Inflationary magnetogenesis with reheating phase from higher curvature coupling

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Abstract. We investigate the generation of magnetic fields from inflation, which occurs via breakdown of the conformal invariance of the electromagnetic (EM) field, when coupled with the Ricci scalar and the Gauss-Bonnet invariant. For the case of instantaneous reheating, the resulting strength of the magnetic field at present is too small and violates the observational constraints. However, the problem is solved provided there is a reheating phase with a non-zero e-fold number. During reheating, the energy density of the magnetic field is seen to evolve as \((a^3H)^{-2}\) and, after that, as \(a^{-4}\) up to the present epoch (here \(a\) is the scale factor and \(H\) the Hubble parameter). It is found that this reheating phase – characterized by a certain e-fold number, a constant value of the equation of state parameter, and a given reheating temperature – renders the magnetogenesis model compatible with the observational constraints. The model provides, in turn, a viable way of constraining the reheating equation of state parameter, from data analysis of the cosmic microwave background radiation.

Keywords: modified gravity, primordial magnetic fields, cosmic flows, inflation

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1 Introduction

Magnetic fields have been observed over the broad range of scales probed so far. They have been detected in galaxies, galaxy clusters and even in intergalactic voids [1–6]. Our understanding of the origin of such large scale magnetic fields can be broadly split along two directions. The first is associated with an astrophysical origin of the fields, which are later amplified by some dynamo mechanism [7–9]. The other possibility is that the magnetic fields have a primordial origin, i.e., a possible generation of magnetic fields during the inflationary epoch [10–37], or in alternative scenarios, as in bouncing cosmology [38–43]. A confirmation of the announced detection of magnetic fields in the large voids might in principle enforce their primordial origin, in front of the other possibility.

Among all the proposals discussed so far, the primordial origin of magnetic fields during inflation has earned a lot of attention, primarily because the inflationary paradigm is able to solve the horizon and flatness problems, and to generate an almost scale invariant power spectrum, which is perfectly consistent with the observational data [44–49]. However, the corresponding inflationary magnetogenesis is riddled with severe difficulties: the most crucial one is how to generate a value of the magnetic strength that is high enough to be compatible with present day observations at the galactic scale. In the standard Maxwell theory, the electromagnetic (EM) field is endowed with a conformal symmetry, so that the electromagnetic field energy density decays as $a^{-4}$ with the universe expansion. Such behavior leads to a very feeble value of the magnetic strength at present, which is fully unable to account for the observational results. This is a clear indication that, in the context of inflationary magnetogenesis, the conformal invariance of the electromagnetic field should be broken at an early
stage, which in turn would allow gauge field production from the quantum vacuum state, thus preventing the electromagnetic field energy from decaying as fast as $a^{-4}$. Several models have been already proposed in the literature to break the conformal invariance of the electromagnetic action, as those including a non-trivial coupling between a scalar field (generally considered as the inflaton) and the gauge field [10–37, 50–55]. Alternative scenarios to break the conformal invariance have been proposed in the shelter of non-linear electrodynamics [56] or 3-form fields [57]. Other important issues that affect inflationary magnetogenesis models are the backreaction and the strong coupling problems [10, 16, 36, 51, 55, 58]. The backreaction issue appears when the strength of the electromagnetic field exceeds the background energy density, which in turn may spoil the inflationary set-up as well, as it suppresses the production of magnetic fields. On the other hand, the strong coupling problem occurs if the effective electric charge becomes high during inflation, rendering the perturbative calculation of the EM field unreliable.

Apart from the inflationary magnetogenesis set-up, one should also add some proposals for magnetic field generation from entirely different perspectives, in particular, from the bouncing scenario [38–43]. Similar to the inflationary theory, bounce cosmology is also able to generate a nearly scale invariant power spectrum, thus becoming compatible with the observational data available. However, most of the bouncing models are plagued with certain difficulties in regard to the cosmological background and the evolution of perturbations; in particular, the BKL instability associated with the anisotropic problem, the violation of the energy conditions at the bounce point, the instability of scalar and tensor perturbations, etc. [59–65]. Here it should be mentioned that these severe problems can be solved, to some extent, in various modified theories of gravity [61, 66–73].

In this paper, we propose an inflationary magnetogenesis model, in which the electromagnetic field couples to the background spacetime curvature, specifically with the Ricci scalar and the Gauss-Bonnet invariant; i.e. we include in our model a higher curvature coupling of the EM field. Such coupling breaks the conformal invariance of the electromagnetic action and allows the production of photons from the Bunch-Davies vacuum. Moreover, being this a higher-order operator, the coupling is suppressed at the Planck scale and, thus, the model becomes free from the strong coupling problem. In regard to the background spacetime, we consider the scalar-Einstein-Gauss-Bonnet gravity theory, which is known to provide viable inflationary models (consistent with the latest Planck results), for suitable choices of the Gauss-Bonnet coupling function and the scalar field potential [74–87]. In such scenario, we try to explore the dynamics of the electric and magnetic fields along with the cosmological expansion of the universe, starting from the inflationary stage. During the first steps of the cosmic expansion, the universe enters a reheating phase, after the end of inflation; and depending on the reheating mechanism, we consider two different scenarios: (1) instantaneous reheating at the end of inflation and (2) a Kamionkowski like reheating model with non-zero e-fold number, in which case the reheating phase is parametrized by a constant effective equation of state (EoS) parameter ($\omega_{\text{eff}}$) [88] (for recent results on the reheating phase, see [89–97]). These two scenarios make qualitative differences in the evolution of electric and magnetic fields. In particular, the presence of the reheating phase with a non-zero e-fold number enhances the strength of the magnetic field, in comparison with the instantaneous reheating case, and this is reflected in the present amplitude of the magnetic field, which is found to differ in the two cases. We should mention that magnetogenesis models with curvature couplings have been proposed earlier, however in quite different contexts [33, 98, 99]. Note that, in our present analysis, we include the higher curvature
Gauss-Bonnet coupling in the set-up and also discuss the effect of the reheating phase in the production of the magnetic field, which makes the present scenario essentially different from earlier ones. Moreover at the time of preparing our manuscript, some authors investigated the effect of reheating phase in an inflationary magnetogenesis model \[100\], however without introducing any higher curvature coupling in the model.

The paper is organized as follows: after describing the model in section 2, we give the general expressions for the EM power spectra in the present context in section 3. The solution for the vector potential and the electromagnetic energy density during inflation are presented in section 4. The corresponding calculations during the reheating phase are carried out in section 6 and section 7, which correspond to the cases of instantaneous reheating and a Kamionkowski like reheating model, respectively. The paper ends with some conclusions.

2 The model

Consider the following action,

\[ S = S_{\text{grav}} + S_{\text{em}}^{(\text{can})} + S_{\text{CB}}, \]

where \( S_{\text{grav}} \) symbolizes the action for the underlying gravity theory which we consider as a scalar coupled Einstein-Gauss-Bonnet theory. In the most general setting, the action for the scalar coupled Einstein-Gauss-Bonnet gravity consists of four terms — the Ricci scalar, the Gauss-Bonnet invariant coupled to an arbitrary function of the scalar field, the kinetic term of the scalar field, and a self-interaction term for the scalar field, such that

\[ S_{\text{grav}} = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi - V(\Phi) - \xi(\Phi) G \right] \]

where \( R \) is the Ricci scalar, \( \kappa^2 = 8\pi G \) (\( G \) is Newton’s constant), \( \Phi \) is the scalar field generally known as inflaton field embedded within the potential \( V(\Phi) \) and \( G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \) is the Gauss-Bonnet invariant. The presence of the coupling function between the scalar field and the Gauss-Bonnet term, symbolized by \( \xi(\Phi) \), ensures the non-topological character of the Gauss-Bonnet term in the above action. The second term of the action (2.1), i.e \( S_{\text{em}}^{(\text{can})} \), denotes the standard electromagnetic field action and given by,

\[ S_{\text{em}}^{(\text{can})} = \int d^4x \sqrt{-g} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] \]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic field tensor of the vector field \( A_\mu \). Finally \( S_{\text{CB}} \) refers to the conformal symmetry breaking part and, in the present context, we consider a non-minimal curvature coupling which breaks the conformal invariance of the electromagnetic field. In particular \( S_{\text{CB}} \) is given by,

\[ S_{\text{CB}} = \int d^4x \sqrt{-g} [f(R, G) F_{\mu\nu} F^{\mu\nu}] \]

where \( f(R, G) \) is an arbitrary analytic function of the Ricci scalar and the GB invariant at the moment and denotes the curvature coupling of the electromagnetic field. \( f(R, G) \), in the present context, is considered to be a polynomial function of the Ricci scalar and the GB invariant, in particular,

\[ f(R, G) = \kappa^{2q} (R^q + G^{q/2}), \]
where \( q \) is the model parameter. The form of \( f(R, G) \) clearly indicates that, as the curvature is significant in the early universe, the curvature coupling introduces then a non-trivial correction to the electromagnetic action; however, at late times (in particular after the end of inflation, as we will show at a later stage), \( S_{CB} \) will not contribute and the electromagnetic field will then behave according to the standard Maxwell’s equations. This is not the case in most of the earlier magnetogenesis models, where a non-minimal coupling between scalar and electromagnetic field is considered in the action in order to break the conformal invariance. Here we would like to stress that the conformal breaking term \( f(R, G) \) is suppressed by \( \kappa^2 q \) and, thus, the present model is free from the strong coupling problem for \( q \sim \mathcal{O}(1) \), what is a remarkable feature of our proposal. Moreover, we will show later that the electromagnetic field has a negligible backreaction on the background inflationary FRW spacetime and, thus, the backreaction problem in the present magnetogenesis scenario will be also resolved naturally.

In regard to the background spacetime evolution, it is worthwhile to mention that the scalar-Einstein-GB theory of gravity leads to an inflationary scenario which is indeed stable with respect to scalar and tensor perturbations of the FRW metric, for suitable choices of \( V(\Phi) \) and \( \xi(\Phi) \). For example, for quadratic choices of \( V(\phi) \) and \( \xi(\Phi) \), one can show that \( S_{\text{grav}} \) provides an acceleration phase of the early universe, which has a graceful exit for numerically interpolated forms of \( V(\Phi) \) and \( \xi(\Phi) \), starting from the quadratic function of the scalar field [82]. The stability of scalar and tensor perturbations in the context of the Gauss-Bonnet gravity theory are ensured due to the presence of the scalar field potential \( V(\Phi) \) in the gravitational action. Furthermore, the speed of the tensor perturbations \( c^2_T \), in general, is not unity in the scalar coupled Einstein-GB theory and the deviation of \( c^2_T \) from unity is proportional to the GB coupling function considered in the model. The result \( c^2_T \neq 1 \) precludes that the gravitational waves propagate with a different speed, compared to the speed of light which is unity in natural units, and thus is not in agreement with the event GW170817. However, there exists a certain class of GB coupling function for which the gravitational wave propagates with \( c^2_T = 1 \) leading to the compatibility of the GB model with GW170817. The inflationary phenomenology and its viability with the latest Planck 2018 results in such Gauss-Bonnet theory that is compatible with GW170817 have been recently discussed in [101–103].

With the action (2.1), particularly with the conformal breaking term \( S_{CB} \), we aim to generate a sufficiently strong magnetic field in the present epoch. The variation of action (2.1) with respect to the gauge field leads to the following equation of motion for \( A_\mu \),

\[
\partial_\alpha \left[ \sqrt{-g} h^2(R, G) \ g^{\mu \alpha} g^{\nu \beta} F_{\mu \nu} \right] = 0 \tag{2.6}
\]

with \( h^2(R, G) = 1 - 4f(R, G) \). The spatially flat FRW metric ansatz will fulfill our purpose i.e we take,

\[
d s^2 = a^2(\eta) \left[ -d\eta^2 + d\vec{x}^2 \right] \tag{2.7}
\]

where \( \eta \) is known as conformal time and \( a(\eta) \) is the scale factor. Owing to the FRW metric ansatz, the temporal and spatial component of eq. (2.6) reduce to

\[
\partial^i [\partial_i A_0 - \partial_0 A_i] = 0 \tag{2.8}
\]

and

\[
\delta^{ij} \partial_0 [h^2(R, G) F_{0j}] - \delta^{ij} \partial^j [h^2(R, G) F_{ij}] = 0 \tag{2.9}
\]
respectively. In the Coulomb gauge ($A_0 = 0$), eq. (2.8) leads to the condition $\partial_i A_i = 0$ which further simplify eq. (2.9) as follows,

$$A''_i(\eta, \vec{x}) + 2 \frac{h'(R, G)}{h(R, G)} A'_i - \partial_l \partial^l A_i = 0. \quad (2.10)$$

As mentioned earlier the gravitational action considered in the present context (see eq. (2.2)) provides a viable inflationary scenario for suitable forms of $V(\Phi)$ and $\xi(\Phi)$. Thus, we consider a quasi de-Sitter inflationary scenario as the background spacetime, where the scale factor is given by

$$a(\eta) = \left( -\frac{\eta}{\eta_0} \right)^{\beta+1} \quad \text{with} \quad \beta = -2 - \epsilon = -3 + \frac{H'}{H^2}. \quad (2.11)$$

