Lower and Upper Bounds on the VC-Dimension of Tensor Network Models

Behnoush Khavari  
DIRO & Mila  
Université de Montréal  
behnoush.khavari@umontreal.ca

Guillaume Rabusseau  
DIRO & Mila, CIFAR AI Chair  
Université de Montréal  
grabus@iro.umontreal.ca

Abstract

Tensor network (TN) methods have been a key ingredient of advances in condensed matter physics and have recently sparked interest in the machine learning community for their ability to compactly represent very high-dimensional objects. TN methods can for example be used to efficiently learn linear models in exponentially large feature spaces [54]. In this work, we derive upper and lower bounds on the VC-dimension and pseudo-dimension of a large class of TN models for classification, regression and completion. Our upper bounds hold for linear models parameterized by arbitrary TN structures, and we derive lower bounds for common tensor decomposition models (CP, Tensor Train, Tensor Ring and Tucker) showing the tightness of our general upper bound. These results are used to derive a generalization bound which can be applied to classification with low-rank matrices as well as linear classifiers based on any of the commonly used tensor decomposition models. As a corollary of our results, we obtain a bound on the VC-dimension of the matrix product state classifier introduced in [54] as a function of the so-called bond dimension (i.e. tensor train rank), which answers an open problem listed by Cirac, Garre-Rubio and Pérez-García in [12].

1 Introduction

Tensor networks (TNs) have emerged in the quantum physics community as a mean to compactly represent wave functions of large quantum systems [43, 4, 50]. Their introduction in physics can be traced back to the work of Penrose [45] and Feynman [14]. Akin to matrix factorization, TN methods rely on factorizing a high-order tensor into small factors and have recently gained interest from the machine learning community for their ability to efficiently represent and perform operations on very high-dimensional data and high-order tensors. They have been for example successfully used for compressing models [41, 67, 40, 28, 68], developing new insights on the expressiveness of deep neural networks [13, 30] and designing novel approaches to supervised [54, 17] and unsupervised [53, 24, 37] learning. Most of these approaches leverage the fact that TN can be used to efficiently parameterize high-dimensional linear maps, which is appealing from two perspectives: it makes it possible to learn models in exponentially large feature spaces and it acts as a regularizer, controlling the capacity of the class of hypotheses considered for learning.

While the expressive power of TN models has been studied recently [16, 2], the focus has mainly been on the representation capacity of TN models, but not on their ability to generalize in the context of supervised learning tasks. In this work, we study the generalization ability of TN models by deriving lower and upper bounds on the VC-dimension and pseudo-dimension of TN models commonly used for classification, completion and regression, from which bounds on the generalization gap of TN models can be derived. Using the general framework of tensor networks, we derive a general upper bound for models parameterized by arbitrary TN structures, which applies to all commonly used tensor decomposition models [19] such as CP [26], Tucker [57] and tensor train (TT) [44], as well
as more sophisticated structures including hierarchical Tucker [18, 22], tensor ring (TR) [71] and projected entangled state pairs (PEPS) [58].

Our analysis proceeds mainly in two steps. First, we formally define the notion of TN learning model by disentangling the underlying graph structure of a TN from its parameters (the core tensors, or factors, involved in the decomposition). This allows us to define, in a conceptually simple way, the hypothesis class $\mathcal{H}_G$ corresponding to the family of linear models whose weights are represented using an arbitrary TN structure $G$. We then proceed to deriving upper bounds on the VC/pseudo-dimension and generalization error of the class $\mathcal{H}_G$. These bounds follow from a classical result from Warren [64] which was previously used to obtain generalization bounds for neural networks [3], matrix completion [52] and tensor completion [39]. The bounds we derive naturally relate the capacity of $\mathcal{H}_G$ to the underlying graph structure $G$ through the number of nodes and effective number of parameters of the TN. To assess the tightness of our general upper bound, we derive lower bounds for particular TN structures (rank-one, CP, Tucker, TT and TR). These lower bounds show that, for completion, regression and classification, our general upper bound is tight up to a log factor for rank-one, TT and TR tensors, and is tight up to a constant for matrices. Lastly, as a corollary of our results, we obtain a bound on the VC-dimension of the tensor train classifier introduced in [54], which answers one of the open problems listed by Cirac, Garre-Rubio and Pérez-García in [12].

**Related work**  Machine learning models using low-rank parametrization of the weights have been investigated (mainly from a practical perspective) for various decomposition models, including low-rank matrices [34, 47, 65], CP [1, 35, 6], Tucker [33, 15, 25, 48], tensor train [46, 9, 42, 54, 17, 51, 10, 63, 66] and PEPS [11]. From a more theoretical perspective, generalization bounds for matrix and tensor completion have been derived in [52, 39] (based on the Tucker format for the tensor case). A bound on the VC-dimension of low-rank matrix classifiers was derived in [65] and a bound on the pseudo-dimension of regression functions whose weights have low Tucker rank was given in [48] (for both these cases, we show that our results improve over these previous bounds, see Section 4.2). To the best of our knowledge the VC-dimension of tensor train classifiers has not been studied in the past, but the statistical consistency of the convex relaxation of the tensor completion problem was studied in [56, 55] for the Tucker decomposition and in [27] for the tensor train decomposition. Lastly, in [36] the authors study the complexity of learning with tree tensor networks using the notion of metric entropy and covering numbers. They provide generalization bounds which are qualitatively similar to ours, but their results only hold for TN structures whose underlying graph is a tree (thus excluding models such as CP, tensor ring and PEPS) and they do not provide lower bounds.

**Summary of contributions**  We introduce a unifying framework for TN-based learning models, which generalizes a wide range of models based on tensor factorization for completion, classification and regression. This framework allows us to consider the class $\mathcal{H}_G$ of low-rank TN models for a given arbitrary TN structure $G$ (Section 3). We provide general upper bounds on the pseudo-dimension and VC-dimension of the hypothesis class $\mathcal{H}_G$ for arbitrary TN structure $G$ for regression, classification and completion. Our results naturally relate the capacity of $\mathcal{H}_G$ to the number of parameters of the underlying TN structure $G$ (Section 4.1). From these results, we derive a generalization bound for TN-based classifiers parameterized by arbitrary TN structures (Theorem 4). We compare our results to previous bounds for specific decomposition models and show that our general upper bound is always of the same order and sometimes even improves on previous bounds (Section 4.2). We derive several lower bounds showing that our general upper bound is tight up to a log factor for particular TN structures (Section 5). A summary of the lower bounds derived in this work, as well as upper bounds implied by our general result for particular TN structures, can be found in Table 1 at the end of the paper.

2 **Preliminaries**

In this section, we present basic notions of tensor algebra and tensor networks as well as generalization bounds based on combinatorial complexity measures. We start by introducing some notations. For any integer $k$ we use $[k]$ to denote the set of integers from 1 to $k$. We use lower case bold letters for vectors (e.g. $\mathbf{v} \in \mathbb{R}^{d_1}$), upper case bold letters for matrices (e.g. $\mathbf{M} \in \mathbb{R}^{d_1 \times d_2}$) and bold calligraphic letters for higher order tensors (e.g. $\mathcal{T} \in \mathbb{R}^{d_1 \times d_2 \times \ldots \times d_k}$). The inner product of two $k$-th order tensors $\mathcal{S}, \mathcal{T} \in \mathbb{R}^{d_1 \times \ldots \times d_k}$ is defined by $\langle \mathcal{T}, \mathcal{S} \rangle = \sum_{i_1=1}^{d_1} \cdots \sum_{i_k=1}^{d_k} \mathcal{T}_{i_1 \ldots i_k} \mathcal{S}_{i_1 \ldots i_k}$. The outer product of
two vectors $\mathbf{u} \in \mathbb{R}^{d_1}$ and $\mathbf{v} \in \mathbb{R}^{d_2}$ is denoted by $\mathbf{u} \otimes \mathbf{v} \in \mathbb{R}^{d_1 \times d_2}$ with elements $(\mathbf{u} \otimes \mathbf{v})_{i,j} = \mathbf{u}_i \mathbf{v}_j$. The outer product generalizes to an arbitrary number of vectors. We use the notation $(\mathbb{R}^{d_i})^{\otimes p}$ to denote the space of $p$-th order hypercubic tensors of size $d \times d \times \cdots \times d$. We denote by $\mathcal{Y}^N$ the space of functions $f : \mathcal{X} \rightarrow \mathcal{Y}$. sign$(\cdot)$ stands for the sign function. Finally, given a graph $G = (V, E)$ and a vertex $v \in V$, we denote by $E_v = \{ e \in E \mid v \in e \}$ the set of edges incident to the vertex $v$.

### 2.1 Tensors and Tensor Networks

**Tensor networks** A tensor network $\mathcal{T} \in \mathbb{R}^{d_1 \times \cdots \times d_p}$ can simply be seen as a multidimensional array $(\mathcal{T}_{i_1, \ldots, i_p} : i_n \in [d_n], n \in [p])$. Complex operations on tensors can be intuitively represented using the graphical notation of tensor network (TN) diagrams [4, 43]. In tensor networks, a $p$-th order tensor is illustrated as a node with $p$ edges (or legs) in a graph $\frac{m}{\overline{m}} \frac{n}{\overline{n}} \frac{x}{\overline{x}}$. An edge between two nodes of a TN represents a contraction over the corresponding modes of the two tensors. Consider the following simple TN with two nodes: $\frac{m}{\overline{m}} \frac{n}{\overline{n}} \frac{x}{\overline{x}}$. The first node represents a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and the second one a vector $\mathbf{x} \in \mathbb{R}^n$. Since this TN has one dangling leg (i.e. an edge which is not connected to any other node), it represents a first order tensor, i.e. a vector. The edge between the second leg of $\mathbf{A}$ and the leg of $\mathbf{x}$ corresponds to a contraction between the second mode of $\mathbf{A}$ and the first mode of $\mathbf{x}$. Hence, the resulting TN represents the classical matrix-product, which can be seen by calculating the $i$-th component of this TN: $i \frac{A}{\overline{A}} \frac{x}{\overline{x}} = \sum_{j} A_{ij} x_j = (\mathbf{A} \mathbf{x})_i$. Other examples of TN representations of common operations on matrices and vectors can be found in Figure 1. A special case of TN is the tensor train decomposition [44] which factorizes a $n$-th order tensor $\mathcal{T}$ in the form $\frac{G_1}{\overline{G_1}} \frac{G_2}{\overline{G_2}} \cdots \frac{G_n}{\overline{G_n}}$. This corresponds to

$$
\mathcal{T}_{i_1, i_2, \ldots, i_n} = \sum_{\alpha_1=1}^{r_1} \cdots \sum_{\alpha_{n-1}=1}^{r_{n-1}} (G_1)_{\alpha_1} (G_2)_{\alpha_1, \alpha_2} \cdots (G_{n-1})_{\alpha_{n-2}, \alpha_{n-1}} (G_n)_{\alpha_{n-1}, i_n}
$$

(1)

where the tuple $(r_i)_{i=1}^{n-1}$ associated with the TT representation is called TT-rank.

