Schwarzian derivative in higher-order Riccati equations

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Abstract. The Sturm–Liouville equation represents the linearised form of the first-order Riccati equation. This provides an evidence for the connection between Schwarzian derivative and this first-order nonlinear differential equation. Similar connection is not obvious for higher-order equations in the Riccati chain because the corresponding linear equations are of order greater than two. With special attention to the second- and third-order Riccati equations we demonstrate that Schwarzian derivative has a natural space in higher Riccati equations. There exist higher-order analogues of the Schwarzian derivative. We demonstrate that equations in the Riccati hierarchy are embedded in these higher-order derivatives.

Keywords. higher-order Riccati equations; linearised form; Schwarzian derivative; higher-order analogue.

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1. Introduction

The Riccati equation given by [1]
\[ \frac{d\omega(x)}{dx} + \alpha(x) + \beta(x)\omega(x) + \omega(x)^2 = 0 \]  
(1)
with \( \alpha(x) \) and \( \beta(x) \) as the arbitrary real valued functions of \( x \) is the only nonlinear differential equation that is of Painlevé type [2]. It appears in many different fields of physics and mathematics [3–5] and can be linearised using the Cole–Hopf transformation [6,7]

\[ \omega(x) = \frac{\phi'(x)}{\phi(x)} \]  
(2)
to write

\[ \phi''(x) + \beta(x)\phi'(x) + \alpha(x)\phi(x) = 0. \]  
(3)
Riccati equation (1) is of first order. Higher-order equations forming the so-called Riccati chain can be obtained by reduction from the matrix Riccati equation [8]. In close analogy with the first-order equation (eq. (1)), all higher-order equations in the Riccati chain can also be linearised via Cole–Hopf transformation.

Studies in symmetries and integrability properties of chains of nonlinear differential equations have a deep root in the physico-mathematical literature [9–11]. Interestingly, using symmetry methods, a non-local transformation was sought to linearise equations not only of the Riccati but also of Abel chain [12–14]. The proposed symmetry-based transformation leads to rather simpler linear differential equations than those provided by the method of Cole and Hopf [6,7].

Our objective in this work is to prove the existence of Schwarzian derivative (SD) [15] in equations of the Riccati chain in general. Admittedly, our approach to the problem will depend crucially on the linearised form of the nonlinear eq. under consideration. In the following, we introduce the apparently intimidating expression for the Schwarzian derivative and present some of its properties. Here we also demonstrate how the linearised form of the Riccati equation plays a role in looking for its connection with the Schwarzian derivative.

Let \( f(x) \) represent a well-behaved function of a single variable \( x \) (real or complex) and primes over it represent the appropriate derivatives with respect to \( x \), then \( S(f(x)) \) written as

\[ S(f(x)) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2 \]  
(4)
is called the Schwarzian derivative of \( f(x) \) or Schwarzian in short [16,17]. Equation (4) arises in a wide variety of mathematical and physical contexts.
ranging from classical complex analysis to conformal field theory [18].

In the theory of complex variables, the linear fractional or Möbius transformation given by
\[ f(x) = \frac{ax + b}{cx + d} \]  
possesses a number of remarkable properties. In particular, the transformation eq. (5) with real coefficients forms a group of symmetries in the real projective space [15,19]. The Schwarzian derivative of \( f(x) \) for the above particular choice is zero. Thus, for any arbitrary function, \( S(f(x)) \) measures how much \( f \) differs from being a Möbius transformation. An important property of the Schwarzian is that \( S(f) = S(g) \) if and only if \( g(x) = (af(x) + b)/(cf(x) + d) \) with \( ad - bc \neq 0 \).

