Exploiting Chordality in Optimization Algorithms for Model Predictive Control

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Abstract In this chapter we show that chordal structure can be used to devise efficient optimization methods for many common model predictive control problems. The chordal structure is used both for computing search directions efficiently as well as for distributing all the other computations in an interior-point method for solving the problem. The chordal structure can stem both from the sequential nature of the problem as well as from distributed formulations of the problem related to scenario trees or other formulations. The framework enables efficient parallel computations.

1 Introduction

Model Predictive Control (MPC) is an important class of controllers that are being employed more and more in industry. [25]. It has its root going back to [7]. The success is mainly because it can handle constraints on control signals and/or states in a systematic way. In the early years its applicability was limited to slow processes, since an optimization problem has to be solved at each sampling instant. Tremendous amount of research has been spent on overcoming this limitation. One avenue has been what is called explicit MPC, [2], where the optimization problem is solved parametrically off-line. Another avenue has been to exploit the inherent structure of the optimization problems stemming from MPC, [12, 30, 27, 31, 26, 15, 16, 28, 18, 1, 8, 4, 29, 17, 9, 10, 24]. Typically this has been to use Riccati recursions to efficiently compute search directions for Interior Point (IP) methods or actives set methods to solve the optimization problem. In this paper we will argue that the important structures that have been exploited can all
be summarized as *chordal structure*. Because of this the same structure exploiting software can be used to speed up all computations for MPC. This is irrespective of what MPC formulation is considered and irrespective of what type of optimization algorithm is used. We assume that the reader is familiar with the receding horizon strategy of MPC and we will only discuss the associated constrained finite-time optimal control problems. We will from now on refer to the associated problem as the MPC problem. We will mostly assume quadratic cost and linear dynamics and inequality constraints. Even if not all problems fall into this category, problems with quadratic objective and linear constraints are often solved as subproblems in solvers.

The remaining part of the paper is organized as follows. We will in Section 2 discuss how chordal sparsity arises and how it can be utilized in general convex optimization problems to obtain computations distributed over a so called clique tree. The presentation is based on [19]. In Section 3 we will then discuss the classical formulation of MPC, and how the problem can be solved using an IP method. Specifically we will discuss how the equations for the search directions can be distributed over the clique tree. The well-known backward dynamic programming solution will be derived as a special case. We will see that we can also do forward dynamic programming, combinations of forward and backward dynamic programming, and even dynamic programming in parallel. In Section 4 we will discuss regularized MPC. In Section 5 we will discuss stochastic MPC. In Section 6 we will discuss distributed MPC, and finally in Section 7 we will give some conclusions, discuss generalizations of our results and directions for future research.

**Notation**

We denote with $\mathbb{R}$ the set of real numbers, with $\mathbb{R}^n$ the set of $n$-dimensional real-valued vectors and with $\mathbb{R}^{m \times n}$ the set of real-valued matrices with $m$ rows and $n$ columns. We denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{N}_n$ the subset $\{1, 2, \ldots, n\}$ of $\mathbb{N}$. For a vector $x \in \mathbb{R}^n$ the matrix $X = \text{diag}(x)$ is a diagonal matrix with the components of $x$ on the diagonal. For two matrices $A$ and $B$ the matrix $A \oplus B$ is a block-diagonal matrix with $A$ as the 1,1-block and $B$ as the 2,2-block. For a symmetric matrix $A$ the notation $A \succeq 0$ is equivalent to $A$ being positive (semi)-definite.

2 **Chordal Sparsity and Convex Optimization**

Consider the following convex optimization problem

$$
\min_x F_1(x) + \cdots + F_N(x),
$$

(1)
where $F_i : \mathbb{R}^n \to \mathbb{R}$ for all $i = 1, \ldots, N$. We assume that each function $F_i$ is only dependent on a small subset of elements of $x$. Let us denote the ordered set of these indexes by $J_i \subseteq \mathbb{N}_n$. We also denote the ordered set of indexes of functions that depend on $x_i$ with $\mathcal{J}_i = \{k \mid i \in J_k\} \subseteq \mathbb{N}_N$. We can then rewrite the problem in (1), as

$$\min_x F_1(E_{J_1}x) + \cdots + F_N(E_{J_N}x),$$  \hfill (2)

where $E_{J_i}$ is a 0–1 matrix that is obtained from an identity matrix of order $n$ by deleting the rows indexed by $\mathbb{N}_n \setminus J_i$. The functions $\bar{F}_i : \mathbb{R}^{|J_i|} \to \mathbb{R}$ are lower dimensional descriptions of $F_i$s such that $\bar{F}_i(x) = F_i(E_{J_i}x)$ for all $x \in \mathbb{R}^n$ and $i \in \mathbb{N}_N$. For instance consider the following optimization problem

$$\min_x F_1(x) + F_2(x) + F_3(x) + F_4(x) + F_5(x) + F_6(x),$$  \hfill (3)

and let us assume that $x \in \mathbb{R}^8$, $J_1 = \{1, 3\}$, $J_2 = \{1, 2, 4\}$, $J_3 = \{4, 5\}$, $J_4 = \{3, 4\}$, $J_5 = \{3, 6, 7\}$ and $J_6 = \{3, 8\}$. We then have $\mathcal{J}_1 = \{1, 2\}$, $\mathcal{J}_2 = \{2\}$, $\mathcal{J}_3 = \{1, 4, 5, 6\}$, $\mathcal{J}_4 = \{2, 3, 4\}$, $\mathcal{J}_5 = \{3\}$, $\mathcal{J}_6 = \{5\}$, $\mathcal{J}_7 = \{5\}$ and $\mathcal{J}_8 = \{6\}$.

This problem can then be written in the same format as in (2) as

$$\min_x F_1(x_1, x_3) + F_2(x_1, x_2, x_4) + F_3(x_4, x_5) + F_4(x_3, x_4) + F_5(x_3, x_6, x_7) + F_6(x_3, x_8).$$  \hfill (4)

The formulation of coupled problems as in (2) enables us to get a more clear picture of the coupling in the problem. Next we describe how the coupling structure in (1) can be expressed graphically using undirected graphs.

### 2.1 Sparsity Graph

A graph $G$ is specified by its vertex and edge sets $V$ and $\mathcal{E}$, respectively. A graph that sheds light on the coupling structure of the problem is the so-called sparsity graph, $G_s$, of the problem. This graph is undirected with vertex set $V_s = \mathbb{N}_n$ and edge set $\mathcal{E}$ with $(i, j) \in \mathcal{E}$ if and only if $\mathcal{J}_i \cap \mathcal{J}_j \neq \emptyset$. Let us now reconsider the example in (4).

