A GLOBAL POINCARÉ INEQUALITY ON GRAPHS VIA A CONICAL CURVATURE-DIMENSION CONDITION

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ABSTRACT. We introduce and study the conical curvature-dimension condition, \( CCD(K,N) \), for graphs. We show that \( CCD(K,N) \) provides necessary and sufficient conditions for the underlying graph to satisfy a sharp global Poincaré inequality which in turn translates to a sharp lower bound for the first eigenvalues of these graphs. Another application of the conical curvature-dimension analysis is finding a sharp estimate on the curvature of complete graphs.

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1. INTRODUCTION

The relation between Ricci curvature bounds and the analytic and geometric properties of a smooth Riemannian manifold is a well studied subject in geometric analysis. Thanks to the seminal work of Sturm [13] [14] and Lott-Villani [11], the notion of lower Ricci curvature bounds can be generalized to the setting of metric and measure spaces.

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A Polish metric measure space that satisfies the Lott-Sturm-Villani’s $CD(K, N)$ curvature-dimension conditions is called a $CD(K, N)$ space. One important aspect of these spaces is that they support both local and global Poincaré inequalities (for a sharp global Poincaré inequality and spectral gap on $CD(K, N)$ metric measure spaces, see [12, Theorem 5.34]). For metric measure spaces that satisfy certain infinitesimal regularity properties, the $CD(K, N)$ curvature-dimension bounds coincide with the Bakry-Émery curvature-dimension bounds (or $BE(K, N)$ for short), see [6]. Also, there is a close relation between the lower Ricci bound of $X$ and the lower Ricci curvature bound(s) of the cone(s) over $X$, when $X$ is a Riemannian manifold with $\text{Ric} \geq (n-1)K$ or more generally an $RCD(K, N)$ space. In particular a Riemannian manifold, $X$, satisfies $\text{Ric} \geq 1$ if and only if the Riemannian cone over $X$ satisfies $\text{Ric} \geq 0$. In the setting of $RCD(K, N)$ metric measure spaces the relation between the weak Ricci curvature bound of $X$ and that of the cone(s) over $X$ has been explored in [7].

There are some disparities between the discrete Laplacian on graphs and the Laplacian on manifolds (or on some more general non-smooth continuous metric measure spaces). Despite these disparities, studying Bakry-Émery type curvature-dimension conditions for the discrete Laplacian has proven fruitful in the sense that in the discrete setting graphs with lower Ricci curvature bounds satisfy some properties that are similar to the ones satisfied by manifolds with lower Ricci curvature bounds, see [10], [9], [4] and [8].

In this paper we acquire partial results relating the curvature of a graph to the curvature of the cone over vertices. In general the paper does not admit a clean cut relation between the lower Bakry-Émery Ricci curvature bound of the base graph and that of the cone over the graph. This is mainly due to the fact that in the discrete setting the distance between any two vertices in a cone is at most two and thus the operator $\Gamma_2$ at any point $x$ (a key ingredient in the definition of curvature-dimension bounds) will depend on the entire graph. So the curvature bound at the cone point over the vertex set of a graph will store the curvature information of the entire graph, see 1.1.

This article is primarily concerned with the properties of the underlying graph $G$ that can be extracted when the cone over $G$ satisfies the $CD(K, N)$ curvature-dimension conditions at the cone point (a property which will be called the conical curvature-dimension, or $CCD(K, N)$ condition). Our main results are a global Poincaré inequality and the spectral gap estimates that follow.

**Definition 1.1 ($CCD(K, N)$ Curvature-Dimension Conditions).** Let $G = (V, E)$ be a finite, connected, undirected, loop-edge free graph and consider the cone over the vertex set of $G$. $G$ is said to satisfy the conical curvature-dimension condition, $CCD(K, N)$ for $K \in \mathbb{R}$ and $N \in (1, \infty]$, if the cone over $G$ satisfies the $CD(K, N)$ curvature-dimension conditions at the vertex $p$, namely if

$$\Gamma_2^p(f)(p) \geq \frac{(\Delta f)^2(p)}{N} + K\Gamma_1^p(f)(p),$$

holds for any function $f$ defined on the cone and $\Delta^c$, $\Gamma_1^c$ and $\Gamma_2^c$ are the usual $\Delta$, $\Gamma_1$ and $\Gamma_2$ operators (see (2), (4) and (5)) except on the cone $C(G)$ over $G$. We note that the second term in (1) is understood to be zero when $N = \infty$.

Now we can state our main theorems and corollaries:

**Theorem 1.1 ($CCD(K, N)$ implies global Poincaré Inequality).** If a graph, $G$, satisfies $CCD(K, N)$ curvature-dimension condition, then for any function $f$ on $G$ one has

$$\sum_{y \in V} \Gamma_1(f)(y) \geq \frac{2 - N}{2N} \left( \sum_{y \in V} f(y) \right)^2 + \frac{2K + |V| - 3}{4} \sum_{y \in V} f^2(y).$$
For functions \( f \) with \( \text{avg}(f) = 0 \), this reduces to the following global Poincaré inequality,
\[
\|f\|_2 \leq \sqrt{\frac{2}{2K + |V| - 3}} \|\nabla f\|_2,
\]
where \( \|\nabla f\|_2 \) is understood in the graph setting to be \( 2 \cdot \sum_{y \in V} \Gamma_1(f)(y) \).

**Corollary 1.2.** If \( G \) satisfies CCD \((K, N)\) condition, then
\[
\lambda_1(G) \geq K + \frac{|V| - 3}{2}.
\]

**Theorem 1.3.** For any graph, \( G \), and a given \( N > 1 \), the conical curvature cannot exceed the following number:
\[
K_{\text{max}}^c = \frac{|V|}{2} + 3 - 2 \frac{|V|}{N}.
\]

**Theorem 1.4** (Curvature Maximizers). Suppose \( G \) satisfies CCD \((K_{\text{max}}^c, N)\). Then any function, \( f \), realizes \( K_{\text{max}}^c \) if and only if \( f \) is either constant or \( f - \text{avg}(f) \) is an eigenfunction corresponding to \( \lambda_1(G) = \frac{N - 2}{N^2}|V| \). Furthermore, when \( G \) is a complete graph, \( f \) must be constant (harmonic).

