Distinct spreads in vector spaces over finite fields

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Abstract
In this short note, we study the distribution of spreads in a point set $P \subseteq \mathbb{F}_q^d$, which are analogous to angles in Euclidean space. More precisely, we prove that, for any $\varepsilon > 0$, if $|P| \geq (1 + \varepsilon)q^{[d/2]}$, then $P$ determines a positive proportion of all spreads. We show that these results are tight, in the sense that there exist sets $P \subseteq \mathbb{F}_q^d$ of size $|P| = q^{[d/2]}$ that determine at most one spread.

1 Introduction
Let $q = p^r$ be a large odd prime power, and $\mathbb{F}_q$ be a finite field of order $q$. For any two vectors $v, u \in \mathbb{F}_q^d$, we define the dot product

$$v \cdot u = v_1u_1 + v_2u_2 + \ldots + v_du_d.$$ 

We further define the distance between any two points $x$ and $y$ by

$$\|x - y\| = (x - y) \cdot (x - y).$$ 

Although it is not a norm, the function $\|x - y\|$ has properties similar to the Euclidean norm (for example, it is invariant under orthogonal matrices and translations).

The Erdős-Falconer distance problem asks for the minimum exponent $\alpha$ such that for any set $P \subseteq \mathbb{F}_q^d$ with $|P| \gg q^\alpha$, the number of distinct distances determined by $P$ is at least $cq$ for some positive constant $c$. Bourgain, Katz, and Tao considered a similar problem on the number of distinct distances determined by a set of points in $\mathbb{F}_2^d$. Iosevich and Rudnev proved that for any $P \subseteq \mathbb{F}_q^d$, if $|P| \geq 2q^{d+1}/2$, then all distances are determined by $P$. The authors of [7] indicated that the exponent $(d + 1)/2$ is the best possible in odd dimensions; the correct exponent is not known in even dimensions.

Let $S_1$ be the unit sphere in $\mathbb{F}_q^d$, i.e. the set of points $x \in \mathbb{F}_q^d$ with $\|x\| = 1$. If $P$ is a subset in the unit sphere $S_1$, the authors of [7] proved that if $|P| \geq Cq^{d/2}$ for some sufficiently large positive constant $C$, then there exists $c > 0$ such that the number of distinct distances determined by points in $P$ is at least $cq$. We note here that their result even can be stated in a stronger form which will be useful for our later applications. The interested reader can find more details in [7, page 15].

1Here and throughout, $X \gg Y$ means that there exists $C > 0$ such that $X \geq CY$. 
Theorem 1 (Hart et al., [7]). For \( P \subseteq S_1 \) in \( \mathbb{F}_q^d \) with \( d \geq 3 \). Suppose that \(|P| \geq Cq^d \) for some positive constant \( C \), then the number of distinct distances determined by points in \( P \) is at least \( \min\{q/2, Cq/4\} \).

In 2014, Bennett, Iosevich and Pakianathan [4] studied a generalization of the Erdős distinct distance problem in vector spaces over finite fields. More precisely, they dealt with the distribution of classes of triangles in a point set in \( \mathbb{F}_q^d \). They proved that for any \( P \subseteq \mathbb{F}_q^2 \), if \(|P| \gg q^{7/4} \) then \( P \) determines at least a positive proportion of all congruence classes of triangles, where two triangles, denoted by \( \Delta(a_1, a_2, a_3) \) and \( \Delta(b_1, b_2, b_3) \), are in the same congruence class if there exist an orthogonal matrix \( M \) and a vector \( z \in \mathbb{F}_q^2 \) such that \( M \cdot a_i + z = b_i \) for \( 1 \leq i \leq 3 \). The threshold \( q^{7/4} \) was improved recently to \( q^{8/5} \) by Bennett et al. [3] by using a clever combination of Fourier analytic techniques and elementary results from group action theory. The authors of [3] also gave a construction of a point set \( P = A \times B \) with \(|A| = q^{1/2-\varepsilon'} \) and \(|B| = q \), and \( P \) determines at most \( cq^3 - \varepsilon'' \) for \( \varepsilon'' > 0 \). Iosevich [10] conjectured that for any \( P \subset \mathbb{F}_q^2 \), if \(|P| \geq Cq^{3/2} \) for a sufficiently large constant \( C \) then \( P \) determines a positive proportion of all congruence classes of triangles. The interested reader can find more discussions and related problems in [4, 3, 5, 13]. There is also a series of papers dealing with similar problems, see for example [2, 3, 4, 6, 9, 12, 14, 15].

