Connection between closeness of classical orbits and the factorization of radial Schrödinger equation

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Abstract

It was shown that the Runge-Lenz vector for a hydrogen atom is equivalent to the raising and lowering operators derived from the factorization of radial Schrödinger equation. Similar situation exists for an isotropic harmonic oscillator. It seems that there may exist intimate relation between the closeness of classical orbits and the factorization of radial Schrödinger equation. Some discussion was made about the factorization of a 1D Schrödinger equation.

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I. INTRODUCTION

The factorization method of Schrödinger \[1,2\] was extended to address the radial Schrödinger equation of a particle in a central potential \(V(r)\)[3-6]. It was shown\[6\] that only for two kinds of potential, the Coulomb potential and isotropic harmonic oscillator, the radial Schrödinger equation can be factorized, and directly from the factorization of
radial equation, two kinds of raising and lowering operators, \( A_\pm(l) \) and \( B_\pm(l) \), can be derived for an isotropic harmonic oscillator, but only one, \( A_\pm(l) \), for a hydrogen atom. This reminds us of the famous Bertrand theorem\(^7\) in classical mechanics, which says: “the only central forces that result in close orbits for all bound particles are the inverse square law and Hooke’s law”. It is guessed that the factorizability of radial Schrödinger equation may be intimately connected with the closeness of classical orbits.

In classical mechanics, the maximum number of functional independent conserved quantities of a closed Hamiltonian system with \( N \) degrees of freedom is \( 2N - 1 \)\(^8\). For a system with independent conserved quantities no less than \( N \) is called integrable\(^9\). An integrable classical system with \( N + \Lambda \) independent conserved quantities (\( 0 \leq \lambda \leq N - 1 \)) is called \( \Lambda \)-fold degenerate, and there exist \( \Lambda \) linear relations with integer coefficients between the \( N \) frequencies \( \omega_i (i = 1, 2, \cdots, N) \) of the system\(^10\). A classical system for \( \Lambda = N - 1 \) is called a completely degenerate system, and there remains only one independent frequency. For example, it is well-known that for a particle in a central potential \( V(r) \), apart from the Hamiltonian, the angular momentum \( l \) is also conserved, and the particle in a general central potential \( V(r) \) is 1-fold degenerate and moves in a plane, but the planar orbits are in general not closed. For a particle in an attractive Coulomb potential (\( V(r) = -\kappa/r \)), it was demonstrated\(^11\) that there exists an additional conserved quantity, the Runge-Lenz vector, \( (m = \kappa = 1) \), \( a = p \times l - r/r \), which guarantees the closeness of a Kepler orbit. In fact, the direction of \( a \) is just that of the major axis of the elliptic orbit and the magnitude of \( a \) is the eccentricity. The existence of Runge-Lenz vector implies that the Coulomb potential has a higher dynamical symmetry \( SO_4 \) than its geometric symmetry \( SO_3 \)\(^12\). However, it is seen that \( a \cdot l = 0 \), and \( a^2 = 2Hl^2 + 1 \), so the number of independent conserved quantities is 5, and the hydrogen atom is a completely degenerate system. Similar situation exists for an isotropic harmonic oscillator.
In Sect. II it will be shown that for a 2-dimensional (2D) hydrogen atom, the Runge-Lenz vector is equivalent to the raising and lowering operators derived from the factorization of radial Schrödinger equation. In Sect. III, it will be shown that, for a 3D hydrogen atom, from the Runge-Lenz vector $a$ and angular momentum $l$, one can construct three kinds of raising and lowering operators ($\Delta n = 0, \Delta l = \pm 1, \Delta m = 0, \pm 1$), which are equivalent to the raising and lowering operators derived from the factorization of radial Schrödinger equation. In Sect. IV we will address isotropic harmonic oscillators. In terms of the two kinds of raising and lowering operators one can construct the conserved quantities characterizing elliptic orbits. Finally, in Sect. V there is some discussion about the factorizability of the Schrödinger equation for a 1D system, which has been investigated extensively in supersymmetric quantum mechanics.

