2-ABSORBING AND STRONGLY 2-ABSORBING SECONDARY SUBMODULES OF MODULES

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Abstract. In this paper, we will introduce the concept of 2-absorbing (resp. strongly 2-absorbing) secondary submodules of modules over a commutative ring as a generalization of secondary modules and investigate some basic properties of these classes of modules.

1. Introduction

Throughout this paper, $R$ will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

Let $M$ be an $R$-module. A proper submodule $P$ of $M$ is said to be prime if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_RM)$ \cite{13}. Let $N$ be a proper submodule of $M$. Then the $M$-radical of $N$, denoted by $M$-rad$(N)$, is defined to be the intersection of all prime submodules of $M$ containing $N$. If $M$ has no prime submodule containing $N$, then the $M$-radical of $N$ is defined to be $M$ \cite{16}. A non-zero submodule $S$ of $M$ is said to be second if for each $a \in R$, the homomorphism $S \rightarrow S$ is either surjective or zero \cite{20}. In this case Ann$_R(S)$ is a prime ideal of $R$.

The notion of 2-absorbing ideals as a generalization of prime ideals was introduced and studied in \cite{8}. A proper ideal $I$ of $R$ is a 2-absorbing ideal of $R$ if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. It has been proved that $I$ is a 2-absorbing ideal of $R$ if and only if whenever $I_1, I_2,$ and $I_3$ are ideals of $R$ with $I_1I_2I_3 \subseteq I$, then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$ \cite{8}. In \cite{9}, the authors introduced the concept of 2-absorbing primary ideal which is a generalization of primary ideal. A proper ideal $I$ of $R$ is called a 2-absorbing primary ideal of $R$ if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

The notion of 2-absorbing ideals was extended to 2-absorbing submodules in \cite{12}. A proper submodule $N$ of $M$ is called a 2-absorbing submodule of $M$ if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_RM)$.

In \cite{5}, the present authors introduced the dual notion of 2-absorbing submodules (that is, 2-absorbing (resp. strongly 2-absorbing) second submodules) of $M$ and investigated some properties of these classes of modules. A non-zero submodule $N$ of $M$ is said to be a 2-absorbing second submodule of $M$ if whenever $a, b \in R$, $L$ is a completely irreducible submodule of $M$, and $abN \subseteq L$, then $aN \subseteq L$ or $bN \subseteq L$ or $ab \in Ann_R(N)$. A non-zero submodule $N$ of $M$ is said to be a strongly 2-absorbing second submodule of $M$ if whenever $a, b \in R$, $L$ is a completely irreducible submodule of $M$, and $abN \subseteq L$, then $aN \subseteq L$ or $bN \subseteq L$ or $ab \in Ann_R(N)$.

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2-absorbing second submodule of $M$ if whenever $a, b \in R$, $K$ is a submodule of $M$, and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in \text{Ann}_R(N)$.

In [18], the authors introduced the notion of 2-absorbing primary submodules as a generalization of 2-absorbing primary ideals of rings and studied some properties of this class of modules. A proper submodule $N$ of $M$ is said to be a 2-absorbing primary submodule of $M$ if whenever $a, b \in R$, $m \in M$, and $abm \in N$, then $am \in M \text{-rad}(N)$ or $bm \in M \text{-rad}(N)$ or $ab \in (N :_R M)$.

The purpose of this paper is to introduce the concepts of 2-absorbing and strongly 2-absorbing secondary submodules of an $R$-module $M$ as dual notion of 2-absorbing primary submodules and obtain some related results.

2. Main results

Let $M$ be an $R$-module. For a submodule $N$ of $M$ the the second radical (or second socle) of $N$ is defined as the sum of all second submodules of $M$ contained in $N$ and it is denoted by $\text{sec}(N)$ (or $\text{soc}(N)$). In case $N$ does not contain any second submodule, the second radical of $N$ is defined to be $(0)$. $N \neq (0)$ is said to be a second radical submodule of $M$ if $\text{sec}(N) = N$ (see [11] and [2]).

A proper submodule $N$ of $M$ is said to be completely irreducible if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of $M$, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of $M$ is an intersection of completely irreducible submodules of $M$ [11].

We frequently use the following basic fact without further comment.

Remark 2.1. Let $N$ and $K$ be two submodules of an $R$-module $M$. To prove $N \subseteq K$, it is enough to show that if $L$ is a completely irreducible submodule of $M$ such that $K \subseteq L$, then $N \subseteq L$.

Definition 2.2. We say that a non-zero submodule $N$ of an $R$-module $M$ is a 2-absorbing secondary submodule of $M$ if whenever $a, b \in R$, $L$ is a completely irreducible submodule of $M$ and $abN \subseteq L$, then $a(\text{sec}(N)) \subseteq L$ or $b(\text{sec}(N)) \subseteq L$ or $ab \in \text{Ann}_R(N)$. By a 2-absorbing secondary module, we mean a module which is a 2-absorbing secondary submodule of itself.

