What can be estimated? Identifiability, estimability, causal inference and ill-posed inverse problems

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Abstract

Here we consider, in the context of causal inference, the basic question: ‘what can be estimated from data?’. We call this the question of estimability. We consider the usual definition adopted in the causal inference literature – identifiability – in a general mathematical setting and show why it is an inadequate formal translation of the concept of estimability. Despite showing that identifiability implies the existence of a Fisher-consistent estimator, we show that this estimator may be discontinuous, and hence unstable, in general. The difficulty arises because the causal inference problem is in general an ill-posed inverse problem.

Inverse problems have three conditions which must be satisfied in order to be considered well-posed: existence, uniqueness, and stability of solutions. We illustrate how identifiability corresponds to the question of uniqueness; in contrast, we take estimability to mean satisfaction of all three conditions, i.e. well-posedness. It follows that mere identifiability does not guarantee well-posedness of a causal inference procedure, i.e. estimability, and apparent solutions to causal inference problems can be essentially useless with even the smallest amount of imperfection. These concerns apply, in particular, to causal inference approaches that focus on identifiability while ignoring the additional stability requirements needed for estimability.

Keywords: identifiability, estimability, causal inference, inverse problems, stability, robust statistics, nonparametric statistics, statistical learning theory, sensitive parameters

1. Introduction

In the causal inference literature, identifiability of a quantity is often taken to be synonymous with ‘estimable from data’. For example Pearl and Bareinboim (2014, p. 583) give the following definition and description:

The following definition captures the requirement that $Q$ be estimable from the data [emphasis ours]:

Definition 1 (Identifiability) Causal query $Q(M)$ is identifiable, given a set of assumptions $A$, if for any two (fully specified) models, $M_1$ and $M_2$, that satisfy $A$, we have

$$P(M_1) = P(M_2) \Rightarrow Q(M_1) = Q(M_2).$$

The implied equivalence between the notion ‘can be estimated from data’ and the formalised concept of identifiability has led, it seems, to a strong focus in the causal literature on identification conditions for various quantities (or ‘causal queries’). The culmination of this consists of proofs of the completeness of the do-calculus for answering identification questions (Shpitser and Pearl, 2006; Huang and Valtorta, 2008). Impressively, these identifiability results are entirely independent of the particular functional form of the causal model (other than e.g. which variables depend on which) and thus can be fully summarised in minimalist ‘nonparametric’ structural equations or graphical terms. The latter consist of directed acyclic graphs (DAGs).

The usual story in the causal literature then is that:

- First we determine what can be estimated in principle from the data using identifiability analysis.
- Then we call up the Statistics department to help design a particular efficient (or otherwise desirable) estimator.

The purpose of the present note is to consider the key mathematical ideas behind the supposed strong connection between identifiability and ‘can be estimated from data’. Contrary to this supposition, we in fact find that the formal concept of identifiability is an inadequate characterisation of the more general concept of ‘can be estimated from the data’ (even in principle), or what we will call estimability. After considering identifiability in detail, we give a more appropriate characterisation of estimability and illustrate the difference between this and identifiability with simple concrete examples. This has the important implication that, while identifiability conditions hold independently of the particular function classes to which causal models belong, estimability (even in principle) requires restrictions on this class. An alternative interpretation is that one can only answer a much more restricted set of causal questions than is usually supposed, where these restrictions are more stringent than those imposed by the requirement of identifiability. This means that we cannot in general trust identifiability results to tell us what can and cannot be estimated, or which causal questions can be answered, without knowing more about the causal functions involved than is usually assumed.

2. Overview and related work

The basic idea in what follows can be summarised, in large part, by the recognition that identifiability is formally characterised as injectivity of a particular mapping, while (we argue) estimability should be characterised by the continuity of the inverse to this mapping. That is, estimability should be related to the (ideal) continuity properties of the associated estimator, considered at the target population distribution. Without continuity, estimation
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is impossible with anything less than perfect knowledge of the data distribution, which we would argue violates the concept of ‘able to be estimated from data’. In short: an identifiability requirement on a ‘forward’ mapping is not sufficient to ensure continuity of its inverse (the estimator), but the latter is essential for estimability even in principle.

Simple examples illustrating the gap between identifiability and estimability have been known for a long time. While we will mostly take an inverse problems perspective on these issues, they are also essentially semi-classical results of statistical estimation theory, with strong origins in robust statistics and the functional-analytic point of view. In particular, the ‘impossibility’ results of Bahadur and Savage (1956, impossibility of estimating the mean), Donoho (1988, impossibility/possibility of estimating upper/lower bounds for functionals of a density measuring ‘complexity’ of some sort) and Tibshirani and Wasserman (1988, characterisation of ‘sensitive parameters’) illustrate the basic issues, and are strongly connected to results from robust statistics (see e.g. Hampel et al., 2011; Huber and Ronchetti, 2011). Importantly, these impossibility results can be considered as defining intrinsic barriers to estimability. In the words of Tibshirani and Wasserman (1988, p. 186), these barriers to estimability are

an intrinsic property of a [quantity], distinct from sampling properties of any estimator.

Alternatively, these barriers may be thought of providing ideal bounds on the possible sampling behaviour of any estimator. Thus, although we frequently use the term ‘estimator’, we are in a sense considering intrinsic ‘sensitivity’ properties of the parameter associated with the population value of the estimator.

In classical inverse problems theory, and the functional analysis literature more generally, these examples amount to a failure to satisfy the third Hadamard condition (Hadamard, 1902) of a well-posed problem: that of stability of the solution to an inverse problem, despite satisfaction the second, and possibly first, conditions of uniqueness and existence, respectively. Here estimability amounts to a problem possessing a ‘well-posed’ solution (i.e. including a stability condition) while identifiability only amounts to requiring uniqueness.

Tikhonov and others (see e.g. Tikhonov and Arsenin, 1977) developed regularisation theory for ill-posed operator equations to overcome these issues. In fact, Donoho (1988) also notes that many of his observations were essentially preceded by work on inverse problems in the geophysical literature. Vapnik, Valiant and others (see e.g. Vapnik, 2013, 1999; Valiant, 1984; Poggio et al., 2004; De Vito et al., 2005) have related and extended some results of (traditionally deterministic) inverse problems to stochastic ill-posed problems and theories of statistical learnability. Similar results have also reappeared recently in the context of the instability of Bayesian conditioning (Owhadi et al., 2015a,b). There is also a rich tradition of considering the link between identification and estimability concepts in the econometrics/economics literature. In this setting, these issues have been studied under the names identification at infinity, irregular identification, and/or ill-posed identification (see e.g. Lewbel, 2016, for a comprehensive review of identification concepts in econometrics).

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1. Which actually preceded equivalent developments in statistics.
2. We thank Pedro H.C. Sant’Anna and a Twitter user named ‘Sam’ for pointing out a number of relevant references from the econometrics literature in a discussion about an earlier preprint version of the present article.
The article by Horowitz (2014) in particular appears to have a very similar philosophy to ours, discussing economics estimation problems from the perspective of ill-posed inverse problems.

As mentioned, we largely take an inverse problems approach in the present work, but relate these ideas back to the statistical and causal inference literature. Importantly, while the inverse problems, statistical, and econometrics communities have long focused on stability and well-posedness results, it appears that the (graphical) causal inference literature is much more lacking in such results. One exception that we are aware of is the recent article by Schulman and Srivastava (2016), which in fact provides a realistic example of the stability issues raised in the present work.

3. Formalism

Evans and Stark (2002) show how much of the machinery of both inverse problems and statistics can subsumed as instances of a general abstract statistical estimation formalism. We largely follow their formalism, though we present a number of basic concepts in terms of elementary category theory as this allows many of the definitions and proofs to be reduced to simple, essentially algebraic, manipulations, while also applying equally to linear and nonlinear problems. This is strongly influenced by the presentation of a ‘unified theory’ of generalised inverses for operator equations defined in various algebraic structures, as developed by Nashed and Votruba (1976). We note in particular, as they do, that the existence and uniqueness results discussed are essentially algebraic in nature, while continuity results require additional (i.e. topological) structure. Thus we first consider the (easier) algebraic results before considering continuity. This also allows us to introduce the concept of identifiability in the most general setting possible, before considering what it leaves out. The reader unfamiliar or uncomfortable with our general formalism can, for the most part, simply imagine the mappings involved to be matrices or linear operators, though the results here are not restricted to this case. The only presentation of statistical concepts in terms of the concepts of category theory that we are aware of is that given by McCullagh (2002) in an article on the question ‘what is a statistical model?’. The present article adopts a slightly different usage of the term ‘model’ than McCullagh (2002), which we hence make explicit below, but apart from some minor terminological differences, it appears that the choice of how to translate basic statistical concepts into categorial concepts is essentially the same in our approach as in theirs.

In what follows we first present our formalism in a general language that, nevertheless, largely follows statistical notation. Then we offer a direct translation of causal identifiability into this setting to show that this language is able to capture the key causal notions of interest. One benefit of our formalism is that it also applies equally well to subjects like inverse problems, where the ‘causal model’ is usually something like a partial differential equation rather than a DAG.

3. Such readers may like to refer to Section 4.4 where we give a concrete translation of our algebraic results to the finite dimensional linear regression setting.
4. See e.g. Mooij et al. (2013); Blom and Mooij (2018); Ackley et al. (2017) for interesting work on bridging the gap between DAG-style causal models and (ordinary) differential equation-based causal models.
3.1 Setting

We consider an arbitrarily-indexed collection of probability distributions of the form

$$P_\theta(X,Y,...),$$

(1)

for observable random variables $X,Y,...$ and where each probability distribution instance is labelled by a value of some otherwise arbitrary (and possibly multi-/infinite-dimensional) indexing ‘parameter’ $\theta \in \Theta$. We make no assumption about the existence or not of densities for probability measures.

The family in (1) above is best thought of as a function, which we call, using the language of inverse problems, the forward mapping:

**Definition 2 (Forward mapping)** The function $P$, where

$$P: \Theta \rightarrow \mathcal{P}, \; \theta \mapsto P_\theta,$$

(2)

from the set $\Theta$ of possible values of our index $\theta$ into the set $\mathcal{P}$ of possible probability measures $\mathbb{P}$ on observable random variables, is called the forward mapping. Unless otherwise specified, $\mathcal{P}$ is the set of all possible probability distributions over our observables.

We emphasise that the distinction between the mapping $P$, the space of distributions $\mathcal{P}$ and specific distributions $\mathbb{P}$ is very important in what follows.

