Superfluid vortex dynamics on a torus and other toroidal surfaces of revolution

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The superfluid flow velocity is proportional to the gradient of the phase of the superfluid order parameter, leading to the quantization of circulation around a vortex core. In this work, we study the dynamics of a superfluid film on the surface of a torus. Such a compact surface allows only configurations of vortices with zero net vorticity. We derive analytic expressions for the flow field, the total energy, and the time-dependent dynamics of the vortex cores. The local curvature of the torus and the presence of non-contractable loops on this multiply connected surface alter both the superfluid flow and the vortex dynamics. Finally we consider more general surfaces of revolution, called toroids.

I. INTRODUCTION

The concept of point vortices in a two-dimensional incompressible and inviscid fluid (also called a perfect fluid) originated in the seminal works of Helmholtz [1]. For such a fluid, the number of point vortices is conserved and, once the effect of the container boundary has been included, the time-dependent evolution of the entire fluid flow field can be reduced completely to the dynamics of such point vortices. Moreover, the vortices can be described in the framework of classical Hamiltonian mechanics [2–4], with the two coordinates of each vortex forming a pair of canonical Hamiltonian variables. These remarkable properties have generated considerable interest not only among physicists but also among mathematicians, with a wide literature available (see for example [5]).

The model of a perfect fluid, initially studied as a simplification of classical fluid dynamics, is almost exactly realized in quantum superfluid phases, and the point-vortex model has found wide application in the context of superfluid helium. The same formalism also applies directly to dilute ultracold superfluid atomic Bose-Einstein condensates (BECs), which have the important advantage of unprecedented control over many experimental parameters, such as interactions and confining potentials [6, 7]. A superfluid state is characterized by a complex scalar order parameter (a condensate wave function) \( \Psi = \sqrt{\rho} e^{i\Phi} \), where \( \rho \) is the condensate number density. Here the phase \( \Phi \) determines the superfluid velocity

\[
\mathbf{v} = \left( \frac{\hbar}{M} \right) \nabla \Phi,
\]

where \( M \) is the mass of the superfluid particles and \( \hbar \) is the reduced Planck’s constant. Equation (1) implies immediately that the flow field is irrotational, apart from singular points.

Although dilute ultracold superfluid BECs are compressible, local changes in the density can be neglected in the Thomas-Fermi (TF) limit, appropriate in many typical experiments. In the incompressible limit the density \( \rho \) is constant, and the condition of current conservation for steady flow \( \nabla \cdot (\rho \mathbf{v}) = 0 \) reduces to the condition \( \nabla \cdot \mathbf{v} = 0 \), namely that the velocity field is divergenceless. In this case, one can define a scalar stream function \( \chi \) such that

\[
\mathbf{v} = \left( \frac{\hbar}{M} \right) \mathbf{n} \times \nabla \chi,
\]

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with \( \mathbf{n} = \mathbf{x} \times \mathbf{y} \) the unit vector normal to the two-dimensional plane. Lines of constant \( \chi \), called streamlines, are the trajectories of particles immersed in the fluid, giving the stream function a direct physical interpretation. The relations between \( \Phi, \chi \) and the flow field
\[
\frac{M v_x}{\hbar} = \frac{\partial \Phi}{\partial x} = - \frac{\partial \chi}{\partial y}, \quad \frac{M v_y}{\hbar} = \frac{\partial \Phi}{\partial y} = \frac{\partial \chi}{\partial x}
\] (3)
can be interpreted as the Cauchy-Riemann equations for the real and imaginary parts of a complex function \( F(z) = \chi + i\Phi \) with complex coordinate \( z = x + iy \). Together, these equations have the complex form
\[
\mathbf{v}_y + i\mathbf{v}_x = \left( \frac{\hbar}{M} \right) F'(z)
\] (4)
implying that \( F(z) \) must be analytic almost everywhere.

The description of two-dimensional flow in terms of a complex potential has a long history in classical hydrodynamics. In a superfluid, however, the dynamics becomes quantized because the velocity potential \( \Phi \) has the physical significance of a condensate phase and must be uniquely defined, up to multiples of \( 2\pi \). The most famous consequence of this requirement is the quantization of vorticity, allowing only multiples of the elementary “charge” of vorticity \( 2\hbar/M \).

For a superfluid container with a nontrivial topology, this quantization condition leads to new constraints on the flow field, as studied first by Ref. [8] (see Ref. [9] for a comprehensive summary). More recently, the vortex dynamics on the surfaces of cylinders and cones has been investigated in Refs. [10, 11].

A cylinder and a cone are both unbounded, similar to the infinite plane. Each of these configurations can support single vortices, because the lines of constant phase can escape to infinity. The situation is very different for a compact surface such as a sphere or a torus. Here the lines of constant phase emerging from a vortex must converge on another vortex with opposite sign, implying that the total vortex charge on a compact surface must vanish [12, 13]. The high symmetry of the sphere means that the dynamics of a vortex dipole is particularly simple. As noted briefly in Sec. 160 of Lamb [12], the problem can be solved straightforwardly via a stereographic projection.

In the present work we study the dynamics of superfluid vortices on the surface of a torus, the simplest compact and multiply connected surface. For such a curved two-dimensional surface, Eqs. (1) and (2) still hold, but now only in the local tangent space of the surface at any given point. In Sec. II, we introduce an isothermal set of coordinates, see Eq. (7), for the torus’ surface, allowing an efficient description of the flow field induced by point vortices. In Sec. III this flow is discussed in the framework of complex potentials. Section IV derives the energy functional for vortex ensembles, and Sec. V obtains the time-dependent dynamics of the vortices themselves. Finally, Sec. VI extends the treatment in the previous sections to superfluid films on generalized toroidal surfaces, deriving an appropriate isothermal coordinate set.