### II. 2D HYDROGEN ATOM

(A) Runge-Lenz vector and a simple algebraic approach to the eigenvalue problem

For a 2D hydrogen atom, the quantum version of the Runge-Lenz vector reads

$$a = \frac{1}{2}(p \times l - l \times p) - e_\rho = p \times l - ip - e_\rho, \quad (1)$$

where $p = p_x i + p_y j, l = l_z k = (xp_y - yp_x) k, \rho = \sqrt{x^2 + y^2}$. It is easily shown that

$$[l_z, a_x] = ia_y, [l_z, a_y] = -ia_x, [a_x, a_y] = -i2Hl_z. \quad (2)$$

In the subspace spanned by the bound states with eigenenergy $E (E < 0)$, $a_x$ and $a_y$ may be replaced by $A_x = \sqrt{-1/2Ea_x}, A_y = \sqrt{-1/2Ea_y}$. Let $A_z = l_z$, then

$$[A_\alpha, A_\beta] = i\varepsilon_{\alpha\beta\gamma}A_\gamma, \quad (3)$$
i.e. \( A_x, A_y \) and \( A_z \) constitute the \( SO_3 \) Lie algebra, and the eigenvalue of \( A^2 = -\frac{1}{4} - \frac{1}{2E} \) is \( m_0(m_0 + 1), m_0 = 0, 1, 2, \cdots \). Therefore, the energy eigenvalue of a 2D hydrogen atom is

\[
E = E_n = -1/2n^2, n = (m_0 + 1/2) = 1/2, 3/2, 5/2, \cdots.
\]

(4)

Alternatively, defining \( a_\pm = a_x \pm ia_y \), it is easily shown that

\[
[l_z, a_\pm] = \pm a_\pm, \tag{5}
\]

\[
a_-a_+ = \frac{H}{2}(2l_z + 1)^2 + 1, \tag{6}
\]

i.e. \( a_\pm \) are just the raising and lowering operators of the magnetic quantum number \( m \).

Assume \( |Em\rangle \) is the eigenstate of \( (H, l_z) \) with eigenvalues \( (E, m) \), then \( a_\pm|Em\rangle \) are also the eigenstates of \( H \) with energy \( E \) and the eigenstates of \( l_z \) with eigenvalues \( (m \pm 1) \).

For a given energy eigenvalue \( E_n \), the allowed \( |m| \) must have an upper limit \( m_0 \), and \( a_+|Em_0\rangle = 0 \). Hence \( a_-a_+|Em_0\rangle = 0 \). Using (6), we get \( \frac{E}{2}(2m_0 + 1)^2 = 1 \), which is just (4). The degenerate states belonging to \( E_n \) may be expressed as \( a^k|E_n m_0\rangle, k = 0, 1, \cdots, 2m_0 \), and the degeneracy is \( 2n = (2m_0 + 1) = 1, 3, 5, \cdots \).

(B) Raising and lowering operators derived from factorization

The energy eigenstate of a 2D hydrogen atom may be chosen as the simultaneous eigenstate of \( (H, l_z) \), i.e. \( \Psi(\rho, \phi) = e^{im\phi}\chi_m(\rho)/\sqrt{\rho}, m = 0, \pm 1, \pm 2, \cdots \), and \( \chi_m(\rho) \) satisfies \( H(m)\chi_m(\rho) = E\chi_m(\rho) \), or

\[
D(m)\chi_m(\rho) = \lambda_m\chi_m(\rho), \lambda_m = -2E, \tag{7}
\]

