Radiative Fermion Mass Hierarchy in a Non-supersymmetric Unified Theory

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Abstract

In non-supersymmetric grand unified models a “radiative fermion mass hierarchy” can be achieved in which the spectrum of quark and lepton masses is determined entirely by physics at the unification scale, with many relations following from the unified gauge symmetry, and with the masses of the lightest family arising from loops. A simple, realistic, and predictive model of this kind is presented. A “doubly lopsided” structure, known to lead to bilarge neutrino mixing, plays a crucial role in the radiative hierarchy.
The masses of the charged quarks and leptons span a large range, the electron mass being only $2.9 \times 10^{-6}$ of the $t$ quark mass. The masses of fermions of the same type in successive families typically differ by about two orders of magnitude. These facts have suggested to many people the possibility of a “radiative hierarchy”, i.e. a hierarchy in which the heavier fermions get mass from tree diagrams while the lighter fermions get mass from loop diagrams [1]. Indeed, this speculation is very old, going back to the observation that $m_e/m_\mu = O(\alpha)$.

In the early 1980s, several papers [2] showed that the idea of a radiative fermion mass hierarchy can be implemented in grand unified theories (GUTS) in such a way that the structure of the observed quark and lepton masses is entirely determined by physics at the unification scale $M_{GUT}$. In these models the loop diagrams that generate the radiative fermion masses contain virtual particles with masses of order $M_{GUT}$. Interest in such models waned with the growing popularity of the idea of low-energy supersymmetry. In supersymmetric models, radiative corrections to the quark and lepton mass matrices would be highly suppressed by the non-renormalization theorem.

There are a number of reasons why one should still take seriously the idea of nonsupersymmetric grand unified models. The main one, of course, is that low-energy supersymmetry has not yet been discovered. Moreover, models with low-energy supersymmetry face several well-known challenges, such as the “$\mu$ problem”, the “SUSY flavor problem”, and the “SUSY CP problem”. While supersymmetry does lead to an impressive unification of gauge couplings and thus seems to go hand-in-hand with the idea of unification, supersymmetric grand unified theories are not without their own difficulties, notably the danger of excessive proton decay mediated by the exchange of Higgsinos. And the main theoretical problem solved by low-energy supersymmetry, namely the tuning of the Higgs mass, may not be so dreadful in the context of the “landscape”, where it may be explained “anthropically” [3].

Another reason that interest in radiative hierarchy schemes waned in the 1980s is that they seemed to require many extra fermions to play the role of virtual particles in loops, and thus it seemed that the models must be rather complicated and unpredictable. For example, in a conference talk in 1982, H. Georgi remarked, “A more serious problem with models of this type is that it is hard to find any without inventing an enormous number of superheavy fermions. I like the abstract idea much better than any of the realizations I
have found.” [4] In this letter a relatively simple non-supersymmetric grand unified model is proposed that leads to a radiative fermion mass hierarchy. Very few superheavy fermions are required, in fact only a family-antifamily pair and a vector multiplet. In this model virtually every qualitative feature of the pattern of quark and lepton masses and mixings is reproduced from a rather simple underlying structure. The interfamily mass ratios and mixing angles, though spanning a large range, are accounted for without the ad hoc introduction of small parameters, since the “loop factors” of $1/16\pi^2$ explain much of the hierarchical structure.

The basic structure of the model discussed here was actually proposed in an earlier paper [5]. However, in that paper it was assumed that all the effective fermion mass operators arise at tree level. No explanation was therefore given why certain elements of the mass matrices were of order $10^{-2}$ compared to the largest elements. Moreover, the existence of the small elements could only be accounted for as tree-level effects by the introduction of a significant amount of additional structure in the Yukawa sector of the theory. Here it is shown that those small elements arise very naturally as radiative corrections, without such additional structure. Thus a simpler model leads to the same results, but with the bonus that both the existence of the small elements and their size is naturally explained.