To compute SD for the Riccati equation, let us write \( f(x) = \phi_1(x)/\phi_2(x) \) where \( \phi_1(x) \) and \( \phi_2(x) \) represent two linearly independent solutions of eq. (3), the linearised form of the nonlinear eq. (1). Making use of eq. (3), the SD for the Riccati equation can be obtained as
\[ S(f(x)) = \frac{1}{2} \beta(x)^2 + 2\alpha(x) - \beta'(x). \]  
This shows that further studies in the linearisation of nonlinear equations using symmetry methods [12–14] are expected to play a role in looking for the existence of SD in other physically important equations. For \( \beta(x) = 0 \), eq. (3) corresponds to time-independent Schrödinger equation with \( \alpha(x) \) as the potential function. Using \( \beta(x) = \beta'(x) = 0 \) in eq. (6) we see that the Schwarzian reconstructs the potential. Assuming
\[ \phi(x) = u(x)v(x) \]  
as a complete solution of eq. (3) it is straightforward to write the equation for \( v(x) \) in the normal form
\[ v''(x) + \lambda(x)v(x) = 0, \]  
where \( \lambda(x) = S(f(x))/2 \). This shows that when the first derivative is removed from the Sturm–Liouville equation, the coefficient of the dependent variable is equal to half the Schwarzian derivative. Apparently, this simple fact has remained unnoticed in the literature. We shall exploit this fact to demonstrate the appearance of SD in higher-order equations in the Riccati chain [20]. We shall further show that higher-order SDs [21] lead, in a rather natural way, to a hierarchy of nonlinear equations and examine if these equations could be converted to the Riccati form.

In §2 we introduce the equations in the Riccati chain and point out that each equation in the chain can be reduced to the linear form. We then show that the set of linear equations so obtained provides a basis to visualise how the Schwarzian derivative enters into higher-order Riccati equations. In §3 we introduce the higher-order SDs for the normalised univalent functions as considered in ref. [21]. In a relatively recent publication [22], the first-order Riccati equation was realised in the context of a Schwarzian equation provided by the derivative as introduced in eq. (4). In the same way, we obtain nonlinear equations associated with higher-order SDs and suggest a method to reduce them to the Riccati form. Finally, in §4 we summarise our outlook on the present work and try to make some concluding remarks.

2. Schwarzian derivative in the Riccati chain

In the most general form, the nth order equation in the Riccati chain can be written as [20]
\[ L^n \omega(x) + \sum_{i=1}^{n} \alpha_i(x)(L^{i-1} \omega(x)) + \alpha_0(x) = 0. \]  
Here \( n \) denotes the order of the Riccati equation and \( L \) stands for a differential operator represented by
\[ L = \frac{d}{dx} + c \omega(x). \]  
The transformation eq. (2) re-written as \( \omega(x) = \phi'(x)/c\phi(x) \) converts eq. (9) into a linear ordinary differential equation given by
\[ \sum_{i=0}^{n} \alpha_i(x) \frac{d^i \phi(x)}{dx^i} + \frac{d^{n+1} \phi(x)}{dx^{n+1}} = 0. \]  
The second-order Riccati equation and the corresponding linear equation obtained by using \( n = 2 \) in eqs (9) and (11) are
\[ \omega''(x) + (3cf(x) + \alpha_2(x))\omega'(x) + c\alpha_2(x)\omega(x)^2 
+ c^2\omega(x)^2 + \alpha_1(x)\omega(x) + \alpha_0(x) = 0 \]  
and
\[ \phi^{(3)}(x) + \alpha_2(x)\phi''(x) + \alpha_1(x)\phi'(x) + c\alpha_0(x)\phi(x) = 0. \]  
The space of solutions of eq. (13) is three dimensional spanned by three linearly independent eigenfunctions, say, \( \phi_1(x) \), \( \phi_2(x) \) and \( \phi_3(x) \). This tends to pose some problem to construct the Schwarzian corresponding to eq. (13). Clearly, the same will be true for other equations in the hierarchy. However, the appearance of SD in the second-order Riccati equation can be seen as follows.
Using eq. (7) in eq. (13) we get
\[ v^{(3)}(x) + a_2^3(x)u''(x) + a_3^3(x)u'(x) + a_4^3(x)u(x) = 0, \]
where
\[
\begin{align*}
a_2^3(x) &= \alpha_2(x) + \frac{3u'(x)}{u(x)} , \\
a_3^3(x) &= \alpha_1(x) + \frac{2\alpha_2(x)u'(x)}{u(x)} + \frac{3u''(x)}{u(x)}
\end{align*}
\] (15a)
and
\[
\begin{align*}
a_4^3(x) &= c\alpha_0(x) + \frac{\alpha_1(x)u'(x)}{u(x)} + \frac{\alpha_2(x)u''(x)}{u(x)} \\
&\quad + \frac{u^{(3)}(x)}{u(x)}.
\end{align*}
\] (15c)
We now choose \( u(x) \) such that \( a_2^2(x) = 0 \). This gives
\[
u(x) = e^{-\int f_2(x)dx}.
\] (16)
From eqs (14) and (16) we find
\[ v^{(3)}(x) + b_3^2(x)u'(x) + b_4^2(x)u(x) = 0, \]
where
\[
b_3^2(x) = \alpha_1(x) - \frac{\alpha_2(x)^2}{3} - \alpha_2'(x)
\] (18a)
and
\[
b_4^2(x) = -\frac{\alpha_2(x)^2}{3} - \frac{2\alpha_2(x)^2}{27} - \frac{\alpha_2'(x)}{3}.
\] (18b)
Equation (17) represents a third-order linear differential equation with the second derivative removed and this result has been found from the second-order Riccati equation. It is easy to see that the coefficient \( b_3^2(x) \) of \( u'(x) \) in eq. (17) stands for the Schwarzian derivative of the Sturm–Liouville equation (3) with \( \beta(x) = 2\alpha_2(x)/3 \) and \( \alpha(x) = \alpha_1(x)/3 \). This is how the SD appears in the second-order Riccati equation.

