Optimal Hardy-Sobolev-Maz’ya inequalities with multiple interior singularities

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\textbf{Dedicated to Prof. Vladimir Maz’ya with esteem}

\textbf{Abstract}

In this article we first establish a complete characterization of Hardy’s inequalities in $\mathbb{R}^n$ involving distances to different codimension subspaces. In particular the corresponding potentials have strong interior singularities. We then provide necessary and sufficient conditions for the validity of Hardy-Sobolev-Maz’ya inequalities with optimal Sobolev terms.

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\section{Introduction}

For $n \geq 3$ we write $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, with $1 \leq k \leq n$. We also introduce the codimension $k$ affine subspace

$$S_k := \{ x = (x_1, \ldots, x_k, \ldots, x_n) \in \mathbb{R}^n : x_1 = \ldots = x_k = 0 \}.$$  

The Euclidean distance of a point $x \in \mathbb{R}^n$ from $S_k$ is then given by

$$d(x) = d(x, S_k) = |X_k|, \quad \text{where} \quad X_k := (x_1, \ldots, x_k, 0, \ldots, 0).$$
The classical Hardy inequality in $\mathbb{R}^n$ when distance is taken from $S_k$, reads
\[
\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq \left( \frac{k-2}{2} \right)^2 \int_{\mathbb{R}^n} \frac{u^2}{|x_k|^2} \, dx, \quad u \in C_0^\infty(\mathbb{R}^n \setminus S_k), \tag{1.1}
\]
where the constant $\left( \frac{k-2}{2} \right)^2$ is the optimal one. This result has been improved and generalized in many different ways, see for example [1, 2, 5, 8, 11, 13, 14, 15, 18, 20, 27, 28] and references therein.

On the other hand the standard Sobolev inequality with critical exponent states that
\[
\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq S_n \left( \int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}} , \quad u \in C_0^\infty(\mathbb{R}^n),
\]
where $S_n = \pi n (n-2) \left( \frac{\Gamma(n/2)}{\Gamma(n)} \right)^{2/n}$ is the best Sobolev constant, see [6, 25]. For versions of Sobolev inequalities involving subcritical exponents and weights see e.g. [4, 7, 12].

Maz'ya, in his book, combined both inequalities when $1 \leq k \leq n-1$, establishing that for any $u \in C_0^\infty(\mathbb{R}^n \setminus S_k)$
\[
\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq \left( \frac{k-2}{2} \right)^2 \int_{\mathbb{R}^n} \frac{u^2}{|x_k|^2} \, dx + c_{k,Q} \left( \int_{\mathbb{R}^n} |x_k|^{1-2n-\frac{n}{2}} |u|^Q \, dx \right)^{\frac{2}{q}}, \tag{1.2}
\]
for $2 < Q \leq 2^* = \frac{2n}{n-2}$; cf. [22], Section 2.1.6/3. Concerning the best constant $c_{k,Q}$, it was shown in [20] that $c_{k,2^*} < S_n$ for $3 \leq k \leq n-1$, $n \geq 4$ or $k = 1$ and $n \geq 4$. Surprisingly, in the case $k = 1$ and $n = 3$ Benguria Frank and Loss [9] (see also Mancini and Sandeep [21]) established that $c_{1,6} = S_3 = 3(\pi/2)^{4/3}$! Maz'ya and Shaposhnikova [23] have recently computed the best constant in the case $k = 1$ and $Q = \frac{2(n+1)}{n-1}$. These are the only cases where the best constant $c_{k,Q}$ is known. For other type of Hardy–Sobolev inequalities see [10, 17, 24].

In case $k = n$, that is, when distance is taken from the origin, inequality (1.2) fails. Brezis and Vazquez [11] considered a bounded domain containing the origin and improved the Hardy inequality by adding a subcritical Sobolev term. It turns out that in a bounded domain one can have the critical Sobolev exponent at the expense however of adding a logarithmic weight. More specifically let
\[
X(t) = (1 - \ln t)^{-1}, \quad 0 < t < 1.
\]
Then the analogue of (1.2) in the case of a bounded domain $\Omega$ containing the origin, for the critical exponent reads:
\[
\int_{\Omega} |\nabla u|^2 \, dx - \left( \frac{n-2}{2} \right)^2 \int_{\Omega} \frac{u^2}{|x|^2} \, dx \geq C_n(\Omega) \left( \int_{\Omega} X^{\frac{n}{n-2}} \left( \frac{|x|}{D} \right) |u|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}}, \quad u \in C_0^\infty(\Omega), \tag{1.3}
\]
where $D = \sup_{x \in \partial \Omega} |x|$; cf [18]. The best constant in (1.3) was recently computed in [3] and is given by
\[
C_n(\Omega) = (n-2) \frac{2(n-1)}{n} S_n.
\]
It is worth noticing that in the case $n = 3$ once again one has $C_3(\Omega) = S_3 = 3(\pi/2)^{4/3}$!

In a recent work [19] we studied Hardy–Sobolev–Maz'ya inequalities that involve distances taken from different codimension subspaces of the boundary. In particular, working in the upper
half space $\mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : x_1 > 0 \}$ and taking distances from $S_k \subset \partial \mathbb{R}^n_+ \equiv S_1$, $k = 1, 2, \ldots, n$, we have established that the following inequality holds true for any $u \in C_0^\infty(\mathbb{R}^n_+)$

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n_+} \left( \frac{\beta_1}{x_1^2} + \frac{\beta_2}{|X_2|^2} + \ldots + \frac{\beta_n}{|X_n|^2} \right) u^2 dx,$$  

(1.4)

if and only if there exist nonpositive constants $\alpha_1, \ldots, \alpha_n$, such that

$$\beta_1 = -\alpha_1^2 + \frac{1}{4}, \quad \beta_m = -\alpha_m^2 + \left( \alpha_{m-1} - \frac{1}{2} \right)^2, \quad m = 2, 3, \ldots, n.$$  

(1.5)

Moreover if $\alpha_n < 0$ one can add in the right hand side the critical Sobolev term, thus obtaining the Hardy–Sobolev–Maz’ya inequality valid for any $u \in C_0^\infty(\mathbb{R}^n_+)$

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n_+} \left( \frac{\beta_3}{|X_3|^2} + \ldots + \frac{\beta_n}{|X_n|^2} \right) u^2 dx + C \left( \int_{\mathbb{R}^n_+} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}};$$  

(1.6)

we refer to [19] for the detailed statements.

In the present work we consider the case where distances are again taken from different codimension subspaces $S_k \subset \mathbb{R}^n$, which however are now placed in the interior of the domain $\mathbb{R}^n$. We consider the cases $k = 3, \ldots, n$ since there is no positive Hardy constant in case $k = 2$ (cf (1.1)) and the case $k = 1$ corresponds to the case studied in [19].

More precisely our first result reads

**Theorem A (Improved Hardy inequality)**

*Suppose $n \geq 3$.*

**i)** Let $\alpha_3, \alpha_4, \ldots, \alpha_n$ be arbitrary real numbers and

$$\beta_3 = -\alpha_3^2 + \frac{1}{4}, \quad \beta_m = -\alpha_m^2 + \left( \alpha_{m-1} - \frac{1}{2} \right)^2, \quad m = 4, \ldots, n.$$  

Then for any $u \in C_0^\infty(\mathbb{R}^n)$ there holds

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n} \left( \frac{\beta_3}{|X_3|^2} + \ldots + \frac{\beta_n}{|X_n|^2} \right) u^2 dx.$$  

**ii)** Suppose that for some real numbers $\beta_3, \beta_4, \ldots, \beta_n$ the following inequality holds

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n} \left( \frac{\beta_3}{|X_3|^2} + \ldots + \frac{\beta_n}{|X_n|^2} \right) u^2 dx,$$

for any $u \in C_0^\infty(\mathbb{R}^n)$. Then, there exists nonpositive constants $\alpha_3, \ldots, \alpha_n$, such that

$$\beta_3 = -\alpha_3^2 + \frac{1}{4}, \quad \beta_m = -\alpha_m^2 + \left( \alpha_{m-1} - \frac{1}{2} \right)^2, \quad m = 4, \ldots, n.$$  

We note that the recursive formula for the $\beta$’s in the above Theorem, is the same as in (1.5). However, since the coefficients in the above Theorem start from $\beta_3$ – and not from $\beta_1$ – the best constants in the case of interior singularities are different from the best constants when singularities
of the same codimension are placed on the boundary. See for instance Corollary 2.3 and compare with Corollary 2.4 of [19].

