YET ANOTHER BASIC ANALOGUE
OF GRAF’S ADDITION FORMULA

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Abstract. An identity involving basic Bessel functions and Al-Salam–Chihara polynomials is proved for which we recover Graf’s addition formula for the Bessel function as the base \( q \) tends to 1. The corresponding product formula is derived. Some known identities for Jackson’s \( q \)-Bessel functions are obtained as limiting cases. As special cases we prove identities for \( q \)-Charlier polynomials.

1. Introduction and formulation of results. A classical result for the Bessel function \( J_\nu(z) \) of order \( \nu \) and argument \( z \) defined by the absolutely convergent series

\[
J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{k! \Gamma(\nu+k+1)}
\]

is the addition formula

\[
J_\nu \left( \sqrt{x^2 + y^2 - 2xy \cos \psi} \right) \left( \frac{x - ye^{-i\psi}}{x - ye^{i\psi}} \right)^{\nu/2} = \sum_{m=-\infty}^{\infty} J_{\nu+m}(x)J_m(y)e^{im\psi},
\]

|ye^{\pm i\psi}| < |x|, due to Graf (1893), cf. [25, §11.3(1)], for general \( \nu \), and due to Neumann (1867) for \( \nu = 0 \), cf. [25, §11.2(1)]. In case \( \nu \in \mathbb{Z} \) the conditions on \( x, y \) in (1.1) can be removed. The corresponding product formula for the Bessel function is

\[
J_{\nu+m}(x)J_m(y) = \frac{1}{2\pi} \int_{0}^{2\pi} J_\nu \left( \sqrt{x^2 + y^2 - 2xy \cos \psi} \right) \left( \frac{x - ye^{-i\psi}}{x - ye^{i\psi}} \right)^{\nu/2} e^{-im\psi} d\psi.
\]
There are many $q$-analogues of the Bessel function and for several of these $q$-analogues of the Bessel function there exist identities which have the Graf addition formula (1.1) as the limit for $q \uparrow 1$. The structure of these identities may be very different from the structure of (1.1). These $q$-analogues of the Graf addition formula often follow from a certain interpretation of a $q$-Bessel function on quantum groups [14], [15], [22], or on quantum algebras [4], [9], [11], or from a certain generating function for a $q$-Bessel function [18], [13], a method closely related to the quantum algebra approach. The first two methods are motivated by the group theoretic proof of the Graf addition formula as given by Vilenkin and Klimyk [24, §4.1.4(2)].

It is the purpose of this note to give analytic proofs of a $q$-analogue of Graf’s addition formula, which was first obtained by a formal calculation using the quantum group of plane motions, and of the corresponding product formula. In the rest of this introduction we formulate the addition and product formula. In the next section we prove the product formula using a connection coefficient formula for the Al-Salam–Chihara polynomials. In §3 we derive the addition formula. In §4 we consider some special and limiting cases, and in particular the limit case $q \uparrow 1$, and we present some links with known results in this direction. In this section we also derive identities for $q$-Charlier polynomials as special cases. Finally, in §5 we remark very shortly on the link with the quantum group of plane motions.

In order to formulate the results we recall the notation for the $q$-shifted factorial;

\[
(a; q)_k = \prod_{i=0}^{k-1} (1 - aq^i), \quad (a_1, \ldots, a_r; q)_k = (a_1; q)_k \cdots (a_r; q)_k, \quad k \in \mathbb{Z}_+ \cup \{\infty\}
\]

and for the $q$-hypergeometric series

\[
\psi_{r,s} \left( \begin{array}{c} \alpha_1, \ldots, \alpha_r \\ \beta_1, \ldots, \beta_s \end{array} \right; q, z) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(q, \beta_1, \ldots, \beta_s; q)_k} \frac{(b_1, \ldots, b_s; q)_k}{(q, \beta_1, \ldots, \beta_s; q)_k} \frac{1}{z^k}.
\]

We always assume $0 < q < 1$. These notations follow the book [6] by Gasper and Rahman, which should be consulted for more information on this subject. For generic values of the parameters the region of convergence of this series is $\infty$, 1, or 0, according to $r < s + 1$, $r = s + 1$, or $r > s + 1$.

