THE TWISTED SATAKE ISOMORPHISM AND CASSELMAN-SHALIKA FORMULA

NADYA GUREVICH

Abstract. For an arbitrary split adjoint group we identify the unramified Whittaker space with the space of skew-invariant functions on the lattice of coweights and deduce from it the Casselman-Shalika formula.

1. Introduction

The Casselman-Shalika formula is a beautiful formula relating values of special functions on a $p$-adic group to the values of finite dimensional complex representations of its dual group. Further, the formula is particularly useful in the theory of automorphic forms for studying $L$-functions.

In this note we provide a new approach to (and a new proof of) Casselman-Shalika formula for the value of spherical Whittaker functions.

To state our results we fix some notations. Let $G$ be a split adjoint group over a local field $F$. We fix a Borel subgroup $B$ with unipotent radical $N$, and consider its Levi decomposition $B = NT$, where $T$ is the maximal split torus. Furthermore, we fix a maximal compact subgroup $K$.

Let $\Psi$ be a non-degenerate complex character of $N$. For an irreducible representation $\pi$ of $G$ it is well known that $\dim \text{Hom}_G(\pi, \text{ind}_N^G \Psi) \leq 1$ and in case it is non-zero we say that the representation $\pi$ is generic. The Whittaker model of such a generic irreducible representation $\pi$ of $G$ is the image of an embedding

$$W : \pi \hookrightarrow \text{Ind}_N^G \Psi.$$ 

Let now $\pi$ be a generic irreducible representation. Recall that $\pi$ is called unramified if $\pi^K \neq \{0\}$ and that in this case $\pi^K = \mathbb{C} \cdot v_0$. Here $v_0$ is a spherical vector. The explicit formula for the function $W(v_0)$ was given in [CS] and is commonly called the Casselman-Shalika formula.

Recall that there is a bijection between irreducible unramified representations of $G$ and the spectrum of the spherical Hecke algebra $H_K = C_c(K \backslash G/K)$. This commutative algebra admits the following description. Let $\Lambda$ be the coweight lattice of $G$. Recall that $\Lambda$ is canonically identified with $T/(T \cap K)$. The Weyl group $W$ acts naturally on the lattice $\Lambda$. Denote by $\mathbb{C}[\Lambda]^W$ the algebra of $W$-invariant elements in $\mathbb{C}[\Lambda]$.

The Satake isomorphism

$$S : H_K \simeq (\text{ind}_{T \cap K}^T 1)^W = \mathbb{C}[\Lambda]^W$$
is defined by
\[ S(f)(t) = \delta_{B}^{-1/2}(t) \int_{N} f(nt) \, dn \]

The main result of the paper is a description of the Whittaker spherical space \( (\text{Ind}_{N}^{G} \Psi)^{K} \) as a concrete \( H_{K} \simeq \mathbb{C}[\Lambda]^{W} \)-module. Namely, it is identified with the space \( \mathbb{C}[\Lambda]^{W,-} \) of functions on the lattice of coweights, that are skew-invariant under the action of the Weyl group \( W \). More formally,

**Theorem 1.1.** There is a canonical isomorphism
\[ j : (\text{ind}_{N}^{G} \Psi)^{K} \to \mathbb{C}[\Lambda]^{W,-}, \]
compatible with Satake isomorphism \( S : H_{K} \simeq \mathbb{C}[\Lambda]^{W} \).

From this result it easily follows that the twisted Satake map
\[ S_{\Psi} : C_{c}(G/K) \to (\text{ind}_{N}^{G} \Psi)^{K}, \quad S_{\Psi}(f)(t) = \int_{N} f(nt) \overline{\Psi(n)} \, dn \]
sends the spectral basis of the spherical Hecke algebra \( H_{K} = C_{c}(K \backslash G/K) \) to the basis of characteristic functions of \( (\text{ind}_{N}^{G} \Psi)^{K} \). In [FGKV], it is explained that this latter result is equivalent to the Casselman-Shalika formula in [CS] (see section 6 for a full account). Thus, we obtain a proof of the Casselman-Shalika formula that does not use the uniqueness of the Whittaker model.