To state our next results we define

\[ \beta_3 = -\alpha_3^2 + \frac{1}{4}, \quad \beta_m = -\alpha_m^2 + \left(\alpha_{m-1} - \frac{1}{2}\right)^2, \quad m = 4, \ldots, n. \quad (1.7) \]

Our next theorem gives a complete answer as to when we can add a Sobolev term.

**Theorem B (Improved Hardy–Sobolev–Maz’ya inequality)**

Let \( \alpha_3, \alpha_4, \ldots, \alpha_n, n \geq 3 \), be arbitrary nonpositive real numbers and \( \beta_3, \ldots, \beta_n \) are given by (1.7). Then, if \( \alpha_n < 0 \) there exists a positive constant \( C \) such that for any \( u \in C_0^\infty(\mathbb{R}^n) \) there holds

\[ \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq \int_{\mathbb{R}^n} \left( \frac{\beta_3}{|X_3|^2} + \cdots + \frac{\beta_n}{|X_n|^2} \right) u^2 \, dx + C \left( \int_{\mathbb{R}^n} |X_2|^{\frac{4n^2 - 2n - Q}{2}} |u|^Q \, dx \right)^{\frac{2}{Q}}, \quad (1.8) \]

for any \( 2 < Q \leq \frac{2n}{n-2} \).

If \( \alpha_n = 0 \) then there is no positive constant \( C \) such that (1.8) holds.

The above result extends considerably the original inequality by Maz’ya (1.2). First by having at the same time, all possible combinations of Hardy potentials involving the distances \( |X_3|, \ldots, |X_n| \). In addition the weight in the Sobolev term is stronger than the weight used in (1.2).

We note that a similar result can be produced in the setting of [19] where singularities are placed on the boundary \( \partial\mathbb{R}^n_+ \). More precisely the following inequality holds true for any \( u \in C_0^\infty(\mathbb{R}^n_+) \)

\[ \int_{\mathbb{R}^n_+} |\nabla u|^2 \, dx \geq \int_{\mathbb{R}^n_+} \left( \frac{\beta_1}{x_1^2} + \frac{\beta_2}{|X_2|^2} + \cdots + \frac{\beta_n}{|X_n|^2} \right) u^2 \, dx + C \left( \int_{\mathbb{R}^n_+} x_1^{\frac{4n^2 - 2n - Q}{2}} |u|^Q \, dx \right)^{\frac{2}{Q}}, \quad (1.9) \]

provided that \( \alpha_n < 0 \), where the constants \( \beta_i \) are given by (1.5) and \( 2 < Q \leq \frac{2n}{n-2} \). In this case the weight in the right hand side is even stronger than the one in (1.8). In the light of (1.9) one may ask whether one can replace the weight \( |X_2| \) in (1.8) by \( |x_1| \). It turns out that this is possible provided we properly restrict the exponent \( Q \). More precisely we have:

**Theorem C (Improved Hardy–Sobolev–Maz’ya inequality)**

Let \( \alpha_3, \alpha_4, \ldots, \alpha_n, n \geq 3 \), be arbitrary nonpositive real numbers and \( \beta_3, \ldots, \beta_n \) are given by (1.7). Then, if \( \alpha_n < 0 \) there exists a positive constant \( C \) such that for any \( u \in C_0^\infty(\mathbb{R}^n) \) there holds

\[ \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq \int_{\mathbb{R}^n} \left( \frac{\beta_3}{|X_3|^2} + \cdots + \frac{\beta_n}{|X_n|^2} \right) u^2 \, dx + C \left( \int_{\mathbb{R}^n} |x_1|^{\frac{2(n-1)}{n-2} - Q} |u|^Q \, dx \right)^{\frac{2}{Q}}, \quad (1.10) \]

for any \( \frac{2(n-1)}{n-2} < Q \leq \frac{2n}{n-2} \).

If \( \alpha_n = 0 \) then there is no positive constant \( C \) such that (1.10) holds.

It is easily seen that the range of the exponent \( Q \) in Theorem C is optimal since otherwise the weight is not locally integrable. In the special case \( \beta_3 = \ldots = \beta_n = 0 \), the corresponding weighted Sobolev inequality in (1.10) was proved by Maz’ya, cf [22] section 2.1.6/2.

An important role in our analysis is played by two weighted Sobolev inequalities, which are of independent interest; see Theorems 3.11 and 5.2.
The paper is organized as follows. In section 2 we give the proof of Theorem A. In section 3 we give the proofs of Theorems B and C. The main ideas are similar to the ones used in [19] to which we refer on various occasions. On the other hand ideas or technical estimates that are different from [19] are presented in detail.

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2 Improved Hardy inequalities with multiple singularities

The following simple lemma may be found in [19].

Lemma 2.1. (i) Let \( F \in C^1(\Omega) \), then

\[
\int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} (\text{div } F - |F|^2) |u|^2 dx + \int_{\Omega} |\nabla u + Fu|^2 dx, \quad \forall u \in C_0^\infty(\Omega).
\]

(ii) Let \( \phi > 0, \phi \in C^2(\Omega) \) and \( u = \phi v \), then we have

\[
\int_{\Omega} |\nabla u|^2 dx = - \int_{\Omega} \nabla \phi \cdot \nabla v |u|^2 dx + \int_{\Omega} \phi |\nabla v|^2 dx, \quad \forall u \in C_0^\infty(\Omega).
\]

Proof. By expanding the square we have

\[
\int_{\Omega} |\nabla u + Fu|^2 dx = \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |F|^2 u^2 dx + \int_{\Omega} F \cdot \nabla u^2 dx.
\]

Identity (2.1) now follows by integrating by parts the last term.

To prove (2.2) we apply (2.1) to \( F = -\nabla \phi \phi \). Elementary calculations now yield the result.

Let us recall our notation

\[ X_k := (x_1, \ldots, x_k, 0, \ldots, 0) \] so that \( |X_k|^2 = x_1^2 + \ldots + x_k^2 \);

in particular \( |X_n| = |x| \). We now give the proof of the first part of Theorem A:

Proof of Theorem A part (i): Let \( \gamma_3, \gamma_4, \ldots, \gamma_n \) be arbitrary real numbers and put

\[ \phi := |X_3|^{-\gamma_3}|X_4|^{-\gamma_4} \cdots |X_n|^{-\gamma_n}, \]

and

\[ F := -\nabla \phi \phi. \]

An easy calculation shows that

\[ F = \sum_{m=3}^{n} \gamma_m \frac{X_m}{|X_m|^2}. \]

With this choice of \( F \), we get

\[ \text{div } F = \sum_{m=3}^{n} \gamma_m \frac{(m-2)}{|X_m|^2}. \]
and
\[ |F|^2 = \sum_{m=3}^{n} \frac{\gamma_m^2}{|X_m|^2} + 2 \sum_{m=3}^{n-1} \sum_{j=1}^{m} \frac{\gamma_m \gamma_j}{|X_m|^2 |X_j|^2} = \sum_{m=3}^{n} \frac{\gamma_m^2}{|X_m|^2} + 2 \sum_{m=3}^{n-1} \sum_{j=1}^{m} \gamma_m \gamma_j. \]

We then get that
\[ -\frac{\Delta \phi}{\phi} = \text{div} F - |F|^2 = \sum_{m=3}^{n} \frac{\beta_m}{|X_m|^2}, \]
where
\[ \beta_3 = -\gamma_3 (\gamma_3 - 1), \]
\[ \beta_m = -\gamma_m (2 - m + \gamma_m + 2 \sum_{j=3}^{m-1} \gamma_j), \quad m = 4, 5, \ldots, n. \]

We next set
\[ \gamma_3 = \alpha_3 + \frac{1}{2}, \]
\[ \gamma_m = \alpha_m - \alpha_{m-1} + \frac{1}{2}, \quad m = 4, 5, \ldots, n. \]

With this choice of \( \gamma \)'s the \( \beta \)'s are given as in the statement of the Theorem.

