Euler number of homology groups of super Lie algebra

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Main changes from the former version:
(1) In the present file, we use the terminology “pre Lie superalgebra” or “Z-graded Lie superalgebra” instead of “super Lie algebra” of the former version.
(2) The title is changed: In the new file, “super Lie algebra” is changed according to (1) and added “and Betti numbers”.
(3) Theorem 4.1 in the former version was not correct not including 0-th chain. The revised one is Lemma 4.2 in this note.
(4) Added new results: For (w, h)-weighted chain complex, m-th Betti numbers are 0 if w ≠ h (Theorem 5.1) and the first Betti number is 0 (Theorem 5.2).

1 Introduction

The well known de Rham cohomology group of a differentiable manifold \( M \) is a cohomology group of the Lie algebra \( \mathfrak{X}(M) \) of smooth vector fields on \( M \) together with the \( \mathfrak{X}(M) \)-module \( C^\infty(M) \) as coefficient. Similarly, the Gel’fand-Fuks cohomology theory is a cohomology theory of infinite dimensional Lie algebras and there are many works on the cohomology of related subject for example, the Lie algebra of volume preserving vector fields, the Lie algebra of formal Hamiltonian vector fields and so on. The notion of these Lie algebra (co)homology groups is easy to understand, but the calculation is hard to complete and one of the reason is the infinity of dimensions. In order to reduce our computation to finite dimensional case, we use an idea of “weight” (c.f. for instance, [5], [4], [3], [2]).

There is (co)homology theory of Lie superalgebras but few works of \( \mathbb{Z} \)-graded version. Among Poisson geometries, \( \sum \Lambda^p \mathfrak{T}(M) \) with the Schouten bracket is known as a prototype of \( \mathbb{Z} \)-graded (pre) Lie superalgebra and it is well-known that a 2-vector field \( \pi \) is Poisson if and only if \([\pi, \pi] = 0\). Then Poisson condition \([\pi, \pi] = 0\) is equivalent to \( \partial(\pi \wedge \pi) = 0 \) in superalgebra homology theory, and \( \sqrt{\ker(\partial)} \) (the square root of cycles) is the space of Poisson structures in some sense and there is some possibility of studying Poisson structures in this direction.

Thus, in this note, we will study homology groups of pre Lie superalgebra and relative homology groups with the coefficient \( g \)-module \( V \) introducing (double) weight by many works of Lie algebra (co)homology theory.

First we recall the definition of Lie superalgebra and pre Lie superalgebra.

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**Definition 1** ((pre) Lie superalgebra) Suppose a real vector space $\mathfrak{g}$ is graded by $\mathbb{Z}$ as $\mathfrak{g} = \sum_{j \in \mathbb{Z}} \mathfrak{g}_j$ and has a bilinear operation $[\ ,\ ]$ satisfying

1. $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$
2. $[X,Y] = (-1)^{i \cdot y}[Y,X]$ where $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$
3. $(−1)^{yz}[[X,Y],Z] + (−1)^yz[[Y,Z],X] + (−1)^{y}z[[Z,X],Y] = 0$ (Jacobi identity).

Then we call $\mathfrak{g}$ a pre (or $\mathbb{Z}$-graded) Lie superalgebra.

A Lie superalgebra $\mathfrak{g}$ is graded by $\mathbb{Z}_2$ as $\mathfrak{g} = \mathfrak{g}_{[0]} \oplus \mathfrak{g}_{[1]}$ and the condition (1.1) is regarded as $[\mathfrak{g}_{[1]}, \mathfrak{g}_{[1]}] \subset \mathfrak{g}_{[0]}$ in modulo 2 sense.

**Remark 1.1** Super Jacobi identity (1.3) above is equivalent to the one of the following.

4. $[[X,Y],Z] = [X, [Y,Z]] + (−1)^{y}[X,Z,Y]$
5. $[X, [Y,Z]] = [[X,Y],Z] + (−1)^{y}[Y,[X,Z]]$

Suppose $\mathfrak{g} = \sum_{j \in \mathbb{Z}} \mathfrak{g}_j$ is a pre Lie superalgebra. Let $\mathfrak{g}_{[0]} = \sum_{i \text{ is even}} \mathfrak{g}_i$ and $\mathfrak{g}_{[1]} = \sum_{i \text{ is odd}} \mathfrak{g}_i$. Then $\mathfrak{g} = \mathfrak{g}_{[0]} \oplus \mathfrak{g}_{[1]}$ holds and this is a Lie superalgebra.

**Example 1.1** Take an $n$-dimensional vector space $V$ and split it as $V = V_0 \oplus V_1$. Define $\mathfrak{g}_{[i]} = \{ A \in \mathfrak{gl}(V) \mid A(V_j) \subset V_{i+j} \}$. For each $A \in \mathfrak{g}_{[i]}$ and $B \in \mathfrak{g}_{[j]}$, define $[A,B] = AB - (-1)^{ij}BA$. Then $\mathfrak{gl}(V) = \mathfrak{g}_{[0]} \oplus \mathfrak{g}_{[1]}$ with this bracket is a Lie superalgebra.

More concretely, we take $n = 2$ and $\dim V_{[i]} = 1$ for $i = 0, 1$. Then $\mathfrak{g}_{[0]} = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$ and $\mathfrak{g}_{[1]} = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$.

Now define $\mathfrak{g}_0 = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}$, $\mathfrak{g}_1 = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$ and $\mathfrak{g}_2 = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$. Then $\mathfrak{gl}(2) = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a pre Lie superalgebra.

We will introduce the notion of double-weighted pre Lie superalgebras (cf. Definition 5) and main results in this note are the calculation of the Euler number of homology groups of double-weighted pre Lie superalgebras of special type.

**Theorem** (cf. Lemma 4.2) For general $n$, the Euler number of chain complex $\{ C_{\bullet,0,h} \}$ is 0 for each $h$.

**Theorem** (cf. Theorem 4.1) For general $n$, the Euler number of chain complex $\{ C_{\bullet,w,h} \}$ is 0 for each $w$ and each $h$.

**Theorem** (cf. Theorem 4.2) The Euler number of $(\overline{C}_{\bullet,w,h}, \partial_V)$ is 0 for each $w$ and $h$.

**Theorem** (cf. Theorem 5.1) The $m$-th Betti number of $\{ C_{\bullet,w,h} \}$ is 0 for each double weight $(w, h)$ if $w \neq h$.

**Theorem** (cf. Theorem 5.2) The first Betti number of $\{ C_{\bullet,w,h} \}$ is 0 for each double weight $w, h$. 

2 Preliminaries, notations and basic facts

In the usual Lie algebra homology theory, \( m \)-th chain space is the exterior algebra \( \Lambda^m g \) of \( g \) and the boundary operator is essentially comes from the operator \( X \wedge Y \mapsto [X,Y] \).

In the case of pre Lie superalgebras, skew-symmetry of bracket operation which yields "exterior algebra" is defined as the quotient of the tensor algebra \( \otimes^m g \) of \( g \) by the two-sided ideal generated by

\[
(2.1) \quad X \otimes Y + (-1)^{\varepsilon_Y} Y \otimes X \quad \text{where} \quad X \in g_x, Y \in g_y,
\]

and we denote the equivalence class of \( X \otimes Y \) by \( X \partial Y \).

Since \( X_{\text{odd}} \Delta Y_{\text{odd}} = Y_{\text{odd}} \Delta X_{\text{odd}} \) and \( X_{\text{even}} \Delta Y_{\text{any}} = -Y_{\text{any}} \Delta X_{\text{even}} \) hold, \( \Delta^m g_k \) is a symmetric algebra with respect to \( \Delta \) for odd \( k \) and is a skew-symmetric algebra with respect to \( \Delta \) for even \( k \).

**Definition 2** Assume that the pre Lie superalgebra \( g \) acts on a module \( V \) as follows: For each homogeneous \( \xi \in g \) we have \( \xi_V \in \text{End}(V) \) and satisfy \( [\xi, \eta]_V = \xi_V \circ \eta_V - (-1)^{|\xi| |\eta|} \eta_V \circ \xi_V \) where \( \xi \in g_{|\xi|} \) and \( \eta \in g_{|\eta|} \). We call \( V \) be a \( g \)-module. We often write \( \xi_V(v) \) by \( \xi \cdot v \).

**Example 2.1** A pre Lie superalgebra \( g \) is itself \( g \)-module by the own bracket \( X \cdot Z = [X, Z] \).

Let \( X, Y \in g \) be homogeneous as \( X \in g_x, Y \in g_y \). \((X \circ Y - (-1)^{\varepsilon_Y} Y \circ X) \cdot Z = [X,Y] \cdot Z \) holds and this is just Jacobi identity.

Suppose we have an exterior product of \( Y_1, \ldots, Y_m \), i.e., \( Y_1 \Delta \cdots \Delta Y_m \). Omitting \( i \)-th element, we have \( Y_1 \Delta \cdots \Delta Y_{i-1} \Delta Y_{i+1} \Delta \cdots \Delta Y_m \). It is often denoted as \( Y_1 \Delta \cdots \Delta Y_i \cdots \Delta Y_m \). Here we denote it by \( \overline{Y}_m[i] \). If we omit \( i \)-th and \( j \)-th elements, we denote the omitted product by \( \overline{Y}_m[i,j] \).

