LOCATING $\mathfrak{A}x$, WHERE $\mathfrak{A}$ IS A SUBSPACE OF $\mathcal{B}(H)$

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ABSTRACT. Given a linear space of operators on a Hilbert space, any vector in the latter determines a subspace of its images under all operators. We discuss, within a Bishop-style constructive framework, conditions under which the projection of the original Hilbert space onto the closure of the image space exists. We derive a general result that leads directly to both the open mapping theorem and our main theorem on the existence of the projection.

1. Introduction

Let $H$ be a real or complex Hilbert space, $\mathcal{B}(H)$ the space of bounded operators on $H$, and $\mathfrak{A}$ a linear subspace of $\mathcal{B}(H)$. For each $x \in H$ write

$\mathfrak{A}x \equiv \{Ax : A \in \mathfrak{A}\},$

and, if it exists, denote the projection of $H$ onto the closure $\overline{\mathfrak{A}x}$ of $\mathfrak{A}x$ by $[\mathfrak{A}x]$. Projections of this type play a very big part in the classical theory of operator algebras, in which context $\mathfrak{A}$ is normally a subalgebra of $\mathcal{B}(H)$; see, for example, [10, 11, 13, 15]. However, in the constructive setting—the one of this paper—we cannot even guarantee that $[\mathfrak{A}x]$ exists. Our aim is to give sufficient conditions on $\mathfrak{A}$ and $x$ under which $[\mathfrak{A}x]$ exists, or, equivalently, the set $\mathfrak{A}x$ is located, in the sense that

$\rho(v, \mathfrak{A}x) \equiv \inf \{\|v - Ax\| : A \in \mathfrak{A}\}$

exists for each $v \in H$.

We require some background on operator topologies. Specifically, in addition to the standard uniform topology on $\mathcal{B}(H)$, we need

$\triangleright$ the strong operator topology: the weakest topology on $\mathcal{B}(H)$ with respect to which the mapping $T \mapsto Tx$ is continuous for all $x \in H$;

$\triangleright$ the weak operator topology: the weakest topology on $\mathcal{B}(H)$ with respect to which the mapping $T \mapsto \langle Tx, y \rangle$ is continuous for all $x, y \in H$.

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1Our constructive setting is that of Bishop [2,3,6], in which the mathematics is developed with intuitionistic, not classical, logic, in a suitable set- or type-theoretic framework [11,12] and with dependent choice permitted.
These topologies are induced, respectively, by the seminorms of the form $T \mapsto \|Tx\|$ with $x \in H$, and $T \mapsto |\langle Tx, y \rangle|$ with $x, y \in H$. The unit ball of $B(H)$ is classically weak-operator compact, but constructively the most we can say is that it is weak-operator totally bounded (see [14]). The evidence so far suggests that in order to make progress when dealing constructively with a subspace or subalgebra $A$ of $B(H)$, it makes sense to add the weak-operator total boundedness of $A_1 \equiv A \cap B_1(H)$ to whatever other hypothesis we are making; in particular, it is known that $A_1$ is located in the strong operator topology—and hence $A_1x$ is located for each $x \in H$—if and only if it is weak-operator totally bounded [7, 14].

Recall that the metric complement of a subset $S$ of a metric space $X$ is the set $-S$ of those elements of $X$ that are bounded away from $X$. When $Y$ is a subspace of $X$, $y \in Y$, and $S \subseteq Y$, we define

$$\rho_Y(y, -S) \equiv \inf \{ \rho(y, z) : z \in Y \cap -S \}$$

if that infimum exists.

We now state our main result.

**Theorem 1.1.** Let $A$ be a uniformly closed subspace of $B(H)$ such that $A_1$ is weak-operator totally bounded, and let $x$ be a point of $H$ such that $Ax$ is closed and $\rho_{Ax}(0, -A_1x)$ exists. Then the projection $[Ax]$ exists.

Before proving this theorem, we discuss, in Section 2, some general results about the locatedness of sets like $Ax$, and we derive, in Section 3, a generalisation of the open mapping theorem that leads to the proof of Theorem 1.1. Finally, we show, by means of a Brouwerian example, that the existence of $\rho_{Ax}(0, -A_1x)$ cannot be dropped from the hypotheses of our main theorem.