The sparsity graph for this problem is illustrated in Figure 1.

As we will see later graph representations of the coupling structure in problems play an important role in designing distributed algorithms for solving coupled problems and gaining insight regarding their distributed implementations. Specifically, chordal graphs and their characteristics play a major role in the design of our proposed algorithm. This is the topic of the next subsection.
2.2 Chordal Graphs and Clique Trees

A graph $G(V, \mathcal{E})$ with vertex set $V$ and edge set $\mathcal{E}$ is chordal if every of its cycles of length at least four has a chord, where a chord is an edge between two non-consecutive vertices in a cycle, [13, Ch. 4]. A clique of $G$ is a maximal subset of $V$ that induces a complete subgraph on $G$. Consequently, no clique of $G$ is entirely contained in any other clique, [5]. Let us denote the set of cliques of $G$ as $C_G = \{C_1, \ldots, C_q\}$. There exists a tree defined on $C_G$ such that for every $C_i, C_j \in C_G$ with $i \neq j$, $C_i \cap C_j$ is contained in all the cliques in the path connecting the two cliques in the tree. This property is called the clique intersection property, and trees with this property are referred to as clique trees. For instance the graph in Figure 1 is chordal and has five cliques, namely $C_1 = \{1, 2, 4\}$, $C_2 = \{1, 3, 4\}$, $C_3 = \{4, 5\}$, $C_4 = \{3, 6, 7\}$ and $C_5 = \{3, 8\}$. A clique tree over these cliques is given in Figure 2. This tree then satisfies the clique intersection property, e.g., notice that $C_2 \cap C_3 = \{4\}$ and the only clique in the path between $C_2$ and $C_3$, that is $C_1$, also includes $\{4\}$.

Chordal graphs and their corresponding clique trees play a central role in our distributed algorithm. For chordal graphs there are efficient methods for computing cliques and clique trees. Sparsity graphs do not have to be chordal. However, there are simple heuristic methods, [6, 21], to compute a chordal embedding of such graphs, where a chordal embedding of a graph $G(V, \mathcal{E})$ is a chordal graph with the same vertex set and an edge set $\mathcal{E}_e$ such that $\mathcal{E} \subseteq \mathcal{E}_e$. For the MPC problems we consider we will derive chordal embeddings manually when required. We will now discuss distributed optimization using message-passing.

2.3 Distributed Optimization Using Message-passing

Consider the optimization problem in (1). Let $G_s(V_s, \mathcal{E}_s)$ denote the chordal sparsity graph for this problem and let $C_s = \{C_1, \ldots, C_q\}$ and $T(V_t, \mathcal{E}_t)$ be its set of cliques and a corresponding clique tree, respectively. It is possible to devise a distributed algorithm for solving this problem that utilizes the clique tree $T$ as its computational graph. This means that the nodes $V_t = N_q$ act as computational agents and collaborate with their neighbors that are defined by the edge set $\mathcal{E}_t$ of the tree. For
example, the sparsity graph for the problem in (4) has five cliques and a clique tree over these cliques is illustrated in Figure 2. This means the problem can be solved distributedly using a network of five computational agents, each of which needs to collaborate with its neighbors as defined by the edges of the tree, e.g., Agent 2 needs to collaborate with agents 1, 4, 5.

![Clique tree](image)

**Fig. 2** Clique tree for the sparsity graph of the problem in (4).

In order to specify the messages exchanged among these agents, we first assign different terms of the objective function in (1) to each agent. A valid assignment is that $F_i$ can only be assigned to agent $j$ if $J_i \subseteq C_j$. We denote the ordered set of indices of terms of the objective function assigned to agent $j$ by $\phi_j$. For instance, for the problem in (4), assigning $\bar{F}_1$ and $\bar{F}_4$ to Agent 2 would be a valid assignment since $J_1, J_4 \subseteq C_2$ and hence $\phi_2 = \{1, 4\}$. Notice that the assignments are not unique and for instance there can exist agents $j$ and $k$ with $j \neq k$ so that $J_j \subseteq C_j$ and $J_k \subseteq C_k$ making assigning $F_i$ to agents $j$ or $k$ both valid. It can be shown that for every term of the objective function there will always exist an agent that it can be assigned to.

We will now express the messages that are exchanged among neighboring agents. Particularly, let $i$ and $j$ be two neighboring agents, then the message sent from agent $i$ to agent $j$, $m_{ij}$, is given by

$$m_{ij}(x_{S_{ij}}) = \min_{x_{C_i \setminus S_{ij}}} \left\{ \sum_{k \in \phi_i} \bar{F}_k(x_k) + \sum_{k \in \text{Ne}(i) \setminus \{j\}} m_{ki}(x_{S_{ik}}) \right\},$$

(5)

where $S_{ij} = C_i \cap C_j$ is the so-called separator set of agents $i$ and $j$. As a result, for agent $i$ to be able to send the correct message to agent $j$ it needs to wait until it has received all the messages from its neighboring agents other than $j$. Hence, the information required for computing a message also sets the communication protocol for this algorithm. Specifically, it sets the ordering of agents in the message-passing procedure in the algorithm, where messages can only be initiated from the leaves of the clique tree and upwards to the root of the tree, which is referred to as an upward pass through the tree. For instance, for the problem in (4) and as can be seen in Figure 2, $\text{Ne}(2) = \{1, 4, 5\}$. Then the message to be sent from Agent 2 to Agent 1 can be written as
\[ m_{21}(x_1, x_4) = \min_{x_3} \{ F_1(x_1, x_3) + F_4(x_3, x_4) + m_{42}(x_3) + m_{52}(x_3) \} . \]  \hspace{1cm} (6)

which can only be computed if Agent 2 has received the messages from agents 4 and 5.

