LOWER BOUNDS FOR UNCENTERED MAXIMAL FUNCTIONS IN ANY DIMENSION

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Abstract. In this paper we address the following question: given $p \in (1, \infty)$, $n \geq 1$, does there exists a constant $A(p, n) > 1$ such that $\|Mf\|_{L^p} \geq A(n, p)\|f\|_{L^p}$ for any nonnegative $f \in L^p(\mathbb{R}^n)$, where $Mf$ is a maximal function operator defined over the family of shifts and dilates of a centrally symmetric convex body. The inequality fails in general for the centered maximal function operator, but nevertheless we give an affirmative answer to the question for the uncentered maximal function operator and the almost centered maximal function operator. In addition, we also present the Bellman function approach of Melas, Nikolidakis and Stavropoulos to maximal function operators defined over various types of families of sets, and in case of parallelepipeds we will show that $A(n, p) = (\frac{p}{p-1})^{1/p}$.

1. Maximal function operators and main results

1.1. Centrally symmetric convex bodies. Fix any centrally symmetric convex body $K$ in $\mathbb{R}^n$ (that is, a compact convex set with non-empty interior). Let $K$ be the family of all shifts of dilations of $K$. For $\lambda \in [0, 1]$, and a centrally symmetric convex body $S$, set $\lambda S$ to be the image of $S$ under the homothety with the center of the $S$, and ratio $\lambda$. If $\lambda = 0$ then $\lambda S \define \{ x \}$ where $x$ is the center of $S$. Given any nonnegative locally integrable function $f$, we define the maximal operator

$$(M_\lambda f)(x) \define \sup_{S \in K : \lambda S \ni x} \frac{1}{|S|} \int_S f,$$

where $|S|$ denotes Lebesgue measure of the set $S$. Notice that $M_0 f$ is the usual centered maximal function operator, while $M_1 f$ is the uncentered maximal function operator. Our first main result is the following theorem.

Theorem 1. Fix $p \in (1, \infty)$, $n \geq 1$, and $\lambda \in (0, 1]$. There exists a constant $A(p, n, \lambda) > 1$ such that

$$\|M_\lambda f\|_{L^p} \geq A(n, p, \lambda)\|f\|_{L^p}, \text{ for all } f \geq 0, f \in L^p(\mathbb{R}^n).$$

This result answers a question raised to the authors by Andrei Lerner during his visit to Kent State. In the case $n = 1$ and $\lambda = 1$, Theorem 1.1 is due to Lerner [11]. There had recently been some activity in understanding the analogous problem for dyadic maximal operators, see A. Melas, E. Nikolidakis, Th. Stavropoulos [13] and Section 1.2 below, but Theorem 1 appears to be new in general for even the uncentered maximal operator if $n > 1$. 

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Theorem 1 does not hold in general for the centered maximal function operator $M_0 f$. Indeed, let $K$ be the unit ball in $\mathbb{R}^n$, $n \geq 3$. Take $f(x) = \min \{|x|^{2-n}, 1\}$ where $p > \frac{n}{n-2}$. Then $f \in L^p(\mathbb{R}^n)$, and, since $f$ is superharmonic, we have $M_0 f = f$ and so $\|M_0 f\|_{L^p} = \|f\|_{L^p}$.

On the other hand if $n = 1, 2$ and $1 < p < \infty$, or $n \geq 3$ and $p \leq \frac{n}{n-2}$, then any function $f \in L^p(\mathbb{R}^n)$ satisfying $\|M_0(f)\|_{L^p} = \|f\|_{L^p}$ (or equivalently $M_0 f = f$) must be zero, see for instance [10, 12, 3]. Hence for this set of exponents we have $\|M_0 f\|_p > \|f\|_p$ for each $f \in L^p$.

We should also mention [9] where (1,1) was investigated in the case $n = 1$, $p = 1$, and $\lambda = 0$, with $\|M_0 f\|_{L^1}$ replaced by the weak (1,1) norm, and it was shown that in that case there is no such $A > 1$.

In Section 2 we provide a simple proof of Theorem 1 in the case when $\lambda = 1$. In Section 3 we adapt the proof to the general case $\lambda \in (0, 1)$.

Let us now make a few remarks about the nature of the constant $A(n, p, \lambda)$. Our proof yields $A(n, p, \lambda) = 1 + \varepsilon(n, p, \lambda)$, where $\varepsilon(n, p, \lambda)$ decays exponentially with the dimension $n$. This dependence is a direct consequence of our use of the Besicovitch covering lemma, and we do not know what the true dependence should be. The constant $\varepsilon(n, p, \lambda)$ must (in general) tend to zero as $\lambda \to 0$, but the dependence on $\lambda$ in our argument is essentially qualitative, as we rely on a compactness argument (Lemma 1 below). Finally, $\varepsilon(n, p, \lambda)$ is comparable with $\frac{1}{p-1}$ as $p \to 1^+$, as it should be.

