Some Results on Circuit Lower Bounds and Derandomization of Arthur-Merlin Problems

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Abstract

We prove a downward separation for $\Sigma_2$-time classes. Specifically, we prove that if $\Sigma_2 E$ does not have polynomial size non-deterministic circuits, then $\Sigma_2 \text{SubEXP}$ does not have fixed polynomial size non-deterministic circuits. To achieve this result, we use Santhanam’s technique [16] on augmented Arthur-Merlin protocols defined by Aydinioglu and van Melkebeek [1]. We show that augmented Arthur-Merlin protocols with one bit of advice do not have fixed polynomial size non-deterministic circuits. We also prove a weak unconditional derandomization of a certain type of promise Arthur-Merlin protocols. Using Williams’ easy hitting set technique [20], we show that $\Sigma_2$-promise $\text{AM}$ problems can be decided in $\Sigma_2 \text{SubEXP}$ with $n^c$ advice, for some fixed constant $c$.

1 Introduction

The power of non-uniform (i.e., circuit) models of computation is a central topic in theoretical computer science. In addition to being intrinsically interesting, proving circuit lower bounds for uniform classes has many important consequences. Indeed, proving that $\text{NP}$ does not have polynomial size Boolean circuits would imply that $\text{P} \neq \text{NP}$.

Circuit lower bounds also have strong connections with the derandomization of probabilistic complexity classes. The so called “hardness vs. randomness” paradigm is based on the idea that if a language has high circuit complexity, we can use the language to derandomize probabilistic classes using pseudorandom generators. Babai, et al. [2] used this idea for “low-end” derandomization of $\text{BPP}$. They showed that if $\text{E}$ does not have polynomial size circuits, then $\text{BPP}$ can be derandomized in subexponential time infinitely often. Subsequently, Impagliazzo and Wigderson [9] gave a “high-end” derandomization of $\text{BPP}$. They proved that, if $\text{E}$ does not have $2^{en}$ size

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Boolean circuits, then $\text{BPP} = P$. We now know that, in certain settings, circuit lower bounds and derandomization are equivalent. Impagliazzo et al, showed that any non-trivial derandomization of the class $\text{MA}$ implies that $\text{NEXP}$ does not have polynomial size deterministic circuits [8]. Kabanets and Impagliazzo subsequently proved that derandomizing the well known Polynomial Identity Testing problem is equivalent to circuit lower bounds [11].

Aydinlioglu and van Melkebeek [1] have recently introduced augmented Arthur-Merlin protocols to extend the equivalence of circuit lower bounds and derandomization to non-deterministic circuits. Using augmented $\text{AM}$, Aydinlioglu and van Melkebeek showed that derandomizing promise $\text{AM}$ in $\Sigma_2\text{SubEXP}$ with $n^c$ bits of advice is equivalent to polynomial size non-deterministic circuit lower bounds for $\Sigma_2\text{E}$.

In this paper, we investigate non-deterministic circuit lower bounds of uniform classes. We prove that non-deterministic circuit lower bounds translate downward for $\Sigma_2\text{-time}$ classes. Specifically, we show that if $\Sigma_2\text{E}$ does not have polynomial size non-deterministic circuits, then $\Sigma_2\text{SubEXP}$ does not have fixed polynomial size non-deterministic circuits. To prove this result, we give fixed polynomial size non-deterministic lower bounds for augmented Arthur-Merlin protocols, which may be of independent interest. To achieve this, we use a technique developed by Santhanam [16] to prove analogous results for $\text{MA}$.

While circuit lower bounds are notoriously hard to prove, there has been important progress in this direction. Kannan proved that $\Sigma_2\text{P} \cap \Pi_2\text{P}$ does not have fixed polynomial size deterministic circuits [12]. Subsequently, Kobler and Watanabe improved this lower bound holds for the weaker class $\text{ZPP}^\text{NP}$ [14]. Cai was able to strengthen this further by showing that $\Sigma_2\text{P}$ does not have fixed polynomial size deterministic circuits [1]. Vinodchandran proved fixed $n^k$ circuit lower bounds for the class $\text{PP}$. Santhanam, using tools from interactive proof protocols [15], [18] and program checking [3], proved that $\text{MA}$ with one bit of advice does not have fixed polynomial size deterministic circuits. There have been fewer unconditional lower bounds for non-deterministic circuits. The smallest class known to have fixed polynomial size non-deterministic circuits is $\Sigma_2\text{P}^\text{NP}$, which follows by relativizing Cai’s result [1]. In this paper, we show that Santhanam’s technique can be applied to the augmented Arthur-Merlin protocols of [1]. This improves the smallest class known to have fixed size non-deterministic circuits.