**Definition 3** Let \( V \) be a \( g \)-module.

For integer \( m > 0 \), define \( \overline{C}_m = \Delta^m g \otimes V = \sum_{i_1 \leq \cdots \leq i_m} g_{i_1} \Delta \cdots \Delta g_{i_m} \otimes V \), called \( m \)-th chain space.

In the case where \( m = 0 \), we define \( \overline{C}_0 = V \).

We define a map \( \partial_V : \overline{C}_m \to \overline{C}_{m-1} \) by

\[
(2.2) \quad \partial_V(Y_1 \Delta \cdots \Delta Y_m \otimes v) = \sum_{i<j} (-1)^{\sum_{s<j}(1+y_s y_i)+\sum_{s<i}(1+y_s y_i)} [Y_j, Y_i] \Delta \overline{\overline{Y}}_m[i,j] \otimes v
\]

\[
(2.3) \quad + (-1)^{m+1} \sum_{i=1}^m (-1)^{\sum_{s<i}(1+y_s y_i)} \overline{Y}_m[i] \otimes Y_i \cdot v
\]

where \( y_i \) is the degree of homogeneous element \( Y_i \), i.e., \( Y_i \in g_{y_i} \).

We have the next basic fact.

**Theorem 2.1** \( \partial_V \circ \partial_V = 0 \) holds. We have \( m \)-th homology group denoted by

\[
H_m(g, V) = \ker(\partial_V : \overline{C}_m \to \overline{C}_{m-1})/\partial_V(\overline{C}_{m+1})
\]

**Remark 2.1** The first term of (2.2) is also expressed as

\[
(2.4) \quad \sum_{i<j} (-1)^{i-1+y_i} \sum_{s<j} y_s Y_1 \Delta \cdots \Delta \underbrace{Y_i \cdots Y_j}_j \Delta \cdots \Delta Y_m \otimes v.
\]
At first, $Y_i$ moves to the left side of $Y_j$, then the parity changes to $(-1)^{\sum_{s=i+1}^{j-1}(1+y_{is})}$. Then the “side effect” of the bracket operation produces $(-1)^{j-1}$. In this note, we have chosen $(-1)^{j-1}$, but it may be possible to choose $(-1)^j$. Then they have just the opposite sign.

**Remark 2.2** If $g$-action on $V$ is trivial, namely $Y \cdot v = 0$ for $\forall Y \in g$ and $\forall v \in V$, then (2.3) is always 0 and we may assume $V = \mathbb{R}$. We call this module the trivial module. Thus, when we essentially deal with the trivial module, the chain space $C_m = \Delta^m g$ and $\partial_V$ is (2.2) without $v$, which we denote $\partial$. It is clear that $\partial \circ \partial = 0$ and we have the homology groups

$$H_m(g, \mathbb{R}) = \ker(\partial : C_m \to C_{m-1})/\partial(C_{m+1}).$$

### 2.1 Homology groups weighted by the first grading

Assume that a $g$-module $V$ is $\mathbb{Z}$-graded, i.e., $V = \sum V_i$, and satisfies $g_i \cdot V_j \subset V_{i+j}$.

**Definition 4** We define a non-zero element in $g_{i_1} \Delta \cdots \Delta g_{i_m} \otimes V_j$ to have $i_1 + \cdots + i_m + j$ as the (first) weight. Define the subspace of $\overline{C}_n$ by $\overline{C}_{m,w} = \sum g_{i_1} \Delta \cdots \Delta g_{i_m} \otimes V_j$, which is the direct sum of different types of spaces of elements with weight $w$.

**Proposition 2.1** The (first) weight $w$ is preserved by $\partial_V$, i.e., we have $\partial_V(\overline{C}_{m,w}) \subset \overline{C}_{m-1,w}$. Thus, we have for a fixed $w, w$-weighted homology groups

$$H_{m,w}(g, V) = \ker(\partial_V : \overline{C}_{m,w} \to \overline{C}_{m-1,w})/\partial(\overline{C}_{m+1,w}).$$

When $V$ is the trivial module, then we have

$$H_{m,w}(g, \mathbb{R}) = \ker(\partial : C_m \to C_{m-1})/\partial(C_{m+1}).$$

### 2.2 Double-weighted homology groups

**Definition 5** (Double-weight) Assume that each subspace $g_i$ of a given pre Lie superalgebra $g$ is directly decomposed by subspaces $g_{i,j}$ as $g_i = \sum g_{i,j}$ and satisfies

$$[X, Y] \in g_{i_1+i_2,j_1+j_2} \quad \text{for each } X \in g_{i_1,j_1}, \; Y \in g_{i_2,j_2}.$$

We say such pre Lie superalgebras are double-weighted. Assume that $g$-module $V$ is also double-weighted $V_{i,j}$ and satisfies $g_{i,j} \cdot V_{i',j'} \subset V_{i+i',j+j'}$. Then we may define double-weighted $m$-th chain space by

$$\overline{C}_{m,w,h} = \sum g_{i_1,h_1} \Delta \cdots \Delta g_{i_m,h_m} \otimes V_{i_h,h_0}$$

**Proposition 2.2** The double-weight $(w, h)$ is preserved by $\partial_V$, i.e., we have $\partial_V(\overline{C}_{m,w,h}) \subset \overline{C}_{m-1,w,h}$. Thus, we have $(w, h)$-weighted homology groups

$$H_{m,w,h}(g, V) = \ker(\partial_V : \overline{C}_{m,w,h} \to \overline{C}_{m-1,w,h})/\partial(\overline{C}_{m+1,w,h}).$$
When $V$ is the trivial module, then we have

$$H_{m,w,h}(g, R) = \ker(\partial : C_{m,w,h} \rightarrow C_{m-1,w,h})/\partial(C_{m+1,w,h}) .$$

# 3 Pre Lie superalgebras with the Schouten bracket

A prototype of pre Lie superalgebra is the exterior algebra of the sections of exterior power of tangent bundle of a differentiable manifold $M$ of dimension $n$

$$g = \sum_{i=1}^{n} \Lambda^i T(M) = \sum_{i=0}^{n-1} g_i , \quad \text{where} \quad g_i = \Lambda^{i+1} T(M)$$

with the Schouten bracket.

There are several ways of defining the Schouten bracket, namely, axiomatic explanation, sophisticated one using Clifford algebra or more direct ones (cf. [6]). Here in the context of Lie algebra homology theory, we introduce the Schouten bracket as follows:

**Definition 6 (Schouten bracket)** For $A \in \Lambda^a T(M)$ and $B \in \Lambda^b T(M)$, we define a binary operation $[\cdot, \cdot]$ by

$$(-1)^{a+1} [A, B] = \partial(A \wedge B) - (\partial A) \wedge B - (-1)^a A \wedge \partial B .$$

In some sense, the Schouten bracket measures how far from the derivation the boundary operator $\partial$ is.

The first chain space is $C_1 = g = \sum_{p=1}^{n} \Lambda^p T(M)$. The second chain space is

$$C_2 = g \Delta g = \sum_{p \geq q} \Lambda^p T(M) \Delta \Lambda^q T(M) = \Lambda^1 T(M) \Delta \Lambda^1 T(M) + \Lambda^1 T(M) \Delta \Lambda^2 T(M) + \cdots$$

$$+ \Lambda^2 T(M) \Delta \Lambda^2 T(M) + \Lambda^2 T(M) \Delta \Lambda^3 T(M) + \cdots$$

**Remark 3.1** Let $\pi \in \Lambda^2 T(M)$. Then $\pi \Delta \pi \in \Lambda^2 T(M) \Delta \Lambda^2 T(M) \subset C_2$ and $\partial(\pi \Delta \pi) = [\pi, \pi] \in C_1$.

Thus, $\pi \in \Lambda^2 T(M)$ is Poisson if and only if $\partial(\pi \Delta \pi) = 0$, and we express it by $\pi \in \sqrt{\ker(\partial)}$ symbolically. It will be interesting to study $\sqrt{\ker(\partial)}$ and also interesting to study specific properties of Poisson structures in $\sqrt{\ker(\partial)C_3}$, which come from the boundary image of the third chain space $C_3$.

In this pre Lie superalgebra, possible weights are non-negative integers. When weight is 0, the chain spaces with trivial action are simply given by $C_{m,0} = \Delta^m g_0 = \Delta^m T(M)$ and the homology is the Lie algebra homology of vector fields for $m = 1, \ldots, n$. For lower weights 1 or 2, the chain spaces are simply given by

$$C_{m,1} = \Delta^{m-1} g_0 \Delta g_1 = \Delta^{m-1} T(M) \Delta \Lambda^2 T(M) \quad \text{for} \quad m = 1, \ldots ,$$

$$C_{m,2} = \Delta^{m-1} g_0 \Delta g_2 + \Delta^{m-2} g_0 \Delta^2 g_1$$

$$= \Delta^{m-1} T(M) \Delta \Lambda^3 T(M) \oplus \Delta^{m-2} T(M) \Delta^2 \Lambda^2 T(M) \quad \text{for} \quad m = 1, \ldots .$$

**Remark 3.2** In particular, $C_{1,2} = \Lambda^3 T(M)$, $C_{2,2} = T(M) \Delta \Lambda^3 T(M) \oplus \Lambda^2 T(M) \Delta \Lambda^2 T(M)$, $C_{3,2} = T(M) \Delta T(M) \Delta \Lambda^3 T(M) \oplus T(M) \Delta \Lambda^2 T(M) \Delta \Lambda^2 T(M)$. Thus, by introducing weight, the chain spaces become smaller and research becomes a little clear and easier.
Given a general weight $w$, the sequences $0 \leq i_1 \leq \cdots \leq i_m \leq n - 1$ with $\sum_{s=1}^{m} i_s = w$ correspond to $1 \leq j_1 \leq \cdots \leq j_m \leq n$ with $\sum_{s=1}^{m} j_s = m + w$ by $j_s = 1 + i_s$. It is known that the non-increasing sequences $j_m, \ldots, j_1$ are Young diagrams of area $w + m$ and length $m$ and we get the original sequences by $i_s = j_s - 1$. For each Young diagram $\{j_m, \ldots, j_1\}$, looking at the ‘multiplicity’ $j_i$ in $\Delta^h \wedge T(M)$, we obtain a sequence $[k_1, k_2, \ldots, k_n]$ consisting of $k_i = \#\{j_s = i\}$.

