Interaction via Reduction and Nonlinear Superconformal Symmetry

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Abstract

We show that the reduction of a planar free spin-$\frac{1}{2}$ particle system by the constraint fixing its total angular momentum produces the one-dimensional Akulov-Pashnev-Fubini-Rabinovici superconformal mechanics model with the nontrivially coupled boson and fermion degrees of freedom. The modification of the constraint by including the particle’s spin with the relative weight $n \in \mathbb{N}$, $n > 1$, and subsequent application of the Dirac reduction procedure (‘first quantize and then reduce’) give rise to the anomaly free quantum system with the order $n$ nonlinear superconformal symmetry constructed recently in [hep-th/0304257]. We establish the origin of the quantum corrections to the integrals of motion generating the nonlinear superconformal algebra, and fix completely its form.

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1 Introduction

The superconformal mechanics was introduced twenty years ago by Akulov and Pashnev \cite{1}, and by Fubini and Rabinovici \cite{2} as a supersymmetric analog of the conformal mechanics of De Alfaro, Fubini and Furlan \cite{3}. It was examined in different aspects in \cite{4, 5}, and nowadays the interest to the conformal and superconformal mechanics is related mainly to the AdS/CFT correspondence conjecture and to the integrable models \cite{6}-\cite{16}.

Recently, the superconformal mechanics model \cite{1, 2} was generalized in \cite{17} following the ideas of nonlinear supersymmetry \cite{18}-\cite{25}. In comparison with the original model \cite{1, 2}, the model \cite{17} is characterized at the classical level by the $n$-fold fermion-boson coupling constant, that gives rise to a radical change of the symmetry: instead of the $osp(2|2)$ Lie superalgebraic structure, the modified system possesses the order $n$ nonlinear superconformal symmetry. The latter is generated by the boson integrals which form, as in the linear $osp(2|2)$ case, the Lie subalgebra $so(1, 2) \oplus u(1)$, while the set of $2(n + 1)$ fermion integrals of motion anticommutes for the order $n$ polynomials of the even integrals. The two essential moments of the construction left, however, unclarified. First, though the quantum analogs of the integrals of motion were found from the requirement of preservation of the symmetry at the quantum level, the origin of the quantum corrections to the integrals remained to be completely unclear. Second, the quantum analog of the classical nonlinear superconformal symmetry algebra was determined entirely only for the simplest case of $n = 2$, whereas for the general case of $n \in \mathbb{N}$ only a part of the anticommutators of the odd integrals was fixed.

In the present paper, we shall clarify the origin of the quantum corrections to the integrals of motion of the model \cite{17}, and shall fix completely the form of the quantum nonlinear superconformal algebra of an arbitrary order $n$. This will be done by the reduction of a planar free spin-$\frac{1}{2}$ particle system to the surface of the constraint fixing linearly the orbital angular momentum in terms of the spin.

The paper is organized as follows. In Section 2 we first show how the one-dimensional model \cite{1, 2} with the nontrivial boson-fermion coupling can be obtained via reduction from the planar system of a free nonrelativistic particle in which spin and translation degrees of freedom are completely decoupled. In Section 3 we consider a modification of the reduction procedure which leads to the generalized model \cite{17}, and fix the form of the order $n$ nonlinear superconformal algebra. In section 4 we shortly summarize the results and discuss possible applications and generalizations of the proposed method of introducing the boson-fermion interaction via a reduction procedure.

2 Superconformal symmetry: linear case

Let us consider a free nonrelativistic spin-$\frac{1}{2}$ particle on the plane ($i = 1, 2$),

$$A = \int L_0 dt, \quad L_0 = \frac{1}{2} \dot{x}_i^2 - \frac{i}{2} \dot{\xi}_i \xi_i. \quad (2.1)$$