### 1.2. Other maximal functions.**

We denote by $\langle f \rangle_A$ the integral average of $f$ over a measurable set $A$, i.e., $\langle f \rangle_A = \frac{1}{|A|} \int_A f$. If $|A| = 0$ then we set $\langle f \rangle_A = 0$. Let $Q$ be some family of convex bodies in $\mathbb{R}^n$. We define the maximal function operator over the family $Q$ as follows:

$$M_Q f(x) = \sup_{Q \ni x : Q \in Q} \langle f \rangle_A.$$

**Definition 1.** We say that the family $Q$ is $\lambda$-dense ($\lambda > 1$) if for any locally integrable $f \geq 0$ and any $Q \in Q$ there exists a filtration $\{F_n\}_{n=0}^\infty$ associated to $Q$ and $f$ such that

1. $F_0 = \{Q\}$.
2. For any $n \geq 0$, $F_n$ consists by at most countable number of sets from $Q$.
3. $Q = \bigcup_{P \in F_n} P$ for any $n \geq 0$.
4. The elements of $F_n$ are almost disjoint, i.e., $|P \cap R| = 0$ for any different $P, R \in F_n$ and for any $n \geq 1$.
5. $F_{n+1}$ is a refinement of $F_n$, i.e., for any $P \in F_n$ there is a family of sets $\text{ch}(P) \subset F_{n+1}$ such that $P = \bigcup_{R \in \text{ch}(P)} R$.
6. $\lim_{n \to \infty} \sup_{P \in F_n} \text{diam}(P) = 0$.
7. $\sup_{R \in \text{ch}(P)} \langle f \rangle_R \leq \lambda \langle f \rangle_P$ for any $P \in F_n$ and any $n \geq 0$.

We will show in Lemma 5 that the set of all parallelepipeds with sides parallel to some fixed linearly independent $n$ vectors in $\mathbb{R}^n$ is $\lambda$-dense for any $\lambda > 1$. In particular the set of all intervals on the real line is $\lambda$-dense for any $\lambda > 1$.

We say that the family $Q$ is exhaustive if for any compact $E \subset \mathbb{R}^n$ there exists $Q \in Q$ such that $E \subset Q$. Our second main result is the following theorem.

**Theorem 2.** If the family $Q$ is exhaustive and $\lambda$-dense for some $\lambda > 1$, then

$$\|M_Q f\|_{L^p} \geq \left( \frac{\lambda^p - 1}{\lambda^p - \lambda} \right)^{1/p} \|f\|_{L^p} \quad \text{for all} \quad f \geq 0, \quad f \in L^p(\mathbb{R}^n).$$
Since $\frac{\lambda^p-1}{\lambda^p-\lambda}$ is decreasing in $\lambda$ for $\lambda > 1$, we want to make $\lambda$ as close to 1 as possible. Also notice that if the family $Q$ is $\lambda$-dense for every $\lambda > 1$ then we can take $\lim_{\lambda \to 1^+} \left( \frac{\lambda-1}{\lambda^p-\lambda} \right)^{1/p} = \left( \frac{p}{p-1} \right)^{1/p}$. Thus by Lemma 5 we obtain the following corollary.

**Corollary 1.** Let $Q$ be the set of all parallelepipeds with sides parallel to some fixed linearly independent $n$ vectors in $\mathbb{R}^n$. Then

$$\|M_Q f\|_{L^p} \geq \left( \frac{p}{p-1} \right)^{1/p} \|f\|_{L^p} \quad \text{for all} \quad f \geq 0, \ f \in L^p(\mathbb{R}^n). \quad (1.2)$$

We wonder if the constant $\left( \frac{p}{p-1} \right)^{1/p}$ is the best possible in Corollary 1. In [11] Lerner obtained (1.2) in one dimensional case, i.e., $n = 1$. We think that the argument presented in [11] (see also Remark 4.1 in Section 4) does not extend to high dimensions $n > 1$.

It is not difficult to see that the set of dyadic cubes $Q$ in $\mathbb{R}^n$ is $2^n$-dense, and this is the smallest $\lambda$ one can choose. In this case we recover the inequality proved by Melas, Nikolidakis and Stavropoulos (see [13]).

**Corollary 2.** Let $Q$ be a set of dyadic cubes in $\mathbb{R}^n$. Then

$$\|M_Q f\|_{L^p} \geq \left( \frac{2^{np}-1}{2^{np}-2^n} \right)^{1/p} \|f\|_{L^p} \quad \text{for all} \quad f \geq 0, \ f \in L^p(\mathbb{R}^n).$$

2. **Uncentered maximal function operator and convex bodies**

In this section we present a simple proof of Theorem 1 in the case $k = 1$.

Assume that $f \geq 0$ is continuous with compact support. Fix $t > 0$ and consider the set

$$K(t) = \left\{ K \in \mathcal{K} : \int_K f = t|K| \right\}.$$

Clearly for each $t > 0$ the set of the centers of $K$ from $\mathcal{K}(t)$ belong to a bounded subset of $\mathbb{R}^n$. We apply the Besicovich covering lemma (see Lemma 3 in the Appendices) to extract a countable subfamily $K_{t,j} \in \mathcal{K}(t)$ so that the function

$$\psi(x,t) = \sum_j \chi_{K_{t,j}}(x)$$

satisfies the following properties:

1) For all $x \in \mathbb{R}^n$, $t > 0$, we have $\psi(x,t) \leq B(n)$ with some constant $B(n)$ depending only on the dimension $n$;

2) If $t > M_1 f(x)$ then $\psi(x,t) = 0$;

3) If $f(x) > t$, then $\psi(x,t) \geq 1$;

4) For every $t > 0$, we have $\int_{\mathbb{R}^n} t\psi(x,t)dx = \int_{\mathbb{R}^n} \psi(x,t)f(x)dx$.

The first property follows from Lemma 3 where $B(n)$ is a Besicovich constant. This number is independent of $t$ and depends only on the dimension $n$.

For the second property, if $t > M_1 f(x)$ then no $K_{t,j}$ contains $x$. Indeed, otherwise if some $K_{t,\ell}$ contains $x$ then $M_1 f(x) \geq \langle f \rangle_{K_{t,\ell}} = t$ by the choice of the family $\mathcal{K}(t)$.

