Ground states for a coupled nonlinear Schrödinger system

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Abstract

We study the existence of ground states for the coupled Schrödinger system

\[
\begin{aligned}
-\Delta u + u &= |u|^{2q-2}u + b|v|^q|u|^{q-2}u \\
-\Delta v + \omega^2 v &= |v|^{2q-2}v + b|u|^q|v|^{q-2}v
\end{aligned}
\]

in $\mathbb{R}^n$, for $\omega \geq 1$, $b > 0$ (the so-called “attractive case”) and $q > 1$ ($q < \frac{n}{n-2}$ if $n \geq 3$). We improve for several ranges of $(q, n, \omega)$ the known results concerning the existence of positive ground state solutions to (1) with non-trivial components. In particular, we prove that for $1 < q < 2$ such ground states exist in all dimensions and for all values of $\omega$, which constitutes a drastic change of behaviour with respect to the case $q \geq 2$. Furthermore, for $q > 2$ and in the one-dimensional case $n = 1$, we improve the results in [14].

Keywords: Non-trivial ground states; Coupled nonlinear Schrödinger Systems; Nehari Manifold.

AMS Subject Classification: 35J20, 35J50, 35J60

1 Introduction

In this paper we consider the system

\[
\begin{aligned}
-\Delta u + \lambda_1 u &= |u|^{2q-2}u + b|v|^q|u|^{q-2}u \\
-\Delta v + \lambda_2 v &= |v|^{2q-2}v + b|u|^q|v|^{q-2}v
\end{aligned}
\]

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with $u, v : \mathbb{R}^n \to \mathbb{R}$ ($n \geq 1$), $q > 1$, $b > 0$ and $\lambda_1, \lambda_2 > 0$, which appears in several physical contexts, namely in nonlinear optics (see [1] and the references therein).

By rescaling the $x$ variable and/or inverting the roles of $u$ and $v$, it is easy to see that (2) can be reduced, without loss of generality, to the system

\[
\begin{align*}
-\Delta u + u &= |u|^{2q-2}u + b|v|^{q} |u|^{q-2}u, \\
-\Delta v + \omega^2 v &= |v|^{2q-2}v + b|u|^{q} |v|^{q-2}v, \quad \omega \geq 1.
\end{align*}
\]

In the last years, this system has been extensively studied by many authors (see for instance [2], [11], [12], [17]). In particular, in [4] and [5] the authors studied the case $q = 2$ and $n = 2, 3$, proving the existence of a constant $\Lambda > 0$ depending on $\omega$ such that for $b < \Lambda$ the system (3) admits a non-trivial radial solution $(u, v) \neq (0, 0)$ (with $u, v > 0$ if $b > 0$). The authors also showed the existence of another constant $\Lambda' \geq \Lambda$ such that for $b > \Lambda'$ the system possesses a radial ground state solution $W_* = (u_*, v_*)$ ($u_*, v_* > 0$), in the sense that $W_*$ minimizes the energy functional associated to (3) among all solutions in $(u, v) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \setminus \{(0, 0)\}$. In [9] Ikoma and Tamaka showed that for $0 < b < \min\{\Lambda, 1\}$, the solutions found in [4], [5] are in fact also least energy solutions.

In [14], following some of the ideas presented in [15], the authors proved the existence of a radial non-trivial ground state solution $(u^*, v^*)$ ($u^*, v^* \geq 0$) for every $b > 0$ and for $(q, n)$ satisfying

\[
1 < q < \begin{cases} 
+\infty & \text{if } n = 1, 2 \\
\frac{n}{n-2} & \text{if } n \geq 3.
\end{cases}
\]

Furthermore, it is shown that for

\[
b \geq c_{\omega, n, q} := \frac{1}{2} \left[1 + \frac{n}{2} \left(1 - \frac{1}{q}\right) + \frac{1}{\omega^2} \left(1 - \frac{n}{2} \left(1 - \frac{1}{q}\right)\right)\right]^{\frac{q}{2}} \omega^{2q-n(q-1)} - 1
\]

there exists a ground state $(u^*, v^*)$ with $u^*, v^* > 0$.

