A topological approach to non-Archimedean mathematics

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December 30, 2014

Abstract

We introduce the notion of Λ-limit for real functions and we use it to construct non-Archimedean extensions of \( \mathbb{R} \). Within this approach, hyperreal fields in the sense of nonstandard analysis are obtained by purely topological considerations as subsets of some Hausdorff spaces. This construction is called Λ-theory and we will compare it with the superstructure approach of Keisler. We prove that within Λ-theory we can construct nonstandard universes in the sense of [19]. Finally we present some applications to the calculus of variations by means of a family of generalized functions called ultrafunctions.

Keywords: non-Archimedean mathematics, Nonstandard Analysis, limits of functions, generalized functions, ultrafunctions.

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1 Introduction

The classical definition of nonstandard reals $\mathbb{R}^*$ within the formalism of Robinson (see e.g. [21]) appears too technical to many researchers, and not directly usable by most mathematicians. Since Robinson’s work first appeared, a simpler semantic approach (due to Elias Zakon) has been developed using purely set-theoretic objects as superstructures, ultrapowers etc. We refer to [14] for a short history of nonstandard analysis (NSA). However, many researcher working in NSA have the feeling that technical notions such as superstructure, ultrapower and transfer principle are not needed in order to carry out calculus with actual infinitesimals, as well as to carry out several applications of NSA. There have been many attempts to simplify and popularize NSA by mean of simplified presentations. We recall here in particular the approaches of Henson [17], Keisler [19] and Nelson [20]; other attempts have been made by Benci, Di Nasso and Forti with algebraic (see [1], [2], [5], [16]) and topological approaches (see [6], [15]). We also suggest [18] where NSA is introduced in a simplified way suitable for many applications.

In this paper we adopt an approach (which we call Λ-theory) that has some features in common with $\alpha$-Theory (see [4]) and also with the topological approaches to NSA presented in [6], [15]. The idea behind our approach is to formalize the intuitive idea of hyperreals as "limits" of real functions. We develop this idea by introducing the notion of Λ-limit for functions, which are limits in particular Hausdorff topological spaces. So our construction of the hyperreals starting from $\mathbb{R}$ is quite similar to the construction of $\mathbb{R}$ as the completion of $\mathbb{Q}$. We also extend our construction to define Λ-limits of bounded functions defined on the superstructure on $\mathbb{R}$ and we prove that within this generalization it is possible to construct a nonstandard universe in the sense of [19]. Moreover, we give a simple example to show how Λ-theory can be applied; this example concerns generalized solutions for certain minimization problems of the calculus of variations.

The basics of Λ-theory will be presented in section 2. In section 3 we confront our approach with the superstructure approach of Keisler, and we show how to construct a nonstandard universe within Λ-theory. Finally, in section 4 we show that Λ-theory gives a nice framework to study a notion of generalized solution for some differential problems which do not have any classical solution.
2 Λ-theory

2.1 The Λ-limit

We call Λ-theory the approach to non-Archimedean mathematics based on the notion of Λ-limit which will be presented below.

Let $\mathcal{L}$ be an infinite set equipped with a free ultrafilter $\mathcal{U}$. Every set $Q \in \mathcal{U}$ will be called qualified set. We will say that a property $P$ is eventually true for the function $\varphi(\lambda)$ if it is true for every $\lambda \in Q$, namely if there exists $Q \in \mathcal{U}$ such that $P(\varphi(\lambda))$ holds for every $\lambda \in Q$. Assume that $\Lambda \notin \mathcal{L}$ and consider the space $\mathcal{L} \cup \{\Lambda\}$. We equip $\mathcal{L} \cup \{\Lambda\}$ with a topology in which the neighborhood of $\Lambda$ are of the form $\{\Lambda\} \cup Q$, $Q \in \mathcal{U}$.

We recall that the limit of a function is defined as follows:

**Definition 1.** Let $(X, \tau)$ be a Hausdorff topological space, let $x_0 \in X$ and let $\varphi : \mathcal{L} \to X$ be a function. We say that $x_0$ is the Λ-limit of the function $\varphi$, and we write

$$\lim_{\lambda \to \Lambda} \varphi(\lambda) = x_0,$$

if for every neighborhood $V$ of $x_0$ the function $\varphi$ is eventually in $V$, namely if there is a qualified set $Q$ such that $\varphi(Q) \subset V$.

