BILINEAR SUMS OF KLOOSTERMAN SUMS,
MULTIPLICATIVE CONGRUENCES AND AVERAGE VALUES
OF THE DIVISOR FUNCTION OVER FAMILIES OF
ARITHMETIC PROGRESSIONS

BRYCE KERR AND IGOR E. SHPARLINSKI

Abstract. We obtain several asymptotic formulas for the sum of the divisor function \( \tau(n) \) with \( n \leq x \) in an arithmetic progression \( n \equiv a \pmod{q} \) on average over \( a \) from a set of several consecutive elements from set of reduced residues modulo \( q \) and on average over arbitrary sets. The main goal is to obtain nontrivial result for \( q \geq x^{2/3} \) with the small amount of averaging over \( a \). We recall that for individual values of \( a \) the limit of our current methods is \( q \leq x^{2/3-\varepsilon} \) for an arbitrary fixed \( \varepsilon > 0 \). Our method builds on an approach due to Blomer (2008) based on the Voronoi summation formula which we combine with some recent results on bilinear sums of Kloosterman sums due Kowalski, Michel and Sawin (2017) and Shparlinski (2017). We also make use of extra applications of the Voronoi summation formulae after expanding into Kloosterman sums and this reduces the problem to estimating the number of solutions to multiplicative congruences.

1. Introduction

1.1. Background. Let

\[ \tau(n) = \sum_{d \mid n} 1, \]

denote the divisor function, where the sum runs over all positive integral divisors \( d \) of an integer \( n \geq 1 \).

For integers \( a \) and \( q \geq 2 \) with \( \gcd(a, q) = 1 \), consider the divisor sum given by:

\[ S(X; a, q) = \sum_{n \leq X \atop n \equiv a \pmod{q}} \tau(n). \]

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Several authors proved independently an asymptotic formula for $S(X, a, q)$ in the range $q \leq X^{2/3-\varepsilon}$ with an arbitrary fixed $\varepsilon > 0$, see discussions and proofs in [3, 8, 13].

To formulate these results more precisely, we need to introduce some notation. Namely, we define the polynomial

$$P(T; q, a) = \sum_{d \mid q} \frac{r_d(a)}{d} (T - 2 \log d + 2\gamma - 1),$$

where $\gamma$ is the Euler-Mascheroni constant

$$r_d(a) = \sum_{e \mid \gcd(a, d)} e\mu(d/e),$$

is the Ramanujan sum, and $\mu(k)$ is the Möbius function.

We now define

$$M(X; a, q) = \frac{X}{q} P(\log X; q, a),$$

which is the expected main term in the asymptotic formula for the sum $S(X; a, q)$, and thus we also define the error term

$$R(X; a, q) = S(X; a, q) - M(X; a, q).$$

It is useful to note that if $\gcd(a, q) = 1$ then

$$M(X; a, q) = \frac{\varphi(q)}{q^2} X (\ln X + 2\gamma - 1) - \frac{2}{q} X \sum_{d \mid q} \frac{\mu(d) \ln d}{d},$$

where $\varphi(k)$ is the Euler function.

Then, uniformly over integers $a$ with $\gcd(a, q) = 1$ we have the bound

$$R(X; a, q) \leq X^{1/3 + o(1)},$$

which given by Blomer [3, Equation (2)] (see also [7]) and generalised to the case of arbitrary $\gcd(a, q)$ by Pongsriiam and Vaughan [13, Theorem 1.1].

Furthermore, Blomer [3, Theorem 1.1], improving the previous result of Banks, Heath-Brown and Shparlinski [2], has shown that

$$\sum_{a=0}^{q-1} R(X; a, q)^2 \leq X^{1+o(1)},$$

which (as also the result of [2]) is nontrivial in the essentially optimal range $q \leq X^{1-\varepsilon}$ with an arbitrary fixed $\varepsilon > 0$.

With respect to a different kind of averaging, namely, over $q$ rather than over $a$, Fouvry [5, Corollary 5] has obtained the following bound:
for any fixed $\varepsilon > 0$ there exists some constant $c > 0$ such that uniformly over integers $a$ with $|a| \leq \exp \left( c \log X \right)$ we have

$$\sum_{X^{2/3+\varepsilon} \leq q \leq X^{1-\varepsilon}} |R(X; a, q)| = O \left( X \exp \left( -c \log X \right) \right).$$

We note that the summation in (1.3) can be extended to $q \leq X^{2/3-\varepsilon}$, however the values of $q$ in the range $X^{2/3-\varepsilon} < q < X^{2/3+\varepsilon}$ have to be avoided. For a class of special moduli, this gap in the range of $q$ has been bridged in [7].

1.2. New set-up and results. Here we consider two apparently new questions, which “interpolate” between obtaining individual bounds like (1.1) and bounds on average like (1.2). Namely, given some subset $A \subseteq \mathbb{Z}^*$ of the reduced residues modulo $q$ we consider the sums

$$D(X; A, q) = \sum_{a \in A} |R(X; a, q)| \quad \text{and} \quad E(X; A, q) = \sum_{a \in A} R(X; a, q).$$

In particular, using (1.2) and the Cauchy-Schwarz inequality, we obtain

$$|E(X; A, q)| \ll D(X; A, q) \ll A^{1/2} X^{1/2+o(1)},$$

which is nontrivial (that is, stronger than the trivial upper bound $AXq^{-1+o(1)}$) provided that $A \geq q^{2+\varepsilon} X^{-1}$ for some fixed $\varepsilon > 0$. Thus in the case $q \sim X^{2/3}$, this becomes $A \geq q^{1/2+\varepsilon}$.

We are interested in obtaining stronger bounds on $D(X; A, q)$ and $D(X; A, q)$ and especially which are nontrivial for small values of $A$ and large values of $q$ (for example, when $A \leq q^{1/2}$ and $q \geq X^{2/3}$).

Our first bound considers the case when $A = \mathcal{I}$ is an interval and depends on a result of Kowalski, Michel and Sawin [11, Theorem 1.1], and thus applies only to prime $q = p$. In particular, as we are mostly interested in the values $q = p \geq X^{2/3}$, to simplify the result we assume that $p \geq X^{4/7}$.

**Theorem 1.1.** For any integers $A$ and $X$, an interval $\mathcal{I}$ of length $A$ and a prime $p$ with

$$A \leq p \quad \text{and} \quad X \geq p \geq X^{4/7},$$

we have,

$$D(X; \mathcal{I}, p) \leq (AX^{1/2}p^{-1/2} + A^{3/2}X^{1/2}p^{-5/8} + A^{1/2}X^{1/2}p^{-1/8} + A^{5/6}X^{5/18}p^{11/72})p^o(1).$$
We now see that the bound of Theorem 1.1 is nontrivial, that is, better than $AX/p$, for

$$Xp^{-3/4+\varepsilon} \geq A \geq \max \{ X^{-13/3}p^{83/12+\varepsilon}, X^{-1}p^{7/4+\varepsilon} \} \quad \text{and} \quad p \leq X^{1-\varepsilon},$$

for some fixed $\varepsilon > 0$. In particular at the critical value $p \sim X^{2/3}$ this condition becomes

$$p^{3/4-\varepsilon} \geq A \geq p^{5/12+\varepsilon}.$$  

Clearly, $|\mathcal{E}(X; I, q)| \leq \mathcal{D}(X; I, q)$, see (1.4), but in general we obtain a stronger result for $\mathcal{E}(X; I, q)$ which does not follow from this trivial inequality and which also applies to composite moduli.