### II. Isothermal Coordinates

A torus has a major radius \( a \) and a minor radius \( b \), with \( a \geq b \), as shown in the top panel of Fig. 1. Every point \( s \) on its surface can be parameterized in terms of two angles \( \phi, \eta \in (-\pi, \pi) \) in Cartesian coordinates:
\[
\begin{align*}
\phi &= (a + b \cos \eta) \cos \phi \\
\eta &= (a + b \cos \eta) \sin \phi \\
z &= b \sin \eta,
\end{align*}
\] (5)
where \( \phi \) is the “toroidal” angle and \( \eta \) is the “poloidal” angle. Correspondingly, the squared line element \( ds^2 = dx^2 + dy^2 + dz^2 \) on the surface becomes
\[
ds^2 = (a + b \cos \eta)^2 d\phi^2 + b^2 d\eta^2 \equiv \lambda_\phi^2 d\phi^2 + \lambda_\eta^2 d\eta^2.
\] (6)
The coordinate set \{\( \phi, \eta \)\} is curvilinear, since physical distances are related to infinitesimal elements \( d\phi \) and \( d\eta \) through the local scale factors \( \lambda_\phi \) and \( \lambda_\eta \). This metric distorts the gradients appearing in Eqs. (1) and (2), precluding a formulation in terms of a complex function \( F(\phi + i\eta) \) of a complex variable \( \phi + i\eta \).

Such formulation is still possible, however, if one finds a suitable set of isothermal coordinates \{\( u, v \)\} with squared line element
\[
ds^2 = \lambda^2 (du^2 + dv^2),
\] (7)
which ensures that the coordinates \{\( u, v \)\} are a conformal parametrization of the surface. Kirchhoff already gave such an isothermal coordinate system for a torus in 1875 [14] in the context of electrohydrodynamics on two-dimensional surfaces:
\[
\phi = \frac{u}{c}, \quad \tan \left( \frac{\eta}{2} \right) = \sqrt{\frac{a + b}{a - b}} \tan \left( \frac{v}{2b} \right),
\] (8)
with \( c = \sqrt{a^2 - b^2} \). Lengthy but elementary algebra shows that \{\( u, v \)\} are indeed isothermal and satisfy Eq. (7) with local scale factor
\[
\lambda = \frac{c}{a - b \cos(u/b)}.
\] (9)

In the \{\( u, v \)\} plane, these coordinates vary in a rectangular cell of dimensions \([-\pi c, \pi c] \times [-\pi b, \pi b]\) with opposite edges identified. The bottom panel of Fig. 1 illustrates this situation. Section VI derives the transformation given in Eq. (8), where we extend the analysis to more general toroidal surfaces of revolution.

### III. Complex Flow Potential of Vortex Configurations

With the isothermal coordinate set \{\( u, v \)\} established in the last section, we can find the flow field in terms of a complex potential function. Define the complex coordinate \( w = u + iv \) describing the point \( s \) on the torus’
FIG. 2. Phase (left panels) and corresponding flow field (right panels) for a torus with major radius $a$ and minor radius $b$ such that $a = \sqrt{2}b$, depicted in the \{u,v\} coordinate set (top panels) and schematically mapped to the surface of the torus (bottom panels). The black lines in the left panels indicate stream lines, the white lines curves of constant phase. For clarity, the right panels omit points too close to the vortex cores, as well as regions with vanishing velocity.

surface and call $F(w)$ the complex flow potential associated with a set of $N$ vortices at positions $w_n$ with charges $q_n = \pm 1$, $n \in \{1, \ldots, N\}$. Close to each vortex core $w_n$ the flow field must behave just as in the flat plane, as described in Chapter IV of Ref. [12]:

$$F(w) = q_n \log(w - w_n) + \mathcal{O}(w - w_n).$$  \hfill (10)

Additionally, the superfluid order parameter $\Psi$ must be periodic across the \{u,v\} cell. This last condition means that the phase $\Phi = \Im\{F\}$ must wind in integer multiples $(k,m)$ of $2\pi$ upon traversing the cell in either direction:

$$\Im\{F(w + 2\pi c)\} = \Im\{F(w)\} + 2\pi k,$$
$$\Im\{F(w + i 2\pi b)\} = \Im\{F(w)\} + 2\pi m. \hfill (11)$$

We can distinguish two classes of functions $F(w)$ fulfilling Eq. (11). The first class contains simple linear functions of the form $F_{k,m}(w) = w(ik/c + m/b)$ with integer coefficients $k,m$. These potentials represent uniform flows with $k$ elementary quanta of circulation around the toroidal direction, and $m$ quanta along the poloidal direction. The second class consists of specific complex functions that are doubly periodic within the elementary cell, called elliptic functions [15]. For our purposes, the following property is crucial: *The sum of all residues of a doubly periodic function at its poles within any elementary cell is zero* (see Sec. 20.12 of Ref. [15]). It follows from Eq. (10) that the net vorticity in the system is zero: $\sum q_n = 0$. In particular the simplest possible configuration is a vortex dipole with \{q_1 = 1, q_2 = -1\}, which we consider in the next section.

### A. Vortex dipoles

As seen in the bottom panel of Fig. 1, the elementary cell in the \{u,v\} plane is a rectangle with periodic boundary conditions. To satisfy these boundary conditions, imagine tiling the whole plane with replicas of the original elementary cell.

For this purpose, it is convenient to consider a specific entire complex function

$$\vartheta_1(\zeta, p) = 2p^{1/4} \sum_{n=0}^{\infty} (-1)^n p^{n+1} \sin[(2n+1)\zeta] \hfill (12)$$

known as the first Jacobi theta function (see Chap. 21 of
It depends on the two dimensionless complex variables \( \zeta \) and \( p \) (with the restriction \( |p| < 1 \)). To ensure the latter constraint, we write \( p = e^{i \pi \tau}, \) where \( 3 \{ \tau \} > 0 \). In the complex \( \zeta \) plane, the function \( \vartheta_1(\zeta, p) \) has a periodic array of zeros at \( \zeta_{km} = k \pi + m \pi \tau, \) with \( k, m \in \mathbb{Z} \). Correspondingly \( \log \vartheta_1(\zeta, p) \) has a periodic array of positive vortices at the same locations \( \zeta_{km} \) [compare Eq. (10)].