In the present paper we will prove the existence of a positive radial decreasing ground state solution to (3) for all $(q, n)$ satisfying the condition (4). Exploring this radial decay, we improve the constant $c_{\omega, n, q}$ derived in [14] for all $q > 1$ and large $\omega$ in the case $n = 1$ and for all $1 < q < 2$ in any dimension, in fact replacing it by 0 in the latter case.
When dealing with the system (3) it is often necessary to treat the case \( n = 1 \) separately due to the lack of compactness of the injection \( H^1_d(\mathbb{R}) \hookrightarrow L^q(\mathbb{R}) \), \( q > 2 \), where \( H^1_d(\mathbb{R}) \) denotes the space of the radially symmetric functions of \( H^1(\mathbb{R}) \). This lack of compactness is, in a sense, a consequence of the inequality
\[
|u(x)| \leq C|x|^{-\frac{n}{2}}\|u\|_{H^1(\mathbb{R}^n)}
\]
for \( u \in H^1_d(\mathbb{R}) \). Indeed, (6) gives no decay in the case \( n = 1 \). However, if \( u \) is also radially decreasing, it is easy to establish that
\[
|u(x)| \leq C|x|^{-\frac{n}{2}}\|u\|_{L^2(\mathbb{R}^n)},
\]
which provides decay in all space dimensions, hence compacity by applying the classical Strauss’ compactness lemma (16). Hence, putting
\[
H^1_{rd}(\mathbb{R}^n) = \{ u \in H^1_d(\mathbb{R}^n) : u \text{ is radially decreasing} \},
\]
we get the compactness of the injection \( H^1_{rd}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) \) for all \( n \geq 1 \) (see the Appendix of [3] for more details). We will use this fact to present a unified approach for the problem of the energy minimization of (3), valid in all space dimensions.

Before stating our results more precisely, and following the functional settings in [4], [5] and [14], let us introduce a few notations: we denote by \( \| \cdot \|_q \) the standard \( L^q(\mathbb{R}^n) \) norm and, for \( (u, v) \in E := H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \), we put
\[
\|(u, v)\|_\omega^2 := \|u\|^2 + \|v\|^2 + \|\nabla u\|^2 + \omega^2\|v\|^2 + \|\nabla v\|^2.
\]
We introduce the energy functional associated to (3),
\[
I(u, v) := \frac{1}{2}\|(u, v)\|_\omega^2 - \frac{1}{2q}\left(\|u\|^{2q}_{2q} + \|v\|^{2q}_{2q} + 2b\|uv\|_q^q\right),
\]
noticing that \( (u, v) \) is a solution of (3) if and only if \( \nabla I(u, v) = 0 \).

We will study the minimization problem
\[
\inf\{I(u, v) : (u, v) \in \mathcal{N}\},
\]
where the so-called Nehari manifold \( \mathcal{N} \) is defined by
\[
\mathcal{N} := \{(u, v) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) : (u, v) \neq (0, 0), \nabla I(u, v) \perp (u, v)\},
\]
that is, \( (u, v) \in \mathcal{N} \) if and only if \( (u, v) \neq (0, 0) \) and
\[
\tau(u, v) := \langle \nabla I(u, v), (u, v) \rangle_{L^2} = \|(u, v)\|_\omega^2 - \left(\|u\|^{2q}_{2q} + \|v\|^{2q}_{2q} + 2b\|uv\|_q^q\right) = 0.
\]
As pointed out in [4] for the case $q = 2$, we notice that
\[
\langle \nabla \tau(u, v), (u, v) \rangle_{L^2} = 2\| (u, v) \|_{L^2}^2 - 2q \left( \| u \|_{L^2}^{2q} + \| v \|_{L^2}^{2q} + 2buv \|_{L^2}^q \right),
\]
and, if $(u, v) \in \mathcal{N}$,
\[
\langle \nabla \tau(u, v), (u, v) \rangle_{L^2} = 2(1 - q)\| (u, v) \|_{L^2}^2 < 0
\]
which shows that $\mathcal{N}$ is locally smooth.

