Geometric Algebra Power Theory in sinusoidal and non-sinusoidal systems: Single-phase circuits

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Abstract

The aim of this work is to present a general geometric algebra power theory (GAPoT) for single-phase circuits in the frequency domain. This theory embodies an interesting new approach with respect to traditionally accepted theories, which addresses the concepts of electrical power in both sinusoidal and non-sinusoidal systems with linear and nonlinear loads for the proper identification of its components to achieve passive compensation of true non-active current. GAPoT outlines the traditional theories based on the apparent power $S$ and confirms these should definitively be reconsidered. It is evidenced that traditional techniques based on the concepts of Budeanu, Fryze and others fail to identify the interactions between voltage and current. Based on the initial proposals of Castro-Núñez, new aspects not previously included are detailed, modified and added in GAPoT. As a result, it is now possible to analyze non-sinusoidal electrical circuits, establish power balances that comply with the principle of energy conservation, and achieve optimal compensation scenarios with both passive and active elements in linear and non-linear loads.

Index terms— geometric algebra, nonsinusoidal power, clifford algebra, circuit theory

1 Introduction

The study of power flow in power systems is a century-old issue. Engineers and scientists around the world have debated about it throughout the XX century and up to the present one. In sinusoidal systems, there is a clear consensus that electrical power can be analysed through a decomposition that takes into account the average power over a period of time $T$, namely the active power $P$, in addition to a quadrature term, the reactive power $Q$. However, in non-sinusoidal systems, there are disagreements among researchers. Several schools have emerged around this topic, each having a different interpretation depending on the approach. In Table 1, relevant theories and their contributions are summarised [1].

There are two primary lines of research for power theory, i.e., time-domain and frequency-domain approach. The former has had a significant impact, especially in three-phase systems, and has a very specific goal, i.e., compensation through active filters. The second one (in its various forms) has been widely used in different of electrical systems, not only for reactive power compensation, for example, but also for electric circuit analysis, power quality analysis, etc. This is why ideas in the frequency domain attract greater interest because they affect much more systems than time-domain theories. In addition, the most accepted theories, based on the formulation by Akagi-Watanabe [2], have been criticised for their lack of coherence and consistency [3] because they cannot explain the energy exchange processes in non-sinusoidal situations. Based on the above reasons, frequency-domain theories seem to be favoured by the scientific community. However, there are two important factors common to all these theories that should be examined in more detail as follows:

1. They all use the apparent power concept as the result of multiplying the RMS voltage $V$ and current $I$, i.e., $S = VI$. 

\[ S = \text{V RMS} \times \text{I RMS} \]
| Author                      | Contribution                                                                 |
|-----------------------------|------------------------------------------------------------------------------|
| Nabae and Tanaka            | Powers based on instantaneous space vector                                    |
| Shepherd and Zakikhani      | Definition of reactive power                                                  |
| Kusters and Moore           | Inductive and capacitive current                                              |
| Depenbrock                  | Fryze-Buchholz-Depenbrock (FBD) Power Theory                                 |
| Sharon                      | Reactive power definitions                                                    |
| Slonim and Van Wyck         | Definition of active, reactive and apparent powers                           |
| Emanuel                    | Definitions of apparent power                                                |
| Czarnecki                   | Current’s Physical components                                                |
| Peng and Lai                | Generalized instantaneous reactive power theory                              |
| Ferrero and Superti-Furga  | The Park power theory                                                        |
| Rossetto and Tenti          | Instantaneous orthogonal currents                                             |
| Peng                        | Generalized non-active power theory                                           |
| Willems                     | Instantaneous voltage and current vectors                                     |
| Fillipski                   | Elucidation of apparent power and power factor                               |
| Watanabe                    | Generalised theory of instantaneous powers $\alpha$-$\beta$-$0$ transformation |
| LaWhite and Ilic            | Vector space decomposition of reactive power                                  |
| Ghassemi                    | Definition of apparent power based on modified voltage                       |
| Cohen and Leon              | Time-domain representation of powers                                          |
| Zhang                       | Universal instantaneous power theory                                          |
| Lev-Ari and Stankovic       | Reactive power definition via local Fourier transform                         |
| Haque                       | Single phases PQ theory                                                       |
| Menti and Zacharias         | Introduced Geometric Algebra to non-sinusoidal power theory                  |
| Castilla and Bravo          | Extended the use of Geometric Algebra in non-sinusoidal power theory          |
| Xianzhong and Guohai        | Generalised theory of instantaneous reactive power for multiphase system     |
| Shin-Kuan and Chang         | Instantaneous power theory based on activefilter                              |
| Dalgerti                    | Concepts based on instantaneous complex power approach                       |

Table 1: Contributions to power theory by main authors. Reproduced from [1].
2. Most of them are supported by complex number algebra, \( S = V I^* \), where \( V \) is the voltage phasor, \( I^* \) is the conjugated current phasor and \( S \) is the complex power.

The complex power arises from a well-known definition that has been traditionally accepted by the community from its inception. It intends, through a pretended analogy of instantaneous power \( p(t) = v(t)i(t) \), to universalise a term that numerous studies have shown does not represent any physical quantity, is not conservative and does not meet the basic principle of conservation of energy (PCoE) [4, 5, 3, 6]. That it delivers correct mathematical results in sinusoidal systems does not imply that it can be properly generalised to non-sinusoidal systems. However, it remains so popular because of the usual decomposition of the current in quadrature terms, which fits better for the representation of physical phenomena. For example, the current physical components (CPC) theory [7] states that the current in single-phase systems can be decomposed as follows:

\[
i(t) = i_a(t) + i_s(t) + i_r(t) + i_G(t)
\]

where \( i_a(t) \) is the active current, \( i_s(t) \) is the scattered current, \( i_r(t) \) is the reactive current and \( i_G(t) \) is the load harmonic generated current. As demonstrated by Czarnecki, all of the above terms are in quadrature, so the RMS values fulfill:

\[
\|I\|^2 = \|I_a\|^2 + \|I_s\|^2 + \|I_r\|^2 + \|I_G\|^2
\]

If the above equation is multiplied by the squared voltage, results in the following:

\[
S^2 = \|V\|^2\|I\|^2 = \|V\|^2\|I_a\|^2 + \|V\|^2\|I_s\|^2 + \|V\|^2\|I_r\|^2 + \|V\|^2\|I_G\|^2
\]

\[
= P^2 + D_s^2 + Q_r^2 + D_G^2
\]

Equation (3) suggests that a power balance is achieved, and the apparent power is composed of other powers caused by certain physical processes. However, this is questionable because the derivation of (3) from (2) cannot mask the physical reality of the problem. Although current \( I \) can be decomposed into certain orthogonal components (following Kirchhoff’s current laws), the above does not necessarily entail that the derived power terms generate a valid decomposition because

\[
S \neq P + D_s + Q_r + D_G
\]

The true physical meaning of the electrical power is in \( p(t) \), i.e., the instantaneous power expressed as energy transferred per unit time as follows:

\[
p(t) = \frac{dW}{dt}
\]

and its mean value \( P \), which represents the average power demanded by a load during a time period \( T \) and transformed into useful work

\[
P = \frac{1}{T} \int_0^T p(t)dt
\]

The efforts of scientists and engineers for years have focused on explaining the difference between \( P \) and \( S \) through various decompositions based on the quadrature of power components, but without any physical meaning, beyond some mathematical parallelism. In our humble opinion, the main task should be focused on a proper decomposition of the current. Even though the multiplication of RMS current and voltage results in a mathematical concept of multi-component power, at the energy level, the active power \( P \) is the sole objective. The rest is completely irrelevant because it does not contribute to any useful work. The decomposition of the current is where we should focus our efforts to detect the part that can be eliminated or compensated for locally to avoid energy losses in power lines or a degradation of power quality. It is therefore essential to find tools or appropriate procedures that fully describe its calculation accurately and consistent with the laws that govern the theory of electrical circuits.

