RELATIONS BETWEEN $L^p$- AND POINTWISE CONVERGENCE OF FAMILIES OF FUNCTIONS INDEXED BY THE UNIT INTERVAL.

Abstract. We construct a variety of mappings from the unit interval $I$ into $L^p([0,1])$ to generalize classical examples of $L^p$-converging sequences of functions with simultaneous pointwise divergence. By establishing relations between the regularity of the functions in the image of the mappings and the topology of $I$, we obtain examples which are $L^p$-continuous but exhibit discontinuity in a pointwise sense to different degrees. We conclude by proving a Lusin-type theorem, namely that if almost every function in the image is continuous, then we can remove a set of arbitrarily small measure from the index set $I$ and establish pointwise continuity in the remainder.

1. Introduction

1.1. Motivation and Overview. Examples of sequences of real functions on a compact domain which have a limit in $L^p$, but do not converge pointwise are well known. Their construction is based on the fact that any interval can be covered infinitely often by a sequence of subintervals of vanishing lengths. Take, for instance, the sequence $(f_i)_{i \in I}$ of characteristic functions $f_i = \chi_{I_i}, i \in \mathbb{N}$, of the intervals $I_i = [\frac{i}{2^k} - 1, \frac{i+1}{2^k} - 1]$, where $k$ is the unique integer with $2^k \leq i < 2^{k+1}$. We can even have all the terms of the sequence being smooth functions, by applying a suitable mollifier to each one of them.

In examples like the above, only the order of the index set is relevant. We can observe, however, that a trivial topology is induced in a natural way by the convergence of the sequence. It is not at all obvious, if these types of examples can be extended to index sets which have more complex topological structure. We wish to address the case of a continuous curve $f$ which maps $I = [0,1]$ into $L^p([0,1])$ and generalize examples like the above. In our setting, the index set $I$ itself has already a non-trivial topological structure, which turns out to interact with the regularity properties of the family $\{f_t, t \in I\}$.

Curves like this often appear in semigroup theory as solutions of PDEs. However, the smoothing properties of the operators in these settings usually result in a high regularity for the solutions for every $t > 0$, and therefore pointwise convergence comes naturally. Even for the more tricky case of $t = 0$, pointwise convergence can often deduced by using tools from harmonic analysis, potential theory, etc. In this paper, no underlying process
is assumed. We are investigating the pointwise behavior of the curves in a purely real analytic way.

By making different assumptions regarding the properties of the functions $f_t$ we construct two example curves in $L^p$ which lack pointwise convergence almost everywhere. The first example is constructed in Section 2, where we assume that $\{f_t, t \in I\} \subset C(\Omega)$ and that almost all $f_t$ are smooth. In Section 3 we show that the criteria on the regularity $f_t$ in Section 2 are optimal by demonstrating that the structure of $I$ makes “everywhere pointwise divergence” impossible and that higher regularity always implies better pointwise convergence properties.

In Section 4 we drop the continuity requirement and construct a curve of very irregular functions. For this curve, not only we have a.e pointwise divergence, but also the divergence is inherited in every subset $T$ of $I$ with positive measure. Finally, Section 5 is devoted to prove that the discontinuity of $f_t$ is necessary to obtain a curve that exhibits such a highly pointwise divergence. In particular, the example in Section 4 motivates a special case of our main result Theorem 5.2., which can be interpreted as a refined version of Lusin’s Theorem in two variables.

1.2. Notation. Throughout, $I$ is the unit interval $[0, 1]$, equipped with the standard norm $|\cdot|$ and the corresponding Borel-$\sigma$-field. Lebesgue measure on $I$ is denoted by $\mu$. We study functions $f : I \times \Omega \rightarrow \mathbb{R}$ of two real variables, we will usually, for $t \in I$, write $f_t$ for the function $f(t, \cdot)$ in one real variable to stress the difference between “time” and “space”, but revert to write $f$ as a function of two variables when it is notationally more convenient. The spacial domain $\Omega \subset \mathbb{R}$ of the functions $f_t, t \in I$, can be chosen to be any interval of $\mathbb{R}$ equipped with its Borel $\sigma$-field and Lebesgue measure. We take $\Omega = [0, 1]$ for convenience in the construction of the examples. To avoid confusion with the "time" interval $I$, whenever we refer to space, i.e. when measuring sets in the domain and range of the real functions $f_t, t \in [0, 1]$, we denote Lebesgue measure by $\lambda$.

