APPROXIMATION AND QUADRATURE BY WEIGHTED LEAST SQUARES POLYNOMIALS ON THE SPHERE

WANTING LU AND HEPING WANG

Abstract. Given a sequence of Marcinkiewicz-Zygmund inequalities in $L_2$ on a usual compact space $M$, Gröchenig in [11] introduced the weighted least squares polynomials and the least squares quadrature from pointwise samples of a function, and obtained approximation theorems and quadrature errors. In this paper we obtain approximation theorems and quadrature errors on the sphere which are optimal. We also give upper bounds of the operator norms of the weighted least squares operators.

1. Introduction

Let
\begin{align*}
S^d := \{x = (x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} \mid |x|^2 = \sum_{k=1}^{d+1} |x_k|^2 = 1\} \quad (d \geq 2)
\end{align*}
be the unit sphere in $\mathbb{R}^{d+1}$ endowed with the rotationally invariant measure $\sigma$ normalized by $\int_{S^d} d\sigma = 1$. We denote by $L_2(S^d)$ the usual Hilbert space of square-integrable functions on $S^d$ with the inner product
\begin{align*}
\langle f, g \rangle := \int_{S^d} f(x)g(x)d\sigma(x)
\end{align*}
and the norm $\|f\|_2 = \langle f, f \rangle^{1/2}$, and by $C(S^d)$ the space of continuous functions on $S^d$ with supremum norm
\begin{align*}
\|f\| := \|f\|_\infty := \sup_{x \in S^d} |f(x)|.
\end{align*}
The space $\Pi_n^d$ of spherical polynomials on $S^d$ of degree $\leq n$ consists of the restrictions to $S^d$ of all polynomials on $\mathbb{R}^{d+1}$ of total degree $\leq n$. The dimension of $\Pi_n^d$ is given by
\begin{align*}
d_n := \dim(\Pi_n^d) = \frac{(2n + d)\Gamma(n + d)}{\Gamma(d + 1)\Gamma(n + 1)}.
\end{align*}
Let $S_n$ be the orthogonal projection from $L_2(S^d)$ onto $\Pi_n^d$, i. e., for $f \in L_2(S^d)$,
\begin{align*}
S_n f(x) = \int_{S^d} f(y)E_n(x, y)d\sigma(y),
\end{align*}
where $E_n(x, y)$ is the reproducing kernel for $\Pi_n^d$ with respect to the inner product $\langle \cdot, \cdot \rangle$. We note that $E_n(x, y)$ satisfies the following properties:
(1) For any $x, y \in S^d$, $E_n(x, y) = E_n(y, x)$;

2010 Mathematics Subject Classification. 41A17; 42A10; 65D30.
Key words and phrases. Marcinkiewicz-Zygmund inequalities; Sampling; Sobolev spaces; Least squares problem; operator norms.
(2) For any fixed \( y \in \mathbb{S}^d \), \( E_n(\cdot, y) \in \Pi_n^d \); 
(3) For any \( p \in \Pi_n^d \) and \( x \in \mathbb{S}^d \), \( p(x) = S_n p(x) = \langle p, E_n(\cdot, x) \rangle \).

This paper is concerned with constructive polynomial approximation on \( \mathbb{S}^d \) that uses function values (the samples) at selected well-distributed points (sometimes called standard information). Here the “well-distributed” points indicate that those points constitute a Marcinkiewicz-Zygmund (MZ) family on \( \mathbb{S}^d \) defined as follows.

**Definition 1.1.** Let \( \mathcal{X} = \{ \mathcal{X}_n \} = \{ x_{n, k} : n = 1, 2, \ldots, k = 1, \ldots, l_n \} \) be a doubly-indexed set of points in \( \mathbb{S}^d \) and \( \tau = \{ \tau_{n, k} : n = 1, 2, \ldots, k = 1, \ldots, l_n \} \subset (0, +\infty) \) be a family of non-negative weights. Then \( \mathcal{X} \) is called a Marcinkiewicz-Zygmund (MZ) family with associated weight \( \tau \), if there exist constants \( A, B > 0 \) independent of \( n \) such that

\[
A \| p \|_2^2 \leq \sum_{k=1}^{l_n} |p(x_{n, k})|^2 \tau_{n, k} \leq B \| p \|_2^2 \quad \text{for all } p \in \Pi_n^d.
\]

The ratio \( \kappa = B/A \) is the global condition number of the Marcinkiewicz-Zygmund family \( \mathcal{X} \), and \( \mathcal{X}_n = \{ x_{n, k} : k = 1, \ldots, l_n \} \) is the \( n \)-th layer of \( \mathcal{X} \).

The MZ inequality (1.3) means that the \( L_2 \)-norm of a spherical polynomial of degree at most \( n \) is comparable to the discrete version given by the weighted \( L_2 \)-norm of its restriction to \( \mathcal{X}_n \). It follows from [13] [2] [7] that such MZ families exist if the families are dense enough. We remark that Fekete points of degree \( [n(1+\varepsilon)] \) (\( \varepsilon > 0 \)) on the sphere are MZ families with the equal weights \( \tau_{n, k} = \frac{1}{d_n} \), see [17], where \( [a] \) is the largest integer not exceeding \( a \in \mathbb{R} \). Also, sufficient conditions and necessary density conditions for MZ families with the equal weights \( \tau_{n, k} = \frac{1}{d_n} \) on the sphere are obtained in [13] and [16], respectively.

Given a MZ family on a usual compact space \( \mathcal{M} \), Gröchenig in [11] introduced the weighted least squares polynomials and the least squares quadrature from pointwise samples of a function, and obtained approximation theorems and quadrature errors. However, the obtained error estimates in [11] are not optimal due to the generality of \( \mathcal{M} \). In this paper we confine the study to the sphere \( \mathbb{S}^d \).

Let \( \mathcal{X} \) be a MZ family with associated weight \( \tau \). Given the samples \( \{ f(x_{n, k}) \} \) of a continuous function \( f \) on \( \mathbb{S}^d \) on the \( n \)-th layer \( \mathcal{X}_n \), we want to approximate \( f \) using only these samples. For this we solve a sequence of weighted least squares problems with samples taken from the samples \( \mathcal{X}_n \):

\[
L_n f = \arg \min_{p \in \Pi_n^d} \sum_{k=1}^{l_n} |f(x_{n, k}) - p(x_{n, k})|^2 \tau_{n, k}.
\]

This procedure yields a sequence \( \{ L_n f \} \) of the best weighted \( L_2 \)-approximation of the data \( \{ f(x_{n, k}) \} \) by a spherical polynomial in \( \Pi_n^d \) for every \( f \in C(\mathbb{S}^d) \). We call \( L_n f \) the weighted least squares polynomial, and \( L_n \) the weighted least squares operator. Clearly, \( L_n \) is the projection onto \( \Pi_n^d \), i.e., \( L_n \) is a bounded linear operator on \( C(\mathbb{S}^d) \) satisfying that \( L_n^2 = L_n \) and the range of \( L_n \) is \( \Pi_n^d \). It follows from (1.3) that for \( p \in \Pi_n^d \), \( p = 0 \) whenever \( p(x_{n, k}) = 0 \). This means that usually \( \mathcal{X}_n \) contains more than \( d_n = \dim \Pi_n^d \) points, so that it is not an interpolating set for \( \Pi_n^d \). Therefore, \( L_n f \) is usually a quasi-interpolant.