Example 2.3. Clearly, every submodule of the $\mathbb{Z}$-module $\mathbb{Z}$ is not secondary. But as $\text{sec}(\mathbb{Z}) = 0$, every submodule of the $\mathbb{Z}$-module $\mathbb{Z}$ is 2-absorbing secondary.

Lemma 2.4. Let $I$ be an ideal of $R$ and $N$ be a 2-absorbing secondary submodule of $M$. If $a \in R$, $L$ is a completely irreducible submodule of $M$, and $IaN \subseteq L$, then $a(\text{sec}(N)) \subseteq L$ if $I(\text{sec}(N)) \subseteq L$ or $Ia \in \text{Ann}_R(N)$.

Proof. Let $a(\text{sec}(N)) \not\subseteq L$ and $Ia \not\in \text{Ann}_R(N)$. Then there exists $b \in I$ such that $abN \neq 0$. Now as $N$ is a 2-absorbing secondary submodule of $M$, $baN \subseteq L$ implies that $b(\text{sec}(N)) \subseteq L$. We show that $I(\text{sec}(N)) \subseteq L$. To see this, let $c$ be an arbitrary element of $I$. Then $(b + c)aN \subseteq L$. Hence, either $(b + c)(\text{sec}(N)) \subseteq L$ or $(b + c)a \in \text{Ann}_R(N)$. If $(b + c)(\text{sec}(N)) \subseteq L$, then since $b(\text{sec}(N)) \subseteq L$ we have $c(\text{sec}(N)) \subseteq L$. If $(b + c)a \in \text{Ann}_R(N)$, then $ca \not\in \text{Ann}_R(N)$. Thus $caN \subseteq L$ implies that $c(\text{sec}(N)) \subseteq L$. Hence, we conclude that $I(\text{sec}(N)) \subseteq L$.

Theorem 2.5. Let $I$ and $J$ be two ideals of $R$ and $N$ be a 2-absorbing secondary submodule of an $R$-module $M$. If $L$ is a completely irreducible submodule of $M$ and $IJN \subseteq L$, then $I(\text{sec}(N)) \subseteq L$ or $J(\text{sec}(N)) \subseteq L$ or $IJ \not\subseteq \text{Ann}_R(N)$. 

Proof. Let $I(\text{sec}(N)) \not\subseteq L$ and $J(\text{sec}(N)) \not\subseteq L$. We show that $IJ \subseteq \text{Ann}_R(N)$. Assume that $c \in I$ and $d \in J$. By assumption, there exists $a \in I$ such that $a(\text{sec}(N)) \not\subseteq L$ and $aJN \subseteq L$. Now Lemma 2.3 shows that $aJ \subseteq \text{Ann}_R(N)$ and so $(I \setminus (L :_R \text{sec}(N)))J \subseteq \text{Ann}_R(N)$. Similarly, there exists $b \in (J \setminus (L :_R \text{sec}(N)))$ such that $Ib \subseteq \text{Ann}_R(N)$ and also $I(J \setminus (L :_R \text{sec}(N))) \subseteq \text{Ann}_R(N)$. Thus we have $ab \in \text{Ann}_R(N)$, $ad \in \text{Ann}_R(N)$, and $cd \in \text{Ann}_R(N)$. As $a + c \in I$ and $b + d \in J$, we have $(a + c)(b + d)N \subseteq L$. Therefore, $(a + c)(\text{sec}(N)) \subseteq L$ or $(b + d)(\text{sec}(N)) \subseteq L$. Hence $c \in I \setminus (L :_R \text{sec}(N))$ which implies that $cd \in \text{Ann}_R(N)$. Similarly, if $(b + d)(\text{sec}(N)) \subseteq L$, we can deduce that $cd \in \text{Ann}_R(N)$. Finally if $(a + c)(b + d) \in \text{Ann}_R(N)$, then $ab + ad + cd \in \text{Ann}_R(N)$ so that $cd \in \text{Ann}_R(N)$. Therefore, $IJ \subseteq \text{Ann}_R(N)$.

Theorem 2.6. Let $N$ be a non-zero submodule of an $R$-module $M$. The following statements are equivalent:

(a) If $abN \subseteq L_1 \cap L_2$ for some $a, b \in R$ and completely irreducible submodules $L_1, L_2$ of $M$, then $a(\text{sec}(N)) \subseteq L_1 \cap L_2$ or $b(\text{sec}(N)) \subseteq L_1 \cap L_2$ or $ab \in \text{Ann}_R(N)$;

(b) If $IJN \subseteq K$ for some ideals $I, J$ of $R$ and a submodule $K$ of $M$, then $I(\text{sec}(N)) \subseteq K$ or $J(\text{sec}(N)) \subseteq K$ or $IJ \in \text{Ann}_R(N)$;

(c) For each $a, b \in R$, we have $a(\text{sec}(N)) \subseteq abN$ or $b(\text{sec}(N)) \subseteq abN$ or $abN = 0$.

