T-Polyform Modules
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Abstract
We introduce the notion of t-polyform modules. The class of t-polyform modules contains the class of polyform modules and contains the class of t-essential quasi-Dedekind.

Many characterizations of t-polyform modules are given. Also many connections between these class of modules and other types of modules are introduced.

Keywords: Polyform, modules, essential submodule, t-essentially, quasi-Dedekind

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Introduction
Throughout the paper, rings will have a nonzero identity element and modules will be unitary right modules. We first briefly review some background materials relevant to the topics discussed in this paper.

Recall that, a submodule N of an R-module M is called essential submodule of M( briefly N≤M ) if for each nonzero submodule W of M , N∩W≠0 [1]. Equivalently N≤M if whenever W≤M, N∩W=(0) [1] A submodule N of M is called closed (denoted by N≤M ) if has no proper essential extension in M; that is, if N≤W≤M, then N=W [1]. Ashari et. al in [2], introduced the concept of t-essential submodule, where a submodule N of M is called t-essential (briefly N≤M ) if whenever W≤M, N∩W⊂Z₂(M), then W⊂Z₂(M) where Z₂(M) is the second singular submodule of M and defined by Z(M) = Z₂(M) = {m : mI=0, for some I≤R} [1].

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Equivalentley \( Z(M) = \{ m \in M : \text{ann}(m) \leq R \} \), [1] where \( \text{ann}(m) = \{ r \in R : mr = 0 \} \). Similarly

\[
Z_{\text{ess}}(M) = \{ m \in M : me = 0, \text{for some } e \leq R \} = \{ m \in M : \ann(m) \leq R \text{ess} \}
\]

Obviously, every essential submodule is t-essential, but not conversely, for example the submodule \((4)\) of the \(Z\)-module \(Z_{12}\) is t-essential but not essential.

However, the two concepts are equivalent if \(M\) is nonsingular (ie. \(Z(M) = 0\)). A module \(M\) is called singular if \(Z(M) = M\) and is called \(Z_2\)-torsion if \(Z_2(M) = 0\). If \(A \leq M\) then \(Z_2(A) = Z_2(M) \cap A\). Asgari, etc., in [2], introduced the concept t-closed submodule where a submodule \(N\) of an \(R\)-module \(M\) is t-closed (denoted by \(N \leq M\)) if \(N\) has no proper t-essential extension in \(M\). It is clear that any \(t\)-closed submodule is closed, but the converse is not true for example (\(0\)) is closed in \(Z_s\) as \(Z\)-module but it is not t-closed. The two concepts closed submodule and \(t\)-closed submodule are coincide in nonsignular modules.

An \(R\)-module \(M\) is called polyform if for each \(L \leq M\) and for each \(\phi : L \rightarrow M\), \(\ker \phi \leq L\) implies \(\phi = 0\) (ie \(\phi \neq 0\), then \(\ker \phi \leq L\)). [3, 4].

Rizvi in [5] introduced the nation of \(k\)-nonsingular module, where an \(R\)-module \(M\) is called \(k\)-nonsingular if \(\phi \in \End(M), \ker \phi \leq M\) implies \(\phi = 0\), where \(\End(M)\) means the ring of endomorphism on \(M\).

It is clear that polyform module implies \(k\)-nonsingular but not conversely see [5].

Thaa’r in [4] gave the notion of essentially quasi-Dedekind modules as a generalization of quasi-Dedekind modules by restricting the definition of quasi-Dedekind modules (which is introduced in [6]) on essential submodules, where an \(R\)-module \(M\) is called essentially quasi-Dedekind if

\[
\Hom(M,N)_{\text{ess}} = 0
\]

for each \(N \leq M\) (that is \(M\) is essentially quasi-Dedekind if every \(N \leq M\), \(N\) is quasi-invertible. Thaa’r in [7] proved that \(k\)-nonsingular modules and essentially quasi-Dedekind are coincided.

F.S and Inaaam in [8] introduced the notion of \(t\)-essentially quasi-Dedekind where an \(R\)-module \(M\) is called \(t\)-essentially quasi-Dedekind (Shortly \(t\)-ess.q-Ded) if \(\Hom(M,N)_{\text{ess}} = 0\) for each \(N \leq M\).

Equivalently \(M\) is \(t\)-ess. q-Ded if for each \(\phi \in \End(M)\) with \(0 \neq \ker \phi \leq M\) implies \(\phi = 0\) [8].

It is obvious that every \(t\)-ess. q-Ded module is ess. q-Ded, but not conversely [8, Rem&Ex.2.2(2)].

In the present paper, motivated by these works, we introduce and study \(t\)-polyform modules as follows: An \(R\)-module \(M\) is called \(t\)-polyform if for each \(L \leq M\), and \(\phi : L \rightarrow M\), \(\ker \phi \leq L\) implies \(\phi = 0\).

