LOW REGULARITY SOLUTIONS FOR THE VLASOV-POISSON-LANDAU/BOLTZMANN SYSTEM

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ABSTRACT. In the paper, we are concerned with the nonlinear Cauchy problem on the Vlasov-Poisson-Landau/Boltzmann system around global Maxwellians in torus or finite channel. The main goal is to establish the global existence and large time behavior of small amplitude solutions for a class of low regularity initial data. The molecular interaction type is restricted to the case of hard potentials for two classical collision operators because of the effect of the self-consistent forces. The result extends the one by Duan-Liu-Sakamoto-Strain [Comm. Pure Appl. Math. 74 (2021), no. 5, 932–1020] for the pure Landau/Boltzmann equation to the case of the VPL and VPB systems.

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1. Introduction

1.1. Equations. We consider the Vlasov-Poisson-Boltzmann (VPB) and Vlasov-Poisson-Landau (VPL) systems describing the motion of plasma particles of two species (e.g. ions and electrons) in a domain \( \Omega \subset \mathbb{R}^3 \):

\[
\begin{align*}
\partial_t F_+ + v \cdot \nabla_x F_+ - \nabla_x \phi \cdot \nabla_v F_+ &= Q(F_+, F_+) + Q(F_+, F_+), \\
\partial_t F_- + v \cdot \nabla_x F_- + \nabla_x \phi \cdot \nabla_v F_- &= Q(F_+, F_-) + Q(F_-, F_-), \\
- \Delta_x \phi &= \int_{\mathbb{R}^3} (F_+ - F_-) \, dv, \\
F_{\pm}(0, x, v) &= F_{0, \pm}(x, v).
\end{align*}
\]

(1.1)

Here, the unknowns \( F_{\pm}(t, x, v) \geq 0 \) stand for the velocity distribution functions for the particles of ions (+) and electrons (−), respectively, at position \( x \in \Omega \) and velocity \( v \in \mathbb{R}^3 \) and time \( t \geq 0 \). For later use we introduce the self-consistent electrostatic field taking the form

\[ E(t, x) = -\nabla_x \phi(t, x), \]

where \( \phi(t, x) \) is the potential function given in terms of the Poisson equation.

For the case of VPL system, the collision operator \( Q \) is given by

\[ Q(G, F) = \nabla_v \cdot \int_{\mathbb{R}^3} \phi(v - v') [G(v') \nabla_v F(v) - F(v) \nabla_v G(v')] \, dv', \]
\[ \sum_{i,j=1}^{3} \partial_{v_i} \int_{\mathbb{R}^3} \phi^{ij}(v-v') \left[ G(v') \partial_{v_j} F(v) - F(v) \partial_{v_j} G(v') \right] dv'. \]  

(1.2)

The non-negative definite matrix-valued function \( \phi = [\phi^{ij}(v)]_{1 \leq i,j \leq 3} \) takes the form of

\[ \phi^{ij}(v) = \left\{ \delta_{ij} - \frac{v_i v_j}{|v|^2} \right\} |v|^\gamma + 2, \]

with \( \gamma \geq -3 \). It is convenient to call it hard potential when \( \gamma \geq -2 \) and soft potential when \(-3 \leq \gamma < -2 \).

The case \( \gamma = -3 \) corresponds to the physically realistic Coulomb interactions; cf. [18].

For the case of VPB system, the collision operator \( Q \) is defined by

\[ Q(G, F) = \int_{\mathbb{R}^3} \int_{S^2} B(v-v, \sigma) \left[ G(v') F(v) - G(v) F(v') \right] d\sigma dv. \]  

(1.3)

Here, \( v, v_* \) and \( v', v_*' \) are velocity pairs given in terms of the \( \sigma \)-representation by

\[ v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v_*' = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma, \quad \sigma \in S^2, \]

that satisfy \( v + v_* = v' + v_*' \) and \( |v|^2 + |v_*|^2 = |v'|^2 + |v_*'|^2 \). The Boltzmann collision kernel \( B(v, v, \sigma) \) depends only on \( |v - v_*| \) and the deviation angle \( \theta \) through \( \cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma \). Without loss of generality we can assume \( B(v, v, \sigma) \) is supported on \( 0 \leq \theta \leq \pi/2 \), since one can reduce the situation with symmetrization: \( \overline{B}(v, v, \sigma) = B(v, v, \sigma) + B(v, v, -\sigma) \). Moreover, \( B(v, v, \sigma) = |v - v_*|^2 b(\cos \theta) \), and we assume that there exist \( C_0 > 0 \) and \( 0 < s < 1 \) such that

\[ \frac{1}{C_0 \theta^{1+2s}} \leq \sin \theta b(\cos \theta) \leq \frac{C_0}{\theta^{1+2s}}, \quad \forall \theta \in (0, \pi/2). \]

It is convenient to call it hard potential when \( \gamma + 2s \geq 0 \) and soft potential when \(-3 < \gamma + 2s < 0 \).

1.2. Reformulation. We will reformulate the problem (1.1) near a global Maxwellian. For this, consider the global Maxwellian equilibrium state:

\[ \mu = \mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}. \]

We will look for a solution to (1.1) of the form

\[ F(t, x, v) = \mu + \mu^{1/2} f(t, x, v). \]

Then \( f = f(t, x, v) \) satisfies

\[ \begin{cases} 
\partial_t f_\pm + v \cdot \nabla_x f_\pm + \frac{1}{2} \nabla_x \phi \cdot v f_\pm + \nabla_x \phi \cdot \nabla_v f_\pm + \nabla \phi \cdot v f_\pm + L_\pm f = \Gamma_\pm(f, f), \\
- \Delta_x \phi = \int_{\mathbb{R}^3} (f_+ - f_-) \mu^{1/2} dv, \\
f(0, x, v) = f_0(x, v),
\end{cases} \]  

(1.4)

where the linearized collision operator \( L = [L_+, L_-] \) and nonlinear collision operator \( \Gamma = [\Gamma_+, \Gamma_-] \) are given respectively by

\[ L_\pm f = \mu^{-1/2} \left\{ 2Q(\mu, \mu^{1/2} f_\pm) + Q(\mu^{1/2} (f_\pm + f_\mp), \mu) \right\}, \]

and

\[ \Gamma_\pm(f, g) = \mu^{-1/2} \left\{ Q(\mu^{1/2} f_\pm, \mu^{1/2} g_\mp) + Q(\mu^{1/2} f_\mp, \mu^{1/2} g_\pm) \right\}. \]

The kernel of \( L \) on \( L^2_\mu \times L^2_\mu \) is the span of

\[ \{ [1, 0] \mu^{1/2}, [0, 1] \mu^{1/2}, [1, 1] \mu^{1/2}, [1, 1] |v|^2 \mu^{1/2} \}, \]

and we define the projection of \( L^2_\mu \times L^2_\mu \) onto \( \text{ker} L \) to be

\[ Pf = \left( a_+(t, x)[1, 0] + a_-(t, x)[0, 1] + v \cdot b(t, x)[1, 1] + (|v|^2 - 3)c(t, x)[1, 1] \right) \mu^{1/2}, \]

or equivalently by

\[ P_\pm f = \left( a_\pm(t, x) + v \cdot b(t, x) + (|v|^2 - 3)c(t, x) \right) \mu^{1/2}. \]

Then for given \( f \), one can decompose \( f \) uniquely as

\[ f = Pf + (I - P)f. \]
The function $a_{\pm}, b, c$ are given by

$$
\begin{align*}
& a_{\pm} = (\mu^{1/2}, f_{\pm})_{L^2_x}, \\
& b_j = \frac{1}{2}(v_j \mu^{1/2}, f_+ + f_-)_{L^2_x}, \\
& c = \frac{1}{12}((|v|^2 - 3)\mu^{1/2}, f_+ + f_-)_{L^2_x}.
\end{align*}
$$

1.3. Spatial domain and boundary condition. In this paper, we focus on two kinds of specific domains $\Omega \subset \mathbb{R}^3$, either torus or finite channel. In what follows we give their definitions and the corresponding conservation laws to equation (1.4).

1.3.1. Case I: Torus. In this case, we let

$$
\Omega = \mathbb{T}^3 = [-\pi, \pi]^3.
$$

Correspondingly, $F(t, x, v)$ is assumed to be spatially periodic in $x$. We also assume $\int_{\mathbb{T}^3} \phi(t, x) dx = 0$ for any $t \geq 0$. It’s well known that if the following

$$
\begin{align*}
& \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_+(t) \mu^{1/2} dv dx = \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f_-(t) \mu^{1/2} dv dx = 0, \\
& \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} (f_+(t) + f_-(t)) v \mu^{1/2} dv dx = 0, \\
& \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} (f_+(t) + f_-(t)) |v|^2 \mu^{1/2} dv dx + \int_{\mathbb{T}^3} |E(t)|^2 dx = 0,
\end{align*}
$$

holds initially at $t = 0$, then we have conservation laws (1.5) for the solution $f(t, x, v)$ for any $t \geq 0$.

1.3.2. Case II: Finite channel. In this case, we set

$$
\Omega = [-1, 1] \times \mathbb{T}^2 = \{ x = (x_1, \hat{x}) : \ x_1 \in [-1, 1], \ \hat{x} := (x_2, x_3) \in \mathbb{T}^2 = [-\pi, \pi]^2 \}.
$$

Correspondingly, $f(t, x, v)$ is assumed to be spatially periodic for $\hat{x}$ and satisfies the following specular reflection boundary condition at $x_1 = \pm 1$:

$$
\begin{align*}
& f(t, -1, \hat{x}, v_1, \bar{v})|_{v_1 > 0} = f(t, -1, \hat{x}, -v_1, \bar{v}), \\
& f(t, 1, \hat{x}, v_1, \bar{v})|_{v_1 < 0} = f(t, 1, \hat{x}, -v_1, \bar{v}),
\end{align*}
$$

where $v = (v_1, \bar{v}) \in \mathbb{R}^3$. For the case of finite channel, we further assume

$$
\partial_{x_1} \phi = 0, \quad \text{on} \ x_1 = \pm 1, \quad (1.7)
$$

for Poisson equation to $\phi$. Due to conservation laws similar to the torus case, we assume that $f(t, x, v)$ satisfies

$$
\begin{align*}
& \int_{[-1, 1] \times \mathbb{T}^2} \int_{\mathbb{R}^3} f_+(t) \mu^{1/2} dv dx = \int_{[-1, 1] \times \mathbb{T}^2} \int_{\mathbb{R}^3} f_-(t) \mu^{1/2} dv dx = 0, \\
& \int_{[-1, 1] \times \mathbb{T}^2} \int_{\mathbb{R}^3} (f_+(t) + f_-(t)) v_i \mu^{1/2} dv dx = 0, \\
& \int_{[-1, 1] \times \mathbb{T}^2} \int_{\mathbb{R}^3} (f_+(t) + f_-(t)) |v|^2 \mu^{1/2} dv dx + \int_{[-1, 1] \times \mathbb{T}^2} |E(t)|^2 dx = 0,
\end{align*}
$$

(1.8)

with all $t \geq 0$.

This work concerns the low regularity global solution and large time asymptotic behavior for the VPB and VPL systems in torus or finite channel as above. The high regularity solution for Boltzmann and Landau equations near a global Maxwellian has been well-studied in the last few decades. For the cutoff Boltzmann case, Caffisch [2, 3] and Ukai-Asano [33] gave the global solution to Cauchy problem in torus. For the non-cutoff Boltzmann case, we refer to the work AMUXY [1] and Gressman-Strain [17]. For Landau case, we refer to Guo [18]. If the Boltzmann and Landau equations are combined with the self-consistent electrical or electromagnetic fields, that is, the Poisson and Maxwell equations, we refer to [10, 13, 19, 20, 22, 29, 30, 32, 34]. For the low regularity solution, one may go back to the classic work by DiPerna-Lions [7], which constructed the renormalized solution by using weak compactness method in $L^1$ framework. For the framework near a global Maxwellian, Duan-Huang-Wang-Yang [9] gave the global well-posedness of cutoff Boltzmann equation with a class of large amplitude data and Duan-Wang [14] generalized it to the case of general bounded domains with diffusive reflection boundaries. Recently, for the non-cutoff Boltzmann and Landau equations, Duan-Liu-Sakamoto-Strain [11] studied the global
mild solutions for small-amplitude initial data in space $L^1_h L^2_w$ with very low regularity. In this work, we expect to generalize this result to the case of the VPB or VPL system. Also, we consider the VPB and VPL system in finite channel with the physically important specular-reflection boundary condition.

For the boundary value theory of collisional kinetic problems such as on Landau and Boltzmann equations, we refer to [4, 5, 8, 15, 21, 23–26, 28, 29, 36]. Since the fundamental work by Guo [21] using an $L^2 - L^\infty$ method, many results have been developed for Boltzmann equation and Landau equation. For instance, Guo-Kim-Tonon-Trescases [24] gave regularity of cutoff Boltzmann equation with several physical boundary conditions. Esposito-Guo-Kim-Marra [15] constructed a non-equilibrium stationary solution and studied the exponential asymptotic stability. Kim-Lee [26] studied cutoff Boltzmann equation with specular boundary condition with external potential. Liu-Yang [27] extended the result in [21] to cutoff soft potential case. Cao-Kim-Lee [4] proved the global existence for Vlasov-Poisson-Boltzmann with diffuse boundary condition. Guo-Hwang-Jang-Ouyang [23] gave the global stability of Landau equation with specular reflection boundary. Duan-Liu-Sakamoto-Strain [11] proved the global existence for Landau and non-cutoff Boltzmann equation in finite channel. Dong-Guo-Ouyang [8] established the global existence for VPB system in general bounded domain with specular boundary condition.

In this work, we would give the global well-posedness of VPB and VPL systems in torus and finite channel in a function space with very low regularity, related to the Wiener space $A(T^3) = L^1_h$ given in (1.9). We shall establish the global existence and exponential time decay for such low regularity initial data in the case of hard potentials.

1.4. Notations. We first give some notations throughout the paper. Let $I$ be the identity mapping. Set $\langle v \rangle = \sqrt{1 + |v|^2}$. $1_S$ is the indicator function on a set $S$. $\langle \cdot \rangle$ denotes the inner product in $\mathbb{C}$. Let $\partial^{\alpha}_{\nu} = \partial_{\nu_1}^{\alpha_1} \partial_{\nu_2}^{\alpha_2} \cdots \partial_{\nu_k}^{\alpha_k}$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ are multi-indices. If each component of $\beta'$ is not greater than that of $\beta$'s, we denote by $\beta' \leq \beta$. The notation $a \approx b$ (resp. $a \lesssim b$, $a \lesssim b$) for positive real function $a$ and $b$ means that there exists $C > 0$ not depending on possible free parameters such that $C^{-1} a \leq b \leq C a$ (resp. $a \geq C^{-1} b$, $a \leq C b$) on their domain. We will write $C > 0$ (large) to be a generic constant, which may change from line to line. Denote the $L^2_v$ and $L^2_{x,v}$, respectively, as

$$|f|^2_{L^2_v} = \int_{\mathbb{R}^3} |f|^2 dv, \quad \|f\|_{L^2_{x,v}}^2 = \int_{\Omega} |f|^2_{L^2_v} dx.$$ 

For any $m \geq 0$, we denote spaces $L^1_k$, $L^2_k$ and their weighted forms $L^1_{k,m}$, $L^2_{k,m}$ respectively by

$$\|\widehat{f}\|_{L^1_k}^2 = \int_{\mathbb{R}^3} |\widehat{f}(k)|^2 dk, \quad \|\widehat{f}\|_{L^2_k}^2 = \int_{\mathbb{R}^2} |\widehat{f}(k)|^2 dk,$$

$$\|\widehat{f}\|_{L^1_{k,m}}^2 = \int_{\mathbb{R}^2} \langle k \rangle^m |\widehat{f}(k)|^2 dk, \quad \|\widehat{f}\|_{L^2_{k,m}}^2 = \int_{\mathbb{R}^2} \langle k \rangle^m |\widehat{f}(k)|^2 dk.$$ 

To capture the dissipation rate, for Landau case, we denote

$$\sigma^{ij}(v) = \phi^{ij} \ast \mu = \int_{\mathbb{R}^3} \phi^{ij}(v - v') \mu(v') dv'$$

$$\sigma^i(v) = \sum_{j=1}^3 \sigma^{ij} \frac{v_j}{2} = \sum_{j=1}^3 \phi^{ij} \ast \left[ \frac{v_j}{2} \right].$$

Define

$$|f|^2_{L^2_{D,w}} = \int_{i,j=1}^3 \int_{\mathbb{R}^3} w^2 (\sigma^{ij}_v \partial_v f \partial_v f + \sigma^{ij}_v \frac{v_j}{2} |f|^2) dv, \quad \|f\|_{L^2_{L^2_{D,w}}}^2 = \int_{\Omega} |f|^2_{L^2_{D,w}} dx,$$

and $|\widehat{f}(k)|^2_{L^2_k} = |\widehat{f}(k)|^2_{L^2_{k,1}}$. Then from [18, Corollary 1, p.399], we have

$$|f|^2_{L^2_{D,w}} = |w(v)P_v \nabla_v f|^2_{L^2_v} + |w(v)\frac{\nabla_v \mu}{\mu} (I - P_v) \nabla_v f|^2_{L^2_v} + |w(v)| \frac{\nabla_v f}{\mu} f|^2_{L^2_v},$$

where $P_v$ is the projection along the direction of $v$.

For Boltzmann case, as in [17], we denote

$$|f|^2_{L^2_D} := \langle v \rangle^\gamma f^2_{L^2_v} + \int_{\mathbb{R}^3} dv \langle v \rangle^{\gamma + 2s + 1} \int_{\mathbb{R}^3} dv' \frac{(f' - f)^2}{d(v,v')^{3+2s}} 1_{d(v,v') \leq 1},$$
and

\[ |f|^2_{L^2_{b,w}} = |w|f|^2_{L^2_{b,w}}, \quad \|f\|^2_{L^2_{b,w}} = \int_\Omega |f|^2_{L^2_{b,w}} \, dx. \]

The fractional differentiation effect is measured using the anisotropic metric on the lifted paraboloid

\[ d(v,v') = \sqrt{\frac{1}{2}(|v|^2 - |v'|^2)^2} \]

Then by \([1, \text{Proposition 2.2}], \) we have

\[ |(v)^2(D_v)^sf|^2_{L^2_v} \leq |f|^2_{L^2_{b,w}} \lesssim \|v\|^2_{L^2_{b,w}} \lesssim |(v)^2(D_v)^sf|^2_{L^2_v}. \]

1.5. **Function space.** To study the global well-posedness of problem (1.4) in different domains, we will consider the following function spaces and energy functionals. We will consider

\[ \gamma \geq -2, \quad \text{for VPL case}, \]

\[ \gamma + 2s \geq 1, \quad \frac{1}{2} \leq s < 1, \quad \text{for VPB case}. \]

We will make use of weight function:

\[ w = w(t,v) := \exp \left( \frac{q(v)}{(1 + t)^N} \right), \]

where \( q \geq 0 \) and we restrict \( 0 < q < 1/8 \) when \( \theta = 2 \). We let \( \theta = -\gamma \) for VPL case when \( -2 \leq \gamma < -1 \) and \( q = 0 \) when \( \gamma \geq -1 \) for VPL case. For the VPB case, we let \( q = 0 \) when \( \gamma + 2s \geq 1 \) and \( \frac{1}{2} \leq s < 1 \). Therefore, such weight \( w \) in (1.10) will be necessarily included only for the VPL case with \( -2 \leq \gamma < -1 \) giving that \( \theta \in [1,2] \).

1.5.1. **Case I: Torus.** For the torus case, to derive the low regularity solution, as in \([11], \) we will consider the Weiner space \( L^1_v = A(T^3), \) where \( k \) is the Fourier variable with respect to \( x. \) In order to close the energy estimate with time integral, we consider

\[ L^1_v L^2_{T_v} L^2_v, \]

with norm

\[ \|\hat{f}\|_{L^1_v L^2_{T_v} L^2_v} := \int_\mathbb{Z}^{3} \sup_{0 \leq t \leq T} \|\hat{f}(t,k)\|_{L^2_v} \, d\Sigma(k) < \infty, \]

where the Fourier transform of \( f(t,x,v) \) with respect to \( x \in \mathbb{T}^3 \) is denoted by

\[ \hat{f}(t,k,v) = \int_{\mathbb{T}^3} e^{-ik \cdot x} f(t,x,v) \, dx, \quad k \in \mathbb{Z}^3. \]

Here we denote \( d\Sigma(k) \) to be the discrete measure on \( \mathbb{Z}^3, \) namely,

\[ \int_{\mathbb{Z}^3} g(k) \, d\Sigma(k) = \sum_{k \in \mathbb{Z}^3} g(k). \]

To obtain the global existence of (1.4), we will denote the “total energy functional” \( E_T \) and “dissipation rate functional” \( D_T \) respectively by

\[ E_T = \|e^{\delta t} \hat{f}\|_{L^1_v L^\infty_{T_v} L^2_v} + \|e^{\delta t} \hat{E}\|_{L^1_v L^\infty_{T_v}}, \]

and

\[ D_T = \|e^{\delta t} \hat{f}\|_{L^1_v L^2_{T_v} L^2_v} + \|e^{\delta t} \hat{E}\|_{L^1_v L^2_{T_v}}, \]

for any \( T > 0 \) and some small constant \( \delta > 0 \) to be chosen in Theorem 1.1. Their weighted form \( E_{T,w} \) and \( D_{T,w} \) are given by

\[ E_{T,w} := \|e^{\delta t} w \hat{f}\|_{L^1_v L^\infty_{T_v} L^2_v}, \]

and

\[ D_{T,w} := \|e^{\delta t} w \hat{f}\|_{L^1_v L^2_{T_v} L^2_v} + \sqrt{qN} \|v\|^\theta (1 + t)^{-\frac{N+1}{2}} \cdot e^{\delta t} w \hat{f}\|_{L^1_v L^2_{T_v} L^2_v}, \]

where \( w \) is given in (1.10).
1.5.2. Case II: Finite channel. For finite channel case, we will also use weight function (1.10) and define function space

\[ L^1_k L^\infty_0 L^2_0, \]

with norm

\[ \| f \|_{L^1_k L^\infty_0 L^2_0} := \int T^2 \sup_{0 \leq t \leq T} \| \hat{f}(t, x, \hat{k}) \|_{L^2_0} \, d\hat{k} < \infty. \]

where the Fourier transform of \( f(t, x, \hat{k}) \) with respect to \( \hat{x} \in T^2 \) is denoted by

\[ \hat{f}(t, x, \hat{k}) = \int T^2 e^{-i\hat{k} \cdot \hat{x}} f(t, x, \hat{x}) \, d\hat{x}, \quad \hat{k} \in T^2. \]

In the case of finite channel, we need to include an extra first-order derivative in \( x \) and thus we define the “total energy functional” \( \mathcal{E}_T \) and “dissipation rate functional” \( \mathcal{D}_T \) respectively by

\[ \mathcal{E}_T = \sum_{|\alpha| \leq 1} \left( \| e^{\delta t} \partial_x^\alpha f \|_{L^1_k L^\infty_0 L^2_0}^2 + \| e^{\delta t} \partial_x^\alpha \hat{E} \|_{L^1_k L^\infty_0 L^2_0}^2 \right), \]

and

\[ \mathcal{D}_T = \sum_{|\alpha| \leq 1} \left( \| e^{\delta t} \partial_x^\alpha f \|_{L^1_k L^\infty_0 L^2_0}^2 + \| e^{\delta t} \partial_x^\alpha \hat{E} \|_{L^1_k L^\infty_0 L^2_0}^2 \right). \]

Here, in the \( x_1 \) direction, we use \( L^2_0 \) space instead of \( L^1_k \). Moreover, their weighted forms are denoted respectively by

\[ \mathcal{E}_{T,w} = \sum_{|\alpha| \leq 1} \| e^{\delta t} w \partial_x^\alpha f \|_{L^1_k L^\infty_0 L^2_0}^2, \]

and

\[ \mathcal{D}_{T,w} = \sum_{|\alpha| \leq 1} \left( \sqrt{qN} \| \langle h \rangle \|_{L^2_0}^2 (1 + t) \| e^{\delta t} w \partial_x^\alpha f \|_{L^1_k L^\infty_0 L^2_0}^2 + \| e^{\delta t} \partial_x^\alpha \hat{E} \|_{L^1_k L^\infty_0 L^2_0}^2 \right) \]

1.6. Main results. In this section, we state our main results on global well-posedness of VPL and VPB systems. We will consider \( \gamma \geq -2 \) for VPL system and \( \gamma + 2s \geq 1, 1/2 \leq s < 1 \) for VPB system for the case of either torus or finite channel.

1.6.1. Case of torus. Here we state our main result on the VPL and VPB systems in torus. In this subsection, we denote \( \gamma \) to be the Fourier transform on \( x \in T^3 \).

**Theorem 1.1** (Existence and large-time behavior). Let \( \Omega = T^3 \) and \( w \) be chosen by (1.10). Assume that \( f_0(x, v) \) satisfies (1.5) with \( t = 0 \). Then the followings hold.

1. For the VPL system with \(-2 \leq \gamma < -1\), there exist \( \varepsilon_0, \delta > 0 \) such that if \( F_0(x, v) = \mu + \mu^{1/2} f_0(x, v) \geq 0 \) and

\[ \| \hat{f}_0 \|_{L^1_k L^2_0} + \| \hat{E}_0 \|_{L^1_k} \leq \varepsilon_0, \]

then there exists a unique global mild solution \( f = f(t, x, v) \) to the problem (1.4) and (1.2) satisfying that \( F(t, x, v) = \mu + \mu^{1/2} f(t, x, v) \geq 0 \) and any \( T > 0 \),

\[ \mathcal{E}_T + \mathcal{D}_T + \mathcal{E}_{T,w} + \mathcal{D}_{T,w} \leq \| \hat{f}_0 \|_{L^1_k L^2_0} + \| \hat{E}_0 \|_{L^1_k}, \]

where \( \mathcal{E}_T, \mathcal{D}_T, \mathcal{E}_{T,w}, \mathcal{D}_{T,w} \) are defined in (1.11), (1.12), (1.13) and (1.14) respectively. In particular, one has the rate of convergence:

\[ \| \hat{f}(t) \|_{L^1_k L^2_0} + \| \hat{E}(t) \|_{L^1_k} \leq e^{-\delta t} \left( \| \hat{f}_0 \|_{L^1_k L^2_0} + \| \hat{E}_0 \|_{L^1_k} \right). \]

2. For the VPL system with \( \gamma \geq -1 \) and VPB system with \( \gamma + 2s \geq 1 \) and \( s \leq s < 1 \), there exist \( \varepsilon_0, \delta > 0 \) such that if \( F_0(x, v) = \mu + \mu^{1/2} f_0(x, v) \geq 0 \) and

\[ \| \hat{f}_0 \|_{L^1_k L^2_0} + \| \hat{E}_0 \|_{L^1_k} \leq \varepsilon_0, \]

then there exists a unique global mild solution \( f(t, x, v) \) to the problem (1.4) satisfying (1.2) or (1.3) with \( F(t, x, v) = \mu + \mu^{1/2} f(t, x, v) \geq 0 \) and for any \( T > 0 \),

\[ \mathcal{E}_T + \mathcal{D}_T \leq \| \hat{f}_0 \|_{L^1_k L^2_0} + \| \hat{E}_0 \|_{L^1_k}. \]

Moreover, one also has the rate of convergence:

\[ \| \hat{f}(t) \|_{L^1_k L^2_0} + \| \hat{E}(t) \|_{L^1_k} \leq e^{-\delta t} \left( \| \hat{f}_0 \|_{L^1_k L^2_0} + \| \hat{E}_0 \|_{L^1_k} \right). \]
Notice that the constants in the above estimates are independent of $T$. For higher spatial regularity, we prove that the initial regularity preserves over time.

