BLOW-UP DYNAMICS FOR
SMOOTH FINITE ENERGY RADIAL DATA SOLUTIONS TO
THE SELF-DUAL CHERN–SIMONS–SCHRÖDINGER EQUATION

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Abstract. We consider the finite-time blow-up dynamics of solutions to the
self-dual Chern–Simons–Schrödinger (CSS) equation (also referred to as the
Jackiw–Pi model) near the radial soliton $Q$ with the least $L^2$-norm (ground
state). While a formal application of pseudoconformal symmetry to $Q$ gives
rise to an $L^2$-continuous curve of initial data sets whose solutions blow up in
finite time, they all have infinite energy due to the slow spatial decay of $Q$.
In this paper, we exhibit initial data sets that are smooth finite energy radial
perturbations of $Q$, whose solutions blow up in finite time. Interestingly, their
blow-up rate differs from the pseudoconformal rate by a power of logarithm.
Applying pseudoconformal symmetry in reverse, this also yields a first exam-
ple of an infinite-time blow-up solution, whose blow-up profile contracts at a
logarithmic rate.

Our analysis builds upon the ideas of previous works of the first two authors
on (CSS) [21, 22], as well as the celebrated works on energy-critical geometric
equations by Merle, Raphaël, and Rodnianski [33, 38]. A notable feature of
this paper is a systematic use of nonlinear covariant conjugations by the co-
vARIANT Cauchy-Riemann operators in all parts of the argument. This not only
overcomes the nonlocality of the problem, which is the principal challenge for
(CSS), but also simplifies the structure of nonlinearity arising in the proof.

1. Introduction

The subject of this paper is the nonrelativistic Chern–Simons gauge field theory
introduced by Jackiw–Pi [17], which is a Lagrangian field theory with the action

$$S[\phi, A] := \frac{1}{2} \int_{\mathbb{R}^{1+2}} A \wedge F + \int_{\mathbb{R}^{1+2}} \frac{1}{2} \text{Im}(\bar{\phi} D_t \phi) + \frac{1}{2} |D_x \phi|^2 - \frac{g}{4} |\phi|^4 dt dx,$$

where $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$ is a complex-valued scalar field, $D_\alpha = \partial_\alpha + i A_\alpha$ ($\alpha = t, 1, 2$) are
the covariant derivatives associated with a real-valued 1-form $A = A_t dt + A_1 dx^1 + A_2 dx^2$ (connection 1-form) and $F = dA$ is the corresponding curvature 2-form.

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Note that (1.1) is simply the sum of the Chern–Simons action, \( \frac{1}{2} \int A \wedge F \), and the action for the (gauge-covariant) cubic nonlinear Schrödinger equation. Following a widespread usage in the mathematical literature, we will refer the resulting Euler–Lagrange equation, written below in Section 1.1, as the Chern–Simons–Schrödinger equation.

The Chern–Simons action has been employed in high energy physics and condensed matter physics to describe interesting planar physics, such as topological massive gauge theories and the quantum Hall effect; we refer to [16,19] for detailed reviews. The model (1.1) under consideration is of particular interest as it is the simplest model that is nonrelativistic (which is the setting of condensed matter physics) and, after a particular choice of the coupling constant \( g \) (namely \( g = 1 \)), self-dual. A remarkable consequence of the self-duality, which was observed in the seminal paper of Jackiw–Pi [17], is the existence of explicit(!) spatially-localized static solutions to the model (also referred to as solitons or nontopological vortices) that are parametrized by the solutions to the (explicitly solvable) Liouville equation. In what follows, we refer to these solutions as Jackiw–Pi vortices.

Most basic among the Jackiw–Pi vortices is the ground state \((Q, A)\), given in the polar coordinates \((r, \theta)\) by

\[
Q(r, \theta) = \sqrt{\frac{1}{1 + r^2}}, \quad A_t = \frac{1}{2} Q^2, \quad A_r = 0, \quad A_\theta = -2 \frac{r^2}{1 + r^2},
\]

which has the minimal charge (i.e., the integral of \(|Q|^2\)) among all Jackiw–Pi vortices. The charge is a natural measure of the size of a solution, as it is invariant under the scaling symmetry of (1.1). The ground state \(Q\) plays a pivotal role in the dynamics of solutions. Indeed, within radial symmetry, it is known that the \(L^2\)-norm of \(Q(x)\) serves as the threshold for global regularity and scattering [27].

An outstanding problem, then, is to understand the dynamics of solutions associated to initial data in the vicinity of \(Q(x)\), with the \(L^2\)-norm greater than or equal to that of \(Q(x)\).

In this regime, an interesting formal dynamics describing finite-time blow up follows from the pseudoconformal symmetry of (1.1). Like the well-known cubic NLS on \(\mathbb{R}^{1+2}\), the Chern–Simons–Schrödinger equation is invariant under the pseudoconformal transformations

\[
(t, x) = (\frac{T}{1 - b^2}, \frac{X}{1 - b^2}), \quad \Phi_b(T, X) = \frac{1}{1 - b^2} e^{-\frac{b^2 T^2}{2}} \phi(\frac{T}{1 - b^2}, \frac{X}{1 - b^2}),
\]

where \(b \in \mathbb{R}\). Applying such transformations with \(b > 0\) to the ground state, we obtain a one parameter family of solutions \((S_b, A_b)\) blowing up in finite time (namely, at \(T = b^{-1}\)). Each \(S_b\) has the same \(L^2\)-norm as \(Q\) and \(S_b(t = 0) \to Q\) in \(L^2\) as \(b \to 0^+\). However, because of the slow spatial decay of \(Q\), each \(S_b\) \((b > 0)\) has infinite \(H^1\)-norm (as well as infinite conserved energy, which is defined below). As a result, if we consider the dynamics of finite energy solutions in the vicinity of \(Q\), the relevance of \(S_b\) and even the possibility of a finite-time blow up are dubious [1].

The main result of this paper is the first construction of finite time blow-up solutions with smooth finite energy radial initial data, which are arbitrarily close to \(Q\) in the \(L^2\)-topology. A detailed description of the blow-up dynamics is given; in particular, we provide a codimension one set of data leading to the blow-up, as well as a sharp description of the rate. The blow-up rate differs from the pseudoconformal rate by a factor of logarithm. This is a sharp contrast to the case of higher equivariance indices \(m \geq 1\), in which case the pseudoconformal blow-up rate is obtained [21]. Interestingly, our blow-up rate is identical to that obtained in the

\footnote{Another standard method to deduce finite-time blow up is using the virial identity à la Glassey, but in the self-dual case, it only leads to a pseudoconformal transform of a static solution; see [22].}
1-equivariant Schrödinger maps \[33\]. Via the pseudoconformal transform, we also construct infinite-time blow-up solutions with the blow-up profile \(Q\), whose scale contracts at a rate logarithmic in \(t\).

Our analysis follows the road map furnished by the seminal works of Rodnianski–Sterbenz \[41\], Raphaël–Rodnianski \[38\], and Merle–Raphaë l–Rodnianski \[33\] in the cases of wave maps, Yang–Mills, and Schrödinger maps. Compared to the previously considered cases, a key challenge in the Chern–Simons–Schrödinger case is the nonlocality of the nonlinearity, which results in a stronger soliton-radiation interaction. Notable features of our proof are a systematic use of nonlinear covariant conjugations, and the treatment of the self-dual Chern–Simons–Schrödinger equation as a coupled system of nonlinearly conjugated variables of varying orders. These ideas provide a simple and efficient way to overcome the nonlocality of the problem. This point of view pervades all steps of our arguments, such as the derivation of modified profiles and sharp modulation laws, decomposition of solutions, and energy estimates. See Section 1.4 for more details.

1.1. The self-dual Chern–Simons–Schrödinger equation. The Euler–Lagrange equation for (1.1) in the self-dual case \(g = 1\) takes the form

\[
\begin{align*}
    D_t \phi &= i(D_1 D_1 + D_2 D_2) \phi + i|\phi|^2 \phi, \\
    F_{11} &= -\text{Im} (\overline{\phi} D_2 \phi), \\
    F_{12} &= \text{Im} (\overline{\phi} D_1 \phi), \\
    F_{12} &= -\frac{1}{2} |\phi|^2.
\end{align*}
\]

We remind the reader that \(\phi : \mathbb{R}^{1+2} \rightarrow \mathbb{C}\) is a complex-valued scalar field, \(D_\alpha = \partial_\alpha + iA_\alpha (\alpha = t, 1, 2)\) are the covariant derivatives associated with a real-valued 1-form \(A = A_t dt + A_1 dx^1 + A_2 dx^2\) (connection 1-form) and \(F = dA\) is the corresponding curvature 2-form. We will refer to this equation as the (self-dual) Chern–Simons–Schrödinger (CSS) equation.

Symmetries and conservation laws. We describe some gauge-covariant symmetries and their associated conservation laws of (1.3) that are of importance in the present work. In this aspect, (1.3) shares many similarities with the cubic NLS

\[i \partial_t \phi + \Delta \phi + |\phi|^2 \phi = 0\]
on \(\mathbb{R}^{1+2}\).

Among the most basic symmetries are the time translation symmetry

\[(t, x) = (t' + t_0, x'), \quad \tilde{\phi} = \phi, \quad (t_0 \in \mathbb{R})\]
and the phase rotation symmetry

\[(t, x) = (t', x'), \quad \tilde{\phi} = e^{i\gamma} \phi, \quad (\gamma \in \mathbb{R})\]

Associated to these symmetries are the conservation laws for the energy and the charge:

\[
E[\phi, A] := \int_{\mathbb{R}^2} \frac{1}{2} |D_x \phi|^2 - \frac{1}{4} |\phi|^4 \, dx
\]
\[
M[\phi] := \int_{\mathbb{R}^2} |\phi|^2 \, dx.
\]

Next, of particular importance in this work are the scaling symmetry,

\[(t, x) = (\lambda^{-2} t', \lambda^{-1} x'), \quad \phi' = \lambda^{-1} \phi, \quad (\lambda > 0)\]

2Each symmetry described here consists of a pre-composition of \(\phi\) with a coordinate transform \((t', x') \rightarrow (t, x)\) and a further transformation of the resulting \(\phi(t', x')\). Gauge covariance refers to the feature that the 1-form \(A\) is simply pulled back by \((t', x') \rightarrow (t, x)\).
under which the $L^2$-norm (or $M[\phi]$) is invariant, and the discrete pseudoconformal symmetry;

$$\tag{1.4} (t, x) = (-\frac{1}{t}, \frac{x}{t}), \quad \phi'(t', x') = \frac{1}{t} e^{i|\phi|^2} \phi.$$  

Associated to these symmetries are the virial identities

$$\begin{cases} 
\partial_t \left( \int_{\mathbb{R}^2} |x|^2 |\phi|^2 dx \right) = 4 \int_{\mathbb{R}^2} x^j \Im(\phi \overline{D}_j \phi) dx, \\
\partial_t^2 \left( \int_{\mathbb{R}^2} x^j \Im(\phi \overline{D}_j \phi) dx \right) = 4E[\phi, A].
\end{cases}$$

Self-duality. The particular choice of the coefficient $g = 1$ in front of $|\phi|^2 \phi$ in (1.3) makes this system self-dual: the minimizers of the Hamiltonian $E[\phi]$, which turn out to coincide with static solutions, are characterized by a first order (as opposed to second order) elliptic equation (see (1.7) below).

We introduce the covariant Cauchy–Riemann operator $D_+$ and its formal $L^2$-adjoint:

$$D_+ := D_1 + iD_2, \quad D_+^* = -D_1 + iD_2.$$  

Observe that

$$D_+^* D_+ = -D_1^2 - D_2^2 - \frac{1}{2} |\phi|^2.$$  

As a consequence, the first equation of (1.3) can be written in the form

$$\tag{1.5} (iD_t + \frac{1}{2} |\phi|^2) \phi - D_+^* D_+ \phi = 0.$$  

Moreover, observe that

$$\frac{1}{2} \int_{\mathbb{R}^2} |D_+ \phi|^2 dx = \frac{1}{2} \int_{\mathbb{R}^2} \Re(\overline{\phi} D_+^* D_+ \phi) dx$$

$$= -\frac{1}{2} \int_{\mathbb{R}^2} \Re(\overline{\phi} (D_1^2 + D_2^2) \phi) dx - \frac{1}{4} \int_{\mathbb{R}^2} |\phi|^4 dx.$$  

After an integration by parts, the last line is exactly the conserved energy of the self-dual(!) Chern–Simons–Schrödinger equation, i.e.,

$$E[\phi, A] = \frac{1}{2} \int_{\mathbb{R}^2} |D_+ \phi|^2 dx.$$  

Therefore, the minimum energy is zero, and the energy minimizers obey the Bogomol’nyi equation

$$\tag{1.6} \begin{cases} 
D_+ \phi = 0, \\
F_{12} = -\frac{1}{2} |\phi|^2.
\end{cases}$$  

The last property is the manifestation of self-duality. Any zero-energy solution (or equivalently, a solution to (1.7)) is a static (i.e., $\partial_t \phi = 0$) solution to (1.3) with $A_t = -\frac{1}{2} |\phi|^2$. Conversely, any static solution with $\phi \in H^1$ and mild conditions on $A_t, A_j$ (e.g., boundedness) necessarily has zero energy and $A_t = -\frac{1}{2} |\phi|^2$ [15].

It was observed by Jackiw–Pi [16] that, at points where $\phi$ is nonzero, (1.7) implies that $|\phi|^2$ solves the Liouville equation $\Delta (\log |\phi|^2) = -|\phi|^2$. The ground state $|Q|^2$ is the unique (up to obvious symmetries) positive finite charge solution to the Liouville equation [6].

---

3 The aforementioned continuous family of pseudoconformal transformations arise by composing the discrete version with the symmetries discussed so far.
Cauchy problem formulation and the Coulomb gauge. The equation (1.3) has gauge invariance, i.e., for any real-valued function $\chi$ (gauge transformation), if $(\phi, A)$ is a solution, then so is its gauge transform $(e^{i\chi}\phi, A - d\chi)$. Accordingly, uniqueness of a solution to the Cauchy problem may be formulated only up to gauge invariance. In order to fix gauge invariance and obtain a (locally) well-posed Cauchy problem, we need to impose a condition on $A$.

In this paper, we impose the Coulomb gauge condition,

$$\partial_t A_1 + \partial_r A_2 = 0,$$

along with a suitable decay condition\(^4\) for $A(t, x)$ as $|x| \to \infty$ (at every $t$) to rule out nontrivial gauge transformations. We mention that (1.3) in Coulomb gauge, viewed along with a suitable decay condition

$$\partial_t A_1 + \partial_r A_2 = 0,$$

where

$$\delta \phi$$

is the Fréchet derivative with respect to the real inner product $\int_{\mathbb{R}^3} \Re(\bar{\psi}\phi)dx$, and $E[\phi]$ is the energy with $A$ determined by $\phi$ and the Coulomb gauge condition.

Equivalence within Coulomb gauge. We begin with a short general discussion of the general equivariance ansatz for (1.3). A complex-valued function $\psi$ on $\mathbb{R}^2$ is said to be $m$-equivariant if

$$\psi(t, r, \theta) = e^{im\theta} v(r)$$

for some radial function $v(r)$, which we refer to as the radial profile of $\psi$. Note that 0-equivariance is equivalent to radiality. By (1.3), if $\phi$ is $m$-equivariant at a fixed $t$, then $F_{ru}$, $F_{t\theta}$ and $F_{r\theta}$ are radial. As the Coulomb gauge condition is also radially symmetric, it follows that, as long as local wellposedness holds, (1.3) in Coulomb gauge preserves $m$-equivariance of $\phi$ for any $m \in \mathbb{Z}$.

Under the $m$-equivariance and Coulomb gauge conditions, $A_r$, $A_\theta$ and the Coulomb gauge condition reduces to $A_r = 0$. The radial profile $u$ of $\phi$, defined by

$$\phi(t, r, \theta) = e^{im\theta} u(t, r),$$

obeys

$$i(\partial_t + A_r[u])u + \partial_r^2 u + \frac{1}{r} \partial_r u - \frac{1}{r^2} (m + A_\theta[u])^2 u + |u|^2 u = 0,$$

where $A_r[u]$, $A_\theta[u]$ are quadratic forms (over $\mathbb{R}$) of $u$ given by

$$A_r[u] = -\int_r^\infty (m + A_\theta) \Re(\bar{u}u) \frac{dr'}{r'}, \quad A_\theta[u] = -\frac{1}{2} \int_0^\infty \Re(\bar{u}u)\theta dr'.$$

We write $A_\theta[u, v]$ for the real bilinear form obtained by polarization. Using $\partial_r A_r = F_{rt}$ and $\partial_r A_\theta = F_{r\theta}$, as well the decay and smoothness properties of $A_r$ and $A_\theta$, it may be easily verified that the connection 1-form $A$ agrees with $A_r[u]dt + A_\theta[u]d\theta$.

Equations (1.10) and (1.11) furnish an evolutionary equation for the radial profile $u$ of an $m$-equivariant solution $\phi$ to (1.3) in Coulomb gauge.

The Cauchy–Riemann operator $D_\phi$ maps $m$-equivariant functions to $(m + 1)$-equivariant functions. Given $A_r = 0$ and $A_\theta = A_\theta[v]$, where $v$ may be the radial profile of an arbitrary $m'$-equivariant function, the radial Cauchy–Riemann operator $(m)D_\phi$ acting on an $m$-equivariant function is defined by the relation

$$D_+(e^{im\theta} w(r)) = e^{i(m+1)\theta} (m)D_\phi w(r),$$

\(^4\)The decay condition will be implicit in the formulae for the components of $A$ in (1.11).
and takes the form
\[(1.12)\quad (m)D_v w = \partial_t w - \frac{1}{r}(m + A_0[v])w.\]

As observed in [22], the nonlinear equation (1.10) can be written in a self-dual form. For radial functions \(v, w\), we also introduce the notation \((m)L_v w\) for the linearization of the (radial) Bogomol’nyi operator \(v \mapsto (m)D_v v\) around \(v\). It may be expressed as
\[(1.13)\quad (m)L_v w = (m)D_v w - \frac{2}{r}A_0[v]w = \partial_t w - \frac{1}{r}(m + A_0[v])w + \frac{v}{r} \int_0^r \text{Re}(\overline{w}w)\,dr'.\]

As an immediate application of (1.6) and (1.13), we see that the Hamiltonian
\[(1.14)\quad \partial_t u + (m)L_u^* (m)D_u u = 0,\]
where \((m)L_u^*\) is the formal \(L^2\)-adjoint of \((m)L_u\).

Finally, for each \(m \geq 0\), there is an explicit \(m\)-equivariant Jackiw-Pi vortex, which is unique up to the symmetries of the equation:
\[Q^{(m)}(r)e^{im\theta} = \sqrt{8}(m + 1)\frac{r^m}{1 + r^{2m+2}}e^{im\theta}.\]

1.2. Known results. A brief discussion of the known results on the Cauchy problem for (1.3) is in order. The well-posedness of (1.3) was first studied in Coulomb gauge; after the earlier works [2,14], Lim [26] proved local well-posedness for (1.3) is in order. The well-posedness of (1.3) was first studied in Coulomb gauge, small data \(H^0\) local well-posedness is proved by Liu-Smith-Tataru [28]. Under equivariance within Coulomb gauge, the equation becomes semilinear and the \(L^2\)-critical local well-posedness can be achieved; see [27].

There are also works on the long-term dynamics. Bergé-de Bouard-Saut [2] used Glassey’s convexity argument [10] to derive a sufficient condition for finite-time blow up. However, this method essentially applies for negative energy solutions, which exist only if \(g > 1\). The same authors [3] carried out a formal computation to derive the log-log blow-up for negative energy solutions. Recently, Oh-Pusateri [30] showed global existence and scattering for small data in weighted Sobolev spaces. Under equivariance within Coulomb gauge, Liu-Smith [27] proved global well-posedness and scattering below the charge of the ground state, \(M[Q^{(m)}]\), for each equivariance class.

Within each equivariance class, a natural question is the dynamics beyond the threshold. At the threshold charge, in addition to the vortex solution \(Q^{(m)}\), there is an explicit finite-time blow-up solution
\[S^{(m)}(t,r) = \frac{1}{|t|}Q^{(m)}\left(\frac{r}{|t|}\right)e^{-\frac{|t|^2}{16}}, \quad t < 0,\]
which is obtained by applying the pseudoconformal transform to \(Q^{(m)}\). Recently, the first and second authors gave a quantitative description of the dynamics in the vicinity of \(S^{(m)}(t)\). When \(m \geq 1\), the authors in [22] constructed pseudoconformal blow-up solutions with a prescribed asymptotic profile. Moreover, they exhibited the rotational instability (see the discussion following (1.12)) of these solutions. This is a backward construction, and an analogue of the construction of Bourgain-Wang solutions and their instability in the NLS context [35].

\[^5\text{We mean by a pseudoconformal blow-up solution, a finite-time blow-up solution} u \text{ that decomposes as} u(t,r) \approx S^{(m)}(t,r) + z(t,r) \text{ with some regular} z(t,r) \text{ near the blow-up time.}\]
On the other hand, when $m \geq 1$, the same authors \cite{21} studied conditional stability of pseudoconformal blow-up solutions in the context of the Cauchy problem. Indeed, they considered the forward construction problem, and constructed a codimension one set of initial data leading to pseudoconformal blow-up, i.e.,

$$u(t,r) = \frac{e^{\gamma^*}}{e^{(T-t)}} Q^{(m)} \left( \frac{r}{e^{(T-t)}} \right) \to u^* \text{ in } L^2$$

for some $\gamma^* \in \mathbb{R}$ and $\ell \in (0, \infty)$ as $t \to T$. The blow-up solutions constructed there are smooth and have finite energy. Moreover, when $m \geq 3$, they constructed a codimension one Lipschitz manifold of initial data yielding pseudoconformal blow-up. In view of \cite{22}, the codimension one condition seems to be optimal.

The aforementioned works \cite{21, 22} only deal with the $m \geq 1$ case. In the current paper, we consider the most physically relevant (and also delicate) case: $m = 0$.

1.3. Main results. Now we specialize to the setting of the present paper. Note, from (1.2), that the ground state $(Q, A)$ is radial ($m = 0$) and obeys the Coulomb gauge condition, with the radial profile

$$Q(r) := Q^{(0)}(r) = \sqrt{\frac{8}{1 + r^2}}.$$

In the remainder of the paper, unless otherwise stated, we assume that $(\phi, A)$ is a radial solution to (1.3) in Coulomb gauge. Namely, we let $m = 0$ and consider

$$\phi(t, x) = u(t, r), \quad A_t[u] = -\int_r^\infty A_0[u]^2 dr', \quad A_\theta[u] = -\frac{1}{2} \int_0^r |u|^2 r' dr'.$$

The equation for $u$ is given by

$$i(\partial_t + iA_t[u])u + \partial_r^2 u + \frac{1}{r} \partial_r u - \frac{1}{r^2} A_\theta^2[u]u + |u|^2 u = 0.$$  

To simplify the notation, we introduce the following shorthands for the first two radial Cauchy–Riemann operators:

$$D_{v,w} := (0) D_{v,w} = \partial_r w - \frac{1}{r} A_{v} w,$$

$$A_{v,w} := (1) D_{v,w} = \partial_r w - \frac{1}{r} (1 + A_\theta[w]) w.$$  

We also use the shorthand

$$L_{v,w} := (0) L_{v,w} = (0) D_{v,w} - \frac{2}{r} A_\theta[v, w] v.$$  

Note that $D_{v}$ and $A_{v}$ are local operators, but $L_{v}$ is a nonlocal operator. The aforementioned self-dual form \cite{14} reads

$$\partial_t u + iL_{u}^* D_{u} u = 0.$$

One of the fundamental differences between the $m \geq 1$ case and the present case $m = 0$ is that $S^{(0)}(t)$ is no longer a finite energy solution, due to the slow decay of $Q$. Though $S^{(0)}(t)$ provides an example of finite-time blow-up $L^2$-solution (with the pseudoconformal blow-up rate $|t|$), it was left open until now whether (CSS) possesses a smooth finite energy blow-up solutions. Our main result answers that such solutions do exist.

By a forward construction, sharper descriptions of the constructed blow-up solutions can be provided. In fact, we show that there exists a codimension one set of initial data yielding finite-time blow-up solutions, whose blow-up rate differs logarithmically from the pseudoconformal blow-up rate.
We define the set $O$ more details. The precise statement of our main result is as follows. It will be shown that the set $O$ data satisfies:

\[(1.20) \tilde{U}_{\text{init}} := \{(\lambda_0, \gamma_0, b_0, \epsilon_0) \in \mathbb{R}^+ \times \mathbb{R} / 2\pi \mathbb{Z} \times \mathbb{R} \times \mathbb{Z}^\perp : b_0 \in (0, b^*), \|\epsilon_0\|_{H_0^3} < b_0^3\} \]

We define the set $U_{\text{init}}$ of coordinates

\[U_{\text{init}} := \{(\lambda_0, \gamma_0, b_0, \epsilon_0) : (\lambda_0, \gamma_0, b_0, \epsilon_0) \in \tilde{U}_{\text{init}}, \, \eta_0 \in (-\frac{b_0}{2|\log b_0|}, \frac{b_0}{2|\log b_0|})\} \]

We define the set $O_{\text{init}}$ by the set of images:

\[O_{\text{init}} := \{e^{\frac{r}{\lambda_0}}[P(\cdot; b_0, \eta_0) + \epsilon_0] \left(\frac{r}{\lambda_0}\right) : (\lambda_0, \gamma_0, b_0, \eta_0, \epsilon_0) \in U_{\text{init}}\} \subseteq H_0^3,\]

where $P(\cdot; b_0, \eta_0)$ is the modified profile defined in Section 4 such that $P(\cdot; 0, 0) = Q$. It will be shown that the set $O_{\text{init}}$ is open, $Q$ lies in the boundary of $O_{\text{init}}$, and the elements of $U_{\text{init}}$ serve as coordinates of the elements of $O_{\text{init}}$. See Lemma $\triangleleft$ for more details. The precise statement of our main result is as follows.

**Theorem 1.1** (Smooth finite energy blow-up solutions). There exists $b^* > 0$ with the following properties. Let $(\lambda_0, \tilde{\gamma}_0, \tilde{b}_0, \tilde{\epsilon}_0) \in \tilde{U}_{\text{init}}$. Then, there exists $\tilde{\gamma}_0 \in (-\frac{\tilde{b}_0}{2|\log \tilde{b}_0|}, \frac{\tilde{b}_0}{2|\log \tilde{b}_0|})$ such that the solution $u(t, r)$ to (CSS) starting from the initial data

\[(1.21) u_0(r) = e^{\frac{r}{\lambda_0}}[P(\cdot; b_0, \eta_0) + \epsilon_0] \left(\frac{r}{\lambda_0}\right) \in O_{\text{init}}\]

satisfies:

- (Finite-time blow up) $u$ blows up in finite time $T = T(u_0) \in (0, \infty)$.
- (Sharp description of the blow up) There exist $\ell = \ell(u_0) \in (0, \infty)$, $\gamma^* = \gamma^*(u_0) \in \mathbb{R}$, and $\gamma^* = \gamma^*(u_0) \in L^2$ such that

  \[u(t, r) - e^{\gamma^*}r^2 (\log(T - t))^2 \left(\frac{t}{(T - t)}\right) Q \left(\frac{|\log(T - t)|^2}{\ell(T - t)}\right) \to u^* \text{ in } L^2\]

  as $t \to T$.

- (Regularity of the asymptotic profile) $u^*$ has the regularity

  \[u^* \in H^1_0.\]

Applying the pseudoconformal transform to the solution constructed in Theorem 1.1 we can construct an infinite-time blow-up solution to (CSS).

**Corollary 1.2** (Infinite-time blow up). There exists a smooth compactly supported (radial) initial data $u_0$ such that the corresponding forward-in-time solution $u$ to (CSS) blows up in infinite time with

\[u(t, r) - (\log t)^2 Q((\log t)^2 r) - e^{it\Delta} u^* \to 0 \text{ in } L^2\]

as $t \to \infty$, for some $u^* \in L^2$.\footnote{We denote by $H^k_m$ the Sobolev space $H^k(\mathbb{R}^2)$ restricted on $m$-equivariant functions. See Section 5.3 for more details.}
Comments on Theorem 1.1 and Corollary 1.2.

1. Finite energy solutions. Not only does the constructed blow-up solutions have finite energy, we can take their initial data to be smooth and compactly supported. Indeed, the profile itself does not have a compact support due to the \( Q \)-part of \( P \). However, by carefully choosing \( \tilde{\epsilon}_0 \) to delete the tail of \( Q \), it is possible to make \( u_0 \) compactly supported.

The deviation by a logarithmic factor from the pseudoconformal blow-up rate stems from the fact that \( S^{(0)}(t) \) has infinite energy. In the context of (NLS), the well-known log-log blow-up rate \([30,32]\), which deviates by a log-log factor from the self-similar blow-up rate, comes from the fact that the exact self-similar solution barely fails to lie in \( L^2 \). A similar remark applies in the wave maps \([38]\).

2. Forward construction. Our method relies on the forward construction and modulation analysis. When \( m \geq 1 \), \( S^{(m)}(t) \) has finite energy, and the forward construction in the previous work \([21]\) yields exactly the pseudoconformal blow-up, which is different from here. See Section 1.4 for more details on the forward construction and novel ideas in the present paper. The arguments used here supercede the old argument in \([21]\). Indeed, we think that one can simplify the proof of \([21]\) for \( m \geq 1 \) by the present method. More precisely, one can further take advantages of the nonlinear ansatz of \([21,22]\) and use the modified profiles for \( w, w_1, w_2 \) (see Section 1.4) as

\[
P = Q^{(n)}_b \chi_{B_0}, \quad P_1 = -(ib + \eta)Q^{(n)}_b \chi_{B_0}, \quad P_2 = (ib + \eta)^2 Q^{(n)}_b \chi_{B_0}.
\]

3. Backward constructions. When \( m \geq 1 \), the first two authors considered the backward construction of blow-up solutions in \([22]\). There, the interaction between the blow-up profile and the asymptotic profile is weak (though some nontrivial nonlocal interactions lead to extra phase rotation of the solution) and the blow-up is given by the pure pseudoconformal blow-up. However, the current \( m = 0 \) case can be viewed as a strongly interacting regime, as can be seen in the logarithmic corrections to the blow-up rates in Theorem 1.1 and Corollary 1.2. We expect that continuous blow-up rates as in \([20,23,24,37]\) might be available in the \( m = 0 \) case by a suitable backward construction.

4. Comparison with the mass-critical NLS: \( \text{(CSS)} \) shares all the symmetries and conservation laws with (NLS). (NLS) has a standing wave solution \( e^{itR(x)} \) with exponentially decaying profile \( R \), but the static solution \( Q \) to \( \text{(CSS)} \) only shows a polynomial spatial decay. Thanks to the pseudoconformal symmetry, there are explicit pseudoconformal blow-up solutions like \( S(t) \) in both cases.

In (NLS), there is a stable blow-up regime, the log-log blow-up for negative energy solutions. However, in \( \text{(CSS)} \), the energy is always non-negative and we believe that stable blow-up regimes do not exist for \( \text{(CSS)} \). Nevertheless, the non-self-dual case \( g > 1 \) is expected to have stable blow-up dynamics as in the (NLS) case; see \([3]\) for a formal derivation of the log-log blow-up for negative energy solutions.

Bourgain-Wang type solutions exist and are unstable in both cases. However, the instability mechanisms differ drastically; we expect the rotational instability for \( \text{(CSS)} \), but the Bourgain-Wang solutions arise as the border of log-log blow-up solutions and global scattering solutions. The difference is due to the different spectral properties of the linearized operator, see \([22]\).

One notable feature only arising in the \( \text{(CSS)} \) case is that we have a log-corrected pseudoconformal rate due to the slow spatial decay of \( Q \).

5. Comparison with Schrödinger and wave maps. \( \text{(CSS)} \) has a remarkable parallel with the Schrödinger and wave maps. Firstly, as observed in \([21]\), after a linear conjugation, the resulting linearized operator is the same as those of wave maps,
Schrödinger maps, and harmonic map heat flows. Secondly, the first correction in the profile construction, which is the source of logarithmic correction to the blow-up rate, is the same as in the wave maps case [38]. However, the modulation equations are quite different from the wave maps case, due to the difference between Schrödinger and wave nature of the equations. In particular, our modulation equations are of the form \( \frac{1}{\lambda^2} + b \approx 0 \) and \( b_s + b^2 + \frac{2b}{\log b} \approx 0 \), while in the wave maps case, the \( b^2 \) term is missing. Interestingly, this equation is the same as that of the Schrödinger maps, which gives arise the same asymptotics for the blow-up rate [33]. However, the higher order terms in the \( b \) and \( \eta \) equations, which are irrelevant to the blow-up rate, are different. Finally, we note that the blow-up dynamics in the higher equivariance case has a completely different story from the Schrödinger maps case: [CSS] has pseudoconformal blow-up solutions for all \( m \geq 1 \) [21][22], but the asymptotic stability is known for \( k \geq 3 \)-equivariant Landau-Lifshitz-Gilbert flows (including both the Schrödinger maps and harmonic map heat flows) [1][12].

6. Connection with moduli space dynamics. As pointed out in [41] in the case of wave maps, the approach of this paper may be thought of as a refinement of the adiabatic approximation by a moduli space dynamics, i.e., approximation of solutions to [CSS] with data close to \( Q \) by a reduced dynamics on the finite dimensional submanifold formed by the static solutions \( \{ \frac{1}{\lambda^2} Q(\gamma) : \lambda \in (0, \infty), \gamma \in \mathbb{R}/2\pi \mathbb{Z} \} \) (moduli space). Our finite-time blow-up solution is formally connected with an incomplete trajectory on the moduli space (along which \( \lambda \to 0 \)). This subject has a rich tradition of its own; we refer to [9] for the study of a model closely related to ours (Manton’s model), and to the monographs [1][29] for generalities.

7. Regularity of the asymptotic profile. We believe, in parallel to the Schrödinger maps case, that the regularity of asymptotic profile in Theorem 1.1 is not better than \( H^1 \). This would require more precise information of the radiation term and careful measuring of the flux as in [31]. In this sense, we further expect that different blow-up rates will be obtained from smooth asymptotic profiles, which is typically assumed in the backward construction problems.

8. Rotational instability. The blow-up solutions constructed in Theorem 1.1 and also in [21] (when \( m \geq 1 \)) are non-generic and obtained in the regime \( |\eta| \ll b \). A natural question is the dynamics near these blow-up solutions.

When \( m \geq 1 \), in view of the modulation equations \( \frac{1}{\lambda^2} + b = 0, \gamma_s = (m + 1)\eta, b_s + b^2 + \eta^2 \approx 0, \) and \( \eta_s \approx 0 \), the regime with \( \eta(t) \approx \eta_0 \neq 0 \) seems to be generic. In this regime, solutions concentrate to the spatial scale up to \( |\eta_0| \), then stop concentrating but exhibit a quick rotation of the phase by \( \text{sgn}(\eta_0)(m + 1)\pi \) on the time interval of length \( \sim |\eta_0| \), and then spread out. This nonlinear scenario is presented by constructing an explicit one-parameter family of solutions [22]. It is conjectured in [21] that the aforementioned rotational instability is universal in the vicinity of pseudoconformal blow-up solutions.

When \( m = 0 \), even constructing an explicit continuous family of solutions exhibiting the instability of blow-up solutions remains an interesting open question. The analysis of the instability mechanism will require more refinements to the modulation equations. In our construction in Theorem 1.1 we found logarithmic corrections to the above modulation equations, but only when \( |\eta| \leq \frac{b}{|\log b|} \), which is valid only in the blow-up regime.

In fact, the rotational instability is also expected in other relevant equations. Authors in [42] present formal computations and numerical evidences for a quick rotation by the angle \( \pi \) for the 1-equivariant Landau-Lifshitz-Gilbert equation.
1.4. Strategy of the proof. We use the forward construction with modulation analysis. We view solutions $u$ of the form

$$u(t, r) = e^{i\gamma(t)} [P(\cdot; b(t), \eta(t)) + \epsilon(t, \cdot)]\left(\frac{r}{\lambda(t)}\right),$$

where $P(\cdot; b, \eta)$ is some modified profile with $P(\cdot; 0, 0) = Q$ and $\epsilon$ is the error term. The main steps of the proof are the construction of the modified profiles $P$ and the control of $\epsilon$. We use the method of tail computations to construct the modified profile $P$ and derive the sharp modulation laws of $\lambda, \gamma, b, \eta$. In order to control $\epsilon$ forward-in-time, we use a robust energy method (with repulsivity) to higher order derivatives of $\epsilon$.

This argument was used to address the forward construction of blow-up dynamics in various contexts. To list a few, we refer to Rodnianski-Sterbenz [41], Raphaël-Rodnianski [38], and Merle-Raphaël-Rodnianski [33] for energy-critical wave maps and Schrödinger maps. We also refer to [13, 39, 40] for other energy-critical equations. The method also extends to the energy-supercritical equations [7, 8, 34]. For (CSS) with $m \geq 1$, the first two authors [21] used this argument for blow-up constructions. This list is not exhaustive. The most relevant ones to this work are [21, 33, 85].

On top of such an existing road map, our main novelty is a systematic use of non-linear covariant conjugation identities for the self-dual Chern–Simons–Schrödinger equation. With this strategy, we are able to overcome most of the difficulties coming from nonlocal nonlinearities. We use this strategy in all steps of the proof.

1. Covariant conjugation identities. The main idea is to view the dynamics not only in the $u$-variable (CSS), but also in its canonical higher order derivatives of $u$. The reader may keep in mind that $u$ has a decomposition of the form (1.22).

Motivated from $D_Q Q = 0$, we look at the variable $u_1 = D_u u$. This nonlinear transform hides (or kills) the modulated $Q$ part, and thus $u_1$ enjoys degeneracy, i.e., $u_1 = 0$ if $u$ coincides with a modulated $Q$. Moreover, the conjugation via $D_u$ behaves very nicely with the original equation (CSS); $u_1$ solves a surprisingly simple equation

$$\partial_t u_1 + iA_u^* A_u u_1 + \left(\int_r^\infty \text{Re}(\pi u_1) dr'\right) iu_1 = 0.$$  

This is the covariant conjugation identity. This shares a similar spirit with [11, 12] in the Schrödinger maps context, where the authors utilized the Hasimoto transform [5] to make the equation semilinear, hide the harmonic map portion of the solution, and be able to look only at the error part. Because the transform hides the soliton part, the modulation laws of $\lambda$ and $\gamma$ can only be dictated from the original equation (CSS) (or at the map level in the Schrödinger maps case).

The equation (1.23) was derived in [21], but it was used in a linearized form,

$$L_Q iL^*_Q = iA^*_Q A_Q.$$  

As opposed to $L_Q$, which is only $R$-linear and nonlocal, the operator $A_Q$ is $C$-linear and local. Remarkably, the second order operator $A^*_Q A_Q = H_Q$ coincides with the linearized operator arising in Schrödinger maps, wave maps, and harmonic map heat flows. Experience from these equations further reveals a repulsivity structure for (CSS), which enabled the analysis in [21].

Although the nonlinear transform $u \mapsto u_1 = D_u u$ kills the modulated $Q$ part, the generalized null modes $i\pi^e Q$ and $\rho$ (see Lemma [5.1]) are still alive, in view of $L_Q i\pi^e Q = \frac{1}{4} i\pi Q$ and $L_Q \rho = \frac{1}{4} \rho$. We now notice that these modes can be also deleted in view of $A_Q (r Q) = 0$ if we take conjugate further by $A_u$ and consider...
\[ u_2 = A_u D_u u. \] This further conjugation also behaves very nicely with the \( u_1 \)-equation (1.23) and yields the following simple equation for \( u_2 \):

\[
\partial_t u_2 + i A_u \lambda u_2 - i\eta(u_1)^2 + \left( \int_r^\infty \Re(\overline{u_1}) dr' \right) i u_2 = 0.
\]

In the following analysis, we will view (CSS) as a system of all the above equations (1.19), (1.23), and (1.25), with compatibility conditions \( u_1 = D_u u \) and \( u_2 = A_u D_u u \). We also take advantages of the degeneracies of the variables \( u_1 \) and \( u_2 \). We note that the derivation of these equations becomes apparent if we formulate (CSS) in terms of Wirtinger derivatives, as is done in Section 2 below.

2. Setup for the modulation analysis. Fix \((\hat{\lambda}_0, \hat{\gamma}_0, \hat{\eta}_0, \hat{e}_0) \in \hat{\mathcal{U}}_{\text{init}}\) and let \( \eta_0 \) vary. Consider the initial data

\[
u_0(r) = \frac{e^{i\hat{\gamma}_0}}{\hat{\lambda}_0} [P(: \hat{\eta}_0) + \hat{e}_0] \left( \frac{r}{\hat{\lambda}_0} \right) \in \mathcal{O}_{\text{init}},
\]

where \( P(: \hat{\eta}_0) \) is our modified profile with \( P(: 0, 0) = Q \) to be introduced in the next step. The set of four parameters \( \lambda, \gamma, b, \eta \) is motivated from the four dimensional generalized null space of the linearized operator.

We let \( u \) be the forward-in-time evolution of \( u_0 \). We will decompose \( u \) as

\[
u(t, r) = \frac{e^{i\gamma(t)}}{\lambda(t)} [P(: b(t), \eta(t)) + \epsilon(t, \cdot)] \left( \frac{r}{\lambda(t)} \right).
\]

Several issues such as the construction of \( P \), fixing the decomposition (parameters and \( \epsilon \)), and the control of \( \epsilon \) forward-in-time, will be explained on the way.

To analyze the blow-up dynamics, we renormalize the variables by introducing

\[
\frac{ds}{dt} = \frac{1}{\lambda^2}, \quad y = \frac{r}{\lambda}, \quad w(s, y) = \lambda e^{-i\gamma} u(t, \lambda y)|_{s=\lambda(t)}.
\]

Moreover, we renormalize \( u_1 \) and \( u_2 \) in the previous step by \( u_1 := D_u w \) and \( u_2 := A_u D_u w \). These renormalized variables satisfy the equations (2.14), (2.15), and (2.16). In these equations, we further introduce the modified phase parameter \( \hat{\gamma} \) with the relation

\[
\hat{\gamma}_s := \gamma_s + \int_s^\infty \Re(\overline{w_1}) dy.
\]

This takes into account some nonlocal interactions leading to an extra phase rotation of the solutions. In particular, it changes the \( \int_r^\infty \)-integral to a \( \int_0^s \)-integral, which is also important to make sense the tail computation in the next step.

The proof of Theorem 1.1 is a combination of bootstrapping and a topological (connectivity) argument. Smallness of \( \epsilon \) will be bootstrapped in the regime \( |\eta| \leq \frac{b}{\log \eta} \). As will be explained later, \( \eta \) is an unstable parameter and the regime \( |\eta| \leq \frac{b}{\log \eta} \) cannot be bootstrapped; we show by a connectivity argument that \( |\eta| \leq \frac{b}{\log \eta} \) on the maximal forward lifespan is guaranteed for some special initial choice \( \eta_0 \). Such special solutions are called trapped solutions, and they will be shown to blow up in finite time as described in Theorem 1.1.

3. Modified profile and sharp modulation equations. The construction of modified profiles and the derivation of sharp modulation laws are among the main challenges of this work. In [21,22], the authors introduced a nonlinear profile ansatz, which was an efficient way to derive pseudoconformal blow-up when \( m \geq 1 \). However, when \( m = 0 \) this profile ansatz produces unacceptable profile error. As we also see a-posteriori from the resulting logarithmically corrected blow-up rate, it seems that
the profiles in [21][22] do not work. Hence we search for sharper modified profiles and modulation laws.

