Conformal Invariance of Black Hole Temperature

Ted Jacobson\textsuperscript{1} and Gungwon Kang\textsuperscript{2}

\textit{Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106}
\textit{Department of Physics, University of Maryland, College Park, MD 20742-4111}

Abstract

It is shown that the surface gravity and temperature of a stationary black hole are invariant under conformal transformations of the metric that are the identity at infinity. More precisely, we find a conformal invariant definition of the surface gravity of a conformal Killing horizon that agrees with the usual definition(s) for a true Killing horizon and is proportional to the temperature as defined by Hawking radiation. This result is reconciled with the intimate relation between the trace anomaly and the Hawking effect, despite the noninvariance of the trace anomaly under conformal transformations.

\textsuperscript{1}jacobson@umdhep
\textsuperscript{2}eunjoo@wam.umd.edu
1 Introduction

Under a conformal transformation $g_{ab} \rightarrow \Omega^2 g_{ab}$ a black hole spacetime remains a black hole with the same event horizon, at least if the conformal factor $\Omega^2$ is regular on the event horizon and goes to unity at null infinity. This is simply because the causal structure is invariant under conformal transformation. Thus it makes sense to study the effect of conformal transformations on the thermodynamical properties of black holes.

However, since the Einstein equation is not conformally invariant, a conformally transformed Einstein black hole solution will not be a solution to the field equations. Nevertheless, the transformed black hole may still serve as a background on which Hawking radiation may occur. Moreover, in gravitation theories (such as Brans-Dicke theory or string theory) which contain a dilaton field $\phi$, both the “Einstein metric” $g_{ab}$ and the metric $e^{2\phi} g_{ab}$ are relevant for different considerations, and one may wish to know how the thermodynamic quantities associated with these two metrics are related.

In particular, we shall focus on the temperature and surface gravity $\kappa$ of a black hole. The surface gravity plays the role of temperature in classical black hole thermodynamics, and it is proportional to the Hawking temperature $T_H = \kappa / 2\pi$ which characterizes the radiation emitted by a black hole via quantum particle production. It is trivial to see that the Hawking radiation of a free, conformally coupled field is invariant under conformal transformations that are the identity at infinity. The form of the radiation can be computed by evaluating the Bogoliubov coefficients that relate the ingoing to the outgoing positive and negative frequency mode functions in the black hole spacetime. This computation involves only the classical propagation of the field, so if the field is conformally coupled, the result is invariant under conformal transformations.

It follows that in those cases where the surface gravity is defined, it too must be conformally invariant. It is not so obvious why this is so however, when phrased simply as a property of surface gravity. Consider for example an asymptotically flat, static spacetime with Killing field $\xi^a$, whose norm $V = (\nabla^a \xi_a)^{1/2}$ goes to unity at infinity. The force that must be exerted at infinity in order to hold a unit mass test particle fixed on an orbit of $\xi^a$ is just $g = (\nabla^a V \nabla_a V)^{1/2}$, evaluated on that orbit. This is the “surface gravity” at the position of the orbit. Under a conformal transformation $g_{ab} \rightarrow \Omega^2 g_{ab}$, one has $g \rightarrow [\Omega^{-2} g^{ab} \nabla_a (\Omega V) \nabla_b (\Omega V)]^{1/2}$, so $g$ is clearly not conformal invariant! However, we shall see below that if the conformal transformation is static, then the surface gravity at the horizon is indeed conformal invariant.

The definition of surface gravity in terms of acceleration of static particles can be generalized to stationary spacetimes, using zero angular momentum particles. One can also define surface gravity of a Killing horizon directly in terms of the Killing field that is null on the horizon (which amounts to the same thing), or by reference to the periodicity of the analytically continued Killing time coordinate that is required in order that the

---

3This is not quite as obvious as it may appear, since $\nabla_a V$ is singular at the horizon where $V \rightarrow 0$. In fact, $2V \nabla_a V = \nabla_a V^2$ is non-vanishing but tangent to $\xi^a$ at the horizon. Thus if $\xi^a \nabla_a \Omega = 0$ at the horizon, $g$ will be invariant there.
Euclidean section of the black hole spacetime be non-singular.