Finite sample data will be represented by the associated empirical measure (essentially implying an I.I.D. assumption). Importantly, the set $\Theta$ is where what we will call ‘models’ live; these lead to, but are not in general equivalent to, probability distributions over observable variables. In causal inference $\Theta$ will contain causal models; in inverse problems $\Theta$ contains e.g. partial differential equation-based models. Hence we adopt the following terminology:

**Definition 3 (Models and model space)** Particular values of $\theta$ will be referred to as models, and $\Theta$ as model space.

These terms are hence used here more in the causal sense than the statistical sense – in the latter (see e.g. McCullagh, 2002) the term ‘model’ often refers to what we call the forward mapping or, at other times, the implied measure. Our usage is consistent with the statistical framework for inverse problems presented by Evans and Stark (2002), however, though they use the alternative term ‘theory’ in the same sense that we use ‘model’. Unless otherwise specified, however, $\Theta$ can be considered an arbitrary set in our mathematical arguments. Functions or functionals of $\theta$, denoted by $q(\theta)$, will often be referred to as ‘parameters’ and represent particular properties of interest of a model $\theta$, i.e.

**Definition 4 (Parameters associated with models)** Functions $q$ of models $\theta$ are called parameters and are defined by

$$q: \Theta \rightarrow \mathcal{V}, \; \theta \mapsto q(\theta)$$

(3)

where $\mathcal{V}$ is some appropriate space of values. We will also consider the identity $q(\theta) = \theta$, with $\mathcal{V} = \Theta$, as defining a parameter representing the full model, and will hence also refer to $\theta$ values as parameters.
In the causal context, discussed further below, parameters like $q(\theta)$ will correspond to so-called ‘causal queries’ of a ‘causal model’ $\theta$. In the present article we take the term estimator to be defined as a map from probability distributions (measures) to the index set (models), i.e.

**Definition 5 (Estimators)** An estimator is a function $T$ satisfying

$$T : \mathcal{P} \to \Theta, \ P \mapsto \theta.$$  

(4)

This includes the case where we have, in the usual (somewhat dubious!) statistical terminology, the ‘true’, ‘full’ or ‘target’ population measure and hence – in the identifiable and exactly-specified case, as we will see – the ‘true’ population parameter.

### 3.2 Translation of causal concepts

Here we translate causal concepts into our language (and vice-versa). We assume the reader is familiar with the usual language used in the graphical causal literature (as used in e.g. Pearl, 2009; Pearl and Bareinboim, 2014), though the mathematical results given in this article apply to much more general classes of systems and hence are largely independent of many of these details.

In the definition of Pearl and Bareinboim (2014) given in the introduction, reference is made to ‘a set of assumptions $A$’. Following these authors, and the equivalent definitions given in Pearl (2009), we take these to be the (qualitative) causal assumptions specified by a causal graph in the form of a DAG. That is, $M_1, M_2$ in this definition refer to fully specified functional causal models, i.e. fully defined models of the ‘data generating mechanism’ with no ‘free parameters’, while $A$ refers to a set of common qualitative graphical properties that are satisfied by both models $M_1, M_2$. This is formalised as:

**Definition 6 (Causal classes)** Fully specified causal models $M_1, M_2$ are said to belong to a class $\mathcal{M}$ iff they satisfy the same graphical causal model, given as a directed acyclic graph (DAG).

**Definition 7 (Causal queries)** A causal query $Q : \mathcal{M} \to \mathcal{V}, \ M \mapsto Q(M)$ is a function mapping fully specified models in a common class $\mathcal{M}$ to some value space $\mathcal{V}$.

Thus in the (more abstract) language of the present work, models $M_1, M_2$ correspond to models $\theta_1, \theta_2$, the causal class $\mathcal{M}$ corresponds to the model space $\Theta$ to which $\theta_1, \theta_2$ belong, and causal queries $Q(M)$ correspond to parameters $q(\theta)$. We also translate the function $^5$ $P$ which maps any fully specified model $M$ to its probability distribution $P(M)$ into our forward mapping $P$. We summarise this in:

**Definition 8 (Translation of basic causal concepts into our abstract framework)**

$$
\begin{align*}
\mathcal{M} & \leftrightarrow \Theta \\
M_1, M_2 \in \mathcal{M} & \leftrightarrow \theta_1, \theta_2 \in \Theta \\
Q(M) & \leftrightarrow q(\theta) \\
P : \mathcal{M} \to \mathcal{P}, \ M \mapsto P(M) & \leftrightarrow P : \Theta \to \mathcal{P}, \ \theta \mapsto P(\theta)
\end{align*}
$$

(5)

5. Which is apparently not explicitly named in the causal setting, and hence is only implicitly distinguished from its value $P(M)$.
This translation implies that any results concerning what we call models and/or parameters in our more abstract general setting also translate directly to the causal setting.

4. Identifiability and the existence of estimators

Here we demonstrate the first step in what would seem to be the direct connection between identifiability and estimability: we show that identifiability implies the existence of Fisher-consistent\(^6\) estimators that can be written as mappings from probability distributions back to the indexing (model or parameter) space. In short, we first show that identifiability of the forward mapping implies the existence of the inverse mapping. Only after this essentially algebraic result do we show what breaks the apparent equivalence between identifiability and estimability: discontinuity of the inverse (estimator) in general.

4.1 Definitions: Identifiability and Fisher consistent estimators

Our definition of identifiability is given as follows:

**Definition 9 (Identifiability of domains, models, and mappings)** The domain \(\Theta\) of a forward mapping \(P : \Theta \to \mathcal{P}\) is called identifiable iff \(P\) is injective, i.e. a 1-1 mapping. We will also use the term identifiable to refer to the models in a common identifiable domain and to the injective mapping \(P\) itself.

We first consider the identifiability of full models (or ‘full parameters’) here; the definition of identifiability of parameters \(q(\theta)\) is given in a following subsection, where we show how our present arguments extend naturally to parameters.

An injective mapping corresponds to the concept of a monomorphism in category theory (see e.g. Lawvere and Rosebrugh, 2003; Lawvere and Schanuel, 2009; Geroch, 2015), and can be given an essentially algebraic characterisation, which we give below\(^7\). Hence, while injectivity of a mapping \(P\) is usually expressed as

\[
P(\theta_1) = P(\theta_2) \implies \theta_1 = \theta_2,
\]

a more general (or more algebraic/categorial) characterisation is

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6. The idea of Fisher consistency is supposed to capture the idea that if these estimators could be evaluated ‘at the true population’, then we would obtain our target quantity, and hence is closely related to the intuition of identifiability. This section demonstrates this tight connection. While we are aware of previous work investigating the links between identifiability and asymptotic consistency (where the former is necessary but not sufficient for the latter, see e.g. Gabrielsen, 1978; Martín and Quintana, 2002), we are unaware of explicit analyses of the link between identifiability and Fisher consistency, as considered here. Based on the equivalence that we demonstrate in what follows, Fisher consistency appears to be the most natural estimation-oriented counterpart to identifiability, though asymptotic consistency is closely related to the topic of the rest of this article, i.e. stable estimation.

7. We adopt the perspective of elementary category theory here for precisely this reason: it provides a general algebra of abstract mappings – e.g. not necessarily linear, or even functions between sets – which means that many results that are often presented in special cases can be shown to hold in much more general settings.
Definition 10 (Injectivity) \( P \) is an injective mapping iff it is left-cancellable, i.e.

\[
P \circ G = P \circ H \implies G = H \tag{7}
\]

for all maps \( G \) and \( H \) with common domain, and with ranges in \( P \)'s domain.

In category theory, elements \( \theta \in \Theta \) can be thought of as special types of maps \( 1 \xrightarrow{\theta} \Theta \), where \( 1 \) is the singleton set, hence \( P(\theta) \) as \( P \circ \theta \), which informally indicates the equivalence of our definition to the usual definition (a formal proof can be found in e.g. Lawvere and Rosebrugh, 2003). Other ideas from category theory that lead to useful cancellation properties include the following. Firstly, an algebraic/categorical characterisation of onto/surjective maps, or epimorphisms in the category of sets and mappings, is that they are right-cancellable. Secondly, an important property of left-inverses – also called retractions – and of right-inverses – also called sections – is that they are epimorphisms and monomorphisms, respectively, and hence are right-cancellable and left-cancellable, respectively. The converse doesn’t hold in general categories, but does hold given the axiom of choice (one form of which can be stated as: ‘all epimorphisms have sections’). Again, in the usual category of sets and mappings the terms ‘epimorphism’ and ‘monomorphism’ can be replaced by the more familiar terms ‘surjective’ and ‘injective’, respectively, and the axiom of choice is usually assumed to hold.

Next we give a categorial characterisation of Fisher consistency. First we recall that an estimator is called Fisher consistent if (Huber and Ronchetti, 2011; Hampel et al., 2011):

\[
T(\mathcal{P}_\theta) = \theta \tag{8}
\]

for all \( \theta \). In terms of the forward mapping this is

\[
T \circ P(\theta) = \theta \tag{9}
\]

for all \( \theta \). This can be summarised as saying

**Definition 11 (Fisher consistency)** An estimator \( T \) is Fisher consistent for a forward mapping \( P \) iff

\[
T \circ P = 1_{\Theta} \tag{10}
\]

where \( 1_{\Theta} \) is the identity map on \( \Theta \).

This means that \( T \) is a left-inverse (retraction) for \( P \) and \( P \) is a right-inverse (section) for \( T \). Although \( T \) is defined on all of \( \mathcal{P} \), or at least a dense subset of it, Fisher consistency only really makes sense for distributions in the range of \( P \), i.e. we only need to check that we can correctly recover a parameter when we know the data was generated from a model with that parameter value.

**4.2 The existence of a Fisher consistent estimator is equivalent to identifiability (given the axiom of choice)**

Here we give an elementary proof of the following result:
Theorem 12 The existence of a Fisher consistent estimator is equivalent to identifiability of the forward mapping, assuming the axiom of choice.

Proof Firstly, suppose that there exists a Fisher-consistent estimator, i.e.

\[ T \circ P = 1_{\Theta}. \] (11)

Now suppose we have \( P(\theta_1) = P(\theta_2) \). In this case

\[ T \circ P(\theta_1) = \theta_1 \] (12)

and

\[ T \circ P(\theta_2) = \theta_2 \] (13)

by Fisher consistency. But \( T \circ P(\theta_1) = T(P(\theta_1)) = T(P(\theta_2)) = T \circ P(\theta_2) \), since \( P(\theta_1) = P(\theta_2) \) by assumption and \( T \) is a function, and thus \( \theta_1 = \theta_2 \), as required to show identifiability.