Its Hamiltonian coincides with that of a free 2D nonrelativistic spinless particle, $H = \frac{1}{2} p_i^2$, and so,

$$I = (X_i \equiv x_i - p_i t, \quad p_i, \quad \xi_i) \quad (2.2)$$
is the set of the integrals of motion,
\[
\frac{d}{dt}I = \frac{\partial}{\partial t}I + \{I, H\} = 0,
\] (2.3)
linear in the phase space variables of the system. The integrals (2.2) form a superextended
2D Heisenberg algebra, \(\{X_i, p_j\} = \delta_{ij}, \{\xi_i, \xi_j\} = -i\delta_{ij}\). The even,
\[
H = \frac{1}{2}p_i^2, \quad K = \frac{1}{2}X_i^2, \quad D = \frac{1}{2}X_ip_i, \quad \tag{2.4}
\]
\[
L = \epsilon_{ij}X_ip_j, \quad \Sigma = -\frac{i}{2}\epsilon_{ij}\xi_i\xi_j, \quad \tag{2.5}
\]
and odd,
\[
Q_1 = p_i\xi_i, \quad Q_2 = \epsilon_{ij}p_i\xi_j, \quad S_1 = X_i\xi_i, \quad S_2 = \epsilon_{ij}X_i\xi_j, \quad \tag{2.6}
\]
quadratic combinations of (2.2) generate the following \(Z_2\)-graded Lie algebra (only the non-
trivial Poisson bracket relations are displayed):
\[
\{D, H\} = H, \quad \{D, K\} = -K, \quad \{K, H\} = 2D, \quad \tag{2.7}
\]
\[
\{D, Q_a\} = \frac{1}{2}Q_a, \quad \{D, S_a\} = -\frac{1}{2}S_a, \quad \tag{2.8}
\]
\[
\{H, S_a\} = -Q_a, \quad \{K, Q_a\} = S_a, \quad \tag{2.9}
\]
\[
\{L, Q_a\} = -\epsilon_{ab}Q_b, \quad \{L, S_a\} = -\epsilon_{ab}S_b, \quad \{\Sigma, Q_a\} = \epsilon_{ab}Q_b, \quad \{\Sigma, S_a\} = \epsilon_{ab}S_b, \quad \tag{2.10}
\]
\[
\{Q_a, Q_b\} = -i\delta_{ab}2H, \quad \{S_a, S_b\} = -i\delta_{ab}2K, \quad \tag{2.11}
\]
\[
\{Q_a, S_b\} = -i\delta_{ab}2D - i\epsilon_{ab}(L + 2\Sigma). \quad \tag{2.12}
\]
The total angular momentum, \(J = L + \Sigma\), commutes with all these quadratic scalar integrals, and (2.7)–(2.12) is identified as the \(osp(2|2) \oplus u(1)\) superalgebra with the \(u(1)\) corresponding
to the centre \(J\). The non-Abelian part of the bosonic subalgebra \(so(1, 2) \oplus u(1)\) of the
conformal superalgebra \(osp(2|2) \cong su(1, 1|1)\) is generated here by the integrals (2.4), and its
Abelian \(u(1)\) subalgebra is associated with the linear combination \(\Sigma + J\).

The boson and fermion degrees of freedom in the system (2.1) are completely decoupled. The interaction between them can be introduced without violating the superconformal
symmetry \(osp(2|2)\) in the following manner. Since \(J\) is the centre, it can be fixed without
changing the structure of the superalgebra (2.7)–(2.12). Let us make this by introducing the
classical constraint
\[
J_1 \equiv L + \Sigma - \alpha \approx 0, \quad \tag{2.13}
\]
where \(\alpha\) is a real parameter. The physical variables are those which commute (in the sense of
the Poisson brackets) with the constraint, and they can immediately be identified with the
Having in mind the quantization and further generalization, it is more convenient first to pass over to the polar coordinates and then to identify the observables.

Let us introduce the two orthonormal vectors

\[ n_i^{(1)} = (\cos \varphi, \sin \varphi), \quad n_i^{(2)} = -\epsilon_{ij} n_j^{(1)} = (- \sin \varphi, \cos \varphi), \quad (2.14) \]

in terms of which

\[ x_i = r n_i^{(1)}, \quad p_i = p_r n_i^{(1)} + r^{-1} L n_i^{(2)}, \quad \xi_i = \xi^{(a)} n_i^{(a)}. \quad (2.15) \]