$L^p(\Omega, \mathbb{R}, \lambda) = L^p$ for $1 \leq p < \infty$ denotes the space of real-valued $p$-integrable functions on $\Omega$, equipped with the topology induced by the seminorm $|| \cdot ||_p$. Furthermore, we write $W^{1,p}$ for the space of all absolutely continuous functions with derivatives belonging to $L^p$ and use the standard notation $C(\Omega)$ and $C^\infty(\Omega)$ for the space of continuous real valued functions on $\Omega$ and the space of real valued smooth functions on $\Omega \setminus \partial \Omega$.

Remark 1.1. Note that we do not identify almost everywhere indentical members of $L^p$, since all our constructions are pointwise. To prove lack of pointwise convergence at a point $t$, we choose a sequence $t_n$ converging to $t$ and assure, that $f_{t_n}$ diverges pointwise on a set of positive measure. Therefore the established irregularity can not be avoided by choosing different
"versions" of \( f_t \) and trivial counterexamples like the continuous transport of a set of measure zero are excluded.

2. Construction of the first example

We begin by showing that there is a \( L^p \) continuous curve of continuous functions, along which pointwise convergence can be established almost nowhere and that this irregularity is achieved while keeping smooth, almost all functions along the curve.

**Theorem 2.1.** Let \( 1 \leq p < \infty \). There is a continuous mapping \( f \) of \([0, 1]\) into \( L^p(\Omega) \), satisfying

(i) \( f_t \) is absolutely continuous for all \( t \in [0, 1] \),

(ii) \( f_t \in C^\infty(\Omega) \) for \( \mu \)-a.e. \( t \),

but also

(A) for \( \mu \)-a.e. \( t \in [0, 1] \) there exists a sequence \( (t_n)_{n \in \mathbb{N}} \) with \( \lim_{n \to \infty} t_n = t \) such that

\[
\lambda \{ x \in \Omega : (f_{t_n}(x))_{n \in \mathbb{N}} \text{ is cauchy} \} = 0.
\]

**Proof.** Fix an increasing sequence \( K_1 \subset K_2 \subset \cdots \subset [0, 1] \) of closed nowhere dense sets such that \( \mu \left( \bigcup_{i=1}^{\infty} K_i \right) = 1 \). For each \( i \), the complement \( K_i^c \) can be represented as a countable union \( \bigcup_{j=1}^{\infty} (r_{i,j}, s_{i,j}) \) of open intervals, whose lengths we denote by \( l_{i,j} = \mu(r_{i,j}, s_{i,j}) \). In this setting define

\[
f^{(i)}(t, x) = \varphi_i(t) \gamma_i(t, x),
\]

where

\[
\varphi_i(t) = \begin{cases} 
\frac{2j}{l_{i,j}} (t - r_{i,j}) & \text{if } t \in (r_{i,j}, r_{i,j} + \frac{l_{i,j}}{2j}), \\
1 & \text{if } t \in [r_{i,j} + \frac{l_{i,j}}{2j}, s_{i,j} - \frac{l_{i,j}}{2j}], \\
\frac{2j}{l_{i,j}} (t - s_{i,j}) & \text{if } t \in (s_{i,j} - \frac{l_{i,j}}{2j}, s_{i,j}), \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
\gamma_i(t, x) = \begin{cases} 
\frac{1}{4^i} \exp \left( -\pi \frac{(x - \frac{t - r_{i,j}}{l_{i,j}})^2}{l_{i,j}^2} \right) & \text{if } t \in (r_{i,j}, s_{i,j}) \text{ for some } j \in \mathbb{N} \\
& \text{and } x \in [0, 1], \\
0 & \text{otherwise}.
\end{cases}
\]

A straightforward calculation shows that \( \| \gamma_i(t, \cdot) \|_p < \frac{1}{4^i} l_{i,j} \). Furthermore, observe that \( f^{(i)}(t, \cdot) \) is uniformly continuous on \( K_i^c \) and \( L^p \)-continuous on the whole \([0, 1]\). We also have that \( \| f^{(i)}(t, \cdot) \|_p \leq 4^{-i} \) for all \( t \in [0, 1], i \in \mathbb{N} \) and \( f^{(i)}(t, x) \leq 4^{-i} \) for all \( t, x \in [0, 1] \times \Omega, i \in \mathbb{N} \). We set

\[
f_t(x) = \sum_{i=1}^{\infty} f^{(i)}(t, x),
\]

