminal results of Nisan [Nis91], Nisan and Szegedy [NS94], and Beals et al. [BBC+01] that for any total Boolean function \( f \), the deterministic query complexity, \( D(f) \), and quantum query complexity, \( Q(f) \), satisfy:

\[
D(f) = O(Q(f)^6). \tag{2}
\]

Grover’s algorithm [Gro96] shows that for the OR function, a quadratic separation between \( D(f) \) and \( Q(f) \) is possible. This was the best known quantum speedup for total functions until Ambainis et al. [ABB+17] constructed a total function \( f \) with

\[
D(f) = \tilde{\Omega}(Q(f)^4). \tag{3}
\]

We show that the quartic separation (up to log factors) in Eq. (3) is actually the best possible.

**Theorem 2.** For all Boolean functions \( f : \{0,1\}^n \to \{0,1\} \), we have \( D(f) = O(Q(f)^4) \).

We deduce Theorem 2 as a corollary of a new tight relationship between \( \deg(f) \) and \( Q(f) \):

**Theorem 3.** For all Boolean functions \( f : \{0,1\}^n \to \{0,1\} \), we have \( \deg(f) = O(Q(f)^2) \).

Observe that Theorem 3 is tight for the OR function on \( n \) variables, whose degree is \( n \) and whose quantum query complexity is \( \Theta(\sqrt{n}) \) [Gro96, BBBV97]. Prior to this work, the best relation between \( \deg(f) \) and \( Q(f) \) was a sixth power relation, \( \deg(f) = O(Q(f)^6) \), which follows from Eq. (2).

As discussed, our proof relies on the restatement of Huang’s result (Theorem 1), showing that \( \deg(f) \leq \lambda(f)^2 \). We show (in Lemma 9) that the measure \( \lambda(f) \) lower bounds the original quantum adversary method of Ambainis [Amb02], which in turn lower bounds \( Q(f) \).

We now show how Theorem 2 straightforwardly follows from Theorem 3 using two previously known connections between complexity measures of Boolean functions.

**Proof of Theorem 2 assuming Theorem 3.** Midrijanis [Mid04] showed that for all total functions \( f \),

\[
D(f) \leq \text{bs}(f) \deg(f), \tag{4}
\]

where \text{bs}(f) is the block sensitivity of \( f \).

Theorem 3 shows that \( \deg(f) = O(Q(f)^2) \). Combining the relationship between block sensitivity and approximate degree from [NS94] with the results of [BBC+01], we get that \( \text{bs}(f) = O(Q(f)^2) \).

(This can also be proved directly using the lower bound method in [BBBV97].)

Combining these three inequalities yields \( D(f) = O(Q(f)^4) \) for all total Boolean functions \( f \). ■

We establish Theorem 3 in Section 3 using Theorem 1 and the spectral adversary method in quantum query complexity [BSS03].

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1. This means that for total functions, quantum query algorithms can only outperform classical query algorithms by a polynomial factor. On the other hand, for partial functions, which are defined on a subset of \( \{0,1\}^n \), exponential and even larger speedups are possible.
**Degree vs. approximate degree.** We also know from the works of Nisan and Szegedy [NS94] and Beals et al. [BBC+01], that for any total Boolean function \( f \),

\[
\deg(f) = O(\widetilde{\deg}(f)^6),
\]  

where \( \deg(f) \) and \( \widetilde{\deg}(f) \) are the exact degree and approximate degree of \( f \) respectively (defined in Section 2). We show that this relationship can be also significantly improved.

**Theorem 4.** For all Boolean functions \( f : \{0, 1\}^n \to \{0, 1\} \), we have \( \deg(f) = O(\widetilde{\deg}(f)^2) \).

This relationship is optimal since it is saturated by the OR function on \( n \) bits that has degree \( n \) and approximate degree \( \Theta(\sqrt{n}) \) [NS94].

Theorem 4 follows by combining \( \deg(f) \leq \lambda(f)^2 \) (Theorem 1) with \( \lambda(f) = O(\widetilde{\deg}(f)) \), which we prove in Section 4. This is the most technically challenging part of this paper, and we provide two proofs of this claim. The first proof (Theorem 17) is arguably simpler, but it is not self contained and uses Sherstov's composition theorem for approximate degree [She13b], and has a large constant hidden in the big Oh. The second proof (Theorem 18) does not rely on this result and achieves the optimal constant inside the big Oh.

Observe that because approximate degree lower bounds quantum query complexity, Theorem 4 also implies Theorem 3 (and hence Theorem 2). Although Theorem 3 is a consequence of this, our proof of Theorem 3 is much simpler and additionally proves that \( \lambda(f) \) lower bounds the positive-weights adversary method [Amb02], which is not implied by the proof of Theorem 4.

**Applications.** In Section 5, we use Theorem 3 to prove the quantum analogue of the famous Aanderaa–Karp–Rosenberg conjecture. Briefly, this conjecture is about the minimum possible query complexity of a nontrivial monotone graph property, for graphs specified by their adjacency matrices.

There are variants of the conjecture for different models of computation. For example, the randomized variant of the Aanderaa–Karp–Rosenberg conjecture, attributed to Karp [SW86, Conjecture 1.2] and Yao [Yao77, Remark (2)], states that for all nontrivial monotone graph properties \( f \), we have \( R(f) = \Omega(n^2) \). Following a long line of work, the current best lower bound is \( R(f) = \Omega(n^{4/3} \log^{1/3} n) \) due to Chakrabarti and Khot [CK01].

The quantum version of the conjecture was raised by Buhrman, Cleve, de Wolf, and Zalka [BCdWZ99], who observed that the best we could hope for is \( Q(f) = \Omega(n) \), because the nontrivial monotone graph property “contains at least one edge” can be decided with \( O(n) \) queries using Grover’s algorithm. Buhrman et al. [BCdWZ99] also showed that all nontrivial monotone graph properties satisfy \( Q(f) = \Omega(\sqrt{n}) \). The current best bound is \( Q(f) = \Omega(n^{2/3} \log^{1/3} n) \), which is credited to Yao in [MSS07]. We resolve this conjecture by showing an optimal \( \Omega(n) \) lower bound.

**Theorem 5.** Let \( f : \{0, 1\}^{(2)} \to \{0, 1\} \) be a nontrivial monotone graph property. Then \( Q(f) = \Omega(n) \).

Theorem 5 follows by combining Theorem 3 with a known quadratic lower bound on the degree of monotone graph properties.

In Section 6, we use Theorem 4 to completely characterize the approximate degree of any read-once formula. It is known that the quantum query complexity of any read-once formula on \( n \) variables is \( \Theta(\sqrt{n}) \) [BS04, Rei11]. It has long been conjectured that the approximate degree of any read-once formula is also \( \Theta(\sqrt{n}) \). It has taken much effort to establish this even for special read-once formulas. For example, the conjecture was proved for the simple depth-two read-once formula \( \text{AND} \circ \text{OR} \) in 2013 [BT13, She13a]. This result was later extended to all constant-depth balanced read-once formulas [BT15] and then to constant-depth unbalanced read-once formulas [BBGK18]. We resolve this question for all read-once formulas.

**Theorem 6.** For any read-once formula \( f : \{0, 1\}^n \to \{0, 1\} \), we have \( \widetilde{\deg}(f) = \Theta(\sqrt{n}) \).
1.1 Known relations and separations

Table 1 summarizes the known relations and separations between complexity measures studied in this paper (and more). This is an update to a similar table that appears in [ABK16] with the addition of $s(f)$ and $\lambda(f)$. Definitions and additional details about interpreting the table can be found in [ABK16].

For all the separations claimed in the table, we provide an example of a separating function or a citation to construction of such a function. All the relationships in the table follow by combining the relationships depicted in Figure 1 and the following inequalities that hold for all total Boolean functions:

- $C(f) \leq \text{bs}(f) s(f)$ [Nis91]
- $D(f) \leq \text{bs}(f) C(f)$ [BBC+01]
- $D(f) \leq \text{bs}(f) \deg(f)$ [Mid04]
- $R(f) = O(\deg(f)^2)$ [KT16]
- $R_0(f) = O(R(f) s(f) \log R(f))$ [KT16]
- $\deg(f) \leq \lambda(f)^2$ [Hua19]
- $s(f) \leq \lambda(f)^2$ (Lemma 34)

Figure 1: Relations between complexity measures. An upward line from a measure $M_1(f)$ to $M_2(f)$ denotes $M_1(f) = O(M_2(f))$ for all total functions.

1.2 Paper organization

Section 2 contains preliminaries and definitions of various complexity measures that appear in this paper. Section 3 reproves Theorem 1, which follows Huang’s original proof [Hua19], and then shows that $\lambda(f) = O(Q(f))$, which establishes Theorem 3. Section 4 establishes $\lambda(f) = O(\deg(f))$, which implies Theorem 4. Section 5 gives some background and motivation for the Aanderaa–Karp–Rosenberg conjecture and proves Theorem 5. Section 6 establishes Theorem 6. We end with some open problems in Section 7. Appendix A describes some properties of $\lambda(f)$, its many equivalent formulations, and its relationship with other complexity measures.

2 Preliminaries

2.1 Query complexity

Let $f : \{0, 1\}^n \to \{0, 1\}$ be a Boolean function. Let $A$ be a deterministic algorithm that computes $f(x)$ on input $x \in \{0, 1\}^n$ by making queries to the bits of $x$. The worst-case number of queries $A$ makes (over choices of $x$) is the query complexity of $A$. The minimum query complexity of any deterministic algorithm computing $f$ is the deterministic query complexity of $f$, denoted by $D(f)$.

We define the bounded-error randomized (respectively quantum) query complexity of $f$, denoted by $R(f)$ (respectively $Q(f)$), in an analogous way. We say an algorithm $A$ computes $f$ with bounded error if $\Pr[A(x) = f(x)] \geq 2/3$ for all $x \in \{0, 1\}^n$, where the probability is over the internal randomness of $A$. Then $R(f)$ (respectively $Q(f)$) is the minimum number of queries required by any randomized (respectively quantum) algorithm that computes $f$ with bounded error. It is clear that $Q(f) \leq R(f) \leq D(f)$. For more details on these measures, see the survey by Buhrman and de Wolf [BdW02].
Table 1: Best known separations between complexity measures

| D  | R₀  | R  | C  | RC | bs  | s  | λ  | Qₑ  | deg | Q  | deg |
|----|-----|----|----|----|-----|----|----|-----|-----|----|-----|
| D  | 2, 2 | 2, 3 | 2, 2 | 2, 3 | 2, 3 | 3, 6 | 4, 6 | 2, 3 | 2, 3 | 4, 4 | 4, 4 |
| R₀ | 1, 1 | 2, 2 | 2, 2 | 2, 2 | 2, 3 | 3, 6 | 4, 6 | 2, 3 | 2, 3 | 3, 4 | 4, 4 |
| R  | 1, 1 | 1, 1 | 2, 2 | 2, 2 | 2, 3 | 3, 6 | 4, 6 | 2, 3 | 2, 3 | 3, 4 | 4, 4 |
| C  | 1, 1 | 1, 1 | 1, 2 | 2, 2 | 2, 2 | 2, 2 | 2, 2 | 1.5, 3 | 1.63, 3 | 2, 4 | 2, 4 |
| RC | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 3, 2 | 2, 4 | 2, 4 | 1.5, 2 | 1.63, 2 | 2, 2 | 2, 2 |
| bs | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 2, 2 | 2, 2 | 1.5, 2 | 1.63, 2 | 2, 2 | 2, 2 |
| s  | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 2, 2 | 2, 2 | 1.5, 2 | 1.63, 2 | 2, 2 | 2, 2 |
| λ  | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 |
| Qₑ | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 |
| deg | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 |
| Q  | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 | 1, 1 |

- An entry a, b in the row M₁ and column M₂ roughly means that there exists a function g with $M_1(g) \geq M_2(g)^{a-\Theta(1)}$, and for all total functions $f$, $M_1(f) \leq M_2(f)^{b+\Theta(1)}$ (see [ABK16] for a precise definition). For example, the 3, 4 entry at row R and column Q means that the maximum possible separation between R and Q is at least cubic and at most quartic.

- The second row of each cell contains an example of a function that achieves the separation (or a citation to an example), where $\oplus = \text{PARITY}$, $\wedge = \text{AND}$, $\lor = \text{OR}$, $\land \lor = \text{AND-OR}$, and $\bar{\lambda}$-tree is the balanced NAND-tree function.

- Cells have a white background if the relationship is optimal and a gray background otherwise.

- Entries with a green background follow from Huang’s result. Entries with a red background follow from this work.

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"[BHT17] exhibited a family of functions satisfying $C(f) = \Omega(UC_{\text{min}}(f)^{1.22})$, and they also gave a transformation which modifies a function $f$ in a way that causes $s_1(f)$ to become $O(1)$, causes $s_o(f)$ to become at most $UC_{\text{min}}(f)$, and does not decrease $C(f)$. Since we have $\lambda(f) \leq \sqrt[3]{s_0(f) s_1(f)}$ by Lemma 31, this implies a power 2.44 separation between $\lambda(f)$ and $C(f)$."

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5
2.2 Sensitivity and block sensitivity

Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function, and let $x \in \{0, 1\}^n$ be a string. A block is a subset of $[n]$. We say that a block $B \subseteq [n]$ is sensitive for $x$ (with respect to $f$) if $f(x \oplus 1_B) \neq f(x)$, where $1_B$ is the $n$-bit string that is 1 on bits in $B$ and 0 otherwise. We say a bit $i$ is sensitive for $x$ if the block $\{i\}$ is sensitive for $x$. The maximum number of disjoint blocks that are all sensitive for $x$ is called the block sensitivity of $x$ (with respect to $f$), denoted by $bs_x(f)$. The number of sensitive bits for $x$ is called the sensitivity of $x$, denoted by $s_x(f)$. Clearly, $bs_x(f) \geq s_x(f)$, since $s_x(f)$ is the minimum number of bits that are sensitive for $x$. We define $s(f) = \max_{x \in \{0, 1\}^n} s_x(f)$ and $bs(f) = \max_{x \in \{0, 1\}^n} bs_x(f)$.