Then we have

If \(M\) is \(t\)-polyform then \(M\) is polyform model and if \(M\) is \(t\)-polyform then \(M\) is \(t\)-ess q-Ded model and none of these implications is reversible (see Rem& Ex.3.2(1),(3)).

We give many properties and characterizations of \(t\)-polyform modules which are analogous to that of polyform modules (See Rem 3.2(3),Th.3.6, Th.4.7)

Also, many connections between \(t\)-polyform module and other types of modules are presented (see Theorems 3.3,3.4,4.1 and 4.4).

Next note that our notion ((\(t\)-polyform modules)) is different from (\(s\)-polyform modules) which is appeared recently in [9] as we explain that in S.3, Note 3.5

2-Preliminaries

We list some known results which are relevant for our work.

**Lemma 2.1** [2]

The following statements are equivalent for a submodule \(A\) of an \(R\)-module \(M\).

1. \(A \leq M\),

2. \(A + Z_{\text{ess}}(M) \leq M\),

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3. \( \frac{A + Z_2(M)}{Z_2(M)} \leq \frac{M}{\text{ess } Z_2(M)} \)

4. \( \frac{M}{A} \) is \( Z_2 \)-torsion (i.e. \( Z_2\left(\frac{M}{A}\right) = \frac{M}{A} \))

**Lemma 2.2** [10]

Let \( A_\lambda \) be a submodule of \( M_\lambda \) for each \( \lambda \in \Lambda \). Then
1. If \( \Lambda \) is a finite set and \( A_{\lambda \in \Lambda} \leq M_{\lambda \in \Lambda} \), then \( \bigcap_{\lambda \in \Lambda} A_{\lambda \in \Lambda} \leq \bigcap_{\lambda \in \Lambda} M_{\lambda \in \Lambda} \).
2. \( \bigoplus_{\lambda \in \Lambda} A_{\lambda \in \Lambda} \leq \bigoplus_{\lambda \in \Lambda} M_{\lambda \in \Lambda} \) if and only if \( A_{\lambda \in \Lambda} \leq M_{\lambda \in \Lambda} \), \( \forall \lambda \in \Lambda \).

**Lemma 2.3** [10]

Let \( A \leq B \leq M \). Then \( A \leq M \) if and only if \( A \leq B \) and \( B \leq M \).

**Lemma 2.4** [2]

Let \( M \) be a \( R \)-module. Then
1. If \( C \leq M \) then \( Z_2(M) \leq C \).
2. \( (0) \leq M \) if and only if \( M \) is nonsingular.
3. If \( A \leq C \leq M \) then \( C \leq M \) if and only if \( \frac{C}{A} \leq \frac{M}{A}_{\text{tes}} \).

**Lemma 2.5** [2]

Let \( C \) be a submodule of an \( R \)-module \( M \). Then the following statements are equivalent:
1. There exists a submodule \( S \) such that \( C \) is a maximal with respect to the property \( C \cap S \) is \( Z_2 \)-torsion.
2. \( C \leq M \).
3. \( C \) contain \( Z_2(M) \) and \( \frac{C}{Z_2(M)} \leq \frac{M}{Z_2(M)} \).
4. \( C \) contains \( Z_2(M) \) and \( C \leq M \).
5. \( C \) is a complement of a nonsingular submodule of \( M \).
6. \( \frac{M}{C} \) is nonsingular.

**Lemma 2.6** [2]

Let \( M = \bigoplus_{\alpha \in \Lambda} M_{\alpha} \) where \( M_{\alpha} \leq M \) for each \( \alpha \in \Lambda \). Then \( Z_2(M) = \bigoplus_{\alpha \in \Lambda} Z_2(M_{\alpha}) \).

### 3- \text{t-polyform Modules}

**Definition 3.1:** An \( R \)-module \( M \) is called \( t \)-polyform if for each \( L \leq M \) and \( \phi : L \rightarrow M \), \( \phi \neq 0 \), then \( \ker \phi \leq L \). A ring \( R \) is said to be right \( t \)-polyform if the module \( R \) is \( t \)-polyform.

#### Remarks and Examples 3.2

1. Every \( t \)-polyform module is polyform, since every essential submodule is \( t \)-essential. However, the converse is not always true for example:
   - Let \( M \) be the \( Z \)-module \( Z_6 \) since \( M \) has no proper essential submodule, then \( M \) is polyform.
   - But \( M \) is singular hence \( Z_2 \)-torsion and so every submodule, \( 0 \neq L \leq M \) is \( Z_2 \)-torsion and hence \( Z_2(L) = L \). Now for each \( 0 \neq \phi : L \rightarrow M, \ker \phi + Z_2(L) = \ker \phi = L \) and hence \( \ker \phi \leq L \) (by Lemma 2.1).

