EINSTEIN METRICS ON COMPACT LIE GROUPS WHICH ARE NOT NATURALLY REDUCTIVE

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Abstract. The study of left-invariant Einstein metrics on compact Lie groups which are naturally reductive was initiated by J. E. D’Atri and W. Ziller in 1979. In the present work we prove existence of non-naturally reductive Einstein metrics on the compact simple Lie groups $SO(n) \ (n \geq 11)$, $Sp(n) \ (n \geq 3)$, $E_6$, $E_7$, and $E_8$.

1. Introduction

A Riemannian manifold $(M, g)$ is called Einstein if the Ricci tensor $\text{r}(g)$ of the metric $g$ satisfies $\text{r}(g) = eg$ for some constant $e$. General existence results are few and difficult to obtain. Among them we mention the works [2], [3], [4], [11], and the survey [10].

For the case of compact Lie groups, left-invariant Einstein metrics have not been widely studied. Even in low dimensional examples such as $SU(3)$ and $SU(2) \times SU(2)$ the number of left-invariant Einstein metrics is still unknown. The only complete work is [7] by J.E. D’Atri and W. Ziller, in which they obtained a large number of left-invariant Einstein metrics that are naturally reductive.

The problem of finding non-naturally reductive left-invariant Einstein metrics on compact Lie groups seems to be harder, and in fact this is stressed in [7] (p. 62). In [9] the second author initiated the study of this problem, and he obtained non-naturally reductive Einstein metrics on the Lie group $SU(n)$ for $n \geq 6$ by using the method of Riemannian submersions (see e.g. [1], Chapter 9). In the present work we prove existence of left-invariant Einstein metrics on several compact Lie groups, which are not naturally reductive.

To every compact simple Lie group $G$ we associate a Kähler C-space, which is a homogeneous space $G/H$ with $H$ the centralizer of a torus in $G$ (also known as generalized flag manifold). It is known that Kähler C-spaces are classified by use of painted Dynkin diagrams, and that each of them can be fibered over an irreducible symmetric space $G/K$ of compact type under the twistor fibration ([6]). We assume that the isotropy representation of the corresponding Kähler C-space $G/H$ decomposes into two irreducible components. It is known that these are mutually non-equivalent as Ad$(H)$-modules. Then these spaces can be classified in terms of painted Dynkin diagrams with one black root, and to simplify our study, we divide these into four types Ia, Ib, IIa, and IIb, depending on whether the black root is next to the negative of the maximal root, and whether the black root separates the Dynkin diagram into one or two components.

It turns out that the left-invariant metrics $< , >$ on $G$ associated to $G/H$ depend on five or four parameters in general. We also consider left-invariant metrics $<< , >>$ on $G$ associated to the symmetric space $G/K$. By comparing these two metrics we obtain the

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components of the Ricci tensor of the metric $\langle , \rangle$ on $G$. In this way the Einstein equation reduces to a more explicit form as an algebraic system of equations. This system reduces further to a polynomial equation of one variable, and we prove existence of left-invariant Einstein metrics on $G$ by proving existence of positive solutions for such a polynomial equation. For certain cases of simple Lie groups it is possible to prove existence of more than one solutions.

For Kähler C-spaces of Type Ia and Ila the solutions correspond to naturally reductive Einstein metrics, whereas non-naturally reductive Einstein metrics on $G$ are obtained from Kähler C-spaces $G/H$ of Types Ib and Iib. The main result is the following:

**Theorem 1.** The compact simple Lie groups $SO(n)$ ($n \geq 11$), $Sp(n)$ ($n \geq 3$), $E_6$, $E_7$, and $E_8$ admit non-naturally reductive Einstein metrics.

We remark that it is still unknown whether the compact simple Lie groups $SU(n)$ ($n = 3, 4, 5$), $SO(n)$ ($n = 5, 7, 8, 9, 10$), $F_4$ and $G_2$ admit a non-naturally reductive Einstein metric.

2. The Ricci tensor of a $G$-invariant metric

Let $(M, g)$ be a Riemannian manifold and $I(M, g)$ the Lie group of all isometries of $M$. Then $(M, g)$ is said to be $K$-homogeneous if a Lie subgroup $K$ of $I(M, g)$ acts transitively on $M$. For a $K$-homogeneous Riemannian manifold $(M, g)$, we fix a point $o \in M$ and identify $M$ with $K/L$ where $L$ is the isotropy subgroup of $K$ at $o$. Let $\mathfrak{k}$ be the Lie algebra of $K$ and $\mathfrak{l}$ the subalgebra corresponding to $L$. Take a vector space $\mathfrak{p}$ complement to $\mathfrak{l}$ in $\mathfrak{k}$ with $\text{Ad}(L)\mathfrak{p} \subseteq \mathfrak{p}$. Then we may identify $\mathfrak{p}$ with $T_o(M)$ in a natural way. We can pull back the inner product $g_o$ on $T_o(M)$ to an inner product on $\mathfrak{p}$, denoted by $\langle , \rangle$. For $X \in \mathfrak{k}$ we will denote by $X_\mathfrak{l}$ (resp. $X_\mathfrak{p}$) the $\mathfrak{l}$-component (resp. $\mathfrak{p}$-component) of $X$. A homogeneous Riemannian metric on $M$ is said to be naturally reductive if there exist $K$ and $\mathfrak{p}$ as above such that

$$\langle [Z, X]_\mathfrak{p}, Y \rangle + \langle X, [Z, Y]_\mathfrak{p} \rangle = 0 \quad \text{for all } X, Y, Z \in \mathfrak{p}.$$

In [7] D’Atri and Ziller have investigated naturally reductive metrics among the left invariant metrics on compact Lie groups, and have given a complete classification in the case of simple Lie groups.

Let $G$ be a compact connected semi-simple Lie group, $H$ a closed subgroup of $G$, and let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{h}$ the subalgebra corresponding to $H$. We denote by $B$ the negative of the Killing form of $\mathfrak{g}$. Then $B$ is an $\text{Ad}(G)$-invariant inner product on $\mathfrak{g}$. Let $\mathfrak{m}$ be an orthogonal complement of $\mathfrak{h}$ with respect to $B$. Then we have

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad \text{Ad}(H)\mathfrak{m} \subseteq \mathfrak{m}.$$

Let $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_p$ be the decomposition into ideals of $\mathfrak{h}$, where $\mathfrak{h}_0$ is the center of $\mathfrak{h}$ and $\mathfrak{h}_i$ ($i = 1, \cdots, p$) are simple ideals of $\mathfrak{h}$. Let $A_0|_{\mathfrak{h}_0}$ be an arbitrary metric on $\mathfrak{h}_0$.

**Theorem 2.** (D’Atri-Ziller [7]) Under the notations above, a left invariant metric on $G$ of the form

$$(1) \quad \langle , \rangle = x \cdot B|_\mathfrak{m} + A_0|_{\mathfrak{h}_0} + u_1 \cdot B|_{\mathfrak{h}_1} + \cdots + u_p \cdot B|_{\mathfrak{h}_p} \quad (x, u_1, \cdots, u_p \in \mathbb{R}_+$$

is naturally reductive with respect to $G \times H$, where $G \times H$ acts on $G$ by $(g, h)yz = gyzh^{-1}$.

Moreover, if a left invariant metric $\langle , \rangle$ on a compact simple Lie group $G$ is naturally reductive, then there exists a closed subgroup $H$ of $G$ and the metric $\langle , \rangle$ is given by the form (1).
Let $m = m_1 \oplus \cdots \oplus m_q$ be a decomposition into irreducible $\mathrm{Ad}(H)$-modules $m_j (j = 1, \cdots, q)$, and assume that the $\mathrm{Ad}(H)$-modules $m_j$ are mutually non-equivalent, and that the ideals $h_i (i = 1, \cdots, p)$ of $h$ are mutually non-isomorphic. In particular, we assume that $\dim h_0 \leq 1$.

We consider the following left invariant metric on $G$ which is $\mathrm{Ad}(H)$-invariant:

\[(2) \quad < , > = u_0 \cdot B|_{h_0} + u_1 \cdot B|_{h_1} + \cdots + u_p \cdot B|_{h_p} + x_1 \cdot B|_{m_1} + \cdots + x_q \cdot B|_{m_q},\]

where $u_0, u_1, \cdots, u_p, x_1, \cdots, x_q \in \mathbb{R}_+$, and the $G$-invariant Riemannian metric on $G/H$:

\[(3) \quad ( , ) = x_1 \cdot B|_{m_1} + \cdots + x_q \cdot B|_{m_q}.\]

To compute the Ricci tensor of the left invariant metric $< , >$ on $G$ and the $G$-invariant Riemannian metric $( , )$ on $G/H$, we use the following notation. We write the decomposition $g = h_0 \oplus h_1 \oplus \cdots \oplus h_p \oplus m_1 \oplus \cdots \oplus m_q$ (resp. $m = m_1 \oplus \cdots \oplus m_q$) as $g = w_0 \oplus w_1 \oplus \cdots \oplus w_p \oplus w_{p+1} \oplus \cdots \oplus w_{p+q}$ (resp. $m = w_{p+1} \oplus \cdots \oplus w_{p+q}$).

Note that the space of left invariant symmetric covariant 2-tensors on $G$ which are $\mathrm{Ad}(H)$-invariant is given by

\[(4) \quad \{v_0 \cdot B|_{w_0} + v_1 \cdot B|_{w_1} + \cdots + v_{p+q} \cdot B|_{w_{p+q}} \mid v_0, v_1, \cdots, v_{p+q} \in \mathbb{R}\}\]

and the space of $G$-invariant symmetric covariant 2-tensors on $G/H$ is given by

\[(5) \quad \{z_{p+1} \cdot B|_{w_{p+1}} + \cdots + z_{p+q} \cdot B|_{w_{p+q}} \mid z_{p+1}, \cdots, z_{p+q} \in \mathbb{R}\}.\]

In particular, the Ricci tensor $r$ of a left invariant Riemannian metric $< , >$ on $G$ is a left invariant symmetric covariant 2-tensor on $G$ which is $\mathrm{Ad}(H)$-invariant and thus $r$ is of the form (4), and the Ricci tensor $\bar{r}$ of a $G$-invariant Riemannian metric on $G/H$ is a $G$-invariant symmetric covariant 2-tensor on $G/H$, and thus $\bar{r}$ is of the form (5).

Let $\{e_\alpha\}$ be a $B$-orthonormal basis adapted to the decomposition of $g$, i.e., $e_\alpha \in w_i$ for some $i$, and $\alpha < \beta$ if $i < j$ (with $e_\alpha \in w_i$ and $e_\beta \in w_j$). We set $A_{\alpha \beta} = B(e_\alpha, e_\beta, e_\gamma)$ so that $[e_\alpha, e_\beta] = \sum_\gamma A_{\alpha \beta} e_\gamma$, and set $\begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A_{\alpha \beta})^2$, where the sum is taken over all indices $\alpha, \beta, \gamma$ with $e_\alpha \in w_i$, $e_\beta \in w_j$, $e_\gamma \in w_k$. Then $\begin{bmatrix} k \\ ij \end{bmatrix}$ is independent of the $B$-orthonormal bases chosen for $w_i, w_j, w_k$, and

\[(6) \quad \begin{bmatrix} k \\ ij \end{bmatrix} = \begin{bmatrix} k \\ ji \end{bmatrix} = \begin{bmatrix} j \\ ki \end{bmatrix}.\]

We write a metric on $G$ of the form (2) as

\[(7) \quad g = y_0 \cdot B|_{w_0} + y_1 \cdot B|_{w_1} + \cdots + y_p \cdot B|_{w_p} + y_{p+1} \cdot B|_{w_{p+1}} + \cdots + y_{p+q} \cdot B|_{w_{p+q}},\]

where $y_0, y_1, \cdots, y_{p+q} \in \mathbb{R}_+$, and a metric on $G/H$ of the form (3) as

\[(8) \quad h = w_{p+1} \cdot B|_{w_{p+1}} + \cdots + w_{p+q} \cdot B|_{w_{p+q}}\]

where $w_{p+1}, \cdots, w_{p+q} \in \mathbb{R}_+$.