Furthermore, it is easy to check that $\mathcal{H} \subset \mathcal{N}$, in view of (8), that is, in view of (8), $0 = \lambda(2 - 2q)\| (u, v) \|_{L^2}^2$, hence $\lambda = 0$ and $\nabla I(u, v) = 0$.

Putting $E_{rd} = H^1_{rd} \times H^1_{rd}$ the cone of symmetric radially decreasing non-negative functions of $E$, we will prove the following result:

**Theorem 1.1** Let $n \geq 1$ and $q > 1$, with $q \leq \frac{n}{n-2}$ if $n > 3$. There exists a minimizing sequence $(u_n, v_n) \in E_{rd}$ for the minimization problem [3]. Furthermore, $(u_n, v_n) \to (u_\ast, v_\ast) \in E_{rd}$ strongly in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. In particular
\[
I(u_\ast, v_\ast) = \min_{\mathcal{N}} I(u, v) = \min_{\mathcal{N} \cap E_{rd}} I(u, v)
\]
\[
= \min \{ I(u, v) : (u, v) \neq (0, 0) \text{ and } \nabla I(u, v) = 0 \}. \quad (9)
\]
Concerning the existence of ground states with non-trivial components, we will show:

**Theorem 1.2** Let $n \geq 1$ and $1 < q < 2$, with $q < \frac{n}{n-2}$ if $n \geq 3$. Then for all $b > 0$ there exists a ground state solution $(u, v) \in E_{rd}$ to (3) with $u > 0$ and $v > 0$.

**Theorem 1.3** Let $n = 1$ and $q \geq 2$. If
\[
b \geq D_{\omega, q} = \frac{2^q - 1}{2} \omega^{1+\frac{q}{2}} - \frac{1}{2} \omega^{-\frac{q}{2}}
\]
there exists a ground state solution $(u, v) \in E_{rd}$ to (3) with $u > 0$ and $v > 0$. 4
Notice that
\[ D_{\omega,q} < C_{\omega,1,q} = \frac{1}{2} \left( \frac{3}{2} - \frac{1}{2q} + \frac{1}{\omega} \left( \frac{1}{2} + \frac{1}{2q} \right) \right)^q \omega^{1+q} - 1 \]
at least for large values of \( \omega \).

2 Proof of Theorem 1.1

We begin by observing that for \((f, g) \in E, (f, g) \neq (0, 0)\), with \(\tau(f, g) \leq 0\), there exists \(t \in [0, 1]\) such that \((tf, tg) \in \mathcal{N}\). Indeed, if \(\tau(f, g) = 0\), we choose \(t = 1\). If \(\tau(f, g) < 0\) we simply notice that \(\tau(tf, tg) = t^2\left(\|f\|_\omega^2 - t^{2q-2}(\|f\|_{2q}^2 + \|g\|_{2q}^2 + 2b\|fg\|_q^q)\right) = t^2 T_{f,g}(t)\),
with \(T_{f,g}(0) > 0\) and \(T_{f,g}(1) < 0\).

Also, we notice that if \((f, g) \in \mathcal{N}\),
\[
I(f, g) = \left(\frac{1}{2} - \frac{1}{2q}\right)\|f\|_\omega \left(\frac{1}{2} - \frac{1}{2q}\right)(\|f\|_{2q}^2 + \|g\|_{2q}^2 + 2b\|fg\|_q^q). \quad (11)
\]

We now take a minimizing sequence \((u_n, v_n) \in \mathcal{N}\) for the problem
\[
m = \inf\{I(u, v) : (u, v) \in \mathcal{N}\}.
\]

From (11), it is clear that \(m \geq 0\) and that \((u_n, v_n)\) is bounded in \(E\).

