ON STRONG SOLUTIONS OF ITÔ’S EQUATIONS WITH $D\sigma$ AND $b$ IN MORREY CLASSES CONTAINING $L_d$

N.V. KRYLOV

Abstract. We consider Itô uniformly nondegenerate equations with time independent coefficients, the diffusion coefficient in $W^{2+\varepsilon,\text{loc}}$ and the drift in a Morrey class containing $L_d$. We prove the unique strong solvability in the class of admissible solutions for any starting point. The result is new even if the diffusion is constant.

1. Introduction

Let $\mathbb{R}^d$ be a $d$–dimensional Euclidean space of points $x = (x^1, \ldots, x^d)$ with $d \geq 3$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, let $\{\mathcal{F}_t\}$ be an increasing filtration of $\sigma$-fields $\mathcal{F}_t \subset \mathcal{F}$, that are complete. Let $w_t$ be a $d_1$-dimensional Wiener process relative to $\{\mathcal{F}_t\}$, where $d_1 \geq d$.

Assume that on $\mathbb{R}^d$ we are given $\mathbb{R}^{d_1}$-valued Borel functions $b, \sigma^k = (\sigma^{ik})$, $k = 1, \ldots, d_1$. We are going to investigate the equation

$$x_t = x + \int_0^t \sigma^k(x_s) \, dw_s^k + \int_0^t b(x_s) \, ds,$$

where and everywhere below the summation over repeated indices is understood.

We are interested in the so-called strong solutions, that is solutions such that, for each $t \geq 0$, $x_t$ is $\mathcal{F}^w_t$-measurable, where $\mathcal{F}^w_t$ is the completion of $\sigma(w_s : s \leq t)$. We present sufficient conditions for the equation to have a strong solution and also for the solution to be unique (strong uniqueness) generalizing those in article [8] which should be read first in order to be able to follow what we are doing here. For this reason we restrict ourselves to a quite short introduction without repeating the one from [8].

A very rich literature on the weak uniqueness problem for (1.1) is beyond the scope of this article. Concerning the strong solutions in the time dependent case probably the best results belong to Röckner and Zhao [10], where, among very many other things, they prove existence and uniqueness of strong solutions of equations like (1.1) with $b = b(t, x)$ and constant $\sigma^k$.

We refer the reader to [10] also for a very good review of the history of the problem.

2020 Mathematics Subject Classification. 60H10, 60J60.

Key words and phrases. Strong solutions, vanishing mean oscillation, singular coefficients, Morrey coefficients.
The coefficient \( b(t, x) \) in [10] belong to the space \( L_q([0, \infty), L_p(\mathbb{R}^d)) \) with
\[
p, q \in [2, \infty], \quad \frac{d}{p} + \frac{2}{q} = 1.
\]
In case \( q = \infty \) they additionally assume that \( b(t, \cdot) \) is a continuous \( L_d \)-valued function. In our case \( b \) is independent of \( t \) and still our results are not covered by [10] not only because we have variable \( \sigma^k \)'s but also because our \( b \) is generally not in \( L_d \).

Here is an example in which we prove existence of strong solutions. Take \( d = 3, d_1 = 12, \) and for some numbers \( \alpha, \beta, \gamma \geq 0 \) let \( \sigma^k \) be the \( k \)th column of the matrix given by \((0/0 := 3^{-1/2})\)
\[
\begin{pmatrix}
\alpha & 0 & 0 \\
0 & \alpha & 0 \\
0 & 0 & \alpha
\end{pmatrix}
+ \frac{\beta}{|x|}
\begin{pmatrix}
x^1 & x^2 & x^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x^1 & x^2 & x^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x^1 & x^2 & x^3
\end{pmatrix},
\]
\[
b(x) = -\frac{\gamma}{|x|} \frac{x}{|x|} I_{0 < |x| \leq 1} + \hat{b}(x),
\]
where \( \hat{b} \) is, for instance, bounded. Our result shows that if \( \alpha = 1 \) and \( \beta \) and \( \gamma \) are sufficiently small, then (1.1) has a strong solution. By the way, if \( \alpha = \gamma = 0 \) and \( \beta = 1 \), there exist strong solutions of (1.1) only if the starting point \( x \neq 0 \) (see [8]). In case \( \alpha = 1 \) and \( \beta = 0 \) strong solutions exist only if \( \gamma \) is sufficiently small. Observe that for \( \beta \neq 0 \) and \( \gamma \neq 0 \) we have \( D\sigma^k, b \in L_{d-\varepsilon, \text{loc}} \) for any \( \varepsilon \in (0, 1) \) but not for \( \varepsilon = 0 \). Recall that the case that \( D\sigma^k, b \in L_{d, \text{loc}} \) is investigated in [8], the results and methods of which are referred to very many times in this article.

Above we were talking only about the existence of strong solutions. The issue of their uniqueness is more delicate and we were able to prove uniqueness only of “admissible” strong solutions (which are shown to exist).

We conclude the introduction by some notation. We set \( u_x = Du \) to be the gradient of \( u \), \( u_{xx} \) to be the matrix of its second-order derivatives,
\[
D_{x^i} u = D_i u = u_{x^i} = \frac{\partial}{\partial x^i} u, \quad u_{x^i \eta^j} = D_{x^i \eta^j} u = D_i D_{\eta^j} u,
\]
\[
\partial_t u = \frac{\partial}{\partial t} u, \quad u_{(\eta)} = \eta^i u_{x^i}.
\]
If \( \sigma(x) = (\sigma^i(x)) \) is vector-valued (column-vector), by \( D\sigma = \sigma_x \) we mean the matrix whose \( ij \)th element is \( \sigma^i_{x^j} \). If \( c \) is a matrix (in particular, vector), we set \( |c|^2 = \text{tr} cc^* \) (= \( \text{tr} c^* c \) if \( c \) is complex-valued).

For \( p \in [1, \infty) \) by \( L_p \) \((L_p(\Gamma))\) we mean the space of Borel (perhaps complex- vector- or matrix-valued) functions on \( \mathbb{R}^d \) (on \( \Gamma \subset \mathbb{R}^d \)) with finite norm given by
\[
\|f\|_{L_p}^p = \int_{\mathbb{R}^d} |f(x)|^p \, dx \quad (\|f\|_{L_p(\Gamma)}^p = \int_{\Gamma} |f(x)|^p \, dx).
\]
By $W^2_p$ we mean the space of Borel functions $u$ on $\mathbb{R}^d$ whose Sobolev derivatives $u_x$ and $u_{xx}$ exist and $u, u_x, u_{xx} \in L_p$. The norm in $W^2_p$ is given by

$$\|u\|_{W^2_p} = \|u_{xx}\|_{L_p} + \|u\|_{L_p}.$$  

Similarly $W^1_p$ is defined. As usual, we write $f \in L^p_{\text{loc}}$ if $f \zeta \in L^p$ for any $\zeta \in C_0^\infty(= C_0^\infty(\mathbb{R}^d))$.

If a Borel $\Gamma \subset \mathbb{R}^d$, by $|\Gamma|$ we mean its Lebesgue measure,

$$\int \Gamma f(x) \, dx = \frac{1}{|\Gamma|} \int \Gamma f(x) \, dx.$$  

Introduce

$$B_R(x) = \{ y \in \mathbb{R}^d : |x - y| < R \}, \quad B_R = B_R(0)$$

and let $B_R$ be the collection of balls of radius $R$.

In the proofs of our results we use various (finite) constants called $N$ which may change from one occurrence to another and depend on the data only in the same way as it is indicated in the statements of the results.

2. Main results

Fix numbers $\delta \in (0, 1)$ and $R_0$, $\|b\|, \|D\sigma\| \in (0, \infty)$. Below $q_0 \in (2, d]$, $d \geq q > d_0 \vee (d/2 + 1)$ (hence $d \geq 3$), where $d_0 = d_0(d, \delta) \in (d/2, d)$ is taken from [7]. Set $a^{ij} = \sigma^{ik} \sigma^{jk}$, $a = (a^{ij})$.

**Definition 2.1.** By an admissible solution of (1.1) we mean any solution $x_t$ such that there exists $p \in (d/2 + 1, q)$ and for each $T \in (0, \infty)$ there exists a constant $N_T$ such that for any nonnegative Borel $f(t, x) \geq 0$

$$E \int_0^T f(t, x_t) \, dt \leq N_T \|f\|_{L^p((0, T) \times \mathbb{R}^d)}.$$  

**Assumption 2.2.** For any $x$ the eigenvalues of $a(x)$ lie between $\delta$ and $\delta^{-1}$, $b \in L_q$, and for any ball $B$ of radius $\rho \leq R_0$

$$\left( \int_B |b|^q \, dx \right)^{1/q} \leq \|b\| \rho^{-1}. \quad (2.1)$$

**Assumption 2.3.** For any ball $B$ of radius $\rho \leq R_0$

$$\left( \int_B |D\sigma|^{q_0} \, dx \right)^{1/q_0} \leq \|D\sigma\| \rho^{-1}, \quad |D\sigma|^2 = \sum_{i,j,k} |\sigma^{ik}_x|.$$  

**Remark 2.4.** The case $q = q_0 = d$ considered in [8] is not excluded and in this case one can take $\|b\|$ as small as one likes on the account of taking $R_0$ sufficiently small. This follows from Hölder’s inequality and the fact that

$$\left( \int_B |b|^d \, dx \right)^{1/d} = N(d) \left( \int_B |b|^d \, dx \right)^{1/d} \rho^{-1}.$$  

The same goes about $\|D\sigma\|$, since $D\sigma^k \in L_d$ in [8]. Adding to this that under the conditions in [8] all solutions are admissible (see Section 4 in [8])
we see that the main result of [8] about the existence and uniqueness of strong solutions follows from the results of the present article.