**Lemma 3.** Let $d_k = \dim w_k$.

(1) The components $r_0, r_1, \cdots, r_{p+q}$ of the Ricci tensor $r$ of the metric $g$ of the form (7) on $G$ are given by

\[(9) \quad r_k = \frac{1}{2y_k} + \frac{1}{4d_k} \sum_{j,i} \frac{y_k}{y_j y_i} \begin{bmatrix} k \\ ji \end{bmatrix} - \frac{1}{2d_k} \sum_{j,i} \frac{y_j}{y_k y_i} \begin{bmatrix} j \\ ki \end{bmatrix} \quad (k = 0, 1, \cdots, p + q),\]
where the sum is taken over all \(i, j = 0, 1, \ldots, p + q\). Moreover, for each \(k\) we have
\[
\sum_{i,j} \left[ \frac{j}{k} \right] = d_k.
\]

(2) The components \(\bar{r}_{p+1}, \ldots, \bar{r}_{p+q}\) of the Ricci tensor \(\bar{r}\) of the metric \(h\) of the form [8] on \(G/H\) are given by
\[
\bar{r}_k = \frac{1}{2w_k} + \frac{1}{4d_k} \sum_{j,i} \frac{w_k}{w_j w_i} \left[ k \right] - \frac{1}{2d_k} \sum_{j,i} \frac{w_j}{w_k w_i} \left[ j \right] \quad (k = p + 1, \ldots, p + q),
\]
where the sum is taken over all \(i, j = p + 1, \ldots, p + q\).

Proof. Let \(\{e^{(k)}_\alpha\}_{\alpha=1}^{d_k}\) be an orthonormal basis of \(\mathfrak{m}_k\) \((k = 0, 1, \ldots, p + q)\) with respect to the inner product \(B\). Put \(X^{(k)}_\alpha = \frac{1}{\sqrt{y_k}} e^{(k)}_\alpha\). Then \(\{X^{(k)}_\alpha\}_{\alpha=1}^{d_k}\) is a \(\langle\cdot,\cdot\rangle\)-orthonormal basis of \(\mathfrak{m}_k\) \((k = 0, 1, \ldots, p + q)\). The Ricci tensor \(r\) of the metric \(g\) is given by the following (cf. [1], pp. 184-185):
\[
r(X, X) = -\frac{1}{2} \sum_{\alpha} \langle [X, X_\alpha], [X, X_\alpha] \rangle + \frac{1}{2} B(X, X) + \frac{1}{4} \sum_{\alpha, \beta} \langle [X_\alpha, X_\beta], X \rangle >^2
\]

for \(X \in \mathfrak{g}\), where \(\{X_\alpha\}\) is an orthonormal basis of \(\mathfrak{g}\) with respect to the metric \(g\). From the above equation, we have that
\[
r_k = r(X^{(k)}_\alpha, X^{(k)}_\alpha) = -\frac{1}{2} \sum_{j,i} \frac{y_j}{y_k y_l} \sum_s B([e^{(k)}_\alpha, e^{(i)}_s]_{\mathfrak{m}_j}, [e^{(k)}_\alpha, e^{(i)}_s]_{\mathfrak{m}_j}) + \frac{1}{2y_k} \sum_{j,i} \frac{y_j}{y_k y_l} \sum_s B([e^{(j)}_s, e^{(i)}_l]_{\mathfrak{m}_k}, e^{(k)}_\alpha)^2.
\]

As we have remarked above,
\[
d_k r_k = \sum_{\alpha=1}^{d_k} r(X^{(k)}_\alpha, X^{(k)}_\alpha) = \frac{d_k}{2y_k} - \frac{1}{2} \sum_{j,i} \frac{y_j}{y_k y_l} \left[ j \right] + \frac{1}{4} \sum_{j,i} \frac{y_k}{y_k y_l} \left[ j \right].
\]

□

3. Decomposition associated to Kähler C-spaces

Let \(G\) be a compact semi-simple Lie group, \(\mathfrak{g}\) the Lie algebra of \(G\) and \(\mathfrak{t}\) a maximal abelian subalgebra of \(\mathfrak{g}\). We denote by \(\mathfrak{g}^\mathbb{C}\) and \(\mathfrak{t}^\mathbb{C}\) the complexification of \(\mathfrak{g}\) and \(\mathfrak{t}\) respectively. We identify an element of the root system \(\Delta\) of \(\mathfrak{g}^\mathbb{C}\) relative to the Cartan subalgebra \(\mathfrak{t}^\mathbb{C}\) with an element of \(\sqrt{-1}\mathfrak{t}\) by the duality defined by the Killing form of \(\mathfrak{g}^\mathbb{C}\). Let \(\Pi = \{\alpha_1, \cdots, \alpha_l\}\) be a fundamental system of \(\Delta\) and \(\{\Lambda_1, \cdots, \Lambda_l\}\) the fundamental weights of \(\mathfrak{g}^\mathbb{C}\) corresponding to \(\Pi\), that is
\[
\frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \quad (1 \leq i, j \leq l).
\]

Let \(\Pi_0\) be a subset of \(\Pi\) and \(\Pi - \Pi_0 = \{\alpha_{i_1}, \cdots, \alpha_{i_r}\}\) \((1 \leq \alpha_{i_1} < \cdots < \alpha_{i_r} \leq l)\). We put \([\Pi_0] = \Delta \cap \{\Pi_0\}_\mathbb{Z}\) where \(\{\Pi_0\}_\mathbb{Z}\) denotes the subspace of \(\sqrt{-1}\mathfrak{t}\) generated by \(\Pi_0\). Consider the root space decomposition of \(\mathfrak{g}^\mathbb{C}\) relative to \(\mathfrak{t}^\mathbb{C}\):
\[
\mathfrak{g}^\mathbb{C} = \mathfrak{t}^\mathbb{C} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha^\mathbb{C}.
\]
We define a parabolic subalgebra $u$ of $g^C$ by 

$$u = t^C + \sum_{\alpha \in [\Pi_0] \cup \Delta^+} g_{\alpha}^C,$$

where $\Delta^+$ is the set of all positive roots relative to $\Pi$. Note that the nilradical $n$ of $u$ is given by 

$$n = \sum_{\alpha \in \Delta^+ - [\Pi_0]} g_{\alpha}^C.$$

We denote by $\tilde{\alpha}$ the highest root of $g^C$.

Let $G^C$ be a simply connected complex semi-simple Lie group whose Lie algebra is $g^C$ and $U$ the parabolic subgroup of $G^C$ generated by $u$. Then the complex homogeneous manifold $G^C/U$ is compact simply connected and $G$ acts transitively on $G^C/U$. Note also that $H = G \cap U$ is a connected closed subgroup of $G$, $G^C/U = G/H$ as $C^\infty$-manifolds, and $G^C/U$ admits a $G$-invariant Kähler metric.

Let $h$ be the Lie algebra of $H$ and $h^C$ the complexification of $h$. Then we have a direct decomposition 

$$u = h^C \oplus n, \quad h^C = t^C + \sum_{\alpha \in [\Pi_0]} g_{\alpha}^C.$$

**Proposition 4.** ([6], Proposition 4.3) Let $z$ be the center of the nilpotent Lie algebra $n$. Then we have $\text{ad}(h^C)(z) \subset z$ and the action of $h^C$ on $z$ is irreducible. Moreover, the $\text{ad}(h^C)$-module $z$ is generated by the highest root space $g_{\tilde{\alpha}}^C$.

Take a Weyl basis $E_{-\alpha} \in g_{\alpha}^C$ ($\alpha \in \Delta$) with 

$$[E_{\alpha}, E_{-\alpha}] = -\alpha (\alpha \in \Delta)$$

$$[E_{\alpha}, E_{\beta}] = \begin{cases} N_{\alpha,\beta}E_{\alpha + \beta} & \text{if } \alpha + \beta \in \Delta \\ 0 & \text{if } \alpha + \beta \notin \Delta, \end{cases}$$

where $N_{\alpha,\beta} = N_{-\alpha,-\beta} \in \mathbb{R}$. Then we have 

$$g = t + \sum_{\alpha \in \Delta} \{ \mathbb{R}(E_{\alpha} + E_{-\alpha}) + \mathbb{R}\sqrt{-1}(E_{\alpha} - E_{-\alpha}) \}$$

and the Lie subalgebra $h$ is given by 

$$h = t + \sum_{\alpha \in [\Pi_0]} \{ \mathbb{R}(E_{\alpha} + E_{-\alpha}) + \mathbb{R}\sqrt{-1}(E_{\alpha} - E_{-\alpha}) \}.$$
by Proposition 3 we have that $\mathfrak{z} = \mathfrak{n}_t$. We define a subspace $\mathfrak{m}_k$ of $\mathfrak{m}$ by

$$
\mathfrak{m}_k = \sum_{\alpha \in \Delta_k} \{ \mathbb{R}(E_\alpha + E_{-\alpha}) + \mathbb{R}\sqrt{-1}(E_\alpha - E_{-\alpha}) \}.
$$

Then $\mathfrak{m}_k (k = 1, \cdots, t)$ are $\text{Ad}(H)$-invariant subspaces of $\mathfrak{m}$ and $\mathfrak{m} = \sum_{j=1}^{t} \mathfrak{m}_j$ is an irreducible decomposition of $\mathfrak{m}$, therefore $t = q$. The following proposition is well known.

**Proposition 5.** ([5]) The Kähler C-space $G^C/U = G/H$ admits a $G$-invariant Kähler-Einstein metric given by

$$
B|\mathfrak{m}_1 + 2B|\mathfrak{m}_2 + \cdots + qB|\mathfrak{m}_q.
$$

In the following we consider the case of $q = 2$, that is $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Then we have a pair $(\Pi, \Pi_0)$ which has an irreducible decomposition

$$
\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2
$$

as $\text{Ad}(H)$-modules, where $\mathfrak{h}_0$ is the center of $\mathfrak{h}$ and $\mathfrak{h}_i (i = 1, 2)$ are simple ideals of $\mathfrak{h}$. In such a decomposition of $\mathfrak{g}$, either one of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ is zero or both are non-zero.

We distinguish Kähler C-spaces with $q = 2$ into types depending on whether the simple root $\alpha_{i_0}$ separates the Dynkin diagram in one or two components, and whether or not it is connected to $-\check{\alpha}$.

| Type | $\mathfrak{g}$ | $(\Pi, \Pi_0)$ | $\mathfrak{dim} \mathfrak{h}_1$ | $\mathfrak{dim} \mathfrak{h}_2$ | $\mathfrak{dim} \mathfrak{m}_1$ | $\mathfrak{dim} \mathfrak{m}_2$ |
|------|------------|----------|--------|--------|--------|--------|
| Ia   | $C_n$      | $1 2 \cdots n-1 n$ | $0$    | $(n-1)(2n-1)$ | $4(n-1)$ | $2$    |
|      | $E_6$      |          | $0$    | $40$    | $35$    | $2$    |
|      | $E_7$      |          | $0$    | $64$    | $66$    | $2$    |
|      | $E_8$      |          | $0$    | $112$   | $133$   | $2$    |
|      | $F_4$      |          | $0$    | $28$    | $21$    | $2$    |
|      | $G_2$      |          | $0$    | $8$     | $3$     | $2$    |

The Dynkin diagram corresponding to $\Pi_0 = \Pi - \{ \alpha_{i_0} \}$ with one component, obtained by removing the vertex $\bullet$, and $\{ \alpha_{i_0} \}$ is next to the negative of the maximal root.
Type Ib

| $\mathfrak{g}$ | $(\Pi, \Pi_0)$ | $\dim \mathfrak{h}_1$ \hspace{1em} $\dim \mathfrak{h}_2$ | $\dim \mathfrak{m}_1$ \hspace{1em} $\dim \mathfrak{m}_2$ |
|----------------|----------------|-------------------|-------------------|
| $E_7$          | ![Dynkin diagram for $E_7$] | 48 \hspace{1em} 0 | 70 \hspace{1em} 14 |
| $E_8$          | ![Dynkin diagram for $E_8$] | 91 \hspace{1em} 0 | 128 \hspace{1em} 28 |
| $F_4$          | ![Dynkin diagram for $F_4$] | 21 \hspace{1em} 0 | 16 \hspace{1em} 14 |

The Dynkin diagram corresponding to $\Pi_0 = \Pi - \{\alpha_{i_0}\}$ with one component, obtained by removing the vertex •, and $\{\alpha_{i_0}\}$ is not next to the negative of the maximal root.