The transformation (2.14), (2.15) to the new variables is canonical,

\[ \{r, p_r\} = 1, \quad \{\varphi, L\} = 1, \quad \{\xi^{(a)}, \xi^{(b)}\} = -i\delta^{ab}, \]

but it is well defined only for \( r = \sqrt{x_i^2} > 0 \). This will be essential for the quantum theory. The scalar variables

\[ q \equiv r, \quad p \equiv p_r, \quad \psi_a \equiv \xi^{(a)} \quad (2.16) \]

commute with the constraint (2.13), and any function of them is an observable. On the surface of the constraint (2.13), the orbital angular momentum is given in terms of \( \Sigma = -i\psi_1 \psi_2, \ L = \alpha - \Sigma \). Then the reduction of the Hamiltonian \( H = \frac{1}{2}(p_r^2 + r^{-2} L^2) \) to the surface (2.13) produces the nontrivial boson-fermion interaction in the resulting one-dimensional system:

\[ H = \frac{1}{2} \left( p^2 + \frac{1}{q^2} \alpha (\alpha + i\psi_1 \psi_2) \right). \quad (2.17) \]

The Hamiltonian (2.17) coincides with that of the classical superconformal model [1, 2], and the quantities \( H, K, D, \Sigma + \alpha, Q_a \) and \( S_a \) being rewritten in terms of observables (2.16), take the form of the generators of the \( osp(2|2) \) superconformal symmetry for the system (2.17) (see below).

The direct quantization of the classical one-dimensional system (2.17) on the half-line \( q > 0 \) reproduces the quantum superconformal model [1, 2]. However, having in mind that the two procedures — ‘first reduce and then quantize’ and ‘first quantize and then reduce’ — generally give different results [26], let us consider shortly the latter procedure. It is this method that will reproduce all the quantum corrections in the quantum analogs of the classical integrals necessary for preserving the nonlinear superconformal symmetry. In ref. [17] the corresponding corrections were introduced by hands, and so, their origin was unclear.

In terms of the polar coordinates the scalar product is

\[ (\Psi_1, \Psi_2) = \int_0^\infty r dr \int_0^{2\pi} \Psi_1^*(r, \varphi) \Psi_2(r, \varphi) d\varphi. \quad (2.18) \]

Here \( \Psi(r, \varphi) \) is a two-component (spinor) wave function, on which the quantum analogs of the Grassmann variables \( \psi_a \) act as the Pauli matrices:

\[ \hat{\psi}_a = \sqrt{\frac{\hbar}{2}} \sigma_a, \quad a = 1, 2. \]
In what follows we put $\hbar = 1$. With respect to the scalar product (2.18) the operators
\[
\hat{p}_r = -i \frac{\partial}{\partial r} - \frac{i}{2r}, \quad \hat{L} = \epsilon_{ij}x_ip_j = -i \frac{\partial}{\partial \varphi}
\]  
(2.19)
are Hermitian. The quantum analog of the constraint (2.13) specifies the physical subspace of the system:
\[
\left( \hat{L} + \frac{1}{2}\sigma_3 - \alpha \right) \Psi_{phys} = 0.
\]  
(2.20)
In (2.20) the second term corresponds to the particle’s spin $\hat{\Sigma} = -\frac{i}{2} [\hat{\psi}_1, \hat{\psi}_2]$. Taking into account the $2\pi$-periodicity of the wave functions, $\Psi(r, \varphi) = \sum_{l=-\infty}^{+\infty} \Phi_l(r)e^{il\varphi}$, we find that eq. (2.20) has non-trivial solution of the form
\[
\Psi_{phys}(r, \varphi) = \begin{pmatrix} \Phi_+(r) \\ \Phi_-(r)e^{i\varphi} \end{pmatrix} e^{ik\varphi}
\]
only when the parameter $\alpha$ takes a half-integer value,
\[
\alpha = k + \frac{1}{2},
\]  
(2.21)
where $k$ is a fixed integer number, $k \in \mathbb{Z}$. The redefinition of the radial wave functions according to $\Phi(r) \rightarrow \phi(r) = \sqrt{2\pi r}\Phi(r)$, and integration in the angular variable reduce (2.18) to the scalar product on the half-line,
\[
(\phi, \phi') = \int_0^{\infty} \phi^*(q)\phi'(q)dq.
\]  
(2.22)
The spinor wave functions $\phi(q)$ are subject now to the boundary condition $\phi(q)|_{q \rightarrow 0} = 0$, and the action of the operator $\hat{p}_r$ is reduced on them to the operator $\hat{p} = -i\partial/\partial q$. Having in mind the quantum relations
\[
\hat{p}_r^2 = \hat{p}_r^2 + \frac{1}{r^2}\left( \hat{L}^2 - \frac{1}{4} \right)
\]  
(2.23)
and (2.20), the reduced quantum Hamiltonian takes the form of the Hamiltonian of the one-dimensional quantum superconformal mechanics model [1, 2],
\[
\hat{H} = \frac{1}{2} \left( -\frac{\partial^2}{\partial q^2} + \frac{1}{q^2}\alpha(\alpha - \sigma_3) \right).
\]  
(2.24)
However, in accordance with relation (2.21), here the quantized parameter $\alpha$ takes only half-integer values. In order it could take any real value, it is necessary to start with a free particle on the punctured plane. In this case the orbital angular momentum operator is changed for
\[
\hat{L} = -i \frac{\partial}{\partial \varphi} + \vartheta, \quad \vartheta \in \mathbb{R}.
\]  
(2.25)
Effectively such a change eliminates the restriction on the $\alpha$ (for the details, see ref. [26]). Note that physically the particle on the punctured plane with the angular momentum (2.25)
corresponds to the system of a point charged particle in a field of the singular magnetic
flux placed at \( x_i = 0 \) \([27, 28]\), and as for the 3D charge-monopole system \([29, 30]\), \((2.25)\) is
the total angular momentum of the particle and electromagnetic field. Further on we shall
suppose that the parameter \( \alpha \) can take any real value.