We use Lemma 2.1 with \( \Omega = \mathbb{R}^n \setminus K_3 \), where \( K_3 := \{ x \in \mathbb{R}^n : x_1 = x_2 = x_3 = 0 \} \). We have
\[ \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq \int_{\mathbb{R}^n} (\text{div} F - |F|^2) u^2 \, dx, \quad u \in C_0^\infty (\mathbb{R}^n \setminus K_3). \quad (2.4) \]

By a standard density argument (2.4) is true even for \( u \in C_0^\infty (\mathbb{R}^n) \). The result then follows from (2.3) and (2.4).

Some interesting cases are presented in the following corollary.

**Corollary 2.2.** Let \( k=3, \ldots, n \), \( n \geq 3 \), and \( u \in C_0^\infty (\mathbb{R}^n) \). Then
\[ \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq \int_{\mathbb{R}^n} \left( \frac{k-2}{2} \right)^2 \frac{1}{|X_k|^2} + \frac{1}{4 |X_{k+1}|^2} \cdots + \frac{1}{4 |X_n|^2} \right) u^2 \, dx. \quad (2.5) \]

Also,
\[ \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq \left( \frac{k-2}{2} \right)^2 \int_{\mathbb{R}^n} \frac{u^2}{|X_k|^2} \, dx + \left( \frac{n-k}{2} \right)^2 \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} \, dx. \quad (2.6) \]

**Proof.** We first prove (2.5). In the case \( k = 3 \) we choose \( \alpha_3 = \alpha_4 = \ldots = \alpha_n = 0 \). In this case all \( \beta_k \)'s are equal to 1/4. In the general case \( k > 3 \) we choose \( \alpha_m = -(m-2)/2 \), when \( m = 3, \ldots, k-1 \) and \( \alpha_m = 0 \), when \( m = k, \ldots, n \).

To prove (2.6) we choose \( \alpha_m = -(m-2)/2 \) when \( m = 3, \ldots, k-1 \), \( a_k = 0 \), \( a_{k+l} = -\frac{1}{2} \), \( l = 1, \ldots, n-k-1 \), \( a_n = 0 \).

We next give the proof of the second part of Theorem A:
Proof of Theorem A, part (ii): We will first prove that \( \beta_3 \leq \frac{1}{4} \), therefore \( \beta_3 = -\alpha_3^2 + \frac{1}{4} \), for suitable \( \alpha_3 \leq 0 \). Then, for this \( \beta_3 \), we will prove that \( \beta_4 \leq (\alpha_3 - \frac{1}{n})^2 \), and therefore \( \beta_4 = -\alpha_4^2 + (\alpha_3 - \frac{1}{4})^2 \) for suitable \( \alpha_4 \leq 0 \) and so on.

**Step 1.** Let us first prove the estimate for \( \beta_3 \). To this end we set

\[
Q_3[u] := \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dx - \sum_{i=4}^{n} \beta_i \int_{\mathbb{R}^n} \frac{u^2}{(x_1^2 + x_2^2 + \ldots + x_i^2)} dx}{\int_{\mathbb{R}^n} \frac{u^2}{x_1^2 + x_2^2 + x_3^2} dx}. 
\]

(2.7)

We clearly have that \( \beta_3 \leq \inf_{u \in C_0^\infty(\mathbb{R}^n)} Q_3[u] \). In the sequel we will show that

\[
\inf_{u \in C_0^\infty(\mathbb{R}^n)} Q_3[u] \leq \frac{1}{4},
\]

(2.8)

whence, \( \beta_3 \leq \frac{1}{4} \).

At this point we introduce a family of cutoff functions for later use. For \( j = 3, \ldots, n \) and \( k_j > 0 \) we set

\[
\phi_j(t) = \begin{cases} 0, & t < \frac{1}{k_j} \\ 1 + \frac{\ln k_j t}{\ln k_j}, & \frac{1}{k_j} \leq t < \frac{1}{k_j} \\ 1, & t \geq \frac{1}{k_j}, \end{cases}
\]

and

\[
h_{k_j}(x) := \phi_j(r_j) \quad \text{where} \quad r_j := |X_j| = (x_1^2 + \ldots + x_j^2)^{\frac{1}{j}}.
\]

Note that

\[
|\nabla h_{k_j}(x)|^2 = \begin{cases} \frac{1}{\ln^2 k_j} \frac{1}{r_j^2}, & \frac{1}{k_j} \leq r_j \leq \frac{1}{k_j}, \\ 0, & \text{otherwise} \end{cases}
\]

We also denote by \( \phi(x) \) a radially symmetric \( C_0^\infty(\mathbb{R}^n) \) function such that \( \phi = 1 \) for \( |x| < 1/2 \) and \( \phi = 0 \) for \( |x| > 1 \).

To prove (2.8) we consider the family of functions

\[
u_{k_3}(x) = |X_3|^{-\frac{1}{2}} h_{k_3}(x) \phi(x).
\]

(2.9)

We will show that as \( k_3 \to \infty \)

\[
\int_{\mathbb{R}^n} |\nabla u_{k_3}|^2 dx - \sum_{i=4}^{n} \beta_i \int_{\mathbb{R}^n} \frac{u^2}{(x_1^2 + x_2^2 + \ldots + x_i^2)} dx = \int_{\mathbb{R}^n} |\nabla u_{k_3}|^2 dx + o(1).
\]

(2.10)

To see this, let us first examine the behavior of the denominator. For \( k_3 \) large we easily compute

\[
\int_{\mathbb{R}^n} |X_3|^{-3} h_{k_3}^2 \phi^2 dx \geq C \int_{\frac{1}{k_3} < x_1^2 + x_2^2 + x_3^2 < \frac{1}{4}} (x_1^2 + x_2^2 + x_3^2)^{-\frac{3}{2}} dx_1 dx_2 dx_3 \geq C \int_0^{\pi} \int_{\frac{1}{k_3}}^{\frac{1}{4}} r^{-1} \sin \theta dr d\theta \geq C \ln k_3.
\]

(2.11)

On the other hand by Lebesgue dominated theorem the terms \( \sum_{i=4}^{n} \beta_i \int_{\mathbb{R}^n_+} \frac{u^2}{(x_1^2 + x_2^2 + \ldots + x_i^2)} dx \) are easily seen to be bounded as \( k_3 \to \infty \). From this we conclude (2.10).
We now estimate the gradient term in (2.10).

\[
\int_{\mathbb{R}^n} |\nabla u_{k_3}|^2 dx = \frac{1}{4} \int_{\mathbb{R}^n} |X_3|^{-3} h_{k_3}^2 \phi^2 dx + \int_{\mathbb{R}^n} |X_3|^{-1} |\nabla h_{k_3}|^2 \phi^2 + \int_{\mathbb{R}^n} |X_3|^{-1} |h_{k_3}|^2 |\nabla \phi|^2 + \text{mixed terms.}
\]  

