THREE-DIMENSIONAL $\mathcal{N} = 4$ SUPERSYMMETRY IN HARMONIC $\mathcal{N} = 3$ SUPERSPACE

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Abstract

We consider the map of three-dimensional $\mathcal{N} = 4$ superfields to $\mathcal{N} = 3$ harmonic superspace. The left and right representations of the $\mathcal{N} = 4$ superconformal group are constructed on $\mathcal{N} = 3$ analytic superfields. These representations are convenient for the description of $\mathcal{N} = 4$ superconformal couplings of the Abelian gauge superfields with hypermultiplets. We analyze the $\mathcal{N} = 4$ invariance in the non-Abelian $\mathcal{N} = 3$ Yang-Mills theory.

Keywords: Harmonic superspace, extended supersymmetry, superconformal symmetry

1 Introduction

The simplest supersymmetric Chern-Simons theory was constructed in the $\mathcal{N}=1, d=3$ superspace with the real coordinates $z = (x^m, \theta^\alpha)$, where the Grassmann coordinate $\theta^\alpha$ has the spinor index $\alpha = 1, 2$ of the group $SL(2, R)$, and $m = 0, 1, 2$ is the three-dimensional vector index [1, 2]. This theory uses the spinor gauge superfield $A_\alpha(z)$. The superfield action of the $\mathcal{N}=1$ Chern-Simons theory was interpreted as a superspace integral of the differential Chern-Simons superform $dA + \frac{2}{3}A^3$ in the framework of the theory of superfield integral forms [3]-[6]. The non-Abelian $\mathcal{N}=2, d=3$ Chern-Simons action was considered in the superspace $z = (x^m, \theta^\alpha, \bar{\theta}^\alpha)$, where $\theta^\alpha$ and $\bar{\theta}^\alpha$ are complex conjugated spinor coordinates [3, 7, 8]. The basic $\mathcal{N}=2$ gauge superfield is $V(z)$, and the gauge group preserves chirality of the matter superfields. The $\mathcal{N} = 3, d = 3$ Chern-Simons theory was first studied by the harmonic-superspace method [9, 10]. The analytic gauge $\mathcal{N} = 3, d = 3$ prepotential is similar to the gauge superfield of the $\mathcal{N} = 2, d = 4$ Yang-Mills theory [11, 12].

The $\mathcal{N} = 5$ and $\mathcal{N} = 6$ Chern-Simons theories were considered also in the harmonic-superspace method [13, 14], but this approach did not succeed in constructing of coupling of the gauge superfield with matter. We note that the $\mathcal{N} = 6$ Chern-Simons supermultiplet has an infinite number of auxiliary fields off mass-shell.

The $\mathcal{N} = 6$ Chern-Simons-matter model for the gauge group $U(N) \times U(N)$ ($ABJM$-model [15]) and the $\mathcal{N} = 8$ BLG-model for the gauge group $SU(2) \times SU(2)$ [16] were investigated in the $\mathcal{N} = 3$ harmonic superspace [17]. The $\mathcal{N} = 3$ supersymmetry is manifest in this formalism, the higher supersymmetry transformations connect different superfields and the corresponding algebra of transformations closes on the mass shell. The quantum aspects of the $\mathcal{N} = 3$ superfield theories were analyzed in [18].

The $\mathcal{N} = 4, d = 3$ superfield theories were studied in our papers [19, 6, 20]. The mirror left and right $\mathcal{N} = 4$ supermultiplets are defined in different harmonic superspaces and this fact is the main obstacle to the construction of the left-right couplings. We review the $\mathcal{N} = 4, d = 3$ superspace formulas in Appendix.
In the next section, we analyze the $\mathcal{N} = 3, d = 3$ superconformal transformations in the standard and harmonic superspaces. The corresponding Killing operator $K_3$ contains the generators of the $\mathcal{N} = 3$ superconformal group $P_m, M_m, D, V_{kl}, Q_{\alpha}^{(kl)}$ and $S_{\alpha}^{(kl)}$ acting as superspace differential operators.

We study the relations between the $\mathcal{N} = 4$ and $\mathcal{N} = 3$ harmonic superfields in section 3. The left $\mathcal{N} = 4$ analytic superfields are decomposed in terms of the $\mathcal{N} = 3$ spinor coordinates $\theta^{++\alpha}, \theta^{0\alpha}$ and the additional spinor coordinate $\theta_{4\alpha}$. We obtain the $\mathcal{N} = 3$ representation of the left $\mathcal{N} = 4$ supermultiplets using the operator $O$ connecting different superspaces. This operator allows us to define the active $\mathcal{N} = 4$ superconformal transformations on the analytic $\mathcal{N} = 3$ superfields. These transformations include the $\mathcal{N} = 3$ superconformal generators and the transformations with the additional even generator $A_{kl}$ and the odd spinor generators $Q_{\alpha}^{4}$ and $S_{\alpha}^{4}$. The mirror map $\mathcal{M}$ from the left to right $\mathcal{N} = 4$ representations is equivalent to the change of the signs of these additional generators. Thus, we study the left and right supermultiplets in the same analytic superspace, the mirror supermultiplets have similar $\mathcal{N} = 3$ transformations, and their additional transformations differ by the signs.

Section 4 is devoted to the construction of the $\mathcal{N} = 4$ superconformal models in the $\mathcal{N} = 3$ harmonic superspace. We consider the analytic Abelian gauge prepotential $V_{0}^{++}$ and its analytic superfield strength $W_{0}^{++}$ and prove that these superfields have the mirror $\mathcal{N} = 4$ transformations. The $\mathcal{N} = 4$ superconformal $BF$ coupling of $W_{0}^{++}$ contains the right Abelian prepotential $A^{++}$. This coupling is equivalent to the difference of two Abelian Chern-Simons interactions. The natural $\mathcal{N} = 4$ superconformal coupling of the left gauge superfield $V_{0}^{++}$ is defined with the left hypermultiplets, while the right $\mathcal{N} = 4$ gauge multiplet $A^{++}$ interacts with the right hypermultiplets. We consider also the nonminimal interaction of $W_{0}^{++}$ with the right hypermultiplet.

The interesting $\mathcal{N} = 4$ superconformal coupling $S^{\beta}(V_{0}^{++}, A^{++})$ of two Abelian superfields contains the $BF$ coupling and the improved forms of the actions for two superfield strengths. Analogous improved forms were considered in the $\mathcal{N} = 2, d = 4$ superspace [12] and in the $\mathcal{N} = 4, d = 3$ superspaces [20]. The $\mathcal{N} = 3$ representation is convenient for the analysis of the quantum properties of this model. We also discuss the action of the nonlinear $\mathcal{N} = 4$ electrodynamics.

In section 5 we study the $\mathcal{N} = 4$ superconformal transformations of the $\mathcal{N} = 3$ gauge superfields. In particular, we obtain the nonlinear fourth supersymmetry transformation of the non-Abelian $\mathcal{N} = 3$ superfield strength and prove the $\mathcal{N} = 4$ invariance of the corresponding non-Abelian action.

## $\mathcal{N} = 3, d = 3$ harmonic superspace

We use the notation of papers [10, 17] in the $\mathcal{N} = 3$ superspace. The real coordinates of the $\mathcal{N} = 3$ superspace are introduced in the central basis

$$z = (x^m, \theta_{(kl)}^\alpha), \quad \theta_{(kl)}^\alpha = \theta^{(kl)\alpha},$$

where $k, l$ are the spinor indices of the automorphism group $SU_V(2)$. The isovector representation for the spinor coordinates contains the Pauli matrices $\theta_B^{\alpha} = g_A^{(kl)}(\tau_B)^{kl}\theta_{(kl)}^\alpha$, $B = 1, 2, 3$. The spinor derivatives in these coordinates are

$$D_{\alpha}^{(kl)} = \partial_{(kl)}^{\alpha} + ig^{(kl)\beta}\partial_{\alpha\beta}, \quad \partial_{\alpha\beta} = (\gamma^m)_{\alpha\beta}\partial_m.$$
\[ \partial_m x^n = \delta_m^n, \quad \partial^{(kl)}_{\alpha} \theta^{\beta}_{(jn)} = \frac{1}{2} (\delta^k_j \delta^l_n + \delta^l_j \delta^k_n), \] (2.2)

where \((\gamma^m)_{\alpha\beta}\) are the three-dimensional \(\gamma\) matrices. The transformations of the \(\mathcal{N} = 3\) superconformal group in the real coordinates (2.1) \([17]\) can be rewritten via the Killing superconformal operator \(K_3\) that contains the generators of the corresponding Lie superalgebra

\[ \delta_{sc} x^m = K_3 x^m, \quad \delta_{sc} \theta^\alpha_{(kl)} = K_3 \theta^\alpha_{(kl)}, \]

\[ K_3(z) = c^m P_m + l^m M_m + b D + k^m K_m + \epsilon^m_{(kl)} V_{kl} + \epsilon^m_{(kl)} \eta^{\alpha}_{(kl)} S^{(kl)}_\alpha, \] (2.3)

where \(c^m, l^m, b, k^m, \epsilon^m_{(kl)}, \eta^{\alpha}_{(kl)}\) are the superconformal parameters. It is easy to construct these superconformal generators in the central basis, for instance, the \(SU_V(2)\) generators have the form

\[ V_{kl} = \varepsilon_{ln} \theta^\alpha_{(kj)} \partial^j_{\alpha} + \varepsilon_{kn} \theta^\alpha_{(lj)} \partial^j_{\alpha}. \] (2.4)

The Killing operator connects the active superconformal transformations of superfields \(\delta^* A\) with the passive superconformal transformations

\[ \delta_{sc} A(z) \equiv A(z + \delta_{sc} z) - A(z) = \delta^* A(z) + K_3 A(z). \] (2.5)

