ON A DIRAC PARTICLE IN AN UNIFORM MAGNETIC FIELD IN 3-DIMENSIONAL SPACES OF CONSTANT CURVATURE

Mosyr State Pedagogical University, Belarus
Institute of Physics, National Academy of Sciences of Belarus
e.ovsiyuk@mail.ru ; redkov@dragon.bas-net.by

There are constructed exact solutions of the quantum-mechanical Dirac equation for a spin S=1/2 particle in Riemannian space of constant negative curvature, hyperbolic Lobachevsky space, in presence of an external magnetic field, analogue of the homogeneous magnetic field in the Minkowski space. A generalized formula for energy levels, describing quantization of the transversal motion of the particle in magnetic field has been obtained. The same problem is solved for spin 1/2 particle in the space of constant positive curvature, spherical Riemann space. A generalized formula for energy levels, describing quantization of the transversal and along the magnetic field motions of the particle on the background of the Riemann space geometry, is obtained.

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1 Introduction

The quantization of a quantum-mechanical particle in the homogeneous magnetic field belongs to classical problems in physics [1] [2] [3] [4]. In 1985 – 2010, a more general problem in a curved Riemannian background, hyperbolic and spherical planes, was extensively studied [5] [6] [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] [19] [20] [21] [22] [23] [24], providing us with a new system having intriguing dynamics and symmetry, both on classical and quantum levels.

Extension to 3-dimensional hyperbolic and spherical spaces was performed recently. In [25] [26] [27], exact solutions for a scalar particle in extended problem, particle in external magnetic field on the background of Lobachevsky $H_3$ and Riemann $S_3$ spatial geometries were found. A corresponding system in the frames of classical mechanics was examined in [28] [29] [30]. In the present paper, we consider a quantum-mechanical problem a particle with spin 1/2 described by the Dirac equation in 3-dimensional Lobachevsky and Riemann space models in presence of the external magnetic field.

The paper is organized as follows. In Section 2, the general covariant Dirac equation is specified in Lobachevsky model in special cylindric coordinate system, in which a generalized concept of an homogeneous magnetic field can be defined straightforwardly. Then the variables in the Dirac equation are separated, and the problem is reduced to a couple of differential equations, describing the motion of the Dirac particle in $z$ and $r$-directions. In Section 3, differential equations in $z$-variable are solved in terms of hypergeometric functions, no quantization arises in accordance with the topological properties of the Lobachevsky model. In Section 4, we examine differential equations in radial variable $r$, producing a generalized quantization rule for transversal energy, due to the presence of external magnetic field.
2 Cylindric coordinates and the Dirac equation in hyperbolic space $H_3$, separation of the variables

In the Lobachevsky space, let us use an extended cylindric coordinates (see [33])

\[ dS^2 = dt^2 - \cosh^2 z (dr^2 + \sinh^2 r \, d\phi^2) - dz^2 , \]

\[ u_1 = \cosh z \, \sinh r \, \cos \phi , \quad u_2 = \cosh z \, \sinh r \, \sin \phi , \]

\[ u_3 = \sinh z , \quad u_0 = \cosh z \, \cosh r ; \quad (1) \]

where $x^j = (r, \phi, z)$: $r \in [0, +\infty)$, $\phi \in [0, 2\pi]$, $z \in (-\infty, +\infty)$; the curvature radius $\rho$ is taken as a unit of the length. An analogue of usual homogeneous magnetic field is defined as [25, 26, 27]

\[ A_\phi = -2B \sinh^2 \frac{r}{2} = -B (\cosh r - 1) . \quad (2) \]

To coordinates (1) there corresponds the tetrad

\[ e^\beta_{(a)} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh^{-1} z & 0 & 0 \\ 0 & 0 & \cosh^{-1} z \sinh^{-1} r & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} . \quad (3) \]

Christoffel symbols $\Gamma_{jk}^r$ and Ricci rotation coefficients $\gamma_{abc}$ are

\[ \Gamma_{rk}^r = \begin{vmatrix} 0 & 0 & \th z \\ 0 & -\sinh r \cosh z & 0 \\ \th z & 0 & 0 \end{vmatrix} , \quad \Gamma_{r}^\phi = \begin{vmatrix} 0 & \cth r & 0 \\ \cth r & 0 & \th z \\ 0 & \th z & 0 \end{vmatrix} , \quad \Gamma_{z}^z = \begin{vmatrix} -\cosh z \sinh z & 0 & 0 \\ 0 & -\sinh z \cosh z \sinh^2 r & 0 \\ 0 & 0 & 0 \end{vmatrix} , \quad \gamma_{122} = \frac{1}{\cosh z \tanh r} , \quad \gamma_{311} = \tanh z , \quad \gamma_{322} = \tanh z . \]

A general covariant Dirac equation (for more detail see [31]) takes the form

\[ \left[ i\gamma^0 \partial_t + \frac{i\gamma^1}{\cosh z} \left( \partial_r + \frac{1}{2} \frac{1}{\th r} \right) + \gamma^2 \frac{i\partial_\phi + eB (\cosh r - 1)}{\cosh z \sinh r} + i\gamma^3 (\partial_z + \th z) - M \right] \Psi = 0 . \quad (4) \]

With the substitution $\Psi = \psi / \cosh z \sqrt{\sinh r}$ eq. (4) becomes simpler

\[ \left[ i\gamma^1 \partial_r + \gamma^2 \frac{i\partial_\phi + eB (\cosh r - 1)}{\sinh r} + \cosh z (i\gamma^0 \partial_t + i\gamma^3 \partial_z - M) \right] \psi = 0 . \quad (5) \]

Solutions of this equation will be searched in the form

\[ \psi = e^{-i\epsilon t} e^{i\epsilon \phi} \begin{vmatrix} f_1(r, z) \\ f_2(r, z) \\ f_3(r, z) \\ f_4(r, z) \end{vmatrix} , \]

where...
so that

\[
\begin{bmatrix}
  i\gamma^1 \partial_r - \gamma^2 \mu(r) + \cosh z \left( \epsilon \gamma^0 + i \gamma^3 \partial_z - M \right)
\end{bmatrix}
\begin{bmatrix}
  f_1(r, z) \\
  f_2(r, z) \\
  f_3(r, z) \\
  f_4(r, z)
\end{bmatrix} = 0,
\]

where

\[
\mu(r) = \frac{m - eB(\cosh r - 1)}{\sinh r}.
\]

Taking the Dirac matrices in spinor basis, we get radial equations for \( f_a(t, z) \)

\[
(\partial_r + \mu) f_4 + \cosh z \partial_z f_3 + i \cosh z (\epsilon f_3 - M f_1) = 0, \\
(\partial_r - \mu) f_3 - \cosh z \partial_z f_4 + i \cosh z (\epsilon f_4 - M f_2) = 0, \\
(\partial_r + \mu) f_2 + \cosh z \partial_z f_1 - i \cosh z (\epsilon f_1 - M f_3) = 0, \\
(\partial_r - \mu) f_1 - \cosh z \partial_z f_2 - i \cosh z (\epsilon f_2 - M f_4) = 0.
\]

(6)

To simplify the system (6) one additional operator should be diagonalized. In flat space, as that can be taken the helicity operator

\[
(\vec{\Sigma} \vec{P}) \Psi_{\text{cart}} = \lambda \Psi_{\text{cart}}, \quad \vec{P} = -i\vec{\nabla}.
\]

Because the cartesian and cylindrical bases are related by spinor gauge transformation over fermion wave functions

\[
\Psi_{\text{cyl}} = S \Psi_{\text{cart}}, \quad S = \begin{bmatrix}
  B & 0 \\
  0 & B
\end{bmatrix}, \quad B = \begin{bmatrix}
  e^{+i\phi/2} & 0 \\
  0 & e^{-i\phi/2}
\end{bmatrix},
\]

for the helicity operator in cylindrical representation

\[
B \vec{\Sigma} \vec{P} B^{-1} = \begin{bmatrix}
  P_3 & e^{+i\phi/2} (P_1 + i P_2) e^{-i\phi/2} \\
  e^{+i\phi/2} (P_1 + i P_2) e^{-i\phi/2} & -P_3
\end{bmatrix},
\]

where

\[
(P_1 \pm i P_2) = -ie^{\pm i\phi/2} \left( \frac{\partial}{\partial r} \pm i \frac{1}{r} \frac{\partial}{\partial \phi} \right),
\]

\[
e^{\mp i\phi/2} (P_1 \pm i P_2) e^{\pm i\phi/2} = [-i(\partial_r + \frac{1}{2r}) \pm i \frac{1}{r} \partial_\phi],
\]

one produces the expression

\[
B \vec{\Sigma} \vec{P} B^{-1} = \gamma^2 \gamma^3 (\partial_r + \frac{1}{2r}) - i\gamma^3 \gamma^1 \frac{i \partial_\phi}{r} + \gamma^1 \gamma^2 \partial_z.
\]

(8)

By the use of the substitution \( \Psi^0 = \psi^0 / \sqrt{r} \), the operator (8) is simplified to

\[
\Sigma^0 = B \vec{\Sigma} \vec{P} B^{-1} = \gamma^2 \gamma^3 \partial_r - i\gamma^3 \gamma^1 \frac{i \partial_\phi}{r} + \gamma^1 \gamma^2 \partial_z.
\]

(9)
In presence of external magnetic field in flat space, the Dirac equation looks
\[
\left(i\gamma^0 \partial_t + i\gamma^1 \partial_r + \gamma^2 \frac{i\partial\phi + eBr^2/2}{r} + i\gamma^3 \partial_z - M\right)\psi^0 = 0 ,
\] (10)
hence an extended helicity operator is
\[
\Sigma_0 = \gamma^2\gamma^3 \partial_r - i\gamma^3\gamma^1 \frac{i\partial\phi + eBr^2/2}{r} + \gamma^1\gamma^2 \partial_z .
\] (11)

We omit all details of calculation proving this.