This distinction between mathematics and physics has already been noted repeatedly by several authors [8, 3, 4, 9]. Mathematical correction certainly is an essential requirement for a theory to be accepted by
the community. However, there are some authors that have considered its physical interpretation as a necessary condition. It is our contention that it is a mistake to attempt to assign physical meaning to something that does not have it, i.e., the non-active power. Only the active power $P$ and instantaneous power $p(t)$ have physical sense, the rest does not. Therefore, the concept that has some methodological logic is the decomposition of the current into components that may be useful for specific objectives, such as the compensation of the non-active current or the improvement of power quality.

In the quest for tools, methods or theories that adequately describe power exchange processes between source and load in any type of system, the geometric algebra (GA) presented by Clifford in 1878, supported by the work of Hamilton (1843) and Grassmann (1844), and then recovered by Hestenes in the 1960’s [10, 11], is the one that fills the gaps detected in the algebra of complex numbers. Numerous studies have demonstrated the success of GA in disciplines such as relativistic physics, electromagnetism and computer vision [12, 13, 14]. In addition, the development of the theory of electrical power based on GA provides a new approach to solving power flow in electrical systems because of its flexibility and capability to represent the multi-component nature of power flow in sinusoidal and non-sinusoidal systems. Specifically, the studies of Menti [15], Castro-Núñez [16, 17, 18, 19], Montoya [20, 21], Castilla and Bravo [22, 23, 24], Lev-Ari [25] and Petroianu [26] demonstrate the capabilities of GA in the analysis of power systems. Based on this approach, a better understanding of power balances can be obtained and, more importantly, compliance with the conservation of energy principle is guaranteed, i.e., Tellegen’s theorem is satisfied.

As shown in previous studies [27, 18, 21], GA applied to sinusoidal, non-sinusoidal, linear and nonlinear circuits is a suitable technique to describe power flow in terms of the energy conservation principle. Thus far, the definition of geometric apparent power $M$ as the geometric product of current and voltage as follows:

$$ M = VI = P + CN = P + CN_d + CN_r $$

has been proposed as an efficient method to describe the net power flow in electrical circuits. In the above equation, $CN$ (geometric non-active power), $CN_d$ (geometric distorted power) and $CN_r$ (geometric reactive power) represent Clifford numbers. Additionally, the decomposition of the current into in-phase and quadrature components as follows:

$$ I = I_p + I_b $$

has contributed to the development of methods for quadrature RMS current compensation [20]. However, some shortcomings in the power formulation have been detected. In this article, several ideas are proposed to correct them along with new decompositions for the total current that takes into account the active current proposed by Fryze. These new ideas and proposals, are the basis for the formulation of the power theory based on GA (GAPoT).

The remainder of this paper is structured as follows. In Section 2, the theoretical basis of most well-known power theories is summarised; in Section 3, the basic concepts of GA are introduced; in Section 4, the proposed analysis of electrical circuits by GA is reviewed; in Section 5, the GAPoT theory is presented; and in Section 6, the main conclusions of this work are presented.

2 Brief review of the main Power Theories

The power theories that have historically influenced electrical engineers the most are briefly reviewed. The purpose is to put into context the contributions of each theory and, more importantly, show its point of weakness and thus highlight why it should be used carefully.

- **Budeanu’s theory.** This is the theory that may have had the greatest impact historically and which any electrical engineer knows and learned in college. It was formulated in 1927 in the frequency domain and establishes the following:
\[
S^2 = \|V\|^2 \|I\|^2 = P^2 + Q^2 + D^2
\]
\[
P = \sum_n V_n I_n \cos \varphi
\]
\[
Q = \sum_n V_n I_n \sin \varphi
\]
\[
D = \sqrt{S^2 - P^2 - Q^2}
\]

The primary issue is that there is no physical phenomenon associated with the reactive power \( Q \), as demonstrated in \([4, 28]\). Similarly, the power \( D \) does not represent a physical phenomenon; it is implicitly defined as a function of \( S \). It is also not possible to correctly compensate the system to minimise the amount of current consumed to maintain a constant \( P \).

- **Fryze’s power theory.** Formulated in 1931 in the time-domain, this theory introduces the important concept of active current, i.e., the minimum current needed to produce the power \( P \) demanded by the load.

\[
i(t) = i_a(t) + i_F(t)
\]
\[
S^2 = P^2 + Q_F^2
\]

Even though it contributes to the decomposition of active current \( i_a(t) \) and in quadrature \( i_F(t) \), the non-active power \( Q_F \) lacks physical meaning and cannot adequately compensate non-sinusoidal systems.

- **Shepherd and Zakikhani’s theory.** Formulated in 1972 in the frequency domain, this theory was the first to provide a reactive current \( i_r(t) \) definition according to the concepts of the voltage and the current in quadrature. The theory provides an effective reduction of the current in the presence of harmonics through an optimal compensator. Unfortunately, their theory does not include the concept of active power \( P \), and therefore, it lacks usability.

- **Akagi-Nabae Instantaneous Reactive Power (IRP) theory.** Introduced in 1984 and defined in the time domain, this theory asserts the possibility of effective current compensation using active elements based on power electronics. To do this, the theory applies the Clarke transform to switch from an \( a-b-c \) three-phase system to an \( \alpha-\beta-0 \) stationary system. This theory was studied by Czarnecki [3], where he showed critical inconsistencies in the compensation of nonsinusoidal circuits. Its scope, therefore, is limited to sinusoidal circuits under specific conditions.

- **Depenbrock’s FBD (Fryze-Buchholz-Depenbrock) theory.** A time-domain theory formulated by Depenbrock in 1993, it generalises the concepts of Fryze into three-phase systems, but it is still based on the non-conservative concept of apparent power \( S \), and therefore fails to satisfy Tellegen’s theorem.