\[
D(m) = -2H(m) = \frac{d^2}{d\rho^2} - \frac{m^2 - 1/4}{\rho^2} + \frac{2}{\rho}.
\]
Directly from the factorization one may derive the raising and lowering operators \[ A_+\(m\) = \frac{d}{d\rho} - \frac{m + 1/2}{\rho} + \frac{1}{m + 1/2}, \]
\[ A_-\(m\) = \frac{d}{d\rho} + \frac{m - 1/2}{\rho} - \frac{1}{m - 1/2}, \]  
whose selection rules are \(\Delta E = 0(\Delta n = 0)\) and \(\Delta m = \pm 1\). Using \(A_\pm\(m\)\), the energy eigenvalues and eigenstates can also be easily obtained. From (7) it is seen that \(E\) depends only on the absolute value of \(m\). Using Hellmann-Feynman theorem
\[ \frac{\partial E}{\partial |m|} = \langle \frac{\partial H(m)}{\partial |m|} \rangle = \frac{|m|}{\rho^2} > 0, \]  
i.e. \(E\) increases monotonically with \(|m|\). Thus, \(m = 0\) for the ground state. On the other hand, for a given \(E, |m|\) must have an upper limit, say, \(m_0(m_0 > 0)\). Then
\[ A_+\(m_0\)\chi_{m_0}(\rho) = \left(\frac{d}{d\rho} - \frac{m_0 + 1/2}{\rho} + \frac{1}{m_0 + 1/2}\right)\chi_{m_0}(\rho) = 0. \]  
So \(\chi_{m_0}(\rho) \sim \rho^{m_0+1/2}e^{-\rho/(m_0+1/2)}\). From (10) we have \(A_-\(m_0 + 1\)A_+\(m_0\)\chi_{m_0}(\rho) = 0\), and using (7) and (8), we get
\[ [D(m_0) - 1/(m_0 + 1/2)^2]\chi_{m_0}(\rho) = [-2E_{m_0} - 1/(m_0 + 1/2)^2]\chi_{m_0}(\rho) = 0. \]  
Thus we get \(E_{m_0} = -1/2(m_0 + 1/2)^2\). Using \(A_-\(m_0\), A_-\(m_0 - 1\), \ldots\), successively operating on \(\chi_{m_0}(\rho)\), one may get all the degenerate eigenstates belonging to \(E_{m_0}, \chi_{m_0, m}(\rho), m = m_0, m_0-1, \ldots, -m_0\). Alternatively, the degenerate eigenstates belonging to \(E_n = -1/2n^2, (n = 1/2, 3/2, 5/2, \ldots)\) may be denoted by \(\chi_{n, m}(\rho), |m| = n - 1/2, n - 3/2, \ldots, 1, 0\), and it can be shown that \(\chi_{n, m}(\rho) \sim \rho^{|m|+1/2}e^{-\rho/n}F(-n, 2|m| + 1, \frac{2n}{n}), \) where \(F\) is the confluent hypergeometric function, and \(n_{\rho} = (n - 1/2) - |m| = 0, 1, 2, \ldots, (n - 1/2)\).

**(C) Equivalence of the Runge-Lenz vector and the raising and lowering operators**
In polar coordinates the Runge-Lenz vector (1) can be expressed as \( \mathbf{a}_\pm = a_x \pm ia_y \)

\[
a_\pm = e^{\pm i\phi}[(\mp \frac{\partial}{\partial \rho} + \frac{1}{\rho} l_z) l_z - \frac{1}{2} (\frac{\partial}{\partial \rho} \mp \frac{1}{\rho} l_z) - 1].
\] (12)

Operating on the eigenfunction \( e^{im\phi} R_m(\rho) \), the raising and lowering of \( m \) in the angular function are accomplished by \( e^{\pm i\phi} \), and \( a_\pm \) are equivalent to operators \( a_\pm(m) \) operating on \( R_m(\rho) \),

\[
a_\pm(m) = [(\mp m \frac{\partial}{\partial \rho} + \frac{m^2}{\rho}) - \frac{1}{2} \frac{\partial}{\partial \rho} \pm \frac{m}{2\rho} - 1] \\
= -(\frac{1}{2} \pm m) \frac{\partial}{\partial \rho} + \frac{m(m \pm 1/2)}{\rho} - 1.
\] (13)

Apart from a trivial constant factor, \( a_\pm(m) \) may be expressed as

\[
a_+(m) = \frac{\partial}{\partial \rho} - \frac{m}{\rho} + \frac{1}{m + 1/2}, \\
a_-(m) = \frac{\partial}{\partial \rho} + \frac{m}{\rho} - \frac{1}{m - 1/2}.
\] (14)

The operators \( a_\pm(m) \) operating on \( R_m(\rho) \) may be replaced by \( A_\pm(m) \) operating on \( \chi_m(\rho) = \sqrt{\rho} R_m(\rho) \)

\[
A_+(m) = \frac{\partial}{\partial \rho} - \frac{m + 1/2}{\rho} + \frac{1}{m + 1/2}, \\
A_-(m) = \frac{\partial}{\partial \rho} + \frac{m - 1/2}{\rho} - \frac{1}{m - 1/2},
\] (15)

which are just the raising and lowering operators (8) derived directly from the factorization of the Schrödinger equation.