Let \( E_n(x, y) \) be the reproducing kernel of \( \Pi_n^d \) with respect to the inner product (1.1). Then the Marcinkiewicz-Zygmund inequalities (1.3) say that every set \( \{ \hat{\mathcal{E}}_{n, k} \):
$k = 1, \ldots, l_n$, $\tilde{e}_{n,k} = \tau_{n,k}^{1/2}E_n(\cdot, x_{n,k})$ is a frame for $\Pi_n^d$ with uniform frame bounds $A, B > 0$, i.e., for all $p \in \Pi_n^d$, we have

$$A\|p\|^2 \leq \sum_{k=1}^{l_n} |\langle p, \tilde{e}_{n,k}\rangle|^2 \leq B\|p\|^2.$$ 

It follows that the associated frame operator

$$T_n p(\cdot) = \sum_{k=1}^{l_n} \langle p, \tilde{e}_{n,k}\rangle \tilde{e}_{n,k}(\cdot) = \sum_{k=1}^{l_n} \tau_{n,k} p(x_{n,k})E_n(\cdot, x_{n,k})$$

is invertible on $\Pi_n^d$ for every $n \in \mathbb{N}$. We obtain the dual frame $\{e_{n,k} = T_n^{-1}(\tilde{e}_{n,k}), k = 1, \ldots, l_n\}$ for $\Pi_n^d$ with uniform frame bounds $B^{-1}$ and $A^{-1}$, and every polynomial $p \in \Pi_n^d$ can be reconstructed from the samples on $X_n$ by

$$p = T_n^{-1}T_n p = \sum_{k=1}^{l_n} \langle p, \tilde{e}_{n,k}\rangle T_n^{-1} \tilde{e}_{n,k} = \sum_{k=1}^{l_n} \tau_{n,k}^{1/2} p(x_{n,k})e_{n,k}.$$ 

The weights for the quadrature rules are defined by

$$w_{n,k} = \langle \tau_{n,k}^{1/2}e_{n,k}, 1 \rangle = \tau_{n,k}^{1/2} \int_{\mathbb{S}^d} e_{n,k}(x) \, d\sigma(x)$$

and the corresponding quadrature rule is defined by

$$I_n(f) = \sum_{k=1}^{l_n} w_{n,k} f(x_{n,k}).$$

Such quadrature rules are usually named least squares quadrature.

Let $\mathcal{X}$ be a MZ family with associated weight $\tau$. For a function $f$ in the Sobolev space $H^\sigma(\mathbb{S}^d)$, $\sigma > d/2$ (see Section 2 for definition of $H^\sigma(\mathbb{S}^d)$), Gröchenig obtained in [11] that

$$\|f - L_n f\|_2 \leq c(1 + \kappa^2)^{1/2} n^{-\sigma+d/2} \|f\|_{H^\sigma},$$

and

$$|\int_{\mathbb{S}^d} f(x) \, d\sigma(x) - I_n(f)| \leq c(1 + \kappa^2)^{1/2} n^{-\sigma+d/2} \|f\|_{H^\sigma},$$

where $c$ depends on $d$, $s$, but not on $f$ or $\kappa$ or the MZ family $\mathcal{X}$.

We remark that the condition $\sigma > d/2$ is the well known necessary and sufficient condition for $H^\sigma(\mathbb{S}^d)$ to be continuously embedded in $C(\mathbb{S}^d)$, and is unavoidable in any approximation scheme that employs function values.

In this paper we improve the above results and obtain the optimal estimates. One of our main results can be formulated as follows.

**Theorem 1.2.** Let $\mathcal{X}$ be a MZ family on $\mathbb{S}^d$ with associated weight $\tau$ and global condition number $\kappa = B/A$, $L_n$ be the weighted least squares operators defined by [11, A], and let $\{I_n | n \in \mathbb{N}\}$ be the sequence of the least squares quadrature. If $f \in H^\sigma(\mathbb{S}^d)$, $\sigma > d/2$, then we have

$$\|f - L_n f\|_2 \leq c(1 + \kappa^2)^{1/2} n^{-\sigma} \|f\|_{H^\sigma},$$

and

$$|\int_{\mathbb{S}^d} f(x) \, d\sigma(x) - I_n(f)| \leq c(1 + \kappa^2)^{1/2} n^{-\sigma} \|f\|_{H^\sigma},$$

where $c$ depends on $d$, $s$, but not on $f$ or $\kappa$ or the MZ family $\mathcal{X}$. 
Remarkably, the convergence rates for the weighted least squares approximation and for the least squares quadrature errors are optimal (as explained in Remark 1.3 below) in a variety of Sobolev space settings. When the global condition number $\kappa = 1$, the weighted least squares operators are just the hyperinterpolation operators on the sphere (see Subsection 2.5). In this case, the inequality \[ \|A\| \leq \kappa \|f\| \] was obtained in [28].