For the present rectangular cell, choose \( \tau = i b / c \) to be purely imaginary, so that \( p = \exp(-\pi b / c) < 1 \). The function \( \vartheta_1(\zeta - \zeta_1, p) \) then represents an infinite array of positive vortices at \( \zeta_1 + \zeta_{km} \). In terms of the complex variable \( w = u + i v \) with dimension of a length, consider a vortex dipole with positive vortex at \( w_1 \) and negative vortex at \( w_2 \) in the elementary cell. It is now clear that what we call the “classical” complex potential for this vortex dipole

\[
F_{cl}(w) = \text{log} \left[ \frac{\vartheta_1 \left( (w - w_1) / 2c, p \right)}{\vartheta_1 \left( (w - w_2) / 2c, p \right)} \right]
\]

(13)

indeed represents the appropriate periodic array of vortex dipoles.

As shown below, however, \( F_{cl}(w) \) does not yield a periodic condensate wave function, similar to the case of a cylinder examined in Ref. [10]. This Jacobi theta function has the following quasi-periodic \( \vartheta_1(\zeta + \pi, p) = -\vartheta_1(\zeta, p) \) and \( \vartheta_1(\zeta + 2 \pi, p) = -p^{-1} e^{-2i \pi \zeta} \vartheta_1(\zeta, p) \).

A detailed analysis shows that \( \mathbb{Z} \{ F_{cl} \} \to \mathbb{Z} \{ F_{cl} \} + \mathbb{R} \{ w_{12} \} / c \) for \( v \to v + 2 \pi b \), namely one revolution along the \( v \) axis, where \( w_{12} = w_1 - w_2 \). This behavior violates Eq. (11). An additional linear term ensures that the order parameter is indeed single valued:

\[
F(w) = \text{log} \left[ \frac{\vartheta_1 \left( (w - w_1) / 2c, p \right)}{\vartheta_1 \left( (w - w_2) / 2c, p \right)} \right] - \frac{\mathbb{R} \{ w_{12} \}}{2 \pi b c} w.
\]

(14)

This added linear term arises from the phase coherence of the superfluid across the entire torus’ surface and is wholly quantum in nature.

The velocity field \( \mathbf{w} \) of the superfluid at a point \( \mathbf{s}(w) \) on the torus is given by Eqs. (1) and (2), evaluated in the local tangent space of the surface at that point. Points at infinitesimally-changed coordinates \( s(w + dw) \) correspond to points at physical distances \( |ds| = \lambda |dw| \) around it; specifically the basis vectors \( 1 \) and \( i \) in the complex plane \( w = u + iv \) are equivalent to the set of local dimensionless tangent vectors \( \mathbf{u} = \partial \mathbf{s} / \partial u \) and \( \mathbf{v} = \partial \mathbf{s} / \partial v \), with norm \( |\mathbf{u}| = |\mathbf{v}| = \lambda \). In this basis, the local gradient \( \nabla \) is \( \nabla = \nabla / \lambda \), where

\[
\nabla = \mathbf{u} \partial_u + \mathbf{v} \partial_v,
\]

(15)

Equations (1) and (2) then become

\[
\mathbf{w} = \frac{\hbar}{M \lambda} \nabla \phi(\mathbf{s})
\]

(16)

\[
\mathbf{w} = \frac{\hbar}{M \lambda} \hat{\mathbf{n}} \times \nabla \chi(\mathbf{s}),
\]

(17)

which can be combined in the complex form

\[
\Omega = \mathbf{w}_v + i \mathbf{w}_u = \frac{\hbar}{M} \frac{F'(w)}{\lambda},
\]

(18)

analogous to Eq. (4). Here \( \Omega \) denotes the complex representation of the velocity field \( \mathbf{w} \). Figure 2 gives an example of a vortex-dipole configuration and the corresponding phase pattern, streamlines and flow field, both in the \( \{u, v\} \) coordinate set (top panels) and mapped to the surface of a torus (bottom panels).

By construction, the imaginary part of Eq. (14) is doubly periodic, as is evident from the colors in the upper left panel of Fig. 2. In contrast, the real part is only quasiperiodic:

\[
F(w + 2 \pi c k + i 2 \pi b m) = F(w) - k \frac{\mathbb{R} \{ w_{12} \}}{b} - m \frac{\mathbb{R} \{ w_{12} \}}{c},
\]

(19)

with integers \( k \) and \( m \). Nevertheless, the physical velocity field in Eq. (18) is doubly periodic, as is clear from its representation in terms of the gradient of the doubly periodic phase function \( \Phi = \mathbb{R} \{ F \} \). Since \( \chi = \mathbb{R} \{ F \} \), streamlines \( \chi = c_1 \) that extend to the boundary of the \( \{u, v\} \) cell must be identified with other streamlines \( \chi = c_1 + k \mathbb{R} \{ w_{12} \} / b + m \mathbb{R} \{ w_{12} \} / c \), with \( k, m \in \mathbb{Z} \). Hence, these curves might wind several times around the torus before closing on themselves, as can be seen in the lower left panel of Fig. 2.