**Remark 2.** We use the notation $\lim_{\lambda \to \Lambda} \varphi(\lambda)$ since one may think to $\Lambda \notin \mathcal{L}$ as a "point at $\infty$" and to the sets in $\mathcal{U}$ as neighborhoods of $\Lambda$; it is similar to the point $\infty$ when one considers $\mathbb{R} \cup \{+\infty\}$. We prefer to use the symbol $\Lambda$ rather than $\infty$ since one may think to $\Lambda$ as a function of $\mathcal{U}$, namely $\Lambda = \Lambda(\mathcal{U})$. Thus the explicit mention of $\Lambda$ reminds that the Λ-limit depends on $\mathcal{U}$.

**Remark 3.** Another way to look at the limit (1) is to consider the Stone-Čech compactification $\beta \mathcal{L}$ of $\mathcal{L}$ with the relative topology and to think to $\Lambda \in \beta \mathcal{L}$ as of a nontrivial element of this compactification.

Our main result is the following:

**Theorem 4.** There exists a Hausdorff topological space $(\mathbb{R}_\mathcal{L}, \tau)$ such that

1. $\mathbb{R}_\mathcal{L} = cl_{\tau} (\mathcal{L} \times \mathbb{R})$;
2. $\mathbb{R} \subseteq \mathbb{R}_\mathcal{L}$ and $\forall c \in \mathbb{R}$

$$\lim_{\lambda \to \Lambda} (\lambda, c) = c;$$
3. for every function $\varphi : \mathcal{L} \to \mathbb{R}$, the limit

$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))$$

exists in $(\mathbb{R}_\mathcal{L}, \tau)$;
4. two functions are eventually equal if and only if

$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) = \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)).$$
Proof We set
\[ I = \{ \varphi \in \mathfrak{F}(\mathcal{L}, \mathbb{R}) \mid \varphi(x) = 0 \text{ in a qualified set} \}. \]

It is not difficult to prove that \( I \) is a maximal ideal in \( \mathfrak{F}(\mathcal{L}, \mathbb{R}) \); then
\[ \mathbb{K} := \frac{\mathfrak{F}(\mathcal{L}, \mathbb{R})}{I} \]
is a field. In the following, we shall identify a real number \( c \in \mathbb{R} \) with the equivalence class of the constant function \([c]_I\). We set
\[ \mathbb{R}_\mathcal{L} = (\mathcal{L} \times \mathbb{R}) \cup \mathbb{K}. \]

We equip \( \mathbb{R}_\mathcal{L} \) with the following topology \( \tau \). A basis for \( \tau \) is given by
\[ b(\tau) = \{ N_{\varphi,Q} \mid \varphi \in \mathfrak{F}(\mathcal{L}, \mathbb{R}), Q \in \mathcal{U} \} \cup \mathcal{P}(\mathcal{L} \times \mathbb{R}) \]
where
\[ N_{\varphi,Q} := \{ (\lambda, \varphi(\lambda)) \mid \lambda \in Q \} \cup \{ [\varphi]_I \} \]
is a neighborhood of \([\varphi]_I\) for every \( Q \in \mathcal{U} \).

In order to show that \( b(\tau) \) is a basis for a topology, we have to show that
\[ \forall A, B \in b(\tau) \exists C \in b(\tau) \text{ such that } C \subset A \cap B. \]

Let \( A, B \in b(\tau) \). If
\[ A \cap B \cap \mathbb{K} = \emptyset, \]
then \( A \cap B \) is open since \( A \cap B \in \mathcal{P}(\mathcal{L} \times \mathbb{R}) \). If
\[ A \cap B \cap \mathbb{K} \neq \emptyset, \]
then \( \exists \xi \in \mathbb{K} \) such that \( \xi \in A \cap B \); therefore there exist \( R, S \in \mathcal{U} \) such that \( A = N_{\varphi,R} \) and \( B = N_{\psi,S} \) with \([\varphi]_I = [\psi]_I = \xi\). Hence there exists \( Q \in \mathcal{U} \) such that
\[ \forall \lambda \in Q, \varphi(\lambda) = \psi(\lambda). \]

Thus if we set \( C := N_{\varphi,R \cap S \cap Q} \) we have that \( C \subset A \cap B \).

