Slow Entropy of Some Parabolic Flows

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Abstract: We study nontrivial entropy invariants in the class of parabolic flows on homogeneous spaces, quasi-unipotent flows. We show that topological complexity (i.e., slow entropy) can be computed directly from the Jordan block structure of the adjoint representation. Moreover using uniform polynomial shearing we are able to show that the metric orbit growth (i.e., slow entropy) coincides with the topological one for quasi-unipotent flows (this also applies to the non-compact case). Our results also apply to sequence entropy. We establish criterion for a system to have trivial topological complexity and give some examples in which the measure-theoretic and topological complexities do not coincide for uniquely ergodic systems, violating the intuition of the classical variational principle.

1. Introduction

The study of dynamical systems typically fits into three paradigms: (partially) hyperbolic, parabolic and elliptic. Quasi-unipotent flows on homogeneous spaces are models for the parabolic regime, exhibiting behaviour such as quantitative equidistribution with polynomial speed, polynomial mixing and controlled polynomial divergence of orbits. The most famous example of a quasi-unipotent flow is the horocycle flow on a constant negative curvature surface. Although quasi-unipotent flows have non-trivial statistical properties, including ergodicity, mixing, logarithm laws, they all have zero topological and metric (Kolmogorov–Sinai) entropy. It is therefore natural to ask for a refinement of the original definition of entropy that would allow one to distinguish such flows and describe their complexity. The main two methods modifying the standard entropy theory have been developed. The first is the method of Kushnirenko [Kus] called sequence entropy (Sect. 1.1.3), in which one allows a growing gaps between “test times” to see if
orbits have diverged. The second was established by Katok and Thouvenot [Kat-Tho] and was called slow entropy (Sect. 1.1), in which one adapts a dimensional characterization of entropy to characterize orbit growth for systems with lower complexity.

To define sequence entropy, we replace the standard wedge \( \bigvee_{n=0}^{N} T^{-n} \mathcal{P} \) with a wedge allowing more time for points to separate, \( \bigvee_{n=0}^{N} T^{-A_n} \mathcal{P} \), where \( A_n \) is an exponentially growing sequence. One then makes the usual definitions through information functions. This method obtains a family of new isomorphism invariants, which was first used in [Kus] to distinguish the horocycle flow from its square, as well as characterize Kronecker systems as those with zero sequence entropy for any sequence. Hulse [Hu], Newton [New1,New2,New3] and Kurg [Kur-New] further developed sequence entropy and Goodman [Go] adapted the definition to the topological setting.

For slow entropy, following the ideas of [Kat1,Kat2], one uses a dimensional characterization of entropy by considering coding spaces and coverings by Hamming balls. This was first done in [Kat-Tho], where a definition for the slow entropy was given in the setting of amenable group actions. Slow entropy was used to give a criterion for the smooth realization of \( \mathbb{Z}^k \) actions. Rather than considering a fixed family of times in which to distinguish orbits, this invariant allows one to still consider all times. Instead, one chooses a new “scale” for which the asymptotic orbit growth rate is computed. In particular, one may use a polynomial family \((n^x)\) instead of an exponential one \((e^{\chi n})\) to obtain a useful invariant for systems with 0 entropy in the usual sense.

We calculate the precise value of slow entropy in polynomial scales for any quasi-unipotent flow on finite-volume homogeneous spaces (Theorems 1.10, 1.11). We also compute the sequence entropy of such flows on compact homogeneous spaces (Theorem 1.12). Furthermore, we also show that the metric and topological slow entropies (and sequence entropies) coincide for quasi-unipotent flows (in the sense that the topological entropies are the supremums of their measure-theoretic entropies over the space of invariant measures). We show that zero topological slow entropy at all scales is equivalent to being topologically conjugate to a translation of a compact Abelian group, a version of a theorem in [Fe] in the topological category. Finally, we give several non-trivial counter-examples for the variational principle of slow entropy.

These results fit into a program of study for systems with low complexity. Ratner [Rat1] adapted the definition of slow entropy to distinguish horocycle flow and its Cartesian square up to Kakutani equivalence by replacing the Hamming distance with another distance function (i.e., \( \bar{f} \)-metric). An alternative approach to the slow entropy type invariant was developed by Blume in [Blu1,Blu2,Blu3], where he studied it in setting with slow growth, such as rank one systems. More recent work on slow entropy for higher rank actions appears in Katok et al. [Kat-Kat-Rod] and Hochman [Ho]. In the case of flows and transformations, one of the few nontrivial explicit calculations was done for smooth surface flows by the first author [Kan], with several applications. The slow entropy of a system also goes by another name, the measure-theoretic complexity. Ferenczi [Fe] proved several results using this terminology, including the characterization of Kronecker systems as those whose measure theoretic complexity is uniformly bounded (i.e., has slow entropy 0 at all scales). The work of Host et al. [HoKraMa] computes the topological complexities for nilsystems and establishes some corollaries. Our work generalizes their computation to arbitrary unipotent actions, as well as answer Question 4 of their paper by also computing the measure-theoretic complexity.

**Plan of the paper.** In Sects. 1.1–1.2.1 we give definitions of metric and topological slow entropy, metric and topological sequence entropy, quasi-unipotent flows, introduce some notations and formulate our main results Theorems 1.10, 1.11 and 1.12. We also
give several examples in which our formula can be explicitly computed. In Sect. 2.1, we recall fundamental structures and tools from the theory of homogenous spaces. In Sect. 2, we calculate the topological slow entropy of quasi-unipotent flows by proving many control lemmas on the decay rates of Bowen balls for compact homogeneous spaces. In Sect. 3, we calculate the measure-theoretic slow entropy of quasi-unipotent flows on compact homogeneous spaces, as well as discuss the relationships between the metric and topological slow entropies (in particular, we show that the variational principle does not hold for general slow entropy results). In both Sects. 2 and 3, we treat only compact homogeneous spaces because they are much easier to handle. In Sect. 5, we extend the results on slow entropy to the case of noncompact homogeneous spaces. We remark that this is more difficult since even though the divergence rates are the same as their compact counterparts, a local analysis is not sufficient because the injectivity radius can tend to 0. In Sect. 4, we calculate the both topological and metric sequence entropy of quasi-unipotent flow.

1.1. Definition of entropy invariants considered. Because we consider many entropy-type invariants, we gather the definitions here for reference throughout the paper.

1.1.1. Topological slow entropy Let \( \varphi^t : X \to X \) or \( f : X \to X \) be a uniformly continuous flow or transformation of a locally compact metric space. The following definitions are for flows, the analogous definitions for transformations are easily deduced. For \( y \in X \) and \( \varepsilon > 0 \), let \( B(y, \varepsilon) := \{ x \in X : d(x, y) < \varepsilon \} \). The Bowen ball of radius \( \varepsilon \) up to time \( T \) (called a \((\varepsilon, T)\)-Bowen ball for short) is defined as:

\[
B^T_{\varphi}(x, \varepsilon) = \bigcap_{t \in [0, T]} \varphi^{-t} \left( B(\varphi^t(x), \varepsilon) \right).
\]

(1)

If \( K \subset X \) is a compact subset of \( X \), let \( N_{\varphi, K}(\varepsilon, T) \) be the minimal number of \((\varepsilon, T)\)-Bowen balls required to cover \( K \) (since \( K \) is compact, this number is finite). Let \( S_{\varphi, K}(\varepsilon, T) \) be the maximal number of \((\varepsilon, T)\)-Bowen balls which can be placed in \( X \) disjointly with centers in \( K \). If \( a : \mathbb{R}_+ \to \mathbb{R}_+ \) is a function such that \( a(T) \to \infty \) as \( T \to \infty \), consider the asymptotics:

\[
\delta^N_{\varphi, K, a} = \limsup_{T \to \infty} \frac{N_{\varphi, K}(\varepsilon, T)}{a(T)}, \quad \delta^S_{\varphi, K, a} = \limsup_{T \to \infty} \frac{S_{\varphi, K}(\varepsilon, T)}{a(T)}.
\]

(2)

We now assume that we have a fixed family of functions \( a_\chi : \mathbb{R}_+ \to \mathbb{R}_+ \), \( \chi \in \mathbb{R}_+ \) such that if \( \chi < \chi' \), then \( a_\chi = o(a_{\chi'}) \).

Definition 1.1. The (slow) topological entropy of \( \varphi^t \) with respect to \( \{a_\chi\} \) is:

\[
h_{\text{top, } a_\chi}(\varphi^t) = \sup_{K} \lim_{\varepsilon \to 0} \left( \sup_{K} \{ \chi : \delta^N_{\varphi, K, a_\chi} > 0 \} \right) = \sup_{K} \lim_{\varepsilon \to 0} \left( \sup_{K} \{ \chi : \delta^S_{\varphi, K, a_\chi} > 0 \} \right).
\]

There are two particularly useful class of functions \( a_\chi \). One is the exponential class \( a_\chi(T) = e^{\chi T} \). In this case the entropy defined above is the usual topological entropy. The other is the polynomial class \( a_\chi(T) = T^\chi \), in which case we call \( R \) the polynomial slow entropy of \( \varphi^t \) and say that the polynomial slow entropy of \( \varphi^t \) gives rate \( T^R \).

\footnote{The second equality in the formula below follows from the following straightforward inequalities: \( N_{\varphi, K}(2\varepsilon, T) \leq S_{\varphi, K}(\varepsilon, T) \leq N_{\varphi, K}(\varepsilon/2, T) \).}
The topological slow entropy is essentially the same as the topological complexity considered in [HoKraMa], however the topological complexity is an equivalence class of rates, whereas the topological slow entropy simplifies the notion by attaching a single number to the topological complexity. One easily deduces the topological complexity from our arguments.

### 1.1.2. Metric slow entropy

Our definition will match the definition of slow entropy following [Kat-Tho] and [Kan]. While not identical, we note that this type of invariant is closely related to the measure-theoretic complexity used in [Fe]. The same remarks made about the difference between topological complexity and topological slow entropy also hold here.

We give definitions for flows, analogous definitions for transformations can be deduced easily. Let \( \phi^t \) act on \( (X, \mathcal{B}, \mu) \) where \( (X, \mathcal{B}, \mu) \) is a standard probability Borel space and \( \mathcal{P} = \{ P_1, \ldots, P_k \} \) be a finite measurable partition of \( X \). For \( T \in \mathbb{R}_+ \) and \( x \in X \) we define the coding of \( x \) to be a function \( \phi_{\mathcal{P},r}(x) : [0, T] \to \{ 1, \ldots, k \} \) defined by

\[
\phi_{\mathcal{P},r}(x) = (x_s)_{s \in [0, T]}, \quad \text{where } x_s = i \quad \text{whenever } t, x \in P_i.
\]  

For any two points \( x, y \in X \) their Hamming distance with respect to \( \mathcal{P} \) is given by

\[
\bar{d}_{\mathcal{P}}(x, y) := \frac{T - |\{ s \in [0, T] : x_s = y_s \}|}{T},
\]

where \( |\cdot| \) denotes the Lebesgue measure on \( \mathbb{R} \).

For \( \varepsilon > 0 \), let \( B_{\mathcal{P}}^T(x, \varepsilon) = \{ y \in X : \bar{d}_{\mathcal{P}}^T(x, y) < \varepsilon \} \) be the \( \varepsilon \)-Hamming ball centered at \( x \). Then as in the definition of topological slow entropy, define:

\[
S(\mathcal{P}, T, \varepsilon) = \min \left\{ \text{card}(F) : \mu \left( \bigcup_{x \in F} B_{\mathcal{P}}^T(x, \varepsilon) \right) > 1 - \varepsilon \right\}.
\]  

Analogously to (2) for a function \( a : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( a(T) \to \infty \) as \( T \to \infty \) we define

\[
A(\phi^t, \mathcal{P}, \varepsilon, a) = \limsup_{T \to \infty} \frac{S(\mathcal{P}, T, \varepsilon)}{a(T)}.
\]  

We now assume that we have a fixed family of functions \( a_\chi : \mathbb{R}_+ \to \mathbb{R}_+ \), \( \chi \in \mathbb{R}_+ \) such that if \( \chi < \chi' \), then \( a_\chi = o(a_{\chi'}) \).

