POLYADIZATION OF ALGEBRAIC STRUCTURES

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ABSTRACT. A generalization of the semisimplicity concept for polyadic algebraic structures is proposed. If semisimple structures can be presented in block diagonal matrix form (resulting in the Wedderburn decomposition), a general form of polyadic structures is given by block-shift matrices. We combine these forms to get a general shape of semisimple nonderived polyadic structures (“double” decomposition of two kinds).

We then introduce the polyadization concept (a “polyadic constructor”) according to which one can construct a nonderived polyadic algebraic structure of any arity from a given binary structure. The polyadization of supersymmetric structures is also discussed. The “deformation” by shifts of operations on the direct power of binary structures is defined and used to obtain a nonderived polyadic multiplication. Illustrative concrete examples for the new constructions are given.

TABLE OF CONTENTS

1. INTRODUCTION
2. PRELIMINARIES
3. POLYADIC SEMISIMPlicity
   3.1. Simple polyadic structures
   3.2. Semisimple polyadic structures
      3.2.1. Supersymmetric double decomposition
4. POLYADIZATION CONCEPT
   4.1. Polyadization of binary algebraic structures
   4.2. Concrete examples of the polyadization procedure
      4.2.1. Polyadization of $GL(2, \mathbb{C})$
      4.2.2. Polyadization of $SO(2, \mathbb{R})$
5. CONCLUSIONS

References

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1. Introduction

I am no poet, but if you think for yourselves, as I proceed, the facts will form a poem in your minds.

"The Life and Letters of Faraday" (1870) by Bence Jones

Michael Faraday

The concept of simple and semisimple rings, modules, and algebras (see, e.g., Erdmann and Holm [2018], Hungerford [1974], Lambek [1966], Rotman [2010]) plays a crucial role in the investigation of Lie algebras and representation theory (Curtis and Reiner [1962], Fulton and Harris [2004], Knapp [1986], as well as in category theory (Harada [1970], Knop [2006], Simson [1977]).

Here we first propose a generalization of this concept for polyadic algebraic structures (Duili [2022a], which can also be important, e.g. in the operad theory (Markl et al. [2002], Loday and Vallette [2012]) and nonassociative structures (Zhevlakov et al. [1982], Sabinin et al. [2006]). If semisimple structures can be presented in the block-diagonal matrix form (resulting to the Wedderburn decomposition (Wedderburn [1908], Herstein [1966], Lam [1991])), a corresponding general form for polyadic rings can be decomposed to a kind of block-shift matrices (Nikitin [1984]). We combine these forms and introduce a general shape of semisimple polyadic structures, which are nonderived in the sense that they cannot be obtained as a successive composition of binary operations, which can be treated as a polyadic ("double") decomposition.

Second, going in the opposite direction, we define the polyadization concept ("polyadic constructor") according to which one can construct a nonderived polyadic algebraic structure of any arity from a given binary structure. Then we briefly describe supersymmetric structure polyadization.

Third, we propose operations "deformed" by shifts to obtain a nonderived $n$-ary multiplication on the direct power of binary algebraic structures.

For these new constructions some illustrative concrete examples are given.

2. Preliminaries

We use notation from [Duili 2022a,b]. In brief, a (one-set) polyadic algebraic structure $\mathcal{A}$ is a set $A$ closed with respect to polyadic operations (or $n$-ary multiplication) $\mu^{[n]} : A^n \rightarrow A$ ($n$-ary magma). We denote polyads (Post [1940]) by bold letters $\mathbf{a} = \mathbf{a}^{(n)} = (a_1, \ldots, a_n)$, $a_i \in A$. A polyadic zero is defined by $\mu^{[n]} \left[ \mathbf{a}^{(n-1)}, z \right] = z, z \in A, \mathbf{a}^{(n-1)} \in A^{n-1}$, where $z$ can be on any place. A (positive) polyadic power $\ell_\mu \in \mathbb{N}$ is $\mathbf{a}^{(\ell_\mu)} = \left( \mu^{[n]} \right)^{\ell_\mu} \left[ \mathbf{a}^{(n-1)\cdot\mu + 1} \right], \ a \in A$. An element of a polyadic algebraic structure $a$ is called $\ell_\mu$-nilpotent (or simply nilpotent for $\ell_\mu = 1$), if there exist $\ell_\mu$ such that $a^{(\ell_\mu)} = z$. A polyadic (or $n$-ary) identity (or neutral element) is defined by $\mu^{[n]} \left[ a, e^{n-1} \right] = a, \ \forall a \in A$, where $a$ can be on any place...
in the l.h.s. A one-set polyadic algebraic structure \( \langle A \mid \mu^{[n]} \rangle \) is totally associative, if \( (\mu^{[n]})^{\circ \, 2} [a, b, c] = \mu^{[n]} [a, \mu^{[n]} [b], c] \) is invariant, with respect to placement of the internal multiplication on any of the \( n \) places, and \( a, b, c \) are polyads of the necessary sizes [DUPLIJ 2018, 2019]. A polyadic semigroup \( S^{(n)} \) is a one-set and one-operation structure in which \( \mu^{[n]} \) is totally associative. A polyadic structure is commutative, if \( \mu^{[n]} = \mu^{[n]} \circ \sigma \), or \( \mu^{[n]} [a] = \mu^{[n]} [\sigma \circ a] \), \( a \in A^n \), for all \( \sigma \in S_n \).

A polyadic structure is solvable, if for all polyads \( b, c \) and an element \( x \), one can (uniquely) resolve the equation (with respect to \( h \)) for \( \mu^{[n]} [b, x, c] = a \), where \( x \) can be on any place, and \( b, c \) are polyads of the needed lengths. A solvable polyadic structure is called a polyadic quasigroup [BELOUSAiov 1972]. An associative polyadic quasigroup is called a \( n \)-ary (or polyadic) group [GAL’MAK 2003]. In an \( n \)-ary group the only solution of

\[
\mu^{[n]} [b, \bar{a}] = a, \quad a, \bar{a} \in A, \quad b \in A^{n-1}
\]

(2.1)

is called a querelement of \( a \) and denoted by \( \bar{a} \) [DÖRNE 1929], where \( \bar{a} \) can be on any place. Any idempotent \( a \) coincides with its querelement \( \bar{a} = a \). The relation (2.1) can be considered as a definition of the unary quereoperation \( \mu^{(1)} [a] = \bar{a} \) [GLIEGCHGWICHT AND GLAŻEK 1967]. For further details and definitions, see [DUPLIJ 2022a].

3. Polyadic semisimplicity

In general, simple algebraic structures are building blocks (direct summands) for the semisimple ones satisfying special conditions (see, e.g., [ERDMANN AND HOLL 2018, LAMBER 1966]).

3.1. Simple polyadic structures. According to the Wedderburn-Artin theorem (see, e.g., [HERSTEIN 1996, LAM 1991, HAZEWINKEL AND GUBARENI 2016]), a ring which is simple (having no two-sided ideals, except zero and the ring itself) and Artinian (having minimal right ideals) \( \mathcal{R}_{\text{simple}} \) is isomorphic to a full \( d \times d \) matrix ring

\[
\mathcal{R}_{\text{simple}} \cong \text{Mat}^{\text{full}}_{d \times d} (\mathcal{D})
\]

(3.1)

over a division ring \( \mathcal{D} \). As a corollary,

\[
\mathcal{R}_{\text{simple}} \cong \text{Hom}_{\mathcal{D}} (V (d \mid \mathcal{D}), V (d \mid \mathcal{D})) \equiv \text{End}_{\mathcal{D}} (V (d \mid \mathcal{D})) ,
\]

(3.2)

where \( V (d \mid \mathcal{D}) \) is a \( d \)-finite-dimensional vector space (left module) over \( \mathcal{D} \). In the same way, a finite-dimensional simple associative algebra \( \mathcal{A} \) over an algebraically closed field \( \mathcal{F} \) is

\[
\mathcal{A} \cong \text{Mat}^{\text{full}}_{d \times d} (\mathcal{F})
\]

(3.3)

In the polyadic case, the structure of a simple Artinian \([2, n]\)-ring \( \mathcal{R}^{[2, n]}_{\text{simple}} \) (with binary addition and \( n \)-ary multiplication \( \mu^{[n]} \)) was obtained in [NIKITIN 1984], where the Wedderburn-Artin theorem for \([2, n]\)-rings was proved. So instead of one vector space \( V (d \mid \mathcal{D}) \), one should consider a direct sum of \( (n - 1) \) vector spaces (over the same division ring \( \mathcal{D} \)), that is

\[
V_1 (d_1 \mid \mathcal{D}) \oplus V_2 (d_2 \mid \mathcal{D}) \oplus \ldots \ldots \oplus V_{n-1} (d_{n-1} \mid \mathcal{D}) ,
\]