It is also worth noting that, generally, condition (2.2) with \( d > q \geq d - 1 \) does not imply that \( b \in L^q_{\delta,\text{loc}} \), no matter how small \( \varepsilon > 0 \) is. Here is an example. Take \( r_n > 0, n = 1, 2, \ldots \), such that the sum of \( \rho_n := r_n^{d-q} \) is 1/2, let \( e_1 \) be the first basis vector, and set \( b(x) = |x|^{-1}I_{|x|<1}, x_0 = 1 \),

\[
x_n = 1 - 2 \sum_{i=1}^{n} r_i^{d-q}, \quad c_n = (1/2)(x_n + x_{n-1})
\]

\[
b_n(x) = r_n^{-1}b(r_n^{-1}(x - c_ne_1)), \quad b = \sum_{i=1}^{\infty} b_n.
\]

Since \( r_n \leq 1 \) and \( d - q \leq 1 \), the supports of \( b_n \)'s are disjoint and for \( p > 0 \)

\[
\int_{B_1} b^p dx = \sum_{i=1}^{\infty} \int_{B_i} b_n^p dx = N(d, p) \sum_{i=1}^{\infty} r_i^{d-p}.
\]

According to this we take the \( r_n \)'s so that the last sum diverges for any \( p > q \). Then observe that for any \( n \geq 1 \) and any ball \( B \) of radius \( \rho \)

\[
\int_{B} b_n^q dx \leq N(d) \rho^{d-q}.
\]

Also, if the intersection of \( B \) with \( \bigcup B_r(c_n) \) is nonempty, the intersection consists of some \( B_{r_i}(c_i), i = i_0, \ldots, i_1 \), and \( B \cap B_{r_{i_0-1}}(c_{i_0-1}) \) if \( i_0 \neq 0 \) and \( B \cap B_{r_{i_1+1}}(c_{i_1+1}) \). In this situation

\[
\sum_{i=i_0}^{i_1} \rho_i \leq 2 \rho,
\]

and therefore,

\[
\int_{B} b^q dx = N(d) \sum_{i=i_0}^{i_1} r_i^{d-q} + \int_{B} b_{i_0-1}^q dx + \int_{B} b_{i_1+1}^q dx \leq N(d)(\rho + \rho^{d-q}),
\]

where the last term is less than \( N(d)\rho^{d-q} \) for \( \rho \leq 1 \) and this yields just a different form of (2.2).

**Theorem 2.5.** Under the above assumptions there is a constant \( N_0 \geq 1 \), depending only on \( d, \delta, q, q_0 \), such that if

\[
N_0(\|D\sigma\| + \|b\|) \leq 1,
\]

then (1.1) has a unique admissible strong solution, and each admissible solution of (1.1) is strong (thus coinciding with the unique one).

**Remark 2.6.** The solution of (1.1) depends on the starting point \( x \): \( x_t = x_t(x) \). In the author’s opinion, it is unlikely that \( x_t(x) \) is Hölder continuous in \( x \) with exponent as close to 1 as we wish unless \( q_0 = q = d \).
How small one can take $q$ is also an interesting question. Most likely, as $\delta \downarrow 0$, $d_0 \uparrow d$ and this also forces $q \uparrow d$. But if $d = d_1$ and we are dealing with the equation

$$x_t = x + w_t + \int_0^t b(x_t)\,dt$$

our conjecture is that one can allow $q > d/2$ as close to $d/2$ as one likes and still have Theorem 2.5 valid.

3. An analytic semigroup

In this section Assumption 2.2 is supposed to be satisfied but with less restrictions on $q$: namely $q \in (1, d]$. Assumption 2.3 is replaced with a weaker Assumption 3.1 which comes after some preparations.

Introduce the uniformly elliptic operators

$$Lu(x) = (1/2)a^{ij}(x)u_{x_ix_j}(x) + b^i(x)u_{x_i}(x) \quad (a = \sigma\sigma^*),$$

$$L_0u(x) = (1/2)a^{ij}(x)u_{x_ix_j}(x)$$

acting on functions given on $\mathbb{R}^d$.

For balls $B$ denote

$$\text{osc} \left(a, B\right) = \frac{1}{|B|^2} \int_{y,z \in B} |a(y) - a(z)| \, dy \, dz,$$

$$a_r^\# = \sup_{B \in B^r} \text{osc} \left(a, B\right).$$

In the rest of the section we impose the following.

**Assumption 3.1.** We have $p \in (1, \infty)$ and $a_{R_0}^\# \leq \theta_0 = \theta_0(d, \delta, p)$, where $\theta_0 > 0$ is taken in a way to accommodate Lemmas 3.3 and 3.4 of [8].

**Remark 3.2.** By Poincaré’s inequality, for $B \in B_\rho$

$$\text{osc} \left(a, B\right) \leq N(d)\rho \int_B |Da| \, dy \leq N(d, \delta)\rho \int_B |D\sigma| \, dy,$$

so that requiring in the future $\|D\sigma\|$ to be sufficiently small will make $a$ satisfy Assumption 3.1. In this sense Assumption 3.1 is weaker than Assumption 2.3.

As a consequence of Assumption 3.1, derived in [8], we have the following. Introduce

$$\Gamma = \{\Re \lambda \geq \lambda_0\} \cup \{\varepsilon_0|3\lambda| \geq -\Re \lambda + \mu_0\},$$

with $\varepsilon_0 > 0$ and $\mu_0 > \lambda_0$ which depend only on $d, \delta, p, R_0$ taken from Lemma 3.4 of [8]. Then this lemma is the following.

**Lemma 3.3.** There exist $\mu_0 > \lambda_0 \geq 1, N_0$, depending only on $d, \delta, p, R_0$, such that, for any $u \in W_2^p$ and $\lambda \in \Gamma$ we have

$$\|u_{xx}\|_{L_p} + |\lambda|\|u\|_{L_p} \leq N_0\|L_0u - \lambda u\|_{L_p}.$$ (3.1)
To replace $L_0$ with $L$ we need to know how to estimate the terms $b^iD_iu$.

Here is the Theorem from [2] in which $d \geq 2$.

**Theorem 3.4.** Let $1 < p < q \leq d$, $v$ be a function on $\mathbb{R}^d$ such that for any $\rho > 0$ and $B \in \mathcal{B}_\rho$

$$\left( \int_B |v|^q \, dx \right)^{1/q} \leq \kappa \rho^{-1},$$

where the constant $\kappa \in (0, \infty)$. Then there exists a constant $N$, depending only on $d, p, q$, such that

$$\int_{\mathbb{R}^d} |v(x)|^p |u(x)|^p \, dx \leq N \kappa^p \int_{\mathbb{R}^d} |u(x)|^p \, dx$$

as long as $u \in C_0^\infty$.

This theorem allows us to do the first step, which generalizes and implies Theorem 3.4 as $R_0 \to \infty$.

**Lemma 3.5.** For $1 < p < q \leq d$ there exists a constant $N = N(p, q, d)$ such that

$$\int_{\mathbb{R}^d} |b(x)|^p |u(x)|^p \, dx \leq N \|b\|^p \int_{\mathbb{R}^d} |u(x)|^p \, dx + NR_0^{-p} \|b\|^p \int_{\mathbb{R}^d} |u|^p \, dx$$

(3.2)

as long as $u \in C_0^\infty$.

Proof. Take $\zeta \in C_0^\infty(B_{R_0})$, $\zeta \geq 0$, such that

$$\int_{B_{R_0}} \zeta^{2p} \, dx = 1, \quad \zeta + R_0 |D\zeta| \leq N(d)R_0^{-d/(2p)}.$$  

(3.3)

We claim that for any $\rho > 0$ and $B \in \mathcal{B}_\rho$ we have

$$\left( \int_B |b\zeta|^q \, dx \right)^{1/q} \leq NR_0^{-d/(2p)} \|b\| \rho^{-1}. \quad (3.4)$$

Indeed, if $\rho \leq R_0$ it suffices to use that $\zeta \leq NR_0^{-d/(2p)}$. In case $\rho > R_0$, it suffices to use that

$$\int_{B_{R_0}} |b\zeta|^q \, dx = NR_0^{-d} \int_{B_{R_0}} |b\zeta|^q \, dx \leq NR_0^{-d/(2p)} \rho^{-d} \int_{B_{R_0}} |b|^q \, dx.$$

$$= NR_0^{-d/(2p)} R_0^d \rho^{-d} \int_{B_{R_0}} |b|^q \, dx \leq NR_0^{-d/(2p)} R_0^d \rho^{-d} \|b\|^q R_0^{-q}.$$

$$= NR_0^{-d/(2p)} R_0^{-d} \rho^{-d} \|b\|^q R_0^{-q} \leq NR_0^{-d/(2p)} \|b\|^q \rho^{-q}.$$  

Now, in light of (3.4) by Theorem 3.4

$$\int_{\mathbb{R}^d} |b\zeta|^p |u|^p \, dx \leq NR_0^{-d/2} \|b\|^p \int_{\mathbb{R}^d} |Du|^p \, dx, \quad u \in C_0^\infty.$$  

We plug in here $\zeta(x+y)$ and $\zeta(x+y)u$ in place of $\zeta$ and $u$, respectively. Then we get

$$\int_{\mathbb{R}^d} \zeta^{2p}(x+y)|b|^p |u|^p \, dx \leq NR_0^{-d/2} \|b\|^p \int_{\mathbb{R}^d} \zeta^p(x+y)|Du|^p \, dx$$

(3.5)
After integrating through with respect to $y$ and using that by Hölder’s inequality and (3.3)
\[ \int_{\mathbb{R}^d} |\zeta|^p \, dy \leq N R_0^{d/2}, \quad \int_{\mathbb{R}^d} |D\zeta|^p \, dy \leq N R_0^{d/2-p}, \]
we come to (3.2). This proves the lemma.

On the basis of Lemmas 3.3 and 3.5 we can repeat what was done in [8] and obtain the first part of the following result about the full operator $L$.

**Theorem 3.6.** Let $p \in (1,q)$. Then under Assumptions 2.2 and 3.1 there exists $N \geq 1$ depending only on $d, \delta, p,$ and $q,$ such that, if $N \parallel b \parallel \leq 1,$
\[ (3.5) \]
then for any $u \in W^2_p$ and $\lambda \in \Gamma$ ($\Gamma$ is introduced before Lemma 3.3),
\[ ||u_{xx}||_{L^p} + ||\lambda||_{L^p} \leq N ||Lu - \lambda u||_{L^p} \] (3.6)
with $N$ depending only on $d, \delta, p, q, R_0$. Furthermore, for any $\lambda \in \Gamma$ and $f \in L^p$ there is a unique $u \in W^2_p$ such that $\lambda u - Lu = f$.

The “existence” part of this theorem, as usual, is proved by the method of continuity.