**Remark 3.3 (3 ways of Young diagram)** A Young diagram $\lambda$ is a non-decreasing sequence of positive integers, say $a_1, \ldots, a_m$. For instance, $\begin{array}{} \lambda \end{array}$ is a sequence of 4, 1, 1, here we denote it as $^\lambda(4, 1, 1)$ where superscript $t$ means “traditional expression”. As explained above, when we focus to multiplicity of elements, we have another sequence, in the concrete example above, 2, 0, 0, 1 and we denote it by $[2, 0, 0, 1]$. Sometimes we have to write many 0 in this expression. The 3rd expression of Young diagram is measuring the height of each column from left to right. Again in the concrete example, we have a sequence 3, 1, 1, 1 and denote it by $\langle 3, 1, 1, 1 \rangle$ and call it tower (vertical) decomposition. It is known in general that the sequence of tower decomposition of $\lambda$ is just the conjugate of $\lambda$, i.e, $\langle \lambda \rangle = ^\lambda$(conjugate of $\lambda$). In detail of relations of those, refer to [6].

**Remark 3.4** We remark that $m$ does not stop at $\dim M$ in general because of property of our new “wedge product” $\Delta$.

### 4 Euler number of homology groups of concrete pre Lie superalgebras

In the previous section, we have pre Lie superalgebras for each differentiable manifold $M$. In this section, we consider the Euclidean space $M = \mathbb{R}^n$ with the Cartesian coordinates $x_1, \ldots, x_n$. Then, we get a pre Lie super subalgebra consisting of multi vector fields of polynomial coefficients. We define

$$g_{i,j} = \mathcal{X}^{i+1}_{j+1}(\mathbb{R}^n) = \{(i + 1)\text{-multi vector fields with } (j + 1)\text{-homogeneous polynomials}\}.$$  

We see easily that $[g_{i_1,j_1}, g_{i_2,j_2}] \subset g_{i_1+i_2,j_1+j_2}$ and so we get a double-weighted pre Lie superalgebra. The spaces $g_{i,j}$ are finite dimensional, precisely $\dim g_{i,j} = \binom{n-1+j+1}{n-1} \binom{n}{i+1}$, and we study each component of $C_{m,w,h}$ in the next subsection.

#### 4.1 Double weighted chain space $C_{m,w,h}$

**Proposition 4.1** The chain space $C_{m,w,h} = \sum \mathcal{X}^{i_1}_{h_1}(\mathbb{R}^n)\Delta \cdots \Delta \mathcal{X}^{i_m}_{h_m}(\mathbb{R}^n)$ of the double-weighted pre Lie superalgebra above is characterized as follows:

1. $(i_s)_{s=1}^{m}$ are non-descending sequences of sum $w + m$ and length $m$. Since each entry is less than $n + 1$, we may count the multiplicity of them as follows: $[k_1, \ldots, k_n]$ where $k_a = \#\{s \mid i_s = a\}$ or denote it by $a:k_a$. 

$$ (i_1, \ldots, i_m) = (\underbrace{1, \ldots, 1}_{k_1}, \ldots, \underbrace{n, \ldots, n}_{k_n}) = (1: k_1, \ldots, n: k_n) = [k_1, \ldots, k_n].$$
we have
\[ \sum_{s=1}^{n} k_s = m, \quad \text{and} \quad \sum_{s=1}^{n} sk_s = w + m. \]

Now denote \( \mathcal{X}^{i_1}_{h_1}(\mathbb{R}^n) \Delta \cdots \Delta \mathcal{X}^{i_m}_{h_m}(\mathbb{R}^n) \) by \( \mathcal{X}^{(i_1, \ldots, i_m)}_{(h_1, \ldots, h_m)} = \mathcal{X}^{[k_1, \ldots, k_m]}_{(h_1, \ldots, h_m)}. \)

2. Each \( h_s \) is non-negative integer and \( \sum_{s=1}^{m}(h_s - 1) = h \), and so \( \sum_{s=1}^{m}(h_s + 1) = h + 2m \). Consider the sequences of Young diagrams of area \( w + 2m \) and length \( m \). Since those \( \{h_s + 1\} \) are not necessarily non-decreasing, we need to get all permutations of them, then shift 1 negatively simultaneously.

3. Assume \( i_p-1 < i_p = \cdots = i_q = k < i_{q+1} \). Then we may relabel so that \( h_p \leq \cdots \leq h_q \), we write
\[
\mathcal{X}^{i_p}_{h_p}(\mathbb{R}^n) \Delta \cdots \Delta \mathcal{X}^{i_q}_{h_q}(\mathbb{R}^n) = \text{SubC}^{(k:(q-p+1))}(h_p, \ldots, h_q)
\]

4. Assume \( i_p = \cdots = i_q \) and \( h_p = \cdots = h_q \). Then
\[
\mathcal{X}^{i_p}_{h_p}(\mathbb{R}^n) \Delta \cdots \Delta \mathcal{X}^{i_q}_{h_q}(\mathbb{R}^n) = \text{SubC}^{(i_p:(q-p+1))}(h_p, \ldots, h_p) = \Delta^{q-p+1} \mathcal{X}^{i_p}_{h_p}(\mathbb{R}^n)
\]

- If \( i_p \) is even, then \( \Delta^{q-p} \mathcal{X}^{i_p}_{h_p}(\mathbb{R}^n) \) is a symmetric algebra and its dimension is
\[
\binom{n}{i_p} \binom{n-i_p+1}{q-p+1} \quad q-p+1
\]

- If \( i_p \) is odd, then \( \Delta^{q-p+1} \mathcal{X}^{i_p}_{h_p}(\mathbb{R}^n) \) is a skew-symmetric algebra and its dimension is
\[
\binom{n}{i_p} \binom{n-i_p+1}{q-p+1} \quad q-p+1
\]

In particular, if \( q-p+1 > \binom{n}{i_p} \binom{n-i_p+1}{q-p+1} \) then the algebra is 0-dimensional.

Introducing a new notation
\[
\text{SubC}^{(k:\ell)}_{[u]} = \bigoplus_{\sum_{s=1}^{\ell}(h_s-1) = u} \text{SubC}^{(k:\ell)}(h_1, \ldots, h_s),
\]
and using the notations in Proposition 4.1, we have

**Corollary 4.1**
\[
\text{C}_{m,w,h} = \sum_{\sum_{s=1}^{n}k_s = m} \sum_{\sum_{s=1}^{n}i_k = w+m} \sum_{\sum_{s=1}^{n}u_l = h} \text{SubC}^{(1:k_1)}_{[u_1]} \Delta \cdots \Delta \text{SubC}^{(n:k_n)}_{[u_n]}
\]

**Proposition 4.2** Assume \( k \) is an odd integer. Let \( [\ell_1, \ell_2, \ldots] \) be the sequence of multiplicities of \( h_1+1, \ldots, h_m+1 \), where \( \ell_b = \#\{i \mid h_i + 1 = b\} \). Then
\[
\text{SubC}^{(k:m)}(h_1, \ldots, h_m) = \Delta^{\ell_1} \mathcal{X}^{k}_{0} \Delta^{\ell_2} \mathcal{X}^{k}_{1} \Delta \cdots
\]
holds. If an inequality \( \ell_i \leq \dim \mathfrak{x}^k_{i-1} = \binom{n}{k} \binom{n-1+i-1}{n-1} \) holds for each \( i \), then \( \text{SubC}^{(k;m)}(h_1, \ldots, h_m) \) is non trivial, and whose dimension is

\[
\prod_i \binom{n}{\ell_i} \binom{n-1+i-1}{n-1}.
\]

**Proof:** Since \( k \) is odd, each algebra \( \Delta^j \mathfrak{x}^k_{i-1} \) is skew-symmetric and the proposition holds comparing the dimension of \( \mathfrak{x}^k_{i-1} \).

The requirements of the chain space \( C_{m,w,h} \) in (4.2) are \( w = \sum_{s=1}^m (s-1)k_s \) and \( \sum_{s=1}^m k_s = m \) for the first weight. Thus, if \( w = 0 \), then \( k_1, k_2, \ldots \) \( = \{m, 0, \ldots\} \) or if \( w = 1 \), then \( k_1, k_2, \ldots \) \( = \{m-1, 1, 0, \ldots\} \). If \( w = 2 \), then \( k_1, k_2, \ldots \) \( = \{m-2, 0, \ldots\} \) or \( m-1, 0, 1, 0, \ldots\).

