Covariant double-null dynamics: (2 + 2)-splitting of the Einstein equations

P R Brady†, S Droz‡, W Israel‡ and S M Morsink‡
† Theoretical Astrophysics 130-33, California Institute of Technology, Pasadena, CA 91125, USA
‡ Canadian Institute for Advanced Research Cosmology Program, Theoretical Physics Institute, University of Alberta, Edmonton, Canada T6G 2J1

Received 31 October 1995, in final form 29 April 1996

Abstract. We develop a (2 + 2)-imbedding formalism adapted to a double foliation of spacetime by a net of two intersecting families of lightlike hypersurfaces. The intersections define a set of 2-spaces with hypersurface-orthogonal lightlike normals. The formalism is two-dimensionally covariant, and leads to simple, geometrically transparent and tractable expressions for the Einstein field equations and the Einstein–Hilbert action, and it should find a variety of applications. It is applied here to elucidate the structure of the characteristic initial-value problem of general relativity.

PACS numbers: 0240, 0420

1. Introduction

The classic analysis of Arnowitt, Deser and Misner (ADM) [1] formulates gravitational dynamics in terms of the evolution of a spatial 3-geometry. The geometrical framework is the imbedding formalism of Gauss and Codazzi for the foliation of spacetime by spacelike hypersurfaces [2].

Quite often, however, one encounters circumstances where a lightlike foliation is especially suitable. Because of the degeneracies that arise in the lightlike case, the imbedding relations are very different and the situation is not quite so familiar and under control. To bypass the degeneracies, one is forced to fall back to a foliation of codimension 2, by spacelike 2-surfaces. It is our aim in this paper to develop a simple (2 + 2)-imbedding formalism of this kind.

Several (2 + 2)-formalisms are extant [3], the earliest and best known being the generalized spin-coefficient formalism of Geroch, Held and Penrose (GHP) [4]. Basically, of course, all such formalisms have the same content, but they take very different forms.

The essential feature of the present approach is that it maintains manifest two-dimensional covariance while operating with objects having direct geometrical meaning. Two-dimensional covariance permits reduction of the Einstein field equations to an especially concise and transparent form: the ten Ricci components are embraced in a set of just three compact, two-dimensionally covariant expressions.

There is a limitation, at least in the version presented here (it applies to most of the formalisms we have listed [3]). The two independent normals to an imbedded 2-surface—conveniently taken as a pair of lightlike vectors, since their directions are uniquely defined—are assumed from the beginning to be hypersurface-orthogonal. This precludes choosing
them as principal null vectors of the Weyl tensor for a twisting geometry like Kerr. In this respect, the formalism is less flexible than GHP, and not as well tailored for the study of algebraically special metrics.

$(2+2)$-formalisms have a wide range of applications: to the analysis of the characteristic initial-value problem [5], the dynamics of strings [6] and of real and apparent horizons [7] and light-cone quantization [8] and gravitational interactions in ultra-high energy collisions [9]. In a separate publication [10], we shall use the present formalism to study the nature of the singularity at the Cauchy horizon in a generic black hole.

We conclude this introduction by briefly outlining the contents of the paper. The basic metrical notions (adapted coordinates, basis vectors and the form of the metric) are defined in section 2. In section 3, we introduce in two-dimensionally covariant form the geometrical information encoded in first derivatives of the metric: the extrinsic curvatures and ‘twist’, as well as the invariant operators which perform differentiation along the two lightlike normals. This comprises the basic formal machinery needed in section 4, which presents the central result of the paper, the tetrad components of the Ricci tensor, as three concise equations (27)–(29). (To make direct access to these results easy, their derivation is deferred to the second half of the paper (sections 9–12), which also provides (section 13) the tetrad components of the full Riemann tensor.)

The contracted Bianchi identities (section 5) are applied in section 7 to analyse the structure of the characteristic initial-value problem after a discussion of various possible choices of coordinate and gauge conditions. In section 8 we sketch the Lagrangian formulation of covariant double-null dynamics.

The Ricci and Riemann components result from the commutation relations for four-dimensional covariant differentiation. Their most efficient derivation calls for a formalism that is both four- and two-dimensionally covariant. Unfortunately, these two requirements do not mesh easily. Four-dimensional covariance tends to clutter the formulae by treating subsidiary two-dimensional quantities, like shift vectors and the two-dimensional connection, as 4-scalars on a par with the primary geometrical properties, extrinsic curvature and twist. Those properties, for their part, are correlated, not with four-dimensional covariant derivatives, but with Lie derivatives, which are non-metric and have no direct link to curvature. To patch up these differences, and thus streamline the derivations, seems to need a certain degree of artifice. In section 10 we address this (purely technical) problem by temporarily working with a ‘rationalized’ covariant derivative which exhibits both four-dimensional and restricted (‘rigid’) two-dimensional covariance.

Some brief remarks (section 14) conclude the paper.

2. $(2+2)$-split of the metric

We shall suppose that we are given a foliation of spacetime by lightlike hypersurfaces $\Sigma^0$ with normal generators $\ell_0^0$, and a second, independent foliation by lightlike hypersurfaces $\Sigma^1$ with generators $\ell_1^0$ nowhere parallel to $\ell_0^0$. The intersections of $\{\Sigma^0\}$ and $\{\Sigma^1\}$ define a foliation of codimension 2 by spacelike 2-surfaces $S$. (The topology of $S$ is unspecified. All our considerations are local.) $S$ has exactly two lightlike normals at each of its points, co-directed with $\ell_0^0$ and $\ell_1^0$.

In terms of local charts, the foliation is described by the imbedding relations

$$x^a = x^a(u^A, \theta^a).$$

Here, $x^a$ are four-dimensional spacetime coordinates (assumed admissible in the sense of Lichnerowicz [11]); $u^0$ and $u^1$ are a pair of scalar fields constant over each of the
hypersurfaces $\Sigma^0$ and $\Sigma^1$, respectively; and $\theta^2$, $\theta^3$ are intrinsic coordinates of the 2-spaces $S$, each characterized by a fixed pair of values $(u^0, u^1)$.

Notation. Our conventions are: Greek indices $\alpha, \beta, \ldots$ run from 0 to 3; upper-case Latin indices $A, B, \ldots$ take values $(0, 1)$; and lower-case Latin indices $a, b, \ldots$ take values $(2, 3)$. We adopt MTW curvature conventions [2] with signature $(-+++)$ for the spacetime metric $g_{\alpha\beta}$. When there is no risk of confusion we shall often omit the Greek indices on 4-vectors like $\ell(A)_{\alpha}$ and $e_{\alpha}(a)$: they are easily identifiable as 4-vectors by their parenthesized labels. Four-dimensional covariant differentiation is indicated either by $\nabla\alpha$ or a vertical stroke: $\nabla_{\beta}A_{\alpha} \equiv A_{\alpha|\beta}$. Four-dimensional scalar products are often indicated by a dot: thus, $\ell(A)_{\alpha} \cdot \ell(B)_{\beta} \equiv g_{\alpha\beta} \ell_{\alpha}(A) \ell_{\beta}(B)$. Further conventions will be introduced as the need arises.

Without essential loss of generality we may assume the functions $x^a(u^A, \theta^a)$ to be smooth (at least thrice differentiable). (We are always free to make the coordinate choice $x^A = u^A$, $x^a = \theta^a$, but at the cost of losing manifest four-dimensional and two-dimensional covariance.)

The lightlike character of the hypersurfaces $\Sigma^A$ is encoded in
$$\nabla u^A \cdot \nabla u^B \equiv g^{\alpha\beta}(\partial_{\alpha}u^A)(\partial_{\beta}u^B) = e^{-\lambda} \eta^{AB},$$
for some scalar field $\lambda(x^a)$, where
$$\eta^{AB} = \text{anti-diag}(-1, -1) = \eta_{AB};$$
and its inverse $\eta_{AB}$ are employed to raise and lower upper-case Latin indices, e.g. $\ell_{(a)} = -\ell^{(1)}$.

The generators $\ell^{(A)}$ of $\Sigma^A$ are parallel to the gradients of $u^A(x^a)$. It is symmetrical and convenient to define $\ell^{(A)} = e^A \nabla u^A$, i.e.
$$\ell_{\alpha}^{(A)} = e^A \partial_{\alpha}u^A,$$
then
$$\ell_{(A)} \cdot \ell_{(B)} = e^A \delta_{AB}.$$

The pair of vectors $e_{(a)}$, defined from (1) by
$$e_{(a)} = \partial x^a / \partial \theta^a,$$
are holonomic basis vectors tangent to $S$. The intrinsic metric $g_{ab} \, d\theta^a \, d\theta^b$ of $S$ is determined by the scalar product:
$$g_{ab} = e_{(a)} \cdot e_{(b)}.$$

Lower-case Latin indices are lowered and raised with $g_{ab}$ and its inverse $g^{ab}$: thus $e^{(a)} \equiv g^{ab} e_{(b)}$ are the dual basis vectors tangent to $S$, with $e^{(a)} \cdot e_{(b)} = \delta_{a}^b$. Since $\ell^{(A)}$ is normal to every vector in $\Sigma^A$, we have
$$\ell_{(A)} \cdot e_{(a)} = 0.$$

In general, $\theta^a$ cannot be chosen so as to remain constant along both sets of generators $\ell^{(A)}$. They are convected (Lie-transported) along the pair of vector fields $\partial x^a / \partial u^A$ (in general, non-lightlike).