All of these definitions of surface gravity are meaningful, and all agree, for a stationary black hole whose event horizon is a Killing horizon. A Killing horizon is a null hypersurface whose null geodesic generators are orbits of a Killing field. A theorem of Hawking\cite{2} shows that in four dimensional Einstein gravity, a stationary black hole event horizon is always a Killing horizon. As far as we are aware, this result has not been generalized to spacetime dimensions other than four or to theories other than Einstein’s. (The physical idea behind it\cite{3} is that if the generators are not Killing orbits, the horizon will be bumpy and will radiate gravitational waves, violating the assumption of stationarity.) Of course it is certainly not true if field equations are not imposed; for instance, it is violated if one subjects a stationary Einstein black hole with a Killing vector $\chi^a$ that is null at the horizon to a stationary conformal transformation with $L_\chi \Omega \neq 0$.

Indeed, suppose $\chi^a$ is a Killing vector of the metric $g_{ab}$, so that $L_\chi g_{ab} = 0$. Then for a conformally related metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$, one has

$$L_\chi \tilde{g}_{ab} = (L_\chi \Omega^2) g_{ab} = (L_\chi \ln \Omega^2) \tilde{g}_{ab}. $$

If the conformal factor is constant along the Killing orbits, $L_\chi \Omega = 0$, then $\chi^a$ is a Killing field for $\tilde{g}_{ab}$ as well. Otherwise, $\chi^a$ is only a conformal Killing field, and a Killing horizon is transformed into what we shall call a conformal Killing horizon.

Conversely, it is easy to see that a spacetime with a conformal Killing field $\chi^a$ is conformal to a spacetime for which $\chi^a$ is a true Killing field. That is, if $L_\chi \tilde{g}_{ab} = 2f \tilde{g}_{ab}$, then the transformed metric $g_{ab} = \Omega^2 \tilde{g}_{ab}$ will satisfy $L_\chi g_{ab} = 0$ provided $\Omega$ is chosen to be a solution to the equation $L_\chi \Omega^2 + 2f \Omega^2 = 0$. The solutions are given along integral curves of $\chi^a$ by $\ln \Omega = -f \, dv$, where $\chi^a \nabla_a v = 1$.

In this letter we shall show that there is a conformal-invariant definition $\kappa_1$ of the surface gravity of a conformal Killing horizon that agrees with all the usual definitions of surface gravity in the case of the Killing horizon of a stationary black hole. Since it is conformal-invariant, $\kappa_1$ is identical to the surface gravity of a conformally related true Killing horizon. Thus it is clear that $\kappa_1/2\pi$ gives the correct Hawking temperature for radiation emitted by a the conformal transform of a stationary black hole.

As a final introductory remark, note that the area of the black hole horizon is not invariant under conformal transformation. This means that for example with the entropy given by one quarter the surface area, the first and second laws are not invariant. Of course we don’t expect that they should be invariant, since the dynamical equations of the theory are not conformal invariant.

## 2 Surface gravity

We would like to define the surface gravity of a conformal Killing horizon in some interesting and useful way. The first question that arises is whether the surface gravity should be thought of as a property of the conformal Killing horizon itself, or whether a particular
conformal Killing field must be selected before the concept of surface gravity even becomes well defined. A simple example demonstrates that in fact a particular conformal Killing field must be specified. The example is two dimensional Minkowski spacetime. Any null line is a Killing horizon with respect to both a null translation Killing field and a boost Killing field. Both of these Killing fields can be used to define a “surface gravity” for the horizon. With the translation, the surface gravity vanishes, whereas with the boost one can obtain any positive value depending on the overall scale of the Killing vector. Thus the surface gravity should not be meaningful until a particular Killing field is selected. In an asymptotically flat spacetime one can select a Killing field, or perhaps a conformal Killing field, by a boundary condition at infinity.

