The octonionic Bergman kernel for the half space

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Abstract: We obtain the octonionic Bergman kernel for half space in the octonionic analysis setting by two different methods. As a consequence, we unify the kernel forms in both complex analysis and hyper-complex analysis.

Keywords: octonions, octonionic analysis, Bergman kernel

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1 Introduction

In a recent paper [16] we derived the octonionic Bergman kernel for the unit ball, based on our newly defined inner product of the octonionic Bergman space. Note that in complex analysis the unit disc and the upper half space are holomorphically equivalent through Cayley transform. In octonionic space there is also a similar Cayley transform mapping the unit ball onto the half space (cf. [17]). So the problems for the octonionic Bergman space on the half space in \( \mathbb{R}^8 \) naturally arise. But unfortunately the octonionic Cayley transform is neither left \( \mathbb{O} \)-analytic not right \( \mathbb{O} \)-analytic by our definition, and the octonions are neither commutative nor associative, which bring barriers to the study of the problems in half space through Cayley transform. Thus we need to investigate the case for half space by other ways.

First we accordingly have the following definition.

Definition 1.1 (the octonionic Bergman space on the half space). Let \( \mathbb{R}^8_+ := \{ x \in \mathbb{O} : \text{Re} x > 0 \} \) be the half space in \( \mathbb{R}^8 \), the octonionic Bergman space \( \mathcal{B}^2(\mathbb{R}^8_+) \) is the class of left octonionic analytic functions \( f \) on \( \mathbb{R}^8_+ \) satisfying

\[
\int_{\mathbb{R}^8_+} |f|^2 dV < \infty,
\]

where \( dV \) is the volume element on \( \mathbb{R}^8_+ \).

The inner product is defined as the usual way.

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Definition 1.2 (inner product on $\mathcal{B}^2(\mathbb{R}^8_+)$). Let $f, g \in \mathcal{B}^2(\mathbb{R}^8_+)$, we define

$$(f, g)_{\mathbb{R}^8_+} := \frac{1}{\omega_8} \int_{\mathbb{R}^8_+} \overline{f} g dV,$$

where $\omega_8 = \frac{\pi^4}{3}$ is the surface area of the unit sphere in $\mathbb{R}^8$.

By density argument and limit argument we prove the following main theorem of this paper:

Theorem 1.1. The octonionic Bergman kernel of $\mathcal{B}^2(\mathbb{R}^8_+)$ exists. Let

$$B(x, a) = -2 \frac{\partial}{\partial x_0} E(x + \overline{a}),$$

where $E(x) = \frac{1}{|x|^2}$ is the octonionic Cauchy kernel, then $B(\cdot, a)$ is the desired octonionic Bergman kernel, i.e., $B(\cdot, a) \in \mathcal{B}^2(\mathbb{R}^8_+)$, and for any $f \in \mathcal{B}^2(\mathbb{R}^8_+)$ and any $a \in \mathbb{R}^8_+$, there holds the following reproducing formula

$$f(a) = (f, B(\cdot, a))_{\mathbb{R}^8_+}.$$

Moreover, the kernel is unique.

The rest of the paper is organized as follows. In Section 2 to make the paper self-contained we briefly review the octonion algebra and octonionic analysis. In Section 3 we give two proofs of Theorem 1.1. In the last section we point out that the Bergman kernels can be unified in one form in both complex analysis and hyper-complex analysis.

2 The octonions and the octonionic analysis

Octonions, which are also called Cayley numbers or the Cayley algebra, were discovered independently by John T. Graves and Arthur Cayley in 1843 and 1845. In 1898, Hurwitz had proved that the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$ are the only normed division algebras over $\mathbb{R}$ ($\mathbb{B}$), with the embedding relation $\mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{H} \subseteq \mathbb{O}$.