Note that a $q$-hypergeometric series with $q^{-n}$, $n \in \mathbb{Z}_+$, as one of the upper parameters terminates, since $(q^{-n}; q)_k = 0$ for $k > n$. If $q^{-n}$ occurs as a lower parameter, then the $q$-hypergeometric series is in general not well-defined. But for $n \in \mathbb{Z}_+$ we use the following convention

\[
(q^{1-n}; q)_\infty \sum_{k=0}^{\infty} \frac{c_k}{(q^{1-n}; q)_k} \frac{(1-n+k; q)_\infty}{(q; q)_k} = (q^{n+1}; q)_\infty \sum_{k=0}^{\infty} \frac{c_{k+n}}{(q^{1+n}; q)_k}.
\]

(This corrects the first identity of the remark following proposition 4.1 in [15].)
In order to formulate the main result of this paper we introduce the Al-Salam–Chihara polynomials;

$$(1.4) \quad S_n(\cos \theta; a, b \mid q) = a^{-n} (ab; q)_n \, \varphi_2 \left( \begin{array}{c} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0 \end{array} \right) ; q, q \right).$$

These polynomials were originally introduced by Al-Salam and Chihara [1] as the most general set of orthogonal polynomials satisfying a certain 'convolution property'. The orthogonality measure for these polynomials has been obtained by Askey and Ismail [2, §3], which is a special case of the more general four-parameter class of Askey-Wilson polynomials, cf. [3]. The definition (1.4) used here gives the Al-Salam–Chihara polynomials as Askey-Wilson polynomials with two parameters set to zero. The orthogonality relations for the Al-Salam–Chihara polynomials are given by, cf. Askey and Ismail [2, §3.8], Askey and Wilson [3, thm. 2.2],

$$\frac{1}{2\pi} \int_0^\pi (S_k S_l)(\cos \theta; a, b \mid q) \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}; q)_\infty} \, d\theta = \delta_{k,l} \frac{(q^{k+1}, abq^k; q)_\infty}{(q; q)_\infty},$$

assuming that $|a| < 1, |b| < 1$. The Al-Salam–Chihara polynomials are symmetric in the parameters $a$ and $b$, cf. [3, p. 6].

The main result of this paper is the following addition formula, valid for $|z| < 1, |a| < 1, |b| < 1$;

$$(1.6) \quad \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} 2\varphi_1 \left( \begin{array}{c} ae^{i\theta}, ae^{-i\theta} \\ q^{\nu+1} \end{array} ; q, z \right) S_m(\cos \theta; aq^{-\nu}, b \mid q) =$$

$$\sum_{n=-m}^\infty (-1)^n a^n z^n q^{(n+1)(n-1)/2} \frac{(q^{1+n}; q)_\infty}{(q; q)_\infty} 1\varphi_1 \left( \begin{array}{c} q^{-m} \\ q^{1+n} ; q, a^2 q^{m+n-\nu} \end{array} \right)$$

$$\times \frac{(q^{\nu+n+1}; q)_\infty}{(q; q)_\infty} 2\varphi_1 \left( \begin{array}{c} abq^{m+n}, 0 \\ q^{\nu+n+1} \end{array} ; q, z \right) S_{n+m}(\cos \theta; a, b \mid q).$$

Of course, this formula can also be considered as a linearisation formula for the product of a $2\varphi_1$-series and an Al-Salam–Chihara polynomial in terms of another set of Al-Salam–Chihara polynomials.

The corresponding product formula is

$$(1.7) \quad \frac{(q^{\nu+1}; q)_\infty}{2\pi(q; q)_\infty} \int_0^\pi 2\varphi_1 \left( \begin{array}{c} ae^{i\theta}, ae^{-i\theta} \\ q^{\nu+1} \end{array} ; q, z \right) S_m(\cos \theta; aq^{-\nu}, b \mid q)$$

$$\times S_{n+m}(\cos \theta; a, b \mid q) \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}; q)_\infty} \, d\theta$$

$$= \frac{(-1)^n a^n z^n q^{(n+1)(n-1)/2}}{(q^{n+m+1}, abq^{m+n}; q)_\infty} \left( \begin{array}{c} q^{1+n} ; q, \end{array} \right)$$

$$\times \frac{(q^{\nu+n+1}; q)_\infty}{(q; q)_\infty} 2\varphi_1 \left( \begin{array}{c} abq^{n+m}, 0 \\ q^{\nu+n+1} \end{array} ; q, z \right)$$

for $|z| < 1, |a| < 1, |b| < 1$. In this product formula we assume $|a| < 1, |b| < 1$, but a similar product formula remains true for any choice of $a$ and $b$ for which the Al-Salam–Chihara polynomials are orthogonal polynomials. In the general case only a finite number of discrete mass points have to be added, cf. Askey and Wilson [3, §2].
2. Proof of the product formula. In order to prove the product formula (1.7) we first prove two lemmas.

