The continuum limit of a 4-dimensional causal set scalar d’Alembertian

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Abstract
The continuum limit of a 4-dimensional, discrete d’Alembertian operator for scalar fields on causal sets is studied. The continuum limit of the mean of this operator in the Poisson point process in 4-dimensional Minkowski spacetime is shown to be the usual continuum scalar d’Alembertian $\Box$. It is shown that the mean is close to the limit when there exists a frame in which the scalar field is slowly varying on a scale set by the density of the Poisson process. The continuum limit of the mean of the causal set d’Alembertian in 4-dimensional curved spacetime is shown to equal $-\frac{1}{2} R$, where $R$ is the Ricci scalar, under certain conditions on the spacetime and the scalar field.

Keywords: causal sets, non-locality, curved spacetimes

(Some figures may appear in colour only in the online journal)

1. Introduction

The existence of a physical Planck scale cutoff is indicated from many different directions in physics, most notably and convincingly by the value of the black hole entropy [1]. Perhaps the simplest response to this is to postulate that spacetime is fundamentally discrete at the Planck scale. Causal sets are discrete spacetimes proposed as the histories in a sum-over-histories approach to quantum gravity which embody the breakdown of continuum spacetime at the Planck scale whilst preserving Lorentz symmetry [2]. Even if one believes that some other substance—strings or loops or something else—will turn out to be more relevant to the physics of quantum gravity at the Planck scale, causal sets can be useful as models of spacetime...
with no structure on scales smaller than the Planck scale which respect a physical symmetry that has been the basis for enormous progress in fundamental physics.

That discreteness can be compatible with Lorentz invariance is welcome news for workers guided by the unity of physics. However, there is a price: the discreteness and Lorentz invariance of causal sets together result in a radical nonlocality [2–4]. This nonlocality, were it incorrigible, could prevent causal sets from being useful phenomenologically and would threaten to derail the causal set programme for quantum gravity. Thus, evidence that the nonlocality of Lorentz invariant discrete structure can be tamed [5, 6] is important to the causal set programme. More recently further evidence was provided by the discovery of a quasi-local, discrete scalar d’Alembertian operator, $B^{(d)}$ for fields on causal sets well-approximated by 2-dimensional Minkowski spacetime [7, 8]. The mean of this operator tends to the exact continuum 2-dimensional flat scalar d’Alembertian in the continuum limit [9]. In [10] this work was extended to 4 dimensions with the introduction of an analogous operator $B^{(d)}$. There it was claimed that for both $d = 2$ and $d = 4$, when $B^{(d)}$ is applied to scalar fields on causal sets which are approximate by $d$-dimensional Lorentzian spacetimes, its mean tends in the continuum limit to the curved space operator, $\frac{1}{2}R$, where $\Box$ is the curved spacetime scalar d’Alembertian and $R$ is the Ricci scalar curvature. In this paper we will prove this result in four dimensions under certain conditions. Note that the equation of motion $B^0 \phi = 0$ for the field on the causal set results in a non-minimal coupling to gravity in the continuum limit in all dimensions dimensions [11–13].

Although these operators (and their generalisations to any dimension [11, 12, 14]) do indeed tame the radical nonlocality referred to above, they do not eliminate it altogether. This remnant of nonlocality is manifest in the dynamics of (scalar) fields on spacetime, which becomes nonlocal. Nonlocal dynamics of exactly this form was recently used in the construction of scalar nonlocal quantum field theories in [15] and [16], potentially leading to novel and interesting phenomenology [17, 18].

Recall that a causal set (or causet) is a locally finite partial order, $(\mathcal{C}, \preceq)$. Local finiteness is the condition that the cardinality of any order interval is finite, where the (inclusive) order interval between a pair of elements $x, y \in \mathcal{C}$ is defined to be $I(x, y) := \{ z \in \mathcal{C} \mid y \preceq z \preceq x \}$. We write $x \prec y$ when $x \preceq y$ and $x = y$. We call a relation $x \prec y$ a link if the order interval $I(x, y)$ contains only $x$ and $y$. We denote by $|\cdot|$ the cardinality of a set and $n(x, y) := |I(x, y)| - 2$.

Given a point $x \in \mathcal{C}$ we define the set of all its past nearest neighbours to be $L_0(x) := \{ y \in \mathcal{C} \mid y \prec x, n(x, y) = 0 \}$. (1)

We refer to this set of elements as the first past layer. We can generalise this by defining the sets of next nearest neighbours, $L_2$, next next nearest neighbours, $L_3$, and so on. In general the $i$-th past layer is defined as $L_i(x) := \{ y \in \mathcal{C} \mid y \prec x \text{ and } n(x, y) = i - 1 \}$. (2)

Consider the discrete retarded operator $B$, on a causet $\mathcal{C}$, defined as follows [10]. If $\phi : \mathcal{C} \to \mathbb{R}$ is a scalar field, then

$$B\phi(x) := \frac{4}{\sqrt{6}l^2}\left[ -\phi(x) + \left( \sum_{y \in L_0(x)} -9 \sum_{y \in L_2(x)} + 16 \sum_{y \in L_4(x)} -8 \sum_{y \in L_6(x)} \right) \phi(y) \right],$$

where $l$ is a length (the analogue of the lattice spacing). The form of the discrete operator $B$ as such a sum-over-layers is dictated by requiring the operator to be a difference operator analogous to the d’Alembertian on a lattice but in addition requiring it to be retarded and Lorentz invariant (see [7] for an explanation in the 2-dimensional case). Four is the minimum number

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of layers in four dimensions and the specific values of the coefficients are required to give the
correct local limit in Minkowski space. Introducing more layers is possible but is analogous
to, say, approximating a second derivative in one dimension by a sum over values of the func-
tion at more than three neighbouring points: nonuniqueness of the coefficients results [14].

$B$ is defined on scalar fields on any causal set but is physically relevant for causal sets that
are well-approximated by a four dimensional Lorentzian manifold, $(\mathcal{M}, g)$. A causet, $(\mathcal{C}, \preceq)$
is well approximated by $(\mathcal{M}, g)$ if there exists a faithful embedding of $\mathcal{C}$ into $\mathcal{M}$ in which
the causal order of the embedded elements respects the order of $\mathcal{C}$ and in which the number
of causet elements embedded in any sufficiently nice, large region of $\mathcal{M}$ approximates the
spacetime volume of that region in fundamental units. These manifold-like, faithfully embed-
dable causets are typical in the random process of sprinkling into $(\mathcal{M}, g)$: a Poisson process
of selecting points in $\mathcal{M}$ with density $\rho$ so that the expected number of points sprinkled in a
region of spacetime volume $V$ is $\rho V$. To do justice to our expectations for quantum gravity, the
density $\rho = l^{-4}$, where $l$ is the fundamental length scale of the order of the Planck length. The
probability for sprinkling $m$ elements into a region of volume $V$ is

$$P(m) = \frac{(\rho V)^m e^{-\rho V}}{m!},$$

This process generates a causet, $\mathcal{C}$ whose elements are the sprinkled points and whose order,
$\preceq$ is that induced by the manifold’s causal order restricted to the sprinkled points.

Let $\phi$ be a real test field of compact support on $\mathcal{M}$ and $x \in \mathcal{M}$. If we sprinkle $\mathcal{M}$ at density
$\rho$, include $x$ in the resulting causet, $\mathcal{C}$, then $L_1(x) \subset \mathcal{C}$ will be a set whose elements lie in the
causal past of $x$, $J^-(x)$, and hug the boundary of $J^-(x)$. Their locus is roughly the hyperboloid
which lies one Planck unit of proper time to the past of $x$. The elements of $L_2(x)$ will also be
distributed down the inside of the boundary of $J^-(x)$, just inside layer 1, and so on. The opera-
tor $B$ can be applied, at point $x$, to the field $\phi$ restricted to the sprinkled causet: $B\phi(x)$ looks
highly nonlocal, involving the value of $\phi$ at enormous numbers of points outside any fixed
neighbourhood of $x$.

The sprinkling process at density $\rho$ produces, for each point $x$ of $\mathcal{M}$, a random variable
whose value is $B\phi(x)$ on the realisation $\mathcal{C}$ of the process. The expectation value of this random
variable is given by the spacetime integral

$$\mathbb{E}(B\phi(x)) = \frac{4 \sqrt{\rho V}}{\sqrt{6}} \left[ -\phi(x) + \rho \int_{\gamma \in J^-(x)} d^4 y \sqrt{-g} \phi(y) e^{-\xi} (1 - 9 \xi + 8 \xi^2 - \frac{4}{3} \xi^3) \right],$$

where $\xi := \rho V(y)$ and $V(y)$ is the volume of the causal interval between $x$ and $y$.

We can see that the integrand is suppressed wherever $\xi$ is large, i.e. wherever the spa-
tetime volume of the causal interval between $x$ and $y$ is larger than a few Planck volumes. However, $\xi$ is small in the part of the region of integration close to the boundary of $J^-(x)$
and, by itself, the exponential factor in the integrand cannot provide the suppression required
to give a value that is approximately a local quantity at $x$. In the following we will show
that, for large enough $\rho$, $B\phi(x)$ is effectively local and is dominated by contributions from
a neighbourhood of $x$: the contributions from far down the boundary of $J^-(x)$ cancel out. Indeed, we will show

$$\lim_{\rho \to \infty} B\phi(x) = \Box \phi(x) - \frac{1}{2} R(x) \phi(x)$$
under certain assumptions about the support of $\phi$ in $(M^4, g)$. We use the conventions of Hawking and Ellis [19].

2. Minkowski spacetime

First we consider the simpler case of a sprinkling in Minkowski spacetime for which we will show that $\lim_{\rho \to \infty} \hat{B}\phi(x) = \Box \phi(x)$. Although this is a special case of the curved space result, it is useful to see this simpler proof first as it will provide the basic framework on which the curved spacetime calculation is built. We will also be able to estimate the corrections to the limiting value, something that will turn out to be harder in the curved case.

Choose $x$ as the origin of cartesian coordinates $\{y^\mu\}$ and in that frame define the usual spatial polar coordinates: $(r, \theta, \varphi)$. Null radial coordinates (past pointing) are defined by $u = \frac{1}{\sqrt{2}}(-t-r)$ and $v = \frac{1}{\sqrt{2}}(-t+r)$ where $t = y^0$. The volume, $V(y)$, of the causal interval between point $y$ with cartesian coordinates $\{y^\mu\}$ and the origin is $V(y) = \frac{\pi}{2} u^2 v^2$.