(2.12)

The first integral of the right hand side behaves exactly as the denominator, that is, it goes to infinity like \(O(\ln k_3)\). The last integral is easily seen to be bounded as \(k_3 \to \infty\). For the middle integral we have

\[
\int_{\mathbb{R}^n} |X_3|^{-1} |\nabla h_{k_3}|^2 \phi^2 \leq \frac{C}{\ln^2 k_3} \int_{\mathbb{R}^n} \frac{1}{\sqrt{(x_1^3 + x_2^3 + x_3^3)^{1/2}} k_3} \bigg| X_3 \bigg|^{-3} dx_1 dx_2 dx_3 \leq \frac{C}{\ln k_3}.
\]

As a consequence of these estimates, we easily get that the mixed terms in (2.12) are of the order \(o(\ln k_3)\) as \(k_3 \to \infty\). Hence, we have that as \(k_1 \to \infty\),

\[
\int_{\mathbb{R}^n} |\nabla u_{k_3}|^2 dx = \frac{1}{4} \int_{\mathbb{R}^n} |X_3|^{-3} h_{k_3}^2 \phi^2 dx + o(\ln k_3).
\]  

(2.13)

From (2.10)-(2.13) we conclude that as \(k_3 \to \infty\)

\[
Q_3[u_{k_3}] = \frac{1}{4} + o(1),
\]

hence \(\inf_{u \in C_0^\infty(\mathbb{R}^n)} Q_3[u] \leq \frac{1}{4}\) and consequently \(\beta_3 \leq \frac{1}{4}\). Therefore for a suitable nonnegative constant \(\alpha_3\) we have that \(\beta_3 = -\alpha_3^2 + \frac{1}{4}\). We also set

\[
\gamma_3 := \alpha_3 + \frac{1}{2}.
\]  

(2.14)

**Step 2.** We will next show that \(\beta_4 \leq (\alpha_3 - \frac{1}{2})^2\). To this end, setting

\[
Q_4[u] := \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dx - (\frac{1}{4} - \alpha_3^2) \int_{\mathbb{R}^n} \frac{u^2}{x_1^3 + x_2^3 + x_3^3} dx - \sum_{i=5}^n \beta_i \int_{\mathbb{R}^n} \frac{u^2}{|X_i|^2} dx}{\int_{\mathbb{R}^n} \frac{u^2}{|X_4|^2} dx},
\]  

(2.15)

will prove that

\[
\inf_{u \in C_0^\infty(\mathbb{R}^n)} Q_4[u] \leq (\alpha_3 - \frac{1}{2})^2.
\]

We now consider the family of functions

\[
u_{k_3,k_4}(x) := |X_3|^{-\gamma_3} |X_4|^{-\alpha_3} \frac{1}{2} h_{k_3}(x) h_{k_4}(x) \phi(x)
\]

\[
= |X_3|^{-\gamma_3} v_{k_3,k_4}(x).
\]  

(2.16)

An easy calculation shows that

\[
Q_4[u_{k_3,k_4}] = \frac{\int_{\mathbb{R}^n} |X_3|^{-2\gamma_3} |\nabla v_{k_3,k_4}|^2 dx - \sum_{i=5}^n \beta_i \int_{\mathbb{R}^n} |X_3|^{-2\gamma_3} |X_i|^{-2\alpha_3} dx}{\int_{\mathbb{R}^n} |X_3|^{-2\gamma_3} |X_4|^{-2\alpha_3} dx}.
\]  

(2.17)

We next use the precise form of \(v_{k_1,k_2}(x)\). Concerning the denominator of \(Q_4[u_{k_3,k_4}]\) we have that

\[
\int_{\mathbb{R}^n} |X_3|^{-2\gamma_3} |X_4|^{-2\alpha_3} dx = \int_{\mathbb{R}^n} (x_1^2 + x_2^2 + x_3^2)^{-1/2 - \alpha_3} (x_1^2 + x_2^2 + x_3^2)^{-2\gamma_3} dx.
\]
Sending $k_3$ to infinity, using the structure of the cutoff functions and then introducing polar coordinates we get

\[ \int_{\mathbb{R}^n} |X_3|^{-2\alpha_3} |X_4|^{-2} v_{\alpha_3,k_4}^2 dx = \int_{\mathbb{R}^n} \left( x_1^2 + x_2^2 + x_3^2 \right)^{-1/2} \left( x_1^2 + x_2^2 + x_3^2 + x_4^2 \right)^{\alpha_3 - 3/2} h_{k_4}^2 \phi^2 dx, \]

\[ \geq C \int_{\frac{k_4}{2} < x_1^2 + x_2^2 + x_3^2} \left( x_1^2 + x_2^2 + x_3^2 \right)^{-1/2 - \alpha_3} \left( x_1^2 + x_2^2 + x_3^2 + x_4^2 \right)^{\alpha_3 - 3/2} dx_1 dx_2 dx_3 dx_4 \]

\[ \geq C \int_{\frac{k_4}{2}}^{\frac{k_1}{2}} r^{-1} dr \]

\[ \geq C \ln k_4. \]

The terms in the numerator that are multiplied by the $\beta_i$'s stay bounded as $k_3$ or $k_4$ go to infinity.

\[ \int_{\mathbb{R}^n} |X_3|^{-2\alpha_3} |\nabla v_{k_3,k_4}|^2 dx = \left( \alpha_3 - \frac{1}{2} \right)^2 \int_{\mathbb{R}^n} |X_3|^{-2\alpha_3} |X_4|^{2\alpha_3 - 3} h_{k_3}^2 h_{k_4}^2 \phi^2 dx \]

\[ + \int_{\mathbb{R}^n} |X_3|^{-2\alpha_3} |X_4|^{2\alpha_3 - 1} |\nabla (h_{k_3} h_{k_4})|^2 \phi^2 \]

\[ + \int_{\mathbb{R}^n} |X_3|^{-2\alpha_3} |X_4|^{2\alpha_3 - 1} h_{k_3}^2 h_{k_4}^2 |\nabla \phi|^2 \]

\[ + \text{mixed terms.} \] (2.18)

The first integral in the right hand side above, is the same as the denominator of $Q_4$, and therefore is finite as $k_3 \to \infty$ and increases like $\ln k_4$ as $k_4 \to \infty$, cf (2.11). The last integral is bounded, no matter how big the $k_3$ and $k_4$ are. Concerning the middle term we have

\[ M[v_{k_3,k_4}] := \int_{\mathbb{R}^n} |X_3|^{-2\alpha_3} |X_4|^{2\alpha_3 - 1} |\nabla (h_{k_3} h_{k_4})|^2 \phi^2 dx \]

\[ = \int_{\mathbb{R}^n} |X_3|^{-2\alpha_3} |X_4|^{2\alpha_3 - 1} h_{k_3}^2 h_{k_4}^2 \phi^2 dx + \int_{\mathbb{R}^n} |X_3|^{-2\alpha_3} |X_4|^{2\alpha_3 - 1} h_{k_3}^2 |\nabla h_{k_4}|^2 \phi^2 dx \]

\[ + \text{mixed term} \]

\[ =: I_1 + I_2 + \text{mixed term}. \] (2.19)

Since

\[ |X_4|^{2\alpha_3 - 1} h_{k_4}^2 = r_4^{2\alpha_3 - 1} \phi_4(r_4) \leq C_{k_4}, \quad 0 < r_4 < 1, \]

we easily get

\[ I_1 \leq \frac{C}{(\ln k_3)^2} \int_{\frac{k_4}{2} < (x_1^2 + x_2^2 + x_3^2)^{1/2} < \frac{k_1}{2}} \left( x_1^2 + x_2^2 + x_3^2 \right)^{-\alpha_3 - 3/2} dx_1 dx_2 dx_3, \]

and therefore, since $\alpha_3 \leq 0$,

\[ I_1 \leq \frac{C}{\ln k_3}, \quad k_3 \to \infty. \] (2.20)