\[ III. \ 3D \ HYDROGEN \ ATOM \]

Now we address the Runge-Lenz vector for a 3D hydrogen atom,

\[
\mathbf{a} = \mathbf{p} \times \mathbf{l} - i\mathbf{p} - \mathbf{r}/r.
\] (16)
Defining \( a_\pm = a_x \pm ia_y \), we get

\[
a_\pm = \mp (\frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y})(l_z \pm 1) \pm \frac{\partial}{\partial z} l_\pm - \frac{x \pm iy}{r}.
\]  

(17)

In the spherical coordinate system \( a_\pm \) may be expressed as

\[
a_\pm = \pm \frac{\partial}{\partial r} [\cos \theta l_\pm - \sin \theta e^{\pm i\phi} (l_z \pm 1)]
\]

\[

\mp \frac{1}{r} [\sin \theta \frac{\partial}{\partial \theta} l_\pm \pm \cos \theta l_\pm (l_z \pm 1) \mp \sin \theta e^{\pm i\phi} l_z (l_z \pm 1)]
\]

\[

- \sin \theta e^{\pm i\phi}.
\]

(18)

Similarly,

\[
a_z = p_x l_y - p_y l_x - ip_z - z/r
\]

\[
= \frac{\partial}{\partial r} \left[ \frac{1}{2} \sin \theta (e^{i\phi} l_- - e^{-i\phi} l_+) - \cos \theta \right]
\]

\[
+ \frac{1}{r} \left[ \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{2} \cos \theta (l_+ l_- + l_- l_+) - \sin \theta (e^{i\phi} l_z l_- + e^{-i\phi} l_z l_+) \right]
\]

\[
- \cos \theta.
\]

(19)

(18) and (19) operating on the simultaneous eigenfunction of \((H, l^2, l_z), \Psi_{nlm}(r, \theta, \phi) = R_{nl}(r) Y_{lm}(\theta, \phi)\), we get

\[
a_\pm \Psi_{nlm} = \pm \frac{d}{dr} R_{nl}(r) [(l + 1)d_{l, \pm m} Y_{l+1, m \pm 1} + ld_{l-1, -(\pm m + 1)} Y_{l-1, m \pm 1}]
\]

\[
+ \frac{1}{r} R_{nl}(r) [l(l + 1)d_{l, \pm m} Y_{l+1, m \pm 1} - l(l + 1)d_{l-1, -(\pm m + 1)} Y_{l-1, m \pm 1}]
\]

\[
\pm R_{nl}(r) [d_{l, \pm m} Y_{l+1, m \pm 1} - d_{l-1, -(\pm m + 1)} Y_{l-1, m \pm 1}],
\]

(20)

\[
a_z \Psi_{nlm} = -\frac{d}{dr} R_{nl}(r) [(l + 1)c_{l, m} Y_{l+1, m} - lc_{l-1, m} Y_{l-1, m}]
\]

\[
+ \frac{1}{r} R_{nl}(r) [l(l + 1)c_{l, m} Y_{l+1, m} + l(l + 1)c_{l-1, m} Y_{l-1, m}]
\]

\[
- R_{nl}(r) [c_{l, m} Y_{l+1, m} + c_{l-1, m} Y_{l-1, m}],
\]

(21)

where

\[
c_{l, m} = \sqrt{\frac{(l + 1)^2 - m^2}{(2l + 1)(2l + 3)}}, \quad d_{l, m} = \sqrt{\frac{(l + m + 1)(l + m + 2)}{(2l + 1)(2l + 3)}}.
\]

(22)
Using (20) and (21), we may get

\[ l^{-a_z}\Psi_{nlm} = \left[ \frac{d}{dr} - \frac{l}{r} + \frac{1}{l+1} \right] R_{nl}(r)(l+1)(l+m+2)c_{l,m}Y_{l+1,m} \]
\[ \quad + \left[ \frac{d}{dr} + \frac{l+1}{r} - \frac{1}{l} \right] R_{nl}(r)l(l-m-1)c_{l-1,m}Y_{l-1,m}, \] (23)

\[ l^+a_z\Psi_{nlm} = -\left[ \frac{d}{dr} + \frac{1}{l+1} \right] R_{nl}(r)(l+1)(l-m+1)d_{l,m}Y_{l+1,m+1} \]
\[ \quad + \left[ \frac{d}{dr} + \frac{l+1}{r} - \frac{1}{l} \right] R_{nl}(r)l(l+m)\theta_{l-1,-(m+1)}Y_{l-1,m+1}, \] (24)

