CO-EULER STRUCTURES ON BORDISMS

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Abstract. Co-Euler structures were studied by Burghelea and Haller on closed manifolds as dual objects to Euler structures. We extend the notion of co-Euler structures to the situation of compact manifolds with boundary. As an application, by studying their variation with respect to smooth changes of the Riemannian metric, co-Euler structures conveniently provide correction terms that can be taken into account when considering the complex-valued analytic torsion on bordisms as a Riemannian invariant.

1. Introduction

In this paper, $M$ is considered to be a compact connected non-necessarily oriented $m$-dimensional manifold with Riemannian metric $g$, and boundary $\partial M$ that inherits its Riemannian metric from that of $M$. Moreover, we assume $\partial M$ to be the disjoint union of two closed (non-necessarily connected) submanifolds $\partial_+ M$ and $\partial_- M$. We write

\[ M := (M, \partial_+ M, \partial_- M) \quad \text{and} \quad M' := (M, \partial_- M, \partial_+ M) \]

(1)

to indicate that $M$ is considered as a bordism from $\partial_+ M$ to $\partial_- M$, and $M'$ for its dual bordism, i.e., $M$ seen as the bordism from $\partial_- M$ to $\partial_+ M$.

The concept of Euler structures was first introduced by Turaev in [Tu90], see also [FT00], for manifolds $M$ with vanishing Euler–Poincaré characteristic $\chi(M)$ to conveniently remove ambiguities in the definition of the Reidemeister torsion. The set of Euler structures $\text{Eul}(M; \mathbb{C})$ is an affine space over the homology group $H_1(M; \mathbb{C})$ in the sense that $H_1(M; \mathbb{C})$ acts freely and transitively on $\text{Eul}(M; \mathbb{C})$. Then, Euler structures were studied on manifolds with arbitrary Euler characteristics at the expense of introducing a base point $x_0 \in M$, see [Bu99].

Co-Euler structures can be considered as dual objects to Euler structures and were introduced by Burghelea and Haller in [BH06a] and [BH06b] and then used in [BH07] to study (variational formulas of) the complex-valued analytic torsion given on closed manifolds. To have the ideas set up, let us recall in the situation of closed manifolds what Co-Euler structures are. Assume that $M$ is closed, connected and that its Euler characteristic $\chi(M) = 0$. If $\Theta_M$ indicates the orientation bundle of $M$ and $\Theta_M^\mathbb{C}$

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\end{itemize}
its complexification, the set of co-Euler structures $\mathfrak{Eul}(M; \mathbb{C})$ is an affine version of cohomology groups $H^{m-1}(M; \Theta^C_M)$. A co-Euler structure is an equivalence class of pairs $(g, \alpha)$, where $g$ is a Riemannian metric on $M$, and $\alpha \in \Omega^{m-1}(M; \Theta^C_M)$ is a $(m - 1)$-smooth differential form over $M$ with $d\alpha = e(g)$ where $e(g) \in \Omega^m(M; \Theta^C_M)$ is the Euler form of $g$. Two such pairs $(g_1, \alpha_1)$ and $(g_2, \alpha_2)$ are equivalent if and only if $\alpha_2 - \alpha_1 = \widetilde{e}(g_1, g_2)$ where $\widetilde{e}(g_1, g_2) \in \Omega^{m-1}(M; \Theta^C_M)/d\Omega^{m-2}(M; \Theta^C_M)$ denotes the Chern–Simons form.

By construction co-Euler structures were well suited to remove the metric ambiguities of the analytic torsion on closed manifolds and finally provide a topological invariant, referred as the modified Ray–Singer torsion, see [BM06].

In Section 3, we define co-Euler structures on a bordism $\mathcal{M}$. As in case for a closed manifold, we start with the case where the relative Euler characteristics $\chi(M, \partial_+ M)$ (or equivalently $\chi(M, \partial_- M)$) vanishes. In this situation the space $\mathfrak{Eul}(\mathcal{M}; \mathbb{C})$ of co-Euler structures on $\mathcal{M}$ can be seen as an affine space over the relative cohomology group $H^{m-1}(M, \partial M; \mathbb{C})$ and depends on the choice of a base point $x_0 \in M$ if the Euler characteristics $\chi(M; \partial M) \neq 0$.

A co-Euler structure on $\mathcal{M}$ is an equivalence class represented by couples $(\varphi, g)$, where $\varphi = (\alpha, \alpha_\partial)$ is a relative form in the relative cochain complex $\Omega^{m-1}(M, \partial M; \Theta^C_M)$, i.e., a pair of differential forms $\alpha \in \Omega^{m-1}(M; \Theta^C_M)$ and $\alpha_\partial \in \Omega^{m-2}(\partial M; \Theta^C_{\partial M})$ with $d\varphi = e(g)$ where $d$ is an appropriate differential on the relative complex, and $e(\mathcal{M}, g) \in \Omega^m(M, \partial M; \Theta^C_M)$ is a relative Euler form associated to $(\mathcal{M}, g)$. Two such pairs $(g_1, \alpha_1)$ and $(g_2, \alpha_2)$ are equivalent if and only if $\alpha_2 - \alpha_1 = \widetilde{e}(\mathcal{M}, g_1, g_2)$ where $\widetilde{e}(\mathcal{M}, g_1, g_2) \in \Omega^{m-1}(M, \partial M; \Theta^C_M)$ modulo exact relative forms $d\Omega^{m-2}(M; \partial M; \Theta^C_M)$.

The relative Euler form and the relative Chern–Simons' forms on $M$ (and on $\partial M$) that we use are based on those worked out by Brüning and Ma in [BM06], which appear in the anomaly formulas for the Ray–Singer metric, see [BM06, Theorem 0.1] and [BM11, Theorem 3.4], and also in the anomaly formulas for the complex-valued Ray–Singer torsion, see [Ma13a, Theorem 2]. For the reader's convenience, we explain in the Appendix how these characteristic forms are constructed.

Moreover, we explain how co-Euler structures on $\mathcal{M}$ are in a one-to-one correspondence with a co-Euler structure on its dual bordism $\mathcal{M}'$, by means of a so-called flip map $\nu^*$, compatible with Poincaré duality and affine over involution in relative cohomology.

In Proposition 1 we derive the infinitesimal variation of representatives of co-Euler structures with respect to smooth changes in the Riemannian metric, which then is used in Section 5 to encode the variation of the complex-valued Ray–Singer torsion.

Then, more generally, we treat the case $\chi(M, \partial_\pm M) \neq 0$, by considering a base point $x_0 \in M$ and we define the space of base-pointed co-Euler structures denoted by $\mathfrak{Eul}_{x_0}(\mathcal{M}; \mathbb{C})$. We obtain Proposition 2 where we study their
infinitesimal variation, by using a regularization procedure for relative forms having a singularity in the interior of $M$ only.

In Section 4 we recall the space of Euler structures on manifolds with boundary defined by Turaev in [Tu90]. We use a relative Mathai–Quillen form, to study Poincaré–Lefschetz duality in terms of a canonical isomorphism relating Euler and co-Euler structures in this setting. The relative Mathai–Quillen form as presented here can be used to compare the complex-valued analytic torsion and the Milnor torsion without the need of (co)-Euler structures.

In Section 5 we define a modified version for the complex-valued Ray–Singer torsion on compact bordisms, by conveniently adding certain correction terms. These correction terms, expressed in terms of co-Euler structures, are incorporated to cancel out the variation of the complex-valued Ray–Singer torsion with respect to smooth variations of the Riemannian metric and bilinear structures, given in [Ma13a, Theorem 2]. In analogy with the situation on closed manifolds, the modified complex-valued analytic torsion depends on the flat connection, the homotopy class of the bilinear form and the co-Euler structure only. Finally, by means of the flip map $\nu^*$, we show naturality of the the modified torsion with respect to Poincaré duality.

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2. Generalities and some conventions

Consider the bordism $\mathbb{M}$ in (1) and denote by $i : \partial M \rightarrow M$ the canonical embedding. Let $\Theta_M \rightarrow M$ be the orientation bundle of $TM$, considered as the real line bundle associated to the frame bundle of $TM$, via the homomorphism $\text{sign} \det : GL_m(\mathbb{R}) \rightarrow O(1) \hookrightarrow GL_1(\mathbb{R})$. Since the structure group $O(1) = \{-1,+1\}$ is discrete, $\Theta_M$ is endowed with a canonical flat connection and a canonical fiber-wise metric which is parallel. We denote by $\Theta_M \mathbb{C}$ the complexification of $\Theta_M$. As usual, $\Omega^q(M)$ is the vector space of smooth differential $q$-forms on $M$ so that $\Omega(M) := \bigoplus_m \Omega^q(M)$ is the de-Rham cochain complex of differential forms with de-Rham differential $d$. Thus, $\Omega(M; \Theta_M)$ is the de-Rham cochain complex of $\Theta_M$-valued differential forms with induced differential still denoted by $d$. Analogously, we denote by $\Theta_{\partial M}$ the orientation bundle of $T\partial M$ and, as real line bundles over $\partial M$, we identify $\Theta_M|_{\partial M} := i^*\Theta_M$ with $\Theta_{\partial M}$ by using the outward normal first convention. The corresponding Levi–Civita connections on $TM$ and $T\partial M$ are denoted by $\nabla$ and by $\nabla^\partial$ respectively. Recall the Hodge $*$-operator $\ast_q := \ast_{g,q} : \Omega^p(M) \rightarrow \Omega^{m-q}(M; \Theta_M)$, i.e., the linear isomorphism defined
by \( \alpha \wedge \alpha' = \langle \alpha, \alpha' \rangle \cdot \text{vol}_q(M) \), for \( \alpha, \alpha' \in \Omega^q(M) \) and \( 0 \leq q \leq m \), where \( \text{vol}_q(M) \in \Omega^m(M; \Theta_M) \) is the volume form of \( M \).

Recall that the relative cohomology group \( H^q(M, \partial M; \Theta_M^C) \) in degree \( q \), can be computed, see [BT82], by means of the \( \mathbb{Z} \)-graded differential cochain complex

\[
\Omega(M, \partial M; \Theta_M^C) := \bigoplus_{q=0}^m \Omega^q(M, \partial M; \Theta_M^C)
\]

where

\[
\Omega^q(M, \partial M; \Theta_M^C) := \Omega^q(M; \Theta_M^C) \oplus \Omega^{q-1}(\partial M; \Theta_M^C)
\]

with \( \Omega^{-1}(\partial M; \Theta_M^C) := 0 \). The space \( \Omega^q(M, \partial M; \Theta_M^C) \) will be referred as the space of relative differential forms. The differential map in (2) is defined by

\[
d : \Omega^q(M, \partial M; \Theta_M^C) \to \Omega^{q+1}(M, \partial M; \Theta_M^C),
\]

where \( i : \partial M \to M \) and \( d^\theta \) is the de-Rham differential at the boundary. Note that \( \Omega(M, \partial M; \Theta_M^C) \) can be considered as a \( \Omega(M) \)-module by setting

\[
\langle (\alpha, \alpha_\theta) \rangle \wedge \langle \beta, \beta_\theta \rangle := \langle \alpha \wedge \beta, \alpha_\theta \wedge i^* \beta \rangle,
\]

for \( \alpha, \beta \in \Omega^q(M, \partial M; \Theta_M^C) \) and \( \beta, \beta_\theta \in \Omega^{q-1}(\partial M; \Theta_M^C) \).

For simplicity, we denote relative forms by

\[
\alpha := (\alpha, \alpha_\theta) \in \Omega(M, \partial M; \Theta_M^C).
\]

Then, we have the graded Leibniz formula

\[
d(\alpha \wedge w) = (d\alpha) \wedge w + (-1)^{q} \alpha \wedge dw,
\]

which holds for each \( \alpha \in \Omega^q(M, \partial M; \Theta_M^C) \) and \( w \in \Omega(M) \).