Thus \( M = Z_6 \) is not \( t \)-polyform.

2. It is known that every semisimple module is polyform but is not necessarily \( t \)-polyform, see the example in (1).

3. It is clear that every \( t \)-polyform module is \( t \)-ess.q. Ded. However, the converse may be noted true in general, for example: Let \( M = Z_p \) as \( Z \)-module, where \( P \) is a prime number. For each \( 0 \neq f : Z_p \rightarrow Z_p \).
Since \( f \neq 0 \), and \( M \) is simple so \( \text{Ker}f = (0) \) and hence by lemma 2.1, \( \text{Ker}f = (0) \leq M \), since \( \text{Ker}f + Z_2(M) = M \leq M \). Thus \( M \) is not t-polyform. But \( M \) is t-ess.q.Ded. Since for each \( f : M \rightarrow M \), with \( 0 \neq \text{Ker}f \), implies \( \text{Ker}f = M \) and so \( f = 0 \).

4. Recall that every nonsingular module \( M \) (i.e. \( Z(M) = 0 \)) is polyform. Also every nonsingular module \( M \) is t-polyform.

**Proof**: Let \( L \leq M, \phi : L \rightarrow M \) and \( \phi \neq 0 \). Since \( M \) is polyform \( \text{Ker} \phi \leq M \). But \( M \) is nonsingular, hence \( \text{Ker} \phi \leq M \). In particular each of the \( Z \)-module: \( Z, Q, Z, Q, Z[X] \) is t-polyform module.

5. Every singular \( M \) (hence \( M \) is \( Z_2 \)-torsion (\( Z_2(M) = M \))) is not t-polyform module.

**Proof**: Let \( L \leq M, \phi : L \rightarrow M \) and \( \phi \neq 0 \). Hence \( \text{Ker} \phi + Z_2(M) = \text{Ker} \phi + M = M \leq M \), so \( \text{Ker} \phi \leq M \) by lemma 2.1. Thus \( M \) is not t-polyform.

6. Prime module need not be t-polyform, for example \( M = Z_2 \oplus Z_2 \) as \( Z \)-module is prime and \( M \) is not t-polyform since \( M \) is singular. However every prime faithful module is nonsingular, hence it is t-polyform by part (4).

7. Every submodule \( N \neq 0 \) of t-polyform module \( M \) is t-polyform.

**Proof**: Let \( 0 \neq L \leq N \) and let \( f : L \rightarrow N, f \neq 0 \). Then, \( 0 \neq \text{iof} : L \rightarrow M \) where \( i \) is the inclusion mapping from \( N \) into \( M \).

Since \( M \) is t-polyform then \( \text{Ker} \phi \leq L \).

But it is easy to check that \( \text{Ker}f = \text{Ker} \phi \) and hence \( \text{Ker}f \leq L \). Thus \( N \) is t-polyform.

In particular if \( \overline{M} \) (quasi-injective hull of \( M \)) or \( E(M) \) (injective hull of \( M \)), then \( M \) is t-polyform.

8. A homomorphic image of t-polyform module is not necessarily t-polyform, for example the \( Z \)-module \( Z \) is t-polyform. Let \( \pi : Z \rightarrow Z/6 \approx Z_6 \) where \( \pi \) is the natural epimorphism, but \( Z_6 \) is not t-polyform by part (1).

9. If \( M \) is a t-polyform \( R \)-module and \( N \leq M \) then \( \frac{M}{N} \) is t-polyform.

**Proof**: Since \( N \leq M \) \( \frac{M}{N} \) is nonsignular by lemma (2.5). Hence \( \frac{M}{N} \) is t-polyform by part (4).

10. Recall that an \( R \)-module is Co-epi-retractable if for each \( N \leq M \), there exists \( K \leq M \) such that \( \frac{M}{N} \approx K \).

If \( M \) is t-polyform and Co-epi-retractable, then \( \frac{M}{N} \) is t-polyform, for each \( N \leq M \).

**Proof**: it follows directly.

The following theorem is a characterization of t-polyform modules.

**Theorem 3.3** An \( R \)-module \( M \) is t-polyform if for each \( 0 \neq L \leq M \) and \( 0 \neq \phi : L \rightarrow M, \text{Ker} \phi \leq L \).

