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Universality of the Abraham-Minkowski dilemma for photon momenta beyond dielectric materials

Damien F. G. Minenna,1,2,3,* Yves Elskens,2 Fabrice Doveil,2 and Frédéric André3

1Centre National d’Études Spatiales, FR-31401 Toulouse cedex 9, France
2Aix-Marseille Université, CNRS, PIIM, UMR 7345, FR-13397 Marseille, France
3Thales, FR-78140 Vélizy, France

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Whenever light is slowed down, for any cause, two different formulas give its momentum. For dielectrics, the coexistence of those momenta was the heart of the century-old Abraham-Minkowski dilemma, recently resolved. We demonstrate that this framework extends to momentum exchange in wave-particle interaction; in particular to Langmuir waves for Landau damping and to vacuum waveguides of electron tubes (metallic slow-wave structures). Focussing on the latter, we show that the dilemma resolution is not limited to discriminating between kinematic and canonical momenta but also involves a non-negligible flowing momentum from Maxwell’s electromagnetic stress. The existence of two momenta in materials, plasmas, and waveguides, for which light velocity modification has entirely different origin, points to the universality of the Abraham-Minkowski dilemma.

The question “what is the momentum of light propagating through glass?” seems simple enough to appear on a high-school quiz [1, 2]. Nevertheless, it took physicists more than a hundred years to answer it in a vigorous debate confronting two rival theories and called the Abraham-Minkowski controversy. For Minkowski [3], respectively Abraham [4], the momentum carried by light is proportional \( p_{\text{Mink}} = n \hbar \omega /c \), respectively inversely proportional \( p_{\text{Abra}} = \hbar \omega / (nc) \), to the refractive index \( n \) of dielectric and magnetic materials (see Table I). The problematic is the same for classical fields or photons. Despite many theoretical and experimental works [5–8], it took a century until recent contributions [9–11] proposed a resolution of the controversy, arguing that both theories are correct. While this resolution is largely accepted, the dilemma is still actively debated in materials [16, 17].

Mathematically speaking, the notion of momentum admits two definitions. The linear (kinematic) momentum of a particle \( i \) is its mass times its velocity \( p_{\text{kin},i} = m_q v_q \), linked to forces), while the conjugate (canonical, in Hamiltonian formalism) momentum is the derivative of the Lagrangian \( p_{\text{can},i} = \partial q_i L \), linked to action). In Newtonian mechanics, there is no difference between both momenta, but this is no more the case in quantum physics or in electrodynamics. In that respect, the canonical momentum [18] of a charged particle \( i \) immersed in a magnetic field is \( p_{\text{can},i} = p_{\text{kin},i} + eA(q_i) \), with its electric charge \( e \), and the vector potential \( A \) such that \( B = \text{rot} A \). The discrepancy between both momenta comes from the involvement of the field. Therefore, we should also expect discrepancies between momenta of light. This is precisely where the controversy originates: the canonical momentum of light is Minkowski’s expression \( p_{\text{Mink}} \) while Abraham’s expression \( p_{\text{Abra}} \) is kinematic. According to Refs. [10, 11], in dielectric and magnetic media, both momenta are linked via the total momentum \( p_{\text{can}} = p_{\text{Mink}} + p_{\text{Mink}} + p_{\text{Abra}} \), with the canonical Table I. Established expressions [8, 12] for the momentum and its density, in dielectric and magnetic materials. A single photon has wavenumber \( k = \omega/v_\phi = n \omega/c \), with angular frequency \( \omega \), phase velocity \( v_\phi \), and refractive index \( n = c/v_\phi \). Classical electrodynamics of media calls on the electric field \( E \), the magnetic field \( H \), the electric displacement \( D = cE \), the magnetic induction \( B = \mu H \), the permittivity \( \epsilon = \epsilon_0 \epsilon_0 \), the magnetic permeability \( \mu = \mu_0 \mu_0 \), and the speed of light in vacuum \( c = (\epsilon_0 \mu_0)^{-1/2} \). The group index of a dispersive medium is \( n_g = c/v_g \), with group velocity \( v_g = \partial n_g /\partial \omega \). Since the group index is \( n_g = n + \omega \partial n_g /\partial \omega \), one can assume \( n_g \approx n \), for low frequencies or if \( n \) is independent of the angular frequency as in non-dispersive media. Consequently, for photons, there is an \( n^2 \) ratio between both momenta, while defining \( n = \sqrt{\epsilon_\mu} \), and \( v_\phi = (\mu_0 \epsilon_0)^{-1/2} \) (for homogeneous, isotropic, non-dispersive, space-time independent media), leads to the same \( n^2 \) ratio for classical fields. In dispersive media, an \( n_g \) ratio is admitted for photons [13] and sometimes inferred [6, 8, 14, 15] for electromagnetic fields though no broad consensus has emerged.

