Keller-Segel model with Logarithmic Interaction and nonlocal reaction term

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Abstract

We investigate the global existence and blow-up of solutions to the Keller-Segel model with nonlocal reaction term \( u (M_0 - \int_{\mathbb{R}^2} u \, dx) \) in dimension two. By introducing a transformation in terms of the total mass of the populations to deal with the lack of mass conservation, we exhibit that the qualitative behavior of solutions is decided by a critical value \( 8\pi \) for the growth parameter \( M_0 \) and the initial mass \( m_0 \). For general solutions, if both \( m_0 \) and \( M_0 \) are less than \( 8\pi \), solutions exist globally in time using the energy inequality, whereas there are finite time blow-up solutions for \( M_0 > 8\pi \) (It involves the case \( m_0 < 8\pi \)) with any initial data and \( M_0 < 8\pi < m_0 \) with small initial second moment. We also show the infinite time blow-up for the critical case \( M_0 = 8\pi \). Moreover, in the radial context, we show that if the initial data \( u_0(r) < \frac{m_0}{M_0 \left( r^{\lambda} + \lambda \right)} \) for some \( \lambda > 0 \), then all the radially symmetric solutions are vanishing in \( L^1_{loc}(\mathbb{R}^2) \) as \( t \to \infty \). If the initial data \( u_0(r) > \frac{m_0}{M_0 \left( r^{\lambda} + \lambda \right)} \) for some \( \lambda > 0 \), then there could exist a radially symmetric solution satisfying a mass concentration at the origin as \( t \to \infty \).

1 Introduction

In this work, we study the Keller-Segel model with logistic sources in dimension two

\[
\begin{align*}
  u_t &= \Delta u - \nabla \cdot (u \nabla c) + u (M_0 - \int_{\mathbb{R}^2} u \, dx), \quad x \in \mathbb{R}^2, \ t \geq 0, \\
  -\Delta c &= u, \quad x \in \mathbb{R}^2, \ t \geq 0, \\
  u(x, 0) &= u_0(x) \geq 0, \quad x \in \mathbb{R}^2.
\end{align*}
\] (1.1)

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Here the initial mass is defined as \( m_0 := \int_{\mathbb{R}^2} u_0(x)dx \). This model is developed to describe the biological phenomenon chemotaxis \([8, 10]\) by introducing nonlocal terms in the logistic growth factor. The first equation states the random (Brownian) diffusion of the cells with a bias directed by the chemoattractant concentration. The chemoattractant \( c \) is directly released by the cells, diffuses on the substrate and can be expressed

\[
c(x, t) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} u(y, t) \log |x - y| dy. \tag{1.2}
\]

The model at hand supposes that the cells production with logistic sources. Here \( M_0 \) can be viewed as the mass capacity \([12]\) and sometimes also called Malthusian parameter, induces an exponential growth for low density populations \([9]\). The nonlocal term \( \int_{\mathbb{R}^2} u dx \) describes the influence of the total mass of the species in the growth of the population, which is a competitive term limiting such growth \([9]\).

As pointed out in \([6, 12]\), in the absence of the reaction term \( u (M_0 - \int_{\mathbb{R}^2} u dx) \), \((1.1)\) has the property that the total mass is conserved \( \int_{\mathbb{R}^2} u_0(x)dx = \int_{\mathbb{R}^2} u(x, t)dx \) and the solution of

\[
\begin{cases}
  u_t = \Delta u - \nabla \cdot (u \nabla c), & x \in \mathbb{R}^2, \ t \geq 0, \\
  -\Delta c = u, & x \in \mathbb{R}^2, \ t \geq 0, \\
  u(x, 0) = u_0(x) \geq 0, & x \in \mathbb{R}^2.
\end{cases} \tag{1.3}
\]

exists globally when the initial mass \( m_0 < 8\pi \), while the solution becomes a singular measure for \( m_0 > 8\pi \). A natural problem is that does the nonlocal reaction change this effect?

To analyze this issue, we consider the total mass

\[
m(t) = \int_{\mathbb{R}^2} u(x, t)dx \tag{1.4}
\]

which satisfies

\[
\frac{dm(t)}{dt} = m(t) (M_0 - m(t)) \tag{1.5}
\]

and we obtain

\[
m(t) = \frac{M_0}{1 + C e^{-M_0 t}}. \tag{1.6}
\]

where

\[
C = \frac{M_0 - m_0}{m_0}. \tag{1.7}
\]
We find that the mass increases in time when $m_0 < M_0$ and versus when $m_0 > M_0$, in both cases it converges to $M_0$ as $t \to \infty$. The aim of this paper is to explore the influence of $M_0, m_0$ on the dynamics of solutions when the nonlocal reaction is present.

In order to put our system in perspective, we recall some related cases. Actually, chemotaxis models with local sources (where $u(M_0 - \int_{\mathbb{R}^2} u \, dx)$ is replaced by $a_0 - u(a_1 - a_2 u^\gamma)$) have been studied recently, which describes the situation where the influence of the nonlocal terms is neglected, and the global existence results have been obtained for different parameters $a_0, a_1, a_2$, see [7, 9, 11, 15] and the references therein. Logistic growth described by nonlocal terms has been used in the context of chemotaxis, which suggests a growth coefficient rate in a competitive system modelling cancer cells behavior. For instance, the nonlocal term in [13] is given in the form
\[
\mu_1 u(x) (1 - \int_\Omega k_1(x,y) u(y) \, dy),
\]
where $u$ denotes the cancer cells density. We refer the reader to [9, 13] for more details.

To illustrate our results, we observe that the striking feature of (1.1) is the absence of mass conservation which makes a difficult task to work with (1.1) directly. To overcome such difficulty, we employ a transformation
\[
\rho(x,t) = \frac{u(x,t)}{m(t)}
\]
where $m(t)$ is defined as (1.6), and it transforms system (1.1) into a diffusion-aggregation equation for $\rho(x,t)$ as follows
\[
\begin{cases}
\rho_t - \Delta \rho - m(t) \nabla \cdot (\rho \nabla w) = \rho, & \text{if } x \in \mathbb{R}^2, \ t \geq 0, \\
-\Delta w = \rho, & \text{if } x \in \mathbb{R}^2, \ t \geq 0, \\
\rho(x,0) = \frac{u_0(x)}{m_0} \geq 0, & \text{if } x \in \mathbb{R}^2.
\end{cases}
\]
(1.9)

Here we should mention that (1.9) admits mass conservation $\int_{\mathbb{R}^2} \rho(x,t) \, dx = \int_{\mathbb{R}^2} \rho(x,0) \, dx = 1$ via the transformation, the price we pay is that the coefficient in front of the concentration is $m(t)$ instead of a constant which also brings barriers to mathematical analysis. We remark that for the case that the concentration with constant sensitivity $\chi$ (without loss of generality, we impose $\chi = 1$), there have been a large amount of results [2, 3, 12] and there is a threshold $8\pi$ on the initial mass separating global existence and finite time blow-up as we mentioned before. Although we can’t take advantage of the initial mass to determine the dynamical behaviors of (1.9) due to $\int_{\mathbb{R}^2} \rho(x,0) \, dx = 1$, this fact still stimulates us to employ the mass conservation of (1.9) to establish global existence and blow-up to system (1.1). Precisely, the main results of the general solutions are stated as follows in connection with $M_0, m_0$ and $8\pi$.

- Global existence: for $m_0 < M_0 < 8\pi$, there exists a weak solution globally in time with bounded initial second moment. Furthermore, the weak solution satisfies the energy inequality
In addition, for $M_0 < m_0 < 8\pi$, all solutions of (1.1) exist globally by the comparison principle.

• $m_0 < M_0 = 8\pi$: solutions exist globally in time, see Theorem 2.3. For $M_0 = 8\pi$, every stationary solution uniquely assumes a radially symmetric form in $\mathbb{R}^2$ up to translation and has infinite second moment. This result provides that the solution blows up as a delta dirac at the center of mass as $t \to \infty$ with finite second moment at any time. We will comment further on these issues in Section 3.

• Finite time blow-up: for $M_0 > 8\pi$, the weak solution blows up at finite time, see Theorem 2.6. For $M_0 < 8\pi < m_0$, if the initial second moment is less than a constant depending on $M_0, m_0$, then there exist solutions blow up at finite time, see Theorem 2.7.

• For $M_0 < m_0 = 8\pi$, we can infer from [2] that solutions might have infinite time blow-up by the comparison principle. However, we can’t exclude the possibility that solutions may be global in time. For $M_0 = 8\pi < m_0$, both global existence and finite time blow-up can occur, although this is still an open question.

Let’s emphasise that in the radial context, there are steady states to (1.1) only for $M_0 = 8\pi$ given by a one-parameter family $U_s(r) = \frac{8\lambda}{(r^2 + \lambda)^2}$ with $\lambda > 0$. The stationary solutions play a critical role on the initial data separating global existence and blow-up. If the initial data $u_0(r) < \frac{m_0}{M_0} \frac{8\lambda}{(r^2 + \lambda)^2}$, then all the radially symmetric solutions are vanishing in $L^1_{loc}(\mathbb{R}^2)$ as $t \to \infty$. If the initial data $u_0(r) > \frac{m_0}{M_0} \frac{8\lambda}{(r^2 + \lambda)^2}$, then there could exist a radially symmetric solution satisfying a mass concentration at the origin as $t \to \infty$. See Section 4.2.1 and Section 4.2.2 for more details.

The results are organized as follows. Section 2 shows the global existence and the finite time blow-up of the weak solution to (1.1). Section 2.1 detects the global existence of solutions with the help of the energy inequality for $M_0 < 8\pi$. The finite time blow-up is considered in Section 2.2 with bounded initial second moment for $M_0 > 8\pi$. Section 3 explores the infinite time blow-up when $M_0 = 8\pi$. Section 4 is devoted to the global existence and the mass concentration at the origin for the radially symmetric solutions.

2 Global existence and finite time blow-up for general solutions

In this section, we prove the global well-posedness for solutions with small $M_0$ and blow-up for large $M_0$. Here an energy inequality (which is derived to prove global existence) introduces a threshold on $M_0$, below which the solution exists globally and above which the finite time blow-up occurs.
with the aid of the second moment. In order to analyse more precisely, we use the usual definition of solutions in distribution sense as in [12].

**Definition 2.1.** Let $u_0(x) \geq 0$ be the initial data satisfying $\int_{\mathbb{R}^2} u_0(x) |\log u_0(x)| dx + \int_{\mathbb{R}^2} |x|^2 u_0(x) dx < \infty$ and $T \in (0, \infty)$. $u$ is a weak solution to system (1.1) if it satisfies:

(i) Regularity:

\[
\begin{align*}
&u \in L^\infty (0, T; L^1_+ \cap L^p (\mathbb{R}^2)) , \text{ for any } 1 \leq p \leq \infty, \\
&\partial_t u \in L^2 (0, T; H^{-1} (\mathbb{R}^2)) , \\
&w^r \in L^2 (0, T; H^1 (\mathbb{R}^2)) \text{ for any } r \geq 1/2.
\end{align*}
\]

(ii) For any $\psi \in C_0^\infty (\mathbb{R}^2)$ and any $0 < t < \infty$,

\[
\begin{align*}
&\int_{\mathbb{R}^2} \psi u(\cdot, t) dx - \int_{\mathbb{R}^2} \psi u_0(x) dx \\
&= \int_0^T \int_{\mathbb{R}^2} \Delta \psi u dx dt - \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{[\nabla \psi(x) - \nabla \psi(y)] \cdot (x - y) u(x, t) u(y, t) dxdydt \\
&+ \int_0^T \int_{\mathbb{R}^2} u(\cdot, t) \psi(x) dx \left( M_0 - \int_{\mathbb{R}^2} u(\cdot, t) dx \right) dt. (2.1)
\end{align*}
\]

2.1 Global existence for $M_0 \leq 8\pi$

In this subsection, we will show the global existence of a weak solution for both cases $M_0 = 8\pi$ and $M_0 < 8\pi$. Firstly we recall the logarithmic HLS inequality which is prepared to the estimates of the weak solution.

**Lemma 2.2 ([12]).** Let $f \geq 0 \in L^1 (\mathbb{R}^2)$ such that $f \log f \in L^1 (\mathbb{R}^2)$. Set $\int_{\mathbb{R}^2} f dx = M$, then

\[
\int_{\mathbb{R}^2} f \log f dx + \frac{2}{M} \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log |x - y| dxdy \geq M \left( \log M - 1 - \log \pi \right). (2.2)
\]

Now we show the main results for $M_0 \leq 8\pi$.