To verify the third property assume $f(x) > t$. Let $K(x)$ be a shift of $K$ centered at $x$. By the intermediate value theorem the continuous function $p(s) = \langle f \rangle_{sK(x)}$ attains value $t$ for some finite positive number $s^*$. Then by the choice of the family $K_{t,j}$ there exists $K_{t,\ell}$ which contains the center of $s^*K(x)$, i.e., $x$, and the property follows.
The fourth property follows immediately from the fact that \( \int_{\mathbb{R}^n} t \chi_{K_{t,j}} = \int_{\mathbb{R}^n} \chi_{K_{t,j}} f(x)dx \) for all \( j \) and \( t > 0 \).

Now the last property, after multiplying both sides by \( t^{p-2} \) and integrating with respect to \( t \) over the ray \((0, \infty)\), yields the equality:

\[
\int_{\mathbb{R}^n} \int_0^\infty t^{p-1} \psi(x, t) dtdx = \int_{\mathbb{R}^n} \int_0^\infty t^{p-2} \psi(x, t) f(x)dtdx.
\] (2.1)

On the left hand side of (2.1) we can restrict the integration with respect to \( t \) to \([0, M_1 f(x)]\) (by property 2). We will estimate the right hand side from below by restricting the inner integration to the interval \([0, f(x)]\). The obtained inequality (together with the properties 2 and 3) justifies the following chain of inequalities:

\[
\frac{B(n)}{p} (\|M_1 f\|_{L^p}^p - \|f\|_{L^p}^p) \geq \int_{\mathbb{R}^n} \int_0^{M_1 f(x)} t^{p-1} \psi(x, t) dtdx - \int_{\mathbb{R}^n} \int_0^{f(x)} t^{p-1} \psi(x, t) dtdx \\
\int_{\mathbb{R}^n} \int_0^{f(x)} t^{p-1} \psi(x, t) \left( \frac{f(x)}{t} - 1 \right) dtdx \geq \int_{\mathbb{R}^n} \int_0^{f(x)} t^{p-1} \left( \frac{f(x)}{t} - 1 \right) dtdx = \|f\|_{L^p}^p.
\]

Thus we obtain (1.1) with the constant \( A(p, n, 1) = \left( 1 + \frac{1}{(p-1)B(n)} \right)^{1/p} \).

3. Almost centered maximal function operator and convex bodies

In this section we work with the operator \( M_\lambda \) for \( \lambda \in (0, 1) \). Assume that \( f \geq 0 \) is continuous with compact support. Fix \( t > 0 \), and consider the family of sets

\( K(t) = \{ K \in \mathcal{K} : \langle f \rangle_K = t \} \)

as before. Once again, we use the Besicovitch covering lemma (Lemma 3 below) to extract the family \( K_{t,j} \) so that the sets \( K_{t,j} \) cover the centers of the sets in \( K(t) \). Set \( \psi(x, t) = \sum_j \chi_{K_{t,j}}(x) \).

In precisely the same manner as in Section 2, we notice the following properties:

1) For all \( x \in \mathbb{R}^n, t > 0 \), we have \( \psi(x, t) \leq B(n) \) with some constant \( B(n) \) depending only on the dimension \( n \);

2) If \( f(x) > t \), then \( \psi(x, t) \geq 1 \);

Unfortunately it is not true that if \( t > M_\lambda f(x) \) then \( \psi(x, t) = 0 \), and therefore we cannot repeat the proof as in the previous section. However, to compensate for the lack of this property we will prove the following dichotomy.

**Lemma 1.** For every \( \varepsilon \in (0, 1) \) there exists \( \eta > 0 \) such that for every \( K \in \mathcal{K} \) and any function \( f \geq 0 \), either

\[
\int_K M_\lambda f(x)dx \geq (1 + \eta) \int_K f(x)dx,
\]

\[
M_\lambda f(x) \geq (1 - \varepsilon) \frac{1}{|K|} \int_K f(x)dx \quad \text{on} \quad (1 - \varepsilon)K.
\]

Before we proceed to the proof of the lemma let us show how it implies the desired estimate. For each \( t > 0 \), the family \( \{ K_{t,j} \} \) can be divided into two subfamilies \( \{ K'_{t,j} \} \) and \( \{ K''_{t,j} \} \) so that the sets \( K'_{t,j} \) satisfy (3.1), and the sets \( K''_{t,j} \) satisfy (3.2). Set

\[
\psi_1(x, t) = \sum_j \chi_{K'_{t,j}}(x), \quad \psi_2(x, t) = \sum_j \chi_{K''_{t,j}}(x) \quad \text{and} \quad \psi_2(x, t) = \sum_j \chi_{(1-\varepsilon)K''_{t,j}}(x).
\]
Clearly \( \psi_1 + \psi_2 \geq 1 \) if \( f(x) > t \), and \( \psi_1, \psi_2 \leq B(n) \) for all \( t > 0 \) and \( x \in \mathbb{R}^n \). We notice that

\[
\text{if } t > \lambda^{-n} M_A f(x) \text{ then } \psi_1(x, t) = 0. \tag{3.3}
\]

Indeed, otherwise there exists a set \( K'_{t,\ell} \in \{K'_{t,j}\} \) containing \( x \) such that

\[
t = \frac{1}{|K'_{t,\ell}|} \int_{K'_{t,\ell}} f(x) dx \leq \frac{\lambda^{-n}}{|\lambda^{-1} K'_{t,\ell}|} \int_{\lambda^{-1} K'_{t,\ell}} \leq \lambda^{-n} M_A f(x).
\]

