Quasiperiodic Sets at Infinity and Meromorphic Extensions of Their Fractal Zeta Functions

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Abstract
We construct an interesting family of unbounded sets at infinity having a quasiperiodic structure in their geometry. We study these sets by analyzing their complex dimensions—a generalization of the classical Minkowski dimension, which are defined analytically—as poles of a suitably defined Lapidus zeta function at infinity introduced in a previous work by the author. We define the tube zeta function at infinity and obtain a functional equation that relates it to the Lapidus zeta function at infinity much as in the classical setting of relative fractal drums. Furthermore, under suitable assumptions, we provide some general results about extending the fractal zeta functions at infinity beyond their abscissae of convergence. We also provide a sufficiency condition for Minkowski measurability as well as a bound from above for the upper Minkowski content directly from the corresponding zeta functions at infinity. We show that complex dimensions of quasiperiodic sets at infinity possess a quasiperiodic structure which can be either algebraic or transcendental. Furthermore, we use the quasiperiodic sets at infinity to construct a maximally hyperfractal set at infinity with prescribed Minkowski dimension, i.e., a set for which the abscissa of convergence of the associated fractal zeta function at infinity becomes its natural boundary.

Keywords  Distance zeta function · Relative fractal drum · Complex dimensions · Minkowski content · Minkowski dimension

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1 Introduction

In this paper, we study degenerated pairs of subsets of the $N$-dimensional Euclidean space $\mathbb{R}^N$, $(A, \Omega)$ where $A := \{\infty\}$ is the point at infinity via their Minkowski dimension at infinity, as well as their complex dimensions which are a far-reaching generalization of the Minkowski dimension. In the classical setting, i.e., when $A$ is just an ordinary nonempty subset and $\Omega$ of finite $N$-dimensional volume, these objects are called relative fractal drums and are a convenient generalization of the notion of a compact subset of $\mathbb{R}^N$. They were studied extensively in [16–18] and the research monograph [21], along with their associated Minkowski dimension, Minkowski content and more importantly their complex dimensions usually defined as poles (or more general singularities; see [15]) of appropriately defined (Lapidus) fractal zeta functions. The distance zeta function of a general RFD $(A, \Omega)$ is defined as

$$\zeta_{A, \Omega}(s) := \int_{\Omega} d(x, A)^{s-N} \, dx; \quad (1.1)$$

initially, for all $s \in \mathbb{C}$ such that $\text{Re} \, s$ is taken to be sufficiently large for the Lebesgue integral to be absolutely convergent. Here, $d(x, A)$ represents the distance from the point $x$ to the set $A$ in the Euclidean sense. It is well-known that absolute convergence is always satisfied for all $s$ such that $\text{Re} \, s > \overline{\text{dim}}_B(A, \Omega)$, where $\overline{\text{dim}}_B(A, \Omega)$ stands for the upper Minkowski dimension of the relative fractal drum $(A, \Omega)$. By a classical result, it is then also holomorphic in that half-plane.

The higher-dimensional theory of complex dimensions (see [19–23]) is a far-reaching generalization of the well-known theory of geometric zeta functions for fractal strings and their complex dimensions due to Michel L. Lapidus and Machiel van Frankenhuijsen and their numerous collaborators (see [24] and the relevant references therein).

As already pointed out, we study here a degenerated type of relative fractal drums $(\infty, \Omega)$ where the set $A$ becomes the point at infinity. The study of such RFDS was started in the authors thesis [32, 33] where the basic notions of the associated Minkowski content and dimension at infinity were introduced, along with the corresponding notion of a fractal zeta function at infinity and the associated complex dimensions. Basic results about these objects were given along with a number of interesting examples.

Here, we will provide further interesting results about the fractal zeta function at infinity, concretely, results about existence of its meromorphic extension beyond the initial half-plane of analyticity as well as about the connection to the notion of Minkowski content and measurability of $(\infty, \Omega)$ based on unpublished results from [32].

The motivation is to construct an interesting family of transcendentally and algebraically quasiperiodic (in the sense of [21]) sets at infinity, which will then be used to construct an example of a maximally hyper-fractal set at infinity in the sense that its zeta function has a natural barrier and thus, cannot be extended beyond the initial half-plane of convergence.
Intuitively, it is clear that the “fractality” of \((\infty, \Omega)\) stems solely from the set \(\Omega\) as opposed to in the classical setting where the “fractality” of the RFD \((A, \Omega)\) stems from the set \(A\) and \(\Omega\) is usually chosen to be metrically associated with the set \(A\) in the sense of [38, 41]. Of course, the set \(A\) is usually the set one wishes to investigate, while the set \(\Omega\) is usually used for localization purposes, i.e., when one wants, for instance, to analyze the part of the set \(A\) “seen” only from the set \(\Omega\). For example, one might understand \(A\) as a boundary of a fractal membrane \(\Omega\), and wishes to investigate the vibrations of \(\Omega\) from the point of view of spectral theory.

Nevertheless, here, as a counterpoint to the classical setting, the aforementioned examples of quasiperiodic sets at infinity show that there exist interesting families of nontrivial RFDs \((\infty, \Omega)\) from the fractal point of view even though the set \(A\) is just one point at infinity. We also note that from the analytical point of view (i.e., fractal zeta functions), it is equivalent, in fact, to study the classical RFD \((\Phi, \Phi(\Omega))\) where \(\Phi: \mathbb{R}^N \to \mathbb{R}^N\) is the standard geometric inversion, i.e., \(\Phi(x) = x/|x|^2\), and \(\Phi := \{0\}\) is the singleton containing the origin of \(\mathbb{R}^N\). Nevertheless, the quasiperiodic sets that we will construct and study here are much easier and more natural to define and study near the point at infinity, and moreover, the original idea on how to obtain such sets comes from looking at infinite domains.

In general, the motivation to study unbounded domains that have fractal properties may be found in problems from oscillation theory [6, 10], automotive industry [36], aerodynamics [5], civil engineering [30] and mathematical applications in biology [26]. Also, unbounded domains are of interest in problems of partial differential equations, for instance, solvability of Dirichlet problems for quasilinear equations in unbounded domains [27] and [28, Section 15.8.1]. See also [1, 9, 13, 31] and [40]. Furthermore, fractal properties of unbounded trajectories of some planar vector fields were studied in [34] in connection to the Hopf bifurcation at infinity. In that paper, the geometric inversion was used to bring the system near zero instead of near infinity and the classical Minkowski dimension of the geometrically inverted system was then studied. Here, we study different objects and work directly at infinity but show, nevertheless, that the connection with geometric inversion is also present.

The paper is organized as follows. In Sect. 2, we recall the most important definitions and results about Minkowski dimension and content of sets at infinity and their associated fractal zeta functions. Most of these results are proved in [33].

In Sect. 3, we provide general results about the possibility of obtaining meromorphic extensions of the fractal zeta functions at infinity. We also define here the tube zeta function at infinity and establish a very important a functional equation relating it to the distance zeta function at infinity in Theorem 2. Next, under suitable hypothesis, Theorem 3 establishes a relation between the residue of the tube zeta function at the point equal to the Minkowski dimension of the associated RFD and its (upper and lower) Minkowski content. Furthermore, Theorems 4 and 8 establish, again under suitable hypotheses, some sufficient conditions from which guarantee the tube zeta function at infinity can be meromorphically extended beyond the initial domain of convergence where one assumes that the RFD is either Minkowski measurable or nonmeasurable, respectively. Moreover, Theorem 5 establishes a sufficient condition for the set to be Minkowski measurable at infinity while Theorem 6 bounds the upper Minkowski content from above in terms of its complex dimensions and the corresponding residues.
Finally, Theorem 7 establishes a relation between the Minkowski measurability of the set at infinity and its geometrically inverted image.

Section 4 is dedicated to establishing some of the more technical but very useful general properties of fractal zeta functions at infinity needed later on, such as the scaling property (Proposition 10) and general behavior under the change of the norm on \( \mathbb{R}^N \) in Theorem 11.

In Sect. 5, we construct maximally hyperfractal sets at infinity of prescribed Minkowski dimension in Theorem 14. Then, we show that these sets can be used to generate algebraically and transcendentally quasiperiodic sets at infinity of any order in Theorems 16 or infinite order in Theorem 17. The construction of classical compact sets and RFDs which are algebraically quasiperiodic is an open problem [21, Problem 6.2.3]. Here, we solve it in the setting of sets at infinity and also explain how they can be used to obtain classical RFDs which are algebraically quasiperiodic by using geometric inversion. The problem of finding algebraically quasiperiodic compact sets is still open.

## 2 Minkowski Dimension and Fractal Zeta Functions of Sets at Infinity

In this section, we recall the most important definitions and results from [33]. We study Lebesgue measurable subsets \( \mathbb{R}^N \) that have finite Lebesgue measure. For such a set \( \Omega \), we let

\[
\Omega := B_t(0)^c \cap \Omega,
\]

where \( |\cdot| \) denotes the \( N \)-dimensional Lebesgue measure, \( t > 0 \) and \( B_t(0)^c \) denotes the complement of the open ball of radius \( t \) centered at 0. For \( r \in \mathbb{R} \), one defines the *upper \( r \)-dimensional Minkowski content of \( \Omega \) at infinity*

\[
\mathcal{M}^r(\infty, \Omega) := \limsup_{t \to +\infty} \frac{|\Omega|}{t^{N+r}},
\]

and, analogously, by taking the lower limit in (2.2) as \( t \to +\infty \), the *lower \( r \)-dimensional Minkowski content of \( \Omega \) at infinity* denoted by \( \underline{\mathcal{M}}^r(\infty, \Omega) \).

One can routinely check that there exists a unique \( r = D \in \mathbb{R} \) at which \( \mathcal{M}^r(\infty, \Omega) \) drops from infinity to zero and similarly for the lower Minkowski content; see Fig. 1. \( D \) is then called the *upper Minkowski dimension of \( \Omega \) at infinity*, \( \overline{\dim}_B(\infty, \Omega) \) or the *upper Minkowski dimension of \( (\infty, \Omega) \)*, i.e., one has

\[
\overline{\dim}_B(\infty, \Omega) := \sup\{ r \in \mathbb{R} : \mathcal{M}^r(\infty, \Omega) = +\infty \}
\]

and similarly for the lower analog denoted by \( \underline{\dim}_B(\infty, \Omega) \). In case one has an equality between the upper and lower Minkowski dimensions, one calls this common value the *Minkowski dimension of \( (\infty, \Omega) \)* and denotes it by \( \dim_B(\infty, \Omega) \).
The graphs of the functions \( r \mapsto M^r(\infty, \Omega) \) and \( r \mapsto \overline{M}^r(\infty, \Omega) \), assuming that \( \Omega \) is Minkowski nondegenerate and nonmeasurable at infinity, that is, \( D := \dim_B(\infty, \Omega) \) exists and \( 0 < M^r(\infty, \Omega) < \infty \) for \( r < D \) and \( \overline{M}^r(\infty, \Omega) = \overline{M}^r(\infty, \Omega) = 0 \) for \( r > D \).

Fig. 1 Graphs of the functions \( r \mapsto M^r(\infty, \Omega) \) and \( r \mapsto \overline{M}^r(\infty, \Omega) \), assuming that \( \Omega \) is Minkowski nondegenerate and nonmeasurable at infinity, that is, \( D := \dim_B(\infty, \Omega) \) exists and \( 0 < M^r(\infty, \Omega) < \infty \) for \( r < D \) and \( \overline{M}^r(\infty, \Omega) = \overline{M}^r(\infty, \Omega) = 0 \) for \( r > D \).