**Definition 1.2.** We say that the (slow) metric entropy of \( \phi^t \) with respect to \( \{ a_\chi \} \) is \( h_{m,a_\chi}(\phi^t) \) if:

\[
h_{m,a_\chi}(\phi^t) = \sup_{\mathcal{P} - \text{finite partition}} h_{m,a_\chi}(\phi^t, \mathcal{P}),
\]

where

\[
h_{m,a_\chi}(\phi^t, \mathcal{P}) := \lim_{\varepsilon \to 0} \left( \sup_{\chi} A(\phi^t, \mathcal{P}, \varepsilon, a_\chi) > 0 \right).
\]

\[2\] Recall that the measure \( \mu \) is regular, hence, for every \( \varepsilon > 0 \) there exists a compact set \( K_\varepsilon \) with \( \mu(K_\varepsilon) > 1 - \varepsilon \).
Recall that a (finite) partition $\mathcal{P}$ is called a generator if the minimal $(T_t)_{t \in \mathbb{R}}$ invariant $\sigma$-algebra containing $\mathcal{P}$ is the whole $\sigma$-algebra $\mathcal{B}$.

**Proposition 1.3** ([Kat-Tho], Proposition 1). If $\mathcal{P}$ is a generator, then

$$h_{m,a}(\varphi^t) = h_{m,a}(\varphi^t, \mathcal{P}).$$

**1.1.3. Measure-theoretic sequence entropy** Recall if $\mathcal{P} = \{P_1, \ldots, P_k\}$ is a partition of a measure space $(X, \mu)$, the entropy of $\mathcal{P}$ is defined as $H(\mathcal{P}) = \sum_{i=1}^{k} \mu(P_i) \log \mu(P_i)$.

**Definition 1.4.** Suppose that $T : (X, \mu) \to (X, \mu)$ is an invertible measure preserving transformation or flow on a Lebesgue space $X$ with probability measure $\mu$. For a given increasing sequence $A = \{t_1, t_2, \ldots, t_n, \ldots\}$ with $t_i \to \infty$, a measurable partition (finite or countable partition) $\mathcal{P}$ such that $H(\mathcal{P}) < \infty$, define

$$h_A(T, \mathcal{P}) = \limsup_{n \to \infty} \frac{1}{n} H \left( \bigvee_{i=1}^{n} T^{-t_i} \mathcal{P} \right),$$

$$h_{A,\mu}(T) = \sup_{\mathcal{P}} h_A(T, \mathcal{P}).$$

(6)

We call $h_{A,\mu}(T)$ the measure-theoretic sequence entropy of $T$ with respect to $A$.

**1.1.4. Topological sequence entropy**

**Definition 1.5.** Suppose that $(X, d)$ is a locally compact metric space and $T : X \to X$ is a flow or transformation. Let $A := \{t_k : k \in \mathbb{N}_0\}$ be any increasing sequence of real numbers (natural numbers in the case of a transformation) tending to $\infty$. Then define $B^n_T(x, \varepsilon; \{t_k\}) = \bigcap_{k=0}^{n} T^{-t_k}(B(T^{t_k}(x), \varepsilon))$. We say a set $E \subset X$ is $(A, n, \varepsilon)-$separated (with respect to $T$) if $B^n_T(x, \varepsilon; \{t_k\}) \cap B^n_T(y, \varepsilon; \{t_k\}) = \emptyset$ for every $x \neq y \in E$. For a compact set $K \subset X$, let $N(A, n, \varepsilon, K)$ denote the largest cardinality of any $(A, n, \varepsilon)-$separated set in $K$. Then define

$$h(A, \varepsilon, K) = \limsup_{n \to \infty} \frac{1}{n} \log N(A, n, \varepsilon, K),$$

$$h_A(T) = \sup_{K} \sup_{\varepsilon > 0} N(A, \varepsilon, K).$$

(7)

And $h_A(T)$ is called the topological sequence entropy of $T$ with respect to $A$.

**1.2. Statement of the main theorem.** Let $\mathfrak{g}$ be a Lie algebra of a connected Lie group $G$. An element $U \in \mathfrak{g}$ is called ad-quasi-unipotent if it can be written as $U = U' + Q$, where $\text{ad}^{N}_{U'} = 0$ for some $N$, $Q$ is ad-compact, and $[Q, U'] = 0$. See Sect. 2.1 for any further undefined terms. A quasi-unipotent flow is a flow on a homogeneous space $G/\Gamma$ defined by $\varphi_t(\mathfrak{g}\Gamma) = (\exp(tU)\mathfrak{g})\Gamma$, where $U$ is an ad-quasi-unipotent element of $\mathfrak{g}$, and $\Gamma$ is a lattice (cofinite volume, discrete subgroup) of $G$. Before we give further definitions, we list some examples of quasi-unipotent flows.
1.2.1. Examples of quasi-unipotent flows  The three main classes of examples of quasi-unipotent flows are representative for our study of entropy invariants and come from the following cases:

(i) $G$ is a semisimple group;
(ii) $G$ is a nilpotent group;
(iii) $G = H \ltimes N$ is the semidirect product of a semisimple group $H$ with a nilpotent group $N$.

In general, the Levi decomposition always gives $G = H \ltimes S$ where $H$ is semisimple and $S$ is solvable. Since the quotient of the solvradical by the nilradical is abelian and acts semisimply on the solvradical, $U$ projects onto a compact element in the quotient $S/N$. As we will see, the action of such compact elements will not contribute to the growth measured in the definition of slow entropy. We give an example of flows for each of the types (i)-(iii).

Example 1.6. Let $G = SL(d, \mathbb{R})$, $\Gamma \subset G$ be any lattice, and

$$
U = \begin{pmatrix}
0 & 1 & & \\
0 & 1 & & \\
& & \ddots & \\
0 & 1 & & 0
\end{pmatrix}, \quad \exp_{\text{alg}}(tU) = \begin{pmatrix}
1 & t & t^2/2 & \cdots & \frac{t^{d-1}}{(d-1)!} \\
1 & t & \cdots & \frac{t^{d-2}}{(d-2)!} & \\
& \ddots & \ddots & \ddots & \ddots \\
& & & 1 & t \\
& & & & 1
\end{pmatrix}.
$$

We call $U$ the **principal** nilpotent element associated to the algebra $\mathfrak{sl}(d, \mathbb{R})$. In the special case of $SL(d, \mathbb{R})$, any nilpotent algebra element is conjugate to a block form element:

$$
U = \begin{pmatrix}
U_1 \\
U_2 \\
\vdots \\
U_n
\end{pmatrix},
$$

where each $U_i \in \mathfrak{sl}(d_i, \mathbb{R})$ is the principal element. Note that this is exactly the Jordan normal form of the matrix $U$. Call each $U_i$ a **principal block** and the sequence $(\dim(U_1), \ldots, \dim(U_n))$ the block sequence of $U$.

Example 1.7. Let $N$ be the matrix Lie group with Lie algebra:

$$
n = \{ U(x, t) : x \in \mathbb{R}^d, t \in \mathbb{R} \}, \quad U(x, t) = \begin{pmatrix}
0 & x_1 & x_2 & \cdots & x_d \\
0 & -t/2 & \cdots & \cdots & (-1)^{d-1}t/d \\
0 & t & \cdots & \cdots & (-1)^{d-1}t/(d-1) \\
& \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & 0
\end{pmatrix}.
$$
Let $\Gamma = N(\mathbb{Z})$ be the $\mathbb{Z}$-points of $N$ with the coordinates described. One may check that $\Gamma$ is a lattice since it contains the $\mathbb{Z}^d$ coming from the $x_i$ coordinates, and that

$$
\exp_{\text{alg}}(U(0, 1)) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\
1 & 1 \\
1 & 1 \\
& \ddots \\
1 & 1 \\
1 & 1 
\end{pmatrix} \in \Gamma.
$$

The unipotent flow induced by the algebra element:

$$U = U((\alpha, -\alpha/2, \ldots, (-1)^{d+1}\alpha/(d + 1)), 1),$$

induces a unipotent flow on $\Gamma \backslash N$ which is transverse to the torus corresponding to the $x$-coordinates. The first return map is exactly the classical affine transformation:

$$F(x_1, \ldots, x_d) = (x_1 + \alpha, x_2 + x_1, \ldots, x_d + x_{d-1}).$$

**Example 1.8.** Let $\rho : SL(d, \mathbb{R}) \to SL(N, \mathbb{R})$ be a rational representation of $SL(d, \mathbb{R})$. Then let $G$ be the group:

$$G = \left\{ \begin{pmatrix} \rho(A) & v \\ 0 & 1 \end{pmatrix} : A \in SL(d, \mathbb{R}), v \in \mathbb{R}^N \right\},$$

$$\Gamma = \left\{ \begin{pmatrix} \rho(A) & v \\ 0 & 1 \end{pmatrix} : A \in SL(N, \mathbb{Z}) \cap \rho(SL(d, \mathbb{R})), v \in \mathbb{Z}^N \right\}.$$

Then take any nilpotent element of $\mathfrak{sl}(d, \mathbb{R})$ gives a flow on $\Gamma \backslash G$. This unipotent flow is a twisted combination of the two previous ones. Indeed, if we factor onto the semisimple component, we get Example 1.6. Furthermore, if $\gamma \in SL(N, \mathbb{Z}) \cap \rho(SL(d, \mathbb{R}))$ corresponds to a closed orbit of $U$ in the semisimple factor, then taking the restriction to that closed orbit gives a system similar to Example 1.7.

One of the main tools for analyzing the dynamics of unipotent flows is to find a good splitting of the tangent bundle. Our splitting is analogous to the Lyapunov splitting for hyperbolic flows, and the Pesin entropy formula in that setting.

Recall that a $d \times d$ matrix $A$ is called ad-	extit{quasi-unipotent} if every eigenvalue of $A$ is purely imaginary. This is equivalent to $\exp(A)$ being quasi-unipotent. Then the standard Jordan normal form for real transformations allows us to assume that up to change of basis,

$$A = \begin{pmatrix} J_{m_1} & & \\
& \ddots & \\
& & J_{m_l} \\
& & \alpha_1 \\
& & \alpha_2 \\
& & \alpha_{l-1} \\
& & \alpha_l \\
& & \beta_{l+1} \\
& & \beta_{l+2} \\
& & \beta_{m_n} \end{pmatrix}.$$
where $J_k$ is the $k \times k$ matrix $J_k = \begin{pmatrix} 0 & 1 \\ \vdots & \vdots \\ \vdots & \vdots \\ 1 & 0 \end{pmatrix}$ and $J^\alpha_k$ is the $2k \times 2k$ matrix,

\[
J^\alpha_k = \begin{pmatrix} Q_\alpha & \text{id} \\ \vdots & \vdots \\ \vdots & \vdots \\ \text{id} & Q_\alpha \end{pmatrix},
\]

with $2 \times 2$ block diagonal entries $J^\alpha_k = \begin{pmatrix} Q_\alpha & \text{id} \\ \vdots & \vdots \\ \vdots & \vdots \\ \text{id} & Q_\alpha \end{pmatrix}$.

We call $(m_1, \ldots, m_l; m_{l+1}, \ldots, m_n)$ the chain structure of $A$.