From the definition of the second weight \( h \), we see that \( \sum s h_s = m + h \), and so \( m \geq h \), in more precise, \( m \geq \max(-h, 1) \). About upper bound of the range \( m \), we discuss later.

### 4.1.1 The first weight \( w = 0 \) case

In this subsection, we assume \( w = 0 \). Then the algebra is just Lie algebra and we see that

\[
\mathcal{C}_{m,0,h} = \text{SubC}^{(1;m)}_{[h]} = \sum_{\sum \ell_t = m} \Delta^t \mathfrak{x}_0^1 \Delta^\ell \mathfrak{x}_1^1 \Delta \cdots.
\]

We have some restrictions from the proposition 4.2.

**Proposition 4.3** Assume \( w = 0 \) and \( \mathcal{C}_{m,0,h} \neq 0 \). Then \( h \geq -\dim \mathfrak{x}_0^1 \), and also \( \max(1,-h) \leq m \leq h + 2 \dim \mathfrak{x}_0^1 + \dim \mathfrak{x}_1^{1} \) holds.

**Proof:** We follow the notation above, then

\begin{align*}
(4.3) & \quad \sum t \ell_t = m \\
(4.4) & \quad \sum t \ell_t = 2m + h
\end{align*}

Since \( (4.4) - 2(4.3) \), we have \( -\ell_1 + \sum_{s>2} (s-2) \ell_s = h \), and \( \sum_{s>2} (s-2) \ell_s = h + \ell_1 \), thus we have \( 0 \leq \ell_1 \).

Applying the requirement \( \ell_1 \leq \dim \mathfrak{x}_0^1 \), we have \( 0 \leq h + \dim \mathfrak{x}_0^1 \).

From \( (4.4) - 3(4.3) \), we have \( -2\ell_1 - \ell_2 + \sum_{s>3} (s-3) \ell_s = -m + h \), thus \( 0 \leq \sum_{s>3} (s-3) \ell_s = -m + h + 2\ell_1 + \ell_2 \). Applying the requirement \( \ell_2 \leq \dim \mathfrak{x}_1^1 \), we have \( m - h \leq 2 \dim \mathfrak{x}_0^1 + \dim \mathfrak{x}_1^{1} \).

**Remark 4.1** In the previous proposition, we have an upper bound of \( m \). If we use the third or more higher comparison, we have more sharp estimate of upper bound of \( m \).

**Example 4.1** Assume \( n = 2 \) for simplicity and we study the chain space

\[
\mathcal{C}_{m,0,h} = \sum_{\sum \ell_t = m} \Delta^t \mathfrak{x}_0^1 \Delta^\ell \mathfrak{x}_1^1 \Delta \cdots.
\]

Assume \( h = -2 \). Then \( m \) starts from 2. The possible Young diagrams are characterized by area \( 2m - 2 \) and length \( m \). We see that the Young diagram \( \langle m, m - 2 \rangle = [1^2, 2^{m-2}] \) is only candidate for our chain space. Thus \( \mathcal{C}_{m,0,-2} = \Delta^2 \mathfrak{x}_0^1(\mathbb{R}^2) \Delta^{m-2} \mathfrak{x}_1^1(\mathbb{R}^2) \) and we get dimension for each space as follows: the Euler number is 0.
Assume $h = -1$. The area is $2m - 1$, and the good Young diagrams are $\langle m,m - 1 \rangle$ or $\langle m,m - 2,1 \rangle$ and so $[1^1, 2^{m-1}]$ or $[1^2, 2^{m-3}, 3^1]$. Thus

$$C_{m,0,-1} = x_0^1(\mathbb{R}^2)\Delta^{m-1}x_1^1(\mathbb{R}^2) \oplus \Delta^2x_0^1(\mathbb{R}^2)\Delta^{m-3}x_1^1(\mathbb{R}^2)\Delta x_2^1(\mathbb{R}^2).$$

So we get dimension for each space as follows: the Euler number is 0.

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|---|
| dim | 2 | 8 | 18 | 32 | 38 | 24 |
| dim $\partial$ | 2 | 6 | 12 | 20 | 18 | 6 |
| Betti | 0 | 0 | 0 | 0 | 0 | 0 |

Assume $h = 0$. The area is $2m$ and good Young diagrams are $\langle m,m \rangle$, $\langle m,m - 1,1 \rangle$, $\langle m,m - 2,1,1 \rangle$ or $\langle m,m - 2,2 \rangle$ and so $[1^0, 2^m]$, $[1^1, 2^{m-2}, 3^1]$, $[1^2, 2^{m-3}, 4^1]$ or $[1^2, 2^{m-4}, 3^2]$. Thus

$$C_{m,0,0} = \Delta^m x_1^1(\mathbb{R}^2) \oplus x_0^1(\mathbb{R}^2)\Delta^{m-2}x_1^1(\mathbb{R}^2)\Delta x_2^1(\mathbb{R}^2) \oplus \Delta^2x_0^1(\mathbb{R}^2)\Delta^{m-3}x_1^1(\mathbb{R}^2)\Delta x_2^1(\mathbb{R}^2) \oplus \Delta^2x_0^1(\mathbb{R}^2)\Delta^{m-4}x_1^1(\mathbb{R}^2)\Delta^2 x_2^1(\mathbb{R}^2).$$

When $h = 0$, zero-th chain space is defined and $C_{0,0,0} = \mathbb{R}$. Thus, the dimension for each space and the rank of $\partial$ are as follows: the Euler number is 0.

| $m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| dim | 1 | 4 | 18 | 60 | 120 | 156 | 134 | 68 | 15 |
| dim $\partial$ | 0 | 4 | 14 | 46 | 74 | 80 | 54 | 13 | 0 |
| Betti | 1 | 0 | 0 | 0 | 0 | 2 | 0 | 1 | 2 |

Assume $h = 1$. The possible Young diagrams have area $2m + 1$ and length $m$ and decompositions are $\langle m,m \rangle$, $\langle m,m - 1,1 \rangle$, $\langle m,m - 1,2 \rangle$, $\langle m,m - 2,1,1 \rangle$, $\langle m,m - 2,2 \rangle$, $\langle m,m - 2,3 \rangle$ and so $[1^0, 2^{m-1}, 3^1]$, $[1^1, 2^{m-2}, 3^0, 4^1]$, $[1^1, 2^{m-3}, 3^2]$, $[1^2, 2^{m-3}, 3^0, 4^0, 5^1]$ or $[1^2, 2^{m-4}, 3^1, 4^1]$, $[1^2, 2^{m-5}, 3^3]$. Thus

$$C_{m,0,1} = \Delta^0x_1^1(\mathbb{R}^2)\Delta^{m-1}x_1^1(\mathbb{R}^2)\Delta x_2^1(\mathbb{R}^2) \oplus x_0^1(\mathbb{R}^2)\Delta^{m-2}x_1^1(\mathbb{R}^2)\Delta^0 x_2^1(\mathbb{R}^2)\Delta x_3^1(\mathbb{R}^2) \oplus \Delta^2x_0^1(\mathbb{R}^2)\Delta^{m-3}x_1^1(\mathbb{R}^2)\Delta x_2^1(\mathbb{R}^2) \oplus \Delta^2x_0^1(\mathbb{R}^2)\Delta^{m-4}x_1^1(\mathbb{R}^2)\Delta^2 x_2^1(\mathbb{R}^2) \oplus \Delta^2x_0^1(\mathbb{R}^2)\Delta^{m-5}x_1^1(\mathbb{R}^2)\Delta^3 x_2^1(\mathbb{R}^2).$$

The dimension for each space is as follows: the Euler number is 0.

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|---|---|---|---|---|---|---|---|---|
| dim | 6 | 40 | 140 | 328 | 522 | 544 | 352 | 128 | 20 |
| dim $\partial$ | 6 | 34 | 106 | 222 | 300 | 244 | 108 | 20 | 0 |
| Betti | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
We discussed in [6] the Euler number of Lie algebra homology groups of given weight \( w \) and homogeneity \( h \) of Poisson tensor where we dealt Young diagrams of area \( w + (2 - h)m \) with length \( m \). By discussion there, we have next result.

**Lemma 4.2** For general \( n \), the Euler number of chain complex \( \{C_{\star,0,h}\} \) is 0.

**Proof:** We use the notations in [6]. Since

\[
C_{m,0,h} = \sum_{\sum_t \ell_t = m, \sum_t t \ell_t = 2m+h} \Delta_{\ell_1} x_0^1 \Delta_{\ell_2} x_1^1 \Delta \cdots ,
\]

we have to deal with Young diagrams \( \nabla_{m}^{2m+h} \) of area \( 2m + h \) with length \( m \). A recursive formula

\[(4.5) \quad \nabla_{m}^{2m+h} = B \cdot \nabla_{m-1}^{2m+h-1} \sqcup T_m \nabla_{m}^{m+h} \]

holds. If \( h = 0 \) then we have

\[
\nabla_{m}^{2m} = B \cdot \nabla_{m-1}^{2m-1} \sqcup T_m \nabla_{m}^{m+h} = T_m \cdot T_m \cup B \cdot \nabla_{m-1}^{2m-1} .
\]

Thus, \( \dim C_{m,0,0} = (\dim x_1^1) + \sum_{\lambda \in \nabla_{m-1}^{2m-1}} \dim(B \cdot \lambda) \).

When we denote each \( \lambda \in \nabla_{m-1}^{2m-1} \) by \( [\ell_1, \ell_2, \ldots] \), \( \sum \ell_t = m - 1 \) and \( \sum t \ell_t = 2m - 1 \) must be satisfied.