Again, as we are mostly interested in the values $q \geq X^{2/3}$, we make a simplifying assumption that $q \geq X^{19/31}$ (and note that $19/31 < 2/3$).

**Theorem 1.2.** For any integers $A$, $X$ and $q$ with

$$A \leq q \quad \text{and} \quad X \geq q \geq X^{19/31},$$

and any interval $I$ of length $A$ we have,

$$|\mathcal{E}(X; I, q)| \leq (AX^{1/2}q^{-1/2} + A^{1/8}X^{1/4}q^{1/2} + A^{1/2}X^{1/4}q^{1/4}) q^{o(1)}.$$

We now see that the bound of Theorem 1.2 is nontrivial, that is, better that $AX/q$ for

$$A \geq \max \{ X^{-6/7}q^{12/7+\varepsilon}, X^{-3/2}q^{5/2+\varepsilon} \} \quad \text{and} \quad q \leq X^{1-\varepsilon},$$

for some fixed $\varepsilon > 0$. In particular, at the critical value $q \sim X^{2/3}$ this condition becomes

$$A \geq q^{3/7+\varepsilon}.$$  

The proofs of the above estimates are based on an approach of Blomer [3] combined with some recent bounds on bilinear sums of Kloosterman sums, see [4, 6, 11, 12, 14–16] for a variety of such bounds.

Our last result considers averaging over an arbitrary set $\mathcal{A}$ and uses extra applications of the Voronoi summation formula to reduce to estimating solutions to multiplicative congruences rather than Kloosterman sums. We use $\mathbb{F}_p$ to denote the field of $p$ elements.

**Theorem 1.3.** Let $p$ be prime and $\mathcal{A} \subseteq \mathbb{F}_p^*$ be any set with cardinality $A$. For any integer $X \geq 1$ we have

$$\mathcal{D}(X; \mathcal{A}, p) \leq \left( A^{3/4}X^{1/4}p^{1/4} + A^{2/3}X^{1/3} \right) X^{o(1)}.$$  

In particular, we see that Theorem 1.3 provides an improvement over (1.1) with trivial summation over $A$, that is, over $AX^{1/3+o(1)}$ once the condition

$$p \leq \min \{ AX^{1/3-\varepsilon}, X^{2/3-\varepsilon} \}$$

...
is satisfied. Furthermore, it improves the bound (1.4) once

\[ p \leq X^{1-\varepsilon}/A. \]

Hence the above two conditions are equivalent to

\[ p \leq \min\{AX^{1/3-\varepsilon}, X^{1-\varepsilon}/A\}. \]

We can now estimate the set of \( a \) for which the bound (1.1) is almost tight. Namely for a fixed \( \kappa > 0 \) we denote by \( \mathcal{A}_\kappa(X; p) A \) the set of \( a \in \mathbb{F}_p^* \) for which \( R(X; a, p) \geq X^{1/3-\kappa} \).

Since \( \mathcal{D}(X; \mathcal{A}_\kappa(X, p), p) \geq \#\mathcal{A}_\kappa(X, p)X^{1/3-\kappa} \), Theorem 1.3 yields:

**Corollary 1.4.** Let \( p \) be prime and \( A \subseteq \mathbb{F}_p^* \) be any set with cardinality \( A \). For any integer \( X \geq 1 \) we have

\[ \#\mathcal{A}_\kappa(X, p) \leq \max\{pX^{-1/3+4\kappa}, X^{3\kappa}\}X^{o(1)}. \]

2. Preliminaries

2.1. Notation. Let \( q \) be a positive integer. We denote the residue ring modulo \( q \) by \( \mathbb{Z}_q \) and denote the group of units of \( \mathbb{Z}_q \) by \( \mathbb{Z}_q^* \).

For integers \( d \geq 1, m \) and \( n \) we define the *Kloosterman sum*

\[ K_d(m, n) = \sum_{x \in \mathbb{Z}_q^*} e_d(m x + n x^\overline{1}), \]

where \( x^\overline{1} \) is the multiplicative inverse of \( x \) modulo \( q \) and

\[ e_d(z) = \exp(2\pi i z/d). \]

We use \( \text{supp} \ F \) to denote the support of a real valued function \( f \), that is, we have \( F(x) \neq 0 \) if and only if \( x \in \text{supp} \ F \).

As usual, \( \{x\} \) denotes the fractional part of a real number \( x \).

We recall that the expressions \( U \ll V \), \( V \gg U \) and \( U = O(V) \) are all equivalent to the statement that \( |U| \leq cV \) for some constant \( c \). Throughout the paper, the implied constants in symbols “\( O \)”, “\( \ll \)” and “\( \gg \)” may occasionally, where obvious, depend on the small positive parameter \( \varepsilon \) and integer parameters \( k \) and \( r \), and are absolute otherwise.

2.2. Error terms and Kloosterman sums. In this section, we collect some useful results that stem from the work of Blomer [3] and link bounds on \( \mathcal{D}(X; I, q) \) and \( \mathcal{E}(X; I, q) \) to bounds on some bilinear sums of Kloosterman sums.

Fix some parameter

\[ 1 \leq Y \leq X/2 \]
and a smooth compactly supported function $w(x)$ satisfying

$$w(x) = \begin{cases} 
1 & \text{if } x \in [2Y, X], \\
0 & \text{if } x \leq Y \text{ or } x \geq X + Y,
\end{cases}$$

and for any integer $j \geq 1$

$$w^{(j)}(x) \ll \frac{1}{Y^{j}}.$$

We now recall the link between the error term $R(X; a, q)$ and some bilinear sums of Kloosterman sums given by Blomer [3]. Namely by [3, Equations (7) and (8)], for any fixed $\varepsilon > 0$ and a parameter $Y$ satisfying (2.1), we have

$$R(X; a, q) = \frac{1}{q} \sum_{d|q} \sum_{\pm n=1}^{\infty} \tau(n) u_{d}^{\pm}(n) K_{d}(\mp n, a) + O((Y/q + 1)(Yq)^{\varepsilon}),$$

with the functions $u_{d}^{\pm}$ defined by

$$u_{d}^{+}(y) = \frac{4}{d} \int_{-\infty}^{\infty} w(x) K_{0} \left( \frac{4\pi(xy)^{1/2}}{d} \right) dx,$$

and

$$u_{d}^{-}(y) = -\frac{2\pi}{d} \int_{-\infty}^{\infty} w(x) Y_{0} \left( \frac{4\pi(xy)^{1/2}}{d} \right) dx,$$

where we use the standard notation for Bessel functions $K_{0}(x)$ and $Y_{0}(x)$ (we note that for typographical simplicity we have replaced the sum $K_{d}(\pm n, -a)$ with the equal sum $K_{d}(\mp n, a)$).