We consider the important commutation relations of the Killing operator with the spinor derivatives, which define the passive superconformal transformations of these derivatives \(\delta_{sc} D^{(kl)}_{\alpha}\)

\[ [K_3, D^{(kl)}_{\alpha}] = -\frac{1}{2} j D^{(kl)}_{\alpha} - \chi^\gamma_{\alpha} D^{(kl)}_{\gamma} + \chi^k_{\alpha} D^{(nl)}_{\alpha} + \chi^l_{\alpha} D^{(kn)}_{\alpha} = \delta_{sc} D^{(kl)}_{\alpha} \] (2.6)

where

\[ \chi^\gamma_{\alpha} = a^\gamma + \frac{1}{4} (x^{\alpha \rho} k^\gamma + x^{\gamma \rho} k^\alpha) - \frac{i}{4} (\theta_{(kl)}^\beta) \theta^{(kl)}_{\gamma} k^\alpha + \frac{i}{2} (\theta_{(kl)}^\alpha) \theta^{(kl)}_{\gamma} + \theta_{(kl)}^\gamma \theta_{(kl)}^\alpha, \]

\[ \chi^k_{\alpha} = a^{kl} - \frac{i}{2} \varepsilon_{jn} \theta^j_{\gamma} \theta^{\gamma \alpha} + \frac{i}{2} \varepsilon_{jn} (\theta^j_{\alpha} \eta^l_{\alpha} + \theta^j_{\alpha} \eta^k_{\alpha}), \]

\[ j = -b - k_m x^m - i \theta^\alpha_{(kl)} \theta^\alpha_{(kl)}, \] (2.7)

are the superfield parameters. We consider the useful relations

\[ D^{(kl)}_{\gamma} \chi^\alpha_{\alpha} = -\delta^\gamma_{\gamma} D^{(kl)}_{\alpha} j + \frac{1}{2} \delta^\rho_{\alpha} D^{(kl)}_{\alpha} j, \]

\[ D^{(nl)}_{\alpha} \chi^{kl}_{\alpha} = \frac{1}{4} \varepsilon^{nk} D^{(il)}_{\alpha} j + \frac{1}{4} \varepsilon^{lk} D^{(nl)}_{\alpha} j + \frac{1}{4} \varepsilon^{ll} D^{(kn)}_{\alpha} j + \frac{1}{4} \varepsilon^{nl} D^{(ik)}_{\alpha} j. \] (2.9)

The primary superfield of the weight \(w\) has the active superconformal transformation

\[ \delta^* A_w = w j A_w - K_3 A_w. \] (2.10)

We study transformations of the spinor derivative of superfield \(A_w\) using the formal relation \([\delta^*, D^{(kl)}_{\alpha}] = 0\) and formula (2.6).

The \(\mathcal{N} = 3\) superconformal linear (tensor) multiplet \(W^{(kl)}\) satisfies the following superfield constraints:

\[ D^{(jn)}_{\alpha} W^{(kl)} = 0, \] (2.11)
where the brackets mean symmetrization in four indices. These constraints are covariant under the superconformal transformations

\[ \delta^* W^{(kl)} = j W^{(kl)} + \lambda^l_n W^{(nl)} + \lambda^k_n W^{(kn)} - K_3 W^{(kl)}. \]  

(2.12)

The superconformal transformation of the $SU_V(2)/U(1)$ harmonics $u_i^\pm$ can be defined as follows:

\[ \delta_{sc} u_i^+ = \lambda^{kl} u^+_k u_i^+ u_i^-, \quad \delta_{sc} u_i^- = 0 \]  

(2.13)

where the matrix $\lambda^{kl}$ is given by (2.7). The harmonic projection of the relation (2.9) yields the condition

\[ u^+_n u^+_i u^+_j D^{(ni)}_\alpha \lambda^{kl} = D^{++}_\alpha \lambda^{++} = 0, \quad \lambda^{++} = \lambda^{kl} u^+_k u^+_i, \quad D^{++} = u^+_n u^+_i D^{(ni)}_\alpha . \]  

(2.14)

The Killing vector $K^C_3 = K_3(z) + \lambda^{++} \partial^{--}$ in the extended central basis $(z, u)$ preserves the Grassmann analyticity condition $D^{++}_\alpha A(z, u) = 0$

\[ [K^C_3, D^{++}_\alpha] = -\lambda^\gamma D^{++}_\gamma - \frac{1}{2} (j - \partial^{--} \lambda^{++}) D^{++}_\alpha . \]  

(2.15)

The Grassmann analyticity is manifest in the analytic basis $z_A = (\zeta, \theta^{--})$

\[ \zeta = (x^m_A, \theta^{++}, \theta^{0\alpha}, u^+_i), \quad x^m_A = x^m + i(\gamma^m)_{\alpha\beta} \theta^{++} \theta^{-\beta}, \quad \theta^{\pm\pm} = U^{\pm\pm kl} \theta^{(kl)}_\alpha, \quad \theta^{0\alpha} = U^{0kl} \theta^{(kl)}_\alpha , \]  

(2.16)

(2.17)

where we define the isovector combinations of the spinor harmonics

\[ U^{++kl} = u^{+k} u^{+l}, \quad U^{--kl} = u^{-k} u^{-l}, \quad U^{0kl} = \frac{1}{2} (u^{+k} u^{-l} + u^{+l} u^{-k}), \]  

(2.18)

\[ \int du U^{++kl} U^{-ij} = -2 \int du U^{0kl} U^0_{ij} = \frac{1}{6} (\delta^k_i \delta^l_j + \delta^k_j \delta^l_i). \]  

(2.19)

The special conjugation $\sim$ is defined in the analytic coordinates

\[ \tilde{u}_i^\pm = u_i^\pm i, \quad \tilde{x}_A^m = x_A^m, \quad \tilde{\theta}_a^{0,\pm\pm} = \theta_a^{0,\pm\pm} . \]  

(2.20)

In the analytic basis, the harmonic and spinor derivatives have the form

\[ D^{++} = \partial^{++} + 2i \theta^{++} \partial^{0\beta} \partial^A_{\alpha\beta} + \theta^{++} \partial^0 + 2 \theta^{0a} \partial^+_a, \quad \partial^A_{\alpha\beta} = (\gamma^m)_{\alpha\beta} \partial^m_A , \]  

(2.21)

\[ D^{--} = \partial^{--} - 2i \theta^{--} \partial^{0\beta} \partial^A_{\alpha\beta} + \theta^{--} \partial^0 - 2 \theta^{0a} \partial^-_a , \]  

\[ D^0 = \partial^0 + 2 \theta^{++} \partial^- - 2 \theta^{--} \partial^+_a, \quad [D^{++}, D^{--}] = D^0 , \]  

\[ D^{++}_\alpha = \partial^{++}_\alpha, \quad D^{--}_\alpha = \partial^{--}_\alpha + 2i \theta^{--} \partial^A_{\alpha\beta}, \quad D^0_\alpha = \frac{1}{2} \partial^0 + i \theta^{0\beta} \partial^A_{\alpha\beta} , \]  

(2.22)

\[ \partial^A_m x^m_A = \delta^m_A, \quad \partial^A_{\alpha\beta} \theta^{0\beta} = \delta^A_{\alpha\beta}, \quad \partial^{++}_\alpha \theta^{\pm\pm} = \delta^A_{\alpha\beta}, \quad [\partial^{++}, \partial^{--}] = \partial^0 , \]  

where $\partial^0, \partial^{\pm\pm}$ are the partial harmonic derivatives. We use the notation

\[ (\theta^{++})^2 = \theta^{++} \theta^{++}, \quad (\theta^0)^2 = \theta^{0\alpha} \theta^0_\alpha, \quad (\theta^{++} \theta^0) = \theta^{++} \theta^0_\alpha, \quad (\theta^{++} \gamma^m \theta^0) = \theta^{++} \theta^{0\beta} (\gamma^m)_{\alpha\beta} , \]  

\[ (D^{++})^2 = D^{+++} D^{++}_\alpha, \quad (D^0)^2 = D^{0\alpha} D^0_\alpha . \]  

(2.23)
The $\mathcal{N} = 3$ analytic superfield $\phi(\zeta)$ satisfies the condition $D^+_\alpha \phi = 0$. The $\mathcal{N} = 3$ superconformal transformations of the analytic coordinates [17] are generated by the Killing operator

$$K^A_3 = K_3(\zeta) + \delta_{sc} \theta^{--\alpha} \partial^{++}_\alpha,$$

(2.24)

where the operator $K_3(\zeta)$ acts in the analytic subspace $\zeta$, for instance,

$$K_3 u_i^+ = \lambda^{++}(\zeta, u) u_i^+, \quad K_3 u_i^- = 0,$$

(2.25)

$$\lambda^{++}(\zeta) = -i(\theta^{+++\gamma m \theta^0}) k^m + i(\theta^{0\alpha} U^{++}_{kl} - \theta^{+++\alpha} U^0_{kl}) \eta^{(kl)} + U^{++}_{kl} a^{kl}.$$

(2.26)

The analytic representation of the even superconformal $\mathcal{N} = 3$ generators has the form

$$P_m = \partial^A_m, \quad M_m = \varepsilon_{mnpl} x^p A \partial^A_l + \frac{1}{2} (\gamma_m)_a^b (\theta^{+++\alpha} \partial^-_{\alpha} + \theta^{0\alpha} \partial^0_{\alpha}),$$

(2.27)

$$D = x^m_A \partial^A_m + \frac{1}{2} (\theta^{+++\alpha} \partial^-_{\alpha} + \theta^{0\alpha} \partial^0_{\alpha}),$$

(2.28)

$$K_m = x^m_A \partial^A_m - \frac{1}{2} (x_A)^2 \partial^A_m - i(\theta^{+++\gamma m \theta^0}) \partial^-$$

$$- \frac{i}{2} (\gamma_m)_{\beta} (\theta^0)^2 \theta^{+++\beta} \partial^-_{\alpha} + \frac{1}{2} x^m_A (\gamma_m)^{\alpha\beta} (\gamma_m)_{\beta\gamma} (\theta^{+++\gamma} \partial^-_{\alpha} + \theta^{0\gamma} \partial^0_{\alpha}),$$

(2.29)

$$V_{kl} = U^{++}_{kl} \partial^- - U^{--}_{kl} \theta^{+++\alpha} (\partial^0_{\alpha} + 2i \theta^{0\alpha} \partial^A_{\alpha}) + 2 U^{0}_{kl} \theta^{+++\alpha} \partial^-_{\alpha}.$$

(2.30)

The odd analytic superconformal generators act also in the analytic superspace

$$Q^{(kl)}_{\alpha} = U^{++}_{kl} \partial^-_{\alpha} - 2i U^{--}_{kl} \theta^{+++\alpha} \partial^A_{\alpha} + U^0_{kl} (\partial^0_{\alpha} + 2i \theta^{0\alpha} \partial^A_{\alpha}),$$

(2.31)

$$S^{(kl)}_{\alpha} = \frac{1}{3} (\gamma_m)_a^b [K_m, Q^{(kl)}_{\beta}].$$

(2.32)

We consider the active realization of the $\mathcal{N} = 3$ superconformal transformations on the analytic superfields

$$\delta^*_s \phi = \delta_{sc} \phi - K_3(\zeta) \phi.$$

(2.33)

The Killing operator satisfies the relations

$$[K^A_3, D^{++}] = -\lambda^{++} D^0 = \delta_{sc} D^{++},$$

$$[K^A_3, D^{--}] = -(D^{--} \lambda^{++}) = \delta_{sc} D^{--},$$

(2.34)

which determine the passive operator transformations.

