Gauge Dependence of Effective Average Action

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Abstract—The gauge dependence of effective average action in the functional renormalization group is studied. The effective average action is considered as non-perturbative solution to the flow equation which is the basic equation of the method. It is proven that at any scale of IR cutoff the effective average action depends on gauges making impossible physical interpretation of all obtained results in this method.

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1. INTRODUCTION

It is a well-known fact that Green functions in gauge theories (and therefore the effective action being the generating functional of one-particle irreducible Green function or vertices) depend on gauges. On the other hand, elements of $S$-matrix should be gauge independent. It means that the gauge dependence of effective action should be a very special form. The gauge dependence is a problem in quantum description of gauge theories beginning with famous papers by Jackiw [1] and Nielsen [2] where the gauge dependence of effective potential in Yang–Mills theories has been found. For Yang–Mills theories in the framework of the Faddeev–Popov quantization method [3] the gauge dependence problem has been found in our papers [4, 5] and for general gauge theories within the Batalin–Vilkovisky formalism [6, 7] in our paper [8] respectively.

Over the past three decades, there has been an increased interest in the nonperturbative approach in Quantum Field Theory known as the functional renormalization group (FRG) proposed by Wet-terich [9, 10]. The FRG approach has got further developments and numerous applications. There are many reviews devoted to detailed discussions of different aspects of the FRG approach and among them one can find [11–13] with qualitative references. As a quantization procedure the FRG belongs to covariant quantization schemes which meets in the case of gauge theories with two principal problems: the unitarity of $S$-matrix and the gauge dependence of results obtained. Solution to the unitarity problem requires consideration of canonical formulation of a given theory on quantum level and use of the Kugo–Ojima method [14] in construction of physical state space. Within the FRG the unitarity problem is not considered at all because main efforts are connected with finding solutions to the flow equation for the effective average action.

In turn, the gauge dependence problem exists for the FRG approach as unsolved ones if one does not take into account the reformulation based on composite operators [15] where the problem was discussed from the point of view of the basic principles of the quantum field theory. Later on the gauge dependence problem in the FRG was discussed in our papers several times for Yang–Mills and quantum gravity theories [16–20] but the reaction from the FRG community was very weak and came down only to mention without any serious study. Situation with the gauge dependence in the FRG is very serious because without solving the problem a physical interpretation of results obtained is impossible. It is main reason to return for discussions of the gauge dependence problem of effective average action in the FRG approach.

The DeWitt’s condensed notations [21] are used. The functional derivatives with respect to fields and sources are understood as right and left ones respectively.

2. GAUGE DEPENDENCE IN YANG–MILLS THEORIES

We start with the action $S_0[A]$ of fields $A$ for given Yang–Mills theory. Generating functional of Green functions, $Z[J]$, can be constructed by the Faddeev–Popov rules [3] in the form of functional integral

$$Z[J] = \int D\phi \exp \left\{ \frac{i}{\hbar} \left( S_{FP}[\phi] + J_i \phi^i \right) \right\},$$

where $\phi = \{ \phi^i \} = \{ A, B, C, \bar{C} \}$ is full set of fields including the ghost $C$ and antighost $\bar{C}$ Faddeev–Popov fields with the Grassmann parities $\varepsilon(\phi^i) =$
ε_i and auxiliary fields B (Nakanishi–Lautrup fields), J = {J_i} are external sources to fields φ, S_FP[φ] is the Faddeev–Popov action

\[ S_{FP}[\phi] = S_0[A] + \Psi[\phi], i R^i(\phi). \]  

(2)

Here in (2) Ψ[φ] is gauge fixing functional (in the simplest case having the form CΩA), and notation \( X_i = \delta X / \delta \phi^i \) is used. The Faddeev–Popov action \( S_{FP}[\phi] \) (2) obeys very important property of invariance under global supersymmetry – BRST symmetry [22, 23],

\[ \delta_B S_{FP}[\phi] = 0, \delta_B \phi^i = R^i(\phi) \mu^i, \mu^2 = 0, \]  

(3)

where \( R^i(\phi) \) are generators of BRST transformations. In what follows the explicit form of \( R^i(\phi) \) is not essential and we omit their presentation.