Type IIa

| $\mathfrak{g}$ | $(\Pi, \Pi_0)$ | $\dim \mathfrak{h}_1$ \hspace{1em} $\dim \mathfrak{h}_2$ | $\dim \mathfrak{m}_1$ \hspace{1em} $\dim \mathfrak{m}_2$ |
|----------------|----------------|-------------------|-------------------|
| $B_n$          | ![Dynkin diagram for $B_n$] | $\begin{cases} 3 & \text{if } p = 1 \\ (n - 2)(2n - 3) & \text{if } 2 \leq p \leq n - 1 \end{cases}$ | $\begin{cases} 4(2n - 3) & \text{if } p = 1 \\ 2 & \text{if } 2 \leq p \leq n - 1 \end{cases}$ |
| $D_n$          | ![Dynkin diagram for $D_n$] | $\begin{cases} 3 & \text{if } p = 1 \\ (n - 2)(2n - 5) & \text{if } 2 \leq p \leq n - 1 \end{cases}$ | $\begin{cases} 8(n - 2) & \text{if } p = 1 \\ 2 & \text{if } 2 \leq p \leq n - 1 \end{cases}$ |

The Dynkin diagram corresponding to $\Pi_0 = \Pi - \{\alpha_{i_0}\}$ with two components, obtained by removing the vertex •, and $\{\alpha_{i_0}\}$ is next to the negative of the maximal root.

Type IIb

| $\mathfrak{g}$ | $(\Pi, \Pi_0)$ | $\dim \mathfrak{h}_1$ \hspace{1em} $\dim \mathfrak{h}_2$ | $\dim \mathfrak{m}_1$ \hspace{1em} $\dim \mathfrak{m}_2$ |
|----------------|----------------|-------------------|-------------------|
| $B_n$          | ![Dynkin diagram for $B_n$] | $\begin{cases} p^2 - 1 & \text{if } p = 1 \\ (n - p)(2n - p + 1) & \text{if } 2 \leq p \leq n - 1 \end{cases}$ | $\begin{cases} 2p(2(n - p) + 1) & \text{if } p = 1 \\ p(p - 1) & \text{if } 2 \leq p \leq n - 1 \end{cases}$ |
| $C_n$          | ![Dynkin diagram for $C_n$] | $\begin{cases} p^2 - 1 & \text{if } p = 1 \\ (n - p)(2n - p + 1) & \text{if } 2 \leq p \leq n - 1 \end{cases}$ | $\begin{cases} 4p(n - p) & \text{if } p = 1 \\ p(p + 1) & \text{if } 2 \leq p \leq n - 1 \end{cases}$ |
| $D_n$          | ![Dynkin diagram for $D_n$] | $\begin{cases} p^2 - 1 & \text{if } p = 1 \\ (n - p)(2n - p - 1) & \text{if } 2 \leq p \leq n - 1 \end{cases}$ | $\begin{cases} 4p(n - p) & \text{if } p = 1 \\ p(p - 1) & \text{if } 2 \leq p \leq n - 1 \end{cases}$ |
| $E_6$          | ![Dynkin diagram for $E_6$] | 24 \hspace{1em} 3 | 40 \hspace{1em} 10 |
| $E_7$          | ![Dynkin diagram for $E_7$] | 45 \hspace{1em} 3 | 64 \hspace{1em} 20 |

The Dynkin diagram corresponding to $\Pi_0 = \Pi - \{\alpha_{i_0}\}$ with two components, obtained by removing the vertex •, and $\{\alpha_{i_0}\}$ is not next to the negative of the maximal root.
Proposition 6. In the decomposition (12) we can take the ideal \( \mathfrak{h}_2 \) so that \( [\mathfrak{h}_2, \mathfrak{m}_2] = \{0\} \).

Proof. We may assume that \( \mathfrak{h}_2 \neq \{0\} \). Then there is only one simple root \( \alpha_{j_0} \) with \( (\alpha_{j_0}, \tilde{\alpha}) \neq 0 \) and thus we can take the ideal \( \mathfrak{h}_2 \) so that \( [\mathfrak{h}_2^c, E_{\tilde{\alpha}}] = \{0\} \). Since \( \mathfrak{m}_2 = [\mathfrak{h}_2^c, E_{\tilde{\alpha}}] \), we have that \( [\mathfrak{h}_2^c, \mathfrak{m}_2] = [\mathfrak{h}_2^c, E_{\tilde{\alpha}}] \subset [\mathfrak{h}_2^c, \mathfrak{h}_2^c, E_{\tilde{\alpha}}] + [\mathfrak{h}_2^c, [\mathfrak{h}_2^c, E_{\tilde{\alpha}}]] = \{0\} \). By the definition of \( \mathfrak{m}_2 \), we get the result. \( \Box \)

In case of the spaces in Table Ia we have that \( \mathfrak{h}_1 = \{0\} \), and for the spaces in Table Ib we have that \( \mathfrak{h}_2 = \{0\} \). Also, for the spaces of Tables IIa and IIb we have that \( \mathfrak{h}_1, \mathfrak{h}_2 \neq \{0\} \).

4. Einstein metrics on compact Lie groups of type II

We consider left invariant metrics

\[
<, > = u_0 \cdot B|_{\mathfrak{h}_0} + u_1 \cdot B|_{\mathfrak{h}_1} + u_2 \cdot B|_{\mathfrak{h}_2} + x_1 \cdot B|_{\mathfrak{m}_1} + x_2 \cdot B|_{\mathfrak{m}_2}
\]

on a compact Lie group \( G \) associated to Kähler C-spaces of Types IIa and IIb. Note that a metric (13) is also Ad(\( H \))-invariant.

Let \( d_1 = \dim \mathfrak{h}_1, d_2 = \dim \mathfrak{h}_2, d_3 = \dim \mathfrak{m}_1 \) and \( d_4 = \dim \mathfrak{m}_2 \). By the relations \( [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{h} + \mathfrak{m}_2, [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{h}, [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1 \), and Proposition 5 we see that \( \left[ \begin{array}{c} k \\ i \\ j \end{array} \right] \) are zero, except

\[
\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ i \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 13 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 22 \\ 14 \end{bmatrix}, \begin{bmatrix} 3 \\ 23 \end{bmatrix}, \begin{bmatrix} 4 \\ 33 \end{bmatrix}.
\]

By Lemma 3 we have that

\[
\begin{align*}
\left[ \begin{array}{c}
3 \\
0 \\
3
\end{array} \right] + \left[ \begin{array}{c}
4 \\
0 \\
4
\end{array} \right] &= 1, \\
\left[ \begin{array}{c}
1 \\
11 \\
13
\end{array} \right] + \left[ \begin{array}{c}
3 \\
14
\end{array} \right] &= d_1, \\
\left[ \begin{array}{c}
2 \\
22 \\
23
\end{array} \right] + \left[ \begin{array}{c}
3 \\
23
\end{array} \right] &= d_2,
\end{align*}
\]

and thus the components of the Ricci tensor \( r \) of the metric

\[
<, > = u_0 \cdot B|_{\mathfrak{h}_0} + u_1 \cdot B|_{\mathfrak{h}_1} + u_2 \cdot B|_{\mathfrak{h}_2} + x_1 \cdot B|_{\mathfrak{m}_1} + x_2 \cdot B|_{\mathfrak{m}_2}
\]

on \( G \) are given by:

\[
\begin{align*}
r_0 &= \frac{u_0}{4x_1^2} \begin{bmatrix} 0 \\ 33 \end{bmatrix} + \frac{u_0}{4x_2^2} \begin{bmatrix} 0 \\ 44 \end{bmatrix}, \\
r_1 &= \frac{1}{4d_1 u_1} \begin{bmatrix} 1 \\ 11 \\
13
\end{bmatrix} + \frac{u_1}{4d_1 x_1^2} \begin{bmatrix} 1 \\
33
\end{bmatrix} + \frac{u_1}{4d_1 x_2^2} \begin{bmatrix} 1 \\
44
\end{bmatrix}, \\
r_2 &= \frac{1}{4d_2 u_2} \begin{bmatrix} 2 \\
22
\end{bmatrix} + \frac{u_2}{4d_2 x_2^2} \begin{bmatrix} 2 \\
33
\end{bmatrix}, \\
r_3 &= \frac{1}{2x_1} - \frac{x_2}{2d_3 x_2^2} \begin{bmatrix} 4 \\
33
\end{bmatrix} - \frac{1}{2d_3 x_1^2} \left( u_0 \begin{bmatrix} 0 \\
33
\end{bmatrix} + u_1 \begin{bmatrix} 1 \\
33
\end{bmatrix} + u_2 \begin{bmatrix} 2 \\
33
\end{bmatrix} \right), \\
r_4 &= \frac{1}{x_2} \left( \frac{1}{2} - \frac{1}{2d_4} \begin{bmatrix} 3 \\
43
\end{bmatrix} \right) + \frac{x_2}{4d_4 x_1^2} \begin{bmatrix} 4 \\
33
\end{bmatrix} - \frac{1}{2d_4 x_2^2} \left( u_0 \begin{bmatrix} 0 \\
44
\end{bmatrix} + u_1 \begin{bmatrix} 1 \\
44
\end{bmatrix} \right). \tag{15}
\end{align*}
\]

We also see that the components of the Ricci tensor \( \bar{r} \) of the metric

\[
<, > = x_1 B|_{\mathfrak{m}_1} + x_2 B|_{\mathfrak{m}_2}
\]
are given by the following:

\[
\begin{align*}
\tilde{r}_1 &= \frac{1}{2x_1} - \frac{x_2}{2d_3x_1^2}\begin{bmatrix} 4 \\ 33 \end{bmatrix} \\
\tilde{r}_2 &= \frac{1}{x_2}\left(\frac{1}{2} - \frac{1}{2d_4}\begin{bmatrix} 3 \\ 43 \end{bmatrix}\right) + \frac{x_2}{4d_4x_1^2}\begin{bmatrix} 4 \\ 33 \end{bmatrix}.
\end{align*}
\]

By Proposition 5 the metric \(B|_{m_1} + 2B|_{m_2}\) is Kähler-Einstein, and thus we have

\[
\frac{1}{2} - \frac{1}{d_3}\begin{bmatrix} 4 \\ 33 \end{bmatrix} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2d_4}\begin{bmatrix} 3 \\ 43 \end{bmatrix}\right) + \frac{1}{2d_4}\begin{bmatrix} 4 \\ 33 \end{bmatrix}.
\]

Thus we get

\[
\begin{bmatrix} 4 \\ 33 \end{bmatrix} = \frac{d_3d_4}{(d_3 + 4d_4)}.
\]

We assume that \(\{\alpha_i_0\}\) is not next to the negative of the maximal root, and that \(\{\alpha_i_0\}\) separates the extended Dynkin diagram in two components, which is the case of Type IIb. The case of spaces of Type IIa will be examined in Section 6.

We set \(\mathfrak{t} = \mathfrak{h} \oplus \mathfrak{m}_2\) and \(\mathfrak{t}_1 = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{m}_2\). Then \(\mathfrak{t}, \mathfrak{t}_1\) are subalgebras of \(\mathfrak{g}\) and \(\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{h}_2\). We also see that \((\mathfrak{g}, \mathfrak{t})\) is an irreducible symmetric pair. Thus we obtain an irreducible decomposition \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}_2 \oplus \mathfrak{m}_1\) as \(\text{Ad}(K)\)-modules, which are mutually non-equivalent.

We consider the following left invariant metrics on \(G\) which are also \(\text{Ad}(K)\)-invariant:

\[
<v_1 \cdot B|_{\mathfrak{t}_1} + v_2 \cdot B|_{\mathfrak{h}_2} + v_3 \cdot B|_{\mathfrak{m}_1}>.
\]

Note that the only non-zero \(\begin{bmatrix} 1 \\ 11 \end{bmatrix}\), \(\begin{bmatrix} 1 \\ 33 \end{bmatrix}\), \(\begin{bmatrix} 2 \\ 22 \end{bmatrix}\), \(\begin{bmatrix} 2 \\ 33 \end{bmatrix}\).