Furthermore, in the case when \( 0 < M^D(\infty, \Omega) \leq \overline{M}^D(\infty, \Omega) < +\infty \), for some \( D \in \mathbb{R} \) (necessarily \( D = \dim_B(\infty, \Omega) \)), we say that \((\infty, \Omega)\) is \textit{Minkowski nondegenerate}. Finally, we call \((\infty, \Omega)\) \textit{Minkowski measurable} if it is Minkowski nondegenerate and its lower and upper Minkowski content coincide.

The next simple facts were proved in [33], and we recall them here for completeness. For any Lebesgue measurable \( \Omega \subseteq \mathbb{R}^N \), one has that \( -\infty \leq \dim_B(\infty, \Omega) \leq \overline{\dim}_B(\infty, \Omega) \leq -N \). Furthermore, both, \( -\infty \) and \( -N \) can be attained; see [33, Example 3 and Proposition 2]. Moreover, if \( \overline{\dim}_B(\infty, \Omega) = -N \), then one always has that \( \overline{M}^{-N}(\infty, \Omega) = 0 \) which follows directly from the definition. This should not be surprising since the set \( A \) is just a point at infinity, hence, one should not expect that the Minkowski dimension of \((\infty, \Omega)\) can be larger than 0. The fact that it cannot be larger than \(-N\) is actually connected to the fact that \( \Omega \) has finite volume. We will show in a future paper that the “dimensional gap interval” \((-N, 0]\) is actually “reserved” for RFDs \((\infty, \Omega)\) where we let \( \Omega \) to have infinite volume. Of course, in that case, the definition of its Minkowski content and the corresponding fractal zeta functions must be modified accordingly since \(|\Omega|\) is infinite.

We point out that also classical RFDs with negative dimension exist; see [21] where this feature is explained by the lack of the so-called cone property. Although the dimension of \((\infty, \Omega)\) is always negative and therefore, seems uninteresting at first, we will show that there exist rich families of unbounded sets \( \Omega \) whose complex dimensions have complicated quasiperiodic structures. Therefore, the source of “fractality” in the sense of Lapidus, i.e., the fact that \((\infty, \Omega)\) possesses non-real complex dimensions and stems solely from the unbounded set \( \Omega \). This also shows that in general, one has to be careful since the source of “fractality” of an RFD \((A, \Omega)\) could be from both sets, \( A \) and \( \Omega \). On the other hand, we conjecture that this cannot happen if \( \Omega \) is metrically associated with \( A \).

The next two examples from [33, Examples 1 and 2] give a glimpse of nontrivial RFDs at infinity.
Example 1 Let $\alpha > 0$ and $\beta > 1$ be fixed and let $a_j := j^\alpha$, $l_j := j^{-\beta}$, $b_j := a_j + l_j$ and $I_j := (a_j, b_j)$ for $j \in \mathbb{N}$. Consider
\[
\Omega(\alpha, \beta) := \bigcup_{j=1}^{\infty} I_j \subseteq \mathbb{R},
\]
then,
\[
D := \dim_B(\infty, \Omega(\alpha, \beta)) = \frac{1 - (\alpha + \beta)}{\alpha} \quad \text{and} \quad M^D(\infty, \Omega(\alpha, \beta)) = \frac{1}{\beta - 1}.
\]
Observe that by varying parameters $\alpha$ and $\beta$, we can obtain any value in $(-\infty, -1)$ for $\dim_B(\infty, \Omega(\alpha, \beta))$.

The next example will be one of the crucial building blocks for the construction of quasiperiodic sets at infinity.

Example 2 For $\alpha > 1$, let $\Omega := \{(x, y) \in \mathbb{R}^2 : x > 1, \ 0 < y < x^{-\alpha}\}$. Then, we have that
\[
D := \dim_B(\infty, \Omega) = -1 - \alpha \quad \text{and} \quad M^D(\infty, \Omega) = \frac{1}{\alpha - 1}.
\]
Observe that $\dim_B(\infty, \Omega) \to -\infty$ and $M^D(\infty, \Omega) \to 0$ as $\alpha \to +\infty$.

In general, the notion of Minkowski dimension at infinity and Minkowski nondegeneracy at infinity do not depend on the choice of the norm on $\mathbb{R}^N$ in which we define the ball $B_t(0)$. More precisely, if $K_t(0)^c$ denotes a complement of a ball in another (necessarily, equivalent) norm $|| \cdot ||$ on $\mathbb{R}^N$, we define the analog of the (upper and lower) Minkowski content at infinity by using $|K_t(0)^c \cap \Omega|$ in (2.2) instead of $|\cdot \cap \Omega|$, and the two Minkowski dimensions of $(\infty, \Omega)$ will coincide. Furthermore, the notion of Minkowski nondegeneracy is invariant by such changes of the norm; see [33, Lemma 2 and Corollary 2].

In the remainder of this section, we recall the definition and basic properties of the Lapidus zeta function at infinity (also called the distance zeta function of $(\infty, \Omega)$) from [33, Section 3] defined by the Lebesgue integral
\[
\zeta_{\infty, \Omega}(s) := \zeta_{\infty, \Omega}(s; T) = \int_{B_T(0)^c \cap \Omega} |x|^{-s-N} \, dx,
\]
for some fixed $T > 0$ and $s$ in $\mathbb{C}$ with Re $s$ sufficiently large.

The dependence on $T > 0$ is not important since changing $T$ amounts to adding an entire function to (2.7), and we are only interested in possible singularities of (2.7). Note that $\zeta_{\infty, \Omega}$ is closely related to the classical distance zeta function of the “geometrically inverted” relative fractal drum $(O, \Phi(\Omega))$. More precisely, they are
connected by a functional equation [33, Theorem 3]: \( \xi_{\infty, \Omega}(s; T) = \xi_{\Omega, \Phi}(s; 1/T) \);
hence, from the point of view of complex dimensions, it is completely equivalent to either study \((\infty, \Omega)\) or its geometric inversion \((\Omega, \Phi(\Omega))\).

We now state a part of the holomorphicity theorem [33, Theorem 5] for the distance zeta function at infinity for the sake of exposition. Recall also that we define the abscissa of convergence of \( \xi_{\infty, \Omega} \) as the infimum of all \( \sigma \in \mathbb{R} \) such that the integral (2.7) is absolutely convergent for all \( s \in \mathbb{C} \) such that \( \text{Re } s > \sigma \), and we denote it by \( D(\xi_{\infty, \Omega}) \).

**Theorem 1** (Holomorphicity theorem [33, Theorem 5]) Let \( \Omega \) be any Lebesgue measurable subset of \( \mathbb{R}^N \) of finite \( N \)-dimensional Lebesgue measure. Assume that \( T \) is a fixed positive number. Then, the following conclusions hold.

(a) The abscissa of convergence of the Lapidus zeta function at infinity

\[
\xi_{\infty, \Omega}(s) = \int_{\Omega} |x|^{-s-N} \, dx
\]  

(2.8)
is equal to the upper box dimension of \( \Omega \) at infinity, i.e.,

\[
D(\xi_{\infty, \Omega}) = \overline{\text{dim}}_B(\infty, \Omega).
\]  

(2.9)

Consequently, \( \xi_{\infty, \Omega} \) is holomorphic on the half-plane \( \{ \text{Re } s > \overline{\text{dim}}_B(\infty, \Omega) \} \).

(b) The half-plane from (a) is optimal.

(c) If \( D = \overline{\text{dim}}_B(\infty, \Omega) \) exists and \( M^D(\infty, \Omega) > 0 \), then \( \xi_{\infty, \Omega}(s) \to +\infty \) for \( s \in \mathbb{R} \) as \( s \to D^+ \).

**Remark 1** In Theorem 1, one may replace the norm appearing in the definition of the distance zeta function at infinity and in the definition of the Minkowski content at infinity by any other norm on \( \mathbb{R}^N \) to obtain a completely analogous result. Furthermore by [33, Proposition 6] the difference \( \xi_{\infty, \Omega}(s; \cdot \cdot \cdot) - \xi_{\infty, \Omega}(s) \) is holomorphic at least on the half-plane \( \{ \text{Re } s > \overline{\text{dim}}_B(\infty, \Omega) - 2 \} \), where \( \xi_{\infty, \Omega}(s; \cdot \cdot \cdot) \) is the distance zeta function of \((\infty, \Omega)\) defined by using the \( \infty \)-norm on \( \mathbb{R}^N \). This result is very practical for obtaining results about the (Euclidean) distance zeta function \( \xi_{\infty, \Omega}(s) \) and determining its poles in that half-plane. Since a complete proof of this result was not given in [33] due to space constrains we give the proof of a more general results; Theorem 11 and Proposition 13 from which [33, Proposition 6] follows directly.

For example, the distance zeta function of \((\infty, \Omega)\) from Example 2 can be computed explicitly (in the \( \cdot \cdot \cdot \)-norm on \( \mathbb{R}^2 \)) and is meromorphic everywhere and given by \( \xi_{\infty, \Omega}(s) = 1/(s + \alpha + 1) \), with a single simple pole at \( s = -1 - \alpha \).

Next, we recall the notion of complex dimensions of \((\infty, \Omega)\) from [33]. Namely, if \( \xi_{\infty, \Omega} \) has a meromorphic extension to some open connected neighborhood \( W \) (usually called the window) of the half-plane \( \{ \text{Re } s \geq \overline{\text{dim}}_B(\infty, \Omega) \} \), one defines the set of visible complex dimensions of \((\infty, \Omega)\) through \( W \) as the set of poles of \( \xi_{\infty, \Omega} \) that are contained in \( W \) and denote it by \( \mathcal{P}(\xi_{\infty, \Omega}, W) := \{ \omega \in W : \omega \text{ is a pole of } \xi_{\infty, \Omega} \} \)

\(^1\) Optimal in the sense that the integral appearing in (2.8) is divergent for real \( s \in (-\infty, 0) \).
which one usually abbreviates to $\mathcal{P}(\zeta_{\infty}, \Omega)$ if there is no ambiguity concerning the choice of $W$ (or when $W = \mathbb{C}$).

The subset of $\mathcal{P}(\zeta_{\infty}, \Omega, W)$ consisting of poles with real part equal to $\dim_B (\infty, \Omega)$ is then called the set of principal complex dimensions of $(\infty, \Omega)$ and denoted by $\dim_{PC} (\infty, \Omega)$.

3 Meromorphic Extensions of Fractal Zeta Functions at Infinity

In general it is difficult to obtain a closed formula for the distance zeta function which would then infer its meromorphic extension to a larger domain, i.e., beyond the original half-plane of holomorphicity.

In this section, we will give sufficient conditions on the Lebesgue measurable set $\Omega \subseteq \mathbb{R}^N$ of finite volume which will ensure that the Lapidus zeta function of $\Omega$ at infinity can be meromorphically extended to some neighborhood of its abscissa of convergence. First, we establish the theorems in terms of the tube zeta function at infinity and then, indirectly, by the functional equation relating the distance and tube zeta functions at infinity (see Theorem 2), we obtain immediately the analogous statements in terms of the distance zeta function at infinity.

Furthermore, we will also give a sufficient condition for a relative fractal drum $(\infty, \Omega)$ to be Minkowski measurable at infinity in terms of its complex dimensions.

In [33, Theorem 6], it was already shown under mild hypotheses that the residue of the distance zeta function of $(\infty, \Omega)$ is closely related to its (upper and lower) Minkowski content at infinity. In particular, under suitable hypotheses, we have that

$$\text{res}(\zeta_{\infty}, \Omega, D) = -(N + D)M^D (\infty, \Omega)$$

where $D = \dim_B (\infty, \Omega)$.