One of the principal interests in proving non-deterministic circuit lower bounds is the derandomization of $\text{AM}$. The work of Klivans and van Melkebeek [13]; Shaltiel and Umans [17] shows that derandomization of $\text{prAM}$ follows from non-deterministic circuit lower bounds. Recently, progress has been made on achieving non-trivial derandomization of $\text{AM}$ in $\Sigma_2\text{-time}$ classes. Kabanets [10] using his “easy witness” technique, and Gutfreund et al [7], gave unconditional derandomization of $\text{AM}$ in pseudo-$\Sigma_2\text{SubEXP}$.\"
Williams’, using his “easy hitting set” technique, recently showed that $\text{AM}$ is contained in $\Sigma_2 \text{SubEXP}$ with fixed $n^c$ advice [20]. In this paper, we investigate derandomization of promise $\text{AM}$. We use Williams easy hitting set technique to show that certain promise $\text{AM}$ protocols can be unconditionally derandomized in $\Sigma_2 \text{SubEXP}$, with fixed $n^c$ bits of advice.

2 Preliminaries

We will assume familiarity with the complexity classes $\text{NP}$, $\Sigma_2 \text{P}$, $\text{PSPACE}$ as well as their exponential- and subexponential-time counterparts. For a language $L$ and integer $n$, we denote the restriction of $L$ to $n$ by $L_n$, consisting of all strings $x \in \{0,1\}^n \cap L$. We denote the complement of a language $L$ by $\overline{L}$. For a language $L$ and a complexity class $C$, we say that $L$ is infinitely often in $C$, denoted $L \in \text{i.o-}C$, if there is a language $A \in C$ such that for infinitely many $n \in \mathbb{N}$, $L_n = A_n$.

2.1 Non-deterministic circuits

A non-deterministic Boolean circuit $C$ is a Boolean circuit which receives two inputs, $x$ of length $n$ and a second input $y$. We say that $C$ accepts input $x$ if there is a string $y$ such that $C(x,y) = 1$. Otherwise, we say that $C$ rejects $x$. The size of a non-deterministic circuit is the number of its connections. For a constant $k \in \mathbb{N}$, the class $\text{NSIZE}(n^k)$ consists of all languages $L$ for which there is a family of non-deterministic circuits $\{C_n\}_{n \in \mathbb{N}}$ such that $C_n$ decides $L_n$ and $\text{size}(C_n) = n^k$. The class $\text{NSIZE}(\text{poly})$ is the union of $\text{NSIZE}(n^k)$ over all constants $k \in \mathbb{N}$.

A partial single-valued nondeterministic (PSV) circuit is a Boolean circuit $C$ which receives two inputs, $x$ of length $n$ and a second input $y$, and has two output gates, value and flag, so that the following holds for every $x \in \{0,1\}^n$.

1. For every $y_1, y_2$, if $C(x,y_1)$ and $C(x,y_2)$ have a 1 at their flag gate, then $C(x,y_1) = C(x,y_2)$.

Circuit $C$ is a total single-valued (TSV) circuit computing the function $f : \{0,1\}^n \rightarrow \{0,1\}$ if the following hold.

1. $C$ is a PSV circuit.
2. For every $x$, there exists some $y$ for which $C(x,y)$ has 1 at its flag gate.

For a constant $k$, the class of $n^k$ size single-valued non-deterministic circuits, $\text{SVSIZE}(n^k)$, consists of all languages $L$ for which there is a family of TSV circuits $\{C_n\}_{n \in \mathbb{N}}$ such that $C_n$ decides $L_n$ and $\text{size}(C_n) = n^k$. The class $\text{SVSIZE}(\text{poly})$ is the union of $\text{SVSIZE}(n^k)$ over all constants $k \in \mathbb{N}$. Note that for any language $L$, if $L, \overline{L} \in \text{NSIZE}(\text{poly})$, then $L$ and $\overline{L}$ are in $\text{SVSIZE}(\text{poly})$. 

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2.2 Arthur-Merlin protocols

Promise problems were first introduced and studied by Even, Selman and Yacobi [5]. They have since been highly useful in complexity theory, and, in particular, probabilistic complexity classes. A promise problem $\Pi = (\Pi_Y, \Pi_N)$ is a pair of disjoint sets $\Pi_Y$ and $\Pi_N$. A language $L$ agrees with a promise problem $\Pi$ if

1. $x \in L$ for every $x \in \Pi_Y$, and
2. $x \notin L$ for every $x \in \Pi_N$.

The class of Promise Arthur-Merlin problems, $\text{prAM}$, is the set of all promise problems $\Pi$ such that there is a polynomial time relation $R(\cdot, \cdot, \cdot)$ such that

$x \in \Pi_Y \implies \Pr_z[(\exists y) R(x, y, z) = 1] \geq 2/3$

$x \in \Pi_N \implies \Pr_z[(\exists y) R(x, y, z) = 1] \leq 1/3$.

The class $\text{AM}$ consists of the problems in $\text{prAM}$ which are languages.