**Tensor network structures** A tensor network (TN) can be fundamentally decomposed in two constituent parts: a tensor network structure, which describes its graphical structure, and a set of core tensors assigned to each node. For example, the tensor in $\mathbb{R}^{d_1 \times d_2 \times d_3 \times d_4}$ represented by the TN $\frac{G_1}{\overline{G_1}} \frac{R}{\overline{R}} \frac{G_2}{\overline{G_2}}$ is obtained by assigning the core tensors $\mathcal{T} \in \mathbb{R}^{d_1 \times d_2 \times R}$ and $\mathcal{S} \in \mathbb{R}^{R \times d_3 \times d_4}$ to the nodes of the TN structure $\frac{G_1}{\overline{G_1}} \frac{R}{\overline{R}} \frac{G_2}{\overline{G_2}}$. Formally, a tensor network structure is given by a graph $G = (V, E, \text{dim})$ where edges are labeled by integers: $V$ is the set of vertices, $E \subseteq V \cup (V \times V)$ is a set of edges containing both classical edges ($e \in V \times V$) and singleton edges ($e \in V$) and $\text{dim} : E \rightarrow \mathbb{N}$ assigns a dimension to each edge in the graph. The set of singleton edges $\delta_G = E \cap V$ corresponds to the dangling legs of a TN. Given a TN structure $G$, one obtains a tensor by assigning a core tensor $\mathcal{T}^v \in \bigotimes_{e \in E_v} \mathbb{R}^{\text{dim}(e)}$ to each vertex $v$ in the graph, where $E_v = \{ e \in E \mid v \in e \}$. The resulting tensor, denoted by $\mathcal{T}(G) \in \bigotimes_{v \in V} \mathbb{R}^{\text{dim}(e)}$, is a tensor of order $|\delta_G|$ in the tensor product space $\bigotimes_{e \in \delta_G} \mathbb{R}^{\text{dim}(e)}$. Given a tensor structure $G = (V, E, \text{dim})$, the set of all tensors that can be obtained by assigning core tensors to the vertices of $G$ is denoted by $\mathcal{T}(G) \subset \bigotimes_{e \in \delta_G} \mathbb{R}^{\text{dim}(e)}$:

$$
\mathcal{T}(G) = \{ \mathcal{T}(G, \{ \mathcal{T}^v \}_{v \in V}) : \mathcal{T}^v \in \bigotimes_{e \in E_v} \mathbb{R}^{\text{dim}(e)}, v \in V \}.
$$

(2)
As an illustration, one can check that the set of $m \times n$ matrices of rank at most $r$ is equal to $T(m \times n \times r)$. Similarly, the set of 4th order $d$-dimensional tensors of TT rank at most $r$ is equal to $T(d \times d \times d \times d \times r)$. Finally, for a given graph structure $G$, the number of parameters of any member of the family $T(G)$ in Equation (2) (which is the total number of entries of the core tensors $\{T_v\}_{v \in V}$) is given by

$$N_G = \sum_{v \in V} \prod_{e \in E_v} \dim(e) \quad (3)$$

This will be a central quantity in the generalization bounds and bounds on the VC-dimension of TN models we derive in Section 4.

Common tensor network structures In Figure 2, we show the tensor network structures associated with classical tensor decomposition models such as CP, Tucker [57] and tensor train (TT) [44], also known as matrix product state (MPS) [43, 50]. For the case of the Candecomp/Parafac (CP) decomposition [26], note that the TN structure is a hyper-graph rather than a graph. We introduced the notion of TN structure focusing on graphs for clarity of exposition in the previous paragraph, but our formalism and results can be straightforwardly extended to hyper-graph TN structures. In addition, we include the tensor ring (TR) [71] (also known as periodic MPS) and PEPS decompositions which have initially emerged in quantum physics and recently gained interest in the machine learning community (see e.g., [11, 60, 61, 69]). We also show the hierarchical Tucker decomposition initially introduced in [18, 22].
Definition 1. Let \( H \subset \{-1, +1\}^X \) be a hypothesis class. The growth function \( \Pi_H : \mathbb{N} \to \mathbb{N} \) of \( H \) is defined by
\[
\Pi_H(n) = \sup_{S = \{x_1, \ldots, x_n\} \subset X} |\{(h(x_1), \ldots, h(x_n)) \mid h \in H\}|.
\]
The VC-dimension of \( H \), \( d_{VC}(H) \), is the largest number of points \( x_1, \ldots, x_n \) shattered by \( H \), i.e., for which \(|\{(h(x_1), \ldots, h(x_n)) \mid h \in H\}| = 2^n \). In other words: \( d_{VC}(H) = \sup\{n \mid \Pi_H(n) = 2^n\} \).

For a real-valued hypothesis class \( H \subset \mathbb{R}^X \), we say that \( H \) pseudo-shatters the points \( x_1, \ldots, x_n \in X \) with thresholds \( t_1, \ldots, t_n \in \mathbb{R} \), if for every binary labeling of the points \( (s_1, \ldots, s_n) \in \{-1, +1\}^n \), there exists \( h \in H \) s.t. \( h(x_i) < t_i \) if and only if \( s_i = -1 \).

The pseudo-dimension of a real-valued hypothesis class \( H \subset \mathbb{R}^X \), \( \text{Pdim}(H) \), is the supremum over \( n \) for which there exist \( n \) points that are pseudo-shattered by \( H \) (with some thresholds).

Pseudo-dimension and VC-dimension are combinatorial measures of complexity (or capacity) which can be used to derive classical uniform generalization bounds over a hypothesis class (see, e.g., [5, 38, 3]). By definition, the pseudo-dimension is related to the notion of VC-dimension by the relation
\[
\text{Pdim}(H) = d_{VC}(\{(x, t) \mapsto \text{sign}(h(x) - t) \mid h \in H\})
\]
which holds for any \( H \subset \mathbb{R}^X \).

3 Tensor Networks for Supervised Learning

In this section, we formalize the general notion of tensor network models. We then show how it encompasses classical models such as low-rank matrix completion [7, 8, 21, 49], classification [34, 47, 65], and tensor train based models [54, 17, 51, 10, 63, 66].

3.1 Tensor Network Learning Models

Consider a classification problem where the input space \( X \) is the space of \( p \)-th order tensors \( \mathbb{R}^{d_1 \times d_2 \times \cdots \times d_p} \). One motivation for TN models is that the tensor product space \( X \) can be exponentially large, thus learning a linear model in this space is often not feasible. Indeed, the number of parameters of a linear classifier \( h : X \mapsto \text{sign}(\langle X, W \rangle) \), where \( W \in \mathbb{R}^{d_1 \times \cdots \times d_p} \) is the tensor weight parameters, grows exponentially with \( p \). TN models parameterize \( W \) as a low-rank TN, thus reducing the number of parameters needed to represent a model \( h \). Our objective is to derive generalization bounds for the class of such hypotheses parameterized by low-rank tensor networks for classification, regression and completion tasks.

Formally, let \( G = (V, E, \text{dim}) \) be a TN structure for tensors of shape \( d_1 \times \cdots \times d_p \), i.e., where the set of singleton edges \( \delta_G = E \cap V = \{v_1, \cdots, v_p\} \) and \( \text{dim}(v_i) = d_i \) for each \( i \in [p] \). We are interested in the class of models whose weight tensors are represented in the TN structure \( G \):
\[
\mathcal{H}^\text{regression}_G = \{ h : X \mapsto \langle W, X \rangle \mid W \in \mathcal{T}(G) \}
\]
\[
\mathcal{H}^\text{classify}_G = \{ h : X \mapsto \text{sign}(\langle W, X \rangle) \mid W \in \mathcal{T}(G) \}
\]
\[
\mathcal{H}^\text{completion}_G = \{ h : (i_1, \cdots, i_p) \mapsto W_{i_1 \cdots i_p} \mid W \in \mathcal{T}(G) \}
\]

In Equation (6) for the completion hypothesis class, \( p \)-th order tensors are interpreted as real-valued functions \( f : [d_1] \times \cdots \times [d_p] \to \mathbb{R} \) over the indices of the tensor. \( \mathcal{H}^\text{completion}_G \) is thus a class of functions over the indices domain, for which the notion of pseudo-dimension is well-defined. This treatment of completion as a supervised learning task was considered previously to derive generalization bounds for matrix and tensor completion [52, 39].

The benefit of TN models comes from the drastic reduction in parameters when the TN structure \( G \) is low-rank, in the sense that the number of parameters \( N_G \) is small compared to \( d_1 d_2 \cdots d_p \). In addition to allowing one to represent linear models in exponentially large spaces, this compression controls the capacity of the corresponding hypothesis class \( \mathcal{H}_G \).

3.2 Examples

To illustrate some TN models, we now present several examples of models based on common TN structures: low-rank matrices and tensor trains.


**Low-rank matrices** As discussed in Section 2.1, if we define the TN structure

\[ G_{\text{mat}}(r) = d_1 \otimes \cdots \otimes d_r, \]

then \( T(G_{\text{mat}}(r)) \) is the set of matrices in \( \mathbb{R}^{d_1 \times d_2} \) of rank at most \( r \). The hypothesis class \( H_{G_{\text{mat}}(r)}^{\text{completion}} \) then corresponds to the classical problem of low-rank matrix completion [7, 8, 21, 49]. Similarly \( H_{G_{\text{mat}}(r)}^{\text{classif}} \) corresponds to the hypothesis class of low-rank matrix classifiers. This hypothesis class was previously considered, notably to compactly represent the parameters of support vector machines for matrix inputs [34, 47, 65]. Lastly, for the regression case, \( H_{G_{\text{mat}}(r)}^{\text{regression}} \) is the set of functions \( \{ h : X \mapsto \text{Tr}(WX^T) \mid \text{rank}(W) \leq r \} \). Learning hypotheses from this class is relevant in, e.g., quantum tomography, where it is known as the low-rank trace regression problem [23, 62, 29, 32].