### 2. Some General Locatedness Results for $Ax$

We now prove an elementary, but helpful, result on locatedness in a Hilbert space.

**Proposition 2.1.** Let $(S_n)_{n \geq 1}$ be a sequence of located, convex subsets of a Hilbert space $H$ such that $S_1 \subseteq S_2 \subseteq \cdots$, let $S_\infty = \bigcup_{n \geq 1} S_n$, and let $x \in H$. For each $n$, let $x_n \in S_n$ satisfy $\|x - x_n\| < \rho(x, S_n) + 2^{-n}$. Then

$$\rho(x, S_\infty) = \inf_{n \geq 1} \rho(x, S_n) = \lim_{n \to \infty} \rho(x, S_n),$$

in the sense that if any of these three numbers exists, then all three do and they are equal. Moreover, $\rho(x, S_\infty)$ exists if and only if $(x_n)_{n \geq 1}$ converges to a limit $x_\infty \in H$; in that case, $\rho(x, S_\infty) = \|x - x_\infty\|$, and $\|x - y\| > \|x - x_\infty\|$ for all $y \in S_\infty$ with $y \neq x_\infty$.

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Note that it is not constructively provable that every element $T$ of $B(H)$ is normed, in the sense that the usual operator norm of $T$ exists. Nevertheless, when we write $\|T\| \leq 1'$, we are using a shorthand for $\|Tx\| \leq \|x\|$ for each $x \in H$. Likewise, $\|T\| < 1'$ means that there exists $c < 1$ such that $\|Tx\| \leq c\|x\|$ for each $x \in H$; and $\|T\| > 1'$ means that there exists $x \in H$ such that $\|Tx\| > \|x\|$. 


Proof. Suppose that \( \rho(x, S_\infty) \) exists. Then \( \rho(x, S_\infty) \leq \rho(x, S_n) \) for each \( n \). On the other hand, given \( \varepsilon > 0 \) we can find \( z \in S_\infty \) such that \( \|x - z\| < \rho(x, S_\infty) + \varepsilon \). Pick \( N \) such that

\[
\rho(x, S_\infty) \leq \rho(x, S_n) \leq \rho(x, S_N) \leq \|x - z\| < \rho(x, S_\infty) + \varepsilon.
\]

The desired conclusion (2.1) now follows.

Next, observe that (by the parallelogram law in \( H \)) if \( m \geq n \), then

\[
\|x_m - x_n\|^2 \leq \|(x_m - m) - (x_m - n)\|^2
\]

\[
= 2\|x - m\|^2 + 2\|x - n\|^2 - 4\left\|x - \frac{1}{2}(x_m + n)\right\|^2
\]

\[
\leq 2\left(\rho(x, S_m) + 2^{-m}\right)^2 + 2\left(\rho(x, S_n) + 2^{-n}\right)^2 - 4\rho(x, S_m)^2,
\]

since \( \frac{1}{2}(x_m + n) \in S_m \). Thus

\[
\|x_m - x_n\|^2 \leq 2\left(\rho(x, S_m) + 2^{-m}\right)^2 - \rho(x, S_m)^2
\]

\[
+ 2\left(\rho(x, S_n) + 2^{-n}\right)^2 - \rho(x, S_n)^2.
\]

(2.2)

If \( \rho(x, S_\infty) \) exists, then, by the first part of the proof, \( \rho(x, S_n) \to \rho(x, S_\infty) \) as \( n \to \infty \). It follows from this and (2.2) that \( \|x_m - x_n\|^2 \to 0 \) as \( m, n \to \infty \); whence \( (x_n)_{n \geq 1} \) is a Cauchy sequence in \( H \) and therefore converges to a limit \( x_\infty \in S_\infty \). Then

\[
\rho(x, S_\infty) = \rho(x, \overline{S_\infty}) \leq \|x - x_\infty\|
\]

\[
= \lim_{n \to \infty} \|x - x_n\|
\]

\[
\leq \lim_{n \to \infty} \left(\rho(x, S_n) + 2^{-n}\right) = \rho(x, S_\infty).
\]

Thus \( \rho(x, S_\infty) = \|x - x_\infty\| \).