In the Lobachevsky space, taking into consideration the explicit form of the Dirac equation
\[
\left[i\gamma^0 \partial_t + \frac{1}{\cosh z} \left(i\gamma^1 \partial_r + \gamma^2 \frac{i\partial\phi + eB\cosh r - 1}{\sinh r}\right) + i\gamma^3 \partial_z - M\right] \psi = 0 ,
\] one may guess the form of a generalized helicity operator \(\Sigma\) in the curved space
\[
\Sigma = \frac{1}{\cosh z} \left(\gamma^2\gamma^3 \partial_r - i\gamma^3\gamma^1 \frac{i\partial\phi + eB\cosh r - 1}{\sinh r}\right) + \gamma^1\gamma^2 \partial_z .
\] (12)

Let us prove the commutation relation by direct calculation. It is convenient to make calculation in several steps. Let it be
\[
\Sigma_1 = \frac{1}{\cosh z} \left(\gamma^2\gamma^3 \partial_r\right), \quad [H, \Lambda_1] = H\Lambda_1 - \Lambda_1 H
\]
\[
= i\gamma^0 \partial_t \frac{1}{\cosh z} \gamma^2\gamma^3 \partial_r + i\gamma^3 \partial_z \frac{1}{\cosh z} \gamma^2\gamma^3 \partial_r
\]
\[
- \frac{1}{\cosh z} \gamma^2 \mu \frac{1}{\cosh z} \gamma^2\gamma^3 \partial_r + i\gamma^3 \partial_z \frac{1}{\cosh z} \gamma^2\gamma^3 \partial_r - M \frac{1}{\cosh z} \gamma^2\gamma^3 \partial_r
\]
\[
- i\gamma^3 \partial_z \frac{1}{\cosh z} \gamma^2\gamma^3 \partial_r + i\gamma^3 \partial_z \partial_t \frac{1}{\cosh z} \gamma^2\gamma^3 \partial_r - \frac{1}{\cosh z} \gamma^2\gamma^3 \partial_r \frac{1}{\cosh z} \gamma^1\gamma^2 \partial_z M =
\]
\[
= - \frac{1}{\cosh^2 z} \gamma^2 \mu \gamma^2\gamma^3 \partial_r + \frac{1}{\cosh^2 z} \gamma^2\gamma^3 \partial_r \gamma^2 \mu
\]
\[
+ i\gamma^3 \partial_z \frac{1}{\cosh z} \gamma^2\gamma^3 \partial_r - \frac{1}{\cosh z} \gamma^2\gamma^3 \partial_r i\gamma^3 \partial_z ,
\]
that is
\[
[H, \Sigma_1] = \frac{1}{\cosh^2 z} \mu \partial_r + \frac{1}{\cosh^2 z} \gamma^3 (\partial_r \mu) + i\gamma^2 (\partial_z \frac{1}{\cosh z}) \partial_r + i\gamma^2 \frac{1}{\cosh z} \partial_r \partial_z .
\]
Now, let it be

$$\Sigma_2 = \frac{1}{\cosh z} i \gamma^2 \gamma^1 \mu(r), \quad [H, \Lambda_2] = H\Lambda_2 - \Lambda_2 H$$

$$= i\gamma^0 \partial_t \frac{1}{\cosh z} i \gamma^2 \gamma^1 \mu + \frac{1}{\cosh z} i\gamma^3 \gamma^1 \mu - i\gamma^3 \gamma^1 \mu$$

$$- i\gamma^1 \gamma^2 \gamma^1 \frac{1}{\cosh^2 z} \mu^2 + i\gamma^3 \partial_z \frac{1}{\cosh z} i \gamma^2 \gamma^1 \mu - M \gamma^3 \gamma^1 \mu$$

$$- \frac{1}{\cosh z} i \gamma^3 \gamma^1 \mu \partial_t - \frac{1}{\cosh z} i \gamma^3 \gamma^1 \mu \frac{1}{\cosh z} i \gamma^1 \partial_r$$

$$+ \frac{1}{\cosh z} i \gamma^3 \gamma^1 \frac{1}{\cosh z} \gamma^2 \mu^2(r) - \frac{1}{\cosh z} i \gamma^3 \gamma^1 \mu \partial_z + \frac{1}{\cosh z} i \gamma^3 \gamma^1 \mu M,$$

that is

$$[H, \Sigma_2] = - \frac{1}{\cosh z} \gamma^3 (\partial_r \mu) - \frac{1}{\cosh z} \gamma^3 \mu \partial_r + \gamma^1 (\partial_z \frac{1}{\cosh z} \mu) + \frac{1}{\cosh z} \gamma^1 \mu \partial_z.$$

Now let it be

$$\Sigma_3 = \gamma^1 \gamma^2 \partial_z, \quad [H, \Lambda_3] = H\Lambda_0 - \Lambda_3 H$$

$$= i\gamma^0 \partial_t \gamma^1 \gamma^2 \partial_z + \frac{1}{\cosh z} i\gamma^1 \partial_r \gamma^1 \gamma^2 \partial_z$$

$$- \frac{1}{\cosh z} \gamma^2 \mu \gamma^1 \gamma^2 \partial_z + i\gamma^3 \partial_z \gamma^1 \gamma^2 \partial_z - M \gamma^1 \gamma^2 \partial_z$$

$$- \gamma^1 \gamma^2 \partial_z i\gamma^0 \partial_t - \gamma^1 \gamma^2 \partial_z \frac{1}{\cosh z} i \gamma^1 \partial_r$$

$$+ \gamma^1 \gamma^2 \partial_z \frac{1}{\cosh z} \gamma^2 \mu - \gamma^1 \gamma^2 \partial_z \gamma^4 \gamma^3 \partial_z + \gamma^1 \gamma^2 \partial_z M,$$

so that

$$[H, \Sigma_3] = - i\gamma^2 \frac{1}{\cosh z} \partial_r \partial_z - i\gamma^2 (\partial_z \frac{1}{\cosh z} \partial_t - \gamma^1 \frac{1}{\cosh z} \mu \partial_z - \gamma^1 \partial_z \frac{1}{\cosh z} \mu.$$

Summing tree commutators, we get a needed one

$$[H, \Sigma] = \gamma^3 \frac{1}{\cosh^2 z} \mu \partial_r + \frac{1}{\cosh^2 z} \gamma^3 (\partial_r \mu)$$

$$+ i\gamma^2 (\partial_z \frac{1}{\cosh z}) \partial_r + i\gamma^2 \frac{1}{\cosh z} \partial_r \partial_z$$

$$- \frac{1}{\cosh^2 z} \gamma^3 (\partial_r \mu) - \gamma^3 \frac{1}{\cosh^2 z} \mu \partial_r$$

$$+ \gamma^1 (\partial_z \frac{1}{\cosh z}) \mu + \frac{1}{\cosh z} \gamma^1 \mu \partial_z$$

$$- i\gamma^2 \frac{1}{\cosh z} \partial_r \partial_z - i\gamma^2 (\partial_z \frac{1}{\cosh z}) \partial_r$$

$$- \gamma^1 \frac{1}{\cosh z} \mu \partial_z - \gamma^1 (\partial_z \frac{1}{\cosh z}) \mu = 0.$$

Let us consider an eigenvalue equation \( \Sigma \psi = \sigma \psi \):

$$\begin{pmatrix} \gamma^2 \gamma^3 \partial_r + i \gamma^3 \gamma^1 \mu(r) \cosh z \gamma^1 \partial_z \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \sigma \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}.$$  

(13)
it leads to
\[
\begin{align*}
(\partial_r + \mu) f_4 + \cosh z \partial_z f_3 - i \cosh z \sigma f_3 &= 0, \\
(\partial_r - \mu) f_3 - \cosh z \partial_z f_4 - i \cosh z \sigma f_4 &= 0, \\
(\partial_r + \mu) f_2 + \cosh z \partial_z f_1 - i \cosh z \sigma f_1 &= 0, \\
(\partial_r - \mu) f_1 - \cosh z \partial_z f_2 - i \cosh z \sigma f_2 &= 0.
\end{align*}
\]

Considering them together with previously obtained system (6)
\[
\begin{align*}
(\partial_r + \mu) f_4 + \cosh z \partial_z f_3 + i \cosh z (\epsilon f_3 - M f_1) &= 0, \\
(\partial_r - \mu) f_3 - \cosh z \partial_z f_4 + i \cosh z (\epsilon f_4 - M f_2) &= 0, \\
(\partial_r + \mu) f_2 + \cosh z \partial_z f_1 - i \cosh z (\epsilon f_1 - M f_3) &= 0, \\
(\partial_r - \mu) f_1 - \cosh z \partial_z f_2 - i \cosh z (\epsilon f_2 - M f_4) &= 0,
\end{align*}
\]
we arrive at a system of linear algebraic equations
\[
\begin{align*}
\sigma f_3 + (\epsilon f_3 - M f_1) &= 0, \\
\sigma f_4 + (\epsilon f_4 - M f_2) &= 0, \\
\sigma f_1 - (\epsilon f_1 - M f_3) &= 0, \\
\sigma f_2 - (\epsilon f_2 - M f_4) &= 0.
\end{align*}
\]

It gives
\[
\sigma = \mp p, \quad p = \sqrt{\epsilon^2 - m^2}, \quad f_3 = \frac{\epsilon \pm p}{M} f_1, \quad f_4 = \frac{\epsilon \pm p}{M} f_2.
\]

With these linear restrictions we get two more simple systems
\[
\sigma = -p,
\]
\[
\begin{align*}
(\partial_r + \mu) f_2 + \cosh z \left( \frac{\partial}{\partial z} + i p \right) f_1 &= 0, \\
(\partial_r - \mu) f_1 - \cosh z \left( \frac{\partial}{\partial z} - i p \right) f_2 &= 0;
\end{align*}
\]
\[
\sigma = +p,
\]
\[
\begin{align*}
(\partial_r + \mu) f_2 + \cosh z \left( \frac{\partial}{\partial z} - i p \right) f_1 &= 0, \\
(\partial_r - \mu) f_1 - \cosh z \left( \frac{\partial}{\partial z} + i p \right) f_2 &= 0.
\end{align*}
\]

For definiteness, let us consider the system (17) (transition to the case (18) is performed by the formal change \( p \mapsto -p \)). Let us search solutions in the form
\[
f_1 = Z_1(z) R_1(r), \quad f_2 = Z_2(z) R_2(r).
\]
Eqs. (17) result in
\[
\begin{align*}
(\partial_r + \mu) Z_2 R_2 + \cosh z \left( \frac{\partial}{\partial z} + i p \right) Z_1 R_1 &= 0, \\
(\partial_r - \mu) Z_1 R_1 - \cosh z \left( \frac{\partial}{\partial z} - i p \right) Z_2 R_2 &= 0.
\end{align*}
\]
Introducing the separating constant $\lambda$, we arrive at two systems, in variables $z$ and $r$ respectively:

$$\cosh z \left( \frac{d}{dz} + ip \right) Z_1 = \lambda Z_2 , \quad \cosh z \left( \frac{d}{dz} - ip \right) Z_2 = \lambda Z_1 ; \quad (21)$$

$$\left( \frac{d}{dr} + \mu \right) R_2 + \lambda R_1 = 0 , \quad \left( \frac{d}{dr} - \mu \right) R_1 - \lambda R_2 = 0 . \quad (22)$$

3 Solution of the differential equations in $z$-variable

From (21) it follows second order differential equations for $Z_1(z)$ and $Z_2(z)$

$$\frac{d^2 Z_1}{dz^2} + \frac{\sinh z}{\cosh z} \frac{dZ_1}{dz} + \left( p^2 + i p \frac{\sinh z}{\cosh z} - \frac{\lambda^2}{\cosh^2 z} \right) Z_1 = 0 , \quad (23)$$

$$\frac{d^2 Z_2}{dz^2} + \frac{\sinh z}{\cosh z} \frac{dZ_2}{dz} + \left( p^2 - i p \frac{\sinh z}{\cosh z} - \frac{\lambda^2}{\cosh^2 z} \right) Z_2 = 0 . \quad (24)$$

Consider the first equation for $Z_1(z)$. In a new variable $y = (1 + \tanh z)/2$, it reads

$$\left[ 4y(1-y) \frac{d}{dy} + 2(1-2y) \frac{d}{dy} + p^2 \left( \frac{1}{1-y} + \frac{1}{y} \right) + ip \left( \frac{1}{1-y} - \frac{1}{y} \right) - 4\lambda^2 \right] Z_1 = 0 .$$