The model is an $SO(10)$ grand unified theory in which the quarks and leptons are contained in the following (left-handed) multiplets: $16_{i=1,2,3} + (16 + \overline{16} + 10)$. The “extra” fermion multiplets in the parentheses form real representations of $SO(10)$, and so the low-energy spectrum consists of just three chiral families, as observed. The Dirac mass matrices of the up-type quarks, down-type quarks, charged leptons, and neutrinos (denoted by $M_U$, $M_D$, $M_L$, and $M_N$, respectively) arise from the following set of Yukawa terms:

$$L_{Yuk} = M(\overline{16} 16) + M_{10}(10 10). + a(\overline{16} 16_3)45_H + \sum_{i=1,2} c_i(10 16_i)16_i H + h_{33}(16_3 16_3)10_H + h_2(16 16_2)10_H + h_3(10 16_3)16_H' + h(16 16)10_H'.$$

The terms on the first line of Eq. (1) are the $O(M_{GUT})$ masses of the extra
fermion multiplets; the terms on the second line contribute $O(M_{GUT})$ masses that mix those extra fermions with the three chiral families $16_i$; the terms on the third line generate the weak-scale $SU(2)_L \times U(1)_Y$-breaking masses; and the last term is needed to give radiative masses to the first family. Higgs multiplets are denoted by the subscript $H$. The Higgs fields $16_{iH}$ obtain vacuum expectation values (VEV) in the $1(16)$ direction. (The expression $\mathbf{p}(\mathbf{q})$ stands for a $\mathbf{p}$ multiplet of $SU(5)$ contained in a $\mathbf{q}$ multiplet of $SO(10)$.) The adjoint Higgs field $45_H$ obtains a VEV that is proportional to the $SO(10)$ generator $B - L$ (i.e. baryon number minus lepton number). These two types of Higgs fields would be enough to break $SO(10)$ down to the Standard Model group $SU(3)_c \times SU(2)_L \times U(1)_Y$; however, there may be other Higgs fields that also contribute to that breaking.

The electroweak gauge symmetry $SU(2)_L \times U(1)_Y$ is spontaneously broken by the Higgs multiplets denoted $10_H$, $10'_H$, and $16'_H$ in Eq. (1), and, more specifically, by the neutral components of the $Y/2 = -1/2$ doublets contained in $\mathbf{5}(10_H)$, $\mathbf{5}(10'_H)$, and $\mathbf{5}(16'_H)$, and the neutral components of the $Y/2 = +1/2$ doublets contained in $\mathbf{5}(10_H)$ and $\mathbf{5}(10'_H)$. Of course, in the low-energy effective theory, which is just the Standard Model, there is only one Higgs doublet, which is some linear combination of these doublets (and their hermitian conjugates).

There are several layers to the quark and lepton mass matrices that arise from Eq. (1). The first layer comes simply from the term $h_{33}(16_3 \ 16_3) 10_H$ and gives contributions of the form

$$M_U^{(1)} = M_N^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} m_U,$$

$$M_D^{(1)} = M_L^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} m_D,$$

where $m_U \equiv h_{33}(\mathbf{5}(10_H))$ and $m_D \equiv h_{33}(\mathbf{5}(10_H))$.

The second layer comes from integrating out the family-antifamily pair $\overline{16} + 16$. The antifamily $\overline{16}$ appears in two mass terms from Eq. (1), which can be combined as follows: $\overline{16}(M 16 + a(45_H) 16)$). These terms have the effect of mixing the $16$ with the $16_3$. One family obtains an $O(M_{GUT})$ mass,
while the other (denoted by the index \(3'\)) remains light. (From now on, primed indices will be used to denote the light families that remain after the superheavy fermions have been integrated out.) Thus, the \(16\) with no index has some of the third light family mixed in with it. The amount of this mixing is proportional to \(B - L\), since \(\langle 45_H \rangle \propto B - L\), and it is thus three times stronger for the leptons than the quarks. As a result, the term \(h_2(16\,16_2)10_H\) from Eq. (1) leads to an effective operator of the form \((16\,3')10_H / M_{\text{GUT}}\), which in turn produces contributions to the effective low-energy mass matrices of the form

\[
M_{U/D}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon/3 \\ 0 & -\epsilon/3 & 0 \end{pmatrix} m_{U/D}, \quad M_{N/L}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\epsilon \\ 0 & \epsilon & 0 \end{pmatrix} m_{U/D}. \tag{3}
\]

The parameter \(\epsilon\) is “naturally” of order one (or slightly smaller, since it is proportional to the sine of the mixing angle between \(16\) and \(16_3\)). To fit the observed masses its actual value must be about 0.15.