Let us take the region of integration $\mathcal{W}$ to be the portion of the causal past of the origin for which $v \leq L$, where $L$ is large enough that the support of $\phi$ is contained in $\mathcal{W}$. $\mathcal{W}$ can be split into 3 parts:

$$W_1 := \{ y \in \mathcal{W} | 0 \leq u \leq v \leq a \}$$

$$W_2 := \{ y \in \mathcal{W} | a \leq v \leq L, \ 0 \leq u \leq \frac{a^2}{v} \}$$

$$W_3 := \mathcal{W} \setminus (W_1 \cup W_2)$$

where $a > 0$ is chosen small enough that the expansions of $\phi$ used in the following calculation are valid. $W_1$ is a neighbourhood of the origin, $W_2$ is a neighbourhood of the past light cone and bounded away from the origin and $W_3$ is a subset of the interior of the causal past that is bounded away from the light cone, see figure 1.

Let

$$I_i = \int_{W_i} d^4 y \ \phi(y) e^{-\rho V(y)} (1 - 9 \rho V(y) + 8(\rho V(y))^2 - \frac{4}{3}(\rho V(y))^3),$$

for $i = 1, 2, 3$ so that

$$\hat{B}\phi(x) := \frac{4\sqrt{\rho}}{\sqrt{6}} [-\phi(x) + \rho(I_1 + I_2 + I_3)].$$

2.1. The ‘deep chronological past’, $W_3$

We first consider first $I_3$. $V(y)$ is bounded away from zero in $W_3$, indeed $V(y) \geq V_{\min} = \frac{\pi}{6} a^4$ so

$$|I_3| \leq e^{-\rho V_{\min}} \int_{W_1} d^4 y \ \left| \phi(y)(1 - 9 \rho V(y) + 8(\rho V(y))^2 - \frac{4}{3}(\rho V(y))^3) \right|$$

which tends to zero faster than any power of $\rho^{-1}$ as $\rho \to \infty$. In what follows, we will often write ‘up to exponentially small terms’, which means we are neglecting terms like $I_3$. 

2.2. Down the light cone, $W_2$

Consider now $I_2$. Note first that

\[ e^{-\xi}(1 - 9\xi + 8\xi^2 - \frac{4}{3}\xi^3) = \hat{\O} e^{-\xi}, \]  

where

\[ \hat{\O} := \frac{4}{3}(H + \frac{1}{2})(H + 1)(H + \frac{3}{2}) \]  

\[ = 1 + 9H_1 + 8H_2 + \frac{4}{3}H_3 \]  

and

\[ H_n := \partial_n^\rho, \partial_{\rho^i}^n \text{ and } H := H_c \]  

The integral we are evaluating can be rewritten as
\[ I_2 = \int d\Omega_2 \int_a^L dv \int_0^\frac{\pi}{2} du \frac{1}{2} (v-u)^2 \phi(y) e^{-au^2}, \]  

where \( \int d\Omega_2 = \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\varphi. \) We have absorbed a factor of \( \pi/6 \) into \( \sigma = \pi \rho/6, \) for convenience. Note that the form of \( \hat{\sigma} \) is unchanged by that manoeuvre:

\[ H = \rho \frac{\partial}{\partial \rho} = \sigma \frac{\partial}{\partial \sigma}. \]  

We will see that \( I_2 = O(\rho^{-1}) \) (equivalently \( O(\sigma^{-1}) \)) and so, when multiplied by \( \rho^{3/2} \), makes no contribution to the limit of \( B(x) \). This can be understood by noticing that \( \hat{\sigma} \) annihilates \( \rho^{-1/2}, \rho^{-1} \) and \( \rho^{-3/2} \). If the function of \( \rho \) on which \( \hat{\sigma} \) acts is well-behaved enough as \( \rho \to \infty \) to be equal to a power series expansion in \( \rho^{-1/2} \), then application of \( \hat{\sigma} \) will eliminate all the terms that would not—after multiplication by \( \rho^{3/2} \)—tend to zero. Another way to understand the result is to notice that the integral over \( W_2 \) involves an integral over the null coordinate \( u \), transverse to the light cone. If the range of the \( u \) integration is small enough—if \( W_2 \) is close enough to the light cone—then \( \phi \) will be approximately constant in \( u \) at fixed values of the other coordinates. The integration over \( u \) for constant \( \phi \) is

\[ \int_0^a \frac{\sigma^2}{u} du \left( 1 - 9 \sigma v^2 u^2 + 8 (\sigma v^2 u^2)^2 - \frac{4}{3} (\sigma v^2 u^2)^3 \right) (v-u)^2 e^{-\sigma v^2 u^2}. \]  

The value of this integral is exponentially suppressed by a factor of \( a^\exp(-\sigma a^2) \). This suggests that the leading, finite \( \rho \) corrections to the limit are set by the \( u \) derivatives of \( \phi \) at \( u = 0 \), and this turns out to be the case.

We assume that \( \sigma \) is chosen small enough and that \( \phi \) is differentiable enough that throughout \( W_2 \), \( \phi(y) \) can be expanded in the transverse coordinate \( u \):

\[ \phi(y) = \phi|_{u=0} + u \phi_u|_{u=0} + \frac{1}{2!} u^2 \phi_{uu}|_{u=0} + \frac{1}{3!} u^3 \Phi(y), \]  

where \( \Phi(y) \) is a continuous function. Let \( I_2 = I_{2,0} + I_{2,1} + I_{2,2} + I_{2,3} \) where

\[ I_{2,i} = \int d\Omega_2 \int_a^L dv F_i \int_0^\frac{\pi}{2} du \frac{1}{2} (v-u)^2 \left( 1 - 9 \sigma u^2 v^2 + 8 \sigma^2 u^2 v^4 - \frac{4}{3} \sigma^3 u^6 v^6 \right) e^{-au^2}, \]  

and \( F_i = \phi|_{u=0}, \phi_u|_{u=0} \) and \( \frac{1}{2} \phi_{uu}|_{u=0} \) for \( i = 0, 1 \) and 2 respectively, and

\[ I_{2,3} = \int d\Omega_2 \int_a^L dv \int_0^\frac{\pi}{2} du \frac{\Phi(y)}{3!} \frac{(v-u)^2}{2} \left( 1 - 9 \sigma u^2 v^2 + 8 \sigma^2 u^2 v^4 - \frac{4}{3} \sigma^3 u^6 v^6 \right) e^{-au^2}. \]  

For \( I_{2,i}, i = 0, 1, 2, \) the \( u \) and \( v \) integrations can be done explicitly and the remaining integral over the angular coordinates can be bounded by bounding \( F_i \) by its uniform norm over the light cone \( u = 0 \). We find that, up to exponentially small contributions, \( I_{2,0} \) vanishes and

\[ |I_{2,1}| \leq \frac{\pi \|\phi_u\|_{H^1}}{3\sigma^2} \left( \frac{1}{a^3} - \frac{1}{L^3} \right). \]
\[ |I_{2,3}| \leq \frac{\pi}{2\sigma^2} \left( \frac{1}{a^2} - \frac{1}{L^2} \right) + O\left( \frac{1}{\sigma^{5/2}} \right) \]  

(24)

where \( \| \cdot \|_{C} \) denotes the uniform norm over the light cone \( u = 0 \) in \( W_2 \).

Finally we must bound \( I_{2,3} \):

\[
\begin{align*}
|I_{2,3}| & \leq \frac{1}{3!} \| \Phi \|_2 \int d\Omega_2 \int_a^L \frac{a^2}{d\nu} \frac{(v-u)^2}{2} u^4 (1 + 9 \sigma u^2 \nu^2 + 8 \sigma^2 u^4 \nu^4 + \frac{4}{3} \sigma^3 \nu^3 \psi_{\nu}(\Phi) e^{-\sigma u^2 \nu^2}) \\
& = \frac{4\pi}{12} \| \Phi \|_2 (1 - 9H_1 + 8H_2 - \frac{4}{3}H_3) \int_a^L \frac{a^2}{d\nu} \frac{(v-u)^2}{2} u^4 e^{-\sigma u^2 \nu^2}. 
\end{align*}
\]

(25)

where \( \| \cdot \|_2 \) is the uniform norm of \( \Phi \) in \( W_2 \) and, is less than or equal to the uniform norm of the third \( u \)-derivative of \( \phi \) in \( W_2 \). We find

\[
|I_{2,3}| \leq \frac{33\pi}{2\sigma^2} \left( \frac{1}{a^2} - \frac{1}{L^2} \right) + O\left( \frac{1}{\sigma^{5/2}} \right) 
\]

(26)

where \( \| \cdot \|_2 \) is the uniform norm in \( W_2 \). The key here is that \( \phi \) has been expanded in an \( u \) far enough that the power of \( u \) in the factor \( u^3 \) in (22) is high enough for the \( u \) integration to bring down enough powers of \( \sigma^{-1} \). We will see the same thing happening in the integral over region \( W_1 \) and again in the curved space case.

Multiplying \( I_2 \) by \( \rho^{1/2} \), we see that the contribution to \( \hat{B}_x \phi(x) \) from the region \( W_2 \) tends to zero in the limit and the leading corrections go like \( \rho^{-1/2} \) and are proportional to the \( u \)-derivatives of \( \phi \) on and close to the light cone.