Augmented Arthur-Merlin protocols were introduced by Aydinlioğlu and van Melkebeek [1]. This definition is similar to $\text{AM}$ protocols, except that there are two verifiers, Arthur and a $\text{coNP}$ verifier $V$.

Definition (Augmented Arthur-Merlin protocols). The class of problems $\text{prAugAM}$ consists of all promise problems $\Pi$ for which there is a constant $c$, a promise problem $\Gamma \in \text{prAM}$ and a language $V \in \text{coNP}$ such that

$x \in \Pi_Y \implies (\exists y)((x, y) \in \Gamma_Y \land (x, y) \in V)$,

$x \in \Pi_N \implies (\forall y)((x, y) \in \Gamma_N \lor (x, y) \notin V)$,

where $x \in \{0, 1\}^n$ and $y \in \{0, 1\}^{nc}$. The class $\text{AugAM}$ consists of the problems in $\text{prAugAM}$ which are languages.

2.3 Pseudorandom Generators

The $\text{SAT}$-relativized hardness $H^{\text{SAT}}(G_{r,n})$ of a pseudorandom generator $G_{r,n} : \{0,1\}^r \rightarrow \{0,1\}^n$ is defined as the minimal $s$ such that there exists an $n$-input $\text{SAT}$ oracle Boolean circuit $C$ of size at most $s$ for which

$|\Pr_{x \in \{0,1\}^r}[C(G_{r,n}(x) = 1] - \Pr_{y \in \{0,1\}^n}[C(y) = 1]| \geq \frac{1}{s}$.

Klivans and van Melkebeek [13] showed that the pseudorandom generator constructions of [2] and [9] relativize. Specifically, they proved the following theorem.

\footnote{Aydinlioğlu and van Melkebeek originally denoted this class by $\text{prM(AM|coNP)}$. We made this change for considerations of length.}
Theorem 1. There is a polynomial-time computable function $F : \{0, 1\}^* \times \{0, 1\}^* \to \{0, 1\}^*$ with the following properties. For every $\epsilon > 0$, there exist $c, d \in \mathbb{N}$ such that

$$F : \{0, 1\}^{cn} \times \{0, 1\}^{d \log n} \to \{0, 1\}^n,$$

and if $r$ is the truth table of a $c \log n$ variable Boolean function of SAT-oracle circuit complexity at least $n^c$, then the function $G_r(s) = F(r, s)$ is a pseudorandom generator with hardness $H^{\text{SAT}}(G_r) > n$.

Klivans and van Melkebeek showed that the existence of pseudorandom generators which are hard for SAT-oracle circuits derandomize prAM \[13\].

3 Non-deterministic Circuit Lower Bounds

We now prove our downward separation for $\Sigma_2$-time classes. We first show that $(\text{coAugAM} \cap \text{AugAM})/1$ does not have fixed polynomial size non-deterministic circuits. We will need the following lemma of Santhanam \[16\], which builds on the ideas of Trevisan and Vadhan \[19\] and Fortnow and Santhanam \[6\].

Lemma 1. There is a PSPACE-complete language $L$ and probabilistic polynomial-time Turing machines $M$ and $M'$ such that for any input $x$ of length $n$ the following hold.

1. $M$ and $M'$ only query their oracle on strings of length $n$.

2. If $M$ (resp. $M'$) is given $L$ as its oracle and $x \in L$ (resp. $x \notin L$), then $M$ (resp. $M'$) accepts with probability 1.

3. If $x \notin L$ (resp. $x \in L$), then irrespective of the oracle, $M$ (resp. $M'$) rejects with probability at least $2/3$.

We will use the complete language of Lemma 1 to define promise Arthur-Merlin problems. Let $L, M$ and $M'$ be as in the definition of Lemma 1. For every PSV circuit $C$ and input $x$, let $\Pr[M^L(C) = 1]$ denote the probability over $M$’s random bits that $M$ accepts when given the language of $C$ as an oracle. We will also make the following assumption on the behavior of $M$ (and $M'$). If $C$ is undefined at some $x'$, and $M$ queries its oracle for $x'$, we will assume that the oracle returns a special symbol ‘?’ and $M$ will immediately halt and reject. Define the promise problem $\Gamma_M^* = (\Gamma_M^Y, \Gamma_M^N)$ by

$$\Gamma_M^Y = \{ \langle x, C \rangle \mid C \text{ is a PSV circuit s.t. } \Pr[M^L(C)(x) = 1] \geq 2/3 \}$$

$$\Gamma_M^N = \{ \langle x, C \rangle \mid C \text{ is a PSV circuit s.t. } \Pr[M^L(C)(x) = 0] \geq 2/3 \}.$$
In a similar manner, and with the same assumption on the behavior of \( M' \), define the promise problem \( \Gamma^{M'} = (\Gamma^{M'}_Y, \Gamma^{M'}_N) \) by

\[
\Gamma^{M'}_Y = \{ \langle x, C \rangle \mid C \text{ is a PSV circuit s.t. } \Pr[M'^{L(C)}(x)] = 1 \geq 2/3 \}
\]

\[
\Gamma^{M'}_N = \{ \langle x, C \rangle \mid C \text{ is a PSV circuit s.t. } \Pr[M'^{L(C)}(x)] = 0 \geq 2/3 \}
\]