**Lemma 2.1.** For the Al-Salam–Chihara polynomials \( S_n(x; a, b | q) \) with \(|a| < 1, |b| < 1\) defined by (1.4), \( m, r \in \mathbb{Z}_+, n \in \mathbb{Z} \) with \( n \geq -m \), we have

\[
\frac{1}{2\pi} \int_0^\pi S_m(\cos \theta; aq^{-\nu}, b | q) S_{n+m}(\cos \theta; a, b | q) \frac{(ae^{i\theta}, ae^{-i\theta}; q)_r (e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}; q)_\infty} d\theta \\
= (-a)^{-n} q^{n(\nu+1)} q^{\frac{1}{2}n(n-1)} (q^{\nu+n+r+1}; q)_\infty (q^{\nu+r+1}; abq^{n+m+r}; q)_\infty \times (q^{-1-n}; q)_\infty 3\varphi_2 \left( q^{-r}, q^{-m-n}, q^{-\nu-n-r}; q, q^{r+1}a/b \right).
\]

**Proof.** We need the connection coefficients for two sets of Al-Salam–Chihara polynomials with one different parameter;

\[
S_n(x; \alpha, b | q) = \sum_{k=0}^n c_{k,n}(\alpha; a)S_k(x; a, b | q)
\]

with

\[
c_{k,n}(\alpha; a) = \frac{(q^{-n}; q)_k}{(q; q)_k} a^{-n-k} (-1)^k q^{nk - \frac{1}{2}k(k-1)(\alpha/a); q}_{n-k},
\]
given by Askey and Wilson [3, (6.4), (6.5) with \( c = d = 0 \)].

Now we start with the left hand side of the statement in the lemma. The term \( (ae^{i\theta}, ae^{-i\theta}; q)_r \) cancels part of the denominator of the weight function. Next use (2.1) to write both Al-Salam–Chihara polynomials in terms of Al-Salam–Chihara polynomials with parameters \( aq^r \) and \( b \). Then we can use (1.5) to see that the left hand side of the statement in the lemma equals

\[
\sum_{k=0}^m \sum_{l=0}^{n+m} \delta_{k,l} \frac{c_{k,m}(aq^{-\nu}; aq^r)c_{l,n+m}(a; aq^r)}{(q^{k+1}, abq^{k+r}; q)_\infty}.
\]

If we use \( k \) as the summation parameter, we can rewrite this as

\[
\frac{(q^{-\nu-r}; q)_m(q^{-r}; q)_{n+m}(aq^r)^{n+2m}}{(q, abq^r; q)_\infty} 3\varphi_2 \left( q^{-m}, q^{-n-m}, abq^r; q^{r+1}a^{-2} \right).
\]

View this as a terminating \( 3\varphi_2 \)-series of degree \( n+m \) to which we apply the series inversion, cf. [6, ex. 1.4(ii)],

\[
3\varphi_2 \left( q^{-p}, a, b; c, d | q \right) = \left( \frac{a, b; q}{b, c; q} \right)_p (-z)^p q^{-\frac{1}{2}p(p+1)} 3\varphi_2 \left( q^{-p}, q^{1-p}/c, q^{1-p}/d; q, cdq^{p+1} \right),
\]

\( p \in \mathbb{Z}_+, \) to obtain a \( 3\varphi_2 \)-series as in the lemma. Some manipulations with \( q \)-shifted factorials finish the proof of the lemma. \( \square \)

The orthogonality relations (1.5) for the Al-Salam–Chihara polynomials show that the integral in lemma 2.1 is zero for \( n > r \). Observe that the right hand side of lemma 2.1 also equals zero for \( n > r \) as follows from (1.3) with \( c_k = 0 \) for \( k > n \) in this case.

The following lemma is straightforward and its proof is left to the reader.
Lemma 2.2. For $|dz| < 1$ we have

$$\left(\frac{q^{\mu+1}; q}{q; q}\right)_{\infty} 1\varphi_1\left(\frac{a}{q^{\mu+1}; q, b z} \right) \left(\frac{q^{\nu+1}; q}{q; q}\right)_{\infty} 2\varphi_1\left(\frac{c, 0}{q^{\nu+1}; q, dz}\right) =$$

$$\left(\frac{q^{\nu+1}; q}{q; q}\right)_{\infty} \sum_{p=0}^{\infty} \frac{(dz)^p (c; q)_p}{(q, q^{\nu+1}; q)_p} \left(\frac{q^{\nu+1}; q}{q; q}\right)_{\infty} 3\varphi_2\left(\frac{q^{-p}, q^{-p-n}, a, b q^{\nu+p+1}}{q^{\mu+1}, q^{1-p}/c; q, d c}\right)$$

the last series being absolutely convergent.