\[ l^-a_z\Psi_{nlm} = -\left[ \frac{d}{dr} - \frac{1}{l+1} \right] R_{nl}(r)(l+1)(l-m+1)d_{l,m}Y_{l+1,m-1} \]
\[ \quad + \left[ \frac{d}{dr} + \frac{l+1}{r} - \frac{1}{l} \right] R_{nl}(r)l(l-m)\theta_{l-1,-m-1}Y_{l-1,m-1}. \] (25)

Now, we may define the operators \( S_\pm \) constructed from the Runge-Lenz vector and angular momentum operator,

\[ S_+ = l^{-a_z} + (l_z - l) a_z, \]
\[ S_- = l^{-a_z} + (l_z + l) a_z. \] (26)

It is easily verified that

\[ S_+\Psi_{nlm} = \left( \frac{d}{dr} - \frac{l}{r} + \frac{1}{l+1} \right) R_{nl}(r)(l+1)(2l+1)c_{l,m}Y_{l+1,m}(\theta, \phi) \]
\[ \propto a_+(l) R_{nl}(r)Y_{l+1,m}(\theta, \phi), \]

\[ S_-\Psi_{nlm} = \left( \frac{d}{dr} + \frac{l+1}{r} - \frac{1}{l} \right) R_{nl}(r)l(2l+1)c_{l-1,m}Y_{l-1,m}(\theta, \phi) \]
\[ \propto a_-(l) R_{nl}(r)Y_{l-1,m}(\theta, \phi), \] (27)

where

\[ a_+(l) = \left( \frac{d}{dr} - \frac{l}{r} + \frac{1}{l+1} \right), \quad a_-(l) = \left( \frac{d}{dr} + \frac{l+1}{r} - \frac{1}{l} \right), \] (28)

which are equivalent to the angular momentum raising and lowering operators \( A_\pm(l) \) derived from the factorization of radial Schrödinger equation for a 3D hydrogen atom\(^6\),

\[ A_+(l) = \left( \frac{d}{dr} - \frac{l+1}{r} + \frac{1}{l+1} \right), A_-(l) = \left( \frac{d}{dr} + \frac{l}{r} - \frac{1}{l} \right), \] (29)
which operate on the radial wavefunction $\chi_{nl}(r) = rR_{nl}(r)$. It is seen that the effect of $S_+(S_-)$ is to increase (decrease) the angular momentum $l$ by 1, but keep the energy and magnetic quantum number $m$ unchanged. Therefore, to clearly indicate the selection rules, $S_+$ and $S_-$ may be relabelled as

$$S_+ \rightarrow S(n, l \uparrow, m) = l_+ a_+ + (l_z - l + 1) a_z,$$
$$S_- \rightarrow S(n, l \downarrow, m) = l_- a_+ + (l_z + l + 2) a_z. \quad (30)$$

Similarly, using the Runge-Lenz vector and angular momentum operator, one may construct the other two kinds of raising and lowering operators

$$S(n, l \uparrow, m \uparrow) = (l_z + l - 1) a_+ - l_+ a_z,$$
$$S(n, l \downarrow, m \uparrow) = (l_z - l - 2) a_+ - l_+ a_z, \quad (31)$$
$$S(n, l \uparrow, m \downarrow) = (l_z - l + 1) a_- - l_- a_z,$n
$$S(n, l \downarrow, m \downarrow) = (l_z + l + 2) a_- - l_- a_z. \quad (32)$$