Furthermore, for \( \alpha \in \Omega^q(M, \partial M; \Theta_M) \) and \( w \in \Omega^{m-q}(M) \), one has the pairing

\[
\langle \alpha, w \rangle := \int_{(M, \partial M)} \alpha \wedge w := \int_{M} \alpha \wedge w - \int_{\partial M} \alpha_\theta \wedge i^* w,
\]

which induces a non-degenerate pairing \( \langle \cdot, \cdot \rangle \) in cohomology:

\[
\langle \cdot, \cdot \rangle : H^*(M, \partial M; \Theta_M^C) \times H^{m-*}(M; \mathbb{C}) \to \mathbb{C}
\]

\[
\langle [\alpha, \alpha_\theta], [\beta] \rangle \mapsto \int_{(M, \partial M)} (\alpha, \alpha_\theta) \wedge \beta.
\]

If in addition \( M \) is connected, then non-degeneracy of \( \langle \cdot, \cdot \rangle \) implies that

\[
H^m(M, \partial M; \Theta_M^C) \cong H^0(M; \mathbb{C}),
\]

We will be also be interested in spaces with a base point. For \( x_0 \in M \setminus \partial M \) a base point in the interior of \( M \), denote by \( \hat{M} := M \setminus \{x_0\} \). Consider

\[
\Omega^q(\hat{M}, \partial M; \Theta_M^C) := \Omega^q(\hat{M}; \Theta_M^C) \oplus \Omega^{q-1}(\partial M; \Theta_M^C)
\]

so that

\[
\Omega(\hat{M}, \partial M; \Theta_M^C) := \bigoplus_{q=0}^m \Omega^q(\hat{M}, \partial M; \Theta_M^C)
\]
endowed with the same differential map $d$ as in [3], is also a $\mathbb{Z}$-graded complex. In analogy with (7), if $M$ is connected, then it is not difficult to show, see [Ma13], that

$$
\begin{align*}
H^m(\partial M, \partial M; \Theta^C_{\mathcal{M}}) & \cong 0 \\
H^{m-1}(M, \partial M; \Theta^C_{\mathcal{M}}) & \cong H^{m-1}(\partial M, \partial M; \Theta^C_{\mathcal{M}}).
\end{align*}
$$

3. Co-Euler structures

In order to construct co-Euler structures on a bordism $\mathcal{M}$, we first need to introduce certain characteristic forms and secondary characteristic forms on the manifold and on its boundary. These characteristic forms are essentially a modified version of those already considered Brüning and Ma in [BM06] when studying the (variation of) Ray–Singer analytic torsion on manifolds with boundary. More precisely, the forms we need on $M$ are the Euler form $e(M, g) \in \Omega^m(M; \Theta^C_{\mathcal{M}})$ associated to the metric $g$, and secondary forms of Chern–Simons type $\tilde{e}(M, g, g') \in \Omega^{m-1}(M; \Theta^C_{\mathcal{M}})$ associated to two (smoothly connected) Riemannian metrics $g$ and $g'$. The characteristic form on $\partial M$ that we need is defined in [BM06, expression (1.17), page 775] denoted by $e_b(\partial M, g)$ and the secondary (Chern–Simons) form is that in [BM06, expression (1.45), page 780] and denoted $\tilde{e}_b(\partial M, g, g') \in \Omega^{m-2}(\partial M; \Theta^C_{\mathcal{M}})$. The forms $e_b(\partial M, g)$ and $\tilde{e}_b(\partial M, g, g')$ were constructed by Brüning and Ma with respect to an inward pointing (unit vector) field along the whole boundary $\partial M$. Here, we want to distinguish the roles of $\partial_+ M$ and $\partial_- M$. We denote by $\varsigma_{\text{in}}$ the unit inward pointing normal vector field on the boundary, and by $\varsigma_{\text{out}} := -\varsigma_{\text{in}}$ the unit outward pointing normal vector field on the boundary. Then, we consider the following vector field

$$
\varsigma := \begin{cases} 
\varsigma_{\text{in}} & \text{on } \in \partial_+ M \\
\varsigma_{\text{out}} & \text{on } \in \partial_- M
\end{cases}
$$

which is inward pointing along $\partial_+ M$ and outward pointing along $\partial_- M$. Then, we use the vector field $\varsigma$ given in (10) to specify a characteristic form $e_0(\partial_+ M, \partial_- M, g) \in \Omega^{m-2}(\partial M; \Theta^C_{\mathcal{M}})$, a slightly modified version of $e_b(\partial M, g)$, and a secondary characteristic form $\tilde{e}_0(\partial_+ M, \partial_- M, g, g') \in \Omega^{m-2}(\partial M, \Theta^C_{\mathcal{M}})$, a slightly modified version of $\tilde{e}_b(\partial M, g, g')$, according to the vector field $\varsigma$. For further details, the reader is strongly referred at this point to the Appendix.

**Definition 1.** Let $\mathcal{M}$ be a Riemannian bordism. Consider the forms from Definition 3 in the Appendix. The relative Euler form is

$$
e(\mathcal{M}, g) := (e(M, g), e_0(\partial_+ M, \partial_- M, g)) \in \Omega^m(M, \partial M; \Theta^C_{\mathcal{M}}).$$
The relative Euler form $\mathbf{e}(\mathbb{M}, g)$ is closed in $\Omega(M, \partial M; \Theta^c_M)$, because of dimensional reasons. From formula (13) in Lemma 1 below, it follows that its cohomology class

$$[\mathbf{e}(\mathbb{M})] := [\mathbf{e}(\mathbb{M}, g)]$$

is independent of $g$.

**Definition 2.** The **secondary relative Euler form** on $\mathbb{M}$ associated to the Riemannian metrics $g_0$ and $g_\tau$ is the relative form

$$\mathbf{e}(\mathbb{M}, g_0, g_\tau) \in \Omega^{m-1}(M, \partial M; \Theta^c_M)$$

given by

$$\mathbf{e}(\mathbb{M}, g_0, g_\tau) := [\mathbf{e}(\mathbb{M}, g_0) - \mathbf{e}(\mathbb{M}, g_{\tau})],$$

where $\mathbf{e}(\mathbb{M}, g_0)$ and $\mathbf{e}_0(\partial_+ M, \partial_- M, g_0, g_\tau)$ are the Chern–Simons forms given in Definition 9 in the Appendix.

**Lemma 1.** (Brüning–Ma) Let $\mathbf{e}(\mathbb{M}, g_0, g_1)$ be the secondary relative Euler form in (12) associated to a couple of Riemannian metrics $g_0, g_1$ in $\mathbb{M}$. If $\{g_s\}$ is a smooth path of Riemannian metrics connecting $g_0$ to $g_1$, then the formula

$$d\mathbf{e}(\mathbb{M}, g_0, g_1) = \mathbf{e}(\mathbb{M}, g_1) - \mathbf{e}(\mathbb{M}, g_0)$$

holds. The secondary relative Euler form $\mathbf{e}(\mathbb{M}, g_0, g_1)$ does depend on the path of metrics, but only up to exact forms, so that, it defines a secondary relative Euler class in the sense of Chern–Simons. Moreover, up to exact forms in relative cohomology, the relations

$$\mathbf{e}(\mathbb{M}, g_0, g_\tau) = -\mathbf{e}(\mathbb{M}, g_\tau, g_0)$$

$$\mathbf{e}(\mathbb{M}, g_0, g_\tau) = \mathbf{e}(\mathbb{M}, g_0, g_s) + \mathbf{e}(\mathbb{M}, g_s, g_\tau)$$

hold.

**Proof.** Since $\partial_+ M$ and $\partial_- M$ are disjoint closed submanifolds, the statements above are exactly [BM06, Theorem 1.9]. The identities in (14) follow from the definition of $\mathbf{e}(\mathbb{M}, g_0, g_\tau)$ in Definition 9 in the Appendix.

3.1. **Co-Euler structures without base point.** We extend the notion of co-Euler structures in [BH07] to the case of bordisms $\mathbb{M}$.

**Lemma 2.** Recall Definitions 1, 9 together with the pairing $\langle \cdot, \cdot \rangle$ from [3]. Assume $M$ is connected. Let $\mathbf{e}(\mathbb{M}, g)$ be the relative form given in Definition 7. We assume first that the relative Euler Characteristics $\chi(\mathbb{M}, \partial_+ M) = 0$. Then the set

$$E^*(\mathbb{M}; \mathbb{C}) := \left\{(g, \alpha) \mid \alpha \in \Omega^{m-1}(M, \partial M; \Theta^c_M), \frac{d\alpha}{\alpha} = \mathbf{e}(\mathbb{M}, g) \right\}$$

is not empty, so that we can define a relation in the space (15) to say that $(g, \alpha) \sim^{cs} (g', \alpha')$ if and only if

$$\alpha' - \alpha = \mathbf{e}(\mathbb{M}, g, g') \in \Omega^{m-1}(M, \partial M; \Theta^c_M)/d\Omega^{m-2}(M, \partial M; \Theta^c_M),$$
where \( \overline{e}(\mathcal{M},g,g') \) is the secondary form defined in (12). The relation \( \sim^{cs} \) is an equivalence relation on \( E^*(\mathcal{M};\mathcal{C}) \).

Proof. By Chern–Gauss–Bonnet formula, see first equality of Lemma 7 below, the relative Euler form \( e(\mathcal{M},g) \) from Definition 1 satisfies
\[
\{\langle e(\mathcal{M},g) \rangle;[1]\} = 0.
\]
Since \( \langle \cdot, \cdot \rangle \) is non-degenerate, the relative form \( e(\mathcal{M},g) \) is exact in relative cohomology. That is, there exists \( \alpha \in \Omega^{m-1}(\mathcal{M},\partial\mathcal{M};\Theta^c_M) \) such that \( d\alpha = e(\mathcal{M},g) \). Hence the space \( E^*(\mathcal{M};\mathcal{C}) \) is not empty. The relation \( \sim^{cs} \) satisfies the reflexivity property, since
\[
\overline{e}(\mathcal{M},g,g) = 0.
\]
Symmetry and transitivity of \( \sim^{cs} \) are implied by Lemma 11.

**Definition 3.** Let \( E^*(\mathcal{M};\mathcal{C}) \) be the space defined in (15). The set of co-Euler structures on a bordism \( \mathcal{M} \) is defined as the quotient
\[
(16) \quad \text{Eul}^*(\mathcal{M};\mathcal{C}) := E^*(\mathcal{M};\mathcal{C})/\sim^{cs};
\]
the equivalence class of \( (g,\alpha) \) will be denoted by \([g,\alpha]\).

**Lemma 3.** Let \( H^{m-1}(\mathcal{M},\partial\mathcal{M};\Theta^c_M) \) be the relative cohomology groups in degree \( m-1 \) with coefficients in \( \Theta^c_M \). For a closed relative form \( \beta \in \Omega^{m-1}(\mathcal{M},\partial\mathcal{M};\Theta^c_M) \), denote by \([\beta]\) its corresponding class in relative cohomology. Consider \( \Upsilon^* \), the action of \( H^{m-1}(\mathcal{M},\partial\mathcal{M};\Theta^c_M) \) on the space of co-Euler structures \( \text{Eul}^*(\mathcal{M};\mathcal{C}) \) from Definition 3 given by
\[
(17) \quad \Upsilon^* : H^{m-1}(\mathcal{M},\partial\mathcal{M};\Theta^c_M) \times \text{Eul}^*(\mathcal{M};\mathcal{C}) \to \text{Eul}^*(\mathcal{M};\mathcal{C})
\]
\[
\left([\beta],[g,\alpha]\right) \mapsto [g,\alpha - \beta].
\]
Then, \( \Upsilon^* \) is well defined, independent of each choice of representatives, free and transitive on \( \text{Eul}^*(\mathcal{M};\mathcal{C}) \).

Proof. For \([\beta] \in H^{m-1}(\mathcal{M},\partial\mathcal{M};\Theta^c_M) \), a class in relative cohomology represented by the closed relative form \( \beta \in \Omega^{m-1}(\mathcal{M},\partial\mathcal{M};\Theta^c_M) \), consider its action on the co-Euler structure \( [g,\alpha] \), represented by the couple \((g,\alpha)\). Remark that \((g,\alpha - \beta) \in E^*(\mathcal{M};\mathcal{C}) \), because \( d(\alpha - \beta) = d\alpha - d\beta = d\alpha = e(\mathcal{M},g) \). Let us prove that \( \Upsilon^* \) does not depend on the choice of representatives. The map \( \Upsilon^* \) is independent of the choice of representative for the co-Euler class. Indeed, let \((g',\alpha')\) represent the same class as \((g,\alpha)\) in the quotient space \( E^*(\mathcal{M};\mathcal{C}) \) for which we have \( \Upsilon^*_\beta((g',\alpha')) = (g',(\alpha' - \beta)) \). Since \((\alpha' - \beta) - (\alpha - \beta) = (\alpha' - \alpha) = \overline{e}(\mathcal{M},g,g') \) modulo relative exact forms, we have \( \Upsilon^*_\beta([g,\alpha]) = \Upsilon^*_\beta([g',\alpha']) \). The map \( \Upsilon^* \) is also independent of the choice of the representative for the class in cohomology \([\beta]\). Indeed, different choices for the cohomology class of \( \beta \) are obtained by adding coboundaries in \( \Omega^{m-1}(\mathcal{M},\partial_+\mathcal{M};\Theta^c_M) \), that is \( \beta + d\gamma \). But for these forms we have \( \Upsilon^*_\beta + d\gamma([g,\alpha]) \), since the equivalence relation \( \sim^{cs} \) is given up
to relative exact forms only, see Lemma 2. So, we have proved \( \Upsilon^* \) is well defined and independent of every choice of representatives.