The message, \( m_{ij} \), that every agent \( j \) receives from a neighboring agent \( i \) in fact summarizes all the necessary information that agent \( j \) needs from all the agents on the \( i \)-side of the edge \((i, j)\). With this description of messages and at the end of an upward-pass through the clique tree, the agent at the root of the tree, indexed \( r \), will have received messages from all its neighbors. Consequently, it will have all the necessary information to compute its optimal solution by solving the following optimization problem

\[ x_{cr}^* = \arg\min_{x_{cr}} \left\{ \sum_{k \in \Phi_r} \tilde{F}_k(x_{kr}) + \sum_{k \in \text{Ne}(r)} m_{kr}(x_{kr}) \right\} . \]  \hspace{1cm} (7)

Then the root sends the computed optimal solution \( (x_{cr}^*)^r \) to its children, i.e., to all agents \( j \in \text{ch}(r) \). Here \( (x_{cr}^*)^r \) denotes the optimal solution computed by agent \( r \). Then all these agents, similar to the agent at the root, will then have received messages from all their neighbors and can compute their corresponding optimal solution as

\[ x_{ci}^* = \arg\min_{x_{ci}} \left\{ \sum_{k \in \Phi_i} \tilde{F}_k(x_{ki}) + \sum_{k \in \text{Ne}(i) \setminus \text{par}(i)} m_{ki}(x_{ki}) \right\} . \]  \hspace{1cm} (8)

The same procedure is executed downward through the tree until we reach the leaves, where each agent \( i \), having received the computed optimal solution by its parent, i.e., \( (x_{ci}^*)^\text{par}(i) \), computes its optimal solution by

\[ x_{ci}^* = \arg\min_{x_{ci}} \left\{ \sum_{k \in \Phi_i} \tilde{F}_k(x_{ki}) + \sum_{k \in \text{Ne}(i) \setminus \text{par}(i)} m_{ki}(x_{ki}) \right\} . \]  \hspace{1cm} (9)

where \( \text{par}(i) \) denotes the index for the parent of agent \( i \).

Notice that in case the optimal solution of (1) is not unique, then we need to modify the algorithm with regularization terms, see [19].

So far we have provided a distributed algorithm to compute optimal solution for convex optimization problems in the form (1). However, this algorithm relies on the fact that we are able to eliminate variables and compute the optimal objective value as a function of the remaining ones in closed form. This capability is essential, particularly for computing the exchanged messages among agents and in turn seemingly limits the scope of problems that can be solved using this algorithm. It turns out that the described algorithm can be incorporated within a primal-dual interior-point method to solve general convex optimization problems, distributedly,
Then the message passing algorithm is used to solve the quadratic subproblems that approximate the overall problem at each and every iterate in order to compute search directions. Hence the messages will be quadratic functions that are easy to represent.

2.4 Interior-Point Methods

All the MPC problems that we encounter in this chapter are special cases of Quadratic Programs (QPs). We will now discuss how such problems can be solved using IP methods. Consider the QP

$$\min \frac{1}{2} z^T Q z + q^T z$$

s.t. $\mathcal{A} z = b$

$$\mathcal{D} z \leq e$$

where $\mathcal{Q} \succeq 0$, i.e. positive semidefinite, where $\mathcal{A}$ has full row rank, and where the matrices and vectors are of compatible dimensions. The inequality in (12) is component-wise inequality. The Karush-Kuhn-Tucker (KKT) optimality conditions for this problem is

$$\begin{bmatrix} Q & \mathcal{D}^T \\ \mathcal{A} & I \end{bmatrix} \begin{bmatrix} z \\ \lambda \end{bmatrix} = \begin{bmatrix} -q \\ b \\ e \end{bmatrix}$$

and $(\mu, s) \geq 0$, where $M = \text{diag}(\mu)$. Blank entries in a matrix are the same as zero entries. Above $\lambda$ and $\mu$ are the Lagrange multipliers for the equality and inequality constraints, respectively. The vector $s$ is the slack variable for the inequality constraints. In IP methods one linearizes the above equations to obtain equations for search directions:

$$\begin{bmatrix} Q & \mathcal{D}^T \\ \mathcal{D} & I \end{bmatrix} \begin{bmatrix} z \\ \lambda \end{bmatrix} = \begin{bmatrix} r_z \\ r_\lambda \end{bmatrix}$$

$$\begin{bmatrix} \mathcal{D} & I \end{bmatrix} \begin{bmatrix} \mu \\ s \end{bmatrix} = \begin{bmatrix} r_\mu \\ r_s \end{bmatrix}$$

where $S = \text{diag}(s)$, and where $r = (r_z, r_\lambda, r_\mu, r_s)$ is some residual vector that depends on what IP method is used. The quantities $r, S$ and $M$ depend on the value of the current iterate in the IP method. From the last two rows above we have $\Delta s = r_\mu - \mathcal{D} \Delta z$ and $\Delta \mu = S^{-1}(r_s - M \Delta s)$. After substitution of these expressions into the first two rows we obtain

$$\begin{bmatrix} Q + \mathcal{D}^T S^{-1} M \mathcal{D} & \mathcal{A}^T \\ \mathcal{A} & I \end{bmatrix} \begin{bmatrix} \Delta z \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} r_z - \mathcal{D}^T S^{-1}(r_s - M r_\mu) \\ r_\lambda \end{bmatrix}$$
We notice that the search directions are obtained by solving an indefinite symmetric linear system of equations. The indefinite matrix is referred to as the KKT matrix, and it is invertible if and only if
\[ Q_{s} = Q + S^T S^{-1} Q \] is positive definite on the nullspace of \( Q \). Notice that the KKT matrix for the search directions can be interpreted as the optimality conditions of a QP with only equality constraints, where the quadratic weight is modified such that it is larger the closer the iterates are to the boundary of the constraints. In case this QP is loosely coupled with chordal structure message passing over a clique tree can be used to compute the search directions in a distributed way as described above. The key to this will be to solve parametric QPs, which is the next topic. Before finishing this section we remark that for active set methods similar QPs also have to be solved. There will however be additional equality constraints depending on what constraints are active at the current iterate.

### 2.5 Parametric QPs

Consider the quadratic optimization problem
\[
\min_{z} \frac{1}{2} z^T M z + m^T z \tag{17}
\]
subject to \( C z = d \) \tag{18}

with \( C \) full row rank and \( M \succeq 0 \). The KKT conditions for the optimal solution are:
\[
\begin{bmatrix} M & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} z \\ \lambda \end{bmatrix} = \begin{bmatrix} -m \\ d \end{bmatrix}
\]

These equations have a unique solution if and only if \( M + C^T C \succ 0 \). Now consider the partitioning of the above problem defined by
\[
M = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} ; \quad C = \begin{bmatrix} A & B \\ D \end{bmatrix} ; \quad d = \begin{bmatrix} e \\ f \end{bmatrix} ; \quad m = \begin{bmatrix} q \\ r \end{bmatrix} ; \quad z = \begin{bmatrix} x \\ y \end{bmatrix}
\]

with \( A \) full row rank. We then assume that the variables related to a specific leaf are \( x \), and we want to solve the problem associated with the leaf, i.e.
\[
\min_{x,y} \frac{1}{2} \begin{bmatrix} x^T & y^T \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + q^T x \tag{19}
\]
subject to \( A x + By = e \) \tag{20}

with respect to all \( y \). The KKT conditions for this problem are
Notice that the solution \( x \) will be affine in \( y \), and hence when it is substituted back into the objective function we obtain a quadratic message in \( y \). The 1,1-block of \( M + C^T C \) is \( Q + A^T A \), which by the Schur complement formula is positive definite. Hence the leaf problem has a unique solution. If we then substitute the solution of the leaf into the overall problem, we will have a unique solution also for this problem, since the overall problem has a unique solution. Because of this, every leaf in the message passing algorithm will have a problem with unique solutions assuming that the overall problem has a unique solution. This also goes for all nodes, since they will become leaves as other leaves are pruned away.