- **Czarnecki’s CPC theory.** This theory was formulated by Czarnecki in 1984 in the frequency domain. According to this theory, the current is decomposed into various quadrature components as follows:

\[
i(t) = i_a(t) + i_r(t) + i_s(t) + i_G(t)
\]

where \( i_a(t) \) is Fryze’s current, \( i_r(t) \) is Shepherd’s current, \( i_s(t) \) is the scattered current caused by the fluctuation of the conductance of the load with frequency and \( i_G(t) \) is the harmonic current generated by the load. Therefore, the apparent power is as follows:

\[
S^2 = P^2 + D_s^2 + Q^2 + D_g^2
\]
The main contribution of the CPC theory is related to the identification of current \( i_s(t) \) and the physical meaning it gives to the decomposition of the currents. Unfortunately, this theory relies on the concept of apparent power \( S \); consequently, it cannot avoid the disadvantages inherent in that proposition. The magnitude of reactive power does not have a specific meaning; therefore, it is not possible to verify the balance of powers. Additionally, the CPC theory does not allow a complete balance of currents and powers in each branch of the circuit.

3 Basics of Geometric Algebra

The early days of GA date back to the XIX century with the studies conducted by Grassmann, Hamilton and particularly, Clifford. Despite the limited relevance it had at the time (attributed to the untimely death of Clifford), it is currently recognized as a unified language for physics and mathematics [11]. The essence of GA lies in the notion of an invertible (geometric) product that captures the geometric relation between two vectors, i.e., the relation between their modules and the angle they form [29]. Many studies [10, 13] have demonstrated that GA, when applied to physics and engineering problems, provides analysis tools far superior to those derived from traditional vector calculus as proposed by Gibbs. For example, complex number algebra, quaternions or even vectors have been proven to be members of GA subspaces [11]. A very interesting feature of GA is that the properties and operators are easily applicable to spaces with any number of dimensions.

The basic principles of GA derive from widely established vector concepts. For example, a vector \( \mathbf{a} = \alpha_1 \sigma_1 + \alpha_2 \sigma_2 \) (that has orientation, sense and magnitude) can be multiplied by another vector \( \mathbf{b} = \beta_1 \sigma_1 + \beta_2 \sigma_2 \) in various ways, such that the result has different meanings. Equation (9) defines the inner or scalar product, and the result is a scalar.

\[
\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| ||\mathbf{b}|| \cos \varphi = \sum \alpha_i \beta_i \tag{9}
\]

Equation (10) defines Grassmann’s or wedge product. This product differs from the traditional vector or outer product (as in Figure 1) primarily because the result is neither a scalar nor a vector, but a new object termed bivector. The bivector is a key concept in GA and does not exist in linear algebra or traditional vector calculus. GA demonstrates that the Gibbs vector product in 3D is simply the dual of the bivector.

\[
\mathbf{a} \wedge \mathbf{b} = ||\mathbf{a}|| ||\mathbf{b}|| \sin \varphi \, \sigma_1 \sigma_2 \tag{10}
\]

Similar to vectors, a bivector has orientation, sense and magnitude. Specifically, the area defined by vectors \( \mathbf{a} \) and \( \mathbf{b} \) is the geometric representation of the bivector (see Figure 2) while the oriented arc represents the sense. An essential property of the wedge product is that it is anticommutative, i.e., \( \mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a} \). Based on the above definitions, bivectors operate similar to vectors, i.e., bivectors can be added, multiplied and their inverse can also be derived. Similar to vectors, bivectors can be expressed as linear combinations of base vectors.
Figure 2: Representation of a bivector $a \wedge b$.

One of the key contributions of GA is the geometric product between two elements. Any geometric entity can be multiplied by another entity through the geometric product, and the result is a vector, bivector, trivector, or in general, a multivector.

$$ab = a \cdot b + a \wedge b$$  \hspace{1cm} (11)

The result of geometric multiplication is a linear combination of the inner or scalar product and the wedge product. If the values of vectors $a$ and $b$ are substituted in (11), we obtain the following:

$$A = ab = (A)_0 + (A)_2 = (\alpha_1\beta_1 + \alpha_2\beta_2) + (\alpha_1\beta_2 - \alpha_2\beta_1)\sigma_1\sigma_2$$  \hspace{1cm} (12)

where $(A)_0$ is the scalar part and $(A)_2$ is the bivector part.

Noninitiated readers of GA may consult either the final Appendix or the specific references [14, 12, 10].

## 4 Circuit analysis and power in Geometric Algebra

The study and analysis of AC electrical circuits in the frequency domain has traditionally been performed using complex number algebra as the fundamental analysis tool. Recently, more advanced techniques such as quaternions [30, 31, 32], have been proposed. However, the method that is causing a major impact in the analysis of electrical systems is the one based on GA because of its inherent robustness and flexibility in representing the multicomponent nature of the current and voltage [15, 1, 27] and its associated energy flow. The simultaneous handling of harmonics in nonsinusoidal and nonlinear environments has been properly demonstrated in the literature [23, 21]. New power terms that comply with the PCoE can be defined using GA. This is possible in none of the power theories formulated so far. Currently, the most comprehensive theory from a formal and mathematical point of view is the theory formulated by Castro-Núñez [27] and extended in his PhD thesis [1], which defines new power concepts such as the geometric reactive power and the degraded power. This thesis presents for the first time an unprecedented form of reactive power caused by the interaction of current and voltage harmonics of different frequency. A more comprehensive analysis is possible and conditions for optimum compensation of nonactive currents, i.e., currents that do not contribute to the active power $P$ in any situation of voltage and current distortion, can be established. The decomposition of the current into an in-phase term and a quadrature term enables the design of passive compensators that contribute to the quasi-optimal compensation.

### 4.1 Castro-Núñez proposal

The basics postulated by Castro-Núñez are defined below. These have been presented in numerous scientific publications and studied through examples scrutinised over time by other authors. First, the base transfor-
formation is presented [17], which is used to transform variables defined in the time domain to the geometric domain as follows:

\[
\begin{align*}
\varphi_{c1}(t) &= \sqrt{2} \cos \omega t \quad \leftrightarrow \quad \sigma_1 \\
\varphi_{s1}(t) &= \sqrt{2} \sin \omega t \quad \leftrightarrow \quad -\sigma_2 \\
\varphi_{c2}(t) &= \sqrt{2} \cos 2\omega t \quad \leftrightarrow \quad \sigma_2 \sigma_3 \\
\varphi_{s2}(t) &= \sqrt{2} \sin 2\omega t \quad \leftrightarrow \quad \sigma_1 \sigma_3 \\
\vdots \\
\varphi_{cn}(t) &= \sqrt{2} \cos n\omega t \quad \leftrightarrow \quad \bigwedge_{i=2}^{n+1} \sigma_i \\
\varphi_{sn}(t) &= \sqrt{2} \sin n\omega t \quad \leftrightarrow \quad \bigwedge_{i=1}^{n+1} \sigma_i \\
\end{align*}
\]  

(13)

where \( \bigwedge_n \sigma_i \) is the product of \( n \) vectors \( \sigma_i \). The subscript \( c \) indicates cosine and the subscript \( s \) indicates sine. For example, a voltage \( v(t) \)

\[
v(t) = \sqrt{2} \left[ 230 \cos \omega t + 20 \sin 4\omega t \right]
\]