In the limit of large major radius, \( a / b \gg 1 \), we expect the torus to approximate the simpler geometry of an infinite cylinder with radius \( b \). The theta function Eq. (12) has the remarkable imaginary transformation \( (-i \tau)^{1/2} \vartheta_1(\zeta, p) = -i \exp(i \tau^* \zeta^2 / \pi) \vartheta_1(\zeta^*, p^*) \), with \( \tau^* = -1 / \tau \) and \( p^* = \exp(i \pi p') = e^{-\pi c / b} \). In the infinite-cylinder limit, where \( c / b \to \infty \), one finds \( p \to 1 \) and correspondingly \( p' \to 0 \). The series Eq. (12) for the transformed theta function with parameter \( p' \) now terminates after the first term and gives

\[
F_{cyl}(w) = \text{log} \left[ \frac{\sin[i(w - w_1) / 2b]}{\sin[i(w - w_2) / 2b]} \right].
\]

(20)

This expression agrees with the flow potential of a vortex dipole on an infinite cylinder of radius \( b \) that we derived in Ref. [10], provided one identifies \( u \) with the coordinate along the axis of the cylinder, and \( v \) with the azimuthal coordinate around its radius. Note that the linear term in Eq. (14) disappears in this limit.

\[\text{B. Flux quanta on the torus and “winding” of single vortices}\]

The function \( F(w) \) also has quasiperiodic properties in terms of the positions \( w_1, w_2 \) of the two vortices. For example, consider \( w_1 \to w_1 + 2 \pi c k + i 2 \pi b m \) with \( k, m \) integer. Using the properties of the Jacobi theta function we find

\[
F(w) \to F(w) + q_1 \left( \frac{m}{c} + i \frac{n}{b} \right) w.
\]

(21)
Winding a single vortex once in the toroidal direction along \( u \) with \( n = 1 \) and \( m = 0 \) introduces an elementary quantum of flux around the poloidal direction along \( v \), and vice versa. When \( w_1 = w_2 \), the complex potential defined in Eq. (14) reduces to \( F(w) = 0 \). According to Eq. (21), if one creates an infinitesimal vortex-antivortex pair and then annihilates it after “winding” one of its members around the torus, the superfluid initially at rest acquires an elementary flux quantum for flow in the orthogonal direction around the torus. When a poloidal winding generates a flux in the toroidal direction, the two-dimensional vortex dipole can be considered the entry and exit point of a single flux line crossing into the inside of the torus. This model realizes the classic picture of phase slippage in ring-shaped superfluids, as discussed by Anderson in Ref. [16].

C. Multipole configurations

The extension to cases with more than two vortices is straightforward. Defining

\[
F(w, w_n) = \log \left[ \vartheta_1 \left( \frac{w - w_n}{2c}, p \right) \right] - \frac{\Re \{ w_n \}}{2\pi bc},
\]

we can write Eq. (14) as

\[
F(w) = \sum_n q_n F(w, w_n). \tag{23}
\]

Every neutral set of \( 2N \) vortices with elementary charges \( q_n = \pm 1 \) can be organized into a set of \( N \) dipoles. In this way, Eq. (23) describes every possible neutral set of vortices on a torus. Correspondingly, we define \( f(w, w_n) = \partial_w F(w, w_n) \), which explicitly gives

\[
f(w, w_n) = \frac{1}{2c} \vartheta_1' \left( \frac{w - w_n}{2c}, \frac{p}{c} \right) - \frac{\Re \{ w_n \}}{2\pi bc}. \tag{24}
\]

The complex velocity field \( \Omega(w) \) then becomes

\[
\Omega(w) = \frac{\hbar}{M} \sum_n q_n f(w, w_n). \tag{25}
\]

An individual term \( q_n F(w, w_n) \) in Eq. (23) can be interpreted as the complex potential of a single vortex on the torus, but this picture applies only for an ensemble of vortices with zero net charge.

IV. ENERGY OF VORTEX CONFIGURATIONS

The energy of a vortex configuration is given by the kinetic energy of the flow field

\[
E = \frac{1}{2} M \rho \int_A du \, dv \, \chi^2 |\mathbf{w}|^2, \tag{26}
\]

with \( A \) the appropriate integration region. Before proceeding, we need to define carefully the region \( A \).

On first impression, \( A \) denotes the \( \{ u, v \} \) elementary cell \( \{ -\pi c, \pi c \} \times \{ -\pi b, \pi b \} \), but close to each vortex core \( w_n \), the squared velocity \( |\mathbf{w}|^2 \sim 1/|w - w_n|^2 \) diverges. As a result, the integral must exclude a small disk around each core center to reflect the finite physical core size \( \xi \) on the torus. As long as \( \xi \) is much smaller than the smallest geometrical scale of the system, the vortex is still effectively point-like, and the torus’ surface can be considered locally flat across the vortex core. In the \( \{ u, v \} \) plane, however, the physical length scale \( \xi \) appears locally rescaled. Specifically, the original disk of physical radius \( \xi \) on the torus becomes a disk with radius \( \xi_n = \xi / \lambda_n \), where \( \lambda_n = \lambda(|u_n, v_n|) \) is the scale factor evaluated at the vortex core. Figure 3 illustrates this transformation for a vortex dipole.

In the curvilinear \( \{ u, v \} \) coordinates, the earlier Eq. (2) acquires a scaling factor: \( \mathbf{w} = \hbar / (M \lambda) \mathbf{n} \times \hat{\nabla} \chi \), where \( \hat{\nabla} \) is defined in Eq. (15). Consequently, the scale factors in Eq. (26) cancel and we find

\[
E = \frac{\rho \hbar^2}{2M} \int_A du \, dv \, (\hat{\nabla} \chi) \cdot (\hat{\nabla} \chi) \\
= \frac{\rho \hbar^2}{2M} \int_A du \, dv \, \left[ \nabla \cdot (\chi \hat{\nabla} \chi) - \chi \hat{\nabla}^2 \chi \right] \\
= \frac{\rho \hbar^2}{2M} \left( \int_{\partial A} \mathbf{ds} \times \mathbf{n} \cdot \chi \hat{\nabla} \chi - \int_A du \, dv \, \chi \hat{\nabla}^2 \chi \right), \tag{27}
\]

where we used integration by parts in the first step and Green’s theorem (essentially the divergence theo-
rem) in two dimensions in the second. We can write \( \chi = \sum_n q_n \chi_n \) with \( \chi_n = \Re\{F(w, w_n)\} \). It is straightforward to confirm that \( \nabla^2 \chi = 2\pi \sum_n q_n \delta(u - u_n) \delta(v - v_n) \). Dirac delta function. Hence the second term in Eq. (27) vanishes because the region \( A \) excludes all vortex centers.