**Proof**: Suppose there exist \( 0 \neq L \leq M \) and \( 0 \neq \phi : L \rightarrow M \), but \( \text{Ker} \phi \leq L \). By definition of t-closed submodule, there exists \( U \leq L \) such that \( U \) is a proper t-essential extension of \( \text{Ker}f \).
Then $\phi \circ i : U \to M$ where $i$ is the inclusion mapping from $U$ into $L$. Clearly $\text{Ker}(i \circ \phi) \leq \text{Ker} \phi$, so that $\text{Ker}(\phi \circ i) \leq U$. Hence $\phi \circ i = 0$ since $M$ is t-polyform. It follows that $\phi(U) = 0$; that is $U \leq \text{Ker} f$ which is a contradiction. Thus $\text{Ker} \phi \leq L$.

Conversely, suppose there exist $L \leq M$ and $0 \neq \phi : L \to M$ with $\text{Ker} \phi \leq L$. But $\text{Ker} \phi \leq L$ by hypothesis, so $\phi = 0$ which is a contradiction. Thus $\text{Ker} f \leq L$ and So $M$ is t-polyform.

The following is another characterization of t-polyform modules

**Theorem 3.4** Let $M$ be an $R$-module. Then $M$ is t-polyform if and only if for each $0 \neq N \leq M$ and for nonzero $f \in \text{Hom}(N,M)$, then $\text{ker} f \not\leq N$.

**Proof**: (i) it is clear

(ii) Let $N \leq M$. If $N \not\leq M$ then nothing to prove if $N \leq M$, let $f : N \to M, f \neq 0$. Since $N \leq M$ then $N \leq M$.

Define $g : N \oplus K \to M$ by $g(n+k) = f(n), n \in N, k \in K$. $g$ is well-defined and $g \neq 0$. By hypothesis, $\text{ker} g \not= N \oplus K$. But $\text{Ker} g = \text{Ker} f \oplus K$ and so that $\text{ker} f \not= N$ by lemma 2.2 (2). Thus $M$ is t-polyform.

The notion of ((st-polyform modules)) appeared in [9], where an $R$-module $M$ is called st-polyform if for each $0 \neq L \leq M, 0 \neq \phi : L \to M$ $\text{ker} f \not\leq L$. A submodule $U$ of $M$ is called st-closed($U \leq M$) if $U$ has no proper semiessential extension of $U$, and a submodule $U$ of $M$ is called semi-essential in $M$ if $U$ has nonzero intersection with any nonzero prime submodule.

**Note 3.5**

The two concepts (t-polyform modules) and (st-polyform modules) are independent as we can see by the following examples.

1. $\mathbb{Z}$ as $\mathbb{Z}$-module is not t-polyform (see Rem 3.2(1)) and it is st-poly by [5, Rem.3(vii)]

2. $\mathbb{Z}$ as $\mathbb{Z}$-module is t-polyform (see Rem 3.2(4)), and it is not st-polyform [see 5, Ex.5(ii)]

[4] gave the following: An $R$-module $M$ is polyform if and only if every essential submodule is rational, where a submodule $N$ of $M$ is called rational in $M$ (briefly $N \leq M$) if $\text{Hom}(V_r, M) = 0$ for each $N \leq V \leq M$ [1].

Note that every rational submodule is essential but not conversely [1].

We give the following:

**Theorem 3.6** An $R$-module $M$ is t-polyform implies every nonzero t-essential submodule of $M$ is rational.

**Proof**: Assume $0 \neq N \leq M$ and $f \in \text{Hom}(V_r, M)$, where $N \leq V \leq M$. Then $f \circ \pi \in \text{Hom}(V, M)$ where $\pi$ is the natural epimorphism from $V$ onto $V_r$. Hence $N \leq \text{ker}(f \circ \pi)$, but $N \leq M$ implies $\text{ker}(f \circ \pi) \leq M$ by lemma (2.3). So that $\text{ker}(f \circ \pi) \leq V$ (since $\text{ker}(f \circ \pi) \subseteq V$). Since $M$ is t-polyform, $f \circ \pi = 0$, and hence $f = 0$. Thus $\text{Hom}(V_r, M) = 0$ that is $N \leq M$.

**Remark 3.7** The converse of theorem (3.6) is not true in general, for example: The $\mathbb{Z}$-module $\mathbb{Z}_6$ is not t-polyform, but $\mathbb{Z}_6$ has only $\mathbb{Z}_6$ as t-essential submodule of $\mathbb{Z}_6$ and $\mathbb{Z}_6 \leq \mathbb{Z}_6$.

However, we have:

**Theorem 3.8** If $M$ is an $R$-module such that every nonzero t-essential submodule is rational, then $M$ is polyform.
Proof: Let \( N \leq M \), hence \( 0 \neq N \leq M \). Then by hypothesis is \( N \leq M \) and every essential submodule is rational. It follows that \( M \) is polyform.

Recall a nonzero \( R \)-module \( M \) is called monoform if for each \( 0 \neq N \leq M \) and for each \( 0 \neq f \in \text{Hom}(N,M) \), then \( \text{ker} \ f = 0 \), [9]. Equivalently a nonzero \( R \)-module \( M \) is monoform if for each nonzero submodule \( N \) of \( M \), \( N \leq M \), [9].