**Theorem 2.3** (Global existence for $M_0 \leq 8\pi$). Assume $\int_{\mathbb{R}^2} |x|^2 u_0(x) dx < \infty$ and $\int_{\mathbb{R}^2} u_0(x) |\log u_0(x)| dx < \infty$. When the initial data satisfies $u_0 \in L^1_+ \cap L^\infty (\mathbb{R}^2)$ and

\[
\int_{\mathbb{R}^2} u_0(x) dx < M_0 \leq 8\pi,
\]

2
then there exists a global weak solution to (1.1) satisfying that for any $0 < t < \infty$
\[
\int_{\mathbb{R}^2} u(\cdot, t) |\log u(\cdot, t)|\, dx + \int_{\mathbb{R}^2} u(\cdot, t)|x|^2\, dx \leq C \left( e^t, \int_{\mathbb{R}^2} u_0(x)|\log u_0(x)|\, dx, \int_{\mathbb{R}^2} u_0(x)|x|^2\, dx \right),
\]
where
\[
\|u(\cdot, t)\|_{L^p(\mathbb{R}^2)} \leq C \left( e^t, \|u_0\|_{L^p(\mathbb{R}^2)} \right), \quad 1 \leq p \leq \infty.
\]
Furthermore, the weak solution satisfies the entropy dissipation
\[
F[u(\cdot, t)] + \int_0^t \int_{\mathbb{R}^2} \frac{u(x, s)}{m(s)} |\nabla \log u(x, s) - \nabla c(x, s)|^2\, dx
ds + \int_0^t \frac{M_0 - m(s)}{2m(s)} \int_{\mathbb{R}^2} u(x, s)c(x, s)\, dx\, ds \leq F[u_0(x)],
\]
where the free energy
\[
F(u) = \frac{1}{m(t)} \int_{\mathbb{R}^2} u \log u\, dx - \frac{1}{2m(t)} \int_{\mathbb{R}^2} u c\, dx - \log m(t)
\]
and $m(t) = \frac{M_0}{1 + \frac{M_0 - m_0}{M_0 - M_0^*} \cdot e^{-M_0^*t}}$.

**Proof.** We decompose the proof into two parts. In Steps 1-8, a regularized equation is constructed and a priori estimates are established to obtain the global existence of a weak solution to (1.1). Furthermore, Steps 9-10 give the second moment and the energy inequality of the weak solution.

Following the method of [14], we take a cut-off function $0 \leq \psi_1(x) \leq 1$
\[
\psi_1(x) = \begin{cases} 
1 & \text{if } |x| \leq 1, \\
0 & \text{if } |x| \geq 2,
\end{cases}
\]
where $\psi_1(x) \in C^\infty_0(\mathbb{R}^d)$. Define
\[
\psi_R(x) := \psi_1(x/R),
\]
as $R \to \infty$, $\psi_R \to 1$, then there exist constants $C_1, C_2$ such that $|\nabla \psi_R(x)| \leq \frac{C_1}{R}$, $|\Delta \psi_R(x)| \leq \frac{C_2}{R^2}$. This cut-off function will be used to derive the existence of the weak solution.

**Step 1 (Approximated problem)** In order to show the existence of a weak solution, we firstly consider the regularized problem for $\varepsilon > 0$
\[
\begin{align*}
\partial_t u_\varepsilon &= \Delta u_\varepsilon - \nabla \cdot (u_\varepsilon c_\varepsilon) + u_\varepsilon \left( M_0 - \int_{\mathbb{R}^2} u_\varepsilon\, dx \right), \quad x \in \mathbb{R}^2, t \geq 0, \\
-\Delta c_\varepsilon &= J_\varepsilon * u_\varepsilon, \quad x \in \mathbb{R}^2, t \geq 0, \\
u_\varepsilon(x, 0) &= u_0(x), \quad x \in \mathbb{R}^2.
\end{align*}
\]
Here $J_\varepsilon$ is the regularizing kernel

$$J_\varepsilon(x) = \frac{1}{\pi} \frac{\varepsilon^2}{(|x|^2 + \varepsilon^2)^2}$$  \hspace{1cm} (2.10)

with $\int_{\mathbb{R}^2} J_\varepsilon dx = 1$. Simple computations show that

$$c_\varepsilon(x,t) = -\frac{1}{4\pi} \int_{\mathbb{R}^2} \log \left( |x - y|^2 + \varepsilon^2 \right) u_\varepsilon(y,t) dy.$$  \hspace{1cm} (2.11)

The regularized initial data $u_{0\varepsilon} \in C^\infty(\mathbb{R}^2)$ is a sequence of approximation for $u_0(x)$ and satisfies

$$\begin{cases}
\|u_{0\varepsilon}\|_{L^p(\mathbb{R}^2)} \leq \|u_0\|_{L^p(\mathbb{R}^2)} \text{ for any } 1 \leq p \leq \infty, \\
u_{0\varepsilon}(x) \rightarrow u_0(x) \text{ in } L^q(\mathbb{R}^2) \text{ for } 1 \leq q < \infty, \\
\int_{\mathbb{R}^2} |x|^2 u_{0\varepsilon}(x) dx \rightarrow \int_{\mathbb{R}^2} |x|^2 u_0(x) dx \text{ as } \varepsilon \rightarrow 0, \\
\int_{\mathbb{R}^2} u_{0\varepsilon} \log u_{0\varepsilon} dx \rightarrow \int_{\mathbb{R}^2} u_0 \log u_0 dx \text{ as } \varepsilon \rightarrow 0. 
\end{cases}$$  \hspace{1cm} (2.12)

Then following the standard parabolic theory, the system on $(u_\varepsilon, c_\varepsilon)$ admits a unique smooth solution with fast decay in space, when the initial data is regularized and truncated. Moreover, there exists $T > 0$ such that for $0 < t < T$ and any $1 \leq r \leq \infty$

$$u_\varepsilon \in L^\infty(0, T; L^r(\mathbb{R}^2)).$$

**Step 2** (Global $L^1$ norm of $u_\varepsilon$ and the boundedness of the second moment) In this step, we will provide some a priori estimates for $u_\varepsilon$. Firstly, multiplying (2.8) with any test function $\psi_R(x)$ gives

$$\begin{align*}
\frac{d}{dt} \int_{\mathbb{R}^2} u_\varepsilon(\cdot,t) \psi_R(x) dx &= \int_{\mathbb{R}^2} u_\varepsilon \Delta \psi_R(x) dx - \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{[\nabla \psi_R(x) - \nabla \psi_R(y)] \cdot (x-y)}{|x-y|^2 + \varepsilon^2} u_\varepsilon(x,t) u_\varepsilon(y,t) dx dy \\
&\quad + \int_{\mathbb{R}^2} u_\varepsilon \psi_R(x) dx \left( M_0 - \int_{\mathbb{R}^2} u_\varepsilon dx \right) .
\end{align*}$$  \hspace{1cm} (2.13)

Letting $\psi_R(x)$ be defined as in (2.8), then

$$\left| \int_{\mathbb{R}^2} u_\varepsilon \Delta \psi_R(x) dx \right| \leq \frac{C}{R^2} \int_{\mathbb{R}^2} u_\varepsilon dx,$$

$$\left| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{[\nabla \psi_R(x) - \nabla \psi_R(y)] \cdot (x-y)}{|x-y|^2 + \varepsilon^2} u_\varepsilon(x,t) u_\varepsilon(y,t) dx dy \right| \leq \frac{C}{R^2} \left( \int_{\mathbb{R}^2} u_\varepsilon dx \right)^2 .$$

Passing to the limit $R \rightarrow \infty$ arrives at

$$\frac{d}{dt} \int_{\mathbb{R}^2} u_\varepsilon dx = \int_{\mathbb{R}^2} u_\varepsilon dx \left( M_0 - \int_{\mathbb{R}^2} u_\varepsilon dx \right)$$  \hspace{1cm} (2.14)
and we compute that for any $t > 0$
\[ m(t) := \int_{\mathbb{R}^2} u_\varepsilon dx = \frac{M_0}{1 + \frac{M_0 - m_0}{m_0} e^{-M_0 t}}. \tag{2.15} \]

Now we will prove that the second moment is bounded in time provided the bounded initial second moment $\int_{\mathbb{R}^2} u_\varepsilon(x, 0) dx$. Consider a test function $\psi_R(x) \in C_0^\infty(\mathbb{R}^2)$ that grows nicely to $|x|^2$ as $R \to \infty$. We can infer from (2.13) that $\int_{\mathbb{R}^2} u_\varepsilon(\cdot, t)\psi_R(x) dx$ is uniformly bounded, thus we may pass to the limit using the Lebesgue monotone convergence theorem to obtain
\[
\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 u_\varepsilon dx = 4 \int_{\mathbb{R}^2} u_\varepsilon dx - \frac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|x - y|^2}{|x - y|^2 + \varepsilon^2} u_\varepsilon(x, t) u_\varepsilon(y, t) dxdy \\
+ \int_{\mathbb{R}^2} |x|^2 u_\varepsilon dx \left( M_0 - \int_{\mathbb{R}^2} u_\varepsilon dx \right) \\
\leq 4 \int_{\mathbb{R}^2} u_\varepsilon dx + M_0 \int_{\mathbb{R}^2} |x|^2 u_\varepsilon dx.
\]
Then one gets by integrating from $0$ to $t$ in time
\[ \int_{\mathbb{R}^2} |x|^2 u_\varepsilon dx \leq C \left( e^t, \int_{\mathbb{R}^2} |x|^2 u_\varepsilon(x, 0) dx \right). \tag{2.16} \]

**Step 3** (Boundedness of $\int_{\mathbb{R}^2} u_\varepsilon \log u_\varepsilon |dx| / M_0 \leq 8\pi$) Recalling (2.15) we shall use the notation
\[ F(u_\varepsilon) = \int_{\mathbb{R}^2} \frac{u_\varepsilon}{m(t)} \log \frac{u_\varepsilon}{m(t)} dx - \frac{1}{2} \int_{\mathbb{R}^2} \frac{u_\varepsilon}{m(t)} c_\varepsilon dx. \tag{2.17} \]

Then a straightforward computation shows that
\[ \frac{d}{dt} F(u_\varepsilon) = - \int_{\mathbb{R}^2} \frac{u_\varepsilon}{m(t)} \left| \nabla \left( \log \frac{u_\varepsilon}{m(t)} - c_\varepsilon \right) \right|^2 dx - \frac{M_0 - m(t)}{2m(t)} \int_{\mathbb{R}^2} u_\varepsilon c_\varepsilon dx. \tag{2.18} \]
Thus for $m_0 < M_0$, differentiating it with respect to $t$ yields
\[ F(u_\varepsilon) \leq F(u_{0\varepsilon}). \tag{2.19} \]
It can be checked that
\[ \int_{\mathbb{R}^2} u_\varepsilon \log u_\varepsilon dx - \frac{1}{2} \int_{\mathbb{R}^2} u_\varepsilon c_\varepsilon dx \leq m(t) F(u_{0\varepsilon}) + m(t) \log m(t). \tag{2.20} \]
Hence from (2.2), we obtain the inequality
\[
\int_{\mathbb{R}^2} u_\varepsilon \log u_\varepsilon dx \\
\leq m(t) F(u_{0\varepsilon}) + m(t) \log m(t) - \frac{1}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} u_\varepsilon(x, t) u_\varepsilon(y, t) \log(|x - y|^2 + \varepsilon^2) dxdy \\
\leq m(t) F(u_{0\varepsilon}) + m(t) \log m(t) - \frac{1}{4\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} u_\varepsilon(x, t) u_\varepsilon(y, t) \log |x - y| dxdy \\
\leq m(t) F(u_{0\varepsilon}) + m(t) \log m(t) - \frac{m^2(t)}{8\pi} (\log m(t) - 1 - \log \pi) + \frac{m(t)}{8\pi} \int_{\mathbb{R}^2} u_\varepsilon \log u_\varepsilon dx, \tag{2.21} \]
For \( m(t) < M_0 \leq 8\pi \), it follows from (2.15) that \( \int_{\mathbb{R}^2} u_\varepsilon \log u_\varepsilon \, dx \) can be estimated respectively

\[
\int_{\mathbb{R}^2} u_\varepsilon \log u_\varepsilon \, dx \leq \begin{cases} 
\frac{8\pi M_0}{m_0(8\pi - M_0)} C \left( F(u_0\varepsilon), M_0, m_0 \right), & \text{for } M_0 < 8\pi, \\
\frac{8\pi (m_0e^{M_0} + M_0 - m_0)}{8\pi (M_0 - m_0)} C \left( F(u_0\varepsilon), M_0, m_0 \right), & \text{for } M_0 = 8\pi.
\end{cases}
\] (2.22)