\( B \cdot \lambda = [1 + \ell_1, \ell_2, \ldots] \) and so \( \dim(B \cdot \lambda) = (\dim x_1^1) (\dim x_1^1) (\dim x_1^1) \ldots \).

We easily see that \( \sum_{m>0} (-1)^m (\dim x_1^1) = -1. \) About the alternating sum of the second term, we have

\[
\sum_{m>0} (-1)^m \sum_{\lambda \in \nabla_{m-1}^{2m-1}} \dim(B \cdot \lambda)
\]

\[
= \sum_{m>0} (-1)^m \sum_{\sum_t \ell_t = m-1, \sum_t t \ell_t = 2m-1} (\dim x_1^1) (\dim x_1^1) \ldots
\]

\[
= \sum_{m>0} (-1)^{1+\sum \ell_t} \sum_{2(1+\sum \ell_t) = 1+\sum \ell_t} (\dim x_1^0) (\dim x_1^1) (\dim x_1^1) \ldots
\]

\[
= \sum_{m>0} (-1)^{\ell_2} \sum_{2(1+\sum \ell_t) = 1+\sum \ell_t} (-1)^{1+\sum \ell_t} (\dim x_1^0) (\dim x_1^1) \ldots
\]

\[
= 0 \quad \text{because} \quad \ell_2 \text{ is free in the condition } 2(1+\sum \ell_t) = 1+\sum \ell_t .
\]

So, \( \sum_{m>0} (-1)^m C_{m,0,0} = \sum_{m>0} (-1)^m (\dim x_1^1) = -1. \) When \( h = 0 \), then 0-th chain space \( C_{0,0,0} \) is defined and trivially 1-dimensional. Thus, the Euler number \( \sum_{m>0} (-1)^m C_{m,0,0} = 0. \)

When \( h < 0 \) then \( (4.5) \) says that \( \nabla_{m}^{2m+h} = B \cdot \nabla_{m-1}^{2m+h-1} \) and we follow the same discussion about \( \dim(B \cdot \lambda) \) and get the conclusion that the Euler number is 0.

When \( h > 0 \) then \( (4.5) \) says that \( \nabla_{m}^{2m+h} = B \cdot \nabla_{m-1}^{2m+h-1} \sqcup T_m \nabla_{m}^{m+h} \) and we know the alternating sum is 0 about the first term. Concerning the second term, take an arbitrary element \( \lambda = [\ell_1, \ell_2, \ldots] \in \nabla_{m}^{m+h} \)
Proposition 4.4

The chain complex $\text{SubC} \wedge [w]$ with the conditions $\sum_s \ell_s = m$ and $\sum_s s \ell_s = m + h$. Then $T_m \cdot \lambda = [0, \ell_1, \ell_2, \ldots]$ and so

$$\begin{align*}
\sum_m (-1)^m \sum_{\ell \in \nabla_m} \dim(T_m \cdot \lambda) = \\
\sum_m (-1)^m \sum_{\ell \in \nabla_m} (\dim x^1_{\ell_1}) (\dim x^1_{\ell_2}) \cdots = \\
\sum_s \sum_{\ell \in \nabla_s} (-1)^{s+1} \ell_s (\dim x^1_{\ell_1}) (\dim x^1_{\ell_2}) \cdots \\
= 0.
\end{align*}$$

4.1.2 The first weight $w = 1$ case

Assume $w = 1$. Then using Corollary 4.1 directly, we have

$$C_{m,1,h} = \sum_{s \in \Delta X} \mathcal{S}^{s} = \sum_{h,m} \chi^{[m-1,1,0,...]}_{(h_1,...,h_m)} = \sum_{h,m} \text{SubC}^{(1:(m-1))}_{[h+1-h_m]} \Delta \text{SubC}^{(2:1)}_{[h]}$$

and $\text{SubC}^{(1:(m-1))}_{[h+1-h_m]}$ is just $C_{m-1,0,h+1-h_m}$. Thus, we have next proposition which gives a rule of expression of $C_{m,1,h}$ by lower weight chain spaces $\text{C}_{m-1,0,h'}$.

Proposition 4.4 The chain complex $\{C_{*,1,h}\}$ is non-trivial if $h \geq -1 \dim X^1_0$, and

$$C_{m,1,h} = \sum_{h'} C_{m-1,0,h-h'+1} \Delta x^2_{h'}$$ for $m \geq 1$.

Each degree $m$ of the chain complex is upper bounded by $h + 2 + 2 \dim X^1_0 + \dim X^1_1$.

For $\mathbb{R}^n$ of general $n$, the Euler number of the chain complex $\{C_{*,1,h}\}$ is always 0 for each $h$.

Proof: (4.6) implies $\dim C_{m,1,h} = \sum_{h'} \dim C_{m-1,0,h-h'+1} \dim x^2_{h'}$ for $m \geq 1$.

$$\begin{align*}
\sum_m (-1)^m \dim C_{m,1,h} = & \sum_{m \geq 1} (-1)^m \sum_{h'} \dim C_{m-1,0,h-h'+1} \dim x^2_{h'} \\
= & - \sum_{h'} \dim x^2_{h'} \sum_m (-1)^m \dim C_{m-1,0,h-h'+1} \\
= & 0 \text{ using Lemma 4.2}.
\end{align*}$$

4.1.3 The first weight $w = 2$ case

Assume $w = 2$. Again, using Corollary 4.1, we have

$$C_{m,2,h} = \sum_{s \in \Delta X} \mathcal{S}^{s} = \sum_{h,m} \chi^{[m-1,0,1,0,...]}_{(h_1,...,h_m)} + \sum_{h,m} \chi^{[m-2,2,0,...]}_{(h_1,...,h_m)}$$

$$= \sum_{h,m} \text{SubC}^{(1:(m-1))}_{[h+1-h_m]} \Delta x^3_{h_m} + \sum_{h,m} \text{SubC}^{(1:(m-2))}_{[h-h']} \Delta \text{SubC}^{(2:2)}_{[h']}.$$
Thus, we have the next proposition which gives a rule of expression of \( C_{m,2,h} \) by lower weight chain spaces.

**Proposition 4.5** The chain complex \( \{C_{\bullet,2,h}\} \) is non-trivial if \( h \geq -(2 + \dim \chi_{1}^{1}) \), and

\[
C_{m,2,h} = \sum_{h'} C_{m-1,0,h+1-h'} \Delta \chi_{h'}^{3} \cup \sum_{a \leq b} C_{m-2,0,h+2-a-b} \Delta \chi_{a}^{2} \Delta \chi_{b}^{2} \quad \text{for} \quad m \geq 2,
\]

and

\[
C_{1,2,h} = \chi_{h+1}^{3}.
\]

The range of degree \( m \) of the chain complex has an upper bound \( h + 4 + 2 \dim \chi_{1}^{1} + \dim \chi_{1}^{1} \).

We can apply Lemma 4.2 for the chain complex \( w = 2 \), we have

**Proposition 4.6** For general \( n \), the Euler number of chain complex \( \{C_{\bullet,2,h}\} \) is always 0 for each \( h \).

**Proof:** We only alternating sum up the dimension of chain spaces. We first sum up the terms which involve \( \chi_{3}^{3} \) as follows:

\[
A = (-1)^{1} \dim \chi_{h+1}^{3} + \sum_{m \geq 2} (-1)^{m} \sum_{h'} \dim C_{m-1,0,h+1-h'} \dim \chi_{h'}^{3} = - \sum_{h'} \dim \chi_{h'}^{3} \sum_{m \geq 0} (-1)^{m} \dim C_{m,0,h+1-h'} = 0 \quad \text{using Lemma 4.2.}
\]

The rest is

\[
B = \sum_{m \geq 2} (-1)^{m} \sum_{a \leq b} \dim C_{m-2,0,h+2-a-b} \dim(\chi_{a}^{2} \Delta \chi_{b}^{2}) = \sum_{a \leq b} \dim(\chi_{a}^{2} \Delta \chi_{b}^{2}) \sum_{m \geq 2} (-1)^{m-2} \dim C_{m-2,0,h+2-a-b} = 0 \quad \text{using again Lemma 4.2.}
\]

\[\square\]

### 4.1.4 General first weight case

Inspired by Propositions 4.4 and 4.6, we have the next general result including those results.

**Theorem 4.1** For general \( n \), the Euler number of chain complex \( \{C_{\bullet,w,h}\} \) is 0 for each \( w \) and \( h \).

**Proof:** We have already seen that it is true for \( w = 0, 1, 2 \). So we may assume \( w > 2 \) and \( m > 0 \). We use the notation (4.1). From Corollary 4.1, we have the chain space is written by

\[
C_{m,w,h} = \bigoplus_{\sum_{i=1}^{n} k_{i} = m} \bigoplus_{\sum_{i=1}^{n} u_{i} = h} \Delta \cdots \Delta \bigoplus_{\sum_{i=1}^{n} v_{i} = w+m} \Delta \bigoplus_{\sum_{i=1}^{n} v_{i} = h} \text{SubC}^{(n;k_{1})}_{[u_{1}]} \Delta \cdots \Delta \text{SubC}^{(n;k_{n})}_{[u_{n}]}.
\]
In this subsection, we first study natural representation of \( M \). Since the Schouten bracket of \( g \) and \( X^0(M) = C^\infty(M) \) lies in \( X^0(M) \oplus \cdots \oplus X^{n-1}(M) \), we regard \( g \) acts on \( X^0(M) = C^\infty(M) \) by

\[
U \cdot f = [U, f] \mod g.
\]

Actually, if \( U \in X^1(M) \) then \( U \cdot f = [U, f] = \langle U, df \rangle \) and if \( U \in X^i(M) \) then \( U \cdot f = 0 \) for \( i > 1 \). So we have a representation space \( V = X^0(M) = C^\infty(M) \) of \( g \) and the action.

