Optimal control of nutrition restricted dynamics model of Microalgae biomass growth model

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Abstract. The biomass of the microalgae is very potential to be proposed as an alternative renewable energy resources because it could be extracted into lipid. Afterward, the lipid could be processed to get the biodiesel or bioethanol. The extraction of the biomass on lipid synthesis process is very important to be studied because the process just gives some amount of lipid. A mathematical model of restricted microalgae biomass growth just gives 1/3 proportion of lipid with respect to the biomass in the synthesis process. An optimal control is designed to raise the ratio between the number of lipid formation and the microalgae biomass to be used in synthesis process. The minimum/ Pontryagin maximum principle is used to get the optimal lipid production. The simulation shows that the optimal lipid formation could be reach by simultaneously controlling the carbon dioxide, in the respiration and photosynthesis process, and intake nutrition rates of liquid waste and urea substrate. The production of controlled microalgae lipid could be increase 6.5 times comparing to the uncontrolled one.

1. Introduction
Microalgae growth were discussed in many studies [1-8]. Its growth models were studied in several papers, i.e Droop model [9], which discusses a simple model that represents its growth restricted by nitrogen. The nitrogen is consider as dependently intracellular growth factor [10]. To optimize the microalgae neutral lipid production, the microalgae growth is controlled by maintaining the dilution rate up to reach steady state condition through the nutrition restriction.

This paper considers two restricted nutrition, i.e liquid waste and urea, as the nutrition of microalgae and optimizes its growth by controlling the ratio of the CO₂ absorption rate due to respiration process and the glucose formation rate due to photosynthesis during the day. The other influenced process are the lipid synthesis process that mobilized the CO₂ and its released because of respiration process in the night. The phenomenon is expressed in a differential equations system that influenced by two added controlled parameters. The first one is due to maintain the ration in order to guarantee the sufficient of CO₂ in glucose formation because of photosynthesis process, while the other is due to nutrition absorption process. Those two parameters control is placed in an appropriate ways and is expected to raise the microalgae lipid production.

The referred model is governed and is analyzed its stability. This paper discuses the optimization of the microalgae lipid production using maximum Pontryagin criteria [11]. The optimum solution is derived for an objectives function that constructed to maximize the number of glucose, biomass and lipid formation and minimize the number of liquid waste and urea nutrition absorption for such Hamiltonian function and dynamics constrain. A numerical simulation is shown to have a better drawing of the optimal growth of microalgae to have the maximum microalgae lipid production.
2. The mathematical model

The mathematical model governed is constructed by consider respectively the growth of \( CO_2 \), glucose formation, microalgae biomass, lipid formation and nutrition. The model is considered to support an optimal strategy of growth and lipid accumulation based on experimental design. In this paper the model is constructed based on the lipid synthesis diagram of microalgae in Figure 1.

\[ \frac{dQ}{dt} = \delta B + \gamma L - \alpha_1 QB \]
\[ \frac{dG}{dt} = \alpha_2 QB - \mu G \]
\[ \frac{dB}{dt} = \mu G + \rho_1 S + \rho_2 N - \beta_1 B \]
\[ \frac{dL}{dt} = \beta_2 B - \gamma L \]
\[ \frac{dS}{dt} = \theta_1 - \rho_1 S \]
\[ \frac{dN}{dt} = \theta_2 - \rho_2 N \]

where parameters \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma, \delta, \mu, \theta_1, \rho_1 \) and \( \rho_2 \) are respectively stated the \( CO_2 \) absorption rate due to respiration process, glucose formation rate due to photosynthesis process, glucose loss and biomass formation rate due to photosynthesis process, biomass growth rate due to the domestic waste nutrition, domestic waste nutrition absorption rate of microalgae, Nitrogen absorption rate of microalgae, lipid formation rate due to biomass synthesis process, \( CO_2 \) release rate due to respiration process and \( CO_2 \) mobilization due to synthesis process. The last two equations of the model are considered separately such that the solutions could be determined directly for such initial value problem, \( S(0) = S_0 \) and \( N(0) = N_0 \) such that the model represents the dynamic interaction between the variables of \( CO_2 \) functional \( Q \), glucose \( G \), biomass \( B \), and lipid \( L \).
2.2. The critical point