Notice that it is always possible to make sure that the matrix \( A \) has full rank for a leaf by pre-processing of the inequality constraints. In case \( A \) does not have full row rank, perform a rank-revealing factorization such that the constraints can be written

\[
\begin{bmatrix}
\bar{A}_1 \\
0
\end{bmatrix} x + \begin{bmatrix}
\bar{B}_1 \\
\bar{B}_2
\end{bmatrix} y = \begin{bmatrix}
\bar{e}_1 \\
\bar{e}_2
\end{bmatrix}
\]

and append the constraint \( \bar{B}_2 y = \bar{e}_2 \) to belong to

\[ Dy = f \]

This can be done recursively over the clique tree so that the parametric QPs for each node satisfies the rank condition, 19.

We will now see how chordal sparsity and distributed computations can be used to solve optimization problems arising in MPC efficiently.

3 Classical MPC

A classical MPC problem can be cast in the form

\[
\begin{align*}
\min_u & \frac{1}{2} \sum_{k=0}^{N-1} \begin{bmatrix} x_k^T \\ u_k \end{bmatrix}^T Q \begin{bmatrix} x_k \\ u_k \end{bmatrix} + \frac{1}{2} x_N^T S x_N \\
\text{s.t.} & \quad x_{k+1} = Ax_k + Bu_k + v_k, \quad x_0 = \bar{x} \\
& \quad C x_k + D u_k \leq e_k
\end{align*}
\]

(21) 

(22) 

(23)

with \( A, B, C, D, Q, S, \bar{x}, e_k, \) and \( v_k \) given, and where \( u = (u_0, u_1, \ldots, u_{N-1}) \) are the optimization variables. The dimensions of the control signal \( u_k \) and the state vector \( x_k \) are \( m \) and \( n \), respectively. The number of inequality constraints are \( q \) for each time index \( k \). The dimensions of all other quantities are defined to be consistent with this. We assume that \( Q \succeq 0 \) and that \( S \succeq 0 \). This is a convex quadratic optimization problem. When the inequality constraints are not present it is a classical Linear Quadratic
(LQ) control problem. It is of course possible to extend the problem formulation to time-varying dynamics, inequality constraints and weights. Also the extension to a linear term in the objective function is straightforward.

3.1 Quadratic Program

The classical formulation in (21–23) is equivalent to (10–12) with $q = 0$,

$$
\begin{align*}
\mathbf{z} &= (x_0, u_0, x_1, u_1, \ldots, x_{N-1}, u_{N-1}, x_N) \\
\mathbf{\lambda} &= (\lambda_0, \lambda_1, \ldots, \lambda_N) \\
\mathbf{b} &= (\bar{x}, v_0, v_1, \ldots, v_{N-1}) \\
\mathbf{e} &= (e_0, e_1, \ldots, e_{N-1})
\end{align*}
$$

and

$$
\mathcal{A} = 
\begin{bmatrix}
1 & & & & \\
-A & -B & I & & \\
& -A & -B & I & \\
& & & \ddots & \\
& & & & -A & -B & I
\end{bmatrix}
$$

$$
\mathcal{D} = [C D] \oplus [C D] \oplus \cdots \oplus [C D]
$$

$$
\mathcal{Q} = Q \oplus Q \oplus \cdots \oplus Q \oplus S
$$

We see that the data matrices are banded. Hence, sparse linear system solvers could be used when solving the KKT equations for search directions in an IP method, but we will see that the structure within the bands can be further utilized. Also we notice that the matrix $\mathcal{Q}$ in (16) has the same structure as $\mathcal{D}$. Therefore the KKT matrix for the search directions can be interpreted as the optimality conditions of an unconstrained LQ control problem for the search directions, where the weights are modified such that they are larger the closer the iterates are to the boundary of the constraints. Since inequality constraints not affect the structure of the KKT matrix we will from now on not consider them when we discuss the different MPC problems.

The classical formulation without any constraints is usually solved using a backward Riccati recursion. We will see how this can be obtained from the general techniques presented above. This is the same derivation that is usually done using backward dynamic programming. We will also investigate forward dynamic programming, and finally we will see how one can obtain parallel computations.
3.2 Backward Dynamic Programming

When looking at (21–22) it can be put in the almost separable formulation in (2) by defining

\[
\bar{F}_1(x_0, u_0, x_1) = I_D(x_0) + \frac{1}{2} [x_0^T Q x_0 + 1]\]
\[
\bar{F}_{k+1}(x_k, u_k, x_{k+1}) = \begin{cases} 
\frac{1}{2} [x_k^T Q x_k + 1], & k = 1, \ldots, N-2 \\
\frac{1}{2} [x_{N-1}^T Q x_{N-1} + 1], & k = N-1
\end{cases}
\]

where \(I_{C_k}(x_k, u_k, x_{k+1})\) is the indicator function for the set

\[
C_k = \{(x_k, u_k, x_{k+1}) \mid x_{k+1} = Ax_k + Bu_k\}
\]

and where \(I_{\bar{D}}(x_0)\) is the indicator function for the set

\[
\bar{D} = \{x_0 \mid x_0 = \bar{x}\}
\]

We have assumed that \(v_k = 0\). It should be stressed that the derivations done below easily can be extended to the general case. The sparsity graph for this problem is depicted in Figure 3 for the case of \(N = 3\). For ease of notation we label the nodes with the states and control signals. The cliques for this graph are

\[
C_{k+1} = \{x_k, u_k, x_{k+1}\}, \quad k = 0, \ldots, N-1
\]

To obtain a backward dynamic problem formulation we define a clique tree by taking \(C_1\) as root as seen in Figure 3. We then assign \(\bar{F}_k\) to \(C_k\). This is the only information that has to be provided to a general purpose software for solving loosely coupled quadratic programs.