In this paper, we study a similar problem on the number of distinct spreads determined by a point set in \( \mathbb{F}_q^d \).

For three points \( \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{F}_q^d \), the spread between two vectors \( \mathbf{a}^\rightarrow \mathbf{b}^\rightarrow \) and \( \mathbf{a}^\rightarrow \mathbf{c}^\rightarrow \) in \( \mathbb{F}_q^d \), which is denoted by \( S(\mathbf{a}^\rightarrow \mathbf{b}^\rightarrow, \mathbf{a}^\rightarrow \mathbf{c}^\rightarrow) \) (or \( S(\mathbf{b}, \mathbf{a}, \mathbf{c}) \) for simplicity), is defined by

\[
S(\mathbf{a}^\rightarrow \mathbf{b}^\rightarrow, \mathbf{a}^\rightarrow \mathbf{c}^\rightarrow) = 1 - \frac{(\mathbf{a}^\rightarrow \mathbf{b}^\rightarrow \cdot \mathbf{a}^\rightarrow \mathbf{c}^\rightarrow)^2}{\|\mathbf{a}^\rightarrow \mathbf{b}^\rightarrow\| \cdot \|\mathbf{a}^\rightarrow \mathbf{c}^\rightarrow\|},
\]

where \( \|\mathbf{x}\| = x_1^2 + \cdots + x_d^2 \). If either term in the denominator is 0, then \( S(\mathbf{a}^\rightarrow \mathbf{b}^\rightarrow, \mathbf{a}^\rightarrow \mathbf{c}^\rightarrow) \) is undefined.

It is clear that this definition is consistent with the square of the sine of the angle between two vectors \( \mathbf{a}^\rightarrow \mathbf{b}^\rightarrow \) and \( \mathbf{a}^\rightarrow \mathbf{c}^\rightarrow \) in Euclidean space

\[
\sin^2(\theta) = 1 - \frac{(\mathbf{a}^\rightarrow \mathbf{b}^\rightarrow \cdot \mathbf{a}^\rightarrow \mathbf{c}^\rightarrow)^2}{\|\mathbf{a}^\rightarrow \mathbf{b}^\rightarrow\| \cdot \|\mathbf{a}^\rightarrow \mathbf{c}^\rightarrow\|}.
\]

The following are some properties of the spread between two vectors \( \mathbf{a}^\rightarrow \mathbf{b}^\rightarrow \) and \( \mathbf{a}^\rightarrow \mathbf{c}^\rightarrow \):

(i) \( S(\mathbf{a}^\rightarrow \mathbf{b}^\rightarrow, \mathbf{a}^\rightarrow \mathbf{c}^\rightarrow) = S\left(r(\mathbf{a}^\rightarrow \mathbf{b}^\rightarrow), s(\mathbf{a}^\rightarrow \mathbf{b}^\rightarrow)\right) \) for any \( r, s \in \mathbb{F}_q^* \);

(ii) \( S(\mathbf{a}^\rightarrow \mathbf{b}^\rightarrow, \mathbf{a}^\rightarrow \mathbf{c}^\rightarrow) = S(\mathbf{a}^\rightarrow \mathbf{c}^\rightarrow, \mathbf{a}^\rightarrow \mathbf{b}^\rightarrow) \);

(iii) \( S(\mathbf{a}^\rightarrow \mathbf{b}^\rightarrow, \mathbf{a}^\rightarrow \mathbf{c}^\rightarrow) = S\left(M \cdot \mathbf{a}^\rightarrow \mathbf{b}^\rightarrow, M \cdot \mathbf{a}^\rightarrow \mathbf{c}^\rightarrow\right) \), where \( M \) is an orthogonal matrix.
In 2015, Bennett [2] made the first investigation on the number of distinct spreads determined by points in $\mathcal{P} \subseteq \mathbb{F}_q^d$. In particular, he obtained the following.