Conversely, suppose that \( x_\infty = \lim_{n \to \infty} x_n \) exists. Let \( 0 < \alpha < \beta \) and \( \varepsilon = \frac{1}{3}(\beta - \alpha) \).

Pick \( N \) such that \( 2^{-N} < \varepsilon \) and \( \|x_\infty - x_n\| < \varepsilon \) for all \( n \geq N \). Either \( \|x - x_\infty\| > \alpha + 2\varepsilon \) or \( \|x - x_\infty\| < \beta \). In the first case, for all \( n \geq N \),

\[
\rho(x, S_n) > \|x - x_n\| - 2^{-n}
\]

\[
\geq \|x - x_\infty\| - \|x_\infty - x_n\| - \varepsilon
\]

\[
> (\alpha + 2\varepsilon) - \varepsilon - \varepsilon = \alpha.
\]

In the other case, there exists \( \nu > N \) such that \( \|x - x_\nu\| < \beta \); we then have

\[
\rho(x, S_\nu) \leq \|x - x_\nu\| < \beta.
\]

It follows from this and the constructive least-upper-bound principle ([6], Theorem 2.1.18) that

\[
\inf \{\rho(x, S_n) : n \geq 1\}
\]

exists; whence, by (2.1), \( d = \rho(x, S_\infty) \) exists.
Finally, suppose that $x_\infty$ exists, and consider any $y \in S_\infty$ with $y \neq x_\infty$. We have
\[
0 < \|y - x_\infty\|^2 = \|y - x - (x_\infty - x)\|^2
\]
\[
= 2\|y - x\|^2 + 2\|x_\infty - x\|^2 - 4\left(\frac{y + x_\infty}{2} - x\right)^2
\]
\[
= 2\left(\|y - x\|^2 - d^2\right) + 2\left(\|x_\infty - x\|^2 - d^2\right) = 2\left(\|y - x\|^2 - d^2\right),
\]
so $\|x - y\| > d$.

For each positive integer $n$ we write
\[
\mathfrak{A}_n \equiv n\mathfrak{A}_1 = \{nA : A \in \mathfrak{A}_1\}.
\]
If $\mathfrak{A}_1$ is weak-operator totally bounded and hence strong-operator located, then $\mathfrak{A}_n$ has those two properties as well.

Our interest in Proposition 2.1 stems from this:

**Corollary 2.2.** Let $\mathfrak{A}$ be a linear subspace of $\mathcal{B}(H)$ with $\mathfrak{A}_1$ weak-operator totally bounded, and let $x, y \in H$. For each $n$, let $y_n \in \mathfrak{A}_n$ satisfy $\|y - y_n\| < \rho(x, \mathfrak{A}_n x) + 2^{-n}$. Then
\[
\rho(y, \mathfrak{A} x) = \inf_{n \geq 1} \rho(y, \mathfrak{A}_n x) = \lim_{n \to \infty} \rho(y, \mathfrak{A}_n x).
\]
Moreover, $\rho(y, \mathfrak{A} x)$ exists if and only if $(y_n)_{n \geq 1}$ converges to a limit $y_\infty \in H$; in which case, $\rho(y, \mathfrak{A} x) = \|y - y_\infty\|$, and $\|y - Ax\| > \|y - y_\infty\|$ for each $A \in \mathfrak{A}$ such that $Ax \neq y_\infty$.

One case of this corollary arises when the sequence $(\rho(y, \mathfrak{A}_n x))_{n \geq 1}$ stabilises:

**Proposition 2.3.** Let $\mathfrak{A}$ be a linear subspace of $\mathcal{B}(H)$ such that $\mathfrak{A}_1$ is weak-operator totally bounded. Let $x, y \in H$, and suppose that for some positive integer $N$, $\rho(y, \mathfrak{A}_N x) = \rho(y, \mathfrak{A}_{N+1} x)$. Then $\rho(y, \mathfrak{A} x)$ exists and equals $\rho(y, \mathfrak{A}_N x)$.