The critical point of the system stated in (1.a) – (1.f), that represents the timeless or stagnant critical point \( T(Q^*, G^*, B^*, L^*) \) where \( Q^* = \frac{\delta + \beta_2}{a_1}, \ G^* = -\frac{SNa_1}{(-\beta_1\alpha_1 + a_2(\delta + \beta_2))\mu}, \ B^* = -\frac{SNa_1}{(-\beta_1\alpha_1 + a_2(\delta + \beta_2))}, \) and \( L^* = -\frac{\beta_2Sn_{\alpha_1}}{(-\beta_1\alpha_1 + a_2(\delta + \beta_2))\mu}. \) The expression of the critical point \( T \) requires \( \frac{a_1}{a_2} < \frac{\delta + \beta_2}{\beta_1} \) that interpreted the importance of maintaining the ratio of the \( CO_2 \) absorption rate due to respiration process and the glucose formation rate due to photosynthesis during the day.

2.3. The stability

Stability appears as the importance of maintaining the nutrition sufficiency. The stability of the critical point is determined from the Jacobian matrix [12] that gives the following fourth order characteristic polynomial of \( \lambda \)

\[
P(\lambda) = A(\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4)
\]

where \( A = \alpha_1\left(-\beta_1\alpha_1 + a_2(\delta + \beta_2)\right) \)

\[
a_1 = -\alpha_1(\alpha_1^2SN + \beta_1(\beta_1 + \mu + \gamma)\alpha_1 - (\delta + \beta_2)(\beta_1 + \mu + \gamma)\alpha_2)
\]

\[
a_2 = -SN(\beta_1 + \mu + \gamma)\alpha_1^3 - \beta_1((\beta_1 + \gamma)\mu + \gamma\beta_1)\alpha_1^2 + (\delta + \beta_2)((\gamma + 2\beta_1)\mu + \gamma\beta_1)\alpha_2\alpha_1 - \mu\alpha_2^2(\delta + \beta_2)^2
\]

\[
a_3 = -SN((\beta_1 + \gamma)\mu + \gamma\beta_1)\alpha_1^3 + \mu(SN\alpha_2\delta - \beta_1^2\gamma)\alpha_1^2 + 2\mu\alpha_2\gamma\beta_1(\delta + \beta_2)\alpha_1
\]

\[
a_4 = \alpha_1\mu A\gamma SN
\]

\[
SN = \rho_1S + \rho_2N
\]

The stability of the critical point is identified using Routh Hurwitz criteria [13] such that the identification is determined from the sign changing of the first coulomb of Table 1 that are \( a_0, a_1, b_1 = \frac{a_1a_2 - a_0a_3}{a_1}, b_2 = a_4, c_1 = \frac{b_1a_3 - a_4b_2}{b_1} \) and \( d_1 = b_2 = a_4. \) The positive sign of \( a_0 \) is guaranteed from the existence requirement, while the positive sign of \( a_1 \) gives \( \frac{a_1}{a_2} < \frac{\beta_1(\beta_1 + \mu + \gamma)(\delta + \beta_2)}{SN\alpha_1 + \beta_1(\beta_1 + \mu + \gamma)}. \) This result comes to a new criteria for \( \frac{a_1}{a_2} \) that is \( \frac{a_1}{a_2} < \min\left(\frac{\delta + \beta_2}{\beta_1}, \frac{\beta_1(\beta_1 + \mu + \gamma)(\delta + \beta_2)}{SN\alpha_1 + \beta_1(\beta_1 + \mu + \gamma)}\right). \) The identification results of the sign of \( b_1, c_1 \) and \( d_1 \) is tabulated in Table 2. The table mentions the critical point, the existence and stability requirements.

| \( \lambda \)   | \( a_0 \) | \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) |
|----------------|----------|----------|----------|----------|----------|
| \( \lambda^4 \) |          |          |          |          |          |
| \( \lambda^3 \) |          |          |          |          |          |
| \( \lambda^2 \) |          |          |          |          |          |
| \( \lambda^1 \) |          |          |          |          |          |
| \( \lambda^0 \) |          |          |          |          |          |
Table 2. Parameter stability identification

| Critical Point | Existence requirement | Stability Requirement | Description |
|---------------|-----------------------|-----------------------|-------------|
| $T(Q^*, G^*, B^*, L^*)$ | $\frac{\alpha_1}{\alpha_2} < \delta + \beta_2 \frac{\beta_1}{\beta_1}$ | $\max \left( \frac{(\mu + \beta_1)(\delta + \beta_2) - \beta_1}{(\beta_1 + \gamma)(\mu + \beta_1)(\delta + \beta_2)} \right) - \frac{\alpha_1}{\alpha_2} < \min \left( \frac{(\delta + \beta_2)(\beta_1 + \mu + \gamma)(\delta + \beta_2)}{\beta_1} \right)$ | unstable |