Octonions are an 8 dimensional algebra over $\mathbb{R}$ with the basis $e_0, e_1, \ldots, e_7$ satisfying

$$e_0^2 = e_0, \ e_i e_0 = e_0 e_i = e_i, \ e_i^2 = -1, \text{ for } i = 1, 2, \ldots, 7.$$

So $e_0$ is the unit element and can be identified with 1. Denote

$$W = \{(1, 2, 3), (1, 4, 5), (1, 7, 6), (2, 4, 6), (2, 5, 7), (3, 4, 7), (3, 6, 5)\}.$$

For any triple $(\alpha, \beta, \gamma) \in W$, we set

$$e_{\alpha} e_{\beta} = e_{\gamma} = -e_{\beta} e_{\alpha}, \ e_{\beta} e_{\gamma} = e_{\alpha} = -e_{\gamma} e_{\beta}, \ e_{\gamma} e_{\alpha} = e_{\beta} = -e_{\alpha} e_{\gamma}.$$
Then by distributivity for any \( x = \sum_{i=0}^{7} x_i e_i, y = \sum_{j=0}^{7} y_j e_j \in \mathcal{O} \), the multiplication \( xy \) is defined to be

\[
xy := \sum_{i=0}^{7} \sum_{j=0}^{7} x_i y_j e_i e_j.
\]

For any \( x = \sum_{i=0}^{7} x_i e_i \in \mathcal{O} \), \( \text{Re} x := x_0 \) is called the scalar (or real) part of \( x \) and \( \overrightarrow{x} := x - \text{Re} x \) is called its vector part. \( \overline{x} := \sum_{i=0}^{7} x_i e_i = x_0 - \overrightarrow{x} \) and \( |x| := (\sum_{i=0}^{7} x_i^2)^{\frac{1}{2}} \) are respectively the conjugate and norm (or modulus) of \( x \), they satisfy: \(|xy| = |x||y|\), \( x\overline{x} = |x|^2\), \( \overline{xy} = y\overline{x} \) \((x, y \in \mathcal{O})\). So if \( x \neq 0 \), \( x^{-1} = \overline{x}/|x|^2 \) gives the inverse of \( x \).

Octonionic multiplication is neither commutative nor associative. But the subalgebra generated by any two elements is associative, namely, The octonions are alternative. \([x, y, z] := (xy)z - x(yz)\) is called the associator of \( x, y, z \in \mathcal{O} \), it satisfies (2.6)

\[
[x, y, z] = [y, z, x] = -[y, x, z], \quad [x, x, y] = [\overline{y}, x, y] = 0.
\]

As a generalization of complex analysis and quaternionic analysis to higher dimensions, the study of octonionic analysis was originated by Dentoni and Sce in 1973 ([4]), and was systematically investigated by Li et al since 1995 ([7]). Octonionic analysis is a function theory on octonionic analytic (abbr. \( \mathcal{O} \)-analytic) functions. Suppose \( \Omega \) is an open subset of \( \mathbb{R}^8 \), \( f = \sum_{j=0}^{7} f_j e_j \in C^1(\Omega, \mathcal{O}) \) is an octonion-valued function, if

\[
Df = \sum_{i=0}^{7} e_i \frac{\partial f_i}{\partial x_i} = \sum_{i=0}^{7} \sum_{j=0}^{7} \frac{\partial f_i}{\partial x_i} e_i e_j = 0
\]

\[
(fD) = \sum_{i=0}^{7} \frac{\partial f_i}{\partial x_i} e_i = \sum_{i=0}^{7} \sum_{j=0}^{7} \frac{\partial f_i}{\partial x_i} e_j e_i = 0,
\]

then \( f \) is said to be left (right) \( \mathcal{O} \)-analytic in \( \Omega \), where the generalized Cauchy–Riemann operator \( D \) and its conjugate \( \overline{D} \) are defined by

\[
D := \sum_{i=0}^{7} e_i \frac{\partial}{\partial x_i}, \quad \overline{D} := \sum_{i=0}^{7} \overline{e_i} \frac{\partial}{\partial x_i}
\]