Proof. $(a) \Rightarrow (b)$. Assume that $IJN \subseteq K$ for some ideals $I, J$ of $R$, a submodule $K$ of $M$, and $IJ \not\subseteq \text{Ann}_R(N)$. Then by Theorem 2.5 for all completely irreducible submodules $L$ of $M$ with $K \subseteq L$ either $I(\text{sec}(N)) \subseteq L$ or $J(\text{sec}(N)) \subseteq L$. If $I(\text{sec}(N)) \subseteq L$ (resp. $J(\text{sec}(N)) \subseteq L$) for all completely irreducible submodules $L$ of $M$ with $K \subseteq L$, we are done. Now suppose that $L_1$ and $L_2$ are two completely irreducible submodules of $M$ with $K \subseteq L_1$, $K \subseteq L_2$, $I(\text{sec}(N)) \not\subseteq L_1$, and $J(\text{sec}(N)) \not\subseteq L_2$. Then $I(\text{sec}(N)) \subseteq L_2$ and $J(\text{sec}(N)) \subseteq L_1$. Since $IJN \subseteq L_1 \cap L_2$, we have either $I(\text{sec}(N)) \subseteq L_1 \cap L_2$ or $J(\text{sec}(N)) \subseteq L_1 \cap L_2$. If $I(\text{sec}(N)) \subseteq L_1 \cap L_2$, then $I(\text{sec}(N)) \subseteq L_1$ which is a contradiction. Similarly from $J(\text{sec}(N)) \subseteq L_1 \cap L_2$ we get a contradiction.

$(b) \Rightarrow (a)$. This is clear.

$(a) \Rightarrow (c)$. By part (a), $N \neq 0$. Let $a, b \in R$. Then $abN \subseteq abN$ implies that $a(\text{sec}(N)) \subseteq abN$ or $b(\text{sec}(N)) \subseteq abN$ or $abN = 0$.

$(c) \Rightarrow (a)$. This is clear.

Definition 2.7. We say that a non-zero submodule $N$ of an $R$-module $M$ is a strongly 2-absorbing secondary submodule of $M$ if satisfies the equivalent conditions of Theorem 2.6. By a strongly 2-absorbing secondary module, we mean a module which is a strongly 2-absorbing secondary submodule of itself.

Let $N$ be a submodule of an $R$-module $M$. Then part (d) of Theorem 2.6 shows that $N$ is a strongly 2-absorbing secondary submodule of $M$ if and only if $N$ is a strongly 2-absorbing secondary module.

Example 2.8. Clearly every strongly 2-absorbing secondary submodule is a 2-absorbing secondary submodule. But the converse is not true in general. For example, consider $M = \mathbb{Z}_6 \oplus \mathbb{Q}$ as a $\mathbb{Z}$-module. Then $M$ is a 2-absorbing secondary
module. But since \(0 \neq 6M \subseteq 0 \oplus \mathbb{Q}\), \(sec(M) = M\), \(2M \nsubseteq 0 \oplus \mathbb{Q}\), and \(3M \nsubseteq 0 \oplus \mathbb{Q}\), \(M\) is not a strongly 2-absorbing secondary module.

**Proposition 2.9.** Let \(N\) be a 2-absorbing second submodule of an \(R\)-module \(M\). Then \(N\) is a strongly 2-absorbing secondary submodule of \(M\).

**Proof.** Let \(a, b \in R\) and \(K\) be a submodule of \(M\) such that \(aN \subseteq K\). Then \(aN \subseteq K\) or \(bN \subseteq K\) or \(abN = 0\) by assumption. Thus \(a(sec(N)) \subseteq aN \subseteq K\) or \(b(sec(N)) \subseteq aN \subseteq K\) or \(abN = 0\), as required. 

The following example shows that the converse of the Proposition 2.9 is not true in general.

**Example 2.10.** Let \(M\) be the \(\mathbb{Z}\)-module \(\mathbb{Z}_{p^\infty}\). Then as \(p^2 \langle 1/p^3 + \mathbb{Z} \rangle \subseteq \langle 1/p + \mathbb{Z} \rangle\), \(p\langle 1/p^3 + \mathbb{Z} \rangle \nsubseteq \langle 1/p + \mathbb{Z} \rangle\), and \(p^2 \langle 1/p^3 + \mathbb{Z} \rangle \neq 0\), we have the submodule \(\langle 1/p^3 + \mathbb{Z} \rangle\) of \(\mathbb{Z}_{p^\infty}\) is not 2-absorbing second submodule. But \(sec(\langle 1/p^3 + \mathbb{Z} \rangle) = \langle 1/p + \mathbb{Z} \rangle\) implies that \(\langle 1/p^3 + \mathbb{Z} \rangle\) is a strongly 2-absorbing secondary submodule of \(M\).

An \(R\)-module \(M\) is said to be a **comultiplication module** if for every submodule \(N\) of \(M\) there exists an ideal \(I\) of \(R\) such that \(N = (0 :_M I)\), equivalently, for each submodule \(N\) of \(M\), we have \(N = (0 :_M Ann_R(N))\) \([1]\).