Let us now quickly describe our proof of the main result. It was inspired by a new simple proof [S] by Savin of the Satake isomorphism of algebras
\[ S : H_{K} \simeq (\text{ind}_{T \cap K}^{T} 1)^{W} = \mathbb{C}[\Lambda]^{W}. \]

Savin has observed that the Satake map \( S : C_{c}(G/K) \to (\text{ind}_{N}^{G} 1)^{K} \) restricted to
\[ C_{c}(I \backslash G/K) = (\text{ind}_{I}^{G} 1)^{K}, \]
where \( I \) is an Iwahori subgroup, defines an explicit isomorphism
\[ (\text{ind}_{I}^{G} 1)^{K} \simeq \text{ind}_{T \cap K}^{T} 1 = \mathbb{C}[\Lambda]. \]

Restricting \( S_{\Psi} \) to \( (\text{ind}_{I}^{G} 1)^{K} \), we prove there there exists an isomorphism
\[ j : (\text{ind}_{N}^{G} \Psi)^{K} \to \mathbb{C}[\Lambda]^{W,-} \]
of \( H_{K} \simeq \mathbb{C}[\Lambda]^{W} \)-modules, making the following diagram
\[
\begin{array}{ccc}
(\text{ind}_{I}^{G} 1)^{K} & \xrightarrow{S_{\Psi}} & (\text{ind}_{N}^{G} \Psi)^{K} \\
\downarrow S & & \downarrow j \\
\mathbb{C}[\Lambda] & \xrightarrow{alt} & \mathbb{C}[\Lambda]^{W,-}
\end{array}
\]
Commutative. Here the space $\mathbb{C}[\Lambda]^W$ is a space of $W$ skew-invariant elements of $\mathbb{C}[\Lambda]$ and the alternating map $alt$ are defined in the section 3.

**Acknowledgements.** I wish to thank Gordan Savin for explaining his proof of the Satake isomorphism and encouraging me to prove the Casselman-Shalika formula. I am most grateful to Joseph Bernstein for his attention and his help in formulating the main result. I thank Roma Bezrukavnikov for answering my questions and Eitan Sayag for his suggestions for improving the presentation. Part of this work was done during the workshop in RIMS. I wish to thank the organizers for their hospitality. My research is partly supported by ISF grant 1691/10.

2. Notations

Let $F$ be a local non-archimedean field and let $q$ be a characteristic of its residue field. Let $G$ be a split adjoint group defined over $F$. Denote by $B$ a Borel subgroup of $G$, by $N$ its unipotent radical, by $\bar{N}$ the opposite unipotent radical, by $T$ the maximal split torus and by $W$ the Weyl group.

Denote by $R$ the set of positive roots of $G$ and by $\Delta$ the set of simple roots. For each $\alpha \in R$ let $x_\alpha : F \to N$ denote the one parametric subgroup corresponding to the root $\alpha$ and $N_\alpha^k = \{x_\alpha(r) : |r| \leq q^{-k}\}$.

Let $\Psi$ be a non-degenerate complex character of $N$ of conductor 1, i.e. for any $\alpha \in \Delta$

$$\Psi|_{N_\alpha^{0}} \neq 1, \Psi|_{N_\alpha^{1}} = 1.$$  

Let $K$ be a maximal compact subgroup of $G$. Then $T_K = T \cap K$ is a maximal compact subgroup of $T$. Choose an Iwahori subgroup $I \subset K$ such that $I \cap N = N_\alpha^1$ for all $\alpha \in R$. In particular $\Psi|_{N \cap I} = 1$, but $\Psi|_{N_\alpha^{0}} \neq 1$ for any $\alpha \in \Delta$.

We fix a Haar measure on $G$ normalized such that the measure of $I$ is one.

The coweight lattice $\Lambda$ of $G$ is canonically identified with $T/T_K$. For any $\lambda \in \Lambda$ denote by $t_\lambda \in T$ its representative. The coweight $\rho$ denotes the half of all the positive coroots. Since $G$ is adjoint one has $\rho \in \Lambda$. We denote by $\Lambda^+$ the set of dominant coweights.