**Remark 3.7.** The use of (3.5) has very much in common with the “form bounded” condition from [3]:
\[ \int_{\mathbb{R}^d} |b\phi|^2 \, dx \leq \delta \int_{\mathbb{R}^d} |D\phi|^2 \, dx + c_\delta \int_{\mathbb{R}^d} |\phi|^2 \, dx \quad \forall \phi \in \mathcal{C}_0^\infty. \] (3.7)
If you take here $\phi = \phi(x/\rho)$, where $\phi \in C_0^\infty$ and $\phi = 1$ on $B_1$ and $\phi = 0$ outside $B_2$, then you get
\[ \int_{B_\rho} |b|^2 \, dx \leq N(d)\delta \rho^{-2} + N(d)c_\delta. \]
This means that (2.2) is satisfied with $q = 2$. Condition (3.7) is used in [3], among very many other things, to construct weak solutions of (1.1) with constant $\sigma^k$’s. We need a stronger condition (2.2) with $q > d/2 + 1$ to prove the existence of strong solutions.

Below in this section we assume that (3.5) holds and denote by $R_\lambda f$ the solution $u$ from Theorem 3.6. Next, exactly as in [8] for complex $t$ in the sector $S := \{|\Re t| < \varepsilon_0 \Re t\}$ with $\varepsilon_0$ introduced before Lemma 3.3 set
\[ T_t = \frac{1}{2\pi i} \int_{\partial S} e^{iz} R_z \, dz, \] (3.8)
where the integral is taken in a counter clockwise direction. Below in this section $p \in (1,q)$.

Here is Theorem 3.8 of [8].
Lemma 3.9. For any ball $b$ approximation of $R$ \(\zeta\) of [8]. Let
\[ N \]
where
\[ Mf \]
holds. We assert the following.
\[ \int \]
where $N$
\[ \epsilon \]
integral equal to one. Define
\[ \zeta \]
where
\[ L \]
erators in $r > 0$ and $a$ constant, depending only on $d, q$
in the proof of Theorem 1 of [1] the left-hand side of (3.11) is dominated by
\[ \sup \]
\[ \parallel \]
\[ \parallel \]
As in Remark 3.9 of [8] the properties of $T_t$ listed in Theorem 3.8 allow us to assert that, if $p > d/2$ and $f \in W^p_2$, then $T_tf$ has a modification that is bounded and continuous in $x$, which we still call $T_tf$. Also $T_tf \to T_s f$ as $t \to s$ in $W^p_2$, and $T_tf(x) \to T_s f(x)$ uniformly on $\mathbb{R}^d$. Therefore, $T_tf(x)$ is a bounded continuous function on $[0, T] \times \mathbb{R}^d$ for any $T \in (0, \infty)$.
Moreover, for $0 < t \leq T$, $f \in L_p$, $q > p > d/2$, and any $x \in \mathbb{R}^d$
\[ |T_tf(x)| \leq \frac{N}{t^{d/(2p)}} \parallel f \parallel L_p, \quad (3.10) \]
where $N$ depends only on $T, d, \delta, p, q, R_0$.
We also need an approximation result which, however, requires special approximation of $b$ and in this respect is more restrictive than Theorem 3.10 of [8]. Let $\zeta \in C^\infty_0(B_1)$ be nonnegative spherically symmetric with integral equal to one. Define $\zeta_n(x) = n^d \zeta(nx)$ and $b_n = b * \zeta_n$.

Lemma 3.9. For any ball $B$ of radius $\rho \leq R_0$ we have
\[ \left( \int_B \sup_{n \geq 1/R_0} |b_n|^q \, dx \right)^{1/q} \leq N(d, q) \parallel b \parallel \rho^{-1}, \quad (3.11) \]
where $N(d, q) \geq 1$.

Proof. If $B = B_\rho(x_0)$, then on $B$
\[ \sup_{n \geq 1/R_0} |b_n| \leq N(d) M(|b| I_{B_2 R_0(x_0)}), \]
where $Mf$ is the Hardy-Littlewood maximal function of $f$. By what is shown in the proof of Theorem 1 of [1] the left-hand side of (3.11) is dominated by a constant, depending only on $d, q$, times $\rho^{-1}$ and times the supremum over $r > 0$ and $B \in \mathcal{B}_r$ of
\[ I := r \left( \int_B |b|^q I_{B_2 R_0(x_0)} \, dx \right)^{1/q}. \quad (3.12) \]
If $r \leq R_0$ then $I$ is dominated by $\|b\|$. For $r \geq R_0$ we have

$$I \leq N(d)r^{1-d/q} \left( \int_{B_{2R_0}(x_0)} |b|^q I_{B_{2R_0}(x_0)} \, dx \right)^{1/q}$$

$$\leq N(d)r^{1-d/q} \sup_{x \in \mathbb{R}^d} \left( \int_{B_{R_0}(x)} |b|^q \, dx \right)^{1/q} \leq N(d)r^{1-d/q} R_0^{d/q-1} \|b\| \leq N(d)\|b\|.$$ 

This proves the lemma.

**Corollary 3.10.** If $u, u_x \in L_p$, then

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} |b_n - b|^p |u|^p \, dx = 0.$$ 

Indeed, $|b_n - b|^p |u|^p \to 0 \text{ (a.e.)}$ and

$$\sup_{n \geq 1/R_0} |b_n - b|^p |u|^p \leq N \sup_{n \geq 1/R_0} |b_n|^p |u|^p,$$

where the last expression is integrable in light of Lemmas 3.9 and 3.5.

**Theorem 3.11.** Let $q > p > d/2$ and let $a_n, n = 1, 2, \ldots$, have the same meaning as $a$. Suppose that, for each $n$, they satisfy Assumptions 2.2 (with the same $\delta$) and 3.1 (with the same $\theta_0$). Suppose that (3.5) is satisfied with $N(d, q)\|b\|$ in place of $\|b\|$, where $N(d, q)$ is taken from (3.11). Assume that $a_n \to a \text{ (a.e.)}$ and take $b_n$ introduced before Lemma 3.9. Denote by $T_{n,t}$ the semigroups constructed on the basis of (3.8) when $R_z$ is replaced with $R_{n,z}$ that is the inverse operator to $z - L_n$, where $L_n = (1/2)a_n^{ij} D_{ij} + b_n^i D_i$. Then for any $t > 0$ and $f \in L_p$ we have $T_{n,t}f \to T_t f$ in $W^2_p$ and, in particular, uniformly on $\mathbb{R}^d$ as $n \to \infty$.

The proof of this theorem is identical to the proof of Theorem 3.10 of [8] up to the point where we show that

$$\lim_{n \to \infty} \|b^n - b\| (R_z f)_x \|_{L_p} = 0,$$

which this time follows from Corollary 3.10.

**Remark 3.12.** If $a$ and $b$ are smooth and $b$ is bounded, then for any $f \in C_0^\infty$ there is a classical solution $u(t, x)$ of the problem $\partial_t u = Lu, u(0, \cdot) = f$. This solution and its derivatives decrease exponentially fast as $|x| \to \infty$ and have all other qualities listed in Theorem 3.8 (iii). Therefore, $u(t, x) = T_t f(x)$. This shows, in particular, that $T_t$ is independent of $p$. Owing to the maximum principle valid for $u$, we have

$$0 \leq T_t f \leq \sup f$$

if an addition $f \geq 0$. In light of (3.10), this also holds for any $f \in L_p$. The semigroup property of $T_t$ and (3.10) imply further that for $t \geq 1$

$$|T_t f| \leq T_t |f| = T_{t-1} T_1 |f| \leq \sup T_1 |f| \leq N \|f\|_{L_p}.$$
Thus, for $t > 0$, $f \in L_p$, $q > p > d/2$, and any $x \in \mathbb{R}^d$
\[ |T_t f(x)| \leq \frac{N}{(t \wedge 1)^{d/(2p)}} \|f\|_{L_p}, \]  
(3.13)
where $N$ depends only on $d$, $\delta, p, q, R_0$.

These conclusions we obtained if the coefficients are smooth and bounded. By using the approximation Theorem 3.11 and mollifying $a$ we get the same conclusions in the general case provided that (3.5) is satisfied with $N(d, q, R_0)\|b\|$ in place of $\|b\|$, where $N(d, q, R_0)$ is taken from (3.11).

4. Stochastic equations with smooth coefficients

Here, in addition to Assumptions 2.2 and 2.3, we suppose that $\sigma$ and $b$ are smooth and bounded. First take $\zeta \in C^\infty_0(\mathbb{R}^{d+1})$, $\zeta(0, x) = 1$ in $(-1, 1) \times B_1$, and, for given $(t_0, x_0) \in \mathbb{R}^{d+1}$, consider the equation
\[ x_t = x_0 + \int_0^t \sigma(x_s) \, dw_s + \int_0^t \zeta(t_0 + s, x_s) b(x_s) \, ds. \]  
(4.1)

According to an obvious modification of Theorem 1.1 of [7] with $p_0 = p = d_0$, $q_0 = q = \infty$ (see, for instance, the proof of Theorem 1.2 in the same paper), there is a constant $\hat{b} = \hat{b}(d, \delta) \in (0, \infty)$ such that if
\[ \left( \int_B |b|^{d_0} \, dx \right)^{1/d_0} \leq \hat{b} \rho^{-1} \]  
(4.2)
for any $\rho \in (0, R_0]$ and $B \in \mathcal{B}_\rho$, then for any $R \in (0, R_0]$, $B \in \mathcal{B}_R$, $x \in \mathbb{R}^d$, and Borel $f \geq 0$
\[ E_x \int_0^{\tau_B} f(x_t) \, dt \leq \hat{N} R^{2-d/d_0} \|f\|_{L_{d_0}(B)}, \]  
(4.3)
where $\hat{N}$ depends only on $d$ and $\delta$, $\tau_B$ is the first exit time of the solution $x_t$ of (4.1) from $B$ and we use the symbol $E_x$ to indicate that we are dealing with solutions of stochastic equations started at $x$.