Here a prime denotes differentiation with respect to $\frac{d}{d\eta}$, $H = \frac{a'}{a}$ is conformal the Hubble parameter and $\epsilon$ is the slow roll parameter having the expression $\epsilon = -\frac{H'}{H^2} + 1$. The scale factor of eq. (2.11) immediately leads to the Hubble parameter, Ricci scalar and the Gauss-Bonnet invariant, as follows

$$H = \frac{\beta + 1}{\eta}, \quad (2.12)$$

and

$$R = \frac{6}{a^2}(H'^2 + H^2) = \frac{6\beta(\beta + 1)}{\eta_0^2} \left( -\frac{\eta}{\eta_0} \right)^{2\epsilon},$$

$$G = \frac{24}{a^4} H^2 H' = -\frac{24(\beta + 1)^3}{\eta_0^4} \left( -\frac{\eta}{\eta_0} \right)^{4\epsilon}, \quad (2.13)$$

respectively. The conformal Hubble parameter is related to the cosmic Hubble parameter by $H = \frac{1}{a} \dot{H}$ and thus we get $H = \frac{1}{\eta_0} (-\eta)'$ or in terms of e-folding number

$$H = \frac{1}{\eta_0} \exp \left( -\frac{\epsilon N}{1 + \epsilon} \right), \quad (2.14)$$

where the e-folding number (up to time $\eta$) is defined as $N = \int^\eta aH d\eta$, i.e the beginning of inflation is designated by $N = 0$ and we consider it to happen when the CMB scale mode crosses the horizon. Eq. (2.14) shows that $\frac{1}{\eta_0}$ specifies the cosmic Hubble parameter at the starting of inflation (we will denote it by $H_0$, i.e $\eta_0^{-1} = H_0$, in the subsequent calculation). Consequently, with the expressions of $R$ and $G$, we determine the curvature coupling function of the electromagnetic field, i.e $f(R, G)$, from eq. (2.5) and is given by,

$$f(R, G) = \kappa^2 q \left\{ \frac{6\beta(\beta + 1)^2}{\eta_0^2} \left[ \frac{24(\beta + 1)^3}{\eta_0^4} \right] \left( -\frac{\eta}{\eta_0} \right)^{2\epsilon} \right\} \left( -\frac{\eta}{\eta_0} \right)^{2\epsilon} \quad (2.15)$$

At this stage we would like to mention that the time dependence of $f(R, G)$ is the sole reason to spoil the conformal symmetry of the electromagnetic field. However the above equation indicates that $f(R, G)$ becomes constant under the condition $\epsilon = 0$ and thus leads to a conformal invariant electromagnetic action. Actually, for $\epsilon = 0$ (i.e a de-Sitter background spacetime), the Ricci scalar and the GB invariant become constant and henceforth $f(R, G)$ will be, too. Thereby the present model where the electromagnetic field is coupled with the background spacetime curvature, requires $\epsilon \neq 0$ in order to break the conformal invariance of the gauge field. Keeping this in mind, we will consider the background inflationary scenario as a quasi de-Sitter evolution, in which case $\epsilon \neq 0$ and $\epsilon < 1$, in the subsequent calculation.
3 Energy density and power spectra for electric and magnetic field

In this section, we aim to calculate the power spectra for both the electric and magnetic fields and in this regard, it may be mentioned that both fields are intrinsically frame dependent. Here we consider the comoving observer for which the four velocity components are given by \( u^\mu = (1/a(\eta), 0, 0, 0) \) and thus the proper time of a comoving observer is defined as \( dt = a(\eta)d\eta \). The computation of the power spectrum requires two ingredients; first we need to know the energy density separately for electric, magnetic fields and, secondly, the vacuum state associated with the electromagnetic field in the background inflationary evolution. Thereby, from the action (2.1), we determine the energy-momentum tensor associated with the electromagnetic field

\[
T_{\alpha\beta} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^\alpha\beta} \left[ -\frac{1}{4} \sqrt{-g} (1 - 4f(R, G)) F_{\mu\nu} F^{\mu\nu} \right]
\]

\[
= -\frac{1}{4} \left\{ g_{\alpha\beta} (1 - 4f(R, G)) F_{\mu\nu} F^{\mu\nu} - 4(1 - 4f(R, G)) g^{\mu\nu} F_{\mu\alpha} F_{\nu\beta} + 8F_{\mu\nu} F^{\mu\nu} \frac{\delta f(R, G)}{\delta g^\alpha\beta} \right\}.
\]

(3.1)

The energy density of the electromagnetic field in the background FRW spacetime is defined as \( T^0_0 = -\frac{1}{a^2} T_{00} \) and thus the above expression of \( T_{\alpha\beta} \) immediately leads to the following form of \( T^0_0 \):

\[
T^0_0 = -\frac{1}{2a^4} (A')^2 \left\{ (1 - 4f(R, G)) + \frac{8}{a^2} \frac{\delta f}{\delta g^{00}} \right\} - \frac{1}{4a^4} F_{ij} F_{ij} \left\{ (1 - 4f(R, G)) - \frac{8}{a^2} \frac{\delta f}{\delta g^{00}} \right\}
\]

(3.2)

where we use the Coulomb gauge condition and also the result \( F_{\mu\nu} F^{\mu\nu} = \frac{1}{a^2} (-2(A')^2 + F_{ij} F_{ij}) \) holds in the spatially flat FRW metric. Furthermore, the term \( \frac{\delta f(R, G)}{\delta g^{00}} \) present in the above expression is determined as

\[
\frac{\delta f(R, G)}{\delta g^{00}} = \frac{\partial f}{\partial R} \frac{\delta R}{\delta g^{00}} + \frac{\partial f}{\partial G} \frac{\delta G}{\delta g^{00}} = -3q\kappa^{2q} \mathcal{H}' \left[R^{q-1} + \frac{2\mathcal{H}^2}{a^2} G^{q-2} \right].
\]

Thereby, the final expression of the electromagnetic energy density in the present context is

\[
T^0_0 = -\frac{1}{2a^4} (A')^2 P(\eta) - \frac{1}{4a^4} F_{ij} F_{ij} Q(\eta),
\]

(3.3)

where \( P(\eta) \) and \( Q(\eta) \) have the following form

\[
P(\eta) = 1 - 4\kappa^{2q}(R^q + G^{q/2}) - \frac{24q\kappa^{2q} \mathcal{H}'}{a^2} \left[R^{q-1} + \frac{2\mathcal{H}^2}{a^2} G^{q-2} \right]
\]

\[
Q(\eta) = 1 - 4\kappa^{2q}(R^q + G^{q/2}) + \frac{24q\kappa^{2q} \mathcal{H}'}{a^2} \left[R^{q-1} + \frac{2\mathcal{H}^2}{a^2} G^{q-2} \right],
\]

(3.4)

with \( R = R(\eta) \) and \( G = G(\eta) \) as shown in eq. (2.13). Thus, using eq. (2.13), one can further determine \( P(\eta) \) and \( Q(\eta) \) as functions of the conformal time

\[
P(\eta) = 1 - \frac{A}{\eta_0^2} \left(\frac{-\eta}{\eta_0}\right)^{2q} - \frac{24q\kappa^{2q} \mathcal{H}'}{a^2} \left\{ \left[\frac{6\beta(\beta+1)}{\eta_0^{2q-2}} \right]^{q-1} \left(\frac{-\eta}{\eta_0}\right)^{2q-2\epsilon} + \frac{2\mathcal{H}^2 \left[-24(\beta+1)^3 q/2-1 \right]}{\eta_0^{2q-4}} \left(\frac{-\eta}{\eta_0}\right)^{2q-4\epsilon} \right\}
\]

(3.5)
and

\[ Q(\eta) = 1 - \frac{4B}{\eta^2} \left( \frac{-\eta}{\eta_0} \right)^{2eq} + \frac{24q\kappa^{2q}}{a^2} \mathcal{H} \left\{ \frac{6\beta(\beta + 1)^{q-1}}{\eta_0^{2q-2}} \left( \frac{-\eta}{\eta_0} \right)^{2eq-2} + \frac{2H^2[-24(\beta + 1)^3]^{q/2-1}}{\eta_0^{2q-4}} \left( \frac{-\eta}{\eta_0} \right)^{2eq-4} \right\} , \]  

(3.6)

respectively, with \( B = \kappa^{2q} \left[ 6\beta(\beta + 1)^q + [-24(\beta + 1)^3]^{q/2} \right] \) and \( \mathcal{H} = \mathcal{H}(\eta) \) as given in eq. (2.12). It is clear that the functions \( P(\eta) \) and \( Q(\eta) \) deviate from unity and become non-trivial solely due to the presence of the conformal breaking term \( f(R, \mathcal{G}) \) in the electromagnetic action. Having determined \( T^0_0 \), we are now in the position to separate the energy density of the electric (\( \hat{E} \)) and magnetic (\( \hat{B} \)) fields, respectively. The first term in eq. (3.3) is obviously the energy density of the electric field, while the other one, depending only on the spatial derivatives of the vector potential, contributes to the magnetic field. Hence, the expectation value of the electric field energy density over the vacuum state \( |0\rangle \) can be written down as

\[ \rho(\hat{E}) = -\frac{P(\eta)}{2a^4} \langle 0 |(A^r_0)^2|0\rangle . \]  

(3.7)

Similarly, the expectation energy density of the magnetic field over \( |0\rangle \) reads

\[ \rho(\hat{B}) = -\frac{Q(\eta)}{4a^4} \langle 0 |F_{ij} F_{ij}|0\rangle . \]  

(3.8)

Here \( |0\rangle \) is the distant past vacuum state for the electromagnetic field and later we will show that the Bunch-Davies state can act as a suitable infinite past vacuum for the gauge field. To evaluate \( \rho(\hat{E}) \) and \( \rho(\hat{B}) \) explicitly, we need to quantize the gauge field (over the inflationary background) by promoting \( A_i(\eta, \vec{x}) \) to a hermitian operator \( \hat{A}_i(\eta, \vec{x}) \) and expanding it in a Fourier basis, as

\[ \hat{A}_i(\eta, \vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3} \sum_{r=1,2} \epsilon_{ri} \left( \hat{b}_r(\vec{k}) A_r(k, \eta) e^{i\vec{k}.\vec{x}} + \hat{b}^+_r(\vec{k}) A^*_r(k, \eta) e^{-i\vec{k}.\vec{x}} \right) , \]  

(3.9)

where \( \vec{k} \) is the Fourier mode momentum (or equivalently the electromagnetic wave vector), \( r \) is the polarization index and runs from \( r = 1, 2 \) with \( \epsilon_{ri} \) being the two polarization vectors. Here we consider the polarization vectors in the standard linear polarization basis, in which case \( \epsilon_{1i} = (1, 0, 0) \) and \( \epsilon_{2i} = (0, 1, 0) \). Clearly in such polarization basis, the Coulomb gauge condition, characterized by \( \partial_i A^i = 0 \), further leads to the condition \( k^i \epsilon_{1i} = k^i \epsilon_{2i} = 0 \), which states that the propagation direction of the electromagnetic wave (or the propagation direction of photon in the quantized language) is perpendicular to the plane spanned by the polarization vectors. Moreover, \( \hat{b}_r(\vec{k}) \) and \( \hat{b}^+_r(\vec{k}) \) are the annihilation and creation operators defined on the distant past vacuum state \( |0\rangle \), i.e the relation \( \hat{b}_r(\vec{k})|0\rangle = 0 \) holds for all \( \vec{k} \). These creation and annihilation operators follow the quantization rule, as

\[ [\hat{b}_p(\vec{k}), \hat{b}^+_q(\vec{k}')] = \delta_{pq} \delta(\vec{k} - \vec{k}') . \]  

(3.10)

With the mode decomposition of \( A_i(\eta, \vec{x}) \) expressed in eq. (3.9) along with the above commutation relation, the expectation energy densities of electric and magnetic fields given in
2.15

while for the magnetic case it comes through the left hand side of eq. (4.1), where
the backreaction problem, which also requires the solution of the electromagnetic mode function.

dependence of the power spectra is also important to understand, in order to investigate the
need to solve the mode function, which is the subject of the next section. Further, the time
depth interval of $\eta$ has been already written down in eq. (4.2), respectively reduce to the following expressions

$$
\frac{\partial \rho(\vec{E})}{\partial \ln k} = P(\eta) \sum_{r=1,2} \frac{k^2}{a^2} |A_r'(k, \eta)|^2, \quad \frac{\partial \rho(\vec{B})}{\partial \ln k} = Q(\eta) \sum_{r=1,2} \frac{k^4}{a^4} |A_r(k, \eta)|^2,
$$

Consequently, the power spectra (defined as the energy density associated to a logarithmic
interval of $k$) of the electric and magnetic fields follow

where $P(\eta)$ and $Q(\eta)$ are shown in eqs. (3.5) and (3.6), respectively. The appearance of
the non-trivial functions $P(\eta)$ and $Q(\eta)$ make the electric and magnetic power spectra in the
present context different in comparison to those of the standard electrodynamic case and,
moreover, as mentioned earlier, $P(\eta)$ and $Q(\eta)$ deviate from unity solely due to the effect
of the conformal breaking coupling $f(R, G)$. The $k$ dependence in the electric and magnetic
power spectra are seemingly different from eq. (3.12), in particular, in the electric power spectrum the $k$ dependence comes through $k^3$ and the time derivative of the mode function
while for the magnetic case it comes through $k^3$ and the mode function itself. Thus, it is very
much likely that, when the electric spectrum becomes scale invariant, the magnetic spectrum
is not so, and vice-versa. In order to reveal the explicit $k$-dependence of the power spectra, we
need to solve the mode function, which is the subject of the next section. Further, the time
dependence of the power spectra is also important to understand, in order to investigate the
backreaction problem, which also requires the solution of the electromagnetic mode function.