The expression for \( b_3^2(x) \), the coefficient of \( u(x) \) in eq. (17), involves the second-order derivative of \( \alpha_2(x) \) and thus it appears that \( b_2^2(x) \) is unlikely to have any connection with the Schwarzian. We, however, found that the combination \( (b_3^2(x) - b_3^2(x)/3)/\alpha_2(x) \) under the constraint \( \alpha_1'(x) = 3\alpha_0(x) \) represents a Schwarzian related to eq. (3) with \( \beta(x) = 2\alpha_2(x)/3 \) and \( \alpha(x) = \alpha_1(x)/3 \).

From eq. (9) we write the third-order Riccati equation as
\[
\begin{align*}
\omega^{(3)}(x) + (4c\omega(x) + \alpha_3(x))\omega''(x) + (6c^2\omega(x)^2 &+ 3c\alpha_3(x)\omega(x) + \alpha_2(x))\omega'(x) + 3c\omega'(x)^2 \\
&+ c^3\omega(x)^4 + c^2\alpha_3(x)\omega(x)^3 \\
&+ c\alpha_2(x)\omega(x)^2 + \alpha_1(x)\omega(x) + \alpha_0(x) = 0.
\end{align*}
\] (19)
The corresponding linear equation reads as
\[
\phi^{(4)}(x) + \alpha_3(x)\phi^{(3)}(x) \\
+ \alpha_2(x)\phi''(x) + \alpha_1(x)\phi'(x) + c\alpha_0(x) = 0.
\] (20)
When the third derivative term in eq. (20) is removed, we get
\[ v^{(3)}(x) + b_3^3(x)\omega''(x) + b_3^3(x)\omega'(x) + b_3^3(x)\omega(x) = 0, \]
where
\[
b_3^3(x) = \alpha_2(x) - \frac{3\alpha_3(x)^2}{8} - \frac{3\alpha'_3(x)}{2},
\] (22a)
\[
b_4^3(x) = \alpha_1(x) - \frac{\alpha_2(x)\alpha_3(x)}{2} + \frac{\alpha_3(x)^3}{8} - \alpha_3''(x)
\] (22b)
and
\[
b_3^3(x) = c\alpha_0(x) - \frac{\alpha_1(x)\alpha_3(x)}{4} + \frac{\alpha_2(x)\alpha_3(x)^2}{16} - \frac{3\alpha_3(x)^4}{256} - \frac{\alpha_2'(x)\alpha_3(x)}{4} + \frac{3\alpha_3(x)^2}{40} + \frac{3\alpha'_3(x)^2}{16} - \frac{\alpha_3(x)^3}{4}.
\] (22c)
The expression for the coefficient \( b_3^3(x) \) of \( \omega''(x) \) in eq. (21) involves only the first derivative of \( \alpha_3(x) \). Thus, as in the case of the second Riccati equation, the result for \( b_3^3(x) \) represents an SD as in eq. (6) with \( \beta(x) = \alpha_3(x)/2 \) and \( \alpha(x) = \alpha_2(x)/6 \). From \( b_3^3(x) \) and \( b_4^3(x) \) we have verified that under the constraint \( \alpha_2'(x) = 3\alpha_1(x)/2 \) the quantity \( (b_3^3(x) - 2b_3^2(x))/\alpha_3(x) \) is a Schwarzian \( -\alpha_2(x) + \alpha_3(x)^2/4 + \alpha'_3(x) \). The method applied for the second- and third-order equations can easily be adapted to look for the presence of SD in still higher-order Riccati equations.