One may of course derive the well-known Riccati-recursion based solution from what has been defined above. The \(k\):th problem to solve is

\[
\min_{u_k} \frac{1}{2} [x_k^T Q x_k + m_{k+1, k}(x_{k+1})]
\]

s.t. \(x_{k+1} = Ax_k + Bu_k\)

for given \(x_k\) starting with \(k = N - 1\) going down to \(k = 0\), where \(m_{N,N-1}(x_N) = \frac{1}{2} x_N^T S x_N\). The optimality conditions are for \(k = N - 1\)

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1 Here we use a supernode for all components of a state and a control signal, respectively. In case there is further structure in the dynamic equations such that not all components of the control signal and the states are coupled, then more detailed modeling could potentially be beneficial.
Fig. 3  Sparsity graph for the problem in (21–22) to the left and its corresponding clique tree to the right.

\[
\begin{bmatrix}
S & 0 & I \\
Q_2 & -B^T & I \\
I & -B & 0
\end{bmatrix}
\begin{bmatrix}
x_N \\
u_{N-1} \\
\lambda_N
\end{bmatrix}
= \begin{bmatrix}
Q_{12}x_{N-1} \\
0 \\
A x_{N-1}
\end{bmatrix}
\]

We notice that no pre-processing of constraints is needed since \([I-B]\) has full row rank. Therefore by the results in Section 2.5, if the overall problem has a unique solution, so does the above problem. It is however not easy to give conditions for when the overall problem has a unique solution. The optimality conditions above are equivalent to \(\lambda_N = -S x_N, x_N = Ax_{N-1} + Bu_{N-1}\) and the equation

\[G_{N-1} u_{N-1} = -H_{N-1}^T x_{N-1}\]

where \(G_{N-1} = Q_2 + B^T S B\) and \(H_{N-1} = Q_{12} + A^T S B\). Let \(F_{N-1} = Q_1 + A^T S A\). Then, if \([A B]\) has full row rank, \(Q\) is positive definite on the nullspace of \([A B]\), and if \(S > 0\), it holds that

\[F_{N-1} H_{N-1} \geq Q + [A B]^T S [A B] > 0\]

\(^2\) Notice that the assumptions are not necessary for the block matrix to be positive definite. Moreover, for the case when it is only positive semidefinite, we still have a solution \(u_{N-1}\), but it is not unique. One may use pseudo inverse to obtain one solution. This follows from the generalization of the Schur complement formula. The full row rank assumption is equivalent to \((A,B)\) not having any uncontrollable modes corresponding to zero eigenvalues. The positive definiteness of \(Q\) on the nullspace of \([A B]\) is equivalent to \(C(zI - A)^{-1}B + D\) not having any zeros at the origin where \(Q = [C D]^T [C D]\) is a full rank factorization.
Therefore is \( G_{N-1} \) positive definite by the Schur complement formula, and hence there is a unique solution

\[
    u_{N-1} = -G_{N-1}^{-1}H_{N-1}^T x_{N-1}
\]

Back-substitution of this into the objective function shows that

\[
    m_{N-1,N-2}(x_{N-1}) = \frac{1}{2} x_{N-1}^T S_{N-1} x_{N-1}
\]

where \( S_{N-1} = F_{N-1} - H_{N-1} G_{N-1}^{-1} H_{N-1} \) which is the first step of the well-known Riccati recursion. Notice that \( S_{N-1} \) is positive definite by the Schur complement formula. Repeating the above steps for the remaining problems shows that the overall solution can be obtained using the Riccati recursion. Notice that a general purpose solver instead factorizes the local optimality conditions at each step. It is well-known that the Riccati recursion provides a factorization of the overall KKT matrix, [28]. It can be shown that the message passing algorithm does the same, [19]. The main point of this chapter is, however, that there is no need to derive Riccati recursions or have any interpretations as factorizations. This becomes even more evident when we look at not so well-studied MPC formulations, where the corresponding structure making it possible to see how Riccati recursions can be used are only revealed after cumbersome manipulations of the KKT equations. In some cases it is not even possible to derive Riccati recursions, which is the point of the next subsection.

### 3.3 Forward Dynamic Programming

Instead of taking \( C_1 \) as root as we did in the previous subsection we can also choose \( C_N \) as root. We then obtain a forward dynamic programming formulation. We assign the functions to the cliques in the same way. The initial problem to solve is

\[
    \min_{u_0} \frac{1}{2} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}^T Q \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \\
    \text{s.t. } x_1 = A x_0 + B u_0, \quad x_0 = \bar{x}
\]

parametrically for all possible values of \( x_1 \). Here we realize that the constraints for \((x_0, u_0)\) do not satisfy the full row rank assumption, i.e.

\[
    \begin{bmatrix} I \\ A B \end{bmatrix}
\]

does not have full row rank. Therefore pre-processing is required, which can be done using e.g. a QR-factorization on \( B \). This will result in constraints on \( x_1 \) that should be passed to the next problem, and then this procedure should be repeated.
Because of this, there is no such clean solution procedure for the forward approach as for the backward approach, and particularly no Riccati recursion based approach. However, the general message passing approach indeed works.

3.4 Parallel Computation

In the previous cases we had a tree that was a chain. It was then possible to let either of the end cliques be the root of the tree. However, nothing stops us from picking up any one of the middle cliques as the root. This would result in two branches, and it would then be possible to solve the problems in the two branches in parallel, one branch using the backward approach, and one using the forward approach. This does however not generalize to more than two parallel branches. If we want to have three or more we need to proceed differently.