On the other hand, the following holds

\[
1_{\{\varphi(x) \leq 1\}} \varphi(x) \log \varphi(x) = 1_{\{\varphi(x) < e^{-2|x|^2}\}} \varphi(x) \log \frac{1}{\varphi(x)} + 1_{\{e^{-2|x|^2} \leq \varphi(x) \leq 1\}} \varphi(x) \log \frac{1}{\varphi(x)} \\
\leq 1_{\{\varphi(x) < e^{-2|x|^2}\}} \sqrt{\varphi(x)} + 1_{\{e^{-2|x|^2} \leq \varphi(x) \leq 1\}} 2\varphi(x)|x|^2 \\
\leq e^{-|x|^2} + 2\varphi(x)|x|^2.
\] (2.24)

Hence combining (2.22) with (2.16) it gives our result for this step

\[
\int_{\mathbb{R}^2} u_\varepsilon |\log u_\varepsilon| \, dx = \int_{\mathbb{R}^2} u_\varepsilon \log u_\varepsilon \, dx + 2 \int_{\mathbb{R}^2} 1_{\{u_\varepsilon \leq 1\}} u_\varepsilon |\log u_\varepsilon| \, dx \\
\leq \int_{\mathbb{R}^2} u_\varepsilon \log u_\varepsilon \, dx + 2 \int_{\mathbb{R}^2} e^{-|x|^2} \, dx + 4 \int_{\mathbb{R}^2} |x|^2 u_\varepsilon \, dx \\
\leq C_1 + C_2(e^t).
\] (2.25)

**Step 4** (Equi-integrability in \( L^p \) norms for \( 1 < p < \infty \)) Testing (2.9) with \( p(u_\varepsilon - K)_+^{p-1} \) we obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^2} (u_\varepsilon - K)^p_+ \, dx + \frac{4(p-1)}{p} \int_{\mathbb{R}^2} |\nabla (u_\varepsilon - K)_+^{p/2}|^2 \, dx \\
= (p-1) \int_{\mathbb{R}^2} J_\varepsilon * u_\varepsilon(u_\varepsilon - K)^p_+ \, dx + Kp \int_{\mathbb{R}^2} J_\varepsilon * u_\varepsilon(u_\varepsilon - K)_+^{p-1} \, dx + p \int_{\mathbb{R}^2} (u_\varepsilon - K)_+^{p-1} \, dx \\
= (p-1) \int_{\mathbb{R}^2} J_\varepsilon * (u_\varepsilon - K + K)(u_\varepsilon - K)_+^{p-1} \, dx + Kp \int_{\mathbb{R}^2} J_\varepsilon * (u_\varepsilon - K + K)_+^{p-1} \, dx \\
\quad + p \int_{\mathbb{R}^2} (u_\varepsilon - K)_+^{p-1} \, dx + K \int_{\mathbb{R}^2} (u_\varepsilon - K)_+^{p-1} \, dx + 2K^2 \int_{\mathbb{R}^2} (u_\varepsilon - K)_+^{p-1} \, dx \\
\quad + M_0 p \int_{\mathbb{R}^2} (u_\varepsilon - K)_+^{p-1} \, dx + KpM_0 \int_{\mathbb{R}^2} (u_\varepsilon - K)_+^{p-1} \, dx.
\] (2.26)

We use GNS inequality to get

\[
\int_{\mathbb{R}^2} (u_\varepsilon - K)_+^{p+1} \, dx \leq C(p) \int_{\mathbb{R}^2} |\nabla (u_\varepsilon - K)_+^{p/2}|^2 \, dx \int_{\mathbb{R}^2} (u_\varepsilon - K)_+ \, dx.
\] (2.27)
By (2.28) one has

$$\int_{u_\varepsilon > K} (u_\varepsilon - K)_+^p dx \leq \int_{u_\varepsilon > K} u_\varepsilon dx \leq \frac{\int_{u_\varepsilon > K} u_\varepsilon |\log u_\varepsilon| dx}{\log K} \leq \frac{\int_{\mathbb{R}^2} u_\varepsilon |\log u_\varepsilon| dx}{\log K} \leq \frac{C(t)}{\log K}. \quad (2.28)$$

Moreover, \( \int_{\mathbb{R}^2} (u_\varepsilon - K)_+^{p-1} dx \) can be estimated as

$$\int_{\mathbb{R}^2} (u_\varepsilon - K)_+^{p-1} dx = \int_{K \leq u_\varepsilon \leq K+1} (u_\varepsilon - K)_+^{p-1} dx + \int_{u_\varepsilon > K+1} (u_\varepsilon - K)_+^{p-1} dx \leq \int_{K \leq u_\varepsilon \leq K+1} 1 dx + \int_{u_\varepsilon > K+1} (u_\varepsilon - K)_+^{p-1} dx \leq \frac{\int_{\mathbb{R}^2} u_\varepsilon dx}{K} + \int_{\mathbb{R}^2} (u_\varepsilon - K)_+^p dx. \quad (2.29)$$

Collecting (2.27), (2.28) and (2.29) we can go further from (2.26) that

$$\frac{d}{dt} \int_{\mathbb{R}^2} (u_\varepsilon - K)_+^p dx + \frac{4(p-1)}{p} \int_{\mathbb{R}^2} |\nabla (u_\varepsilon - K)_+^{p/2}|^2 dx \leq C(p) \frac{C(t)}{\log K} \int_{\mathbb{R}^2} |\nabla (u_\varepsilon - K)_+^{p/2}|^2 dx + C(K,p) \int_{\mathbb{R}^2} u_\varepsilon dx \leq C(K,p) \int_{\mathbb{R}^2} (u_\varepsilon - K)_+^p dx. \quad (2.30)$$

Choosing \( K \) large enough such that \( \frac{C(p)C(t)}{\log K} < \frac{4(p-1)}{p} \), we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^2} (u_\varepsilon - K)_+^p dx + C(p,K) \int_{\mathbb{R}^2} |\nabla (u_\varepsilon - K)_+^{p/2}|^2 dx \leq C_1 + C_2 \int_{\mathbb{R}^2} (u_\varepsilon - K)_+^p dx \quad (2.31)$$

and subsequently for any \( 0 < t < T \)

$$\int_{\mathbb{R}^2} (u_\varepsilon - K)_+^p dx \leq C(t) \int_{\mathbb{R}^2} u_\varepsilon^p dx. \quad (2.32)$$

**Step 5** (Boundedness of \( L^p \) norm for \( 1 < p \leq \infty \)) Now we can go further to prove that \( \int_{\mathbb{R}^2} u_\varepsilon^p dx \) is bounded as follows.

$$\int_{\mathbb{R}^2} u_\varepsilon^p dx = \int_{u_\varepsilon \leq K} u_\varepsilon^p dx + \int_{u_\varepsilon > K} u_\varepsilon^p dx \leq K^{p-1} \int_{\mathbb{R}^2} u_\varepsilon dx + \int_{u_\varepsilon > K} (u_\varepsilon - K)_+^{p-1} u_\varepsilon dx \leq K^{p-1} M_0 + \max(2^{p-2}, 1) \left( \int_{u_\varepsilon > K} (u_\varepsilon - K)_+^{p-1} u_\varepsilon dx + K^{p-1} M_0 \right) \leq K^{p-1} M_0 + \max(2^{p-2}, 1) \left( K \left( \int_{\mathbb{R}^2} u_\varepsilon - K)_+^{p-1} dx + \int_{\mathbb{R}^2} (u_\varepsilon - K)_+^p dx + K^{p-1} M_0 \right) \right) \leq K^{p-1} M_0 + \max(2^{p-2}, 1) \left( \int_{\mathbb{R}^2} u_\varepsilon dx + K \left( \int_{\mathbb{R}^2} u_\varepsilon - K)_+^p dx + \int_{\mathbb{R}^2} (u_\varepsilon - K)_+^p dx + K^{p-1} M_0 \right) \right),$$
where the last line is given by (2.29). Therefore, (2.32) guarantees that for any $0 < t < T$

$$
\int_{\mathbb{R}^2} u_\varepsilon^p dx \leq C(M_0, e^t, K, p) \tag{2.33}
$$

for any $1 < p < \infty$. Furthermore, integrating (2.31) from 0 to $T$ we have that for any $T > 0$

$$
\nabla u_\varepsilon^p \in L^2(0, T; L^2(\mathbb{R}^2)) \quad \text{for any} \quad 1 < p < \infty. \tag{2.34}
$$

Finally, making use of the iterative method in the spirit of [1] we have the uniformly boundedness

$$
u_\varepsilon \in L^\infty(0, T; L^\infty(\mathbb{R}^2)) \tag{2.35}
$$

\textbf{Step 6 (Time regularity)} It directly follows from (2.33), (2.34) and (2.35) that

$$
\| \nabla u_\varepsilon^r \|_{L^2(0, T; L^2(\mathbb{R}^2))} \leq C, \quad \text{for any} \quad r > 1/2, \tag{2.36}
$$

$$
\| u_\varepsilon \nabla c_\varepsilon \|_{L^\infty(0, T; L^{\bar{p}}(\mathbb{R}^2))} \leq C, \tag{2.37}
$$

$$
\| \partial_t u_\varepsilon \|_{L^2(0, T; H^{-1}(\mathbb{R}^2))} \leq C. \tag{2.38}
$$

Then by Lemma 4.23 in [1] one has that for any bounded domain $\Omega$, there exists a subsequence $u_\varepsilon$ without relabeling such that

$$
u_\varepsilon \to u \quad \text{in} \quad L^2(0, T; L^{\bar{p}}(\Omega)), \quad \text{for any} \quad 1 \leq \bar{p} < \infty. \tag{2.39}
$$

By a standard diagonal argument, the following uniform strong convergence holds true that for any $R > 0$

$$
u_\varepsilon \to u \quad \text{in} \quad L^2(0, T; L^{\bar{p}}(B_R)) \tag{2.40}
$$

Let us turn back to sketch the proof of (2.37) and (2.38). Actually, the term

$$
\nabla c_\varepsilon = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x - y}{|x - y|^2 + \varepsilon^2} u_\varepsilon(y, t) dy \tag{2.41}
$$

can be estimated from the Young inequality that

$$
\| \nabla c_\varepsilon \|_{L^\infty(\mathbb{R}^2)} \leq \frac{1}{2\pi} \left\| \int_{0 < |x - y| \leq 1} \frac{u_\varepsilon(y, t)}{|x - y|} dy + \int_{|x - y| > 1} \frac{u_\varepsilon(y, t)}{|x - y|} dy \right\|_{L^\infty(\mathbb{R}^2)}
$$

$$
\leq \frac{1}{2\pi} \left( \| u_\varepsilon \|_{L^\infty(\mathbb{R}^2)} \left\| \frac{1}{|x|} \right\|_{L^1(0 < |x| \leq 1)} + \| u_\varepsilon \|_{L^1(\mathbb{R}^2)} \right)
$$

$$
\leq C \left( \| u_\varepsilon \|_{L^\infty(\mathbb{R}^2)} + \| u_\varepsilon \|_{L^1(\mathbb{R}^2)} \right). \tag{2.42}
$$
The bound of \( u_\varepsilon \nabla c_\varepsilon \in L^\infty(0,T;L^\infty(\mathbb{R}^2)) \) follows by using \( u_\varepsilon \in L^\infty(0,T;L^1(\mathbb{R}^2)) \).