**Remark 1.3.** For $N \in \mathbb{N}$, the sampling numbers (or the optimal recovery) of the Sobolev classes $BH^s(\mathbb{S}^d)$ (the unit ball of the space $H^s(\mathbb{S}^d)$) in $L_2(\mathbb{S}^d)$ are defined by

$$ g_N(BH^s(\mathbb{S}^d), L_2(\mathbb{S}^d)) := \inf_{\xi_1, \ldots, \xi_N \in \mathbb{S}^d} \sup_{f \in BH^s(\mathbb{S}^d)} \|f - \varphi(f(\xi_1), \ldots, f(\xi_N))\|_2, $$

where the infimum is taken over all $N$ points $\xi_1, \ldots, \xi_N$ in $\mathbb{S}^d$ and all mappings $\varphi$ from $\mathbb{R}^N$ to $L_2(\mathbb{S}^d)$. The optimal quadrature errors of the Sobolev classes $BH^s(\mathbb{S}^d)$ are defined by

$$ e_N(\text{Int}; BH^s(\mathbb{S}^d)) := \inf_{\lambda_1, \ldots, \lambda_N \in \mathbb{R}} \sup_{\xi_1, \ldots, \xi_N \in \mathbb{S}^d} \left| \int_{\mathbb{S}^d} f(x) \, d\sigma(x) - \sum_{j=1}^N \lambda_j f(\xi_j) \right|. $$

It follows from [31], [1] and [13] that

$$ g_N(BH^s(\mathbb{S}^d), L_2(\mathbb{S}^d)) \asymp N^{-s/d} \quad \text{and} \quad e_N(\text{Int}; BH^s(\mathbb{S}^d)) \asymp N^{-s/d}, $$

where the notation $A(N) \asymp B(N)$ means that $A(N) \lesssim B(N)$ and $A(N) \gtrsim B(N)$, $A(N) \ll B(N)$ means that there exists a positive constant $c$ independent of $N$ such that $A(N) \leq cB(N)$, and $A(N) \gg B(N)$ means $B(N) \ll A(N)$.

It follows from [16, 17, 18] that there exist MZ families with $l_n \asymp d_n \asymp N \asymp n^d$. For such MZ family it follows from (1.7) that

$$ \sup_{f \in BH^s(\mathbb{S}^d)} \|f - l_n(f)\|_2 \asymp N^{-s/d} \lesssim g_N(BH^s(\mathbb{S}^d), L_2(\mathbb{S}^d)), $$

which implies that the weighted least squares operators $l_n$ are asymptotically optimal algorithms in the sense of optimal recovery. Also, we have

$$ \sup_{f \in BH^s(\mathbb{S}^d)} \left| \int_{\mathbb{S}^d} f(x) \, d\sigma(x) - l_n(f) \right| \asymp N^{-s/d} \lesssim e_N(\text{Int}; BH^s(\mathbb{S}^d)), $$

which means that the least squares quadrature rules are the asymptotically optimal quadrature formulas.

For a linear operator $L$ on $C(\mathbb{S}^d)$, the operator norm $\|L\|$ of $L$ is given by

$$ \|L\| := \sup\{\|Lf\| : f \in C(\mathbb{S}^d), \|f\| \leq 1\}. $$

The quantity $\|L\|$ of a projection $L$ is also called the Lebesgue constant of $L$ and the estimation of $\|L\|$ is extremely important in numerical computation. We use the Christoffel functions to get the following estimation.

**Theorem 1.4.** Let $X$ be a MZ family on $\mathbb{S}^d$ with associated weight $\tau$ and global condition number $\kappa = B/A$, and let $l_n$ be the weighted least squares operators defined by (1.8). Then we have

$$ n^{(d-1)/2} \ll \|L_n\| \ll \kappa^{1/2} n^{d/2}. $$
Remark 1.5. When $\kappa = 1$, the weighted least squares operators are just the hyperinterpolation operators on the sphere (see Subsection 2.5). In this case, we have (see [23, 10, 20])

$$\|L_n\| \approx n^{(d-1)/2}.$$  

Hence, we conjecture that the upper bound of the operator norm of $L_n$ is $n^{(d-1)/2}$ multiplied by a positive constant independent of $n$. However, we have not been able to prove it.

Remark 1.6. Since the weighted least squares operators $L_n$ is the projection onto $\Pi_n^d$, by the Lebesgue theorem the hyperinterpolation error in the uniform norm can be estimated as

$$\|f - L_n(f)\| \leq (1 + \|L_n\|) E_n(f) \ll \kappa^{1/2} n^{d/2} E_n(f),$$

where

$$E_n(f) := \inf_{p \in \Pi_n^d} \|f - p\|$$

is the best approximation of $f$ by $\Pi_n^d$.

2. Preliminary

This section is devoted to giving some basic facts about spherical harmonics (see for example, [7]) and some preliminary knowledge.

2.1. Harmonic analysis on the sphere.

Let $S^d = \{x \in \mathbb{R}^{d+1} \mid |x| = 1\}$ denote the unit sphere in $\mathbb{R}^{d+1}$, where $(x, y) = x \cdot y$ is the usual inner product and $|x| = (x, x)^{1/2}$ is the Euclidean norm. We denote by $H^d_\ell$ the space of all spherical harmonics of degree $\ell$ on $S^d$, i.e., the space of the restrictions to $S^d$ of all harmonic homogeneous polynomials of exact degree $\ell$ on $\mathbb{R}^{d+1}$. It is well known that the dimension of $H^d_\ell$ is

$$N(d, \ell) = \dim H^d_\ell = \left\{ \begin{array}{ll}
1, & \text{if } \ell = 0,
\frac{(2\ell+d-1)(\ell+d-2)!}{(d-1)!}, & \text{if } \ell = 1, 2, \ldots .
\end{array} \right.$$  

The spaces $H^d_k$, $k = 0, 1, 2, \ldots$ are just the eigenspaces corresponding to the eigenvalues $-k(k+d-1)$ of the Laplace-Beltrami operator $\triangle$ on the sphere $S^d$ and are mutually orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{S^d} f(x)g(x)d\sigma(x).$$

Let

$$\{Y_{\ell,k} = Y_{\ell,k}^d \mid k = 1, \ldots, N(d, \ell)\}$$

be a fixed orthonormal basis for $H^d_\ell$. We have the addition theorem for the spherical harmonics of degree $\ell$:

$$\sum_{k=1}^{N(d,\ell)} Y_{\ell,k}(x)Y_{\ell,k}(y) = \frac{\ell + \lambda}{\lambda} C^\lambda_\ell (x \cdot y), \quad x, y \in S^d, \quad \lambda = \frac{d-1}{2}, \quad d \geq 2,$$

where $C^\lambda_\ell(t)$ is the usual ultraspherical (Gegenbauer) polynomial of order $\lambda$ normalized by $C^\lambda_\ell(1) = \frac{\Gamma((\ell+2\lambda)/2)}{\Gamma(2\lambda)\Gamma(\ell+1)}$ and generated by

$$\frac{1}{(1 - 2zt + z^2)^{\lambda}} = \sum_{k=0}^{\infty} C^\lambda_\ell(t)z^k \quad (0 \leq z < 1).$$
(See [20, p. 81]).