For the other direction, we suppose that the forward mapping is identifiable, i.e. is a monomorphism (1-1). We will demonstrate the existence of a Fisher-consistent estimator.

In order to do so, we will use a condition that turns out to be equivalent to the axiom of choice. This does not hold in arbitrary categories, or even in arbitrary toposes, but is usually considered to hold in the usual category of sets and mappings. Here it amounts to the assumption of the existence of a generalised inverse for all maps in the category, known elsewhere (see e.g. Nashed and Votruba, 1976) as the assumption that all maps are (von Neumann) regular or quasi-invertible\(^8\). This form of the axiom is likely most familiar from the characterisation of generalised inverses in linear algebra, but the key idea also carries over to nonlinear maps.

The most convenient form of the required axiom can be expressed as (Lawvere and Rosebrugh, 2003):

Definition 13 (Axiom of choice.) For all maps \( P \) there exists a \( G \) such that

\[ P \circ G \circ P = P, \] (14)

where \( G \) is called an inner inverse to \( P \).

As discussed by Nashed and Votruba (1976), inner inverses are not generally outer inverses as well, where a map \( G \) is called an outer inverse for \( P \) if it satisfies:

\[ G \circ P \circ G = G. \] (15)

Generalised inverses in the usual sense are required to be both inner and outer inverses. Given an inner inverse, however, a map which is both an inner inverse and an outer inverse

\(^8\) While a seemingly technical matter, the need to assume this to establish the existence of an estimator given identifiability points to one source of the difficulty in establishing a stable connection between estimability and identifiability: it allows for an existence result to hold while providing no guarantee on the further properties (e.g. stability) of the object or on any procedure of construction of the object.
can always be constructed from an inner inverse $G$ by taking $G^\dagger = G \circ P \circ G$. Hence for simplicity we will assume that this construction has been carried out and that we are working with a map that is also an outer inverse, i.e. is a proper generalised inverse.

Now, if, in (14), we simply take $T = G$ and take $P$ to be our usual forward mapping, we guarantee the existence of a $T$ satisfying

$$P \circ T \circ P = P. \quad (16)$$

From the identifiability assumption we know that $P$ is 1-1 and hence left-cancellable, which implies

$$T \circ P = 1_\Theta \quad (17)$$

as required to show Fisher consistency.

Thus we conclude: identifiability is equivalent to the existence of a (Fisher consistent) estimator (in categories with the axiom of choice).

### 4.3 Note on idempotent maps

While $T \circ P$ is the identity in the above setting, $P \circ T$ is not the identity in general, but is an idempotent map since

$$(P \circ T) \circ (P \circ T) = (P \circ T \circ P) \circ T = P \circ T. \quad (18)$$

In linear regression problems in statistics this is often called the *hat matrix* (Huber and Ronchetti, 2011; Hoaglin and Welsch, 1978), while in inverse problems it is sometimes called the *data resolution operator* (Aster et al., 2013): it maps the actual data distribution to a ‘smoothed’ or estimated data distribution. This map will be the identity only if $P$ is surjective (onto); in most statistical estimation problems it is not surjective and a reduction of data (see e.g. Fisher, 1990) occurs.

To (hopefully!) help the reader grasp the relatively abstract discussion so far, we give a brief illustration of how these ideas work in the classical statistical context of linear regression next, before continuing the translation of these abstract ideas to the *causal* setting.

### 4.4 Illustration of basic concepts

Here we give an elementary illustration of the above ideas in terms of the linear regression problem.

**Example 1 (Linear regression)** In the basic linear regression context we wish to ‘solve’ the linear problem

$$A\theta = y \quad (19)$$

9. i.e. a general projection mapping satisfying $A^2 = A$. 

10
Where $A$ is an $m \times n$ real matrix mapping parameters to data, $\theta \in \mathbb{R}^n$ is the parameter (model) vector, and $y \in \mathbb{R}^m$ is the data vector and is assumed to be given. The goal is to estimate (i.e. solve for) $\theta$ given $A$ and $y$. Here injectivity of $A$, which is equivalent to identifiability of $\theta$ as shown above, means that the columns of $A$ are linearly independent. We do not assume that $A$ is surjective – in fact in regression problems it will not be. Thus to ‘invert’ $A$, for a general $y$ not necessarily in the range of $A$, we need to find a generalised inverse, here denoted by $A^\dagger$, such that it satisfies our inner inverse condition, i.e.

$$AA^\dagger A = A.$$  \hfill (20)

From basic linear algebra we know that linear independence of $A$’s columns means that, while $A$ is not invertible in general, $A^TA$ is positive definite and invertible, where $A^T$ is the transpose of $A$. This means that the following is well-defined as a possible choice for $A^\dagger$:

$$A^\dagger = (A^TA)^{-1}A^T.$$  \hfill (21)

Substituting this into the left-hand side of our inner inverse condition gives:

$$A(A^TA)^{-1}A^TA = AI = A$$  \hfill (22)

which verifies that $A^\dagger$ is in fact an inner inverse. Furthermore, because $A$ is injective/identifiable, then by the general argument above $A^\dagger$ should also a left inverse of $A$. This can be directly verified:

$$A^\dagger A = (A^TA)^{-1}A^TA = I.$$  \hfill (23)

$A^\dagger$ is not a right inverse in general, however, i.e. $AA^\dagger \neq I$ in general. Instead $AA^\dagger$ is a projection (idempotent) matrix since

$$(AA^\dagger)(AA^\dagger) = (A(A^TA)^{-1}A^T)(A(A^TA)^{-1}A^T) = A(A^TA)^{-1}A^T = AA^\dagger$$  \hfill (24)

Putting this together, we can take $A^\dagger$ as our estimator (denoted by $\hat{T}$ in the general arguments of the previous section), with estimated model/parameter

$$\hat{\theta} = (A^TA)^{-1}A^Ty = A^\dagger y$$  \hfill (25)

and estimated/smoothed data

$$\hat{y} = A(A^TA)^{-1}A^Ty = AA^\dagger y = \hat{A}y$$  \hfill (26)

where $\hat{A} = AA^\dagger$ defines the so-called hat matrix. Our estimator is Fisher-consistent because $A^\dagger$ is a left inverse of $A$ and hence, if our data was generated by $\theta$ i.e. $y = A\theta$, then

$$A^\dagger y = A^\dagger A\theta = (A^TA)^{-1}A^TA = I\theta = \theta$$  \hfill (27)

for any $\theta$, i.e. our estimator recovers the true value.

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10. Here we are following the standard linear regression setting where parameters are mapped to data vectors, rather than mapped to data distributions. Either setting is covered by the general formalism. See also Evans and Stark (2002) for the relation between these settings.
All of the above is standard statistics, and essentially just linear algebra. Nothing is said about e.g. the stability of the above – as is well known, such regression estimates can be extremely unstable in the presence of outliers (Hampel et al., 2011; Huber and Ronchetti, 2011). The point of the more abstract formalism adopted in the rest of this article is that the same essentially algebraic concepts carry over to the general nonlinear setting as well. Furthermore, as we will show next, because these results only rely on general algebraic concepts they can also be carried over to the causal inference setting using an appropriate interpretation of the relevant domains, codomains and mappings. Again, however, these results neglect stability and/or topological considerations. The final parts of the present article will thus be concerned with these topics.

4.5 Parameters and causal queries

Here we extend our previous proof to the case of parameters $q(\theta)$, and then show that our concept of parameter identifiability translates, in the special case of causal models, to causal identifiability. We then consider Fisher consistency for parameters.

4.5.1 Identifiability for parameters

The usual definition of identifiability of a parameter function $q(\theta)$ (see e.g. Evans and Stark, 2002) is:

**Definition 14 (Identifiability of parameters – standard)** A parameter $q: \Theta \rightarrow \mathcal{V}$ is identifiable with respect to $P$ iff for all $\theta_1 \neq \theta_2$

$$P(\theta_1) = P(\theta_2) \implies q(\theta_1) = q(\theta_2)$$

(28)

or, equivalently, iff for all $\theta_1 \neq \theta_2$

$$q(\theta_1) \neq q(\theta_2) \implies P(\theta_1) \neq P(\theta_2).$$

(29)

This is clearly equivalent to the definition of causal identifiability given in the introduction, in the special case of the causal interpretation of our more abstract framework. A more general set-theoretical definition of (sub-) parameter identifiability is given by McCullagh (2002), which we can translate into our notation as:

**Definition 15 (Identifiability of parameters – set theoretical)** A parameter $q: \Theta \rightarrow \mathcal{V}$ is identifiable with respect to $P$ iff for all $v_1, v_2 \in \mathcal{R}(q)$

$$v_1 \neq v_2 \implies P[q^{-1}\{v_1\}] \cap P[q^{-1}\{v_2\}] = \emptyset$$

(30)

where $q^{-1}\{v\}$ denotes the inverse set image of the singleton set $\{v\}$ (and is hence set-valued), and $P[A]$ denotes the forward set image of a set $A$. Or, equivalently, iff for all $v_1, v_2 \in \mathcal{R}(q)$

$$P[q^{-1}\{v_1\}] \cap P[q^{-1}\{v_2\}] \neq \emptyset \implies v_1 = v_2.$$

(31)

This also means that if $q$ is identifiable then, for all $v_1, v_2 \in \mathcal{R}(q)$,

$$P[q^{-1}\{v_1\}] \cap P[q^{-1}\{v_2\}] \neq \emptyset \implies P[q^{-1}\{v_1\}] = P[q^{-1}\{v_2\}].$$

(32)
We adopt a slightly different formalisation again, which we will show that is equivalent to the standard definition. This amounts to considering all possible 'representatives' of the inverse images of points \( v \) in the range space \( \mathcal{R}(q) \) of \( q \) in McCullagh (2002)'s definition. We state our definition as follows:

**Definition 16 (Identifiability of parameters – choice of representatives)** A parameter \( q : \Theta \to V \) is identifiable with respect to \( P \) iff

\[
\forall s : \mathcal{R}(q) \to \Theta \text{ such that } q \circ s = 1_V, \text{ then } \\
\forall \theta_1, \theta_2 [s \circ q(\theta_1) \neq s \circ q(\theta_2) \implies P \circ s \circ q(\theta_1) \neq P \circ s \circ q(\theta_2)]
\]

or, equivalently, iff

\[
\forall s : \mathcal{R}(q) \to \Theta \text{ such that } q \circ s = 1_V, \text{ then } \\
\forall \theta_1, \theta_2 [P \circ s \circ q(\theta_1) = P \circ s \circ q(\theta_2) \implies s \circ q(\theta_1) = s \circ q(\theta_2)].
\]

As mentioned above this characterisation amounts to working with arbitrary choices of 'representative' parameters in \( q \)-induced equivalence classes, defined by\(^{11} \theta^* = s \circ q(\theta) \) for any \( s \), and their images \( P(\theta^*) = P(s \circ q(\theta)) \) under \( P \), rather than the entire inverse image \( q^{-1}\{q(\theta)\} \). In short, \( P \) is 1-1 over all possible subsets of \( \Theta \) containing only 'representatives' of 'sub-parameter' equivalence classes having the same \( q \) value.