**Proof.** By Theorem 4.3.1 of [6], there exists a unique $z \in \mathfrak{A}_N x$ such that $\rho(y, \mathfrak{A}_N x) = \|y - z\|$. We prove that $y - z$ is orthogonal to $\mathfrak{A} x$. Let $A \in \mathfrak{A}$, and consider $\lambda \in \mathbb{C}$ so small that $\lambda A \in \mathfrak{A}_1$. Since,
\[
z - \lambda Ax \in \overline{\mathfrak{A}_N x},
\]
we have
\[
\langle y - z - \lambda Ax, y - z - \lambda Ax \rangle \geq \rho(y, \mathfrak{A}_{N+1} x)^2
\]
\[
= \rho(y, \mathfrak{A}_N x)^2 = \langle y - z, y - z \rangle.
\]
This yields
\[
|\lambda|^2 \|Ax\|^2 + 2 \text{Re} (\lambda \langle y - z, Ax \rangle) \geq 0.
\]
Suppose that $\text{Re} \langle y - z, Ax \rangle \neq 0$. Then by taking a sufficiently small real $\lambda$ with
\[
\lambda \text{Re} \langle y - z, Ax \rangle < 0,
\]
we obtain a contradiction. Hence $\text{Re} \langle y - z, Ax \rangle = 0$. Likewise, $\text{Im} \langle y - z, Ax \rangle = 0$. Thus $\langle y - z, Ax \rangle = 0$. Since $A \in \mathfrak{A}$ is arbitrary, we conclude that $y - z$ is orthogonal to $\mathfrak{A} x$ and hence to $\overline{\mathfrak{A} x}$. It is well known that this implies that $z$ is the unique closest point to $y$ in the closed linear subspace $\overline{\mathfrak{A} x}$. Since $\mathfrak{A} x$ is dense in $\overline{\mathfrak{A} x}$, it readily follows that $\rho(y, \mathfrak{A} x) = \rho(y, \overline{\mathfrak{A} x}) = \|y - z\|$.

\[\square\]
The final result in this section will be used in the proof of our main theorem. 

**Proposition 2.4.** Let $\mathfrak{A}$ be a linear subspace of $\mathcal{B}(H)$ with weak-operator totally bounded unit ball, and let $x \in H$. Suppose that there exists $r > 0$ such that 

$$\mathfrak{A}_1x \supset B_{\mathfrak{A}x}(0, r) \equiv \mathfrak{A}x \cap B(0, r).$$

Then $\mathfrak{A}x$ is located in $H$; in fact, for each $y \in H$, there exists a positive integer $N$ such that $\rho(y, \mathfrak{A}x) = \rho(y, \mathfrak{A}_N x)$.

**Proof.** Fixing $y \in H$, compute a positive integer $N > 2 \|y\|/r$. Let $A \in \mathfrak{A}$, and suppose that 

$$\|y - Ax\| < \rho(y, \mathfrak{A}_N x).$$

We have either $\|Ax\| < Nr$ or $\|Ax\| > 2 \|y\|$. In the first case, $N^{-1}Ax \in B_{\mathfrak{A}x}(0, r)$, so there exists $B \in \mathfrak{A}_1$ with $N^{-1}Ax = Bx$ and therefore $Ax = NBx$. But $NB \in \mathfrak{A}_N$, so 

$$\|y - Ax\| = \|y - NBx\| \geq \rho(y, \mathfrak{A}_N x),$$

a contradiction. In the case $\|Ax\| \geq Nr > 2 \|y\|$, we have 

$$\|y - Ax\| \geq \|Ax\| - \|y\| > \|y\| \geq \rho(y, \mathfrak{A}_N x),$$

another contradiction. We conclude that $\|y - Ax\| \geq \rho(y, \mathfrak{A}_N x)$ for each $A \in \mathfrak{A}$. On the other hand, given $\varepsilon > 0$, we can find $A \in \mathfrak{A}_N$ such that $\|y - Ax\| < \rho(y, \mathfrak{A}_N x) + \varepsilon$. It now follows that $\rho(y, \mathfrak{A}x)$ exists and equals $\rho(y, \mathfrak{A}_N x)$. 

\[\square\]

3. GENERALISING THE OPEN MAPPING THEOREM

The key to our main result on the existence of projections of the form $[\mathfrak{A}x]$ is a generalisation of the open mapping theorem from functional analysis ([6], Theorem 6.6.4). Before giving that generalisation, we note a proposition and a lemma.