Since the space $\Pi_n^d$ can be expressed as a direct sum of

$$\Pi_n^d = \mathcal{H}^d_0 \bigoplus \mathcal{H}^d_1 \bigoplus \cdots \bigoplus \mathcal{H}^d_n,$$

we obtain that

$$\{Y_{\ell,k} \mid k = 1, \ldots, N(d, \ell), \ \ell = 0, 1, \ldots, n\}$$

forms an $L_2$-orthonormal basis of $\Pi_n^d$. Hence, the $L_2$-reproducing kernel $E_n(x, y)$ of $\Pi_n^d$ has the explicit expression:

$$E_n(x, y) = \sum_{\ell=0}^{n} \sum_{k=1}^{N(d, \ell)} \frac{\ell + \lambda}{\lambda} C_{\ell}^\lambda(x \cdot y), \ \lambda = \frac{d-1}{2}, \ x, y \in \mathbb{S}^d.$$

Specifically, we have

$$E_n(x, x) = \sum_{\ell=0}^{n} \sum_{k=1}^{N(d, \ell)} \frac{\ell + \lambda}{\lambda} \Gamma(\ell + 2\lambda) \Gamma(\ell + 1) \approx n^d, \ \lambda = \frac{d-1}{2}, \ x \in \mathbb{S}^d.$$  

We remark that

$$\{Y_{\ell,k} \mid k = 1, \ldots, N(d, \ell), \ \ell = 0, 1, \ldots\}$$

is an orthonormal basis for the Hilbert space $L_2(\mathbb{S}^d)$. Thus any $f \in L_2(\mathbb{S}^d)$ can be expressed by its Fourier (or Laplace) series:

$$f = \sum_{\ell=0}^{\infty} \sum_{k=1}^{Z(d, \ell)} \langle f, Y_{\ell,k} \rangle Y_{\ell,k},$$

where

$$\langle f, Y_{\ell,k} \rangle = \int_{\mathbb{S}^d} f(x) Y_{\ell,k}(x) \, d\sigma(x)$$

are the Fourier coefficients of $f$. The Sobolev space $H^\sigma(\mathbb{S}^d)$, where $\sigma > 0$, is defined as the space of all functions in $L_2(\mathbb{S}^d)$ for which the norm

$$\|f\|_{H^\sigma} := \left( \sum_{\ell=0}^{\infty} \left( 1 + \ell(\ell + d - 1) \right)^\sigma \sum_{k=1}^{N(d, \ell)} |\langle f, Y_{\ell,k} \rangle|^2 \right)^{1/2}$$

is finite. The space $H^\sigma(\mathbb{S}^d)$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{H^\sigma} := \sum_{\ell=0}^{\infty} \left( 1 + \ell(\ell + d - 1) \right)^\sigma \sum_{k=1}^{N(d, \ell)} \langle f, Y_{\ell,k} \rangle \langle g, Y_{\ell,k} \rangle,$$

which induces the norm $\| \cdot \|_{H^\sigma}$. It is easily seen that for $f \in H^\sigma(\mathbb{S}^d)$,

$$\|f - S_n f\|_2 \leq cn^{-\sigma} \|f\|_{H^\sigma},$$

where $S_n f$ is given by (2.2).

If $\sigma > d/2$, then the space $H^\sigma(\mathbb{S}^d)$ is continuously embedded in $C(\mathbb{S}^d)$ and is a reproducing kernel Hilbert space with the reproducing kernel

$$K_\sigma(x, y) = \sum_{\ell=0}^{\infty} \left( 1 + \ell(\ell + d - 1) \right)^{-\sigma} \sum_{k=1}^{N(d, \ell)} Y_{\ell,k}(x) Y_{\ell,k}(y).$$
2.2. Two examples of MZ families.

We shall give two examples of MZ families with equal weights. Let $X$ be a finite subset of $S^d$ with cardinality $d_n := \dim \Pi_n^d$. The points $\xi_1, \ldots, \xi_{d_n}$ in $X$ maximizing a Vandermonde-type determinant

$$|\Delta(\xi_1, \ldots, \xi_{d_n})| = |\det (Q_i(\xi_j))_{i,j=1}^{d_n}|$$

are called Fekete points of degree $n$ for $S^d$ (these points are sometimes called extremal fundamental systems of points, as in [24]), where $Q_i$, $i = 1, \ldots, d_n$ are a basis of $\Pi_n^d$. It is known that Fekete points are independent of the choice of the polynomial basis. For any fixed $\varepsilon > 0$, if $X = \{X_n\}$ and $X_n$ are Fekete points of degree $[n(1 + \varepsilon)]$ for $S^d$, then $X$ is a MZ family with equal weights $\tau_{n,k} = \frac{1}{d_n}$ (see [17]).

Let $X$ be a finite subset of $S^d$. The mesh norm of $X$ is

$$\rho(X) = \sup_{u \in S^d} \inf_{z \in X} d(u, z),$$

where $d(x, y)$ denotes the standard geodesic distance $\arccos x \cdot y$ between two points $x$ and $y$ on $S^d$. Let $\mathcal{X} = \{\mathcal{X}_n\} = \{x_{n,k} : n = 1, 2, \ldots, k = 1, \ldots, l_n\}$ be a doubly-indexed set of points in $S^d$. We say that $\mathcal{X}$ is uniformly separated if there is a positive number $\varepsilon > 0$ such that

$$d(x_{n,k}, x_{n,l}) \geq \frac{\varepsilon}{n + 1} \quad \text{if} \quad k \neq l$$

for all $n \in \mathbb{N}$. Let $\mathcal{X} = \{\mathcal{X}_n\}$ be a uniformly separated array in $S^d$ such that for all $n \geq 1$,

$$\rho(\mathcal{X}_n) \leq \frac{\eta}{n},$$

where $\eta < \pi/2$. Then $\mathcal{X}$ is an MZ family with the equal weights $\tau_{n,k} = \frac{1}{d_n}$ (see [18]).

However, in [17] and [18], the authors do not give estimates of the global condition numbers of two MZ families on the sphere.