It is known that every monoform is polyform. Now we ask the following: Is there any relation between \( t \)-polyform modules and moniform?

Consider the following remarks

Remarks 3.9
1. \( t \)-polyform modules need not be monoform, for example: The \( Z \)-module \( Z \oplus Z \) is \( t \)-polyform (Rem 3.2.(4)), but it is not monoform since there exists \( f : Z \oplus 2Z \to Z \oplus Z \) such that \( f(x,y) = (y,0) \) for each \( x \in Z \), \( y \in 2Z \) then \( \text{ker} f = Z \oplus (0) \neq \text{zero submodule} \).
2. Monoform module may be not \( t \)-polyform module, for example: The \( Z \)-module \( Z_p \), where \( p \) is a prime number, is monoform but it is not \( t \)-polyform.

Therefore, we introduce the following

Definition 3.10 An \( R \)-module \( M \) is called \( t \)-essential monoform (shortly \( t \text{-ess- mono} \)) if for each \( 0 \neq N \leq M \) and \( 0 \neq f \in \text{Hom}(N,M) \) then \( \text{ker} f = 0 \).

Every simple module is \( t \)-ess mono and every monoform module is \( t \)-ess. mono.

Proposition 3.11: Let \( M \) be a \( t \)-ess- mono. module. Then \( M \) is quasi-Dedekind and hence \( M \) is \( t \text{-ess.q.-Ded} \).

Proof: Since \( M \leq M \) and \( M \) is a \( t \)-ess- mono. the for each \( 0 \neq f \in \text{End} (M) \) implies \( \text{ker} f = 0 \). Thus \( M \) is quasi-Dedekind by [6, Th1.5, p.26] and hence \( M \) is \( t \text{-ess.q.-Ded} \).

By th.(3.6), We have: If \( M \) is \( t \)-polyform, then for each \( 0 \neq N \leq M \) implies \( N \leq M \).

Now we give the following

Proposition 3.12: If \( M \) is \( t \)-ess- mono. \( R \)-module, then for each \( 0 \neq N \leq M \) implies \( N \leq M \).

Proof: Suppose there exists \( 0 \neq N \leq M \) but \( N \leq M \) Hence there exists \( V \nmid N \) such that \( \text{Hom}(V, N^{\circ}) = 0 \) \( \text{V \nmid N} \), so \( f \in \text{Hom}(V, M) \), \( f \neq 0 \). It follows that \( f \circ \pi \in \text{Hom}(V,M) \), where \( \pi \) is natural epimorphism from \( V \) onto \( M \), and \( f \circ \pi \neq 0 \) (Since \( f \neq 0 \)). But \( N \preceq V \), \( V \leq M \) and since \( M \) is \( t \)-ess- mono, \( \text{ker}(f \circ \pi) = 0 \). Since \( N \preceq \text{ker}(f \circ \pi) = 0 \) thus \( N = 0 \) which is a contradiction therefore \( N \leq M \).

Corollary 3.13: Let \( M \) be a \( t \)-ess- mono. Then \( M \) is polyform.

Proof: It follows by prop.(3.12) and Th.(3.8)

Proposition 3.14: Let \( M \) a quasi-injective \( R \)-module. If \( M \) is \( t \text{-ess.q.-Ded} \), then for each \( 0 \neq N \leq M \) implies \( N \leq M \).

Proof: Let \( 0 \neq N \leq M \) Since \( M \) is \( t \text{-ess.q.-Ded} \), \( \text{Hom}(M, N^{\circ}) = 0 \); that is \( N \) is a quasi-invertible submodule of \( M \). Since \( M \) is quasi-injective, then by \( [6, \text{Th3.5 p.16}] \), \( M \) is a rational extension of \( N \); that is \( N \leq M \).

Corollary 3.15: Let \( M \) be a quasi-injective if \( M \) is \( t \text{-ess.q.-Ded} \), then \( M \) is polyform

Proof: It follows by prop. (3.14) and Th.(3.7)

We can Summarize results of S.3 by the following tables
$4$ More about $t$-polyform module

It is known that, for an $R$-module $M$, the following are equivalent:

1. Every essential submodule is rational (i.e. $M$ is polyform)
2. For each $0 \neq N \leq M$, $f: N \to M$, $f \neq 0$, then $\ker f \leq_c N$ (i.e. All partial endomorphism of $M$ have closed kernels in their domains)
3. $\text{End}(M)$ is vonneuman regular
4. For each $N \leq M$, $\text{Hom}\left(\frac{M}{N}, \frac{M}{M}\right)=0$

Proof

$(1) \iff (2) \iff (3)$ [2 , 4.9.P.34].
$(2) \iff (3) \iff (4)$ [13].