From definitions (1) and (2) it follows that the functional \( Z[J] \) depends on fields. To study the character of this dependence, let us consider an infinitesimal variation of gauge fixing functional \( \Psi[\phi] \to \Psi[\phi] + \delta \Psi[\phi] \) in the functional integral for \( Z[J] \). Then we obtain \( (\partial_{\delta} J_i / \delta) \)

\[ \delta Z[J] = \frac{i}{\hbar} \int D\phi \delta \Psi[i \phi] R^i(\phi) \]

\[ \times \exp \left\{ \frac{i}{\hbar} \left( S_{FP}[\phi] + J_i \phi^i \right) \right\} \]

\[ = \frac{i}{\hbar} \delta \Psi[i \phi] \left[ -i \hbar \partial J_i \right] R^i \left( -i \hbar \partial J_i \right) Z[J]. \]  

(4)

There exists an equivalent presentation of the variation for \( Z[J] \) under variations of gauge conditions. Indeed, making use of the change of integration variables in the functional integral for \( Z[J] \) with the choice \( \Psi[\phi] + \delta \Psi[\phi] \) in the form of the BRST transformations,

\[ \delta \phi^i = R^i(\phi) \mu^i[\phi], \]  

(5)

taking into account that the corresponding Jacobian, \( J_i \), is equal to

\[ J = \exp \{-\mu[\phi], i R^i(\phi) \}, \]  

(6)

choosing the functional \( \mu[\phi] \) in the form \( \mu[\phi] = (i / \hbar) \delta \Psi[\phi] \), then we have

\[ \delta Z[J] = \frac{i}{\hbar} \int D\phi J_i R^i(\phi) \delta \Psi[\phi] \]

\[ \times \exp \left\{ \frac{i}{\hbar} \left( S_{FP}[\phi] + J_i \phi^i \right) \right\} \]

\[ = \frac{i}{\hbar} J_i R^i \left( -i \hbar \partial J_i \right) \delta \Psi \left[ -i \hbar \partial J_i \right] Z[J]. \]  

(7)

Both relations are equivalent due to the evident equality

\[ \int D\phi \partial_{\delta} \phi \left( \Psi[\phi] R^i(\phi) \right) \]

\[ \times \exp \left\{ \frac{i}{\hbar} \left( S_{FP}[\phi] + J_i \phi^i \right) \right\} = 0, \]  

(8)

where the following equations

\[ S_{FP,i}[\phi] R^i(\phi) = 0, \]

\[ R^i(\phi) R^i(\phi) = 0, \]  

(9)

should be used.

In terms of the functional \( W[J] = -i \hbar \ln Z[J] \) the above relations rewrite as

\[ \delta W[J] = J_i R^i \left( \partial J W - i \hbar \partial J_i \right) \]

\[ \times \delta \Psi \left( \partial J W - i \hbar \partial J_i \right) \cdot 1, \]  

(10)

\[ \delta W[J] = \delta \Psi \left( \partial J_i W - i \hbar \partial J_i \right) \]

\[ \times R^i \left( \partial J W - i \hbar \partial J_i \right) \cdot 1. \]  

(11)

Introducing the effective action, \( \Gamma = \Gamma[\Phi] \), through the Legendre transformation of \( W[J] \),

\[ \Gamma[\Phi] = W[J] - J_i \Phi^i, \]

\[ \Phi^i = \frac{\delta W}{\delta J_i}, \delta \Gamma / \delta \Phi^i = -J_i, \]  

(12)

the gauge dependence of effective action is described by the equivalent relations

\[ \delta \Gamma[\Phi] = - \frac{\delta \Gamma}{\delta \Phi^i} R^i(\Phi) \delta \Psi(\Phi) \cdot 1, \]  

(13)

\[ \delta \Gamma[\Phi] = \delta \Psi \left( \Phi \right) R^i(\Phi) \cdot 1, \]  

(14)

where the notations

\[ \Phi^i = \Phi^i + i (\Gamma'^{-1})^i_{jk} \frac{\delta}{\delta \Phi^j} \]

\[ \left( \Gamma'^{-1} \right)^{ik} \left( \Gamma'^{-1} \right)_{kj} = \delta^i_j, \]  

(15)

are used.

From the presentation (13) it follows the important statement that the effective action does not depend on the gauge conditions at the extremals,

\[ \delta \Gamma \big|_{\delta \Phi^i = 0} = 0, \]  

(16)

making possible the physical interpretation of results obtained in the Faddeev–Popov method for Yang–Mills theories.