4 Solving for the mode function and scale dependence of the power spectra
during inflation

In this section we will determine the solution of the mode function and for this purpose we
need the evolution equation for the vector potential in terms of the conformal time, which
has been already written down in eq. (2.10). More explicitly, we need to recast eq. (2.10) in
Fourier space, which leads to the evolution equation of the mode function as

$$
A''_r(k, \eta) + \frac{2k'}{h} A'_r(k, \eta) + k^2 A_r(k, \eta) = 0,
$$

where $h^2(R, G) = 1 - 4f(R, G)$ and $f(R, G)$ is expressed in eq. (2.15). Introducing Mukhanov-
Sasaki like variable for the electromagnetic field as $\tilde{A}_r(k, \eta) = h(\eta)A_r(k, \eta)$, the above equation transforms to

$$
\tilde{A}''_r(k, \eta) + \left( k^2 - \frac{h''}{h} \right) \tilde{A}_r(k, \eta) = 0.
$$

The factor $\frac{h''}{h}$ entirely depends on the background spacetime evolution and, thus, the last term
in the left hand side of eq. (4.2) depicts how the electromagnetic perturbation couples with
the background spacetime curvature. Using eq. (2.15), the $h(R, G)$ is determined as $h(R, G) =
(1 - 4f(R, G))^{1/2} = \left[ 1 - \frac{4B}{9\eta^2} \left( -\frac{q}{\rho} \right) \right]^{2\eta^{-1/2}}$ with recall, $B = \eta^{2\eta} \left[ (6\beta(\beta+1))^q + [ -24(\beta+1)^3 ]^{q/2} \right]$. 

---

- 8 -
At this stage, it deserves mentioning that the conformal breaking (CB) coupling \( f(R, G) \) is suppressed by the Planck mass over the exponent \( 2q \) and, thus, a higher value of \( q \) leads to a larger suppression of \( f(R, G) \) by the Planck scale. Hence, we consider \( 0 < q < 1 \) in the present work (the negative values of \( q \) will lead to a divergence at \( R \to 0 \) and, thus, we exclude the case \( q < 0 \).

Such consideration of \( q \) along with \( \epsilon < 1 \) leads to the condition \( f(R, G) < 1 \) for a wide range of conformal time, starting deeply from the sub-Hubble regime; as an example for \( q = 0.5 \) and \( \epsilon = 0.01 \), then \( f(R, G) \) becomes less than unity in the regime \(-k\eta < 10^{332}\) which is in the deep sub-Hubble radius. This safely allows to expand \( h(R, G) = \left[ 1 - 4f(R, G) \right]^{1/2} \) as a binomial expansion, which leads to the following expression

\[
h(R, G) = 1 - \frac{2B}{\eta_0^q} \left( \frac{-\eta}{\eta_0} \right)^{2q},
\]

where we retain up to the first binomial order. This expression of \( h(R, G) \) immediately transforms eq. (4.2) as

\[
\tilde{A}'_v(k, \eta) + \left(k^2 - \frac{4Bcq}{\eta_0^q (1 - \frac{2B}{\eta_0^q}) \eta^2} \right) \tilde{A}_v(k, \eta) = 0,
\]

and solving it for \( \tilde{A}_v(k, \eta) \), we get

\[
\tilde{A}_v(k, \eta) = \sqrt{-k\eta} \left[ D_1 J_\nu(-k\eta) + D_2 J_{-\nu}(-k\eta) \right],
\]

where \( \nu^2 = \frac{1}{4} - \frac{4Bcq}{\eta_0^q \left( \frac{4B}{\eta_0^q} - 1 \right)} \), \( J_\nu \) is the Bessel function of the first kind and it may be observed that the mode function for both polarizations has the same solution. The solution of eq. (4.3) can also be expressed in terms of \( J_\nu \) and \( Y_\nu \) where \( Y_\nu \) is the modified Bessel function, however here it is advantageous to replace the modified Bessel function by \( J_\nu \) and \( J_{-\nu} \) and thus the mode function solution has the form of eq. (4.4). Moreover, \( D_1, D_2 \) are two integration constants which can be further determined from the initial condition of \( \tilde{A}_v(k, \eta) \) and as an initial state of the mode function, we consider the Bunch-Davies vacuum. Actually, in the deep sub-Hubble regime, i.e. in the regime of \( |k\eta| \gg 1 \), the Mukhanov-Sasaki equation can be approximated to \( \tilde{A}'_v(k, \eta) + k^2 \tilde{A}_v(k, \eta) = 0 \) which possesses the Bunch-Davies solution like \( \tilde{A}_v(k, \eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta} \) and, thus, the Bunch-Davies initial condition is well justified in the present context.

In the sub-Hubble region of the distant past, the Bessel functions have the following limit

\[
\lim_{|k\eta| \gg 1} J_\nu(-k\eta) = \sqrt{\frac{2}{\pi(-k\eta)}} \cos \left( -k\eta - \frac{\pi}{2} \left( \nu + \frac{1}{2} \right) \right);
\]

\[
\lim_{|k\eta| \gg 1} J_{-\nu}(-k\eta) = \sqrt{\frac{2}{\pi(-k\eta)}} \sin \left( -k\eta + \frac{\pi}{2} \left( \nu + \frac{1}{2} \right) \right),
\]

and plugging the above expressions into eq. (4.4), we get the sub-Hubble limit of the electromagnetic mode function as

\[
\lim_{|k\eta| \gg 1} \tilde{A}_v(k, \eta) = \sqrt{\frac{2}{\pi}} \left[ D_1 \cos \left( -k\eta - \frac{\pi}{2} \left( \nu + \frac{1}{2} \right) \right) + D_2 \sin \left( -k\eta + \frac{\pi}{2} \left( \nu + \frac{1}{2} \right) \right) \right] \]

\[
\quad = \sqrt{\frac{2}{\pi}} \left[ e^{-ik\eta} \left( D_1 \frac{1}{2} e^{-i\frac{\pi}{2} (\nu + \frac{1}{2})} + D_2 \frac{1}{2i} e^{i\frac{\pi}{2} (\nu + \frac{1}{2})} \right) + e^{ik\eta} \left( D_1 \frac{1}{2} e^{i\frac{\pi}{2} (\nu + \frac{1}{2})} - D_2 \frac{1}{2i} e^{-i\frac{\pi}{2} (\nu + \frac{1}{2})} \right) \right],
\]
where in the second line we expand the vector potential in terms of \( \exp(\pm ik\eta) \). Owing to the Bunch-Davies initial condition, the coefficient of the positive frequency mode function is given by \( \frac{1}{\sqrt{2k}} \) and that of the negative frequency mode function becomes zero. This leads to the following forms of \( D_1 \) and \( D_2 \)

\[
D_1 = \frac{1}{2} \sqrt{\frac{\pi}{k}} \frac{e^{-\frac{1}{2}q(\nu + \frac{1}{2})}}{\cos[\pi(\nu + 1/2)]}, \quad D_2 = \frac{1}{2} \sqrt{\frac{\pi}{k}} \frac{e^{\frac{1}{2}q(\nu + \frac{1}{2})}}{\cos[\pi(\nu + 1/2)]},
\]

and, consequently, the final solution of the mode function becomes

\[
A_r(k, \eta) = \frac{1}{h(\eta)} \tilde{A}_r(k, \eta) = \sqrt{-k\eta} \frac{\sqrt{-k\eta}}{2 \left[ 1 - \frac{4B}{\eta^2_0} \left( -\frac{\eta}{\eta_0} \right)^2 \right]^{1/2}} \times \left\{ \sqrt{\frac{\pi}{k}} \frac{e^{-\frac{1}{2}q(\nu + \frac{1}{2})}}{\cos[\pi(\nu + 1/2)]} J_\nu(-k\eta) + \sqrt{\frac{\pi}{k}} \frac{e^{\frac{1}{2}q(\nu + \frac{1}{2})}}{\cos[\pi(\nu + 1/2)]} J_{-\nu}(-k\eta) \right\}.
\]

In the superhorizon limit, in which case the modes are outside of the Hubble radius i.e. \( k < \frac{1}{\pi} \) (recall \( H \) is the Hubble parameter), the mode function can be expressed as

\[
\lim_{|k\eta| \ll 1} A_r(k, \eta) = \frac{1}{[1 - \frac{4B}{\eta^2_0} \left( -\frac{\eta}{\eta_0} \right)^2]^{1/2}} \left\{ D_1 \frac{2\nu\Gamma(\nu + 1)}{2\nu\Gamma(-\nu)} \right\} (-k\eta)^{\nu + \frac{1}{2}} + D_2 \frac{2\nu\Gamma(-\nu + 1)}{2\nu\Gamma(-\nu)} (-k\eta)^{-\nu + \frac{1}{2}},
\]

where we have used the power law expansion of the Bessel function given by \( \lim_{|k\eta| \ll 1} J_\nu(-k\eta) = \frac{(-1)^\nu}{2\nu!(\nu + 1)\pi} (-k\eta)^\nu \), and similarly for \( J_{-\nu}(-k\eta) \). Eq. (4.8) clearly indicates that \( A_r(k, \eta) \) depends on the parameter \( q \) and the inflationary energy scale determined by \( \eta_0^{-1} \) (recall \( \eta_0^{-1} = H_0 \) actually specifies the cosmic Hubble parameter at the beginning of inflation, as in eq. (2.14)). As mentioned earlier, \( q \) is considered to lie within \( 0 < q < 1 \) and, moreover, \( H_0 \) is generally considered to be \( H_0 = 10^{-3} M_{Pl} \approx 10^{14}\text{GeV} \).

Having derived the vector potential, let us now turn to the electric and magnetic power spectra and their respective scale dependencies on \( k \). For this purpose, first we need to recall, \( \nu^2 = \frac{1}{4} - \frac{4Bq}{\eta^2_0 (2B/\eta^2_0 - 1)} \) with \( B = \kappa^2 q \{ [\beta(\beta + 1)]^q + [-24(\beta + 1)^3]^{q/2} \} \); which depicts that \( \nu \) must be positive as the parameter \( q \) is greater than zero. Thereby, in the superhorizon limit, it is evident that \( (-k\eta)^{\nu + \frac{1}{2}} \ll (-k\eta)^{-\nu + \frac{1}{2}} \), and thus the term containing \( D_1 \) in the right hand side of eq. (4.8) is negligible, as compared to the other one containing \( D_2 \). Consequently, from eq. (3.12) we have the following power spectra for electric and magnetic fields as,

\[
\frac{\partial \rho(\bar{E})}{\partial \ln k} = \frac{P(\eta) (\nu - \frac{1}{2})^2 H_0^4}{4\pi \left[ 1 - \frac{4B}{\eta^2_0} \left( -\frac{\eta}{\eta_0} \right)^2 \right]^{2q}} \left\{ 2^{\nu}\Gamma(-\nu + 1) \cos[\pi(\nu + 1/2)] \right\}^2 (-k\eta)^{3 - 2\nu},
\]

and

\[
\frac{\partial \rho(\bar{B})}{\partial \ln k} = \frac{Q(\eta) H_0^4}{4\pi \left[ 1 - \frac{4B}{\eta^2_0} \left( -\frac{\eta}{\eta_0} \right)^2 \right]^{2q}} \left\{ 2^{\nu}\Gamma(-\nu + 1) \cos[\pi(\nu + 1/2)] \right\}^2 (-k\eta)^{5 - 2\nu},
\]
Figure 1. $\nu(q)$ vs $q$: in the Blue curve, $H_0 = 10^{-5} M_{Pl}$, $\epsilon = 0.01$ and in the Yellow Curve, $H = 10^{-5} M_{Pl}$, $\epsilon = 0.1$. Both the curves start from $\nu = 0.5$ at $q = 0$, which is also expected from the expression of $\nu(q)$.

respectively. The above two expressions help us to explore the possibility of a scale invariant magnetic field spectrum. We may note that the scale invariance for the magnetic field spectrum does not imply the scale invariance of the electric field spectrum, in particular the electric power spectrum becomes scale invariant for $\nu = 3/2$ while, in the magnetic case, $\nu = 5/2$ leads to a scale invariant power spectrum. However one can easily investigate that for any possible value of $q$, i.e. within $0 < q < 1$, the function $\nu(q)$ does not acquire either the value $3/2$ or $5/2$ and thus the electric and magnetic power spectra are not scale invariant in the present context. The above two expressions help us to explore the possibility of a scale invariant magnetic field spectrum. We may note that the scale invariance for the magnetic field spectrum does not imply the scale invariance of the electric field spectrum, in particular the electric power spectrum becomes scale invariant for $\nu = 3/2$ while, in the magnetic case, $\nu = 5/2$ leads to a scale invariant power spectrum. However one can easily investigate that for any possible value of $q$, i.e. within $0 < q < 1$, the function $\nu(q)$ does not acquire either the value $3/2$ or $5/2$ and thus the electric and magnetic power spectra are not scale invariant in the present context. The above two expressions help us to explore the possibility of a scale invariant magnetic field spectrum. We may note that the scale invariance for the magnetic field spectrum does not imply the scale invariance of the electric field spectrum, in particular the electric power spectrum becomes scale invariant for $\nu = 3/2$ while, in the magnetic case, $\nu = 5/2$ leads to a scale invariant power spectrum. However one can easily investigate that for any possible value of $q$, i.e. within $0 < q < 1$, the function $\nu(q)$ does not acquire either the value $3/2$ or $5/2$ and thus the electric and magnetic power spectra are not scale invariant in the present context.