2.3. The weighted least squares polynomials $L_nf$.

Now let $X$ be a MZ family with associated weight $\tau$. We use the weighted discretized inner product

$$(f, g)_{(n)} := \sum_{k=1}^{l_n} f(x_{n,k})g(x_{n,k})\tau_{n,k}$$

and the discretized norm

$$\|f\|_{(n)}^2 = (f, f)_{(n)}.$$

We consider the corresponding orthogonal polynomial projection $L_n$ onto $\Pi_n^d$ with respect to the weighted discretized inner product $\langle \cdot, \cdot \rangle_{(n)}$, namely the weighted least squares polynomial $L_nf$ defined by

$$L_nf = \arg \min_{p \in \Pi_n^d} \|f - p\|_{(n)}^2 = \arg \min_{p \in \Pi_n^d} \sum_{k=1}^{l_n} |f(x_{n,k}) - p(x_{n,k})|^2 \tau_{n,k}.$$
We shall give a formal expression for $L_n f$. Let $\varphi_i, i = 1, \ldots, d_n$ be an orthonormal basis of $\Pi^d_n$ with respect to the weighted discrete scalar product (2.3), i.e.,

$$\langle \varphi_i, \varphi_j \rangle_{(n)} = \delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} 1 \leq i, j \leq d_n.$$ 

Such $\varphi_i, i = 1, \ldots, d_n$ can be obtained by applying Gram-Schmidt orthogonalization process to the basis $\{Y_{\ell,k} \mid k = 1, \ldots, N(d, \ell), \ell = 0,1,\ldots,n\}$ of $\Pi^d_n$. We set

$$D_n(x, y) = \sum_{k=1}^{d_n} \varphi_k(x) \varphi_k(y).$$

Clearly, the weighted least squares polynomial $L_n f$ is just the orthogonal projection of $f$ onto $\Pi^d_n$ with respect to the weighted discrete scalar product (2.3), i.e.,

$$L_n f(x) = \langle f, D_n(\cdot, x) \rangle_{(n)} = \sum_{k=1}^{l_n} f(x_{n,k}) D(x_{n,k}, x) \tau_{n,k},$$

and $D_n(x, y)$ is the reproducing kernel for $\Pi^d_n$ with respect to the weighted discrete scalar product (2.3) satisfying the following properties:

1. For any $x, y \in \mathbb{S}^d$, $D_n(x, y) = D_n(y, x)$;
2. For any fixed $y \in \mathbb{S}^d$, $D_n(\cdot, y) \in \Pi^d_n$;
3. For any $p \in \Pi^d_n$ and $x \in \mathbb{S}^d$,

$$p(x) = L_n p(x) = \langle p, D_n(\cdot, x) \rangle_{(n)} = \sum_{k=1}^{l_n} p(x_{n,k}) D(x_{n,k}, x) \tau_{n,k}.$$ 

According to the definition of the operator norm and (2.4), by the standard argument we have

$$\|L_n\| = \max_{x \in \mathbb{S}^d} \sum_{k=1}^{l_n} |D(x_{n,k}, x)| \tau_{n,k}.$$ 

2.4. The least squares quadrature.

Let $\mathcal{X}$ be a MZ family with a weight $\tau$ and let $E_n(x, y)$ be the reproducing kernel of $\Pi^d_n$. Then the frame operator

$$T_n p(\cdot) = \sum_{k=1}^{l_n} \langle p, \tilde{e}_{n,k} \rangle \tilde{e}_{n,k}(\cdot) = \sum_{k=1}^{l_n} \tau_{n,k} p(x_{n,k}) E_n(\cdot, x_{n,k})$$

is invertible on $\Pi^d_n$, where $\tilde{e}_{n,k} = \tau_{n,k}^{1/2} E_n(\cdot, x_{n,k})$. We set

$$e_{n,k} = T_n^{-1}(\tilde{e}_{n,k}) \text{ and } w_{n,k} = \langle \tau_{n,k}^{1/2} e_{n,k}, 1 \rangle = \tau_{n,k}^{1/2} \int_{\mathbb{S}^d} e_{n,k}(x) d\sigma(x).$$

Then the least squares quadrature is defined by

$$I_n(f) = \sum_{k=1}^{l_n} w_{n,k} f(x_{n,k}).$$

In this subsection we shall show that

$$I_n(f) = \int_{\mathbb{S}^d} L_n f(x) d\sigma(x) = \sum_{k=1}^{l_n} f(x_{n,k}) \tau_{n,k} \int_{\mathbb{S}^d} D(x_{n,k}, x) d\sigma(x).$$
In order to prove (2.7) it suffices to show that
\[ \tau_n^{1/2} e_{n,k} = \tau_n D(x_{n,k}, \cdot). \]
Indeed, by (2.4) and (2.5) we have for \( x, y \in S^d \)
\[
T_n(D_n(\cdot, y))(x) = \sum_{k=1}^{l_n} \tau_{n,k} D(x_{n,k}, y) E_n(x, x_{n,k}) = L_n(E(\cdot, x))(y) = E_n(x, y).
\]
It follows that
\[
\tau_n^{1/2} e_{n,k} = \tau_n T_n^{-1}(E_n(\cdot, x_{n,k})) = \tau_n D(x_{n,k}, \cdot).
\]
Hence, (2.7) holds. By (2.7) and the Hölder inequality we obtain that
\[
(2.8) \left| \int_{S^d} f(x) d\sigma(x) - I_n(f) \right| \leq \int_{S^d} |f(x) - L_n f(x)| d\sigma(x) \leq \|f - L_n f\|_2.
\]

2.5. Hyperinterpolation and the weighted least squares polynomials.

Hyperinterpolation was originally introduced by I. H. Sloan (see [22]). It uses the Fourier orthogonal projection of a function which can be expressed in the form of integrals, but approximates the integrals used in the expansion by means of a positive cubature formula. Hence, hyperinterpolation is a discretized orthogonal projection on polynomial subspaces and provides a polynomial approximation which relies only on a discrete set of data. In recent years, hyperinterpolation has attracted much interest, and a great number of interesting results have been obtained (see [3, 4, 5, 6, 10, 12, 14, 15, 20, 21, 22, 23, 25, 27, 28, 29, 30, 32]).

Assume that \( Q_n(f) = \sum_{k=1}^{l_n} \tau_{n,k} f(x_{n,k}) \), \( n = 1, 2, \ldots \) is a sequence of positive quadrature formulas on \( S^d \) which are exact for \( \Pi_{2n}^d \), i.e., \( \tau_{n,k} > 0 \), and for all \( f \in \Pi_{2n}^d \),
\[
\int_{S^d} f(x) d\sigma(x) = Q_n(f) = \sum_{k=1}^{l_n} \tau_{n,k} f(x_{n,k}).
\]

For any \( p \in \Pi_{2n}^d \), using \( f = p^2 \) in the above equality we obtain that the family \( X \) is a MZ family with the constants \( A = B = 1 \) and the global condition number \( \kappa \) is equal to 1.