Before passing over to the generalization of the construction for the case of nonlinear
superconformal symmetry, we note that the quantum constraint \((2.20)\) can be represented
equivalently in the form
\[
\hat{J}_{1}\Psi_{\text{phys}} = \left( \hat{L} + \Pi_{+} - \frac{1}{2} \right) \Psi_{\text{phys}} = 0,
\]
where the term \( \frac{1}{2} \) is of the quantum origin (it includes the factor \( \hbar = 1 \)), and \( \Pi_{+} = \frac{1}{2}(\sigma_{3} + 1) \)
being a projector, \( \Pi_{+}^{2} = \Pi_{+} \), is the fermion quantum number operator, \( \Pi_{+} = \hat{\psi}_{+}\hat{\psi}_{-}, \hat{\psi}_{\pm} = \frac{1}{\sqrt{2}}(\sigma_{1} \pm i\sigma_{2}) \).

### 3 Nonlinear superconformal symmetry

Now we are in position to be able to generalize the construction for the nonlinear
superconformal symmetry case. We start, again, from the system of a free spin-\( \frac{1}{2} \) particle on the
(punctured) plane, but instead of \((2.26)\), we postulate the quantum constraint
\[
\hat{J}_{n}\Psi_{\text{phys}} = \left( \hat{L} + n\Pi_{+} - \frac{1}{2} \right) \Psi_{\text{phys}} = 0, \tag{3.1}
\]
where \( n \) is an arbitrary integer, which for definiteness is supposed to be positive. The formal
sense of the change of the quantum condition \((2.26)\) for \((3.1)\) is clear: eq. \((2.26)\) singles out
the two eigenstates of \( \hat{L} \) with the eigenvalues shifted for 1, while the \( \hat{L} \)-eigenvalues of the
upper and lower components of the spinor satisfying eq. \((3.1)\) are shifted relatively in \( n \).

Let us demonstrate that the nonlinear superconformal symmetry is realized in the system
reduced by the quantum equation \((3.1)\). To identify the symmetry generators, we begin with
the analysis of the corresponding classical system. The classical analog of the quantum
equation \((3.1)\) is the constraint
\[
J_{n} = L + n\Pi_{+} - \alpha \approx 0, \tag{3.2}
\]
where \( \Pi_{+} = \xi_{+}\xi_{-}, \xi_{\pm} = \frac{1}{\sqrt{2}}(\xi_{1} \pm i\xi_{2}), \{\xi_{+}, \xi_{-}\} = -i \), and we have omitted the quantum term \( \frac{1}{2} \).
Classically \( \Pi_{+} \) coincides with \( \Sigma \), and so, at \( n = 1 \) the constraint \((3.2)\) takes the form of the
constraint \((2.13)\). The constraint \((3.2)\) appears from the Lagrangian
\[
\mathcal{L}_{n} = \mathcal{L}_{0} - \frac{1}{2\alpha^{2}_{i}}(\epsilon_{jk}x_{j}\dot{x}_{k} + n\xi_{+}\xi_{-} - \alpha)^{2}, \tag{3.3}
\]
with \( \mathcal{L}_{0} \) given by eq. \((2.1)\), as the unique (primary) constraint, while \( H = \frac{1}{2}p_{i}^{2} \) is generated
by \((3.3)\) as the canonical Hamiltonian.