One can take $\zeta(t/n, x/n)$ in place of $\zeta$ in the above arguments and, since any cylinder for large $n$ is absorbed in $(-n, n) \times B_n$ where the coefficients of (4.1) coincide with the ones in (1.1), we convince ourselves that (4.3) holds for solutions of (1.1) once (4.2) is satisfied (and the coefficients are smooth). By plugging in $|b|$ in place of $f$ in (4.3) and using that
\[ \left( \int_B |b|^{d_0} \, dx \right)^{1/d_0} \leq \left( \int_B |b|^q \, dx \right)^{1/q}, \]
we get that
\[ \hat{b}_{R_0} := \sup_{\rho \leq R_0} \frac{1}{\rho} \sup_{B \in \mathcal{B}_\rho} E_x \int_0^{\tau_B} |b(x_s)| \, ds \leq \hat{N}(d, \delta)\|b\|. \]
It follows that there is a constant $\bar{N} = \bar{N}(d, \delta) \geq 1$ such that, if
\[ \bar{N}\|b\| \leq 1, \tag{4.4} \]
then (4.2) holds for any $\rho \in (0, R_0]$ and $B \in B_\rho$ and $\bar{N}\bar{b}_R \leq 1$, where this $\bar{N} = \bar{N}(d, \delta)$ is taken from Theorem 2.3 of [7]. In this case all conclusions of Theorem 2.3 of [7] hold true for any $R \leq R_0$. Everywhere below in this section we suppose that (4.4) holds. We thus have the following result, in which
\[ \tau_R = \inf\{t \geq 0 : x_t \notin B_R\}, \quad \gamma_R = \inf\{t \geq 0 : x_t \in \bar{B}_R\}. \tag{4.5} \]

Theorem 4.1. There is a constant $\bar{\xi} = \bar{\xi}(d, \delta) \in (0, 1)$ such that for any $R \leq R_0$ and $x$
\[ P_x(\tau_R \geq R^2) \leq 1 - \bar{\xi}. \tag{4.6} \]
Moreover for $n = 1, 2, \ldots$
\[ P_x(\tau_R \geq nR^2) \leq (1 - \bar{\xi})^n, \tag{4.7} \]
so that
\[ E_x\tau_R \leq N(d, \delta)R^2. \tag{4.8} \]
Furthermore, the probability starting from a point in $\bar{B}_{9R/16}$ to reach the ball $\bar{B}_{R/16}$ before exiting from $B_R$ is bigger than $\bar{\xi}$: for any $x$ with $|x| \leq 9R/16$
\[ P_x(\tau_R > \gamma_{R/16}) \geq \bar{\xi}. \tag{4.9} \]

Once this result is established we can use all results from [6] based on Theorem 2.3 from there. In particular, here is a particular case of Theorem 2.6 of [6]. We set $\tau'_R = \tau_R \wedge R^2$.

Theorem 4.2. For any $\lambda, R > 0$ satisfying $\lambda \geq R_0^{-2}$ we have
\[ E e^{-\lambda \tau'_R} \leq e^{\check{\xi}/2} e^{-\sqrt{\lambda}R\check{\xi}/2}. \tag{4.10} \]
In particular, for any $R > 0$ and $t \leq RR_0\check{\xi}/4$ we have
\[ P(\tau'_R \leq t) \leq e^{\check{\xi}/2} \exp\left(-\frac{\check{\xi}^2 R^2}{16t}\right). \tag{4.11} \]

This theorem implies Corollary 2.8 of [6].

Corollary 4.3. For any $m > 0$ and $0 \leq s \leq t$ we have
\[ E \sup_{r \in [s, t]} |x_r - x_s|^m \leq N(|t - s|^{m/2} + |t - s|^m), \tag{4.12} \]
where $N = N(m, R_0, \check{\xi})$.

Since $\sigma$ and $b$ are smooth, from the classical theory we know that $E_x f(x_t) = T_t f(x)$ for any $f \in L_p$ with $p \in [1, \infty]$. In particular, (3.13) implies that for $q > p > d/2$, $t > 0$
\[ E_x |f(x_t)| \leq N(t \wedge 1)^{-d/(2p)} \|f\|_{L_p}, \tag{4.13} \]
where $N$ depends only on $d, \delta, p, q, R_0$. As an obvious consequence of this estimate we also have that for any $q > p > d/2$, $\lambda \geq 1$

$$
E_x \int_0^\infty e^{-\lambda t} |f(t, x_t)| dt = \int_0^\infty e^{-\lambda t} E_x |f(t, x_t)| dt
$$

$$
\leq N \int_0^\infty e^{-\lambda (t \wedge 1)} - d/(2p) \|f(t, \cdot)|_{L^p} dt \leq N \lambda^{(d+2)/(2p)-1} \|f\|_{L^p(R^{d+1})},
$$

(4.14)

where $N$ depends only on $d, \delta, p, q, R_0$.

5. Properties of admissible solutions

Recall that

$$W^{1,2}_p([0, T] \times \mathbb{R}^d) = \{u : u, u_x, u_{xx}, \partial_t u \in L^p([0, T] \times \mathbb{R}^d)\}.$$

It is known that if $u \in W^{1,2}_p([0, T] \times \mathbb{R}^d)$ and $p > d/2 + 1$, then $u$ has a modification which is bounded and continuous on $[0, T] \times \mathbb{R}^d$. Therefore, talking about $u$ of class $W^{1,2}_p([0, T] \times \mathbb{R}^d)$ we will always mean this modification. Below by $x_t$ we mean an admissible solution of (1.1), corresponding to a $p \in (d/2 + 1, q)$, starting at $x_0$ and assuming that it exists.

**Theorem 5.1 (Ito’s formula).** Let $u \in W^{1,2}_p([0, T] \times \mathbb{R}^d)$. Then with probability one for all $t \in [0, T]$ we have

$$u(t, x_t) = u(0, x_0) + \int_0^t (\partial_t + L)u(s, x_s) ds + \int_0^t \sigma^{ik} D_i u(s, x_s) dw^k_s, \quad (5.1)$$

where the stochastic integral is a square integrable martingale on $[0, T]$.

This theorem is proved by using (2.1) in the same way as Theorem 1.3 of [5] is proved on the basis of Theorem 2.6 of [5]. We only outline the main points which are

$$E \int_0^T \|b(x_t) D_i u(t, x_t)\| dt \leq N \|D^2 u\|_{L^p((0,T) \times \mathbb{R}^d)}, \quad (5.2)$$

$$E \int_0^T |D u(t, x_t)|^2 dt \leq N \|\partial_t u, D^2 u\|_{L^p((0,T) \times \mathbb{R}^d)}, \quad (5.3)$$

where the constants $N$ are independent of $u$.

Estimate (5.2) immediately follows from (2.1) and Theorem 3.4. To prove (5.3) observe that

$$1 - (d + 2) \left(\frac{1}{p} - \frac{1}{2p}\right) \geq 0$$

so that by embedding theorems (see, for instance, Lemma 2.3.3 in [9])

$$\|D u\|_{L^p((0,T) \times \mathbb{R}^d)} \leq \|D^2 u\|^2_{L^{2p}((0,T) \times \mathbb{R}^d)} \leq N \|\partial_t u, D^2 u\|_{L^p((0,T) \times \mathbb{R}^d)}.$$

This and (2.1) imply (5.3).

Here is a modification of Theorem 4.4 of [8] in our situation.
Theorem 5.2. Let $T \in (0, \infty)$ and $f \in L_p \cap L_{2p}$. Then

(i) For each $t > 0$ we have $E[f(x_t)] = T_t f(x_0)$. In particular,

$$E|f(x_t)| \leq N(t \wedge 1)^{-d/(2p)} \|f\|_{L_p},$$

(5.4)

where $N$ depends only on $d, \delta, p, q, R_0$;

(ii) For each $t > 0$ with probability one we have

$$f(x_t) = T_t f(x_0) + \int_0^t \sigma^i D_i T_{t-s} f(x_s) \, dw_s^k,$$

(5.5)

where $\sigma^i D_i T_{t-s} f(x) = (\sigma^i D_i T_{t-s} f)(x)$ and similar notation is also used below;

(iii) For each $t > 0$

$$T_t f^2(x_0) = (T_t f(x_0))^2 + \sum_k \int_0^t T_s \left[ \left( \sum_i \sigma^i D_i T_{t-s} f \right)^2 \right] (x_0) \, ds.$$  

Proof. If $f \in W^p_\infty$, then $u(s, x) := T_{t-s} f(x)$, $s \leq t$, satisfies the condition of Theorem 5.1 and we get (5.5) by that theorem. By taking the expectations of both sides we get that $E[f(x_t)] = T_t f(x_0)$. Then (5.4) follows from (3.13).

By taking the expectations of the squares of both sides of (5.5) we obtain (5.6). Thus, all assertions of the theorem are true if $f \in W^2_p$.

Assertion (i) holds for any $f \in L_p$, which is seen from the fact that both $T_t f(x_0)$ and $E[f(x_t)]$ are bounded linear functionals on a dense subset $W_p^2$ of $L_p$.

Then, as $f^n \in W^2_p$ tend to $f$ in $L_p \cap L_{2p}$, $T_{t-s} f^n \to T_{t-s} f$ in $W^2_p$ for $s < t$ (see (3.9)). By embedding theorems ($p \geq d/2$) $DT_{t-s} f^n \to DT_{t-s} f$ in $L_{2p}$ and in light of (3.13)

$$T_s \left[ \left( \sum_i \sigma^i D_i T_{t-s} f^n \right)^2 \right] (x_0) \to T_s \left[ \left( \sum_i \sigma^i D_i T_{t-s} f \right)^2 \right] (x_0)$$

for any $0 < s < t$. Furthermore, $(f^n)^2 \to f^2$ in $L_p$ and, due to (3.13),

$$T_t (f^n)^2(x_0) \to T_t f^2(x_0).$$

It follows by Fatou’s lemma (and (5.6)) that

$$T_t f^2(x_0) \geq (T_t f(x_0))^2 + \sum_k \int_0^t T_s \left[ \left( \sum_i \sigma^i D_i T_{t-s} f \right)^2 \right] (x_0) \, ds.$$  

(5.7)

Hence, the right-hand side of (5.5) is well defined. Furthermore,

$$E \left| \int_0^t \sigma^i D_i T_{t-s} f(x_s) \, dw_s^k - \int_0^t \sigma^i D_i T_{t-s} f^n(x_s) \, dw_s^k \right|^2$$

$$= \sum_k \int_0^t T_s \left[ \left( \sum_i \sigma^i D_i T_{t-s} (f - f^n) \right)^2 \right] (x_0) \, ds$$

$$\leq T_t (f - f^n)^2(x_0) - (T_t (f - f^n)(x_0))^2 \leq T_t (f - f^n)^2(x_0) = E[f(x_t) - f^n(x_t)]^2,$$

where the first inequality is due to (5.4). The last expression tends to zero in light of (5.4), which allows us to get (5.5) by passing to the limit in its
version with $f^n$ in place of $f$. After that (5.6) follows as above. The theorem is proved.