Moreover, (2.38) can be deduced by using the previous estimates that for any test function \( \psi \in L^2(0,T;H^1(\mathbb{R}^2)) \)

\[
\left| \int_0^T \int_{\mathbb{R}^2} \partial_t u_\varepsilon \psi \, dx \, dt \right| \leq \int_0^T \int_{\mathbb{R}^2} |\nabla u_\varepsilon \cdot \nabla \psi| \, dx \, dt + \int_0^T \int_{\mathbb{R}^2} |u_\varepsilon \nabla c_\varepsilon \cdot \nabla \psi| \, dx \, dt \\
\leq \left( \| \nabla u_\varepsilon \|_{L^2(0,T;\mathbb{R}^2)} + \| \sqrt{u_\varepsilon \nabla c_\varepsilon \|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))}} \sqrt{T} \| u_\varepsilon \|_{L^1(\mathbb{R}^2)} \right) \| \nabla \psi \|_{L^2(0,T;L^2(\mathbb{R}^2))} \\
\leq C \| \nabla \psi \|_{L^2(0,T;L^2(\mathbb{R}^2))}.
\]

(2.43)

**Step 7 (Strong convergence of \( u_\varepsilon \))** Furthermore, we will take advantage of the second moment estimate (2.16) to establish the strong convergence of \( u_\varepsilon \) in \( L^2(0,T;L^q(\mathbb{R}^2)) \) and extend (2.40) to the whole space. We compute that for any \( 1 \leq q < \infty \)

\[
\int_0^T \| u_\varepsilon \|_{L^q(|x| > R)}^2 \, dt \leq \int_0^T \| u_\varepsilon \|_{L^\infty(|x| > R)}^{2(q-1)/q} \| u_\varepsilon \|_{L^1(|x| > R)}^{2/q} \, dt \\
\leq \int_0^T \| u_\varepsilon \|_{L^\infty(|x| > R)}^{2(q-1)/q} \left( \int_{\mathbb{R}^2} \| u_\varepsilon \|_{L^2(|x| > R)}^2 \, dx \right)^{2/q} \, dt \to 0 \quad \text{as} \quad R \to \infty,
\]

(2.44) and the weak semi-continuity of \( L^2(0,T;L^q(\mathbb{R}^2)) \) implies

\[
\int_0^T \| u \|_{L^q(|x| > R)}^2 \, dt \leq \liminf_{\varepsilon \to 0} \int_0^T \| u_\varepsilon \|_{L^q(|x| > R)}^2 \, dt \to 0 \quad \text{as} \quad R \to \infty.
\]

(2.45)

Therefore, the following inequality is derived that for any \( 1 \leq q < \infty \), as \( R \to \infty, \varepsilon \to 0 \),

\[
\int_0^T \| u_\varepsilon - u \|_{L^q(\mathbb{R}^2)}^2 \, dt = \int_0^T \left( \| u_\varepsilon - u \|_{L^q(|x| > R)} + \| u_\varepsilon - u \|_{L^q(|x| \leq R)} \right)^2 \, dt \\
\leq C(q) \left( \int_0^T \| u_\varepsilon \|_{L^q(|x| > R)}^2 \, dt + \int_0^T \| u_\varepsilon - u \|_{L^q(|x| \leq R)}^2 \, dt \right) \to 0
\]

(2.46)

In the last inequality, the first term goes to zero due to (2.41), the second term is given by (2.45) and (2.40) provides the third term. Hence one has

\[ u_\varepsilon \to u \quad \text{in} \quad L^2(0,T;L^r(\mathbb{R}^2)), \quad \text{for any} \quad 1 \leq r < \infty. \]

(2.47)

**Step 8 (Existence of the weak solution)** Now multiplying (2.29) by \( \psi \in C_0^\infty(\mathbb{R}^2) \) and integrating it with respect to \( x \) and \( t \), we get the weak formulation for \( u_\varepsilon \)

\[
\int_{\mathbb{R}^2} \psi u_\varepsilon(\cdot,t) \, dx \cdot dt = \int_{\mathbb{R}^2} \psi u_{0\varepsilon} \, dx \\
= \int_0^T \int_{\mathbb{R}^2} \Delta \psi u_\varepsilon \, dx \, dt - \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \psi(x) \cdot \nabla \psi(y)}{|x-y|^2 + \varepsilon^2} u_\varepsilon(x,t) u_\varepsilon(y,t) \, dx \, dy \, dt \\
+ \int_0^T \int_{\mathbb{R}^2} u_\varepsilon(\cdot,t) \psi(x) \, dx \left( M_0 - \int_{\mathbb{R}^2} u_\varepsilon(\cdot,t) \, dx \right) \, dt.
\]

(2.48)
It’s to be noticed that (2.40) directly yields
\[
\int_0^T \int_{\mathbb{R}^2} \Delta \psi u \psi_0 dx dt \to \int_0^T \int_{\mathbb{R}^2} \Delta \psi u dx dt, \quad \varepsilon \to 0. \tag{2.49}
\]

As to the second term of the right side of (2.48), note that
\[
\left| \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left[ \nabla \psi(x) - \nabla \psi(y) \right] \cdot (x - y) \left( \frac{1}{|x - y|^2} - \frac{1}{|x - y|^2 + \varepsilon^2} \right) u_\varepsilon(x,t) u_\varepsilon(y,t) dx dy dt \right|
\leq C\varepsilon \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{u_\varepsilon(x,t) u_\varepsilon(y,t)}{|x - y|} dx dy dt \leq C\varepsilon \int_0^T \| u_\varepsilon \|_{L^4(\mathbb{R}^2)}^2 dt \leq C\varepsilon. \tag{2.50}
\]

In addition, by Cauchy inequality we have
\[
\left| \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left[ \nabla \psi(x) - \nabla \psi(y) \right] \cdot (x - y) \left( \frac{u_\varepsilon(x,t) u_\varepsilon(y,t)}{|x - y|^2} - \frac{u(x,t) u(y,t)}{|x - y|^2} \right) dx dy dt \right|
\leq C \left( \int_0^T \int_{\Omega \times \Omega} |u_\varepsilon(x) - u(x)| u_\varepsilon(y) dx dy + \int_{\Omega \times \Omega} |u_\varepsilon(y) - u(y)| |u(x)| dx dy dt \right)
\leq C \int_0^T \| u_\varepsilon - u \|_{L^2(\Omega)}^2 dt \int_0^T \| u_\varepsilon \|_{L^2(\Omega)}^2 dt, \tag{2.51}
\]

where the last line is given by the semi-continuity of \( \| u \|_{L^2(\mathbb{R}^2)} \leq \liminf_{\varepsilon \to 0} \| u_\varepsilon \|_{L^2(\mathbb{R}^2)} \). Thus taking the limit \( \varepsilon \to 0 \) and combining (2.50) and (2.51) we conclude
\[
\int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left[ \nabla \psi(x) - \nabla \psi(y) \right] \cdot (x - y) \left( \frac{u_\varepsilon(x,t) u_\varepsilon(y,t)}{|x - y|^2 + \varepsilon^2} - \frac{u(x,t) u(y,t)}{|x - y|^2} \right) dx dy dt \to 0. \tag{2.52}
\]

Thanks to (2.49), (2.52) and (2.47), passing to the limit \( \varepsilon \to 0 \) in (2.48) one has that for any \( 0 < t < T \)
\[
\int_{\mathbb{R}^2} \psi u(\cdot,t) dx - \int_{\mathbb{R}^2} \psi u_0(x) dx
= \int_0^T \int_{\mathbb{R}^2} \Delta \psi u dx dt - \frac{1}{4\pi} \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left[ \nabla \psi(x) - \nabla \psi(y) \right] \cdot (x - y) u(x,t) u(y,t) dx dy dt
+ \int_0^T \int_{\mathbb{R}^2} u(\cdot,t) \psi(x) dx \left( M_0 - \int_{\mathbb{R}^2} \psi(\cdot,t) dx \right) dt. \tag{2.53}
\]

This gives the existence of a global weak solution.

**Step 9** (The second moment of the weak solution)  Consider a test function \( \psi_R(x) \in C_0^\infty(\mathbb{R}^2) \) and \( \psi_R(x) = |x|^2 \) for \( |x| < R, \psi_R(x) = 0 \) for \( |x| \geq 2R \), letting \( \psi = \psi_R \) in (2.53) arrives at
\[
\int_{\mathbb{R}^2} \psi_R u(\cdot,t) dx - \int_{\mathbb{R}^2} \psi_R u_0(x) dx
= \int_0^T \int_{\mathbb{R}^2} \Delta \psi_R u dx dt - \frac{1}{4\pi} \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left[ \nabla \psi_R(x) - \nabla \psi_R(y) \right] \cdot (x - y) u(x,t) u(y,t) dx dy dt
+ \int_0^T \int_{\mathbb{R}^2} u(\cdot,t) \psi_R(x) dx \left( M_0 - \int_{\mathbb{R}^2} u(\cdot,t) dx \right) dt. \tag{2.54}
\]
Moreover, we follow the same lines as (2.13) to (2.15) to claim that
\[
\int_{\mathbb{R}^2} u(x,t) dx = m(t).
\] (2.55)

As before, since \(\Delta \psi_R(x)\) and \(|\nabla \psi_R(x) - \nabla \psi_R(y)|/(x-y)^2|\) are bounded, thus the first two terms in the right-hand side of (2.54) are bounded. As a consequence, as \(R \to \infty\), we may pass to the limit using the Lebesgue monotone convergence theorem with \(u \in L^1(\mathbb{R}^2)\) and obtain that for any \(t > 0\)
\[
\int_{\mathbb{R}^2} |x|^2 u(x,t) dx = \int_{\mathbb{R}^2} |x|^2 u_0(x) dx + \frac{1}{2\pi} \int_0^t m(s) ds - \frac{1}{2m(s)} \int_0^t m^2(s) ds
\]
\[
+ \int_0^t \int_{\mathbb{R}^2} |x|^2 u(x,s) dx (M_0 - m(s)) ds.
\] (2.56)

Then by Gronwall’s inequality we have
\[
\int_{\mathbb{R}^2} |x|^2 u(x) dx \leq C.
\] (2.57)

**Step 10** (The energy inequality of the weak solution) Integrating (2.18) in time from 0 to \(t\) follows
\[
\frac{1}{m(t)} \int_{\mathbb{R}^2} u_\varepsilon \log u_\varepsilon dx - \frac{1}{2m(t)} \int_{\mathbb{R}^2} u_\varepsilon c_\varepsilon dx - \log m(t) + \int_0^t \int_{\mathbb{R}^2} u_\varepsilon |\nabla \log u_\varepsilon - \nabla c_\varepsilon|^2 dxds
\]
\[
+ \int_0^t \frac{M_0 - m(s)}{2m(s)} \int_{\mathbb{R}^2} u_\varepsilon c_\varepsilon dx ds = \frac{1}{m_0} \int_{\mathbb{R}^2} u_0 \log u_0 dx - \frac{1}{2m_0} \int_{\mathbb{R}^2} u_0 c_0 dx - \log m_0.
\] (2.58)

The aim of the final step is taking limit \(\varepsilon \to 0\) in (2.58) to deduce the energy inequality (2.6).