Our aim is to give analogize property for $t$-polyform module.

In S.3 we prove that an $R$- module $M$ is $t$-polyform if and only if for each $0 \neq N \leq M$, $f: N \to M$, $f \neq 0$ implies $\ker f \leq M$.

Now we prove the following:

Theorem 4.1 An $R$- module $M$ is $t$-polyform if and only if for each $0 \neq N \leq M$, $\text{Hom}\left(\frac{M}{N}, \frac{M}{M}\right)=0$.

Proof: Suppose there exists $(N \leq M) \neq 0$ such that $\text{Hom}\left(\frac{M}{N}, \frac{M}{M}\right)=0$. Hence there exists $f: \frac{M}{N} \to \frac{M}{M}$ and $f \neq 0$, and so there exists $m + N \in \frac{M}{N}$, $m + N \neq 0$ such that $f(m + N) = m' \neq 0$. Since $M \leq \frac{M}{M}$, there exists $r \in R$ with $0 \neq m' \in M$ let $m' = x$. Define $\phi: N + Rm \to Rx \subseteq M$, $\phi(n + tm) = tx$ for each $n \in N$, $t \in R$. To show that $\phi$ is well - defined: if $n_1 + t_1m = n_2 + t_2m$, then $n_1 - n_2 = ...$
(t_2 - t_1)m \in N. Hence (t_2 - t_1)f(m + N) = f(t_2 - t_1)m + N] = 0, this implies (t_2 - t_1)m' = 0 and so (t_2 - t_1)x = 0. Thus (t_2 - t_1)x = 0.

So that a_2x = a_1x. It is clear that \phi \neq 0.

Now \psi: N + Rm \to M where i: R \to M is the inclusion \psi \neq 0. Hence Ker(\psi) = Ker(\phi) = 0. But N \subseteq Ker(\phi) and N \subseteq M implies Ker(\phi) \subseteq M. Therefore \psi \circ \phi \neq 0. Hence Ker(\psi \circ \phi) \subseteq N + Rm.

Finally, suppose that A is not t-polyform. Then there exists M \subseteq K, f \in \text{Hom}(K, M), f \neq 0 and Kerf \subseteq K. Since M is quasi-injective there exist g \in \text{End}(\bar{M}) such that g \circ i = j \circ f where i: K \to \bar{M}, j: M \to \bar{M} be the inclusion mappings.

Since f \neq 0, then g \neq 0. It is clear that Kerf \subseteq \text{kerg}.

Define by \overline{g}: \overline{M} \to M by \overline{g}(m + \text{kerf}) = g(m) for each \overline{m} \in M. Then it is easy to see that \overline{g} is well-defined it follows that g \circ i \in \text{Hom}(\overline{M}/\text{kerf}, M), where i: \overline{M}/\text{kerf} \to \overline{M}/\text{kerf} by hypothesis \overline{g} \circ i = 0.

That is for each \overline{m} \in M, \overline{g} \circ i(m + \text{kerf}) = \overline{g}(m + \text{kerf}) = g(m) = 0. Thus g = 0 which is a contradiction. Therefore Kerf \subseteq K and M is a t-polyform module.

Recall that an R-module M is called Rickart if for each f \in \text{End}(M), Kerf \subseteq M [14, Def. 2.11, P. 20]. The following results is given in [14, Lemma 2.4.21, P. 59].

**Lemma 4.2**

The following condition are equivalent for a right R-module M:
1. M is a polyform module.
2. M is K-nonsingular (where \overline{M} is the quasi-injective hull of M).
3. M is a Rickart module.

We prove the following characterization for t-polyform modules.

**Theorem 4.3**

An R-module M is t-polyform if and only if \overline{M} is t-ess. q-Ded.

**Proof:**

(\rightarrow) Suppose there exists K \subseteq M and \phi \in \text{Hom}(K, M) with ker\phi \subseteq K. To prove \phi = 0. Since \overline{M} \subseteq \overline{M}, hence M \subseteq \overline{M}. Thus Ker\phi \subseteq K \subseteq M \subseteq \overline{M} which implies Ker\phi \subseteq \overline{M} and K \subseteq \overline{M} by Lemma (2.30).

Now K = K \cap E(K) \subseteq \overline{M} \cap E(K) by Lemma 2.2(1), where E(K) is the injective hull of K. Hence \overline{M} \cap E(K) \subseteq \overline{M} \cap E(K) for some X \subseteq \overline{M}. Define \psi: K \cap X \to \overline{M} by \psi = \phi on k and \phi = 0 on X. Since \overline{M} is quasi-injective, there exist \overline{\psi}: \overline{M} \to \overline{M} such that \overline{\psi} \circ i = \psi where i: K \cap X \to M be the inclusion mapping. Since \overline{\psi} = \psi on K \cap X, then \overline{\psi} = \phi on K and \overline{\psi} = 0 on X.