2.3 Degree measures

A polynomial $q \in \mathbb{R}[x_1, \ldots, x_n]$ is said to represent the function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ if $q(x) = f(x)$ for all $x \in \{0, 1\}^n$. A polynomial $q$ is said to $\epsilon$-approximate $f$ if $q(x) \in [0, \epsilon]$ for all $x \in f^{-1}(0)$ and $q(x) \in [1 - \epsilon, 1]$ for all $x \in f^{-1}(1)$. The degree of $f$, denoted by $\deg(f)$, is the minimum degree of a polynomial representing $f$. The $\epsilon$-approximate degree, denoted by $\deg_{\epsilon}(f)$, is the minimum degree of a polynomial $\epsilon$-approximating $f$. We will omit $\epsilon$ when $\epsilon = 1/3$. We know that $D(f) \geq \deg(f)$, $R(f) \geq \deg(f)$, and $Q(f) \geq \deg(f)/2$.

The degree of $f$ as a polynomial is also called the Fourier-degree of $f$, which equals $\max\{|S| : |\hat{f}(S)| \neq 0\}$ where $\hat{f}(S) := \mathbb{E}_x[f(x) \cdot (-1)^{\sum_{i \in S} x_i}]$. In particular, $\deg(f) < n$ if and only if $f$ agrees with the Parity function, $\text{PARITY}_n(x) = \oplus_{i=1}^n x_i$, on exactly half of the inputs.

3 Degree, spectral sensitivity, and quantum query complexity

Before proving Theorem 3, which is based on Huang’s proof, we reinterpret his result in terms of a new complexity measure of Boolean functions that we call $\lambda(f)$: the spectral norm of the sensitivity graph of $f$.

**Definition 7** (Sensitivity Graph $G_f$, Spectral Sensitivity $\lambda(f)$). Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function. The sensitivity graph of $f$, $G_f = (V, E)$ is a subgraph of the Boolean hypercube, where $V = \{0, 1\}^n$, and $E = \{(x, x \oplus e_i) \in V \times V : i \in [n], f(x) \neq f(x \oplus e_i)\}$. That is, $E$ is the set of edges between neighbors on the hypercube that have different $f$-value. Let $A_f$ be the adjacency matrix of the graph $G_f$. We define the spectral sensitivity of $f$ as $\lambda(f) = \|A_f\|$.

Note that since $G_f$ is bipartite, the largest and smallest eigenvalues of $A_f$ are equal in magnitude [GR01, Theorem 8.8.2], and because $A_f$ is a real symmetric matrix, $\lambda(f)$ is also the maximum eigenvalue of $A_f$.

Huang’s proof of the sensitivity conjecture [Hua19] can be divided into two steps:

1. $\forall f : \deg(f) \leq \lambda(f)^2$
2. $\forall f : \lambda(f) \leq s(f)$

The second step is the simple fact that the spectral norm of an adjacency matrix is at most the maximum degree of any vertex in the graph, which equals $s(f)$ in this case.

We reprove the first claim of Huang’s proof [Hua19], i.e., $\deg(f) \leq \lambda(f)^2$, for completeness. This is Theorem 1 from the introduction.

**Theorem 1.** For all Boolean functions $f : \{0, 1\}^n \rightarrow \{0, 1\}$, we have $\deg(f) \leq \lambda(f)^2$.  

Proof. Without loss of generality we can assume that $\text{deg}(f) = n$ since otherwise we can restrict our attention to a subcube of dimension $\text{deg}(f)$ in which the degree remains the same and the top eigenvalue is at most $\lambda(f)$. Specifically, we can choose any monomial in the polynomial representing $f$ of degree $\text{deg}(f)$ and set all the variables not appearing in this monomial to 0.

For $f$ with $\text{deg}(f) = n$, let $V_0 = \{x \in \{0, 1\}^n : f(x) = \text{parity}_n(x)\}$ and $V_1 = \{x \in \{0, 1\}^n : f(x) \neq \text{parity}_n(x)\}$. By the fact that $\text{deg}(f) = n$ we know that $|V_0| \neq |V_1|$ as otherwise $f$ would have 0 correlation with the $n$-variate parity function, implying that $f$’s top Fourier coefficient is 0.

We also note that any edge in the hypercube that goes between $V_0$ and $V_0$ is an edge in $G_f$ since it changes the value of $f$. This holds since for such an edge, $(x, x + e_i)$, we have $f(x) = \text{parity}_n(x) \neq \text{parity}_n(x + e_i) = f(x + e_i)$. Similarly, any edge in the hypercube that goes between $V_1$ and $V_1$ is an edge in $G_f$.

Assume without loss of generality that $|V_0| > |V_1|$. Thus, $|V_0| \geq 2^{n-1} + 1$. We will show that there exists a nonzero vector $v'$ supported only on the entries of $V_0$, such that $\|A_f \cdot v'\| \geq \sqrt{n} \cdot \|v'\|$.

Let $G = (V,E)$ be the complete $n$-dimensional Boolean hypercube. That is, $V = \{0,1\}^n$ and $E = \{(x,x + e_i) : x \in \{0,1\}^n, i \in [n]\}$. Take the following signing of the edges of the Boolean hypercube, defined recursively.

$$B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } B_i = \begin{pmatrix} B_{i-1} & I \\ I & -B_{i-1} \end{pmatrix} \text{ for } i \in \{2, \ldots, n\}. \quad (6)$$

This gives a new matrix $B_n \in \{-1,0,1\}^{V \times V}$ where $B_n(x,y) = 0$ if and only if $x$ is not a neighbor of $y$ in the hypercube.

Huang showed that $B_n$ has $2^n/2$ eigenvalues that equal $-\sqrt{n}$ and $2^n/2$ eigenvalues that equal $+\sqrt{n}$. To show this, he showed that $B_n^2 = n \cdot I$ by induction on $n$ and thus all eigenvalues of $B_n$ must be either $+\sqrt{n}$ or $-\sqrt{n}$. Then, observing that the trace of $B_n$ is 0, as all diagonal entries equal 0, we see that we must have an equal number of $+\sqrt{n}$ and $-\sqrt{n}$ eigenvalues.

Thus, the subspace of eigenvectors for $B_n$ with eigenvalue $\sqrt{n}$ is of dimension $2^n/2$. Using $|V_1| < 2^n/2$, there must exists a nonzero eigenvector for $B_n$ with eigenvalue $\sqrt{n}$ that vanishes on $V_1$. Fix $v$ to be any such vector.

Let $v'$ be the vector whose entries are the absolute values of the entries of $v$. We claim that $\|A_f \cdot v'\| \geq \sqrt{n} \cdot \|v'\|$. To see so, note that for every $x \in V_0$ we have

$$\begin{align*}
(A_f \cdot v')_x &= \sum_{y ~ x \neq f(y)} v'_y = \sum_{y ~ x = x \in V_0} v'_y = \sum_{y ~ x} v'_y \\
&\geq \sum_{y \in \{0,1\}^n} |B_{x,y} v_y| \geq \sum_{y \in \{0,1\}^n} B_{x,y} v_y = \sqrt{n} \cdot |v_x| = \sqrt{n} \cdot v'_x.
\end{align*} \quad (7)$$

On the other hand, for $x \in V_1$ we have $(A_f \cdot v')_x = 0 = v'_x$. Thus the norm of $A_f \cdot v'$ is at least $\sqrt{n}$ times the norm of $v'$, and hence $\lambda(f) = \|A_f\| \geq \sqrt{n} = \sqrt{\text{deg}(f)}$.

Finally, we prove that $\lambda(f) = O(Q(f))$. This proof goes via the spectral adversary method, $\text{SA}(f)$, introduced by Barman, Saks, and Szegedy [BSS03], which has many other equivalent formulations described in [SS06]. This result also follows from the equivalent characterization of $\lambda(f)$ as $K(f)$, a complexity measure defined by Koutsoupias [Kou93] that we describe in more detail in Appendix A, and the known result that $K(f) \leq \text{SA}(f)$ due to Laplante, Lee, and Szegedy [LLS06, Theorem 5.2]. We provide a self-contained proof here for completeness.

**Definition 8** (Spectral Adversary method). Let $\{D_i\}_{i \in [n]}$ and $F$ be matrices of size $\{0,1\}^n \times \{0,1\}^n$ with entries in $\{0,1\}$ satisfying $D_i[x,y] = 1$ if and only if $x_i \neq y_i$, and $F[x,y] = 1$ if and
only if \( f(x) \neq f(y) \). Let \( \Gamma \) denote a \( \{0,1\}^n \times \{0,1\}^n \) nonnegative symmetric matrix such that \( \Gamma \circ F = \Gamma \) (i.e., the nonzero entries of \( \Gamma \) are a subset of the the nonzero entries of \( F \)). Then \( \text{SA}(f) = \max_{\Gamma} \frac{||\Gamma||}{\max_{i \in [n]} ||\Gamma \circ D_i||} \).

Barnum, Saks, and Szegedy [BSS03] proved that \( \text{Q}(f) = \Omega(\text{SA}(f)) \).

**Lemma 9.** For all partial Boolean functions \( f \), \( \lambda(f) \leq \text{SA}(f) = O(\text{Q}(f)) \).

**Proof.** We first prove that \( \lambda(f) \leq \text{SA}(f) \). Indeed, one can take \( \Gamma \) to be simply the adjacency matrix of \( G_f \). That is, for any \( x, y \in \{0,1\}^n \) put \( \Gamma[x,y] = 1 \) if and only if \( y \sim x \) in the hypercube and \( f(x) \neq f(y) \). We observe that \( ||\Gamma|| = \lambda(f) \). On the other hand, for any \( i \in [n] \), \( \Gamma \circ D_i \) is the restriction of the sensitive edges in direction \( i \). The maximum degree in the graph represented by \( \Gamma \circ D_i \) is 1 hence \( ||\Gamma \circ D_i|| \) is at most 1. Thus we have

\[
\text{SA}(f) \geq \frac{||\Gamma||}{\max_{i \in [n]} ||\Gamma \circ D_i||} \geq \lambda(f). \tag{8}
\]

Combining this with \( \text{Q}(f) = \Omega(\text{SA}(f)) \) [BSS03], we get \( \lambda(f) \leq \text{SA}(f) = O(\text{Q}(f)) \). \( \square \)

From Theorem 1 and Lemma 9 we immediately get Theorem 3.

## 4 Degree vs. approximate degree

In this section we establish that for all total functions, spectral sensitivity is lower bounded by approximate degree. We first prove the simpler result for exact degree and then provide two proofs of the result for approximate degree.

### 4.1 Spectral sensitivity lower bounds degree

We first show the simpler result that spectral sensitivity lower bounds (exact) degree.

**Theorem 10.** For all total Boolean functions \( f : \{0,1\}^n \rightarrow \{0,1\} \), \( \lambda(f) \leq \text{deg}(f) \).

We start by expressing \( \lambda(f) \) in a way that allows us to relate it to a polynomial representing \( f \). In the following let \( H \) denote the Hadamard matrix of size \( 2^n \times 2^n \), defined as \( H_{xy} = (-1)^{x \cdot y} 2^{-n/2} \). For any function \( f : \{0,1\}^n \rightarrow \mathbb{R} \), we let \( \text{diag}(f) \) be the diagonal matrix that satisfies \( \text{diag}(f)_{xx} = f(x) \). We also let \( X_{xx} \) be the diagonal matrix satisfying \( X_{xx} = |x| \), where \( |x| \) is the Hamming weight of \( x \).

**Lemma 11.** Let \( f : \{0,1\}^n \rightarrow \{0,1\} \) be a total Boolean function and \( g : \{0,1\}^n \rightarrow \{-1,1\} \) be defined as \( g = 1 - 2f \). Then \( \lambda(f) = \max_{v:||v||=1} v^T (RXR - X)v \), where \( R = H \text{ diag}(g)H \).

**Proof.** Following Definition 7, let the sensitivity graph of \( f \) be \( G_f \), its adjacency matrix be \( A_f \), and its spectral sensitivity be \( \lambda(f) = ||A_f|| \). Since \( A_f \) is symmetric, we have \( \lambda(f) = \max_{v:||v||=1} ||v^TA_fv|| \). Furthermore, \( G_f \) is a bipartite matrix, as there are no edges between vertices of odd and even Hamming weight, which means the spectrum of \( A_f \) is symmetric about 0 [GR01, Theorem 8.8.2]. Thus \( \lambda(f) = \max_{v:||v||=1} v^T A_f v \).

Let \( A_H \) be the adjacency matrix of the hypercube graph \( (V, E) \) with \( V = \{0,1\}^n \) and edges \((x, x \oplus e_i)\) for all \( x \in \{0,1\}^n \) and \( i \in [n] \). Then we can express \( A_f \) as

\[
2A_f = A_H - \text{diag}(g)A_H \text{ diag}(g), \tag{9}
\]
Consequently, $R = H A_H$, where we have defined $c$ without changing its degree, note that $\deg x H v$, vectors $g \in \{x, y\}$. Let $v$ be in $g$ and we establish the second property now. Note that this lemma does not require the output of $x \neq y$ and otherwise the expression evaluates to $n - 2|x|$ showing that $HA_HH = (n\mathbb{I} - 2X)$. Furthermore, since $H$ is an involution (i.e., $H^2 = \mathbb{I}$), we have $HA_HH = A_H$. Using these two identities, we have

$$\lambda(f) = \max_{v: \|v\| = 1} v^T A_f v = \max_{v: \|v\| = 1} v^T H A_f H v$$

$$= \frac{1}{2} \max_{v: \|v\| = 1} v^T (H A_HH - H \text{ diag}(g)A_H \text{ diag}(g)H)v$$

$$= \frac{1}{2} \max_{v: \|v\| = 1} v^T (n\mathbb{I} - 2X - H \text{ diag}(g)H(n\mathbb{I} - 2X)H \text{ diag}(g)H)v$$

$$= \max_{v: \|v\| = 1} v^T (H \text{ diag}(g)H - (X + (H \text{ diag}(g)H)X(H \text{ diag}(g)H)v$$

$$= \max_{v: \|v\| = 1} v^T (RXR - X)v,$$

where Eq. (11) uses $H^T = H$ and $\|Hv\| = \|v\|$.