With the substitution $Z_1 = y^A (1-y)^B \tilde{Z}_1(y)$, it leads to

$$4y(1-y) \frac{d^2 Z_1}{dy^2} + 4 \left( \frac{2A + 1}{2} - (2A + 2B + 1) y \right) \frac{d\tilde{Z}_1}{dy}$$

$$+ \left[ \frac{2A(2A-1) + p(p-i)}{y} + \frac{2B(2B-1) + p(p+i)}{1-y} \right]$$

$$- 4(A+B)^2 - 4\lambda^2 \right] \tilde{Z}_1 = 0 .$$

With restrictions

$$A = -\frac{ip}{2}, \quad B = \frac{1+ip}{2}, \quad \frac{1-4\lambda^2}{2}, \quad (25)$$

the equation for $\tilde{Z}_1$ becomes of hypergeometric type

$$y(1-y) \frac{d^2 \tilde{Z}_1}{dz^2} + \left[ 2A + 1 \frac{1}{2} - (2A + 2B + 1) y \right] \frac{d\tilde{Z}_1}{dz} - \left[ (A+B)^2 + \lambda^2 \right] \tilde{Z}_1 = 0 \quad (26)$$

with parameters given by

$$a = A + B + i \lambda , \quad b = A + B - i \lambda , \quad c = 2A + \frac{1}{2} , \quad Z_1 = y^A (1-y)^B F(a, b, c; y) \quad (27)$$

For definiteness, let us choose

$$\frac{1}{2} + \frac{ip}{2}, \quad \frac{1}{2} + \frac{ip}{2}, \quad A + B = ip + 1/2 ; \quad a = ip + 1/2 + i \lambda , \quad b = ip + 1/2 - i \lambda , \quad c = ip + 3/2 , \quad (28)$$
Two linearly independent solutions are

\[ Z_1^{(1)} = y^{A}(1 - y)^{B}U_1(y) = y^{A}(1 - y)^{B}F(a, b, c, y) = y^{+ip/2+1/2}(1 - y)^{+ip/2}F(a, b, c; y) = y^{+ip/2+1/2}(1 - y)^{-ip/2+1/2}F(c - a, c - b, c; y) , \]

(29)

\[ Z_1^{(5)} = y^{A}(1 - y)^{B}U_5(y) = y^{-ip/2}(1 - y)^{+ip/2}F(a + 1 - c, b + 1 - c, 2 - c, y) = y^{-ip/2}(1 - y)^{-ip/2+1/2}F(1 - a, 1 - b, 2 - c, y) . \]

(30)

At \( z \to -\infty \) \( (y \to 0) \) they behave as follows

\[ Z_1^{(1)} = y^{+ip/2+1/2} \to 0 , \quad Z_1^{(5)} = y^{-ip/2} . \]

(31)

To find behavior of these solutions in the region \( z \to +\infty \) \( (y \to 1) \) one should make use of Kummer’s relations

\[ U_1 = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} U_2 + \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} U_6 , \]

\[ U_5 = \frac{\Gamma(2 - c)\Gamma(c - a - b)}{\Gamma(1 - a)\Gamma(1 - b)} U_2 + \frac{\Gamma(2 - c)\Gamma(a + b - c)}{\Gamma(a + 1 - c)\Gamma(b + 1 - c)} U_6 , \]

(32)

where two couples of linearly independent solutions are involved

\[ U_1(y) = F(a, b, c; y) , \]

\[ U_5 = y^{1-c}F(a + 1 - c, b + 1 - c, 2 - c, y) ; \]

\[ U_2(y) = F(a, b, a + b - c + 1; 1 - y) , \]

\[ U_6(y) = (1 - y)^{c-a-b}F(c - a, c - b, c - a - b + 1; 1 - y) . \]

(33)

Thus, for \( Z_1^{(1)} \) we get the following expansions

\[ Z_1^{(1)} = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \times y^{+ip/2+1/2}(1 - y)^{+ip/2} F(a, b, a + b - c + 1; 1 - y) \]

\[ + \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} y^{+ip/2+1/2}(1 - y)^{+ip/2} \times (1 - y)^{c-a-b}F(c - a, c - b, c - a - b + 1; 1 - y) , \]

(34)

Allowing for identity \( c - a - b = -ip - 1/2, \) we derive asymptotic formula at \( z \to +\infty \) \( (y \to 1) \)

\[ Z_1^{(1)} \sim \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} (1 - y)^{+ip/2} + \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} (1 - y)^{-ip/2+1/2} . \]

(35)
Similarly, for $Z_1^{(5)}$ we produce

$$Z_1^{(5)} = \frac{\Gamma(2 - c)\Gamma(c - a - b)}{\Gamma(1 - a)\Gamma(1 - b)} 
\times y^{-ip/2}(1 - y)^{+ip/2} \times F(a, b, a + b - c + 1; 1 - y) + \frac{\Gamma(2 - c)\Gamma(a + b - c)}{\Gamma(a + 1 - c)\Gamma(b + 1 - c)} 
\times y^{-ip/2}(1 - y)^{+ip/2}(1 - y)^c-a-b F(c - a, c - b, c - a - b + 1; 1 - y).$$

(36)

Allowing for identity $c - a - b = -ip - 1/2$, we derive an asymptotic formula at $z \to +\infty$ ($y \to 1$)

$$Z_1^{(5)} \sim \frac{\Gamma(2 - c)\Gamma(c - a - b)}{\Gamma(1 - a)\Gamma(1 - b)} (1 - y)^{+ip/2} + \frac{\Gamma(2 - c)\Gamma(a + b - c)}{\Gamma(a + 1 - c)\Gamma(b + 1 - c)} (1 - y)^{-ip/2+1/2}. 
\text{(37)}$$

In the same manner, let us consider the equation for $Z_2(z)$. In the new variable $y = (1 + \tanh z)/2$, it reads

$$\left[4y(1-y)\frac{d}{dy} + 2(1-2y)\frac{d}{dy} + p^2\left(\frac{1}{1-y} + \frac{1}{y} - ip\left(\frac{1}{1-y} - \frac{1}{y}\right) - 4\lambda^2\right)\right]Z_2 = 0.$$ 

With the substitution $Z_2 = y^K(1-y)^L \tilde{Z}_2(y)$, it leads to

$$4y(1-y)\frac{d^2\tilde{Z}_2}{dy^2} + 4\left[2K + \frac{1}{2} - (2K + 2L + 1) y\right]\frac{d\tilde{Z}_2}{dy} + \left[\frac{2K(2K - 1) - p(-p - i)}{y} + 2L(2L - 1) - p(-p + i)\right] \frac{1}{1-y} - 4(K + L)^2 - 4\lambda^2 \right] \tilde{Z}_2 = 0.$$ 

With restrictions

$$K = \frac{ip}{2}, \quad \frac{1 - ip}{2}, \quad L = -\frac{ip}{2}, \quad \frac{1 + ip}{2}, \quad \text{(38)}$$

the equation for $\tilde{Z}$ is reduced to that of hypergeometric type

$$y(1-y)\frac{d^2\tilde{Z}_2}{dz^2} + \left[2K + \frac{1}{2} - (2K + 2L + 1) y\right]\frac{d\tilde{Z}_2}{dz} - \left[(K + L)^2 + \lambda^2\right] \tilde{Z}_2 = 0 \quad \text{(39)}$$

with parameters given by

$$\alpha = K + L + i\lambda, \quad \beta = K + L - i\lambda, \quad \gamma = 2K + \frac{1}{2}, \quad Z_2 = y^K(1-y)^L F(\alpha, \beta, \gamma; y). \quad \text{(40)}$$

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For definiteness, let us choose
\[ K = \frac{+ip}{2}, \quad L = \frac{ip + 1}{2}, \quad K + L = ip + 1/2, \]
\[ \alpha = ip + 1/2 + i\lambda, \quad \beta = ip + 1/2 - i\lambda, \quad \gamma = ip + 1/2. \] \tag{41}

Two linearly independent solutions are
\[ Z_2^{(1)} = y^K(1 - y)^L U_1(y) = y^K(1 - y)^L F(\alpha, \beta, \gamma; y) \]
\[ = y^{+ip/2}(1 - y)^{+ip/2+1/2} F(\alpha, \beta, \gamma; y), \] \tag{42}
\[ Z_2^{(5)} = y^K(1 - y)^L U_5(y) \]
\[ = y^{-ip/2+1/2}(1 - y)^{-ip/2+1/2} F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma, y). \] \tag{43}

Among the functions \( Z_2^{(1)}(y), Z_2^{(5)}(y), Z_2^{(1)}(y), Z_2^{(5)}(y) \) there must exist pairs related by the first order system (see (21))
\[ \left( y(1 - y) \frac{d}{dy} + \frac{ip}{2} \right) Z_1 = \lambda \sqrt{y(1 - y)} Z_2, \]
\[ \left( y(1 - y) \frac{d}{dy} - \frac{ip}{2} \right) Z_2 = \lambda \sqrt{y(1 - y)} Z_1. \] \tag{44}

Let us start with
\[ Z_2^{(1)} = z_2^{(1)} y^{+ip/2+1/2}(1 - y)^{+ip/2} F(a, b, c; y), \]
\[ Z_2^{(5)} = z_2^{(5)} y^{-ip/2+1/2}(1 - y)^{-ip/2+1/2} F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma, y), \]
\[ a = \alpha = ip + 1/2 + i\lambda, \quad b = \beta = ip + 1/2 - i\lambda, \]
\[ c = \gamma + 1 = ip + 3/2. \] \tag{45}

Substituting these expressions into the second equation
\[ \left( y(1 - y) \frac{d}{dy} - \frac{ip}{2} \right) Z_2^{(1)} = \lambda \sqrt{y(1 - y)} Z_2^{(1)}, \]
we get
\[ z_2^{(1)} \left( y(1 - y) \frac{d}{dy} - \frac{ip}{2} \right) y^{+ip/2}(1 - y)^{+ip/2+1/2} F(\alpha, \beta, \gamma; y) \]
\[ = z_2^{(1)} \lambda y^{+ip/2+1}(1 - y)^{+ip/2+1/2} F(a, b, c; y), \]
or
\[ z_2^{(1)} \left( \frac{ip}{2} (1 - y) - \left( \frac{ip}{2} + \frac{1}{2} \right) y + y(1 - y) \frac{d}{dy} - \frac{ip}{2} \right) F(a, b, c - 1; y) \]
\[ = z_2^{(1)} \lambda y F(a, b, c; y). \]
From whence it follows
\[
z_{2}^{(1)} \left(-ip - \frac{1}{2} + (1 - y) \frac{d}{dy}\right) F(a, b, c - 1; y) = z_{1}^{(1)} \lambda F(a, b, c; y) ,
\]
or differently
\[
z_{2}^{(1)} \left(1 - c + (1 - y) \frac{d}{dy}\right) F(a, b, c - 1; y) = z_{1}^{(1)} \lambda F(a, b, c; y) .
\] (46)