Let us show that \( \tau \) is a Hausdorff topology. Clearly it is sufficient to check it for points in \( \mathbb{K} \), so let \( \xi \neq \zeta \in \mathbb{K} \). Since \( \xi \neq \zeta \), there exists \( \varphi, \psi \in \mathfrak{F}(\mathcal{L}, \mathbb{R}), Q \in \mathcal{U} \) such that
\[ \xi = [\varphi]_I, \quad \zeta = [\psi]_I \quad \text{and} \quad \forall \lambda \in Q, \varphi(\lambda) \neq \psi(\lambda). \]
Therefore
\[ N_{\varphi,Q} \cap N_{\psi,Q} = \emptyset. \]

Let us observe that, by construction, for every function \( \varphi : \mathcal{L} \rightarrow \mathbb{R} \) we have that
\[ \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) = [\varphi]_I. \quad (2) \]
In fact, given a neighborhood \( N_{\varphi,Q} \) of \([\varphi]_I\), we have that \( \{ \varphi(\lambda) \mid \lambda \in Q \} \subseteq N_{\varphi,Q} \), so \([\varphi]_I\) is a \( \Lambda \)-limit of the function \((\lambda, \varphi(\lambda))\). Since the space is Hausdorff, the limit is unique, so \( \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) = [\varphi]_I \).

Let us prove that \((\mathbb{R}_\mathcal{L}, \tau)\) has the desired properties:
• property (1) follows directly by the definition of $\tau$;

• property (2) follows by the identification of every real number $c \in \mathbb{R}$ with the equivalence class of the constant function $[c]$;

• properties (3) and (4) follow by equation (2).  \hfill $\square$

A natural question is the following: if we assume that $\mathcal{U}$ is only a free filter, does a result similar to Thm 4 hold? The answer is no:

**Proposition 5.** Let us assume that there exists a Hausdorff topological space $(X, \tau)$ such that condition (3) of Thm 4 hold. Then $\mathcal{U}$ is an ultrafilter.

**Proof** Let us assume that $\mathcal{U}$ is not an ultrafilter. Let $A \subseteq \mathcal{L}$ be such that $A \notin \mathcal{U}$, $A^c \notin \mathcal{U}$. Let $\varphi : \mathcal{L} \to \mathbb{R}$ be such that

$$\varphi(\lambda) = \begin{cases} 1, & \text{if } \lambda \in A; \\ 0, & \text{if } \lambda \notin A. \end{cases}$$

Let $L = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))$. Let $O_1$, $O_2$, $O_3$ be open sets such that $0 \in O_0$, $1 \in O_1$, $L \in O_2$, $O_0 \cap O_1 = \emptyset$. Let us also suppose that $O_2 \cap O_1 \neq \emptyset$ iff $L = 1$. Let $B \in \mathcal{U}$ be such that $\varphi(\lambda) \in O_2$ for every $\lambda \in B$. Since $A, A^c \notin \mathcal{U}$, necessarily $A \cap B \neq \emptyset$ and $A^c \cap B \neq \emptyset$. Therefore $\{0, 1\} \subseteq O_2$ and this is absurd: if $L \neq 1$ then, by construction, $1 \notin O_2$; if $L = 1$ then, by construction, $0 \notin O_2$.  \hfill $\square$

Motivated by the philosophical similarity between the properties expressed in Thm 4 and the construction of $\mathbb{R}$ as the completion of $\mathbb{Q}$, we introduce the following definition:

**Definition 6.** A Hausdorff topological space $(\mathbb{R}_L, \tau)$ that satisfies conditions (2-4) of Thm 4 will be called a $(\mathcal{L}, \mathcal{U})$-completion of $\mathbb{R}$.

The topology introduced in the proof of Thm 4 plays an important role. So we introduce the following definition.

**Definition 7.** Let $(\mathbb{R}_L, \tau)$ be a $(\mathcal{L}, \mathcal{U})$-completion of $\mathbb{R}$. We call **slim topology**, and we denote by $\tau_{\mathcal{U}}$, the topology on $\mathbb{R}_L$ generated by the family of open sets

$$\{ N_{\varphi, Q} \mid \varphi \in \mathcal{F}(\mathcal{L}, \mathbb{R}), Q \in \mathcal{U} \} \cup \mathcal{P}(\mathcal{L} \times \mathbb{R})$$

where, for every $\varphi \in \mathcal{F}(\mathcal{L}, \mathbb{R})$, $Q \in \mathcal{U}$ we set

$$N_{\varphi, Q} := \{ (\lambda, \varphi(\lambda)) \mid \lambda \in Q \} \cup \{ \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \}.$$ 