If we take $a = 0$ in lemma 2.2 and we replace $d$ by $d/c$ and we let $c \to \infty$, then we essentially obtain the product formula for the Hahn-Exton $q$-Bessel function, cf. Swarttouw [21, (3.1)]. If we take $c = 0$ in lemma 2.2 and we replace $b$ by $b/a$ before taking $a \to \infty$, then we obtain the product formula for Jackson’s $q$-Bessel functions, cf. Rahman [20, (2.1)].

The proof of the product formula (1.7) can now be given. Use the series representation for the $2\varphi_1$-series to see that for $|z| < 1$

$$\left(\frac{q^{\nu+1}; q}{q; q}\right)_{\infty} \int_0^\pi 2\varphi_1\left(\frac{(ae^{i\theta}, ae^{-i\theta}; q^{\nu+1}}{q}; z\right) S_m(\cos \theta; a q^{-\nu}, b | q)$$

$$\times S_{n+m}(\cos \theta; a, b | q) \left(\frac{\cos 2i\theta, -\cos 2i\theta; q}{ae^{i\theta}, -ae^{-i\theta}; q}\right)_{\infty} d \theta$$

$$= \frac{(-a)^{-n} q^{n(\nu+1)} q^{\frac{1}{2} n(n-1)}}{(q^{m+1}, ab q^{n+m}; q)_{\infty}} \left(\frac{q^{\nu+n+1}; q}{q; q}\right)_{\infty} \sum_{r=0}^{\infty} z^r \left(\frac{ab q^{n+m}}{q^{1-n}}, q\right)_r$$

$$\times \left(\frac{q^{1-n}; q}{q; q}\right)_{\infty} 3\varphi_2\left(\frac{q^{-r}, q^{-m-n}, q^{-\nu-n-r}}{q^{1-n}, q^{1-m-n-r}/(ab); q, q^{1+r} q^{-1}/b}\right)$$

by lemma 2.1 and some straightforward simplifications. Interchanging summation and integration is allowed for $|z| < 1$ by use of the estimate $|(ae^{i\theta}, ae^{-i\theta}; q)_r| \leq (-|a| q)_{\infty}$. By lemma 2.2 this expression is equal to

$$\frac{(-a)^{-n} q^{n(\nu+1)} q^{\frac{1}{2} n(n-1)}}{(q^{m+1}, ab q^{n+m}; q)_{\infty}} \left(\frac{q^{1-n}; q}{q; q}\right)_{\infty} 1\varphi_1\left(\frac{q^{-n-m}}{q^{1-n}} ; q, a^2 q^{m-\nu} z\right)$$

$$\times \left(\frac{q^{\nu+n+1}; q}{q; q}\right)_{\infty} 2\varphi_1\left(\frac{ab q^{n+m}}{q^{\nu+n+1}} , 0 ; q, z\right)$$

Now use that (1.3) implies that

$$\left(\frac{q^{1-n}; q}{q; q}\right)_{\infty} 1\varphi_1\left(\frac{a}{q^{1-n}}; q, z\right) = z^n (-1)^n q^{\frac{1}{2} n(n-1)} \left(\frac{a q^{1+n}; q}{aq^n, q; q}\right)_{\infty} 1\varphi_1\left(\frac{a q^n}{q^{1+n}}; q, z q^n\right),$$

which holds for $n \in \mathbb{Z}$. This finishes the proof of the product formula (1.7).
3. Proof of the addition formula. The hard work for the proof of the addition formula (1.6) has been done in the previous section. Let \( L^2((-1, 1), w(x)dx) \) denote the space of quadratically integrable functions on \((-1, 1)\) with respect to the weight function \( w(x) \) defined by

\[
w(\cos \theta) = \frac{1}{2\pi \sin \theta} \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}; q)_\infty}.
\]

The Al-Salam–Chihara polynomials \( S_n(x; a, b \mid q) \) form a basis for this \( L^2\)-space.