From (4) and (5) one finds that $\partial x^a / \partial u^A - \ell_{(a)}^{(A)}$ is orthogonal to $\ell_{(B)}$, i.e. tangent to $S$. This validates the decomposition
$$\frac{\partial x^a}{\partial u^A} = \ell_{(a)}^{(A)} + s_{a}^{A} e_{(a)},$$
thus defining a pair of ‘shift vectors’ $s_\alpha^A$ tangent to $S$ (see figure 1).

An arbitrary displacement $dx^\alpha$ in spacetime is, according to (6) and (9), decomposable as

$$dx^\alpha = \ell^\alpha(A) du^A + e^\alpha(a) (d\theta^a + s^\alpha_A du^A).$$

From (5), (7) and (8) we read off the completeness relation

$$g_{\alpha\beta} = e^{-\lambda} \eta_{AB} \ell^\alpha(A) \ell^\beta(B) + g_{ab} e^\alpha(a) e^\beta(b).$$

Combining (10) and (11) shows that the spacetime metric is decomposable as

$$g_{\alpha\beta} dx^\alpha dx^\beta = e^{\lambda} \eta_{AB} du^A du^B + g_{ab} (d\theta^a + s^\alpha_A du^A)(d\theta^b + s^\beta_B du^B).$$

3. Two-dimensionally covariant objects embodying first derivatives of the metric: extrinsic curvatures $K_{Aab}$, twist $\omega^a$ and normal Lie derivatives $D_A$

Absolute derivatives of four-dimensional tensor fields with respect to $u^A$ and $\theta^a$ are projections of the four-dimensional covariant derivative $\nabla_\alpha$, and are denoted by

$$\frac{\delta}{\delta u^A} = \frac{\partial x^\alpha}{\partial u^A} \nabla_\alpha, \quad \frac{\delta}{\delta \theta^a} = e_{(a)} \cdot \nabla.$$  

From (6) and the symmetry of the mixed partial derivatives and the affine connection,

$$\frac{\delta \epsilon_{(a)}}{\delta u^A} = \frac{\delta}{\delta \theta^a} \left( \frac{\partial x^\alpha}{\partial u^A} \right), \quad \frac{\delta \epsilon_{(a)}}{\delta \theta^b} = \frac{\delta \epsilon_{(b)}}{\delta \theta^a}.$$  

The object

$$\Gamma^c_{ab} = e^{(c)} \cdot \frac{\delta \epsilon_{(a)}}{\delta \theta^b}$$

is, as the notation suggests, the Christoffel symbol associated with $g_{ab}$, as is easily verified by forming $\partial_c g_{ab}$, recalling (7) and applying Leibnitz’s rule.
Covariant double-null dynamics

Associated with its two normals $\ell_{(A)}$, $S$ has two extrinsic curvatures $K_{Aab}$, defined by

$$K_{Aab} = e_{(a)} \cdot \delta \ell_{(A)}/\delta \theta^b = \ell_{(A)\alpha\beta} e_{(a)}^\alpha e_{(b)}^\beta.$$  (16)

(Since we are free to rescale the null vectors $\ell_{(A)}$, a certain scale-arbitrariness is inherent in this definition.) Because of (8), we can rewrite this as

$$K_{Aab} = -\ell_{(A)} \cdot \delta e_{(a)}/\delta \theta^b,$$  (17)

which exhibits symmetry in $a, b$.

A further basic geometrical property of the double foliation is given by the Lie bracket of $\ell_{(B)}$ and $\ell_{(A)}$, i.e. the 4-vector

$$[\ell_{(B)}, \ell_{(A)}] = 2(\ell_{(B)} \cdot \nabla)\ell_{(A)}.$$  (18)

Noting (9) and the fact that the Lie bracket of the vectors $\partial x^a/\partial u^B$ and $\partial x^a/\partial u^A$ vanishes identically, and recalling (14), we find

$$[\ell_{(B)}, \ell_{(A)}] = \epsilon_{AB} \omega^a e_{(a)},$$  (19)

where the 2-vector $\omega^a$ is given by

$$\omega^a = \epsilon_{AB} (\partial B s^a_A - s^b_A s^a_B),$$  (20)

the semicolon indicates two-dimensional covariant differentiation associated with the metric $g_{ab}$, and $\epsilon_{AB}$ is the two-dimensional permutation symbol, with $\epsilon_{01} = +1$. (Note that raising indices with $\eta^{AB}$ to form $\epsilon^{AB}$ yields $\epsilon^{10} = +1$.)

The geometrical significance of the ‘twist’ $\omega^a$ can be read off from (19): the curves tangent to the generators $\ell_{(0)}, \ell_{(1)}$ mesh together to form 2-surfaces (orthogonal to the surfaces $S$) if and only if $\omega^a = 0$. In this case, it would be consistent to allow the coordinates $\theta^a$ to be dragged along both sets of generators, and thus to gauge both shift vectors to zero.

We denote by $D_A$ the two-dimensionally invariant operator associated with differentiation along the normal direction $\ell_{(A)}$. Acting on any two-dimensional geometrical object $X^{a...b...}$, $D_A$ is formally defined by

$$D_A X^{a...b...} = (\partial_A - L_{s^A}) X^{a...b...}.$$  (21)

Here, $\partial_A$ is the partial derivative with respect to $u^A$ and $L_{s^A}$ the Lie derivative with respect to the 2-vector $s^A_a$.

As examples of (21), we have for a 2-scalar $f$ (this includes any object bearing upper-case, but no lower-case, Latin indices):

$$D_A f = (\partial_A - s^A_a \partial_a) f = \ell_{(A)\beta} \partial_{\beta} f$$  (22)

(in which the second equality follows at once from (9)); and for the 2-metric $g_{ab}$:

$$D_A g_{ab} = \partial_A g_{ab} - 2 s_{A(a;b)} = 2K_{Aab},$$  (23)

in which the second equality is derivable from (7), (14) and (9). (For the detailed derivation, see (75) below, or appendix B.) The geometrical meaning of $D_A$ is quite generally the following (see appendix B): $D_A X^{a...b...}$ is the projection onto $S$ of the Lie derivative of the equivalent tangential 4-tensor

$$X^{a...b...}_{\rho...} \equiv X^{a...b...}_{(a)} e_{(a)}^\rho e_{(b)}^\beta,$$

with respect to the 4-vector $\ell_{(A)}$:

$$D_A X^{a...b...}_{\rho...} = e_{(a)}^\rho e_{(b)}^\beta \ell_{(A)\mu} X^{a...b...}_{\mu...}.$$  (24)
The objects $K_{Aab}$, $\omega^a$ and $D_A$ comprise all the geometrical structure that is needed for a succinct two-dimensionally covariant expression of the Riemann and Ricci curvatures of spacetime. According to (16), (19) and (24), all are simple projections onto $S$ of four-dimensional geometrical objects. Consequently, they transform very simply under two-dimensional coordinate transformations. Under the arbitrary reparametrization
\[
\theta^a \to \theta'^a = f^a(\theta^b, u^A)
\] (which leaves $u^A$ and hence the surfaces $\Sigma^A$ and $\Sigma$ unchanged), $\omega_a$ and $K_{Aab}$ transform cogrediently with
\[
epsilon^{(a)} \to \epsilon'^{(a)} = \epsilon^{(b)} \partial \theta^b / \partial \theta'^a
\] (see (6)), and $D_A$ is invariant. By contrast, $\partial x^a / \partial u_A$, and hence the shift vectors $s^a_A$ (see (9)), undergo a more complicated gauge-like transformation, arising from the $u$-dependence in (25).

4. The Ricci tensor

The geometrical and notational groundwork laid in the previous sections allows us now to simply display the components of the Ricci tensor, deferring derivations to sections 9–12. Our notation for the tetrad components is typified by
\[
R_{(ab)} = R_{a(b)} e^a_{(a)} e^b_{(b)}, \quad R_{aA} = R_{a(b)} e^a_{(a)} \ell^b_A.
\]

The results are
\[
R_{(ab)} = \frac{1}{2} (R_{(ab)} - \epsilon^{-\lambda}(D_A + K_A) K_{Aab}) + 2 \epsilon^{-\lambda} K_{A(a} D^{b)} K_{A} - \frac{1}{2} \epsilon^{-2\lambda} \omega_a \omega_b - \lambda_{,ab} - \frac{1}{2} \lambda_{,a} \lambda_{,b}
\]
(27)

\[
R_{AB} = -D(A K_B) - K_{Aab} K^c_{A} + K(A D_B) \lambda - \frac{1}{2} \eta_{AB} \left( (D^E + K^E) D_E \lambda - e^{-\lambda} \omega^a \omega_a + (\epsilon^{(a)} \epsilon^{(a)} a ) \right)
\]
(28)

\[
R_{aA} = K_{Aa} \cdot b - \partial_a K_A = \frac{1}{2} \delta_a D_A \lambda + \frac{1}{2} K_A \partial_a \lambda + \frac{1}{2} \epsilon_{AB} \omega^a [(D^B + K^B) \omega_a - \omega_a D^B \lambda] \]
(29)

where $(2)R$ is the curvature scalar associated with the 2-metric $g_{ab}$, and $K_A \equiv K_{Aa}^a$.