The first term in Eq. (27) is a boundary integral that separates into a counter-clockwise (positive) loop around the outer boundary of the elementary cell and clockwise (negative) loops of radius \( \xi_n \) around each vortex center \( w_n \). The integral around the outer boundary of the elementary cell vanishes because of the quasiperiodic properties of the complex potential in Eq. (23). The remaining parts are the small circles \( C_n \) around each vortex core:

\[
E = \frac{\hbar^2}{2M} \sum_n \int_{C_n} ds \times \nabla \chi \\
= \frac{\hbar^2}{2M} \sum_{n,m,k} q_m q_k \int_{C_n} ds \times \nabla \chi_m \nabla \chi_k, \tag{28}
\]

where \( \nabla \) acts on the first variable of \( \chi_k = \Re\{F(w, w_k)\} \).

Near the \( n \)th vortex, it is convenient to introduce the local vector \( \delta = (u - u_n) \mathbf{u} + (v - v_n) \mathbf{v} \), of length \( |\delta| = \xi_n \), for points \( w \) on \( C_n \). Correspondingly, the local stream function behaves like \( \chi_n \approx \log \delta \) apart from an additional constant. Most terms appearing in Eq. (28) will be negligible because of the small length \( 2\pi \xi_n \) of each curve \( C_n \). The dominant contributions come from terms with \( k = n \), since the singular behavior of \( \nabla \chi_n \approx \delta / \delta \) cancels the small circumference on \( C_n \).

For \( m \neq n \), the function \( \chi_m \) reduces to the constant

\[
\chi_{nm} = \Re\{F(w_n, w_m)\}, \tag{29}
\]

apart from small corrections of order \( \xi / b \). Terms with \( m = n \) require special care, however, because of the logarithmic behavior of \( \chi_n \approx \log \delta \). On the curve \( C_n \), we can set \( \delta = \xi_n \cos \theta \mathbf{u} + \xi_n \sin \theta \mathbf{v} \) and define

\[
\chi_{nn} = \log \left( \frac{\vartheta'_1 \xi_n}{2c} \right) - \frac{\Re\{w_n\}^2}{2\pi bc}, \tag{30}
\]

where \( \vartheta'_1 = \partial \vartheta_1(\zeta, p)/\partial \zeta |_{\zeta = 0} \) is a constant. We can then write the dominant contributions in Eq. (28) in the compact form

\[
\frac{2ME}{\hbar^2} = -2\pi \sum_{n,m} q_n q_m \chi_{nm} + O(\xi / b). \tag{31}
\]

Note that this expression involving a double sum over \( \chi_{nm} \) takes the same form as that for a set of vortices in the plane. Omitting the constant core-energy term in Eq. (30), we can group the terms in Eq. (31) into three categories according to their physical origin:

\[
\frac{2ME}{\hbar^2} = E_{\text{class}} + E_{\text{curv}} + E_{\text{quant}} \tag{32}
\]

\[
= -2\pi \sum_{n,m} q_n q_m \log \left| \partial_1 \left( \frac{w_n - w_m}{2c}, p \right) \right| + 2\pi \sum_n \log \lambda_n \\
+ \sum_{n,m} q_n q_m \frac{u_n u_m}{bc},
\]

where the primed double sum omits terms with \( n = m \). The classical term \( E_{\text{class}} \) gives the energy of point vortices in a classical inviscid, incompressible fluid contained in the elementary cell with a constant metric, which we may call a “flat” torus. The curvature term \( E_{\text{curv}} \) represents changes in the kinetic energy due to the curvature of the torus induced by the locally distorted metric Eq. (7) and would not be present for a flat metric. In the context of simply connected two-dimensional superfluids, these terms have been discussed extensively in [13]. Finally, as discussed in the previous section, the quantum term \( E_{\text{quant}} \) arises from the need to ensure that the superfluid order parameter is single valued and hence represents a purely quantum contribution. The next section shows how these terms generate distinct contributions to the vortex-core dynamics.

V. VORTEX DYNAMICS

The time-dependent evolution of the flow field of a two-dimensional inviscid, incompressible fluid containing point vortices can be reduced entirely to the motion of the vortex cores, as first established through Helmholtz’s famous vorticity theorems [1]. Each vortex moves with the fluid at the “local” velocity \( \mathbf{w}_n \) at its core position, regularized to omit the divergent contribution coming from the vortex itself. A detailed calculation in the Appendix gives [see Eq. (A7)]

\[
\mathbf{w}_n = \frac{\hbar}{M \lambda(s_n)} \hat{n} \times \nabla \left( \chi_n^{\text{reg}}(s) - \frac{q_n}{2} \log \lambda(s) \right) \bigg|_{s \to s_n} \tag{33}
\]

for any isothermal coordinate set \( s = \{u, v\} \) with a general scaling factor \( \lambda(s) \). Here

\[
\chi_n^{\text{reg}}(s) = \Re\{F(w) - q_n \log(w - w_n)\} \tag{34}
\]

is the regular part of the stream function near \( w_n \).