**Theorem 1.2** (Propagation of spatial regularity). Let all conditions in Theorem 1.1 be satisfied and $T, m \geq 0$. Then the followings hold.

(1) For the VPL systems with $-2 \leq \gamma < -1$, there exist $\varepsilon_0, \delta > 0$ such that if

$$||w\tilde{f}_0||_{L^1_{k,m}L^2_x} + ||\tilde{E}_0||_{L^1_{k,m}} \leq \varepsilon_0,$$

then the solution $f$ to (1.4) and (1.2) established in Theorem 1.1 satisfies

$$\|e^{\delta t}w\tilde{f}\|_{L^1_{k,m}L^2_x} + \|e^{\delta t}\tilde{E}\|_{L^1_{k,m}L^2_x} \lesssim \|w\tilde{f}_0\|_{L^1_{k,m}L^2_x} + \|\tilde{E}_0\|_{L^1_{k,m}}.$$

(1.20)

(2) For the VPL system with $\gamma \geq -1$ and VPB system with $\gamma + 2s \geq 1$ and $\frac{1}{2} \leq s < 1$, there exist $\varepsilon_0, \delta > 0$ such that if

$$\|\tilde{f}_0\|_{L^1_{k,m}L^2_x} + \|\tilde{E}_0\|_{L^1_{k,m}} \leq \varepsilon_0,$$

then the solution $f$ to the problem (1.4) satisfying (1.2) or (1.3) established in Theorem 1.1 satisfies

$$\|e^{\delta t}\tilde{f}\|_{L^1_{k,m}L^\infty_x} + \|e^{\delta t}\tilde{E}\|_{L^1_{k,m}L^\infty_x} \lesssim \|\tilde{f}_0\|_{L^1_{k,m}L^\infty_x} + \|\tilde{E}_0\|_{L^1_{k,m}}.$$

**Theorem 1.3.** Let $\Omega = [-1, 1] \times T^2$ and $w$ be chosen by (1.10). Assume that $f_0(x, v)$ satisfies (1.8) with $t = 0$. Then the followings hold.

(1) For the VPL system with $-2 \leq \gamma < -1$, there exist $\varepsilon_0, \delta > 0$ such that if $F_0(x_1, \tilde{x}, v) = \mu + \mu^{1/2}f_0(x_1, \tilde{x}, v) \geq 0$ and

$$\sum_{|\alpha| \leq 1} \left( \|\tilde{w}^{\partial^\alpha}f_0\|_{L^1_{k,m}L^2_{x,v}}^2 + \|\partial^\nu\tilde{E}_0\|_{L^1_{k,m}L^2_{x,v}}^2 \right) \leq \varepsilon_0,$$

then there exists a unique mild solution $f(t, x_1, \tilde{x}, v)$ to the initial boundary value problem with specular reflection boundary (1.4), (1.6), (1.7) and (1.2), satisfying that $F(t, x_1, \tilde{x}, v) = \mu + \mu^{1/2}f(t, x_1, \tilde{x}, v) \geq 0$ and for any $T > 0$,

$$\mathcal{E}_{T,w} + D_{T,w} + E_T + D_T \lesssim \sum_{|\alpha| \leq 1} \left( \|\tilde{w}^{\partial^\alpha}f_0\|_{L^1_{k,m}L^2_{x,v}}^2 + \|\partial^\nu\tilde{E}_0\|_{L^1_{k,m}L^2_{x,v}}^2 \right),$$

(1.22)

where $\mathcal{E}_T, D_T, \mathcal{E}_{T,w}, D_{T,w}$ are defined in (1.15), (1.16), (1.17) and (1.18), respectively. In particular, one has the rate of convergence:

$$\sum_{|\alpha| \leq 1} \left( \|\tilde{w}^{\partial^\alpha}f_0\|_{L^1_{k,m}L^2_{x,v}}^2 + \|\partial^\nu\tilde{E}_0\|_{L^1_{k,m}L^2_{x,v}}^2 \right) \lesssim e^{-\delta t} \sum_{|\alpha| \leq 1} \left( \|\tilde{w}^{\partial^\alpha}f_0\|_{L^1_{k,m}L^2_{x,v}}^2 + \|\partial^\nu\tilde{E}_0\|_{L^1_{k,m}L^2_{x,v}}^2 \right).$$

(2) For the VPL system with $\gamma \geq -1$ and VPB system with $\gamma + 2s \geq 1$ and $\frac{1}{2} \leq s < 1$, there exist $\varepsilon_0, \delta > 0$ such that if $F_0(x_1, \tilde{x}, v) = \mu + \mu^{1/2}f_0(x_1, \tilde{x}, v) \geq 0$ and

$$\sum_{|\alpha| \leq 1} \left( \|\tilde{w}^{\partial^\alpha}f_0\|_{L^1_{k,m}L^2_{x,v}}^2 + \|\partial^\nu\tilde{E}_0\|_{L^1_{k,m}L^2_{x,v}}^2 \right) \leq \varepsilon_0,$$

(1.23)

then there exists a unique mild solution $f(t, x_1, \tilde{x}, v)$ to the initial boundary value problem with specular reflection boundary (1.4), (1.6) and (1.7) satisfying (1.2) or (1.3) such that $F(t, x_1, \tilde{x}, v) = \mu + \mu^{1/2}f(t, x_1, \tilde{x}, v) \geq 0$ and for any $T > 0$,

$$\mathcal{E}_T + D_T \lesssim \sum_{|\alpha| \leq 1} \left( \|\tilde{w}^{\partial^\alpha}f_0\|_{L^1_{k,m}L^2_{x,v}}^2 + \|\partial^\nu\tilde{E}_0\|_{L^1_{k,m}L^2_{x,v}}^2 \right).$$

In particular, one has the large time behavior:

$$\|\tilde{w}^{\partial^\alpha}f\|_{L^1_{k,m}L^2_{x,v}} + \|\partial^\nu\tilde{E}\|_{L^1_{k,m}L^2_{x,v}} \lesssim e^{-\delta t} \sum_{|\alpha| \leq 1} \left( \|\tilde{w}^{\partial^\alpha}f_0\|_{L^1_{k,m}L^2_{x,v}} + \|\partial^\nu\tilde{E}_0\|_{L^1_{k,m}L^2_{x,v}} \right).$$

Similar to Theorem 1.2, we have the propagation of spatial regularity in variable $\tilde{x}$. 

Theorem 1.4 (Propagation of spatial regularity in \( \hat{x} \)). Let all the assumptions in Theorem 1.3 be satisfied and \( T, m \geq 0 \).

(1) For the VPL system with \(-2 \leq \gamma < -1\), there exist \( \varepsilon_0, \delta > 0 \) such that if
\[
\|w \hat{f}_0\|_{L^1_{k,m}L^2_{x}L^2_{v}} + \|\tilde{E}_0\|_{L^1_{k,m}L^2_{x}L^2_{v}} \leq \varepsilon_0,
\]
then the solution \( f \) to (1.4) and (1.2) established in Theorem 1.1 satisfies
\[
\|e^{\delta t}w \hat{f}\|_{L^1_{k,m}L^2_{x}L^2_{v}} + \|e^{\delta t}\tilde{E}\|_{L^1_{k,m}L^2_{x}L^2_{v}} \lesssim \|w \hat{f}_0\|_{L^1_{k,m}L^2_{x}L^2_{v}} + \|\tilde{E}_0\|_{L^1_{k,m}L^2_{x}L^2_{v}}.
\]

(2) For the VPL system with \( \gamma \geq -1 \) and VPB system with \( \gamma + 2s \geq 1 \) and \( 1/2 \leq s < 1 \), there exist \( \varepsilon_0, \delta > 0 \) such that if
\[
\|\hat{f}_0\|_{L^1_{k,m}L^2_{x}L^2_{v}} + \|\tilde{E}_0\|_{L^1_{k,m}L^2_{x}L^2_{v}} \leq \varepsilon_0,
\]
then the solution \( f \) to the problem (1.4) satisfying (1.2) or (1.3) established in Theorem 1.1 satisfies
\[
\|e^{\delta t}\hat{f}\|_{L^1_{k,m}L^2_{x}L^2_{v}} + \|e^{\delta t}\tilde{E}\|_{L^1_{k,m}L^2_{x}L^2_{v}} \lesssim \|\hat{f}_0\|_{L^1_{k,m}L^2_{x}L^2_{v}} + \|\tilde{E}_0\|_{L^1_{k,m}L^2_{x}L^2_{v}}.
\]

Noticing that we can obtain the “full” dissipation terms as indicated in (1.12), (1.16) and hence, the exponential time decay on \((f, E)\) can be derived. By applying the weight \( \langle k \rangle^m \) after the Fourier transform \( \hat{\cdot} \), one can obtain the propagation of initial regularity over time.

The paper is organized as follows. In Section 2, we give some basic estimates on collision operators. In Section 3, we give the macroscopic dissipation estimates for VPL and VPB systems in torus. In Section 4, we derive the macroscopic dissipation estimates in finite channel. In Sections 5 and 6, we prove the global existence and large-time behavior with the help of macroscopic estimates for torus and finite channel, respectively. To complete the arguments, we give in Section 7 the proof for the local-in-time existence of the Vlasov-Poission-Landau equation with the specular reflection boundary condition in the finite channel.
In this section, we give some basic estimate on collision operator \( L \) and \( \Gamma(\cdot, \cdot) \). We begin with splitting \( L_\pm \). For the Landau case, let \( \varepsilon > 0 \) small and choose a smooth cutoff function \( \chi(|v|) \in [0, 1] \) such that \( \chi(|v|) = 1 \) if \( |v| < \varepsilon \); \( \chi(|v|) = 0 \) if \( |v| > 2\varepsilon \). Then we split \( L_\pm f = -A_\pm f + K_\pm f \) as in [35, Section 4.2], where

\[
-A_\pm f = 2\partial_{\alpha_i}(\sigma^{ij}\partial_{v_j}f)_{\pm} - 2\sigma^{ij}\frac{v_i v_j}{2}f_{\pm} + 2\partial_{\alpha_i}\sigma^{ij}1_{|v_i| > R}f_{\pm} + A_1 f \\
+ (K_1 - 1_{|v_i| \leq r}K_11_{|v_i| \leq r})f ,
\]

\[
K_\pm f = 2\partial_{\alpha_i}\sigma^{ij}1_{|v_i| \leq r}f_{\pm} + 1_{|v| \leq r}K_11_{|v| \leq r}f ,
\]

where \( R > 0 \) is to be chosen large, \( \varepsilon > 0 \) is to be chosen small, and \( A_1 \) and \( K_1 \) are given respectively by

\[
A_1 f = -\sum_{\pm} \mu^{-1/2}\partial_{\alpha_i}\left\{ \mu \left[ \left( \phi^{ij} \chi \right) * \left( \mu \frac{v_i}{|v|} \frac{v_j}{|v|} f_{\pm} \right) \right] \right\},
\]

\[
K_1 f = -\sum_{\pm} \mu^{-1/2}\partial_{\alpha_i}\left\{ \mu \left[ \left( \phi^{ij} (1 - \chi) \right) * \left( \mu \frac{v_i}{|v|} \frac{v_j}{|v|} f_{\pm} \right) \right] \right\},
\]

with the convolution taken with respect to the velocity variable \( v \). Then [35, eq. (4.33) and eq. (4.32)] shows that

\[
\sum_{\pm} \left( A_\pm f, f \right)_{L_\pm^2} \geq c_0 |f|_{L_\pm^2}^2 ,
\]

and

\[
\left| (K_1 g, h)_{L_\pm^2} \right| \lesssim |\mu|^{1/10} g |L_\pm^2| |\mu|^{1/10} h |L_\pm^2| . \tag{2.1}
\]

From [18, Lemma 3], we know

\[
|\partial_{\beta} \sigma^{ij}(v)| + |\partial_{\beta} \sigma^i(v)| \leq C_\beta (1 + |v|^2)^{\gamma-1} |v| . \tag{2.2}
\]

Thus, (2.1) and (2.2) imply that \( L_\pm \) is a bounded operator on \( L_v^2 \) with estimate

\[
[K f]_{L_v^2} \lesssim |\mu|^{1/10} f |L_v^2| .
\]

For Boltzmann case, we split \( L_\pm f = -A_\pm f + K f \) with

\[
-A_\pm f = 2\mu^{-1/2}Q(\mu, \mu^{1/2} f_{\pm} ) ,
\]

\[
K f = \mu^{-1/2}Q(\mu^{1/2}(f_+ + f_-), \mu ) .
\]

Then by [1, Lemma 2.15], we have

\[
[K f]_{L_v^2} \lesssim |\mu|^{1/10} f |L_v^2| .
\]

Moreover, we have the following Lemma on the estimates of \( L_\pm \) and \( \Gamma_\pm \).

**Lemma 2.1.** Let \( w \) be given by (1.10). Assume \( \gamma > \max\{-3, -2s - \frac{3}{2}\} \) for Boltzmann case and \( \gamma \geq -3 \) for Landau case. Then

\[
\sum_{\pm} (L_\pm f, f)_{L_v^2} \gtrsim |(I - P) f|_{L_v^2}^2 , \tag{2.3}
\]

and

\[
\sum_{\pm} (w^2 L_\pm g, g)_{L_v^2} \gtrsim c_0 |g|_{L_v^2(Bc)}^2 - C|g|_{L_v^2(Bc)}^2 . \tag{2.4}
\]

There exists decomposition \( L_\pm = -A_\pm + K_\pm \) such that \( K_\pm \) is a bounded linear operator on \( L_v^2 \) and \( A_\pm \) satisfies

\[
\sum_{\pm} (w^2 A_\pm g, g) \geq c_0 |g|_{L_v^2(Bc)}^2 - C|g|_{L_v^2(Bc)}^2 , \tag{2.5}
\]

Moreover, for any \( |\alpha| + |\beta| \leq 3 \), we have

\[
(w^2 \partial_{\beta}^\alpha \Gamma_\pm (g_1, g_2), \partial_{\beta}^\alpha \Gamma_\pm g_3)_{L_v^2} \lesssim \sum_{\pm} \left( (w^2 \partial_{\beta}^\alpha g_1 |L_v^2| \partial_{\beta}^\alpha g_2 |L_v^2|, + |\partial_{\beta}^\alpha g_1 |L_v^2|, |w^2 \partial_{\beta}^\alpha g_2 |L_v^2|) |\partial_{\beta}^\alpha g_3|_{L_v^2} \right) . \tag{2.6}
\]

where the summation is taken over \( \alpha_1 + \alpha_2 = \alpha, \beta_1 + \beta_2 = \beta \).
Proof. The proof of (2.3) and (2.4) can be found in [31, Lemma 5, Lemma 9] for Landau case and [12, Lemma 2.7] for the Boltzmann case. Note that the proof in [12] is also valid for hard potential. The proof of (2.5) can be found in [31, Lemma 7 and Lemma 8] for Landau case and [12, Lemma 2.6] for Boltzmann case. The proof of (2.6) is given by [31, Lemma 10] for Landau case and [12, 16, Lemma 2.4 and Lemma 2.4]. □

3. Macroscopic estimates on torus

In this section we will derive the a priori estimates for the macroscopic part of the solution to equation:

\[ \partial_t f_\pm + v \cdot \nabla_x f_\pm \pm \nabla_x \phi \cdot v \mu^{1/2} - L_\pm f = g_\pm, \]  

(3.1)

where

\[ g_\pm = \pm \nabla_x \phi \cdot \nabla_v f_\pm + \frac{1}{2} \nabla_x \phi \cdot v f_\pm + \Gamma_\pm(f, f), \]

(3.2)

and \( \phi \) is given by

\[ - \Delta_x \phi = a_+ - a_- \]  

(3.3)

To capture the macroscopic dissipation, we take the following velocity moments

\[ \mu^j, v_j \mu^j, \frac{1}{6} (|v|^2 - 3) \mu^j, (v_j v_m - 1) \mu^j, \frac{1}{10} (|v|^2 - 5) v_j \mu^j \]

with \( 1 \leq j, m \leq 3 \) for the equation (3.1). By taking the average and difference on \( \pm \) of the resultant equations, one sees that the coefficient functions \( [a_\pm, b, c] = [a_\pm, b, c](t, x) \) satisfy the fluid-type system

\[
\begin{align*}
\partial_t \left( \frac{a_+ + a_-}{2} \right) + \nabla_x \cdot b &= 0, \\
\partial_t b_j + \partial_{x_j} \left( \frac{a_+ + a_-}{2} + 2c \right) + \frac{1}{2} \sum_{m=1}^{3} \partial_{x_m} \Theta_{jm}([1 \mathbf{P}] f \cdot [1, 1]) &= \frac{1}{2} \sum_{\pm} (g_\pm, v_j \mu^{1/2})_{L^2}, \\
\partial_c + \frac{1}{3} \nabla_x \cdot b + \frac{5}{6} \sum_{m=1}^{3} \partial_{x_m} \Lambda_j([1 \mathbf{P}] f \cdot [1, 1]) &= \frac{1}{12} \sum_{\pm} (g_\pm, (|v|^2 - 3) \mu^{1/2})_{L^2}, \\
\frac{1}{2} \partial_t \Theta_{jm}([1 \mathbf{P}] f \cdot [1, 1]) + \partial_{x_j} b_m + \partial_{x_m} b_j &= \frac{1}{2} \sum_{\pm} \Theta_{jm}(g_\pm + h_\pm), \\
\frac{1}{2} \partial_t \Lambda_j([1 \mathbf{P}] f \cdot [1, 1]) + \partial_{x_j} c &= \frac{1}{2} \partial_t \Lambda_j(g_+ + g_- + h_+ + h_-),
\end{align*}
\]

(3.4)

for \( 1 \leq j, m \leq 3 \), where

\[ h_\pm = -v \cdot \nabla_x (\mathbf{I}_\pm - \mathbf{P}_\pm) f + L_\pm f, \]

\[ \Theta_{jm}(f_\pm) = (f_\pm, (v_j v_m - 1) \mu^{1/2})_{L^2}, \quad \Lambda_j(f_\pm) = \frac{1}{10} (f_\pm, (|v|^2 - 5) v_j \mu^{1/2})_{L^2}, \]

and

\[
\begin{align*}
\partial_t (a_+ - a_-) + \nabla_x \cdot G &= 0, \\
\partial_t G + \nabla_x (a_+ - a_-) - 2E + \nabla_x \cdot \Theta([1 \mathbf{P}] f \cdot [1, -1]) &= ((g + L f ) \cdot [1, -1], v \mu^{1/2})_{L^2},
\end{align*}
\]

(3.5)

where

\[ G = ([1 \mathbf{P}] f \cdot [1, -1], v \mu^{1/2})_{L^2}. \]  

(3.6)
For further application in the case of torus, we take Fourier transform on $x \in \mathbb{T}^3$ of (3.4) (3.5) and (3.3) to obtain
\[
\begin{align*}
&\partial_t \left( \frac{a_+ + a_-}{2} \right) + ik \cdot \hat{b} = 0, \\
&\partial_t \hat{b}_j + ik_j \left( \frac{a_+ + a_-}{2} + 2 \hat{c} \right) + \frac{1}{2} \sum_{m=1}^{3} ik_m \Theta_{jm}(\{ I - P \} \hat{f} \cdot [1, 1]) = \frac{1}{2} \sum_{\pm} (g_{\pm}, v_j \mu^{1/2}) L^2_{\xi}, \\
&\partial_t \hat{c} + \frac{1}{3} ik \cdot \hat{b} + \frac{5}{6} \sum_{j=1}^{3} ik_j \Lambda_j(\{ I - P \} \hat{f} \cdot [1, 1]) = \frac{1}{12} \sum_{\pm} (g_{\pm}, \|v\|^2 - 3) \mu^{1/2}) L^2_{\xi}, \\
&\frac{1}{2} \partial_t \Lambda_j(\{ I - P \} \hat{f} \cdot [1, 1]) + ik_j \hat{b}_m + ik_m \hat{b}_j = \frac{1}{2} \sum_{\pm} \Theta_{jm}(g_{\pm} + h_{\pm}),
\end{align*}
\]
and
\[
\begin{align*}
\partial_t (\hat{a}_+ - \hat{a}_-) + ik \cdot \hat{G} &= 0, \\
\partial_t \hat{G} + ik(\hat{a}_+ - \hat{a}_-) - 2 \hat{F} + ik \cdot \Theta(\{ I - P \} \hat{f} \cdot [1, -1]) = (\hat{g} + L \hat{f}) \cdot [1, -1], \mu^{1/2}) L^2_{\xi},
\end{align*}
\]
\end{equation}

With the above preparation, we are ready to obtain the following macroscopic estimate in isotropic case.

**Theorem 3.1.** Let $\gamma \geq -3$ for Landau case, $\gamma > \max\{-3, -2s - 3/2\}$ for Boltzmann case and $T > 0$. Let $f$ be the solution of (1.4) in torus with initial data satisfying (1.5). Then
\[
\left\| (a_+, a_-, b, c) \right\|_{L_T^1 L_x^2} + \| \hat{F} \|_{L_T^1 L_x^2} \lesssim \| F \|_{L_T^1 L_x^2} + \| \hat{F}_0 \|_{L_T^1 L_x^2} + \| \{ I - P \} \hat{f} \|_{L_T^1 L_x^2} L^2_{\xi} + \| \{ I - P \} \hat{f} \|_{L_T^1 L_x^2} (\| \hat{F} \|_{L_T^1 L_x^2} + \| \hat{F}_0 \|_{L_T^1 L_x^2}) \lesssim \| \hat{F} \|_{L_T^1 L_x^2} L^2_{\xi}. \quad \text{(3.9)}
\]

**Proof.** We will prove the desired estimate by dividing norm $L^2_k$ into cases $k = 0$ and $k \neq 0$.

**Case $k = 0$:** Using (3.7) with $k = 0$, we have
\[
\begin{align*}
&\partial_t \left( \frac{a_+ + a_-}{2} \right) |_{k=0} = 0, \\
&\partial_t \hat{b}_j |_{k=0} = \frac{1}{2} \sum_{\pm} (\pm \nabla_x \phi \cdot \nabla_x f_{\pm} + \frac{1}{2} \nabla_x \phi \cdot v_{f_{\pm}, \mu^{1/2}}) L^2_{\xi}, \\
&\partial_t \hat{c} |_{k=0} = \frac{1}{12} \sum_{\pm} (\pm \nabla_x \phi \cdot \nabla_x f_{\pm} + \frac{1}{2} \nabla_x \phi \cdot v_{f_{\pm}, \|v\|^2 - 3} \mu^{1/2}) L^2_{\xi}.
\end{align*}
\]

Taking integration by parts on $\nabla_x$ and using (3.3), we have
\[
\begin{align*}
&\partial_t \hat{b}_j |_{k=0} = \frac{1}{2} \sum_{\pm} \left( (\pm \nabla_x \phi \cdot \epsilon_j \mu^{1/2} - \frac{1}{2} v_j \mu^{1/2}) L^2_{\xi} + \frac{1}{2} (\nabla_x \phi \cdot v_{f_{\pm}, \mu^{1/2}}) L^2_{\xi} \right) \\
&= \frac{1}{2} \int_{\mathbb{T}^3} \partial^\phi \phi \Delta_x \phi \, dx = 0,
\end{align*}
\]

where $\epsilon_j$ is unit vector in $\mathbb{R}^3$ with the $j$’s component being 1. By using (3.8) and integration by parts on $\nabla_x$,
\[
\begin{align*}
&\partial_t \hat{c} |_{k=0} = -\frac{1}{12} \sum_{\pm} (\pm \nabla_x \phi f_{\pm} + 2 v_{f_{\pm}, \mu^{1/2}}) L^2_{\xi} = -\frac{1}{6} \int_{\mathbb{T}^3} \nabla_x \phi \cdot \mathcal{U} d\Sigma(k) = -\frac{1}{12} \int_{\mathbb{T}^3} |E|^2 d\Sigma(k).
\end{align*}
\]

These equations give the conservation laws (1.5). Since the initial data $f_0$ satisfies (1.5), using Plancherel’s Theorem, we have
\[
\begin{align*}
&\hat{a}_+ |_{k=0} \equiv \hat{a}_- |_{k=0} \equiv \hat{b}_j |_{k=0} \equiv 0, \\
&\hat{c} |_{k=0} \equiv -\frac{1}{12} \int_{\mathbb{T}^3} |E|^2 \, dx.
\end{align*}
\]

Taking $L^2_2$ norms on $\hat{c} |_{k=0}$, we have
\[
\left( \int_0^T |\mathcal{E}_{k=0}|^2 \, dt \right)^{1/2} \lesssim \left( \int_0^T \|E\|_{L^2_x}^4 \, dt \right)^{1/2} \lesssim \left( \int_0^T \|\mathcal{E}\|_{L^2_x}^4 \, dt \right)^{1/2} \lesssim \mathop{\sup}_{0 \leq t \leq T} \|\mathcal{E}\|_{L^1_t L^\infty_x} \|\mathcal{E}\|_{L^1_t L^2_x}. \tag{3.10}
\]

**Case** \(|k| \geq 1\): To obtain the estimate for \(|k| \geq 1\), we use the trick in [11, Theorem 5.1]. In order to carry out the estimate in a unified way, we take a general test function as

\[
\widehat{\Phi}(t, k, v) \in C^1((0, \infty) \times \mathbb{Z}^3 \times \mathbb{R}^3).
\]

Also, we denote \(\zeta(v)\) to be a smooth function satisfying

\[
\zeta(v) \lesssim e^{-\lambda|v|^2},
\]

for some \(\lambda > 0\). The function \(\zeta(v)\) may change from line to line. The following integration will be used frequently: if \(p > -1\) is an even number, then

\[
\int_{\mathbb{R}} z^p e^{-|z|^2} \, dz = (p-1)! \sqrt{2\pi}.
\tag{3.11}
\]

