Connecting wall modes and boundary zonal flows in rotating Rayleigh–Bénard convection

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Using direct numerical simulations, we study rotating Rayleigh-Bénard convection in a cylindrical cell for a broad range of Rayleigh, Ekman, and Prandtl numbers from the onset of wall modes to the geostrophic regime, an extremely important one in geophysical and astrophysical contexts. We connect linear wall-mode states that occur prior to the onset of bulk convection with the boundary zonal flow that coexists with turbulent bulk convection in the geostrophic regime through the continuity of length and time scales and of convective heat transport. We quantitatively collapse drift frequency, boundary length, and heat transport data from numerous sources over many orders of magnitude in Rayleigh and Ekman numbers. Elucidating the heat transport contributions of wall modes and of the boundary zonal flow are critical for characterizing the properties of the geostrophic regime of rotating convection in finite, physical containers and is crucial for connecting the geostrophic regime of laboratory convection with geophysical and astrophysical systems.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Phase diagram of states of rotating Rayleigh–Bénard convection: \(Ra/Ra_w\) vs. Ek. Symbols for different data sets: new data reported here (solid circle - red), \textsuperscript{17-18} (solid circle - black), \textsuperscript{19} (solid square - black), \textsuperscript{22} (solid triangles - black), \textsuperscript{12} (open squares - black), and \textsuperscript{1} (open diamonds - black).}
\end{figure}

Rayleigh–Bénard convection with rotation (RRBC) about a vertical axis is a prototypical laboratory realization of geophysical and astrophysical systems that combines buoyancy forcing and rotation \textsuperscript{19}. Much recent experimental \textsuperscript{8} \textsuperscript{10} \textsuperscript{12} and theoretical/numerical interest \textsuperscript{13} \textsuperscript{15} in rotating convection has focused on the geostrophic regime where rotation dominates. In particular, one is interested in the scaling of the normalized heat transport \(Nu\) with \(Ra\) to compare with theoretical predictions of asymptotic models that provide insight into broader geo- and astrophysical situations. There are significant experimental challenges \textsuperscript{16} for making a compelling comparison including reaching smallEk number with correspondingly large Ra. Consequently, the geometry of experimental convection cells have tended towards small aspect ratio \(\Gamma = D/H < 1\), where \(D\) and \(H\) are the cell diameter and height, respectively. Recently several investigations \textsuperscript{17} \textsuperscript{19} have revealed a boundary zonal flow (BZF) that contributes strongly to convective heat transport. The BZF has features reminiscent of wall modes and of the BZF are the normalized precession frequency \(\omega_d\), the radial length scale \(\delta_0\), and the heat transport \(Nu\), that there is a continuous evolution of the wall mode states into the BZF which coexists with the geostrophic convection modes. We also find that the forgotten wall mode contribution to the heat transport plays an important role in determining the scaling of \(Nu\) in the geostrophic regime, a crucial element in a proper comparison among experiment, DNS, and theory.

The dimensionless control parameters in RRBC are the Rayleigh number \(Ra = \alpha g \Delta H^3/(\kappa \nu)\), Prandtl number \(Pr = \nu/\kappa\), Ekman number \(Ek = \nu/(2\Omega H^2)\), and cell aspect ratio \(\Gamma\) where \(\alpha\) is isobaric thermal expansion coefficient, \(\nu\) kinematic viscosity, \(\kappa\) fluid thermal diffusivity, \(g\) acceleration of gravity, \(\Omega\) angular rotation rate, and \(\Delta\) the temperature difference between horizontal confining plates. The global response of the system is the normalized heat transport \(Nu\), and the time and length scales of the wall modes and of the BZF are the normalized precession frequency \(\omega_d = \omega/\Omega\) and radial localization length scale \(\delta_0/H\). We present data for \(Ek = 10^{-6}\), \(Pr = 0.8\), \(\Gamma = 1/2\), and \(2 \times 10^7 \leq Ra \leq 5 \times 10^9\) that spans the wall mode onset at \(Ra_w = 2.8 \times 10^7\) through the onset of bulk convection.
convection at $Ra_c \approx 9 \times 10^8$. We use our results on this system over wider ranges of $Ek$ and $Ra$ with data from other experiments and DNS to test our proposed power-law scalings.

The regimes of rotating convection in finite containers are wall-mode states at the lowest $Ra$, followed by a transition to the geostrophic state of rotating convection, and finally to a transition to weakly-rotating states at the highest $Ra$. The first instability from the no-convection base state is to wall modes with critical Rayleigh number $Ra_w \approx 31.8Ek^{-1} + 46.6Ek^{-2/3}$. To emphasize the role of these wall modes, we plot the boundaries of rotation and wall mode precession are shown.