We will study $(\mathcal{L}, \mathcal{U})$-completions of $\mathbb{R}$ in detail in the following section.
2.2 The hyperreal field

Let \((\mathbb{R}_\mathcal{L}, \tau)\) be a \((\mathcal{L}, \mathcal{U})\)-completion of \(\mathbb{R}\). Let us fix some notation: we will denote by \(K\) the set 
\[ K = \left\{ \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \mid \varphi \in \mathcal{F}(\mathcal{L}, \mathbb{R}) \right\}. \]

The aim of this section is to study a few properties of \(K\).

**Proposition 8.** \(\mathbb{R}_\mathcal{L} = (\mathcal{L} \times \mathbb{R}) \cup K\).

**Proof** By condition (1) in Thm 4 we know that \(\mathcal{L} \times \mathbb{R} \subseteq \mathbb{R}_\mathcal{L}\). By condition (3) we deduce that \(\mathbb{R}_\mathcal{L} = (\mathcal{L} \times \mathbb{R}) \cup K\). We conclude by observing that \((\mathcal{L} \times \mathbb{R}) \cap K = \emptyset\): in fact, let us suppose that there exists \(\varphi : \mathcal{L} \to X\) such that 
\[ \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) = (\lambda_0, r) \in \mathcal{L} \times \mathbb{R}. \]
Since \(\{(\lambda_0, r)\}\) is open, by definition there exists \(Q \in \mathcal{U}\) such that \(\forall \lambda \in Q, (\lambda, \varphi(\lambda)) = (\lambda_0, r)\). Therefore \(Q = \{\lambda_0\}\), and this is absurd since \(\mathcal{U}\) is free. \(\Box\)

By condition (2) in the definition of \((\mathcal{L}, \mathcal{U})\)-completions it follows that \(\mathbb{R} \subseteq K\). Moreover we have the following result:

**Proposition 9.** For every finite subset \(F \subseteq \mathbb{R}\), for every function \(\varphi : \mathcal{L} \to \mathbb{R}\) we have that 
\[ \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \in F. \]

**Proof** Let \(F = \{x_1, \ldots, x_n\}\). For every \(i \leq n\) let 
\[ A_i = \{\lambda \in \mathcal{L} \mid \varphi(\lambda) = x_i\}. \]
Since $\mathcal{U}$ is an ultrafilter, there exists exactly one index $i_0 \leq n$ such that $A_{i_0} \in \mathcal{U}$. Now let $\xi = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))$. Let us suppose that $\xi \neq x_{i_0}$. Let $O_1, O_2$ be disjoint open sets such that $\xi \in O_1, x_{i_0} \in O_2$. Since $x_{i_0}$ is the limit of the constant sequences with value $x_{i_0}$, there exists $B \in \mathcal{U}$ such that
$$\{(\lambda, x_{i_0}) \mid \lambda \in B\} \subseteq O_2.$$ Let $C \in \mathcal{U}$ be such that $\{(\lambda, \varphi(\lambda)) \mid \lambda \in C\} \subseteq O_1$. Then by construction we have that
$$\forall \lambda \in A_{i_0} \cap B \cap C \ (\lambda, \varphi(\lambda)) = (\lambda, x_{i_0}) \in O_1 \cap O_2,$$ and this is a contradiction since $O_1 \cap O_2 = \emptyset$. Therefore $\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) = x_{i_0} \in F$. □

Sum and product of elements on $\mathbb{R}$ can be extended to $\mathbb{K}$ as follows:

**Definition 10.** We set
$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) + \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)) := \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda) + \psi(\lambda));$$
$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \cdot \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)) := \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda) \cdot \psi(\lambda)).$$