In the proof below we use two main properties of the matrix $R$. First, $R$ is a symmetric, orthonormal matrix. Second, that $R_{xy} = 0$ if $|x \oplus y| > \deg(g)$, where $|x \oplus y|$ is the Hamming distance between $x$ and $y$. Since any polynomial representing $f$ can be transformed into one representing $g$ without changing its degree, note that $\deg(g) = \deg(f)$. The first property follows straightforwardly and we establish the second property now. Note that this lemma does not require the output of $g$ to be in $\{-1, 1\}$, a fact we use in the next section.

**Lemma 12.** Let $g : \{0, 1\}^n \rightarrow \mathbb{R}$ have real degree $d$ and let $R = H \text{ diag}(g)H$. Then for all $x, y \in \{0, 1\}^n$, $R_{xy} = \hat{g}(x \oplus y)$, where for all $z \in \{0, 1\}^n$, $\hat{g}(z) = \frac{1}{2^n} \sum_{y \in \{0, 1\}^n} (-1)^{z \cdot y} g(y)$. Consequently, $R_{xy} = 0$ if $|x \oplus y| > d$.

**Proof.** From the definition of $R$, we have

$$R_{xy} = \frac{1}{2^n} \sum_{z \in \{0, 1\}^n} (-1)^{(x,z)} (-1)^{(z,y)} g(z) = \frac{1}{2^n} \sum_{z \in \{0, 1\}^n} (-1)^{(x \oplus y,z)} g(z) = \hat{g}(x \oplus y).$$

Since $g$ can be represented by a polynomial of degree $d$, all Fourier coefficients $\hat{g}(z)$ with $|z| > d$ are 0 and hence if $|x \oplus y| > d$ we have $R_{xy} = \hat{g}(x \oplus y) = 0$.

From Lemma 11, we know that $\lambda(f)$ is the maximum value of $v^T (RXR - X)v$ over all unit vectors $v$, which can be written as

$$v^T (RXR - X)v = \sum_{x \in \{0, 1\}^n} |x|(Rv)_x^2 - \sum_{x \in \{0, 1\}^n} |x|v_x^2 = \sum_{i=1}^{n} ic_i - \sum_{j=1}^{n} jb_j,$$

where we have defined $c_i := \sum_{x:|x|=i} (Rv)_x^2$ and $b_j := \sum_{x:|x|=j} v_x^2$. 

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To upper bound this expression, we need to relate the \( c_i \) and \( b_j \) quantities. We can establish a relationship using the fact that because \( R \) is sparse, if the input vector \( v \) is concentrated on one Hamming weight, then \( Rv \) will be concentrated on nearby Hamming weights (up to distance \( d \)). We formalize this idea in the lemma below. Note that this lemma does not use all the properties of \( R \) that we have established and we will need this in the next section.

**Lemma 13.** Let \( R \) be a matrix with \( \| R \| \leq 1 \) satisfying \( R_{xy} = 0 \) when \( |x \oplus y| > d \). For any vector \( v \), define \( c_i := \sum_{x:|x|=i} (Rv)_x^2 \) and \( b_j := \sum_{x:|x|=j} v_x^2 \). Then for any \( r \in \{d+1, \ldots, n\} \), we have

\[
\sum_{i=r}^{n} c_i \leq \sum_{j=r-d}^{n} b_j. \tag{18}
\]

**Proof.** By expanding the definition of \( c_i \), we have

\[
\sum_{i=r}^{n} c_i = \sum_{y:|y|\geq r} (Rv)_y^2 = \sum_{y:|y|\geq r} \left( \sum_{x \in \{0,1\}^n} R_{xy}v_x \right)^2 = \sum_{y:|y|\geq r} \left( \sum_{x \in \{0,1\}^n} R_{xy} \Pi_{(r-d)} v_x \right)^2, \tag{19}
\]

where we define \( \Pi_{(r)} \) to be the diagonal projector that satisfies the following for any vector \( v \):

\[
(\Pi_{(r)}v)_x = \begin{cases} v_x & \text{if } |x| \geq r \\ 0 & \text{otherwise} \end{cases}. \tag{20}
\]

The last equality holds because \( R(x,y) = 0 \) if \( |x \oplus y| > d \) so we can restrict the sum over \( x \) to be over those \( x \) with \( |x| \geq r - d \), and thus the only entries \( v_x \) that appear in the sum have \( |x| \geq r - d \). Thus we have

\[
\sum_{i=r}^{n} c_i = \sum_{y:|y|\geq r} ((R(\Pi_{(r-d)}v)_y)^2 \leq \sum_{y \in \{0,1\}^n} ((R(\Pi_{(r-d)}v)_y)^2 = \| R(\Pi_{(r-d)}v) \|^2, \tag{21}
\]

where the inequality holds because we only added more positive numbers to the sum by relaxing the sum over \( y \). Finally,

\[
\| R(\Pi_{(r-d)}v) \|^2 \leq \| \Pi_{(r-d)} v \|^2 = \sum_{x:|x|\geq r-d} v_x^2 = \sum_{j=r-d}^{n} b_j. \tag{22}
\]

where the inequality uses \( \| R \| \leq 1 \). This yields the claimed result. \( \square \)

We can now formally show the upper bound on \( v^T(RXR - X)v \) for a unit vector \( v \).

**Lemma 14.** Let \( R \) be a matrix with \( \| R \| \leq 1 \) satisfying \( R_{xy} = 0 \) when \( |x \oplus y| > d \). For any unit vector \( v \), we have

\[
v^T(RXR - X)v \leq d. \tag{23}
\]

**Proof.** As shown in Eq. (17),

\[
v^T(RXR - X)v = \sum_{i=1}^{n} i c_i - \sum_{j=1}^{n} j b_j, \tag{24}
\]

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where \( c_i := \sum_{x:|x|=i}(Rv)^2 \) and \( b_j := \sum_{x:|x|=j}v^2 \). Lemma 13 already established the following inequalities for all \( r \in \{d+1, \ldots, n\} \):

\[
\sum_{i=r}^n c_i \leq \sum_{j=r-d}^n b_j ,
\]

(25)

If we sum up these inequalities for all values of \( r \in \{d+1, \ldots, n\} \), we don’t quite get the sums that appear in Eq. (24). So let’s throw in some additional inequalities. For \( r \in \{1, \ldots, d\} \), we have

\[
\sum_{i=r}^n c_i \leq 1 ,
\]

(26)

which uses the fact that \( \sum_{i=r}^n c_i \leq \sum_{i=0}^n c_i = \|Rv\|^2 \leq 1 \) because \( \|R\| \leq 1 \) and \( \|v\| = 1 \). We also have for all \( k \in \{0, \ldots, d-1\} \)

\[
0 \leq \sum_{j=n-k}^n b_j ,
\]

(27)

using the fact that all \( b_j \geq 0 \).

If we sum up all the inequalities in Eq. (25) for \( r \in \{d+1, \ldots, n\} \), the inequalities in Eq. (26) for \( r \in \{1, \ldots, d\} \), and the inequalities in Eq. (27) for \( k \in \{0, \ldots, d-1\} \), we get

\[
\sum_{i=1}^n ic_i \leq \sum_{j=1}^n jb_j + d ,
\]

(28)

which shows that \( v^T(RXR - X)v \leq d \).

We can now establish Theorem 10.

**Proof of Theorem 10.** From Lemma 11, we have that \( \lambda(f) = \max_{\|v\|=1} v^T(RXR - X)v \) and from Lemma 14 we know that for any unit vector \( v \), \( v^T(RXR - X)v \leq d = \deg(g) = \deg(f) \).

### 4.2 Spectral sensitivity lower bounds approximate degree

We will now strengthen the previous result to show that spectral sensitivity also lower bounds approximate degree. As an intermediate result, we first establish this bound with a log factor. The stronger result without a log factor will follow from this in a completely black box way.

**Lemma 15.** For all total Boolean functions \( f : \{0,1\}^n \to \{0,1\} \), \( \lambda(f) = O(\deg(f) \log n) \).

We use the same notation as in the previous section, where \( f : \{0,1\}^n \to \{0,1\} \) is the total Boolean function under consideration and \( g(x) = 1 - 2f(x) \).

Let \( \tilde{g} \) be a minimum degree polynomial that \( \epsilon \)-approximates the function \( g : \{0,1\}^n \to \{-1,1\} \) for some \( \epsilon < 1/2 \) to be chosen later. We know that there exists a polynomial that \( 1/3 \)-approximates \( f \) and has degree \( \tilde{\deg}(f) \). It is also known that \( \tilde{\deg}(f) = O(\deg(f) \log(1/\epsilon)) \) for any (total or partial) function \( f \). See [BNRdW07] for an explicit construction using “amplification polynomials.” Hence there is a \( \epsilon \)-approximating polynomial for \( f \), and hence for \( g \), of degree at most \( O(\deg(f) \log(1/\epsilon)) \). Let \( \tilde{g} \) be such a polynomial with degree \( d = O(\deg(f) \log(1/\epsilon)) \). Specifically, we have for all \( x \in \{0,1\}^n \) that \( g(x) = -1 \iff \tilde{g}(x) \in [-1, -1 + \epsilon] \) and \( g(x) = 1 \iff \tilde{g}(x) \in [1 - \epsilon, 1] \).

We start by proving a statement analogous to Lemma 11, but using a polynomial that \( \epsilon \)-approximates \( f \).
**Lemma 16.** Let $f : \{0,1\}^n \to \{0,1\}$ be a total Boolean function and $g : \{0,1\}^n \to \{-1,1\}$ be defined as $g = 1 - 2f$. Let $\tilde{g}$ be a degree $O(\deg(f) \log(1/\epsilon))$ polynomial that $\epsilon$-approximates $g$. Then $\lambda(f) = \max_{v:||v||=1} v^T (RXR - X)v + 3\epsilon n$, where $R = H \text{diag}(\tilde{g}) H$.

**Proof.** Using Lemma 11, we have

$$\lambda(f) = \max_{v:||v||=1} v^T (RXR - X)v$$

$$= \max_{v:||v||=1} v^T (H \text{diag}(g) H X H \text{diag}(g) H - X)v$$

$$= \max_{v:||v||=1} v^T (H(\text{diag}(\tilde{g}) + \text{diag}(g - \tilde{g})) H X H (\text{diag}(\tilde{g}) + \text{diag}(g - \tilde{g})) H - X)$$

$$\leq \max_{v:||v||=1} v^T (H \text{diag}(\tilde{g}) H X H \text{diag}(\tilde{g}) H - X)v + 3\epsilon n$$

$$= \max_{v:||v||=1} v^T (RX\tilde{R} - X)v + 3\epsilon n,$$

where the inequality follows from the fact that $||\text{diag}(g - \tilde{g})|| \leq \epsilon$, $||H|| = 1$, and $||X|| = n$. ■

As before, let us examine the matrix $\tilde{R} = H \text{diag}(\tilde{g}) H$. It follows from the definition that $\tilde{R}$ is a symmetric matrix. It is also nearly orthonormal and satisfies $||\tilde{R}|| \leq 1$ because

$$||\tilde{R}|| = \max_x |\tilde{g}(x)| \leq 1. \quad (34)$$

Finally, we still have $\tilde{R}_{xy} = 0$ if $|x \oplus y| > d = \deg(\tilde{g})$ as before from Lemma 12, since the lemma did not assume that the output of $g$ was in $\{-1,1\}$. We are now ready to prove Lemma 15.

**Proof of Lemma 15.** From Lemma 16 we have that $\lambda(f) = \max_{v:||v||=1} v^T (RX\tilde{R} - X)v + 3\epsilon n$, where $\tilde{R} = H \text{diag}(\tilde{g}) H$ and $\tilde{g}$ is an $\epsilon$-approximating polynomial for $g$ of degree $d = O(\deg(f) \log(1/\epsilon))$.

The matrix $\tilde{R}$ satisfies the assumptions of Lemma 14, so we have $\lambda(f) \leq d + 3\epsilon n$. Choosing $\epsilon = 1/3n$ gives us $\lambda(f) = O(\deg(f) \log n)$. ■

Our main result now follows from this weaker statement in a black box way.

**Theorem 17.** For all total Boolean functions $f : \{0,1\}^n \to \{0,1\}$, $\lambda(f) = O(\widetilde{\deg}(f))$.

**Proof.** This proof uses Boolean function composition and the tensor power trick [Tao08, 1.9.4]. It relies on the composition properties of the complexity measures $\lambda$ and $\widetilde{\deg}$.

First, it is not too hard to show that for all Boolean functions $f$ and $g$,

$$\lambda(f \circ g) = \lambda(f) \lambda(g). \quad (35)$$

This essentially follows from known results on the quantum adversary bound, but we include a proof the appendix (Theorem 29) for completeness. We only need the $\geq$ direction in this proof.

Second, we need the fact that approximate degree composes with at most a constant factor overhead in the upper bound direction. Sherstov [She13b] showed that for all total functions $f$ and $g$, we have

$$\widetilde{\deg}(f \circ g) \leq c \widetilde{\deg}(f) \widetilde{\deg}(g), \quad (36)$$

for some universal constant $c \geq 1$.

Now from Lemma 15, we know that there exists a constant $c'$ such that for all $f : \{0,1\}^n \to \{0,1\}$, we have

$$\lambda(f) \leq c' \widetilde{\deg}(f) \log n. \quad (37)$$
Let \( f^k : \{0,1\}^n \rightarrow \{0,1\} \) denote the function \( f \) composed with itself \( k \) times. Then we have for all \( k \in \mathbb{N} \),
\[
\lambda(f)^k = \lambda(f^k) \leq c' \deg(f^k) \log(n^k) \leq c' e^{k-1} \deg(f)^k (k \log n).
\] (38)
Taking the \( k \)th root on both sides gives us
\[
\lambda(f) \leq (c' k \log n)^{1/k} \deg(f).
\] (39)
Since this equation holds for arbitrarily large \( k \), we must have
\[
\lambda(f) \leq c \deg(f),
\] (40)
which completes the proof.