![Instantaneous temperature fields](image)

FIG. 2. Instantaneous temperature fields (left – horizontal at $z = H/2$ with streamlines; right – vertical at $r = 0.98R$) for $Ek = 10^{-6}$. Corresponding $Ra$ and $\epsilon$ and $Ro$: (a) $3 \times 10^7$, 0.071, (b) $5 \times 10^8$, 17, (c) $1 \times 10^9$, 35. Directions of rotation and wall mode precession are shown.

role of these wall modes, we plot the boundaries of rotating convection regimes in Fig. 4 in a parameter space of $Ra/Ra_w$ and $Ek$. The transition to bulk rotating convection would occur in an infinite system via linear instability at $Ra_c \approx (8.7 - 9.6Ek^{1/6})Ek^{-4/3}$. In the presence of sidewalls, however, the transition to a bulk convection state depends on $\Gamma$ and on the nonlinear state of the wall modes because of the non-zero base state [4] with $Nu > 1$.

Whereas at modest $Ek \gtrsim 10^{-5}$ the onset of bulk convection $Ra_c/Ra_w \lesssim 10$, for smaller $Ek$ there is an expanding and more nonlinear range of wall modes. For large enough $Ra$ at fixed $Ek$, buoyancy dominates over rotation, and the transition to this regime for $Ek \lesssim 10^{-5}$ and $Pr < 1$ is identified empirically as $Ra_t \sim Ek^{-2}[12, 13]$. The intermediate regime of bulk rotation-dominated convection is known as the geostrophic regime where convective Taylor columns, vortical plumes, and the condition of geostrophy are of great interest. To understand experiments and DNS in realistic, confined convection cells, it is crucial to characterize the role of wall modes on the nonlinear evolution from no convection into the geostrophic regime and to connect the wall modes with the recently discovered boundary zonal flow (BZF) that exists in the turbulent geostrophic regime [17, 19, 22]. That is our task here.

A qualitative understanding of the evolution of the state of rotating convection can be gained by considering instantaneous mid-plane horizontal cross sections of temperature fields and associated streamlines and corresponding vertical temperature fields at the sidewall boundary ($r = 0.98R$). Fig. 2 shows these fields for $Ek = 10^{-6}$ and for several $Ra$; also labeled is the reduced $Ra$ defined as $\epsilon = Ra/Ra_w - 1$ where we take the experimental value $Ra_w = 2.8 \times 10^7$. Very close to onset ($\epsilon \approx 0.07$), the flow is organized as a mode-1 state with symmetric upwelling warmer (red) and downwelling cooler regions (blue), an overall anti-cyclonic rotation at the mid-plane, and a sinusoidal mean-temperature isotherm in the vertical field as shown in Fig. 2(a) (the retrograde direction of precession in the vertical profiles is to the left). Because of the confined geometry with $\Gamma = 1/2$, the wall mode thermal field is largest near the boundary but extends significantly into the cell interior; this has important implications for the heat transport crossover from wall modes to bulk modes described later.

With increasing $Ra$, the wall-mode state becomes more nonlinear but time-independent (in a frame co-rotating with the retrograde traveling wall mode) for $Ra \lesssim 4 \times 10^8$. The state presented in Fig. 2(b) for $Ra = 5 \times 10^8$ shows the more complex horizontal temperature field and flow circulation and the strongly nonlinear square-wave-like vertical profile with forward/backward (left/right) asymmetry; it is also weakly time dependent suggesting a wall-mode transition to an oscillatory state. For larger $Ra$, Fig. 2(c), the streamlines are irregular, indicating unsteady flow and thermal inhomogeneity appears in the interior. One sees vertical striations arising from the influence of aperiodic time-dependent bulk modes interacting with the wall mode; a weak BZF has appeared.

The wall mode state is characterized by four main properties that we consider here: the heat transport $Nu$, the precession frequency $\omega$, the azimuthal mode num-
FIG. 3. Nu vs. Ra for Pr = 0.8, Ek = 10^{-6} and Γ = 1/2. (a) Region of pure wall modes Ra < 5 \times 10^8 and onset of bulk convection for Ra ≥ 9 \times 10^9. Vertical bars are standard deviations of Nu fluctuations. Nu_{off} is the amount contributed by wall modes at bulk convection onset. (b),(c) Larger range of Ra with total Nu (solid, black) and contributions averaged over regions defined by (a) r/R ≤ r_0 (bulk modes) and (b) r/R > r_0 (wall modes/BZF).