where $E, F, G, H, J, K, L$ and $P$ are in a complex following expression of parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma, \delta, \mu, \theta_1, \rho_1$ and $\rho_2$

$$E = SN(\beta_1 + \mu + \gamma)\alpha_1^5 + SN(\beta_1 + \mu + \gamma)^2 \alpha_2 + (\mu + \gamma)(\delta + \beta_2)\alpha_1^2 + \beta_1(\beta_1 + \gamma)(\mu + \beta_1)(\mu + \gamma)\alpha_1^2 + \alpha_2(\delta + \beta_2)(SN\mu\beta_2\alpha_2)\alpha_1^3 + \alpha_2^2((3\mu + \gamma)\beta_1 + \gamma(\mu + \gamma))(\delta + \beta_2)^2(\mu + \beta_1)\alpha_1$$

$$F = SN(\delta + \beta_2)\alpha_1^3 + ((2\delta + 3\beta_2)\mu + 2\gamma(\delta + \beta_2))\beta_1)\alpha_1^3$$

$$G = \mu(SN\mu\beta_2\alpha_2)\alpha_1^2 + 2\mu\alpha_2\gamma(\delta + \beta_2)\alpha_1$$

$$H = SN(\beta_1 + \gamma)\mu + \gamma\beta_1)\alpha_3 + \mu\beta_1^2\gamma\alpha_1^2 + \alpha_2^2\gamma(\mu + \gamma)\beta_1^2$$

$$J = SN\mu\beta_2\alpha_2 - 2\beta_1(\frac{3}{2}\mu + \gamma)\beta_1 + \gamma(\mu + \gamma)$$

$$K = SN(\delta + \beta_2)\beta_1^2 + ((2\delta + 3\beta_2)\mu + 2\gamma(\delta + \beta_2))\beta_1) - (\mu + \gamma)^2(\delta + \beta_2)\beta_2 - \beta_1^2$$

3. The optimization model

This paper proposes an optimal control to maintain the internal and external nutrition to have the maximal production of microalgae lipid. Its production is formulated in an objective function such that its optimal value is solved from the Hamiltonian function that restricted by the controlled nonlinear dynamical system.

To reach the maximal of microalgae lipid production two programs are proposed. The first one due to maximize the $CO_2$ consumption needed in photosynthesis and respiration process, while the other one due to minimizing the nutrition by restricting the liquid waste content. The controlled nonlinear dynamical system is designed to optimize the lipid production by two controlled parameters, that are $u_1$ and $u_2$. Both parameters respectively represent mixotrophic by providing light with organic carbon [14] and Nitrogen nutrition inlet control [15-16]. The controlled model is stated

$$\frac{dQ}{dt} = \delta B + \gamma L - u_1\alpha_1 Q B$$  
$$\frac{dG}{dt} = u_1\alpha_2 Q B - \mu G$$  
$$\frac{dB}{dt} = \mu G + \rho_1 S + \rho_2 N - \beta_1 B$$  
$$\frac{dL}{dt} = \beta_2 B - \gamma L$$  
$$\frac{dS}{dt} = \theta_1 - (1 - u_2)\rho_1 S$$  
$$\frac{dN}{dt} = \theta_2 - (1 - u_2)\rho_2 N$$
The controlled model in equations (2.a) – (2.f) is solved using Pontryagin maximum principle [11] by defining an objective function
\[ f(G, B, L, S, N, u_1, u_2, t) = \max_0^t \left( G(t) + B(t) + L(t) - S(t) - N(t) + \frac{1}{2} u_1^2 + \frac{1}{2} (1 - u_2)^2 \right) dt \]

The optimal solution of the controlled model is derived by defining the following Hamiltonian system
\[ H = f(G, B, L, S, N, u_1, u_2, t) + \lambda g(G, B, L, S, N, u_1, u_2, t) \]