### 4.3 An alternate self-contained proof

We now reprove the previous result, that spectral sensitivity lower bounds approximate degree, without using Sherstov’s composition theorem for approximate degree [She13b]. Our alternate proof also has the advantage of yielding a tighter upper bound by a constant factor.

As in the previous section, \( f : \{0,1\}^n \rightarrow \{0,1\} \) is the total function under consideration and we want to relate \( \lambda(f) \) to \( \deg_{\epsilon}(f) \), where for any \( \epsilon \in [0,1/2] \), \( \deg_{\epsilon}(f) \) is the minimum degree of a polynomial \( q \) such that for all \( x \in \{0,1\}^n \), \( f(x) = 1 \implies q(x) \in [1 - 2\epsilon, 1] \) and \( f(x) = 0 \implies q(x) \in [-1, -1 + 2\epsilon] \).

The main result of this section is the following, which also implies Theorem 10 by setting \( \epsilon = 0 \).

**Theorem 18.** For all total Boolean functions \( f : \{0,1\}^n \rightarrow \{0,1\} \) and \( \epsilon \in [0,1/2] \), \( \lambda(f) \leq \frac{1}{1 - 2\epsilon} \deg_{\epsilon}(f) \).

We start by upper bounding \( \lambda(f) \) by the norm of a matrix \( B_q \) that is derived from the polynomial \( q \). For any \( \epsilon \), let \( q \) be an \( \epsilon \)-approximating polynomial for \( f \) (as defined above). We then define for all \( x, y \in \{0,1\}^n \),
\[
(B_q)_{xy} = \begin{cases} \frac{1}{2}(q(x) - q(y)) & \text{if } |x \oplus y| = 1, \\ 0 & \text{if } |x \oplus y| \neq 1. \end{cases}
\] (41)

**Lemma 19.** For any total Boolean function \( f : \{0,1\}^n \rightarrow \{0,1\} \) and \( B_q \) as defined in Eq. (41) from an \( \epsilon \)-approximating polynomial \( q \), we have \( \lambda(f) \leq \frac{1}{1 - 2\epsilon} \|B_q\| \).

**Proof.** We know that \( \lambda(f) = \|A_f\| \), where \( (A_f)_{xy} = 1 \) if and only if \( |x \oplus y| = 1 \) and \( f(x) \neq f(y) \), and otherwise \( (A_f)_{xy} = 0 \). Because \( A_f \) has nonzero entries only when \( f(x) \neq f(y) \), if we reorder the basis \( \{0,1\}^n \) such that all inputs with \( f(x) = 0 \) come first and all inputs with \( f(y) = 1 \) come after, then \( A_f \) will be a block matrix of the form
\[
A_f = \begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix}.
\] (42)
It is easy to see that \( \|A_f\| = \|A\| \). Let us now write \( B_q \) in the same reordered basis, and call the bottom left matrix \( B \). For \( x \in f^{-1}(1) \) and \( y \in f^{-1}(0) \), \( B_{xy} = \frac{1}{2}(q(x) - q(y)) \) if \( |x \oplus y| = 1 \). For these inputs, we know that \( q(x) \in [1 - 2\epsilon, 1] \) and \( q(y) \in [-1, -1 + 2\epsilon] \), and thus \( B_{xy} \in [1 - 2\epsilon, 1] \) if \( |x \oplus y| = 1 \). All other entries in \( B \) equal 0. The matrix \( A \) satisfies \( A_{xy} = 1 \) if \( |x \oplus y| = 1 \). Thus we observe that \( B_{xy} \geq (1 - 2\epsilon)A_{xy} \) for all \( x \in f^{-1}(1) \) and \( y \in f^{-1}(0) \). Let \( u \) and \( v \) be unit vectors such
that $u^T Av = \|A\|$. Since $A$ is a nonnegative matrix, we may assume without loss of generality that $u, v \geq 0$.

Using these vectors, we see that $\|B\| \geq u^T B v \geq (1 - 2\varepsilon) u^T Av = (1 - 2\varepsilon) \|A\|$.

Since $B$ is a submatrix of $B_q$, we have $\|B_q\| \geq \|B\| \geq (1 - 2\varepsilon) \|A\| = (1 - 2\varepsilon) \|A_f\| = (1 - 2\varepsilon) \lambda(f)$.

We now express $B_q$ in terms of matrices that are easier to work with. The first matrix, which was also defined in the previous section, is $\tilde{R} = H \text{diag}(q)H$, where $\text{diag}(q)$ denotes the $2^n \times 2^n$ matrix with $q(x)$ on the diagonal. The second matrix is $W$ of size $2^n \times 2^n$, defined as $W_{xy} = |x| - |y|$ for all $x, y \in \{0, 1\}^n$.

In the following, for two $m \times n$ matrices $A$ and $B$, we denote by $A \odot B$ the Hadamard product (i.e., entrywise product) of $A$ and $B$. $A \odot B$ is an $m \times n$ matrix, with elements given by $(A \odot B)_{i,j} = A_{i,j}B_{i,j}$.

**Lemma 20.** For the matrix $B_q$ defined in Eq. (41), and $W$ and $\tilde{R}$ defined above, $\|B_q\| = \|W \odot \tilde{R}\|$.

**Proof.** We have $B_q = (\text{diag}(q)A_H - A_H \text{diag}(q))/2$, where $A_H$ is the adjacency matrix of the Boolean hypercube. This can be verified by observing that at position $(x, y)$, both sides are 0 if $|x \oplus y| \neq 1$, and otherwise both sides are $(q(x) - q(y))/2$. Now, recall that $A_H = H(n\mathbb{1} - 2X)H$, where $\mathbb{1}$ is the identity matrix, $X$ is the matrix satisfying $X_{xy} = |x|$ and $H$ is the Hadamard matrix. Hence

$$HB_qH = \frac{H \text{diag}(q)H(nI - 2X) - (nI - 2X)H \text{diag}(q)H}{2} = X\tilde{R} - \tilde{R}X. \quad (43)$$

Then for any $x, y \in \{0, 1\}^n$,

$$(HB_qH)_{xy} = (X\tilde{R} - \tilde{R}X)_{xy} = |x|\tilde{R}_{xy} - |y|\tilde{R}_{xy} = (|x| - |y|)\tilde{R}_{xy} = (W \odot \tilde{R})_{xy}. \quad (44)$$

Since $H$ is unitary, $\|B_q\| = \|HB_qH\| = \|W \odot \tilde{R}\|$.

We now need to upper bound $\|W \odot \tilde{R}\|$. First, we observe that Lemma 12 shows that the matrix $\tilde{R}$ satisfies $\tilde{R}_{xy} = 0$ if $|x \oplus y| \geq d = \deg_c(f)$. The lemma was established for a different matrix $R$, but $\tilde{R}$ satisfies the assumptions of the lemma as well. Then, due to the above observation, $W \odot \tilde{R} = V \odot \tilde{R}$ for any matrix $V$ that satisfies $V_{xy} = W_{xy}$ for $x, y$ with $|x \oplus y| \leq d$. Thus, it suffices to bound $\|V \odot \tilde{R}\|$ for any matrix $V$ as above. In particular, we could design a matrix $V$ as above all whose entries are bounded by $d$ in absolute value. It is tempting to say that since $V$’s entries are bounded by $d$ and since $\|\tilde{R}\| \leq 1$ then $\|V \odot \tilde{R}\| \leq d$ however this is not well-justified since matrix norms can increase dramatically even if we just change the signs of some entries in a matrix. Nevertheless, we would show that there exists a choice of $V$ for which the above assertion hold. Moreover we would show that for a certain choice of $V$ as above, it holds that $\|V \odot A\| \leq d \|A\|$ for any matrix $A$. Indeed, such a property is captured by the $\gamma_2$-norm of $V$. This $\gamma_2$-norm arises in communication complexity [LMSS07, LSS08] and is also known as the Schur product norm [Wal86]. It is defined as follows.

**Definition 21 ($\gamma_2$ norm).** For any $m \times n$ matrix $A$, we define

$$\gamma_2(A) = \min_{X, Y: X^T Y = A} c(X)c(Y), \quad (45)$$

where $c(X)$ is the maximum $\ell_2$ norm of any column of $X$.\footnote{Otherwise, by taking point-wise absolute values on $u$ and $v$ we would get two unit vectors $\tilde{u}$ and $\tilde{v}$ with $\tilde{u}^T \tilde{v} = \sum_{i,j} \tilde{u}_i \tilde{v}_j \geq \sum_{i,j} u_i A_{i,j} v_j = u^T Av$.}
A crucial property of the $\gamma_2$-norm is that $\|A \odot B\| \leq \gamma_2(A) \cdot \|B\|$ for any two $m \times n$ matrices $A$ and $B$.\footnote{This inequality can also be used to define the $\gamma_2$ norm. As [LSS08, Theorem 9] shows, we can equivalently define $\gamma_2(A)$ as the maximum of $\|A \odot B\|$ over all matrices $B$ with $\|B\| \leq 1$.} We use it to prove the next lemma.

**Lemma 22.** Let $B_q$ and $W$ be as defined above. Let $V$ be any matrix that satisfies $V_{xy} = W_{xy}$ when $|x| - |y| \leq d$. Then $\|B_q\| \leq \gamma_2(V)$.

**Proof.** In Lemma 20 we showed that $\|B_q\| = \|W \odot \tilde{R}\|$. Since $\tilde{R}_{xy} = 0$ if $|x \oplus y| \geq d$, $W \odot \tilde{R} = V \odot \tilde{R}$ since $V$ and $W$ agree on inputs where $|x| - |y| \leq d$, which is implied by the condition $|x \oplus y| \leq d$. Thus $\|B_q\| = \|V \odot \tilde{R}\|$. We then use the relationship $\|A \odot B\| \leq \gamma_2(A)\|B\|$, which is not hard to show (see, e.g., [AHJ87, Section 5] or [LSS08, Theorem 9]). This gives us $\|B_q\| \leq \gamma_2(V)\|\tilde{R}\|$. Finally, since $R = H \text{diag}(q)H$ and for any $x$, $|q(x)| \leq 1$, we have $\|\tilde{R}\| \leq 1$. ■

Now we can upper bound $\|B_q\|$ by $\gamma_2(V)$ for any matrix $V$ that satisfies $V_{xy} = |x| - |y|$ when $|x| - |y| \leq d$. Instead of working with $V$, which is a $2^n \times 2^n$ matrix, the following lemma will allow us to work with an $(n+1) \times (n+1)$ matrix.

**Lemma 23.** Let $M$ be any $(n+1) \times (n+1)$ matrix that satisfies $M_{st} = s - t$ for all $s, t \in \{0, \ldots, n\}$ with $|s - t| \leq d$. Then there exists a matrix $V$ satisfying the conditions of Lemma 22, such that $\gamma_2(V) \leq \gamma_2(M)$.

**Proof.** If $M$ is such a matrix, then we can get an appropriate matrix $V$ simply by duplicating rows and columns of $M$. That is, index the rows and columns of $M$ by $\{0, 1, \ldots, n\}$, and let $V$ be the $2^n \times 2^n$ matrix defined by $V_{x,y} = M_{|x|,|y|}$. Then we can get $V$ from $M$ by duplicating row $j$ of $M$ to create $\binom{n}{j}$ copies of it, for each $j \in \{0, 1, \ldots, n\}$, and then repeating the duplication for the columns.

To ensure that $\gamma_2(V) \leq \gamma_2(M)$, all we need to show is that duplicating a row or column of a matrix does not increase $\gamma_2(\cdot)$. This is easy to see: if the matrix we start with is $M = X^TY$ with $c(X)c(Y) = \gamma_2(M)$, then duplicating a row of $M$ gives the matrix $M'$, which can be factored $M' = (X')^TY$, where $X'$ is the matrix we get by duplicating the corresponding row of $X^T$ (which is a column of $X$). Since duplicating a column of $X$ does not affect $c(X)$, we get a factorization of $M'$ which certifies that $\gamma_2(M') \leq c(X)c(Y) = \gamma_2(M)$. A similar argument shows that duplicating a column of $M$ also does not increase $\gamma_2(\cdot)$.

■

**Lemma 24.** There is an $(n+1) \times (n+1)$ matrix $M$ such that $\gamma_2(M) \leq d$ and $M_{st} = s - t$ for all $s, t \in \{0, 1, \ldots, n\}$ with $|s - t| \leq d$.

**Proof.** We start with a slightly informal “picture” proof. To explain the matrix $M$, it will be simplest to give an example for the case $d = 3, n = 12$. We pick $M$ to be the following matrix:

\[
\begin{pmatrix}
0 & -1 & -2 & -3 & -2 & -1 & 0 & +1 & +2 & +3 & +2 & +1 & 0 \\
+1 & 0 & -1 & -2 & -3 & -2 & -1 & 0 & +1 & +2 & +3 & +2 & +1 \\
+2 & +1 & 0 & -1 & -2 & -3 & -2 & -1 & 0 & +1 & +2 & +3 & +2 \\
+3 & +2 & +1 & 0 & -1 & -2 & -3 & -2 & -1 & 0 & +1 & +2 & +3 \\
+2 & +3 & +2 & +1 & 0 & -1 & -2 & -3 & -2 & -1 & 0 & +1 & +2 \\
+1 & 2 & +3 & +2 & +1 & 0 & -1 & -2 & -3 & -2 & -1 & 0 & +1 \\
0 & +1 & +2 & +3 & +2 & +1 & 0 & -1 & -2 & -3 & -2 & -1 & 0 \\
-1 & 0 & +1 & +2 & +3 & +2 & +1 & 0 & -1 & -2 & -3 & -2 & -1 \\
-2 & -1 & 0 & +1 & +2 & +3 & +2 & +1 & 0 & -1 & -2 & -3 & -2 \\
-3 & -2 & -1 & 0 & +1 & +2 & +3 & +2 & +1 & 0 & -1 & -2 & -3 \\
-2 & -3 & -2 & -1 & 0 & +1 & +2 & +3 & +2 & +1 & 0 & -1 & -2 \\
-1 & -2 & -3 & -2 & -1 & 0 & +1 & +2 & +3 & +2 & +1 & 0 & -1 \\
0 & -1 & -2 & -3 & -2 & -1 & 0 & +1 & +2 & +3 & +2 & +1 & 0
\end{pmatrix}
\]  

(46)

Here the matrix $M$ is a $13 \times 13$ matrix which satisfies $M_{st} = s - t$ when $|s - t| \leq 3$: that is, within distance 3 of its diagonal, the entries of $M$ are equal to the distance to the diagonal (and they are positive below the diagonal and negative above it). Next, we write $M$ as a sum of $d = 3$
matrices that have entries in \{-1, 0, +1\}. Note that each of the following \(d\) matrices have \(d \times d\) blocks of \(-1\) just above the diagonal and \(d \times d\) blocks of \(+1\) just below the diagonal.