We note that in [3] a slightly more complicated notation $\tilde{w}(n)$ is used, however with the dependence on $d$ suppressed. In turn, we also suppress the dependence on $X$ in the notation for the functions $u_{d}^{\pm}(y)$, as well as we do for the following quantities

$$(2.3) \quad U(d) = d^{2}X^{-1} \quad \text{and} \quad V(d) = d^{2}X^{1+\varepsilon}Y^{-2}.$$  

We now recall, by [3, Equations (11)] we have

$$(2.4) \quad u_{d}^{\pm}(n) \ll \begin{cases} 
X^{1+\varepsilon}d^{-1}, & \text{if } n \leq U(d), \\
X^{1/4}d^{1/2}n^{-3/4}, & \text{if } U(d) < n \leq V(d).
\end{cases}$$

Furthermore,

$$(2.5) \quad u_{d}^{\pm}(n) \ll \frac{1}{n^{\varepsilon}}, \quad \text{if } n > V(d),$$

for any fixed $c > 0$, with the implied constant depending on $c$. In particular, the contribution to $R(X; a, q)$ from $n \geq V(d)$ is negligible.
and we can limit the summation over \( n \) up to \( V(d) \) and absorb the difference in the already present error term. We then have

\[
R(X; a, q) = \frac{1}{q} \sum_{d|q} \sum_{\pm} \sum_{n=1}^{V(d)} \tau(n) u_d^+(n) K_d(\mp n, a) + O((Y/q + 1)(Yq)^{\varepsilon}).
\]

(2.6)

Hence, changing the order of summations, we derive from (2.6) that

\[
D(X; \mathcal{I}, q) \leq D^*(X; \mathcal{I}, q) + O(A(Y/q + 1)(Yq)^{\varepsilon}),
\]

(2.7) where

\[
D^*(X; \mathcal{I}, q) = \frac{1}{q} \sum_{d|q} \sum_{\pm} \sum_{n=1}^{V(d)} \tau(n) u_d^+(n) K_d(\mp n, a).
\]

Hence for some complex numbers \( \alpha_{a,d} \) with \( |\alpha_{a,d}| = 1 \) we can write

\[
D^*(X; \mathcal{I}, q) = \frac{1}{q} \sum_{d|q} \sum_{\pm} \sum_{a \in \mathcal{I}} \sum_{n=1}^{V(d)} \alpha_{a,d} \tau(n) u_d^+(n) K_d(\mp n, a).
\]

(2.8)

For \( E(X; \mathcal{I}, q) \) there is no need to introduce the weights \( \alpha_{a,d} \), so we have

\[
|E(X; \mathcal{I}, q)| \leq E^*(X; \mathcal{I}, q) + O(A(Y/q + 1)(Yq)^{\varepsilon}),
\]

(2.9) where

\[
E^*(X; \mathcal{I}, q) = \frac{1}{q} \sum_{d|q} \sum_{\pm} \sum_{a \in \mathcal{I}} \sum_{n=1}^{V(d)} \tau(n) u_d^+(n) K_d(\mp n, a).
\]

(2.10)

2.3. Bilinear sums of Kloosterman sums. Given two intervals

\[ \mathcal{I} = \{B + 1, \ldots, B + A\}, \quad \mathcal{J} = \{M + 1, \ldots, M + N\} \subseteq [1, q - 1] \]

of \( A \) and \( N \) consecutive integers, respectively and two sequence of weights \( \alpha = \{\alpha_a\}_{a \in \mathcal{I}} \) and \( \nu = \{\nu_n\}_{n \in \mathcal{J}} \), we define the following bilinear sums of Kloosterman sums

\[
S_d(\alpha, \nu; \mathcal{I}, \mathcal{J}) = \sum_{a \in \mathcal{I}} \sum_{n \in \mathcal{J}} \alpha_a \nu_n K_d(n, a),
\]

\[
S_d(\nu; \mathcal{I}, \mathcal{J}) = \sum_{a \in \mathcal{I}} \sum_{n \in \mathcal{J}} \nu_n K_d(n, a).
\]

We now collect some bounds on these sums slightly simplifying and adjusting them to our notation; more bounds can be found in [4, 6, 11, 12, 14–16].

For general bilinear sums, we recall a bound of Kowalski, Michel and Sawin [11, Theorem 1.1], which improves that of Fouvry, Kowalski and
Michel [6, Theorem 1.17], see also [4, Theorem 5.1] near \( A \sim N \sim p^{1/2} \), see also [4, Theorem 5.1], which however is known only for prime moduli \( q = p \).

**Lemma 2.1.** For a prime \( p \geq 1 \), integers \( 1 \leq A, N \leq p \), an initial interval \( \mathcal{J} = \{1, \ldots, N\} \), weights \( \alpha \) and \( \nu \) with
\[
|\alpha_a| \leq 1, \quad a \in \mathcal{I}, \quad \text{and} \quad |\nu_n| \leq 1, \quad n \in \mathcal{J},
\]
and \( p, A, N \) satisfying
\[
p^{1/4} \leq AN \leq p^{5/4}, \quad N \leq Ap^{1/4},
\]
we have,
\[
|S_p(\alpha, \nu; \mathcal{I}, \mathcal{J})| \leq \left( AN^{1/2} p^{1/2} + A^{13/16} N^{13/16} p^{43/64} \right) p^{o(1)}.
\]
For the sums \( S_d(\nu; \mathcal{I}, \mathcal{J}) \) we recall the bound from [14]:

**Lemma 2.2.** For integers \( 1 \leq A, N \leq d \), and weights \( \nu \) with
\[
|\nu_n| \leq 1, \quad n \in \mathcal{J},
\]
we have,
\[
|S_d(\nu; \mathcal{I}, \mathcal{J})| \leq N^{3/4} \left( A^{1/8} d + A^{1/2} d^{3/4} \right) d^{o(1)}.
\]
Note that for \( q \geq N \geq A^{3/2} \) the following bound
\[
|S_d(\nu; \mathcal{I}, \mathcal{J})| \leq N^{1/2} A^{1/2} d^{1+o(1)},
\]
from [15, Theorem 3.2] is stronger (analysing the proof one can see that the result holds without any changes for composite moduli). However in our case it does not seem to bring any new results.

Furthermore, when \( q = p \) is prime the bound of Blomer, Fouvry, Kowalski, Michel and Milićević [4, Theorem 5.1(2)]
\[
|S_p(\nu; \mathcal{I}, \mathcal{J})| \leq A^{7/12} N^{5/6} p^{3/4+o(1)},
\]
which holds under the conditions
\[
A, N \leq p, \quad AN \leq p^{3/2}, \quad N \leq A^2,
\]
and extends the range of non-triviality of Lemma 2.2 but is weaker near \( A \sim N \sim p^{1/2} \) which is a crucial range for our argument. We note that in [4] it is formulated only for the initial interval \( \mathcal{I} = \{1, \ldots, A\} \), but seems to extend to arbitrary intervals \( \mathcal{I} = \{B + 1, \ldots, B + A\} \).

Finally, recent bounds on both \( S_p(\alpha, \nu; \mathcal{I}, \mathcal{J}) \) and \( S_p(\nu; \mathcal{I}, \mathcal{J}) \), due to Kowalski, Michel and Sawin [12], are nontrivial in wider ranges but again apply only to prime moduli \( q = p \).

To conclude we stress that the aforementioned bounds from [4, 6, 12] do not seem to improve the results of Theorems 1.1 and 1.2 in
interesting ranges, that is when $A \leq q^{1/2}$ and $q \geq X^{2/3}$ (even for prime $q = p$).