By definition, $U \in g$ induces a quasi-unipotent flow if and only if $\text{ad}U$ is an ad-quasi-unipotent transformation of $g \cong \mathbb{R}^N$. We say that the chain structure of $U$ is the chain structure of $\text{ad}U$.

Example 1.9. When $g = \text{sl}(3, \mathbb{R})$ and $U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, the chain structure will be $(5, 3)$ as we have one chain with depth 5 and one chain of depth 3, illustrated below:

\[
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{\text{ad}_U} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{\text{ad}_U} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{ad}_U} \begin{pmatrix} 0 & -3 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{ad}_U} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

Theorem 1.10. Let $a_\chi(t) = t^\chi$ and $\varphi^t$ be a quasi-unipotent flow on a finite volume homogeneous space with chain structure $(m_1, \ldots, m_l; m_{l+1}, \ldots, m_n)$. Then the topological slow entropy of $\varphi^t$ with respect to $a_\chi$ is $R$, where:

\[
R = \sum_{i=1}^l \binom{m_i}{2} + 2 \sum_{i=l+1}^n \binom{m_i}{2}.
\]

Theorem 1.11. Let $a_\chi(t) = t^\chi$ and $\varphi^t$ be a quasi-unipotent flow on a finite volume homogeneous space with chain structure $(m_1, \ldots, m_l; m_{l+1}, \ldots, m_n)$. Then the measure-theoretic slow entropy of $\varphi^t$ with respect to $a_\chi$ is $R$, where:

\[
R = \sum_{i=1}^l \binom{m_i}{2} + 2 \sum_{i=l+1}^n \binom{m_i}{2}.
\]

As a corollary to Theorems 1.10 and 1.11, the topological slow entropy of a quasi-unipotent flow is the supremum of the measure-theoretic slow entropies taken over all invariant measures. We will see that this property is fairly special (see the “Appendix”).

Theorem 1.12. Let $\varphi^t$ be a quasi-unipotent flow on a compact homogeneous space. Then with respect to the sequence $A_n = C\lambda^n$, the measure-theoretic sequence entropy and topological sequence entropy of $\varphi^t$ is $R \log \lambda$, with $R$ as in Theorems 1.10 and 1.11.
Remark 1.13. The compactness assumption allows one to guarantee that any separation of orbits happens at a local level: separation for the flow implies the points also separate in the universal cover. In the non cocompact setting it might happen that points separate on the universal cover, but don’t separate on the quotient space. This happens exactly when the orbits escape towards the cusp (where the injectivity radius becomes arbitrarily small). Therefore one needs to show that for a large measure set of points, the time that the orbits spend in the neighborhood of the cusp is a small proportion of the time interval under consideration. For Sects. 2.1 to 3 we concentrate on cocompact lattice case and extend our result to non-compact lattice in Sect. 5 for slow entropy.

2. Topological Slow Entropy of Quasi-unipotent Flows

2.1. Preliminaries on homogeneous spaces

2.1.1. Metrics and measures on homogeneous spaces. Let $\Gamma \subset G$ be a (discrete) subgroup. We introduce a metric on the homogeneous space $\Gamma \backslash G$ by first introducing a left-invariant metric on $G$. Fix an inner product $\langle \cdot, \cdot \rangle_0$ on $\mathfrak{g}$, and define for $v, w \in T_g G$:

$$\langle v, w \rangle = \langle dL_{g^{-1}}v, dL_{g^{-1}}w \rangle_0.$$

By construction, $\langle \cdot, \cdot \rangle$ is left-invariant, so it induces a Riemannian metric on the space $\Gamma \backslash G$. The Riemannian metric also has an associated exponential mapping $\exp_\text{geom} : \mathfrak{g} \to G$, which is $C^\infty$ and satisfies

$$d_0 \exp_\text{geom} = \text{id}. \quad (9)$$

Like the algebraic exponential, there is a local inverse of $\exp_\text{geom}$ which we will denote by $\log_\text{geom}$. The following is immediate from the definition of the inner product.

**Lemma 2.1.** The Riemannian volume is a (left) Haar measure on $G$. In particular, it is independent of the metric $\langle \cdot, \cdot \rangle_0$ when determining a probability measure on a homogeneous space.

2.1.2. Uniformities and topological entropy. It is classical that the topological entropy of a uniformly continuous dynamical system on a compact metric space does not depend on the Lipschitz class of metric chosen. In fact, in the compact case, it is a purely topological invariant. One may further generalize this phenomenon by considering *uniform topological spaces*. Such spaces are natural generalizations of metric spaces. Uniform spaces have canonical bases associated to them, which generalize balls in metric spaces. In a metric space $(X, d)$, the uniformity on $X$ induced by the metric is the collection of open sets of which contain a set of the form $B^\varepsilon(x, y) = \{(x, y) : d(x, y) < \varepsilon, x, y \in X\} \subset X \times X$. Notice that the balls $B(x, \varepsilon)$ are exactly $B^\varepsilon(x) = (\{x\} \times X) \cap B^\varepsilon$. More generally, if $V$ is an element of the uniformity $\mathcal{U}$, we let $V(x) = (\{x\} \times X) \cap V$. Uniformities axiomitize certain properties of these subsets; for basics on uniform topological spaces, we refer the reader to [Kelley, Chapter 6]. Let $\mathcal{U}$ be the uniformity induced by the metric on $G$ and $\Gamma \backslash G$, which we abusively denote by the same letter.

Recall that if $U, V \in \mathcal{U}$, then

$$U * V = \{(x, z) : \text{there exists } y \text{ such that } (x, y) \in U, (y, z) \in V\}.$$
Let $G$ be a Lie group and $\Gamma$ be a discrete subgroup. Notice that for every $\tilde{x} = \Gamma x \in \Gamma \backslash G$, there exists $\varepsilon > 0$ such that the map $X \mapsto \tilde{x} \exp(X) = \Gamma x \exp(X)$ is injective on $\{X \in \mathfrak{g} : ||X|| < \varepsilon\}$. This follows easily from the discreteness of $\Gamma$, and the fact that $\gamma_i x \exp(X_i) \to \gamma X \exp(X)$ implies that the distance between $\gamma^{-1} \gamma_i$ and $x \exp(X) \exp(-X_i) x^{-1}$ tends to zero. Hence if $X_i$ and $X$ are both sufficiently close to $0$ (with the closeness depending on the conjugation action of $x$), $\gamma_i$ is eventually constant, and we may assume without loss of generality it is id. Therefore, $X_i$ converges to $X$ and the map is locally injective on a sufficiently small neighborhood of $x$ (which may vary with $x$).

Fix some compact set $K \subset \Gamma \backslash G$. Since the size of the neighborhood above depends continuously on $x$, there exists some $\varepsilon > 0$ such that $X \mapsto \tilde{x} \exp(X)$ is injective on $\{X \in \mathfrak{g} : ||X|| < \varepsilon\}$ for every $\tilde{x} \in K$. The supremum over all such $\varepsilon$ is called the injectivity radius of $K$, denoted $\text{inj}(K)$.

Suppose that there is a fixed norm $||\cdot||$ on $\mathfrak{g}$. The following Lemmas will be useful to us later:

**Lemma 2.2.** If $G$ is a Lie group with Lie algebra $\mathfrak{g}$, then for any sufficiently small $\varepsilon_0$, the collection $\mathcal{B}$ of sets $V^{(\varepsilon)} = \{(g, \exp(X)) : g \in G, ||X|| < \varepsilon, 0 < \varepsilon < \varepsilon_0\}$ is a base of the uniformity of $G$ and $\Gamma \backslash G$ induced by any left-invariant metric for any norm $||\cdot||$ on $\mathfrak{g}$.

**Proof.** Recall that the sets $B^{(\varepsilon)} = \{(g, h) : d(g, h) < \varepsilon, g, h \in G\}$ is a base of the uniformity on $G$ and $\Gamma \backslash G$. Therefore, it suffices to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $V^{(\delta)} \subset B^{(\varepsilon)}$. Notice that $\exp : B(0, \varepsilon_0) \to G$ is a diffeomorphism, hence there exists $\delta > 0$ such that if $||X|| < \delta, d(g, \exp(X)) < \varepsilon$. \qed

We will work with the base $\mathcal{B}$, quite extensively, and therefore establish convenient notation related to it. If $x, y \in G$ and $\tilde{x}, \tilde{y} \in \Gamma \backslash G$, then $y \in V^{(\varepsilon)}(x)$ implies that $y = x \exp(X)$ for some $X = X(x, y) \in \mathfrak{g}$ with $||X|| < \varepsilon$. Notice that $X(x, y)$ is well-defined for points $x, y \in G$ or $\tilde{x}, \tilde{y} \in \Gamma \backslash G$ if they are sufficiently close.

**Remark 2.3.** In fact, even locally, the function $(x, y) \mapsto ||X(x, y)||$ is not necessarily a metric for any norm on $\mathfrak{g}$. Indeed, one may check directly on, for instance, $SL(2, \mathbb{R})$, that this candidate for a distance fails the triangle inequality. However, it does satisfy a weaker form of the triangle inequality: for every $\varepsilon > 0$, there exists $\delta > 0$ such that $V^{(\delta)} \ast V^{(\delta)} \subset V^{(\varepsilon)}$ (in the case of a metric, one may take $\delta = \varepsilon/2$). This is sufficient to study the entropy properties of the homogeneous systems, and is exactly the property of a uniformity that replaces the usual triangle inequality for metrics.

Given some $V \in \mathcal{U}$ and $K \subset X$ compact, we define a $V$-separated set of $K$ to be a set of points $\{x_i\}$ such that $(x_i, x_j) \not\in V$ if $i \neq j$ (notice that if $V = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}$, then this is the usual notion of an $\varepsilon$-separated set). We similarly define a $V$-cover. Given a uniformly continuous flow $\psi^t$ and $V \in \mathcal{U}$, let $V_T = \cap_{t \in [0, T]} (\psi^t \times \psi^{-t})^{-1}(V) \in \mathcal{U}$ (notice that if $V$ is the set corresponding to the metric ball as above, then $V_T(x) = V_T \cap (\{x\} \times X) \subset X$ is the set corresponding to the Bowen ball). Given $K \subset X$ compact, let $N_{\psi, K}(V, T)$ be the minimal cardinality of a $V_T$-cover of $X$, and $S_{\psi, K}(V, T)$ be the maximal cardinality of a $V_T$-separated subset of $K$.

**Lemma 2.4 ([Hood]).** If $\psi^t : X \to X$ is a uniformly continuous transformation of a uniform space $X$, then if $V \in \mathcal{U}$ and $W \ast W \subset V$, then $N_{\psi, K}(V, T) \leq S_{\psi, K}(V, T) \leq N_{\psi, K}(W, T)$.
Corollary 2.5. If \( V^{(e)} \in \mathcal{U} \) is a nested sequence of subsets which is a base of the uniformity \( \mathcal{U} \), then one may replace \( N_{\psi,K}(e,T) \) in the definition of topological slow entropy with \( N_{\psi,K}(V^{(e)}, T) \).

2.1.3. The adjoint representation  \( G \) acts on itself by conjugation \( C_G : h \mapsto g^{-1}hg \), and taking the derivative at the identity in the coordinate \( h \) gives the adjoint representation of \( G \) on \( \mathfrak{g} = T_eG, \text{ Ad} : G \to GL(\mathfrak{g}). \) Taking the derivative of this map in the \( g \) coordinate yields the Adjoint representation of the Lie algebra \( \mathfrak{g} \), \( \text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g}), \) which coincides with the Lie bracket: \( \text{ad}(X)Y = [X, Y] \). The following are standard tools from the theory of Lie groups, which we write as a Lemma to reference.