The same argument is used to see that \( H^{m-1}(M, \partial M; \Theta^C_M) \) acts freely on \( \mathfrak{Eul}^*(M; \mathbb{C}) \). Indeed, if \( \beta \) is such that \( [g, \alpha - \beta] = [g, \alpha] \), then \( \beta = \tilde{e}(M, g, g') + d\tilde{\beta}' \), but, since the first term on the right hand side in the equality above vanish, the relative form \( \tilde{\beta} \) is necessarily exact.

We show this action is transitive on \( \mathfrak{Eul}^*(M; \mathbb{C}) \): for two classes \([g, \alpha]\) and \([g', \alpha']\), we can choose the relative form \( \beta := (\alpha - \alpha') + \tilde{e}(M, g, g') \). By Lemma 1, the relative form \( \beta \) is closed, since \( d\beta = e(M, g) - e(M, g') + d\tilde{e}(M, g, g') = 0 \). Finally, we have \( \Upsilon^*_g ([g, \alpha]) = [g, \alpha - \beta] = [g', \alpha'] \). \( \square \)

3.1.1. The flip map for co-Euler Structures. Let us consider the spaces of co-Euler structures \( \mathfrak{Eul}^*(M; \mathbb{C}) \) and \( \mathfrak{Eul}^*(M'; \mathbb{C}) \), from Definition 3, corresponding to the mutually dual bordisms \( M \) and \( M' \) respectively. In view of Lemma 9 in the Appendix, there is a natural map

\[
\nu^* : \mathfrak{Eul}^*(M; \mathbb{C}) \to \mathfrak{Eul}^*(M'; \mathbb{C}) \quad [g, \alpha] \mapsto [g, \alpha (-1)^m C \alpha]
\]

which is affine over the involution in relative cohomology

\[
(-1)^m : id : H^{m-1}(M, \partial M; \Theta^C_M) \to H^{m-1}(M, \partial M; \Theta^C_M).
\]

Remark 1. If \( M \) is a closed manifold, i.e., \( \partial_+ M = \partial_- M = \emptyset \), then clearly \( \mathfrak{Eul}^*(M; \mathbb{C}) = \mathfrak{Eul}^*(M'; \mathbb{C}) \), are affine over \( H^{m-1}(M; \Theta^C_M) \) and coincide with \( \mathfrak{Eul}^*(M; \mathbb{C}) \), the set of co-Euler structures on a manifold without boundary (see \( \text{BH07} \)). If \( M \) is closed and of odd dimension, then the involution \( \nu^* \), being affine over \( -id \), possesses a unique fixed point in \( \mathfrak{Eul}^*(M; \mathbb{C}) \), which corresponds to the canonical co-Euler structure

\[
\epsilon^*_\text{can} := [g, (\alpha_{\text{can}} = 0, \alpha_\partial = 0)]
\]

where \( \alpha_{\text{can}} = 0 \), because for odd dimensional closed manifolds \( e(M, g) = 0 \) and forms \( \alpha_\partial = 0 \), see \( \text{BH07} \) Section 2.2).

3.1.2. Infinitesimal variation of co-Euler structures without base point. The following result generalizes \( \text{BH07} \) (56).

Proposition 1. Let \( M \) be a bordism and assume that the relative Euler characteristics \( \chi(M, \partial_+ M) = 0 \). Consider \( \{(g_u, \alpha_u)\}_u \) a smooth one-parameter family of Riemannian metrics \( g_u \) and relative forms \( \alpha_u \), representing the same co-Euler structure \([g_u, \alpha_u] \in \mathfrak{Eul}^*(M; \mathbb{C})\). For each Riemannian metric \( g_u \) consider the forms

\[
e(M, g_u) \in \Omega^m(M, \partial M; \Theta^C_M)
\]

and

\[
B(\partial_+ M, \partial_- M, g_u) \in \Omega^{m-1}(\partial M; \Theta^C_M)
\]

from Definition 8 as well as the relative Chern-Simon’s form

\[
\bar{\mathcal{E}}(M, g_u, g_w) \in \Omega^{m-1}(M, \partial M; \Theta^C_M)
\]
from Definition \([8]\). Let \(E\) be a complex flat vector bundle over \(M\) with flat connection \(\nabla^E\), endowed with a smooth family of non-degenerate symmetric bilinears forms \(b_u\). If

\[
\omega(\nabla^E, b_u) := -\frac{1}{2} \text{Tr}(b_u^{-1} \nabla^E b_u) \in \Omega^1(M; \mathbb{C})
\]

denotes the Kamber–Tondeur form associated to \(b_u\) and \(\nabla^E\), see [BH07], and the integral \(\int (M, \partial M)\) is the pairing from \([3]\). Then, the formulas

\[
\frac{\partial}{\partial u} \int (M, \partial M) 2\alpha_u \wedge \omega(\nabla^E, b_u)
\]

\[
= -(-1)^m \int (M, \partial M) \mathcal{E}(M, g_u) \text{Tr}\left(b_u^{-1} b_u\right)
\]

\[
+ 2 \int (M, \partial M) \frac{\partial}{\partial \tau} \bigg|_{\tau=0} \bar{\mathcal{E}}(M, g_u, g_u + \tau \tilde{g}_u) \wedge \omega(\nabla^E, b_u)
\]

and

\[
\frac{\partial}{\partial u} \int_{\partial M} B(\partial_+ M, \partial_- M, g_u) = \int_{\partial M} \frac{\partial}{\partial \tau} \bigg|_{\tau=0} B(\partial_+ M, \partial_- M, g + \tau \tilde{g}_u).
\]

hold.

**Proof.** First, remark that \(\frac{\partial}{\partial u} \omega_u = \frac{\partial}{\partial u} \omega_u - \frac{\partial}{\partial u} \omega_u = \frac{\partial}{\partial u} \mathcal{E}(M, g_u + \tau \tilde{g}_u)\), and also that \(\frac{\partial}{\partial u} B(\partial_+ M, \partial_- M, g_u) = \frac{\partial}{\partial u} B(\partial_+ M, \partial_- M, g + \tau \tilde{g}_u)\). From [BH07] we have the identity \(\frac{\partial}{\partial u} \text{Tr}(b_u^{-1} \nabla^E b_u) = d \text{Tr}(b_u^{-1} \nabla^E b_u)\). Therefore, since for each \(u\), the couple \([g_u, \alpha_u]\) represents the same co-Euler structure, we obtain, modulo exact relative forms

\[
\frac{\partial}{\partial u} \int (M, \partial M) 2\alpha_u \wedge \omega(\nabla^E, b_u)
\]

\[
= \int (M, \partial M) \partial_u \alpha_u \wedge \omega(\nabla^E, b_u) + \int_{(M, \partial M)} \omega(\nabla^E, b_u) - \int (M, \partial M) \omega(\nabla^E, b_u)
\]

\[
= 2 \int (M, \partial M) \frac{\partial}{\partial \tau} \mathcal{E}(M, g_u, g_u + \tau \tilde{g}_u) \wedge \omega(\nabla^E, b_u) + \int (M, \partial M) - \partial_u \wedge \partial \text{Tr}(b_u^{-1} b_u).
\]

\((*)\)

with \(\alpha_u = (\alpha_u, \sigma_u)\), \(d\alpha_u = \mathcal{E}(M, g_u)\) and Stokes' Theorem, the second term on the right above becomes

\((*)=\)

\[-(-1)^m \int_M \alpha_u \text{Tr}(b_u^{-1} b_u) - \int_{\partial M} \alpha_u \text{Tr}(b_u^{-1} b_u)\]

\[-(-1)^m \int_{\partial M} (\alpha_u \text{Tr}(b_u^{-1} b_u)) - \int_M (d^\alpha_u \alpha_u \text{Tr}(b_u^{-1} b_u))\]

\[-(-1)^m \int_{(M, \partial M)} d\alpha_u \text{Tr}(b_u^{-1} b_u)\]

\[-(-1)^m \int_{(M, \partial M)} \mathcal{E}(M, g_u) \text{Tr}(b_u^{-1} b_u)\]

\[= -(-1)^m \int_{(M, \partial M)} \mathcal{E}(M, g_u) \text{Tr}(b_u^{-1} b_u)\]

\[= -(-1)^m \int_{(M, \partial M)} \mathcal{E}(M, g_u) \text{Tr}(b_u^{-1} b_u)\]
Since $M$ is assumed to be connected, in view of the first equality in (9), the space in (22) is non-empty. Then, as for the case without base point, we have the relation: $(g, \alpha) \sim_{cs} (g', \alpha')$ in $E^*_x(M; \mathbb{C})$ if and only if

$$\alpha' - \alpha = \overline{c}(M, g, g') \in \Omega^{m-1}(\hat{M}, \partial M; \Theta_M^C)/d\Omega^{m-2}(\hat{M}, \partial M; \Theta_M^C).$$

The relation in (23) is an equivalence relation for the same reasons as in the case without base point.

**Definition 4.** The quotient space

$$Eul^*_x(M; \mathbb{C}) := E^*_x(M; \mathbb{C})/\sim_{cs}$$

is called the space of co-Euler structures based at $x_0$ on $M$ and the equivalence class of the pair $(g, \alpha)$ is denoted by $[g, \alpha]$.

The action of $H^{m-1}(M, \partial M; \Theta_M^C)$ on $Eul^*_x(M; \mathbb{C})$ defined by

$$\Upsilon^* : H^{m-1}(M, \partial M; \Theta_M^C) \times Eul^*_x(M; \mathbb{C}) \to Eul^*_x(M; \mathbb{C})$$

$$([\beta], [g, \alpha]) \mapsto [g, \alpha - \beta]$$

is well defined and independent of each choice of representatives, see Lemma 3. In addition, the action specified by (24) is free and transitive since $H^{m-1}(M, \partial M) \cong H^{m-1}(\hat{M}, \partial M)$, see (9).

Finally, the flip map

$$\nu^* : Eul^*_x(M; \mathbb{C}) \to Eul^*_{x_0}(M'; \mathbb{C})$$

$$[g, \alpha] \mapsto [g, (-1)^m \alpha]$$

intertwines the spaces $Eul^*_x(M, \partial M, \partial \pm M, \partial \pm M; \mathbb{C})$ and it is affine over the involution in relative cohomology

$$(-1)^m \text{id} : H^{m-1}(\hat{M}, \partial M; \Theta_M^C) \to H^{m-1}(\hat{M}, \partial M; \Theta_M^C).$$

### 3.2.1. Variational formula for co-Euler structures with base point

We give an analog to Proposition 1 in the case of co-Euler structures with base point. Let $\alpha \in \Omega^{m-1}(\hat{M}; \Theta_M^C)$ be a smooth differential form on $M$, with possible singularity $x_0 \in \text{int}(M)$, the interior of $M$ and $\hat{M} := M \setminus \{x_0\}$. For $\omega$ a closed 1-form on $M$, we make sense of integrals of the type $\int_{\hat{M}} \alpha \wedge \omega$, by means of a regularization procedure as described in the remaining of this section.

First, recall that the local degree of $\alpha$ at the singularity $x_0$, see for instance [BT82, Chapter II.11], is given by

$$\deg_{x_0}(\alpha) := \lim_{\delta \to 0} \int_{\partial(B^m(\delta, x_0))} i^* \alpha,$$

where $\partial(B^m(\delta, x))$ indicates the boundary of the $m$-dimensional closed ball $B^m(\delta, x)$ centered at $x_0$ and radius $\delta > 0$. With the standard sign convention involved in Stokes’ Theorem, $\partial(B^m(\delta, x))$ is oriented with respect to the unit outwards pointing vector field normal to $B^m(\delta, x)$. 