Let \(f_1 = \dim \mathfrak{t}_1\), \(f_2 = \dim \mathfrak{h}_2\) and \(f_3 = \dim \mathfrak{m}_1\). By Lemma 3(1) the components of the Ricci tensor \(\tilde{r}\) of the metric \(v_1 \cdot B|_{\mathfrak{t}_1} + v_2 \cdot B|_{\mathfrak{h}_2} + v_3 \cdot B|_{\mathfrak{m}_1}\) on \(G\) are given by the following:

\[
\begin{align*}
\tilde{r}_1 &= \frac{1}{4f_1v_1}\begin{bmatrix} 1 \\ 11 \end{bmatrix} + \frac{v_1}{4f_1v_3^2}\begin{bmatrix} 1 \\ 33 \end{bmatrix} \\
\tilde{r}_2 &= \frac{1}{4f_2v_2}\begin{bmatrix} 2 \\ 22 \end{bmatrix} + \frac{v_2}{4f_2v_3^2}\begin{bmatrix} 2 \\ 33 \end{bmatrix} \\
\tilde{r}_3 &= \frac{1}{2v_3} - \frac{1}{2f_3v_3^2}\left(v_1\begin{bmatrix} 1 \\ 33 \end{bmatrix} + v_2\begin{bmatrix} 2 \\ 33 \end{bmatrix}\right).
\end{align*}
\]

Note that equations (18) are obtained from equations (15) by setting \(v_1 = u_0 = u_1 = x_2\), \(v_2 = u_2\) and \(v_3 = x_1\). In fact, for these values the metrics < > and < < > > on \(G\) coincide, so the components of the corresponding Ricci tensors are equal. Therefore, it follows that
Lemma 7. For the metric $\langle \ , \rangle$ on $G$, the non-zero numbers $\begin{bmatrix} k \\ ij \end{bmatrix}$ are given as follows:

$$
\begin{bmatrix}
0 \\ 33
\end{bmatrix} = \frac{d_3}{(d_3 + 4d_4)}, \quad \begin{bmatrix} 0 \\ 44 \end{bmatrix} = \frac{4d_4}{(d_3 + 4d_4)}
$$

$$
\begin{bmatrix}
1 \\ 11
\end{bmatrix} = \frac{2d_4(2d_1 + 2 - d_4)}{(d_3 + 4d_4)}, \quad \begin{bmatrix} 1 \\ 33 \end{bmatrix} = \frac{d_1d_3}{(d_3 + 4d_4)}
$$

$$
\begin{bmatrix}
1 \\ 44
\end{bmatrix} = \frac{2d_4(d_4 - 2)}{(d_3 + 4d_4)}, \quad \begin{bmatrix} 2 \\ 22 \end{bmatrix} = d_2 - \frac{d_3(d_3 + 2d_4 - 2d_1 - 2)}{2(d_3 + 4d_4)}
$$

$$
\begin{bmatrix}
2 \\ 33
\end{bmatrix} = \frac{d_3(d_3 + 2d_4 - 2d_1 - 2)}{2(d_3 + 4d_4)}, \quad \begin{bmatrix} 4 \\ 33 \end{bmatrix} = \frac{d_3d_4}{(d_3 + 4d_4)}.
$$

Thus we have

Proposition 8. The components of the Ricci tensor $r$ of the metric

$$
\langle \ , \rangle = u_0 \cdot B|_{b_0} + u_1 \cdot B|_{b_1} + u_2 \cdot B|_{b_2} + x_1 \cdot B|_{m_1} + x_2 \cdot B|_{m_2}
$$
on $G$ are given by

$$
\begin{cases}
    r_0 = \frac{u_0 d_3}{4x_1^2(d_3 + 4d_4)} + \frac{u_0 d_4}{x_2^2(d_3 + 4d_4)}, \\
    r_1 = \frac{1}{2d_1 u_1} \frac{d_4(2d_1 + 2 - d_4)}{(d_3 + 4d_4)} + \frac{u_1 d_3}{4x_1^2(d_3 + 4d_4)} + \frac{u_1 d_4(d_4 - 2)}{2d_1 x_2^2(d_3 + 4d_4)}, \\
    r_2 = \frac{1}{4d_2 u_2} \left( \frac{d_2}{2} - \frac{d_3(d_3 + 2d_4 - 2d_1 - 2)}{2(d_3 + 4d_4)} \right) + \frac{u_2}{4d_2 x_1^2} \frac{d_3(d_3 + 2d_4 - 2d_1 - 2)}{2(d_3 + 4d_4)}, \\
    r_3 = \frac{1}{2x_1} - \frac{x_2}{2x_1^2(d_3 + 4d_4)} \frac{d_4}{2d_1 u_1} \left( \frac{u_0}{d_3 + 4d_4} + \frac{u_1 d_1}{(d_3 + 4d_4)} + \frac{u_2 d_3}{2(d_3 + 4d_4)} \right), \\
    r_4 = \frac{1}{x_2} \frac{2d_4}{d_3 + 4d_4} + \frac{x_2}{4x_1^2(d_3 + 4d_4)} \frac{d_3}{2} - \frac{1}{x_2^2} \left( \frac{u_0}{2d_3 + 4d_4} + \frac{u_1 d_4}{d_3 + 4d_4} \right).
\end{cases}
$$

Now a metric
\[ <, > = u_0 \cdot B|_{\bar{h}_0} + u_1 \cdot B|_{\bar{h}_1} + u_2 \cdot B|_{\bar{h}_2} + x_1 \cdot B|_{m_1} + x_2 \cdot B|_{m_2} \]

on \( G \) is Einstein if and only if there exists a positive solution \( \{u_0, u_1, u_2, x_1, x_2, e\} \) of the system of equations

\[
(21) \quad r_0 = e, \quad r_1 = e, \quad r_2 = e, \quad r_3 = e, \quad r_4 = e.
\]

We normalize the system of equations by putting \( x_1 = 1 \). From (20), we have

\[
(22) \quad -4d_4u_0 + 4(d_3 + 4d_4)e x_2^2 - d_3u_0x_2^2 = 0,
\]
\[
(23) \quad 2d_4(2 - d_4)u_1^2 - 2d_4(2 + 2d_1 - d_4)x_2^2 \\
\quad + 4d_1(d_3 + 4d_4)e u_1x_2^2 - d_1d_3u_1^2x_2^2 = 0,
\]
\[
(24) \quad -2d_3 - 2d_4d_3 - 2d_2d_3 + d_3^2 - 8d_2d_4 + 2d_3d_4 \\
\quad + 8d_2(d_3 + 4d_4)e u_2 + d_3(2 + 2d_1 - d_3 - 2d_4)u_2^2 = 0,
\]
\[
(25) \quad -2d_3 - 8d_4 + 4(d_3 + 4d_4)e + 2u_0 + 2d_1u_1 \\
\quad + (-2 - 2d_1 + d_3 + 2d_4)u_2 + 2d_4x_2 = 0,
\]
\[
(26) \quad 8u_0 - 4(2 - d_4)u_1 - 8d_4x_2 + 4(d_3 + 4d_4)e x_2^2 - d_3x_2^3 = 0.
\]

By solving the linear equations (22), (25) and (26) with respect to \( u_0, u_1 \) and \( e \), we obtain that

\[
(27) \quad u_0 = (x_2^2(-8d_3 - 32d_4 + 4d_3d_4 + 16d_4^2 + (-8 - 8d_1 + 4d_3 + 12d_4 + 4d_1d_4 \\
\quad - 2d_3d_4 - 4d_4^2)u_2 + (8d_4 - 8d_1d_4 - 4d_4^2)x_2 - d_1d_3x_2^3))/8(-2 + d_4)d_4 \\
\quad + (-8 - 8d_1 - 4d_3 + 4d_4 - 4d_1d_4 + 2d_3d_4)x_2^2 - d_1d_3x_2^4),
\]

\[
(28) \quad u_1 = (x_2(-32d_4^2 + 4(2 + d_4)(2d_3 + 8d_4 + 2u_2 + (2d_1 - d_3 - 2d_4)u_2)x_2 \\
\quad - 4d_4(8 + 3d_3 + 2d_4)x_2^2 + 3(2d_3 + 8d_4 + (2 + 2d_1 - d_3 - 2d_4)u_2)x_2^3 \\
\quad - d_3(2 + d_3 + 2d_4)x_2^4))/(2(8(2 - d_4)d_4 + (8 + 8d_1 + 4d_3 - 4d_4 \\
\quad + 4d_1d_4 - 2d_3d_4)x_2^2 + d_1d_3x_2^4)),
\]

\[
(29) \quad e = ((4d_4 + d_3x_2^2)(8d_3 + 32d_4 - 4d_3d_4 - 16d_4^2 + (8 + 8d_1 - 4d_3 - 12d_4 - 4d_1d_4 \\
\quad + 2d_3d_4 + 4d_4^2)u_2 + (-8d_4 + 8d_1d_4 + 4d_4^2)x_2 + d_1d_3x_2^3))/((4d_3 + 4d_4 \\
\quad \times (8(2 - d_4)d_4 + (8 + 8d_1 + 4d_3 - 4d_4 + 4d_1d_4 - 2d_3d_4)x_2^2 + d_1d_3x_2^4))).
\]

From (23), (28) and (29), we get a quadratic equation with respect to \( u_2 \). By solving this equation with respect to \( u_2 \), we get

\[
(30) \quad u_2 = \frac{(4d_4 - (2d_3 + 8d_4)x_2 + (2 + 2d_1 + d_3 + 2d_4)x_2^2)}{((2 + 2d_1 - d_3 - 2d_4)x_2)}
\]
or
\begin{equation}
\begin{aligned}
 u_2 &= -\left(128 (-2 + d_4) d_4^2 (4 + 4 d_1 - 4 d_4 - 2 d_1 d_4 - 3 d_4^2) + 64 (-2 + d_4) d_4 (2 + d_4) \right.
  \\
 & \times (2 + 2 d_1 + d_4) (d_3 + 4 d_4) x_2 + 32 d_4 (-16 - 32 d_1 - 16 d_1^2 - 8 d_3 - 8 d_1 d_3 + 24 d_4 d_4
  \\
 & + 40 d_4 - 16 d_1^2 d_4 + 12 d_3 d_4 + 8 d_3 d_4 + 12 d_4^2 - 4 d_1^2 d_4^2 + 10 d_3^2 d_4^2 - 10 d_1 d_3 d_4^2
  \\
 & - 10 d_4^4 - 2 d_1 d_4^3 - 7 d_3 d_4^3 - 2 d_4^4) x_2^2 + 32 d_3 d_4 (d_3 + 4 d_4) (-4 + 2 d_1 + 3 d_1 d_4 + d_4^2)
  \\
 & \times x_2^3 + 8 d_3 d_4 (-16 d_1 - 16 d_1^2 + 32 d_4 - 8 d_1^2 d_4 + 10 d_3 d_4 - 16 d_1 d_3 d_4
  \\
 & - 8 d_4^2 - 8 d_1 d_4^2 - 5 d_3 d_4^2 - 4 d_4^3) x_2^4 + 4 d_3^2 (d_3 + 4 d_4) (4 d_1 - 2 d_4 + 6 d_1 d_4 + 4 d_4^2) x_2^5
  \\
 & + 2 d_3^2 d_4 (4 - 28 d_4 - 4 d_1^2 + 2 d_3 - 10 d_1 d_3 + 2 d_4 - 10 d_1 d_4 - d_3 d_4 - 2 d_4^2) x_2^6
  \\
 & + 2 d_1 d_3^2 (d_3 + 4 d_4) x_2^7 - d_1 d_3^2 (2 + d_3 + 2 d_4) x_2^8) / ((2 + 2 d_1 - d_3 - 2 d_4) x_2
  \\
 & \times (2 + 2 d_1 - d_3 - 2 d_4) x_2 (8 + 4 d_4 + d_3 x_2^2) (8 (-2 + d_4) d_4 (2 + 2 d_1 + d_4)
  \\
 & + 2 d_3 d_4 (-2 + 4 d_1 + d_4) x_2^2 d_3 d_4 x_2^4),
\end{aligned}
\end{equation}
provided
\begin{equation}
d_1 d_3 x_2^4 + 2 (4 + 4 d_1 + d_3 - 2 d_4 + 2 d_1 d_4 - d_3 d_4) x_2^5 - 8 d_4 (-2 + d_4) \neq 0.
\end{equation}

**Proposition 9.** *If a left invariant metric \( < , > \) of the form \([13]\) on \( G \) for Type \( \Pi b \) is naturally reductive with respect to \( G \times L \) for some closed subgroup \( L \) of \( G \), then one of the following holds:*

1. \( x_1 = x_2 \), 
2. \( u_0 = u_1 = x_2 \), 
3. \( u_0 = u_1 = u_2 = x_1 = x_2 \), that is \([13]\) is a bi-invariant metric.