We put \(u_n^*\) and \(v_n^*\) the decreasing radial rearrangements of \(|u_n|\) and \(|v_n|\) respectively. It is well-known that this rearrangement preserves the \(L^p\) norm \((1 \leq p \leq +\infty)\). Furthermore, the Pólya-Szego inequality
\[
\|\nabla f^*\|_2 \leq \|\nabla f\|_2
\]
in addition with the inequality \(\|\nabla f\|_2 \leq \|\nabla f\|_2\) (see [13]) shows that
\[
\|(u_n^*, v_n^*)\|_2^\omega \leq \|(u_n, v_n)\|_2^\omega.
\]

On the other hand, the Hardy-Littlewood inequality
\[
\int |fg| \leq \int f^* g^*
\]
combined with the monotonicity of the map \(\lambda \to \lambda^q\) (see for instance [8] for details) yields \(\|fg\|_q \leq \|f^* g^*\|_q\) and, finally,
\[
\tau(u_n^*, v_n^*) \leq \tau(u_n, v_n) = 0.
\]
Next, let \( t_n \in [0, 1] \) such that \((t_n u^*_n, t_n v^*_n) \in \mathcal{N}\). We obtain
\[
I(t_n u^*_n, t_n v^*_n) = t_n^2 \left( \frac{1}{2} - \frac{1}{2q} \right) \| (u^*_n, v^*_n) \|_{\tilde{\omega}}^2 \leq \left( \frac{1}{2} - \frac{1}{2q} \right) \| (u_n, v_n) \|_{\tilde{\omega}}^2 = I(u_n, v_n)
\]
and we obtained a minimizing sequence \((t_n u^*_n, t_n v^*_n)\) in \(E_{rd}\), denoted again, in what follows, by \((u_n, v_n)\). Since this sequence is bounded in \(H^1(\mathbb{R}^n)\), up to a subsequence, \((u_n, v_n) \to (u_*, v_*)\) in \(H^1(\mathbb{R}^n)\) weak. Also, since the injection \(E_{rd} \to L^{2q}(\mathbb{R}^n)\) is compact, up to a subsequence, \((u_n, v_n) \to (u_*, v_*)\) in \(L^{2q}(\mathbb{R}^n)\) strong.
Hence, since \(\|u_n\|_{2q} + \|v_n\|_{2q} + 2b|u_n v_n|_q \to \|u_*\|_{2q} + \|v_*\|_{2q} + 2b|u_* v_*|_q^q\), we deduce that \(\tau(u_*, v_*) \leq \liminf \tau(u_n, v_n) = 0\).
Once again, let \( t \in [0, 1] \) such that \((tu_*, tv_*) \in \mathcal{N}\).
\[
m \leq I(tu_*, tv_*) = t^2 \left( \frac{1}{2} - \frac{1}{2q} \right) \| (u_*, v_*) \|_{\tilde{\omega}}^2 \leq \left( \frac{1}{2} - \frac{1}{2q} \right) \liminf \| (u_n, v_n) \|_{\tilde{\omega}}^2 \leq \liminf I(u_n, v_n) = m.
\]
This implies that \((tu_*, tv_*)\) is a minimizer. In particular, all inequalities above are in fact equalities: \( t = 1, (u_*, v_*) \in \mathcal{N}, \| (u_*, v_*) \|_{\tilde{\omega}} = \lim \| (u_n, v_n) \|_{\tilde{\omega}}, \|u_n\|_{H^1} \to \|u_*\|_{H^1}, \|v_n\|_{H^1} \to \|v_*\|_{H^1}\) and \((u_n, v_n) \to (u_*, v_*)\) in \(H^1(\mathbb{R}^n)\) strong.
Finally, it is clear that \((u_*, v_*)\) is a ground state: if \((w_1, w_2) \neq (0, 0)\) is a critical point of \(I\) such that \(I(w_1, w_2) < I(u_*, v_*)\), taking once again \(w^*_1\) and \(w^*_2\) the decreasing radial rearrangements of \(|w_1|\) and \(|w_2|\), there exists \( t \in [0, 1] \) such that \((tw^*_1, tw^*_2) \in \mathcal{N}\) and \(I(tw^*_1, tw^*_2) \leq I(w_1, w_2)\), which leads to a contradiction. This completes the proof of Theorem \[\square\]