Lemma 4. Let $\alpha$ be a smooth form in $\Omega^{m-1}(M; \Theta^C_M)$ such that $d\alpha$ and $\alpha_\theta$ are smooth and without singularities in $M$. For $\omega$ a smooth closed 1-form on $M$, choose a smooth function $f \in C^\infty(M)$ such that the 1-form

$$\omega' := \omega - df$$

is smooth on $M$ and vanishes on a small neighborhood of $x_0$. Then the complex-valued function

$$(27) \quad S(\alpha, \omega, f) := \int_{(M, \partial M)} \alpha \wedge \omega' + (-1)^m \int_{(M, \partial M)} df \wedge f(x_0) \text{deg}_{x_0}(\alpha),$$

does not depend on the choice of $f$ and satisfies the following assertions.

1. If $\beta \in \Omega^{m-1}(M, \partial M; \Theta^C_M)$, i.e., without singularities, then

$$S(\beta, \omega) = \int_{(M, \partial M)} \beta \wedge \omega.$$  

In particular, $S(d\gamma, \omega) = 0$ for all $\gamma \in \Omega^{m-2}(M, \partial M; \Theta^C_M)$.

2. $S(\omega, \alpha)$ is linear in $\alpha$ and in $\omega$.

3. $S(d\omega, dh) = (-1)^m \int_{(M, \partial M)} d\alpha \wedge h - h(x) \text{deg}_{x_0}(\alpha)$.

Proof. Without loss of generality assume $X(\alpha) = \{x\}$. We want to know how the function $\int_{(M, \partial M)} \alpha \wedge \omega'$ changes, with respect to $f$. Let us take $f_1, f_2 \in C^\infty(M)$ two functions as above, such that the corresponding one forms $\omega_1', \omega_2'$ vanish on a small open neighborhood of $x_0$, so that $d(f_2 - f_1) = 0$ locally around $x_0$; that means $f_2 - f_1$ is constant\footnote{If we choose $f_2(x) = f_1(x) = 0$, then $f_2 - f_1 = 0$ around $x_0$.} on a small neighborhood of $x_0$. Now, consider the variation

$$\Delta = \int_{(M, \partial M)} \alpha \wedge (w_2' - w_1') = \int_{M \setminus \{x\}} \alpha \wedge (w_2' - w_1') - \int_{\partial M} \alpha_\theta \wedge (w_2' - w_1')$$

$$= -\int_{M \setminus \{x\}} \alpha \wedge d(f_2 - f_1) + \int_{\partial M} d\alpha \wedge \alpha_\theta \wedge (d(f_2 - f_1)).$$

We develop both terms on the right of the last equality above. The first one, the integral over $M$, can be re written as

$$-\int_{M \setminus \{x\}} \alpha \wedge d(f_2 - f_1) = -(-1)^{m-1} \int_{M \setminus \{x\}} \alpha \wedge (f_2 - f_1) + (-1)^m \int_{M \setminus \{x\}} d\alpha \wedge (f_2 - f_1),$$

whereas the second one, the integral over the boundary becomes

$$\int_{\partial M} d\alpha \wedge \alpha_\theta \wedge (f_2 - f_1) = (-1)^{m-2} \int_{\partial M} d^0(\alpha_\theta \wedge (f_2 - f_1)) - (1)^{m-2} \int_{\partial M} d^0 \alpha_\theta \wedge (f_2 - f_1)$$

$$= (-1)^{m-1} \int_{\partial M} d^0 \alpha_\theta \wedge (f_2 - f_1)$$

and therefore

$$\Delta = -(-1)^{m-1} \left( \int_{M \setminus \{x\}} \alpha \wedge (f_2 - f_1) - \int_{M \setminus \{x\}} d\alpha \wedge (f_2 - f_1) - \int_{\partial M} d^0 \alpha_\theta \wedge (f_2 - f_1) \right)$$

$$= -(-1)^{m-1} \left( \int_{M \setminus \{x\}} d(\alpha(f_f - f_1)) - \int_{M \setminus \{x\}} d\alpha \wedge (f_2 - f_1) - \int_{\partial M} d^0 \alpha_\theta \wedge (f_2 - f_1) \right),$$

where we have used

$$\int_{M \setminus \{x\}} d\alpha \wedge (f_2 - f_1) = \int_M d\alpha \wedge (f_2 - f_1).$$
since by assumption, the form \( da \) does not have singularities on \( M \). Hence, to make sense of \( \Delta \), we now make sense of the integral \( \int_{M \setminus \{x\}} d(\alpha(f_2 - f_1)) \).

This integral can be computed as the limit:

\[
\int_{M \setminus \{x\}} d(\alpha(f_2 - f_1)) = \lim_{\delta \to 0} \int_{M \setminus \mathbb{B}(\delta, x)} d(\alpha(f_2 - f_1))
\]

where \( \mathbb{B}(\delta, x) \) is the closed ball centered at \( x_0 \) of radius \( \delta > 0 \) and with boundary \( \partial(\mathbb{B}(\delta, x)) \) endowed with the orientation specified by the unit outwards pointing vector field normal to \( \mathbb{B}(\delta, x) \). Then, by using Stokes’ Theorem with the standard convention, the limit above can be computed as

\[
\int_{M \setminus \{x\}} d(\alpha(f_2 - f_1)) = \lim_{\delta \to 0} \int_{M \setminus \mathbb{B}(\delta, x)} d(\alpha(f_2 - f_1)) = \int_{\partial M} i^* (\alpha(f_2 - f_1)) + \lim_{\delta \to 0} \int_{\partial(\mathbb{B}(\delta, x))} i^* (\alpha(f_2 - f_1)).
\]

where \( -\partial(\mathbb{B}^m(\delta, x)) \) indicates the sphere with opposite orientation as that of \( \partial(\mathbb{B}(\delta, x)) \). Now, we look at the second term on the right of the equality above. Since \( f_2 - f_1 \) is constant on a small neighborhood of \( x_0 \), we have, for \( \delta' > 0 \) small enough,

\[
\lim_{\delta \to 0} \int_{\partial(\mathbb{B}^m(\delta, x))} i^* (\alpha(f_2 - f_1)) = (f_2 - f_1)(x) \lim_{\delta \to 0} \int_{\partial(\mathbb{B}(\delta, x))} i^* \alpha = \text{for all } x' \in B(\delta', x),
\]

where the sign \((-1)^m\) above comes from the standard convention taken for the Stokes’ Theorem. Hence

\[
\int_{M \setminus \{x\}} d(\alpha(f_2 - f_1)) = \int_{\partial M} i^* (\alpha(f_2 - f_1)) + (-1)^m (f_2 - f_1)(x) \deg_{x_0}(\alpha).
\]

Therefore the variation \( \Delta \) becomes

\[
\Delta = (-1)^{m-1} \left[ \int_{\partial M} i^* (\alpha(f_2 - f_1)) + (-1)^m (f_2 - f_1)(x) \deg_{x_0}(\alpha) \right] - \int_M d\alpha \wedge (f_2 - f_1) - \int_{\partial M} d\alpha \wedge (f_2 - f_1) = (-1)^{m-1} \left[ \int_{\partial M} i^* (\alpha \wedge d\alpha) \wedge i^* (f_2 - f_1) - \int_M d\alpha \wedge (f_2 - f_1) + (-1)^m (f_2 - f_1)(x) \deg_{x_0}(\alpha) \right]
\]

and so

\[
S_{f_2}(\omega) - S_{f_1}(\omega) = \Delta + \left( -1 \right)^m \int_{M \setminus \partial M} d\alpha \wedge (f_2 - f_1) - (f_2 - f_1)(x) \deg_{x_0}(\alpha) = 0,
\]

so \( S_{f}(\omega) \) does not depend on the choice of \( f \). Remark the linearity of \( S(\omega) \) with respect to \( \omega \) immediately follows also from its independance of \( f \). The remaining assertions in (1) and (2) follow from similar considerations as above, we omit the details. Let us turn to assertion (3). For a smooth
function \( h \), we compute
\[
S_f(\omega + \alpha) = \int_{(M,\partial M)} h(\omega + df) + (-1)^m \int_{(M,\partial M)} df \wedge f(x_0) \deg x_0(\alpha),
\]
\[
= \int_{(M,\partial M)} h(\omega + dh) + (-1)^m \int_{(M,\partial M)} df \wedge (f-h) \deg x_0(\alpha),
\]
that is,
\[
(-1)^m \int_{(M,\partial M)} df \wedge h(x) \deg x_0(\alpha) = S_f(\omega + dh) - S_{f-h}(\omega).
\]
where the second equality above holds, since \( S \) is independent of \( f \) and the third one because \( S \) is linear on \( \omega \).

**Corollary 1.** Let \( \alpha \) be as in Lemma 4 Then, we have the formula
\[
\deg x_0(\alpha) = (-1)^m \int_{(M,\partial M)} d\alpha.
\]

**Proof.** Let \( \omega \), \( f \), \( \alpha \) and \( \mathcal{X}(\alpha) \) be as above and consider \( f_0 \) to be a constant function on \( M \). Then we compute
\[
S_{f+f_0}(\alpha, \omega) = \int_{(M,\partial M)} \alpha \wedge \omega' + (-1)^m \int_{(M,\partial M)} df \wedge f - \int_{(M,\partial M)} df \wedge (f(x_0) \deg x_0(\alpha) + f_0 \deg x_0(\alpha)),
\]
\[
= S_f(\alpha, \omega) + f_0 \left( (-1)^m \int_{(M,\partial M)} df \wedge \deg x_0(\alpha) \right).
\]
But, from Lemma 4 above, we know \( S_{f+f_0}(\alpha, \omega) = S_f(\alpha, \omega) \), and hence the last term on the right above vanishes, so that the desired relation between the total degree of the form \( \alpha \) and \( \alpha \) follows.

The formula obtained in Corollary 1, which computes the total degree of \( \alpha \) in terms of the relative form \( \alpha \), is used to conclude the following, and hence generalizing formula (20) in Proposition 1.

**Proposition 2.** Consider a bordism \( M \), together with the relative Euler form \( e(M, g) \) from Definition 7. For \( x_0 \) a base point in the interior of \( M \), consider the space \( \text{Eul}^*_x(M; \mathbb{C}) \) of co-Euler structures at \( x_0 \) from Definition 4. Let \( e \in \text{Eul}^*_x(M; \mathbb{C}) \) be represented by \( (g, \alpha) \), where \( \alpha := (\alpha, \alpha_\partial) \) is a relative form with \( \alpha \in \Omega^{m-1}(M; \Theta_C^M) \) with unique singularity at \( x_0 \) and assume that \( \alpha \) and \( \alpha_\partial \) are smooth, i.e. without singularities in \( M \). For \( \omega \in \Omega^1(M) \), a smooth closed 1-form on \( M \), choose a smooth function \( f \in C^\infty(M) \) such
that $\omega' := \omega - df \in \Omega^1(M)$ is a smooth 1-form that vanishes on a small neighborhood of $x_0$. Then

$$ S_f(\alpha, \omega) = \int_{(M, \partial M)} \alpha \wedge \omega' + (-1)^m \int_{(M, \partial M)} e(M, g) \wedge f - f(x_0) \chi(M, \partial M) $$

In particular, if $e^*$ is represented by $(g, \alpha)$ and $(g', \alpha')$, then

$$ S(\alpha', \omega) - S(\alpha, \omega) = \int_{(M, \partial M)} \tilde{e}(M, g, g') \wedge \omega. $$

Proof. Under these assumptions, from Corollary 1, we have

$$ \deg x_0(\alpha) = (-1)^m \int_{(M, \partial M)} d\alpha = (-1)^m \int_{(M, \partial M)} e(M, g) = \chi(M, \partial_- M), $$

where the last equality follows from Gauss–Bonnet–Chern Theorem. Therefore, (28) follows from the definition of $S$ in (27). Finally, formula (29) follows from (28) and the defining relation (23).

\[\square\]

4. Poincaré duality for (co)-Euler Structures

4.1. Euler Structures on bordisms. Let $M$ be a compact Riemannian bordism of dimension $m$. Euler structures were introduced by Turaev in [Tu90] in order to remove the metric ambiguities in the definition of the Reidemeister torsion. In this section, we recall a possible definition adapted to our conventions. For the sake of brevity, we assume that $\chi(M, \partial_- M) = 0$ and we restrict to the case of Euler structure without base point. The general case, without any assumption on $\chi(M, \partial_- M)$, leads to the definition of Euler structures with a base point $x_0$ in the interior of $M$ in an analogous manner as in the situation for closed manifolds, see [BH06a] and [BH06b].