### 2.3. The near region, \( W_1 \)

Now consider

\[
I_1 = \hat{O} \int_{W_1} d^4y \, \phi(y) e^{-\rho \psi(y)} 
\]

(27)

\[
= \hat{O} \int_0^a d\nu \int_0^\nu d\nu' \int d\Omega_2 \int d\nu \int d\nu' \left( \frac{v-u)^2}{2} \phi(y) e^{-\rho \psi(y)}. 
\]

(28)

We assume we can expand the field in \( W_1 \),

\[
\phi(y) = \phi(0) + y^\mu \phi_\mu(0) + \frac{1}{2} y^\mu y^\nu \phi_{\mu\nu}(0) + y^\mu y^\nu y^\alpha \psi_{\mu\nu\alpha}(y), 
\]

(29)

where \( \psi_{\mu\nu\alpha}(y) \) are \( C^1 \)-functions of the \( y^\mu \) (they are not components of a tensor, the indices just label the functions). The first two terms of the above expansion of \( \phi \) contribute to \( I_1 \)

\[
\hat{O} \int_0^a d\nu \int_0^\nu d\nu' \int d\Omega_2 \int d\nu \int d\nu' \frac{(v-u)^2}{2} \phi(0) e^{-\rho \psi(y)} 
\]

(30)

\[
= \frac{1}{\rho} (1 - e^{-\rho \nu^2}) \phi(0), 
\]

(31)

and

\[
\hat{O} \int_0^a d\nu \int_0^\nu d\nu' \int d\Omega_2 \int d\nu \int d\nu' \frac{(v-u)^2}{2} y^\mu \phi_\mu(0) e^{-\rho \psi(y)} 
\]

(32)
\[
- \frac{6\sqrt{2}}{a^2\pi\rho^2} + \left( \frac{6\sqrt{2}a}{a^2\pi\rho^2} \right) e^{-\frac{\pi}{6}\sigma^2} \phi_{\sigma}(0).
\]

The first term (31) cancels with the term \( \phi(x) \) in the expression for \( \tilde{B}(x) \) (5), while the second contributes nothing in the limit \( \rho \to \infty \). The leading correction at finite \( \rho \) behaves as

\[
\frac{l^2}{a^2} \phi_{\sigma}(0).
\]

The third term of the expansion of \( \phi \) (29) is of most interest to us: it contributes to \( \tilde{B}(x) \)

\[
\frac{4}{\sqrt{6}} \rho \pi \hat{\mathcal{O}} \int_0^a dv \int_0^v du \int d\Omega_2 \frac{1}{2}(v-u)^2 y^\nu y^\mu \phi_{\mu\nu}(0)e^{-\frac{\pi}{6}\sigma^2}
\]

\[
= \Box \phi(0) - \frac{4\sqrt{6}}{a^2\pi\sqrt{\rho}} \phi_{\sigma}(0) + \frac{9}{a^2\pi\rho} \left( \phi_{\sigma}(0) + 3\phi_{\sigma}(0) \right)
\]

up to exponentially small terms (there is a sum on \( i \) implied in \( \phi_{i\sigma} \)). The leading correction at finite \( \rho \) is, up to a factor of order one,

\[
\frac{l^2}{a^2} \phi_{\sigma}(0).
\]

Finally we need to show that the integral

\[
\hat{\mathcal{O}} \int_0^a dv \int_0^v du \int d\Omega_2 \frac{(v-u)^2}{2} y^\mu y^\nu \phi_{\mu\nu}(y)e^{-\frac{\pi}{6}\sigma^2}
\]

does not contribute in the limit, where \( y^\nu = (\tau, r \cos \theta, r \sin \theta \cos \varphi, r \sin \theta \sin \varphi) \). At the end of the calculation, the uniform norm of the integrand over the region of integration will be used to bound the integral, and the angular dependent factors of \( \cos \theta \) etc will make no difference to the result and we can drop them now, for convenience. We therefore need to show that each integral of the form

\[
\hat{\mathcal{O}} \int d\Omega_2 \int_0^a dv \int_0^v du (v-u)^2 u^\mu v^\nu \psi e^{-\frac{\pi}{6}\sigma^2}, \quad m + n = 3,
\]

tends to zero faster than \( \rho^{-3/2} \), where \( \psi \) stands for one of the \( \phi_{\mu\nu}(y) \) and is a function of \( u, v, \theta \) and \( \phi \), and, again, for convenience we have defined \( \sigma = \pi\rho/6 \).

Leaving the integration over the angles for later, consider

\[
K_{m,n} := \hat{\mathcal{O}} \int_0^a dv \int_0^v du (v-u)^2 u^\mu v^\nu \psi e^{-\frac{\pi}{6}\sigma^2}, \quad m + n = 3.
\]

Note first that

\[
e^{-\sigma^2} = \sqrt{\pi} \frac{\partial}{\partial u} \text{erf}(\sqrt{\sigma} uv)
\]

Using this identity and integrating (40) by parts in \( u \) we find

\[
K_{m,n} = -\hat{\mathcal{O}} \int_0^v du \int_0^v dv \frac{\partial}{\partial u} ((v-u)^2 u^\mu v^\nu) \sqrt{\pi} \frac{\text{erf}(\sqrt{\sigma} uv)}{\sqrt{\sigma}},
\]

since the boundary terms vanish. The following identity
\[ \hat{\mathcal{O}} \left( \frac{\operatorname{erf}(\sqrt{\sigma} u)}{\sqrt{\sigma}} \right) = \frac{2}{\sqrt{\pi}} u v \hat{P} e^{-\sigma u^2}, \]  

(43)

where \( \hat{P} = \frac{2}{\sqrt{3}} (H + 1)(H + \frac{3}{2}) \) allows one to rewrite the integral as

\[ K_{m,n} = -\hat{P} \int_0^a dv \int_0^v du \frac{\partial}{\partial u} ((v - u)^2 u^m v^n \psi) u e^{-\sigma u^2}. \]  

(44)

Integrating by parts again in \( u \) we find

\[ -\hat{P} \int_0^a dv \int_0^v du \frac{\partial^2}{\partial u^2} ((v - u)^2 u^m v^n \psi) e^{-\sigma u^2}. \]  

(45)

Using (41) and integrating by parts in \( u \) again we find

\[ K_{m,n} = \hat{P} \int_0^a dv \int_0^v du \frac{\partial^3}{\partial u^3} ((v - u)^2 u^m v^n \psi) \frac{\operatorname{erf}(\sqrt{\sigma} u v)}{\sigma^{3/2}}. \]  

(46)

where \( \tilde{\psi} := \tilde{\psi}(v, \theta, \phi) = \psi_{u=v} \). Using the following identity

\[ -\hat{P} \left( \frac{\operatorname{erf}(\sqrt{\sigma} z)}{\sigma^{3/2}} \right) = -\frac{2}{3\sqrt{\pi}} z^3 e^{-\sigma z^2}, \]  

(47)

gives

\[ K_{m,n} = -\frac{1}{6} \int_0^a dv \int_0^v du u^3 \frac{\partial^3}{\partial u^3} ((v - u)^2 u^m v^n \psi) e^{-\sigma u^2} + \frac{1}{3} \int_0^a dv \tilde{\psi} b^6 u^m v^n e^{-\sigma v^4}. \]  

(48)

Including now the integration over angles, the contribution of the second term of (48) to \( I_1 \), is bounded by

\[ \int d\Omega_2 \left| \frac{1}{3} \int_0^a dv \tilde{\psi} b^6 u^m v^n e^{-\sigma v^4} \right| \leq 4\pi \frac{\| \tilde{\psi} \|}{\sigma^{3/2}} \frac{\Gamma(5/2)}{12} \]  

(49)

up to exponentially small terms, where \( \| \cdot \| \) is the uniform norm over region \( W_1 \), we have used that \( m + n = 3 \), and we recall that \( \sigma = \pi \rho/6 \). When multiplied by \( \rho^{3/2} \) this term therefore gives a correction of \( O(\rho^{-1}) \).

Consider now the first integral in (48), and denote a term in the expansion of \((v - u)^2 u^m v^n\) by \( u^i v^j \), \( i + j = 5 \). Then

\[ u^3 \frac{\partial^3}{\partial u^3} (u^i v^j) = i(i - 1)(i - 2) u^{i-1} v^j + 3i(i - 1) u^i v^{j-1} + 3i u^i v^{j-2} \psi' + 3u^i v^{j-3} \psi'' + u^i v^{j-4} \psi''' \]  

(50)

where \( \psi' \) denotes differentiation with respect to \( u \). We can write any such term as \( u^{i+k} v^j \psi^{(k)} \) where \( \psi^{(n)} \ = \frac{\partial^n}{\partial u^n} \psi, i + k \geq 3 \) and \( k = 0, 1, 2, 3 \).

Then, the contribution of each of these terms to \( I_1 \) is bounded—up to exponentially small terms—by \( 4\pi \) (from the integration over the angles) times

\[ \frac{\| \psi^{(k)} \|}{2(i + k - j)} \left( \frac{1}{\sigma^{i+k+2}} \Gamma \left( \frac{i + j + k + 2}{2} \right) \right) \left( \frac{a^{i+j+k-2}}{\sigma^{i+k+1}} \Gamma \left( \frac{i + k + 1}{2} \right) \right) \]  

(51)
for $i + k = j$, and
\[
\frac{\|\psi^{(k)}\|_1}{8\sigma^{|i+k+1|}} \left( \frac{i + k + 1}{2} \right) \left( \log(\sigma a^k) - \psi \left( \frac{i + k + 1}{2} \right) \right)
\]
(52)
for $i + k = j$ where $\Psi(z) = d\ln \Gamma(z)/dz$ is the Euler $\Psi$-function.

All these terms tend to zero in the limit and the leading order, finite $\rho$ correction occurs when $i + j = 5, k = 0$ and (after being multiplied by $\rho^{3/2}$) is $O(\rho^{-1/2})$.

All contributions to $B\phi(x)$ have now been accounted for and we see that its limit is $\Box \phi(x)$ as $\rho \to \infty$.

2.4. Finite $\rho$ corrections

The correct continuum value for the limit of the mean is a good sign. However, for causal sets the important question is how $B\phi(x)$ behaves when the discreteness length $l$ is of order the Planck length so that $l^4 \rho = 1$ is large but finite.

The above calculations show that $B\phi$ is a good approximation to $\Box \phi$ at finite $\rho$ whenever there exists a coordinate frame and a length scale $a$ such that $\rho a^4 \gg 1$, i.e. $l \ll a$—so that the ‘exponentially small terms’ referred to in the calculations are indeed small—and such that the following conditions on the derivatives of $\phi$ in that frame hold.

The leading order corrections from $W_1$, the neighbourhood around the origin of size $a$, give conditions
\[
\frac{l^2}{a^4} \phi_u(0), \frac{l^2}{a^2} \phi_{uu}(0) \ll \Box \phi(0), \quad \text{and}
\]
\[
l \|\psi\|_1, l^2 a^2 \|\psi^{(2)}\|_1, l^2 a^2 \|\psi^{(3)}\|_1, l^2 \log \left( \frac{a}{l} \right) \|\psi^{(1)} \|_1, l^4 \log \left( \frac{a}{l} \right) \|\psi^{(3)}\|_1 \ll \Box \phi(0),
\]
(54)
where $\psi^{(k)}$ denotes the $k$-th derivative of $\psi$ with respect to $u$. Recall that $\psi$ stands for a third derivative of the field $\phi$ with respect to RNC and so a term like $\psi^{(3)}$ is a sixth derivative of the field in RNC in the neighbourhood of the origin. We see that, if $l \ll a$ and $a < \lambda$, where $\lambda$ is the characteristic scale on which $\phi$ varies, then these conditions hold.

The leading order corrections from $W_2$, close to the light cone, give conditions
\[
\frac{l^2}{a^4} \|\phi_u\|_{L^c}, \frac{l^2}{a^2} \|\phi_{uu}\|_{L^c}, \frac{l^2}{a} \|\phi_{uuu}\|_2 \ll \Box \phi(0).
\]
(55)
Note that these conditions apply to derivatives of the field $\phi$ with respect to $u$ on the light cone and in its neighbourhood $W_2$. If $l \ll a$ and $a < \lambda$, then, these conditions will be satisfied.