IV. ISOTROPIC HARMONIC OSCILLATORS

It is well-known that an nD isotropic harmonic oscillator has the dynamical symmetry $SU_n$. For a 3D isotropic harmonic oscillator, apart from the Hamiltonian $H$ and angular momentum $l$, there exist five additional conserved quantities which constitute a quadruple tensor

$$Q_{xy} = xy + p_x p_y, Q_{yz} = yz + p_y p_z, Q_{zx} = zx + p_z p_x,$$
$$Q_1 = \frac{1}{2}[(x^2 - y^2) + (p_x^2 - p_y^2)],$$
$$Q_0 = \frac{1}{2\sqrt{3}}[(x^2 + y^2 - 2z^2) + (p_x^2 + p_y^2 - 2p_z^2)]. \quad (33)$$
It can be shown that there exist four relations among the nine conserved quantities, so a 3D isotropic harmonic oscillator is also a completely degenerate system and moves, in general, along an elliptic orbit and the direction of the semi-axes and eccentricity are characterized by the quadruple tensor. It has been shown\textsuperscript{[6]} that the radial Schrödinger equation of an isotropic harmonic oscillator as well as a hydrogen atom can be factorized. Nevertheless, it was noted that for an isotropic harmonic oscillator, two (rather than one) kinds of raising and lowering operators can be derived from factorization, and these operators themselves are not conserved quantities (unlike the situation of the hydrogen atom). However, it can be shown that in terms of the two kinds of raising and lowering operators one can construct the conserved quantities characterizing an elliptic orbit.

For simplicity, we take a 2D isotropic harmonic oscillator as example. A 2D isotropic harmonic oscillator has the dynamical symmetry $SU_2$, which is locally isomorphic to $SO_3$ (the dynamical symmetry of a 2D hydrogen atom). For a 2D isotropic harmonic oscillator in classical mechanics, one may construct two conserved quantities, $Q_{xy} = xy + p_x p_y$ and $Q_1 = \frac{1}{2}[(x^2 - y^2) + (p_x^2 - p_y^2)]$, and the intersection angle $\gamma$ of the major axis with the $x$ axis is determined by $\tan 2\gamma = Q_{xy}/Q_1$, and the eccentricity $\propto [Q_{xy}^2 + Q_1^2]^{1/2}$.

In ref. \textsuperscript{[6]}, it was shown that from the factorization of radial Schrödinger equation for a 2D isotropic oscillator, two kinds of raising and lowering operators, $A_\pm$ and $B_\pm$, operating on the radial wave function $\chi_m(\rho)$, can be derived

$$A_\pm(m) = \frac{d}{d\rho} \mp \frac{m \pm 1/2}{\rho} \pm \rho,$$

$$B_\pm(m) = \frac{d}{d\rho} \mp \frac{m \pm 1/2}{\rho} \mp \rho,$$

which may be replaced by $a_\pm(m)$ and $b_\pm(m)$ operating on $R_m(\rho) = \chi_m(\rho)/\sqrt{\rho}$,

$$a_\pm(m) = \frac{d}{d\rho} \mp \frac{m}{\rho} \pm \rho,$$

$$b_\pm(m) = \frac{d}{d\rho} \mp \frac{m}{\rho} \mp \rho.$$
When operating on the whole wave function, $e^{im\phi}R_m(\rho)$, $a_\pm(m)$ and $b_\pm(m)$ may be replaced by $a_\pm$ and $b_\pm$,

\[
\begin{align*}
a_+ &= e^{i\phi} \left[ \frac{\partial}{\partial \rho} - \frac{1}{\rho} l_z + \rho \right], \\
a_- &= e^{-i\phi} \left[ \frac{\partial}{\partial \rho} + \frac{1}{\rho} l_z - \rho \right], \\
b_+ &= e^{i\phi} \left[ \frac{\partial}{\partial \rho} - \frac{1}{\rho} l_z - \rho \right], \\
b_- &= e^{-i\phi} \left[ \frac{\partial}{\partial \rho} + \frac{1}{\rho} l_z + \rho \right].
\end{align*}
\] (36)

It is easily shown that, similar to (5) for a 2D hydrogen atom, for a 2D isotropic oscillator we have

\[
[l_z, a_\pm] = \pm a_\pm, [l_z, b_\pm] = \pm b_\pm,
\] (37)

i.e. both $a_\pm$ and $b_\pm$ are also angular momentum raising and lowering operators, but unlike the hydrogen atom, here $a_\pm$ and $b_\pm$ are not conserved quantities,

\[
[H, a_\pm] = \mp a_\pm, [H, b_\pm] = \pm b_\pm.
\] (38)