However, in order to ensure that the backreaction of the electromagnetic field stays small during inflation, we shall first consider the energy stored in the electric field at a given time $\eta_c$, which is

$$
\rho(\vec{E}, \eta_c) = \int_{k_i}^{k_c} \frac{\partial \rho(\vec{E})}{\partial \ln k} d\ln k = \frac{P(\eta_c) (\nu - \frac{1}{2})^2 H_0^4 (1 - e^{-2N_c})}{4\pi \left[ 1 - \frac{4B}{\eta_0^2 \left( \frac{\eta_c}{\eta_0} \right)^{2e\nu}} \right]^{2} \left\{ 2^{-\nu} \Gamma(-\nu + 1) \cos\left[ \pi (\nu + 1/2) \right] \right\}^2},$

(4.11)

where we used eq. (4.9), $k_i$ and $k_c$ denote the mode-momenta that cross the horizon at the beginning of inflation and at $\eta = \eta_c$, respectively, and thus $k_i$ is safely considered to be the same as for the CMB scale. Moreover, $N_c$ is the number of inflationary e-foldings up to the time $\eta = \eta_c$ and, thereby, $N_c$ is given by $N_c = \ln \left( \frac{a_i}{a_c} \right)$, with $a_i = a(\eta_i)$ and $a_c = a(\eta_c)$. We also have the relation $|\eta_c| = k_i^{-1}$ and $k_c$ is obviously greater than the CMB scale momentum $k_{CMB} \approx 10^{-40}\text{GeV} \approx 0.02\text{Mpc}^{-1}$. Similarly, using eq. (4.10), we determine the energy
density from the magnetic fields, which yields

\[ \rho(B, \eta_c) = \int_{k_i}^{k_f} \frac{\partial \rho(B)}{\partial \ln k} d\ln k = \frac{Q(\eta_c) H_0^4 (1 - e^{-4N_c})}{4\pi \left[ 1 - 4\frac{B}{\eta_0^2} \left( \frac{\eta_c}{\eta_0} \right)^{2q} \right]^{2} \left\{ 2^{-\nu} \Gamma(-\nu + 1) \cos [\pi(\nu + 1/2)] \right\}^2}. \]  \hspace{1cm} (4.12)

The total electromagnetic energy density at \( \eta = \eta_c \) becomes

\[ \rho_{em}(\eta_c) = \rho(E, \eta_c) + \rho(B, \eta_c) = \frac{H_0^4 \left[ P(\eta_c) (\nu - \frac{1}{2})^2 (1 - e^{-2N_c}) + Q(\eta_c) (1 - e^{-4N_c}) \right]}{4\pi \left[ 1 - 4\frac{B}{\eta_0^2} \left( \frac{\eta_c}{\eta_0} \right)^{2q} \right]^{2} \left\{ 2^{-\nu} \Gamma(-\nu + 1) \cos [\pi(\nu + 1/2)] \right\}^2}. \]  \hspace{1cm} (4.13)

where \( P(\eta) \) and \( Q(\eta) \) are shown in eqs. (3.5) and (3.6), respectively. In order to avoid the backreaction issue, we have to ensure that the electromagnetic energy density is less than that of the background energy density; in particular, we have to show \( \rho_{em} < 3M_{Pl}^2 H^2 \) during inflation. In the context of the scalar-Einstein-Gauss-Bonnet theory, which is considered to be the background gravity theory in the present work, the background energy density gets contributions from the scalar field and also from the higher curvature terms through the GB coupling function. In regard to the explicit expressions of the functions \( P(\eta) \) and \( Q(\eta) \), plugging the conformal Hubble parameter from eq. (2.12) to eqs. (3.5), (3.6) and after simplifying a bit, we obtain

\[ P(\eta_c) = 1 - 4 \left( 1 + \frac{q}{2} \right) (12^q + 24^{q/2}) \left( \frac{H_0}{M_{Pl}} \right)^{2q} \left( - \frac{\eta_c}{\eta_0} \right)^{2q}, \]

\[ Q(\eta_c) = 1 - 4 \left( 1 + \frac{q}{2} \right) (12^q + 24^{q/2}) \left( \frac{H_0}{M_{Pl}} \right)^{2q} \left( - \frac{\eta_c}{\eta_0} \right)^{2q}. \]  \hspace{1cm} (4.14)

where we used \( \eta_0^{-1} = H_0 \). The above expressions, along with the consideration of \( 0 < q < 1 \) and \( \epsilon < 1 \), eq. (4.13), clearly indicate that the electromagnetic energy density during inflation is of the order of \( H_0^4 \), i.e.

\[ \rho_{em}(\eta_c) \sim H_0^4. \]  \hspace{1cm} (4.15)

The inflationary energy scale is less than the Planck scale; in particular, we consider \( H_0 = 10^{-5} M_{Pl} \) and, thus, eq. (4.15) leads to the inequality \( \rho_{em} \ll M_{Pl}^2 H^2 \). This confirms that the electromagnetic field has a negligible backreaction on the background inflationary spacetime, leading to the resolution of the backreaction problem in the present magnetogenesis scenario.

5 Curvature perturbation sourced by EM field during inflation

The production of the gauge field during inflation may source the curvature perturbation in the super Hubble scales [108–114]. The power spectrum of the induced curvature perturbations or the induced non-Gaussianities should satisfy the Planck constraints. Thereby in the present context where the EM field non-minimally couples with the background spacetime curvature, it is important to investigate whether the curvature perturbations induced by the EM field obeys the Planck constraints or not. Earlier, in [108–114], the authors discussed such kind of induced curvature perturbations and the corresponding constraints on the model parameters in a magnetogenesis scenario where the EM field couples with a scalar field (say,
the inflaton field). However in the present paper, the EM field couples with the background Ricci scalar and the Gauss-Bonnet invariant, unlike to the case where the EM field couples with a scalar field.

The curvature perturbation $\zeta(\eta, \vec{x})$ is defined as the perturbation of the scale factor $a(\eta, \vec{x})$ on the uniform density slice, i.e $\zeta(\eta, \vec{x}) = \ln[a(\eta, \vec{x})/a(\eta)]$ where $\eta$ is the cosmic time. Then the curvature perturbation that is sourced by the EM field is given by [108],

$$
\zeta_{em}(\eta, \vec{x}) = -\frac{2H}{\epsilon \rho_{inf}} \int_{\eta_m}^{\eta} \frac{d\eta'}{a(\eta')} \rho_{em}(\eta', \vec{x})
$$

(5.1)

where $H$ is the cosmic time Hubble parameter during inflation, $\epsilon$ is the slow roll parameter, $\rho_{inf}$ is the background inflaton energy density (recall, the scalar coupled Gauss-Bonnet curvature is responsible for the inflation in the present context) and $\rho_{em}$ denotes the EM field energy density. The lower limit $\eta_m$ in the integral corresponds to the time after which the gauge field production starts and the mode which crosses the horizon at $\eta = \eta_m$ will be symbolized by $k_{min}$ in the later calculation. Here we assume $k_{CMB} > k_{min}$, i.e the generation of electromagnetic fields is considered to begin earlier than the horizon-crossing of CMB modes. At this stage it deserves mentioning that the electromagnetic anisotropic stress which can also source the curvature perturbation is not taken into account in eq. (5.1). This is due to the fact that the contribution from the electromagnetic anisotropic stress is suppressed during slow roll inflation, in particular by the inverse of the slow roll parameter $\epsilon$, in comparison to the contribution written in the right hand side of eq. (5.1) [114]. The EM field energy density can be expressed as $\rho_{em}(\eta, \vec{x}) = \rho_E(\eta, \vec{x}) + \rho_B(\eta, \vec{x})$, however eqs. (4.9) and (4.10) clearly indicate that the ratio of magnetic to electric power spectrum in the superhorizon scale is given by: $P_B/P_E \sim (-k\eta)^2/(\nu - \frac{1}{2})^2$ which depicts that the magnetic field strength is much lower than the electric field strength in the superhorizon limit. Thereby we can consider the EM field energy density as $\rho_{em} \approx \rho_E = \frac{1}{2}E^2$. Such consideration allows to express the EM field energy density in Fourier space as follows,

$$
\rho_{em}(\eta, \vec{k}) = \frac{1}{2} \int \int \frac{d^3p_1 d^3p_2}{(2\pi)^6} \delta(\vec{p}_1 + \vec{p}_2 - \vec{k}) E(\eta, \vec{p}_1) E(\eta, \vec{p}_2),
$$

(5.2)

where the electric field is defined as $|E(\eta, k)| = \frac{1}{\sqrt{2}}|A'(k, \eta)|$. Using eq. (4.8), we determine the electric field during inflation as,

$$
|E(\eta, k)| = \frac{(\nu - \frac{1}{2})}{\sqrt{2}} H^2 k^{-\nu} (\eta)^{3-\nu}
$$

(5.3)

with, recall, $\nu^2 = \frac{1}{4} - \frac{4Beq}{m_0^2(2B/m_0^2 - 1)}$ and $B = \kappa^{2q}\{[6\beta(\beta + 1)]^q + [-24(\beta + 1)^3]^q/2\}$. Using the above expression of electric field and following the procedure of [108], we evaluate the 2-point correlator of $\zeta_{em}(\eta, \vec{k})$ at $\eta = \eta_f$ (i.e at the end of inflation) in the present context as,

$$
\langle \zeta_{em}(\eta_f, \vec{k}_1) \zeta_{em}(\eta_f, \vec{k}_2) \rangle = 2\delta(\vec{k}_1 + \vec{k}_2) G_2 \int_{k_{min}}^{k_f} d^3p_1 d^3p_2 \delta(\vec{p}_1 - \vec{x}_1) \delta(\vec{p}_2 - \vec{x}_2) p_1^{-2\nu} p_2^{-2\nu}
$$

(5.4)

$$
\times \left( \delta_{j_1 j_2} - \langle \vec{p}_1 \rangle_{j_1} \langle \vec{p}_1 \rangle_{j_2} \right) \left( \delta_{j_1 j_2} - \langle \vec{p}_2 \rangle_{j_1} \langle \vec{p}_2 \rangle_{j_2} \right) \times \left\{ \prod_{i=1,2} \int_{\eta_m}^{\eta_f} d\eta_i (\eta_i)^{2-2\nu} \right\}
$$
where \( k_f \) is the mode that crosses the horizon at the end of inflation i.e at \( \eta = \eta_f \) and \( k_{\text{min}} \) is defined after eq. (5.1). Moreover the factor \( G_2 \), present in the above expression, is given by,

\[
G_2 = \left[ \frac{H_f^2 (\nu - \frac{1}{2})^2}{6 \epsilon M_{\text{Pl}}^2} \right] .
\]

To derive \( G_2 \), we use \( \rho_{\text{inf}}(\eta_f) = 3H_f^2 M_{\text{Pl}}^2 \) which holds true due to the fact that the EM field has negligible backreaction on the background inflationary spacetime. Performing the \( p_2 \) and the \( \eta \) integral of eq. (5.4), we get

\[
\langle \zeta_{\text{em}}(\eta_f, \vec{k}_1) \zeta_{\text{em}}(\eta_f, \vec{k}_2) \rangle = \frac{32\pi}{3} \delta(\vec{k}_1 + \vec{k}_2) G_2 \int_{k_{\text{min}}}^{k_f} dp_1 \ p_1^{2-2\nu} (p_1 + k_2)^{-2\nu} \times \left\{ \frac{(-\eta_f)^{3-2\nu} - (-\eta_m)^{3-2\nu}}{3 - 2\nu} \right\}^2 .
\]

where we use the integral \( \int d\Omega \hat{k}_i \hat{k}_j = \frac{4\pi}{3} \delta_{ij} \). Here we would like to mention that the quantity \( \nu = \frac{1}{2} \left[ 1 - \frac{16 B_{\text{eq}}}{3(2B/\nu^2 - 1)} \right]^{1/2} \) is less than 1/2, which can also be ensured from the figure 1. Thereby the integral in eq. (5.6) will get the maximum contribution from the upper limit \( k_f \) and as a result, the final form of the two point correlator comes as,

\[
\langle \zeta_{\text{em}}(\vec{k}_1) \zeta_{\text{em}}(\vec{k}_2)(\eta_f) \rangle = \frac{32\pi}{3k_f^3} \delta(\vec{k}_1 + \vec{k}_2) G_2 \left\{ \frac{1}{3 - 2\nu} \right\} \left\{ \frac{(-k_1 \eta_f)^{3-2\nu} - (-k_1 \eta_m)^{3-2\nu}}{3 - 2\nu} \right\} \times \left\{ \frac{1 + (\hat{k}_1 \hat{k}_2)^2}{(k_1 k_2)^3} + \frac{1 + (\hat{k}_1 \hat{k}_3)^2}{(k_1 k_3)^3} + \frac{1 + (\hat{k}_2 \hat{k}_3)^2}{(k_2 k_3)^3} \right\} .
\]