Let $^LG$ be the complex dual group of $G$. Then $\Lambda$ is also identified with the lattice of weights of $^LG$. For a dominant weight $\lambda$ we denote by $V_\lambda$ the highest weight module of $^LG$ and by $wt(V_\lambda)$ the multiset of all the weights of this module.

3. Functions on Lattices

Consider the algebra $\mathbb{C}[\Lambda] = \text{Span}\{e^\nu : \nu \in \Lambda\}$. The Weyl group $W$ acts naturally on the lattice $\Lambda$. We denote by $\mathbb{C}[\Lambda]^W$ the algebra of $W$-invariant elements in $\mathbb{C}[\Lambda]$. The character map defines an isomorphism of algebras

$$\text{Rep}(^LG) \simeq \mathbb{C}[\Lambda]^W.$$
For an irreducible module $V_\lambda$ denote
\[ a_\lambda = \text{char}(V_\lambda) = \sum_{\nu \in \text{wt}(V_\lambda)} e^\nu. \]
The elements $\{a_\lambda | \lambda \in \Lambda^+\}$ form a basis of $\mathbb{C}[\Lambda]^W$. The algebra $\mathbb{C}[\Lambda]^W$ acts on the space $\mathbb{C}[\Lambda]$ by multiplication.

The element $f \in \mathbb{C}[\Lambda]$ is called skew-invariant if $w(f) = (-1)^{l(w)} f$, where $l(w)$ is the length of the element $w$. Denote by $\mathbb{C}[\Lambda]^{W,-}$ the space of $W$ skew-invariant elements. The algebra $\mathbb{C}[\Lambda]^W$ acts on $\mathbb{C}[\Lambda]^{W,-}$ by multiplication. Note that the action is torsion free.

Define the alternating map
\[ \text{alt} : \mathbb{C}[\Lambda] \to \mathbb{C}[\Lambda]^{W,-}, \quad \text{alt}(e^\mu) = \sum_{w \in W} (-1)^{l(w)} e^{w\mu} : \mu \in \Lambda \]

It is a map of $\mathbb{C}[\Lambda]^W$ modules. The elements
\[ \{r_{\mu+\rho} = \sum_{w \in W} (-1)^{l(w)} e^{w(\mu+\rho)} : \mu \in \Lambda^+\} \]
form a basis of $\mathbb{C}[\Lambda]^{W,-}$. Note that for any $\lambda \in \Lambda^+$
\[ \text{alt}(e^{\lambda+\rho}) = r_{\lambda+\rho} = r_\rho \cdot a_\lambda = \text{alt}(a_\lambda), \]
where the second equality is the Weyl character formula.

4. HECKE ALGEBRAS

4.1. The spherical Hecke algebra. The spherical Hecke algebra $H_K = C_c(K\backslash G/K)$ is the algebra of locally constant compactly supported bi-$K$ invariant functions with the multiplication given by convolution $\ast$. It has identity element $1_K$ - the characteristic function of $K$ divided by $[K : I]$.

Consider the Satake map
\[ S : C_c(G/K) \to C(N\backslash G/K) = C(T/T_K) \]
defined by
\[ S(f)(t) = \delta_B^{-1/2}(t) \int_N f(nt)dn. \]
The famous Satake theorem claims that the restriction of $S$ to $H_K$ defines an isomorphism of algebras $S : H_K \simeq \mathbb{C}[\Lambda]^W$. Denote by $A_\lambda$ the element of $H_K$ corresponding to $a_\lambda$ under this map. Thus $H_K = \text{Span}\{A_\lambda : \lambda \in \Lambda^+\}$.

4.2. The Iwahori-Hecke algebra. The Iwahori-Hecke algebra $H_I = C_c(I\backslash G/I)$ is the algebra of locally constant compactly supported bi-$I$ invariant functions with the multiplication given by convolution. Below we remind the list of properties of $H_I$, all can be found in [HKP].
(1) The algebra $H_{I}$ contains a commutative algebra $A \cong \mathbb{C}[\Lambda]$.

\[ A = \text{Span}\{\theta_{\mu} | \mu \in \Lambda\}, \]

where

\[ \theta_{\mu} = \begin{cases} \delta_{B}^{1/2} 1_{I_\mu} & \mu \in \Lambda^+; \\ \theta_{\mu_1} \ast \theta_{\mu_2}^{-1} & \mu = \mu_1 - \mu_2, \quad \mu_1, \mu_2 \in \Lambda^+ \end{cases} \]

The center $Z_{I}$ of the algebra $H_{I}$ is $A^{W} \cong \mathbb{C}[\Lambda]^{W}$.