Lemma 2.6. If \( X, Y \in \mathfrak{g} \),
\[
\exp(-X)\exp(Y)\exp(X) = \exp(\text{Ad}(X)Y),
\]
\[
\exp(\text{ad}(X)) := \sum_{k=0}^{\infty} \frac{\text{ad}(X)^k}{k!} = \text{Ad}(\exp(X)).
\]

2.1.4. Polynomial decay rates

Lemma 2.7. Let \( p(t) = \sum_{k=0}^{d} a_k t^k \) be a polynomial of degree \( d \). There exists \( C(d) \) such that if \( |p(t)| < \varepsilon \) for all \( t \in [0, T] \), then \( |a_k| < C(d)T^{-k}\varepsilon \) for all \( k = 0, \ldots, d \). Conversely, if \( |a_k| < C(d)^{-1}T^{-k}\varepsilon \) for all \( k = 0, \ldots, d \), then \( |p(t)| < \varepsilon \) for all \( t \in [0, T] \).

Proof. Let \( \mathcal{P}_d \) denote the space of polynomials of degree at most \( d \). Then \( \mathcal{P}_d \) carries a family of norms \( ||p||_T = \sup_{t \in [0, T]} |p(t)| \), as well as the norm \( ||p||_\infty = \max_k |a_k| \), where \( p(t) = \sum k a_k t^k \). Since \( \mathcal{P}_d \cong \mathbb{R}^d \), and all norms on \( \mathbb{R}^d \) are equivalent, we conclude that there exists \( C(d) \) such that \( C(d)^{-1} ||p||_\infty \leq ||p||_1 \leq C(d) ||p||_\infty \).

Now we compute:
\[
||p||_T = \sup_{0 \leq t \leq T} \left| \sum_{k=0}^{d} a_k t^k \right| = \sup_{0 \leq t \leq T} \left| \sum_{k=0}^{d} (a_k T^k) t^k \right| \geq C(d)^{-1} \max_{k \in \{0, \ldots, d\}} |a_k T^k|.
\]

Therefore for \( k \in \{0, \ldots, d\} \), we get
\[
|a_k| \leq C(d)T^{-k} ||p||_T.
\]

The reverse computation follows similarly. □

Given a matrix \( A \) as in (8), we introduce a norm \( ||\cdot||_\infty \) on \( \mathbb{R}^N \). We write \( v \in \mathbb{R}^N \) in the standard basis \( v = \sum_{i=1}^{L} t_i e_i + \sum_{i=1}^{N'} x_i e_{L+2l-1} + y_i e_{L+2l} \), so that each \( e_i, 1 \leq i \leq L \) is part of a standard Jordan block \( J_k \), and each \( e_i, e_{i+1} \) corresponds to a \( 2 \times 2 \) block in a block of the form \( J_k \), and \( N' \) is the number of \( 2 \times 2 \) blocks. Then we let \( ||v||_\infty = \max \left\{ |t_i|, \sqrt{x_i^2 + y_i^2} \right\} \). Notice that this is the usual \( L_\infty \) norm if \( A \) does not have any \( 2 \times 2 \) blocks (i.e., \( A \) is ad-unipotent). If \( v \in \mathbb{R}^N \), let \( v_t = \exp(tA)v \).

Corollary 2.8. Let \( A \) be as in (8) and \( B_T(0, \varepsilon) \) denote the set of all \( v \in \mathbb{R}^N \) such that \( ||v_t||_\infty < \varepsilon \) for all \( t \in [0, T] \). Then there exists \( C > 0 \) such that \( T^{-R}/C \leq \text{vol}(B_T(0, \varepsilon)) \leq CT^{-R} \), where \( R = \sum_{i=1}^{L} \binom{m_i}{2} + \sum_{i=L+1}^{n} 2\binom{m_i}{2} \).

The reverse computation follows similarly. □
Proof. First consider the case of a single Jordan block $A = J_m$. Then by direct computation, one sees that $(\exp(tA)v)_1 = p(t) := \sum_{j=0}^{m-1} \frac{t^j}{j!} v_{j+1}$ for every $v \in \mathbb{R}^m$. Therefore, if the $L^\infty$ norm is bounded by $\varepsilon$, then the polynomial $|p(t)| \leq \varepsilon$ for all $t \in T$. So, by Lemma 2.7, $|v_j| \leq C(m)T^{-j}\varepsilon$. Hence,

$$B_T(0, \varepsilon) \subset \prod_{j=0}^{m-1} [-C(m)T^{-j}\varepsilon, C(m)T^{-j}\varepsilon],$$

and we obtain the upper bound.

Similarly, by using the converse of Lemma 2.7, one obtains a rectangular subset which is contained in $B_T(0, \varepsilon)$,

$$\prod_{j=0}^{m-1} [-C(m)^{-1}T^{-j}\varepsilon, C(m)^{-1}T^{-j}\varepsilon] \subset B_T(0, \varepsilon).$$

Therefore, we obtain the lower bound.

To obtain the general form, notice that each Jordan block gives an invariant subspace with similar estimates. In the case of a block of $2 \times 2$ matrices, let $(x_1, y_1, \ldots, x_m, y_m) \in \mathbb{R}^{2N}$. Notice that $J^\alpha_m = \text{diag}(Q, \ldots, Q) + J_0^m$, and $\text{diag}(Q, \ldots, Q)$ and $J_0^m$ commute. Furthermore, $J_0^m$ has Jordan normal form $\begin{pmatrix} J_m & -J_m \\ 0 & 0 \end{pmatrix}$ and $\exp\text{diag}(Q, \ldots, Q)$ is a compact matrix that preserves the norm we have constructed (it rotates each $2 \times 2$ block. Therefore, the points which stay within $\varepsilon$ for $J^\alpha_m$ are exactly those which stay within $\varepsilon$ for $J_0^m$, or equivalently, two separate Jordan blocks. Therefore, we may treat each block of the form $J^\alpha_m$ as two separate blocks. \qed

Proof of Theorem 1.10. We use Corollary 2.5 to compute the topological entropy of an ad-quasi-unipotent flow $\varphi'$. Since $\varphi'$ is ad-quasi-unipotent, it is the translation action by some one parameter subgroup $\exp(tU)$. We may therefore assume that $g$ has canonically been identified with $\mathbb{R}^n$ so that $ad_U$ is given by the matrix in (8). The base of the uniformity we use is $V^{(\varepsilon)} = \{(g \exp(X), g) : ||X||_\infty < \varepsilon\}$, as described in Lemma 2.2. Notice that by Lemma 2.6, $V_T^{(\varepsilon)}$ is exactly $\{(g \exp(X), g) : X \in B_T(0, \varepsilon)\}$, where $B_T(0, \varepsilon)$ is as in Corollary 2.8.

Fix a compact set $K$ and $0 < \varepsilon_0 < \text{inj}(K)$. Then since the exponential mapping is a smooth diffeomorphism from $B(0, \varepsilon_0)$ to $G/\Gamma$, there exists some constant $\delta = \delta(\varepsilon_0)$ such that $\delta \leq \text{Jac}(d \exp_X) \leq 1/\delta$ for every $X \in B(0, \varepsilon_0)$. Therefore, if $\mu$ is the Haar measure on $G/\Gamma$,

$$(\delta/C) \cdot T^{-R} \leq \delta \text{vol}(B_T(0, \varepsilon_0)) \leq \mu(V_T^{(\varepsilon_0)}(x)) \leq \text{vol}(B_T(0, \varepsilon_0)) / \delta \leq (C/\delta) \cdot T^{-R}.$$ \hspace{1cm} (12)

Therefore, at most $(C/\delta) \cdot T^R$ disjoint sets of the form $V_T^{(\varepsilon)}(x_i)$ can be placed in $X$, giving that the topological slow entropy is at most $R$. Similarly, one needs at least $(\delta/C) \cdot T^R$ such sets to cover the space, which gives that the topological slow entropy is at least $R$. \qed
3. Topological and Metric Slow Entropy of Quasi-unipotent Flows

In this section we will prove some relationships between the metric and topological slow entropies of topological dynamical systems.

3.1. Equality of topological and metric slow entropy for quasi-unipotent flows. Throughout this section we assume that $\Gamma \backslash G$ is compact. See Sect. 5 for a treatment of flows on noncompact homogeneous spaces. The arguments for the noncompact spaces also apply to the compact ones, but the argument for the compact spaces is significantly simpler and the proof in the noncompact case uses similar ideas, so we include both. We introduce following notion to describe the partition we will work at first:

**Definition 3.1 (Well-partitionable).** A metric space $X$ is **well-partitionable** if it is $\sigma$-compact and for any Borel probability measure $\mu$, compact set $K$ and $\varepsilon > 0$, there exists a finite partition $P$ of $K$ whose atoms have diameter less than $\varepsilon$ and such that $\mu \left( \bigcup_{\xi \in P} \partial_\varepsilon \xi \right) < \varepsilon$, where $\partial_\varepsilon \xi = \{ y \in X : B(y, \varepsilon) \cap \xi \neq \emptyset \text{ but } B(y, \varepsilon) \not\subset \xi \}$.

**Remark 3.2.** Note that any smooth manifold is well-partitionable.

We begin our proof by recalling a corollary of Proposition 2 from [Kat-Tho]. In [Kat-Tho] the authors consider a compact space $X$ but their proof of Proposition 2 generalizes easily to the case where $X$ is well-partitionable:

**Theorem 3.3 (Slow Goodwyn’s Theorem).** Suppose that $f$ is a continuous, invertible flow or transformation on a well-partitionable $X$. Then for any invariant measure $\mu$ and family of scales $a_x$:

$$h_{\mu, a_x}(f) \leq h_{\text{top}, a_x}(f).$$

We now compute the metric slow entropy for the flows considered in the previous sections. We first prove several lemmas. We will use the following classical result on polynomials, which is a form of the Chebyshev inequality, see for instance [Bru].

**Lemma 3.4 (Brudnyi–Ganzburg inequality).** Let $V \subset \mathbb{R}$ be an interval and $\omega \subset V$ a measurable subset. Then, for any polynomial $p$ of degree at most $k$, we have

$$\sup_{V} |p| \leq \left( \frac{4|V|}{|\omega|} \right)^k \sup_{\omega} |p|.$$

For any subset $A \subset \mathbb{R}$, let $|A|$ denote the Lebesgue measure of $A$. Recall that if $X \in \mathfrak{g}$, we let $X_t = \text{Ad}(\exp(tU))X$.

**Lemma 3.5.** Let $U \in \mathfrak{g}$ be an ad-quasi-unipotent element inducing the flow $\varphi^t$ and $X \in \mathfrak{g}$ be an element such that $||X||_\infty < \eta$ (where $||\cdot||_\infty$ is the norm for which $\text{ad}_U$ is put in the form (8)). If $||Xs||_\infty = \eta$ and $||X_t||_\infty < \eta$ for all $t \in [0, S)$, then there exists $c_\varphi > 0$ such that

$$\left| \{ t \in [0, S] : ||\text{Ad}(\exp(tU))X||_\infty < c_\varphi \eta \} \right| < S/10.$$
Proof. Let $\omega := \{t \in [0, S] : \|\text{Ad}(\exp(tU))X\|_\infty < c_\varphi \eta\}$ and $V = [0, S]$. We wish to show that $|\omega| < S/10$.