For the sake of passing limit in (2.58), we split it into three parts. Firstly it is derived from Lemma 4.5 in the "Appendix" that
\[
\int_{\mathbb{R}^2} u(x,t) \log u(x,t) dx \leq \liminf_{\varepsilon \to 0} \int_{\mathbb{R}^2} u_\varepsilon(x,t) \log u_\varepsilon(x,t) dx.
\] (2.59)

Secondly, the lower semi-continuity of the energy dissipation is followed from Lemma 4.9 of [4] that for any \(t > 0\)
\[
\int_0^t \frac{1}{m(s)} \int_{\mathbb{R}^2} |2\nabla \sqrt{u} - \sqrt{u} \nabla c_\varepsilon|^2 dx ds \leq \liminf_{\varepsilon \to 0} \int_0^t \frac{1}{m(s)} \int_{\mathbb{R}^2} |2\nabla \sqrt{u} - \sqrt{u} \nabla c_\varepsilon|^2 dx ds.
\] (2.60)

It remains to verify the strong convergence of \(\int_{\mathbb{R}^2} u_\varepsilon c_\varepsilon dx\) that
\[
\int_{\mathbb{R}^2} u_\varepsilon(x,t) c_\varepsilon(x,t) dx \to \int_{\mathbb{R}^2} u(x,t) c(x,t) dx \text{ a.e. in } (0,T).
\] (2.61)
Moreover, we deal with
\[ I = I_1 + I_2 + I_3. \]

Getting (2.64) and (2.65) together and handling \( I_3 \) similarly yield that as \( \varepsilon \to 0, \)
\[ I_2 \to 0, \quad I_3 \to 0. \] (2.66)
Hence combining (2.59), (2.60) and (2.61), letting \( \varepsilon \to 0 \) in (2.58) leads to

\[
\int_{\mathbb{R}^2} \frac{u(x,t)}{m(t)} \log \frac{u(x,t)}{m(t)} dx - \frac{1}{2m(t)} \int_{\mathbb{R}^2} u(x,t)c(x,t)dx + \int_0^t \int_{\mathbb{R}^2} \frac{u(x,s)}{m(s)} |\nabla \log u(x,s) - \nabla c(x,s)|^2 dxds
+ \int_0^t M_0 - m(s) \int_{\mathbb{R}^2} u(x,s)c(x,s)dxds \leq \frac{1}{m_0} \int_{\mathbb{R}^2} u_0(x) \log u_0(x)dx - \frac{1}{2m_0} \int_{\mathbb{R}^2} u_0(x)c_0dx - \log m_0,
\]

which is the desired inequality (2.6) and thus completes the proof.

\[\square\]

**Remark 2.4.** For \( M_0 < m_0 < 8\pi \), then \( M_0 - m(t) < 0 \) for all \( t > 0 \). Therefore, \( u(x,t) \) is a sub-solution of the Keller-Segel system \( v_t = \Delta v - \nabla \cdot (v \nabla w) \), \( -\Delta w = v \) which admits a global solution when \( \int_{\mathbb{R}^2} v(x,0)dx < 8\pi \). By the comparison principle, all solutions of (1.1) exist globally.

### 2.2 Finite time blow-up for \( M_0 > 8\pi \)

In this subsection, we will show the blow-up statement with finite initial second moment. We start with the case \( M_0 > 8\pi \) in which we use the standard argument relying on the evolution of the second moment of solutions as done in [3, 12]. By passing to the limit from Steps 1-8 and adapting the argument (2.56) in Step 9 of Theorem 2.6 without further computation we obtain

**Lemma 2.5 (The Second Moment).** Assume \( \int_{\mathbb{R}^2} |x|^2 u_0(x)dx < \infty \) and \( u_0(x) \in L^1_+ \cap L^\infty(\mathbb{R}^2) \). Let \( u(x,t) \) be the weak solution to system (1.1), then the second moment satisfies

\[
m_2(t) := \int_{\mathbb{R}^2} |x|^2 u(x,t)dx = m(t) \left[ 4t - \frac{1}{2\pi} \ln \left( \frac{m_0}{M_0} e^{M_0 t} + \frac{M_0 - m_0}{M_0} \right) + \int_{\mathbb{R}^2} |x|^2 u_0(x)dx \right] m_0^{-1}.
\]

where \( m(t) = \left( \frac{1}{M_0} - \left( \frac{m_0 - M_0}{M_0 m_0} \right) e^{-M_0 t} \right)^{-1} \).

We notice that the second moment evolution is more complicated than in the classical system corresponding to mass conservation where the time derivative of the second moment is a constant. An easy consequence of the previous lemma is the following blow-up result.

**Theorem 2.6 (Finite time blow-up for \( M_0 > 8\pi \)).** Assume \( \int_{\mathbb{R}^2} |x|^2 u_0(x)dx < \infty \). If \( M_0 > 8\pi \), then the solution of (1.1) blows up in finite time and there exists a \( 0 < T < \infty \) such that

\[
\limsup_{t \to T} \|u(\cdot,t)\|_p = \infty, \quad \text{for any } \quad p > 1.
\]

**Proof.** Our proof hinges on the second moment evolution. First, for \( M_0 > 8\pi \), the second moment \( \int_{\mathbb{R}^2} |x|^2 u(x,t)dx \) can be estimated that

\[
\int_{\mathbb{R}^2} |x|^2 u(x,t)dx \leq \begin{cases} (4 - \frac{M_0}{2\pi}) M_0 t + \frac{M_0}{m_0} \int_{\mathbb{R}^2} |x|^2 u_0(x)dx - \frac{M_0}{2\pi} \ln \frac{m_0}{M_0} & \text{for } m_0 \leq M_0, \\ (4 - \frac{M_0}{2\pi}) m_0 t + \int_{\mathbb{R}^2} |x|^2 u_0(x)dx, & \text{for } m_0 > M_0. \end{cases}
\]
Consider $M_0 > 8\pi$, it enables us to get that the second moment of the weak solution will become negative after some time and contradicts the non-negativity of $u$. Therefore, there is a $T^* > 0$ such that $\lim_{t \to T^*} m_2(t) = 0$, and using Hölder’s inequality one has
\[
\int_{\mathbb{R}^2} u(x,t) \, dx = \int_{|x| \leq R} u(x,t) \, dx + \int_{|x| > R} u(x,t) \, dx \leq CR^{(p-1)/p} \|u\|_{L^p} + \frac{1}{R^2} m_2(t), \quad \text{for all } p > 1.
\] (2.70)
Choosing $R = (\frac{C m_2(t)}{||u||_{L^p}})^{1/(a+2)}$ with $a = 2(p - 1)/p$, we obtain
\[
||u||_{L^1} \leq C ||u||_{L^p}^{\frac{2}{a+2}} m_2(t)^{\frac{a+2}{a+2}},
\] (2.71)
which induces that
\[
\limsup_{t \to T^*} ||u||_{L^p} \geq \lim_{t \to T^*} \frac{||u(x,t)||_{L^1}^{\frac{a+2}{a+2}}}{C m_2(t)^{\frac{a+2}{a+2}}} = \infty.
\] (2.72)
Thus the proof is completed. $\blacksquare$

In addition, we know that the solution exists globally for $m_0 < M_0 < 8\pi$. On the contrary, we could obtain the finite time blow-up result for $M_0 < 8\pi < m_0$.

**Theorem 2.7** (Finite time blow-up for $M_0 < 8\pi < m_0$). For $M_0 < 8\pi < m_0$, assume $\int_{\mathbb{R}^2} |x|^2 u_0(x) \, dx < C(M_0, m_0)$, where $C(M_0, m_0)$ is given by
\[
C(M_0, m_0) := -\frac{(8\pi - M_0) m_0}{2M_0\pi} \ln \frac{m_0 - M_0}{8\pi - M_0} + \frac{4m_0}{M_0} \ln \frac{m_0}{8\pi},
\]
then the solution of (1.1) blows up in finite time.

**Proof.** Denote $h(t) = \frac{1}{m(t)} \int_{\mathbb{R}^2} |x|^2 u(x,t) \, dx$, from (2.67) we find,
\[
\frac{dh(t)}{dt} = 4 - \frac{m(t)}{2\pi}, \quad \frac{d^2h(t)}{dt^2} = -\frac{m(t)(M_0 - m(t))}{2\pi}.
\]
For $M_0 < 8\pi < m_0$, we are able to find that $h(t)$ firstly decreases from the initial data and reaches the global minimum $\int_{\mathbb{R}^2} |x|^2 u_0(x) \, dx - C(M_0, m_0)/8\pi$. Thus we conclude that $m_2(t)$ should become negative in finite time when the initial second moment is small, namely $\int_{\mathbb{R}^2} |x|^2 u_0(x) \, dx < C(M_0, m_0)$. This suggests that the solution blows up in finite time as proved in Theorem 2.6. Therefore the blow-up statement for the case $M_0 < 8\pi < m_0$ is completed without further comment. $\blacksquare$
3 Infinite time blow-up for $m_0 < M_0 = 8\pi$

As mentioned earlier, the solution exists globally for $0 < m_0 < M_0 = 8\pi$. In this section, we will firstly show that nontrivial steady state solutions can exist only in the case $M_0 = 8\pi$. The infinite second moment of the steady states could serve to give a hint on the further proof of infinite time blow-up.

3.1 Steady states

This subsection is primarily devoted to the analysis on the steady solutions of (1.1). Keeping (1.6) in mind we say that $m(t) \to M_0$ as $t \to \infty$, and the stationary equation is followed in the sense of distribution

$$\begin{cases}
\Delta U_s(x) - \nabla \cdot (U_s(x) \nabla C_s(x)) = 0, & x \in \mathbb{R}^2, \\
-\Delta C_s(x) = U_s(x), & x \in \mathbb{R}^2
\end{cases}$$

with

$$\int_{\mathbb{R}^2} U_s(x) dx = M_0. \quad (3.2)$$

Now three equivalent statements for the stationary solutions are shown and one has the constant chemical potential inside the support of the steady solutions.

**Proposition 3.1** (Three equivalent statements for the steady states). Let $\Omega \in \mathbb{R}^2$ be a connected open set. Assuming that $U_s \log U_s \in L^1(\mathbb{R}^2)$ and $U_s \in L^1_1(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} U_s dx = M_0$, $U_s \in C(\bar{\Omega})$ and $U_s > 0$ in $\Omega$, $U_s = 0$ in $\mathbb{R}^2 \setminus \Omega$. Moreover, if $\Omega$ is unbounded, assume that $U_s$ decays at infinity.

Assume also $C_s \in C^2(\mathbb{R}^2)$ is the Newtonian potential satisfying

$$\begin{align*}
\Delta U_s - \nabla \cdot (U_s \nabla C_s) &= 0, & \text{in } \mathbb{R}^2, \\
\mu_s &= \log U_s - C_s, & \text{in } \mathbb{R}^2
\end{align*} \quad (3.3)$$

in the sense of distribution. Then the following three statements are equivalent:

(i) No dissipation: $\int_\Omega U_s |\nabla \mu_s|^2 dx = 0$.

(ii) $U_s$ is the minimizer of the total interaction energy $F(u) = \frac{1}{m(t)} \int_{\mathbb{R}^2} u \log u dx - \frac{1}{2m(t)} \int_{\mathbb{R}^2} u^2 dx - \log m(t)$.

(iii) The chemical potential satisfies

$$\mu_s(x) = \bar{C}, \quad \forall x \in \text{Supp}(U_s)$$

$\bar{C}$ is a constant.
where $\text{Supp}(U_s) = \mathbb{R}^2$ and

$$
\bar{C} = \frac{1}{M_0} \left( \int_{\mathbb{R}^2} U_s \log U_s - U_s c_s dx \right). \tag{3.6}
$$

**Proof.** Firstly we prove (i) $\Leftrightarrow$ (iii): (i) is directly from (iii). (i) $\Rightarrow$ (iii): Suppose $\int_{\Omega} U_s |\nabla \mu_s|^2 dx = 0$. It follows from $U_s > 0$ at any point $x_0 \in \Omega$ that $\nabla \mu_s = 0$ in a neighborhood of $x_0$ and thus $\mu_s$ is a constant in this neighborhood. By the connectedness of $\Omega$ we deduce that $\mu_s \equiv \bar{C}$ in $\Omega$.