2.4. **Characters and multiplicative congruences.** Let $\mathcal{X}_q$ be the set of *multiplicative* characters of the residue ring modulo $q \geq 1$ and let $\mathcal{X}_q^\ast = \mathcal{X}_q \setminus \{\chi_0\}$ be the set of *nonprincipal* characters; we refer the reader to [9, Chapter 3] for the relevant background. In particular, we make use of the following orthogonality property of characters, see [9, Section 3.2],

\[
\frac{1}{\varphi(q)} \sum_{\chi \in \mathcal{X}_q} \chi(a) = \begin{cases} 1, & \text{if } a \equiv 1 \pmod{q}, \\ 0, & \text{otherwise,} \end{cases}
\]

which holds for any integer $a$ with $\gcd(a, q) = 1$.

We need the following result of Ayyad, Cochrane and Zheng [1, Theorem 1], see also [10] for sharper error terms in the case of prime $q = p$.

**Lemma 2.3.** For any integers $K$ and $H$ we have

\[
\sum_{\chi \in \mathcal{X}_q^\ast} \left| \sum_{x=K}^{K+H} \chi(x) \right|^4 \ll H^2 q^{o(1)}.
\]

We note that there is no restriction $H < q$ in the statement of Lemma 2.3 and this is important for the proof of Theorem 1.3.

**Lemma 2.4.** Let $p$ be prime and let $\mathcal{H}_i$ be intervals of lengths $H_i$, $i = 1, 2, 3, 4$. Then we have

\[
\# \{(x_1, x_2, x_3, x_4) \in \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \times \mathcal{H}_4 : x_1 \ldots x_4 \not\equiv 0 \pmod{p}, x_1 x_2 \equiv x_3 x_4 \pmod{p}\} \ll \frac{H_1 H_2 H_3 H_4}{p} + O \left( (H_1 H_2 H_3 H_4)^{1/2} p^{o(1)} \right).
\]

2.5. **Reduction to smooth sums.** For a a smooth function $g(x, y)$ we define the Fourier transform

\[
\hat{g}(u, v) = \int_{\mathbb{R}^2} g(x, y) e(ux + vy) dx dy,
\]

where $e(z) = \exp(2\pi iz)$.

The following is [9, Proposition 4.11].

**Lemma 2.5.** Let $q$ and $z$ be integers with $\gcd(z, q) = 1$. For a smooth function $g(x, y)$ with a compact support we define

\[
\tau_g(n) = \sum_{m_1 m_2 = n} g(m_1, m_2) \quad \text{and} \quad \tau_h(n) = \sum_{m_1 m_2 = n} h(m_1, m_2),
\]
where
\[ h(x, y) = \frac{1}{q} \hat{g} \left( \frac{x}{q}, \frac{y}{q} \right), \]
and
\[ \tau_h(0) = \int_{\mathbb{R}^2} \left( \frac{1}{q} + \left\{ \frac{x}{q} \right\} \frac{\partial}{\partial x} + \left\{ \frac{y}{q} \right\} \frac{\partial}{\partial y} \right) g(x,y) dxdy. \]

Then we have
\[ \sum_{m \in \mathbb{Z}} \tau_g(m) e_q(zm) = \sum_{n \in \mathbb{Z}} \tau_h(n) e_q(-z^{-1}n). \]

We require a variant of Lemma 2.5 with twists by multiplicative characters.

**Lemma 2.6.** With notation as in Lemma 2.5, let \( \chi \) be a primitive character mod \( q \). Then we have
\[ \sum_{m \in \mathbb{Z}} \tau_g(m) \chi(m) = \eta(\chi) \sum_{n \in \mathbb{Z}} \tau_h(n) \overline{\chi}(n), \]
where
\[ \eta(\chi) = \frac{\tau(\chi)}{\tau(\chi)} \]
and
\[ \tau(\chi) = \sum_{z=1}^{q} \overline{\chi}(z) e_q(z), \]
denotes the Gauss sum.

**Proof.** We have
\[ \sum_{m \in \mathbb{Z}} \tau_g(m) \chi(m) = \frac{1}{\tau(\chi)} \sum_{z=1}^{q-1} \overline{\chi}(z) \sum_{m \in \mathbb{Z}} \tau_g(m) e_q(zm), \]
and hence by Lemma 2.5
\[ \sum_{m \in \mathbb{Z}} \tau_g(m) \chi(m) = \frac{1}{\tau(\chi)} \sum_{z=1}^{q-1} \overline{\chi}(z) \sum_{n \in \mathbb{Z}} \tau_h(n) e_q(-z^{-1}n) \]
\[ = \frac{1}{\tau(\chi)} \sum_{n \in \mathbb{Z}} \tau_h(n) \sum_{z=1}^{q-1} \overline{\chi}(z) e_q(-z^{-1}n), \]
which simplifies to
\[ \sum_{m \in \mathbb{Z}} \tau_g(m) \chi(m) = \frac{\chi(-1) \tau(\overline{\chi})}{\tau(\chi)} \sum_{n \in \mathbb{Z}} \tau_h(n) \overline{\chi}(n). \]
and the result follows since 
\[ \overline{\tau(\chi)} = \chi(-1)\tau(\chi). \]

For a proof of the following, see [5, Lemma 2].

**Lemma 2.7.** There exists a sequence of smooth functions \( \Psi_\ell, \ell = 0, 1, \ldots \) satisfying
\[ \text{supp } \Psi_\ell \subseteq \left[ 2^{\ell-1}, 2^{\ell+1} \right] \quad \text{and} \quad \Psi_\ell^{(k)}(x) \ll \frac{1}{x^k}, \]
with the implied constant depending on \( k \), such that for any \( x \geq 1 \) we have
\[ \sum_{\ell \geq 0} \Psi_\ell(x) = 1. \]

3. Proofs of Main Results

3.1. **Proof of Theorem 1.1.** We now set
\[ Y = \max\{A^{1/2}X^{1/2+\varepsilon/2}p^{3/8}, A^{-1/2}X^{1/2+\varepsilon/2}p^{7/8}, A^{-1/6}X^{5/18}p^{83/72}\} \]
and also
\[ U = U(p) = p^2X^{-1} \quad \text{and} \quad V = V(p) = p^2X^{1+\varepsilon}Y^{-2}. \]
So, using our assumption that \( p \geq X^{4/7} \), we easily verify that
\[ p^{1/4} \leq AU \leq AV \leq p^{5/4} \quad \text{and} \quad U \leq V \leq Ap^{1/4} < p. \]
We also verify that
\[ A^{13/16}X^{5/16}Y^{-1/8}p^{19/64} \leq AY/p. \]
We now fix some \( \varepsilon > 0 \) and define the integer \( \ell \) by the conditions
\[ 2^{\ell-1}U \leq V < 2^\ell \]
and set
\[ V_i = \min \left\{ 2^iU, V \right\}, \quad i = 0, \ldots, \ell. \]
Hence, we derive from (2.8) that for a prime \( q = p \) we have
\[ \mathcal{D}^*(X; \mathcal{I}, p) = \frac{1}{p} \sum_{\pm} \sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{V(p)} \alpha_{\alpha,n} \tau(n)u_p^\pm(n)K_p(\mp n, a). \]
Therefore,
\[ \mathcal{D}^*(X; \mathcal{I}, p) \leq \frac{1}{p} \sum_{\pm} \left( D_1^\pm + D_2^\pm \right), \]
where

\[
D_1^\pm = \sum_{a \in \mathcal{I}} \sum_{1 \leq n \leq U} \alpha_{a,p} \tau(n) u_p^\pm(n) K_p(\mp n, a),
\]

\[
D_2^\pm = \sum_{i=0}^{t-1} \sum_{a \in \mathcal{I}} \sum_{V_i \leq n < V_{i+1}} \alpha_{a,p} \tau(n) u_p^\pm(n) K_p(\mp n, a).
\]

For \(D_1^\pm\), using the well-known bound on the divisor function (see, for example, [9, Equation (1.81)]) and recalling (2.4), we have

\[
|\tau(n) u_p^\pm(n)| \leq X^{1+\varepsilon+o(1)} p^{-1}
\]

for \(1 \leq n \leq U\).