To this end, let us consider a simple example where $N = 6$. Let us also assume that we want to solve this problem using two computational agents such that each would perform independently, and hence in parallel. For this, we define dummy variables $\bar{u}_0$ and $\bar{u}_1$ and constrain them as

$$\bar{u}_0 = x_3, \quad \bar{u}_1 = x_6$$

This is similar to what is done in [24] to obtain parallel computations. We also define the following sets

$$\mathcal{C}_{-1} = \{x_0 : x_0 = \bar{x}\}$$

$$\mathcal{C}_k = \{(x_k, u_k, x_{k+1}) : x_{k+1} = Ax_k + Bu_k; k = 0, 1\}$$

$$\mathcal{C}_2 = \{(x_2, u_2, \bar{u}_0) : \bar{u}_0 = Ax_2 + Bu_2\}$$

$$\mathcal{C}_k = \{(x_k, u_k, x_{k+1}) : x_{k+1} = Ax_k + Bu_k; k = 3, 4\}$$

$$\mathcal{C}_5 = \{(x_5, u_5, \bar{u}_1) : \bar{u}_1 = Ax_5 + Bu_5\}$$

$$\mathcal{D}_0 = \{(x_3, \bar{u}_0) : \bar{u}_0 = x_3\}$$

$$\mathcal{D}_1 = \{(x_6, \bar{u}_1) : \bar{u}_1 = x_6\}$$

Then the problem in (21-22) can be equivalently written as
\[
\min_{u} \frac{1}{2} \sum_{k=0}^{1} \sum_{j=2}^{1} \begin{bmatrix} x_{k} \\ u_{k} \end{bmatrix}^{T} Q \begin{bmatrix} x_{k} \\ u_{k} \end{bmatrix} + \mathcal{J}_{c_{1}} \{ x_{k}, u_{k}, x_{k+1} \} + \\
\frac{1}{2} \begin{bmatrix} x_{2} \\ u_{2} \end{bmatrix}^{T} Q \begin{bmatrix} x_{2} \\ u_{2} \end{bmatrix} + \mathcal{J}_{c_{2}} \{ x_{2}, u_{2}, \bar{u}_{0} \} + \\
\frac{1}{2} \sum_{k=3}^{1} \begin{bmatrix} x_{k} \\ u_{k} \end{bmatrix}^{T} Q \begin{bmatrix} x_{k} \\ u_{k} \end{bmatrix} + \mathcal{J}_{c_{3}} \{ x_{k}, u_{k}, x_{k+1} \} + \\
\frac{1}{2} \begin{bmatrix} x_{5} \\ u_{5} \end{bmatrix}^{T} Q \begin{bmatrix} x_{5} \\ u_{5} \end{bmatrix} + \mathcal{J}_{c_{5}} \{ x_{5}, u_{5}, \bar{u}_{1} \} + \frac{1}{2} \bar{u}_{1}^{T} S \bar{u}_{1} + \\
\mathcal{J}_{c_{-1}} \{ x_{0} \} + \mathcal{J}_{\mathcal{D}_{0}} \{ x_{3}, \bar{u}_{0} \} + \mathcal{J}_{\mathcal{D}_{1}} \{ x_{6}, \bar{u}_{1} \}
\] (25)

where \( \mathcal{J}_{x} (x) \) is the indicator function for the set \( \mathcal{X} \). Notice that it is important to define \( C_{2} \) in terms of \( \bar{u}_{0} \) and not in terms of \( x_{3} \), and similarly for \( C_{5} \). This trick will allow us to have two independent computational agents. The reason for this will be clear later on.

Let us consider the sparsity graph for the problem in (25), which is depicted in Figure 4, marked with solid lines. In order to take obtain a clique tree that facilitates parallel computations, we first add edges, marked with dotted lines, between \( x_{0}, \bar{u}_{0}, x_{3}, \bar{u}_{1} \) and \( x_{6} \) such that they form a maximal complete subgraph in the graph. The original graph was chordal, but adding the dotted edges destroyed this. Therefore we make a chordal embedding by adding the dashed edges. We actually add even more edges, which corresponds to merging cliques. These are the dash-dotted edges. The reason we do this is that we do not need computational agents for more cliques than the ones we get after the merging. A clique tree which corresponds to the modified sparsity graph in Figure 4 is illustrated in Figure 5. This clique tree obviously enables parallel computations. The different terms in (25) are assigned such that rows one and two are assigned to the left branch, rows three and four to the right branch and the last row to the root.

Notice that in this particular example we obtained a clique tree with two parallel branches. However, we can generalize to several parallel branches by introducing more dummy variables and constraints. Also it is worth pointing out that the subproblem which is assigned to the root of clique tree can be seen as an LQ problem and hence we can use the procedure discussed above recursively. This is similar to what is presented in [24]. This is obtained in the example above by not connecting all of \( x_{0}, \bar{u}_{0}, x_{3}, \bar{u}_{1} \) and \( x_{6} \). Instead one should connect all of \( x_{0}, \bar{u}_{0} \) and \( x_{3} \) and then all of \( x_{3}, \bar{u}_{1} \) and \( x_{6} \) separately. This will then split the clique \( c_{1} \) into two cliques in the clique tree. The dynamics for this LQ problem is defined by the sets \( \mathcal{D}_{k}, k = 0, 1 \). The incremental costs for this problem will be the messages sent by the children of these cliques. Notice that we do not really have to know that the resulting problem will be an LQ problem—that is just an interpretation. We only need to know how

---

3 The added edges corresponds to saying that terms in the objective function are functions of variables which they are actually not.
to split the root clique into one for each parallel branch. If we want four parallel branches the clique tree will be like in Figure 6.

Fig. 4 A modified sparsity graph for the problem in (25). The initial sparsity graph, without any modification, is marked with solid lines.

\[ C_1 = \{ x_0, \bar{u}_0, x_3, \bar{u}_1, x_6 \} \]

\[ C_2 = \{ x_0, u_0, x_1, \bar{u}_0 \} \]

\[ C_3 = \{ x_1, u_1, x_2, \bar{u}_0 \} \]

\[ C_4 = \{ x_2, u_2, \bar{u}_0 \} \]

\[ C_5 = \{ x_3, u_3, x_4, \bar{u}_1 \} \]

\[ C_6 = \{ x_4, u_4, x_5, \bar{u}_1 \} \]

\[ C_7 = \{ x_5, u_5, \bar{u}_1 \} \]

Fig. 5 Corresponding clique tree for the modified sparsity graph shown in Figure 4.
3.5 Merging of Cliques

It is not always the case that one has one agent or processor available for each and every clique in the clique tree. What then can be done is to merge cliques until there are as many cliques as there are processors. Let us consider the clique tree in Figure 6. We can merge the cliques in each and every parallel branch into one clique. The resulting clique tree will then be a chain as depicted in Figure 7. This could have been done even before the clique tree was constructed. However, it is beneficial for each of the four agents in the example to utilize the additional structure within their cliques, i.e. that they have an internal chain structure. This information would have been lost in case the cliques were merged before the clique tree was formed.