(2) The finite dimensional Hecke algebra $H_{f} = \mathbb{C}(I \backslash K/I)$ is a subalgebra of $H_{I}$. The elements $t_{w} = 1_{IwI}$, where $w \in W$ form a basis of $H_{f}$. Multiplication in $H$ induces a vector space isomorphism

\[ H_{f} \otimes_{\mathbb{C}} A \rightarrow H_{I} \]

In particular the elements $t_{w}\theta_{\mu}$ where $w \in W, \mu \in \Lambda$ form a basis of $H_{I}$.

(3) The algebra $H_{K}$ is embedded naturally in $H_{I}$. One has $H_{K} = Z_{I} \ast 1_{K}$ and

\[ A_{\lambda} = \left( \sum_{\nu \in \text{wt}(V_{\lambda})} \theta_{\nu} \right) \ast 1_{K} \]

(4) For the simple reflection $s \in W$ corresponding to a simple root $\alpha$ and a coweight $\mu$ one has

\[ t_{s}\theta_{\mu} = \theta_{s\mu}t_{s} + (1-q)\frac{\theta_{s\mu} - \theta_{\mu}}{1-\theta_{-\alpha}}. \]

In particular $t_{s}$ commutes with $\theta_{k\alpha} + \theta_{-k\alpha}$ for any $k \geq 0$. In addition $t_{s}$ commutes with $\theta_{\mu}$ whenever $s\mu = \mu$.

4.3. The intermediate algebra. Finally consider the space $H_{I,K}$ defined by

\[ H_{K} \subset H_{I,K} = H_{I} \ast 1_{K} = C_{c}(I \backslash G/K) \subset H_{I}. \]

It has a structure of right $H_{K}$ module. The space $H_{I,K}$ plays a crucial role in Savin’s paper [S]. The Satake map restricted to it is the isomorphism of $H_{K} \cong \mathbb{C}[\Lambda]^{W}$ modules:

\[ S : H_{I,K} \cong \mathbb{C}[\Lambda], \quad S(\theta_{\mu} \ast 1_{K}) = e^{\mu}. \]

In particular, it is shown that the elements $\{\theta_{\mu}^{K} = \theta_{\mu} \ast 1_{K}, \mu \in \Lambda\}$ form a basis of $H_{I,K}$.

5. The Whittaker space $(\text{ind}_{N}^{G} \Psi)^{K}$

Let $\Psi$ be a non-degenerate character of conductor 1. Consider the space $(\text{ind}_{N}^{G} \Psi)^{K}$ of complex valued functions on $G$ that are $(N, \Psi)$-equivariant on the left, right $K$-invariant functions and are compactly supported modulo $N$. 
The space \((\text{ind}_{N}^{G}\Psi)^{K}\) has a structure of right \(H_{K}\) module by
\[
(\phi \ast f)(x) = \int_{G} \phi(xy^{-1})f(y)dy, \quad \phi \in (\text{ind}_{N}^{G}\Psi)^{K}, f \in H_{K}.
\]

Any function \(\phi\) on \((\text{ind}_{N}^{G}\Psi)^{K}\) is determined by its values on \(t_{\lambda}: \lambda \in \Lambda\) and \(\phi(t_{\lambda}) = 0\) unless \(\lambda \in \Lambda^{+} + \rho\).

The space \((\text{ind}_{N}^{G}\Psi)^{K}\) has a basis of characteristic functions \(\{\phi_{\lambda}: \lambda \in \Lambda^{+} + \rho\}\) where
\[
\phi_{\lambda}(ntk) = \begin{cases} 
\delta_{B}^{1/2}(t)\Psi(n) & t \in Nt_{\lambda}K, \lambda \in \Lambda^{+} + \rho; \\
0 & \text{otherwise}
\end{cases}
\]

The main theorem of this paper is the description of \((\text{ind}_{N}^{G}\Psi)^{K}\) as \(H_{K}\) module.