We will eventually consider the two point correlator at the CMB scales, i.e \( k_1 = k_{\text{CMB}}, \) and since, as mentioned earlier, the EM field generation starts earlier than the horizon-crossing of the CMB modes, we have \( k_f \gg k_1 = k_{\text{CMB}} \gg k_{\text{min}} \). Furthermore, by using eq. (5.3) along with the described procedure in [108], we calculate the 3-point and 4-point correlators of the induced curvature perturbation and they are given by the following expressions,

\[
\langle \zeta_{\text{em}}(\vec{k}_1) \zeta_{\text{em}}(\vec{k}_2) \zeta_{\text{em}}(\vec{k}_3)\rangle = \frac{32\pi}{3k_f^3} \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) G_3 \left\{ \frac{(-k_1 \eta_f)^{3-6\nu}}{3 - 2\nu} \right\} \times \left\{ \frac{1 + (\hat{k}_1 \hat{k}_2)^2}{(k_1 k_2)^3} + \frac{1 + (\hat{k}_1 \hat{k}_3)^2}{(k_1 k_3)^3} + \frac{1 + (\hat{k}_2 \hat{k}_3)^2}{(k_2 k_3)^3} \right\} .
\]

and

\[
\langle \zeta_{\text{em}}(\vec{k}_1) \zeta_{\text{em}}(\vec{k}_2) \zeta_{\text{em}}(\vec{k}_3) \zeta_{\text{em}}(\vec{k}_4) \rangle = \frac{64\pi}{3} \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) G_4 \left\{ \frac{(-k_1 \eta_f)^{3-8\nu}}{3 - 2\nu} \right\} \times \left\{ \frac{1 + (\hat{k}_1 \hat{k}_2)^2 + (\hat{k}_1 \hat{k}_3)^2 + (\hat{k}_2 \hat{k}_3)^2 - (\hat{k}_1 \hat{k}_4)^2 (\hat{k}_2 \hat{k}_4)^2 (\hat{k}_3 \hat{k}_4)^2}{(k_1 k_2 k_3 k_4)^3} \right\} + 11 \text{ perms.}
\]
respectively, with $\vec{k}_{1:3} = \vec{k}_1 + \vec{k}_3$. Moreover $G_3$ and $G_4$ have the following forms,

$$G_3 = \left[ \frac{H_f^2(\nu - \frac{1}{2})^2}{6\epsilon M_{Pl}^2} \right]^3, \quad G_4 = \left[ \frac{H_f^2(\nu - \frac{1}{2})^2}{6\epsilon M_{Pl}^2} \right]^4. \quad (5.10)$$

The 2-point correlator in eq. (5.7) immediately leads to the power spectrum of the curvature perturbation induced by the EM field as,

$$\mathcal{P}(\zeta_{em}) = \frac{2}{3} \left[ \frac{H_f^2(\nu - \frac{1}{2})^2}{6\epsilon M_{Pl}^2} \right]^2 \left\{ \left( \frac{k_f/k_1}{3 - 2\nu} \right)^{3-4\nu} \left( \frac{-k_1\eta_f}{3 - 2\nu} \right)^{3-2\nu} \right\}^2. \quad (5.11)$$

Similarly the 3-point and 4-point correlators provide the induced non-linear parameters $f_{NL}^{em}$ and $\tau_{NL}^{em}$ in the present magnetogenesis model as,

$$f_{NL}^{em} = \frac{10}{27} \left[ \frac{H_f^2(\nu - \frac{1}{2})^2}{6\epsilon M_{Pl}^2} \right]^3 \left( \frac{1}{\mathcal{P}(\zeta_{em})} \right)^2 \left\{ \left( \frac{k_f/k_1}{3 - 2\nu} \right)^{3-6\nu} \left( \frac{-k_1\eta_f}{3 - 2\nu} \right)^{3-2\nu} \right\}^3, \quad (5.12)$$

and

$$\tau_{NL}^{em} = \frac{1}{3} \left[ \frac{H_f^2(\nu - \frac{1}{2})^2}{6\epsilon M_{Pl}^2} \right]^4 \left( \frac{1}{\mathcal{P}(\zeta_{em})} \right)^3 \left\{ \left( \frac{k_f/k_1}{3 - 2\nu} \right)^{3-8\nu} \left( \frac{-k_1\eta_f}{3 - 2\nu} \right)^{3-2\nu} \right\}^4. \quad (5.13)$$

respectively. As mentioned earlier, the mode $k_1$ is identified with the CMB scale and thus we have the relations like $-k_1\eta_f = e^{-N_f}$ (where $N_f$ is the e-fold number from the horizon crossing of $k_{CMB}$ to the horizon-crossing of $k_f$, i.e. $N_f$ is the inflationary e-fold number) and $-k_1\eta_m = e^{(N_{em}-N_f)}$ where $N_{em}$ is the e-fold from the horizon crossing of $k_{min}$ to the horizon crossing of $k_f$, i.e $N_{em}$ denotes the e-fold during which the EM field production occurs. Eqs. (5.11), (5.12) and (5.13) clearly reveal that $\mathcal{P}(\zeta_{em})$, $f_{NL}^{em}$ and $\tau_{NL}^{em}$ depend on the quantities: $H_0/M_{Pl}$, $H_f/M_{Pl}$, $\epsilon$, $q$, $N_f$ and $N_{em} - N_f$. Out of these quantities, $H_f$ is related to $H_0$, $\epsilon$ and $N_f$ via $H_f = H_0 \exp \left[ \frac{-N_f}{1+\epsilon} \right]$, as shown in eq. (2.14). In particular, $H_0 = 10^{-5}M_{Pl}$, $\epsilon = 0.1$, $N_f = 58$ immediately leads to the Hubble parameter at the end of inflation as $H_f = 5.1 \times 10^{-8}M_{Pl}$. Due to such parametric regime along with $q = 0.5$ (which we will also consider in the next section during the determination of magnetic field’s strength at present epoch), eqs. (5.11), (5.12) and (5.13) become,

$$\mathcal{P}(\zeta_{em}) \approx 10^{-27} \times \exp \left[ 4(N_{em} - N_f) \right]$$

$$f_{NL}^{em} \approx 10^{-24} \times \exp \left[ 6(N_{em} - N_f) \right]$$

$$\tau_{NL}^{em} \approx 10^{-23} \times \exp \left[ 8(N_{em} - N_f) \right]. \quad (5.14)$$

Having the theoretical predictions of $\mathcal{P}(\zeta_{em})$, $f_{NL}^{em}$ and $\tau_{NL}^{em}$ in hand, we now confront the model with the Planck results which put certain constraints on such quantities given by,

$$\mathcal{P}^{obs}(\zeta) \approx 2.1 \times 10^{-9}, \quad f_{NL} \leq 14.3 = f_{NL}^{obs} \text{ (say)}, \quad \tau_{NL} \leq 2800 = \tau_{NL}^{obs} \text{ (say)}. \quad (5.15)$$

The restriction, that the theoretical predictions of $\mathcal{P}(\zeta_{em})$, $f_{NL}^{em}$ and $\tau_{NL}^{em}$ do not exceed their respective observed values provided by the Planck results, in turn lead to corresponding
constraint on \( N_{em} - N_f \), in particular,

\[
\mathcal{P}(\zeta_{em}) < \mathcal{P}^{obs}(\zeta) \implies N_{em} - N_f < \frac{1}{4} \ln \left[ 10^{27} \times \mathcal{P}^{obs}(\zeta) \right] \approx 10.4
\]

\[
\mathcal{J}_{em} < \mathcal{J}^{obs} \implies N_{em} - N_f < \frac{1}{6} \ln \left[ 10^{24} \times \mathcal{J}^{obs} \right] \approx 9.6
\]

\[
\mathcal{T}_{em} < \mathcal{T}^{obs} \implies N_{em} - N_f < \frac{1}{8} \ln \left[ 10^{23} \times \mathcal{T}^{obs} \right] \approx 7.6 \quad (5.16)
\]

Eq. (5.16) clearly evidents that the allowed space of \( N_{em} - N_f \) becomes tighter due to the restriction \( \tau^{em}_{NL} < \tau^{obs}_{NL} \) compared to the other two restrictions. Thus as a whole, eq. (5.16) provides the constraints from the curvature perturbation induced by the electromagnetic field during inflation in the present magnetogenesis scenario where the EM field couples with the background Ricci scalar and the Gauss-Bonnet curvature.

6 Present magnetic strength for the instantaneous reheating case

In this section we will concentrate on the strength of the magnetic field in the present epoch, as it is important to know whether the model can generate magnetic fields of sufficient strength. For that purpose we need to know the conductivity of the universe, both during the inflationary epoch and immediately after it. During inflation the universe was a poor electrical conductor, however after inflation it became a very good conductor and hence the electric currents became important during this phase. The high electrical conductivity in the post inflationary epoch lies on the consideration that at the end of inflation, we assume *instantaneous reheating* (the reheating e-folding number is taken to be zero), i.e., the universe makes a sudden jump from the inflationary phase to a radiation dominated epoch during which the cosmic Hubble parameter goes as \( H \propto t^{-1} \) with \( t \) being the cosmic time (at a later stage, we will relax this assumption and will consider a reheating epoch with a non-zero e-folding number after inflation). Due to the large conductivity in the post inflationary phase, one can write the current density as \( J^i = \sigma E^i \) where \( \sigma \) is the conductivity and \( E^i \) are the components of the electric field. Then, the corresponding vector potential has two sorts of solutions: one independent of time and the other behaving as \( \exp(-\sigma t) \), which is vanishingly small. Thus, the vector potential remains constant with time and suggests that the electric field becomes soon negligible, while the magnetic field remains as the dominant piece. Moreover, we mentioned earlier that at late time the spacetime curvature becomes low enough that the conformal breaking term \( S_{CB} \) will not contribute and the electromagnetic field follows the standard Maxwell’s equations. In particular, we consider \( f(R,G) \) to be zero during the post inflationary phase, which can be also connected with the other point of view, as eq. (2.15) clearly shows that the conformal coupling \( f(R,G) \) goes to zero (and, consequently, also \( P(\eta) = Q(\eta) = 1 \) as \( k\eta \to 0 \), i.e. at the end of inflation). The conformal symmetry of the electromagnetic field is thus restored after inflation, and the electromagnetic energy density decays as \( 1/a^4 \) or, equivalently, the magnetic field energy density evolves as \( 1/a^4 \), as the electric field is practically zero in the post inflationary epoch. Hence, the magnetic field strength at the present epoch is related with that at the end of inflation by the expression

\[
\frac{\partial \rho(\vec{B})}{\partial \ln k} \bigg|_0 = \left( \frac{a_f}{a_0} \right)^4 \frac{\partial \rho(\vec{B})}{\partial \ln k} \bigg|_{\eta_f}, \quad (6.1)
\]
In the case of instantaneous reheating, which we have considered in section 7, the present magnetic strength for a Kamionkowski like reheating model is obtained around the CMB scales. Therefore the theoretical prediction of $B_0$ coming from the present model lies far below the range of the observational constraints; and this argument is not just confined to $q = 0.5$, but also valid for the whole parametric regime that we consider in the present context, i.e. for $0 < q < 1$.

7 Present magnetic strength for a Kamionkowski like reheating model with non-zero e-folding number

In the case of instantaneous reheating, which we have considered in section 6, the conductivity turns on already after the end of inflation and, as a result, the electric field quickly goes to zero. However, if we consider the reheating phase with a non-zero e-folding number, then there is no reason to keep the assumption that the conductivity becomes large immediately after inflation. Indeed, the conductivity just remains non-zero and, consequently, the strong electric field induces the magnetic field evolution during the epoch between the end of inflation and the end of reheating [23]. This yields less redshift of the magnetic field in the reheating epoch, as compared to $1/a^4$, and thus the magnetic field’s present strength may become to be much larger than what has been estimated in eq. (6.3). In such situation, our next aim is to calculate the current magnetic strength in the present magnetogenesis model by considering the reheating phase with a certain, non-zero e-folding number.
Concerning the reheating dynamics, we follow the conventional reheating mechanism given by Kamionkowski et al. [88], where the inflaton energy is supposed to instantaneously convert into radiation at the end of reheating. In this process, the main idea is to parametrize the reheating phase by an effective equation of state $\omega_{\text{eff}}$, in particular the Hubble parameter during reheating is connected to that at the end of inflation by the EoS parameter $\omega_{\text{eff}}$. Moreover, the duration of the reheating phase, characterized by the respective e-fold number $N_{\text{re}}$, and the reheating temperature ($T_{\text{re}}$) can be expressed in terms of $\omega_{\text{eff}}$ and of some inflationary parameters by the following relations [88, 104],

\begin{equation}
N_{\text{re}} = \frac{4}{(1 - 3\omega_{\text{eff}})} \left[ -\frac{1}{4} \ln \left( \frac{45}{\pi^2 g_{\text{re}}} \right) - \frac{1}{3} \ln \left( \frac{11g_{s,\text{re}}}{43} \right) \right. \\
\left. - \ln \left( \frac{k}{a_0 T_0} \right) - \ln \left( \frac{(3H_0^2M_{\text{Pl}}^2)^{1/4}}{H_0} \right) - N_f \right],
\end{equation}

\begin{equation}
T_{\text{re}} = H_0 \left( \frac{43}{11g_{s,\text{re}}} \right)^{1/7} \left( \frac{a_0 T_0}{k} \right) \exp \left[ - (N_f + N_{\text{re}}) \right],
\end{equation}

where the present CMB temperature is $T_0 = 2.725$K, the pivot scale $\frac{k}{a_0} \approx 0.02$Mpc$^{-1}$ (taken as CMB scale) and $a_0$ is the present cosmological scale factor. Here, for simplicity, we have taken both the values of the degrees of freedom for entropy at reheating ($g_{s,\text{re}}$) and the effective number of relativistic species upon thermalization ($g_{\text{re}}$) to be the same and of the order of 100, i.e., $g_{s,\text{re}} = g_{\text{re}} \approx 100$. With this reheating model, we are now going to evaluate the electromagnetic mode function and, consequently, the power spectrum during the reheating epoch, in the next subsection.

