Neutral regular black hole solution in Rastall gravity

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In this work, we investigate the static, spherically symmetric black hole solutions in a generalized Rastall gravity. In particular, the regular black hole solution is uncharged. The properties of the regular black hole solutions are discussed. Moreover, the parameter \(C_M\) is shown to be related to the degree of violation of energy-momentum conservation in Rastall theory.

\textbf{Keywords}: regular black hole, generalized Rastall gravity, static spherically symmetrical spacetime
INTRODUCTION

The black hole is an interesting prediction of general relativity, a compact celestial body with enormous mass and immense gravity. Since 2015, LIGO and Virgo collaborations have detected various gravitational waves emanated from the coalescence of black holes [1–6], which serves as one of the most crucial evidence for the existence of black holes. According to general relativity, singularity might be found accompanying the black hole solution. In the vicinity of the singularity, the spacetime curvature approaches infinite, and all classical theories become invalid. Fortunately, according to the cosmic censorship hypothesis proposed by Penrose, it is understood that the singularity is typically hidden behind the event horizon. The latter is a one-way membrane which prevents the singularity from affecting any observer located outside of the horizon. Alternatively, in 1968, Bardeen proposed a regular black holes metric, where a region of nonsingular spacetime substitutes the central singularity [7]. On the other hand, the metric asymptotically coincides with that of the Schwarzschild solution at infinity. Subsequently, following this line of thought, various models for regular black holes were proposed [10–14]. Later, it was argued that the black hole solutions can be physically interpreted in terms of nonlinear electrodynamics with magnetic monopole [8, 9]. Further investigations concerning the quasinormal modes of regular black hole metrics have also been carried out [15–19], and the corresponding metrics are shown to be stable against various types of perturbations. However, nonlinear electrodynamics encounters several potential difficulties. Firstly, astronomical objects are, by and large, electrically neutral, or do not carry a substantial amount of charges. Secondly, while the (linear) quantum electrodynamics is one of the most accurately validated theories in physics, there is still a lack of strong experimental support for nonlinear electrodynamics. Lastly, the constructed regular black hole solution usually requires the concept of the magnetic monopole, which is a hypothetical elementary particle. Moreover, owing to cosmic inflation, only an insignificant amount of magnetic monopoles might persist in the observable universe.

Regarding the preceding discussions, the present study involves an attempt to derive an electrically and magnetically neutral, regular black hole solution in the framework of Rastall theory. Rastall’s gravity was proposed in 1972 by Rastall [20], as a generalization of Einstein’s general relativity. It is proposed that the conservation of energy-momentum tensor in curved spacetime can be relaxed, and it attains the form

\[ T_{\mu\nu} = a_{\mu} , \] (1.1)

where \( a_{\mu} \) should vanish in flat spacetimes so that in this case the theory restores the Einstein’s gravity.

In this original work, Rastall assumes

\[ a_{\mu} = \lambda \nabla_{\mu} R , \] (1.2)

and therefore the field equation becomes

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa (T_{\mu\nu} - \lambda g_{\mu\nu} R) , \] (1.3)

where \( \kappa = 8\pi G/c^4 \) and \( \lambda \) is a constant. As a theory of modified gravity, Rastall’s gravity has received increasing attention lately [21–39], particularly due to recent findings in cosmology [40–49].
In a recent study [39], it is pointed out that in accordance to Rastall’s original proposal, $a_\mu$ can adopt various forms besides Eq. (1.2). This is because the only requirement is that $a_\mu$ vanishes in flat spacetimes [20], as one does not lead to any confliction with the present observations. In this context, we assume

$$a_\mu = \nabla^\nu A_{\nu\mu}, \quad (1.4)$$

where $A_{\nu\mu} = A_{\mu\nu}$, and the field equation can be written as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa(T_{\mu\nu} - A_{\mu\nu}). \quad (1.5)$$

Here $A_{\mu\nu}$ as well as its derivatives must be sufficiently small where the curvature of the spacetime vanishes. It is straightforward to show that one can formally express various modified theories of gravity, such as $f(R)$ gravity and quadratic gravity, regarding the above-generalized form of Rastall’s gravity [36, 39].