(3.4)

where \( V_i (d_i \mid \mathcal{D}) \) is a \( d_i \)-dimensional polyadic vector space [DUPLIJ 2019], \( i = 1, \ldots, n - 1 \). Then, instead of (3.2) we have the cyclic direct sum of homomorphisms

\[
\mathcal{R}^{[2, n]}_{\text{simple}} \cong \text{Hom}_{\mathcal{D}} (V_1 (d_1 \mid \mathcal{D}), V_2 (d_2 \mid \mathcal{D}) \oplus \text{Hom}_{\mathcal{D}} (V_2 (d_2 \mid \mathcal{D}), V_3 (d_3 \mid \mathcal{D})) \oplus \ldots \ldots \oplus \text{Hom}_{\mathcal{D}} (V_{n-1} (d_{n-1} \mid \mathcal{D}), V_1 (d_1 \mid \mathcal{D})) .
\]

(3.5)

This means that after choosing a suitable basis in terms of matrices (when the ring multiplication \( \mu^{[n]} \) coincides with the product of \( n \) matrices) we have
Theorem 3.1. The simple polyadic ring \( R_{\text{simple}}^{[2,n]} \) is isomorphic to the \( d \times d \) matrix ring (cf. \( (3.1) \))
\[
R_{\text{simple}}^{[2,n]} \cong Mat_{d \times d}^{\text{shift}(n)} (D) = \left\{ M^{\text{shift}(n)} (d \times d) \mid \nu^{[2]}, \mu^{[n]} \right\},
\]
where \( \nu^{[2]} \) and \( \mu^{[n]} \) are binary addition and ordinary product of \( n \) matrices, \( M^{\text{shift}} \) is the block-shift (traceless) matrix over \( D \) of the form (which follows from \( (3.5) \))
\[
M^{\text{shift}(n)} (d \times d) = \begin{pmatrix}
0 & B_1 (d_1 \times d_2) & \cdots & 0 \\
0 & 0 & B_2 (d_2 \times d_3) & \cdots & 0 \\
& & & \ddots & \ddots & \vdots \\
& & & & 0 & B_{n-2} (d_{n-2} \times d_{n-1}) \\
B_{n-1} (d_{n-1} \times d_1) & \cdots & 0 & 0
\end{pmatrix},
\]
where \( (n - 1) \) blocks are nonsquare matrices \( B_i (d' \times d'') \in Mat_{d' \times d''}^{\text{full}} (D) \) over the division ring \( D \), and \( d = d_1 + d_2 + \ldots + d_{n-1} \).

Remark 3.2. The set of the fixed size blocks \( \{ B_i (d' \times d'') \} \) does not form a binary ring, because \( d' \neq d'' \).

Assertion 3.3. The block-shift matrices of the form \( (3.7) \) are closed with respect to \( n \)-ary multiplication and binary addition, and we call them \( n \)-ary matrices.

Taking distributivity into account we arrive at the polyadic ring structure \( (3.6) \).

Corollary 3.4. In the limiting case \( n = 2 \), we have
\[
M^{\text{shift}(n=2)} (d \times d) = B_1 (d_1 \times d_1)
\]
and \( d = d_1 \), giving a binary ring \( (3.1) \).

Assertion 3.5. A finite-dimensional simple associative \( n \)-ary algebra \( A^{(n)} \) over an algebraically closed field \( F \) is isomorphic to the block-shift \( n \)-ary matrix \( (3.7) \) over \( F \)
\[
A^{(n)} \cong Mat_{d \times d}^{\text{shift}(n)} (F).
\]

3.2. Semisimple polyadic structures. The Wedderburn-Artin theorem for semisimple Artinian rings \( R_{\text{semispl}} \) states that \( R_{\text{semispl}} \) is a finite direct product of \( k \) simple rings, each of which has the form \( (3.1) \).

Using \( (3.2) \), for each component, we decompose the \( d \)-finite-dimensional vector space (left module) into a direct sum of length \( k \)
\[
V (d) = W^{(1)} (q^{(1)} | D^{(1)}) + W^{(2)} (q^{(2)} | D^{(2)}) + \ldots + W^{(k)} (q^{(k)} | D^{(k)}),
\]
where \( d = q^{(1)} + q^{(2)} + \ldots + q^{(k)} \). Then, instead of \( (3.2) \) we have the isomorphism
\[
R_{\text{semispl}} \cong \text{End}_{\mathcal{D}^{(1)}} W^{(1)} (q^{(1)} | D^{(1)}) \oplus \text{End}_{\mathcal{D}^{(2)}} W^{(2)} (q^{(2)} | D^{(2)}) \oplus \ldots \oplus \text{End}_{\mathcal{D}^{(k)}} W^{(k)} (q^{(k)} | D^{(k)}).
\]

In a suitable basis the Wedderburn-Artin theorem follows

Theorem 3.6. A semisimple Artinian (binary) ring \( R_{\text{semispl}} \) is isomorphic to the \( d \times d \) matrix ring
\[
R_{\text{semispl}} \cong Mat_{d \times d}^{\text{diag}(k)} (D) = \left\{ M^{\text{diag}(k)} (d \times d) \mid \nu^{[2]}, \mu^{[2]} \right\},
\]

\footnote{We enumerate simple components by an upper index in round brackets \((k)\), block-shift components by lower index without brackets, and the arity is an upper index in square brackets \([n]\).}
where $\nu^{[2]}$ and $\mu^{[2]}$ are binary addition and binary product of matrices, $M^{\text{diag}(k)}(d \times d)$ are block-diagonal matrices of the form (which follows from (3.11))

$$M^{\text{diag}(k)}(d \times d) = \begin{pmatrix}
A^{(1)}(q^{(1)} \times q^{(1)}) & 0 & \ldots & 0 \\
0 & A^{(2)}(q^{(2)} \times q^{(2)}) & \ddots & \vdots \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & \ldots & A^{(k)}(q^{(k)} \times q^{(k)})
\end{pmatrix},$$  \hspace{1cm} (3.13)

where $k$ square blocks are full matrix rings over division rings $D^{(j)}$

$$A^{(j)}(q^{(j)} \times q^{(j)}) \in \text{Mat}^{\text{full}}_{q^{(j)} \times q^{(j)}}(D^{(j)}), \quad j = 1, \ldots, k, \quad d = q^{(1)} + q^{(2)} + \ldots + q^{(k)}. \hspace{1cm} (3.14)$$

The same matrix structure has a finite-dimensional semisimple associative algebra $\mathcal{A}$ over an algebraically closed field $F$ (see (3.3)). For further details, see, e.g., [HERSTEIN 1996], [LAM 1991], [HAZEWINKEL AND GUBAREN].

General properties of semisimple Artinian $[2, n]$-rings were considered in [NIKITIN 1984] (for ternary rings, see [LISTER 1971], [PROFERA 1982]). Here we propose a new manifest matrix structure for them.

Thus, our task is to decompose each of the $V_i(d_i)$, in (3.4) into components as in (3.10)

$$V_i(d_i) = W_i^{(1)}(d_i^{(1)} | D^{(1)}) \oplus W_i^{(2)}(d_i^{(2)} | D^{(2)}) \oplus \ldots \oplus W_i^{(k)}(d_i^{(k)} | D^{(k)}), \quad i = 1, \ldots, n-1. \hspace{1cm} (3.15)$$

In matrix language this means that each block $B_{d_i \times d_i}$ from the polyadic ring (3.7) should have the semisimple decomposition (3.13), i.e. be a block-diagonal square matrix of the same size $p \times p$, where $p = d_1 = d_2 = \ldots = d_{n-1}$ and the total matrix size becomes $d = (n-1)p$. Moreover, all the blocks $B$’s should have diagonal blocks $A$’s of the same size, and therefore $q^{(j)} = q_1^{(j)} = q_2^{(j)} = \ldots = q_{n-1}^{(j)}$ for all $j = 1, \ldots, k$ and $p = q^{(1)} + q^{(2)} + \ldots + q^{(k)}$, where $k$ is the number of semisimple components. In this way the cyclic direct sum of homomorphisms for the semisimple polyadic rings becomes (we use
different division rings for each semisimple component as in (3.14)

\[ R_{\text{semispl}}^{[2,n]} \cong \text{Hom}_D(1) \left( W_{1}^{(1)} (q^{(1)} | D^{(1)}), W_{2}^{(1)} (q^{(1)} | D^{(1)}) \right) \]
\[ \oplus \text{Hom}_D(2) \left( W_{1}^{(2)} (q^{(2)} | D^{(2)}), W_{2}^{(2)} (q^{(2)} | D^{(2)}) \right) \oplus \ldots \]
\[ \ldots \oplus \text{Hom}_D(k) \left( W_{1}^{(k)} (q^{(k)} | D^{(k)}), W_{2}^{(k)} (q^{(k)} | D^{(k)}) \right) \]
\[ \oplus \text{Hom}_D(1) \left( W_{n-2}^{(1)} (q^{(1)} | D^{(1)}), W_{n-1}^{(1)} (q^{(1)} | D^{(1)}) \right) \]
\[ \oplus \text{Hom}_D(2) \left( W_{n-2}^{(2)} (q^{(2)} | D^{(2)}), W_{n-1}^{(2)} (q^{(2)} | D^{(2)}) \right) \oplus \ldots \]
\[ \ldots \oplus \text{Hom}_D(k) \left( W_{n-2}^{(k)} (q^{(k)} | D^{(k)}), W_{n-1}^{(k)} (q^{(k)} | D^{(k)}) \right) \]
\[ \oplus \text{Hom}_D(1) \left( W_{n-1}^{(1)} (q^{(1)} | D^{(1)}), W_{1}^{(1)} (q^{(1)} | D^{(1)}) \right) \]
\[ \oplus \text{Hom}_D(2) \left( W_{n-1}^{(2)} (q^{(2)} | D^{(2)}), W_{1}^{(2)} (q^{(2)} | D^{(2)}) \right) \oplus \ldots \]
\[ \ldots \oplus \text{Hom}_D(k) \left( W_{n-1}^{(k)} (q^{(k)} | D^{(k)}), W_{1}^{(k)} (q^{(k)} | D^{(k)}) \right). \quad (3.16) \]

After choosing a suitable basis we obtain a polyadic analog of the Wedderburn-Artin theorem for semisimple Artinian \([2, n]\)-rings \( R_{\text{semispl}}^{[2,n]} \), which can be called as the double decomposition (of the first kind or shift-diagonal).