Let \( \Omega_n = \mathbf{w}_{n,v} + i\mathbf{w}_{n,u} \) be the complex velocity of the \( n \)th vortex core. For a set of \( N \) vortices on a torus, the Appendix shows that Eq. (33) has the equivalent complex
form

\[ \Omega_n = \frac{\hbar}{M} \frac{1}{\lambda(v_n)} \left( \sum_{m \neq n} q_m f(w_n, w_m) \right) + i \frac{q_n}{2} \frac{\lambda'(v_n)}{\lambda(v_n)} - q_n \frac{u_n}{2} \frac{q_n}{bc}, \]  

with \( f(w_n, w_m) \) defined by Eq. (24).

Note that Eq. (35) is the physical velocity of the vortex. In the curvilinear coordinate set \( s = \{u,v\} \), however, the equations of motion for each vortex take the form \( \dot{u}_n = \Re\{\Omega_n\}/\lambda_n \) and \( \dot{v}_n = \Im\{\Omega_n\}/\lambda_n \), where \( \lambda_n = \lambda(s_n) \).

Kirchhoff [2] introduced the Hamiltonian formulation for classical point vortices on a plane (see also Sec. 157 of [12]) and Refs. [3, 4] extended the description to include more general planar geometries. Here, a detailed analysis shows that vortex dynamics on a torus can be recast in a Hamiltonian form, where the coordinates \( u_n \) and \( v_n \) serve as conjugate variables and Eq. (31) gives the energy

\[
\rho \hbar 2 \pi q_n \dot{u}_n = \frac{1}{\lambda_n^2} \frac{\partial}{\partial u_n} E(s_1, \ldots, s_N)
\]

\[
\rho \hbar 2 \pi q_n \dot{v}_n = -\frac{1}{\lambda_n^2} \frac{\partial}{\partial u_n} E(s_1, \ldots, s_N),
\]

This formulation ensures that the time evolution conserves the energy in Eq. (31), as it must because the system has no dissipation.

The first set of terms in Eq. (35) describes the velocity induced by all other vortices at the position of the \( n \)th vortex core. The second term corresponds to the curvature term \( E_{\text{curv}} \) in Eq. (32); it induces a purely toroidal motion. The last term is the effect of the quantum term \( E_{\text{quant}} \) in Eq. (32) of the \( n \)th vortex on itself, inducing a purely poloidal motion.

In a flat two-dimensional plane, vortex dipoles always move rigidly in the direction perpendicular to the dipole axis given by the right-hand rule. On the torus, however,
the possible trajectories are much more complicated, as depicted in Fig. 4, where Eq. (36) was integrated numerically for a representative sample of initial vortex positions. The “classical” rigid dipole motion occurs only for a “poloidal” initial vortex dipole aligned along the poloidal axis and symmetric with respect to the \( v = 0 \) line in the \((u, v)\) plane as shown in Fig. 4a. In this case, the motion is purely toroidal, analogous to that on an infinite cylinder. In contrast, the poloidal movement of a toroidal dipole induces stretching of the toroidal distance between vortices (Fig. 4b) due to the \( v \) dependence of the curvature terms Eq. (32). Indeed, the vortices can even execute counter-rotating loops (Fig. 4c) for sufficiently large initial separation. The remainder of Fig. 4 illustrates various other initial configurations.

Of interest are also static dipole configurations, where \( \Omega_{1,2} = 0 \). Symmetric configurations with either a purely toroidal or poloidal dipole axis allow for such solutions, depending on the geometric scales \( a \) and \( b \). Interestingly, for a “classical” fluid without the \( E_{\text{quant}} \) term, a toroidal dipole would be static in the case of \( w_1 = w_2 = \pm \pi e/2 \), when the vortex cores are separated by the maximal distance possible on the torus. The quantum term \( E_{\text{quant}} \) however forces this configuration to move (see Fig. 4c) and the equilibrium distance varies.

VI. GENERALIZED TOROIDAL SURFACES OF REVOLUTION

The discussion above can be extended to surfaces similar to the torus by finding an appropriate isothermal coordinate set where the complex potential formalism applies. Consider a two-dimensional curve, described parametrically as \( f(\eta) = (f_1(\eta), f_2(\eta)) \), \( \eta \in (-\pi, \pi) \), such that \( f \) is closed and never crosses itself. A corresponding surface of revolution has the parametrization [compare Eq. (5)]

\[
\mathbf{s}(\phi, \eta) = \begin{cases} 
  x = [a + f_1(\eta)] \cos \phi \\
  y = [a + f_1(\eta)] \sin \phi \\
  z = f_2(\eta) 
\end{cases}, \quad \phi, \eta \in (-\pi, \pi),
\]

(37)

where we restrict to cases \( a + \min_\eta f_1(\eta) > 0 \), such that the resulting surface has a central hole and hence is multiply connected. Such surfaces are known as toroids. The line element corresponding to Eq. (37) is

\[
ds^2 = \left[ a + f_1(\eta) \right]^2 d\phi^2 + \left[ f_1'(\eta)^2 + f_2'(\eta)^2 \right] d\eta^2,
\]

(38)

defining \( \lambda_\phi \) and \( \lambda_\eta \). When \( f \) describes a circle with radius \( b \), these expressions yield the parametrization of a torus in Eq. (5).