In this work, we investigate Rastall’s gravity with

$$A_{\mu\nu} = \lambda g_{\mu\nu} H(R), \quad (1.6)$$

where $H = H(R)$ is an arbitrary func of the Ricci scalar. According to the above discussions, one requires that $H = 0$ in flat spacetime, where $R = 0$. In the remainder of my paper, it is shown that one may derive regular black hole solutions for the Rastall’s gravity determined by Eq. (1.6).

The present paper is organized as follows. In the next section, we discuss the general properties of regular black holes. The specific form of the line element is given explicitly. In section III, we provide a detailed account for constructing the regular black hole solution in Rastall’s gravity which assumes Eq. (1.6). Further discussions and concluding remarks are given in the last section.

**PROPERTIES OF REGULAR BLACK HOLES**

In the section, we discuss the general properties of the metric for a regular black hole and derive the relevant requirements to be fulfilled. We start by considering the following form of static, spherically symmetric metric in four-dimensional spacetimes,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (2.1)$$

where

$$f(r) = 1 - \frac{2M(r)}{r} = 1 - \frac{2M_0}{r} C_M(r). \quad (2.2)$$

Here $M_0$ is considered as the mass of black hole as measured by an inertial observer sitting at infinity, and the parameter $C_M$ is introduced in the last equality which satisfies $C_M \to 1$ as $r \to \infty$. In what follows, we derive the condition for the black hole solution to be regular at $r = 0$ in terms of that of $C_M(r)$.

The avoidance of singularity implies that the curvature of the spacetime does not diverge. In practice, it is achived by investigating the convergence of the relevant scalar quantites,
namely, \( R = g^{\mu\nu} R_{\mu\nu} \), \( \mathcal{R} = R_{\alpha\beta\gamma\sigma} R^{\alpha\beta\gamma\sigma} \) and \( \mathcal{R} = R_{\alpha\beta} R^{\alpha\beta} \). For a static regular black hole, we substitute Eq. (2.1) and Eq. (2.2) into the above quantities and find

\[
R = \frac{2M_0}{r^2} \left( 2C' + rC'' \right),
\]

\[
\mathcal{R} = \frac{2M_0^2}{r^4} \left( 4C'_M + r^2 C''_M \right),
\]

\[
\mathcal{R} = \frac{4M_0^2}{r^6} \left[ 12C'^2 + 4rC_M (rC'' - 4C') + r^2 \left( r^2 C''_M - 8C'_M + 8C''_M \right) \right],
\]

(2.3)

where \( C'_M \equiv \frac{dC_M}{dr} \) and \( C''_M \equiv \frac{d^2 C_M}{dr^2} \). In the vicinity of \( r = 0 \), we assume \( C_M \sim \frac{r^\alpha}{r^0} + o(r^0) \).

By substituting this condition into the above scalars and expanding around \( r = 0 \), one finds that the requirement of convergence implies a condition \( \alpha \geq 3 \). On the other hand, one has \( C_M \to 1 \) as \( r \to \infty \), which does not lead to any divergence. The above condition must be satisfied for any regular black hole solution.

Now, let us discuss the properties of the metric near the black hole horizon. If one denote the horizon by \( r_p \), for \( f(r_p) = 0 \) one has

\[
f(r) = 1 - \frac{r_p C_M(r)}{r C_M(r_p)}.
\]

(2.4)

Therefore, the condition \( r_p > 0 \) implies \( C_M(r_p) > 0 \). The temperature at the horizon is

\[
T_H = \frac{1 - r_p C_p}{4\pi r_p}.
\]

(2.5)

where \( C_p = C'_M/C_M |_{r=r_p} \).

In literature, in order to derive a metric for the regular black hole, one introduces nonlinear electrodynamic field to the system. Alternatively, in the following section, it is shown that a static regular black hole solution can be found in the Rastall’s gravity in terms of \( H(R) \).

**REGULAR BLACK HOLES IN RASTALL’S GRAVITY**

In static, spherically symmetric black hole metric, the energy-momentum tensor of a given type of fluid surrounding the black hole can be written as \[ [21, 50] \]

\[
T^t_t = -\rho(r),
\]

\[
T^i_j = -\rho(r) \alpha \left[ \beta \delta^i_j - (1 + 3\beta) \frac{r_i r_j}{r_n r_n} \right],
\]

(3.1)

where \( \rho \) and \( p \) are the energy density and pressure of the matter field. When averaged over the angles for an isotropic system, the spatial components read

\[
\langle T^i_j \rangle = \frac{\alpha}{3} \rho \delta^i_j = p \delta^i_j,
\]