3. Riccati equations from higher-order Schwarzians

The higher-order Schwarzians are defined inductively [21] such that
\[
s_{n+1}(f) = s_n(f) - (n - 1)\frac{f'''}{f'} s_n(f), \quad n \geq 3
\] (23)
with \( s_3(f) = S(f(x)) \) as given in eq. (4). From eq. (22c) we can easily deduce
\[
s_4(f) = \frac{f^{(4)}}{f'} - 6\frac{f'''}{f'^2} + 6\left(\frac{f''}{f'}\right)^3,
\] (24a)
\[
\sigma_5(f) = \frac{f^{(5)}}{f'} - 10 \frac{f^{(4)} f''}{f'^2} - 6 \left( \frac{f'''}{f'} \right)^2 \\
+ 48 \frac{f'''' f'''^2}{f'^3} - 36 \left( \frac{f'''}{f'} \right)^4,
\]

(24b)

\[
\sigma_6(f) = \frac{f^{(6)}}{f'} - 15 \frac{f^{(5)} f''}{f'^2} - 22 \frac{f^{(4)} f'''}{f'^2} \\
+ 108 \frac{f'''' f'^2}{f'^3} + 132 \frac{f'''^2 f''}{f'^4} \\
- 480 \frac{f''''^2 f'^3}{f'^4} + 288 \left( \frac{f'''}{f'} \right)^5
\]

(24c)

and similar results for still higher-order Schwarzian derivatives.

If we denote the pre-Schwarzian derivative \( f''''(x) / f'(x) \) by \( y(x) \), the Schwarzian equation [22]

\[
S(f(x)) = -g(x)
\]

(25)

provided by \( \sigma_3(f) \) with \( g(x) \) as an external source leads to the nonlinear equation

\[
y'(x) - \frac{1}{2} y(x)^2 + g(x) = 0.
\]

(26)

Equation (26) was identified as Riccati equation presumably because it could be linearised by the Cole–Hopf transformation [6,7].

We feel that it will be interesting to examine if equations similar to eq. (25) written using higher-order Schwarzians lead to higher-order Ricatti equations. To that end, we first begin with eq. (24a)

\[
y''(x) - 3y'(x)y(x) + y(x)^3 + g_1(x) = 0
\]

(27)

written in terms of pre-Schwarzian derivative \( y(x) \). It is straightforward to verify that the Cole–Hopf transformation \( y(x) = -\psi'(x) / \psi(x) \) reduces eq. (27) in the linear form such that this nonlinear equation is, in fact, the second-order Riccati equation. In respect of this the nonlinear equations constructed by using \( \sigma_i(f) \) for \( i \geq 4 \) tend to pose problems. In the following we demonstrate this by dealing with \( \sigma_5(x) \).

Equation (24b) leads to the nonlinear equation

\[
y''''(x) - 6y''(x)y(x) - 3y'(x)^2 \\
+ 12y'(x)y(x)^2 - 3y(x)^4 + g_2(x) = 0.
\]

(28)

The transformation used for linearising eq. (27) reduces eq. (28) to

\[
\frac{\psi^{(4)}(x)}{\psi(x)} + \frac{2\psi'(x)\psi'''(x)}{\psi(x)^2} + g_2(x) = 0.
\]

(29)