Solving the state and co-state equations simultaneously

The state equations:
\[
\begin{align*}
\dot{Q} &= \frac{\partial H}{\partial \lambda_1} = \delta B + \gamma L - u_1 \alpha_1 QB \\
\dot{G} &= \frac{\partial H}{\partial \lambda_2} = u_1 \alpha_2 QB - \mu G \\
\dot{B} &= \frac{\partial H}{\partial \lambda_3} = \mu G + \rho_1 S + \rho_2 N - \beta_2 B \\
\dot{L} &= \frac{\partial H}{\partial \lambda_4} = \beta_2 B - \gamma L \\
\dot{S} &= \frac{\partial H}{\partial \lambda_5} = \theta_1 - (1 - u_2) \rho_1 S \\
\dot{N} &= \frac{\partial H}{\partial \lambda_6} = \theta_2 - (1 - u_2) \rho_2 N 
\end{align*}
\]

The co-State equations:
\[
\begin{align*}
\dot{\lambda}_1 &= \frac{\partial H}{\partial Q} = -u_1 \alpha_1 B \lambda_1 + u_1 \alpha_2 B \lambda_2 \\
\dot{\lambda}_2 &= \frac{\partial H}{\partial G} = -u_1 \alpha_2 Q \lambda_1 + u_1 \alpha_2 Q \lambda_2 - \beta_1 \lambda_3 + \beta_2 \lambda_4 \\
\dot{\lambda}_3 &= \frac{\partial H}{\partial B} = -1 + u_1 \alpha_1 Q \lambda_1 + u_1 \alpha_2 Q \lambda_2 - \beta_1 \lambda_3 + \beta_2 \lambda_4 \\
\dot{\lambda}_4 &= \frac{\partial H}{\partial L} = 1 + \gamma \lambda_1 - \gamma \lambda_4 \\
\dot{\lambda}_5 &= \frac{\partial H}{\partial S} = -1 + \rho_1 \lambda_3 - (1 - u_2) \rho_1 \lambda_5 \\
\dot{\lambda}_6 &= \frac{\partial H}{\partial N} = -1 + \rho_2 \lambda_3 - (1 - u_2) \rho_2 \lambda_6 
\end{align*}
\]

with the stationer condition
\[ \frac{dH}{d\theta} = 0 \]
gives the following optimal parameter control
\[
\begin{align*}
-1 + \alpha_1 QB \lambda_1 + \alpha_2 B \lambda_2 &= 0 \Rightarrow u_1 = -\frac{\alpha_1 QB \lambda_1 - \alpha_2 B \lambda_2}{1 + \rho_2 \lambda_3} \\
-1 + (1 - u_2) + \lambda_5 (u_2 \rho_1 S) + \lambda_6 (u_2 \rho_2 N) &= 0 \Rightarrow u_2 = -\frac{1}{1 + \rho_1 \lambda_5 + \rho_2 \lambda_6} 
\end{align*}
\]

4. The numerical solution and discussion

The optimization solution is solved numerically for some given parameter values and initial values of variables. The initial values, i.e. \( Q(0) = 1, G(0) = 1, B(0) = 0.5 \) and \( L(0) = 1 \), shows that, in the beginning process, the number of CO\(_2\) is assumed in 100% portion as well as the glucose and no lipid formation. The value of parameters refers to [3].
Figure 2. The Dynamic of Glucose, Biomass, Lipid, Urea and Liquid Waste

Figure 2 shows that the portions of glucose, biomass and lipid are successfully to be increased. The successes of the increasing value of the variables with respect to time is also supported by figure 3 that describe the greater value of controlled variables comparing to the related uncontrolled glucose, biomass and Lipid. A similar characteristic is also figured for the CO\textsubscript{2} formation in figure 4 that comparison of it in the beginning and time of intervals. At the beginning the controlled CO\textsubscript{2} still less than the uncontrolled one, but after that it turns to be greater. The production of controlled microalgae lipid could be increase 6.5 times comparing to the uncontrolled one.
Figure 3. The Comparison of The Controlled and Uncontrolled of (a) Glucose, (b) Biomass and (c) Lipid
Figure 4. The Comparison of The Controlled and Uncontrolled of CO$_2$ formation
(a). In the beginning time of interval  (b). In the end time of interval

5. The concluding remarks
The mathematical model of microalgae biomass growth that its nutrition is restricted externally and internally had been derived. The optimal solution of the model that solved using maximum Pontryagin principle shows that the microalga lipid production could be maximized increase. This result is potentially to be further studied by consider the cost production of the lipid synthesis process.

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