**Proposition 3.1.** If $C$ is a balanced, convex subset of a normed space $X$, then $V \equiv \bigcup_{n \geq 1} nC$ is a linear subspace of $X$.

**Proof.** Let $x \in V$ and $\alpha \in C$. Pick a positive integer $n$ and an element $c$ of $C$ such that $x = nc$. If $\alpha \neq 0$, then since $C$ is balanced, $|\alpha|^{-1}ac \in C$, so 

$$\alpha x = anc = |\alpha| n |\alpha|^{-1}ac \in |\alpha| nC \subset (1 + |\alpha|) nC.$$

In the general case, we can apply what we have just proved to show that 

$$(1 + \alpha) x \in (1 + |1 + \alpha|) nC \subset (2 + |\alpha|) nC.$$

Now, since $C$ is balanced, 

$$-x = n (-c) \in nC \subset (2 + |\alpha|) nC.$$

Hence, by the convexity of $(2 + |\alpha|) nC$, 

$$\alpha x = 2 \frac{(1 + \alpha)x - x}{2} \in 2(2 + |\alpha|) nC.$$

Taking $N$ as any integer $> 2(2 + |\alpha|) n$, we now see that $\alpha x \in NC \subset V$. In view of the foregoing and the fact that $(nC)_{n \geq 1}$ is an ascending sequence of sets, if $x'$ also belongs to $V$
we can take $N$ large enough to ensure that $\alpha x$ and $x'$ both belong to $NC$. Picking $c, c' \in C$ such that $\alpha x = Nc$ and $x' = Nc'$, we obtain
\[
\alpha x + x' = 2N \left( \frac{c + c'}{2} \right) \in 2NC,
\]
so $\alpha x + x' \in V$. \hfill \Box

We call a bounded subset $C$ of a Banach space $X$ superconvex if for each sequence $(x_n)_{n \geq 1}$ in $C$ and each sequence $(\lambda_n)_{n \geq 1}$ of nonnegative numbers such that $\sum_{n=1}^{\infty} \lambda_n$ converges to 1 and the series $\sum_{n=1}^{\infty} \lambda_n x_n$ converges, we have $\sum_{n=1}^{\infty} \lambda_n x_n \in C$. In that case, $C$ is clearly convex.

**Lemma 3.2.** Let $C$ be a located, bounded, balanced, and superconvex subset of a Banach space $X$, such that $X = \bigcup_{n \geq 1} nC$. Let $y \in X$ and $r > \|y\|$. Then there exists $\xi \in 2C$ such that if $y \neq \xi$, then $\rho(z, C) > 0$ for some $z$ with $\|z\| < r$.

**Proof.** Either $\rho(y, C) > 0$ and we take $z = y$, or else, as we suppose, $\rho(y, C) < r/2$. Choosing $x_1 \in 2C$ such that $\|y - \frac{1}{2} x_1\| < r/2$ and therefore $\|2y - x_1\| < r$, set $\lambda_1 = 0$. Then either $\rho(2y - x_1, C) > 0$ or $\rho(2y - x_1, C) < r/2$. In the first case, set $\lambda_k = 1$ and $x_k = 0$ for all $k \geq 2$. In the second case, pick $x_2 \in 2C$ such that $\|2y - x_1 - \frac{1}{2} x_2\| < r/2$ and therefore $\|2^2 y - 2x_1 - x_2\| < r$, and set $\lambda_2 = 0$. Carrying on in this way, we construct a sequence $(x_n)_{n \geq 1}$ in $2C$, and an increasing binary sequence $(\lambda_n)_{n \geq 1}$ with the following properties.