In a similar way, effective operators arise from integrating out the \(SO(10)\)-vector multiplet of quarks and leptons, \(10\). This multiplet contains a \(\mathbf{5} + 5\) of \(SU(5)\). The \(\mathbf{5}(10)\) appears in several mass terms from Eq. (1), which can be combined as \(\mathbf{5}(10)[M_{10}\mathbf{5}(10) + \sum_{i=1,2} c_i (1(16_H)\mathbf{5}(16_i))].\) These terms have the effect of mixing the \(\mathbf{5}(10)\) with the \(\mathbf{5}(16_1)\) and \(\mathbf{5}(16_2)\). One linear combination of them obtains an \(O(M_{\text{GUT}})\) mass, while the other two linear combinations are in the light families and denoted \(\mathbf{5}_1'\) and \(\mathbf{5}_2'\). Consequently, the \(SO(10)\)-vector of fermions \(10\) has mixed in with it some of the first two families of right-handed down quarks and left-handed leptons (these being the particle types contained in a \(\mathbf{5}\) of \(SU(5)\)). That means that the term \(h_3(10\,16_3)16'_H\) in Eq. (1) leads to effective mass terms of the form \((C_1\mathbf{5}_1' + C_2\mathbf{5}_2')10_3 m_D\). This gives

\[
M_N^{(3)} = M_U^{(3)} = 0, \quad M_D^{(3)} = M_L^{(3)T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_1 & C_2 & 0 \end{pmatrix} m_D. \tag{4}
\]

Note the crucial fact that only \(M_D\) and \(M_L\) get these contributions. The parameters \(C_1\) and \(C_2\) are naturally of order one. The actual fit to the data require \(C_1 \simeq 0.8\) and \(C_2 \simeq 1.2\).
The full tree-level mass matrices, which are obtained by adding the three layers given in Eqs. (2)-(4), have the form

\[
M_U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon/3 \\ 0 & -\epsilon/3 & 1 \end{pmatrix} m_U, \quad M_D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon/3 \\ C_1 & C_2 - \frac{\epsilon}{3} & 1 \end{pmatrix} m_D,
\]

\[
M_N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\epsilon \\ 0 & \epsilon & 1 \end{pmatrix} m_U, \quad M_L = \begin{pmatrix} 0 & 0 & C_1 \\ 0 & 0 & C_2 - \epsilon \\ 0 & \epsilon & 1 \end{pmatrix} m_D.
\]

The convention here is that the mass matrices are multiplied from the left by the left-handed fermions and from the right by the right-handed fermions. These equations for the quark and lepton mass matrices are approximate. The exact expressions involve factors, such as \(1/\sqrt{1 + (a\langle 45_H \rangle/M)^2}\) and \(1/\sqrt{1 + (\sum_i c_i\langle 16'_H \rangle/M_{10})^2}\), which are essentially just the cosines of angles describing the mixing between the extra fermions \(16 + \overline{16} + 10\) and the three chiral families \(16_i\). If one assumes these mixing angles are small, these cosine factors may be set to one. (The fact that \(\epsilon \simeq 0.15\) would most simply be explained by saying that \(a\langle 45_H \rangle/M\) is of that order.)