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\]
which defines a function \( f : [0, 1] \longrightarrow L^p(\Omega) \) for which the properties stated in the theorem can be verified. To show that \( f \) is a continuous mapping of the unit interval into \((L^p(\Omega), \| \cdot \|_p)\) we fix \( \varepsilon > 0 \), choose \( l \in \mathbb{N} \) such that \( 2^{-l} < \frac{\varepsilon}{2} \) and estimate for \( t, u \in \mathcal{I} \), using the triangle inequality and Fatou’s Lemma

\[
\left( \int_{\Omega} |f_u(x) - f_t(x)|^p \, dx \right)^{\frac{1}{p}}
= \left( \int_{\Omega} \sum_{i=1}^{\infty} \left( |f^{(i)}(u, x) - f^{(i)}(t, x)| \right)^p \, dx \right)^{\frac{1}{p}}
\leq \sum_{i=1}^{\infty} \left( \int_{\Omega} |f^{(i)}(u, x) - f^{(i)}(t, x)|^p \, dx \right)^{\frac{1}{p}}
= \sum_{i=1}^{l} \left( \int_{\Omega} |f^{(i)}(u, x) - f^{(i)}(t, x)|^p \, dx \right)^{\frac{1}{p}}
+ \sum_{i=l}^{\infty} \left( \int_{\Omega} |f^{(i)}(u, x) - f^{(i)}(t, x)|^p \, dx \right)^{\frac{1}{p}}
\leq \frac{\varepsilon}{2}.
\]

The finite left hand side summand can be made smaller than \( \frac{\varepsilon}{2} \) by choosing \( u \) close to \( t \) and the right hand side summand is bounded by \( 2^{-l} \) and therefore by \( \frac{\varepsilon}{2} \). Since \( t \) and \( \varepsilon \) can be chosen arbitrarily, we conclude that \( L^p \)-continuity holds for every \( t \in [0, 1] \).

Now set \( K := \bigcup_{i=1}^{\infty} K_i \), then for \( t \in K \) there exists an index \( i \) for which \( t \in K_i \). Hence \( f(t, \cdot) \) is a finite sum of \( C^\infty(\Omega) \) functions and therefore smooth, so (ii) holds. For (i), absolute continuity only needs to be verified for \( t \in K \). Clearly all \( f^{(i)}(t, \cdot) \) are absolutely continuous and so are the finite sums \( \sum_{i=1}^{k} f^{(i)}(t, \cdot) \). For existence of the limits, we note that

\[
\frac{d}{dx} \exp \left( -\frac{\pi (x - t - r_{i,j})^2}{l_{i,j}^{2p}} \right)
= -\frac{1}{4} \frac{2\pi}{l_{i,j}^{2p}} \left( x - t - r_{i,j} \right) \exp \left( -\frac{\pi (x - t - r_{i,j})^2}{l_{i,j}^{2p}} \right),
\]

which is still summable in \( i \). Hence the first derivative of \( f_t \) exists for all \( t \) a.e. on \( \Omega \), which is equivalent to absolute continuity.

The final step is to show that (A) holds by constructing for every \( t \in K \) a sequence \( (t_n)_{n \in \mathbb{N}} \) such that \( f(t_n, \cdot) \) has the desired property. Let \( t \in K \)
Proposition 3.1. Let $\delta > n \in I_{\max}$ be fixed and let $i = i(t) = \min\{j \in \mathbb{N} : t \in K_j\}$. Since $K_i$ is nowhere dense, there exists a subsequence of intervals $\left((r_{i,j_n}, s_{i,j_n})\right)_{n \in \mathbb{N}}$ indexed by $(j_n) = (j_n(t))$ with endpoints $r_{i,j_n}, s_{i,j_n}$ converging to $t$. For each one of these intervals we can find an integer $k = k(n)$ and a $k$-tuple $\tau(n) = (\tau^1(n), \tau^2(n), ... , \tau^k(n))$, where $\tau^i(n) \in (r_{i,j_n}, s_{i,j_n})$, with the property that for every $x \in \left[\frac{1}{j_n(t)}, 1 - \frac{1}{j_n(t)}\right]$ there exists an index $l = l(x, n) \in \{1, \ldots, k\}$ such that $f^{(i)}(\tau^l(n), x) > \frac{2}{\delta 4^{-i}}$.