Taking the inner product of (3.1) with \(\Phi\) in \(L^2_t\) and integrating over \([0, T]\), we have

\[
(\hat{f}_\pm(T), \hat{\Phi}(T))_{L^2_x} + (\hat{f}_\pm(0), \hat{\Phi}(0))_{L^2_x} - \int_0^T (\hat{f}_\pm, \partial_t \hat{\Phi})_{L^2_x} \, dt - \int_0^T (\hat{f}_\pm, iv \cdot k \hat{\Phi})_{L^2_x} \, dt \\
\pm \int_0^T (\nabla_x \phi \cdot v \mu^{1/2}, \hat{\Phi})_{L^2_x} \, dt + \int_0^T (-L \pm \hat{f}, \hat{\Phi})_{L^2_x} \, dt = \int_0^T (\hat{g}_\pm, \hat{\Phi})_{L^2_x} \, dt.
\]

Then we decompose \(f = P_\pm f + (I_\pm - P_\pm) f\) to obtain

\[
- \int_0^T (P_\pm f, iv \cdot k \hat{\Phi})_{L^2_x} \, dt = \sum_{j=1}^{5} S_j,
\]

where

\[
S_1 = (\hat{f}_\pm, \hat{\Phi})_{L^2_x}|_{t=0} - (\hat{f}_\pm, \hat{\Phi})_{L^2_x}|_{t=T},
S_2 = \int_0^T (\hat{f}_\pm, \partial_t \hat{\Phi})_{L^2_x} \, dt,
S_3 = \int_0^T ((I_\pm - P_\pm) f, iv \cdot k \hat{\Phi})_{L^2_x} \, dt + \int_0^T (L \pm \hat{f}, \hat{\Phi})_{L^2_x} \, dt,
S_4 = \mp \int_0^T (\nabla_x \phi \cdot v \mu^{1/2}, \hat{\Phi})_{L^2_x} \, dt,
S_5 = \int_0^T (\hat{g}_\pm, \hat{\Phi})_{L^2_x} \, dt.
\]

**Step 1. Estimate of \(\hat{c}(t, k, v)\).** Firstly, we consider the estimate on \(\hat{c}\) for \(|k| \geq 1\). We choose

\[
\hat{\Phi} = \hat{\phi}_c = (|v|^2 - 5)iv \cdot k \hat{\phi}_c(t, k) \mu^{1/2},
\]

where \(\phi_c\) is defined by

\[
|k|^2 \hat{\phi}_c(t, k) = \hat{c}(t, k).
\tag{3.12}
\]

Note that here we assume \(|k| \geq 1\) and write \(\hat{\phi}_c(t, k) = \hat{c}(t, k)/|k|^2\). By this choice, we can deduce that

\[
- \int_0^T (P_\pm f, iv \cdot k \hat{\Phi})_{L^2_x} \, dt \\
= \sum_{j,n=1}^3 \int_0^T \left( (a_{\pm} \pm \hat{b} \cdot v + (|v|^2 - 3)\hat{c}) \mu^{1/2}, (|v|^2 - 5)v_j v_n (-k_jk_n) \mu^{1/2} \hat{\phi}_c \right) dt \\
= 10 \int_0^T (\hat{c}, |k|^2 \hat{\phi}_c) dt = 10 \int_0^T |\hat{c}(t, k)|^2 dt,
\]

where

\[
\hat{b} = \mathcal{E}_{k=0}, \hat{\phi}_c = \mathcal{E}_{k=1} \phi_c.
\]
where we used the orthogonality of the different integrands for the second equality. Now we estimate the $S_j$’s. Since $|k| \geq 1$, it holds that
\[ |\Phi_c(t, k)| \lesssim |\Phi_c(t, k)| \lesssim |\Phi_c(t, k)| = \mu^{1/4}|\tilde{c}(t, k)|. \]
Thus,\[ |S_1| \leq |\tilde{f}(k, T)|_{L^2_\mu}^2 + |\tilde{f}_0(k)|_{L^2_\mu}^2. \]
For $S_2$, we first notice that from the third equation of (3.7),
\[ |\partial_2 \tilde{c}| \lesssim |\tilde{b}(t, k)| + |\{I - P\} \tilde{f}(t, k)| + |(\tilde{g}_+ + \tilde{g}_-, \zeta)_{L^2_\mu}|. \]
Thus, noticing $|k| \geq 1$,
\[ |S_2| = |\int_0^T (\tilde{f}_\pm, \partial_2 \Phi_c)_{L^2_\mu} dt| \leq |\int_0^T |\{I - P\} \tilde{f}_\pm|_{L^2_\mu} dt| \leq \eta \int_0^T |\partial_2 \tilde{c}|^2 |k| dt + C_\eta \int_0^T |\{I - P\} \tilde{f}_\pm|_{L^2_\mu}^2 dt \leq \eta \int_0^T |\tilde{b}(t, k)|^2 dt + \eta \int_0^T |(\tilde{g}_+ + \tilde{g}_-, \zeta)_{L^2_\mu}|^2 dt + C_\eta \int_0^T |\tilde{b}(t, k)|^2 dt, \]
where the second equality follows from orthogonality. For $S_3$, by Hölder’s inequality, (3.12) and (3.13), we have
\[ |S_3| \leq \eta \int_0^T |\tilde{c}(t, k)|^2 dt + C_\eta \int_0^T |\{I - P\} \tilde{f}(t, k)|_{L^2_\mu}^2 dt. \]
For the term $S_4$, noticing $|k| \geq 1$ and (3.12),
\[ |S_4| \lesssim \int_0^T |\nabla \tilde{c}|^2 dt + \eta \int_0^T |\tilde{c}(t, k)|^2 dt. \]
For the term $S_5$, noticing (3.12), we have
\[ |\sum_i S_5| \lesssim \eta \int_0^T |\tilde{c}|^2 dt + C_\eta \int_0^T |(\tilde{g}_+ + \tilde{g}_-, \zeta)_{L^2_\mu}|^2 dt. \]
Combining the above estimates and choosing $\eta > 0$ sufficiently small, we have
\[ \int_0^T |\tilde{c}(t, k)|^2 dt \leq |\tilde{f}(k, T)|_{L^2_\mu}^2 + |\tilde{f}_0(k)|_{L^2_\mu}^2 + \eta \int_0^T |\tilde{b}(t, k)|^2 dt + \eta \int_0^T |(\tilde{g}_+ + \tilde{g}_-, \zeta)_{L^2_\mu}|^2 dt + C_\eta \int_0^T |\tilde{b}(t, k)|_{L^2_\mu}^2 dt + C_\eta \int_0^T |(\tilde{g}_+ + \tilde{g}_-, \zeta)_{L^2_\mu}|^2 dt. \]
\[ \text{Step 2. Estimate of } \tilde{b}(t, k, v). \text{ Now we consider the estimate of } \tilde{b}. \text{ For this purpose we choose} \]
\[ \tilde{\Phi} = \tilde{\Phi}_b = \sum_{m=1}^3 \tilde{\Phi}_b^{j, m}, j = 1, 2, 3, \]
where
\[ \tilde{\Phi}_b^{j, m} = \left\{ \begin{array}{ll}
((|v|^2 v_m v_j k_m \phi_{\overline{j}} - \frac{7}{2} (v_j^2 - 1) i k_j \phi_{\overline{j}}) \mu^{1/2}, & j \neq m, \\
\frac{7}{2} (v_j^2 - 1) i k_j \phi_{\overline{j}} \mu^{1/2}, & j = m, \end{array} \right. \]
and
\[ |k|^2 \phi_{\overline{j}}(t, k) = \tilde{b}_j(t, k). \]
Under this choice we have
\[ - \sum_{m=1}^3 \int_0^T (P_{\pm} \tilde{f}, iv \cdot k \tilde{\Phi}_b^{j, m})_{L^2_\mu} dt \]
\[ = - \sum_{m=1}^3 \int_0^T ((a_{\pm} + \tilde{b} \cdot v + (|v|^2 - 3) \tilde{c}) \mu^{1/2}, iv \cdot k \tilde{\Phi}_b^{j, m})_{L^2_\mu} dt \]
\[- \sum_{m=1,m \neq j}^{3} \int_{0}^{T} (v_{m} v_{j} \mu^{1/2} \hat{b}_{j}, |v|^{2} v_{m} v_{j} \mu^{1/2} (-k_{m}^{2} \hat{\phi}_{j})_{L_{2}}^{2} dt \\
- \sum_{m=1,m \neq j}^{3} \int_{0}^{T} (v_{m} v_{j} \mu^{1/2} \hat{b}_{m}, |v|^{2} v_{m} v_{j} \mu^{1/2} (-k_{m} k_{j} \hat{\phi}_{j})_{L_{2}}^{2} dt + \frac{7}{2} \sum_{m=1,m \neq j}^{3} \int_{0}^{T} (v_{m}^{2} \mu^{1/2} \hat{b}_{m}, (v_{m}^{2} - 1) \mu^{1/2} (-k_{m} k_{j} \hat{\phi}_{j})_{L_{2}}^{2} dt \\
- \frac{7}{2} \int_{0}^{T} (v_{m}^{2} \mu^{1/2} \hat{b}_{m}, (v_{m}^{2} - 1) \mu^{1/2} (-k_{m}^{2} \hat{\phi}_{j})_{L_{2}}^{2} dt \\
= -7 \sum_{m=1}^{3} \int_{0}^{T} (\hat{b}_{j}, (-k_{m}^{2} \hat{\phi}_{j})_{L_{2}}^{2} dt) = 7 \int_{0}^{T} \hat{b}_{j}(t, k)^{2} dt.
\]

Note that \(\int_{\mathbb{R}^{3}} v_{m}^{2} (v_{m}^{2} - 1) \mu \, dv = 2, \int_{\mathbb{R}^{3}} v_{m} v_{j}^{2} \mu \, dv = 0, \int_{\mathbb{R}^{3}} v_{m}^{2} v_{j}^{2} \mu \, dv = 7 \) when \(m \neq j \). Since \(|k| \geq 1\), it holds that
\[
|\Phi_{b}^{1,m}(t, k)| \lesssim \mu^{1/4} |k| |\hat{\phi}_{j}(t, k)| \lesssim \mu^{1/4} |k|^{2} |\hat{\phi}_{j}(t, k)| = \mu^{1/4} |\hat{b}(t, k)|,
\]
and then we have \(|\Phi_{b}^{1,m}(t, k)|_{L_{2}} \lesssim |\hat{b}(t, k)|\). Thus,
\[
|S_{1}| \lesssim |\hat{f}(k, T)|_{L_{2}}^{2} + |\hat{f}_{0}(k)|_{L_{2}}^{2}.
\]

For \(S_{2}\), we first notice that from the second equation of (3.7),
\[
|\partial_{k} \hat{b}_{j} | \lesssim |k| (|\hat{\alpha}_{+} + \hat{\alpha}_{-}| + |\hat{\xi}| + |\{I - P\} \hat{f}(t, k)| + |\hat{\alpha}_{+} + \hat{\alpha}_{-}, \xi|_{L_{2}}^{2}.
\]

Thus, noticing \(|k| \geq 1\),
\[
|S_{2}| \leq \int_{0}^{T} \left| [(I_{\pm} - P \pm) f, \partial_{k} \Phi_{b}^{1,m}]_{L_{2}} \right| dt + \int_{0}^{T} \left| (P \pm \hat{f}, \partial_{k} \Phi_{b}^{1,m})_{L_{2}} \right| dt \\
\lesssim \eta \int_{0}^{T} \frac{|\partial_{k} \hat{b}_{j}|^{2}}{|k|^{2}} dt + C_{\eta} \int_{0}^{T} \left| \{I - P\} \hat{f}_{D}\right|^{2} dt + C_{\eta} \int_{0}^{T} |\hat{e}(t, k)|^{2} dt \\
\lesssim \eta \int_{0}^{T} |\hat{\alpha}_{+} + \hat{\alpha}_{-}|^{2} dt + \eta \int_{0}^{T} |\hat{\alpha}_{+} + \hat{\alpha}_{-}, \xi|_{L_{2}}^{2} dt \\
+ C_{\eta} \int_{0}^{T} \left| \{I - P\} \hat{f}_{D}\right|^{2} dt + C_{\eta} \int_{0}^{T} |\hat{e}(t, k)|^{2} dt.
\]

Here, in the second inequality, we use the orthogonality for the term \(\int_{0}^{T} |(P \pm \hat{f}, \partial_{k} \Phi_{b}^{1,m})_{L_{2}}| dt\) in the v-integration. For \(S_{3}\), by Hölder’s inequality, (3.15) and (3.16), we have
\[
|S_{3}| \lesssim \eta \int_{0}^{T} \hat{b}(t, k)^{2} dt + C_{\eta} \int_{0}^{T} \left| \{I - P\} \hat{f}(t, k) \right|_{D}^{2} dt.
\]

For the term \(S_{4}\), noticing \(|k| \geq 1\) and (3.15),
\[
|S_{4}| \lesssim \int_{0}^{T} \langle \nabla_{x} \phi, |\phi|^{2} \hat{\phi}_{j} \rangle dt \lesssim C_{\eta} \int_{0}^{T} \langle |\nabla_{x} \phi|^{2} \rangle dt + \eta \int_{0}^{T} |\hat{b}(t, k)|^{2} dt.
\]

For the term \(S_{5}\), noticing (3.15), we have
\[
|S_{5} \| \lesssim \eta \int_{0}^{T} |\hat{\alpha}|^{2} dt + C_{\eta} \int_{0}^{T} (|\hat{\alpha}_{+} + \hat{\alpha}_{-}, \xi|_{L_{2}}^{2} dt.
\]

Combining the above estimates and choosing \(\eta > 0\) sufficiently small, we have
\[
\int_{0}^{T} \hat{b}(t, k)^{2} dt \lesssim \int_{0}^{T} \hat{f}(k, T)|^{2} dt + \int_{0}^{T} \hat{f}_{0}(k)|^{2} dt + \eta \int_{0}^{T} |\hat{\alpha}_{+} + \hat{\alpha}_{-}|^{2} dt + C_{\eta} \int_{0}^{T} |\hat{e}(t, k)|^{2} dt \\
+ C_{\eta} \int_{0}^{T} \left| \{I - P\} \hat{f}(t, k) \right|_{D}^{2} dt + C_{\eta} \int_{0}^{T} \langle \nabla_{x} \phi|^{2} \rangle dt + C_{\eta} \int_{0}^{T} (|\hat{\alpha}_{+} + \hat{\alpha}_{-}, \xi|_{L_{2}}^{2} dt. \quad (3.17)
\]
Step 3. Estimate of $\bar{a}_+ + \bar{a}_-$. Now we consider $\bar{a}_+ + \bar{a}_-$. We set

$$\hat{\Phi} = \hat{\Phi}_a = (|v|^2 - 10)iv \cdot k\hat{\phi}_a \mu^{1/2},$$

where $\hat{\phi}_a(t, k) = (\hat{\phi}_a + (t, k), \hat{\phi}_a - (t, k))$ is a solution to

$$|k|^2 \hat{\phi}_a(t, k) = |k|^2 \hat{\phi}_a - (t, k) = \bar{a}_+ + \bar{a}_-.$$

For this choice we have

$$- \sum_{\pm} \int_0^T (P_{\pm} f, iv \cdot k\hat{\Phi}_{\pm})_{L^2} dt$$

$$= - \sum_{\pm} \sum_{j,n} \int_0^T ((\hat{a}_+ + \hat{b} \cdot v + (|v|^2 - 3)\hat{c})\mu^{1/2}, v_j v_n (|v|^2 - 10)\mu^{1/2}(-k_j k_n)\hat{\phi}_a)_{L^2} dt$$

$$= 5 \sum_{\pm} \int_0^T \hat{a}_\pm |k|^2 \hat{\phi}_a_{\pm} dt = 5 \int_0^T |\bar{a}_+ + \bar{a}_-|^2 dt.$$

Note that $((|v|^2 - 3)\mu^{1/2}, v_j v_n (|v|^2 - 10)\mu^{1/2})_{L^2} = 0$. The estimate of $S_1$ is similar to the previous case. For $S_2$, we first notice that from the second equation of (3.7),

$$|\partial_t (\bar{a}_+ + \bar{a}_-)| \lesssim |k|\hat{b}|.$$

Thus, noticing $|k| \geq 1$,

$$|S_2| \lesssim \int_0^T |(I \pm - \hat{P}_\pm) f, \partial_t \hat{\Phi}_a|_{L^2} dt + \int_0^T |(P_{\pm} \hat{f}, \partial_t \hat{\Phi}_a)_{L^2}| dt$$

$$\lesssim \int_0^T |\partial_t (\hat{a}_+ + \hat{a}_-)|^2 dt + \int_0^T |(I - \hat{P}) \hat{f}|_{L^2}^2 dt + \int_0^T |\hat{b}(t, k)|^2 dt$$

$$\lesssim \int_0^T |\hat{b}(t, k)|^2 dt + \int_0^T |(I - \hat{P}) \hat{f}|_{L^2}^2 dt.$$

Here in the second line we used the orthogonality in the $v$-integration for the term $\int_0^T (P_{\pm} \hat{f}, \partial_t \hat{\Phi}_a)_{L^2} dt$. Similar to the previous case, we have

$$|S_3| \lesssim \eta \int_0^T |\hat{a}_+ + \hat{a}_-|^2 dt + C_\eta \int_0^T |(I - \hat{P}) \hat{f}(t, k)|_{L^2}^2 dt.$$}

For the term $S_4$, noticing the sign $\pm$, we have $\sum_{\pm} S_4 = 0$. For $S_5$, we have

$$|S_5| \lesssim \eta \int_0^T |\hat{a}_+ + \hat{a}_-|^2 dt + C_\eta \int_0^T |(\hat{g}_+ + \hat{g}_-, \zeta)|_{L^2}^2 dt.$$

Combining the above estimates and choosing $\eta > 0$ sufficiently small, we have

$$\int_0^T \sum_{\pm} |\hat{a}_+ + \hat{a}_-|^2 dt \lesssim \int_0^T |\hat{f}(k, T)|_{L^2}^2 + \int_0^T |\hat{f}_n(k)|_{L^2}^2 + \int_0^T |\hat{b}(t, k)|^2 dt$$

$$+ \int_0^T |(I - \hat{P}) \hat{f}(t, k)|_{L^2}^2 dt + \int_0^T |(\hat{g}_+ + \hat{g}_-, \zeta)|_{L^2}^2 dt. \quad (3.18)$$

Step 4. Estimate of $\bar{a}_+ - \bar{a}_-$ and $\hat{E}$. Set

$$\hat{\Phi} = \hat{\Phi}_a = (|v|^2 - 10)iv \cdot k\hat{\phi}_a \mu^{1/2},$$

where $\hat{\phi}_a(t, k) = (\hat{\phi}_a + (t, k), \hat{\phi}_a - (t, k))$ is a solution to

$$|k|^2 \hat{\phi}_a(t, k) = |k|^2 \hat{\phi}_a - (t, k) = \bar{a}_+ - \bar{a}_-,$$

$$|k|^2 \hat{\phi}_a - (t, k) = \bar{a}_+ - \bar{a}_-.$$

Then the macroscopic part reduces to

$$- \sum_{\pm} \int_0^T (P_{\pm} f, iv \cdot k\hat{\Phi}_{\pm})_{L^2} dt$$

$$= - \sum_{\pm} \sum_{j,n} \int_0^T ((\hat{a}_+ + \hat{b} \cdot v + (|v|^2 - 3)\hat{c})\mu^{1/2}, v_j v_n (|v|^2 - 10)\mu^{1/2}(-k_j k_n)\hat{\phi}_a)_{L^2} dt$$
For the part $S_2$, by (3.8), we have
\[ |\partial_t \phi_{a_\pm}| \leq |k|^{-2}|\nabla_x \cdot \tilde{G}| \leq |k|^{-1}|\tilde{G}|, \]
and hence,
\begin{align*}
\sum_{\pm} |S_2| & \leq \sum_{\pm} \int_0^T \left| (P_{\pm} \tilde{f}, (|v|^2 - 10)iv \cdot k \partial_t \phi_{a_\pm} \mu^{1/2}) \right|_{L^2_x} dt \\
& \quad + \sum_{\pm} \int_0^T \left| ((\mathbf{I}_\pm - P_{\pm}) \tilde{f}, (|v|^2 - 10)iv \cdot k \partial_t \phi_{a_\pm} \mu^{1/2}) \right|_{L^2_x} dt \\
& \lesssim \eta^2 \int_0^T |\tilde{b}|^2 dt + C\eta \int_0^T |(\mathbf{I} - P) \tilde{f}| \tilde{b}^T dt.
\end{align*}

For the part $S_4$, we observe from (3.3) that
\begin{align*}
\sum_{\pm} S_4 &= \sum_{\pm} \sum_{j,n} \int_0^T (\partial \tilde{\phi} \mu^{1/2}, (|v|^2 - 10)iv_j v_n k n \phi_{a_\pm} \mu^{1/2})_{L^2_x} dt \\
&= 5 \sum_{\pm} \sum_{j} \int_0^T (\partial \tilde{\phi} | ik_j \phi_{a_\pm}) dt \\
&= 10 \sum_{j} ik_j \int_0^T (\partial \tilde{\phi} | a_+ - a_-) dt \\
&= -10 \int_0^T (\tilde{\phi} | -\Delta x \tilde{\phi}) dt = -10 \int_0^T |\nabla_x \tilde{\phi}|^2 dt.
\end{align*}

This gives the dissipation rate of $\nabla_x \phi(t, k)$ when $|k| \geq 1$. Notice that when $k = 0$, $\nabla_x \tilde{\phi} = ik \tilde{\phi} = 0$. The estimates on terms $S_1, S_3, S_5$ are similar to Step 3 and we omit it for brevity. Therefore, combining the above estimates we have
\begin{align*}
\int_0^T |a_+ - a_-|^2 dt + \int_0^T |\nabla_x \tilde{\phi}|^2 dt & \lesssim |\tilde{f}(T, k)|_{L^2_x}^2 + |\tilde{f}_0(k)|_{L^2_x}^2 + \eta^2 \int_0^T |\tilde{b}|^2 dt \\
& \quad + C\eta \int_0^T |(\mathbf{I} - P) \tilde{f}(t, k)| \tilde{b}^T dt + \int_0^T |(\tilde{g}_+ + \tilde{g}_-, \zeta)|_{L^2_x}^2 dt. \quad (3.19)
\end{align*}

**Step 5. Energy estimate.** Now taking combination (3.19) + $\kappa \times (3.14) + \kappa^2 \times (3.17) + \kappa^3 \times (3.18)$ and choosing $\eta << \kappa << 1$, we have for $|k| \geq 1$ that
\begin{align*}
\int_0^T |a_+ - a_-|^2 dt + \int_0^T |\nabla_x \tilde{\phi}|^2 dt & \lesssim |\tilde{f}(k, T)|_{L^2_x}^2 + |\tilde{f}_0(k)|_{L^2_x}^2 \\
& \quad + \|\mathbf{I} - P\| \int_0^T |(\tilde{g}_+ + \tilde{g}_-, \zeta)|_{L^2_x}^2 dt.
\end{align*}

Note that $2|a_+|^2 + 2|a_-|^2 = |a_+ + a_-|^2 + |a_+ - a_-|^2$. Together with (3.10), taking the square root and summation over $k \in \mathbb{Z}^d$, we have
\begin{align*}
\|(a_+, a_-, b, \zeta)\|_{L^1 L^2_x} + \|\tilde{E}\|_{L^1 L^2_x} & \lesssim \left( \int_0^T \left( \left| (\nabla_x \tilde{\phi} \cdot \nabla_x \tilde{f}_{\pm}) \right| \zeta(v) \right|_{L^2_x}^2 dt \right)^{1/2} d\Sigma(k). \quad (3.20)
\end{align*}

For the last term, we will use the trick as (3.10). Noticing (3.2), we estimate the term in $g_{\pm}$ one by one. For the term $(\nabla_x \phi \cdot \nabla_x f_{\pm})^\zeta$, noticing $\zeta(v)$ is smooth and has exponential decay, we take integration by parts with respect to $v$ to obtain
\begin{align*}
\int_{\mathbb{Z}^d} \left( \int_0^T \left| (\nabla_x \phi \cdot \nabla_x f_{\pm})^\zeta, \zeta(v) \right|_{L^2_x}^2 dt \right)^{1/2} d\Sigma(k)
\end{align*}
\[
\begin{align*}
&\lesssim \int_{\mathbb{R}^L_k} \left( \int_0^T \left( \int_{\mathbb{R}^3} \left| \hat{E}(k-l) \right| |\hat{f}(l)| L_2 \right) \right)^{1/2} d\Sigma(k) \\
&\lesssim \int_{\mathbb{R}^L_k} \int_{\mathbb{R}^3} \left( \int_0^T \left| \hat{E}(k-l) \right|^2 |\hat{f}(l)|^2 L_2 \right)^{1/2} d\Sigma(l) d\Sigma(k) \\
&\lesssim \int_{\mathbb{R}^L_k} \int_{\mathbb{R}^3} \sup_{0 \leq T} \left| \hat{E}(k-l) \right| \left( \int_0^T |\hat{f}(l)|^2 L_2 \right)^{1/2} d\Sigma(l) d\Sigma(k) \\
&\lesssim \|\hat{E}\|_{L^2_k L^2} \|\hat{f}\|_{L^4_k L^4}. 
\end{align*}
\]

Similarly, for the term \(\frac{1}{2} |(\nabla_x \phi \cdot v f_\pm)^\alpha|\), we have
\[
\int_{\mathbb{R}^L_k} \left( \int_0^T \left( \frac{1}{2} |(\nabla_x \phi \cdot v f_\pm)^\alpha|, \zeta(e) \right) L_2 \right)^2 dt)^{1/2} d\Sigma(k) \lesssim \|\hat{E}\|_{L^2_k L^2} \|\hat{f}\|_{L^4_k L^4}. 
\]

For the term \(\Gamma_{\pm}(f, f)\), we have from (2.6) that
\[
\begin{align*}
&\int_{\mathbb{R}^L_k} \left( \int_0^T \left( |\Gamma_{\pm}(f, f), \zeta(e) \right) L_2 \right)^2 dt)^{1/2} d\Sigma(k) \\
&= \int_{\mathbb{R}^L_k} \left( \int_0^T \left( (\Gamma_{\pm}(f(k-l), f(l)), \zeta(e) \right) L_2 d\Sigma(l) \right)^2 dt)^{1/2} d\Sigma(k) \\
&\lesssim \int_{\mathbb{R}^L_k} \left( \int_0^T \left( |\hat{f}(k-l)| L_2 \right| \hat{f}(l)| L_2 \right) d\Sigma(l) \right)^2 dt)^{1/2} d\Sigma(k) \\
&\lesssim \int_{\mathbb{R}^L_k} \int_{\mathbb{R}^3} \sup_{0 \leq T} \left| \hat{f}(k-l) \right| L_2 \left( \int_0^T \left| \hat{f}(l)|^2 L_2 \right) dt \right)^{1/2} d\Sigma(l) d\Sigma(k) \\
&\lesssim \|\hat{f}\|_{L^4_k L^4} L_2 \|\hat{f}\|_{L^4_k L^4}. 
\end{align*}
\]

Plugging the above estimate into (3.20), we obtain (3.9) and complete the proof of Theorem 3.1. \(\square\)

4. Macroscopic estimates for finite channel

In this section, we write \(\hat{\cdot}\) to denote the Fourier transform with respect to \(\hat{x} \in \mathbb{T}^2\). We will derive the macroscopic estimates in the case of finite channel. Consider the following problem
\[
\partial_t f_\pm + v_l \partial_x f_\pm + \vec{v} \cdot \nabla_x f_\pm \pm \nabla_x \phi \cdot v f_\pm + L_\pm f = g_\pm, 
\]
with initial data \((f_0, E_0)\) and boundary condition
\[
\begin{align*}
\hat{f}(t, 1, \vec{k}, v_1, v) |_{v_1 > 0} &= \hat{f}(t, 1, \vec{k}, -v_1, \vec{v}) \\
\hat{f}(t, 1, \vec{k}, v_1, \vec{v}) |_{v_1 < 0} &= \hat{f}(t, 1, \vec{k}, -v_1, \vec{v}),
\end{align*}
\]
where
\[
g_\pm = \pm \nabla_x \phi \cdot \nabla_x f_\pm + \frac{1}{2} \nabla_x \phi \cdot v f_\pm + \Gamma_{\pm}(f, f),
\]
and the potential is determined by Poisson equation:
\[
-\Delta_x \phi = a_+ - a_-, 
\]
with the zero Neumann boundary condition
\[
\partial_x \phi = 0, \text{ on } x_1 = \pm 1.
\]

As in the torus case, we denote \(\zeta(e)\) to be a smooth function satisfying \(\zeta(e) \lesssim e^{-\lambda |v|^2}\), for some \(\lambda > 0\). The function \(\zeta(e)\) may change from line to line. Then we have the following macroscopic estimate in anisotropic case.