A. Isothermal coordinate set

As before, for a description of a flow field via a complex potential, we must find an isothermal coordinate set. Let us define \( \{u, v\} \) such that \( \eta'(u) = \eta(v) \) only and \( \phi'(u) = \phi(v) \) only. Inserting this ansatz into Eq. (38), we find

\[
ds^2 = \lambda_\phi(u)^2 \left( \frac{d\phi}{du} \right)^2 \left( du^2 + \frac{\lambda_\eta(v)(d\eta}{dv})^2 dv^2 \right) \]

(39)

For the coordinate set \( \{u, v\} \) to be isothermal, the factor in the second term in Eq. (39) must equal unity:

\[
\lambda_\phi(\eta) \frac{d\phi}{du} = \lambda_\eta(\eta) \frac{d\eta}{dv}.
\]

(40)

Since \( \lambda_\phi, \lambda_\eta \) depend on \( \eta \) only, we assume \( \phi = u/\gamma \), with \( \gamma \) some arbitrary length scale. Then Eq. (40) can be solved for \( v(\eta) \) by separation of variables

\[
\frac{\lambda_\phi(\eta)}{\lambda_\eta(\eta)} d\eta = \frac{1}{\gamma} dv, \quad \text{which gives}
\]

\[
v(\eta) = \gamma \int_0^\eta \sqrt{f_1'(\eta')^2 + f_2'(\eta')^2} d\eta'.
\]

(41)

The resulting line element is

\[
ds^2 = \lambda_f(v)^2 \left( du^2 + dv^2 \right), \quad \lambda_f(v) = \frac{a + f_1[\eta(v)]}{\gamma}
\]

(42)

where \( \eta(v) \) is the inversion of Eq. (41). The integrand in Eq. (41) is always positive, so that \( v(\eta) \) is bijective and the inverse \( \eta(v) \) well defined. In practice, as long as the length \( \int_{-\pi}^{\pi} |f(\eta)| d\eta \) is finite, both \( v(\eta) \) and \( \eta(v) \) can easily be evaluated numerically.

For a standard torus with \( f(\eta) = b \cos \eta, \sin \eta \) and \( \gamma = c = \sqrt{a^2 - b^2} \), one recovers Kirchhoff’s transformation Eq. (8).

B. Flow potential

In the \( \{u, v\} \) coordinate set, the toroidal surface of revolution is a doubly periodic cell with dimensions \( 2\pi \gamma \times \Delta v \), with \( \Delta v = v(\pi) - v(-\pi) \). Considering a superfluid confined to the surface described by Eq. (37), the problem of finding the flow potential of vortex configurations in the \( \{u, v\} \) plane is identical to that of the torus:

\[
F_f(u, w) = \sum_n \vartheta_n F_f(u, w_n)
\]

\[
F_f(w, w_n) = \log \vartheta_n \left( \frac{w - w_n}{2\gamma}, p_f \right) - \frac{\Re \{w_n\}}{\gamma \Delta v} w,
\]

(43)

with \( p_f = \exp(-\Delta v/2\gamma) \).

All results obtained in the previous section still apply, with the generalized scaling factor \( \lambda_f \) in Eq. (42) replacing the scaling factor \( \lambda(v) \). Figure 5 gives an example of a toroidal surface of revolution and the corresponding phase pattern of a vortex dipole on its surface.
The superfluid order parameter must be single valued, which leads to the quantization of vortices in two-dimensional planar superfluids. On a cylinder, torus and similar closed surfaces, the same mechanism quantizes flow along any closed path (possibly encircling holes of the surface), significantly altering the flow field of vortices compared to both a classical fluid and a thin cylindrical superfluid film. Correspondingly, these restrictions greatly affect the vortex-vortex interaction energy and the dynamics of the vortices themselves. A spatially-varying metric generates additional contributions to the vortex energy and dynamics.

The inter-vortex energy plays a large role in the transition to the superfluid phase at low temperatures, which is of Berezinskii-Kosterlitz-Thouless (BKT) type in two dimensions [17]. The thermodynamics of vortex excitations in non-planar geometries merits further study [18]. Since the quantization of point vortices requires phase coherence on a length scale much shorter than that required for the quantization of flow around the torus circumferences, the vortices on a torus might prove interesting for studying finite-temperature effects and the predicted coherence length in BKT-type superfluids.

Experimentally, the realization of a toroidal superfluid seems difficult. In the context of cold gases, Bose-Einstein Condensates (BECs) have been created in a multitude of trap shapes with unprecedented control over the geometry [19–21]. Toroidal shell traps can be engineered for the gas, but the effect of gravity may lead to inhomogeneities in the thickness of the fluid. A possible approach would be to do the experiment under microgravity. Such conditions may be obtained, for example, in a free-falling “Einstein elevator” [22], or aboard the International Space Station, where a cold-atom experiment was recently deployed. Current proposals indeed envision the creation of a thin ellipsoidal shell of BEC in space [21]. Following our discussion in Sec. VI, we expect the quantum effect on vortex dynamics to be robust with regards to the specific shape of the trap geometry, as long as it is toroidal.

Various techniques for optical imprinting of vortices have been proposed [23, 24]. In a toroidal shell BEC, these techniques should still apply. Since an imprinting beam will always pierce the BEC twice, it will create a vortex anti-vortex pair, consistent with the condition of zero net vorticity. Otherwise, vortices may be formed by rapidly lowering the temperature of the fluid below the critical temperature for superfluidity [25].

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In this appendix, we first study vortex dynamics for general isothermal coordinates $s = \{u, v\}$ with metric parameter $\lambda(s)$. A torus then represents a simple example with $\lambda$ depending only on $v$.

1. Isothermal coordinates in two dimensions

Consider a two-dimensional surface with an isothermal parametrization $s = \{u, v\}$ [compare Eq. (7)] and metric parameter $\lambda(s)$. To make contact with Sec. V, it is helpful to focus on the flow near a single vortex core at position $s_n$ with charge $q_n$ [included in the definition of the total stream function $\chi(s)$].

It is convenient to start from the corresponding full complex potential $F(w)$, which is logarithmically singular near the complex coordinate $w_n = u_n + iv_n$. Separating out this singular part gives the residual regular part

$$F^{\text{reg}}(w) = F(w) - q_n \log(w - w_n) \quad (A1)$$

near $w_n$.