**Theorem 5.1.** Let \(\Psi\) be a character of conductor 1. Then there is an isomorphism
\[
j: (\text{ind}_{N}^{G}\Psi)^{K} \simeq \mathbb{C}[\Lambda]^{W,-}
\]
compatible with \(H_{K} \simeq \mathbb{C}[\Lambda]^{W}\).

5.1. **The twisted Satake isomorphism.** For a fixed character \(\Psi\) of \(N\), consider a twisted Satake map
\[
S_{\Psi}: C_{c}(G/I) \rightarrow (\text{ind}_{N}^{G}\Psi)^{I}
\]
defined by
\[
S_{\Psi}(f)(t) = \int_{N} f(nt)\overline{\Psi(n)}dn.
\]

**Corollary 5.2.** The restriction of \(S_{\Psi}\) to the right \(H_{K}\) submodule \(\theta_{\rho}^{K} \ast H_{K}\) defines an isomorphism
\[
S_{\Psi}: \theta_{\rho}^{K} \ast H_{K} \simeq (\text{ind}_{N}^{G}\Psi)^{K}
\]
such that \(S_{\Psi}(\theta_{\rho}^{K} \ast A_{\lambda}) = \phi_{\lambda+\rho}\).

**Proof.** By Weyl character formula \(r_{\lambda+\rho} = r_{\rho} \cdot a_{\lambda}\). Hence
\[
j(\phi_{\lambda+\rho}) = r_{\lambda+\rho} = r_{\rho} \cdot a_{\lambda} = j(\phi_{\rho} \ast A_{\lambda}),
\]
and thus \(\phi_{\rho} \ast A_{\lambda} = \phi_{\lambda+\rho}\).

Restricting \(S_{\Psi}\) to \(H_{I,K}\), we obtain
\[
S_{\Psi}(\theta_{\rho}^{K} \ast A_{\lambda}) = S_{\Psi}(\theta_{\rho}^{K}) \ast A_{\lambda} = \phi_{\rho} \ast A_{\lambda} = \phi_{\lambda+\rho}.
\]

Since \(A_{\lambda}\) and \(\phi_{\lambda+\rho}\) are bases of \(H_{K}\) and \((\text{ind}_{N}^{G}\Psi)^{K}\) respectively, the map \(S_{\Psi}\) is an isomorphism.

To prove the theorem we shall need two lemmas. The first one ensures surjectivity of the map \(S_{\Psi}\) and the second one describes its kernel.
Lemma 5.3. $S_{\Psi}(\theta_{\mu}^{K}) = \phi_{\mu}$ for all $\mu \in \Lambda^{+} + \rho$. In particular the map

$$S_{\Psi} : H_{I,K} \rightarrow (\text{ind}_{N}^{G} \Psi)^{K}$$

is surjective.

Proof. It is enough to compute $S_{\Psi}(\theta_{\mu} * 1_{K})(t_{\gamma})$ for $\gamma \in \Lambda^{+}$.

Since $\mu$ is dominant one has

$$\theta_{\mu}^{K} = \delta_{B}^{1/2}(t_{\mu})1_{It_{\mu}K}$$

and hence

$$S_{\Psi}(\theta_{\mu}^{K})(t_{\gamma}) = \delta_{B}^{1/2}(t_{\gamma}) \int_{N_{\gamma,\mu}} \overline{\Psi(n)} dn,$$

where

$$N_{\gamma,\mu} = \{n \in N : nt_{\gamma} \in It_{\mu}K\}.$$  

The set

$$N_{\gamma,\mu} = \begin{cases} \emptyset & \gamma \neq \mu \\ N \cap K & \gamma = \mu \end{cases}$$