**Corollary 1.5** (Ricci Curvature of Complete Graphs). Suppose \( G \) is the complete graph on \( n \) vertices, then the \( CD(K, N) \) property coincides with the CCD \((K^c, N)\) condition on the complete subgraph with \( n - 1 \) vertices and the curvature of \( G \) is \( \frac{3}{2} + 1 - 2 \frac{(n-1)}{N} \). Furthermore any function that realizes this curvature bound is constant (harmonic).

**Remark 1.6.** When \( N = \infty \), our bound \( K_{\text{max}}^c = 1 + \frac{n}{2} \) coincides with the maximum Ricci curvature of complete graphs as found in [8].

The following theorem illustrates an applications of our \( \Gamma \)-calculus on cones:

**Theorem 1.7.** Suppose \( G \) satisfies \( CD(K, \infty) \) for \( K \leq \frac{1}{2} \) then the subgraph \( G \subset C(G) \) satisfies \( CD(K + \frac{1}{2}, \infty) \).

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### 2. Preliminaries

Let \( G = (V, E) \) be an undirected, unweighted, connected, locally finite graph without any loop-edges. Let \( f : V \rightarrow \mathbb{R} \) and consider the space of square-summable functions on the vertex set.

The graph Laplacian is given by
\[
\Delta f(x) = \sum_{y \sim x} (f(y) - f(x)), \tag{2}
\]
where \( y \sim x \) means that \( (y, x) \in E \). Also, note that the graph Laplacian is a real valued, self-adjoint linear operator (for a thorough treatment of the graph Laplacian see [5]).

Let \( F \subset V \), then the boundary of \( F \) is
\[
\partial F := \{ (x, y) \in E \mid x \in F \text{ and } y \in V \setminus F \}.
\]

The isoperimetric constant (or Cheeger’s constant) is then defined as
\[
b(G) := \inf \left\{ \frac{|\partial F|}{\min\{|F|, |V \setminus F|\}} : 0 < |F| < \infty \right\}.
\]
A well known generalization of Cheeger’s and Buser’s results for Riemannian manifolds is the following theorem due to Dodziuk [5] and Alon-Milman [3].

**Theorem 2.1** ([5], [3]). Let $G = (V, E)$ be a finite, connected, edge-loop free graph. Let $d_{\text{max}} = \sup_{v \in V} \{\deg(v)\}$ and let $\lambda_1$ be the first non-trivial eigenvalue of $\Delta$, then

$$\frac{\lambda_1}{2} \leq h(G) \leq \sqrt{2d_{\text{max}} \lambda_1}. \tag{3}$$

The $\Gamma$ operators of Bakry-Émery associated to the graph Laplacian, $\Delta$, are:

$$\Gamma_1(f, g)(x) = \frac{1}{2} \left[ \Delta(fg)(x) - g(x)\Delta f(x) - f(x)\Delta g(x) \right]. \tag{4}$$

$$\Gamma_2(f, g)(x) = \frac{1}{2} \left[ \Delta \Gamma_1(f, g)(x) - \Gamma_1(\Delta f, g)(x) - \Gamma_1(f, \Delta g)(x) \right]. \tag{5}$$

It is straightforward to check that

$$\Gamma_1(f, g)(x) = \frac{1}{2} \sum_{y \sim x} (f(y) - f(x))(g(y) - g(x)) = \frac{1}{2} \langle \nabla f, \nabla g \rangle. \tag{6}$$

Throughout these notes $\Gamma_1(f) := \Gamma_1(f, f)$ and similarly $\Gamma_2(f) := \Gamma_2(f, f)$. Thus $\Gamma_1(f)(x) = \frac{1}{2} |\nabla f(x)|^2$ and one can verify the following useful divergence-type identity:

$$\frac{1}{2} \|\nabla f\|^2 = \sum_{y \in V} \Gamma_1(f)(y) = - \sum_{y \in V} f(y)\Delta f(y). \tag{7}$$

**Definition 2.2** (Bakry-Émery Curvature-Dimension Condition). Suppose $K \in \mathbb{R}$ and $N \in (1, \infty]$. We say that a graph $G = (V, E)$ satisfies the curvature-dimension conditions, $\text{CD}(K, N)$, if for every $x \in V$ and every $f \in \ell^2(V)$,

$$\Gamma_2(f)(x) \geq \frac{(\Delta f)^2(x)}{N} + K \Gamma_1(f)(x). \tag{8}$$

Note when $N = \infty$, the second term in the inequality above is understood to be 0.

**Definition 2.1** (Uniform and Pointwise Ricci Curvatures). We define the dimensional (respectively, dimensionless) Ricci curvature of the graph $G$, $\text{Ric}_N(G)$ (respectively, $\text{Ric}_\infty(G)$), by

$$\text{Ric}_N(G) := \sup \{ K : G \text{ satisfies } \text{CD}(K, N) \}$$

and

$$\text{Ric}_\infty(G) := \sup \{ K : G \text{ satisfies } \text{CD}(K, \infty) \}.$$ 

Similarly, we define the pointwise curvatures by

$$\text{Ric}_N(y) := \sup \{ K : \Gamma_2(f)(y) \geq \frac{1}{N} (\Delta f)^2(y) + K \Gamma_1(f)(y), \forall f \}$$

and

$$\text{Ric}_\infty(y) := \sup \{ K : \Gamma_2(f)(y) \geq K \Gamma_1(f)(y), \forall f \}.$$ 

**Definition 2.3** (Conical Ricci Curvatures). We define the conical Ricci curvature by

$$CR\text{ic}_N(G) := \sup \{ K : G \text{ satisfies } \text{CCD}(K, N) \text{ as in (1)} \}$$

and

$$CR\text{ic}_\infty(G) := \sup \{ K : G \text{ satisfies } \text{CCD}(K, \infty) \text{ as in (1)} \}.$$ 

We close this section by recalling that the first non-zero eigenvalue of the Laplacian may be computed via the Rayleigh quotient:

$$\lambda_1 = \inf \left\{ \frac{\|\nabla f\|^2}{\|f\|^2} : \text{avg}(f) = 0 \right\}. \tag{9}$$
3. Cones over Graphs and Their $\Gamma$–Calculus

The complete cone, $C(G)$, over a finite graph $G$ is constructed by taking the graph Cartesian product of $G$ and $H$, $G \square H$, where $H = (\{q, p\}, \{(q, p)\})$ is the complete graph on two vertices $q$ and $p$, and then identifying all the vertices whose second component is $p$. In this paper $p$ refers to the cone point of $C(G)$.