We can easily see that: Ker\psi \subseteq Ker\phi \cap X but Ker\phi \subseteq K and X \subseteq \overline{X} . hence by lemma 2.2(1), Ker\psi \subseteq K \cap X . On the other hand, K \subseteq \overline{M}, so K \cap X \subseteq \overline{M} . Therefore Ker\phi \subseteq \overline{M} . Also Ker\overline{\psi} \subseteq Ker\psi .

It follows that Ker\overline{\psi} \subseteq \overline{M} , hence \overline{\psi} = 0 since \overline{M} is t-ess. q-Ded. However \overline{\psi} = 0 implies \phi = 0.

Thus M is t-polyform.

(\leftarrow) To prove \overline{M} is t-ess. q-Ded. Let f \in \text{End}(M) and f \neq 0. To show that Kerf \subseteq \overline{M}, we shall prove that Kerf \subseteq \overline{M} and hence Kerf \subseteq \overline{M} By [3, Lemma 2.3], there exists K \subseteq \overline{M} such that Kerf \subseteq K. Hence K \subseteq \overline{M} by Lemma 2.5, so that \overline{M} = K \cap A for some A \subseteq \overline{M}. Define h: \overline{M} \to \overline{M} by h|_A = 0 and h|_{\overline{M}} = f|_{\overline{M}} . Hence Kerh = Kerf \cap A. But Kerf \subseteq K, A \subseteq \overline{M}, implies Kerh = Kerf \cap A \subseteq K \cap A by

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Lemma 2.2(2). Now for any \( \alpha \in \text{End} \, \overline{M} \), Ker(\( h\circ\alpha \)) = \( \alpha^{-1}(\text{Ker}\, h) \). Since \( \text{Ker}\, h \leq \overline{M} \), then \( \alpha^{-1}\text{Ker}\, h \leq \overline{M} \) by [10, 2014, cor. 1.2]. Thus \( \text{Ker}(h \circ \alpha) \)

\[ \cap \text{M} \leq \overline{M} \cap \text{M} = \text{M} \] by Lemma (2.2). Then by Theorem (4.1), \( 0 = \text{Hom}(\frac{\text{M}}{\text{Ker}(h \circ \alpha) \cap \text{M}}, \overline{M}) = \text{Hom}((h \circ \alpha)(\text{M}, \overline{M}) \) and so \( h(\alpha(M)) = 0 \). Since \( \alpha \in \text{End} \, (\text{M}) \) is arbitrary, \( \text{h}(\overline{M}) = \sum_{\alpha \in \text{End}(\text{M})} h(\alpha(M)) = 0 \). Thus h=0 and

\[ \text{Kerf} = K \leq \overline{M} \] Thus \( \text{Kerf} \leq \overline{M} \).

Corollary 4.4

Let \( \text{M} \) be a quasi-injective module then \( \text{M} \) is t-poly form if and only if \( \text{M} \) is t-ess-q-Ded

Proof: It follows directly by Th. (4.3).

Recall that an \( R \)-module \( \text{M} \) is called a t-Rickart if \( t_{\text{M}}(\phi) = \phi^{-1}(Z_{2}(\text{M})) \) is a direct summand of \( \text{M} \) for every \( \phi \in \text{End} \, (\text{M}) \)[1, Def2.1].

Note that every nonsingular Rickart module is t-Rickart, every extending module and every \( Z_{2} - \text{tor}sion \) module (i.e a module \( \text{M} \) for which \( Z_{2}(\text{M}) = \text{M} \)) is t-Rickart. A Rickart module need not be t-Rickart, see [1, Ex.2.10]

We prove that

Theorem 4.5

If \( \text{M} \) is a t-polyform module, then \( \overline{M} \) is t-Rickart

Proof:

Since \( \overline{M} \) is nonsingular, \( Z_{2}(\overline{M}) \leq \overline{M} \) and hence \( Z_{2}(\overline{M}) \leq \overline{M} \). But \( \overline{M} \) is quasi-injective (hence extending) so that \( Z_{2}(\overline{M}) \) is a direct summand of \( \overline{M} \). Thus \( \overline{M} = Z_{2}(\overline{M}) \oplus C \) for some \( C \leq \overline{M} \). But \( C \approx \frac{\overline{M}}{Z_{2}(\overline{M})} \) which is nonsingular, so \( C \) is nonsingular. But \( M \) is t-polyform, hence \( \overline{M} \) is t-ess. Quasi-Ded by Theorem 4.3. Thus \( \overline{M} \) is K-nonsingular (i.e ess. q-Ded) . On other, \( \overline{M} \) is quasi-injective, so \( \overline{M} \) is extending. But \( \overline{M} \) is K-nonsingular extending module implies \( \overline{M} \) is Baer which implies Rickart by [15, Lemma 2.2.4, r.13].