5. Bianchi identities and Bondi’s lemma

The Ricci components are linked by four differential identities, the contracted Bianchi identities
\[
\nabla_{a} R_{b}^{a} = \frac{1}{2} \delta_{a} R,
\]
(30)

where the four-dimensional curvature scalar $R = R^a_a$ is given by
\[
R = e^{-\lambda} R_{A}^{A} + R_{a}^{a},
\]
(31)

according to (11).

As we show in section 12, projecting (30) onto $e^{(a)}$ leads to
\[
(D_A + K_A) R_{A}^{a} = \frac{1}{2} \delta_{a} R_{A}^{A} + \frac{1}{2} \epsilon^{(a)} \delta_{a} (4) R_{b}^{b} - (e^{2\lambda} (4) R_{A}^{b} )_{,b}.
\]
(32)

Projection of (30) onto $\ell_{(A)}$ similarly yields
\[
D_B (R_{A}^{B} - \frac{1}{2} \delta_{A} R_{D}^{b} ) - \frac{1}{2} \epsilon^{(a)} D_{A} (4) R_{a}^{a}
\]
\[
= \epsilon^{(a)} (4) R_{ab} K_{A}^{ab} - R_{A}^{B} K_{B} - (e^{2\lambda} R_{A}^{A})_{,a} + \epsilon_{AB} \omega^a R_{B}^{a}.
\]
(33)
Equations (32) and (33) express the four Bianchi identities in terms of the tetrad components of the Ricci tensor.

We now look at the general structure of these equations. For $A = 0$ in (33), $R^0_0$ does not contribute to the first (parenthesized) term, since

$$-R_{01} = R^0_0 = R^1_1 = \frac{1}{2} R^A_A.$$ (34)

This equation therefore takes the form

$$D_1 R_{00} + \frac{1}{2} e^\lambda D_0 (4) R^a_a = -K_0 R_{01} + L(4) R_{ab}, R_{00}, R_{0a}, \partial_a),$$ (35)

in which the schematic notation $L$ implies that the expression is linear homogeneous in the indicated Ricci components and their two-dimensional spatial derivatives $\partial_a$.

The other ($A = 1$) component of (33) has the analogous structure

$$D_0 R_{11} + \frac{1}{2} e^\lambda D_1 (4) R^a_a = -K_1 R_{01} + L(4) R_{ab}, R_{11}, R_{1a}, \partial_a).$$ (36)

The form of the remaining two Bianchi identities (32) is

$$D_0 R_{1a} + D_1 R_{0a} = L(4) R_{ab}, R_{01}, R_{Aa}, \partial_a).$$ (37)

It is noteworthy that the appearance of $R_{01}$ in (35) and (36) is purely algebraic: its vanishing would be a direct consequence of the vanishing of just six of the other components. Bondi et al [13] and Sachs [5] therefore refer to the $R_{01}$ field equation as the ‘trivial equation’.

The structure of (35)–(37) provides an insight into how the field equations propagate initial data given on a lightlike hypersurface. Let us (arbitrarily) single out $u^0$ as ‘time’, and suppose that the six ‘evolutionary’ vacuum equations

$$(4) R_{ab} = 0, \quad R_{00} = R_{0a} = 0$$ (38)

are satisfied everywhere in the neighbourhood of a hypersurface $u^0 = constant$. (Bondi and Sachs refer to $R_{ab}$ as the ‘main equations’ and to $R_{00}, R_{0a}$ as ‘hypersurface equations’. $R_{00}$ is, in fact, the Raychaudhuri focusing equation [12], governing the expansion of the lightlike normal $\ell(0) = -e^{(1)}$ to the transverse hypersurface $u^1 = constant$, and $R_{0a}$ similarly governs its shear.)

Then (35) shows that the trivial equation $R_{01} = 0$ is satisfied automatically. From (36) and (37) it can be further inferred that if $R_{11}$ and $R_{1a}$ vanish on one hypersurface $u^0 = constant$, then they will vanish everywhere as a consequence of the six evolutionary equations (38). This is the content of the Bondi–Sachs lemma [5, 13], which identifies the three conditions $R_{11} = R_{1a} = 0$ as constraints—on the expansion and shear of the generators $\ell(1)$ of an initial hypersurface $u^0 = constant$—which are respected by the evolution.

6. Coordinate conditions and gauge fixing

The characteristic initial-value problem [5] involves specifying initial data on a given pair of lightlike hypersurfaces $\Sigma^0, \Sigma^1$ intersecting in a 2-surface $S_0$.

It is natural to choose our parameters $u^A$ so that $u^0 = 0$ on $\Sigma^0$ and $u^1 = 0$ on $\Sigma^1$. The requirement (2) that $u^A$ be globally lightlike already imposes two coordinate conditions on $(u^A, \theta^a)$, considered as coordinates of spacetime. Two further global conditions may be imposed. We may, for instance, demand that $\theta^a$ be convected (Lie-propagated) along the lightlike curves tangent to $e^a_0$ from values assigned arbitrarily on $\Sigma^0$. According to (9), this means the corresponding shift vector is zero everywhere:

$$e^a_0 = -e^a_0 \partial_a \theta^a = 0.$$ (39)
In this case, (20) shows that
\[ \omega^a = \partial_0 s_1^a \]  

is just the ‘time’-derivative of the single remaining shift vector.

These global coordinate conditions can still be supplemented by appropriate initial conditions. We are still free to require that \( \theta^a \) be convected along generators of \( \Sigma^0 \) from assigned values on \( S_0 \); then
\[ s_1^a = 0 \quad (u^0 = 0) \]  

in addition to (39).

In addition to (or independently of) (39) and (41), we are free to choose \( u^1 \) along \( \Sigma^0 \) and \( u^b \) along \( \Sigma^1 \) to be affine parameters of their generators. On \( \Sigma^0 \), for instance, this means, by virtue of (9) and (4),
\[ \ell^a = \left( \frac{dx^a}{dt} \right)_{\text{gen}} = -g^{a\beta} \partial_\beta u^0 = -e^{-\lambda} g^{a\beta} \epsilon^0_\beta \]

so that \( \lambda \) vanishes over \( \Sigma^0 \). There is a similar argument for \( \Sigma^1 \). Thus, we can arrange that
\[ \lambda = 0 \quad (\Sigma^0 \text{ and } \Sigma^1). \]  

Alternatively, in place of (42), the coordinate condition
\[ D_1 \lambda = \frac{1}{2} K_1 \quad \text{on } \Sigma^0 \]  

could be imposed to normalize \( u^1 \). (A corresponding condition on \( \Sigma^1 \) would normalize \( u^0 \).)

The Raychaudhuri equation (28) for \( R_{11} \) on \( \Sigma^0 \) would then become linear in the expansion rate \( \overline{K}_1 = \partial_1 \ln g^{1/2} \), which facilitates its integration (cf Hayward [3], Brady and Chambers [7]).

7. The characteristic initial-value problem

We are now ready to address the question of what initial data are needed to prescribe a unique vacuum solution of the Einstein equations in a neighbourhood of two lightlike hypersurfaces \( \Sigma^0 \) and \( \Sigma^1 \) intersecting in a 2-surface \( S_0 \) [5].

We arbitrarily designate \( u^0 \) as ‘time’, and shall refer to \( \Sigma^0 \) \( (u^0 = 0) \) as the ‘initial’ hypersurface and to \( \Sigma^1 \) \( (u^1 = 0) \) as the ‘boundary’.

We impose the coordinate conditions (39), (41) and (42) to tie down \( \theta^a \) and \( u^A \). While (39) and (41) control the way \( \theta^a \) are carried off \( S_0 \), onto \( \Sigma^0 \) and into spacetime, the choice of \( \theta^a \) on \( S_0 \) itself is unrestricted. Thus, our procedure retains covariance under the group of two-dimensional transformations \( \theta^a \to \theta'^a = f^a(\theta^b) \).

In the 4-metric \( g_{ab} \), given by (12), the following six functions of four variables are then left undetermined:
\[ g_{ab}, \lambda, s_1^a. \]  

(In place of \( s_1^a \), it is completely equivalent to specify \( \omega^a = \partial_0 s_1^a \), since the ‘initial’ value of \( s_1^a \) is pegged by (41).)