Note that (3.1) implies the following inequality

\[
\int_{\mathbb{R}^n} \int_0^\infty (M_A f(x) - f(x)) \psi_1(x, t) t^{p-2} dtdx \geq \eta \int_{\mathbb{R}^n} \int_0^\infty f(x) \psi_1(x, t) t^{p-2} dtdx. \tag{3.4}
\]

Since \( \psi_1 \leq B(n) \), and \( M_A f \geq f \), we have that

\[
\frac{B(n)}{(p-1)\lambda^{-n(p-1)}} \left( \|M_A f\|_p^p - \|f\|_p^p \right) \geq B(n) \int_{\mathbb{R}^n} [M_A f(x) - f(x)] \frac{(M_A f)^{p-1}}{(p-1)\lambda^{-n(p-1)}} dx
\]

\[
\geq \int_{\mathbb{R}^n} \int_0^\infty \lambda^{-n} M_A f(x) [M_A f(x) - f(x)] \psi_1(x, t) t^{p-2} dtdx.
\]

Notice that (3.3) enables us to extend the integration over \( t \) in the inner integral on the right hand side, and so we derive from (3.4) that

\[
\frac{B(n)}{\eta(p-1)\lambda^{-n(p-1)}} \left( \|M_A f\|_p^p - \|f\|_p^p \right) \geq \frac{1}{\eta} \int_{\mathbb{R}^n} \int_0^\infty [M_A f(x) - f(x)] \psi_1(x, t) t^{p-2} dtdx
\]

\[
\geq \int_{\mathbb{R}^n} \int_0^\infty f(x) \psi_1(x, t) t^{p-2} dtdx. \tag{3.5}
\]

Unfortunately we do not know that \( \psi_1 \geq 1 \) on the set \( \{ t < f(x) \} \), but instead only that \( \psi_1 + \psi_2 \geq 1 \), and so we need to invoke the function \( \psi_2 \).

We have \( \int_{K'_{t,\ell}} f(x) dx = \int_{K''_{t,\ell}} f(x) dx \), and so \( (1 - \varepsilon)^{-n} \int_{(1 - \varepsilon)K''_{t,\ell}} f(x) dx = \int_{K''_{t,\ell}} f(x) dx \). Tonelli’s theorem therefore yields that

\[
(1 - \varepsilon)^{-n} \int_{\mathbb{R}^n} \int_0^\infty \psi_2^2(x, t) t^{p-1} dtdx = \int_{\mathbb{R}^n} \int_0^\infty f(x) \psi_2^2(x, t) t^{p-2} dtdx. \tag{3.6}
\]

Since for each \( x \in (1 - \varepsilon)K''_{t,j} \) we have \( \frac{M_A f(x)}{1 - \varepsilon} > t \) then (3.6) can be rewritten as follows

\[
(1 - \varepsilon)^{-n} \int_{\mathbb{R}^n} \int_0^{M_A f(x)/(1 - \varepsilon)} \psi_2^2(x, t) t^{p-1} dtdx = \int_{\mathbb{R}^n} \int_0^\infty f(x) \psi_2^2(x, t) t^{p-2} dtdx. \tag{3.7}
\]

In the left hand side of (3.7) we estimate \( \psi_2^2 \) from above by \( \psi_1 + \psi_2 \). After combining the resulting inequality together with (3.5) we obtain

\[
\frac{B(n)}{(p-1)\lambda^{-n(p-1)}} \left( \|M_A f\|_p^p - \|f\|_p^p \right) + (1 - \varepsilon)^{-n} \int_{\mathbb{R}^n} \int_0^{M_A f(x)/(1 - \varepsilon)} t^{p-1}(\psi_2 + \psi_1)
\]

\[
\geq \int_{\mathbb{R}^n} \int_0^\infty f(x) (\psi_1 + \psi_2) t^{p-2} dtdx \geq \int_{\mathbb{R}^n} \int_0^\infty f(x) (\psi_1 + \psi_2) t^{p-2} dtdx.
\]
We now subtract the term \( \int_{\mathbb{R}^n} \int_0^{f(x)} t^{p-1}(\psi_1 + \psi_2)dt \) from both sides of previous inequality, which yields

\[
\frac{B(n)}{\eta(p - 1)\lambda^{-n(p-1)}}\left(\|M_{\alpha}f\|_p^p - \|f\|_p^p\right) + (1 - \varepsilon)^{-n} \int_{\mathbb{R}^n} \left(\int_{f(x)}^{M_{\alpha}(x)} t^{p-1}(\psi_1 + \psi_2)dt\right)dx
\]

\[+ ((1 - \varepsilon)^{-n} - 1) \int_{\mathbb{R}^n} \int_0^{f(x)} t^{p-1}(\psi_1 + \psi_2)dt \geq \int_{\mathbb{R}^n} \int_0^{f(x)} \left(\frac{f(x)}{t} - 1\right)(\psi_1 + \psi_2)t^{p-1}dt.
\]

As in the previous section, we estimate the left hand side of the previous display from above using the inequality \( B(n) \geq \psi_1 + \psi_2 \), and the right hand side from below using that estimate \( \psi_1 + \psi_2 \geq 1 \) in the domain of integration (notice that \( t < f(x) \)). Finally we obtain

\[
\frac{B(n)}{\eta(p - 1)\lambda^{-n(p-1)}}\left(\|M_{\alpha}f\|_p^p - \|f\|_p^p\right) + \frac{B(n)(1 - \varepsilon)^{-n}}{p} \left(\|M_{\alpha}f\|_p^p - \|f\|_p^p\right)
\]

\[+ ((1 - \varepsilon)^{-n} - 1)\frac{B(n)}{p} \|f\|_{L^p}^p \geq \frac{\|f\|_p^p}{p(p - 1)},
\]

which, when rearranged, becomes

\[
\|M_{\alpha}f\|_p^p \geq \left(1 + \frac{1 - (1 - \varepsilon)^{-n-p} + \frac{1}{B(n)(p-1)}}{\eta(p-1)\lambda^{-n(p-1)} + (1 - \varepsilon)^{-n-p}}\right)\|f\|_p^p.
\]

Choosing \( \varepsilon > 0 \) sufficiently small so that \( 1 - (1 - \varepsilon)^{-n-p} + \frac{1}{B(n)(p-1)} > 0 \) we obtain the desired estimate.

