GENERALIZED HELMHOLTZ CONDITIONS FOR LAGRANGIAN SYSTEMS WITH NON-CONSERVATIVE FORCES

IOAN BUCATARU AND OANA CONSTANTINESCU

Abstract. In this paper we provide generalized Helmholtz conditions, in terms of a semi-basic 1-form, which characterize when a given system of second order ordinary differential equations is equivalent to the Lagrange equations for some given arbitrary non-conservative forces. These generalized Helmholtz conditions, when expressed in terms of a multiplier matrix, reduce to those obtained by Mestdag, Sarlet and Crampin \cite{25} for the particular case of dissipative or gyroscopic forces. When the involved geometric structures are homogeneous with respect to the fiber coordinates, we show how one can further simplify the generalized Helmholtz conditions. We provide examples where the generalized Helmholtz conditions can be integrated and the corresponding Lagrangian and Lagrange equations can be found.

1. Introduction

The classic inverse problem of Lagrangian mechanics requires to find the necessary and sufficient conditions, which are called Helmholtz conditions, such that a given system of second order ordinary differential equations (SODE) is equivalent to the Euler-Lagrange equations of some regular Lagrangian function. The problem has a long history and the literature about the subject is vast. There are various approaches to this problem, using different techniques and mathematical tools, \cite{1, 3, 8, 10, 13, 14, 20, 23, 26, 30, 31}.

In this work we discuss the inverse problem of Lagrangian systems with non-conservative forces. Locally, the problem can be formulated as follows. We consider a SODE in normal form

\begin{equation}
\frac{d^2 x^i}{dt^2} + 2G^i (x, \dot{x}) = 0
\end{equation}

and an arbitrary covariant force field $\sigma_i(x, \dot{x})dx^i$. We will provide necessary and sufficient conditions, which we will call generalized Helmholtz conditions, for the existence of a Lagrangian $L$ such that the system (1.1) is equivalent to the Lagrange equations

\begin{equation}
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \right) - \frac{\partial L}{\partial x^i} = \sigma_i(x, \dot{x}).
\end{equation}

When the covariant forces are of dissipative or gyroscopic type, the problem has been studied recently in \cite{12, 25}. In these two papers the authors provide generalized Helmholtz conditions, in terms of a multiplier matrix, for a SODE (1.1) to represent Lagrange equations with non-conservative forces of dissipative or gyroscopic type.

While, in this paper, we generalize the work of Crampin, Mestdag and Sarlet \cite{12, 25}, the approach we use is different. We will reformulate the inverse problem of Lagrangian systems with non-conservative forces in terms of a semi-basic 1-form, using the Frölicher-Nijenhuis formalism,
and by extending the techniques developed in [3]. The main contribution of this paper is to provide, in Theorem 3.2, generalized Helmholtz conditions in terms of semi-basic 1-forms for the most general case of Lagrangian systems with non-conservative forces. These generalized Helmholtz conditions are equivalent, when expressed in terms of a multiplier matrix, to those obtained by Mestdag, Sarlet and Crampin [25] for the particular case of dissipative or gyroscopic forces.

The structure of the paper is as follows. In Section 2 we use the Frölicher-Nijenhuis formalism [15, 17, 19] to provide a geometric setting associated to a given system (1.1). This framework includes: a nonlinear connection, dynamical covariant derivative and curvature type tensors. In Section 3 we use this geometric setting to reformulate the inverse problem of Lagrangian systems and provide generalized Helmholtz conditions in terms of a semi-basic 1-form. This semi-basic 1-form will represent the Poincaré-Cartan 1-form of the sought after Lagrangian. In our approach, in the most general context, the covariant force field $\sigma$ is a priori given, together with the SODE (1.1). This differs from the semi-variational equations studied by Rossi in [29, Section 2], where one has to search for the Lagrangian function and the covariant force field as well. In the particular case when the covariant force field $\sigma$ is zero, the generalized Helmholtz conditions $GH_1 - GH_3$ of Theorem 3.2 reduce to, and simplify, the Helmholtz obtained in [3, Theorem 4.1].

In the next two sections we show that the generalized Helmholtz conditions $GH_1 - GH_3$ of Theorem 3.2 reduce to those obtained by Mestdag, Sarlet and Crampin [12, 25] for the particular case of dissipative or gyroscopic forces. Theorem 4.2 provides three equivalent sets of conditions, in terms of semi-basic 1-forms, for a SODE to be of dissipative type. One advantage of formulating the generalized Helmholtz conditions in terms of forms is discussed in Proposition 4.3, where we study the formal integrability of such conditions. An important consequence of Proposition 4.3 is that any SODE on a 2-dimensional manifold is of dissipative type. See also [5, 9] for the formal integrability of the Helmholtz conditions expressed in terms of a semi-basic 1-form. Theorem 5.2 provides two equivalent sets of generalized Helmholtz conditions, in terms of semi-basic 1-forms, which characterize Lagrangian systems of gyroscopic type.

In section 6 we discuss the inverse problem of Lagrangian systems when all the involved geometric objects are homogeneous with respect to the velocity coordinates. If the degree of homogeneity is not equal to 1, in Theorem 6.3, we prove that the generalized Helmholtz condition $GH_2$ is a consequence of the other two. When the covariant force field is zero, depending on the degree of homogeneity, the problem reduces to the Finsler metrizability problem or the projective metrizability problem. By restricting the generalized Helmholtz conditions to these particular cases we obtain in Corollary 6.4 a reformulation of the Finsler metrizability problem and in Corollary 6.6 a reformulation of the projective metrizability problem.

In the last section we show how the techniques developed throughout the paper can be used to discuss various examples. For these examples, the generalized Helmholtz conditions can be integrated and therefore we can find the corresponding Lagrangian and Lagrange equations. The examples we analyse consist of non-variational projectively metrizable sprays that are of dissipative type and a class of gyroscopic semisprays.

2. The geometric framework

2.1. A geometric setting for semisprays. A system of second order ordinary differential equations on some configuration manifold can be identified with some special vector field on the tangent space of the manifold, which will be called a semispray.

In this section, we use the Frölicher-Nijenhuis formalism [15, 19] to associate a geometric setting to a given semispray, by developing a calculus of derivations on the tangent bundle of the configuration manifold, [2, 3, 16, 17]. Alternatively, one can derive a calculus of derivations on
forms along the tangent bundles projection \(24\) and use it to study various problems associated to a given semispray \([12, 25]\).

For an \(n\)-dimensional smooth manifold \(M\), denote by \(TM\) its tangent bundle. Local coordinates \((x^i)\) on \(M\) induce local coordinates \((x^i, y^j)\) on \(TM\). The real algebra of smooth functions on \(M\) will be denoted by \(C^\infty(M)\), while the Lie algebra of smooth vector fields on \(M\) will be denoted by \(\mathfrak{X}(M)\).

Consider \(C \in \mathfrak{X}(TM)\) the Liouville (dilation) vector field and \(J\) the tangent structure (vertical endomorphism). Throughout this paper we use the summation convention over covariant and contravariant repeated indices. With this convention, the Liouville vector field and the tangent structure are locally given by:

\[
C = y^j \frac{\partial}{\partial y^j}, \quad J = \frac{\partial}{\partial y^i} \otimes dx^i.
\]

The canonical submersion \(\pi : TM \to M\) induces a natural foliation on \(TM\). The tangent spaces to the leaves of this foliation determine a regular \(n\)-dimensional distribution, \(VTM : u \in TM \to V_uTM = \text{Ker} \, d_u\pi \subset T_uTM\), which is called the vertical distribution. The forms dual to the vertical vector fields will play an important role in this work. These are semi-basic (vector valued) forms on \(TM\), with respect to the canonical projection \(\pi\). In the homogenous context, semi-basic forms and a corresponding differential calculus have been used to discuss various problems in Lagrangian mechanics by Klein in \([18]\).

A form on \(TM\) is called semi-basic if it vanishes whenever one of its arguments is vertical. A form \(\omega\) on \(TM\) is called a basic form if both \(\omega\) and \(d\omega\) are semi-basic forms. A vector valued form on \(TM\) is called semi-basic if it takes vertical values and it vanishes whenever one of its arguments is vertical. For example, the tangent structure \(J\) is a vector valued semi-basic 1-form.

In order to develop a geometric setting, which we will need to discuss some problems associated to a given semispray, we will make use of the Frölicher-Nijenhuis formalism. Within this formalism one can identify derivations to vector valued forms on \(TM\). We briefly recall the Frölicher-Nijenhuis formalism following \([15, 17, 19]\).

Consider a vector valued \(l\)-form \(L\) on \(TM\). We will denote by \(i_L : \Lambda^k(TM) \to \Lambda^{k+l-1}(TM)\) the derivation of degree \((l-1)\), given by

\[
i_L \xi (X_1, \ldots, X_{k+l-1}) = \frac{1}{l!(k-1)!} \sum_{\sigma \in S_{k+l-1}} \text{sign}(\sigma) \alpha \left( L(X_{\sigma(1)}, \ldots, X_{\sigma(l)}), X_{\sigma(l+1)}, \ldots, X_{\sigma(k+l-1)} \right),
\]

where \(S_{k+l-1}\) is the permutation group of \(\{1, \ldots, k+l-1\}\). We denote by \(d_L : \Lambda^k(TM) \to \Lambda^{k+l}(TM)\) the derivation of degree \(l\), given by

\[
d_L = [i_L, d] = i_L \circ d - (-1)^{l-1} d \circ i_L.
\]

The derivation \(i_L\) is trivial on functions and hence it is uniquely determined by its action on \(\Lambda^1(TM)\). It follows that \(i_L\) is a derivation of \(i_*\)-type \([15, 17]\), or an algebraic derivation \([19]\). The derivation \(d_L\) commutes with the exterior derivative \(d\) and hence it is uniquely determined by its action on \(C^\infty(TM)\). It follows that \(d_L\) is a \(d_*\)-type \([15, 17]\), or a Lie derivation \([19]\).

For two vector valued forms \(K\) and \(L\) on \(TM\), of degrees \(k\) and \(l\), we consider the Frölicher-Nijenhuis bracket \([K, L]\), which is the vector valued \((k+l)\)-form, uniquely determined by

\[
d_{[K, L]} = [d_K, d_L] = d_k \circ d_L - (-1)^{kl} d_L \circ d_K.
\]

For various commutation formulae, within the Frölicher-Nijenhuis formalism, we will use the Appendix A of the book by Grifone and Muzsnay \([17]\). We will use some vector valued forms on \(TM\) to associate a differential calculus on \(TM\). Some of these are naturally defined on \(TM\), like the
tangent structure \( J \) and the Liouville vector field \( \mathcal{C} \). The differential calculus associated to these is very useful to discuss some geometric aspects related to semi-basic forms. For a more detailed discussion about semi-basic forms and a vertical calculus on \( TM \) see [3] Section 2.2. Directly from the definition of the tangent structure \( J \) it follows that \([J,J]=0\) and according to formula (2.1) it follow that \( d_J^2 = 0 \). Therefore, any \( d_J \)-exact form is \( d_J \)-closed and according to a Poincaré-type Lemma [34], any \( d_J \)-closed form is locally \( d_J \)-exact. The derivation \( d_J \) corresponds to the operator \( d \) used by Klein in [18], where it is shown the equivalence of \( d_J \)-closed and \( d_J \)-exact semi-basic and homogeneous forms.