Before turning to the radiative effects that generate the masses and mixings of the first family, let us see how the simple tree-level structure of Eq. (5) reproduces the known pattern of masses and mixings of the second and third families. The first thing to notice is that in the limit of \(\epsilon \to 0\) only one family has mass. The mass of the \(t\) quark is just \(m_U\), whereas the \(b\) quark and \(\tau\) lepton both have mass \(\sqrt{1 + |C_1|^2 + |C_2|^2} m_D\). The well-known successful \(SU(5)\) relation \(m_b^0 \simeq m_\tau^0\) is thus obtained in this model. The superscript 0 refers to relations at the scale \(M_{GUT}\). The large off-diagonal elements \(C_1\) and \(C_2\) that appear asymmetrically in \(M_D\) and \(M_L\) are the characteristic features of “lopsided” models [6]. Having both large \(C_1\) and \(C_2\) makes this a “doubly lopsided” model. Such a structure, as is well-known [5], leads to the “bi-large” pattern of neutrino mixing that has been observed. This happens by the diagonalization of the charged lepton mass matrix \(M_L\). In diagonalizing \(M_L\), the large element \(C_1\) must be eliminated by a rotation from the left by an angle \(\theta_{sol}\) in the 12 plane, where \(\tan \theta_{sol} = -C_1/C_2\). In the process, \(C_2\) is replaced by \(C \equiv \sqrt{|C_1|^2 + |C_2|^2}\). This element must then be eliminated by a
rotation acting on \( M_L \) from the left by an angle \( \theta_{atm} \) in the 23 plane, where \( \tan \theta_{atm} = -C \). The net result, if no other rotations of leptons were done, would be a neutrino mixing matrix of the form

\[
U_{MNS} = \begin{pmatrix}
c_s & s_s & 0 \\
-c_a s_s & c_a c_s & s_a \\
s_a s_s & -s_a c_s & c_a
\end{pmatrix},
\]  

(6)

where \( s_a \equiv \sin \theta_{atm} \), \( c_a \equiv \cos \theta_{atm} \), \( s_s \equiv \sin \theta_{sol} \), and \( c_s \equiv \cos \theta_{sol} \). As can be seen from the form of \( M_N \) in Eq. (5), there are rotations in the 23 plane of \( O(\epsilon) \) required to diagonalize the neutrino mass matrix. And there will be a further rotation of order \( \sqrt{m_e/m_\mu} \) in the 12 plane required to complete the diagonalization of the charged lepton mass matrix \( M_L \). So the neutrino mixing matrix will not be of exactly the form shown in Eq. (6). In particular, the 13 angle will not be exactly zero, but of order 0.1.

There are no corresponding large contributions to the CKM mixing angles of the quarks, since the rotations required to eliminate the large off-diagonal elements \( C_1 \) and \( C_2 \) from \( M_D \) involve only the right-handed quarks. This is, of course, the way that lopsided models account for the fact that the observed quark mixings are much smaller than the neutrino mixings. These rotations acting on \( M_D \) from the right, bring it to the form

\[
M'_D = \begin{pmatrix}
0 & 0 & 0 \\
0 & -s_a \epsilon/3 & c_a \epsilon/3 \\
0 & 0 & 1/c_a
\end{pmatrix} m_D.
\]  

(7)

We have taken the rotation angles here also to be to be \( \theta_{sol} \) and \( \theta_{atm} \), thus neglecting the slight difference between the 23 element of \( M_L \) and the 32 element of \( M_D \), which is an effect higher order in \( \epsilon \). One sees that a 22 element is induced in \( M'_D \), thus giving the strange quark a mass of \( O(\epsilon) \). In particular, \( m_s^0 / m_b^0 \simeq \sin \theta_{atm} \cos \theta_{atm} (\epsilon/3) \simeq \epsilon/6 \). The corresponding rotations acting on \( M_L \) induce a 22 element that is 3 times larger (because of the factor of \( (B - L) \) that multiplies the parameter \( \epsilon \) in these matrices). This gives the famous Georgi-Jarlskog relation \( m_\mu^0 \simeq 3 m_s^0 \) [7], which is known to work reasonably well.