respectively. A function \( f \) is \( \mathcal{O} \)-analytic means that \( f \) is meanwhile left \( \mathcal{O} \)-analytic and right \( \mathcal{O} \)-analytic. From

\[
\overline{D}(Df) = (\overline{D}D)f = \triangle f = f(DD) = (fD)\overline{D},
\]

we know that any left (right) \( \mathcal{O} \)-analytic function is always harmonic. In the sequel, unless otherwise specified, we just consider the left \( \mathcal{O} \)-analytic case as the right \( \mathcal{O} \)-analytic case is essentially the same. A Cauchy-type integral formula for this setting is:
Lemma 2.1 (Cauchy’s integral formula, see [4,9]). Let \( M \subset \Omega \) be an \( 8 \)-dimensional, compact differentiable and oriented manifold with boundary. If \( f \) is left \( \Omega \)-analytic in \( \Omega \), then

\[
f(x) = \frac{1}{\omega_8} \int_{y \in \partial M} E(y-x)(d\mathbf{s}_y f(y)), \quad x \in \mathcal{M}^0,
\]

where \( E(x) = \frac{1}{|x|^8} \) is the octonionic Cauchy kernel, \( d\mathbf{s}_y = n(y)dS \), \( n(y) \) and \( dS \) are respectively the outward-pointing unit normal vector and surface area element on \( \partial M \), \( \mathcal{M}^0 \) is the interior of \( \mathcal{M} \).

Octonionic analytic functions have a close relationship with the Stein–Weiss conjugate harmonic systems. If the components of \( F \) consist a Stein–Weiss conjugate harmonic system on \( \Omega \subset \mathbb{R}^8 \), then \( F \) is \( \Omega \)-analytic on \( \Omega \). But conversely this is not true ([5]). For more information and recent progress about octonionic analysis, we refer the reader to [7,10,15,17].

3 Two proofs of Theorem 1.1

3.1 Proof of Theorem 1.1 by density

For \( \delta > 0 \) we let \( \mathbb{R}^8_\delta = \{ x \in \mathbb{R}^8 : \text{Re} x > -\delta \} \). The set of functions that are left octonionic analytic and square integrable on \( \mathbb{R}^8_\delta \) is denoted by \( \mathcal{B}^2(\mathbb{R}^8_\delta) \). Similar to the harmonic Bergman space (see [1]), we have

Lemma 3.1. \( \bigcup_{0<\delta<1} \mathcal{B}^2(\mathbb{R}^8_\delta) \) is dense in \( \mathcal{B}^2(\mathbb{R}^8) \).

Proof. Let \( u \in \mathcal{B}^2(\mathbb{R}^8) \). For \( \delta > 0 \), the function \( x \mapsto u(x+\delta) \) belongs to \( \mathcal{B}^2(\mathbb{R}^8_\delta) \). But the functions \( u(x+\delta) \) converge to \( u(x) \) in \( L^2(\mathbb{R}^8_\delta) \) as \( \delta \to 0 \). \( \square \)

Proof of Theorem 1.1. By lemma 3.1 we only need to check the theorem for \( \mathcal{B}^2(\mathbb{R}^8_\delta) \) for any \( \delta > 0 \). Now suppose \( u \in \mathcal{B}^2(\mathbb{R}^8_\delta) \), we have

\[
\frac{1}{\omega_8} \int_{\mathbb{R}^7} \frac{\partial}{\partial x_0} E(x + \overline{\alpha})u(x)dV_x
= \frac{1}{\omega_8} \int_{\mathbb{R}^7} \int_0^{-\infty} \frac{\partial}{\partial x_0} E(x_0 + \overline{x} + \alpha)u(x_0 + \overline{x})dx_0d\overline{x}.
\]

(3.1)