We shall formally verify that a vacuum 4-metric is uniquely determined by the following initial data:

(a) On \( S_0 \), seven functions of two variables \( \theta^a \):
\[ g_{ab}, \omega^a, \overline{K}_A = \partial_A \ln g^{1/2} \quad (S_0); \]  

(b) On \( \Sigma^0 \), six functions of two variables \( \theta^a \):
\[ g_{ab}, \omega^a, \overline{K}_A = \partial_A \ln g^{1/2} \quad (\Sigma^0); \]  

(c) On \( \Sigma^1 \), five functions of two variables \( \theta^a \):
\[ g_{ab}, \omega^a, \overline{K}_A = \partial_A \ln g^{1/2} \quad (\Sigma^1); \]  

(d) On \( \partial \Sigma^0 \), four functions of one variable \( \theta^a \):
\[ g_{ab}, \omega^a, \overline{K}_A = \partial_A \ln g^{1/2} \quad (\partial \Sigma^0); \]  

(e) On \( \partial \Sigma^1 \), four functions of one variable \( \theta^a \):
\[ g_{ab}, \omega^a, \overline{K}_A = \partial_A \ln g^{1/2} \quad (\partial \Sigma^1); \]  

(f) On \( S \), three functions of one variable \( \theta^a \):
\[ g_{ab}, \omega^a, \overline{K}_A = \partial_A \ln g^{1/2} \quad (S). \]
(b) On \( \Sigma^0 \) and \( \Sigma^1 \), two independent functions of three variables which specify the intrinsic conformal 2-metric:

\[
g^{-\frac{1}{2}} g_{ab} \quad (\Sigma^0 \quad \text{and} \quad \Sigma^1). \tag{46}
\]

Instead of (46), it is equivalent to give the shear rates of the respective generators,

\[
\sigma_{1a} \quad \text{on} \quad \Sigma^0, \quad \sigma_{0a} \quad \text{on} \quad \Sigma^1, \tag{47}
\]

defined as the trace-free extrinsic curvatures:

\[
\sigma_{Aab} = K_{Aab} - \frac{1}{2} g_{ab} K_A = \frac{1}{2} g^{\frac{1}{2}} \partial_A (g^{-\frac{1}{2}} g_{ab}), \tag{48}
\]

(where \( K_{Aab} \) is defined in (53) below). These two functions correspond to the physical degrees of freedom (‘radiation modes’) of the gravitational field [5, 14].

To build a vacuum solution from the initial data (45), (47), we begin by noting that (39) implies that \( \partial_0 = \partial_\lambda, K_{0ab} = K_{0ab} \) everywhere. Hence the general expression (28) for \( R_{00} \) reduces here to

\[
-R_{00} = (\partial_0 + \frac{1}{2} K_0 - \lambda,0)K_0 + \sigma_{0ab}\sigma_{0}^{ab}. \tag{49}
\]

On \( \Sigma^1 \), we have \( \lambda = \lambda,0 = 0 \) by (42). Thus, (49) becomes an ordinary differential equation for

\[
K_0 = K_0 = \partial_0 \ln g^{-\frac{1}{2}}
\]

as a function of \( a^0 \). This can be integrated along the generators, using the given data for \( \sigma_{0ab} \) on \( \Sigma^1 \), and the initial value of \( K_0 \) on \( S_0 \), to obtain \( g^\frac{1}{2} \), hence the full 2-metric \( g_{ab} \) (hence also \( K_{0ab} \)) over \( \Sigma^1 \).

Expression (29) for \( R_{0a} \) reduces similarly to

\[
R_{0a} = -\frac{1}{2} e^{-\lambda} (\partial_0 + K_0 - \lambda,0)\omega_a - \frac{1}{2} (\partial_0 - K_0)\lambda,0 + K_{0ab} \omega^b - \partial_a K_0 \tag{50}
\]

in a spacetime neighbourhood of \( \Sigma^0 \) and \( \Sigma^1 \). On \( \Sigma^1 \), since \( K_{0ab} \) is now known, and \( \lambda = \lambda,0 = 0 \), (50) is a linear ordinary differential equation for \( \omega_a \) which may be integrated along generators, with initial condition (45), to find \( \omega_a \) (hence \( s^1 \)).

Thus, our knowledge of the six metric functions (44) has been extended to all of \( \Sigma^1 \) with the aid of the evolutionary equations \( R_{00} = R_{0a} = 0 \).

A similar procedure, applied to the constraint equations \( R_{11} = R_{1a} = 0 \), determines the functions (44) (hence also \( K_{1ab} \)) over the initial hypersurface \( \Sigma^0 \). (Here we exploit (41)—implying \( D_1 = \partial_1 \)—which holds on \( \Sigma^0 \) only. This limitation is of little practical consequence, since the Bianchi identities (section 7) relieve us of the need to recheck the constraints off \( \Sigma^0 \).)

Thus, the data (44), together with their tangential derivatives \( \partial_1, \partial_a \), which we denote schematically by

\[
D = \{g_{ab}, \lambda, \omega_a, s^1_1, \partial_1, \partial_a\}, \tag{51}
\]

are now known all over the initial hypersurface \( \Sigma^0 \), \( u^0 = 0 \) (note that \( D \) includes \( K_{1ab} \)).

We now proceed recursively. Suppose that \( D \) is known over some hypersurface \( \Sigma : u^0 = \text{constant} \). We show that the six evolutionary equations \( ^{(4)}R_{ab} = 0, R_{00} = R_{0a} = 0 \), together with the known boundary values of \( g_{ab}, K_{0ab}, \omega_a \), and \( s^1_1 \) on \( \Sigma^1 \), determine all first-order time derivatives \( \partial_0 \) of \( D \), and hence the complete evolution of \( D \).

Expression (27) for the evolutionary equations \( ^{(4)}R_{ab} = 0 \) can be written more explicitly, with the aid of the identity

\[
D_A K^A_{ab} - 2K_{A(a} d^A K^A_{b)d} = -2D_1 K_{0ab} + 4K_{0(a} d^b K_{b)d} + \omega_{(a;b)}, \tag{52}
\]
2220

P R Brady et al

which is rooted in the symmetry

\[ \partial_B K_{Aab} = 0, \quad K_{Aab} \equiv K_{Aab} + s_{A(a;b)} \]  (53)

(see (23) and appendix B).

The equations \( (4) R_{ab} = 0 \) are thus seen to reduce to a system of three linear ordinary
differential equations for \( K_{0ab} \) as functions of \( u^1 \) on \( \Sigma \), whose coefficients are concomitants
of the known data \( D \) on \( \Sigma \). Together with the boundary conditions on \( K_{0ab} \) at \( u^1 = 0 \)
(i.e. the intersection of \( \Sigma \) with \( \Sigma^1 \)), they determine a unique solution for \( K_{0ab} \) on \( \Sigma \).

We next turn to (49) and (50) to read off the values of \( \partial_0 \lambda \) and \( \partial_0 \omega_a \) on \( \Sigma \). Since the
remaining time derivatives are known trivially from \( \partial_0 s_{A(a)} = \omega_a, \quad \frac{1}{2} \partial_0 g_{ab} = K_{0ab} = K_{0ab} \),
we are now in possession of the first time derivatives of all the data \( D \) on \( \Sigma \).

This completes our formal demonstration that the initial conditions (45) and (46), or
(45) and (47), determine (at least locally) a unique vacuum spacetime.

8. The Lagrangian

According to (12) and (31), the Einstein–Hilbert Lagrangian density \( \mathcal{L} = (-4)^{\frac{1}{2}} R_a^a \)
decomposes as

\[ \mathcal{L} = (4/e^\lambda)^{\frac{1}{2}} e^\lambda R^A_A + (4/e^\lambda)^{\frac{1}{2}} R_a^a, \]  (54)

in which \( g^{\frac{1}{2}} \) refers to the determinant of \( g_{ab} \). Substitution from (27) and (28) yields the
explicit form

\[ g^{\frac{1}{2}} \mathcal{L} = e^{\lambda} \left( (2) R - D_A (2 K^A + D^A \lambda) - K_A K^A - K_{ab} K^A_{ab} \right. \]

\[ + \frac{1}{2} e^{-\lambda} \omega_a \omega_{a} - e^\lambda \left( 2 \lambda_{a}^a + \frac{3}{2} \lambda_{a} \lambda_{a}^a \right) \]  (55)

Second derivatives of the metric in (55) can be isolated in the form of a pure divergence
by calling on the identities

\[ g^{\frac{1}{2}} D_A X^A = \partial_a \left[ (-4)^{\frac{1}{2}} e^{-\lambda} X^A \epsilon^a_{(A)} \right] - g^{\frac{1}{2}} X^A K_A, \]  (56)

\[ A_{a}^a + A^a \lambda_{a} = \nabla_a (A^a \epsilon^a_{(a)}), \]  (57)

which follow from (102) below, and hold for any scalars \( X^A \) and 2-vector \( A^a \). We thus obtain

\[ \mathcal{L} = -\partial_a \left[ (4)^{\frac{1}{2}} e^\lambda (2 K^A + D^A \lambda) \epsilon^a_{(A)} + 2 (-4)^{\frac{1}{2}} \lambda_{a}^a \epsilon^a_{(a)} \right] \]

\[ + g^{\frac{1}{2}} \left[ e^{\lambda} (2) R + K_A K^A - K_{ab} K^A_{ab} + \frac{1}{2} e^{-\lambda} \omega_a \omega_{a} + K_A D_A \lambda + \frac{1}{2} e^\lambda \lambda_{a} \lambda_{a}^a \right]. \]  (58)

The divergence term integrates as usual to a surface term in the action \( S = \int \mathcal{L} d^4 x \), and
has no influence on the classical equations of motion.