**Lemma 2.** Let \( L, M \) and \( M' \) be as in the definition of Lemma 1. Let \( \Gamma^M \) and \( \Gamma^{M'} \) be the promise problems defined above. Then \( \Gamma^M \) and \( \Gamma^{M'} \) are in \( \text{prAM} \).

**Proof.** We give the Arthur-Merlin protocol for \( \Gamma^M \). The protocol for \( \Gamma^{M'} \) is identical. On input \( \langle x, C \rangle \), the Arthur-Merlin protocol works as follows. Arthur guesses a random string \( r \), and sends \( r \) to Merlin. Merlin responds with a sequence of witnesses \( w_1, \ldots, w_{p(n)} \). Arthur then simulates \( M \) with \( L(C) \) as its oracle by using the provided witnesses. That is, for every query \( q_j \), Arthur simulates \( C(q_j, w_j) \). If for any \( j \), \( C(q_j, w_j) \) does not have a 1 at its flag gate, Arthur immediately halts and rejects. Otherwise, Arthur uses the value of \( C(q_j, w_j) \) as the oracle response and continues.

From the definition of \( M \) and \( M' \), it is clear that \( \Gamma^M \) and \( \Gamma^{M'} \) are in \( \text{prAM} \).

The usefulness of the \( \text{coNP} \) verifier in the definition of augmented AM protocols is that it allows for us to simulate interactive proof protocols. In the deterministic circuit setting, we are able to prove that \( \text{PSPACE} \subseteq \text{P/poly} \) implies that \( \text{PSPACE} = \text{MA} \). This follows from the fact that Merlin can send Arthur a Boolean circuit claiming to compute the prover’s strategy, and Arthur simply simulates the interactive proof protocol using this circuit as the oracle. In the non-deterministic setting, however, this method breaks down. The essential difficulty is that Arthur cannot know if the non-deterministic circuit returns “no” on every path, or just the one Merlin gives. The inclusion of a \( \text{coNP} \) verifier allows the proof for deterministic circuits to extend to the non-deterministic setting. Using this strategy, Aydınlioğlu and van Melkebeek [1] proved the following Lemma.

**Theorem 2.** If \( \text{PSPACE} \subseteq \text{NP/poly} \), then \( \text{PSPACE} \subseteq \text{AugAM} \).

For the sake of clarity, we will break the proof of our main theorem into two parts. The first part uses Santhanam’s technique [10] to show that augmented Arthur-Merlin protocols with one bit of advice do not have fixed polynomial size \( \text{SV-circuits} \). We then modify this proof slightly to achieve the stronger statement, that class \( (\text{AugAM} \cap \text{coAugAM})/1 \) does not have fixed size non-deterministic circuits.

**Theorem 3.** For every \( k \in \mathbb{N} \) there is a language \( A \in \text{AugAM}/1 \) such that \( L \notin \text{SVSIZE}(n^k) \).
Proof. First assume that \( \text{PSPACE} \) has polynomial size SV circuits. Then, by Theorem 2, \( \text{PSPACE} \subseteq \text{AugAM} \), and the conclusion follows. So we may assume that \( \text{PSPACE} \not\subseteq \text{SVSIZE}(\text{poly}) \).

Let \( k \in \mathbb{N} \) and \( L \) be the \( \text{PSPACE} \)-complete language of Lemma \( \text{I} \). By our assumption, \( L \not\in \text{SVSIZE}(\text{poly}) \). For every \( n \in \mathbb{N} \), define the \( \text{Min}(L_n) \in \mathbb{N} \) to be the size of the smallest \( \text{SV} \)-circuit computing \( L_{-n} \). Define the language \( A \) by

\[
A = \{ x1^y \mid x \in L, \ 0 < |x| \leq y, \ y \text{ is a power of } 2 \text{ and } (y + |x|)^{k+1} \leq \text{Min}(L_n) < (2y + |x|)^{k+1} \}.
\]

We first show that \( A \in \text{AugAM}/1 \). Define the \( \text{AugAM} \) protocol with one bit of advice as follows. On input \( w \), if the advice is set to 0, Arthur halts and rejects. If the advice bit is set to 1, Arthur verifies that \( w = x1^y \), where \( 0 < |x| \leq y \) and \( y \) is a power of 2. If the input is not of this form, Arthur halts and rejects. Otherwise, if the input is of the correct form, Merlin sends a non-deterministic circuit \( C_L \) claiming to compute \( L_{-n} \) to both verifiers. The \( \text{coNP} \) verifier \( V \) checks that \( C_L \) is a PSV circuit. That is, \( V \) checks that for every string \( x' \) and every two witnesses \( w_1, w_2 \), \( C_L(x', w_1) = C_L(x', w_2) \) whenever the flag gates of both are set to 1. It is clear that this can be done in \( \text{coNP} \). For the Arthur-Merlin phase, we run the protocol \( \Gamma^M \) of Lemma 2.