Let $\Omega$ be a Lebesgue measurable subset of $\mathbb{R}^N$ and $|\Omega| < \infty$. Similarly as in the case of standard relative fractal drums [21], we define the tube zeta function of $\Omega$ at infinity and denote it with $\tilde{\zeta}_{\infty, \Omega}$:

$$\tilde{\zeta}_{\infty, \Omega}(s; T) := \int_T^{+\infty} t^{-s-N-1} |t\Omega| \, dt,$$

where $T > 0$ is fixed. The next theorem establishes the aforementioned functional equation, from which the analyticity of the tube zeta function will follow.

**Theorem 2** (Tube-distance functional equation at infinity) Assume $\Omega \subseteq \mathbb{R}^N$ with $|\Omega| < \infty$ and let $T > 0$ be fixed. Let $s \in \mathbb{C}$ such that $\text{Re } s > \overline{\dim}_B (\infty, \Omega)$. Then,

$$\int_{T\Omega} |x|^{-s-N} \, dx = T^{-s-N} |T\Omega| - (s + N) \int_T^{+\infty} t^{-s-N-1} |t\Omega| \, dt,$$

i.e., the following functional equation holds:

$$\zeta_{\infty, \Omega}(s; T) = T^{-s-N} |T\Omega| - (s + N) \tilde{\zeta}_{\infty, \Omega}(s; T).$$
Proof} Firstly, if we assume that \( s \) is real, then from [33, Proposition 4], we have that (3.2) is valid whenever \( s > \dim B(\infty, \Omega) \). To establish the equality for any complex \( s \) such that \( \Re s > \dim B(\infty, \Omega) \), it is now enough to show that both sides of Equation (3.2) are analytic for such \( s \). Observe that the left-hand side of (3.2) is analytic for \( s \) such that \( \Re s > \dim B(\infty, \Omega) \) according to Theorem 1. Exactly, the same is also true for the right-hand side of (3.2) since this is a Dirichlet-type integral where we let \( \varphi(t) = t^{-s} \) and \( d\mu(t) = t^{-N-1}|_{\Omega}|dr \). Note that according to [21, Theorem 2.1.44(a)], it is sufficient to show that this integral is absolutely convergent for \( \Re s > \dim B(\infty, \Omega) \).

For \( \overline{D} := \dim B(\infty, \Omega) \) and \( s \in \mathbb{C} \) such that \( \Re s > \overline{D} \), let us choose \( \varepsilon > 0 \) sufficiently small such that \( \Re s > \overline{D} + \varepsilon \). The fact that \( \overline{M}_\infty^{\overline{D}+\varepsilon}(\Omega) = 0 \) yields an existence of a constant \( C_T > 0 \) for which \( \overline{D} t^{N+\overline{D}+\varepsilon} \) for every \( t \in (T, +\infty) \).

Next, we complete the proof as follows:

\[
|\tilde{\zeta}_{\infty, \Omega}(s; T)| \leq \int_T^{+\infty} t^{-\Re s-N-1}|_{\Omega}| dt \leq C_T \int_T^{+\infty} t^{-\Re s-N-1}t^{N+\overline{D}+\varepsilon} dt
\]

\[
= C_T \int_T^{+\infty} t^{\overline{D}+\varepsilon-\Re s-1} dt = C_T \frac{T^{\overline{D}+\varepsilon-\Re s}}{\Re s - (\overline{D} + \varepsilon)} < +\infty. \tag{3.4}
\]

\( \square \)

Remark 2 The tube-distance functional equation (3.3) in Theorem 2 enables one to state the definition of (principal) complex dimensions also via the tube zeta function \( \tilde{\zeta}_{\infty, \Omega} \) at infinity since their poles share the same locations.

The next theorem is now an immediate consequence and analog of [33, Theorem 6] in terms of the tube zeta function of \((\infty, \Omega)\), its residue at \( s = \dim B(\infty, \Omega) \) and its upper and lower Minkowski contents.

**Theorem 3** (Residue and Minkowski content connection) Let \( \Omega \subseteq \mathbb{R}^N \) be of finite Lebesgue measure and such that \( \dim B(\infty, \Omega) = D < -N \) with \( 0 < M_D^D(\infty, \Omega) \leq \overline{M}^D(\infty, \Omega) < \infty \). If \( \tilde{\zeta}_{\infty, \Omega} \) can be meromorphic extended to some neighborhood of \( s = D \), then \( D \) is a necessarily a simple pole and it holds that

\[
\overline{M}^D(\infty, \Omega) \leq \operatorname{res}(\tilde{\zeta}_{\infty, \Omega}, D) \leq \underline{M}^D(\infty, \Omega). \tag{3.5}
\]

Moreover, if \( \Omega \) is Minkowski measurable at infinity, then we have

\[
\operatorname{res}(\tilde{\zeta}_{\infty, \Omega}, D) = M^D(\infty, \Omega). \tag{3.6}
\]

**Proof** Using the fact that \( \zeta_{\infty, \Omega}(s) = T^{-s-N}|_{\Omega}| - (s+N)\tilde{\zeta}_{\infty, \Omega}(s) \) for every \( s \in \mathbb{C} \) such that \( \Re s > D \) (proved in Theorem 2) and by using [33, Theorem 6], we immediately have

\[
\operatorname{res}(\zeta_{\infty, \Omega}, D) = \lim_{s \to D} (s-D) \left[ T^{-s-N}|_{\Omega}| - (s+N)\tilde{\zeta}_{\infty, \Omega}(s) \right],
\]
\[ \text{res}(\zeta_{\infty, \Omega}, D) = -(N + D) \text{res}(\tilde{\zeta}_{\infty, \Omega}, D). \]

Now, we proceed by giving a first result which relates the asymptotics of the tube function \( t \mapsto |t| \Omega | \) as \( t \to +\infty \) and the existence of a meromorphic extension of the tube zeta function at infinity.

**Theorem 4** (Meromorphic extension—Minkowski measurable case) Assume that \( \Omega \subseteq \mathbb{R}^N \) is of finite Lebesgue measure. Assume also that there exist constants \( \alpha > 0 \), \( M \in (0, +\infty) \) and \( D < -N \) which satisfy the following asymptotics:

\[ |t| \Omega | = Mt^{N+D} + O(t^{N+D-\alpha}) \quad \text{as} \ t \to +\infty. \quad (3.7) \]

Then, \( \text{dim}_B(\infty, \Omega) \) exists and \( \text{dim}_B(\infty, \Omega) = D \). Furthermore, \( \Omega \) is then Minkowski measurable at infinity with Minkowski content \( M^{D}(\infty, \Omega) = \mathcal{M} \). Moreover, \( \tilde{\zeta}_{\infty, \Omega} \) has a unique meromorphic extension to the open half-plane \( \{ \text{Re} s > D - \alpha \} \), and the only pole in this half-plane is simple and located at \( s = D \) with residue \( \text{res}(\tilde{\zeta}_{\infty, \Omega}, D) = M \).

**Proof** We fix \( T > 0 \), and then have

\[
\tilde{\zeta}_{\infty, \Omega}(s) = \int_T^{+\infty} t^{-s-N-1}|t| \Omega | dt = \int_T^{+\infty} t^{-s-N-1}t^{N+D}(\mathcal{M} + O(t^{-\alpha})) dt
= \mathcal{M} \int_T^{+\infty} t^{D-s-1} dt + \int_T^{+\infty} t^{-s} O(t^{D-\alpha-1}) dt
= \frac{\mathcal{M}T^{D-s}}{s-D} + \int_T^{+\infty} t^{-s} O(t^{D-\alpha-1}) dt
\]

provided that \( \text{Re} s > D \). The function \( \zeta_1 \) is obviously meromorphic everywhere. On the other hand, we have that

\[ |\zeta_2(s)| \leq K \int_T^{+\infty} t^{D-\text{Re} s-\alpha-1} dt < \infty \]

whenever \( \text{Re} s > D - \alpha \) and some positive constant \( K \) from, and hence, the claim of the theorem now follows immediately. \( \square \)

One would like to show that Minkowski measurability of \( \Omega \) at infinity can be characterized by its fractal zeta function at infinity similarly to [21, Chapter 5] for classical relative fractal drums. One direction of this result is a consequence of the Wiener–Pitt Tauberian theorem. The other direction is partially resolved by Theorem 4, where one has to add the additional assumption on the asymptotics of the tube formula of \((\infty, \Omega)\). For the general case, one would need to express the (fractal) tube formula of \((\infty, \Omega)\) in terms of its complex dimensions at infinity. This will be a topic for...
future work where we expect that the technique of inverse Mellin transform from [21, Chapter 5] will give the desired result.

Next, we state and prove the announced sufficiency condition for Minkowski measurability at infinity.

**Theorem 5** (Sufficient condition for Minkowski measurability at infinity) Assume $\Omega \subseteq \mathbb{R}^N$ has finite Lebesgue measure and $\dim_B(\infty, \Omega) = D < -N$. Assume also that $\tilde{\zeta}_{\infty, \Omega}$ of $(\infty, \Omega)$ is meromorphic in some neighborhood $U$ of the critical line $\{\text{Re} \, s = D\}$. Furthermore, assume that $D$ is its only pole in $U$ and that it is simple. Then, $D = \dim_B(\infty, \Omega) = F$ and the RFD $(\infty, \Omega)$ is Minkowski measurable and

$$\mathcal{M}^D(\infty, \Omega) = \text{res}(\tilde{\zeta}_{\infty, \Omega}, D).$$

(3.8)

Equivalently, in terms of $\xi_{\infty, \Omega}$, one has

$$\mathcal{M}^D(\infty, \Omega) = \frac{\text{res}(\xi_{\infty, \Omega}, D)}{-(N + D)}.$$  

(3.9)

**Sketch of proof** We change the variable by $v = \log t$ in the integral for $\tilde{\zeta}_{\infty, \Omega}(\cdot ; T)$ (choosing $T = 1$ without loss of generality) to obtain

$$\tilde{\zeta}_{\infty, \Omega}(s + D) = \int_{1}^{+\infty} t^{s-D-1-N} |t, \Omega| \, dt$$

$$= \int_{0}^{+\infty} e^{-sv} e^{-v(D+N)} |e^v, \Omega| \, dv$$

$$= \{\mathcal{L}\sigma\}(s).$$

(3.10)

where $\sigma(v) := e^{-v(D+N)} |e^v, \Omega|$ and $\mathcal{L}$ denotes the Laplace transform. One then defines the function

$$H(s) := \tilde{\zeta}_{\infty, \Omega}(s + D) - \frac{\text{res}(\tilde{\zeta}_{\infty, \Omega}, D)}{s}$$

(3.11)

which is holomorphic on the neighborhood $\tilde{U} := U - \{D\}$ of the critical line $\{\text{Re} \, s \geq 0\}$ so that one can apply the Wiener–Pitt Tauberian theorem [12, Chapter III, Lemma 9.1 and Proposition 4.3], and one then proceeds analogously as in the proof of [21, Theorem 5.4.2]. We omit the details.

An analog of [21, Theorem 5.4.4] can also be obtained when, in addition to $D$, there are other singularities on the critical line $\{\text{Re} \, s = D\}$ of the relative fractal drum $(\infty, \Omega)$. We state the theorem without the proof which is similar to the proof of [21, Theorem 5.4.4].