Suppose that a conformal Killing vector $\chi^a$ is null on some conformal Killing horizon. Then, at the horizon, we can consider the following three candidate definitions of surface gravity:

$$\nabla_a (\chi^b \chi_b) = -2\kappa_1 \chi_a$$

(1)

$$\chi^b \nabla_b \chi^a = \kappa_2 \chi^a$$

(2)

$$(\kappa_3)^2 = -\frac{1}{2} (\nabla^a \chi^b)(\nabla_{[a} \chi_{b]})$$

(3)

The first quantity, $\kappa_1$, is well-defined since $\chi^2 = 0$ everywhere on the horizon, so its gradient must be proportional to the normal to the horizon, which is $\chi_a$ itself. The second quantity, $\kappa_2$, is well-defined since the horizon is a null hypersurface whose null generators are therefore geodesics. The third quantity, $\kappa_3$, is obviously well-defined, but what needs explanation is the antisymmetrization of the $ab$ index pair. When $\chi^a$ is a true Killing vector, $\nabla_a \chi_b$ is already antisymmetric, and can be thought of as the infinitesimal generator of an isometry (Lorentz transformation) in the tangent space. In the case of a Killing horizon this isometry is a boost. For a conformal Killing vector, one has

$$2\nabla_{(a} \chi_{b)} = L_\chi g_{ab} = 2f g_{ab}.$$ 

(4)

This symmetric part can be thought of as the infinitesimal generator of a dilatation in the tangent space. If one wants a definition that captures only the quantity related to the local acceleration along the conformal Killing flow, it makes sense to discard this symmetric part.

It is easy to determine the relationship between these three quantities. In terms of the function $f$ defined in (4) above, one has

$$\kappa_1 = \kappa_2 - 2f = \kappa_3 - f.$$ 

(5)

(In relating $\kappa_3$ to the others one uses the fact that $\chi_{[a} \nabla_{b]} \chi_{c]} = 0$ at the horizon, which holds because $\chi^a$ is orthogonal to a hypersurface (the horizon) there.) For a true Killing field, $f$ vanishes, and all definitions agree.
Now let us consider the effect of making a conformal transformation. Of course $\chi^a$ remains a conformal Killing field and the conformal Killing horizon remains such. We wish to determine how the three quantities $\kappa_i$ transform, assuming that they are computed with respect to the original conformal Killing vector $\chi^a$. (In the asymptotically flat case, the original conformal Killing field can be determined by its “initial data” at infinity, which can be considered fixed if the conformal factor goes to unity at infinity.)

Since in general $f$ changes under a conformal transformation, at most one of the quantities $\kappa_i$ can be conformally invariant. In fact, $\kappa_1$ is the winner:

$$\tilde{\nabla}_a(g_{bc}\chi^b\chi^c) = \nabla_a(\Omega^2 g_{bc}\chi^b\chi^c)$$
$$= -2\Omega^2\kappa_1 g_{ab}\chi^b + (\nabla_a\Omega^2) g_{bc}\chi^b\chi^c$$
$$= -2\kappa_1\tilde{g}_{ab}\chi^b,$$

(6)

so that $\tilde{\kappa}_1 = \kappa_1$, provided that $\nabla_a\Omega^2$ is nonsingular at the horizon.

Conformal invariance of $\kappa_1$ implies via (3) invariance of $\kappa_2$ and $\kappa_3$ under those a conformal transformations that are constant along the Killing field, since $\mathcal{L}_\chi\Omega = 0$ implies that the function $f$ in (4) is unchanged.

Two more definitions of black hole surface gravity are referred to commonly, which are equivalent to $\kappa_{1,2,3}$ for the case of stationary black holes with Killing horizons. The force per unit mass $\kappa_4$ that must be applied at infinity to hold a zero angular momentum particle “at rest” just outside the horizon is one of these. The other is $\kappa_5 = 2\pi/\beta$, where $\beta$ is the period of the imaginary time coordinate required by regularity at the horizon of the Euclidean section obtained by analytic continuation. Both of these definitions make sense only when the spacetime is stationary.