Indeed, since $\mu \in \Lambda^{+}$ one has $It_{\mu}K = (N \cap I)t_{\mu}K$. One inclusion is obvious. For another inclusion use the Iwahori factorization

$$I = (I \cap N)T_{K}(I \cap \overline{N})$$

to represent any $g \in It_{\mu}K$ as

$$g = na_{0}\overline{n}t_{\mu}k = nt_{\mu}a_{0}(t_{\mu}^{-1}\overline{n}t_{\mu})k,$$

where $n \in N \cap I, a_{0} \in T_{K}, \overline{n} \in \overline{N}, k \in K$. Since $\mu$ is dominant one has $(t_{\mu}^{-1}\overline{n}t_{\mu}) \in K$. So $g \in (N \cap I)t_{\mu}K$. Hence $N_{\gamma,\mu} = \emptyset$ unless $\gamma = \mu$ and $N_{\mu,\mu} = (N \cap I)t_{\mu}(N \cap K)t_{\mu}^{-1} = N \cap I$ since $\mu \in \Lambda^{+} + \rho$. In particular $\Psi|_{N_{\mu,\mu}} = 1$. Hence

$$S_{\Psi}(\theta_{\mu} * 1_{K}) = \phi_{\mu}.$$

Lemma 5.4. Let $\alpha \in \Delta, s$ be a simple reflection corresponding to $\alpha$ and $\iota_{\alpha} = 1_{I} + t_{s}$ be the characteristic function of a parahoric subgroup $I_{\alpha}$ corresponding to $\alpha$.

1) $S_{\Psi}(\iota_{\alpha}) = 0$.
2) $S_{\Psi}(\theta_{\mu}^{K} + \theta_{s,\mu}^{K}) = 0$ for all $\mu \in \Lambda$.

Proof. 1) $S_{\Psi}(\iota_{\alpha})(tw) = \int_{N} \iota_{\alpha}(ntw)\overline{\Psi(n)} dn = \int_{N \cap I_{\alpha}(tw)^{-1}} \overline{\Psi(n)} dn$

The set $N \cap I_{\alpha}(tw)^{-1}$ is empty unless $w \in \{e, s\}$ and $t \in T_{K}$, in which case

$$S_{\Psi}(\iota_{\alpha})(tw) = \int_{N \cap I_{\alpha}} \overline{\Psi(n)} dn = 0$$
since the integral contains an inner integral over $N^0_{\alpha}$ on which $\Psi$ is not trivial.

2) Let us represent any $\mu = \mu' + k\alpha$ where $(\mu', \alpha) = 0$. Then $s\mu = \mu' - k\alpha$.

In particular

$$\theta^K_{\mu'} + \theta^K_{s\mu} = \theta_{\mu'}(\theta_{k\alpha} + \theta_{-k\alpha})i_{\alpha}1_K$$

By the results in 4.2 the element $i_{\alpha}$ commutes with $\theta_{\mu'}(\theta_{k\alpha} + \theta_{-k\alpha})$ and hence the above equals

$$i_{\alpha}\theta_{\mu'}(\theta_{k\alpha} + \theta_{-k\alpha})1_K = i_{\alpha}(\theta^K_{\mu'} + \theta^K_{s\mu})$$

By part (1) it follows that $S_{\Psi}(\theta^K_{\mu'} + \theta^K_{s\mu}) = 0$. \hfill $\square$

**Proof.** of 5.1. We have shown that the map $S_{\Psi}$ is surjective onto $(\text{ind}_{N}^{G}\Psi)^K$ and

$$\text{Ker} S_{\Psi} = \text{Span}\{\theta_{\mu} - (-1)^{l(w)}\theta_{w\mu}|\mu \in \Lambda, w \in W\}.$$ 

Another words

$$(\text{ind}_{N}^{G}\Psi)^K \simeq H_{I,K}/\text{Ker} S_{\Psi} = \mathbb{C}[\Lambda]^{W,-}$$

as $H_{K} \simeq \mathbb{C}[\Lambda]^{W}$-modules. \hfill $\square$

6. **Casselman-Shalika Formula**

Let $(\pi, G, V)$ be an irreducible smooth generic unramified representation and denote by $\gamma \in LT/W$ its Satake conjugacy class. Choose a spherical vector $v_0$ and normalize the Whittaker functional $W_{\gamma} \in \text{Hom}_{G}(\pi, \text{Ind}_{N}^{G}\overline{\Psi})$ such that $W_{\gamma}(t_{\rho}v_0) = 1$.