**Theorem 11.** $(\mathbb{K}, +, \cdot, 0, 1)$ is a field which contains $\mathbb{R}$.

**Proof** That $\mathbb{R} \subseteq \mathbb{K}$ follows by condition (2) of the definition of $(\mathcal{L}, \mathcal{U})$-completion. The only non-trivial property that we have to prove to show that $\mathbb{K}$ is a field is the existence of a multiplicative inverse for every $x \neq 0$. Let $x \in \mathbb{K}$, $x \neq 0$. Since the topology is Hausdorff and $x \neq 0$, there is a set $Q \in \mathcal{U}$ such that
$$\forall \lambda \in Q, \varphi(\lambda) \neq 0.$$ Let $\phi : \mathcal{L} \to \mathbb{R}$ be defined as follows:
$$\phi(\lambda) = \begin{cases} 1 & \text{if } \lambda \notin Q; \\ \frac{1}{Q(\lambda)} & \text{if } \lambda \in Q. \end{cases}$$ Then $\varphi(\lambda) \cdot \phi(\lambda) = 1$ for every $\lambda \in Q$, thus $\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \cdot \lim_{\lambda \to \Lambda} (\lambda, \phi(\lambda)) = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda) \cdot \phi(\lambda)) = 1$, namely
$$x^{-1} := \lim_{\lambda \to \Lambda} (\lambda, \phi(\lambda))$$ is the inverse of $x$. □

The ordering of $\mathbb{R}$ can be extended to $\mathbb{K}$ by setting
$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) < \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)) \iff \varphi(\lambda) < \psi(\lambda) \text{ eventually,}$$ namely iff $\{\lambda \in \mathcal{L} \mid \varphi(\lambda) < \psi(\lambda)\}$ is qualified. This ordering is clearly an extension of the ordering relation defined on $\mathbb{R}$ since, for every $x, y \in \mathbb{R}$, if $x \leq y$ and $\varphi_x, \varphi_y : \mathcal{L} \to \mathbb{R}$ are the constant sequences with values resp. $x, y$ then
$$\{\lambda \in \mathcal{L} \mid \varphi_x(\lambda) < \varphi_y(\lambda)\} = \mathcal{L},$$ which is qualified.
Remark 12. Usually, the inclusion \( \mathbb{R} \subseteq \mathbb{K} \) is proper: e.g., let \( \mathcal{U} \) be a countably incomplete ultrafilter. Let \( \langle A_n \mid n \in \mathbb{N} \rangle \) be a family of elements of \( \mathcal{U} \) such that \( \bigcap_{n \in \mathbb{N}} A_n = \emptyset \), let \( B_n = \bigcap_{i \leq n} A_n \) for all \( n \in \mathbb{N} \) and let \( \phi : \mathcal{L} \rightarrow \mathbb{R} \) be defined as follows: for every \( \lambda \in \mathcal{L} \),

\[
\phi(\lambda) = n \iff \lambda \in B_n \setminus B_{n+1}.
\]

Then \( \lim_{\lambda \to \Lambda} (\lambda, \phi(\lambda)) \notin \mathbb{R} \): in fact, \( \lim_{\lambda \to \Lambda} (\lambda, \phi(\lambda)) > n \) for every \( n \in \mathbb{N} \) (and so, in particular, this limit is infinite). This holds since, for every \( n \in \mathbb{N} \), by construction we have that

\[
\{ \lambda \in \mathcal{L} \mid \phi(\lambda) \geq n \} = B_n \in \mathcal{U}.
\]

When the inclusion \( \mathbb{R} \subseteq \mathbb{K} \) is proper we have that \( \mathbb{K} \) is a superreal non Archimedean field. In this case, it will be called a hyperreal field. The terminology will be motivated by Cor 14.

We want to study the relationships between \( \mathbb{K} \) and the ultrapower \( \mathbb{R}^\mathcal{U} \).

Definition 13. Let \( \equiv_{\mathcal{U}} \) be the equivalence relation on \( \mathbb{R}^{\mathcal{L}} \) defined as follows: for every \( \varphi, \psi : \mathcal{L} \rightarrow \mathbb{R} \)

\[
\varphi \equiv_{\mathcal{U}} \psi \iff \{ \lambda \in \mathcal{L} \mid \varphi(\lambda) = \psi(\lambda) \} \in \mathcal{U}.
\]

The ultrapower \( \mathbb{R}^\mathcal{U} \) is the quotient \( \mathbb{R}^{\mathcal{L}} / \equiv_{\mathcal{U}} \).

Corollary 14. \( \mathbb{K} \) and \( \mathbb{R}^\mathcal{U} \) are isomorphic as fields.