Since we view the system of $w$, $w_1$, $w_2$ equations, we construct modified profiles $P$, $P_1$, $P_2$ for $w$, $w_1$, $w_2$, respectively, and derive sharp modulation equations using the tail computation (under the adiabatic ansatz $\frac{\lambda}{\gamma} + b = 0$ and $\gamma_s = -\eta$). This strategy, one of our novelties, remarkably simplify the rest of the analysis. From the degeneracies of $w_1$ and $w_2$, $P_1$ and $P_2$ have the same order of degeneracies: $P_1 = O(b)$, $P_2 = O(b^2)$. As a result, the following simple profile expansions turn out to be sufficient:

$$
P := Q + \chi_{B_t} \{ -ib \frac{\log}{\log b} Q - \eta \rho \},
$$

$$
P_1 := \chi_{B_t} \{ -(ib + \eta) \frac{\log}{\log b} Q \} + \chi_{B_0} \{ b^2 T_{2,0} \},
$$

$$
P_2 := \chi_{B_0} \{ (b^2 - 2ib\eta - \eta^2) U_2 + ib^3 U_{3,0} \},
$$

for some profiles $T_{2,0}, U_2, U_{3,0}$, and cutoffs $\chi_{B_t}, \chi_{B_0}$. When we derive $T_{2,0}$ and $U_2$, we will see that the zero resonance $yQ \notin L^2$ to the linearized operator $H_Q = A_Q^* A_Q$ leads to a logarithmic correction in the modulation laws, as in [33][38][39]. In our setting, this is observed in the $w_1$-equation and yields the sharp modulation laws:

$$
b_s + b^2 + \eta^2 + c_b(b^2 - \eta^2) = 0, \quad \eta_s + 2c_b b \eta = 0,
$$

where $c_b \approx \frac{2}{\log b}$. We remark that it is necessary to expand $P_2$ up to the $b^2$-order. However, again thanks to the degeneracies of $P_1$, $P_2$, cruder expansions for $P$ and $P_1$ suffice.

In order to guarantee a finite-time blow up, we need $|\eta| \ll b$. However, in view of $\eta_s + 2c_b b \eta = 0$, the trapped regime $|\eta| \lesssim \frac{b}{\log b}$ is non-generic. Thus we view $\eta$ as an unstable parameter.

4. Decomposition and propagation of smallness of $\epsilon$. Having defined our profiles $P$, $P_1$, $P_2$, we decompose our solutions as

$$
w = P + \epsilon, \quad w_1 = P_1 + \epsilon_1, \quad w_2 = P_2 + \epsilon_2,
$$

so that $\epsilon$ satisfies certain orthogonality conditions. Here the main novelty is to use $\epsilon_1$ and $\epsilon_2$ that are defined via higher order (nonlinearly) conjugated variables. Although $\epsilon_1 \approx LQ \epsilon$ and $\epsilon_2 \approx A_Q LQ \epsilon$ at the leading order, $\epsilon_1$ and $\epsilon_2$ are defined in a nonlinear fashion. We call them nonlinear adapted derivatives. Linear adapted derivatives such as $LQ \epsilon$ and $A_Q LQ \epsilon$ were used in [21], whose idea goes back to the works [8][33][34][35]. Here, by using nonlinear adapted derivatives, the error terms arising in $\epsilon_1$ and $\epsilon_2$ equations are significantly simplified than the ones obtained by linear adapted derivatives. As we will see in Section 5.5 the equation of $\epsilon_2$ contains only a few error terms of critical size, which simplifies the energy estimates as well as the Morawetz corrections.

To fix the modulation parameters $\lambda, \gamma, b, \eta$, we impose four orthogonality conditions. We make a non-standard choice: we impose two orthogonality conditions on $\epsilon$, and two on $\epsilon_1$. The first two are used to detect the modulation equations of $\lambda$ and $\gamma$; and the other two are used to detect the modulation equations of $b$ and $\eta$. For the latter, we can take advantages from the degeneracy in the $\epsilon_1$-equation. In this way, the roles of $\epsilon$ and $\epsilon_1$-equations are to detect the modulation laws.

We apply the energy method to propagate the smallness of $\epsilon$ (and $\epsilon_1$ and $\epsilon_2$) in a bootstrap argument. The main part is to control a $H^3$-level quantity of $\epsilon$. Thus we use the $\epsilon_2$-equation, whose associated energy functional is $(\epsilon_2, A_Q A_Q^* \epsilon_2)_r = \|A_Q \epsilon_2\|_{L^2}^2 = \|\epsilon_3\|_{L^2}^2$. Here we can use the repulsivity from the operator $A_Q A_Q^*$ (5.5) and also the full degeneracy $P_2 = O(b^3)$. In fact, the sole use of the energy functional $\|\epsilon_3\|_{L^2}^2$ is not sufficient to close the bootstrap, due to some non-perturbative
terms in the $\epsilon_2$-equation. To overcome this difficulty, we use a Morawetz-type correction derived from the repulsivity of $A_Q A_Q^*$. A similar technique was used in \cite{33}.

In the energy/Morawetz estimates, we benefit from the use of the $\epsilon_2$-variable in a significant way. If one merely proceeds with linear adapted derivatives, there appear a lot of errors of critical size $O(be)$ in the equation; see for example the $R_{i,-1}$ term in \cite{21}. Thanks to our approach of covariant conjugations, we significantly reduced the critical errors. In fact, our variable $\epsilon_2$ is $A_Q L_Q \epsilon$ at the leading order, but a lot of $O(be)$ terms are hidden in $\epsilon_2$. This enables us to choose a Morawetz correction in a simple form.

5. After bootstrapping. As mentioned above, $\eta$ is an unstable parameter. We find a special $\eta_0$ ensuring that the solution remains trapped by a soft connectivity argument. The sharp blow-up rates are obtained by testing against a better approximation of the generalized kernel elements. The argument in this step is very similar to that in \cite{33}.

**Notation.** For $A \in \mathbb{C}$ and $B > 0$, we use the standard asymptotic notation $A \lesssim B$ or $A = O(B)$ to denote the relation $|A| \leq CB$ for some positive constant $C$. The dependencies of $C$ is specified by subscripts, e.g., $A \lesssim B \Leftrightarrow A = O_B(B) \Leftrightarrow |A| \leq C(E)B$. We also introduce the shorthands

$$\langle \rangle = (1 + (\cdot)^2)^{\frac{1}{2}}, \quad \log_+ (\cdot) = \max\{0, \log(\cdot)\}, \quad \log_- (\cdot) = \max\{0, -\log(\cdot)\}.$$

Given a function $f : (0, \infty) \to \mathbb{C}$, we introduce the shorthand

$$\int f = \int_{\mathbb{R}^2} f(|x|)dx = 2\pi \int f(r) rdr.$$

For functions $f, g : (0, \infty) \to \mathbb{C}$, their real $L^2$ inner product is given by

$$(f, g)_r := \int \text{Re}(\overline{f}g).$$

For $s \in \mathbb{R}$, let $\Lambda_s$ be the infinitesimal generator of the $H^s$-invariant scaling, i.e.,

$$\Lambda_s f := \left. \frac{d}{d\lambda} \right|_{\lambda=1} \lambda^{1-s} f(\lambda) = (1-s + r\partial_r) f.$$

For a nonnegative integer $k$ and a function $f : (0, \infty) \to \mathbb{C}$, we define

$$|f|_k(r) := \sup_{0 \leq \ell \leq k} |r^{\ell} \partial^{\ell}_r f(r)|,$$

$$|f|_{-k}(r) := \sup_{0 \leq \ell \leq k} |r^{-\ell} \partial^{\ell}_r f(r)| = r^{-k}|f|_{-k}.$$

For $f : (0, \infty) \to \mathbb{C}$, $B > 0$ and a norm $\| \cdot \|$, we write $f = O_X(B)$ to denote $\|f\|_X \lesssim B$.

We will use the Laplacian acting on $m$-equivariant functions: $\Delta_m = \partial_{rr} + \frac{1}{r} \partial_r - \frac{m^2}{r^2}$. We will also denote $\partial_+ = \partial_1 + i\partial_2$. If $\partial_+$ acts on $m$-equivariant functions $f(r)e^{im\phi}$, then $\partial_+[f(r)e^{im\phi}] = [\partial^{(m)}_+ f] e^{i(m+1)\phi}$, where $\partial^{(m)}_+ := \partial_r - \frac{m}{r}$. When the equivariance index $m$ is clear from the context, we use an abuse of notation $\partial_+ f = \partial^{(m)}_+ f$.

We will use two different localization radii for the modified profiles:

$$(1.26) \quad B_0 := b^{-\frac{1}{2}}, \quad B_1 := b^{-\frac{1}{2}} |\log b|.$$


**Organization of the paper.** In Section 2 we introduce covariant conjugation identities, which provide the algebraic foundation of the paper. In Section 3 we review the linearization of (CSS), study outgoing Green’s functions for linearized operators, and construct adapted function spaces. In Section 4 we construct the modified profiles. In Section 5 we set up the bootstrap procedure and prove Theorem 1.1 and Corollary 1.2. In Appendix A we prove various facts regarding the adapted function spaces.

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## 2. Covariant conjugation identities

As alluded to in Introduction, we will use higher order variables $u$, $D_u u$, and $A_v D_u u$ obtained by covariant conjugations. Our goal in this section is to derive the equations satisfied by $u$, $D_u u$, and $A_v D_u u$, which provide the starting point for our analysis. We will also need the renormalized variables $\bar{w}$, $w_1$, and $w_2$ of $u$, $D_u u$, and $A_v D_u u$, respectively, and the equations satisfied by them. To achieve this goal, we employ a reformulation of (2.3) in terms of the Wirtinger derivatives (see (2.4)–(2.5)), which is an elegant way to make the self-dual nature of (2.3) manifest.

To make clear the generality of the algebraic manipulations we perform, we proceed in a gauge-covariant, non-radial fashion in Section 2.1, and only in Section 2.2 do we re-impose the Coulomb gauge condition and radial symmetry.

### 2.1. Self-dual Chern–Simons–Schrödinger in terms of Wirtinger derivatives and covariant conjugation

To make the self-dual nature of (2.3) manifest, it is expedient to rewrite (2.3) in terms of the Wirtinger derivatives

$$D_\tau = \frac{1}{2} \partial_1 + \frac{1}{2i} \partial_2, \quad \partial_\tau = \frac{1}{2} \partial_1 - \frac{1}{2i} \partial_2.$$  

Accordingly, given any connection 1-form (i.e., a real-valued 1-form) $A$, we define

$$A_\tau = A(\partial_1) = \frac{1}{2} A_1 + \frac{1}{2i} A_2, \quad A_\tau = A(\partial_2) = \frac{1}{2} A_1 - \frac{1}{2i} A_2,$$

$$D_\tau = \partial_1 + i A_1 = \frac{1}{2} D_1 + \frac{1}{2i} D_2, \quad D_\tau = \partial_2 + i A_2 = \frac{1}{2} D_1 - \frac{1}{2i} D_2.$$  

Since the 1-form $A$ is real-valued, we have $A_\tau = A_\tau$. For any complex-valued smooth functions $\phi, \psi$, we have

$$\partial_\tau (\bar{\psi} \phi) = \bar{\psi} D_\tau \phi + D_\tau \psi \phi, \quad \partial_\tau (\psi \phi) = \bar{\psi} D_\tau \phi + D_\tau \psi \phi.$$  

The Cauchy–Riemann operator $D_+$ and its adjoint $D_+^*$ are expressed as

$$D_+ = 2D_\tau, \quad D_+^* = -2D_\tau.$$  

We note the following anti-commutator relations:

$$\partial_\tau \partial_\tau + \partial_\tau \partial_\tau = 2 \partial_\tau \partial_\tau = \frac{1}{2}(\partial_1^2 + \partial_2^2),$$

$$D_\tau D_\tau + D_\tau D_\tau = \frac{1}{2}(D_1^2 + D_2^2).$$  

---

7Geometrically, we are simply complexifying the tangent, co-tangent and the associated tensor bundle over $\mathbb{R}^2$, and using the basis $(dz, \bar{dz}) = (dx_1 + i dx_2, dx_1 - i dx_2)$ for the complexified co-tangent bundle $T^* \mathbb{R}^2$. The Wirtinger derivatives arise as the dual basis on the complexified tangent bundle $T \mathbb{R}^2$. 

---
On the other hand, the commutator of two covariant derivatives is expressed in terms of the curvature tensor. At the level of the curvature 2-form $F$, we introduce
\begin{align*}
F_{iz} := F(\partial_t, \partial_z) &= \frac{1}{2} F_{t1} + \frac{1}{2} F_{t2}, \\
F_{i\tau} := F(\partial_z, \partial_\tau) &= -\frac{1}{2} F_{t2}.
\end{align*}
Since $F$ is real-valued, we have $F_{iz} = F_{i\tau}$ and $F_{i\tau} = -F_{iz}$. Clearly,
\begin{align*}
D_\alpha D_\beta - D_\beta D_\alpha = iF_{\alpha\beta}, \\
\partial_\alpha A_\beta - \partial_\beta A_\alpha = F_{\alpha\beta},
\end{align*}
for $\alpha, \beta \in \{t, \tau, z\}$.

We now write (1.3) in terms of the Wirtinger derivatives. The curvature relations in (1.3) may be rewritten in the form
\begin{align}
\begin{cases}
F_{t\tau} = \frac{i}{2} \tilde{\phi} D_{t\tau} \phi - \frac{1}{2} \tilde{\phi} D_{\tau} \phi - \frac{1}{2} \tilde{\phi} |\phi|^2, \\
F_{z\tau} = \frac{1}{2} \tilde{\phi} |\phi|^2,
\end{cases}
\end{align}
(2.2)
At this point, observe that $F_{i\tau}$ cleanly splits into a term involving $D_{\tau} \phi$ and a total derivative $-\frac{1}{2} \partial_t |\phi|^2$. The latter term can be removed by introducing a modified connection 1-form $\tilde{A}$,
\begin{align}
\tilde{A} = \tilde{A}_t dt + \tilde{A}_1 dx^1 + \tilde{A}_2 dx^2 = (A_t - \frac{1}{2} |\phi|^2) dt + A_1 dx^1 + A_2 dx^2
\end{align}
(2.3)
after which, for the associated curvature $\tilde{F} = d\tilde{A}$, (2.2) simplifies to
\begin{align}
\begin{cases}
\tilde{F}_{t\tau} = \tilde{\phi} D_{t\tau} \phi, \\
\tilde{F}_{z\tau} = \frac{1}{4} |\phi|^2,
\end{cases}
\end{align}
(2.4)
where $\tilde{D}_\alpha = \partial_\alpha + i \tilde{A}_\alpha$ is covariant derivative associated with $\tilde{A}$. Remarkably, with (2.1), (2.3) and (2.4), (1.5) simplifies to
\begin{align}
i\tilde{D}_1 \phi + 4 \tilde{D}_z \tilde{D}_\tau \phi = 0.
\end{align}
(2.5)
Equations (2.4) and (2.5) constitute the self-dual Chern–Simons–Schrödinger equation expressed in terms of the Wirtinger derivatives. By (2.4) and (2.5), the Bogomol’nyi equation may be written as
\begin{align}
\begin{cases}
D_{t\tau} \phi = 0, \\
F_{z\tau} = \frac{1}{4} |\phi|^2.
\end{cases}
\end{align}
(2.6)
In this formulation, it is straightforward to derive the following covariant conjugation identities, which will play a key role in the remainder of this paper:

**Proposition 2.1** (Covariant conjugation identities). Let $\phi, \tilde{A}$ obey (2.4) and (2.5). Then
\begin{align}
\begin{cases}
i\tilde{D}_1 \tilde{D}_1 \phi + 4 \tilde{D}_z \tilde{D}_\tau \tilde{D}_1 \phi = 0, \\
i\tilde{D}_1 \tilde{D}_2 \tilde{D}_1 \phi + 4 \tilde{D}_2 \tilde{D}_2 \tilde{D}_1 \phi + \tilde{\phi} (\tilde{D}_1 \phi)^2 = 0.
\end{cases}
\end{align}
(2.7, 2.8)

**Remark 2.2.** The extensive use of the equations in Proposition 2.1 is one of the key ideas in this work. An immediate advantage of working with (2.4) and (2.5) is that, thanks to (2.6), $D_{t\tau} \phi$ vanishes when $\phi$ is a modulated soliton. As a consequence, the linearization of $(i\tilde{D}_1 + 4 \tilde{D}_z \tilde{D}_\tau) (\tilde{D}_1 \phi)$ at a modulated soliton does not contain any nonlocal terms in the corresponding linearization of $D_{t\tau} \phi$, which is a huge simplification over the case of $(iD_1 + 4 D_z D_\tau) \phi$. Moreover, the simplicity of (2.8) already suggests that $D_{t\tau} \tilde{D}_1 \phi$ is a very convenient “nonlinear” high-order variable to prove energy estimate for. See also Remark 1.10 below for a further important cancellation that occurs for $D_{t\tau} \tilde{D}_1 \phi$ at the linearized level.

\footnote{Note that the spatial components of $\tilde{A}$ and $\tilde{A}$ are the same, so $\tilde{D}_\tau = D_{t\tau}$ and $\tilde{D}_z = D_z$.}
We remark that (2.7) was first proved and used in [21] in the context of proving higher order energy estimates. In this work, the use of (2.7) and (2.8) pervades the whole argument, namely in the modified profile construction, the modulation estimate and the key third-order energy estimate.

Proof. To prove (2.7), we simply apply $\tilde{D}_z$ to (2.5), then use (2.3) to commute $\tilde{D}_z$ inside. As a result, we obtain

$$0 = i\hat{D}_t \hat{D}_z \phi + 4\hat{D}_r \hat{D}_z \phi + i[\hat{D}_z, \hat{D}_t] \phi + 4[\hat{D}_z, \hat{D}_r] \phi$$

$$(\hat{D}_z \phi - 2|\phi|^2 \hat{D}_z \phi - |\phi|^2 \hat{D}_z \phi) = 0,$$

where the last two terms cancel. To prove (2.8), we apply $\tilde{D}_z$ to (2.7) and commute it with $\tilde{D}_t$ using (2.3).

Finally, for the convenience of the reader, we restate the identities in Proposition 2.1 in terms of $D_+$ and the original connection using (2.1) and (2.3):

$$(iD_t + \frac{1}{2} |\phi|^2)D_+ \phi - D_+^* D_+ \phi = 0,$$

$$(iD_t + \frac{1}{2} |\phi|^2)D_+ \phi - D_+^* D_+ \phi + \overline{\phi}(D_+^* \phi) = 0.$$

Note that in the original variables, $D_+ \phi = D_\phi = 2D_+ \phi$ and $D_- \tilde{D}_z \phi = 4D_+ D_+ \phi$. In our analysis, we use $D_+ \phi$ and $D_- D_+ \phi$ as our conjugated variables.

2.2. Equations in renormalized variables. Starting from (2.4)–(2.5), we now impose the Coulomb gauge condition $\partial_1 A_1 + \partial_2 A_2 = 0$ and the radial symmetry ansatz $\phi(t, r, \theta) = u(t, r)$. Since $\tilde{A}_j = A_j$ ($j = 1, 2$), we have, as before,

$$\tilde{A}_r = 0, \quad \tilde{A}_\theta = A_\theta[u] = \frac{1}{2} \int_0^r |u|^2 r' dr'.$$

By $\tilde{A}_r = 0$, the relation $\partial_1 = e^{-i\theta} \partial_r + e^{i\theta} \partial_2$, and (2.4), we have

$$\partial_r \tilde{A}_t = F_{tr} = e^{-i\theta} F_{\tilde{r}1} + e^{i\theta} F_{\tilde{r}2} = -2 \frac{\Re(\overline{\phi}(e^{-i\theta} \tilde{D}_z \phi))}{\Re(\overline{\phi} D_+ \phi)}.$$

Since $\tilde{A}_t \to 0$ as $r \to \infty$, we may integrate from $0$ to $\infty$ to obtain

$$\tilde{A}_t = \int_0^\infty \Re(\overline{\phi} D_+ \phi) dr'.$$

In this setting, (2.7) and (2.8) take the form

$$\left( i\partial_t - \int_0^\infty \Re(\overline{\phi} D_+ \phi) dr' \right) D_+ u - A_+^* A_+ D_+ u = 0,$$

$$\left( i\partial_t - \int_0^\infty \Re(\overline{\phi} D_+ \phi) dr' \right) A_+ D_+ u - A_+ A_+^* A_+ D_+ u + \overline{\phi} D_+ \phi)^2 = 0,$$

Next, given modulation parameters $\lambda : I \to (0, \infty)$ and $\gamma : I \to \mathbb{R}$, which we assume to be $C^1$, consider the renormalized independent variables $(s, y)$ and dependent variable $w$ defined by

$$\frac{ds}{dt} = \frac{1}{\lambda^2}, \quad y = \frac{r}{\lambda}, \quad w(s, y) = \lambda e^{-i\gamma} u(t, \lambda y)|_{t=t(s)}.$$

To simplify the notation, in what follows we write $\lambda(s) = \lambda(t(s)), \gamma(s) = \gamma(t(s))$ and so on. The associated nonlinear higher order variables are defined by

$$w_1 = (0) D_+ w = D_+ w = \lambda^2 e^{-i\gamma}(D_\phi u)(t, \lambda y)|_{t=t(s)},$$

$$w_2 = (1) D_+ w = A_+ w_1 = \lambda^3 e^{-i\gamma}(A_+ D_+ u)(t, \lambda y)|_{t=t(s)}.$$
By applying a simple change of variables to \((1.19), (2.9)\) and \((2.10)\), and rewriting
\[-\int_0^\infty \text{Re}(\overline{w} D_w w) \, dy' = -\int_0^\infty \text{Re}(\overline{w} D_w w) \, dy' + \int_0^y \text{Re}(\overline{w} D_w w) \, dy',\]
we obtain the equations of the renormalized variables \(w, w_1, \) and \(w_2\):

**Proposition 2.3** (Equations in renormalized variables). Let \((\phi, A)\) be a solution on \(I \times \mathbb{R}^2\) obeying the Coulomb gauge condition and radial symmetry (see Section 1.1). Given \(C^1(I)\) modulation parameters \(\lambda(t) > 0\) and \(\gamma(t) \in \mathbb{R}\) for \(t \in I\), consider the renormalized variables \((s, y, w)\) and \(w_1, w_2\) defined by \((2.11), (2.12), \) and \((2.13)\).

Then the renormalized variables \(w, w_1, \) and \(w_2\) obey the following equations:

\[(2.14) \quad (\partial_s - \frac{\lambda_s}{\Lambda} \Lambda + \gamma_s i) w + i L_s^* D_w w = 0,\]

\[(2.15) \quad (\partial_s - \frac{\lambda_s}{\Lambda} \Lambda - 1 + \gamma_s i) w_1 + i A_w^* A_w w_1 - \left(\int_0^y \text{Re}(\overline{w} w_1) \, dy'\right) i w_1 = 0,\]

\[(2.16) \quad (\partial_s - \frac{\lambda_s}{\Lambda} \Lambda - 2 + \gamma_s i) w_2 + i A_w A_w^* w_2 - \left(\int_0^y \text{Re}(\overline{w} w_1) \, dy'\right) i w_2 - i \overline{w} w_1^2 = 0,\]

where

\[\tilde{\gamma}_s := \gamma_s + \int_0^\infty \text{Re}(\overline{w} w_1) \, dy.\]

**Remark 2.4.** At the technical level, the reason for the introduction of the correction \(\tilde{\gamma}_s\) is to switch the domain of the integration in the nonlocal term \(\int \text{Re}(\overline{w} w_1) \, dy'\) from \([y, \infty)\) to \([0, y]\), which is crucial in the ensuing analysis. Conceptually, the correction \(\tilde{\gamma}_s\) contains the dominant nonlocal effect of the radiation on the soliton, which results in extra phase rotation of the soliton in the similar spirit of \([21, 22]\).

3. Linearized operators at \(Q\) and adapted function spaces

Our goal is to construct a blow-up solution staying close to the modulated soliton profiles \(Q\). After rescaling our solutions, it is necessary to study the linearized dynamics of \((\text{CSS})\) around \(Q\). In Section 3.1, we first review the linearization of \((\text{CSS})\). In Section 3.2, we construct right inverses of some linear operators that will be used for the construction of modified profiles. In Section 3.3, we introduce adapted function spaces and associated coercivity estimates to be used in the modulation and higher order energy estimates.

3.1. Linearization of the Bogomol’nyi equation and \((\text{CSS})\) at \(Q\). In this subsection, we briefly collect facts about the linearization of \((\text{CSS})\) around \(Q\), which already appeared in \([22, \text{Section 3]}\) and \([21, \text{Section 2.1}]\) (for the case of higher equivariance case \(m \geq 1\)). As we have seen in \((1.19)\), we firstly linearize the Bogomol’nyi operator \(w \mapsto D_w w\) and then linearize \((\text{CSS})\).

Consider the (radial Coulomb-gauge) Bogomol’nyi operator \(w \mapsto D_w w\). We write

\[(3.1) \quad D_{w+\epsilon}(w + \epsilon) = D_w w + L_w \epsilon + N_w(\epsilon),\]

where (cf. \((1.13)\))

\[L_w \epsilon := D_w \epsilon + w B_w \epsilon,\]

\[N_w(\epsilon) := \epsilon B_w \epsilon + \frac{1}{4} w B_w \epsilon + \frac{1}{2} \epsilon B_w \epsilon,\]

\[B_w \epsilon := -\frac{2}{\eta} A_\eta [w, \epsilon] = \frac{1}{\eta} \int_0^y \text{Re}(\overline{w} \epsilon) y' \, dy'.\]
The $L^2$-adjoint $L^\ast_w$ of $L_w$ takes the form

$$L^\ast_w v = D^\ast_w v + B^\ast_w(\bar{v}),$$

$$B^\ast_w v = w \int_y^\infty \Re v \, dy'.$$

Next, we linearize (CSS), which we write in the self-dual form (1.19): $\partial_t u + i L^\ast_w D_w u = 0$. We write

$$i L^\ast_{w+\epsilon} D_{w+\epsilon} (w + \epsilon) = i L^\ast_w D_w w + L_w \epsilon + (\text{h.o.t.} \, \epsilon),$$

where

$$L_w \epsilon := L^\ast_w L_w \epsilon + D_w w (B_w \epsilon) + B^\ast_w (\bar{\epsilon} D_w w) + B^\ast_w (\bar{\epsilon} D_w w).$$

In particular, from $D_Q Q = 0$, we observe the self-dual factorization of $i L_Q$:

$$i L_Q = i L^\ast_Q L_Q. \tag{3.2}$$

This identity was first observed in [25]. Thus, the linearization of (1.19) at $Q$ is

$$\partial_t \epsilon + i L_Q \epsilon = 0, \quad \text{or} \quad \partial_t \epsilon + i L_Q^\ast L_Q \epsilon = 0. \tag{3.3}$$

Next, observe that if we linearize (2.9) at $Q$, then we obtain

$$\partial_t L_Q \epsilon + i A_Q^\ast A_Q L_Q \epsilon = 0. \tag{3.4}$$

Comparing this equation with the application of $L_Q$ to (3.3) (as well as the right-invertibility of $L_Q$ from Proposition 3.4 below), we arrive at the remarkable linearized conjugation identity

$$i A_Q^\ast A_Q = L_Q i L^\ast_Q. \tag{3.5}$$

This identity was first observed in [21]. Note that while $L_Q$ and $L^\ast_Q$ are separately nonlocal operators, the left-hand side is manifestly local. We introduce the notation

$$H_Q := A_Q^\ast A_Q. \tag{3.6}$$

Note that while $L_Q$, $L^\ast_Q$ and $L_Q$ are only $\Re$-linear, $D_Q$, $A_Q$, $H_Q$ and their adjoints are $\mathbb{C}$-linear. We further remark that $A_Q$ and $H_Q$ are exactly same as those in the wave maps and Schrödinger maps problems, see [35] (2.4) and (2.5) and [33] (2.11)]. See also [39, 40] for the related harmonic map heat flows.

Finally, we linearize (2.10) at $Q$ to arrive at

$$\partial_t A_Q L_Q \epsilon + i A_Q A_Q^\ast A_Q L_Q \epsilon = 0. \tag{3.7}$$

A crucial fact is that $A_Q A_Q^\ast$ has a positive repulsive potential:

$$A_Q A_Q^\ast = -\partial_{yy} - \frac{1}{y} \partial_y + \frac{\tilde{V}}{y^2},$$

where

$$\tilde{V} = (2 + A_y [Q])^2 + y^2 Q^2 \geq 0 \quad \text{and} \quad -y \partial_y \tilde{V} \geq 0. \tag{3.8}$$

The repulsivity of $A_Q A_Q^\ast$ was first used in [41]. This is also used in the Chern–Simons–Schrödinger setting [21]. Note that the positivity of $\tilde{V}$ is much weaker than that of the higher equivariance case. Indeed, we have $\tilde{V} \sim |y|^{-2}$ when $m = 0$ but $\tilde{V} \sim 1$ when $m \geq 1$. See Remark 5.11 for related discussions.

Next, we consider the kernels of the above linearized operators at $Q$. Via the phase rotation and scaling symmetries of the Bogomol’nyi operator, we have

$$L_Q (A Q) = 0, \quad L_Q (i Q) = 0. \tag{3.9}$$

Despite the presence of a nonlocal term, it can be shown that the $L^2$-kernel of $L_Q$ is indeed span$_\mathbb{R} \{ A Q, i Q \}$; see [22, Section 3].
For $D_Q$, we have
\[
(3.10) \quad D_Q Q = 0.
\]
Since $D_Q$ is first-order, local, and $C$-linear, its $L^2$-kernel is given by span$_C\{Q\}$.

For $A_Q$, due to $A_Q(yv) = yD_v v$, it follows that
\[
(3.11) \quad A_Q(yQ) = 0.
\]
As $A_Q$ is also first-order, local, and $C$-linear, its formal (smooth) kernel is span$_C\{yQ\}$. Moreover, by (3.6), it follows that
\[
(3.12) \quad H_Q(yQ) = 0.
\]
However, $yQ \notin L^2$; in fact, it is a resonance at zero for the operator $H_Q$.

We turn to the formal generalized kernel of $iL_Q$. By (3.2), it follows that
\[
(3.13) \quad iL_Q(Q) = 0, \quad iL_Q(iQ) = 0,
\]
and that the $L^2$-kernel of $iL_Q$ is span$_Q\{Q, iQ\}$. Concerning the formal kernel of $(iL_Q)^2$, which is a part of the formal generalized kernel of $iL_Q$, we have
\[
(3.14) \quad iL_Q(iyQ^2) = Q, \quad iL_Q iQ = iQ,
\]
where the first identity is easy to verify and $iQ$ is given in Lemma 3.1 below.

**Lemma 3.1** (Generalized nullspace element $\rho$). There exists a unique smooth function $\rho : (0, \infty) \to \mathbb{R}$ satisfying the following properties:

1. (Smoothness on the ambient space) The $m$-equivariant extension $\rho(x) := \rho(y)e^{im\theta}, x = ye^{\theta}$, is smooth on $\mathbb{R}^2$.
2. (Equation) $\rho(r)$ satisfies
   \[
   L_Q \rho = \frac{1}{2} Q \quad \text{and} \quad L_Q \rho = Q.
   \]
3. (Pointwise bounds) We have
   \[
   (3.15) \quad |\rho|_k \lesssim_k y^2 Q, \quad \forall k \in \mathbb{N}.
   \]

For the construction of $\rho$ including the $m = 0$ case, see [22, Lemma 3.6]. Further properties of $\rho$ can be proved by following the proof of [21, Lemma 2.1] ($m \geq 1$ case) with a suitable modification. Alternatively, we may construct $\rho$ and prove the preceding lemma by taking $\rho = (\text{out}) L_Q^{-1}(\frac{1}{2} Q)$, where $(\text{out}) L_Q^{-1}$ is defined by (3.17) and Proposition 3.4. We omit the proof.

When $m \geq 2$, the following spaces
\[
N_g(iL_Q) := \text{span}_Q\{Q, iQ, iy^2 Q, \rho\} \subseteq L^2,
\]
\[
N_g(L_Q i) \perp := \{i\rho, y^2 Q, Q, iQ\} \perp \subseteq L^2
\]
are formally invariant under the flow $\partial_t + iL_Q$. Moreover, we have a clean splitting of $L^2$ by
\[
L^2 = N_g(iL_Q) \oplus N_g(L_Q i) \perp.
\]
In the previous work [21], one was motivated by this splitting to make a decomposition of the form
\[
u(r) = e^{\frac{\gamma t}{1}} \mathcal{P}(\lambda; b, \eta) + \epsilon \left( \frac{T}{\lambda} \right),
\]
where the four modulation parameters $\lambda, \gamma, b, \eta$ take into account the generalized null space elements $(P(\lambda; 0, 0) = Q$, $\partial_t P \approx -i\frac{\gamma}{m} Q$, and $\partial_\eta P \approx -(m + 1)\rho$) and $\epsilon$ belongs to (a truncated version of) $N_g(L_Q i) \perp$. When $m \in \{0, 1\}$, the above decomposition does not make sense rigorously, but still suggests a similar decomposition. It also provides a starting point of the construction of modified profiles $P$.

\[\text{In the sense that } i\frac{\gamma}{m} Q, \rho \notin L^2.\]
The following relation among the formal generalized kernel elements of $iL_Q$ and the kernel of $A_Q$, which may be read off of (3.13), is useful:

$$L_Q\rho = \frac{1}{2}yQ, \quad L_Q(i\rho^2) = \frac{1}{2}iyQ.$$  

(3.16)

3.2. Outgoing Green’s functions. In this subsection, we construct right inverses of the (radial) linear operators $L_Q$, $A_Q$, and $H_Q = A_Q^*A_Q$. These can be used in the construction of modified profiles $P$ (more precisely, the Taylor expansions) [10].

Since $L_Q$, $A_Q$, and $H_Q$ have nontrivial kernels, their right inverses are not unique. To fix them, we simply impose a good behavior (degeneracy) near $y = 0$. Concretely, for $T \in \{L_Q, i^{-1}L_Q, A_Q, H_Q\}$ we construct the outgoing Green’s function $(^{(T)}G)(y, y')$, which is characterized by the properties

$$T\left(\left(^{(T)}G(y, y')\right)\right) = \delta_{y'}(y),$$

(3.17)

$$^{(T)}G(y, y') = 0 \quad \text{for} \ 0 < y < y',$$

for a linear operator $T$ acting on real-valued functions of the variable $y$. The second property of (3.17) concerning the support is the outgoing property. Recall that $L_Q$ is only $\mathbb{R}$-linear. When $L_Q$ acts on complex-valued functions, we need to separate the real and imaginary parts. For the $\mathbb{C}$-linear operators $A_Q$ and $H_Q$, the same Green’s functions still work for complex-valued functions. The desired right inverse may then be defined as

$$^{[\text{out}]T^{-1}}f(y) = \int_0^\infty ^{(T)}G(y, y')f(y')\,dy'.$$

(3.18)

By the outgoing property, the domain of the integral on the RHS would be restricted to $y'$, which is the good behavior we need.

Outgoing Green’s function for $A_Q$. We start with $A_Q$, which is the simplest.

**Proposition 3.2.** The outgoing Green’s function for $A_Q$ takes the form

$$(^{A_Q}G)(y, y') = 1_{(0, \infty)}(y - y')\frac{yQ(y)}{y'Q(y')},$$

where

$$J(y) = yQ, \quad \Gamma(y) = J\int_1^y J^{-2}(y')\,dy'.$$

For any nonnegative integer $k$, we have

$$|J(y)|_k \lesssim_k \begin{cases} y & \text{if } y \leq 1, \\ y^{-1} & \text{if } y \geq 1, \end{cases} \quad |\Gamma(y)|_k \lesssim_k \begin{cases} y^{-1} & \text{if } y \leq 1, \\ y & \text{if } y \geq 1. \end{cases}$$

(3.19)

In fact, it turns out that outgoing Green’s functions for $L_Q$ are not used in this work. However, we include their construction as it may be of independent interest. It should be used when one expands the modified profile $P$ in higher order.
This is simply the standard construction of Green’s function for the second-order differential operator \( A_Q^2A_Q = -\frac{\partial^2}{\partial y^2} + \frac{1}{y} \partial_y + \frac{1}{y^2} \) using the fundamental basis consisting of \( J \) (recall that \( A_Q^2A_QJ = 0 \)) and \( \Gamma \), where the latter is obtained by integrating the Wronskian relation \( I^\nu J - J^\nu T = 0 \). For details, we refer to [38, Appendix A] (see also [33,49]), where exactly the same operator (in the case \( k = 1 \)) is considered.

**Outgoing Green’s function for \( L_Q \).** Finally, we turn to the first-order operator \( L_Q \), which is most involved due to its nonlocality. Unlike \( A_Q \) and \( H_Q \), the operator \( L_Q \) is not \( C \)-linear; nevertheless, it is \( \mathbb{R} \)-linear and preserves the real and imaginary parts. Hence, in order to invert \( L_Qu = f \) for a complex-valued function \( f \), we need Green’s functions for \( L_Q \) and \( i^{-1}L_Qi \).

**Proposition 3.4.** The outgoing Green’s functions for \( L_Q \) and \( i^{-1}L_Qi \) are

\[(L_Q)G(y, y') = 1_{(0, \infty)}(y - y') \frac{Q(y)}{Q(y')} f(y, y'),\]

\[(i^{-1}L_Qi)G(y, y') = 1_{(0, \infty)}(y - y') \frac{Q(y)}{Q(y')},\]

where \( I(y, y') \) is smooth on \( \{(y, y') : 0 < y' < y\} \) and obeys the following properties:

1. **(Upper bounds) For any nonnegative integer \( k \), we have**

   \[
   |(y\partial_y)^k I(y, y')| \lesssim_k \begin{cases} 
   1 + (y')^{-2} \log \left( 2 + \frac{(y)}{(y')} \right) & \text{if } k = 0, \\
   \frac{2(y')^2}{y^2} \min \left\{ \frac{y^2}{(y')^2}, (y')^{-2} \right\} & \text{if } k = 1, \\
   \frac{y^2}{1 + y^2} \left( 1 + (y')^{-2} \log \left( 2 + \frac{(y)}{(y')} \right) \right) & \text{if } k \geq 2.
   \end{cases}
   \]

2. **(Behavior near the diagonal) We have**

   \[
   \lim_{y - y' \to 0+} I(y, y') = 1, \quad \lim_{y - y' \to 0+} y\partial_y I(y, y') = 0.
   \]

Moreover, for any nonnegative integer \( k \), define \( I^{(k)}(y) := \lim_{y' \to y} (y\partial_y)^k I(y, y') \). For \( k \geq 2 \) and any nonnegative integer \( \ell \), we have

\[
|I^{(k)}(y)| \lesssim_{k, \ell} \frac{y^2}{1 + y^2}.
\]

**Proof.** We begin with the simpler case of \( i^{-1}L_Qi \). For a real-valued function \( u \),

\[i^{-1}L_Qiu = D_Qu = \partial_y u - \frac{1}{y}A_Q|Q|u.\]

In particular, \( i^{-1}L_Qi \) is a local operator (acted on real-valued functions). Moreover, recall from (3.10) that \( D_QQ = 0 \). Substituting \((i^{-1}L_Qi)G(y, y') = g_{y'}(y) \frac{Q(y)}{Q(y')} \) becomes

\[
\partial_y g_{y'} = \delta_{y'}(y),
\]

\[g_{y'}(y) = 0 \quad \text{for} \quad 0 < y < y',
\]

from which the desired expression for \((i^{-1}L_Qi)G(y, y') \) follows.

Next, we turn to \( L_Q \). While \( \ker L_Q = \{AQ\} \), variation of constants does not work due to the nonlocal integral term. Instead, we simply make the same substitution \((L_Q)G(y, y') = I(y, y') \frac{Q(y)}{Q(y')} \) as before, after which (3.17) becomes

\[
\partial_y I(y, y') + \frac{1}{y} \int_0^y zQ^2(z)I(z, y')dz = \delta_{y'}(y),
\]

\[I(y, y') = 0 \quad \text{for} \quad 0 < y < y'.
\]

\[\text{Note that the Wronskian relation is derived from } \partial_y(y^\nu J - yJ^\nu T) = 0. \text{ This also motivates the formula of } \Gamma.\]
Integrating from \( y = 0 \) and applying Fubini’s theorem, we arrive at the Volterra-type equation

\[
I(y, y') = 1_{(0, \infty)}(y - y') - \int_0^y z Q^2 \log \frac{y}{z} I(z, y') \, dz.
\]

By a standard Picard iteration argument applied to (3.20), the existence of a unique solution \( I(y, y') \) for \( y \in (y', y_+) \) for some \( y_+ > y' \) follows. Moreover, it is clear that \( \lim_{y' \to 0^+} I(y, y') = 1 \). Finally, observe that \( I(y, y') \) may be extended past \( y_+ \) as long as \( \lim \sup_{y \to y_+^{-}} |I(y, y')| < \infty \).

In order to construct and estimate \( I(y, y') \) on the whole interval \((y', \infty)\), we introduce a parameter \( C_0 > 1 \) to be fixed below, and split the argument into the following two cases:

**Case 1:** \( y < 2C_0 \). We may assume that \( y' \leq y < \min\{2C_0, y_+\} \), since \( I(y, y') \) is zero otherwise. Then by (3.20),

\[
|I(y, y')| \leq 1 + \int_{y'}^{y} z Q^2 \log \frac{y}{z} |I(z, y')| \, dz,
\]

so by Gronwall’s inequality,

\[
|I(y, y')| \leq \exp \left( \int_{y'}^{y} z Q^2 \log \frac{y}{z} \, dz \right) \lesssim C_0.
\]

In particular, if we take \( C_0 \to \infty \), it already follows that \( I(y, y') \) exists for all \( y \in (y', \infty) \). However, the resulting bound for large \( y \)'s is bad, so we give a separate argument in that case as follows.

**Case 2:** \( y > C_0 \), where \( C_0 \) is a parameter to be fixed below. By the preceding remark, we may assume that \( I(y, y') \) exists on \( y \in (y', \infty) \). In this case, for \( \max\{y', C_0\} < y \), we rewrite (3.20) as

\[
I(y, y') = 1 - \int_0^{C_0} z Q^2 \log \frac{y}{z} I(z, y') \, dz - \int_{\max\{y', C_0\}}^y z Q^2 \log \frac{y}{z} I(z, y') \, dz.
\]

Consider the norm

\[
\|g\| := \sup_{y > \max\{y', C_0\}} \left( 1 + \langle C_0^{-1} y' \rangle^{-2} \log \left( 2 + \frac{(C_0^{-1} y')^{-1}}{(C_0^{-1} y')^{1/2}} \right) \right)^{-1} |g(y)|.
\]

Observe that \( g_0 = 1 \) if \( y' > C_0 \) and \( |g_0| \lesssim C_0 \) by Case 1 otherwise; hence \( \|g\| \lesssim C_0 \). On the other hand, we claim that

\[
\|Tg\| \lesssim C_0^{-2} \|g\|.
\]

To verify (3.21), we may normalize \( \|g\| = 1 \). For simplicity, we only consider the case \( y' > C_0 \); the alternative case may be handled similarly. Since \( z Q^2 \lesssim z^{-3} \) on the domain of integration, we have

\[
\left| \int_{y'}^{y} z Q^2 \log \frac{y}{z} g(z) \, dz \right| \lesssim \int_{y'}^{y} \frac{1}{z^3} \log \frac{y}{z} \left( 1 + \langle C_0^{-1} y' \rangle^{-2} \log \left( 2 + \frac{(C_0^{-1} y')^{-1}}{(C_0^{-1} y')^{1/2}} \right) \right) \, dz
\]

\[
\lesssim (y')^{-2} \log \frac{y}{y'} (1 + \langle C_0^{-1} y' \rangle^{-2})
\]

\[
\lesssim C_0^{-2} \langle C_0^{-1} y' \rangle^{-2} \log \left( 2 + \frac{(C_0^{-1} y')^{-1}}{(C_0^{-1} y')^{1/2}} \right),
\]

which proves (3.21).

By (3.21), \( T \) is a contraction with respect to \( \|\cdot\| \) once we fix a large enough \( C_0 > 1 \). By the contraction mapping principle, it follows that,

\[
|I(y, y')| \lesssim C_0 1 + \langle C_0^{-1} y' \rangle^{-2} \log \left( 2 + \frac{(C_0^{-1} y')^{-1}}{(C_0^{-1} y')^{1/2}} \right) \lesssim C_0 \, \langle y' \rangle^{-2} \log \left( 2 + \frac{(y')^{-1}}{(y')^{1/2}} \right),
\]
which is the desired bound for $I(y, y')$.