In order to prove (ii) $\Leftrightarrow$ (iii), we define the minimizer of $F(u)$: for any $\varphi \in C_0^\infty(\Omega)$, let $\overline{\Omega}_0 = \text{supp } \varphi$ with $\int_{\overline{\Omega}_0} \varphi(x) dx = 0$, $\Omega_0 \subset \subset \Omega$. There exists

$$
\varepsilon_0 := \min_{y \in \overline{\Omega}_0} \frac{U_s(y)}{\max_{y \in \overline{\Omega}_0} |\varphi(y)|} > 0,
$$

such that $U_s + \varepsilon \varphi \geq 0$ in $\Omega$ for $0 < \varepsilon < \varepsilon_0$. Now $U_s$ is a critical point of $F(u)$ in $\Omega$ if and only if

$$
\frac{d}{d\varepsilon} \bigg|_{\varepsilon = 0} F(U_s + \varepsilon \varphi) = 0, \ \forall \varphi \in C_0^\infty(\Omega). \tag{3.7}
$$

The above definition yields that for any $\varphi \in C_0^\infty(\Omega)$ with $\int_{\overline{\Omega}_0} \varphi(x) dx = 0$

$$
\int_{\Omega} (\mu_s - \int_{\Omega} \frac{U_s}{M_0} dx) \varphi dx = 0. \tag{3.8}
$$

For any $\psi \in C_0^\infty(\Omega)$, denoting

$$
\varphi = \psi - \frac{U_s}{M_0} \int_{\Omega} \psi dx, \tag{3.9}
$$

(3.8) becomes

$$
\int_{\Omega} \left( \mu_s - \int_{\Omega} \frac{U_s}{M_0} \mu_s dx \right) \psi dx = 0, \text{ for any } \psi \in C_0^\infty(\Omega). \tag{3.10}
$$

This implies

$$
\mu_s = \log U_s - C_s = \bar{C}, \text{ a.e. in } \Omega, \tag{3.11}
$$

where

$$
\bar{C} = \frac{1}{M_0} \int_{\Omega} U_s \log U_s - U_s c_s dx. \tag{3.12}
$$

In addition, it follows from (2.2) that

$$
2 \int_{\mathbb{R}^2} U_s \log U_s dx + 2 M_0 (1 + \log \pi - \log M_0) \geq \int_{\mathbb{R}^2} U_s C_s dx \geq \int_{\Omega} U_s C_s dx. \tag{3.13}
$$
Recalling $U_s \log U_s \in L^1(\mathbb{R}^2)$ we can further deduce that
\[
\bar{C} \geq \frac{1}{M_0} \left( \int_{\Omega} U_s \log U_s \, dx - 2 \int_{\mathbb{R}^2} U_s \log U_s \, dx - 2M_0(1 + \log \pi - \log M_0) \right),
\] (3.14)

Now we claim $\Omega = \mathbb{R}^2$. If $\Omega$ is bounded, then (3.11) implies $\bar{C} = -\infty$ at the boundary of $\Omega$. This contradiction with (3.14) implies $\Omega$ is unbounded and the connected unbounded open set is $\mathbb{R}^2$. Hence we complete the proof for (i) $\iff$ (iii) and (ii) $\iff$ (iii). □

From Proposition 3.1, we can obtain that the stationary equation is
\[
\begin{cases}
\log U_s - C_s = \bar{C}, & \text{in } \mathbb{R}^2, \\
-\Delta C_s = U_s, & \text{in } \mathbb{R}^2.
\end{cases}
\] (3.15)

Letting $\phi = \log U_s$ in (3.15), the steady equation reduces to
\[
-\Delta \phi = e^\phi, \quad \text{in } \mathbb{R}^2.
\] (3.16)

It has been proved in [5, 16] that all the solutions $\phi(x) \in C^2(\mathbb{R}^2)$ of (3.16) uniquely assume the radial form up to translation
\[
\phi(x) = \log \frac{32\lambda^2}{(4 + \lambda^2|x|^2)^2}, \quad \lambda > 0.
\] (3.17)

It provides us with the explicit expression of the stationary solutions
\[
U_s(x) = e^{\phi(x)} = \frac{32\lambda^2}{(4 + \lambda^2|x|^2)^2}, \quad \lambda > 0,
\] (3.18)

with $\int_{\mathbb{R}^2} U_s(x) \, dx = 8\pi$. One readily check that there are nontrivial solutions to (3.1) only in the case of a special value of the total mass. Combining with (3.2) it gives our final estimates.

**Lemma 3.2.** There are nontrivial steady states to (1.1) only for
\[
M_0 = 8\pi,
\] (3.19)

given by a one-parameter family with $\lambda > 0$
\[
U_s(x) = \frac{32\lambda^2}{(4 + \lambda^2|x|^2)^2}
\] (3.20)

with infinite second moment $\int_{\mathbb{R}^2} |x|^2 U_s(x) \, dx = \infty$. 

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3.2 Infinite time blow-up

For \( m_0 < M_0 = 8\pi \), combining (1.8) and Theorem 2.3 we deduce that solutions to system (1.9) exist globally for any \( 0 < t < \infty \) and satisfies

\[
\int_{\mathbb{R}^2} \rho(\cdot, t) \log \rho(\cdot, t) \, dx + \int_{\mathbb{R}^2} |\rho(\cdot, t)|^2 \, dx < \infty. \tag{3.21}
\]

In this subsection, we will further show that any solution of (1.1) will converge towards a delta dirac at the center of mass via system (1.9). The main tool are the free energy functional

\[
F[\rho](t) + \int_0^t \left( \int_{\mathbb{R}^2} \rho(\log \rho - m(s)w) \, dx + \frac{m'(s)}{2} \int_{\mathbb{R}^2} \rho \, dx \right) ds \leq F[\rho_0]. \tag{3.22}
\]

with \( F[\rho] := \int_{\mathbb{R}^2} \rho \log \rho \, dx - m(t) \int_{\mathbb{R}^2} \rho \, dx \). To prove this result, applying the mass conservation of \( \rho \) we follow a procedure analogous to Lemma 3.6 and Corollary 3.2 in [2] to obtain

Lemma 3.3. Assume \( \int_{\mathbb{R}^2} (|x|^2 + |\log \rho_0(x)|) \rho_0(x) \, dx < \infty \). If \( \rho_0 \in L^1_+ \cap L^\infty(\mathbb{R}^2) \) and \( m_0 < M_0 = 8\pi \), given any solution \( \rho \) to (1.9), we have

\[
\lim_{t \to \infty} \int_{\mathbb{R}^2} \rho(x, t) \log \rho(x, t) \, dx = +\infty \tag{3.23}
\]

and

\[
\lim_{t \to \infty} \rho(x, t) = \delta_{M_1} \quad \text{weakly --* as measures.}
\]

where \( M_1 \) is the center of mass in infinite time.

Proof. Without loss of generality, let \( \int_{\mathbb{R}^2} x \rho(x, t) \, dx = 0 \). Now we assume by contradiction that there exists an increasing sequence of time \( \{t_k\} \to \infty \) such that

\[
\int_{\mathbb{R}^2} \rho(x, t_k) \log \rho(x, t_k) \, dx < \infty.
\]

Combining with the boundedness of the second moment of \( \rho \), we can find a subsequence of \( \rho(x, t_k) \) (without relabelling for simplicity) that converges weakly to \( \rho_\infty(x) \) in \( L^1(\mathbb{R}^2) \) by Dunford-Pettis theorem. In addition, by \( \rho = \frac{u(x,t)}{m(t)} \) and (2.67), one has that the second moment of the limiting density \( \rho_\infty(x) \) satisfies

\[
0 < \int_{\mathbb{R}^2} |x|^2 \rho_\infty(x) \, dx < \infty. \tag{3.24}
\]
On the other hand, it follows from the definition (1.6) of \( m(t) \) and the logarithmic H-L-S inequality (2.2) that \( F(\rho) \) is bounded from below:

\[
F(\rho) = \int_{\mathbb{R}^2} \rho(x,t) \log \rho(x,t) dx - \frac{m(t)}{2} \int_{\mathbb{R}^2} \rho(x,t) w(x,t) dx \\
\geq \int_{\mathbb{R}^2} \rho(x,t) \log \rho(x,t) dx - 4\pi \int_{\mathbb{R}^2} \rho(x,t) w(x,t) dx \\
\geq \int_{\mathbb{R}^2} \rho(x,t) \log \rho(x,t) dx + 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho(x) \rho(y) \log |x - y| dx dy \\
\geq C,
\]

where the last line holds due to \( \int_{\mathbb{R}^2} \rho(\cdot, t) dx = 1 \) for any \( 0 < t < \infty \). So from (3.22) we have

\[
0 \leq \lim_{t \to \infty} \int_0^t \left( \int_{\mathbb{R}^2} \rho \nabla \log \rho - m(s) \nabla w \right)^2 dx + \frac{m'(s)}{2} \int_{\mathbb{R}^2} \rho w dx \right) ds \leq F[\rho_0] - \liminf_{t \to \infty} F[\rho](t).
\]

As a consequence,

\[
\lim_{t \to \infty} \int_0^\infty \left( \int_{\mathbb{R}^2} \rho \nabla \log \rho - m(s) \nabla w \right)^2 dx + \frac{m'(s)}{2} \int_{\mathbb{R}^2} \rho w dx \right) ds = 0.
\]

Noting that \( \lim_{t \to \infty} m(t) = 8\pi, \lim_{t \to \infty} m'(t) = 0 \), we have, up to the extraction of subsequences, that the limit \( \varrho_\infty(s, x) \) of \( (s, x) \mapsto \varrho_\infty(t + s, x) \) when \( t \) goes to infinity satisfies

\[
\nabla \log \varrho_\infty - 8\pi \nabla w_\infty = 0, \quad w_\infty = -\frac{1}{2\pi} \log |\cdot| * \varrho_\infty.
\]

(3.25)

Now reviewing Section 3.1 we find that \( \varrho_\infty \) are the family of stationary solutions to (1.9) with infinite second momentum by virtue of the transformation (1.8) and Lemma 3.2. This fact contradicts with (3.24) and thus the first result holds true.

Moreover, by making use of the mass conservation of \( \rho \), a careful reading of the proof of Lemma 3.1 and Corollary 3.2 in [2] gives the latter desired result. \( \Box \)

Applying (1.8) we are able to show our main result of system (1.1) in this section.

**Theorem 3.4** (Infinite Time Blow-up). Assume \( \int_{\mathbb{R}^2} (|\log u_0(x)| + |x|^2) u_0(x) dx < \infty \) and \( u_0 \in L^1_+ \cap L^\infty(\mathbb{R}^2) \). If \( \int_{\mathbb{R}^2} u_0(x) dx < M_0 = 8\pi \), the the weak solution \( u \) to (1.1) exists globally in time and converges to a delta dirac of mass \( M_0 \) concentrated at the center of mass as \( t \to \infty \).

## 4 Radially symmetric solutions

In this section, we firstly study the radially symmetric solutions of (1.9). The single equation we derive from (1.9) with radial symmetry satisfies a comparison principle. We can reach the steady...
state of the reduced system and have a new intuition of the relationship among $M_0, m_0$ and $8\pi$ besides the second moment. Subsequently, we divide it into $M_0 < 8\pi$ and $M_0 > 8\pi$. In these cases, the comparison principle plays an important role to capture the full structures of the reduced system. Finally, we transfer the system (1.9) back to the original system (1.1) and summarize the dynamical behaviors of radial symmetric solutions in different cases.

4.1 Reduced system and steady states

Radially symmetric solutions of the system (1.9) on $\rho(t,r), w(t,r)$ is equivalent to
\[
\begin{cases}
\frac{\partial}{\partial t}(r\rho) = (r\rho)' - m(t)(rw')', & t \geq 0, \ r > 0, \\
-(rw')' = r\rho, & t \geq 0, \ r > 0, \\
\rho'(t,r = 0) = 0, & t \geq 0, \\
\rho(t = 0,r) = \rho_0(r) \geq 0, & r \geq 0.
\end{cases}
\]  

(4.1)

This can be reduced to a single equation on $M(t,r)$ which is defined as
\[
M(t,r) = 2\pi \int_r^0 \sigma \rho(t,\sigma) d\sigma = -2\pi r w'(t,r).
\]  

(4.2)

Integrating (4.1) we arrive at
\[
\begin{cases}
\frac{\partial}{\partial t}M(t,r) = r \left( \frac{M'}{r} \right)' + \frac{m(t)}{2\pi r} M'M, & t \geq 0, \ r > 0, \\
M(t,0) = 0, \ M(t,\infty) = 1, & t \geq 0, \\
M(0,r) = 2\pi \int_r^0 \sigma \rho_0(\sigma) d\sigma, & r \geq 0.
\end{cases}
\]  

(4.3)

Next we write for time independent solutions
\[
\begin{cases}
\frac{r^2}{M'} \left( \frac{M'}{r} \right)' + \frac{M_0}{2\pi} \bar{M}' \bar{M} = r \bar{M}'' - \bar{M}' + \frac{M_0}{4\pi} \left( \bar{M}^2 \right)', & r > 0, \\
\bar{M}(0) = 0, \ \bar{M}'(r) > 0.
\end{cases}
\]  

(4.4)

A simple manipulation leads to
\[
\bar{M}_\lambda(r) = \frac{8\pi}{M_0 (1 + \lambda r^{-2})} \text{ for some } \lambda > 0.
\]  

(4.5)

Coming back to the density we obtain
\[
\bar{\rho}_\lambda(r) = \frac{8\lambda}{M_0 (r^2 + \lambda)^2}.
\]  

(4.6)

It is worthwhile to note that (4.6) is consistent with (3.20) up to a parameter. Recalling $\bar{M}(\infty) = 1$, we need to emphasize that there are radial steady states to (4.3) only for $8\pi = M_0$, given by $\bar{M}_\lambda(r) = \frac{1}{1 + \lambda r^{-2}}$. 