Also, since

\[ |X_3|^{-2\alpha_3} h_{k_3}^2 = r_3^{2\alpha_3 - 1} \phi_3(r_3) \leq C_{k_3}, \quad 0 < r_3 < 1, \]
we similarly get (for any $k_3$)

$$I_2 \leq \frac{C}{(\ln k_4)^2} \int_{k_4^{1/2} < (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2} < k_4} (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{-1/2} dx_1 dx_2 dx_3 dx_4$$

(2.21)

$$\leq \frac{C}{\ln k_4}, \quad k_4 \to \infty. \quad (2.22)$$

From (2.19)–(2.21) we have that as $k_4 \to \infty$,

$$M[u_{\infty,k_4}] = o(1).$$

Returning to (2.18) we have that as $k_4 \to \infty$,

$$\int_{R^n} |X_3|^{-2\gamma_3} |\nabla u_{\infty,k_4}|^2 dx = \left(\alpha_3 - \frac{1}{2}\right)^2 \int_{R^n} |X_4|^{-2\gamma_3} |X_4|^{-2} v_{\infty,k_4}^2 dx + o(\ln k_4).$$

(2.23)

We then have that as $k_4 \to \infty$,

$$Q_4[u_{\infty,k_4}] = \left(\alpha_3 - \frac{1}{2}\right)^2 + o(1),$$

(2.24)

consequently, $\beta_4 \leq (\alpha_3 - \frac{1}{2})^2$, and therefore $\beta_4 = -\alpha_4^2 + (\alpha_3 - \frac{1}{2})^2$ for suitable $\alpha_4 \leq 0$. We also set

$$\gamma_4 = \alpha_4 - \alpha_3 + \frac{1}{2}.$$  

**Step 3.** The general case. At the $(q-1)$th step, $3 \leq q \leq n$, we have already established that

$$\beta_3 = -\alpha_3^2 + \frac{1}{4},$$

$$\beta_m = -\alpha_m^2 + \left(\alpha_{m-1} - \frac{1}{2}\right)^2, \quad m = 4, 5, \ldots, q-1,$$

for suitable nonpositive constants $\alpha_i$. Also, we have defined

$$\gamma_3 = \alpha_3 + \frac{1}{2},$$

$$\gamma_m = \alpha_m - \alpha_{m-1} + \frac{1}{2}, \quad m = 4, 5, \ldots, q-1.$$

Our goal for the rest of the proof is to show that $\beta_q \leq (\alpha_{q-1} - \frac{1}{2})^2$. To this end we consider the quotient

$$Q_q[u] := \frac{\int_{R^n} |\nabla u|^2 dx - \sum_{q \neq q=3}^n \beta_i \int_{R^n} |X_i|^2 dx}{\int_{R^n} |u|^2 dx}. \quad (2.25)$$

The test function is now given by

$$u_{k_3,k_q}(x) := |X_3|^{-\gamma_3} |X_4|^{-\gamma_4} \ldots |X_{q-1}|^{-\gamma_{q-1}} |X_q|^{-\gamma_q-\frac{1}{2}} h_{k_4}(x) h_{k_q}(x) \phi(x)$$

$$=: |X_3|^{-\gamma_3} |X_4|^{-\gamma_4} \ldots |X_{q-1}|^{-\gamma_{q-1}} v_{k_q}(x). \quad (2.26)$$

The proof is analogous to the case $q = 4$ and goes along the lines of [19].

The following corollary is a direct consequence of the above theorem and shows that the constants obtained in Corollary [2.22] are sharp.
Corollary 2.3. For $3 \leq k \leq n$,
\[
\inf_{u \in C^1_0(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} \nabla u^2 \, dx}{\int_{\mathbb{R}^n} \frac{|u|}{X_k}^2 \, dx} = \left( \frac{k - 2}{2} \right)^2 \tag{2.27}
\]
\[
\inf_{u \in C^1_0(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} \nabla u^2 \, dx - \frac{(k - 2)^2}{2} \int_{\mathbb{R}^n} \frac{|u|^2}{X_k} \, dx - \cdots - \frac{1}{4} \int_{\mathbb{R}^n} \frac{|u|^2}{X_{m+1}} \, dx}{\int_{\mathbb{R}^n} \frac{|u|^2}{X_{m+1}} \, dx} = \frac{1}{4} \tag{2.28}
\]
for $k \leq m < n$. And
\[
\inf_{u \in C^1_0(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} \nabla u^2 \, dx - \frac{(k - 2)^2}{2} \int_{\mathbb{R}^n} \frac{|u|^2}{X_k} \, dx}{\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2} \, dx} = \left( \frac{n - k}{2} \right)^2 \tag{2.29}
\]

Proof. All are consequences of Theorem A. For (2.27) we take $\alpha_l = -\frac{k-2}{2}$, $l = 1, \ldots, k - 1$. For (2.28) and (2.29) we take the $\alpha$’s of Corollary 2.2.
\[
\square
\]

3. Hardy-Sobolev-Maz’ya inequalities

We first establish the following result that will be used for Theorem B.

Theorem 3.1. (weighted Sobolev inequality) Let $\sigma_2, \sigma_3, \ldots, \sigma_n$ be real numbers, with $n \geq 2$. We set $c_l := \sigma_2 + \ldots + \sigma_l + l - 1$, for $2 \leq l \leq n$. We assume that
\[
c_l > 0 \quad \text{whenever} \quad \sigma_l \neq 0,
\]
for $l = 2, \ldots, n$. Then, there exists a positive constant $C$ such that for any $w \in C^1_0(\mathbb{R}^n)$ there holds
\[
\int_{\mathbb{R}^n} |X_2|^{\sigma_2} \cdots |X_n|^{\sigma_n} |\nabla w| \, dx \geq C \left( \int_{\mathbb{R}^n} \left( |X_2|^{b} |X_3|^{\sigma_3} \cdots |X_n|^{\sigma_n} |w| \right)^{\frac{q}{n}} \, dx \right)^{\frac{1}{q}} \tag{3.1}
\]
where
\[
b = \sigma_2 - 1 + \frac{q - 1}{q} n \quad \text{and} \quad 1 < q \leq \frac{n}{n - 1}.
\]

Proof: For
\[
1 < q \leq n/(n - 1) \quad \text{and} \quad b = \sigma_2 - 1 + \frac{q - 1}{q} n,
\]
we easily obtain the following $L^1$ interpolation inequality
\[
|||X_2|^b v||_q \leq c_1 |||X_2|^{2\sigma_2} v||_{n/(n-1)} + c_2 |||X_2|^{\sigma_2 - 1} v||_1.
\]
Using the inequality
\[
\left| \int_{\mathbb{R}^n} \text{div}\mathbf{F} |v| \, dx \right| \leq \int_{\mathbb{R}^n} |\mathbf{F}| |\nabla v| \, dx, \tag{3.2}
\]
with the vector field $\mathbf{F} = |X_2|^{\sigma_2 - 1} X_2$ one obtains
\[
|\sigma_2 + 1| \int_{\mathbb{R}^n} |X_2|^{\sigma_2 - 1} |v| \, dx \leq \int_{\mathbb{R}^n} |X_2|^{\sigma_2} |\nabla v| \, dx.
\]
Here we have to restrict ourselves to $\sigma_2 + 1 > 0$ to ensure that $|X_2|^{\sigma_2-1} \in L^1_{loc}(\mathbb{R}^n)$. Also, by combining this inequality with the standard $L^1$ Sobolev inequality we get

$$|||X_2|^{\sigma_2}v||_{\frac{n}{\sigma_2-1}} \leq |||X_2|^{\sigma_2}\nabla v||_1.$$ 