Let us define the bosonic counterparts of the mutually conjugate Grassmann variables \( \xi_{\pm} \),
\[
X_{\pm} = \frac{1}{\sqrt{2}}(X_{1} \pm iX_{2}), \quad P_{\pm} = \frac{1}{\sqrt{2}}(p_{1} \pm ip_{2}), \tag{3.4}
\]
satisfying the nontrivial Poisson bracket relations \( \{X_+, P_-\} = \{X_-, P_+\} = 1 \). In terms of these variables, we can identify the even observables defined as those having zero Poisson brackets with the constraint (3.2). These are the same quadratic quantities (2.4), (2.5) taking the form
\[
H = P_+P_-, \quad K = X_+X_-, \quad D = \frac{1}{2}(X_+P_- + P_+X_-),
\]
\[
L = i(X_+P_- - X_-P_+), \quad \Sigma = \xi_+\xi_-. \tag{3.5}\]

To identify the odd observables being integrals of motion in the sense of eq. (2.3), it is sufficient to note that
\[
\{\Sigma, \xi_\pm\} = \mp i\xi_\pm, \quad \{L, X_\pm\} = \mp iX_\pm, \quad \{L, P_\pm\} = \mp iP_\pm.
\]
Therefore, the set of odd independent integrals of motion commuting with the constraint (3.2) is
\[
S^+_{n,l} = 2^{n/2}(i)^{n-l}(P_-)^{n-l}(X_-)^l\xi_+, \quad S^-_{n,l} = 2^{n/2}(-i)^{n-l}(P_+)^{n-l}(X_+)^l\xi_-, \quad l = 0, \ldots, n.
\]
At \( n=1 \) these are the linear combinations of the odd integrals (2.6).

Since on the surface of the constraint (3.2) the relation
\[
C \equiv 4(KH - D^2) + 2n\Sigma = \alpha^2, \tag{3.8}
\]
is valid, there the quantity \( C \) commutes with all the set of the integrals (3.5), (3.6), (3.7). The even integrals (3.5) and \( \Sigma \) form, as before, the Lie algebra \( so(1, 2) \oplus u(1) \). Then, treating the constraint (3.2) as that fixing the orbital angular momentum \( L \), and taking into account the relation (3.8), we find that on the surface (3.2) the integrals (3.5), \( \Sigma \) and (3.7) form the nonlinear superalgebra given in addition to eq. (2.7) by the following nontrivial Poisson bracket relations:
\[
\{D, S^\pm_{n,l}\} = \left(\frac{n}{2} - l\right)S^\pm_{n,l}, \quad \{\Sigma, S^\pm_{n,l}\} = \mp iS^\pm_{n,l}, \tag{3.9}
\]
\[
\{H, S^\pm_{n,l}\} = \pm iS^\pm_{n,l-1}, \quad \{K, S^\pm_{n,l}\} = \pm i(n-l)S^\pm_{n,l+1}, \tag{3.10}
\]
\[
\{S^+_{n,m}, S^-_{n,l}\} = -i(2H)^{n-m}(2K)^l(\alpha - 2iD)^{m-l} - i\Sigma(2H)^{n-m-1}(2K)^{l-1} \times
\]
\[
(\alpha - 2iD)^{m-l}(n(m-l)(\alpha - 2iD) + 4al(n-m)), \quad m \geq l. \tag{3.11}
\]
The brackets between the odd integrals for the case \( m < l \) can be obtained from (3.11) by a complex conjugation. The relations (2.7), (3.9)-(3.11) give a nonlinear generalization of the superconformal algebra \( osp(2|2) \) with the Casimir element (3.8).

In terms of the polar coordinates (2.14), (2.15), we have the relations
\[
P_\pm = \frac{1}{\sqrt{2}}(p_r \pm iv^{-1}L)e^{\pm iv}, \quad X_\pm = \frac{1}{\sqrt{2}}r e^{\pm iv} - P_\pm t, \tag{3.12}
\]
while the variables
\[
\psi_\pm = \xi_\pm e^{\mp iv}, \quad \{\psi_+, \psi_-\} = -i, \tag{3.13}
\]
are the odd observables commuting with the constraint (3.2), which at \( n = 1 \) are transformed to the linear combinations of the odd variables defined by eq. (2.15). Using the notation \( q = r, p = p_r, \) and the constraint (3.2), we obtain the reduced 1D classical Hamiltonian,

