Spinor Representations of Orthogonal and Symplectic Yangians

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Abstract—We consider the explicit spinor matrices of low rank orthogonal algebras and the corresponding RTT algebras using the new approach to calculation of spinorial R matrix. We also discuss the Algebraic Bethe Ansatz for spinor and vector monodromy matrices.

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1. INTRODUCTION

Yangians of the orthogonal and symplectic symmetry types allow for the linear evaluation only by additional constraints on the universal enveloping of the corresponding Lie algebra. The spinor representations, where the Yangin generators are constructed from the underlying algebra, obey these constraints, i.e. the Yang–Baxter RLL relation involving the fundamental matrix with orthogonal or symplectic symmetry and the spinor L operators holds [5].

Reshetikhin [6] proposed to use the spinor representation and the fusion procedure to approach the ABA in the orthogonal case. The ABA for the so(3) case has been shown to be related to the well known sl(2) case. Our investigation goes along this concept.

We consider the Yang–Baxter RLL relation acting in the tensor product of the fundamental and two copies of the spinor representations. It is used as the defining relation for the involved R operator intertwining the spinor representations. The spinorial R operator has been obtained in the orthogonal case in [9] and considered in [5, 10], then the uniform formulation of the orthogonal and symplectic cases and of the ortho-symplectic case have been given in [11].

In this paper we propose an alternative approach using instead the invariant built from the product of one generator out of each of the spinor algebras and its powers and turns out much simpler. We obtain the spinorial operator in terms of the Euler Beta function. We consider low rank orthogonal cases. The general result of the spinorial R operator is reduced to a finite sum, which is derived from the spectral decomposition of the invariant.

The structure of the spinorial R in even dimensions (so(2m), Dm series) appears to be much simpler than the odd dimensional case (so(2m + 1), Bm series).

In the even–dimensional cases the separation of the tensor product of spinor representation spaces, where R acts, into chiral parts results in the corresponding separation of R. Both parts of R obey the Yang–Baxter relations separately.

2. THE SPINORIAL R-MATRIX

The Yang–Baxter relation for the fundamental R matrix reads

\[ R^{\alpha\beta}_{\gamma\delta}(u - v)R^{\gamma\delta}_{\epsilon\zeta}(v)R^{\epsilon\zeta}_{\alpha\beta}(u) = R^{\epsilon\zeta}_{\gamma\delta}(v)R^{\gamma\delta}_{\alpha\beta}(u)R^{\alpha\beta}_{\epsilon\zeta}(u). \] (2.1)

R looks similar for the cases of orthogonal and symplectic symmetry [7, 8],

\[ R^{\alpha\beta}_{\gamma\delta}(u) = u \left( u + \frac{n}{2} - \epsilon \right) I^{\alpha\beta}_{\gamma\delta} + \left( u + \frac{n}{2} - \epsilon \right) P^{\alpha\beta}_{\gamma\delta} - \epsilon u K^{\alpha\beta}_{\gamma\delta}, \] (2.2)

where

\[ I^{\alpha\beta}_{\gamma\delta} = \delta^{\alpha}_{\beta} \delta^{\gamma}_{\delta}, \quad P^{\alpha\beta}_{\gamma\delta} = \delta^{\alpha}_{\gamma} \delta^{\beta}_{\delta}, \quad K^{\alpha\beta}_{\gamma\delta} = \epsilon^{\alpha\beta} \epsilon^{\gamma\delta}. \] (2.3)

The choices \( \epsilon + 1 \) and \( \epsilon = -1 \) correspond to the so(n) and sp(2m) cases respectively. The index range is \( a, b, \cdots, n \) or \( 1, \ldots, 2m \). \( \epsilon^{ab} \) denotes the metric tensor which is symmetric in the orthogonal and anti-symmetric in the symplectic case.

The Yang–Baxter RLL relation with the above R

\[ R^{\alpha\beta}_{\gamma\delta}(u)I^{\gamma\delta}_{\epsilon\zeta}(u + v)I^{\epsilon\zeta}_{\alpha\beta}(v) = I^{\gamma\delta}_{\epsilon\zeta}(v)I^{\epsilon\zeta}_{\alpha\beta}(u + v)R^{\alpha\beta}_{\gamma\delta}(u), \] (2.4)

is fulfilled by the linear form of the L-operator

\[ I^{\alpha\beta}_{\gamma\delta}(u) = u \delta^{\alpha}_{\beta} + G^{\alpha}_{\beta}, \quad G^{\alpha}_{\beta} = \frac{1}{2} \left( \epsilon^{a} \epsilon^{b} - \epsilon^{c} \epsilon^{d} \right) \] (2.5)
generated by $c_a$ obeying the commutation relations
\[ e^a c^b + e^b c^a = \epsilon^{ab}, \] (2.6)
or \( (e^a)^\gamma (e^b)^\beta + e (e^b)^\gamma (e^a)^\beta = \epsilon^{ab} \delta^\alpha_\beta. \)