To verify that nothing is lost, we will prove the equivalence to the standard definition.

**Theorem 17** The parameter \( q \) is identifiable, with respect to the forward mapping \( P \), according to the standard definition iff it is identifiable, with respect to \( P \), according to our alternative definition.

**Proof** We will prove each direction by contrapositive, i.e. we will show non-identifiability according to one definition implies non-identifiability according to the other.

Firstly, suppose that \( q \) is not identifiable according to our alternative definition. Then there exists an \( s : \mathcal{R}(q) \to \Theta \), such that \( q \circ s = 1_V \), and such that there exists \( \theta_1, \theta_2 \) with \( P \circ s \circ q(\theta_1) = P \circ s \circ q(\theta_2) \) but \( s \circ q(\theta_1) \neq s \circ q(\theta_2) \). Define \( \theta'_1 = s \circ q(\theta_1) \) and \( \theta'_2 = s \circ q(\theta_2) \). Then \( q(\theta'_1) = q \circ s \circ q(\theta_1) = q(\theta_1) \) (by the key property of \( s \)) and \( q(\theta'_2) = q \circ s \circ q(\theta_2) = q(\theta_2) \). Hence \( q(\theta'_1) \neq q(\theta'_2) \) but \( P(\theta'_1) = P \circ s \circ q(\theta_1) = P \circ s \circ q(\theta_2) = P(\theta'_2) \), which demonstrates non-identifiability according to the usual definition.

Next, suppose that \( q \) is non-identifiable according to the usual definition. Then there exists \( \theta_1 \) and \( \theta_2 \) for which \( P(\theta_1) = P(\theta_2) \) but \( q(\theta_1) \neq q(\theta_2) \). We simply need to show that there always exists an \( s \) for which the previous construction works 'in reverse'. Since \( q(\theta_1) \neq q(\theta_2) \) we know that the range of \( q \), i.e. the domain of \( s \), contains at least two distinct points, \( v_1 = q(\theta_1) \) and \( v_2 = q(\theta_2) \). We will construct an \( s \), starting from these two points, and extend it to the rest of the range of \( q \) (if any). Define \( s \) on these two points by \( s(v_1) = \theta_1 \) and \( s(v_2) = \theta_2 \). Then clearly \( q \circ s \) and \( s \circ q \) are the identity mappings when restricting attention to \( \{v_1, v_2\} \) and \( \{\theta_1, \theta_2\} \), respectively, i.e. \( s \) defines a bijection when restricted to these two sets. Next consider \( q' \) mapping \( \Theta \setminus q^{-1}\{v_1, v_2\} \) to \( \mathcal{R}(q) \setminus \{v_1, v_2\} \).

This is clearly surjective since \( q \) is surjective on \( \mathcal{R}(q) \). Thus, by the axiom of choice, there

\(^{11}\) Note that \( (s \circ q) \circ (s \circ q) = s \circ (q \circ s) \circ q = s \circ q \) since \( q \circ s = 1_V \) and hence \( s \circ q \) is idempotent.
exists an injective mapping $s'$ defined on $\mathcal{R}(q) \setminus \{v_1, v_2\}$ such that $q' \circ s' = 1_{\mathcal{R}(q) \setminus \{v_1, v_2\}}$. Defining $s$ to be equal to this $s'$ over $\mathcal{R}(q) \setminus \{v_1, v_2\}$, and defined as above for $\{v_1, v_2\}$, thus ensures the existence of a mapping satisfying $q \circ s = 1_Y$. Furthermore, we have $P \circ s \circ q(\theta_1) = P(s(q(\theta_1))) = P(\theta_1) = P(\theta_2) = P((q(\theta_2))) = P \circ s \circ q(\theta_2)$, while we know $q(\theta_1) \neq q(\theta_2)$ by assumption. Thus $q$ is non-identifiable according to our definition. 

To make clearer the (equivalence of the) above definitions of identifiability we give a simple example of when a parameter, $q$, is not identifiable with respect to a given forward mapping $P$. We will, for simplicity, consider a forward mapping $P$ which is vector-valued rather than measure-valued\footnote{This example can be directly related to the probabilistic case when the model is based on an additive error assumption, or a location model. As shown in Evans and Stark (2002), a parameter in this setting is identifiable if it is identifiable using just the associated deterministic component of the forward mapping.}.

**Example 2 (A non-identifiable mapping and parameter)** Here we consider a simple example of a non-identifiable mapping and parameter. For simplicity, here we take all of our domains and ranges to be (real-valued) $n$-spaces and the associated mappings $q : \mathbb{R}^2 \to \mathbb{R}$ and $P : \mathbb{R}^2 \to \mathbb{R}$ as linear transformations. Hence we have $\Theta = \mathbb{R}^2$, $V = \mathbb{R}$ and $\mathcal{P} = \mathbb{R}$. Then, for any $\theta \in \Theta$ we can write $\theta = (x, y)^T$ where $x, y \in \mathbb{R}$, and define the parameter of interest by $q(\theta) = q\theta = x$, i.e., $q = (1, 0)$. Furthermore, we define $P$ by $P(\theta) = P\theta = x + y$, i.e., $P = (1, 1)$.

We now show that $q$ is not identifiable with respect to $P$ in terms of Definitions 14 through 16. To do this, let $\theta_1 = (x_1, y_1)^T$, for some $x_1, y_1 \in \mathbb{R}$ with $x_1 \neq y_1$ and take $\theta_2 = (y_1, x_1)^T$. Then $P(\theta_1) = x_1 + y_1 = y_1 + x_1 = P(\theta_2)$, while $q(\theta_1) = x_1 \neq y_1 = q(\theta_2)$. Hence $q$ is not identifiable with respect to $P$ according to the standard Definition 14.

Now, we show that $q$ is not identifiable according to the set-theoretical Definition 15. Let $v_1 = q(\theta_1) = x_1$ and $v_2 = q(\theta_2) = y_1$ so that $v_1 \neq v_2$. Then the inverse set images of $v_1$ and $v_2$ are $q^{-1}\{v_1\} = \{(x_1, z) | z \in \mathbb{R}\}$ and $q^{-1}\{v_2\} = \{(y_1, w) | w \in \mathbb{R}\}$ respectively and $P[q^{-1}\{v_1\}] \cap P[q^{-1}\{v_2\}] = \mathbb{R} \neq \emptyset$. Thus $q$ is not identifiable with respect to $P$ according to Definition 15.

Finally, let $s$ be defined by:

$$s(q(\theta)) = \begin{cases} 
\theta_1, & q(\theta) = q(\theta_1) \\
\theta_2, & q(\theta) = q(\theta_2) \\
(x, 0), & q(\theta) \notin \{q(\theta_1), q(\theta_2)\}.
\end{cases}$$

This can be thought of as, for example, $s : x \mapsto (x, 0)$, i.e. $s = (1, 0)^T = q^\dagger$ (the Moore-Penrose pseudo-inverse), for $x \in \mathbb{R} \setminus \{x_1, y_1\}$, while for $\theta'_1 \in q^{-1}\{v_1\}$, $s(q(\theta'_1)) = \theta_1$ and for $\theta'_2 \in q^{-1}\{v_2\}$, $s(q(\theta'_2)) = \theta_2$. It is easily verified that for all $v \in \mathbb{R}$, $q \circ s(v) = v$, i.e. $q \circ s = 1_\mathbb{R} = 1$. Furthermore, we have $s \circ q(\theta_1) = x_1 \neq y_1 = s \circ q(\theta_2)$, while $P \circ s \circ q(\theta_1) = P(\theta_1) = x_1 + y_1 = y_1 + x_1 = P(\theta_2) = P \circ s \circ q(\theta_2)$. Therefore $q$ is not identifiable with respect to $P$ according to Definition 16.

A schematic of the above example is given in Figure 1.
What can be estimated?

Figure 1: Illustration of Example 2, showing a non-identifiable parameter \( q \), relative to a forward mapping \( P \). The section \( s \) 'picks out' the representatives of the non-identifiable \( q \) values, while for simplicity is equal to the minimum norm representatives in the other cases. The values of \( s \) are indicated by the dark lines/points in the middle diagram.

Next we will extend the key theorem relating identifiability and the existence of Fisher-consistent estimators from the previous section.

4.5.2 Fisher consistency for parameters

First we define estimators of parameters \( q \), and Fisher consistency for these parameters:

**Definition 18 (Fisher consistency for parameter estimators)** A estimator for a parameter \( q : \Theta \to \mathcal{V} \) is a function \( t : P \to \mathcal{V} \). An estimator \( t \) is said to be Fisher-consistent for \( q \) iff \( t \) satisfies:

\[
\forall s : \mathcal{R}(q) \to \Theta \text{ such that } q \circ s = 1_{\mathcal{V}}, \\
\forall \theta \left[ t \circ P \circ s = 1_{\mathcal{V}} \right].
\] (35)

Importantly, \( t \) does not depend on the choice of \( s \), i.e. it does not depend on the choice of \( \theta \) representatives for the \( q \) values. As noted in the previous section, although \( t \) is defined over all \( P \), or a dense subset of this, Fisher consistency only really makes sense over the range of \( P \), i.e. we only need to check that we can correctly recover a parameter when we know the data was generated from a model with that parameter value.