The conclusion is that if a frame and a scale $a \gg l$ exist for which $\phi$ is slowly varying on the scale of $a$ in a neighbourhood of $x$ and is slowly varying on that scale transverse to the past light cone of $x$, in a neighbourhood of the light cone, then $B\phi$ is a good approximation to $\Box \phi$.

That there is a global frame in which these conditions hold in a neighbourhood of the whole past light cone is a strong condition. It is possible that it can be weakened. For example, if a neighbourhood of the past light cone can be covered by patches in each of which the field varies slowly in a null direction transverse to the light cone, it might be possible to show that the contribution from each patch vanishes in the limit and thus to prove a more powerful result.
3. Curved spacetime

We assume again that the field $\phi$ is of compact support and we will again split the region of integration $J'(x) \cap \text{Support}(\phi)$ into three parts: the deep chronological past, $W_1$, bounded away from $\partial J'(x)$; $W_2$, a neighbourhood of $\partial J'(x)$ bounded away from $x$; and the near region, $W_3$, a neighbourhood of $x$. We assume certain differentiability and other conditions on $\phi$ and the metric which will be stated as they are used during the calculation.

Let $N$ be a Riemann normal neighbourhood of $x$ with Riemann Normal Coordinates (RNC) $\{ y^\mu \}$ centred on the origin $x$ and, as before, we define spatial polar coordinates: $r = \sqrt{\sum_{i=1}^{3} (y^i)^2}$, $\theta$ and $\varphi$. We also again define radial null coordinates $u = \frac{1}{\sqrt{2}} (-y^0 - r)$ and $v = \frac{1}{\sqrt{2}} (-y^0 + r)$ in $N$ where $u$ and $v$ increase into the past.

We define $LC := \partial J'(x) \cap \text{Support}(\phi)$ and assume that every point of $LC$ lies on a unique past directed null geodesic from $x$. This is a strong condition: generally there will be caustics on $LC$. Each null geodesic generator of $LC$, $\gamma(\theta, \varphi)$, is labelled by the polar angles $\theta$ and $\varphi$ and has tangent vector, $T(\theta, \varphi)$, at $x$ with components in RNC: $T(\theta, \varphi)^\mu = \frac{1}{\sqrt{2}} (-1, \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$. The past pointing null tangent vectors at $x$ come in antipodal pairs, $(T(\theta, \varphi), T(\pi - \theta, \varphi + \pi))$, such that $T(\theta, \varphi)^\mu T(\pi - \theta, \varphi + \pi)_\mu = -1$. From this we define Null Gaussian Normal Coordinates (NGNC) $\{ V, U, \theta, \varphi \}$ [20] in a neighbourhood, $N_{LC}$, of $\partial J'(x)$ which contains $LC$ and is bounded away from $x$. The coordinates $\theta$ and $\varphi$ are the labels of the null geodesic generators of $\partial J'(x)$, $\gamma(\theta, \varphi)$. The coordinate $V$ is the affine parameter along each $\gamma(\theta, \varphi)$ and is equal to $v$ (the RNC) along the generators in the overlap of the RNC and NGNC patches. The transverse null coordinate $U$ is the affine parameter along past pointing, ingoing null geodesics from every point on $LC$ such that the tangent vector to the ingoing null geodesic at point $p$ on $\gamma(\theta, \varphi)$ is the vector $T(\pi - \theta, \varphi + \pi)$ at $x$, parallally transported to $p$ along $\gamma(\theta, \varphi)$.

In calculating the integral (5), we will need to know the behaviour of the function $V(y)$, which is the volume of the causal interval between $x$ and $y$. For $y$ in the near region close to $x$, we can use the results of Myrheim and Gibbons and Solodukhin [21, 22] to expand $V(y)$ in RNC. For the region down the light cone, we show in appendix that for $y$ in $N_{LC}$ with NGNC $\{ V, U, \theta, \varphi \}$ the limit of $U^{-2}V(y)$ as $U \rightarrow 0$ is finite and we denote $\lim_{U \rightarrow 0} U^{-2}V(y) = f_0(V, \theta, \varphi)$. Indeed, if the causal interval between $x$ and $y$ is contained in a tubular neighbourhood of null geodesic $\gamma(\theta, \varphi)$ on which there are Null Fermi Normal Coordinates (NFNC) [23] then $V(y) = U^2 f_0(V, \theta, \varphi) + U^2 G(V, U, \theta, \varphi)$, where $G$ is a continuous function. Furthermore, $f_0$ is an increasing function of $V$ and so therefore is $V(y)$, for small enough $U$.

Using this information we now define the regions $W_i$, $i = 1, 2, 3$.

Let the near region, $W_1$, be a subregion of $N$:

$$W_1 := \{ y \in N \mid 0 \leq u \leq v \leq a \} \quad (56)$$

for some $a > 0$ such that $W_1$ is approximately flat, i.e. the metric in RNC everywhere in $W_1$ is close to the Minkowski metric $\eta_{\mu\nu}$ in inertial coordinates.

The down-the-light-cone region, $W_2$, is defined by

$$W_2 := \{ y \in N_{LC} \mid 0 < a'(\theta, \varphi) \leq V \leq L, \quad 0 \leq U \leq \frac{b^2}{\sqrt{f_0(V, \theta, \varphi)}} \} \quad (57)$$

$^4$To avoid confusion between $V(y)$ and NGNC coordinate $V$ we always write the volume function with its argument $y$. 


where the cutoff $L$ is large enough that $W_2$ includes the whole of $LC$ outside $W_1$. The topology of $W_2$ is $I \times I \times S^2$, where $I$ is the unit interval. $b > 0$ is assumed to be small enough that the entire causal interval between the origin $x$ and any point with NGNC $(V, U, \theta, \varphi) \in W_2$ lies in a tubular neighbourhood of null geodesic, $\gamma(\theta, \varphi)$, on which Null Fermi Normal Coordinates (NFNC) exist. It is also assumed that $b$ is small enough that the correction to $V(y)$ for $y \in W_2$ is small compared to the leading contribution, i.e. $U^G(V, U, \theta, \varphi) \ll U_0(V, \theta, \varphi)$ in $W_2$. When the spacetime is flat, $u = U$ and $v = V$ on the intersection of $N$ and $N_{1C}$ and taking $b = a' = a$ we recover the regions defined in the previous section for Minkowski space. When there is curvature, $u \neq U$ and $v \neq V$ on the intersection of $N$ and $N_{1C}$ and there will be a mis-alignment between the boundaries of $W_1$ and $W_2$ for any choice of $a'$. However, if the normal neighbourhood $N$ is approximately flat, then $u \approx U$ and $v \approx V$. The mismatch can be made as small as we like by taking $a$ to zero as the density $\rho$ increases. We will keep $a' = a$, whilst bearing in mind that they will be almost equal. This is sketched in figure 2.

The deep chronological past, $W_3$, is

$$W_3 := \left(\text{Supp}(\phi) \cap J^-(x)\right) \setminus (W_1 \cup W_2)$$

and is bounded away from $LC$.

As before we define

$$I_i = \hat{O} \int_{W_i} d^4 y \sqrt{-g(y)} \phi(y)e^{-\rho V(y)},$$

for $i = 1, 2, 3$ so that

$$\bar{B}\phi(x) := \frac{4\sqrt{\bar{\rho}}}{\sqrt{6}} [-\phi(x) + \rho(I_1 + I_2 + I_3)].$$
3.1. Deep chronological past

Consider first

\[ I_3 = \int_{W_3} d^4y \sqrt{-g} \phi(y) e^{-\rho V(y)} \times (1 - 9\rho V(y) + 8(\rho V(y))^2 - \frac{4}{3}(\rho V(y))^3). \]  

(61)

\( V(y) \) is only zero on \( LC \) and since \( W_3 \) is bounded away from \( LC \), \( V(y) \) is bounded away from zero on \( W_3 \): \( V(y) \geq V_{\text{min}} > 0 \). So

\[ \left| \int_{W_3} d^4y \sqrt{-g} \phi(y) e^{-\rho V(y)}(1 - 9\rho V(y) + 8(\rho V(y))^2 - \frac{4}{3}(\rho V(y))^3) \right| \leq e^{-\rho V_{\text{min}}} \int_{W_3} d^4y \sqrt{-g} \phi(y)(1 - 9\rho V(y) + 8(\rho V(y))^2 - \frac{4}{3}(\rho V(y))^3) \]  

(62)

which tends to zero faster than any power of \( \rho^{-1} \) as \( \rho \to \infty \).

The conditions on \( W_1 \) and \( W_2 \) given above mean that, on \( W_3 \), \( V(y) \) attains its minimum on the boundary between \( W_3 \) and \( W_1 \cap W_2 \), and its approximate value is \( V_{\text{min}} \approx \pi a^2 b^2 \).

3.2. Down the light cone

We work in NGNC, \( \{ V, U, \theta, \varphi \} \), in this region.

We showed in appendix that in curved space the limit of \( U^{-2}V(y) \) as \( U \to 0 \) is finite and we now assume enough differentiability of the metric so that \( V(y) \) has the following expansion in \( W_2 \):

\[ V(y) = U^2 f_0(V, \theta, \varphi) + U^3 f_1(V, \theta, \varphi) + U^4 f_2(V, \theta, \varphi) + U^5 F(y). \]  

(63)

We further assume enough differentiability of the metric and field that \( \sqrt{-g(y)} \) and \( \phi \) have the following expansions in \( W_2 \):

\[ \sqrt{-g(y)} = h_0(V, \theta, \varphi) + Uh_1(V, \theta, \varphi) + U^2 h_2(V, \theta, \varphi) + U^3 H(y) \]  

(64)

\[ \phi(y) = \phi \big|_{U=0} + U \phi \big|_{U=0} \big|_{U=0} + \frac{1}{2} U^2 \phi \big|_{U=0} \big|_{U=0} + U^3 \Phi(y). \]  

(65)

The functions \( F, H \) and \( \Phi \) are continuous and we have adopted a notation convention that a function denoted by a lower case letter is independent of \( U \) and a function denoted by an upper case letter may depend on \( U \).