(14)
can be transformed to the geometric domain as:

\[
V = \begin{bmatrix} 230 \sigma_1 + 20 \sigma_{1345} \end{bmatrix} 
\]

(15)

One of the significant contributions by Castro-Núñez is the definition of impedance and admittance in the geometric domain. To do this, Kirchhoff’s laws are set out in the time domain and then completely transferred to the geometric domain, such that a relation between voltage and current is found. For example, consider the circuit in Figure 3 and a voltage source \( v(t) = \sqrt{2} V \cos(\omega t) \). According to [14], a rotating vector \( n(t) \) can be represented in \( \mathcal{G}_2 \) domain as follows:

\[
n(t) = e^{-\frac{1}{2} \omega t \sigma_{12}} N e^{\frac{1}{2} \omega t \sigma_{12}} = R^\dagger N R
\]

(16)

where \( R \) is a rotor and \( N \) is a vector or geometric phasor. If the analysis equation is applied to the proposed circuit as follows:

\[
v(t) = Ri(t) + L \frac{di(t)}{dt}
\]

(17)

and is combined with (16), the result is

\[
e^{-\frac{1}{2} \omega t \sigma_{12}} V e^{\frac{1}{2} \omega t \sigma_{12}} = Re^{-\frac{1}{2} \omega t \sigma_{12}} I e^{\frac{1}{2} \omega t \sigma_{12}} + L \frac{di(e^{-\frac{1}{2} \omega t \sigma_{12}} I e^{\frac{1}{2} \omega t \sigma_{12}})}{dt}
\]

(18)
In the previous equation, we must first perform the derivative and choose a specific instant for the steady state, for example $t = 0$, so we get

$$V = IR + I \cdot L\sigma_{12}$$  \hspace{1cm} (19)$$

Observe the scalar product between the current $I$ and the term $L\sigma_{12}$. If the terms are reordered, the result is the geometric impedance expression as follows:

$$Z = VI^{-1} = R - L\omega\sigma_{12} = R + X_L\sigma_{12}$$  \hspace{1cm} (20)$$

where $X_L$ is the geometric inductive reactance. It should be noted that $\frac{V}{I}$ is ambiguous in GA, so it should be avoided. Instead, left or right multiplication by the inverse is the proper choice. Similarly, a geometric capacitive reactance can be obtained for a circuit with capacitors. In equation (20), the inverse of the current was multiplied from the right, which results in a negative geometric reactance. If the multiplication is performed from the left, then a positive reactance is obtained, although the author prefers the first method. Nevertheless, in terms of practical results, the choice of method has a negligible effect [19].

Admittance is defined as follows:

$$Y = Z^{-1} = \frac{Z^\dagger}{Z} = \frac{Z^\dagger}{\|Z\|^2} = G + B\sigma_{12}$$  \hspace{1cm} (21)$$

In general, the impedance of a load at the frequency of harmonic $n$ is as follows:

$$Z_n = R + \left(\frac{1}{n\omega C} - n\omega L\right)\sigma_{12}$$  \hspace{1cm} (22)$$

Therefore, any nonsinusoidal voltage, such as the following:

$$v(t) = \sum_{i=1}^{n} v_i(t) = D_1 \cos(\omega t) + E_1 \sin(\omega t) + \sum_{h=2}^{d} D_h \cos(h\omega t) + \sum_{h=2}^{k} E_h \sin(h\omega t)$$  \hspace{1cm} (23)$$

can be transformed to the geometric domain as the following:

$$V = D_1\sigma_1 - E_1\sigma_2 + \sum_{h=2}^{d} \left[ D_h \bigwedge_{i=1}^{h+1} \sigma_i \right] + \sum_{h=2}^{k} \left[ E_h \bigwedge_{i=1, i \neq 2}^{h+1} \sigma_i \right]$$  \hspace{1cm} (24)$$

Similarly, the resulting current for a load will be as follows:

$$I = G_1 D_1\sigma_1 - G_1 E_1\sigma_2 + \sum_{h=2}^{d} \left[ G_h D_h \bigwedge_{i=2}^{h+1} \sigma_i \right] + \sum_{h=2}^{k} \left[ G_h E_h \bigwedge_{i=1, i \neq 2}^{h+1} \sigma_i \right] -$$

$$- B_1 E_1\sigma_1 - B_1 D_1\sigma_2 + \sum_{h=2}^{d} \left[ B_h D_h \bigwedge_{i=1, i \neq 2}^{h+1} \sigma_i \right] - \sum_{h=2}^{k} \left[ B_h E_h \bigwedge_{i=2}^{h+1} \sigma_i \right]$$  \hspace{1cm} (25)$$

For each harmonic, the admittance is $Y_n = G_n + B_n\sigma_{12}$. In equation (25), the current has been decomposed into the following two components:

$$I = I_\parallel + I_\perp = I_g + I_b$$  \hspace{1cm} (26)$$

where $I_g$ is the in-phase current caused by the conductance $G_n$ of each harmonic and $I_b$ is the quadrature current caused by the susceptance $B_n$. 

Once the voltage and current are obtained, the geometric apparent power is defined as the product of both magnitudes as follows:

\[
M = VI = \langle M_g \rangle_0 + \sum_{i=1}^{n+1} \langle M_g \rangle_i + CN_{r(ps)} + CN_{r(hi)}
\]

where

- \( M_g \) is the parallel geometric apparent power
- \( M_b \) is the quadrature geometric apparent power
- \( P \) is the active power
- \( CN_d \) is the degraded power
- \( CN_r \) is the quadrature geometric power or reactive geometric power
- \( CN_{r(ps)} \) is the reactive geometric power due to voltage and current phase shift of same components
- \( CN_{r(hi)} \) is the reactive geometric power due to voltage and current cross products

In sinusoidal conditions, \([16, 17]\) demonstrates that equation (27) is reduced to the well-known equation \( S = P + jQ \), since \( CN_{r(hi)} = CN_d = 0 \), and \( CN_{r(ps)} = Q\alpha_12 \). Additionally, Castro-Núñez demonstrated the following properties of \( M \):

- \( \|M\| \neq \|V\|\|I\| \)
- \( \|M\| = \sqrt{P^2 + CN_{r(ps)}^2 + CN_{r(hi)}^2 + CN_d^2} \)
- \( M \) is a conservative quantity that takes into account the net direction of power flows in the branches of any circuit. The conservation of energy principle and Tellegen’s Theorem \([19]\) are both satisfied.

It should be noted that Castro-Núñez had to include a correction factor to adjust the balance of active power as follows:

\[
f = (-1)^{\frac{k(k-1)}{2}}
\]

With its current formulation, \( P = \sum P_i \) is not always satisfied because not all \( k \)-vectors square to +1. This is a serious shortcoming in this theory. Additionally, it does not include a method to handle the presence of interharmonics or subharmonics. Finally, the decomposition of currents into an in-phase component \( I_g \) and a quadrature component \( I_b \) has a limited scope since it does not provide a deeper insight to achieve minimal current with constant power \( P \). Therefore, only a partial compensation can be performed through passive elements.