There exist a formula, relating derivatives of hypergeometric functions with contiguous hypergeometric functions
\[
\left(1 - c + (1 - y) \frac{d}{dy}\right) F(a, b, c - 1, y) = \frac{(a - c + 1)(b - c + 1)}{c - 1} F(a, b, c, y).
\]
Hence relation (46) gives
\[
z_{2}^{(1)} \frac{(a - c + 1)(b - c + 1)}{c - 1} F(a, b, c, y) = z_{1}^{(1)} \lambda F(a, b, c; y),
\]
from whence it follow
\[
z_{2}^{(1)} \frac{(a - c + 1)(b - c + 1)}{c - 1} = z_{1}^{(1)} \lambda .
\] (47)

Let us consider the second pair of solutions, starting with
\[
Z_{1}^{(5)} = y^{-ip/2}(1 - y)^{+ip/2} F(a + 1 - c, b + 1 - c, 2 - c, y),
\]
\[
a = ip + 1/2 + i\lambda , \quad b = ip + 1/2 - i\lambda , \quad c = ip + 3/2 ,
\]
and
\[
Z_{2}^{(5)} = y^{-ip/2+1/2}(1 - y)^{+ip/2+1/2} F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma, y)
\]
\[
\alpha = ip + 1/2 + i\lambda , \quad \beta = ip + 1/2 - i\lambda , \quad \gamma = ip + 1/2 .
\]
Firstly, it should be noted
\[
a' = a + 1 - c = +i\lambda , \quad \alpha + 1 - \gamma = +i\lambda + 1 = a' + 1 ,
\]
\[
b' = b + 1 - c = -i\lambda , \quad \beta + 1 - \gamma = i\lambda + 1 = b' + 1 ,
\]
\[
2 - c = -ip + 1/2 = c' , \quad 2 - \gamma = -ip + 3.2 = c' + 1 ,
\]
Substituting the above expressions into the first equation
\[
z_{1}^{(5)} \left(y(1 - y) \frac{d}{dy} - \frac{ip}{2}\right) y^{+ip/2}(1 - y)^{+ip/2+1/2} F(a', b', c'; y)
\]
\[
= z_{2}^{(5)} \lambda y^{+ip/2+1}(1 - y)^{+ip/2+1/2} F(a' + 1, b' + 1, c' + 1; y),
\]
we get
\[
z_{1}^{(5)} \left(\frac{ip}{2} - (1 - y) - \frac{ip}{2} y + y(1 - y) \frac{d}{dy} + \frac{ip}{2}\right) F(a', b', c'; y)
\]
\[
= z_{2}^{(5)} \lambda y(1 - y) F(a' + 1, b' + 1, c' + 1; y).
\]
It reduces to
\[ z_1^{(5)} y(1-y) \frac{d}{dy} F(a',b',c';y) = z_2^{(5)} \lambda y(1-y) F(a'+1,b'+1,c'+1;y), \]
so we arrive at
\[ z_1^{(5)} \frac{a'b'}{c'} = z_2^{(5)} \lambda. \] (48)

4 Solution of the equations in \( r \)-variable

From radial equations (22) it follows second order equations for \( R_1 \) and \( R_2 \)
\begin{align*}
\left( \frac{d^2}{dr^2} - \frac{d\mu}{dr} - \mu^2 + \lambda^2 \right) R_1 &= 0, \\
\left( \frac{d^2}{dr^2} + \frac{d\mu}{dr} - \mu^2 + \lambda^2 \right) R_2 &= 0. \quad (49)
\end{align*}

Remembering on the meaning of \( \mu(r) \) we obtain their explicit form (for shortness let us note \( eB \) as \( B \))
\begin{align*}
\left( \frac{d^2}{dr^2} + \frac{m \cosh r + B (\cosh r - 1)}{\sinh^2 r} - \frac{[m - B (\cosh r - 1)]^2}{\sinh^2 r} + \lambda^2 \right) R_1 &= 0, \\
\left( \frac{d^2}{dr^2} - \frac{m \cosh r + B (\cosh r - 1)}{\sinh^2 r} - \frac{[m - B (\cosh r - 1)]^2}{\sinh^2 r} + \lambda^2 \right) R_2 &= 0.
\end{align*}

Two equations are related by the formal change
\[ m \Rightarrow -m, \quad B \Rightarrow -B, \quad R_1 \Rightarrow R_2. \]

With the variable \( y = (1 + \cosh r)/2 \), making the substitution
\[ R_1 = y^A (1-y)^C \tilde{R}_1(y), \]
one gets an equation for \( \tilde{R}_1 \)
\begin{align*}
y(1-y) \frac{d^2 \tilde{R}_1}{dy^2} + \left[ 2A + \frac{1}{2} - (2A + 2C + 1) y \right] \frac{d \tilde{R}_1}{dy} \\
+ \frac{A^2 - A/2 - m^2/4 - m/4 - mB - B^2 - B/2}{y} \\
+ \frac{C^2 - C/2 - m^2/4 + m/4}{1-y} - (A + C)^2 - \lambda^2 + B^2 \right] \tilde{R}_1 &= 0. \quad (50)
\end{align*}

With restrictions
\[ A = -\frac{2B + m}{2}, \quad \frac{2B + m + 1}{2}, \quad C = \frac{m}{2}, \quad \frac{1-m}{2}, \]
we arrive at an equation of hypergeometric type
\begin{align*}
y(1-y) \frac{d^2 \tilde{R}_1}{dy^2} + \left[ 2A + \frac{1}{2} - (2A + 2C + 1) y \right] \frac{d \tilde{R}_1}{dy} \\
- \left[ (A + C)^2 + \lambda^2 - B^2 \right] \tilde{R}_1 &= 0, \]

(51)
so that

\[ R_1 = y^A (1 - y)^C F(\alpha, \beta, \gamma; y), \quad \gamma = 2A + \frac{1}{2}, \]
\[ \alpha = A + C + \sqrt{B^2 - \lambda^2}, \quad \beta = A + C - \sqrt{B^2 - \lambda^2}. \]  

(52)

To have solutions finite in the origin \( r = 0 \) \((y \to 1)\) (corresponding geometrical points belong to the axis \( z: u_0 = \cosh z, u_3 = \sinh z, u_1 = 0, u_2 = 0 \)) and in infinity \( r \to \infty \) \((y \to \infty)\), we must take positive \( C \) and negative \( A \), such that

\[ C > 0, \quad A < 0, \quad C + A < 0, \]
\[ R_1 = y^A (1 - y)^C F(\alpha, \beta, \gamma; y). \]  

(53)

Let us write down all four possibilities to choose the parameters (for definiteness let it be \( B > 0 \))

1. \( C = \frac{m}{2}, \quad A = -\frac{2B + m}{2}, \quad C + A = -B; \)
2. \( C = \frac{1 - m}{2}, \quad A = -\frac{2B + m}{2}, \quad C + A = -B - m + \frac{1}{2}; \)
3. \( C = \frac{m}{2}, \quad A = \frac{2B + m + 1}{2}, \quad C + A = B + m + \frac{1}{2}; \)
4. \( C = \frac{1 - m}{2}, \quad A = \frac{2B + m + 1}{2}, \quad C + A = B + 1. \)  

(54)

Regarding \( (53) \) we conclude that only variants 1 and 2 are acceptable for describing bound states (they coincide when \( m = +1/2 \))

\begin{align*}
\text{Variant 1,} \quad m > 0 & \quad (m = +1/2, +3/2, \ldots); \\
\text{Variant 2,} \quad -B + 1/2 < m < 1 & \quad (m = m_{\text{min}}, \ldots, -1/2, +1/2).
\end{align*}  

(55)

Respective expressions for radial functions are

\begin{align*}
\text{Variant 1,} \quad m = +1/2, +3/2, \ldots, \\
C = m/2, \quad A = -B - m/2 < 0, \quad R_1 = y^{B - m/2} (1 - y)^{m/2} F(a, b, c; y), \\
a = -B + \sqrt{B^2 - \lambda^2}, \quad b = -B - \sqrt{B^2 - \lambda^2}, \quad c = -2B - m + \frac{1}{2}; \\
\text{with the quantization rule} \\
a = -n \implies \sqrt{B^2 - \lambda^2} = B - n \implies \lambda^2 = B^2 - (B - n)^2; \quad (56)
\end{align*}

to have radial function finite at the infinity \( r \to \infty \), the following inequality must be imposed

\[ A + C + n < 0 \implies B - n > 0; \]  

(57)

which insures the positive square root \( +\sqrt{B^2 - \lambda^2} \) in \( (56) \).
\textbf{Variant 2}. \(-B + 1/2 < m < 1\) \((m = m_{\text{min}}, \ldots, -1/2, +1/2)\),

\[ C = 1/2 - m/2, \quad A = -B - m/2 < 0, \]

\[ R_1 = y^{-B-m/2} (1 - y)^{1/2-m/2} F(a', b', c'; y), \]

\[ a' = -B - m + 1/2 + \sqrt{B^2 - \lambda^2}, \]

\[ b' = -B - m + 1/2 - \sqrt{B^2 - \lambda^2}, \quad c' = -2B - m + 1/2; \]

the quantization rule is

\[ a' = -n \implies +\sqrt{B^2 - \lambda^2} = B + m - 1/2 - n \]

\[ \implies \lambda^2 = B^2 - (B + m - 1/2 - n)^2; \]

an inequality must be fulfilled

\[ A + C + n < 0 \implies B + m - 1/2 - n > 0, \]

which make positive the root \(+\sqrt{B^2 - \lambda^2}\) in (59). Note that when \(m = +1/2\), the formula (59)
give the same result as (56).

Thus, the energy spectrum for spin \(1/2\) particle in the magnetic field in the Lobachevsky
space model is given by two formulas

\textbf{Variant 1}. \quad \lambda^2 = B^2 - (B - n)^2, \quad m = +1/2, +3/2, \ldots, \quad n < B; \]

\textbf{Variant 2}. \quad \lambda^2 = B^2 - (B + m - 1/2 - n)^2, \quad -B + 1/2 < m_{\text{min}}, \ldots, +1/2, \quad n < B + m - 1/2. \]

Now we should construct explicit form of radial functions \(R_2(y)\). Analysis is similar (but
with significant differences). In the variable \(y = (1 + \cosh r)/2\), making the substitution

\[ R_2 = y^K (1 - y)^L \bar{R}_2(y), \]

one gets equation \(\bar{R}_2\)

\[ y(1 - y) \frac{d^2 \bar{R}_2}{dy^2} + \left[ 2K + \frac{1}{2} - (2K + 2L + 1) y \right] \frac{d\bar{R}_2}{dy} \]

\[ + \left[ K^2 - K/2 - m^2/4 + m/4 - mB - B^2 - B/2 \right] \frac{\bar{R}_2}{y} \]

\[ + \frac{L^2 - L/2 - m^2/4 - m/4}{1 - y} - (K + L)^2 - \lambda^2 + B^2 \right] \bar{R}_2 = 0. \]