Because there are no large off-diagonal elements (like \( C_1 \), \( C_2 \)) in the up quark mass matrix \( M_U \) there is no \( O(\epsilon) \) 22 element induced in \( M_U \). Rather, \( m_c^0 / m_t^0 \simeq (\epsilon/3)^2 \). This accords well with the observed mass ratios, which do
approximately satisfy $m_c/m_t \sim (2m_s/m_b)^2$. The mixing between the second and third families of quarks, $V_{cb}$, arises from the mismatch between the 23 rotation required to diagonalize $M_U$, which is $\epsilon/3$, and the corresponding angle for $M_D$, which one sees from Eq. (7) is approximately $\cos^2 \theta_{atm} (\epsilon/3) \sim \epsilon/6$. This gives $V_{cb} = \sin^2 \theta_{atm} (\epsilon/3) \sim m_s/m_b$, which is qualitatively correct. However, as will be seen, there are corrections to $V_{cb}$ coming from the radiative contributions to $M_D$ that cannot be neglected, so one does not expect a perfect fit from the tree-level form in Eqs. (5),(7).

The qualitative features of the masses and mixings of the two heavy families are strikingly reproduced: $\theta_{atm}, \theta_{sol} \sim 1; \theta_{13} \ll 1; V_{cb} \sim m_s^0/m_b^0 \sim m_\mu/3m_\tau^0 \sim \epsilon/6; and m_\nu_e^0/m_\nu_\mu^0 \sim \epsilon^2/9$.

In order for the first family of quarks and leptons to obtain mass, there must be additional non-zero entries in the mass matrices shown in Eq. (5). The additional entries must make $\det M_D \cong \det M_L \neq 0$. The reason is that the famous and well-satisfied Georgi-Jarlskog relation $m_\mu^0/m_\mu^0 = 1/3 m_d^0/m_s^0$ follows from $m_b^0 = m_\tau^0$ and the other Georgi-Jarlskog relation $m_s^0 = m_\mu^0/3$ only if $\det M_D = \det M_L$. Consequently, a non-zero 11 element of $M_D$ would require a 11 element of $M_L$ that was 1/3 as large in order to contribute equally to $\det M_D$ and $\det M_L$, as can be seen from Eq. (5). However, that pattern does not arise from any simple effective operators. Similarly, a non-zero 12 element of $M_D$ would require a non-zero 21 element of $M_L$ that was 1/3 as large, which is not easy to achieve for the same reason. Therefore, given that obviously some non-zero element is needed in the first row of $M_D$ and first column of $M_L$ to give mass to the electron and $d$ quark, it must be that $(M_D)_{13} \cong (M_L)_{31} \neq 0$. Such contributions can be arranged to exist at tree-level by introducing further fermion and scalar multiplets and appropriate couplings for them, as was done in [5]. As will now be seen, however, precisely these needed non-zero elements automatically arise from one-gauge-boson-loop diagrams, while such diagrams leave all the other zeros in the mass matrices unaffected.

The one-gauge-boson-loop diagram in question is shown in Fig. 1(a). The gauge boson in this diagram is in a 10 of SU(5) (of course, it is in the adjoint 45 of SO(10)), so that it turns 10’s of SU(5) into 5’s and vice versa. That means that it interchanges $d_L \leftrightarrow d_R$ and $\ell_L \leftrightarrow \ell_R$ in the diagrams. Consequently, as can be seen from Fig. 1(a), the one-loop 13 element of $M_D$ comes from the large tree-level 31 element. In the same way, the 23 element gets a one-loop contribution that is an echo of the large tree-level 32 element.
(Loops involving gauge bosons that are in 1 or 24 of SU(5), do not change the SU(5) representation of the fermions they couple to, and can be shown, therefore, not to give non-zero contributions to the mass matrices where there were zeros at tree level.) The diagram in Fig 1(a) superficially looks divergent. However, the accidental symmetry (discussed later) that makes $(M_D)_{13}$ vanish at tree level guarantees that the loop is finite, as an exact calculation indeed shows. The finiteness of this diagram is more obvious if we write it in the form shown in Fig. 1(b). Neglecting higher order effects in $\epsilon$, one finds that $(M_D)_{13} = (M_L)_{13} = \frac{g_U^2}{32\pi^2} I(M_g^2/M_f^2) (M_D)_{31}$, where $g_U$ is the gauge coupling at the unification scale, $I(x) \equiv \ln x/(x - 1)$, $M_g$ is the mass of the superheavy gauge boson in Fig. 1(a), and $M_f$ is the mass of the heaviest fermion in the loop. (This expression is valid in the limit of small mixing between the chiral families 16 and the extra fermion multiplets $16 + \overline{16} + 10$.) The finite one-loop contributions to $(M_D)_{23}$ and $(M_L)_{32}$ are larger by a factor $C_2/C_1$. The other zero elements in the mass matrices get no contributions from these one-gauge-boson-loop diagrams, because there is no non-zero tree-level entry for them to “echo”.