There exists another description of gauge dependence of effective action for Yang–Mills theories: The effective action can be presented in the form of gauge independent functional in which all gauge dependence contains in their arguments [4, 5].
3. EFFECTIVE AVERAGE ACTION

The main idea of the functional renormalization group is to modify the behavior of propagators in IR region with the help of a regulator action \( S_k[\phi] \) being quadratic in fields. In the case of Yang–Mills theories it leads to action

\[
S_{W_k}[\phi] = S_{FP}[\phi] + S_k[\phi],
\]

\[
S_k[\phi] = \frac{1}{2} R_{k[ij]} \phi^i \phi^j. \tag{17}
\]

Standard choice of regulators \( R_{k[ij]} \) is

\[
R_{k[ij]} = z_{ij} \frac{\Box \exp \left( -\frac{\Box}{k^2} \right)}{1 - \exp \left( -\frac{\Box}{k^2} \right)}, \quad \Box = \partial^\mu \partial_\mu,
\]

with properties

\[
\lim_{k \to 0} R_{k[ij]} = 0. \tag{19}
\]

The action \( S_{W_k}[\phi] \) is not invariant under the BRST transformations

\[
\delta_B S_{W_k}[\phi] = \delta_B S_k[\phi] = R_{k[ij]} \phi^j \frac{\partial}{\partial \phi^i}(\phi) \mu \neq 0. \tag{20}
\]

The generating functional of Green functions has the form

\[
Z_k[J] = \int D\phi \exp \left\{ \frac{i}{\hbar} \left[ S_{W_k}[\phi] + J_\phi \phi^i \right] \right\} = \exp \left\{ \frac{i}{\hbar} W_k[J] \right\}. \tag{21}
\]

Variation of \( \delta Z_k[J] \) under change of gauge condition can be presented as

\[
\delta Z_k[J] = \frac{i}{\hbar} \left( J_i - i \hbar R_{k[ij]} \partial J_j \right) \times R^i(\partial J W - i \hbar \partial J) \delta \Psi[\partial J W - i \hbar \partial J] Z_k[J]. \tag{22}
\]

In terms of \( W_k[J] \) we have

\[
\delta W_k[J] = \left( J_i + R_{k[ij]} \partial J_j - i \hbar \partial J_i \right) \times R^i(\partial J W - i \hbar \partial J) \delta \Psi[\partial J W - i \hbar \partial J] \cdot 1. \tag{23}
\]

Introducing the effective average action, \( \Gamma_k = \Gamma_k[\Phi] \), being the main quantity in the FRG through the Legendre transformation of \( W_k[J] \),

\[
\Gamma_k[\Phi] = W_k[J] - J_\Phi \Phi^i,
\]

\[
\Phi^i = \frac{\delta W_k}{\delta J_i}, \quad (\Gamma_k)_{ij} = \frac{\delta^2 \Gamma_k}{\delta \Phi_i \delta \Phi^j}, \tag{24}
\]

the gauge dependence of effective average action is described as

\[
\delta \Gamma_k[\Phi] = - \left( \frac{\delta \Gamma_k}{\delta \Phi^i} - R_{k[ij]} \Phi^j \right) R^i(\hat{\Phi}) \delta \Psi[\phi] \cdot 1, \tag{25}
\]

with the notations

\[
\hat{\Phi}^i = \Phi^i + i \hbar (\Gamma_k^{-1})^{ij} i_j \delta \Phi^j, \quad (\Gamma_k^\prime)_{ij} = \frac{\delta^2 \Gamma_k}{\delta \Phi_i \delta \Phi^j}, \tag{26}
\]

Due to (25) the effective average action remains gauge dependent even on their extremals

\[
\delta \Gamma_k[\Phi] \bigg|_{\partial \phi \Gamma_k = 0} \neq 0 \tag{27}
\]

making impossible physical interpretation of results obtained in the FRG.

4. GAUGE DEPENDENCE OF FLOW EQUATION

The above analysis of gauge dependence of effective average action \( \Gamma_k[\Phi] \) does not convince people from the FRG community because it is based on theorems in Quantum Field Theory formulated within the standard perturbation approach while it is assumed that the flow equation for \( \Gamma_k[\Phi] \) in the FRG is considered non-perturbatively.