The linear ansatz for $L$-operator (2.5) implies the so(n) or sp(2m) Lie algebra relations,
\[ \left[ G_\alpha^a, G_\beta^b \right] = \delta^b_\beta G_\alpha^a - \delta^a_\alpha G_\beta^b + \epsilon^{ab} e_{\alpha\beta} c^\alpha G_\beta^a \]
\[ + \epsilon^{\alpha\beta} e_{\alpha\beta} G_\alpha^a - e G_\alpha^b e^{\alpha\beta} e_{\beta\alpha} b^\alpha b^\beta, \]
and symmetry condition,
\[ e^{\alpha\beta} e_{\alpha\beta} \left( \delta^a_\alpha G_\beta^a + \delta^a_\beta G_\alpha^a \right) \]
\[ = \left( \delta^a_\alpha G_\beta^a + \delta^a_\beta G_\alpha^a \right) e^{\alpha\beta} e_{\beta\alpha}, \] (2.8)
as well as the additional constraint
\[ e^{\alpha\beta} e_{\alpha\beta} \left( G_\alpha^a - \beta \delta^a_\beta \right) G_\beta^a = G_\alpha^a \left( G_\beta^a - \beta \delta^a_\alpha \right) e^{\alpha\beta} e_{\beta\alpha}, \] (2.9)
which specifies the Yangian linear evaluation.

The Yang–Baxter relation is formulated in the ten-dimensional oscillator Fock space, in the symplectic (or 2-dimensional) and the spinor space concerned with the fundamental (vector) space oror product of three representation spaces. We are which specifies the Yangian linear evaluation.

Along with this Yang–Baxter relation we consider the auxiliary space factors, $R_{\alpha\beta}$. For example, the modified Yang–Baxter relation as
\[ R_{\alpha\beta}(u) L_{\alpha\gamma}(u + v) L_{\beta\gamma}(v) = L_{\beta\gamma}(v) L_{\beta\gamma}(u + v) R_{\alpha\beta}(u). \] (2.10)

Equation (2.11) takes the form
\[ \widetilde{\mathcal{R}}_{12}(u) \left( (u + v + \frac{\epsilon}{2}) \delta^a_\gamma - c^a_1 c^a_0 \right) \]
\[ \times \left( (v + \frac{\epsilon}{2}) \delta^a_\gamma - c^a_2 c^a_0 \right) = \left( (u + v + \frac{\epsilon}{2}) \delta^a_\gamma - c^a_1 c^a_0 \right) \]
\[ \times \left( (u + v + \frac{\epsilon}{2}) \delta^a_\gamma - c^a_2 c^a_0 \right) \] (2.14)

The spinorial $R$ operator is an element of $\mathcal{C}_1 \otimes \mathcal{C}_2$ (2.6).

We consider it as the defining relation for the wanted spinorial $R$ matrix. The terms proportional to $v^2$ are canceled, linear terms lead to the symmetry condition
\[ \mathcal{R}_{12}(u) (c^a_1 c^a_0 + c^a_2 c^a_0) (c^a_1 c^a_1 + c^a_2 c^a_2) \mathcal{R}_{12}(u), \] (2.15)
from which one deduces that $\mathcal{R}_{12}(u)$ depends on $c_1$ and $c_2$ through the invariant combination
\[ z = -i z_{12} = -i c^a_0 e_{ab} c^b_2 = -i e c_{1e} c^e. \] (2.16)
The last relation from $v^0$ has the form
\[ \mathcal{R}_{12}(u) (-u c^a_0 c^a_1 + c^a_0 c^a_1 c^a_1 c^a_2) \] (2.17)

Introducing
\[ c^a_0 = c_1 \pm i c_2, \quad z c^a = c^a_0 (z \pm 1), \]
and multiplying (2.15) by $c^a_{0} e_{ab}$ from the left and by $c^b_{0}$ from the right one obtains
\[ \mathcal{R}_{12}(u | z \mp 1) A = A \mathcal{R}_{12}(u | z \pm 1), \]
where
\[ A = c_x^d e_{da} \left( c_i^a c_i^b + c_i^c c_i^c \right) e_{ob} c_{pz} \left( d^d / 2 \mp i \varepsilon z \right) \]
\[ + \left( \pm i e^d / 2 + i z \right) \left( \pm i d^d / 2 - i e z \right) = 0, \]
here \( d \equiv e_{ab} c^a c^b = \delta_a^a \).

For the analogous projection with the opposite choice of signs one has
\[ \hat{N}_{12}(u | z \mp 1) = B \hat{N}_{12}(u | z \mp 1), \]
where
\[ B = c_x^d e_{da} \left( c_i^a c_i^b + c_i^c c_i^c \right) e_{ob} c_{pz} \left( d^d / 2 \pm i \varepsilon z \right) \]
\[ + \left( \pm i e^d / 2 + i z \right) \left( \pm i d^d / 2 - i e z \right) = 2 e \left( d^d / 2 \pm i \varepsilon z \right)^2. \]

The analogous consideration of the defining relation (2.17) leads to the following equation for the opposite signs,
\[ \hat{N}_{12}(u | z \mp 1) C_L = C_R \hat{N}_{12}(u | z \mp 1), \]
where
\[ C_L = e_x^d e_{da} \left( -u c_i^a c_i^b + c_i^c i e c_{pz} \right) e_{ob} c_{pz} \]
\[ = (-u e \pm z) \left( d^d / 2 \pm i \varepsilon z \right)^2, \]
\[ C_R = e_x^d e_{da} \left( -u c_i^a c_i^b + c_i^c i e c_{pz} \right) e_{ob} c_{pz} \]
\[ = (-u e \mp z) \left( d^d / 2 - i \varepsilon z \right)^2. \]