**Theorem 2.11.** Let \(M\) be a finitely generated comultiplication \(R\)-module. If \(N\) is a strongly 2-absorbing secondary submodule of \(M\), then \(Ann_R(N)\) is a 2-absorbing primary ideal of \(R\).

**Proof.** Let \(a, b, c \in R\) be such that \(abc \in Ann_R(N)\), \(ac \notin \sqrt{Ann_R(N)}\), and \(bc \notin \sqrt{Ann_R(N)}\). Since by \([1]\ 2.12]\), \(Ann_R(sec(N)) = \sqrt{Ann_R(N)}\), there exist completely irreducible submodules \(L_1\) and \(L_2\) of \(M\) such that \(ac(sec(N)) \nsubseteq L_1\) and \(bc(sec(N)) \nsubseteq L_2\). But \(abcN = 0 \subseteq L_1 \cap L_2\) implies that \(abN \subseteq (L_1 \cap L_2 :_M c)\). Now as \(N\) is a strongly 2-absorbing secondary submodule of \(M\), we have \(a(sec(N)) \subseteq (L_1 \cap L_2 :_M c)\) or \(b(sec(N)) \subseteq (L_1 \cap L_2 :_M c)\) or \(abN = 0\). If \(a(sec(N)) \subseteq (L_1 \cap L_2 :_M c)\) (resp. \(b(sec(N)) \subseteq (L_1 \cap L_2 :_M c)\)), then \(ac(sec(N)) \subseteq L_1\) (resp. \(bc(sec(N)) \subseteq L_2\)) contradicted. Hence \(abN = 0\), as needed.

**Theorem 2.12.** Let \(N\) be a submodule of a comultiplication \(R\)-module \(M\). If \(Ann_R(N)\) is a 2-absorbing primary ideal of \(R\), then \(N\) is a strongly 2-absorbing secondary submodule of \(M\).

**Proof.** Let \(abN \subseteq K\) for some \(a, b \in R\) and some submodule \(K\) of \(M\). As \(M\) is a comultiplication module, there exists an ideal \(I\) of \(R\) such that \(K = (0 :_M I)\). Hence \(Iab \subseteq Ann_R(N)\) which implies that either \(Ia \subseteq \sqrt{Ann_R(N)}\) or \(Ib \subseteq \sqrt{Ann_R(N)}\) or \(ab \in Ann_R(N)\). If \(ab \in Ann_R(N)\), we are done. If \(Ia \subseteq \sqrt{Ann_R(N)}\), as \(\sqrt{Ann_R(N)} \subseteq Ann_R(sec(N))\), we have \(Ia(sec(N)) = 0\). This implies that \(a(sec(N)) \subseteq K\) because \(M\) is a comultiplication module. Similarly, if \(Ib \subseteq \sqrt{Ann_R(N)}\), we get \(b(sec(N)) \subseteq K\). This completes the proof. 

The following example shows that Theorem 2.12 is not satisfied in general.

**Example 2.13.** Consider the \(\mathbb{Z}\)-module \(M = \mathbb{Z}_p \oplus \mathbb{Z}_q \oplus \mathbb{Q}\), where \(p \neq q\) are two prime numbers. Then \(M\) is not a comultiplication \(\mathbb{Z}\)-module and \(Ann_{\mathbb{Z}}(M) = 0\) is a 2-absorbing primary ideal of \(R\). But since \(0 \neq pqM \subseteq 0 \oplus 0 \oplus \mathbb{Q}\), \(sec(M) = M\), \(pM \nsubseteq 0 \oplus 0 \oplus \mathbb{Q}\), and \(qM \nsubseteq 0 \oplus 0 \oplus \mathbb{Q}\), \(M\) is not a strongly 2-absorbing secondary module.
In [13, 2.6], it is shown that, if $M$ is a finitely generated multiplication $R$-module and $N$ is a 2-absorbing primary submodule of $M$, then $M\text{-rad}(N)$ is a 2-absorbing submodule of $M$. In the following lemma, we see that some of this conditions are redundant.

**Lemma 2.14.** Let $N$ be a 2-absorbing primary submodule of an $R$-module $M$. Then $M\text{-rad}(N)$ is a 2-absorbing submodule of $M$.

**Proof.** This follows from the fact that $M\text{-rad}(M\text{-rad}(N)) = M\text{-rad}(N)$ by Proposition 2. □

**Proposition 2.15.** Let $M$ be an $R$-module. Then we have the following.

(a) If $N$ is a 2-absorbing (resp. strongly 2-absorbing) secondary submodule of an $R$-module $M$, then $\text{sec}(N)$ is a 2-absorbing (resp. strongly 2-absorbing) second submodule of $M$.

(b) If $N$ is a second radical submodule of $M$, then $N$ is a 2-absorbing (resp. strongly 2-absorbing) second submodule if and only if $N$ is a 2-absorbing (resp. strongly 2-absorbing) secondary submodule.