We are going to study the gauge dependence of effective average action found as a solution to the flow equation. To do this we find first of all the partial derivative of \( Z_k[J] \) with respect to IR cutoff parameter \( k \). The result reads

\[
\partial_k Z_k[J] = \frac{i}{\hbar} \int D\phi \partial_k S_k[\phi] \times \exp \left\{ \frac{i}{\hbar} \left[ S_{W_k}[\phi] + J_A \Phi^i \right] \right\} = \frac{i}{\hbar} \partial_k S_k[-i \hbar \partial J] Z_k[J]. \tag{28}
\]

In deriving this result, the existence of functional integral is only used.

In terms of generating functional of connected Green functions we have

\[
\partial_k W_k[J] = \partial_k S_k[\partial J W - i \hbar \partial J] \cdot 1. \tag{29}
\]

The basic equation (flow equation) of the FRG approach has the form

\[
\partial_k \Gamma_k[\Phi] = \partial_k S_k[\hat{\Phi}] \cdot 1, \tag{30}
\]

where \( \hat{\Phi} = \{ \hat{\Phi}^i \} \) is defined in (26). It is assumed that solutions to the flow equations present the effective average action \( \Gamma_k[\Phi] \) beyond the usual perturbation calculations.

Now, we analyze the gauge dependence problem of the flow equation. Note that up to now this problem has never been discussed in the literature. To do this we consider the variation of \( \partial_k Z_k[J] \) under an infinitesimal change of gauge fixing functional, \( \Psi[\phi] \rightarrow \Psi[\phi] + \delta \Psi[\phi] \). Taking into account that \( \partial_k S_k[\phi] \) does not depend on gauge fixing procedure, we obtain

\[
\delta \partial_k Z_k[J] = \left( \frac{i}{\hbar} \right)^2 \partial_k S_k[-i \hbar \partial J] \cdot 1. \tag{31}
\]
of the functional $W_k[J]$ we have
\[ \delta \partial_k W_k[J] = \delta_k S_k[\delta_j W_k - i h \partial_j] \times \delta \Psi,[\delta_j W_k - i h \partial_j] R^i(\partial_j W_k - i h \partial_j) \cdot 1. \] (32)

Finally, the gauge dependence of the flow equation is described by the equation
\[ \delta \partial_k \Gamma_k[\Phi] = \delta_k S_k[\Phi] \delta \Psi,[\Phi] R^i(\Phi) \cdot 1. \] (33)

Therefore, at any finite value of $k$ the flow equation depends on gauges. The same conclusion is valid for the effective average action.

But what is about the case when $k \rightarrow 0$? Usual argument used by the FRG community to argue gauge independence is related to the statement that due to the property
\[ \lim_{k \rightarrow 0} \Gamma_k = \Gamma, \] (34)

where $\Gamma$ is the standard effective action constructed by the Faddeev–Popov rules, the gauge dependence of average effective action disappears at the fixed point. In our opinion this property is not sufficient to claim the gauge independence at the fixed point. The reason to think so is the flow equation which includes the differential operation with respect to the IR parameter $k$.

Indeed, let us present the effective average action in the form
\[ \Gamma_k = \Gamma + k H_k, \] (35)

where functional $H_k$ obeys the property
\[ \lim_{k \rightarrow 0} H_k = H_0 \neq 0. \] (36)

Then we have the relations
\[ \partial_k \lim_{k \rightarrow 0} \Gamma_k = 0, \quad \lim_{k \rightarrow 0} \partial_k \Gamma_k = H_0. \] (37)

These two operation do not commute and the statement of gauge independence at the fixed point seems groundless within the FRG approach. Due to this reason it seems as very actual problem for the FRG community to fulfill calculations of the effective average action at the fixed point using, for example, a family of gauges with one parameter and choice of two different values of the parameter.

5. DISCUSSIONS

The gauge dependence problem in the framework of FRG approach has been analyzed. The standard quantization of Yang–Mills theories within the Faddeev–Popov method is characterized by the BRST symmetry which governs gauge independence of $S$-matrix elements. In turn the BRST symmetry is broken in the FRG approach with all negative consequences for physical interpretation of results. But usual reaction of the FRG community with this respect is that they are only interested in the effective average action evaluated at the fixed point where the gauge independence is expected. One of the goals of this investigation was to study the gauge dependence of the effective average action as a solution of the flow equation. For the first time the equation describing the gauge dependence of flow equation has been explicitly derived. It was found the gauge dependence of flow equation at any finite value of the IR parameter $k$. The FRG cannot be considered as a suitable quantization method of gauge theories given physically meaningful results.

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