The Casselman-Shalika formula reads as follows:

**Theorem 6.1.**

$$W_{\gamma}(v_0)(t_{\lambda+\rho}) = \begin{cases} 
\delta_{B}^{1/2}(t_{\lambda+\rho}) \text{tr} V_{\lambda}(t_{\gamma}) & \lambda \in \Lambda^+ \\
0 & \text{otherwise}
\end{cases}$$

It is shown in [FGKV], that Theorem 5.2 that the formula (6.1) implies the Corollary 5.2 and it is mentioned that the two statements are equivalent. Let us now prove the other direction.

**Proof.** We deduce the formula 6.1 from 5.2. Let $\pi$ be a generic unramified representation with the Satake parameter $\gamma \in LT$ and a spherical vector $v_0$ and the Whittaker model $W_{\gamma} : \pi_{\gamma} \rightarrow \text{Ind}_{N}^{G}\overline{\Psi}$ such that $W_{\gamma}(v_0)(t_{\rho}) = 1$.

Define the map $\chi_{\gamma} : H_{K} \rightarrow \mathbb{C}$ by

$$\pi(f)v_0 = \int_{G} f(g)\pi(g)v_0 \, dg = \chi_{\gamma}(f)v_0$$

and the map $r_{\gamma} : \text{ind}_{N}^{G}\Psi \rightarrow \mathbb{C}$ by

$$r_{\gamma}(\phi) = \int_{N\backslash G} W_{\gamma}(v_0)(g)\phi(g) \, dg.$$
Then
\[
\begin{align*}
 r_{\gamma}(S_{\Psi}(\theta_{\rho}^{K} * A_{\lambda})) & = \int_{N \backslash G} \int_{N} (\theta_{\rho}^{K} * A_{\lambda})(ng)\overline{\Psi}(n)W_{\gamma}(v_{0})(g) \, dn \, dg = \\
 & = \int_{G} \int_{G} (\theta_{\rho}^{K} * A_{\lambda})(g)W_{\gamma}(v_{0})(g) \, dg = \\
 & = \int_{G} \int_{G} \theta_{\rho}^{K}(gx^{-1})A_{\lambda}(x)W_{\gamma}(gx^{-1} \cdot x \cdot v_{0})(1) \, dx \, dg = \chi_{\gamma}(A_{\lambda})W_{\gamma}(v_{0})(t_{\rho}).
\end{align*}
\]

Under the identification \( H_{K} \simeq \text{Rep}(^{L}G) \) the homomorphism \( \chi_{\gamma} \) sends an irreducible representation \( V \) to \( \text{tr} \, V(\gamma) \). In particular \( \chi_{\gamma}(A_{\lambda}) = \text{tr} \, V_{\lambda}(\gamma) \).

\[
\text{tr} \, V_{\lambda}(\gamma) = \chi_{\gamma}(A_{\lambda})W(v_{0})(t_{\rho}) = r_{\gamma}(S_{\Psi}(\theta_{\rho}^{K} * A_{\lambda})) = r_{\gamma}(\phi_{\lambda+\rho}) = \delta_{B}^{-1/2}(t_{\lambda+\rho})W(\gamma)(v_{0})(t_{\lambda+\rho})
\]

Hence
\[
W_{\gamma}(v_{0})(t_{\lambda+\rho}) = \delta_{B}^{1/2}(t_{\lambda+\rho}) \text{tr} \, V_{\lambda}(\gamma).
\]

\[\square\]

REFERENCES

[CS] W. Casselman, J. Shalika The unramified principal series of \( p \)-adic groups. II. The Whittaker function. Compositio Math. 41 (1980), no. 2, 207-231.

[FGKV] E. Frenkel, D. Gaitsgory, D. Kazhdan, K. Vilonen Geometric realization of Whittaker functions and the Langlands conjecture. J. Amer. Math. Soc. 11 (1998), no. 2, 451-484.

[HKP] T. Haines, R. Kottwitz, A. Prasad Iwahori-Hecke Algebras, J. Ramanujan Math. Soc. 25 (2010), no. 2, 113-145.

[S] G. Savin The tale of two Hecke algebras, arXiv:1202.1486

School of Mathematics, Ben Gurion University of the Negev, POB 653, Beer Sheva 84105, Israel
E-mail address: ngur@math.bgu.ac.il