\[
H = \frac{1}{2} \left( p^2 + \frac{1}{q^2} \alpha (\alpha - 2n \Sigma) \right),
\]

the even,

\[
\Sigma = \psi_+ \psi_-, \quad D = \frac{1}{2} q p - H t, \quad K = \frac{1}{2} q^2 - 2 D t - H t^2,
\]

and the odd,

\[
S^+_{n,l} = 2^{n/2} r^{n-l} (p - i \alpha q^{-1})^{n-l} (q - (p - i \alpha q^{-1}) t)^l \psi_+, \quad S^-_{n,l} = (S^+_{n,l})^*,
\]

integrals of motion, generating the nonlinear generalization of the superconformal symmetry \( osp(2|2) \). Therefore, the reduction of the nonrelativistic 2D free spin-\( \frac{1}{2} \) particle system by the constraint (3.2) produces the 1D classical system of ref. [17] with nontrivially coupled boson and fermion degrees of freedom, which possesses the nonlinear superconformal symmetry.

In ref. [17], it was showed that the quantum nonlinear superconformal symmetry is generated by the set of quantum operators

\[
\hat{H} = \frac{1}{2} \left( -\frac{\partial^2}{\partial q^2} + \frac{1}{q^2} (a_n + b_n \sigma_3) \right), \tag{3.14}
\]

with

\[
a_n = \alpha_n^2 + \frac{1}{4} (n^2 - 1), \quad b_n = -n \alpha_n, \quad \alpha_n = \alpha - \frac{1}{2} (n - 1), \tag{3.15}
\]

\[
\hat{\Sigma} = \frac{1}{2} \sigma_3, \quad \hat{D} = -\frac{i}{2} \left( q \frac{\partial}{\partial q} + \frac{1}{2} \right) - \hat{H} t, \quad \hat{K} = \frac{1}{2} q^2 - 2 \hat{D} t - \hat{H} t^2, \tag{3.16}
\]

\[
\hat{S}^+_{n,l} = (q + it \mathcal{D}_{a-n+1}) (q + it \mathcal{D}_{a-n+2}) \ldots (q + it \mathcal{D}_{a-n+l}) \mathcal{D}_{a-n+l+1} \ldots \mathcal{D}_a \hat{\psi}_+, \quad \hat{S}^-_{n,l} = (\hat{S}^+_{n,l})^*, \tag{3.17}
\]

where

\[
\mathcal{D}_\gamma = \frac{\partial}{\partial q} + \frac{\gamma}{q}. \tag{3.18}
\]

The second terms in \( a_n \) and \( \alpha_n \) in eq. (3.15) (proportional to \( (n^2 - 1) \) and \( (n - 1) \)) include the quantum factors \( \hbar^2 \) and \( \hbar \), respectively, while the term \( \frac{\gamma}{q} \) in (3.18) includes the factor \( \hbar (= 1) \).

These quantum corrections in the quantum analogs of the corresponding classical quantities were found in [17] from the requirement of preservation of the nonlinear superconformal symmetry. However, their origin remained to be completely unclear. Now we shall show that the application of the reduction procedure ‘first quantize and then reduce’ to the system (2.1), (3.2) produces exactly the anomaly free quantum system with the nonlinear superconformal symmetry generators given by eqs. (3.14)–(3.18).
Proceeding from the relations (2.14), (2.15) and (3.1), we construct the quantum analogs of the classical quantities (3.12):

\[ \hat{P}_r = \frac{1}{\sqrt{2}} \left( \hat{p}_r + \frac{i}{r} \left( \hat{L} - \frac{1}{2} \right) \right) e^{i\phi}, \quad \hat{P}_- = \left( \hat{P}_r \right)^\dagger, \quad \hat{X}_\pm = \frac{1}{\sqrt{2}} r e^{\pm i\phi} - \hat{P}_\pm t. \]

Putting these relations into the quantum analogs of the odd integral (3.7), transporting subsequently the phase factor \( e^{-i\phi} \) to the right (to the left), defining, in accordance with relation (3.13), the operators \( \hat{\psi}_\pm = e^{-i\phi} \sigma_\pm, \hat{\psi}_- = e^{i\phi} \sigma_-, \sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2) \), and passing over to the scalar product (2.22) in the manner described above, we obtain the fermion operators (3.17). The quantum Hamiltonian (3.14) is obtained proceeding from the relation (2.23) as before, but now using the quantum analog (3.1) of the classical constraint (3.2). This procedure correctly reproduces the described quantum corrections in (3.15).