Now we consider \( M = \mathbb{R}^n \) and we may study relative homology groups of the chain spaces \( \Delta^n g \otimes V \) with the boundary operator \( \partial V \) as introduced in the section 2.

Now we introduce double-weighted chain spaces using the specialty of our base space \( M = \mathbb{R}^n \). The first component \( \text{SubC}^{(1:k_i)}_{[u_1]} \) is equal to the chain space \( C_{k_1,0,u_1} \) with the first weight 0. Thus

\[
\sum_{m>0} (-1)^m \dim C_{m,w,h} = \sum_{\sum_{i=1}^n (i-1)k_i = w} (-1)^{\sum_{i=1}^n k_i} \sum_{u_j} \dim C_{k_1,0,u_1} \dim \left( \text{SubC}^{(2:k_2)}_{[u_2]} \Delta \cdots \Delta \text{SubC}^{(n:k_n)}_{[u_n]} \right)
\]

we used that the condition \( \sum_{i=1}^n (i-1)k_i = w \) does not involve \( k_1 \), and now we use Lemma 4.2

\[
\sum_{\sum_{i=1}^n (i-1)k_i = w} (-1)^{\sum_{i=1}^n k_i} \sum_{u_j} 0 = 0.
\]

\[
\Box
\]

### 4.1.5 Extended case

\( X^0(M) \oplus X^1(M) \oplus \cdots \oplus X^n(M) \) is also a pre Lie superalgebra including \( X^1(M) \oplus \cdots \oplus X^n(M) \) which we dealt so far.

In this subsection, we first study natural representation of \( g = X^1(M) \oplus \cdots \oplus X^n(M) \) for general manifold \( M \). Since the Schouten bracket of \( g \) and \( X^0(M) = C^\infty(M) \) lies in \( X^0(M) \oplus \cdots \oplus X^{n-1}(M) \), we regard \( g \) acts on \( X^0(M) = C^\infty(M) \) by

\[
U \cdot f = [U, f] \mod g.
\]

Actually, if \( U \in X^1(M) \) then \( U \cdot f = [U, f] = \langle U, df \rangle \) and if \( U \in X^i(M) \) then \( U \cdot f = 0 \) for \( i > 1 \). So we have a representation space \( V = X^0(M) = C^\infty(M) \) of \( g \) and the action.

Now we consider \( M = \mathbb{R}^n \) and we may study relative homology groups of the chain spaces \( \Delta^n g \otimes V \) with the boundary operator \( \partial V \) as introduced in the section 2.

Now we introduce double-weighted chain spaces using the specialty of our base space \( M = \mathbb{R}^n \). The
chain spaces are given by

\[
\overline{C}_{m,w,h} = \sum_{\sum_{j=1}^{k} (i_j - 1) = w} \chi^{i_1}_{h_1}(\mathbb{R}^n) \Delta \cdots \Delta \chi^{i_k}_{h_k}(\mathbb{R}^n) \otimes \chi^0_{h_0}(\mathbb{R}^n) \quad (m \geq 0)
\]

\[
= \sum_{h_0} C_{m,w,h+1-h_0} \otimes \chi^0_{h_0}(\mathbb{R}^n) \quad (m \geq 0),
\]

where \( C_{*,w,h} \) are the chain spaces in the trivial module. We easily see the next proposition.

**Proposition 4.8** The double-weight is invariant by \( \partial_V \), i.e., \( \partial_V(\overline{C}_{m,w,h}) \subset \overline{C}_{m-1,w,h} \). Thus, we have the double-weighted homology groups \( H_{m,w,h}(g,V) \) with \( g \)-module \( V \).

**Proof:** We know that \([\chi^{i_1}_{h_1}, \chi^{i_r}_{h_r}] \subset \chi^{i_1+\cdot+\cdot+\cdot+i_r}_{h_1+\cdot+\cdot+\cdot+h_r} \) and in particular, \([\chi^0_{h_0}, \chi^0_{h_0}] \subset \chi^{i_1-1}_{h_1} \). Thus, we directly see that the double-weight of the first part \([\chi^{i_1}_{h_1}, \chi^{i_q}_{h_q}] \Delta \cdots \Delta \chi^{i_q}_{h_q} \otimes \chi^0_{h_0} \) does not change. About the second part, \( \chi^{i_1}_{h_1} \Delta \cdots \Delta \chi^{i_q}_{h_q} \otimes [\chi^{i_p}_{h_p}, \chi^0_{h_0}] \) is 0 if \( i_p \neq 1 \). When \( i_p = 1 \), the first weight is \( \sum_{s \neq p} (i_s - 1) = \sum_{s=1}^{m} (i_s - 1) = w \) and the second weight is \( \sum_{s \neq p} (h_s - 1) + (h_p + h_0 - 1 - 1) = \sum_{s=0}^{m} (h_s - 1) = h \).

Thus, \( \sum_{m \geq 0} (-1)^m \text{dim } \overline{C}_{m,w,h} = \sum_{m \geq 0} (-1)^m \sum_{h_0} \text{dim } C_{m,w,h+1-h_0} \text{dim } \chi^0_{h_0} = \sum_{h_0} \text{dim } \chi^0_{h_0} \sum_{m \geq 0} (-1)^m \text{dim } C_{m,w,h+1-h_0} \)

using Theorem 4.1

\[
= \sum_{h_0} \text{dim } \chi^0_{h_0} \times 0 = 0.
\]

**4.3 Example of pre Lie superalgebra related to a Lie superalgebra**

In Example 1.1, we saw toy models of Lie superalgebra and pre Lie superalgebra. We study the chain complexes of those.

\( g = gl(2) = g_0 \oplus g_1 \oplus g_2 \). Take a basis \( u_1 \in g_0, u_2, u_3 \in g_1, u_4 \in g_2 \) with the following bracket relation:
Given a weight \( w \), the chain spaces are given by \( C_{m,w} \) for \( m = w-1, w, w+1 \) and we get dimension, rank and Betti numbers as right above.

Suppose \( i' = i + Ip \) and \( j' = j + Jp \), i.e., \( i \equiv i' \) and \( j \equiv j' \mod p \). Then

\[
i' + j' = i + j + (I + J)p \equiv i + j \mod p
\]

and

\[
i'j' = (i + Ip)(j + Jp) = ij + iJp + jIp + IJp^2 \equiv ij \mod p
\]

but \( i'j' - ij = iJp + jIp + IJp^2 \equiv 0 \mod 2 \) if \( p \) is even. So Lie super algebras should be divided into even number of subspaces. Thus, our example \( g = \mathfrak{gl}(2) = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \) is not Lie superalgebra. The Lie superalgebra in Example 1.1 is sometimes denoted as \( \mathfrak{gl}(1|1) = \mathfrak{g}[0] \oplus \mathfrak{g}[1] \).

\( \mathfrak{g}[0] \) is spanned by \( u_1 \) and \( u_2 \), and \( \mathfrak{g}[1] \) is spanned by \( u_3 \) and \( u_4 \). Those basis satisfy the next bracket relations:

\[
\begin{array}{c|cccc}
\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \\
\hline
\mathbf{u}_1 & 0 & 2\mathbf{u}_2 & -2\mathbf{u}_3 & 0 \\
\mathbf{u}_2 & -2\mathbf{u}_2 & 0 & \mathbf{u}_4 & 0 \\
\mathbf{u}_3 & 2\mathbf{u}_3 & \mathbf{u}_4 & 0 & 0 \\
\mathbf{u}_4 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\mathbf{m} & w-1 & w & w+1 \\
\hline
C_{m,w} & \Delta^{w-2}\mathfrak{g}_1\mathfrak{g}_2 & \Delta^w\mathfrak{g}_1 \oplus \mathfrak{g}_0\Delta^{w-2}\mathfrak{g}_1\mathfrak{g}_2 & \mathfrak{g}_0\Delta^w\mathfrak{g}_1 \\
\hline
\text{dim} & w-1 & 2w & w+1 \\
\text{dim} \partial & w-1 & w+1 & 0 \\
\text{Betti} & 0 & 0 & 0 \\
\end{array}
\]

Depending on the weight to be even or odd, we have

\[
\begin{align*}
C_{2m,[0]} &= \Delta^{2m}\mathfrak{g}[1] \oplus \Delta^2\mathfrak{g}[0]\Delta^{2m-2}\mathfrak{g}[1] , \\
C_{2m,[1]} &= \mathfrak{g}[0]\Delta^{2m-1}\mathfrak{g}[1] , \\
C_{2m+1,[0]} &= \mathfrak{g}[0]\Delta^{2m}\mathfrak{g}[1] , \\
C_{2m+1,[1]} &= \Delta^{2m+1}\mathfrak{g}[1] \oplus \Delta^2\mathfrak{g}[0]\Delta^{2m-1}\mathfrak{g}[1] .
\end{align*}
\]