### 7.1 Electromagnetic mode function and power spectrum during the reheating epoch

After the end of inflation the curvature coupling function $f(R, G)$ is considered to be zero, which can be also connected with the continuity point of view as eq. (2.15) clearly shows that the conformal breaking coupling $f(R, G)$ goes to zero as $k\eta \to 0$, i.e. at the end of inflation. This indicates that the conformal coupling of the electromagnetic field is restored in the post inflationary phase and, hence, the evolution of the electromagnetic field becomes standard Maxwellian [23]. Thus, the gauge field production from the quantum vacuum ceases to exist, in particular the absolute value of the Bogoliubov coefficient after inflation becomes constant (with respect to time) and equal to that at the end inflation. During the stage between the end of inflation and reheating, the electromagnetic mode function will follow the Maxwell equations in vacuum, i.e., the equation of motion (4.1) with $f(R, G) = 0$ (or equivalently $h(R, G) = 1$), given by

\begin{equation}
A^{\text{(re)}}(k, \eta) + k^2 A^{\text{(re)}}(k, \eta) = 0,
\end{equation}

where $A^{\text{(re)}}(k, \eta)$ denotes the electromagnetic mode function during reheating and, moreover, recall that $A(k, \eta)$ (i.e. without any superscript) symbolizes the electromagnetic mode function in the inflationary phase (see section 4). Eq. (7.3), which is free from source term, indicates that we consider the Universe to be a bad conductor during the reheating phase. However, due to the Schwinger production, the assumption of zero conductivity demands a proper justification, which we will perform in section 8. Solving eq. (7.3), one gets

\begin{equation}
A^{\text{(re)}}(k, \eta) = \frac{1}{\sqrt{2k}} \left[ c_k e^{-ik(\eta - \eta_f)} + d_k e^{ik(\eta - \eta_f)} \right],
\end{equation}
with \( c_k, d_k \) two integration constants and \( \eta_f \) the end instant of inflation. The integration constants can be determined by matching \( A^{(re)}(k, \eta) \) and \( A'(k, \eta) \) at the end of inflation; in particular,

\[
A^{(re)}(k, \eta_f) = A(k, \eta_f) \quad \text{and} \quad A'(k, \eta_f) = A'(k, \eta_f),
\]

respectively, where \( A(k, \eta) \) represents the mode function during inflation and follows eq. (4.7). Therefore, the integration constants turn out to be,

\[
c_k = \sqrt{\frac{k}{2}} A(k, \eta_f) + \frac{i}{\sqrt{2k}} A'(k, \eta_f)
\]

\[
d_k = \sqrt{\frac{k}{2}} A(k, \eta_f) - \frac{i}{\sqrt{2k}} A'(k, \eta_f)
\]

with \( A(k, \eta_f) \) can be obtained from eq. (4.7) by putting \( \eta = \eta_f \), i.e,

\[
A(k, \eta_f) = \frac{\sqrt{-k \eta_f}}{\left[1 - \frac{4B}{\eta_0^2} \left(-\frac{\eta_f}{\eta_0}\right)^{2\nu} \right]^{1/2}} \left\{ D_1 J_\nu(-k \eta_f) + D_2 J_{-\nu}(-k \eta_f) \right\},
\]

and, consequently, \( A'(k, \eta_f) \) is given by

\[
A'(k, \eta_f) = \frac{k}{\left[1 - \frac{4B}{\eta_0^2} \left(-\frac{\eta_f}{\eta_0}\right)^{2\nu} \right]^{1/2}} \left\{ \sqrt{-k \eta_f} \left[D_1 J_{1+\nu}(-k \eta_f) + D_2 J_{-1-\nu}(-k \eta_f) \right] \right.
\]

\[
+ \frac{1}{\sqrt{-k \eta_f}} \left[D_1 \left(\nu - \frac{1}{2}\right) J_\nu(-k \eta_f) - D_2 \left(\nu + \frac{1}{2}\right) J_{-\nu}(-k \eta_f) \right] \right\}.
\]

Moreover, due to the interaction between the electromagnetic field and the background time dependent FRW spacetime, the electromagnetic field vacuum, starting from the Bunch-Davies vacuum at \( \eta \to -\infty \), changes with time and, as a result, particles are produced from this vacuum. Correspondingly, the Bogoliubov coefficients \( (\alpha_k(\eta) \text{ and } \beta_k(\eta)) \) at time \( \eta \) during reheating are given by

\[
\alpha_k(\eta) = \sqrt{\frac{k}{2}} A^{(re)}(k, \eta) + \frac{i}{\sqrt{2k}} A'(k, \eta),
\]

\[
\beta_k(\eta) = \sqrt{\frac{k}{2}} A^{(re)}(k, \eta) - \frac{i}{\sqrt{2k}} A'(k, \eta).
\]

With the solution of \( A^{(re)}(k, \eta) \), the above expressions boil down to the following

\[
\alpha_k(\eta) = c_k \ e^{-ik(\eta-\eta_f)}, \quad \beta_k(\eta) = d_k \ e^{ik(\eta-\eta_f)},
\]

which in turn relate \( c_k \) and \( d_k \) with the Bogoliubov coefficients defined at \( \eta = \eta_f \), in particular \( c_k = \alpha_k(\eta_f) \) and \( d_k = \beta_k(\eta_f) \). Therefore, eq. (7.10) demonstrates that the absolute value of the Bogoliubov coefficients during reheating are time independent and equal to those at the end of inflation. Now, \( |\beta_k(\eta)| \) represents the total number of produced particles (having momentum \( k \)) at time \( \eta \) from the Bunch-Davies vacuum defined at \( \eta \to -\infty \). Hence, the time
independency of the Bogoliubov coefficients during the reheating phase is a direct consequence of the fact that the conformal symmetry of the electromagnetic field is restored after inflation. With \( c_k = \alpha_k(\eta_f) \) and \( d_k = \beta_k(\eta_f) \), eq. (7.4) can be alternatively expressed as

\[
A^{(re)}(k, \eta) = \frac{1}{\sqrt{2k}} \left[ \alpha_k(\eta_f) e^{-i k(\eta - \eta_f)} + \beta_k(\eta_f) e^{i k(\eta - \eta_f)} \right].
\] (7.11)

By plugging back this solution of \( A^{(re)}(k, \eta) \) into eq. (3.12) and by putting \( P(\eta) = Q(\eta) = 1 \) (as \( f(R, G) = 0 \) in the post inflationary phase), we determine the magnetic and electric power spectra during the reheating epoch, as follows

\[
\frac{\partial \rho(\vec{B})}{\partial \ln k} = \frac{1}{2\pi^2} \sum_{r=1,2} \frac{k^5}{a^4} |A_r^{(re)}(k, \eta)|^2
\]

\[
= \frac{1}{2\pi^2} \left( \frac{k^4}{a^4} \right) \left[ |\alpha_k(\eta_f)|^2 + |\beta_k(\eta_f)|^2 + 2|\alpha_k(\eta_f)\beta_k(\eta_f)| \cos \{\theta_1 - \theta_2 - 2k(\eta - \eta_f)\} \right]
\] (7.12)

and

\[
\frac{\partial \rho(\vec{E})}{\partial \ln k} = \frac{1}{2\pi^2} \sum_{r=1,2} \frac{k^2}{a^4} |A_r^{(re)}(k, \eta)|^2
\]

\[
= \frac{1}{2\pi^2} \left( \frac{k^4}{a^4} \right) \left[ |\alpha_k(\eta_f)|^2 + |\beta_k(\eta_f)|^2 - 2|\alpha_k(\eta_f)\beta_k(\eta_f)| \cos \{\theta_1 - \theta_2 - 2k(\eta - \eta_f)\} \right],
\] (7.13)

respectively, with \( \theta_1 = \text{Arg}[\alpha_k(\eta_f)] \) and \( \theta_2 = \text{Arg}[\beta_k(\eta_f)] \). Consequently, the total electromagnetic power spectrum is

\[
\frac{\partial \rho_{em}}{\partial \ln k} = \frac{\partial \rho(\vec{B})}{\partial \ln k} + \frac{\partial \rho(\vec{E})}{\partial \ln k} = \frac{1}{\pi^2} \left( \frac{k^4}{a^4} \right) \left[ |\alpha_k(\eta_f)|^2 + |\beta_k(\eta_f)|^2 \right].
\] (7.14)

It may be observed from the above equation that the comoving electromagnetic power spectrum is independent of time, which is due to the fact that the conformal symmetry of the electromagnetic field is restored or, equivalently, the Bogoliubov coefficients become time-independent in the reheating phase. Coming back to eq. (7.12), the magnetic power spectrum at time \( \eta \) is found to depend on \( \alpha_k(\eta_f) \), \( \beta_k(\eta_f) \) and \( \eta - \eta_f \). So, in the following, we will explicitly evaluate these quantities.

- **Determination of \( \alpha_k(\eta_f) \) and \( \beta_k(\eta_f) \)**: from eqs. (7.6) and (7.10), one can determine \( \alpha_k(\eta_f) \) and \( \beta_k(\eta_f) \) in terms of \( k\eta_f \), \( \eta_0 \) and the model parameter \( q \), as follows

\[
\alpha_k(\eta_f) = \frac{\sqrt{k^2}}{\left[ 1 - \frac{4B}{\eta_0^q} \left( \frac{-\eta_f}{\eta_0} \right) \right]^{1/2} \left[ \sqrt{-k\eta_f} \left[ D_1 J_\nu(-k\eta_f) + D_2 J_{-\nu}(-k\eta_f) \right] + i \sqrt{-k\eta_f} \left[ D_1 J_{-1+\nu}(-k\eta_f) + D_2 J_{-1-\nu}(-k\eta_f) \right] \right]}{\left[ -i + \frac{i}{\sqrt{-k\eta_f}} \left[ D_1 \left( e^{i/2} - \frac{1}{\eta_f} \right) J_\nu(-k\eta_f) - D_2 \left( e^{i/2} + \frac{1}{\eta_f} \right) J_{-\nu}(-k\eta_f) \right] \right]}
\] (7.15)
and

\[ \beta_k(\eta_f) = \frac{\sqrt{k/2}}{\left[ 1 - \frac{4B}{\eta_0} \left( \frac{-n_0}{\eta_0} \right) \right]^{2\eta}} \left\{ \sqrt{-k\eta_f} \left[ D_1 J_\nu(-k\eta_f) + D_2 J_{-\nu}(-k\eta_f) \right] \right\} \]

\[ -i\sqrt{-k\eta_f} \left[ D_1 J_{1+\nu}(-k\eta_f) + D_2 J_{-\nu}(-k\eta_f) \right] \]

\[ \frac{i}{\sqrt{-k\eta_f}} \left[ D_1 \left( \nu + \frac{1}{2} \right) J_\nu(-k\eta_f) - D_2 \left( \nu + \frac{1}{2} \right) J_{-\nu}(-k\eta_f) \right] \]  

respectively. For the modes around the CMB mode (on which we are interested eventually, to determine the current magnetic strength), we have the relation \(-k\eta_f = e^{-N_f}\) with \(N_f\) being the inflation e-folding number. Therefore, the Bessel function \(J_\nu(-k\eta_f)\) present in the above expression has the following asymptotic form

\[ \lim_{|k\eta_f| \ll 1} J_\nu(-k\eta) = \frac{1}{2\nu T(\nu + 1)} (-k\eta)^\nu \]

and also similar asymptotic forms hold for \(J_{-\nu}(-k\eta_f), J_{1-\nu}(-k\eta_f)\). Hence, it is evident from eqs. (7.15) and (7.16) that \(\alpha_k(\eta_f)\) and \(\beta_k(\eta_f)\) contain terms like \((-k\eta_f)^{-\nu - 1/2}\) and, due to the fact that \(\nu\) is positive (see figure 1), the presence of \((-k\eta_f)^{-\nu - 1/2}\) makes \(|\alpha_k(\eta_f)|, |\beta_k(\eta_f)|\) much larger than one.

\* Determination of “\(\eta - \eta_f\)” during reheating: \* the term \(k(\eta - \eta_f)\) in eq. (7.12) actually leads to the non-conventional dynamics of the magnetic field. Due to the constant equation of state during reheating dynamics this special term boils down to the following simple form:

\[ k(\eta - \eta_f) = \frac{2k}{(3\omega_{\text{eff}} + 1)} \left( \frac{1}{aH} - \frac{1}{a_f H_f} \right), \]

where we used \(\eta - \eta_f = \int_0^\infty \frac{da}{a^2 H^2} \).