We now show that this protocol correctly decides \( A \) given correct advice. First assume that \( w = x1^y \in A \), so \( x \in L \). Then there is a TSV-circuit \( C \) of size \( s \), where \( (y + |x|)^{k+1} \leq s < (2y + |x|)^{k+1} \). When Merlin gives both verifiers this circuit, the \( \text{coNP} \) verifier \( V \) will accept. Since \( C \) computes \( L_{-n} \), by the property of the probabilistic TM \( M \) of Lemma \( \text{II} \), \( M^L(C)(x) \) accepts with probability 1. Therefore \( \langle x, C \rangle \in \Gamma^M_N \), and the protocol accepts.

Assume that \( w = x1^y \notin A \). If \( y \) is not of the correct form, then, given the correct advice, the above protocol immediately rejects. If \( y \) is of the correct form, then \( x \notin L \). Let \( C \) be a circuit of size \( s \), where \( (y + |x|)^{k+1} \leq s < (2y + |x|)^{k+1} \). If \( C \) is not PSV, then the \( \text{coNP} \) verifier \( V \) will reject and the protocol is correct. Otherwise, \( C \) is a PSV circuit. By the property of the probabilistic TM \( M \) of Lemma \( \text{II} \), \( M^L(C)(x) \) must reject with probability at least 2/3. Hence \( \langle x, C \rangle \in \Gamma^M_N \). Since \( C \) was arbitrary, the protocol correctly decides \( A \).

We now prove that \( A \) does not have SV non-deterministic circuits of size \( n^k \). Assume otherwise, and let \( C_1, C_2, \ldots \) be a sequence of SV non-deterministic circuits such that \( C_m \) decides \( A_{-m} \) and \( C_m \) is of size \( m^k \). Let \( s(m) \) be the minimum circuit size of \( L_{-m} \). By our assumption, there is an infinite number of input lengths \( m \) such that \( s(m) > (m + 1)^{k+1} \). For any such \( m \), define the following circuit \( C'_m \) deciding \( L_{-m} \). First, the unique value \( y \) such that \( y \) is a power of 2 and \( (m + y)^{k+1} \leq s(m) < (m + 2y)^{k+1} \) is hardcoded into \( C'_m \). On input \( x \) of length \( m \), \( C'_m \) simulates \( C_{m+y}(x1^y) \). Since the size of \( C'_m \) is at most the size of \( C_{m+y} \), we have that the size of
\(C'_m\) is less than \(s(m)\). This contradicts our assumption, and the proof is complete.

We now modify the proof of Theorem 3 slightly to achieve the following theorem.

**Theorem 4.** For every \(k \in \mathbb{N}\) there is a language \(A \in (\text{coAugAM} \cap \text{AugAM})/1\) such that \(A \notin \text{NSIZE}(n^k)\).

**Proof.** The proof is similar to that of Theorem 3. If \(\text{PSPACE}\) has polynomial size SV circuits, then by Theorem 2, \(\text{PSPACE} = \text{coAugAM} \cap \text{AugAM}\), and the claim follows.

Assume that \(\text{PSPACE} \not\subseteq \text{SVSIZE}(\text{poly})\). Let \(k \in \mathbb{N}\), and \(L, \overline{L}\) be the \(\text{PSPACE}\) complete languages of Lemma 1. For every \(n \in \mathbb{N}\), define the \(\text{Min}(L_n) \in \mathbb{N}\) to be the size of the smallest non-deterministic circuit computing \(L_n = n\). Recall the definition of language \(A\), \(A = \{x1^y \mid x \in L, 0 < |x| \leq y, y\text{ is a power of 2 and } (y + |x|)^{k+1} \leq \text{Min}(L_n) < (2y + |x|)^{k+1}\}\).