- If $\lambda_n = 0$, then
  \[
  \rho \left( 2^{n-1} y - \sum_{i=1}^{n} 2^{n-i-1} x_i, C \right) < \frac{r}{2}
  \]
  and
  \[
  \left\| 2^n y - \sum_{i=1}^{n} 2^{n-i} x_i \right\| < r.
  \]

- If $\lambda_n = 1 - \lambda_{n-1}$, then
  \[
  \rho \left( 2^{n-1} y - \sum_{i=1}^{n} 2^{n-i-1} x_i, C \right) > 0
  \]
  and $x_k = 0$ for all $k \geq n$.

Compute $\alpha > 0$ such that $\|x\| < \alpha$ for all $x \in 2C$. Then the series $\sum_{i=1}^{\infty} 2^{-i} x_i$ converges, by comparison with $|\alpha| \sum_{i=1}^{\infty} 2^{-i}$, to a sum $\xi$ in the Banach space $X$. Since $\sum_{i=1}^{\infty} 2^{-i} = 1$ and $C$ is superconvex, we see that
\[
\sum_{i=1}^{\infty} 2^{-i} x_i = 2 \sum_{i=1}^{\infty} 2^{-i} \left( \frac{1}{2} x_i \right) \in 2C.
\]
If $y \neq \xi$, then there exists $N$ such that
\[
\left\| y - \sum_{i=1}^{N} 2^{-i} x_i \right\| > 2^{-N} r.
\]
and therefore
\[ \left\| 2^N y - \sum_{i=1}^{N} 2^{N-i} x_i \right\| > r. \]

It follows that we cannot have \( \lambda_N = 0 \), so \( \lambda_N = 1 \) and therefore there exists \( \nu \leq N \) such that \( \lambda_\nu = 1 - \lambda_{\nu-1} \). Setting
\[ z \equiv 2^{\nu-1} y - \sum_{i=1}^{\nu-1} 2^{\nu-i-1} x_i, \]
we see that \( \rho(z, C) > 0 \) and \( \|z\| < r \), as required.

We now prove our generalisation of the open mapping theorem.

**Theorem 3.3.** Let \( X \) be a Banach space, and \( C \) a located, bounded, balanced, and superconvex subset of \( X \) such that \( \rho(0, -C) \) exists and \( X = \bigcup_{n\geq1} nC \). Then there exists \( r > 0 \) such that \( B(0, r) \subset C \).

**Proof.** Consider the identity
\[ X = \bigcup_{n\geq1} nC. \]

By Theorem 6.6.1 of [6] (see also [3]), there exists \( N \) such that the interior of \( NC \) is inhabited. Thus there exist \( y_0 \in NC \) and \( R > 0 \) such that \( B(y_0, R) \subset NC \). Writing \( y_1 = N^{-1} y_0 \) and \( r = (2N)^{-1} R \), we obtain \( B(y_1, 2r) \subset C \). It follows from Lemma 6.6.3 of [6] that \( B(0, 2r) \subset C \). Now consider any \( y \in B(0, 2r) \). By Lemma 3.2, there exists \( \xi \in 2C \) such that if \( y \neq \xi \), then there exists \( z \in B(0, 2r) \) with \( \rho(z, C) > 0 \). Since \( B(0, 2r) \subset C \), this is absurd. Hence \( y = \xi \in 2C \). It follows that \( B(0, 2r) \subset 2C \) and hence that \( B(0, r) \subset C \).

Note that in Lemma 3.2 and Theorem 3.3 we can replace the superconvexity of \( C \) by these two properties: \( C \) is convex, and for each sequence \( (x_n)_{n\geq1} \) in \( C \), if \( \sum_{n=1}^{\infty} 2^{-n} x_n \) converges in \( H \), then its sum belongs to \( C \).

We now derive two corollaries of Theorem 3.3.