**Tensor train tensors** The tensor train (TT) decomposition model [44] also known as matrix product state (MPS) in the quantum physics community [43, 50], has a number of parameters that grows only linearly with the order of the tensor. This makes the TT format an appealing model for compressing the parameters of ML models [54, 42, 16, 41]. We now present the tensor train classifier model which was introduced in [54] and subsequently explored in [17]. Given a vector input \( x \in \mathbb{R}^p \), Stoudenmire and Schwab [54] propose to map \( x \) into a high-dimensional space of \( p \)-th order tensors \( X = \mathbb{R}^{d_1 \times \cdots \times d_2} \) by applying a local feature map \( \phi : \mathbb{R} \rightarrow \mathbb{R}^d \) to each component of the vector \( x \) and taking their outer product: \( \Phi(x) = \phi(x_1) \otimes \phi(x_2) \otimes \cdots \otimes \phi(x_p) \in (\mathbb{R}^d)^\otimes p \).

Instead of relying on the so-called kernel trick, Stoudenmire and Schwab propose to directly learn the parameters \( W \) of a linear model \( h : x \mapsto \text{sign}(\langle W, \Phi(x) \rangle) \) in the exponentially large feature space \( X \). The learning problem is made tractable by parameterizing \( W \) as a low-rank TT tensor (see Equation (1)). Letting

\[ G_{\text{TT}}(r_1, \cdots, r_{p-1}) = \]

\[ \]

the hypothesis class considered in [54] is \( H_{G_{\text{TT}}(r_1, \cdots, r_{p-1})}^{\text{classif}} \). In addition to the approach of [54], which was extended in [17] and [51], tensor train classifiers were also previously considered in [10, 63, 66]. Similarly, the hypothesis class \( H_{G_{\text{TT}}(r_1, \cdots, r_{p-1})}^{\text{completion}} \) corresponds to the low-rank TT completion problem [20, 46, 59].

**Other TN models** Lastly, we mention that our formalism can be applied to any tensor models having a low-rank structure, including CP, Tucker, tensor ring and PEPS. As mentioned previously, for the case of the CP decomposition, the graph \( G \) of the TN structure is in fact a hyper-graph with \( |V| = p \) nodes and \( N_G = pd^r \) parameters for a weight tensor in \( (\mathbb{R}^d)^\otimes p \) with CP rank at most \( r \). Several TN learning models using these decomposition models have been proposed previously, including [25, 48] for regression in the Tucker format, [11] for classification using the PEPS model, [35, 6] for classification with the CP decomposition and [60, 70] for tensor completion with TR.

4 **Pseudo-dimension and Generalization Bounds for Tensor Network Models**

In this section, we give a general upper bound on the VC-dimension and pseudo-dimension of hypothesis classes parameterized by arbitrary TN structures for regression, classification and completion. We then discuss corollaries of this general upper bound for common TN models including low-rank matrices and TT tensors, and compare them with existing results. Examples of particular upper bounds that can be derived from our general result can be found in Table 1.

4.1 **Upper Bounds on the VC-dimension, Pseudo-dimension and Generalization Gap**

The following theorem states one of our main results which upper bounds the VC and pseudo-dimension of models parameterized by arbitrary TN structures.

**Theorem 2.** Let \( G = (V, E, \text{dim}) \) be a tensor network structure and let \( H_G^{\text{regression}}, H_G^{\text{classif}}, H_G^{\text{completion}} \) be the corresponding hypothesis classes defined in Equations (4-6), where each model has \( N_G \) parameters (see Equation (3)).

Then, \( \text{Pdim}(H_G^{\text{regression}}), \ d_{\text{VC}}(H_G^{\text{classif}}) \) and \( \text{Pdim}(H_G^{\text{completion}}) \) are all upper bounded by \( 2N_G \log(12|V|) \).
These bounds naturally relate the capacity of the TN classes $H_G^{\text{regression}}$, $H_G^{\text{classification}}$, $H_G^{\text{completion}}$ to the number of parameters $N_G$ of the underlying TN structure $G$. Following the analysis of [52] for matrix completion and its extension to the Tucker decomposition model presented in [39], the proof of this theorem leverages Warren’s theorem which bounds the number of sign patterns a system of polynomial equations can take.

**Theorem 4.** Let

$$S = \{v \in V \mid deg(v) \leq 2\}$$

This bound improves on the one given in [65] where the VC-dimension of $H_G^{\text{classification}}$ is bounded by $r(d_1 + d_2) log(r(d_1 + d_2))$ (see Theorem 2 in [65]). For the matrix completion case, our upper bound
We now present lower bounds on the VC and pseudo-dimensions of standard TN models: rank-one, with our general upper bound (Theorem 2), the bounds from [36] cannot be applied to TN structures. We will derive lower bounds showing that the upper bounds on the VC/pseudo-dimension of Corollary 5 are tight up to the constant factor 10 for matrix completion, regression and classification.

**Tensor train** Let $G_{TT}(r) = \mathcal{T}(\mathcal{T}(\cdots \mathcal{T}(d_1) \circ \cdots \circ d_k)_{G})$, and $\mathcal{T}(G_{TT}(r))$ be the set of tensors of TT rank at most $r$. In this case, we have $|V| = p$ and $N_G = O(dpr^2)$ where $d = \max_i d_i$. For this class of hypotheses, Theorems 2 and 4 give the following result.

**Corollary 6.** $\text{Pdim}(H_{G_{TT}(r)}^{\text{regression}})$, $d_{\text{VC}}(H_{G_{TT}(r)}^{\text{classification}})$ and $\text{Pdim}(H_{G_{TT}(r)}^{\text{completion}})$ are all in $O(dpr^2 \log(p))$, where $d = \max_i d_i$. Moreover, with high probability over the choice of a sample $S$ of size $n$ drawn i.i.d. from a distribution $D$, the generalization gap $R(h) - \tilde{R}_S(h)$ of any hypothesis $h \in H_{G_{TT}(r)}^{\text{classification}}$ is in $O\left(\sqrt{dpr^2 \log(n)/n}\right)$.

This result applies for the MPS model introduced in [54] and thus answers the open problem listed as Question 13 in [12]. To the best of our knowledge, the VC-dimension of tensor train classifier models has not been studied previously and our work is the first to address this open question. The lower bounds we derive in Section 5 show that the upper bounds on the VC/pseudo-dimension of Corollary 6 are tight up to a $O(\log(p))$ factor.

**Tucker** We briefly compare our result with the ones proved in [39] for tensor completion and in [48] for tensor regression using the Tucker decomposition. For a Tucker decomposition with maximum rank $r$ for tensors of size $d_1 \times \cdots \times d_p$, with maximal dimension $d = \max_i d_i$, the number of parameters is in $O(r^p + dpr)$ and the number of vertices in the TN structure is $p + 1$. In this case, Theorems 2 and 4 show that the VC/pseudo-dimensions are in $O((r^p + dpr) \log(p))$ and the generalization gap is in $O\left(\sqrt{(r^p + dpr) \log(n)/n}\right)$ with high probability for any classifier parameterized by a low-rank Tucker tensor. It is worth observing that in contrast with the tensor train decomposition, all bounds have an exponential dependency on the tensor order $p$. In [39], the authors give an upper bound on the analogue of the growth function for tensor completion problems which is equivalent to ours. In [48], the pseudo-dimension of regression functions whose weight parameters have low Tucker rank is upper-bounded by $O\left((r^p + dpr) \log(pd^{p-1})\right)$, which is looser than our bound due to the term $d^{p-1}$ (though a similar argument to the one we use in the proof of Theorem 4 can be used to tighten the bound given in [48]).

**Tree tensor networks** Lastly, we compare our result with the ones presented in [36] where the authors study the complexity of learning with tree tensor networks using metric entropy and covering numbers. The results presented in [36] only hold for TN structures whose underlying graph $G$ is a tree. Let $G$ be a tree and $\ell$ be a loss function which is both bounded and Lipschitz. Under these assumptions, it is shown in [36] that, for any $h \in H_{G}^{\text{regression}}$, with high probability over the choice of a sample $S$ of size $n$ drawn i.i.d. from a distribution $D$, the generalization gap $R(h) - \tilde{R}_S(h)$ is in $\tilde{O}(\sqrt{N_G/n})$. Theorem 4 gives a similar upper bound in $\tilde{O}(\sqrt{N_G/n})$ on the generalization gap of low-rank tensor classifiers. However, our results hold for any TN structure $G$. Thus, in contrast with our general upper bound (Theorem 2), the bounds from [36] cannot be applied to TN structures containing cycles such as tensor ring and PEPS.

## 5 Lower Bounds

We now present lower bounds on the VC and pseudo-dimensions of standard TN models: rank-one, CP, Tucker, TT and TR.

**Theorem 7.** The VC-dimension and pseudo-dimension of the classification, regression and completion hypothesis classes defined in Equations (4-6) for the rank-one, CP, Tucker, TT and TR tensor network structures satisfy the lower bounds presented in Table 1.

These lower bounds show that the general upper bound of Theorem 2 is tight up to a $O(\log(p))$ factor for rank-one, TT and TR tensors and is tight up to a constant for low-rank matrices.
We showed that this general bound can be applied to obtain bounds on the complexity of relevant regression and classification.

Future directions include deriving tighter upper bounds and/or lower bounds for the specific TN structures. This includes tightening up our general upper bound to remove the log factor in the number of vertices of the TN structure, deriving a stronger lower bound for CP (we conjecture our lower bound can be improved by a factor \( p \) for CP), and loosening the condition under which our stronger lower bound holds for TT and TR (for TR, we conjecture that a lower bound of \( \tilde{\Omega}(pr^2d) \) holds for any \( p \geq 3 \) and \( r \leq d^k \) for some value of \( k > 1 \)). Lastly, studying other complexity measures of general TN models (e.g. Rademacher complexity and covering numbers), and deriving a structural risk minimization framework for TN models from our results are interesting future directions.

### Acknowledgements

This research is supported by the Canadian Institute for Advanced Research (CIFAR AI chair program) and the Natural Sciences and Engineering Research Council of Canada (Discovery program, RGPIN-2019-05949).
References

[1] Evrim Acar, Daniel M Dunlavy, Tamara G Kolda, and Morten Mørup. Scalable tensor factorizations for incomplete data. Chemometrics and Intelligent Laboratory Systems, 106(1):41–56, 2011.

[2] Sandesh Adhikary, Siddarth Srinivasan, Jacob Miller, Guillaume Rabusseau, and Byron Boots. Quantum tensor networks, stochastic processes, and weighted automata. In The 24th International Conference on Artificial Intelligence and Statistics, volume 130, 2021.