**Theorem 6** (Bound on the upper Minkowski content) Assume $\Omega \subseteq \mathbb{R}^N$ has finite Lebesgue measure and let $D := \dim_B(\infty, \Omega) < -N$. Assume also that $\tilde{\zeta}_{\infty, \Omega}$ can be meromorphically extended to some open connected domain $U$ containing critical line
\[ \text{Re } s = \overline{D} \] and having \( \overline{D} \) as its simple pole. Furthermore, assume there is at least one other pole of \( \tilde{\zeta}_{\infty, \Omega} \) (different than \( \overline{D} \)) located on the line \( \{ \text{Re } s = \overline{D} \} \) and let

\[
\lambda_{(\infty, \Omega)} := \inf \{ |\overline{D} - \omega| : \omega \in \dim_{PC}(\infty, \Omega) \setminus \{ \overline{D} \} \} \tag{3.12}
\]

Then, the upper \( \overline{D} \)-dimensional Minkowski content of \( \Omega \) at infinity is bounded as follows:

\[
\mathcal{M}^D(\infty, \Omega) \leq \frac{-3(N + D)\lambda_{(\infty, \Omega)}}{2\pi \left(1 - e^{2\pi(N + D)/\lambda_{(\infty, \Omega)}}\right)} \text{res}(\tilde{\zeta}_{\infty, \Omega}, \overline{D}), \tag{3.13}
\]

or, equivalently,

\[
\mathcal{M}^D(\infty, \Omega) \leq \frac{3\lambda_{(\infty, \Omega)}}{2\pi \left(1 - e^{2\pi(N + D)/\lambda_{(\infty, \Omega)}}\right)} \text{res}(\zeta_{\infty, \Omega}, \overline{D}). \tag{3.14}
\]

As was already mentioned in the introduction, there is a deep connection between \((\infty, \Omega)\) and its image under the geometric inversion. In light of [33, Theorem 3] and Theorem 3, we obtain a more precise connection between these two RFDs assuming \((\infty, \Omega)\) is Minkowski measurable at infinity and assuming also that \(\zeta_{\infty, \Omega}\) possesses a meromorphic continuation to some neighborhood of \(D = \dim_{B}(\infty, \Omega)\).

**Theorem 7** (Geometric inversion and Minkowski measurability) Let \( \Omega \subseteq \mathbb{R}^N \) be of finite \( N \)-volume and \( \dim B(\infty, \Omega) = D < -N \). Assume also that \( \Omega \) is Minkowski measurable at infinity. Furthermore, assume also that \(\zeta_{\infty, \Omega}\) is meromorphically extendable to some neighborhood of the point \( s = D \). Then, the (classical) RFD \((0, \Phi(\Omega))\) obtained by the geometric inversion is (classically) Minkowski measurable, and we have:

\[
\mathcal{M}^D(0, \Phi(\Omega)) = -\frac{N + D}{N - D} \mathcal{M}^D(\infty, \Omega). \tag{3.15}
\]

**Proof** Since, for a fixed \( T > 1 \) from [33, Theorem 3], one has

\[
\zeta_{\infty, \Omega}(s; T) = \zeta_{0, \Phi(\Omega)}(s; 1/T); \tag{3.16}
\]

it follows that the relative distance zeta function of \((0, \Phi(\Omega))\) satisfies [21, Theorem 4.1.14], and we have that

\[
\dim_B(0, \Phi(\Omega)) = D(\zeta_{0, \Phi(\Omega)}) = D(\zeta_{\infty, \Omega}) = \dim_B(\infty, \Omega) = D.
\]

From the functional equation (3.16), we now conclude that \(\zeta_{0, \Phi(\Omega)}\) satisfies the analog of Theorem 5 for classical relative fractal drums [21, Theorem 5.4.2], and therefore,
(0, Φ(Ω)) is Minkowski measurable. Furthermore, D is a pole of first order and its residue is independent of T which together with [33, Theorem 6] yields

\[(N - D)\mathcal{M}^D(0, \Phi(\Omega)) = \text{res}(\zeta_{0,\Phi(\Omega)}, D) = \text{res}(\zeta_{\infty,\Omega}, D) = - (N + D)\mathcal{M}^D(\infty, \Omega),\]  
\quad (3.17)

which yields the desired result. \(\square\)

**Remark 3** It is clear that a reverse of Theorem 7 can also be stated and proved, i.e., when one first imposes the analogous assumptions on (0, Ω) and then concludes about (∞, Φ(Ω) although one need an additional assumption requiring Φ(Ω) to have finite Lebesgue measure. We omit a detailed statement here.

We will now state a version of Theorem 4 where (∞, Ω) is not Minkowski measurable and its tube function satisfies a log-periodic asymptotic formula. The theorem demonstrates the relation between the ‘inner geometric oscillations’ of (∞, Ω) and its principal complex dimensions. If \(G: \mathbb{R} \to \mathbb{R}\) is a periodic function with minimal period \(T\), we use the following notation:

\[G_0(\tau) := \chi_{[0,T]}(\tau)G(\tau)\]  
\quad (3.18)

where \(\chi_A\) is the characteristic function of a set \(A\). Furthermore, we denote the Fourier transform of \(G\) with \(\mathcal{F}G\) or \(\hat{G}\), i.e.,

\[\mathcal{F}G(s) = \hat{G}(s) := \int_{-\infty}^{+\infty} e^{-2\pi s\sqrt{-1}}G(\tau)\,d\tau.\]  
\quad (3.19)

**Theorem 8** (Meromorphic extension—Minkowski nonmeasurable case) Let \(\Omega\) be of finite N-volume. Let also \(D < -N, \alpha > 0\) be fixed constants and \(G: \mathbb{R} \to (0, +\infty)\) be a nonconstant periodic function of minimal period \(T\). Assume that the tube function of \(\Omega\) at infinity satisfies the following asymptotics:

\[|t\Omega| = t^{N+D}(G(\log t) + O(t^{-\alpha})) \quad \text{as} \quad t \to +\infty.\]  
\quad (3.20)

Then, \(G\) is continuous and the Minkowski dimension of (∞, Ω) exist and equals to \(D\). Furthermore, the following equalities hold:

\[\mathcal{M}^D(\infty, \Omega) = \min G, \quad \mathcal{M}^D(\infty, \Omega) = \max G.\]  
\quad (3.21)

Moreover, \(\tilde{\zeta}_{\infty,\Omega}\) has a meromorphic extension to \(\Pi_\alpha := \{\text{Re} s > D - \alpha\}\). Additionally, the poles of \(\zeta_{\infty,\Omega}\) located in \(\Pi_\alpha\) are all simple and given by

\[\mathcal{P}_\alpha(\zeta_{\infty,\Omega}) = \left\{ s_k = D + \frac{2\pi}{T}ik : \hat{G}_0\left(\frac{k}{T}\right) \neq 0, \quad k \in \mathbb{Z}\right\},\]  
\quad (3.22)
with residues
\[ \text{res}(\tilde{\zeta}_\infty, \Omega, s_k) = \frac{1}{T} \hat{G}_0 \left( \frac{k}{T} \right) \] (3.23)
that satisfy the following:
\[ |\text{res}(\tilde{\zeta}_\infty, \Omega, s_k)| \leq \frac{1}{T} \int_0^T G(\tau) \, d\tau, \quad \lim_{k \to \pm \infty} \text{res}(\tilde{\zeta}_\infty, \Omega, s_k) = 0. \] (3.24)

Finally, the residue of \( \tilde{\zeta}_\infty, \Omega \) at \( D \) lies between the lower and the upper Minkowski content of \( (\infty, \Omega) \):
\[ M^D(\infty, \Omega) < \text{res}(\tilde{\zeta}_\infty, \Omega, D) < \overline{M}^D(\infty, \Omega) < \infty. \] (3.25)

**Proof** The fact that \( G \) is continuous follows directly from [21, Lemma 2.3.30] by applying it to the function \( F(t) := |t, \Omega| t^{N+D} \), namely note that it is continuous for \( t > 0 \). By using (3.20), we have
\[ \tilde{\zeta}_\infty(s) = \int_p^{+\infty} t^{-s-N-1} |t, \Omega| \, dt = \int_p^{+\infty} t^{-s-N-1} t^{N+D}(G(\log t) + O(t^{-\alpha})) \, dt \]
\[ = \int_p^{+\infty} t^{D-s-1} G(\log t) \, dt + \int_p^{+\infty} t^{-s} O(t^{D-\alpha-1}) \, dt \]
for some fixed \( P > 0 \). As in the proof of Theorem 4, we have that \( \zeta_2 \) is analytic for \( s \) such that \( \text{Re} \, s > D - \alpha \), and hence, it remains to prove that \( \zeta_1 \) is meromorphic in the whole complex plane. This can be shown directly by obtaining a closed form for \( \zeta_1 \). Since \( G \) is \( T \)-periodic, we have that
\[ \zeta_1(s) = \int_p^{+\infty} t^{D-s-1} G(\log t + T) \, dt. \]

Let us introduce a new variable \( u \) such that \( \log u = \log t + T \), i.e., \( u = e^T t \), to obtain
\[ \zeta_1(s) = \int_{e^T P}^{+\infty} e^{-T(D-s-1)} u^{D-s-1} G(\log u) e^{-T} \, du \]
\[ = e^{-T(D-s)} \int_{e^T P}^{+\infty} u^{D-s-1} G(\log u) \, du \]
\[ = e^{-T(D-s)} \left( \int_{e^T P}^{+\infty} u^{D-s-1} G(\log u) \, du + \int_{e^T P}^P u^{D-s-1} G(\log u) \, du \right) \]
\[ = e^{-T(D-s)} \left( \zeta_1(s) + \int_{e^T P}^P u^{D-s-1} G(\log u) \, du \right) \]
which gives us a closed form for $\zeta_1$:

$$
\zeta_1(s) = \frac{e^{-T(D-s)}}{e^{-T(D-s)} - 1} \int_{t^P} e^{TP(t^{D-s-1}) G(\log t) \, dt} \\
= \frac{e^{T(s-D)}}{e^{T(s-D)} - 1} \int_{\log P}^{\log P+T} e^{-\tau(s-D)} G(\tau) \, d\tau.
$$

Observe now that $I(s)$ is entire by a classical argument since the integrand is not singular along the path of integration. This implies that $\zeta_1$ is meromorphic in all of $\mathbb{C}$ with poles equal to the zeroes $s_k$ of $\exp(T(s-D)) - 1$ but for which, in addition, we have that $I(s_k) \neq 0$ since otherwise one has a removable singularity at $s_k$. Note that from $\exp(T(s_k-D)) = 1$, one has $\exp(-\tau(s_k-D)) = \exp(-2\pi i k \tau / T)$, and hence,

$$
I(s_k) = \int_{\log P}^{\log P+T} e^{-2\pi i k \tau} G(\tau) \, d\tau = \hat{G}_0 \left( \frac{k}{T} \right),
$$

since the integrand is $T$-periodic. This provides the description of the complex dimensions of $\Omega$ at infinity that are contained in $\{ \text{Re } s > D - \alpha \}$ given by (3.22). Moreover, we observe that this set contains $D$ since

$$
I(D) = I(s_0) = \int_0^T G(\tau) \, d\tau > 0.
$$

Namely, since $G$ is continuous and periodic observe that

$$
G([0, T]) = [\mathcal{M}^D(\infty, \Omega), \mathcal{M}^D(\infty, \Omega)]
$$

by Equation (3.20) and since $G$ is also nonconstant, again from (3.20), it follows that $0 < \mathcal{M}^D(\infty, \Omega) < \mathcal{M}^D(\infty, \Omega) < \infty$.