**Proof.** (a) This follows from the fact that $\text{sec}(\text{sec}(N)) = \text{sec}(N)$ by [4, 2.1]. □

(b) This follows from part (a) □

Let $N$ and $K$ be two submodules of an $R$-module $M$. The **coproduct** of $N$ and $K$ is defined by $(0:_M \text{Ann}_R(N)\text{Ann}_R(K))$ and denoted by $C(NK)$ [6].

**Theorem 2.16.** Let $N$ be a submodule of an $R$-module $M$ such that $\text{sec}(N)$ is a second submodule of $M$. Then we have the following.

(a) $N$ is a strongly 2-absorbing secondary submodule of $M$.

(b) If $M$ is a comultiplication $R$-module, then $C(N^t)$ is a strongly 2-absorbing secondary submodule of $M$ for every positive integer $t \geq 1$, where $C(N^t)$ means the coproduct of $N$, $t$ times.

**Proof.** (a) Let $a, b \in R$, $K$ be a submodule of $M$ such that $abN \subseteq K$, and $b(\text{sec}(N)) \nsubseteq K$. Then as $\text{sec}(N)$ is a second submodule and $a(\text{sec}(N)) \subseteq aN \subseteq (K:_M b)$, we have $a(\text{sec}(N)) = 0$ by [3, 2.10]. Thus $a(\text{sec}(N)) \subseteq K$, as needed.

(b) Let $M$ be a comultiplication $R$-module. Then there exists an ideal $I$ of $R$ such that $N = (0:_M I)$. Thus by [4, 2.1],

$$\text{sec}(c(N^t)) = \text{sec}((0:_M I^t)) = \text{sec}((0:_M I)) = \text{sec}(N).$$

Now the results follows from to the proof of part (a). □

**Theorem 2.17.** Let $M$ be a comultiplication $R$-module. Then we have the following.

(a) If $N_1, N_2, ..., N_n$ are strongly 2-absorbing secondary submodules of $M$ with the same second radical, then $N = \sum_{i=1}^{n} N_i$ is a strongly 2-absorbing secondary submodule of $M$.

(b) If $N_1, N_2, ..., N_n$ are 2-absorbing secondary submodules of $M$ with the same second radical, then $N = \sum_{i=1}^{n} N_i$ is a 2-absorbing secondary submodule of $M$.

(c) If $N_1$ and $N_2$ are two secondary submodules of $M$, then $N_1 + N_2$ is a strongly 2-absorbing secondary submodule of $M$. 
(d) If \( M \) is finitely generated, \( N \) is a submodule of \( M \) which possess a secondary representation, and \( sec(N) = K_1 + K_2 \), where \( K_1 \) and \( K_2 \) are two minimal submodules of \( M \), then \( N \) is a strongly 2-absorbing secondary submodule of \( M \).

Proof. (a) Let \( a, b \in R \) and \( K \) be a submodule of \( M \) such that \( abN \subseteq K \). Thus for each \( i = 1, 2, \ldots, n \), \( abN_i \subseteq K \). If there exists \( 1 \leq j \leq n \) such that \( a(sec(N_j)) \subseteq K \) or \( b(sec(N_j)) \subseteq K \), then \( a(sec(N)) \subseteq K \) or \( b(sec(N)) \subseteq K \) (note that \( sec(N) = sec(\sum_{i=1}^n N_i) = \sum_{i=1}^n sec(N_i) = sec(N) \) by [11 2.6]). Otherwise, \( abN_i = 0 \) for each \( i = 1, 2, \ldots, n \). Hence \( abN = 0 \), as desired.

(b) The proof is similar to the part (a).

(c) As \( N_1 \) and \( N_2 \) are secondary submodules of \( M \), \( Ann_R(N_1) \) and \( Ann_R(N_2) \) are primary ideals of \( R \). Hence \( Ann_R(N_1 + N_2) = Ann_R(N_1) \cap Ann_R(N_2) \) is a 2-absorbing primary ideal of \( R \) by [9 2.4]. Thus by Theorem 2.12, \( N_1 + N_2 \) is a strongly 2-absorbing secondary submodule of \( M \).

(d) Let \( N = \sum_{i=1}^n N_i \) be a secondary representation. By [4 2.6], \( sec(N) = \sum_{i=1}^n sec(N_i) \). Since \( sec(N_i)'s \) are secondary submodules of \( M \) by [4 2.13], we have
\[
\{ sec(N_1), sec(N_2), \ldots, sec(N_n) \} = \{ K_1, K_2 \}.
\]
Without loss of generality, we may assume that for some \( 1 \leq t < n \), \( \{ sec(N_1), \ldots, sec(N_t) \} = \{ K_1 \} \) and \( \{ sec(N_{t+1}), \ldots, sec(N_n) \} = \{ K_2 \} \). Set \( H_1 := N_1 + \ldots + N_t \) and \( H_2 := N_{t+1} + \ldots + N_n \). By [11 2.12], \( H_1 \) and \( H_2 \) are secondary submodules of \( M \). Therefore, by part (c), \( N = H_1 + H_2 \) is a strongly 2-absorbing secondary submodule of \( M \). \( \square \)

The following example shows that the direct sum of two strongly 2-absorbing secondary \( R \)-modules is not a strongly 2-absorbing secondary \( R \)-module in general.