The harmonic projection of the $\mathcal{N} = 3$ linear multiplet (2.11) $W^{++} = u_k^+ u_l^+ W^{(kl)}$ satisfies the constraints

$$D^{++}_\alpha W^{++} = 0, \quad D^{++} W^{++} = 0$$

(2.35)

and has the superconformal transformation

$$\delta^{++}_3 W^{++} = 2 \lambda W^{++} - K_3 W^{++},$$

$$\lambda = -\frac{1}{2} b - \frac{1}{2} k_m x^m_A + i(\theta^{0\alpha} U^0_{kl} - \theta^{+++\alpha} U^{++}_{kl}) \eta^{(kl)} + U^0_{kl} a^{kl}.$$

(2.36)
The superfield superconformal parameters $j$ and $\lambda^{kl}$ (2.12) can be expressed via the analytic superconformal parameters

$$\lambda^{kl} = U^{--kl} \lambda^{++} - U^{0kl} D^{--} \lambda^{++} + \frac{1}{2} U^{++kl} (D^{--})^2 \lambda^{++},$$

$$j = 2 \lambda - D^{--} \lambda^{++}, \quad \lambda = -\frac{1}{2} b + U^0_{kl} \lambda^{kl}. \quad (2.37)$$

The Abelian scalar analytic prepotential $V_0^{++}(\zeta)$ has the gauge transformation

$$\delta_A V_0^{++} = -D^{++} \Lambda, \quad \tilde{V}_0^{++} = -V_0^{++}, \quad \tilde{\Lambda} = -\Lambda. \quad (2.38)$$

The gauge-invariant real analytic superfield strength can be defined via this gauge superfield [10, 17]

$$W_0^{++}(\zeta) = -\frac{1}{4} (D^{++})^2 V_0^{--} = IV_0^{++} = \int d\zeta_1^{-3} I(\zeta, \zeta_2) V_0^{++}(\zeta_2),$$

$$I(\zeta, \zeta_2) = \frac{1}{16} (D^{++})^2 (D_2^{++})^2 \delta^9(z - z_2) \frac{1}{(u^+ u_2^-)^2}, \quad (2.39)$$

where $V_0^{--}$ is the harmonic connection, and $I$ is the integral operator. We use the relations

$$D^{++} V_0^{--} = D^{++} V_0^{++}, \quad D^{++} W_0^{++} = 0. \quad (2.40)$$

We note that the superfields $V_0^{++}$ and $W_0^{++}$ have opposite $P$-parities with respect to the transformation

$$P(x_A^0, x_A^1, x_A^2) = (x_A^0, -x_A^1, x_A^2), \quad P \theta_\alpha^{0,\pm \pm} = - (\gamma_1)^\beta \theta_\beta^{0,\pm \pm}. \quad (2.41)$$

The component fields of the vector multiplet $\phi^{kl}, A_m, \lambda^\alpha, \chi^{kl}_\alpha$ and $X^{kl}$ can be determined in the $WZ$-gauge

$$V_{WZ}^{++} = 3(\theta^{++})^2 U^{-\alpha}_{kl} \phi^{kl} + 2(\theta^{++} \gamma^m \theta^0) A_m + 2i(\theta^0)^2 \theta^{+++\alpha} \lambda^\alpha + 3i(\theta^{++})^2 \theta^{0\alpha} U_{kl}^{-\alpha} \chi^{kl}_\alpha$$

$$+ 3i(\theta^{++})^2 (\theta^0)^2 U_{kl}^{-\alpha} X^{kl}. \quad (2.42)$$

The $P$-even superfield $V_0^{++}$ contains the pseudoscalar field $\phi^{kl}$, the scalar $X^{kl}$ and the vector field $A_m$.

The $\mathcal{N} = 3$ supersymmetry transformations of the component fields

$$\delta \phi^{kl} = -i e^{(kl)\alpha} \lambda_\alpha - \frac{i}{2} \epsilon^{(kl)\alpha} (\lambda_j^0 + e^{(lj)\alpha} \lambda_j^k), \quad (2.43)$$

$$\delta \lambda^\alpha = -\frac{i}{2} (\epsilon^{(kl)\gamma} \gamma^m \chi_{kl}^\alpha), \quad \delta \lambda^\alpha = -\frac{i}{2} (\epsilon^{(kl)\gamma} \partial^{(k \gamma)} - \epsilon^{(kl)\gamma} X_{kl}^\alpha), \quad (2.44)$$

$$\delta \chi_{kl}^\alpha = \epsilon^{(kl)\gamma} \partial^{(k \gamma)} \phi_j^l + \epsilon^{(lj)\gamma} \partial^{(l \gamma)} \phi_k^j + (\gamma_m)_{\alpha \beta} F^m \epsilon^{(kl)\beta} - \epsilon^{(kl)\beta} X_j^l - \epsilon^{(lj)\beta} X_k^j, \quad (2.45)$$

$$\delta F^m = \epsilon^{mpq} (\partial_p A_q - \partial_q A_p), \quad \delta X^{kl} = -i e^{(kl)\alpha} \partial^{(k \gamma)} \lambda_\gamma + i (\epsilon^{(kl)\alpha} \partial^{(l \gamma)} \lambda_j^\alpha + e^{(lj)\alpha} \partial^{(j \gamma)} \lambda_j^k). \quad (2.46)$$

can be obtained from the superfield transformation of $V_{WZ}^{++}$. 

6
The Abelian superfield strength has the simple form in this gauge

\[
W_0^{++} = -(D^0)^2 + \frac{1}{2}(D^0)^2D--D^{++} + \frac{1}{2}(D^0D--)D^{++} - \frac{1}{12}(D^0)^2(D--)^2(D^{++})^2
- \frac{1}{6}(D^0D--)D--(D^{++})^2 - \frac{1}{24}(D--)^2(D^{++})^2]\ V_0^{++}.
\]

(2.47)

The component decomposition of the superfield strength contains the fields and their derivatives

\[
W_0^{++} = U^{++}_{kl}\phi^l - 2i(\theta^{++}\gamma^m\theta^0)U^0_{kl}\partial_m\phi^l + (\theta^{++})^2(\theta^0)^2U^{-}_{kl}\Box\phi^l
- i(\theta^{++}\gamma_m\theta^0)F^m + i\theta^{++}\lambda_\alpha - (\theta^0)^2\theta^{++}\partial^A\lambda^\gamma
+ i(\theta^{++}U^0_{kl} - i\theta^0\theta^0U^{++}_{kl})\lambda^\alpha + [(\theta^0)^2\theta^{++}\theta^0U^0_{kl} + (\theta^{++})^2\theta^0\theta^0U^{-}_{kl}]\partial^A\lambda^\gamma
+ i[(\theta^{++})^2U^{-}_{kl} + (\theta^0)^2U^{++}_{kl} - 2(\theta^{++}\theta^0)U^0_{kl}]X^L_{kl},
\]

(2.48)

where \( \Box = \eta^{mn}\partial_m^A\partial_n^A \) and \( F^m \) is the Abelian field strength.

3 Relations between \( \mathcal{N} = 4 \) and \( \mathcal{N} = 3 \) harmonic superfields

3.1 Left \( \mathcal{N} = 4 \) representations

The extended \( \mathcal{N} = 3 \) analytic basis includes coordinates (2.17) and the additional spinor coordinate \( \theta^4_4 \)

\[
x^m_A, \theta^{\pm\alpha}, \theta^{0\alpha}, \theta^4_4, u^{\pm}_k.
\]

(3.1)

We analyze the relation between the left analytic \( \mathcal{N} = 4 \) basis (A.15) and the extended \( \mathcal{N} = 3 \) basis

\[
\theta^{+\alpha} = \theta^{0\alpha} + \frac{1}{2}\theta^4_4, \quad \theta^{-+\alpha} = \theta^{0\alpha} - \frac{1}{2}\theta^4_4,
\]

(3.2)

\[
x^m_L = x^m_A + i(\gamma^m)_{\alpha\beta}\theta^{0\alpha}\theta^4_4.
\]

(3.3)

We consider the relations between the differential operators of two bases

\[
\frac{\partial}{\partial x^m_L} = \frac{\partial}{\partial x^m_A}, \quad \partial^+_\alpha = \frac{1}{2}\partial^\alpha - i\theta^{0\beta}\partial^A_{\alpha\beta} - \partial^4_\alpha - \frac{i}{2}\theta^4_4\partial^A_{\alpha\beta},
\]

(3.4)

\[
\partial^+_\alpha = \frac{1}{2}\partial^\alpha + i\theta^{0\beta}\partial^A_{\alpha\beta} + \partial^4_\alpha - \frac{i}{2}\theta^4_4\partial^A_{\alpha\beta},
\]

(3.5)

\[
D^{++}_L = D^{++}, \quad D^{--}_L = D^{--}.
\]

(3.6)

The partial harmonic derivatives of different bases are identical, because the coordinate transformations (3.2)-(3.3) do not contain harmonics manifestly. The \( \mathcal{N} = 4 \) supersymmetry generators (A.22) can be rewritten in the basis (3.1)

\[
Q^i_\alpha = \partial^i_\alpha - \frac{i}{2}\theta^4_4\partial^A_{\alpha\beta}.
\]

(3.7)
and operators $Q^{(kl)}_{\alpha}$ are given by the standard formula (2.31).

The left analytic $\mathcal{N} = 4$ superfield $\Phi_L$ in the extended $\mathcal{N} = 3$ analytic basis

$$\Phi_L(x^m_L, \theta^{+\alpha}, \theta^{-\alpha}, u) = O\phi_L(x^m_A, \theta^{+\alpha}, \theta^{0\alpha}, u),$$  

$$O = \exp(-\theta^a_4 D^0_\alpha) = 1 - \theta^a_4 D^0_\alpha - \frac{1}{4}(\theta_4)^2(D^0)^2$$  

is connected by the invertible operator $O$ to the corresponding $\mathcal{N} = 3$ analytic superfield $\phi_L(\zeta)$. It is evident, that the $\mathcal{N} = 4$ and $\mathcal{N} = 3$ analytic superspaces have equal dimensions, and their differential operators are connected by the $O$-map

$$O^{-1}\partial^{\pm}_\alpha = -\partial^{\pm}_\alpha, \quad O^{-1}\partial^{\pm}_\alpha = \partial^{\pm}_\alpha, \quad O^{-1}D^L_m = \delta^L_m, \quad O^{-1}\partial^{\pm\pm}_\alpha = \partial^{\pm\pm}_\alpha,$$

$$O^{-1}x^m_L = x^m_A, \quad O^{-1}\theta^{+\alpha} = \theta^{0\alpha}, \quad O^{-1}\theta^{-\alpha} = \theta^{0\alpha} - \theta^a_4.$$  

(3.10)

The representation (3.8) has the manifest left analyticity.