| Momentum of a single photon\(^a\) | Momentum density of classical fields |
|---------------------------------|--------------------------------------|
| Minkowski’s \( p_{\text{Mink}} = \hbar k = n \hbar \omega /c \) \( g_{\text{Mink}} = D \times B \) |
| Abraham’s \( p_{\text{Abra}} = \hbar k /n_g = \hbar \omega / c \) \( g_{\text{Abra}} = E \times H \) |

\( ^a \) Some authors set \( k = \omega/c \) with \( p_{\text{Mink}} = n\hbar k, p_{\text{Abra}} = \hbar k/n_g \).
the dilemma was only investigated in matter, this letter presents two systems with starkly different causes for light speed modification. In both cases, we obtain Abraham’s and Minkowski’s momenta. The first system considers electromagnetic fields propagating inside the dispersive metallic waveguide of a vacuum electron tube. The second system considers Langmuir waves within the Landau damping (or amplification) context in plasmas. Our main goal is to demonstrate the universality of the Abraham-Minkowski dilemma beyond materials, so we give priority to the first system since forces and energies in vacuum are well established. In addition, we have simulations to assess the extent of the dilemma to waveguides with an algorithm benchmarked against experimental measurements (see below). Evidences that the dilemma occurs in plasmas (using an analogous treatment) are raised at the end of this letter. The resolution of the dilemma for both systems goes beyond the resolution (3) ensures their conjugate momenta” and \( \mathcal{I}_k \) the “generalized coordinates”. The normalisation (3) ensures their Poisson brackets are canonical. The dynamics of charged particles \( (i) \) in electromagnetic fields is dictated by the Lorentz force \( \sum_i m_i \dot{q}_i = -e\dot{A}(q_i) - e(\ddot{q}_i \cdot \nabla)A(q_i) + \sum_i e\dot{q}_i \times \mu_0H \), leading to the well-known particles energy \( \mathcal{H} = \mathcal{H}_{\text{em}} + \mathcal{H}_{\text{cpl}} \). Relativistic expressions leave unchanged the field momentum. From the total energy (self-consistent Hamiltonian) \( H = H_{\text{tot}} + H_{\text{cpl}} \), the Hamilton evolution equations are
relations (4)-(5) if \(-i\omega_k \mathbf{A}_k = \mathbf{E}_k^*\), in agreement with the Weyl gauge, and on setting \(\Omega_k = \omega_k\).

The Lagrangian of the system is obtained with the Legendre transformation of the total Hamiltonian, as \(L_{\text{tot}} = \sum_i p_{\text{can},i} \cdot \dot{q}_i + \sum_i V_k \dot{I}_k - H_{\text{tot}}\). This Lagrangian is invariant under space translations \(q_i \rightarrow q_i + \delta r\) and \(I_k \rightarrow I_k e^{-i k \delta r}\). Hence, the infinitesimal variation of the Lagrangian (a.k.a. the generalised force on the system)

\[
\frac{\partial L_{\text{tot}}}{\partial \dot{q}_i} = \frac{d}{dt} \sum_i \mathbf{p}_{\text{can},i} + \sum_k i k \mathbf{V}_k \dot{I}_k,
\]

must vanish. Applying Noether’s theorem [20] leads to a first formulation of the conserved total momentum in the waveguide

\[
\mathbf{p}_{\text{tot}} = \sum_i \mathbf{p}_{\text{can},i} + \sum_k i \mathbf{V}_k \dot{I}_k \mathbf{k},
\]

as the sum of canonical momenta of the particles and the fields, with Minkowski’s momentum in the waveguide

\[
\mathbf{p}_{\text{Mink}}^\text{wg} = \sum_k i \mathbf{V}_k \dot{I}_k \mathbf{k}.
\]

In addition, the derivative of Eq. (3) enables the calculation of the group velocity \(v_g = \partial_k \omega_k\), with \(\omega_k = \Omega_k\). Using the derivative of Helmholtz’ equations, the identity \(\mathfrak{B} \cdot (\nabla \times \mathfrak{A}) = \nabla \cdot (\mathfrak{A} \times \mathfrak{B}) + \mathfrak{A} \times (\nabla \times \mathfrak{B})\), for arbitrary \(\mathfrak{A}, \mathfrak{B}\), and the divergence theorem, we obtain

\[
v_g = \frac{1}{\omega_k} \int_V \mathbf{E}_k^*(\mathbf{r}) \times \mathbf{H}_k(\mathbf{r}) \, dV + \text{c.c.},
\]

for the monochromatic (a.k.a. continuous waveform, CW) regime where group and refractive indices are considered as constant. Then, Eq. (9) yields the Minkowski momentum density

\[
g_{\text{Mink}}^\text{wg}(\mathbf{r}, t) = \frac{mn_g}{c^2} \mathbf{E} \times \mathbf{H},
\]

for dispersive waveguides in the monochromatic regime (see the difference with materials in Table 1).