For $y \partial_y I(y, y')$, we use the equation
\begin{equation}
(3.22) \quad y \partial_y I(y, y') = -\int_0^y z Q^2(z) I(z, y') dz \quad \text{for } y' < y,
\end{equation}
which immediately follows from (3.19). From (3.22), $\lim_{y' \to 0^+} y \partial_y I(y, y') = 0$ is immediate. To verify the asserted bound for $\lim_{y' \to 0^+} I(y, y')$, which immediately follows from (3.19), From (3.22),
\begin{equation}
|y \partial_y I(y, y')| \lesssim \begin{cases} (y - y') y' & y' < y \leq 2y', y' \leq 2 \\ 2y' - y & y' < y \leq 2y', y' > 2 \\ (y')^{-2} & 2y' < y, y' > 2, \end{cases}
\end{equation}
each of which is a straightforward consequence of (3.22), $|z Q^2| \lesssim \frac{1}{r^2}$ and the preceding bound for $I(y, y')$. Finally, the assertions concerning $(y \partial_y)^k I(y, y')$ follow in an inductive manner from
\begin{equation}
(y \partial_y)^2 I(y, y') = -y^2 Q^2(y) I(y, y') \quad \text{for } y' < y,
\end{equation}
which is obtained by differentiating (3.22); we omit the details. \hfill \Box

3.3. Adapted function spaces. In this subsection, we briefly review the definitions of equivariant Sobolev spaces $H^s_m$ and construct adapted function spaces $H^1_m$, $H^2_m$, and $H^3_m$. These function spaces are designed to have (sub-)coercivity estimates of the linear operators $L_Q$, $A_Q$, and $A^Q$ at various levels of regularity. Moreover, since $L_Q$ and $A_Q$ shifts the equivariance index by 1, and $A^Q$ shifts the equivariance index by $-1$ when viewed as acting on functions on the ambient space $\mathbb{R}^2$, we need to handle various equivariance indices too.

Equivariant Sobolev spaces. Perhaps a natural starting point is to consider equivariant Sobolev spaces. Let $m \geq 0$. Given an $m$-equivariant function $f$, we will often identify it with its radial part $g : \mathbb{R}^+ \to \mathbb{C}$, i.e. $f(x) = g(r) e^{i m \theta}$, under the usual polar coordinates relation $x_1 + i x_2 = r e^{i \theta}$. We often consider $g$ as an $m$-equivariant function, i.e. we say that $g$ belongs to some $m$-equivariant function space if its $m$-equivariant extension belongs to that.

For $s \geq 0$, we denote by $H^s_m$ the set of $m$-equivariant $H^s(\mathbb{R}^2)$ functions. The set of $m$-equivariant Schwartz functions is denoted by $\mathcal{S}_m$. The $H^s_m$-norm and $\dot{H}^s_m$-norm mean the usual $H^s(\mathbb{R}^2)$-norm and $\dot{H}^s(\mathbb{R}^2)$-norm, but the subscript $m$ indicates the equivariance index. When $0 \leq k \leq m$, we have generalized Hardy’s inequality \cite{22} Lemma A.7:
\begin{equation}
(3.23) \quad \|f|^{-k}\|_{L^2} \sim_{k,m} \|f\|_{H^k_m}, \quad \forall f \in \mathcal{S}_m.
\end{equation}
In addition, when $m \geq 1$ and $k = 1$, we have the Hardy-Sobolev inequality \cite{22} Lemma A.6:
\begin{equation}
(3.24) \quad \|r^{-1} f\|_{L^2} + \|f\|_{L^\infty} \lesssim \|f\|_{H^1_m}.
\end{equation}
As is well known, (3.24) fails when $m = 0$, but we can have a logarithmically weakened version of it; see \cite{A1}. The generalized Hardy’s inequality (3.23) allows us define the space $\dot{H}^k_m$ when $0 \leq k \leq m$ by taking the completion of $\mathcal{S}_m$ under the $H^k_m$-norm, with the embedding properties
\begin{equation}
\mathcal{S}_m \hookrightarrow H^k_m \hookrightarrow \dot{H}^k_m \hookrightarrow L^2_{\text{loc}}.
\end{equation}
Adapted function spaces. As alluded to above, we will track the dynamics of \(w\), \(w_1 = D_ww\), and \(w_2 = Aww_1 = \dot{A}_ww\): see the equations (2.14), (2.15), and (2.16). The related linearized equations are (3.12), (3.13), and (3.17), respectively. Thus we need to handle adapted derivatives \(LQ\epsilon, A_QLQ\epsilon\), and so on. Here we investigate how these derivatives control the original \(\epsilon\). The preceding equivariant Sobolev spaces do not work very well with those adapted derivatives. We need to introduce new adapted function spaces \(\mathcal{H}_m^k\), which are slightly modified from the original equivariant Sobolev spaces \(\mathcal{H}_m^k\). More precisely, we will obtain (sub-)coercivity properties of \(LQ\), \(A_Q\), and \(A_Q^\star\) in terms of \(\mathcal{H}_m^k\)-norms.

We define the \(\mathcal{H}_m^k\)-norms for \((k, m) \in \{(1, 0), (1, 2), (2, 1), (3, 0)\}\) by
\[
\|v\|_{\mathcal{H}_m^k} := \|\partial_r v\|_{L^2} + \|r^{-1}(\log r)^{-1}v\|_{L^2},
\]
\[
\|v\|_{\mathcal{H}_m^k_t} := \|\partial_r v\|_{L^2} + \|r^{-1}(\log r)^{-1}v\|_{L^2},
\]
\[
\|v\|_{\mathcal{H}_m^k_0} := \|\partial_r v\|_{L^2} + \|r^{-1}(\log r)^{-1}|v|_{-1}\|_{L^2} + \|r^{-1}(r)^{-2}(\log r)^{-1}v\|_{L^2}.
\]

The space \(\mathcal{H}_m^k\) is defined by the completion of the space \(\mathcal{S}_m\) of \(m\)-equivariant Schwartz functions under the \(\mathcal{H}_m^k\)-norms. It turns out that \(\mathcal{H}_m^1 \rightarrow \mathcal{H}_m^2, \mathcal{H}_m^2 \rightarrow \mathcal{H}_m^3,\) and \(\mathcal{H}_m^3 \rightarrow \mathcal{H}_m^4\). But we have a reverse embedding for \(\mathcal{H}_m^2: \mathcal{H}_m^2 \rightarrow \mathcal{H}_m^1\). Note that the norms \(\mathcal{H}_m^k\) are same as \(\mathcal{H}_m^k\) norms for high frequency pieces. In particular, one has \(\mathcal{H}_m^k \cap L^2 = H_k^m\). See Appendix A for more details.

The spaces \(\mathcal{H}_m^k\) are constructed in order to have boundedness and subcoercivity estimates of \(LQ\), \(A_Q\), and \(A_Q^\star\). Actually this is how we chose the weights in the definitions of the \(\mathcal{H}_m^k\)-norms. For more details, we refer to [21 Section 2.3]. Since \(LQ\) and \(A_Q\) have nontrivial kernels, it is impossible to have a coercivity estimate like \(\|LQv\|_{L^2} \sim \|v\|_{\mathcal{H}_m^1}\). Instead, we can find a subcoercivity estimate as like
\[
\|LQv\|_{L^2} + \|1_{r^{-1}}v\|_{L^2} \sim \|v\|_{\mathcal{H}_m^1}^2.
\]

The associated coercivity can be obtained by ruling out the kernel elements of \(LQ\). The same remark applies to \(A_Q\). For \(A_Q^\star\), due to the positivity (3.8) of \(A_Q^\star\) and the unconditionality of \(\mathcal{H}_m^k\), the unconditionality coercivity estimate for \(A_Q^\star\) holds. As a result, we have the following coercivity estimates (see Appendix A for the proof).

**Proposition 3.5** (Linear coercivity estimates).

1. (Coercivity of \(LQ\) at \(H^1\)-level) Let \(\psi_1, \psi_2 \in (\mathcal{H}_m^1)^\star\) be such that the \(2 \times 2\) matrix \((a_{ij})\) defined by \(a_{11} = \langle \psi_1, LQ\rangle_r\) and \(a_{12} = \langle \psi_1, iQ\rangle_r\) has nonzero determinant. Then, we have a coercivity estimate
\[
\|v\|_{\mathcal{H}_m^1} \lesssim_{\psi_1, \psi_2} \|LQv\|_{L^2} \lesssim \|v\|_{\mathcal{H}_m^1}, \quad \forall v \in \mathcal{H}_m^1 \cap \{\psi_1, \psi_2\}^\perp.
\]

2. (Coercivity of \(LQ\) at \(H^3\)-level) Let \(\psi_1, \psi_2 \in (\mathcal{H}_m^3)^\star\) be such that the \(2 \times 2\) matrix \((a_{ij})\) defined by \(a_{11} = \langle \psi_1, LQ\rangle_r\) and \(a_{12} = \langle \psi_1, iQ\rangle_r\) has nonzero determinant. Then, we have a coercivity estimate
\[
\|v\|_{\mathcal{H}_m^3} \lesssim_{\psi_1, \psi_2} \|LQv\|_{\mathcal{H}_m^3} \lesssim \|v\|_{\mathcal{H}_m^3}, \quad \forall v \in \mathcal{H}_m^3 \cap \{\psi_1, \psi_2\}^\perp.
\]

3. (Coercivity of \(A_Q\) at \(H^2\)-level) Let \(\psi_1, \psi_2 \in (\mathcal{H}_m^2)^\star\) be such that the \(2 \times 2\) matrix \((a_{ij})\) defined by \(a_{11} = \langle \psi_1, rQ\rangle_r\) and \(a_{12} = \langle \psi_1, irQ\rangle_r\) has nonzero determinant. Then, we have a coercivity estimate
\[
\|v\|_{\mathcal{H}_m^2} \lesssim_{\psi_1, \psi_2} \|A_Qv\|_{\mathcal{H}_m^2} \lesssim \|v\|_{\mathcal{H}_m^2}, \quad \forall v \in \mathcal{H}_m^2 \cap \{\psi_1, \psi_2\}^\perp.
\]

4. (Unconditional coercivity of \(A_Q^\star\) at \(H^1\)-level) We have
\[
\|A_Q^\star v\|_{L^2} \sim \|v\|_{\mathcal{H}_m^1}, \quad \forall v \in \mathcal{H}_m^1.
\]
In later applications, we will use orthogonality conditions depending on a large truncation parameter \( M \). Thus in the above coercivity estimates \( \lesssim_{\psi_1, \psi_2} \) becomes \( \lesssim_M \).

We will later decompose \( w, w_1, w_2 \) as

\[
w = P + \epsilon, \quad w_1 = P_1 + \epsilon_1, \quad w_2 = P_2 + \epsilon_2,
\]

where \( P, P_1, P_2 \) are some modified profiles, and \( \epsilon, \epsilon_1, \epsilon_2 \) are the errors. Thus \( \epsilon, \epsilon_1, \epsilon_2 \) are constructed in a nonlinear fashion (later called nonlinear adapted derivatives), we approximately have \( \epsilon_1 \approx L_{\epsilon} \) and \( \epsilon_2 \approx A_{\epsilon} \). We will also use \( \epsilon_3 = A_{\epsilon} \). In bootstrap analysis, we want to control \( \|\epsilon\|_{L^2}, \|\epsilon_1\|_{L^2}, \) and \( \|\epsilon_2\|_{L^2} \). In view of the above coercivity estimates, \( \|\epsilon_1\|_{L^2} \) will control \( \|\epsilon\|_{H^1}, \) and \( \|\epsilon_2\|_{L^2} \) will control \( \|\epsilon_2\|_{H^2}; \|\epsilon_1\|_{H^1}; \|\epsilon\|_{H^2} \).

See Lemma 5.6.

Finally, for technical reasons, we will need an auxiliary norm \( \|\cdot\|_{X} \)

\[
(3.29) \quad \|f\|_{X} := \|\langle y \rangle^{-2} \log_{+} y f\|_{L^2}.
\]

This will be used in the Morawetz correction (Section 5.3), e.g. in the estimate

\[
(i\epsilon_2, yQ^2\epsilon_1)_r \lesssim \|\epsilon_2\|_{H^2} \|\langle y \rangle^{-2} \log_{+} y \epsilon_1\|_{L^2} \sim \|\epsilon_3\|_{L^2} \|\epsilon_1\|_{X}.
\]

4. Modified Profiles

This section is devoted to the construction of modified profiles and the derivation of a sharp logarithmic correction to the pseudoconformal blow-up rate, which are one of the novelties of this work.

As we have seen in Section 3.3, the information on the generalized nullspace of \( i\mathcal{L}_Q \) suggests a decomposition of the form

\[
u(t, r) = \frac{e^{\gamma(t)}}{\lambda(t)} \left[ P(\cdot; b(t), \eta(t)) + \epsilon(t, \cdot) \right] \left( \frac{r}{\lambda(t)} \right),
\]

where \( P(\cdot; 0, 0) = Q, \partial_b P \approx -i\frac{\omega}{4} Q, \partial_{\eta} P \approx -(m + 1)\rho \). Here we focus on the modulated blow-up profile \( P \).

The case considered here \( (m = 0) \) is significantly different from the case \( m \geq 1 \).

When \( m \geq 1 \), the authors in [21] constructed pseudoconformal blow-up solutions using the modified profiles

\[
Q^{(q)}_b(y) = \chi_{B_{\epsilon}}(y)Q^{(q)}(y)e^{-ib\frac{y}{\epsilon}},
\]

where \( Q^{(q)} \) some profile satisfying \( Q^{(0)} = Q \) and \( \partial_{\eta} Q^{(q)} \approx -(m + 1)\rho \). Moreover, \( Q^{(q)} \) is obtained by solving the modified Bogomol’nyi equation [22]

\[
D_{Q^{(q)}} Q^{(q)} = -\eta^2 Q^{(q)}
\]

in the region \( y \ll |\eta|^{-\frac{1}{b}} \). This profile suggests the modulation equation of the form

\[
(4.1) \quad \frac{\lambda}{\lambda} b = 0, \quad \gamma_{\eta} = (m + 1)\eta, \quad b + b^2 + \eta^2 = 0, \quad \eta_{\eta} = 0.
\]

This nonlinear profile ansatz was a quick and efficient way to derive the above modulation equation. Moreover, when \( m \geq 1 \), the profile error \( \Psi \) (generated by the truncation \( \chi_{B_{\epsilon}} \)) is sufficiently small to guarantee pseudoconformal blow up.

Moreover, the rotational instability for \( m \geq 1 \) can be read off from (4.1). Setting \( \eta \) as a fixed small constant \( \eta_0 \), (4.1) has solutions

\[
b(t) = |t|, \quad \lambda(t) = (t^2 + \eta^2)^{\frac{1}{2}}, \quad \eta(t) = \eta_0,
\]

\[
\gamma(t) = \begin{cases} 0 & \text{if } \eta_0 = 0, \\ \text{sgn}(\eta)(m + 1) \tan^{-1}(\frac{1}{\eta_0}) & \text{if } \eta_0 \neq 0. 
\end{cases}
\]
When \( \eta_0 = 0 \), the solution blows up in the pseudoconformal regime and shows no phase rotation. However, when \( \eta_0 \neq 0 \), regardless how much \( |\eta_0| \) is small, the solution is global and shows an abrupt phase rotation on the short time interval \( |t| \lesssim |\eta_0| \), by the fixed amount of angle \( \theta_n^{(m+1)} \pi \). In [22], an explicit family of solutions for \( \eta_0 \geq 0 \) was constructed to establish the (one-sided) rotational instability for \( m \geq 1 \).

Unfortunately when \( m = 0 \), the above nonlinear profile ansatz does not work; it generates a profile error \( \Psi \) of critical size. Hence we search for a more refined profile. Experiences from other critical equations such as wave maps, Schrödinger maps, and harmonic map heat flows [33,38,39] tell us that there might be a logarithmic correction to the blow-up rate, driven by the zero resonance for the linearized operator \( H_Q \).

The authors in [21] found a remarkable conjugation identity [85], which bridges (1.3) to the above critical equations. This connection is observed when we proceed to the variable \( L_Q \epsilon \) in the linearized equation

\[
\partial_t L_Q \epsilon + i H_Q L_Q \epsilon = 0, \quad H_Q = A_Q A_Q.
\]

As mentioned earlier, this \( H_Q \) is the same as the one appearing in the above critical equations and has the resonance \( yQ \not\in L^2 \). This connection motivates us to look at the \( w_1 \)-equation, instead of the original equation for \( w \). Moreover, we are able to extract, from the \( w_1 \)-equation, logarithmic corrections to \( b + b^2 + \eta^2 = 0 \), which results in a logarithmic correction to the pseudoconformal blow-up rate.

Motivated from the previous discussion, we not only track the dynamics of \( w \), but also its canonical higher order variables \( w_1 = D_w w \) and \( w_2 = A_w w_1 \). Using the conjugation identities, we derived evolution equations of \( w_1 \) and \( w_2 \). We view (CSS) as a system of evolution equations of \( w, w_1, w_2 \) under the compatibility conditions \( w_1 = D_w w \) and \( w_2 = A_w w_1 \). We are about to construct modified profiles \( P, P_1, P_2 \) for \( w, w_1, w_2 \), respectively.

Of course one can try to set \( P_1 = D_p P \) and \( P_2 = A_p P_1 \), but this choice is nothing but looking at only the \( w \)-equation. One of the main novelties here is to construct \( P, P_1, P_2 \) that approximately solve the evolution equations as well as the compatibility conditions. Here, the point is that we also relax the compatibility conditions: \( D_p P \approx P_1 \) and \( A_p P_1 \approx P_2 \).

In this setting, we have another advantage. It turns out that we do not need to expand \( P \) and \( P_1 \) to very higher orders. In fact, it suffices to expand \( P \) only up to linear order and \( P_1 \) up to quadratic order. This is because the degeneracy of the profiles \( P_1 \) and \( P_2 \), which ultimately relies on the facts that \( D_Q Q = 0 \) and \( A_Q L_Q \) kills all the elements of \( \{ A_Q, iQ, i\frac{\sqrt{2}}{\rho} Q, \rho \} \). As we will apply the energy estimate for the variable \( w_2 \), \( P_2 \) should be constructed to the highest order compared to \( P \) and \( P_1 \). However, thanks to the degeneracy of \( P_2 \), it contains only the quadratic and cubic order terms, which are still quite simple.

Finally, we remark that we are able to observe logarithmic corrections in the modulation laws from the \( w_1 \)-equation. As explained above, at the linear level, the \( w_1 \)-equations solves a similar equation to the Schrödinger map case. The effects of the logarithmic corrections can be seen in the quadratic terms of \( P_1 \) and \( P_2 \) expansions.

4.1. Formal derivation of the profiles. Our starting points are the evolution equations (2.11), (2.15), and (2.16) for \( w, w_1 = D_w w \), and \( w_2 = A_w w_1 \), derived in Proposition 2.3. After substitutions \( w_1 = D_w w \) and \( w_2 = A_w w_1 \), they are written

\[\text{for } m\text{-equivariant functions, the phase rotation by angle } (m+1)\pi \text{ corresponds to the spatial rotation by angle } \left( \frac{m+1}{m+1} \right) \pi.\]
\begin{align*}
(\partial_x - \frac{\lambda_s}{\lambda} + \gamma_s i) w + i L_\omega^+ w_1 &= 0, \quad (4.3) \\
(\partial_x - \frac{\lambda_s}{\lambda} \Lambda_{-1} + \gamma_s i) w_1 + i A_{w}^+ w_2 &= \left( \int_0^y \text{Re}(\overline{w} w_1) dy \right) i w_1 = 0, \quad (4.4) \\
(\partial_x - \frac{\lambda_s}{\lambda} \Lambda_{-2} + \gamma_s i) w_2 + i A_{w} A_{w}^+ w_2 &= \left( \int_0^y \text{Re}(\overline{w} w_1) dy \right) i w_2 - i w_1^2 = 0, \quad (4.5)
\end{align*}

where
\[\tilde{\gamma}_s = \gamma_s + \int_0^\infty \text{Re}(\overline{w} w_1) dy.\]

Recall that the role of the phase correction \(\gamma_s \mapsto \tilde{\gamma}_s\) is to replace the above \(\int_y^\infty\) integral by \(\int_y^w\). Note that \(\int_y^\infty\) has the technical problem that it cannot be defined for functions with growing tails, which typically arise in the Taylor expansion of the profiles. See also Remark 2.4. Assume the adiabatic ansatz
\[\lambda_s + b = 0 \quad \text{and} \quad \tilde{\gamma}_s = -\eta.\]

We will construct an approximation solution of the form
\[(w, w_1, w_2) = (P, P_1, P_2),\]

and the compatibility conditions \(w_1 = D_w w\) and \(w_2 = A_w w_1\). Here, \(P, P_1,\) and \(P_2\) will be suitable localizations of \(\hat{P}\)
\begin{align*}
\hat{P} &:= Q - ib\frac{\eta}{4} Q - \eta \rho, \\
\hat{P}_1 := -(ib + \eta)\frac{\eta}{2} Q + b^2 T_{2,0}, \\
\hat{P}_2 := (b^2 - 2ib\eta - \eta^2) U_2 + ib^4 U_{3,0},
\end{align*}

where \(T_{2,0}, U_2, U_{3,0}\) are real-valued. The profiles \(T_{2,0}, U_2, U_{3,0}\), as well as the laws for \(b_s\) and \(\eta_s\), are unknowns and will be chosen subsequently to minimize the profile error.

The profiles up to the first order in \(\hat{P}, \hat{P}_1,\) and \(\hat{P}_2\) are easily derived from the generalized nullspace relations and the adiabatic ansatz. Indeed, if we start from \(w = Q\), then \(D_w Q = 0\) and the compatibility conditions suggest that zeroth order terms of \(w_1\) and \(w_2\) should vanish. Next, from (4.3) and the adiabatic ansatz \(\lambda_s + b = 0\) and \(\gamma_s \approx \eta\), we are led to
\[L_Q^+ w_1 \approx 1 ib \Lambda Q - \eta Q.\]

This suggests the choice \(w_1 \approx -(ib + \eta)\frac{\eta}{2} Q\). By linearizing the compatibility relation \(w_1 = D_w w\), we have
\[L_Q (w - Q) \approx -(ib + \eta)\frac{\eta}{2} Q,\]

which motivates the choice \(w \approx 1 \quad Q - ib\frac{\eta}{4} Q - \eta \rho\). Finally, \(A_Q (y Q) = 0\) and the compatibility relation \(w_2 = A_w w_1\) suggest \(w_2 \approx 0\). In summary, we are led to
\begin{align*}
\hat{P} &\approx 1 \quad Q - ib\frac{\eta}{4} Q - \eta \rho, \\
\hat{P}_1 &\approx -(ib + \eta)\frac{\eta}{2} Q, \\
\hat{P}_2 &\approx 0.
\end{align*}

\textsuperscript{13}This approach works for general \(m \geq 0\). In that case, assume \(\tilde{\gamma}_s = -(m + 1)\eta\) and \(\hat{P} = Q - ib\frac{\eta}{4} Q - \eta(m + 1)\rho\) instead.

\textsuperscript{14}In the sense that they are equal up to the first order.
We now search for higher order expansions for $\hat{P}$, $\hat{P}_1$, and $\hat{P}_2$. In the following, we will also assume
\[ |\eta| \leq \frac{b}{\log b}. \]
Although our sharp modulation equation will be slightly different from (3.1) of the $m \geq 1$ case, (4.1) still motivates us to assume $|\eta| \ll b$ to guarantee the blow-up.

Remark 4.1. In order to obtain the sharp energy estimate (4.30) under $|\eta| \leq \frac{b}{\log b}$, it is necessary to expand $\hat{P}_2$ up to $b^3$-order terms. Thus one may start from considering a general expansion
\[
\hat{P} = Q - ib\frac{\xi}{2}Q - \eta \rho + b^2 \tilde{S}_{2,0} + b\eta \tilde{S}_{1,1} + \eta^2 \tilde{S}_{0,2} + \cdots,
\]
\[
\hat{P}_1 := -(ib + \eta)\frac{\xi}{2}Q + b^2 \tilde{T}_{2,0} + b\eta \tilde{T}_{1,1} + \eta^2 \tilde{T}_{0,2} + \cdots,
\]
\[
\hat{P}_2 := b^2 \tilde{U}_{2,0} + b\eta \tilde{U}_{1,1} + \eta^2 \tilde{U}_{0,2} + b^3 \tilde{U}_{3,0},
\]
for some complex-valued profiles $\tilde{S}_{1,1}$, $\tilde{T}_{1,1}$, and $\tilde{U}_{1,1}$. Due to (4.30) and $|\eta| \leq \frac{b}{\log b}$, it is enough to stop at $b^3 \tilde{U}_{3,0}$; our main goal is to construct $\tilde{U}_{3,0}$.

In the following, we will use the ansatz (4.6) for the simplicity of presentation. On the way, the reader may see that the linear expansion is enough for $\hat{P}$, and the expansion up to the $b^2$-term is enough for $\hat{P}_1$. The other quadratic terms $b\eta \tilde{T}_{1,1}$ and $\eta^2 \tilde{T}_{0,2}$ are not necessary, due to $|\eta| \leq \frac{b}{\log b}$. Moreover, the coefficients in the ansatz (4.6) naturally appear in the derivation.

**Derivation of $U_2$ and $T_{2,0}$.**

Here we search for the quadratic terms of the expansions. For this, we look at the $w_1$-equation (4.4). We collect the $O(b^2, b\eta, \eta^2)$-terms (not including $O(1, b, \eta)$ terms) in the equation (4.4):
\[
\partial_s w_1 \to (-ib_s - \eta_s)(\frac{\xi}{2}Q),
\]
\[
b\Lambda_{-1} w_1 \to (-ib^2 - b\eta)\Lambda_{-1}(\frac{\xi}{2}Q),
\]
\[
-\eta iw_1 \to (b\eta + i\eta^2)(\frac{\xi}{2}Q),
\]
\[
iA^*_0 w_2 \to (ib^2 + 2b\eta - i\eta^2)A^*_0 U_2,
\]
\[
-(\int_0^1 \text{Re}(\overline{w}w_1)dy')iw_1 \to (b\eta - i\eta^2)(2 - \Lambda)(\frac{\xi}{2}Q),
\]
where in the last one we used
\[
(4.7) \quad (\int_0^1 \overline{w}'Qdy')\frac{\xi}{2}Q = -A_0[\frac{\xi}{2}Q] = \frac{\xi}{2}Q + (yA_Q - y\partial_y)\frac{\xi}{2}Q = (2 - \Lambda)(\frac{\xi}{2}Q).
\]
Summing up, we arrive at
\[
(-i(b_s + b^2 + \eta^2) - \eta_s)(\frac{\xi}{2}Q) + (ib^2 + 2b\eta - i\eta^2)(A^*_0 U_2 - \Lambda(\frac{\xi}{2}Q)) = 0.
\]
Here, the key point is that $\Lambda(\frac{\xi}{2}Q)$ exhibits better spatial decay (by order 2) compared to the main term $yQ$, which is grouped together with the modulation differentials $b_s$, $\eta_s$. Roughly speaking, the term with the worst growth $yQ$ is cancelled by choosing $b_s$, $\eta_s$ appropriately, whereas we attempt to introduce profile $U_2$ (and also $T_{2,0}$ below) to solve away the remaining better decaying terms. This is the tail computation due to $\Sigma_0, \Sigma_3, \Sigma_5$.

This motivates us to formally set
\[ b_s + b^2 + \eta^2 = 0 \quad \text{and} \quad \eta_s = 0 \]
up to quadratic terms. For the profile $U_2$, a naive choice would be to solve $A^*_0 U_2 - \Lambda(\frac{\xi}{2}Q) = 0$. However, with this choice we cannot avoid the profile error $\Pi_2$ of critical size. Indeed, solving $A^*_0 U_2 - \Lambda(\frac{\xi}{2}Q) = 0$, we have $U_2 \sim 1$ near infinity. This lack of decay is due to the violation of the $L^2$-solvability condition $(\Lambda(\frac{\xi}{2}Q), \frac{\xi}{2}Q), r = 2\pi \neq 0$,
which in turn is due to $yQ \notin L^2$. Continuing the expansion with this $U_2$, we would arrive at $U_{3,0} \sim y^2$ near infinity. In the computation of the profile error $\Psi_2$, with any cutoff at some $y = B$, $\|\Psi_2\|_{H^1}$ would see the cutoff error of $U_{3,0}$ at $y = B$, which is

$$b^3\|1_{y \sim B}|U_{3,0}|_{-3}\|_{L^2} \sim b^3\|1_{y \sim B^\frac{1}{2}}\|_{L^2} \sim b^3.$$  

This error is of critical size, in the sense that we would not be able to make $\|\cdot\|_{H^1} \ll b^2$ in the energy argument because of it. This also explains why we cannot use the profile ansatz used in the case $m \geq 1$.

To overcome this issue, we follow [58] and use the fact that $\frac{Q}{2}Q$ is a resonance to the operator $A_0^*A_2$. From the compatibility condition $A_w w_1 = w_2$ (compare $l^2$-order terms), we choose $T_{2,0}$ such that

$$A_Q T_{2,0} = U_2.$$  

Thus if $A_0^*U_2 = \Lambda(\frac{Q}{2}Q)$, then $T_{2,0}$ should satisfy $A_0^*A_Q T_{2,0} = \Lambda(\frac{Q}{2}Q)$. Note again that the $L^2$-solvability condition does not hold because $\frac{Q}{2}Q \notin L^2$:  

$$\Lambda(\frac{Q}{2}Q), \frac{Q}{2}Q), r = 2\pi \neq 0.$$  

As in [58] p.31 Step 6], we introduce

$$c_b := \frac{(\Lambda(\frac{Q}{2}Q), \frac{Q}{2}Q)_r}{(\frac{Q}{2}Q, \frac{Q}{2}Q)_r} = \frac{2}{|\log b|} + O\left(\frac{1}{|\log b|^2}\right)$$

and solve instead \footnote{When $m \geq 1$, the solvability condition $(\Lambda(\frac{Q}{2}Q), \frac{Q}{2}Q)_r = 0$ holds because $\frac{Q}{2}Q \notin L^2$. Thus one may define $U_2$ and $T_{2,0}$ by solving $A_0^*U_2 = \Lambda(\frac{Q}{2}Q)$ and $A_Q T_{2,0} = U_2$ instead. Note that one can find explicit formulas $U_2 = -\frac{Q}{2}Q$ and $T_{2,0} = -\frac{Q}{2}Q$, as motivated from the Taylor expansion of the pseudoconformal phase $e^{-ib\frac{Q}{2}}$. This leads to the pseudoconformal blow-up rate.}

$$A_0^* A_Q T_{2,0} = \Lambda(\frac{Q}{2}Q) - c_b \frac{Q}{2}Q \chi_{B_0} =: g_2.$$  

For the choice of the radius $B_0$, see Remark 4.8. We remark that the power $-\frac{1}{2}$ of $B_0 = b^{-\frac{1}{2}}$ is tied to the sharp blow-up rate.

Therefore, we will choose $U_2$ and $T_{2,0}$ such that

\begin{align}
(4.8) & \quad A_0^* U_2 = \Lambda(\frac{Q}{2}Q) - c_b \frac{Q}{2}Q \chi_{B_0} = g_2,
(4.9) & \quad A_Q T_{2,0} = U_2.
\end{align}

With this $U_2$, it turns out that one has a logarithmic gain $\frac{1}{|\log b|}$ in the region $y \gtrsim B_0$, so the previous issue is overcome. On the other hand, the equation (4.3) is solved up to quadratic terms with the additional error

$$(ib^2 + 2b\eta - i\eta^2)c_b \frac{Q}{2}Q \chi_{B_0}.$$  

This will give rise to additional terms of order $O\left(\frac{b^2}{|\log b|}, \frac{b\eta}{|\log b|}, \frac{\eta^2}{|\log b|}\right)$ in the equations for $b_\lambda$ and $\eta_\lambda$, which in turn cause the logarithmic correction to the blow-up rate. As a result, we get the formal parameter law:

\begin{align}
(4.10) & \quad \frac{\lambda_\lambda}{\lambda} + b = 0, \quad \bar{\gamma}_\lambda = -\eta, \quad b_\lambda + b^2 + \eta^2 + c_b(b^2 - \eta^2) = 0, \quad \eta_\lambda + 2c_b b\eta_\lambda = 0,
\end{align}

where $c_b \approx \frac{2}{|\log b|}$ defined above.

\textbf{Remark 4.2 (Full quadratic expansion for $\tilde{P}_1$).} By the same way, but using $A_0^* A_w w_1$ instead of $A_0^* w_2$ in (4.3) and collecting the quadratic terms $O(b^2, b\eta, \eta^2)$, one can derive the full quadratic expansion of $\tilde{P}_1$:

$$\tilde{P}_1 = -(ib + \eta)\frac{Q}{2}Q + (b^2 - 2ib\eta - \eta^2)T_{2,0} + (ib\eta + \eta^2)\tilde{T}_2,$$
where $T_{2,0}$ is as above and $T_2$ solves $A_2 T_2 = A_2 [Q, \rho] Q$. As mentioned in the previous remark, $O(b\eta)$ and $O(\eta^2)$ terms are not necessary in the derivation of $U_{3,0}$ and later analysis.

**Derivation of $U_{3,0}$.**

We finally search for the $b^3$ term of the $P_3$-expansion. We again look at the $w_1$-equation. We collect $b^3$-terms of the error.

\[
\begin{align*}
\partial_t w_1 & \to -2b^3 T_{2,0}, \\
 b \Lambda w_1 & \to b^3 \Lambda T_{2,0}, \\
 -\eta i w_1 & \to 0, \\
 i \Lambda^* w_2 & \to -b^3 A_2^* U_{3,0}, \\
 -(\int_0^y \text{Re}(\bar{w} w_1) dy') i w_1 & \to -b^3 (\int_0^y (QT_{2,0} + \frac{(\omega')^2}{8} Q^2) y' dy') \frac{\sqrt{2}}{2} Q.
\end{align*}
\]

Summing these up, we are motivated to choose $U_{3,0}$ by solving

\[(4.11) \quad A_2^* U_{3,0} = \Lambda T_{2,0} - (\int_0^y (QT_{2,0} + \frac{(\omega')^2}{8} Q^2) y' dy') \frac{\sqrt{2}}{2} Q =: g_{3,0}.
\]

Taking $A_2^* g_{3,0} = A_2^* \Lambda T_{2,0} - (QT_{2,0} + \frac{\omega^2}{8} Q^2)(\frac{\sqrt{2}}{2} Q)$.

### 4.2. Estimates of profiles in Taylor expansions

In the previous subsection, we discussed how we choose the higher order profiles $T_{2,0}, U_2, U_{3,0}$ used in the definitions $P_1$ and $P_2$. Here we construct these profiles satisfying \((4.8), (4.9), \) and \((5.9)\), using the outgoing Green’s function discussed in Section 3.2.

**Lemma 4.3** (Profiles $T_{2,0}, U_2, U_{3,0}$). Let $0 < b < b^\ast$. Define the smooth functions on $(0, \infty)$ by

\[
\begin{align*}
T_{2,0}(y; b) & := \text{(out)} H_Q^{-1} g_2, \\
U_2(y; b) & := A_2 T_{2,0} = -(A_2 \Gamma) \int_0^y g_2 Jy' dy' = (A_2 \Gamma) \int_0^\infty g_2 Jy' dy', \\
U_{3,0}(y; b) & := A_2^* \text{(out)} H_Q^{-1} g_{3,0} = -(A_2 \Gamma) \int_0^y g_{3,0} Jy' dy'.
\end{align*}
\]

where

\[
\begin{align*}
g_2(y; b) & = \Lambda(\frac{\sqrt{2}}{2} Q) - c_b \frac{\sqrt{2}}{2} Q \chi_{B_0}, \\
g_{3,0}(y; b) & = \Lambda T_{2,0} - \frac{(\int_0^y (QT_{2,0} + \frac{(\omega')^2}{8} Q^2) y' dy') (\frac{\sqrt{2}}{2} Q)}{(\int_0^\infty Q\chi_{B_0} Q).}
\end{align*}
\]

Then for any nonnegative integer $k$, the following properties hold:

1. (Rough pointwise estimates, only sharp in the compact regions $y \sim 1$) We have

\[(4.13) \quad |U_2| + \frac{1}{y} |T_{2,0}| + \frac{1}{y} |U_{3,0}| \lesssim_k 1.
\]

2. (Sharp pointwise estimates)

(a) In the region $1 \leq y \leq B_0$, we have

\[
|U_2| + \frac{1}{y} |T_{2,0}| + \frac{1}{y} |U_{3,0}| \lesssim_k \frac{1}{\log y^2} |\log(b^\ast y)|,
\]

\[
|b \partial_y U_2| + \frac{1}{y} |b \partial_y T_{2,0}| + \frac{1}{y} |b \partial_y U_{3,0}| \lesssim_k \frac{1}{\log y^2} |\log(b^\ast y)|.
\]

(b) In the region $B_0 \leq y \leq 2B_0$, we have

\[
|U_2| + \frac{1}{y} |T_{2,0}| + \frac{1}{y} |U_{3,0}| \lesssim_k \frac{1}{\log y^2},
\]

\[
|b \partial_y U_2| + \frac{1}{y} |b \partial_y T_{2,0}| + \frac{1}{y} |b \partial_y U_{3,0}| \lesssim_k \frac{1}{\log y^2}.
\]
(c) In the region \( y \leq 1 \), we have
\[
\frac{1}{y^3}[U_2] + \frac{1}{y^2}|T_{2,0}| + \frac{1}{y}|U_{3,0}| \lesssim k 1.
\]
Moreover, the profile \( T_{2,0} \) has smooth 1-equivariant extension on \( \mathbb{R}^2 \); and the profiles \( U_2, U_{3,0} \) have smooth 2-equivariant extension on \( \mathbb{R}^2 \).

**Remark 4.4.** An important point is that one has logarithmic gain in the region \( y \sim B_0 \). In the region \( y \lesssim 1 \), we do not have any logarithmic gain.

**Remark 4.5.** The rough pointwise estimates are sharp only in the region \( y \sim 1 \), and not sharp in far regions \( y \gtrsim B_0 \). Thus rough pointwise estimates will be effective when the main contributions to errors come from the compact region \( y \sim 1 \). Of course, the rough pointwise estimates are easy to implement.

**Proof.** Bounds of \( U_2 \) are immediate from the bounds of \( T_{2,0} \). Henceforth, we focus on \( T_{2,0} \) and \( U_{3,0} \).

For \( T_{2,0} \), thanks to the cancellation property near the infinity
\[
1_{[1,\infty)}|\Lambda(yQ)| \lesssim 1_{[1,\infty]}y^{-3},
\]
g_2 satisfies (use \( \partial_b g_2 \lesssim \frac{1}{y \log y} \) for \( \partial_b g_2 \))
\[
|g_2|_k \lesssim_k 1_{[0,1]}y + 1_{[1,2B_0]} \frac{1}{y \log y} + 1_{[2B_0,\infty]} \frac{1}{y},
\]
\[
|\partial_b g_2|_k \lesssim_k \frac{1}{y \log y} (1_{[0,1]} \frac{y}{\log y} + 1_{[0,1]} \frac{1}{y \log y} + 1_{[B_0,2B_0]} \frac{1}{y}).
\]
In particular, by Proposition 3.3, it easily follows that
\[
1_{[0,1]}|T_{2,0}|_k \lesssim_k y^3, \quad 1_{[0,1]}|\partial_b T_{2,0}|_k \lesssim_k \frac{1}{y \log y} y^3.
\]
Because \( g_2 \) satisfies the solvability condition \( (g_2, yQ)_r = 0 \) (thus by differentiating it \( (\partial_b g_2, yQ)_r = 0 \)), we can rewrite (see Proposition 3.3)
\[
T_{2,0} = \int_y^b g_2 y' dy' + \Gamma_y^\infty y^2 Jy' dy',
\]
\[
\partial_b T_{2,0} = \int_y^b \partial_b g_2 y' dy' + \Gamma_y^\infty \partial_b g_2 J y' dy'.
\]
Substituting the pointwise estimates of \( g_2 \) shows the bounds of \( T_{2,0} \):
\[
1_{[1,\infty)}|T_{2,0}|_k \lesssim_k 1_{[1,2B_0]} \frac{1}{y \log y} (1_{[0,1]} \frac{y}{\log y} + \log y),
\]
\[
1_{[1,\infty)}|\partial_b T_{2,0}|_k \lesssim_k 1_{[1,2B_0]} \frac{1}{y \log y} (1_{[0,1]} \frac{y}{\log y} + \log y).
\]
From these estimates, the sharp pointwise estimates for \( T_{2,0} \) follow.

Finally, we bound \( U_{3,0} \). We start from estimating \( g_3,0 \). By the nonsharp bounds
\[
|QT_{2,0}|_k \lesssim_k \frac{y^3}{y + \log y}, \quad |Q\partial_b T_{2,0}|_k \lesssim_k 1_{[0,1]} \frac{1}{y \log y} y^3 + 1_{[1,\infty)} \frac{1}{y \log y} y^3,
\]
we obtain
\[
|g_3,0 - A_1 T_{2,0}|_k \lesssim_k 1_{[0,1]} y^6 + 1_{[1,\infty)} y^6 + 1_{[1,\infty)} \frac{1}{y \log y},
\]
\[
|\partial_b (g_3,0 - A_1 T_{2,0})|_k \lesssim_k 1_{[0,1]} \frac{1}{y \log y} y^6 + 1_{[1,\infty)} \frac{1}{y \log y} y^6.
\]
Hence, using the sharp \( T_{2,0} \)-estimates for \( A_1 T_{2,0} \), it follows that
\[
|g_3,0|_k \lesssim_k 1_{[0,1]} y^3 + 1_{[1,2B_0]} y^3 (\frac{y}{\log y} + 1_{[2B_0,\infty]} \frac{1}{y \log y} y) + 1_{[2B_0,\infty]} y^3 (\frac{y}{\log y} + 1_{[1,\infty)} \frac{1}{y \log y}) + 1_{[1,\infty)} \frac{1}{y \log y} (y^3 + \log y),
\]
\[
|\partial_b g_3,0|_k \lesssim_k 1_{[0,1]} \frac{1}{y \log y} y^3 + 1_{[1,2B_0]} \frac{1}{y \log y} y^3 \frac{y}{\log y} + 1_{[2B_0,\infty]} \frac{1}{y \log y} y^3 (\frac{y}{\log y} + 1_{[1,\infty)} \frac{1}{y \log y})
\]
Substituting these bounds to
\[
U_{3,0} = -(A_Q \Gamma) \int_0^y g_3,0 y dy', \quad \partial_b U_{3,0} = -(A_Q \Gamma) \int_0^y \partial_b g_3,0 J y dy',
\]
and using $|A_Q|_k \lesssim 1_{(0,1)} \frac{1}{\log y} + 1_{[1,\infty)}$, we have

$$
|U_{3,0}|_k \lesssim 1_{(0,1)} y^2 + 1_{[1,2B_0]} \frac{1}{\log b} y^2 (\log(b^{\frac{1}{2}})) \\
+ 1_{[2B_0,\infty)} \left( \frac{1}{\log b} (\log(b^{\frac{1}{2}})) + y \right)
$$

$$
|\partial_b U_{3,0}|_k \lesssim 1_{(0,1)} b \frac{1}{\log b} y^4 + 1_{[1,2B_0]} \frac{1}{\log b} y^2 (\log(b^{\frac{1}{2}})) \\
+ 1_{[B_0,\infty)} \frac{1}{b \log b} (\log(b^{-\frac{1}{2}})) + y).
$$

Thus the $U_{3,0}$ estimate follows.

We finally note that the smoothness (analyticity) of the profiles at the origin follow from the explicit formula of the involved functions. This completes the proof.

4.3. Modified profiles. We are now ready to define the modified profiles $P$, $P_1$, and $P_2$ by adding suitable truncations. Then we will show that $P$, $P_1$, and $P_2$ solve the evolution equations (4.3), (4.4), (4.5) under the formal parameter evolution laws (4.10), and the compatibility conditions $P_1 \approx D_P P$ and $P_2 \approx A_P P_1$ up to admissible errors.