[3] Martin Anthony and Peter L. Bartlett. Neural network learning: Theoretical foundations. cambridge university press, 2009.

[4] Jacob Biamonte and Ville Bergholm. Tensor networks in a nutshell. arXiv preprint arXiv:1708.00006, 2017.

[5] Olivier Bousquet, Stéphane Boucheron, and Gábor Lugosi. Introduction to statistical learning theory. In Summer School on Machine Learning, pages 169–207. Springer, 2003.

[6] Deng Cai, Xiaofei He, Ji-Rong Wen, Jiawei Han, and Wei-Ying Ma. Support tensor machines for text categorization. Technical report, 2006.

[7] Emmanuel J Candès and Benjamin Recht. Exact matrix completion via convex optimization. Foundations of Computational mathematics, 9(6):717–772, 2009.

[8] Emmanuel J Candès and Terence Tao. The power of convex relaxation: Near-optimal matrix completion. IEEE Transactions on Information Theory, 56(5):2053–2080, 2010.

[9] Chuan Chen, Zhe-Bin Wu, Zi-Tai Chen, Zi-Bin Zheng, and Xiong-Jun Zhang. Auto-weighted robust low-rank tensor completion via tensor-train. Information Sciences, 567:100–115, 2021.

[10] Zhongming Chen, Kim Batselier, Johan AK Suykens, and Ngai Wong. Parallelized tensor train learning of polynomial classifiers. IEEE transactions on neural networks and learning systems, 29(10):4621–4632, 2017.

[11] Song Cheng, Lei Wang, and Pan Zhang. Supervised learning with projected entangled pair states. arXiv preprint arXiv:2009.09932, 2020.

[12] Juan Ignacio Cirac, José Garre-Rubio, and David Pérez-García. Mathematical open problems in projected entangled pair states. Revista Matemática Complutense, 32(3):579–599, 2019.

[13] Nadav Cohen, Or Sharir, and Amnon Shashua. On the expressive power of deep learning: A tensor analysis. In Proceedings of the 29th Conference on Learning Theory, volume 49, pages 698–728, 2016.

[14] Richard P Feynman. Quantum mechanical computers. Foundations of physics, 16(6):507–531, 1986.

[15] Silvia Gandy, Benjamin Recht, and Isao Yamada. Tensor completion and low-n-rank tensor recovery via convex optimization. Inverse Problems, 27(2):025010, 2011.

[16] Ivan Glasser, Ryan Sweke, Nicola Pancotti, Jens Eisert, and J Ignacio Cirac. Expressive power of tensor-network factorizations for probabilistic modeling. In Advances in Neural Information Processing Systems, pages 1496–1508, 2019.

[17] Ivan Glasser, Nicola Pancotti, and J Ignacio Cirac. From probabilistic graphical models to generalized tensor networks for supervised learning. IEEE Access, 8:68169–68182, 2020.

[18] Lars Grasedyck. Hierarchical singular value decomposition of tensors. SIAM Journal on Matrix Analysis and Applications, 31(4):2029–2054, 2010.

[19] Lars Grasedyck, Daniel Kressner, and Christine Tobler. A literature survey of low-rank tensor approximation techniques. GAMM-Mitteilungen, 36(1):53–78, 2013.

[20] Lars Grasedyck, Melanie Kluge, and Sebastian Kramer. Variants of alternating least squares tensor completion in the tensor train format. SIAM Journal on Scientific Computing, 37(5):A2424–A2450, 2015.

[21] David Gross. Recovering low-rank matrices from few coefficients in any basis. IEEE Transactions on Information Theory, 57(3):1548–1566, 2011.

[22] Wolfgang Hackbusch and Stefan Kühn. A new scheme for the tensor representation. Journal of Fourier analysis and applications, 15(5):706–722, 2009.
[23] Nima Hamidi and Mohsen Bayati. On low-rank trace regression under general sampling distribution. arXiv preprint arXiv:1904.08576, 2019.

[24] Zhao-Yu Han, Jun Wang, Heng Fan, Lei Wang, and Pan Zhang. Unsupervised generative modeling using matrix product states. Physical Review X, 8(3):031012, 2018.

[25] Zhi He, Jie Hu, and Yiwen Wang. Low-rank tensor learning for classification of hyperspectral image with limited labeled samples. Signal Processing, 145:12–25, 2018.

[26] Frank L Hitchcock. The expression of a tensor or a polyadic as a sum of products. Journal of Mathematics and Physics, 6(1-4):164–189, 1927.

[27] Masaaki Imaizumi, Takanori Maehara, and Kohei Hayashi. On tensor train rank minimization: Statistical efficiency and scalable algorithm. In Advances in Neural Information Processing Systems, pages 3930–3939, 2017.

[28] Pavel Izmailov, Alexander Novikov, and Dmitry Kropotov. Scalable gaussian processes with billions of inducing inputs via tensor train decomposition. In International Conference on Artificial Intelligence and Statistics, volume 84, 2018.

[29] Hachem Kadri, Stéphane Ayache, Riikka Huusari, Alain Rakotomamonjy, and Liva Ralaivola. Partial trace regression and low-rank kraus decomposition. In Proceedings of the 37th International Conference on Machine Learning, volume 119, 2020.

[30] Valentin Khrulkov, Alexander Novikov, and Ivan V. Oseledets. Expressive power of recurrent neural networks. In Proc. of ICLR, 2018.

[31] Tamara G Kolda and Brett W Bader. Tensor decompositions and applications. SIAM review, 51(3):455–500, 2009.

[32] Vladimir Koltchinskii and Dong Xia. Optimal estimation of low rank density matrices. J. Mach. Learn. Res., 16(53):1757–1792, 2015.

[33] Ji Liu, Przemyslaw Musialski, Peter Wonka, and Jieping Ye. Tensor completion for estimating missing values in visual data. In IEEE 12th International Conference on Computer Vision, 2009.

[34] Luo Luo, Yubo Xie, Zhihua Zhang, and Wu-Jun Li. Support matrix machines. In Proceedings of the 32nd International Conference on Machine Learning, volume 37, 2015.

[35] Konstantinos Makantasis, Anastasios D Doulamis, Nikolaos D Doulamis, and Antonis Nikitakis. Tensor-based classification models for hyperspectral data analysis. IEEE Transactions on Geoscience and Remote Sensing, 56(12):6884–6898, 2018.

[36] Bertrand Michel and Anthony Nouy. Learning with tree tensor networks: complexity estimates and model selection. arXiv preprint arXiv:2007.01165, 2020.

[37] Jacob Miller, Guillaume Rabusseau, and John Terilla. Tensor networks for probabilistic sequence modeling. In The 24th International Conference on Artificial Intelligence and Statistics, volume 130, 2021.

[38] Mehryar Mohri, Afshin Rostamizadeh, and Ameet Talwalkar. Foundations of machine learning. MIT press, 2018.

[39] Maximilian Nickel and Volker Tresp. An analysis of tensor models for learning on structured data. In Joint European Conference on Machine Learning and Knowledge Discovery in Databases, pages 272–287. Springer, 2013.

[40] Alexander Novikov, Anton Rodomanov, Anton Osokin, and Dmitry P. Vetrov. Putting mrfs on a tensor train. In Proceedings of the 31th International Conference on Machine Learning, volume 32, 2014.

[41] Alexander Novikov, Dmitry Podoprikhin, Anton Osokin, and Dmitry P. Vetrov. Tensorizing neural networks. In Advances in Neural Information Processing Systems 28, 2015.

[42] Alexander Novikov, Mikhail Trofimov, and Ivan Oseledets. Exponential machines. arXiv preprint arXiv:1605.03795, 2016.

[43] Román Orús. A practical introduction to tensor networks: Matrix product states and projected entangled pair states. Annals of Physics, 349:117–158, 2014.

[44] Ivan V Oseledets. Tensor-train decomposition. SIAM Journal on Scientific Computing, 33(5):2295–2317, 2011.
[45] Roger Penrose. Applications of negative dimensional tensors. *Combinatorial mathematics and its applications*, 1:221–244, 1971.

[46] Ho N Phien, Hoang D Tuan, Johann A Bengua, and Minh N Do. Efficient tensor completion: Low-rank tensor train. *arXiv preprint arXiv:1601.01083*, 2016.

[47] Hamed Pirsiavash, Deva Ramanan, and Charless C. Fowlkes. Bilinear classifiers for visual recognition. In *Advances in Neural Information Processing Systems*, 2009.

[48] Guillaume Rabusseau and Hachem Kadri. Low-rank regression with tensor responses. In *Advances in Neural Information Processing Systems*, 2016.

[49] Benjamin Recht, Maryam Fazel, and Pablo A Parrilo. Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. *SIAM review*, 52(3):471–501, 2010.

[50] Ulrich Schollwöck. The density-matrix renormalization group in the age of matrix product states. *Annals of physics*, 326(1):96–192, 2011.

[51] Raghavendra Selvan and Erik B Dam. Tensor networks for medical image classification. *arXiv preprint arXiv:2004.10076*, 2020.

[52] Nathan Srebro, Noga Alon, and Tommi S. Jaakkola. Generalization error bounds for collaborative prediction with low-rank matrices. In *Advances in Neural Information Processing Systems*, 2004.

[53] E. Miles Stoudenmire. Learning relevant features of data with multi-scale tensor networks. *Quantum Science and Technology*, 3(3):034003, 2018.

[54] Edwin Miles Stoudenmire and David J. Schwab. Supervised learning with tensor networks. In *Advances in Neural Information Processing Systems*, 2016.

[55] Ryota Tomioka and Taiji Suzuki. Convex tensor decomposition via structured schatten norm regularization. In *Advances in Neural Information Processing Systems*, 2013.

[56] Ryota Tomioka, Taiji Suzuki, Kohei Hayashi, and Hisashi Kashima. Statistical performance of convex tensor decomposition. In *Advances in Neural Information Processing Systems*, 2011.

[57] Ledyard R Tucker. Some mathematical notes on three-mode factor analysis. *Psychometrika*, 31(3):279–311, 1966.

[58] Frank Verstraete, Valentin Murg, and J Ignacio Cirac. Matrix product states, projected entangled pair states, and variational renormalization group methods for quantum spin systems. *Advances in Physics*, 57(2):143–224, 2008.

[59] Wenqi Wang, Vaneet Aggarwal, and Shuchin Aeron. Tensor completion by alternating minimization under the tensor train (tt) model. *arXiv preprint arXiv:1609.05587*, 2016.