On the other hand, Abraham’s momentum is found exactly as in vacuum, using the classic reformulation [18, 27] of the Lorentz force density

\[
f = \nabla \cdot \mathbf{\sigma} - \partial_t (\mathbf{E} \times \mathbf{H}) / c^2,
\]

obtained from Maxwell equations in vacuum with sources, with the Maxwell stress tensor \(\mathbf{\sigma}\), and the Abraham momentum density

\[
g_{\text{Abra}}^\text{wg}(\mathbf{r}, t) = \frac{1}{c^2} \mathbf{E} \times \mathbf{H},
\]

as for materials (see Table 1). For waveguides, we use decomposition (1)-(2) [20] and Eq. (10) for the wavevector to re-express Eq. (13) as the Abraham momentum

\[
g_{\text{Abra}}^\text{wg} = \sum_k \frac{1}{m_{ng}} i \mathbf{V}_k \dot{I}_k \mathbf{k}.
\]

The last item needed to resolve the Abraham-Minkowski dilemma for waveguides is the link between both momenta, Eqs (9) and (14). This is done by comparing the two balances of forces of the system: a first one from the Noether approach Eq. (7) \(\sum_i \mathbf{p}_{\text{can},i} + \mathbf{p}_{\text{Mink}} = 0\), and a second one from the Lorentz force Eq. (12) \(\sum_i \mathbf{p}_{\text{kin},i} + \mathbf{p}_{\text{Abra}} - \int_V \nabla \cdot \mathbf{\sigma} \, dV = 0\). Comparing both leads to a second formulation for the conserved total momentum

\[
\mathbf{p}_{\text{tot}} = \sum_i \mathbf{p}_{\text{can},i} + \mathbf{p}_{\text{Mink}} = \sum_i \mathbf{p}_{\text{kin},i} + \mathbf{p}_{\text{Abra}} + \mathbf{p}_{\text{flow}},
\]

with the sum running on particles \((i)\), which is the crucial relation needed to solve the dilemma. In our field representation, this gives the flowing momentum

\[
\mathbf{p}_{\text{flow}} = \sum_k \left(1 - \frac{1}{m_{ng}}\right) i \mathbf{V}_k \dot{I}_k \mathbf{k} + \sum_i \sum_k c_i \mathbf{I}_k \mathbf{A}_k(q_i),
\]

as the difference ensuring Eq. (15).

Most importantly, the use of the Lorentz force introduces the time derivative [27] \(\dot{p}_{\text{flow}} = - \int_V \nabla \cdot \mathbf{\sigma} \, dV\) of the flowing momentum leaving volume \(V\). As soon as amplification or attenuation occurs, the system is not spatially homogeneous since the fields exert a stress and thereby the force balance needs to be completed by the Maxwell stress. Indeed, the time derivative of Eq. (16) coincides with the integral of the divergence of Maxwell’s stress tensor [18, 27] \(\nabla \cdot \mathbf{\sigma} = \epsilon_0 (\nabla \cdot \mathbf{E}) \mathbf{E} + \mu_0 (\nabla \times \mathbf{H}) \times \mathbf{H} + \epsilon_0 (\nabla \times \mathbf{E}) \times \mathbf{E}\).

The key difference of Eq. (15) with the resolution of the dilemma in materials [10, 11] is the inclusion of the flowing momentum which becomes crucial for waveguide-based amplifiers (while it vanishes for mere propagation in a passive medium). Figure 1 shows an example of momentum exchange in a traveling-wave tube with a large \(m_{ng}\). The algorithm used is based on an N-body approach [20, 24] with a decomposition similar to Eqs (1)-(2), specialized for the traveling-wave tube geometry and combined with a degrees-of-freedom reduction. It has shown an excellent agreement with real tube measurements [28-30]. For Figure 1, we removed specific tube features (waveguide losses, industrial adjustments, imperfect adaptations) to limit the interaction only to the momentum exchange (15). In helix traveling-wave tubes, fields can be assumed almost longitudinal, leading to only a small difference between canonical and kinematic momenta. Since Abraham’s momentum is much smaller than Minkowski’s, and because, in typical electron tubes,
the difference between canonical and kinematic momenta of electrons can be really small, we immediately see that the flowing momentum can obviously not be ignored.