One can now easily obtain the residues of $\tilde{\zeta}_{\infty, \Omega}$ at $s_k = D + \frac{2\pi i k}{T}$ for any $k \in \mathbb{Z}$, from (3.26) and by using L’Hospital’s rule:

$$
\text{res}(\tilde{\zeta}_{\infty, \Omega}, s_k) = \text{res}(\zeta_1, s_k) = \lim_{s \to s_k} \frac{s - s_k}{e^{T(s-D)} - 1} e^{T(s-D)} I(s) = \frac{1}{T} \hat{G}_0 \left( \frac{k}{T} \right).
$$

Substituting $k = 0$ in the above expression, we obtain the inequalities in (3.25).

---

2 By classical argument, see e.g., [25].
Finally, since $G_0 \in L^1(\mathbb{R})$, one has that $|\hat{G}_0(\tau)| \leq ||G_0||_{L^1(\mathbb{R})} = ||G||_{L^1(0,T)}$ and $\lim_{t \to \infty} \hat{G}_0(t) = 0$ follows from the Riemann–Lebesgue lemma; see e.g., [35] or [29], which yields (3.24) directly from (3.28).

4 Properties of Fractal Zeta Functions at Infinity

We begin the section with a lemma from which we will derive the scaling property of fractal zeta functions at infinity. Recall that for a parameter $\lambda > 0$ and a subset $\Omega$ of $\mathbb{R}^N$, we define

$$\lambda \Omega := \{ \lambda x : x \in \Omega \}. \quad (4.1)$$

Lemma 9 Let $\Omega \subseteq \mathbb{R}^N$ of finite $N$-volume. For any $\lambda > 0$ and $t > 0$, we have:

$$|B_t(0)^c \cap \lambda \Omega| = \lambda^N |B_{t/\lambda}(0)^c \cap \Omega| \quad (4.2)$$

and

$$\mathcal{M}^r(\infty, \lambda \Omega) = \lambda^{-r} \mathcal{M}^r(\infty, \Omega), \quad \mathcal{M}^r(\infty, \lambda \Omega) = \lambda^{-r} \mathcal{M}^r(\infty, \Omega), \quad (4.3)$$

for every real number $r$.

Proof We have that $\lambda(B_{t/\lambda}(0)^c \cap \Omega) = B_t(0)^c \cap \lambda \Omega$ from which the first part of the lemma follows directly. For the second part, we observe that

$$\mathcal{M}^r(\infty, \lambda \Omega) = \limsup_{t \to +\infty} \frac{|B_t(0)^c \cap \lambda \Omega|}{t^{N+r}} = \limsup_{t \to +\infty} \frac{\lambda^{-r} |B_{t/\lambda}(0)^c \cap \Omega|}{(t/\lambda)^{N+r}} = \lambda^{-r} \mathcal{M}^r(\infty, \Omega)$$

and also completely analogously for the lower limit.

The next result will prove useful in examples, as well as in the construction of quasiperiodic sets at infinity.

Proposition 10 (Scaling property at infinity) Let $\Omega$ be a Lebesgue measurable subset of $\mathbb{R}^N$ with finite Lebesgue measure, $T > 0$ and $\lambda > 0$. Then, $D(\zeta_{\infty, \lambda \Omega}) = D(\zeta_{\infty, \Omega}) = \dim_B(\infty, \Omega)$. Furthermore, the functional equation,

$$\zeta_{\infty, \lambda \Omega}(s; \lambda T) = \lambda^{-s} \zeta_{\infty, \Omega}(s; T) \quad (4.4)$$

is valid in the open half-plane $\Re s > \dim_B(\infty, \Omega)$.

Proof From Lemma 9, we know that $\dim_B(\infty, \lambda \Omega) = \dim_B(\infty, \Omega)$. The scaling formula (4.4) is proven by a change of variable formula with $y = x/\lambda$ and recalling that $dx = \lambda^N dy$.

\(\square\) Springer
\[ \zeta_{\infty, \Omega}(s; \lambda T) = \int_{B_{T}(0) \cap \lambda \Omega} |x|^{-s-N} \, dx \]
\[ = \int_{B_{T}(0) \cap \Omega} |\lambda y|^{-s-N} \lambda^N \, dy \tag{4.5} \]
\[ = \lambda^{-s} \int_{B_{T}(0) \cap \Omega} |y|^{-s-N} \, dy = \lambda^{-s} \zeta_{\infty, \Omega}(s; T) \]

for \( s \in \mathbb{C} \) such that \( \text{Re} \, s > \dim_{B_{\infty}}(\infty, \Omega) \).

We will now prove a result which will be very useful for almost all examples that we will look at. Namely, to obtain closed forms for the distance zeta function of \( \Omega \) at infinity, it is often more practical to use, for instance, the max norm instead of the classical norm on \( \mathbb{R}^2 \).

**Definition 1** Let \( || \cdot ||_1 \) and \( || \cdot ||_2 \) be two (necessarily equivalent) norms on \( \mathbb{R}^N \) and let \( \Omega \subseteq \mathbb{R}^N \). We will say that \( || \cdot ||_1 \) and \( || \cdot ||_2 \) are equivalent of order \( \alpha \in \mathbb{R} \) for \( (\infty, \Omega) \) if

\[ ||x||_1 = ||x||_2 + O\left(||x||_1^{\alpha}\right), \quad \text{as} \quad ||x||_1 \to +\infty, \ x \in \Omega. \tag{4.6} \]

In this case, we will write

\[ || \cdot ||_1 \overset{\alpha}{\sim} || \cdot ||_2. \tag{4.7} \]

This equivalence is well-defined since the two norms are equivalent in the standard sense. More precisely, since there exist \( m \), \( M > 0 \) such that \( m|| \cdot ||_1 \leq || \cdot ||_2 \leq M|| \cdot ||_1 \), we have that \( O\left(||x||_1^{\alpha}\right) = O\left(||x||_2^{\alpha}\right) \) for every \( \alpha \in \mathbb{R} \) when \( ||x||_1 \to +\infty \) or \( ||x||_2 \to +\infty \). From this, one gets symmetry and transitivity easily.

Note that if the set \( \Omega \subseteq \mathbb{R}^N \) contains a half-line \( p \) through the origin, then by using homogeneity of norms, one can easily deduce that (4.6) implies \( ||x||_1 = ||x||_2 \) for every \( x \in p \cap \Omega \). The sets \( \Omega \) we are interested usually contain only a finite number of such half-lines (usually just one) as is the case in Sect. 5, and the fine information introduced by Definition 1 then plays a crucial role in determining complex dimensions of such sets. The following simple example illustrates the spirit of Definition 1.

**Example 3** Consider the set \( \Omega := \{ (t, \sqrt{t}) : t \geq 1 \} \subseteq \mathbb{R}^2 \) and two norms on \( \mathbb{R}^2 \) namely, the 1-norm and the max norm. Then for \( x = (t, t') \in \Omega \), one has the following

\[ |x|_1 = |t| + |t'| = |t| + \sqrt{|t|} = |x|_\infty + \sqrt{|x|_\infty} \leq |x|_\infty + \sqrt{|x|_1}, \]

which implies that

\[ || \cdot ||_1 \overset{1/2}{\sim} || \cdot ||_\infty. \]
Theorem 11  (Behavior under change of norm) Let $\Omega$ be a Lebesgue measurable subset of $\mathbb{R}^N \setminus \{0\}$ with finite Lebesgue measure and assume $\overline{D} := \text{dim}_B(\infty, \Omega) < -N$. Furthermore, assume that $\| \cdot \|$ is a norm in $\mathbb{R}^N$ such that for some $\alpha \in (-\infty, 1)$, we have

$$|x|^\alpha \sim \|x\|. \quad (4.8)$$

Then, the difference

$$\xi_{\infty, \Omega}(\cdot) - \xi_{\infty, \Omega}(\cdot; \| \cdot \|) \quad (4.9)$$

is analytic in the half-plane

$$\{ \text{Re } s > (\overline{D} - (1 - \alpha)) \}. \quad (4.10)$$

Proof  We observe that for every $s \in \mathbb{C}$ the function $f_s(z) := z^{-s-N}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and define $F(s, x) := f_s(|x|) - f_s(||x||)$. Then, from Corollary 19 applied to $f_s$, we conclude that there exists a function $r : \mathbb{C} \times (\Omega \setminus X) \to (0, +\infty)$ such that

$$|F(s, x)| = |x|^{-s-N} - \|x||^{-s-N} \leq \sqrt{2}|s + N|r(s, x)^{-\text{Re } s - N - 1}|x| - \|x||, \quad (4.11)$$

where $X := \{ x \in \Omega : |x| = \|x\| \}$. Let $m$ and $M$ be the positive constants such that

$$m|x| \leq \|x\| \leq M|x| \text{ for } x \in \mathbb{R}^N$$

and denote

$$C_m := \min\{1, m\}, \quad C_M := \max\{1, M\}. \quad (4.12)$$

Furthermore, since

$$|x| < r(s, x) < \|x\| \text{ or } \|x\| < r(s, x) < |x|, \quad (4.13)$$

we have that

$$C_m|x| \leq r(s, x) \leq C_M|x|, \quad (4.14)$$

which implies that

$$r(s, x)^{-\text{Re } s - N - 1} \leq |x|^{-\text{Re } s - N - 1} \max\{C_m^{-\text{Re } s - N - 1}, C_M^{-\text{Re } s - N - 1}\}. \quad (4.15)$$

Furthermore, by taking $T > 1$ sufficiently large, we can assume that $||x| - \|x||| \leq c|x|^\alpha$ which together with (4.15) and (4.11) yields

$$|F(s, x)| \leq c\sqrt{2}|s + N| \max\{C_m^{-\text{Re } s - N - 1}, C_M^{-\text{Re } s - N - 1}\}|x|^{-\text{Re } s - N - 1 + \alpha}. \quad (4.16)$$
Suppose now that \( K \) is a compact subset in \( \{ \text{Re} s > \overline{D} - (1 - \alpha) \} \), and let
\[
C_K := \max_{s \in K} \left\{ c \sqrt{2} |s + N| \max\{ C_m^{\text{Re} s - N - 1}, C_M^{\text{Re} s - N - 1} \} \right\}
\]
and define the function \( g_K \) as follows:
\[
g_K(x) := C_K |x|^{-(\min\{\text{Re} s : s \in K\} - \alpha + 1) - N}
\]
so that we have \( |F(s, x)|_K \leq g_K(x) \) for \( x \in T \Omega \setminus X \). We observe that \( g_K \) is in \( L^1(T \Omega) \), since if \( s \in K \), then \( \text{Re} s > D - (1 - \alpha) \) so that
\[
\min\{\text{Re} s : s \in K\} - \alpha + 1 > D - (1 - \alpha) - \alpha + 1 = \overline{D},
\]
which, in turn, implies that
\[
\int_{T \Omega} g_K(x) \, dx = C_K \xi_\infty \left( \min\{\text{Re} s : s \in K\} - \alpha + 1 ; \Omega \right) < \infty.
\]

Finally, we have shown that \( F(s, x) \) satisfies the hypotheses of [21, Theorem 2.1.47 and Remark 2.1.48] (see also [25]), and therefore,
\[
\int_{T \Omega \setminus X} F(s, x) \, dx = \xi_{\infty, \Omega \setminus X}(s) - \xi_{\infty, \Omega \setminus X}(s; || \cdot ||)
\]
is holomorphic on \( \{ \text{Re} s > \overline{D} - (1 - \alpha) \} \). On the other hand, \( \xi_{\infty, \Omega \setminus X}(s) = \xi_{\infty, \Omega \setminus X}(s) + \xi_{\infty, X}(s) \) and analogously for \( \xi_{\infty, \Omega}(s; || \cdot ||) \) which completes the proof.