Thus, we infer that the nonlinear equation (28) when appended by \(-2\psi'(x)\psi'''(x) / \psi(x)^2 \) (expressed in terms of \( y(x) \) and derivatives of \( y(x) \)) will give the third equation in the Riccati chain. Similarly, the fourth-order Riccati equation can be obtained (following from \( \sigma_6(f) \)) from the nonlinear equation

\[
y^{(4)}(x) - 10y(x)y''''(x) - 12y'(x)y''(x) \\
+ 36y(x)^2y''(x) + 36y(x)y'(x)^2 \\
+ 60y'(x)y(x)^3 + 12y(x)^5 + g_3(x) = 0
\]

(30)

by adding \( 4\psi'(x)^2\psi'''(x)/\psi(x)^3 + 2\psi'(x)\psi''''(x)/\psi(x)^3 + 5\psi'(x)\psi(4)(x)/\psi(x)^2 \) to it.

The higher-order SDs used in this work were proposed by Schippers [21] to fit the flow that satisfies the Löwner differential equation [23]. In addition, Aharanov [24] and Tamanoi [25] gave two different definitions to introduce higher-order analogues of SD. In particular, Aharanov provided the necessary and sufficient conditions for a non-constant meromorphic function on the unit disc to be univalent in terms of his Schwarzian. On the other hand, Tamanoi studied combinatorial structures of the higher-order Schwarzian introduced by him. We have checked that the results for higher-order derivatives appearing in the scheme of Schippers are more suitable for the present work.

In the above we have seen that the Schwarzian equations formed by using \( \sigma_n \) for \( n \geq 5 \) tend to exhibit some deviation from the corresponding Riccati equations. However, one can choose appropriate weight factors for the numerical coefficients in higher-order Schwarzian equations to reduce them in the Riccati form. For example, such reduced equations corresponding to eqs (28) and (30) are given by

\[
y''''(x) - 4y(x)y''''(x) - 3y'(x)^2 + 12y(x)^2y'(x) \\
- y(x)^4 + g_2(x) = 0
\]

(31)

and

\[
y^{(4)}(x) - 5y(x)y'''(x) - 10y'(x)y''(x) \\
+ 10y(x)^2y''(x) + 15y(x)y'(x)^2 \\
- 10y(x)^2y'(x) + y(x)^5 + g_3(x) = 0.
\]

(32)

Equations (31) and (32) can be linearised by using \( y(x) = -\psi'(x) / \psi(x) \) to read

\[
\psi^{(4)}(x) + g_2(x)\psi(x) = 0
\]

(33)

and

\[
\psi^{(5)}(x) + g_3(x)\psi(x) = 0
\]

(34)

respectively such that eqs (31) and (32) indeed represent the third and fourth equations in the Riccati chain.

4. Conclusion

In this work, we examined the connection between the SD and higher-order equations in the Riccati chain. We
achieved this by removing the $n$th-order derivative term from the $(n + 1)$th order linear differential equation obtained by linearising the $n$th order Riccati equation using the Cole–Hopf transformation [6,7]. More than 50 years ago, such a reduced equation was considered by Kim [26] to look for some generalisation of the SD. We note that the generalised derivatives so obtained led to highly complicated expressions for the coefficients of first- and zeroth-order derivative terms in a third-order equation. Here we derived a method to demonstrate how the so-called traditional SD enters into the higher-order equations in the Riccati chain.

There are three well-known methods [21,23,24] to introduce higher-order SDs. But we found it convenient to work with the definition of Schippers [21] to demonstrate how equations in the Riccati chain are embedded in higher-order SDs.

We have seen that all results presented in this paper depend crucially on the appropriate use of Cole–Hopf transformation. This point transformation, however, does not appear to be related to the recently proposed non-local transformation [12–14] which can linearise equations in both Riccati and Abel chains. In a remarkable paper, Sachdev [27] sought a generalisation of the Cole–Hopf transformation in the form $F(u(x,t)) = k(x,t)[\log \varphi(x,t)]_x$ [28] to construct nonlinear parabolic equations from a linear parabolic equation. Relatively recently, as opposed to the inverse problem treated in ref. [27], a similar transformation $\varphi(x,t) = -2\alpha\omega_x(x,t)/\omega(x,t)$ was introduced [29] to linearise the Burgers equation. Heuristically, the integrals of these transformations appear to resemble the non-local transformation of Pradeep et al [12].

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