[60] Wenqi Wang, Vaneet Aggarwal, and Shuchin Aeron. Efficient low rank tensor ring completion. In *IEEE International Conference on Computer Vision*, 2017.

[61] Wenqi Wang, Yifan Sun, Brian Eriksson, Wenlin Wang, and Vaneet Aggarwal. Wide compression: Tensor ring nets. In *2018 IEEE Conference on Computer Vision and Pattern Recognition*, 2018.

[62] Yazhen Wang et al. Asymptotic equivalence of quantum state tomography and noisy matrix completion. *The Annals of Statistics*, 41(5):2462–2504, 2013.

[63] Yongkang Wang, Weicheng Zhang, Zhuliang Yu, Zhenghui Gu, Hao Liu, Zhaokuan Cai, Congjun Wang, and Shihan Gao. Support vector machine based on low-rank tensor train decomposition for big data applications. In *2017 12th IEEE Conference on Industrial Electronics and Applications (ICIEA)*, pages 850–853. IEEE, 2017.

[64] Hugh E Warren. Lower bounds for approximation by nonlinear manifolds. *Transactions of the American Mathematical Society*, 133(1):167–178, 1968.

[65] Lior Wolf, Hueihan Jhuang, and Tamir Hazan. Modeling appearances with low-rank SVM. In *IEEE Computer Society Conference on Computer Vision and Pattern Recognition*, 2007.

[66] Xiaowen Xu, Qiang Wu, Shuo Wang, Ju Liu, Jiande Sun, and Andrzej Cichocki. Whole brain fmri pattern analysis based on tensor neural network. *IEEE Access*, 6:29297–29305, 2018.

[67] Yinchong Yang, Denis Krompass, and Volker Tresp. Tensor-train recurrent neural networks for video classification. In *Proceedings of the 34th International Conference on Machine Learning*, volume 70, 2017.
[68] Rose Yu, Max Guangyu Li, and Yan Liu. Tensor regression meets gaussian processes. In *International Conference on Artificial Intelligence and Statistics*, volume 84, 2018.

[69] Longhao Yuan, Jianting Cao, Xuyang Zhao, Qiang Wu, and Qibin Zhao. Higher-dimension tensor completion via low-rank tensor ring decomposition. In *2018 Asia-Pacific Signal and Information Processing Association Annual Summit and Conference (APSIPA ASC)*, pages 1071–1076. IEEE, 2018.

[70] Longhao Yuan, Chao Li, Danilo Mandic, Jianting Cao, and Qibin Zhao. Tensor ring decomposition with rank minimization on latent space: An efficient approach for tensor completion. In *Proc. of AAAI*, volume 33, pages 9151–9158, 2019.

[71] Qibin Zhao, Guoxu Zhou, Shengli Xie, Liqing Zhang, and Andrzej Cichocki. Tensor ring decomposition. *arXiv preprint arXiv:1606.05535*, 2016.
A Proofs

A.1 Upper Bound and Generalization Bound

A.1.1 Proof of Theorem 2

Theorem. Let \( G = (V, E, \dim) \) be a tensor network structure and let \( \mathcal{H}_{G}^{\text{regression}}, \mathcal{H}_{G}^{\text{classification}}, \mathcal{H}_{G}^{\text{completion}} \) be the corresponding hypothesis classes defined in Equations (4), (5), (6) where each model has \( N_G \) parameters (see Equation (3)).

Then, \( \text{Pdim}(\mathcal{H}_{G}^{\text{regression}}), d_{VC}(\mathcal{H}_{G}^{\text{classification}}) \) and \( \text{Pdim}(\mathcal{H}_{G}^{\text{completion}}) \) are all upper bounded by \( 2N_G \log(12|V|) \).

Proof. We start with the pseudo-dimension introduced in Definition 1. Consider \( n \) input tensors \( \mathcal{X}_1, \cdots, \mathcal{X}_n \) and arbitrary threshold values \( t_1, \cdots, t_n \). To upper-bound \( \text{Pdim}(\mathcal{H}_{G}^{\text{regression}}) \), it is enough to show that for any set \( S = \{ \mathcal{X}_1, \cdots, \mathcal{X}_n \} \) and threshold values \( t_1, \cdots, t_n \), the number of relative sign patterns realized by the class of functions \( \mathcal{H}_{G}^{\text{regression}} \), is bounded by a value depending only on \( n \) and tensor network structure \( G \). Formally, we define the maximal number of sign patterns as follows:

\[
f(n, G) := \sup_{\mathcal{X}_1, \cdots, \mathcal{X}_n \in S} \left\{ \left| \left( \frac{\text{sign}(h(\mathcal{X}_1) - t_1)}{\cdots} \frac{\text{sign}(h(\mathcal{X}_n) - t_n)}{h \in \mathcal{H}_{G}^{\text{regression}}} \right) \right| \right\}
\]  

(8)

Let \( G = (V, E) \) be an arbitrary TN structure. For \( h \in \mathcal{H}_{G}^{\text{regression}} \), by definition, \( h : \mathcal{X} \mapsto (\mathcal{W}, \mathcal{X}) \) for some weight tensor \( \mathcal{W} \in T(G) \). Consequently, there exists a collection of core tensors \( T \in \bigotimes_{e \in E} \mathbb{R}^{\dim(e)} \) such that \( \mathcal{W} = TN(G, \{ T \}_{v \in V}) \) (see Equation (2)) and it follows that \( h(\mathcal{X}) \) is a polynomial of degree \( |V| \) over \( N_G \) variables. The variables of the polynomial are the entries of the core tensors \( \{ T \}_{v \in V} \).

Now, given a set of input tensors \( S = \{ \mathcal{X}_1, \cdots, \mathcal{X}_n \} \), the value \( f(n, N_G) \) in Equation (8) is thus bounded by the number of sign patterns that a system of \( n \) polynomial equations (one for each input data point) of order \( |V| \) over \( N_G \) variables can take. It then follows from Warren’s theorem (Theorem 3) that

\[
f(n, G) \leq \left( \frac{4en|V|}{N_G} \right)^{N_G}.
\]

(9)

Bound on the pseudo-dimension. To extract a bound on the pseudo-dimension from the above bound on the number of relative sign patterns, we follow the line of the proof of Theorem 8.3 in [3].

First observe that by the definition of the pseudo-dimension, if \( f(n, N_G) < 2^n \) for some \( n \), then \( \text{Pdim}(\mathcal{H}_{G}^{\text{regression}}) < n \). Using the bound on \( f(n, N_G) \), we have \( f(n, N_G) \leq \left( \frac{4en|V|}{N_G} \right)^{N_G} < 2^n \) if and only if

\[
N_G \left( \log n + \log \frac{4e|V|}{N_G} \right) < n.
\]

(10)

Using the classical inequality \( \ln n \leq nb + \ln \frac{1}{b} - 1 \), or equivalently \( \log n \leq \frac{nb}{\ln 2} + \log \frac{1}{b} \), it follows that

\[
\log n \leq \frac{n}{2N_G} + \log \frac{2N_G}{e \ln 2}.
\]

Consequently, Equation (10) is implied by \( n > 2N_G \log \frac{8|V|}{\ln 2} \), which is in turn implied by \( n > 2N_G \log(12|V|) \).
We thus have shown that $\text{Pdim}(H_{\text{regression}}^{G}) \leq 2N_{G} \log(12|V|)$. Since $\text{Pdim}(H) = d_{\text{VC}}(\{(x, t) \mapsto \text{sign}(h(x) - t) \mid h \in H\})$ for any hypothesis class $H$, this upper bound implies that there exists no set of $k \geq 2N_{G} \log(12|V|)$ points that are shattered by the hypothesis class

$$\{(\mathbf{x}, t) \mapsto \text{sign}(h(\mathbf{x}) - t) \mid h \in H_{\text{regression}}^{G}\} = \{(\mathbf{x}, t) \mapsto \text{sign}(V(\mathbf{x}) - t) \mid V \in \mathcal{T}(G)\}.$$ 

In particular, no set of $k$ points with thresholds $t_{1} = \cdots = t_{k} = 0$ is shattered by $H_{\text{regression}}^{G}$, which is equivalent to no set of $k$ points being shattered by $H_{\text{classif}}^{G}$, hence $\text{Pdim}(H_{\text{classif}}^{G}) \leq 2N_{G} \log(12|V|)$.

Similarly, for the completion case we argue that the maximum number of multiples of indices shattered by the function class $H_{\text{complete}}^{G}$ is bounded by the same value as $\text{Pdim}(H_{\text{regression}}^{G})$. The *Pseudo-dimension* of $H_{\text{complete}}^{G}$ is by definition, the maximum number of indices, i.e., the maximum number of the entries of the tensor, that could be pseudo-shattered (with thresholds zero) by the class of tensors $H_{\text{complete}}^{G}$. Each component of the tensor $T_{i_{1},\ldots,i_{p}}$ can be written as the following inner product

$$T_{i_{1},\ldots,i_{p}} = \langle T, e_{i_{1}}^{(1)} \otimes e_{i_{2}}^{(2)} \otimes \cdots \otimes e_{i_{p}}^{(p)} \rangle$$

where each $e_{i}^{(j)} \in \mathbb{R}^{d_{j}}$ is the $j$th vector of the canonical basis of $\mathbb{R}^{d_{j}}$. Thus, no set of more than $2N_{G} \log(12|V|)$ indices can be shattered by $H_{\text{complete}}^{G}$, since this would mean that the corresponding set of points $e_{i_{1}}^{(1)} \otimes \cdots \otimes e_{i_{p}}^{(p)}$ would be shattered by $H_{\text{regression}}^{G}$. Hence, $\text{Pdim}(H_{\text{complete}}^{G}) \leq 2N_{G} \log(12|V|)$.

**A.1.2 Proof of Theorem 4**

**Theorem.** Let $S$ be a sample of size $n$ drawn from a distribution $D$. Then, for any $\delta > 0$, with probability at least $1 - \delta$ over the choice of $S$, we have for any $h \in H_{\text{classif}}^{G}$

$$R(h) < \hat{R}(S) + 2 \sqrt{\frac{2}{n} \left( N_{G} \log \frac{8\text{en}|V|}{N_{G}} + \log \frac{4}{\delta} \right)}. \quad (11)$$

**Proof.** We use a symmetrization lemma and a corollary of Hoeffding’s inequality. For this part, let $H = H_{\text{classif}}^{G}$.