**Corollary 12** Let \( \Omega \) be a measurable subset in \( \mathbb{R}^N \) with \( |\Omega| < \infty \) such that \( \dim_B(\infty, \Omega) = D \) exists. Furthermore, suppose that \( \xi_{\infty, \Omega} \) can be meromorphically extended to an connected and open neighborhood \( U \) of the half-plane \( \{ \text{Re} s \geq \overline{D} \} \).

Take \( || \cdot || \) to be another norm in \( \mathbb{R}^N \) for which \( ||x|| \geq \langle \infty, \Omega \rangle ||x|| \) for some \( \alpha \in (-\infty, 1) \). Then, \( \tilde{\xi}_{\infty, \Omega}(\cdot ; || \cdot ||) \) can also be meromorphically extended to \( V := U \cap \{ \text{Re} s > D - (1 - \alpha) \} \). Furthermore, the locations of poles in \( V \) and their multiplicities does not depend on the choice of the norm in this case. Moreover, the principal parts of the Laurent expansion around each pole in \( V \) also do not depend on the choice of the norm.

**Proof** Since, by hypothesis, \( \tilde{\xi}_{\infty, \Omega} \) is meromorphic on \( V = U \cap \{ \text{Re} s > D - (1 - \alpha) \} \), the corollary follows directly from Theorem 11 which states that the difference of these two distance zeta functions is holomorphic on \( V \).

**Remark 4** It is clear that the above corollary is still valid if we interchange the roles of the two distance zeta functions.
An important special case of the above theorem, which we will be using in almost all examples considered, is when the set \( \Omega \subseteq \mathbb{R}^N \) is contained in a cylinder of finite radius. This is in fact [33, Proposition 6] for which the proof was omitted in [33]. We restate it here and give a short proof by using Theorem 11.

**Proposition 13** Let \( \Omega \subseteq \mathbb{R}^N \) with \( |\Omega| < \infty \) be such that it is contained in a cylinder

\[
x_2^2 + x_3^2 + \cdots + x_N^2 \leq C
\]

for some constant \( C > 0 \) where \( x = (x_1, \ldots, x_N) \). Furthermore, let \( \bar{D} := \mathbb{d} \dim_B(\infty, \Omega) \) and \( T > 0 \). Then,

\[
\zeta_{\infty, \Omega}(s; T) - \zeta_{\infty, \Omega}(s; T; |\cdot|_{\infty})
\]

is analytic in the open half-plane \( \{ \Re s > \bar{D} - 2 \} \).

Consequently, if any of the two functions can be meromorphically extended to some open connected neighborhood \( U \) of the of the initial half-plane of convergence \( \{ \Re s > \bar{D} \} \), the sets of poles in \( U \cap \{ \Re s > \bar{D} - 2 \} \) coincide as well as their orders.

**Proof** We observe that for \( T > 0 \) sufficiently large, we have

\[
|x| - |x|_{\infty} = |x| - |x_1| = \frac{\sum_{i=2}^{N} x_i^2}{|x| + |x_1|} \leq C|x|^{-1}, \quad x \in \tau \Omega.
\]

In other words, \( |x|_{(\infty, \Omega)}^{-1} \) \( |x|_{\infty} \), and the conclusion now follows by applying Theorem 11. \( \square \)

## 5 Maximally Hyperfractal and Quasiperiodic Sets at Infinity

In this section, we will construct quasiperiodic subsets of \( \mathbb{R}^2 \) with prescribed box dimension at infinity. We will use the Cantor-like two parameter sets \( \Omega_{\infty}^{(a, b)} \) introduced in [33] which will be our building blocks for the construction of a maximally hyperfractal set at infinity; that is, according to the terminology of [21], a set \( \Omega \) with its distance zeta function at infinity having the critical line \( \{ \Re s = \mathbb{d} \dim_B(\infty, \Omega) \} \) as a natural boundary. This construction will also give examples of algebraically and transcendently quasiperiodic sets at infinity. One of the open problems in [21] was the question of existence of algebraically quasiperiodic bounded sets and RFDs. Here, we give a positive answer in the case of unbounded sets at infinity. Furthermore, a similar construction can be done in the setting of classical RFDs. More precisely, one could look into \( (0, \Phi(\Omega)) \) obtained by geometric inversion where \( \Omega \) is the quasiperiodic set at infinity constructed here since the two distance zeta functions are essentially the same by [33, Theorem 3]. On the other hand, it is not clear immediately whether \( (0, \Phi(\Omega)) \) is quasiperiodic since we do not have a general result about geometric inversion which would ensure that. The reason for this is in the fact that we would need an asymptotic
formula which relates \( t \mapsto |B_t(0)^c \cap \Omega| \) to the function \( t \mapsto |B_{1/t}(0) \cap \Phi(\Omega)| \) when \( t \) tends to infinity. We do not know if such formula can be derived in the general case, but it is reasonable to expect that \((0, \Phi(\Omega))\) will be quasiperiodic having the same quasiperiods as \((\infty, \Omega)\).

Another idea to construct an algebraically quasiperiodic RFDSs is to use the geometric inversion in one coordinate; that is, \( \Phi_1(x, y) := (1/x, y) \) and apply it to quasiperiodic sets at infinity constructed here. We leave this, as well as other properties of the ‘partial geometric inversion’ for future work. One more approach would be to consider a ‘radial version’ of RFDs considered here since then the geometric inversion of such an RFD would be much more easier to handle.

We now recall the Cantor-like sets at infinity from [33, Section 5]. Namely, one ‘stacks’ the translated images of the two-parameter unbounded sets \( \Omega_m(a, b) \) along the \( y \)-axis on top of each other, where

\[
\Omega_m(a, b) := \{(x, y) \in \mathbb{R}^2 : x > a^{-m}, 0 < y < x^{-b}\}, \quad m \geq 1.
\]

More precisely, for each \( m \geq 1 \) one takes \( 2^{m-1} \) copies of \( \Omega_m(a, b) \) and arranges all of them by vertical translations so that they are pairwise disjoint and lie in the strip \( \{0 \leq y \leq S\} \) where \( S \) denotes the finite sum of widths of all of these sets. Under the condition that \( b > 1 + \log_1/a 2 > \) the disjoint union of all these sets, \( \Omega_\infty(a, b) \) is of finite volume and also \( S \) is finite; see [33] and Fig. 2.

In [33, Example 6], the following closed formula (meromorphic everywhere) for the distance zeta function (using the \( | \cdot |_\infty \) norm on \( \mathbb{R}^2 \)) has been obtained:

\[
\zeta_{\infty, \Omega_\infty(a, b)}(s; | \cdot |_\infty) = \frac{1}{s + b + 1} \cdot \frac{1}{a^{-(s+b+1)} - 2}.
\]
Furthermore, it was shown that the complex dimensions of \( \Omega_{\infty}^{(a,b)} \) at infinity visible through \( W := \{ \Re s > \log_{1/a} b - 3 \} \) are given by

\[
\{- (b + 1) \} \cup \left( \log_{1/a} 2 - (b + 1) + \frac{2\pi i}{\log(1/a)} \mathbb{Z} \right) \tag{5.2}
\]

and hence

\[
\text{dim}_B(\infty, \Omega_{\infty}^{(a,b)}) = \log_{1/a} 2 - (b + 1). \tag{5.3}
\]

Furthermore, note that \( 2\pi / \log(1/a) \) tends to 0 as \( a \to 0^+ \) making the complex dimensions more dense on the critical line when the parameter \( a \) becomes smaller.

**Example 4** We will compute \( \text{dim}_B(\Omega_{\infty}^{(a,b)}) \) directly, and for it to be easier, we will measure the neighborhoods of infinity in the \( | \cdot |_\infty \) norm. As \( \Omega_{\infty}^{(a,b)} \) is contained in a horizontal strip of finite width, according to [33, Lemma 3], this will not affect the value of the Minkowski content of \( \Omega_{\infty}^{(a,b)} \) at infinity; see also Lemma 18 in Appendix. Now, for \( t > 1/a \), we have

\[
|K_t(0)^c \cap \Omega_{\infty}^{(a,b)}| = \sum_{n=1}^{\lfloor \log_{1/a} t \rfloor} 2^{n-1} \int_t^{+\infty} x^{-b} \, dx + \sum_{n>\lfloor \log_{1/a} t \rfloor} \int_{a^n}^{+\infty} x^{-b} \, dx
\]

\[
= \frac{1}{b-1} \left[ 1^{1-b} \sum_{n=1}^{\lfloor \log_{1/a} t \rfloor} 2^{n-1} + \sum_{n>\lfloor \log_{1/a} t \rfloor} 2^{n-1} (a^{-b-1})^n \right]
\]

\[
= \frac{1}{b-1} \left[ 1^{1-b} \left( 2^{\lfloor \log_{1/a} t \rfloor} - 1 \right) + \frac{1}{a^{1-b} - 2} \cdot 2^{\lfloor \log_{1/a} t \rfloor} a^{-b-1} \right].
\]

Using the fact that \( \lfloor \log_{1/a} t \rfloor = \log_{1/a} t - \{ \log_{1/a} t \} \) and \( 2^{\log_{1/a} t} = t^{\log_{1/a} 2} \), we then have that

\[
|K_t(0)^c \cap \Omega_{\infty}^{(a,b)}| = \frac{1^{1-b} + \log_{1/a} 2}{b-1} \left[ 2^{-\lfloor \log_{1/a} t \rfloor} + \frac{1}{a^{1-b} - 2} \cdot \left( \frac{a^{1-b}}{2} \right)^{\lfloor \log_{1/a} t \rfloor} \right] - \frac{1^{1-b}}{b-1}.
\]

From this, we deduce that for \( D := \log_{1/a} 2 - (b + 1) \), we have

\[
|K_t(0)^c \cap \Omega_{\infty}^{(a,b)}| = t^{-b+D} \left( G(\log t) - \frac{t^{-\log_{1/a} 2}}{b-1} \right) \quad \text{as } t \to +\infty \quad (5.4)
\]

\[^3\text{Although expected, here we are not certain if these are indeed all of the complex dimensions in } \mathbb{C} \text{ of } (\infty, \Omega_{\infty}^{(a,b)}) \text{ since one has to use Proposition 13 since here we are using the } | \cdot |_\infty \text{-norm in the definition of the distance zeta function of } (\infty, \Omega_{\infty}^{(a,b)}).\]
with $G$ being the $T$-periodic function

$$G(\tau) := 2^{-\left\lfloor \frac{\tau}{\log(1/a)} \right\rfloor} \left( 1 + \left( \frac{a^{1-b}}{a^{1-b} - 2} \right) \right), \quad (5.5)$$

where $T := \log(1/a)$. Furthermore, this result implies that

$$\dim_B(\infty, \Omega_\infty^{(a,b)}) = \log_{1/a} 2 - (b + 1).$$

Note that $\dim_B(\infty, \Omega_\infty^{(a,b)}) \to -\infty$ as $b \to +\infty$ and $\dim_B(\infty, \Omega_\infty^{(a,b)}) \to -(b + 1) < -2$ as $a \to 0^+$ but can be made as close to $-2$ as desirable. Moreover, $\Omega_\infty^{(a,b)}$ is not Minkowski measurable at infinity. Namely,

$$\overline{\mathcal{M}}^D(\infty, \Omega_\infty^{(a,b)}) = \max G = G(0) = \frac{1}{b - 1} \cdot \frac{a^{1-b} - 1}{a^{1-b} - 2} \quad (5.6)$$

and

$$\underline{\mathcal{M}}^D(\infty, \Omega_\infty^{(a,b)}) = \min G = G(\tau_{\min}), \quad (5.7)$$

where $\tau_{\min}$ is the unique point of the global minimum of the function $G$ on the interval $[0, 1]$ which can be explicitly computed:

$$\tau_{\min} = \frac{\log(1 + (b - 1) \log_2 a) - \log(2 - a^{1-b})}{(b - 1) \log a}.$$

In the next theorem we will construct a maximal hyperfractal set $\Omega$ at infinity. More precisely, we will now construct a set with a prescribed Minkowski dimension $D < -2$ at infinity such that every point on the abscissa of convergence $\{\Re s = D\}$ represents a nonremovable singularity of the $\zeta_{\infty,\Omega}$. By the definitions introduced in [21], we will call such sets maximally hyperfractal at infinity.