The hyperinterpolation operator \( H_n \) on \( S^d \) is defined by
\[
H_n f(x) = \sum_{k=1}^{l_n} \tau_{n,k} f(x_{n,k}) E_n(x, x_{n,k}), \quad f \in C(S^d).
\]

We use the discretized inner product
\[
\langle f, g \rangle_{(n)} := \sum_{k=1}^{l_n} f(x_{n,k}) g(x_{n,k}) \tau_{n,k}
\]
and the discretized norm \( \|f\|^2_{(n)} = \langle f, f \rangle_{(n)} \). Since the quadrature formula \( Q_n \) is exact for \( \Pi_{2n}^d \), we get for all \( p, q \in \Pi_{2n}^d \),
\[
\langle p, q \rangle = \int_{S^d} p(x) q(x) d\sigma(x) = Q_n(p q) = \langle p, q \rangle_{(n)}.
\]
Hence,
\[ \{ Y_{\ell,k} \mid k = 1, \ldots, N(d, \ell), \ell = 0, 1, \ldots, n \} \]
forms an orthonormal basis of $\Pi_n^d$ with respect to $\langle \cdot, \cdot \rangle_{(n)}$. It follows that
\[ H_n f(x) = \langle f, E_n(\cdot, x) \rangle_{(n)} = \sum_{\ell=0}^{n} \sum_{k=1}^{N(d,\ell)} \langle f, Y_{\ell,k} \rangle_{(n)} Y_{\ell,k}(x), \]
which means that the hyperinterpolation operator $H_n$ is just the discretized orthogonal projection on the polynomial subspace $\Pi_n^d$ with respect to $\langle \cdot, \cdot \rangle_{(n)}$, and satisfies
\[ \| f - H_n f \|_{(n)}^2 = \min_{p \in \Pi_n^d} \| f - p \|_{(n)}^2 = \min_{p \in \Pi_n^d} \sum_{k=1}^{l_n} |f(x_{n,k}) - p(x_{n,k})|^2 \tau_{n,k}. \]
Hence, the hyperinterpolation $H_n f$ is just the weighted least squares polynomial for a sequence of positive quadrature formulas on $S^d$ which are exact for $\Pi_n^d$.

On the other hand, if the global condition number $\kappa$ of a MZ family $X$ with the weight $\tau$ is equal to 1, then for all $p \in \Pi_n^d$, we have
\[ A\|p\|_2^2 = \sum_{k=1}^{l_n} |p(x_{n,k})|^2 \tau_{n,k}. \]
It follows that for $f, g \in \Pi_n^d$,
\[ A\langle f, g \rangle = \sum_{k=1}^{l_n} f(x_{n,k})g(x_{n,k})\tau_{n,k}. \]

Since any function in $\Pi_{2n}^d$ can be expressed as a linear combination of the product of the functions $x^\alpha$ and $x^\beta$, we obtain that $X$ determines a sequence of positive quadrature formulas $Q_n(f) = \frac{1}{A} \sum_{k=1}^{l_n} \tau_{n,k}f(x_{n,k})$ on $S^d$ which are exact for $\Pi_{2n}^d$,

where, $x \in S^d$, $\alpha, \beta \in \mathbb{N}_{0}^{d+1}$, $|\alpha| \leq n$, $|\beta| \leq n$, $x^\alpha := x_1^{\alpha_1} \cdots x_{d+1}^{\alpha_{d+1}}$, $|\alpha| = \alpha_1 + \cdots + \alpha_{d+1}$. This means that the weighted least squares operators $L_n$ are just the hyperinterpolation operators $H_n$.

Hence, the hyperinterpolation is just the weighted least squares polynomial for a MZ family with the global condition number $\kappa = 1$, and the weighted least squares polynomial may be viewed as a generalization of the hyperinterpolation.

3. Proofs of Theorems 1.2 and 1.4

In order to prove Theorem 1.2, we shall use the following lemma.

Lemma 3.1. [2] Theorem 2.1] Let $\Gamma$ be a finite subset of $\mathbb{S}^d$, and let $\{ \mu_\omega : \omega \in \Gamma \}$ be a set of positive numbers satisfying
\[ \sum_{\omega \in \Gamma} \mu_\omega |f(\omega)|^{p_0} \leq C_1 \int_{\mathbb{S}^d} |f(x)|^{p_0} \, d\sigma(x), \quad \forall f \in \Pi_n^d, \]
for some $0 < p_0 < \infty$ and some positive integer $N$. If $0 < q < \infty$, $M \geq N$ and $f \in \Pi_M^d$, then
\[ \sum_{\omega \in \Gamma} \mu_\omega |f(\omega)|^q \leq CC_1 \left( \frac{M}{N} \right)^d \int_{\mathbb{S}^d} |f(y)|^q \, d\sigma(y), \]
where $C > 0$ depends only on $d$ and $q$. 
Proof of Theorem 1.2.
We rewrite the orthonormal basis \( \{ Y_{\ell,k} \mid k = 1, \ldots, N(d, \ell), \ \ell = 0, 1, \ldots, n \} \) of \( \Pi^d_n \) as \( \{ \phi_k \mid k = 1, \ldots, d_n \} \). Let
\[
\mathbf{y}_n = (\tau_{n,1}^{1/2} f(x_{n,1}), \ldots, \tau_{n,d_n}^{1/2} f(x_{n,d_n}))^* = (\mathbf{y}_{n,1}, \ldots, \mathbf{y}_{n,d_n})^*.
\]
be the given sampling data column vector, \( U_n \) be the \( l_n \times d_n \) matrix with entries
\[
(U_n)_{kl} = \tau_{n,k}^{1/2} \phi_l(x_{n,k}), \quad k = 1, \ldots, l_n, \ l = 1, \ldots, d_n,
\]
and let \( R_n = U_n^* U_n \), where \( U_n^* \) is the conjugate transpose of the matrix \( U_n \). Suppose that the solution to the weighted least squares problem (1.4) is the polynomial
\[
L_n f = \sum_{k=1}^{d_n} a_{n,k} \phi_k \in \Pi^d_n
\]
with coefficient column vector \( \mathbf{a}_n \). According to the standard formula for the solution of a least squares problem by means of the Moore-Penrose inverse \( U_n^+ = (U_n^* U_n)^{-1} U_n^* = R_n^{-1} U_n^* \), we obtain that
\[
\mathbf{a}_n = U_n^+ \mathbf{y}_n = R_n^{-1} U_n^* \mathbf{y}_n.
\]
For any \( c \in \mathbb{R}^{d_n}, p = \sum_{k=1}^{d_n} c_k \phi_k \), we have
\[
\|p\|_2^2 = |c|^2 \quad \text{and} \quad (R_n c, c) = (U_n c, U_n c) = \sum_{k=1}^n |p(x_n,k)|^2 \tau_{n,k}.
\]
It follows from (1.3) that
\[
A|c|^2 \leq (R_n c, c) \leq B|c|^2,
\]
which means that the spectrum of every \( R_n \) is contained in the interval \([A, B]\). We obtain that the operator norm of \( R_n^{-1} \) is bounded by \( A^{-1} \) and the operator norm of \( U_n^* \) is bounded by
\[
\|U_n^*\| = \|U_n\| = \|U_n^* U_n\|^{1/2} = \|R_n\|^{1/2} \leq B^{1/2},
\]
where the operator norm of a matrix \( A \) is defined by \( \|A\| = \sup_{\|x\|=1} |Ax| \).