With restrictions

\[ K = \frac{2B + m}{2}, \quad -2B - m + 1, \quad L = \frac{-m}{2}, \quad \frac{1 + m}{2}, \quad \]

we arrive at an equation of hypergeometric type

\[ y(1 - y) \frac{d^2 \bar{R}_2}{dy^2} + \left[ 2K + \frac{1}{2} - (2K + 2L + 1) y \right] \frac{d\bar{R}_2}{dy} \]

\[-\left[ (K + L)^2 + \lambda^2 - B^2 \right] \bar{R}_2 = 0, \]
so that

\[ R_2 = y^K (1 - y)^L F(\alpha', \beta', \gamma'; y), \quad \gamma' = 2K + \frac{1}{2}, \]
\[ \alpha' = K + L + \sqrt{B^2 - \lambda^2}, \quad \beta' = K + L - \sqrt{B^2 - \lambda^2}. \] (64)

To have solutions finite at \( r = 0 \) (\( y \to 1 \)) and at \( r \to \infty \) (\( y \to \infty \)), we must take positive \( L \) and negative \( K \), such that

\[ L > 0, \quad K < 0, \quad L + K < 0, \]
\[ R_2(y) = y^K (1 - y)^L F(\alpha', \beta', \gamma'; y). \] (65)

Let us write down all four possibilities to choose the parameters (remembering that \( B > 0 \))

1’. \[ L = \frac{-m}{2}, \quad K = \frac{2B + m}{2}, \quad L + K = B \ (NO); \]
2’. \[ L = \frac{1 + m}{2}, \quad K = \frac{2B + m}{2}, \quad L + K = B + m + \frac{1}{2} \ (NO); \]
3’. \[ L = \frac{-m}{2}, \quad K = \frac{-2B - m + 1}{2}, \quad L + K = -B - m + \frac{1}{2}; \]
4’. \[ L = \frac{1 + m}{2}, \quad K = \frac{-2B - m + 1}{2}, \quad L + K = -B + 1. \] (66)

Regarding (63) we conclude that only two variants, 3’ and 4’, are acceptable for describing bound states

\[ \text{Variant 4'}, \quad m > -1, \quad B > 1 \quad (m = -1/2, +1/2, +3/2,...); \]
\[ \text{Variant 3'}, \quad -B + 1/2 < m < 0 \quad (m = m_{min}, ..., -1/2). \] (67)

Respective expressions for radial functions are

\[ \text{Variant 4'}, \quad m = -1/2, +1/2, +3/2,..., \]
\[ L = m/2 + 1/2, \quad K = -B - m/2 + 1/2 < 0, \]
\[ \gamma = -B - m + \frac{3}{2}, \quad \alpha = -B + 1 + \sqrt{B^2 - \lambda^2}, \quad \beta = -B - 1 - \sqrt{B^2 - \lambda^2}, \]
\[ R_2 = y^{-B-m/2+1/2} (1 - y)^{m/2+1/2} F(\alpha, \beta, \gamma; y), \] (68)

with the quantization rule

\[ \alpha = -N \quad \Rightarrow \quad \sqrt{B^2 - \lambda^2} = B - N - 1 \quad \Rightarrow \]
\[ \lambda^2 = B^2 - (B - N - 1)^2; \] (69)

to have radial function finite at the infinity \( r \to \infty \), the following inequality must be imposed

\[ K + L + N < 0 \quad \Rightarrow \quad B - N - 1 > 0, \] (70)

which insures the positive square root \( +\sqrt{B^2 - \lambda^2} \) in (69).
Variant 3’, \(-B + 1/2 < m < 0\) \((m = m_{\text{min}}, ..., -1/2)\),

\[
L = -m/2, \quad K = -B - m/2 + 1/2 < 0, \\
K + L = -B - m + 1/2, \quad \gamma' = -2B - m + 3/2, \\
\alpha' = -B - m + 1/2 + \sqrt{B^2 - \lambda^2}, \\
\beta' = -B - m + 1/2 - \sqrt{B^2 - \lambda^2}, \\
R_2 = y^{-B - m/2 + 1/2} (1 - y)^{-m/2} F(\alpha', \beta', \gamma'; y); \\
\]

(71)

the quantization rule is

\[
\alpha' = -N \quad \implies \quad +\sqrt{B^2 - \lambda^2} = B + m - 1/2 - n \\
\implies \quad \lambda^2 = B^2 - (B + m - 1/2 - n)^2; \\
\]

(72)

the inequality must be fulfilled

\[
A + C + n < 0 \quad \implies \quad B + m - 1/2 - n > 0, \\
\]

(73)

which make positive the root \(+\sqrt{B^2 - \lambda^2}\) in (72).

Analysis of the bound states from equation for \(R_2\) can be summarized by the formula

Let us collect results on energy level obtained from analysis of \(R_1(y)\) and \(R_2(y)\):

\[
R_1(y), \quad 1. \quad \lambda^2 = B^2 - (B - n)^2, \\
\quad m = +1/2, +3/2, ..., \quad n < B; \\
2. \quad \lambda^2 = B^2 - (B + m - 1/2 - n)^2, \\
\quad -B + 1/2 < m = m_{\text{min}}, ..., -1/2, +1/2, \quad n < B + m - 1/2; \\
\]

(74)

and

\[
R_2(y), \quad 4'. \quad \lambda^2 = B^2 - (B - N - 1)^2, \\
\quad m = -1/2, +1/2, +3/2, ..., \quad N + 1 < B; \\
3'. \quad \lambda^2 = B^2 - (B + m - 1/2 - n)^2, \\
\quad -B + 1/2 < m = m_{\text{min}}, ..., -1/2, \quad n < B + m - 1/2. \\
\]

(75)

We see evident correlations between the cases 1 and 4’, as well as between the cases 2 and 3’. To detail them, let us turn to the system (22) relating functions \(R_1\) and \(R_2\)

\[
\left(\frac{d}{dr} + \mu\right) R_2 + \lambda R_1 = 0, \quad \left(\frac{d}{dr} - \mu\right) R_1 - \lambda R_2 = 0. \\
\]

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After translating to the variable \( y = (1 + \cosh r)/2 \)

\[
\frac{d}{dr} = \sqrt{-y(1-y)} \frac{d}{dr}, \quad \sinh r = 2\sqrt{-y(1-y)},
\]

\[
\mu = \frac{m - B(\cosh r - 1)}{\sinh r} = \frac{m - 2B(y - 1)}{2\sqrt{-y(1-y)}},
\]

it assumes the form

\[
\sqrt{-y(1-y)} \left( \frac{d}{dy} - \frac{m/2 + B}{y} - \frac{m/2}{1-y} \right) R_2 + \lambda R_1 = 0,
\]

\[
\sqrt{-y(1-y)} \left( \frac{d}{dy} + \frac{m/2 + B}{y} + \frac{m/2}{1-y} \right) R_1 - \lambda R_2 = 0.
\] (77)

First, let us consider the cases 1 and 4’:

\[
R_1 = r_1 y^{-B-m/2} (1-y)^{m/2} F(a, b, c; y),
\]

\[
a = -B + \sqrt{B^2 - \lambda^2}, \quad b = -B - \sqrt{B^2 - \lambda^2}, \quad c = -2B - m + \frac{1}{2};
\]

\[
R_2 = r_2 y^{-B-m/2+1/2} (1-y)^{m/2+1/2} F(\alpha, \beta, \gamma; y),
\]

\[
\alpha = -B + 1 + \sqrt{B^2 - \lambda^2} = a + 1, \quad \beta = -B - 1 + \sqrt{B^2 - \lambda^2} = b + 1, \quad \gamma = -2B - m + \frac{3}{2} = c + 1.
\]

Substituting expressions for \( R_1 \) and \( R_2 \) into the second equation

\[
\left( \frac{d}{dy} + \frac{m/2 + B}{y} + \frac{m/2}{1-y} \right) R_1 - \lambda \frac{\lambda}{\sqrt{-y(1-y)}} R_2 = 0,
\]

that is

\[
\left( \frac{d}{dy} + \frac{m/2 + B}{y} + \frac{m/2}{1-y} \right) r_1 y^{-B-m/2} (1-y)^{m/2} F(a, b, c; y)
\]

\[\quad -i\lambda r_2 y^{-B-m/2} (1-y)^{m/2} F(a + 1, b + 1, c + 1; y) = 0,
\]

we arrive at

\[
r_1 \frac{d}{dy} F(a, b, c, y) - i\lambda r_2 F(a + 1, b + 1, c + 1, y) = 0,
\]

which means

\[m a b c - i\lambda r_2 = 0. \] (78)
Now consider the cases 2 and 3':

\[ R_1 = r_1 y^{-B-m/2} (1 - y)^{1/2-m/2} F(a', b', c'; y), \]
\[ a' = -B - m + \frac{1}{2} + \sqrt{B^2 - \lambda^2}, \]
\[ b' = -B - m + \frac{1}{2} - \sqrt{B^2 - \lambda^2}, \]
\[ c' = -2B - m + \frac{1}{2}; \]

\[ R_2 = r_2 y^{-B-m/2+1/2} (1 - y)^{-m/2} F(\alpha', \beta', \gamma'; y), \]
\[ \alpha' = -B - m + \frac{1}{2} + \sqrt{B^2 - \lambda^2} = a', \]
\[ \beta' = -B - m + \frac{1}{2} - \sqrt{B^2 - \lambda^2} = b', \]
\[ \gamma' = -2B - m + \frac{3}{2} = c' + 1. \]

Substituting expressions for \( R_1 \) and \( R_2 \) into the second equation

\[
\left( \frac{d}{dy} + \frac{m/2 + B}{y} + \frac{m/2}{1 - y} \right) r_1 y^{-B-m/2}(1 - y)^{1/2-m/2} F(a', b', c' + 1; y) \\
- \lambda r_2 y^{-B-m/2+1/2} (1 - y)^{-m/2} F(a', b', c'; y) = 0.
\]

we get

\[
 r_1 \left( (m - 1/2) + (1 - y) \frac{d}{dy} \right) F(a', b', c'; y) - ir_2 F(a', b', c' + 1; y) = 0.
\]

And further, with the use of identity

\[
- (a' + b' - c') F(a', b', c', y) + (1 - y) \frac{d}{dy} F(a', b', c', y) = \frac{(a' - c')(b' - c')}{c'} F(a', b', c' + 1, y)
\]

we arrive at a needed relationship

\[
r_1 \frac{(a' - c')(b' - c')}{c'} - i\lambda r_2 F(a', b', c' + 1; y) = 0.
\]

In the next part of the paper, we consider an analogous problem for a particle with spin 1/2 described by Dirac equation in spherical Riemann space. Though the treatment is similar, spherical geometry noticeably changes the task and final results.
5 Cylindric coordinates and the Dirac equation in spherical space \( S_3 \), separation of the variables

In the spherical Riemann space \( S_3 \), let us use an extended cylindric coordinates (see [33])

\[
dS^2 = dt^2 - \cos^2 z(dr^2 + \sin^2 r d\phi^2) + dz^2,
\]

\( z \in [-\pi/2, +\pi/2] , \quad r \in [0, +\pi] , \quad \phi \in [0, 2\pi] , \)

\[
u_1 = \cos z \sin r \cos \phi , \quad \nu_2 = \cos z \sin r \sin \phi , \quad \nu_3 = \sin z , \quad \nu_0 = \cos z \cos r . \tag{81}
\]