We consider now the sequence $(t_m)_{m \in \mathbb{N}}$ obtained by concatenating the $k(n) + 2$-tuples $(r_{i,j_n}, \tau^1(n), \tau^2(n), ... , \tau^k(n), s_{i,j_n})$ in increasing order in $n$. Fix $x_0 \in [0, 1], n_0 \in \mathbb{N}$ and $\varepsilon = \frac{1}{6} 4^{-i}$. Since $t \notin K_h$ for any $h < i$ we know that the functions $f^{(h)}(\tau)$ are uniformly continuous around $t$ every $h < i$, i.e there is a $\delta > 0$ such that $\sum^{i-1}_{h=1} (f^{(h)}(t, \cdot) - f^{(h)}(\tau, \cdot)) < \frac{1}{6} 4^{-i}$ for every $\tau$ with $|t - \tau| \leq \delta$. Since $t_n$ converges to $t$ there is $n_1 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n > n_1$ we have $|t_n - t| < \delta$ and by construction of $t_n$ there are $n, m > \max\{n_0, n_1\}$ such that $f^{(i)}(t_m, x_0) - f^{(i)}(t_n, x_0) > \frac{2}{\delta}$. For these $n, m$ we have

$$f(t_m, x) - f(t_n, x) = \sum_{h=1}^{\infty} (f^{(h)}(t_m, x) - f^{(h)}(t_n, x))$$

$$= \sum_{h=1}^{i-1} (f^{(h)}(t_m, x) - f^{(h)}(t_n, x)) + f^{(i)}(t_m, x) - f^{(i)}(t_n, x)$$

$$+ \sum_{h=i}^{\infty} (f^{(h)}(t_m, x) - f^{(h)}(t_n, x))$$

$$> \frac{2}{3 \cdot 4^i} - \frac{1}{4 \cdot 4^i} - \sum_{h=1}^{\infty} \frac{1}{4^h} = \frac{2}{3 \cdot 4^i} - \frac{1}{4 \cdot 4^i} - \frac{1}{4 \cdot 4^i} = \varepsilon,$$

so $(f(t_n, x))_{n \in \mathbb{N}}$ is not cauchy. \qed

3. Optimality of the conditions in Theorem 2.1

In this section we show that Theorem 2.1 is sharp in two senses. Firstly, we cannot obtain divergence on the whole of $I$.

**Proposition 3.1.** Let $f$ be a continuous map of $[0, 1]$ into $\mathcal{L}^p(\Omega)$. If $f_t$ is continuous for every $t$, then there is a comeagre subset $T \subset [0, 1]$ such that for any $t \in T$ and any sequence $(t_n)_{n \in \mathbb{N}}$ with limit $t$

$$\lim_{n \to \infty} f_{t_n}(x) = f_t(x) \quad \text{for all } x \in \Omega.$$

**Proof.** Define for $0 < q < p$ the sets

$$T_{pq} = \{ t \in [0, 1] : \exists x(t) \in \Omega \text{ with } f_t(x(t)) < q < p < \limsup_{s \to t} f_s(x(t)) \}.$$
We first want to prove by contradiction that $T_{pq}$ are nowhere dense sets. Let us assume that there are $q < p$ such that $T_{pq}$ is dense in an open ball $B(t_0, r_0)$, with $t_0 \in T_{pq}$. We then have that no open subset $S$ of the ball $B(t_0, r_0)$ is disjoint from $T_{pq}$.

We will first show, that for any such $S$ and any choice of $\delta > 0$ and sufficiently small $\rho > 0$, there exist $t \in S$ and $r < \rho$ such that for every $s \in B(t, r)$ we have $\omega(s, \delta) > q - p$, where $\omega(s, \delta) = \sup\{|f_s(x) - f_s(y)| : |x - y| < \delta\}$.