The $O$-transformations of the $\mathcal{N} = 4$ supersymmetry generators and the harmonic derivatives give us

$$O^{-1}Q^{(kl)}_{\alpha} O = Q^{(kl)}_{\alpha}, \quad O^{-1}Q^4_{\alpha} O = Q^4_{\alpha} + \partial^4_{\alpha}, \quad Q^4_{\alpha} = -D^0_\alpha,$$  

$$O^{-1}D^L_{\alpha} = D^L_{\alpha} = D^{\pm\pm}_\alpha - \theta^a_4 D^{\pm\pm}_\alpha, \quad [D^L_{\alpha}, (Q^4_{\alpha} + \partial^4_{\alpha})] = 0.$$  

(3.11)

(3.12)

The $O$-transformations of the $SU_L(2) \times SU_R(2)$ generators (A.24),(A.25) can be obtained analogously

$$O^{-1}L_{kl} O\phi_L = L_{kl}\phi_L, \quad O^{-1}R_{kl} O\phi_L = R_{kl}\phi_L,$$  

(3.13)

where the analytic parts of the operators have the form

$$L_{kl} = U^{++}_{kl}\partial^{--} + U^{+0}_{kl}\theta^{0\alpha}\partial^{-\alpha} + U^{00}_{kl}(\theta^{++\alpha}\partial^{-\alpha} + \theta^{0\alpha}\partial^-_\alpha) + 2iU^{--}_{kl}\theta^{++\alpha}\theta^{0\alpha}\partial^{A}_{\beta},$$  

(3.14)

$$R_{kl} = -U^{++}_{kl}\theta^{0\alpha}\partial^{-\alpha} + U^{00}_{kl}\theta^{++\alpha}\partial^-_\alpha - U^{00}_{kl}\theta^{0\alpha}\partial^{\alpha}_\alpha + U^{++}_{kl}\theta^{0\alpha}\partial^{\alpha}_\alpha.$$  

(3.15)

The map (3.8) defines the representation of the $\mathcal{N} = 4$ superconformal group on the $\mathcal{N} = 3$ superfield $\phi_L(\zeta)$

$$K_4\Phi_L(\zeta) \to K_4\phi_L(\zeta) = O^{-1}K_4 O \phi_L,$$  

$$K_L = K_3(\zeta) + K_{4/3}(\zeta), \quad K_{4/3} = \epsilon^\alpha L^4_{\alpha} + \eta^\alpha S^4_{\alpha} + b^{kl}A_{kl}$$  

(3.16)

(3.17)

where $K_3(\zeta)$ is the analytic representation of the $\mathcal{N} = 3$ Killing operator (2.33), and the additional operators in $K_{4/3}$ have the form

$$Q^4_{\alpha} = -D^0_\alpha, \quad S^4_{\alpha} = \frac{1}{3}(\gamma^m)_{\alpha\beta}[K_m, Q^{A\beta}_{\alpha}],$$  

$$A_{kl} = L_{kl} - R_{kl} = U^{++}_{kl}(\partial^{--} + 2\theta^{0\alpha}\partial^{-\alpha}) + 2U^{00}_{kl}\theta^{0\alpha}\partial^{\alpha}_\alpha - U^{--}_{kl}\theta^{++\alpha}(\theta^{0\alpha} + 2i\theta^{0\beta}\partial^{A}_{\alpha\beta}).$$  

(3.18)

(3.19)

It is easy to check the relation

$$[A_{kl}, Q^4_{\alpha}] = -[A_{kl}, D^0_\alpha] = -Q_{(kl)\alpha}. $$  

(3.20)
We use the commutation relation

\[ [\mathcal{K}_{4/3}, D^{++}] = -\lambda_4^{++}D^0, \]
\[ \lambda_4^{++} = -\frac{i}{2}\eta_4^a\theta_\alpha^{++} + b^{kl}U_\alpha^{++} = D^{++}\lambda_4, \quad \lambda_4 = -\frac{i}{2}\eta_4^a\theta_\alpha^{0} + b^{kl}U_\alpha^{0}. \]  

(3.21)

In the \( \mathcal{N} = 3 \) representation, the \( \mathcal{N} = 4 \) superconformal transformation of the left tensor multiplet \( L^{++} \) has the form

\[ \delta_4^{++}L^{++} = 2(\lambda + \lambda_4)L^{++} - \mathcal{K}_L L^{++}. \]

(3.22)

### 3.2 Right \( \mathcal{N} = 4 \) representations

The alternative basis in the right \( \mathcal{N} = 4 \) superspace (A.30) contains \( x_m^R \) and the spinor coordinates \( \theta_\alpha^{\pm\pm} = u_\alpha^{\pm} u_\alpha^{\mp} \theta_\alpha^{(kl)} \), \( \theta_\alpha^{\pm\mp} = u_\alpha^{\mp} u_\alpha^{\pm} \theta_\alpha^{(kl)} \). We consider the relation between the right \( \mathcal{N} = 4 \) superspace and the extended \( \mathcal{N} = 3 \) basis (3.1)

\[ x_m^R = \mathcal{M} x_m^L = x_m^A - i(\gamma^m)_{\alpha\beta} \theta_\alpha^{0} \theta_\beta^{4}. \]

(3.23)

The action of the map \( \mathcal{M} \) on the left \( \mathcal{N} = 4 \) analytic superfield in the extended \( \mathcal{N} = 3 \) basis changes the sign of \( \theta_\alpha^{4} \) in the representation (3.8) and gives the formula for the right analytic superfield

\[ O^{-1}\phi^R(\zeta) = \exp(\theta_4^\alpha D^0_\alpha) = 1 + \theta_4^\alpha D^0_\alpha - \frac{1}{4}(\theta_4^\alpha)^2(D^0)^2, \]

(3.25)

where \( \phi^R(\zeta) \) is the corresponding \( \mathcal{N} = 3 \) analytic superfield.

The representation (3.24) guarantees the manifest right analyticity

\[ \partial_\alpha^{++}[O^{-1}\phi^R(\zeta)] = 0. \]

(3.26)

The mirror map of formulae (3.13) yields the right representation of the \( SU_L(2) \times SU_R(2) \) generators

\[ O\hat{L}_{k\ell}O^{-1}\phi_R = \mathcal{M}\mathcal{L}_{k\ell}\phi_L = \mathcal{R}_{k\ell}\phi_R, \]
\[ O\hat{R}_{k\ell}O^{-1}\phi_R = \mathcal{M}\mathcal{R}_{k\ell}\phi_L = \mathcal{L}_{k\ell}\phi_R, \]

(3.27)

where the analytic operators are given in (3.14) and (3.15).

The mirror map of the \( \mathcal{N} = 4 \) superconformal group has the form

\[ K_4\phi^R(\zeta_R) \to \mathcal{K}_R\phi^R(\zeta), \]
\[ \mathcal{K}_R = \mathcal{M}\mathcal{K}_L = K_3(\zeta) - K_{4/3}, \]

(3.28)

(3.29)

where \( \mathcal{K}_{4/3}(\zeta) \) includes the additional \( \mathcal{N} = 4 \) generators (3.17). Thus, the map \( \phi^R(\zeta) = \mathcal{M}\phi^L \) changes the signs in transformations with the additional \( \mathcal{N} = 4 \) generators, for instance, \( \mathcal{M}\epsilon_4^\alpha D^0_\alpha\phi_L = -\epsilon_4^\alpha D^0_\alpha\phi_R. \)
4 \( \mathcal{N} = 4 \) superconformal models in \( \mathcal{N} = 3 \) superspace

The manifestly supersymmetric \( \mathcal{N} = 4 \) models were investigated in the \( \mathcal{N} = 4 \) superspace [20]. We reformulate these models in the \( \mathcal{N} = 3 \) basis (3.1). The corresponding representation of the left Abelian \( \mathcal{N} = 4 \) gauge superfield is

\[
V_{4L}^{++} = [1 - \theta_4^0 \partial_0^0 - \frac{1}{4}(\theta_4^0)^2(D_0^0)^2]V_0^{++}(\zeta),
\]

where \( V_0^{++}(\zeta) \) is the Abelian \( \mathcal{N} = 3 \) analytic prepotential (2.38). The nonanalytic \( \mathcal{N} = 4 \) Abelian harmonic connection satisfies the equation

\[
D_{L+}^{++}V_{4L}^{--} = D_{L-}^{--}V_{4L}^{++}.
\]

We define the left representation of the fourth supersymmetry generator \( Q_4^\alpha \) on the \( \mathcal{N} = 3 \) gauge \( U(1) \) prepotential (2.38)

\[
Q_4^\alpha V_0^{++} = -D_0^0 V_0^{++}.
\]

In the gauge (2.42), the supersymmetry transformation contains an additional term with the composite parameter \( \Lambda_4(\epsilon_4) \)

\[
\delta^*(\epsilon_4)V_{WZ}^{++} = \epsilon_4^0 D_0^0 V_{WZ}^{++} - \partial^{++} \Lambda_4(\epsilon_4)
\]

and the \( Q_4^\alpha \)-transformations of the component fields have the form

\[
\begin{align*}
\delta(\epsilon_4)\phi^{kl} &= \frac{i}{2}\epsilon_4^\alpha \chi^{kl}_\alpha, \\
\delta(\epsilon_4)A_m &= \frac{i}{2}(\epsilon_4^\gamma \gamma_m \lambda), \\
\delta(\epsilon_4)\chi^{k\ell}_\alpha &= \frac{1}{2}\epsilon_4^\beta (\gamma_m)_{\alpha\beta} F^{m}, \\
\delta(\epsilon_4)\chi^{kl}_\alpha &= \epsilon_4^\beta \partial_{\alpha \beta} \phi^{kl} + \epsilon_4^\alpha X^{kl}, \\
\delta(\epsilon_4)X^{kl} &= \frac{i}{2}\epsilon_4^\alpha \partial_{\alpha \beta} \chi^{\beta kl}.
\end{align*}
\]

The \( \mathcal{N} = 4 \) Abelian superfield strength can be expressed via the superfield \( V_{4L}^{--} \)

\[
W_{4R}^{++}(V_{L}^{++}) = -\frac{1}{4}(D^{++})^2V_{4L}^{--} = [1 + \theta_4^0 D_0^0 - \frac{1}{4}(\theta_4^0)^2(D_0^0)^2]W_0^{++}(\zeta),
\]

and it can be connected with the Abelian \( \mathcal{N} = 3 \) analytic superfield strength \( W_0^{++} \) (2.39). By definition, the superfield strength of the left gauge superfield \( W_{4R}^{++} \) satisfies the right analyticity condition

\[
D_{\alpha}^{++}W_{4R}^{++} = 0, \quad D_{\alpha}^{--}W_{4R}^{++} = 0
\]

and the harmonic condition \( D_{L+}^{++}W_{4R}^{++} = D_{R-}^{++}W_{4R}^{++} = 0 \). The superconformal transformation of this right analytic superfield has the form

\[
\begin{align*}
\delta_4^\alpha W_{4R}^{++} &= 2\lambda R W_{4R}^{++} - \mathcal{K}_4 W_{4R}^{++} , \\
\lambda_R &= \lambda(\zeta) + \frac{i}{2} k_m (\theta^0 \gamma^m \theta_4) - U_{kl}^0 b^{kl} - \frac{i}{2} U_{kl}^0 \eta_{\alpha}^{(kl)} + \frac{i}{2} \left( \theta^0 \alpha_4 - \frac{1}{2} \theta^0_4 \right) n_{\alpha}^4,
\end{align*}
\]
where the parameter $\lambda(\zeta)$ is defined in (2.36).