Under stationary conformal transformations that go to unity at infinity, $\kappa_5$ is clearly invariant, since the regularity condition imposed at a point of the Euclidean horizon merely states that circumference of an infinitesimal circle with center on the horizon is $2\pi$ times its radius, and both of these simply scale with the value of the conformal factor at the center of the circle.

Conformal invariance of the force per unit mass definition $\kappa_4$ is not quite so obvious. Indeed, as mentioned in the introduction, it is not conformal invariant, except in the limit where the test particle approaches the horizon. In the static case, the test particle follows an orbit of the Killing field $\chi^a$, so has velocity $u^a = (-\chi^d\chi_d)^{-1/2}\chi^a$ and acceleration $a^c = u^b\nabla_b u^c$. Thus one has $\kappa_2^3 = \lim\{-\chi^a\chi_a/(a^c a_c)\} = \lim\{(\chi^b\nabla_b\chi^c)(\chi^a\nabla_a\chi_c)/(\chi^d\chi_d)\}$ as the horizon is approached. As shown for example in Ref. [4], this expression is equal to $\kappa_3^2$. As shown above, $\kappa_3$ is invariant under a static conformal transformation.

More generally in the stationary but nonstatic case, one has a time translation Killing field $\xi^a$ and a rotation Killing field $\psi^a$. Let the constant $\omega_\gamma$ be chosen so that $\zeta^a = \xi^a + \omega_\gamma\psi^a$ satisfies $\zeta^a\psi_a = 0$ at some radius $r$, so the integral curve of $\zeta^a$ at $r$ corresponds to that of a zero angular momentum test particle. One can show that the force per unit mass that must be applied at infinity to hold such a test particle on this worldline is given by $F_\infty = \{(\zeta^a\zeta_a)/(a^c a_c)\}^{1/2}$, where $a^c$ is the acceleration of the worldline. In the limit as the horizon is approached, $\zeta^a$ approaches the Killing field $\chi^a$ that is null on the horizon, and one has
κ_4 \equiv \lim_{F_\infty} = \lim\{((\zeta^b \nabla_b \zeta^c)(\zeta^a \nabla_a \zeta^c)/(-\zeta^d \zeta_d))\} = \lim\{((\chi^b \nabla_b \chi^c)(\chi^a \nabla_a \chi^c)/(-\chi^d \chi_d))\} = \kappa_4^2.

It follows that κ_4 is invariant under stationary axisymmetric conformal transformations. More general conformal transformations will in general destroy the physical interpretation of κ_4 as the force per unit mass exerted at infinity to hold a zero angular momentum particle just outside the horizon.

### 3 Hawking radiation and the trace anomaly

As mentioned in the introduction, the Hawking radiation at infinity in a conformally coupled free field is invariant under conformal transformations. But even for a nonconformally coupled field, the Hawking temperature (as opposed to the scattering of the radiation) is conformal invariant. This already follows from conformal invariance of the surface gravity, together with the known relation \( T_H = \kappa/2\pi \) between surface gravity and Hawking temperature. However, it is instructive to understand this fact directly in terms of the derivation of the Hawking effect.

To deduce the Hawking effect, one can compare the free-fall frequencies of outgoing modes near the horizon with those in the asymptotic future, rather than by comparing in and out modes. For this purpose, suppose the line element on a timelike surface has the form \( ds^2 = C\, du\, dv \), where \( C \) goes to 0 at the horizon and to 1 at future null infinity, and \( u \) goes to infinity as the horizon is approached. Then \( u \) is the retarded Killing time coordinate at future null infinity, and can be used to define positive frequency there. The relevant “free-fall” notion of positive frequency for defining the Hawking state near the horizon can be taken with respect to the affine parameter \( \lambda \) along a (null) line of constant \( v \). This affine parameter satisfies \( d\lambda = C\, du \). As \( u \to \infty \) at the horizon, \( \lambda \) runs over only a finite range. Sufficiently near the horizon, the effect of a non-singular conformal transformation \( C \to \tilde{C} = \Omega^2 C \) is thus simply to rescale \( \lambda \) to \( \tilde{\lambda} = \Omega^2_H \lambda \). For the very high affine-frequency wavepackets near the horizon that are relevant in the Hawking effect, the notions of positive \( \lambda \)-frequency and positive \( \tilde{\lambda} \)-frequency thus agree, so the Hawking temperature is unchanged.