Hence we arrive at

$$\left(\int_{\mathbb{R}^n} (|X_2|^{b}|v|)^q dx \right)^{1/q} \leq c \int_{\mathbb{R}^n} |X_2|^{\sigma_2}|\nabla v| dx.$$ 

Now let $v = |X_3|^{\sigma_3}w$ in the above inequality. This gives

$$|||X_2|^{b}|X_3|^{\sigma_3}w||_q \leq c \int_{\mathbb{R}^n} |X_2|^{\sigma_2}|X_3|^{\sigma_3} w| \nabla w| dx + |\sigma_3| \int_{\mathbb{R}^n} |X_2|^{\sigma_2}|X_3|^{\sigma_3-1} w| dx.$$ 

Letting $F = |X_2|^{\sigma_2}|X_3|^{\sigma_3-1}X_3$ in (3.2), we get

$$|\sigma_2 + \sigma_3 + 2| \int_{\mathbb{R}^n} |X_2|^{\sigma_2} |X_3|^{\sigma_3-1} w| dx \leq \int_{\mathbb{R}^n} |X_2|^{\sigma_2} |X_3|^{\sigma_3} \nabla w| dx. \quad (3.3)$$

Here we have to assume $\sigma_2 + \sigma_3 + 2 > 0$ to guarantee that $|X_2|^{\sigma_2}|X_3|^{\sigma_3-1} \in L^1_{loc}(\mathbb{R}^n)$. The two previous estimates give us

$$|||X_2|^{b}|X_3|^{\sigma_3}w||_q \leq c \int_{\mathbb{R}^n} |X_2|^{\sigma_2} |X_3|^{\sigma_3} \nabla w| dx.$$ 

If we would have $\sigma_3 = 0$, we have our result immediately and we do not have to check whether the constant $\sigma_2 + \sigma_3 + 2$ is positive or not. We may repeat this procedure iteratively. In the $l$-th step we use the vector field

$$F = |X_2|^{\sigma_2}|X_3|^{\sigma_3} \ldots |X_1|^{\sigma_{l-1}}X_l,$$

in (3.2) to get:

$$|c_l| \ |||X_2|^{\sigma_2}|X_3|^{\sigma_3} \ldots |X_1|^{\sigma_{l-1}}w||_1 \leq \int_{\mathbb{R}^n} |X_2|^{\sigma_2} |X_3|^{\sigma_3} \ldots |X_1|^{\sigma_l} \nabla w| dx.$$ 

As before, we note that we do not need this inequality in the case $\sigma_l = 0$ and if $\sigma_l \neq 0$ we have to assume $c_l = \sigma_2 + \ldots + \sigma_l + l - 1 > 0$ to ensure the integrability of the integrand on the left hand side. From this it then analogously follows that

$$c \ |||X_2|^{b}|X_3|^{\sigma_3} \ldots |X_1|^{\sigma_l}w||_q \leq \int_{\mathbb{R}^n} |X_2|^{\sigma_2} |X_3|^{\sigma_3} \ldots |X_1|^{\sigma_l} \nabla w| dx,$$

which is (3.1).

For the proof of Theorem C we will use the following variant of Theorem 3.1.

**Theorem 3.2. (Weighted Sobolev inequality)** Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be real numbers, with $n \geq 2$. We set $\overline{c}_l := \sigma_1 + \ldots + \sigma_l + l - 1$, for $1 \leq l \leq n$. We assume that

$$\overline{c}_l > 0 \quad \text{whenever} \quad \sigma_l \neq 0,$$
for \( l = 1, 2, \ldots, n \). Then, there exists a positive constant \( C \) such that for any \( w \in C_0^{\infty}(\mathbb{R}^n) \) there holds

\[
\int_{\mathbb{R}^n} |x_1|^{\sigma_1} |x_2|^{\sigma_2} \cdots |x_n|^{\sigma_n} |\nabla w| dx \geq C \left( \int_{\mathbb{R}^n} \left( |x_1|^{\frac{b}{q}} |x_2|^{\sigma_2} \cdots |x_n|^{\sigma_n} |w| \right)^q dx \right)^{\frac{1}{q}},
\]

where

\[
b = \sigma_1 - 1 + \frac{q-1}{q} \quad \text{and} \quad 1 < q \leq \frac{n}{n-1}.
\]

**Proof:** Let

\[
1 < q \leq n/(n-1) \quad \text{and} \quad b = \sigma_1 - 1 + \frac{q-1}{q} n.
\]

We first consider the case \( \sigma_1 > 0 \). We will use the following \( L^1 \) interpolation inequality

\[
|||x_1|^b \nu||_q \leq c_1 |||x_1|^{\sigma_2} \nu||_{\frac{n}{n-1}} + c_2 |||x_1|^{\sigma_2-1} \nu||_1.
\]

Working similarly as in the proof of Theorem D we end up with

\[
\left( \int_{\mathbb{R}^n} (|x_1|^b |\nu|)^q dx \right)^{1/q} \leq c \int_{\mathbb{R}^n} |x_1|^{\sigma_1} |\nabla \nu| dx.
\]

(3.5)

In case \( \sigma_1 = 0 \), inequality (3.5) is still valid, see [22], Section 2.1.6/1.

The rest of the proof goes as in Theorem D. That is, we apply (3.5) to \( v = |X_2|^\nu w \) to get

\[
|||x_1|^b |X_2|^{\sigma_2} |w||_q \leq c \int_{\mathbb{R}^n} |x_1|^{\sigma_1} |X_2|^{\sigma_2} |\nabla w| dx + |\sigma_2| c \int_{\mathbb{R}^n} |x_1|^{\sigma_1} |X_2|^{\sigma_2-1} |w| dx.
\]

Letting \( F = |x_1|^{\sigma_1} |X_2|^{\sigma_2-1} X_2 \) in (3.6), we get

\[
|\sigma_1 + \sigma_2 + 1| \int_{\mathbb{R}^n} |x_1|^{\sigma_1} |X_2|^{\sigma_2-1} |w| dx \leq \int_{\mathbb{R}^n} |x_1|^{\sigma_1} |X_2|^{\sigma_2} |\nabla w| dx.
\]

(3.6)

The condition \( \tau_2 = \sigma_1 + \sigma_2 + 1 > 0 \) guarantees that \( |x_1|^{\sigma_1} |X_2|^{\sigma_2-1} \in L^1_{\text{loc}}(\mathbb{R}^n) \) and leads to

\[
|||x_1|^b |X_2|^{\sigma_2} |w||_q \leq c \int_{\mathbb{R}^n} |x_1|^{\sigma_1} |X_2|^{\sigma_2} |\nabla w| dx.
\]

We omit further details.