By assumption, there is $t_1 \in S \cap T_{pq}$, hence there is a point $x(t_1) \in \Omega$ with $f_{t_1}(x(t_1)) < q < p < \limsup_{s \to t_1} f_s(x(t_1))$. Since $f_{t_1}(x(t_1)) < q$, there exists $0 < \delta_1 < \delta$ such that $f_{t_1}(y) < q$, for all $y \in B(x(t_1), \delta_1)$, by continuity of $f_{t_1}$. Moreover, $f$ is $\mathcal{L}^p$-continuous, hence there is $r_1 > 0$ such that for every $s \in B(t_1, r_1)$, there exists $x^{s_n}(s) \in B(x(t_1), \delta_1)$ for which $f_s(x^{s_n}(s)) < q$. We choose now a second point $t_2 \in B(t_1, r_1)$ with $f_{t_2}(x(t_1)) > p$ and by continuity of $f_{t_2}$ we can fix $\delta_2 > 0$ such that $f_{t_2}(y) > p$ for all $y \in B(x(t_1), \delta_2)$. Using $\mathcal{L}^p$-continuity again, we can find $r_2 > 0$ such that for every $s \in B(t_2, r_2)$ there exists $x'(s) \in B(x(t_2), \delta_2)$ with $f_s(x'(s)) > p$. The above assertion now holds for the choices $t = t_2, r = \min\{r_1, r_2, \sup\{|t_2 - s|, s \in \partial B(t_1, r_1)\}\}$ and $\delta = \delta_1$.

Applying the above construction to vanishing sequences $(\rho_n)_{n \in \mathbb{N}}, (\delta_n)_{n \in \mathbb{N}}$, we can find points $t_n$ and radii $r_n < \rho_n$ with $B(t_{n+1}, r_{n+1}) \subset B(t_n, r_n)$ and $\omega(t_n, \delta_n) > q - p$ for every $s \in B(t_n, r_n)$. Since $\lim_{n \to \infty} r_n = 0$, we have that $\lim_{n \to \infty} t_n = t_{\infty}$ for some $t_{\infty} \in [0, 1]$. Moreover, we have that $\omega(t_{\infty}, \delta_n) > p - q$ for every $n \in \mathbb{N}$, which contradicts the assumption that $f_t$ is continuous for every $t \in [0, 1]$, hence our initial assumption that $T_{pq}$ is not nowhere dense cannot be true.

We can apply the same argument to the sets

$$S_{pq} = \{ t : \exists x(t) \in \Omega \text{ such that } f_t(x(t)) > q > p > \liminf_{s \to t} f_s(x(t)) \},$$

and the comeager set $T$ mentioned in the theorem is the complement of

$$\bigcup_{p,q \in \mathbb{Q}} (T_{pq} \cup S_{pq}).$$

Secondly, we can prove that we cannot make the regularity requirement (i) in Theorem 2.1 stronger.

**Proposition 3.2.** Let $f$ be a continuous mapping of $[0, 1]$ into $\mathcal{L}^p(\Omega) \cap W^{1,q}$, where $1 \leq p < \infty$ and $q > 1$. Then there is an open dense set $T \subset [0, 1]$ such that for all $t \in T$ and any sequence $(t_n)_{n \in \mathbb{N}}$ with limit $t$,

$$\lim_{n \to \infty} f_{t_n}(x) = f_t(x) \text{ for all } x \in \Omega.$$
For the proof of Proposition 3.2 we need to establish an auxiliary lemma about the relation between $\mathcal{L}^p$-continuity and pointwise continuity.

**Lemma 3.3.** Let $f$ be $\mathcal{L}^p$-continuous and $S \subset [0,1]$ an open interval. If $f(t) \in W^{1,q}(\Omega)$ for some $q > 1$ and $\{f_t; t \in S\}$ is bounded in $W^{1,q}$, then $f$ is pointwise continuous for every $t \in S$, i.e. $\lim_{n \to \infty} f_{t_n}(x) = f_t(x)$ for every sequence $(t_n)_{n \in \mathbb{N}}$ converging to $t$ and every $x \in \Omega$.