The projection of (2.17) with the same signs leads to
\[ \hat{N}_{12}(u | z \mp 1) D_L = D_R \hat{N}_{12}(u | z \mp 1), \]
where
\[ D_L = e_x^d e_{da} \left( -u c_i^a c_i^b + c_i^c i e c_{pz} \right) e_{ob} c_{pz} \]
\[ = (u e \mp z) \left( d^d / 4 - z^2 \right), \]
\[ D_R = e_x^d e_{da} \left( -u c_i^a c_i^b + c_i^c i e c_{pz} \right) e_{ob} c_{pz} \]
\[ = (-u e \mp z) \left( d^d / 4 - z^2 \right). \]

Canceling the common factor \( d^d / 4 - z^2 \) in both sides one obtains
\[ \hat{N}_{12}(u | z \mp 1) (u e \mp z) = (-u e \mp z) \hat{N}_{12}(u | z \pm 1), \]
from which one deduces the wanted spinorial operator
\[ \hat{N}_{12}(u | z) = r(u) \frac{\Gamma \left( \frac{1}{2} (z + 1 - u e) \right)}{\Gamma \left( \frac{1}{2} (z + 1 + u e) \right)}. \]

2.1. Symplectic Cases of Low Rank

It is instructive to specify the general solution for spinorial \( R \)-matrix for symplectic algebras of low rank. For \( sp(2) \) one can realize the algebra \( C \) in terms of a pair of operators of multiplication and differentiation as
\[ c_1 = \partial = c_1^1, \quad c_1 = x = -c_1^1, \quad z = -i(x \partial_2 - x \partial_1). \]
Thus we have
\[ \hat{N}_{12}^{sp(2)}(u) = r(u) \frac{\Gamma \left( u + 1 - \frac{1}{2} (x \partial_2 - x \partial_1) \right)}{\Gamma \left( 1 - u - \frac{1}{2} (x \partial_2 - x \partial_1) \right)}. \]
The \( sp(4) \) case corresponds to two such pairs.
\[ c_2 = c_4 = e, \quad c_2 = d = f, \quad c_2 = c_2^1 = d_2^1 = y, \quad c_2 = c_2^2 = d_2^2 = y, \]
\[ z = -i(x \partial_2 - x \partial_1 + y \partial_2 - y \partial_1) = z_x + z_y, \]
\[ \hat{N}_{12}^{sp(4)}(u) = r(u) \times \frac{\Gamma \left( u + 1 - \frac{1}{2} (x \partial_2 - x \partial_1 + y \partial_2 - y \partial_1) \right)}{\Gamma \left( 1 - u - \frac{1}{2} (x \partial_2 - x \partial_1 + y \partial_2 - y \partial_1) \right)}. \]

2.2. The Orthogonal Case

In the orthogonal case, due to the Clifford relation for \( c^a \) (Dirac gamma matrices), the general expression for the spinor-spinor \( R \)-matrix in form of the Euler Beta-function can be transformed to a polynomial in \( z \) as well as to a finite expansion in the invariants
\[ I_k = \gamma^0 \cdots \gamma^k, \quad k = 0, 1, \ldots n. \]
In this case it is convenient to modify slightly the definition of the invariant \( z \):
\[ z = \frac{1}{2} z_x^e z_x^e, \]
Besides of the absence of the imaginary unit in this definition, we suppose here that the Dirac gamma matrices \( \gamma^0 \) and \( \gamma^1 \) related to different spaces \textit{commute}, in contrast to the previous convention when we suppose their \textit{anticommutation} in order to have the unified description with the symplectic case.

Having modified the definition of \( z \) we go through the steps of the above derivation in order to check that the result is not changed.

Consider the RLL-relation
\[ \hat{N}_{12}(u | z) \left( v + \frac{1}{2} \right) \gamma_1 \gamma_2 + \left( u + v + \frac{1}{2} \right) \gamma_2 \gamma_3 - \gamma_1 z \gamma_2 = \left( v + \frac{1}{2} \right) \gamma_1 \gamma_2 + \left( u + v + \frac{1}{2} \right) \gamma_2 \gamma_3 - \gamma_1 z \gamma_2 \hat{N}_{12}(u | z). \]
We multiply by \( \gamma_x^e = \gamma_x^1 \pm \gamma_x^2 \) from the left and by \( \gamma_x^e \) or \( \gamma_x^f \) from the right and use
\[ z \gamma_x^e = \gamma_x^e (-z \pm 1), \]
(note that here an additional minus sign appears due to the change the commutativity convention). We obtain
\[ \mathcal{R}_{12}(u) - z \pm 1)(u + 2v + 1 \mp z)(d \pm 2z)^2 \]
\[ = (u + 2v + 1 \mp z)(d \pm 2z)^2 \mathcal{R}_{12}(u) - z \pm 1), \]
and
\[ \mathcal{R}_{12}(u) - z \pm 1)(-u \pm 2z)(d^2 - 4z^2) \]
\[ = \left(u \pm \frac{1}{2}z\right)(d^2 - 4z^2) \mathcal{R}_{12}(u) - z \mp 1), \]
from which we deduce the general solution for the so(n) case:
\[ \frac{\mathcal{R}_{12}(u) - z \pm 1}{\mathcal{R}_0(u)} = \frac{\Gamma\left(z + 1 - u\right)}{\Gamma\left(z + 1 + u\right)} \]
\[ = B \left(z + 1 - u, u\right). \tag{2.23} \]

2.3. Spinor and Vector Monodromy Matrices

We show how the monodromy matrices with spinor auxiliary space (\( \mathcal{F}_0 \)) and vector auxiliary space (\( \mathcal{V}_0 \)) are related by fusion. The gamma matrices are known to intertwine the vector and the product of spinor representations. Formally, the following analysis can be extended to the symplectic case. However, that case is connected with the problem of defining infinite sums over the spinor indices. We restrict ourselves to the orthogonal case, which will be considered in the remaining part of the paper.