**Theorem 4.1.** Let \(\gamma \geq -3\) for Landau case, \(\gamma > \max\{-3, -2s - 3/2\}\) for Boltzmann case and \(T > 0\). Let \(f\) be the solution of (4.1), (4.2), (4.3) and (4.4) in finite channel with initial data satisfying (1.8). Then
\[
\sum_{|\alpha| \leq 1} \|\partial^\alpha(a_+, a_-, \hat{\phi}, \hat{\phi})\|_{L^4_k L^4 L^4} + \sum_{|\alpha| \leq 1} \|\partial^\alpha E\|_{L^4_k L^4 L^4},
\]
\begin{equation}
\lesssim \sum_{|\alpha| \leq 1} (\|\tilde{\partial}^\alpha f(T)\|_{L^2_1 L_{2,1,0}^2} + \|\tilde{\partial}^\alpha \tilde{f}_0\|_{L^2_1 L_{2,1,0}^2}) + \sum_{|\alpha| \leq 1} \|\{I - \mathbf{P}\} \tilde{\partial}^\alpha f\|_{L^2_1 L_{2,1,0}^2, L^2_2} + \sum_{|\alpha| \leq 1} (\|\tilde{\partial}^\alpha E\|_{L^2_1 L_{2,1}^2} + \|\tilde{\partial}^\alpha f\|_{L^2_1 L_{2,1}^2, L^2_2}) \sum_{|\alpha| \leq 1} \|\tilde{\partial}^\alpha g\|_{L^2_1 L_{2,1,0}^2, L^2_2} + \|\tilde{E}\|_{L^2_1 L_{2,1}^2} \|\tilde{E}\|_{L^2_1 L_{2,1}^2}.
\end{equation}

Proof. Let $|\alpha| \leq 1$. Acting $\partial := \tilde{\partial}^\alpha$ to (4.1) and taking the Fourier transform with respect to $\tilde{x}$, we have

\begin{equation}
\partial_t \tilde{f}^\pm = v_1 \partial_{x_1} \tilde{f}^\pm + i \tilde{v} \cdot k \tilde{f}^\pm \pm (\partial \nabla_x \phi)^\wedge \cdot v \mu^{1/2} - L_k \partial_t \tilde{f} = \tilde{g}^\pm,
\end{equation}

with the specular reflection condition

\begin{equation}
\tilde{f}(t, -1, \tilde{k}, \tilde{v}), |v| > 0 = \tilde{f}(t, -1, \tilde{k}, -v), \tilde{v}),
\end{equation}

\begin{equation}
\tilde{f}(t, 1, \tilde{k}, \tilde{v}), |v| < 0 = \tilde{f}(t, 1, \tilde{k}, -v).
\end{equation}

Let $\hat{\Phi}(t, x_1, \tilde{k}, v) \in C^1((0, +\infty) \times (-1, 1) \times \mathbb{R}^3)$ with $\tilde{k} = (k_2, k_3) \in \mathbb{Z}^2$ be a test function. Taking the inner product of $\hat{\Phi}(t, x_1, \tilde{k}, v)$ and (4.5) with respect to $(x_1, v)$ and integrating the resulting identity with respect to $t$ over $[0, T]$ for any $T > 0$, we obtain

\begin{equation}
(\partial \tilde{f}^\pm, \hat{\Phi})_{L^2_1, \mathbf{P}} = (\partial \tilde{f}^\pm, \hat{\Phi})_{L^2_1, (0)} - \int_0^T (\partial \tilde{f}^\pm, \partial_t \hat{\Phi})_{L^2_1, \mathbf{P}} dt
\end{equation}

\begin{equation}
- \int_0^T (\partial \tilde{f}^\pm, v \cdot \nabla_{x_1, x} \Phi)_{L^2_1, \mathbf{P}} dt + \int_0^T (v_1 \partial \tilde{f}^\pm(1), \hat{\Phi}(1))_{L^2_1} dt - \int_0^T (v_1 \partial \tilde{f}^\pm(-1), \hat{\Phi}(-1))_{L^2_1} dt
\end{equation}

\begin{equation}
\pm \int_0^T ((\partial \nabla_x \phi)^\wedge \cdot v \mu^{1/2}, \hat{\Phi})_{L^2_1, \mathbf{P}} dt - \int_0^T (L_k \partial f^\pm, \hat{\Phi})_{L^2_1, \mathbf{P}} dt = \int_0^T (\tilde{g}^\pm, \hat{\Phi})_{L^2_1, \mathbf{P}} dt.
\end{equation}

Using the decomposition $\tilde{f}^\pm = \mathbf{P} \tilde{f}^\pm + (I - \mathbf{P}) \tilde{f}^\pm$, we have

\begin{equation}
- \int_0^T (\partial \mathbf{P}^\pm f, v \cdot \nabla_{x_1, x} \Phi)_{L^2_1, \mathbf{P}} dt = \sum_{j=1}^6 S_j,
\end{equation}

where $S_j$ are defined by

\begin{equation}
S_1 = - (\partial \tilde{f}^\pm, \hat{\Phi})_{L^2_1, (T)} + (\partial \tilde{f}^\pm, \hat{\Phi})_{L^2_1, (0)} - \int_0^T (\partial \tilde{f}^\pm, \partial_t \hat{\Phi})_{L^2_1, \mathbf{P}} dt,
\end{equation}

\begin{equation}
S_3 = \int_0^T ((\partial (I - \mathbf{P}) \pm f)^\wedge, v \cdot \nabla_{x_1, x} \Phi)_{L^2_1, \mathbf{P}} dt,
\end{equation}

\begin{equation}
S_4 = \int_0^T (L_k \partial \tilde{f}^\pm, \hat{\Phi})_{L^2_1, \mathbf{P}} dt + \int_0^T (\partial \tilde{g}^\pm, \hat{\Phi})_{L^2_1, \mathbf{P}} dt,
\end{equation}

\begin{equation}
S_5 = \pm \int_0^T ((\partial \nabla_x \phi)^\wedge \cdot v \mu^{1/2}, \hat{\Phi})_{L^2_1, \mathbf{P}} dt,
\end{equation}

\begin{equation}
S_6 = \int_0^T (v_1 \partial \tilde{f}^\pm(1), \hat{\Phi}(1))_{L^2_1} dt + \int_0^T (v_1 \partial \tilde{f}^\pm(-1), \hat{\Phi}(-1))_{L^2_1} dt.
\end{equation}

Step 1. Estimate on $\tilde{c}(t, x_1, \tilde{k})$. We choose the following test function

$\hat{\Phi} = \hat{\Phi}_C = (|v|^2 - 5)(v \cdot \nabla_{x_1, x} \phi_c(t, x_1, \tilde{k})) \mu^{1/2}$,

where $\phi_c$ solves

\begin{equation}
\begin{cases}
- \partial_{x_1}^2 \tilde{\phi}_c + |\tilde{k}|^2 \tilde{\phi}_c = \tilde{\partial}^\alpha, \\
\tilde{\phi}_c(\pm 1, \tilde{k}) = 0, & \text{if } \partial = \partial_{x_1}, \\
\tilde{\phi}_c(\pm 1, \tilde{k}) = 0, & \text{if } \partial = \partial_{x_2}, \partial_{x_3}.
\end{cases}
\end{equation}

Then by standard elliptic estimate, we have

\begin{equation}
\|\partial_{x_1}^2 \tilde{\phi}_c\|_{L^2_1} + |\tilde{k}| \|\partial_{x_1} \tilde{\phi}_c\|_{L^2_1} + |\tilde{k}| \|\tilde{\phi}_c\|_{L^2_1} \lesssim \|\tilde{\partial}^\alpha\|_{L^2_1}.
\end{equation}
When $\partial = I$ and $k = 0$, (4.8) is a pure Neumann boundary problem. However, the mean of $\hat{c}|_{\vec{k}=0}$ doesn’t vanish: $\int_{-1}^{1} \hat{c}(x_1, 0) \, dx_1 \neq 0$ and, we can’t find a solution. In order to derive the dissipation estimates for $\hat{c}|_{\vec{k}=0}$, we apply Poincaré’s inequality and (1.8) to obtain that

$$
\|\hat{c}|_{\vec{k}=0}\|_{L^2_{x_1}} \leq \|\partial x_1 \hat{c}|_{\vec{k}=0}\|_{L^2_{x_1}} + \left| \int_{-1}^{1} \hat{c}|_{\vec{k}=0} \, dx_1 \right| \leq \|\partial x_1 \hat{c}|_{\vec{k}=0}\|_{L^2_{x_1}} + \| \hat{E} \|_{L^2_{x_1}}^2.
$$

Similar to the case of torus, we can estimate the term $\|\hat{E}\|_{L^2_{x_1}}^2$ by

$$
\int_{0}^{T} \|\hat{E}\|_{L^2_{x_1}}^2 \, dt \leq \int_{0}^{T} \|\hat{E}\|_{L^2_{x_1}}^2 \, dt \leq \int_{0}^{T} \|\hat{E}\|_{L^2_{x_1}}^2 \, dt \leq \sup_{0 \leq t \leq T} \|\hat{E}\|_{L^2_{x_1}}^2 \int_{0}^{T} \|\hat{E}\|_{L^2_{x_1}}^2 \, dt.
$$

Thus, by Young’s inequality $\| \cdot \|_{L^pL^q} \leq \| \cdot \|_{L^pL^r}$ ($p \geq q \geq 1$), we have

$$
\int_{0}^{T} \|\hat{c}|_{\vec{k}=0}\|_{L^2_{x_1}}^2 \, dt \leq \int_{0}^{T} \|\partial x_1 \hat{c}|_{\vec{k}=0}\|_{L^2_{x_1}}^2 \, dt + \|\hat{E}\|_{L^2_{x_1}}^2 \|\bar{E}\|_{L^2_{x_1}}^2 \|\hat{E}\|_{L^2_{x_1}}^2 \|\hat{E}\|_{L^2_{x_1}}^2.
$$

(4.10)

To the end of this step, we assume that either $\partial \neq I$ or $|\vec{k}| \neq 0$. Next we let $\partial = I, \partial x_1, \partial x_2$. Taking inner product of (4.8) with $\hat{\phi}_c$ over $x_1 \in [-1, 1]$, we have

$$
\|\partial x_1 \hat{\phi}_c\|_{L^2_{x_1}} + |\hat{c}|^2 \|\hat{\phi}_c\|_{L^2_{x_1}} \leq \|\hat{\phi}_c\|_{L^2_{x_1}},
$$

The above estimate implies that for any $\vec{k}$,

$$
\|\partial x_1 \hat{\phi}_c\|_{L^2_{x_1}} + |\hat{c}|^2 \|\hat{\phi}_c\|_{L^2_{x_1}} \leq \|\hat{\phi}_c\|_{L^2_{x_1}}.
$$

(4.11)

Similarly, since derivative on $t$ doesn’t affect the boundary value, we have

$$
\|\partial t \partial x_1 \hat{\phi}_c\|_{L^2_{x_1}} + |\hat{c}|^2 \|\hat{\phi}_c\|_{L^2_{x_1}} \leq \|\hat{\phi}_c\|_{L^2_{x_1}}.
$$

(4.12)

On the other hand, when $\partial = \partial x_1$, (4.8) is a Dirichlet boundary problem. Then taking inner product of (4.8) with $\hat{\phi}_c$ over $x_1 \in [-1, 1]$, we have

$$
\|\partial x_1 \hat{\phi}_c\|_{L^2_{x_1}} + |\hat{c}|^2 \|\hat{\phi}_c\|_{L^2_{x_1}} \leq \|\hat{\phi}_c\|_{L^2_{x_1}} = (\hat{c}, \partial x_1 \hat{\phi}_c)_{L^2_{x_1}} \leq \|\hat{c}\|_{L^2_{x_1}} \|\partial x_1 \hat{\phi}_c\|_{L^2_{x_1}}.
$$

This implies that

$$
\|\partial x_1 \hat{\phi}_c\|_{L^2_{x_1}} + |\hat{c}| \|\partial x_1 \hat{\phi}_c\|_{L^2_{x_1}} \leq \|\hat{\phi}_c\|_{L^2_{x_1}}.
$$

(4.13)

Similarly,

$$
\|\partial t \partial x_1 \hat{\phi}_c\|_{L^2_{x_1}} + |\hat{c}| \|\partial t \partial x_1 \hat{\phi}_c\|_{L^2_{x_1}} \leq \|\partial t \hat{\phi}_c\|_{L^2_{x_1}}.
$$

(4.14)

Now we can compute (4.7). For the left hand side of (4.7), we have

$$
-\int_{0}^{T} (\hat{\partial P}_{\pm} f, \nabla x_1 \hat{\Phi}_c)_{L^2_{x_1,v}} \, dt
= -\sum_{j,m=1}^{3} \int_{0}^{T} (\hat{\partial P}_{\pm} f, \nabla x_1 \hat{\Phi}_c, v_j v_m (|v|^2 - 3) \hat{c}, v_j v_m (|v|^2 - 5) \mu (\partial x_1 \partial x_m \hat{c})^v)_{L^2_{x_1,v}} \, dt
= 10 \sum_{j=1}^{3} \int_{0}^{T} (\hat{\partial c}^j, -\hat{\partial c}^j \hat{\Phi}_c)_{L^2_{x_1,v}} \, dt = 10 \int_{0}^{T} \|\hat{\partial c}\|_{L^2_{x_1}}^2 \, dt.
$$

For the right hand side of (4.7), by Hölder’s inequality and the elliptic estimate (11.1) and (14.13), we have

$$
|S_1| \leq \sum_{|\alpha| \leq 1} \left( \|\hat{\partial \alpha f}(T)\|_{L^2_{x_1,v}} + \|\hat{\partial \alpha f}_0\|_{L^2_{x_1,v}} \right).
$$

For $S_2$, we see from (4.12) and (4.14) that

$$
|S_2| \leq \int_{0}^{T} |\hat{\partial f}, \hat{\partial t} \Phi_c|_{L^2_{x_1,v}} \, dt = \int_{0}^{T} \|((I - \hat{P}) \hat{\partial f}, \hat{\partial t} \Phi_c)|_{L^2_{x_1,v}} \, dt.
$$
Therefore, when
\[ \therefore \text{Using the Neumann boundary condition (4.11).} \]

For the term \( S_4 \), applying (4.11) and (4.13), we have
\[ |S_4| \leq \eta \int_0^T \| \tilde{\nabla} \|_{L^2_t} dt + C_\eta \int_0^T \| (I - \mathbf{P}) \tilde{\nabla} f \|_{L^2_t} dt. \]

For the term \( S_5 \), using Cauchy-Schwarz’s inequality, we have
\[ |S_5| \leq \eta \int_0^T \| \tilde{\nabla} \|_{L^2_t} dt + C_\eta \int_0^T \| (I - \mathbf{P}) \tilde{\nabla} f \|_{L^2_t} dt + C_\eta \int_0^T \| (\partial g, \zeta) \|_{L^2_t} dt. \]

For \( S_6 \), we need to use the specular reflection boundary condition (4.6):
\[ \tilde{f}(1, v_1) \big|_{v_1 \neq 0} = \tilde{f}(1, -v_1), \quad \tilde{f}(-1, v_1) \big|_{v_1 \neq 0} = \tilde{f}(-1, v_1). \tag{4.15} \]

Using the Neumann boundary condition (4.4), we know that
\[ \tilde{\partial_x} \phi(\pm 1, \bar{k}) = 0. \]

By the above boundary value for \( f \) and \( \phi \), we have \( g_{\pm}(-1, -v_1) \big|_{v_1 \neq 0} = g_{\pm}(-1, v_1) \) and \( g_{\pm}(1, -v_1) \big|_{v_1 \neq 0} = g_{\pm}(1, v_1) \). Thus, using the equation (4.11) to define derivative \( \tilde{\partial_x} f_{\pm} \), one has
\[ -v_1 \tilde{\partial_x} f(-1, \bar{k}, -v_1) = \left( \tilde{\partial_x} f_{\pm} + \bar{v} \cdot \nabla_x f_{\pm} \pm \nabla_x \phi \cdot v \mu^{1/2} - L_{\pm} f \right)(-1, \bar{k}, v_1) \]
\[ = v_1 \tilde{\partial_x} f(-1, \bar{k}, v_1), \tag{4.16} \]
and similarly,
\[ -v_1 \tilde{\partial_x} f(1, \bar{k}, -v_1) = v_1 \tilde{\partial_x} f(1, \bar{k}, v_1). \tag{4.17} \]

For the case \( \partial = \partial_x \), by (4.8) with boundary condition \( \tilde{\phi}_c(\pm 1, \bar{k}) = 0 \), we know that \( \tilde{\partial_x} \phi(\pm 1, \bar{k}) = 0 \).

Thus by the definition of \( \tilde{\Phi}_c \), we have
\[ \tilde{\Phi}_c(-1, \bar{k}, -v_1, \bar{v}) = -\tilde{\Phi}_c(-1, \bar{k}, v_1, \bar{v}), \quad \tilde{\Phi}_c(1, \bar{k}, -v_1, \bar{v}) = -\tilde{\Phi}_c(1, \bar{k}, v_1, \bar{v}). \]

Therefore, when \( \partial = \partial_x \), by change of variable \( v_1 \mapsto -v_1 \), (4.16) and (4.17), we have
\[ S_6 = -\int_0^T (v_1 \tilde{\partial_x} f_{\pm}(1, v_1), \tilde{\Phi}_c(1, v_1))_{L^2_x} dt \]
\[ + \int_0^T (v_1 \tilde{\partial_x} f_{\pm}(-1, v_1), \tilde{\Phi}_c(-1, v_1))_{L^2_x} dt \]
\[ = -\int_0^T (v_1 \tilde{\partial_x} f_{\pm}(1, v_1), \tilde{\Phi}_c(1, v_1))_{L^2_x} dt \]
\[ + \int_0^T (v_1 \tilde{\partial_x} f_{\pm}(-1, v_1), \tilde{\Phi}_c(-1, v_1))_{L^2_x} dt \]
\[ = \int_0^T (v_1 \tilde{\partial_x} f_{\pm}(1, v_1), \tilde{\Phi}_c(1, v_1))_{L^2_x} dt - \int_0^T (v_1 \tilde{\partial_x} f_{\pm}(-1, v_1), \tilde{\Phi}_c(-1, v_1))_{L^2_x} dt = 0. \tag{4.19} \]

Here \( S_6 \) equal to zero because (4.18) and (4.19) are the same except the sign.

For the case \( \partial = \partial_{x_2}, \partial_{x_3} \), by boundary condition \( \tilde{\partial_{x_2}} \tilde{\phi}_c(\pm 1, \bar{k}) = 0 \), we know that
\[ \tilde{\Phi}_c(-1, \bar{k}, -v_1, \bar{v}) = \tilde{\Phi}_c(-1, \bar{k}, v_1, \bar{v}), \quad \tilde{\Phi}_c(1, \bar{k}, -v_1, \bar{v}) = \tilde{\Phi}_c(1, \bar{k}, v_1, \bar{v}). \]
On the other hand, from (4.15) we have
\[ v_1 \partial f(-1, k, -v_1) = v_1 \partial f(-1, k, v_1), \quad v_1 \partial f(1, k, -v_1) = v_1 \partial f(1, k, v_1). \]

Therefore, when \( \partial = 1, \partial_{x_2}, \partial_{x_3} \), by change of variable \( v_1 \rightarrow -v_1 \), we have
\[
S_0 = - \int_0^T (v_1 \partial f_{\pm}(1, v_1), \Phi_c(1, v_1))_{L^2_0} dt + \int_0^T (v_1 \partial f_{\pm}(-1, v_1), \Phi_c(-1, v_1))_{L^2_0} dt
= \int_0^T (v_1 \partial f_{\pm}(1, v_1), \Phi_c(1, v_1))_{L^2_0} dt - \int_0^T (v_1 \partial f_{\pm}(-1, v_1), \Phi_c(-1, v_1))_{L^2_0} dt
= 0.
\]

Combining the above estimates for \( S_j \)'s (1 \( \leq j \leq 6 \)), applying (4.10) for the case of \( \alpha = 0 \) and \( \bar{k} = 0 \), taking summation of (4.7) for \( \alpha = 1 \) for the remaining cases and then taking the square root and summation over \( k \in \mathbb{Z}^2 \), and finally letting \( \eta \) suitably small, we obtain
\[
\sum_{|\alpha| \leq 1} ||\partial^\alpha c||^2_{L^2_0 L^2_0 L^2_0} \lesssim \sum_{|\alpha| \leq 1} \left( ||\partial^\alpha f(T)||_{L^1 L^2_{x_1,v}}^2 + ||\partial^\alpha f_0||_{L^1 L^2_{x_1,v}}^2 \right)
+ C \eta \sum_{|\alpha| \leq 1} \left( ||(1 - P)\partial^\alpha f||_{L^1 L^2_{x_1,v} L^2_{x_2} L^2_{x_3}} + ||\partial^\alpha E||_{L^1 L^2_{x_1,v} L^2_{x_2} L^2_{x_3}} + ||(\partial^\alpha g, \zeta)||_{L^1 L^2_{x_1,v} L^2_{x_2} L^2_{x_3}} \right)
+ \eta^{1/2} ||\nabla_k \hat{b}||_{L^1 L^2_{x_1,v} L^2_{x_2} L^2_{x_3}} + ||\hat{E}||_{L^1 L^2_{x_1,v} L^2_{x_2} L^2_{x_3}} ||\hat{E}||_{L^1 L^2_{x_1,v} L^2_{x_2} L^2_{x_3}}. \tag{4.22}
\]

Step 2. Estimate of \( \hat{b}(t, x_1, \bar{k}) \). Now we consider the estimate of \( \hat{b} \). For this purpose we choose
\[
\hat{\Phi} = \hat{\Phi}_b = \sum_{m=1}^3 \Phi_b^{j,m}, \quad j = 1, 2, 3,
\]
where
\[ \Phi_b^{j,m} = \left\{ \begin{array}{ll}
(v^2 v_m v_j \partial_{x_m} \phi_j - \frac{7}{2} (v^2_m - 1) \partial_{x_j} \phi_j) \mu^{1/2}, & j \neq m, \\
\frac{7}{2} (v^2_j - 1) \partial_{x_j} \phi_j \mu^{1/2}, & j = m.
\end{array} \right. \]

Also, \( \phi_j (1 \leq j \leq 3) \) solves
\[
\left\{ \begin{array}{l}
- \partial^2_{x_1} \hat{\phi}_j + |\bar{k}|^2 \hat{\phi}_j(k) = \partial \bar{b}_j(k), \\
\partial_{x_1} \hat{\phi}_j(\pm 1, \bar{k}) = \hat{\phi}_3(\pm 1, \bar{k}) = 0,
\end{array} \right. \tag{4.23}
\]
for the case \( \partial = \partial_{x_1} \) and
\[
\left\{ \begin{array}{l}
- \partial^2_{x_1} \hat{\phi}_j + |\bar{k}|^2 \hat{\phi}_j(k) = \partial \bar{b}_j(k), \\
\hat{\phi}_j(\pm 1, \bar{k}) = \partial_{x_1} \hat{\phi}_2(\pm 1, \bar{k}) = \partial_{x_1} \hat{\phi}_3(\pm 1, \bar{k}) = 0,
\end{array} \right. \tag{4.24}
\]
for the case \( \partial = 1, \partial_{x_2}, \partial_{x_3} \). When \( \bar{k} = 0 \), \( j = 1 \) and \( \partial = \partial_{x_1} \), (4.23) is Neumann boundary problem. When \( j = 2, 3 \) and \( \partial = I \), (4.24) is also Neumann boundary problem. Their existence are guaranteed by the fact that \( \int_0^1 \int_{x_1} \partial_{x_1} b_1 \, d\bar{x}dx_1 = b_1(1) - b_1(1) = 0 \) and \( \int_0^1 \int_{x_1} b_j \, d\bar{x}dx_1 = 0 \) for \( j = 2, 3 \), which follows from (4.6) and (1.8) respectively. Under this choice we have
\[
- \sum_{m=1}^3 \int_0^T (\mathbf{P}_\pm \partial f, iv \cdot k \Phi_b^{j,m})_{L^1_{x_1,v}} dt
= - \sum_{m=1, m \neq j}^3 \int_0^T (v_m v_j \mu^{1/2} \partial \bar{b}_j, v^2 v_m v_j \mu^{1/2} \partial_{x_m}^2 \phi_j)_{L^1_{x_1,v}} dt
- \sum_{m=1, m \neq j}^3 \int_0^T (v_m v_j \mu^{1/2} \partial \bar{b}_m, v^2 v_m v_j \mu^{1/2} \partial_{x_m} \phi_j)_{L^1_{x_1,v}} dt
+ 7 \sum_{m=1, m \neq j}^3 \int_0^T (\partial \bar{b}_m, \partial_{x_m} \phi_j)_{L^2_0} dt - 7 \int_0^T (\partial \bar{b}_m, \partial^2_{x_j} \phi_j)_{L^2_0} dt
\]
\[ = -7 \sum_{m=1}^{3} \int_{0}^{T} (\partial b_{j}, \partial^{2} m \phi_{j})_{L^{2}_{x}} dt = 7 \int_{0}^{T} \| \partial b_{j} \|_{L^{2}_{x}} dt. \]