Recall the unlocalized modified profiles

$$
\hat{P} = Q - ib^2 y^2 Q - \eta \rho,
$$

$$
\hat{P}_1 = -(ib + \eta)b^2 y^2 Q + b^2 T_{2,0},
$$

$$
\hat{P}_2 = (b^2 - 2ib\eta - \eta^2)U_2 + ib^3 U_{3,0}.
$$

We define the localized modified profiles with $B_0 = b^{-\frac{1}{2}}$ and $B_1 = b^{-\frac{1}{2}} \log b$ by

$$
P := Q + \chi_{B_1} \{ -ib^2 y^2 Q - \eta \rho \}
$$

$$
P_1 := \chi_{B_1} \{ -(ib + \eta)b^2 y^2 Q \} + \chi_{B_0} \{ b^2 T_{2,0} \},
$$

$$
P_2 := \chi_{B_0} \{ (b^2 - 2ib\eta - \eta^2)U_2 + ib^3 U_{3,0} \}.
$$

We truncated linear terms at $B_1$, but higher order terms at $B_0$. It is crucial to take $B_1 \gg B_0$; see Remark 4.8 below for the motivation. To incorporate the logarithmic corrections to the modulation equations, we introduce

$$
\text{Mod} := \left( \frac{\lambda}{\Lambda} + b, \gamma, -\eta, b_s + b^2 + \eta^2, \eta_s \right)^t,
$$

$$
\overline{\text{Mod}} := \left( \frac{\lambda}{\Lambda} + b, \gamma, -\eta, b_s + b^2 + \eta^2 + c_0 (b^2 - \eta^2), \eta_s + 2c_0 \eta \right)^t,
$$

$$
v_k := (\Lambda^{-1} P_k, -iP_k, -\partial_b P_k, -\partial_y P_k)^t, \quad \forall k \in \{ 0, 1, 2 \},
$$

where we write $f = f_0$ in short.

**Proposition 4.6** (Modified profile). Assume

$$
|\eta| \leq \frac{b}{\log b} \quad \text{and} \quad 0 < b < b^*.
$$

(1) (Estimates for modulation vectors) For $v = v_0$, we have

$$
1_{(0,2M)}(|\Lambda P - \Lambda Q| + |iP - iQ|) \lesssim b,
$$

$$
1_{(0,2M)}(|\partial_b P + i\frac{\eta}{2} Q| + |\partial_y P + \rho|) = 0.
$$

For $v_1$, we have $b$-degeneracy for scalings/phase: (recall the $X$-norm [37,29])

$$
||\Lambda^{-1} P_1||_{X} + ||i P_1||_{X} \lesssim b,
$$

$$
||\partial_b P_1 + \chi_{B_1} i\frac{\eta}{2} Q||_{X} + ||\partial_y P_1 + \chi_{B_1} \frac{\eta}{2} Q||_{X} \lesssim b \log b C.
$$
For $v_2$, we have full degeneracy
\begin{equation}
\|\Delta_{-2}P_2\|_{H^1_T} + \|iP_2\|_{H^1_T} \lesssim b^2,
\end{equation}
\begin{equation}
\|\partial_b P_2\|_{H^1_T} + \|\partial_{\eta} P_2\|_{H^1_T} \lesssim b.
\end{equation}

(2) Compatibility relations of $P, P_1, P_2$ We have
\begin{align}
\|D_b P - P_1\|_{L^2} &\lesssim b, \\
\|D_{\eta} P - P_1\|_{H^1} &\lesssim b^2, \\
\|A_{\eta} P - P_2\|_{H^1} &\lesssim \frac{b^3}{|\log b|}.
\end{align}

(3) Equation for $P$ We can write
\begin{equation}
(\partial_s - \frac{\lambda}{\chi} \Delta + \gamma_s i) P + i L'_{\eta} P_1 = -\text{Mod} \cdot v + i\Psi
\end{equation}
such that
\begin{equation}
1_{(0,2M)} \|\Psi\| \lesssim b^2 |\log b|.
\end{equation}

(4) Equation for $P_1$ We can write
\begin{equation}
(\partial_s - \frac{\lambda}{\chi} \Delta_{-1} + \gamma_{s} i) P_1 + i A_{P} P_2 - \left(\int_{0}^{y} \text{Re}(\overline{P} P_1) dy'\right) i P_1 = -\text{Mod} \cdot v_1 + i\Psi_1
\end{equation}
such that we have
\begin{equation}
\|\Psi_1\|_{X} \lesssim b^3 |\log b|^C.
\end{equation}

(5) Equation for $P_2$ We can write
\begin{equation}
(\partial_s - \frac{\lambda}{\chi} \Delta_{-2} + \gamma_{s} i) P_2 + i A_{P} A_{P} P_2 - \left(\int_{0}^{y} \text{Re}(\overline{P} P_1) dy'\right) i P_2 - i\overline{P} P_1^2
= -\text{Mod} \cdot v_2 + i\Psi_2
\end{equation}
such that we have a sharp $H^1_T$-estimate
\begin{equation}
\|\Psi_2\|_{H^1_T} \lesssim \frac{b^3}{|\log b|}.
\end{equation}

Remark 4.7. We make the general remark that, in order to close the energy estimate in the main bootstrap argument in the following section, the second line of (4.21) needs to be sharp even up to the power of $|\log b|$, and (4.30) seems to have a very little room, namely, only a small power of $|\log b|$\footnote{In the energy estimate, the size of (4.30) limits the size of bootstrap assumption on $\|\epsilon_3\|_{L^2}$, which is a $H^1_T$-like quantity of $\epsilon$. In view of the gain $\|\epsilon_3\|_{L^2}$ in the refined modulation estimate (Lemma 5.6) and that the size of $\|\epsilon_3\|_{L^2}$ should not disturb this modulation estimate, it seems that we have a room of a small power of $|\log b|$ for (4.30).} For the remaining error estimates at the same level, we have more room; for instance (4.21) only needs to be of size $o(b^3)$ as $b \to 0$ in order to compensate for an implicit constant depending on $M$ arising from Lemma 5.6.

Remark 4.8. We note that the larger localization scale $B_1 = b^{-\frac{1}{2}}|\log b|$ for the first-order profiles is needed for the localization errors in (4.23) and (4.24), actually, in view of Remark 4.7, truncating at $y \sim b^{-\frac{3}{2}}|\log b|^\alpha$ for any $\alpha > 0$ is enough.

All the localization scales should be $b^{-\frac{1}{2}}$ up to some logarithmic powers. For example, if one uses a smaller radius $B' = b^{-\alpha}$ for some $0 < \alpha < \frac{1}{2}$, then the profile error $\Psi_2$ arising from applying the cutoff $\chi_{B'}$ to $U_{3,0}$ cannot satisfy (4.30). On the other hand, if one uses a larger scale $B' = b^{-\alpha}$ for some $\alpha > \frac{1}{2}$, then the cutoff error measured in lower Sobolev norms might be harmful; e.g., the second line of
proof. Thus much error is acceptable for the profile errors equation (4.29) will be used in the sharp third energy estimate. These tell us how \( P \) we view the RHS as errors and substitute the pointwise bounds from the energy estimate compared to the general case without self-duality, in which a higher modulation estimates of \( \lambda \). This shows (4.19).

Multiplying \( \partial \) thus the claim for \( \partial P_1 \) in lower order than \( P_2 \).

\[ (3.15) \]

Remark 4.9. As we will see in Section 5, the \( P \)-equation (4.23) will be used in the modulation estimates of \( \lambda \) and \( \gamma \); the \( P_1 \)-equation (4.27) will be used in the modulation estimates of \( b \) and \( \eta \), and also in the Morawetz corrections; the \( P_2 \)-equation (4.29) will be used in the sharp third energy estimate. These tell us how much error is acceptable for the profile errors \( \Psi \), \( \Psi_1 \), and \( \Psi_2 \). It is necessary for \( \Psi \) and \( \Psi_1 \) to be small in order not to disturb the modulation laws (4.10). This says that it is only necessary to have \( \Psi = o(b) \) and \( \Psi_1 = o(b^2) \). This also explains why it suffices to expand \( P \) and \( P_1 \) in lower order than \( P_2 \).

Remark 4.10. The full degeneracy estimate (4.21) for \( \nu_2 \) holds thanks to the fact that \( P_2 \approx A_Q L_Q P \) at the linear level, while \( A_Q L_Q(\log Q) = A_Q L_Q(\rho) = A_Q(\frac{\nu}{2}Q) = 0 \). This cancellation allows for an easier treatment of the term \( \text{Mod} \cdot \nu_2 \) in the energy estimate compared to the general case without self-duality, in which a higher derivative of \( P \) is not expected to possess such a degeneracy [13].

Proof. Step 1: Estimates for the modulation vectors.

We first show (4.19). Note that \( 1_{(0,2M]} \) is insensitive to the localizations \( \chi_{B_0}, \chi_{B_1} \). Thus

\[ 1_{(0,2M]}(\Lambda P - \Lambda Q) = 1_{(0,2M]}(-ib\Lambda(\frac{\nu}{2}Q) - \eta \Lambda \rho), \]

\[ 1_{(0,2M]}(iP - iQ) = 1_{(0,2M]}(b\frac{\nu}{2}Q - i\rho \eta), \]

\[ 1_{(0,2M]}(\partial_b P + i\frac{\nu}{2}Q) = 0, \]

\[ 1_{(0,2M]}(\partial_b P + \rho) = 0. \]

We view the RHS as errors and substitute the pointwise bounds from the \( \rho \)-estimates (4.13). This shows (4.19).

We turn to show (4.20). We will use the rough estimates (4.13): \( |P_1| \lesssim 1_{(0,2M]}(b\frac{1}{\eta} + b^2y) \). In view of the \( X \)-norm (4.20), we multiply by \( \langle y \rangle^{-2} \langle \log + y \rangle \) and take the \( L^2 \)-norm to get the claims for \( \Lambda_{-1} P_1 \) and \( i P_1 \). For \( \partial_b P_1 \), we compute

\[ \partial_b P_1 + i\frac{\nu}{2}Q \chi_{B_1} \]

\[ = \chi_{B_0} (2bT_{2,0} + b^2 \partial_b T_{2,0}) + (\partial_b \chi_{B_0})(b^2 T_{2,0}) + (\partial_b \chi_{B_1})(-(ib + \eta)\frac{\nu}{2}Q). \]

Multiplying \( \langle y \rangle^{-2} \langle \log + y \rangle \) to the RHS and taking \( L^2 \) yield the claim for \( \partial_b P_1 \). For \( \partial_y P_1 \), we in fact have

\[ \partial_y P_1 + \frac{\nu}{2}Q \chi_{B_1} = 0, \]

thus the claim for \( \partial_y P_1 \) follows trivially.

We turn to show (4.21). Due to the positivity of \( A_Q A_Q^* \) (3.23), it suffices to estimate \( ||A_Q^* \nu_2||_{L^2} \). We will need to use the logarithmic gain induced by taking \( A_Q^* \). From the definitions of \( U_2 \) and \( U_{3,0} \), we have

\[ A_Q^* U_2 = g_2, \quad A_Q^* U_{3,0} = g_{3,0}. \]
We also have the scaling identity
\[ A_0^* \Lambda^{-2} P_2 = \Lambda^{-3} A_0^* P_2 + \frac{1}{2} \langle y Q^2 \rangle P_2. \]
Thus the desired claim
\[ \| A_0^* \Lambda^{-2} P_2 \|_{L^2} + \| A_0^* i P_2 \|_{L^2} \lesssim \| A_0^* P_2 \|_{L^2} + \| \langle y \rangle^{-3} P_2 \|_{L^2} \lesssim b^2 \]
follows from
\[
\begin{align*}
|A_0^* P_2|_1 & \lesssim 1_{\{0,2B_0\}}(b^2 |g_2|_1 + b^3 |g_{3,0}|_1) + 1_{\{B_0,2B_0\}} \frac{1}{b} |\hat{P}_2|_1, \\
|\langle y \rangle^{-3} P_2| & \lesssim 1_{\{0,2B_0\}}(b^3 \langle y \rangle^{-3} + b^3 \langle y \rangle^{-1}),
\end{align*}
\]
and (4.11), (4.17). For \( \partial_b P_2 \), note that
\[
\begin{align*}
\partial_b P_2 &= \chi_{B_0}(2b - 2i \eta) U_2 + 3ib^2 U_{3,0} + (b^2 - 2ib \eta - \eta^2) \partial_b U_2 + ib \partial_b U_{3,0} \\
&\quad + \left( \partial_b \chi_{B_0} \right) \hat{P}_2.
\end{align*}
\]
For the first line, we take \( A_0^* \), measure the \( L^2 \)-norm and proceed as before, where we also use (4.15) and (4.18) for \( A_0^* \partial_b U_2 = \partial_b g_2 \) and \( A_0^* \partial_b U_{3,0} = \partial_b g_{3,0} \), respectively. For the second line, we have \( \| (\partial_b \chi_{B_0}) \hat{P}_2 \|_{-1} \lesssim \frac{1}{|\log \eta|} \) by Lemma 4.3. For \( \partial_b P_2 \), note that
\[ \partial_b P_2 = \chi_{B_0}^{-1}(2b - 2i \eta) U_2. \]
Again, we take \( A_0^* \), measure the \( L^2 \)-norm and proceed as before.

**Step 2:** The relations between \( P, P_1, P_2 \).

We first show (4.19) and (4.23). From the linearization of the Bogomol’nyi operator, we have
\[
D_P P = L_Q(P - Q) - \frac{1}{y} A_0 [P - Q] Q - \frac{1}{y} (A_0 [P] - A_0 [Q]) (P - Q)
= \chi_{B_1} \{ -(ib + \eta) \frac{\partial^2}{\partial \eta^2} Q \} + [L_Q, \chi_{B_1}] \{ -(ib + \eta) \frac{\partial^2}{\partial \eta^2} Q - \eta \rho \}
- \frac{1}{y} A_0 \chi_{B_1} \{ -(ib + \eta) \frac{\partial^2}{\partial \eta^2} Q - \eta \rho \} [Q - \frac{1}{y} (A_0 [P] - A_0 [Q]) (P - Q)].
\]
By the definition of \( P_1 \), we see that the first term \( \chi_{B_1} \{ -(ib + \eta) \frac{\partial^2}{\partial \eta^2} Q \} \) cancels:
\[
D_P P - P_1 = [L_Q, \chi_{B_1}] \{ -(ib + \eta) \frac{\partial^2}{\partial \eta^2} Q - \eta \rho \}
- \frac{1}{y} A_0 \chi_{B_1} \{ -(ib + \eta) \frac{\partial^2}{\partial \eta^2} Q - \eta \rho \} [Q - \frac{1}{y} (A_0 [P] - A_0 [Q]) (P - Q)]
- \chi_{B_0} (b^2 T_{2,0}).
\]
It suffices to measure the \( L^2 \)-difference and \( \mathcal{H}_2 \)-difference of the RHS.

We now estimate each line on the RHS of (4.31). For the first line, notice that
\[
[L_Q, \chi_{B_1}] f = (\partial_b \chi_{B_1}) f + \frac{Q}{y} \left( \int_0^y \chi_{B_1} \Re f Q y' dy' - \chi_{B_1} \int_0^y \Re f Q y' dy' \right).
\]
In particular, \( [L_Q, \chi_{B_1}] f \) only uses the information of \( f \) on \( [B_1, 2B_1] \) with the estimates
\[
\|[L_Q, \chi_{B_1}] f \|_2 \lesssim 1_{\{B_1,2B_1\}} \frac{1}{b} |f|_2 + 1_{\{B_1,1\}} \frac{1}{b^2} \int_{2B_1}^B |f| \left( \frac{y'}{y} \right)^2 \| \chi_{B_1} \|_2.
\]
Substituting \( f = -ib \frac{\partial^2}{\partial \eta^2} Q - \eta \rho \), Lemma 4.3 implies that
\[
\|[L_Q, \chi_{B_1}] \{ -(ib + \eta) \frac{\partial^2}{\partial \eta^2} Q - \eta \rho \} \|_{L^2} \lesssim b,
\]
\[
\|[L_Q, \chi_{B_1}] \{ -(ib + \eta) \frac{\partial^2}{\partial \eta^2} Q - \eta \rho \} \|_{-2} \lesssim \frac{\frac{\varepsilon^2}{|\log \eta|}}{b}.
\]
We remark that while the contribution of the second term in (4.32) is nonlocal, thanks to the fast decay \( \frac{1}{y} \), its contribution is better by \( b |\log |b| \) compared to the first term.
For the second line of (4.31), using the bound
\[ |A_0[\chi_{B_1}(-ib\frac{2}{3}Q - \eta \rho)]|_2 \lesssim b^2 \min\{y^2, B_1^2\} \]
we have
\[ \|\frac{1}{b}A_0[\chi_{B_1}(-ib\frac{2}{3}Q - \eta \rho)]|Q|_{L^2} \lesssim b^{-2}, \]
\[ \|\frac{1}{b}A_0[\chi_{B_1}(-ib\frac{2}{3}Q - \eta \rho)]|Q|_{L^2} \lesssim b^2. \]

For the third line of (4.31), we note the bound
\[ 1_{(0,2B_1)}|\frac{1}{b}A_0[P] - A_0[Q]|_2 \lesssim \frac{b}{|\log b|} \left( \frac{\log y}{y(\log y)} \right) + b^2 y, \]
which follows from
\[ A_0[P] - A_0[Q] = -\int_0^y \text{Re}(P - Q)Qy'dy' - \frac{1}{2} \int_0^y |P - Q|^2 y'dy', \]
and the easy bounds
\[ |\text{Re}(P - Q)|_2 \lesssim 1_{(0,2B_1)} y \leq 1_{(0,2B_1)} b, \]
\[ |\text{Im}(P - Q)|_2 \lesssim 1_{(0,2B_1)} b. \]
Thus
\[ \|\frac{1}{b}(A_0[P] - A_0[Q])(P - Q)|L^2 \lesssim b^2, \]
\[ \|\frac{1}{b}(A_0[P] - A_0[Q])(P - Q)|_{L^2} \lesssim \frac{b^2}{|\log b|}. \]

For the last line of (4.31), the sharp estimates show
\[ \|\chi_{B_1} b^2 T_{2,0}2,0\|_{L^2} \lesssim \frac{b}{|\log b|}. \]
For the \( \overline{\mathcal{H}}_2 \) estimate, crudely estimating the \( \|\cdot\|_{L^2} \)-norm will give only \( b^2 \log b \frac{1}{y} \), so we will elaborate a little bit more. In view of the subcoercivity estimates (4.2) and (A.5), it suffices to estimate
\[ \|A_Q^* A_Q(\chi_{B_1} b^2 T_{2,0})\|_{L^2}. \]
Since \( A_Q^* A_Q T_{2,0} = g_2 \), after commuting \( A_Q^* A_Q \) with \( \chi_{B_0} \), this can be estimated by
(\text{using Lemma 1.1 and (4.14)})
\[ \|A_Q^* A_Q(\chi_{B_1} b^2 T_{2,0})\|_{L^2} \lesssim \|1_{(0,2B_0)} b^2 g_2\|_{L^2} + \|1_{(B_0,2B_0)} b^2 |T_{2,0}| - 1\|_{L^2} \lesssim b^2. \]
Thus (4.22) and (4.23) are proved.

We turn to show (4.24). Using \( A_Q(yQ) = 0 \) and \( A_Q T_{2,0} = U_2 \), we obtain \( A_P P_1 = A_Q P_1 + (A_P - A_Q) P_1 \)
\[ = \chi_{B_0} b^2 U_2 + (\partial_y \chi_{B_1}) (b^2 T_{2,0}) + (\partial_y \chi_{B_1}) \{( -ib - \eta ) \frac{2}{3} Q\} + (A_P - A_Q) P_1. \]
Therefore, we have
(4.35)
\[ A_P P_1 - P_2 = \chi_{B_0} \{(2ib\eta + \eta^2) U_2 - ib^3 U_{3,0}\}
+ (\partial_y \chi_{B_1}) \{b^2 T_{2,0}\} + (\partial_y \chi_{B_1}) \{( -ib - \eta ) \frac{2}{3} Q\} + (A_P - A_Q) P_1. \]
It remains to estimate the RHS of (4.35) in the \( \overline{\mathcal{H}}_2 \)-norm. For the first term, we use (4.28) and \( |\eta| \leq \frac{b}{|\log b|} \) to estimate
\[ \|A_Q^* (\chi_{B_0} \{(2ib\eta + \eta^2) U_2 - ib^3 U_{3,0}\})\|_{L^2} \lesssim \|\chi_{B_0} \frac{b^2}{|\log b|} g_2 + b^3 |g_{3,0}|\|_{L^2} + \|\frac{b^2}{y (\log y)} |U_2| + b^3 |U_{3,0}|\|_{L^2} \lesssim \frac{b^2}{|\log b|}. \]

\footnote{One needs to add the contribution in a compact region \( y \sim 1 \), strictly speaking. But this contribution can be easily estimated by \( b^2 \) because we do not need to worry about growing tails of the profiles.}
where in the last inequality we used (4.11) and (4.17). Using the sharp estimates in Lemma 4.3 and \(|\eta| \leq \frac{b}{\log b}\), we have
\[
\|[(\partial_y \chi_{B_0})] b^2 T_{2, 0}]|_{-1}\|_{L^2} \lesssim \frac{b^2}{\log b},
\]
\[
\|[(\partial_y \chi_{B_1})] \{(-ib - \eta) \frac{b}{Q}\}]|_{-1}\|_{L^2} \lesssim \frac{b^2}{\log b},
\]
where we used the logarithmic improvement \(\frac{1}{\log b}\) of \(T_{2, 0}\) in the region \([B_0, 2B_0]\) and \(B_1 = B_0 \log b\). Finally, we use (4.34) and \(|P_1| \approx 1\) to estimate
\[
\||A_P - A_Q|P_1|_{-1}\|_{L^2} \lesssim \|1_{(0, 2B_1]}(\frac{b}{\log b} \log \frac{y}{b}) + b^2 y) \cdot b \frac{1}{\log b}\|_{L^2} \lesssim \frac{b^2}{\log b}.
\]
This completes the proof of (4.24).

Step 3: Equation for \(P\).
Here, as our aim is to measure \(\Psi\) in the region \([0, 2M]\), in many cases (only except the \(L_\gamma\)-part) the error computations are simple and profile localization has no effect.

Firstly, we note the computations
\[
1_{(0, 2M]}(b_s + b^2 + \eta^2) \partial_b P + \eta_s \partial_y P + O(b^2)),
\]
(4.36) \(1_{(0, 2M]}(\frac{\lambda}{\chi} \Lambda P) = 1_{(0, 2M]}(b \Lambda Q - \left(\frac{\lambda}{\chi} + b\right) \Lambda Q + O(b^2))\),
\(1_{(0, 2M]}(\gamma_s iP = 1_{(0, 2M]}(\gamma_s iP + \gamma_s \eta + O(\frac{b^2}{\log b}))\),
which easily follow from
\[
1_{(0, 2M]}(b_s + b^2) \partial_b P = O(b^2),
\]
\(1_{(0, 2M]}(b \Lambda Q - \Lambda Q) = O(b^2)\),
\(1_{(0, 2M]}(\gamma_s iP - iQ) = O(\frac{b^2}{\log b})\).

Next, we claim that
(4.37) \(1_{(0, 2M]}iL^*_P P_1 = 1_{(0, 2M]}(-b\Lambda Q - \eta \Lambda Q + O(b^2|\log b|)).\)

To see this, let us write
\[
iL^*_P P_1 = iL^*_Q(-ib - \eta)\frac{b}{Q}\]
\[+ iL^*_Q((ib + \eta)(1 - \chi_{B_1})\frac{b}{Q}) + i(L^*_P - L^*_Q)(-ib + \eta)\frac{b}{Q}\chi_{B_1}]
\[+ iL^*_P(\chi_{B_0} b^2 T_{2, 0}).
\]
For the first term, we use \(L^*_Q(-ib + \eta)\frac{b}{Q} = -ib Q\) and \(L^*_Q(\frac{b}{Q}) = Q\) to get
\[iL^*_Q(-ib + \eta)\frac{b}{Q} = -b\Lambda Q - \eta Q.
\]
For the second term, we have
\[
1_{(0, 2M]}iL^*_Q((ib + \eta)(1 - \chi_{B_1})\frac{b}{Q}) \lesssim 1_{(0, 2M]}bQ \int_{B_1}^2 Q^2 y' dy' \lesssim 1_{(0, 2M]}\frac{b^2}{\log b^2}.
\]
For the third term, we note that
\[
1_{(0, 2M]}(L^*_P - L^*_Q)(-ib + \eta)\frac{b}{Q}\chi_{B_1})
\lesssim 1_{(0, 2M]}b\{|A_0[P] - A_0[Q]| + |P - Q| \int_0^{2B_1} Q^2 y' dy' + |P| \int_0^{2B_1} |P - Q| Q y' dy'\}).
\]
Using \(1_{(0, 2M]}|A_0[P] - A_0[Q]| + |P - Q| \lesssim b\), which follow from (4.34) and (4.33), we see that
\[
1_{(0, 2M]}i(L^*_P - L^*_Q)(-ib + \eta)\frac{b}{Q}\chi_{B_1}) \lesssim b^2|\log b|.
\]
For the fourth term, note that
\[L^*_P(\chi_{B_0} f) \lesssim \frac{1}{b} |f|_1 + \frac{1}{b}|A_0[P]| |f| + |P| \int_0^{2B_0} |P - Q||f| dy' + |P| \int_0^{2B_0} Q|f| dy'.
\]
By the rough pointwise bounds (4.13), we have \(1_{(0,2M)} \frac{1}{r} |A_0| + |P| \lesssim 1\). Then using (4.13) again for \(T_{2,0}\), we see that
\[1_{(0,2M)} L^p_r (\chi B_0 b^2 T_{2,0}) \lesssim b^3 |\log b|.
Thus the claim (4.37) is proved.

Summing up the claims (4.36) and (4.37), we have
\[(\partial_s - \frac{\lambda s}{\lambda} \Lambda + \gamma_s i)P + iL^* P_1 = -\text{Mod} \cdot v + i\Psi
\]
with
\[1_{(0,2M)}|\Psi| \lesssim b^3 |\log b|.

**Step 4:** Equation for \(P_1\) and refined modulation equations.

Although we motivated the profile \(U_{3,0}\) using the \(w_1\)-equation by solving up to \(O(b^3)\) correctly, here it is not necessary to keep track of \(O(b^3)\) because the asserted claim (4.28) only requires \(O_X(b^3 |\log b|^C)\). Thus we will only keep up to quadratic terms. However, in Step 5, we need to keep track of the \(O(b^3)\)-terms in order to get the sharp estimate (4.30).

Firstly, we claim that
\[
\begin{align*}
\partial_s P_1 &= \chi B_0 \{ (ib^2 + i\eta^2 + ic_0(b^2 - \eta^2) + 2c_0 \eta) \frac{\chi}{b} \} \\
&+ (b_s + b^2 + \eta^2 + c_0(b^2 - \eta^2)) \partial_b P_1 + (\eta_s + 2c_0 \eta) \partial_b P_1 \\
&+ O_X(b^3 |\log b|^C).
\end{align*}
\]
This would follow from
\[
\begin{align*}
\partial_b P_1 + \chi B_0 (i \frac{\chi}{b} Q) &= O_X(b |\log b|^C), \\
\partial_b P_1 + \chi B_0 \frac{\chi}{b} Q &= O_X(b |\log b|^C).
\end{align*}
\]
These follow from (4.20) and
\[
|| (\chi B_1 - \chi B_0) y Q \|_X \lesssim ||1_{[B_0,2B_1]} \frac{1}{|b^2|^6} (|\log y|) \|_{L^2} \lesssim b |\log b|^C.
\]
Next, we claim that
\[
\begin{align*}
\frac{\lambda s}{\lambda} \Lambda^{-1} P_1 &= \chi B_0 \{ (-ib^2 - b\eta) \Lambda^{-1} (\frac{\chi}{b} Q) \} - \left( \frac{\lambda s}{\lambda} + b \right) \Lambda^{-1} P_1 \\
&+ O_X(b^3 |\log b|^C).
\end{align*}
\]
This would follow from
\[
\begin{align*}
\Lambda^{-1} P_1 + \chi B_0 \{ (ib + \eta) \Lambda^{-1} (\frac{\chi}{b} Q) \} &= O_X(b^2 |\log b|^C),
\end{align*}
\]
in turn follows from applying the rough estimates (4.13) to
\[
\begin{align*}
\Lambda^{-1} P_1 + \chi B_0 \{ (ib + \eta) \Lambda^{-1} (\frac{\chi}{b} Q) \} \\
&= \Lambda^{-1} \{ \chi B_0 b^2 T_{2,0} \} - (\chi B_1 - \chi B_0) (ib + \eta) \Lambda^{-1} (\frac{\chi}{b} Q) - (y \partial_y \chi B_1)((ib + \eta) \frac{\chi}{b} Q).
\end{align*}
\]
Next, we claim that
\[
\begin{align*}
\Lambda^{-1} P_1 &= \chi B_0 \{ (-ib + i\eta^2) \frac{\chi}{b} Q \} + (\gamma_s + \eta) i P_1 + O_X(b^3 |\log b|^C).
\end{align*}
\]
This would follow from
\[
\begin{align*}
P_1 + \chi B_0 (ib + \eta) \frac{\chi}{b} Q &= O_X(b^2 |\log b|^C),
\end{align*}
\]
which follows from applying the rough estimates to:
\[
\begin{align*}
P_1 + \chi B_0 (ib + \eta) \frac{\chi}{b} Q &= \chi B_0 \{ b^2 T_{2,0} \} - (\chi B_1 - \chi B_0) (ib + \eta) \frac{\chi}{b} Q.
\end{align*}
\]
Next, we claim that
\[
\begin{align*}
\Lambda^{-1} P_1 &= \chi B_0 (b\eta - i\eta^2) (2 - \Lambda) \frac{\chi}{b} Q + O_X(b^3 |\log b|^C).
\end{align*}
\]
To show this, we begin with the bounds
\[
\mathcal{P} = Q + \chi_{B_\varepsilon}\{ib\frac{\partial}{\partial y}Q - \eta\rho\},
\]
\[
P_1 = \chi_{B_\varepsilon}\{-ib + \eta\frac{\partial}{\partial y}Q\} + 1_{[0,2B_\varepsilon]}O(b^2y),
\]
which follow from (4.13). It then follows that
\[
\text{Re}(\mathcal{P}P_1) = 1_{[0,2B_\varepsilon]}(-\eta\frac{\partial}{\partial y}Q^2) + 1_{[0,2B_\varepsilon]}O(b^2\frac{1}{9}).
\]
Hence,
\[
1_{[0,2B_\varepsilon]}\int_0^y \text{Re}(\mathcal{P}P_1)dy' = \chi_{B_\varepsilon}\int_0^y (-\eta\frac{\partial}{\partial y}Q^2)dy' + O(1_{[0,B_\varepsilon]}b^2|\log b| + 1_{[B_\varepsilon,2B_\varepsilon]}\frac{1}{|\log b|}).
\]
Thus
\[
-\int_0^y \text{Re}(\mathcal{P}P_1)dy' = \chi_{B_\varepsilon}\{(bn - i\eta^2)(\int_0^y \frac{\partial}{\partial y}Q^2dy')\frac{1}{2}Q\} + O(1_{[0,B_\varepsilon]}\frac{b^2}{|\log b|}y + 1_{[B_\varepsilon,2B_\varepsilon]}\frac{b^2}{|\log b|}).
\]
The last term contributes to the error \(O_X(b^3|\log b|^C)\) as desired. The proof of the claim (4.41) follows from the computation
\[
(\int_0^y \frac{\partial}{\partial y}Q^2dy')\frac{1}{2}Q = (2 - \Lambda)(\frac{1}{2}Q),
\]
where we have used \(A_Q(\frac{1}{2}Q) = 0\).

Finally, we claim that
\[
iA_P^*P_2 = \chi_{B_\varepsilon}\{(ib^2 + 2bn - i\eta^2)g_2\} + O_X(b^3|\log b|^C).
\]
In fact, we will prove a stronger estimate for later use in Step 5:
\[
iA_P^*P_2 = \chi_{B_\varepsilon}\{(ib^2 + 2bn - i\eta^2)g_2 - b^3g_{3,0}\} + 1_{[0,2B_\varepsilon]}O(||p||_2L^2)(\frac{b^3}{|\log b|}).
\]
To see this, we start from
\[
iA_P^*P_2 = \chi_{B_\varepsilon}i(A_Q^*\hat{P}_2) + (\partial_y\chi_{B_\varepsilon})i\hat{P}_2 + i(A_P^* - A_Q^*)P_2.
\]
We keep the first term in the form
\[
\chi_{B_\varepsilon}i(A_Q^*\hat{P}_2) = \chi_{B_\varepsilon}\{(ib^2 + 2b\eta - i\eta^2)A_Q^*U_2 - b^3A_Q^*U_{3,0}\}
\]
\[
= \chi_{B_\varepsilon}\{(ib^2 + 2b\eta - i\eta^2)g_2 - b^3g_{3,0}\}.
\]
For the second term, we use \(\partial_y\chi_{B_\varepsilon}|_2 \lesssim 1_{[0,2B_\varepsilon]}\frac{1}{y}\) and \(1_{[0,2B_\varepsilon]}|\hat{P}_2|_2 \lesssim \frac{b^2}{|\log b|}\) to get
\[
||((\partial_y\chi_{B_\varepsilon})i\hat{P}_2)|| \lesssim \frac{b^3}{|\log b|}.
\]
For the last term, we note that, by (4.33),
\[
|(A_P^* - A_Q^*)f||_2 \lesssim \frac{1}{2}(b\eta|P| - A_Qf)\lesssim (\frac{b}{|\log b|} + b^2y)|f||_2.
\]
Using also \(|P_2|_2 \lesssim 1_{[0,2B_\varepsilon]}b^2\), which follows from (4.13), we have
\[
||(A_P^* - A_Q^*)P_2|| \lesssim \frac{b^3}{|\log b|}.
\]
Summing up the above claims (4.33)–(4.43), we have
\[
(\partial_x - \frac{\lambda}{\chi}A_{-1} + \tilde{\gamma}_s i)P_1 + iA_P^*P_2 - \left(\int_0^y \text{Re}(\mathcal{P}P_1)dy'\right)iP_1
\]
\[
= -\tilde{\text{Mod}} \cdot \nu_1 + \chi_{B_\varepsilon}\{(ib^2 + 2b\eta - i\eta^2)(g_2 - \Lambda(\frac{1}{2}Q) + c_b\frac{1}{2}Q)\} + O_X(b^3|\log b|^C).
\]
By the definition of \(g_2\), the quadratic order terms almost vanish:
\[
\chi_{B_\varepsilon}\{(ib^2 + 2b\eta - i\eta^2)(g_2 - \Lambda(\frac{1}{2}Q) + c_b\frac{1}{2}Q)\}
\]
\[
= \chi_{B_\varepsilon}(1 - \chi_{B_\varepsilon})(ib^2 + 2b\eta - i\eta^2)c_b\frac{1}{2}Q = O_X(b^3|\log b|^C).
\]
Therefore, we can rearrange the above display as

$$
\mathcal{M} \cdot \mathbf{v} + O_\chi(b^3 \lvert \log b \rvert^C) =: \mathcal{M} \cdot \mathbf{v} + i \Psi_1.
$$

This completes the proof of (4.28).

**Step 5:** Equation for $P_2$ and sharp energy estimates.

Firstly, we claim that

$$
\partial_t P_2 = -2b^3 U_2 \chi_{B_0} + (b_s + b^2 + \eta^2 + c_5(b^2 - \eta^2)) \partial_b P_2
$$

(4.45)

$$
+ i(\eta_s + 2c_5b\eta) \partial_b P_2 + O_{H^2_2}(\frac{b^5}{\log b}).
$$

We note that the terms including $c_5$ can be considered as an error, but we include them to match the formula for $\mathcal{M}$. By (4.11) and (5.28), the claim would follow from

$$
\|A_\chi^*(\partial_b P_2 - 2bU_2\chi_{B_0})\|_{L^2} \lesssim \frac{b}{\log b}.
$$

We compute

$$
\partial_t P_2 = 2bU_2\chi_{B_0} = \chi_{B_0}(-2i\eta \tilde{U}_2 + 3ib^2 U_{3,0} + (b^2 - \eta^2 - 2ib\eta)\partial_b \tilde{U}_2 + ib^3 \partial_b U_{3,0}) + (\partial_b \chi_{B_0})\tilde{P}_2.
$$

Taking $A_Q^*$, we have

$$
A_Q^*(\partial_b P_2 - 2bU_2\chi_{B_0})
$$

$$
= \chi_{B_0}(-2i\eta g_2 + 3ib^2 g_{3,0} + (b^2 - \eta^2 - 2ib\eta)\partial_b g_2 + ib^3 \partial_b g_{3,0})
$$

$$
+ (\partial_b \chi_{B_0})(-2i\eta U_2 + 3ib^2 U_{3,0} + (b^2 - \eta^2 - 2ib\eta)\partial_b U_2 + ib^3 \partial_b U_{3,0})
$$

$$
+ A_Q^*((\partial_b \chi_{B_0})\tilde{P}_2).
$$

Using $|\eta| \leq \frac{b}{\log b}$ and the sharp bounds (4.14), (4.15), (4.17) and (4.18), we have

$$
\|\chi_{B_0}(-2i\eta g_2 + 3ib^2 g_{3,0} + (b^2 - \eta^2 - 2ib\eta)\partial_b g_2 + ib^3 \partial_b g_{3,0})\|_{L^2} \lesssim \frac{b}{\log b}.
$$

Next, using the logarithmic gain at $y \sim B_0$ in Lemma 4.3 we also have

$$
\|\chi_{B_0}(-2i\eta U_2 + 3ib^2 U_{3,0} + (b^2 - \eta^2 - 2ib\eta)\partial_b U_2 + ib^3 \partial_b U_{3,0})\|_{L^2} \lesssim \frac{b}{\log b},
$$

$$
\|A_Q^*((\partial_b \chi_{B_0})\tilde{P}_2)\|_{L^2} \lesssim \frac{b}{\log b}.
$$

Next, we claim that

$$
-\frac{\Lambda}{\Lambda} \Lambda_{-2} P_2 = \chi_{B_0}b^3 \Lambda_{-2} U_2 - \left(\frac{\Lambda}{\Lambda} + b\right) \Lambda_{-2} P_2 + O_{H^2_2}(\frac{b^5}{\log b}).
$$

(4.46)

By (6.28), it suffices to show

$$
\|A_Q^*((y\partial_y \chi_{B_0})\tilde{P}_2 + \chi_{B_0} \Lambda_{-2} (\tilde{P}_2 - b^2 U_2))\|_{L^2} \lesssim \frac{b^5}{\log b}.
$$

For this, further using $A_Q^* \Lambda_{-2} = \Lambda_{-2} A_Q^* - \frac{b\partial b}{2}$, it suffices to show

$$
\|1_{(0,2B_0)}((-2ib\eta - \eta^2)g_2 + ib^3 g_{3,0})\|_{L^2} \lesssim \frac{b^5}{\log b},
$$

$$
\|1_{(0,2B_0)}(\frac{1}{2\Lambda})(-2ib\eta - \eta^2)U_2 + ib^3 U_{3,0}\|_{L^2} \lesssim \frac{b^5}{\log b},
$$

$$
\|1_{(B_0,2B_0)}(\frac{1}{2\Lambda}(2b^2 |U_2| + b^3 |U_{3,0}|))\|_{L^2} \lesssim \frac{b^5}{\log b}.
$$

These are now immediate consequences of $|\eta| \leq \frac{b}{\log b}$, the sharp pointwise bounds in Lemma 4.3 as well as (4.14) and (4.17).

Next, we claim that

$$
\tilde{\gamma}_b i P_2 = (\tilde{\gamma}_b + \eta) i P_2 + O_{H^2_2}(\frac{b^5}{\log b}).
$$

(4.47)

This immediately follows from $|\eta| \leq \frac{b}{\log b}$ and (4.21).
Next, we claim that
\[(4.48) \quad \| - (f_0^y \text{Re}(\mathcal{P}P_1) dy') i P_2 \|_{\mathcal{H}_2} \lesssim \frac{b^3}{|\log b|} .\]

By (3.28), it suffices to show
\[\| \text{Re}(\mathcal{P}P_1) P_2 \|_{L^2} + \| (f_0^y \text{Re}(\mathcal{P}P_1) dy') A_Q^* P_2 \|_{L^2} \lesssim \frac{b^3}{|\log b|} .\]

Since $P_2$ is supported in $(0, 2B_0)$, it suffices to estimate on that region. Note that $\text{Re}(\mathcal{P}P_1) = O(|\eta| + b^2)$, at least in $y \lesssim 1$. By (4.44) and the rough bound $|P_2| \lesssim b^2$, the first one $\| \text{Re}(\mathcal{P}P_1) P_2 \|_{L^2} \lesssim \frac{b^3}{|\log b|}$ follows. The second one follows from
\[1_{(0, 2B_0)} |f_0^y \text{Re}(\mathcal{P}P_1) dy'| \lesssim \frac{b}{|\log b|},\]
which is proved using $|\eta| \lesssim \frac{b}{|\log b|}$ and (4.12), as well as
\[A_Q^* P_2 \lesssim 1_{(0, 2B_0)} (b^2 |g_2| + b^3 |g_3, 0|) + 1_{(B_0, 2B_0)} \frac{1}{y} |P_2| \lesssim \frac{b^3}{|\log b|},\]
where we used Lemma 4.3.11 and (4.17).

Next, we claim that
\[(4.49) \quad -i \mathcal{P}(P_1)^2 = \chi_{B_0} \{ (ib^2 + 2b\eta - i\eta^2) (\frac{b^2}{4} Q^3) - b^3 (y Q^2 T_{2, 0} + \frac{1}{16} \tilde{Q}^3) \} + O_{\mathcal{H}_2} (\frac{b^3}{|\log b|}).\]

To see this, it suffices to use the rough estimates (4.13) and $|\eta| \lesssim \frac{b}{|\log b|}$, by which we have
\[\mathcal{P} P_1 = -\chi_{B_1} (ib + \eta) \frac{b^2}{4} Q + b^2 (\chi_{B_0} Q T_{2, 0} + \chi_{B_1} \frac{b^2}{16} Q^3) + O(1_{(0, 2B_1)} \frac{b^3}{|\log b| (y)} + 1_{(0, 2B_0)} b^3).\]

Thus
\[-i \mathcal{P}(P_1)^2 = \chi_{B_1} (ib^2 + 2b\eta - i\eta^2) \frac{b^2}{4} Q^3 - b^3 (\chi_{B_0} y Q^2 T_{2, 0} + \chi_{B_1} \frac{b^2}{16} Q^3) + O(1_{(0, 2B_1)} \frac{b^3}{|\log b| (y)} + 1_{(0, 2B_0)} b^4) = \chi_{B_0} \{ (ib^2 + 2b\eta - i\eta^2) \frac{b^2}{4} Q^3 - b^3 (y Q^2 T_{2, 0} + \frac{1}{16} \tilde{Q}^3) \} + O(1_{(B_0, 2B_1)} \frac{b^3}{y} + 1_{(0, 2B_1)} \frac{b^3}{|\log b| (y)} + 1_{(0, 2B_0)} b^4).\]

Taking the $\| \cdot \|_{L^2}$ norm, the claim follows.