We use the orthogonal decomposition
\[
\|f - L_n f\|_2^2 = \|f - S_n f\|_2^2 + \|S_n f - L_n f\|_2^2.
\]
We recall that
\[
S_n f = \sum_{k=1}^{d_n} f_{n,k} \phi_k
\]
with coefficient column vector \( \mathbf{f}_n \), where \( f_{n,k} = \langle f, \phi_k \rangle \). By the Parseval equality we obtain that
\[
\|S_n f - L_n f\|_2^2 = |\mathbf{f}_n - \mathbf{a}_n|^2 = |\mathbf{f}_n - R_n^{-1} U_n^* \mathbf{y}_n|^2
\]
\[
= |R_n^{-1} U_n^* (U_n \mathbf{f}_n - \mathbf{y}_n)|^2
\]
\[
\leq A^{-2} B |U_n \mathbf{f}_n - \mathbf{y}_n|^2.
\]
Finally, we have
\[
(U_n \mathbf{f}_n)_k = \tau_{n,k}^{1/2} S_n f(x_{n,k}).
\]
Hence,

\[ |U_n f_n - y_n|^2 = \sum_{k=1}^{l_n} |f(x_{n,k}) - S_n f(x_{n,k})|^2 \tau_{n,k} = \| f - S_n f \|^2_{(n)}. \]

It follows that

\[(3.1) \quad \| S_n f - L_n f \|_2 \leq A^{-2} B \| f - S_n f \|_2^{(n)} \]

For \( f \in H^\sigma(S^d), \ \sigma > d/2, \) we define

\[ A_0 f = S_n f, \quad A_k f = S_{2^k n} f - S_{2^{k-1} n} f \quad \text{for} \quad k \geq 1. \]

Then for \( f \in H^\sigma(S^d), \ \sigma > d/2, \) the series \( \sum_{k=0}^{\infty} A_k f(x) \) converges to \( f(x) \) uniformly in \( S^d. \) By (2.2) we have

\[(3.2) \quad \| A_k f \|_2 \leq \| f - S_{2^k n} f \|_2 + \| f - S_{2^{k-1} n} f \|_2 \ll 2^{-k \sigma} n^{-\sigma} \| f \|_{H^\sigma}. \]

Note that

\[ f(x) - S_n f(x) = \sum_{k=1}^{\infty} A_k f(x), \]

and the right series converges uniformly on \( S^d. \) Hence, by (3.1) and the trigonometric inequality we have

\[ \| S_n f - L_n f \|_2 \leq A^{-1} B^{1/2} \| f - S_n f \|_{(n)} \leq A^{-1} B^{1/2} \sum_{k=1}^{\infty} \| A_k f \|_{(n)}. \]

Note that \( A_k f \in \Pi_{2^k n}. \) Using Lemma 3.1 with \( p_0 = q = 2 \) and (3.2), we obtain that

\[ \| A_k f \|^2_{(n)} = \sum_{k=1}^{l_n} |A_k f(x_{n,k})|^2 \tau_{n,k} \]

\[ \leq c B \left( \frac{2^{k n}}{n} \right)^d \int_{S^d} |A_k f(x)|^2 d\sigma(x) \ll B 2^{kd/2} 2^k \sigma n^{-2\sigma} \| f \|^2_{H^\sigma}. \]

We have

\[ \| S_n f - L_n f \|_2 \leq A^{-1} B^{1/2} \sum_{k=1}^{\infty} \| A_k f \|_{(n)} \ll A^{-1} B \sum_{k=1}^{\infty} 2^{kd/2} 2^{-k \sigma} n^{-\sigma} \| f \|_{H^\sigma} \ll A^{-1} B n^{-\sigma} \| f \|_{H^\sigma}. \]

It follows that

\[ \| f - L_n f \|_2 = \left( \| f - S_n f \|_2^2 + \| S_n f - L_n f \|_2^2 \right)^{1/2} \ll (1 + \kappa^2)^{1/2} n^{-\sigma} \| f \|_{H^\sigma}. \]

This completes the proof of (1.5). Inequality (1.6) follows from (1.5) and (2.8) directly.

The proof of Theorem 1.2 is now finished. \( \square \)
Remark 3.2. The inequality (3.1) was proved in [11]. For the convenience of the reader, we provide details of the proof.

Proof of Theorem 1.4.

In order to give the lower estimate of the operator norm $\|L_n\|$, we use the Daugavet theorem to obtain that

$$\|L_n\| \geq \|S_n\| \approx n^{(d-1)/2}.$$ 

So it suffices to get the upper estimate of $\|L_n\|$.

Let the functions

$$\Phi_n(x) = \inf_{p \in \Pi_n^d, \|p\| = 1} \left\| f \right\|_2^2, \ x \in S^d,$$

and

$$\Psi_n(x) = \inf_{p \in \Pi_n^d, \|p\| = 1} \left\| f \right\|_{(n)}^2, \ x \in S^d$$

be the Christoffel functions with respect to the inner $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{(n)}$, respectively. We remark that estimates for Christoffel functions are useful in comparing different norms of functions in $\Pi_n^d$, and they are also basic tools in the theory of orthogonal polynomials.

It follows from (1.3) that

$$(3.3) \quad A \Phi_n(x) \leq \Psi_n(x) \leq B \Phi_n(x), \ x \in S^d.$$ 

Now let $E_n(x,y)$ and $D_n(x,y)$ be the reproducing kernels of $\Pi_n^d$ with respect to the inner $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{(n)}$, respectively. According to [11] Theorem 3.5.6, we have

$$E_n(x,x)^{-1} = \Phi_n(x) \quad \text{and} \quad D_n(x,x)^{-1} = \Psi_n(x).$$

It follows from (3.3) that

$$(3.4) \quad B^{-1} E_n(x,x) \leq D_n(x,x) \leq A^{-1} E_n(x,x), \ x \in S^d.$$ 

Using $p = 1$ in (1.3) we get that

$$(3.5) \quad A \leq \sum_{k=1}^{l_n} \tau_{n,k} \leq B.$$ 

Using (3.4), the Cauchy-Schwartz inequality, (2.5), (3.5), (3.4), and (2.1), we have