**Proof.** Let $f$ satisfy the assumptions of the lemma and fix $\varepsilon > 0$. Invoking the Sobolev Embedding Theorem (see e.g. [1, Theorem 4.12]), we can assume that $f_t$ is Hölder-continuous with exponent $q' = 1 - \frac{1}{q}$ and constant $C > 0$ independent of $t$, i.e. we have for all $t \in S$,

$$|f_t(x) - f_t(y)| \leq C|x - y|^{q'}, \quad \text{for all } x, y \in \Omega.$$  \hspace{1cm} (1)

Now fix $x \in \Omega$ and any $s, t \in S$, then applying the triangle inequality and \ref{1} we obtain for all $y \in \Omega$,

$$|f_t(x) - f_s(x)| \leq |f_t(x) - f_t(y)| + |f_t(y) - f_s(y)| + |f_s(x) - f_s(y)| \leq 2C|x - y|^{q'} + |f_t(y) - f_s(y)|.$$  

Integrating both sides in $y$ on the interval $B(x, \eta) = (x - \frac{\eta}{2}, x + \frac{\eta}{2})$, where $0 < \eta < \min\{\frac{\varepsilon}{2}, 2 \sqrt[4]{\text{diameter} \Omega}\}$, yields

$$\left(\int_{B(x, \eta)} |f_t(x) - f_s(x)|^p dy\right)^{\frac{1}{p}} \leq \left(\int_{B(x, \eta)} (2C|x - y|^{q'})^p dy\right)^{\frac{1}{p}} + \left(\int_{B(x, \eta)} |f_t(y) - f_s(y)|^p dy\right)^{\frac{1}{p}}$$

and thus

$$\eta^\frac{1}{p} |f_t(x) - f_s(x)| \leq \eta^\frac{1}{p} \varepsilon + \|f_t - f_s\|_p.$$  

This implies

$$|f_t(x) - f_s(x)| \leq \frac{\varepsilon}{2} + \|f_t - f_s\|_p \eta^\frac{1}{p}$$

and using $\mathcal{L}^p$ continuity of $f$ we derive that

$$|f_t(x) - f_s(x)| \leq \varepsilon,$$

for all $s$ sufficiently close to $t$. \hfill \Box

We are now ready to prove Proposition 3.2.

**Proof of Proposition 3.2.** Let $f_t \in W^{1,q}(\Omega), q > 1$ for every $t \in [0,1]$. Since $f_t$ is absolutely integrable for all $t$, we can expand each $f_t$ into a fourier series

$$f^{(m)}(t, x) = \sum_{n=-m}^{n=m} c_n(t) e^{in\pi(x-\frac{1}{2})}$$
for which we have \( \lim_{m \to \infty} f^{(m)}(t, \cdot) = f_t \), w.r.t. \( \| \cdot \|_{L^q} \), see e.g. [2] p. 78.

Using \( L^p \)-continuity of \( f \) we obtain that the coefficients \( c_n(t) = \int_{\Omega} f_t \exp \left( -i \pi (\cdot - \frac{1}{n}) \right) d\lambda \) are continuous in \( t \) and furthermore \( g^{(m)}(t) = \| f^{(m)}(t, \cdot) \|_{L^q} \) is a continuous function. Hence \( g(t) = \| f_t \|_{L^q} \) can be represented as a limit of continuous functions and therefore is continuous on the complement of a meager set, see e.g. [3, Theorem 7.3]. This implies that \( g(t) \) is locally bounded on an open dense set. Thus the assumptions of Lemma 3.3 are satisfied and its application yields the statement of the theorem. \( \square \)

4. Construction of the second example

If the requirement (i) in Theorem 2.1 is dropped, then it is possible to create an example in which pointwise divergence is obtained on every subset of \( I \) of positive measure.

**Theorem 4.1.** There exists a continuous function \( f : [0, 1] \to L^p(\Omega) \), such that for all measurable sets \( T \subset [0, 1] \) with \( \mu(T) > 0 \) and almost every \( t \in T \) there exists a sequence \( (t_n)_{n \in \mathbb{N}} \subset T \) with \( \lim_{n \to \infty} t_n = t \) and \( \lambda(A_t) = 1 \) where

\[ A_t = \{ x : \lim_{n \to \infty} f_{t_n}(x) \neq f_t(x) \} \]

**Proof.** Let \( \{ q_m, m \in \mathbb{N} \} \subset I \) be dense and assigned to each \( q_m \) a sequence \( (s_{m,k})_{k \in \mathbb{N}} \) defined by

\[ s_{m,k} = q_m - \frac{1}{k + r(m)} , \text{ where } r(m) = \min \left\{ r : q_m - \frac{1}{r} \geq 0 \right\} . \]