Next, we claim that
\[(4.50) \quad i A_P A_P P_2 = \chi_{B_0} \{ (ib^2 + 2b\eta - i\eta^2) A_Q g_2 - b^3 \chi_{g_3, 0} \} + O_{\mathcal{H}_2} (\frac{b^3}{|\log b|}).\]

Recall from (4.44) that
\[A_P^* P_2 = \chi_{B_0} \{ (ib^2 + 2b\eta - i\eta^2) g_2 - b^3 \chi_{g_3, 0} \} + 1_{(0, 2B_0)} \cdot O_{\| \cdot \|_{L^2}} (\frac{b^3}{|\log b|}).\]

Thus
\[i A_Q A_P^* P_2 = \chi_{B_0} \{ (ib^2 + 2b\eta - i\eta^2) A_Q g_2 - b^3 \chi_{g_3, 0} \} + O_{\mathcal{H}_2} (\frac{b^3}{|\log b|}).\]

On the other hand, using $(A_P - A_Q) f = -\frac{1}{y} (A_0 [P] - A_0 [Q]) f$, (4.33), (4.44), (4.11) and (4.47), we have
\[\| (A_P - A_Q) A_P^* P_2 \|_{\mathcal{H}_2} \lesssim 1_{(0, 2B_0)} \left( \frac{b}{|\log b| (y)} + b^2 y \right) |A_P^* P_2|_{L^2} \lesssim \frac{b^3}{|\log b|} .\]

Thus the claim is shown.
Summing up the above claims (4.35) – (4.50) yield
\[
(\partial_s - \frac{\lambda}{\chi} \Lambda_{-2} + \tilde{\gamma} s) P_2 + i A_P A_P' P_2 - \left( \int_0^y \text{Re}(PP')dy' \right) i P_2 - i \overline{P'}(P_1)^2
\]
\[
= -\text{Mod} \cdot v_2 + \chi_Bc_i \{(ib^2 + 2b\eta - i\eta^2)(\frac{2}{\chi} Q^3 + A_Qg_2) + b^3(\Lambda U_2 - yQ^2 T_{2,0} - \frac{6}{16} Q^3 - A_Q g_{3,0})\} + O_{\tilde{H}^2} \left( \frac{b^3}{|\log| \eta|} \right).
\]
In fact, the \(b^3\)-order term vanishes, by the definition of \(U_{3,0}\). To see this, we rearrange the \(b^3\)-order term as
\[
\Lambda U_2 - yQ^2 T_{2,0} - \frac{6}{16} Q^3 - A_Q g_{3,0}
\]
\[
= \Lambda(A_Q T_{2,0}) - \frac{1}{4} yQ^2 T_{2,0} - (QT_{2,0} + \frac{1}{8} Q^2)(\frac{2}{\chi} Q) - A_Q g_{3,0}.
\]
Using the scaling identity \(\Lambda A_Q T_{2,0} - \frac{1}{4} yQ^2 T_{2,0} = A_Q A_1 T_{2,0}\) and (4.12), the above display continues as
\[
= A_Q A_1 T_{2,0} - (QT_{2,0} + \frac{1}{8} Q^2)(\frac{2}{\chi} Q) - A_Q g_{3,0} = 0.
\]
Next, by the definition of \(g_2\), the quadratic order term almost vanishes. Indeed, using the scaling identity \(A_\chi A = \Lambda - 1 AQ - \frac{\chi^2}{\chi^2} Q^2\) and \(A_Q(yQ) = 0\), we have
\[
\chi Bc_i \{(ib^2 + 2b\eta - i\eta^2)(\frac{2}{\chi} Q^3 + A_Qg_2)\}
\]
\[
= -\chi Bc_i \{c_\chi (ib^2 + 2b\eta - i\eta^2)(\partial_\eta \chi Bc_i)\frac{2}{\chi} Q\} = O_{\tilde{H}^2} \left( \frac{b^3}{|\log| \eta|} \right).
\]
Therefore,
\[
(\partial_s - \frac{\lambda}{\chi} \Lambda_{-2} + \tilde{\gamma} s) P_2 + i A_P A_P' P_2 - \left( \int_0^y \text{Re}(PP')dy' \right) i P_2 - i \overline{P'}(P_1)^2
\]
\[
= -\text{Mod} \cdot v_2 + O_{\tilde{H}^2} \left( \frac{b^3}{|\log| \eta|} \right) = -\text{Mod} \cdot v_2 + \Psi_2.
\]
The proof of (4.30) is now completed. \(\square\)

5. Trapped solutions

So far, we constructed the modified profiles \(P, P_1, P_2\), and derived the formal modulation equations (4.10). Applying the modulation parameters satisfying (4.10) to the modified profiles give approximate finite-time blow-up solutions to (1.10). In this section, we hope to construct a full nonlinear solution \(u\) to (1.10), whose evolution closely follows that of the approximate solution.

To achieve this, we will decompose our solution \(u\) of the form
\[
u(t, r) = e^{\epsilon(t)} \frac{\lambda(t)}{\chi(t)} [P(\cdot; b(t), \eta(t))] + \epsilon(t, \cdot) \left( \frac{r}{\lambda(t)} \right),
\]
where \(\epsilon(t, y)\) is the error part of \(u\). We will fix the decomposition by imposing certain orthogonality conditions. We then apply a robust energy method with a bootstrap argument to show that \(\epsilon\) is sufficiently small (and goes to 0 at the blow-up time), guaranteeing that the modulation parameters \(\lambda, \gamma, b, \eta\) evolve as in (4.10).

As mentioned earlier, we carry out the analysis on the hierarchy of equations for \(w, w_1, w_2\): (2.11), (2.15), and (2.16). As our modified profiles \(P, P_1, P_2\) are motivated from this hierarchical structure, the decomposition of \(u\) will also be based on this structure. Indeed, we use the decompositions
\[
w = e^{-i\gamma} \lambda u(\lambda) = P(\cdot; b, \eta) + \epsilon,
\]
\[
w_1 = D_b w = P_1(\cdot; b, \eta) + \epsilon_1,
\]
\[
w_2 = A_b w_1 = P_2(\cdot; b, \eta) + \epsilon_2,
\]
and impose four orthogonality conditions to fix the decomposition.
In this hierarchy, \( \epsilon_1 \) or \( \epsilon_2 \) are the same as \( L_Q \epsilon \) or \( A_Q L_Q \epsilon \), respectively, at the leading order. In the previous work [21], the authors used linear adapted derivatives such as \( L_Q \epsilon, A_Q L_Q \epsilon, \) or \( A_Q^3 A_Q L_Q \epsilon \). Such adapted derivatives were used in the earlier works [33],[34],[35]. In this paper, however, we proceed to nonlinear adapted derivatives. Compared to that the linear adapted derivatives are chosen to respect the linear flows, our nonlinear adapted derivatives are chosen to respect the nonlinear flows. It turns out that going up to higher order by nonlinear adapted derivatives is more efficient, in the sense that error terms in the evolution equations are much simpler.

The roles of the equations at different levels are all distinct. The evolution equations of \( \lambda \) and \( \gamma \) are derived at the level of the \( u \)-equation. The \( w_1 \)-equation detects the sharp evolution equations of \( b \) and \( \eta \), from which we observe the logarithmic corrections in the blow-up rate \((5.59)\). Finally, the energy method will be applied to \( \epsilon_2 \), where we observe the repulsivity \((5.8)\), and the full degeneracy of \( P_2 \) \((1.21)\).

5.1. Decompositions of solutions. In this subsection, we explain in detail how we decompose our solutions. We use the decomposition

\[
\begin{align*}
&u(t, \tau) = \frac{e^{v(t)}(t)}{\lambda(t)} \{ P(\epsilon; b(t), \eta(t)) + \epsilon(t, \cdot) \}
\end{align*}
\]

For each time \( t \), there are four degrees of freedom to choose the parameters \( \lambda, \gamma, b, \eta \). We determine them by imposing four orthogonality conditions on \( \epsilon \). What follows is a fixed-time analysis and we omit the time variable \( t \).

We note that in the hierarchy of the variables \( w, w_1, w_2 \), the modulation parameters \( \lambda, \gamma, b, \eta \) and the error parts \( \epsilon, \epsilon_1, \epsilon_2 \) are determined according to the decomposition \((5.2)\):

\[
\begin{align*}
w &:= e^{v(t)} \lambda u(\lambda), \quad w_1 := D_w w, \quad w_2 := A_{w} w_1, \\
\epsilon &:= w - P(\epsilon; b, \eta), \quad \epsilon_1 := w_1 - P_1(\epsilon; b, \eta), \quad \epsilon_2 := w_2 - P_2(\epsilon; b, \eta).
\end{align*}
\]

We will consider two different decompositions, corresponding to two different orthogonality conditions. Perhaps a standard decomposition would require \( \epsilon \) to lie in \( N_0(L_Q)^\perp \). However, due to slow decay of the generalized null elements, we use truncated orthogonality conditions \((5.3)\):

\[
(\epsilon, Z_1)_r = (\epsilon, Z_2)_r = (\epsilon, Z_3)_r = (\epsilon, Z_4)_r = 0,
\]

where

\[
\begin{align*}
Z_1 &:= y^2 Q \chi_M - 2(\rho, y^2 Q \chi_M)_r L_Q^* (y Q \chi_M), \\
Z_2 &:= i \rho \chi_M - \frac{(y^2 Q, \rho \chi_M)_r}{2(y Q, y Q \chi_M)_r} L_Q^* (i y Q \chi_M), \\
Z_3 &:= L_Q^* (i y Q \chi_M), \\
Z_4 &:= L_Q^* (y Q \chi_M).
\end{align*}
\]

Another way of putting this is to say \( \epsilon \in \mathcal{Z}^\perp \), where \( \mathcal{Z}^\perp \) is a codimension four linear subspace of \( H_0^3 \) defined by

\[
\mathcal{Z}^\perp := \{ \epsilon \in H_0^3 : (\epsilon, Z_1)_r = (\epsilon, Z_2)_r = (\epsilon, Z_3)_r = (\epsilon, Z_4)_r = 0 \}.
\]

We call this decomposition the rough decomposition. We will use it as a preliminary decomposition, for instance when we describe the initial data set and its coordinates. The choices of \((5.4)\) is motivated from the transversality condition; see \((5.7)\) below.

However, we will use a different decomposition that detects sharper modulation equations for \( b \) and \( \eta \). In view of the hierarchical structure, these are well-detected from the \( \epsilon_1 \)-equation instead of the \( \epsilon \)-equation. One may observe the error for a
more refined modulation equation $\hat{\text{Mod}}$ in the $P_1$-equation (4.27). Thus we replace the third and fourth orthogonality conditions in (5.3) by orthogonality conditions for $\epsilon_1$:

\[(5.5) \quad (\epsilon, Z_1)_r = (\epsilon, Z_2)_r = (\epsilon_1, \tilde{Z}_3)_r = (\epsilon_1, \tilde{Z}_4)_r = 0,\]

where

\[\tilde{Z}_3 := iyQM\chi, \quad \tilde{Z}_4 := yQM\chi.\]

In view of $\epsilon_1 \approx LQ\epsilon$ up to the leading order, this is a slight modification of the rough decomposition. We will call this the nonlinear decomposition, as $\epsilon$ does not belong to a fixed codimension four linear subspace. More precisely, after writing (5.5) in terms of $b, \eta, \epsilon$, we see that $\epsilon$ belongs to some codimension four manifold depending on $b$ and $\eta$. The nonlinear decomposition does not in general mean that $\epsilon$ belongs to $Z^\perp$.

**Lemma 5.1** (Estimates of $Z_k$’s). The following estimates hold.

1. (Logarithmic divergence)

\[(5.6) \quad (yQ, yQM)_r = 16\pi \log M + O(1).\]

2. (Pointwise estimates)

\[|Z_1|_1 + |Z_2|_1 \lesssim M^2 Q_1(0,2M),\]
\[|Z_3|_1 + |Z_4|_1 \lesssim Q_1(0,2M),\]
\[|\tilde{Z}_3|_1 + |\tilde{Z}_4|_1 \lesssim yQ_1(0,2M).\]

3. (Transversality) For $k \in \{1, 2, 3, 4\}$, we have

\[(5.7) \quad (\Lambda Q, Z_k)_r = (-yQ, yQ_1)_r + O(1)\delta_{1k},\]
\[(-iQ, Z_k)_r = (-1/4)(yQ, yQ_1)_r + O(1)\delta_{2k},\]
\[(iy^2 Q, Z_k)_r = 1/4(yQ, yQ_1)_r \delta_{3k},\]
\[(\rho, Z_k)_r = 1/2(yQ, yQ_1)_r \delta_{4k}.\]

For $k \in \{3, 4\}$, we have

\[(5.8) \quad (i\frac{y}{4} Q, \tilde{Z}_k)_r = \frac{1}{2}(yQ, yQ_1)_r \delta_{3k},\]
\[(\frac{y^2}{4} Q, \tilde{Z}_k)_r = \frac{1}{2}(yQ, yQ_1)_r \delta_{4k}.\]

**Proof.**

1. This is immediate from the explicit formula (1.14) of $Q$.

2. The pointwise estimates for $Z_1$ and $Z_2$ follow from (5.15) and

\[|L^*_Q(yQ_1)| + |L_Q^*(iyQ_1)| \lesssim Q_1(0,2M),\]
\[|(\rho, y^2 Q_1)_r| \lesssim M^2,\]
\[(yQ, yQ_1)_r \sim \log M.\]

The pointwise estimates for $\tilde{Z}_3$ and $\tilde{Z}_4$ are immediate.

3. Let $k \in \{1, 2\}$. Since $Z_1$ is real, $Z_2$ is imaginary, and $L_Q\Lambda Q = L_QiQ = 0$, we have

\[(\Lambda Q, Z_k)_r = (\Lambda Q, y^2 Q_1)_r \delta_{1k},\]
\[(-iQ, Z_k)_r = -(Q, \rho Q)_r \delta_{2k}.\]
We then compute
\[(\Lambda Q, y^2 Q_{XM})_r = \frac{1}{2}([y^2 Q_{XM}, \Lambda Q, Q])_r = -(y^2 Q_{XM}, Q)_r + O(1),\]
\[(Q, \rho X_M)_r = \frac{1}{2}(y Q, L_Q(\rho X_M))_r = \frac{1}{2}(y Q, y Q X_M)_r + O(1).\]

Next, using \(L_Q \rho = \frac{1}{2} y Q \) and \(L_Q iy^2 Q = 2iy Q\), we see that the additional terms in the definition of \(Z_1\) and \(Z_2\) are chosen to satisfy
\[(iy^2 Q, Z_k)_r = (\rho, Z_k)_r = 0.\]

Let \(k \in \{3, 4\}\). Since \(L_Q \Lambda Q = L_Q iQ = 0\), we have
\[(\Lambda Q, Z_k)_r = (-iQ, Z_k)_r = 0.\]

Since \(Z_3\) is imaginary, \(Z_4\) is real, \(L_Q \rho = \frac{1}{2} y Q\), and \(L_Q iy^2 Q = 2i y Q\), we have
\[
(i \frac{\delta \rho}{\delta} Q, Z_k)_r = \frac{1}{2}(y Q, y Q X_M)_r \delta_{3k},
\]
\[(\rho, Z_k)_r = \frac{1}{2}(y Q, y Q X_M)_r \delta_{4k}.\]

Thus (5.7) is proved. Finally, (5.8) for \(\tilde{Z}_3\) and \(\tilde{Z}_4\) are immediate from the fact that \(\tilde{Z}_3\) is imaginary and \(\tilde{Z}_4\) is real. \(\square\)

We will define an open set \(O_{\text{dec}} \subseteq H^3_0\) near the soliton tube, on which both the above decompositions can be made. The set of coordinates \((\lambda, \gamma, b, \eta, \epsilon)\) will be denoted by \(U_{\text{dec}}\). For \(\delta_{\text{dec}} > 0\) to be chosen, we define \(U_{\text{dec}} \subseteq \mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \times \mathbb{Z}^+\) by the set of \((\lambda, \gamma, b, \eta, \epsilon)\) satisfying
\[0 < b < \delta_{\text{dec}}, \quad |\eta| < \frac{2}{\log k}, \quad ||\epsilon||_{H^3_0} < \delta_{\text{dec}}.\]

The set \(O_{\text{dec}}\) is defined by the set of images
\[O_{\text{dec}} := \{ \frac{e^{i \gamma}}{\lambda} |P(\cdot; b, \eta) + \epsilon(\lambda) : (\lambda, \gamma, b, \eta, \epsilon) \in U_{\text{dec}} \} \).

**Lemma 5.2 (Decompositions).** For sufficiently large \(M\), there exist \(\delta_1 > \delta_{\text{dec}} > \delta_{\text{dec}} > 0\) such that the following holds.

1. **(The set \(O_{\text{dec}}\) and rough decomposition)** The set \(O_{\text{dec}}\) is open in \(H^3_0\). Moreover, the map
   \[
   \Phi(\lambda, \gamma, b, \eta, \epsilon)(r) := \frac{e^{i \gamma}}{\lambda} |P(\cdot; b, \eta) + \epsilon(\lambda) ;
   \]
   is a homeomorphism from \(U_{\text{dec}}\) to \(O_{\text{dec}}\). We denote by \(G^{(1)}\) the \((\lambda, \gamma, b, \eta, \epsilon)\)-components of \(\Phi^{-1}\).

2. **(Nonlinear decomposition)** For any \(u \in O_{\text{dec}}\), there exists unique \((G^{(2)}(\epsilon), \epsilon) = (\lambda, \gamma, b, \eta, \epsilon) \in \mathbb{R}_+ \times \mathbb{R}/2\pi \mathbb{Z} \times B_3(0) = B_3(0) \times B_3(0) = 0\) such that
   \[
   (\epsilon, Z_1)_r = (\epsilon, Z_2)_r = (\epsilon, \tilde{Z}_3)_r = (\epsilon, \tilde{Z}_4)_r = 0.
   \]

3. **(C1-regularity)** The map \(u \mapsto (\lambda, \gamma, b, \eta)\) for each decomposition is \(C^1\), i.e. the maps \(G^{(1)}\) and \(G^{(2)}\) are \(C^1\).

4. **(Difference estimate)** For \(u \in O_{\text{dec}}\), we have
   \[
   \text{dist}(G^{(1)}(u), G^{(2)}(u)) \lesssim ||(\epsilon_1, \tilde{Z}_3)_r|| + ||(\epsilon_1, \tilde{Z}_4)_r||,
   \]
   where \(\epsilon_1\) is computed using the rough decomposition and the formula (5.9).

5. **(Initial data set)** We have \(U_{\text{init}} \subseteq U_{\text{dec}}\) and \(O_{\text{init}} \subseteq O_{\text{dec}}\). Moreover, the statements of (1) also hold when we replace \(U_{\text{dec}}\) and \(O_{\text{dec}}\) by \(U_{\text{init}}\) and \(O_{\text{init}}\), respectively.

---

\^18Since we are using two different decompositions, we have two different \((\lambda, \gamma, b, \eta, \epsilon)\) for the same \(u \in O_{\text{dec}}\). We will use the same notation \((\lambda, \gamma, b, \eta, \epsilon)\) when no confusion will arise.
Proof. The proof is an extension of [21] Lemma 4.2. We include the full proof for the reader’s convenience.

Let us introduce some notation to be used in this proof. For \( \lambda \in \mathbb{R}_+ \) and \( \gamma \in \mathbb{R}/2\pi \mathbb{Z} \), let us denote

\[
f_{\lambda, \gamma}(y) := \frac{e^{\gamma y}}{\lambda} f\left(\frac{y}{\lambda}\right), \quad X_{\lambda, \gamma} := \{f_{\lambda, \gamma} : f \in X\}.
\]

We equip \( \mathbb{R}_+ \) with the metric \( \text{dist}(\lambda_1, \lambda_2) = |\log(\lambda_1/\lambda_2)| \), and equip \( \mathbb{R}/2\pi \mathbb{Z} \) with the induced metric from \( \mathbb{R} \). We will choose small parameters \( \delta_1, \delta'_1, \delta_2, \delta_{dec} > 0 \) on the way, with the dependence \( 0 < \delta^* \ll \delta_{dec} \ll \delta_2 \ll \delta'_1 \ll \delta_1 \ll M^{-1} \).

**Step 1: Extension of the profiles \( P \) and \( P_1 \).

Notice that in Section 3 the profiles \( P \) and \( P_1 \) are considered only for \( (b, \eta) \) with \( |\eta| < b \) (specifically \( |\eta| \leq \frac{b}{\log b} \) with \( b > 0 \) small), not for all \( |(b, \eta)| \ll 1 \). As we want to apply the implicit function theorem at \( Q = P(0; 0, 0) \), we will consider artificial extensions \( \tilde{P}(y; b, \eta) \) and \( \tilde{P}_1(y; b, \eta) \) of \( P(y; b, \eta) \) and \( P_1(y; b, \eta) \) defined for all \( (b, \eta) \) in a neighborhood of \( (0, 0) \), respectively.

Firstly, we extend \( P(y; b, \eta) \) and \( P_1(y; b, \eta) \) for \( |\eta| \leq \frac{2b}{\log b} \) and \( |b| < \delta_1 \). If \( b = 0 \) (hence \( \eta = 0 \)), then we set \( P(0; 0) = Q \) and \( P_1(0; 0) = 0 \). If \( b \neq 0 \), then define \( P \) and \( P_1 \) via the formulae (4.25) and (4.27) with \( B_0 = |b|^{-\frac{1}{2}} \), \( B_1 = |b|^{-\frac{1}{2}} \log |b|^{-1} \), \( \delta_{2, 0}(y; b, \eta) := \delta_{2, 0}(y; |b|, \eta) \), and similarly for \( \tilde{T}_2, T_{3, 0} \). We remark that the estimates (4.19) and (4.20) are still valid for \( |b| < \delta_1 \). In particular \( \partial_b P = -i \frac{\bar{b}}{Q} P \) and \( \partial_b P_1 = -i \frac{\bar{b}}{Q} P \) when \( (b, \eta) = (0, 0) \).

Next, in order to define the extensions \( \tilde{P} \) and \( \tilde{P}_1 \) for all \( |(b, \eta)| \ll 1 \), we will introduce a suitable cutoff function for \( \eta \). Choose a smooth function \( \psi : \mathbb{R} \to \mathbb{R} \) such that \( \psi(\tilde{\eta}) = \tilde{\eta} \) for \( \tilde{\eta} \leq 2 \) and \( \sup |\psi| \lesssim 1 \). For \( |b| < \delta_1 \), we define \( \psi_b(\tilde{\eta}) = \frac{1}{\log |b|} |\tilde{\eta}|^{1/2} \psi\left(\frac{1}{\log |b|} |\tilde{\eta}|\right) \) if \( b \neq 0 \) and \( \psi_0(\tilde{\eta}) = 0 \). Thus \( \partial_b \psi_b(\tilde{\eta}) = -\text{sgn}(b)\left(\frac{1}{\log |b|} + \frac{1}{1/\log |b|}\right) \frac{1}{2} A_2 \tilde{\eta} \right) \) if \( b \neq 0 \) and \( \partial_b \psi_b(\tilde{\eta}) = 0 \). In particular, \( \|\partial_b \psi_b\|_{L^\infty} \lesssim \frac{1}{\log |b|} \).

Finally, we define

\[
\tilde{P}(\cdot; b, \eta) := P(\cdot; b, \psi_b(\eta)) - (\eta - \psi_b(\eta))(\psi_2 Q)e^{2\lambda M},
\]

\[
\tilde{P}_1(\cdot; b, \eta) := P_1(\cdot; b, \psi_b(\eta)) - (\eta - \psi_b(\eta))(\psi_2 Q)e^{2\lambda M},
\]

for \( |\eta|, |b| < \delta_1 \). By the definition, \( \tilde{P}(\cdot; b, \eta) = P(\cdot; b, \eta) \) for \( |(b, \eta)| \leq \frac{2b}{\log b} \).

**Step 2: Setting for the implicit function theorem.

The main part of the proof is to use the implicit function theorem. Define the maps

\[
\mathbf{F}^{(1)}, \mathbf{F}^{(2)} : \mathbb{R}_+ \times \mathbb{R}/2\pi \mathbb{Z} \times B_{\delta_1}(0) \times B_{\delta_1}(0) \times L^2 \to \mathbb{R}^4
\]

with variables \( \lambda, \gamma, b, \eta, u \) and components \( \mathbf{F}_1^{(j)}, \mathbf{F}_2^{(j)}, \mathbf{F}_3^{(j)}, \mathbf{F}_4^{(j)} \), by

\[
\mathbf{F}_1^{(1)}(\lambda, Z_1) = (\epsilon, Z_1), \quad \mathbf{F}_2^{(1)}(\lambda, Z_2) = (\epsilon, Z_2), \quad \mathbf{F}_3^{(1)}(\lambda, Z_3) = (\epsilon, Z_3), \quad \mathbf{F}_4^{(1)}(\lambda, Z_4) = (\epsilon, Z_4),
\]

\[
\mathbf{F}_1^{(2)}(\lambda, Z_1) = (\epsilon, Z_1), \quad \mathbf{F}_2^{(2)}(\lambda, Z_2) = (\epsilon, Z_2), \quad \mathbf{F}_3^{(2)}(\lambda, Z_3) = (\epsilon_1, Z_3), \quad \mathbf{F}_4^{(2)}(\lambda, Z_4) = (\epsilon_1, Z_4),
\]

where \( w := e^{-\gamma} \lambda u(\lambda), \quad w_1 := D_w w, \quad \epsilon := w - \tilde{P}(\cdot; b, \eta), \quad \epsilon_1 := w_1 - \tilde{P}_1(\cdot; b, \eta). \)

Here, \( \mathbf{F}^{(1)} \) and \( \mathbf{F}^{(2)} \) correspond to the rough and nonlinear decomposition, respectively.

We first consider \( \mathbf{F}^{(1)} \). In order to use the implicit function theorem, we will check that \( \mathbf{F}^{(1)} \) is \( C^1 \) and \( \partial_{\lambda, \gamma, b, \eta, u} \mathbf{F}^{(1)} \) is invertible at \( (\lambda, \gamma, b, \eta, u) = (1, 0, 0, 0, Q) \).

\[\text{To be more precise, one may consider the senses of } \partial_b \text{ and } \partial_\eta. \text{ We note that our computations of } \partial_b \text{ and } \partial_\eta \text{ hold pointwise and also in } L^2_{\text{loc}}.\]
For different \((\lambda, \gamma)\), we will apply scale/phase invariances in Step 3. For \((\lambda, \gamma, b, \eta, u)\) near \((1, 0, 0, 0, Q)\), we compute using (5.7)

\[
\partial_k F^{(1)}_k = (AQ)_k |Z_k|_{\lambda, \gamma} - (u - Q, |AQ|_{\lambda, \gamma}),
\]

\[
= (-yQ, yQ\chi_M)O(1))\delta_{1k} + M^{\infty}O(||u - Q||_L^2),
\]

\[
\partial_k F^{(1)}_k = (-\partial_{\eta}Z_k)_{\lambda, \gamma} + (u - Q, |Z_k|_{\lambda, \gamma}),
\]

\[
= (-\frac{1}{4}(yQ, yQ\chi_M)O(1))\delta_{2k} + M^{\infty}O(||u - Q||_L^2).
\]

Next, by the pointwise estimates (4.19) and \(|\partial_a \psi_b|_{L^\infty} \lesssim \frac{1}{|\log |b||}\), we have

\[
1_{(0,2M)}|\partial_a \tilde{P} (0; b, \eta) + i\frac{2Q}{Q}| = 1_{(0,2M)} \left| \partial_a \psi_b (\eta) \cdot \partial_{\tilde{\eta}} = \psi_b (\eta) P(\tau; b, \bar{\eta}) + \partial_a \psi_b (\eta) \rho \right| \lesssim 1_{(0,2M)}|b|^2 + \frac{1}{|\log |b||},
\]

Combining this with (5.7), we have

\[
\partial_k F^{(1)}_k = (-\partial_{\eta} \tilde{P}, Z_k), \quad \partial_k F^{(1)}_k = \frac{1}{4}(yQ, yQ\chi_M), \partial_{3k} + M^{\infty}O(1_{(0,2M)}|b|^2).
\]

Next, again by pointwise estimates (4.19), we have

\[
1_{(0,2M)}|\partial_a \tilde{P} (\tau; b, \eta) + \rho| = 1_{(0,2M)} \left| \partial_{\tilde{\eta}} = \psi_b (\eta) \right| \lesssim 1_{(0,2M)}|b|^2.
\]

Combining this with (5.7), we have

\[
\partial_k F^{(1)}_k = (-\partial_{\eta} \tilde{P}, Z_k), \quad \partial_k F^{(1)}_k = \frac{1}{4}(yQ, yQ\chi_M), \partial_{4k} + M^{\infty}O(|b|).
\]

Finally, we have

\[
\frac{\delta F^{(1)}_k}{\delta u} = (Z_k)_{\lambda, \gamma} \in L^2.
\]

In summary, \(F^{(1)}\) is \(C^1\) and \(\partial_{\lambda, \gamma, \eta, \eta} F^{(1)}\) is invertible at \((\lambda, \gamma, b, \eta, u) = (1, 0, 0, 0, Q)\) since the nonzero leading terms are on the diagonal.

We turn to \(F^{(2)}\). We check that \(F^{(2)}\) is \(C^1\) and \(\partial_{\lambda, \gamma, \eta, \eta} F^{(2)}\) is invertible at \((\lambda, \gamma, b, \eta, u) = (1, 0, 0, 0, Q)\). As \(F^{(2)}_1 = F^{(1)}_1\) and \(F^{(2)}_2 = F^{(1)}_2\), it suffices to consider \(F^{(2)}_k\) for \(k \in \{3, 4\}\). Let us temporarily denote \(f_{\lambda, \gamma} := e^{\gamma f} (\tau)\) (the \(H^1\)-scaling). For \((\lambda, \gamma, b, \eta, u)\) near \((1, 0, 0, 0, Q)\), we compute using \(D\chi Q = 0\) and the linearization of the Bogomol'nyi operator (5.1) that

\[
\partial_k F^{(2)}_k = -(D\chi u, [\lambda_1 \tilde{Z}_k]_{\lambda, \gamma}) + (u - Q, L^2 \partial_{\tilde{\eta}} [\lambda_1 \tilde{Z}_k]_{\lambda, \gamma}) - (N\partial (u - Q), [\lambda_1 \tilde{Z}_k]_{\lambda, \gamma}) + M^{\infty}O(||u - Q||_L^2).
\]

Similarly,

\[
\partial_{\eta} F^{(2)}_k = (D\chi u, [\tilde{Z}_k]_{\lambda, \gamma}) = M^{\infty}O(||u - Q||_L^2).
\]

For \(\partial_b\) and \(\partial_{\eta}\), by \(|\partial_b \psi_b|_{L^\infty} \lesssim \frac{1}{|\log |b||}\) we have

\[
1_{(0,2M)}|\partial_b \tilde{P}_1 (0; b, \eta) + i\frac{2Q}{Q}| = 1_{(0,2M)} \left| \partial_b \psi_b (\eta) \cdot \partial_{\tilde{\eta}} = \psi_b (\eta) P(\tau; b, \bar{\eta}) + \partial_b \psi_b (\eta) \rho \right| \lesssim 1_{(0,2M)}|b|^2 + \frac{1}{|\log |b||}.
\]

Combining this with (5.8), we have

\[
\partial_k F^{(2)}_k = (-\partial_{\eta} \tilde{P}, \tilde{Z}_k), \quad \partial_k F^{(2)}_k = \frac{1}{4}(yQ, yQ\chi_M), \partial_{3k} + M^{\infty}O(1_{(0,2M)}|b|^2).
\]
Similarly, we have
\[
1_{(0,2M)}|\partial_{\eta} \tilde{P}_1(\cdot;b,\eta) + \frac{\delta}{2} Q| = 1_{(0,2M)}|\psi_{\eta}(\eta) \left( \partial_{\eta=\psi_{\eta}(\eta)} P_1(\cdot;b,\eta) + \frac{\delta}{2} Q \right)| \lesssim 1_{(0,2M)}|b|y
\]
so
\[
\partial_{\eta} f^{(2)}_k = (-\partial_{\eta} \tilde{P}_1, \tilde{Z}_k), = \frac{1}{2}(yQ, yG_{\lambda,\gamma}, \delta_{4k} + M^C O(|b|)).
\]
Finally, we have
\[
\frac{\delta f^{(2)}_k}{\delta u} = L^*_{u,\lambda} [\tilde{Z}_k]_{\lambda,\gamma} \in L^2.
\]
This shows that \( F^{(2)} \) is \( C^1 \) and \( \partial_{\lambda,\gamma,b,\eta} F^{(2)} \) is invertible at \((\lambda, \gamma, b, \eta, u) = (1, 0, 0, 0, Q)\).

Therefore, by the implicit function theorem, provided that \( M \gg 1 \), there exist \( 0 < \delta_2 \ll \delta_1^2 \)
and \( C^1 \)-maps \( G^{(j)}_{1,0} : B_{\delta_2}(Q) \to B_{\delta_1}(1, 0, 0, 0) \) such that for given \( u \in B_{\delta_2}(Q) \subseteq L^2 \), \( G^{(j)}_{1,0}(u) \) is a unique solution to \( F^{(j)}(G^{(j)}_{1,0}(u), u) = 0 \) in \( B_{\delta_1}(1, 0, 0, 0) \). Moreover, we have a Lipschitz estimate
\[
\text{dist}(G^{(j)}_{1,0}(u), (1, 0, 0, 0)) \lesssim \|u - Q\|_{L^2}.
\]
The proof of the implicit function theorem also guarantees the difference estimate:
\[
\text{dist}(G^{(1)}_{1,0}(u), G^{(2)}_{1,0}(u)) \lesssim |F^{(2)}(G^{(1)}_{1,0}(u), u) - F^{(2)}(G^{(2)}_{1,0}(u), u)| = |F^{(2)}(G^{(1)}_{1,0}(u), u)|.
\]

**Step 3: Definition and uniqueness of \( G^{(j)} \).**
We now apply scale/phase invariances to cover the \( \delta_2 \)-neighborhood of \( \{Q_{\lambda,\gamma} : \lambda \in \mathbb{R}_+, \gamma \in \mathbb{R}/2\pi\mathbb{Z}\} \) in \( L^2 \). For \( \lambda \in \mathbb{R}_+ \) and \( \gamma \in \mathbb{R}/2\pi\mathbb{Z} \), apply the scale/phase invariances to \( G^{(j)}_{1,0} \) to define \( G^{(j)}_{\lambda,\gamma} : B_{\delta_2}(Q)_{\lambda,\gamma} \to B_{\delta_1}(\lambda, \gamma, 0, 0) \) in the obvious way. Thus uniqueness property of \( G^{(j)}_{\lambda,\gamma} \) holds for values in \( B_{\delta_1}(\lambda, \gamma, 0, 0) \) and there holds the difference estimate
\[
(5.10) \quad \text{dist}(G^{(1)}_{\lambda,\gamma}(u), G^{(2)}_{\lambda,\gamma}(u)) \lesssim |F^{(2)}(G^{(1)}_{\lambda,\gamma}(u), u)|.
\]
We claim that
\[
G^{(j)} := \bigcup_{\lambda,\gamma} G^{(j)}_{\lambda,\gamma} : \bigcup_{\lambda,\gamma} B_{\delta_2}(Q)_{\lambda,\gamma} \to \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z} \times B_{\delta_1}(0) \times B_{\delta_1}(0)
\]
is well-defined, i.e. the family \( \{G^{(j)}_{\lambda,\gamma}\}_{\lambda,\gamma} \) is compatible. Indeed, if \( u \in B_{\delta_2}(Q)_{\lambda_1,\gamma_1} \cap B_{\delta_2}(Q)_{\lambda_2,\gamma_2} \), then \( \text{dist}((\lambda_1, \gamma_1), (\lambda_2, \gamma_2)) \lesssim \delta_2 \) thus \( \text{dist}(G^{(j)}_{\lambda_1,\gamma_1}(u), (\lambda_1, \gamma_1, 0, 0)) \lesssim \delta_2 \ll \delta_1 \). Since \( G^{(j)}_{\lambda,\gamma}(u) \) satisfies the equation \( F^{(j)}(G^{(j)}_{\lambda,\gamma}(u), u) = 0 \), we have \( G^{(j)}_{\lambda,\gamma}(u) = G^{(j)}_{\lambda_1,\gamma_1}(u) \) by the uniqueness of \( G^{(j)}_{\lambda_1,\gamma_1}(u) \) in \( B_{\delta_1}(\lambda_1, \gamma_1, 0, 0) \).

Having defined \( G^{(j)} \), we can define the map
\[
e^{(j)} : \bigcup_{\lambda,\gamma} B_{\delta_2}(Q)_{\lambda,\gamma} \to B_{\delta_1}(0)
\]
by \( e^{(j)}(u) = u_{\lambda_1,\gamma_1} - \tilde{P}(\cdot;b,\eta) \), where \((\lambda, \gamma, b, \eta, u) = G^{(j)}(u)\).

Next, we claim the uniqueness property of \( G^{(j)} \) : given \( u \in \bigcup_{\lambda,\gamma} B_{\delta_2}(Q)_{\lambda,\gamma}, G^{(j)}(u) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z} \times B_{\delta_1}(0) \times B_{\delta_1}(0) \) is the unique solution to \( F^{(j)}(G^{(j)}(u), u) = 0 \) such that \( \|e^{(j)}\|_{L^2} < \delta_1' \). To see this, let \( G' = (\lambda', \gamma', b', \eta') \) be a solution to \( F^{(j)}(G', u) = 0 \) such that \( e' = u_{\lambda_1,\gamma_1} - P(\cdot;b',\eta') \) satisfies \( \|e'\|_{L^2} < \delta_1' \). If \( \text{dist}(G', G^{(j)}(u)) < \delta_1 \), then \( G^{(j)}(u) = G' \) by the uniqueness of \( G^{(j)}(u) \). If \( \text{dist}(G', G^{(j)}(u)) \geq \delta_1 \), then \( ||P(\cdot;b,\eta')_{\lambda,\gamma} - \tilde{P}(\cdot;b,\eta)_{\lambda,\gamma}||_{L^2} \gtrsim \delta_1 \) but \( \|e'\|_{L^2}, \|e^{(j)}\|_{L^2} < \delta_1' \ll \delta_1 \), contradicting \( \|P(\cdot;b,\eta')_{\lambda,\gamma} + e'_{\lambda,\gamma}\|_{L^2} = u = [P(\cdot;b,\eta) + e_{\lambda,\gamma}] \).

**Step 4: Coordinate system of the rough decomposition.**

\(^{20}\)We fix \( \delta_1 \) here, but we will freely shrink \( \delta_2 \) if necessary.

\(^{21}\)Here we choose \( \delta_1' \) with \( \delta_1' \ll \delta_1 \) and then shrink \( \delta_2 \) to define the maps \( e^{(j)} \).
From now on, we work with the $H^3_0$-topology and $j = 1$. Note that $\epsilon^{(1)}$ is continuous on the $H^3_0$-topology, i.e.

$$\epsilon^{(1)} : \bigcup_{j, \gamma} B^{H^3_0}_{\delta_j}(Q)_{\lambda, \gamma} \to B^{\mathbb{Z}^+}_{\delta_j}(0)$$

is continuous. By the definition of $\epsilon^{(1)}$, the map

$$(G^{(1)}, \epsilon^{(1)}) : \bigcup_{j, \gamma} B^{H^3_0}_{\delta_j}(Q)_{\lambda, \gamma} \to \mathbb{R}_+ \times \mathbb{R}/2\pi \mathbb{Z} \times B_{\delta_j}(0) \times B_{\delta_j}(0) \times B^{\mathbb{Z}^+}_{\delta_j}(0)
$$

has a continuous left inverse

$$\Phi : \mathbb{R}_+ \times \mathbb{R}/2\pi \mathbb{Z} \times B_{\delta_j}(0) \times B_{\delta_j}(0) \times B^{\mathbb{Z}^+}_{\delta_j}(0) \to H^3_0
$$

$$(\lambda, \gamma, b, \eta, \epsilon) \mapsto [P(\cdot; b, \eta) + \epsilon]_{\lambda, \gamma}.$$ 

Moreover, the uniqueness of $G^{(1)}$ implies that $\text{Im}(G^{(1)}, \epsilon^{(1)}) = \Phi^{-1}(\bigcup_{j, \lambda, \gamma} B^{H^3_0}_{\delta_j}(Q)_{\lambda, \gamma})$ (and in particular it is open) and $\Phi|_{\text{Im}(G^{(1)}, \epsilon^{(1)})}$ is a right inverse of $(G^{(1)}, \epsilon^{(1)})$. Therefore, the restriction

$$\Phi|_{\text{Im}(G^{(1)}, \epsilon^{(1)})} : \text{Im}(G^{(1)}, \epsilon^{(1)}) \to \bigcup_{j, \lambda, \gamma} B^{H^3_0}_{\delta_j}(Q)_{\lambda, \gamma}$$

is a homeomorphism with the inverse $(G^{(1)}, \epsilon^{(1)})$.

**Step 5: Completion of the proof.**

We finish the proof of this lemma.

1. We further restrict to the sets $\mathcal{U}_{\text{dec}}$ and $\mathcal{O}_{\text{dec}}$. Since $\overline{\mathcal{U}_{\text{dec}}}$ lies in the domain of $\Phi$ and $\overline{\mathcal{O}_{\text{dec}}} \subseteq \bigcup_{j, \lambda, \gamma} B^{H^3_0}_{\delta_j}(Q)_{\lambda, \gamma}$, we have $\overline{\mathcal{U}}_{\text{dec}} \subseteq \text{Im}(G^{(1)}, \epsilon^{(1)})$ due to the uniqueness of $G^{(1)}$. Therefore, restricting the homeomorphism $\Phi|_{\text{Im}(G^{(1)}, \epsilon^{(1)})}$ on $\overline{\mathcal{U}}_{\text{dec}}$ implies that $\mathcal{O}_{\text{dec}}$ is open, $\Phi(\overline{\mathcal{U}}_{\text{dec}}) = \overline{\mathcal{O}}_{\text{dec}}$, and $\Phi|_{\overline{\mathcal{U}}_{\text{dec}}} : \overline{\mathcal{U}}_{\text{dec}} \to \overline{\mathcal{O}}_{\text{dec}}$ is a homeomorphism.

2. This is merely a summary of the properties of $G^{(2)}$ shown above.

3. We showed above that $G^{(2)}$ is $C^1$ with respect to the $L^2$-topology. The $C^1$ property of $G^{(1)}$ on the $H^3_0$-topology is immediate from the embedding $H^3_0 \hookrightarrow L^2$.

4. $(5.9)$ follows from $G^{(2)} = \bigcup_{j, \lambda, \gamma} G^{(j)}_{\lambda, \gamma}$, the difference estimate $(5.10)$ for $G^{(1)}_{\lambda, \gamma}$ and $G^{(2)}_{\lambda, \gamma}$, and the definition of $F^{(2)}$. Note that $F_k(2) = F_k(1) = 0$ for $k \in \{1, 2\}$.

5. This follows from the parameter dependence $b^* \ll \delta_{\text{dec}}(M)$.

5.2. **Trapped solutions and reduction of Theorem 1.1**

In this subsection, we reduce Theorem 1.1 to Propositions 5.3, 5.4, and 5.5. We also prove Corollary 1.2. Among these, the main ingredient is a bootstrap argument, Proposition 5.3. We will call solutions satisfying the bootstrap conditions the **trapped solutions**. By bootstrapping (Proposition 5.3) with a connectivity argument (Proposition 5.4), we show the existence of trapped solutions. We then show that (Proposition 5.5) those solutions are finite-time blow-up solutions as described in Theorem 1.1. Such an argument is standard in the literature.

Roughly speaking, trapped solutions are required to satisfy $|\eta| \ll b$ and certain smallness conditions on $\epsilon$ on its maximal forward lifespan, to guarantee the blow-up derived in Section 4. To describe more precisely, we quantify $|\eta| \ll b$ and the smallness conditions on $\epsilon$ in terms of the nonlinear decomposition (see Lemma 5.2) and nonlinear adapted derivatives of $\epsilon$. Namely, for a function $u \in \mathcal{O}_{\text{dec}}$, we decompose it as

$$u(r) = e^{i\gamma}(P(\cdot; b, \eta) + \epsilon)
$$
with the orthogonality conditions (5.3) according to Lemma 5.2. We recall the nonlinear adapted derivatives, which are given by
\[
\begin{align*}
  w &:= e^{-\gamma t}u(\lambda t), \\
  w_1 &:= D_w w, \\
  w_2 &:= A_w w_1, \\
  \epsilon &:= w - P(\cdot; b, \eta), \\
  \epsilon_1 &:= w_1 - P_1(\cdot; b, \eta), \\
  \epsilon_2 &:= w_2 - P_2(\cdot; b, \eta).
\end{align*}
\]

We further define \( \epsilon_3 \) by
\[
\epsilon_3 := A^* w_2 \epsilon_2.
\]

Here it suffices to use (or define) a linear adapted derivative \( \epsilon_3 \) of \( \epsilon_2 \), as opposed to \( \epsilon_1 \) or \( \epsilon_2 \). With these adapted derivatives, we can rigorously state our bootstrap hypothesis. For a large universal constant \( K > 1 \) to be chosen later, we set the bootstrap assumptions
\[
0 < b < b^*, \quad |\eta| < \frac{b}{10 \log b},
\]
\[
\epsilon \in L^2, \quad \epsilon_1 \in L^2, \quad \epsilon_2 \in L^2, \quad \epsilon_3 \in L^2.
\]

Let \( u \) be a solution to (1.19) with the initial data \( u_0 \in O_{\text{init}} \) and maximal forward-in-time lifespan \([0, T)\). This \( u \) is called a trapped solution if it admits the nonlinear decomposition for each time \( t \in [0, T) \) and satisfies the bootstrap assumptions (5.11).