More generally for a subset, $X \subset V(G)$, the partial cone, $C(X, G)$, is a subgraph of $C(G)$ containing $G$ and all edges $(x, p)$, $x \in X$. For brevity we will use a superscript $e$ to denote any operation that is taking place in a partial cone over $G$. Notice that any vertex $v \in V(G)$ can be thought of as the cone point over the 1-sphere based at $v$, i.e. $S_1^e := \{ y \in V \mid d_G(y, v) = 1 \} = X$ in the above construction. In this way partial cones can be useful in studying cliques.

The first subsection is devoted to proving a few lemmas that calculate the $\Delta$ and $\Gamma$ operators of a partial cone in terms of the similar operators on the base graph. The last subsection is devoted to an immediate result.

3.1. $\Gamma$-Calculus on a Cone. Since $\Delta$ and $\Gamma$ operators agree for functions that differ by a constant we may assume, without loss of generality, that $f(p) = 0$. Denote by $S_p^e$ and $B_p^e$ the metric spheres and balls (resp.) with radius $n$ and center $p$ in the cone. For any subset $B \subset V$, the notation $v \in B \sim x$ means $v \in B$ and $v \sim p$.

**Remark 3.1.** Note that $\Delta$ and $\Gamma_1$ only depend on vertices that are at most one away. Thus, $\Delta^e f(x) = \Delta f(x)$ and $\Gamma_1^e(f)(x) = \Gamma_1(f)(x)$ when $x \sim p$.

**Lemma 3.2.** Let $f$ be a function on the cone with $f(p) = 0$ then,

$$\Delta^e f(x) = \begin{cases} \Delta f(x) - f(x); & x \sim p \\ \sum_{y \in S_p^e} f(y); & x = p \end{cases}$$

**Proof.**

(1) If $x \sim p$, then

$$\Delta^e f(x) = \sum_{y \in C \sim x} (f(y) - f(x)) = \sum_{y \in V \sim x} (f(y) - f(x)) + (f(p) - f(x)) = \Delta f(x) - f(x).$$

(2) If $x = p$, then

$$\Delta^e f(p) = \sum_{y \in C \sim p} (f(y) - f(p)) = \sum_{y \in S_p^e} f(y).$$

**Lemma 3.3.** Let $f$ be a function on the cone with $f(p) = 0$ then,

$$\Gamma_1^e(f)(x) = \begin{cases} \Gamma_1(f)(x) + \frac{1}{2} f^2(x); & x \sim p \\ \frac{1}{2} \sum_{y \in S_p^e} f^2(y); & x = p \end{cases}$$

**Proof.**

(1) If $x \sim p$, then using (6)

$$\Gamma_1^e(f)(x) = \frac{1}{2} \sum_{y \in C \sim x} (f(y) - f(x))^2 = \frac{1}{2} \sum_{y \in V \sim x} (f(y) - f(x))^2 + \frac{1}{2} (f(p) - f(x))^2 = \Gamma_1(f)(x) + \frac{1}{2} f^2(x).$$

(2) If $x = p$, then using (6)

$$\Gamma_1^e(f)(p) = \frac{1}{2} \sum_{y \in V} (f(y) - f(p))^2 = \frac{1}{2} \sum_{y \in S_p^e} f^2(y).$$
In the next few lemmas we calculate the constituent parts that appear in the definition of $\Gamma_2^c$.

**Remark 3.4.** Note that $\Gamma_2^c$ depends on vertices at most two away. Thus $\Gamma_2^c$ coincides with $\Gamma_2$ when $x \in V \setminus B_p^2$.

**Lemma 3.5.** Let $f$ be a function defined on the cone, and suppose $f(p) = 0$, then

$$
\Gamma_1^c(f, \Delta^c f)(x) = \begin{cases} \\
\Gamma_1(f, \Delta f)(x) - \frac{1}{2} \sum_{y \in S_{1,x}^c} f(y) (f(y) - f(x)); & x \in S_p^2 \\
\Gamma_1(f, \Delta f)(x) - \frac{1}{2} \sum_{y \in S_{1,x}^c} (f(y) - f(x))^2 + \frac{1}{2} f(x) \sum_{y \in S_{p,x}^c} (f(y) - f(x)) \\
- \frac{1}{2} f(x) \sum_{y \in S_{p,x}^c} f(y) + \frac{1}{2} f(x) \Delta f(x) - \frac{1}{2} f^2(x) & x \sim p \\
\frac{1}{2} \sum_{y \in S_{p,x}^c} f(y) \Delta f(y) - \frac{1}{2} \sum_{y \in S_{p,x}^c} f^2(y) - \frac{1}{2} \sum_{y \in S_{p,x}^c} f(y))^2 & x = p
\end{cases}
$$

**Proof.** (1) If $x \in S_p^2$, then using (6)

$$
\Gamma_1^c(f, \Delta^c f)(x) = \frac{1}{2} \sum_{y \in V \setminus S_{1,x}^c} (f(y) - f(x))(\Delta^c f(y) - \Delta^c f(x))
$$

$$
= \frac{1}{2} \sum_{y \in V \setminus S_{1,x}^c} (f(y) - f(x))(\Delta^c f(y) - \Delta^c f(x))
$$

$$
+ \frac{1}{2} \sum_{y \in S_{1,x}^c} (f(y) - f(x))(\Delta^c f(y) - \Delta^c f(x))
$$

$$
= \frac{1}{2} \sum_{y \in V \setminus S_{1,x}^c} (f(y) - f(x))(\Delta f(y) - \Delta f(x))
$$

$$
+ \frac{1}{2} \sum_{y \in S_{1,x}^c} (f(y) - f(x))(\Delta f(y) - f(y) - \Delta f(x))
$$

$$
= \frac{1}{2} \sum_{y \in V \setminus x} (f(y) - f(x))(\Delta f(y) - \Delta f(x)) - \frac{1}{2} \sum_{y \in S_{1,x}^c} f(y)(f(y) - f(x))
$$

$$
= \Gamma_1(f, \Delta f)(x) - \frac{1}{2} \sum_{y \in S_{1,x}^c} f(y)(f(y) - f(x)).
$$
(2) If $x \sim p$, then using (6)