Variation of \( S \) with respect to

\[ -e^\lambda = g_{(0)(1)} \equiv \ell_{(0)} \cdot \ell_{(1)} \]

reproduces the expression obtained from (27) for \( G^{01} = \frac{1}{2} e^\lambda R_a^a \). Similarly, variation with
respect to \( s_{a}^a \) yields the expression (29) for \( G_{a}^a = R_{aa} \), if we take account of the implicit
dependence of \( K_{Aab}, D_A \) and \( \omega_{a} \) on \( s_{a}^a \) through

\[ K_{Aab} = \frac{1}{2} \partial_A g_{ab} - s_{A(a;b)}, \]  (59)
(22) and (20). Finally, variation with respect to \( \gamma_{ab} \) yields \((4)\gamma_{ab}\), if we note the identity
\[
g^{-\frac{1}{2}} \frac{\delta}{\delta \gamma_{ab}} \int \psi (2)R g^{\frac{1}{2}} \delta \theta = \gamma_{ab} \psi^{c} \psi - \psi_{;ab}.
\]

Thus, variation of the action (58) yields eight of the ten Einstein equations. The remaining two equations—the Raychaudhuri equations for \( R_{00} \) and \( R_{11} \)—cannot be retrieved directly from (58), because the \textit{a priori} conditions \( \eta^{00} = \eta^{11} = 0 \) (expressing the lightlike character of \( u^{0} \) and \( u^{1} \)) which are built into (58), preclude us from varying with respect to these ‘variables’. The two Raychaudhuri equations can, however, be effectively recovered from the other eight equations via the Bianchi identities.

The Hamiltonian formulation of the dynamics has been discussed in detail by Torre [3]. We hope to pursue this topic elsewhere.

9. Gauss–Weingarten (first-order) relations

In this second half of the paper, we return to the beginning and to the task of laying a more complete geometrical foundation for the Ricci and Bianchi formulae which we quoted without derivation in (27)–(29) and (32), (33). We begin with the first-order imbedding relations for the 2-surface \( S \) as a subspace of spacetime.

Equations (15) and (17) allow us to decompose the 4-vector \( \delta e^{a}_{(b)} / \delta \theta^{b} \) in terms of the basis \((\ell(A), e(a))\). Recalling (5), we find
\[
\delta e^{a}_{(b)} / \delta \theta^{b} = -e^{-\lambda} K^{A}_{ab} \ell(A) + \Gamma^{c}_{ab} e(c).
\]

Similarly, in view of (16), we may make the decomposition
\[
\delta \ell(A) / \delta \theta^{a} = L_{ABa} \ell(B) + K_{Aa}^{b} e(b),
\]
where the first coefficient is given by
\[
L_{ABa} = e^{-\lambda} \ell(B) \cdot \delta \ell(A) / \delta \theta^{a}.
\]

This coefficient can be reduced to a much simpler form: its symmetric part is
\[
L_{(AB)a} = \frac{1}{2} e^{-\lambda} \partial_{a} (\ell(A) \cdot \ell(B)) = \frac{1}{2} \eta_{AB} \partial_{a} \lambda.
\]

To obtain the skew part, we note first that
\[
\delta^{a}_{(a)} \ell^{b}_{(B) \ell_{(A) [\alpha] \beta} = 0}
\]
since \( \ell(A) \) is proportional to a lightlike gradient (see (4)). With the aid of (65) and (19) we now easily derive
\[
L_{[AB]a} = e^{-\lambda} \ell^{\beta}_{(B) \ell_{(A)} \beta} \delta^{a}_{(a)} = \frac{1}{2} e^{-\lambda} [\ell_{(B)} \ell_{(A)}]^{a} e_{(a)a}
= \frac{1}{2} e_{AB} \omega_{a} e^{-\lambda}.
\]

Combining (64) and (66), we arrive at the simple expression
\[
2 L_{ABa} = \eta_{AB} \partial_{a} \lambda + e^{-\lambda} e_{AB} \omega_{a}
\]
for the first coefficient in (62).

The Gauss–Weingarten equations (61) and (62) govern the variation of the 4-vectors \( \ell(A) \) and \( e(a) \) along directions tangent to \( S \). We now turn to their variation along the two normals.
We have from (4),
\[ \nabla_\beta \ell_{(A)\alpha} = 2 \ell_{(A)[\alpha} \partial_{\beta]} \lambda + \nabla_\alpha \ell_{(A)\beta}, \]
(68)
multiplying by \( \ell^B_\land \) and symmetrizing in \( A, B \) gives
\[ (\ell_{(B)} \cdot \nabla) \ell_{(A)} + (\ell_{(A)} \cdot \nabla) \ell_{(B)} = 2 \ell_{(A)D_B} \lambda - \eta_{AB} \epsilon^{\land} \nabla \lambda, \]
(69)
where \( D_A \lambda \) is defined as in (22); it follows that
\[ \nabla \lambda = e^{-\lambda} \ell^{(A)} \lambda + \epsilon^{(a)} \partial_{\mu} \lambda. \]
(70)
On the other hand, the difference of the two terms on the left of (69) is given by (19) as \( \epsilon_{AB} \omega_{a} \epsilon^{(a)} \). Adding finally yields
\[ (\ell_{(B)} \cdot \nabla) \ell_{(A)} = N_{ABC} \ell_{(C)} - \eta_{AB} \epsilon_{(a)} \epsilon^{(a)}, \]
(71)
where
\[ N_{ABC} = D(A \lambda) \eta_{BC} - \frac{1}{2} \eta_{AB} D_{C} \lambda, \]
(72)
and \( L \) was defined in (67).

Proceeding finally to the transverse variation of \( \epsilon_{(a)} \), we have from (14) and (9),
\[ \frac{\delta \epsilon_{(a)}}{\delta u^A} = \frac{\delta}{\delta \theta^a} (\ell_{(A)} + \epsilon_{(a)} \epsilon_{(b)}. \]
Substituting from (61) and (62), it is straightforward to reduce this to
\[ \frac{\delta \epsilon_{(a)}}{\delta u^A} = (L_{AB} - \eta_{AB} \epsilon_{(a)} + \tilde{K}_{ab} \epsilon_{b}) \]
(73)
where
\[ \tilde{K}_{ab} = K_{ab} + s_{ab}. \]
(74)

Applying Leibnitz’s rule to
\[ \partial_{A} g_{ab} = \frac{\delta}{\delta u^A} (e_{(a)} \cdot e_{(b)}), \]
we read off from (73) the result
\[ \frac{1}{2} \partial_{A} g_{ab} = \mathcal{K}_{ab} \equiv \tilde{K}_{(ab)}, \]
(75)
which gives direct geometrical meaning to the extrinsic curvature in terms of transverse variation of the 2-metric.

The normal absolute derivatives of \( \epsilon_{(a)} \) are given by (recalling (9) and (13))
\[ (\ell_{(A)} \cdot \nabla) \epsilon_{(a)} = \frac{\delta \epsilon_{(a)}}{\delta u^A} - s_{ab} \delta \epsilon_{(a)} \delta \theta^b. \]

With the help of (73) and (61) this reduces to
\[ (\ell_{(A)} \cdot \nabla) \epsilon_{(a)} = L_{AB} \epsilon_{(b)} + (\mathcal{K}_{ab} - s_{ab} \epsilon_{(b)}) \]
(76)
Correspondingly, the two normal derivatives of \( g_{ab} \) are
\[ (\ell_{(A)} \cdot \nabla) g_{ab} = 2 \mathcal{K}_{ab} - s_{ab} \partial_{\mu} g_{ab}. \]
(77)

The two-dimensionally noncovariant terms which appear in (76) and (77) are not a mistake. They arise because the normal gradient \( \ell_{(A)} \cdot \nabla \), applied to objects carrying lower-case Latin indices—let us say \( g_{ab} \)—does not preserve manifest two-dimensional covariance, since it contains (see (22)) a piece \(-s_{a}^{\alpha} \partial_{\alpha} g_{ab} \) involving ordinary (rather than two-dimensional covariant) derivatives with respect to \( \theta^c \). Although not incorrect, this is a formal impediment: it threatens to clutter our formulae with terms in the shift vectors \( s_{A}^{a} \) which are, to boot, noncovariant. In the following section, we explain how this can be remedied by introducing a ‘rationalized’ gradient operator \( \tilde{\nabla} \).
10. Rationalized operators $\tilde{\nabla}$, $\tilde{D}_A$, $\nabla_a$

The rationalized operator $\tilde{\nabla}_a$ avoids the two-dimensionally noncovariant terms which appear when $\nabla_a$ is applied to objects bearing lower-case Latin indices, as in (61), (76) and (77).

Applied to scalar fields or to 4-tensors not bearing lower-case Latin indices, $\tilde{\nabla}$ is identical with $\nabla$. If the object does carry such indices, there are supplementary terms involving the two-dimensional connection $\Gamma^a_{bc}$.