Conversely, 1) if \( x_1 = x_2 \), then the metric \( < , > \) is given by \( u_0 \cdot B |_{\mathfrak{h}_0} + u_1 \cdot B |_{\mathfrak{h}_1} + u_2 \cdot B |_{\mathfrak{h}_2} + x_1 \cdot B |_{\mathfrak{m}_1 \oplus \mathfrak{m}_2} \) and is naturally reductive with respect to \( G \times H \), and 2) if \( u_0 = u_1 = x_2 \), then the metric \( < , > \) is given by \( u_0 \cdot B |_{\mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{m}_2} + u_2 \cdot B |_{\mathfrak{h}_2} + x_1 \cdot B |_{\mathfrak{m}_1} \) and is naturally reductive with respect to \( G \times K \), where the Lie algebra \( \mathfrak{k} \) is given by \( (\mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{m}_2) \oplus \mathfrak{h}_2 \).

**Proof.** Let \( \mathfrak{l} \) be the Lie algebra of \( L \). Then we have that either \( \mathfrak{l} \subset \mathfrak{h} \) or \( \mathfrak{l} \not\subset \mathfrak{h} \). First we consider the case when \( \mathfrak{l} \not\subset \mathfrak{h} \). Let \( \mathfrak{k} \) be the subalgebra of \( \mathfrak{g} \) generated by \( \mathfrak{l} \) and \( \mathfrak{h} \). Since \( \mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \) is an irreducible decomposition as \( \text{Ad}(H) \)-modules, we see that the Lie algebra \( \mathfrak{k} \) contains \( \mathfrak{m}_1 \) or \( \mathfrak{m}_2 \). Note that \( \mathfrak{m}_1, \mathfrak{m}_2 \subset \mathfrak{h} \oplus \mathfrak{m}_2, [\mathfrak{m}_1, \mathfrak{m}_1] \cap \mathfrak{m}_2 \neq \{0\}, [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{h} \) and \( [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1 \). If \( \mathfrak{k} \) contains \( \mathfrak{m}_1 \), then \( \mathfrak{k} \) also contains \( \mathfrak{m}_2 \), and hence \( \mathfrak{k} = \mathfrak{g} \). Thus the metric is bi-invariant. If \( \mathfrak{k} \) contains \( \mathfrak{m}_2 \), then \( \mathfrak{k} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{m}_2 \). Put \( \mathfrak{t}_1 = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{m}_2 \). Then \( \mathfrak{k} = \mathfrak{t}_1 \oplus \mathfrak{h}_2 \) is an ideal decomposition of simple ideals. Thus we have that \( u_0 = u_1 = x_2 \).

Now we consider the case of \( \mathfrak{l} \subset \mathfrak{h} \). Since the orthogonal complement \( \mathfrak{l}^\perp \) of \( \mathfrak{l} \) with respect to \( B \) contains the orthogonal complement \( \mathfrak{h}^\perp \) of \( \mathfrak{h} \), we see that \( \mathfrak{l}^\perp \supset \mathfrak{m}_1 \oplus \mathfrak{m}_2 \). By Theorem 1, since the invariant metric \( < , > \) is naturally reductive with respect to \( G \times L \), it follows that \( x_1 = x_2 \). The converse is a direct consequence of Theorem 1.

If \( u_2 \) is given by \([30]\), then from \([27]\), \([28]\) and \([29]\) and by using a computer algebra system, we see that
\begin{equation}
 u_0 = x_2, \quad u_1 = x_2, \quad e = (4 d_4 + d_3 x_2^2)/(4(d_3 + 4 d_4)x_2).
\end{equation}
Thus by Proposition \([4]\), the metric \( < , > \) is naturally reductive with respect to \( G \times K \). Note that \( G/K \) is an irreducible symmetric space and these Einstein metrics \( < , > \) have been studied by D’Atri-Ziller \([7]\).
Therefore from now on we consider the case when $u_2$ is given by (31).

1) Case $G$ is of $B_n$-type.

We consider the case of $n \geq 5$ and $p = 3$. Then we have that $d_1 = 8$, $d_2 = (n-3)\cdot(2n-5)$, $d_3 = 2 \cdot 3 \cdot (2n - 5)$ and $d_4 = 3 \cdot 2$. From (31), (27), (28) and (29) we obtain that

\[(33) \quad u_2 = \left(-512 + 256(-1 + 2n)x_2 - 32(-75 + 38n)x_2^2ight)
+ 192(-5 + 2n)(-1 + 2n)x_2^3 - 6(-5 + 2n)(-125 + 74n)x_2^4
+ 43(-5 + 2n)^2(-1 + 2n)x_2^5 - (-5 + 2n)^2(-94 + 63n)x_2^6
+ 3(-5 + 2n)^3(-1 + 2n)x_2^7 - (-5 + 2n)^3(-4 + 3n)x_2^8)/((-3 + n)x_2
\times(16 + 3(-5 + 2n)x_2^2)(16 + 9(-5 + 2n)x_2^2 + (-5 + 2n)^2x_2^4)),
\]

\[(34) \quad u_1 = (x_2(16 + 3(-5 + 2n)x_2^2))/(16 + 9(-5 + 2n)x_2^2 + (-5 + 2n)^2x_2^4),
\]

\[(35) \quad u_0 = (x_2(256 + 240(-5 + 2n)x_2^2 + 51(-5 + 2n)^2x_2^4 + 3(-5 + 2n)^3x_2^6))/((16 + 3(-5 + 2n)x_2^2)(16 + 9(-5 + 2n)x_2^2 + (-5 + 2n)^2x_2^4)),
\]

\[(36) \quad e = ((4 + (-5 + 2n)x_2^2)(256 + 240(-5 + 2n)x_2^2 + 51(-5 + 2n)^2x_2^4
+ 3(-5 + 2n)^3x_2^6))/((4(-1 + 2n)x_2(16 + 3(-5 + 2n)x_2^2)
\times(16 + 9(-5 + 2n)x_2^2 + (-5 + 2n)^2x_2^4))).
\]

From (21), (33) and (36), we get the following equation for $x_2$ :

\[(37) \quad -524288n + 262144(3n)(-1 + 2n)x_2 + 65536(27 + 49n - 43n^2 - 2n^3)x_2^2
+ 16384(-1 + 2n)(-345 + 31n + 62n^2)x_2^3
+ 2048(-6480 - 1023n + 8284n^2 - 2332n^3 - 192n^4)x_2^4
+ 1605(-5 + 2n)(-1 + 2n)(-1605 + 191n + 382n^2)x_2^5
+ 256(-5 + 2n)(-30240 + 5521n + 30262n^2 - 9444n^3 - 920n^4)x_2^6
+ 64(-5 + 2n)^2(-1 + 2n)(-30240 + 5521n + 30262n^2 - 9444n^3 - 920n^4)x_2^7
+ 8(-5 + 2n)^2(-15567 + 2449n + 4898n^2)x_2^8
+ 12(-5 + 2n)^3(-1 + 2n)(-14017 + 2967n + 5934n^2)x_2^9
+ (-5 + 2n)^3(-300735 + 34744n + 377253n^2 - 126480n^3 - 12004n^4)x_2^{10}
+ 3(-5 + 2n)^4(-1 + 2n)(-5155 + 1539n + 3078n^2)x_2^{11}
+ 3(-5 + 2n)^4(-4442 - 4565n + 13645n^2 - 4422n^3 - 344n^4)x_2^{12}
+ 6(-5 + 2n)^5(-1 + 2n)(-113 + 53n + 106n^2)x_2^{13}
+ 3(-5 + 2n)^5(311 - 830n + 845n^2 - 252n^3 - 12n^4)x_2^{14}
+ 9(1 + n)(-5 + 2n)^6(-1 + 2n)^2x_2^{15}
+ 3(-5 + 2n)^6(-4 + 3n)(-7 + 5n - 2n^2)x_2^{16} = 0.
\]

We denote by $f(x_2)$ the left-hand side of equation (37). We consider the values of $f(x_2)$ at $x_2 = 1$ and $x_2 = 17/10$. Using a computer algebra system, we see that

\[f(1) = 8(-7 + 2n)(-5 + 2n)(1 + 2n)^2(6 + 17n + 7n^2 - 40n^3 + 12n^4)\]
which is positive if \( n \geq 4 \), and by expanding \( f(17/10) \) as a function of \( n \) into series at \( n = 5 \), we see that

\[
f(17/10) = (-2375459471975900057437500 - 374334767070688128502678125 (5 + n) - 103651929030368084523415625 (5 + n)^2 - 131885489711907058331076250 (5 + n)^3 - 95931514181594436085898500 (5 + n)^4 - 43274741600923805795069960 (5 + n)^5 - 12373465769695851958925104 (5 + n)^6 - 2189118636501094094792672 (5 + n)^7 - 219129014907392089654464 (5 + n)^8 - 9504591553625063640192 (5 + n)^9)/10^{16},
\]

which is negative if \( n \geq 5 \). Thus we see that, for \( n \geq 5 \), the equation \( f(x_2) = 0 \) has a solution \( x_2 = x_2^0 \) between \( 1 < x_2 < 17/10 \).

We claim that the solution \( x_2^0 \) of \( f(x_2) = 0 \) with \( 1 < x_2^0 < 17/10 \) satisfies the property (32). We denote by \( q(x_2) \) the left-hand side of (32). Then we have

\[
q(x_2) = 48 (-5 + 2 n) x_2^4 + 2 (120 - 24 (-5 + 2 n)) x_2^2 - 192
\]

and

\[
q'(x_2) = 192 x_2 ((2n - 5)x_2^2 - (n - 5)) > 0.
\]

Thus \( q(x_2) \) is monotone increasing for \( x_2 \geq 1 \). Since \( q(1) = 48 \), we see that \( q(x_2) \geq 48 \) for \( x_2 \geq 1 \).

Hence, we obtain a solution \( \{u_0, u_1, u_2, x_1, x_2, e\} = \{u_0^0, u_1^0, u_2^0, 1, x_2^0, e^0\} \) of equations (21) from (33), (34), (35) and (36). It is obvious that \( u_0^0 > 0, u_1^0 > 0, e^0 > 0 \) from (34), (35) and (36).

Now we claim that \( u_2^0 > 0 \). From (33), it is enough to show that the numerator

\[
h(x_2) = -512 + 256 (-1 + 2 n) x_2 - 32 (-75 + 38 n) x_2^2 + 192 (-5 + 2 n) (-1 + 2 n) x_2^3 + (-125 + 74 n) x_2^4 + 43 (-5 + 2 n)^2 (-1 + 2 n) x_2^5 - (5 + 2 n)^2 (-94 + 63 n) x_2^6 + 3 (-5 + 2 n)^3 (-1 + 2 n) x_2^7 - (5 + 2 n)^3 (-4 + 3 n) x_2^8
\]

of \( u_2 \) is positive for \( 1 < x_2 < 17/10 \). By expanding \( h(x_2) \) into series at \( x_2 = 1 \), we see that

\[
h(x_2) = -(2n - 5)^3(3n - 4)(x_2 - 1)^8 - (2n - 5)^3(18n - 29)(x_2 - 1)^7 - (2n - 5)^2 (84n^2 - 329n + 361) (x_2 - 1)^6 - 4(2n - 5)^2 (21n^2 - 60n + 71) (x_2 - 1)^5 - (2n - 5) (330n^2 - 1021n + 850) (x_2 - 1)^4 + 3(2n - 5) (56n^3 - 388n^2 + 726n - 489) (x_2 - 1)^3 + (336n^4 - 2468n^3 + 5956n^2 - 5433n + 1155) (x_2 - 1)^2 + (144n^4 - 784n^3 + 1148n^2 - 232n - 226) (x_2 - 1) + 2 (12n^4 - 40n^3 + 7n^2 + 21n - 4).
\]
Using that $0 \leq (x_2 - 1)^4 \leq (7/10)^4$, we see that

$$h(x_2) \geq - (2n - 5)^3 (3n - 4) (7/10)^8 - (2n - 5)^3 (18n - 29) (x_2 - 1)^3 (7/10)^4$$

$$- (2n - 5)^2 (84n^2 - 329n + 361) (x_2 - 1)^2 (7/10)^4$$

$$- 4 (2n - 5)^2 (21n^2 - 60n + 71) (x_2 - 1) (7/10)^4$$

$$- (2n - 5) (330n^2 - 1021n + 850) (7/10)^4$$

$$+ 3 (2n - 5) (56n^3 - 388n^2 + 726n - 489) (x_2 - 1)^3$$

$$+ (336n^4 - 2468n^3 + 5956n^2 - 5433n + 1155) (x_2 - 1)^2$$

$$+ (144n^4 - 784n^3 + 1148n^2 - 232n - 226) (x_2 - 1)$$

$$+ 2 (12n^4 - 40n^3 + 7n^2 + 21n - 4).$$

We denote by $K(x_2)$ the right-hand side of inequality (39). By using a computer algebra system we see that

$$K(x_2) = \frac{(x_2 - 1)^3}{10^4} - (2n - 5) (1507128 (n - 5)^3 + 12109796 (n - 5)^2 + 27370330 (n - 5))$$

$$+ 9568475 + \frac{(x_2 - 1)^2}{10^4} (2553264 (n - 5)^4 + 33578676 (n - 5)^3 + 155942816 (n - 5)^2)$$

$$+ 300412905 (n - 5) + 194919600 \cdot \frac{(x_2 - 1)}{2500} \cdot (158316 (n - 5)^4 + 2790980 (n - 5)^3)$$

$$+ 16163691 (n - 5)^2 + 37902330 (n - 5) + 30517600 \cdot \frac{1}{10^8} \cdot (2261644776 (n - 5)^4)$$

$$+ 2260843332 (n - 5)^3 + 85946990890 (n - 5)^2 + 141092427975 (n - 5) + 67673648625),$$

hence $K(x_2)$ is positive for $1 < x_2 < 17/10$ and $n \geq 5$.