**Theorem 2** (Theorem 6.5, [2]). Let $\mathcal{P}$ be a set of points in $\mathbb{F}_q^2$. If $|\mathcal{P}| \geq 2q - 1$ then the number of distinct spreads determined by points in $\mathcal{P}$ is $q$.

It is clear that Theorem 2 is sharp up to the coefficient of $q$, since the number of spreads determined by points in a line of $q$ points is at most one. For higher dimensional cases, Bennett [2] had an observation on a connection between spreads and distances:

**A connection between spreads and distances on a sphere:** Suppose $\mathcal{P}_1$ is a subset in the unit sphere $S_1$, it is easily to check that $S(\overrightarrow{0a}, \overrightarrow{0b}) = S(\overrightarrow{0c}, \overrightarrow{0d})$ with $a, b, c, d \in \mathcal{P}_1$ if and only if either $\|a - b\| = \|c - d\|$ or $\|a - b\| = \|c + d\|$. Thus if $\mathcal{P}_1$ determines a positive proportion of all distances then $\mathcal{P}_1$ determines a positive proportion of all spreads. Therefore if we have a set $\mathcal{P} \subseteq \mathbb{F}_q^d$ satisfying $|\mathcal{P}| \gg q^{d+2}$ then there exists a sphere of radius $t \neq 0$ such that $|S_t \cap \mathcal{P}| \gg q^{d/2}$. From the first property of spread, we may assume that $S_t$ is the unit sphere. It follows from Theorem [1] that $S_1 \cap \mathcal{P}$ determines a positive proportion of all distances, therefore $S_1 \cap \mathcal{P}$ determines a positive proportion of all spreads. In other words, we have proved the following.

**Theorem 3** (Theorem 6.3, [2]). Let $\mathcal{P}$ be a set of points in $\mathbb{F}_q^d$, with $d \geq 3$. If $|\mathcal{P}| \gg q^{(d+2)/2}$ then $\mathcal{P}$ determines a positive proportion of all spreads.

We remark here that if $\mathcal{P}$ is a subset in the unit sphere $S_1$, the third listed author [14] showed that for $\mathcal{P} \subseteq \mathbb{F}_q^3$ with $|\mathcal{P}| \gg q^{3/2}$, the number of occurrences of a fixed spread $\gamma$ among $\mathcal{P}$ is $\Theta \left( \frac{|\mathcal{P}|^2}{q} \right)$ if $1 - \gamma$ is not a square in $\mathbb{F}_q$.

The main purpose of this short note is to give sharp results on the number of distinct spreads determined by a large set in $\mathbb{F}_q^d$. Our main idea is to prove that for $\mathcal{P} \subseteq \mathbb{F}_q^d$ of size $|\mathcal{P}| = q^{[d/2]}$, there exists a point $p$ in $\mathcal{P}$ such that it is incident to at least $(1 - o(1)) \frac{\alpha}{1 + \epsilon} q^{d/2}$ lines spanned by $\mathcal{P}$.

**Statement of main results:** Our first result gives us the number of distinct spreads determined by $\mathcal{P} \subseteq \mathbb{F}_q^d$ with $d$ even.

**Theorem 4.** For any $\epsilon > 0$, there exists $c > 0$ such that the following holds. Let $\mathcal{P}$ be a set of points in $\mathbb{F}_q^d$ with $d \geq 2$ even. If $|\mathcal{P}| \geq (1 + \epsilon)q^{d/2}$, then the number of distinct spreads determined by $\mathcal{P}$ is at least $cq$.

If $\mathcal{P}$ be a subset in $\mathbb{F}_q^d$ with $d$ odd, then we can embed $\mathcal{P}$ in $\mathbb{F}_q^{d+1}$ with the last coordinate of 0. Therefore, as a direct consequence of Theorem 4, we obtain the following result.

**Theorem 5.** For any $\epsilon > 0$, there exists $c > 0$ such that the following holds. Let $\mathcal{P}$ be a set of points in $\mathbb{F}_q^d$ with $d \geq 3$ odd. If $|\mathcal{P}| \geq (1 + \epsilon)q^{(d+1)/2}$, then the number of distinct spreads determined by $\mathcal{P}$ is at least $cq$. 