To partition the spacial domain, set

\[ b_{m,k} = \max \left\{ 0, \frac{2(t - s_{m,k})}{s_{m,k+1} - s_{m,k}} - \frac{1}{4k+1} \right\} \text{ and } c_{m,k} = \min \left\{ 1, \frac{2(t - s_{m,k})}{s_{m,k+1} - s_{m,k}} \right\}. \]

Furthermore, denoting by \( \chi_{T \times A} \) the characteristic function of a subset \( T \times A \subset I \times \Omega \) and setting \( S_{m,k} = [s_{m,k}, s_{m,k+1}] \), define functions \( f^{(m,k)}(t, \cdot) : \Omega \to L^p(\Omega) \) by

\[ f^{(m,k)}(t, x) = 2^m \chi_{S_{m,k} \times [b_{m,k}, c_{m,k}]}(t, x) . \]

These functions satisfy \( \| f_{m,k}(t, \cdot) \|_p \leq \frac{1}{2^{m+k}} \), for all \( t \in I \) and one also checks easily that \( f^{(m,k)}(t, \cdot) \) is \( L^p \)-continuous in the first variable for all \( t \in I \). We can now set

\[ f_t(x) = \sum_{k,m=1}^{\infty} f^{(m,k)}(t, x) \]

and the limit \( f \) is a well defined real function, since \( f^{(m,k)}(t, x) \geq 0 \) and

\[ \sum_{m,k=1}^{\infty} \| f^{(m,k)}(t, \cdot) \|_p < \infty . \]

Now let \( S \subset I \) be of positive measure \( \lambda(S) > 0 \), and let \( t \in S \) have density 1 with respect to \( S \). We inductively construct a sequence \( (t_n)_{n \in \mathbb{N}} \) such that

\[ \limsup_{n \to \infty} f(t_n, x) \neq \liminf_{n \to \infty} f(t_n, x), \text{ for almost all } x \in [0, 1] . \]
To this end, initiate the construction with \( t_1 = 0, n_0 = m_0 = 1 \) and let \((\gamma_i)_{i \in \mathbb{N}} \subset [0,1] \) increase to 1. Assume, we have already constructed the initial members of the sequence \( t_1, \ldots, t_{n_1}, \ldots, t_{n_2}, \ldots, t_{n_{i-1}} \). Since \( t \) is a point of density 1 in \( S \), we can fix \( \rho_i \in (0,1) \setminus \{l^{-1}; l \in \mathbb{N} \} \) such that for every \( \rho < \rho_i \),
\[
\lambda(S \cap B(t,\rho)) \geq \gamma_i \rho \quad \text{and} \quad 1 - \rho^2 \geq \gamma_i.
\]
Let now \( \lambda \) be a continuous function in \([0,1] \) such that the restriction \( (\gamma_i)_{i \in \mathbb{N}} \) converges to 1.

\[\square\]

5. Necessity of discontinuous \( f_t \) in Theorem 4.1

In this section we prove our final and most general result, namely that dropping continuity with respect to \( t \) for almost every \( t \) is essential in order to be able to construct an extremely irregular curve like in Theorem 4.1. We show, that if the function \( f_t \) is continuous for every \( t \), then a refined version of Lusin's theorem in two variables holds.

**Theorem 5.1.** Let \( f : [0,1] \times \Omega \rightarrow L^p \) be Borel measurable such that \( f_t \) is a continuous function for \( \mu \)-a.e \( t \in \Omega \). Then, for every \( \varepsilon > 0 \), there is a set \( T_\varepsilon \subset [0,1] \) with \( \mu(T_\varepsilon^C) < \varepsilon \) such that the restriction
\[
f_{|T_\varepsilon \times \Omega} : (T_\varepsilon \times \Omega, | \cdot | \otimes | \cdot |) \rightarrow (\mathbb{R}, | \cdot |)
\]
is a continuous function.

It is noticeable that in Theorem 5.1 only the fact that \( f \) is a measurable function in \([0,1] \times \Omega \) is needed and there is no \( L^p \)-continuity assumed. However \( L^p \)-continuity guarantees that \( f \) is a measurable function in \([0,1] \times \Omega \) and therefore the claimed necessity in Theorem 4.1 is a straightforward consequence.