2) Case $G$ is of $C_n$-type.

We consider the case of $n \geq 3$ and $p = 2$. Then we have that $d_1 = 3$, $d_2 = (n - 2) \cdot (2n - 3)$, $d_3 = 4 \cdot 2 \cdot (n - 2)$ and $d_4 = 3 \cdot 2$. From (31), (27), (28) and (29) we obtain that

$$u_2 = (-912 + 448 (1 + n) x_2 - 4 (-397 + 256 n) x_2^2 + 368 (-2 + n) (1 + n) x_2^3$$

$$-24 (-2 + n) (-19 + 17 n) x_2^4 + 96 (-2 + n)^2 (1 + n) x_2^5$$

$$- (-2 + n)^2 (-47 + 68 n) x_2^6 + 8 (-2 + n)^3 (1 + n) x_2^7 - (-2 + n)^3 (1 + 4 n) x_2^8$$

$$/(2 (-3 + 2 n) x_2 (4 + (-2 + n) x_2^2) (14 + 8 (-2 + n) x_2^2 + (-2 + n)^2 x_2^4)),$$

$$u_1 = (x_2 (4 + (-2 + n) x_2^2)) / (14 + 8 (-2 + n) x_2^2 + (-2 + n)^2 x_2^4),$$

$$u_0 = (x_2 (4560 (-2 + n) x_2^2 + 14 (-2 + n)^2 x_2^4 + (-2 + n)^3 x_2^6))$$

$$/((4 + (-2 + n) x_2^2) (14 + 8 (-2 + n) x_2^2 + (-2 + n)^2 x_2^4)),$$

$$e = ((3 + (-2 + n) x_2^2) (76 + 60 (-2 + n) x_2^2 + 14 (-2 + n)^2 x_2^4 + (-2 + n)^3 x_2^6))$$

$$/(4 (1 + n) x_2 (4 + (-2 + n) x_2^2) (14 + 8 (-2 + n) x_2^2 + (-2 + n)^2 x_2^4)).$$
From (24), (40) and (43), we get the following equation for $x_2$:

$$
\begin{align*}
(44) \quad &207936 (1 + 2 n) - 102144 (1 + n) (5 + 2 n) x_2 + 16 (-16577 - 41122 n + 64640 n^2 + 1568 n^3) x_2^2 - 64 (1 + n) (-24590 + 3103 n + 6206 n^2) x_2^3 + 16 (-29251 + 29870 n - 136972 n^2 + 61176 n^3 + 2576 n^4) x_2^4 - 128 (-2 + n) (1 + n) (-7475 + 1264 n + 2528 n^2) x_2^5 + 4 (-2 + n) (-155306 + 25437 n - 248456 n^2 + 133792 n^3 + 6920 n^4) x_2^6 - 128 (-2 + n)^2 (1 + n) (-2207 + 559 n + 1118 n^2) x_2^7 + 8 (-2 + n)^2 (-34571 + 3055 n - 35132 n^2 + 21996 n^3 + 1216 n^4) x_2^8 - 16 (-2 + n)^3 (1 + n) (-2324 + 1159 n + 2318 n^2) x_2^9 + 2 (-2 + n)^3 (-31006 + 5873 n - 25762 n^2 + 17808 n^3 + 944 n^4) x_2^{10} - 32 (-2 + n)^4 (1 + n) (-5 + 88 n + 176 n^2) x_2^{11} + (-2 + n)^4 (-7229 + 3419 n - 6086 n^2 + 4328 n^3 + 192 n^4) x_2^{12} - 8 (-2 + n)^5 (1 + n) (56 + 29 n + 58 n^2) x_2^{13} + 2 (-2 + n)^5 (-191 + 229 n - 215 n^2 + 144 n^3 + 4 n^4) x_2^{14} - 8 (-2 + n)^6 (1 + n) x (4 + n + 2 n^2) x_2^{15} + (-2 + n)^6 (-1 + 4 n) (5 - 3 n + 2 n^2) x_2^{16} = 0.
\end{align*}
$$

By using a similar method as for $B_n$-type, we see that for $n \geq 3$ the equation (44) has a solution $x_2 = x_2^0$ between $1 < x_2 < 5/4$. Then we obtain a solution $\{u_0, u_1, u_2, x_1, x_2, e\} = \{u_0^0, u_1^0, u_2^0, 1, x_2^0, e^0\}$ of equations (21) from (40), (41), (42) and (43), and we also see that $u_0^0 > 0$.

3) Case $G$ is of $D_n$-type.

We consider the case of $n \geq 6$ and $p = 3$. Then we have that $d_1 = 8$, $d_2 = (n - 3) \cdot (2n - 7)$, $d_3 = 4 \cdot 3 \cdot (n - 3)$ and $d_4 = 3 \cdot 2$. From (51), (27), (28) and (29) we obtain that

$$
\begin{align*}
(45) \quad &u_2 = (-256 + 256 (-1 + n) x_2 - 32 (-47 + 19 n) x_2^2 + 384 (-3 + n) (-1 + n) x_2^3 - 12 (-3 + n) (-81 + 37 n) x_2^4 + 172 (-3 + n)^2 (-1 + n) x_2^5 - (-3 + n)^2 (-251 + 126 n) x_2^6 + 24 (-3 + n)^3 (-1 + n) x_2^7 - 2 (-3 + n)^3 (-11 + 6 n) x_2^8) /
(\mathbf{0} - 72 n) x_2 (8 + 3 (-3 + n) x_2^2) (8 + 9 (-3 + n) x_2^2 + 2 (-3 + n^2) x_2^4)),
\end{align*}
$$

$$
\begin{align*}
(46) \quad &u_1 = (x_2 (8 + 3 (-3 + n) x_2^2)) / (8 + 9 (-3 + n) x_2^2 + 2 (-3 + n^2) x_2^4),
\end{align*}
$$

$$
\begin{align*}
(47) \quad &u_0 = (x_2 (64 + 120 (-3 + n) x_2^2 + 251 (-3 + n)^2 x_2^4 + 6 (-3 + n)^3 x_2^6)) / ((8 + 3 (-3 + n) x_2^2) (8 + 9 (-3 + n) x_2^2 + 2 (-3 + n^2) x_2^4)),
\end{align*}
$$

$$
\begin{align*}
(48) \quad &e = ((2 + (-3 + n) x_2^2) (64 + 120 (-3 + n) x_2^2 + 51 (-3 + n)^2 x_2^4 + 6 (-3 + n)^3 x_2^6)) / (4 (-1 + n) x_2 (8 + 3 (-3 + n) x_2^2) (8 + 9 (-3 + n) x_2^2 + 2 (-3 + n^2) x_2^4)).
\end{align*}
$$
From (24), (15) and (48), we get the following equation for $x_2$:

$$u_2 = - \left( 186 x_2^8 - 480 x_2^7 + 967 x_2^6 - 1616 x_2^5 + 1592 x_2^4 - 1728 x_2^3 + 956 x_2^2 - 576 x_2 + 144 \right) / \left( x_2 (2 x_2^2 + 3) (3 x_2^2 + 2) (5 x_2^2 + 6) \right),$$

$$u_1 = \frac{x_2 (2 x_2^2 + 3) (3 x_2^2 + 2)}{(2 x_2^2 + 3) (3 x_2^2 + 2)},$$

$$u_0 = \frac{x_2 (30 x_2^6 + 125 x_2^4 + 140 x_2^2 + 36)}{(2 x_2^2 + 3) (3 x_2^2 + 2) (5 x_2^2 + 6)},$$

$$e = \frac{x_2 (2 x_2^2 + 3) (3 x_2^2 + 2) (5 x_2^2 + 6)}{8 x_2 (2 x_2^2 + 3) (3 x_2^2 + 2) (5 x_2^2 + 6)}.$$
By using a similar method as for $B_n$-type, we see that for $n \geq 6$ equation (54) has a solution $x_2 = x_2^0$ between $1 < x_2 < 5/3$. Then we obtain a solution \( \{u_0, u_1, u_0, x_2, e\} = \{u_0^0, u_1^0, u_0^0, 1, x_2^0, e^0\} \) of equations (21) from (50), (51), (52) and (53), and we also see that $u_2^0 > 0$.

This gives the solution
\[
\{u_0^0, u_1^0, u_0^0, 1, x_2^0, e^0\} \approx \{1.88908, 0.379243, 0.140912, 1.62965, 0.32505\},
\]
and, similarly, we obtain three more solutions given by
\[
\begin{align*}
\{u_0^0, u_1^0, u_2^0, 1, x_2^0, e^0\} &\approx \{0.393637, 0.308385, 0.103143, 0.361629, 0.425457\}, \\
\{u_0^0, u_1^0, u_2^0, 1, x_2^0, e^0\} &\approx \{0.547238, 0.370178, 1.60644, 0.483835, 0.360612\}, \\
\{u_0^0, u_1^0, u_2^0, 1, x_2^0, e^0\} &\approx \{1.52202, 0.418588, 1.31967, 1.27928, 0.306505\}.
\end{align*}
\]

5) Case $G$ is of $E_7$-type.

In this case we have that $d_1 = 45$, $d_2 = 3$, $d_3 = 64$ and $d_4 = 20$. From (31), (27), (28) and (29), we obtain
\[
\begin{align*}
(55) \quad u_2 &= \left(-1696x_2^8 - 4608x_2^7 + 10904x_2^6 - 19008x_2^5 + 22140x_2^4 - 25488x_2^3 \\
&\quad + 16849x_2^2 - 11088x_2 + 3620\right) / \left(6x_2 \left(x_2^2 + 1\right) \left(4x_2^2 + 7\right) \left(8x_2^2 + 11\right)\right),
\end{align*}
\]
\[
\begin{align*}
(56) \quad u_1 &= \frac{x_2 \left(8x_2^2 + 11\right)}{2 \left(x_2^2 + 1\right) \left(4x_2^2 + 7\right)}, \\
(57) \quad u_0 &= \frac{x_2 \left(64x_2^6 + 336x_2^4 + 480x_2^2 + 181\right)}{2 \left(x_2^2 + 1\right) \left(4x_2^2 + 7\right) \left(8x_2^2 + 11\right)}, \\
(58) \quad e &= \frac{\left(4x_2^2 + 5\right) \left(64x_2^6 + 336x_2^4 + 480x_2^2 + 181\right)}{72x_2 \left(x_2^2 + 1\right) \left(4x_2^2 + 7\right) \left(8x_2^2 + 11\right)}.
\end{align*}
\]

From (21), (55), and (58), we get the following equation for $x_2$:
\[
(59) \quad 24313856x_2^{16} - 128581632x_2^{15} + 482637824x_2^{14} - 1357332480x_2^{13} \\
\quad + 3043447808x_2^{12} - 580442112x_2^{11} + 9347615296x_2^{10} - 13107483648x_2^9 \\
\quad + 15962982496x_2^8 - 16875749376x_2^7 + 15608426188x_2^6 - 12310144128x_2^5 \\
\quad + 8333330528x_2^4 - 4638529008x_2^3 + 2039329151x_2^2 - 672320880x_2 + 114663500 = 0.
\]

By using a similar method as for $E_6$-type, we obtain four solutions \( \{u_0^0, u_1^0, u_2^0, 1, x_2^0, e^0\} \) of equations (21) which are approximately given by
\[
\begin{align*}
\{u_0^0, u_1^0, u_2^0, 1, x_2^0, e^0\} &\approx \{0.633451, 0.328931, 0.0705205, 0.509298, 0.409568\}, \\
\{u_0^0, u_1^0, u_2^0, 1, x_2^0, e^0\} &\approx \{0.819745, 0.377972, 1.54275, 0.649661, 0.360839\}, \\
\{u_0^0, u_1^0, u_2^0, 1, x_2^0, e^0\} &\approx \{1.56687, 0.432465, 1.3115, 1.25338, 0.312624\}, \\
\{u_0^0, u_1^0, u_2^0, 1, x_2^0, e^0\} &\approx \{1.8899, 0.414278, 0.0931131, 1.55163, 0.319015\}.
\end{align*}
\]

Thus we have proved the following.:

**Proposition 10.** (1) The compact Lie groups $SO(2n + 1)$ ($n \geq 5$), $Sp(n)$ ($n \geq 3$), and $SO(2n)$ ($n \geq 6$) admit at least one left-invariant Einstein metric which is not naturally reductive.