Definition 5. Let $X : M \to TM$ be a vector field on $M$, which is transverse to the zero section, inward pointing along $\partial_+ M$ and outward pointing along $\partial_- M$. We call such a vector field $X$ to be adapted to the bordism $M$.

Let $X = X^{-1}(0)$ be the set of zeros of $X$. The transversality condition means each $x \in X$ is non-degenerate with Hopf index $\text{Ind}_X(x) \in \{\pm 1\}$. Consider the singular 0-chain in $M$

$$ e(X) := \sum_{x \in X} \text{Ind}_X(x) x \in C^\text{sing}_0(M; \mathbb{C}) $$

If $\chi(M, \partial_- M) = 0$, then by the Hopf–Poincaré theorem, see for instance [BP01, Lemma 1.2 and Proposition 1.1], $X$ admits a (smooth) singular 1-chain $c \in C^\text{sing}_1(M; \mathbb{C})$ such that

$$ \partial c = e(X); $$

this singular 1-chain will be called an Euler chain, see [BH06b, Section 4].

Let $X_0$ and $X_1$ be two adapted vector fields to $M$. Then, there exists a smooth one-parameter family of vector fields $X_t$ connecting $X_0$ to $X_1$, with the property that $X_t$ is inward pointing along $\partial_+ M$ and outward pointing
along $\partial_- M$, for each $t \in I := [0, 1]$. For $p_M : M \times I \to M$, the canonical projection, consider the bundle $p_M^*TM \to M \times I$ and denote by $\tilde{X} \in \Gamma(p_M^*TM)$ the section corresponding to the smooth family of vector fields $X_t$. With the help of small perturbations, we may assume that $\tilde{X}$ is also transversal to the zero section; in other words that $X_t$ is adapted to $\mathbb{M}$ for each $t \in I$. Therefore, its zero set $\tilde{X} := \tilde{X}^{-1}(0) \subset M \setminus \partial M \times I$ is a canonically oriented one dimensional submanifold with boundary $\partial\tilde{X} = \tilde{X} \cap (M \times \partial I)$. Let
\begin{equation}
\tilde{e}(X_0, X_1) \in C^1_{\text{sing}}(M; \mathbb{C}) / \partial C^2_{\text{sing}}(M; \mathbb{C})
\end{equation}
be the equivalence class, called the Chern–Simons’ class, obtained by projecting a representative of the fundamental class of $\tilde{X}$ into $M$, by means of the projection $p_M$. The class $\tilde{e}(X_0, X_1)$ depend neither on the representative of the fundamental class of $\tilde{X}$ nor on the homotopy of vector fields connecting $X_0$ to $X_1$, see [BH06a]. The class $\tilde{e}(X_0, X_1)$ is represented by the 0-set of a generic homotopy connecting $X_0$ to $X_1$, by a smooth family of vector fields $X_t$ adapted to $\mathbb{M}$ for each $t \in [0, 1]$, so that the integral
\begin{equation}
\int_{\tilde{e}(X_0, X_1)} \omega = \int_{\tilde{X}} p_M^* \omega
\end{equation}
is well defined, for every closed one form $\omega \in \Omega^1(M)$. Moreover, for the Chern–Simons’ classes in (32), and the singular 0-chains in (30), the relations
\begin{align}
\tilde{e}(X, X) &= 0, \\
\tilde{e}(X_0, X_1) + \tilde{e}(X_1, X_2) &= \tilde{e}(X_0, X_2) \\
\partial \tilde{e}(X_0, X_1) &= \tilde{e}(X_1) - \tilde{e}(X_0),
\end{align}
hold.

Now, for simplicity assume $\chi(M, \partial_- M) = 0$. Let $(X_0, c_0)$ and $(X_1, c_1)$ be two pairs of adapted vector fields $X_0$ and $X_1$ with corresponding singular 1-chain $c_0$ and $c_1$ as in (31), respectively. We call such pairs to be equivalent if and only if
\begin{equation}
c_1 - c_0 = \tilde{e}(X_0, X_1) \in C^1_{\text{sing}}(M; \mathbb{C}) / \partial C^2_{\text{sing}}(M; \mathbb{C}).
\end{equation}
This is an equivalence relation because of the identities in (34) and we denote by $[X, c]$ the corresponding equivalence classes.

**Definition 6.** Assume $\chi(M; \partial_- M) = 0$. The space of Euler structures $\mathfrak{Eul}(\mathbb{M}; \mathbb{C})$ is defined as the set of equivalence classes $[X, c]$ of pairs $(X, c)$ under the equivalence relation in (35).

There is an action $\Upsilon$ of $H_1(M; \mathbb{C})$ on $\mathfrak{Eul}(\mathbb{M}; \mathbb{C})$, given by
\[ \Upsilon : ([X, c], [\sigma]) \mapsto [X, c + \sigma], \]
for each $\sigma \in H_1(M; \mathbb{C})$ on $[X, c] \in \mathfrak{Eul}(\mathbb{M}; \mathbb{C})$. This action is well defined, free and transitive, because of the relations in (34), see also [BH06a], [BP01] and [Tu90].
Recall that, see [BT82, Chapter II.11], under the involution
\[(36) \quad \xi : TM \to TM \text{ given by } \xi(Y) = -Y\]
the Hopf index of \( x \in X \) satisfies \( \text{ind}_- X(x) = (-1)^m \text{ind}_X(x) \), and hence
\[(37) \quad e(-X) = (-1)^m e(X) \quad \text{ and } \quad \tilde{e}(-X_1, -X_2) = (-1)^m \tilde{e}(X_1, X_2),\]
so that, we obtain a flip map, between Euler structures on dual bordisms
\[(38) \quad \nu : \text{Eul}(\mathbb{M}; \mathbb{C}) \to \text{Eul}(\mathbb{M}'; \mathbb{C})\]
\[
[X, c] \mapsto [-X, (-1)^m c],
\]
which is affine over the involution in homology
\[(39) \quad (-1)^m \text{id} : H_1(M, \mathbb{C}) \to H_1(M, \mathbb{C}).\]

4.2. A relative Mathai–Quillen form. Let \( \mathbb{M} \) be a bordism of dimension \( m \) and Riemannian metric \( g \). For \( \pi : TM \to M \), recall that the Mathai–Quillen form
\[(40) \quad \psi(M, g) \in \Omega^{m-1}(TM \setminus M; \pi^* \Theta_M)\]
associated to the Levi–Civita connection on \( TM \), satisfies
\[(41) \quad \text{d} \psi(M, g) = \pi^* e(M, g)\]
where \( e(M, g) \) is the Euler form of \( M \), and for \( \xi \)
\[(42) \quad \psi(M, g) = (-1)^m \xi^* \psi(M, g),\]
see [MQ86] and [BZ92].

Definition 7. Let \( Q \subseteq TM|_{\partial M} \) be the subset of all vectors over \( \partial_+ M \) which are inward pointing and all vectors over \( \partial_- M \) which are outward pointing. We define a relative Mathai–Quillen form by
\[
\underpsi(M, g) := (\psi(M, g), \psi_{\partial}(\partial_+ M, \partial_- M, g)) \in \Omega^{m-1}(TM \setminus M, Q; \pi^* \Theta_M)
\]
where its boundary component
\[(43) \quad \psi_{\partial}(\partial_+ M, \partial_- M, g) := \int_0^1 \text{inc}_s \iota_\partial h^* \psi(M, g) ds \in \Omega^{m-2}(Q; \pi^* \Theta_M),\]
is defined by using the homotopy
\[
h : Q \times [0, 1] \to Q \subseteq TM \setminus M \text{ given by } h_s := s \cdot \text{id} + (1 - s)(\varsigma \circ \pi),
\]
with \( \varsigma \) being the unit vector field in \([10]\), and \( \text{inc} : Q \to Q \times [0, 1] \), canonical inclusion, and \( \iota_\partial h_s \) indicates the contraction with respect to the vector field \( \partial_s \).

Lemma 5. In analogy with \([41]\) and \([42]\), the relative Mathai–Quillen form from Definition 7 satisfies
\[(44) \quad \text{d} \underpsi(M, g) = \pi^* e(M, g),\]
where \( \text{d} \) is the differential given in \([3]\) and
\[(45) \quad \underpsi(M, g) = (-1)^m \xi^* \underpsi(M', g),\]
where $\xi$ is the involution in \cite{36}. Moreover, if $g_0$ and $g_1$ be two Riemannian metrics on $M$, then
\begin{equation}
\psi(M, g_1) - \psi(M, g_0) = \pi^*\tilde{e}(\mathbb{M}, g_0, g_1)
\end{equation}
modulo $d(\Omega^{m-2}(TM\setminus M, Q; \pi^*\Theta_M))$.

Proof. First, from (41), it follows modulo
\begin{equation}
\int_{\mathbb{M}} \mathcal{L}_\xi \omega = 0,
\end{equation}
where the last equality holds since $\pi^*\omega(M, g)$ being a $m$-form, its pull-back by $h$ to the boundary must vanish for dimensional reasons. Then, by applying the exterior derivative to (43), using its naturality with respect to pull-backs, the (Lie) derivative $\frac{d}{ds} = d \circ \iota_{\partial_s} - \iota_{\partial_s} \circ d$, formula (41), Stokes’ Theorem and $\text{inc}_0^* = \text{inc}_1^* = \text{id}$, we obtain
\begin{equation}
d\psi(M, g) = h^*d\psi(M, g) = h^*\pi^*\omega(M, g) = 0,
\end{equation}
where $\xi$ follows immediately from (42).

Then, using $\psi(M, g) = \omega(M, g)$ given in \cite{BM06, Formula (2.10)}, the first claim follows. The behavior of the relative Mathai–Quillen form with respect to the involution $\xi$ follows immediately from \cite{12}.

We now prove \cite{16}. Consider the transgressed Euler form $\tilde{e}(M, g_0, g_1)$ from Definition \cite{9} in the Appendix. With (41), which in this case translates as $d\psi(M, g) = \pi^*\omega(M, g)$, we obtain
\begin{equation}
\psi(M, g_1) - \psi(M, g_0) = \pi^*\tilde{e}(M, g_0, g_1) + d\int_0^1 \text{inc}_0\iota_{\partial_t}\psi(M, g) dt.
\end{equation}
Now, consider the homotopy $\tilde{h} : I \times I \times \partial M \to Q$ given by $\tilde{h}_{s,t} := s \cdot \text{id} + (1 - s)(\tilde{\xi} \circ \pi)$ and remark that
\begin{equation}
\iota_{\partial_t}d\tilde{h}^*\psi(M, g_1) = \iota_{\partial_t}\pi^*\omega(M, g) = 0
\end{equation}
Analogously, consider the Chern–Simons’ form $\tilde{e}(\partial_+ M, \partial_- M, g_0, g_1)$ from Definition \cite{9} Then, we have
\begin{equation}
\psi(M, g_1) - \psi(M, g_0) = \pi^*\tilde{e}(\partial_+ M, \partial_- M, g_0, g_1)
\end{equation}
Combining (49) and (50), we obtain formula (46) expressing the dependence of the Mathai–Quillen form on the metric.

Let $X$ be adapted to $M$ as in Definition \cite{5} Then, we have a smooth map of pairs $X : (M \setminus \mathcal{X}) \to (TM\setminus M, Q)$, where $Q$ is as in Definition \cite{7} and hence
\begin{equation}
X^*\psi(M, g) \in \Omega^{m-1}(M \setminus \mathcal{X}, \partial M, \Theta_M^C).
\end{equation}
Moreover the integral $\int_{X(M \setminus \mathcal{X})} X^*\psi(M, g) \wedge \omega$ is absolute convergent for each $\omega \in \Omega^1_c(M \setminus \mathcal{X}; \mathbb{C})$ vanishing on a neighborhood of $\mathcal{X}$.
Lemma 6. For every smooth function \( f \) on \( M \), being locally constant on a neighborhood of \( X \), we have
\[
(−1)^m \int_{(M \setminus X, \partial M)} X^*ψ(M, g) \wedge df = \int_{(M, \partial M)} e(M, g) f - \sum_{x \in X} \text{Ind}_X(x) f(x)
\]

Proof. This follows from fully developing the integral
\[
\int_{(M \setminus X, \partial M)} d(X^*ψ(M, g) \wedge f)
\]
by using the graded Leibniz formula in (4), the paring (5), the identity (44), Stokes’ Theorem and that
\[
\lim_{δ \to 0} \int_{−∂B^n(δ, x)} X^*ψ(M, g) = \text{Ind}_X(x)
\]
for every zero \( x \in X \). The signs conventions when using Stokes’ Theorem are taken with the standard convention as in the proof of Lemma 4. □

The following Lemma gives a relative version of the Hopf’s formula at the same time.