**Corollary 5.2.** Let \( f \) be a continuous function from \([0,1] \) to \( L^p(\Omega) \). If \( f_t \in L^p(\Omega) \) is continuous for \( \mu \)-a.e \( t \in [0,1] \), then the assertion in Theorem 5.1 holds.

Before we proceed with the proof of Theorem 5.1 we would like to point out the difference between Theorem 5.1 and the classical result of Lusin. In Lusin's theorem an arbitrarily small set of \( \mathcal{I} \times \Omega \) is removed in order to establish continuity on the remainder. In our case the stronger assumption of continuity with respect to one variable entails the information that this small set is of the form \( T_\varepsilon^C \times \Omega \), so it is only necessary to remove a "slice"
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in the space-time domain.

For the proof of Theorem 5.1 we also need the following preliminary result.

**Lemma 5.3.** Let $F = \{f_t; t \in I\}$ be a family of continuous functions. Then, for every $\varepsilon > 0$, there is a set $S_\varepsilon \subset I$, such that $\mu(S^C_\varepsilon) < \varepsilon$, with the property that $F_\varepsilon := \{f_t; t \in S_\varepsilon\}$ is equicontinuous.

**Proof.** Let $\omega_\delta$ denote the $\delta$-oscillation functional,

$$\omega_\delta(g) := \sup\{|g(x) - g(y)| : |x - y| < \delta\} \quad (3)$$

and set $\omega_n(t) = \omega_\frac{1}{n}(f_t)$ on $F$. We have $\lim_{n \to \infty} \omega_n(t) = 0$ for every $t$, since all $f_t$ are continuous. From Egorov’s Theorem (see, e.g. [3, Theorem 8.3]) we deduce that for every $\varepsilon > 0$, there exists a set $S_\varepsilon$ with $\mu(S^C_\varepsilon) < \varepsilon$ such that $\omega_n|_{S_\varepsilon}$ converges uniformly. Now, fixing $\varepsilon > 0$, we wish to show that the uniform convergence of $\omega_n$ to zero implies equicontinuity.

Let $\eta > 0$. Since $\lim_{n \to \infty} \omega_n = 0$ uniformly on $S_\varepsilon$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\omega_n < \eta \text{ for every } t \in S_\varepsilon.$$  

Hence, choosing $\delta = \frac{1}{n_0}$ in (3) and evaluating $\omega_\delta$ on $F_\varepsilon$, we obtain

$$\sup\{|f_t(x) - f_t(y) : |x - y| < \delta|\} < \eta.$$  

This means $F_\varepsilon$ is equicontinuous, since $\eta$ was chosen arbitrarily. \qed

From Lemma 5.3 and Lusin’s Theorem we can finally deduce Theorem 5.1.

**Proof of Theorem 5.1.** Since $f$ is Borel measurable, we have that $f^x$, where $f^x(t) := f(t, x)$, is a Borel measurable function for almost every $x \in [0, 1]$. We can therefore choose a dense countable subset $X = \{x_n; n \in \mathbb{N}\} \subset \Omega$ such that $f^{x_n}$ is Borel measurable for every $n \in \mathbb{N}$. By Lusin’s theorem, for every function $f^{x_n}, n \in \mathbb{N}$, and any fixed small parameter $\varepsilon$ there is a set $U_n \subset [0, 1]$ such that $\mu((U_n)^C) < \varepsilon$ and $f^{x_n}|_{U_n}$ is continuous. Now define $V_\varepsilon := \bigcup_{n=1}^\infty U_n$ and apply Lemma 5.3 to the family $\{f(t, \cdot); t \in [0, 1]\}$ to obtain a set $S_\varepsilon$ such that $\{f(t, \cdot); t \in S_\varepsilon\}$ is equicontinuous. Now, setting $T_\varepsilon := S_\varepsilon \cap V_\varepsilon$, we conclude the construction, since $\{f(t, \cdot); t \in T_\varepsilon\}$ is equicontinuous and for fixed $\eta > 0, n \in \mathbb{N}$,

$$|f(s, x_n) - f(t, x_n)| < \eta \text{ for all } t, s \in T_\varepsilon \text{ with } |t - s| < \delta_n.$$  

It remains to check, that the restriction $f|_{T_\varepsilon \times \Omega}$ is continuous, which can be done by a straightforward diagonalisation argument. \qed

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