It remains to prove Lemma 1.

Proof of Lemma 1. Since the maximal function \( M_{\alpha}f(x) \) commutes with dilations and shifts, i.e., \( (M_{\alpha}f(\alpha \cdot + \beta))(x) = (M_{\alpha}f(\cdot))(\alpha x + \beta) \), it is enough to prove the lemma for some fixed \( K_0 \in \mathcal{K} \) with \( |K_0| = 1 \). Assume to the contrary that there exists \( \varepsilon_0 \in (0, 1) \) and a sequence of non-negative functions \( f_j \) that satisfy \( \int_{K_0} M_{\alpha}f_j(x)dx < (1 + \frac{1}{j}) \int_{K_0} f_j dx \) while, for every \( j \), the inequality \( M_{\alpha}f_j \geq (1 - \varepsilon_0)\frac{\int_{K_0} f_j dx}{|K_0|} \) does not hold on \( (1 - \varepsilon_0)K_0 \). By considering \( \frac{\lambda_{K_0} f_j}{\int_{K_0} f_j} \), we can assume that \( \int_{K_0} f_j = 1 \) and \( f_j \) is supported in \( K_0 \).

With a view to passing to a limit, we first claim that the sequence \( f_j \) is uniformly integrable on \( K_0 \).

To this end, recall Stein’s inequality [19], which states that there is a constant \( C > 0 \) depending only on \( n \), such that if \( f \) is a non-negative function with \( \int_{K_0} f = 1 \), then

\[
\int_{K_0} f \ln(\max\{1, f\})dx \leq C \int_{K_0} M_{\alpha}(f)dx.
\]

For the benefit of the reader we include a proof in an appendix (see Lemma 4).

Returning to our sequence \( f_j \), we find that \( \int_{K_0} f_j \ln(\max\{1, f_j\})dx \leq C(1 + \frac{1}{j}) \leq 2C \) and this readily yields that the sequence \( \{f_j\} \) is uniformly integrable.

Consequently, with the aid of the Dunford–Pettis theorem, we may (by passing to a subsequence if necessary) find \( f \in L^1(K_0) \) such that \( f_j \rightarrow f \) in the \( \sigma(L^1(K_0), L^\infty(K_0)) \) topology (i.e. the sequence \( f_j \) converges weakly to \( f \) over bounded functions).

It is clear that \( \int_{K_0} f = 1 \). Further we have that \( \liminf_j M_{\alpha}f_j \geq M_{\alpha}f \) a.e. on \( K_0 \), and so Fatou’s lemma yields that \( \int_{K_0} M_{\alpha}f \leq 1 \).

The properties \( \int_{K_0} f = 1 \) and \( \int_{K_0} M_{\alpha}f \leq 1 \) imply that \( M_{\alpha}f = f \) almost everywhere on \( K_0 \). We will show that \( f = 1 \) almost everywhere on \( K_0 \) and that this will contradict our
assumption that the inequality $M_j f_j \geq (1 - \varepsilon_0)$ fails to hold on $(1 - \varepsilon_0)K_0$ for sufficiently large $j$.

Fix $r > 0$. Set $f_r = f * \varphi_r$, where $\varphi \geq 0$ is a smooth bump function supported on $B(0, 1)$ such that $\int \varphi = 1$, and $\varphi_r(x) = r^{-n} \varphi(\frac{x}{r})$. If $f$ is non-constant (a.e.) on $K_0$, then we can find for arbitrarily small $r > 0$ a point $x_0 \in K_r = \{x \in K_0 : \text{dist}(x, \partial K_0) > r\}$ so that $\nabla f_r(x_0) \neq 0$.

Notice that $M_j f_r \leq \varphi_r * M_j f = \varphi_r * f = f_r$ on $K_r$, and therefore $M_j f_r = f_r$ on $K_r$.

Take any set $K' \in \mathcal{K}$ centered at $x_0$. Then (for instance by expanding $f_r$ in a Taylor series), we see that there is a constant $C > 0$, that may depend on $r$, such that $|\langle f_r \rangle_{K'} - f_r(x_0)| \leq C\|f_r\|_{L^\infty(K_0)}s^2$ for all sufficiently small $s$. But then provided that $s$ is small enough to ensure that $\lambda s K' \subset K_r$, we have that $f_r(z) = M_j f_r(z) \geq \langle f \rangle_{sK'}$ for all $z \in \lambda s K'$. Therefore, for sufficiently small $s$, we have

$$f_r(z) \geq \langle f_r \rangle_{sK'} \geq f_r(x_0) - C\|f_r\|_{L^\infty(K_0)}s^2 \text{ for every } z \in s \lambda K'. \quad (3.8)$$

This inequality already contradicts to our assumption that the gradient is not zero at point $x_0$. We therefore conclude that $f = 1$ on $K_0$.