Lemma 7. For the bordism \( M \) consider an adapted vector field \( X \) as in Definition 5. Let \( \chi(M, \partial - M) \) be the Euler characteristic relative to \( \partial - M \). Then, with \( e(M, g) \) the relative Euler form given in Definition 1 and the paring in (5), we have
\[
\chi(M, \partial - M) = \int_{(M, \partial M)} e(M, g) = \sum_{x \in X} \text{Ind}_X(x)
\]

Proof. The first equality is a restatement of the (Chern–Gauss–Bonne) formula in [BM11, Theorem 3.4], see also [Ma13, Theorem 6.1.14], in terms of the relative form from Definition 1. The second equality, directly follows from Lemma 6 using the function \( f = 1 \). □

4.3. Poincaré duality. For simplicity, assume \( \chi(M, \partial - M) = 0 \). Consider the spaces of Euler and co-Euler structures on \( M \). The following generalizes Proposition 5 in [BH06a].

Theorem 1. There is a natural isomorphism of affine spaces
\[
P : \mathfrak{Eul}(M; C)^* \to \mathfrak{Eul}(M; C),
\]
which intertwines the flip map \( ν^* \) with \( ν \) and is affine over the Poincaré–Lefschetz duality
\[
PD : H^{n-1}(M, \partial M; \Theta^C_M) \to H_1(M; C).
\]
In other words, for every \( β \in H^{n-1}(M, \partial M; \Theta^C_M) \) and every co-Euler structure \( ε^* \in \mathfrak{Eul}(M; C) \) we have
\[
P(ε^* + β) = P(ε^*) + PD(β).
\]
Proof. Let \((g, \alpha)\) be a pair representing the co-Euler structure \(\epsilon^*\) and \(\psi(M, g)\) the relative Mathai–Quillen form from Definition 7. Choose a vector field \(X\) which is transverse to the zero section, inward pointing along \(\partial_+ M\) and outward pointing along \(\partial_- M\) and with set of isolated singularities \(X\) in the interior of \(M\). Since \(d\alpha = e(M, g)\) and Lemma 5, the relative form \(\psi(M, g) - \alpha\) is closed and therefore defines a relative cohomology class in \(H^{m-1}(M \setminus X, \partial M; \Theta_M)\). Now, we identify the relative cohomology class \([X^\ast \psi(M, g) - \alpha]\) to its dual Poincaré–Lefschetz class \([c]\in H_1(M, X; \mathbb{C})\) represented by a singular 1-chain \(c\), by the requirement

\[
\int_{(M \setminus X, \partial M)} (X^\ast \psi(M, g) - \alpha) \wedge \omega = \int_c \omega
\]

to hold for all closed 1-forms \(\omega \in \Omega^1(M; \mathbb{C})\) compactly supported on \(M \setminus X\). Moreover, it is possible to choose a singular 1-chain \(c\) which is an Euler chain, i.e. \(\partial c = e(X)\) with \(e(X)\) is the 0-chain from \((30)\). Indeed, in the case \(\chi(M, \partial M) = 0\), this follows by setting \(\omega = df\) for an arbitrary smooth function \(f\), developing the left hand side of the identity in \((54)\) with (Gauss–Bonnet Theorem in) Lemma 7 and using Stokes’ Theorem on the right hand side of the identity in \((54)\).

The assignment \(P : (g, \alpha) \mapsto (X, c)\) specified by the condition \((54)\) induces the map \((51)\). This follows from \((35)\), formula \((33)\), Lemma 5 and Proposition 2 and using the same strategy as that in the situation of closed manifolds, see \([BH06b, Lemma 2 and (19)]\). That is, \(P\) does depend on neither representative of Euler structure, co-Euler structure and cohomology classes in \(H^1(M; \mathbb{C})\) and one obtains the pairing

\[
\mathbb{T} : \text{Eul}^*(M; \mathbb{C}) \times \text{Eul}(M; \mathbb{C}) \to H_1(M; \mathbb{C}),
\]

with the property

\[
\mathbb{T}(\epsilon^* + \beta, \epsilon + \sigma) = \mathbb{T}(\epsilon^*, \epsilon) - \sigma + \text{PD}(\beta)
\]

for every \(\epsilon^* \in \text{Eul}^*(M; \mathbb{C})\), \(\epsilon \in \text{Eul}(M; \mathbb{C})\), \(\sigma \in H_1(M; \mathbb{C})\) and relative form \(\beta \in H^{m-1}(M, \partial M; \Theta_M)\). Using \((55)\) and that \(\text{Eul}^*(M; \mathbb{C})\) and \(\text{Eul}(M; \mathbb{C})\) are affine spaces over relative cohomology and homology groups respectively, one obtains that \(P\) is affine over the homomorphism \(\text{PD}\) expressing the Poincaré–Lefschetz duality in \((52)\), and hence formula \((53)\) holds. Since \(\text{PD}\) is an isomorphism, \(P\) is so. Finally because of the properties of the relative Mathai–Quillen form and definition of the involution \(\nu\), it is clear that \(P\) intertwines the flip maps \(\nu^*\) and \(\nu\) on the spaces of co-Euler and Euler structures respectively.

\(\square\)
5. Co-Euler structures and the complex-valued analytic torsion

In this section, we extend [BH07, Theorem 4.2] to the situation of a bordism \( M \). We refer the reader to [Ma13a] for details, since we use the definitions, notation and results therein.

Let \( E \) be a complex flat vector bundle over \( M \). Assume \( E \) is endowed with a fiber-wise non-degenerate symmetric bilinear form \( b \). Consider the bilinear Laplacian

\[
\Delta_{E,g,b} := d_E d_E^\sharp + d_E^\sharp d_E
\]

acting on smooth \( E \)-valued forms satisfying absolute boundary conditions on \( \partial_+ M \) and relative boundary conditions on \( \partial_- M \), see [Ma13a].

Consider \([\tau(0)]_{E,g,b}^{M}\) the bilinear form induced in the determinant line \( \det H^*(M,\partial_- M; E) \) by the restriction of \( b \) to 0-generalized eigenspace of \( \Delta_{E,g,b} \), with the use of a Hodge–de-Rham theorem and the Knudson–Munford isomorphism, see [KM76] and [Ma13a]. Then, the complex-valued Ray–Singer torsion is the bilinear form on \( \det H(M,\partial_- M; E) \) defined by

\[
[\tau_{RS}]_{E,g,b}^{M} := [\tau(0)]_{E,g,b}^{M} \cdot \prod_q \left( \det^\prime (\Delta_{E,g,b,q}) \right)^{(-1)^q},
\]

where \( \det^\prime (\Delta_{E,g,b,q}) \) is the \( \zeta \)-regularized determinant of \( \Delta_{E,g,b,q} \) defined as

\[
\det^\prime (\Delta_{E,q}) := \exp \left( - \left. \frac{\partial}{\partial s} \right|_{s=0} \text{Tr}(\Delta_{E,g,b,q}^c)^{-s} \right)
\]

with

\[
\Delta_{E,g,b,q}^c := \Delta_{E,g,b,q}\big|_{\Omega_q^{\Delta_{E,g,b,q}(M;E)(0)^c|_B}}
\]

being the restriction of \( \Delta_{E,g,b} \) to the space of smooth differential forms of degree \( q \) which are not in 0-generalized eigenspace of \( \Delta_{E,g,b} \) but satisfy the boundary conditions above.

The generalized complex-valued Ray–Singer torsion on closed manifolds was constructed in [BH07, Theorem 4.2], by adding appropriate correction terms to the complex-valued torsion in order to cancel out the infinitesimal variation to the complex-valued analytic torsion. These correction terms were introduced using co-Euler structures, once the anomaly formulas for the torsion were computed. The procedure in the situation on a compact bordism is carried out in a similar fashion. In fact, the required correction terms are constructed by using this time co-Euler structures on compact bordisms, see Section 3 and the anomaly formulas in [Ma13a, Theorem 3].

Theorem 2. Let \( M \) be a bordism with Riemannian metric \( g \). Assume \( \chi(M,\partial_- M) = 0 \). Let \( e^* \in \text{Eul}^*(M;\mathbb{C}) \) a the co-Euler structure (without base point), see Section 3.7. Let \( E \) be a complex flat vector bundle over \( M \), with flat connection \( \nabla^E \). Assume \( E \) is endowed with a complex non-degenerate symmetric bilinear form \( b \). Then,

\[
[\tau]_{E,e^*,[b]}^{M} := [\tau_{RS}]_{E,g,b}^{M} \cdot e \left( 2 \int_{(M,\partial_- M)} \omega(E,b) - \text{rank}(E) \int_{\partial_+ M} B(\partial_+ M,\partial_- M,g) \right)
\]
where

- $[\tau^{RS}]_{E,g,b}$ is the torsion on $\mathbb{M}$ in (56),
- $(g, \alpha)$ is a representative of the co-Euler structure $e^*$,
- $B(\partial_+ M, \partial_- M, g)$ is the characteristic form from Definition 8,
- $[b]$ indicates the homotopy class of $b$,
- $\omega(E, b)$ is the Kamber–Tondeur form for $\nabla_E$ and $b$ in (19),

is well defined as bilinear form on $\det(H(M, \partial_\pm M))$.

Proof. We have to prove that $[\tau^{E, e^*}]$ is independent of the choice of representatives for the co-Euler structure and it depends on $\nabla_E$ and the homotopy class $[b]$ of $b$ only. For $(g_w, \alpha_w) \in U$ a real one-parameter smooth path of Riemannian metrics $g_w$ on $M$ relative forms $\alpha_w \in \Omega^{m-1}(M, \partial M; \Theta^C_M)$ representing the same co-Euler structure $e^* \in \mathfrak{E}ul^*(\mathbb{M}; \mathbb{C})$ and $\{b_w\}$ a real one-parameter smooth path of non-degenerate symmetric bilinear forms on $E$, consider the family $[\tau^{E,g,b_w}]_{E,g,b}$ of bilinear forms.

We claim that $[\tau^{E,g,b_w}]_{E,g,b}$ is independent of the parameter $w$, or equivalently its corresponding logarithmic derivative vanishes. To prove the claim, fix $u \in U$, consider the complex number $[\tau^{E,g,b_u}]_{E,g,b}$ and remark that its logarithm derivative with respect to $w$, is the sum of two contributions:

(a) the logarithmic derivative w.r.t. $w$ of the exponential depending on the co-Euler structures:

$$\exp \left( 2 \int_{(M, \partial M)} \alpha_w \wedge \omega(E, b_w) - \text{rank}(E) \int_{\partial M} B(\partial_+ M, \partial_- M, g_w) \right)$$

(b) the logarithmic derivative w.r.t. $w$ of $[\tau^{E,g,b_w}]/[\tau^{E,g,b_u}]$, which corresponds to the anomaly formulas for the complex-valued Ray–Singer torsion.

The logarithmic derivative in (b) has been computed [Ma13a, Theorem 2] in terms of the characteristic forms, as defined by Brüning and Ma,

$$B(\partial M, g_w), \ e(M, g_w), \ e_b(\partial M, g_w) \text{ and } e_b(\partial M, g_u, g_w). \quad (58)$$

The logarithmic derivative (a), computed in Proposition 1 expresses the variation of the representatives of the co-Euler structures, see Definition 3 with respect to smooth variations of $w$. The corresponding formulas in (20) and (21) from Proposition 1 are written (see Definitions 8 and 8 and Definition 11 (12)) in terms of the characteristic forms

$$B(\partial_+ M, \partial_- M, g_w) := B_\zeta(\partial M, g_w)$$

$$e(M, g_w), \ e_b(\partial M, g_w) \quad \text{and} \quad e_b(\partial M, g_u, g_w).$$

where $\zeta$ is the unit vector field at the boundary defined in (10). But the construction of the forms in (59) is compatible with the forms from Brüning.
and Ma in \((58)\). More precisely,

\[
B(\partial_+ M, \partial_- M, g_w)|_{\partial_+ M} = (\pm 1)^{m-1}B(\partial M, g_w)|_{\partial_+ M}
\]

\[
e_\beta(\partial_+ M, \partial_- M, g_w)|_{\partial_+ M} = (\pm 1)^m e_\beta(\partial M, g_w)|_{\partial_+ M}
\]

\[
\tilde{e}_\beta(\partial_+ M, \partial_- M, g_u, g_w)|_{\partial_+ M} = (\pm 1)^m \tilde{e}_\beta(\partial M, g_u, g_w)|_{\partial_+ M}
\]

see also Lemma \([4]\). Therefore, with \((60)\), the contribution from (a) and (b) are the same up to \(-1\) factor. The proof is complete.