Example 2.18. Clearly, the \( \mathbb{Z} \)-modules \( \mathbb{Z}_6 \) and \( \mathbb{Z}_{10} \) are strongly 2-absorbing secondary \( \mathbb{Z} \)-modules. Let \( M = \mathbb{Z}_6 \oplus \mathbb{Z}_{10} \). Then \( M \) is not a strongly 2-absorbing secondary \( \mathbb{Z} \)-module. By [3 2.1], \( sec(M) = M \). Thus \( M \) is not a strongly 2-absorbing secondary \( \mathbb{Z} \)-module by Proposition 2.15.

Lemma 2.19. Let \( f : M \rightarrow \hat{M} \) be a monomorphism of \( R \)-modules. Then we have the following.

(a) If \( N \) is a submodule of \( M \), then \( sec(f(N)) = f(sec(N)) \).

(b) If \( \hat{N} \) is a submodule of \( \hat{M} \) such that \( \hat{N} \subseteq f(M) \), then \( sec(f^{-1}(\hat{N})) = f^{-1}(sec(\hat{N})) \).

Proof. (a) Let \( \hat{S} \) be a second submodule of \( f(N) \). Then one can see that \( f^{-1}(\hat{S}) \) is a secondary submodule of \( N \). Thus \( f(f^{-1}(\hat{S})) \subseteq f(sec(N)) \). Therefore, \( sec(f(N)) \subseteq f(sec(N)) \). The reverse inclusion is clear.

(b) Let \( S \) be a secondary submodule of \( f^{-1}(\hat{N}) \). Then one can see that \( f(S) \) is a secondary submodule of \( \hat{N} \). Thus \( f^{-1}(S) \subseteq f^{-1}(sec(\hat{N})) \). Therefore, \( sec(f^{-1}(\hat{N})) \subseteq f^{-1}(sec(\hat{N})) \). To see the reverse inclusion, let \( \hat{S} \) be a secondary submodule of \( \hat{N} \). Then \( f^{-1}(\hat{S}) \) is a second submodule of \( f^{-1}(\hat{N}) \). This implies that \( f^{-1}(sec(\hat{N})) \subseteq sec(f^{-1}(\hat{N})) \). \( \square \)

Theorem 2.20. Let \( f : M \rightarrow \hat{M} \) be a monomorphism of \( R \)-modules. Then we have the following.

(a) If \( N \) is a strongly 2-absorbing secondary submodule of \( M \), then \( f(N) \) is a strongly 2-absorbing secondary submodule of \( \hat{M} \).
Lemma 2.23. Let \( \hat{N} \) be a strongly 2-absorbing secondary submodule of \( \hat{M} \) and \( \hat{N} \subseteq f(M) \), then \( f^{-1}(\hat{N}) \) is a strongly 2-absorbing secondary submodule of \( M \).

Proof. (a) Since \( N \neq 0 \) and \( f \) is a monomorphism, we have \( f(N) \neq 0 \). Let \( a, b \in R \), \( K \) be a submodule of \( \hat{M} \), and \( abf(N) \subseteq K \). Then \( abN \subseteq f^{-1}(K) \). As \( N \) is strongly 2-absorbing secondary submodule, \( a(\text{sec}(N)) \subseteq f^{-1}(K) \) or \( b(\text{sec}(N)) \subseteq f^{-1}(K) \) or \( abN = 0 \). Therefore, by Lemma 2.19 (a),

\[
a(\text{sec}(f(N))) = a(f(\text{sec}(N))) \subseteq f(f^{-1}(K)) = f(M) \cap \hat{K} \subseteq \hat{K}
\]
or\[
b(\text{sec}(f(N))) = b(f(\text{sec}(N))) \subseteq f(f^{-1}(K)) = f(M) \cap \hat{K} \subseteq \hat{K}
\]
or \( abf(N) = 0 \), as needed.

(b) If \( f^{-1}(\hat{N}) = 0 \), then \( f(M) \cap \hat{N} = f(f^{-1}(\hat{N})) = f(0) = 0 \). Thus \( \hat{N} = 0 \), a contradiction. Therefore, \( f^{-1}(\hat{N}) \neq 0 \). Now let \( a, b \in R \), \( K \) be a submodule of \( M \), and \( abf^{-1}(\hat{N}) \subseteq K \). Then

\[
ab\hat{N} = abf(M) \cap \hat{N} = abf^{-1}(\hat{N}) \subseteq f(K).
\]

As \( \hat{N} \) is strongly 2-absorbing secondary submodule, \( a(\text{sec}(\hat{N})) \subseteq f(K) \) or \( b(\text{sec}(\hat{N})) \subseteq f(K) \) or \( ab\hat{N} = 0 \). Hence by Lemma 2.19 (b),

\[
a(\text{sec}(f(N))) = a(f(\text{sec}(N))) \subseteq f^{-1}(f(K)) = K \text{ or } b(\text{sec}(f(N))) = b(f(\text{sec}(N))) \subseteq f^{-1}(f(K)) = K \text{ or } abf^{-1}(\hat{N}) = 0,
\]
as desired. \( \square \)

Corollary 2.21. Let \( M \) be an \( R \)-module and let \( N \subseteq K \) be two submodules of \( M \). Then \( N \) is a strongly 2-absorbing secondary submodule of \( K \) if and only if \( N \) is a strongly 2-absorbing secondary submodule of \( M \).