We note that the assumptions (5.11) are initially satisfied at \( t = 0 \). In other words, any elements of \( O_{\text{init}} \) satisfy (5.11). Indeed, if we are given \((\lambda, \gamma, \tilde{b}, \tilde{\eta}, \tilde{\epsilon}) \in U_{\text{init}}\) and denote \( \tilde{w} = P(\cdot; \tilde{b}, \tilde{\eta}) + \tilde{\epsilon} \) and \( \tilde{\epsilon}_1 = D_w \tilde{w} - P_1(\cdot; \tilde{b}, \tilde{\eta}) \), then we have for \( k \in \{3, 4\} \)
\[
(\tilde{\epsilon}_1, \tilde{\epsilon}_2) = (\tilde{\epsilon}_1, \tilde{\epsilon}_2) + (D_w \tilde{w} - P_1 - L_Q \tilde{\epsilon}, Z_k)_r
\]
\[
\lesssim M^C (\|D_P P - P_1\|_{H^1} + \|L_P - L_Q\|_{H^2} + \|N_P(\tilde{\epsilon})\|_{H^2}) \lesssim M^C \|\tilde{\epsilon}\|^2,
\]
where the last inequality can be proved by the proof of (5.15) below. Therefore, by the difference estimate (5.9), the rough decomposition \((\lambda, \gamma, \tilde{b}, \tilde{\eta}, \tilde{\epsilon}) \in U_{\text{init}}\) and the nonlinear decomposition \((\lambda, \gamma, b, \eta, \epsilon) \) only differ by \( O(M^C \|\tilde{\epsilon}\|^2) \) for data in \( O_{\text{init}} \).

In the sequel, we will see that all the assumptions except the bound \( |\eta| < \frac{b}{10 \log b} \) can be bootstrapped. In view of the modulation equations \( \eta_s \approx 0 \) and \( b_s + b^2 + \frac{b^3}{\log b} \approx 0 \), the \( \eta \)-bound cannot be bootstrapped. Thus the trapped solutions are non-generic, and this is the source of codimension one illustrated before. We will construct these non-generic solutions using a soft connectivity argument.

We conclude this subsection by reducing the proof of Theorem 1.1 into three propositions: main bootstrap (Proposition 5.3), a proposition for the connectivity argument (Proposition 5.4), and a sharp description of the trapped solutions (Proposition 5.5). The heart of the proof is the main bootstrap, Proposition 5.3.

**Proof of Theorem 1.1 assuming Propositions 5.3, 5.4, and 5.5.** Let \((\tilde{\lambda}_0, \tilde{\gamma}_0, \tilde{b}_0, \tilde{\epsilon}_0) \in \tilde{U}_{\text{init}}\) and consider \( \tilde{\eta}_0 \) which varies in the range \((-\frac{\tilde{\eta}_0}{2 \log \tilde{\eta}_0}, \frac{\tilde{\eta}_0}{2 \log \tilde{\eta}_0})\). Define \( u_0 \in O_{\text{init}} \) via (1.21) and let \( u \) be the forward-in-time maximal solution to (1.11) with the initial data \( u_0 \) and lifespan \([0, T)\).

Our main goal is to show that \( u \) is a trapped solution for a well-chosen \( \tilde{\eta}_0 \). Notice that \( u_0 \) is formed by the rough decomposition. Define the exit time of \( O_{\text{dec}} \):
\[
T_{\text{dec}} := \sup\{t \in [0, T) : u(t') \in O_{\text{dec}}, \text{ for } t' \in [0, t]\} \in (0, T).
\]

Thus \( u(t) \) for \( t \in [0, T_{\text{dec}}) \) admits the nonlinear decomposition \((\lambda(t), \gamma(t), b(t), \eta(t), \epsilon(t))\) according to Lemma 5.2. Moreover, if \( T_{\text{dec}} < T \), then \( u(T_{\text{dec}}) \in O_{\text{dec}} \setminus O_{\text{dec}} \) and it also admits the nonlinear decomposition at time \( t = T_{\text{dec}} \). Next, thanks to (5.12),
the nonlinear decomposition \((\lambda_0, \gamma_0, b_0, \eta_0, \epsilon_0)\) at \(t = 0\) satisfies the bootstrap assumption (5.11). Thus we can also define the exit time of the bootstrap hypotheses:

\[ T_{\text{exit}} := \sup \{ \tau \in [0, T_{\text{dec}}] : (5.11) \text{ holds for all } \tau' \in [0, \tau] \} \in (0, T_{\text{dec}}]. \]

Thus our goal is to show that \(T_{\text{exit}} = T_{\text{dec}} = T\) for some \(\hat{\eta}_0\). Then \(u\) is a trapped solution with this \(\hat{\eta}_0\).

In fact, it suffices to show that \(T_{\text{exit}} = T_{\text{dec}}\) for some \(\hat{\eta}_0\). Indeed, if \(T_{\text{exit}} = T_{\text{dec}}\) but \(T_{\text{dec}} < T\), then \(u(T_{\text{dec}}) \in \Gamma_{\text{dec}} \setminus O_{\text{dec}}\) but \((\lambda, \gamma, b, \eta, \epsilon)\) at \(t = T_{\text{exit}}\) lies in the closure of the bootstrap hypotheses. Since \(u(T_{\text{exit}}) = u(T_{\text{dec}})\), we must have \(b = \eta = 0\) and \(\epsilon = 0\) at \(t = T_{\text{exit}}\). In other words, \(u\) is a rescaled \(Q\), which is a static solution. This contradicts the assumption \(u_0 \in O_{\text{exit}}\).

To show that \(T_{\text{exit}} = T_{\text{dec}}\) for some \(\hat{\eta}_0\), assume for the sake of contradiction that \(T_{\text{exit}} < T_{\text{dec}}\) for all \(\hat{\eta}_0\). The following proposition is shown in Section 5.6 and is of the proof of Theorem 1.1.

**Proposition 5.3 (Main bootstrap).** Let \(u\) have the nonlinear decomposition \((\lambda, \gamma, b, \eta, \epsilon)\). If the bootstrap hypotheses (5.11) hold for \(t \in [0, \tau]\) for some \(\tau > 0\), then the following hold for \(t \in [0, \tau]\):

\[ b \in (0, b_0], \quad \|\epsilon\|_{L^2} < \frac{1}{2}(b^*)^\frac{1}{2}, \quad \|\epsilon_1\|_{L^2} < \frac{b}{2\log b_0}, \quad \|\varepsilon_3\|_{L^2} < \frac{b}{2\log b_0}. \]

The fact that \(T_{\text{exit}} < T_{\text{dec}}\) together with Proposition 5.3 imply that \(\eta = \frac{b}{2\log b_0}\) at \(t = T_{\text{exit}}\). To derive a contradiction, we use a basic connectivity argument. Let \(\mathcal{I}_{\pm}\) be the set of initial \(\hat{\eta}_0\) such that \(\eta = \frac{b}{2\log b_0}\) at \(t = T_{\text{exit}}\). Note that \(\mathcal{I}_{\pm}\) partitions \((-\frac{b_0}{2\log b_0}, \frac{b_0}{2\log b_0})\). The following proposition is shown in Section 5.6.

**Proposition 5.4 (The sets \(\mathcal{I}_{\pm}\)).** The sets \(\mathcal{I}_{\pm}\) are nonempty and open.

We have a contradiction from the connectivity of \((-\frac{b_0}{2\log b_0}, \frac{b_0}{2\log b_0})\). Thus our claim, \(T_{\text{exit}} = T_{\text{dec}}\) for some \(\hat{\eta}_0\), is proved. Therefore, there exists a trapped solution \(u\) with this \(\hat{\eta}_0\).

The remaining part of the proof is the sharp description of this trapped solution. The following is proved in Section 5.6.

**Proposition 5.5 (Sharp description).** Let \(u\) be a trapped solution. Then, it blows up in finite time as described in Theorem 1.1. □

This ends the proof of Theorem 1.1 assuming Propositions 5.3, 5.4, and 5.5.

Using Theorem 1.1 and the pseudoconformal transform, we prove Corollary 1.2.

**Proof of Corollary 1.2.** Let \(v\) be a finite-time blow-up solution with smooth compactly supported initial data \(v_0\), constructed in Theorem 1.1 (see also Comments on Theorem 1.1). Applying scaling, phase rotation, and time translation symmetries, we may assume that \(v\) is defined on \([-T, 0)\) with \(v(-T) = v_0\) and blows up at time 0 with the decomposition

\[ v(t) = \frac{|\log |t||^2}{|t|} Q \left( \frac{|\log |t||^2}{|t|} r \right) - v^* \to 0 \text{ in } L^2 \]

as \(t \to 0^-\). For convenience, we rewrite this as

\[ v(t) = \frac{|\log |t||^2}{|t|} Q \left( \frac{|\log |t||^2}{|t|} r \right) + e^{it\Delta} v^* + \text{err}(t) \]

with \(\|\text{err}(t)\|_{L^2} \to 0\) as \(t \to 0^-\). We now apply the pseudoconformal transform \(C\) (1.4) to obtain the solution \(u\) on \([1/T, \infty)\) defined by

\[ u(t) := [Cv](t). \]
Note that the initial data \( u(1/T) \) is smooth and compactly supported. Since \( \mathcal{C} \) preserves the \( L^2 \)-norm, the contribution of \( \text{err}(t) \) is negligible: \( \|\mathcal{C}(\text{err})(t)\|_{L^2} \to 0 \) as \( t \to \infty \). Moreover, since \( \mathcal{C} \) preserves linear Schrödinger waves, \[
|\mathcal{C}(e^{it\Delta} u^*)(t)| = e^{it\Delta} u^*
\]
for some \( u^* \in L^2 \) with \( \|u^*\|_{L^2} = \|v^*\|_{L^2} \). Finally, we have
\[
|\mathcal{C}\left(\frac{|\log|t||^2}{|t|}Q\left(\frac{|\log|t||^2}{|t|}r\right)\right)(t)| = e^{it\Delta} |\log(t)||^2Q\left(\|\log(t)\|^2r\right).
\]

We can remove \( e^{it\Delta} \) by applying the dominated convergence theorem (after rescaling):
\[
\left\| (e^{it\Delta} - 1)\left(\frac{|\log(t)||^2Q\left(\|\log(t)\|^2r\right)}\right) \right\|_{L^2} \to 0
\]
as \( t \to \infty \). Therefore,
\[
u(t) - \|\log(t)||^2Q\left(\|\log(t)\|^2r\right) - e^{it\Delta} u^* \to 0 \text{ in } L^2
\]
as \( t \to \infty \).

5.3. Coercivity for nonlinear adapted derivatives. Recall that we decomposed our solution \( u \) according to the nonlinear decomposition. That is,
\[
\begin{align*}
\epsilon &= e^{-\imath \gamma} \lambda u(\lambda), & w_1 := D_w w, & w_2 := A_w w_1, \\
\epsilon &= w - P(\gamma; b, \eta), & \epsilon_1 := w_1 - P_1(\gamma; b, \eta), & \epsilon_2 := w_2 - P_2(\gamma; b, \eta),
\end{align*}
\]
and the orthogonality conditions
\[
\langle \epsilon, Z_1 \rangle = \langle \epsilon, Z_2 \rangle = \langle \epsilon_1, \tilde{Z}_3 \rangle = \langle \epsilon_1, \tilde{Z}_4 \rangle = 0
\]
are satisfied. We defined \( \epsilon_3 \) by \( \epsilon_3 := A_{\gamma}^\ast \epsilon_2 \).

The goal of this section is to transfer the linear coercivity estimates to the nonlinear adapted derivatives \( \epsilon_1, \epsilon_2, \epsilon_3 \), under the bootstrap assumptions (5.11). By the linearization of the Bogomol’nyi operator (see (5.11)), \( D_F P \approx P_1 \) (see (4.22) and (4.23)), and \( A_F P_1 \approx P_2 \) (see (4.24)), we see that \( \epsilon_1 \approx L_{Q}\epsilon \) and \( \epsilon_2 \approx A_{Q}\epsilon_1 \).

As mentioned earlier, we will take advantages of nonlinear adapted derivatives over linear ones in various places. With nonlinear adapted derivatives, one can observe that error terms are simplified in the evolution equations of \( \epsilon_1, \epsilon_2, \epsilon_3 \) in Sections 5.4 and 5.5. The following estimates are the trade-offs. We need additional arguments to establish the coercivity relations of the nonlinear adapted derivatives.

Lemma 5.6 (Nonlinear coercivity estimates). The following estimates hold.

\( (H^1\text{-level}) \)
\[
\|\epsilon\|_{\dot{H}^1_0} \lesssim M K b|\log(b)|^2 \lesssim b|\log(b)|^2.
\]
The second term is estimated by
\[ \|e_2\|_{\dot{H}^1} \sim \|e_3\|_{L^2}, \]
(5.14)
and
\[ \|e_1\|_{\dot{H}^1} \lesssim_M K \frac{b^2}{|\log b|} \lesssim \frac{b^2}{|\log b|^4}. \]
(5.15)
(Interpolation estimates at \(\dot{H}^1\)-level)
(5.16)
\[ \|e\|_{\dot{H}^1} \lesssim_M K b^2 \lesssim b^2 |\log b|^{4+}. \]

Proof. (1) From the relation
\[ P_1 + e_1 = w_1 = D_w w = D_P P + L_Q e + (L_P - L_Q)e + N_P(e), \]
the coercivity estimates (3.28) implies that
\[ \|e\|_{\dot{H}^1} \lesssim_M \|L_Q e\|_{L^2} \lesssim \|e_1\|_{L^2} + \|D_P P - P_1\|_{L^2} + \|L_P - L_Q\|e\|_{L^2} + \|N_P(e)\|_{L^2}. \]

The second term is estimated by \(b\) due to (4.12). We claim that the last two terms are estimated by
\[ \|L_P - L_Q\|e\|_{L^2} + \|N_P(e)\|_{L^2} \lesssim (o_{b \to 0}(1) + \|\dot{H}^1\| \|\dot{H}^1\|_{L^2}. \]

Assuming this, these terms are absorbed into the LHS which we have
\[ \|e\|_{\dot{H}^1} \lesssim_M \|e_1\|_{L^2} + b. \]

The bootstrap hypothesis on \(e_1\) and parameter dependence yield (5.14).

Henceforth, we show the claim (5.18). Notice that \((L_P - L_Q)e\) and \(N_P(e)\) are linear combinations of \(\frac{1}{y} A \theta[\psi_1, \psi_2] \bar{\psi}_3\), which we estimate by
\[ \|\frac{1}{y} A \theta[\psi_1, \psi_2] \bar{\psi}_3\|_{L^2} \lesssim \min_{\{1,2,3\}} \|\langle y \rangle \psi_j \|_{L^2} \|\langle \frac{\rho}{y} \rangle^{\frac{3}{2}} \psi_j \|_{L^2} \|\langle \frac{\rho}{y} \rangle^{\frac{3}{2}} \bar{\psi}_3\|_{L^2}. \]

For \((L_P - L_Q)e\), we can assume \(\psi_1 = e, \psi_2 = P - Q\), and \(\psi_3 \in \{P, Q\}\) so
\[ \|L_P - L_Q\|e\|_{L^2} \lesssim b^{\frac{3}{2}} \|\|\dot{H}^1\|_{L^2}. \]

For \(N_P(e)\), we can assume \(\psi_1 = \psi_2 = e\) and \(\psi_3 \in \{P, e\}\) so
\[ \|N_P(e)\|_{L^2} \lesssim ((\|\dot{H}^1\|_{L^2}) \|\dot{H}^1\|_{L^2}) \|\dot{H}^1\|_{L^2} \lesssim (\|\dot{H}^1\|_{L^2} + o_{b \to 0}(1)) \|\dot{H}^1\|_{L^2}. \]

This shows the claim (5.18).

(2) The equivalence (5.14) follows from the positivity of \(A^*_Q A^*_Q\) (3.38).

We turn to (5.15). We simultaneously consider the relations
\[ P_2 + e_2 = w_2 = A_w w_1 = A_P P_1 + A_Q e_1 + (A_w - A_Q) e_1 + (A_w - A_P) P_1, \]
\[ P_1 + e_1 = w_1 = D_w w = D_P P + L_Q e + (L_P - L_Q)e + N_P(e). \]

By the coercivity estimates (3.27) and (3.38), we have
\[ \|e_1\|_{\dot{H}^1} \lesssim_M \|e_2\|_{\dot{H}^1} + \|A_P P_1 - P_2\|_{\dot{H}^1} + \|A_w - A_Q\| e_1 \|\dot{H}^1\|_{\dot{H}^1} + \|A_w - A_P\| P_1 \|\dot{H}^1\|_{\dot{H}^1}, \]
\[ \|e\|_{\dot{H}^1} \lesssim_M \|e\|_{\dot{H}^1} + \|D_P P - P_1\|_{\dot{H}^1} + \|L_P - L_Q\| e\|_{\dot{H}^1} + \|N_P(e)\|_{\dot{H}^1}. \]

Here, we have \(\|e_2\|_{\dot{H}^1} \lesssim \|e_3\|_{L^2}\) by (5.14), and the \(\epsilon\)-independents terms \(D_P P - P_1\) and \(A_P P_1 - P_2\) are estimated in (4.23) and (4.24). Therefore,
\[ \|e_1\|_{\dot{H}^1} \lesssim_M \|e_3\|_{L^2} + \frac{b^3}{|\log b|} \|\dot{H}^1\|_{\dot{H}^1} + \|\dot{H}^1\|_{\dot{H}^1} \|\dot{H}^1\|_{L^2} \|\dot{H}^1\|_{L^2} + \frac{b^3}{|\log b|} \|L_P - L_Q\| e\|_{\dot{H}^1} + \frac{b^3}{|\log b|} \|N_P(e)\|_{\dot{H}^1}. \]

\[ \text{One may proceed by a bootstrap argument to do this rigorously.} \]
We claim the estimates
\begin{align}
\|(A_w - A_Q)\psi\|_{\dot{H}^2_y} + \|(A_w - A_P)P_1\|_{\dot{H}^2_y} & \lesssim b^3 + (a_y - \theta_0(1) + \|\varepsilon\|_{\dot{H}^2_y}^2 + \|\varepsilon\|_{\dot{H}^2_y}^2)\left(\|\psi\|_{\dot{H}^2_y} + \frac{1}{\log b} \|\varepsilon\|_{\dot{H}^2_y}\right), \\
\|(L_P - L_Q)\varepsilon\|_{\dot{H}^2_y} + \|N_P(\varepsilon)\|_{\dot{H}^2_y} & \lesssim b^3 + (a_y - \theta_0(1) + \|\varepsilon\|_{\dot{H}^2_y}^2)\|\varepsilon\|_{\dot{H}^2_y}^2.
\end{align}

Assuming these claims, we have
\[\|\varepsilon\|_{\dot{H}^2_y} + \frac{1}{\log b} \|\varepsilon\|_{\dot{H}^2_y} \lesssim M \|\varepsilon\|_{L^2} + \frac{b^2}{\log b},\]
which implies (5.15) and (5.16) after substituting the bootstrap hypothesis for \(\varepsilon_2\).

Henceforth, we show the claims (5.19) and (5.20). Firstly, we show (5.19). For the first term on the LHS of (5.19), we use the definition of \(H^1_y\) to have
\[\|(A_w - A_Q)\psi\|_{\dot{H}^2_y} \lesssim \|(w_y^2 - Q^2)\psi\|_{L^2} + \|A_0[w] - A_0[Q]\|_{L^\infty} \|\partial_y \psi\|_{L^2} + \|\log y\| \|A_0[w] - A_0[Q]\|_{L^\infty} \|\psi\|_{\dot{H}^1_y}.
\]

Using the estimates
\[\|(w_y^2 - Q^2)\psi\|_{L^2} \lesssim \|(w_y^2 - |P|^2)\|_{L^\infty} \|\psi\|_{L^2} + \||P|^2 - Q^2\|_{L^\infty} \|\psi\|_{L^2} \lesssim \|(w_y^2 - |P|^2)\|_{L^\infty} \|\psi\|_{L^2} + b^2 - \|\psi\|_{\dot{H}^1_y},\]

weighted \(L^\infty\)-estimates (see Lemma (A.14))
\[\|\varepsilon\|_{L_y^2} \lesssim \|\varepsilon\|_{H^0_y}^2 + \|\varepsilon\|_{\dot{H}^2_y}^2,\]

and substituting (5.13) and \(\|\varepsilon\|_{L^2} = o_y - \theta_0(1)\), we obtain
\[\|(A_w - A_Q)\varepsilon\|_{\dot{H}^2_y} \lesssim b^3 + (a_y - \theta_0(1) + \|\varepsilon\|_{H^0_y}^2 + \|\varepsilon\|_{\dot{H}^2_y}^2)\left(\|\psi\|_{\dot{H}^2_y} + \frac{1}{\log b} \|\varepsilon\|_{\dot{H}^2_y}\right),\]
as desired in (5.19). Next, the second term on the LHS of (5.19) is estimated by
\[\|(A_w - A_P)P_1\|_{\dot{H}^2_y} \lesssim \|(A_w - A_P)\psi\|_{L^\infty} \|P_1\|_{L^2} \lesssim b^2 - \|w_y^2 - |P|^2\|_{L^\infty} \lesssim b^2 + \|P\|_{L^\infty},\]
Recalling how we dealt with \(\|\varepsilon\|_{L_y^2} + \|P\|_{L^\infty}\) above, this bound suffices. This completes the proof of (5.19).

Next, we show (5.20). Recall that \((L_P - L_Q)\varepsilon\) and \(N_P(\varepsilon)\) are linear combinations of \(\frac{1}{b} A_0[\psi_1, \psi_2, \psi_3]\). In view of (A.3) (see also its proof), we have
\[\|\frac{1}{b} A_0[\psi_1, \psi_2, \psi_3]\|_{\dot{H}^2_y} \lesssim \|\Delta_1(\frac{1}{b} A_0[\psi_1, \psi_2, \psi_3])\|_{L^2} + \|1_{y^2 - 1} \frac{1}{b} A_0[\psi_1, \psi_2] \psi_3\|_{L^2},\]

\[\lesssim \|1_{y^2 - 1} A_0[\psi_1, \psi_2, \psi_3]\|_{L^2} + \left(\int \right) \Delta_y (\frac{1}{b} A_0[\psi_1, \psi_2]) y\, dy \partial_y \psi_3\|_{L^2} + \|\partial_y \psi_3\|_{\dot{H}^2_y}\]

We will only consider choices of \(\psi_1, \psi_2, \psi_3\) that can contribute to \((L_P - L_Q)\varepsilon\) or \(N_P(\varepsilon)\). That is, the set of \(\psi_1, \psi_2, \psi_3\) contains at least two \(\varepsilon\)'s or one \(\varepsilon\) and one
This completes the proof of (5.20). The first two terms can be estimated using weighted $L^\infty$-estimates (Lemma A.15): 

$$
\|1_{y > 1} A_0 [\psi_1, \psi_2] \|_{L^2} + \| \langle \int_y^0 \text{Re}(\psi_1 \psi_2) y' dy' \rangle (\partial_y - \frac{1}{y}) \partial_y \psi_3 \|_{L^2} 
\lesssim \left\{ \begin{array} {l}
\|\psi_1 \psi_2\|_{L^2} \| \frac{1}{y} (\partial_y - \frac{1}{y}) \partial_y e\|_{L^2} \lesssim \|b\| \|\| e \|_{L^2} \| \| e \|_{L^2} \| \| e \|_{L^2} \| \| e \|_{L^2} \| \| e \|_{L^2} 
\quad \text{if } \psi_3 = e,
\|\langle \psi_1 \rangle - 3 + \psi_2 \|_{L^\infty} \lesssim \|b\| \|\| e \|_{L^2} \| \| e \|_{L^2} \| \| e \|_{L^2} 
\quad \text{if } \psi_3 \in \{P, Q\},
\|b\| \|\| e \|_{L^2} \| \| e \|_{L^2} \| \| e \|_{L^2} 
\quad \text{if } \psi_3 = P - Q,
\end{array} \right. 
\right.
\]
Subtracting the second from the first, we get the equation for \( \epsilon \):

\[
(\partial_s - \frac{\lambda_s}{\lambda} \Lambda + \gamma_s i) \epsilon + (i L^*_w w_1 - i L^*_P P_1) = \text{Mod} \cdot v - i \Psi.
\]

From the identity

\[
i L^*_w w_1 - i L^*_P P_1 = i L^*_Q \epsilon_1 + (i L^*_P - i L^*_Q) \epsilon_1 + (i L^*_w - i L^*_P) w_1,
\]

the first term \( i L^*_Q \epsilon_1 \) can be considered as the leading term of \( i L^*_w w_1 - i L^*_P P_1 \).

Next, we derive the equation for \( \epsilon_1 \). Recall that

\[
(5.22) \quad \tilde{\gamma}_s = \gamma_s + \int_0^\infty \text{Re}(\overline{\psi} w_1) dy.
\]

Recall also (2.15) and (4.27):

\[
(\partial_s - \frac{\lambda_s}{\lambda} \Lambda_{-1} + \bar{\gamma}_s i) w_1 + i A^*_w w_2 - \left( \int_0^y \text{Re}(\overline{\psi} w_1) dy \right) i w_1 = 0,
\]

\[
(\partial_s - \frac{\lambda_s}{\lambda} \Lambda_{-1} + \bar{\gamma}_s i) P_1 + i A^*_P P_2 - \left( \int_0^y \text{Re}(\overline{\psi} P_1) dy \right) i P_1 = -\text{Mod} \cdot v_1 + i \Psi_1.
\]

Subtracting the second from the first, we get the equation for \( \epsilon_1 \):

\[
(5.23) \quad (\partial_s - \frac{\lambda_s}{\lambda} \Lambda_{-1} + \bar{\gamma}_s i) \epsilon_1 + i A^*_Q \epsilon_2 = -(i A^*_w - i A^*_P) w_2 - (i A^*_P - i A^*_Q) \epsilon_2 + \left( \int_0^y \text{Re}(\overline{\psi} w_1) dy \right) i \epsilon_1 + \left( \int_0^y \text{Re}(\overline{\psi} P_1) dy \right) i P_1 + \text{Mod} \cdot v_1 - i \Psi_1.
\]

**Lemma 5.7** (Modulation estimates). We have

\[
|b_s + b^2 + \eta^2 + c_3(b^2 - \eta^2)| + |\eta_0 + 2c_3 \eta| \lesssim \frac{\|\epsilon\|_{L^2}}{\sqrt{|\log \mu^2|}}.
\]

**Proof.** In this proof, we freely use the bootstrap hypotheses (5.11), as well as Lemmas 5.6 and 4.2, to estimate \( \epsilon, \epsilon_1 \), and \( \epsilon_2 \). We also abuse the notation and identify the operator \( A^*_w - A^*_P \), which is simply the multiplication by a function (namely, the difference of the zeroth order terms), with that function.

We note that the estimate of \( |\gamma_s - \eta| \) will follow from the estimate of \( |\bar{\gamma}_s + \eta| \) and the claim

\[
(5.26) \quad \gamma_s - \gamma_s = \int_0^\infty \text{Re}(\overline{\psi} w_1) dy = -2\eta + O(b^2).\]

The claim can be obtained from the computations

\[
\int_0^\infty \text{Re}(\overline{\psi} P_1) dy = \eta \int_0^\infty (-\frac{b}{2} Q^2) dy + O(b^2) = -2\eta + O(b^2).
\]

and

\[
\left| \int_0^\infty \text{Re}(\overline{\psi} P_1) dy \right| \lesssim \|\epsilon\|_{L^2} \|\frac{\log y}{y}\| P_1 \|L^2 \lesssim b^{2-},
\]

\[
\left| \int_0^\infty \text{Re}(\overline{\psi} \epsilon_1) dy \right| \lesssim \|\frac{\log y}{y}\| \|L^2\| \|\epsilon_1\| \|L^2\| \lesssim b^{2-},
\]

\[
\left| \int_0^\infty \text{Re}(\overline{\psi} \epsilon_1) dy \right| \lesssim \|\epsilon\|_{L^\infty} \|\frac{\log y}{y}\| \|L^2\| \|\epsilon_1\| \|L^2\| \lesssim b^{2-},
\]

where in the last inequality we used (A.11).

In order to derive the modulation estimates for \( \lambda \) and \( \bar{\gamma} \), we differentiate the orthogonality conditions \( \langle \epsilon, Z_k \rangle_r = 0 \) for \( k \in \{1, 2\} \). It is convenient to rearrange the equation (5.21) as

\[
\text{Mod} \cdot (v + (\lambda \epsilon, -i \epsilon, 0, 0)')
\]

\[
= \partial_s \epsilon + b \Lambda \epsilon + \eta \epsilon + (i L^*_w w_1 - i L^*_P P_1) + i \Psi
\]

\[
- (\bar{\gamma}_s - \gamma_s + 2\eta) i w - c_3(b^2 - \eta^2)(\partial s P) - 2c_3b \eta (\partial_y P).
\]
Taking the inner product with $Z_k$ with $k \in \{1, 2\}$, we get
\[
\sum_{j=1}^{4} \{(v_j, Z_k)_r + O(M^C \|e\|_{\tilde{H}_k^0})\} \mod_j
\]
(5.27) 
\[
= (i L^*_w w_1 - i L^*_P P_1, Z_k)_r - b(\epsilon, \Lambda Z_k)_r - \eta(\epsilon, i Z_k)_r - (\Psi, i Z_k)_r
\]
\[+ (\tilde{\gamma}_s - \gamma_s + 2\eta)(w, i Z_k)_r - c_b (b^2 - \eta^2)(\partial_b P, Z_k)_r - 2c_b b\eta(\partial_b P, Z_k)_r.
\]
We first look at the matrix structure of the LHS of (5.27). By the transversality computation (5.7) and the fact that $Z_k$ is supported in the region $y \leq 2M$, we have
\[
\{ (v_j, Z_k)_r + O(M^C \|e\|_{\tilde{H}_k^0}) \}_{1 \leq k \leq 4}
\]
(5.28) 
\[
= \left( -\left\langle yQ, yQ \chi_M \right\rangle + O(1) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + O(M^C b) \right)
\]
Note that this matrix has logarithmic divergence due to $(yQ, yQ \chi_M)_r \sim \log M$ by (5.6).

We turn to estimate the RHS of (5.27). We claim that
\[
|\text{RHS of } (5.27) | \lesssim b^2.
\]
(5.29)
For the first term on the RHS of (5.27), we have
\[
|\langle (L^*_w w_1 - i L^*_P P_1, Z_k)_r | \lesssim \|w_1, (L_w - L_P) i Z_k_r |
\]
\[
\lesssim \|y\|^{-2} \|e\|_{L^2} \left\langle y \right\rangle^2 \left\langle L_P i Z_k \right\rangle_{L^2} + \|w_1\|_{L^2} \| (L_w - L_P) i Z_k \|_{L^2}
\]
\[
\lesssim b^2 \|y\|^{-2} \left\langle L_P i Z_k \right\rangle_{L^2} + b^2 \| (L_w - L_P) i Z_k \|_{L^2},
\]
since it suffices to show
\[
\|L_P i Z_k \|_{L^2} \lesssim M^C,
\]
\[
\| (L_w - L_P) i Z_k \|_{L^2} \lesssim b^{-1}.
\]
The estimate for $L_P i Z_k$ follows from
\[
\|L_P i Z_k \|_{L^2} \lesssim M^C \|y\|^{-3}.
\]
The estimate for $(L_w - L_P) i Z_k$ follows from
\[
\| (L_w - L_P) i Z_k \|_{L^2} \lesssim \frac{1}{b} \| (A_0 [w] - A_0 [P]) Z_k \| + \frac{1}{b} \| A_0 [e, i Z_k] w \| + \frac{1}{b} \| A_0 [P, i Z_k] e \|
\]
and
\[
\frac{1}{b} \| (A_0 [w] - A_0 [P]) Z_k \|_{L^2} + \frac{1}{b} \| A_0 [e, i Z_k] w \|_{L^2} \lesssim M^C \|e\|_{\tilde{H}_k^0} \lesssim b^2,
\]
\[
\frac{1}{b} \| A_0 [P, i Z_k] e \|_{L^2} \lesssim M^C \|e\|_{\tilde{H}_k^0} \lesssim b^{-1}.
\]
The remaining terms on the RHS of (5.27) can be estimated using (1.26) and (5.20); we have
\[
|b(\epsilon, \Lambda Z_k)_r | + |\eta(\epsilon, i Z_k)_r | \lesssim b M^C \|e\|_{\tilde{H}_k^0} \lesssim b^3,
\]
\[
| (\Psi, i Z_k)_r | \lesssim M^C b^2 | \log b | \lesssim b^2,
\]
\[
|(\tilde{\gamma}_s - \gamma_s + 2\eta)(w, i Z_k)_r | \lesssim M^C |\tilde{\gamma}_s - \gamma_s + 2\eta | \lesssim b^2,
\]
and
\[
|c_b (b^2 - \eta^2)(\partial_b P, Z_k)_r | + |2c_b b\eta(\partial_b P, Z_k)_r | \lesssim b^2 \left( |(\partial_b P, Z_k)_r | + |(\partial_b P, Z_k)_r | \right) \lesssim b^2 M^C \lesssim b^{-2}.
\]
Therefore, the claim (5.29) is proved.
Next, in order to derive the modulation estimates for $b$ and $\eta$, we differentiate the orthogonality conditions $(\epsilon_1, \tilde{Z}_k)_r = 0$ for $k \in \{3, 4\}$. We rearrange the equation (5.30) as

\[
\text{Mod} \cdot (v_1 + (A_{-1} \epsilon_1, -i \epsilon_1, 0, 0)^T)
\]

\[
= \partial_t \epsilon_1 + i A^w_0 \epsilon_2 + b A_{-1} \epsilon_1 - \eta i \epsilon_1 + (i A^w_0 - i A^p_0) w_2 + (i A^p_0 - i A^Q_0) \epsilon_2
\]

\[-(\int_0^y \text{Re}(\overline{w_1})dy') i \epsilon_1 - (\int_0^y \text{Re}(\overline{w_1} - \overline{P} P_1)dy') i P_1 + i \Psi_1.
\]

Taking the inner product with $\tilde{Z}_k$ with $k \in \{3, 4\}$, we get

(5.31)

\[
\sum_{j=1}^4 \{((v_1)_j, \tilde{Z}_k)_r + O(M^G \|\epsilon_1\|_{\mathcal{H}^1})\} \text{Mod}_j = (i A^Q_0 \epsilon_2, \tilde{Z}_k)_r
\]

\[+ (b A_{-1} \epsilon_1 - \eta i \epsilon_1, \tilde{Z}_k)_r + ((i A^w_0 - i A^p_0) w_2 + (i A^p_0 - i A^Q_0) \epsilon_2, \tilde{Z}_k)_r
\]

\[-(\int_0^y \text{Re}(\overline{w_1})dy') i \epsilon_1 + (\int_0^y \text{Re}(\overline{w_1} - \overline{P} P_1)dy') i P_1, \tilde{Z}_k)_r + (i \Psi_1, \tilde{Z}_k)_r.
\]

We first look at the matrix structure of the LHS of (5.31). By the structure of $v_1$ (4.20) (in particular the degeneracy $A_{-1} P_1 = O(b) = i P_1$), the transversality computation (5.8), and the fact that $\tilde{Z}_k$ is supported in $[0, 2M]$, we obtain

(5.32)

\[
\{((v_1)_j, \tilde{Z}_k)_r + O(M^G \|\epsilon_1\|_{\mathcal{H}^1})\} 3 \leq k \leq 4, 1 \leq j \leq 4
\]

\[
= \begin{pmatrix}
0 & 0 & (\frac{1}{4} y \Psi_1, y Q \chi_M)_r & 0
\end{pmatrix} + O(M^G b).
\]

As before, $(\frac{1}{4} y \Psi_1, y Q \chi_M)_r \sim \log M$.

We turn to estimate the RHS of (5.31). We claim that

(5.33)

\[
|\text{RHS of (5.31)}| \lesssim \sqrt{\log M} \|\epsilon_3\|_{L^2} + b^3.
\]

For the first term, since $A^Q_0 \epsilon_2 = \epsilon_3$, we estimate as)

\[
|\langle i A^Q_0 \epsilon_2, \tilde{Z}_k\rangle_r| \lesssim \|\epsilon_3\|_{L^2} \|\tilde{Z}_k\|_{L^2} \lesssim \sqrt{\log M} \|\epsilon_3\|_{L^2}.
\]

For the remaining terms, we claim the following weighted $L^2$-estimates (this is also for a later use in the Morawetz correction; see the proof of Lemma 5.12):

(5.34)

\[
\|\epsilon_1\|_{L^2} \lesssim b^2.
\]

(5.35)

\[
\|A^w_0 - A^p_0\|_{L^2} \|\epsilon_2\|_{L^2} \lesssim b^3.
\]

(5.36)

\[
\|\int_0^y \text{Re}(\overline{w_1})dy'\|_{L^2} \|\epsilon_1\|_{L^2} \lesssim \|\epsilon_1\|_{\mathcal{H}^1} \|\epsilon_1\|_{L^2} \lesssim b^2.
\]

(5.37)

\[
\|\Psi_1\|_{L^2} \lesssim b^3.
\]

Here, we recall from (5.29) that the $X$-norm is given by $\|f\|_X = ||y||^{-1} ||\log y|| f \|_{L^2}$.

We note that (5.33) follows from combining (5.31) - (5.36) with $||y||^{-1} ||\log y|| \|\tilde{Z}_k\|_{L^2} \lesssim M^G$. Henceforth, we focus on proving (5.34) - (5.36).

The estimate (5.34) follows from

\[
\|\epsilon_1\|_X = ||y||^{-2} ||\log y|| \|\epsilon_1\|_{L^2} \lesssim \|\epsilon_1\|_{\mathcal{H}^1} \|\epsilon_1\|_{L^2} \lesssim b^{-2}.
\]

\[\text{The way of estimating this contribution is quite different from the } m \geq 1 \text{ case. When } m \geq 1, \text{ the inner product matrix } (5.33) \text{ has no logarithmic divergence in } M. \text{ Instead, the smallness factor in } M \text{ of (5.29) comes from } A_Q \tilde{Z}_k = 0 \text{ for } k \in \{3, 4, 5\} \text{ and } \|\epsilon_3 \tilde{Z}_k\|_{L^2} \sim \|\epsilon_3\|_{L^2}.
\]

When $m = 0$, the smallness factor in $M$ of (5.29) simply comes from $(\log M)^{\frac{1}{2}} / (\log M)$, where $(\log M)^{\frac{1}{2}}$ and $\log M$ come from $\|\tilde{Z}_k\|_{L^2} \sim (\log M)^{\frac{1}{2}}$ and the inner product matrix (5.32), respectively.
For \((5.35)\), since \(\|\epsilon_2\|_{H^1_k} + \|P_2\|_{H^1_k} \lesssim b^2\), it suffices to show
\[
\|y(y)^{-2}(\log_+ y)^2(A^*_0 - A^*_P) + |A^*_P - A^*_Q|\|_{L^\infty} \lesssim b.
\]
The estimate for \(A^*_0 - A^*_P\) follows from the observation that \(A^*_0 - A^*_P\) is a linear combination of \(\frac{1}{y}A_0[\psi_1, \psi_2]\), where \(\psi_1 \in \{P, \epsilon\}\) and \(\psi_2 = \epsilon\) and the estimate
\[
\|\langle y \rangle^{-2}(\log_+ y)^2 A_0[\psi_1, \psi_2]\|_{L^\infty} \lesssim \|\psi_1\|_{L^2}(\|\langle y \rangle^{-1} - 2\epsilon\|_{L^2} \lesssim \|\epsilon\|_{H^1_k}\|\epsilon\|_{L^2} \lesssim b^{\frac{3}{4}}.
\]
The estimate for \(A^*_P - A^*_Q = -\frac{1}{y}(A_0[P] - A_0[Q])\) follows from \((5.33)\). The estimate \((5.36)\) follows from
\[
\|\int_0^y \text{Re}(\overline{w_1} dy')\epsilon_1\|_X \lesssim \int_0^y \text{Re}(\overline{w_1} dy')\|\epsilon_1\|_X
\leq \|w\|_{L^2}\|y^{-1}w_1\|_{L^2}\|\epsilon_1\|_{H^1_k}\|\epsilon_1\|_{L^2}^{\frac{3}{2}} \lesssim b^{-3}.
\]
and
\[
\|\langle f^y \rangle \text{Re}(\overline{w_1} - TP_1 dy')P_1\|_X
\leq \|\langle (\log_+ y)^2 P_1 \|_{L^2}\|y^{-1}(P\epsilon_1 + |\epsilon_1|)\|_{L^1}
\lesssim b^{1-}(\|P\|_{L^2}\|y^{-1}\epsilon_1\|_{L^2} + \|\langle y \rangle^{-2}\epsilon_1\|_{L^2} + \|\langle y \rangle^{-3}\|_{L^2}) \lesssim b^{-3}.
\]
Finally, the claim \((5.37)\) is proved in \((1.28)\). Thus the claims \((5.34)-(5.37)\) and hence \((5.30)\) are proved.

To complete the proof, we use the structures of the matrices \((5.28)\) and \((5.32)\) and the logarithmic divergence \((5.6)\) to find
\[
|\frac{A}{\lambda} + b| + |\gamma_0 + \eta| \lesssim \frac{1}{\log M}(5.29) + M^C b(5.33),
\]
\[
|b_s + b^2 + \eta^2 + c_0(b^2 - \eta^2)| + |\eta_s + 2c_0 b\eta| \lesssim \frac{1}{\log M}(5.33) + M^C b(5.29).
\]
Substituting the claims finishes the proof. □

The estimates \((5.24)\) and \((5.25)\) suffice to close our bootstrap procedure and derive finite-time blow up. However, these do not suffice to derive the sharp blow-up rates. Substituting \(\|\epsilon_3\|_{L^2} \leq K \frac{b^2}{\log |t|}\), the estimate \((5.29)\) would only yield
\[
|b_s + b^2 + \frac{\delta^2}{\log |t|}| \lesssim K \sqrt{\frac{\delta^2}{\log |t|}},
\]
which would not be enough to determine the precise coefficient of \(\frac{b^2}{\log |t|}\). The sharp blow-up rate depends on the coefficient of \(\frac{b^2}{\log |t|}\).

To overcome this issue, we note that the estimates are saturated by the contribution of \((iA^*_Q \epsilon_2, \tilde{Z}_k)\). To make this term smaller, we test \((5.30)\) against better approximations of the kernel elements \(yQ, iyQ\) of \(A_Q\) instead of \(\tilde{Z}_k\) \((k = 3, 4)\). With this correction, we improve the bound \(\frac{1}{\sqrt{\log |t|}}\|\epsilon_3\|_{L^2}\) of \((5.25)\) by a logarithmic factor \(\sqrt{\log |t|}\). From this, we can see that the sharp coefficient of \(\frac{b^2}{\log |t|}\) is 2.

The same argument was previously used in \((5.33)\).

For a small universal constant \(\delta > 0\) we introduce
\[
B_3 = b^{-\delta}, \quad \tilde{Z}_{3, \delta} = yQ\chi_{B_3}, \quad \tilde{Z}_{4, \delta} = iyQ\chi_{B_3}.
\]
The refined modulation estimates will be derived from differentiating \((\epsilon_1, \tilde{Z}_{k, \delta})\). We remark that we do not use \((\epsilon_1, \tilde{Z}_{k, \delta})\) as orthogonal conditions from the beginning. If \((\epsilon_1, \tilde{Z}_{k, \delta})\) were used, then the implicit constants of the coercivity relations would depend on \(b\) and create serious complications.