Note that from (3.11), we have \( \int_{\mathbb{R}^{3}} v_{m}^{2} (v_{m} - 1) \mu dv = 2, \int_{\mathbb{R}^{3}} v_{m}^{2} (v_{m} - 1) \mu dv = 0, \int_{\mathbb{R}^{3}} v_{m}^{2} \mu dv = 7 \) when \( m \neq j \). Similar to the calculation for deriving (4.9), (4.11), (4.12), (4.13) and (4.14), we have that for \( |k| \geq 0, \)

\[ \| \partial_{x} \partial_{x} \phi_{j} \|_{L^{2}_{x}} + |k| \| \partial \phi_{j} \|_{L^{2}_{x}} \lesssim \| \partial b_{j} \|_{L^{2}_{x}}, \]

\[ \| \partial_{x} \phi_{j} \|_{L^{2}_{x}} + |k| \| \phi_{j} \|_{L^{2}_{x}} \lesssim \| b_{j} \|_{L^{2}_{x}}, \]

and

\[ \| \partial_{x}^{2} \phi_{j} \|_{L^{2}_{x}} + |k| \| \partial_{x} \phi_{j} \|_{L^{2}_{x}} + |k| \| \phi_{j} \|_{L^{2}_{x}} \lesssim \| \partial b_{j} \|_{L^{2}_{x}}. \]

As a consequence,

\[ |S_{2}| \leq \eta \int_{0}^{T} \| \partial t \nabla_{x} \phi_{j} \|_{L^{2}_{x}}^{2} dt + C_{0} \int_{0}^{T} \| (I - P) \partial f \|_{L_{x}^{2}}^{2} dt + C_{0} \int_{0}^{T} \| \phi \|_{L_{x}^{2}}^{2} dt \]

\[ \leq \eta \int_{0}^{T} \| \partial t \nabla_{x} \phi_{j} \|_{L^{2}_{x}}^{2} dt + C_{0} \int_{0}^{T} \| (I - P) \partial f \|_{L_{x}^{2}}^{2} dt + C_{0} \int_{0}^{T} \| \phi \|_{L_{x}^{2}}^{2} dt \]

\[ \leq \eta \int_{0}^{T} \| (\nabla_{x} (a_{+} + a_{-})) \|_{L^{2}_{x}}^{2} dt + C_{0} \sum_{|\alpha| \leq 1} \int_{0}^{T} \| (I - P) \phi_{\alpha} \|_{L_{x}^{2}}^{2} dt \]

\[ + C_{0} \int_{0}^{T} \| \phi \|_{L_{x}^{2}}^{2} dt, \]

where we used the second equation of (3.4) in the last inequality. By the same argument as in the case of \( \tilde{c}(t,x_{1},k) \), we have

\[ |S_{1}| + |S_{3}| + |S_{4}| + |S_{5}| \lesssim \sum_{|\alpha| \leq 1} \| \phi_{\alpha} \|_{L_{x}^{2}}^{2} \]

\[ + C_{0} \int_{0}^{T} \| \phi \|_{L_{x}^{2}}^{2} dt + C_{0} \int_{0}^{T} \| (I - P) \nabla f \|_{L_{x}^{2}}^{2} dt + C_{0} \int_{0}^{T} \| (\phi_{\alpha}, \zeta) \|_{L_{x}^{2}}^{2} dt. \]

Now we consider the boundary term \( S_{6} \). For the case \( \partial = \partial_{x_{1}} \), by boundary condition (4.23), we have

\[ \Phi_{b}(-1,k,-v_{1},\bar{v}) = -\Phi_{b}(-1,k,v_{1},\bar{v}), \quad \Phi_{b}(1,k,-v_{1},\bar{v}) = -\Phi_{b}(1,k,v_{1},\bar{v}). \]

Together with (4.16) and (4.17), by changing of variable \( v_{1} \mapsto -v_{1} \), we know that

\[ S_{0} = -\int_{0}^{T} (v_{1} \partial_{x_{1}} \tilde{f}_{\pm}(1,v_{1}), \Phi_{b}(1,v_{1}))_{L_{x}^{2}} dt + \int_{0}^{T} (v_{1} \partial_{x_{1}} \tilde{f}_{\pm}(-1,v_{1}), \Phi_{b}(-1,v_{1}))_{L_{x}^{2}} dt \]

\[ = \int_{0}^{T} (v_{1} \partial_{x_{1}} \tilde{f}_{\pm}(1,v_{1}), \Phi_{b}(1,v_{1}))_{L_{x}^{2}} dt - \int_{0}^{T} (v_{1} \partial_{x_{1}} \tilde{f}_{\pm}(-1,v_{1}), \Phi_{b}(-1,v_{1}))_{L_{x}^{2}} dt = 0. \]

For the case \( \partial = I, \partial_{x_{2}}, \partial_{x_{3}}, \) by (4.24), we know that \( \partial_{x_{2}} \phi_{j}(\pm 1) = \partial_{y} \phi_{j}(\pm 1) = 0 \) for \( j = 2, 3 \). Thus

\[ \Phi_{b}(-1,k,v_{1},\bar{v}) = \Phi_{b}(1,k,v_{1},\bar{v}), \quad \Phi_{b}(1,k,v_{1},\bar{v}) = \Phi_{b}(1,k,v_{1},\bar{v}), \]

\[ \Phi_{b}(-1,k,v_{1},\bar{v}) = \Phi_{b}(1,k,v_{1},\bar{v}), \quad \Phi_{b}(1,k,v_{1},\bar{v}) = \Phi_{b}(1,k,v_{1},\bar{v}), \]

and hence by (4.20) and change of variable \( v_{1} \mapsto -v_{1} \), we have

\[ S_{6} = -\int_{0}^{T} (v_{1} \partial_{x_{1}} \tilde{f}_{\pm}(1,v_{1}), \Phi_{b}(1,v_{1}))_{L_{x}^{2}} dt + \int_{0}^{T} (v_{1} \partial_{x_{1}} \tilde{f}_{\pm}(-1,v_{1}), \Phi_{b}(-1,v_{1}))_{L_{x}^{2}} dt \]

\[ = \int_{0}^{T} (v_{1} \partial_{x_{1}} \tilde{f}_{\pm}(1,v_{1}), \Phi_{b}(1,v_{1}))_{L_{x}^{2}} dt - \int_{0}^{T} (v_{1} \partial_{x_{1}} \tilde{f}_{\pm}(-1,v_{1}), \Phi_{b}(-1,v_{1}))_{L_{x}^{2}} dt = 0. \]

Combining the above estimates, taking summation of (4.7) over \( |\alpha| \leq 1 \), and then taking the square root and summation over \( k \in \mathbb{Z}^{2} \), and finally letting \( \eta \) sufficiently small, we have

\[ \sum_{|\alpha| \leq 1} \| \phi_{\alpha} \|_{L_{x}^{2}L_{x}^{2}} \lesssim \sum_{|\alpha| \leq 1} \left( \| \phi_{\alpha} \|_{L_{x}^{2}L_{x}^{2}} + \| \phi_{\alpha} \|_{L_{x}^{2}L_{x}^{2}} \right) \]
\[+ \eta^{1/2} \| (\nabla_x (a_+ + a_-))^2 \|_{L_1^2 L_2^1 L_4^1} + C_0 \sum_{|\alpha| \leq 1} \left( \| (I - P) \partial^{\alpha} f \|_{L_1^1 L_2^1 L_4^1} 
abla_x f \right)_{L_1^1 L_2^1 L_4^1} + \| \partial^{\alpha} c \|_{L_1^1 L_2^1 L_4^1} + \| (\partial^{\alpha} g, \zeta) \|_{L_1^1 L_2^1 L_4^1} + \| \partial^{\alpha} \tilde{E} \|_{L_1^1 L_2^1 L_4^1} \right). \] (4.25)

**Step 3.** Estimate on \( \hat{u}_+ (t, x_1, \tilde{k}) - \hat{u}_- (t, x_1, \tilde{k}) \) and \( \hat{E} (t, x_1, \tilde{k}) \). We choose the following two test functions

\[ \hat{\phi} = \hat{\phi}_{a \pm} = (|v|^2 - 1)(v \cdot \nabla_{x_1, \tilde{k}} \hat{\phi}_{a \pm} (t, x_1, \tilde{k})) \mu^{1/2}, \]

where \( \phi_a = (\phi_{a+} (x_1, \tilde{k}), \phi_{a-} (x_1, \tilde{k})) \) solves

\[\begin{align*}
- \partial_{x_1}^2 \phi_{a+} + |\tilde{k}|^2 \phi_{a+} &= \tilde{u}_+ - \tilde{u}_-, \\
- \partial_{x_1}^2 \phi_{a-} + |\tilde{k}|^2 \phi_{a-} &= \tilde{u}_- - \tilde{u}_+,
\end{align*}\] (4.26)

with boundary condition \( \hat{\phi}_a (\pm 1, \tilde{k}) = 0 \) for the case \( \tilde{\vartheta} = \partial_{x_1} \), and \( \hat{\vartheta}_{x_1} \hat{\phi}_a (\pm 1, \tilde{k}) = 0 \) for the case \( \tilde{\vartheta} = \tilde{I}, \tilde{\vartheta}_{x_2}, \tilde{\vartheta}_{x_3} \). When \( |\tilde{k}| = 0 \) and \( \tilde{\vartheta} = \tilde{I} \), (4.26) is a pure Neumann boundary problem and we need \( \int_{-1}^1 \int_{\mathbb{S}^2} a_+ - a_- \, dx \, dt \) to ensure the existence for (4.26), which follows from (1.8). In particular, as in the case of \( \hat{\vartheta} \), we add zero mean condition \( \int_{-1}^1 \phi_a (x_1, 0) \, dx_1 = 0 \) for Neumann boundary case. Thus, similar to the calculation from (4.9) to (4.14), we have

\[ \| \partial_{x_1} \phi_{a+} \|_{L_2^1, v} + |\tilde{k}| \| \partial_{x_1} \phi_{a-} \|_{L_2^1, v} + |\tilde{k}|^2 \| \phi_{a+} \|_{L_2^1, v} \lesssim \| \tilde{u}_+ - \tilde{u}_- \|_{L_2^1, v}, \]

and

\[ \| \partial_{x_1} \phi_{a+} \|_{L_2^1, v} + |\tilde{k}| \| \phi_{a-} \|_{L_2^1, v} \lesssim \| \tilde{u}_+ - \tilde{u}_- \|_{L_2^1, v}. \] (4.27)

Now we compute (4.7) with summation on \( \pm \). For the left hand side, taking summation on \( \pm \), we have

\[ - \sum_{\pm} \int_{0}^{T} \left( \partial_t (\tilde{f} \cdot v \cdot \nabla_{x_1, \tilde{k}} \hat{\phi}_{a \pm}) \right)_{L_2^1, v} \, dt \]

\[= - \sum_{\pm} \sum_{j, m = 1}^{3} \int_{0}^{T} \left( \partial_t \phi_{a \pm} + \tilde{b} \cdot v + (|v|^2 - 3) \tilde{c}, v_j v_m (|v|^2 - 10) \mu (\partial_{x_1} \partial_{x_m} \phi_{a \pm}) \right)_{L_2^1, v} \, dt \]

\[= \sum_{\pm} \sum_{j = 1}^{3} \int_{0}^{T} \left( \partial_t \phi_{a \pm} - \partial_{x_1}^2 \phi_{a \pm} \right)_{L_2^1, v} \, dt = \int_{0}^{T} \| \tilde{u}_+ - \tilde{u}_- \|_{L_2^1, v}^2 \, dt. \]

For \( S_2 \), we decompose \( f_\pm \) into \( P_\pm f \) and \( (I - P) \pm f \) and apply (4.27) to obtain

\[ \| \sum_{\pm} S_2 \| \lesssim \int_{0}^{T} \| (I - P) \partial_t \tilde{f} \partial_{x_1} \hat{\phi}_{a \pm} \|_{L_2^1, v} \, dt + \int_{0}^{T} \| (P \partial_t \tilde{f}, \partial_{x_1} \hat{\phi}_{a \pm})_{L_2^1, v} \| \, dt \]

\[\lesssim \int_{0}^{T} \| \tilde{c} \|_{L_2^1, v} \, dt + \int_{0}^{T} \| (I - P) \partial_t \tilde{f} \|_{L_2^1, v}^2 + \int_{0}^{T} \| \tilde{b} \|_{L_2^1, v}^2 \, dt \]

\[\lesssim \sum_{|\alpha| \leq 1} \int_{0}^{T} \| (I - P) \partial^{\alpha} \tilde{f} \|_{L_2^1, v}^2 dt + \int_{0}^{T} \| \tilde{b} \|_{L_2^1, v}^2 dt, \]

where we used the first equation of (3.5) For \( S_1, S_3 \) and \( S_4 \), we apply the same argument as in Step 1 to obtain

\[ |S_1| + |S_3| + |S_4| \lesssim \sum_{|\alpha| \leq 1} \left( \| \partial^{\alpha} \tilde{f} (T) \|_{L_2^1, v} + \| \partial^{\alpha} \tilde{f}_0 \|_{L_2^1, v} \right) + \eta \sum_{|\alpha| \leq 1} \int_{0}^{T} \| \tilde{u}_+ - \tilde{u}_- \|_{L_2^1, v} \, dt \]

\[+ C_0 \sum_{|\alpha| \leq 1} \int_{0}^{T} \| (I - P) \partial^{\alpha} \tilde{f} \|_{L_2^1, v}^2 dt + C_0 \int_{0}^{T} \| \tilde{g}, \zeta \|_{L_2^1, v}^2 dt. \]

For \( S_5 \), notice from (4.26) that \( \phi_{a+} = - \phi_{a-} \). Thus, by direction calculations,

\[ S_5 = \sum_{\pm} \int_{0}^{T} \left( (\partial \nabla_x \phi)^\gamma \cdot v \mu^{1/2}, \hat{\phi}_{a \pm} \right)_{L_2^1, v} \, dt \]

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Consequently, applying the same arguments in (4.1), boundary condition \( \hat{\phi}_0(\pm 1, \bar{k}) = 0 \) and Neumann boundary condition (4.4), we deduce from integration by parts that

\[
\sum_{j=1}^{3} ((\partial_{x_j} \phi)^\wedge, \partial_{x_j} \hat{\phi}_0)_{L^2_{\bar{k}}} = (\partial_{x_j} \phi, -\Delta_{x} \hat{\phi}_0)_{L^2_{\bar{k}}} = (\partial_{x_j} \phi, -\partial_{x_j} \Delta_{x} \phi)_{L^2_{\bar{k}}}
\]

\[
= \| \partial_{x_j} \nabla \phi \|_{L^2_{\bar{k}}}. 
\]

Similarly, when \( \partial = I, \partial_{x_j}, \partial_x \), note that \( \partial_{x_j} \hat{\phi}_0(\pm 1) = 0 \) and \( \partial \partial_{x_j} \phi(\pm 1) = 0 \). Then

\[
\sum_{j=1}^{3} ((\partial_{x_j} \phi)^\wedge, \partial_{x_j} \hat{\phi}_0)_{L^2_{\bar{k}}} = (\partial \phi, \partial (\hat{a}_+ - \hat{a}_-))_{L^2_{\bar{k}}} = (\partial \phi, -\partial \Delta_{x} \phi)_{L^2_{\bar{k}}}
\]

\[
= \| \partial \nabla \phi \|_{L^2_{\bar{k}}}. 
\]

Consequently,

\[
S_5 = -10 \int_0^T \| \partial \nabla \phi \|_{L^2_{\bar{k}}} dt. 
\]

Applying the same arguments in (4.19) and (4.21), one can obtain \( S_6 = 0 \). Therefore, combining the above estimates, taking summation of (4.7) over \( |\alpha| \leq 1 \), and then taking the square root and summation over \( \bar{k} \in \mathbb{Z}^2 \), and finally letting \( \eta > 0 \) suitably small, we have

\[
\sum_{|\alpha| \leq 1} \left( \| \partial^\alpha a_+ - \partial^\alpha a_- \|_{L^1_T L^2_{\bar{k}}} + \| \partial^\alpha \nabla \phi \|_{L^1_T L^2_{\bar{k}}} \right)
\]

\[
\leq \sum_{|\alpha| \leq 1} \left( \| \partial^\alpha f(T) \|_{L^1_T L^2_{\bar{k}}} + \| \partial^\alpha f_0 \|_{L^1_T L^2_{\bar{k}}} + \| \partial^\alpha b \|_{L^1_T L^2_{\bar{k}}} + \| (1-\mathbf{P}) \partial^\alpha f \|_{L^1_T L^2_{\bar{k}}} + \| \partial^\alpha g \|_{L^1_T L^2_{\bar{k}}} \right). 
\]

(4.28)

**Step 4. Estimate on \( \hat{a}_+((t, x_1, \bar{k}) + \hat{a}_-((t, x_1, \bar{k}) \).** Similar to the above case, we choose the following two test functions

\[
\hat{\Phi} = \hat{\Phi}_\pm = (|\nu|^2 - 10)(\nu \cdot \nabla \hat{\phi}_\pm(\nu \cdot \nu_{\pm}(t, x_1, \bar{k})))^{1/2}, 
\]

where \( \phi_\pm = (\phi_{a_+}(x_1, \bar{k}), \phi_{a_-}(x_1, \bar{k})) \) solves

\[
-\partial^2_{x_1} \hat{\phi}_\pm + |\bar{k}|^2 \hat{\phi}_\pm = \hat{a}_+ + \hat{a}_-, 
\]

\[
-\partial^2_{x_1} \hat{\phi}_\pm + |\bar{k}|^2 \hat{\phi}_\pm = \hat{a}_+ + \hat{a}_-, 
\]

(4.29)

with boundary condition \( \hat{\phi}_0(\pm 1, \bar{k}) = 0 \) for the case \( \partial = \partial_{x_1} \), and \( \partial_{x_j} \phi_0(\pm 1, \bar{k}) = 0 \) for the case \( \partial = I, \partial_{x_2}, \partial_{x_3} \). When \( |\bar{k}| = 0 \) and \( \partial = I \), (4.29) is a pure Neumann boundary problem and we need \( \int_{-1}^1 \int_{\mathbb{T}^2} a_+ - a_- \partial_{x_1} \partial\bar{k} \) to ensure its existence and assume \( \phi_\pm \) has zero mean in this case.

For the left hand side of (4.7), taking summation on \( \pm \), we have

\[
- \sum_{\pm} \int_0^T (\partial \hat{\Phi}_\pm f, v \cdot \nabla \hat{\phi}_\pm)_{L^2_{\bar{k}}} dt 
\]

\[
= - \sum_{\pm} \sum_{j,m=1}^{3} \int_0^T (\partial \phi_{a_\pm} + \partial \bar{b} \cdot v + (|\nu|^2 - 3)\partial \bar{c}, v_j v_m (|\nu|^2 - 10) \mu (\partial_{x_j} \partial_{x_m} \phi_{a_\pm})^\wedge)_{L^2_{\bar{k}}} dt 
\]

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Following the same argument as in the Step 3 and using the first equation of (3.4), we have

\[
| S_2 | \leq \sum_{|\alpha| \leq 1} \int_0^T \| \{ I - P \} \partial^\alpha f \|^2_{L^2_t L^2_x} + \sum_{|\alpha| \leq 1} \int_0^T \| \partial^\alpha b \|^2_{L^2_t L^2_x} dt,
\]

and

\[
| S_1 | + | S_3 | + | S_4 | + | S_5 | \leq \| \partial f(T) \|^2_{L^2_t L^2_x} + \| \partial f_0 \|^2_{L^2_t L^2_x} + \eta \int_0^T \| \partial a_+ + \partial a_- \|_{L^2_t} dt + C_\eta \int_0^T \| \partial E \|_{L^2_t} dt + C_\eta \int_0^T \| \partial (\phi, \zeta) \|_{L^2_t} dt.
\]

Also, similar to Step 3, \( S_5 \) vanishes by using the boundary condition for \( \phi_0 \). Thus,\n
\[
\sum_{|\alpha| \leq 1} \| \partial^\alpha a_+ + \partial^\alpha a_- \|_{L^2_t L^2_x} \leq \sum_{|\alpha| \leq 1} \left( \| \partial^\alpha f(T) \|_{L^1_t L^2_x} + \| \partial f_0 \|_{L^1_t L^2_x} + \| \{ I - P \} \partial^\alpha f \|_{L^1_t L^2_x} \right) + \| \{ I - P \} \partial^\alpha f \|_{L^1_t L^2_x} + \| \{ I - P \} \partial^\alpha f \|_{L^1_t L^2_x} + \| \{ I - P \} \partial^\alpha f \|_{L^1_t L^2_x}.
\]

**Step 5. Energy estimate.** Now we take the linear combination (4.28) + \( \kappa \times (4.22) + \kappa^2 \times (4.25) \) and let \( \kappa, \eta \) sufficiently small, then

\[
\sum_{|\alpha| \leq 1} \left( \| \partial^\alpha (\phi_+, \phi_-, \tilde{b}, \tilde{c}) \|_{L^2_t L^2_x} + \int_0^T \| \partial^\alpha \phi \|_{L^2_t L^2_x} \right) \leq \sum_{|\alpha| \leq 1} \left( \| \partial^\alpha f(T) \|_{L^1_t L^2_x} + \| \partial f_0 \|_{L^1_t L^2_x} + \| \{ I - P \} \partial^\alpha f \|_{L^1_t L^2_x} \right) + \| \{ I - P \} \partial^\alpha f \|_{L^1_t L^2_x} + \| \{ I - P \} \partial^\alpha f \|_{L^1_t L^2_x} + \| \{ I - P \} \partial^\alpha f \|_{L^1_t L^2_x}.
\]

Note that \( | \partial a_+ + \partial a_- |^2 + | \partial a_+ - \partial a_- |^2 = 2 | \partial a_+ |^2 + 2 | \partial a_- |^2 \).

Next we estimate \( g_\pm \). Using a similar argument as (3.21), we have

\[
\int_{\mathbb{Z}^2} \left( \int_0^T \left( \int_{\mathbb{Z}^2} \| \left( \partial^\alpha \phi_l \cdot \nabla v \right) \|_{L^2_t} \right) dt \right)^{1/2} d\Sigma(k)
\]

\[
\leq \int_{\mathbb{Z}^2} \left( \int_0^T \left( \int_{\mathbb{Z}^2} \| \partial E(k - l) \|_{L^2_t} \right) dt \right)^{1/2} d\Sigma(k)
\]

\[
\leq \sum_{|\alpha| \leq 1} \int_{\mathbb{Z}^2} \left( \int_0^T \| \partial^\alpha \phi_l \|_{L^2_t} \right) dt \right)^{1/2} d\Sigma(k)
\]

**Similar to (3.22) and (3.23), we have**

\[
\int_{\mathbb{Z}^2} \left( \int_0^T \left( \frac{1}{2} \left( (\partial^\alpha (\phi, f) \right)^2 \right) \right) dt \right)^{1/2} d\Sigma(k)
\]

\[
\leq \sum_{|\alpha| \leq 1} \int_{\mathbb{Z}^2} \left( \int_0^T \| \partial^\alpha \phi_l \|_{L^2_t} \right) dt \right)^{1/2} d\Sigma(k)
\]

and

\[
\int_{\mathbb{Z}^2} \left( \int_0^T \left( \| \partial^\alpha (f, f) \|^2 \right) \right) dt \right)^{1/2} d\Sigma(k)
\]

\[
\leq \sum_{|\alpha| \leq 1} \int_{\mathbb{Z}^2} \left( \int_0^T \| \partial^\alpha f \|_{L^2_t} \right) dt \right)^{1/2} d\Sigma(k)
\]
Plugging the above three estimates into (4.31), we have
\[
\sum_{|\alpha| \leq 1} \| \partial^{\alpha} (a_{+1}, a_{-1}, b, \bar{c}) \|_{L_t^1 L_x^2}, \sum_{|\alpha| \leq 1} \| \partial^{\alpha} E \|_{L_t^1 L_x^2} \leq \sum_{|\alpha| \leq 1} (\| \partial^{\alpha} f(T) \|_{L_t^1 L_x^2}, \| \partial^{\alpha} f_0 \|_{L_t^1 L_x^2}) + \sum_{|\alpha| \leq 1} \| (I - P) \partial^{\alpha} f \|_{L_t^1 L_x^2} + \sum_{|\alpha| \leq 1} \| \partial^{\alpha} f \|_{L_t^1 L_x^2} \| E \|_{L_t^1 L_x^2} \| \bar{E} \|_{L_t^1 L_x^2}.
\]
This completes the proof of Theorem 4.1.

5. Proof of the main result in torus

In this section, we shall obtain the global existence and large time behavior for system (1.4) in torus. Assume \( \gamma \geq -2 \) for Landau case and \( \gamma + 2s \geq 1, 1/2 < s < 1 \) for Boltzmann case.

Recall that for VPL case, we let \( \vartheta = -\gamma \) when \(-2 \leq \gamma < -1 \) and \( \vartheta = 0 \) when \( \gamma \geq -1 \). For VPB case, we let \( q = 0 \) when \( \gamma + 2s \geq 1, \frac{1}{2} < s < 1 \). Then \( \vartheta \in [1, 2] \) and we define weight \( w \) as in (1.10), where \( q \geq 0 \) and we restrict \( 0 < q < 1 \) when \( \vartheta = 2 \). To obtain the microscopic estimate, we take Fourier transform of (1.4) on \( x \) to obtain
\[
\partial_t f_{\pm} + i\vartheta \cdot k f_{\pm} \pm \frac{1}{2} (\nabla_x \varphi \cdot v f_{\pm})^\pm + (\nabla_x \varphi \cdot \nabla_v f_{\pm})^\pm \pm \nabla_x \varphi \cdot v \mu^{1/2} - L_{\pm} f = \Gamma_{\pm}(f, f).
\]
(5.1)

Denote \( h = e^{\delta t} f \). Multiplying (5.1) with \( e^{\delta t} \) and \( w \), we have
\[
\partial_t (w h_{\pm}) + \frac{qN}{(1 + t)^{N+1}} w^2 h_{\pm} + i\vartheta \cdot k w h_{\pm} \pm \frac{1}{2} (\nabla_x \varphi \cdot w v h_{\pm})^\pm + (\nabla_x \varphi \cdot w \nabla_v h_{\pm})^\pm
\pm e^{\delta t} \nabla_x \varphi \cdot w \mu^{1/2} - w L_{\pm} \dot{h} = e^{-\delta t} w \Gamma_{\pm}(h, h) + \delta w h_{\pm}.
\]
(5.2)

Taking the inner product of (5.2) with \( w h_{\pm} \) over \( \mathbb{R}^3 \) and taking the real part, we have
\[
\begin{align*}
\frac{1}{2} \partial_t |w h_{\pm}|_{L_x^2}^2 &+ \frac{qN}{(1 + t)^{N+1}} |w v h_{\pm}|_{L_x^2}^2 \pm \frac{1}{2} \text{Re} \left( (\nabla_x \varphi \cdot w v h_{\pm})^\pm, w h_{\pm} \right)_{L_x^2}^\pm \\
&\pm \text{Re} \left( (\nabla_x \varphi \cdot w \nabla_v h_{\pm})^\pm, w h_{\pm} \right)_{L_x^2}^\pm \pm \text{Re} \left( e^{\delta t} \nabla_x \varphi \cdot w \mu^{1/2}, w h_{\pm} \right)_{L_x^2}^\pm - \text{Re} \left( (w^2 L_{\pm} \dot{h}, h_{\pm}) \right)_{L_x^2}^\pm \\
&= \text{Re} \left( e^{-\delta t} w^2 T_{\pm}(h, h), h_{\pm} \right)_{L_x^2}^\pm + \delta |w h_{\pm}|_{L_x^2}^2.
\end{align*}
\]
(5.3)

We will take summation on (5.3) over \( \pm \), integration over \( t \in [0, T] \), the square root and integration over \( k \in \mathbb{Z}^3 \). So, we write the following estimates to control the trouble terms.