Note that, with these new forms for Abraham’s and Minkowski’s expressions, we can oversimplify the quantization by enforcing $i\hbar \nabla \hat{Z}_k \to \hat{N}_k \hbar$, in Eq. (9) and Eq. (14), to quantize classical fields in terms of photon counting operators $\hat{N}_k$. Though crude, this procedure reflects on the debate on photon momentum. Using this transform, Minkowski’s momentum, respectively Abraham’s, for a single photon becomes the familiar $\hat{p}_{\text{Mink}} = \hbar \hat{k}$, respectively $\hat{p}_{\text{Abra}} = \hbar k/(\eta n_k)$ (see Table I).

The dilemma does not only extend to vacuum waveguides but is relevant to every wave-particle system (with $v_0 \neq c$) involving momentum exchange. Arguably the most famous one is Landau damping [31, 32] (and Landau growth, a.k.a. bump-on-tail instability, with the same origin [21]) occurring in plasma physics (beam-plasma system). Here, the propagation medium of the Langmuir waves is a plasma and the dispersion relation derives from the (bulk) electron velocity and position distribution functions. The interaction between waves and (beam) electrons is the key to Landau damping. To investigate the latter, the $N$-body Hamiltonian approach has shown [19, 33] that the associated dynamics conserves a total wave-particle momentum. The conservation of this total momentum implies a non-linear synchronization between the particles and the wave, leading to the physical result that Landau damping or growth is linked to the slope of the particle velocity distribution function $f(v_i)$ near the phase velocity of the wave. Although it may seem surprising to relate Landau damping and vacuum waveguides, this connection was already fruitfully made decades ago [34, 35]. Beside the transition to chaos [36], Landau damping (more precisely the nonlinear synchronization between an electron beam and waves) has been accurately investigated [37] using a traveling-wave tube.

Because the wave-particle interactions for traveling-wave tubes and for Landau damping are analogous [19, 37], a similar spectral decomposition to Eqs (1)-(2), applies to Langmuir waves in plasma. However, there are some critical differences. In particular, Langmuir waves can be considered as electrostatic, which implies equality of the canonical and kinematic momenta of particles ($\hat{p}_{\text{can},i} = \hat{p}_{\text{kin},i}$). Yet, the Abraham-Minkowski dilemma still applies, thanks to the crucial flowing momentum: A complete demonstration that Langmuir waves have two momenta will be the subject of a forthcoming publication. Indeed, conservation of the total momentum, as in Eq. (8), has already been demonstrated experimentally [37] and theoretically [19, 21] (without identifying Minkowski). On the other hand, when expressing the electric force $-e\nabla \varphi$, with the electrostatic potential $\varphi$ (proportional to the square of the wave intensity), Abraham’s momentum arises complemented with the flowing momentum to ensure momentum balance as well, in agreement with Eq. (15).

In many ways, the Abraham-Minkowski controversy looks like a modern version, involving electrodynamics, of the vis viva controversy [38] (during the 17th and 18th centuries, both “linear momentum” ($mv$) and “kinetic energy” ($mv^2$) were discussed as the same concept of “force” as it was called). The problem originates from the difference between definitions one uses.

For either physical system, one obtains two different equations for the momentum conservation: one from the Hamilton properties leading to Minkowski’s (canonical) expression, and one from the force balance leading to Abraham’s (kinematic). This leads to the conclu-
sion that the Abraham-Minkowski dilemma arises in vacuum waveguides of electron tubes and in Landau damping/amplification.

Finding two momenta of light not only in dielectrics and magnetic materials, but also in waveguides and in plasmas, implies that the difference between Abraham’s and Minkowski’s is a universal issue, relevant to many chapters of physics. Investigating momentum exchange in wave-particle systems requires taking notice of the dilemma, because it results from the same fundamental processes: we expect the debate to arise in every domains where momentum exchange occurs with slowed down light. It is worth noting that simulations presented in Figure 1 were made with an algorithm [28–30] used for industrial applications. Consequently, the distinction between Abraham’s and Minkowski’s approaches and considering the Maxwell stress, is relevant to actual devices.

Unlike for materials, solving the Abraham-Minkowski dilemma for waveguides and plasmas implies taking account of the flowing momentum (which cannot be ascribed to a supporting material medium), meaning this dilemma also highlights a difference about the impact of the electromagnetic stress on the expression of the light momenta.

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* Corresponding author.
damien.minenna@univ-amu.fr

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