A semispray, or a second order vector field, is a globally defined vector field on \( TM, S \in \mathfrak{X}(TM) \), that satisfies \( S S = \mathcal{C} \). Locally, a semispray can be expressed as follows:

\[
(2.2) \quad S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y) \frac{\partial}{\partial y^i}.
\]

A curve \( c : t \in I \subset \mathbb{R} \to c(t) = (x^i(t)) \in M \) is called a geodesic of the semispray \( S \) if its natural lift to \( TM, c^\ast : t \in I \subset \mathbb{R} \to c^\ast(t) = (x^i(t), dx^i/\partial t) \in TM \), is an integral curve of \( S \). Locally, a curve \( c(t) = (x^i(t)) \) is a geodesic of the semispray \( S \) if it satisfies the system of second order ordinary differential equations (1.1).

A semispray \( S \) induces a geometric framework on \( TM \), whose main objects are listed below. The horizontal and vertical projectors, \( h \) and \( v \) are given by, [16],

\[
h = \frac{1}{2} (\text{Id} - [S,J]), \quad v = \frac{1}{2} (\text{Id} + [S,J]).
\]

The two projectors determine two regular \( n \)-dimensional, supplementary, distributions on \( TM \), a horizontal distribution \( HTM = \text{Im} h = \text{Ker} v \) and the vertical distribution \( VTM = \text{Im} J = \text{Ker} h \). As it can be seen from the above formulae, the Frölicher-Nijenhuis bracket \( \Gamma = [J,S] = h - v \) is an almost product structure, which can be directly checked as in [16]. Locally, the two projectors \( h \) and \( v \) can be expressed as follows

\[
h = \frac{\delta}{\delta x^i} \otimes dx^i, \quad v = \frac{\partial}{\partial y^i} \otimes \delta y^i, \quad \frac{\delta}{\delta x^i} \otimes \delta y^i - N^j_i \frac{\partial}{\partial y^j}, \quad \delta y^i = dy^i + N^j_i dx^j, \quad N^j_i = \frac{\partial G^i}{\partial y^j}.
\]

Another important Frölicher-Nijenhuis bracket, for the geometric setting induced by a semispray, is \([S,h]\). It induces two geometric structures, the almost complex structure \( \mathcal{F} \), and the Jacobi endomorphism \( \Phi \), which are given by

\[
(2.3) \quad \mathcal{F} = h \circ [S,J] - J, \quad \Phi = v \circ [S,h].
\]

Locally, the almost complex structure can be expressed as follows

\[
\mathcal{F} = \frac{\delta}{\delta x^i} \otimes \delta y^i - \frac{\partial}{\partial y^i} \otimes dx^i.
\]

One can immediately see that \( \mathcal{F} + J = h \circ [S,h] \) is an almost product structure on \( TM \). The Jacobi endomorphism is a vector valued semi-basic 1-form and it has the following local expression

\[
(2.4) \quad \Phi = R^j_i \frac{\partial}{\partial y^i} \otimes dx^j, \quad \frac{\partial}{\partial y^i} \otimes \delta y^j - S(N^j_i) - N^j_i N^j_l.
\]

The horizontal distribution induced by a semispray is, in general, non-integrable. The obstruction to its integrability is given by the curvature tensor \( R \). This is a vector valued semi-basic 2-form and it is defined as the Nijenhuis tensor of the horizontal projector:

\[
(2.5) \quad R = \frac{1}{2} [h,h] = \frac{1}{2} R^i_{jk} \frac{\partial}{\partial y^i} \otimes dx^j \wedge dx^k, \quad R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}.
\]
The Jacobi endomorphism $\Phi$ and the curvature tensor $R$ are closely related by the following formula

$$3R = [J, \Phi],$$

$$3R^i_{jk} = \frac{\partial R^i_j}{\partial y^k} - \frac{\partial R^i_k}{\partial y^j}.$$

From the above formula and (2.1) it follows that the commutator of the derivations of degree 1, $d_J$ and $d_\Phi$, is a derivation of degree 2, given by

$$[d_J, d_\Phi] = 3d_R.$$

We introduce now the dynamical covariant derivative associated to a semispray. The origins of this concept go back to the work of Kosambi [20], where a tensorial operator of differentiation, called the bi-derivative, has been introduced to provide a geometric treatment to systems of differential equations. The term dynamical covariant derivative has been introduced by Cariñena and Martín in [8] as a derivation of degree zero along the tangent bundle projection, see also [12, 21, 25, 32].

In this work we follow the approach from [2, 3]. We define the dynamical covariant derivative as a tensor derivation $\nabla$ on $TM$, whose action on functions and vector fields is given as follows. For $f \in C^\infty(TM)$ and $X \in \mathfrak{X}(TM)$, we define

$$\nabla f = S f, \quad \nabla X = h\circ hS + v\circ hS + v = hS + F + J - \Phi.$$

Therefore, the action of $\nabla$ on the exterior algebra of $TM$ is given by

$$\nabla = hS - i_{J - \Phi}.$$

The action of $\nabla$ on forms is a derivation of degree zero and hence it has a unique decomposition as a derivation of $d_*$-type and a derivation of $i_*$-type. Formula (2.10) gives such decomposition for $\nabla$, the $d_*$-type derivation is $hS$ and the $i_*$-type derivation is $i_{J - \Phi}$. The following commutation formula can be shown using items iii) and iv) of [3, Theorem 3.5]

$$[d_J, \nabla] = d_h + 2i_R.$$

For more properties of the dynamical covariant derivative and some commutation formulae with some other geometric structures, we refer to [3, Section 3.2].

2.2. Lagrange systems and non-conservative covariant forces. Consider $L : TM \to \mathbb{R}$ a Lagrangian, which is a smooth function on $TM$ whose Hessian with respect to the fibre coordinates

$$g_{ij} = \frac{\partial^2 L}{\partial y^i \partial y^j}$$

is non-trivial. We say that $L$ is a regular Lagrangian if the Poincaré-Cartan 2-form $dd_J L$ is a symplectic form on $TM$. Locally, the regularity condition of a Lagrangian $L$ is equivalent to the fact that the Hessian (2.12) of $L$ has maximal rank $n$ on $TM$.

For a Lagrangian $L$, we consider $E_L = \mathcal{C}(L) - L$ its Lagrangian energy.
For an arbitrary semispray $S$ and a Lagrangian $L$, the following 1-form (called the Euler-Lagrange 1-form, or the Lagrange differential by Tulczyjew [33]) is a semi-basic 1-form:

\[
(2.13) \quad \delta_S L = L_S \delta_d L - dL = \delta_i L_S - 2d_h L = \nabla \delta_d L - d_h L \tag{2.13}
\]

\[
= \left\{ S \left( \frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} \right\} dx^i = \left\{ \frac{\delta S(L)}{\delta y^i} - 2 \frac{\delta L}{\delta x^i} \right\} dx^i = \left\{ \nabla \left( \frac{\partial L}{\partial y^i} \right) - \frac{\delta L}{\delta x^i} \right\} dx^i.
\]

The inverse problem of Lagrangian mechanics requires, for a given semispray $S$, to decide whether or not there exists a Lagrangian $L$ with vanishing Lagrange differential, which means $\delta_S L = 0$. In this case we will say that the semispray $S$ is Lagrangian. Locally it means that the solutions of the system \((1.1)\) are among the solutions of the Euler-Lagrange equations of some Lagrangian $L$. Necessary and sufficient conditions for the existence of such Lagrangian are called Helmholtz conditions and were expressed in terms of a multiplier matrix [11, 13, 20, 23, 50, 31], a semi-basic 1-form [3, 11], or a 2-form [1, 10, 18, 21, 22, 26].

In this work we study the more general problem, when for a given semispray $S$ and a semi-basic 1-form $\sigma$, we ask the existence of a Lagrangian $L$, whose Lagrange differential is $\sigma$.

**Definition 2.1.** Consider $S$ a semispray and $\sigma \in \Lambda^1(TM)$ a semi-basic 1 form. We say that $S$ is of Lagrangian type with covariant force field $\sigma$ if there exists a (locally defined) Lagrangian $L$ such that $\delta_S L = \sigma$.

The above definition expresses the fact that the solutions of the system \((1.1)\) are among the solutions of the Lagrangian equations \((1.2)\). If the Lagrangian $L$, which we search for, is regular, then the two systems \((1.1)\) and \((1.2)\) are equivalent.

**Proposition 2.2.** Consider $S$ a semispray of Lagrangian type with covariant force field $\sigma$ and Lagrangian function $L$. Then, the variation of the Lagrangian energy $E_L$ along the geodesics of the semispray $S$ is given by

\[
(2.14) \quad S(E_L) = i_S \sigma.
\]

**Proof.** Directly from the Definition 2.1 we have the semispray $S$, the Lagrangian function $L$ and the semi-basic 1-form $\sigma$ satisfy $\delta_S L = \sigma$. We apply to both sides of this formula the inner product $i_S$, which gives $i_S \delta_S L = i_S \sigma$. For the left hand side of this last formula we have $i_S \delta_S L = L_S i_S \delta_d L - i_S dL = L_S \mathcal{C}(L) - L_S L = L_S (E_L)$, which implies that formula \((2.14)\) is true. \hfill $\Box$

There is an important aspect of Definition 2.1 that we want to emphasize, if we do not make any requirement about the covariant force field $\sigma$. For an arbitrary semispray $S$ there is always a Lagrangian $L$ and a semi-basic 1-form $\sigma$ such that $\delta_S L = \sigma$. This case corresponds to the semi-variational equations studied by Rossi in [29, Section 2]. In our analysis, we start with a given semispray $S$ and a given semi-basic 1-form $\sigma$ on $TM$ and search for a Lagrangian $L$ such that $\delta_S L = \sigma$. We study these aspects in Section 3. More exactly, for a given semispray $S$ and a semi-basic 1-form $\sigma$, we provide necessary and sufficient conditions, which we will call generalized Helmholtz conditions, for the existence of a semi-basic 1-form $\theta$ that represents the Poincaré-Cartan 1-form of a Lagrangian $L$ such that $\delta_S L = \sigma$. We show that for the case $\sigma = 0$, these generalized Helmholtz conditions reduce to the classic Helmholtz conditions, expressed in terms of a semi-basic 1-form, as in [3, Theorem 4.1].

When the covariant force field $\sigma$ is of special type (dissipative, gyroscopic or homogeneous), there are two problems we pay attention to. First we specialize the problem mentioned above to the particular case of our semi-basic 1-form $\sigma$. Secondly, we will not require $\sigma$ to be given together with the semispray. Hence, we will start with a semispray $S$ and study the existence of a
Lagrangian \( L \) and a semi-basic 1-form \( \sigma \) of some special type such that \( \delta_S L = \sigma \). We study these aspects and the close relationship between the two problems in Sections 4, 5 and 6.

3. Generalized Helmholtz Conditions

In this section we will provide necessary and sufficient conditions for a given semispray \( S \) to be of Lagrangian type for a given covariant force field \( \sigma \). These conditions, which we will refer to as generalized Helmholtz conditions, will be expressed in terms of a semi-basic 1-form, generalizing the classic Helmholtz conditions obtained in [3]. We will prove that for some particular cases of the covariant force field (dissipative and gyroscopic) the generalized Helmholtz conditions reduce to those obtained by Mestdag, Sarlet and Crampin in [25] in terms of a multiplier matrix.