3  Ground states with non-trivial components

Let \((u_*, v_*) \in E_{rd}\) the ground state mentionned in Theorem \[\square\]. If \(v_* = 0\), \(u_* = u_0\) is the unique positive radially symmetric solution of the elliptic equation \(-\Delta u + u = u^{2q-1}\) (see \[10\]).

Also, if \(u_* = 0\), \(v_* = v_0\) is the unique positive radially symmetric solution of \(-\Delta v + \omega^2 v = v^{2q-1}\), which relates to \(u_0\) by the relation \(v_0(x) = \omega^{\frac{n-1}{n}} u_0(\omega x)\). Hence, to show the existence of a ground state with nontrivial components, we only have to exhibit an element \((f, g) \in \mathcal{N} \cap E_{rd}, f \neq 0, g \neq 0\), such that
\[
I(f, g) \leq \min\{ I(u_0, 0), I(0, v_0) \}. \tag{12}
\]
Hence, we obtain
\[ I(u_0, 0) = \left( \frac{1}{2} - \frac{1}{2q} \right) \| u_0 \|_{2q}^2, \quad I(0, v_0) = \omega^{\frac{2q}{q-1} - n} \left( \frac{1}{2} - \frac{1}{2q} \right) \| u_0 \|_{2q}^2 \]
and \[ \frac{2q}{q-1} - n > 0, \] for \( \omega \geq 1 \) the inequality (12) reduces to
\[ I(f, g) \leq I(u_0, 0). \] (13)

We first compute \( x > 0 \) such that \((f, g) := (x u_0, x \theta v_0) \in \mathcal{N}, \) where \( \theta > 0 \) will be chosen later (see [6] and [7] for a recent application of a related technique to the Schrödinger-KdV system):
\[ \tau(f, g) = x^2 \| (u_0, \theta v_0) \|_{\omega}^2 - x^{2q} \left( \| u_0 \|_{2q}^2 + \theta^2 q \| v_0 \|_{2q}^2 + 2b \theta^q \| u_0 v_0 \|_{q}^q \right) = 0. \]

Since
\[ \| \theta v_0 \|_{\omega}^2 = \omega^{2 + \frac{q}{q-1} - n} \| \theta u_0 \|_{2q}^2 + \omega^{2 + \frac{q}{q-1} - n} \| \theta \nabla u_0 \|_{2q}^2 = \omega^{\frac{2q}{q-1} - n} \| \theta^2 \| u_0 \|_{2q}^2 \]
and
\[ \| v_0 \|_{2q}^2 = \omega^{\frac{2q}{q-1} - n} \| u_0 \|_{2q}^2, \]
we obtain
\[ x^{2q-2} = \frac{(1 + \theta^2 \omega^{\frac{2q}{q-1} - n}) \| u_0 \|_{2q}^2}{(1 + \theta^{2q} \omega^{\frac{2q}{q-1} - n}) \| u_0 \|_{2q}^2 + 2b \theta^q \| u_0 v_0 \|_{q}^q} = \frac{1 + \theta^2 \omega^{\frac{2q}{q-1} - n}}{1 + \theta^{2q} \omega^{\frac{2q}{q-1} - n} + 2b \theta^q \| u_0 v_0 \|_{q}^q}. \]