**Corollary 3.4 (The open mapping theorem)** [6, Theorem 6.6.4]. Let \( X, Y \) be Banach spaces, and \( T \) a sequentially continuous linear mapping of \( X \) onto \( Y \) such that \( T \left( B(0, 1) \right) \) is located and \( \rho \left( 0, -T \left( B(0, 1) \right) \right) \) exists. Then there exists \( r > 0 \) such that \( B(0, r) \subset T \left( B(0, 1) \right) \).

**Proof.** In view of Theorem 3.3 it will suffice to prove that \( C \equiv T \left( B(0, 1) \right) \) is superconvex. But if \( (x_n)_{n\geq1} \) is a sequence in \( B(0, 1) \) and \( (\lambda_n)_{n\geq1} \) is a sequence of nonnegative numbers such that \( \sum_{n=1}^{\infty} \lambda_n = 1 \), then \( \|\lambda_n x_n\| \leq \lambda_n \) for each \( n \), so \( \sum_{n=1}^{\infty} \lambda_n x_n \) converges in \( X \); moreover,
\[ \left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\| \leq \sum_{n=1}^{\infty} \lambda_n = 1, \]

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This is but one version of the open mapping theorem; for another, see [5].
so, by the sequential continuity of $T$,

$$T \left( \sum_{n=1}^{\infty} \lambda_n x_n \right) \in C.$$ 

Thus $C$ is superconvex.

Theorem 3.3 also leads to the proof of Theorem 1.1:

Proof. Taking $C \equiv \mathfrak{A}_1 x$, we know that $C$ is located (since $\mathfrak{A}_1$ is weak-operator totally bounded and hence, by [7, 14], strong-operator located), as well as bounded and balanced. To prove that $C$ is superconvex, consider a sequence $(A_n)_{n \geq 1}$ in $\mathfrak{A}_1$, and a sequence $(\lambda_n)_{n \geq 1}$ of nonnegative numbers such that $\sum_{n=1}^{\infty} \lambda_n$ converges to 1. For $k \geq j$ we have

$$\left\| \sum_{n=j}^{k} \lambda_n A_n \right\| \leq \sum_{n=j}^{k} \lambda_n,$$

so $\sum_{n=1}^{\infty} \lambda_n A_n$ converges uniformly to an element $A$ of $\mathcal{B}_1(H)$. Since $\mathfrak{A}$ is uniformly closed, $A \in \mathfrak{A}_1$, so $\sum_{n=1}^{\infty} \lambda_n A_n x = Ax \in \mathfrak{A}_1 x$. Thus $C$ is superconvex. We can now apply Theorem 3.3 to produce $r > 0$ such that $B_{\mathfrak{A}_1}(0, r) \subset C$. The locatedness of $\mathfrak{A}_1 x$, and the consequent existence of the projection $[\mathfrak{A}_1 x]$, now follow from Proposition 2.4.

We now discuss further the requirement, in Theorem 1.1, that $\rho_{\mathfrak{A}_1 x}(0, \mathfrak{A}_1 x)$ exist, where $\mathfrak{A}_1$ is weak-operator totally bounded. We begin by giving conditions under which that requirement is satisfied.

If $\mathfrak{A}_1 x$ has positive, finite dimension—in which case it is both closed and located in $H$—then $\mathfrak{A}_1 x - \mathfrak{A}_1 x$ is inhabited, so Proposition (1.5) of [9] can be applied to show that $\mathfrak{A}_1 x - \mathfrak{A}_1 x$ is located in $\mathfrak{A}_1 x$. In particular, $\rho_{\mathfrak{A}_1 x}(0, -\mathfrak{A}_1 x)$ exists. On the other hand, if $P$ is a projection in $\mathcal{B}(H)$ and $\mathfrak{A} \equiv \{ PTP : T \in \mathcal{B}(H) \}$, then $\mathfrak{A}$ can be identified with $\mathcal{B}(P(H))$, so $\mathfrak{A}_1$ is weak-operator totally bounded. Moreover, if $x \neq 0$, then $\mathfrak{A}_1 x = P(H)$ and so is both closed and located, $\mathfrak{A}_1 x = \overline{B}(0, \|Px\|) \cap P(H)$, and $\rho_{\mathfrak{A}_1 x}(0, -\mathfrak{A}_1 x) = \|Px\|$.