The general vector-vector monodromy matrix is defined by the fundamental \( R \)-matrix (2.2):
\[ T_{abc, \ldots \, an}^{\beta, \ldots \, a_n}(u) R_{\alpha \beta}(u) \ldots R_{\alpha_n \beta_n}(u). \tag{2.24} \]
We are interested in the diagonalization of the trace of this matrix:
\[ t_{\beta_1 \ldots \beta_n}(u) = \mathbb{T}_{\alpha_1 \beta_1 \ldots \beta_n}(u), \tag{2.25} \]
or \( t(u) = t_0 \mathbb{T}(u) \), because it is the generating function of the integrals of motion.

Using the \( L \)-operator (2.5) one can construct another monodromy matrix
\[ T_{\beta_1 \ldots \beta_n, \ldots \, \alpha_n}^{\beta, \ldots \, a_n}(u) = \left(L_{\alpha \beta}(u) \ldots L_{\alpha_n \beta_n}(u)\right)^\alpha, \tag{2.26} \]
acting in the tensor product of the same quantum space \( \mathcal{V}_1 \otimes \ldots \otimes \mathcal{V}_N \) and the spinor auxiliary space \( \mathcal{F}_0 \).

We consider also the related monodromy matrix
\[ T_{\beta_0 \ldots \beta_n, \ldots \, \alpha_n}^{\gamma, \ldots \, a_n}(u) \left(-u - \frac{1}{2}\right) = \left(L_{\alpha \beta}(u) \ldots L_{\alpha_n \beta_n}(u)\right)^\gamma, \tag{2.27} \]
Because of the inversion relation
\[ L_{\alpha \beta}(u + \beta + \frac{1}{2}) L_{\gamma \delta}(u - \frac{1}{2}) = -u(u + \beta + 1) \delta_{\alpha \gamma} \delta_{\beta \delta}, \tag{2.28} \]
it can be expressed by the inverse of the previous monodromy matrix
\[ T_{\beta_0 \ldots \beta_n, \ldots \, \alpha_n}^{\gamma, \ldots \, a_n}(u) \left(-u - \frac{1}{2}\right) = \left(-u(u + \beta + 1)\right)^N, \tag{2.29} \]
We have used that in the orthogonal case (2.5) takes the form
\[ L^a_b(u) = u \delta^{ab} - \frac{1}{2} \gamma^{ab}. \tag{2.29} \]
Consider the product
\[ T_{\beta_1 \ldots \beta_n, \ldots \, \alpha_n}^{\alpha, \ldots \, a_n}(u + \beta - \frac{1}{2})(\gamma_{\beta_0}^a \beta_{\gamma}^a \ldots \beta_n) \left(-u - \frac{1}{2}\right) \]
\[ \equiv L_{\alpha \beta_0, \ldots \, \alpha_n}^{a, \ldots \, a_n}(u + \beta - \frac{1}{2}) \alpha_1 \ldots \alpha_n \beta_1 \ldots \beta_n \ldots \beta_n \]
\[ \times \left(u + \beta - \frac{1}{2}\right)(\gamma_{\beta_0}^a \beta_{\alpha_0}^a \ldots \beta_n) \left(-u - \frac{1}{2}\right) \]
\[ \times L_{\beta_1 \ldots \beta_n, \ldots \, \alpha_n}^{\gamma, \ldots \, a_n}(u) \left(-u - \frac{1}{2}\right) \]
\[ = (-1)^N T_{\alpha_1 \ldots \alpha_n}^{a, \ldots \, a_n}(u) (\gamma_{\beta_0}^a \beta_{\alpha_0}^a \ldots \beta_n). \tag{2.30} \]
In this way we obtain the fusion relation between spinorial and vector monodromy matrices. The fusion of two spinorial monodromy matrices by trace over the auxiliary spinor space \( \mathcal{F}_0 \) results in the vector monodromy matrix with \( \mathcal{V}_0 \) as auxiliary space.

\[ T_{\beta_0 \ldots \beta_n, \ldots \, \alpha_n}^{\gamma, \ldots \, a_n}(u + \beta - \frac{1}{2})(\gamma_{\beta_0}^a \beta_{\gamma}^a \ldots \beta_n) \]
\[ \times \left(u + \beta - \frac{1}{2}\right) \mathbb{T}_{\alpha_1 \ldots \alpha_n}(u) (\gamma_{\beta_0}^a \beta_{\alpha_0}^a \ldots \beta_n). \tag{2.31} \]
3. EVEN-DIMENSIONAL ORTHOGONAL ALGEBRAS

In Section 3 we have obtained the spinorial $\hat{R}$-matrix in form of the Euler Beta-function for an arbitrary orthogonal algebra. Now we shall consider low rank examples corresponding to the $D_4$ series. The universal expression (2.23) results in explicit forms using the corresponding characteristic polynomial in the invariant $z$ and the resulting spectral decomposition, considered in Appendix.