4 Regularized MPC

A regularized MPC problem is obtained from the above MPC problem by adding a regularization term to the objective function. Typically this is term is either proportional to the squared $\ell^2$ (Euclidian) norm, so-called Tikhonov-regularization, or proportional to the $\ell^1$ norm, so-called Lasso-regularization. In both cases convexity
is preserved, since the sum of two convex functions is convex. In the former case a quadratic objective function will remain quadratic. This is however not the case for Lasso-regularization. A fairly general Lasso-regularized problem is:

$$\min \frac{1}{2} \sum_{k=0}^{N-1} y_k^T Q y_k + \frac{1}{2} x_k^T S x_k + \sum_{k=0}^{N-1} \|y_k\|_1$$ (26)

s.t. $x_{k+1} = Ax_k + Bu_k + v_k, \quad x_0 = \bar{x}$ (27)

$Cx_k + Du_k \leq e_k$ (28)

$Ex_k + Fu_k = y_k$ (29)

### 4.1 Equivalent QP

An equivalent problem formulation for the case of no inequality constraints and $v_k = 0$ is:
the different values of $A$ consider stochastic events to take place. The outcome of the stochastic events are $k$ place at each time stage $M$ of scenarios is $\text{ming from scenario trees can be expolitd, e.g.} [14, 23, 22, 11]$. The total number

$$W$$

will in this section consider a stochastic MPC problem based on a scenario

5 Stochastic MPC

We will in this section consider a stochastic MPC problem based on a scenario tree description. Several other authors have investigated how the structure stemming from scenario trees can be exploited, e.g. [14, 23, 22, 11]. The total number of scenarios is $M = d^r$, where $d$ is the number of stochastic events that can take place at each time stage $k$, and where $r$ is the number of time stages for which we consider stochastic events to take place. The outcome of the stochastic events are the different values of $A^k, B^k_l$ and $v^k_l$. Notice that for values of $k < r$ several of these

$$\min_u \frac{1}{2} \sum_{k=0}^{N-1} \left[ x_k^T Q x_k + \frac{1}{2} x_k^T S x_k + \sum_{k=0}^{N-1} t_k \right]$$

s.t. $x_{k+1} = Ax_k + Bu_k$, $x_0 = \bar{x}$

$$t_k \geq Ex_k + Fu_k$$

$$t_k \geq -Ex_k - Fu_k$$

This can be put in the almost separable formulation in (2) by defining

$$\tilde{F}_1(x_0, u_0, t_0, x_1) = \mathcal{F}_1(x_0) + \frac{1}{2} \left[ x_0^T Q x_0 \right] + t_0 + \mathcal{F}_0(x_0, u_0, t_0, x_1)$$

$$\tilde{F}_{k+1}(x_k, u_k, t_k, x_{k+1}) = \frac{1}{2} \left[ x_k^T Q x_k + t_k + \mathcal{F}_{k+1}(x_k, u_k, t_k, x_{k+1}) \right], \quad k \in \mathbb{N}_{N-2}$$

$$\tilde{F}_N(x_{N-1}, u_{N-1}, t_{N-1}, x_N) = \frac{1}{2} \left[ x_{N-1}^T Q x_{N-1} + t_{N-1} \right] + \mathcal{F}_{N-1}(x_{N-1}, u_{N-1}, t_{N-1}, x_N)$$

$$+ \mathcal{F}_{N-1}(x_{N-1}, u_{N-1}, t_{N-1}, x_N) + \frac{1}{2} x_N^T S x_N$$

where $\mathcal{F}_{k+1}(x_k, u_k, t_k, x_{k+1})$ is the indicator function for the set

$$\mathcal{G}_k = \{(x_k, u_k, x_{k+1}) \mid x_{k+1} = Ax_k + Bu_k; \quad t_k \geq Ex_k + Fu_k; \quad t_k \geq -Ex_k - Fu_k\}$$

and where $\mathcal{F}_1(x_0) = \{x_0 \mid x_0 = \bar{x}\}$. It should be stressed that the derivations done below easily can be extended to the general case. The sparsity graph for this problem is very similar to the one for the classical formulation. The cliques for this graph are

$$C_{k+1} = \{x_k, u_k, t_k, x_{k+1}\}, \quad k = 0, \ldots, N - 1$$

To obtain a backward dynamic problem formulation we define a clique tree by taking $\mathcal{C}_1$ as root similarly as for the classical formulation. We then assign $\tilde{F}_k$ to $\mathcal{C}_k$. This is the only information that has to be provided to a general purpose software for solving loosely coupled quadratic programs. We can do the forward dynamic programming formulation as well as a parallel formulation.

$$\min_u \frac{1}{2} \sum_{k=0}^{N-1} \left[ x_k^T Q x_k + \frac{1}{2} x_k^T S x_k + \sum_{k=0}^{N-1} t_k \right]$$

s.t. $x_{k+1} = Ax_k + Bu_k$, $x_0 = \bar{x}$

$$t_k \geq Ex_k + Fu_k$$

$$t_k \geq -Ex_k - Fu_k$$

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5 Stochastic MPC

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quantities are the same. The optimization problem is

\[
\min \sum_{j=1}^{M} \omega_j \left( \frac{1}{2} \sum_{k=0}^{N-1} [x_k^j]^T Q [x_k^j] + \frac{1}{2} (x_N^j)^T S x_N^j \right)
\]

subject to

\[
x_k^j = A_k^j x_k^j + B_k^j u_k^j + v_k^j, \quad x_0^j = \bar{x}
\]

where the index \( j \) refers to the \( j \)th scenario. Here we define \( u = (u^1, u^2, \ldots, u^M) \)
with \( u^j = (u_0^j, u_1^j, \ldots, u_{N-1}^j) \), and

\[
\bar{C} = \begin{bmatrix}
C_{1,2} & -C_{1,2} & C_{2,3} & -C_{2,3} & \cdots & \cdots & C_{M-1,M} & -C_{M-1,M}
\end{bmatrix}
\]

with

\[
C_{j,j+1} = [I \ 0]
\]

where \( I \) is an identity matrix of dimension \( m \) times the number of nodes that scenarios \( j \) and \( j+1 \) have in common. The value of \( \omega_j \) is the probability of scenario \( j \). The constraint \( \bar{C}u = 0 \) is the so-called non-ancipativity constraint. Instead of saying that each initial state \( x_0^j \) is equal to \( \bar{x} \) we instead consider the equivalent formulation \( x_0^j = \bar{x} \) and \( x_0^j = x_{0,j+1} \) for \( j \in N_{M-1} \).