**Theorem 14** (Maximally hyperfractal set at infinity) *For any $D < -2$, there exists a set $\Omega \subseteq \mathbb{R}^2$ of finite $N$-volume which is maximally hyperfractal with $\dim_B(\infty, \Omega) = D$ and Minkowski nondegenerate at infinity.*

**Proof** Let us fix $D < -2$ and choose a nonincreasing sequence $(a_n)_{n \geq 1}$ such that $0 < a_n < 1/2$ for every $n \in \mathbb{N}$ and $a_n \to 0^+$ as $n \to +\infty$. Furthermore, we define the sequence $b_n := \log_{1/a_n} 2 - D - 1$ and observe that for $D < -2$ the condition $b_n > 1 + \log_{1/a_n} 2$ is fulfilled. For the two parameter unbounded set $\Omega_\infty^{(a_n,b_n)}$ from Example 4, we have that $\dim_B(\infty, \Omega_\infty^{(a_n,b_n)}) = D$. The next step is to scale every one of these sets with a suitable parameter, namely we define the sets $\tilde{\Omega}_n$ for every $n \in \mathbb{N}$ as follows:

$$\tilde{\Omega}_n := \frac{1}{2^n} \Omega_\infty^{(a_n,b_n)}.$$
Finally, we construct the sets $\Omega_n$ by translating each set $\tilde{\Omega}_k$ vertically for the amount $l_n$ which is equal to the sum of the heights of each $\tilde{\Omega}_k$ for $k < n$, i.e., $l_1 := 0$ and

$$l_n := \sum_{k=1}^{n-1} \frac{a_k b_k}{2^k (1 - 2a_k b_k)}$$

for $n > 1$. We next let the desired set $\Omega$ be a disjoint union of the sets $\Omega_n$ and observe that the scaling factor in the definition of the sets $\tilde{\Omega}_k$ ensures that $\Omega$ is of finite volume and that it lies in a horizontal strip of finite width.

Similarly as before, this ensures us that calculating the tube formula of $\Omega$ using the $| \cdot |_\infty$-norm on $\mathbb{R}^2$ will not affect the Minkowski content of $\Omega$ at infinity. For $t > 1$, we have that

$$|K_t(0)^c \cap \Omega| = \sum_{n=1}^\infty \sum_{n=1}^\infty |K_t(0)^c \cap \Omega_n| + \sum_{n=1}^\infty |K_{2n}^c(0)^c \cap \Omega_{\infty}(a_n, b_n)|$$

$$= \sum_{n=1}^\infty 2^{-2n} |K_{2n}^c(0)^c \cap \Omega_{\infty}(a_n, b_n)|$$

$$= \sum_{n=1}^\infty \frac{t^{2n + D}}{2^{-Dn}} \left( G_n \left( \log(2n t) \right) - \frac{t^{-\log_2(1/a_n)^2}}{2^n \log_2(1/a_n)^2 (b_n - 1)} \right)$$

where we have used (4.2) with $N = 2$ and $G_n$ is the $\log(1/a_n)$-periodic function defined by (5.5) with $a$ and $b$ replaced by $a_n$ and $b_n$, respectively. In other words, we have:

$$|K_t(0)^c \cap \Omega| = t^{2n + D} \left( G(\log t) - \sum_{n=1}^\infty \frac{t^{-\log_2(1/a_n)^2}}{(b_n - 1)2^n(-D + \log_2(1/a_n)^2)} \right), \quad (5.8)$$

where

$$G(\tau) := \sum_{n=1}^\infty 2^n D G_n (\tau + n \log 2). \quad (5.9)$$

The convergence of the sum for every $t > 1$ in (5.8) follows from the facts that $\log_2(1/a_n)^2 \in (0, 1)$, $-D > 2$ and $b_n - 1 > -D - 2 > 0$ for all $n \in \mathbb{N}$, i.e.,

$$\sum_{n=1}^\infty \frac{t^{-\log_2(1/a_n)^2}}{(b_n - 1)2^n(-D + \log_2(1/a_n)^2)} \leq -\frac{1}{D + 2} \sum_{n=1}^\infty \frac{1}{(2^{-D + \log_2(1/a_n)^2})^n} < \infty.$$ 

Furthermore, the series defining the function $G$ is also convergent for $\tau > 0$. To see this, we observe that from (5.6), we have:

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for all \( n \in \mathbb{N} \). The last inequality above can be easily shown from the conditions on \( a_n \) and \( b_n \). Furthermore, from this, we obtain

\[
G(\tau) = \sum_{n=1}^{\infty} 2^{nD} G_n(\tau + n \log 2) \leq \sum_{n=1}^{\infty} \frac{1}{(2-D)^n} < \infty,
\]

from which it follows that

\[
\mathcal{M}^D(\infty, \Omega) \leq \sum_{n=1}^{\infty} \frac{1}{(2-D)^n} < \infty.
\]

On the other hand, since \( \Omega \supseteq \Omega_1 \), we have

\[
\mathcal{M}^D(\infty, \Omega) \geq \mathcal{M}^D(\infty, \Omega_1) = \mathcal{M}^D(\infty, 2^{-1}\Omega^{(a_1, b_1)}) = 2^D \mathcal{M}^D(\infty, \Omega^{(a_1, b_1)}) > 0.
\]

The last equality above is a consequence of Lemma 9 with \( r = D \), while the conclusion of positivity follows from (5.7).

Let us now show that \( \{ \text{Re } s = D \} \) represents a natural boundary for the associated distance zeta function at infinity. Using Proposition 10, one has

\[
\zeta_{\infty, \Omega}(s) = \sum_{n=1}^{\infty} \zeta_{\infty, 2^{-n}\Omega^{(a_n, b_n)}}(s; 2^n; | \cdot |_{\infty}) = \sum_{n=1}^{\infty} 2^{ns} \cdot \zeta_{\infty, \Omega^{(a_n, b_n)}}(s; 2^n)
\]

and it is holomorphic on \( \{ \text{Re } s > D \} \). Furthermore, according to [33, Example 6], for every \( n \in \mathbb{N} \), the zeta function \( \zeta_{\infty, \Omega^{(a_n, b_n)}}(s; 2^n; | \cdot |_{\infty}) \) is meromorphic everywhere and its poles are located at \( D + \frac{2\pi i}{\log(1/a_n)} \mathbb{Z} \), each one being simple. Since \( \Omega^{(a_n, b_n)} \) is contained in a strip, i.e., a cylinder of finite width, according to Proposition 13, \( \zeta_{\infty, \Omega}(s; 2^n) \) is meromorphic at least on \( \{ \text{Re } s > D - 2 \} \) and its poles in that half-plane coincide with that of \( \zeta_{\infty, \Omega^{(a_n, b_n)}}(s; 2^n; | \cdot |_{\infty}) \). Hence, the poles of \( \zeta_{\infty, \Omega}(s) \) are dense on \( \{ \text{Re } s = D \} \) because one has that \( \log(1/a_n) \to +\infty \) as \( n \to +\infty \). Hence, \( \Omega \) is maximally hyperfractal at infinity. \( \square \)

Now, we are ready to show that a careful choice of the parameters in the two parameter set \( \Omega^{(a, b)}_\infty \) gives examples of algebraically and transcendentally quasiperiodic sets at infinity. First, we recall some needed definitions from number theory for the convenience of the reader.

In order to define quasiperiodic sets at infinity, we will use a definition of quasiperiodic functions adapted in [21, Definitions 3.1.9.]—the case of finitely many periods and [21, Definitions 4.6.6]—the case of countably many quasiperiods. 4 Roughly speaking,

4 There exist different notions of quasiperiodic functions in the literature; e.g., [4, 11, 24, 37], [14, Appendix F] The definition we use—[21, Definition 3.1.9.]—is adapted from the one in [39].
we say that the function is algebraically quasiperiodic if its quasiperiods are independent over rational numbers and transcendently quasiperiodic if the quasiperiods are independent over the algebraic numbers, while the number of quasiperiods is referred to as the order of the quasiperiodic function, which is also allowed to be countably infinite.

We now define quasiperiodic sets at infinity which complements the analogous definition of quasiperiodic bounded sets from [21].

**Definition 2** (Quasiperiodic set at infinity) Assume \( \Omega \subseteq \mathbb{R}^N \) is of finite Lebesgue measure and such that it has the following tube formula at infinity:

\[
|t \Omega| = t^{N+D} (G(\log t) + o(1)) \quad \text{as} \quad t \to +\infty,
\]

(5.10)

where \( G(\tau) \geq 0 \) for all \( \tau \in \mathbb{R} \) satisfying \( 0 < \lim \inf_{\tau \to +\infty} G(\tau) \leq \lim \sup_{\tau \to +\infty} G(\tau) < +\infty \) where \( D \in (-\infty, -N] \) is fixed.\(^5\)

We will then call \( \Omega \) an algebraically or transcendentally \( n \)-quasiperiodic set at infinity if \( G = G(\tau) \) is algebraically or transcendentally \( n \)-quasiperiodic, respectively.

Furthermore, we also say that \( \Omega \) is \( \infty \)-quasiperiodic set at infinity when the function \( G \) is \( \infty \)-quasiperiodic. Moreover, we make the same distinction between the algebraic and the transcendental case for such sets \( \Omega \).

We denote the families of algebraically and transcendentally \( n \)-quasiperiodic sets at infinity, by \( \mathcal{D}_{aqp}(n) \) and \( \mathcal{D}_{tqp}(n) \), respectively, where \( n \geq 2 \) is an integer or \( n = \infty \).

**Theorem 15** The families \( \mathcal{D}_{tqp}(2) \) and \( \mathcal{D}_{aqp}(2) \) are infinite.

**Proof** We note that in the construction of the set \( \Omega \) in the proof of Theorem 14 if we only take two sets \( \Omega_{(a_1, b_1)}^\infty \) and \( \Omega_{(a_2, b_2)}^\infty \) instead of infinitely many, we can construct an algebraically or a transcendentally \( 2 \)-quasiperiodic unbounded set at infinity with prescribed box dimension at infinity equal to \( D < -2 \). We point out here that the set \( \Omega \) constructed from sets \( \Omega_{(a_1, b_1)}^\infty \) and \( \Omega_{(a_2, b_2)}^\infty \) has the following tube formula at infinity

\[
|K_t \cap \Omega| = t^{2+D} (G(\log t) + O(t^{-\log_{1/a_1}^2})) \quad \text{as} \quad t \to +\infty,
\]

(5.11)

where

\[
G(\tau) = 2^D G_1(\tau + \log 2) + 2^{2D} G_2(\tau + 2 \log 2)
\]

(5.12)

is a \( 2 \)-quasiperiodic function with

\[
G_i(\tau) = 2^{-\left\lfloor \frac{\tau}{\log(1/a_i)} \right\rfloor} \left( 1 + \frac{a_i^{-b_i}}{a_i^{-b_i} - 2} \right)^{\left\lfloor \frac{\tau}{\log(1/a_i)} \right\rfloor}
\]

(5.13)

for \( i = 1, 2 \). As we can see the set \( \Omega \) is then \( 2 \)-quasiperiodic at infinity but in the sense of the ‘cube’ tube function at infinity \( t \mapsto |K_t(0)^c \cap \Omega| \). To get a ‘proper’

\(^5\) Implicitly, then \( \dim B(\infty, \Omega) = D \), \( M^D(\infty, \Omega) = \lim \inf_{\tau \to +\infty} G(\tau) \) and \( \overline{M}^D(\infty, \Omega) = \lim \sup_{\tau \to +\infty} G(\tau) \).
2-quasiperiodic set at infinity, one should mimic this construction in a radial way, i.e., use an analog of sets \( \Omega_{\infty}^{(a_i, b_i)} \) that are ‘arranged’ around radial rays emanating from the origin. We will not get into the details of this construction, but on the other hand, we can use Lemma 18 from Appendix to deduce that if we choose \( D \in (-3, -2) \), we do get ‘proper’ 2-quasiperiodic sets at infinity even in the present construction. More precisely, since \( \Omega \) is contained in a strip of finite width, by Lemma 18, we have that

\[
|t_\Omega| = |K_\infty(0)^c \cap \Omega| + O(t^{-1}) = t^{2+D}(G(\log t) + O(t^{-\log_1 a_1^2}) + O(t^{-2-D-1}))
\]

when \( t \to +\infty \), that is, \( \Omega \) is 2-quasiperiodic at infinity.