Proof The isomorphism is given by the map \( \Psi : \mathbb{K} \rightarrow \mathbb{R}^\mathcal{U} \) such that, for every \( \varphi : \mathcal{L} \rightarrow \mathbb{R} \),

\[
\Psi\left( \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \right) = [\varphi]_{\mathcal{U}}.
\]

Condition (4) in the definition of \( (\mathcal{L}, \mathcal{U}) \)-completion entails that \( \Psi \) is injective, whereas the definition of \( \mathbb{K} \) as the set of all possible \( \Lambda \)-limits entails that \( \Psi \) is surjective. Since it is immediate to see that \( \Psi \) also preserves the operations, we have that it is an isomorphism.

We will strengthen Cor 14 in Thm 36. By Cor 14 it clearly follows that, as sets, \( \mathbb{R}^{\mathcal{L}} \cong (\mathcal{L} \times \mathbb{R}) \cup \mathbb{R}^\mathcal{U} \).

Remark 15. Let us note that \( (\mathcal{L} \times \mathbb{R}) \cup \mathbb{R}^\mathcal{U}, \tau \) is a \( (\mathcal{L}, \mathcal{U}) \)-completion of \( \mathbb{R} \) for different choices of \( \tau \). One such choice is \( \tau_{\mathcal{U}} \): a different topology can be constructed as follows: let us fix a function \( \varphi \) with \( \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \notin \mathbb{R} \), a nonempty infinite set \( B \notin \mathcal{U} \), a free filter \( \mathcal{F} \) on \( B \) and let us consider the

\[\text{An ultrafilter } \mathcal{U} \text{ is countably incomplete if there exists a family } \langle A_n \mid n \in \mathbb{N} \rangle \text{ of elements of } \mathcal{U} \text{ such that } \bigcap_{n \in \mathbb{N}} A_n = \emptyset.\]

\[\text{A superreal non Archimedean field is an ordered field that properly contains } \mathbb{R}.\]
following topology $\tau$ on $\mathcal{L} \times \mathbb{R} \cup \mathbb{R}_U^\mathbb{R}$: if $\xi \neq \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))$ then a family of open neighborhoods of $\xi$ is 

$$\{O_{\varphi,Q} \mid Q \in \mathcal{U}, \psi \text{ function with } \xi = \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda))\};$$

if $\xi = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))$ then a family of open neighborhoods of $\xi$ is

$$\{O_{F,Q} \mid F \in \mathcal{F}, Q \in \mathcal{U}\}$$

where, for every $F \in \mathcal{F}, Q \in \mathcal{U}$ we set

$$O_{F,Q} = O_{\varphi,Q} \cup \{(\lambda, x) \mid \lambda \in F, x \in \mathbb{R}\}.$$ 

By construction, $((\mathcal{L} \times \mathbb{R}) \cup \mathbb{R}_U^\mathbb{R}, \tau)$ is a $((\mathcal{L}, \mathcal{U})$-completion of $\mathbb{R}$.

The next two propositions characterize the slim topology.

**Proposition 16.** The slim topology $\tau_U$ is finer than any topology $\tau$ that makes $((\mathcal{L} \times \mathbb{R}) \cup \mathbb{R}_U^\mathbb{R}, \tau)$ a $(\mathcal{L}, \mathcal{U})$-completion of $\mathbb{R}$.

**Proof** Let $\tau$ be given, let $O$ be an open set in $\tau$ and let $x \in O$. If $x \in \mathcal{L} \times \mathbb{R}$ then $\{x\}$ is an open neighborhood of $x$ in $\tau_U$ contained in $O$; if $x = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))$ for some function $\varphi : \mathcal{L} \to \mathbb{R}$ then let $B \in \mathcal{U}$ be such that $\{(\lambda, \varphi(\lambda)) \mid \lambda \in B\} \subseteq O$; therefore, by construction, $O_{\varphi,B}$ is an open neighborhood of $x$ in $\tau_U$ entirely contained in $O$. This proves that $O$ is an open set in $\tau_U$, therefore $\tau_U$ is finer than $\tau$. $\square$

**Proposition 17.** Let $((\mathcal{L} \times \mathbb{R}) \cup \mathbb{R}_U^\mathbb{R}, \tau)$ a $(\mathcal{L}, \mathcal{U})$-completion of $\mathbb{R}$. The following facts are equivalent:

1. $\tau = \tau_U$;
2. for every set $B \subseteq (\mathcal{L} \times \mathbb{R})$ we have that 

$$\text{cl}_\tau(B) = B \cup \left\{\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \mid \exists A \in \mathcal{U} \forall \lambda \in A \ (\lambda, \varphi(\lambda)) \in B\right\}.$$ 