Specifically, we define

$$\tilde{\nabla}_a = \nabla_a + p^{(a)}_\alpha \left( \nabla_a - e^{(a)} \cdot \nabla \right)$$

in which $e^{(a)} \cdot \nabla \equiv \delta / \delta \theta^a$ is the absolute derivative introduced in (13), and the operator $\nabla_a$ will be specified in a moment. We have introduced the pair of 4-vectors $p^{(a)} = \nabla_\theta a$, i.e.

$$p^{(a)}_\alpha = \partial x^\alpha / \partial u^a. \quad (79)$$

Their projections onto the basis vectors are, according to (6) and (9),

$$e^{(a)}_b = \delta^a_b, \quad e^{(a)}_\alpha \cdot \ell(A) = -s^a A, \quad (80)$$

from which follows the identity

$$\delta^a_a - p^{(a)}_\alpha e^{\beta}_\alpha = e^{(a)}_\beta e^{(a)}_\alpha = e^{-\lambda} \ell(A) \partial x^\alpha / \partial u^A. \quad (81)$$

Hence (78) can be recast in terms of the absolute derivative $\delta / \delta u^A$:

$$\tilde{\nabla} = e^{-\lambda} \ell(A) \partial / \partial u^A + p^{(a)} \nabla_a. \quad (82)$$

We next introduce the differential operator

$$\tilde{D}_A \equiv \ell(A) \cdot \tilde{\nabla} = \ell(A) \cdot \nabla - s^a A \left( \nabla_a - \delta / \delta \theta^a \right). \quad (83)$$

An alternative form of which,

$$\tilde{D}_A = \delta / \delta u^A - s^a A \nabla_a, \quad (84)$$

follows from (82).

Since $e^{(a)} \cdot \tilde{\nabla} = \nabla_a$, we can reconstruct $\tilde{\nabla}$ from (83) in yet another form:

$$\tilde{\nabla} = e^{-\lambda} \ell(A) \tilde{D}_A + e^{(a)} \nabla_a. \quad (85)$$

We now specify the operator $\nabla_a$. It is defined so as to act as a two-dimensional covariant derivative on all lower-case Latin indices (including parenthesized ones), and at the same time as an absolute derivative $\delta / \delta \theta^a$ on Greek indices. Upper-case Latin indices are treated as inert.

As an example,

$$\nabla_b e^{(a)} = \delta e^{(a)} / \delta \theta^b - \Gamma^a_{bc} e^{(c)}. \quad (86)$$

It is evident that, quite generally, the ‘correction’ $\nabla_a - \delta / \delta \theta^a$ in (83) and (78) is linear and homogeneous in the two-dimensional connection $\Gamma^a_{bc}$.

Examples of how $\nabla_a$ and $\tilde{D}_A$ act on scalars and 2-tensors are

$$\nabla_a f = \partial_a f, \quad \tilde{D}_A f = (\partial_A - s^a A \partial_a) f = D_A f, \quad (87)$$

$$\nabla_a X^{b...c...} = X^{b...c...}, \quad \tilde{D}_A X^{b...c...} = (\partial_A - s^a A \nabla_a) X^{b...c...}. \quad (87)$$

For the 2-metric $g_{ab}$, we have from (84),

$$\tilde{D}_A g_{ab} = \partial_A g_{ab}, \quad (88)$$

since $\nabla_c g_{ab} \equiv g_{ab; c} = 0$. Thus, (75) can be expressed as

$$\frac{1}{2} \tilde{D}_A g_{ab} = K_{Aab}, \quad (89)$$
which should be contrasted with (77).

Similarly, with the aid of (86) and (83), the noncovariant expressions (61) and (76) become

\[ \nabla_b e(a) = \nabla_a e(b) = -e^{-\lambda} K_{Aab} \ell(A), \quad (90) \]

\[ \tilde{D}_A e(a) = L_{ABA} \ell(B) + \tilde{K}_{Aab} e(b). \quad (91) \]

Quite generally, \( \tilde{D}_A \), \( \nabla_a \) and \( \tilde{\nabla} \) preserve both four-dimensional covariance and covariance under 'rigid' two-dimensional coordinate transformations \( \theta^a \to \theta^a = f^a(\theta^b) \), with no dependence on \( u^A \). (\( u \)-dependence of \( f^a \) would induce 'gauge' transformations of the shift vectors \( s^a \), see the remarks following (26).)

With the aid of (85), the last two results can be put together to form the rationalized covariant derivative of \( e(a) \):

\[ e^\lambda \tilde{\nabla}_\beta e(a)_\alpha = (L_{BAa} \ell(A) + \tilde{K}_{Bab} e(b)_\alpha) \ell(B)_\beta - K_{Aab} \ell(A)_\alpha e(b)_\beta. \quad (92) \]

Equations (62) and (71) similarly combine to produce

\[ \nabla_\beta \ell(A)_\alpha = (e^{-\lambda} N_{ABC} \ell(C) - L_{BAa} e(a)_\alpha) \ell(B)_\beta + (L_{ABA} \ell(B) + K_{Aab} e(a)_\alpha) e(b)_\beta. \quad (93) \]

(There is no distinction here between \( \tilde{\nabla} \) and \( \nabla \), since \( \ell(A) \) carries no lower-case Latin indices.)

To sum up: equations (92) and (93) encapsulate the full set of first-order (Gauss–Weingarten) equations, which control tangential and normal variations of the basis vectors \( e(a) \), \( \ell(A) \). The coefficients in these equations are given by (16), (67), (20), (72) and (74). Their geometrical meaning emerges from (75), (19) and remarks following those equations.

11. Rationalized Ricci commutation rules

The usual commutation relations need to be modified for \( \tilde{\nabla}_a \). To derive the modified form, consider the action of \( \tilde{\nabla}_a \) on any field object \( X_a \) bearing just one lower-case Latin and an arbitrary set of other indices.

From (78) and (86),

\[ \tilde{\nabla}_\gamma \tilde{\nabla}_\beta X_a = (\delta^\alpha_a \nabla_\gamma - p^{(n)}_\gamma \Gamma^C_{na}) (\delta^\beta_b \nabla_\beta - p^{(m)}_\beta \Gamma^B_{mb}) X_b. \]

Skew-symmetrizing with respect to \( \beta \) and \( \gamma \), and noting from (79) that \( \nabla_\gamma p^{(m)}_\beta = 0 \), leads to

\[ \tilde{\nabla}_\gamma \tilde{\nabla}_\beta X_a = \nabla_\gamma \nabla_\beta X_a + p^{(m)}_\beta \left( p^{(n)}_\gamma \frac{1}{2} R^b_{ama} - e^{-\lambda} \ell(A)_\gamma \partial_A \Gamma^b_{ma} \right) X_b. \quad (94) \]

in which \( \partial_\gamma \Gamma^b_{ma} \) has been expanded using

\[ \partial_\gamma = e^{-\lambda} \ell(A)_\gamma \partial_A + p^{(m)}_\gamma \partial_m, \quad (95) \]

which is a special case of (82).

The right-hand side of (94) can be further reduced: \( \partial_A \Gamma^b_{ma} \) is a 2-tensor, given by

\[ \partial_A \Gamma^b_{ma} = 2K_{A(a)} {^b}^{l} A^{,m} : ^b - K_{Aa} {^b}^{l} A^{,m} : ^b \quad (96) \]

according to (75); and in two dimensions we have

\[ R^b_{ama} = (\partial_A \Gamma^b_{ma} - \partial_m \Gamma^b_{ma}). \quad (97) \]

If \( e^{(a)}_\alpha \) is substituted for \( X_a \), (92)–(94) can be used to express the projection onto \( e^{(a)}_\alpha \) of the four-dimensional Riemann tensor in terms of the first-order Gauss–Weingarten variables
Covariant double-null dynamics

\[ K, L, N \text{ and their derivatives. If our interest is primarily in the Ricci tensor, the contracted form } (\gamma = \alpha) \text{ of (94) suffices:} \]

\[
(\tilde{\nabla}_\alpha \tilde{\nabla}_\beta - \tilde{\nabla}_\beta \tilde{\nabla}_\alpha) e^\alpha_{(a)} = e^\alpha_{(a)} R^A_{\alpha \beta} - \frac{1}{2} (^{(2)}R p_{(a)\beta} + e^{-\lambda}(\partial_a \mathcal{K}_A)\ell^\beta_{(a)} \right)
\]

where

\[
\mathcal{K}_A \equiv \mathcal{K}_A^a = \partial_A \ln \frac{1}{2}
\]

and \( g \equiv \det g_{ab} \).

12. Contracted Gauss–Codazzi (second-order) relations and the Ricci tensor

The Gauss–Codazzi relations are the integrability conditions of the system of first-order (Gauss–Weingarten) differential equations (92), (93). As just noted, they express projections of the four-dimensional Riemann tensor in terms of \( K, L, N \) and their first derivatives.

Contraction of these equations gives frame components of the Ricci tensor. The most concise way of deriving these components in practice is through recourse to a generalized form of Raychaudhuri’s equation \([12]\).