\textsuperscript{24}For \(C\) used in \(M^C\) bounds, \(\delta = \frac{1}{\log b}\) suffices.
Lemma 5.8 (Refined modulation estimates for \(b\) and \(\eta\)). Define
\[
\tilde{b} := b - \frac{(\epsilon_1, \tilde{Z}_{k,\delta})_r}{(\frac{1}{2} y Q, y Q \chi B_3)_r} \quad \text{and} \quad \tilde{\eta} := \frac{(\epsilon_1, \tilde{Z}_{k,\delta})_r}{(\frac{1}{2} y Q, y Q \chi B_3)_r}.
\]
Then,
\[
\tilde{b} - b + |\tilde{\eta} - \eta| \lesssim b^{2-C_5},
\]
(5.38)
\[
|\tilde{b}_s + b^2 + \eta^2 + c_b(b^2 - \eta^2)| + |\tilde{\eta}_s + 2c_b\eta| \lesssim \frac{1}{|\log b|} \| \epsilon_3 \|_{L^2} + b^{3-C_5}.
\]
(5.39)
In particular,
\[
|\tilde{b}_s + b^2 + \frac{\eta^2}{|\log b|} + |\tilde{\eta}| \lesssim \frac{b^3}{|\log b|^2}.
\]
(5.40)

Proof. In the following, we will compute \((\partial_x \epsilon_1, \tilde{Z}_{k,\delta})_r\). We take the inner product of (5.30) and \(\tilde{Z}_{k,\delta}\) to obtain a variant of (5.31):
\[
\sum_{j=1}^4 \left\{ \left( (\nabla_1)_{j}, \tilde{Z}_{k,\delta} \right)_r + O(b^{-C_5} \| \epsilon_1 \|_{H^1}^2) \right\} \tilde{\text{Mod}}_k = \left( \partial_x \epsilon_1, \tilde{Z}_{k,\delta} \right)_r + (iA^*_Q \epsilon_2, \tilde{Z}_{k,\delta})_r
\]
+ \((b\Lambda - \epsilon_1 - \eta \epsilon_1, \tilde{Z}_{k,\delta})_r + ((iA^*_w - iA^*_p)w_2 + (iA^*_p - iA^*_Q)\epsilon_2, \tilde{Z}_{k,\delta})_r
\]
- \((\int_0^y \text{Re}(\bar{\eta} w_1) dy')\epsilon_1 + (\int_0^y \text{Re}(\bar{\eta} P_1)) dy')i P_1, \tilde{Z}_{k,\delta})_r + (i \Psi, \tilde{Z}_{k,\delta})_r.
\]

We remark that there is an additional term \((\partial_x \epsilon_1, \tilde{Z}_{k,\delta})_r\) on the RHS of (5.41). The matrices on the RHS of (5.41) satisfies (c.f. (5.32))
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & (\frac{1}{2} y Q, y Q \chi B_3)_r & 0 \\
0 & 0 & 0 & (\frac{1}{2} y Q, y Q \chi B_3)_r
\end{pmatrix} + O(b^{-C_5}).
\]
(5.42)

For the terms on the RHS of (5.41), estimates are very similar to those in Lemma 5.7 with replacing \(M\) by \(B_3\). We use \(\|Z_{k,\delta}\|_{L^2} \lesssim \sqrt{\log B_3}\) and \(\|(y)^2 \log (y+1)\|_{L^2} \lesssim b^{-C_5}\), and follow the proof of (5.35) to obtain
\[
\text{RHS of (5.41)} = (\partial_x \epsilon_1, \tilde{Z}_{k,\delta})_r + O(\sqrt{\log B_3} \| \epsilon_3 \|_{L^2} + b^{3-C_5}).
\]
(5.43)
Summing up (5.42) and (5.43), and then applying the previous modulation estimates (Lemma 5.7) to treat the term \(O(b^{1-C_5} \text{ Mod})\), we arrive at
\[
\text{Mod}_k = \left( \partial_x \epsilon_1, \tilde{Z}_{k,\delta} \right)_r + O\left( \frac{1}{\sqrt{\log B_3}} \| \epsilon_3 \|_{L^2} + b^{3-C_5} \right).
\]
(5.44)

for \(k \in \{3,4\}\).

We now differentiate \((\partial_x \epsilon_1, \tilde{Z}_{k,\delta})_r\), by parts:
\[
\left| \left( \partial_x \epsilon_1, \tilde{Z}_{k,\delta} \right)_r - \partial_x \left( \frac{(\epsilon_1, \tilde{Z}_{k,\delta})_r}{(\frac{1}{2} y Q, y Q \chi B_3)_r} \right) \right| \leq \left| \left( \partial_x \epsilon_1, \tilde{Z}_{k,\delta} \right)_r \right| + \left| \frac{(\epsilon_1, \tilde{Z}_{k,\delta})_r}{(\frac{1}{2} y Q, y Q \chi B_3)_r} \right| \lesssim \left| \left( \partial_x \epsilon_1, \tilde{Z}_{k,\delta} \right)_r \right| \lesssim b^{3-C_5}.
\]
(5.45)

Using \(|b_s \partial_x B_3| \lesssim b \| b \partial_x B_3 \| \lesssim b_1\|_{H^1(B_2 \chi B_3)}\), we obtain
\[
\left| \left( \partial_x \epsilon_1, \tilde{Z}_{k,\delta} \right)_r - \partial_s \left( \frac{(\epsilon_1, \tilde{Z}_{k,\delta})_r}{(\frac{1}{2} y Q, y Q \chi B_3)_r} \right) \right| \lesssim b^{3-C_5}.
\]

The definitions of \(\tilde{b}\) and \(\tilde{\eta}\) are motivated in view of (5.44) and (5.45), and the estimates (5.38), (5.39), and (5.40) are immediate. \(\square\)
5.5. Energy estimate in $\mathcal{H}^3_0$. In this subsection, we propagate the control of $\epsilon$ forward-in-time. The main idea is the energy method in higher derivatives with repulsivity. More precisely, we proceed to higher order derivatives by adapted derivatives, say $\epsilon_k$. We then apply the energy method with correction terms. The correction terms are designed to exploit the repulsivity observed in the variable $\epsilon_2$. Such an idea appeared in [33,38,41] in the context of wave maps and Schrödinger maps.

We will apply the energy method to $\epsilon_2$ with the energy functional $\|A_Q^*\epsilon_2\|_{L^2}^2 = \|\epsilon_2\|_{H^2}^2$. Indeed, we need to work at least in the $H^3$-level due to scaling reasons. More precisely, as we are in the situation $\lambda \sim b|\log b|^2$ (which is dictated by the formal parameter law (4.10)), we can expect at best $\|\epsilon_k\|_{L^2} \lesssim \lambda^k \sim b^k|\log b|^{2k}$. In order to guarantee the modulation equation $b_s + b^2 + \frac{2b^2}{|\log b|} \approx 0$, we need $k > 2$ in view of Lemma 5.7. On the other hand, when $k = 3$, a toy model

$$(\partial_s - \frac{\lambda}{\lambda} \Lambda_{-3}) \epsilon_3 \approx -iA_Q^*\Psi_2$$

implies

$$(\partial_s - 3\frac{\lambda}{\lambda})\|\epsilon_3\|_{L^2} \lesssim \|A_Q^*\Psi_2\|_{L^2} \lesssim \frac{b^0}{|\log b|^3}$$

by (1.30). Integrating this loses $b$, which yields $\|\epsilon_3\|_{L^2} \lesssim \frac{b^2}{|\log b|^3}$.

In view of Lemma 5.7, this bound suffices to guarantee the modulation equation $b_s + b^2 + \frac{2b^2}{|\log b|} \approx 0$. Moreover, this motivates the bootstrap hypothesis for $\|\epsilon_3\|_{L^2}$.

In the energy estimate, there appear two non-perturbative contributions in $\frac{1}{2}(\partial_s - 6\frac{\lambda}{\lambda})\|\epsilon_3\|_{L^2}^2$. One is from the commutator of the scaling operator $\Lambda_{-3}$ and $A_Q^*$ acting on $\epsilon_3$. In the energy estimate, we will see that this contribution has the good (negative) sign, thanks to the repulsivity (4.3) of the operator $A_Q^*A_Q^*$, i.e.

$-\partial_\lambda(A_Q^*A_Q^*) = \frac{yQ_{\lambda}}{\lambda} \leq 0$ where $Q_{\lambda} = \lambda^{-1}Q(\lambda^{-1})$. Another non-perturbative contribution comes from the cubic nonlinearity. This will be treated by both a Morawetz correction and the above repulsivity.

We start by deriving the equation for $\epsilon_2$. Recall (2.10) and (4.19):

$$(\partial_s - \frac{\lambda}{\lambda} \Lambda_{-2} + \bar{y}_s i)w_2 + iA_wA_w^*w_2 - \left(\int_0^y \text{Re}(\bar{w}w_2)dy\right)iw_2 - i\bar{w}w_2^2 = 0,$$

$$(\partial_s - \frac{\lambda}{\lambda} \Lambda_{-2} + \bar{y}_s i)P_2 + iA_PA_P^*P_2 - \left(\int_0^y \text{Re}(\bar{P}P_2)dy\right)iP_2 - i\bar{P}P_2^2 = -\text{Mod} \cdot v_2 + i\Psi_2.$$

Subtracting the second from the first and using the identity

$$iA_wA_w^*w_2 - iA_PA_P^*P_2 = iA_Q^*A_Q^*\epsilon_2 + (iA_wA_w^* - iA_PA_P^*)w_2 + (iA_PA_P^* - iA_Q^*A_Q^*)\epsilon_2,$$

we obtain the equation for $\epsilon_2$:

$$(\partial_s - \frac{\lambda}{\lambda} \Lambda_{-2} + \bar{y}_s i)\epsilon_2 + iA_Q^*A_Q^*\epsilon_2 - (i\bar{w}w_2^2 - i\bar{P}P_2^2)$$

$$= -(iA_wA_w^* - iA_PA_P^*)w_2 - (iA_PA_P^* - iA_Q^*A_Q^*)\epsilon_2$$

$$+ \left(\int_0^y \text{Re}(\bar{w}w_1 - \bar{P}P_1)dy\right)iw_2 + \left(\int_0^y \text{Re}(\bar{P}P_1)dy\right)i\epsilon_2 + \text{Mod} \cdot v_2 - i\Psi_2.$$

Here we wrote the cubic difference term $-(i\bar{w}w_2^2 - i\bar{P}P_2^2)$ on the LHS, because it is a non-perturbative term. This term will be handled using a Morawetz correction. All the terms on the RHS are perturbative.
Lemma 5.9 (Energy identity of $\epsilon_3$). We have

$$
\frac{1}{2}(\partial_s - 6\frac{\Lambda}{\sqrt{y}})\|\epsilon_3\|_{L^2}^2 = b(\epsilon_3, \frac{y}{2}Q^2\epsilon_2 + A_Q^*(yQ^2\epsilon_1))_r \\
+ b\|\epsilon_3\|_{L^2} \cdot O\left(\frac{1}{\sqrt{\log M}}\|\epsilon_3\|_{L^2} + \frac{b^2}{|\log \eta|}\right).
$$

Remark 5.10. We remind the reader that the relations between $\epsilon$, $\epsilon_1$, and $\epsilon_2$ are highly nonlinear. If one were to proceed to higher order derivatives in a linear fashion, e.g. $\epsilon_2 = A_QL\epsilon$, then one would encounter a lot of non-perturbative errors $O(b^2)$ in the energy identity. Such errors would contain nonlocal expressions from $A_0$ or $A_1$, thus it would be very difficult to find correction terms. However, as we proceed with nonlinear adapted derivatives, we are able to take advantages from the degeneracies $P_1 = O(b)$ and $P_2 = O(b^2)$ to simplify the non-perturbative terms significantly. In this sense, we believe that using nonlinear adapted derivatives is more efficient and describes the blow-up regime more precisely than using the linear ones.

Remark 5.11. When $m \geq 1$, the situation is simpler than here. In that case, the authors in [21] were able to close the argument using linear adapted derivatives. This is mainly due to the stronger repulsivity of $A_QA_Q^*$ and better decay of $Q$. The stronger repulsivity enables (a localized version of) the monotonicity from the virial functional $(\epsilon_2, iA_2)r$, see [21] (2.8) and (5.36)]. Moreover, thanks to the better decay of $Q$, many nonlocal contributions of size $O(b^2)$ can in fact be estimated by some local norms of $\epsilon$. See [21] (5.33) and Lemma 5.1.

In contrast, the case $m = 0$ has serious problems from the slower decay of $Q$ and weaker repulsivity of $A_QA_Q^*$. In fact, $A_QA_Q^* \approx -\Delta_0$ near the spatial infinity, as the potential $\frac{\Lambda}{\sqrt{y}}$ decays faster than $\frac{\Lambda}{y}$. Thus the argument using a localized virial functional as in [21] meets a serious difficulty from the fact that $-\Delta_0$ (on 2D) has zero resonance. Thus we do not rely on the virial functional in this paper, but rather construct a precise correction term to handle non-perturbative terms. To find such corrections, it is also crucial to proceed with nonlinear adapted derivatives, to simplify the structure of non-perturbative terms significantly.

Proof. As before, in this proof, we freely use the bootstrap hypotheses (5.11), as well as Lemmas 5.6, A.14, and A.15 to estimate $\epsilon$, $\epsilon_1$, and $\epsilon_2$. We also abuse the notation and identify the operator $A_wA_w^* - A_w^*A_w^*$, which is simply the multiplication by a function (namely, the difference of the zeroth order terms), with that function.

The equation for $\epsilon_3 = A_Q^*\epsilon_2$ is given as

$$
(\partial_s - 6\frac{\Lambda}{\sqrt{y}} + \tilde{\gamma}_s + \frac{y}{2})\epsilon_3 + iA_Q^*A_Q\epsilon_3 \\
= \frac{\Lambda}{y}((\partial_s A_Q^*)\epsilon_2 + A_Q^*(iy\bar{w}^2 - i\overline{\mathcal{P}}\mathcal{P}^2)) + A_Q^*(\text{RHS of (5.46)}).
$$

As opposed to $\epsilon_1$ or $\epsilon_2$, we take a linear adapted derivative to get $\epsilon_3$. Taking the inner product with $\epsilon_3$, we have the energy identity

$$
\frac{1}{2}(\partial_s - 6\frac{\Lambda}{\sqrt{y}})\|\epsilon_3\|_{L^2}^2 = \frac{\Lambda}{y}(\epsilon_3, (\partial_s A_Q^*)\epsilon_2)_r + (\epsilon_3, A_Q^*(iy\bar{w}^2 - i\overline{\mathcal{P}}\mathcal{P}^2))_r \\
+ \|\epsilon_3\|_{L^2} \cdot O(\|\text{RHS of (5.46)}\|_{H^1}).
$$

The first and second terms have non-perturbative contributions. For the first term, using $\partial_s A_Q^* = -\frac{\Lambda}{\sqrt{y}}Q^2$ and the modulation estimate,

$$
\frac{\Lambda}{y}(\epsilon_3, (\partial_s A_Q^*)\epsilon_2)_r = b(\epsilon_3, \frac{y}{2}Q^2\epsilon_2)_r + O(b^2)\|\epsilon_3\|_{L^2}^2.
$$

For the second term, we first write

$$
i\bar{w}\bar{w}^2 - i\overline{\mathcal{P}}\mathcal{P}^2 = byQ^2\epsilon_1 + (2i\overline{\mathcal{P}}\mathcal{P} - byQ^2)\epsilon_1 + i\overline{\mathcal{P}}\mathcal{P}\epsilon_1^2 + i\bar{w}\bar{w}^2.
$$
We keep $byQ^2\epsilon_1$ and estimate the rest: (we use the weighted $L^\infty$-bounds from Lemma A.13 for $L^\infty$ terms)
\[
\|(2i\mathcal{T}P_1 - byQ^2)\epsilon_1\|_{\dot{H}^1_x} \lesssim \|(|y|yQ^2 + b^2y^3Q^2)|\epsilon_1|_{L^2} \lesssim \frac{b}{\log b} \|\epsilon_1\|_{\dot{H}^1_x} \lesssim_M b^3 \frac{K}{\log b} \lesssim \frac{b^3}{\log b},
\]
\[
\|\mathcal{T}\epsilon_1\|_{\dot{H}^1_x} \lesssim \|\langle y \rangle^{-2}\epsilon_1\|_{L^2} \lesssim \|\langle y \rangle^{-1}\epsilon_1\|_{L^\infty} \|\langle y \rangle^{-1}|\epsilon_1\|_{L^2} \lesssim b^4, 
\]
\[
\|\mathcal{T}(w_1^2 - \epsilon_1^2)\|_{\dot{H}^1_x} \lesssim \|P_1|\mathcal{T}(w_1 + \epsilon_1)\|_{L^2} \lesssim b^1 \|\langle y \rangle^{-1}\mathcal{T}(P_1 + 2\epsilon_1)\|_{L^2} \lesssim b^1 \|\langle y \rangle^{-2}\partial_y\epsilon\|_{L^2} \|\langle y \rangle P_1\|_{L^\infty} + \|\langle y \rangle^{-2}\epsilon\|_{L^2} \|\langle y \rangle\|_{L^\infty} |\epsilon_1|_{L^2} \lesssim b^4, 
\]
\[
\|\mathcal{T}\epsilon_1^2\|_{\dot{H}^1_x} \lesssim \|\partial_y\epsilon\|_{L^2}|\epsilon_1|_{\dot{H}^1_x} + \|\epsilon\|_{L^\infty} \|\langle y \rangle\|_{L^\infty} \|\langle y \rangle\|_{L^\infty} \|\epsilon\|_{L^\infty} |\epsilon_1|_{L^2} \lesssim b^4.
\]
Therefore, we have
\[
(\epsilon_3, A_Q^2(\mathcal{T}w_1^2 - i\mathcal{T}P_1^2))_2 = b(\epsilon_3, A_Q^2(yQ^2\epsilon_1))_2 + b|\epsilon_3|_{L^2} \cdot O(\frac{b^3}{\log b}).
\]
The remaining terms are all treated as errors; we claim
(5.48) \[
\|\text{RHS of } 5.40 \|_{\dot{H}^1_x} \lesssim \frac{b}{\sqrt{\log M}} \|\epsilon_3\|_{L^2} + \frac{b^3}{\sqrt{\log b}}.
\]
In fact, we will see that $\frac{b}{\sqrt{\log M}} \|\epsilon_3\|_{L^2}$ is saturated by the modulation term and $\frac{b^3}{\sqrt{\log b}}$ is saturated by $\Psi_2$.

Firstly, we show
\[
\|(A_wA_{w} - A_P A_P^*)w_2\|_{\dot{H}^1_x} \lesssim b^4. 
\]
We first note that
\[
\|(A_w A_w^* - A_P A_P^*)w_2\|_{\dot{H}^1_x} \lesssim \|A_w A_w^* - A_P A_P^*\|_{L^\infty} |w_2|_{\dot{H}^1_x} + \|\partial_y(A_w A_w^* - A_P A_P^*)\|_{L^2} |w_2|_{L^\infty} \lesssim b^2 \|A_w A_w^* - A_P A_P^*\|_{L^2} + b^2 \|\partial_y(A_w A_w^* - A_P A_P^*)\|_{L^2}.
\]
Recall that
\[
A_w A_w^* = -\partial_{yy} - \frac{b}{y^2} \partial_y + \frac{b^2}{y^2}((2 + A_\theta[w])^2 + \frac{b^2}{y^2} |w|^2).
\]
Thus
\[
\|A_w A_w^* - A_P A_P^*\|_{L^\infty} \lesssim \|w^2 - |P|^2\|_{L^\infty} \lesssim \|\langle y \rangle^{-2}\epsilon\|_{L^\infty} + \|\epsilon\|_{L^\infty}^2 \lesssim b^2.
\]
On the other hand, using $|w| \lesssim 1$ and $|A_\theta[w]| + |A_\theta[P]| \lesssim y^2 \langle y \rangle^{-2}$, we have the pointwise bound
\[
|\partial_y(A_w A_w^* - A_P A_P^*)| \lesssim \frac{1}{y^2} |A_\theta[w]| + \frac{b^2}{y^2} |w|^2 - |P|^2| + |P_\partial_y w - \mathcal{T}\partial_y P|.
\]
We estimate the $L^2$ norms by
\[
\|\frac{1}{y^2} |A_\theta[w]| + \frac{b^2}{y^2} |w|^2 - |P|^2\|_{L^2} \lesssim \|\langle y \rangle^{-2}\epsilon\|_{L^2} + \|\epsilon\|_{L^2} \lesssim b^2
\]
and
\[
\|P_\partial_y w - \mathcal{T}\partial_y P\|_{L^2} \lesssim \|\langle y \rangle^{-2}\epsilon\|_{L^2} + \|\epsilon\|_{L^2} \lesssim b^2.
\]
Next, we show
\[
\|(A_P A_P^* - A_Q A_Q^*)\|_{\dot{H}^1_x} \lesssim \frac{b^3}{\log b} \|\epsilon_3\|_{L^2}.
\]
This follows from
\[ \|(A_P A_P^* - A_Q A_Q^*) \epsilon_2\|_{L^2} \lesssim \|\log |y|\| A_P A_P^* - A_Q A_Q^*\|_{L^\infty} \|\epsilon_2\|_{L^2} \lesssim \frac{b}{|\log |y||} \|\epsilon_3\|_{L^2}. \]

Note that \( \frac{b}{|\log |y||} \) comes from \( \|P|^2 - Q^2\| \lesssim 1_{(0,2B)}(|\eta|Q + b^2 y^2Q)\).

Next, we show
\[ \|(\int_0^y \text{Re}(\overline{w} w_1 - TP_1) dy') i w_2\|_{L^2} \lesssim b^{5-}. \]

If \( \partial_y \) does not hit the integral term, we estimate this by (using the estimates shown in the proof of \( (5.26) \))
\[ \|\frac{b}{y} \text{Re}(\overline{w} w_1 - TP_1)\|_{L^1} \|w_2\|_{L^2} \lesssim b^{2-} \|w_2\|_{L^2} \lesssim b^{4-}. \]

If \( \partial_y \) hits the integral term, we would like to put \( w_2 \in L^\infty \), but here we have a technical problem that \( \epsilon_2 \notin L^\infty \). Instead, we put \( w_2 \) in \( L^\infty \) using \( (5.17) \)
\[ \|w_2\|_{L^\infty} \lesssim \|w_2\|_{L^2} \|\partial_y w_2\|_{L^2} \lesssim b^{2-}. \]

Thus we estimate this contribution as
\[ \|\text{Re}(\overline{w} w_1 - TP_1) i w_2\|_{L^2} \lesssim (b) (y)^{-1} \|\epsilon\|_{L^2} + \|\epsilon\|_{L^2} + \|\epsilon\|_{L^\infty} \|\epsilon_2\|_{L^2} \|w_2\|_{L^\infty} \lesssim b^{4-}. \]

Next, it is easy to see that
\[ \|\text{Re}(TP_1) i \epsilon_2\|_{L^2} \lesssim \frac{b}{|\log |y||} \|\epsilon_2\|_{L^2} \lesssim \frac{b}{|\log |y||} \|\epsilon_3\|_{L^2}. \]

Next, by the modulation estimates (Lemma \( 5.7 \)) and cancellation estimates \( (5.21) \), we have
\[ \|\text{Mod} \cdot v_2\|_{L^2} \lesssim b(\frac{1}{\sqrt{\log |y|}} \|\epsilon_3\|_{L^2} + b^{3-}). \]

Lastly, we use the sharp energy estimate \( (4.30) \):
\[ \|\Psi_2\|_{L^2} \lesssim \frac{b}{|\log |y||}. \]

This completes the proof. \( \Box \)

We now aim to handle the non-perturbative contribution \( b(\epsilon_3, \frac{y}{2} Q^2 \epsilon_2 + A_Q^*(y Q^2 \epsilon_1)) \).

To motivate this, we write
\[ b(\epsilon_3, \frac{y}{2} Q^2 \epsilon_2 + A_Q^*(y Q^2 \epsilon_1)) = 3b(\epsilon_3, \frac{y}{2} Q^2 \epsilon_2) + b((A_Q \epsilon_3, y Q^2 \epsilon_1) - (y Q^2 \epsilon_2, A_Q^* \epsilon_2)). \]

The first term is \emph{nonpositive}, thanks to the repulsivity:
\[ (\epsilon_3, y Q^2 \epsilon_2) = -2(A_Q^* \epsilon_2, (\partial_t A_Q^*) \epsilon_2), \]
\[ = -(\epsilon_2, \partial_t (A_Q^* A_Q^*) \epsilon_2) = (\epsilon_2, \frac{\omega_0}{y} \epsilon_2) \leq 0. \]

The second term can be deleted by a \emph{Morawetz correction}:
\[ (A_Q \epsilon_3, y Q^2 \epsilon_1) - (y Q^2 \epsilon_2, A_Q^* \epsilon_2) \approx \partial_t (i \epsilon_2, y Q^2 \epsilon_1), \]
from \( i \partial_t \epsilon_2 \approx A_Q \epsilon_3 \) and \( i \partial_t \epsilon_1 \approx A_Q^* \epsilon_2 \). More precisely, we have the following.

\textbf{Lemma 5.12} (Morawetz correction). We have
\[ |b(i \epsilon_2, y Q^2 \epsilon_1)| \lesssim b^{5-}, \]
\[ (\partial_t - 6 \frac{b}{|\log |y||}) \{ b(i \epsilon_2, y Q^2 \epsilon_1) \} = b(\epsilon_3, A_Q^*(y Q^2 \epsilon_1) - y Q^2 \epsilon_2) + O(\frac{b}{\sqrt{\log |y|}} \|\epsilon_3\|_{L^2} + b^{5-}). \]
Proof. The first bound \(5.50\) is immediate from \(5.44\) and the bootstrap hypothesis:

\[ b(i\epsilon_2, yQ^2\epsilon_1), \lesssim b\|\epsilon_2\|_{\tilde{H}_2^1}(y)^{-2}\langle \log_+ y \rangle \epsilon_1 \|L^2 \lesssim b^3 \|\epsilon_3\|_{L^2} \lesssim b^5. \]

We turn to the derivative estimate \(5.51\). We compute

\begin{equation}
(\partial_s - \frac{\Delta}{2})(b(i\epsilon_2, yQ^2\epsilon_1)), \tag{5.52}
\end{equation}

\[ = b(AQ\epsilon_3, yQ^2\epsilon_1), - b(yQ^2\epsilon_2, A_Q^*\epsilon_2), + (b_s - 6\frac{\Delta}{2}b)(i\epsilon_2, yQ^2\epsilon_1), \]

\[ + b(i\partial_s\epsilon_2 - AQ\epsilon_3, yQ^2\epsilon_1), - b(yQ^2\epsilon_2, i\partial_s\epsilon_1 - A_Q^*\epsilon_2). \]

As illustrated in the above, the first two terms of \(5.52\) are the desired corrections:

\[ b(AQ\epsilon_3, yQ^2\epsilon_1), - b(yQ^2\epsilon_2, A_Q^*\epsilon_2), = \epsilon_3, A_Q^*(yQ^2\epsilon_1) - yQ^2\epsilon_2),. \]

The remaining terms of \(5.52\) are all treated as errors. The third term is easily estimated by

\[ \|(b_s - 6\frac{\Delta}{2}b)(i\epsilon_2, yQ^2\epsilon_1), \| \lesssim b^5\|\epsilon_2, yQ^2\epsilon_1,\| \lesssim b^6. \]

For the fourth term, by the estimate

\[ \|b(i\partial_s\epsilon_2 - AQ\epsilon_3, yQ^2\epsilon_1), \| \lesssim b\|y^{-1}(\log_+ y)^{-1}(i\partial_s\epsilon_2 - AQ\epsilon_3)\|L^2\|\epsilon_1\|L^2, \]

\[ \lesssim b^3\|y^{-1}(\log_+ y)^{-1}(i\partial_s\epsilon_2 - AQ\epsilon_3)\|L^2, \]

it suffices to prove

\[ \|y^{-1}(\log_+ y)^{-1}(\partial_s\epsilon_2 + iAQ\epsilon_3)\|L^2 \lesssim b^3. \]

To show this, rewrite the equation \(5.46\) as

\[ \partial_s\epsilon_2 + iAQ\epsilon_3 = \frac{\Delta}{2} \Delta_2\epsilon_2 - \tilde{\gamma}_s i\epsilon_2 + (\eta\sigma + i\tilde{\mathcal{P}}^2) + (\text{RHS of }5.46). \]

It only suffices to estimate the first three terms on the RHS above, because we know from the proof of energy estimate \(5.48\) that \(\|\text{RHS of }5.46\|_{\tilde{H}_2^1} \lesssim \frac{b^3}{\log y^3}\). We now estimate

\[ \|y^{-1}(\log_+ y)^{-1}(\Delta_2\epsilon_2 - \tilde{\gamma}_s i\epsilon_2)\|L^2 \lesssim b\|\epsilon_2\|_{\tilde{H}_2^1} \lesssim b^3. \]

Next, from the proof of \(5.48\)

\[ \|y^{-1}(\log_+ y)^{-1}(\eta\sigma + i\tilde{\mathcal{P}}^2)\|L^2 \lesssim \|y^{-1}(\log_+ y)^{-1}byQ^2\|L^2 + \frac{\|y\|}{\log y} \lesssim b\|\epsilon_1\|_{\tilde{H}_2^1} + \frac{b^3}{\log y} \lesssim b^3. \]

Finally, the last term of \(5.52\) can be estimated by

\[ \|b(yQ^2\epsilon_2, i\partial_s\epsilon_1 - A_Q^*\epsilon_2)\| \lesssim b\|\epsilon_2\|_{\tilde{H}_2^1}\langle \log_+ y \rangle\langle \partial_s\epsilon_1 + iA_Q^*\epsilon_2 \rangle \|L^2 \lesssim b\|\epsilon_2\|_{L^2}\|\partial_s\epsilon_1 + iA_Q^*\epsilon_2 \|L^2, \]

so it suffices to establish the bound

\[ \|\partial_s\epsilon_1 + iA_Q^*\epsilon_2 \|L^2 \lesssim \frac{1}{\sqrt{\log y}}\|\epsilon_3\|_{L^2} + b^3. \]

To show this, rewrite the equation \(5.23\) of \(\epsilon_1\) as

\[ \partial_s\epsilon_1 + iA_Q^*\epsilon_2 = \frac{\Delta}{2} \Delta_1\epsilon_1 - \tilde{\gamma}_1 i\epsilon_1 - (iA_Q^* - iA_Q^*)\sigma_2 - (iA_Q - iA_Q^*)\epsilon_2 \]

\[ + (\int_0^y \text{Re}(\sigma_1 dy')i\epsilon_1 + (\int_0^y \text{Re}(\sigma_1 - \text{Re}(\tilde{\mathcal{P}})) dy')iP_1 \]

\[ + \text{Mod} \cdot v_1 - i\tilde{\Psi}_1. \]

Recall that all terms except \(\text{Mod} \cdot v_1\) on the RHS are already estimated in the proof of the modulation estimates; see \(5.34\) - \(5.37\). Thus these terms contribute to the
error $O(b^{3-})$. The term $\widetilde{\text{Mod}} \cdot v_1$ can be estimated by the modulation estimates (Lemma 5.7) and estimates for $v_1$ (4.20):

$$\|\widetilde{\text{Mod}} \cdot v_1\|_X \lesssim \frac{1}{\sqrt{\log M}} \|\epsilon_3\|_{L^2} + b^{3-}.$$ 

This completes the proof. □

Define the modified third energy by

$$\mathcal{F}_3 := \frac{1}{2}\|\epsilon_3\|_{L^2}^2 - b(i\epsilon_2, yQ^2\epsilon_1)_r.$$ 

**Proposition 5.13** (The modified energy inequality). We have

$$|\mathcal{F}_3 - \frac{1}{2}\|\epsilon_3\|_{L^2}^2| \leq b^{5-}$$

(5.53)

$$\left(\partial_s - \frac{6\lambda}{\sqrt{b}}\right)\mathcal{F}_3 \leq b(\frac{1}{\sqrt{\log M}} \|\epsilon_3\|_{L^2} + C_{\frac{b^4}{\log b^2}}),$$

(5.54)

where $C$ is some universal constant.

**Proof.** The coercivity (5.53) follow from (5.50). For the monotonicity (5.54), we combine (5.47) and (5.49) to have

$$|\left(\partial_s - \frac{6\lambda}{\sqrt{b}}\right)\mathcal{F}_3 - \frac{1}{2}b(\epsilon_3, yQ^2\epsilon_2)_r|$$

$$\leq Cb\left(\frac{1}{\sqrt{\log M}} \|\epsilon_3\|_{L^2}^2 + \frac{b^7}{2} \|\epsilon_3\|_{L^2}^2 + \frac{b^7}{2} \|\epsilon_3\|_{L^2}^2 + C_{\frac{b^4}{\log b^2}}\right)$$

$$\leq b\left(\left(\frac{C}{\sqrt{\log M}} + \frac{1}{\sqrt{B}}\right) \|\epsilon_3\|_{L^2}^2 + \frac{C_{\frac{b^4}{\log b^2}}}{\sqrt{B}}\right).$$

By the repulsivity (5.49) and $M \gg 1$, we have

$$\left(\partial_s - \frac{6\lambda}{\sqrt{b}}\right)\mathcal{F}_3 \leq b(\frac{1}{\sqrt{\log M}} \|\epsilon_3\|_{L^2} + C_{\frac{b^4}{\log b^2}}).$$

This completes the proof. □

### 5.6. Proofs of Propositions 5.3, 5.4, and 5.5

In this last subsection, we finish the proofs of Propositions 5.3, 5.4, and 5.5. The arguments here are very similar to the Schrödinger map case [33]. We include the proofs for the sake of completeness. We note that there are some simplifications in our case, thanks to the conservation of mass and energy.

**Lemma 5.14** (Consequences of modulation estimates). We have

$$\int_0^t \frac{b}{\lambda^2} \frac{b^4}{\lambda^6 \log b^2} d\tau \leq \frac{b^4(t)}{\lambda^6(t) |\log b(t)|^2} - \frac{1}{2} \frac{b^4(t)}{\lambda^6(t) |\log b(t)|^2},$$

(5.55)

$$\frac{b(t) |\log b(t)|^2}{\lambda(t)} = \left(1 + O\left(\frac{1}{|\log b_0|^2}\right)\right) \frac{b(t) |\log b(t)|^2}{\lambda_0},$$

(5.56)

$$\frac{\lambda(t)}{\lambda_0} \leq \left(\frac{b(t)}{b_0}\right)^\frac{2}{5}.$$ 

(5.57)

**Proof.** The estimate (5.55) follows from $\frac{b}{\lambda^2} = -\frac{b}{\lambda^2} + O\left(\frac{b^2}{\lambda^2}\right)$ and integration by parts:

$$\int_0^t \frac{b}{\lambda^2} \frac{b^4}{\lambda^6 \log b^2} d\tau$$

$$= \frac{1}{6} \left[\frac{b^4}{\lambda^6 \log b^2}\right]_0^t - \frac{4}{6} \int_0^t \frac{b^4 b^3}{\lambda^6 \log b^2} d\tau + O\left(\frac{1}{|\log b^2|}\right) \int_0^t \frac{b}{\lambda^2} \frac{b^4}{\lambda^6 \log b^2} d\tau$$

$$\leq \frac{1}{6} \frac{b^4(t)}{\lambda^6(t) |\log b(t)|^2} + \frac{4}{6} \int_0^t \frac{b}{\lambda^2} \frac{b^4}{\lambda^6 \log b^2} d\tau + O\left(\frac{1}{|\log b^2|}\right) \int_0^t \frac{b}{\lambda^2} \frac{b^4}{\lambda^6 \log b^2} d\tau.$$ 

To show the estimate (5.56), we need the refined modulation estimates (Lemma 5.8). We compute using (5.38), (5.40) and \(|b_s + b^2| + |\tilde{b}_s + \tilde{b}^2| \lesssim \frac{b^2}{\log b}|_{\text{log} b}|^2\) to get
\[
\partial_s \log \left(\frac{\lambda}{b|\log b|^2}\right) = \left(\frac{\lambda_s}{\lambda} + \tilde{b} - \left(\frac{\tilde{b}_s + \tilde{b}^2 + \frac{2\tilde{b}^2}{\log b}}{b}\right) + O\left(\frac{b}{|\log b|^2}\right) = O\left(\frac{b}{|\log b|^2}\right).
\]
Integrating this, we have
\[
\left(\frac{\tilde{b}(t)|\log b(t)|^2}{\lambda(t)}\right)^{-1} \frac{\tilde{b}_0|\log \tilde{b}_0|^2}{\lambda_0} - 1 \leq \int_0^t \frac{b}{\lambda^2} \frac{1}{|\log b|^2} d\tau.
\]
The error term (the RHS) can be estimated using \(\frac{b}{\lambda} = -\frac{b}{\lambda} + O\left(\frac{1}{\log b}\right)\):
\[
\left(1 + O\left(\frac{1}{\log b}\right)\right) \int_0^t \frac{b}{\lambda^2} \frac{1}{|\log b|^2} d\tau = -\int_0^t \frac{b_s}{b|\log b|^2} d\tau \lesssim \frac{1}{|\log b_0|^2}.
\]
Finally replacing \(\tilde{b}\) by \(b\) using (5.38) completes the proof of (5.56).

The estimate (5.57) follows from
\[
\partial_s \log \left(\frac{\lambda}{b}\right) = \frac{\lambda_s}{3\lambda} + \left(\frac{\lambda}{\lambda} + b\right) - \left(b_s + b^2\right) = -\frac{b}{3} + O\left(\frac{b}{|\log b|}\right) \leq 0.
\]
This completes the proof.

We are now ready to prove the main bootstrap Proposition 5.3.

**Proof of the main bootstrap Proposition 5.3.** Note that \(b(t) \leq b_0\) is immediate from \(b_s \approx -b^2 < 0\).

We first close the \(||\epsilon_3||_{L^2}\)-bound. By the modified energy inequality (Proposition 5.13), we have
\[
\frac{1}{2} \frac{||\epsilon_3(t)||_{L^2}}{|\lambda(t)|} \leq \frac{1}{2} \frac{||\epsilon_3(0)||_{L^2}}{|\lambda_0|} + \frac{b_0^{9/2}}{\lambda^7} + \frac{\int_0^{1/5} K^2 + B}{100} \frac{b^4}{|\log b|^2} d\tau.
\]
Applying the claims (5.38) and (5.37) yields
\[
||\epsilon_3(t)||_{L^2}^2 \leq \left(\frac{b(t)}{b_0}\right)^{\frac{9}{2}} ||\epsilon_3(0)||_{L^2}^2 + \frac{K^2}{50} \frac{b^4(t)}{|\log b(t)|^2}.
\]
Applying the initial bound (1.20) and \(K \gg 1\), this closes the \(||\epsilon_3||_{L^2}\)-bound.

We now close the \(||\epsilon_1||_{L^2}\)-bound. Thanks to the energy conservation, we have
\[
\frac{||w_1(t)||_{L^2}}{|\lambda(t)|} = \frac{||w_1(0)||_{L^2}}{|\lambda_0|}.
\]
Thus we have
\[
||\epsilon_1(t)||_{L^2} \leq ||w_1(t)||_{L^2} + ||P_1(t)||_{L^2} \\
\leq \frac{\lambda(t)}{\lambda_0} \left(||\epsilon_1(0)||_{L^2} + Cb_0|\log b_0|^2 + Cb(t)|\log b(t)|\right) \\
\leq \frac{\lambda(t)}{\lambda_0} \left(||\epsilon_1(0)||_{L^2} + b_0|\log b_0|^2 + b(t)|\log b(t)|\right)^2.
\]
Applying the initial bound (1.20), (5.56), and \(K \gg 1\), this closes the \(||\epsilon_1||_{L^2}\)-bound.

We now close the \(||\epsilon||_{L^2}\)-bound. Thanks to the mass conservation,
\[
||w(t)||_{L^2} = ||w(0)||_{L^2}.
\]
Therefore,
$$
\| \epsilon(t) \|_{L^2} \leq \| \epsilon(0) \|_{L^2} + O(b^*). 
$$
Applying the initial bound 1.20 and $b^* \ll 1$, this closes the $\| \epsilon \|_{L^2}$-bound.

We turn to the proof of Proposition 5.5. Let us recall the situation in the proof of Theorem 1.1. For a fixed $(${\tilde{\lambda}}, {\tilde{\gamma}}, {\tilde{b}}, {\tilde{\epsilon}}$)$ \in \mathcal{U}_{init}$, we were considering the one-parameter family of solutions $u^{(\tilde{\eta})}$ starting from the initial data formed by the rough decomposition, i.e. $u_0^{(\tilde{\eta})} = \frac{a^{(\tilde{\eta})}}{\lambda_0} [P(\cdot, \tilde{b}, \tilde{\eta}) + \tilde{\epsilon}]$, $\tilde{\eta} \in (-\frac{b_0}{2 \log b_0}, \frac{b_0}{2 \log b_0})$. Here we added a superscript $(\tilde{\eta})$ for clarification. We then changed the decomposition into the nonlinear decomposition ($\lambda, \gamma, b_0, \eta_0, \epsilon_0$), and denote by $(\lambda(t), \gamma(t), b(t), \eta(t), \epsilon(t))$ the nonlinear decomposition of $u^{(\tilde{\eta})}(t)$ at time $t$. We also recall by (5.12) that the difference of $(\tilde{\lambda}, \tilde{\gamma}, \tilde{b}, \tilde{\eta}, \tilde{\epsilon})$ and $(\lambda, \gamma, b_0, \eta_0, \epsilon_0)$ is bounded by $\tilde{b}_0^2$. Finally, we assumed (for a contradiction argument) that for any $\tilde{\eta}$ the solution $u^{(\tilde{\eta})}$ exits the trapped regime by violating the $\eta$-bound:
$$
|\eta(T^{(\tilde{\eta})}_{exit})| = \frac{b(T^{(\tilde{\eta})}_{exit})}{\log b(T^{(\tilde{\eta})}_{exit})}. 
$$

Proof of Proposition 5.5 We need to show that $I_{\pm}$ are nonempty open sets.

To show that $I_{\pm}$ is nonempty, we show $\pm \frac{1}{5} \frac{b_0}{\log b_0} \in I_{\pm}$. We compute the variation of the ratio $\frac{\eta|\log b|}{b}$ using the modulation estimates 5.25:
$$
\partial_s \left( \frac{\eta|\log b|}{b} \right) = \frac{\eta|\log b|}{b} \left( - \frac{b_0}{b} \right) + \frac{\eta}{b} \left( 1 + O\left( \frac{1}{\log b} \right) \right) + O\left( \frac{Kb}{\sqrt{\log M}} \right).
$$

Thus if $\left| \frac{\eta|\log b|}{b} \right| \geq \frac{1}{10}$ holds at some time, $|\frac{\eta|\log b|}{b}|$ starts to increase, thanks to $\frac{K}{\sqrt{\log M}} \ll 1$. In particular, if $\tilde{\eta}_0 \leq \frac{1}{20} \frac{b_0}{\log b_0}$, then $\eta_0 \geq \frac{1}{10} \frac{b_0}{\log b_0}$ so $\eta$ must have same sign with $\eta_0$ at $T^{(\eta_0)}_{exit}$, saying that $\pm \frac{1}{5} \frac{b_0}{\log b_0} \in I_{\pm}$.

We turn to show that $I_{\pm}$ is open. Since $\tilde{\eta}_0 \in I_{\pm}$, there exists $t^{(\tilde{\eta}_0)} \in [0, T^{(\tilde{\eta}_0)}_{exit})$ such that $\pm \eta(t^{(\tilde{\eta}_0)}(t^{(\tilde{\eta}_0)})) > \frac{1}{2} \frac{b_0}{\log b_0} \left( t^{(\tilde{\eta}_0)} \right)$. By the continuous dependence, obtained by combining the local well-posedness and Lemma 5.2 for all $\eta_0$ near $\tilde{\eta}_0$ we have $t^{(\eta_0)} \in [0, T^{(\eta_0)}_{exit})$ and $\pm \eta(t^{(\eta_0)}(t^{(\eta_0)})) > \frac{1}{2} \frac{b_0}{\log b_0} \left( t^{(\eta_0)} \right)$. Such $\eta_0$ belongs to $I_{\pm}$ due to the argument in the previous paragraph. This completes the proof.