From the estimate \(|(ae^{i\theta}, ae^{-i\theta}; q)_r| \leq (-|a|; q)_\infty^2\) we obtain that the left hand side of (1.6) as a function of \( x = \cos \theta \) is an element of this \( L^2\)-space. Consequently, we may develop it in the basis of Al-Salam–Chihara polynomials

\[
\frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} 2\varphi_1 \left( \frac{ae^{i\theta}, ae^{-i\theta}}{q^{\nu+1}} ; q, z \right) S_m(\cos \theta; aq^{-\nu}, b \mid q) = \sum_{n=-m}^{\infty} A_n S_{n+m}(\cos \theta; a, b \mid q)
\]

with

\[
A_n \int_{-1}^{1} \left( S_{n+m}(x; a, b \mid q) \right)^2 w(x) \, dx = \frac{(q^{\nu+1}; q)_\infty}{2\pi (q; q)_\infty} \int_{0\pi} 2\varphi_1 \left( \frac{ae^{i\theta}, ae^{-i\theta}}{q^{\nu+1}} ; q, z \right) \times S_m(\cos \theta; aq^{-\nu}, b \mid q) S_{n+m}(\cos \theta; a, b \mid q) \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}; q)_\infty} \, d\theta.
\]

Now use the orthogonality relations for the Al-Salam–Chihara polynomials (1.5) and the product formula (1.7) to find the correct value for \( A_n \) as in the addition formula (1.6). This proves the addition formula (1.6) as an identity in \( L^2((-1, 1), w(x)dx) \).

The left hand side of (1.6) is a continuous function of \( \cos \theta \), so it suffices to show that the right hand side is continuous as well. For this we have to show that the convergence of the right hand side is uniform with respect to \( \cos \theta \). This follows from the estimates

\[
\left| \frac{(q^{1+n}; q)_\infty}{(q; q)_\infty} \varphi_1 \left( \frac{q^{-m}}{q^{1+n}} ; q, a^2 q^{m+n-\nu} z \right) \right| \leq (-|a^2 z| q^{m-\nu}; q)_\infty
\]

and the asymptotic behaviour of the Al-Salam–Chihara polynomials given by

\[
S_n(\frac{1}{2} (\xi + \xi^{-1}); a, b \mid q) = \xi^{-n} A(\xi) + \mathcal{O}(\xi^{-n}), \quad n \to \infty, \ |\xi| < 1,
\]

with \( A(\xi) = (a\xi, b\xi; q)_\infty / (\xi^2; q)_\infty \) off the spectrum and by

\[
S_n(\cos \theta; a, b \mid q) = 2|A(e^{i\theta})| \cos(n\theta - \phi) + \mathcal{O}(q^{n/2}), \quad n \to \infty, \ 0 < \theta < \pi, \ \phi = \arg A(e^{i\theta}),
\]

on the spectrum, cf. Askey and Ismail [2, §3.1], Ismail and Wilson [7, (1.11), (1.13), §3].
4. Special and limiting cases. The first special case of interest of the addition and product formula is the case \( m = 0 \), which gives the decomposition of the \( 2\varphi_1 \)-series involved in terms of Al-Salam–Chihara polynomials. Some other special and limiting cases are described in the rest of this section.

The Koornwinder-Swarttouw addition formula as a limit case. Formally we can obtain the \( q \)-analogue of Graf’s addition formula for the Jackson \( q \)-Bessel function derived by Koornwinder and Swarttouw [18, (4.10)] as a special case of the addition formula (1.6) by use of the asymptotic behaviour of the Al-Salam–Chihara polynomials off the spectrum, cf. (3.1), as follows. Let \( m \to \infty \) in (1.6) and use that formally

\[
\lim_{m \to \infty} \frac{S_{n+m}(\frac{1}{2}(\xi + \xi^{-1}); a, b \mid q)}{S_{m}(\frac{1}{2}(\xi + \xi^{-1}); aq^{-\nu}, b \mid q)} = \xi^{-n} \frac{(a\xi; q)_\infty}{(aq^{-\nu}\xi; q)_\infty}, \quad |\xi| < 1,
\]

by (3.1). Replace in the resulting formula \( z, a, \) and \( \xi \) by \(-y^2, q^{\nu/2}x/y,\) and \( q^{\nu/2}/s\) to obtain formally

\[
(4.1) \quad y^{\nu}(xy^{-1}s^{-1}, q^{n+1}; q)_\infty 2\varphi_1 \left( \frac{q^{\nu}xy^{-1}s^{-1}, xy^{-1}s}{q^{\nu+1}}, q; -y^2 \right) = \sum_{n=-\infty}^{\infty} s^n y^{\nu+n} \times \frac{(q^{\nu+n+1}; q)_\infty}{(q; q)_\infty} 2\varphi_1 \left( \frac{0, 0}{q^{\nu+n+1}; q, -y^2}, x^n q^{\frac{n(n-1)}{2}} \frac{(q^{n+1}; q)_\infty}{(q; q)_\infty} 0\varphi_1 \left( \frac{q^{n+1}; q, -x^2 q^n}{q^{\nu+1}; q, -y^2} \right) \right),
\]

which has been proved rigorously by Koornwinder and Swarttouw [18, (4.10)] using generating function techniques for \( \nu \in \mathbb{Z} \).