$$\Gamma_1(f, \Delta^c f)(x) = \frac{1}{2} \left[ \sum_{y \in C \sim x} (f(y) - f(x)) (\Delta^c f(y) - \Delta^c f(x)) \right]$$

$$= \frac{1}{2} \left[ \sum_{y \in S_p \sim x} (f(y) - f(x)) (\Delta f(y) - \Delta f(x) + f(x)) \right]$$

$$+ \frac{1}{2} \left[ \sum_{y \in S_p^{1} \sim x} (f(y) - f(x)) (\Delta f(y) - f(y) - \Delta f(x) + f(x)) \right]$$

$$+ \frac{1}{2} (f(p) - f(x)) \left( \sum_{y \in S_p^{1}} f(y) - \Delta f(x) + f(x) \right)$$

$$= \frac{1}{2} \left[ \sum_{y \in S_p \sim x} (f(y) - f(x)) (\Delta f(y) - \Delta f(x)) \right] + \frac{1}{2} f(x) \left[ \sum_{y \in S_p^{1} \sim x} (f(y) - f(x)) \right]$$

$$+ \frac{1}{2} \left[ \sum_{y \in S_p^{1} \sim x} (f(y) - f(x)) (\Delta f(y) - f(y) - \Delta f(x)) \right] - \frac{1}{2} \sum_{y \in S_p^{1} \sim x} (f(y) - f(x))^2$$

$$= \frac{1}{2} \left[ \sum_{y \in V \sim x} (f(y) - f(x)) (\Delta f(y) - \Delta f(x)) \right] - \frac{1}{2} \sum_{y \in S_p^{1} \sim x} (f(y) - f(x))^2$$

$$+ \frac{1}{2} f(x) \left[ \sum_{y \in S_p^{1} \sim x} (f(y) - f(x)) \right] - \frac{1}{2} f(x) \left[ \sum_{y \in S_p^{1} \sim x} f(y) + \frac{1}{2} f(x) \Delta f(x) - \frac{1}{2} f^2(x) \right]$$

$$= \Gamma_1(f, \Delta f)(x) - \frac{1}{2} \sum_{y \in S_p^{1} \sim x} (f(y) - f(x))^2 + \frac{1}{2} f(x) \sum_{y \in S_p^{1} \sim x} (f(y) - f(x))$$

$$- \frac{1}{2} f(x) \sum_{y \in S_p^{1}} f(y) + \frac{1}{2} f(x) \Delta f(x) - \frac{1}{2} f^2(x).$$

(3) If $x = p$, then using (6)

$$\Gamma_1(f, \Delta^c f)(p) = \frac{1}{2} \left[ \sum_{y \in C \sim p} (f(y) - f(p)) (\Delta^c f(y) - \Delta^c f(p)) \right]$$

$$= \frac{1}{2} \left[ \sum_{y \in S_p^{1}} (f(y) - f(p)) (\Delta f(y) - f(y) - \sum_{z \in S_p^{1}} f(z)) \right]$$

$$= \frac{1}{2} \left[ \sum_{y \in S_p^{1}} \left[ f(y) \Delta f(y) - f^2(y) - f(y) \sum_{z \in S_p^{1}} f(z) \right] \right]$$

$$= \frac{1}{2} \left[ \sum_{y \in S_p^{1}} f(y) \Delta f(y) - \frac{1}{2} \sum_{y \in S_p^{1}} f^2(y) - \frac{1}{2} \left( \sum_{y \in S_p^{1}} f(y) \right)^2 \right].$$

Lemma 3.6. Let $f$ be a function defined on the cone, and suppose $f(p) = 0$, then

$$\Delta^c \Gamma_1(f)(x) = \begin{cases} 
\Delta \Gamma_1(f)(x) + \frac{1}{2} \sum_{y \in S_p^{1} \sim x} f^2(y); & x \in S_p^2 \\
\Delta \Gamma_1(f)(x) - \Gamma_1(f)(x) & x \sim p \\
+ \frac{1}{2} \left[ \sum_{y \in S_p^{1} \sim x} f^2(y) + \sum_{y \in S_p^{1}} f^2(y) \right] - \frac{1}{2} \deg(x) f^2(x) & x = p \\
\sum_{y \in S_p^{1}} \Gamma_1(f)(y) - \frac{|S_p^{1}| - 1}{2} \sum_{y \in S_p^{1}} f^2(y); & x = p
\end{cases}$$
Proof.  

(1) If \( x \in S_p^0 \), then 
\[
\Delta^c \Gamma_1^c(f)(x) = \sum_{y \in C \sim x} \left[ \Gamma_1^c(f)(y) - \Gamma_1^c(f)(x) \right] 
\]
\[
= \sum_{y \in V \setminus S_p^1 \sim x} \left[ \Gamma_1(f)(y) - \Gamma_1(f)(x) \right] + \sum_{y \in S_p^1 \sim x} \left[ \Gamma_1^c(f)(y) - \Gamma_1^c(f)(x) \right] 
\]
\[
= \sum_{y \in V \sim x} \left[ \Gamma_1(f)(y) - \Gamma_1(f)(x) \right] + \sum_{y \in S_p^1 \sim x} \left[ \Gamma_1(f)(y) + \frac{1}{2} f^2(y) - \Gamma_1(f)(x) \right] 
\]
\[
= \sum_{y \in V \sim x} \left[ \Gamma_1(f)(y) - \Gamma_1(f)(x) \right] + \frac{1}{2} \sum_{y \in S_p^1 \sim x} f^2(y) 
\]
\[
= \Delta \Gamma_1(f)(x) + \frac{1}{2} \sum_{y \in S_p^1 \sim x} f^2(y). 
\]