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 & +2 & +3 & +2 & +1 & 0 \\
+1 & 0 & 1 & 0 & 1 & 0 & +1 & +2 & +3 & +2 & 0 \\
+3 & +2 & +1 & 0 & +2 & +3 & +2 & 0 & +1 & 0 & 0 \\
+2 & +3 & +2 & +1 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\
+1 & 0 & 1 & 0 & 1 & 0 & +1 & 0 & 0 & 0 & 0 \\
0 & 1 & +2 & +3 & +2 & +1 & 0 & +1 & 0 & 0 & 0 \\
+1 & 0 & +1 & +2 & +3 & +2 & +1 & +2 & +3 & +2 & +1 \\
+1 & 0 & 1 & +2 & +3 & +2 & +1 & 0 & +1 & 0 & 0 \\
+1 & 0 & +1 & +2 & +3 & +2 & +1 & +2 & +3 & +2 & +1 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & +1 & +1 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & +1 & +1 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & +1 & +1 & 0 \\
+1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
+1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Finally, we use two additional properties of \(Q\) blocks along the main diagonal. Finally, whether the value is positive or negative depends whether \(d\) is the length of the intersection of the line segment from \((i, j)\) to \((i + d - 1, j + d - 1)\) and the \(d \times d\) blocks along the main diagonal. Finally, whether the value is positive or negative depends whether the line segment additionally intersects a block of \(1\)'s (if \(i > j\)) or a block of \(-1\)'s (if \(i < j\)) and thus \(C_0 + \cdots + C_{d-1}\) is in fact equal to \(M\).

For a general \(d\), \(M\) will be a sum of \(d\) matrices that we call \(C_0, C_1, \ldots, C_{d-1}\). We now explain how to upper bound \(\gamma_2(C_k)\) and obtain an upper bound on \(\gamma_2(M)\). Define \(Q\) to be the matrix

\[
Q = J_{[n/4d] + 1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \end{pmatrix} \otimes J_d,
\]

where \(\otimes\) denotes the Kronecker (tensor) product, and where \(J_m\) denotes the \(m \times m\) all-ones matrix. Further, this middle matrix is \((\begin{smallmatrix} +1 & -1 \\ -1 & +1 \end{smallmatrix}) \otimes \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\).

For \(k = 0, \ldots, d - 1\), we let \(C_k\) be the \(n \times n\) submatrix defined by \((C_k)_{i,j} = Q_{i + k, j + k}\). In other words, \(C_0\) is the top left \(n \times n\) corner of \(Q\), and for each \(k \geq 1\), we obtain \(C_k\) from \(C_{k-1}\) by shifting the submatrix diagonally.

Thus, the \((i,j)\) entry of \(C_0 + \cdots + C_{d-1}\) is equal to \(Q_{i,j} + Q_{i+1,j+1} + \cdots + Q_{i+d-1,j+d-1}\). Because \(Q\) is a block matrix whose blocks are of size \(d \times d\), this is equal in magnitude to \(|i - j|\), as \(d - |i - j|\) is the length of the intersection of the line segment from \((i, j)\) to \((i + d - 1, j + d - 1)\) and the \(d \times d\) blocks along the main diagonal. Finally, whether the value is positive or negative depends whether the line segment additionally intersects a block of \(1\)'s (if \(i > j\)) or a block of \(-1\)'s (if \(i < j\)) and thus \(C_0 + \cdots + C_{d-1}\) is in fact equal to \(M\).

Now that we have argued that all the matrices \(C_k\) are submatrices of shifted versions of \(Q\), let us compute \(\gamma_2(Q)\). We now claim that \(\gamma_2(Q) = 1\). This follows from a few easy-to-verify facts about the \(\gamma_2(\cdot)\) norm:

1. \(\gamma_2(A \otimes B) = \gamma_2(A) \gamma_2(B)\).
2. \(\gamma_2(J_m) = 1\) for all \(m\).
3. \(\gamma_2(\begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}) = 1\) and \(\gamma_2(\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}) = 1\).

Since \(Q\) decomposes into a Kronecker product of matrices with \(\gamma_2(\cdot) = 1\) we get that \(\gamma_2(Q) = 1\). Finally, we use two additional properties of \(\gamma_2(\cdot)\): first, that it is subadditive (indeed, it is a norm), so \(\gamma_2(M) \leq \sum_{k=0}^{d} \gamma_2(C_k)\); and second, that it is non-increasing under restriction to a submatrix, so \(\gamma_2(C_k) \leq \gamma_2(Q)\) for each \(k = 0, 1, \ldots, d - 1\). Together, these properties imply that \(\gamma_2(M) \leq d\), completing the picture proof.
We also provide a more explicit and formal way of showing that there is an appropriate matrix $M$ satisfying $\gamma_2(M) \leq d$; this method also avoids using any properties of $\gamma_2(\cdot)$ by directly giving a factorization $M = ST$. This factorization still corresponds to writing $M$ as the sum of $C_0 + C_1 + \cdots + C_{d-1}$, together with the observation that each of the matrices $C_k$ has rank 2 (as can be seen from their Kronecker product decomposition).

Specifically, the matrices $S$ and $T$ will have dimensions $2d \times (n + 1)$, and will be defined as follows. For any $s \in \{0, 1, \ldots, n\}$ and any $j \in \{0, \ldots, 2d - 1\}$, we can write $s + j = a + b(2d)$ for unique integers $a \in \{0, 1, \ldots, 2d - 1\}$ and $b \in \mathbb{N}$; that is, $a$ is the remainder of $s + j$ modulo $2d$, and $b = [(s + j)/2d]$. If $a \in [0, d - 1]$, we define $S_{js} = (-1)^b$ and $T_{js} = 0$. Otherwise (i.e., if $a \in [d, 2d - 1]$), we define $S_{js} = 0$ and $T_{js} = -(-1)^b$. We will use $S_j$ to denote row $j$ of $S$, and similarly for $T_j$. (For reference, in terms of the picture proof, we will have $S_j^T T_j + S_j^T d T_j + d = C_j$ for all $j \in \{0, 1, \ldots, d - 1\}$.)

In each column of $S$ and $T$, half the entries are zero and half are $\pm 1$, and hence we have $c(S) = c(T) = \sqrt{d}$. To complete the proof of the lemma, all we need to do is to set $M = ST$ and to show that $M_{st} = s - t$ whenever $|s - t| \leq d$. To this end, observe that $(S^T T_j)_{st}$ is nonzero if and only if $s + j \mod 2d \in [0, d - 1]$ and $t + j \mod 2d \in [d, 2d - 1]$. When $|s - t| \leq d$, it’s easy to see that the number of different values of $j \in \{0, 1, \ldots, 2d - 1\}$ for which this happens is exactly $|s - t|$. Moreover, if $s < t$, then it’s not hard to see that $(S^T T_j)_{st}$ will be negative if it is nonzero, while if $s > t$, it will be positive if it is nonzero. Hence $(ST)_{st} = \sum_{j=0}^{2d-1} (S_j^T T_j)_{st} = s - t$ whenever $|s - t| \leq d$.

Finally, we can put the pieces together to prove Theorem 18.

**Proof of Theorem 18.** From Lemma 19, we have $\lambda(f) \leq \frac{1}{1 - 2x} \|B_q\|$. Lemma 22 and Lemma 23 then give us $\|B_q\| \leq \gamma_2(M)$, for any matrix $M$ that satisfies the conditions in Lemma 24. Lemma 24 constructs such a matrix $M$ with $\gamma_2(M) \leq d = \deg_{\mathcal{G}_c}(f)$, completing the proof.

## 5 Monotone graph properties

The Aanderaa–Karp–Rosenberg conjectures are a collection of conjectures related to the query complexity of deciding whether an input graph specified by its adjacency matrix satisfies a given property in various models of computation.

Specifically, let the input be an $n$-vertex undirected simple graph specified by its adjacency matrix. This means we can query any unordered pair $\{i, j\}$, where $i, j \in [n]$, and learn whether there is an edge between vertex $i$ and $j$. Note that the input size is $\binom{n}{2} = \Theta(n^2)$.

A function $f$ on $\binom{n}{2}$ variables is a graph property if it treats the input as a graph and not merely a string of length $\binom{n}{2}$. Specifically, the function must be invariant under permuting vertices of the graph. In other words, the function can only depend on the isomorphism class of the graph, not the specific labels of the vertices. A function $f$ is monotone (increasing) if for all $x, y \in \{0, 1\}^n$, $x \leq y \iff f(x) \leq f(y)$, where $x \leq y$ means $x_i \leq y_i$ for all $i \in [n]$. For a monotone function, negating a 0 in the input cannot change the function value from 1 to 0. In the context of graph properties, if the input graph has a certain monotone graph property, then adding more edges cannot destroy the property.

Examples of monotone graph properties include “$G$ is connected,” “$G$ contains a clique of size $k$,” “$G$ contains a Hamiltonian cycle,” “$G$ has chromatic number greater than $k$,” “$G$ is not planar”, and “$G$ has diameter at most $k$.” Many commonly encountered graph properties (or their negation) are monotone graph properties. Finally, we say a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is nontrivial if there exist inputs $x$ and $y$ such that $f(x) \neq f(y)$.
The deterministic Aanderaa–Karp–Rosenberg conjecture, also called the *evasiveness conjecture*, states that for all nontrivial monotone graph properties \( f \), \( D(f) = \binom{n}{2} \). This conjecture remains open to this day, although the weaker claim that \( D(f) = \Omega(n^2) \) was proved over 40 years ago by Rivest and Vuillemin [RV76]. Several works have improved on the constant in their lower bound, and the best current result is due to Scheidweiler and Triesch [ST13], who prove a lower bound of \( D(f) \geq (1/3 - o(1)) \cdot n^2 \). The evasiveness conjecture has been established in several special cases including when \( n \) is prime [KSS84] and when restricted to bipartite graphs [Yao88].

The randomized Aanderaa–Karp–Rosenberg conjecture asserts that all nontrivial monotone graph properties \( f \) satisfy \( R(f) = \Omega(n^2) \). A sequence of increasingly stronger lower bounds, starting with a lower bound of \( \Omega(n \log^{1/2} n) \) due to Yao [Yao91], a lower bound of \( \Omega(n^{5/4}) \) due to King [Kin88], and a lower bound of \( \Omega(n^{4/3}) \) due to Hajnal [Haj91], has led to the current best lower bound of \( \Omega(n^{4/3} \log^{1/3} n) \) due to Chakrabarti and Khot [CK01]. There are also two lower bounds due to Friedgut, Kahn, and Wigderson [FKW02] and O’Donnell, Saks, Schramm, and Servedio [OSSS05] that are better than this bound for some graph properties.

The quantum Aanderaa–Karp–Rosenberg conjecture states that all nontrivial monotone graph properties \( f \) satisfy \( Q(f) = \Omega(n) \). This is the best lower bound one could hope to prove since there exist properties with \( Q(f) = O(n) \), such as the property of containing at least one edge. In fact, for any \( \alpha \in [1, 2] \) it is possible to construct a graph property with quantum query complexity \( \Theta(n^\alpha) \) using known lower bounds for the threshold function [BBC+01].

As stated in the introduction, the question was first raised by Buhrman, Cleve, de Wolf, and Zalka [BCdWZ99], who showed a lower bound of \( \Omega(\sqrt{n}) \). This was improved by Yao to \( \Omega(n^{2/3} \log^{1/6} n) \) using the technique in [CK01] and Ambainis’ adversary bound [Amb02]. Better lower bounds are known in some special cases, such as when the property is a subgraph isomorphism property, where we know a lower bound of \( \Omega(n^{3/4}) \) due to Kulkarni and Podder [KP16].

As stated in Theorem 5, we resolve the quantum Aanderaa–Karp–Rosenberg conjecture and show an optimal \( \Omega(n) \) lower bound. The proof combines Theorem 3 with a quadratic lower bound on the degree of nontrivial monotone graph properties. With some work, the original quadratic lower bound on the deterministic query complexity of nontrivial monotone graph properties by Rivest and Vuillemin [RV76] can be modified to prove a similar lower bound for degree. We were not able to find such a proof in the literature, and instead combine the following two claims to obtain the desired claim.

First, we use the result of Dodis and Khanna [DK99, Theorem 2]:

**Theorem 25.** For all nontrivial monotone graph properties, \( \deg_2(f) = \Omega(n^2) \).

Here \( \deg_2(f) \) is the minimum degree of a Boolean function when represented as a polynomial over the finite field with two elements, \( \mathbb{F}_2 \). We combine this with a standard lemma that shows that this measure lower bounds \( \deg(f) \). A proof can be found in [O’D09, Proposition 6.23]:

**Lemma 26.** For all Boolean functions \( f : \{0, 1\}^n \rightarrow \{0, 1\} \), we have \( \deg_2(f) \leq \deg(f) \).

Combining these with Theorem 3, we get that all nontrivial monotone graph properties \( f \) satisfy \( Q(f) = \Omega(n) \), which is the statement of Theorem 5.