Now we iterate (5.5) and by repeating literally what is done in [8] we come to the following conclusions in which (as in [8])

$$Q^k f(x) = \sigma^{ik}(x) D_t T f(x),$$

for $s_1, \ldots, s_n > 0$ we define

$$Q_{s_n, \ldots, s_1} f(x) = \sum_{k_1, \ldots, k_n} [Q_{s_n}^{k_n} \cdots Q_{s_1}^{k_1} f]^2(x), \quad (5.8)$$

and $\mathcal{F}_t^w$ is the completion of $\sigma(w_s : s \leq t)$.

**Theorem 5.3.** Let $f \in L_p \cap L_{2p}$, $t > 0$. Then

$$E \left( f(x_t) \mid \mathcal{F}_t^w \right) = T_t f(x_0)$$

$$+ \sum_{m=1}^{\infty} \int_{t_1 > \cdots > t_m > 0} T_{t_m} Q_{t_{m-1} - t_m}^{k_m} \cdots Q_{t_{t_1}}^{k_1} f(x_0) \, du_{t_m}^{k_m} \cdots du_{t_1}^{k_1},$$

where the series converges in the mean square sense.

**Theorem 5.4.** Let $f \in L_p \cap L_{2p}$, $t_0 > 0$. Then $f(x_{t_0})$ is $\mathcal{F}_{t_0}^w$-measurable iff

$$\lim_{m \to \infty} \int_{t_0 > t_1 > \cdots > t_m > 0} T_{t_m} Q_{t_{m-1} - t_m - \cdots - t_1} f(x_0) \, dt_m \cdots dt_1 = 0. \quad (5.9)$$

Furthermore, under either of the above equivalent conditions

$$f(x_t) = T_t f(x_0)$$

$$+ \sum_{m=1}^{\infty} \int_{t_1 > \cdots > t_m > 0} T_{t_m} Q_{t_{m-1} - t_m}^{k_m} \cdots Q_{t_{t_1}}^{k_1} f(x_0) \, dw_{t_m}^{k_m} \cdots dw_{t_1}^{k_1}. \quad (5.10)$$

**Theorem 5.5.** If equation (1.1) has two admissible solutions which are not indistinguishable, then it does not have any admissible strong solution. In particular, if (1.1) has an admissible strong solution, then it is a unique admissible solution.

**Theorem 5.6.** If equation (1.1) has a strong admissible solution on one probability space then it has a strong admissible solution on any other probability space carrying a $d_1$-dimensional Wiener process.

Simple manipulations with (5.9) as in [8] using (3.13) lead to the following particular case of Theorem 5.9 of [8].

**Theorem 5.7.** Let $f \in L_p \cap L_{2p}$. Then $f(x_t)$ is $\mathcal{F}_t^w$-measurable for any $t > 0$ if there exists a $\nu > 0$ such that

$$\left\| \int_{R^n} e^{-\nu(s_{m-1} + \cdots + s_0)} Q_{s_{m-1}, \ldots, s_0} f \, ds_{m-1} \cdots ds_0 \right\|_{L_p}^p \to 0 \quad (5.11)$$

as $m \to \infty$, where $R^m_+ = (0, \infty)^m$. 

We are going to prove that (5.11) holds under Assumptions 2.3 and 2.2 and assuming that (2.3) holds for an appropriate $N_0$, by showing that the series composed of the left-hand sides of (5.11) converges.

6. SOME ESTIMATES IN THE CASE OF $C^\infty$ COEFFICIENTS

We suppose that $\sigma^k, b$ satisfy Assumption 2.2 and are infinitely differentiable with each derivative bounded.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, let $\{\mathcal{F}_t\}$ be an increasing filtration of $\sigma$-fields $\mathcal{F}_t \subset \mathcal{F}$, that are complete. Let $w_t$ be a $d_1$-dimensional Wiener process relative to $\{\mathcal{F}_t\}$. We also assume that there is a $(d + 1)$ independent $d$-dimensional Wiener, relative to $\{\mathcal{F}_t\}$, process $B_t^{(0)}, \ldots, B_t^{(d)}$ independent of $w_t$. Take $x, \eta \in \mathbb{R}^d$, a nonnegative bounded infinitely differentiable $K_0$, the role of which will be emphasized later, with each derivative bounded given on $\mathbb{R}^d$, and consider the following system

$$x_t = x + \int_0^t \sigma^k(x_s) \, dw^k_s + \int_0^t b(x_s) \, ds, \quad (6.1)$$

$$\eta_t = \eta + \int_0^t \sigma^k_{(\eta_s)}(x_s) \, dw^k_s + \int_0^t b_{(\eta_s)}(x_s) \, ds$$

$$+ \int_0^t K_0(x_s) \, dB^{(0)}_s + \int_0^t K_0(x_s) \eta^k_s \, dB^{(k)}_s. \quad (6.2)$$

As is well known, (6.1) has a unique solution which we denote by $x_t(x)$. By substituting it into (6.2) we see that the coefficients of (6.2) grow linearly in $\eta$ and hence (6.2) also has a unique solution which we denote by $\eta_t(x, \eta)$. By the way, observe that equation (6.2) is linear with respect to $\eta_t$. Therefore $\eta_t(x, \eta)$ is an affine function of $\eta$. For the uniformity of notation we sometimes set $x_t(x, \eta) = x_t(x)$.

For $t \geq 0$ and $(x, \eta) \in \mathbb{R}^{2d}$ consider the equation

$$\partial_t u(t, x, \eta) = (1/2)\sigma^{ik} \sigma^{jk} (x) u_{x_i x_j} (t, x, \eta) + \sigma^{ik} \sigma_{(\eta)}^{jk} (x) u_{x_i \eta_j} (t, x, \eta)$$

$$+ (1/2) \sigma^{ik}_{(\eta)} \sigma^{jk}_{(\eta)} (x) u_{\eta_i \eta_j} (t, x, \eta) + (1/2) K_0^2 (x) (1 + |\eta|^2) \hat{\delta}_{ij} u_{\eta_i \eta_j} (t, x, \eta)$$

$$+ b_j (x, \eta) u_{x_i} (t, x, \eta) + b^i_{(\eta)} (x) u_{\eta_i} (t, x, \eta) =: \hat{L}(x, \eta) u(t, x, \eta) \quad (6.3)$$

naturally related to system (6.1)-(6.2).

Here is Lemma 6.3 of [8].

**Lemma 6.1.** Let $x, \eta \in \mathbb{R}^d$ and let $f(x)$ be infinitely differentiable with bounded derivatives. Then for any $t \in (0, \infty)$ ($t_0 = t$)

$$E \left[ f(\eta(x, \eta)) (x_t(x)) \right]^2 \geq \left[ (T_t f(x)) (\eta) \right]^2$$

$$+ \sum_{m=1}^\infty \sum_{k_1, \ldots, k_m} \int_{t > t_1 > \ldots > t_m > 0} \left[ (T_{t_m} Q_{t_{m-1} - t_m} \cdots Q_{t_1 - t_2} f(x)) (\eta) \right]^2 \, dt_m \cdots \, dt_1. \quad (6.4)$$
Next, we want to estimate the left-hand side of (6.4) which according to Lemma 6.1 of \[8\] satisfies (6.3).

In the future we need a more precise information than that provided in Lemma 6.1 of \[8\].

**Lemma 6.2.** Take \( f \in C^\infty_0 \) and set

\[ u(t, x, \eta) = E[f(\eta_t(x, \eta))(x_t(x))]^2. \]

Then \( u \) is infinitely differentiable in \((x, \eta)\) and each of its derivatives is continuous in \( t \) and

\[
|u(t, x, \eta)| + |u_x(t, x, \eta)| + |u_\eta(t, x, \eta)|
+ |u_{xx}(t, x, \eta)| + |u_{x\eta}(t, x, \eta)| + |u_{\eta\eta}(t, x, \eta)|
\leq Ne^{N_t - \kappa|x|}(1 + |\eta|^2),
\]

where \( N, \kappa > 0 \) are independent of \( x, \eta \).

**Proof.** We are going to use the terminology and results from Sections 2.7 and 2.8 of \[4\]). Take unit \( \mu, \nu \in \mathbb{R}^d \). As it follows from \[4\], the solution \( x_t(x) \) of (6.1) is infinitely \( LB \)-differentiable in the direction of \( \mu \) and the equations for the derivatives can be obtained by formal differentiation of (6.1). This provides a sufficient information to assert that the solution \( \eta_t(x, \eta) \) of (6.2) is infinitely \( LB \)-differentiable in the direction of \( \mu \) in the variable \( x \) and the equations for the derivatives can be obtained by formal differentiation of (6.2). Similar assertion is true for the derivatives of \( \eta_t(x, \eta) \) with respect to \( \eta \) just because it is an affine function of \( \eta \). It follows, in particular, that \( u \) is infinitely differentiable in \((x, \eta)\).

By Theorem 2.8.8 of \[4\] for any \( T, r \in (0, \infty) \)

\[
E\left( \sup_{t \leq T} \left| \frac{\partial}{\partial \mu} x_t(x) \right|^r + \sup_{t \leq T} \left| \frac{\partial^2}{\partial \mu^2} x_t(x) \right|^r \right) \leq Ne^{NT},
\]

where \( N \) is independent of \( x, \eta, \mu, \nu \). The derivative of \( \eta_t(x, \eta) \) with respect to \( \eta \) satisfies the same equation (6.2) but without the stochastic integral of \( K_0(x_s) dB_s^{(0)} \). Therefore this derivative admits an estimate similar to (6.6). Of course, the second-order derivative of \( \eta_t(x, \eta) \) with respect to \( \eta \) is zero. The mixed derivative

\[ LB - \frac{\partial^2}{\partial \mu \partial \nu} \eta_t(x, \eta) \]

satisfies the same equation as \( \beta_t := LB - (\partial/\partial \nu) \eta_t(x, \eta) \) but with zero initial data and a free term

\[
\int_0^t \sigma^k_{(\beta_s)}(\alpha_s)(x_s) dw^k_s + \int_0^t b_{(\beta_s)}(\alpha_s)(x_s) ds + \int_0^t K_{(\alpha_s)}(x_s) \beta_s^k dB_s^{(k)},
\]

where \( \alpha_t := LB - (\partial/\partial \mu) x_t(x) \). It follows very easily from \[4\] that this derivative also admit an estimate like (6.6).