(2) If \( x \sim p \), then 
\[
\Delta^c \Gamma_1^c(f)(x) = \sum_{y \in C \sim x} \left[ \Gamma_1^c(f)(y) - \Gamma_1^c(f)(x) \right] 
\]
\[
= \sum_{y \in S_p^1 \sim x} \left[ \Gamma_1(f)(y) - \Gamma_1(f)(x) - \frac{1}{2} f^2(x) \right] 
\]
\[
+ \sum_{y \in S_p^1 \sim x} \left[ \Gamma_1(f)(y) - \Gamma_1(f)(x) + \frac{1}{2} \left( f^2(y) - f^2(x) \right) \right] + \frac{1}{2} \sum_{y \in S_p^1 \sim x} f^2(y) - \Gamma_1(f)(x) 
\]
\[
= \sum_{y \in S_p^1 \sim x} \left[ \Gamma_1(f)(y) - \Gamma_1(f)(x) \right] - \frac{1}{2} \deg(x) f^2(x) + \sum_{y \in S_p^1 \sim x} \left[ \Gamma_1(f)(y) - \Gamma_1(f)(x) \right] 
\]
\[
+ \frac{1}{2} \sum_{y \in S_p^1 \sim x} f^2(y) + \frac{1}{2} \sum_{y \in S_p^1 \sim x} f^2(y) - \Gamma_1(f)(x) 
\]
\[
= \sum_{y \in V \sim x} \left[ \Gamma_1(f)(y) - \Gamma_1(f)(x) \right] - \Gamma_1(f)(x) 
\]
\[
+ \frac{1}{2} \sum_{y \in S_p^1 \sim x} f^2(y) + \frac{1}{2} \sum_{y \in S_p^1 \sim x} f^2(y) - \frac{1}{2} \deg(x) f^2(x) 
\]
\[
= \Delta \Gamma_1(f)(x) - \Gamma_1(f)(x) + \frac{1}{2} \left[ \sum_{y \in S_p^1 \sim x} f^2(y) + \sum_{y \in S_p^1 \sim x} f^2(y) \right] - \frac{1}{2} \deg(x) f^2(x). 
\]

(3) If \( x = p \), then 
\[
\Delta^c (\Gamma_1^c(f))(p) = \sum_{y \in S_p^1} \left[ \Gamma_1^c(f)(y) - \Gamma_1^c(f)(p) \right] 
\]
\[
= \sum_{y \in S_p^1} \left[ \Gamma_1(f)(y) + \frac{1}{2} f^2(y) - \frac{1}{2} \sum_{y \in S_p^1} f^2(y) \right] 
\]
\[
= \sum_{y \in S_p^1} \left[ \Gamma_1(f)(y) + \frac{1}{2} \sum_{y \in S_p^1} f^2(y) - \frac{|S_p^1| - 1}{2} \sum_{y \in S_p^1} f^2(y) \right] 
\]
\[
= \sum_{y \in S_p^1} \left[ \Gamma_1(f)(y) - \frac{|S_p^1| - 1}{2} \sum_{y \in S_p^1} f^2(y) \right]. 
\]
Lemma 3.7. Let $f$ be a function defined on the cone, and suppose $f(p) = 0$, then

$$
\Gamma_2^c(f)(x) = \begin{cases}
\Gamma_2(f)(x) + \frac{3}{4} \sum_{y \in S_p^1 \sim x} f^2(y) - \frac{1}{2} f(x) \sum_{y \in S_p^1 \sim x} f(y); & x \in S_p^2 \\
\Gamma_2(f)(x) - \frac{1}{2} \Gamma_1(f)(x) + \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 \\
\quad + \frac{1}{4} \left[ \sum_{y \in S_p^1 \sim x} f^2(y) - \deg(x) f^2(x) \right] - \frac{1}{2} f(x) \Delta f(x) \\
\quad - \frac{1}{2} f(x) \sum_{y \in S_p^1 \sim x} (f(y) - f(x)) + \frac{1}{4} \sum_{y \in S_p^1} f^2(y) + \frac{1}{2} f^2(x); & x \sim p \\
\frac{1}{2} \sum_{y \in S_p^1} \Gamma_1(f)(y) - \frac{1}{2} \sum_{y \in S_p^1} f(y) \Delta f(y) \\
\quad - \frac{|S_p^1| - 3}{4} \sum_{y \in S_p^1} f^2(y) + \frac{1}{2} \left( \sum_{y \in S_p^1} f(y) \right)^2; & x = p
\end{cases}
$$

Proof.  

(1) If $x \in S_p^2$, then

$$
\Gamma_2^c(f)(x) = \frac{1}{2} \Delta^c \Gamma_1^c(f)(x) - \Gamma_1^c(f, \Delta^c f)(x)
$$

$$
= \frac{1}{2} \Delta \Gamma_1(f)(x) + \frac{1}{4} \sum_{y \in S_p^1 \sim x} f^2(y) - \Gamma_1(f, \Delta f)(x) + \frac{1}{2} \sum_{y \in S_p^1 \sim x} f(y)(f(y) - f(x))
$$

$$
= \Gamma_2(f)(x) + \frac{3}{4} \sum_{y \in S_p^1 \sim x} f^2(y) - \frac{1}{2} f(x) \sum_{y \in S_p^1 \sim x} f(y).
$$

(2) If $x \sim p$, then

$$
\Gamma_2^c(f)(x) = \frac{1}{2} \Delta^c \Gamma_1^c(f)(x) - \Gamma_1^c(f, \Delta^c f)(x)
$$

$$
= \frac{1}{2} \Delta \Gamma_1(f)(x) - \frac{1}{2} \Gamma_1(f)(x) + \frac{1}{4} \sum_{y \in S_p^1 \sim x} f^2(y) + \frac{1}{4} \sum_{y \in S_p^1} f^2(y) - \frac{1}{4} \deg(x) f^2(x)
$$

$$
\quad - \Gamma_1(f, \Delta f)(x) + \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 - \frac{1}{2} f(x) \sum_{y \in S_p^1 \sim x} (f(y) - f(x))
$$

$$
\quad + \frac{1}{2} f(x) \sum_{y \in S_p^1} f(y) - \frac{1}{2} f(x) \Delta f(x) + \frac{1}{2} f^2(x).
$$

$$
= \left[ \frac{1}{2} \Delta \Gamma_1(f)(x) - \Gamma_1(f, \Delta f)(x) \right] - \frac{1}{2} \Gamma_1(f)(x) + \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2
$$

$$
\quad + \frac{1}{4} \left[ \sum_{y \in S_p^1 \sim x} f^2(y) - \deg(x) f^2(x) \right] + \frac{1}{4} \sum_{y \in S_p^1} f^2(y) - \frac{1}{2} f(x) \sum_{y \in S_p^1 \sim x} (f(y) - f(x))
$$

$$
\quad - \frac{1}{2} f(x) \Delta f(x) + \frac{1}{2} f^2(x)
$$

$$
= \Gamma_2(f)(x) - \frac{1}{2} \Gamma_1(f)(x) + \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 - \frac{1}{2} f(x) \sum_{y \in S_p^1 \sim x} (f(y) - f(x))
$$

$$
\quad + \frac{1}{4} \left[ \sum_{y \in S_p^1 \sim x} f^2(y) - \deg(x) f^2(x) \right] - \frac{1}{2} f(x) \Delta f(x) + \frac{1}{4} \sum_{y \in S_p^1} f^2(y) + \frac{1}{2} f^2(x).
$$