For the third term on the left hand side of (5.3), recalling \( E = -\nabla_x \varphi \), we have
\[
\left| \int_0^T \frac{1}{2} (\nabla_x \varphi \cdot v h_{\pm})^\pm, w^2 h_{\pm} \right|_{L_x^2} dt \leq \int_0^T \int_{\mathbb{R}^3} \left| e^{\delta t} \bar{E}(k - l) \right| \left| e^{\delta t} w(v) \bar{h}(l) \right|_{L_x^2} d\Sigma(l) \left| e^{\delta t} w(v) \bar{h}(k) \right|_{L_x^2} dt \leq \int_0^T C_q \left( \int_{\mathbb{R}^3} \left| e^{\delta t} \bar{E}(k - l) \right| \left| e^{\delta t} w(v) \bar{h}(l) \right|_{L_x^2} d\Sigma(l) \right)^2 + \eta^2 \left| e^{\delta t} w(v) \bar{h}(k) \right|_{L_x^2}^2 dt.
\]
Taking the square root and summation over \( k \in \mathbb{Z}^3 \), we have
\[
\begin{align*}
\int_{\mathbb{R}^3} \left( \int_0^T \frac{1}{2} (\nabla_x \varphi \cdot v h_{\pm})^\pm, w^2 h_{\pm} \right)_{L_x^2} dt \left| \bar{E}(k) \right|_{L_x^2} dt \leq & C_q \int_{\mathbb{R}^3} \left( \int_0^T \left| e^{\delta t} \bar{E}(k - l) \right| \left| e^{\delta t} w(v) \bar{h}(l) \right|_{L_x^2} d\Sigma(l) \right)^2 + \eta \left| e^{\delta t} w(v) \bar{h}(k) \right|_{L_x^2}^2 dt \leq C_q \int_{\mathbb{R}^3} \left( \int_0^T \left| e^{\delta t} w(v) \bar{h}(l) \right|_{L_x^2} d\Sigma(l) \right)^2 + \eta \left| e^{\delta t} w(v) \bar{h}(k) \right|_{L_x^2}^2 dt.
\end{align*}
\]
\[ \lesssim C_0 \| e^{\text{dt}} \tilde{E} \|_{L^1_t L^\infty_x} \| e^{-\frac{\text{dt}}{2}} w(v) \hat{\mathcal{N}} h \|_{L^1_t L^2_x L^2_v} + \eta \| e^{-\frac{\text{dt}}{2}} w(v) \hat{\mathcal{N}} h \|_{L^1_t L^2_x L^2_v}, \]  

(5.4)

where we used Young’s inequality for integration and Fubini’s Theorem. For Landau case, the forth term on the left hand of (5.3) with integration on \( t \in [0, T] \) can be controlled by

\[
\left| \int_0^T \left( (\nabla_x \phi \cdot \nabla_v h)_{\pm}, w^2 h_{\pm} \right)_{L^2_t} dt \right|
\]

\[
\lesssim \int_0^T \int_{\mathbb{R}^3} |e^{\text{dt}} \tilde{E}(k - l)||w(v)^{\gamma/2} \nabla_v \hat{h}(l)|_{L^2_v} d\Sigma(l) |e^{-\text{dt}} w(v)^{-\gamma/2} \hat{h}(k)|_{L^2_v} dt
\]

\[
\lesssim \int_0^T C_0 \left( \int_{\mathbb{R}^3} |e^{\text{dt}} \tilde{E}(k - l)||e^{-\frac{\text{dt}}{2}} w(v)^{\gamma/2} \nabla_v \hat{h}(l)|_{L^2_v} d\Sigma(l) \right)^2 + \eta^2 \| e^{-\frac{\text{dt}}{2}} w(v)^{-\gamma/2} \tilde{f}(k) \|_{L^2_v}^2 dt.
\]

Taking the square root and integration over \( k \in \mathbb{Z}^3 \), similar to (5.4), we have

\[
\int_{\mathbb{R}^3} \left| \int_0^T \frac{1}{2} \left( (\nabla_x \phi \cdot \nabla_v h)_{\pm}, w^2 h_{\pm} \right)_{L^2_t} dt \right|^{1/2} d\Sigma(k)
\]

\[
\lesssim C_0 \| e^{\text{dt}} \tilde{E} \|_{L^1_t L^\infty_x} \| e^{-\frac{\text{dt}}{2}} h \|_{L^1_t L^2_x L^2_v} + \eta \| e^{-\frac{\text{dt}}{2}} w(v)^{-\gamma/2} \hat{h} \|_{L^1_t L^2_x L^2_v}.
\]

(5.5)

For Boltzmann case, we deal with the forth term on the left hand of (5.3) by interpolation. Indeed, since \( q = 0 \) for Boltzmann case, we have

\[
\left| \int_0^T \left( (\nabla_x \phi \cdot \nabla_v h)_{\pm}, h_{\pm} \right)_{L^2_t} dt \right|
\]

\[
\lesssim \int_0^T \int_{\mathbb{R}^3} \left( \tilde{E}(k - l) \nabla_v \hat{h}(l), \hat{h}(k) \right)_{L^2_v} d\Sigma(l) dt
\]

\[
\lesssim \int_0^T \int_{\mathbb{R}^3} |e^{\text{dt}} \tilde{E}(k - l)||e^{-\frac{\text{dt}}{2}} (v) \tilde{f}(D_v)^{\alpha} \hat{h}(l)_{L^2_v}|e^{-\frac{\text{dt}}{2}} (v) \tilde{f}(D_v)^{1-\gamma} \hat{h}(k)|_{L^2_v} d\Sigma(l) dt
\]

(5.6)

Here we shall deal with the term \( |e^{-\frac{\text{dt}}{2}} (v) \tilde{f}(D_v)^{1-\gamma} \hat{h}(k)|_{L^2} \) in (5.6). By Young’s inequality,

\[
\langle v \rangle^{\frac{\gamma}{2} - \frac{\alpha}{2}} \langle \eta \rangle^{1-\gamma} \lesssim \langle v \rangle^{\frac{\gamma}{2} - \frac{\alpha}{2} - \frac{\gamma}{2}} + \langle v \rangle^{\frac{\gamma}{2} - \frac{\alpha}{2} - \frac{\gamma}{4}},
\]

where \( \eta \) is the Fourier variable of \( v \). Similar calculation can be applied on derivatives of \( \langle v \rangle^{\frac{\gamma}{2} - \frac{\alpha}{2}} \langle \eta \rangle^{1-\gamma} \). Thus, \( \langle v \rangle^{\frac{\gamma}{2} - \frac{\alpha}{2}} \langle \eta \rangle^{1-\gamma} \) belongs to symbol class \( S(\langle v \rangle^{\frac{\gamma}{2} - \frac{\alpha}{2} - \frac{\gamma}{4}}, \langle v \rangle^{\frac{\gamma}{2} - \frac{\alpha}{2} - \frac{\gamma}{2}}) \). Applying [6, Lemma 2.4 and Corollary 2.5], we have

\[
\| |v|^{\frac{\gamma}{2} - \frac{\alpha}{2}} (D_v)^{1-\gamma} \hat{h}(k) \|_{L^2_v} \lesssim \langle |v| \rangle^{\frac{\gamma}{2} - \frac{\alpha}{2}} (D_v)^{1-\gamma} \hat{h}(k)_{L^2_v} + \langle |v| \rangle^{\frac{\gamma}{2} - \frac{\alpha}{2} - \frac{\gamma}{4}} \hat{h}(k)_{L^2_v}.
\]

Recall that \( \frac{\gamma}{2} \geq -\frac{\alpha}{2(2\gamma - 1)} \) for Boltzmann case in our setting. Thus, substituting the above estimate into (5.6), taking square root and summation over \( k \in \mathbb{Z}^3 \) of the resultant estimate, we have

\[
\int_{\mathbb{R}^3} \left| \int_0^T \left( (\nabla_x \phi \cdot \nabla_v h)_{\pm}, h_{\pm} \right)_{L^2_t} dt \right|^{1/2} d\Sigma(k)
\]

\[
\lesssim \int_{\mathbb{R}^3} \left( \int_0^T \left( e^{\text{dt}} \tilde{E}(k - l)||e^{-\frac{\text{dt}}{2}} (v) \tilde{f}(D_v)^{\alpha} \hat{h}(l)_{L^2_v} + \eta^2 |e^{-\frac{\text{dt}}{2}} (v) \tilde{f}(D_v)^{1-\gamma} \hat{h}(k)|_{L^2_v} dt
\]

\[
\lesssim \| e^{\text{dt}} \tilde{E} \|_{L^1_t L^\infty_x} \| e^{-\frac{\text{dt}}{2}} h \|_{L^1_t L^2_x L^2_v} + \eta \| e^{-\frac{\text{dt}}{2}} w(v) \hat{h} \|_{L^1_t L^2_x L^2_v}.
\]

(5.7)

For the fifth term on the left hand of (5.3) when \( q = 0 \), we take the summation over \( \pm \) to obtain

\[
\sum_{\pm} \pm \text{Re}(e^{\text{dt}} \nabla_x \phi \cdot \nu_{\pm}^{1/2}, \nabla_h_{\pm})_{L^2_t} = \text{Re}(e^{\text{dt}} \tilde{E} \cdot \nu, \nabla_x \phi) = \partial_k |e^{\text{dt}} \tilde{E}|^2 - \delta |e^{\text{dt}} \tilde{E}|^2.
\]

(5.9)
For the fifth left-hand term of (5.3) when \( q > 0 \), we write an upper bound:
\[
|\text{Re}(e^{st\nabla_x \phi} \cdot v \mu^{1/2}, w^2 \hat{h}_\pm)|_{L^2_x}^2 \lesssim |e^{st\hat{E}}|^2 + |\langle v \rangle|^{2+}\hat{h}_\pm|^2_{L^2_x}.
\] (5.10)

For the sixth term on the left hand of (5.3), when \( q = 0 \), we take the summation over \( \pm \) to obtain
\[
\sum_{\pm} (L_x \hat{h}_\pm, \hat{h}_\pm)_{L^2_x} \geq \lambda \| (I - P) \hat{h} \|_{L^2_x}^2.
\] (5.11)

When \( q > 0 \), we use (2.4) to deduce that
\[
\sum_{\pm} (w^2 L \pm \hat{h}_\pm, \hat{h}_\pm)_{L^2_x} \geq \lambda \| \hat{h} \|_{L^2_{x,y}}^2 - C \| \hat{h} \|_{L^2_{x,y}}^2,
\] (5.12)
for some \( \lambda, C > 0 \). For the first term on the right hand of (5.3), taking integration over \( t \in [0, T] \), by (2.6), we have
\[
\int_0^T \left| \int_{\mathbb{R}^3} \left( e^{-stw^2} \Gamma \langle h, \hat{h}_\pm \rangle_{L^2_x} \right) dt \right|^2 \lesssim C_0 \| \hat{h} \|_{L^1_{t,x} L^2_y L^{4/3} T} \| \hat{h} \|_{L^1_{t,x} L^2_y L^{4/3} T}^2 + \eta \| \hat{h} \|_{L^1_{t,x} L^2_y L^{4/3} T}^2.
\] (5.13)

Now we take the summation on (5.3) over \( \pm \), integration over \( t \in [0, T] \), the square root and then integration over \( k \in \mathbb{Z}^3 \). If \( q = 0 \) in (1.10), then combining estimates (5.4), (5.5), (5.7), (5.9), (5.11) and (5.13), we have
\[
\| \hat{h} \|_{L^1_{t,x} L^2_y L^{4/3} T} \| \hat{h} \|_{L^1_{t,x} L^2_y L^{4/3} T} + \| (I - P) \hat{h} \|_{L^1_{t,x} L^2_y L^{4/3} T} \lesssim \int_0^T \left( \int_{\mathbb{R}^3} \left( e^{-stw^2} \Gamma \langle h, \hat{h}_\pm \rangle_{L^2_x} \right) dt \right)^2 \leq C_0 \| \hat{h} \|_{L^1_{t,x} L^2_y L^{4/3} T} \| \hat{h} \|_{L^1_{t,x} L^2_y L^{4/3} T}^2 + \eta \| \hat{h} \|_{L^1_{t,x} L^2_y L^{4/3} T}^2.
\] (5.14)

If \( q \neq 0 \), then combining estimates (5.4), (5.5), (5.7), (5.10), (5.12) and (5.13), we have
\[
\| \hat{h} \|_{L^1_{t,x} L^2_y L^{4/3} T} \| \hat{h} \|_{L^1_{t,x} L^2_y L^{4/3} T} \lesssim \| \hat{f} \|_{L^1_{t,x} L^2_y L^{4/3} T} \| \hat{f} \|_{L^1_{t,x} L^2_y L^{4/3} T}^2 + \| (I - P) \hat{h} \|_{L^1_{t,x} L^2_y L^{4/3} T}^2.
\] (5.15)

Noticing \(-\gamma \leq \vartheta\), we take combination (5.14) \( + \delta^{1/2} \times (1.15) \) and let \( \eta \ll \delta < 1 \) sufficiently small to obtain
\[
\| \hat{h} \|_{L^1_{t,x} L^2_y L^{4/3} T} \| \hat{h} \|_{L^1_{t,x} L^2_y L^{4/3} T} \lesssim \| \hat{f} \|_{L^1_{t,x} L^2_y L^{4/3} T} \| \hat{f} \|_{L^1_{t,x} L^2_y L^{4/3} T}^2 + \| (I - P) \hat{h} \|_{L^1_{t,x} L^2_y L^{4/3} T}^2.
\] (5.16)
The terms with velocity weight such as \( \|e^{-\frac{t}{2}}w(v)^{1/2}\hat{h}\|_{L^1_tL^2_xL^2_{\gamma}} \) are controlled by extra dissipation term \( \|e^{\delta t}w(v)^{1/2}\hat{h}\|_{L^1_tL^2_xL^2_{\gamma}} \).

Now we are in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** Recall that \( h = e^{\delta t}f \). Adding the weight \( e^{\delta t} \) in (3.1), we have
\[
\partial_\nu h + iv \cdot k h = e^{\delta t}\nabla_x \phi \cdot \mu^{1/2} - L_{\pm}h = e^{\delta t}\tilde{g}_{\pm} + \delta h_{\pm}.
\]

Note that, compared to (3.1), the only extra term is \( \delta h_{\pm} \). Then following the same argument in Theorem 3.1, we deduce the macroscopic estimate:
\[
\|e^{\delta t}(a^+, a^-, b, c)\|_{L^1_tL^2_x} + \|e^{\delta t}\hat{E}\|_{L^1_tL^2_x} \lesssim \|\hat{h}\|_{L^1_tL^2_xL^2_{\gamma}} + \|\hat{f}_0\|_{L^1_tL^2_x} \\
+ \|\hat{E}\|_{L^1_tL^2_x} + \|e^{\delta t}\hat{E}\|_{L^1_tL^2_x} \lesssim \|\hat{h}\|_{L^1_tL^2_xL^2_{\gamma}} + \|\hat{f}_0\|_{L^1_tL^2_x} \\
+ \|\hat{E}\|_{L^1_tL^2_x} + \|e^{\delta t}\hat{E}\|_{L^1_tL^2_x} + \delta^{1/2}\|\hat{h}\|_{L^1_tL^2_xL^2_{\gamma}}.
\]

where we have an extra term \( \delta^{1/2}\|\hat{h}\|_{L^1_tL^2_xL^2_{\gamma}} \) compared to Theorem 3.1. Now we take the combination (5.16) + \kappa \times (5.17) with sufficiently small \( \kappa, \delta, \eta > 0 \) to obtain
\[
\|\hat{h}\|_{L^1_tL^2_xL^2_{\gamma}} + \|e^{\delta t}\hat{E}\|_{L^1_tL^2_xL^2_{\gamma}} + \|\hat{h}\|_{L^1_tL^2_xL^2_{\gamma}} + \|e^{\delta t}\hat{E}\|_{L^1_tL^2_xL^2_{\gamma}} \\
+ \delta^{1/2}\|w\|_{L^1_tL^2_xL^2_{\gamma}} + \|\hat{h}\|_{L^1_tL^2_xL^2_{\gamma}} + \|e^{\delta t}\hat{E}\|_{L^1_tL^2_xL^2_{\gamma}} \\
\lesssim \|\hat{f}_0\|_{L^1_tL^2_xL^2_{\gamma}} + \|\hat{E}\|_{L^1_tL^2_xL^2_{\gamma}} + \|\hat{h}\|_{L^1_tL^2_xL^2_{\gamma}} + \|e^{\delta t}\hat{E}\|_{L^1_tL^2_xL^2_{\gamma}} + \|\hat{h}\|_{L^1_tL^2_xL^2_{\gamma}}.
\]

Next we discuss the result in the following two cases.

**Case 1:** \( \gamma \geq -1 \) for VPL case and \( \gamma + 2s \geq 1, 1/2 \leq s < 1 \) for VPB case. In this case, we have \( \|w(v)^{1/2}\|_{L^2_x} \lesssim \|w\|_{L^2_{\nu}L^4_x} \) and \( \|\hat{h}\|_{L^1_tL^2_xL^2_{\gamma}} \lesssim \|\hat{h}\|_{L^1_tL^2_xL^2_{\gamma}} \),)

Choosing \( q = 0 \) in (1.10) and \( \delta \) in (5.17) sufficiently small, we obtain
\[\mathcal{E}_T + \mathcal{D}_T \lesssim \|\hat{f}_0\|_{L^1_tL^2_xL^2_{\gamma}} + \|\hat{E}\|_{L^1_tL^2_x} + \mathcal{E}_T \mathcal{D}_T,\]

where \( \mathcal{E}_T \) and \( \mathcal{D}_T \) are defined by (1.11) and (1.12) respectively. Under the smallness of \( \|\hat{f}_0\|_{L^1_tL^2_xL^2_{\gamma}} + \|\hat{E}\|_{L^1_tL^2_xL^2_{\gamma}} \), it’s now standard to apply the continuity argument to obtain
\[\mathcal{E}_T + \mathcal{D}_T \lesssim \|\hat{f}_0\|_{L^1_tL^2_xL^2_{\gamma}} + \|\hat{E}\|_{L^1_tL^2_x}.
\]

This completes the case \( \gamma \geq -1 \) for VPL systems and \( \gamma + 2s \geq 1, 1/2 \leq s < 1 \) for VPB systems.

**Case 2:** \( -2 \leq \gamma < -1 \) for VPL case. We need to apply the estimate with time-velocity weight. In this case, we have \( \|w\|_{L^1_tL^2_xL^2_{\gamma}} \leq \|\hat{h}\|_{L^1_tL^2_xL^2_{\gamma}} \) and
\[
\|e^{-\frac{t}{2}}w(v)^{1/2}\hat{h}\|_{L^1_tL^2_xL^2_{\gamma}} \lesssim \sqrt{\delta q N} \|\hat{h}\|_{L^1_tL^2_xL^2_{\gamma}}.
\]

Then letting \( \delta > 0 \) small enough in (5.18), we obtain
\[\mathcal{E}_T + \mathcal{D}_T + \mathcal{E}_{T,w} + \mathcal{D}_{T,w} \lesssim \|\hat{f}_0\|_{L^1_tL^2_xL^2_{\gamma}} + \|\hat{E}\|_{L^1_tL^2_x} + (\mathcal{E}_{T,w} + \mathcal{E}_T)(\mathcal{D}_{T,w} + \mathcal{D}_T),\]

where \( \mathcal{E}_{T,w} \) and \( \mathcal{D}_{T,w} \) are defined by (1.13) and (1.14) respectively. Under the smallness of \( \|\hat{f}_0\|_{L^1_tL^2_xL^2_{\gamma}} \) and \( \|\hat{E}\|_{L^1_tL^2_xL^2_{\gamma}} \), we obtain (1.19) by using continuity argument and local existence from Section 7. □

Next we prove the propagation of initial regularity.

**Proof of Theorem 1.2.** Following closely the proofs of from (5.4) to (5.7) and Theorem 3.1, one can show the analogous higher order trilinear estimates. That is, for \( m \geq 0 \),
\[
\int_{\mathbb{R}^3} \left| \int_0^T \left( (\nabla_x \phi \cdot w(v_{x}f_{x})^\wedge, (k)^{2m}w(f_{x})^\wedge_{L^2_{x}} dt \right)^{1/2} \right| d\Sigma(k) \lesssim C_{\eta'} \|e^{\delta t}\hat{E}\|_{L^1_tL^2_xL^2_{\gamma}} \\|v \cdot \hat{h}\|_{L^1_tL^2_xL^2_{\gamma}} + \eta \|e^{-\frac{t}{2}}w(v)^{1/2}\hat{f}\|_{L^1_tL^2_xL^2_{\gamma}}.
\]

\[
\int_{\mathbb{R}^3} \left| \int_0^T (\nabla_x \phi \cdot w\nabla_x f_{x})^\wedge, (k)^{2m}w(f_{x})^\wedge_{L^2_{x}} dt \right|^{1/2} d\Sigma(k)
\]

29
\[
\begin{align*}
\langle k \rangle^m \sum \partial_t \overline{w h} + \frac{q N N^{(0)}}{(1 + t)^N} \overline{w h} + v \cdot \nabla w \partial h = e^{\partial_t f} w h = e^{\partial_t f} (\partial_t \langle h \rangle, h) + \delta w h.
\end{align*}
\]

Taking Fourier transform \(\hat{\overline{w h}}\) over \(\mathbb{T}^3\) and the inner product with \(\hat{w} \overline{h}\) over \([-1, 1] \times \mathbb{R}_v^3\) and the real part, we have

\[
\begin{align*}
\left( \frac{1}{2} \right) \int_{\mathbb{T}^3} \overline{w h} (1) - v \overline{w h} (-1) dv
\end{align*}
\]

Following carefully the argument from (5.4) to (5.13), by replacing Fourier transform on torus \(\mathbb{T}^3\) by Fourier transform on \(\mathbb{T}^2\), we have the following estimates.
Lemma 6.1. Assume $\gamma \geq -2$ in Landau case and $\gamma + 2s \geq 1$, $\frac{1}{2} \leq s < 1$ in Boltzmann case. Let $\partial = \partial^\ast$ with $|\alpha| \leq 1$. For any $T > 0$, $\eta > 0$, we have

$$
\int_{\mathbb{Z}^2} \left| \int_0^T \frac{1}{2} \left( w^2 (\partial (\nabla_x \phi \cdot v h) + \nabla_t h) \right) \cdot d\Sigma(k) \right|^2 dt \leq C_\eta \sum_{|\alpha| \leq 1} \| e^{\delta t} \partial^\ast \nabla_x \phi \|_{L_2^1} \sum_{|\alpha| \leq 1} \| e^{-\delta t} \langle v \rangle \langle w \partial^\ast h \rangle \|_{L_2^1},
$$

Moreover, for the term $\nabla_x \phi \cdot \nabla_t h$, we have

$$
\int_{\mathbb{Z}^2} \left| \int_0^T (\partial (\nabla_x \phi \cdot v h) + \nabla_t h) \cdot dt \right|^2 d\Sigma(k) \leq C_\eta \sum_{|\alpha| \leq 1} \| w \partial^\ast f \|_{L_2^1} \sum_{|\alpha| \leq 1} \| \nabla \phi \|_{L_2^1} \sum_{|\alpha| \leq 1} \| e^{-\delta t} \langle v \rangle \langle w \partial^\ast h \rangle \|_{L_2^1},
$$

for Landau case, and

$$
\int_{\mathbb{Z}^2} \left| \int_0^T (\nabla_x \phi \cdot \nabla_t h, h) \cdot dt \right|^2 d\Sigma(k) \leq C_\eta \sum_{|\alpha| \leq 1} \| e^{\delta t} \partial^\ast \nabla_x \phi \|_{L_2^1} \sum_{|\alpha| \leq 1} \| e^{-\delta t} \langle v \rangle \langle w \partial^\ast h \rangle \|_{L_2^1} + \eta \| e^{-\delta t} \langle v \rangle \langle w \partial^\ast h \rangle \|_{L_2^1}.
$$

for Boltzmann case.

Proof. Here we only prove (6.2), since the other estimates are similar to the argument from (5.4) to (5.13) by using the Banach algebra structure of $H_{1}^{1}$. We also assume $\partial = \partial_{z_1}$, since the other cases are similar. Noticing

$$
\partial (\nabla_x \phi \cdot v f) = \partial \nabla_x \phi \cdot v f_{\pm} + \nabla_x \phi \cdot v \partial f_{\pm},
$$

one has

$$
\left| \int_0^T \frac{1}{2} \left( w^2 (\partial (\nabla_x \phi \cdot v f_{\pm}) + \nabla_t f_{\pm}) \cdot dt \right) \right|^2 \leq \int_0^T \int_{\mathbb{Z}^2} \int_0^1 \int_{\mathbb{Z}^2} \left| \nabla \nabla_x \phi (k-l) \langle v \rangle \nabla f (l) \right| + |\nabla \phi (k-l) \langle v \rangle w \partial f (l) | \cdot d\Sigma(l) \times |w \partial f (k)| \cdot dx_1 dv dt \leq \int_0^T \int_{\mathbb{Z}^2} \int_{-1}^1 \left| e^{\delta t} \partial^\ast \nabla_x \phi (k-l) \langle v \rangle \langle w \partial^\ast f (l) \rangle \right|_{L_2^1} + \eta \left| e^{-\delta t} \langle v \rangle \langle w \partial^\ast f (l) \rangle \right|_{L_2^1} d\Sigma(l) \cdot dx_1 dv dt.
$$
Taking the square root and summation over $k \in \mathbb{Z}^2$, we have

\[
\int_{\mathbb{Z}^2} \left| \int_0^T \frac{1}{2} \left( w(\partial(\nabla_x \phi \cdot v f_\pm)) \wedge w\partial h_\pm \right)_{L^1_v} dt \right|^{1/2} d\Sigma(k)
\]

\[
\leq C_0 \int_{\mathbb{Z}^2} \int_{\mathbb{Z}^2} \left( \int_0^T \int_{-1}^1 |e^{i\theta T} \nabla_x \phi(\hat{k} - \bar{l})|^2 |e^{-i\omega T} (v) + v_1|^2 w(\phi f_\pm) \partial h_\pm \wedge \partial h_\pm dx_1 \right) d\Sigma(\bar{l}) d\Sigma(\hat{k})
\]

\[
+ C_0 \int_{\mathbb{Z}^2} \int_{\mathbb{Z}^2} \left( \int_0^T \int_{-1}^1 |e^{i\theta T} \nabla_x \phi(\hat{k} - \bar{l})|^2 |e^{-i\omega T} (v) \partial f(\phi f_\pm) \partial h_\pm \wedge \partial h_\pm dx_1 \right) d\Sigma(\bar{l}) d\Sigma(\hat{k})
\]

\[
+ \eta \|e^{-i\omega T} (v) \partial f(\phi f_\pm) \wedge \partial h_\pm \|_{L^1_v L^2_v L^1_v L^2_v}.
\]

Here we have used Young's inequality and Fubini's Theorem.