Consider $(1 - \varepsilon_0)K_0$. There are a finite number of sets $K_\ell \subset K_0$ with $K_\ell \in \mathcal{K}$ such that the sets $\lambda K_\ell$ cover $(1 - \varepsilon_0)K_0$. Since the sequence $f_j$ converges weakly to the constant function 1 over $L^\infty(K_0)$, we have that $\langle f_j \rangle_{K_\ell} \to 1$ as $j \to \infty$ for each $\ell$. Consequently, we can choose sufficiently large $j_0 > 0$ such that $\langle f_j \rangle_{K_\ell} \geq 1 - \varepsilon_0$ for every $\ell$ and every $j \geq j_0$. Thus $M_j f_j \geq (1 - \varepsilon_0)$ on $(1 - \varepsilon)K_0$ for all $j \geq j_0$. This final contradiction completes the proof of the lemma. \qed

4. Bellman function approach

4.1. The proof of Theorem 2. In this section we will prove Theorem 2. Given compactly supported continuous $f \geq 0$, for any $Q \in \mathcal{Q}$ we set

$$(x_Q(f), y_Q(f), z_Q(f)) \overset{\text{def}}{=} (\langle f \rangle_Q, \langle f \rangle_Q^p, \sup_{R \in \mathcal{Q} : R \supset Q} \langle f \rangle_R).$$

Sometimes we will omit the variable $f$ and we just write $(x_Q, y_Q, z_Q)$. For a real number $a$ we set $(a)_+ \overset{\text{def}}{=} \max\{a, 0\}$. First we will prove the following lemma.

Lemma 2. If the family $\mathcal{Q}$ is $\lambda$-dense, then for any $S \in \mathcal{Q}$ and any $p > 1$ we have

$$\langle (M_Q f)^p \rangle_S \geq z_S^p(f) + \frac{\lambda^p - 1}{\lambda^p} (y_S(f) - x_S(f)z_S^{p-1}(f))_+.$$

First let us explain that the lemma implies Theorem 2. Indeed, since $\mathcal{Q}$ is exhaustive we can find a sequence of sets $S_0 \subset S_1 \subset \ldots$ so that $S_j \in \mathcal{Q}$ for all $j \geq 0$, and for any compact set $E \in \mathbb{R}^n$ there exists $S_\ell$ such that $E \subseteq S_\ell$. We apply the lemma to the sets $S_j$:

$$\int_{S_j} (M_Q f)^p \geq |S_j| z_{S_j}^p(f) + \frac{\lambda^p - 1}{\lambda^p} \left( \int_{S_j} f^p - |S_j| x_{S_j}(f) z_{S_j}^{p-1}(f) \right)_+. \quad (4.1)$$

As $j \to \infty$, the left hand side of (4.1) tends to $\|M_Q f\|_{L^p}^p$. On the right hand side of (4.1) we have $|S_j| z_{S_j}^p(f) \to 0$, $|S_j| x_{S_j}(f) z_{S_j}^{p-1} \to 0$, and $\int_{S_j} f^p \to \|f\|_{L^p}^p$ and the Theorem follows. Now we return to the proof of Lemma 2.
Proof. For $0 \leq x \leq z, \lambda > 1$ and $p > 1$ we define the Bellman function as follows

$$B(x, y, z; \lambda) = z^p + \frac{\lambda^p - 1}{\lambda^p - 1}(y - xz^{p-1})_+.$$ 

Fix some locally integrable $f \geq 0$ such that $\int f \neq 0$. Pick any $Q \in Q$, $|Q| > 0$. By $\lambda$-density we have $Q = \cup_{P \in ch(Q)} P$ and $\langle f \rangle_P \leq \lambda \langle f \rangle_Q$. We show the following main inequality:

$$B(x_Q, y_Q, z_Q; \lambda) \leq \sum_{P \in ch(Q)} \frac{|P|}{|Q|} B(x_P, y_P, z_P; \lambda). \tag{4.2}$$

First notice that $z_P \geq \max\{x_P, z_Q\}$ for any $P \in ch(Q)$. Since $B$ is increasing in $z$ in its domain, it is enough to prove (4.2) when the quantities $z_P$ are replaced by $\max\{x_P, z_Q\}$. If $y_Q - x_Qz_Q^{p-1} \leq 0$, then the inequality is obvious because $B(x_P, y_P, \max\{x_P, z_Q\}; \lambda) \leq z_Q^P$. So we assume $y_Q - x_Qz_Q^{p-1} > 0$. Then we will prove the stronger inequality where all expressions of the form $(\cdots)_+$ on the right hand side of (4.2) are replaced by the lower bounds $(\cdots)$. Since $y_Q = \sum_P \frac{|P|}{|Q|} y_P$ and $x_Q = \sum_P \frac{|P|}{|Q|} x_P$, the inequality we wish to prove takes the following form after dividing by $z_Q^P$

$$1 - \frac{\lambda^p - 1}{\lambda^p - 1} \sum_{P \in ch(Q)} \frac{|P|}{|Q|} z_P \leq \sum_{P \in ch(Q)} \frac{|P|}{|Q|} \left[ \max\left\{ \frac{x_P}{z_Q}, 1 \right\}^p - \frac{\lambda^p - 1}{\lambda^p - 1} \max\left\{ \frac{x_P}{z_Q}, 1 \right\}^{p-1} \right].$$

This can be rewritten as follows

$$\frac{\lambda^p - 1}{\lambda^p - 1} \left[ \sum_{P \in ch(Q)} \frac{|P|}{|Q|} \frac{x_P}{z_Q} \left( \max\left\{ \frac{x_P}{z_Q}, 1 \right\}^{p-1} - 1 \right) \right] \leq \sum_{P \in ch(Q)} \frac{|P|}{|Q|} \max\left\{ \frac{x_P}{z_Q}, 1 \right\}^p - 1.$$