3
Sharpness of results: We show that the conditions on the size of $P$ in Theorem 4 and Theorem 5 are sharp.

Theorem 6. Suppose that either $q \equiv 1 \mod 4$ and $d$ even, or $q \equiv 3 \mod 4$ and $d \equiv 0 \mod 4$. Then there exists a subset $P$ in $\mathbb{F}^d$ such that $|P| = q^{d/2}$ and there is no spread determined by points in $P$.

Theorem 7. Suppose that either $q \equiv 1 \mod 4$ and $d$ odd, or $q \equiv 3 \mod 4$ and $d \equiv 1 \mod 4$. Then there exists a subset $P$ in $\mathbb{F}^d$ with such that $|P| = q^{(d+1)/2}$ and the number of distinct spreads determined by points in $P$ is at most one.

The dependence of these constructions on the arithmetic properties of the underlying field is necessary, and the authors believe that it may be possible to improve Theorem 4 in the case that $q \equiv 3 \mod 4$ and $d \equiv 2 \mod 4$. See section 3 for details.

The rest of this note is organized as follows: in Section 2 we give a proof of Theorem 4, in Section 3, we give proofs of Theorems 6 and 7. In Section 4, we define spreads among more than three points, and leave the study of such spreads for future work.

2 Proof of Theorem 4

To prove Theorem 4, we make use of the following theorem due to the first listed author and Saraf in [8].

Theorem 8 (Corollary 5, [8]). For any $\epsilon > 0$ and $P \subseteq \mathbb{F}^d$ with $|P| \geq (1 + \epsilon)q^{d-1}$, the number of lines spanned by $P$ is bounded below by $\alpha \epsilon q^{2d-2}$, where $\alpha = \epsilon^2(1 + \epsilon + \epsilon^2)^{-1}$.

For the reader’s convenience, we include a brief sketch of the proof of Theorem 8. It is based on the expander mixing lemma for bipartite graphs (e.g. Lemma 8 in [8]). The expander mixing lemma states that, if $G$ is a bipartite graph such that there is a large gap between the largest and second-largest eigenvalues of the adjacency matrix of $G$, and $V_L, V_R$ are sufficiently large subsets of the left and right vertices of $G$, then the number of edges between $V_L$ and $V_R$ is approximately the number that would be expected if $V_L, V_R$ were chosen uniformly at random. To prove Theorem 8, we apply the expander mixing lemma to the incidence graph of points and lines in $\mathbb{F}_q^d$, with $V_L = P$ and $V_R$ the set $L$ of lines that each contain at most one point of $P$. The fact that each line of $L$ contains at most one point of $P$ gives an upper bound on the number of incidences between $P$ and $L$, and a lower bound follows from the expander mixing lemma. Combining these bounds gives Theorem 8.

By using Theorem 8, we are able to show in our following theorem that if the cardinality of $P$ is much smaller than $q^{d-1}$, we still have many distinct lines spanned by $P$.

Theorem 9. For any $0 < \epsilon < q-1$, let $P \subseteq \mathbb{F}_q^d$ with $|P| \geq (1 + \epsilon)q^{k-1}$. Then, the number of lines spanned by $P$ is bounded below by $(1 - o(1))\alpha \epsilon q^{2k-2}$. 

4
Proof. Let $\pi'$ be a uniformly random projection from $\mathbb{F}_q^d$ to $\mathbb{F}_q^k$.

Let $a, b$ be two arbitrary distinct points in $\mathbb{F}_q^d$. We claim that the probability that $\pi'(a) = \pi'(b)$ is less than $q^{-k}$. Note that, if $\pi'(a) = \pi'(b)$, then $\pi'(a - x) = \pi'(b - x)$ for an arbitrary translation vector $x$. Hence, we may without loss of generality assume that $a = 0$. Then, the question of whether $\pi'(a) = \pi'(b)$ reduces to the question of whether $b$ lies in the kernel of $\pi'$, which is a uniformly random $(d - k)$-dimensional linear subspace. This probability is $(q^{d-k} - 1)/q^d < q^{-k}$.