This argument showing the conformal invariance of the Hawking temperature for non-conformally coupled fields holds also for nonstationary conformal transformations. Note however that without stationarity, the Hawking radiation will be distorted by a time-dependent “potential”, and there will in general be particle production over and beyond the Hawking radiation.

The conformal invariance of the Hawking radiation at first seems to contradict the fact\(^7, 8, 9\) that the Hawking energy flux in a conformally coupled field is determined in two spacetime dimensions by the non-conformal-invariant trace anomaly \( \langle T \rangle = R/24\pi \) (and is intimately related to the trace anomaly in any spacetime dimension). Here we shall briefly reconcile these two facts.

Assuming the expectation value of the energy-momentum tensor \( \langle T^{ab} \rangle \) is conserved \( (\nabla_a \langle T^{ab} \rangle = 0) \) and finite, it follows without further assumptions (not even stationarity
that the flux at infinity is given in the limit $u \to \infty$ by

$$T_{uu}(\infty) = \int_{v_0}^{\infty} C \langle T \rangle_{,u} \, dv, \quad (7)$$

where the metric is $ds^2 = C \, du \, dv$ as in the previous section, and $v_0$ is any fixed value of $v$. (Actually, $T_{uu}$ signifies the net outgoing energy flux only if there is no incoming flux at late advanced times $v$.) Now $R = 2C^{-1}(\ln C)_{,uv}$, so we have $CR_{,u} = 2\{(\ln C)_{,uu} - \frac{1}{2}(\ln C)_{,u}^2\}v$. Thus in fact the above formula (7) for $T_{uu}$ can be integrated quite generally, yielding

$$T_{uu}(\infty) = -\frac{1}{12\pi}[(\ln C)_{,uu} - \frac{1}{2}(\ln C)_{,u}^2](u = \infty, v_0). \quad (8)$$

Under a regular conformal transformation $C \to \tilde{C} = \Omega^2 C$, the right hand side of (8) only changes by $u$-derivatives of $\ln \Omega^2$. Regularity of $\Omega$ implies that $\Omega_u = (d\Omega/d\lambda)(d\lambda/du)$ vanishes at the horizon, because $d\Omega/d\lambda$ is finite there and $d\lambda/du$ vanishes since an infinite range of the $u$ coordinate is covered in a finite range of affine parameter $\lambda$. Thus the flux $T_{uu}(\infty)$ is in fact unchanged by a regular conformal transformation, as expected.

It is a pleasure to thank David Garfinkle and Ulvi Yurtsever for a helpful discussion on conformal Killing fields and Killing horizons, and D.G. for a correction and suggestions that improved the presentation. T.J. was supported by NSF grant PHY91-12240. Research at the ITP, UCSB was supported by NSF Grant PHY89-04035.

**References**

[1] S.W. Hawking, *Comm. Math. Phys.* **43** (1975) 199.

[2] S.W. Hawking, *Comm. Math. Phys.* **25** (1972) 152; S.W. Hawking and G.R.F. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, 1973).

[3] We thank K.S. Thorne for explaining this to us.

[4] R.M. Wald, *General Relativity* (University of Chicago Press, 1984).

[5] R. Geroch, *Comm. Math. Phys.* **13** (1969) 180, Appendix B.

[6] G.W. Gibbons and M.J. Perry, *Phys. Rev. Lett.* **36** (1976) 985.

[7] P.C.W. Davies, S.A. Fulling, and W.G. Unruh, *Phys. Rev. D* **13**, 2720 (1976).

[8] S.M. Christensen and S.A. Fulling, *Phys. Rev. D* **15**, 2088 (1977).

[9] See for example: N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space* (Cambridge University Press, 1982).

[10] T. Jacobson, *Phys. Rev.* **D44** (1991) 1731.