5 GAPoT methodology

GAPoT tries to address several deficiencies in existing theories. From advanced systems with non-sinusoidal and nonlinear sources to simpler circuits made up of linear loads and sinusoidal sources, the different alternatives have failed to provide an accurate and detailed explanation about how energy flows or an interpretation in practical engineering terms.
GAPoT is firmly based on the principles of Castro-Núñez discussed above, but with relevant changes that reveal some baseline issues. The first and main overhaul, is related to the definition of geometric apparent power \( M = VI \). We propose in GAPoT a slightly but different new definition:

\[
M = VI^* \tag{29}
\]

This is necessary because if the reverse of the current is not performed, the result is: a) having to include \( f = (-1)^{\frac{(k-1)}{2}} \), an unnatural corrective term to calculate active power \( P \) and b) miscalculating the rest of nonactive power terms. Of course, this is one of the main drawbacks of Castro-Núñez theory.

The above can be verified with a very simple example by supposing that a voltage at a fundamental frequency \( V_1 = \alpha_1 \sigma_1 + \alpha_2 \sigma_2 \) (for simplicity, we consider \( \omega = 1 \)) is applied to a linear load, so a current \( I_1 = \beta_1 \sigma_1 + \beta_2 \sigma_2 \) is obtained. The product of both quantities is as follows:

\[
M_1 = V_1 I_1 = (\alpha_1 \beta_1 + \alpha_2 \beta_2) \sigma_{12} + (\alpha_1 \beta_2 - \alpha_2 \beta_1) \sigma_{12} \tag{30}
\]

which matches \( V_1 I_1^* \) because the reverse of a vector is the vector itself. However, supposing the same voltage at twice the frequency used above, i.e., \( \omega = 2 \), results in the following:

\[
V_2 = \alpha_1 \sigma_{13} + \alpha_2 \sigma_{23} \\
I_2 = \beta_1' \sigma_{13} + \beta_2' \sigma_{23}
\]

Remember that the impedance/admitance is different for \( \omega = 2 \), so the coefficients \( \beta_1 \) and \( \beta_2 \) changes to \( \beta_1' \) and \( \beta_2' \), respectively. The new power expression is:

\[
M_2 = V_2 I_2^* = -(\alpha_1 \beta_1' + \alpha_2 \beta_2') + (\alpha_1 \beta_2' - \alpha_2 \beta_1') \sigma_{12}
\]

which is clearly different to (30). Moreover, after the correction factor \( f \) proposed by Castro-Núñez is applied, the value of the geometric reactive/quadrature power remains different and thus incorrect. However, if we apply the new definition

\[
M_2 = V_2 I_2^* = (\alpha_1 \beta_1' + \alpha_2 \beta_2') + (\alpha_1 \beta_2' - \alpha_2 \beta_1') \sigma_{12}
\]

it now agrees with (30). The necessity of performing the reverse of the current to obtain the right value for the geometric apparent power, seems to be clear.

The second contribution of GAPoT is the addition of interharmonics and subharmonics in the transformation from time domain to geometric domain [33]. In addition to (13), noninteger multiples of fundamental (interharmonics) are also defined

\[
X_{c_{n,pk}} = \left( \bigwedge_{i=2}^{n+1} \sigma_i \right) \sigma_{(k+n+2)} \\
X_{s_{n,pk}} = \left( \bigwedge_{i=1, \ i \neq 2}^{n+1} \sigma_i \right) \sigma_{(k+n+2)} \tag{31}
\]

where \( p_k \) is the interharmonic \( k \) that exists between harmonic \( n \) and harmonic \( n + 1 \).

The third contribution is related to the decomposition of the total current and its allocation to physical phenomena that have engineering significance. The original decomposition of Castro-Núñez into in-phase current and quadrature current is performed according to equation (26) as follows:

\[
I_g = \sum_n G_n(V)_n \\
I_b = \sum_n B_n \sigma_{12}(V)_n
\]
However, this decomposition can be expanded to include other interesting terms that have been described in the scientific literature: active or Fryze’s current $I_a$, which is the minimum current necessary to obtain the active power $P$ of the load; and the harmonic current generated by a load $I_G$, which is the current that has frequency components that are not present in the supply. The active current is obtained using the concept of Fryze’s equivalent load, i.e., a load equivalent conductance $G_e$ that demands the same power $P$ as the original load when the same voltage $v(t)$ with RMS $\|V\|$ is applied as follows:

$$G_e = \frac{P}{\|V\|^2} \tag{32}$$

The active current can then be defined as follows:

$$I_a = G_e V \tag{33}$$

This current is already included in $I_g$, and therefore, it can be inferred that there is another additional current component as follows:

$$I_s = I_g - I_a \tag{34}$$

which coincides with the scattered current defined in the CPC theory proposed by Czarnecki. The term scattered current will be used for $I_s$ to avoid the introduction of new terms in addition to those already used in the existing literature. Once more, GA demonstrates its potential by naturally describing the basic components of electrical interest. The complete decomposition of the current then would be as follows:

$$I = I_a + I_s + I_b + I_G \tag{35}$$

with the following:

- $I_a$: minimum current for active power $P$ in the load
- $I_s$: current due to changes in conductance with frequency
- $I_b$: quadrature current with voltage
- $I_G$: harmonic current generated by the load

It can be readily demonstrated (not performed due to lack of space) that the four components of the current are orthogonal, in addition to the already well-known quadrature between $I_g$ and $I_b$ (25) as follows:

$$I_a \cdot I_s = 0$$
$$I_a \cdot I_b = 0$$
$$I_s \cdot I_b = 0$$
$$I_a \cdot I_G = 0$$
$$I_s \cdot I_G = 0$$
$$I_b \cdot I_G = 0$$

and therefore, the following is also satisfied:

$$\|I\|^2 = \|I_a\|^2 + \|I_s\|^2 + \|I_b\|^2 + \|I_G\|^2 \tag{36}$$

Once the decomposition of currents is presented, note that left multiplying the reverse of equation (35) by $V$, the geometric power equation is derived:

$$M = V I^T = M_a + M_s + M_b + M_G \tag{37}$$

which is totally unrelated to the following traditional apparent power formula:

$$\|V\|^2 \|I\|^2 = \|V\|^2 \|I_a\|^2 + \|V\|^2 \|I_s\|^2 + \|V\|^2 \|I_b\|^2 + \|V\|^2 \|I_G\|^2 \tag{38}$$
where the expression (36) multiplied by $\|V\|^2$ is used.

Finally, the power factor is defined based on the geometric apparent power as

$$p_f = \frac{P}{\|M\|}$$  \hspace{1cm} (39)

### 6 Examples

Different circuits will be solved to validate the proposed power theory. Additionally, the results will be compared with other theories to demonstrate the superiority of GAPoT. The Matlab environment has been chosen to facilitate the resolution of the proposed circuits. Specifically, the Clifford Multivector Toolbox [34] developed by Sangwine and Hitzer has been used.