Denote \( \Delta^a\mathbf{u}_3\Delta^b\mathbf{u}_4 \) by \( F(a,b) \). Then

\[
\begin{align*}
\partial(F(a,b)) &= ab(\mathbf{u}_1 + \mathbf{u}_2)\Delta F(a-1, b-1) \\
\partial(\mathbf{u}_1\Delta F(a,b)) &= -ab\mathbf{u}_1\Delta\mathbf{u}_2\Delta F(a-1, b-1) + (a-b)F(a,b) \\
\partial(\mathbf{u}_2\Delta F(a,b)) &= ab\mathbf{u}_1\Delta\mathbf{u}_2\Delta F(a-1, b-1) - (a-b)F(a,b) \\
\partial(\mathbf{u}_1\Delta\mathbf{u}_2\Delta F(a,b)) &= (a-b)\mathbf{u}_1\Delta\mathbf{u}_2\Delta F(a,b)
\end{align*}
\]


5 Betti numbers of homology groups of concrete pre Lie superalgebras

So far, we studied the chain spaces \( C_{m,w,h} \) for fixed space dimension \( n \) and double weight \((w, h)\). As stated in Remark 3.1, we may find all Poisson structures in the second homology group of pre Lie superalgebra of tangent bundle of \( M \) with the Schouten bracket. Thus, it is interesting to study the second and/or the third homology group. But, it seems hard to attack to general manifold \( M \). So, again we deal with the pre Lie superalgebra of homogeneous polynomial coefficients multi vector fields on \( \mathbb{R}^n \). In this section, we study not only the second Betti number but also Betti numbers of general degree.

In general pre Lie superalgebra theory, recursive formulae of the boundary operator is given as below:

\[
\partial(A_1 \Delta \cdots \Delta A_{m+1}) = \partial(A_1 \Delta \cdots \Delta A_{m}) \Delta A_{m+1} + (-1)^m A_1 \Delta \cdots \Delta A_{m} A_{m+1} + (-1)^{m+1} (A_1 \Delta \cdots \Delta A_{m+1}) A_{m+1}
\]

where

\[
(A_1 \Delta \cdots \Delta A_{m}) A_{m+1} = (A_1 \Delta \cdots \Delta A_{m-1}) [A_m, A_{m+1}] + (-1)^{m} A_1 \Delta \cdots \Delta A_{m-1} A_{m+1} \Delta A_m
\]

\[
\sum_{i=1}^{m} (-1)^{a_{m+1} \sum_{i=1}^{m} a_i} A_1 \Delta \cdots \Delta A_i, A_{m+1} \Delta \cdots \Delta A_m
\]

\[
\partial(A_0 \Delta A_1 \Delta \cdots \Delta A_{m}) = -A_0 \Delta \partial(A_1 \Delta \cdots \Delta A_{m}) + A_0 \cdot (A_1 \Delta \cdots \Delta A_{m})
\]

where

\[
A_0 \cdot (A_1 \Delta \cdots \Delta A_{m}) = [A_0, A_1] \Delta (A_2 \Delta \cdots \Delta A_{m}) + (-1)^{a_{1}a_{2}} A_1 \Delta A_0 \cdot (A_2 \Delta \cdots \Delta A_{m})
\]

\[
\sum_{i=1}^{m} (-1)^{a_{1} \sum_{i=1}^{m} a_i} A_1 \Delta \cdots \Delta A_i [A_0, A_1] \Delta \cdots \Delta A_m
\]

for each homogeneous elements \( A_i \in \mathfrak{g}_{a_i} \). In lower degree, the boundary operator is given as bellows:

\[
\partial(A \Delta B) = [A, B]
\]

\[
\partial(A \Delta B \Delta C) = -A \Delta [B, C] + [A, B] \Delta C + (-1)^{ab} B \Delta [A, C]
\]

for each homogeneous elements \( A \in \mathfrak{g}_a, B \in \mathfrak{g}_b, C \in \mathfrak{g}_c \).

If we will handle Poisson structures on \( \mathbb{R}^n \) by homology theory of pre Lie superalgebra, then Remark 3.1 says we will deal with \( \{C_{\bullet, w=2, h}\} \), where

\[
C_{1,2,h} = \mathfrak{x}_{h+1}^3,
\]

\[
C_{2,2,h} = \sum_{a+b=h+2} \mathfrak{x}_a^1 \mathfrak{x}_b^3 + \sum_{a+b=h+2} \mathfrak{x}_a^2 \mathfrak{x}_b^2,
\]

\[
C_{3,2,h} = \sum_{c+a+b=h+2+1} \mathfrak{x}_c^1 \mathfrak{x}_a^1 \mathfrak{x}_b^3 + \sum_{c+a+b=h+2+1} \mathfrak{x}_c^1 \mathfrak{x}_a^2 \mathfrak{x}_b^2,
\]

\[\vdots\]

Remark 5.1 Since \( C_{m,w,h} = \sum_{\sum_{i=1}^{m} a_i = w+m, \sum_{i=1}^{m} b_i = h+m} \mathfrak{x}_{b_1}^{d_1} \mathfrak{x}_{b_2}^{d_2} \Delta \cdots \Delta \mathfrak{x}_{b_m}^{d_m} \) in general, if \( C_{m,w,h} \neq 0 \) then \( a_i \leq n \)
for each $i$, and so $\sum_{i=1}^{m} a_i \leq mn$. Thus, $w \leq m(n - 1)$. Namely, $w$ is bounded from above by the dimension $n$ and the degree $m$ of the chain space.

Since $M = \mathbb{R}^n$, we have a special vector field $E = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \in \mathfrak{X}_1$ (called Euler vector field). It is known that if $f$ is $h$-homogeneous polynomial, then $[E, f] = hf$. If $D \in \mathfrak{X}_0^p$, then $[E, D] = -pD$.

We have the next lemma in general:

**Lemma 5.1** For each $U \in \mathfrak{X}_h^p$,

(5.9) \[ E \cdot U = [E, U] = (-p + h)U \]

holds. In fact, the action of $E$ is divided into two parts:

(5.10) \[ \sum_{k=1}^{n} x_k[\frac{\partial}{\partial x_k}, U] = hU, \]

and

(5.11) \[ \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \wedge [x_k, U] = -pU. \]

Thus,

(5.12) \[ E \cdot W = (-w + h)W \quad (\forall W \in C_{m,w,h}). \]

**Proof of (5.12):** Using (5.5), we have

\[
E \cdot \sum_{i} A_i^1 \Delta \cdots \Delta A_i^m = \sum_{i} \sum_{k=1}^{m} A_i^1 \Delta \cdots \Delta [E, A_k^i] \Delta \cdots \Delta A_i^m = \sum_{i} \sum_{k=1}^{m} A_i^1 \Delta \cdots \Delta (-w(A_k^i) + \tilde{h}(A_k^i)) \Delta \cdots \Delta A_i^m \\
= \sum_{i} \sum_{k=1}^{m} (-w(A_k^i) + \tilde{h}(A_k^i)) A_i^1 \Delta \cdots \Delta A_k^i \Delta \cdots \Delta A_i^m \\
= \sum_{i} (-w + h) A_i^1 \Delta \cdots \Delta A_k^i \Delta \cdots \Delta A_i^m \\
= (-w + h) \sum_{i} A_i^1 \Delta \cdots \Delta A_k^i \Delta \cdots \Delta A_i^m.
\]

Using Lemma above, we have the next proposition:

**Proposition 5.1** Define a map $\phi : C_{m,w,h} \to C_{m+1,w,h}$ by $\phi(U) = E \Delta U$. Then we have

(5.13) \[ \partial \circ \phi + \phi \circ \partial = (-w + h) id. \]

**Proof:** Take $\forall W \in C_{m,w,h}$.

\[
\partial(\phi W) = \partial(E \Delta W) = E \Delta \partial W + E \cdot W = -\phi(\partial W) + (-w + h)W
\]

Directly from this proposition we have the next theorem.
**Theorem 5.1 (m-th Betti number)** Each $m$-th Betti numbers of $(w, h)$-weighted chain complex $\{C_{*,w,h}\}$ is 0 if $w \neq h$.

**Proof:** Take a general cycle $W \in C_{m,w,h}$. The proposition above yields

$$(-w+h)W = \partial(\phi(W)) + \phi(\partial U) = \partial(\phi(W)) \quad \text{and} \quad W = \frac{1}{-w+h}\partial(E\Delta W) \quad \text{if} \quad w \neq h.$$ 



**Remark 5.2** When $w = h$, Theorem 5.1 says $E\Delta U$ is a cycle if $U$ is a cycle in $C_{m,w,h}$.

**Remark 5.3** We have the table of Betti numbers of $\{C_{*,0,0}\}$ of $\mathbb{R}^2$ in Example 4.1, which shows non-trivial Betti numbers: $b_0 = 1, b_5 = 2, b_7 = 1, b_8 = 2$.

Theorem 5.1 being concerned with $m$-th Betti numbers makes sense for $m = 1$ with assumption $w \neq h$. But we have the next result without any restriction for the first Betti number.

**Theorem 5.2 (1st Betti number)** The first Betti number of $(w, h)$-weighted chain complex $\{C_{*,w,h}\}$ is 0 for each double weight $(w, h)$.

**Proof:** Fix general weight $(w, h)$. $C_{1,w,h} = \mathcal{X}^{w+1}_{h+1}$ and $C_{2,w,h} = \sum_{p+q=2+w, a+b=2+h} x_a^p y_b^q$.