Since \( u_0 \) is radial and nonincreasing and \( \omega \geq 1, \)
\[ \| u_0 v_0 \|_{q}^q = \omega^{\frac{q}{q-1}} \int u_0^q(x) u_0^q(\omega x)dx \leq \omega^{\frac{q}{q-1}} \int u_0^q(x) u_0^q(x)dx = \omega^{\frac{q}{q-1}} \| u_0 \|_{2q}^2. \]

Also,
\[ \| u_0 v_0 \|_{q}^q \geq \omega^{\frac{q}{q-1}} \int u_0^q(\omega x)dx = \omega^{\frac{q}{q-1} - n} \| u_0 \|_{2q}^2. \]

Hence, we obtain
\[ \frac{1 + \theta^2 \omega^{\frac{2q}{q-1} - n}}{1 + \theta^{2q} \omega^{\frac{2q}{q-1} - n} + 2b \theta^q \omega^{\frac{q}{q-1} - n}} \leq x^{2q-2} \leq \frac{1 + \theta^2 \omega^{\frac{2q}{q-1} - n}}{1 + \theta^{2q} \omega^{\frac{2q}{q-1} - n} + 2b \theta^q \omega^{\frac{q}{q-1} - n}} \] (14)
and
\[ I(f, g) = x^2 \left( \frac{1}{2} - \frac{1}{2q} \right) \| (u_0, \theta v_0) \|_{\omega}^2 = x^2 \left( \frac{1}{2} - \frac{1}{2q} \right) (1 + \theta^2 \omega^{\frac{2q}{q-1} - n}) \| u_0 \|_{2q}^2. \]
The condition (13) then becomes $x^2(1 + \theta^2 \omega^{2q - n}) \leq 1$, and in view of (14), a sufficient condition is

$$\frac{(1 + \theta^2 \omega^{2q - n})^q}{1 + \theta^2 \omega^{2q - n} + 2b \theta \omega^{2q - n}} \leq 1,$$

that is,

$$b \geq \frac{(1 + \theta^2 \omega^{2q - n})^q - 1 - \theta^2 \omega^{2q - n}}{2 \theta \omega^{2q - n}}.$$

We now put $\theta^2 = \epsilon^2 \omega^{2q - n}$, for $\epsilon > 0$, obtaining the condition

$$b \geq \frac{(1 + \epsilon^2)^q - 1}{2 \epsilon^q} \omega^{q-\frac{n}{2}(q-2)} - \frac{1}{2} \epsilon^q \omega^{(q-1)n}.$$

For $1 < q < 2$, $\lim_{\epsilon \to 0} \frac{(1 + \epsilon^2)^q - 1}{2 \epsilon^q} = 0$.

Hence, the arbitrary value of $\epsilon$ establishes the sufficient condition $b > 0$.

For $n = 1$, putting $\epsilon = 1$, we obtain the bound

$$b \geq \frac{2q - 1}{2} \omega^{1+\frac{q}{2}} - \frac{1}{2} \omega^{-\frac{q}{2}},$$

as stated in Theorem 1.3.

We finish by making a few remarks:

**Remark 3.1** For $\omega = 1$ and $\theta = 1$, we obtain, for all $n \geq 1$, the bound $2^{q-1} - 1$ which is known to be optimal for $q \geq 2$, in the sense that for $b < 2^{q-1} - 1$ all ground states of (3) have one null component (see [14], Theorem 2.5).

**Remark 3.2** The bound in (13) can be slightly improved for large values of $\omega$ by replacing the quantity $\frac{2q - 1}{2}$ by the minimum of $\frac{(1 + \epsilon^2)^q - 1}{2 \epsilon^q}$ for $\epsilon > 0$.

**Remark 3.3** For $n \geq 4$ we have $1 < q < 2$, hence the problem of the existence of ground states with non-trivial components is completely solved for these spatial dimensions.

**Acknowledgment:** This article was partially supported by Fundação para a Ciência e Tecnologia, through contract PEst-OE/MAT/UI0297/2015.
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