#### 6.1 Sinusoidal case

The first example is a linear circuit supplied by a sinusoidal source. The aim is to demonstrate that simple circuits can be solved with the proposed theory. Once it’s accomplished, it is possible to move on to larger and more complex circuits.

The circuit in Figure 4 represents a system with linear RLC elements, an ideal voltage source $v(t) = 50\sqrt{2}\cos\omega t$ and an ideal current source $J(t) = 20\sqrt{2}\cos(\omega t + 90)$. KCL/KVL (mesh current method) and phasors in complex algebra are used to solve it:

$$\begin{pmatrix} 2 + j & j & 0 \\ j & 2 - j & 0 \\ -2j & -2 & -1 \end{pmatrix} \begin{pmatrix} \vec{I}_a \\ \vec{I}_b \\ \vec{U}_{\vec{J}} \end{pmatrix} = \begin{pmatrix} -40 \\ 50 + 40j \\ -40j + 40 \end{pmatrix}$$

If the unknown vector is solved, the result is as follows:

$$\begin{pmatrix} \vec{I}_a \\ \vec{I}_b \\ \vec{U}_{\vec{J}} \end{pmatrix} = \begin{pmatrix} -6.66 - 1.66j \\ 10 + 28.33j \\ -63.33 - 3.33j \end{pmatrix}$$

In Table 2, the full results for the current, the voltage and the apparent power are shown. The data in the table show how the balance of complex apparent powers is achieved with sinusoidal sources as expected.

If the same circuit is solved using GA, applying Kirchhoff’s laws results in the following:

$$\begin{pmatrix} 2 - \sigma_{12} & -\sigma_{12} & 0 \\ -\sigma_{12} & 2 + \sigma_{12} & 0 \\ 2\sigma_{12} & -2 & -1 \end{pmatrix} \begin{pmatrix} I_a \\ I_b \\ U_{\vec{J}} \end{pmatrix} = \begin{pmatrix} -40\sigma_1 \\ 50\sigma_1 + 40\sigma_2 \\ 40\sigma_1 - 40\sigma_2 \end{pmatrix}$$
The transformed voltage and current to the geometric domain are as follows:

$$\begin{array}{ccc}
I & \vec{V} & \vec{S} \\
R_1 & -6.67 - 1.67j & -13.33 - 3.33j & 94.47 \\
R_2 & 10.00 + 8.33j & 20.00 + 16.67j & 388.86 \\
C & 16.67 + 30.00j & 30.00 - 16.67j & -1177.90j \\
L & -6.67 - 21.67j & 43.33 - 13.33j & 1027.90j \\
\end{array}$$

To verify this, the circuit in Figure 3 is considered again with $R = 1\Omega$, $L = 1H$, but with a nonsinusoidal voltage $v(t) = \sqrt{2}[100\sin(\omega t - 45) + 30\sin(2\omega t + 30)]$. The transformed voltage and current to the geometric domain are as follows:

$$\begin{align*}
V &= -70.71\sigma_1 - 70.71\sigma_2 + 25.98\sigma_{13} + 15.00\sigma_{23} \\
I &= -42.43\sigma_1 + 14.14\sigma_2 + 5.06\sigma_{13} - 5.23\sigma_{23}
\end{align*}$$

| $R_1$  | $\sigma_1$ | $\sigma_2$ | $\sigma_{12}$ | $\sigma_{13}$ | $\sigma_{23}$ | $\sigma_{11}$ | $\sigma_{22}$ | $\sigma_{33}$ |
|-------|------------|------------|---------------|---------------|---------------|--------------|--------------|--------------|
| $R_2$ | $\sigma_1$ | $\sigma_2$ | $\sigma_{12}$ | $\sigma_{13}$ | $\sigma_{23}$ | $\sigma_{11}$ | $\sigma_{22}$ | $\sigma_{33}$ |
| $C$   | $\sigma_1$ | $\sigma_2$ | $\sigma_{12}$ | $\sigma_{13}$ | $\sigma_{23}$ | $\sigma_{11}$ | $\sigma_{22}$ | $\sigma_{33}$ |
| $L$   | $\sigma_1$ | $\sigma_2$ | $\sigma_{12}$ | $\sigma_{13}$ | $\sigma_{23}$ | $\sigma_{11}$ | $\sigma_{22}$ | $\sigma_{33}$ |

Table 2: Summary table for RLC circuit (phasor solution).

| $I$ | $V$ | $M$ |
|-----|-----|-----|
| $R_1$ | $\sigma_1$ | $\sigma_2$ | $\sigma_{12}$ | $\sigma_{13}$ | $\sigma_{23}$ | $\sigma_{11}$ | $\sigma_{22}$ | $\sigma_{33}$ |
| $R_2$ | $\sigma_1$ | $\sigma_2$ | $\sigma_{12}$ | $\sigma_{13}$ | $\sigma_{23}$ | $\sigma_{11}$ | $\sigma_{22}$ | $\sigma_{33}$ |
| $C$ | $\sigma_1$ | $\sigma_2$ | $\sigma_{12}$ | $\sigma_{13}$ | $\sigma_{23}$ | $\sigma_{11}$ | $\sigma_{22}$ | $\sigma_{33}$ |
| $L$ | $\sigma_1$ | $\sigma_2$ | $\sigma_{12}$ | $\sigma_{13}$ | $\sigma_{23}$ | $\sigma_{11}$ | $\sigma_{22}$ | $\sigma_{33}$ |

Table 3: Summary table for RLC circuit (GA solution).

Solving again for the unknown vector, the result is as follows:

$$\begin{pmatrix}
I_a \\
I_b \\
U_J
\end{pmatrix} = \begin{pmatrix}
2 - \sigma_{12} & -\sigma_{12} & 0 \\
-\sigma_{12} & 2 + \sigma_{12} & 0 \\
2\sigma_{12} & -2 & -1
\end{pmatrix}^{-1} \begin{pmatrix}
-40\sigma_1 \\
50\sigma_1 + 40\sigma_2 \\
40\sigma_1 - 40\sigma_2
\end{pmatrix} = \begin{pmatrix}
-6.66\sigma_1 - 1.66\sigma_2 \\
10\sigma_1 + 28.33\sigma_2 \\
-63.33\sigma_1 - 3.33\sigma_2
\end{pmatrix}$$

Table 3 shows the same results as Table 2 but transferred to the geometric domain. Compliance with PCoE is observed for the geometric power $M$.