Take $U \in \mathcal{X}^{w+1}_{h+1}$. Then $(\frac{\partial}{\partial x_k})D(x_k U) \in \mathcal{X}_0^1 \mathcal{X}_{h+2}^{n+1} \subset C_{2,w,h}$. Now we see

$$\partial((\frac{\partial}{\partial x_k})D(x_k U)) = [\partial (x_k U)] = U + x_k [\partial (x_k U)]$$

$$\sum_{k=1}^n \partial((\frac{\partial}{\partial x_k})D(x_k U)) = nU + \sum_{k=1}^n x_k [\partial (x_k U)] = (n + 1 + h)U .$$

We have result about the second Betti number when $w = h = 0$.

**Proposition 5.2** The 2nd Betti number is zero when $w = h = 0$.

**Proof:** Since $C_{2,0,0} = \sum_{a+b=0, a \leq b} x_a^1 \Delta x_b^1 = x_0^1 \Delta x_2^1 + x_1^1 \Delta x_1^1$, a general element $T \in C_{2,0,0}$ is given by $T = \sum_i A_i \Delta B_i + \sum_{j<\ell} p^{i,\ell} Y_j \Delta Y_\ell$ where

$$A_i \in \mathcal{X}_0^1, B_i \in \mathcal{X}_2^1, Y_j \in \mathcal{X}_1^1, p^{i,\ell} + p^{\ell,j} = 0$$

and satisfies cycle condition $\partial(\sum_i A_i \Delta B_i + \sum_{j<\ell} p^{i,\ell} Y_j \Delta Y_\ell) = 0$, i.e., $\sum_i [A_i, B_i] + \sum_{j<\ell} p^{i,\ell} [Y_j, Y_\ell] = 0$. Consider

$$\sum_i (\frac{\partial}{\partial x_k})D(A_i \Delta (x_k B_i)) + \sum_{j<\ell} p^{i,\ell} (\frac{\partial}{\partial x_k})D(Y_j \Delta (x_k B_i)) = C_{3,0,0} .$$
$$\partial\left(\sum_i (\frac{\partial}{\partial x_k}) A_i \Delta(x_k B_i) + \sum_{j<\ell} p^{j,\ell}(\frac{\partial}{\partial x_k}) \Delta A_j \Delta(x_k Y_{\ell})\right)$$

$$= - \sum_i (\frac{\partial}{\partial x_k}) \Delta[A_i, x_k B_i] + \sum_i \left(\frac{\partial}{\partial x_k}\right) A_i \Delta(x_k B_i) + \sum_i A_i \Delta(\frac{\partial}{\partial x_k}, x_k B_i)$$

$$- \sum_{j<\ell} p^{j,\ell}(\frac{\partial}{\partial x_k}) \Delta A_j \Delta(x_k Y_{\ell}) + \sum_{j<\ell} p^{j,\ell}(\frac{\partial}{\partial x_k}) \Delta(x_k Y_{\ell}) + \sum_{j<\ell} p^{j,\ell} Y_j \Delta(\frac{\partial}{\partial x_k}, x_k Y_{\ell})$$

$$= - \sum_i (\frac{\partial}{\partial x_k}) \Delta[I_i, x_k B_i] + \sum_i \left(\frac{\partial}{\partial x_k}\right) A_i \Delta(x_k B_i) + \sum_i A_i \Delta(\frac{\partial}{\partial x_k}, x_k B_i)$$

$$- \sum_{j<\ell} p^{j,\ell}(\frac{\partial}{\partial x_k}) \Delta X_j \Delta(x_k Y_{\ell}) + \sum_{j<\ell} p^{j,\ell}(\frac{\partial}{\partial x_k}) \Delta(x_k Y_{\ell}) + \sum_{j<\ell} p^{j,\ell} Y_j \Delta(\frac{\partial}{\partial x_k}, x_k Y_{\ell})$$

from cycle condition, we have

$$= - \sum_i (\frac{\partial}{\partial x_k}) \Delta[I_i, x_k B_i] + \sum_i \left(\frac{\partial}{\partial x_k}\right) A_i \Delta(x_k B_i) + \sum_i A_i \Delta(\frac{\partial}{\partial x_k}, x_k B_i)$$

$$- \sum_{j<\ell} p^{j,\ell}(\frac{\partial}{\partial x_k}) \Delta X_j \Delta(x_k Y_{\ell}) + \sum_{j<\ell} p^{j,\ell}(\frac{\partial}{\partial x_k}) \Delta(x_k Y_{\ell}) + \sum_{j<\ell} p^{j,\ell} Y_j \Delta(\frac{\partial}{\partial x_k}, x_k Y_{\ell}) + \sum_{j<\ell} p^{j,\ell} Y_j \Delta(\frac{\partial}{\partial x_k}, x_k Y_{\ell})$$

since $[A_i, x_k]$ are constant number, we have

$$= - \sum_i [A_i, x_k]\left(\frac{\partial}{\partial x_k}\right) \Delta B_j + 0 + \sum_i A_j \Delta(\frac{\partial}{\partial x_k}, x_k B_j)$$

$$- \sum_{j<\ell} p^{j,\ell}(\frac{\partial}{\partial x_k}) \Delta X_j \Delta(x_k Y_{\ell}) + \sum_{j<\ell} p^{j,\ell}(\frac{\partial}{\partial x_k}) \Delta(x_k Y_{\ell}) + \sum_{j<\ell} p^{j,\ell} Y_j \Delta(\frac{\partial}{\partial x_k}, x_k Y_{\ell}) \cdot$$

Now, summing up by $k$, we have

$$\sum_k \partial\left(\sum_i (\frac{\partial}{\partial x_k}) A_i \Delta(x_k B_i) + \sum_j (\frac{\partial}{\partial x_k}) \Delta A_j \Delta(x_k Y_{\ell})\right)$$

$$= - \sum_i A_i \Delta B_i + \sum_i A_i \Delta(B_i + x_k[\left(\frac{\partial}{\partial x_k}, B_i\right)])$$

$$+ \sum_{j,k} (\frac{\partial}{\partial x_k}) \Delta([X_j, x_k Y_{\ell}] + [\left(\frac{\partial}{\partial x_k}, Y_j\right) \Delta(x_k Y_{\ell}) + Y_j \Delta(Y_{\ell} + x_k[\left(\frac{\partial}{\partial x_k}, Y_{\ell}\right)])$$

$$= - \sum_i A_i \Delta B_i + \sum_i A_i \Delta((n + 2) B_i)$$

$$- \sum_{j<\ell,k} p^{j,\ell}(\frac{\partial}{\partial x_k}) \Delta([X_j, x_k Y_{\ell}] + \sum_{j<\ell,k} p^{j,\ell}(\left(\frac{\partial}{\partial x_k}, Y_j\right) \Delta(x_k Y_{\ell}) + \sum_{j<\ell} p^{j,\ell} Y_j \Delta((n + 1) Y_{\ell})$$

$$=(n + 1)(\sum_i A_i \Delta B_i + \sum_{j<\ell} p^{j,\ell} Y_j \Delta Y_{\ell})$$

$$= - \sum_{j<\ell,k} p^{j,\ell}(\frac{\partial}{\partial x_k}) \Delta([X_j, x_k Y_{\ell}] + \sum_{j<\ell,k} p^{j,\ell}(\left(\frac{\partial}{\partial x_k}, Y_j\right) \Delta(x_k Y_{\ell}) \cdot$$

We show the sum of the last two terms is zero as follows: Since $X^1 \ni Y_j = \sum_{k,\ell} Y_{k,\ell} x_{\ell}(\frac{\partial}{\partial x_k})$ where
\[ Y^{k,\ell}_j \text{ are constant.} \]

\[ [Y_j, x_k]Y_\ell = \sum_i Y^{k,\ell}_j x_i Y_\ell, \quad [(\frac{\partial}{\partial x_k}), Y_j] = \sum_s [(\frac{\partial}{\partial x_k}), Y^s_j] (\frac{\partial}{\partial x_s}) = \sum_s Y^{s,k}_j (\frac{\partial}{\partial x_s}) \]

we get 2nd term + 3rd term is

\[ -\sum_{j<\ell,t,k} p^{j,\ell}_t (\frac{\partial}{\partial x_k}) \Delta (\sum_i Y^{k,\ell}_j x_i Y_\ell) + \sum_{j<\ell,t,k} p^{j,\ell}_t \sum_s Y^{s,k}_j (\frac{\partial}{\partial x_s}) \Delta (x_k Y_\ell) \]

\[ = -\sum_{j<\ell} \sum_k (\frac{\partial}{\partial x_k}) \Delta (\sum_i Y^{k,\ell}_j x_i Y_\ell) + \sum_{j<\ell} \sum_s Y^{s,k}_j (\frac{\partial}{\partial x_s}) \Delta (x_k Y_\ell) \]

we get

\[ = -\sum_{j<\ell} \sum_k (\frac{\partial}{\partial x_k}) \Delta (\sum_i Y^{k,\ell}_j x_i Y_\ell) + \sum_{j<\ell} \sum_s (\frac{\partial}{\partial x_s}) \Delta (Y^{s,k}_j x_i Y_\ell) \]

\[ = 0. \]

\[ \blacksquare \]

**Remark 5.4** We expect to know the second Betti number of the chain complex \( \{C_{*,w,w}\} \) for \( w > 0 \). We know the second Betti number is 0 for lower \( n \).

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