5.1. **Without conditions on** \(\chi(M, \partial\pm M)\). Let \(M\) be a bordism and \(E\) a complex flat vector bundle over \(M\) with flat connection \(\nabla^E\). We assume it is endowed with a complex non-degenerate symmetric bilinear form \(b\) and \(\omega(E, b)\) the corresponding closed 1-form of Kamber—Tondeur, see \((19)\). For \(x_0 \in \text{int}(M)\), let \(e^* \in \text{Eu}^r_0(M; \mathbb{C})\) be a co-Euler structures based at \(x_0\), see Definition \([4]\) represented by \((g, \alpha)\), where \(\alpha := (\alpha, \alpha_0)\) is a relative form with \(\alpha \in \Omega^{m-1}(M; \Theta^n_{M})\) and \(M := M \setminus \{x_0\}\).

Let \(b_{(\det E_{x_0})^{-\chi(M, \partial_- M)}}\) be the induced bilinear form on \((\det E_{x_0})^{-\chi(M, \partial_- M)}\).

Consider \(\tau^{\text{RS}}_{E, g, b}\) the function regularizing \(\int_{(M, \partial M)}\) studied in Proposition \([2]\).

**Theorem 3.** The formula

\[
\tau^{\text{an}}_{E, e^*_0, [b]} := \tau^{\text{RS}}_{E, g, b} \cdot e^{2S(\alpha, \omega(E, b)) - \text{rank}(E)} \int_{\partial M} B(\partial_+ M, \partial_- M, g) \otimes b_{(\det E_{x_0})^{-\chi(M, \partial_- M)}},
\]

defines a bilinear form on \((\det H(M, \partial_- M)) \otimes (\det E_{x_0})^{-\chi(M, \partial_- M)}\), which is independent of the choice of representative for the co-Euler structure and depends on the connection and the homotopy class \([b]\) of \(b\) only.

**Proof.** On the one hand, if \(b\) is fixed and we only look at changes of the metric, then the variation of \(\tau^{\text{an}}_{E, (g, \omega, \theta), [b]}\) with respect to the metric compensates the variation of the function \(S(\alpha, \omega(E, b))\), which is explicitly given by formula \((29)\) in Proposition \([2]\). On the other hand, when \(g\) and \(e^*_0\) are kept constant and we allow \(b\) to smoothly change from \(b_1\) to \(b_2\), then the variation of the Kamber—Tondeur form is given by

\[
\omega(E, b_2) - \omega(E, b_2) = -\frac{1}{2} \det((b^{-1}_1 b_2)^{-1}) d \det(b_1^{-1} b_2) = -\frac{1}{2} d \log \det((b^{-1}_1 b_2)^{-1}),
\]

where the last equality holds, since \(b_2\) and \(b_1\) are homotopic and therefore the function

\[
\det((b^{-1}_1 b_2)^{-1}) : M \to \mathbb{C} \setminus \{0\},
\]

is homotopic to the constant function 1, which in turn allows to find a function

\[
\log \det((b^{-1}_1 b_2)^{-1}) : M \to \mathbb{C},
\]

with

\[
d \log \det((b^{-1}_1 b_2)^{-1}) = \det((b^{-1}_1 b_2)^{-1}) d \det(b_1^{-1} b_2).
\]
This, with \( f = \text{Tr}(b_1^{-1}b_2)^{-1} \) and Lemma 34 implies that
\[
2S_j(\omega(E,b_2)) - 2S_j(\omega(E,b_1)) = 2S_j(\alpha d\log \det ((b_1^{-1}b_2)^{-1}))
\]
\[
= -(-1)^m \int_{(M,\partial M)} \omega(M,g) \log \det ((b_1^{-1}b_2)^{-1}) + \log \det ((b_1^{-1}b_2)^{-1})(x_0)\chi(M,\partial_- M)
\]
\[
= -(-1)^m \int_{(M,\partial M)} \omega(M,g) \text{Tr}((b_1^{-1}b_2)^{-1}) + \text{Tr}((b_1^{-1}b_2)^{-1})(x_0)\chi(M,\partial_- M),
\]
where the additional term \( \text{Tr}((b_1^{-1}b_2)^{-1})(x_0)\chi(M,\partial_- M) \) cancels the variation of the induced bilinear form on \( (\det E_{x_0})^{-\chi(M,\partial_- M)} \) given by
\[
(b_1(\det E_{x_0})^{-\chi(M,\partial_- M)})^{-1}b_2(\det E_{x_0})^{-\chi(M,\partial_- M)} = \det ((b_1^{-1}b_2)^{-1})^{-\chi(M,\partial_- M)}.
\]

5.2. Complex-valued analytic torsion and Poincaré duality. Let us consider the bordism \( \mathcal{M}' \) in (1) dual to \( \mathcal{M} \), \( E' \) the dual complex vector bundle of \( E \) endowed with the corresponding dual connection and \( b' \) the non-degenerate symmetric bilinear form dual to \( b \) on \( E \). By Poincaré–Lefschetz duality, we have
\[
H^p(M,\partial_+ M; E' \otimes \Theta_M) \cong H^{m-p}(M,\partial_- M; E)^!\]
and hence there is a canonical isomorphism of determinant line bundles
\[
\det(H(M,\partial_+ M; E' \otimes \Theta_M)) \cong (\det(H(M,\partial_- M; E)))^{-1}^{m+1},\]
see for instance [KM76], [Mi62] and [Mi66]. The bilinear Laplacians \( \Delta_{E,g,b,q} \) and \( \Delta'_{E' \otimes \Theta_M,b',b',m,q} \) as well as the corresponding boundary conditions are intertwined by the isomorphism \( s_g \otimes b : \Omega^q(M;E) \rightarrow \Omega^{m-q}(M;E' \otimes \Theta_M) \). This implies that their \( L^2 \)-realizations of \( \Delta_{E,g,b,q} \) and \( \Delta'_{E,g,b,m,q} \) are isospectral, and therefore
\[
\det'(\Delta_{E,g,b,q}) = \det'(\Delta'_{E' \otimes \Theta_M,b',m,q}).
\]

By definition of the torsion in (17), the isomorphism in (63), the identity in (62), the formula \( \Pi_q(\det'(\Delta_{E,g,b,q}))^{-1} = 1 \) see [Ma13], the relation between the forms \( B(\partial_+ M,\partial_- M,g) \) and \( B(\partial_- M,\partial_+ M,g) \) from Lemma 3 and
\[
\omega(E' \otimes \Theta_M,b') = -\omega(E,b),\]
see [BH07] Section 2.4, we obtain
\[
[\tau]_{\mathcal{M}'}^{E' \otimes \Theta_M,\nu^*(\epsilon^*)}[b'] = \left( [\tau]_{\mathcal{M}}^{E,\epsilon^*,[b]} \right)^{-1}^{m+1},
\]
where \( \nu^* : \text{Eu}^*(\mathcal{M};\mathbb{C}) \rightarrow \text{Eu}^*(\mathcal{M}';\mathbb{C}) \) is the map in (15), intertwining the corresponding co-Euler structures. The formula in (65) exhibits the behavior.
of generalized complex-valued torsion on the bordism $\mathbb{M}$ under Poincaré–Lefschetz duality, generalizing this situation in the case without boundary, see [BH07] (31).

6. Appendix

In this section, for the reader’s convenience, we stay close to the notation in [BM06] (see also [BZ92, Chapter 3]).

6.1. The Berezin integral and Pfaffian. For $A$ and $B$ two unital $\mathbb{Z}_2$-graded algebras, with respective unities $1_A$ and $1_B$, we consider their $\mathbb{Z}_2$-graded tensor product denoted by $A \widehat{\otimes} B$. The map $w \mapsto w \widehat{\otimes} 1_B$ provides a canonical isomorphism between $A$ and the subalgebra $A \widehat{\otimes} 1_B \subset A \widehat{\otimes} B$, whereas with the map $w \mapsto \hat{\omega} := 1_A \hat{\otimes} w$ we canonically identify $B$ with the subalgebra $\hat{B} := 1_A \hat{\otimes} B \subset A \widehat{\otimes} B$. As $\mathbb{Z}_2$-graded algebras, one has $A \widehat{\otimes} \hat{B} \cong A \widehat{\otimes} B$.

Let $W$ and $V$ be finite dimensional vector spaces of dimension $n$ and $l$ respectively, with $W'$ and $V'$ their corresponding dual spaces. We denote by $\Theta_W$ the orientation line of $W$. Assume $W$ is endowed with a Hermitian product $\langle \cdot, \cdot \rangle$, fix $\{w_i\}_{i=1}^n$ an orthonormal basis of $W$ and use the metric to fix $\{w^i\}_{i=1}^n$ the corresponding dual basis in $W'$. Then, each antisymmetric endomorphism $K$ of $W$ can be uniquely identified with the section $K$ of $\hat{\Lambda}(W')$ given by

$$K := \frac{1}{2} \sum_{1 \leq i,j \leq n} \langle w_i, Kw_j \rangle \hat{w}^i \wedge \hat{w}^j.$$ 

The Berezin integral

$$\int^B : \Lambda V' \otimes \Lambda(W') \to \Lambda V' \otimes \Theta_W$$

is the linear map given by $\alpha \hat{\otimes} \beta \mapsto C_B \beta_{g,b}(w_1, \ldots, w_n)$, with constant $C_B := (-1)^{n(n+1)/2} \pi^{-n/2}$. Then, $\text{Pf}(K/2\pi)$, the Pfaffian of $K/2\pi$, is defined by

$$\text{Pf}(K/2\pi) := \int^B \exp(K/2\pi).$$

Remark that $\text{Pf}(K/2\pi) = 0$, if $n$ is odd. By standard fiber-wise considerations the map $\text{Pf}$ is extended for vector bundles over $M$.

6.2. Certain characteristic forms on the boundary. Let $M$ be a $m$-dimensional compact Riemannian manifold with boundary $\partial M$ and denote by $i : \partial M \hookrightarrow M$ the canonical embedding. We denote by $g := g^{TM}$ (resp. $g^\partial := g^{T\partial M}$) the Riemannian metric on $TM$ (resp. on $T\partial M$ and induced by $g$), by $\nabla$ (resp. $\nabla^\partial$) the corresponding Levi-Civita connection and by $R^{TM}$ (resp. $R^{T\partial M}$) its curvature. Let $\{e_i\}_{i=1}^m$ be an orthonormal frame of $TM$ with the property that near the boundary, $e_m = \varsigma_{in}$, i.e., the inward pointing unit normal vector field on the boundary. The corresponding induced orthonormal local frame on $T\partial M$ will be denoted by $\{e_\alpha\}_{\alpha=1}^{m-1}$. As usual, the
metric is used to fix \( \{ e^i \}_{i=1}^m \) (resp. \( \{ e^\alpha \}_{\alpha=1}^{m-1} \)) the corresponding dual frame of \( T^*M \) (resp. \( T^*\partial M \)).

With the notation in Appendix B, a smooth section \( w \) of \( \Lambda T^*M \) is identified with the section \( w \otimes 1 \) of \( \Lambda T^*M \otimes \Lambda T^*M \), whereas \( \tilde{w} \) denotes the corresponding section \( 1 \otimes w \) of \( \Lambda T^*M \otimes \Lambda T^*M \).

Here, the Berezin integrals \( \int_{\Lambda M} : \Lambda T^*M \otimes \Lambda T^*M \to \Lambda T^*M \otimes \Theta_M \) and \( \int_{\partial M} : \Lambda T^*\partial M \otimes \Lambda (T^*\partial M) \to \Lambda T^*\partial M \otimes \Theta_{\partial M} \) can be compared under the given convention for the induced orientation bundle on the boundary, see Section 2.