We will show that \(\overline{A} \in \text{AugAM}/1\), where the single bit of advice is the same as the bit to compute \(A\). If the advice is set to 0, then Arthur accepts. If \(y\) is not of the correct form, then Arthur accepts. Otherwise, Merlin will send a non-deterministic circuit \(C\) to Arthur and the \(\text{coNP}\) verifier \(V\). If \(C\) is not a PSV circuit, then \(V\) rejects. Otherwise, Arthur and Merlin run the protocol for \(\Gamma_{M'}\) on \(\langle x, C \rangle\). Assume that \(x1^y \in \overline{A}\). If \(y\) is not of the correct form then the above protocol will accept given the correct advice. If \(y\) is of the correct form then \(x \in \overline{L}\). Therefore, by the property of \(\overline{L}\) and \(M'\), there is a TSV circuit \(C\) such that \(\langle x, C \rangle \in \Gamma_{M'}^N\) and the protocol accepts. Assume that \(x1^y \notin \overline{A}\), so \(x \notin \overline{L}\). Then for every circuit \(C\) Merlin gives to the verifiers, either \(C\) is not PSV, and \(V\) will reject, or \(\langle x, C \rangle \in \Gamma_{M'}^N\), and Arthur will reject. Hence \(\overline{A} \in \text{AugAM}/1\). Finally, we note that the protocols for \(A\) and \(\overline{A}\) are given the same bit of advice.

The proof that \(\overline{A}\) does not have single-valued circuits of size \(n^k\) is nearly identical to that of Theorem 3.

Therefore, for every \(k \in \mathbb{N}\), there is a language \(A \in (\text{coAugAM} \cap \text{AugAM})/1\) such that \(A \notin \text{SVSIZE}(n^k)\). We now extend this to non-deterministic circuits. Assume that for some \(c \in \mathbb{N}\),

\[(\text{coAugAM} \cap \text{AugAM})/1 \subseteq \text{NSIZE}(n^c)\]

It suffices to show that we can construct a TSV circuit of size \(O(n^c)\) computing any language in \((\text{coAugAM} \cap \text{AugAM})/1\) as follows. Let \(A \in (\text{coAugAM} \cap \text{AugAM})/1\), and let \(C_A\) and \(C_{\overline{A}}\) be size \(n^c\) non-deterministic circuits computing \(A\) and \(\overline{A}\), respectively. Define the circuit \(C\) which, given \(x \in \{0, 1\}^n\) and \(y \in \{0, 1\}^{n^c}\), simulates \(C_A(x, y)\) and \(C_{\overline{A}}(x, y)\). \(C\) accepts if \(C_A(x, y)\) accepts.
with its flag bit set, and rejects if $C(x, y)$ accepts with its flag bit set. Then $C$ is a TSV circuit of size $O(n^c)$ computing $A$ and $\overline{A}$, a contradiction.

Essentially the same proof as Theorem 4 shows that prAugAM does not have fixed polynomial size non-deterministic circuits.

**Theorem 5.** For every $k \in \mathbb{N}$ there is a language $A \in (pr - coAugAM \cap prAugAM)$ such that $A \notin NSIZE(n^k)$.

**Proof.** The proof is similar to that of Theorem 4 except that we eliminate the one bit of advice through the use of a promise. Recall the language

$$A = \{x1^y \mid x \in L, 0 < |x| \leq y, y \text{ is a power of } 2 \text{ and } (y + |x|)^{k+1} \leq \text{Min}(L_{n}) < (2y + |x|)^{k+1}\}.$$  

Our promise consists of all strings $x1^y$ such that $y$ is of the correct form. The remainder of the proof follows the proof of Theorem 4.

We are now able to prove our downward separation result for non-deterministic circuit size. Note that, with the infinitely often and almost everywhere reversed, the converse is true using standard arguments. That is, if $\Sigma_2\text{SubEXP}$ does not have fixed polynomial size circuits almost everywhere, then $\Sigma_2E$ does not have polynomial size non-deterministic circuits infinitely often. We will need the following theorem due to Aydinliöğlu and van Melkebeek [1].

**Theorem 6.** The following are equivalent.

1. $\text{prAM} \subseteq \Sigma_2\text{TIME}(2^n)/n^\epsilon$ for every constant $\epsilon > 0$.

2. $\Sigma_2E \subseteq \text{i.o.-NP/poly}$.

The following lemma is implicit in [1], which we prove for completeness.

**Lemma 3.** $\text{prAM} \subseteq \Sigma_2\text{TIME}(2^n)/n^\epsilon$ for every $\epsilon > 0$ if and only if $\text{prAugAM} \subseteq \Sigma_2\text{TIME}(2^n)/n^\epsilon$ for every $\epsilon > 0$.