### 6.2 Non sinusoidal case

One of the major drawbacks with existing power theories is their inability to properly handle nonsinusoidal systems because they cannot verify the principle of conservation of energy or Tellegen’s theorem for the apparent power $S$. GAPoT overcomes this problem through GA by using a conservative quantity, the geometric apparent power $M$. To verify this, the circuit in Figure 3 is considered again with $R = 1\Omega$, $L = 1H$, but with a nonsinusoidal voltage $v(t) = \sqrt{2}[100\sin(\omega t - 45) + 30\sin(2\omega t + 30)]$. The transformed voltage and current to the geometric domain are as follows:

$$\begin{align*}
V &= -70.71\sigma_1 - 70.71\sigma_2 + 25.98\sigma_{13} + 15.00\sigma_{23} \\
I &= -42.43\sigma_1 + 14.14\sigma_2 + 5.06\sigma_{13} - 5.23\sigma_{23}
\end{align*}$$
The power is obtained using the reverse of the current as follows:

\[ M = VI^\dagger = 2052.9 + 877.90\sigma_3 - 4211.76\sigma_{12} - 1731.32\sigma_{123} \]

\[ \|M\| = 5071.7 \]

In Table 4, a more detailed analysis of the power balance is shown. To stress the power concept defined by Castro-Núñez against GAPoT, the term \( M_{CN} = VI \) is also included. The data show how GAPoT properly adds the contribution of the first and second harmonic (\( M_1 \) and \( M_2 \)), while Castro-Núñez fails to subtract the two terms. This is a clear indication that power must be defined using the reversed current.

In Table 5, the analysis of currents considering the decomposition described in the equation (35) is shown. It can be verified that the sum of the active current \( I_a \) plus the scattered current \( I_s \) results in the in-phase current \( I_g \). The total current \( I \) is obtained by adding the quadrature or reactive current \( I_b \). It can also be shown that all components are in quadrature. GA makes it possible to simultaneously solve the system for all currents and all harmonic components, demonstrating that the principle of superposition is embedded in GA itself.

For comparison, the same circuit can be solved with a more distorted voltage, for example, with three harmonics as follows:

\[ v(t) = 100\sqrt{2}\sin(\omega t - 30) + 50\sqrt{2}\sin(2\omega t + 45) + 10\sqrt{2}\sin(3\omega t + 75) \]

The transformation of the voltage and current to the geometric domain is as follows:

\[ V = -50.00\sigma_1 - 86.60\sigma_2 + 35.35\sigma_{13} + 35.35\sigma_{23} + 2.59\sigma_{134} + 9.66\sigma_{234} \]
\[ I = -44.64\sigma_1 + 2.68\sigma_2 + 10.40\sigma_{13} - 6.24\sigma_{23} + 1.64\sigma_{134} + 0.16\sigma_{234} \]

Tables 6 and 7 show the decomposition of the current and power. The data demonstrate how the power balance is achieved and, therefore, the PCoE is also satisfied. In addition, a decomposition based on the engineering usefulness of the current components is presented.
### Table 6: Current summary in amperes for non sinusoidal case (3 harmonics)

| k-vector          | $\sigma_1$ | $\sigma_2$ | $\sigma_{13}$ | $\sigma_{23}$ | $\sigma_{134}$ | $\sigma_{234}$ | $\| \cdot \|$ |
|-------------------|------------|------------|----------------|---------------|----------------|---------------|----------------|
| $I_s$             | -1.4692    | -2.5447    | -3.9525        | -3.9525       | -0.3716        | -1.3870       | 6.4761         |
| $I_a$             | -8.5308    | -14.7758   | 6.0322         | 6.0322        | 0.4416         | 1.6480        | 19.1516        |
| $I_g$             | -10.0000   | -17.3205   | 2.0797         | 2.0797        | 0.0700         | 0.2611        | 20.2169        |
| $I_b$             | -34.6410   | 20.0000    | 8.3189         | -8.3189       | 1.5664         | -0.4197       | 41.7257        |
| $I$               | -44.6410   | 2.6795     | 10.3986        | -6.2392       | 1.6363         | -0.1586       | 46.3655        |

### Table 7: Power summary for non sinusoidal case (3 harmonics)

| k-vector          | $\sigma_0$ | $\sigma_3$ | $\sigma_4$ | $\sigma_{12}$ | $\sigma_{34}$ | $\sigma_{123}$ | $\sigma_{124}$ | $\sigma_{1234}$ | $\| \cdot \|$ |
|-------------------|------------|------------|------------|---------------|---------------|----------------|---------------|----------------|----------------|
| $M_s$             | -398.0023  | -110.5834  | -167.0791  | 182.6958      | -7.9491       | -29.5584       |               |               |                |
| $M_a$             | 2149.7615  | 1648.0240  | 147.7578   |               |               |               |               |               | 88.3173        |
| $M_g$             | 2149.7615  | 1250.0217  | 37.1744    | -167.0791     | 182.6958      | -7.9491        | 58.7589       |               |                |
| $M_b$             | 2149.7615  | 1463.1668  | 18.8914    | -4604.4515    | -21.5808      | -2885.5392     | -180.0527     | 288.4926       | 6033.70        |
| $M$               | 2149.7615  | 1463.1668  | 18.8914    | -4604.4515    | -21.5808      | -2885.5392     | -180.0527     | 288.4926       | 6033.70        |

The benefit of this proposal is evident when compensation scenarios are proposed. In this situation, certain specific targets are pursued, such as the minimisation of losses in power lines maintaining a constant active power flow $P$ or the elimination of harmonic currents generated by the load. In this situation, it is necessary to reduce the current to a minimum in such manner that it transfers the power to be converted into useful work. A compensator can be built with passive elements and active elements, such as controllable current sources that supply current or voltage sources that compensate distorted voltages. These elements can function independently or in a coordinated manner to form what is known as hybrid filters.

GAPoT makes it possible to identify which current components can be compensated, and through which elements, to maximise the power factor based on currents. Specifically, given the current defined in (35), the following can be shown:

- A passive compensator composed of reactors can only compensate $I_b$ by suitably choosing its susceptance $B$.
- $I_s$ and $I_G$ can only be compensated by an active compensator based on nonlinear elements.

The formulation to carry out the compensation using passive elements is discussed in detail by the authors in [20]. With the publication of GAPoT, new techniques can be developed for the definition of the reference current in active power filters and, consequently, build more advanced compensation strategies. Most importantly, the techniques will be based on a complete and consistent power theory.

### 6.3 Circuits with harmonic generating loads

The ability of GAPoT to analyse electrical circuits with nonlinear loads is demonstrated. The circuit in Figure 5, proposed by Czarnecki [35] and analysed with GA by Castro-Núñez in [27] is revisited; the data show that the use of the present definition of apparent power $S$ leads to discrepancies in the apparent and reactive power.

The voltage and current source are as follows:

$$v(t) = 100\sqrt{2}\sin \omega t$$

$$j_c(t) = 50\sqrt{2}\sin 2\omega t$$
which are transformed to the geometric domain as:

\[ V = -100 \sigma_2 \]
\[ J_c = 50 \sigma_{13} \]

Applying Kirchhoff’s laws, setting \( R_s = R_l = 1 \Omega \) and noting that there are no reactors, the following values are obtained:

\[ V_x = -80 \sigma_2 - 40 \sigma_{13} \]
\[ V_{Rs} = -20 \sigma_2 + 40 \sigma_{13} \]
\[ I = -20 \sigma_2 + 40 \sigma_{13} \]
\[ I_{Rl} = -20 \sigma_2 - 10 \sigma_{13} \]

Applying the decomposition of currents suggested by GAPoT, the total current \( I \) is as follows:

\[ I = I_a + I_s + I_b + I_G = \frac{-20 \sigma_2 + 40 \sigma_{13}}{I_a} \]

In Table 8, the apparent geometric power of each element and the one that flows to the load is shown. The data show that the sum of the geometric power generated by the voltage source and that generated by the current source correspond to the geometric power consumed by the passive elements. In this case, each source generates 2000 W of active power and 4000 VA of nonactive power, but of opposite sign, thereby canceling the effects. The power that flows from the source to the load is that same non-active power, as reflected correctly by the term \( M_x \). Note that there is no geometric reactive power term \((\sigma_{12})\). In addition, this verifies the determination of the flow direction of the net power, which in this case, is from the load to the source.