After integrating by parts in the inner integral, (3.1) becomes

\[
\frac{1}{\omega_8} \int_{\mathbb{R}^7} \left( E(x_0 + \overline{x} + \alpha)u(x_0 + \overline{x}) \right)_0^{+\infty} - \int_0^{+\infty} E(x_0 + \overline{x} + \alpha) \frac{\partial}{\partial x_0} u(x_0 + \overline{x})dx_0 \right) d\overline{x}
= \frac{1}{\omega_8} \left( -\int_{\mathbb{R}^7} E(\overline{x} + \alpha)u(\overline{x})d\overline{x} - \int_{\mathbb{R}^7} \int_0^{+\infty} \frac{\partial}{\partial x_0} E(x_0 + \overline{x} + \alpha)u(x_0 + \overline{x})dx_0d\overline{x} \right)
= -u(a) - \frac{1}{\omega_8} \int_0^{+\infty} \left( \int_{\mathbb{R}^7} E(x_0 + \overline{x} + \alpha) \frac{\partial}{\partial x_0} u(x_0 + \overline{x})d\overline{x} \right) dx_0,
\]

(3.2)
where in the first equality we use the fact that \( u(x_0 + x) \to 0 \) as \( x_0 \to +\infty \), in the second equality we apply the Cauchy integral formula to \( u \) on half space \( \mathbb{R}^8_+ \). After applying again the Cauchy integral formula to the function \( w \mapsto \frac{\partial u}{\partial x_0} \big|_{x=x_0+w} \), \((3.2)\) becomes
\[
- u(a) - \frac{1}{\omega_8} \int_0^{+\infty} \left( \int_{\mathbb{R}^7} E(\overline{\mathbf{x}} + x_0 + a) \frac{\partial u}{\partial x_0} \big|_{x=x_0+\overline{\mathbf{x}}} \, d\overline{\mathbf{x}} \right) \, dx_0 \\
= - u(a) - \int_0^{+\infty} \frac{\partial u}{\partial x_0} \big|_{x=a+2x_0} \, dx_0 \\
= - u(a) - \frac{1}{2} u(a + 2x_0) \big|_0^{+\infty} \\
= - \frac{1}{2} u(a).
\]

So the octonionic Bergman kernel \( B(x, a) \) exists and equals \( -2 \frac{\partial}{\partial x_0} E(x + \overline{\mathbf{a}}) \). The proof of the uniqueness of the kernel is similar to the case for the unit ball (see [16]) which we omit here. The proof of Theorem 1.1 is therefore complete. \( \square \)

### 3.2 Proof of Theorem [1.1] by taking limits

Let \( B_{p,r} \) be the ball centered at the point \( p \in \mathbb{R}^8 \) of radius \( r \), \( B_{p,r}(x, a) \) be the octonionic Bergman kernel for \( B_{p,r} \). The inner product on the octonionic Bergman space \( \mathcal{B}^2(B_{p,r}) \) is defined by
\[
(f, g)_{B_{p,r}} := \frac{1}{\omega_8} \int_{B_{p,r}} \left( g(x) \frac{\overline{x - p}}{|x - p|} \right) \left( \frac{x - p}{|x - p|} f(x) \right) \, dV,
\]
Then we have

**Lemma 3.2.** The octonionic Bergman kernel \( B_{p,r}(x, a) \) is given by
\[
B_{p,r}(x, a) = \frac{1}{r^8} B_{0,1} \left( \frac{x - p}{r}, \frac{a - p}{r} \right),
\]
where
\[
B_{0,1}(x, a) = \frac{6(1 - |a|^2|x|^2) + 2(1 - \overline{\mathbf{a}}) (1 - \overline{\mathbf{a}})}{|1 - \overline{\mathbf{a}}|^2}
\]
is the octonionic Bergman kernel for the unit ball given in [10].