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![Fig. 1(a)](image-url)
Figure 1. The diagram in (a) shows how the 13 element of $M_D$ (or $M_L^T$) arises radiatively from the tree-level 31 element, shown as the blob in the center. The $10(45_g)$ in the loop is a superheavy gauge boson. The diagram in (b) is more detailed and shows why the loop is finite.

The non-zero 13 element of $M_D$ and 31 element of $M_L$ induced by the one-gauge-boson-loop diagrams are obviously not enough to make these matrices have non-zero determinant. There must also be a non-zero 22 (or 21) element for $M_D$ with an equal non-zero 22 (or 12) element for $M_L$. The gauge loops do not produce these. They can arise, however, in a rather simple way from the one-Higgs-boson-loop diagram shown in Fig. 2.
Figure 2. A diagram showing how the 22 elements of the mass matrices can arise radiatively through Higgs-boson loops.

Whereas the gauge-loop diagrams discussed previously must exist, the diagram in Fig. 2 only exists if certain couplings are present. In particular, there must be the last term in Eq. (1), namely $h(16 16) 10^H_1$, and a Higgs-mass term of the form $10^H_1 10^H$. The diagram in Fig. 2 also gives a 22 element for the up-quark mass matrix $M_U$. However, if one assumes that the $Y/2 = +1/2$ and $Y/2 = -1/2$ VEVs in $\langle 10^H_1 \rangle$ are of the same order, then the contribution of Fig. 2 to $m_c$ is only a few percent and thus negligible.

The full mass matrices, including the contributions from one-loop diagrams, have the form

$$M_U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon/3 \\ 0 & -\epsilon/3 & 1 \end{pmatrix} m_U, \quad M_D = \begin{pmatrix} 0 & 0 & \delta_g \\ 0 & \delta_H & \epsilon/3 + \delta'_g \\ C_1 & C_2 - \frac{\epsilon}{3} & 1 \end{pmatrix} m_D,$$

$$M_N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\epsilon \\ 0 & \epsilon & 1 \end{pmatrix} m_U, \quad M_L = \begin{pmatrix} 0 & 0 & C_1 \\ 0 & \delta_H & C_2 - \epsilon \\ \delta_g & \epsilon + \delta'_g & 1 \end{pmatrix} m_D,$$

where $\delta'_g = (C_2/C_1)\delta_g$. A fit of the first family masses and mixings requires that $\delta_g$ and $\delta_H$ be approximately equal to $10^{-2}$, which is consistent with the magnitude of one-loop effects. Note that $M_U$ still has rank = 2, so that the $u$ quark is massless even at one-loop level. This is quite in accord with the fact that $m_u/m_t \ll m_d/m_b$ and $m_e/m_\mu$. In order to reproduce the observed mass of the $u$ quark, there has to be either a 11 element in $M_U/m_U$ that is $\sim 3 \times 10^{-5}$ or 12 and 21 elements that are $\sim 4 \times 10^{-4}$. In either case, these are much smaller than one would expect from a typical one-loop diagram, but quite consistent with a two-loop effect.