**Lemma 8.** *(Symmetrization Lemma) [5, Lemma 2]* Let $S$ and $S'$ be two random samples of size $n$ drawn from a distribution $D$. Then for any $t > 0$ such that $nt^{2} \geq 2$, we have

$$\mathbb{P}_{S \sim D} \left[ \sup_{h \in H} \left( R(h) - \hat{R}(S) \right) \geq t \right] \leq 2 \mathbb{P}_{S,S' \sim D} \left[ \sup_{h \in H} \left( \hat{R}(S') - \hat{R}(S) \right) \geq \frac{t}{2} \right], \quad (12)$$

**Corollary 9.** If $Z_{1},\ldots,Z_{n},Z'_{1},\ldots,Z'_{n}$ are $2n$ i.i.d. random variables with values in $[0,1]$, then for all $\varepsilon > 0$ we have

$$\mathbb{P} \left[ \frac{1}{n} \sum_{i=1}^{n} Z_{i} - \frac{1}{n} \sum_{i=1}^{n} Z'_{i} > \varepsilon \right] \leq 2 \exp \left( -\frac{n\varepsilon^{2}}{2} \right). \quad (13)$$

This statement is proved by rewriting $\mathbb{P}[\frac{1}{n} \sum_{i=1}^{n} Z_{i} - \frac{1}{n} \sum_{i=1}^{n} Z'_{i} > \varepsilon]$ as $\mathbb{P}[\frac{1}{n} \sum_{i=1}^{n} (Z_{i} - \mathbb{E}[Z_{i}]) - \frac{1}{n} \sum_{i=1}^{n} (Z'_{i} - \mathbb{E}[Z'_{i}]) > \varepsilon] \leq \mathbb{P}[\frac{1}{n} \sum_{i=1}^{n} (Z_{i} - \mathbb{E}[Z_{i}]) > \frac{\varepsilon}{2}] + \mathbb{P}[\frac{1}{n} \sum_{i=1}^{n} (Z'_{i} - \mathbb{E}[Z'_{i}]) > \frac{\varepsilon}{2}]$. Then by using Hoeffding’s inequality, Equation (13) is proved.

From Lemma 8 we have

$$\mathbb{P}_{S \sim D} \left[ \sup_{h \in H} \left( R(h) - \hat{R}(S) \right) \geq 2\varepsilon \right] \leq 2 \mathbb{P}_{S \sim D, S' \sim D} \left[ \max_{h \in H_{S,S'}} (\hat{R}(S') - \hat{R}(S)) \geq \varepsilon \right] \leq 2 \mathbb{P}_{S,S' \sim D} \left[ \exists h \in H_{S,S'} \mid (\hat{R}(S') - \hat{R}(S)) \geq \varepsilon \right], \quad (14)$$

where $H_{S,S'}$ is the projection of the hypothesis class $H$ onto the subset $S \cup S'$. Then, by applying the union bound followed by Corollary 9 (by taking the bounded loss as the random variable $Z$) and
recalling the notion of the growth function from Definition 1, we get
\[
\mathbb{P}_{S \sim D} \left[ \sup_{h \in \mathcal{H}} (R(h) - \hat{R}_S(h)) \geq 2\varepsilon \right] \leq 2\Pi_{\mathcal{H}}(2n) \left( \mathbb{P}_{S,S' \sim D} \left[ \hat{R}_{S'}(h) - \hat{R}_S(h) \geq \varepsilon \right] \right)
\leq 4 \Pi_{\mathcal{H}}(2n) \exp \left( -\frac{n\varepsilon^2}{2} \right),
\] (15)

In order to upper bound the growth function of the hypothesis class \( \mathcal{H}_G^{\text{class}} \) we can use the same argument as we did for the pseudo-dimension, which results in an upper bound similar to the one for the number of relative sign patterns in Equation (9)
\[
\Pi_{\mathcal{H}_G^{\text{class}}}(n) \leq \left( \frac{4en|V|}{N_G} \right)^{N_G}
\] (16)

Combining Equations (15) and (16), we get
\[
\mathbb{P}_{S \sim D} \left[ \sup_{h \in \mathcal{H}} (R(h) - \hat{R}_S(h)) \geq \varepsilon \right] \leq 4 \left( \frac{8en|V|}{N_G} \right)^{N_G} e^{-\frac{n\varepsilon^2}{2}}
\] (17)

Equation (11) then directly follows from setting the failure probability equal to \( \delta \) and solving for \( \varepsilon \).

\[ \square \]

A.2 Proof of Theorem 7

In this section, we give the proofs of all the lower bounds appearing in Table 1. All the proofs rely on the following lemma which gives a useful way for jointly deriving lower bounds on the pseudo-dimension and VC-dimension of the hypothesis classes of linear models for regression, completion and classification defined in Eq. (4-6).

**Lemma 10.** Let \( V \subset \mathbb{R}^d \) and define the hypothesis classes
\[
\mathcal{H}^{\text{completion}} = \{ h : i \mapsto w_i \mid w \in V \}
\]
\[
\mathcal{H}^{\text{regression}} = \{ h : x \mapsto \langle w, x \rangle \mid w \in V \}
\]
\[
\mathcal{H}^{\text{class}} = \{ h : x \mapsto \text{sign}(\langle w, x \rangle) \mid w \in V \}.
\]

If there exist \( k \) indices \( i_1, \ldots, i_k \in [d] \) that are shattered by \( V \), i.e., such that
\[
|\{(\text{sign}(w_{i_1}), \text{sign}(w_{i_2}), \ldots, \text{sign}(w_{i_k})) \mid w \in V \}| = 2^k,
\]
then \( d_{\text{VC}}(\mathcal{H}^{\text{class}}), \text{Pdim}(\mathcal{H}^{\text{regression}}) \) and \( \text{Pdim}(\mathcal{H}^{\text{completion}}) \) are all lower bounded by \( k \).

**Proof.** Let \( e_1, \ldots, e_d \) be the canonical basis of \( \mathbb{R}^d \) and let \( i_1, \ldots, i_k \in [d] \) be a set of indices shattered by \( V \). Since \( \langle w, e_i \rangle = w_i \) for all \( i \in [d] \), the points \( e_{i_1}, \ldots, e_{i_k} \) are shattered by \( \mathcal{H}^{\text{class}} \) and thus \( d_{\text{VC}}(\mathcal{H}^{\text{class}}) \geq k \).

Similarly, since \( \text{Pdim}(\mathcal{H}) = d_{\text{VC}}(\{(x,t) \mapsto \text{sign}(h(x) - t) \mid h \in \mathcal{H}) \}) \) for any hypothesis class \( \mathcal{H} \), the set of points \( e_{i_1}, \ldots, e_{i_k} \) with thresholds \( t_1 = t_2 = \cdots = t_k = 0 \) is shattered by the hypothesis class \( \{(x,t) \mapsto \text{sign}(\langle w, x \rangle - t) \mid w \in V \} \), and thus \( \text{Pdim}(\mathcal{H}^{\text{regression}}) \geq k \).

Lastly, the set of indices \( i_1, \ldots, i_k \) with thresholds \( t_1 = t_2 = \cdots = t_k = 0 \) is shattered by the class \( \{(i,t) \mapsto \text{sign}(w_i - t) \mid w \in V \} \), and thus \( \text{Pdim}(\mathcal{H}^{\text{completion}}) \geq k \).

\[ \square \]

A.2.1 Rank One Tensors

**Theorem 11.** Let \( G_{\text{rank-one}} = \{ \underbrace{i_1 \otimes \cdots \otimes i_1}_{d} \} \) be the tensor network structure corresponding to \( p \)-th order rank one tensors, i.e., \( T(G_{\text{rank-one}}) = \{ u_1 \otimes u_2 \otimes \cdots \otimes u_p \mid u_1, u_2, \ldots, u_p \in \mathbb{R}^d \} \).

The VC-dimension and pseudo-dimensions \( d_{\text{VC}}(\mathcal{H}^{\text{class}}_{G_{\text{rank-one}}}), \text{Pdim}(\mathcal{H}^{\text{regression}}_{G_{\text{rank-one}}}), \text{Pdim}(\mathcal{H}^{\text{completion}}_{G_{\text{rank-one}}}) \) are all lower bounded by \( (d - 1)p \).
Proof. We show that the set of indices
\[ S = \left\{ (d, \ldots, d, j, d, \ldots, d) \mid i \in [p], j \in [d-1] \right\} \]
is shattered by \( \mathcal{T}(G_{\text{rank-one}}) \), the result then follows from Lemma 10. More precisely, we show that \( S \) is shattered by the set of rank one tensors
\[ A = \left\{ \left( \frac{v_1}{1} \right) \otimes \left( \frac{v_2}{1} \right) \otimes \cdots \otimes \left( \frac{v_p}{1} \right) \mid v_1, v_2, \ldots, v_p \in \mathbb{R}^{d-1} \right\} \subset \mathcal{T}(G_{\text{rank-one}}). \]

Indeed, for any multi-index \((d, \ldots, d, j, d, \ldots, d)\) \( S \) and any rank one tensor \( \mathcal{T} = \left( \frac{v_1}{1} \right) \otimes \left( \frac{v_2}{1} \right) \otimes \cdots \otimes \left( \frac{v_p}{1} \right) \) \( A \), we have
\[ \mathcal{T}_{d, \ldots, d, j, d, \ldots, d} = \left( \frac{v_1}{1} \right) \otimes \left( \frac{v_2}{1} \right) \otimes \cdots \otimes \left( \frac{v_p}{1} \right) = (v_i)_j. \]
It follows that the \((d-1)p\) components \( \mathcal{T}_{i_1, \ldots, i_k} \) for \( \mathcal{T} \in A \) and \((i_1, \ldots, i_k) \in S \) can take any arbitrary values (the entries of the vectors \( v_1, v_2, \ldots, v_p \)) \( \mathbb{R}^{d-1} \) and thus that \( S \) is shattered by \( A \). The result then directly follows from Lemma 10.

\( \Box \)

A.2.2 Tensor Train and Tensor Ring

Theorem 12. Let \( r \leq d^{\frac{k-1}{k-1}} \), let \( G_{TT}(r) = \) \( \) be the tensor network structure corresponding to \( p \) th order tensors of tensor train rank at most \( r \), and let \( G_{TR}(r) = \) \( \) be the tensor network structure corresponding to \( p \) th order tensors of tensor ring rank at most \( r \).