Let \( A^\alpha \) be an arbitrary 4-vector (which may bear arbitrary label indices) and \( B^\alpha \) a second vector free of lower-case Latin indices, so that \( \tilde{\nabla}_\beta B^\alpha = \nabla_\beta B^\alpha \) and the standard commutation rules apply. Then it is easy to check the identity

\[
R^A_{\alpha \beta} A^\alpha B^\beta = \tilde{\nabla}_\beta (A^\alpha \tilde{\nabla}_\alpha B^\beta) - A^\alpha \tilde{\nabla}_\alpha (\nabla_\beta B^\beta) - (\tilde{\nabla}_\beta A^\alpha) (\tilde{\nabla}_\alpha B^\beta).
\]

(100)

If, on the other hand, \( B \) is replaced by \( e^{(b)} \), then we call upon the commutation law (98) for \( \tilde{\nabla} \), with the result

\[
R^A_{\alpha \beta} A^\alpha e^\beta_{(b)} = \tilde{\nabla}_\beta (A^\alpha \tilde{\nabla}_\alpha e^\beta_{(b)}) - A^\alpha \tilde{\nabla}_\alpha (\tilde{\nabla}_\beta e^\beta_{(b)}) - (\tilde{\nabla}_\beta A^\alpha) (\tilde{\nabla}_\alpha e^\beta_{(b)})
\]

\[
+ \frac{1}{2} (^{(2)}R (A \cdot p_{(b)}) - e^{-\lambda}(\partial_b \mathcal{K}_B)(A \cdot \ell^{(B)})).
\]

(101)

With the choices \( A = \ell(a) \) and \( e^{(a)} \), \( B = \ell(b) \) we can recover all frame components of the Ricci tensor from these equations in tandem with (92) and (93). Some details of these calculations are recorded in appendix A. The final results have already been listed in (27)–(29).

We next turn to the contracted Bianchi identities. The projection of (30) onto \( e^{(a)} \) yields

\[
\tilde{\nabla}_\beta (R^A_{\alpha \beta} e^\alpha_{(a)}) - R^A_{\alpha \beta} \tilde{\nabla}_\beta e^\alpha_{(a)} = \frac{1}{2} \partial_a R.
\]

The second term is evaluated with the aid of (92). In the first term, we expand

\[
e^\alpha_{(a)} R^A_{\alpha \beta} = R^A_{\alpha \beta} e^\beta_{(b)} + e^{-\lambda} R^A_{\alpha \beta} e^\beta_{(a)},
\]

and note the (often used) results

\[
\tilde{\nabla}_\beta e^\beta_{(b)} = \partial_b \lambda, \quad \nabla_\beta \ell_{(a)} = K_A + D_A \lambda,
\]

(102)

which follow from (92) and (93). The result is (32), and (33) is obtained similarly.

13. The Riemann tensor

We list here the tetrad components of the Riemann tensor, obtainable from the uncontracted Ricci commutation rules (see, e.g., (94)). The notation for the tetrad components is as in
section 4,

\[
(4) R_{\alpha\beta} = (2) R^{[\alpha} A^{\beta]} - 2 e^{-\lambda} K A[\alpha K^{\beta]}
\]

\[
R_{ABCD} = \frac{1}{4} \epsilon_{AB} \epsilon_{CD} (2 e^\lambda E D E + 3 \omega^{\mu} \omega_\mu + e^{2\lambda} \lambda^{,\mu} \lambda^{,\mu})
\]

\[
R_{\alpha\beta\gamma\delta} = 2 K_{\alpha\beta\gamma\delta} - K_{\alpha\beta\gamma\delta} - e^{-\lambda} \epsilon_{\alpha\beta} K^{[\gamma} \Psi_{\delta]}
\]

\[
R_{\alpha\beta\gamma\delta} = \frac{1}{4} \epsilon_{\alpha\beta} \epsilon_{\gamma\delta} (D A \omega_\mu + K_{\alpha\beta} \omega^\mu - e^{\lambda} \epsilon_{\alpha\beta} (D E D + K E_{,\mu} \lambda_{,\mu}) - \omega_\mu D A \lambda)
\]

\[
R_{A B}^{a b} = - D^{(A} K^{B)}_{a b} + K^{A}_{,a} K^{B}_{,b} + D^{(A} \lambda K^{B)}_{a b} - \frac{1}{2} \eta^{A B} D E \lambda K E_{a b}
\]

\[
- \frac{1}{4} \eta^{A B} (e^{\lambda} \lambda_{,a} \lambda_{,b} + e^{2\lambda} \lambda_{,a} \lambda_{,b} + 2 e^\lambda \lambda_{,a} \lambda_{,b}) - \frac{1}{4} \epsilon_{A B} g^\frac{1}{2} \epsilon_{a b} \tau.
\]

We have here defined

\[ e^{-\lambda} \tau = g^{-\frac{1}{4}} \epsilon^{a b} \partial_a (e^{-\lambda} \omega_b). \]

14. Concluding remarks

The (2 + 2) double-null imbedding formalism developed in this paper leads to simple and geometrically transparent expressions for the Einstein field equations (27)–(29) and the Einstein–Hilbert action (96). It should find ready application in a variety of areas, as indicated in the introduction.

(2 + 2) formalisms are certainly not new [3, 4], but they have languished on the relativist’s back-burner. We hope that this exposition will play a role in promoting these versatile methods from the realm of esoterica into an everyday working tool.

Acknowledgments

PRB is grateful to Werner Israel for his kind hospitality during visits to the University of Alberta. This work was supported by the Canadian Institute for Advanced Research and by NSERC of Canada. PRB was also supported by EPSRC of Great Britain.

Appendix A. Computing Ricci components: some intermediate details

For the convenience of enterprising readers who wish to derive the Ricci components (64)–(66) for themselves, we record here some intermediate steps of the computations.

Computation of \((4) R_{ab}\) from (101) requires evaluation of

\[
\tilde{\nabla}_\beta (e^{a}_{(\alpha)} e^{\beta}_{(\beta)}) = - e^{-\lambda} (\tilde{D} A + K A) K^{a b}
\]

\[
(\tilde{\nabla}_\beta e^{a}_{(\alpha)}) (\tilde{\nabla}^{\alpha} e^{\beta}_{(\beta)}) = \frac{1}{2} (\lambda_{,a} \lambda_{,b} + e^{-2\lambda} \omega_a \omega_b) - 2 e^{-\lambda} K A^{a} d \tilde{K}^{b} d
\]

which can be verified from (92) and (67).

Computation of \(R_{AB}\) from (100) requires

\[
\ell_{(A)\alpha} \epsilon^{\beta}_{(B)} = (D A \lambda) (D \lambda) - \frac{1}{2} \eta_{A B} (D E \lambda) (D E \lambda)
\]

\[
+ K_{a b} K^{a b} - \frac{1}{2} \epsilon_{a b} \eta_{A B} (\lambda_{,a} \lambda_{,b} + e^{-2\lambda} \omega_a \omega_b)
\]

which follows from (93), (67) and (72).

Finally, computation of \(R_{Aa}\) requires

\[
(\tilde{\nabla}_\beta e^{a}_{(\alpha)}) e^{\beta}_{(A)\alpha} = \frac{1}{2} \lambda_{,a} D A \lambda + (K_{a b} + \frac{1}{2} \Delta K A a b) \lambda_{,b} + \frac{1}{2} e_{A B} e^{-\lambda} (\omega_a D B \lambda + \Delta K B_{a b} \omega_b)
\]
Covariant double-null dynamics

in which

$$\Delta K_{Aab} \equiv \tilde{K}_{Aab} - K_{Aab} = s_{Aab}.$$ 

**Appendix B. The operator $D_A$: commutation rules and other properties**

In section 3 we gave two definitions—(21) and (24)—for the operator $D_A$. It is straightforward to show their equivalence. We have

\[ [e_{(a)}, e_{(b)}] = 0, \quad [\partial x / \partial u^a, e_{(b)}] = 0, \]

since the Lie bracket of two holonomic vectors vanishes (cf (14)). In combination with (9), this yields

\[ \ell_{(A)}, e_{(b)} = -s^a_{A} e_{(a)}, e_{(b)} = (\partial b s^a_{A}) e_{(a)}. \]  

Hence, for any 2-vector $X^b$,

\[ \mathcal{L}_{\ell_{(A)}}(X^b) e_{(b)} = \left( (\partial A - \mathcal{L} s^a_{A}) X^b \right) e_{(b)} \]  

which proves the equivalence of (24) and (21) when applied to $X^b$.

To prevent possible confusion, we emphasize that $X^b$ transform as components of a 2-vector under transformations (25) of the two-dimensional coordinates $\theta^a$, but they are simultaneously scalars under transformations of the four-dimensional coordinates $x^a$. Therefore $X^b$ assume their role as 4-scalars in the four-dimensional Lie derivative on the left-hand side of (B2), but in the two-dimensional Lie derivative on the right-hand side they play their alternate role as 2-vector components. Since this point has caused difficulty for at least one reader, we repeat this derivation in appendix C in the language of modern differential geometry.