(3) If $x = p$, then

$$
\Gamma_2^c(f)(p) = \frac{1}{2} \Delta^c \Gamma_1^c(f)(p) - \Gamma_1^c(f, \Delta^c f)(p)
$$

$$
= \frac{1}{2} \sum_{y \in S_p^1} \Gamma_1(f)(y) - \frac{|S_p^1| - 1}{4} \sum_{y \in S_p^1} f^2(y) - \frac{1}{2} \sum_{y \in S_p^1} f(y) \Delta f(y) + \frac{1}{2} \sum_{y \in S_p^1} f^2(y) + \frac{1}{2} \left( \sum_{y \in S_p^1} f(y) \right)^2
$$

$$
= \frac{1}{2} \sum_{y \in S_p^1} \Gamma_1(f)(y) - \frac{1}{2} \sum_{y \in S_p^1} f(y) \Delta f(y) - \frac{|S_p^1| - 3}{4} \sum_{y \in S_p^1} f^2(y) + \frac{1}{2} \left( \sum_{y \in S_p^1} f(y) \right)^2.
$$

3.2. $\Gamma_{2}^{c}$ for $C(G)$. When $C = (V^c, E^c)$ is the full cone over $V(G)$, then $S_{p}^{1} = V$ and so Lemma 3.7 reduces to

Lemma 3.8.

\[ \Gamma_{2}^{c}(f)(x) = \begin{cases} \frac{1}{2} \Gamma_{1}(f)(x) + \frac{1}{2} \sum_{y \in V} f^2(y) + \frac{1}{2} f^2(x); & x \sim p \\ \sum_{y \in V} \Gamma_{1}(f)(y) - \frac{|V|-3}{4} \sum_{y \in V} f^2(y) + \frac{1}{2} \left( \sum_{y \in V} f(y) \right)^2; & x = p. \end{cases} \]

Proof. Since $S_{p}^{1} = V$ and $S_{p}^{2} = \emptyset$ the first case in Lemma 3.7 disappears. In case 2 notice that when $S_{p}^{1} = V$, then $\frac{1}{2} \sum_{y \in S_{p}^{1}/x} f(y) - f(x) = \Gamma_{1}(f)(x)$ and $\frac{1}{2} \sum_{y \in S_{p}^{1}/x} (f^2(y) - f^2(x)) = (\Delta f^2)(x)$. Since $\frac{1}{2} \Gamma_{1}(f)(x) = \frac{1}{2} (\Delta f^2)(x) - \frac{1}{2} f(x) \Delta f(x)$ the case when $x \sim p$ follows. When $x = p$, applying the identity (7) gives the desired result. \[ \square \]

This leads to the following result regarding the curvature of the cone,

Theorem 1.7. Suppose $G$ satisfies $CD(K, \infty)$ for $K \leq \frac{1}{2}$ then the subgraph $G \subset C(G)$ satisfies $CD(K + \frac{1}{2}, \infty)$.

Proof of Theorem 1.7. Suppose $G$ satisfies $CD(K, \infty)$ for $K \leq \frac{1}{2}$. Since $G$ satisfies $CD(K, \infty)$ then by Lemma 3.8 for $x \sim p$,

\[ \Gamma_{2}^{c}(f)(x) \geq (K + 1) \Gamma_{1}(f)(x) + \frac{1}{4} \sum_{y \in V} f^2(y) + \frac{1}{2} f^2(x). \]

Therefore,

\[ \Gamma_{2}^{c}(f)(x) \geq (K + 1) \Gamma_{1}(f)(x) + \frac{1}{4} \sum_{y \in V} f^2(y) - \frac{K}{2} f^2(x). \]

Since $K \leq \frac{1}{2}$, then $\frac{1}{2} \sum_{y \in V} f^2(y) - \frac{K}{2} f^2(x) \geq 0$. Hence we may drop both terms from the inequality and $C(G)$ satisfies $CD(K + 1, \infty)$ for $x \sim p$. \[ \square \]

4. $CCD(K, N)$ AND GLOBAL POINCARÉ INEQUALITY

If the cone $C$ satisfies the $CD(K, N)$ inequality at the vertex, $p$, then by Lemmas 3.3 and 3.2, we get

\[ \Gamma_{2}^{c}(f)(p) \geq \frac{1}{N} \left( \sum_{y \in V} f(y) \right)^2 + \frac{K}{2} \sum_{y \in V} f^2(y). \]

(10)

This leads to the following,

Theorem 1.1. If a graph, $G$, satisfies $CCD(K, N)$ curvature-dimension condition, then for any function $f$ on $G$ one has

\[ \sum_{y \in V} \Gamma_{1}(f)(y) \geq \frac{2 - N}{2N} \left( \sum_{y \in V} f(y) \right)^2 + \frac{2K + |V| - 3}{4} \sum_{y \in V} f^2(y). \]

(11)