The limit \( q \uparrow 1 \). In (4.1) we replace \( x \) and \( y \) by \((1 - q)x \) and \((1 - q)y \) before we take the limit \( q \uparrow 1 \). If we use the \( q \)-gamma function \( \Gamma_q(x) = (q; q)_\infty (q^{x}; q)_\infty^{-1}(1 - q)^{1-x} \), cf. [6, §1.10], and the limit relation \( \Gamma_q(x) \to \Gamma(x) \) as \( q \uparrow 1 \), we see that (4.1) goes over into Graf’s addition formula (1.1). Since (4.1) is a limiting case of (1.6), we have shown that (1.6) is a \( q \)-analogue of Graf’s addition formula (1.1).

It is also possible to use the techniques of Van Assche and Koornwinder [23, thm. 1] to treat the limit case \( q \uparrow 1 \) of the addition formula (1.6) to Graf’s addition formula (1.1). To this end observe that from (2.1) we have

\[
\frac{S_{n+m}(x; a, b \mid q)}{S_{m}(x; aq^{-\nu}, b \mid q)} = \sum_{k=0}^{n+m} \frac{(q^{\nu}; q)_k}{(q; q)_k} (aq^{-\nu})^k (q^{m+n-k+1}; q)_k \frac{S_{n+m-k}(x; aq^{-\nu}, b \mid q)}{S_{m}(x; aq^{-\nu}, b \mid q)}.
\]

In the right hand side we replace \( q = e^{1/m}, c \in (0, 1) \), and let \( m \to \infty \). Then [23, thm. 1] can be used to evaluate this limit and the rest of the limit transition is a straightforward exercise using the binomial formula. See also [17, §4] for the details of a similar limit transition.

The limit transition of the product formula (1.7) to the product formula (1.2) for the Bessel function as \( q \uparrow 1 \) is treated by use of theorem 2 of Van Assche en Koornwinder [23]. This time we have to use the connection coefficient formula (2.1) in the form

\[
S_{m}(x; aq^{-\nu}, b \mid q) = \sum_{k=0}^{m} \frac{(q^{\nu}; q)_k}{(q; q)_k} a^k (q^{m-k+1}; q)_k S_{m-k}(x; a, b \mid q).
\]
before replacing \( q = e^{1/m}, c \in (0, 1) \), and letting \( m \to \infty \). Invoking [23, thm. 2] and the binomial theorem shows that (1.7) tends (1.2). See also [17, §5] for the details of a similar limit transition.

**Orthogonality relations for \( q \)-Charlier polynomials.** The addition formula (1.6) is a \( q \)-analogue of Graf’s addition formula for the Bessel function from which the Hansen-Lommel orthogonality relations for the Bessel functions, \( \sum_{k=-\infty}^{\infty} J_n(z)J_{n+p}(z) = \delta_{0,p}, p \in \mathbb{Z}, z \in \mathbb{C} \), can be derived, cf. Watson [25, §§2.4, 2.5, 11.2, 11.3]. Here we can also specialise the parameters to obtain orthogonality relations from the addition formula. Take \( a = b = e^{i\theta} = q^{\frac{1}{2}}, \nu = p \in \mathbb{Z} \) and observe that

\[
\frac{(q^{p+1};q)_\infty}{(q;q)_\infty} 2\varphi_1 \left( \begin{array}{c} 1, q \\ q^{p+1}; q, z \end{array} \right) = \begin{cases} 0, & p < 0, \\ (q;q)_p^{-1}, & p \geq 0, \end{cases}
\]

\[
S_m \left( \frac{1}{2} (q^{\frac{1}{2}} + q^{-\frac{1}{2}}); q^{\frac{1}{2}} - p, q^{\frac{1}{2}} | q \right) = q^{-\frac{1}{4}m} (1^{-p}; q)_m = \begin{cases} (q;q)_m q^{-\frac{1}{4}m}, & p = 0, \\ 0, & 0 < p \leq m, \\ (q^{1-p}; q)_m q^{-\frac{1}{4}m}, & p > m, \end{cases}
\]

\[
S_{n+m} \left( \frac{1}{2} (q^{\frac{1}{2}} + q^{-\frac{1}{2}}); q^{\frac{1}{2}}, q^{\frac{1}{2}} | q \right) = q^{-\frac{1}{4}(n+m)} (q;q)_{n+m}
\]

to find for \( p \leq m \) the orthogonality relations

\[
\delta_{0,p}(q; q)_m = \lim_{n \to -m} (-z)^n q^{\frac{1}{2}(n-1)} (q; q)_{n+m} \frac{(q^{n+1};q)_\infty}{(q;q)_\infty} 1\varphi_1 \left( \begin{array}{c} q^{-m} \\ q^{n+1}; q, 1+m+n-p \end{array} \right) 
\times \frac{(q^{n+p+1};q)_\infty}{(q;q)_\infty} 2\varphi_1 \left( \begin{array}{c} q^{n+m+1}, 0 \\ q^{n+p+1}; q, z \end{array} \right).
\]