We are ready to prove the result in finite channel.

**Proof of Theorem 1.3.** Define $h = e^{i\theta T} f$ and let $\partial = \partial^\alpha$ with $|\alpha| \leq 1$. Applying $\partial$ on (1.4) and multiplying with $w e^{i\theta T}$, we have

\[
\partial_t w \partial h_\pm + \frac{qN(v)\hat{\theta}}{(1 + t)^{N+1}} w \partial h_\pm + v \cdot \nabla_x w (\partial h_\pm) + \frac{1}{2} w \partial(\nabla_x \phi \cdot v h_\pm) + w \partial(\nabla_x \phi \cdot \nabla_x h_\pm)
\]

\[
\pm e^{i\theta T} \partial^\alpha \phi \cdot \nu \mu^{1/2} - w L_\pm \partial h = w e^{-i\theta T} (\partial^\alpha (h, h)) + \delta w h_\pm.
\]

With time weight $e^{i\omega T}$, the only extra term is $\delta w h_\pm$. Then following the argument we used in Theorem 4.1, we have

\[
\sum_{|\alpha| \leq 1} \|e^{i\theta T} \partial^\alpha (\hat{\sigma}^\alpha, \hat{\sigma}^\alpha, \hat{\sigma}^\alpha, \hat{\sigma}^\alpha, \hat{\sigma}^\alpha, \hat{\sigma}^\alpha)\|_{L^1_v L^2_v L^2_v} + \sum_{|\alpha| \leq 1} \|e^{i\theta T} \partial^\alpha E\|_{L^1_v L^2_v L^2_v}
\]

\[
\leq \sum_{|\alpha| \leq 1} \|\partial^\alpha h(T)\|_{L^1_v L^2_v L^2_v} + \|\partial^\alpha f_0\|_{L^1_v L^2_v L^2_v}
\]

\[
+ \sum_{|\alpha| \leq 1} \left( \|e^{i\theta T} \partial^\alpha E\|_{L^1_v L^2_v L^2_v} + \|\partial^\alpha \hat{h}\|_{L^1_v L^2_v L^2_v L^2_v} \right) \sum_{|\alpha| \leq 1} \|\partial^\alpha \hat{h}\|_{L^1_v L^2_v L^2_v L^2_v}
\]

\[
+ \sum_{|\alpha| \leq 1} \left( \|\{I - P\} \partial^\alpha \hat{h}\|_{L^1_v L^2_v L^2_v L^2_v} + \delta^{1/2} \|\partial^\alpha \hat{h}\|_{L^1_v L^2_v L^2_v L^2_v} \right).
\]

(6.3)

Note from (4.15) and (4.16) that

\[
|\partial f_\pm(-1, \hat{k}, -v_1, \hat{v})| = |\partial f_\pm(-1, \hat{k}, v_1, \hat{v})|, \quad |\partial f_\pm(1, \hat{k}, -v_1, \hat{v})| = |\partial f_\pm(1, \hat{k}, v_1, \hat{v})|
\]

on $v_1 \neq 0$. Thus, by change of variable $v_1 \mapsto -v_1$,

\[
\int_{\mathbb{R}^3} |v_1| \partial h(1)|^2 - v_1 |\partial h(-1)|^2 dv = 0.
\]

(6.5)

Then (6.1) becomes

\[
\frac{1}{2} \partial_t \|w \partial h_\pm\|^2_{L^2_v L^1_v} + \frac{qN(v)}{(1 + t)^{N+1}} \|w \partial h_\pm\|_{L^2_v L^1_v}^2 \leq \text{Re}(w \partial h_\pm \partial h_\pm)_{L^2_v L^1_v}
\]

\[
+ \frac{1}{2} \text{Re}(w(\partial(\nabla_x \phi \cdot v h_\pm)) \wedge w \partial h_\pm)_{L^2_v L^1_v} \pm \text{Re}(w(\partial(\nabla_x \phi \cdot \nabla_x h_\pm)) \wedge w \partial h_\pm)_{L^2_v L^1_v}
\]

\[
\pm \text{Re}(e^{i\theta T} w(\partial(\nabla_x \phi \cdot v h_\pm)) \wedge w \partial h_\pm)_{L^2_v L^1_v} + \text{Re}(e^{-i\theta T} w(\partial(\nabla_x \phi \cdot h_\pm)) \wedge w \partial h_\pm)_{L^2_v L^1_v}.
\]
If \( q \neq 0 \), the second to fifth terms on the right hand of (6.6) can be estimated by using Lemma 6.1. If \( q = 0 \), we further look at the forth term of the right hand of (6.6):

\[
\sum_{\pm} \pm \text{Re}(\partial_{x} \phi \cdot v_{\mu}^{1/2}, \partial f_{\pm})_{L^{2}_{2}}, = \text{Re}(\partial_{x} \phi \cdot \partial_{x} G)_{L^{2}_{2}},
\]

where \( G \) is defined in (3.6). Note from (1.7), (4.15) and (4.16) we have the following. \( \partial \phi(\pm 1) = 0 \) when \( \partial = \partial_{x_{1}} \). For the case \( \partial = 1, \partial_{x_{2}}, \partial_{x_{3}} \), we know that \( v_{1} \mu^{1/2} \) is odd with respect to \( v_{1} \) and hence, \( \partial G(\pm 1) = 0 \). In any cases, we have \( (\partial \phi(1) | \partial G(1)) = (\partial \phi(1) | \partial G(1)) = 0 \). Then multiplying \( e^{2Lt} \), we have from (3.5) and (3.3) that

\[
e^{2Lt} \sum_{\pm} \pm \text{Re}(\partial_{x} \phi \cdot v_{\mu}^{1/2}, \partial f_{\pm})_{L^{2}_{2}}, = -e^{2Lt} \text{Re}(\partial_{x} \phi, (\partial_{t} \phi(a + a - a)) \gamma)_{L^{2}_{2}},
\]

\[
e^{2Lt} \text{Re}(\partial_{x} \phi, (\partial_{t} \phi(a + a - a)) \gamma)_{L^{2}_{2}} = \frac{1}{2} \delta t \| e^{Lt} \partial_{x} \phi \|_{L^{2}_{2}}^{2} - \delta \| e^{Lt} \partial_{x} \phi \|_{L^{2}_{2}},
\]

Together with Lemma 6.1, taking summation on \( \pm \) of (6.6), integration on \( t \in [0, T] \), absolute value, square root and summation over \( k \in \mathbb{Z}^{2} \) and \( |\alpha| \leq 1 \), we have that when \( q = 0 \),

\[
\sum_{|\alpha| \leq 1} \| \partial^{\alpha} h_{0} \|_{L^{1}_{1} L^{\infty}_{T} L^{2}_{2}}, + \|1 - P\| \partial^{\alpha} h_{0} \|_{L_{T}^{1} L_{2}^{2}}, + \| e^{Lt} \partial^{\alpha} E_{0} \|_{L^{1}_{1} L^{2}_{2}}, + \eta \sum_{|\alpha| \leq 1} \| e^{-\frac{\eta}{2}} (v) \gamma^{1/2} \partial^{\alpha} h_{0} \|_{L^{1}_{1} L^{2}_{2}}, + D_{T},
\]

\[
+ \| e^{Lt} \partial^{\alpha} \nabla_{x} \phi \|_{L^{2}_{2}}, (6.7),
\]

where \( E_{T} \) and \( D_{T} \) are given by (1.15) and (1.16) respectively. Similarly, when \( q \neq 0 \), we use (2.4) to find that

\[
\sum_{|\alpha| \leq 1} \left( \| e^{-\frac{\eta}{2}} (v) \gamma^{1/2} \partial^{\alpha} h_{0} \|_{L^{1}_{1} L^{2}_{2}}, + \| e^{Lt} \partial^{\alpha} E_{0} \|_{L^{1}_{1} L^{2}_{2}}, + \| \partial^{\alpha} h_{0} \|_{L_{T}^{2}} \right) + \eta \sum_{|\alpha| \leq 1} \| e^{-\frac{\eta}{2}} (v) \gamma^{1/2} \partial^{\alpha} h_{0} \|_{L^{1}_{1} L^{2}_{2}}, + D_{T},
\]

\[
+ \| e^{Lt} \partial^{\alpha} \nabla_{x} \phi \|_{L^{2}_{2}}, (6.8),
\]

where \( E_{T,w} \) and \( D_{T,w} \) are given by (1.17) and (1.18). Notice that \( \| \partial^{\alpha} h_{\pm} \|_{L^{1}_{1} L^{2}_{2}}, \lesssim \| \partial^{\alpha} h_{\pm} \|_{L^{1}_{1} L^{2}_{2}}, L_{2} \). Then taking linear combination (6.7) + \( \kappa \times (6.3) + \kappa^{2} \times (6.8) \) and letting \( \delta, \eta, \kappa > 0 \) suitably small, we have

\[
E_{T} + D_{T} + E_{T,w} + D_{T,w} \lesssim \sum_{|\alpha| \leq 1} \left( \| e^{-\frac{\eta}{2}} (v) \gamma^{1/2} \partial^{\alpha} h_{0} \|_{L^{1}_{1} L^{2}_{2}}, + \| e^{Lt} \partial^{\alpha} E_{0} \|_{L^{1}_{1} L^{2}_{2}}, \right) + \left( C_{\eta} (E_{T} + E_{T,w}) + \eta \right) \sum_{|\alpha| \leq 1} \| e^{-\frac{\eta}{2}} (v) \gamma^{1/2} \partial^{\alpha} h_{0} \|_{L^{1}_{1} L^{2}_{2}}, + D_{T} + D_{T,w} \right) + \eta \sum_{|\alpha| \leq 1} \| e^{-\frac{\eta}{2}} (v) \gamma^{1/2} \partial^{\alpha} h_{0} \|_{L^{1}_{1} L^{2}_{2}}, (6.9)
\]

Then we discuss the \textit{a priori} estimates in two cases.
Case I: $\gamma \geq -1$ for VPL case and $\gamma + 2s \geq 1$, $1/2 \leq s < 1$ for VPB case. In this case, we have
\[
\| e^{-\frac{t}{2}} \langle v \rangle^{\gamma/2} \partial^\alpha f \|_{L^1_k L^2_t L^2_{x,v}} \leq \| e^{-\frac{t}{2}} \langle v \rangle^{1/2} \partial^\alpha f \|_{L^1_k L^2_t L^2_{x,v}} \lesssim \mathcal{D}_T.
\]
Letting $q = 0$ and $\eta > 0$ small enough in (6.9), we have
\[
\mathcal{E}_T + \mathcal{D}_T \lesssim \sum_{|\alpha| \leq 1} \left( \| \partial^\alpha \mathcal{F}_0 \|_{L^1_k L^\infty_t L^2_{x,v}} + \| \partial^\alpha \mathcal{E}_0 \|_{L^1_k L^2_{x,v}}^2 \right) + \mathcal{E}_T \mathcal{D}_T.
\]
This concludes the proof when $\gamma \geq -1$ in Landau case and $\gamma + 2s \geq 1$, $1/2 \leq s < 1$ in Boltzmann case by using the standard continuity argument under the smallness of (1.23).

Case II: $-2 \leq \gamma < -1$ for VPL case. In this case, recall that we choose $\vartheta = -\gamma$. Then we have
\[
\| e^{-\frac{t}{2}} w(v) \langle v \rangle^{\gamma/2} \partial^\alpha h \|_{L^1_k L^2_t L^2_{x,v}} \lesssim \| e^{-\frac{t}{2}} w(v)^{1/2} \partial^\alpha h \|_{L^1_k L^2_t L^2_{x,v}} \lesssim \| \langle v \rangle^{\frac{\gamma}{2}} (1 + t) \cdot \frac{\partial^\alpha}{\partial^\alpha h} \|_{L^1_k L^2_t L^2_{x,v}} \lesssim \mathcal{D}_T.
\]
Letting $\eta > 0$ in (6.9) small enough, we have
\[
\mathcal{E}_{T,w} + \mathcal{D}_{T,w} + \mathcal{E}_T + \mathcal{D}_T \lesssim \sum_{|\alpha| \leq 1} \left( \| w \partial^\alpha \mathcal{F}_0 \|_{L^1_k L^2_{x,v}}^2 + \| \partial^\alpha \mathcal{E}_0 \|_{L^1_k L^2_{x,v}}^2 \right) + (\mathcal{E}_T + \mathcal{E}_{T,w})(\mathcal{D}_T + \mathcal{D}_{T,w}).
\]
Once (6.10) is obtained, then (1.22) follows from the standard continuity argument and local existence from Section 7 under the smallness of $\sum_{|\alpha| \leq 1} \left( \| w \partial^\alpha \mathcal{F}_0 \|_{L^1_k L^2_{x,v}} + \| \partial^\alpha \mathcal{E}_0 \|_{L^1_k L^2_{x,v}}^2 \right)$; cf. [18]. This concludes the proof of the global existence and large-time behavior of mild solutions.

The uniqueness of the initial boundary value problem (1.4) can be proved by applying the similar method as the previous energy estimates and is now quite standard. Also the local solution we extend from section 7.1 is unique. So we omit these analogues details. The positivity of the solutions to VPL systems can be obtained from [22, Lemma 12, page 800]. The positivity of solutions to VPB systems can be guaranteed by [19]. This completes the proof of Theorem 1.3.

Proof of Theorem 1.4. We will show that the regularity of the initial data can propagate along time. Let $m \geq 0$. Since $k$ doesn’t concern the boundary $x_1 \in [-1, 1]$, following closely the argument proving Theorem 4.1, Lemma 6.1, we have the following estimates:
\[
\sum_{|\alpha| \leq 1} \left( \| e^{\delta T} \partial^\alpha (\mathcal{A} + \mathcal{B} + \mathcal{C}) \|_{L^1_k L^2_t L^2_{x,v}} + \| e^{\delta T} \partial^\alpha E \|_{L^1_k L^2_t L^2_{x,v}} \right) \lesssim \sum_{|\alpha| \leq 1} \left( \| \partial^\alpha h(T) \|_{L^1_k L^2_{x,v}} + \| \partial^\alpha \mathcal{F}_0 \|_{L^1_k L^2_{x,v}} + \sum_{|\alpha| \leq 1} \| (\mathbf{I} - \mathbf{P}) \partial^\alpha h \|_{L^1_k L^2_{x,v}} \right) \sum_{|\alpha| \leq 1} \left( \| \partial^\alpha \mathcal{E}_0 \|_{L^1_k L^2_t L^2_{x,v}} \right)
\]
and
\[
\int_{\mathbb{R}^2} \left[ \int_0^{T} \left( (\partial T \langle \nabla x \phi \cdot w \partial x \mathbf{h}) \hat{\mathbf{n}} \right)^\wedge, (\hat{k})^{2m} (w \partial x \mathbf{h}) \right) dt \right]^{1/2} d\Sigma(\hat{k}) \lesssim \mathcal{C}_\gamma \sum_{|\alpha| \leq 1} \| e^{\delta T} \partial^\alpha \nabla x \phi \|_{L^1_k L^2_t L^2_{x,v}} \sum_{|\alpha| \leq 1} \| e^{-\frac{t}{2}} \langle v \rangle^{1/2} \partial^\alpha \mathcal{F}_0 \|_{L^1_k L^2_t L^2_{x,v}} \lesssim \sum_{|\alpha| \leq 1} \| e^{-\frac{t}{2}} \langle v \rangle^{1/2} \partial^\alpha \mathcal{F}_0 \|_{L^1_k L^2_t L^2_{x,v}} \lesssim \mathcal{D}_T.
\]

\[
\int_{\mathbb{R}^2} \left[ \int_0^{T} \left( \langle \partial (\nabla \phi \cdot w \nabla x \mathbf{h}) \rangle, (\hat{k})^{2m} (w \partial x \mathbf{h}) \right) \right]^{1/2} d\Sigma(\hat{k}) \lesssim \mathcal{C}_\gamma \sum_{|\alpha| \leq 1} \| e^{\delta T} \partial^\alpha \nabla x \phi \|_{L^1_k L^2_t L^2_{x,v}} \sum_{|\alpha| \leq 1} \| e^{-\frac{t}{2}} \langle v \rangle^{\gamma/2} (w \nabla \phi \partial^\alpha h \rangle \|_{L^1_k L^2_t L^2_{x,v}} \lesssim \sum_{|\alpha| \leq 1} \| e^{-\frac{t}{2}} \langle v \rangle^{\gamma/2} (w \nabla \phi \partial^\alpha h \rangle \|_{L^1_k L^2_t L^2_{x,v}} \lesssim \mathcal{D}_T.
\]
For the boundedness of $\Gamma \pm$, we have
\[
\int_{\mathbb{S}^2} \left( \int_0^T \left( w^2 \partial (\Gamma \pm (f, g))^n, (\hat{k})^{2m} h \right)_{L^2_{1, \pm}} \right) dt \right)^{1/2} d\Sigma(\hat{k}) 
\leq C_\eta \| \partial \xi k \|_{L^1_{k,m} L^2_{1, \pm}} + \eta \| \hat{w} \hat{h} \|_{L^1_{k,m} L^2_{1, \pm}}^{1/8} \| L^1_{k,m} L^2_{1, \pm} \|_{L^2_{1, \pm}.}
\]

Consequently, noticing $(\hat{k})^m$ doesn’t affect the boundary terms, one can apply similar calculation for deriving (6.10) to obtain that
\[
\sum_{|\alpha| \leq 1} \left( \| \hat{w} \hat{\xi} k \|_{L^1_{k,m} L^2_{1, \pm}} + \| \hat{\xi} \hat{w} \|_{L^1_{k,m} L^2_{1, \pm}} + \| \hat{\xi} \hat{w} \|_{L^1_{k,m} L^2_{1, \pm}} \right) 
\leq \sum_{|\alpha| \leq 1} \left( \| \hat{w} \hat{\xi} k \|_{L^1_{k,m} L^2_{1, \pm}}^{2} + \| \hat{\xi} \hat{w} \|_{L^1_{k,m} L^2_{1, \pm}}^{2} \right),
\]

provided that $\epsilon_0 > 0$ in (1.21) is suitably small. This completes the proof of Theorem 1.4 for VPL systems when $-2 \leq \gamma < -1$. The proof of other cases follows similarly and we omit the details for brevity. \hfill \Box

7. LOCAL EXISTENCE

In this section, we are concerned with the local-in-time existence of solutions to problem (1.4). For brevity of presentation, we only give the proof for Vlasov-Poisson-Landau equation with the specular reflection boundary condition in the finite channel, since the other cases are similar.

**Theorem 7.1.** Let $\gamma \geq -2$ and $w$ be given by (1.10). Then there exists $\epsilon_0 > 0$, $T_0 > 0$ such that if $F_0(x_1, x, v) = \mu + v^{1/2} f_0(x_1, x, v) \geq 0$ and
\[
\sum_{|\alpha| \leq 1} \left( \| \hat{w} \hat{\xi} k \|_{L^1_{k,m} L^2_{1, \pm}} + \| \hat{\xi} \hat{w} \|_{L^1_{k,m} L^2_{1, \pm}} \right) \leq \epsilon_0,
\]

then the specular reflection boundary problem for VPL systems (1.4), (1.7), (1.6) and (1.2) admits a unique solution $f(t, x, v)$ on $t \in [0, T_0], x \in \Omega = [-1, 1] \times T^2, v \in T^3$, satisfying estimate
\[
E_{T_0} + D_{T_0} + E_{T_0,w} + D_{T_0,w} \leq \sum_{|\alpha| \leq 1} \left( \| \hat{w} \hat{\xi} k \|_{L^1_{k,m} L^2_{1, \pm}} + \| \hat{\xi} \hat{w} \|_{L^1_{k,m} L^2_{1, \pm}} \right),
\]

where $E_{T_0}, D_{T_0}, E_{T_0,w}$ and $D_{T_0,w}$ are defined by (1.15), (1.16), (1.17) and (1.18) respectively.

We begin with the following linear inhomogeneous problem on the finite channel:
\[
\begin{aligned}
\partial_t f_{\pm} + v \cdot \nabla_x f_{\pm} &\pm \frac{1}{2} \nabla_x \phi \cdot v f_{\pm} \pm \nabla_x \phi \cdot \nabla_v f_{\pm} \\
- \Delta_x \phi &\pm \frac{1}{2} v f_{\pm}^{1/2} - A_{\pm} f = \Gamma_{\pm} (g, h) + K h,
\end{aligned}
\]

for a given $h = h(t, x, v)$ and $\psi = \psi(t, x)$.

**Lemma 7.2.** There exist $\epsilon_0 > 0$, $T_0 > 0$ such if
\[
\sum_{|\alpha| \leq 1} \left( \| \hat{w} \hat{\xi} k \|_{L^1_{k,m} L^2_{1, \pm}} + \| \hat{\xi} \hat{w} \|_{L^1_{k,m} L^2_{1, \pm}} \right)
\]
and similarly, 

\[ \| \omega \partial^\alpha h \|_{L^2_{1/2}T^2_{-1/2}L^2_{-1/2}} + \| \partial^\beta \nabla \psi \|_{L^2_{1/2}T^2_{-1/2}L^2_{-1/2}} \leq \varepsilon_0, \quad (7.3) \]

then the initial boundary value problem (7.2) admits a unique weak solution \( f = f(t,v) \) on \([-1,1] \times T^2 \times \mathbb{R}^3 \) satisfying

\[ E_{T_0} + D_{T_0} + E_{T_0,w} + D_{T_0,w} \]

\[ \lesssim \sum_{|\alpha| \leq 1} \left( \| \omega \partial^\alpha f_0 \|_{L^2_{1/2}T^2_{-1/2}L^2_{-1/2}} + \| \partial^\beta E_0 \|_{L^2_{1/2}T^2_{-1/2}L^2_{-1/2}} \right) + T_0^{1/2} \sum_{|\alpha| \leq 1} \| \partial^\alpha h \|_{L^2_{1/2}T^2_{-1/2}L^2_{-1/2}} \quad (7.4) \]

where \( E_{T_0}, D_{T_0}, E_{T_0,w} \) and \( D_{T_0,w} \) are defined by (1.15), (1.16), (1.17) and (1.18) respectively. The weak solution \( f \) is defined by

\[ \int_0^{T_0} (\partial_t \tilde{f}_\pm + v \cdot \nabla_x \tilde{f}_\pm + \frac{1}{2} (\nabla_x \psi \cdot v \tilde{f}_\pm)^\gamma + (\nabla_x \psi \cdot \nabla_v \tilde{f}_\pm)^\gamma \mp \nabla_x \hat{\psi} \cdot v \mu^{1/2} \mp \nabla_x \hat{\psi} \cdot g_{\pm} \tilde{f}_\pm)_{L^2_{1/2,k,v}} \, dt \]

\[ - \int_0^{T_0} (A_\pm f, \tilde{g}_{\pm})_{L^2_{1/2,k,v}} \, dt = \int_0^{T_0} (\Gamma_\pm (h,f), \tilde{g}_{\pm})_{L^2_{1/2,k,v}} + (K h, \tilde{g}_{\pm})_{L^2_{1/2,k,v}} \, dt, \]

for any \( g \) belonging to

\[ \{ g = (g_+, g_-) \in L^\infty_T L^\infty_{x,v} : \tilde{g}(t,-1,k,v,1) \rangle_{v_1 < 0} = \tilde{g}(t,-1,k,+v,1), \]

\[ \tilde{g}(t,1,k,v,1) \rangle_{v_1 < 0} = \tilde{g}(t,1,k,-v,1) \}. \]

Proof. Let \( \eta_0 \) and \( \eta_x \) be the standard mollifier in \( \mathbb{R}^3 \) and \([-1,1] \times T^2 : \eta_0, \eta_x \in C^\infty_c, 0 \leq \eta_0, \eta_x \leq 1, \]

\[ \int \eta_0 dx = 1 \]. For \( \varepsilon > 0 \), let \( \eta^\varepsilon_0(v) = \varepsilon^{-3} \eta_0(\varepsilon^{-1} v) \) and \( \eta^\varepsilon_x(x) = \varepsilon^{-3} \eta_x(\varepsilon^{-1} x) \). Then we mollify the initial data as \( \tilde{f}_0^\varepsilon = f_0 \ast \eta^\varepsilon_0 \ast \eta^\varepsilon_x, E_0^\varepsilon = E_0 \ast \eta^\varepsilon_x \). Note that \( \tilde{f}_0^\varepsilon, E_0^\varepsilon \) are still periodic with respect to \( x \in T^2 \).