Throughout this work, we make the following assumption about the semi-basic 1-form \( \theta \) that will be involved in expressing the generalized Helmholtz conditions. We say that a semi-basic 1-form \( \theta = \theta_i(x, y)dx^i \) is non-trivial if the matrix \( g_{ij} = \partial \theta_i/\partial y^j \) is non-trivial. If \( \theta = d_J L \) is the Poincaré-Cartan 1-form of some function \( L \), then \( \theta \) is non-trivial if and only if \( L \) is a Lagrangian function.

**Theorem 3.1.** Consider \( S \) a semispray and \( \sigma \in \Lambda^1(TM) \) a semi-basic 1-form. The semispray \( S \) is of Lagrangian type with covariant force field \( \sigma \) if and only if there exists a non-trivial, semi-basic 1-form \( \theta \in \Lambda^1(TM) \) such that \( \mathcal{L}_S \theta - \sigma \) is a closed 1-form on \( TM \).

**Proof.** For the direct implication, from Definition 2.1 it follows that there exists a Lagrangian \( J \) such that \( \mathcal{L}_S d_J L - dL = \sigma \). We take \( \theta = d_J L \), the Poincaré-Cartan 1-form of \( L \). It follows that \( \mathcal{L}_S \theta - \sigma = dL \), which is exact and hence a closed 1-form. Since \( L \) is a Lagrangian function, we have that the semi-basic 1-form \( \theta \) is non-trivial.

For the converse implication, we assume that there exists \( \theta \in \Lambda^1(TM) \) a non-trivial, semi-basic 1-form, such that \( \mathcal{L}_S \theta - \sigma \) is a closed 1-form on \( TM \). Therefore, there exists a (locally defined) function \( L \) on \( TM \) such that

\[
\mathcal{L}_S \theta - \sigma = dL.
\]

We apply the derivation \( i_J \), to both sides of this formula. In the right hand side we have \( i_J dL = d_J L \). We evaluate now the left hand side. Since \( \theta \) and \( \sigma \) are semi-basic 1-forms, it follows that \( i_J \sigma = 0 \). For a vector valued 1-form \( K \) on \( TM \), we use the commutation formula, see [17, A.1, page 205],

\[
i_K \mathcal{L}_S = \mathcal{L}_S i_K + i_{[K,S]}.
\]

If \( K = J \), the tangent structure, we use above formula and \([J,S] = h - v \). It follows that \( i_J \mathcal{L}_S \theta = i_{[J,S]} \theta = i_h \theta = \theta \) and hence \( \theta = d_J L \). Therefore, the non-trivial, semi-basic 1 form \( \theta \) is the Poincaré-Cartan 1-form of \( L \), and hence \( L \) is a Lagrangian function. We replace this in formula (3.1) and obtain that the semispray \( S \) is of Lagrangian type with the covariant force field \( \sigma \). □

In Theorem 3.1 if we search for a regular Lagrangian \( L \), then the corresponding semi-basic 1-form \( \theta \) has to be regular as well, in the following sense. A semi-basic 1-form \( \theta \in \Lambda^1(TM) \) is said to be regular if \( d\theta \) is a symplectic 2-form on \( TM \). If \( \theta = d_J L \) is the Poincaré-Cartan 1-form of some function \( L \), then the regularity condition of \( \theta \) is equivalent to the regularity of the Lagrangian function \( L \).

Next theorem provides necessary and sufficient conditions for the existence of the semi-basic 1-form, which we discussed in Theorem 3.1 using the differential operators associated to a given semispray.
Theorem 3.2. A semispray $S$ is of Lagrangian type with covariant force field $\sigma$ if and only if there exists a non-trivial, semi-basic 1-form $\theta \in \Lambda^1(TM)$ such that the following generalized Helmholtz conditions are satisfied:

$\text{GH}_1$ $d_j \theta = 0;$

$\text{GH}_2$ $d_k \theta = \frac{1}{2} \nabla d_j \sigma - d_k \sigma;$

$\text{GH}_3$ $\nabla d_i \theta = d_0 \sigma - \frac{1}{2} i_{x+z} d_j \sigma.$

Proof. We fix a semispray $S$ and a semi-basic 1-form $\theta \in \Lambda^1(TM)$. According to Theorem 3.1 we have that $S$ is of Lagrangian type with covariant force field $\sigma$ if and only if there exists a non-trivial, semi-basic 1-form $\theta \in \Lambda^1(TM)$ such that

\begin{equation}
(3.3) \quad \mathcal{L}_S \theta = d \sigma.
\end{equation}

We will prove now that formula (3.3) is equivalent to the three generalized Helmholtz conditions $\text{GH}_1 - \text{GH}_3$.

For the direct implication, we consider $\theta$ a non-trivial, semi-basic 1-form on $TM$ that satisfies formula (3.3). We apply both sides of this formula the derivation $i_j$. Using commutation formula (3.2) for $K = J$ and the fact that $i_{[J,S]} d \theta = i_{\theta \circ J} d \theta = i_{2h - 1d} d \theta = 2i_h d \theta - 2d \theta = 2d \theta$, we obtain

\begin{equation}
(3.4) \quad \mathcal{L}_S d_j \theta + 2d \theta = d_j \sigma.
\end{equation}

We apply again the derivation $i_j$ to both sides of the above formula and we use that $d_h \theta$ and $d_j \sigma$ are semi-basic 2-forms, which implies that $i_j d_h \theta = 0$ and $i_j d_j \sigma = 0$. Using again the commutation formula for $i_j$ and $\mathcal{L}_S$, it follows that $i_{[J,S]} d_j \theta = 0$. Since $i_{[J,S]} d_j \theta = i_{\theta \circ J} d \theta = 2d \theta$ we obtain that $d_j \theta = 0$, which is first generalized Helmholtz condition $\text{GH}_1$. We replace this in formula (3.4) and obtain the following formula

\begin{equation}
(3.5) \quad d_h \theta = \frac{1}{2} d_j \sigma.
\end{equation}

Since $\theta$ is a semi-basic 1-form, it follows that $i_h \theta = \theta$ and $i_\theta \theta = 0$. Therefore we have

\begin{equation}
(3.6) \quad d \theta = 2d \theta - d \theta = i_{\theta \circ J} d \theta - d \theta = i_h d \theta + i_\theta d \theta - d \theta = d_h \theta + d_\theta \theta.
\end{equation}

The condition $d_j \theta = 0$ reads $d_j \theta(X, Y) = d \theta(JX, Y) + d \theta(X, JY) = 0$, for all $X, Y \in X(TM)$. It follows that for any two vertical vector fields $V, W$ on $TM$, we have $d \theta(V, W) = 0$.

In order to show that the next two generalized Helmholtz conditions are satisfied, we will prove first that

For $K = J$ and $\mathcal{L}_S$, we obtain

\begin{equation}
(3.7) \quad i_{\theta \circ J} d \theta = 0.
\end{equation}

Consider $X_1, Y_1 \in X(TM)$. It follows that there exist $X_2, Y_2 \in X(TM)$ such that $(\mathcal{F} + J)(X_1) = hX_2$ and $(\mathcal{F} + J)(Y_1) = hY_2$. Moreover, if we compose to the left these two equalities with the tangent structure $J$ and use the fact that $J \circ \mathcal{F} = \nu$ and $J \circ h = J$, we obtain $\nu X_1 = JX_2$ and $\nu Y_1 = JY_2$. Using these equalities we obtain

\begin{align*}
(iv + J d_\theta) (X_1, Y_1) &= d \theta ((\mathcal{F} + J)(X_1), \nu Y_1) + d \theta (\nu X_1, (\mathcal{F} + J)(Y_1)) \\
&= d \theta (hX_2, JY_2) + d \theta (JX_2, hY_2) = d_j \theta(X_2, Y_2) = 0.
\end{align*}

We apply now the derivation $i_h$ to both sides of formula (3.3) and use the commutation rule (3.2) for $K = h$. It follows

\begin{equation}
\mathcal{L}_S i_h d \theta + i_{[h, S]} d \theta = i_h d \sigma.
\end{equation}
We use the fact that $\theta$ and $\sigma$ are semi-basic forms, which implies that $i_{h}d\theta = d_{h}\theta - d\theta$ and $i_{h}d\sigma = d_{h}\sigma - d\sigma$, and formula (3.3) again to obtain
\begin{equation}
\mathcal{L}_{S}d_{h}\theta + i_{[h,S]}d\theta = d_{h}\sigma.
\end{equation}
From formula (2.3) we obtain that $[h,S] = -F - J - \Phi$. Now using formula (3.6) we obtain $i_{[h,S]}d\theta = -i_{F+j}d\theta - i_{\Phi}d\theta = -i_{F+j}d_{h}\theta - d_{h}\theta$. With this formula we go back to (3.7), where we use the fact that $\mathcal{L}_{S}d_{h}\theta - i_{F+j}d_{h}\theta = \nabla d_{h}\theta$. It follows
\begin{equation}
\nabla d_{h}\theta - d_{h}\theta = d_{h}\sigma.
\end{equation}
If we make use of formula (3.5) to substitute $d_{h}\theta$, we obtain that the second generalized Helmholtz condition $GH_{2}$ is true as well. Using formula (2.10) we obtain
\begin{equation}
\mathcal{L}_{S}d\theta = \nabla d\theta + i_{F+j}d\theta - i_{\Phi}d\theta.
\end{equation}
If we replace this in (3.3), we obtain that the second generalized Helmholtz condition $GH_{2}$ is true as well. Using formula (2.10) we obtain
\begin{equation}
\mathcal{L}_{S}d\theta = \nabla d\theta + i_{F+j}d\theta - i_{\Phi}d\theta.
\end{equation}
If we replace this in (3.3), we obtain
\begin{equation}
\nabla d\theta + d_{h}\sigma = d_{h}\sigma + d_{\sigma}.
\end{equation}
In the above formula we use (3.8) and formula (3.5) and obtain the last generalized Helmholtz condition $GH_{3}$.

We will prove now the converse, which means that the three generalized Helmholtz conditions $GH_{1} - GH_{3}$ imply the condition (3.3). We will prove first that the existence of a non-trivial, semi-basic 1-form $\theta$ that satisfies the three conditions $GH_{1} - GH_{3}$ implies the existence of a non-trivial, semi-basic 1-form $\tilde{\theta}$ that satisfies $GH_{1} - GH_{3}$ and (3.5) as well.

Consider $\theta$ a non-trivial, semi-basic 1-form that satisfies the generalized Helmholtz conditions $GH_{1} - GH_{3}$.