This and the fact that

\[
E \sup_{t \leq T} |\eta_t, x|^r \leq N(1 + |\eta|^r)e^{NT}
\]
and $\sigma$ and $b$ are bounded allows us to argue as before Theorem 6.4 of [8] and obtain (6.5) by using that $f$ has compact support. The lemma is proved.

In the future we might be interested in estimating not only the left-hand side of (6.4) but a slightly more general quantity. Therefore, we take an infinitely differentiable $f(x, \eta) \geq 0$ such that for an $m > 0$ and a constant $N$

$$
(|f| + |f_x| + |f_\eta| + |f_{xx}| + |f_{x\eta}| + |f_{\eta\eta}|)(x, \eta) \leq N (1 + |\eta|)^m
$$

for all $x, \eta$ and such that $f(x, \eta) = 0$ for all $\eta$ if $|x| \geq R$ for some $R > 0$. Then denote $u(t, x, \eta) = Ef[(x_t, \eta_t)(x, \eta)]$. According to [8], there exist constants $\mu > 0$, $\kappa = \kappa(m) \geq 0$, and a function $M(t)$ bounded on each time interval $[0, T]$ such that for all $t, x, \eta$ we have

$$
|u(t, x, \eta)| + |u_x(t, x, \eta)| + |u_\eta(t, x, \eta)| + |u_xx(t, x, \eta)| + |u_{x\eta}(t, x, \eta)| + |u_{\eta\eta}(t, x, \eta)| \leq M(t) e^{-\mu|x|} (1 + |\eta|^2)^\kappa. 
$$

(6.7)

This justifies the integrations by parts we perform below.

Introduce

$$
h = (1 + |\eta|^2)^{-\kappa - d}
$$

and observe that for a constant $N = N(d, \kappa)$ we have

$$
|\eta| \cdot |h_\eta| \leq Nh, \quad |((1 + |\eta|^2) h)_{\eta\eta}| \leq Nh.
$$

**Theorem 6.3.** Let $r \geq 2$ and suppose that the above $u \geq 0$. Then there is a constant $\tilde{N} = \tilde{N}(d, \delta, q, q_0, \kappa) \geq 1$ such that if

$$
r \tilde{N} (\|D\sigma\| + \|b\|) \leq 1, 
$$

then there exists a constant $N$, depending only on $d$, $\delta$, $q$, $q_0$, $\kappa$, $r$, $R_0$, and there is a function $K_0$ such that for any $t \geq 0$

$$
\int_{\mathbb{R}^{2d}} h(\eta) u^r(t, x, \eta) dxd\eta \leq e^{Nt} \int_{\mathbb{R}^{2d}} h(\eta) f^r(x, \eta) dxd\eta.
$$

(6.9)

The proof of this theorem proceeds as usual by multiplying (6.3) by $h(\eta) u^{r-1}(t, x, \eta)$ and integrating by parts over $[0, t] \times \mathbb{R}^{2d}$. The integral of the left-hand side is

$$
r^{-1} \int_{\mathbb{R}^{2d}} h(\eta) u^r(t, x, \eta) dxd\eta - r^{-1} \int_{\mathbb{R}^{2d}} h(\eta) f^r(x, \eta) dxd\eta.
$$

Therefore, in light of Gronwall’s inequality, to prove the theorem it suffices to prove the following estimate.

**Lemma 6.4.** Let $\kappa \geq 0$, $r \in [2, \infty)$. Then there is a constant $\tilde{N} \geq 1$ depending only on $d$, $\delta$, $q$, $q_0$, $\kappa$, such that if

$$
r \tilde{N} (\|D\sigma\| + \|b\|) \leq 1, 
$$

(6.10)

then there exists a constant $N$, depending only on $d$, $\delta$, $q$, $q_0$, $\kappa$, $r$, $R_0$, and there is a function $K_0$ such that for any smooth function $v(x, \eta) \geq 0$
(independent of $t$), for which condition (6.7) is satisfied with $v$ in place of $u$ and some $M$, we have

$$\int_{\mathbb{R}^d} h(\eta)v^{r-1}(x, \eta)\hat{L}v(x, \eta)\, dx d\eta \leq N\int_{\mathbb{R}^d} h(\eta)v^r(x, \eta)\, dx d\eta. \quad (6.11)$$

Proof. We basically repeat the proof of Lemma 6.5 of [8] with some changes caused by the weaker assumptions on $\sigma$ and $b$. For simplicity of notation we drop the arguments $x, \eta$. We also write $U \sim V$ if their integrals over $\mathbb{R}^d$ coincide, and $U \ll V$ if the integral of $U$ is less than or equal to that of $V$. Below the constants called $N$, sometimes with indices, depend only on $d, \delta, q, q_0, \kappa, r, R_0$ unless specifically noted otherwise. Constants called $\bar{N}$ depend only on $d, \delta, q, q_0, \kappa$.

Set $w = v^{r/2}$ and note simple formulas:

$$v^{r-1}v_x = (2/r)w'_x, \quad v^{r-2}v_{x_j} = (4/r^2)w_xw_{x_j}. \quad \text{(6.10)}$$

Then denote by $\bar{L}_1$ the sum of the first-order terms in $\bar{L}$ and observe that integrating by parts shows that

$$hv^{r-1}b_{(q)}'v_{q'}' \sim -\frac{1}{r}h_{q}b_{(q)}v^r - \frac{1}{r}hb_{x}v^r \quad \sim \frac{2}{r}\eta^k b_{q}w_{x^k} + \frac{2}{r}hb_{x}w_{q_i}. \quad \text{(6.11)}$$

Hence,

$$hv^{r-1}\bar{L}_1v \sim \frac{2}{r}\eta^k b_{q}w_{x^k} + \frac{4}{r}hb_{x}w_{q_i}. \quad \text{(6.12)}$$

Observe that by Lemma 3.5

$$\int_{\mathbb{R}^d} |b_{i}w_{x^k}| \, dx \leq \left(\int_{\mathbb{R}^d} |w_x|^2 \, dx\right)^{1/2} \left(\int_{\mathbb{R}^d} |b|^2 |w|^2 \, dx\right)^{1/2} \leq \bar{N}|b|\int_{\mathbb{R}^d} |w_x|^2 \, dx + N\int_{\mathbb{R}^d} |w|^2 \, dx, \quad \text{(6.12)}$$

where $\bar{N}$ depends only on $d, q$ and $N$ depends only on $d, q, R_0$, and, formally, $\|b\|$. But we suppress its dependence on $\|b\|$ because, in light of (6.10) we assume from the start that $\|b\|, \|D\sigma\| \leq 1$.

Since $|\eta| |h_{q}| \leq N(\kappa, d)h$, it follows that

$$\eta^k h_{q}b_{i}w_{x^k} \ll \bar{N}|b|\|h|w_x|^2 + Nh|w|^2. \quad \text{(6.13)}$$

Similarly, $(4/r)hb_{i}w_{x^i} \ll \bar{N}|b|\|h|w_x|^2 + Nh|w|^2$ and we conclude that

$$hv^{r-1}\bar{L}_1v \ll \bar{N}|b|\|h|w_x|^2 + Nh|w|^2. \quad \text{(6.13)}$$

Starting to deal with the second order derivatives note that

$$hv^{r-1}(1/2)\sigma^ik\sigma^jkv_{x^i x^j} \sim -(r - 1)/2v^{r-2}h\sigma^ikv_{x^i}\sigma^jkv_{x^j} - \frac{1}{2}h[\sigma^ik\sigma^jk + \sigma^ik\sigma^jk]v^{r-1}v_{x^j} = -(2r - 2)/r^2h\sigma^ikw_x\sigma^jkw_{x^j} - \frac{1}{r}h[\sigma^ik\sigma^jk + \sigma^ik\sigma^jk]w_{x^j} \leq -(1/r)h\sigma^ikw_x\sigma^jkw_{x^j} + h[\sigma^ik\sigma^jk + \sigma^ik\sigma^jk]w_{x^j}, \quad \text{(6.14)}$$