**Theorem 3.7.** The semisimple polyadic Artinian ring \( R_{\text{semispl}}^{[2,n]} \) (of the first kind) is isomorphic to the \( d \times d \) matrix ring

\[ R_{\text{semispl}}^{[2,n]} \cong \text{Mat}^{\text{shift-diag}(n,k)}_{d \times d} (D) = \langle \{ N^{\text{shift-diag}(n,k)} (d \times d) \} | \nu^{[2]}, \mu^{[n]} \rangle, \quad (3.17) \]

where \( \nu^{[2]}, \mu^{[n]} \) are binary addition and ordinary product of \( n \) matrices, \( N^{\text{shift-diag}(n,k)}_{d \times d} \) (\( n \) is arity of \( N \)'s and \( k \) is number of simple components of \( N \)'s) are the block-shift \( n \)-ary matrices with block-diagonal
square blocks (which follows from (3.16))

\[
N^{\text{shift-diag}(n,k)} (d \times d) = \\
\left( \begin{array}{cccc}
0 & B_1^{(k)} (p \times p) & \cdots & 0 \\
0 & 0 & B_2^{(k)} (p \times p) & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & 0 & B_{n-2}^{(k)} (p \times p) \\
B_{n-1}^{(k)} (p \times p) & 0 & \cdots & 0 & 0 \\
\end{array} \right), \quad (3.18)
\]

\[
B_i^{(k)} (p \times p) = \\
\left( \begin{array}{ccc}
A_i^{(1)} (q^{(1)} \times q^{(1)}) & 0 & \cdots & 0 \\
0 & A_i^{(2)} (q^{(2)} \times q^{(2)}) & \ddots & \vdots \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & \cdots & A_i^{(k)} (q^{(k)} \times q^{(k)}) \\
\end{array} \right), \quad (3.19)
\]

where

\[
d = (n - 1) p, \quad (3.20)
\]

\[
p = q^{(1)} + q^{(2)} + \ldots + q^{(k)}. \quad (3.21)
\]

and the \( k \) square blocks \( \Lambda \)'s are full matrix rings over the division rings \( \mathcal{D}^{(j)} \)

\[
A_i^{(j)} (q^{(j)} \times q^{(j)}) \in \text{Mat}^{\text{full}}_{q^{(j)} \times q^{(j)}} (\mathcal{D}^{(j)}), \quad j = 1, \ldots, k; \; i = 1, \ldots, n - 1. \quad (3.22)
\]

Remark 3.8. By analogy with (3.8), in the limiting case \( n = 2 \), we have in (3.18) one block \( B_1^{(k)} (p \times p) \) only and (3.19) gives its standard (binary) semisimple ring decomposition.

This allows us to introduce another possible double decomposition in the opposite sequence to (3.18)-(3.19), we call it of second kind or reverse, or diagonal-shift. Indeed, in a suitable basis we first provide the standard block-diagonal decomposition (3.13), and then each block obeys the block-shift decomposition (3.7). Here we do not write the “reverse” analog of (3.16) and arrive directly to

**Theorem 3.9.** The semisimple polyadic Artinian ring \( \hat{\mathcal{R}}_{\text{semispl}}^{[2,n]} \) (of the second kind) is isomorphic to the \( d \times d \) matrix ring

\[
\hat{\mathcal{R}}_{\text{semispl}}^{[2,n]} \cong \text{Mat}^{\text{diag-shift}(n,k)}_{d \times d} (\mathcal{D}) = \left\{ \hat{N}^{\text{diag-shift}(n,k)} (d \times d) \mid \nu^{[2]}, \mu^{[n]} \right\}, \quad (3.23)
\]

where \( \nu^{[2]}, \mu^{[n]} \) are binary addition and ordinary product of \( n \) matrices, \( \hat{N}^{\text{diag-shift}(n,k)}_{d \times d} \) ( \( n \) is arity of \( \hat{N} \)'s and \( k \) is number of simple components of \( \hat{N} \)'s) are the block-diagonal \( n \)-ary matrices with block-shift
nonsquare blocks

\[ \hat{N}^{\text{diag-shift}(n,k)}(d \times d) = \begin{pmatrix} \hat{A}^{(1)}(q^{(1)} \times q^{(1)}) & 0 & \ldots & 0 \\ 0 & \hat{A}^{(2)}(q^{(2)} \times q^{(2)}) & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \hat{A}^{(k)}(q^{(k)} \times q^{(k)}) \end{pmatrix}, \quad (3.24) \]

\[ \hat{A}^{(j)}(q^{(j)} \times q^{(j)}) = \begin{pmatrix} 0 & \hat{B}_1^{(j)}(p_1^{(j)} \times p_2^{(j)}) & \ldots & 0 \\ 0 & 0 & \hat{B}_2^{(j)}(p_2^{(j)} \times p_3^{(j)}) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \hat{B}_{n-1}^{(j)}(p_{n-1}^{(j)} \times p_1^{(j)}) & 0 & \ldots & 0 \end{pmatrix}, \quad (3.25) \]

where
\[ q^{(j)} = p_1^{(j)} + p_2^{(j)} + \ldots + p_{n-1}^{(j)}, \quad (3.26) \]
\[ d = q^{(1)} + q^{(2)} + \ldots + q^{(k)}, \quad (3.27) \]
and the \((n-1)k\) blocks \(\hat{B}\)’s are nonsquare matrices over the division rings \(D^{(j)}\)
\[ \hat{B}_i^{(j)}(p_i^{(j)} \times p_{i+1}^{(j)}) \in \text{Mat}^{\text{full}}_{p_i^{(j)} \times p_{i+1}^{(j)}}(D^{(j)}), \quad j = 1, \ldots, k, \quad i = 1, \ldots, n-1. \quad (3.28) \]

**Definition 3.10.** The ring obeying the double decomposition of the first kind \([3.13]-[3.19]\) (of the second kind \([3.24]-[3.29]\)) is called polyadic ring of the first kind (resp. of the second kind).

**Proposition 3.11.** The polyadic rings of the first and second kind are not isomorphic.

**Proof.** This follows from the manifest forms \([3.13]-[3.19]\) and \([3.24]-[3.29]\). Also, in general case, the \(\hat{B}\)-matrices can be nonsquare \([3.28]\).

Thus the two double decompositions introduced above can lead to a new classification for polyadic analogs of semisimple rings.

**Example 3.12.** Let us consider the double decomposition of two kinds for ternary \((n = 3)\) rings with two semisimple components \((k = 2)\) and blocks as full \(q \times q\) matrix rings over \(\mathbb{C}\). Indeed, we have for the ternary nonderived rings \(\mathcal{R}^{[2,3]}_{\text{semispl}}\) and \(\mathcal{R}^{[2,3]}_{\text{semispl}}\) of the first and second kind, respectively, the following block structures

\[ N^{\text{shift-diag}(3,2)}(4q \times 4q) = \begin{pmatrix} 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_2 \\ B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \end{pmatrix}, \quad (3.29) \]

\[ \hat{N}^{\text{diag-shift}(3,2)}(4q \times 4q) = \begin{pmatrix} 0 & \hat{A}_1 & 0 & 0 \\ \hat{A}_2 & 0 & 0 & 0 \\ 0 & 0 & \hat{B}_1 & 0 \\ 0 & 0 & \hat{B}_2 & 0 \end{pmatrix}, \quad (3.29) \]
where \( A_i, B_i, \hat{A}_i, \hat{B}_i \in \text{Mat}_{q \times q}^{\text{full}}(\mathbb{C}) \). In terms of component blocks, the ternary multiplications in the rings \( \mathcal{R}_{\text{semi}}^{[2,3]} \) and \( \mathcal{R}_{\text{semi}}^{[2,3]} \) are

**kind I:***

\[
A'_1 B''_1 A''_1 = A_1, \quad A'_2 B''_2 A''_2 = A_2, \\
B'_1 A''_1 B''_1 = B_1, \quad B'_2 A''_2 B''_2 = B_2. 
\] (3.30)

**kind II:***

\[
\hat{A}'_1 \hat{A}''_1 \hat{A}''_1 = \hat{A}_1, \quad \hat{A}'_2 \hat{A}''_2 \hat{A}''_2 = \hat{A}_2, \\
\hat{B}'_1 \hat{B}''_1 \hat{B}''_1 = \hat{B}_1, \quad \hat{B}'_2 \hat{B}''_2 \hat{B}''_2 = \hat{B}_2. 
\] (3.32)

It follows from (3.30)–(3.31) and (3.32)–(3.33) that \( \mathcal{R}_{\text{semi}}^{[2,3]} \) and \( \mathcal{R}_{\text{semi}}^{[2,3]} \) are not ternary isomorphic.