As described above, it is possible to verify that GAPoT is proven regardless of the value and order of the harmonics chosen. If, e.g., the value of the current source is \( j_c(t) = 50\sqrt{2}\sin 3\omega t \), then the new power
values are those shown in Table 9. The data show that the power balances continue to satisfy the PCoE, as expected. In this case, the flows change slightly to accommodate the change in current source.

7 Conclusions

This paper presents a completely new formulation of the power theory based on GA (GAPoT) for single-phase circuits with linear and nonlinear loads under sinusoidal and nonsinusoidal conditions. This new tool refines, corrects and improves the results obtained in previous studies. The definition of geometric apparent power as the product of voltage and reversed current ensures a correct determination of the flow of active power $P$ and nonactive power. Additionally, the optimal decomposition of the load current into meaningful engineering terms, enables the development of compensation strategies not easily performed previously. The use of GA makes it possible to analyse electrical circuits in a unified manner and in compliance with traditional laws that govern circuit theory. GAPoT reiterates the concerns raised by other renowned authors about continuing to use the concept of apparent power $S$, since it is an controversial term that lacks physical sense and that is not applicable in nonsinusoidal or nonlinear systems.

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Appendix

A General concepts

Given an ortho-normal base $\sigma = \{\sigma_1, \sigma_2, ..., \sigma_n\}$ for a vector space in $\mathbb{R}^n$, it is possible to define a new space called geometric algebra $\mathcal{G}_n$. In this new space we not only have vector bases $\sigma$, but there are also other bases that define the products among the vectors themselves. For example, in the case of $\mathbb{R}^3$ (Euclidean space), you have a base formed by 3 unitary vectors $\sigma = \{\sigma_1, \sigma_2, \sigma_3\}$. These unitary vectors fulfill that $\sigma_k \sigma_k = \sigma_k^2 = 1$ and $\sigma_i \wedge \sigma_j = 0$ for $i \neq j$. Taking advantage of this property and using the Grassmann wedge product $\sigma_i \wedge \sigma_j = -\sigma_j \wedge \sigma_i$, it can be verified that

\[
(\sigma_i \wedge \sigma_j)^2 = (\sigma_i \sigma_j)(\sigma_i \sigma_j) = \sigma_i(\sigma_j \sigma_i)\sigma_j = \sigma_i(-\sigma_j \sigma_i)\sigma_j = -\sigma_i \sigma_j \sigma_i \sigma_j = -(\sigma_i)^2(\sigma_j)^2 = -(1)(1) = -1
\]

$\sigma_i \sigma_j = \sigma_{ij}$ squares to $-1$, so we can conclude that we are dealing with a new element, namely a bivector.

In the same way, the product wedge of 3 vectors is called trivector, and in general, the product of $k$ vectors

| $k$-vector       | $\sigma_0$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\| \cdot \|$ |
|------------------|------------|------------|------------|------------|----------------|
| $M_u$            | 2000       | -4000      | 4472.1     |            |                |
| $M_{Rs}$         | 2000       | -1600      | 2561.2     |            |                |
| $M_{Rl}$         | 2000       | 1600       | 2561.2     |            |                |
| $M_{Je}$         | 2000       | 4000       | 4472.1     |            |                |
| $M_x$            | -2400      | 2400       |            |            |                |

Table 9: Power decomposition for non linear load with $j_c(t) = 50\sqrt{2}\sin 3\omega t$
is called $k$-vector. It is therefore concluded that the most general basis for $\mathcal{G}_3$ is

$$\{1, \sigma_1, \sigma_2, \sigma_3, \sigma_{12}, \sigma_{13}, \sigma_{23}, \sigma_{123}\} \quad (41)$$

Generally speaking, the elements of a geometric algebra are called multivectors ($\mathbf{A}$) and are expressed as a linear combination of their different bases.

$$\mathbf{A} = \langle \mathbf{A} \rangle_0 + \langle \mathbf{A} \rangle_1 + \langle \mathbf{A} \rangle_2 + \ldots + \langle \mathbf{A} \rangle_n = \sum_{k=0}^{n} \langle \mathbf{A} \rangle_k \quad (42)$$

where each $\langle \mathbf{A} \rangle_k$ is an element of grade $k$, representing scalars (grade 0), vectors (grade 1), bivectors (grade 2) or in general, $k$-vectors (grade $k$).

**B Geometric operations**

The geometric product is one of the fundamental contributions to the geometric algebra of Grassmann and Clifford. It is defined as the sum of the inner or scalar product and the wedge or Grassmann product. For the case of 2 vectors $\mathbf{a}$ and $\mathbf{b}$.

$$\mathbf{a}\mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \quad (43)$$

For the special case that the vectors are unitary bases $\sigma_i$ and $\sigma_j$ with $i \neq j$, the result obtained is a bivector.

$$\sigma_i\sigma_j = \sigma_i \cdot \sigma_j + \sigma_i \wedge \sigma_j = \sigma_i \wedge \sigma_j = \sigma_{ij} \quad (44)$$

In addition, since the wedge product is anticommutative, we also have

$$\sigma_{ij} = \sigma_i\sigma_j = \sigma_i \wedge \sigma_j = -\sigma_j \wedge \sigma_i = -\sigma_{ji} \quad (45)$$

Unlike vectors, whose square is 1, bivectors square to $-1$.

$$\sigma_{ij}\sigma_{ij} = \sigma_i\sigma_j\sigma_i\sigma_j = -\sigma_j\sigma_i\sigma_i\sigma_j = -\sigma_j\sigma_j = -1 \quad (46)$$

Finally, an important operation that applies to multivectors is detailed, namely reversion, which consists of

$$\mathbf{A}^\dagger = \sum_{k=0}^{n} \langle \mathbf{A}^\dagger \rangle_k = (-1)^{k(k-1)/2} \langle \mathbf{A} \rangle_k \quad (47)$$

The norm of a multivector $\|\mathbf{A}\|$ is always a scalar and can be obtained

$$\|\mathbf{A}\| = \sqrt{\langle \mathbf{A}^\dagger \mathbf{A} \rangle_0} = \sqrt{\langle \mathbf{A}^\dagger \mathbf{A} \rangle_0} = \sum_{k=0}^{n} \langle \langle \mathbf{A} \rangle_k \langle \mathbf{A}^\dagger \rangle_k \rangle_0 \quad (48)$$

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