Since the entries of $X_t = \text{Ad}(\exp(tU))X$ are polynomials in Corollary 2.8 (or in the case of $2 \times 2$ blocks, the norms $\sqrt{x_i^2 + y_i^2}$ are absolute values of polynomials), the entry for which $\text{Ad}(\exp(tU))X$ achieves absolute value $\eta$ is a polynomial in $t$, call it $p(t)$. On $[0, S]$, all other entries are bounded by $\eta$, so by Lemma 3.4, if $k = \max \{m_i\}$, we have

$$\sup_V |p(t)| \leq \left(\frac{4|V|}{|\omega|}\right)^k \sup_\omega |p(t)|. \quad (13)$$

By construction, $\sup_\omega |p(t)| \leq \sup_\omega ||X_t||_\infty \leq c\eta$ and $\sup_V ||X_t|| = \sup_V |p(t)| = \eta$. Then

$$\eta \leq \left(\frac{4S}{|\omega|}\right)^k c_\varphi \eta.$$ 

Therefore,

$$|\omega| \leq 4c_\varphi^{1/k}S.$$ 

By picking $c_\varphi$ such that $4c_\varphi^{1/k} < \frac{1}{10}$, we know that $|\omega| < \frac{S}{10}$. □

For $\delta > 0$ let $P_\delta$ be a finite partition of $\Gamma \setminus G$ with atoms contained in sets of the form $V(\delta)$ for some fixed $\delta > 0$. Notice that as $\delta \to 0$, the partitions $\{P_\delta\}_{\delta \in (0, 1)}$ converge to the full $\sigma$-algebra in the limit. Therefore it suffices to compute the slow entropy (and show it is equal to $R$) for partitions $\{P_n\}_{n \in \mathbb{N}}$ along any sequence $\delta_n \to 0$. Throughout this section, we fix such a partition $P := P_{c_\varphi \eta}$ (i.e. $\delta = c_\varphi \eta$) where $\eta < \text{inj}(\Gamma \setminus G)$. In the following lemmas, the parameters $\varepsilon_0 > 0$ and $T_0 > 0$ depend on $\eta > 0$. Then the result follows by taking any sequence $\eta_n \to 0$.

**Lemma 3.6.** There exists $\varepsilon_0$, $T_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ and every $T > T_0$, the following holds. If the Hamming distance satisfies the inequality $d_{\varphi, P}(\bar{x}, \bar{y}) < \varepsilon$, then there exists an interval $I \subset [0, T]$ with $|I| \geq 4T/5$ such that $\varphi^t(\bar{y}) \in V(\eta)(\varphi^t(\bar{x}))$ for every $t \in I$.

Note that as an immediate corollary, we get:

**Corollary 3.7.** There exists $\varepsilon_0$, $T_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ and every $T > T_0$, we have $B_{\varphi, P}(\bar{x}, \varepsilon) \subset \varphi^{-T/5} \left(V_{4T/5}^{(c_\varphi \eta)}(\varphi^{T/5}\bar{x})\right)$.

**Proof.** Let $\varepsilon < \varepsilon_0$, $T > T_0$ and $\bar{y} \in B_{\varphi, P}(\bar{x}, \varepsilon)$ (see (1)). Then by Lemma 3.6 there exists $I = [a, b]$, $b - a \geq \frac{4}{5}T$ such that for $t \in [0, \frac{4}{5}T] \subset [a, b - a]$

$$\varphi^t(\varphi^a(\bar{y})) \in V^{(c_\varphi \eta)}(\varphi^t(\varphi^a(\bar{x}))).$$

By $b - a \geq \frac{4}{5}T$, we know $a \leq \frac{1}{5}T$ and $b \geq \frac{4}{5}T$. This is exactly the containment claimed. □
Proof of Lemma 3.6. Fix \( \tilde{x}, \tilde{y} \in \Gamma \setminus G \) such that \( \tilde{d}^T_{\varphi, \mathcal{P}}(\tilde{x}, \tilde{y}) < \varepsilon \) and \( \varepsilon < \frac{1}{10} \). To simplify notation we write \( \tilde{x}_t = \varphi^t \tilde{x}, \tilde{y}_t = \varphi^t \tilde{y} \). Divide the interval \([0, T]\) by the following procedure: let \( t_0 \in [0, T] \) be the smallest for which \( \tilde{y}_{t_0} \in V^{(c_{\varphi} \eta)}(\tilde{x}_{t_0}) \). Notice that such \( t_0 \) always exists as \( \tilde{d}^T_{\varphi, \mathcal{P}}(\tilde{x}, \tilde{y}) < \varepsilon \) and the atoms of \( \mathcal{P} \) have diameter less than \( c_{\varphi} \eta \) (in fact there is at least \((1 - \varepsilon)T\) of such \( t_0 \)'s). Let then \( S_0 \geq 0 \) be the smallest number such that \( \varphi^{S_0}(\tilde{y}_{t_0}) \not\in V^{(\eta)}(\varphi^{S_0}(\tilde{x}_{t_0})) \) (if such \( S_0 \) does not exist, we set \( S_0 = +\infty \)). Notice that since \( c_{\varphi} < 1 \) it follows by the definition of \( t_0 \) that \( S_0 > 0 \).

Now inductively we define \( t_{i+1} > t_i + S_i \) to be the smallest for which \( \tilde{y}_{t_i} \in V^{(c_{\varphi} \eta)}(\tilde{x}_{t_i}) \), and \( S_{i+1} \) be such the smallest number for which \( \varphi^{S_{i+1}}(\tilde{y}_{t_i}) \not\in V^{(\eta)}(\varphi^{S_{i+1}}(\tilde{x}_{t_i})) \) (or \( S_{i+1} = +\infty \) if \( S_{i+1} \) does not exist). We continue until \( t_{J+1} \geq T \) or \( S_J = \infty \). By the definition of \( t_i \) it follows that \( S_{i+1} > 0 \) and hence \( (t_i) \) is a strictly increasing sequence.

By the definition of \((t_j)\) and \((S_j)\) it follows that

\[
\tilde{y}_t \in V^{(c_{\varphi} \eta)}(\tilde{x}_t) \text{ for every } t \in [t_J, \tilde{T}],
\]

where \( \tilde{T} := \min(t_J + S_J, T) \). Moreover, since \( t_{J+1} \geq T \) it follows that

\[
\tilde{y}_t \not\in V^{(c_{\varphi} \eta)}(\tilde{x}_t) \text{ for every } t \in [\tilde{T}, T],
\]

Notice that as \( \tilde{y}_t \in V^{(c_{\varphi} \eta)}(\tilde{x}_t) \) for \( t \in [t_i, t_i + S_i] \) and \( \eta < \text{inj}(\Gamma \setminus G) \), we can use coordinates of the neighborhood \( V^{(c_{\varphi} \eta)}(\tilde{x}_t) \) to conclude that \( y \in \{ x: \| x \|_\infty < \eta, t \in [t_i, t_i + S_i] \} \). Therefore, by the choice of norm, Lemmas 2.6 and 3.5, for every \( i = 0, ..., J \)

\[
| \{ t \in [t_i, t_i + S_i] : \tilde{y}_t \not\in V^{(c_{\varphi} \eta)}(\tilde{x}_t) \} | \geq \frac{9}{10} S_i.
\]

Notice that

\[
\{ t \in [0, t_J] : \tilde{y}_t \in V^{(c_{\varphi} \eta)}(\tilde{x}_t) \} = \{ t \in [0, t_{J-1} + S_{J-1}] : \tilde{y}_t \in V^{(c_{\varphi} \eta)}(\tilde{x}_t) \}
\]

\[
\subset \bigcup_{i=0}^{J-1} \{ t \in [t_i, t_i + S_i] : \tilde{y}_t \in V^{(c_{\varphi} \eta)}(\tilde{x}_t) \},
\]

Using (15) and the fact that \( \tilde{d}^T_{\varphi, \mathcal{P}}(\tilde{x}, \tilde{y}) < \varepsilon \) (where \( \varepsilon < \varepsilon_0 := \frac{1}{2} c_{\varphi} \eta \)), we get

\[
(1 - \varepsilon)T \leq | \{ t \in [0, T] : \tilde{y}_t \in V^{(c_{\varphi} \eta)}(\tilde{x}_t) \} | = | \{ t \in [0, \tilde{T}] : \tilde{y}_t \in V^{(c_{\varphi} \eta)}(\tilde{x}_t) \} | = | \{ t \in [0, t_J] : \tilde{y}_t \in V^{(c_{\varphi} \eta)}(\tilde{x}_t) \} | + | \{ t \in [t_J, \tilde{T}] : \tilde{y}_t \in V^{(c_{\varphi} \eta)}(\tilde{x}_t) \} | \leq \frac{t_J}{10} + (\tilde{T} - t_J),
\]

where the last inequality follows from (16) and (17). Define \( I := [t_J, \tilde{T}] \). Then (since \( t_J \leq T \), \( \tilde{T} - t_J \geq (1 - \varepsilon)T - \frac{T}{10} \geq \frac{4T}{5} \)). This and (14) finishes the proof. □

Using Lemma 3.6 we can prove Theorem 1.11.

Proof of Theorem 1.11, Cocompact \( \Gamma \). By Corollary 3.7,

\[
\mu \left( B^T_{\varphi, \mathcal{P}}(\tilde{x}, \varepsilon) \right) \leq \mu \left( \varphi^{-T/5} \left( V_3^{(c_{\varphi} \eta)}(\varphi^{T/5} \tilde{x}) \right) \right).
\]
Notice that by Corollary 2.8 and the fact that $\mu$ is equivalent to $\exp_*(\text{vol})$ with bounded Jacobian,

$$
\mu \left( V^{(c_\varphi \eta)}_{3T/5} (\varphi^{T/5} \bar{x}) \right) \leq C_0(c_\varphi \eta, \varphi') T^{-R},
$$

hence the metric growth is larger or equal than $T^R$. By Theorems 3.3 and 1.10, the metric growth is asymptotically equal to $T^R$. \( \square \)

### 4. Sequence Entropy of Quasi-unipotent Flows

Recall the definitions of sequence entropy (see Sect. 1.1.4). We prove the following analogy of Corollary 2.8. We first establish a parallel version lemma with Lemma 2.7:

**Lemma 4.1.** Fix $L > 0$ and $\lambda > 1$. Let $p(t) = \sum_{k=0}^d a_k t^k$ be a polynomial of degree $d$ and also suppose $n > d$ is sufficiently large. There exists $C(d)$ such that if $|p(t)| < \varepsilon$ for all $t \in \{L, L\lambda, \ldots, L\lambda^n\}$, then $|a_k| < C(\lambda, d)\lambda^{-kn}\varepsilon$ for all $k = 0, \ldots, d$. Conversely, if $|a_k| < C(\lambda, d)^{-1}\lambda^{-kn}\varepsilon$ for all $k$, then $|p(t)| < \varepsilon$ for all $t \in \{0, L, L\lambda, \ldots, L\lambda^n\}$.

**Proof.** Recall (as in Lemma 2.7) that $P_d$ is the set of all polynomial functions of degree at most $d$.

Now for any element $f = \sum_{k=0}^d f_i x^i \in P_d$, we define following two norms $\| \cdot \|_1$ and $\| \cdot \|_2$, respectively.

$$
\| f \|_1 = \max_{s=0,1,\ldots,d+1} |f(L\lambda^s)|,
$$

$$
\| f \|_2 = \max_{0 \leq k \leq d} |f_k|.
$$

(18)

While $f$ takes on only finitely many values (in fact, exactly $d + 2$ values), $f$ has at most degree $d$, so $\| \cdot \|_1$ is a genuine norm (and not a seminorm). Since $P_d$ is a finite dimensional vector space and thus $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent.

Assume that $|p(L\lambda^s)| < \varepsilon$ for $s = 0, 1, \ldots, n$, and let $g(t) = p(\lambda^{n-d-1}t)$. Then $g_k = \lambda^{(n-d-1)k} p_k$. But the assumption on the smallness of $p(L\lambda^s)$ implies that $|g(L\lambda^s)| < \varepsilon$ for $s = 0, \ldots, d + 1$. Therefore, since we have equivalence of $\| \cdot \|_1$ and $\| \cdot \|_2$, there exists $C = C(\lambda, d)$ such that $|g_k| \leq C \| g \|_2 < C \varepsilon$. Then $|p_k| = \lambda^{-(n-d-1)k} |g_k| < C' \lambda^{-nk} \varepsilon$, where $C' = C \lambda^{d+1}$.