This argument is easily extended. For instance, for the 2-metric $g_{ab}$, the definition (24) gives

\[ D_A g_{ab} = e^a_{(a)} e^b_{(b)} \mathcal{L}_{\ell_{(A)}} g_{a\beta} \]

\[ = 2 e^a_{(a)} e^b_{(b)} \ell_{(A)}(a | b) \]

\[ = 2 K_{Aab} \]

by (16), which agrees with the form (23) obtained from the definition (21). (Strictly speaking, (24) requires that the projector $\Delta_{a\beta} \equiv e_{(a)\alpha} e^b_{(a)}$ should replace $g_{a\beta}$ in the first equality above. But, according to the completeness relation (11), the difference involves the Lie derivative of $\ell_{(A)}(a | b)$, which is linear homogeneous in $\ell_{(A)}$ and projects to zero.)

Commutation relations for $D_A$ follow most easily from the definition (24). For a scalar field $f$,

\[ 2D_B D_A f = [\mathcal{L}_{\ell_{(A)}}, \mathcal{L}_{\ell_{(B)}}] f = \mathcal{L}_{\ell_{(B)}(A)} f = \epsilon_{AB} \omega^a \partial_a f, \]

where we have recalled the well known result that the commutator of two Lie derivatives is the Lie derivative of the commutator (i.e. the Lie bracket), and made use of (19).

Consider next the operation on a 2-vector $X^a$. We have from (24),

\[ D_B D_A X^a = e^a_{(a)} \mathcal{L}_{\ell_{(B)}}(e^b_{(b)} D_A X^b) \]

\[ = e^a_{(a)} \mathcal{L}_{\ell_{(B)}}(\Delta^a_{\beta} \mathcal{L}_{\ell_{(A)}}(e^\beta_{(b)} X^b)). \]
The projection tensor $\Delta_\alpha^\beta$ can be replaced by $\delta_\alpha^\beta$, because the Lie derivative, operating on the difference, gives terms proportional to $\ell_{(E)}^\beta$ or $\ell_{(E,\alpha)}$, which project to zero, noting (B2). Thus,

$$D_B D_A X^a = \epsilon_\beta^a \mathcal{L}_{(E)}^\xi \mathcal{L}_{(E)}^\eta (\epsilon_\beta^a X^b).$$

We can now proceed exactly as for the scalar case to derive the commutator. The result (generalized to an arbitrary 2-tensor) is

$$[D_B, D_A]X_{a\ldots b\ldots} = \epsilon_{AB} \mathcal{L}_{(a)} X_{b\ldots}.$$

In particular,

$$[D_B, D_A]g_{ab} = 2\epsilon_{AB} \omega_{(a;b)}.$$

Recalling (B3), this may be written as

$$D_{(B} K_{A)b} = \frac{1}{2} \epsilon_{AB} \omega_{(a; b)},$$

which was used in (52), and contracts to

$$D_{(B} K_{A)} = \frac{1}{2} \epsilon_{AB} \omega^{a;} a.$$

These last two identities also play a role in symmetrizing—or, more properly, recognizing the implicit symmetry of—the raw expressions for $R_{AB}$ and $R_{AaBb}$ that emerge from the Ricci commutation relations. The manifestly symmetric expressions listed in (28) and section 13 have been symmetrized with the aid of these identities.

To conclude, we note the rule for commuting $D_A$ and the two-dimensional covariant derivative $\nabla_a$. The commutator $[D_A, \nabla_a]$, applied to any 2-tensor, is formed by a pattern similar to its two-dimensional covariant derivative, but with $\Gamma^a_{bc}$ replaced by

$$D_A \Gamma^a_{bc} = 2K_{A(b}^a c) - K_{A(b;}^a c.$$

As examples:

$$[D_A, \nabla_a]X^b = X^d D_A \Gamma^b_{da},$$

$$[D_A, \nabla_a g_{bc}] = -2(D_A \Gamma^d_{(a(b} g_{c)d) = -2K_{A(b;} a).$$

The justification for the rule is that the partial derivative $\partial_a$ (applied to any two-dimensional geometrical object) commutes with both $\partial_A$ and the two-dimensional Lie derivative $\mathcal{L}_{(d)}$, so that, by (21),

$$[D_A, \partial_a] X_{b\ldots c\ldots} = 0.$$

Appendix C. A dictionary for differential geometers

Readers more at home with modern notation may find the following derivation of (B2) more congenial. (We thank an anonymous board member of the journal for suggesting inclusion of this appendix.)

Consider the metric given by (12) and the basis of 1-forms, defined by (11),

$$e^{(A)} \equiv e^\lambda d\mu^A \quad \text{(C1)}$$

$$e^{(a)} \equiv d\theta^a + s^a_A d\mu^A. \quad \text{(C2)}$$
The dual vector fields to (C1) and (C2) are
\[ e^{(a)} = \frac{\partial}{\partial \theta^a} \]
\[ e^{-\lambda} \ell^{(A)} \equiv e^{-\lambda} \left( \frac{\partial}{\partial u^A} - s^a_A \frac{\partial}{\partial \theta^a} \right) . \]
To derive (21) from (24) (in the vector field case), consider a vector field
\[ X = X^a \frac{\partial}{\partial \theta^a} . \]
Then the Lie derivative of \( X \) with respect to \( \ell^{(A)} \) evaluates to
\[ \mathcal{L}_{\ell^{(A)}} X = [\ell^{(A)}, X] \]
\[ = \left[ \frac{\partial}{\partial u^A} - s^a_A \frac{\partial}{\partial \theta^a} , X^b \frac{\partial}{\partial \theta^b} \right] \]
\[ = \left( \frac{\partial}{\partial u^A} X^b \right) \frac{\partial}{\partial \theta^b} - \left[ s^a_A \frac{\partial}{\partial \theta^a} , X^b \frac{\partial}{\partial \theta^b} \right] . \]  
(C3)
Following classical tensor calculus notation, write the components of the last bracket in (C3) as \( \mathcal{L}_{\ell^{(A)}} X \) so that
\[ \mathcal{L}_{\ell^{(A)}} X = (\partial_A X^b - \mathcal{L}_{\ell^{(A)}} X^b) \frac{\partial}{\partial \theta^b} . \]  
(C4)
Contracting (C3) and (C4) gives
\[ e^{(a)} \cdot \mathcal{L}_{\ell^{(A)}} X = \partial_A X^b - \mathcal{L}_{\ell^{(A)}} X^b . \]  
(C5)
The definitions of \( D_A X^a \) in (24) and (21) correspond, respectively, to the left- and right-hand sides of (C5).

References

[1] Arnowitt R, Deser S and Misner C W 1962 Gravitation: an Introduction to Current Research, ed L Witten (New York: Wiley) ch 7
[2] Misner C W, Thorne K S and Wheeler J A 1973 Gravitation (San Francisco, CA: Freeman) ch 21
[3] d’Inverno R A and Smallwood J 1980 Phys. Rev. D 22 1233
Smallwood J 1983 J. Math. Phys. 24 599
Torre C G 1986 Class. Quantum Grav. 3 773
McManus D 1992 Gen. Rel. Grav. 24 65
Hayward S A 1993 Class. Quantum Grav. 10 779
d’Inverno R A and Vickers J A G 1995 Class. Quantum Grav. 12 753
[4] Geroch R, Held A and Penrose R 1973 J. Math. Phys. 14 874
[5] Sachs R K 1962 J. Math. Phys. 3 908
Dautcourt G 1963 Ann. Phys., Lpz 12 202
Penrose R 1980 Gen. Rel. Grav. 12 225
Friedrich H 1981 Proc. R. Soc. A 375 169
Stewart J M and Friedrich H 1982 Proc. R. Soc. A 384 427
Rendall A D 1990 Proc. R. Soc. A 427 221
Stewart J M 1990 Advanced General Relativity (Cambridge: Cambridge University Press) ch 4
[6] Unruh W G, Hayward G, Israel W and McManus D 1989 Phys. Rev. Lett. 62 2897
[7] Israel W 1986 Phys. Rev. Lett. 56 789; 57 397; 1986 Can. J. Phys. 64 120
Hayward S A 1994 Phys. Rev. D 49 6467; 1994 Class. Quantum Grav. 11 3025
Brady P R and Chambers C M 1995 Phys. Rev. D 51 4177
[8] Dirac P A M 1949 Rev. Mod. Phys. 21 392
Rohrlich F 1971 Acta Phys. Aust. Suppl. 8 277
Frolov V P 1978 Fortschr. Phys. 26 455
Goldberg J N 1985 Found. Phys. 15 439
Convery M E, Taylor C C and Jun J W 1995 Phys. Rev. D 51 4445
[9] Verlinde E and Verlinde H 1993 String Quantum Gravity and Physics at the Planck Energy Scale ed N Sanchez
(Singapore: World Scientific) p 262
Kallosh R 1992 Phys. Lett. 275B 284
[10] Brady P R, Droz S and Morsink S M 1996 Paper submitted for publication
[11] Synge J L 1960 Relativity: The General Theory (Amsterdam: North-Holland) p 1
[12] Wald R M 1984 General Relativity (Chicago, IL: University of Chicago Press) p 218
[13] Bondi H, van der Burg M G J and Metzner A W K 1962 Proc. R. Soc. A 269 21
[14] d’Inverno R A and Stachel J 1978 J. Math. Phys. 19 2447