Since \( \int_1^{T_1} |\eta^\varepsilon_0|^2 \, dx \leq \int_{T_1} T_2 \eta^\varepsilon_0(\varepsilon x) \, d\varepsilon x = 1 \) and \( |\eta^\varepsilon_0| \leq \int |\eta^\varepsilon_0| \, dx = 1 \), we have

\[ \| \tilde{f}_0^\varepsilon \|_{L^1_{1/2}T^2_{-1/2}L^2_{-1/2}} \leq \| f_0 \ast \eta^\varepsilon_0 \ast \eta^\varepsilon_x \|_{L^1_{1/2}T^2_{-1/2}L^2_{-1/2}} \lesssim \| \eta_0 \|_{L^1_{1/2}} \| \eta^\varepsilon_x \|_{L^1_{1/2}} \| \tilde{f}_0 \|_{L^1_{1/2}T^2_{-1/2}L^2_{-1/2}}, \]

and similarly,

\[ \| \nabla_x \tilde{f}_0^\varepsilon \|_{L^1_{1/2}T^2_{-1/2}L^2_{-1/2}} \leq \| \nabla_x f_0 \|_{L^1_{1/2}T^2_{-1/2}L^2_{-1/2}}, \]

\[ \| E_0^\varepsilon \|_{L^1_{1/2}T^2_{-1/2}L^2_{-1/2}} \leq \| E_0 \|_{L^1_{1/2}T^2_{-1/2}L^2_{-1/2}}, \]

\[ \| \nabla_x E_0^\varepsilon \|_{L^1_{1/2}T^2_{-1/2}L^2_{-1/2}} \leq \| \nabla_x E_0 \|_{L^1_{1/2}T^2_{-1/2}L^2_{-1/2}}. \]

We would like to add the vanishing term \( \varepsilon \langle v \rangle^{30-8|\beta|} \partial^\beta \partial f \) to the first equation of (7.2). Here we pick 30 as a merely large constant. But since there's boundary on \( x_1 \), in order to eliminate the boundary effect, we directly consider the weak form of the solution. For any \( g_{\pm} \), we consider

\[ (\partial_t f_{\pm}, g_{\pm})_{L^2_{-1/2}} + \varepsilon \sum_{|\alpha| + |\beta| \leq 3} \langle \langle v \rangle^{30-8|\beta|} \partial^\beta g_{\pm} \rangle_{L^2_{-1/2}} + (v \cdot \nabla_x f_{\pm}, g_{\pm})_{L^2_{-1/2}} \]

\[ - (A_\pm f, g_{\pm})_{L^2_{-1/2}} = (\Gamma_{\pm}(h,f, g_{\pm})_{L^2_{-1/2}} + (K h, g_{\pm})_{L^2_{-1/2}}. \]

Denote (7.5) by

\[ (\partial_t f_{\pm}, g_{\pm})_{L^2_{-1/2}} + B_{\pm} \langle f, g \rangle = (K h, g_{\pm})_{L^2_{-1/2}}. \]

Then \( B = (B_+, B_-) \) is a bilinear operator on \( \mathcal{H} \times \mathcal{H} \), where \( \mathcal{H} = \{ f = (f_+, f_-) \in L^2_{-1/2} : \langle \langle v \rangle^{15-4|\beta|} \partial^\beta f \rangle_{L^2_{-1/2}}, \forall |\alpha| + |\beta| \leq 3, \]

\[ \tilde{f}(t,-1,k,v,1) \rangle_{v_1 < 0} = \tilde{f}(t,-1,k,-v,1), \]

\[ \tilde{f}(t,1,k,v,1) \rangle_{v_1 < 0} = \tilde{f}(t,1,k,-v,1) \}. \]

Note that the terms involving both \( \psi \) and \( h \) can be controlled as

\[ \langle 1/2 n_x \cdot v f_{\pm}, g_{\pm} \rangle_{L^2_{-1/2}} + (\nabla_x \psi \cdot \nabla_v f_{\pm}, g_{\pm})_{L^2_{-1/2}} + (\Gamma_{\pm}(h,f), g_{\pm})_{L^2_{-1/2}} \]

\[ \lesssim \| \nabla_x \psi \|_{L^\infty_x} \| \langle v \rangle^{1/2} f \|_{L^2_{-1/2}} \| \langle v \rangle^{1/2} g \|_{L^2_{-1/2}} + \| (\nabla_x \psi \cdot \nabla_v f_{\pm}, g_{\pm})_{(\langle v \rangle^{1/2} f)^{-1/2}} \|_{L^2_{-1/2}} \]

\[ + \| (K_{\pm} h, g_{\pm})_{L^2_{-1/2}} \|_{L^2_{-1/2}} \]

\[ + \| (h_{L^\infty_x} \| \langle v \rangle^{1/2} f \|_{L^2_{-1/2}} \| \langle v \rangle^{1/2} g \|_{L^2_{-1/2}} \]

\[ + \| (h_{L^\infty_x} \| \langle v \rangle^{1/2} f \|_{L^2_{-1/2}} \| \langle v \rangle^{1/2} g \|_{L^2_{-1/2}} \]

\[ + \| (K_{\pm} h, g_{\pm})_{L^2_{-1/2}} \|_{L^2_{-1/2}} \]. \]
\[
\begin{align*}
\lesssim \|\nabla_x \psi\|_{L^2_{x,v}} \|g\psi\|_{L^2_{x,v}} & + \|\nabla_x \psi\|_{L^2_{x,v}} \|g\|_{L^2_{x,v}} + \|\nabla_x \psi\|_{L^2_{x,v}} \|g\|_{L^2_{x,v}}.
\end{align*}
\]

Then using (7.3) to control the upper bound of \(\|\nabla_x \psi\|_{L^2_{x,v}}\) and \(\|\nabla_x \psi\|_{L^2_{x,v}}\), it’s direct to obtain from (2.5) that
\[
\begin{align*}
\int_0^T \sum_{\pm} B_{\pm}[f,f] \, dt & \geq \epsilon \sum_{|\alpha|+|\beta| \leq \delta} \int_0^T \|\langle v \rangle^{1/2} f \|_{L^2_{x,v}} \|g \|_{L^2_{x,v}} \, dt \\
& \quad + \lambda \int_0^T \|f\|_{L^2_{x,v}}^2 \, dt - C \sup_{0 \leq t \leq T} \|f\|_{L^2_{x,v}}^2,
\end{align*}
\]
for some \(\lambda > 0\). Notice that the boundary term generating from \((v \cdot \nabla_x f_\pm, f_\pm)\) vanishes by using the same argument as (6.5). Also, \(f_\pm^c\) is smooth and \(K\) is a linear bounded operator \(L^2_{x,v}\). Then by the standard theory of linear evolution equations on Hilbert space \(\mathcal{H}\), there exists \(T_0 > 0\) and unique solution \(f^c \in \mathcal{H}\) to equation (7.5) and (7.2) on time \([0, T_0]\), which is smooth on \((t, x, v)\), where \(\phi\) is solved by (7.2) and (7.6). Then \(f^c\) solves
\[
\begin{align*}
\int_0^{T_0} \left( (\partial_t f_\pm^c, g_\pm)_{L^2_{x,v}} + \epsilon \sum_{|\alpha|+|\beta| \leq \delta} \|\langle v \rangle^{30-8|\beta|} \partial_3^\alpha f_\pm^c, \partial_3^\beta g_\pm \|_{L^2_{x,v}} \right) \, dt \\
& \quad + \int_0^{T_0} (v \cdot \nabla_x f_\pm^c \pm \frac{1}{2} \nabla_x \psi \cdot v f_\pm^c \pm \nabla_x \psi \cdot \nabla_x f_\pm^c \pm v \mu^{1/2} - A_\pm f_\pm^c, g_\pm)_{L^2_{x,v}} \, dt \\
& \quad = \int_0^{T_0} \left( \Gamma_\pm (h, f^c), g_\pm \right)_{L^2_{x,v}} + (K h, g_\pm)_{L^2_{x,v}} \, dt,
\end{align*}
\]
for any \(g \in \mathcal{H}\). Using identity \((\cdot, \cdot)_{L^2_{x,v}} = \langle \cdot , \cdot \rangle_{L^2_{x,v}}\) with Fourier transform \(\hat{\cdot}\) over \(\mathbb{T}^2\), we have
\[
\begin{align*}
\int_0^{T_0} (\partial_t f_\pm^c, \hat{g}_\pm, \hat{f}_\pm)_{L^2_{x,v}} \, dt & \geq \epsilon \sum_{|\alpha|+|\beta| \leq \delta} \int_0^{T_0} \|\langle v \rangle^{30-8|\beta|} \partial_3^\alpha f_\pm^c, \partial_3^\beta \hat{g}_\pm \|_{L^2_{x,v}} \, dt \\
& \quad + \int_0^{T_0} (v \cdot \nabla_x f_\pm^c \pm \frac{1}{2} \nabla_x \psi \cdot v f_\pm^c \hat{\psi} \pm \nabla_x \psi \cdot \nabla_x f_\pm^c \hat{\psi} \pm v \mu^{1/2} - A_\pm f_\pm^c, \hat{g}_\pm)_{L^2_{x,v}} \, dt \\
& \quad = \int_0^{T_0} \left( \Gamma_\pm (h, f^c), \hat{g}_\pm \right)_{L^2_{x,v}} + (K h, \hat{g}_\pm)_{L^2_{x,v}} \, dt,
\end{align*}
\]
and with weight \(w\) involved, we have
\[
\begin{align*}
\int_0^{T_0} (\partial_t w f_\pm^c, w \hat{g}_\pm, \hat{f}_\pm)_{L^2_{x,v}} \, dt & \geq \epsilon \sum_{|\alpha|+|\beta| \leq \delta} \int_0^{T_0} \|\langle v \rangle^{30-8|\beta|} w \partial_3^\alpha f_\pm^c, w \partial_3^\beta \hat{g}_\pm \|_{L^2_{x,v}} \, dt \\
& \quad + \epsilon \int_0^{T_0} \sum_{|\alpha|+|\beta| \leq \delta} \left( \langle v \rangle^{30-8|\beta|} \partial_3^\alpha f_\pm^c \right) \sum_{\beta' < \beta} (\partial_{3-\beta'} w^2 \partial_3^\beta g_\pm)_{L^2_{x,v}} \, dt \\
& \quad + \int_0^{T_0} (v \cdot \nabla_x f_\pm^c \pm \frac{1}{2} (\nabla_x \psi \cdot v w f_\pm^c) \hat{\psi} \pm (\nabla_x \psi \cdot \nabla_x w f_\pm^c) \hat{\psi} \pm v \mu^{1/2} - A_\pm f_\pm^c, \hat{g}_\pm)_{L^2_{x,v}} \, dt \\
& \quad = \int_0^{T_0} \left( (w \Gamma_\pm (h, f^c), \hat{g}_\pm)_{L^2_{x,v}} + w (K h, \hat{g}_\pm)_{L^2_{x,v}} \right) \, dt,
\end{align*}
\]
for any \(g \in \mathcal{H}\), where \(\alpha = (\alpha_1, \alpha_2, \alpha_3)\). Note that after Fourier transform on \(\mathbb{T}^2\), we take only inner product \(L^2_{x,v}\).

Next we derive the identities on derivatives. Let \(|\alpha_1| = 1\). We consider the equation
\[
\begin{align*}
\partial_t f_\pm^{\alpha_1} + v \cdot \nabla_x f_\pm^{\alpha_1} & \pm \frac{1}{2} \nabla_x \psi \cdot v f_\pm^{\alpha_1} \pm \nabla_x \psi \cdot \nabla_x f_\pm^{\alpha_1} \pm E^{\alpha_1} \cdot v \mu^{1/2} - A_\pm \partial^{\alpha_1} f \\
& = \Gamma_\pm (\partial^{\alpha_1} h, f) + \Gamma_\pm (h, f^{\alpha_1}) \pm \frac{1}{2} \partial^{\alpha_1} \nabla_x \psi \cdot v f_\pm \pm \partial^{\alpha_1} \nabla_x \psi \cdot \nabla_x f_\pm + K \partial^{\alpha_1} h,
\end{align*}
\]
\[37\]
with initial data \( f^{\alpha_1}(0, x, v) = \partial^{\alpha_1} f_0(x, v) \), \( E^{\alpha}(0, x) = \partial^{\alpha} E_0(x) \) and (7.2)\(_5\)–(7.2)\(_6\). Using the same argument we used to derive (7.6), there exists \( f^{\varepsilon, \alpha_1} \in \mathcal{H} \) such that

\[
\int_0^T (\partial_t f^{\varepsilon, \alpha_1}_{\pm}, g_{\pm})_{L^2_{t,v}} dt + \varepsilon \sum_{|\alpha|+|\beta| \leq 3} \int_0^T \left( (v)^{30-8|\beta|} \partial_{\alpha} f^{\varepsilon, \alpha_1}_{\pm} \partial_{\beta} g_{\pm} \right)_{L^2_{t,v}} dt \\
+ \int_0^T (v \cdot \nabla_x f^{\varepsilon, \alpha_1}_{\pm} \pm \frac{1}{2} \nabla_x \psi \cdot v f^{\varepsilon, \alpha_1}_{\pm} \pm \nabla_x \psi \cdot \nabla_v f^{\varepsilon, \alpha_1}_{\pm} \mp E^{\varepsilon, \alpha_1} \cdot v)^{1/2} g_{\pm} \right)_{L^2_{t,v}} dt \\
= \int_0^T (A_{\pm} f^{\varepsilon, \alpha_1} + \Gamma_{\pm} (\partial^{\alpha_1} h, f^{\varepsilon} + \Gamma_{\pm} (h, f^{\varepsilon, \alpha_1}), g_{\pm})_{L^2_{t,v}} dt \\
+ \int_0^T \left( \mp \frac{1}{2} \partial^{\alpha_1} \nabla_x \psi \cdot v f^{\varepsilon, \alpha_1}_{\pm} \pm \partial^{\alpha_1} \nabla_x \psi \cdot v f^{\varepsilon, \alpha_1}_{\pm} + K \partial^{\alpha_1} h, g_{\pm} \right)_{L^2_{t,v}} dt,
\]

(7.9)

for \( g \in \mathcal{H} \). Then \( f^{\varepsilon, \alpha_1} = \partial^{\alpha_1} f^{\varepsilon} \), \( E^{\varepsilon, \alpha_1} = \partial^{\alpha_1} E^{\varepsilon} \) in the weak sense by using (7.6) and (7.9). Then similar to (7.7) and (7.8), we have

\[
\int_0^T (\partial_t f^{\varepsilon, \alpha_1}_{\pm}, g_{\pm})_{L^2_{t,v}} dt + \varepsilon \sum_{|\alpha|+|\beta| \leq 3} \int_0^T \left( (v)^{30-8|\beta|} \partial_{\alpha} f^{\varepsilon, \alpha_1}_{\pm} \partial_{\beta} g_{\pm} \right)_{L^2_{t,v}} dt \\
+ \int_0^T (v \cdot \nabla_x f^{\varepsilon, \alpha_1}_{\pm} \pm \frac{1}{2} \nabla_x \psi \cdot v f^{\varepsilon, \alpha_1}_{\pm} \hat{\nabla} \nabla_v f^{\varepsilon, \alpha_1}_{\pm} \hat{\nabla} g_{\pm} \right)_{L^2_{t,v}} dt \\
+ \int_0^T (\pm \partial^{\alpha_1} \nabla_x \psi \cdot v f^{\varepsilon, \alpha_1}_{\pm})_{L^2_{t,v}} dt \\
+ \int_0^T (\mp \partial^{\alpha_1} \nabla_x \psi \cdot v f^{\varepsilon, \alpha_1}_{\pm})_{L^2_{t,v}} dt \\
+ \int_0^T \left( \mp \frac{1}{2} \partial^{\alpha_1} \nabla_x \psi \cdot v f^{\varepsilon, \alpha_1}_{\pm} + K \partial^{\alpha_1} h, g_{\pm} \right)_{L^2_{t,v}} dt,
\]

(7.10)

and the weighted form is

\[
\int_0^T (\partial_t w \partial^{\alpha_1} f^{\varepsilon, \alpha_1}_{\pm}, \hat{w} g_{\pm})_{L^2_{t,v}} dt + \varepsilon \sum_{|\alpha|+|\beta| \leq 3} \int_0^T \left( (v)^{30-8|\beta|} w \partial_{\alpha} \partial_{\beta} f^{\varepsilon, \alpha_1}_{\pm}, \hat{w} \partial_{\beta} g_{\pm} \right)_{L^2_{t,v}} dt \\
+ \varepsilon \int_0^T \sum_{|\alpha|+|\beta| \leq 3} \left( (v)^{30-8|\beta|} \partial_{\alpha} \partial_{\beta} f^{\varepsilon, \alpha_1}_{\pm}, \sum (\partial_{\beta-\beta' \cdot w^2 \partial^{\alpha_1}} g_{\pm}) \hat{w} \right)_{L^2_{t,v}} dt \\
+ \int_0^T (v \cdot (w \nabla_x \partial^{\alpha_1} f^{\varepsilon, \alpha_1}_{\pm}) \hat{w} + \frac{1}{2} (\nabla_x \psi \cdot v w \partial^{\varepsilon_1} f^{\varepsilon, \alpha_1}_{\pm}) \hat{w} \right)_{L^2_{t,v}} dt \\
+ \int_0^T (\pm w \partial^{\alpha_1} \nabla_x \psi \cdot v^{1/2} - w A_{\pm} \partial^{\alpha_1} f^{\varepsilon, \alpha_1}_{\pm}, \hat{w} g_{\pm})_{L^2_{t,v}} dt \\
= \int_0^T (\Gamma_{\pm} (\partial^{\alpha_1} h, f^{\varepsilon} + \Gamma_{\pm} (h, \partial^{\alpha_1} f^{\varepsilon}), g_{\pm})_{L^2_{t,v}} dt \\
+ \int_0^T (\pm \frac{1}{2} (w \partial^{\alpha_1} \nabla_x \psi \cdot v f^{\varepsilon, \alpha_1}_{\pm}) \hat{w} \right)_{L^2_{t,v}} dt \\
+ \int_0^T (w K \partial^{\alpha_1} h, \hat{w} g_{\pm})_{L^2_{t,v}} dt.
\]

(7.11)

Following the arguments from (6.4) to (6.10), choosing \( g = f \) in (7.7), (7.8), (7.10) and (7.11), we can obtain the similar energy estimate to (6.10). It suffices to deal with the third terms in (7.8) and (7.11). Noticing \( |\partial^{\beta} w^2| \leq (v)^{2|\beta|} w^2 \), we have

\[
\varepsilon \int_{\mathbb{R}^2} \left( \int_0^T \sum_{|\alpha|+|\beta| \leq 3} \left( (v)^{30-8|\beta|} \partial_{\alpha} \partial_{\beta} f^{\varepsilon, \alpha_1}_{\pm}, \sum (\partial_{\beta-\beta' \cdot w^2 \partial^{\alpha_1}} g_{\pm}) \hat{w} \right)_{L^2_{t,v}} dt \right)^{1/2} d\Sigma(k) \\
= \varepsilon \sum_{|\alpha|+|\beta| \leq 3} \| (v)^{15-4|\beta|} \partial_{\alpha} \partial_{\beta} f^{\varepsilon, \alpha_1}_{\pm} \|_{L^2_{t,v}} \sum_{|\alpha'|+|\beta'| \leq 2} \| (v)^{15-4|\beta'|} - \partial_{\alpha'} \partial_{\beta'} f^{\varepsilon, \alpha_1}_{\pm} \|_{L^2_{t,v}}.
\]
where we used Definition (7.11) and (7.7) respectively. Therefore, we take the linear combination (7.7) + \( \kappa \times (7.8) \) + \( \sum_{|\alpha| \leq 1} \left( (7.10) + \kappa \times (7.11) \right) \) with \( g = f \) and \( \kappa \) suitably small. Following the arguments from (6.4) to (6.10) and taking summation over \( \pm \), square root and summation on \( \pm k \in \mathbb{Z}^2 \) of the resultant estimate, we have the following estimate (Note that we use (2.5) to estimate \( A_{\pm} \) while \( K \) is linear bounded on \( L^2_{x,v} \)):

\[
\mathcal{E}_{T_0,w}(f^\varepsilon) + \mathcal{D}_{T_0,w}(f^\varepsilon) + \mathcal{E}_{T_0}(f^\varepsilon) + \mathcal{D}_{T_0}(f^\varepsilon) + \frac{\varepsilon}{2} \sum_{|\alpha| + |\beta| \leq 1} \sum_{|\alpha| + |\beta| \leq 3} \int_0^{T_0} \| \langle t, x, v \rangle \left\{ \left( \frac{\sqrt{q}}{2} \right) (\nabla_x \psi \cdot v) \right\} \|^2_{L^2_{x,v}} dt \right) \right)^{1/2} d\Sigma(k)
\]

\[
\leq \sum_{|\alpha| \leq 1} \left( \| \sqrt{q} \nabla \cdot \sqrt{q} f_0 \|^2_{L^2_{x,v}} + \| \sqrt{q} \nabla \cdot \sqrt{q} \phi_0 \|^2_{L^2_{x,v}} + \right) + T_0^{1/2} \sum_{|\alpha| \leq 1} \| \sqrt{q} \nabla \cdot \sqrt{q} \phi_0 \|^2_{L^2_{x,v}}.
\]

Thus, \( f^\varepsilon \) is uniformly bounded with respect to norms:

\[
\sum_{|\alpha| \leq 1} \left( \| \sqrt{q} \nabla \cdot \sqrt{q} f_0 \|^2_{L^2_{x,v}} + \right)
\]

Denote the weak limit of \( \{ f^\varepsilon \} \) as \( \varepsilon \to 0 \) to be \( f \). Taking limit \( \varepsilon \to 0 \) in (7.7), then \( f \) solves

\[
\int_0^{T_0} \left( \partial_t f \pm v \cdot \nabla f \pm \frac{1}{2} (\nabla \cdot \sqrt{q} f) + \nabla \cdot (\sqrt{q} f) \right) dt = T_0 \left( \Gamma_{\pm}(h,f) \right)_{L^2_{x,v}} + \left( \sqrt{q} \nabla \cdot \sqrt{q} \phi_0 \right)_{L^2_{x,v}} dt,
\]

with initial data \( f(0) = f_0 \), for any \( g \) belonging to

\[
\left\{ g \in L^\infty_{k} L^2_{T_0} L^2_{x,v} : \right\} = \left\{ g(t,-1,k,v) \right\}_{v > 0} = \left\{ g(t,1,k,-v) \right\}_{v < 0}.
\]

Taking limit \( \varepsilon \to 0 \) in (7.12), we obtain (7.4). This completes the proof of Lemma 7.2. \( \Box \)

**Proof of Theorem 7.1.** Write \( (f^*_0, E^*_0) \) to be the mollification of \( (f_0, E_0) \). We now construct the approximation solution sequence which is denoted by

\[
\left\{ (f^n(t,x,v), \phi^n(t,x)) \right\}_{n=0}^{\infty}
\]
using the following iterative scheme:

\[
\begin{aligned}
\frac{\partial f_n^{n+1}}{\partial t} + v \cdot \nabla_x f_n^{n+1} \pm \frac{1}{2} \nabla_x \phi_n \cdot v f_n^{n+1} \mp \nabla_x \phi_n \cdot \nabla_y f_n^{n+1} \\
\pm \nabla_x \phi^{n+1} \cdot v \mu^{1/2} - A_{\pm} f_n^{n+1} = \Gamma_\pm (f^n, f^{n+1}) + K f^n,
\end{aligned}
\]

\[
\begin{aligned}
- \Delta_x \phi^{n+1} = \int_{\mathbb{R}^2} (f^{n+1}_+ - f^{n+1}_-) \mu^{1/2} dx,
\end{aligned}
\]

\[
\begin{aligned}
f^{n+1}(0, x, v) = f_0^+(x, v), & \quad E^{n+1}(0, x) = E_0^+(x),
\end{aligned}
\]

\[
\begin{aligned}
\left| f^{n+1}(t, -1, k, v_1, \bar{v}) \right|_{v_1 > 0} = f^{n+1}(t, -1, k, -v_1, \bar{v}),
\end{aligned}
\]

\[
\begin{aligned}
\left| f^{n+1}(t, 1, k, v_1, \bar{v}) \right|_{v_1 < 0} = f^{n+1}(t, 1, k, -v_1, \bar{v}),
\end{aligned}
\]

\[
\begin{aligned}
\partial_{x_1} \phi^{n+1} = 0, & \quad \text{on } x_1 = \pm 1,
\end{aligned}
\]

for \( n = 0, 1, 2, \ldots \), where we set \( f^0(t, x, v) = f_0(x, v) \). With Lemma 7.2, it is a standard procedure to apply the induction argument to show that there exists \( \varepsilon_0 > 0 \) such that if

\[
\sum_{|\alpha| \leq 1} \left( \left\| \frac{\partial^{\alpha} f_0}{\partial t} \right\|_{L^2_t L^2_x L^2_v}^2 + \left\| \frac{\partial^{\alpha} E_0}{\partial t} \right\|_{L^2_t L^2_x L^2_v}^2 \right) \leq \varepsilon_0,
\]

then the approximate solution sequence \( \{ f^n \} \) is well-defined in the sense of the following norms are finite:

\[
\sum_{|\alpha| \leq 1} \left( \left\| \frac{\partial^{\alpha} f}{\partial t} \right\|_{L^2_t L^2_x L^2_v}^2 + \left\| \frac{\partial^{\alpha} E}{\partial t} \right\|_{L^2_t L^2_x L^2_v}^2 \right) \leq \varepsilon_0.
\]

Notice that \( f^{n+1} - f^n \) solves

\[
\begin{aligned}
\partial_t (f^{n+1}_- - f^n_-) + v \cdot \nabla_x (f^{n+1}_- - f^n_-) + \frac{1}{2} \nabla_x \phi_0 \cdot v (f^{n+1}_- - f^n_-) & \pm \frac{1}{2} \nabla_x \phi_0 \cdot v (f^{n+1}_+ - f^n_+) + (\nabla_x \phi_0 - \nabla_x \phi_0^n) \cdot \nabla_v f^n_+ \pm (\nabla_x \phi_0 - \nabla_x \phi_0) \cdot \nabla_v f^n_+ + \frac{1}{2} (\nabla_x \phi_0 - \nabla_x \phi_0^n) \cdot v \mu^{1/2} \\
- A_{\pm} (f^{n+1}_- - f^n_-) = \Gamma_\pm (f^n, f^{n+1} - f^n) + K (f^n - f^{n-1}),
\end{aligned}
\]

for \( n = 1, 2, 3, \ldots \), Using the method for deriving (7.4), we know that \( (f^{n+1}_- - f^n_-) \) is a Cauchy sequence with respect to norms in (7.13) Then the limit function \( f(t, x, v) \) is indeed a solution to (1.4), (1.6) and (1.7) satisfying estimate (7.1). For the positivity, we can use the argument from [22, Lemma 12, page 800]; the details are omitted for brevity. If \( (g, \psi) \) is another solution to (1.4), (1.6) and (1.7) satisfying (7.1), then similar to (7.14), \( f - g \) satisfies

\[
\begin{aligned}
\partial_t (f^{n+1}_- - f^n_-) + v \cdot \nabla_x (f^{n+1}_- - g^n_-) & \pm \frac{1}{2} \nabla_x \phi_0 \cdot v (f^{n+1}_- - g^n_+) + \frac{1}{2} (\nabla_x \phi_0 - \nabla_x \psi^n) \cdot v g^n_+ \\
- A_{\pm} (f^{n+1}_- - g^n_-) = \Gamma_\pm (g, f - g) + \Gamma_\pm (f - g, g) + K (f - g),
\end{aligned}
\]

with zero initial data. Applying the similar calculations for deriving (7.4) and noticing the zero initial data, we deduce that \( f = g \) by choosing \( T_0 \) sufficiently small. The proof of Theorem 7.1 is complete. \( \square \)

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