We apply the derivation $d_{J}$ to both sides of formula $GH_{2}$ to obtain
\begin{equation}
d_{J}d_{h}\theta = \frac{1}{2}d_{J}\nabla d_{J}\sigma - d_{J}d_{h}\sigma.
\end{equation}
We evaluate first each of the two sides of the above formula. For the left hand side, using the commutation formula (2.7), as well as the fact that $d_{J}\theta = 0$, we have
\begin{equation}
d_{J}d_{h}\theta = d_{[J,h]}\theta = 3d_{h}\theta = 3d_{h}d_{h}\theta.
\end{equation}
Using the commutation formula (2.11), the fact $d_{J}^{2} = 0$ and the fact that $i_{R}d_{J}\sigma = 0$, since $d_{J}\sigma$ is a semi-basic 2-form and $R$ is a vertically-valued form, we can express the first term of the right hand side of formula (3.9) as follows
\begin{equation}
d_{J}\nabla d_{J}\sigma = d_{h}d_{J}\sigma.
\end{equation}
Since $[J,h] = 0$ it follows that $d_{h}d_{J} + d_{J}d_{h} = 0$. Now, if we replace everything in both sides of formula (3.9) we obtain
\begin{equation}
3d_{h}d_{h}\theta = \frac{3}{2}d_{h}d_{J}\sigma,
\end{equation}
which can be further written as follows
\begin{equation}
d_{h}\left(d_{h}\theta - \frac{1}{2}d_{J}\sigma\right) = 0.
\end{equation}
If we apply the derivation $d_{J}$ to both sides of formula $GH_{3}$ we obtain
\begin{equation}
d_{J}\nabla d_{e}\theta = d_{J}d_{e}\sigma - \frac{1}{2}d_{J}i_{F+j}d_{J}\sigma.
\end{equation}
To evaluate the left hand side of formula (3.11) we use the commutation formula (2.11)
\[ d_j \nabla d \theta = \nabla d_j d \theta + d_h d \theta + 2i_{R}d \theta. \]

We use the fact that \([J, v] = 0\), which implies that \(d_J d \theta + d_v d_J \theta = 0\), to obtain that \(d_J d \theta = 0\).

Since \(2R = [h, h] = [h, \text{Id} - v] = -[h, v]\) it follows that
\[ d_h d \theta + d_v d_h \theta = d_{[h, v]} = -2d_R \theta = -2i_{R}d \theta. \]

We use all these calculations to write the left hand side of formula (3.11) as follows
\[ d_J \nabla d \theta = -d_v d_h \theta. \]

Finally we have to evaluate the right hand side of formula (3.11). For its second term, we will use the following commutation formula, [17, A.1, page 205], for two vector valued 1-forms \(K\) and \(L\)
\[ i_{K}d_{L} = d_{L}i_{K} + d_{LK} - i_{[K, L]}. \]

For \(L = J\) and \(K = F + J\) we have
\[ i_{F} + i_{J}d_{J}d_{\sigma} = d_{J}i_{F} + i_{J}d_{\sigma} + d_{J}[F + J]d_{J}d_{\sigma} - i_{[F + J, J]}d_{J}d_{\sigma}. \]

Since \(J \circ (F + J) = v\), \([F + J, J] = [F, J] = -R\) and \(i_{[F + J, J]}d_{J}d_{\sigma} = -i_{R}d_{J}d_{\sigma} = 0\), from the above formula, we obtain
\[ d_{J}i_{F} + d_{J}d_{\sigma} = -d_{v}d_{J}d_{\sigma}. \]

Using these calculations we can write the right hand side of formula (3.11) as follows
\[ d_{J}d_{v}d_{\sigma} - \frac{1}{2}d_{J}i_{F} + d_{J}d_{\sigma} = -d_{v}d_{J}d_{\sigma} + \frac{1}{2}d_{v}d_{J}d_{\sigma} = -\frac{1}{2}d_{v}d_{J}d_{\sigma} \]

It follows that one can write formula (3.11) as follow
\[ (3.13) \quad d_{v} \left( d_{h} \theta - \frac{1}{2}d_{J}d_{\sigma} \right) = 0. \]

From the two formulae (3.10) and (3.13), it follows that
\[ (3.14) \quad d \left( d_{h} \theta - \frac{1}{2}d_{J}d_{\sigma} \right) = 0. \]

From the above formula, we obtain that there exists a locally defined basic 1-form \(\beta\) such that
\[ (3.15) \quad d_{h} \theta - \frac{1}{2}d_{J}d_{\sigma} = d_{\beta}. \]

The semi-basic 1-form \(\tilde{\theta} = \theta - \beta\) satisfies all three generalized Helmholtz condition \(GH_1 - GH_3\) and formula (3.5) as well. Since \(\beta\) is a basic 1-form, we have that the semi-basic 1-form \(\tilde{\theta}\) is non-trivial if and only if the semi-basic 1-form \(\theta\) is non-trivial.

For this non-trivial, semi-basic 1-form \(\tilde{\theta}\), we will prove that formula (3.3) is true. Formula (3.6), which we proved in the first part of our proof, is still true for \(\tilde{\theta}\) since for this we need that \(\tilde{\theta}\) is a semi-basic 1-form that satisfies \(d_{J} \tilde{\theta} = 0\). Using formula (2.10) we obtain
\[ (3.16) \quad \mathcal{L}_{\tilde{\theta}}d_{\tilde{\theta}} = \nabla d_{h} \tilde{\theta} + \nabla d_{v} \tilde{\theta} + i_{F + J}d_{h} \tilde{\theta} - d_{h} \tilde{\theta}. \]

In the right hand side of formula (3.16) we replace \(d_{h} \tilde{\theta}, \nabla d_{v} \tilde{\theta}\) and \(d_{h} \tilde{\theta}\) in terms of \(\sigma\), from (3.5) and the conditions \(GH_2 - GH_3\). It follows that formula (3.3) is true.
In the absence of the exterior force, which means that $\sigma = 0$, the generalized Helmholtz conditions of Theorem 3.2 reduce to the Helmholtz conditions in [3, Theorem 4.1]. We note that Theorem 4.1 of [3] contains four conditions that are necessary and sufficient for a semispray $S$ to be Lagrangian. In 2009, at a Conference in Levico-Terme, where the results of [3] were presented, Willy Sarlet told us that he believes that one of the four conditions of [3, Theorem 4.1] is redundant. Five years latter, we have the proof that Willy Sarlet was right. Next corollary, which is a consequence of Theorem 3.2, for $\sigma = 0$, reformulates and improves, in the spirit of the discussion with Willy Sarlet, the result of [3, Theorem 4.1].

**Corollary 3.3.** A semispray $S$ is Lagrangian if and only if there exists a non-trivial, semi-basic 1-form $\theta$ that satisfies the following Helmholtz conditions

\[
\begin{align*}
H_1) \quad & d_J \theta = 0; \\
H_2) \quad & d_\Phi \theta = 0; \\
H_3) \quad & \nabla d_v \theta = 0.
\end{align*}
\]

In [3, Theorem 4.1] there is an extra condition $d_h \theta = 0$ that has been used. As we have seen in the proof of the second part of Theorem 3.2, it can be shown that the three Helmholtz conditions $H_1 - H_3$ imply that $d_h \theta = d\beta$, for a basic 1-form $\beta$. Therefore, the new semi-basic 1-form $\tilde{\theta} = \theta - \beta$ satisfies the three Helmholtz conditions $H_1 - H_3$ as well as the fourth condition $d_h \tilde{\theta} = 0$, which was used in [3, Theorem 4.1].

We will provide now a local description of the three generalized Helmholtz conditions $GH_1 - GH_3$.

Consider a semispray, locally given by formula (2.2), and let $\theta = \theta_i dx^i$, $\sigma = \sigma_i dx^i$ be two semi-basic 1-forms on $TM$. We have

\[
\begin{align*}
LGH_H_1) \quad & g_{ij} = g_{ji}, \\
LGH_H_2) \quad & g_{ik} R^k_j - g_{jk} R^k_i = \frac{1}{2} \nabla \left( \frac{\partial \sigma_i}{\partial y^j} - \frac{\partial \sigma_j}{\partial y^i} \right) - \left( \frac{\partial \sigma_i}{\partial x^j} - \frac{\partial \sigma_j}{\partial x^i} \right), \\
LGH_H_3) \quad & \nabla g_{ij} = \frac{1}{2} \left( \frac{\partial \sigma_i}{\partial y^j} + \frac{\partial \sigma_j}{\partial y^i} \right).
\end{align*}
\]

Condition $LGH_3$ and the local expression of $d_h \theta = 0$ appear also in [4, Theorem 3.1], as conditions (1) and (2) that uniquely fix the nonlinear connection of a Lagrange space and a non-conservative force. If $\sigma = 0$, the conditions $LGH_1 - LGH_3$ represent the classic four Helmholtz conditions in terms of the multiplier matrix $g_{ij}$.

### 4. The dissipative case

In this section we restrict our results from the previous section to the particular case when the covariant force field is given by a semi-basic 1-form $\sigma$ that is $d_J$-closed. This slightly extends the dissipative case studied in [29], where the semi-basic 1-form $\sigma$ is $d_J$ exact, which means that it is given by $\sigma = d_J D$, for a given function $D$ on $TM$. We note that this case includes the classic dissipation of Rayleigh type, when the function $D$ is negative definite, quadratic in the velocities.
In Theorem 3.2, the semi-basic 1-form $\sigma$ is given, together with the semispray $S$. We will address now the case when for a given semispray $S$, the semi-basic 1-form $\sigma$, being of special type, is not a priori given, and we have to search for it, together with the Lagrangian $L$.

**Definition 4.1.** A semispray $S$ is said to be of dissipative type if there exists a Lagrangian $L$ and a $d_J$-closed semi-basic 1-form $\sigma$ on $TM$ such that $\delta_S L = \sigma$.

Consider $S$ a semispray of dissipative type. For the classic Rayleigh dissipation, we have that $i_S \sigma = i_S d_J D = \mathcal{C}(D) = 2D$. According to Proposition 2.2, we obtain that the variation of the energy function $E_L$, along the geodesics of the semispray $S$, is given by formula (2.14), which in this case reads $S(E_L) = 2D < 0$.

By reformulating the results of Theorem 3.2, we obtain a characterization of semisprays that are of dissipative type in terms of a non-trivial, semi-basic 1-form $\theta$, which will be the Poincaré-Cartan 1-form of the sought after Lagrangian, and a $d_J$-closed semi-basic 1-form $\sigma$, in which case it is not given, but we have to search for.

The freedom we have in the dissipative case, in searching for a $d_J$-closed semi-basic 1-form $\sigma$, will allow us to obtain also characterizations in terms of the semi-basic 1-form $\theta$ only. See the conditions $D_1$ and $D_3$ below.

Next theorem corresponds to, and was inspired by 25 Corollary 1, Theorem 1, Theorem 3).

**Theorem 4.2.** A semispray $S$ is of dissipative type if and only if there exist a non-trivial, semi-basic 1-form $\theta$ and a $d_J$-closed semi-basic 1-form $\sigma$ on $TM$ that satisfy one of the following equivalent sets of conditions

\begin{align*}
D_1 & : d_J \theta = 0, d_\theta \theta = 0;
D_2 & : d_J \theta = 0, d_\theta \theta = -d_h \sigma, \nabla d_\theta \theta = d_\sigma \sigma;
D_3 & : d_J \theta = 0, d_\theta \theta = 0, d_h d_\theta \theta = 0.
\end{align*}

**Proof.** We will prove the following implications: $D_1 \Rightarrow S$ is dissipative $\Rightarrow D_2 \Rightarrow D_3 \Rightarrow D_1$.