**Proof.** The backward direction is immediate. Let $\Pi = (\Pi_Y, \Pi_N)$ be a promise problem in prAugAM. Let $\Gamma = (\Gamma_Y, \Gamma_N)$ be the corresponding prAM problem for $\Pi$, and $V$ be the coNP problem for $\Pi$. Let $c \in \mathbb{N}$ be the constant for $\Pi$, and $\epsilon > 0$. By our assumption, there is a $\Sigma_2\text{TIME}(2^{n^\epsilon/c})$ machine $M$ taking $n^{\epsilon/c}$ bits of advice which is consistent with the promise $\Gamma$. Define the $\Sigma_2\text{TIME}(2^{n^\epsilon})$ machine $N$ taking $n^\epsilon$ bits of advice as follows. On input $x \in \{0, 1\}^n$, guess a string $y \in \{0, 1\}^{n^\epsilon}$, and check if $\langle x, y \rangle \in V$ using the NP oracle. If it is not, reject. Otherwise, simulate $M$ on $\langle x, y \rangle$ with the given advice string. It is clear that $N$ is a $\Sigma_2\text{TIME}(2^{n^\epsilon})$ time machine taking $n^\epsilon$ bits of advice. Assume that $x \in \Pi_Y$. Then there is a $y \in \{0, 1\}^{n^\epsilon}$ such that $\langle x, y \rangle \in V$ and $\langle x, y \rangle \in \Gamma_Y$. Therefore, given the correct advice
string $\alpha \in \{0,1\}^n$, $N$ accepts. Assume that $x \in \Pi_N$. Let $y \in \{0,1\}^{n^c}$ be any string guessed by $N$. If $(x,y) \notin V$, then $N$ will reject. Otherwise, $(x,y) \in \Pi_N$. Therefore, given the correct advice string $\alpha \in \{0,1\}^n$, $N$ rejects. $\square$

We are now able to prove the downward separation results for non-deterministic circuit size. First, we have the following “low-end” separation.

**Theorem 7.** If $\Sigma^2 E \not\subseteq \text{i.o.-NP/poly}$, then for every $k \in \mathbb{N}$, there is a language $A \in \Sigma^2 \text{SubEXP}$ such that $A \not\in \text{NSIZE}(n^k)$.

**Proof.** Let $k \in \mathbb{N}$. If $\Sigma^2 \text{EXP} \not\subseteq \text{i.o.-NP/poly}$, then by Theorem 6 and Lemma 3, prAug$\text{AM} \subseteq \Sigma^2 \text{TIME}(2^n^\epsilon)/n^\epsilon$ for every $\epsilon > 0$. By Theorem 5, there is a language $A \in \text{prAugAM}$ such that $A \not\in \text{SVSIZE}(n^{2k})$. Let $\epsilon > 0$, and let $M$ be the $\Sigma^2 \text{TIME}(2^n)$ machine deciding $A$ given $n^\epsilon$ bits of advice. We can encode advice into the input as follows. Define the language $A' = \{x, \alpha | M \text{ accept } x \text{ given } \alpha \in \{0,1\}^n \text{ as advice}\}$.

It is clear that $A' \in \Sigma^2 \text{TIME}(2^n)$. For sufficiently large $n$, $(O(n + n^\epsilon))^k < n^{2k}$. We therefore have that $A' \not\in \text{NSIZE}(n^k)$. As $k$ and $\epsilon$ were chosen arbitrarily, we see that $\Sigma^2 \text{SubEXP} \not\subseteq \text{NSIZE}(n^k)$. $\square$

Theorem 4 also implies that derandomizing pr$\text{AM}$ in $\Sigma^2 \text{P}$ gives fixed polynomial size lower bounds for $\Sigma^2 \text{P}$.

**Corollary 1.** If pr$\text{AM} \subseteq \Sigma^2 \text{P}$, then $\Sigma^2 \text{P} \not\subseteq \text{NSIZE}(n^k)$ for any fixed $k \in \mathbb{N}$.

**Proof.** Assume that pr$\text{AM} \subseteq \Sigma^2 \text{P}$. Then prAug$\text{AM} \subseteq \Sigma^2 \text{P}$. By Theorem 5, prAug$\text{AM} \not\subseteq \text{NSIZE}(n^k)$ for any fixed $k$, and the conclusion follows. $\square$

### 4 Mild Derandomization of Promise AM

**Definition.** A $\Sigma^2$-promise problem is a promise problem $\Gamma = (\Gamma_Y, \Gamma_N)$ such that there is a language $L \in \Sigma^2 \text{P}$ which decides the promise $\Gamma_Y \cup \Gamma_N$. That is, for every length $n$ and all strings $x \in \{0,1\}^n$,

$$x \in \Gamma_Y \cup \Gamma_N \text{ if and only if } x \in L.$$  

The class $\Sigma^2$-pr$\text{AM}$ consists of all $\Sigma^2$-promise problems in pr$\text{AM}$.

A polynomial size hitting set for a $\Sigma^2$-pr$\text{AM}$ problem $\Gamma = (\Gamma_Y, \Gamma_N)$ is a polynomial size set $S$ of $n^k$-bit strings that will take the role of Arthur in the AM protocol. Formally, $S$ is a hitting set if, for every $x \in \Gamma_Y \cup \Gamma_N$,

$$x \in \Gamma_Y \implies (\forall y \in S)(\exists z) R(x,y,z) = 1$$  

$$x \in \Gamma_N \implies (\forall z)(\exists y \in S) R(x,y,z) = 0,$$
where $R$ is a deterministic polynomial time computable relation for $\Gamma$. Note that we do not worry about the instances which are not in the promise $\Gamma$.