By above inequality, we finish the first part of the lemma and second part of the lemma follows similarly. \( \square \)

Recall that the base of the uniformity is $V^{(\varepsilon)} = \{(g \exp(X), g) : \|X\|_\infty < \varepsilon\}$. In order to describe the sequence Bowen ball, we define $V^{(\varepsilon)}_{\{L\lambda^k\}, N} = \bigcap_{i=0}^N (\varphi^{L\lambda^i} \times \varphi^{L\lambda^i})^{-1}(V)$ and $V^{(\varepsilon)}_{\{L\lambda^k\}, N}(\bar{x}) = V^{(\varepsilon)}_{\{L\lambda^k\}, N} \bigcap \{(\bar{x}) \times \Gamma \setminus G\}$.

**Corollary 4.2.** Fix $L > 0$, $\lambda > 1$ and $\varepsilon > 0$ sufficiently small. Then for any $\bar{x} \in \Gamma \setminus G$, there exists $C_0(\varepsilon, L, \lambda, \varphi') > 0$ such that $\lambda^{-NR}/C_0 \leq \mu(V^{(\varepsilon)}_{\{L\lambda^k\}, N}(\bar{x})) \leq C_0 \lambda^{-NR}$, where $R = \sum_{i=1}^{l} \binom{m_i}{2} + \sum_{i=l+1}^{n} 2\binom{m_i}{2}$. 

Proof. Due to \( V_{L\lambda}^{(e)}(\bar{x}) \subset V_{[L\lambda^k]}^{(e)}(\bar{x}) \), the lower bound of the measure of \( V_{[L\lambda^k],N}^{(e)}(\bar{x}) \) will follow from (12).

The upper bound will follow if we show that \( V_{[L\lambda^k],N}^{(\delta)}(\bar{x}) \subset V_{L\lambda}^{(e)}(\bar{x}) \) for some \( \delta, \varepsilon > 0 \) independent of \( N \). We prove this inclusion by induction on \( N \). For the base step, notice that for any \( \varepsilon > 0 \) we may choose \( \delta > 0 \) so that this holds simply by continuity of \( \varphi^L \). For the inductive step, we suppose the statement is true for \( n \).

Recall that \( V_{L\lambda}^{(e)}(\bar{x}) = \{ \bar{x} \exp(X) : X \in B_{L\lambda}(0, \varepsilon) \} \) and \( X \in B_{L\lambda}(0, \varepsilon) \) implies that \( \| \Ad(\exp(tU))X \|_\infty < \varepsilon \) for \( 0 \leq t \leq L\lambda n \). Combining the facts that the entries of \( \Ad(\exp(tU))X \) are polynomials and Lemma 2.7, we have \( \| \Ad(\exp(tU))X \|_\infty < C_3 \varepsilon \) for \( 0 \leq t \leq L\lambda^{n+1} \), where \( C_3 \) is a global constant. This implies that if \( \bar{y} \in V_{L\lambda}^{(e)}(\bar{x}) \), then \( y \in V_{L\lambda}^{(\delta)}(\bar{x}) \) for \( \varepsilon \) small enough, where \( x \) and \( y \) are lifts chosen to minimize the distance. By inductive assumption, we have \( V_{[L\lambda^k],n}^{(\delta)}(\bar{x}) \subset V_{L\lambda}^{(e)}(\bar{x}) \) and thus above implies for two points in \( V_{[L\lambda^k],n}^{(\delta)}(\bar{x}) \), their lifts (which minimize the distance at \( t = 0 \)) will also stay \( C_3 \varepsilon \) close in the universal cover up to time \( L\lambda^{n+1} \) if \( \varepsilon < \frac{\inf(G \setminus \Gamma)}{C_3} \).

Let \( B_{n}^{(\varepsilon)}(L\lambda^k) \) denote the set of all \( v \in \mathbb{R}^N \) such that \( ||v_t||_\infty < \varepsilon \) for all \( t \in \{ L, L\lambda, \ldots, L\lambda^n \} \), where \( v_t = \Ad(\exp(tU))v \). Due to points in \( V_{[L\lambda^k],n}^{(\delta)}(\bar{x}) \) are also \( C_3 \varepsilon \) close in the universal cover up to time \( L\lambda^{n+1} \) for lifts which minimize the distance at \( t = 0 \), this implies that \( V_{[L\lambda^k],n+1}^{(\delta)}(\bar{x}) = \{ \bar{x} \exp(X) : X \in B_{n+1}^{(\varepsilon)}(L\lambda^k) \} \). By Lemma 4.1 and same reasoning as to get (10), we obtain

\[
B_{n+1}^{(\varepsilon)}(L\lambda^k)(0, \delta) \subset \bigcap_{j=0}^{m-1} \{ -C(\lambda)\lambda^{-j(n+1)} \delta, C(\lambda)\lambda^{-j(n+1)} \delta \}.
\]

The (11) implies that if \( \delta = \frac{\varepsilon}{C(\lambda)} \), we will have \( B_{n+1}^{(\varepsilon)}(L\lambda^k)(0, \delta) \subset B_{L\lambda^{n+1}}(0, \varepsilon) \) and thus \( V_{[L\lambda^k],n+1}^{(\delta)}(\bar{x}) \subset V_{L\lambda^{n+1}}^{(e)}(\bar{x}) \), which finishes the induction. \( \square \)

Proof of Theorem 1.12. With Corollary 4.2 in hand, the computation for topological sequence entropy follows from an argument virtually identical to that of the proof of Theorems 1.10 and 1.11. We give a proof here for completeness.

By Corollary 4.2, at most \( C_0 \cdot \lambda^{NR} \) disjoint sets of the form \( V_{[L\lambda^k],N}(x_i) \) can be placed in \( \Gamma \setminus G \), giving the upper that the topological sequence entropy is at most \( R \log \lambda \). Similarly, one needs at least \( (1/C_0) \cdot \lambda^{NR} \) such sets to cover the space, which gives that the topological sequence entropy is at least \( R \log \lambda \).

We proceed to the proof in the metric category. In order to simplify the notation, we denote \( H = \varphi^L \). By considering a family of partitions \( \{ P_\alpha \} \) whose atoms are small hypercubes in the coordinates given by the chain structure and whose diameters converge to zero as they index \( \alpha \) diverges to infinity. We pick \( \alpha \) large enough to guarantee that the diameter of each atom is less than \( \varepsilon/2 \) and we write \( P = P_\alpha \) to simplify notation. Let us compute the measure of atoms of \( P^{-n} = \bigcup_{k=0}^{n} H^{-k} P \). Notice that if two points \( \bar{x}, \bar{y} \) belong to the same atom of \( P^{-n} \), they stay close for iterations 1, \( \lambda, \ldots, \lambda^n \) of \( H \), which is equivalent to \( \bar{y} \) belongs to \( V_{[L\lambda^k],n}^{(\delta)}(\bar{x}) \). Thus the maximum of measures of \( P^{-n} \)'s atoms should be no larger than the supremum of \( \mu(V_{[L\lambda^k],n}^{(\delta)}(\bar{x})) \) over all \( \bar{x} \). Then by Corollary 4.2, we have
\[
\mu(V_{(L\lambda^k),n}(\bar{x})) \leq C_0 \lambda^{-nR}.
\]

This immediately yields that \(h_{A,\mu}(\phi') \geq R \log \lambda\).

For the upper bound, by the Theorem 3.1 in [Go], we have \(h_{A,\mu}(\phi') \leq h_{A,\text{top}}(\phi') = R \log \lambda\). \(\Box\)

## 5. Slow Entropy of Flows on Non-compact Spaces

Although in previous sections we only deal with the cocompact lattice, our method in fact can be applied to non-cocompact lattice case by following arguments.

### 5.1. Hamming balls estimates in noncompact homogeneous spaces

In this section, we suppose that the space \(\Gamma \backslash G\) is not compact. Let \(K \subset \Gamma \backslash G\) be such that \(\mu(K) > 1 - \delta\) for some \(\delta > 0\). Given \(\epsilon > 0\), choose a partition \(\mathcal{P}_\epsilon\) of \(\Gamma \backslash G\) such that \(K^c\) is one atom of partition and the diameter of the remaining atoms is less than \(\epsilon\). In the following proposition, the flow \(\phi^t\) is the lift of \(\phi^t\) to the universal cover.

**Proposition 5.1.** Let \(K\) be a compact subset of \(\Gamma \backslash G\) such that \(\mu(K) > 1 - \delta\) for some \(1/100 > \delta > 0\) and let \(\eta = \text{inj}(K)\). Choose a partition \(\mathcal{P} := \mathcal{P}_{c \phi^t \eta}\) as above, where \(0 < c_\phi < 1\) is as in Lemma 3.5 applied to the flow on the universal cover. Then there exists \(T_0, \epsilon_0 > 0\) and a set \(L \subset K\) such that \(\mu(L) > 1 - 2\delta\) such that the following property holds true: for \(0 < \epsilon < \epsilon_0\), \(\bar{x} \in L\) and \(\bar{y} \notin \bar{x}C_G(U)\), if \(\bar{d}_{\phi^t \mathcal{P}}(\bar{x}, \bar{y}) < \epsilon\), then \(\phi^t(\bar{y}) \in V^{(n)}(\phi^t(\bar{x}))\), where \(0 \leq t \leq \frac{7}{10}T\) and \(x^T, y^T\) are lifts of \(\phi^t \bar{x}, \phi^t \bar{y}\) minimizing the distance in the universal cover.

**Proof.** Let \(L_0(T_0, \xi) \subset X\) denote the set of all points \(\bar{x} \in \Gamma \backslash G\) such that

\[
\{t \in [0, T]: \phi^t(\bar{x}) \in K\} \geq (1 - \xi)\mu(K)T,
\]

for all \(T \geq T_0\). Then by the ergodic theorem, for any \(\xi > 0\), we may find \(T_0\) sufficiently large such that \(\mu(L(T_0, \xi)) > 1 - \delta\). We will specify \(\eta\) later, and take \(L = L_0 \cap K\), so that \(\mu(L) \geq 1 - 2\delta\).