and observe that

$$\int_{\mathbb{R}^d} h(\eta)v^{r-1}(x, \eta)\hat{L}v(x, \eta)\, dx d\eta \leq N\int_{\mathbb{R}^d} h(\eta)v^r(x, \eta)\, dx d\eta. \quad \text{(6.11)}$$
where the inequality (to simplify the writing) is due to the fact that \( r \geq 2 \).
In this inequality the first term on the right is dominated in the sense of \( \prec \)
by
\[-(1/r)\delta h|w_x|^2\]
(see Assumption 2.2). The remaining term contains \( w w_{x,i} \) and we treat it as above. Then we get
\[
h v^{r-1} (1/2) \sigma_{x,i}^{ik} \sigma_{x,j}^{jk} v_{x^n} \prec -(1/r) \delta - \hat{N} \| D \sigma \| h |w_x|^2 + Nh|w|^2. \quad (6.14)
\]
Next,
\[
h v^{r-1} \sigma_{x,i}^{ik} \sigma_{x,j}^{jk} v_{x^n} \prec -(r - 1) h \sigma_{x,i}^{ik} v^{r-2} \sigma_{x,j}^{jk} v_{x^n} - v^{r-1} v_{x,i} \left[ h \sigma_{x,i}^{ik} \sigma_{x,j}^{jk} + h \sigma_{x,i}^{ik} \sigma_{x,j}^{jk} \right]
= -((4 - 4)/r^2) h \sigma_{x,i}^{ik} v_{x^n} \sigma_{x,j}^{jk} w_x
- (2/r) w w_{x,i} \left[ h \sigma_{x,i}^{ik} \sigma_{x,j}^{jk} + h \sigma_{x,i}^{ik} \sigma_{x,j}^{jk} \right].
\]
We estimate the first term on the right roughly using
\[
|\sigma_{x,i}^{ik} w_{x^n} \sigma_{x,j}^{jk} w_x| \leq \varepsilon |w_x|^2 + \hat{N} \varepsilon^{-1} |\eta| \sum_k |\sigma_{x,i}^{ik}|^2 |w_{x^n}|^2.
\]
The second term contains \( w w_{x,i} \) and allows the same handling as before. Therefore,
\[
h v^{r-1} \sigma_{x,i}^{ik} \sigma_{x,t}^{jk} v_{x^n} \prec (\varepsilon + \hat{N} \| D \sigma \|) h |w_x|^2 + Nh|w|^2 + \hat{N} \varepsilon^{-1} h |\eta| \sum_k |\sigma_{x,i}^{ik}|^2 |w_{x^n}|^2. \quad (6.15)
\]
The last term in \( h v^{r-1} L v \) containing \( \sigma \) is
\[
h v^{r-1} (1/2) \sigma_{x,i}^{ik} \sigma_{x,j}^{jk} v_{x^n} \prec -(1/r) h \sigma_{x,i}^{ik} v^{r-2} \sigma_{x,j}^{jk} v_{x^n}
- (1/2) v^{r-1} \sigma_{x,i}^{ik} \sigma_{x,j}^{jk} h \sigma_{x,i}^{ik} + h \sigma_{x,i}^{ik} \sigma_{x,j}^{jk}
\prec \hat{N} h (|\eta|^2 |w_{x^n}|^2 + w^2) \sum_k |\sigma_{x,i}^{ik}|^2 + I,
\]
where
\[
I = -(1/(2r)) h (w^2) \sigma_{x,i}^{ik} \sigma_{x,j}^{jk}.
\]
To estimate the last term observe that by Lemma 3.5
\[
\int_{\mathbb{R}^d} |\sigma_{x,i}^{ik}|^2 w^2 dx \leq \hat{N} \| D \sigma \|^2 \int_{\mathbb{R}^d} |w_x|^2 dx + N \int_{\mathbb{R}^d} |w|^2 dx. \quad (6.16)
\]
Above we had terms with \( \| D \sigma \| \) and now we have \( \| D \sigma \|^2 \). To make formulas
somewhat easier observe that \( \hat{N} \), we are after, is bigger than one, so that \( \| D \sigma \| \leq 1 \) and hence,
\[
I \prec \hat{N} h \| D \sigma \||w_x|^2 + Nh|w|^2
\]
and
\[
h v^{q-1} (1/2) \sigma_{x,i}^{ik} \sigma_{x,j}^{jk} v_{x^n} \prec \hat{N} h |\eta|^2 |w_{x^n}|^2 \sum_k |\sigma_{x,i}^{ik}|^2.
\]
\begin{align*}
   +hw^2 \left( N + \tilde{N} \sum_k |\sigma_k|^2 \right) + \tilde{N}h\|D\sigma\||w_x|^2. \quad (6.17)
\end{align*}

Finally,
\begin{align*}
   hv^{r-1}(1/2)K_0^2(1 + |\eta|^2)\delta^{ij}v_{\eta^i\eta^j} & \sim -((2r - 2)/r^2)hK_0^2(1 + |\eta|^2)|w_\eta|^2 \\
   & - (2/r)K_0^2(h(1 + |\eta|^2))_{\eta^i}w_{\eta^i} \\
   \sim -((2r - 2)/r^2)hK_0^2(1 + |\eta|^2)|w_\eta|^2 \quad & + (1/r)w^2 K_0^2 \delta^{ij}(h(1 + |\eta|^2))_{\eta^i\eta^j} \\
   \leq - (1/r)hK_0^2(1 + |\eta|^2)|w_\eta|^2 & + \tilde{N}w^2K_0^2h. \quad (6.18)
\end{align*}

By combining (6.13), (6.14), (6.15), (6.17), and (6.18), and using that $|\eta| \leq 1 + |\eta|^2$, we see that for any $\varepsilon \in (0, 1)$
\begin{align*}
   hv^{a-1}Lv \leq & \left[ \tilde{N}_1(\varepsilon + \|b\| + \|D\sigma\|) - \delta/r \right] h|w_x|^2 \\
   & + \tilde{N}_2\varepsilon^{-1}h(1 + |\eta|^2)\sum_k |\sigma_k|^2|w_\eta|^2 + Nh^2 + \tilde{N}_3hw^2 \left( K_0^2 + \sum_k |\sigma_k|^2 \right) \\
   & - (1/r)hK_0^2(1 + |\eta|^2)|w_\eta|^2. \quad (6.19)
\end{align*}

Here one sees clearly why introducing $K_0$, which in no way helped us in (6.4), is actually crucial. With $K_0 \equiv 0$ we would not be able to estimate the term with $|w_\eta|^2$. Now, take and fix $\varepsilon$ so that $\tilde{N}_1\varepsilon \leq \delta/(2r)$. After that set
\begin{align*}
   K_0^2 = 1 + \tilde{N}_2r\varepsilon^{-1}\sum_k |\sigma_k|^2
\end{align*}
(1 is added to guarantee the smoothness of $K_0$) and observe that according to (6.16)
\begin{align*}
   \tilde{N}_3hw^2 \left( K_0^2 + \sum_k |\sigma_k|^2 \right) = Nh^2 + \tilde{N}hw^2 \sum_k |\sigma_k|^2 \\
   \leq \tilde{N}_4\|D\sigma\||w_x|^2 + Nh^2.
\end{align*}

Then (6.19) becomes
\begin{align*}
   hv^{r-1}Lv \leq & Nh|w|^2 - \left[ (1/(2r))\delta - (\tilde{N}_1 + \tilde{N}_4)(\|b\| + \|D\sigma\|) \right] h|w_x|^2.
\end{align*}

We can certainly believe that $\tilde{N}_1 \geq 1$, take $\tilde{N}$ in (6.10) to be equal to $(2/\delta)(\tilde{N}_1 + \tilde{N}_4) \geq 1$, and conclude that if (6.10) holds, then
\begin{align*}
   hv^{r-1}Lv \sim Nh|w|^2.
\end{align*}

The lemma is proved.
7. Proof of Theorem 2.5

Set \( p = (1/2)(d/2 + 1 + q) \), take \( \zeta_n \) introduced before Lemma 3.9 and set \( b_n = b * \zeta_n, \sigma_n = \sigma * \zeta_n, a_n = \sigma_n \sigma_n^* \). Define

\[
\|D\sigma_n\| = \sup_{\rho \leq R_0} \rho \left( \int_B |D\sigma_n|^q dx \right)^{1/q}, \quad \|b_n\| = \sup_{\rho \leq R_0} \rho \left( \int_B |b_n|^q dx \right)^{1/q}.
\]

Lemma 7.1. There is a constant \( N_0 = N_0(d, \delta, q_0, q) \geq 1 \) such that (2.3) is satisfied with this \( N_0 \), then for sufficiently large \( n \)

a) We have

\[
p\tilde{N}((\|D\sigma_n\| + \|b_n\|)) \leq 1,
\]

where \( \tilde{N} = \tilde{N}(d, \delta/2, q_0, 2) \) is taken from Theorem 6.3;

b) We have \( a_{n,R_0}^{\#} \leq \theta_0(d, \delta/2, \rho) \) and \( \tilde{N}(d, \delta/2, p, q)N(d, q)\|b_n\| \leq 1 \), where \( \theta_0 \) is taken from Assumption 3.1, \( \tilde{N} \) is the maximum of \( \tilde{N}(d, \delta/2, p, q) \) from (3.11) and \( \tilde{N}(d, \delta/2) \) from (4.4), and \( N(d, q) \) is taken from (3.11);

c) The eigenvalues of \( a_n \) are between \( \delta/2 \) and \( 2\delta \).

Proof. a) The possibility to find \( N_0 = N_0(d, \delta, q, \tilde{N}) \) such that, (2.3) would imply that \( q\tilde{N}\|b_n\| \leq 1/2 \), follows from Lemma 3.9. This lemma has an obvious counterpart applicable to \( D\sigma \) and this proves a).

b) The above argument and Remark 3.2 also take care of b).

c) Denote by \( \sigma \) the \( d \times d_1 \)-matrix whose columns are the \( \sigma^k \)'s and observe that

\[
|\sigma_n^k(x)\lambda| \leq \zeta_n(x) * |\sigma^k(x)\lambda| \leq \delta^{-1/2} |\lambda|.
\]

Therefore we need only prove that for sufficiently large \( n \)

\[
|\sigma_n^k(x)\lambda| \geq |\lambda|\delta^{1/2}/\sqrt{2}.
\]

For any \( y \) we have

\[
|\sigma_n^k(x)\lambda| \geq |\sigma^k(y)\lambda| - |(\sigma_n^k(x) - \sigma^k(y))\lambda| \geq |\lambda|\delta^{1/2} - |(\sigma_n^k(x) - \sigma^k(y))\lambda|
\]

\[
\geq |\lambda|\left(\delta^{1/2} - |\sigma_n^k(x) - \sigma^k(y)|\right)
\]

Furthermore,

\[
\int_{\mathbb{R}^d} |\sigma_n^k(x) - \sigma^k(x - y)| \zeta_n(y) dy
\]

\[
\leq \int_{B_1} \int_{B_1} |\sigma^k(x - z/n) - \sigma^k(x - y/n)| \zeta(y) \zeta(z) dy dz
\]

\[
\leq N(d, q_0)\|D\sigma\|,
\]

where the last inequality is due to Poincaré. We see that to obtain c) it suffices to have an appropriate \( N_0 = N_0(d, \delta, q_0) \). The lemma is proved.

In the rest of the section we suppose that (2.3) is satisfied with \( N_0 \) from Lemma 7.1 and first prove the existence of solutions.