**Proof.** If \( f(x) \in \mathcal{B}^2(B_{p,r}) \), then \( f(rx + p) \in \mathcal{B}^2(B_{0,1}) \). So for any \( b \in B_{0,1} \) we have
\[
f(rb + p) = \frac{1}{\omega_8} \int_{B_{0,1}} \left( B_{0,1}(x, b) \frac{\overline{x}}{|x|} \right) \left( \frac{x}{|x|} f(rx + p) \right) \, dV_x.
\]
Let \( rb + p = a, \, rx + p = y \), then \( a, y \in B_{p,r} \), and the above reproducing formula becomes
\[
f(a) = \frac{1}{r^8 \omega_8} \int_{B_{p,r}} \left( B_{0,1} \left( \frac{y - p}{r}, \frac{a - p}{r} \right) \right) \left( \frac{y - p}{|y - p|} f(y) \right) \, dV_y.
\]
\( \square \)
Note that $\mathbb{R}_+^8 = \bigcup_{r=1}^{\infty} B_{r,r}$. Similar to the harmonic Bergman space (see \[1\]), we have

**Lemma 3.3.** The octonionic Bergman kernel $B(x, a)$ for $\mathbb{R}_+^8$ satisfies

$$B(x, a) = \lim_{r \to \infty} B_{r,r}(x, a)$$

for all $x, y \in \mathbb{R}_+^8$.

**Proof of Theorem 1.1.** By Lemma 3.2 and Lemma 3.3 we get

$$B(x, a) = \lim_{r \to \infty} \frac{1}{r^8} B_{0,1} \left( \frac{x-r}{r}, \frac{a-r}{r} \right)$$

$$= \lim_{r \to \infty} \left( \frac{6}{r^8} \left( 1 - \frac{|a-r|^2|x-r|^2}{r^4} \right) + \frac{2}{r^8} \left( 1 - \frac{x-r(a-r)}{r^2} \right) \right) \left( 1 - \frac{x-r(a-r)}{r^2} \right)$$

$$= \lim_{r \to \infty} \frac{6}{r^8} \left( r - \frac{|a-r|^2|x-r|^2}{r^4} \right) + \frac{2}{r^8} \left( r - \frac{x-r(a-r)}{r^2} \right) \left( r - \frac{x-r(a-r)}{r^2} \right).$$

But

$$\lim_{r \to \infty} \left( r - \frac{x-r(a-r)}{r} \right) \right)$$

$$= \lim_{r \to \infty} \frac{r^2 - x - r(a-r)}{r}$$

$$= \lim_{r \to \infty} \frac{(a+\bar{a})r - \overline{xa}}{r}$$

$$= a + \bar{a},$$

and

$$\lim_{r \to \infty} \left( r - \frac{|a-r|^2|x-r|^2}{r^4} \right)$$

$$= \lim_{r \to \infty} \frac{r^4 - |(a-r)(x-r)|^2}{r^3}$$

$$= \lim_{r \to \infty} \frac{r^4 - |r^2 - (a+x)r + ax|^2}{r^3}$$

$$= \lim_{r \to \infty} \frac{r^4 - (r^2 - (a+x)r + ax)(r^2 - (\bar{a} + \overline{a})r + \overline{ax})}{r^3}$$

$$= a + x + \overline{a} + \bar{a}$$

$$= 2\text{Re}(a + \overline{a}).$$

Hence,

$$B(x, a) = \frac{(12\text{Re}(a + \overline{a}) + 2(a + \overline{a}) \left( a + \overline{a} \right))}{|a + \overline{a}|^{10}} = -2 \frac{\partial}{\partial x_0} \left( \frac{x + \overline{a}}{|x + \overline{a}|^8} \right).$$
4 Final remarks

The arguments in the above proofs can be also applied to both complex analysis and Clifford analysis, we therefore can unify the Bergman kernels and reproducing formulas in complex and hyper-complex contexts. Let \( \mathcal{A} \) denote the complex algebra or hyper-complex algebra, i.e., \( \mathcal{A} \) may refer to complex numbers \( \mathbb{C} \), quaternions \( \mathbb{H} \), octonions \( \mathbb{O} \), or Clifford algebra \( \mathcal{C} \). Assume that the dimension of the underlying space of \( \mathcal{A} \) is \( m \), \( \mathbb{R}^m_+ := \{ x \in \mathcal{A} : \text{Re} x > 0 \} \) is the half space in \( \mathbb{R}^m \). Then for any function \( f \) which belongs to the Bergman space \( B^2(\mathbb{R}^m_+) \) and any point \( a \in \mathbb{R}^m_+ \), there holds