FIG. 4. (a) Nusselt number, (b) (ω_d − ω_{dc}) Ek^{-5/3}, and (c) (δ_0/H) Ek^{-2/3} vs. ε, for Pr = 0.8 and Ek = 10^{-6}. Also plotted (open triangles) in (b) and (c) data from [22], and (open diamonds) in (c) data from [4, 5, 7] for wall-mode radial width from shadowgraph for different Γ = 2, 5, 10, Pr = 6.4 with Ek = 2.3 \times 10^{-4}, 2.3 \times 10^{-3}, 1.8 \times 10^{-3} and ε = 0.84, 1, 0.22, respectively.

FIG. 5. Scaled (a) boundary mode drift frequency (ω_d − ω_{dc}) Ek^{-5/3} Pr^{4/3} ≈ 0.022 (Ra − Ra_w); (b) sidewall boundary length scale (δ_0/H) (Ra − Ra_w)^{-1/6} ≈ 4.7 Ek^{2/3}; and (c) heat transport (Nu − 1) Ek^{2/3} vs. Ra Ek^2.
ber, and the radial distribution of convective amplitude (as measured by heat transport or azimuthal velocity $v_\phi$). The azimuthal mode number is 1 because of small $\Gamma = 1/2$. Previously we demonstrated that for the BZF $m = 1$ for $\Gamma \leq 3/4$ and $m = 2\Gamma$ for $\Gamma = 1$ or 2 [13]. Our data show continuity from wall mode to BZF but multiple mode-number states are stable for larger $\Gamma$ and further study is necessary to elucidate this relationship.

We first consider the heat transport and its contributions from the wall mode, from the bulk state, and from the BZF. In Fig. 3(a), we show $\text{Nu}$ versus $\text{Ra}$ that covers the wall mode regime $3 \times 10^7 < \text{Ra} < 5 \times 10^8$, a transition region $5 \times 10^8 < \text{Ra} < 9 \times 10^9$, and the onset of strong bulk modes coexisting with remnant wall-localized modes, i.e., a BZF. The inset shows linear growth (with quadratic corrections) of the wall mode heat transport near onset consistent with the expected scaling $\text{Nu} - 1 = a\epsilon + b\epsilon^2$. The fit gives $a = 1.5$ and $b = -0.08$ where $\text{Ra}_w = 2.8 \times 10^7$ (compared to the theoretical value $3.2 \times 10^7$ for an insulating wall and a planar wall [20, 23]). As the wall modes become more nonlinear, $\text{Nu}$ increases less rapidly and approaches an insulating wall mode using $\text{Ra}_w$.

The collapse is $\text{Nu}_0 = 1 = 0.022 (\text{Ra} - \text{Ra}_w)$ (consistent with [18]) and $(\text{b}_0/\text{d}) (\text{Ra} - \text{Ra}_w)^{-1/6} \approx 4.7\text{Ek}^{2/3}$, where we have combined data reported here and from [17, 18] with data from [4, 5, 7, 12, 19, 22]. The Ra and $\omega_d$ dependences are corrected for their finite values at the onset of wall modes using $\text{Ra} - \text{Ra}_w$ and $\omega_d$ - $\omega_d$, respectively. The collapse over almost 10 decades in $\text{Ra} - \text{Ra}_w$ (Fig. 5) showing $\omega_d - \omega_d \sim \text{Ra} - \text{Ra}_w$ and over 4 decades in Ek showing $\delta_0 \sim \text{Ek}^{2/3}$, see Fig. 5a, unambiguously establishes the connection between the wall modes and the BZF.

Finally, we consider the scaling of $\text{Nu}$ where we plot $(\text{Nu} - 1)\text{Ek}^{2/3}$ versus $\text{Ra}\text{Ek}^2$ in Fig. 5(c). The data for $\text{Pr} \approx 1$ collapse very nicely. We choose this scaling arrangement to reveal the $\text{Nu} - 1 \sim \text{Ra}_1/3$ independent of Ek for large Ra, and $\text{Nu}_0 \sim \text{Ra}\text{Ek}^{4/3}$ at smaller Ra. As demonstrated earlier in Fig. 3(b), a linear dependence of the geostrophic contribution to the total heat transport is obtained by subtracting the wall mode contribution $\text{Nu}_0$ and taking the bulk reduced Ra as $\epsilon_b = \text{Ra}/\text{Ra}_w - 1$, similar to analysis in [4, 5]. Other scalings can be confusing when ignoring the wall mode contribution which has mostly been overlooked in recent experiments on the geostrophic regime. Further investigation of the scalings presented here, the nature of the nonlinear wall mode states, the imperfect crossover to bulk RRBC modes, and the role of $\Gamma$ on the properties reported here are important to fully characterize RRBC.

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