We are now ready to give the proof of Theorem B:

**Proof of Theorem B:** As a first step we will establish that for any \( v \in C_0^{\infty}(\mathbb{R}^n) \):

\[
\int_{\mathbb{R}^n} |X_2|^{2\sigma_2} |X_3|^{2\sigma_3} \cdots |X_n|^{2\sigma_n} |\nabla v|^2 dx \geq C \left( \int_{\mathbb{R}^n} \left( |X_2|^{2\sigma_2} |X_3|^{2\sigma_3} \cdots |X_n|^{2\sigma_n} |v| \right)^q dx \right)^{\frac{2}{q}},
\]

(3.7)

provided that \( c_l := \sigma_2 + \cdots + \sigma_l + (l-1) > 0 \), if \( \sigma_l \neq 0 \), \( 2 \leq l \leq n \) where

\[
B = \sigma_2 - 1 + \frac{Q-2}{2Q} n \quad \text{and} \quad 2 < Q \leq \frac{2n}{n-2}.
\]
To show (3.7) we apply Theorem 3.1 to the function $w = |v|^{s}$, with $s = \frac{Q+2}{2}$, $sq = Q$ and $b = B$. Trivial estimates give

$$C \left( \int_{\mathbb{R}^n} |X_2|^{\sigma q} |X_3|^{\sigma q} \ldots |X_n|^{\sigma q} |v|^{s q} dx \right)^{1/q} \leq s \int_{\mathbb{R}^n} |X_2|^{\sigma_3} |X_3|^{\sigma_3} \ldots |X_n|^{\sigma_n} |v|^{s-1} |\nabla v| dx.$$  

We apply Cauchy-Schwartz to the right hand side and the result follows.

We will use (3.7) with $\sigma_2 = \frac{1}{4}((Q - 2)n - 2Q)$, so that $2\sigma_2 - \frac{2QB}{Q+2} = 0$. We notice that the requirement

$$c_2 = \sigma_2 + 1 = \frac{1}{4}(Q - 2)(n - 2) > 0,$$  

is equivalent to $Q > 2$ and therefore is satisfied.

To continue we will use Lemma 2.1. We recall that for $\phi > 0$ and $u = \phi v$ with $v \in C_0^\infty(\mathbb{R}^n \setminus S_2)$, we have that

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} \frac{\Delta \phi}{\phi} |u|^2 dx = \int_{\mathbb{R}^n} \phi^2 |\nabla v|^2 dx. \quad (3.8)$$

We will choose for $\phi$,

$$\phi(x) = |X_3|^{\frac{2\alpha_3}{Q+2}} |X_4|^{\frac{2\alpha_4}{Q+2}} \ldots |X_n|^{\frac{2\alpha_n}{Q+2}}$$

$$= |X_3|^{-\gamma_3} |X_4|^{-\gamma_4} \ldots |X_n|^{-\gamma_n}, \quad (3.9)$$

where,

$$\gamma_3 = \alpha_3 + \frac{1}{2},$$

$$\gamma_m = \alpha_m - \alpha_{m-1} + \frac{1}{2}, \quad m = 3, \ldots, n.$$  

Therefore

$$\sigma_m = -\frac{Q+2}{2} \gamma_m, \quad m = 3, \ldots, n.$$  

We now apply (3.7) to obtain that

$$\int_{\mathbb{R}^n} \phi^2 |\nabla v|^2 dx \geq C \left( \int_{\mathbb{R}^n} |X_2|^{\frac{Q+2}{2} n - Q} |\phi v|^{q} dx \right)^{\frac{q}{Q}}, \quad (3.10)$$

provided that for $3 \leq l \leq n$,

$$c_l := \sigma_2 + \ldots + \sigma_l + l - 1 > 0, \quad \text{whenever} \quad \sigma_l \neq 0. \quad (3.11)$$

Combining (3.10) with (3.8) we get

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} \frac{\Delta \phi}{\phi} |u|^2 dx \geq C \left( \int_{\mathbb{R}^n} |X_2|^{\frac{Q+2}{2} n - Q} |u|^{Q} dx \right)^{\frac{Q}{Q}}.$$  

On the other hand, by Theorem A(i),

$$- \frac{\Delta \phi}{\phi} = \frac{\beta_3}{|X_3|^2} + \ldots + \frac{\beta_n}{|X_n|^2},$$

and the desired inequality follows.
It remains to check condition (3.11). For \( l = 2 \) we have already checked it. For \( 3 \leq l \leq n \), after some calculations we find that

\[
c_l = \sigma_2 + \ldots + \sigma_l + l - 1
\]

\[
= \frac{1}{4}(Q - 2)(n - 2) - \frac{Q + 2}{2}(\gamma_l + \ldots + \gamma_l) + l - 1
\]

\[
= \frac{Q + 2}{2} \left( -\alpha_l + \frac{(Q - 2)(n - l)}{2(Q + 2)} \right).
\]

Recalling that \( \alpha_l \leq 0 \), we conclude that if \( l \leq n - 1 \) then \( c_l > 0 \), whereas if \( l = n \), then \( c_n > 0 \) if and only if \( \alpha_n < 0 \). This proves (1.8) for \( u \in C_0^\infty(\mathbb{R}^n \setminus S_2) \) and by a density argument the result holds for any \( u \in C_0^\infty(\mathbb{R}^n) \).

In the rest of the proof we will show that (1.8) fails in case \( \alpha_n = 0 \). To this end we will establish that

\[
\inf_{u \in C_0^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\nabla u|^2 dx - \beta_3 \int_{\mathbb{R}^n} \frac{|u|^2}{|X_3|^2} dx - \ldots - \beta_n \int_{\mathbb{R}^n} \frac{|u|^2}{|X_n|^2} dx = 0, \tag{3.12}
\]

where \( \beta_n = (\alpha_{n - 1} - \frac{1}{2})^2 \). Let

\[
u(x) = |X_3|^{-\gamma_3} \ldots |X_{n - 1}|^{-\gamma_{n - 1}} v(x).
\]

A straightforward calculation, quite similar to the one leading to (2.17), shows that the infimum in (3.12) is the same as the following infimum

\[
\inf_{v \in C_0^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} \left( \frac{\beta_3}{2} |v|^2 \prod_{j=3}^{n-1} |X_j|^{-2\gamma_j} |X_n|^{-2\gamma_n} |X_3|^{-2\gamma_3} dx \right) \left( \frac{\int_{\mathbb{R}^n} \frac{2Q}{2\alpha_3^2} n^{-1} \prod_{j=3}^{n-1} |X_j|^{-\gamma_j} (|v|^Q dx) \right)^{\frac{1}{Q}} = 0. \tag{3.13}
\]

We now choose the following test functions

\[
v_{k_3,\varepsilon} = |X_n|^{-\gamma_n + \varepsilon} h_{k_3}(x) \phi(x), \quad \varepsilon > 0, \tag{3.14}
\]

where \( h_{k_3}(x) \) and \( \phi(x) \) are the same test functions as in the first step of the proof of Theorem A(ii). For this choice, after straightforward calculations, quite similar to the ones used in the proof of Theorem A(ii), we obtain the following estimate for the numerator \( N \) in (3.13):

\[
N[v_{\infty,\varepsilon}] = \left( \left( \alpha_{n - 1} - \frac{1}{2} + \varepsilon \right)^2 - \left( \alpha_{n - 1} - \frac{1}{2} \right)^2 \right) \int_{\mathbb{R}^n} \prod_{j=3}^{n-1} |X_j|^{2\gamma_j} |X_n|^{-2\gamma_n + 2 + \varepsilon} \phi^2(x) dx + O_\varepsilon(1),
\]

\[
= C\varepsilon \int_{\mathbb{R}^n} r^{-1 + 2\varepsilon} \sin \theta_2 \prod_{j=3}^{n-1} (\sin \theta_j)^{1 - 2\alpha_j} \phi^2(r) d\theta_1 \ldots d\theta_{n - 1} dr + O_\varepsilon(1)
\]

\[
= C\varepsilon \int_0^1 r^{-1 + \varepsilon} dr + O_\varepsilon(1).
\]

In the above calculations we have taken the limit \( k_3 \to \infty \) and we have used polar coordinates in \((x_1, \ldots, x_n) \to (\theta_1, \ldots, \theta_{n - 1}, r)\). We then conclude that

\[
N[v_{\infty,\varepsilon}] < C, \quad \text{as} \quad \varepsilon \to 0. \tag{3.15}
\]
Similar calculations for the denominator $D$ in (3.13) reveal that

$$D[v_{\infty, \varepsilon}] = C \left( \int_{\mathbb{R}^n} r^{-1+\varepsilon} Q \prod_{j=2}^{n-1} (\sin \theta_j)^{-n-1+Q(\frac{n-1}{2} - \alpha_j)} \vartheta^Q d\theta \right) \frac{2^\varepsilon}{\varepsilon}$$

$$\geq C \left( \int_{0}^{1} r^{-1+\varepsilon} Q dr \right) \frac{2^\varepsilon}{\varepsilon}$$

$$= C \varepsilon^{-\frac{2^\varepsilon}{\varepsilon}}.$$

We then have that

$$\frac{N[v_{\infty, \varepsilon}]}{D[v_{\infty, \varepsilon}]} \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

and therefore the infimum in (3.15) or (3.12) is equal to zero. This completes the proof of the Theorem.