\[
  f(a) = (f, B(\cdot, a))_{\mathbb{R}^m_+} = \frac{1}{\omega_m} \int_{\mathbb{R}^m_+} B(x, a) f(x) dV = \frac{1}{\omega_m} \int_{\mathbb{R}^m_+} -2 \frac{\partial}{\partial x_0} \left( \frac{x + \bar{a}}{|x + \bar{a}|^m} \right) f(x) dV = \frac{1}{\omega_m} \int_{\mathbb{R}^m_+} \frac{(2(m-2)\text{Re}(x + \bar{a}) + 2(x + \bar{a}) (x + \bar{a})) (x + \bar{a})}{|x + \bar{a}|^{m+2}} f(x) dV,
\]

where \( \omega_m \) is the surface area of the unit sphere in \( \mathbb{R}^m \), \( dV \) is the volume element on \( \mathbb{R}^m_+ \).

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References

[1] S. Axler, P. Bourdon, W. Ramey, Harmonic Function Theory (2nd edition), Springer, New York, 2001.
[2] J.C. Baez, The octonions, Bull. Amer. Math. Soc. 39 (2) (2002) 145–205.
[3] F. Brackx, R. Delanghe, F. Sommen, Clifford Analysis, Research Notes in Math., vol. 76, Pitman Advanced Publishing Program, Boston, 1982.
[4] P. Dentoni, M. Sce, Funzioni regolari nell’algebra di Cayley, Rend. Sem. Mat. Univ. Padova. 50 (1973) 251–267.
[5] A. Hurwitz, Über die Composition der quadratischen Formen von beliebig vielen Variablen, Nachr. Ges. Wiss. Göttingen (1898) 309–316.
[6] N. Jacobson, Basic Algebra I (2nd edition), W. H. Freeman and Company, New York, 1985.
[7] X.M. Li, Octonionic analysis, PhD Thesis, Peking University, 1998.

[8] X.M. Li, L.Z. Peng, On Stein–Weiss conjugate harmonic function and octonion analytic function, Approx. Theory & its Appl. 16 (2) (2000) 28–36.

[9] X.M. Li, L.Z. Peng, The Cauchy integral formulas on the octonions, Bull. Belg. Math. Soc. 9 (1) (2002) 47–64.

[10] X.M. Li, L.Z. Peng, T. Qian, Cauchy integrals on Lipschitz surfaces in octonionic spaces, J. Math. Anal. Appl. 343 (2) (2008) 763–777.

[11] X.M. Li, L.Z. Peng, T. Qian, The Paley–Wiener theorem in the non-commutative and non-associative octonions, Sci. China Ser. A 52 (1) (2009) 129–141.

[12] X.M. Li, J.X. Wang, Orthogonal invariance of the Dirac operator and the critical index of subharmonicity for octonionic analytic functions, Adv. Appl. Clifford Algebras 24 (1) (2014) 141–149.

[13] X.M. Li, K. Zhao, L.Z. Peng, The Laurent series on the octonions, Adv. Appl. Clifford Algebras 11 (S2) (2001) 205–217.

[14] X.M. Li, K. Zhao, L.Z. Peng, Characterization of octonionic analytic functions, Complex Variables 50 (13) (2005) 1031–1040.

[15] J.Q. Liao, X.M. Li, J.X. Wang, Orthonormal basis of the octonionic analytic functions, J. Math. Anal. Appl. 366 (1) (2010) 335–344.

[16] J.X. Wang, X.M. Li, The octonionic Bergman kernel for the unit ball, Adv. Appl. Clifford Algebras 28:60 (2018).

[17] J.X. Wang, X.M. Li, J.Q. Liao, The quaternionic Cauchy–Szegö kernel on the quaternionic Siegel half space, arXiv:1210.5086v1.