We will also use this expansion of the exponential factor in the integrand:

\[ e^{-\rho V(y)} = e^{-\rho U^2 F} \left( 1 - \rho U^2 (f_1 + U^2 f_2 + U^3 F) + \frac{1}{2} \rho^2 U^4 (f_1 + U^2 f_2 + U^3 F)^2 \right) \]

\[ + \sum_{k=3}^{\infty} \frac{(-\rho)^k}{k!} U^{3k} (f_1 + U^2 f_2 + U^3 F)^k. \]  

(66)

We want to calculate

\[ I_2 = \hat{\mathcal{O}} \int d^{4} \Omega_2 \int_{a}^{b} dV \int_{0}^{\sqrt{\frac{b^2}{\kappa^2 (V, \theta, \varphi)}}} dU \sqrt{-g(y)} \phi(y) e^{-\rho V(y)}. \]  

(67)
Substituting the expansions (63)–(66) into (67) one finds three types of integrals. Integrals of the first kind, denoted by $I_{21}$, involve only $U$-independent unknown functions and do not have a factor of the infinite sum. A general such term can be written as

$$I_{21} := \hat{O} \left\{ \rho^q \int d\Omega_2 \int_{\tau}^{\tau'} dV \int_0^{\beta_0} \frac{b^2}{\sqrt{\delta(V,\theta,\varphi)}} dU \, U^{n+3q} e^{-\rho t_0^2} \right\},$$

(68)

where $\hat{O}$ denotes one (or a product) of the unknown functions independent of $U$, $q = 0, 1, 2$ and $0 \leq n \leq 4 + q$. These terms can be dealt with straightforwardly since the $U$-integration can be done explicitly:

$$I_{21} = \hat{O} \left\{ \frac{1}{2^{n+q} \rho} \int d\Omega_2 \int_{\tau}^{\tau'} dV \int_0^{\beta_0} \frac{\gamma(U,\theta,\varphi)}{f_0(V,\theta,\varphi)} \left( \Gamma \left( \frac{n+3q+1}{2} \right) - \Gamma \left( \frac{n+3q+1}{2}, \rho t_0^2 \right) \right) \right\}.$$  

(69)

The second term is exponentially small. The first term is annihilated by $\hat{O}$ for $n + q = 0, 1, 2$, and (after being multiplied by $\rho^{3/2}$) contributes a term that goes to zero in the limit for $n + q > 2$.

Integrals of the second kind, denoted by $I_{22}$, involve $U$-dependent unknown functions—recall, these are denoted by capital letters—and do not have a factor of the infinite sum in the integrand. Each can be written generically as

$$I_{22} := \hat{O} \left\{ \rho^q \int d\Omega_2 \int_{\tau}^{\tau'} dV \int_0^{\beta_0} \frac{b^2}{\sqrt{\delta(V,\theta,\varphi)}} dU \, U^{n+3q} \Xi(U, V, \theta, \varphi) e^{-\rho t_0^2} \right\},$$

(70)

where $n \geq 3$ when $q = 0$, and $n \geq 2$ when $q = 1, 2$. Acting with $\hat{O}$ on $\rho^q e^{-\rho t_0^2}$ we find

$$I_{22} = \hat{O} \left\{ \frac{\rho^q}{3} \int d\Omega_2 \int_{\tau}^{\tau'} dV \int_0^{\beta_0} \sqrt{\delta(V,\theta,\varphi)} dU \, U^{n+3q} \Xi(U, V, \theta, \varphi) e^{-\rho t_0^2} \right\} \times \left( 3 + 11q + 12q^2 + 4q^3 - 3 \rho t_0^2 (3 + 2q)^2 + 12 \rho^2 U^4 f_0^2 (2 + q) - 4 \rho^3 U t_0^3 \right).$$

(71)

each term of which can be bounded. We show one example here:

$$\left| \rho^q \int d\Omega_2 \int_{\tau}^{\tau'} dV \int_0^{\beta_0} \sqrt{\delta(V,\theta,\varphi)} dU \, U^{n+3q} \Xi(U, V, \theta, \varphi) e^{-\rho t_0^2} \right| \leq \rho^q \int d\Omega_2 \int_{\tau}^{\tau'} dV \|\Xi\|_{U}(V, \theta, \varphi) \int_0^{\beta_0} \sqrt{\delta(V,\theta,\varphi)} dU \, U^{n+3q} e^{-\rho t_0^2} \]

$$= \frac{1}{2^{n+q+1}} \int d\Omega_2 \int_{\tau}^{\tau'} dV \|\Xi\|_{U}(V, \theta, \varphi) \int_0^{\beta_0} \sqrt{\delta(V,\theta,\varphi)} dU \, U^{n+3q} e^{-\rho t_0^2} \]

$$= \frac{1}{2^{n+q+1}} \int d\Omega_2 \int_{\tau}^{\tau'} dV \|\Xi\|_{U}(V, \theta, \varphi) \int_0^{\beta_0} \sqrt{\delta(V,\theta,\varphi)} dU \, U^{n+3q} e^{-\rho t_0^2} \]

where $\|\Xi\|_{U}(V, \theta, \varphi)$ is the uniform norm of $\Xi$ over the $U$ coordinate. After being multiplied by $\rho^{3/2}$, this goes to zero in the limit, since $n + 3q \geq 3$. The other terms are similar.
Finally, the remaining term in $I_2$ is

$$I_{23} := \hat{Q} \int d\Omega_2 \int_{\nu}^{L} dV \int_0^{\frac{b^2}{\sqrt{6}\ell_0}} dU \sqrt{-g(y)} \phi(y) e^{-\rho \ell_0} \sum_{k=3}^{\infty} \frac{(-\rho)^k}{k!} U_{2k} G(y)^k,$$

(73)

where $G(y) = (f_1 + U_{2} + U^2 F)$.

We will see that each term in $I_{23}$ arising from the action of $\hat{Q} = (1 + 9H_1 + 8H_2 + \frac{4}{3}H_3)$ on the integrand, is $O(\rho^{-2})$. First we note that

$$\left| \sum_{k=3}^{\infty} \frac{(-\rho)^k}{k!} U_{2k} G^k \right| \leq \frac{\rho^3}{6} U^3 |G|^3 e^{\rho |G|},$$

(74)

where $G := f_1 + U_{2} + U^2 F$. Recall that in defining $W_2$, $b$ was chosen small enough that $U^3 |G| \ll U^2 b_0$ in $W_2$, so we have

$$\left| \int d\Omega_2 \int_{\nu}^{L} dV \int_0^{\frac{b^2}{\sqrt{6}\ell_0}} dU \sqrt{-g(y)} \phi(y) e^{-\rho \ell_0} \sum_{k=3}^{\infty} \frac{(-\rho)^k}{k!} U_{2k} G^k \right|$$

$$\leq \frac{\rho^3}{6} \int d\Omega_2 \int_{\nu}^{L} dV \int_0^{\frac{b^2}{\sqrt{6}\ell_0}} dU \sqrt{-g(y)} \phi(y) G^3 [U^3 e^{\rho |G|}]$$

$$\leq \frac{\rho^3}{6} \rho^3 \int d\Omega_2 \int_{\nu}^{L} dV \sqrt{-g} \phi G^3 \||U^3 e^{\rho |G|}$$

$$= \frac{2\rho^6}{\rho^2} \int d\Omega_2 \int_{\nu}^{L} dV \sqrt{-g} \phi G^3 \||U$$

(75)

neglecting terms proportional to $e^{-\rho b^{3/2}}$. Exchanging the order of summation and integration is justified as the range of integration is finite and the partial sums of the series are uniformly integrable.

After being multiplied by $\rho^{3/2}$ this term is of order $\rho^{-1/2}$ and hence goes to zero in the limit. The terms arising from the action of each $H_i$, $i = 1, 2, 3$ on the integrand can be bounded similarly and are also of order $\rho^{-1/2}$.

### 3.3. The near region

Now that it has been demonstrated that the contributions to the mean from the region of integration bounded away from the origin vanish in the limit, we can conclude that the value of $\lim_{\rho \to \infty} B(x)$, if it is finite, must be local since as $\rho \to \infty$, we can choose $a$ to be arbitrarily small. The only local scalar quantities of the correct dimensions are $\Box \phi(x)$ and $R \phi(x)$. In this section we show that the limit is finite and that the precise linear combination is (6).

In the near region, $W_1$, we work with Riemann normal coordinates $\{y^i\}$ centred on $x = 0$ and the usual spatial polar coordinates: $r = \sqrt{\sum_{i=3}^{3}(y^i)^2}$, $\theta$ and $\varphi$, and radial null coordinates $u = \frac{1}{\sqrt{2}}(-y^0 - r)$ and $v = \frac{1}{\sqrt{2}}(-y^0 + r)$. We will show that

$$\lim_{\rho \to \infty} (\rho^{3/2} \rho H_1 - \rho^{1/2} \phi(x)) = \frac{\sqrt{6}}{4} \left( \Box \phi(x) - \frac{1}{2} R(x) \phi(x) \right).$$

(76)
where
\[
I_1 = \hat{\mathcal{O}} \int_{W_1} d^4y \sqrt{-g(y)} \phi(y) e^{-\rho V(y)}
\]
\[
= \hat{\mathcal{O}} \int_0^a dv \int_0^v du \int d\Omega_2 (v-u)^2 \sqrt{-g(y)} \phi(y) e^{-\rho V(y)}.
\]
(77)

We note that above and throughout this section \(\sqrt{-g(y)}\) will denote the square root of minus the determinant of the metric in RNC.

In \(W_1\), we have expansions \([21, 22]\):
\[
\sqrt{-g} = 1 - \frac{1}{6} y^\mu y^\nu R_{\mu\nu}(0) + y^\mu y^\nu y^\rho T_{\mu\nu\rho}(y).
\]
(78)

\[
\phi(y) = \phi(0) + y^\mu \phi_{\mu}(0) + \frac{1}{2} y^\mu y^\nu \phi_{\mu\nu}(0) + y^\mu y^\nu y^\rho \Psi_{\mu\nu\rho}(y).
\]
(79)

\[
V(y) = \frac{\pi}{24} \tau^4 - \frac{\pi}{4320} \tau^6 R(0) + \frac{\pi}{720} \tau^4 y^\mu y^\nu R_{\mu\nu}(0) + \tau^4 y^\mu y^\nu y^\rho S_{\mu\nu\rho}(y)
\]
\[
= V_0(y) + \delta V(y)
\]
(80)

where \(\tau^2 = 2a^2 y^2\), \(V_0(y) = \frac{\pi}{24} \tau^4 = \frac{\pi}{12} a^2 y^2\) and \(\delta V(y)\) is the rest. \(T_{\mu\nu\rho}(y), \Psi_{\mu\nu\rho}(y)\) and \(S_{\mu\nu\rho}(y)\) are \(C^1\)-functions.