Here is a consequence of Theorem B.

**Corollary 3.3.** Let $3 \leq k < n$ and $2 < Q \leq \frac{2n}{n-2}$. Then, for any $\beta_n < \frac{1}{4}$, there exists a positive constant $C$ such that for all $u \in C_0^\infty(\mathbb{R}^n)$ there holds

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n} \left( \frac{k-2}{2} \right)^2 \frac{1}{|X_k|^2} + \frac{1}{4 |X_{k-1}|^2} + \ldots + \frac{1}{4 |X_{n-1}|^2} + \frac{\beta_n}{|X_n|^2} \right) |u|^2 dx$$

$$+ C \left( \int_{\mathbb{R}^n} |X_2|^\frac{2-2n}{2} |u|^2 dx \right) \frac{2^\varepsilon}{\varepsilon}.$$

If $\beta_n = \frac{1}{4}$ the previous inequality fails.

In case $k = n$ we have that for any $\beta_n < \frac{(n-2)^2}{4}$, there exists a positive constant $C$ such that for all $u \in C_0^\infty(\mathbb{R}^n)$ there holds

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \beta_n \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx + C \left( \int_{\mathbb{R}^n} |X_2|^\frac{2-2n}{2} |u|^2 dx \right) \frac{2^\varepsilon}{\varepsilon}.$$

The above inequality fails for $\beta_n = \frac{(n-2)^2}{4}$.

**Proof.** In Theorem B we make the following choices: In the case $k = 3$ we choose $\alpha_3 = \alpha_4 = \ldots = \alpha_{n-1} = 0$. In this case $\beta_3 = 1/4$, $k = 1, \ldots, n-1$. The condition $\alpha_n < 0$ is equivalent to $\beta_n < \frac{1}{4}$.

In the case $3 < k \leq n-1$ we choose $\alpha_m = -m/2$, when $m = 1, 2, \ldots, k-1$ and $\alpha_m = 0$, when $m = k, \ldots, n-1$. Finally, in case $k = n$, we choose $\alpha_m = -(m-2)/2$, for $m = 3, 4, \ldots, n-1$.

We finally give the proof of Theorem C:

**Proof of Theorem C:** We first prove that the following inequality holds for any $v \in C_0^\infty(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} |x_1|^{2\sigma_1 - \frac{2QB}{Q+2}} |X_2|^{\frac{4\sigma_2}{Q+2}} \ldots |X_n|^{\frac{4\sigma_n}{Q+2}} |\nabla v|^2 dx \geq C \left( \int_{\mathbb{R}^n} |x_1|^{\frac{2QB}{Q+2}} |X_2|^{\frac{2Q_2}{Q+2}} \ldots |X_n|^{\frac{2Q_n}{Q+2}} |v|^Q dx \right) \frac{2^\varepsilon}{\varepsilon},$$

(3.16)
provided that \( \overline{\alpha}_l := \sigma_1 + \ldots + \sigma_l + (l - 1) > 0 \), if \( \sigma_l \neq 0 \), \( 1 \leq l \leq n \) where

\[
B = \sigma_1 - 1 + \frac{Q - 2}{2Q} n \quad \text{and} \quad \frac{2(n - 1)}{n - 2} < Q \leq \frac{2n}{n - 2}.
\]

To show (3.16) we apply Theorem 3.2 to the function \( \nu^s \), with \( s = \frac{Q + 2}{2} \), \( sq = Q \) and \( b = B \), and then use Cauchy-Schwartz inequality.

We will use (3.16) with \( \sigma_1 = \frac{1}{2}(Q - 2)n - 2Q \) and \( \sigma_2 = 0 \). In this case \( 2\sigma_1 - \frac{2QB}{Q + 2} = 0 \). The choice of \( \phi \) stays the same as in the proof of Theorem B. Eventually, we arrive at

\[
\int_{\mathbb{R}^n} |\nabla u|^2 dx - \int_{\mathbb{R}^n} \left( \frac{\beta_3}{|X_3|^2} + \ldots + \frac{\beta_n}{|X_n|^2} \right) |u|^2 dx \geq C \left( \int_{\mathbb{R}^n} |x_1|^\frac{Q - 2}{2}n - Q |u|^Q dx \right)^\frac{2}{Q},
\]

provided that the \( \overline{\alpha}_l \)'s satisfy our assumptions of Theorem 3.2. However it turns out that

\[
\overline{\alpha}_l = \frac{Q + 2}{2} \left( -\alpha_l + \frac{(Q - 2)(n - l)}{2(Q + 2)} \right), \quad 1 \leq l \leq n,
\]

and our assumptions are satisfied in case \( \alpha_n < 0 \).

In remains to prove that (1.10) fails in case \( \alpha_n = 0 \). To this end we will establish that

\[
\inf_{u \in C_0^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dx - \beta_3 \int_{\mathbb{R}^n} |u|^2 dx - \ldots - \beta_n \int_{\mathbb{R}^n} |u|^2 dx}{\left( \int_{\mathbb{R}^n} |x_1|^\frac{Q - 2}{2}n - Q |u|^Q dx \right)^\frac{2}{Q}} = 0,
\tag{3.17}
\]

where \( \beta_n = (\alpha_{n-1} - \frac{1}{2})^2 \). The test functions used in the proof of Theorem B can also be used here since they belong in the proper function space. The result follows by observing that the weight here is stronger than in Theorem B.

\[\square\]

An easy consequence of the above Theorem is the following:

**Corollary 3.4.** Let \( 3 \leq k \leq n \) and \( \frac{2(n - 1)}{n - 2} < Q \leq \frac{2n}{n - 2} \). Then, for any \( \beta_n < \frac{1}{4} \), there exists a positive constant \( C \) such that for all \( u \in C_0^\infty(\mathbb{R}^n) \) there holds

\[
\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n} \left( \left( \frac{k - 2}{2} \right)^2 \frac{1}{|X_k|^2} + \frac{1}{4} \frac{1}{|X_{k-1}|^2} + \ldots + \frac{1}{4} \frac{1}{|X_{n-1}|^2} + \frac{\beta_n}{|X_n|^2} \right) |u|^2 dx
\]

\[+ C \left( \int_{\mathbb{R}^n} |x_1|^\frac{Q - 2}{2}n - Q |u|^Q dx \right)^\frac{2}{Q}.
\]

If \( \beta_n = \frac{1}{4} \) the previous inequality fails.

In case \( k = n \) we have that for any \( \beta_n < \frac{(n - 2)^2}{4} \), there exists a positive constant \( C \) such that for all \( u \in C_0^\infty(\mathbb{R}^n) \) there holds

\[
\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \beta_n \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx + C \left( \int_{\mathbb{R}^n} |x_1|^\frac{Q - 2}{2}n - Q |u|^Q dx \right)^\frac{2}{Q}.
\]

The above inequality fails for \( \beta_n = \frac{(n - 2)^2}{4} \).
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