Then, the VC-dimension and pseudo-dimensions \( d_{VC}(\mathcal{H}_{G_{TT}(r)}^{\text{classif}}), d_{VC}(\mathcal{H}_{G_{TT}(r)}^{\text{regression}}), \text{Pdim}(\mathcal{H}_{G_{TT}(r)}^{\text{regression}}), \text{Pdim}(\mathcal{H}_{G_{TT}(r)}^{\text{completion}}) \) and \( \text{Pdim}(\mathcal{H}_{G_{TR}(r)}^{\text{completion}}) \) are all lower bounded by \( r^2d \).

Moreover, in the particular case where \( r = d \) and \( p = 3k \) for some \( k \in \mathbb{N} \), the VC-dimension and pseudo-dimensions \( d_{VC}(\mathcal{H}_{G_{TT}(r)}^{\text{classif}}), d_{VC}(\mathcal{H}_{G_{TR}(r)}^{\text{classif}}), \text{Pdim}(\mathcal{H}_{G_{TT}(r)}^{\text{completion}}), \text{Pdim}(\mathcal{H}_{G_{TR}(r)}^{\text{completion}}) \) are all lower bounded by \( \frac{p(r^2d-1)}{3} \).

Proof. We start with the tensor train case, the tensor ring case will be handled similarly.

Let \( r \leq d^{\frac{k-1}{k-1}} \). We will show that there exists a set of \( r^2d \) indices \((i_1, \ldots, j_1), \ldots, (i_{r^2d}, \ldots, j_{r^2d})\) that is shattered by \( \mathcal{T}(G_{TT}(r)) \) the set of tensors of tensor train rank at most \( r \), i.e., such that
\[ \left| \{(\text{sign}(\mathcal{W}_{i_1, \ldots, j_1}), \text{sign}(\mathcal{W}_{i_2, \ldots, j_2}), \ldots, \text{sign}(\mathcal{W}_{i_{r^2d}, \ldots, j_{r^2d}})) \mid \mathcal{W} \in \mathcal{T}(G_{TT}(r)) \} \right| = 2^{r^2d}. \]

In order to do so, we will consider a tensor train tensor \( \mathcal{T} \) with cores \( \mathcal{G}^{(1)}, \ldots, \mathcal{G}^{(p)} \), where the \((k+1)\)th core \( \mathcal{G}^{(k+1)} \) will be free while the other cores are fixed in such a way that each component of \( \mathcal{G}^{(k+1)} \) appears exactly once in the entries of \( \mathcal{T} \).

Let \( e_1, \ldots, e_r \) be the canonical basis of \( \mathbb{R}^r \) and let \( e_i = 0 \) for any \( i > r \). Let \( k = \left\lfloor \frac{r}{2} \right\rfloor \) and let \( \mathcal{G}^{(k)} \) be the \( k \)th core of the tensor train tensor \( \mathcal{T} \) (i.e., the middle core). The other cores of \( \mathcal{T} \) are defined as follows: for each \( j \in [d] \),
\begin{align*}
\mathcal{G}^{(1)}_{j} &= e_j^T \\
\mathcal{G}^{(s)}_{i,j} &= e_i (e_{(j-1)d^r+1}^T + e_{2(j-1)d^r+2}^T + \cdots + e_e (e_{(j-1)d^r+r}^T) \quad \text{for } s = 2, \ldots, k-1 \\
\mathcal{G}^{(s)}_{i,j} &= e_1 \mathcal{E}_{(j-1)d^r-s+1}^T + e_2 \mathcal{E}_{(j-1)d^r-s+2}^T + \cdots + e_e ((j-1)d^r-s+r)^T \quad \text{for } s = k+1, \ldots, p-1 \\
\mathcal{G}^{(p)}_{j} &= e_j.
\end{align*}
With these definitions, one can check that
\[ G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(1)} G_{j_1; j_2 \cdots; j_{k-1}; j_k}^{(2)} \cdots G_{k-1; k}^{(k-1)} = e_1^T \]
for any \( i_1, \cdots, i_{k-1} \in [d] \) and
\[ G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(k+1)} G_{j_1; j_2 \cdots; j_{k-1}; j_k}^{(k+2)} \cdots G_{k-1; k}^{(p-1)} G_{j_1; j_2 \cdots; j_{k-1}; j_k}^{(p)} = e_{i_p}^T \]
for any \( i_k, \cdots, i_p \in [d] \). Letting \( j_0, \cdots, j_t = j_0 + \sum_{r=1}^t (j_r - 1) d^r \) for any \( j_0, \cdots, j_t \in [d] \), it follows that for any \( i_1, \cdots, i_p \in [d] \),
\[ T_{i_1; \cdots; i_p} = \begin{cases} G_{[i_1; i_2 \cdots; i_{k-1}; i_k; i_p; i_{p+1}; \cdots; i_{k+1}]}^{(k)} & \text{if } i_1, i_2, \cdots, i_k, i_{k+1} \leq r \text{ and } i_p, i_{p+1}, \cdots, i_k, i_{k+1} \leq r \\ 0 & \text{otherwise.} \end{cases} \]

Since \( r \leq d^{\left\lceil \frac{2k}{p} \right\rceil} \) and \( k = \left\lceil \frac{2}{p} \right\rceil \), this implies that for any \( k \)-th core \( G^{(k)} \), the tensor train tensor \( T \) contains all the \( r^2 d \) entries of \( G^{(k+1)} \). Thus, the set of \( r^2 d \) indices \( \{(i_1, \cdots, i_p) \mid [i_1, i_2, \cdots, i_k, i_{k+1}] \leq r \} \) is shattered by \( T(G_T(r)) \) and the first part of the theorem follows from Lemma 10.

We now prove the second part of the theorem for the TT case, using a different construction. Let \( r = d = 3k \) for some \( k \in \mathbb{N} \). We will construct a family of tensors in \( T(G_T(r)) \) where a third of the \( p = 3k \) cores will be free while the other cores are fixed in such a way that the resulting tensor \( T \) can be seen as the outer product of \( k \) 3rd order tensor of size \( d \times d \times d \). By observing that such tensors can be interpreted as rank one \( k \)-th order tensors in \( \mathbb{R}^{d^3 \times d^3 \times \cdots \times d^3} \), the second part of the theorem will follow from Theorem 11.

Let \( G^{(1)} \cdots G^{(p)} \) be the core tensors of the TT decomposition. The core tensors \( G^{(3s+2)} \in \mathbb{R}^{d^3 \times d^3 \times d^3} \) for \( s = 0, \cdots, k - 1 \) are free while the other cores are defined as follows: for any \( j \in [d] \),
\[ G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(3s+2)} = e_j^T \]
\[ G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(3s+1)} = e_i e_j^T \]
\[ G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(3s)} = e_i e_j e_k^T \]
\[ G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(p)} = e_i e_j e_k e_{j+k}^T \]

for any \( i_1, \cdots, i_k \in [d] \). We have
\[ T_{i_1; \cdots; i_p} = \begin{cases} G_{i_1; i_2 \cdots; i_{k-1}; i_k; i_p; i_{p+1}; \cdots; i_{k+1}}^{(3s+2)} & \text{if } i_1, i_2, \cdots, i_k, i_{k+1} \leq r \\ 0 & \text{otherwise.} \end{cases} \]

It follows that, for any \( i_1, \cdots, i_p \in [d] \), we have
\[ T_{i_1; \cdots; i_p} = \begin{cases} G_{i_1; i_2 \cdots; i_{k-1}; i_k; i_p}^{(1)} G_{i_1; i_2 \cdots; i_{k-1}; i_k; i_p}^{(2)} \cdots G_{i_1; i_2 \cdots; i_{k-1}; i_k; i_p}^{(p-1)} G_{i_1; i_2 \cdots; i_{k-1}; i_k; i_p}^{(p)} \\ \quad \quad = (e_1)_{i_1} (G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(2)}) (e_1 e_i)_{i_1} (G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(5)}) (e_i e_i)_{i_1} (G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(p)}) \cdots (e_i e_i e_j)_{i_1} (G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(p-1)}) (e_i e_i e_j e_{i+k}^T) \\ \quad \quad = G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(2)} G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(5)} G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(p)} \end{cases} \]

which implies that \( T = G_{i_1; i_2 \cdots; i_{k-1}; i_k; i_p}^{(2)} G_{i_1; i_2 \cdots; i_{k-1}; i_k; i_p}^{(5)} \cdots G_{i_1; i_2 \cdots; i_{k-1}; i_k; i_p}^{(p-1)} = \bigotimes_{s=0}^{k-1} G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(3s+2)} \). By reshaping the set of tensors constructed in this way into \( k \)-th order tensors in \( \mathbb{R}^{d^3 \times d^3 \times \cdots \times d^3} \), one can see that this set of tensors is exactly the set of rank one \( k \)-th order tensors of size \( d^3 \times d^3 \times d^3 \), for which the corresponding VC dimension and pseudo dimensions are lower bounded by \( k(d - 1) = p(r^2 d - 1)/3 \) from Theorem 11.

The proof for the tensor ring case uses the exact same constructions with the difference in the definition of the first and last core tensors which are defined by \( G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(1)} = e_1 e_j^T \) and \( G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(p)} = e_1 e_j^T \) for each \( j \in [d] \). With these definitions, one can check that
\[ G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(1)} G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(2)} G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(3)} \cdots G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(k-1)} = e_1 e_j^T \]
for any \( i_1, \cdots, i_{k-1} \in [d] \) and
\[ G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(k+1)} G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(k+2)} \cdots G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(p-1)} G_{i_1; i_2 \cdots; i_{k-1}; i_k}^{(p)} = e_1 e_j e_{i+k}^T \]
for any \(i_{k+1}, \ldots, i_p \in [d]\). It follows that for any \(i_1, \ldots, i_p \in [d]\),
\[
\mathcal{T}_{i_1, \ldots, i_p} = \text{Tr} \ g^{(1)}_{i_1, \ldots, i_p} g^{(2)}_{i_2, \ldots, i_{p-1}} \cdots g^{(k-1)}_{i_{k-1}, \ldots, i_p} g^{(k)}_{i_{k+1}, \ldots, i_{p-1}} \cdots g^{(p-1)}_{i_{p-1}, \ldots, i_p} g^{(p)}_{i_1, \ldots, i_p} = \begin{cases} g^{(k)}_{i_1, i_2, \ldots, i_{k-1}, i, i_{k+1}, \ldots, i_p} & \text{if } [i_1, i_2, \ldots, i_k] \leq r \text{ and } [i_p, i_{p-1}, \ldots, i_{k+1}] \leq r \\
0 & \text{otherwise.} \end{cases}
\]

The proof of the first part of the theorem then follows the exact same argument as for the TT case. The second part of the theorem for TR is proved exactly as the one for TT by replacing the first and last cores again by \(g^{(1)}_{i,j} = e_1 e_j^\top\) and \(g^{(p)}_{i,j} = e_j e_1^\top\) for each \(j \in [d]\).

\[ \square \]

### A.2.3 Tucker

**Theorem 13.** Let \(r \leq d\) and let \(G_{\text{Tucker}}(r) = A_{\text{Tucker}} \) be the tensor network structure corresponding to \(p\)th order tensors of Tucker rank at most \(r\).