The superfields $V^{++}_{4L}$ and $W^{++}_{4L}$ are defined in different $\mathcal{N} = 4$ superspaces, so it is convenient to use their representations $V^{++}_0$ (4.1) and $W^{++}_0$ (4.8) in the $\mathcal{N} = 3$ superspace. The corresponding supersymmetry transformation of the pseudoscalar superfield strength $W^{++}_0$ can be obtained from the $\mathcal{N} = 4$ transformations $W^{++}_{4L}$

$$Q^{A}_{\alpha}W^{++}_0(V^{++}_0) = D^{A}_{\alpha}W^{++}_0(V^{++}_0).$$

(4.12)

This transformation has a mirror form in comparison with (4.3). This formula can be checked with the help of the relation

$$\delta^*(\epsilon_4)V^{--}_0 = -\frac{1}{2}\epsilon_4^{\alpha}D^{--}D^{++}V^{--}_0 = -\epsilon_4^{\alpha}D^{0\alpha}V^{--}_0 - \frac{1}{2}\epsilon_4^{\alpha}D^{++}D^{--}V^{--}_0.$$

(4.13)

The active right $\mathcal{N} = 4$ superconformal transformation of the superfield strength

$$\delta^*_4W^{++}_0 = 2(\lambda - \lambda_4)W^{++}_0 - \mathcal{K}_R W^{++}_0$$

(4.14)

contains the operator $\mathcal{K}_R$, the $\mathcal{N} = 3$ parameter $\lambda$ (2.36) and the additional parameter $\lambda_4$ (3.21).

The analytical Abelian $\mathcal{N} = 3$ Chern-Simons action

$$\int d\zeta^{-4}V^{++}W^{++}_0 = \int d^6zd\zeta V^{++}_0V^{--}_0$$

(4.15)

is not invariant under the $\mathcal{N} = 4$ supersymmetry and the $P$-parity.

We consider the Abelian analytic pseudoscalar gauge superfield $A^{++}(\zeta)$

$$\delta_A A^{++} = -\mathcal{D}^{++}A_A, \quad PA^{++} = -A^{++}, \quad \widetilde{A}^{++} = -A^{++}.$$

(4.16)

The $WZ$-gauge for this superfield

$$A^{++}_{WZ} = 3(\theta^{++})^2U_{kl}^{--}A^{++}_A + 2(\theta^{++})^m\theta^0B_m + 2i(\theta^0)^2\theta^{++}\alpha\xi_\alpha + 3i(\theta^{++})^2\theta^0\alpha U_{kl}^{--}B_m^{kl} + 3i(\theta^{++})^2(\theta^0)^2U_{kl}^{--}Y_{kl},$$

(4.17)

includes the scalar $A^{kl}_A$, pseudoscalar $Y_{kl}$, pseudovector field $B_m$, and spinor fields $\xi_\alpha, \rho_{kl}$. 

By definition, this $\mathcal{N} = 3$ superfield transforms as the right $\mathcal{N} = 4$ supermultiplet

$$\delta^*_4 A^{++} = -\mathcal{K}_R A^{++}.$$

(4.18)

The corresponding scalar superfield strength describes the left tensor $\mathcal{N} = 4$ multiplet (3.22)

$$L^{++}_A(A^{++}) = IA^{++},$$

(4.19)

where the linear integral operator $I$ is defined in (2.39).

Now we can construct the $BF$ interaction with the coupling constant $\beta$

$$S_{BF}(V^{++}_0, A^{++}) = -i\beta\int d\zeta^{-4}V^{++}_0L^{++}_A = -i\beta\int d\zeta^{-4}d\zeta A^{++}W^{++}_0,$$

(4.20)
which is invariant under the $\mathcal{N} = 4$ superconformal transformations and the $P$-parity transformation. The equivalent interaction was defined in the $\mathcal{N} = 4$ superspace [20]. The $\mathcal{N} = 3$ $BF$ interaction can be expressed as the difference of two Abelian Chern-Simons terms

$$S_{BF} = -i\beta \int d\zeta^{-4}(V_{L}^{++}W_{L}^{++} - V_{R}^{++}W_{R}^{++}),$$
$$V_{0}^{++} = V_{L}^{++} + V_{R}^{++}, \quad A^{++} = V_{L}^{++} - V_{R}^{++}. \quad (4.21)$$

We note that each Chern-Simons term transforms non-trivially under the fourth supersymmetry

$$\delta^*(\epsilon_4)V_{L}^{++} = \epsilon_4^a D_0^a V_{R}^{++}, \quad \delta^*(\epsilon_4)V_{R}^{++} = \epsilon_4^a D_0^a V_{L}^{++}. \quad (4.22)$$

The left hypermultiplet $q^+(\zeta)$ has the $\mathcal{N} = 4$ transformation

$$\delta^*_4 q^+ = (\lambda + \lambda_4)q^+ - K_Lq^+. \quad (4.23)$$

The $\mathcal{N} = 4$ superconformal minimal interaction of the left superfields $q^+$ and $V_{0}^{++}$ has the form

$$\int d\zeta^{-4} q^+ \nabla^{++} q^+ = \int d\zeta^{-4} q^+ (D^{++} q^+ + V_{0}^{++} q^+). \quad (4.24)$$

It is not difficult to construct the nonminimal $\mathcal{N} = 4$ superconformal coupling of the superfield strength $W_{0}^{++}(V_{0}^{++})$ with the right real hypermultiplet $\Omega$

$$\int d\zeta^{-4} (D^{++} \Omega)^2 + \Omega^{-2}(W_{0}^{++})^2, \quad (4.25)$$
$$\delta^*_4 \Omega = (\lambda - \lambda_4)\Omega - K_R\Omega. \quad (4.26)$$

The interaction of the gauge superfield $A^{++}$ with two complex hypermultiplets $q^{+a}(a = 1, 2)$ was studied [17]

$$S(q, \bar{q}, A) = \int d\zeta^{-4} \bar{q}_a^+ (D^{++} + A^{++}) q^{+a}, \quad (4.27)$$
$$\tilde{q}^{+a} = \tilde{q}_a^+ = \epsilon_{ab} q^{+b}, \quad P q^{+a} = q^{+a}, \quad P(\bar{q}_a^+ q^{+a}) = -\bar{q}_a^+ q^{+a}. \quad (4.28)$$

The action of the Abelian ABJM model $S_{0_{ABJM}} = S_{BF}(V_0, A) + S(q, \bar{q}, A)$ is invariant under the three nonlinear supersymmetry transformations [17]

$$\delta_\epsilon V_{0}^{++} = \frac{2}{\beta} \epsilon^{aob} \theta_0^a q_0^+ q_b^+, \quad \delta_\epsilon A^{++} = 0, \quad (4.29)$$
$$\delta_\epsilon q^{+a} = i\epsilon^{aob} [D_0^a + \frac{1}{2} D_{++} A^{--} + \theta^{--} L^{++}_A] q_b^+, \quad \delta_\epsilon \bar{q}_a^+ = i \epsilon_{ab} [D_0^a - \frac{1}{2} D_{++} A^{--} - \theta^{--} L^{++}_A] q_b^+, \quad (4.29)$$

which form the $\mathcal{N} = 6$ supersymmetry together with the $\mathcal{N} = 3$ transformations. The algebra of these nonlinear transformations closes on the corresponding equations of motion.
The action \( S^0_{ABJM} \) is also invariant under the additional off-shell supersymmetry

\[
\delta^*(\epsilon_4)V^{++}_0 = \epsilon^a_4 D^a_\alpha V^{++}_0, \quad \delta^*(\epsilon_4)A^{++} = -\epsilon^a_4 D^a_\alpha A^{++}, \quad \delta^*(\epsilon_4)q^{+a} = -\epsilon^a_4 D^a_\alpha q^{+a}.
\] (4.30)

These transformations do not commute with the nonlinear transformations (4.29)

\[
\begin{align*}
[\delta^*(\epsilon_4), \delta_\epsilon]V^{++}_0 &= -\frac{1}{\beta} \epsilon^a_4 \epsilon^{ab}_\alpha q^{+}_a q^{+}_b, \quad [\delta^*(\epsilon_4), \delta_\epsilon]A^{++} = 0, \\
[\delta^*(\epsilon_4), \delta_\epsilon]q^{+a} &= \epsilon^a_4 \epsilon^{ab}_\alpha [-i \partial^A_{\alpha \beta} + \frac{1}{2} D^0_\alpha D^{++}_\beta A^{--} - \theta^- \theta^- D^0_\alpha L^{++}_A] q^{+}_b.
\end{align*}
\] (4.31)

The improved \( \mathcal{N} = 3 \) left linear multiplet \( w^{++} \) can be defined via the Abelian superfield

\[
W^{++}_0(V^{++}_0) = \gamma (u^+_k u^+_l C^{kl} + w^{++}), \quad C^{kl} C_{kl} = 2,
\] (4.32)

using the constant \( \gamma \) of dimension one and dimensionless constants \( C_{kl} \) describing the spontaneous symmetry breaking. We introduce the analogous improved Abelian prepotential \( v^{++} \)

\[
V^{++}_0 = 3 \gamma C^{--} (\theta^{++})^2 + \gamma v^{++}, \quad w^{++}(v^{++}) = I v^{++},
\] (4.33)

where \( C^{--} = u^-_k u^-_l C^{kl} \).