Theorem 7.2. There exists a probability space and a \( d_1 \)-dimensional Wiener process on it such that equation (1.1) has a solution for which estimate (4.14) holds.
Proof. As usual we apply Skorokhod’s method. In light of Lemma 7.1, for sufficiently large \( n \), \( \sigma_n \) and \( b_n \) satisfy Assumptions 2.2, 2.3 and (4.4) with \( \delta/2 \) in place of \( \delta \). Therefore, for the solutions \( x^n_t \) of

\[
x^n_t = x + \int_0^t \sigma_n(x^n_s) \, dw_s + \int_0^t b_n(x^n_s) \, ds
\]

estimates (4.12) and (4.14) hold. After that we repeat the proof of Theorem 2.6.1 of [4] and see that to finish proving the existence part of the current theorem it suffices to show that for any \( T \in (0, \infty) \)

\[
\int_0^T |b_n(x^n_t) - b(x_t)| \, dt \to 0
\]

in probability as \( n \to \infty \) provided that \( x^n_t \) are solutions of (7.3) (with perhaps different Wiener precesses for each \( n \)) and \( x_t \) is a continuous process such that \( x^n_t \to x_t \) in probability for any \( t \in [0, \infty) \).

Due to the convergence of \( x^n_t \) to \( x_t \) estimate (4.14) holds if \( f \) is, in addition, bounded and continuous. Then, of course, this estimate is extended to all \( f \in L_p \). Also obviously, estimate (4.12) is true. This shows that the probability of

\[
\{ \sup_{t \leq T} |x^n_t| \geq R \} \cup \{ \sup_{t \leq T} |x_t| \geq R \}
\]

can be made as small as we like for all \( n \) if \( R \) is large enough. It follows that to prove (7.4) it suffices to prove that

\[
\lim_{n \to \infty} E \int_0^T |\zeta(x^n_t)b_n(x^n_t) - \zeta(x_t)b(x_t)| \, dt = 0
\]

for any \( \zeta \in C_0^\infty \).

Observe that for any bounded and continuous \( \mathbb{R}^d \)-valued \( g \) the above limit is dominated by

\[
\lim_{n \to \infty} E \int_0^T |\zeta(x^n_t)b_n(x^n_t) - g(x^n_t)| \, dt + E \int_0^T |g(x_t) - \zeta(x_t)b(x_t)| \, dt,
\]

where both terms can be made as small as we like because of estimate (4.14) valid for \( x^n_t \) and \( x_t \) and of the fact that \( \zeta b_n \to \zeta b \) in \( L_p \) (even in \( L_q \)). This proves (7.4) and establishes the existence of solution. It turns out that in the above argument \( x_t \) is exactly a solution for which, as we have seen, estimate (4.14) is valid. The theorem is proved.

Next, we prove that any admissible solution of (1.1) is strong. Let \( f \in C_0^\infty \). First we deal with smooth coefficients and develop necessary estimates. Come back to Section 6 and consider the system (6.1)-(6.2) in which replace \( \sigma, b \) with \( \sigma_n, b_n \) with \( n \) so large that the assertions a)-c) of Lemma 7.1 are valid. Denote by \( (x_{n,t}, \eta_{n,t})(x, \eta) \) the solution of the new system and let \( u_n(t, x, \eta) = E[f(\eta_{n,t}(x, \eta))(x_{n,t}(x))]^2 \). Owing to Lemma 6.2 estimate (6.7) holds with \( \kappa = 1 \) and Lemma 7.1 a) allows us to use the
conclusion of Theorem 6.3 with \( r = p \) and \( u_n \) in place of \( u \): There exists 
\[ N = N(d, \delta, q, q_0, R_0) \]

such that
\[
\int_{\mathbb{R}^d} h(\eta)u^p_n(t, x, \eta) \, d\eta dx \leq e^{Nt} \int_{\mathbb{R}^d} h(\eta)f^p(x, \eta) \, d\eta dx. \tag{7.6}
\]

By Lemma 6.1 estimate (7.6) implies that for \( t \geq 0 \) we have
\[
\int_{\mathbb{R}^d} h(\eta)v^p_n(t, x, \eta) \, d\eta dx \leq Ne^{Nt}, \tag{7.7}
\]
where (and below) \( N \) depends only on \( f, d, \delta, q, q_0, \) and \( R_0). 

\[
v_n(t, x, \eta) := \sum_{m=1}^{\infty} \sum_{k_1, \ldots, k_m} \int_{t>t_1>\ldots>t_m>0} \left( T_{n, t_m} Q_{n,t_m-1-t_m}^{k_m} \cdots Q_{n,t-t_1}^{k_1} f(x) \right)(\eta) \, dt_m \cdots dt_1,
\]
and \( T_{n, t}, Q_{n,t}^{k} \) are constructed from \( \sigma_n, b_n \) in the same way as \( T_t, Q_t^{k} \) are constructed from \( \sigma, b \). By the way this construction is possible thanks to Lemma 7.1 b).

Obviously, \( v_n(t, x, \eta) \) is a quadratic function of \( \eta \). Hence, (7.7) implies that, for any \( R \in (0, \infty) \)
\[
\int_{\mathbb{R}^d} \sup_{|\eta| \leq R} v^p_n(t, x, \eta) \, dx \leq Ne^{Nt} R^{2p}. \tag{7.8}
\]

Observe that in notation (5.8) naturally modified for \( \sigma_n, b_n \)
\[
\sum_k v_n(t, x, \sigma^k) = \sum_{m=1}^{\infty} \int_{t>t_1>\ldots>t_m>0} Q_{n,t_m,t_{m-1}-t_m} \cdots Q_{n,t-t_1} f(x) \, dt_m \cdots dt_1
\]
\[
= \sum_{m=1}^{\infty} \int_{S_m(t)} Q_{n,s_m,s_{m-1},\ldots,s_1-t_1} f(x) \, ds_m \cdots ds_1 =: \sum_{m=1}^{\infty} I_{n,m}(t, x),
\]
where \( S_m(t) = \{(s_1, \ldots, s_m) : s_k > 0, s_1 + \ldots + s_m < t\} \). Next, for \( \nu > 0 \) by Hölder’s inequality
\[
\sum_{m=1}^{\infty} \int_{\mathbb{R}^d} \left( \int_0^\infty e^{-\nu t} I_{n,m}(t, x) \, dt \right)^p \, dx \leq \nu^{1-p} \int_0^\infty e^{-\nu t} \left( \sum_{m=1}^{\infty} \int_{\mathbb{R}^d} I_{n,m}^p(t, x) \, dx \right) dt
\]
\[
\leq \nu^{1-p} \int_0^\infty e^{-\nu t} \int_{\mathbb{R}^d} \left( \sum_k v_n(t, x, \sigma^k) \right)^p \, dxdt,
\]
which thanks to (7.8) implies that for appropriate \( \nu \), depending only on \( f, d, \delta, q, q_0, \) and \( R_0 \),
\[
\sum_{m=1}^{\infty} \int_{\mathbb{R}^d} \left( \int_0^\infty e^{-\nu t} I_{n,m}(t, x) \, dt \right)^p \, dx \leq N, \tag{7.9}
\]
where $N$ depends only on $f$, $d$, $\delta$, $q$, $q_0$, and $R_0$.

Now we let $n \to \infty$ in (7.9). Observe that since $\sigma_n \to \sigma$, $b_n \to b$ (a.e.) we have $a_R^{#} \leq \theta_0(d, \delta/2, p)$ and $\tilde{N}(d, \delta/2, p, q)N(d, q)\|b\| \leq 1$, where $\theta_0$ is taken from Assumption 3.1, $\tilde{N}$ is taken from (3.5), and $N(d, q)$ is taken from (3.11). Therefore, the semigroup $T_t$ is well defined as in Section 3.

Also note that in light of Theorem 3.11 for any $\eta \in \mathbb{R}^d$, $t > t_1 \ldots > t_m > 0$

$$
(T_{n,t_m}^{k_m} \cdots Q_{n,t-t_1}^{k_1} f(x))_{(\eta)} \to (T_{m,t}^{k_m} Q_{m-1,t_m}^{k_m} \cdots Q_{1,t-t_1}^{k_1} f(x))_{(\eta)}
$$

in $L_p$. It follows by Fatou’s lemma that

$$
\lim_{n \to \infty} I_{n,m} \geq I_m, \quad \sum_{m=1}^{\infty} \int_{\mathbb{R}^d} \left( \int_0^\infty e^{-\nu t} I_m(t, x) \, dt \right)^p \, dx < \infty.
$$

Finally, by observing that

$$
\int_0^\infty e^{-\nu t} I_m(t, x) \, dt = \int_{\mathbb{R}_{m+1}} e^{-\nu(s_0+\ldots+s_m)} Q_{s_m} \cdots Q_{s_0} f(x) \, ds_m \cdots \, ds_0
$$

and referring to Theorem 5.7, we conclude that $f(x_t)$ is $F^w_t$-measurable for any $t \geq 0$. The arbitrariness of $f$ and $t$ finishes the proof.

REFERENCES

[1] F. Chiarenza and M. Frasca, Morrey spaces and Hardy-Littlewood maximal function, Rend. Mat. Appl. (7) 7 (1987), No. 3-4, 273–279, (1988).

[2] F. Chiarenza and M. Frasca, A remark on a paper by C. Fefferman, Proc. Amer. Math. Soc., Vol. 108 (1990), No. 2, 407–409.

[3] D. Kinzebulatov, Regularity theory of Kolmogorov operator revisited, Canad. Math. Bull. 2020, pp. 1–12, http://dx.doi.org/10.4153/S0008439520000697

[4] N.V. Krylov, “Controlled diffusion processes”, Nauka, Moscow, 1977 in Russian; English translation by Springer, 1980.

[5] N.V. Krylov, On stochastic equations with drift in $L_d$, Ann. Probab., Vol. 49 (2021), No. 5, 2371–2398.

[6] N.V. Krylov, On potentials of Itô’s processes with drift in $L_{d+1}$, http://arxiv.org/abs/2102.10694

[7] N.V. Krylov, On diffusion processes with drift in a Morrey class containing $L_{d+2}$, http://arxiv.org/abs/2104.05603

[8] N.V. Krylov, On strong solutions of Itô’s equations with $a \in W^2_d$ and $b \in L_d$, http://arxiv.org/abs/2007.06040

[9] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uraltseva, “Lineinye i Kvazilineinye Uravneniya Parabolicheskogo Tipa,” (Russian) [Linear and Quasi-Linear Equations of Parabolic Type], “Nauka”, Moscow, 1968; English translation, Amer. Math. Soc., Providence, RI, 1968.

[10] M. Röckner and G. Zhao, SDEs with critical time dependent drifts: strong solutions, arXiv:2103.05803, 2021

Email address: nkrylov@umn.edu

127 Vincent Hall, University of Minnesota, Minneapolis, MN, 55455