**Remark 3.13.** Consider a binary sum of the block matrices of the first and second kind (3.22):

\[
\mathcal{P}^{(3,2)}(4q \times 4q) = N^{\text{shift-diag}(3,2)}(4q \times 4q) + \hat{N}^{\text{diag-shift}(3,2)}(4q \times 4q) = \begin{pmatrix} 0 & \hat{A}_1 & A_1 & 0 \\ \hat{A}_2 & 0 & 0 & A_2 \\ B_1 & 0 & 0 & \hat{B}_1 \\ 0 & B_2 & \hat{B}_2 & 0 \end{pmatrix}. 
\] (3.34)

The set of matrices (3.34) forms the nonderived \([2,3]\)-ring \( \mathcal{P}^{[2,3]} \) over \( \mathbb{C} \)

\[
\mathcal{P}^{[2,3]} = \left\{ \{\mathcal{P}^{(3,2)}(4q \times 4q) \mid \nu^{[2]}, \mu^{[3]} \} \right\}, 
\] (3.35)

where \( \nu^{[2]}, \mu^{[3]} \) are binary addition and ordinary product of 3 matrices (3.34).

Notice that the \( \mathcal{P} \)-matrices (3.34) are the block-matrix version of the circle matrices \( M_{\text{circ}} \) which were studied in [DUIJL AND VOGL 2021] in connection with 8-vertex solutions to the constant Yang-Baxter equation [LAMBE AND RADFORD 1997] and the corresponding braiding quantum gates [KAUFFMAN AND LOMONACO 2004]. [MELNIKOV ET AL. 2018].

### 3.2.1. Supersymmetric double decomposition

Let us generalize the above double decomposition (of the first kind) to superrings and superalgebras. For that, we first assume that the constituent vector spaces (entering in (3.16)) are super vector spaces (\( \mathbb{Z}_2 \)-graded vector spaces) obeying the standard decomposition into even and odd parts

\[
W^{(j)}(q^{(j)} \mid \mathcal{D}^{(j)}) = W^{(j)}(q^{(j)}_{\text{even}} \mid \mathcal{D}^{(j)})_{\text{even}} \oplus W^{(j)}(q^{(j)}_{\text{odd}} \mid \mathcal{D}^{(j)})_{\text{odd}}, \quad i = 1, \ldots, n - 1, \quad j = 1, \ldots, k, 
\] (3.36)

where \( q^{(j)}_{\text{even}} \) and \( q^{(j)}_{\text{odd}} \) are dimensions of the even and odd spaces, respectively, \( q^{(j)} = q^{(j)}_{\text{even}} + q^{(j)}_{\text{odd}} \).

The parity of a homogeneous element of the vector space \( v \in W^{(j)}(q^{(j)} \mid \mathcal{D}^{(j)}) \) is defined by \( |v| = \overline{0} \) (resp. \( \overline{1} \)), if \( v \in W^{(j)}(q^{(j)}_{\text{even}} \mid \mathcal{D}^{(j)})_{\text{even}} \) (resp. \( W^{(j)}(q^{(j)}_{\text{odd}} \mid \mathcal{D}^{(j)})_{\text{odd}} \)), and \( \overline{0}, \overline{1} \in \mathbb{Z}_2 \). For details, see [BEREZIN 1987], [LEITES 1983]. In the graded case, the \( k \) square blocks \( A \)'s in (3.22) are full supermatrix rings of the size \( (q^{(j)}_{\text{even}} \mid q^{(j)}_{\text{odd}}) \times (q^{(j)}_{\text{even}} \mid q^{(j)}_{\text{odd}}) \), while the square \( B \)'s (3.19) are block-diagonal supermatrices, and the block-shift \( n \)-ary supermatrices have a nonstandard form (3.19).

We assume that in super case a polyadic analog of the Wedderburn-Artin theorem for semisimple Artinian superrings (of the first kind) is also valid, with the form of the double decomposition (3.18)–(3.19) being the same, however now the blocks \( A \)'s and \( B \)'s are corresponding supermatrices.
4. POLYADIZATION CONCEPT

Here we propose a general procedure for how to construct new polyadic algebraic structures from binary (or lower arity) ones, using the “inverse” (informally) to the block-shift matrix decomposition \([3.7]\). It can be considered as a polyadic analog of the inverse problem of the determination of an algebraic structure from the knowledge of its Wedderburn decomposition [Dietzel and Mittal 2021].

4.1. Polyadization of binary algebraic structures. Let a binary algebraic structure \(\mathcal{X}\) be represented by \(p \times p\) matrices \(B_y \equiv B_y (p \times p)\) over a ring \(\mathcal{R}\) (a linear representation), where \(y\) is the set of \(N_y\) parameters corresponding to an element \(x\) of \(\mathcal{X}\). Because the binary addition in \(\mathcal{R}\) transfers to the matrix addition without restrictions (as opposed to the polyadic case, see below), we will consider only the multiplicative part of the resulting polyadic matrix ring. In this way, we propose a special block-shift matrix method to obtain \(n\)-ary semigroups \((n\text{-ary groups})\) from the binary ones, but the former are not derived from the latter [Gal’mak 2003, Duplij 2022a]. In general, this can lead to new algebraic structures that were not known before.

**Definition 4.1.** A (block-matrix) polyadization \(\Phi_{pol}\) of a binary semigroup (or group) \(\mathcal{X}\) represented by square \(p \times p\) matrices \(B_y\) is an \(n\)-ary semigroup (or an \(n\)-ary group) represented by the \(d \times d\) block-shift matrices (over a ring \(\mathcal{R}\)) of the form \([3.7]\) as follows

\[
Q_{y_1,\ldots,y_{n-1}} = Q_{y_1,\ldots,y_{n-1}}^{B_{\text{shift}(n)}} (d \times d) = \begin{pmatrix}
0 & B_{y_1} & \cdots & 0 & 0 \\
0 & 0 & B_{y_2} & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & 0 & B_{y_{n-2}} \\
B_{y_{n-1}} & 0 & \cdots & 0 & 0
\end{pmatrix},
\]

(4.1)

where \(d = (n-1)p\), and the \(n\)-ary multiplication \(\mu^{[n]}\) is given by the product of \(n\) matrices \([4.1]\).

In terms of the block-matrices \(B\)’s the multiplication

\[
\mu^{[n]} \left[ \underbrace{Q_{y_1,\ldots,y_{n-1}}, Q_{y_1,\ldots,y_{n-1}}, \ldots, Q_{y_1,\ldots,y_{n-1}}, Q_{y_1,\ldots,y_{n-1}}}^{n} \right] = Q_{y_1,\ldots,y_{n-1}}
\]

(4.2)

has the cyclic product form (see [Duplij 2021])

\[
B_{y_1} B_{y_2} \cdots B_{y_{n-1}} B_{y_n} = B_{y_1},
\]

(4.3)

\[
B_{y_1} B_{y_2} \cdots B_{y_{n-1}} B_{y_n} = B_{y_2},
\]

(4.4)

\cdots

\[
B_{y_1} B_{y_2} \cdots B_{y_{n-1}} B_{y_n} = B_{y_{n-1}}.
\]

(4.5)

**Remark 4.2.** The number of parameters \(N_y\) describing an element \(x \in \mathcal{X}\) increases to \((n-1)N_y\), and the corresponding algebraic structure \(\langle Q_{y_1,\ldots,y_{n-1}} \mid \mu^{[n]} \rangle\) becomes \(n\)-ary, and so \((4.1)\) can be treated as a new algebraic structure, which we denote by the same letter with the arities in double square brackets \(\mathcal{X}^{[[n]]}\).

We now analyze some of the most general properties of the polyadization map \(\Phi_{pol}\) which are independent of the concrete form of the block-matrices \(B\)’s and over which algebraic structure (ring, field, etc...) they are defined. We then present some concrete examples.
Definition 4.3. A unique polyadization $\Phi_{Upol}$ is a polyadization where all sets of parameters coincide

$$y = y_1 = y_2 \ldots = y_{n-1}.$$  \hfill (4.6)

Proposition 4.4. The unique polyadization is an $n$-ary-binary homomorphism.