To see the pattern of masses and mixings for the first family, it is convenient to make the same transformation of $M_D$ that eliminates the large off-diagonal elements $C_1$ and $C_2$, and which at tree-level led to Eq. (7). At one loop, it leads to
It is well-known that the empirical relation \( \tan \theta_C \simeq \sqrt{m_d/m_s} \) is obtained if the following conditions are satisfied: \( (M_U)_{12}, (M_U)_{21} \simeq 0 \), \( (M_D)_{11} \simeq 0 \), and \( |(M_D)_{12}| \simeq |(M_D)_{21}| \). In order to satisfy the last of these conditions in this model, all that is required (as can be seen from Eq. (9)) is that \( |\delta_H \sin \theta_{sol}| \simeq |\delta_g \sin \theta_{atm}| \), which is natural, given that \( \delta_g \) and \( \delta_H \) are both one loop effects and that \( \theta_{atm} \) and \( \theta_{sol} \) are both of order one.

The only other parameter that relates to the first family that remains to be accounted for is \( V_{ub} \). From Eq. (9), one sees that, because \( |V_{ub}| \simeq |(M_D')_{13}/m_b| \) and \( |V_{us}| \simeq |(M_D')_{12}/m_s| \), one obtains the relation \( |V_{ub}| \simeq |V_{us}|(m_s/m_b) \cot \theta_{atm} \). Using the facts that \( m_s/m_b \simeq 0.02 \), \( V_{us} \simeq 0.2 \), and \( \theta_{atm} \simeq \pi/4 \), one obtains \( |V_{ub}| \simeq 0.004 \), which is quite close to the measured value.

One can invert this equation to get a value for \( \theta_{atm} \): using the best fit values for \( |V_{ub}| \) and \( |V_{us}| \) and the Georgi-Jarlskog value for \( m_s/m_b \) (i.e. \( m_\mu/3m_\tau \)), one finds \( \tan \theta_{atm} \simeq 1.47 \), corresponding to \( \theta_{atm} \simeq 56^\circ \). (Considering that there are contributions to the atmospheric angle of \( O(\epsilon) \) of about \( 10^\circ \) coming from the diagonalization of the neutrino mass matrix, this value is reasonable.) Since \( |V_{ub}| \simeq \delta_g \cos^2 \theta_{atm} \) one obtains \( \delta_g \simeq 0.94 \times 10^{-2} \). From the relation \( \delta_H \sin \theta_{sol} \simeq \delta_g \sin \theta_{atm} \) and the measured value \( \tan \theta_{sol} \simeq 0.66 \), one then obtains \( \delta_H \simeq 1.4 \times 10^{-2} \). The values of \( \theta_{atm} \) and \( \theta_{sol} \) directly yield \( C_1 \simeq 0.8 \) and \( C_2 \simeq 1.23 \). If one substitutes \( C_1 = 0.8 \) into the expression for the loop diagram in Fig 1(a), and uses for the unified gauge coupling \( \alpha_U = 0.025 \), one finds \( \delta_g \simeq 0.72 \times 10^{-2} I(x) \). Since \( I(x) \) is a slowly varying function and \( I(1) = 1 \), this is nicely in accord with the value \( \delta_g \simeq 0.94 \times 10^{-2} \) obtained by fitting the quark and lepton masses. Thus the masses of the electron and \( d \) quark are very well explained as radiative effects.