An analogue of usual homogeneous magnetic field is defined as [25, 26, 27]

\[
A_\phi = -2B \sin^2 r/2 = B (\cos r - 1) . \tag{82}
\]

To coordinates (81) there corresponds the tetrad

\[
e^\beta_{(a)}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^{-1} z & 0 & 0 \\ 0 & 0 & \cos^{-1} z \sin^{-1} r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{83}
\]

Christoffel symbols \( \Gamma^r_{jk} \) and Ricci rotation coefficients \( \gamma_{abc} \) are

\[
\Gamma^r_{jk} = \begin{pmatrix} 0 & 0 & -\tan z \\ -\tan z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \Gamma^\phi_{jk} = \begin{pmatrix} 0 & \cot r & 0 \\ \cot r & 0 & -\tan z \\ 0 & -\tan z & 0 \end{pmatrix} ,
\]

\[
\Gamma^z_{jk} = \begin{pmatrix} \sin z \cos z & 0 & 0 \\ 0 & \sin z \cos z \sin^2 r & 0 \\ 0 & 0 & 0 \end{pmatrix} ,
\]

\[
\gamma_{122} = \frac{1}{\cos z \tan r} , \quad \gamma_{311} = -\tan z , \quad \gamma_{322} = -\tan z . \tag{84}
\]

A general covariant Dirac equation (for more detail see [10]) takes the form

\[
\left[ i\gamma^0 \partial_t + \frac{i\gamma^1}{\cos z} (\partial_r + \frac{1}{2\tan r}) + \gamma^2 \frac{i\partial_\phi - eB(\cos r - 1)}{\cos z \sin r} + i\gamma^3 (\partial_z - \tan z - M) \right] \Psi = 0 . \tag{85}
\]

With the substitution \( \Psi = \psi/\cos z \sqrt{\sin r} \) eq. (85) becomes simpler

\[
\left[ i\gamma^1 \partial_r + \gamma^2 \frac{i\partial_\phi - eB(\cos r - 1)}{\sin r} + \cos z (i\gamma^0 \partial_t + i\gamma^3 \partial_z - M) \right] \psi = 0 . \tag{86}
\]

Solutions of this equation will be searched in the form

\[
\psi = e^{-i\epsilon t} e^{i m \phi} \begin{pmatrix} f_1(r, z) \\ f_2(r, z) \\ f_3(r, z) \\ f_4(r, z) \end{pmatrix} ;
\]
so that
\[
[i\gamma^1\partial_r - \mu(r)\gamma^2 + \cos z(\epsilon\gamma^0 + i\gamma^3\partial_z - M)] \begin{pmatrix} f_1(r, z) \\ f_2(r, z) \\ f_3(r, z) \\ f_4(r, z) \end{pmatrix} = 0,
\]
where
\[
\mu(r) = \frac{m - eB(1 - \cos r)}{\sin r}.
\]
Taking the Dirac matrices in spinor basis, we get radial equations for \( f_a(t, z) \)
\[
\begin{align*}
(\partial_r + \mu) f_1 + \cos z \partial_z f_3 + i \cos z (\epsilon f_3 - M f_1) &= 0, \\
(\partial_r - \mu) f_3 - \cos z \partial_z f_1 + i \cos z (\epsilon f_1 - M f_3) &= 0, \\
(\partial_r + \mu) f_2 + \cos z \partial_z f_1 - i \cos z (\epsilon f_1 - M f_2) &= 0, \\
(\partial_r - \mu) f_1 - \cos z \partial_z f_2 - i \cos z (\epsilon f_2 - M f_1) &= 0.
\end{align*}
\]
With linear restriction \( f_3 = A f_1, f_4 = A f_2 \), where
\[
\epsilon - \frac{M}{A} = -\epsilon + M A \quad \implies \quad A = A_{1,2} = \frac{\epsilon \pm p}{M}
\]
eqs. (88) give
\[
\begin{align*}
(\partial_r + \mu) f_2 + \cos z \partial_z f_1 + i \cos z (-\epsilon + M A) f_1 &= 0, \\
(\partial_r - \mu) f_1 - \cos z \partial_z f_2 + i \cos z (-\epsilon + M A) f_2 &= 0.
\end{align*}
\]
Thus, we have two possibilities
\[
A = (\epsilon + p)/M,
\]
\[
\begin{align*}
(\partial_r + \mu) f_2 + \cos z (\partial_z + ip) f_1 &= 0, \\
(\partial_r - \mu) f_1 - \cos z (\partial_z - ip) f_2 &= 0;
\end{align*}
\]
\[
A = (\epsilon - p)/M,
\]
\[
\begin{align*}
(\partial_r + \mu) f_2 + \cos z (\partial_z - ip) f_1 &= 0, \\
(\partial_r - \mu) f_1 - \cos z (\partial_z + ip) f_2 &= 0.
\end{align*}
\]
Evidently, such a simplification can be rationalized through diagonalization of extended helicity operator \( \Lambda \) (compare with \( \square \))
\[
\Lambda = \frac{1}{\cos z} \left( \gamma^2 \gamma^3 \partial_r - i \gamma^3 \gamma^4 \frac{i\partial_\phi - eB(\cos r - 1)}{\sin r} \right) + \gamma^1 \gamma^2 \partial_z.
\]
For definiteness, let us consider the system \( \square \) (transition to the case \( \square \) is performed by the formal change \( p \mapsto -p \)). Let us search solutions in the form
\[
f_1 = Z_1(z) R_1(r), \quad f_2 = Z_2(z) R_2(r).
\]
Introducing a separating constant \( \lambda \), we get two systems
\[
\begin{align*}
\cos z \left( \frac{d}{dz} + ip \right) Z_1 &= \lambda Z_2, \\
\cos z \left( \frac{d}{dz} - ip \right) Z_2 &= \lambda Z_1, \\
(\frac{d}{dr} + \mu) R_2 + \lambda R_1 &= 0, \\
(\frac{d}{dr} - \mu) R_1 - \lambda R_2 &= 0.
\end{align*}
\]
6 Solution of the equations in $z$-variable

From (94) it follows the second-order differential equations for $Z_1(z)$ and $Z_2(z)$

\[
\begin{align*}
\left(\frac{d^2}{dz^2} - \frac{\sin z}{\cos z} \frac{d}{dz} + p^2 - ip \frac{\sin z}{\cos z} - \frac{\lambda^2}{\cos^2 z}\right) Z_1 &= 0, \\
\left(\frac{d^2}{dz^2} - \frac{\sin z}{\cos z} \frac{d}{dz} + p^2 + ip \frac{\sin z}{\cos z} - \frac{\lambda^2}{\cos^2 z}\right) Z_2 &= 0.
\end{align*}
\]

(96)

In a new variable $y = (1 + i \tan z)/2$, with the use of the substitution $Z_1 = y^A(1 - y)^C Z(y)$, eq. (96) gives

\[
\begin{align*}
4y (1 - y) \frac{d^2 Z}{dz^2} + 4\left[ 2A + \frac{1}{2} - (2A + 2C + 1) y \right] \frac{dZ}{dz} \\
+ \left[ \frac{2A(2A-1) - p (p+1)}{y} + \frac{2C(2C-1) - p (p-1)}{1-y} \right] \\
- 4 (A + C)^2 + 4\lambda^2 \right] Z = 0.
\end{align*}
\]

With restrictions

\[
\begin{align*}
A &= -\frac{p}{2}, \quad \frac{p+1}{2}, \quad C = \frac{p}{2}, \quad \frac{1-p}{2},
\end{align*}
\]

(97)

for $Z$ we get an equation of hypergeometric type

\[
y(1 - y) \frac{d^2 Z}{dz^2} + \left[ 2A + \frac{1}{2} - (2A + 2C + 1) y \right] \frac{dZ}{dz} - [(A + C)^2 - \lambda^2] Z = 0,
\]

where

\[
a = A + C + \lambda, \quad b = A + C - \lambda, \quad c = 2A + \frac{1}{2},
\]

\[
Z_1 = y^A (1 - y)^C F(a, b, c, y).
\]

(98)

Because $p > 0$, to get solutions that are finite at the point $z = \pm \pi/2$ (that correspond to the points $u_a = (0, 0, 0, \pm 1)$ in the spherical space $S_3$), we should take negative values for $A$ and $C$:

\[
A = -\frac{p}{2}, \quad C = \frac{1-p}{2},
\]

\[
a = -p + \frac{1}{2} + \lambda, \quad b = -p + \frac{1}{2} - \lambda, \quad c = -p + \frac{1}{2},
\]

\[
Z_1 = y^A(1-y)^C U_1(y) = y^{-p/2}(1-y)^{-p/2+1/2} F(a, b, c; y).
\]

(99)

Quantization condition is

\[
a = -n \quad \Rightarrow \quad p = \lambda + n + \frac{1}{2}.
\]

(100)

Besides we must require that

\[
A + B - n < 0 \quad \Rightarrow \quad -p + n + \frac{1}{2} < 0 \quad \Rightarrow \quad \lambda > 0.
\]
There exist symmetrical quantization condition

\[ b = -n \quad \Rightarrow \quad p = -\lambda + n + \frac{1}{2}, \]  

(101)

and

\[ A + B - n < 0 \quad \Rightarrow \quad -p + n + \frac{1}{2} < 0 \quad \Rightarrow \quad \lambda < 0. \]

These two possibilities are equivalent, for definiteness we will take the variant with \( \lambda > 0 \).

Let us turn to the equation for second function \( Z_2 \)

\[ \left( \frac{d^2}{dz^2} - \frac{\sin z}{\cos z} \frac{d}{dz} + p^2 + i p \frac{\sin z}{\cos z} - \frac{\lambda^2}{\cos^2 z} \right) Z_2 = 0. \]  

(102)

In a new variable \( y = (1 + i \tan z)/2 \), with the use of the substitution \( Z_2 = y^K (1 - y)^L \hat{Z}(y) \), eq. (102) gives

\[
4y(1-y) \frac{d^2 \hat{Z}}{dz^2} + 4\left[ 2K + \frac{1}{2} - (2K + 2L + 1)y \right] \frac{d \hat{Z}}{dz} \\
+ \left[ \frac{2K(2K-1) + p(-p+1)}{y} + \frac{2L(2L-1) + p(-p-1)}{1-y} \\
- 4(K + L)^2 + 4\lambda^2 \right] \hat{Z} = 0.
\]

Requiring

\[ K = \frac{p}{2}, \quad L = -\frac{p + 1}{2}, \]  

(103)

for \( \hat{Z} \) we get an equation of hypergeometric type

\[ y(1-y) \frac{d^2 \hat{Z}}{dz^2} + \left[ 2K + \frac{1}{2} - (2K + 2L + 1)y \right] \frac{d \hat{Z}}{dz} - [(K + L)^2 - \lambda^2] \hat{Z} = 0, \]

where

\[ \alpha = K + L + \lambda, \quad \beta = K + L - \lambda, \quad \gamma = 2K + \frac{1}{2}; \]

\[ Z_2 = y^K (1 - y)^L F(\alpha, \beta; y). \]  

(104)