We define the improved coupling of \( V^{++}_0 \) in the \( \mathcal{N} = 3 \) superspace by analogy with the \( \mathcal{N} = 2, d = 4 \) and \( \mathcal{N} = 4, d = 3 \) cases [12, 20]

\[
S^L_0(V^{++}_0) = -\frac{1}{\gamma} \int d\zeta^{-4} \left( \frac{w^{++}}{1 + \sqrt{1 + w^{++} C^{--}}} \right)^2 = -\frac{1}{\gamma} \int dz du \frac{v^{--} v^{++}}{(1 + \sqrt{1 + w^{++} C^{--}})^2}.
\] (4.34)

This coupling is invariant under the \( \mathcal{N} = 4 \) superconformal transformations

\[
\delta^* w^{++} = 2(\lambda - \lambda_4)(w^{++} + C^{++}) - 2(\lambda^{++} - \lambda^+_4) C^0 - K_R w^{++},
\] (4.35)

which are equivalent to transformations (4.14). The \( \mathcal{N} = 2, d = 3 \) representation of the action (4.34) was studied in [22].

The variation of this action in \( v^{++} \) can be expressed via the nonlinear function \( B_C^{++}(w^{++}) \)

\[
\delta S^L_0(V^{++}_0) = -\frac{1}{\gamma} \int dz du \delta v^{--} B_C^{++}(w^{++}) = -\frac{1}{\gamma} \int d\zeta^{-4} \delta v^{++} F_C^{++}(w^{++}),
\]

\[
B_C^{++}(w^{++}) = \frac{w^{++}}{(1 + \sqrt{1 + w^{++} C^{--}}) \sqrt{1 + w^{++} C^{--}}},
\]

\[
F_C^{++}(w^{++}) = IB_C^{++}(w^{++}), \quad D^{++} F_C^{++} = 0,
\] (4.36, 4.37)

where the operator \( I \) was many times used above. The action \( S^L_0 \) gives the linear equation for the function \( B_C^{++} \).
The similar $\mathcal{N} = 4$ superconformal coupling of the right gauge superfield $A^{++}$ arises from the improved superfield $l^{++}$

$$L_A^{++}(A^{++}) = \gamma(u_i^+ u^+_l e^{kl} + l^{++}), \quad c^{kl}c_{kl} = 2, \quad (4.38)$$

$$S_0^R(A^{++}) = -\frac{1}{\gamma} \int d\zeta^{-4} \left( \frac{l^{++}}{1 + \sqrt{1 + l^{++}e^{-}}\zeta} \right)^2. \quad (4.39)$$

The combined $\mathcal{N} = 3$ action

$$S^3(V^{++}, A^{++}) = S_{BF} + S_L^0 + S_R^0 \quad (4.40)$$
describes the nontrivial superconformal interaction of two Abelian gauge superfields. The quantum properties of this model can be studied by the method of the $\mathcal{N} = 3$ supergraphs [18].

The equivalent superconformal model was earlier considered in the $\mathcal{N} = 4$ superspace [20]. The corresponding Abelian $\mathcal{N} = 4$ prepotentials were defined in different mirror superspaces. At the field-component level this model describes nonlinear couplings of two topologically massive gauge fields with spinor, scalar and pseudoscalar fields.

The action of the $\mathcal{N} = 4$ electrodynamics contains the constant $g$ of the dimension 1/2

$$S_2^E = -\frac{1}{g^2} \int d\zeta^{-4}(W^{++}_0)^2. \quad (4.41)$$

It is invariant under the $P$-parity and $\mathcal{N} = 4$ supersymmetry, but breaks the scale invariance.

We define the dimensionless uncharged analytic function of the Abelian superfield strength, which has the right $\mathcal{N} = 4$ transformation

$$K = \xi^2(D^{++})^2(D^{--})^2(D^{--})^2(W^{++}_0)^2, \quad \delta^*(\epsilon_4)K = -\epsilon_4^a D^0_\alpha K, \quad (4.42)$$

where the constant $\xi$ has the dimension $-2$. The analytic $\mathcal{N} = 3$ superfield density of the nonlinear electrodynamics action

$$S_N^E = -\frac{1}{g^2} \int d\zeta^{-4}(W^{++}_0)^2[1 + f(K)], \quad (4.43)$$

is proportional to the density of the quadratic action (4.41) and to the nonlinear function of the superfield $K$. This action has the $\mathcal{N} = 4$ supersymmetry. The component Lagrangian contains the nonlinear terms $(F_m F^n)^n, (\partial_m \phi k l \delta^m \phi k l)^n$.

## 5 Non-Abelian gauge theory

We consider the representation $V^{++}_L(\hat{\zeta}_L) = [1 - \theta_\alpha^a D^0_\alpha - \frac{1}{4}(\theta_4)^2(D^0)^2]V^{++}(\zeta)$ for the non-Abelian gauge superfields. In this case, the superconformal $\mathcal{N} = 4$ symmetry operator $K_L$ (3.17) acts linearly on the non-Abelian prepotential $V^{++}$, for instance, the fourth supersymmetry transformation has the form $Q^4_\alpha V^{++} = -D^0_\alpha V^{++}$. The corresponding
nonlinear fourth supersymmetry transformation of the non-Abelian $\mathcal{N} = 3$ connection
\[ \delta^* (\epsilon_4) V^{--} = - \epsilon_4^4 \hat{Q}_\alpha \hat{Q}_\alpha V^{--} \]
arises from the harmonic zero-curvature equation \cite{21}
\[ \hat{Q}_\alpha V^{--} = \frac{1}{2} \nabla^{--} D_\alpha^{++} V^{--} = D_\alpha^0 V^{--} - \frac{1}{2} [D_\alpha^{++} V^{--}, V^{--}] + \frac{1}{2} T_\alpha^{++} D^{--}, \]
\[ \nabla^{++} \hat{Q}_\alpha V^{--} = \nabla^{--} Q_\alpha^{++} V^{++}, \quad \nabla^{--} = D^{--} + V^{--}, \quad [\nabla^{++}, \nabla^{--}] = D^0. \] (5.1)

The operator $K_3^A (2.33),(2.34)$ acts linearly on the superfield $V^{--}$. The nonlinear action of the special conformal supersymmetry generator on $V^{--}$ arises from the commutator $\frac{1}{2} (\gamma^m)^{\alpha\beta} [K_m, \hat{Q}_\beta^4]$.

We consider the non-Abelian $\mathcal{N} = 3$ superfield strength
\[ W^{++} = -\frac{1}{4} (D^{++})^2 V^{--}. \] (5.2)

The fourth supersymmetry transformation of this superfield has the form
\[ \hat{Q}_\alpha^4 W^{++} = -\frac{1}{4} (D^{++})^2 \hat{Q}_\alpha^4 V^{--} = D_\alpha^0 W^{++} - [D_\alpha^{++} V^{--}, W^{++}], \] (5.3)

it preserves analyticity
\[ D_\beta^{++} \hat{Q}_\alpha^4 W^{++} = 0, \quad D_\beta^{++} D_\alpha^{++} V^{--} = -2 \varepsilon_{\beta\alpha} W^{++}. \] (5.4)

We note that the nonlinear terms in the transformations (5.1) and (5.4) differ by the coefficient 2.

The action of the $\mathcal{N} = 3$ Yang-Mills theory has the form
\[ S_{SYM} = \frac{1}{4g^2} \int d\zeta^{-4} (D^{++})^2 \text{Tr} V^{--} W^{++} = -\frac{1}{g^2} \int d\zeta^{-4} \text{Tr} (W^{++})^2. \] (5.5)

The gauge-invariant analytic density of the action $L^{(4)} = \text{Tr} (W^{++})^2$ transforms linearly by analogy with the Abelian quantity $(W_{0}^{++})^2 (4.14)$
\[ \delta_4^* L^{(4)} = 4 (\lambda - \lambda_4) L^{(4)} - K_R L^{(4)}, \] (5.6)

so the action $S_{SYM}$ is invariant under the fourth supersymmetry $Q_\alpha^4 L^{(4)} = D_\alpha^0 L^{(4)}$, although it breaks the conformal invariance.

Using the improved Abelian right tensor multiplet $w^{++}(V_0^{++}) (4.35)$ we can construct the right analytic density $F(w) = (1 + w^{++} C^{--})^{-3/2}$,
\[ \delta_4^* F(w) = -2 (\lambda - \lambda_4) F(w) + D^{++} A^{--} - K_R F(w), \] (5.7)

where $A^{--}$ is some analytic superfield, which is series in degrees of $w^{++} C^{--}$ and is linear in superconformal parameters $\lambda - \lambda_4$ and $\lambda^{++} - \lambda^{++}_4$. This density allows us to define the superconformal generalization of the non-Abelian gauge action
\[ S(V^{++}, V_0^{++}) = -\frac{1}{g^2} \int d\zeta^{-4} F(w) \text{Tr} (W^{++})^2, \] (5.8)
which contains also the Abelian gauge superfield $w^{++}$. To prove the superconformal symmetry we use transformations of $F(w)$ and $L^{(+4)}$, and formulas

$$-\int d\zeta^{-4}K_{R}L^{(+4)} = -2\int d\zeta^{-4}(\lambda - \lambda_{4})L^{(+4)}, \quad D^{++}L^{(+4)} = 0. \quad (5.10)$$

The superconformal generalization of the non-Abelian theory can also be constructed with the help of the right hypermultiplet $\Omega(\zeta) = \gamma C^{kl}U^{0}_{kl} + \omega(\zeta) \quad (4.26)$, then the superconformal density is $\Omega^{-2}\text{Tr}(W^{++})^{2}$. The kinetic term for the superfield $\Omega$ has a standard form. The parameters $\gamma$ and $C^{kl}$ describe the spontaneous breaking of the superconformal symmetry in this interaction.

6 Conclusions

We review the superconformal transformations in the standard and analytic harmonic superspaces with the $\mathcal{N} = 3, d = 3$ supersymmetry. The active local form of the $\mathcal{N} = 3$ transformation is defined via the Killing operator $K_{3}$ which contains the superconformal generators. The commutators of $K_{3}$ with the flat $\mathcal{N} = 3$ spinor derivatives determine matrices of the superconformal transformations. These matrices are used in the superconformal transformations of the standard $\mathcal{N} = 3$ superfields. The $\mathcal{N} = 3$ analytic superspace is convenient for the description of the hypermultiplet couplings with the gauge Chern-Simons or Yang-Mills superfields. Analogous superconformal structures were considered in the $\mathcal{N} = 4, d = 3$ superspaces [20], but this formalism has difficulties in the analysis of the left-right $\mathcal{N} = 4$ supermultiplet couplings.