Replace \( z \) by \( -z^2 q^p/4 \) and let \( m \to \infty \) in (4.2) to find Hansen-Lommel orthogonality relations for the Jackson \( q \)-Bessel function, cf. [12, thm. 3.1], which can also be obtained from the Koornwinder-Swarttouw \( q \)-analogue of Graf’s addition formula (4.1), cf. [12, rem. 1, p. 432].

In the orthogonality relations (4.2) we use the limiting case \( b \to 0 \) of Heine’s transformation formula [6, (1.4.6)], cf.

\[
2\varphi_1 \left( \begin{array}{c} a, 0 \\ c, z \end{array} \right) = \frac{1}{(z;q)_\infty} 1\varphi_1 \left( \begin{array}{c} c/a \\ c, az \end{array} \right).
\]

Next we replace \( n, p \) and \( z \) by \( h - m, m - r, -aq^{-r} \) to see that (4.2) is equivalent to the orthogonality relations for the \( q \)-Charlier polynomials, cf. [6, ex. 7.13, with the squared norm replaced by its reciprocal],

\[
\sum_{h=0}^{\infty} \frac{c^h q^{\frac{1}{2}(h-1)}}{(q;q)_h} (c_m c_r) (q^{-h}; a; q) = \delta_{m,r} q^{-m} (-a^{-1} q; q)_m (-a; q)_\infty
\]
where the \( q \)-Charlier polynomials are defined by

\[
c_m(x; a; q) = 2 \varphi_1 \left( \frac{q^{-m} x}{a}; q, -\frac{q^{m+1}}{a} \right) = (-a)^{-m} q^{m^2} x^m (x^{-1} q^{1-m}; q)_m 1 \varphi_1 \left( \frac{q^{-m}}{x^{-1} q^{1-m}; q, -\frac{aq^{1-m}}{x}} \right).
\]

The last equality follows by series inversion. So the \( q \)-Charlier polynomials are related to Moak’s \( q \)-Laguerre polynomials \([19]\) by

\[
c_m(q^{-\alpha-m}; a; q) = (-aq^\alpha)^{-m} (q; q)_m L^{(\alpha)}_m (aq^m; q),
\]

\[L^{(\alpha)}_n (x; q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} 1 \varphi_1 \left( \frac{q^{-n}}{q^{\alpha+1}; q, -x q^{\alpha+n+1}} \right).
\]

The orthogonality relations (4.4) for the \( q \)-Charlier polynomials have been obtained from the quantum algebra approach by Kalnins, Miller and Mukherjee \([10, (3.2)]\) and Floreanini and Vinet \([5, (59)]\) using representations of the \( q \)-oscillator algebra. The orthogonality relations are also a byproduct of the quantum group theoretic proof of an addition formula for the big \( q \)-Legendre polynomial, cf. \([16, \text{cor. 4.2}]\). In the limit case \( m \to \infty \) of (4.2) we know that the dual orthogonality relations also hold, cf. \([12, \text{thm. 3.3}]\), but the orthogonality relations dual to the orthogonality relations (4.4) for the \( q \)-Charlier polynomials do not hold, cf. \([16, \text{prop. 4.1, cor. 4.2}]\). This is not correct in \([5, (60)]\).

More identities for \( q \)-Charlier polynomials. The transformation formula (4.3) can also be applied in the general addition formula (1.6). If we next replace \( a, b, z, n, \) and \( \nu \) by \( q^{\frac{1}{2}(\mu+1)} \sqrt{\alpha/\beta}, q^{\frac{1}{2}(\mu+1)} \sqrt{\beta/\alpha}, -\beta q^{-r}, h-m, \) and \( m-r+\mu, \) we get the following extension of the orthogonality relations for the \( q \)-Charlier polynomials;