For the first implication, we assume that there is a non-trivial, semi-basic 1-form $\theta$ such that $d_J \theta = 0$ and $d_\theta \theta = 0$. From the first condition $D_1$, it follows that there exists a Lagrangian $L$, locally defined on $TM$, such that $\theta = d_J L$. Now, the second condition $D_1$ reads $0 = d_h d_J L = -d_J d_h L$. This means that the semi-basic 1-form $d_\theta L$ is $d_J$-closed and hence it is locally $d_J$-exact. Therefore, there exists a function $f$, locally defined on $TM$, such that $d_\theta L = d_J f$. Consider now the function $D = S(L) - 2f$. It follows that $d_J S(L) - 2d_h L = d_J D$. In view of the second expression (2.13) of $\delta_S L$, it follows that $\delta_S L = d_J D$, which shows that the semispray $S$ is of dissipative type, with the $d_J$-exact (and hence $d_J$-closed) semi-basic 1-form $\sigma = d_J D$.

For the second implication, we assume that the semispray $S$ is of dissipative type. By definition, it follows that there exist a Lagrangian $L$ and a $d_J$-closed semi-basic 1-form $\sigma$ on $TM$ such that $\delta_S L = \sigma$. Last condition implies that formula (3.3) is true, where $\theta = d_J L$ and $d_J \sigma = 0$. Therefore, the three generalized Helmholtz conditions $GH_1 - GH_3$ of Theorem 3.2 are satisfied. If we use the fact that $d_J \sigma = 0$, we have that the three conditions $GH_1 - GH_3$ imply the three conditions $D_2$.

For the third implication, we consider $\theta$ a non-trivial, semi-basic 1-form and $\sigma$ a $d_J$-closed semi-basic 1-form on $TM$ such that the three conditions $D_2$ are satisfied.

Using formulae (2.3) and (2.6), as well as the first two conditions $D_2$, it follows

\begin{equation}
3d_\theta h \theta = d_{[J,K]} \theta = d_J d_\theta \theta + d_\theta d_J \theta = -d_J d_\theta \sigma = d_h d_\theta \sigma = 0.
\end{equation}

We apply the derivation $d_J$ to both sides of last the condition $D_2$. Since $[J,v] = 0$, using formula (2.1), it follows that $d_J d_\sigma \sigma = -d_J d_\theta \sigma = 0$. Using the commutation formula (2.11), we have

\begin{equation}
0 = d_J d_\sigma \sigma = d_J \nabla d_\sigma \theta = \nabla d_J d_\sigma \theta + d_h d_\sigma \theta + 2i_R d_\sigma \theta.
\end{equation}
In the above formula we use that \( i_{R}d_{c}\theta = i_{R}d\theta = d_{R}\theta = 0 \) and \( d_{J}d_{c}\theta = 0 \). It follows that \( d_{h}d_{c}\theta = 0 \).

For the last implication, the idea of the proof has been borrowed from the proof of [25, Theorem 1]. Our proof is different and the idea is to show that \( d_{h}\theta \) is a closed, basic 2-form.

The second condition \( D_{3}, d_{R}\theta = 0 \), is equivalent to \( d_{h}d_{h}\theta = 0 \). Using (2.1), we have \( d_{h}d_{c}\theta + d_{c}d_{h}\theta = d_{h,v}\theta = -2d_{h}\theta = 0 \). In this last formula we use the last condition \( D_{3} \) and obtain \( d_{c}d_{h}\theta = 0 \). Since \( d_{h}\theta \) is a semi-basic 2-form it follows that

\[
(4.2) \quad dd_{h}\theta = d_{h}d_{h}\theta + d_{c}d_{h}\theta = 0.
\]

Above formula implies that \( d_{h}\theta \) is a basic 2-form and it is closed. It follows that it is locally exact. Therefore, there exists a basic 1-form \( \eta \) such that \( d_{h}\theta = d\eta \). We consider now the semi-basic 1-form

\[
(4.3) \quad \theta = \eta.
\]

Since \( \theta \) is a basic 1-form, we have that the semi-basic 1-forms \( \tilde{\theta} \) and \( \theta \) are simultaneously non-trivial. We have that \( d_{J}\tilde{\theta} = d_{J}\theta = 0 \) and \( d_{h}\tilde{\theta} = 0 \), which means that the non-trivial, semi-basic 1-form \( \tilde{\theta} \) satisfies the two conditions \( D_{1} \).

As we have seen in the proof of Theorem 4.2 see formula 4.3, semi-basic 1-forms \( \theta \) that satisfy different sets of conditions \( D_{1}, D_{2} \) or \( D_{3} \) might differ by a basic 1-form.

The three conditions \( D_{2} \) that we use in Theorem 4.2 are equivalent to the four conditions of Theorem 1 in the work of Mestdag, Sarlet and Crampin [25]. First condition \( D_{2} \) is equivalent to the symmetry of the \((0,2)\)-type tensors \( g \) and \( D^{V}g \), last condition \( D_{2} \) is equivalent to [24] (24)], while the second condition \( D_{2} \) is equivalent to [25] (25)). The correspondence between the semi-basic 1-form \( \theta = \theta_{j}dx^{j} \) in our formulation and the the \((0,2)\)-type tensor \( g = (g_{ij}) \) in the work of Mestdag, Sarlet and Crampin [25] is the following:

\[
g_{ij} = \theta_{j}/\partial y^{i}.
\]

The three conditions \( D_{3} \) of Theorem 4.2 are equivalent to the four conditions of Theorem 3 in the work of Mestdag, Sarlet and Crampin [25]. With the above correspondence between the semi-basic 1-form \( \theta \) and the \((0,2)\)-type tensor \( g \), we have that the condition \( d_{R}\theta \) is equivalent to condition [25] (44}) and the condition \( d_{h}d_{c}\theta = 0 \) is equivalent to the symmetry of the \((0,2)\)-type tensor \( D^{H}g \) in [24, Theorem 3].

Locally, the three equivalent sets of conditions \( D_{1}, D_{2} \) and \( D_{3} \) can be expressed as follows

\[
LD_{1}) \quad \frac{\partial \theta_{i}}{\partial y^{j}} = \frac{\partial \theta_{j}}{\partial y^{i}}; \quad \delta \theta_{i} = \delta \theta_{j} \quad \text{by} \quad \delta \theta_{i} = \frac{\partial \theta_{i}}{\partial x^{j}}.
\]

\[
LD_{2}) \quad g_{ij} = g_{ji}, \quad \frac{\partial g_{ij}}{\partial y^{k}} = \frac{\partial g_{ik}}{\partial y^{j}}, \quad g_{ik}R_{j}^{k} = \frac{\delta}{\delta x^{j}} \left( \frac{\partial D}{\partial y^{k}} \right) - \frac{\delta}{\delta x^{j}} \left( \frac{\partial D}{\partial y^{k}} \right), \quad \nabla g_{ij} = \frac{\partial^{2}D}{\partial y^{i}\partial y^{j}}.
\]

\[
LD_{3}) \quad g_{ij} = g_{ji}, \quad \frac{\partial g_{ij}}{\partial y^{k}} = \frac{\partial g_{ik}}{\partial y^{j}}, \quad g_{il}R_{jk}^{l} + g_{k}R_{lj}^{k} + g_{j}R_{ki}^{l} + g_{i}R_{jl}^{k} = 0; \quad g_{ij} = 0.
\]

In the last set of conditions \( LD_{3} \), \( g_{ijk} = \delta g_{ij}/\delta x^{k} - g_{i}^{\gamma}g_{j}^{\delta}g_{k}^{\gamma} \), where \( \Gamma_{j}^{i}_{k} = \partial N_{j}^{i}/\partial y^{k} \), is the \( h \)-covariant derivative of the tensor \( g \) with respect to the Berwald connection. The conditions in the third set \( LD_{3} \) represent conditions (45)-(46) in [25]. The conditions in the second set \( LD_{2} \) represent conditions (30)-(32) in [25].

For the formal integrability of the system \( D_{1} \): \( d_{J}\theta = 0 \) and \( d_{h}\theta = 0 \), we can follow the results of [5, 9], where the formal integrability of very similar systems was investigated. See the first-order partial differential operator \( P_{1} \) in [5, (4.2)] or the first-order partial differential operator \( P \) in [9, (13)]. Using a very similar proof as the one of Theorems 4.2 and 4.3 in [5] and Theorems 3 and 4 in [9], we can state the following result.
Proposition 4.3. The only obstruction for the formal integrability of the system $D_1$ is given by the condition
\[ d_R \theta = 0. \]

The condition (4.4) has been considered also in [25] as condition (44) of Theorem 3.

An important consequence of the previous proposition appears in dimension 2. In this case, the obstruction $d_R \theta = 0$ is automatically satisfied, being a semi-basic 3-form on a 2-dimensional manifold. Therefore, we obtain the following corollary.

Corollary 4.4. Any semispray on a 2-dimensional manifold is of dissipative type.

The first example of Section 5 in [25], as well as the example (7.2) that we discuss in Section 7 agree with the conclusion of the above corollary.

5. The gyroscopic case

In this section, we restrict the general results obtained in Section 3 to the particular case when the exterior covariant force field $\sigma$ is of the type $i S \omega$, for $\omega$ a basic 2-form. This corresponds to the gyroscopic case, studied in [25].

Definition 5.1. A semispray $S$ is said to be of gyroscopic type if there exists a Lagrangian $L$ and a basic 2-form $\omega$ on $T M$ such that $\delta_S L = i \sigma \omega$.

In this case, using Proposition 2.2 we obtain that $S(E_L) = 0$ and hence the Lagrangian energy is conserved along the integral curves of the Lagrangian system.

Next theorem provides a characterization of gyroscopic systems.

Theorem 5.2. A semispray $S$ is of gyroscopic type if and only if there exist a non-trivial, semi-basic 1-form $\theta$ and a basic 2-form $\omega$ on $T M$ that satisfy one of the following equivalent sets of conditions
\[ G_1 \) $d_J \theta = 0$, $d_R \theta = i \sigma d \omega$, $\nabla d \theta = 0$;
\[ G_2 \) $d_J \theta = 0$, $d_R \theta = i \sigma d_R \theta$, $\nabla d \theta = 0$.

Proof. We will prove the following implications: $S$ is gyroscopic $\implies G_1 \implies G_2 \implies S$ is gyroscopic.

For the first implication, we assume that $S$ is of gyroscopic type and therefore there exists a Lagrangian $L$ and a basic 2-form $\omega$ such that $\delta_S L = i \sigma \omega$. We will provide here a direct proof, which is a little bit simpler then the direct proof of Theorem 3.2 due to the particular form of the covariant force field $\sigma = i \sigma \omega$. Using third form of $\delta_S L$ in formula (2.13), we can write the condition that $S$ is of gyroscopic type as follows
\[ \nabla d_J L - d_h L = i \sigma \omega. \]

If we apply the derivation $d_J$ to the both sides of the above formula, use the commutation formula (2.11), as well as the commutation $d_h d_J = - d_J d_h$, we obtain
\[ 2 d_h d_J L + 2 i \eta d_J L = d_J i \sigma \omega. \]

To evaluate the right hand side of the above formula we will use the following commutation formula, [17 A.1, page 205],
\[ i_X d_K = - d_K i_X + L_K X + i_{[K,X]}, \]

\[ 2 d_h d_J L + 2 i \eta d_J L = d_J i \sigma \omega. \]
where $K$ is a vector valued 1-form and $X$ is a vector field. For $K = J$ and $X = S$, this commutation formula reads

\[(5.3) \quad d_J i_S \omega = -i_S d_J \omega + \mathcal{L}_C \omega + i_{[J,S]} \omega = 2 \omega. \]

In the above formula we did use the following calculations. Since $\omega$ is a basic 2-form it follows that both $\omega$ and $d \omega$ vanish whenever an argument is vertical. It follows that $i_J \omega = 0$ and $i_J d \omega = 0$, which implies $d_J \omega = 0$. For a vertical vector field $V$ on $TM$, we have that $i_V \omega = 0$ and $i_V d \omega = 0$ and hence $\mathcal{L}_V \omega = 0$. In particular, for $V = C$, we have $\mathcal{L}_C \omega = 0$. Using the fact that $[J, S] = h - v = \text{Id} - 2v$ it follows that $i_{[J, S]} \omega = i_{4d} \omega = 2 \omega$.