3.1. The $so(4)$ Case

In this case we have the characteristic polynomial

$$W_4 = z(z^2 - 1)(z^2 - 4) = 0.$$  \hspace{1cm} (3.32)

In other words, $d = 4$ any function of $z$ is represented by a polynomial of fourth degree. The roots of $W_4$ are the eigenvalues of $z$ and we are lead to the spectral decomposition

$$\hat{R}_{so(4)}(z|u) = \sum_{k=2}^2 \frac{1}{k+1} B\left(\frac{k+1-u}{2}, u\right) P_k.$$  \hspace{1cm} (3.33)

Here the sum goes over the roots of $W_4$: $z = 0, \pm 1, \pm 2$ and $P_k$ are projection operators on the corresponding eigenspace,

$$P_0 = \frac{1}{4}(z^2 - 1)(z^2 - 4), \quad P_{\pm 1} = \frac{1}{3!}(z \pm 1)(z^2 - 4),$$

$$P_{\pm 2} = \frac{1}{4!}(z^2 - 1)(z \pm 2).$$  \hspace{1cm} (3.34)

Using (3.32) one can check the properties $P_k P_l = P_k \delta_{k,l}$, $\sum_{k=2}^2 P_k = 1$. Also using the functional equation $B(x + 1, y) = \frac{x}{x + y} B(x, y)$, one deduces the explicit form of the spinorial $R$ matrix

$$\hat{R}_{so(4)}(z|u) = B\left(\frac{1-u}{2}, \frac{1-u}{1+u}\right) \left(P_2 + P_{-2} + P_0\right)$$

$$+ B\left(\frac{1-u}{2}, \frac{1-u}{1+u}\right) \left(P_1 - P_{-1}\right) = \hat{R}_{so(4)}^{(1)}(u|z) + \hat{R}_{so(4)}^{(2)}(u|z)$$

separating it into two parts given by even and odd functions of $z$, respectively. Both parts satisfy the Yang–Baxter relation separately.

$z$ is acting on $F_1 \otimes F_2$ and the permutation operator is given by

$$P_{12} = P_0 + P_1 - P_{-1} - P_2 - P_{-2}$$

$$= \frac{1}{6}(z^4 - 2z^2 - 7z^2 + 8z + 6) = \frac{1}{4}(1 + \frac{1}{6!} \gamma_1 \gamma_2^5)$$

$$+ \frac{1}{(2\gamma)^2} \gamma_1^{ab} \gamma_2^{ab} - \frac{1}{(3\gamma)^2} \gamma_1^{abc} \gamma_2^{abc} + \frac{1}{(4\gamma)^2} \gamma_1^{ab} \gamma_2^{ab}.$$  \hspace{1cm} (3.35)

Consider now both parts of the spinorial $R$ matrix in more detail. The simpler part

$$\hat{R}_{so(4)}^{(1)}(u|z) = -P_1 + P_{-1} = -\frac{z^2}{3}(z^2 - 4)$$

$$= -\frac{1}{144} \gamma_1^{abc} c_2 + \frac{3}{4} \gamma_1^{ab} x_2,$$

is given by an odd function of $z$. The corresponding $R$-matrix without check ($\hat{R}^{(1)} = P_1 \hat{R}^{(1)}$) becomes even in $z$,

$$\hat{R}_{so(4)}^{(1)}(u|z) = -P_1 - P_{-1} = -\frac{z^2}{3}(z^2 - 4) = \frac{1}{2}(1 + \gamma_1^5 \gamma_2^5),$$

$$\gamma_1^5 = \gamma_2^5 \gamma_3 \gamma_4,$$

and is trivial, i.e. diagonal and independent of the spectral parameter, with the following non-vanishing entries:

$$\hat{R}_{12}^{(1)} = \hat{R}_{13}^{(1)} = \hat{R}_{21}^{(1)} = \hat{R}_{24}^{(1)} = \hat{R}_{31}^{(1)} = \hat{R}_{34}^{(1)} = \hat{R}_{43}^{(1)} = 1.$$  \hspace{1cm} (3.36)

The RLL Yang–Baxter relation

$$\hat{R}_{so(4)}^{(1)}(u|z) \hat{R}_{so(4)}^{(1)}(v|z) = \hat{R}_{so(4)}^{(1)}(w|z) \hat{R}_{so(4)}^{(1)}(v|z)$$

reduces to the identity:

$$\left(1 + \gamma_1^5 \gamma_2^5\right) \gamma_1^{ab} \gamma_2^{bc} = \gamma_1^{ab} \gamma_2^{bc} \left(1 + \gamma_1^5 \gamma_2^5\right).$$

We have also the Yang–Baxter RRR relation of the trivial form

$$\left(1 + \gamma_1^5 \gamma_2^5\right) \left(1 + \gamma_1^5 \gamma_2^5\right) = \gamma_1^5 \gamma_2^5 \left(1 + \gamma_1^5 \gamma_2^5\right).$$

One deduces also the useful formulas:

$$-\frac{1}{3} \gamma_1^5 (z^2 - 4) = \frac{1}{2} (1 + \gamma_1^5 \gamma_2^5) \equiv \Pi_+,$$

$$\frac{1}{3} (z^2 - 1) (z^2 - 3) = \frac{1}{2} (1 - \gamma_1^5 \gamma_2^5) \equiv \Pi_-.$$  \hspace{1cm} (3.37)