We also use the expansion of the exponential factor,
\[
e^{-\rho V(y)} = e^{-\rho V(0)} e^{-\rho V(y)} = e^{-\rho V(0)} \left(1 - \rho \delta V(y) + \sum_{k=2}^{\infty} \frac{(-\rho)^k}{k!} (\delta V)^k \right).
\]
(81)

Using (78)-(81) we expand the integrand in (77) and collect the terms in 4 groups:
\[
\sqrt{-g(y)} \phi(y) e^{-\rho V} = (A(y) + B(y) + C(y) + D(y)) e^{-\rho V},
\]
(82)

where

\[
A(y) = \phi + \frac{1}{2} y^\mu y^\nu \phi_{\mu\nu} - \frac{1}{6} \phi y^\mu y^\nu R_{\mu\nu} + \frac{\rho \tau^4}{4320} \phi (\tau^2 R - 6 y^\mu y^\nu R_{\mu\nu});
\]
(83)

\[
B(y) = \left(1 - \frac{1}{6} y^\mu y^\nu R_{\mu\nu}\right) y^\alpha \phi_{\alpha} - \frac{1}{12} y^\mu y^\nu y^\rho R_{\mu\nu\rho} \phi_{\alpha\beta} + \rho \left( y^\alpha \phi_{\alpha} + \frac{1}{2} y^\alpha y^\beta \phi_{\alpha\beta} - \frac{1}{6} \phi y^\alpha y^\beta R_{\alpha\beta} - \frac{1}{6} y^\mu y^\nu y^\rho R_{\mu\nu\rho}\phi_{\alpha}\right)
\]
\[- \frac{1}{12} y^\mu y^\nu y^\rho R_{\mu\nu\rho} \phi_{\alpha\beta} \left(\frac{\pi}{4320} \tau^6 R - \frac{\pi}{720} \tau^4 y^\alpha R_{\alpha\beta}\right);
\]
(84)
\[ C(y) = \left( 1 - \frac{1}{6} y^\nu y^\rho R_{\nu\rho} + y^\nu y^\rho y^\alpha T_{\nu\rho\alpha} \right) y^\sigma y^\nu \Phi_{\sigma\nu} \\
+ y^\nu y^\rho y^\alpha T_{\nu\rho\alpha} \left( \phi(0) + y^\nu \phi_{\nu}(0) + \frac{1}{2} y^\nu y^\rho \phi_{\nu\rho}(0) \right) \\
- \rho \left( y^\nu y^\rho y^\alpha \Psi_{\nu\rho\alpha} - \frac{1}{6} y^\nu y^\rho R_{\nu\rho} y^\alpha y^\alpha T_{\nu\rho\alpha} \phi + y^\nu y^\rho \phi_{\nu\rho}(0) + \frac{1}{2} y^\nu y^\rho y^\sigma \Phi_{\sigma\nu\rho} \right) \\
+ \left( 1 - \frac{1}{6} y^\nu y^\rho R_{\nu\rho} \right) (y^\nu y^\rho y^\gamma S_{\nu\rho\gamma} + y^\nu y^\rho y^\alpha T_{\nu\rho\alpha}) \\
\times \left( \frac{-\pi}{4320} \tau^\nu + \frac{\pi}{720} \tau^\nu y^\rho R_{\nu\rho} + \tau^\nu y^\rho y^\alpha S_{\nu\rho\alpha} \right) ; \tag{85} \]

\[ D(y) = -g(y) \phi(y) \sum_{k=2}^{\infty} \frac{(-\rho)^k}{k!}(\delta V)^k. \tag{86} \]

Then,
\[ I_1 = I_A + I_B + I_C + I_D \tag{87} \]

where
\[ I_A := \hat{O} \int_0^a dv \int_0^v du \int d\Omega (v-u)^2 A(y) e^{-\rho |l|}, \tag{88} \]

and similarly for \( I_B, I_C \) and \( I_D \). \( I_A \) is the only integral involving no unknown functions.

We will see that \( \rho^{3/2} I_A \) gives the nonzero contributions in the limit and \( \rho^{3/2} I_B, \rho^{3/2} I_C \) and \( \rho^{3/2} I_D \) vanish in the limit. Consider \( I_A \). Integrating over the angular coordinates gives
\[ I_A = \hat{O} \int_0^a dv \int_0^v du \frac{(v-u)^2}{2} \left( 4 \pi \phi + \pi (u+v)^2 \left( \phi_{00} - \frac{1}{3} (1 + \frac{\pi}{30} \rho u^3 v^3) \phi R_{00} \right) \right) \\
+ \frac{\pi}{3} (v-u)^2 \left( \phi_{00} - \frac{1}{3} (1 + \frac{\pi}{30} \rho u^3 v^3) \phi R_{00} \right) + \frac{\pi^2}{135} \rho u^3 v^3 \phi R_{00} \right) e^{-\frac{\pi}{5} u^3 v^3}. \tag{89} \]

Expanding out the brackets in the integrand gives a sum of terms each of which is a constant times something of the form
\[ \hat{O} \int_0^a dv \int_0^v du \ u^m v^n (\sigma u^2 v^2) e^{-\sigma u^2 v^2} \tag{90} \]

where \( \sigma = \rho \pi/6, q = 0, 1 \) and \( m + n = 2, 4 \). \( H \) commutes with \( \hat{O} \) and does not change the power of \( \sigma \) so we only need look at case \( q = 0 \).

Let
\[ Z_{m,n} := \int_0^a dv \int_0^v du \ u^m v^n e^{-\sigma u^2 v^2}, \quad m, n \in \mathbb{N}. \tag{92} \]
Up to exponentially small terms we have, for \( n \neq m, \)
\[
Z_{m,n} = \frac{1}{2(m-n)} \left[ \frac{1}{\sigma^{m+n+2/4}} \Gamma \left( \frac{m+n+2}{4} \right) - \frac{a^{n-m}}{\sigma^{m+1/2}} \Gamma \left( \frac{m+1}{2} \right) \right]
\]
(93)
and, for \( n = m, \)
\[
Z_{m,m} = \frac{1}{8\sigma^{m+1/2}} \left( \ln(\sigma a^4) - \Psi \left( \frac{m+1}{2} \right) \right) \Gamma \left( \frac{m+1}{2} \right).
\]
(94)

The terms in \( Z_{m,n} \) with \( n \neq m \) are powers of \( \sigma \) and since \( \hat{\mathcal{O}} \) kills \( \sigma^{-\frac{s}{2}} \), the correction to the limit of \( \rho^2 I_B \) from such terms is \( O(\rho^{-\frac{s}{2}}) \). The contributions that are nonzero in the limit come from terms proportional to \( Z_{m,m} \) for \( m = 1, 2 \) and arise from the action of \( \hat{\mathcal{O}} \) on \( \log(\sigma) \sigma^{m+1/2} \). We find that
\[
\lim_{\rho \to \infty} \frac{4}{\sqrt{6}} \left( \rho^3 I_A - \sqrt{\rho} \phi(0) \right) = (\Box - \frac{1}{2} R(0)) \phi(0).
\]
(95)

Consider \( I_B \). Again there are no unknown functions, the integration over the angles can be done and \( I_B \) becomes a sum of terms, each of which is a constant times something of the form (91) where \( q = 0, 1 \) and \( m + n = 3, 5, 6, 7, 8 \). Using (93) and (94) we find that the leading contribution to \( \rho^3 I_B \) is a sum of terms that are \( O(\log(\rho) \rho^{\frac{s}{2}}) \) which vanishes in the limit.

Consider \( I_C \). It is a sum of terms, each of the form
\[
\hat{\mathcal{O}} \left\{ \int_0^a dv \int_0^v du \int d\Omega_2 \frac{(v-u)^2}{2} (y^a)(\sigma u^2 v^2)^s \Xi(y) e^{-\sigma u^2 v^2} \right\}
\]
(96)
\[
= (-1)^q H^q \hat{\mathcal{O}} \left\{ \int_0^a dv \int_0^v du \int d\Omega_2 \frac{(v-u)^2}{2} (y^a)^s \Xi(y) e^{-\sigma u^2 v^2} \right\}
\]
(97)
where \( q = 0, 1, s = 3, \ldots, 9 \), \( (y^a)^s \) stands for a product of \( s \) coordinates and \( \Xi \) stands for one of the functions of \( y \) in the expansion of the brackets in term \( C(y) \). We have used \( \tau^4 = 4u^2 v^2 \) and \( \sigma = \rho \pi l_6 \).

The operator \( H \) does not change any power of \( \sigma \) and we see that what it acts on is the same as (38), except that in that earlier case \( s = 3 \) only. Thus we need to calculate terms like (39) with \( m + n = s = 3, \ldots, 9 \). Using the calculations in section 2.3 we find that \( I_C \) vanishes in the limit and the leading corrections are of order \( \rho^{-1/4} \).

Finally we deal with \( I_D \),
\[
I_D = \hat{\mathcal{O}} \left\{ \int_0^a dv \int_0^v du \int d\Omega_2 (v-u)^2 f(y) \sum_{k=2}^{\infty} \frac{(-\rho^2 V)^k}{k!} e^{-\sigma u^2 v^2} \right\}
\]
(98)
where \( f(y) := \sqrt{-g(y)} \phi(y)/2 \). We use \( \sigma = \rho \pi l_6 \), as before, and define \( \xi(y) \) and \( \Lambda_{\mu\nu}(y) \) by
\[
\rho^2 V(y) = \sigma u^2 v^2 \xi(y)
\]
(99)
\[
\xi(y) = y^\mu y^\nu \Lambda_{\mu\nu}(y).
\]
(100)
We have assumed that in \( W_1 \) the metric is approximately flat so that \( \delta V \ll V_0 \) and \( ||\xi|| \ll 1 \) in \( W_1 \). This will be important later. Now, leaving the angular integration to be done at the end, we have
\[ \hat{O} \int_0^a dv \int_0^v du \sum_{k=2}^{\infty} \frac{1}{k!} (v-u)^2 f(y) \xi^k (-\sigma u^2 v^2)^k e^{-\sigma u^2 v^2} \]

\[ = \hat{O} \int_0^a dv \int_0^v du \sum_{k=2}^{\infty} \frac{1}{k!} (v-u)^2 f(y) \xi^k H_k e^{-\sigma u^2 v^2} \]

\[ = \sum_{k=2}^{\infty} \frac{1}{k!} H_k \hat{O} \int_0^v dv \int_0^u du (v-u)^2 f(y) \xi^k e^{-\sigma u^2 v^2} \]

\[ = -\frac{1}{6} \int_0^a dv \int_0^v du \sum_{k=2}^{\infty} \frac{1}{k!} H_k \left\{ u^2 \frac{\partial^3}{\partial u^3} ((v-u)^2 f(y) \xi^k) e^{-\sigma u^2 v^2} \right\} \]

\[ + \frac{1}{3} \int_0^a dv \sum_{k=2}^{\infty} \frac{1}{k!} H_k \left\{ v f (2u^2 \tilde{L}_u)^k e^{-\sigma v^2} \right\}. \quad (101) \]

where we have used that \( \hat{O} \) and \( H_k \) commute and for the last step we used the same integration by parts done in equations (40)–(48). The tilde means setting \( u = v \), so that \( \xi^k \sim \Lambda \). The exchanging of the orders of summation, integration and differentiation by \( \sigma \) is justified as the partial sums are uniformly integrable.