Using (3.3), we see that Lemma 2.1 applies with \(N = U\). Hence after rescaling the weights in order to apply the bound (3.6), we obtain

\[
D_1^\pm \leq X^{1+\varepsilon} p^{-1} \left( AU^{1/2} p^{1/2} + A^{13/16} U^{13/16} p^{43/64} \right) p^o(1).
\]

Thus, recalling the choice of \(U\) and \(V\) from (3.2), we see that

\[
D_1^\pm \leq X^{1+\varepsilon} p^{-1} \left( AX^{-1/2} p^{3/2} + A^{13/16} X^{-13/16} p^{147/64} \right) p^o(1)
\]

\[
= \left( AX^{1/2+\varepsilon} p^{1/2} + A^{13/16} X^{3/16+\varepsilon} p^{83/64} \right) p^o(1).
\]

To estimate \(D_2^\pm\), we note that by (2.4) we have

\[
|\tau(n) u_p^\pm(n)| \leq X^{1/4} V_i^{-3/4} p^{1/2+o(1)}
\]

for \(V_i \leq n < V_{i+1}\). We also recall (3.3), hence by Lemma 2.1 (after rescaling the weights again) with \(N = V_{i+1}\), as in (3.7), we obtain

\[
D_2^\pm \leq \sum_{i=0}^{t-1} X^{1/4} V_i^{-3/4} p^{1/2+o(1)} \left( A V_i^{1/2} p^{1/2} + A^{13/16} V_i^{13/16} p^{43/64} \right)
\]

\[
= AX^{1/4} p^{1+o(1)} \sum_{i=0}^{t-1} V_i^{-1/4} + A^{13/16} X^{1/4} p^{75/64+o(1)} \sum_{i=0}^{t-1} V_i^{1/16}
\]

\[
\leq \left( AX^{1/4} p U^{-1/4} + A^{13/16} X^{1/4} p^{75/64 V^{1/16}} \right) p^o(1).
\]

Thus, recalling the choice of \(U\) and \(V\) from (3.2), we see that

\[
D_2^\pm \leq AX^{1/2} p^{1/2+o(1)} + A^{13/16} X^{5/16+\varepsilon/16} Y^{-1/8} p^{83/64+o(1)}.
\]

Substituting (3.7) and (3.8) in (3.5) we obtain

\[
\mathcal{D}^*(X; \mathcal{I}, p) \leq AX^{1/2} p^{-1/2+o(1)} + A^{13/16} X^{5/16+\varepsilon/16} Y^{-1/8} p^{19/64+o(1)}.
\]
Hence, by (2.7) and also using that \( Y/p \geq 1 \), we obtain
\[
\mathcal{D}(X; \mathcal{I}, p) \leq AX^{1/2}p^{-1/2+o(1)} + A^{13/16}X^{5/16+\varepsilon/16}Y^{-1/8}p^{19/64+o(1)} + AY^{1+\varepsilon}p^{1+\varepsilon} \ll (AX^{1/2}p^{-1/2} + A^{13/16}X^{5/16}Y^{-1/8}p^{19/64} + AY/p)X^{2\varepsilon}.
\]
Furthermore, from (3.4) we conclude that
\[
\mathcal{D}(X; \mathcal{I}, p) \ll (AX^{1/2}p^{-1/2} + AY/p)X^{2\varepsilon}.
\]
Using that \( \varepsilon > 0 \) is arbitrary and recalling the choice of \( Y \) in (3.1), we obtain
\[
\mathcal{D}(X; \mathcal{I}, p) \leq (AX^{1/2}p^{-1/2} + A^{3/2}X^{1/2}p^{-5/8} + A^{1/2}X^{1/2}p^{-1/8} + A^{5/6}X^{5/18}p^{11/72})p^{o(1)},
\]
which concludes the proof.

3.2. Proofs of Theorem 1.2. We now set
\[
Y = \sqrt{qX^{1+\varepsilon}},
\]
so recalling (2.3), we see that we always have
\[
U(d) \leq V(d) \leq d
\]
for every \( d \mid q \).

We fix some \( \varepsilon > 0 \). For each positive \( d \mid q \) we define the integer \( \ell(d) \) by the conditions
\[
2^{\ell(d)-1}U(d) \leq V(d) < U(d)2^{\ell(d)}
\]
and set
\[
V_i(d) = \min \{2^iU(d), V(d)\}, \quad i = 0, \ldots, \ell(d).
\]
Hence, we derive from (2.10) that
\[
\mathcal{E}^*(X; \mathcal{I}, q) = \frac{1}{q} \sum_{d \mid q} \sum_{\pm} \left( E_1^\pm(d) + E_2^\pm(d) \right),
\]
where
\[
E_1^\pm(d) = \sum_{a \in \mathcal{I}} \sum_{1 \leq n \leq U(d)} \tau(n)u_d^\pm(n)K_d(\mp n, a),
\]
\[
E_2^\pm(d) = \sum_{i=0}^{\ell(d)-1} \sum_{a \in \mathcal{I}} \sum_{V_i(d) \leq n < V_{i+1}(d)} \tau(n)u_d^\pm(n)K_d(\mp n, a).
\]
For $E_1^\pm(d)$, from the well-known bound on the divisor function (see, for example, [9, Equation (1.81)]) and recalling (2.4), we have

$$|\tau(n)u_d^\pm(n)| \leq X^{1+\varepsilon+o(1)}d^{-1}$$

for $1 \leq n \leq U(d)$.

Using (3.10), we see that Lemma 2.2 applies with $N = U(d)$. Hence after rescaling the weights, we obtain

$$E_1^\pm(d) \leq X^{1+\varepsilon+o(1)}d^{-1}U(d)^{3/4} \left( A^{1/8}d + A^{1/2}d^{3/4} \right)$$

$$= X^{1+\varepsilon+o(1)}U(d)^{3/4} \left( A^{1/8} + A^{1/2}d^{-1/4} \right).$$

Thus recalling the definition of $U(d)$ in (2.3), we see that for any $d | q$ we have

$$E_1^\pm(d) \leq X^{1/4+\varepsilon} \left( A^{1/8}q^{3/2} + A^{1/2}q^{5/4} \right) q^{\circ(1)}. \tag{3.12}$$

To estimate $E_2^\pm(d)$, we note that by (2.4) we have

$$|\tau(n)u_d^\pm(n)| \leq d^{1/2}X^{1/4}V_i(d)^{-3/4}q^{\circ(1)}$$

for $V_i(d) \leq n < V_{i+1}(d)$. We also recall (3.10), hence by Lemma 2.2 (after rescaling the weights again) with $N = V_{i+1}(d)$, as in (3.12), we obtain

$$E_2^\pm(d) \leq d^{1/2}X^{1/4} \left( A^{1/8}d + A^{1/2}d^{3/4} \right) q^{\circ(1)}$$

$$\leq X^{1/4+\varepsilon} \left( A^{1/8}q^{3/2} + A^{1/2}q^{5/4} \right) q^{\circ(1)}. \tag{3.13}$$