In ref. [17] the quantum analog of the nonlinear superconformal algebraic relations (3.11) was obtained in a complete form only for the particular case \( n = 2 \) by a direct calculation of the anticommutation relations of the operators (3.17). The knowledge of the origin of the odd operators (3.17) allows us to fix the form of the nonlinear superconformal symmetry in general case of \( n \in \mathbb{N} \). To this end, we proceed from the quantum generators presented in the form corresponding to the classical expressions (3.5)-(3.7). It is obtained via a direct substitution of the classical quantities \( X_\pm, \xi_\pm \) for their quantum analogs satisfying the nontrivial (anti)commutation relations

\[ [\hat{X}_+, \hat{P}_-] = [\hat{X}_-, \hat{P}_+] = i, \quad [\hat{\xi}_+, \hat{\xi}_-] = 1. \]

Then a simple calculation gives the following nontrivial commutation relations

\[ [\hat{H}, \hat{K}] = -2i\hat{D}, \quad [\hat{D}, \hat{H}] = i\hat{H}, \quad [\hat{D}, \hat{K}] = -i\hat{K}, \quad (3.19) \]

\[ [\hat{\Sigma}, \hat{S}^\pm_{n,l}] = \pm \hat{S}^\pm_{n,l}, \quad [\hat{D}, \hat{S}^\pm_{n,l}] = i \left( \frac{n}{2} - l \right) \hat{S}^\pm_{n,l}, \quad (3.20) \]

\[ [\hat{H}, \hat{S}^\pm_{n,l}] = \mp l \hat{S}^\pm_{n,l-1}, \quad [\hat{K}, \hat{S}^\pm_{n,l}] = \mp (n - l) \hat{S}^\pm_{n,l+1}, \quad (3.21) \]

together with the Casimir operator

\[ \hat{C} \equiv 2(\hat{H} \hat{K} + \hat{K} \hat{H}) - 4\hat{D}^2 + 2n\alpha_n \hat{\Sigma} = \alpha_n^2 + \frac{1}{4} n^2 - 1, \quad (3.22) \]

where \( \hat{\Sigma} = \frac{1}{2} \sigma_3 \). To obtain the last relation, we have also used the quantum constraint (3.1). The quantum analog of the classical relation (3.11) is given by the anticommutator

\[ [\hat{S}^+_{n,m}, \hat{S}^-_{n,l}] = \sum_{s=0}^{\min(n, m)} 2^s s! C^s_{n-m} C^s_{l} \times ((2\hat{K})^{l-s}(2\hat{H})^{n-m-s}\mathcal{P}_{m-l+s}(-2i\hat{D} + c_s)\Pi_+ + (-1)^{m-l}(2\hat{H})^{n-m-s}(2\hat{K})^{l-s}\mathcal{P}_{m-l+s}(2i\hat{D} + d_s)\Pi_-), \quad (3.23) \]

where \( \Pi_\pm = \frac{1}{2} \pm \Sigma, min(a, b) = a \) (or, \( b \)) when \( a \leq b \) (or, \( b \leq a \), \( C^s_{l} = \frac{n!}{s!(l-s)!}, \mathcal{P}_k(z) \) is a polynomial of order \( k \), \( \mathcal{P}_0(z) = 1, \quad \mathcal{P}_k(z) = z(z+2)\ldots(z+2(k-1)), \quad k > 0, \)
and
\[ c_s = \alpha + \frac{3}{2} + n - 2(m + s), \quad d_s = -\alpha + \frac{1}{2} + 2(l - s). \]

In (3.23) we suppose \( m \geq l \), while the case corresponding to \( m \leq l \) is obtained from it by the Hermitian conjugation. To calculate the anticommutator (3.23), we have used the relation
\[
\hat{X}_+^l \hat{P}_-^k = \sum_{s=0}^{\min(l,k)} i^s s! C_s^k \hat{P}_-^{k-s} \hat{X}_+^{l-s},
\]
and the analogous relation with the \( \hat{X}_+ \) and \( \hat{P}_- \) exchanged in their places and with the \( i \) changed for \(-i\). Besides, the product \( \hat{X}_- \hat{P}_+ = \hat{D} + \frac{1}{2}(\hat{L} + 1) \) has been presented in the equivalent form using the relation \( \hat{L} = \alpha + \frac{1}{2} - n\Pi_+ \) following from the equation of the quantum constraint (3.1).