For functions $f$ with $\text{avg}(f) = 0$, this reduces to the following global Poincaré inequality,

\[ \|f\|_2 \leq \sqrt{\frac{2}{2K + |V| - 3}} \|\nabla f\|_2, \]

where $\|\nabla f\|_2$ is understood in the graph setting to be $2 \cdot \sum_{y \in V} \Gamma_{1}(f)(y)$. 
Proof of Theorem 1.1. Suppose a graph $G$ satisfies $CCD(K,N)$ condition. By Lemma 3.8, 
\[
\Gamma_2(f)(p) = \sum_{y \in V} \Gamma_1(f)(y) - \frac{|V| - 3}{4} \sum_{y \in V} f^2(y) + \frac{1}{2} \left( \sum_{y \in V} f(y) \right)^2,
\] (12)
Upon combining (12) and (10), we will arrive at 
\[
\sum_{y \in V} \Gamma_1(f)(y) - \left( \frac{|V| - 3}{4} \sum_{y \in V} f^2(y) + \frac{1}{2} \left( \sum_{y \in V} f(y) \right)^2 + \frac{K}{2} \sum_{y \in V} f^2(y) \right) \geq \frac{1}{N} \sum_{y \in V} f(y)^2 + \frac{K}{2} \sum_{y \in V} f^2(y),
\] which simplifies to 
\[
\sum_{y \in V} \Gamma_1(f)(y) \geq \frac{2}{N} \sum_{y \in V} f^2(y) + \frac{2K + |V| - 3}{4} \sum_{y \in V} f^2(y).
\] (13)
For $f$ with $\text{avg}(f) = 0$, the above reduces to 
\[
\frac{1}{2} \sum_{y \in V} |\nabla f(y)|^2 \geq \frac{2K + |V| - 3}{4} \sum_{y \in V} f^2(y).
\]
By the definition in 6 this yields the Poincaré inequality, 
\[
\|f\|_2 \leq \sqrt{\frac{2}{2K + |V| - 3}} \|\nabla f\|_2, \text{ when } \text{avg}(f) = 0.
\]

Theorem 1.3. For any graph, $G$, and a given $N > 1$, the conical curvature cannot exceed the following number: 
\[
K_{c,\max}^N = \frac{|V|}{2} + \frac{3}{2} - \frac{2}{N}.
\]
Proof of Theorem 1.3. Suppose a finite graph $G$ satisfies the $CCD(K,N)$ and $f$ is a non-zero harmonic function, then one has $\sum_{y \in V} \Gamma_1(f)(y) = 0$ (i.e. $f$ is constant on connected components). Thus, 
\[
\frac{2K + |V| - 3}{4} \sum_{y \in V} f^2(y) \leq \frac{N - 2}{2N} \left( \sum_{y \in V} f(y) \right)^2.
\]
By the Cauchy-Schwarz inequality, 
\[
\left( \sum_{y \in V} f(y) \right)^2 \leq |V| \cdot \sum_{y \in V} f^2(y),
\]
which implies 
\[
\frac{2K + |V| - 3}{4} \sum_{y \in V} f^2(y) \leq \frac{N - 2}{2N} |V| \cdot \sum_{y \in V} f^2(y).
\]
Since, $f$ is not constant zero, 
\[
K \leq \frac{|V|}{2} - \frac{2|V|}{N^2} + \frac{3}{2}.
\]
Having established an upper bound for the curvature at the cone point over the vertex set of the graph $G$ we now turn to an investigation of when the maximum curvature value is achieved.

Lemma 4.1. For any finite graph, $G$, the Ricci curvatures $Ric_\infty(G)$, $Ric_N(G)$, $CRic_\infty(G)$ and $CRic_N(G)$ are realized by some functions, i.e. there are functions that achieve the equality in the (corresponding) defining Bakry-Émery curvature-dimension inequalities.
Range without loss of generality, we may assume that\( \text{Range}(g_i) \subset [-1, 1] \), for all \( i \). Now since \( V(G) \) is finite then by a diagonal argument one can find a subsequence \( g_j \) of the \( g_i \)'s that converge to a function \( g \). Taking the limit of (14) as \( j \to \infty \) shows that \( g \) achieves \( \text{Ric}_N(G) \).

5. Functions That Maximize the Conical Curvature

In this section we show that

**Theorem 1.4.** Suppose \( G \) satisfies \( \text{CCD}(K_{\text{max}}^c, N) \). Then any function, \( f \), realizes \( K_{\text{max}}^c \) if and only if \( f \) is either constant or \( f - \text{avg}(f) \) is an eigenfunction corresponding to \( \lambda_1(G) = \frac{\sum f^2}{N} \). Furthermore, when \( G \) is a complete graph, \( f \) must be constant (harmonic).

**Proof.** Suppose \( G \) satisfies \( \text{CCD}(K_{\text{max}}^c, N) \), then for any \( f \)

\[
\sum_{y \in V} \Gamma_1(f)(y) \geq 2 - \frac{N}{2N} \left( \sum_{y \in V} f(y) \right)^2 + \frac{2K_{\text{max}}^c}{4} + \frac{|V| - 3}{4N} \sum_{y \in V} f^2(y).
\]

Since \( K_{\text{max}}^c = \frac{|V|}{2} + \frac{3}{2} - 2\frac{|V|}{N} = N|V| + 3N - 4|V| \), this simplifies to

\[
\sum_{y \in V} \Gamma_1(f)(y) \geq \frac{2 - N}{2N} \left( \sum_{y \in V} f(y) \right)^2 + \frac{N - |V| - 2|V|}{N} \sum_{y \in V} f^2(y)
\]

\[
\geq \frac{2 - N}{2N} \left( \sum_{y \in V} f(y) \right)^2 + \frac{N - 2}{2N} \cdot |V| \sum_{y \in V} f^2(y)
\]

\[
\geq \frac{N - 2}{2N} \left[ |V| \sum_{y \in V} f^2(y) - \left( \sum_{y \in V} f(y) \right)^2 \right].
\]

Take \( \varphi : V \to \mathbb{R} \) to be any variational function on the vertex set of \( G \) and let \( t \in \mathbb{R} \), then

\[
\sum_{y \in V} \Gamma_1(f + t\varphi)(y) \geq \frac{N - 2}{2N} \left[ |V| \sum_{y \in V} (f + t\varphi)^2(y) - \left( \sum_{y \in V} (f + t\varphi)(y) \right)^2 \right].
\]

Suppose now that \( f \) achieves \( K_{\text{max}}^c \), then for any \( \varphi \) the above inequality becomes an equality (i.e. \( \frac{d}{dt} \big|_{t=0} \) of both sides must be equal for any variation \( \varphi \)). Hence a straightforward calculation yields the linearized equation,

\[
\sum_{y \in V} \sum_{z \sim y} (f(z) - f(y))(\varphi(z) - \varphi(y)) = \frac{N - 2}{2N} \left[ |V| \sum_{y \in V} f(y)\varphi(y) - \sum_{y \in V} f(y) \sum_{y \in V} \varphi(y) \right].
\]