Hence, by linearity of expectation, the expected number of pairs $a, b \in P$ such that $\pi'(a) = \pi'(b)$, denoted by $E_{\text{coll}}$, is $E_{\text{coll}} < \binom{|P|}{2}q^{-k} = (1-o(1))(1+\varepsilon)^2 q^{k-2}/2$. In particular, there exists a projection $\pi$ from $\mathbb{F}_q^d$ to $\mathbb{F}_q^k$ such that the number of collisions is at most $E_{\text{coll}}$. By a Bonferroni inequality, the image $\pi(P)$ of $P$ has size at least $|\pi(P)| \geq |P| - E_{\text{coll}}$. Thus $|\pi(P)| = (1-o(1))|P|$. The conclusion of the theorem follows from Theorem 8 and the observation that $\pi(P)$ does not span more lines than $P$.

Corollary 10. Let $P$ be a set of points in $\mathbb{F}_q^d$ with $d$ even, and $L$ be the set of spanned lines by $P$. Suppose that $|P| = (1+\varepsilon)q^{d/2}$, $\varepsilon > 0$, then there exists a point $p$ in $P$ such that it is incident to at least $(1-o(1))\frac{a_\varepsilon}{1+\varepsilon}q^{d/2}$ lines from $L$.

Proof. It follows from Theorem 9 that the number of lines spanned by $P$ is bounded below by $(1-o(1))a_\varepsilon q^d$. By the pigeonhole-principle, there exists a point $p$ in $P$ such that it is incident to at least $(1-o(1))\frac{a_\varepsilon}{1+\varepsilon}q^{d/2}$ lines, and the corollary follows.

Proof of Theorem 4: By Corollary 10 if $|P| \geq (1+\varepsilon)q^{d/2}$, then there exists a point $p$ in $P$ such that it is incident to at least $c q^{d/2}$ lines that are spanned by $P$ for some positive constant $c$ depending on $\varepsilon$.

Suppose $d = 2$. Then, if $\sqrt{-1} \in \mathbb{F}_q$, then there are $q - 1$ points of $\mathbb{F}_q^2$ at distance 0 from $p$, lying on a single isotropic line with slope $\sqrt{-1}$. If $\sqrt{-1} \notin \mathbb{F}_q$, then there is no point distinct from $p$ at zero distance from $p$. If $a, b, c \in P$ such are in three distinct, non-isotropic lines incident to $p$, then an easy calculation shows that $S(a, p, b) \neq S(a, p, c)$, which proves Theorem 4 in the case $d = 2$.

Suppose $d > 2$. We denote the set of lines incident to $p$ by $L'$. One can check that there exists a sphere $S_t$ of radius $t \neq 0$ such that $|S_t \cap L'| \geq \frac{cq^{d/2}}{2}$. Without loss of generality, we assume that $a = 0$ and $t = 1$. Theorem 4 implies that $S_1 \cap L'$ determines a positive proportion of all distances. Thus Theorem 4 follows from the connection between spreads and distances given in the introduction.

3 Proofs of Theorems 6 and 7

In this section, we give constructions showing that our results are generally tight.

Vectors $v, u$ are orthogonal if $v \cdot u = 0$. If $v \cdot v = 0$, then $v$ is isotropic. Our construction relies on the existence of totally isotropic subspaces of dimension $d/2$. Such subspaces have been considered before in a similar context, for example see [21 Lemma 5.1].

We consider two different cases, depending on whether $q \equiv 1 \mod 4$ or $q \equiv 3 \mod 4$. 


Lemma 11. If \( q \equiv 1 \mod 4 \) and \( d \) is even, then there exist \( d/2 \) mutually orthogonal, linearly independent, isotropic vectors in \( \mathbb{F}_q^d \).

Proof. Since \( q \equiv 1 \mod 4 \), we have that \( i = \sqrt{-1} \) is an element of \( \mathbb{F}_q \). Let \( \mathbf{v}_1 = (1, i, 0, \ldots, 0), \mathbf{v}_2 = (0, 0, 1, i, 0, \ldots, 0), \ldots, \mathbf{v}_{d/2} = (0, 0, \ldots, 1, i) \). It is easy to verify that these vectors satisfy the conclusion of the lemma.

Lemma 12. If \( q \equiv 3 \mod 4 \) and \( d \equiv 0 \mod 4 \), then there exist \( d/2 \) mutually orthogonal, linearly independent, isotropic vectors in \( \mathbb{F}_q^d \).