Substituting (3.12) and (3.13) in (3.11) we obtain

$$\mathcal{E}^*(X; I, q) \leq X^{1/4+\varepsilon} \left( A^{1/8}q^{1/2} + A^{1/2}q^{1/4} \right) q^{\circ(1)}.$$

Hence, recalling (2.9) and also using that $Y/q \geq 1$, we obtain

$$\mathcal{E}(X; I, q) \ll X^{1/4+\varepsilon} \left( A^{1/8}q^{3/2} + A^{1/2}q^{5/4} \right) q^{\circ(1)} + AY^{1+\varepsilon}q^{-1+\varepsilon}.$$

Using that $\varepsilon > 0$ is arbitrary and recalling the choice of $Y$ in (3.9), we obtain

$$\mathcal{E}(X; I, q) \leq \left( A^{1/8}X^{1/4}q^{3/2} + A^{1/2}X^{1/4}q^{5/4} + AX^{1/2}q^{-1/2} \right) q^{\circ(1)},$$

which concludes the proof.

3.3. Proof of Theorem 1.3. Let $U$ and $V$ be as in (3.2) and apply (2.2) with

$$Y = \frac{X^{1/3}p}{A^{1/3}}, \tag{3.14}$$
to obtain
\[
D(X; A, p) \ll \frac{1}{p} \sum_{\pm A} \left| \sum_{n=1}^{\infty} \tau(n) u_p^\pm(n) K_p(\mp n, a) \right| + A \left( \frac{Y}{p} + 1 \right) (qY)^\varepsilon
\]
(3.15)
\[
= \frac{1}{p} (S^+ + S^-) + A \left( \frac{Y}{p} + 1 \right) (pY)^\varepsilon,
\]
where
\[
S^\pm = \sum_{\pm A} \left| \sum_{n=1}^{\infty} \tau(n) u_p^\pm(n) K_p(\mp n, a) \right|.
\]
We consider \( S^- \), a similar argument applies to \( S^+ \). With notation as in Lemma 2.7, we have
\[
S^- = \sum_{\pm A} \left| \sum_{m,n=1}^{\infty} u_p^-(mn) K_p(mn, a) \right|
\]
\[
= \sum_{\pm A} \left| \sum_{m,n=1}^{\infty} u_p^-(mn) K_p(mn, a) \sum_{j,k=0}^{\infty} \Psi_j(m) \Psi_k(n) \right|
\]
\[
\leq \sum_{j,k=0}^{\infty} \sum_{\pm A} \left| \sum_{m,n=\mathbb{Z}} \Psi_j(m) \Psi_k(n) u_p^-(mn) K_p(mn, a) \right|.
\]
We note that we extend the summation over \( m \) and \( n \) to \( \mathbb{Z} \) to be able to apply Lemma 2.5 (since the functions \( \Psi_\ell \) are supported only on positive integers this does not change the sum). Hence
\[
S^- \ll \sum_{j,k=0}^{\infty} S^-(j, k),
\]
where
\[
S^-(j, k) = \sum_{\pm A} \left| \sum_{m,n=\mathbb{Z}} \Psi_j(m) \Psi_k(n) u_p^-(mn) K_p(mn, a) \right|,
\]
and we recall that for each integer \( \ell \geq 0 \), \( \Psi_\ell \) is a smooth function satisfying
\[
\text{supp} \Psi_\ell \subseteq [2^{\ell-1}, 2^{\ell+1}] \quad \text{and} \quad \Psi_\ell(x) \ll 1.
\]
Let
\[
Q = \max \left\{ U, \frac{p}{A} \right\},
\]
define the integer $s$ by

$$2^{s-1}Q \leq V \leq 2^s Q,$$

and set

$$V_i = \min\{2^i Q, V\}, \quad i = 0, \ldots, s.$$  

Recalling (3.14), we note that $U \leq Q \leq V$.

We partition

$$(3.18) \quad S^- \ll \sum_{j,k \atop 2^{j+k} \leq Q} S^-(j, k) + \sum_i^s \sum_{j,k = 0 \atop V_i < 2^{j+k} \leq 2V}^\infty S^-(j, k) + \sum_{j,k = 0 \atop 2^{j+k} \geq 2V}^\infty S^-(j, k).$$

By (2.5), as in the derivation of (2.6), the contribution from the terms with $2^{j+k} \geq 2V$ is negligible, and in particular we have

$$\sum_{j,k = 0 \atop 2^{j+k} \geq 2V}^\infty S^-(j, k) = O(1),$$

which after substitution to (3.18) gives

$$(3.19) \quad S^- \ll \sum_{j,k = 0 \atop 2^{j+k} \leq Q}^\infty S^-(j, k) + \sum_i^s \sum_{j,k = 0 \atop V_i < 2^{j+k} \leq 2V}^\infty S^-(j, k) + O(1).$$

We first estimate the contribution to $S^-$ from terms with $2^{j+k} \leq Q$. With $U = U(q)$ as in (2.3), suppose first that $2^{j+k} \leq U$. By (2.4), (3.16) and the Weil bound for Kloosterman sums, see for example [9, Theorem 11.11], we have

$$S^-(j, k) \ll \sum_{a \in A} \sum_{m,n \in \mathbb{Z}} |\Psi_j(m)||\Psi_k(n)||u_p^-(mn)||K_p(mn, a)| \ll A p^{3/2} X^{\varepsilon}.$$  

If $U \leq 2^{j+k} \leq Q$ then

$$S^-(j, k) \ll \sum_{a \in A} \sum_{m,n \in \mathbb{Z}} |\Psi_j(m)||\Psi_k(n)||u_p^-(mn)||K_p(mn, a)| \ll 2^{(j+k)/4} A X^{1/4} p^{1+\varepsilon} \ll A Q^{1/4} X^{1/4} p^{1+\varepsilon},$$

and hence by (3.19)