Proof. This follows from Theorem 2.20 by using the natural monomorphism \( K \rightarrow M \). \( \square \)

Proposition 2.22. Let \( M \) be a cocyclic \( R \)-module with minimal submodule \( K \) and \( N \) be a submodule of \( M \) such that \( rN \neq K \) for each \( r \in R \). If \( N/K \) is a strongly 2-absorbing secondary submodule of \( M/K \), then \( N \) is a strongly 2-absorbing secondary submodule of \( M \).

Proof. Let \( a, b \in R \) and \( H \) be a submodule of \( M \) such that \( abN \subseteq H \). Then \( ab(N/K) \subseteq H/K \) implies that \( a(\text{sec}(N/K)) \subseteq H/K \) or \( b(\text{sec}(N/K)) \subseteq H/K \) or \( ab(N/K) = 0 \). If \( ab(N/K) = 0 \), then \( abN = 0 \) because \( rN \neq K \) for each \( r \in R \). Otherwise, since \( a(\text{sec}(N))/K \subseteq \text{sec}(N/K) \), we have \( a(\text{sec}(N)) \subseteq H \) or \( b(\text{sec}(N)) \subseteq H \) as required. \( \square \)

Let \( R_i \) be a commutative ring with identity and \( M_i \) be an \( R_i \)-module, for \( i = 1, 2 \). Let \( R = R_1 \times R_2 \). Then \( M = M_1 \times M_2 \) is an \( R \)-module and each submodule of \( M \) is in the form of \( N = N_1 \times N_2 \) for some submodules \( N_1 \) of \( M_1 \) and \( N_2 \) of \( M_2 \). In addition, \( M_i \) is a comultiplication \( R_i \)-module, for \( i = 1, 2 \) if and only if \( M \) is a comultiplication \( R \)-module by \([19], 2.1\). \( \square \)

Lemma 2.23. Let \( R = R_1 \times R_2 \) and \( M = M_1 \times M_2 \), where \( M_1 \) is an \( R_1 \)-module and \( M_2 \) is an \( R_2 \)-module. If \( N = N_1 \times N_2 \) is a submodule of \( M \), then we have the following.

(a) \( N \) is a second submodule of \( M \) if and only if \( N = S_1 \times 0 \) or \( N = S_2 \times 0 \), where \( S_1 \) is a second submodule of \( N_1 \) and \( S_2 \) is a second submodule of \( M_2 \).
Lemma 2.25. \( \square \)

Theorem 2.24. Let \( R = R_1 \times R_2 \) and \( M = M_1 \times M_2 \), where \( M_1 \) is a comultiplication \( R_1 \)-module and \( M_2 \) is a comultiplication \( R_2 \)-module. Then we have the following.

(a) If \( M_1 \) be a finitely generated \( R_1 \)-module, then a non-zero submodule \( K_1 \) of \( M_1 \) is a strongly 2-absorbing secondary submodule if and only if \( N = K_1 \times 0 \) is a strongly 2-absorbing secondary submodule of \( M \).

(b) If \( M_2 \) be a finitely generated \( R_2 \)-module, then a non-zero submodule \( K_2 \) of \( M_2 \) is a strongly 2-absorbing secondary submodule if and only if \( N = 0 \times K_2 \) is a strongly 2-absorbing secondary submodule of \( M \).

(c) If \( K_1 \) is a secondary submodule of \( M_1 \) and \( K_2 \) is a secondary submodule of \( M_2 \), then \( N = K_1 \times K_2 \) is a strongly 2-absorbing secondary submodule of \( M \).