The part even $z$ of the spinorial $R$ matrix

$$\hat{R}_{so(4)}^{(2)}(u|z) = \frac{u + 2}{4} (1 + \gamma_1^5 \gamma_2^5) - \frac{u}{16} \gamma_1^{ab} \gamma_2^{ab},$$
corresponds to the $R$-matrix without check also even in $z$

$$R_{so(4)}(u|z) = (u-1)(P_2 + P_{-2}) + (1 + u)P_0$$

$$= \frac{u-1}{12}z^2 (z^2 - 1) + \frac{u+1}{4}(z^2 - 1)(z^2 - 4)$$

$$= (u + \frac{1}{2} \frac{1}{3}(z^4 - 4z^2 + 3) - \frac{1}{2}(z^2 - 1)$$

$$= \frac{2u+1}{4}(1 + \gamma_1^5 \gamma_2^5) - \frac{1}{16} \gamma_1^\rho \gamma_2^\rho.$$ 

It is represented by the matrix with the following non-vanishing entries:

$$R_{11}^{11}(u) = R_{22}^{22}(u) = R_{33}^{33}(u) = R_{44}^{44}(u) = u + 1,$$

$$R_{14}^{14}(u) = R_{23}^{23}(u) = R_{32}^{32}(u) = R_{41}^{41}(u) = u,$$  

$$R_{14}^{14}(u) = R_{23}^{23}(u) = 1 = R_{32}^{32}(u) = R_{41}^{41}(u).$$

So one sees that this matrix has the block form: its entries $R_{ij}^{ab}(u)$ differ from zero only if $a$, $b$, $c$ and $d$ belong to sets $(1, 4)$ or $(2, 3).

It is important to notice that the two different parts $R_{so(4)}^{11}(u)$ and $R_{so(4)}^{22}(u)$, which are odd and even functions of $z$ correspondingly, are distinguished by chirality. Indeed these solutions are proportional to the chiral projectors $\Pi_+$ and $\Pi_-$, respectively, where

$$\Pi_+ = \frac{1}{2}(1 + \gamma_1^5 \gamma_2^5).$$

One deduces also

$$\Pi_+ = \frac{1}{3}z^2 (z^2 - 4), \quad \Pi_- = \frac{1}{3}(z^2 - 1)(z^2 - 3).$$

This chiral property ensures the consistency of these solutions: both of them intertwine a pair of $L$-operators i.e. obey the RLL-relation linear in $L$, but also satisfy the trilinear RRR-relation not only separately, but also in arbitrary combination, due to their orthogonality.

The chiral projectors $\Pi_\pm$ separate the 16-dimensional representation space $\mathcal{F}_1 \otimes \mathcal{F}_2$ into two eight-dimensional chiral subspaces. In particular, the subspace corresponding to $\Pi_+$ is spanned by eight eigenvectors of $\gamma$ corresponding to the eigenvalues $\pm 1$, while the six eigenvectors, corresponding to the zero eigenvalue of $\gamma$ as well as vectors corresponding to the eigenvalues $\pm 2$. In terms of the projectors of the eigen- spaces this reads as

$$\Pi_+ P_{\pm 1} = P_{\pm 1} \Pi_+ = P_{\pm 1}, \quad \Pi_- P_{\pm 1} = 0 = P_{\pm 1} \Pi_-,$$

$$\Pi_+ P_0 = P_0 \Pi_+ = P_0, \quad \Pi_- P_2 = P_2 \Pi_- = P_2,$$

$$\Pi_+ P_0 = 0 = P_0 \Pi_+, \quad \Pi_- P_2 = 0 = P_2 \Pi_+.$$ 

### 3.2. ABA for the $so(4)$ Case

The observed simple structure of the spinorial $R_{so(4)}$ matrix is helpful for the diagonalization of the trace of the spinorial monodromy matrix

$$T_{so(4)}^{so(4)}(u) = R_{so(4)}^{so(4)}(u) R_{so(4)}^{so(4)}(u) ... R_{so(4)}^{so(4)}(u),$$  

or in components

$$T^{\alpha_0, \alpha_1, ... \alpha_n}_{\beta_0, \beta_1, ... \beta_n}(u) = R^{\alpha_0, \alpha_1, ... \alpha_n}_{\beta_0, \beta_1, ... \beta_n}(u).$$

The explicit form of the matrix $R_{so(4)}^{11}(u)$ results in the following representation as a $4 \times 4$ matrix in auxiliary space $\mathcal{F}_0 \otimes \mathcal{F}_0$ with operator valued elements acting in the $k$'s quantum space $\mathcal{F}_k$

$$R_{so(4)}^{11}(u) = \begin{pmatrix} R_1^1(u) & 0 & 0 & R_4^1(u) \\ 0 & R_2^2(u) & R_3^2(u) & 0 \\ 0 & R_3^3(u) & R_3^3(u) & 0 \\ R_4^4(u) & 0 & 0 & R_4^4(u) \end{pmatrix}.$$  