Proof. In the case of (4.6) all $(n - 1)$ relations (4.3–4.5) coincide

$$\prod_{j=1}^{n} B_{y_j} B_{y''_j} \ldots B_{y''''_j} B_{y'''_j} = B_{y},$$  \hfill (4.7)

which means that the ordinary (binary) product of $n$ matrices $B_y$'s is mapped to the $n$-ary product of matrices $Q_{y_j}$'s (4.2)

$$\mu[[n]] \left[ Q_{y_1}, Q_{y''_1}, \ldots, Q_{y'''_1} \right] = Q_y,$$ \hfill (4.8)

as it should be for an $n$-ary-binary homomorphism, but not for a homomorphism. \hfill $\square$

Assertion 4.5. If matrices $B_y \equiv B_y (p \times p)$ contain the identity matrix $E_p$, then the $n$-ary identity $E_d^{(n)}$ in $\langle \{Q_y (d \times d)\} \mid \mu[[n]] \rangle$, $d = (n - 1) p$ has the form

$$E_d^{(n)} = \begin{pmatrix} 0 & E_p & \ldots & 0 & 0 \\ 0 & 0 & E_p & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & E_p \\ E_p & 0 & \ldots & 0 & 0 \end{pmatrix}.$$ \hfill (4.9)

Proof. It follows from (4.1), (4.2) and (4.7). \hfill $\square$

In this case the unique polyadization maps the identity matrix to the $n$-ary identity $\Phi_{Upol} : E_p \rightarrow E_d^{(n)}$.

Assertion 4.6. If the matrices $B_y$ are invertible $B_y B^{-1}_y = B^{-1}_y B_y = E_p$, then each $Q_{y_1, \ldots, y_{n-1}}$ has a querelement

$$\overline{Q}_{y_1, \ldots, y_{n-1}} = \begin{pmatrix} 0 & \overline{B}_{y_1} & \ldots & 0 & 0 \\ 0 & 0 & \overline{B}_{y_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \overline{B}_{y_{n-2}} \\ \overline{B}_{y_{n-1}} & 0 & \ldots & 0 & 0 \end{pmatrix},$$ \hfill (4.10)

satisfying

$$\mu[[n]] \left[ Q_{y_1, \ldots, y_{n-1}}, Q_{y_1, \ldots, y_{n-1}}, \ldots, Q_{y_1, \ldots, y_{n-1}} \overline{Q}_{y_1, \ldots, y_{n-1}} \right] = Q_{y_1, \ldots, y_{n-1}}.$$  \hfill (4.11)

where $\overline{Q}_{y_1, \ldots, y_{n-1}}$ can be on any places and

$$\overline{B}_{y_1} = B^{-1}_{y_1} B^{-1}_{y_2} \ldots B^{-1}_{y_{n-2}} B^{-1}_{y_{n-1}} B^{-1}_{y_1} B^{-1}_{y_2} \ldots B^{-1}_{y_{n-2}} B^{-1}_{y_{n-1}} B^{-1}_{y_1} B^{-1}_{y_2} \ldots B^{-1}_{y_{n-2}} B^{-1}_{y_{n-1}} B^{-1}_{y_1} B^{-1}_{y_2} \ldots B^{-1}_{y_{n-2}} B^{-1}_{y_{n-1}} B^{-1}_{y_1}.$$ \hfill (4.12)

Proof. This follows from (4.10–4.11) and (4.3–4.5), then consequently applying $B_{y_i}^{-1}$ (with suitable indices) on both sides, we obtain (4.12). \hfill $\square$
Let us suppose that on the set of matrices \( \{B_y\} \) over a binary ring \( \mathcal{R} \) one can consider some analog of a multiplicative character \( \chi : \{B_y\} \to \mathcal{R} \), being a (binary) homomorphism, such that
\[
\chi(B_{y_1}) \chi(B_{y_2}) = \chi(B_{y_1} B_{y_2}). \quad (4.13)
\]

For instance, in case \( B \in GL(p, \mathbb{C}) \), the determinant can be considered as a (binary) multiplicative character. Similarly, we can introduce

**Definition 4.7.** A polyadized multiplicative character \( \chi : \{Q_{y_1,\ldots,y_{n-1}}\} \to \mathcal{R} \) is proportional to a product of the binary multiplicative characters of the blocks \( \chi(B_y) \)
\[
\chi(Q_{y_1,\ldots,y_{n-1}}) = (-1)^n \chi(B_{y_1}) \chi(B_{y_2}) \cdots \chi(B_{y_{n-1}}). \quad (4.14)
\]

The normalization factor \((-1)^n\) in (4.14) is needed to be consistent with the case when \( \mathcal{R} \) is commutative, and the multiplicative characters are determinants. It can also be consistent in other cases.

**Proposition 4.8.** If the ring \( \mathcal{R} \) is commutative, then the polyadized multiplicative character \( \chi \) is an \( n \)-ary-binary homomorphism.

**Proof.** It follows from (4.7)–(4.8), (4.14) and the commutativity of \( \mathcal{R} \). \( \square \)

**Proposition 4.9.** If the ring \( \mathcal{R} \) is commutative and unital with the unit \( E_p \), then the algebraic structure \( \langle \{Q_{y_1,\ldots,y_{n-1}}\} | \mu^{[\ell]} \rangle \) contains polyadic (\( n \)-ary) idempotents satisfying
\[
B_{y_1} B_{y_2} \cdots B_{y_{n-1}} = E_p. \quad (4.15)
\]

**Proof.** It follows from (4.8) and (4.9). \( \square \)

### 4.2. Concrete examples of the polyadization procedure.

#### 4.2.1. Polyadization of \( GL(2, \mathbb{C}) \).
Consider the polyadization procedure for the general linear group \( GL(2, \mathbb{C}) \). We have for the 4-parameter block matrices \( B_{y_i} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in GL(2, \mathbb{C}), \ y_i = (a_i, b_i, c_i, d_i) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}, \ i = 1, 2, 3. \) Thus, the 12-parameter 4-ary group \( GL^{[4]}(2, \mathbb{C}) = \langle \{Q_{y_1,y_2,y_3}\} | \mu^{[4]} \rangle \) is represented by the following 6 \( \times 6 \) Q-matrices
\[
Q_{y_1,y_2,y_3} = \begin{pmatrix} 0 & B_{y_1} & 0 \\ 0 & 0 & B_{y_2} \\ B_{y_3} & 0 & 0 \end{pmatrix} \in GL^{[4]}(2, \mathbb{C}), \quad B_{y_i} \in GL(2, \mathbb{C}), \quad i = 1, 2, 3, \quad (4.16)
\]
obeying the 4-ary multiplication
\[
\mu^{[4]}[Q_{y_1,y_2,y_3}, Q_{y_1',y_2',y_3'], Q_{y_1'',y_2'',y_3''}, Q_{y_1''',y_2''',y_3'''}] = Q_{y_1,y_2,y_3} Q_{y_1',y_2',y_3'} Q_{y_1'',y_2'',y_3''} Q_{y_1''',y_2''',y_3'''} Q_{y_{1''''},y_{2''''},y_{3''''}} = Q_{y_1,y_2,y_3}. \quad (4.17)
\]

In terms of the block matrices \( B_{y_i} \), the multiplication \( (4.17) \) becomes (see (4.2)–(4.5))
\[
B_{y_1} B_{y_2} B_{y_3} B_{y_1'} B_{y_1'} = B_{y_1}, \quad (4.18)
B_{y_2} B_{y_2'} B_{y_2'} = B_{y_2}, \quad (4.19)
B_{y_3} B_{y_3'} B_{y_3'} = B_{y_3}, \quad (4.20)
\]
which can be further expressed in the B-matrix entries (its manifest form is too cumbersome to give here).

For \( \{Q_{y_1,y_2,y_3}\} \) to be a \( 4 \)-ary group each \( Q \)-matrix should have the unique querelement determined by the equation (see (4.11))
\[
Q_{y_1,y_2,y_3} Q_{y_1,y_2,y_3} Q_{y_1,y_2,y_3} Q_{y_1,y_2,y_3} = Q_{y_1,y_2,y_3}. \quad (4.21)
\]
Concrete examples of the polyadization procedure

which has the solution

\[
\bar{Q}_{y_1,y_2,y_3} = \begin{pmatrix}
0 & \bar{B}_{y_1} & 0 \\
0 & 0 & \bar{B}_{y_2} \\
\bar{B}_{y_3} & 0 & 0
\end{pmatrix},
\]  

(4.22)

where (see (4.12))

\[
\bar{B}_{y_1} = B_{y_1}^{-1} B_{y_2}^{-1}, \quad \bar{B}_{y_2} = B_{y_2}^{-1} B_{y_3}^{-1}, \quad \bar{B}_{y_3} = B_{y_3}^{-1} B_{y_1}^{-1}.
\]  

(4.23)

In the manifest form the querelements of \(GL^{[4]}(2, \mathbb{C})\) are (4.22), where

\[
\bar{B}_{y_1} = \frac{1}{\Delta_3 \Delta_2} \begin{pmatrix}
b_2 c_2 + d_3 d_2 & -b_3 a_2 - d_3 b_2 \\
-a_3 c_2 - c_3 d_2 & a_3 a_2 + c_3 b_2
\end{pmatrix},
\]  

(4.24)

\[
\bar{B}_{y_2} = \frac{1}{\Delta_2 \Delta_3} \begin{pmatrix}
b_1 c_3 + d_1 d_3 & -b_1 a_3 - d_1 b_3 \\
-a_1 c_3 - c_1 d_3 & a_1 a_3 + c_1 b_3
\end{pmatrix},
\]  

(4.25)

\[
\bar{B}_{y_3} = \frac{1}{\Delta_2 \Delta_1} \begin{pmatrix}
b_2 c_1 + d_2 d_1 & -b_2 a_1 - d_2 b_1 \\
-a_2 c_1 - c_2 d_1 & a_2 a_1 + c_2 b_1
\end{pmatrix},
\]  

(4.26)

where \(\Delta_i = a_i d_i - b_i c_i \neq 0\) are the (nonvanishing) determinants of \(B_{yi}\).