$$(3.20) \quad S^- \ll A p^{3/2+\varepsilon} (\log p)^2 + Q^{1/4} A X^{1/4} p^{1+\varepsilon} + \sum_i^s \sum_{V_i < 2^{j+k} \leq 2V} S^-(j, k).$$
Fix some integer \( t \leq s \), some pair \((j, k)\) satisfying
\[
3.21 \quad V_t < 2^{j+k} \leq 2V_t,
\]
and consider \( S^-(j, k) \). Expanding the Kloosterman sum and interchanging summation, we have
\[
S^-(j, k) = \sum_{a \in A} \left| \sum_{y=1}^{p-1} e_p(ay^{-1}) \sum_{m,n \in \mathbb{Z}} \Psi_j(m)\Psi_k(n)u_p^-(mn) e_p(mny) \right|.
\]
Applying Lemma 2.5 with
\[
3.22 \quad g(x, y) = \Psi_j(m)\Psi_k(n)u_p^-(mn),
\]
gives
\[
S^-(j, k) = \sum_{a \in A} \left| \sum_{n \in \mathbb{Z}} \tau_h(n) \sum_{y=1}^{p-1} e_p((a-n)y^{-1}) \right|,
\]
with notation as in Lemma 2.5 and \( h \) is given by
\[
h(x, y) = \frac{1}{p} \hat{g} \left( \frac{x}{p}, \frac{y}{p} \right).
\]
For some sequence of complex numbers \( \vartheta(a) \) satisfying \( |\vartheta(a)| = 1 \) we have
\[
3.23 \quad S^-(j, k) = \sum_{y=1}^{p-1} \sum_{a \in A} \sum_{n \in \mathbb{Z}} \tau_h(n) \vartheta(a) e_p((a-n)y^{-1}),
\]
and hence
\[
S^-(j, k) = p \sum_{a \in A, n \in \mathbb{Z}} \frac{\vartheta(a) \tau_h(n)}{n \equiv a \pmod{p}} - \sum_{a \in A} \vartheta(a) \sum_{n \in \mathbb{Z}} \tau_h(n).
\]
Since \( 0 \notin A \) and \( p \) is prime, using (2.11) to control the condition \( n \equiv a \pmod{p} \) via multiplicative characters, we have
\[
\sum_{a \in A, n \in \mathbb{Z} \atop n \equiv a \pmod{p}} \vartheta(a) \tau_h(n) = \frac{1}{p-1} \sum_{\chi} \left( \sum_{a \in A} \vartheta(a) \overline{\chi(a)} \right) \left( \sum_{n \in \mathbb{Z}} \tau_h(n) \chi(n) \right),
\]
and isolating the contribution from the trivial character gives
\[
\sum_{a \in A, n \in \mathbb{Z} \atop n \equiv a \pmod{p}} \vartheta(a) \tau_h(n) = \frac{1}{p-1} \sum_{\chi \neq \chi_0} \left( \sum_{a \in A} \vartheta(a) \overline{\chi(a)} \right) \left( \sum_{n \in \mathbb{Z}} \tau_h(n) \chi(n) \right)
+ \frac{1}{p-1} \sum_{a \in A} \vartheta(a) \sum_{n \in \mathbb{Z}} \tau_h(n) - \frac{1}{p-1} \sum_{a \in A} \vartheta(a) \sum_{n \equiv 0 \pmod{p}} \tau_h(n).
Hence by (3.23)

(3.24) \[ S^-(j, k) \ll \Sigma_1 + \frac{1}{p} \Sigma_2 + \Sigma_3, \]

where

\[
\begin{align*}
\Sigma_1 &= \sum_{\chi \neq \chi_0} \left( \sum_{a \in A} \vartheta(a) \overline{\chi}(a) \right) \left( \sum_{n \in \mathbb{Z}} \tau_h(n) \chi(n) \right), \\
\Sigma_2 &= \sum_{a \in A} \vartheta(a) \sum_{n \in \mathbb{Z}} \tau_h(n), \\
\Sigma_3 &= \sum_{a \in A} \vartheta(a) \sum_{n \equiv 0 \pmod{p}} \tau_h(n).
\end{align*}
\]

Considering first \( \Sigma_2 \), we have

\[
\sum_{n \in \mathbb{Z}} \tau_h(n) = \frac{1}{p} \sum_{m, n \in \mathbb{Z}} \hat{g} \left( \frac{m}{p}, \frac{n}{p} \right).
\]

With \( g \) as in (3.22) we have

(3.25) \[ \sum_{m, n \in \mathbb{Z}} \hat{g} \left( \frac{m}{p}, \frac{n}{p} \right) = p^2 \sum_{m, n \in \mathbb{Z}} g(pm, pn). \]

Indeed, to see this it is enough to notice that for \( G(x, y) = g(px, py) \) we have

\[
\hat{G}(u, z) = \int_{\mathbb{R}^2} G(x, y) e(u \cdot x + z \cdot y) \, dx \, dy
\]

\[
= \frac{1}{p^2} \int_{\mathbb{R}^2} g(px, py) e((u/p)(px) + (v/p)(py)) \, d(px) \, d(py)
\]

\[
= \frac{1}{p^2} \int_{\mathbb{R}^2} g(x, y) e((u/p)x + (v/p)y) \, dx \, dy = \frac{1}{p^2} \hat{g} \left( \frac{m}{p}, \frac{n}{p} \right),
\]

and we obtain (3.25) by Poission summation. Hence, from the equations (3.22) and (3.25), we see that