Then, the VC-dimension and pseudo-dimensions \(d_{\text{VC}}(\mathcal{H}_{\text{Tucker}}^{\text{classif}}), \text{Pdim}(\mathcal{H}_{\text{Tucker}}^{\text{regression}})\) and \(\text{Pdim}(\mathcal{H}_{G_{\text{Tucker}}(r)}^{\text{completion}})\) are all lower bounded by \(r^p\)

**Proof.** Let \(r \leq d\). We show that there exists a set of \(r^p\) indices \((i_1, \ldots, j_1), \ldots, (i_{p+1}, \ldots, j_{p+1})\) that is shattered by \(\mathcal{T}(G_{\text{Tucker}}(r))\) (the set of tensors of Tucker rank at most \(r\)), i.e., such that
\[
|\{(\text{sign}(\mathcal{W}_{i_1, \ldots, j_1}), \text{sign}(\mathcal{W}_{i_2, \ldots, j_2}), \ldots, \text{sign}(\mathcal{W}_{i_{p+1}, \ldots, j_{p+1}})) \mid \mathcal{W} \in \mathcal{T}(G_{\text{Tucker}}(r))\}| = 2^{r^p}.
\]

Let \(P = (I_r \times r_0 \times (d-r)) \in \mathbb{R}^{d \times r}\). We consider the following subset of \(\mathcal{T}(G_{\text{Tucker}}(r))\):
\[
A = \{ g \times_1 P \times_2 P \times_3 \cdots \times_p P \mid g \in \mathbb{R}^{r \times r \times \cdots \times r} \} \subset \mathcal{T}(G_{\text{Tucker}}(r))
\]
where \(\times_k\) denotes the mode-\(k\) product (see, e.g., [31]). It is easy to see that any tensor \(\mathcal{T} = g \times_1 P \times_2 P \times_3 \cdots \times_p P \in A\) will have entries \(g_{i_1, \ldots, i_p} = \mathcal{G}_{i_{k+1}, \ldots, i_p}\) for any \(i_1, \ldots, i_p \in [r]\). Hence the set of \(r^p\) indices \([r] \times [r] \times \cdots \times [r] \subset [d] \times [d] \times \cdots \times [d]\) is shattered by \(\mathcal{T}(G_{\text{Tucker}}(r))\) and the result directly follows from Lemma 10.

\[ \square \]

### A.2.4 CP

**Theorem 14.** Let \(r \leq d^p - 1\) and let \(G_{\text{CP}}(r) = A_{\text{CP}}\) be the tensor network structure corresponding to \(p\)th order tensors of CP rank at most \(r\).

Then, the VC-dimension and pseudo-dimensions \(d_{\text{VC}}(\mathcal{H}_{\text{CP}}^{\text{classif}}), \text{Pdim}(\mathcal{H}_{\text{CP}}^{\text{regression}})\) and \(\text{Pdim}(\mathcal{H}_{G_{\text{CP}}(r)}^{\text{completion}})\) are all lower bounded by \(rd\).

**Proof.** Let \(r \leq d^p - 1\). We show that there exists a set of \(rd\) indices \((i_1, \ldots, j_1), \ldots, (i_{p+1}, \ldots, j_{p+1})\) that is shattered by \(\mathcal{T}(G_{\text{CP}}(r))\) (the set of tensors of CP rank at most \(r\)), i.e., such that
\[
|\{(\text{sign}(\mathcal{W}_{i_1, \ldots, j_1}), \text{sign}(\mathcal{W}_{i_2, \ldots, j_2}), \ldots, \text{sign}(\mathcal{W}_{i_{p+1}, \ldots, j_{p+1}})) \mid \mathcal{W} \in \mathcal{T}(G_{\text{CP}}(r))\}| = 2^{rd}.
\]

We construct a tensor \(\mathcal{T}\) of CP rank at most \(r\) such that each component of a matrix \(A \in \mathbb{R}^{d \times r}\) appears at least once in the entries of \(\mathcal{T}\). Similarly to the previous proofs, \(A\) will be a free parameter allowed to take any value while the other components of the parametrization of \(\mathcal{T}\) will be fixed.

Let \(A \in \mathbb{R}^{d \times r}\), we define \(p\) tensors \(A^{(1)}, \ldots, A^{(p)} \in \mathbb{R}^{d \times \cdots \times d}\) of order \(p\) as follows: for all \(i_1, \ldots, i_p, \tau_1, \ldots, \tau_{p-1} \in [d]\),
\[
A^{(1)}_{i_1, \tau_1, \ldots, \tau_{p-1}} = \begin{cases} \delta_{\tau_1 + (\tau_2 - 1)d + \cdots + (\tau_{p-1} - 1)d, \tau_{p-1} - 1} & \text{if } \tau_1 + (\tau_2 - 1)d + \cdots + (\tau_{p-1} - 1)d^{p-2} \leq r \\
0 & \text{otherwise} \end{cases}
\]
\[
A^{(s)}_{i_1, \tau_1, \ldots, \tau_{p-1}} = \delta_{\tau_1, \tau_{s-1}} \quad \text{for } s = 2, \ldots, p
\]
where $\delta$ is the Kronecker symbol. Let $S = \{(\tau_1, \cdots, \tau_{p-1}) \in [d] \times \cdots \times [d] \mid \tau_1 + (\tau_2 - 1)d + \cdots + (\tau_{p-1} - 1)d^{p-2} \leq r\}$. Let $T \in \mathbb{R}^{d \times \cdots \times d}$ be the $p$th order tensor defined by

$$T_{i_1, i_2, \cdots, i_p} = \sum_{\tau_1=1}^{d} \sum_{\tau_2=1}^{d} \cdots \sum_{\tau_{p-1}=1}^{d} A^{(1)}_{i_1, \tau_1, \tau_2, \cdots, \tau_{p-1}} A^{(2)}_{\tau_2, \tau_1, \tau_2, \cdots, \tau_{p-1}} \cdots A^{(p)}_{i_p, \tau_1, \tau_2, \cdots, \tau_{p-1}}$$

for all $i_1, \cdots, i_p \in [d]$. It can easily be checked that $T$ is a tensor of CP rank at most $r$, i.e., $T \in T(G_{CP}(r))$. Indeed, from the definition of $A^{(1)}$, we have

$$T_{i_1, i_2, \cdots, i_p} = \sum_{(\tau_1, \cdots, \tau_{p-1}) \in S} A^{(1)}_{i_1, \tau_1, \tau_2, \cdots, \tau_{p-1}} A^{(2)}_{\tau_2, \tau_1, \tau_2, \cdots, \tau_{p-1}} \cdots A^{(p)}_{i_p, \tau_1, \tau_2, \cdots, \tau_{p-1}}$$

where the sum is over at most $r$ terms (from the definition of $S$). At the same time, we have

$$T_{i_1, i_2, \cdots, i_p} = \sum_{(\tau_1, \cdots, \tau_{p-1}) \in S} A^{(1)}_{i_1, \tau_1, \tau_2, \cdots, \tau_{p-1}} A^{(2)}_{\tau_2, \tau_1, \tau_2, \cdots, \tau_{p-1}} \cdots A^{(p)}_{i_p, \tau_1, \tau_2, \cdots, \tau_{p-1}}$$

$$= \sum_{(\tau_1, \cdots, \tau_{p-1}) \in S} \delta_{i_2, \tau_1} \delta_{i_3, \tau_2} \cdots \delta_{i_p, \tau_{p-1}}$$

$$= \begin{cases} A_{i_1, i_2 + (i_3 - 1)d + \cdots + (i_p - 1)d^{p-2}} & \text{if } i_2 + (i_3 - 1)d + \cdots + (i_p - 1)d^{p-2} \leq r \\ 0 & \text{otherwise} \end{cases}$$

Hence, each one of the components of $A$ appears exactly once in $T$. In particular, this implies that the set of indices

$$\{(i_1, \cdots, i_p) \in [d] \times \cdots \times [d] \mid i_2 + (i_3 - 1)d + \cdots + (i_p - 1)d^{p-2} \leq r\}$$

of size $rd$ is shattered by $T(G_{CP}(r))$. The theorem then directly follows from Lemma 10.

\[\square\]

### B Experiments

To evaluate the theoretical upper bound provided in Theorem 4, we perform a simple binary classification experiment with synthetic data. We draw a random low rank TT target tensor $\mathcal{W} \in \mathbb{R}^{4 \times 4 \times 4 \times 4}$ of rank 8 by drawing the components of the cores of the TT decomposition i.i.d. from a uniform distribution between -1 and 1. Input-output data is generated with $y_i = \text{sign}(\langle \mathcal{W}, \mathcal{X}_i \rangle)$ for training and testing, where the components of $\mathcal{X}_i$ are drawn i.i.d. from a normal distribution. Using the cross-entropy as loss function, we optimize the empirical risk using stochastic gradient descent with a learning rate of $10^{-2}$ to learn a TT hypothesis of rank $r$.

In Figure 3, we report the log generalization gap of the learned hypothesis $h$, $\log(R(h) - \hat{R}_G(h))$, where the true risk $R(h)$ is estimated on a test set of size 4,000 for different scenarios. In Figure 3 (left), we show how the sample size affects the generalization gap for learned hypothesis of rank $r = 2$ and $r = 4$. As expected, the generalization gap decreases as the sample size grows, and is smaller for $r = 2$ than $r = 4$ which is also expected from the Theorem 4. In Figure 3 (right), we show how the rank $r$ of the learned hypothesis affects the generalization for samples sizes 2,000 and 4,000. As expected, the higher the rank of the TT weight tensor, the larger the model complexity and hence the generalization gap. In both figures, we observe that the theoretical upper bound and the experimental results follow a similar trend as a function of sample size and hypothesis rank.
Figure 3: Dashed lines represent the theoretical bound, full lines represent the log generalization gap (averaged over 20 runs for both experiments), and shaded areas show the standard deviation. (left) Generalization error for two models with ranks $r = 2$ and $r = 4$ as a function of training size. (right) Generalization error for two sample sizes $n = 2000$ and $n = 4000$ as a function of the rank of the learned hypothesis.