**Theorem 19** A parameter \( q \) is identifiable iff there exists a Fisher consistent estimator for \( q \).

**Proof** Firstly, suppose that \( q \) is identifiable. Then for any \( s \) with \( q \circ s = 1_{\mathcal{V}} \), \( P \circ s \) is a 1-1 function on \( \mathcal{V} \). By the previous section, this implies the existence of a \( t \), possibly depending on the choice of \( s \), such that \( t \circ (P \circ s) = 1_{\mathcal{V}} \). We will show that \( t : P \to \mathcal{V} \) is in fact independent of \( s \), at least when considered as an estimator \( t' \) defined over distributions in the range of \( P \), i.e. as \( t' : \mathcal{R}(P) \to \mathcal{V} \) (distributions outside of the range of \( P \) cannot make \( t \) Fisher inconsistent, since there is no 'true' parameter to recover).
To do this, suppose we have two pairs \( s_1, t_1 \) and \( s_2, t_2 \) satisfying the above and consider, for arbitrary \( v \in \mathcal{V} \), an element \( P_v \) of \( \mathcal{R}(P) \subseteq \mathcal{P} \), obtained by two different choices of \( s, s_1 \) and \( s_2 \), respectively. That is, assume that we have

\[
P_v = P \circ s_1(v) = P \circ s_2(v),
\]

regardless of whether \( s_1 = s_2 \). We show that \( t_1(P \circ s_1(v)) = t_2(P \circ s_2(v)) = v \), i.e. that they define the same function \( t' : \mathcal{R}(P) \to \mathcal{V} \) from distributions in the range of the forward mapping to parameter values. By Fisher consistency, we know that for all \( v \) we have

\[
t_1(P \circ s_1(v)) = t_2(P \circ s_2(v)) = v,
\]

and hence

\[
t_1(P_v) = t_2(P_v) = v.
\]

Thus we have a unique \( t' : \mathcal{R}(P) \to \mathcal{V} \) independent of the choice of \( s \). Clearly if \( q \) is an identifiable parameter then we can extend \( t' \) to a Fisher consistent estimator \( t : \mathcal{P} \to \mathcal{V} \) defined on all of \( \mathcal{P} \), since \( t \) can take any value for distributions not in the range of \( P \) without sacrificing Fisher consistency.

Finally we prove that if a Fisher consistent estimator for a parameter \( q \) exists, then \( q \) is identifiable. Suppose that a Fisher consistent estimator, \( t \), exists for \( q \). Now suppose that for arbitrary \( s \) such that \( q \circ s = 1_\mathcal{V} \), and for arbitrary \( \theta_1, \theta_2 \), we have \( P \circ s \circ q(\theta_1) = P \circ s \circ q(\theta_2) \).

Then we have, by Fisher consistency,

\[
t \circ P \circ s \circ q(\theta_1) = s \circ q(\theta_1)
\]

and

\[
t \circ P \circ s \circ q(\theta_2) = s \circ q(\theta_2).
\]

But \( t \circ P \circ s \circ q(\theta_1) = t(P \circ s \circ q(\theta_1)) = t(P \circ s \circ q(\theta_2)) = t \circ P \circ s \circ q(\theta_1) \), since \( P \circ s \circ q(\theta_1) = P \circ s \circ q(\theta_2) \) by assumption and \( t \) is a function. Thus \( s \circ q(\theta_1) = s \circ q(\theta_2) \), and hence we have identifiability.

5. Continuity considerations: the break between identifiability and estimability

So far we have shown, under very minimal assumptions, that identifiability leads to the existence of a suitable Fisher-consistent estimator. Now we show that, despite appearances, this mere existence result is not sufficient to guarantee what we will call estimability, even in principle. The key (missing) requirement is a stability condition: we should still be able to estimate the quantity of interest with just an ‘infinitesimal’ amount of uncertainty about the population probability distribution. This requires more structure than identifiability and Fisher consistency, and is hence more difficult to discuss in general terms. Without this, however, estimation – in any sense connected to the real world – is impossible. Furthermore,
stability conditions lead to a need to reconsider the ‘existence’ part of the above argument, in terms of the domain of definition of the generalised inverse/estimator. Thus we will see that identifiability does not guarantee estimability, and there are many cases where any estimator of a given identified quantity has unbounded sensitivity and hence is inestimable in principle.

5.1 Well-posed and ill-posed problems

A number of concepts have appeared in the inverse problems and statistical literature, all related in some way to the realisation that, in addition to existence and/or uniqueness, stability, continuity and sensitivity considerations are required. We briefly recap a few general concepts here, before giving specific examples and references in the following subsections.

Firstly, in the inverse problems literature, and in the mathematical literature more generally, it is common to follow Hadamard (1902) and call a problem well-posed if the following three semi-formal criteria hold (see also Engl et al., 1996; Tikhonov and Arsenin, 1977):

- For all admissible problem data, a solution exists
- For all admissible problem data, the solution is unique
- The solution depends continuously on the problem data, i.e. is stable.

The problem then is called ill-posed if at least one of the above conditions does not hold. To make these conditions fully formal requires specific definitions to be given to e.g. the admissible problem data of interest, the relevant topology for continuity, and so on, but these capture the basic idea over a range of cases. Engl et al. (1996); Tikhonov and Arsenin (1977) provide good overviews of classical, mostly deterministic, inverse problems theory. Other references that we draw on extensively include Nashed and Votruba (1976) and Groetsch (1977).

In general failure to satisfy either of the first two criteria can essentially be solved in an algebraic sense by the concept of a generalised inverse, discussed earlier, which amounts to solving the problem ‘as well as can be’. Aside from the condition of being an inner and outer inverse, a unique generalised inverse requires a choice of up to two idempotent maps (projections) in general; a choice of these can be shown to define a unique algebraic generalised inverse (Nashed and Votruba, 1976). In the case of an injective map, we saw above that failure of surjectivity leads to the idempotent but non-identity map \( P \circ T \) on data distributions. Since the ‘model resolution’ operator is the identity in this case, only the data resolution projection operator needs to be chosen to give a unique solution. In statistics a common choice leads to satisfying the data in the least squares sense. Similarly, failure of injectivity (i.e. failure of identifiability) can be ‘solved’ by considering the associated idempotent (but now non-identity) mapping \( T \circ P \) defined on the parameter/model space. In statistics and inverse problems, a common choice to resolve non-uniqueness is to take the model with the smallest norm, and so on. For a general forward mapping \( P \), which is neither an injective nor surjective mapping, one must choose two idempotent mappings – one on the data distribution space, one on the parameter space. Combined with the inner
and outer inverse conditions, this leads to a unique algebraic generalised inverse (again, see Nashed and Votruba, 1976).

5.2 Continuity, stability, and sensitivity

5.2.1 Ideas from inverse problems and functional analysis

The third Hadamard condition given above is typically more difficult and subtle to satisfy, and is not resolved by the usual, essentially algebraic, concept of a generalised inverse. Furthermore, as discussed by Nashed and Votruba (1976), once continuity conditions come into play, a distinction between algebraic generalised inverses and topological generalised inverses must be introduced to handle the surprisingly subtle interplay between existence, uniqueness, and continuity. In particular, much more attention must be given to the relevant domains and codomains of definition, and a distinction between algebraic subspace complements and topological subspace complements becomes important.

First we give the standard definitions of algebraic and topological complementary subspaces. Again we largely follow Nashed and Votruba (1976) and, in places, Groetsch (1977).

**Definition 20 (Algebraic and topological complementary subspaces)** Given a vector space $\mathcal{A}$, we say that two subspaces of $\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{A}_2$, are complementary subspaces, or algebraic complements, in $\mathcal{A}$ iff

$$\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$$  \hspace{1cm} (41)

where $\oplus$ denotes the algebraic direct sum, i.e. if every $a \in \mathcal{A}$ can be uniquely written as the sum $a_1 + a_2$ of an element $a_1 \in \mathcal{A}_1$ and an element $a_2 \in \mathcal{A}_2$. These are called topological complementary subspaces, or topological complements, if, in addition to being algebraic complements, any associated linear projector (idempotent map) $P$ with $\mathcal{R}(P) = \mathcal{A}_1$ and $\mathcal{R}(I - P) = \mathcal{A}_2$ is continuous. Or, equivalently, if these subspaces are both closed.

The existence of topological complements for any closed subspace is not guaranteed in arbitrary Banach spaces, but is guaranteed in any Hilbert space. In Banach spaces we must in general add the supposition that topological complements exist, and we will assume this in what follows.

Now we consider the relevant domains and codomains, given $P : \Theta \to \mathcal{P}$. In particular we suppose that $P$'s codomain can be written as

$$\mathcal{P} = \overline{\mathcal{R}(P)} \oplus \mathcal{C}$$  \hspace{1cm} (42)

where $\mathcal{C}$ is the topological complement of the closure of the range $\mathcal{R}(P)$ of $P$. From the definition of topological complementary subspaces, this complement is always closed. In a Hilbert space we have $\mathcal{C} = \mathcal{R}(P)^ot$ and so

$$\mathcal{P} = \overline{\mathcal{R}(P)} \oplus \mathcal{R}(P)^ot.$$  \hspace{1cm} (43)

On the other hand, we take the natural domain of definition $\mathcal{D}(T)$ of our generalised inverse (estimator) $T$ to be

$$\mathcal{D}(T) = \mathcal{R}(P) \oplus \mathcal{C},$$  \hspace{1cm} (44)
where $\mathcal{C}$ is the same as before (i.e. the topological complement of the closure of the range). Again in a Hilbert space we have $\mathcal{C} = \mathcal{R}(P)^\perp$ and so

$$D(T) = \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp.$$ \hfill (45)

Note that unless $\mathcal{R}(P) = \overline{\mathcal{R}(P)}$, i.e. unless the range of $P$ is closed, then $D(T) \neq \mathcal{P}$. This seemingly subtle distinction will play an important role in establishing the continuity, or not, of the generalised inverse.

5.2.2 Conditions for continuous/discontinuous generalised inverses

Here we consider various special cases of the general requirement that (the generalised inverse) $T$ be continuous. The linear theory is relatively standard and well-developed, while the nonlinear case less so. Nevertheless, we can also give various general conditions for when a nonlinear problem is ill- or well-posed.

Firstly, consider the case where the forward mapping $P$ is an everywhere-defined, linear and continuous operator which maps between two Hilbert spaces\textsuperscript{13}. Many of the following results are essentially consequences of the famous Open Mapping Theorem (Banach and Schauder), extended to generalised inverses. The standard theorem can be stated in our setting as:

**Theorem 21 (Open Mapping Theorem)** Consider a linear operator $P : \Theta \to \mathcal{P}$. If $P$ has closed range $\mathcal{R}(P)$, then the image $P[\mathcal{O}]$ of any open set $\mathcal{O} \subset \Theta$ is an open set in $\mathcal{R}(P)$. Similarly, the image of any closed subset is closed. If $P$ is also injective and surjective, then $P^{-1} : \mathcal{P} \to \Theta$ exists and is continuous (bounded).

Proofs may be found in textbooks on functional analysis. This result implies the Closed Graph Theorem:

**Theorem 22 (Closed Graph Theorem)** Let $P : \Theta \to \mathcal{P}$ be a graph-closed operator, i.e. have closed graph where the graph of an operator is given by

$$\mathcal{G}(P) = \{ (\theta, P(\theta)) \mid \theta \in \Theta \}.$$ \hfill (46)

Then $P$ is continuous.