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4.2 Refined existence and blow-up

In terms of (4.1) and (4.5), we can use a comparison argument to construct sub-or super-solutions in the spirit of [12]. The analysis for the case $M_0 > 8\pi$ and $M_0 < 8\pi$ are quite different in view of the expression of (4.5). Hence, in the following we shall separate these two cases to discuss.

4.2.1 Global existence for $M_0 < 8\pi$

Lemma 4.1 (Global solutions). Let $m_0 < M_0$ and $\bar{M}_{\lambda_0}(r)$ is defined as (4.5) for some $\lambda_0 > 0$. We assume that

$$1 < \frac{8\pi}{M_0}, \quad M(0, r) < \bar{M}_{\lambda_0}(r)$$

(4.7)

for any $r > 0$, then the solution of (4.1) vanishes in $L^1(\mathbb{R}^2)$ locally. That’s

$$M(t, r) \to 0 \quad \text{as} \quad t \to \infty$$

(4.8)

uniformly in interval $0 \leq r < R$ for any $R > 0$.

Proof. Due to $M_0 < 8\pi$ and $M(0, r) < \bar{M}_{\lambda_0}(r)$, we may choose $0 < \mu < 1$ such that

$$\mu \frac{8\pi}{M_0} > 1 \quad \text{and} \quad M(0, r) < \mu \bar{M}_{\lambda_0}(r), \ \forall r > 0.$$ 

(4.9)

In view of (1.6) and $m_0 < M_0$, we also have

$$m_0 < m(t) < M_0$$

(4.10)

for any $0 < t < \infty$. Then we look for a supersolution to (4.1) as follows:

$$\bar{N}(t, r) = \min \left(1, \mu \frac{8\pi}{M_0 (1 + \lambda(t)r^{-2})} \right)$$

(4.11)

where

$$\lambda(t) = \lambda_0 + \frac{M_0}{\mu \pi} (1 - \mu) \left( \mu \frac{8\pi}{M_0} - 1 \right) t.$$ 

(4.12)

Furthermore, we may define $R(t)$ as

$$1 = \mu \frac{8\pi}{M_0 (1 + \lambda(t)R(t)^{-2})}$$

(4.13)

which gives rise to

$$\left\{
\begin{array}{ll}
    r \geq R(t), & \bar{N} = 1, \\
    r \leq R(t), & \bar{N} = \mu \frac{8\pi}{M_0 (1 + \lambda(t)r^{-2})}.
\end{array}
\right.$$ 

(4.14)
Now we claim that $\tilde{N}(t, r)$ is a supersolution to (4.1). Denote the operator

$$L(\tilde{N}) := \frac{\partial}{\partial t} \tilde{N} - \frac{\tilde{N}'}{r} - \frac{m(t)}{2\pi r} \tilde{N}' \tilde{N}.$$  \hfill (4.15)

For $r \leq R(t)$, with the fact (4.10) we derive that

$$L(\tilde{N}) = L(\mu \frac{8\pi}{M_0 (1 + \lambda(t)r^{-2})}) = \tilde{N} \frac{r^{-2}}{1 + \lambda(t)r^{-2}} \left(-\lambda'(t) + \frac{8\lambda(t)r^{-2}}{1 + \lambda(t)r^{-2}} \left(1 - \mu \frac{m(t)}{M_0}\right)\right) \geq \tilde{N} \frac{r^{-2}}{1 + \lambda(t)r^{-2}} \left(-\lambda'(t) + \frac{M_0}{\mu \pi} \left(1 - \mu \frac{m(t)}{M_0}\right) \left(\mu \frac{8\pi}{M_0} - 1\right)\right) \geq \tilde{N} \frac{r^{-2}}{1 + \lambda(t)r^{-2}} \left(-\lambda'(t) + \frac{M_0}{\mu \pi} (1 - \mu) \left(\mu \frac{8\pi}{M_0} - 1\right)\right) = 0.$$  \hfill (4.16)

Here the last line holds by the choice (4.12) of $\lambda(t)$. We also find that for $r > R(t)$, $L(\tilde{N}) = L(1) = 0$. Hence recalling $M(0, r) \leq M(0, \infty) = 1$ and (4.9) respectively, the comparison principle implies that

$$M(t, r) \leq \min \left(1, \mu \frac{8\pi}{M_0 (1 + \lambda(t)r^{-2})}\right) = \tilde{N}(t, r)$$  \hfill (4.17)

for all $0 \leq t < \infty$ and $0 \leq r < \infty$. Moreover, by (4.12) we find

$$\lambda(t) \to \infty,$$  \hfill (4.18)

then $R(t) \to \infty$ for $t$ large enough. Therefore, for any given interval $r \in (0, R)$ we have

$$M(t, r) \leq \mu \frac{8\pi}{M_0 (1 + \lambda(t)r^{-2})} \to 0 \text{ as } t \to \infty.$$  \hfill (4.19)

Thus completes the proof. \hfill $\square$

4.2.2 Infinite time blow-up for $M_0 > 8\pi$

Next we prove a refined blow-up result (with a weaker condition than the second moment condition) which is close enough to exhibit the shape of the chemotactic collapse solution.

Lemma 4.2 (Blow-up). Assume $M_0 < m_0$ and $\bar{M}_{\lambda_0}(r)$ is defined as (4.15) for some $\lambda_0 > 0$. If

$$\frac{8\pi}{M_0} < 1, \quad M(0, r) > \bar{M}_{\lambda_0}(r)$$  \hfill (4.20)
for any \( r > 0 \) and there is no finite time blow-up, then the solution to (4.1) blows up in infinite time. Furthermore, the solution has a mass concentration at the origin, that’s

\[
M(t, r(t)) > \frac{8\pi}{M_0}
\]  

for \( r(t) \to 0 \) as \( t \to \infty \).

**Proof.** Since \( \frac{8\pi}{M_0} < 1 \), We firstly choose \( 1 < \mu_0 < \mu_1 \) such that

\[
\mu_0 \frac{8\pi}{M_0} < \mu_1 \frac{8\pi}{M_0} < 1 \quad \text{and} \quad M(0, r) > \mu_1 \bar{M}_{\lambda_0}(r) > \mu_0 \bar{M}_{\lambda_0}(r), \quad \forall r > 0.
\]  

(4.22)

Combining \( M_0 < m_0 \) with (1.6) also leads to

\[
M_0 < m(t) < m_0.
\]  

(4.23)

We consider the function

\[
N(t, r) = \max \left( \mu_1 \frac{8\pi}{M_0 (1 + \lambda_0 r^{-2})}, \mu_0 \frac{8\pi}{M_0 (1 + \lambda(t) r^{-2})} \right),
\]  

(4.24)

where \( \lambda(t) = \lambda_0 e^{At} \) for some \( A < 0 \), we will prove that \( N \) is a subsolution as it is the maximum of two subsolutions.

Firstly we denote \( N_1 := \mu_1 \frac{8\pi}{M_0 (1 + \lambda_0 r^{-2})} \) and compute

\[
\frac{\partial}{\partial t} N_1 - r \left( \frac{N_1'}{r} \right)' - \frac{m(t)}{2\pi r} N_1' N_1 = N_1 \frac{r^{-2}}{1 + \lambda_0 r^{-2}} \frac{8\lambda_0}{\lambda_0 + r^2} \left( 1 - \frac{m(t)}{M_0} \right)
\]

\[
\leq N_1 \frac{r^{-2}}{1 + \lambda_0 r^{-2}} \frac{8\lambda_0}{\lambda_0 + r^2} (1 - \mu_1)
\]

\[
< 0.
\]  

(4.25)

Then it follows from (4.22) that \( N_1 \) is a subsolution to (4.1). Secondly, we have to prove that \( \mu_0 \frac{8\pi}{M_0 (1 + \lambda(t) r^{-2})} \) is a subsolution in the interval \( 0 < r \leq R(t) \) where it achieves the maximum, i.e.

\[
\mu_1 \frac{8\pi}{M_0 (1 + \lambda_0 R(t)^{-2})} = \mu_0 \frac{8\pi}{M_0 (1 + \lambda(t) R(t)^{-2})}.
\]  

(4.26)

By the definition of \( R(t) \), there exists a constant \( R_0 \) such that \( r \leq R(t) \leq R_0 \) with \( \mu_1 \frac{8\pi}{M_0 (1 + \lambda_0 R_0^{-2})} = \)
\[ \mu_0 \frac{8 \pi}{M_0}, \] then using (4.23) and a direct calculation provide

\[
\frac{\partial}{\partial t} N - r \left( \frac{N'}{r} \right)' - \frac{m(t)}{2 \pi r} N' N = N_0 \frac{r^{-2}}{1 + \lambda(t)r^{-2}} \left( -\lambda'(t) + \frac{8 \lambda(t)}{\lambda(t) + r^2} \left( 1 - \mu_0 \frac{m(t)}{M_0} \right) \right) \leq N_0 \frac{r^{-2}}{1 + \lambda(t)r^{-2}} \left( -\lambda'(t) + \frac{8 \lambda(t) R_0^{-2}}{\lambda_0 R_0^{-2} + 1} \left( 1 - \mu_0 \right) \right) \leq N_0 \frac{r^{-2}}{1 + \lambda(t)r^{-2}} \left( -\lambda'(t) + \frac{8 \mu_0 R_0^{-2} \lambda(t)}{\mu_1} \left( 1 - \mu_0 \right) \right) = 0 \quad (4.27)

by choosing

\[ \lambda(t) = \lambda_0 e^{-\frac{8 \mu_0 R_0^{-2} (\mu_0 - 1)t}{\mu_1}} = \lambda_0 e^{At}. \quad (4.28) \]

We notice that \( \underline{N} = \mu_1 \frac{8 \pi}{M_0 (1 + \lambda_0 r^{-2})} \) for \( r > R(t) \). Now keeping (4.22) in mind, the comparison principle gives

\[ M(t, r) \geq \underline{N}(t, r) \geq \mu_0 \frac{8 \pi}{M_0 (1 + \lambda(t)r^{-2})} = \mu_0 \frac{8 \pi}{M_0 (1 + \lambda_0 e^{At} r^{-2})}. \quad (4.29) \]

Therefore, at \( r = e^{\frac{4t}{\lambda}} \) we find

\[ M(t, r) \geq \mu_0 \frac{8 \pi}{M_0 \left( 1 + \lambda_0 e^{\frac{4t}{\lambda}} \right)} \quad (4.30) \]

and the subsequent mass concentration at \( r = 0 \)

\[ M(t, r(t)) > \frac{8 \pi}{M_0} \quad (4.31) \]

for \( r(t) \to 0 \) as \( t \to \infty \). This is our desired estimate. \( \square \)

### 4.3 Main results for the radially symmetric solutions of (1.1)

Now we are ready to summarize the main results of (4.1) as follows.

**Theorem 4.3.** Assume that the initial data \( \rho(0, r) \) is radially symmetric, \( \bar{\rho}_{\lambda_0}(r) \) is defined as (4.6) for some \( \lambda_0 > 0 \).
(1) If $M_0 < 8\pi$ and
\[ \rho(0, r) < \bar{\rho}_{\lambda_0}(r) \] (4.32)
for any $r > 0$, then the radially symmetric solution of (4.1) vanishes in $L^1(\mathbb{R}^2)$ locally as $t \to \infty$.

(2) If $M_0 > 8\pi$ and
\[ \rho(0, r) > \bar{\rho}_{\lambda_0}(r) \] (4.33)
for any $r > 0$, then there could exist one solution to (4.1) which blows up in finite time or has a mass concentration at the origin such that
\[ 2\pi \int_0^{r(t)} \rho(t, \sigma) \sigma d\sigma > \frac{8\pi}{M_0} \] (4.34)
for $r(t) \to 0$ as $t \to \infty$.

By using (1.8), we transfer Theorem 4.3 to the original system (1.1).

**Theorem 4.4.** Assume that $u_0(r)$ is the radially symmetric initial data to (1.1).

(1) If $m_0 < M_0 < 8\pi$ and
\[ u_0(r) < \frac{8m_0}{M_0} \frac{\lambda_0}{(\lambda_0 + r^2)^2}, \quad \forall r > 0 \] (4.35)
for some $\lambda_0 > 0$, then any radially symmetric solution of (1.1) vanishes in $L^1(\mathbb{R}^2)$ locally as $t \to \infty$.