Because \( p > 0 \), to get solutions that are finite at the point \( z = \pm \pi/2 \), we should take negative values for \( K \) and \( L \):

\[ K = -\frac{p + 1}{2}, \quad L = -\frac{p}{2}, \]  

\[ \alpha = -p + \frac{1}{2} + \lambda, \quad \beta = -p + \frac{1}{2} - \lambda, \quad \gamma = -p + \frac{3}{2}; \]

\[ Z_2 = y^{-p/2 + 1/2}(1 - y)^{-p/2} F(\alpha, \beta, \gamma; y). \]

Quantization condition is the old one

\[ \alpha = -n \quad \Rightarrow \quad p = \lambda + n + \frac{1}{2}. \]
Now, it is the point to find a relative factor between $Z_1(y)$ and $Z_2(y)$ with the help of the main first order system

$$i \left( y(1 - y) \frac{d}{dy} + \frac{p}{2} \right) Z_1 = \lambda \sqrt{y(1 - y)} Z_2,$$

$$i \left( y(1 - y) \frac{d}{dy} - \frac{p}{2} \right) Z_2 = \lambda \sqrt{y(1 - y)} Z_1. \quad (105)$$

Starting with

$$A = -\frac{p}{2}, \quad C = \frac{1}{2} - \frac{p}{2},$$

$$Z_1 = z_1 y^{-p/2} (1 - y)^{1/2 - p/2} F(a, b, c; y),$$

$$a = -p + \frac{1}{2} + \lambda, \quad b = -p + \frac{1}{2} - \lambda, \quad c = -p + \frac{1}{2};$$

$$K = -\frac{p}{2} + \frac{1}{2}, \quad L = -\frac{p}{2},$$

$$Z_2 = z_2 y^{-p/2 + 1/2} (1 - y)^{-p/2} F(\alpha, \beta, \gamma; y),$$

$$\alpha = -p + \frac{1}{2} + \lambda = a, \quad \beta = -p + \frac{1}{2} - \lambda = b, \quad \gamma = -p + \frac{3}{2} = c + 1,$$

and substituting functions $Z_1$ and $Z_2$ into the first equation in (105), we get

$$z_1 \left[ (1 - y) \frac{d}{dy} - (a + b - c) \right] F(a, b, c; y) z_1 + i\lambda z_2 F(a, b, c + 1; y) = 0. \quad (106)$$

With the use of the known relationship

$$-(a + b - c) F(a, b, c, y) + (1 - y) \frac{d}{dz} F(a, b, c, y) = \frac{(a - c)(b - c)}{c} F(a, b, c + 1, y), \quad (106)$$

it gives

$$z_1 \frac{(a - c)(b - c)}{c} F(a, b, c + 1, y) + i\lambda z_2 F(a, b, c + 1; y) = 0,$$

that is

$$z_1 \frac{(a - c)(b - c)}{c} + i\lambda z_2 = 0. \quad (107)$$

### 7 Solution of the equations in $r$-variable

From eqs. (105) it follows a second-order differential equation for $R_1$ (for brevity let $eB$ be noted as $B$)

$$\frac{d^2 R_1}{dr^2} + \left[ \frac{m \cos r - B (\cos r - 1)}{\sin^2 r} - \frac{[m + B (\cos r - 1)]^2}{\sin^2 r} + \lambda^2 \right] R_1 = 0. \quad (108)$$
With a new variable \( y = (1 + \cos r)/2 \), eq. (108) reads

\[
y(1-y) \frac{d^2 R_1}{dy^2} + \left[ \frac{1}{2} - y \right] \frac{dR_1}{dy} - \left[ -\lambda^2 + \frac{m^2}{4} \left( \frac{1}{y} + \frac{1}{1-y} \right) \right] R_1 = 0.
\]

With the substitution \( R_1 = y^A(1-y)^C \tilde{R}_1(y) \), we get

\[
y(1-y) \frac{d^2 \tilde{R}_1}{dy^2} + \left[ 2A + \frac{1}{2} - (2A + 2C + 1)y \right] \frac{d\tilde{R}_1}{dy}
+ \left[ \frac{A^2 - A/2 - m^2/4 - m/4 + mB - B^2 + B/2}{y} \right. \\
+ \left[ \frac{C^2 - C/2 - m^2/4 + m/4}{1-y} \right] - (A + C)^2 + \lambda^2 + B^2 \right] \tilde{R}_1 = 0.
\]

Requiring

\[ A = \frac{2B - m}{2}, \quad 1 - \left( \frac{2B - m}{2} \right), \quad C = \frac{m}{2}, \quad \frac{1-m}{2}, \quad (109) \]

we arrive at a differential equation of hypergeometric type \([32]\)

\[
y(1-y) \frac{d^2 \tilde{R}_1}{dy^2} + [2A + \frac{1}{2} - (2A + 2C + 1)y] \frac{d\tilde{R}_1}{dy}
- [ (A + C)^2 - \lambda^2 - B^2 ] \tilde{R}_1 = 0. \quad (110)
\]

where

\[
\alpha = A + C - \sqrt{B^2 + \lambda^2}, \quad \beta = A + C + \sqrt{B^2 + \lambda^2}, \quad \\
\gamma = 2A + 1/2, \quad R_1 = y^A(1-y)^C F(\alpha, \beta, \gamma; y). \quad (111)
\]

In order to have a finite solution at the origin \( r = 0 \) (that corresponds to the half-curve \( u_0 = + \cos z, u_3 = \sin z, u_1 = 0, u_2 = 0 \)) and at \( r = \pi \) (that corresponds to the other part of the curve, \( u_0 = - \cos z, u_3 = \sin z, u_1 = 0, u_2 = 0 \)), we must take positive values for \( A \) and \( C \) (\( A > 0, C > 0 \))

\[
R_1 = y^A(1-y)^C F(\alpha, \beta, \gamma; y). \quad (112)
\]

There are possible four variants (for definiteness assuming \( B > 0 \)):

variant 1, \[ A = \frac{2B - m}{2}, \quad C = \frac{1-m}{2}, \]
variant 2, \[ A = \frac{2B - m}{2}, \quad C = \frac{m}{2}, \]
variant 3, \[ A = \frac{m + 1 - 2B}{2}, \quad C = \frac{m}{2}, \]
variant 4, \[ A = \frac{m + 1 - 2B}{2}, \quad C = \frac{1-m}{2}. \]
For definiteness, let us consider $B > 0$. There arise the following three types of solutions

**Variant 1**, $m = \ldots, -3/2, -1/2, +1/2, +3/2$,

$$R_1 = y^{(2B-m)/2} (1-y)^{(1-m)/2} F(a, b, c; y),$$

$$a = B - m + 1/2 - \sqrt{\lambda^2 + B^2} = -n,$$

$$b = B - m + 1/2 + \sqrt{\lambda^2 + B^2}, \quad c = 2B - m + 1/2;$$

**Variant 2**, $m = +1/2, \ldots, \mu < 2B$,

$$R_1 = y^{(2B-m)/2} (1-y)^{m/2} F(a', b', c'; y),$$

$$a' = B - \sqrt{\lambda^2 + B^2} = -n,$$

$$b' = B + \sqrt{\lambda^2 + B^2}, \quad c' = 2B - m + 1/2;$$

**Variant 3**, $m > 2B - 1$,

$$R_1 = y^{(m+1-2B)/2} (1-y)^{m/2} F(a'', b'', c''; y),$$

$$a'' = m + 1/2 - B - \sqrt{\lambda^2 + B^2} = -n'',$$

$$b'' = m + 1/2 - B + \sqrt{\lambda^2 + B^2}, \quad c'' = -2B + m + 3/2.$$

From eqs. (95), it follows a second-order differential equation for $R_2$ (again let $eB$ be noted as $B$)

$$\left(\frac{d^2 R_2}{dy^2} - \frac{m \cos r - B (\cos r - 1)}{\sin^2 r} - \frac{(m + B (\cos r - 1))^2}{\sin^2 r} + \lambda^2\right) R_2 = 0. \quad (113)$$

With a new variable $y = (1 + \cos r)/2$, and the substitution $R_2 = y^A (1-y)^C \tilde{R}_2(y)$, we get

$$y(1-y)\frac{d^2 \tilde{R}_2}{dy^2} + \left[ 2K + 1/2 - (2K+2L+1)y \right] \frac{d \tilde{R}_2}{dy}$$

$$+ \left[ \frac{K^2 - K/2 - m^2/4 + m/4 + mB - B^2 - B/2}{1-y} \right] \tilde{R}_2 = 0.$$

With restrictions

$$K = -\frac{2B-m}{2}, \quad \frac{1+(2B-m)}{2}, \quad L = -\frac{m}{2}, \quad \frac{1+m}{2}, \quad (117)$$

we arrive at a differential equation of hypergeometric type

$$y(1-y)\frac{d^2 \tilde{R}_2}{dy^2} + \left[ 2K + 1/2 - (2K+2L+1)y \right] \frac{d \tilde{R}_2}{dy}$$

$$- \left[ (K+L)^2 - \lambda^2 - B^2 \right] \tilde{R}_2 = 0, \quad (118)$$

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where \( \gamma = 2K + 1/2 \) and

\[
\alpha = K + L - \sqrt{B^2 + \lambda^2}, \quad \beta = K + L + \sqrt{B^2 + \lambda^2}, \quad R_2 = y^K (1-y)^L F(\alpha, \beta, \gamma; y). \tag{119}
\]

To have a finite solution at the origin \( r = 0 \) (that corresponds to the half-curve \( u_0 = +\cos z, \ u_3 = \sin z, u_1 = 0, u_2 = 0 \)) and at \( r = \pi \) (that corresponds to the other part of the curve, \( u_0 = -\cos z, u_3 = \sin z, u_1 = 0, u_2 = 0 \)), we must take positive values for \( K \) and \( L \) \((K > 0, \ L > 0) \)

\[
R_1 = y^A (1-y)^C F(\alpha, \beta, \gamma; y). \tag{120}
\]

There are possible four variants (assuming \( B > 0 \)):

- variant 1', \( K = \frac{-2B + m}{2}, \quad L = \frac{1 + m}{2} \),
- variant 2', \( K = \frac{-2B + m}{2}, \quad L = \frac{-m}{2} \),
- variant 3', \( K = \frac{-m + 1 + 2B}{2}, \quad L = \frac{-m}{2} \),
- variant 4', \( K = \frac{-m + 1 + 2B}{2}, \quad L = \frac{1 + m}{2} \).

There are possible three of them.

**Variant 1'**, \( m > 2B \),

\[
R_2 = y^{-B+m/2}(1-y)^{m/2+1/2} F(\alpha', \beta', \gamma'; y),
\]

\[
\alpha' = -B + m + 1/2 - \sqrt{B^2 + \lambda^2},
\]

\[
\beta' = -B + m + 1/2 + \sqrt{B^2 + \lambda^2},
\]

\[
\gamma' = -2B + m + 1/2, \tag{121}
\]

with quantization condition

\[
\alpha' = -n, \quad \sqrt{B^2 + \lambda^2} = n + m + 1/2 - B > 0, \tag{122}
\]

which correlates with the variant 3 for \( R_1(y) \).