We consider the non-trivial, semi-basic 1-form $\theta = d_J L$, and use the fact that $i_R d_J L = 0$. Then, from formula $[5.1]$ it follows $d_J \theta = \omega$ and hence $\mathcal{L}_S d_J \theta = \mathcal{L}_S \omega$. The condition $\delta_S \omega = i_S \omega$ implies $\mathcal{L}_S d \theta = di_S \omega$, which can be further written as follows

\[\mathcal{L}_S d_J \theta + \mathcal{L}_S d_h \theta = \mathcal{L}_S \omega - i_S d \omega \iff \mathcal{L}_S d_J \theta = -i_S d \omega.\]

In the last formula above, if we use $[2.10]$ and $\delta \Phi$ preserves $d \omega$ then we obtain

\[\nabla d_J \theta - d \Phi \delta \theta = -i_S d \omega.\]

In both sides of this formula we separate the semi-basic 2-forms and obtain $d \Phi \delta \theta = i_S d \omega$, and what remains is $\nabla d_J \theta = 0$. Therefore, we have shown that all three conditions $G_1$ are satisfied.

For the second implication, we apply the derivation $d_J$ to both sides of the second condition $G_1$. Using the commutation formula $[2.7]$ and $[5.2]$ it follows

\[3d_J \theta = d_{[J, S]} \theta = d_J d \Phi \delta \theta = d_J i_S d \omega = i_{[J, S]} d \omega = 3d \omega.\]

This implies $d_J \theta = d \omega$ and hence the condition $G_2$ is satisfied.

For the last implication, we assume now that there is a non-trivial, semi-basic 1-form $\theta$ that satisfies the set of conditions $G_2$. We apply the derivation $d_J$ to both sides of the last condition $G_2$ and use the commutation formula $[2.11]$. It follows

\[0 = d_J \nabla d_J \theta = d_J d_h \theta + 2i_R d_h \theta = d_J d_h \theta = d_h d_J \theta.\]

Last formula expresses the fact that $d_h \theta$ is a basic 2-form. We denote it $d_h \theta = \omega$ and have that

\[d \omega = dd_h \theta = d_h^2 \theta = d_R \theta.\]

From the first condition $G_2$, and the fact that the semi-basic 1-form $\theta$ is non-trivial, it follows that there exists a locally defined Lagrangian $L$ such that $\theta = d_J L$. We have now $d_h d_J L = \omega$. Since $\omega$ is a basic 2-form, it follows that $2 \omega = d_J i_S \omega$ and hence we obtain

\[-d_J d_h L = \frac{1}{2} d_J i_S \omega \iff d_J (2d_h L + i_S \omega) = 0.\]

From the last formula it follows that there is a locally defined function $f$ on $TM$ such that $2d_h L + i_S \omega = d_J f$. We consider now the function $g = S(L) - f$ and obtain

\[(5.4) \quad \delta_S \omega = d_J S(L) - 2d_h L = i_S \omega + d_J g.\]

Let us note that so far, for this implication, we have used first and third conditions of the set $G_2$ to obtain formula $[5.4]$. This implication corresponds to Proposition 3. However, in formula $[5.4]$ $\beta = d_J g$ is a semi-basic 1-form, while in Proposition 3 the authors claim that $\beta$ is a basic 1-form. We will prove that is due to the second condition of the set $G_2$ that will imply the fact that $\beta$ is a basic 1-form and closed.

From formula $[5.4]$ it follows

\[(5.5) \quad \mathcal{L}_S d \theta = di_S d_h \theta + dd_J g,\]
which can be written further as follows

\[ \mathcal{L}_g d\theta + \mathcal{L}_S d\gamma = \mathcal{L}_g d\theta - i_S d\eta + dd_f g. \]

Using the last two conditions of the set \( G_2 \) and the above formula, it follows that \( dd_f g = 0 \). We use this in formula (5.3) and obtain

\[ \mathcal{L}_S d\eta = di_S \omega, \]

which in view of Theorem 3.1 it follows that \( \delta_S L = i_S \omega \). Last formula shows that the spray \( S \) is of gyroscopic type.

The characterization of a semispray of gyroscopic type given by conditions \( G_1 \) in terms of a semi-basic 1-form corresponds to [25, Theorem 2], where the conditions where expressed in terms of a multiplier matrix, see [25, (36)-(37)]. The characterization using the second set of conditions \( G_2 \) corresponds to [25, Theorem 4], in terms of a multiplier matrix, [25, (47)].

Locally, the two sets of equivalent conditions \( G_1 \) and \( G_2 \) can be written as follows.

\[ LG_1) \quad g_{ij} = g_{ji}, \quad \frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ij}}{\partial y^k}, \quad g_{ik} R^k_j - g_{jk} R^k_i = \left( \frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{jk}}{\partial x^k} + \frac{\partial \omega_{ki}}{\partial x^k} \right) y^k, \quad \nabla g_{ij} = 0, \]

\[ LG_2) \quad g_{ij} = g_{ji}, \quad \frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ij}}{\partial y^k}, \quad g_{ik} R^k_j - g_{jk} R^k_i = \left( g_{il} R^l_{jk} + g_{kl} R^l_{ij} + g_{jl} R^l_{ki} \right) y^k, \quad \nabla g_{ij} = 0. \]

Last two conditions \( LG_1 \) represent conditions [25, (38)-(39)], while second condition \( LG_2 \) represents condition [25, (49)].

In the case when the basic 2-form \( \omega \) is closed, which includes the 2-dimensional case, we have that the three conditions \( G_1 \) coincide with the classic Helmholtz conditions \( H_1 \) - \( H_3 \) of Corollary 3.3. This equivalence, expressed in terms of the corresponding multiplier matrix, has been discussed in [25], see the remark on page 62. Alternatively, we can explain this as follows. Since \( \omega \) is closed, it follows that there exists a locally defined basic 1-form \( \alpha \) such that \( \omega = d\alpha \). Using the fact that \( \alpha = d_j i_S \alpha \), we have that

\[ i_S \omega = i_S d\alpha = \mathcal{L}_S \alpha - di_S \alpha = \mathcal{L}_S d_j i_S \alpha - di_S \alpha = \delta_S (i_S \alpha). \]

Therefore we have that \( \delta_S L = i_S \omega \) is equivalent to \( \delta_S (L - i_S \alpha) = 0 \), with the new Lagrangian \( L - i_S \alpha \). Therefore, the set of conditions \( G_1 \) for \( \theta \) and \( \omega \) is equivalent to the set of conditions \( H_1 \) - \( H_3 \) for the non-trivial, semi-basic 1-form \( \theta - \alpha \).

6. **The homogeneous case**

In this section we study the case when all the involved geometric objects are homogeneous with respect to the fibre coordinates. Consequently, we will restrict all our geometric structures to the slit tangent bundle \( T_0 M = TM \setminus \{0\} \).

A semispray \( S \in \mathfrak{X}(T_0 M) \) is called a *spray* if it is 2-homogeneous, which means that \([C,S] = S\). The coefficients \( G^j(x,y) \) in formula (2.2), which are locally defined functions on \( T_0 M \), are homogeneous of order 2 in the fibre coordinates.

**Definition 6.1.** Consider \( S \) a spray and \( \sigma \in \Lambda^1(T_0 M) \) a semi-basic 1-form, homogeneous of order \( p \in \mathbb{N}^* \). We say that the spray \( S \) is of *Finslerian type with covariant force field \( \sigma \) if there exists a Lagrangian \( L \in C^\infty(T_0 M) \), homogeneous of order \( p \), such that \( \delta_S L = \sigma \).

We will discuss now some particular cases of Definition 6.1 in the Finslerian context. For this, we recall the notion of a Finsler function, Finsler metrizability and projective metrizability.

**Definition 6.2.** A positive function \( F : TM \to \mathbb{R} \) is called a *Finsler function* if it satisfies
i) $F$ is smooth on $T_0M$ and continuous on the null section;
ii) $F$ is positive homogeneous with respect to the fibre coordinates: $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda \geq 0$;
iii) $F^2$ is a regular Lagrangian on $T_0M$.

When $\sigma = 0$, depending on the values of $p$, we obtain two important particular cases of the inverse problem of Lagrangian mechanics. The case $\sigma = 0$, $p = 1$, and $L = F$, for a Finsler function $F$, is known as the projective metrizability problem [5, 6, 17, 18]. The case $\sigma = 0$, $p = 2$, and $L = F^2$, for a Finsler function $F$, is known as the Finsler metrizability problem [7, 27]. For the general case, when $\sigma \neq 0$, we will see that we also have to make distinction between the two cases $p = 1$ and $p > 1$.

Within this homogeneous context we will also pay attention to the dissipative and gyroscopic cases. For the gyroscopic case, we have that $\sigma = i\omega$ is homogeneous of order 1 and therefore we will analyse this only in the case when the order of homogeneity is $p = 1$.

6.1. The case $p > 1$. Being a particular case of the inverse problem of Lagrangian mechanics, we can use the three Helmholtz conditions $H_1 - H_3$ of the Corollary 3.3 to characterize the Finsler metrizability problem in terms of a semi-basic 1-form, homogeneous of order 1. In this case, it has been shown recently by Prince [28, Lemma 3.4] and Rossi [29, Theorem 3.4], with different local techniques and using the multiplier matrix expression of the Helmholtz conditions, that the Helmholtz condition which involves the Jacobi endomorphism is a consequence of the other Helmholtz conditions. Our next theorem provides a similar result, in the general context of a $p$-homogeneous covariant force field $\sigma$, for $p > 1$. More exactly, we will prove that, in this homogeneous context, the generalized Helmholtz condition $GH_2$ is a consequence of the other two generalized Helmholtz conditions $GH_1$ and $GH_3$ of Theorem 3.2.

**Theorem 6.3.** Consider $S$ a spray and $\sigma \in \Lambda^1(T_0M)$ a semi-basic 1-form, homogeneous of order $p > 1$. The spray $S$ is of Finslerian type with covariant force field $\sigma$ if and only if there exists a non-trivial, semi-basic form $\theta \in \Lambda^1(T_0M)$, homogeneous of order $(p - 1)$ that satisfies the two generalized Helmholtz conditions $GH_1$ and $GH_3$ of Theorem 3.2.

**Proof.** Consider $\theta$ a non-trivial, semi-basic 1-form, homogeneous of order $(p - 1)$ that satisfies the two generalized Helmholtz conditions $GH_1$ and $GH_3$ of Theorem 3.2. We will prove that $\delta_S(i\omega \theta/p) = \sigma$, which means that the spray $S$ is of Finslerian type with covariant force field $\sigma$. In this case, the Lagrangian is given by $L = i\omega \theta/p$ and it is homogeneous of order $p$.