6 Approximate degree of read-once formulas

A read-once formula over the De Morgan basis, which consists of AND gates, OR gates, and NOT gates, is a formula in which each variable appears exactly once. Examples of read-once
formulas include the AND_2 and OR_2 functions themselves, and compositions of these functions such as AND_2 ∘ OR_2.

While it was already established in [NS94] that \( \widetilde{\deg}(\text{AND}_2) = \widetilde{\deg}(\text{OR}_2) = \Theta(\sqrt{n}) \), the approximate degree of AND_\sqrt{n} ∘ OR_\sqrt{n} remained open until 2013 when it was shown that AND_\sqrt{n} ∘ OR_\sqrt{n} = \Theta(\sqrt{n}) [BT13, She13a] using a linear programming characterization of approximate degree. Notice that in both cases the approximate degree is the square root of the number of variables. This was later extended to constant-depth balanced read-once formulas [BT15] and constant-depth unbalanced read-once formulas [BBGK18]. We finally resolve this question for all read-once formulas by establishing Theorem 6 from the introduction:

**Theorem 6.** For any read-once formula \( f : \{0, 1\}^n \to \{0, 1\} \), we have \( \widetilde{\deg}(f) = \Theta(\sqrt{n}) \).

Note that we already knew that for read-once formulas \( f, Q(f) = \Theta(\sqrt{n}) \). The lower bound was established by Barnum and Saks [BS04] and the upper bound was established by Reichardt [Rei11].

The upper bound in Theorem 6 follows straightforwardly from Reichardt’s upper bound [Rei11], since approximate degree lower bounds quantum query complexity [BBC+01]. The lower bound is a consequence of Theorem 4 because the degree of a read-once formula is equal to the number of variables.

**Lemma 27.** For any read-once formula \( f : \{0, 1\}^n \to \{0, 1\} \), we have \( \deg(f) = n \).

**Proof.** By using De Morgan’s laws, we can assume that the read-once formula only contains AND and NOT gates. The base case of a formula with \( n = 1 \) variable is easy, since the only such formulas as \( x_1 \) and \( \overline{x_1} \), which have degree 1. More generally, \( \deg(f) = \deg(f) \), since if a polynomial \( p(x) \) equals \( f(x) \) for all \( x \in \{0, 1\}^n \), then the polynomial \( 1 - p \) equals \( f \).

All that remains to be shown is that for read-once formulas \( f : \{0, 1\}^n \to \{0, 1\} \) and \( g : \{0, 1\}^m \to \{0, 1\} \), we have \( \deg(f \land g) = \deg(f) + \deg(g) \), where \( f \land g : \{0, 1\}^{n+m} \to \{0, 1\} \) is the function that evaluates to \( f(x) \land g(y) \) for \( x \in \{0, 1\}^n \) and \( y \in \{0, 1\}^m \). The upper bound is obvious since multiplying the polynomials that represent \( f \) and \( g \) gives us a polynomial for \( f \land g \) with degree equal to the sum of their degrees. However, since the polynomial representation of a Boolean function is unique, and there is no way of cancelling out higher degree terms by multiplying these polynomials (since the polynomials involve different sets of variables), we get that \( \deg(f \land g) = \deg(f) + \deg(g) \). Induction on the structure of the formula completes the proof. ■

## 7 Open questions

We saw that \( \lambda(f) \) lower-bounds all the complexity measures in Figure 1, and is polynomially related to all of them. We know that \( \deg(f) \leq \lambda(f)^2 \) and \( s(f) \leq \lambda(f)^2 \), and these relationships are optimal, but the optimal relationships between all other complexity measures and \( \lambda(f) \) remain open. For example, perhaps \( \bs(f) = O(\lambda(f)^2) \) or even \( \RC(f) = O(\lambda(f)^2) \)? The best relationship between block sensitivity and sensitivity also remains open.

It may also be possible to relate \( \lambda(f) \) to the rational degree of \( f \), which is the minimum degree of polynomials \( p \) and \( q \) such that for all \( x \in \{0, 1\}^n \), \( q(x) \neq 0 \) and \( f(x) = p(x)/q(x) \). It is unknown if rational degree is polynomially related to the complexity measures in Figure 1, although the question has been open for a long time [NS94].

Another longstanding open problem is to show a quadratic relation between deterministic query complexity and block sensitivity:

**Conjecture 1.** For all Boolean functions \( f : \{0, 1\}^n \to \{0, 1\} \), we have \( D(f) = O(\bs(f)^2) \).
If this conjecture were true, it would optimally resolve several relationships in Table 1, and would imply, for example, $D(f) = O(R(f)^2)$.

After settling the best relation between $D(f)$ and $Q(f)$, the next pressing question is to settle the best relation between $R(f)$ and $Q(f)$. Recently, two independent works [BS20, SSW20] showed a power 3 separation between $R(f)$ and $Q(f)$, while the best known relationship is a power 4 relationship (from this work). We conjecture that the upper bound can be improved.

**Conjecture 2.** For all Boolean functions $f : \{0,1\}^n \rightarrow \{0,1\}$, we have $R(f) = O(Q(f)^3)$.

Finally, for the special case of monotone total Boolean functions $f$, Beals et al. [BBC+01] already showed in 1998 that $D(f) = O(Q(f)^4)$. It would be interesting to know whether this can be improved, perhaps all the way to $D(f) = O(Q(f)^2)$.

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We show that the measure $\lambda(f)$ satisfies various elegant properties. First, it can be defined in multiple ways, one of which was introduced by Koutsoupias back in 1993 [Kou93]. It also has a formulation as a special case of the quantum adversary bound and hence can be expressed as a semidefinite program closely related to that of the quantum adversary bound. Due to this characterization, $\lambda(f)$ can be viewed as both a maximization problem and a minimization problem. These equivalent formulations are described in Appendix A.1.

Second, we show that $\lambda(f)$ satisfies perfect composition: $\lambda(f \circ g) = \lambda(f) \lambda(g)$. Third, we show that $\lambda(f) \leq \sqrt{s_0(f) s_1(f)}$, which was already observed by Laplante, Lee, and Szegedy [LLS06] (though we give a slightly different proof). Finally, we show lower bounds on $\lambda(f)$ and an optimal quadratic separation between $\lambda(f)$ and $s(f)$.

### A.1 Equivalent formulations

**Theorem 28.** For all Boolean functions $f : \{0,1\}^n \to \{0,1\}$, we have

$$\lambda(f) = K(f) = \text{Adv}_1(f) = \text{Adv}_2^\perp(f),$$

where the measures $K(f)$, $\text{Adv}_1(f)$, and $\text{Adv}_2^\perp(f)$ are defined below. Furthermore, $\text{Adv}_1(f)$ itself has several equivalent formulations: $\text{Adv}_1(f) := \text{SA}_1(f) = \text{SWA}_1(f) = \text{MM}_1(f) = \text{GSA}_1(f)$.

We now define all these measures before proving this theorem.

**Koutsoupias complexity $K(f)$**. For a Boolean function $f$, let $A \subseteq f^{-1}(0)$, and let $B \subseteq f^{-1}(1)$. Let $Q$ be the matrix with rows and columns labeled by $A$ and $B$ respectively, with $Q[i,j] = 1$ if the Hamming distance of $i$ and $j$ is 1, and $Q[i,j] = 0$ otherwise. Koutsoupias [Kou93] observed that $\|Q\|^2$ is a lower bound on formula size, for every such choice of $A$ and $B$. We define $K(f)$ to be the maximum value of $\|Q\|$ over choices of $A$ and $B$. Thus $K(f)^2$ is a lower bound on the formula size of $f$.

**Single-bit positive adversary $\text{Adv}_1(f)$**. We define $\text{Adv}_1(f)$ as a version of the adversary bound where we are only allowed to put nonzero weight on input pairs $(x,y)$ where $f(x) \neq f(y)$ and the Hamming distance between $x$ and $y$ is exactly 1. We will define $\text{Adv}_1(f)$ in terms of the spectral adversary version, which we also denote by $\text{SA}_1(f)$. $\text{Adv}_1(f) = \text{SA}_1(f)$ is defined as the maximum of

$$\frac{\|\Gamma\|}{\max_{i \in [n]} \|\Gamma \circ D_i\|}$$

(50)

over matrices $\Gamma$ of a special form. We require $\Gamma$ satisfy the following: (1) its entries are nonnegative reals; (2) its rows and columns are indexed by $\text{Dom}(f)$; (3) $\Gamma[i,j] = 0$ whenever $f(x) = f(y)$; (4) $\Gamma[i,j] = 0$ whenever the Hamming distance of $x$ and $y$ is not 1; and (5) $\Gamma$ is not all 0. In the above expression, $\circ$ refers to the Hadamard (entrywise) product, $\text{Dom}(f)$ is the domain of $f$, and $D_i$ is the $\{0,1\}$-valued matrix with $D_i[x,y] = 1$ if and only if $x_i \neq y_i$.
Single-bit negative adversary $\text{Adv}_1^+(f)$. We define $\text{Adv}_1^+(f)$ using the same definition as $\text{Adv}_1(f)$ above, except that the matrix $\Gamma$ is allowed to have negative entries. Note that since this is a relaxation of the conditions on $\Gamma$, we clearly have $\text{Adv}_1^+(f) \geq \text{Adv}_1(f)$.

Single-bit strong weighted adversary $\text{SWA}_1(f)$. We define $\text{SWA}_1(f)$ as a single-bit version of the strong weighted adversary method $\text{SWA}(f)$ from [SS06]. For this definition, we say a weight function $w: \text{Dom}(f) \times \text{Dom}(f) \to [0, \infty)$ is feasible if it is symmetric (i.e., $w(x, y) = w(y, x)$) and if it satisfies the conditions on $\Gamma$ above (i.e., it places weight 0 on a pair $(x, y)$ unless both $f(x) \neq f(y)$ and the Hamming distance between $x$ and $y$ is 1). We view such a feasible weight scheme $w$ as the weights on a weighted bipartite graph, where the left vertex set is $f^{-1}(0)$ and the right vertex set is $f^{-1}(1)$. We let $\text{wt}(x) := \sum_y w(x, y)$ denote the weighted degree of $x$ in this graph, i.e., the sum of the weights of its incident edges. Then $\text{SWA}_1(f)$ is defined as the maximum, over such feasible weight schemes $w$, of

$$\min_{x,i: w(x,i) > 0} \frac{\sqrt{\text{wt}(x)\text{wt}(x^i)}}{\text{wt}(x, x^i)}.$$  \hspace{1cm} (51)

Here $x$ ranges over $\text{Dom}(f)$, $i$ ranges over $[n]$, and $x^i$ denotes the string $x$ with bit $i$ flipped.\footnote{Readers familiar with the adversary bound should note that this definition is analogous a weighted version of Ambainis’s original adversary method; in the original method, the denominator was the geometric mean of (a) the weight of the neighbors of $x$ with disagree with $x$ at $i$, and (b) the weight of the neighbors of $x'$ which disagree with $x'$ at $i$; but in our case, both (a) and (b) are simply $w(x, x')$, since $x'$ is the only string that disagrees with $x$ on bit $i$ and is connected to $x$ in the bipartite graph.}

Single-bit minimax adversary $\text{MM}_1(f)$. Unlike the other forms, we define $\text{MM}_1(f)$ as a minimization problem rather than a maximization problem. We say a weight function $w: \text{Dom}(f) \times [n] \to [0, \infty)$ is feasible if for all $x, y \in \text{Dom}(f)$ with $f(x) \neq f(y)$ and Hamming distance 1, we have $w(x, i)w(y, i) \geq 1$, where $i$ is the bit on which $x$ and $y$ disagree. $\text{MM}_1(f)$ is defined as the minimum, over such feasible weight schemes $w$, of

$$\max_{x \in \text{Dom}(f)} \sum_{i \in [n]} w(x, i).$$ \hspace{1cm} (52)

Semidefinite program version $\text{GSA}_1(f)$. We define $\text{GSA}_1(f)$ to be the optimal value of the following semidefinite program.

$$\begin{align*}
\text{maximize} & \quad \langle Z, A_f \rangle \\
\text{subject to} & \quad \Delta \text{ is diagonal} \\
& \quad \text{tr} \Delta = 1 \\
& \quad \Delta - Z \circ D_i \succeq 0 \quad \forall i \in [n] \\
& \quad Z \succeq 0
\end{align*}$$  \hspace{1cm} (53)

Here $Z$ and $\Delta$ are variable matrices with rows and columns indexed by $\text{Dom}(f)$, $A_f$ is the $\{0, 1\}$-matrix with $A_f[x, y] = 1$ if and only if both $f(x) \neq f(y)$ and $(x, y)$ have Hamming distance 1, and $D_i$ is the $\{0, 1\}$-matrix with $D_i[x, i] = 1$ if and only if $x_i \neq y_i$.

We now prove Theorem 28.

Proof. Recall that in the definition of $K(f)$, we picked $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$ and defined the resulting matrix $Q$. Since the spectral norm of a submatrix is always smaller than or equal to the spectral norm of the original matrix, we can always assume without loss of generality that
\[A = f^{-1}(0) \text{ and } B = f^{-1}(1).\] Then \(K(f) = \|Q\|\) for the resulting matrix \(Q\) with rows and columns indexed by \(f^{-1}(1)\) and \(f^{-1}(0)\) respectively. Now, recall that \(A_f\) was the adjacency matrix of the graph \(G_f\), which has an edge between \(x\) and \(y\) if \(f(x) \neq f(y)\) and the Hamming distance between \(x\) and \(y\) is 1. The rows and columns of \(A_f\) are each indexed by \(\text{Dom}(f)\). By rearranging them, we can make \(A_f\) be block diagonal with blocks equal to \(Q\) and \(Q^\top\). From there it follows that \(\|A_f\| = \|Q\|\), so \(\lambda(f) = K(f)\).