**Variant 3'**, \( m < 0 \),

\[
R_2 = y^{B-m/2+1/2}(1-y)^{-m/2} F(\alpha'', \beta'', \gamma''; y),
\]

\[
\alpha'' = B - m + 1/2 - \sqrt{B^2 + \lambda^2},
\]

\[
\beta'' = B - m + 1/2 + \sqrt{B^2 + \lambda^2},
\]

\[
\gamma'' = 2B - m + 3/2, \tag{123}
\]

with quantization condition

\[
\alpha'' = -n, \quad \sqrt{B^2 + \lambda^2} = n - m + 1/2 + B > 0 \tag{124}
\]
which correlates with the **variant 1** for \( R_1(y) \).

\[
\begin{align*}
\text{Variant } 4', \quad & -1 < m < 1 + 2B, \\
R_2 &= y^{B-m/2+1/2} (1 - y)^{m/2+1/2} F(\alpha'', \beta'', \gamma'', y), \\
\alpha'' &= B + 1 - \sqrt{B^2 + \lambda^2}, \\
\beta'' &= B + 1 + \sqrt{B^2 + \lambda^2}, \\
\gamma'' &= 2B - m + 3/2,
\end{align*}
\] (125)

with quantization condition

\[
\alpha'' = -n, \quad \sqrt{B^2 + \lambda^2} = n + 1 + B,
\] (126)

which correlates with the **variant 2** for \( R_1(y) \).

Now we should find relative factor between \( R_1 \) and \( R_2 \). In the variable \( y = (1 + \cos r)/2 \), the relevant radial system assumes the form

\[
-\sqrt{y(1-y)} \left( \frac{d}{dy} + \frac{B - m/2}{y} - \frac{m/2}{1-y} \right) R_2 + \lambda R_1 = 0,
\]

\[
-\sqrt{y(1-y)} \left( \frac{d}{dy} - \frac{B - m/2}{y} + \frac{m/2}{1-y} \right) R_1 - \lambda R_2 = 0.
\] (127)

Let us find relative factor for \( R_1, R_2 \) in the variants 1 -- \( 3' \):

**Variant 1**, \( m < 0 \),

\[
R_1 = y^{(2B-m)/2} (1 - y)^{(1-m)/2} F(a, b, c; y),
\]

\[
a = B - m + 1/2 - \sqrt{\lambda^2 + B^2},
\]

\[
b = B - m + 1/2 + \sqrt{\lambda^2 + B^2},
\]

\[
c = 2B - m + 1/2;
\]

**Variant 3'**, \( m < 0 \),

\[
R_2 = y^{B-m/2+1/2} (1 - y)^{-m/2} F(\alpha'', \beta'', \gamma'', y),
\]

\[
\alpha'' = B - m + 1/2 - \sqrt{B^2 + \lambda^2} = a,
\]

\[
\beta'' = B - m + 1/2 + \sqrt{B^2 + \lambda^2} = b,
\]

\[
\gamma'' = 2B - m + 3/2 = c + 1.
\]

Substituting these functions into the second equation

\[
-\sqrt{y(1-y)} \left( \frac{d}{dy} - \frac{B - m/2}{y} + \frac{m/2}{1-y} \right) \times r_2 y^{(2B-m)/2} (1 - y)^{(1-m)/2} F(a, b, c; y)
\]

\[
-\lambda r_2 y^{B-m/2+1/2} (1 - y)^{-m/2} F(a, b, c + 1, y) = 0.
\]
we get
\[ r_1 \left( (B - \frac{m}{2}) \frac{1}{y} (1 - y) - \left( \frac{1}{2} - \frac{m}{2} \right) + (1 - y) \frac{d}{dy} \right) \]
\[ - (B - \frac{m}{2}) \frac{1}{y} (1 - y) + \frac{m}{2} \right) F(a, b, c; y) + \lambda r_2 F(a, b, c + 1, y) = 0 . \]

It reads differently as
\[ r_1 \left[ (c - a - b) + (1 - y) \frac{d}{dy} \right] F(a, b, c; y) + \lambda r_2 F(a, b, c + 1, y) = 0 ; \]

from whence with the use of the formula
\[ (c - a - b) F(a, b, c, y) + (1 - y) \frac{d}{dz} F(a, b, c, y) = \frac{(a - c)(b - c)}{c} F(a, b, c + 1, y) , \]

we arrive at a needed relationship
\[ \frac{(a - c)(b - c)}{c} r_1 + \lambda r_2 = 0 . \]

Let us find relative factor for \( R_1, R_2 \) in the variants 2 − 4':

**Variant 2**, \( m = +1/2, ..., \mu < 2B, \)
\[ R_1 = y^{(2B - m)/2} (1 - y)^{m/2} F(a', b', c'; y) , \]
\[ a' = B - \sqrt{\lambda^2 + B^2} , \]
\[ b' = B + \sqrt{\lambda^2 + B^2} , \]
\[ c' = 2B - m + 1/2 ; \]

**Variant 4',** \( -1 < m < 1 + 2B, \)
\[ R_2 = y^{B - m/2 + 1/2} (1 - y)^{m/2 + 1/2} F(a'', b'', c'', y) , \]
\[ a'' = B + 1 - \sqrt{B^2 + \lambda^2} = a' + 1 , \]
\[ \beta'' = B + 1 + \sqrt{B^2 + \lambda^2} = b' + 1 , \]
\[ c'' = 2B - m + 3/2 = c' + 1 . \]

Substituting these functions into the second equation
\[ - \sqrt{y(1 - y)} \left( \frac{d}{dy} - \frac{B - m/2}{y} + \frac{m/2}{1 - y} \right) \]
\[ \times r_1 y^{(2B - m)/2} (1 - y)^{m/2} F(a', b', c'; y) \]
\[ - \lambda r_2 y^{B - m/2 + 1/2} (1 - y)^{m/2 + 1/2} F(a' + 1, b' + 1, c' + 1; y) = 0 . \]

we get
\[ r_1 \left( \frac{B - m/2}{y} - \frac{m/2}{1 - y} + \frac{d}{dy} - \frac{B - m/2}{y} + \frac{m/2}{1 - y} \right) F(a', b', c'; y) \]
\[ + \lambda r_2 y^{B - m/2 + 1/2} (1 - y)^{m/2 + 1/2} F(a' + 1, b' + 1, c' + 1; y) = 0 ; \]
from whence it follows

\[ r_1 \frac{a'b'}{c'} + r_2 \lambda = 0. \quad (129) \]

Let us find a relative factor for \( R_1, R_2 \) in the variants \( 3 - 4' \):

**Variant 3,** \( m > 2B - 1 \),

\[ R_1 = y^{(m+1-2B)/2}(1-y)^{m/2} F(a'', b'', c''; y), \]
\[ a'' = m + 1/2 - B - \sqrt{\lambda^2 + B^2} = \alpha', \]
\[ b'' = m + 1/2 - B + \sqrt{\lambda^2 + B^2} = \beta', \]
\[ c'' = -2B + m + 3/2 = \gamma' + 1. \]

**Variant 1',** \( m > 2B \),

\[ R_2 = y^{-B+m/2}(1-y)^{m/2+1/2} F(\alpha', \beta', \gamma', y), \]
\[ \alpha' = -B + m + 1/2 - \sqrt{B^2 + \lambda^2}, \]
\[ \beta' = -B + m + 1/2 + \sqrt{B^2 + \lambda^2}, \]
\[ \gamma' = -2B + m + 1/2. \quad (130) \]

Substituting these functions into the first equation

\[-\sqrt{y(1-y)} \left( \frac{1}{y} \frac{d}{dy} + \frac{B-m/2}{y} - \frac{m/2}{1-y} \right)\]
\[ \times r_2 y^{-B+m/2}(1-y)^{m/2+1/2} F(\alpha', \beta', \gamma', y) \]
\[ + \lambda r_1 y^{(m+1-2B)/2} (1-y)^{m/2} F(\alpha', \beta', \gamma' + 1; y) = 0, \]

we get

\[- r_2 \left( \frac{(B-m/2)}{1-y} - \frac{m/2}{y} - \frac{1-y}{y} \right) \frac{d}{dy} F(\alpha', \beta', \gamma', y) \]
\[ + \lambda r_1 F(\alpha', \beta', \gamma' + 1; y) = 0, \]

or differently

\[- r_2 \left( \frac{(\gamma' - \alpha' - \beta')}{\gamma'} + (1-y) \frac{d}{dy} \right) F(\alpha', \beta', \gamma', y) \]
\[ + \lambda r_1 F(\alpha', \beta', \gamma' + 1; y) = 0, \]

which results in

\[- r_2 \frac{(\alpha' - \gamma')(\beta' - \gamma')}{\gamma'} + \lambda r_1 = 0 \quad (131) \]
8 Conclusions

Let us summarize results.

First, note that in Lobachevsky space, the formulas (74) describe finite number of discrete energy levels, governed by the magnitude of magnetic field $B$. The whole situation can be clarified by Fig.1.

$$m \pm |2B + m| + 2n < 0$$

Fig. 1. $H_3$-model, bound states at $B > 0, -B < m$

Besides, results from (74) may be presented by an unifying relation

$$A + C + \sqrt{B^2 - \lambda^2} = -n \implies \sqrt{B^2 - \lambda^2} = -\frac{|2B + m|}{2} + \frac{|m|}{2} + n.$$ (132)

This form is valid for negative $B$ as well, though Fig. 1 should be modified for $B < 0$.

Note here the way of transition to the limit of the flat Minkowski space – it is realized as follows

$$\lambda^2 \to \frac{P^2 \rho^2}{\hbar^2} = \lambda_0^2 \rho^2, \quad B \to \frac{eB}{\hbar c} \rho^2,$$

1. $\lambda_0^2 = \frac{2eB}{\hbar} n$; 2. $\lambda_0^2 = \frac{2eB}{\hbar} (n - m + \frac{1}{2})$. (133)

In the spherical Riemann model, similarly results on quantization (113)–(??) can be presented by an unifying formula

$$A + C - \sqrt{B^2 - \lambda^2} = -n \implies \sqrt{B^2 - \lambda^2} = \frac{|2B - m|}{2} + \frac{|m|}{2} + n.$$ (134)
and Fig. 2

\[ |m| - |2B - m| + 2n > 0 \]

\[ + |2B - m| \]

\[ +2B \]

\[ -2B \]

\[ m \]

**Fig. 2.** $S_3$-model, bound states at $B > 0$

Evidently, in the case of negative $B$ this Fig. 2 should be slightly modified.

There should be given a clarifying remarks about quantum number $m$. In fact, the above used relationship $-i\partial_\phi \Psi = m \Psi$ (in both curved models) represents transformed from Cartesian coordinates to cylindrical an eigenvalue equation for the third projection of the total angular momentum of the Dirac particle

\[
\hat{J}_3 \Psi_{Cart} = (-i \frac{\partial}{\partial \phi} + \Sigma_3) \Psi_{Cart} = m \Psi_{Cart}; \quad (135)
\]

this means that for the quantum number $m$ are permitted only half-integer values $m = \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots$

Note that some other new problems concerning quantum mechanical particles in external magnetic field in flat Minkowski space and in curved spaces of constant curvature, were considered in [34]. More wide discussion of the different quantum mechanical problems in spaces of Lobachevsky $H_3$ and Riemann $S_3$ may be found in [35].

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