**Definition 4.10.** We call \(GL^{[4]}(2, \mathbb{C})\) a polyadic (4-ary) general linear group.

If we take the binary multiplicative characters to be determinants \(\chi(B_{yi}) = \Delta_i \neq 0\), then the polyadized multiplicative character in \(GL^{[4]}(2, \mathbb{C})\) becomes

\[
\chi(Q_{y_1,y_2,y_3}) = \Delta_1 \Delta_2 \Delta_3,
\]  

(4.27)

which is a 4-ary-binary homomorphism, because (see (4.18)–(4.20))

\[
\chi(Q_{y_1',y_2',y_3'}) \chi(Q_{y_1'',y_2'',y_3''}) \chi(Q_{y_1''',y_2''',y_3'''}) \chi(Q_{y_1''''y_2''''y_3''''}) = (\Delta_1' \Delta_2'' \Delta_3''') (\Delta_1'' \Delta_2''' \Delta_3''') (\Delta_1''' \Delta_2'''' \Delta_3''''') (\Delta_1'''' \Delta_2''''' \Delta_3''''''')
\]  

(4.28)

The 4-ary identity \(E_6^{(4)}\) of \(GL^{[4]}(2, \mathbb{C})\) is unique and has the form (see (4.9))

\[
E_6^{(4)} = \begin{pmatrix}
0 & E_2 & 0 \\
0 & 0 & E_2 \\
E_2 & 0 & 0
\end{pmatrix},
\]  

(4.29)

where \(E_2\) is the identity of \(GL(2, \mathbb{C})\). The 4-ary identity \(E_6^{(4)}\) satisfies the 4-ary idempotence relation

\[
E_6^{(4)} E_6^{(4)} E_6^{(4)} E_6^{(4)} = E_6^{(4)}.
\]  

(4.30)

In general, the 4-ary group \(GL^{[4]}(2, \mathbb{C})\) contains an infinite number of 4-ary idempotents \(Q_{y_1,y_2,y_3}^{\text{idemp}}\) defined by the system of equations

\[
Q_{y_1,y_2,y_3}^{\text{idemp}} Q_{y_1',y_2',y_3'}^{\text{idemp}} Q_{y_1'',y_2'',y_3''}^{\text{idemp}} Q_{y_1''',y_2''',y_3'''''}^{\text{idemp}} = Q_{y_1,y_2,y_3}^{\text{idemp}},
\]  

(4.31)

which gives

\[
B_{y_1}^{\text{idemp}} B_{y_2}^{\text{idemp}} B_{y_3}^{\text{idemp}} = E_2,
\]  

(4.32)
or manifestly
\begin{align}
a_1a_2a_3 + a_1b_2c_3 + a_3b_1c_2 + b_1c_3d_2 &= 1, \\
a_2b_3c_1 + b_2c_1d_3 + b_3c_2d_1 + d_1d_2d_3 &= 1, \\
a_1a_2b_3 + a_1b_2d_3 + b_1b_3c_2 + b_1d_2d_3 &= 0, \\
a_2a_3c_1 + a_3c_2d_1 + b_2c_1c_3 + c_3d_1d_2 &= 0.
\end{align}

The infinite set of idempotents in $GL^{[4]} (2, \mathbb{C})$ is determined by $12 - 4 = 8$ complex parameters, because one block-matrix (with 4 complex parameters) can always be excluded using the equation \eqref{4.32}.

**Remark 4.11.** The above example shows, how “far” polyadic groups can be from ordinary (binary) groups: the former can contain infinite number of 4-ary idempotents determined by \eqref{4.33}–\eqref{4.36}, in addition to the standard idempotent in any group, the 4-ary identity \eqref{4.29}.

### 4.2.2. Polyadization of $SO (2, \mathbb{R})$

Here we provide a polyadization for the simplest subgroup of $GL (2, \mathbb{C})$, the special orthogonal group $SO (2, \mathbb{R})$. In the matrix form $SO (2, \mathbb{R})$ is represented by the one-parameter rotation matrix

\[ B (\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \in SO (2, \mathbb{R}), \quad \alpha \in \mathbb{R} / 2\pi \mathbb{Z}, \]  

satisfying the commutative multiplication

\[ B (\alpha) B (\beta) = B (\alpha + \beta), \]

and the (binary) identity $E_2$ is $B (0)$. Therefore, the inverse element for $B (\alpha)$ is $B (-\alpha)$.

The 4-ary polyadization of $SO (2, \mathbb{R})$ is given by the 3-parameter 4-ary group of $Q$-matrices $SO^{[4]} (2, \mathbb{R}) = \langle \{ Q (\alpha, \beta, \gamma) \} \mid \mu^{[4]} \rangle$, where (cf. \eqref{4.16})

\[ Q (\alpha, \beta, \gamma) = \begin{pmatrix} 0 & B (\alpha) & 0 \\ 0 & 0 & B (\beta) \\ B (\gamma) & 0 & 0 \end{pmatrix} \]  

\[ = \begin{pmatrix} 0 & 0 & \cos \alpha & -\sin \alpha \\ 0 & 0 & \sin \alpha & \cos \alpha \\ 0 & 0 & 0 & 0 \\ \cos \gamma & -\sin \gamma & 0 & 0 \end{pmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R} / 2\pi \mathbb{Z}, \]  

and the 4-ary multiplication is

\[ \mu^{[4]} [ Q (\alpha_1, \beta_1, \gamma_1), Q (\alpha_2, \beta_2, \gamma_2), Q (\alpha_3, \beta_3, \gamma_3), Q (\alpha_4, \beta_4, \gamma_4) ] 
= Q (\alpha_1, \beta_1, \gamma_1) Q (\alpha_2, \beta_2, \gamma_2) Q (\alpha_3, \beta_3, \gamma_3) Q (\alpha_4, \beta_4, \gamma_4) 
= Q (\alpha_1 + \beta_2 + \gamma_3 + \alpha_4, \beta_1 + \gamma_2 + \alpha_3 + \beta_4, \gamma_1 + \alpha_2 + \beta_3 + \gamma_4) = Q (\alpha, \beta, \gamma), \]  

which is noncommutative, as opposed to the binary product of $B$-matrices \eqref{4.38}.

The querelement $\overline{Q} (\alpha, \beta, \gamma)$ for a given $Q (\alpha, \beta, \gamma)$ is defined by the equation (see \eqref{4.21})

\[ Q (\alpha, \beta, \gamma) Q (\alpha, \beta, \gamma) \overline{Q} (\alpha, \beta, \gamma) = Q (\alpha, \beta, \gamma), \]

which has the solution

\[ \overline{Q} (\alpha, \beta, \gamma) = Q (-\beta - \gamma, -\alpha - \gamma, -\alpha - \beta). \]
Definition 4.12. We call \( SO^{[4]}(2, \mathbb{R}) \) a polyadic (4-ary) special orthogonal group, and \( Q(\alpha, \beta, \gamma) \) is called a polyadic (4-ary) rotation matrix.

Informally, the matrix \( Q(\alpha, \beta, \gamma) \) represents the polyadic (4-ary) rotation. There are an infinite number of polyadic (4-ary) identities (neutral elements) \( E(\alpha, \beta, \gamma) \) which are defined by

\[
E(\alpha, \beta, \gamma) E(\alpha, \beta, \gamma) E(\alpha, \beta, \gamma) Q(\alpha, \beta, \gamma) = Q(\alpha, \beta, \gamma),
\]

and the solution is

\[
E(\alpha, \beta, \gamma) = Q(\alpha, \beta, \gamma), \quad \alpha + \beta + \gamma = 0.
\]

It follows from (4.44) that \( E(\alpha, \beta, \gamma) \) are 4-ary idempotents (see (4.30) and Remark 4.11).

The determinant of \( B(\alpha) \) and \( Q(\alpha, \beta, \gamma) \) are 1, and therefore the corresponding multiplicative characters and polyadized multiplicative characters (4.14) are also equal to 1.

Comparing with the successive products of four \( B \)-matrices (4.37)

\[
B(\alpha_1) B(\alpha_2) B(\alpha_3) B(\alpha_4) = B(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4),
\]

we observe that 4-ary multiplication (4.41) gives a shifted sum of four angles.