Therefore, the operators $a_\pm$ and $b_\pm$ themselves can not be directly equivalent to the conserved quantities characterizing the elliptic orbit. However, it is easily verified that the conserved quantities $l_z$, $Q_1$ and $Q_{xy}$ can be constructed in terms of $a_\pm$ and $b_\pm$,

\[
\begin{align*}
l_z &= \frac{1}{4}(a_- a_+ - b_+ b_-), \\
Q_1 &= -\frac{1}{4}(a_- b_- + b_+ a_+), \\
Q_{xy} &= -\frac{i}{4}(a_- b_- - b_+ a_+).
\end{align*}
\] (39)

V. SOME DISCUSSION ABOUT 1D SYSTEMS

The Schrödinger’s factorization method and the concept of raising and lowering operators were extended extensively in supersymmetric quantum mechanics\cite{13-16} to treat
the Schrödinger equation for a particle in a general 1D potential $V(x)$. It was shown that for a potential $V(x)$, provided the ground bound state energy $E_0$ is finite ($E_0 \neq -\infty$) and the ground state wave function $\Psi_0(x)$ is differentiable, one can construct the corresponding raising and lowering operators, $A^+$ and $A$, and the Schrödinger equation can always be factorized. The supersymmetric partner Hamiltonian, $H_\pm = A^+ A$ and $H_+ = AA^+$, have the same energy spectra, $E_n^{(+)} = E_{n+1}^{-}$, ($n = 0, 1, 2, \cdots$), except the ground state energy of $H_-$ ($E_0^{(-)} = 0$), and the eigenstates with the same eigenvalue of $H_-$ and $H_+$ are connected with each other by $A^+$ and $A$. It was shown that this is due to the shape invariance\[17\] of $V(x)$, which may also be considered as a special kind of dynamical symmetry.

It is interesting to note that the classical orbits of all bound particles in a regular 1D potential are always closed. However, no classical close orbits exist for some singular potentials, e.g., for a 1D hydrogen atom ($e = 1$)\[17\]

$$V(x) = -\frac{1}{|x|}, (-\infty < x < +\infty).$$

On the other hand, for a quantum 1D hydrogen atom, the ground state energy $E_0 = -\infty$, and $\Psi_0(x) \sim \sqrt{\delta(x)}$ is not differentiable at the origin. Thus, it seems understandable why the Schrödinger equation of a 1D hydrogen atom can not be factorized.

As for a 1D harmonic oscillator, the energy eigenvalues and eigenstates are well known, which are quite similar to those for a 3D isotropic harmonic oscillator ($l = 0$ case). The raising and lowering operators derived from factorization are $a^+ = \frac{1}{\sqrt{2}}(x - \frac{d}{dx})$ and $a = \frac{1}{\sqrt{2}}(x + \frac{d}{dx})$, which connect the neighboring eigenstates with opposite parity ($\Delta N = 1$). However, it should be noted that the 1D harmonic oscillator potential formally corresponding to a 3D isotropic oscillator ($V(r) = r^2/2, r \geq 0$) is

$$V(x) = \begin{cases} x^2/2 & x \geq 0 \\ \infty & x < 0 \end{cases},$$

whose energy levels are $E_N = (N + 1/2)$, $N = 1, 3, 5, \cdots$. The usually adopted 1D harmonic
oscillator is \( V(x) = \frac{x^2}{2} \) \((-\infty < x < +\infty\)) with reflection symmetry, whose levels are \( E_n = n + 1/2, \) \( n = 0, 1, 2, 3, \cdots \) and the neighboring eigenstates are of opposite parity. From this, one can understand why there exist two kinds of raising and lowering operators, \( A_{\pm} \) and \( B_{\pm} \), for an \( n \)D \((n \geq 2)\) isotropic harmonic oscillator, and \( A_{\pm} \) and \( B_{\pm} \) are different in form from the operators \( a^+ \) and \( a \) for a 1D harmonic oscillator. However, one may use the product operator of \( A \) and \( B^{[6]} \), i.e. the operator \( C \) as the raising and lowering operators connecting the neighboring eigenstates with the same parity. In fact, for a 3D isotropic harmonic oscillator\(^6\)

\[
C(l = 0, N \uparrow\uparrow) = \frac{d^2}{dr^2} + r^2 - 2r \frac{d}{dr} - 1
\]

is the same form as \( 2a^+a = \frac{d^2}{dx^2} + x^2 - 2x \frac{d}{dx} - 1 \) for a 1D harmonic oscillator, and both have the selection rule \( \Delta N = 2 \).

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