Since \( C \leq \overline{M} \), then \( C \) is Rickart. Thus \( \overline{M} \) is t-Rickart by [11, Th2.6.1 (1→2)]

Remarks 4.6

1. The converse of Th.(4.5) is not true if \( Z_{2}(\overline{M}) \neq 0 \)

Proof: Since \( \overline{M} \) is t-Rickart, \( \overline{M} = Z_{2}(\overline{M}) \oplus C \), for some nonsingular Rickart submodule \( C \) of \( \overline{M} \). If \( Z_{2}(\overline{M}) \neq 0 \), then there \( i: Z_{2}(\overline{M}) \to \text{M} \), where \( i \) is the inclusion mapping, and \( i \neq 0 \). Thus ker \( f \) = (0). But

\[ (0) \leq Z_{2}(\overline{M}) \leq Z_{2}(\overline{M}) \text{ t-ess } \]

That is \( \text{Kerf} \leq Z_{2}(\overline{M}) \) and so \( \overline{M} \) is not t-polyform therefore \( (\overline{M}) \) is not t-ess.q-Ded by cor (4.4)

2. If \( Z_{2}(\overline{M}) = 0 \) and \( \overline{M} \) is t-Rickart, then \( M \) is t-polyform.

Proof: As in (1), \( \overline{M} = Z_{2}(\overline{M}) \oplus C \) where \( C \) is nonsingular Rickart. Since \( Z_{2}(\overline{M}) = 0 \), then \( \overline{M} = C \); that is \( \overline{M} \) is nonsingular, hence \( \overline{M} \) is t-polyform thus \( M \) is t-polyform by Rem & Ex.2.2.(7)

Now we have:

Theorem 4.7

Let \( M \) be a t-polyform extending module. Then \( \overline{M} \oplus M \) is t-Rickart module.

Proof: Since \( M \) is extending, \( M \) is t-Rickart. Also \( M \) is t-polyform implies \( \overline{M} \) is t-Rickart by (4.4). By [1, Th2.6.1] \( M = Z_{2}(M) \oplus A \), \( A \) is nonsingular Rickart sub-module of \( M \), \( \overline{M} = Z_{2}(\overline{M}) \oplus B \), \( B \) is a nonsingular Rickart submodule of \( \overline{M} \). Hence \( \overline{M} \oplus M = Z_{2}(\overline{M}) \oplus Z_{2}(M) \oplus (B \oplus A) = Z_{2}(\overline{M}) \oplus M \oplus (B \oplus A) \) by Lemma 2.6 hence \( B \oplus A \) is a nonsingular submodule of \( \overline{M} \oplus M \) since \( A \leq M \), then \( A \) is t-polyform and extending and so \( A \) is polyform and
extending $B \leq M$ and $\overline{M}$ is quasi-injective, hence $B$ is a quasi-injective. On the other hand, $M = Z_2(M) \oplus A$ implies $\overline{M} = Z_2(\overline{M}) \oplus \overline{A}$. But $\overline{M} = Z_2(\overline{M}) \oplus B$. So $B = \overline{A}$. Thus $B \oplus A = \overline{A} \oplus A$ and hence by [14, prop 2.4.22, p.60], $B \oplus A$ is Rickart and then by [11, Th 2.6.1], $\overline{M} \oplus M$ is t-Rickart.

It is well-known that a submodule $N$ of $M$ is fully invariant if for each $f \in \text{End}(M)$, $f(N) \leq N$. Also recall the following basic fact: if $N$ is a fully invariant sub module of $M = M_1 \oplus M_2$ then $N = (N \cap M_1) \oplus (N \cap M_2)$

**Proposition 4.8**

For an $R$-module $M$, if $E(M)$ (injective hull of $M$) is t-poly form, then $Z_2(M)$ is a direct summand of $M$.

**Proof:** Since $E(M)$ is t-polyform, then $\overline{E(M)}$ is t-Rickart by Th.(4.5). But $E(M) = E(M)$, hence $E(M)$ is t-Rickart. Then by [1, Th.2.6.1] $E(M) = Z_2(E(M)) \oplus A$. A is a nonsingular Rickart submodule of $E(M)$ since $M$ is a fully invariant submodule of $E(M)$, then $M = (Z_2(E(M) \cap M)) \oplus (A \cap M)$, but $Z_2(M) = Z_2(E(M)) \cap M$. Thus $M = Z_2(M) \oplus (A \cap M)$ therefore $Z_2(M) \preceq M$.

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