Replacing $\frac{x_P}{z_Q}$ by $\max\left\{ \frac{x_P}{z_Q}, 1 \right\}$ we see that it is enough to prove the following stronger (in fact equivalent) inequality

$$\frac{\lambda^p - 1}{\lambda^p - 1} \left[ \sum_{P \in ch(Q)} \frac{|P|}{|Q|} \left( \max\left\{ \frac{x_P}{z_Q}, 1 \right\}^p - \max\left\{ \frac{x_P}{z_Q}, 1 \right\} \right) \right] \leq \sum_{P \in ch(Q)} \frac{|P|}{|Q|} \max\left\{ \frac{x_P}{z_Q}, 1 \right\}^p - 1. \tag{4.3}$$

Now notice that if $1 \leq s \leq \lambda$ then

$$\frac{\lambda^p - 1}{\lambda^p - 1} (s^p - s) \leq s^p - 1. \tag{4.4}$$

This is a consequence of the fact that the function $s \mapsto s^p$ is convex and its graph between $s = 1$ and $s = \lambda$ lies below corresponding chord. Since $1 \leq \max\left\{ \frac{x_P}{z_Q}, 1 \right\} \leq \lambda$ the inequality (4.3) follows by averaging (4.4) at the points $\max\left\{ \frac{x_P}{z_Q}, 1 \right\}$ with the weights $\frac{|P|}{|Q|}$. The main inequality (4.2) is proved.

We start with $S$ and iterate the main inequality $m$ times. We obtain

$$B(x_S(f), y_S(f), z_S(f); \lambda) \leq \sum_{P \in F_m} \frac{|P|}{|S|} B(x_P(f), y_P(f), z_P(f); \lambda).$$
Recall that by the definition of $\lambda$-density we have
\[
\lim_{m \to \infty} \sup_{P \in F_m} \text{diam}(P) = 0.
\]
The family $Q$ consists of convex sets so the Lebesgue differentiation theorem implies that in the limit we will obtain the desired result:
\[
B(x_S(f), y_S(f), z_S(f); \lambda) \leq \langle B(f, f^p, M_Q(f)) \rangle_S = \langle (M_Q(f))^p \rangle_S.
\]
\[\square\]

Remark 1. We would like to mention that in one dimensional case there is a simple proof to obtain the constant \((p - 1)/p\) as the lower bound for the uncentered maximal function operator defined over the intervals. The argument is due the Lerner [11] (see also [5]). Indeed, for \(f \geq 0\) we consider \((M_R f)(x) = \sup_{b > x} \frac{1}{b - x} \int_x^b f(t) dt\). Then notice that
\[
t |\{x : (M_R f)(x) > t\}| = \int_{\{x : (M_R f)(x) > t\}} f(s) ds \tag{4.5}
\]
(for example, see Lemma 1 in [5]). Now multiplying both sides of (4.5) by \(t^{p-2}\) and integrating from 0 to $\infty$ with respect to $dt$ we obtain $\frac{\|M_R f\|_p^p}{p} = \frac{1}{p-1} \int_{\mathbb{R}} f(M_R f)^p$. Therefore $\frac{\|M_R f\|_p^p}{p} \geq \frac{1}{p-1} \int_{\mathbb{R}} f^p$ and the desired estimate follows. Unfortunately it is unclear how to use this argument to obtain the lower bounds for the maximal function operator defined over the $\lambda$-dense family in \(\mathbb{R}^n\), \(n \geq 2\).

Remark 2. We omit the explanation of why we consider this special function \(B(x, y, z; \lambda)\) because it is not necessary for the formal proof. However, the reason lies in the geometry of the solution of the relevant homogeneous Monge–Ampère equation. The function appeared for the first time in [13]. It can be derived by using the methods recently developed in works of N. Osipov, D. Stolyarov, V. Vasyunin, P. Zatitskiy and P. Ivanisvili (see e.g. [7, 8, 6]).

APPENDICES

4.2. Besicovitch covering lemma.

Lemma 3. Let $K_1 \subset K$, and let $E$ be the set of centers of the sets in $K_1$. Assume that $E$ is a bounded subset of \(\mathbb{R}^n\), and $\sup_{S \in K_1} \text{diam}(S) < \infty$. Then there exists a constant \(B(n) > 0\) depending only on the dimension $n$, and at most countable collection of sets $K_j \in K_1$ such that $E \subset \bigcup K_j$ and each point of $x \in \mathbb{R}^n$ is covered by at most $B(n)$ sets from the family $\{K_j\}$.

Notice that the above formulation of the Besicovitch covering lemma with origin symmetric convex sets is equivalent to the more commonly found formulation concerning a collection of balls, only with the usual Euclidean norm replaced by some other norm in $\mathbb{R}^n$. A proof of the covering lemma in a general finite dimensional normed space can be found (with a bit of work) in [1, 15, 2]. For a simple proof, the reader can see Füredi and Loeb [4], where it is moreover shown that \((1.001)^n \leq B(n) \leq 5^n\) for any choice of origin symmetric convex body.

We shall use the Besicovitch covering lemma to prove the general form of Stein’s inequality.
Lemma 4. There is a constant $C = C(n) > 0$ such that if $K_0 \in \mathcal{K}$ and $f \geq 0$ satisfies $(f)_{K_0} = 1$, then
\[ \int_{K_0} f \ln(\max\{1, f\})dx \leq C \int_{K_0} M_0(f)dx. \]

Notice that the lemma is proved for the centered maximal function (i.e., the smallest maximal function).