(2) If $m_0 > M_0 > 8\pi$ and
\[ u_0(r) > \frac{8m_0}{M_0} \frac{\lambda_0}{(\lambda_0 + r^2)^2}, \quad \forall r > 0 \] (4.36)
for some $\lambda_0 > 0$, then any radially symmetric solution of (1.1) has finite time blow-up or has a mass concentration at the origin such that
\[ 2\pi \int_0^{r(t)} u(t, \sigma) \sigma d\sigma > 8\pi \] (4.37)
for $r(t) \to 0$ as $t \to \infty$.

we comment that, in the blow-up result, the limitation on the second moment are in fact useless in the radially symmetric case and they can be relaxed by the comparison condition with the steady state (4.5) which corresponds to an infinite second moment. In contrast to Section 2.2, infinite initial second moment can produce infinite time blow-up.
Appendix

Lemma 4.5. Let $u_{\varepsilon}, u \in L^1_+(\mathbb{R}^2)$ satisfy

\begin{align}
\int_{\mathbb{R}^2} |x|^2 u_{\varepsilon} \, dx &< \infty, \quad \int_{\mathbb{R}^2} |x|^2 u \, dx < \infty, \\
\int_{\mathbb{R}^2} u_{\varepsilon} \log u_{\varepsilon} \, dx &< \infty, \quad (4.38) \\
\int_{\mathbb{R}^2} u_{\varepsilon} \log u_{\varepsilon} \, dx &< \infty, \quad (4.39) \\
u_{\varepsilon} \rightharpoonup u \quad (\varepsilon \to 0) \text{ weakly in } L^1(\mathbb{R}^2). \quad (4.40)
\end{align}

Then

\begin{align}
\int_{\mathbb{R}^2} u(x) \log u(x) \, dx &< \infty, \quad (4.41)
\end{align}

and

\begin{align}
\int_{\mathbb{R}^2} u(x) \log u(x) \, dx &\leq \liminf_{\varepsilon \to 0} \int_{\mathbb{R}^2} u_{\varepsilon}(x) \log u_{\varepsilon}(x) \, dx. \quad (4.42)
\end{align}

**Proof.** In the following, $C, C_1, C_2$ stand for finite positive constants that are independent of $\varepsilon$.

We will give the proof step by step.

(i) We firstly show that

\begin{align}
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} (1 + |x|^\beta) u_{\varepsilon}(x) \, dx = \int_{\mathbb{R}^2} (1 + |x|^\beta) u(x) \, dx, \quad \text{for any } 0 < \beta < 2. \quad (4.43)
\end{align}

In fact, for any $R > 1$,

\begin{align*}
&\left| \int_{\mathbb{R}^2} (1 + |x|^\beta)(u_{\varepsilon}(x) - u(x)) \, dx \right| \\
\leq &\int_{\mathbb{R}^2} (1 + |x|^\beta) 1_{\{|x| \leq R\}}(u_{\varepsilon}(x) - u(x)) \, dx + \int_{|x| > R} (1 + |x|^\beta) u_{\varepsilon}(x) \, dx + \int_{|x| > R} (1 + |x|^\beta) u(x) \, dx.
\end{align*}

We deal with the last two terms as follows.

\begin{align*}
&\int_{|x| > R} (1 + |x|^\beta) u_{\varepsilon}(x) \, dx + \int_{|x| > R} (1 + |x|^\beta) u(x) \, dx \\
\leq &2 \left( \int_{|x| > R} |x|^\beta u_{\varepsilon}(x) \, dx + \int_{|x| > R} |x|^\beta u(x) \, dx \right) \\
\leq &\frac{2}{R^{2-\beta}} \left( \int_{\mathbb{R}^2} |x|^2 u_{\varepsilon}(x) \, dx + \int_{\mathbb{R}^2} |x|^2 u(x) \, dx \right) \leq C \frac{1}{R^{2-\beta}}.
\end{align*}
Since, by weak convergence,
\[
\left| \int_{\mathbb{R}^2} (1 + |x|^\beta)1_{\{|x| \leq R\}}(u_\varepsilon(x) - u(x)) \, dx \right| \to 0, \quad \varepsilon \to 0.
\]

It follows that
\[
\limsup_{\varepsilon \to 0} \left| (1 + |x|^\beta)(u_\varepsilon(x) - u(x)) \right| \leq \frac{C}{R^{2-\beta}}, \quad \forall R > 1.
\]

Letting \( R \to \infty \) leads to (4.43).

(ii) Let’s prove
\[
\int_{\mathbb{R}^2} u(x) \log u(x) \, dx < \infty. \quad (4.44)
\]

Actually, by convexity of \( y \mapsto y \log y, y \geq 0 \), one has
\[
u_\varepsilon \log u_\varepsilon \geq u \log u + (1 + \log u)(u_\varepsilon - u).
\]

From this we have, for any \( m \in \mathbb{N} \),
\[
1_{\{1 \leq u(x) \leq m\}} u_\varepsilon \log u_\varepsilon \\
\geq 1_{\{1 \leq u(x) \leq m\}} u \log u + 1_{\{1 \leq u(x) \leq m\}}(1 + \log u)(u_\varepsilon - u).
\]

Hence
\[
\int_{\mathbb{R}^2} 1_{\{1 \leq u(x) \leq m\}} u \log u \, dx + \int_{\mathbb{R}^2} 1_{\{1 \leq u(x) \leq m\}}(1 + \log u)(u_\varepsilon - u) \, dx \\
\leq \int_{\mathbb{R}^2} 1_{\{1 \leq u(x) \leq m\}} u_\varepsilon \log u_\varepsilon \, dx \\
\leq \sup_{\varepsilon > 0} \int_{\mathbb{R}^2} u_\varepsilon \log u_\varepsilon \, dx =: C_1(< \infty).
\]

Since, by weak convergence,
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} 1_{\{1 \leq u(x) \leq m\}}(1 + \log u)(u_\varepsilon - u) \, dx = 0,
\]

it follows by passing to the limit \( \varepsilon \to 0 \) that
\[
\int_{\mathbb{R}^2} 1_{\{1 \leq u(x) \leq m\}} u \log u \, dx \leq C_1.
\]

Then letting \( m \to \infty \) gives by Levi monotone convergence
\[
\int_{\mathbb{R}^2} 1_{\{u(x) \geq 1\}} u \log u \, dx \leq C_1.
\]
This together with (2.24) and the boundedness of $\int_{\mathbb{R}^2} |x|^2 u(x) dx$ yields
\[
\int_{\mathbb{R}^2} u(x) |\log u(x)| dx = \int_{\mathbb{R}^2} 1_{\{u(x) < 1\}} u(x) |\log u(x)| dx + \int_{\mathbb{R}^2} 1_{\{u(x) \geq 1\}} u(x) |\log u(x)| dx \\
\leq \int_{\mathbb{R}^2} e^{-|x|^2} dx + 2 \int_{\mathbb{R}^2} u(x) |x|^2 dx + C_1 < \infty.
\]
Hence we close the proof of (4.44).

(iii) Next, to prove (4.42), we consider the truncation
\[
g_m(x) = u(x) \wedge m + \frac{1}{m} e^{-|x|}.
\]
By convexity we have
\[
u \geq g_m(x) \log g_m(x) + (1 + \log g_m(x))(u - g_m),
\]
i.e.
\[
g_m(x) \log g_m(x) \leq u \log u + (1 + \log g_m(x))(g_m - u). \tag{4.45}
\]

We firstly use (4.44) to check the condition of $g_m \log g_m$ for using Lebesgue dominated convergence, i.e.
\[
\int_{\mathbb{R}^2} g_m |\log g_m| dx \leq C_2. \tag{4.46}
\]
By convexity of $y \mapsto y \log y$ on $y \geq 0$ one has
\[
(a + b) \log (a + b) = \left(\frac{1}{2}a + \frac{1}{2}b\right) \log \left(\frac{1}{2}a + \frac{1}{2}b\right) \\
\leq \frac{1}{2} a \log 2a + \frac{1}{2} b \log 2b, \quad a, b \geq 0. \tag{4.47}
\]
We also have
\[
(a + b) \log \frac{1}{a + b} \leq a \log \frac{1}{a} + b \log \frac{1}{b}, \quad a, b \geq 0. \tag{4.48}
\]
Now we use (4.47) to see that if $g_m \geq 1$, then, because $m \geq 4$, we must have $u(x) > 1/2$, so that
\[
g_m(x) |\log g_m(x)| = g_m(x) \log g_m(x) \\
\leq (u(x) \wedge m) \log (2u(x) \wedge m) + \frac{1}{m} e^{-|x|} \log \left(\frac{2}{m} e^{-|x|}\right) \\
\leq u(x) \log (2u(x)) + \frac{1}{m} e^{-|x|} \frac{2}{m} e^{-|x|} \\
\leq u(x) + u(x) |\log u(x)| + e^{-2|x|}.
\]
While if \( g_m(x) < 1 \), then \( u(x) < 1 \), then by (4.48) we have
\[
g_m(x) \log g_m(x) = \left( u(x) + \frac{1}{m} e^{-|x|} \right) \log \left( \frac{1}{u(x) + \frac{1}{m} e^{-|x|}} \right)
\leq u(x) \log u(x) + \frac{1}{m} e^{-|x|} \log(m e^{|x|})
\leq u(x) \log u(x) + \frac{2}{\sqrt{m}} e^{-|x|/2},
\]
where we used \( \log y = 2 \log \sqrt{y} \leq 2 \sqrt{y} \) for \( y \geq 0 \). So we have proved an \( L^1 \) control
\[
g_m(x) \log g_m(x) \leq u(x) + u(x) \log u(x) + e^{-|x|/2} + e^{-2|x|}.
\quad (4.49)
\]

The aim next is to deal with the second term in the right hand side of (4.45). Using
\[
u(x) \land m = u(x) - (u(x) - m)_+,
\]
we handle
\[
(1 + \log g_m(x))(g_m(x) - u_\epsilon(x))
=(1 + \log g_m(x))(u(x) - u_\epsilon(x)) - (1 + \log g_m(x))(u(x) - m)_+ + (1 + \log g_m(x)) \frac{1}{m} e^{-|x|}.
\quad (4.50)
\]
By considering \( g_m(x) \geq 1 \) and \( g_m(x) < 1 \) we have
\[
|1 + \log g_m(x)| \leq 1 + (\log(m + 1) \lor (|x| + \log m)) \leq 2(1 + |x|) \log(m + 1).
\]
Therefore we have
\[
|1 + \log g_m(x)| \frac{1}{m} e^{-|x|} \leq 2 \frac{\log(m + 1)}{m} (1 + |x|) e^{-|x|},
\quad (4.51)
\]
and by (4.43),
\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^2} (1 + \log g_m(x))(u(x) - u_\epsilon(x)) = 0.
\quad (4.52)
\]
Moreover we have
\[
-(1 + \log g_m(x))(u(x) - m)_+ \leq 0.
\quad (4.53)
\]
In fact the case \( u(x) \leq m \) is obvious. For the case \( u(x) > m \) we have \( g_m(x) > m > 1 \), so that \( \log g_m(x) > 0 \). So (4.53) holds true.
From (4.45), collecting (4.50)-(4.53) we have by taking lower limit $\varepsilon \to 0$ that
\[
\int_{\mathbb{R}^2} g_m(x) \log g_m(x) \, dx \leq \liminf_{\varepsilon \to 0} \int_{\mathbb{R}^2} u_\varepsilon(x) \log u_\varepsilon(x) \, dx + \frac{2 \log(m+1)}{m} \int_{\mathbb{R}^2} (1 + |x|) e^{-|x|} \, dx, \quad \forall m \geq 4.
\] (4.54)

Finally by continuity one has
\[
g_m(x) \log g_m(x) \to u(x) \log u(x) \quad (m \to \infty).
\] (4.55)

Then it follows (4.49) and Lebesgue dominated convergence and (4.54) that
\[
\int_{\mathbb{R}^2} u(x) \log u(x) \, dx = \lim_{m \to \infty} \int_{\mathbb{R}^2} g_m(x) \log g_m(x) \, dx \leq \liminf_{\varepsilon \to 0} \int_{\mathbb{R}^2} u_\varepsilon(x) \log u_\varepsilon(x) \, dx.
\] (4.56)

Thus we close the proof. $\Box$

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