(3.2)

where we have used \( \langle r^i r_j \rangle = \frac{1}{3} \delta^i_j r_n r_n \). When considering barotropic equation of state, we have \[ [21, 50] \]

\[
p = \omega \rho,
\]

\[
\omega = \frac{\alpha}{3},
\]

\[
\beta = -\frac{1 + 3\omega}{6\omega}.
\]

(3.3)
The corresponding energy-momentum tensor takes the following form

\[
T^t_t = T^r_r = -\rho(r),
\]

\[
T^\theta_\theta = T^\phi_\phi = \frac{1}{2}(1 + 3 \omega)\rho(r).
\]  
(3.4)

Now, by substituting the static black hole metric, Eq.(2.1), into the field equation of Rastall’s theory, Eq.(1.5) and Eq.(1.6), one obtains

\[
rf'(r) + f(r) - 1 + \kappa \left[r^2 \rho(r) + \lambda r^2 H\right] = 0,
\]

\[
r f''(r) + 2 f'(r) + \kappa \left[2\lambda rH - (1 + 3\omega)r\rho(r)\right] = 0.
\]  
(3.5)

While the equation for the energy-momentum tensor, Eq.(1.1), becomes

\[
\rho'(r) + 3 \frac{1 + \omega}{r} \rho(r) + \lambda \frac{dH}{dr} = 0.
\]  
(3.6)

We note Eq.(3.6) is not an independent equation since it is implied by Eq.(3.5). Here, \(H = H(R)\) is a function of the Ricci scalar, which determines the specific form of the metric while satisfying Eq.(3.6).

By solving Eq.(3.5-3.6), one finds

\[
H = \frac{(1 + 3\omega)(f(r) - 1) + 3(1 + \omega)rf'(r) + r^2 f''(r)}{3r^2\lambda(1 + \omega)},
\]

\[
\rho(r) = \frac{r^2 f''(r) - 2f(r) + 2}{3\kappa(1 + \omega)r^2}.
\]  
(3.7)

It can be rewritten in terms of \(C_M\) by making use of Eq.(2.2)

\[
H = 2M_0(1 + 3\omega)C'_M(r) + rC''_M(r)
\]

\[
\frac{3\kappa\lambda(1 + \omega)r^2}{3\kappa(1 + \omega)r^2},
\]

\[
\rho(r) = 2M_0 \frac{2C'_M(r) - rC''_M(r)}{3\kappa(1 + \omega)r^2}.
\]  
(3.8)

Now are in the position to study the conditions under which the above solution is indeed regular. According to the discussions above, in flat spacetime, \(H(R)\) satisfies the condition

\[
H(R \to 0) \to 0.
\]  
(3.9)

This also implies that \(H(R) \to 0\) as \(r \to \infty\). On the other hand, one requires that \(\rho(r)\) does not possess any singularity in the entire range \(r \in [0, +\infty)\). By making use of the properties of the curvature scalars and \(C_M(r)\) discussed previously, we find the desired conditions

\[
C_M(r \to 0) \to r^{\alpha_{\text{center}}} \quad \text{with} \quad \alpha_{\text{center}} \geq 3,
\]

\[
C_M(r \to \infty) \to 1 + r^{\alpha_{\text{infinity}}} \quad \text{with} \quad \alpha_{\text{infinity}} < 0.
\]  
(3.10)

A simple solution satisfying the above conditions reads

\[
C_M(r) = \frac{r^3}{r^3 + 2\sigma^2}.
\]  
(3.11)
In this case, we have
\[ f(r) = 1 - \frac{2M_0 r^2}{r^3 + 2\sigma^2}, \]
\[ \rho(r) = \frac{24 M_0 \sigma^2 r^3}{\kappa(1 + \omega)(r^3 + 2\sigma^2)^3}, \]
\[ H = 12 M_0 \sigma^2 \frac{\kappa \lambda(1 + \omega)r^3 + 2(\omega + 1)\sigma^2}{r^3 + 2\sigma^2}, \]
\[ R = 24 M_0 \sigma^2 \frac{4\sigma^2 - r^3}{r^3 + 2\sigma^2}. \]  

(3.12)

The above solution can easily be identified as a Hayward regular black hole [10]. Moreover, the above results imply that \( r = r(R) \). Subsequently, one finds the following relations
\[ H(R) = 3B^2 R^2 \left( \omega - 1 \right) \frac{B^2 M_0^{1/3} - 2RM_0^{2/3}}{\kappa \lambda(1 + \omega)(B^2 - 2RM_0^{1/3})}, \]  

(3.13)

where \( B = \left( R^{3/2} \sqrt{81\sigma^2 R + 8M_0} - 9\sigma R^2 \right)^{1/3} \). Obviously, Eq. (3.13) satisfies the condition that \( H(R) \to 0 \) as \( R \to 0 \). The obtained results indicate that we have constructed regular black hole solutions in Rastall’s gravity, and in particular, we note that \( \omega \neq -1 \).