For the $(p - 1)$-homogeneous, non-trivial, semi-basic 1-form $\theta$ that satisfies the condition $GH_1$, we use [3, Proposition 4.2]. It follows that

$$L = \frac{1}{p} i\omega \theta \tag{6.1}$$

is the only $p$-homogeneous Lagrangian that satisfies $d_j L = \theta$.

We use the same argument that we used in the proof of Theorem 3.2 from formula (3.11) to formula (3.15). It follows that the condition $GH_3$ implies that the 2-form $d_j \theta - d_j \sigma/2$ is a basic form and hence it is homogeneous of order 0. Since the semi-basic 1-form $\theta$ is homogeneous of order $p - 1$ and the semi-basic 1-form $\sigma$ is homogeneous of order $p > 1$ it follows that the 2-form $d_j \theta - d_j \sigma/2$ is homogeneous of order $p - 1 \neq 0$. Consequently we obtain that this 2-form has to vanish and therefore we have

$$d_j \theta = \frac{1}{2} d_j \sigma. \tag{6.2}$$
In formula (6.2) we apply to the left the inner product \( i_S \) and use the commutation formula (5.2) to evaluate \( i_S d_h \theta \) and \( i_S d_J \sigma \). It follows

\[
-d_h i_S \theta + \mathcal{L}_S \theta + i_{[h,S]} \theta = \frac{1}{2} \left( -d_J i_S \sigma + \mathcal{L}_C \sigma + i_{[J,S]} \sigma \right).
\]

Using the fact that \( d_J L = \theta \) it follows that \( i_{[h,S]} \theta = i_{[S]h} \theta = -d_v L \). Now, we use the \( p \)-homogeneity of \( \sigma \) and \( i_{[J,S]} \sigma = i_{h-\nu} \sigma = \sigma \) and hence we obtain

\[
-pd_h L + \mathcal{L}_S d_J L - d_v L = -\frac{1}{2} d_J i_S \sigma + \frac{p+1}{2} \sigma.
\]

From the above formula we can express the Lagrange differential as follows

\[
(6.3) \quad \delta_S L = (p-1)d_h L - \frac{1}{2} d_J i_S \sigma + \frac{p+1}{2} \sigma.
\]

We will evaluate now the two 2-forms in both sides of formula \( GH_3 \) on the pair of vectors \((C, S)\). First we have

\[
i_C d_v \theta = i_{i_C} d \theta = \mathcal{L}_C \theta = (p-1)\theta.
\]

From this formula, using also formula (6.1) we have

\[
i_S i_C d_v \theta = (p-1)i_S \theta = p(p-1)L.
\]

According to [3, Proposition 3.6], in the homogeneous case, the dynamical covariant derivative \( \nabla \) commutes with the inner products \( i_S \) and \( i_C \). From the above formula, it follows that the value of the 2-form \( \nabla d_v \theta \) on the pair of vectors \((C, S)\) is given by

\[
(6.4) \quad i_S i_C \nabla d_v \theta = p(p-1)S(L).
\]

Since \((F+J)(C) = S\), it follows that \((i_{F+J}d_J \sigma)(C, S) = d_J \sigma(S, S) = 0\). Using the fact that \( \sigma \) is semi-basic and \( p \)-homogeneous, we have \( i_C d_v \sigma = i_C d \sigma = \mathcal{L}_C \sigma = p \sigma \). It follows that the value of the 2-form from the right hand side of \( GH_3 \) on the pair of vectors \((C, S)\) is given by

\[
(6.5) \quad i_S i_C \left( d_v \sigma - \frac{1}{2} i_{F+J} d_J \sigma \right) = pi_S \sigma.
\]

From the two formulae (6.4) and (6.5) it follows that

\[
(6.6) \quad (p-1)S(L) = i_S \sigma.
\]

If we use now the commutation formula (2.1) for \( J \) and \( S \) we obtain

\[
d_{[J,S]} L = d_J S(L) - \mathcal{L}_S d_J L.
\]

In the formula above we replace \( S(L) \) from (6.6) and use the fact that \([J, S] = h - v\). It follows

\[
d_h L - d_v L = \frac{1}{p-1} d_J i_S \sigma - \mathcal{L}_S d_J L.
\]

From the above formula we can express the Lagrange differential as follows

\[
(6.7) \quad \delta_S L = \frac{1}{p-1} d_J i_S \sigma - 2d_h L.
\]

From the two formulae (6.3) and (6.7) it follows that \( \delta_S L = \sigma \) and hence the spray \( S \) is of Finslerian type with covariant force field \( \sigma \).

From Theorem 6.3, in the particular case when \( \sigma = 0 \), we obtain the following corollary that provides a characterization of the Finsler metrizability problem in terms of a semi-basic 1-homogeneous 1-form.
Corollary 6.4. A spray $S$ is Finsler metrizable if and only if there exists a semi-basic 1-form $\theta \in \Lambda^1(T_0M)$ that satisfies the following sets of conditions

FMA) $d\theta$ is a symplectic form, $i_\theta \theta > 0$;
FMD) $d_J\theta = 0$, $\nabla d_\theta \theta = 0$, $\mathcal{L}_C\theta = \theta$.

The first two differential conditions, FMD, for Finsler metrizability, represent the two Helmholtz conditions $H_1$ and $H_3$ of Corollary 3.3. As we have seen in the proof of Theorem 6.3, these two conditions plus the homogeneity condition $\mathcal{L}_C\theta = \theta$ assure the existence of a 2-homogeneous Lagrangian $L = i_\theta \theta/2$ such that $\delta_S L = 0$. The algebraic conditions FMA assure that the Lagrangian $L$ is regular and it can be written as $L = F^2$, for a Finsler function $F$.

Another characterization for the Finsler metrizability problem has been obtained by Muzsnay in [27]. More exactly, in [27, Theorem 1], Muzsnay has shown that a spray $S$ is Finsler metrizable if and only if there exists a 2-homogeneous regular Lagrangian $L$ that satisfies the condition $d_h L = 0$.

In the next theorem we provide a similar result in the general context of an $p$-homogeneous force field, with $p > 1$.

Theorem 6.5. Consider a spray and $\sigma \in \Lambda^1(T_0M)$ a semi-basic 1-form, homogeneous of order $p > 1$. The spray $S$ is of Finslerian type with covariant force field $\sigma$ if and only if there exists a $p$-homogeneous Lagrangian $L \in C^\infty(T_0M)$ such that

$$
2(p-1)d_h L = (1-p)\sigma + d_J i_\sigma \sigma.
$$

(6.8)

Proof. For the given spray $S$ and the semi-basic 1-form $\sigma$, homogeneous of order $p > 1$, we assume that there is a Lagrangian $L$, homogeneous of order $p$, such that $\delta_S L = \sigma$. We apply to both sides of this formula the inner product $i_\sigma$. It follows

$$
i_\sigma \sigma = i_\sigma \delta_S L = \mathcal{L}_S i_\sigma d_J L - S(L) = \mathcal{L}_S C(L) - S(L) = (p-1)S(L).
$$

In the second expression of the Lagrange differential $\delta_S L$ from formula (2.13) we replace the expression of $S(L)$ from the above formula. It follows

$$
\frac{1}{p-1} d_J i_\sigma \sigma - 2d_h L = \sigma,
$$

which is formula (6.8).

For the converse we assume that there is a $p$-homogeneous Lagrangian $L$ that satisfies formula (6.8). We apply to both sides of this formula the inner product $i_\sigma$. For the left hand side we have $2(p-1)i_\sigma d_h L = 2(p-1)hS(L) = 2(p-1)S(L)$, since $hS = S$. For the right hand side we have $i_\sigma d_J i_\sigma \sigma + (1-p)i_\sigma \sigma = C(i_\sigma \sigma) + (1-p)i_\sigma \sigma = (p+1)i_\sigma \sigma + (1-p)i_\sigma \sigma = 2i_\sigma \sigma$. It follows that formula (6.9) is true. We continue the argument we used in the proof of Theorem 6.3 after formula (6.6) and obtain formula (6.7). In this one we replace $d_h L$ from formula (6.8) and obtain $\delta_S L = \sigma$, which completes the proof. $\square$

We will analyze formula (6.8) in the dissipative case. In this case, the covariant force field $\sigma$ is of the form $\sigma = d_J D$. Since $\sigma$ is homogeneous of order $p$, then the potential function $D$ (if we want it to be homogeneous as well) is homogeneous of order $(p+1)$. In formula (6.8), we evaluate the right hand side. First we have

$$
d_J i_\sigma \sigma = d_J i_\sigma d_J D = d_J C(D) = (p+1) d_J D = (p+1)\sigma.
$$

From Theorem 6.5 we obtain the following characterization of Finslerian dissipative sprays.
Corollary 6.6. A spray $S$ is of Finslerian type with a dissipative covariant force field $\sigma \in \Lambda^1(T_0 M)$, homogeneous of order $p > 1$, if and only if there exists a Lagrangian $L \in C^\infty(T_0 M)$, homogeneous of order $p$, such that

$$
(p - 1)d_hL = \sigma.
$$

In the particular case when $\sigma = 0$, the result of the above corollary provides a characterization of Finsler metrizable sprays and it reduces to the result obtained by Muzsnay in [27, Theorem 1].

6.2. The case $p = 1$. A set of Helmholtz conditions for the projective metrizability problem, in terms of a degenerate multiplier matrix (the angular matrix) has been proposed by Crampin, Mestdag and Saunders in [11]. The equivalence of these Helmholtz conditions with other conditions expressed in terms of a semi-basic 1-form has been showed by the same authors in [11] Theorem 4. Next theorem gives a similar characterization, in the more general context, in the presence of a covariant force field $\sigma$, homogeneous of order 1. In this general context, the semi-basic 1-form $\sigma$ has to satisfy the condition $i_\sigma \sigma = 0$. Let us explain this. Consider a spray $S$ and a 1-homogeneous, semi-basic 1-form $\sigma$. We assume that there is a 1-homogeneous Lagrangian $L$ such that $\delta S = \sigma$. If we apply $i_\sigma$ to both sides of this formula we obtain $S(C(L) - L) = i_\sigma \sigma$ and hence we necessarily should have $i_\sigma \sigma = 0$.

Theorem 6.7. Consider $S$ a spray and a semi-basic 1-form $\sigma \in \Lambda^1(T_0 M)$, homogeneous of order 1 that satisfies the condition $i_\sigma \sigma = 0$. The spray $S$ is of Finslerian-type, with covariant force field $\sigma$ if and only if there exists a non-trivial, 0-homogeneous semi-basic 1-form $\theta \in \Lambda^1(T_0 M)$ that satisfies one of the following two equivalent sets of conditions

i) $d_J \theta = 0$, $d_h \theta = \frac{1}{2} d_J \sigma$;

ii) $GH_1), GH_2), GH_3)$.

Proof. In view of Theorem 5.2, it remains to prove the equivalence of the two sets of conditions.

For the first implication, we assume that there exists a non-trivial, 0-homogeneous, semi-basic 1-form $\theta$ that satisfies the set of conditions i). Since $d_J \theta = 0$, using [3] Proposition 4.2, it follows that $L = i_\sigma \theta$ is the only 1-homogeneous Lagrangian such that $\theta = d_J L$. The second condition i) can be written as follows $d_J (2d_h L + \sigma) = 0$, which implies that there exists a locally defined 2-homogeneous function $f$ such that $2d_h L + \sigma = -d_J f$. We consider now the 2-homogeneous function $g = f - S(L)$. It follows $\delta_S L = \sigma + d_J g$. We apply $i_\sigma$ to both sides of the last formula and obtain $0 = C(g) = 2g$. Therefore $\delta_S L = \sigma$, which in view of Theorem 3.2 we have that the set of conditions ii) are satisfied.