Next, recall that \(\text{Adv}_1(f)\) is defined as the maximum ratio \(\|\Gamma\|/\max_i \|\Gamma \circ D_i\|\) over valid choices of \(\Gamma\). Note that since \(\Gamma[x, y]\) can only be nonzero if \(x\) and \(y\) disagree on one bit, \(\Gamma \circ D_i\) is nonzero only on pairs \((x, y)\) which disagree exactly on bit \(i\). In other words, if \(P_i\) denotes the \(\{0, 1\}\)-valued matrix with \(P_i[x, y] = 1\) if and only if \(x\) and \(y\) disagree on bit \(i\) and only on \(i\), then \(\Gamma \circ D_i\) is nonzero only in entries where \(P_i\) is 1. Now, note that \(P_i\) is a permutation matrix. Hence, by rearranging the rows and columns of \(\Gamma \circ D_i\), we can get it to be diagonal. This means \(\|\Gamma \circ D_i\|\) is the maximum entry of \(\Gamma \circ D_i\), and hence \(\max_i \|\Gamma \circ D_i\|\) is the maximum entry of \(\Gamma\). It follows that \(\text{Adv}_1(f)\) is the maximum of \(\|\Gamma\|\) over feasible matrices \(\Gamma\) with \(\max(\Gamma) \leq 1\), where \(\max(\Gamma) = \max_{ij} [\Gamma]_{ij}\). This argument also holds for \(\text{Adv}_1^\pm(f)\), which is the maximum of \(\|\Gamma\|\) over feasible (possibly negative) matrices \(\Gamma\) with \(\max(\Gamma) \leq 1\).

Next, observe that negative weights never help for maximizing \(\|\Gamma\|\); indeed, if we had \(\Gamma\) with negative entries maximizing \(\|\Gamma\|\), then we would have vectors \(u\) and \(v\) with \(\|u\|_2 = \|v\|_2 = 1\) and \(u^\top \Gamma v = \|\Gamma\|\); but then replacing \(u\) and \(v\) with their entry-wise absolute values, and replacing \(\Gamma\) with its entry-wise absolute value \(\Gamma'\), we clearly get that \(\|\Gamma'\| \geq \|\Gamma\|\). However, \(\max(\Gamma') = \max(\Gamma)\), so \(\Gamma'\) remains feasible. This means we can always take the maximizing matrix \(\Gamma\) to be nonnegative, so \(\text{Adv}_1^\pm(f) = \text{Adv}_1(f)\). We can similarly assume that the unit vectors \(u\) and \(v\) maximizing \(u^\top \Gamma v\) are nonnegative.

Finally, consider the maximizing matrix \(\Gamma\) and the maximizing unit vectors \(u\) and \(v\), all nonnegative, and satisfying \(\max(\Gamma) \leq 1\). Note that the expression \(u^\top \Gamma v\) is nondecreasing in the entries of \(\Gamma\), since everything is nonnegative. Hence to maximize \(u^\top \Gamma v\), we can always take every nonzero entry of \(\Gamma\) to be 1, since this maintains \(\max(\Gamma) \leq 1\). In other words, the matrix maximizing \(\|\Gamma\|\) will always simply be \(A_f\), and hence \(\text{Adv}_1(f)\) is always exactly equal to \(\lambda(f)\).

It remains to show that \(\text{SA}_1(f) = \text{SWA}_1(f) = \text{MM}_1(f) = \text{GSA}_1(f)\). The proof of this essentially follows the arguments in [SS06] for the regular positive adversary, though some steps are a little simpler. To start, we’ve seen that \(\text{SA}_1(f) = \lambda(f)\). Since \(A_f\) is symmetric, we have \(\lambda(f) = v^\top A_f v\) for some unit vector \(v\), which we’ve established is nonnegative; this vector is also an eigenvector, so \(A_f v = \lambda(f)v\). Consider the weight scheme \(w(x, y) = v[x]v[y]A_f[x, y]\). Then \(w_t(x) = \sum_y v[x]v[y]A_f[x, y] = v[x](A_f v)[x] = \lambda(f)v[x]^2\). Hence if \(w(x, x^t) > 0\), we have

\[
\frac{\sqrt{w_t(x)w_t(x^t)}}{w(x, x^t)} = \frac{\lambda(f)v[x]v[x^t]}{v[x]v[x^t]A_f[x, x^t]} = \lambda(f).
\]  

This means \(\text{SWA}_1(f) \geq \text{SA}_1(f)\). In the other direction, let \(w\) be a feasible weight scheme for \(\text{SWA}_1(f)\), let \(\Gamma[x, y] = w(x, y)/\sqrt{w_t(x)w_t(y)}\), and let \(v[x] = \sqrt{w_t(x)/W}\), where \(W = \sum_x w_t(x)\). Then \(\|v\|_2^2 = \sum_x w_t(x)/W = 1\), and

\[
v^\top \Gamma v = \sum_{x,y} \sqrt{w_t(x)w_t(y)w(x, y)/W} \sqrt{w_t(x)w_t(y)} = (1/W) \sum_{x,y} w(x, y) = 1.
\]  

Hence \(\|\Gamma\| \geq 1\). On the other hand, we have \(\max(\Gamma) = \max_{x,y} w(x, y)/\sqrt{w_t(x)w_t(y)}\). This means that the ratio \(\|\Gamma\|/\max(\Gamma)\) equals \(\min_{x,y:w(x, y) > 0} \sqrt{w_t(x)w_t(y)}/w(x, y)\), which is \(\text{SWA}_1(f)\); thus \(\text{SA}_1(f) \geq \text{SWA}_1(f)\).
Next we examine $\text{GS}_A^1(f)$. Consider a solution $(Z, \Delta)$ to this semidefinite program and define $\Gamma = Z \circ M \circ A_f$, where $M$ is defined as $M = uu^T$ and $u$ is defined by $u[x] = 1/\sqrt{\Delta[x,x]}$ when $\Delta[x,x] > 0$ and $u[x] = 0$ otherwise. Recall that $\Delta$ is diagonal and that $\Delta - Z \circ D_i \succeq 0$ for all $i$. Since positive semidefinite matrices are symmetric, $Z \circ D_i$ must be symmetric for all $i$, so $Z$ is symmetric. Moreover, the diagonal of $Z \circ D_i$ is all zeros, so we must have $\Delta \succeq 0$. Further, if $\Delta[x,x] = 0$ for some $x$, we must have the corresponding row and column of $Z$ be all zeros. If we let $\Delta'$ and $Z'$ be $\Delta$ and $Z$ with the all-zero rows and columns deleted, then it is clear that $\Delta' - Z' \circ D_i \succeq 0$ if and only if $\Delta - Z \circ D_i \succeq 0$. Defining $M'$ as $M$ with those rows and columns deleted and $u'$ as $u$ with those entries deleted, we have $M' = u'(u')^T > 0$. Observe that $\Delta' - Z' \circ D_i \succeq 0$ if and only if $v^T(\Delta' - Z' \circ D_i)v \geq 0$ for all vectors $v$, which is if and only if $(v \circ u')^T(\Delta' - Z' \circ D_i)(v \circ u') \geq 0$ for all vectors $v$ (since we have $u' > 0$). This, in turn, is equivalent to $M' \circ (\Delta' - Z' \circ D_i) \succeq 0$. Since $M' \circ \Delta' = I$, this is equivalent to $I - M' \circ Z' \circ D_i \succeq 0$, which is in turn equivalent to $I - M \circ Z \circ D_i \succeq 0$. Since $Z \succeq 0$ and we are maximizing $\langle Z, A_f \rangle$, it never helps for $Z$ to have nonzero entries in places where $A_f = 0$. Hence we can assume without loss of generality that $Z = Z \circ A_f$, which means the constraint becomes $I - \Gamma \circ D_i \succeq 0$, where we defined $\Gamma = M \circ Z \circ A_f$. We thus have $\|\Gamma \circ D_i\| \leq 1$.

On the other hand, letting $v[x] = \sqrt{\Delta[x,x]}$, we have

$$v^T \Gamma v = \sum_{x,y} v[x]v[y]M[x,y]Z[x,y]A_f[x,y] = \sum_{x \neq y; \Delta[x,x], \Delta[y,y] > 0} Z[x,y]A_f[x,y] = \langle Z, A_f \rangle.$$  

(56)

Hence $\text{SA}_A^1(f) \succeq \text{GS}_A^1(f)$. The reduction in the other direction works similarly: start with an adversary matrix $\Gamma$ with $\text{max}(\Gamma) \leq 1$, and let $v$ be its principle eigenvector. Then set $Z = \Gamma \circ (v^Tv)$ and $\Delta = I \circ (v^Tv)$. Then $I - \Gamma \circ D_i \succeq 0$, which implies that $\Delta - Z \circ D_i \succeq 0$. We also have $\text{tr} \\Delta = 1$, $Z \succeq 0$, and $\langle Z, A_f \rangle = \|\Gamma\|$.

Finally, we handle $\text{MM}_1^1(f)$. To do so, we first take the dual of the semidefinite program for $\text{GS}_A^1(f)$. This dual has the form

$$\text{minimize} \quad \alpha$$

subject to

$$\sum_i R_i \circ I \leq \alpha I$$

$$\sum_i R_i \circ D_i \geq A_f$$

$$R_i \succeq 0 \quad \forall i \in [n]$$

(57)

where the variables are $\alpha$ (a scalar) and matrices $R_i$, each with rows and columns indexed by $\text{Dom}(f)$. Strong duality follows since when $A_f$ is not all zeros, and the semidefinite program in $\text{GS}_A^1(f)$ has a strictly feasible solution (just take $Z$ to equal $\epsilon A_f$ for a small enough positive constant $\epsilon$, and take $\Delta = I/\|\text{Dom}(f)\|$). This means the optimal solution of the minimization problem above equals $\text{Adv}_1(f)$. It remains to show that this optimal solution $T$ also equals $\text{MM}_1^1(f)$.

Let $\alpha$ and $\{R_i\}_i$ be a feasible solution to the semidefinite program above. Since $R_i \succeq 0$, we have $R_i = X_iX_i^T$ for some matrix $X_i$. Define $w(x,i) = R_i[x,x]$. Note that we also have $w(x,i) = \sum_a X_i[x,a]^2$. Then by Cauchy–Schwarz, $w(x,i)w(y,i) \geq (\sum_a X_i[x,a]X_i[y,a])^2 = (X_iX_i^T)[x,y]^2 = R_i[x,y]^2$. If $x$ and $y$ are such that $A_f[x,y] = 1$, then they disagree in only one bit $i$, and hence $D_i[x,y] = 1$ for that $i$ and $D_j[x,y] = 0$ for all $j \neq i$. Since we have $\sum_i R_i \circ D_i \geq A_f$, we conclude that for all such pairs $(x,y)$, we have $w(x,i)w(y,i) \geq R_i[x,y]^2 \geq A_f[x,y]^2 = 1$ on the bit $i$ where $x$ and $y$ differ; hence the weight scheme $w$ is feasible. Furthermore, for any $x$, $\sum_i w(x,i) = \sum_i R_i[x,x] \leq \alpha I[x,x] = \alpha$. Hence $\text{MM}_1^1(f)$ is at most the optimal value of this semidefinite program.

In the other direction, consider a feasible weight scheme $w$, and define $R_i[x,y] = \sqrt{w(x,i)w(y,i)}$. Then $R_i = w(\cdot,i)w(\cdot,i)^T$, where we treat $w(\cdot,i)$ as a vector; hence $R_i \succeq 0$. Moreover, $R_i \succeq 0$, and for a pair $(x,y)$ with $A_f[x,y] = 1$, there is some $i$ which is the unique bit they disagree on, and
hence \( w(x, i)w(y, i) \geq 1 \); but this means that \( R_i[x, y] \geq 1 \), and so \( (R_i \cdot D_i)[x, y] \geq 1 = A_f[x, y] \).

Finally, \( \sum_i R_i[x, x] = \sum_i w(x, i) \), which means that \( \sum_i R_i \circ I \leq MM_1(f) \cdot I \), as desired.

### A.2 Composition theorem

Just like we have perfect composition theorems for degree (i.e., \( \deg(f \circ g) = \deg(f) \deg(g) \)) and deterministic query complexity (i.e., \( D(f \circ g) = D(f) D(g) \)), we can show one for \( \lambda(f) = \Adv_1(f) \).

**Theorem 29.** For all (possibly partial) functions \( f \) and \( g \), we have \( \Adv_1(f \circ g) = \Adv_1(f) \Adv_1(g) \).

**Proof.** For the lower bound direction, note that \( A_f \) is the matrix with \( A_f[x, y] = 1 \) if \( f(x) \neq f(y) \) and the Hamming distance between \( x \) and \( y \) is 1 (with \( A_f[x, y] = 0 \) otherwise). \( A_g \) is defined similarly. We wish to lower bound \( \|A_{f \circ g}\| \). To do so, we first introduce some notation. Let \( n \) be the input size of \( f \) and let \( m \) be the input size of \( g \). For \( nm \)-bit strings \( x \) and \( y \), we write \( x = x^{(1)}x^{(2)} \ldots x^{(n)} \) and \( y = y^{(1)}y^{(2)} \ldots y^{(n)} \), where \( x^{(i)} \) and \( y^{(i)} \) are \( m \)-bit strings. Write \( g(x) \) as shorthand for the string \( g(x^{(1)})g(x^{(2)}) \ldots g(x^{(n)}) \), and similarly for \( g(y) \). For a string \( x \in \{0, 1\}^m \), let \( x^{(i,j)} \) denote the string \( x \) with the bit at position \((i, j)\) flipped; in particular, the Hamming distance between \( x \) and \( x^{(i,j)} \) is 1. We will also use \( s_f(z) \) to denote the set of sensitive bits of the string \( z \) with respect to function \( f \).