These theorems extend to similar results concerning generalised inverses, i.e. Fisher-consistent estimators, e.g.:

**Theorem 23** The generalised inverse (estimator) $T : \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp \subseteq \mathcal{P} \to \Theta$, associated with the linear, continuous forward operator $P : \Theta \to \mathcal{P}$, is a linear operator which is bounded (continuous) if and only if the range of $P$, $\mathcal{R}(P)$, is closed. In this case $\mathcal{R}(P) \oplus \mathcal{R}(P)^\perp = \mathcal{P}$ and hence $T$ is defined on all of $\mathcal{P}$. Furthermore, since $\mathcal{R}(P \circ T) = \mathcal{R}(P)$, we have that $\mathcal{R}(P)$ is closed, and hence $T$ is continuous, iff the range $\mathcal{R}(P \circ T)$ of the idempotent operator $P \circ T$ is closed.

\textsuperscript{13} Many of these results can also be extended to closed, densely-defined but possibly unbounded linear operators with domains and codomains that are Banach spaces, or at least Banach spaces that have topological complements.
Proof A sketch of a proof goes as follows. Recall that a function is continuous iff its inverse set image of any closed set is also closed. The inverse set image of the closed set $\Theta$ under the generalised inverse is $\mathcal{R}(P) \oplus \mathcal{R}(P)^\perp \subseteq \mathcal{P}$, which cannot be closed if $\mathcal{R}(P)$ is non-closed. Hence continuity implies closed range. The converse, that closed range implies continuous generalised inverse, follow directly from the Closed Graph Theorem.

This theorem is stated for arbitrary Hilbert space domains and codomains in e.g. (Nashed and Votruba, 1976; Engl et al., 1996; Groetsch, 1977); here we have simply translated this to the special case where the domain of $P$ is $\Theta$ and the codomain of $P$ is $\mathcal{P}$. Importantly, Hilbert spaces are often very natural settings for the domains and codomains in inverse problems and statistical problems (Small and McLeish, 2011; Evans and Stark, 2002; Vapnik, 2013; Groetsch, 1977; Engl et al., 1996; Ramsay and Silverman, 2005), so this is no real restriction, though more general results in e.g. Banach spaces also exist.

The following stability condition is often more convenient to work with than the closed range condition (Drabek and Milota, 2009; Groetsch, 1977):

**Theorem 24** If $P : \Theta \to \mathcal{P}$ is a linear, continuous, injective (identifiable) forward operator between Hilbert (or Banach) spaces, then its range $\mathcal{R}(P)$ is closed iff there is a positive constant $c$ such that for all $\theta \in \Theta$:

$$\|P(\theta)\| \geq c\|\theta\|.$$  \hfill (47)

See e.g. (Drabek and Milota, 2009; Groetsch, 1977) for proofs. In such a case we say the forward mapping $P$ is bounded away from zero. Importantly, one can have operators that are invertible and yet not bounded away from zero – in such cases one has a sequence of singular values with zero as a limit point.

We can also see how the above gives a stability condition on the estimator/estimates in the sense that it means there exists a $c > 0$ such that

$$\|\theta_1 - \theta_2\| \leq \frac{1}{c}\|P(\theta_1) - P(\theta_2)\|$$  \hfill (48)

which follows directly from linearity and taking $\theta = \theta_1 - \theta_2$. That is, defining $P_1 = P(\theta_1)$ and $P_2 = P(\theta_2)$, we have

$$\|T(P_1) - T(P_2)\| \leq \frac{1}{c}\|P_1 - P_2\|$$  \hfill (49)

for a Fisher consistent estimator $T$, which is hence a bounded/continuous linear operator. On the other hand, since $P$ is bounded, we also have, for some $b > 0$ and for all $\theta$,

$$\|P(\theta)\| \leq b\|\theta\|.$$  \hfill (50)

This means that we can write the continuity of $T$ as

$$\frac{\|T(P_1) - T(P_2)\|}{\|T(P_2)\|} \leq \frac{b}{c}\frac{\|P_1 - P_2\|}{\|P_2\|}.$$  \hfill (51)
For finite-dimensional linear problems, the range is in fact always closed and hence all finite-dimensional linear problems are formally stable in the above sense. As is well-known in the applied mathematics literature, however, finite-dimensional problems can exhibit a form of instability essentially equivalent to ill-posedness, called \textit{ill-conditioning}. To see this, consider the stability theorem for finite-dimensional inverse problems in the form:

**Theorem 25** If $P : \Theta \to \mathcal{P}$ is a linear, continuous, injective (identifiable) and finite-dimensional forward operator then

$$\frac{\|T(P_1) - T(P_2)\|}{\|T(P_2)\|} \leq \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}} \frac{\|P_1 - P_2\|}{\|P_2\|},$$

(52)

where $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ are the maximum and minimum singular values of $P$, respectively (see e.g. Aster et al., 2013, for an elementary derivation). The ratio $\frac{\sigma_{\text{max}}}{\sigma_{\text{min}}}$ is known as the \textit{condition number}. Thus in such problems, while they are technically well-posed, they can be ill-posed for any achievable precision, e.g. to machine zero, or at least exhibit essentially uncontrollable instability. This leads to the following semi-formal definition:

**Definition 26 (Ill-conditioned inverse problems)** A finite-dimensional linear inverse problem is known as ill-conditioned (c.f. ill-posed in the infinite-dimensional case) if the condition number of the forward mapping is unacceptably large.

Finally, we briefly consider nonlinear problems. Linear operators have the property that they are either continuous everywhere or nowhere, and that continuity and boundedness are equivalent. This makes continuity easy to characterise in terms of the (mapping from the domain to) range only. Nonlinear operators do not have this property, however, so are much more difficult to deal with in general. We will not attempt to survey the literature on nonlinear functional analysis here, but note that the following (uni-directional) result does still hold (we assume for convenience here that $\Theta$ and $\mathcal{P}$ are Hilbert spaces):

**Theorem 27** The generalised inverse (estimator) $T : \mathcal{R}(P) \oplus \mathcal{R}(P)^\perp \subseteq \mathcal{P} \to \Theta$, associated with the (possibly nonlinear) continuous forward operator $P : \Theta \to \mathcal{P}$, has at least one point of discontinuity if the range of $P$, $\mathcal{R}(P)$, is non-closed.

**Proof** A function is continuous iff its inverse set image of any closed set is also closed. The inverse set image of the closed set $\Theta$ under the generalised inverse is $\mathcal{R}(P) \oplus \mathcal{R}(P)^\perp \subseteq \mathcal{P}$, which cannot be closed if $\mathcal{R}(P)$ is non-closed.

Furthermore, we have the following key result which relies on the idea of restricting models to compact subsets\textsuperscript{14} of the full parameter space (Tikhonov and Arsenin, 1977; Zeidler, 1985):

**Theorem 28** Suppose $P : M \to \mathcal{R}(P)$ is a continuous, injective forward mapping defined on a compact subset $M \subseteq \Theta$. Then $P^{-1} : \mathcal{R}(P) \to M$ is continuous.

\textsuperscript{14} Recall that the general topological characterisation of a compact set is that every open cover has a finite subcover. In $\mathbb{R}^n$ the concept of a compact subset is the same as a closed and bounded subset, but this does not hold in more general spaces.
For a proof and natural extensions to generalised inverses see (Tikhonov and Arsenin, 1977).

The concept of compactness plays a key role in nonlinear functional analysis, and provides the means to extend many results that hold for finite-dimensional spaces to infinite-dimensional spaces and to express key approximation methods (Zeidler, 1995, 1985). Compactness also underlies the concept of regularisation, in which an ill-posed problem defined on a non-compact domain is replaced by a sequence of nested restricted problems with the solutions of each problem restricted to one of a sequence of nested, compact subsets of the domain, \( M_1 \subset M_2 \subset \ldots \subset M_n \subset \ldots \), and where the closure of their union equals \( \Theta \), i.e. \( \bigcup_{n=1}^{\infty} M_n = \Theta \). Each solution provides a stable approximation of the solution to the full problem, and the art of regularisation lies in choosing the trade-off between the goodness of approximation and stability. In reality, this choice must also take into account ill-conditioning in addition to formal ill-posedness, as discussed above.

A good discussion of classical regularisation theory, stochastic ill-posed problems and the link between compactness and the concept of ‘capacity control’ in statistical learning theory is given by Vapnik (2013). Instability of nonlinear estimators has also been considered in the standard statistical literature, which we consider next.

### 5.2.3 Ideas from statistics

Essentially the same continuity concepts as discussed above have arisen in the statistical literature. In contrast to the inverse problems literature, however, much of this literature tends to assume the existence of (Fisher) consistent estimators \( a \text{ priori} \), and hence assume (typically implicitly) identifiability. The main focus is then the further properties of these estimators, such as continuity or robustness. Again, perhaps counter to some intuitions, we find the implication that mere existence of a (Fisher) consistent estimator, and hence identifiability, does not imply estimability even in principle.

An early example is the illustration by Bahadur and Savage (1956) of the impossibility of fully nonparametric estimation of the mean. As shown there, it is impossible to obtain a nontrivial confidence interval for the mean, when considered as a functional defined over the set of all distributions for which this exists and is finite. Subsequent work of particular relevance here is that of Donoho (1988) on one-sided inference and Tibshirani and Wasserman (1988) on sensitive parameters. These nicely illustrate the general concepts underlying the impossibility result of Bahadur and Savage (1956), and are also closely related to work in robust statistics (see e.g. Huber and Ronchetti, 2011; Hampel et al., 2011), in particular the general notion of qualitative robustness and the related (local) robustness measures derived from the influence function, both introduced by Hampel (1971). From a slightly different but related direction, Vapnik (2013) discusses the key role that classical inverse problems theory played in the development of his and colleagues’ work in the theory of statistical machine learning. More recent work on stability and statistical machine learning, with strong connections to inverse problems theory, includes that by Poggio et al. (2004); De Vito et al. (2005).

First we briefly consider the concept of sensitive parameters, introduced by Tibshirani and Wasserman (1988), and building on Bahadur and Savage (1956). They use the term functional parameter for what we call the estimator \( T \) mapping (ideal) distributions back to parameter/indexing space. We will hence call these sensitive estimators; following Tibshirani and Wasserman
(1988) we emphasise, however, that these are functions of population distributions, and are not finite sample quantities.