We use Williams easy hitting set technique to give a nontrivial derandomization of $\Sigma^2_{\text{prAM}}$. This is an analog of Williams’ result for $\text{AM}$ [20]. We will consider hitting sets for $\Sigma^2_{\text{prAM}}$ which are computable by polynomial size circuits with oracle access to $\text{NP}$. There are two cases. Either there is a constant such that, for every problem $\Gamma$ in $\Sigma^2_{\text{prAM}}$, there is a $n^c$ size hitting set for $\Gamma$ and $\Sigma^2_{\text{prAM}}$ can be computed in $P^{\text{NP}}/O(n^c)$. Otherwise, for every $c$, there is a problem in $\Sigma^2_{\text{prAM}}$ which has no small hitting sets. We can use this fact to find a string of high complexity, and use a pseudorandom generator to derandomize $\Sigma^2_{\text{prAM}}$.

**Theorem 8.** At least one of the following holds.

1. There is a constant $c \in \mathbb{N}$ such that $\Sigma^2_{\text{prAM}} \subseteq P^{\text{NP}}/O(n^c)$.
2. $\text{prAM} \subseteq \text{i.o.-} \Sigma^2_{\text{TIME}}(2^{n^c})/n^c$ for every $\epsilon > 0$.

In particular, there is a constant $c$ such that

$$\Sigma^2_{\text{prAM}} \subseteq \text{i.o.-} \Sigma^2_{\text{SubEXP}}/n^c.$$

**Proof.** First assume that there exists a constant $c \in \mathbb{N}$ such that, for every $\Sigma^2_{\text{prAM}}$ promise problem $\Gamma$ there is a circuit with oracle access to $\text{SAT}$ of size $n^c$ computing a hitting set for $\Gamma$. Then $\Sigma^2_{\text{prAM}} \subseteq P^{\text{NP}}/O(n^c)$. This follows, since the advice is simply the oracle circuit computing the hitting set.

Otherwise, for every constant $c$, there is a $\Sigma^2_{\text{prAM}}$ problem $\Gamma$ and a polynomial time relation $R$ such that, for infinitely many input lengths, every set hitting set for $\Gamma$ has circuit complexity at least $n^c$. We will show that this fact allows us to compute a string of hard $\text{SAT}$-oracle complexity. Once we have such a string, we use the pseudorandom generator of Klivans and van Melkebeek [13] to derandomize $\text{prAM}$. Let $\Pi = (\Pi_Y, \Pi_N)$ be the $\text{prAM}$ promise problem we wish to derandomize. Let $k \in \mathbb{N}$ be the number such that the number of random bits Arthur uses is at most $n^k$. Finally, let $\epsilon > 0$. We show how to compute $\Pi$ in $\Sigma^2_{\text{TIME}}(2^{n^c})/n^c$. Let $R$ be a polynomial time relation for a $\Sigma^2_{\text{prAM}}$ problem $\Gamma$ such that, for infinitely many input lengths $n$, every hitting set of $\Gamma$ has circuit complexity of at least $n^{2k/\epsilon}$. On input $x \in \{0, 1\}^n$, the $\Sigma^2_{\text{TIME}}(2^{n^c})/n^c$ first guesses a hitting set $S$ for $\Gamma$ on inputs of size $n^c$. The advice is the cardinality of $\Gamma_Y$. The machine then guesses three sets of strings, $U_Y, U_N, U_O$ such that $|U_Y| = |\Gamma_Y|$, and $|U_Y| + |U_N| + |U_O| = 2^{n^c}$. For each string in $U_O$, the machine verifies that it is not in the promise $\Gamma$. By our assumption of the promise, this can be done in $\Sigma^2_{\text{P}}$ time. For each string $x' \in U_Y$, the machine uses its oracle to verify that for every $y \in S$, there is a $z$ such that $R(x', y, z)$ accepts. Finally, for each string $x' \in U_N$, the machine verifies that there is some $y \in S$ such that...
every string $z$ satisfies $R(x', y, z) = 0$. Once the machine has verified each of these items, it then uses the guessed hitting set $S$ and a pseudorandom generator of $[13]$ to derandomize $\Pi$, and accepts if and only if $x \in \Pi_y$.

Therefore, we have that $\Sigma_2$-$\text{prAM}$ is in either $\text{P}^{\text{NP} / O(n^c)}$ for some fixed constant $c$ or $\text{prAM}$ is in $\Sigma_2\text{TIME}(2^{n^\epsilon})/n^{\epsilon}$ for every $\epsilon > 0$, and the claim follows. \hfill \Box

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