Let \(T \geq T_0\), \(\bar{x} \in L\), \(\bar{y}\) be such that \(\bar{d}_{\phi^t \mathcal{P}}(\bar{x}, \bar{y}) < \epsilon\) and \(x_t, y_t \in G\) be lifts of \(\phi^t(\bar{x}), \phi^t(\bar{y})\) to \(G\) such that \(d_G(x_t, y_t) = d_G(\phi^t(\bar{x}), \phi^t(\bar{y}))\). Divide the interval \([0, T]\) into subintervals by following method. We inductively define a sequence \(S_i\) and \(T_i\) as follows. Let \(T_0 = 0\), and

\[
S_i = \min\{t \geq T_i: \phi^t \bar{x} \in K \text{ and } \phi^t \bar{y} \in V^{(n)}(\phi^t \bar{x})\},
\]

\[
T_{i+1} = \min\{t \geq S_i: \phi^{t-S_i} \bar{y} \notin V^{(n)}(\phi^{t-S_i} \bar{x})\}.
\]

Note that \(S_0 < \infty\), since \(\bar{x} \in L\) implies that on a large measure subset of \([0, T]\) the orbit of \(\bar{x}\) returns to \(K\), and \(\bar{d}_{\phi^t \mathcal{P}}(\bar{x}, \bar{y}) < \epsilon\) implies that \(\bar{x}\) and \(\bar{y}\) belong to the same atom of \(\mathcal{P}\) (and hence are \(c_{\phi^t \eta}\)-close) on a large measure subset of \([0, T]\). Then by Lemma 3.5 for universal cover, we have the following claim.
Claim 5.2. For any \( i \geq 1 \) such that \( T_i \leq T \), we may choose \( C_1 > 0 \) (as in the definition of \( \varepsilon_0 \)) independent of \( i \) such that for set \( B_i = \{ t \in [S_i-1, T_i] : \Phi^{t-S_{i-1}} y_{S_i-1} \notin V(\alpha')(\Phi^{t-S_{i-1}} x_{S_i-1}) \} \), where \( x_{S_i-1}, y_{S_i-1} \) are the lifts of \( \phi^{S_{i-1}} \bar{x}, \phi^{S_{i-1}} \bar{y} \) minimize the distance at time \( S_{i-1} \) in the universal cover, then we have

\[
|B_i| \geq \frac{9}{10} (T_i - S_{i-1}).
\]

Now denote \( A = \{ t \in [0, T] : \phi^t \bar{x} \in K \text{ and } \phi^t \bar{y} \in V(\alpha'(\Phi^t \bar{x})) \} \). Recall that the atoms (other than \( K^c \)) of \( \mathcal{P}_{c \alpha} \) have diameter less than \( c_{\phi \eta} \). Therefore, by choosing \( \varepsilon_0 \) (which bounds \( d_{\phi, \mathcal{P}}(\bar{x}, \bar{y}) \)) and \( \xi \) (as in the definition of \( L \)) small enough, we can guarantee \( |A| \geq \frac{9}{10} T \). If \( t \in (T_i, S_i) \), either \( \phi^t \bar{x} \notin K \) or \( \phi^t \bar{y} \notin V(\alpha'(\Phi^t \bar{x})) \). In particular, if \( S_i < T \)

\[
(T_i, S_i) \cap A = \emptyset,
\]

and thus

\[
\left| \bigcup_i (T_i, S_i) \right| < \frac{1}{10} T.
\]

From the Claim 5.2, we know that whenever \( T_i < T \):

\[
\left| \bigcup_i B_i \right| \geq \frac{9}{10} \left| \bigcup_i [S_i, T_i] \right|.
\]

We claim that \( A \cap B_i = \emptyset \) for every \( i \). Indeed, if \( t \in A \cap B_i \), then

\[
\Phi^{t-S_i} y_{S_i} \notin V(\alpha'(\Phi^{t-S_i} x_{S_i})), \quad \Phi^{t-S_i} y_{S_i} \in V(\eta)(\Phi^{t-S_i} x_{S_i}).
\]

Therefore, \( \phi^t \bar{y} \notin V(\alpha'(\Phi^t \bar{x})) \) and thus contradicts to the definition of \( t \in A \). Then we have following inequality,

\[
|A| \leq |A \cap \bigcup_i B_i| + |A \cap \bigcup_i ([S_i-1, T_i] \setminus B_i)| + |A \cap \bigcup_i (T_i, S_i)| + |A \cap I|.
\]

(21)

where \( I \) is either \([T_{i_0}, T]\) or \([S_{i_0}, T]\). Recall that the first and third term on the right equal to 0 (due to (19) and paragraph after (20)) and second term is less than \( \frac{4}{10} T \) (due to (20)), thus we know \( |A \cap I| \geq \frac{7}{10} T \). In particular, \( I \) takes the form \([S_{i_0}, T]\) (since \( A \cap (T_i, S_i) = \emptyset \) for every \( i \)) and has length at least \( 7/10 \). By setting \([a, b] = [S_{i_0}, T]\), we conclude the proposition. \( \square \)
5.2. Proof of Theorems 1.10 and 1.11: noncompact case. We show that the measure-theoretic polynomial entropy is at least $R$ and the topological polynomial entropy is at most $R$. Then by Theorem 3.3 (which can be applied by Remark 3.2 and $\Gamma \backslash G$ is a smooth manifold (Theorem 21.13 in [Lee]), we know $\Gamma \backslash G$ is well-partitionable.), we conclude the main theorem. We first show the lower bound. By Proposition 5.1, we know that the Hamming ball with radius $\epsilon$, center $\bar{x}$ and partition $\mathcal{P}$ will be contained in

$$\varphi^{-\frac{3}{10}T} \pi \left( \bigcup_{x' \in \pi^{-1}(\varphi^{\frac{3}{10}\bar{x}})} V^{(\eta)}_{7T/10}(x') \right).$$

Recall that $\Gamma$ acts on left, quasi-unipotent flow acts on right and our metric is left invariant, thus for any $x_1, x_2 \in \pi^{-1}(\varphi^{\frac{3}{10}\bar{x}})$, we have

$$\pi(V^{(\eta)}_{7T/10}(x_1)) = \pi(V^{(\eta)}_{7T/10}(x_2)).$$

Hence since $\text{inj}(x) = \infty$ for the flow $\Phi$ on the universal cover, by (12) applied to $\Phi$:

$$\mu(B^T_{\nu,\mathcal{P}}(\bar{x}, \epsilon/2)) \leq (C/\delta)(7T/10)^{-R}.$$

Given any cover of a large measure subset of $L$ by Hamming balls, we can replace the centers with points of $L$, which will still cover $L$ if we double the radius $\epsilon/2$ to $\epsilon$. So we need at least $CT^R(\epsilon/2, T)$-Hamming balls to cover $L$.

Now we turn to the topological category. Notice that the lower bound of (12) applies without the assumption on $\text{inj}(\Gamma \backslash G)$. So as in the proof of the compact case, we get the upper bound on slow entropy is $T^R$.

6. Computation of Slow Entropy for Quasi-unipotent Flows

We give a way of practically calculating the slow entropy of $U$ in the case of a semisimple group. If $U = U' + Q \in \mathfrak{g}$ is an ad-quasi-unipotent element of a semisimple Lie algebra, an $\mathfrak{sl}(2, \mathbb{R})$-triple is a triple $(V, X, U')$ such that:

$$[X, U'] = 2U', \quad [X, V] = -2V, \quad [U', V] = X.$$

The well-known Jacobson–Morozov theorem ensures the existence of $\mathfrak{sl}(2, \mathbb{R})$-triples for arbitrary unipotent elements. Let $G$ be semisimple, $U' \in \mathfrak{g}$ be unipotent, $C(U')$ be the centralizer of $U'$ in $\mathfrak{g}$ and $(V, X, U')$ be an $\mathfrak{sl}(2, \mathbb{R})$-triple associated to $U'$. Then $\text{ad}_X(C(U')) = C(U')$, and all eigenvalues of $\text{ad}_X$ on $C(U') \subset \mathfrak{g}$ are non-negative integers. Let $d_n$ denote the dimension of the eigenspace for the eigenvalue $n$.

**Corollary 6.1.** The topological and metric slow entropy of the quasi-unipotent flow induced by $U$ with respect to the polynomial family is $R$, where:

$$R = \sum_{n=0}^{\infty} d_n \cdot \binom{n+1}{2}.$$
Proof. Let $G$, $U$ and $(V, X, U)$ be as in the discussion preceding the statement. The bracket relations on $V$, $X$ and $U$ show that they correspond to an algebra homomorphism $\varphi : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}$ defined via:

$$\varphi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = U, \quad \varphi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = X, \quad \varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = V.$$ 

This defines a representation of $\mathfrak{sl}(2, \mathbb{R})$ by sending $Y \mapsto \text{ad}_Y : \mathfrak{g} \to \mathfrak{g}$. Since $\mathfrak{sl}(2, \mathbb{R})$ is semisimple, this representation can be decomposed into a sum of irreducible ones. The finite-dimensional irreducible representations of $\mathfrak{sl}(2, \mathbb{R})$ are well-classified, and are indexed by $\mathbb{N}_0$. If $n \in \mathbb{N}_0$, let $V_n$ be the space spanned by $\{X_0, \ldots, X_n\}$, so that $\dim(V_n) = n + 1$. Then the action of $\mathfrak{sl}(2, \mathbb{R})$ is given by:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot X_j = c_{n,j} X_{j+1}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot X_j = (2j-n)X_j,$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot X_j = c'_{n,j} X_{j-1},$$

for some nonzero universal constants $c_{n,j}$ and $c'_{n,j}$ and we make the convention that $X_{-1} = 0$ and $X_{n+1} = 0$, where the existence of $c_{n,j}$ and $c'_{n,j}$ are a straightforward consequence of the existence of chain structure. In particular, each $V_n$ gives a Jordan block of length $n + 1$ for $U$. Note that if $[U, Y] = 0$, we may decompose $Y$ via a decomposition into the irreducible sub-representations. Thus, the centralizer of $U$ is exactly the sum of the centralizers in each irreducible subspace. But only the $X_n$ terms in each irreducible subspace commute with $U$, and here the eigenvalue of $X$ is $n$. So for each eigenvector of $X$ with eigenvalue $n$, there is a corresponding chain of length $n + 1$, and this gives a basis. Applying Theorems 1.10 and 1.11, we obtain the Corollary. □

We get the following corollary of Theorems 1.10, 1.11 and Corollary 6.1:

Corollary 6.2. The topological and metric slow entropy of Examples 1.6, 1.7 and 1.8 give rate $T^R$, where:

(i) $R = \sum_{i=1}^{m} \frac{1}{6} k_i (4k_i + 1)(k_i - 1) + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \frac{1}{6} k_i \left( k_i^2 + 3k_j^2 - 3k_j - 1 \right)$, where $(k_1, \ldots, k_m)$, $k_i \leq k_{i+1}$ is the block sequence of $U$;

(ii) $R = (\ell_0^2)$;

(iii) $R = R_{ss} + R_p$, where $R_{ss}$ is the rate coming from Part (6.2) and $R_p = \sum \ell_i (\ell_i - 1)/2$.

Here, $\ell_i$ are the lengths of the Jordan blocks of $d \rho(U)$ acting on $\mathbb{R}^N$.

Proof. Calculation for Example 1.6

Let us begin by assuming that we have a single block in $SL(d, \mathbb{R})$ (in this case the nilpotent element $U$ is sometimes called principal). According to Corollary 6.1, we may identify an $\mathfrak{sl}(2, \mathbb{R})$ triple and the centralizer of $U$. Direct computation shows an $\mathfrak{sl}(2, \mathbb{R})$ triple can be constructed by taking:

$$X = \begin{pmatrix} d - 1 & d - 3 & \cdots & \cdots & \cdots \\ d - 3 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ d - 1 & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad V = \begin{pmatrix} 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$
Let $E_{i,j}$ denote the matrix whose entries are all zero, except for the $(i, j)$th entry which is 1. Since the eigenvalues of $X$ on the centralizer of $U$ must be positive integers, know that the centralizer lies completely in the upper triangular matrices and any element must be in the sums of $W_n = \bigoplus_{i=0}^{d-k} \mathbb{R} E_{i,i+k}$ for a fixed $k \geq 0$ (since these are the eigenspaces of $X$). From direct computation one sees that each such subspace, there is a unique element (up to scaling) commuting with $U$ (it is exactly $\sum_{i=1}^{d-k} E_{i,i+k}$). Hence, the eigenvalues of $\text{ad}_X$ on $C(U)$ are all simple, and equal to $2, 4, \ldots, 2d - 2$. So by Corollary 6.1, the topological slow entropy is:

$$\sum_{k=1}^{d-1} 2k(2k + 1)/2 = \frac{d}{6}(d - 1)(4d + 1).$$

This shows the result for a single block. To consider multiple blocks, note that there exists a $\mathfrak{sl}(2, \mathbb{R})$ triple respecting the block form of $U$, by taking the block forms of $X$ and $V$. So in each $k_i \times k_j$ block, we may apply our analysis from the previous sections for elements of the centralizer in each block. This accounts for the first sum in the expression. Each off-diagonal block is preserved by the $\mathfrak{sl}(2, \mathbb{R})$ triple, so we will find a basis of the centralizer by considering each block independently. Furthermore, since the upper triangular blocks are dual to the lower triangular blocks and all $\mathfrak{sl}(2, \mathbb{R})$ representations are self-dual, we may consider only the upper triangular ones and double the result. Again, an element of the centralizer contributing to the slow entropy must lie in a positive eigenspace of $X$, so if we consider a block of $k_i \times k_j$ matrices (we suppose that $k_j \geq k_i$), elements of the centralizer must lie in $\bigoplus_{a=1}^{k_j-b} \mathbb{R} E_{a,a+b}$ for a fixed $0 \leq b \leq k_j - 1$. Direct computation shows that an element of the centralizer exists in such a subspace (and is unique up to scalar) if and only if $k_j - k_i \leq b \leq k_j - 1$ (and is given by $\sum_{a=1}^{k_j-b} E_{a,a+b}$). The eigenvalue of $X$ on such a subspace is given by $\{k_j - k_i + 2\ell : 0 \leq \ell \leq k_i - 1\}

Hence each block, indexed by $1 \leq i < j \leq m$, we have a contribution to slow entropy equal to:

$$\sum_{\ell=0}^{k_i-1} (k_j - k_i + 2\ell)(k_j - k_i + 2\ell + 1)/2 = \frac{1}{6} k_i \left(k_i^2 + 3k_j^2 - 3k_j - 1\right).$$

Recalling that we must double the contribution for the lower-triangular blocks, we get the formula.