Tibshirani and Wasserman (1988) define a sensitive parameter $T$, in our terminology called a sensitive estimator, for $\mathcal{P}$ a complete, separable metric space with distance $\delta$, and $\Theta = \mathcal{R}(T) \subseteq \mathbb{R}^n$ as:

**Definition 29 (Sensitive estimator)** An estimator $T : \mathcal{P} \rightarrow \Theta$ is called sensitive with respect to metric $\delta$ iff for all $\mathbb{P} \in \mathcal{P}, \theta \in \Theta, \epsilon > 0$, there exists $\mathbb{Q} \in \mathcal{P}$ such that

$$\delta(\mathbb{P}, \mathbb{Q}) \leq \epsilon \text{ and } T(\mathbb{Q}) = \theta.$$  \hspace{1cm} (53)

This means that, given a probability distribution $\mathbb{P} \in \mathcal{P}$ we can always find a distribution $\mathbb{Q} \in \mathcal{P}$ that is arbitrarily close to $\mathbb{P}$, in terms of the probability distribution distance $\delta$, and yet will give an arbitrary estimator value. In particular, given $\mathbb{P}$, an arbitrarily close $\mathbb{Q}$ can always be found that nevertheless makes $T(\mathbb{Q})$ arbitrarily different to $T(\mathbb{P})$. Tibshirani and Wasserman (1988) primarily take $\delta$ to be the total-variation distance, but also consider the Prohorov and contamination distances.\(^{15}\) They show that sensitivity with respect to the contamination distance implies sensitivity with respect to the others. Similarly to Bahadur and Savage (1956), a notable example of a sensitive parameter is the mean.

Hampel considered stability earlier, in the context of robust statistics and finite samples. Tibshirani and Wasserman (1988) show, however, that we can consider the $n \rightarrow \infty$ limit and obtain essentially the same conclusions. In particular, they show by considering the limiting case of the definition of qualitative robustness given by Hampel (1971), that qualitative robustness for a real-valued estimator implies the estimator is continuous with respect to the Euclidean distance between the estimator values.\(^{16}\) Additional discussion of the relationships between qualitatively robust sequences of estimators, continuous sequences of estimators, and continuity of functional estimators is given by Hampel et al. (2011); Huber and Ronchetti (2011). For now we introduce a key heuristic tool from robust statistics – the influence function (Huber and Ronchetti, 2011; Hampel et al., 2011), which essentially amounts to the Gateaux derivative of $T$ at $F$ in the direction of the point mass $\Delta_y$ at $y$ (though it exists under weaker conditions).

**Definition 30 (Influence function)** The influence function is defined by

$$\text{IF}(y; T, F) = \lim_{\epsilon \downarrow 0} \frac{T((1 - \epsilon)F + \epsilon \Delta_y) - T(F)}{\epsilon}.$$  \hspace{1cm} (54)

where $\Delta_y$ is the point mass at $y$.

It can be shown that the asymptotic variance of the estimator is, under some regularity conditions, given by

$$V(T, F) = \int \text{IF}(y; T, F)^2 dF(y)$$  \hspace{1cm} (55)
Smaller variance corresponds to greater (asymptotic) efficiency, and one use of the influence function is hence to help construct efficient estimators. There is, however, a trade-off between efficiency and robustness to perturbations or sensitivity, e.g., infinitesimal contamination. As we will see, the mean is maximally efficient for location estimates under the normal family, but is also maximally sensitive to distributional contamination. The influence function also aids in these considerations. A measure of robustness in terms of the influence function is the so-called gross-error sensitivity of $T$ at $F$:

**Definition 31 (Gross-error sensitivity)** The gross-error sensitivity is defined by:

$$
\gamma^* = \sup |IF(y; T, F)|,
$$

i.e. the maximal value of the influence function, where this is taken over all $y$ for which this exists.

This measures the worst influence that an infinitesimal contamination can have, and can be considered as giving an upper bound on the asymptotic bias (Hampel et al., 2011). Unbounded influence functions thus give unbounded gross-error sensitivity and unbounded asymptotic bias. As discussed by Tibshirani and Wasserman (1988), functionals with unbounded gross error sensitivity are sensitive in their sense.

The key message for our purposes is simply that there are strong connections between the statistical notions of ‘sensitivity’ and ‘robustness’, and continuity of estimators\(^{17}\).

Finally, and again following Bahadur and Savage (1956), Donoho (1988) gives conditions for nonlinear statistical functionals to be inestimable, at least in the two-sided sense. In particular, Donoho considers the ‘truly’ or ‘strongly’ nonparametric setting, which he describes in intuitive terms as\(^ {18}\):

Intuitively, a truly nonparametric family of distributions has the property that, when it contains a distribution $\mathcal{P}$ it also contains all other distributions which cannot be reliably distinguished from $\mathcal{P}$ at a given sample size based on any empirical test...Following this line of reasoning, we arrive at the requirement that, for $\mathcal{P}$ to be nonparametric, it should contain at least a small $\tau$ neighbourhood around essentially every point.

He demonstrates, also by introducing the topological concepts necessary to characterise continuity and related notions, that a number of functionals of distributions are badly discontinuous\(^ {19}\) in this setting. This includes the mean, as well as several measures of complexity such as the number of modes of a density. Interestingly, while the mean cannot...

\(^{17}\) These connections are particularly strong when identifiability is assumed or generalised inverses are used.

\(^{18}\) E.g. the median is robust and insensitive but is technically discontinuous unless it is uniquely defined e.g. as the minimum value for which the cumulative distribution is equal to 0.5. Identifiability and/or the concept of the generalised inverse naturally lead to such an additional condition and hence continuity of the median.

\(^{19}\) Which he characterises in terms of a dense graph condition, closely related to the idea of sensitivity in Tibshirani and Wasserman (1988), as they note.
be given either lower or upper bounds, he demonstrates that many measures of complexity
can be given lower bounds, just not upper bounds – e.g. that one can say a distribution
must be ‘at least this complex’, but cannot rule out more complex distributions based
on any empirical test. Again the important point for our purposes is that additional, e.g.
topological, concepts and continuity must be considered in order to characterise estimability,
even in the identifiable scenario.

6. So, what is estimability?

Which the above in mind, we can give a tentative and semi-formal characterisation of
estimability, with respect to our current setting:

**Definition 32 (Estimability)** Given a forward mapping \( P : \Theta \to \mathcal{P} \), between models
\( \theta \in \Theta \) and observable distributions \( \mathcal{P} \in \mathcal{P} \), a parameter \( q : \Theta \to \mathcal{V} \) is estimable if it is
identifiable and, in addition, the implied Fisher-consistent estimator \( t : \mathcal{P} \to \mathcal{V} \) is sufficiently
continuous.

This includes the case where, for example, one would say a formally well-posed but ill-
conditioned estimator leads to inestimability. We have also only stated this condition as
a sufficient condition, as one may even consider dropping the strict identifiability require-
ment and consider instead e.g. estimation subject to bounded asymptotic bias (see e.g.
Evans and Stark, 2002).

7. Examples of identifiable but inestimable quantities

Here we consider, using special cases of our more general setting, simple examples of prob-
lems that arise in statistics, causal inference and inverse problems. In each subsection we
first formulate the basic problems, then we consider stability. We largely follow Vapnik
(2013) in converting statistical questions to operator equations, and Hampel et al. (2011);
Huber and Ronchetti (2011) in our discussion of influence functions and robustness. Simi-
lar examples have appeared in the econometrics literature (see e.g. Khan and Tamer, 2010;
Lewbel, 2016; Escanciano, 2018; Horowitz, 2014, and references therein).

7.1 Basic estimation problems

7.1.1 Conditional density estimation

Consider the causal model described in Figure 7.1.1, where \( U \) is a (potentially unmeasured)
mediator for the \( X \rightarrow Y \) relationship.

In this case we have

\[
p(y|do(x)) = p(y|x)
\]

and hence our causal estimation problem reduces to a statistical estimation problem of
determining \( p(y|x) \). We will consider observed data \((x, y)_i, \ i = 1, .., n\), in the infinite
data limit \( n \to \infty \), as is standard in identifiability analysis in causal inference, but will
also consider the stability at this ideal point. This can also be considered as analysing the
approach to the limit, in addition to the behaviour at the limit, but we again emphasise that this is essentially an intrinsic limiting property related to estimability in principle.

First, and following Vapnik (2013) as noted above, we consider the definition of the conditional density \( p(y|x) \) as the solution \( f(x, y) \) to:

\[
\int_{-\infty}^{y} \int_{-\infty}^{x} f(x, y) dF(x') dy = F(x, y),
\]

(58)

where \( F(x) \) and \( F(x, y) \) are the cumulative distribution functions of \( x \) and \( x, y \) respectively. If the relevant densities exist we can write (informally) \( dF(x) = p(x) dx \).

As noted by Vapnik (2013); Vapnik and Izmailov (2015), as well as e.g. Donoho (1988); Davies (2014); Lewbel (2016), the cumulative distributions and the empirical data are naturally directly related in the sense that data generated by two ‘close’ \( F \) will also be close, and that the empirical \( F \) converges consistently and rapidly to the ‘true’ \( F \) (e.g. via the famous Glivenko-Cantelli theorem and related bounds on convergence rates). See for example Van der Vaart and Wellner (2013). Furthermore, there is a 1-1 correspondence between \( F \) and the associated probability measure. Hence it is natural to take \( F \) as the given and \( p(y|x) \) as determined from this via the above integral equation. If a unique solution exists, then \( p(y|x) \) is uniquely determined.

The above can be written as a Fredholm integral equation (i.e. with unrestricted integration limits) via:

\[
\int \int H(y - y')H(x - x')f(y', x')dF(x')dy' = F(x, y),
\]

(59)

where \( H \) is the Heaviside (step) function with \( H(z) = 0 \) for \( z \geq 1 \) and \( H(z) = 0 \) otherwise. This means that, abstractly, we have

\[
Kf = F,
\]

(60)

where the operator \( K \) has the kernel \( k \):

\[
k(x, x', y, y') = k(x - x', y - y') = H(y - y')H(x - x').
\]

(61)

Thus to determine \( p(y|do(x)) \) in the identifiable case, and when \( p(y|do(x)) = p(y|x) \), we must solve the statistical problem defined by the above integral equation.
What can be estimated?

By making the identification \( p(y|do(x)) \leftrightarrow \theta \), and since as mentioned above the cumulative distribution function \( F(x, y) \) uniquely determines the probability distribution \( P(X, Y) \), we can consider \( Kf = F \) to represent our forward mapping, i.e. we make the correspondence

\[
Kf = F \leftrightarrow P(\theta) = \mathbb{P}_\theta.
\]

Thus we see that the problem of determining the basic causal quantity \( p(y|do(x)) \) can be formulated in terms of solving a (Fredholm) integral equation.