For the converse, assume that there exists a non-trivial, 0-homogeneous, semi-basic 1-form $\theta$ that satisfies the three condition $GH_1 - GH_3$. As we have seen in the proof of Theorem 5.2, the two conditions $GH_2$ and $GH_3$ imply that there exists a basic 1-form $\beta$ such that formula (5.13) is satisfied. The 0-homogeneous, semi-basic 1-form $\theta = \theta - \beta$ is non-trivial, satisfies the set of conditions i) and this completes the proof.

When $\sigma = 0$, the two conditions i) were used to characterize the projective metrizability of a spray in [3, Theorem 3.8]. Again, when $\sigma = 0$, the three conditions ii) were expressed in terms of the angular metric of some Finsler function as classic Helmholtz conditions in [11] and their equivalence with the set of conditions i) was proven in [11, Theorem 4].

In the particular case when the covariant force field $\sigma$ is of gyroscopic type, we obtain, using Theorem 6.7, the following characterization for Finslerian gyroscopic sprays.
Corollary 6.8. A spray $S$ is of Finslerian gyroscopic type if and only if there exists a non-trivial, semi-basic 1-form $\theta \in \Lambda^1(T_0M)$ that satisfies the following conditions

$$(6.10) \quad \mathcal{L}_C \theta = 0, \quad d_J \theta = 0, \quad d_h \theta = \omega,$$

for $\omega$ a basic 2-form.

Proof. If $\omega$ is a basic 2-form, using formula (5.3), it follows that $d_J S \omega = 2 \omega$. Therefore, the last condition i) of Theorem 6.7 is equivalent to the last condition (6.10). □

As we have seen in the proof of Theorem 5.2, the last two conditions (6.10) are necessary conditions for an arbitrary semispray to be of gyroscopic type. Above Corollary 6.8 states that in the homogeneous case, these two conditions are also sufficient.

In the particular case when $\omega = 0$, the equations (6.10) coincide with the differential equations [5, (3.8)], which together with some algebraic equations, provide a characterization of projectively metrizable sprays.

7. Examples

7.1. Non-variational, projectively metrizable sprays that are dissipative. In this subsection we will provide examples of projectively metrizable sprays which are not Finsler metrizable and hence not variational but are of dissipative type.

A spray $S$ is projectively metrizable if there exists a Finsler function $F$ such that $\delta S F = 0$. We underline the fact that $F$ is a Finsler function and it is not a regular Lagrangian. However, for a Finsler function $F$, we have that $L = F^2$ is a regular Lagrangian. A spray $S$ is Finsler metrizable if there exists a Finsler function $F$ such that $\delta S F^2 = 0$. In this case, we say that $S$ is the geodesic spray of the Finsler function $F$.

In [6] it has been shown that for a given geodesic spray, its projective class contains infinitely many sprays that are not Finsler metrizable. More exactly, in [6, Theorem 5.1] it has been shown that if $S_F$ is the geodesic spray of a Finsler function $F$, then there are infinitely many values of $\lambda \in \mathbb{R}$ such that the projectively related spray $S = S_F - 2\lambda FC$ is not Finsler metrizable. We consider such a spray $S$, which is projectively metrizable and hence satisfies $\delta S F = 0$, and we will prove that the spray $S$ is of dissipative type. Using the fact that $S_F$ is the geodesic spray of $F$ we have that $S_F(F) = 0$ and hence $S(F) = S_F(F) - 2\lambda FC(F) = -2\lambda F^2$. Therefore

$$(7.1) \quad \delta S F^2 = 2S(F)d_J F + 2F \delta S F = 2S(F)d_J F = -4\lambda F^2 d_J F.$$

The semi-basic 1-form $\sigma = -4\lambda F^2 d_J F$ is $d_J$-closed and from formula (7.1) it follows that the spray $S$ is of dissipative type.

Within the same class of sprays that we have discussed above, we will consider now a concrete example in dimension 2. We consider the spray $S \in \mathfrak{X}(\mathbb{R}^2 \times \mathbb{R}^2)$, given by

$$(7.2) \quad S = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} - \left((y^1)^2 + (y^2)^2\right) \frac{\partial}{\partial y^1} - 4y^1 y^2 \frac{\partial}{\partial y^2}.$$

In [1, Example 7.2] it has been shown that the system of second order ordinary differential equations corresponding to this spray is not variational. Using different techniques, in [7, §5.3] it has been shown that the spray $S$ is not Finsler metrizable and hence it cannot be variational as well. Since it is a 2-dimensional spray, it follows that $S$ is projectively metrizable. According to Corollary 4.4, it follows that the sprays $S$ is of dissipative type. We will show here directly, that the spray $S$ is of dissipative type by using the characterization $D_1$ of Theorem 4.2. We will prove that there exists
a semi-basic 1-form $\theta = \theta_1(x, y)dx^1 + \theta_2(x, y)dx^2$ that satisfies the two conditions $D_1$, which can be written as follows

$$
(7.3) \quad \frac{\partial \theta_1}{\partial y^2} - \frac{\partial \theta_2}{\partial x^1} = y^2 \left( \frac{\partial \theta_1}{\partial y^1} - 2 \frac{\partial \theta_2}{\partial y^2} \right) + y^1 \frac{\partial \theta_1}{\partial y^2}.
$$

For the given spray \( \mathcal{L} \), the coefficients of the canonical nonlinear connection are given by

$$
N_1^i = y^1, \ N_2^i = y^2, \ N_1^2 = 2y^2, \ N_2^2 = 2y^1.
$$

Using first condition \( (7.3) \), the second condition \( (7.3) \) can be written as follows

$$
(7.4) \quad \frac{\partial \theta_1}{\partial x^2} - \frac{\partial \theta_2}{\partial x^1} = y^2 \left( \frac{\partial \theta_1}{\partial y^1} - 2 \frac{\partial \theta_2}{\partial y^2} \right) + y^1 \frac{\partial \theta_1}{\partial y^2}.
$$

It is easy to see that the PDE system \( (7.4) \) has solutions. For example $\theta_1(x^1, y^1)dx^1 + \theta_2(x^2, y^2)dx^2$ is a solution to the system \( (7.4) \) if and only if $\partial \theta_1/\partial y^1 = 2\partial \theta_2/\partial y^2$. A Riemannian solution to this system is provided by $\partial \theta_1/\partial y^1 = 2\partial \theta_2/\partial y^2 = 2$. In this case the Lagrangian function and the dissipative function are given by

$$
L = \frac{1}{2} \left( 2(y^1)^2 + (y^2)^2 \right), \quad D = -\frac{2}{3}(y^1)^3 - 2y^1(y^2)^2.
$$

It is easy to check directly that for the spray $S$ given by formula \( (7.2) \), the Lagrangian $L$ and the dissipative function $D$ given by formula \( (7.5) \), we have $\delta S L = d_j D$, which means that $S$ is of dissipative type.

### 7.2. A class of gyroscopic semisprays.

Historically, gyroscopic systems are systems of second order ordinary differential equations in $\mathbb{R}^n$, of the form

$$
(7.6) \quad \frac{d^2 x^i}{dt^2} = A_j^i \frac{dx^j}{dt} + B_j^i x^i,
$$

where $A_j^i$ is a constant, skew symmetric matrix and $B_j^i$ is a constant, symmetric matrix.

We will consider now a generalization of the above system, on some open domain $\Omega \subset \mathbb{R}^n$.

$$
(7.7) \quad \frac{d^2 x^i}{dt^2} + 2N_j^i(x) \frac{dx^j}{dt} + V^i(x) = 0.
$$

Consider $g_{ij}$ a scalar product in $\mathbb{R}^n$. Using the conditions $G_1$ of the Theorem \( 5.2 \) we will prove that the system \( (7.7) \) is of gyroscopic type, with the Lagrangian function $L(x, y) = g_{ij} y^i y^j / 2$, if and only if the functions $N_j^i(x)$ and $V^i(x)$ satisfy the following conditions

$$
(7.8) \quad g_{ik} N_j^k + g_{jk} N_i^k = 0, \quad g_{ik} \frac{\partial V^k}{\partial x^j} - g_{jk} \frac{\partial V^k}{\partial x^i} = 0.
$$

In the particular case when $g_{ij} = \delta_{ij}$ is the Euclidean metric, the functions $N_j^i$ are constant and $V^i(x)$ are linear functions, the conditions \( (7.8) \) assure that \( (7.6) \) is a gyroscopic system.

We will use the local version $LG_1$ of the set of conditions $G_1$ of Theorem \( 1.2 \) to test when the system \( (7.7) \) is of gyroscopic type. For the considered Lagrangian $L = g_{ij} y^i y^j / 2$, its Poincaré-Cartan 1-form is given by $\theta = d_j L = g_{ij} y^j dx^i$. Therefore, we want to find the necessary and sufficient conditions for the existence of a basic 2-form $\omega \in \Lambda^2(\Omega)$ that satisfies the conditions $G_1$ of Theorem \( 4.2 \). As we have seen in the proof of Theorem \( 4.2 \) the basic 2-form $\omega$ is necessarily given by the following formula $d_h \theta = \omega$. Locally, the components $\omega_{ij}$ of the 2-form $\omega$ are given by

$$
(7.9) \quad \omega_{ij} = N_t^k g_{kj} - N_j^k g_{ki}.
$$
We pay attention now to the second condition \(LG_1\), which reads \(\nabla g_{ij} = 0\) and implies
\[
N_i^k g_{kj} + N_j^k g_{ki} = 0,
\]
which is first condition \(\text{7.8}\). From the two formulae (7.9) and (7.10) it follows
\[
N_j^i(x) = \frac{1}{2} g^{ik} \omega_{jk}(x).
\]
We use formula (2.4) to compute the local components \(R^i_j\) of the Jacobi endomorphism. These are given by
\[
R^i_j = 2 \frac{\partial N^k_j}{\partial x^i} y^j + \frac{\partial v^k_j}{\partial x^i} - \frac{\partial N^k_i}{\partial x^j} y^j - N^k_i N^j_l.
\]
From this formula, and using formula (7.11), it follows
\[
R^k_j g_{ik} - g_{kj} R^k_i = \left( \frac{\partial \omega_{ij}}{\partial x^l} + \frac{\partial \omega_{lj}}{\partial x^i} + \frac{\partial \omega_{jl}}{\partial x^i} \right) y^j + g_{ik} \frac{\partial v^k_j}{\partial x^i} - g_{jk} \frac{\partial v^k_i}{\partial x^i}.
\]
Therefore, last condition \(LG_1\) is satisfied if and only if the second condition (7.8) is satisfied. It follows that the system (7.7) is gyroscopic if and only if the two conditions (7.8) are satisfied.

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IOAN BUCATARU, FACULTY OF MATHEMATICS, ALEXANDRU IOAN CUZA UNIVERSITY, IASI, ROMANIA
URL: http://www.math.uaic.ro/~bucataru/

OANA CONSTANTINESCU, FACULTY OF MATHEMATICS, ALEXANDRU IOAN CUZA UNIVERSITY, IASI, ROMANIA
URL: http://www.math.uaic.ro/~oanacon/