More exactly, for the triple \((\alpha, \beta, \gamma)\) we introduce the circle (left) shift operator by

\[
s\alpha = \beta, \quad s\beta = \gamma, \quad s\gamma = \alpha
\]

with the property \( s^3 = \text{id} \). Then the 4-ary multiplication (4.41) becomes

\[
\mu^{[4]}[Q(\alpha_1, \beta_1, \gamma_1), Q(\alpha_2, \beta_2, \gamma_2), Q(\alpha_3, \beta_3, \gamma_3), Q(\alpha_4, \beta_4, \gamma_4)] = Q(\alpha_1 + s\alpha_2 + s^2\alpha_3 + \alpha_4, \beta_1 + s\beta_2 + s^2\beta_3 + \beta_4, \gamma_1 + s\gamma_2 + s^2\gamma_3 + \gamma_4).
\]

The quarelement has the form

\[
Q(\alpha, \beta, \gamma) = Q(-s\alpha - s^2\alpha, -s\beta - s^2\beta, -s\gamma - s^2\gamma).
\]

The multiplication (4.48) can be (informally) expressed in terms of a new operation, the 4-ary “cyclic shift addition” defined on \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) by (see (4.41))

\[
\nu^{[4]}[\alpha_1, \alpha_2, \alpha_3, \alpha_4] = s^0\alpha_1 + s^1\alpha_2 + s^2\alpha_3 + s^3\alpha_4 = \alpha_1 + s\alpha_2 + s^2\alpha_3 + \alpha_4,
\]

where \( \nu^{[4]} \) is (informally)

\[
\nu^{[4]}[\alpha_1, \alpha_2, \alpha_3, \alpha_4] = s^0\alpha_1 + s^1\alpha_2 + s^2\alpha_3 + s^3\alpha_4 = \alpha_1 + s\alpha_2 + s^2\alpha_3 + \alpha_4,
\]

and \( s^0 = \text{id} \). This can also be treated as some “deformation” of the repeated binary additions by shifts. It is seen that the 4-ary operation \( \nu^{[4]} \) (4.50) is not derived and cannot be obtained by consequent binary operations on the triples \((\alpha, \beta, \gamma)\) as (4.48).

In terms of the 4-ary cyclic shift addition the 4-ary multiplication (4.48) becomes

\[
\mu^{[4]}[Q(\alpha_1, \beta_1, \gamma_1), Q(\alpha_2, \beta_2, \gamma_2), Q(\alpha_3, \beta_3, \gamma_3), Q(\alpha_4, \beta_4, \gamma_4)] = Q(\nu^{[4]}[(\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2), (\alpha_3, \beta_3, \gamma_3), (\alpha_4, \beta_4, \gamma_4)]).
\]

The binary case corresponds to \( s = \text{id} \), because in (4.37) we have only one angle \( \alpha \), as opposed to three angles in (4.47).

Thus, we conclude that just as the binary product of \( B \)-matrices corresponds to the ordinary angle addition (4.38), the 4-ary multiplication of polyadic rotation \( Q \)-matrices (4.39) corresponds to the 4-ary cyclic shift addition (4.51) through (4.52).
4.3. "Deformation" of binary operations by shifts. The concrete example from the previous subsection shows the strong connection (4.52) between the polyadization procedure and the shifted operations (4.51). Here we generalize it to an \( n \)-ary case for any semigroup.

Let \( \mathcal{A} = \langle A \mid (+) \rangle \) be a binary semigroup, where \( A \) is its underlying set and \( (+) \) is the binary operation (which can be noncommutative). The simplest way to construct an \( n \)-ary operation \( \nu^{[n]} : A^n \to A \) is the consequent repetition of the binary operation (see (4.46))

\[
\nu^{[n]}[\alpha_1, \alpha_2, \ldots, \alpha_n] = \alpha_1 + \alpha_2 + \ldots + \alpha_n,
\]

where the \( n \)-ary multiplication \( \nu^{[n]} \) (4.58) is called derived [Dörnte 1929, Zupnik 1967].

To construct a nonderived operation, we now consider the (external) \( m \)-th direct power \( \mathcal{A}^m \) of the semigroup \( \mathcal{A} \) by introducing \( m \)-tuples

\[
a \equiv a^{(m)} = (\alpha, \beta, \ldots, \gamma), \quad \alpha, \beta, \ldots, \gamma \in A, \quad a \in A^m.
\]

The \( m \)-th direct power becomes a binary semigroup by endowing \( m \)-tuples with the componentwise binary operation (\( + \)) as

\[
a_1 + a_2 = \left( \alpha_1, \beta_1, \ldots, \gamma_1 \right) + \left( \alpha_2, \beta_2, \ldots, \gamma_2 \right) = \left( \alpha_1 + \alpha_2, \beta_1 + \beta_2, \ldots, \gamma_1 + \gamma_2 \right).
\]

The derived \( n \)-ary operation for \( m \)-tuples (on the \( m \)-th direct power) is then defined componentwise by analogy with (4.53)

\[
\nu^{[n]}[a_1, a_2, \ldots, a_n] = a_1 + a_2 + \ldots + a_n.
\]

Now using shifts, instead of (4.56) we construct a nonderived \( n \)-ary operation on the direct power.

**Definition 4.13.** A cyclic \( m \)-shift operator \( s \) is defined for the \( m \)-tuple (4.54) by

\[
s^m = \text{id}.
\]

For instance, in this notation, if \( m = 3 \) and \( a = (\alpha, \beta, \gamma) \), then \( sa = (\gamma, \alpha, \beta) \), \( s^2a = (\beta, \gamma, \alpha) \), \( s^3a = a \) (as in the previous subsection).

To obtain a nonderived \( n \)-ary operation, by analogy with (4.53), we deform by shifts the derived \( n \)-ary operation (4.56).

**Definition 4.14.** The shift deformation by (4.57) of the derived operation \( \nu^{[n]} \) on the direct power \( \mathcal{A}^m \) is defined noncomponentwise by

\[
\nu^{[n]}_s[a_1, a_2, \ldots, a_n] = \sum_{i=1}^{n} s^{i-1} a_i = a_1 + sa_2 + \ldots + s^{n-1} a_n,
\]

where \( a \in A^m \) (4.54) and \( s^0 = \text{id} \).

Note that till now there exist no relations between \( n \) and \( m \).

**Proposition 4.15.** The shift deformed operation \( \nu^{[n]}_s \) is totally associative, if

\[
s^{n-1} = \text{id},
\]

\[
m = n - 1.
\]
Proof. We compute
\[
\left[\nu_s^{[n]}[a_1, a_2, \ldots, a_n] : a_{n+1}, a_{n+2}, \ldots, a_{2n-1}\right] = (a_1 + sa_2 + \ldots + s^{n-1}a_{n+1}) + sa_{n+1} + s^2a_{n+2} + \ldots + s^{n-1}a_{2n-1}
\]
\[
= a_1 + s(a_2 + sa_3 + \ldots + s^{n-1}a_{n+1}) + s^2a_{n+2} + s^3a_{n+3} + \ldots + s^{n-1}a_{2n-1}
\]
\[
\vdots
\]
\[
a_1 + sa_2 + \ldots + s^{n-2}a_n + s^{n-1}(a_{n+1} + sa_{n+2} + s^2a_{n+3} + \ldots + s^{n-1}a_{2n-1})
\]
\[
\nu_s^{[n]}[a_1, a_2, \ldots, a_{n-1}, a_n])^{[n]}[a_{n+1}, a_{n+2}, \ldots, a_{2n-1}] \right],
\]
which satisfy in all lines, if \(s^{n-1} = \text{id}(4.59)\).}

Corollary 4.16. The set of \((n-1)\)-tuples \((4.54)\) with the shift-deformed associative operation \((4.58)\) is a non-derived \(n\)-ary semigroup \(S_{\text{shift}}^{[n]} = \langle \{a\} | \nu_s^{[n]} \rangle\) constructed from the binary semigroup \(A\).

Proposition 4.17. If the binary semigroup \(A\) is commutative, then \(S_{\text{shift}}^{[n]}\) becomes a non-derived \(n\)-ary group \(G_{\text{shift}}^{[n]} = \langle \{a\} | \nu_s^{[n]}, \nu_s^{[1]} \rangle\), such that each element \(a \in A^{n-1}\) has a unique querelement \(\bar{a}\) (an analog of inverse) by
\[
\bar{a} = \nu_s^{[1]}[a] = -(sa + s^2a + \ldots + s^{n-2}a),
\]
where \(\nu_s^{[1]} : A^{n-1} \rightarrow A^{n-1}\) is an unary quereoperation.

Proof. We have the definition of the querelement
\[
\nu_s^{[n]}[\bar{a}, a_1, a_2, \ldots, a_n] = a,
\]
where \(\bar{a}\) can be on any place. So \((4.58)\) gives the equation
\[
\bar{a} + sa + s^2a + \ldots + s^{n-2}a + a = a,
\]
which can be resolved for the commutative and cancellative semigroup \(A\) only, and the solution is \((4.62)\). If \(\bar{a}\) is on the \(i\)th place in \((4.63)\), then it has the coefficient \(s^{i-1}\), and we multiply both sides by \(s^{i-1}\) to get \(\bar{a}\) without any shift operator coefficient using \((4.55)\), which gives the same solution \((4.62)\).}
Proof. The definition of polyadic identity in terms of the deformed \( n \)-ary product in the direct power is
\[
\nu_s^{[n]} \left[ \hat{e}, \hat{e}, \ldots, \hat{e}, a \right] = a, \quad \forall a \in \mathcal{A}^{n-1}. \tag{4.69}
\]
Using (4.53) we get the equation
\[
e^{\hat{e} + s \hat{e} + s^2 \hat{e} + \ldots + s^{n-2} \hat{e} + a} = a. \tag{4.70}
\]
After cancellation by \( a \) we obtain (4.68).
\( \square \)

For \( n = 4 \) and \( e = (\alpha_0, \beta_0, \gamma_0) \) we obtain an infinite set of identities satisfying
\[
e = (\alpha_0, \beta_0, \gamma_0), \quad \alpha_0 + \beta_0 + \gamma_0 = 0. \tag{4.71}
\]
To see that they are \( 4 \)-ary idempotents, insert \( a = e \) into (4.69).