With the above expressions of \(\alpha_k(\eta_f), \beta_k(\eta_f)\) and \(k(\eta - \eta_f)\), the magnetic power spectrum during the reheating phase in eq. (7.12) becomes [23]

\[ \frac{\partial \rho(\vec{B})}{\partial \ln k} = \frac{1}{\pi^2} \left( \frac{k^4}{a^4} \right) |\beta_k(\eta_f)|^2 \left\{ \text{Arg}[\alpha_k(\eta_f) \beta_k^*(\eta_f)] - \pi - \left( \frac{4k}{3\omega_{\text{eff}} + 1} \right) \left( \frac{1}{aH} - \frac{1}{a_f H_f} \right) \right\}^2. \]

Thereby, the evolution of the magnetic power in the reheating epoch is controlled by two different terms, namely the conventional one associated with the redshift factor \(\propto a^{-4}\), emerging from the first term in the right-hand side of the eq. (7.18), and another term associated with the redshift factor \(\propto (a^3 H)^{-2}\), which has emerged out from \(\frac{1}{aH}\). Since the universe expands with deceleration, i.e. \(\omega_{\text{eff}} > -\frac{1}{3}\), the magnetic power would be eventually dominated by the component \(\propto (a^3 H)^{-2}\). The term proportional to \((a^3 H)^{-2}\) carries the main difference in inflationary magnetogenesis between the two cases: (i) instantaneous reheating and (ii) a reheating phase with non-zero e-fold number. Actually, in the context of instantaneous reheating, the magnetic power goes down as \(a^{-4}\) after inflation and until today, unlike in the case of the reheating phase with non-zero e-fold number, where the magnetic power goes as
from the end of inflation to the end of reheating, and only then as $a^{-4}$ until the present epoch. Therefore as a whole, due to the presence of the reheating phase, the magnetic field’s present strength will be larger in comparison to that for instantaneous reheating. However the electric power in the reheating epoch goes by the conventional way $\propto a^{-4}$, in particular,

$$\left. \frac{\partial \rho(\vec{E})}{\partial \ln k} \right|_{\eta_f} = \frac{1}{\pi^2} \left( \frac{k^4}{a_f^4} \right) |\beta_k(\eta_f)|^2$$

Hence the electric and magnetic power evolve differently in the reheating phase, in particular, the electric power goes down as $a^{-4}$ while the magnetic power as $(a^3 H)^{-2}$.

### 7.2 Current magnetic strength and constraints on $\omega_{\text{eff}}$

The presence of a reheating phase with non-zero e-fold number leads to the electric field continuing to exist during post-inflationary era until the universe becomes purely conductive and this generally happens after the end of the reheating epoch. The strong electric field during reheating induces the magnetic field evolution and can support the production of sufficient strength of magnetic field to survive at present time. For the Kamionkowski reheating model (considered in the present work), the universe is supposed to expand with some constant EoS parameter ($\omega_{\text{eff}} > -\frac{1}{3}$ such that the expansion decelerates) in the reheating phase. Thus, the Hubble parameter at the end of reheating ($H_{\text{re}}$) can be related to that at the end of inflation ($H_f$) as

$$H_{\text{re}} = H_f \left( \frac{a_{\text{re}}}{a_f} \right)^{-\frac{3}{2}(1+\omega_{\text{eff}})}$$

where the suffix ‘re’ denotes the end point of reheating and the scale factor $a_{\text{re}}$ can be identified as $\frac{a_{\text{re}}}{a_f} = e^{N_{\text{re}}}$ with $N_{\text{re}}$ being the e-fold number of the reheating epoch and given in eq. (7.1). Consequently, from eq. (7.18) we evaluate the magnetic power spectrum at the end instant of reheating, as

$$\left. \frac{\partial \rho(\vec{B})}{\partial \ln k} \right|_{\text{re}} = \frac{1}{\pi^2} \left( \frac{k^4}{a_{\text{re}}^4} \right) |\beta_k(\eta_f)|^2$$

and, as a result, eq. (7.21) leads to the current magnetic strength, as

$$B_0 = \frac{\sqrt{2}}{\pi} \left( \frac{k}{a_0} \right)^2 |\beta_k(\eta_f)| \left\{ \text{Arg} [\alpha_k(\eta_f) \beta_k^*(\eta_f)] - \pi - \left( \frac{4}{3\omega_{\text{eff}} + 1} \right) \left( \frac{k}{a_f H_f} \right) \left[ \left( \frac{H_f}{H_{\text{re}}} \right)^{3\omega_{\text{eff}}+1} - 1 \right] \right\}^2$$
where we recall that the Bogoliubov coefficients have been obtained in eqs. (7.15) and (7.16), respectively. From the above expression, we may observe that the magnetic field’s present amplitude explicitly depends on the reheating parameters ($\omega_{\text{eff}}$ and $H_{\text{re}}$) as well as on some inflationary parameters. Thus the current magnetic strength encodes the information of various cosmological epochs of the universe, in particular the reheating and the inflationary epoch. As a consequence, probing $B_0$ opens up in turn a window for probing the early stage of the universe, particularly the reheating phase through the current observational amplitude of the magnetic field. Actually the expression in eq. (7.23) shows a direct one-to-one correspondence between the current magnetic strength $B_0$ and the effective reheating EoS parameter $\omega_{\text{eff}}$. Interestingly, the effective equation of state is no longer a free parameter as it is fixed by the $B_0$ via CMB.

Having obtained the final expression of $B_0$, now we confront our model with the CMB observations which put a constraint on the current magnetic strength, as $10^{-10} \text{G} \lesssim B_0 \lesssim 10^{-22} \text{G}$. For this purpose, we need $\frac{H_f}{H_{\text{re}}}$, which depends on $N_{\text{re}}$, as discussed after eq. (7.20). The reheating e-fold number ($N_{\text{re}}$) has the expression shown in eq. (7.1) and in turn requires $H_0$ and $H_f$ (i.e. the Hubble parameter at the beginning and at the end of inflation, respectively). We consider $H_0 = 10^{-5} M_{\text{Pl}}$, $\epsilon = 0.1$ and $N_f = 58$, by which $H_f$ can be estimated from eq. (2.14) as $H_f = H_0 \exp \left( \frac{\epsilon N_f}{1+\epsilon} \right) = 5.1 \times 10^{-8} M_{\text{Pl}}$. Moreover, the model parameter $q$ is taken as $q = 0.5$ and recall $\eta_0^{-1} = H_0$. With such considerations, we plot $B_0$ versus $\omega_{\text{eff}}$ by using eq. (7.23) (see figure 2).

Figure 2 clearly demonstrates that the theoretical prediction of $B_0$ lies well within the observational constraints for a certain range of values of the reheating EoS parameter, given by $0.28 \lesssim \omega_{\text{eff}} \lesssim 0.30$. The upper label of the $x$-axis of figure 2 shows the respective reheating temperature ($T_{\text{re}}$) for different values of $\omega_{\text{eff}}$, obtained from eq. (7.2). Moreover, due to the BBN constraint, $T_{\text{re}}$ is limited by $T_{\text{re}} \gtrsim 10^{-2} \text{GeV}$, which is shown by the vertical line in figure 2. Such constraint on $T_{\text{re}}$, in turn, sets a lower permissible limit on $\omega_{\text{eff}}$, in particular $\omega_{\text{eff}} \gtrsim 0.2965$, as shown in the same figure. Therefore, in order to make the present model compatible both with the CMB observations on $B_0$ and the BBN constraint on $T_{\text{re}}$, the viable regime of $\omega_{\text{eff}}$ turns out to be: $0.28 \lesssim \omega_{\text{eff}} \lesssim 0.2965$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{figure2.png}
\caption{$B_0$ (in Gauss units) vs $\omega_{\text{eff}}$, corresponding to $\frac{k}{m_0} = 0.02 \text{Mpc}^{-1}$. The reheating temperature for different values of $\omega_{\text{eff}}$ is shown in the upper label of the $x$-axis.}
\end{figure}
Therefore, the presence of the reheating phase with a non-zero e-fold number does in fact enhance the strength of the magnetic fields surviving in the present epoch and makes the theoretical predictions of the model fully compatible with the most recent observations. This is to be compared to the usual instantaneous reheating case, where the magnetic field’s current strength is always found to be far below the present observational constraints.

8 Constraints from Schwinger backreaction

In the earlier section, we consider the universe to be a bad conductor during the reheating phase, which means that the EM field during the reheating obeys the standard Maxwell’s equations in vacuum. However, due to the Schwinger production, the assumption of zero conductivity demands a proper investigation, which is the subject of the present section.

Eqs. (7.18) and (7.19) clearly indicate that the ratio of magnetic to electric power spectrum goes by,

\[
P_B \over P_E = \left\{ \alpha_k(\eta) \beta_k^*(\eta) \right\} - \pi \left( \frac{4k}{3\omega_{\text{eff}} + 1} \left( \frac{1}{aH} - \frac{1}{a_fH_f} \right) \right)^2 \tag{8.1}
\]

with \(P_B = \frac{\partial p_B}{\partial \ln k}\), \(P_E = \frac{\partial p_E}{\partial \ln k}\). Moreover, recall, the Bogoliubov coefficients at \(\eta_f\) are given by,

\[
\alpha_k(\eta_f) = \sqrt{\frac{k}{2}} A(k, \eta_f) + \frac{i}{\sqrt{2k}} A'(k, \eta_f), \quad \beta_k(\eta_f) = \sqrt{\frac{k}{2}} A(k, \eta_f) - \frac{i}{\sqrt{2k}} A'(k, \eta_f). \tag{8.2}
\]

with \(A(k, \eta_f)\) is shown in eq. (7.7). The presence of \(J_{-\nu}(-k\eta)\) in the superhorizon solution of the EM mode function makes \(\frac{dA(k,\eta)}{d(-k\eta)|_{\eta_f}}\) much larger than \(A(k,\eta_f)\). Consequently the Bogoliubov coefficients from eq. (8.2) can be expressed as,

\[
\alpha_k(\eta_f) = \frac{i}{\sqrt{2k}} A'(k, \eta_f), \quad \beta_k(\eta_f) = -\frac{i}{\sqrt{2k}} A'(k, \eta_f), \tag{8.3}
\]

which further leads to the relative phase factors of the Bogoliubov coefficients at \(\eta = \eta_f\) as \(\text{Arg}[\alpha_k(\eta_f) \beta_k^*(\eta_f)] \simeq \pi\). As a result, the ratio \(P_B \over P_E\) from eq. (8.1) takes the following form,

\[
P_B \over P_E = \left\{ \left( \frac{4k}{3\omega_{\text{eff}} + 1} \left( \frac{1}{aH} - \frac{1}{a_fH_f} \right) \right)^2 \right\}. \tag{8.4}
\]

Here it may be mentioned that we are interested in determining the magnetic strength around the CMB scales which, in fact, lies within the superhorizon regime in the reheating phase, in particular the mode \(k\) in the above expression satisfies \(k < aH\). Hence the above equation implies that the electric power spectrum is stronger compared to the magnetic power spectrum in the reheating epoch. Such a strong electric field can give rise to Schwinger production of charged particles, which in turn may backreact the magnetogenesis scenario \[23, 105–107\]. Moreover the Authors of \[23\] themselves acknowledged in their paper that it is not at all obvious that the conductivity is negligible during preheating; indeed it might be so for a certain time but eventually it may catch up with the enormous values we see in the thermalised
plasma. Therefore it is important to discuss the Schwinger backreaction in the present context by considering a non-zero electrical conductivity of the universe during the reheating phase.