Its diagonal $4 \times 4$ matrix components are

$$R_1^1(u) = diag(u+1, 0, 0, 0),$$

$$R_2^2(u) = diag(0, u+1, 0, 0),$$

$$R_3^3(u) = diag(0, u, u+1, 0),$$

$$R_4^4(u) = diag(u, 0, 0, u+1)$$

and have to be regarded as the elements of the Cartan subalgebra. The off-diagonal elements have to be regarded as lowering

$$R_4^1(u) = \mathbf{e}_3, \quad R_4^2(u) = \mathbf{e}_2,$$

and rising

$$R_4^3(u) = \mathbf{e}_1, \quad R_4^4(u) = \mathbf{e}_1,$$

generators correspondingly. Consequently, the monodromy matrix (3.38) defined as an ordered matrix product of the factors (3.33) preserves the block form,

$$\mathcal{X}_{so(4)}(u) = \begin{pmatrix} \mathcal{X}_1^1(u) & 0 & 0 & \mathcal{X}_1^4(u) \\ 0 & \mathcal{X}_2^2(u) & \mathcal{X}_2^2(u) & 0 \\ 0 & \mathcal{X}_3^3(u) & \mathcal{X}_3^3(u) & 0 \\ \mathcal{X}_4^4(u) & 0 & 0 & \mathcal{X}_4^4(u) \end{pmatrix},$$  

with Cartan elements $\mathcal{X}_1(u)$ and rising $C_1(u) = \mathcal{X}_1^1(u)$, $C_2(u) = \mathcal{X}_2^2(u)$, and lowering $B_2(u) = \mathcal{X}_2^2(u)$, $B_2(u) = \mathcal{X}_1^1(u)$ generators. We have obtained a representation of the spinorial RRTT algebra and its decomposition into a representation of two subalgebras of the $s(2)$ type Yangian.
This becomes more evident by a similarity transformation with the $4 \times 4$ matrix $V$

$$ V = e_{11} + e_{24} + e_{33} + e_{42}, \quad V^{-1} = V. \quad (3.41) $$

One calculates easily

$$ V \mathcal{H}^{so(4)}(u)V = \begin{pmatrix} \mathcal{X}_1^*(u) & \mathcal{X}_4^*(u) & 0 & 0 \\ \mathcal{X}_1^*(u) & \mathcal{X}_4^*(u) & 0 & 0 \\ 0 & 0 & \mathcal{X}_3^*(u) & \mathcal{X}_3^*(u) \\ 0 & 0 & \mathcal{X}_5^*(u) & \mathcal{X}_5^*(u) \end{pmatrix}, \quad (3.42) $$

and

$$ (V \otimes V) \mathcal{H}^{so(4)}(u)(V \otimes V) = \begin{pmatrix} \mathcal{R}^{I}_{11}(u) & 0 & 0 & 0 \\ 0 & \mathcal{R}^{I}_{14}(u) & \mathcal{R}^{I}_{44}(u) & 0 \\ 0 & \mathcal{R}^{I}_{14}(u) & \mathcal{R}^{I}_{44}(u) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.43) $$

with $8 \times 8$ block-matrices

$$ \mathcal{R}^{I}(u) = \begin{pmatrix} \mathcal{R}^{I}_{11}(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{R}^{I}_{14}(u) & 0 & 0 & \mathcal{R}^{I}_{44}(u) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{R}^{I}_{14}(u) & \mathcal{R}^{I}_{44}(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.44) $$

Combining all these relations one deduces, that the general $so(4)$ spinor monodromy matrix has the form (3.40), i.e.

$$ T^I_1(u) = T^I_4(u) = T^I_4(u) = T^I_4(u) = 0, $$

and

$$ T^I_2(u) = T^I_3(u) = T^I_4(u) = T^I_4(u) = 0. $$

Thus also in the case of arbitrary representations in the quantum space the $so(4)$ monodromy matrix generates an algebra equivalent to two independent $s(2)$ Yangian algebras and the spectral problem for its trace leads to the ordinary $s(2)$ ABA.

In particular, this applies to the spinor-vector monodromy matrix (2.26). Recall, that it produces the $so(4)$ vector-vector monodromy matrix by fusion (2.31),

$$ T_{a_1 a_2 \ldots a_8}^{a_9 a_{10} \ldots a_{15}}(u) = \frac{1}{4} \text{tr} T^{a_9 \ldots a_8}_{a_1 a_2 \ldots a_8} \left( u + \frac{1}{2} \right) T_{a_1 a_2 \ldots a_8}^{a_9 a_{10} \ldots a_{15}} \left( u + \frac{3}{2} \right). \quad (3.45) $$

We have to substitute $\beta = 1$ for $so(4)$. Then taking into account that $T$ and $\bar{T} = T^{-1}$ have the form (3.40) in spinor space, one can calculate explicit the matrix elements of the vector monodromy in terms of products of the matrix elements of the spinor monodromy. In particular for the trace we have

$$ T^{a_9}_{a_9}(u) = \frac{1}{2} \left( (T^I_1(u) + T^I_4(u))(T^I_2(u) + T^I_3(u)) \right. $$

$$ \left. + (T^I_2(u) + T^I_3(u))(T^I_1(u) + T^I_4(u)) \right), \quad (3.46) $$

where we have abbreviated $T^{a_9}_{a_9}(u + \frac{1}{2}) = \left( L_1(u + \frac{1}{2}) \ldots L_N(u + \frac{1}{2}) \right)^{a_9}_{a_9}$ and $T = T^{-1}(u + \frac{3}{2})$.

The vector-vector $so(4)$ monodromy matrix is given by sums of products of the spinor-vector transfer matrix elements which obey $s(2)$ type Yangain relations.