For a general system of well-separated vortices, we now assume that the complex potential $F(w)$ is a sum over individual vortices, as in Eq. (23). In the vicinity of the $n$th vortex $w \approx w_n$, all terms for $m \neq n$ remain finite.

Thus the regular contribution to the complex potential near $w_n$ has the intuitive form

$$F^{\text{reg}}_n(w) = \sum_{m \neq n} q_m F(w, w_m) + q_n [F(w, w_n) - \log(w - w_n)] \quad (A2)$$

By construction, $F^{\text{reg}}_n(w)$ remains finite as $w \to w_n$, with a well-defined value $F^{\text{reg}}_n(w_n)$ that includes the contribution from all the other vortices.
Similarly, \( \chi(s) = \Re \{ F(w) \} \) in the vicinity of \( s_n \) separates into a singular part \( q_n \log|s - s_n| \) and a regular part

\[
\chi_{n, reg}(s) = \sum_{m \neq n} q_m \chi_m(s) + q_n [\chi_n(s) - \log|s - s_n|] \quad (A3)
\]

where \( \chi_m(s) = \Re \{ F(w, w_m) \} \). As for the complex potential, Eq. \( (A3) \) includes the contribution from all the other vortices. For a flow field described by such a stream function \( \chi(s) \), the associated hydrodynamic velocity field is

\[
w(s) = \frac{\hbar}{M \lambda(s)} \hat{n} \times \vec{\nabla} \chi(s), \quad (A4)
\]

where \( \vec{\nabla} = u \partial_u + v \partial_v \), as in Eq. \( (15) \).

Near the vortex core at \( s_n \), we write \( \hat{n} \times \vec{\nabla} \chi(s) = q_n \hat{n} \times (s - s_n)/|s - s_n|^2 + \hat{n} \times \vec{\nabla} \chi_{n, reg}(s) \). An expansion of \( w(s) \) in powers of \( s - s_n \) gives

\[
w(s) = \frac{\hbar}{M} \left[ \frac{1}{\lambda(s_n)} \hat{n} + \vec{\nabla} \left( \frac{1}{\lambda(s)} \right)_{s \to s_n} (s - s_n) \right] \times \nabla \chi_{n, reg}(s) + O(s - s_n).
\]

Introducing the normalized vector \( \hat{s} = (s - s_n)/|s - s_n| \), we have the leading terms

\[
w(s) = \frac{\hbar}{M} \frac{1}{\lambda(s_n)} \hat{n} \times \left[ q_n \hat{s} + \hat{n} \times \vec{\nabla} \chi_{n, reg}(s) \right] - q_n \hat{s} \frac{\hat{s}}{\lambda(s_n)} \left( \vec{\nabla} \lambda(s_n) \cdot \hat{s} \right) + O(s - s_n).
\]

Since the flow field in Eq. \( (A6) \) diverges for \( s \to s_n \), we cannot directly identify \( w(s_n) \) as the velocity \( w_n \) of the core. Instead, we define the velocity of the vortex core as the angular average on a circle of small radius \( |s - s_n| \) around the core \( s_n \), with \( \langle \cdots \rangle = (2\pi)^{-1} \int_0^{2\pi} d\theta \cdots \).

This angular average can be applied term by term to Eq. \( (A6) \). The first term vanishes because \( \langle \hat{s}_k \hat{s}_l \rangle = 0 \), the second term is simply a constant, and the identity \( \langle \hat{s}_k \hat{s}_l \rangle = \frac{1}{2} \delta_{k,l} \) simplifies the last term. In this way we find the final result for the translational velocity

\[
w_n = \frac{\hbar}{M} \frac{1}{\lambda(s_n)} \hat{n} \times \left[ \vec{\nabla} \chi_{n, reg}(s) - \frac{q_n}{2\lambda(s_N)} \vec{\nabla} \lambda(s) \right]_{s \to s_n} = \frac{\hbar}{M} \frac{1}{\lambda(s_n)} \hat{n} \times \left[ \vec{\nabla} \chi_{n, reg}(s) - \frac{q_n}{2} \log \lambda(s) \right]_{s \to s_n}.
\]

In connection with the study of vortex dynamics in Sec. V, the last equation has the equivalent form

\[
w_n = \frac{\hbar}{M} \frac{1}{\lambda(s_n)} \left[ \left( \frac{v}{\partial u} - u \frac{\partial v}{\partial v} \right) \left( \chi_{n, reg}(s) - \frac{q_n}{2} \log \lambda(s) \right) \right]_{s \to s_n}.
\]

Here, the Hamiltonian structure identifies \( u_n \) and \( v_n \) as the canonical variables.

2. Application to a torus

For a torus, the second part of \( F_{n, reg}(w) \) in Eq. \( (A2) \) is \( q_n \log \left( \frac{\theta_1[(w - w_m)/2\pi]}{(w - w_n)} \right) - q_n u_n w/(2\pi bc) \). Here, the first term is even in its first argument \( w - w_n \) and hence does not contribute to the vortex velocity. As a result, the regular part of the complex potential can be taken as

\[
F_{n, reg}(w) = \sum_{m \neq n} q_m \log \left( \frac{w - w_m}{2\pi} \right), p - \sum_m q_m u_m w.
\]

Correspondingly, the regular part of the stream function can be taken as \( \chi_{n, reg}(s) = \Re \{ F_{n, reg}(w) \} \).

As for a plane, the two real vector components in Eq. \( (A8) \) combine to give the complex expression in Eq. \( (35) \).

\[
\Omega_n = w_{n,v} + i w_{n,u} = \frac{\hbar}{M} \frac{1}{\lambda(v_n)} \left( \sum_{m \neq n} q_m f(w_n, w_m) + i q_n \frac{\lambda'(v_n)}{2 \lambda(v_n)} - q_n \frac{u_n}{2\pi bc} \right).
\]

with \( f(w_n, w_m) \) defined by Eq. \( (24) \).