Eqs. (3.19)–(3.21), (3.23) give a general form of the (anti)commutation relations of the order \( n \) nonlinear generalization of the superconformal algebra \( osp(2|2) \), which can be denoted as \( osp(2|2)_n \). From them, in particular, it is easy to reproduce the simplest nonlinear case of the \( osp(2|2)_2 \) superalgebra found in ref. [17].

4 Discussion and outlook

We have showed that the one-dimensional models corresponding to the cases of linear, \( osp(2|2) \), and nonlinear, \( osp(2|2)_n \), superconformal symmetries can be obtained by the reduction of the planar free spin-\( \frac{1}{2} \) particle system by the constraint (3.1) with \( n = 1 \) or \( n > 1 \). The reduction produces the nontrivial coupling of the boson and fermion degrees of freedom with conservation of the corresponding (linear or nonlinear) superconformal symmetry of the initial system. This method not only has given a natural explanation of the origin of the quantum corrections necessarily to be included in the generators of the \( osp(2|2)_n \) with \( n > 1 \) for preservation of the symmetry at the quantum level, but also has allowed us to fix the form of the quantum \( osp(2|2)_n \) superalgebra.

The reduction procedure with the constraint (3.1) can alternatively be treated as a reduction of the planar free spin-\( \frac{n}{2} \) particle system. Indeed, the quantum constraint (3.1) can be changed for the system of the quantum equations
\[
\left( \hat{L} + \hat{\Pi}_+^{(n)} - \alpha - \frac{1}{2} \right) \Psi_{phys} = 0, \quad (4.1)
\]
\[
\left( \hat{\Pi}_{1+} - \hat{\Pi}_{j+} \right) \Psi_{phys} = 0, \quad j = 2, \ldots, n, \quad (4.2)
\]
with
\[
\hat{\Pi}_+^{(n)} = \sum_{k=1}^{n} \hat{\Pi}_{k+}, \quad \hat{\Pi}_{k+} = \frac{1}{2} \left( \sigma_k^3 + 1 \right),
\]
and \( \frac{1}{2} \sigma_k^3 \), \( k = 1, \ldots, n \), corresponding to the set of the \( n \) independent spins of the value \( \frac{1}{2} \). Then eq. (4.1) fixes the value of the total angular momentum, while the set of the equations (4.2) prescribing the constituent spins to be polarized in the same direction. With taking

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into account the quantum constraints (4.2), the operator in eq. (4.1) reduces to the operator $\hat{L} + n\hat{\Pi}_+ - \alpha - \frac{1}{2}$, and after changing the notation $\sigma_3^1 \rightarrow \sigma_3^3$, eq. (4.1) takes a form of eq. (3.1). The corresponding classical Lagrangian for such a planar system is

$$\mathcal{L}^{(n)} = \frac{1}{2} \dot{x}_i^2 - \frac{i}{2} \xi_k^i \dot{\xi}_i^k - \frac{1}{2 \mu^2}(x_1 \dot{x}_2 - x_2 \dot{x}_1 + \xi^k \dot{\xi}_i^k - \alpha)^2 + \sum_{j=2}^{n} v_j (\xi_i^1 \xi_j^1 - \xi_i^j \xi_j^i).$$

Here $v_j$, $j = 2, \ldots, n$, is the set of Lagrange multipliers, $\xi_i^k$, $k = 1, \ldots, n$, is the set of $n$ planar Grassmann vectors, and $\xi_{\pm}^k = \frac{1}{\sqrt{2}}(\xi_i^k \pm i \xi_j^k)$.

To conclude, we enumerate shortly possible applications and generalizations of the results.

It would be interesting to generalize the described method of introduction of the boson-fermion coupling for other systems. We hope that this, on the one hand, could clarify the nature of the nontrivial quantum corrections appearing generally under attempt of quantization of the systems possessing nonlinear supersymmetry [19, 20, 24]; on the other hand, the method could be useful for the analysis of the quasi exactly solvable systems [31, 32, 33], to which the nonlinear supersymmetry is intimately related [20–24]. If the method admits a generalization for higher dimensions, it could be applied to investigation of the supersymmetric many-particle integrable systems.

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