Now, for the algebraical case, it suffices to choose \( a_1 \in (0, 1/2) \) and define, for instance, \( a_2 := a_1^{\sqrt{m}} \) where \( m \geq 2 \) is an integer that is not a perfect square. Then, we have that \( b_1 = \log_1 a_1^2 - D - 1 \) and \( b_2 = \log_1 a_2^2 - D - 1 \). Furthermore for the periods, we have that

\[
T_1 = \log(1/a_1) \quad \text{and} \quad T_2 = \log(1/a_2) = \sqrt{m} \log(1/a_1),
\]

i.e., \( T_2/T_1 = \sqrt{m} \).

On the other hand, if we choose, for instance, \( a_1 = 1/3 \) and \( a_2 = 1/k \) where \( k > 3 \) is an integer that is not a power of 3, we have that

\[
\frac{T_1}{T_2} = \frac{\log 3}{\log k} = \log k^3
\]

which is a transcendental number, a fact that follows from the Gel’fond–Schneider theorem [8].

\( \square \)

**Remark 5** As a consequence of (5.2), we have that the complex dimensions at infinity of the set \( \Omega \) from Theorem 15 visible through \( W := \{ \text{Re } s > D - 2 \} \) are given by

\[
\{ D - \log_1 a_i^2 : i = 1, 2 \} \cup (D + p(a_1)iz) \cup (D + p(a_2)iz) \quad (5.14)
\]

where \( p(a_i) = 2\pi/\log(1/a_i) \) for \( i = 1, 2 \) are the oscillatory quasiperiods of \( \Omega \).

**Theorem 16** (Construction of \( n \)-quasiperiodic sets at infinity) The families \( \mathcal{D}_{i\text{qp}}^\infty(n) \) and \( \mathcal{D}_{a\text{qp}}^\infty(n) \) are infinite for every integer \( n \geq 2 \).

**Proof** One proceeds similarly as in Theorem 15, the difference is that one now has to take \( n \) sets \( \Omega_{\infty}^{(a_i, b_i)} \), for \( i = 1, \ldots, n \) instead of only two. In that way, we construct a set \( \Omega \) with \( n \) quasipersiods at infinity which will be ‘proper’ \( n \)-quasiperiodic if we additionally restrict ourselves to \( D \in (-3, -2) \). (See the discussion in the proof of Theorem 15.) For the algebraically \( n \)-quasiperiodic case, we may choose \( a_1 \in (0, 1/2) \).
and define \( a_{i+1} := a_i^{1/\sqrt{p_i}} \) where \( p_i \) is the \( i \)-th prime number for \( i \geq 1 \). Then for the quasiperiods of \( \Omega \), we have that

\[
T_1 = \log(1/a_1) \quad \text{and} \quad T_{i+1} = \log(1/\sqrt{a_i^{1/\sqrt{p_i}}}) = T_1\sqrt{p_i}
\]

for \( i \geq 1 \). It is obvious that the quasiperiods \( T_1, \ldots, T_n \) are algebraically dependent. On the other hand, they are rationally independent. Namely suppose that there are \( \lambda_1, \ldots, \lambda_n \in \mathbb{Q} \) such that

\[
\lambda_1 T_1 + \lambda_2 T_2 + \ldots + \lambda_n T_n = 0.
\]

This is equivalent to

\[
\lambda_1 + \lambda_2 \sqrt{2} + \cdots + \lambda_n \sqrt{p_{n-1}} = 0
\]

which is possible only if \( \lambda_1 = \cdots = \lambda_n = 0 \) according to a result of Besicovitch [3]. This proves that the set \( \Omega \) indeed is algebraically \( n \)-quasiperiodic at infinity.

Let us now construct a transcendentally \( n \)-quasiperiodic set at infinity. We choose now \( a_i := 1/p_{i+1} \) with \( p_i \) being the \( i \)-th prime number for \( i \geq 1 \). Note that now \( T_i = \log(1/a_i) = \log p_{i+1} \), and these numbers are rationally independent. Reasoning by contradiction, if there were rational numbers \( \lambda_1, \ldots, \lambda_n \) such that \( \sum_{i=1}^{n} \lambda_i \log p_{i+1} = 0 \), one would have \( \prod_{i=1}^{n} p_i^{\lambda_i} = 1 \) which contradict the Fundamental theorem of algebra. Next, Baker’s Theorem [2, Theorem 2.1] implies that the numbers \( T_1, \ldots, T_n \) are also algebraically independent; that is, the set \( \Omega \) is transcendentally \( n \)-quasiperiodic.

\[\square\]

**Remark 6**  Similarly as in Remark 5, the set \( \Omega \) constructed in Theorem 16 will have the following set of complex dimensions visible through \( W = \{ \Re s > D - 2 \} \):

\[
\bigcup_{i=1}^{n} \left( \{D - \log_1/a_i, 2\} \cup (D + \mathbf{p}(a_i) \mathbb{Z}) \right)
\]

where \( \mathbf{p}(a_i) = 2\pi / \log(1/a_i) \) for \( i = 1, \ldots, n \) are the oscillatory quasiperiods of \( \Omega \) at infinity.

**Remark 7**  It is clear that one can construct somewhat more general examples of \( n \)-quasiperiodic sets at infinity than the ones from the proof of Theorem 16 by choosing other admissible values for the parameters \( a_i \).

Let us conclude this section by defining the notion of \( \infty \)-quasiperiodic sets at infinity and showing that the maximally hyperfractal set \( \Omega \) at infinity from Theorem 14 gives an example of such a set. Moreover, by carefully choosing the parameters \( a_i \), we can construct an infinite number of algebraically and transcendentally \( \infty \)-quasiperiodic sets at infinity.
In much the same way as before, if we denote with $\mathcal{D}_{qp}^\infty(\infty)$ the family of all $\infty$-quasiperiodic sets at infinity, then it is clear that this family is a disjoint union of $\mathcal{D}_{ap}^\infty(\infty)$ and $\mathcal{D}_{tp}^\infty(\infty)$; that is, the algebraically $\infty$-quasiperiodic subfamily and the transcendently $\infty$-quasiperiodic subfamily, respectively.

**Theorem 17** (Construction of $\infty$-quasiperiodic sets at infinity) *The families $\mathcal{D}_{ap}^\infty(\infty)$ and $\mathcal{D}_{tp}^\infty(\infty)$ are infinite.*

**Proof** For a fixed $D < -2$, a member of each subfamily is the maximal hyperfractal $\Omega$ at infinity constructed in Theorem 14 for a specifically chosen sequence of parameters $a_i$. More precisely, to get a ‘proper’ $\infty$-quasiperiodic set at infinity, we have to choose $D \in (-3, -2)$. (See the discussion in the proof of Theorem 15.) We can generalize the proof of Theorem 16; that is, we take $(p_i)_{i \geq 1}$ be any increasing sequence of prime numbers. For the algebraically $\infty$-quasiperiodic set at infinity, we may choose $a_1 \in (0, 1/2)$ and define $a_{i+1} := a_1^{\sqrt{p_i}}$ for $i \geq 1$. Similarly as before, from [3], we easily conclude that the sequence of quasiperiods $T_i = \log(1/a_i)$, $i \geq 1$ is rationally independent.

On the other hand, for the transcendally $\infty$-quasiperiodic set at infinity, we may choose $a_i := 1/p_{i+1}$ for $i \geq 1$ and, similarly as before, [2, Theorem 2.1] assures that the sequence of quasiperiods $T_i = \log(1/a_i)$, $i \geq 1$ is algebraically independent. \(\square\)

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Appendix A: Some Technical Results

Here, we state and prove a result that is needed when constructing quasiperiodic sets at infinity in Sect. 5. Recall that the volume of the $N$-dimensional unit ball is given by

$$\omega_N := \frac{\pi^{N/2}}{\Gamma\left(\frac{N}{2} + 1\right)}$$

where $\Gamma$ denotes the Gamma function.

Lemma 18 Let $\Omega \subseteq \mathbb{R}^N$ with $|\Omega| < \infty$ be such that it is contained in a cylinder

$$x_1^2 + x_2^2 + \cdots + x_N^2 \leq C$$

for some constant $C > 0$ where $x = (x_1, \ldots, x_N)$. Then,

$$|t\Omega| = |K_t(0)^c \cap \Omega| + O(t^{-1}) \quad \text{as} \quad t \to +\infty,$$

where $K_t(0)$ is the ball of radius $t$ centered at 0 in the max norm, $|\cdot|_\infty$ on $\mathbb{R}^N$.

Proof We note that for $t$ sufficiently large the difference $|t\Omega| - |K_t(0)^c \cap \Omega|$ is less than the volume of the $N$-dimensional cylinder of height $h := t - \sqrt{t^2 - C^2}$ with base of radius $C$. In other words, we have that

$$\left| |t\Omega| - |K_t(0)^c \cap \Omega| \right| \leq h\omega_{N-1}C^{N-1} = \frac{\omega_{N-1}C^{N+1}}{t + \sqrt{t^2 + C^2}} = O(t^{-1}).$$

We also state here a simple and useful corollary of the complex mean value theorem [7, Theorem 2.2] that is needed in the proof of theorem 11.

Corollary 19 Let $f$ be a holomorphic function defined on an open convex subset $U_f$ of $\mathbb{C}$. Furthermore, let $a$ and $b$ be two distinct points in $U_f$. Then,

$$|f(b) - f(a)| \leq \sqrt{2}|b - a| \max_{s \in [a,b]} |f'(s)|,$$

where $[a, b]$ denotes the segment connecting $a$ and $b$.

Proof From [7, Theorem 2.2], we have that there are $s_1, s_2 \in (a, b)$ such that

$$\left| \frac{f(b) - f(a)}{b - a} \right|^2 = \left| \operatorname{Re}(f'(s_1)) \right|^2 + \left| \operatorname{Im}(f'(s_2)) \right|^2 \leq |f'(s_1)|^2 + |f'(s_2)|^2 \leq 2 \max_{s \in [a,b]} |f'(s)|^2.$$

Complete the proof by taking the square root of both sides and multiplying by $|b - a|$.

\footnotetext[6]{Here, we also use the obvious inequalities $|\operatorname{Re} s|, |\operatorname{Im} s| \leq |s|$ for $s \in \mathbb{C}$.}
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