We study the $\mathcal{N} = 4$ superconformal models in the framework of more flexible $\mathcal{N} = 3$ harmonic superspace. Left and right superfields from the mirror $\mathcal{N} = 4$ superspaces are connected by the operator transformations with the corresponding $\mathcal{N} = 3$ harmonic superfields in the same superspace. The representations of the additional $\mathcal{N} = 4$ superconformal generators $Q_{a}^{4}$, $S_{a}^{4}$ and $A_{kl}$ are constructed in the $\mathcal{N} = 3$ analytic superspace. The mirror map changes signs of the corresponding $\mathcal{N} = 4$ superconformal transformations in the $\mathcal{N} = 3$ superspace.

It is easy to reformulate the $\mathcal{N} = 4$ superfield models [20] in the $\mathcal{N} = 3$ superfield representation analyzing the additional supersymmetry. We prove that the gauge prepotential $V^{++}_{0}$ and its superfield strength are the mirror $\mathcal{N} = 4$ supermultiplets. The superfield Abelian $\mathcal{N} = 3 BF$ coupling $S_{BF}$ (4.20) connects the left scalar gauge prepotential with the superfield strength of the right pseudoscalar gauge prepotential. This $BF$ coupling is part of the $U(1) \times U(1)$ ABJM model in the $\mathcal{N} = 3$ superspace which has the on-shell $\mathcal{N} = 6$ supersymmetry [17]. The improved superconformal forms of the left and right tensor multiplets $S^{L}_{0}$ (4.34) and $S^{R}_{0}$ (4.39) are defined in the same $\mathcal{N} = 3$ superspace. We derive the $\mathcal{N} = 3$ representation of the superfield equations of motion for the interesting model based on the action $S_{BF} + S_{L}^{L} + S_{L}^{R}$.

We consider the $\mathcal{N} = 4$ supersymmetry transformations on the non-Abelian $\mathcal{N} = 3$ superfields and construct the superconformal coupling of these superfields with the Abelian gauge superfield. A similar coupling was considered earlier in the $\mathcal{N} = 4$ superspace [20].
Appendix

The \( \mathcal{N} = 4, d = 3 \) superspace is covariant with respect to the Lorentz group \( SO(2,1) \sim SL(2,R) \) and the automorphism group \( SU_L(2) \times SU_R(2) \). The important property of the \( \mathcal{N} = 4 \) superspace is the discrete symmetry with respect to the mirror map

\[
\mathcal{M} : \quad SU_L(2) \leftrightarrow SU_R(2) \quad \text{(A.1)}
\]

We consider the coordinates of the \( d = 3, \mathcal{N} = 4 \) superspace in the central basis \([10, 19, 20]\):

\[
z = (x^m, \theta^a_{\alpha}), \quad (A.2)
\]

where \( i \) and \( a \) are the two-component indices of the automorphism groups \( SU_L(2) \) and \( SU_R(2) \), respectively.

In this paper, we identify indices of two \( SU(2) \) groups and consider the following decomposition of the \( \mathcal{N} = 4 \) spinor coordinates:

\[
\theta^a_{\alpha} \rightarrow \theta^a_{\alpha(kl)} = \theta^a_{(kl)} + \frac{1}{2} \varepsilon_{kl} \theta^a_4, \quad (A.3)
\]

where \( \theta^a_{(kl)} \) are the \( \mathcal{N} = 3 \) CB coordinates and \( \theta^a_4 \) is an additional fourth spinor coordinate. The \( \mathcal{N} = 3 \) central basis in the \( \mathcal{N} = 4 \) superspace has the form

\[
\hat{z} = (x^m, \theta^a_{(kl)}, \theta^a_4). \quad (A.4)
\]

We define the corresponding decomposition of the partial spinor derivatives

\[
\partial^a_{\alpha} = \partial^a_{(kl)} - \varepsilon^{a}{}_{b} \partial^{b}_{\alpha}, \quad \partial^a_{\alpha} \theta^b_{\beta} = \delta^a_{\beta} \delta^b_{\gamma}, \quad \partial^a_{\alpha} \theta^b_{\gamma} = \delta^b_{\alpha}, \quad (A.5)
\]

and the \( \mathcal{N} = 4 \) spinor derivatives

\[
D^a_{\alpha} = D^a_{(kl)} - \varepsilon^{a}{}_{b} D^{b}_{\alpha}, \quad D^4_{\alpha} = \partial^4_{\alpha} + \frac{i}{2} \theta^a_{4} \partial_{\alpha \gamma}. \quad (A.6)
\]

The \( \mathcal{N} = 3 \) superspace is invariant under the mirror map

\[
\mathcal{M} \theta^a_{\alpha} = \theta^a_{i k}, \quad \mathcal{M} \theta^a_{(kl)} = \theta^a_{(kl)}, \quad \mathcal{M} \theta^a_{4} = -\theta^a_{4}. \quad (A.7)
\]

The \( \mathcal{N} = 4 \) Killing operator contains the corresponding superconformal parameters and generators

\[
K_4 = c^m P_m + l^m \hat{M}_m + b \hat{D} + k^m \hat{K}_m + \omega^{kl} L_{kl} + \Omega^{kl} R_{kl} + \epsilon^a_{(kl)} Q^a_{\alpha} + \eta^a_{(kl)} \hat{S}^a_{\alpha(kl)} + \eta^a_4 \hat{S}^a_4. \quad (A.8)
\]

Generators of the \( SU_L(2) \times SU_R(2) \) transformations \( L_{kl} = \frac{1}{2} (V_{kl} + A_{kl}) \) and \( R_{kl} = \frac{1}{2} (V_{kl} - A_{kl}) \) can be written in terms of the \( SU_V(2) \) generator \( V_{kl} \) (2.4) and the additional generator \( A_{kl} = \theta^a_{4} \partial^a_{(kl)} - 2 \theta^a_{(kl)} \partial^a_{4} \) which connects spinor coordinates \( \theta^a_{(kl)} \) and \( \theta^a_4 \). Now we can separate
the operator of the $\mathcal{N} = 3$ superconformal transformations $K_3$ and the operator of additional transformations $K_{4/3}$

$$K_4 = K_3 + K_{4/3}, \quad K_{4/3} = b^{kl} A_{kl} + \epsilon^i_\alpha q^i_\alpha + \eta^i_\alpha S^i_\alpha. \quad (A.9)$$

The $\mathcal{N} = 4$ superconformal transformations of the spinor coordinates have the form

$$\delta \theta^\alpha_{kl} = \epsilon^\alpha_{kl} + l^\alpha_{kl} - \theta^\alpha_{kl} - \omega^j_{kl} \theta^\alpha_j - \Omega^j_{k} \theta^\alpha_j - \frac{1}{2} k^\beta \chi^\alpha_{kl} - \frac{i}{4} [\Theta + \frac{1}{2} (\theta^4)^2] \eta^\alpha_{kl} = K_4 \theta^\alpha_{kl} \quad (A.10)$$

where $\Theta = \theta^\alpha_{(kl)} \theta^k_\alpha$, $\Theta + \frac{1}{2} (\theta^4/2)^2 = \eta^k_\alpha \eta^k_\alpha$.

The fourth supersymmetry generator contains $\theta^4_\alpha$ and the corresponding spinor derivative

$$Q^4_\alpha = \partial^4_\alpha - \frac{i}{2} \theta^4_\alpha \partial_\alpha. \quad (A.11)$$

The mirror map (A.7) yields the automorphism of the $\mathcal{N} = 4$ superconformal Lie superalgebra

$$\mathcal{M} A_{kl} = -A_{kl}, \quad \mathcal{M} Q^4_\alpha = -Q^4_\alpha, \quad \mathcal{M} S^4_\alpha = -S^4_\alpha, \quad \mathcal{M} K_3 = K_3. \quad (A.12)$$

The left analytic $\mathcal{N} = 4$ basis [20] uses the left $SU(2)_L/U(1)$ harmonics $u^\pm_k$ and the coordinates

$$\zeta_L = (x^m_L, \theta^\alpha_{L}, u^\pm_L), \quad \theta^\mp_\alpha = -u^\mp k \theta^\alpha_{L}, \quad \partial^\mp_\alpha \theta^\pm_\alpha = \delta^\pm_\alpha \delta^\mp_\alpha, \quad (A.13)$$

$$x^m_L = x^m - i u^i L^m, \quad \theta^\pm_\alpha = u^\pm k \theta^\alpha_{L}, \quad \theta^\mp_\alpha = u^\mp k \theta^\alpha_{L}, \quad (A.14)$$

We consider the alternative representation of the left analytic basis

$$\hat{\zeta}_L = (x^m, \theta^\alpha_{L}, \theta^\pm_\alpha, u^\pm_L), \quad \theta^\pm_\alpha = u^\pm k \theta^\alpha_{L}, \quad \theta^\mp_\alpha = u^\mp k \theta^\alpha_{L}, \quad (A.15)$$

It is easy to connect the partial derivatives in different representations, for instance,

$$\partial^+ = -u^{-} \partial^{-} - u^{+} \partial^{-}, \quad \hat{\partial}^+_L = \partial^+ + \theta^{-} \partial^+_\alpha + \theta^{+} \partial^-_\alpha. \quad (A.18)$$

The alternative representations of the $\mathcal{N} = 4$ spinor and harmonic derivatives have the form

$$D^{++}_\alpha = \partial^+_\alpha, \quad D^{--}_\alpha = -\partial^-_\alpha, \quad \text{etc.} \quad (A.19)$$

$$D^{++}_L = \partial^+ + 2 i \theta^{-} \partial^+_{\alpha L} + \theta^{+} (\partial^+_\alpha + \partial^-_\alpha) + (\theta^{-} \theta^+ + \theta^{+} \theta^-) \partial^+_\alpha, \quad (A.20)$$

$$D^0_L = \partial^0 + 2 \theta^+ \partial^-_{\alpha L} - 2 \theta^- \partial^+_{\alpha L}. \quad (A.20)$$
where we use the partial derivatives in the new coordinates

\[ \partial^\pm \pm^\mp \beta = \partial^\mp \pm^\mp \beta = \delta^\mp \beta, \quad \partial^\pm \pm^\mp \beta = 0. \] (A.21)

In the left basis, the \( \mathcal{N} = 4 \) supersymmetry generators \( Q^k_L = Q^{(kl)}_L - \varepsilon^{kl}Q^L_4 \) have the form

\begin{align*}
Q^{(kl)}_L &= U^{++kl} \partial_{\alpha}^{--} + U^{--kl} (\partial_{\alpha}^{++} - 2i \theta^{+\alpha} \partial^L_{\alpha \beta} + U^{0kl} (\partial_{\alpha}^{+-} + \partial_{\alpha}^{-+} + 2i \theta^{+\alpha} \partial^L_{\alpha \beta}), \\
Q^L_4 &= \frac{1}{2} (\partial_{\alpha}^{+-} - \partial_{\alpha}^{-+}) - i \theta^{+\alpha} \partial^L_{\alpha \beta}.
\end{align*} (A.22)

It is also easy to construct the generator of the special conformal transformations \( K_m = K^L_m + K^R_m \)

\begin{align*}
K^L_m &= -i (\theta^{++} \gamma_4 \theta^{+-}) \partial^{--} + x_m \partial^L_{\alpha} - \frac{1}{2} (x_L)^2 \partial^L_{\alpha} - i (\gamma_m)^{\alpha \beta} \theta^{+\gamma} \partial^{--} + \theta^{-\gamma} \partial^{+-} + 2 \theta^{+\gamma} \partial^{L}_{\alpha \beta}, \\
K^R_m &= \frac{1}{2} x^n_L (\gamma_m)^{\alpha \beta} (\gamma_m)_{\beta \gamma} [\theta^{+\gamma} \partial^{--} + \theta^{-\gamma} \partial^{+-} - i (\gamma_m)^{\alpha \beta} \theta^{+\gamma} (\theta^{-})^2 \partial^{--} + i (\gamma_m)^{\alpha \beta} \theta^{+\gamma} (\theta^{-})^2 \partial^{+-},
\end{align*} (A.23)

where the operator \( K^L_m \) acts on the left analytic superfields.