In view of Propositions 5.3 and 5.4 we have constructed a trapped solution $u$. The remaining task is to show that $u$ blows up in finite time as described in Theorem 1.1.

Proof of Proposition 5.5 The proof is very similar to Section 6].

\footnote{In fact this $b_0$ also depends on $\tilde{\eta}_0$, because the transition from the rough decomposition to the nonlinear decomposition may change the value of $b$.}

\footnote{Strictly speaking, since $b_0$ depends on the choice of $\tilde{\eta}_0$, $b_0$ for $\tilde{\eta}_0$ and $b_0$ for $\eta_0$ in general differ. However, the continuous dependence still holds for $b_0$.}
(1) By the claim (5.57), we have
\[
\partial_t \lambda^\frac{\cdot}{\cdot} = -\frac{b}{3\lambda^\frac{\cdot}{\cdot}} + \frac{1}{3\lambda^\frac{\cdot}{\cdot}} \left( \frac{\lambda_s}{\lambda} + b \right) = -\frac{b}{3\lambda^\frac{\cdot}{\cdot}} \left( 1 + O((b^*)^{1-}) \right) \leq -\frac{b_0}{4\lambda^\frac{\cdot}{\cdot}}
\]
This implies the finite-time blow up, \( T < +\infty \). By the standard blow-up criterion, we have \( \lambda(T) := \lim_{t \to T} \lambda(t) = 0 \). Moreover, due to (5.50) and \( |\eta| < \frac{b}{|\log b|} \), we have \( b(T) := \lim_{t \to T} b(t) = 0 \) and \( \eta(T) := \lim_{t \to T} \eta(t) = 0 \).

(2) We start by rewriting the claim (5.50) as
\[
(5.58) \quad b(1 + O(1)) = \ell \left( 1 + O(\frac{1}{|\log b|}) \right), \quad \ell := \lim_{t \to T} \frac{b(t)|\log b(t)|^2}{\lambda(t)} \in (0, \infty),
\]
where the existence of \( \ell \in (0, \infty) \) follows from (5.50) (on \([t, T]\) instead of on \([0, t]\)) and \( b(T) = 0 \).

We now claim the asymptotics of the parameters \( \lambda \) and \( b \):
\[
(5.59) \quad \lambda(t) = \ell, \quad T - t \left( 1 + o_{t \to T}(1) \right),
\]
\[
(5.60) \quad b(t) = \ell^2, \quad T - t \left( 1 + o_{t \to T}(1) \right).
\]
To see this, we first derive the asymptotics of \( \lambda \) and \( b \) in the \( s \)-variable. We integrate the refined modulation estimate (5.41) in the \( s \)-variable from \([s, \infty)\) to obtain
\[
\tilde{b}(s) = \frac{1}{s} - \frac{2}{s \log s} + O \left( \frac{1}{s \log s} \right).
\]
By (5.38), the same asymptotics apply to \( b(s) \). Thus (5.58) yields
\[
\ell \lambda(s) = \frac{(\log s)^2}{s} (1 + o_{s \to \infty}(1))
\]
and hence
\[
(5.61) \quad b = \frac{\ell \lambda}{|\log(\ell \lambda)|^2} (1 + o_{s \to \infty}(1)).
\]
In the original time variable \( t \), the sharp \( \lambda \)-asymptotics (5.59) follow from integrating
\[
\lambda_t = b \lambda \left( 1 + o_{t \to T}(1) \right) = -\ell \left( 1 + o_{t \to T}(1) \right)
\]
backwards in time from \( T \) to \( t \) with \( \lambda(T) = 0 \). The sharp \( b \)-asymptotics (5.60) follow from substituting the sharp \( \lambda \)-asymptotics into (5.61).

Next, we claim that \( \gamma(t) \) converges to some \( \gamma^* \) as \( t \to T \). Indeed, from the refined modulation estimate (5.41) and \( \eta \to 0 \), we have
\[
|\tilde{\eta}(s)| \lesssim \frac{1}{s (\log s)^2}.
\]
By (5.38), the same bound holds for \( \eta(s) \). Thus the modulation estimate (5.24) says that \( \gamma_s \) is integrable in \([s, \infty)\):
\[
|\gamma_s| \lesssim \frac{1}{s (\log s)^2}.
\]
Hence \( \gamma(t) \) converges to some \( \gamma^* \) as \( t \to T \).

(3) It now remains to show that \( u \) decomposes as in Theorem 1.1

\footnote{A \( H^1 \)-solution blows up at a finite time \( T < +\infty \) only if \( \lim_{t \to T} ||u(t)||_{H^1} = \infty \).}
We first claim the outer $L^2$-convergence: $1_{(R,\infty)}(t)u(t)$ converges in $L^2$ for any $R > 0$. To show this, choose any $R > 0$ and we show that $1_{(R,\infty)}(t)u(t)$ converges in $L^2$. In view of $i\partial_t(1_{(R,\infty)}u) = 1_{(R,\infty)}L^2u(t)$, it suffices to show that $t \mapsto \|1_{(R,\infty)}L^2u(t)\|_{L^2}$ is integrable. By scaling, we observe that

$$\|1_{(\lambda^{-1}R,\infty)}L^2u(t)\|_{L^2} = \lambda^{-2}(t)\|1_{(1,R,\infty)}L^2u(t)\|_{L^2}.$$  

Since

$$1_{(\lambda^{-1}R,\infty)}L^2u(t) \lesssim 1_{(\lambda^{-1}R,\infty)}(|w_1| + |w|_\infty \int_0^\infty |w| \, dy'),$$

we have

$$\|1_{(\lambda^{-1}R,\infty)}L^2u(t)\|_{L^2} \lesssim \|1_{(\lambda^{-1}R,\infty)}|w_1| - 1\|_{L^2} (1 + \|w\|_2\),$$

Because $P_1$ is supported in $(0, 2B_1)$ and $2B_1 < \lambda^{-1}R$ for $t$ sufficiently close to $T$, we have by Lemma \ref{lem:A.1}

$$\|1_{(\lambda^{-1}R,\infty)}L^2u(t)\|_{L^2} \lesssim \|1_{(\lambda^{-1}R,\infty)}|w_1| - 1\|_{L^2} \lesssim b^2 \log b^4.$$  

Using the sharp asymptotics \eqref{eq:5.59} and \eqref{eq:5.60}, $\lambda^{-2}b^4 \log b^4$ is integrable, and hence the claim is proved.

The above claim says that there exists a function $u^*$ such that $1_{(R,\infty)}(t)u(t) \to 1_{(R,\infty)}u^*$ in $L^2$ for any $R > 0$. We show that this $u^*$ satisfies the statement of Theorem 1.1. Let

$$\tilde{e}^\gamma(t, r) := \frac{e^{\gamma(t)}}{\lambda(t)} e^\left(t, \frac{r}{\lambda(t)}\right)$$

Since $(\gamma, b, \eta) \to (\gamma^*, 0, 0)$ and $\frac{t}{\lambda(t)\log(T-t)r^2} \to 1$, we have

$$\frac{e^{\gamma(t)}}{\lambda(t)} P\left(\frac{r}{\lambda(t)}; b(t), \eta(t)\right) - e^{\gamma^*} \frac{t\log(T-t)^2}{t(T-t)Q\left(\frac{\log(T-t)^2}{t(T-t)}\right)} \to 0 \text{ in } L^2.$$  

Thus it suffices to show that $u^* \in H^1_0$ and $\tilde{e}^\gamma(t) \to u^*$ in $L^2$ as $t \to T$. On one hand, $1_{(R,\infty)}(t)\tilde{e}^\gamma(t) \to 1_{(R,\infty)}u^*$ in $L^2$ for any $R > 0$, as the outer convergence is insensitive to the concentrating bubble. On the other hand, due to the boundedness of $\frac{b(t)\log b(t)}{\lambda(t)}$ (see \eqref{eq:5.59} and $\|e^*\|_{H^1_0} = \lambda^{-1}||e||_{H^1_0}$, we see that $\tilde{e}^\gamma(t)$ is uniformly bounded in $H^1_0$. Therefore, $u^* \in H^1_0$ and $\tilde{e}^\gamma(t) \to u^*$ weakly in $H^1_0$. By the Rellich-Kondrachov compactness theorem, $\tilde{e}^\gamma(t) \to u^*$ in $L^2_{\text{loc}}$. Combining this with outer $L^2$-convergence shows that $\tilde{e}^\gamma(t) \to u^*$ in $L^2$. This finishes the proof. \hfill \Box

**Appendix A. Adapted function spaces**

In this section, we prove the facts regarding to the adapted function spaces introduced in Section \ref{sec:5.3}. Our main focuses are on (sub-)coercivity estimates of Proposition \ref{prop:3.5}. On the way, we compare the adapted function spaces with the usual equivariant Sobolev spaces and prove various $L^\infty$-estimates and interpolation estimates.

Our main tools are weighted Hardy’s inequalities:

**Lemma A.1** (Weighted Hardy’s inequality for $\partial_r$; see \cite[Lemma A.1]{21}). Let $0 < r_1 < r_2 < \infty$; let $\varphi : [r_1, r_2] \to \mathbb{R}$ be a $C^1$ weight function such that $\partial_r \varphi$ is nonvanishing and $\varphi \lesssim |r\partial_r \varphi|$. Then, for smooth $f : [r_1, r_2] \to \mathbb{C}$, we have

$$\int_{r_1}^{r_2} \left| \frac{f}{r} \right|^2 |r\partial_r \varphi| \, dr \lesssim \int_{r_1}^{r_2} |\partial_r f|^2 \varphi \, r \, dr + \begin{cases} \varphi(r_2)|f'(r_2)|^2 & \text{if } \partial_r \varphi > 0, \\ \varphi(r_1)|f'(r_1)|^2 & \text{if } \partial_r \varphi < 0. \end{cases}$$

By carefully choosing $\varphi$, we also have logarithmic Hardy’s inequality:
Define the space \( \dot{H}^1 \), \( \dot{H}^2 \), \( \dot{H}^3 \), and \( \dot{H}^0 \). These are all different from \( H^1, H^2, H^3 \), and \( H^0 \), but are essentially same for functions with high frequency. As a result, their inhomogeneous versions are the same: \( \dot{H}^k_{\text{inh}} \cap L^2 = H^k_{\text{inh}} \).

The adapted function spaces are motivated to have boundedness and subcoercivity estimates for the linear adapted derivatives, e.g. \( L_Q \), \( \dot{A}_Q L_Q \), and \( \dot{A}_Q^* A_Q L_Q \) with various levels of regularity. The first one \( \dot{H}^1 \) is designed to control \( \epsilon \), provided that \( \epsilon_1 \approx L_Q \epsilon \in L^2 \). On the other hand, the spaces \( \dot{H}^2 \), \( \dot{H}^3 \), and \( \dot{H}^0 \) are designed to control \( \epsilon_2, \epsilon_1 \), and \( \epsilon \), provided that \( \epsilon_3 = A_Q^* \epsilon_2 \in L^2 \).

For \( \theta \)-equivariant Schwartz functions \( f \), define
\[
\|f\|_{\dot{H}^1} := \|\partial_r f\|_{L^2} + \|r^{-1} \log r \|^{-1} f\|_{L^2}.
\]
Define the space \( \dot{H}^0 \) by taking the completion of \( S_0 \) under this norm. This is the adapted function space at \( \dot{H}^1 \)-level. We note that \( \dot{H}^0 \) is stronger than \( \dot{H}^1 \), due to its control at infinity. Nevertheless, \( L^2 \cap \dot{H}^0 = H^0 \).

Lemma A.3 (Boundedness and subcoercivity of \( L_Q \)). For \( v \in \dot{H}^0 \), we have
\[
\|L_Q v\|_{L^2} + \|v\|_{L^2} \approx \|v\|_{\dot{H}^0}.
\]
Moreover, the kernel of \( L_Q : \dot{H}^1 \to L^2 \) is \( \text{span}_Q \{ A_Q, iQ \} \).

Proof. By density, we may assume \( v \in S_0 \). Recall that \( L_Q = D_Q + Q B_Q \). Firstly, \( Q B_Q \) is perturbative in the sense that
\[
\|Q B_Q v\|_{L^2} \lesssim \|\frac{1}{r} \int_0^r \rho^{-3} \rho' \rho^2 \|_{L^2} \lesssim \|\rho^{-3} v\|_{L^2} \lesssim \|v\|_{L^2} + r_0^{-1} \|v\|_{\dot{H}^1},
\]
for any \( r_0 \geq 1 \). Therefore, it suffices to show
\[
\|D_Q v\|_{L^2} + \|v\|_{L^2} \approx \|v\|_{\dot{H}^0}.
\]
We note that the boundedness \( (\lesssim) \) is obvious. Henceforth, we focus on the subcoercivity \( (\gtrsim) \) of \( D_Q \). We use the operator identity \( D_Q = Q \partial_r Q^{-1} \) and try to apply weighted Hardy’s inequality (Lemma A.1) for \( Q^{-1} v \). In the region \( r \geq 10 \), we have \(-r \partial_r (Q^2) \sim Q^2\), so applying Lemma A.1 for \( f = Q^{-1} v \) with \( \varphi = Q^2 \) yields
\[
\|1_{[0, \infty)} v\|_{L^2}^2 \lesssim \|1_{[0, \infty)} D_Q v\|_{L^2}^2 + \|v\|_{L^2}^2,
\]
provided that \( r_0 \geq 10 \). Averaging over \( r_0 \in [10, 20] \), we get
\[
\|1_{[20, \infty)} v\|_{L^2}^2 \lesssim \|1_{[10, \infty)} D_Q v\|_{L^2}^2 + \|1_{[10, 20]} v\|_{L^2}^2.
\]
In the region \( r \leq \frac{10}{11} \), we have \( Q \sim 1 \). We choose \( \varphi : (0, \frac{1}{10}) \to \mathbb{R}_+ \) such that \( r \partial_r \varphi = Q^2 (\log r) r^{-2} \) and \( \lim_{r \to 0^+} \varphi(r) = 0 \). It satisfies \( \varphi \lesssim r \partial_r \varphi \), so applying Lemma A.1 for \( f = Q^{-1} v \) with this \( \varphi \) and averaging the boundary term yield
\[
\|1_{[0, \frac{1}{10}]} v\|_{L^2}^2 \lesssim \|1_{[0, \frac{1}{11}]} D_Q v\|_{L^2}^2 + \|1_{[\frac{1}{11}, \frac{1}{10}]} v\|_{L^2}^2.
\]
Therefore, we have proved that
\[
\|D_Q v\|_{L^2}^2 + \|v\|_{L^2}^2 \gtrsim \|1_{y \leq 1} + 1_{y > 1} \|_{r (\log r)^{-1}} v\|_{L^2}^2.
\]
\[\text{This } \varphi \text{ is very similar to that used in the proof of logarithmic Hardy’s inequality.}\]
Adding both sides by $\|1_{y^{-1}} v\|_{L^2}^2$, we get
$$\|D_O v\|_{L^2}^2 + \|1_{y^{-1}} v\|_{L^2}^2 \gtrsim \|\frac{1}{r \log r} v\|_{L^2}^2.$$ Combining this with $\|D_O v\|_{L^2} = \|\partial_r v\|_{L^2} + O(\|\frac{1}{r \log r} v\|_{L^2})$ yields the conclusion.

For the kernel characterization, we refer to [22, Lemma A.5]. The argument there still works for $m = 0$ with a slight modification. □

**Lemma A.4** (Coercivity of $L_Q$ at $\dot{H}^1$-level). Let $\psi_1, \psi_2$ be elements of the dual space $(\dot{H}_0^1)^*$. If the $2 \times 2$ matrix $(a_{ij})$ defined by $a_{11} = (\psi_1, \Lambda Q)_r$ and $a_{12} = (\psi_1, iQ)_r$, has nonzero determinant, then we have a coercivity estimate
$$\|v\|_{\dot{H}_0^1} \lesssim_{\psi_1, \psi_2} \|L_Q v\|_{L^2} \lesssim \|v\|_{\dot{H}_0^1}, \quad \forall v \in \dot{H}_0^1 \cap \{\psi_1, \psi_2\}^\perp.$$

**Proof.** We omit the proof and refer to [21, Lemma A.6]. □

The space $\dot{H}_2^1$.

Define the space $\dot{H}_2^1$ by taking the completion of $\mathcal{S}_2$ under the norm for 2-equivariant functions
$$\|v\|_{\dot{H}_2^1} := \|\partial_r v\|_{L^2} + \|r^{-1} \log r \|_{r^{-1}} v\|_{L^2}.$$ Note that $\dot{H}_2^1$ is weaker than $\dot{H}_2^1$ at infinity. Nevertheless, we have $\dot{H}_2^1 \cap L^2 = H_2^1$.

**Lemma A.5** (Coercivity of $A_Q^*$). For $v \in \dot{H}_2^1$, we have
$$\|A_Q^* v\|_{L^2} \sim \|v\|_{\dot{H}_2^1}.$$

**Proof.** By density, we may assume $v \in \mathcal{S}_2$. From
$$A_Q A_Q^* = -\partial_{rr} - \frac{1}{r} \partial_r + \frac{\nabla}{r},$$

$$\tilde{V} = (2 + A_Q[Q])^2 + r^2 Q^2 \sim \langle r \rangle^{-2},$$

we have
$$\|A_Q^* v\|_{L^2}^2 \sim \|\partial_r v\|_{L^2}^2 + \|r^{-1} (\log r)^{-1} v\|_{L^2}^2.$$ Applying the logarithmic Hardy’s inequality (A.1), we have
$$\|1_{r \geq 1} r^{-1} (\log r)^{-1} v\|_{L^2} \lesssim \|\partial_r v\|_{L^2} + \|1_{r \geq 1} v\|_{L^2}.$$ Absorbing $\|1_{r \geq 1} v\|_{L^2}$ into $\|r^{-1} (\log r)^{-1} v\|_{L^2}$, the conclusion follows. □

The space $\dot{H}_1^2$.

Define the space $\dot{H}_1^2$ by taking the completion of $\mathcal{S}_1$ under the norm for 1-equivariant functions
$$\|v\|_{\dot{H}_1^2} := \|\partial_r v\|_{L^2} + \|r^{-1} \log r \|_{r^{-1}} v\|_{L^2}.$$ It turns out that $\dot{H}_1^2$ is stronger than $\dot{H}_2^1$ and $\dot{H}_1^2 \cap L^2 = H_1^2$.

**Lemma A.6** (Comparison of $\dot{H}_1^2$ and $\dot{H}_2^1$). For $v \in \mathcal{S}_1$, we have
$$\|v\|_{\dot{H}_1^2} \sim \|v\|_{\dot{H}_2^1} + \|1_{r \geq 1} v\|_{L^2}.$$ Moreover, one cannot remove $\|1_{r \geq 1} v\|_{L^2}$ in the estimate (A.3).

**Proof.** For the $(\tilde{r})$-direction, due to $\|v\|_{\dot{H}_1^2} \sim \|\partial_r v\|_{\dot{H}_2^1} \sim \|\partial_r v\|_{L^2}$ by (5.20), it suffices to establish
$$\|\partial_r v\|_{\dot{H}_2^1} \sim \|\partial_r v\|_{r^{-1}} \lesssim \|v\|_{\dot{H}_1^2}.$$ To show (A.4), we recognize that $\partial_r = (\partial_r + \frac{1}{r})(\partial_r - \frac{1}{r})$ and $\partial_r - \frac{1}{r}$ is the radial part of $\partial_r$ acting on 1-equivariant functions. Applying Hardy’s inequality (Lemma A.1)
to the operator $\partial_r + \frac{1}{r} = \frac{1}{r} \partial_r r$ with $f = r(\partial_r + \frac{1}{r})v$, $\varphi = \frac{1}{r}$, $r_1 \to 0$, and $r_2 \to \infty$ we obtain
\[
\|\frac{1}{r} (\partial_r + \frac{1}{r})v\|_{L^2} \lesssim \|\partial_r v\|_{L^2}.
\]
Since $\partial_r = (\partial_r + \frac{1}{r}) - \frac{1}{r}$, it is also possible to upgrade the above as
\[
\|\partial_r v\|_{L^2} \lesssim \|\partial_r v\|_{L^2} \lesssim \|\partial_r v\|_{L^2}.
\]
This shows (A.4) and hence the $(\gtrsim)$-direction of (A.3).

For the $(\lesssim)$-direction, we note that
\[
\|\partial_r v\|_{L^2} = \|\partial_r + \frac{1}{r} \partial_r v\|_{L^2} \lesssim \|\partial_r v\|_{L^2} \lesssim \|\partial_r v\|_{L^2} \lesssim \|v\|_{H^2}.
\]
Next, by the logarithmic Hardy’s inequality (A.1), we have
\[
\|r^{-2} (\log r)^{-1} v\|_{L^2} \lesssim \|\partial_r (\frac{1}{r} v)\|_{L^2} + \|1_{r \sim 1} v\|_{L^2} \lesssim \|\partial_r + \frac{1}{r} v\|_{L^2} + \|1_{r \sim 1} v\|_{L^2}.
\]
Using $\partial_r v = (\partial_r + \frac{1}{r}) v + \frac{1}{r} v$, we further deduce that
\[
\|r^{-1} (\log r)^{-1} v\|_{L^2} \lesssim \|\partial_r + \frac{1}{r} v\|_{L^2} + \|1_{r \sim 1} v\|_{L^2} \lesssim \|v\|_{H^2} + \|1_{r \sim 1} v\|_{L^2}.
\]
This completes the proof of (A.3).

To see why $\|1_{r \sim 1} v\|_{L^2}$ in (A.3) cannot be removed, consider $v(x) = (x_1 + ix_2) \sum_{n=1}^{N} \chi_{2^n} (x)$ with $N \in \mathbb{N}$ sufficiently large. Then $\|v\|_{H^2} \gtrsim N$ but $\|v\|_{H^2} \lesssim N^{\frac{1}{2}}$.

We turn to the subcoercivity estimate. We want to control $v$, provided that $A_Q v \in \tilde{H}^2_{\chi}$.

**Lemma A.7** (Boundedness and subcoercivity of $A_Q$). For $v \in \tilde{H}^2_{\chi}$, we have
\[
\|A_Q v\|_{\tilde{H}^1_{\chi}} + \|1_{r \sim 1} v\|_{L^2} \sim \|v\|_{\tilde{H}^2_{\chi}}.
\]
Moreover, the kernel of $A_Q : \tilde{H}^2_{\chi} \to \tilde{H}^1_{\chi}$ is span$_C \{RQ\}$.

**Remark A.8.** The log weight in the definition of $\tilde{H}^1_{\chi}$ cannot be improved (or, removed). Indeed, if one considers $v(x) = (x_1 + ix_2) \chi_R (x)$ for large $R$, then $\|A_Q v\|_{\tilde{H}^2_{\chi}}$ is uniformly bounded in $R$, but both $\|r^{-2} v\|_{L^2}$ and $\|r^{-1} \partial_r v\|_{L^2}$ diverge as $R \to \infty$.

**Proof.** By density, we may assume $v \in S_1$. We note that
\[
A_Q A_Q = -\partial_r - \frac{1}{r} \partial_r + \frac{1}{r} - Q^2 = -\Delta_1 - Q^2,
\]
\[
\|Q^2 v\|_{L^2} \lesssim \|1_{[r_0^{-1}, r_0]} v\|_{L^2} + r_0^{-2} \|v\|_{\tilde{H}^2_{\chi}},
\]
\[
\|v\|_{\tilde{H}^2_{\chi}} \sim r_0 \|\Delta_1 v\|_{\tilde{H}^1_{\chi}} + \|1_{[r_0^{-1}, r_0]} v\|_{L^2},
\]
for $r_0 \geq 10$. Taking $r_0$ sufficiently large, we obtain
\[
\|v\|_{\tilde{H}^2_{\chi}} \sim r_0 \|A_Q^* A_Q v\|_{L^2} + \|1_{[r_0^{-1}, r_0]} v\|_{L^2}.
\]
Applying the coercivity (A.2) shows the subcoercivity estimate. For the kernel characterization, notice that $A_Q$ is a first-order differential operator such that $A_Q (rQ) = 0$. A standard ODE theory concludes the proof.

**Lemma A.9** (Coercivity of $A_Q$ at $H^2$-level). Let $\psi_1, \psi_2$ be elements of $(\tilde{H}^2_{\chi})^*$, which is the dual space of $\tilde{H}^2_{\chi}$. If the $2 \times 2$ matrix $(a_{ij})$ defined by $a_{11} = (\psi_1, rQ)_\psi$ and $a_{12} = (\psi_1, rQ)_\psi$ has nonzero determinant, then we have a coercivity estimate
\[
\|v\|_{\tilde{H}^2_{\chi}} \lesssim \psi_1, \psi_2 \|A_Q^* v\|_{\tilde{H}^2_{\chi}} \lesssim \|v\|_{\tilde{H}^2_{\chi}}, \quad \forall v \in \tilde{H}^2_{\chi} \cap \{\psi_1, \psi_2\}.
\]

**Proof.** We omit the proof as it can be proved in a similar manner to Lemma A.4.
The space $\hat{H}^3_0$.

Define the space $\hat{H}^3_0$ by taking the completion of $S_0$ under the norm for 0-equivariant functions

$$\|v\|_{\hat{H}^3_0} = \|\partial_{rrr}v\|_{L^2} + \|r^{-1}(\log r)^{-1}\partial_r v|_{-1}\|_{L^2} + \|r^{-1}(r)^{-2}(\log r)^{-1}v\|_{L^2}.$$ 

It turns out that $\hat{H}^3_0$ is stronger than $\hat{H}^3_0$ but $\hat{H}^3_0 \cap L^2 = H^3_0$.

**Lemma A.10** (Comparison of $\hat{H}^3_0$ and $\hat{H}^3_0$). For $v \in S_0$, we have

$$(A.6) \quad \|v\|_{\hat{H}^3_0} \sim \|v\|_{\hat{H}^3_0} + \|1_{r^\alpha v}\|_{L^2}.$$ 

Moreover, $\|1_{r^\alpha v}\|_{L^2}$ cannot be removed.

**Proof.** For the $(\gtrsim)$-direction, it suffices to establish

$$(A.7) \quad \|\partial_+\partial_+v\|_{\hat{H}^3_0} \sim \|\partial_+\partial_+v\|_{L^2} \lesssim \|v\|_{\hat{H}^3_0},$$

due to $\|v\|_{\hat{H}^3_0} \sim \|\partial_+\partial_+v\|_{\hat{H}^3_0} \sim \|\partial_+\partial_+v\|_{L^2}$. To show (A.7), we recognize that $\partial_{rrr} = (\partial_r + \frac{1}{r})(\partial_r - \frac{1}{r})\partial_r$ and $(\partial_r - \frac{1}{r})\partial_r$ is the radial part of $\partial_r \partial_r$ acting on 0-equivariant functions. Therefore, we use Hardy’s inequality for $\partial_r + \frac{1}{r} = \frac{1}{r} \partial_r r$ in the proof of (A.3) to have

$$\|\partial_r \partial_r v\|_{L^2} \lesssim \|\partial_{rrr}v\|_{L^2}.$$ 

This shows (A.7) and hence the $(\gtrsim)$-direction of (A.6).

For the $(\lesssim)$-direction, we use (A.3) and weighted logarithmic Hardy’s inequality (A.1) to have

$$\|\partial_{rrr}v\|_{L^2} + \|r^{-1}(\log r)^{-1}\partial_r v\|_{L^2} \lesssim \|\partial_r v\|_{\hat{H}^3_0} \lesssim \|\partial_+ v\|_{\hat{H}^3_0},$$

$$\|r^{-1}(r)^{-2}(\log r)^{-1}v\|_{L^2} \lesssim (r)^{-2}(\log r)^{-1}\partial_r v\|_{L^2} + \|1_{r^\alpha v}\|_{L^2},$$

$$\|\partial_r v\|_{\hat{H}^3_0} \lesssim \|\partial_+ v\|_{\hat{H}^3_0} + \|1_{r^\alpha v}\|_{L^2} \lesssim \|v\|_{H^3_0} + \|1_{r^\alpha v}\|_{L^2}.$$ 

Thus

$$\|v\|_{\hat{H}^3_0} \lesssim \|v\|_{\hat{H}^3_0} + \|1_{r^\alpha v}\|_{L^2} \lesssim \|v\|_{\hat{H}^3_0} \lesssim \|1_{r^\alpha v}\|_{L^2}.$$ 

In order to remove $\|1_{r^\alpha v}\|_{L^2}$ in (A.6), we consider $v(x) = |x|^2 \sum_{m=1}^N \chi_{2^n}(x)$ with $N \in \mathbb{N}$ sufficiently large. Then $\|v\|_{\hat{H}^3_0} \gtrsim N$ but $\|v\|_{\hat{H}^3_0} \lesssim N^2$.

We turn to the subcoercivity estimates of $L_Q$.

**Lemma A.11** (Boundedness and subcoercivity of $L_Q$ at $\hat{H}^3_0$-level). For $v \in \hat{H}^3_0$, we have

$$(A.8) \quad \|L_Qv\|_{\hat{H}^3_0} + \|1_{r^\alpha v}\|_{L^2} \sim \|v\|_{\hat{H}^3_0}.$$ 

Moreover, the kernel of $L_Q : \hat{H}^3_0 \to \hat{H}^3_0$ is spanned by $\{\Delta Q, iQ\}$.

**Remark A.12.** The log weight in the definition of $\hat{H}^3_0$ cannot be improved (or, removed), by arguing similarly as in Remark A.8 with the function $v(x) = |x|^2 \chi_{2^n}(x)$.

**Proof.** By density, we may assume $v \in S_0$. Recall $L_Q = D_Q + QB_Q$. We first claim that the contribution of $QB_Q$ is perturbative:

$$(A.9) \quad \|QB_Qv\|_{\hat{H}^3_0} \lesssim \|(r)^{-5}v\|_{L^2} + \|(r)^{-4}\partial_r v\|_{L^2}.$$ 

To see this, we estimate using (A.3)

$$\|QB_Qv\|_{\hat{H}^3_0} \lesssim \|QB_Qv\|_{\hat{H}^3_0} + \|1_{r^\alpha} QB_Qv\|_{L^2} \lesssim \|\Delta_1(QB_Qv)\|_{L^2} + \|1_{r^\alpha} QB_Qv\|_{L^2}.$$
The RHS can be bounded by
\[ \|\Delta_1(Q B_{Qv})\|_{L^2} = \|\partial_r(\partial_r + \frac{1}{r})(Q B_{Qv})\|_{L^2} = \|\partial_r\{\partial_r Q B_{Qv} + Q Re(Qv)\}\|_{L^2} = \|\{(\partial_r - \frac{1}{r})(\partial_r Q B_{Qv} + 2(\partial_r Q) Re(Qv) + Q \partial_r Re(Qv)\}\|_{L^2} \lesssim \|\langle r \rangle^{-5} v\|_{L^2} + \|\langle r \rangle^{-4} \partial_r v\|_{L^2} \]
and
\[ \|1_{r \leq 1} Q B_{Qv}\|_{L^2} \lesssim 1_{r \leq 1} v\|_{L^2}. \]

Next, we show the \((\lesssim)\)-direction of \((A.8)\). By \((A.9)\), it suffices to show
\[ \|D_Q v\|_{H^1} \lesssim \|v\|_{H^1}. \]
In view of \((A.3)\), we have
\[ \|D_Q v\|_{H^1} \leq \|D_Q v\|_{H^1} + \|1_{r \leq 1} D_Q v\|_{L^2} \lesssim \|\partial_r D_Q v\|_{L^2} + \|1_{r \leq 1} v\|_{L^2}. \]

In the region \(r \ll 1\), we have \(D_Q \approx \partial_r\), so
\[ \|1_{(0,1]}(\partial_r D_Q v)\|_{L^2} \lesssim \|1_{(0,1]}(\partial_r + \partial_r v)\|_{L^2} + \|1_{(0,1]}(\partial_r - \frac{1}{r})(\frac{A_r(Q)}{r}v)\|_{L^2}. \]

In the region \(r \gg 1\), we have \(D_Q \approx \partial_r + \frac{\dot{v}}{r}\), so
\[ \|1_{[1,\infty]}(\partial_r D_Q v)\|_{L^2} \lesssim \|1_{[1,\infty]}(\partial_r - \frac{1}{r})(\partial_r + \frac{\dot{v}}{r}) v\|_{L^2} + \|1_{[1,\infty]}(\partial_r - \frac{1}{r})(\frac{2 + A_r(Q)}{r}v)\|_{L^2}. \]

One crucial observation is that \((\partial_r - \frac{1}{r})(\partial_r + \frac{\dot{v}}{r}) v\|_{L^2} - 1\) can be controlled by \(|\partial_r + \partial_r v| - 1\) in view of
\[ (A.10) \quad (\partial_r + \frac{\dot{v}}{r})(\partial_r - \frac{1}{r})(\partial_r + \frac{\dot{v}}{r}) v = (\partial_r + \frac{\dot{v}}{r})(\partial_r - \frac{1}{r})\partial_r v = (\partial_r + \frac{\dot{v}}{r})\partial_r + v \]
and Hardy’s inequality:
\[ \|1_{[1,\infty]}(\partial_r - \frac{1}{r})(\partial_r + \frac{\dot{v}}{r}) v\|_{L^2} \lesssim \|1_{[1,\infty]}(\partial_r + \partial_r v)\|_{L^2} + \|1_{[1,\infty]}(\partial_r - \frac{1}{r})(\frac{2 + A_r(Q)}{r}v)\|_{L^2}. \]

Combining the above estimates, we arrive at
\[ \|D_Q v\|_{H^1} \lesssim \|\partial_r + v\|_{L^2} + \|1_{(0,1]}(\partial_r - \frac{1}{r})(\frac{A_r(Q)}{r}v)\|_{L^2} + \|1_{[1,\infty]}(\partial_r - \frac{1}{r})(\frac{2 + A_r(Q)}{r}v)\|_{L^2} \]
Applying the estimates
\[ \|1_{(0,1]}(\partial_r - \frac{1}{r})(\frac{A_r(Q)}{r}v)\|_{L^2} + \|1_{[1,\infty]}(\partial_r - \frac{1}{r})(\frac{2 + A_r(Q)}{r}v)\|_{L^2} \lesssim \|\langle r \rangle^{-3} \partial_r v\|_{L^2} + \|\langle r \rangle^{-4} \partial_r v\|_{L^2} + \|\langle r \rangle^{-5} v\|_{L^2} \]
and \((A.7)\), we get
\[ \|D_Q v\|_{H^1} \lesssim \|\partial_r + v\|_{L^2} + \|\langle r \rangle^{-3} \partial_r v\|_{L^2} + \|\langle r \rangle^{-4} \partial_r v\|_{L^2} + \|\langle r \rangle^{-5} v\|_{L^2} \lesssim \|v\|_{H^1}. \]
Combining this with \((A.9)\), the \((\lesssim)\)-direction of \((A.8)\) is proved.

Next, we show the \((\gtrsim)\)-direction of \((A.8)\). By \((A.6)\), we have
\[ \|v\|_{H^1} \lesssim \|v\|_{H^1} + \|1_{r \leq 1} v\|_{L^2} \lesssim \|\partial_r + v\|_{L^2} + \|1_{r \leq 1} v\|_{L^2}. \]

Thus we aim to control \(|\partial_r + v\|_{L^2}| - 1\) in terms of \(|D_Q v|_{H^1}| - 1\). Again, we separately consider the regions \(r \leq 1\) and \(r \geq 1\). In the region \(r \leq 1\),
\[ \|1_{(0,1]}(\partial_r + v)\|_{L^2} \lesssim \|1_{(0,1]}(\partial_r + D_Q v)\|_{L^2} + \|1_{(0,1]}(\partial_r - \frac{1}{r})(\frac{A_r(Q)}{r}v)\|_{L^2}. \]
In the region $r \geq 1$, we use (A.10) with Hardy’s inequality that
\[
\|1_{[1,\infty)}[\partial_{r,r}v]_{-1}\|_{L^2}
\lesssim \|1_{[1,\infty)}[\partial_{r} - \frac{1}{r})(\partial_{r} + \frac{2}{r})v]_{-1}\|_{L^2} + \|1_{[1,\infty)}[v]_{-2}\|_{L^2}
\lesssim \|1_{[1,\infty)}[\partial_{r},DQv]_{-1}\|_{L^2} + \|1_{[1,\infty)}[\partial_{r}v]_{-2}\|_{L^2} + \|1_{[1,\infty)}[\partial_{r} - \frac{1}{r}](\Delta_{r}Q)v]_{-1}\|_{L^2}.
\]

Therefore,
\[
\|v\|_{\dot{H}^0_{r}} \lesssim \|\partial_{r}v\|_{-1}\|_{L^2} + \|1_{r\rightarrow 1}v\|_{L^2}
\lesssim \|\partial_{r}DQv\|_{-1}\|_{L^2} + \|\partial_{r}v\|_{L^2} + \|v\|_{L^2} + \|v\|_{L^2}
\lesssim \|DQv\|_{\dot{H}^0_{r}} + \|(r)^{-3}\partial_{rr}v\|_{L^2} + \|\partial_{r}v\|_{L^2} + \|\partial_{r}v\|_{L^2} + \|\partial_{r}v\|_{L^2},
\]
where in the last inequality we used (A.4). Combining this with (A.9), we have proved that
\[
\|LQv\|_{\dot{H}^0_{r}} + \|\partial_{r}v\|_{L^2} + \|\partial_{r}v\|_{L^2} + \|\partial_{r}v\|_{L^2} + \|\partial_{r}v\|_{L^2} \lesssim \|v\|_{\dot{H}^0_{r}}.
\]
It now remains to replace the perturbative terms by $\|1_{r\rightarrow 1}v\|_{L^2}$. For this, we have
\[
\|\partial_{r}v\|_{L^2} + \|\partial_{r}v\|_{L^2} + \|\partial_{r}v\|_{L^2} \lesssim \|1_{[r_0,\infty)}[v]_{-2}\|_{L^2} + \|v\|_{\dot{H}^0_{r}}
\]
and choose $r_0$ large enough to obtain
\[
\|LQv\|_{\dot{H}^0_{r}} + \|1_{r\rightarrow 1}v\|_{L^2} \gtrsim \|v\|_{\dot{H}^0_{r}}.
\]
Finally applying an interpolation bound
\[
\|1_{r\rightarrow 1}v\|_{L^2} \lesssim \|1_{r\rightarrow 1}v\|_{L^2} + \|1_{r\rightarrow 1}v\|_{L^2} + \|1_{r\rightarrow 1}v\|_{L^2},
\]
completes the proof of the $(\gtrsim)$-direction of (A.8).

The kernel characterization can be proved by a slight modification of the argument in [27] Lemma A.13. \qed

Lemma A.13 (Coercivity of $L_Q$ at $\dot{H}^3$-level). Let $\psi_1, \psi_2$ be elements of the dual space $(\dot{H}^0_{r})^\ast$. If the $2 \times 2$ matrix $(a_{ij})$ defined by $a_{11} = \langle \psi_1, LQ \rangle_r$ and $a_{12} = \langle \psi_1, iQ \rangle_r$ has nonzero determinant, then we have a coercivity estimate
\[
\|v\|_{\dot{H}^3_{r}} \lesssim_{\psi_1, \psi_2} \|LQv\|_{\dot{H}^3_{r}} \lesssim \|v\|_{\dot{H}^3_{r}}, \quad \forall v \in \dot{H}^3_{r} \cap \{\psi_1, \psi_2\}^\perp.
\]

Proof. We omit the proof and refer to [27] Lemma A.15. \qed

Interpolation and $L^\infty$ estimates.

Lemma A.14 (Interpolation estimates). Let $v_2$ be a radial function and $v_1 \in H^1_r$. We have
\[
(A.11) \quad \|v_2\|_{L^\infty} \lesssim \|v_2\|_{L^2}^{1/2} \|\partial_r v_2\|_{L^2}^{1/2},
\]
\[
(A.12) \quad \|v_1\|_{L^1} \lesssim \|v_1\|_{L^2} \|v_1\|_{L^2}^{1/2}.
\]

Proof. For the estimate (A.11), we will in fact show
\[
\|v_2\|_{L^{2p}} \lesssim \|v_2\|_{L^2}^{1/p} \|\partial_r v_2\|_{L^2}^{1/p}, \quad \forall p \in [1, \infty).
\]
As the case $p = 1$ is immediate, it suffices to show for $p \in [2, \infty)$ by interpolation. Applying the FTC to the expression $\partial_r |v_2|^p(r) \lesssim_{p} |v_2|^{p-1} \partial_r v_2$ and using
Minkowski’s inequality, we get

\[ \|v_2\|_{L^2}^p \lesssim_p \int_r^\infty \|v_2\|^{p-1} |\partial_r v_2| \, dr \lesssim_p \int_r^\infty \|1_{r \leq r'}\|_{L^2} \|v_2\|^{p-1} |\partial_r v_2| \, dr' \lesssim_p \int_0^\infty \|v_2\|^{p-1} |\partial_r v_2| \, r' \, dr'. \]

Therefore,

\[ \|v_2\|_{L^p}^p \lesssim_p \|v_2\|_{L^2}^{p-1} \|\partial_r v_2\|_{L^2} \lesssim_p \|v_2\|_{L^2}^{p-1} \|\partial_r v_2\|_{L^2}. \]

Rearranging this completes the proof of (A.11).

The estimate (A.12) follows from

\[ \|v_1\|_{L^\infty} \lesssim \|v_1\|_{\mathcal{H}_{l,1}^1}, \]

near the origin, we have

\[ \|v_1\|_{L^2} \lesssim \|v_1\|_{\mathcal{H}_{l,2}^1}, \]

\[ \|v_2\|_{L^2} \lesssim \|v_2\|_{\mathcal{H}_{l,2}^1}. \]

where in the last inequality we used (A.3). □

Lemma A.15 (Weighted \( L^\infty \)-estimates). Let \( v_m \) be \( m \)-equivariant functions, \( m \in \{0, 1, 2\} \). Near the origin, we have

\[ \|1_{(0,1)} v\|_{L^\infty} \lesssim \|v\|_{\mathcal{H}_{l,0}^1}, \]

\[ \|1_{(0,1)} v_1\|_{L^\infty} \lesssim \|v_1\|_{\mathcal{H}_{l,2}^1}, \]

\[ \|1_{(0,1)} v_2\|_{L^\infty} \lesssim \|v_2\|_{\mathcal{H}_{l,2}^1}. \]

Near infinity, we have

\[ \|1_{(1,\infty)} v\|_{L^\infty} \lesssim \|v\|_{\mathcal{H}_{l,0}^1}, \]

\[ \|1_{(1,\infty)} v_1\|_{L^\infty} \lesssim \|v_1\|_{\mathcal{H}_{l,2}^1}, \]

\[ \|1_{(1,\infty)} v_2\|_{L^\infty} \lesssim \|v_2\|_{\mathcal{H}_{l,2}^1}. \]

Proof. Near the origin, \( L^\infty \)-estimates for \( v \) and \( v_1 \) follow from \( \mathcal{H}_{l,0}^3 \to H_{l,0}^3 \) and \( \mathcal{H}_{l,2}^2 \to H_{l,2}^2 \), and the Sobolev embeddings. For \( v_2 \), we use the FTC argument:

\[ \|1_{(0,1)} v_2\|_{L^\infty} \lesssim \int_0^1 \|\frac{1}{2} v_2\|_{L^2} |\partial_r v_2| \, r' \, dr' \lesssim \|1_{(0,1)} v_2\|_{L^2} \|1_{(0,1)} \partial_r v_2\|_{L^2} \lesssim \|v_2\|_{\mathcal{H}_{l,2}^1}. \]

Near infinity, all the estimates except \( \|1_{(1,\infty)} v_1\|_{L^\infty} \) follow from the FTC arguments and the definitions of our adapted function spaces. We omit their proofs. For \( \|1_{(1,\infty)} v_1\|_{L^\infty} \), we use (A.23) and (A.12) instead:

\[ \|1_{(1,\infty)} v_1\|_{L^\infty} \lesssim \|v_1\|_{\mathcal{H}_{l,1}^1} \lesssim \|v_1\|_{L^2} \|v_1\|_{\mathcal{H}_{l,2}^1}. \]

This completes the proof. □

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