(2) The compact Lie groups $E_6$ and $E_7$ admit at least four left-invariant Einstein metrics which are not naturally reductive.
5. Einstein metrics on compact Lie groups of type I

We assume that \( \{\alpha_{i_0}\} \) is not next to the negative of the maximal root, and by removing \( \{\alpha_{i_0}\} \) from the extended Dynkin diagram the resulting diagram is connected, which is the case of Type Ib. The case of spaces of Type Ia will be examined in Section 6.

We consider left invariant metrics

\[
<, > = u_0 \cdot B|_{\mathfrak{h}_0} + u_1 \cdot B|_{\mathfrak{h}_1} + x_1 \cdot B|_{\mathfrak{m}_1} + x_2 \cdot B|_{\mathfrak{m}_2}
\]

on a compact Lie group \( G \) associated to Kähler C-spaces of Type Ib. Note that a metric (60) is also \( Ad(H) \)-invariant.

Let \( d_1 = \dim \mathfrak{h}_1 \), \( d_3 = \dim \mathfrak{m}_1 \) and \( d_4 = \dim \mathfrak{m}_2 \). By the relations \([\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h} \oplus \mathfrak{m}_2, \] \([\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{h}, \mathfrak{m}_1, \mathfrak{m}_2 \subset \mathfrak{m}_1 \), we see that \([k]_{ij}\) are zero, except \([3]_{03}, [4]_{04}, [1]_{11}, [3]_{13}, [4]_{14}, [33]_{44}\). By Lemma 3 we have that

\[
\begin{align*}
\begin{bmatrix} 3 \\ 03 \end{bmatrix} + \begin{bmatrix} 4 \\ 04 \end{bmatrix} &= 1, \\
\begin{bmatrix} 1 \\ 11 \end{bmatrix} + \begin{bmatrix} 3 \\ 13 \end{bmatrix} + \begin{bmatrix} 4 \\ 14 \end{bmatrix} &= d_1, \\
2 \begin{bmatrix} 0 \\ 33 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 33 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 33 \end{bmatrix} &= d_3, \\
2 \begin{bmatrix} 0 \\ 44 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 44 \end{bmatrix} + \begin{bmatrix} 3 \\ 43 \end{bmatrix} &= d_4.
\end{align*}
\]

and thus the components of the Ricci tensor \( r \) of the metric (60) are given by the following:

\[
\begin{align*}
\begin{align*}
\begin{bmatrix} r_0 \\ 03 \end{bmatrix} &= \frac{u_0}{4x_1^2} \begin{bmatrix} 0 \\ 33 \end{bmatrix} + \frac{u_0}{4x_2^2} \begin{bmatrix} 0 \\ 44 \end{bmatrix}, \\
\begin{bmatrix} r_1 \\ 11 \end{bmatrix} &= \frac{1}{4d_1 u_1} \begin{bmatrix} 1 \\ 11 \end{bmatrix} + \frac{u_1}{4d_1 x_1^2} \begin{bmatrix} 1 \\ 33 \end{bmatrix} + \frac{u_1}{4d_1 x_2^2} \begin{bmatrix} 1 \\ 44 \end{bmatrix}, \\
\begin{bmatrix} r_3 \\ 33 \end{bmatrix} &= \frac{1}{2x_1} - \frac{x_2}{2d_3 x_1^2} \begin{bmatrix} 4 \\ 33 \end{bmatrix} - \frac{1}{2d_3 x_1^2} \left( \begin{bmatrix} 0 \\ 33 \end{bmatrix} + \begin{bmatrix} 1 \\ 33 \end{bmatrix} \right), \\
\begin{bmatrix} r_4 \\ 44 \end{bmatrix} &= \frac{1}{x_2} \left( \frac{1}{2} - \frac{1}{2d_4} \begin{bmatrix} 3 \\ 43 \end{bmatrix} \right) + \frac{x_2}{4d_4 x_1^2} \begin{bmatrix} 4 \\ 33 \end{bmatrix} - \frac{1}{2d_4 x_2^2} \left( \begin{bmatrix} 0 \\ 44 \end{bmatrix} + \begin{bmatrix} 1 \\ 44 \end{bmatrix} \right).
\end{align*}
\end{align*}
\]

By the same method as in Section 4, we can compute the numbers \([k]_{ij}\) and we obtain:

**Lemma 11.** For the metric \(<, > \) on \( G \), the non-zero numbers \([k]_{ij}\) are given as follows:

\[
\begin{align*}
\begin{bmatrix} 0 \\ 33 \end{bmatrix} &= \frac{d_3}{(d_3 + 4d_4)}, \\
\begin{bmatrix} 0 \\ 44 \end{bmatrix} &= \frac{4d_4}{(d_3 + 4d_4)}, \\
\begin{bmatrix} 1 \\ 11 \end{bmatrix} &= \frac{2d_4(2d_1 + 2 - d_4)}{(d_3 + 4d_4)}, \\
\begin{bmatrix} 1 \\ 33 \end{bmatrix} &= \frac{d_1d_3}{(d_3 + 4d_4)}, \\
\begin{bmatrix} 1 \\ 44 \end{bmatrix} &= \frac{2d_4(d_4 - 2)}{(d_3 + 4d_4)}, \\
\begin{bmatrix} 4 \\ 33 \end{bmatrix} &= \frac{d_3d_4}{(d_3 + 4d_4)}.
\end{align*}
\]
Thus we have

**Proposition 12.** The components of the Ricci tensor $r$ of the metric

$$< , > = u_0 \cdot B|_{b_0} + u_1 \cdot B|_{b_1} + x_1 \cdot B|_{m_1} + x_2 \cdot B|_{m_2}$$
on $G$ are given by

$$r_0 = \frac{u_0}{4 x_1^2 (d_3 + 4 d_4)} + \frac{d_3}{x_2^2 (d_3 + 4 d_4)},$$
$$r_1 = \frac{1}{2 d_1 u_1} \frac{d_4 (2 d_1 + 2 - d_4)}{(d_3 + 4 d_4)} + \frac{u_1}{4 x_1^2 (d_3 + 4 d_4)} + \frac{d_3}{2 d_1 x_2^2 (d_3 + 4 d_4)},$$
$$r_3 = \frac{1}{2 x_1} - \frac{x_2}{2 x_1^2 (d_3 + 4 d_4)} - \frac{1}{2 x_1^2} \left( u_0 \frac{1}{(d_3 + 4 d_4)} + u_1 \frac{d_1}{(d_3 + 4 d_4)} \right),$$
$$r_4 = \frac{1}{x_2 (d_3 + 4 d_4)} + \frac{x_2}{4 x_1^2 (d_3 + 4 d_4)} - \frac{1}{x_2^2} \left( u_0 \frac{2}{(d_3 + 4 d_4)} + u_1 \frac{d_4 - 2}{d_3 + 4 d_4} \right).$$

(63)

Now a metric

$$< , > = u_0 \cdot B|_{b_0} + u_1 \cdot B|_{b_1} + x_1 \cdot B|_{m_1} + x_2 \cdot B|_{m_2}$$
on $G$ is Einstein if and only if there exists a positive solution $\{u_0, u_1, x_1, x_2, e\}$ of the system of equations

$$r_0 = e, \quad r_1 = e, \quad r_3 = e, \quad r_4 = e.$$  

(64)

We normalize the system of equations by putting $x_1 = 1$. From (63) we have that

$$-4 d_4 u_0 + 4 (d_3 + 4 d_4) e x_2^2 - d_3 u_0 x_2^2 = 0,$$

(65)

$$2 d_4 (2 - d_4) u_1^2 - 2 d_4 (2 + 2 d_1 - d_4) x_2^2 + 4 d_1 (d_3 + 4 d_4) e u_1 x_2^2$$

$$- d_3 u_1^2 x_2^2 = 0,$$

(66)

$$- d_3 - 4 d_1 + 2 (d_3 + 4 d_4) e + u_0 + d_1 u_1 + d_4 x_2 = 0,$$

(67)

$$8 u_0 - 4 (2 - d_4) u_1 - 8 d_4 x_2 + 4 (d_3 + 4 d_4) e x_2^2 - d_3 x_2^3 = 0,$$

(68)

By solving the linear equations (65), (67) and (68) with respect to $u_0, u_1$ and $e$, we have that

$$u_0 = (x_2^2 (-8 d_3 - 32 d_1 + 4 d_3 d_4 + 16 d_4^2 + 4 d_4 (2 - 2 d_1 - d_4) x_2 - d_1 d_3 x_2^3))/\left(8(-2 + d_4) d_4 + 2 (-4 - 4 d_1 - 2 d_3 + 2 d_4 - 2 d_1 d_4 + d_3 d_4) x_2^2 - d_1 d_3 x_2^4\right),$$

(69)

$$u_1 = (x_2 - 2) x_2 (d_3 (d_3 + 2 d_4 + 2) x_2 - 4 d_4 (d_3 - 1) x_2^2$$

$$+ 4 (2 d_4^2 + d_4 d_4 + 8 d_4 + 2 d_3) x_2 - 16 d_4^2)/\left(2 (8 (2 - d_4) d_4 + 4 (2 + 2 d_1 + d_3 - d_4) + 4 d_1 d_4 - 2 d_3 d_4) x_2^2 + d_1 d_3 x_2^4\right),$$

(70)

$$e = ((4 d_4 + d_3 x_2^2) (4 (2 d_3 + 8 d_4 - d_3 d_4 - 4 d_4^2) + 4 (-2 d_4 + 2 d_1 d_4 + d_4^2) x_2$$

$$+ d_1 d_3 x_2^3)) / (4 (d_3 + 4 d_4) \times (8 (2 - d_4) d_4$$

$$+ (8 + 8 d_1 + 4 d_3 - 4 d_4 + 4 d_1 d_4 - 2 d_3 d_4) x_2^2 + d_1 d_3 x_2^4)).$$

(71)
We substitute (69), (70) and (71) to equation (66) and obtain that

\[(72) \quad (2d_1 + d_3 + 2d_4 + 2)x_2^2 - 2(d_3 + 4d_4)x_2 + 4d_4 = 0\]
or

\[(73) \quad d_1d_3^3(d_3 + 2d_4 + 2)x_2^8 - 2d_1d_3^3(d_3 + 4d_4)x_2^7 + 2d_3^2d_4(4d_1^2 + 10d_3d_4 + 10d_d_4 + 28d_1 + 2d_4^2 - 2d_3 + 3d_4 - 2d_4 - 4)x_2^6 - 4d_3^2(d_3 + 4d_4)(d_4^2 + 6d_1d_4 - 2d_4 + 4d_1)x_2^5 + 8d_3d_4(4d_1^3 + 8d_1^2d_4^2 + 5d_3d_4^2 + 8d_4^2 + 8d_4^2d_4 + 24d_1d_4 + 16d_1d_3d_4 - 10d_3d_4 - 32d_4 + 16d_1^2 + 16d_1)x_2^4 - 32d_3d_4(d_3 + 4d_4)(d_4^2 + 3d_1d_4 + 2d_4 - 4)x_2^3 + 32d_4^2d_4^2 + 7d_3d_4^3 + 10d_4^3 + 4d_1^2d_4^2 + 10d_1d_3d_4^2 - 10d_3d_4^2 - 12d_4^2 + 16d_1^2d_4 - 8d_3d_4 - 12d_1d_4 - 40d_4 + 16d_1^2 + 32d_1 + 8d_1d_3 + 8d_3 + 16)x_2^2 - 64(d_4 - 2)d_4(d_4 + 2)(2d_1 + d_4 + 2)(d_3 + 4d_4)x_2 + 128(d_4 - 2)d_4^2(3d_4^2 + 2d_1d_4 + 4d_4 - 4d_1 - 4) = 0,\]
provided

\[(74) \quad d_1d_3x_2^4 + 2(2d_4d_1 + 4d_1 + 2d_3 - d_3d_4 + 4d_4 + 4)x_2^2 - 8(d_4 - 2)d_4 \neq 0.\]

**Proposition 13.** If a left invariant metric \(\langle , \rangle\) of the form (60) on \(G\) for Type I b is naturally reductive with respect to \(G \times L\) for some closed subgroup \(L\) of \(G\), then one of the following holds:

1) \(x_1 = x_2\), 2) \(u_0 = u_1 = x_2\), 3) \(u_0 = u_1 = x_1 = x_2\), that is (60) is a bi-invariant metric.