Let us now consider the first term in (101). Acting with the \( H_k \) we find it equals

\[ -\frac{1}{6} \int_0^a dv \int_0^v du \sum_{k=2}^{\infty} \frac{(-\sigma)^k}{k!} u^{2k} v^{2k} u^{\partial^3}{\partial u^3} ((v-u)^2 f(y) \xi^k) e^{-\sigma u^2 v^2} \]

\[ = -\frac{\sigma^2}{6} \int_0^a dv \int_0^v du u^{2q} \sum_{k=2}^{\infty} \frac{(-\sigma)^k}{k!} u^{2k} v^{2k} \frac{\partial^3}{\partial u^3} [(v-u)^2 f(y) \xi^k] e^{-\sigma u^2 v^2} \quad (102) \]

Our strategy will be to bound the infinite sum by something times the exponential \( e^{\sigma u^2 v^2/2} \), which is possible because \( |\xi| \ll 1 \). Then the resulting Gaussian integral can be done.

To this end we expand

\[ \frac{\partial^3}{\partial u^3} (h \xi^{k+2}) = [h'' \xi^2 + 3(k+2)h' \xi^3 + 3(k+2)(k+1)h \xi^4 + 3(k+2)h \xi^5 + 3(k+2)(k+1)h \xi^6] \xi^{k+1}, \quad (103) \]

where \( h = (v-u)^2 f \) and \( \xi \) denotes differentiation with respect to \( u \). Recalling that \( \xi(y) = v^y \frac{\partial^y}{\partial y} \), we see that each of the terms multiplying \( \xi^k \) has a factor of \( u^{2q} \) with \( 3 \leq i+j \leq 6 \) and each term multiplying \( \xi^{k+1} \) has a factor of \( u^{2q} \), \( 5 \leq i+j \leq 8 \).

Thus, every term in (102) is of the form

\[ \sigma^{2+q} \int_0^a dv \int_0^v du \Theta(y) u^{2+2i} v^{2+2j+1} \sum_{k=0}^{\infty} \frac{(-\sigma)^k}{(k+2+q)!} u^{2k} v^{2k} \xi^k e^{-\sigma u^2 v^2} \quad (104) \]

where \( 3 \leq i+j \leq 6 \) when \( q = 0 \) and \( 5 \leq i+j \leq 8 \) when \( q = 1 \).

As \( |\xi| \ll 1 \) in \( W \), it is certainly less than 1/2 and we can bound the infinite sum

\[ \sum_{k=0}^{\infty} \frac{\xi^k}{(k+2+q)!} (-\sigma)^k u^{2k} v^{2k} \ll e^{\frac{1}{2} \sigma u^2 v^2} \quad (105) \]

\( \forall y \in W_1 \). Then (104) for \( q = 0 \) can be bounded:
\[
\int \int_0^a \int_0^v \int_0^u \int_0 \Theta(y) u^{7+i+\frac{4}{3} j} \sum_{k=0}^\infty \frac{(-\sigma)^k}{(k+2)!} u^{2k+2k} \xi^k e^{-\sigma u^2 x^2}
\]

\[
< ||\Theta|| \int_0^a \int_0^v \int_0 \Theta(y) u^{7+i \frac{4}{3} j} \xi^k e^{-\sigma u^2 x^2}
\]

\[
(106)
\]

\[
= \left\{ \begin{array}{ll}
2 ||\Theta|| (\Gamma((i+3)+j)/4) - \frac{a^{j-i-3} \Gamma(4+i/2)}{\sigma^{2+i/2}}, & i-j+3 \neq 0 \\
\frac{\ln \sigma}{\sigma^{2+i/2}} (\ln(\sigma^2 a^4) - \Psi(4+i/2)) \Gamma(4+i/2), & i-j+3 = 0
\end{array} \right.
\]

\[
(107)
\]

up to exponentially small terms, where \( \sigma = \sigma/2 \). The integral over the angles contributes a factor of \( 4\pi \).

Since \( i+j \geq 3 \), these terms, after multiplication by \( \rho^{3/2} \), vanish in the limit and the leading correction to the limit is \( O(\ln(\rho)/\sqrt{\rho}) \). The terms for \( q = 1 \) also tend to zero and the leading correction is again \( O(\ln(\rho)/\sqrt{\rho}) \).

Let us now turn to the second term in (101):

\[
\left| \int_0^a \int_0^v \int_0^u \int_0 \Theta(y) u^{7+i \frac{4}{3} j} \sum_{k=0}^\infty \frac{(-\sigma)^k}{(k+2)!} u^{2k+2k} \xi^k e^{-\sigma u^2 x^2} \right|
\]

\[
\leq \int_0^a \int_0^v \int_0^u \int_0 \Theta(y) u^{7+i \frac{4}{3} j} \sum_{k=0}^\infty \frac{(-\sigma)^k}{(k+2)!} u^{2k+2k} \xi^k e^{-\sigma u^2 x^2}
\]

\[
\leq 4 \sigma^2 \| \bar{f} \|_1 \| \bar{\Lambda} \|_2 \| \bar{\Lambda} \|_2 \int_0^a \int_0^v \int_0 \Theta(y) u^{7+i \frac{4}{3} j} \sum_{k=0}^\infty \frac{(-\sigma)^k}{(k+2)!} u^{2k+2k} \xi^k e^{-\sigma u^2 x^2}
\]

\[
\leq 4 \sigma^2 \| \bar{f} \|_1 \| \bar{\Lambda} \|_2 \| \bar{\Lambda} \|_2 \int_0^a \int_0^v \int_0 \Theta(y) u^{7+i \frac{4}{3} j} e^{-\sigma u^2 x^2}
\]

\[
\leq 4 \sigma^2 \| \bar{f} \|_1 \| \bar{\Lambda} \|_2 \| \bar{\Lambda} \|_2 \int_0^a \int_0^v \int_0 \Theta(y) u^{7+i \frac{4}{3} j} e^{-\sigma u^2 x^2}
\]

where the second to last inequality follows from \( 2\sigma^2 |\bar{\Lambda}| \ll 1 \). The final integral gives a contribution of order \( \rho^{-3/4} \), after multiplication by \( \rho^{3/2} \) and is not one of the leading order corrections.

3.4. Finite \( \rho \) corrections

As in the flat spacetime case, we want to know how \( \delta \phi(x) \) behaves when \( \rho = l^{-4} \) is large but finite. Unlike in flat spacetime however, since we lack an explicit expression for the expansion of the volume of long skinny intervals ‘down the light cone’ in \( W_2 \), the finite \( \rho \) corrections to the limit from \( W_2 \) can only be given in terms of integrals of unknown functions, such as \( f_0(V, \theta, \phi) \) in (72), and are not very enlightening. In appendix it is shown that \( f_0(V, \theta, \phi) \) depends only on the curvature components along the null geodesic on the light cone labelled by \( (\theta, \phi) \) and it may be possible to find \( f_0(V, \theta, \phi) \) and further terms in the expansion of the volume as expressions involving integrals of these curvature components along the null geodesic.

There is one case in which we do not need to know the behaviour of the volume of long skinny intervals to bound the corrections and that is when a neighbourhood of the whole of the light cone is covered by RNC in which the metric is approximately flat. That means that
all corrections to the metric in RNC are small: $R_{\mu\nu}(0) y^\mu y^\nu \ll 1$ and $\nabla_{\rho} R_{\mu\nu}(0) y^\mu y^\nu \ll 1$ etc throughout $W_2$. In particular, $RL^2 \ll 1$ where $R$ stands for any component of the curvature at the origin and $L$ is the cutoff on the coordinate $v$ in $W_2$.

Then, the NGNC coordinates $U$ and $V$ are replaced by their RNC versions $u$ and $v$ in the integrals for the corrections and the expansions of all functions are well approximated by their flat space versions. For example, function $f_0(V^2, \theta, \phi)$ is well-approximated by $\frac{v^2}{n}$. The integrals for the corrections in region $W_2$ can be bounded and the corrections take the same form as they do in the flat space case, up to factors of order one. In this case, however, the condition that the metric be approximately flat in RNC in the whole of $W_2$ means that the Ricci scalar times $\phi$, $R\phi$, at the origin is negligible compared with $\Box \phi$—assuming that the scale on which $\phi$ varies is small compared to the cutoff—and the limiting value in this case is close to $\Box \phi$, as one would expect.

Then, the same conditions from the flat space apply, namely that $\phi$ should be slowly varying in the direction transverse to the light cone over the scale $a$ and $a \gg l$. And the same conditions as in the flat space case will also result from considering corrections from the near region, $W_1$, as one can verify by examining the various terms. The conclusion is that the situation in which we can estimate the corrections—without further knowledge about long skinny intervals—is when the region is approximately flat in some frame, and the field is slowly varying on the discreteness scale in that same frame. In which case, the result is close to the flat space result and $B\phi$ is close to $\Box \phi$.

Consider now the case when $\phi = 1$. Then the limit of $B\phi$ equals $-\frac{1}{2} R$ and we can ask when the value of $B\phi$ is close to that limit. Again, our lack of knowledge about $V(y)$ for long skinny intervals means we can only answer the question under conditions that the metric is approximately flat in RNC in the whole of $W_2$. In that case, if all curvature length scales are large compared with the discreteness scale, then the value of $B\phi$ is close to $-\frac{1}{2} R$.

4. Discussion

In [11, 12], causal set d’Alembertians were defined for dimensions $d = 3$ and $d > 4$, and it was shown that if the mean of these d’Alembertians has a local limit as $\rho \to \infty$ then that limit will be $\Box - R/2$ in all dimensions. We expect the argument for the existence of the local limit under certain conditions given above for $d = 4$, to be able to be generalised for $d = 3$ and $d > 4$. In two dimensions, the conformal flatness of spacetime should make the proof more straightforward.