At the end of this subsection we consider the $RTT$ algebra generated by $\mathcal{H}^{so(4)}(u)$. Due to its chirality and the diagonal character it leads to a trivial solution for $T(u)$ of the RTT-relation. For instance,
\((\alpha_1, \alpha_2) = (1,1), \ (\gamma_1, \gamma_2) = (1, 2)\) and \((\gamma_1, \gamma_2) = (2, 1)\) by (3.36) leads to
\[
T^i_1(v)T^i_2(u) = 0 = T^i_2(v)T^i_1(u).
\]

The remaining RTT-relations imply that the most general monodromy matrix \(T(u)\) intertwined by \(N^{(0)}_{so(4)}(u)\) is given by the diagonal matrix:
\[
T(u) = \begin{pmatrix}
T^1_1(u) & T^1_2(u) & T^1_3(u) & T^1_4(u) \\
T^2_1(u) & T^2_2(u) & T^2_3(u) & T^2_4(u) \\
T^3_1(u) & T^3_2(u) & T^3_3(u) & T^3_4(u) \\
T^4_1(u) & T^4_2(u) & T^4_3(u) & T^4_4(u)
\end{pmatrix},
\]

which has vanishing rising and lowering generators and does not produce dynamics.

**Proposition.** The spinorial \(R\) matrix with \(so(4)\) symmetry \(N^{(2)}\) (acting in the chiral subspace of the \(\Pi\) projection) generates the spinorial RTT algebra decomposing into two subalgebras of the \(s\ell_2\) Yangian type. The spinorial RTT algebra generated by \(N^{(1)}\) (acting in the chiral subspace of the \(\Pi\) projection) is a trivial commuting algebra. In this way, the ordinary \(s\ell_2\) ABA allows to construct solutions of the ABA of for the spinorial Yangian of \(so(4)\) type.

### 3.3. Algebraic Bethe Ansatz for the \(so(3)\) Case

The spinorial RTT relation for the \(so(3)\) monodromy matrix (2.26) is coincides with the one for the vector \(s\ell(2)\) monodromy matrix. Indeed, denoting
\[
T(u) = \begin{pmatrix}
A(u) & B(u) \\
C(u) & D(u)
\end{pmatrix},
\]
one obtains that commutation relations between \(A, B, C\) and \(D\) are the same as in \(s\ell(2)\) case, because the spinor-spinor \(R\)-matrix (2.30) intertwining them, up to \(u \to 2u\).

The coincidence of the spinorial \(R\) matrix of \(so(3)\) symmetry with the well known Yang formula for \(s\ell(2)\) was a central point in the classical paper by Reshetikhin [6]. The fusion relation (2.31) results in expression of the 9 elements of the vector monodromy matrix \(\mathcal{T}\) in terms of the 4 spinorial RTT generators in \(T(u)\) for all representations admitting this relation. The study of [14] resulted in relations among the matrix elements of \(\mathcal{T}\), in particular the three elements in the upper triangle are expressed in terms of one of them. This confirms that (2.31) establishes the equivalence of the spinorial \(s\ell(2)\) type RTT algebra and the ordinary Yangian of \(so(3)\) type.

Using identity
\[
(\sigma^\delta \gamma^\beta \sigma^\delta \gamma^\beta)_{\alpha} = 2\delta^\beta_\alpha \delta^\beta_{\alpha u},
\]
one can transform (2.31) for \(so(3)\) case to form
\[
\mathcal{T}^{e,\alpha_1 \ldots \alpha_k}_{\beta,\beta_1 \ldots \beta_k} (u) = \mathcal{T}^{e,\alpha_1 \ldots \alpha_k}_{\beta,\beta_1 \ldots \beta_k} (u)(\mathcal{T}^{-1})^{\gamma_1 \ldots \gamma_k}_{\delta_1 \ldots \delta_k} (u),
\]

\[
\mathcal{T}^{e,\gamma_1 \ldots \gamma_k}_{b,\gamma_1 \ldots \gamma_k} (u) = \mathcal{T}^{e,\gamma_1 \ldots \gamma_k}_{b,\gamma_1 \ldots \gamma_k} (u)(\mathcal{T}^{-1})^{\gamma_1 \ldots \gamma_k}_{\delta_1 \ldots \delta_k} (u).
\]

It is well known that in \(2 \times 2\) case the inverse monodromy matrix entries are given by:
\[
T^1_1(u) = T^2_2(u - 1)\Delta^1(u), \quad T^1_2(u) = -T^2_1(u - 1)\Delta^1(u),
\]
\[
T^3_1(u) = T^4_2(u - 1)\Delta^3(u), \quad T^3_2(u) = T^4_1(u - 1)\Delta^3(u),
\]
where \(T(u)\) stand for \(T^{-1}(u)\) and \(\Delta(u) = T^1_1(u)T^2_2(u - 1) - T^1_2(u)T^2_1(u - 1)\) is quantum determinant, belonging to the center of RTT-algebra.

So taking into account these formulas one obtains
\[
\mathcal{T}^{e,\alpha_1 \ldots \alpha_k}_{\beta,\beta_1 \ldots \beta_k} (u) = \mathcal{T}^{e,\alpha_1 \ldots \alpha_k}_{\beta,\beta_1 \ldots \beta_k} (u)\mathcal{T}^{\gamma,\gamma_1 \ldots \gamma_k}_{b,\gamma_1 \ldots \gamma_k} (u)\Delta^1(u) - 1. \quad (3.49)
\]

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