Conversely, 1) if \(x_1 = x_2\), then the metric \(\langle , \rangle\) is given by \(u_0 \cdot B|_{\mathfrak{h}_0} + u_1 \cdot B|_{\mathfrak{h}_1} + x_1 \cdot B|_{\mathfrak{m}_1 \oplus \mathfrak{m}_2}\) and is naturally reductive with respect to \(G \times H\), and 2) if \(u_0 = u_1 = x_2\), then the metric \(\langle , \rangle\) is given by \(u_0 \cdot B|_{\mathfrak{h}_0 \oplus \mathfrak{h}_2 + \mathfrak{m}_2} + x_1 \cdot B|_{\mathfrak{m}_1}\) and is naturally reductive with respect to \(G \times K\), where the Lie algebra \(\mathfrak{k}\) is given by \(\mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{m}_2\).

The proof is similar to Proposition 9.

1) Case \(G\) is of \(E_7\)-type.

Then we have that \(d_1 = 48, d_3 = 70, d_4 = 14\). Equation (72) becomes \(28(x_2 - 1)(7x_2 - 2) = 0\). For \(x_2 = 1\), equations (69), (70) and (71) give \(u_0 = 1, u_1 = 1\) and \(\epsilon = \frac{1}{4}\), which is a bi-invariant metric. For \(x_2 = \frac{2}{7}\), equations (69), (70) and (71) give that \(u_0 = \frac{2}{7}, u_1 = \frac{2}{7}\) and \(\epsilon = \frac{3}{7}\). By Proposition 13 this is a naturally reductive Einstein metric on \(G\).

Equation (73) reduces to

\[263424(6250x_2^8 - 15750x_2^7 + 27125x_2^6 - 41175x_2^5 + 36030x_2^4 - 34560x_2^3 + 17248x_2^2 - 9216x_2 + 2048) = 0.\]

This equation has two positive solutions \(x_2 \approx 0.319422\) and \(x_2 \approx 1.62088\). Note that these solutions satisfy (74). For \(x_2 \approx 0.319422\), equations (69), (70) and (71) give \(u_0 \approx 0.348835\), \(u_1 \approx 0.275827\) and \(\epsilon \approx 0.428332\). For \(x_2 \approx 1.62088\), equations (69), (70) and (71) give \(u_0 \approx 1.86993, u_1 \approx 0.334612\) and \(\epsilon \approx 0.338795\). By Proposition 13 these are two non-naturally reductive Einstein metrics on \(G\).
2) Case $G$ is of $E_8$-type.

Then we have that $d_1 = 91$, $d_3 = 128$, $d_4 = 28$. Analogously, equations (72) and (73) become $16(x_2 - 1)(23x_2 - 7) = 0$ and

$$11904x_2^8 - 30720x_2^7 + 56144x_2^6 - 86400x_2^5 + 80752x_2^4 - 79440x_2^3 + 42853x_2^2 - 23850x_2 + 6293 = 0.$$  

From these we obtain two naturally reductive Einstein metrics

$$\{u_0, u_1, x_2, e\} = \{1, 1, 1, 1/4\}, \quad \{u_0, u_1, x_2, e\} = \{7/23, 7/23, 7/23, 39/92\},$$

and two non-naturally reductive Einstein metrics given by

$$\{u_0, u_1, x_2, e\} \approx \{0.475824, 0.282007, 0.39314, 0.422612\}$$

$$\{u_0, u_1, x_2, e\} \approx \{1.88246, 0.345485, 1.59071, 0.337789\}.$$

3) Case $G$ is of $F_4$-type.

Then we have that $d_1 = 21$, $d_3 = 16$, $d_4 = 14$. Equations (72) and (73) become $8(x_2 - 1)(11x_2 - 7) = 0$ and

$$86016(46x_2^8 - 144x_2^7 + 767x_2^6 - 1728x_2^5 + 4116x_2^4 - 6696x_2^3 + 8119x_2^2 - 8352x_2 + 4004) = 0.$$  

From the first equation we obtain two naturally reductive Einstein metrics

$$\{u_0, u_1, x_2, e\} = \{1, 1, 1, 1/4\}, \quad \{u_0, u_1, x_2, e\} = \{7/23, 7/23, 7/23, 39/92\},$$

but the second equation has no real solutions.

Thus we have proved the following:.

**Proposition 14.** The compact Lie groups $E_7$ and $E_8$ admit at least two left-invariant Einstein metrics which are not naturally reductive.

Theorem 1 now follows from Propositions 10 and 14.

6. EINSTEIN METRICS ON COMPACT LIE GROUPS WHICH ARE NATURALLY REDUCTIVE

Now we consider compact Lie groups associated to Kähler C-spaces of Types Ia and IIa. Note that $d_4 = 2$ in these cases.

In case of Type IIa we set $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_2$ and $\mathfrak{k}_1 = \mathfrak{h}_0 \oplus \mathfrak{m}_2$. Then $\mathfrak{k}, \mathfrak{k}_1$ are subalgebras of $\mathfrak{g}$, $\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2$ and $(\mathfrak{g}, \mathfrak{k})$ is an irreducible symmetric pair. We also have an irreducible decomposition $\mathfrak{g} = \mathfrak{t}_1 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{m}_1$ as $\text{Ad}(K)$-modules, which are mutually non-equivalent.

**Proposition 15.** If a left invariant metric $<,>$ of the form (13) on $G$ for Type IIa is naturally reductive with respect to $G \times L$ for some closed subgroup $L$ of $G$, then one of the following holds:

1) $x_1 = x_2$, 2) $u_0 = x_2$, 3) $u_0 = u_1 = u_2 = x_1 = x_2$, that is (13) is a bi-invariant metric.

Conversely, 1) if $x_1 = x_2$, then the metric $<,>$ is given by $u_0 \cdot B|_{\mathfrak{h}_0} + u_1 \cdot B|_{\mathfrak{h}_1} + u_2 \cdot B|_{\mathfrak{h}_2} + x_1 \cdot B|_{\mathfrak{m}_1 \oplus \mathfrak{m}_2}$ and is naturally reductive with respect to $G \times H$, and 2) if $u_0 = x_2$, then the metric $<,>$ is given by $u_0 \cdot B|_{\mathfrak{h}_0 \oplus \mathfrak{m}_2} + u_1 \cdot B|_{\mathfrak{h}_1} + u_2 \cdot B|_{\mathfrak{h}_2} + x_1 \cdot B|_{\mathfrak{m}_1}$ and is naturally reductive with respect to $G \times K$, where the Lie algebra $\mathfrak{k}$ is given by $(\mathfrak{h}_0 \oplus \mathfrak{m}_2) \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2$. 
The proof is similar to Proposition 9.

Note that the number \( \left[ \frac{1}{44} \right] \) in Lemma 8 is zero, so the first and the fifth equation of the system \((71)\) simplify and give rise to the relation \( u_0 = x_2 \). Hence, by Proposition 15 we only obtain Einstein metrics which are naturally reductive.

In case of Type Ia we consider the metric \( < , > \) on \( G \) given by
\[
(75) \quad u_0 \cdot B|_{h_0} + u_2 \cdot B|_{h_2} + x_1 \cdot B|_{m_1} + x_2 \cdot B|_{m_2},
\]
and set \( \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_2 \) and \( \mathfrak{k}_1 = \mathfrak{h}_0 \oplus \mathfrak{m}_2 \). Then \( \mathfrak{k}, \mathfrak{k}_1 \) are subalgebras of \( \mathfrak{g}, \mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{h}_2 \) and \( (\mathfrak{g}, \mathfrak{k}) \) is an irreducible symmetric pair. We also have an irreducible decomposition \( \mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{m}_1 \) as \( \text{Ad}(K) \)-modules, which are mutually non-equivalent.

**Proposition 16.** If a left invariant metric \( < , > \) of the form \((75)\) on \( G \) for Type Ia is naturally reductive with respect to \( G \times L \) for some closed subgroup \( L \) of \( G \), then one of the following holds:

1) \( x_1 = x_2, \) 2) \( u_0 = x_2, \) 3) \( u_0 = u_2 = x_1 = x_2, \) that is \((75)\) is a bi-invariant metric.

Conversely, 1) if \( x_1 = x_2 \), then the metric \( < , > \) is given by \( u_0 \cdot B|_{h_0} + u_2 \cdot B|_{h_2} + x_1 \cdot B|_{m_1} \oplus m_2 \) and is naturally reductive with respect to \( G \times H \), and 2) if \( u_0 = x_2 \), then the metric \( < , > \) is given by \( u_0 \cdot B|_{h_0}^{\oplus m_2} + u_2 \cdot B|_{h_2} + x_1 \cdot B|_{m_1} \) and is naturally reductive with respect to \( G \times K \), where the Lie algebra \( \mathfrak{k} \) is given by \( (\mathfrak{h}_0 \oplus \mathfrak{m}_2) \oplus \mathfrak{h}_2 \).

The proof is similar to Proposition 9.

By the same method as in Section 4, we have

**Proposition 17.** The components of the Ricci tensor \( r \) of the metric \( < , > \) on \( G \) are given by
\[
\begin{align*}
 r_0 &= \frac{u_0}{4 x_1^2 (d_3 + 8)} + \frac{u_2}{x_2^2 (d_3 + 8)}, \\
 r_2 &= \frac{1}{4 d_2 u_2} \left( d_2 - \frac{d_3 (d_3 + 2)}{2(d_3 + 8)} \right) + \frac{u_2}{4 d_2 x_1^2} \frac{d_3 (d_3 + 2)}{2(d_3 + 8)}, \\
 r_3 &= \frac{1}{2} \frac{x_2}{x_1} - \frac{2}{2 x_1^2 (d_3 + 8)} - \frac{1}{2} \frac{x_2}{x_1^2} \left( \frac{u_0}{(d_3 + 8)} \right) + \frac{u_2}{x_2^2} \frac{d_3 (d_3 + 2)}{2(d_3 + 8)}, \\
 r_4 &= \frac{1}{x_2} \frac{4}{(d_3 + 8)} + \frac{x_2}{4 x_1^2 (d_3 + 8)} - \frac{u_0}{x_2^2} \frac{2}{(d_3 + 8)}.
\end{align*}
\]

Now a metric
\[
< , > = u_0 \cdot B|_{h_0} + u_2 \cdot B|_{h_2} + x_1 \cdot B|_{m_1} + x_2 \cdot B|_{m_2}
\]
on \( G \) is Einstein if and only if there exists a positive solution \( \{ u_0, u_2, x_1, x_2, e \} \) of the system of equations
\[
(77) \quad r_0 = e, \quad r_2 = e, \quad r_3 = e, \quad r_4 = e.
\]

We normalize the system of equations by putting \( x_1 = 1 \). Then the equation \( r_0 = r_4 \) give rise to the relation \( u_0 = x_2 \). Hence, by Proposition 16 we only obtain Einstein metrics which are naturally reductive.
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