$$\|L_n\| = \max_{x \in S^d} \sum_{k=1}^{l_n} \tau_{n,k} |D_n(x,x_{n,k})|$$

$$\leq \max_{x \in S^d} \left( \sum_{k=1}^{l_n} \tau_{n,k} (D_n(x,x_{n,k}))^2 \right)^{1/2} \left( \sum_{k=1}^{l_n} \tau_{n,k} \right)^{1/2}$$

$$\leq B^{1/2} \max_{x \in S^d} D_n(x,x)^{1/2}$$

$$\ll \kappa^{1/2} \max_{x \in S^d} E_n(x,x)^{1/2}$$

$$\ll \kappa^{1/2} n^{d/2}.$$ 

This completes the proof of Theorem 1.4. □
References

[1] J. S. Brauchart, K. Hesse, Numerical integration over spheres of arbitrary dimension, Constr. Approx. 25(1) (2007) 41-71.
[2] G. Brown, F. Dai, Approximation of smooth functions on compact two-point homogeneous spaces, J. Funct. Anal. 220(2) (2005) 401-423.
[3] M. Caliari, S. De Marchi, R. Montagna, M. Vianello, Hyper2d: A numerical code for hyperinterpolation on rectangles, Appl. Math. Comput. 183 (2006) 1138-1147.
[4] M. Caliari, S. De Marchi, M. Vianello, Hyperinterpolation on the square, J. Comput. Appl. Math. 210 (1-2) (2007) 78-83.
[5] M. Caliari, S. De Marchi, M. Vianello, Hyperinterpolation in the cube, Comput. Math. Appl. 55 (2008) 2490-2497.
[6] F. Dai, On generalized hyperinterpolation on the sphere, Proc. Amer. Math. Soc. 134 (10) (2006), 2931–2941.
[7] F. Dai, Y. Xu, Approximation theory and harmonic analysis on spheres and balls, Springer Monographs in Mathematics, Springer, New York, 2013.
[8] I. K. Daugavet, Some applications of the Marcinkiewicz-Berman identity, Vestnik Leningrad Univ. Math. 1 (1974) 321-327.
[9] C.F. Dunkl, Y. Xu, Orthogonal Polynomials of Several Variables, Cambridge Univ. Press, 2001.
[10] T. Le Gia and I. H. Sloan, The uniform of hyperinterpolation on the unit sphere in an arbitrary number of dimension, Constr. Approx. 17(2) (2001) 249-265.
[11] K. Gröchenig, Sampling, Marcinkiewicz-Zygmund inequalities, approximation, and quadrature rules, J. Approx. Theory 257 (2020) 105455, 20pp.
[12] O. Hansen, K. Atkinson, D. Chien, On the norm of the hyperinterpolation operator on the unit disc and its use for the solution of the nonlinear Poisson equation,IMA J. Numer. Anal. 29 (2) (2009) 257-283.
[13] K. Hesse, A lower bound for the worst-case cubature error on spheres of arbitrary dimension, Numer. Math. 103(3) (2006) 413-433.
[14] K. Hesse, I. H. Sloan, Hyperinterpolation on the sphere, in: "Frontiers in Interpolation and Approximation (Dedicated to the memory of Ambikeshwar Sharma)" (editors: N.K.Govil, H.N.Mhaskar, R.N.Mohapatra, Z.Nashed, J.Szabados), Chapman & Hall/CRC, 2006 pp. 213-248.
[15] S. De Marchi, M. Vianello, Y. Xu, New quadrature formulas and hyperinterpolation in three variables, BIT Numer. Math. 49 (1) (2009) 55-73.
[16] J. Marzo, Marcinkiewicz-Zygmund inequalities and interpolation by spherical harmonics, J. Funct. Anal. 250 (2) (2007), 559-587.
[17] J. Marzo, J. Ortega-Cerdà, Equidistribution of Fekete points on the sphere, Constr. Approx. 32(3) (2010), 513-521.
[18] J. Marzo, B. Pridhnani, Sufficient conditions for sampling and interpolation on the sphere, Constr. Approx. 40 (2014), 241-257.
[19] H. N. Mhaskar, F. J. Narcowich, J. D. Ward, Spherical Marcinkiewicz-Zygmund inequalities and positive quadrature, Math. Comp. 70 (2001), 1113-1130 (Corrigendum: Math. Comp. 71 (2001) 453-454).
[20] M. Reimer, Hyperinterpolation on the sphere at the minimal projection order, J. Approx. Theory 104(2) (2000) 272-286.
[21] M. Reimer, Generalized hyperinterpolation on the sphere and the Newman-Shapiro operators, Constr. Approx. 18 (2002) 183-203.
[22] I. H. Sloan, Interpolation and hyperinterpolation over general regions, J. Approx. Theory 83 (1995) 238-254.
[23] I. H. Sloan, R. S. Womersley, Constructive polynomial approximation on the sphere, J. Approx. Theory 103(1) (2000) 91-118.
[24] I. H. Sloan, R. S. Womersley, Extremal systems of points and numerical integration on the sphere, Adv. Comput. Math. 21 (12) (2004) 107-125.
[25] I. H. Sloan, R. S. Womersley, Filtered hyperinterpolation: a constructive polynomial approximation on the sphere, Int. J. Geomath 3 (2012) 95-117.
[26] G. Szegő, Orthogonal polynomials, American Mathematical Society Colloquium Publications 23; Revised ed., American Mathematical Society, Providence, RI, 1959.
[27] J. Wade, On hyperinterpolation on the unit ball, J. Math. Anal. Appl. 401 (1) (2013) 140-145.
[28] H. Wang, Optimal lower estimates for the worst case quadrature error and the approximation by hyperinterpolation operators in the Sobolev space setting on the sphere, Int. J. Wavelets Multiresolut. Inf. Process, 7 (6) (2009) 813-823.
[29] H. Wang, Z. Huang, C. Li, L. Wei, On the norm of the hyperinterpolation operator on the unit ball, J. Approx. Theory 192 (2015) 132-143.
[30] H. Wang, I. H. Sloan, On filtered polynomial approximation on the sphere, J. Fourier Anal. Appl. 23 (2017) 863-876.
[31] H. Wang, K. Wang, Optimal recovery of Besov classes of generalized smoothness and Sobolev classes on the sphere, J. Complexity 32 (2016) 40-52.
[32] H. Wang, K. Wang, X. Wang, On the norm of the hyperinterpolation operator on the d-dimensional cube, Comput. Math. Appl. 68 (2014) 632-638.

School of Mathematical Sciences, Capital Normal University, Beijing 100048, China.
Email address: luwanting1234@163.com

School of Mathematical Sciences, Capital Normal University, Beijing 100048, China.
Email address: wanghp@cnu.edu.cn.