We end with a Brouwerian example showing that we cannot drop the existence of $\rho_{\mathfrak{A}_1 x}(0, -\mathfrak{A}_1 x)$ from the hypotheses of Theorem 1.1. Consider the case where $H = \mathbb{R} \times \mathbb{R}$, and let $\mathfrak{A}$ be the linear subspace (actually an algebra) of $\mathcal{B}(H)$ comprising all matrices of the form

$$T_{a,b} \equiv \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

with $a, b \in \mathbb{R}$. It is easy to show that $\mathfrak{A}$ is uniformly closed: if $(a_n), (b_n)$ are sequences in $\mathbb{R}$ such that $(T_{a_n,b_n})_{n \geq 1}$ converges uniformly to an element $T \equiv \begin{pmatrix} a_\infty & p \\ q & b_\infty \end{pmatrix}$, then

$$a_n = T_{a_n,b_n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a_\infty,$$

Likewise, $b_n \rightarrow b_\infty$, $p = 0$, and $q = 0$. Hence $T = T_{a_\infty,b_\infty} \in \mathfrak{A}$.
Now, if \((x, y)\) is in the unit ball of \(H\), then
\[
\|T_{a,b} \left( \begin{array}{c} x \\ y \end{array} \right) \| = \left\| \begin{array}{c} ax \\ by \end{array} \right\| = a^2 x^2 + b^2 y^2
\]
\[
= a^2 (x^2 + y^2) + (b^2 - a^2) y^2
\]
\[
= a^2 + (b^2 - a^2) y^2.
\]
We see from this that if \(a^2 \geq b^2\), then \(\|T_{a,b}\| = a^2\); moreover, \(T_{a,b}(1,0) = a\), so \(\|T_{a,b}\| = a^2\).
If \(a^2 < b^2\), then a similar argument shows that \(\|T_{a,b}\| = b^2\). It now follows that \(\|T_{a,b}\|\)
exists and equals \(\max\{\|a\|, |b|\}\). Also, since, relative to the uniform topology on \(B(H)\), \(\mathfrak{A}_1\)
is homeomorphic to the totally bounded subset
\[
\{ (a, b) : \max\{\|a\|, |b|\} \leq 1 \}
\]
of \(\mathbb{R}^2\), it is uniformly, and hence weak-operator, totally bounded.

Consider the vector \(\xi \equiv (1, c)\), where \(c \in \mathbb{R}\). If \(c = 0\), then \(\mathfrak{A}_1 = \mathbb{R} \times \{0\}\), the projection
of \(H\) on \(\mathfrak{A}_1\) is just the projection on the \(x\)-axis, and \(\rho((0,1), \mathfrak{A}_1) = 1\). If \(c \neq 0\), then
\[
\mathfrak{A}_1 = \{ (a, cb) : a, b \in \mathbb{R} \} = \mathbb{R} \times \mathbb{R},
\]
the projection of \(H\) on \(\mathfrak{A}_1\) is just the identity projection \(I\), and \(\rho((0,1), \mathfrak{A}_1) = 0\). Suppose, then, that the projection \(P\) of \(H\) on \(\mathfrak{A}_1\) exists. Then either \(\rho((0,1), \mathfrak{A}_1) > 0\) or
\(\rho((0,1), \mathfrak{A}_1) < 1\). In the first case, \(c = 0\); in the second, \(c \neq 0\). Thus if \([\mathfrak{A}_1 x]\) exists for each
\(x \in H\), then we can prove that
\[
\forall x \in \mathbb{R} (x = 0 \lor x \neq 0),
\]
a statement constructively equivalent to the essentially nonconstructive omniscience principle \(\mathbf{LPO}\):

For each binary sequence \((a_n)_{n \geq 1}\), either \(a_n = 0\) for all \(n\) or else there exists
\(n\) such that \(a_n = 1\).

It follows from this and our Theorem 1.1 that if \(\rho_{\mathfrak{A}_1}(0, -\mathfrak{A}_1 x)\) exists for each \(x \in H\), then
we can derive \(\mathbf{LPO}\).

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