\[
\sum_{n \in \mathbb{Z}} \tau_h(n) = p \sum_{m, n \in \mathbb{Z}} \Psi_j(pm) \Psi_k(pn) u_p(p^2mn).
\]
By (2.4) and (3.21)
\[ \sum_{m,n \in \mathbb{Z}} |\Psi_j(pm)||\Psi_k(pn)||u^-_p(p^2mn)| \]
\[ \ll \sum_{2^j/p \leq m \leq 2^{j+1}/p} \sum_{2^{k-1}/p \leq m \leq 2^{k+1}/p} |u^-_p(p^2mn)| \]
\[ \ll \frac{2^{j+k} X^{1/4} p^{1/2}}{p^2} \ll X^{1/4} V_t^{1/4} p^{-3/2}, \]
so that
\[ (3.26) \sum_{n \in \mathbb{Z}} \tau_h(n) \ll X^{1/4} V_t^{1/4} p^{-1/2}, \]
which implies
\[ (3.27) \Sigma_2 \ll AX^{1/4} V_t^{1/4} p^{-1/2}. \]
Considering \(\Sigma_3\), by (3.26) we have
\[ \sum_{n \equiv 0 \pmod{p}} \sum_{n \in \mathbb{Z}} \tau_h(n) \]
\[ = \frac{1}{p} \sum_{z=1}^{p-1} \sum_{n \in \mathbb{Z}} \tau_h(n) e_p(zn) + \frac{1}{p} \sum_{n \in \mathbb{Z}} \tau_h(n) \]
\[ = \frac{1}{p} \sum_{z=1}^{p-1} \sum_{n \in \mathbb{Z}} \tau_h(n) e_p(zn) + O \left( X^{1/4} V_t^{1/4} p^{-3/2} \right). \]
By Lemma 2.5 and (3.22)
\[ \sum_{n \in \mathbb{Z}} \tau_h(n) e_p(zn) = \sum_{n \in \mathbb{Z}} \tau_g(n) e_p(-z^{-1}n), \]
so that
\[ \sum_{z=1}^{p-1} \sum_{n \in \mathbb{Z}} \tau_h(n) e_p(zn) = \sum_{n \in \mathbb{Z}} \tau_g(n) \sum_{z=1}^{p-1} e_p(-z^{-1}n) \ll p \sum_{n \equiv 0 \pmod{p}} \tau_g(n) + \sum_{n \in \mathbb{Z}} \tau_g(n). \]
We have
\[ \sum_{n \equiv 0 \pmod{p}} \tau_g(n) \ll \sum_{n,m \in \mathbb{Z}} |\Psi_j(pm)||\Psi_k(pn)||u^-_p(pmn)| \]
\[ + \sum_{n,m \in \mathbb{Z}} |\Psi_j(m)||\Psi_k(pn)||u^-_p(pmn)|. \]
For the first sum above
\[
\sum_{n,m \in \mathbb{Z}} \Psi_j(pm) \Psi_k(n) u_p^{-}(pmn) \ll \sum_{2^{j-1} \leq m \leq 2^{j+1} / p} \sum_{2^{k-1} \leq n \leq 2^{k+1}} |u_p^{-}(pmn)|
\ll \frac{2^{j+k} X^{1/4} p^{1/2}}{p} \ll X^{1/4} V_t^{1/4} p^{-1/2},
\]
and similarly for the second sum
\[
\sum_{n,m \in \mathbb{Z}} |\Psi_j(m)||\Psi_k(pm)||u_p^{-}(pmn)| \ll X^{1/4} V_t^{1/4} p^{-1/2}.
\]
A similar argument shows
\[
\sum_{n \in \mathbb{Z}} \tau_g(n) \ll X^{1/4} V_t^{1/4} p^{1/2},
\]
so that
\[
\sum_{z=1}^{p-1} \sum_{n \in \mathbb{Z}} \tau_h(n) e_p(zn) \ll X^{1/4} p^{1/2} V_t^{1/4}.
\]
Hence by (3.28)
\[
(3.29) \quad \Sigma_3 \ll A X^{1/4} V_t^{1/4} p^{-1/2}.
\]
It now remains to estimate \( \Sigma_1 \). B Lemma 2.6 we have
\[
(3.30) \quad \Sigma_1 \ll \sum_{\chi \neq \chi_0} \left| \sum_{a \in A} \vartheta(a) \chi(a) \right| \left| \sum_{n \in \mathbb{Z}} \tau_g(n) \chi(-n) \right|,
\]
and hence by the Cauchy-Schwarz inequality
\[
\Sigma_1^2 \ll \sum_{\chi} \left| \sum_{a \in A} \vartheta(a) \chi(a) \right|^2 \sum_{\chi} \left| \sum_{n \in \mathbb{Z}} \tau_g(n) \chi(-n) \right|^2
\ll p^2 A \sum_{\substack{m,n \in \mathbb{Z} \\ \gcd(mn,p)=1 \\ n \equiv m \pmod{p}}} \tau_g(m) \tau_g(n).
\]
We have
\[
\sum_{\substack{m,n \in \mathbb{Z} \\ \gcd(mn,p)=1 \\ n \equiv m \pmod{p}}} \tau_g(m) \tau_g(n) \ll \sum_{2^{j-1} \leq m_1, m_2 \leq 2^{j+1}} \sum_{2^{k-1} \leq n_1, n_2 \leq 2^{k+1}} \sum_{\substack{m_1 \equiv n_2 m_2 \pmod{p} \\ \gcd(m_1 m_2 n_1 n_2, p)=1}} |u_p^{-}(m_1 n_1)||u_p^{-}(m_2 n_2)|.
\]
By (2.4) and Lemma 2.4
\[
\sum_{m,n \in \mathbb{Z}, \gcd(mn,p) = 1 \atop n \equiv m \pmod{p}} \tau_g(m)\tau_g(n) \leq \frac{X^{1/2}p^{5/2}}{2^{3(i+j)/2}} \left( \frac{2^{2(i+j)}}{p} + 2^{i+j} \right) p^{o(1)}
\]
and hence by (3.30)
\[
\Sigma_1 \leq A^{1/2} \left( X^{1/4}V_t^{1/4} + \frac{X^{1/4}p^{1/2}}{V_t^{1/4}} \right) p^{1+o(1)}.
\]
Since \( A < p \), from (3.24), (3.27) and (3.29)
\[
S^-(j,k) \leq A^{1/2} \left( X^{1/4}V_t^{1/4} + \frac{X^{1/4}p^{1/2}}{V_t^{1/4}} \right) p^{1+o(1)},
\]
and hence by (3.20)
\[
S^- \ll Ap^{3/2+\varepsilon+o(1)} + Q^{1/4}AX^{1/4}p^{1+\varepsilon} + \sum_{i=0}^{s} \sum_{V_i < 2^{j+k} \leq 2V_t} A^{1/2} \left( X^{1/4}V_t^{1/4} + \frac{X^{1/4}p^{1/2}}{V_t^{1/4}} \right) p^{1+o(1)}.
\]
Since the above sum contains \( O((\log X)^3) \) terms, in the negative powers of \( V_t \) we replace \( V_t \) with its smallest possible value \( Q \), while in the positive powers of \( V_t \) we replace \( V_t \) with its largest possible value \( V \) and derive
\[
S^- \ll Ap^{3/2+\varepsilon+o(1)} + Q^{1/4}AX^{1/4}p^{1+\varepsilon} + A^{1/2}X^{1/4}V^{1/2}p^{1+o(1)} + \frac{A^{1/2}X^{1/4}}{Q^{1/4}p^{5/2+o(1)}}.
\]
Recalling (3.17) we see that \( Q \geq p/A \) thus
\[
\frac{A^{1/2}X^{1/4}}{Q^{1/4}p^{5/2}} \leq \frac{A^{1/2}X^{1/4}p^{3/2+o(1)}}{(p/A)^{1/4}} = A^{3/4}X^{1/4}p^{5/4}.
\]
Moreover
\[
Q^{1/4}AX^{1/4}p \leq U^{1/4}AX^{1/4}p + A^{3/4}X^{1/4}p^{5/4} = Ap^{3/2} + A^{3/4}X^{1/4}p^{5/4}.
\]
Therefore, we obtain
\[
S^- \ll Ap^{3/2+\varepsilon+o(1)} + A^{3/4}X^{1/4}p^{5/4+\varepsilon} + A^{1/2}X^{1/4}V^{1/4}p^{1+o(1)}.
\]
A similar estimate holds for \( S^+ \) and hence by (2.3) and (3.15)
\[
D(X; A, p) \ll Ap^{1/2+\varepsilon+o(1)} + A^{3/4}X^{1/4}p^{1/4+\varepsilon} + \frac{A^{1/2}X^{1/2}p^{1/2+o(1)}}{Y^{1/2}}
\]
\[
+ A \left( \frac{Y}{p} + 1 \right) (pY)^\varepsilon,
\]
which by (3.14) implies
\[
D(X; A, p) \leq (Ap^{1/2} + A^{3/4}X^{1/4}p^{1/4} + A^{2/3}X^{1/3}) (pX)^{\varepsilon+o(1)}.
\]
Since \( \varepsilon > 0 \) is arbitrary, we obtain
\[
D(X; A, p) \leq (Ap^{1/2} + A^{3/4}X^{1/4}p^{1/4} + A^{2/3}X^{1/3}) X^{o(1)}.
\]
Clearly, if \( p > X/A \) then \( Ap^{1/2} > A^{1/2}X^{1/2} \) and thus the bound (1.4) is stronger. On the other hand, for \( p \leq X/A \) we have \( Ap^{1/2} \leq A^{3/4}X^{1/4}p^{1/4} \), which this completes the proof.

4. Remarks

For almost all \( q \) a stronger version of Lemma 2.2 has also been given in [14]. In turn, this can be used to improve Theorem 1.2 for almost all moduli \( q \). In fact, limiting this set of moduli to only prime \( q = p \) simplifies this question significantly. For composite values of \( q \) one also has to eliminate \( q \) having an “undesirable” divisor \( d \mid q \), yet there is little doubt that this can be done.

It is also interesting to extend Theorem 1.1 to arbitrary integer moduli \( q \). Unfortunately, the analogues of the bounds on bilinear sums \( S_d(\alpha, \nu; I, J) \) from [4] are known only for prime \( q \), while the method of [14, 15] seems to work only for the sums \( S_d(\nu; I, J) \).

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Department of Pure Mathematics, University of New South Wales, Sydney, NSW 2052, Australia

E-mail address: bryce.kerr89@gmail.com

Department of Pure Mathematics, University of New South Wales, Sydney, NSW 2052, Australia

E-mail address: igor.shparlinski@unsw.edu.au