Proof. It is a classical fact that there exists an isotropic vector in \( \mathbb{F}_q^d \) for \( d \geq 3 \); for example, it is a special case of the Chevalley-Warning theorem. Let \( \mathbf{v}_1 = (a, b, c, 0, \ldots, 0), \mathbf{v}_2 = (0, -c, b, a, 0, \ldots, 0), \mathbf{v}_3 = (0, 0, 0, 0, a, b, c, 0, \ldots, 0), \ldots, \mathbf{v}_{d/2} = (0, \ldots, 0, -c, b, a) \). It is easy to verify that these vectors satisfy the conclusion of the lemma.

Note that the assumption that \( d \equiv 0 \mod 4 \) is necessary in Lemma 12. For example, it can be verified by a direct calculation that there is no set of three mutually orthogonal, linearly independent, isotropic vectors in \( \mathbb{F}_6^3 \). It is an interesting open question whether Theorem 4 is tight when \( q \equiv 3 \mod 4 \) and \( d \equiv 2 \mod 4 \).

Proof of Theorem 6. Suppose \( d = 2m \) with \( d \geq 1 \). If \( q \equiv 3 \mod 4 \), then further suppose that \( m \) is even. Let \( \mathcal{P} \) be the subspace spanned by the mutually orthogonal isotropic vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_m \) given by Lemma 11 or 12. Since \( \|\mathbf{v}_i\| = 0 \) for all \( 1 \leq i \leq m \), it follows from the definition of spread that there is no spread determined by three vectors in \( \mathcal{P} \). On the other hand, the size of \( \mathcal{P} \) is \( q^{d/2} \), which ends the proof of the theorem.

Proof of Theorem 7. Suppose \( d = 2m + 1 \) with \( m \geq 1 \). If \( q \equiv 3 \mod 4 \), then further suppose that \( m \) is even. Let \( \mathbf{v}_1, \ldots, \mathbf{v}_m \) be the mutually orthogonal, isotropic vectors given by Lemma 11 or 12 in the \((d-1)\)-dimensional subspace of \( \mathbb{F}_q^d \) consisting of points whose last coordinate is 0, and let \( \mathbf{v}_{m+1} = (0, 0, \ldots, 0, 1) \). Let \( \mathcal{P} \) be the subspace spanned by \( \mathbf{v}_1, \ldots, \mathbf{v}_{m+1} \). The size of \( \mathcal{P} \) is \( q^{(d+1)/2} \). It is easy to check that the spread determined by any triple of points in \( \mathcal{P} \) is either undefined or one. Thus the number of distinct spreads determined by \( \mathcal{P} \) is at most one. This concludes the proof of the theorem.

4 Higher order spreads

As mentioned in the introduction, spreads are analogous to the sine of angles in Euclidean space. It may be interesting to consider a generalization of this notion to spreads between more than two different vectors. This is an analog of polar sine in Euclidean space.

In more detail, Let \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) be vectors in \( \mathbb{F}_q^d \), with \( d \geq k \). Let \( V = [\mathbf{v}_1, \ldots, \mathbf{v}_k] \) be the \( d \times k \) matrix formed by concatenating the column vectors of \( \mathbf{v}_1, \ldots, \mathbf{v}_k \). Define

\[
\text{psin}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \frac{\text{det}(V^T V)}{\prod_{i=1}^k \|\mathbf{v}_i\|}.
\]
For points $x_1, \ldots, x_{k+1}$, we define the $k$-spread to be
\[
S_k(x_1, \ldots, x_{k+1}) = \text{psin}(x_2 - x_1, \ldots, x_{k+1} - x_1).
\]
It is straightforward to verify that, for $k = 2$, this matches the definition of spread given in the introduction.

A natural question is, for any fixed $k$ and $c > 0$, how large a subset of $\mathbb{F}_q^d$ can determine fewer than $cq$ $k$-spreads? We leave this for future work.

5 Acknowledgments

Research of the first listed author was supported by NSF grants CCF-1350572 and DMS-1344994. The second listed author was partially supported by Swiss National Science Foundation grants 200020-162884 and 200020-144531. The research of the third listed author is funded by the National Foundation for Science and Technology Development Project. 101.99-2013.21.

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