\[ \begin{align*}
\text{Proof.} & \quad \text{(a) Let } K_1 \text{ be a strongly 2-absorbing secondary submodule of } M_1. \text{ Then } \text{Ann}_{R_1}(K_1) \text{ is a 2-absorbing primary ideal of } R_1 \text{ by Theorem 2.11.} \text{ Now since } \\
& \quad \quad \text{Ann}_R(N) = \text{Ann}_{R_1}(K_1) \times R_2, \text{ we have } \text{Ann}_R(N) \text{ is a 2-absorbing primary ideal of } R \text{ by [9] 2.23.} \text{ Thus the result follows from Theorem 2.12.} \text{ Conversely, let } N = K_1 \times 0 \text{ be a strongly 2-absorbing secondary submodule of } M. \text{ Then } \\
& \quad \quad \text{Ann}_R(N) = \text{Ann}_{R_1}(K_1) \times R_2 \text{ is a primary ideal of } R \text{ by Theorem 2.11.} \text{ Thus } \text{Ann}_{R_1}(K_1) \text{ is a primary ideal of } R_1 \text{ by [9] 2.23.} \text{ Thus by Theorem 2.12, } K_1 \text{ be a strongly 2-absorbing secondary submodule of } M_1. \\
& \quad \quad \text{(b) We have similar arguments as in part (a).} \\
& \quad \quad \text{(c) Let } K_1 \text{ be a secondary submodule of } M_1 \text{ and } K_2 \text{ be a secondary submodule of } M_2. \text{ Then } \text{Ann}_{R_1}(K_1) \text{ and } \text{Ann}_{R_2}(K_2) \text{ are primary ideals of } R_1 \text{ and } R_2, \text{ respectively. Now since } \\
& \quad \quad \text{Ann}_R(N) = \text{Ann}_{R_1}(K_1) \times \text{Ann}_{R_2}(K_2), \text{ we have } \text{Ann}_R(N) \text{ is a 2-absorbing primary ideal of } R \text{ by [9] 2.23.} \text{ Thus the result follows from Theorem 2.12.} \quad \square
\end{align*} \]

Lemma 2.25. Let \( N \) be a submodule of a comultiplication \( R \)-module \( M \). Then \( N \) is a secondary submodule if and only if \( \text{Ann}_R(N) \) be a primary ideal of \( R \).

\[ \begin{align*}
\text{Proof.} & \quad \text{The necessity is clear. For converse, let } r \in R. \text{ As } M \text{ is a comultiplication module, } rN = (0 :_M I) \text{ for some ideal } I \text{ of } R. \text{ Now } rI \subseteq \text{Ann}_R(N) \text{ implies that } \\
& \quad \quad I \subseteq \text{Ann}_R(N) \text{ or } r^t \in \text{Ann}_R(N) \text{ for some positive integer } t. \text{ Thus as } M \text{ is a comultiplication } R \text{-module, } N = rN \text{ or } r^tN = 0 \text{ for some positive integer } t. \quad \square
\end{align*} \]

Theorem 2.26. Let \( R = R_1 \times R_2 \) be a decomposable ring and \( M = M_1 \times M_2 \) be a finitely generated comultiplication \( R \)-module, where \( M_1 \) is an \( R_1 \)-module and \( M_2 \) is an \( R_2 \)-module. Suppose that \( N = N_1 \times N_2 \) is a non-zero submodule of \( M \). Then the following conditions are equivalent:

(a) \( N \) is a strongly 2-absorbing secondary submodule of \( M \); 

(b) Either \( N_1 = 0 \) and \( N_2 \) is a strongly 2-absorbing secondary submodule of \( M_2 \) or \( N_2 = 0 \) and \( N_1 \) is a strongly 2-absorbing secondary submodule of \( M_1 \) or \( N_1, N_2 \) are secondary submodules of \( M_1, M_2 \), respectively.

\[ \begin{align*}
\text{Proof.} & \quad (a) \Rightarrow (b). \text{ Let } N = N_1 \times N_2 \text{ be a strongly 2-absorbing secondary submodule of } M. \text{ Then } \text{Ann}_R(N) = \text{Ann}_{R_1}(N_1) \times \text{Ann}_{R_2}(N_2) \text{ is a 2-absorbing primary ideal}
\end{align*} \]
of \( R \) by Theorem 2.11. By [9, 2.23], we have \( \text{Ann}_{R_1}(N_1) = R_1 \) and \( \text{Ann}_{R_2}(N_2) \) is a 2-absorbing primary ideal of \( R_2 \) or \( \text{Ann}_{R_1}(N_2) = R_2 \) and \( \text{Ann}_{R_1}(N_1) \) is a 2-absorbing primary ideal of \( R_1 \) and \( \text{Ann}_{R_2}(N_2) \) are primary ideals of \( R_1 \) and \( R_2 \), respectively. Suppose that \( \text{Ann}_{R_1}(N_1) = R_1 \) and \( \text{Ann}_{R_2}(N_2) \) is a 2-absorbing primary ideal of \( R_2 \). Then \( N_1 = 0 \) and \( N_2 \) is a strongly 2-absorbing secondary submodule of \( M_2 \) by Theorem 2.12. Similarly, if \( \text{Ann}_{R_2}(N_2) = R_2 \) and \( \text{Ann}_{R_1}(N_1) \) is a 2-absorbing primary ideal of \( R_1 \), then \( N_2 = 0 \) and \( N_1 \) is a strongly 2-absorbing secondary submodule of \( M_1 \). If the last case hold, then as \( M_1 \) (resp. \( M_2 \)) is a comultiplication \( R_1 \) (resp. \( R_2 \)) module, \( N_1 \) (resp. \( N_2 \)) is a secondary submodule of \( M_1 \) (resp. \( M_2 \)) by Lemma 2.25.

\((b) \Rightarrow (a)\). This follows from Theorem 2.25. □

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