We show in Figure 8 the sparsity graph for the case of \( d = r = 2 \) and \( N = 3 \). Then \( M = 4 \). We realize that this graph is not chordal. A chordal embedding is obtained by adding edges such that \( C_0 = \{x_0^1, x_0^2, x_0^3, x_0^4\} \) is a complete graph. Also edges should be added such that \( C_1^1 = \{x_0^1, u_0^1, x_1^1, x_0^2, x_1^2\} \) and \( C_1^3 = \{x_0^3, u_0^3, x_0^4, x_0^4, x_1^4\} \) are complete graphs. A clique tree for this chordal embedding is shown in Figure 9 where \( C_{k+1}^j = \{x_k^j, u_k^j, x_{k+1}^j\} \) with \( k \in N_{M-1} \). The assignments of functions are for \( C_0 = \{x_0^1, x_0^2, x_0^3, x_0^4\} \)

\[
\bar{F}_0(x_0^1, x_0^2, x_0^3, x_0^4) = \bar{F}_0(x_0^1) + \sum_{j=1}^{M-1} \bar{F}_0(x_0^j, x_0^{j+1})
\]

where \( \bar{F}_0 = \{x \mid x = \bar{x}\} \) and \( \bar{F}_0 = \{(x,y) \mid x = y\} \). For \( C_1^1 \) we assign

\[
F_1^1(x_0^1, u_0^1, x_0^1, x_0^2, u_0^2, x_1^1) = \frac{1}{2} \sum_{j=1}^{2} \omega_j \left( x_0^j \right)^T Q \left( x_0^j \right) + \bar{F}_0(x_0^j, u_0^j, x_1^1) + + \bar{F}_0(u_0^j, u_0^j),
\]

for \( C_1^3 \) we assign...
\[ F_k^j(x_k, u_k', x_{k+1}) = \omega_k \frac{1}{2} \left[ \begin{array}{c} x_k' \\ u_k' \\ x_{k+1} \\ u_0 \\ x_0 \\ u_0 \\ x_0 \\ u_0 \\ x_0 \\ u_0 \end{array} \right]^T Q \left[ \begin{array}{c} x_k' \\ u_k' \\ x_{k+1} \\ u_0 \\ x_0 \\ u_0 \\ x_0 \\ u_0 \\ x_0 \\ u_0 \end{array} \right] + \mathcal{J}_k(x_k', u_k', x_{k+1}) + \mathcal{J}_0(u_0, u_0), \]

for \( C_{k+1} \), where \( k \in \{ N \} \) and \( j \in \{ M \} \), we assign

\[ F_k^j(x_k, u_k', x_{k+1}) = \omega_k \frac{1}{2} \left[ \begin{array}{c} x_k' \\ u_k' \\ x_{k+1} \\ u_0 \\ x_0 \\ u_0 \\ x_0 \\ u_0 \\ x_0 \\ u_0 \end{array} \right]^T Q \left[ \begin{array}{c} x_k' \\ u_k' \\ x_{k+1} \\ u_0 \\ x_0 \\ u_0 \\ x_0 \\ u_0 \\ x_0 \\ u_0 \end{array} \right] + \mathcal{J}_k(x_k', u_k', x_{k+1}) + \mathcal{J}_0(u_0, u_0), \]

and for \( C_N' \), where \( j \in \{ M \} \), we assign

\[ \bar{F}_k^j(x_0, u_0', x_1, x_0, u_0, x_1) = \sum_{j=3}^{4} \omega_j \frac{1}{2} \left[ \begin{array}{c} x_0' \\ u_0' \end{array} \right]^T Q \left[ \begin{array}{c} x_0' \\ u_0' \end{array} \right] + \mathcal{J}_k(x_0', u_0', x_1) + \mathcal{J}_0(x_0', u_0'), \]
\[
\bar{F}_N(x_{N-1}, u_{N-1}, x_N) = \omega \frac{1}{2} \begin{bmatrix} x_{N-1} \\ u_{N-1} \end{bmatrix}^T Q \begin{bmatrix} x_{N-1} \\ u_{N-1} \end{bmatrix} + \omega \frac{1}{2} (x_N)^T S x_N + \mathcal{J}_{N-1} (x_{N-1}, u_{N-1}, x_N)
\]

where \( \mathcal{J}_{N-1} (x_{N-1}, u_{N-1}, x_N) \) is the indicator function for the set

\[
\mathcal{J}_{N-1} = \{ (x_{N-1}, u_{N-1}, x_N) \mid x_{N-1} = A x_{N-1} + B u_{N-1} \}
\]

It is possible to introduce even further parallelism by combining the above formulation with a parallel formulation in time as described in Section 3.4.

### 6 Distributed MPC

There are many ways to define distributed MPC problems. We like to think of them in the following format:

\[
\min u \sum_{i=1}^{M} \sum_{k=0}^{N-1} \begin{bmatrix} x_i^k \\ u_i^k \end{bmatrix}^T \mathcal{Q} \begin{bmatrix} x_i^k \\ u_i^k \end{bmatrix} + \frac{1}{2} (x_i^N)^T S x_i^N \\
\text{s.t. } x_{i+1} = A^{(i)} x_i^k + B^{(i)} u_i^k + \sum_{j \in \mathcal{N}_i} A^{(i,j)} x_j^k + B^{(i,j)} u_j^k + v_i^k, \quad x_0 = \bar{x}_i
\]

for \( i \in \mathcal{N}_M \), where \( \mathcal{N}_i \subset \mathcal{N}_M \setminus \{i\} \). We see that the only coupling in the problem is in the dynamic constraints through the summation over \( \mathcal{N}_i \), which typically contains few elements. If one consider a sparsity graph for the above problem one will realize that it is not necessarily chordal. Some heuristic method, such as presented in [6, 21], can most likely be applied successfully in many cases to obtain a sparse chordal embedding of the sparsity graph. From this a clique tree can be computed using other algorithms presented in [6, 21]. See also [19] for a more detailed discussion on how to compute clique trees.

We may consider distributed problems that are stochastic as well. Also extensions to parallelism in time is possible.

### 7 Conclusions

We have in this chapter shown how it is possible to make use of the inherent chordal structure of many MPC formulations in order to exploit IP methods that make use of any chordal structure to distribute its computations over several computational agents that can work in parallel. We have seen how the classical backward Riccati recursion can be seen as a special case of this, albeit not a parallel recursion, but
We have also discussed distributed MPC and stochastic MPC over scenario trees. The latter formulation can probably be extended also to robust MPC over scenario trees. Then the subproblems will be quadratic feasibility problems and not quadratic programs. Also it should be possible to consider sum-of-norms regularized MPC. We also believe that it is possible to exploit structure in MPC coming from spatial discretization of PDEs using chordal sparsity. How to carry out these extensions is left for future work.

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24 Anders Hansson and Sina Khosheftrat Pakazad

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