We consider representations of the \( SU_L(2) \times SU_R(2) \) superconformal generators in this basis

\begin{align*}
L^{kl} &= U^{++kl} \partial^{--} + U^{++kl} \partial^{+-} - U^{++kl} \partial ^{-+} - U^{++kl} \partial^{--} \\
&+ U^{0kl} (\theta^{+\alpha} \partial^{--} + \theta^{-\alpha} \partial^{+-} + \theta^{+\alpha} \partial^{--} + \theta^{-\alpha} \partial^{+-} - U^{0kl} (\theta^{+\alpha} \partial^{--} + \theta^{+\alpha} \partial^{+-} + \theta^{-\alpha} \partial^{--} + \theta^{+\alpha} \partial^{+-}) \\
&+ 2i U^{++kl} (\theta^{+\alpha} \partial^{--} + \theta^{-\alpha} \partial^{+-}).
\end{align*} (A.24)

\begin{align*}
R^{kl} &= -U^{++kl} \theta^{+\alpha} \partial^{--} + U^{0kl} (\theta^{+\alpha} \partial^{--} + \theta^{-\alpha} \partial^{+-} - U^{0kl} (\theta^{+\alpha} \partial^{--} + \theta^{+\alpha} \partial^{+-} + \theta^{-\alpha} \partial^{--} + \theta^{+\alpha} \partial^{+-}) \\
&- U^{++kl} (\theta^{+\alpha} \partial^{--} + \theta^{-\alpha} \partial^{+-} + \theta^{+\alpha} \partial^{--} + \theta^{-\alpha} \partial^{+-}).
\end{align*} (A.25)

We can study the active \( \mathcal{N} = 4 \) superconformal transformations of the left analytic superfields \( \Phi_L(\zeta_L) \), for instance,

\[ \delta^S \Phi_L = -[\omega^{kl} L_{kl} + \Omega^{kl} R_{kl} + \epsilon^{kl} Q^{kl}_L] \Phi_L \]

\[ = -[\alpha^k V_{kl} + b^k A_{kl} + \epsilon^{kl} Q^{k}_{\alpha} + \epsilon^4 Q^{(kl)}_4] \Phi_L. \] (A.26)

The superconformal operators \( R_{kl} \) do not act on the left even coordinates

\[ R_{kl} x^m_L = 0, \quad R_{kl} u^\pm_L = 0, \quad \left[[R_{kl}, D^\pm_L]ight] = 0. \] (A.27)

The \( \mathcal{N} = 4 \) Killing operator \( K_4 \) satisfies the following relations in the left basis:

\begin{align*}
[K_4, D^{++}_L] &= -\lambda^{++}_L D^0_L, \quad [K_4, D^{--}_L] = -(D^{--}_L \lambda^{++}_L) D^{--}_L, \\
\lambda^{++}_L &= (a^{kl} + b^{kl}) u^+_k u^+_l - i (\theta^{+\alpha} \gamma^m \theta^{++})(k_m \\
+ i (\theta^{+\alpha} u^+_k u^+_l - \theta^{+\alpha} u^-_k u^-_l) \eta^{(lk)}_\alpha - \frac{i}{2} \theta^{+\alpha} \eta^4_L. \] (A.28)

\begin{align*}
\lambda^{--}_L &= (a^{kl} + b^{kl}) u^-_k u^-_l - i (\theta^{+\alpha} \gamma^m \theta^{--})(k_m \\
+ i (\theta^{+\alpha} u^-_k u^-_l - \theta^{+\alpha} u^+_k u^+_l) \eta^{(lk)}_\alpha - \frac{i}{2} \theta^{+\alpha} \eta^4_L.
\end{align*} (A.29)
We obtain the coordinates and partial derivatives of the right analytic \( N = 4 \) basis \( \hat{\zeta}_R = \mathcal{M} \hat{\zeta}_L \) using the mirror map from the left basis (A.15)

\[
\hat{\zeta}_R = \mathcal{M} \hat{\zeta}_L = (x^m_R, \theta^{++\alpha}, \theta^{-+\alpha}, u^\pm_i),
\]

(A.30)

\[
x^m_R = M x^m_L = x^m + i u^+ j u^- k \varepsilon^{kn}(\gamma^m)_{\alpha\beta} \theta^\alpha_{\alpha\beta} \theta^\beta_{\alpha\beta}
\]

\(
= x^m_L + 2i (\gamma^m)_{\alpha\beta} \theta^{-+\alpha} \theta^{++\beta},
\)

(A.31)

\[
\mathcal{M} \theta^{\pm\pm\alpha} = \theta^{\pm\pm\alpha}, \quad \mathcal{M} \theta^{\mp\mp\alpha} = \theta^{\mp\mp\alpha}, \quad \mathcal{M} u^\pm_k = u^\pm_k,
\]

(A.32)

\[
\mathcal{M} \partial^L_m = \partial^R_m, \quad \mathcal{M} \partial^R_{\alpha\pm} = \partial^L_{\alpha\pm}, \quad \mathcal{M} \partial^R_{\alpha\mp} = \partial^L_{\alpha\mp}.
\]

The right basis in [20] contains the independent right harmonics \( v^a_\alpha \). We consider the active transformations of superfields and, therefore, can formally use the same harmonics and partial harmonic derivatives in the left and right bases. In the right basis, the spinor and harmonic derivatives have the form

\[
\hat{D}^{++}_\alpha = \hat{D}^{--}_\alpha = \mathcal{M} D^{++}_\alpha, \quad \hat{D}^{-+}_\alpha = -\hat{D}^{+-}_\alpha = \mathcal{M} D^{-+}_\alpha,
\]

(A.33)

\[
\hat{D}^{--}_\alpha = -\hat{D}^{+-}_\alpha = 2i \theta^{--\beta} \partial^R_{\alpha\beta} = \mathcal{M} D^{+-}_\alpha, \quad \hat{D}^{+-}_\alpha = \hat{D}^{--}_\alpha = 2i \theta^{-+\beta} \partial^R_{\alpha\beta} = \mathcal{M} D^{-+}_\alpha
\]

(A.34)

\[
D^{++}_R = \partial^{++} + 2i \theta^{++\alpha} \partial^R_{\alpha\beta} + \theta^{++\alpha} (\hat{D}^{++}_{\alpha} + \hat{D}^{-+}_{\alpha}) + (\theta^{++\alpha} + \theta^{-+\alpha}) \hat{D}^{++}_\alpha = \mathcal{M} D^{+-}_L,
\]

\[
D^{-+}_R = \partial^{-+} - 2i \theta^{-+\alpha} \partial^{-+}_{\alpha\beta} \partial^R_{\alpha\beta} + \theta^{-+\alpha} (\hat{D}^{++}_{\alpha} + \hat{D}^{-+}_{\alpha}) + (\theta^{++\alpha} + \theta^{-+\alpha}) \hat{D}^{++}_\alpha.
\]

(A.35)

The right analytic part of the generator for the special conformal transformations has the form

\[
\mathcal{K}^R_m = i (\theta^{++\gamma} \partial^R_{m\gamma} - \partial^{-+\gamma}) \partial^{-+} + x_m R x^R_R \partial^R_m - \frac{1}{2} (x^R_R)^2 \partial^R_m - \frac{i}{2} (\gamma^m)_{\alpha\gamma} \partial^{++\alpha} (\theta^{--\gamma})^2 \hat{\partial}^{-+}_\gamma
\]

\[
+ \frac{1}{2} x^R_R (\gamma^m)_{\alpha\beta} (\gamma^m)_{\gamma\delta} [\theta^{++\gamma} \hat{\partial}^{-+}_\gamma + \theta^{-+\gamma} \hat{\partial}^{+-}_\gamma].
\]

(A.36)

The right representation of the \( SU_L(2) \times SU_R(2) \) generators and the supersymmetry generators can be obtained by the mirror map of the left representation

\[
\hat{L}_{kl} = \mathcal{M} R_{kl}, \quad \hat{A}_{kl} = -\mathcal{M} A_{kl} = \mathcal{M} (R_{kl} - L_{kl}),
\]

(A.37)

\[
\hat{Q}^R_{\alpha} = \mathcal{M} Q^R_{\alpha} = \frac{1}{2} (\hat{D}^{++}_\alpha - \hat{D}^{-+}_\alpha) + i \theta^{++\alpha} \partial^R_{\alpha\gamma}.
\]

(A.38)

The right representation of the Killing operator \( \hat{K}_4 = K_3 - K_{4/3} \) satisfies the relation

\[
[\hat{K}_4, D^{++}_R] = -\lambda^{++}_R D^0 R,
\]

(A.39)

\[
\lambda^{++}_R = \mathcal{M} \lambda^{++}_L = (a^{kl} - b^{kl}) u^+ k u^+_l + i (\theta^{++\gamma} \partial^{-+\gamma} m) k^m + i (\theta^{-+\alpha} u^+_k u^+_l - \theta^{++\alpha} u^+_k u^+_l) \eta^{(lk)}_{\alpha} + \frac{i}{2} \theta^{++\alpha} \eta^4_{\alpha}.
\]

(A.40)

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