The curvature \( R^{TM} \) associated to \( \nabla \), considered as a smooth section of \( \Lambda^2(T^*M) \otimes \Lambda^2(T^*M) \to M \), can be expanded in terms of the frame above as

\[
R^{TM} := \frac{1}{2} \sum_{1 \leq k, l \leq m} g^{TM}(e_k, R^{TM} e_l) e^\wedge e^l \in \Gamma(M; \Lambda^2(T^*M) \otimes \Lambda^2(T^*M))
\]

In the same way, consider the forms

\[
i^{TM} := \frac{1}{2} \sum_{1 \leq k, l \leq m} g^{TM}(e_k, i^{TM} e_l) e^\wedge e^l \in \Gamma(\partial M; \Lambda^2(T^*\partial M) \otimes \Lambda^2(T^*\partial M))
\]

\[
R^{TM}|_{\partial M} := \frac{1}{2} \sum_{1 \leq \alpha, \beta \leq m-1} g^{TM}(e_\alpha, R^{TM} e_\beta) e^\wedge e^\beta \in \Gamma(\partial M; \Lambda^2(T^*\partial M) \otimes \Lambda^2(T^*\partial M))
\]

\[
R^{TM} e_\alpha := \frac{1}{2} \sum_{k,l=1}^{m-1} \sum_{k,l=1}^{m-1} g^{TM}(e_\alpha, e^k \wedge e^l) e^\wedge e^\beta \in \Gamma(\partial M; \Lambda^2(T^*\partial M) \otimes \Lambda^2(T^*\partial M))
\]

\[
\mathbf{S}_c := \frac{1}{n-1} \sum_{\beta=1}^{n-1} g^{TM}((i^* \nabla^{TM}) e_\beta) e^\beta \in \Gamma(\partial M; \Lambda^2(T^*\partial M))
\]

\[
\mathbf{S} := \frac{1}{n-1} \sum_{\beta=1}^{n-1} g^{TM}((i^* \nabla^{TM}) S_m e_\beta) e^\beta \in \Gamma(\partial M; \Lambda^2(T^*\partial M))
\]

to define

\[
e(M, \nabla^{TM}) := \int_{\partial M} \exp \left(-\frac{1}{2} R^{TM} \right),
\]

\[
e(\partial M, \nabla^{\partial M}) := \int_{\partial M} \exp \left(-\frac{1}{2} R^{\partial M} \right),
\]

\[
e_{\partial M} (\partial M, \nabla^{TM}) := (-1)^{m-1} \int_{\partial M} \exp \left(-\frac{1}{2} R^{TM} |_{\partial M} \right) \sum_{k=0}^{\infty} \frac{S_k^{m-1}}{2(\frac{m}{2}+1)^k},
\]

\[
B_\alpha (\partial M, \nabla^{TM}) := -\int \frac{du}{u} \int_{\partial M} \exp \left(-\frac{1}{2} R^{TM} |_{\partial M} - u^2 S_m^2 \right) \sum_{k=1}^{\infty} \frac{(uS_m)^k}{2(\frac{m}{2}+1)^k},
\]

\[
B(\partial M, \nabla^{TM}) := -\int \frac{du}{u} \int_{\partial M} \exp \left(-\frac{1}{2} R^{TM} |_{\partial M} - u^2 S_m^2 \right) \sum_{k=1}^{\infty} \frac{(uS_m)^k}{2(\frac{m}{2}+1)^k}.
\]

Lemma 8.

\[
e_{\partial M} (\partial M, \nabla^{TM}) = (-1)^{m-1} e_{\partial M} (\partial M, \nabla^{TM})
\]

\[
B_\alpha (\partial M, \nabla^{TM}) = (-1)^{m-1} B_\alpha (\partial M, \nabla^{TM}).
\]
Proof. First, note that $S_\chi = -S_{-\chi}$. We compute $e_{b,\chi}(\partial M, \nabla^{TM})$ by recalling that Berezin integrals see top degrees terms only:

$$e_{b,\chi}(\partial M, \nabla^{TM}) = (-1)^{m-1} \int_{BM} \exp \left( \frac{1}{2} (\frac{R^{TM}}{\partial M}) \right) \sum_{k=0}^\infty \frac{S_k^{b,\chi}}{2(\frac{m}{2}+1)^k},$$

$$= (-1)^{m-1} \int_{BM} \sum_{l=0}^\infty \frac{(-1)^l \Pi^{TM}_{l} \partial M}{l!} \sum_{k=0}^\infty \frac{(-1)^k S_k^{b,\chi}}{2(\frac{m}{2}+1)^k},$$

$$= (-1)^{m-1} \int_{BM} \sum_{l,k=0}^\infty \frac{(-1)^l \Pi^{TM}_{l} \partial M}{l!} \frac{(-1)^k S_k^{b,\chi}}{2(\frac{m}{2}+1)^k},$$

$$= (-1)^{m-1} \int_{BM} \sum_{l=0}^\infty \frac{(-1)^l \Pi^{TM}_{l} \partial M}{l!} \frac{(-1)^m s^{n-(2l+1)}}{2(\frac{m}{2}+1)^k},$$

$$= (-1)^{m-1} \int_{BM} \sum_{l,k=0}^\infty \frac{(-1)^l \Pi^{TM}_{l} \partial M}{l!} \frac{(-1)^k s^{n-(2l+1)}}{2(\frac{m}{2}+1)^k},$$

and analogously for the forms $B_{\pm\chi}(\partial M, \nabla^{TM})$. \hfill \Box

**Definition 8.** Define the functions $\Pi_{\pm} : \partial M \to \mathbb{R}$ respectively by

$$\Pi_{+}(y) := \begin{cases} 1 & \text{if } y \in \partial_+ M \\ 0 & \text{if } y \in \partial_- M \end{cases} \quad \text{and} \quad \Pi_{-}(y) := \begin{cases} 0 & \text{if } y \in \partial_+ M \\ 1 & \text{if } y \in \partial_- M. \end{cases}$$

and set

$$e_{b}(\partial_{+}M, \partial_{-}M, \nabla^{TM}) := i_{*}^{+} (e_{b,\chi}(\partial M, \nabla^{TM})) \Pi_{+} - i_{*}^{-} (e_{b,\chi}(\partial M, \nabla^{TM})) \Pi_{-},$$

$$e_{b}(\partial_{-}M, \partial_{+}M, \nabla^{TM}) := i_{*}^{+} (e_{b,\chi}(\partial M, \nabla^{TM})) \Pi_{+} - i_{*}^{-} (e_{b,\chi}(\partial M, \nabla^{TM})) \Pi_{-},$$

$$B(\partial_{+}M, \partial_{-}M, \nabla^{TM}) := B_{\chi}(\partial M, \nabla^{TM})$$

$$B(\partial_{-}M, \partial_{+}M, \nabla^{TM}) := B_{-\chi}(\partial M, \nabla^{TM}).$$

**Lemma 9.** For the forms given in Definition 8, the relations

$$e_{b}(\partial_{+}M, \partial_{-}M, \nabla^{TM}) = (-1)^{m} e_{b}(\partial_{-}M, \partial_{+}M, \nabla^{TM})$$

$$B(\partial_{+}M, \partial_{-}M, \nabla^{TM}) = (-1)^{m-1} B(\partial_{-}M, \partial_{+}M, \nabla^{TM})$$

hold.

Proof. This is clear from construction and Lemma 8. \hfill \Box

6.3. Secondary characteristic forms. Let $\{g_s := g_s^{TM}\}_{s \in \mathbb{R}}$ (resp. $\{g^{\partial} := g^{\partial TM}\}_{s \in \mathbb{R}}$) be a smooth family of Riemannian metrics on $TM$ (resp. the induced family of metrics on $T\partial M$). We sketch the construction in [BM06] (see also [BZ92] (4.53)) for the (secondary) Chern–Simons forms $e_{b}(M, g_0, g_s)$ and $e_{b}(\partial M, g_0, g_s)$.

Let $\nabla := \nabla_{g_s}$ and $R := R^{TM}_{g_s}$ (resp. $\nabla^{\partial} := \nabla^{\partial TM}_{g_s}$ and $R^{\partial} := R^{\partial TM}_{g_s}$) be the Levi-Civitā connections and curvatures on $TM$ (resp. on $T\partial M$) associated to the metrics $g_s$ (resp. $g^{\partial}_s$). Consider the deformation spaces
\[ \tilde{M} := M \times \mathbb{R} \text{ (resp. } \partial \tilde{M} := \partial M \times \mathbb{R} \text{)} \] with projection \( \pi_M: \tilde{M} \to \mathbb{R} \) and \( p_M: \tilde{M} \to M \), its canonical projections (resp. \( \pi_{\partial \tilde{M}}: \partial \tilde{M} \to \mathbb{R} \) and \( p_{\partial \tilde{M}}: \partial \tilde{M} \to \partial M \)). If \( \tilde{i} := i \times \text{id}_{\mathbb{R}}: \tilde{M} \to \tilde{M} \) is the natural embedding induced by \( i: \partial M \to M \), then \( \pi_{\partial \tilde{M}} = \pi_M \circ \tilde{i} \). The vertical bundle of the fibration \( \tilde{M} \to \mathbb{R} \) (resp. \( \pi_{\partial \tilde{M}}: \partial \tilde{M} \to \mathbb{R} \)) is the pull-back of the tangent bundle \( TM \to M \) along \( p_M: \tilde{M} \to M \) (resp. the pull-back of \( T\partial M \to \partial M \) along \( p_{\partial \tilde{M}}: \partial \tilde{M} \to \partial M \)), i.e.,

\[ (69) \quad \mathcal{T}M := p_M^* TM \to \tilde{M}, \quad (\text{resp. } \mathcal{T}\partial M := p_{\partial \tilde{M}}^* T\partial M \to \partial \tilde{M}) \]

and it is considered as a subbundle of \( T\tilde{M} \) (resp. \( T\partial \tilde{M} \)). The bundle \( \mathcal{T}M \) (resp. \( \mathcal{T}\partial M \)) in (69) is naturally equipped with a Riemannian metric \( g_{\mathcal{T}M} \) which coincides with \( g_s \) (resp. \( g_{\mathcal{T}\partial M}^s \)) at \( M \times \{s\} \) (resp. \( \partial M \times \{s\} \)), for which there exists a unique natural metric connection \( \nabla^{\mathcal{T}M} \) (resp. \( \nabla^{\mathcal{T}\partial M} \)) whose curvature tensor is denoted by \( R^{\mathcal{T}M} \) (resp. \( R^{\mathcal{T}\partial M} \)); for more details, see [BM06, Section 1.5, (1.44) and Definition 1.1], and also [BZ92, (4.50) and (4.51)]. Near the boundary, consider orthonormal frames of \( \mathcal{T}M \) such that \( e_m(y, s) = \zeta \) for each \( y \in \partial M \) with respect to the metric \( g_s \). Finally, by using the formalism described above associated to \( R^{\mathcal{T}M} \) and \( R^{\mathcal{T}\partial M} \) to define \( e \), if \( \text{inc}_s: M \to \tilde{M} \) is the inclusion map given by \( \text{inc}_s(x) = (x, s) \) for \( x \in M \) and \( s \in \mathbb{R} \), then one defines

**Definition 9.**

\[
\begin{align*}
\tilde{e}(M, g_0, g_\tau) &:= \int_0^\tau \text{inc}_s^* \left( \mu \left( \frac{\partial}{\partial s} \right) e(M, \nabla^{\mathcal{T}M}) \right) ds \\
&\in \Omega^{m-1}(M, \Theta_M) \\
\tilde{e}_\theta(\partial_+ M, \partial_- M, g_0, g_\tau) &:= \int_0^\tau \text{inc}_s^* \left( \mu \left( \frac{\partial}{\partial s} \right) e_{\theta}(\partial_+ \tilde{M}, \partial_- \tilde{M}, \nabla^{\mathcal{T}M}) \right) ds \\
&\in \Omega^{m-2}(\partial M, \Theta_M) \\
\tilde{e}_b(\partial M, g_0, g_\tau) &:= \int_0^\tau \text{inc}_s^* \left( \mu \left( \frac{\partial}{\partial s} \right) e_{b}(\partial \tilde{M}, \nabla^{\mathcal{T}M}) \right) ds \\
&\in \Omega^{m-2}(\partial M, \Theta_M)
\end{align*}
\]

where \( \mu(X) \) indicates the contraction with respect to the vector field \( X \).

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