\[
\sum_{h=0}^{\infty} \frac{(\alpha \beta)^{\frac{1}{2} h} q^{\frac{1}{2} h(\mu+1)} q^{\frac{1}{2} h(h-1)} (q^{1+h+\mu}; q)_\infty}{(q; q)_h} c_m (q^{-h}; \alpha ; q) c_r (q^{-h-\mu}; \beta ; q) \\
\times S_h (\cos \theta; q^{\frac{1}{2}(\mu+1)} \sqrt{\alpha/\beta}, q^{\frac{1}{2}(\mu+1)} \sqrt{\beta/\alpha} \mid q) = (-1)^{m+r} q^{\frac{1}{2} h(m+\mu)} q^{r(m-\mu)} \alpha^{-\frac{1}{2} m} \beta^{\frac{1}{2} m-r} \\
\times (-\beta q^{-r}; q) \infty \frac{(q^{1+m-r+\mu}; q)_\infty}{(q; q)_\infty} 2 \varphi_1 \left( q^{\frac{1}{2}(\mu+1)} \sqrt{\alpha/\beta} e^{i \theta}, q^{\frac{1}{2}(\mu+1)} \sqrt{\beta/\alpha} e^{-i \theta} \mid q \right) \\
\times S_m (\cos \theta; q^{r-m+\frac{1}{2}(1-\mu)} \sqrt{\alpha/\beta}, q^{\frac{1}{2}(\mu+1)} \sqrt{\beta/\alpha} \mid q).
\]

Using (4.5) we can also rewrite this as an identity for \( q \)-Laguerre polynomials, which then gives an alternative for the addition formulas for \( q \)-Laguerre polynomials derived by Kalnins, Manocha and Miller \([8, (7.14)]\) and Kalnins and Miller \([9, (4.13)]\) using representations of the \( q \)-oscillator algebra.

For particular choices of \( \cos \theta \) it is possible to evaluate the Al-Salam–Chihara polynomials, which then simplifies the formula for the \( q \)-Charlier polynomials. In particular, for
\[ e^{i\theta} = q^{\frac{1}{2}(\mu+1)} \sqrt{\beta/\alpha} \] the Al-Salam–Chihara polynomials reduce to \( q \)-shifted factorials as in the proof of (4.2) and we obtain

\[
\sum_{h=0}^{\infty} \frac{\alpha^h q^{\frac{1}{2}h(h-1)}}{(q; q)_h} c_m(q^{-h}; \alpha; q)c_r(q^{-h-\mu}; \beta; q) = (-1)^{m+r} q^{\frac{1}{2}m(m-1)} q^{r(m-\mu)} \beta^{-r} \\
\times (q^{1+r-m}; q)_m(-\beta q^{-r}; q)_{\infty} \left( \frac{q^{1+m-r+\mu}; q}_{q^{1+\mu}; q}_{\infty} \right) 2\varphi_1 \left( \frac{q^{\mu+1}, \alpha/\beta}{q^{1+m-r+\mu}; q, -\beta q^{-r}} \right)
\]

as a nice extension of the orthogonality relations of the \( q \)-Charlier polynomials.

5. Concluding remark. As already said in the introduction, the addition formula (1.6) is originally derived from the interpretation of certain \( q \)-Bessel functions on the quantum group of plane motions. Actually, the addition formula follows from the identity [15, (4.6) with \( r = \infty \)], which reflects the homomorphism property of a representation of a group which classically gives addition formulas if the matrix elements are known in terms of special functions. As indicated in remark 4.3 of [15] the elements in this identity are explicitly (but only on a formal level) known in terms of a certain non-commutative algebra \( A_q \). Then we apply the infinite dimensional \( \tau \otimes \tau \) as defined in [14, §5] to [15, (4.6) with \( r = \infty \)] to obtain an operator identity in \( \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}) \). By letting it act on suitable vectors in the representation space we can convert the operator identity to the addition formula (1.6) for special values of \( a, b \) and \( \nu \).

A certain \( q \)-analogue of Graf’s addition formula for the Hahn-Exton \( q \)-Bessel function can also be proved from a similar quantum group theoretic interpretation, cf. [14, §6]. (See [11, §3] for a quantum algebra theoretic proof and [17] for an analytic proof as well as for the limit case \( q \uparrow 1 \).) In this case the starting point is [14, (6.1)], which can be considered as the limit case \( s, t \to \infty \) of the starting point [15, (4.6) with \( r = \infty \)] for the addition formula in this paper. So it is tempting to think that the \( q \)-analogue of Graf’s addition formula for the Hahn-Exton \( q \)-Bessel functions ([14, thm. 6.3], [11, §3], [17, (1.4)]) can be obtained as a limiting case of the addition formula (1.6). However, I have not been able to show this.

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