**Calculation for Example 1.7**

In this case it is easy to find a basis consisting of two chains, since one may check that $\text{ad}_U^d = 0$ but $\text{ad}_U^{d-1} \neq 0$. As a result, there is at least one chain of dimension $d$. But since $\dim(\mathfrak{n}) = d + 1$, the only possibility is to have one chain of length $d$ and another of length one. This gives the formula immediately.

**Calculation of Example 1.8**

Observe that because $U \in \mathfrak{sl}(d, \mathbb{R})$, $\text{ad}_U$ preserves the subspaces $\mathfrak{sl}(d, \mathbb{R})$ and $\mathbb{R}^N$. We can hence find a basis subordinate to this splitting putting $\text{ad}_U$ in Jordan normal form. We may use the calculations for Example 1.6 to get the first part. Note that $\text{ad}_U$ acts on $\mathbb{R}^N$ as $d\rho(U)$, a nilpotent matrix. This gives the formula. \(\square\)

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A. Appendix: Failure of the Variational Principle

We now show the failure of the variational principal in a general setting. We do so even in the smooth category. We first show some certain characterizations of transformations with zero entropy at every scale $a$.

A.1. Characterization of Kronecker systems. We recall the following result from [Fe]:

**Proposition A.1** [Fe]. A measure-preserving transformation or flow $f$ has $h_{\mu,a_x}(f) = 0$ with respect to every family of scales $a_x$ if and only if $f$ is measurably conjugate to an action by translations on a compact abelian group.

We prove an analogue in the topological category, namely:

**Proposition A.2.** A minimal homeomorphism $f : X \to X$ of a compact metric space has $h_{\text{top},a_x}(f) = 0$ for every family of scales $a_x$ if and only if $f$ is topologically conjugate to a translation on a compact abelian group.

We use the following folklore characterization:

**Lemma A.3.** Let $f : X \to X$ be a transitive homeomorphism or a transitive flow of a compact metric space. Then $f$ is topologically conjugate to translations on a compact abelian group if and only if $\{f^t\}_{t \in \mathbb{R}}$ is uniformly equicontinuous.

**Proof of Proposition A.2.** First, we show that if $f$ is topologically conjugate to a translation on a compact abelian group, it has zero topological slow entropy at all scales. If $f$ is topologically conjugate to a translation of a compact abelian group, we are free to use the bi-invariant metric for which $f$ is an isometry. Hence, the Bowen balls are equal to the standard balls for arbitrary $n$. In particular, $N_{f,X}(\varepsilon,T)$ is independent of $T$ for any $\varepsilon$. Since $a(T)$ must tend to $\infty$, we get the result.

For the converse, first fix $\varepsilon > 0$. That is, we show that if we have zero entropy at all scales, then $f$ is topologically conjugate to a translation on a compact abelian group. Since $f$ is assumed to have 0 entropy at all scales, $N_{f,X}(\varepsilon,T) \leq N_0$ for some fixed $N_0 = N_0(\varepsilon)$ and all $T > 0$. Let $Y_t(\varepsilon) \subset X^{N_0}$ be the set of $N_0$-tuples such that $(x_i) \in Y_t$ if and only if $\bigcup_i B_{f^t}(x_i,\varepsilon) = X$. Then $Y_t(\varepsilon)$ is nonempty for every $t$ and $Y_t(\varepsilon) \subset Y_s(\varepsilon)$ for $s < t$, so $\bigcap_{t > 0} Y_t(\varepsilon/2) \neq \emptyset$.

We claim that $Y_t(\varepsilon/2) \subset Y_t(\varepsilon)$. Indeed, fix $t$ and suppose that $(x_i^k) \in Y_t(\varepsilon/2)$ is a sequence of $N_0$-tuples converging as $k \to \infty$ to a $N_0$ tuple $(x_i)$. The relation $(x_i^k) \in Y_t(\varepsilon/2)$ is equivalent to the property that if $y \in X$, there exists some $i$ such that $d(f^t(x_i^k), f^t(y)) < \varepsilon/2$ for every $0 \leq t' \leq t$. Since $d$ and $f^{t'}$ are continuous for each $t'$, $d(f^{t'}(x_i), f^{t'}(y)) \leq \varepsilon/2 < \varepsilon$. This proves the claim.

We may thus find an $N_0$-tuple $(x_i)$ so that $\bigcup_i B_f(x_i,\varepsilon) = X$ for every $t$. Since the sets $B_f^t(x_i,\varepsilon)$ are nested with respect to $t$, and each point must lie in such a neighborhood for every $t$, the sets $B_f^\infty(x_i,\varepsilon) = \bigcap_{t > 0} B_f^t(x_i,\varepsilon)$ still cover $X$. They are closed, but...
since finitely many cover the space $X$, at least one must have nonempty interior. That is, for some $z \in Z$ and $0 < \gamma < \varepsilon/2$, we have $B(z, \gamma) \subset \overline{B^t_f(x_i, \varepsilon)} \subset B^t_f(x_i, 2\varepsilon)$ for every $t > 0$.

Fix $z \in X$. Since $f$ is assumed to be minimal, for every $\gamma > 0$ there exists a $T_\gamma > 0$ such that $\bigcup_{t=0}^{T_\gamma} f^{-t}z$ is $\gamma$-dense ($T_\gamma$ depends on $z$, but since $z$ is an apriori fixed point, we drop it from the notation). Equivalently, $\bigcup_{t=0}^{T_\gamma} f^{-t}(B(z, \gamma)) = X$. This means that for every $x \in X$ there exists $T_x \in [0, T_\gamma]$ such that $f^{T_x}(x) \in B(z, \gamma)$. Notice that the family $\{f^t\}_{t \in [0, T_\gamma]}$ is compact (by compactness of $[0, T_\gamma]$). Since any compact family of homeomorphisms of a compact space is uniformly equicontinuous, we may choose $\delta > 0$ so that $f^t(B(x, \delta)) \subset B(f^t(x), \gamma)$ for $t \leq T_\gamma$.

Now, given any $x \in X$, by choice of $T$, we may find $t \leq T$ such that $d(f^t(x), z) < \gamma$. Furthermore, by choice of $\delta$, we conclude that $f^t(B(x, \delta)) \subset B(f^t(x), \gamma) \subset B(z, 2\gamma) \subset B^s_f(x_i, \varepsilon)$ for every $s > 0$. Then $f^{t+s}(B(x, \delta)) \subset B(f^s(x_i), \varepsilon)$, and if $d(x, y) < \delta$, $d(f^t(x), f^t(y)) < 2\varepsilon$ for every $t \geq 0$. Since $\varepsilon$ was arbitrary, we conclude that $\{f^t\}$ is uniformly equicontinuous, and the result. \(\square\)

A.2. Failure of the variational principle. If $f : X \to X$ is a homeomorphism of a compact metric space, let $\mathcal{M}(f)$ denote the space of invariant measures. Recall that a system is Kronecker if it has pure-point spectrum. The results of the previous section show that if we can find a minimal topological system $f : X \to X$ such that $\mathcal{M}(f)$ is finite dimensional and every $\mu \in \mathcal{M}(f)$ is Kronecker, but which is not topologically conjugate to a translation on a compact abelian group, then the variational principle for the usual entropy theory will fail for slow entropy. Such systems can be found in non-standard realization theory, and the approximation-by-conjugation method first used by Anosov and Katok in [AK]. We document the conclusion here:

**Proposition A.4.** Let $M$ be a manifold with a free circle action. Then there exists a uniquely ergodic, volume preserving, $C^\infty$ diffeomorphism $f : M \to M$ which is measurably conjugate to a translation on a torus $\mathbb{T}^d$, $d \geq 1$.

Since a toral translation has entropy zero at all scales and since in the above example we may assume that the diffeomorphism is not uniformly equicontinuous, we obtain from the previous result the following corollary:

**Corollary A.5.** Let $M$ be a manifold with a free circle action. Then there exists a $C^\infty$ diffeomorphism $f : M \to M$ and family of scales $\{a_\chi\}$ such that:

\[ \sup_{\mu \in \mathcal{M}(f)} h_{\mu, a_\chi}(f) < h_{\text{top}, a_\chi}(f). \]

A similar example can be found by using an example due to Furstenberg [Fur], found in [Kat-Has]:

**Proposition A.6.** There exists a minimal $C^\infty$ diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ which is measurably conjugate to $(x, y) \mapsto (x + \alpha, y)$ for some $\alpha \in S^1$. In particular,

\[ \sup_{\mu \in \mathcal{M}(f)} h_{\mu, a_\chi}(f) < h_{\text{top}, a_\chi}(f). \]

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3 It is not enough to say that every $\mu \in \mathcal{M}(f)$ is Kronecker, since $(x, y) \mapsto (x, y + x)$ has this property but nontrivial slow entropy.
Another interesting class we note are the *Sturmian sequences*. These are zero entropy closed subshifts on two symbols which are uniquely ergodic, and whose unique measure gives a transformation measurably isomorphic to a circle rotation. They can be obtained in the smooth category by taking the non-wandering set of the $C^1$ Denjoy examples of a nontransitive diffeomorphism. Again, because the only translations on compact abelian groups which are Cantor sets are the odometers, we get the following result:

**Proposition A.7.** Any Sturmian subshift $\sigma$ preserving an invariant measure $\mu$ satisfies

$$0 = h_{\mu,a_\chi}(\sigma) < h_{\text{top},a_\chi}(\sigma),$$

for some family of scales $\{a_\chi\}$.

These examples motivate the following questions:

**Question A.8.** For which continuous transformations $T : X \to X$ the variational principle

$$\sup_{\mu \in \mathcal{M}(T)} h_{\mu,a_\chi} = h_{\text{top},a_\chi},$$

holds true for every family of scales $a_\chi$?

Another observation on these examples is the following: for a given Kronecker system, one may find an “ideal” realization which links the measurable orbit growth structure with the topological orbit growth structure. In particular, we ask the following question:

**Question A.9.** Which measure preserving transformations $T$ of a probability space admit, as a measurable factor, a (unique) topological system $(Y, S)$ such that the pushforward measure has full support in $Y$ and the equality of entropies $h_{\mu,a_\chi}(T) = h_{\text{top},a_\chi}(S)$ holds at all scales $a_\chi$?

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