Now fix \( r \in V \) and let \( \varphi(y) = \delta_r(y) \). Notice that \( \sum_{z \sim y} (f(z) - f(y))(\delta_r(z) - \delta_r(y)) \) is zero except when \( y = r \) or \( y \sim r \). If \( y = r \) the result is \( -\sum_{z \sim r} (f(z) - f(r)) \). When \( y \sim r \) there is exactly one non-zero term in the sum, \( (f(r) - f(y)) \) and summing over all \( y \sim r \) we get \( \sum_{y \sim r} (f(r) - f(y)) \). Thus

\[
\sum_{y \in V} \sum_{z \sim y} (f(z) - f(y))(\delta_r(z) - \delta_r(y)) = -2\Delta f(r),
\]

where

\[
\Delta f(r) = \sum_{y \sim r} (f(r) - f(y)).
\]
Hence, when \( N \) and \( \phi \) are fixed, which is equivalent to
\[
\Delta f(r) = \frac{N - 2}{4N} \Delta f(r),
\]
where \( \Delta \) denotes the Laplacian for the graph completion, \( \bar{G} \), of \( G \). This right away implies that \( \Delta f(r) = 0 \) and we are done. Furthermore when \( G \) is a complete graph, \( f \) is harmonic on \( G \).

Now suppose \( G \) is arbitrary. By equation (17)
\[
\Delta (f - \text{avg}(f))(r) = -\frac{N - 2}{4N} |V|(f - \text{avg}(f))(r).
\]
Now if \( f \) is not constant then by the Rayleigh quotient, (9), we see that \( \lambda_1(G) = \frac{N - 2}{4N} |V| \) and \( f - \text{avg}(f) \) is an eigenfunction for \( \lambda_1 \).

For the "if" direction suppose for some non-constant function, \( f \), that \( f - \text{avg}(f) \) is an eigenfunction for \( \lambda_1 = \frac{N - 2}{4N} |V| \). Tracing back the above computations one has (15) holds for \( \varphi = \delta_y \)'s. Then since (15) is linear in \( \varphi \), one can use \( f = \sum_{y \in V} f(y) \delta_y \) instead of \( \varphi \) which will translate to \( f \) realizing \( K_{\max}^N \).

6. **CCD(\( K, N \)) and Lower Bounds on \( \lambda_1(G) \)**

In this section we assume that a given graph, \( G \), satisfies the CCD(\( K, N \)) condition. We will use the resulting global Poincaré inequality along with the results of the last section to find lower bounds on the first non-zero eigenvalues of such graphs.

**Lemma 6.1.** Suppose \( G \) satisfies CCD(\( K, N \)) and \( N \geq 2 \). Then Cheeger's isoperimetric constant, \( h(G) \), satisfies
\[
h(G) \geq \frac{2|V| + 4NK + N|V| - 6N}{8N},
\]
and,
\[
\lambda_1(G) \geq \frac{(2|V| + 4NK + N|V| - 6N)^2}{128N^2d_{\max}}.
\]

**Proof.** Since \( G \) satisfies the CCD(\( K, N \)) condition for any \( f \), we have the global Poincaré inequality from Theorem 1.1,
\[
\sum_{y \in V} \Gamma_1(f)(y) \geq \frac{2 - N}{2N} \left( \sum_{y \in V} f(y) \right)^2 + \frac{2K + |V| - 3}{2} \sum_{y \in V} f^2(y).
\]
Suppose \( F \subset V \) and \( |F| \leq \frac{|V|}{2} \). Let \( f = \chi_F \) be the characteristic function of \( F \), then (11) becomes
\[
\frac{2 - N}{2N} |F|^2 + \frac{2K + |V| - 3}{4} |F| \leq 2|\partial F|.
\]
Hence, when \( N \geq 2 \),
\[
\frac{|\partial F|}{|F|} \geq \frac{2 - N}{4N} |F| + \frac{2K + |V| - 3}{4} \geq \frac{2 - N |V|}{4N} + \frac{2K + |V| - 3}{4} = 2|V| + 4NK + N|V| - 6N.
\]
Now applying Cheeger's inequality (3) (see [1] and [2]), we know that \( \lambda_1(G) \geq \frac{h^2(G)}{2d_{\max}} \), where \( d_{\max} \) is the maximum degree in the graph, \( G \). Hence,
\[
\lambda_1 \geq \frac{h^2(G)}{2d_{\max}} \geq \frac{(2|V| + 4NK + N|V| - 6N)^2}{128N^2d_{\max}}.
\]
In the rest of this section we show that any lower bound, \( \lambda \), for \( \lambda_1(G) \) will imply that \( G \) satisfies \( CCD(K, N) \) for some \( K \) and \( N \) (depending on \( \lambda \)).

**Theorem 6.1.** Suppose \( \lambda_1(G) \geq \lambda \), then \( G \) satisfies \( CCD(K, N) \) for any \( K \) and \( N \) with

\[
N \geq \frac{2|V|}{|V| - \lambda}, \quad \text{and} \quad K \geq \frac{\lambda - |V| + 3}{2}.
\]

**Proof.** Since \( \lambda_1(G) \geq \lambda \) then by the Rayleigh quotient, (9), we get

\[
\sum_{y \in V} \Gamma_1(f)(y) \geq \frac{\lambda}{2} \sum_{y \in V} (f - \text{avg}(f))^2(y) = \frac{\lambda}{2} \left[ \sum_{y \in V} f^2(y) + |V| \text{avg}(f)^2 - 2 \text{avg}(f) \sum_{y \in V} f(y) \right] = \frac{\lambda}{2} \sum_{y \in V} f^2(y) + \frac{\lambda}{2|V|} \left( \sum_{y \in V} f(y) \right)^2 - \frac{\lambda}{|V|} \left( \sum_{y \in V} f(y) \right)^2.
\]

Comparing (19) to the global Poincaré inequality, (11) due to the \( CCD(K, N) \) condition, and one observes that \( G \) satisfies \( CCD(K, N) \) for any \( K \), and \( N \) where

\[
\frac{\lambda}{|V|} \leq \frac{N - 2}{N}, \quad \text{and} \quad \lambda \leq 2K + |V| - 3.
\]

The conclusion follows by noticing that one always have \( \lambda_1(G) \leq |V| \). 

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