So far we have ignored the important question of the fluctuations about the mean. These fluctuations grow with the sprinkling density. In order to tame the fluctuations, Sorkin introduced, in two dimensions, for each fixed physical non-locality length scale, $l_k \geq l$, a causal set operator, $B_k^{(2)}$, whose mean over sprinklings at density $\rho$ does not depend on $\rho$ but is equal to the mean of $B_k^{(2)}$ with $\rho$ replaced by $l_k^{-2}$. This was extended to four dimensions [10]:

$$B_k^{(4)} \phi(x) = \frac{4}{\sqrt{6} l^2_k} \left[ -\phi(x) + \epsilon \sum_{y < x} f(n(x, y), \epsilon) \phi(y) \right],$$  

(109)

where $\epsilon = (ll_k)^2$ and

$$f(n, \epsilon) = (1 - \epsilon)^n \left[ 1 - \frac{9n}{1 - \epsilon} + \frac{8\epsilon^2 n!}{(n - 2)! (1 - \epsilon)^2} - \frac{4\epsilon^3 n!}{3(n - 3)! (1 - \epsilon)^3} \right].$$  

(110)
and also to all other dimensions [11, 12]. In each dimension, \( d \), the mean of \( B_k^{(d)} \) over sprinkling at density \( \rho = l^{-d} \) takes the same form as for the original \( \text{d}^\prime \) but with the discreteness scale \( \rho \) replaced by \( \rho_k = l_k^{-d} \). Therefore, the mean, \( \bar{B}_k^{(d)} \), of \( B_k^{(d)} \) is given by (5) with \( \rho \) replaced by \( \rho_k^{-d} \). Thus, results about \( B \) at finite \( \rho \) can be applied to \( B_k \). Simulations of \( B_k^{(d)} \) for a selection of test functions in 2-, 3- and 4-dimensional flat space indicate that its fluctuations do die away as \( l \to 0 \) [7, 10, 11] but this remains to be more thoroughly tested.

In calculating the limit of the mean of the causal set \( \text{d}^\prime \) in curved space, we made the assumption that between \( x \) and every point of \( \partial J^x_\phi \) there is a unique null geodesic. This is a strong assumption and it is possible that it can be weakened. The assumption is made so that \( \partial J^x_\phi \) can be treated as a null geodesic congruence, guaranteeing the existence of null Gaussian normal coordinates in a neighbourhood of \( \partial J^x_\phi \). When the assumption fails and there are caustics on \( \partial J^x_\phi \) it is nevertheless the case, in a globally hyperbolic spacetime, that every point on \( \partial J^x_\phi \) lies on at least one null geodesic from \( x \). Moreover, the set of points on \( \partial J^x_\phi \) which are not connected to \( x \) by a single null geodesic consists of those points that lie on caustics and is a set of measure zero in \( \partial J^x_\phi \). It might be possible to construct a proof in the general case by covering the region of integration down the light cone with appropriate finite collections of subregions in each of which Null Gaussian Normal Coordinates can be constructed. If the integral can be performed in each subregion and shown to be equal to zero in the limit, one might be able to argue that the whole integral also tends to zero.

The applicability of our result is limited by the fact that we have not been able to estimate the finite \( \rho \) corrections to the limit from the down-the-light-cone region in terms of physically interpretable quantities. In order to estimate these corrections one needs an explicit asymptotic expansion for the volume of causal intervals which hug the past light cone, the long skinny intervals. These intervals have small volume but are not necessarily approximately flat. If, however, there is an approximately flat Riemann normal neighbourhood of the whole of \( \partial J^x_\phi \) \( \cap \text{Supp}(\phi) \) of thickness \( a \gg l \), then \( \bar{B}_\phi(x) \) is a good approximation to \( \Box \phi(x) \) at finite density \( \rho = l^{-4} \). Moreover, if \( \phi = 1 \) and there is an approximately flat Riemann normal neighbourhood of the whole of \( \partial J^x_\phi \) \( \cap \text{Supp}(\phi) \) of thickness \( a \gg l \), then \( \bar{B}_\phi(x) \) is a good approximation to \( -\frac{1}{2}R(x) \) at finite density \( \rho = l^{-4} \).

The value of the continuum limit of the mean of the causal set \( \text{d}^\prime \) was used to propose a causal set action in \( d = 2, 4 \) [10] and in other dimensions [11, 12]. Actions derived from \( B_k \) have also been defined. Our results suggest that the action in \( d = 4 \) evaluated on a causal set that is a sprinkling of an approximately flat region of a four dimensional spacetime will be approximately equal to the Einstein–Hilbert (EH) action of the spacetime—if the fluctuations around the mean are small. We do not know what the value would be for a causal set that is a sprinkling into a spacetime which is large compared to the scale set by the curvature and it is possible it is not close to the EH action. If this turns out to be the case, we would have a discrete action whose mean is close to the EH action for regions small compared to any curvature length scale, but starts to deviate from it as the size of the region, \( L \), approaches the curvature length scale. If the proposed causal set action is relevant in a continuum regime of full quantum gravity, this might indicate that the continuum approximation to quantum gravity is general relativity for approximately flat regions of spacetime but deviates from it on scales large compared to the curvature scale.

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Appendix

Consider a point \(y\) in \(N_{LC}\) with Null Gaussian Normal Coordinates \((U, V, \theta, \phi)\) and the volume, \(V(y)\) of the causal interval, \(\text{Int}(x, y)\), between \(x\) and \(y\).

We use the null geodesic \(\gamma(\theta, \phi)\) defined in the text to define Null Fermi Normal Coordinates (NFNC) \(\gamma_{x_1 x_2}^{\perp}\). They are defined using a pseudo-orthonormal tetrad at \(x \equiv \{E_{\theta \phi}, E_{\pi \theta \phi}, E_{\pi \theta \phi}, E_1, E_2\}\) where \(E_1 = T(\theta, \phi)\), \(E_2 = T(\pi - \theta, \phi + \pi)\) and \(E_1\) and \(E_2\) are spacelike unit vectors, orthogonal to each other and to both \(T(\theta, \phi)\) and \(T(\pi - \theta, \phi + \pi)\). The affine parameter along \(\gamma(\theta, \phi)\) is \(x^+\). This tetrad is parallel transported along \(\gamma(\theta, \phi)\):

\[
\begin{align*}
E_1 & = E_1^0, \\
E_2 & = E_2^0 + E_1 E_1^0 - E_2 E_1^0,
\end{align*}
\]

where all the curvature components are evaluated on the null geodesic, the barred indices, \(\vec{a}, \vec{b}\) etc, run over the three transverse directions \(-\), 1, 2 and unbarred, \(a, b\) etc, over the spatial transverse directions 1 and 2 only. Note: there is a sign difference between the \(x^-\) coordinate of Blau et al and our \(x^-\) coordinate here.

Fixing \(y\) with its NGNC \(\{U, V, \theta, \phi\}\), we assume that the causal interval \(\text{Int}(x, y)\) between \(x\) and \(y\) lies in the tubular neighbourhood of \(\gamma(\theta, \phi)\) on which the NFNC are defined. We want to evaluate

\[
V(y) = \int_{\text{Int}(x, y)} dx^- dx^0 dx^1 dx^2 \sqrt{-g(x^+, x^-, x^1, x^2)},
\]

the volume of \(\text{Int}(x, y)\) as an expansion in \(U\) as \(U \to 0\). In other words we want to consider the limit as the interval tends to the segment of the null geodesic \(\gamma(\theta, \phi)\) between \(x\) and the point with NGNC \(\{0, V, \theta, \phi\}\). This is related to the Penrose limit.

Rescaling the coordinates \(z^- = x^- / U\), \(z^+ = x^+\) and \(z^a = x^a / \sqrt{U}\), in the limit \(U \to 0\) the metric becomes

\[
ds^2 = U(\ldots) + O(U^{-1/2})
\]

and the next terms proportional to \(U^{3/2}\) and \(U^2\) in the expansion can be found in appendix A of [23]. Using this one can show that \(\sqrt{-g(z^+, z^-, z^1, z^2)} = U^2[1 + U f(U, z^+, z^-, z^1, z^2)]\), where, we assume, \(f\) is a differentiable function of \(U\).

References

A. Belenchia et al. Class. Quantum Grav. 33 (2016) 245018
$$V(y) = \int_{\text{Int}(x,y)} dz^+ dz^- dz^0 dz^2 \sqrt{-g(z^+, z^-, z^1, z^2)} \quad (A.4)$$

$$= U^2 \int_{\text{Int}(x,y)} dz^+ dz^- dz^0 dz^2 [1 + U f(U, z^+, z^-, z^1, z^2)]. \quad (A.5)$$

The interval $\text{Int}(x,y)$ is defined by the causal structure of the metric and is the same for $ds^2$ as for the conformally rescaled metric

$$\tilde{ds}^2 = U^{-1} ds^2 = -2dz^+ dz^- + \delta_{ab} dz^a dz^b - R_{+a+b}(z^+) z^a b^b (dz^+)^2 + O(U^2). \quad (A.6)$$

In the $U \to 0$ limit, therefore, the integral

$$\int_{\text{Int}(x,y)} dz^+ dz^- dz^0 dz^2 \quad (A.7)$$
tends to the volume of the causal interval between the origin and the point with coordinates $(z^+ = V, z^- = 1, z^1 = 0, z^2 = 0)$ in the Penrose limit metric

$$ds^2 = -2dz^+ dz^- + \delta_{ab} dz^a dz^b - R_{+a+b}(z^+) z^a b^b (dz^+)^2. \quad (A.8)$$

Note that $R_{+a+b}(z^+)$ are the curvature components of the original metric along $\gamma(\theta, \phi)$ in the original, unscaled NFNC. We denote this limit volume by $f_0(V, \theta, \phi)$. It is an open question whether $f_0$ can be expressed more concretely in terms of (integrals of) the curvature components along $\gamma(\theta, \phi)$.

We conclude that $U^{-2} V(y) \to f_0(V, \theta, \phi)$ as $U \to 0$ and so $V(y) = U^2 f_0(V, \theta, \phi) + U^3 G(V, U, \theta, \varphi)$ with $G$ a continuous function of $U$. In fact, we need to assume more differentiability than this for $V(y)$ for the proof but the crucial fact established here is the $U^2$ behaviour of the leading term in the expansion.

We can also show that $f_0$ is a monotonic increasing function of $V$. Consider two points $p$ and $p'$ in the Penrose limiting geometry (A.8) with coordinates $(z^+ = V, z^- = 1, z^1 = 0, z^2 = 0)$ and $(z^+ = V', z^- = 1, z^1 = 0, z^2 = 0)$ respectively, where $V < V'$. There is a future pointing null geodesic, along which $z^- = 1, z^1 = 0, z^2 = 0$, from $p'$ to $p$ and so the causal interval from $p'$ to $x$ contains the causal interval from $p$ to $x$. Then, if $U$ is small enough, it follows that $V(y)$ is monotonic increasing in $V$.

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