The above covers the case where the data contains variability/errors in \( x \) as well as in \( y \) and hence naturally leads to considering perturbations to both the operator \( K \) and the right-hand side \( F \), i.e. to consideration of solutions to the perturbed equation

\[
K_\delta f = F_\delta.
\]

This case is considered explicitly in a statistical context by e.g. Vapnik (2013); Vapnik and Izmailov (2015), though the idea of replacing the operator itself is also at the heart of classical regularisation methods. For simplicity, however, we will instead consider the ‘conditional’ case of estimating \( p(y|x) \) for fixed/known/error-free \( x \), i.e. we will essentially restrict attention to equations of the form \( Kf = F_\delta \), where the perturbations are to the right-hand side only. Explicitly, in this setting we again have an integral equation and \( p(y|x) \) is, by definition, the solution to

\[
\int H(y - y')f(y', x)dy' = F_{Y|X}(y|x),
\]

where \( F_{Y|X}(y|x) = \mathbb{P}(Y \leq y, X = x) \) and \( x \) is considered known. The problem then is to determine the solution \( f(y, x) = p(y|x) \) to this equation, given \( F \) and \( H \). Again, we must solve a (Fredholm) integral equation. We also have the following normalisation and non-negativity constraints for probability densities:

\[
\int f(y, x)dy = \int p(y|x)dy = 1 \text{ for all } x, \text{ and } f(y, x) = p(y|x) \geq 0 \text{ for all } x, y.
\]

It is entirely possible for the above equations to possess unique but unstable solutions, as we will demonstrate with a simple concrete example, following Vapnik (2013); Vapnik and Izmailov (2015). This example is also very similar to the example of numerical differentiation given by Engl et al. (1996) and the examples considered by Horowitz (2014).

**Example 3** Consider the case where \( F_{Y|X}(y|x) = y \) over \( y \in [0, 1] \), i.e. we must solve

\[
\int H(y - y')f(y', x)dy' = y
\]

subject to the normalisation condition over \( y \in [0, 1] \). It is easy to verify that the solution is:

\[
f(y, x) = p(y|x) = 1.
\]
Consider next the solution to the equation with a small perturbation to the right-hand side:

\[
\int H(y - y')f_\delta(y', x)dy' = y + \delta \sin\left(\frac{1}{\delta}y\right)
\]

for \( \delta = \frac{1}{2n\pi}, n \in \mathbb{N} \). In this case the solution is:

\[
f(y, x) = p(y|x) = 1 + \cos\left(\frac{1}{\delta}y\right).
\]

From this, we see that as \( \delta \to 0 \), i.e. \( n \to \infty \), we have that the perturbed right-hand side approaches the unperturbed right-hand side, but the solution to the perturbed equation does not approach the solution to the unperturbed equation. This example is illustrated in Figure 3.

The above example demonstrates the ill-posedness of the problem of estimating \( p(y|x) \), and hence \( p(y|do(x)) \) in this case. Of course, ill-posedness can be addressed via regularisation methods, but this amounts to requiring additional restrictions on the model space in which solutions are sought, i.e. restrictions on the causal questions and answers. This is true even despite identifiability, i.e. the solution can be unique but arbitrarily unstable.

### 7.1.2 Regression

Regression models can also be written directly as solutions to Fredholm integral equations in a similar manner to the above, or obtained by first estimating the conditional distribution function (Vapnik, 2013; Vapnik and Izmailov, 2015). Here we simply directly consider the regression function, for fixed \( x \), as a functional of the conditional cumulative distribution function. This makes the arguments of Tibshirani and Wasserman (1988); Huber and Ronchetti (2011); Hampel et al. (2011); Hampel (1971) concerning sensitive functionals and robust statistics directly applicable.
What can be estimated?

We again emphasise that, in general, regression functions are not equal to what is sometimes called causal regression functions (Wasserman, 2013). This latter function captures the ‘response’ $Y$ to ‘treatment’ $X = x$ in the continuous setting. When $X$ is randomly assigned or when e.g. the DAG in Figure 7.1.1 holds, the statistical and causal regression functions are numerically equal, however. For simplicity, we will again assume this relationship holds. Hence we have identifiability, but need to further consider stability in order to assess estimability.

The regression function is defined, for each $x$, by

$$r(x) = \int yp(y|x)dy = \int ydF_{Y|X}(y|x),$$

(70)

which defines $r(x)$ as a linear functional of the conditional distribution function of the form $T(F_{Y|X}) = \int \psi dF_{Y|X}$, i.e. we have

$$r(x) = T(F_{Y|X=x}).$$

(71)

It can be shown that linear functionals of the form $T(F) = \int \psi dF$ are bounded iff $\psi$ is bounded and continuous (Huber and Ronchetti, 2011).

**Example 4** In our case we have $\psi = y$, which is clearly unbounded, and hence $T$ is unbounded (discontinuous). More concretely, it is easy to show that, given any $\epsilon > 0$, the mean of the $\epsilon$-contaminated distribution $Q = (1-\epsilon)F + \epsilon G$ can be made arbitrarily different to that of $F$ by appropriate choice of $G$, despite these distributions being arbitrarily close in terms of contamination distance (Tibshirani and Wasserman, 1988; Huber and Ronchetti, 2011; Hampel et al., 2011). By taking $G$ to be the delta measure at $y$, $\Delta_y$, we obtain the influence function $IF(y;T,F)$, defined above. It is straightforward to show that for $T$ representing the mean, $IF(y;T,F) = y$. Hence the gross-error sensitivity, which we recall from above is the supremum of the influence function and is a measure of asymptotic bias, is equal to $+\infty$, i.e. the mean is an arbitrarily sensitive parameter in the nonparametric setting. Thus the mean is essentially inestimable in the general setting.

As discussed above, similar results include those of Bahadur and Savage (1956); Donoho (1988). In the causal estimation context this means that any causal quantity corresponding to the mean is a sensitive causal query in the truly nonparametric setting. Such queries can be said to be identifiable but inestimable.

**7.1.3 Other examples: continuous confounding, average treatment effects, ill-conditioning**

The purpose of the present work is to point out basic conceptual issues in causal inference as seen from an inverse problems perspective. The above examples are intended as elementary and illustrative. Much of (graphical) causal inference work to date appears to be split into conceptual abstract work on topics like identifiability, and concrete implementations in terms of restricted model classes e.g. linear regression. We note, however, that there is increasing interest in truly nonparametric causal inference. One example that was recently brought to our attention is the work by Miao et al. (2018) which discusses the case of a continuous confounder. In their article, the need to solve Fredholm integral equations
appears in exactly the same way as we have considered here. They analyse and solve this by the usual tools of inverse problems theory, e.g. by using the singular value decomposition. We emphasise that this leads to additional regularity conditions of exactly the sort discussed in the present article.

A natural extension to Example 4 has been considered in the econometrics literature: estimating the average treatment effect (ATE) using propensity scores (see e.g. Rosenbaum and Rubin, 1983, for early work). As discussed by Khan and Tamer (2010), this leads to a functional of the form \( \int \psi dF \) where \( \psi \) is unbounded unless the propensity score is bounded away from zero and one, i.e. does not just lie in \((0,1)\) but instead lies in \((b_l,b_u)\) for some \(b_l > 0, b_u < 1\). They specifically note that this leads to a subtle interplay between identification and estimability ‘that limits the practical usefulness of estimators based on these models’. As mentioned previously, Lewbel (2016) provides a comprehensive overview of identification concepts in econometrics and the (often subtle) link between identifiability and estimation. This particularly true of cases Lewbel (2016) labels ‘identification concepts that affect inference’. Horowitz (2014) considers econometrics estimation problems from the perspective of ill-posed inverse problems and gives a number of examples similar to those considered here.

From a slightly different direction Schulman and Srivastava (2016) consider the ‘sensitivity of causal identification to small perturbations in the input’. They calculate a condition number for the identification algorithm for identifiable semi-Markovian models, and find that this can be extremely large, i.e. the identification algorithm can be extremely numerically unstable. This illustrates that the issues we have raised here are not restricted to the continuous ill-posed case, but also appear in the form of ill-conditioning in discrete problems. Though we have not done so yet, it would certainly be of interest to consider whether the tools of inverse problems theory can aid in determining which models lead to bad condition numbers and, potentially, how one might introduce appropriate regularisation in such cases to stabilise the algorithm. This would, of course, involve replacing the ill-conditioned problem by a ‘similar’ well-conditioned problem and hence modifying the causal question addressed.

8. Discussion

We have considered the concept of estimability and its relation to the concept of identifiability. The (graphical) causal inference literature, in contrast to the inverse problems, statistical, and econometrics literature, takes identifiability as synonymous with estimability. In our view this is a mistake, despite its intuitive appeal. This is supported by the numerous examples of ill-posed problems in the broader literature on inverse problems, statistics, statistical machine learning, and econometrics, of which we have presented only a small number in the present work. In addition to simple examples, we have noted that these issues appear to arise in real causal inference problems (Schulman and Srivastava, 2016; Miao et al., 2018; Khan and Tamer, 2010; Lewbel, 2016; Escanciano, 2018).

\[\int \psi dF\text{ where } \psi \text{ is unbounded unless the propensity score is bounded away from zero and one, i.e. does not just lie in } (0,1) \text{ but instead lies in } (b_l,b_u) \text{ for some } b_l > 0, b_u < 1.\] This functional is potentially nonlinear, but it can be considered as a linear functional of the distribution when the propensity score function is taken as a ‘given’, with values in \((0,1)\), but otherwise arbitrary. In this case the result from Huber and Ronchetti (2011) – that linear functionals are bounded iff \( \psi \) is bounded and continuous – applies directly.
The source of the disconnect between identifiability and estimability (even in principle) is a failure to consider stability to perturbations, even infinitesimally-small ones. In terms of the Hadamard (1902) conditions for well-posed problems, identifiability represents uniqueness, but ignores stability. We propose that true estimability requires additional stability considerations, i.e. corresponds to considering whether the problem is well-posed.

These mathematical considerations also raise interesting philosophical questions concerning what types of causal questions are truly answerable in the general setting. For example, Donoho (1988) notes that the types of quantities that are estimable in the truly nonparametric setting concern lower bounds on measures of ‘model complexity’. Similarly, Vapnik (2013) relates his learning theory, regularisation via restriction to compact subsets, and restrictions on model class ‘capacity’ (informally a measure of model class complexity) to Popper’s concept falsifiability (Popper, 2005). In short, one might say that we can potentially rule out simple causal explanations, but cannot rule out more complex causal explanations. Regardless of this interpretation, however, thinking about stability and estimability, even in principle, requires us to go beyond mere identifiability.

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