**Proof.** Without loss of generality we may assume that $f$ is supported in $K_0$. Let $t > 1$. For each $x \in \{f > t\}$, choose some set $K_x \in \mathcal{K}$ centered at $x$ that satisfies $(f)_{K_x} = t$. Notice that for each $t > 0$ the diameter of such sets $K_x$ are uniformly bounded. Now use the Besicovitch covering lemma to extract a sequence $K_j$ from the collection $\{K_x : x \in \{f > t\}\}$ such that
- $\bigcup_j K_j \supset \{f > t\}$, and
- the sets $K_j$ have bounded overlap (with overlap number $B(n) \leq 5^n$).

For each $x \in K_j$, we have that the set in $\mathcal{K}$ given by the concentric double of $K_j$ shifted to be centred at $x$ contains the set $K_j$. From this we deduce that $M_0(f) \geq \frac{1}{t^2}$ on $\bigcup_j K_j$. Finally, notice that $|K_j| \leq |K_0|$ (because $t > 1 = (f)_{K_0}$). Since each set $K_j$ is centrally symmetric and centered in $K_0$, it is a subset of $2K_0$ and $|K_j \cap K_0| \geq c(n)|K_j|$ for a constant $c(n)$ depending only on $n$. Therefore,
\[ |\{x \in K_0 : M_0(f)(x) > 2^{-n}t\}| \geq \frac{1}{B(n)} \sum_j |K_j \cap K_0| = \frac{c(n)}{tB(n)} \sum_j \int_{K_j} f dx \geq \frac{c(n)}{tB(n)} \int_{\{f > t\}} f dx. \]
Integrating this expression over $t > 1$ we obtain the desired estimate. \qed

4.3. $\lambda$-density of parallelepipeds.

**Lemma 5.** The set of all parallelepipeds in $\mathbb{R}^n$ with nonzero volume, such that the sides are parallel to the fixed $n$ linearly independent vectors, is $\lambda$-dense for any $\lambda > 1$.

**Proof.** Clearly if the family is $\lambda_1$-dense then it is $\lambda_2$ dense for any $\lambda_2 \geq \lambda_1$. Therefore we consider the case when $2 \geq \lambda > 1$. Let $P$ be a parallelepiped from the family. Take any hyperplane which is parallel to a facet $L^+$ of the parallelepiped and call it $H$. The facet $L^+$ has an opposite facet $L^-$. We consider those $H$ that intersect the parallelepiped and divide it into two parts which we denote by $P^H$ and $P^H_+$ correspondingly ($P^H_+$ contains the facet $L_+$). First choose $H$ so that $|P^H| = |P^H_+|$. If $(f)_{P^H} = (f)_{P^H_+}$ then $\max\{\langle f \rangle_{P^H}, \langle f \rangle_{P^H_+}\} \leq \lambda(\langle f \rangle_P)$, and we say that the partition $P = P^H_+ \cup P^H$ is a *good partition*. Otherwise, consider the case $\langle f \rangle_{P^H} < \langle f \rangle_{P^H_+}$ (the case of the opposite inequality is similar). Then we start moving the hyperplane $H$ closer to the facet $L^-$. Let $H^*$ be the hyperplane parallel to $L^+$ such that $\lambda = \frac{|P^H_+|}{|P^H|}$. We move $H$ toward $H^*$ while $\langle f \rangle_{P^H} < \langle f \rangle_{P^H_+}$. If at some position of $H$ the equality happens then we stop and say that the hyperplane $H$ in that position gives a *good partition* of $P$, $P = P^H_+ \cup P^H$. If the equality never happens then we just choose $H = H^*$. Notice that in this case we have
\[ \lambda \langle f \rangle_P = \lambda \left( \frac{|P^H|}{|P|} \langle f \rangle_{P^H} + \frac{|P^H_+|}{|P|} \langle f \rangle_{P^H_+} \right) \geq \lambda \frac{|P^H_+|}{|P|} \langle f \rangle_{P^H_+} = \max\{\langle f \rangle_{P^H}, \langle f \rangle_{P^H_+}\}. \]
The last equality follows from the fact that all the time we had $\langle f \rangle_{P^H} < \langle f \rangle_{P^H_+}$.

The advantage of this partition is that the hyperplane $H$ which splits the parallelepiped $P$ into two parallelepipeds $P = P_+ \cup P_-$ satisfies the following properties
1. We have \( \max\{\langle f \rangle_{P_-}, \langle f \rangle_{P_+}\} \leq \lambda \langle f \rangle_P \).

2. None of the parts \( P_- \) and \( P_+ \) is too large, i.e., \( \max\left\{ \frac{|P_-|}{|P|}, \frac{|P_+|}{|P|} \right\} \leq \frac{1}{\lambda} \).

If we iterate the partition then it almost gives us \( \lambda \)-density, except it might happen that the diameters of the smaller parallelepipeds will not tend to zero.

In order to chop the parallelepiped so that the diameters of \( P_{\pm} \) tend to zero uniformly in the process of iteration, sometimes we need to change the direction of the hyperplane \( H \) (it should be parallel to different facets of \( P \)). On each step, we consider the largest side (face of dimension 1) of the parallelepiped \( P \) (if they are several, we pick any of them). We choose the direction of the hyperplane \( H \) so that it is parallel to the facets transversal to the largest side. Since on each step we are cutting the largest side of the parallelepiped with the ratio separated uniformly from zero, it is clear that if we set \( F_0 = \{P\}, F_1 = \{P_-, P_+\}, F_2 = \{P_-, P_-, P_+, P_+\} \) etc., we will have
\[
\lim_{n \to \infty} \sup_{P \in F_n} \text{diam}(P) = 0.
\]

\[\square\]

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