The value of \( \epsilon \) is fixed to be 0.15 directly by \( m_c/m_t \) and the value of \( m_U/m_D \) is fixed by \( m_t/m_b \), so that all that remains to be determined of the parameters appearing in Eq. (8) are the complex phases.
Most of the complex phases can be rotated away from the matrices by phase redefinitions of complete Standard Model multiplets, and so have no low-energy implications. However, three phases cannot be rotated away. These can be taken to be the phases of $\delta_H$, $\delta'_g$, and the phase of the $\epsilon$ that appears with $C_2$ in the 32 element of $M_D$ and $M_U^T$. Call these phases $\alpha_H$, $\alpha_g$, and $\alpha_\epsilon$. The last of these has an effect only at subleading order in $\epsilon$ and so we will ignore it. The phases $\alpha_H$ and $\alpha_g$, on the other hand, enter importantly in the expressions for $m_s$ and $V_{cb}$. Fixing these phases by these two measured quantities gives a definite prediction for the CKM CP-violating phase $\delta_{CKM}$. A fit gives roughly $\alpha_H \sim 0$, $\alpha_g \sim \pi/2$. The angle $\phi_3$ in the unitarity triangle then comes out to be about $32^\circ$, which is too small. It should be noted, however, that the CP-violating phase is more sensitive to the values of the model parameters than most of the other observables. Moreover, two-loop effects can significantly affect it. For example, the type of two-loop effects needed to generate the mass of the $u$ quark could well generate a 12 element for $M_U$ of order $10^{-4}$. That would have an effect on $V_{us}$, and thus on the best-fit value of $\delta_g$, of order 20%. That would in turn affect the CP-violating phase (which comes predominantly from the phase of $\delta'_g$) by a factor of the same order.

The foregoing numbers are based on a rough fit, and a more careful analysis must be done. This would require (a) doing the renormalization group running of the mass and mixing angles from the unification scale down to low scales within a breaking scheme for $SO(10)$ that reproduces the unification of gauge couplings, and (b) doing a global fit to all the masses and mixings. In the limit we are taking (of small mixing between the chiral families $16_i$ and the real extra fermions $16 + \overline{16} + 10$), there are 9 dimensionless model parameters: $m_U/m_D$, $C_1$, $C_2$, $\epsilon$, $\delta_g$, $\delta_H$, and the phases $\alpha_g$, $\alpha_H$, and $\alpha_\epsilon$, the last being relatively unimportant. These must fit 14 dimensionless observables: 7 mass ratios of the charged fermions (not counting the $u$ quark, which is still massless at one-loop level), and 7 mixing parameters.

Finally, something must be said about the structure of the Yukawa sector given in Eq. (1). It is easily seen that the terms given in Eq. (1) leave accidental global abelian symmetries (some of which may be gauged) that prevent other Yukawa terms that could be dangerous. It is these accidental symmetries that make the zeros in the mass matrices in Eq. (5) “technically natural”, and guarantee that the loop effects $\delta_g$, $\delta'_g$ and $\delta_H$ are “finite and calculable.”
In conclusion, we have shown that it is possible to construct a rather simple and predictive model of quark and lepton masses based on the idea of a “radiative hierarchy”. All the dimensionless parameters in this model take values of order one, except $m_U/m_D$. That ratio determines the overall scale of $Y/2 = +1/2$ masses relative to $Y/2 = -1/2$ masses (i.e. “up” to “down”), and may have an “anthropic” or “landscape” explanation [8]. The large interfamily hierarchies are achieved without ad hoc tuning of parameters to small values. A striking aspect of the model is the way that the “lopsided” structure (i.e. $C_1 \sim C_2 \sim 1$) explains so many features of the light fermion spectrum. In particular, it explains (a) the large solar and atmospheric mixing angles and the small value of $\theta_{13}$ (the so-called “bilinear” pattern of neutrino mixing), (b) the emergence of the Georgi-Jarlskog factor of 3 (which would be $\approx 9$ if it were not for the fact that $C_2 \gg \epsilon$), (c) the largeness of $m_s/m_b$ and $m_\mu/m_\tau$ compared to $m_c/m_t$, (d) the fact that $V_{cb}$ is of order $m_s/m_b$ rather than $\sqrt{m_s/m_b}$, as it is in models with symmetric mass matrices, (e) the fact that $V_{ab}$ comes out with the correct magnitude ($\approx V_{us}(m_s/m_b) \approx 0.004$), and (f) the fact that $m_\mu/m_t \ll m_d/m_b, m_c/m_\tau$, which is a consequence of the fact that the lopsided entries do not appear in $M_U$, so that the one-gauge loop diagrams cannot change its rank, and the $u$ quark must obtain mass from two-loop order. In other words many of the peculiar features of the spectrum are traceable to a single simple feature of the mass matrices.

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