Let \( v \) be the principal eigenvector of \( A_f \) and let \( u \) be the principal eigenvector of \( A_g \). We can assume they are nonnegative. Then \( \|v\|_2 = \|u\|_2 = 1 \), \( A_f v = \lambda(f) v \), and \( A_g u = \lambda(g) u \). Let \( v_0 \) denote the component of \( v \) on 0-inputs of \( f \) and let \( v_1 \) be the component for 1-inputs, so that \( v = [v_0, v_1] \). Define \( u_0 \) and \( u_1 \) similarly. Then since \( A_f \) never has a 1 in a position \((x, y)\) where \( f(x) = f(y) \), it decomposes into blocks of the form \([0, B_f; B_f^T, 0] \), and we must have \( A_f v_0 = \lambda(f) v_0 \) and \( A_f v_1 = \lambda(f) v_0 \). Similarly, \( A_g u_0 = \lambda(g) u_1 \) and \( A_g u_1 = \lambda(g) u_0 \). We can assume without loss of generality that \( \|u_0\|_2^2 = \|u_1\|_2^2 = \|v_0\|_2^2 = \|v_1\|_2^2 = 1/2 \), because otherwise, rebalancing the weights of \( v_0 \) and \( v_1 \) could increase \( v^T A_f v \) without increasing \( \|v\| \) (and similarly for \( u_0 \) and \( u_1 \)).

Define the vector \( \alpha \) with one entry for each input to \( f \circ g \) by

\[
\alpha[x] := 2^{n/2}v[g(x)]u[x^{(1)}]u[x^{(2)}] \ldots u[x^{(n)}].
\]

Then

\[
\|\alpha\|^2 = \sum_x \alpha[x]^2 = 2^n \sum_{z \in \Dom(f)} \sum_{y_1 \in g^{-1}(z_1)} \ldots \sum_{y_n \in g^{-1}(z_n)} v[z]^2 u[y_1]^2 \ldots u[y_n]^2
\]

\[
= 2^n \sum_{z \in \Dom(f)} v[z]^2 \|u_{z_1}\|_2^2 \ldots \|u_{z_n}\|_2^2 = 1.
\]

We also have

\[
\alpha^T A_{f \circ g} \alpha = \sum_{x, x'} \alpha[x] \alpha[x'] A_{f \circ g}[x, x']
\]

\[
= \sum_x \sum_{i \in [n], j \in [m]} \alpha[x] \alpha[x^{(i,j)}] A_{f \circ g}[x, x^{(i,j)}]
\]

\[
= \sum_{z \in \Dom(f)} \sum_{y_1 \in g^{-1}(z_1)} \ldots \sum_{y_n \in g^{-1}(z_n)} \sum_{i \in \Dom(z)} \alpha[y_1 \ldots y_n] \alpha[y_1 \ldots y_i^f \ldots y_n]
\]

\[
= \sum_{z \in \Dom(f)} \sum_{y_1 \in g^{-1}(z_1)} \ldots \sum_{y_n \in g^{-1}(z_n)} \sum_{i \in [n]} \sum_{j \in [m]} \alpha[y_1 \ldots y_n] \alpha[y_1 \ldots y_i^f \ldots y_n] A_f[z, z^i] A_g[y_i, y_j^f]
\]

\[
= \sum_{z \in \Dom(f)} \sum_{i \in [n]} v[z]v[z^i] A_f[z, z^i] \gamma(z, i),
\]

29
where

\[
\gamma(z, i) = \sum_{y_1 \in y^{-1}(z_1)} \cdots \sum_{y_n \in y^{-1}(z_n)} \sum_{j \in [m]} 2^m u[y_1] \ldots u[y_n] \cdot u[y_1'] \ldots u[y_n'] A_g[y_1, y_1']
\]

\[
= \sum_{y_1 \in y^{-1}(z_1)} \sum_{j \in [m]} 2 u[y_1] u[y_1'] A_g[y_1, y_1'] \prod_{k \neq i} 2 \|u_{z_k}\|^2
\]

\[
= 2 u_i^T B_g u_i
\]

\[
= \lambda(g).
\]

Hence

\[
\alpha^T A_{f \circ g} \alpha = \lambda(g) \sum_{z \in \text{Dom}(f)} \sum_{i \in [n]} v[z] v[z'] A_f[z, z'] = \lambda(g) v^T A_f v = \lambda(g) \lambda(f).
\] (61)

This shows that \( \lambda(f \circ g) \geq \lambda(f) \lambda(g) \).

For the upper bound direction, we use MM1. Let \( w_f \) be a feasible weight scheme for \( f \), and let \( w_g \) be a feasible weight scheme for \( g \). Define weight scheme \( w \) for \( f \circ g \) by \( w(x, (i, j)) = w_f(g(x), i) \cdot w_g(x^{(i)}, j) \). Then clearly \( w \) is nonnegative and for each \( z \),

\[
\sum_{i, j} w(x, (i, j)) = \sum_i w_f(g(x), i) \sum_j w_g(x^{(i)}, j) \leq wt_f(x) \max_i wt_g(x^{(i)}),
\] (62)

where we use \( wt_f(z) \) to denote \( \sum_i w_f(z, i) \) and similarly for \( wt_g(z) \). Hence the objective value of \( w \) is at most the product of the objective values of \( w_f \) and \( w_g \). Finally, note that

\[
w(x, (i, j)) w(x^{(i)}, (i, j)) = w_f(g(x), i) w_f(g(x)^{i}, i) w_g(x^{(i)}, j) w_g((x^{(i)})^j, j).
\] (63)

If \((i, j)\) is sensitive for \( x \), then \( i \) is sensitive for \( g(x) \) and \( j \) is sensitive for \( x^{(i)} \), and hence we have \( w_f(g(x), i) w_f(g(x)^{i}, i) \geq 1 \) and \( w_g(x^{(i)}, j) w_g((x^{(i)})^j, j) \geq 1 \), which means \( w(x, (i, j)) w(x^{(i)}, (i, j)) \geq 1 \). Therefore, \( w \) is feasible, and we have \( \text{MM1}(f \circ g) \leq \text{MM1}(f) \text{MM1}(g) \), as desired.

### A.3 Upper bounds

We now show a slightly better upper bound on \( \lambda(f) \), that it is upper bounded by the geometric mean of the 0-sensitivity and 1-sensitivity, which can be a better upper bound than \( s(f) \).

We provide two proofs of this. The first uses the \( \lambda(f) \) formulation and uses a linear algebra argument about norms. This proof is due to Laplante, Lee, and Szegedy [LLS06], who observed this about the measure \( K(f) \).

To describe this proof, we briefly need to describe some matrix norms. For a vector \( v \in \mathbb{R}^n \), the \( p \)-norm for a positive integer \( p \) is defined as \( \|v\|_p = \left( \sum_{i \in [n]} |v_i|^p \right)^{1/p} \). We also define \( \|v\|_{\infty} = \max_{i \in [n]} |v_i| \). Note that \( \|v\|_1 \) is simply the sum of the absolute values of all the entries of the vector.

Similarly, for a matrix \( A \in \mathbb{R}^{n \times m} \), we define the induced \( p \)-norm of \( A \) to be

\[
\|A\|_p = \max \{ \|Ax\|_p : \|x\|_p = 1 \}. \quad (64)
\]

The spectral norm \( \|A\| \) is the induced 2-norm \( \|A\|_2 \). The 1-norm \( \|A\|_1 \) is simply the maximum sum of absolute values of entries in any column of the matrix. The \( \infty \)-norm \( \|A\|_\infty \) is the maximum sum of absolute values of entries in any row of the matrix.

Lastly, we need a useful relationship between these norms sometimes called Hölder’s inequality for induced matrix norms (see [GL13, Corollary 2.3.2] for a proof):
Proposition 30. For all matrices $A \in \mathbb{R}^{n \times m}$, we have $\|A\| \leq \sqrt{\|A\|_1} \|A\|_\infty$.

We can now prove the upper bound:

Lemma 31. For all (possibly partial) functions $f$, we have $\lambda(f) \leq \sqrt{s_0(f)} s_1(f)$.

Proof. We know that $\lambda(f) = \|A_f\|$ and $A_f$ is a matrix of the form $\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$ if we rearrange the rows and columns so that all 0-inputs come first and are followed by 1-inputs, since $A_f$ only connects inputs with different $f$-values. Thus we have

$$\lambda(f) = \|A_f\| = \|B\| \leq \sqrt{\|A_f\|_1 \|B\|_\infty} = \sqrt{s_0(f)} s_1(f), \tag{65}$$

where we used Hölder’s inequality (Proposition 30) and the fact that the maximum row and column sum of $B$ are precisely $s_0(f)$ and $s_1(f)$, respectively.

Our second proof of this claim uses the MM$_1(f)$ formulation which yields an arguably simpler proof.

Lemma 32. For all (possibly partial) functions $f$, we have $\text{Adv}_1(f) \leq \sqrt{s_0(f)} s_1(f)$.

Proof. Using the MM$_1(f)$ version of $\text{Adv}_1(f)$, set $w(x, i) = \sqrt{s_0(f)} / \sqrt{s_1(f)}$ if $f(x) = 1$, and set $w(x, i) = \sqrt{s_1(f)} / \sqrt{s_0(f)}$ if $f(x) = 0$. Then if $x$ and $y$ differ in a single bit $i$, we clearly have $w(x, i) w(y, i) = 1$. On the other hand, $\sum_i w(x, i) \leq s_1(f) \cdot \sqrt{s_0(f)} / \sqrt{s_1(f)} = s_0(f) s_1(f)$ for 1-inputs $x$, and analogously $\sum_i w(y, i) \leq s_0(f) s_1(f)$ for 0-inputs $y$.

Using this better bound on $\lambda(f)$ and Huang’s result, we also get that for all total Boolean functions $f$,

$$\deg(f) \leq s_0(f) s_1(f). \tag{66}$$

This result was also recently observed by Laplante, Naserasr, and Sunny [LNS20]. Unlike their proof, the following uses Huang’s theorem in a completely black-box way.

Proposition 33. Assume that $\deg(f) \leq s(f)^2$ for all total Boolean functions $f$. Then we also have $\deg(f) \leq s_0(f) s_1(f)$.

Proof. Let $s_0(f) = k$ and $s_1(f) = \ell$. We know that $\deg(f) \leq \max\{k, \ell\}$ by assumption. Let $\text{AND}_k \circ \text{OR}_\ell$ be the AND function on $k$ bits composed with the OR function on $\ell$ bits. Clearly $s_0(\text{AND}_k \circ \text{OR}_\ell) = \ell$ and $s_1(\text{AND}_k \circ \text{OR}_\ell) = k$. Furthermore, because the function is monotone, the sensitive bits for a 0-input are bits set to 0, and the sensitive bits for a 1-input are bits set to 1. This means that composing this function with $f$ with yield a function where the one-sided sensitivity will be upper bounded by the product of one-sided sensitivity of the individual functions. Hence for all $b \in \{0, 1\}$, we have

$$s_0(\text{AND}_k \circ \text{OR}_\ell \circ f) \leq s_0(\text{AND}_k \circ \text{OR}_\ell) s_0(f) \leq k\ell. \tag{67}$$

Using the assumption on the function $\text{AND}_k \circ \text{OR}_\ell \circ f$, we get

$$\deg(\text{AND}_k \circ \text{OR}_\ell \circ f) \leq (s(\text{AND}_k \circ \text{OR}_\ell \circ f))^2 \leq (k\ell)^2. \tag{68}$$

Finally, it is well known that $\deg(f \circ g) = \deg(f) \deg(g)$ (see, e.g., [Tal13]), and hence $\deg(\text{AND}_k \circ \text{OR}_\ell \circ f) = k\ell \deg(f)$, which implies $\deg(f) \leq k\ell$. \qed
A.4 Lower bounds

Finally, we describe some lower bounds on $\lambda(f)$. These follow from known results, but we reproduce them here for completeness.

**Lemma 34.** For all (possibly partial) functions $f$, $s(f) \leq \lambda(f)^2$.

**Proof.** Consider any input $x$ with sensitivity $s(f)$. This means $x$ has $s(f)$ neighbors on the hypercube with different $f$ value. The sensitivity graph restricted to these $s(f) + 1$ inputs is a star graph centered at $x$. The spectral norm of the adjacency matrix of the star graph on $k + 1$ vertices is $\sqrt{k}$. Since the spectral norm of $A_f$ is lower bounded by that of a submatrix, we have $\lambda(f) \geq \sqrt{s(f)}$.

This relationship is tight for the $\text{OR}_n$ function which has $s(\text{OR}_n) = n$ and $\lambda(\text{OR}_n) = \sqrt{n}$. Although $\text{OR}_n$ has unbalanced sensitivities, with $s_0(\text{OR}_n) = n$ and $s_1(\text{OR}_n) = 1$, there are functions $f$ with $s(f) = s_0(f) = s_1(f) = n$ and $\lambda(f) = \sqrt{n}$. One example of such a function is $x_1 \oplus \text{OR}(x_2, \ldots, x_n)$. Another example of such a function with a quadratic gap between $s(f)$ and $\lambda(f)$ is the function that is 1 if and only if the input string has Hamming weight 1. This function has $s_0(f) = n$ since the all zeros string is fully sensitive and $s_1(f) = n$ since every Hamming weight 1 string is also fully sensitive. But we know that this problem can be solved by Grover’s algorithm with $O(\sqrt{n})$ queries, and hence $\lambda(f) = O(Q(f)) = O(\sqrt{n})$.

We can also lower bound $\|A_f\|$ by $\|A_f\| \geq |v^T A_f v|$ for any vector $v$ with $\|v\| = 1$. If we take $v$ to be the normalized all ones vector, this is just the average sensitivity.

**Lemma 35.** For all (possibly partial) functions $f$, $\lambda(f) \geq E_x[s_x(f)]$.

For example, this shows that $\lambda(\text{parity}_n) = n$. The bound in Lemma 35 can be improved by only taking the expectation on the right over a subset of the inputs of $f$, which then equals another complexity measure originally defined by Khrapchenko [Khr71]. See [Kou93] for more on this relationship and Khrapchenko’s bound.