In presence of non-zero conductivity, symbolized by $\sigma$, the Ampere-Maxwell equation of the EM field reads as,

$$
\epsilon_{ijl} \partial_j B_l = E_i' + \left( a\sigma + \frac{a'}{a} \right) E_i \tag{8.5}
$$

where $\epsilon_{ijl}$ is the 3-Levi Civita symbol. Thereby the condition under which the Schwinger backreaction can be neglected is given by,

$$
|a\sigma| < \frac{a'}{a} aH \quad \text{or} \quad |\sigma| < H. \tag{8.6}
$$

To progress further, we need a certain form of $\sigma$ in the FRW cosmological background where the Hubble parameter evolves as $H \propto a^{-\frac{3}{2}(1+\omega_{de})}$. A complete analysis of $\sigma$ in a generic FRW spacetime is beyond the scope of this paper. However the electrical conductivity in FRW background can be estimated from that of in the Minkowski background by replacing the elapsed time of the electric field in the Minkowski spacetime to $H^{-1}$ in FRW spacetime, as explained in [23]. This results the conductivity in FRW spacetime as,

$$
\sigma = \frac{1}{12\pi^3} \frac{|e^3E|}{H} \exp \left\{- \frac{\pi m^2}{|eE|} \right\} \tag{8.7}
$$

where $m$ and $e$ are the mass and the electrical charge of the produced charged particles respectively. Here we would like to mention that the above form of $\sigma$ is also valid for a de-Sitter background spacetime under the strong electric field approximation [105]. Eq. (8.7) can be solved for the electric field as,

$$
\frac{|E|}{H^2} = \frac{12\pi^3}{e^3} \left( \frac{\sigma}{H} \right) \exp \left\{ W \left( \frac{e^2 m^2 H}{12\pi^2 H^2 / \sigma} \right) \right\} \tag{8.8}
$$

where $W(x)$ is known as the Lambert W-function and corresponds to the solution of $We^W = x$. For low massive charged particles i.e for $m^2 \ll H^2$, the W-function can be approximated as $W(0 < x \ll 1) \approx x$ and thus the above equation can be expressed as,

$$
\frac{|E|}{H^2} = \frac{12\pi^3}{e^3} \left( \frac{\sigma}{H} \right). \tag{8.9}
$$

Thereby the above expression translates the condition of eq. (8.6) to an upper bound of the electric field during the reheating phase as,

$$
\frac{|E|}{H^2} < \frac{12\pi^3}{e^3} \approx 12\pi^3, \tag{8.10}
$$

where the electric field ($E$) during the reheating phase depends on the Bogoliubov coefficient $\beta_k(\eta_f)$, as indicated by eq. (7.19). The superhorizon solution of the EM mode function (see eq. (7.7)) immediately leads to the following expression of $\beta_k(\eta_f)$,

$$
|\beta_k(\eta_f)| = \left| \frac{2Beq}{(\eta_0^{2q}(\frac{2B}{\eta_0^{2q}} - 1))} \right| (-k\eta_f)^{-\nu - \frac{1}{2}} \tag{8.11}
$$
where we use $\nu^2 = \frac{1}{3} - \frac{4Bc_0}{q_0^2 (2B/6_q^2 - 1)}$ with $B = \kappa^2 \{ [6\beta(\beta + 1)]^q + [-24(\beta + 1)^3]^q/2 \}$ and, recall, $q$ is the model parameter which is generally taken as $q = 0.5$ during the determination of the present magnetic strength in section 7. Moreover, as mentioned earlier, the reheating phase is dominated by the constant EoS $\omega_{\text{eff}}$ and thus the Hubble parameter in eq. (8.10) follows the evolution as,

$$H^2 = H_f^2 \left( \frac{a}{a_f} \right)^{-3(1+\omega_{\text{eff}})}$$

(8.12)

with $H_f$ being the Hubble parameter at the end of inflation and given by $H_f = H_0 \exp \left[-\frac{cN_f}{1+2}\right]$. Using the above expressions of $\beta_k(\eta_f)$ and $H_f^2$, the inequality of eq. (8.10) turns out to be,

$$\frac{1}{12\pi^3} \frac{2Bc_0}{q_0^2 (2B/6_q^2 - 1)} (-k\eta_f)^{1+3\omega_{\text{eff}}} \left( \frac{a}{a_f} \right)^{1+3\omega_{\text{eff}}} < 1.$$  

(8.13)

As evident, the quantity in the left hand side of the above inequality increases with the scale factor during the reheating phase and thus it attains the maximum value at the end of reheating i.e at $a = a_{\text{re}}$. Thereby we define,

$$\mathcal{M} = \frac{1}{12\pi^3} \frac{2Bc_0}{q_0^2 (2B/6_q^2 - 1)} (-k\eta_f)^{3-\nu} \left( \frac{a_{\text{re}}}{a_f} \right)^{1+3\omega_{\text{eff}}}$$

(8.14)

which is the maximum value of the left hand side quantity of eq. (8.13). The factors $k\eta_f$ and $\frac{a_{\text{re}}}{a_f}$ present in the above expressions are connected with the e-folding number of inflation and reheating era respectively, in particular, they are given by: $-k\eta_f = e^{-N_f}$ and $\frac{a_{\text{re}}}{a_f} = e^{N_{\text{re}}}$, where $k$ is considered to be around the CMB scale and the $N_{\text{re}}$ (in terms of reheating EoS and $N_f$) is shown in eq. (7.1). Plugging back such forms of $k\eta_f$ and $\frac{a_{\text{re}}}{a_f}$ into eq. (8.14), we get

$$\mathcal{M} = \left( \frac{c_0(12q + 24q/2)}{6\pi^3} \right) \left( \frac{H_0}{M_{Pl}} \right)^{2q} \exp \left\{ - \left( \frac{3}{2} - \nu \right) N_f + (1 + 3\omega_{\text{eff}}) N_{\text{re}} \right\}$$

(8.15)

where we use the form of $B$ (as shown earlier after eq. (8.11)). With the expression of $N_{\text{re}}$ in eq. (7.1), we determine the argument within the exponential term as follows,

$$- \left( \frac{3}{2} - \nu \right) N_f + (1 + 3\omega_{\text{eff}}) N_{\text{re}} = -N_f \left[ \frac{3}{2} - \nu \right] + 4 \left( 1 + 3\omega_{\text{eff}} \right)$$

(8.16)

$$- \frac{4 (1 + 3\omega_{\text{eff}})}{(1 - 3\omega_{\text{eff}})} \left\{ - \frac{1}{4} \ln \left( \frac{45}{\pi^2 g_{\text{re}}} \right) + \frac{1}{3} \ln \left( \frac{11g_{s,\text{re}}}{43} \right) + \ln \left( \frac{k}{a_0 T_0} \right) + \ln \left( \frac{(3H_0^2 M_{Pl}^2)^{1/4}}{H_0} \right) \right\}.$$  

Thereby eqs. (8.15) and (8.16) clearly indicate that $\mathcal{M}$ depends on some inflationary parameters like $H_0/M_{Pl}$, $\epsilon, N_f$; the reheating EoS i.e $\omega_{\text{eff}}$; and the model parameter $q$. Recall, the parametric regime that we have considered in determining the current magnetic strength ($B_0$) in the previous section are given by: $H_0 = 10^{-5} M_{Pl}$, $\epsilon = 0.1$, $N_f = 58$ and $q = 0.5$ respectively, for which the Hubble parameter at the end of inflation comes as $H_f = H_0 \exp \left[ -\frac{cN_f}{1+2} \right] = 5.1 \times 10^{-8} M_{Pl}$. With this same parametric space and by using
Figure 3. $M$ vs $\omega_{\text{eff}}$ by the solid curve, corresponding to $k_{a0} = 0.02\text{Mpc}^{-1}$. Moreover we consider $g_{rc} = g_{s,rc} = 100$ and $T_0 = 2.3 \times 10^{-13}\text{GeV}$. The dashed horizontal curve denotes the constant value $= 1$.

Figure 3 clearly demonstrates that the quantity $M$ lies below unity for $\omega_{\text{eff}} \lesssim 0.299$, which in turn confirms the inequality of eq. (8.13) during the entire reheating epoch. Thereby we may argue that in the regime $\omega_{\text{eff}} \lesssim 0.299$, the Schwinger backreaction or equivalently the electrical conductivity in the reheating phase may be neglected and consequently the evolution equation of the electromagnetic field can be regarded as the standard Maxwell’s equation in vacuum. Here we need to recall from figure 2 that in order to make the magnetic field’s current strength ($B_0$) compatible with the CMB observations, the viable regime of $\omega_{\text{eff}}$ is found to be: $0.28 \lesssim \omega_{\text{eff}} \lesssim 0.2965$ which is, in fact, a sub-part of $\omega_{\text{eff}} \lesssim 0.299$. Thereby the regime of the reheating EoS that makes the model viable in regard to the CMB observations on $B_0$ as well as leads to a negligible Schwinger backreaction is given by $0.28 \lesssim \omega_{\text{eff}} \lesssim 0.2965$.

Before concluding, it deserves mentioning that the cosmological models characterized by vectors non-minimally coupled to curvature, in particular the action that contain an effective potential term for the vector field (i.e $V(A^2)$) may lead to ghost in the model. The presence of $V(A^2)$ in the action spoils the $U(1)$ invariance of the electromagnetic field, due to which the massive vector field gets three physical degrees of freedom: two of them are usual transverse modes and the other one is the longitudinal mode. The longitudinal mode with momentum $p^2 > |M^2|$ (where $M^2$ is the effective mass squared of the vector field) appears as a ghost field in the model, i.e a field with negative kinetic energy, whenever $M^2 < 0$ [115–118]. One of such example where the vector field has a negative $M^2$, is given by the following action [33]:

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{12} RA^2 \right], \quad (8.17)$$

where $V(A^2) = -\frac{1}{12} RA^2$ and thus the effective mass squared of the vector field comes as $M^2 = -R/6$ which is indeed negative.

However on contrary, in the present work, it is the kinetic term of the electromagnetic field that gets coupled with the spacetime curvature, in particular the EM Lagrangian is $L \sim f(R, \mathcal{G}) F_{\mu\nu}F^{\mu\nu}$ (see eq. (2.4)), and hence there is no potential term of the electromagnetic field in this work.
field appearing in the action. Thus the U(1) invariance is preserved in the present model, and consequently the problematic longitudinal mode is absent. Thereby we may argue that in the present model where the kinetic term of the EM field couples with the background Ricci scalar and the Gauss-Bonnet curvature, is free from ghost fields. However such kind of argument needs a study of full perturbation analysis, which is beyond the scope of this paper and thus expected to study in future.

9 Conclusion

We have constructed a viable inflationary magnetogenesis model where the electromagnetic field couples with the spacetime curvature, specifically, with the Ricci scalar and the Gauss-Bonnet invariant. The background spacetime is controlled by the well-studied, and mathematically well-grounded, scalar-Einstein-Gauss-Bonnet gravity theory, which is known to provide a good inflationary model for suitable choices of the Gauss-Bonnet coupling function and the scalar field potential. The model has some quite remarkable features. First, the non-minimal coupling between the electromagnetic field and the curvature breaks down the conformal invariance of the field and, therefore, it does not require any additional coupling to the scalar field. A second key feature is that, as the curvature is significant in the early universe, the conformal breaking term \( S_{CB} \) introduces a non-trivial correction to the electromagnetic action; however, at late times (in particular after the end of inflation), \( S_{CB} \) does not actually contribute, and the electromagnetic field behaves according to the standard Maxwell equations. Thirdly, the conformal breaking coupling is suppressed by \( \kappa^2 q \) (with \( q \) being the model parameter) and, thus, it does not lead to the strong coupling problem for \( q \sim O(1) \), which affects so many models, as this value also moves within a viable parametric regime, in regard to the compatibility of the magnetogenesis model with the most reliable observational data. As last strong point of our model, the electromagnetic field is found to have a negligible backreaction on the background spacetime, and thus the backreaction issue in the present magnetogenesis model is naturally resolved.

In such scenario, we have explored the evolution of the electric and magnetic fields with the expansion of the universe, starting from the inflationary era. During the cosmic evolution, the universe enters into a reheating phase after the inflation epoch, and depending on the reheating mechanism, we have considered two different cases: (i) as the first one, we assumed an instantaneous reheating where the universe makes a sudden jump from the inflationary epoch to a radiation dominated stage, during which the cosmic Hubble parameter goes as \( H \propto t^{-1} \), with \( t \) being the cosmic time; and (ii) a case where the universe experiences a reheating phase, with non-zero e-fold number; in particular, we have considered the conventional reheating mechanism proposed by Kamionkowski et al. [88], where the main idea is to parametrize the reheating phase by a constant effective equation of state parameter \( (\omega_{\text{eff}}) \).

We have shown above that, in the instantaneous reheating case, the conductivity becomes large immediately after inflation and, consequently, the electric field dies out quite rapidly. This, along with the fact that the electromagnetic field acquires conformal symmetry after inflation, leads to the evolution of the magnetic field energy density to go as \( a^{-4} \) during the post inflationary epoch. As a result, the theoretical predictions for the present amplitude of the magnetic field \( (B_0) \) was found to lie in fact far below the observational constraint given by \( 10^{-22} \text{G} \lesssim B_0 \lesssim 10^{-10} \text{G} \).

However, we have seen that the scenario becomes completely different when the reheating phase is considered to have a non-zero e-fold number. In this case, the conductivity
remains non-zero and, consequently, the strong electric field induces a magnetic field evolution during the epoch between the end of inflation and until the end of reheating. On the other hand, after the reheating phase, the universe becomes a good conductor and, hence, the electric field goes to zero. Specifically, the magnetic field energy density evolves as \((a^3H)^{-2}\) during the reheating era, and later, as \(a^{-4}\), as usual (this means, from the end of the reheating stage up to the present epoch). Such evolution clearly indicates that the presence of a reheating phase with non-vanishing e-fold number enhances the amplitude of the magnetic field’s current, as compared to ordinary instantaneous reheating, where the magnetic field energy density decays as \(a^{-4}\) from the very end of inflation.

As a consequence, the present strength of the magnetic field falls within the range dictated by the observational constraints, for a suitable regime of reheating parameters, thus overcoming the severe problems of the instantaneous reheating case. Moreover, the evolution of the magnetic field through the reheating phase allows to encode very valuable information about the reheating EoS parameter (\(\omega_{\text{eff}}\)) and about some inflationary parameters on the current magnetic field strength \((B_0)\). Therefore, probing \(B_0\) opens up, in turn, a window for probing the early stage of the universe, in particular, the reheating phase, through the current observational amplitude of the magnetic field. It has been shown in the paper that, in order to make compatible the present magnetogenesis model both with the CMB observations on \(B_0\) and with the BBN constraint on the reheating temperature, the viable regime of \(\omega_{\text{eff}}\) has to be the following: \(0.28 \lesssim \omega_{\text{eff}} \lesssim 0.2965\). This provides a viable constraint on the reheating EoS parameter from CMB observations.

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