Thus, starting from a binary semigroup \( \mathcal{A} \), using our polyadization procedure we have obtained a nonderived \( n \)-ary group on \((n - 1)\)th direct power \( \mathcal{A}^{n-1} \) with the shift deformed multiplication. This construction reminds the Post-like associative quiver from [DUPLI] [2018, 2022a], and allows us to construct a nonderived \( n \)-ary group from any semigroup in the unified way presented here.

4.4. Polyadization of binary supergroups. Here we consider a more exotic possibility, when the B-matrices are defined over the Grassmann algebra, and therefore can represent supergroups (see (3.36) and below). In this case B’s can be supermatrices of two kinds, even and odd, which have different properties [BEREZIN 1987], [LEITES 1983]. The general polyadization procedure remains the same, as for ordinary matrices considered before (see Definition 4.1), and therefore we confine ourselves to examples only.

Indeed, to obtain an \( n \)-ary matrix (semi)group represented now by the Q-supermatrices (4.1) of the nonstandard form, we should take \((n - 1)\) initial B-supermatrices which present a simple \((k = 1 \) in (3.19)) binary (semi)supergroup, which now have different parameters \( B_{y_i} \equiv B_{y_i} \left( (p_{\text{even}} \mid p_{\text{odd}}) \times (p_{\text{even}} \mid p_{\text{odd}}) \right) \), \( i = 1, \ldots, n - 1 \), where \( p_{\text{even}} \) and \( p_{\text{odd}} \) are even and odd dimensions of the B-supermatrix. The closure of the Q-supermatrix multiplication is governed by the closure of B-supermatrix multiplication (4.3)–(4.5) in the initial binary (semi)supergroup.

4.4.1. Polyadization of \( GL \) \( (1 \mid 1, \Lambda) \). Let \( \Lambda = \Lambda_{\text{even}} \oplus \Lambda_{\text{odd}} \) be a Grassmann algebra over \( \mathbb{C} \), where \( \Lambda_{\text{even}} \) and \( \Lambda_{\text{odd}} \) are its even and odd parts (it can be also any commutative superalgebra). We provide (in brief) the polyadization procedure of the general linear supergroup \( GL \) \( (1 \mid 1, \Lambda) \) for \( n = 3 \). The 4-parameter block (invertible) supermatrices become \( B_{y_i} = \begin{pmatrix} a_i & \alpha_i \\ \beta_i & b_i \end{pmatrix} \in GL \) \( (1 \mid 1, \Lambda) \), where the parameters are \( y_i = (a_i, b_i, \alpha_i, \beta_i) \in \Lambda_{\text{even}} \times \Lambda_{\text{even}} \times \Lambda_{\text{odd}} \times \Lambda_{\text{odd}}, i = 1, 2 \). Thus, the 8-parameter ternary supergroup \( GL^{[3]} \) \( (1 \mid 1, \Lambda) \) is represented by the following \( 4 \times 4 \) Q-supermatrices
\[
Q_{y_1, y_2} = \begin{pmatrix} 0 & B_{y_1} \\ B_{y_2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_1 & \alpha_1 \\ 0 & 0 & \beta_1 & b_1 \\ a_2 & \alpha_2 & 0 & 0 \\ \beta_2 & b_2 & 0 & 0 \end{pmatrix} \in GL^{[3]} \) \( (1 \mid 1, \Lambda) \), \tag{4.72}
\]
which satisfy the ternary (nonderived) multiplication
\[
\mu^{[3]} \left[ Q_{y_1, y_2}, Q_{y_1', y_2'}, Q_{y_1'', y_2''} \right] = Q_{y_1, y_2} Q_{y_1', y_2'} Q_{y_1'', y_2''} = Q_{y_1, y_2}. \tag{4.73}
\]
In terms of the block matrices \( B_{y_i} \), the multiplication (4.17) becomes (see (4.2)–(4.5))
\[
B_{y_1} B_{y_2} B_{y_1'} = B_{y_1}, \tag{4.74}
\]
\[
B_{y_2} B_{y_1'} B_{y_2'} = B_{y_2}. \tag{4.75}
\]
In terms of the $B$-supermatrix parameters the supergroup $GL^{[3]} (1 | 1, \Lambda)$ is defined by
\[
\begin{align*}
\alpha_1^t \beta_2^m a_1^m + a_1^t \alpha_2^m b_1^m + a_1^t \beta_2^m b_1^m + a_1^t \alpha_2^m a_1^m &= a_1, \\
\beta_1^t \alpha_2^m b_1^m + a_1^t \beta_2^m a_1^m + b_1^t \beta_2^m b_1^m + b_1^t \alpha_2^m a_1^m &= b_1, \\
\alpha_2^t \beta_1^m a_1^m + a_2^t \alpha_2^m b_1^m + a_2^t \beta_1^m b_1^m + a_2^t \alpha_2^m a_1^m &= a_2, \\
\beta_2^t \alpha_2^m b_1^m + a_2^t \beta_1^m b_1^m + b_2^t \beta_1^m a_1^m + b_2^t \alpha_2^m a_1^m &= b_2, \\
\alpha_2^t \alpha_2^m a_1^m + a_2^t \alpha_2^m b_1^m + a_2^t \beta_1^m b_1^m + a_2^t \beta_1^m a_1^m &= \beta_1, \\
\beta_2^t \beta_2^m a_1^m + a_2^t \beta_2^m b_1^m + a_2^t \alpha_2^m a_1^m + a_2^t \alpha_2^m b_1^m &= \beta_2.
\end{align*}
\]

(4.76)

The unique querelement in $GL^{[3]} (1 | 1, \Lambda)$ can be found from the equation (see (4.11))
\[
Q_{y_1,y_2} Q_{y_1,y_2} \overline{Q}_{y_1,y_2} = Q_{y_1,y_2},
\]
where the solution is
\[
\overline{Q}_{y_1,y_2} = \left( \begin{array}{cc}
0 & B_{y_1}^{-1} \\
B_{y_2}^{-1} & 0
\end{array} \right),
\]
with (see (4.12))
\[
B_{y_1}^{-1}, B_{y_2}^{-1} \in GL (1 | 1, \Lambda).
\]

**Definition 4.19.** We call $GL^{[3]} (1 | 1, \Lambda)$ a polyadic (ternary) general linear supergroup obtained by the polyadization procedure from the binary linear supergroup $GL (1 | 1, \Lambda)$.

The ternary identity $E_4^{(3)}$ of $GL^{[3]} (1 | 1, \Lambda)$ has the form (see (4.3))
\[
E_4^{(3)} = \left( \begin{array}{cc}
0 & E_2 \\
E_2 & 0
\end{array} \right),
\]
where $E_2$ is the identity of $GL (1 | 1, \Lambda)$, and is ternary idempotent
\[
E_4^{(3)} E_4^{(3)} E_4^{(3)} = E_4^{(3)}.
\]

(4.81)

The ternary supergroup $GL^{[3]} (1 | 1, \Lambda)$ contains the infinite number of ternary idempotents $Q_{y_1,y_2}^{idemp}$ defined by the system of equations
\[
Q_{y_1,y_2}^{idemp} Q_{y_1,y_2}^{idemp} Q_{y_1,y_2}^{idemp} = Q_{y_1,y_2}^{idemp},
\]
which gives
\[
B_{y_1}^{idemp} B_{y_2}^{idemp} = E_2.
\]

(4.83)

Therefore, the idempotents are determined by $8 - 4 = 4$ Grassmann parameters. One of the ways to realize this is to exclude from (4.83) the $2 \times 2 B$-supermatrix. In this case, the idempotents in the supergroup $GL^{[3]} (1 | 1, \Lambda)$ become
\[
Q_{y_1,y_2}^{idemp} = \left( \begin{array}{cc}
0 & B_{y_1}^{-1} \\
(B_{y_2})^{-1} & 0
\end{array} \right),
\]
where $B_{y_i} \in GL (1 | 1, \Lambda)$ is an invertible $2 \times 2$ supermatrix of the standard form (see Remark 4.11).

In the same way one can polyadize any supergroup that can be presented by supermatrices.
5. Conclusions

In this paper we have given answers to the following important questions: how to obtain nonderived polyadic structures from binary ones, and what would be a matrix form of their semisimple versions? First, we introduced a general matrix form for polyadic structures in terms of block-shift matrices. If the blocks correspond to a binary structure (a ring, semigroup, group or supergroup), this can be treated as a polyadization procedure for them. Second, the semisimple blocks which further have a block-diagonal form give rise to semisimple nonderived polyadic structures. For a deeper and expanded understanding of the new constructions introduced, we have given clarifying examples. The polyadic structures presented can be used, e.g. for the further development of differential geometry and operad theory, as well as in other directions which use higher arity and nontrivial properties of the constituent universal objects.

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