**DISCUSSIONS AND CONCLUDING REMARKS**

Following the original ideal proposed by Rastall, in this work, the conservation of the energy-momentum tensor is generalized to the form
\[ \nabla_\mu T^\mu_\nu = \lambda \nabla_\nu H(R), \]  

(4.1)

while the corresponding field equation reads
\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa (T_{\mu\nu} - \lambda g_{\mu\nu} H(R)), \]  

(4.2)

where \( H(R) \) vanishes in flat spacetime.

It has been shown that regular black hole solution can be derived in the framework of the Rastall’s gravity. As discussed above, the obtained black hole spacetimes can be electrically neutral, which does not involve nonlinear electrodynamics as well as the associated theoretical speculations. In this context, the present study provides a meaningful alternative to the existing models.

The constructed black hole spacetime is surrounded by a matter field, described by energy-momentum tensor, Eq. (3.4). However, to guarantee that the solution is nonsingular, one finds that the equation of state of the matter field has \( \omega \neq -1 \). In other words, the dark energy model in terms of cosmological constant cannot be a candidate for hosting a regular black hole solution in question.

On the other hand, when assuming \( T_{\mu\nu} = 0 \), the static black hole solution found in the present model indicates that the corresponding metric is an equivalent (anti-)de Sitter
spacetime. To demonstrate this point, one first contracts both sides of Eq. (4.2) by $g^{\mu\nu}$ to obtain

$$4\lambda\kappa H(R) - R = 0 .$$  \hspace{1cm} (4.3)

This is an algebraic equation, and in general, it possesses a nonvanishing root of $R$, besides the root at the origin. Thus one may rewrite Eq. (4.2) as follows

$$R_{\mu\nu} = g_{\mu\nu} \left( \frac{1}{2} R - \kappa \lambda H(R) \right) \equiv g_{\mu\nu} \Lambda_{\text{eff}} ,$$  \hspace{1cm} (4.4)

where $R$ is a non-zero root of Eq. (4.3). For the reason which will shortly become clear, $\Lambda_{\text{eff}}$ is defined as the effective cosmological constant, and one has

$$R = 4\Lambda_{\text{eff}} ,$$
$$H = \frac{\Lambda_{\text{eff}} \kappa \lambda}{\kappa \lambda} .$$  \hspace{1cm} (4.5)

As a result, the metric can be written as

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda_{\text{eff}}}{3} r^2 .$$  \hspace{1cm} (4.6)

Even though the cosmological constant is not an ad hoc assumption in the field equation, the asymptotical behaviour of the resulting metric indicates that it arises naturally from the consistency of the theory. For instance, let us consider the case where $\Lambda_{\text{eff}} > 0$. The de Sitter background is realized in the context of the present generalized Rastall gravity in terms of the effective cosmological constant $\Lambda_{\text{eff}} = \frac{R}{4}$. If one chooses $H(R) = R^n$ with $n > 1$ and $\lambda > 0$, we find

$$R = 4\Lambda_{\text{eff}} = \left( \frac{1}{4\kappa \lambda} \right)^{\frac{1}{n-1}} ,$$
$$H = \frac{\Lambda_{\text{eff}} \kappa}{\kappa \lambda} = \left( \frac{1}{4\kappa \lambda} \right)^{\frac{n}{n-1}} ,$$
$$\Lambda_{\text{eff}} = \frac{1}{4} \left( \frac{1}{4\kappa \lambda} \right)^{\frac{1}{n-1}} .$$  \hspace{1cm} (4.7)

The above-obtained de-Sitter solution is physically intriguing, since it merges entirely from the vacuum Rastall equation without the cosmological constant, which probably may lead to further implications in cosmology. More effort along this line of thought is under progress.

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