**Proof** (1) $\Rightarrow$ (2) Let $\varphi$ be a function. Let $A = \{\lambda \in \mathcal{L} \mid (\lambda, \varphi(\lambda))\} \in B$. If $A \in \mathcal{U}$ then for every open neighborhood $O$ of $\xi = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))$ we have that $O \cap B \neq \emptyset$ by construction, so $\xi \in \text{cl}_\tau(B)$; if $A \notin \mathcal{U}$ then $O_{\varphi,A}$ is a neighborhood of $\xi$ such that $O_{\varphi,A} \cap B = \emptyset$, therefore $\xi \notin \text{cl}_{\tau_U}(B)$.

(2) $\Rightarrow$ (1) Let $A \in \mathcal{U}$, let $\varphi : \mathcal{L} \to \mathbb{R}$ and let $\xi = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))$. Let us consider $B = (\mathcal{L} \times \mathbb{R}) \setminus O_{A,\varphi}$. By hypothesis and construction

$$\text{cl}_\tau(B) = [(\mathcal{L} \times \mathbb{R}) \cup \mathbb{R}_U^\mathbb{R}] \setminus O_{A,\varphi}.$$ 

Therefore $O_{A,\varphi}$ is open for every $A \in \mathcal{U}$, $\varphi : \mathcal{L} \to \mathbb{R}$, so $\tau$ is finer than $\tau_U$ which, as a consequence of Proposition 16 entails that $\tau = \tau_U$. $\square$

**Definition 18.** We will call $((\mathcal{L} \times \mathbb{R}) \cup \mathbb{R}_U^\mathbb{R}, \tau_U)$ the canonical $(\mathcal{L}, \mathcal{U})$-completion of $\mathbb{R}$.

From now on we will work only with the canonical $(\mathcal{L}, \mathcal{U})$-completion of $\mathbb{R}$. The more general notion of $(\mathcal{L}, \mathcal{U})$-completions of a generic set (for $\mathcal{U}$ generic free filter on $\mathcal{L}$) will be studied in a forthcoming paper.
2.3 Natural extension of sets and functions

From now on, \( (\cdot) \) will denote the closure operator in the canonical \((\mathcal{L}, \mathcal{U})\)-completion of \( \mathbb{R} \).

**Definition 19.** For every \( E \subseteq \mathbb{R} \) we set
\[
E_{\mathcal{L}} := \mathcal{L} \times E.
\]

A different and related (as we will show in Prop 21) extension of \( E \) is the following:

**Definition 20.** Given a set \( E \subset \mathbb{R} \), we set
\[
E^* := \left\{ \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)) \mid \psi(\lambda) \in E \right\};
\]

\( E^* \) is called the **natural extension** of \( E \).

Let us observe that by property (2) of the definition of \((\mathcal{L}, \mathcal{U})\)-completions it follows that \( E \subseteq E^* \). Following the notation introduced in Def 20 from now on we will denote \( K \) by \( \mathbb{R}^* \).

It is easy to modify the proof of Prop 8 to obtain the following result:

**Proposition 21.** For every \( E \subseteq \mathbb{R} \) we have that \( E_{\mathcal{L}} = (\mathcal{L} \times E) \cup E^* \).

It is also possible to extend functions to \( \mathbb{R}_{\mathcal{L}} \). To this aim, given a function
\[ f : A \to B \]
we will denote by
\[ f_{\mathcal{L}} : \mathcal{L} \times A \to \mathcal{L} \times B \]
the function defined as follows:
\[
f_{\mathcal{L}} (\lambda, x) = (\lambda, f(x)).
\]

**Lemma 22.** For every \( A, B \subseteq \mathbb{R} \), for every function \( f : A \to B \), \( f \) can be extended to a continuous function
\[
\overline{f_{\mathcal{L}}} : A_{\mathcal{L}} \to B_{\mathcal{L}}.
\]
Moreover, the restriction of \( \overline{f_{\mathcal{L}}} \) to \( A \) coincides with \( f \).

**Proof** To extend \( f_{\mathcal{L}} \) it is sufficient to define it on \( A^* \). Let \( \varphi \in A^\mathcal{L} \). We set
\[
\overline{f_{\mathcal{L}}} \left( \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \right) = \lim_{\lambda \to \Lambda} (\lambda, f (\varphi(\lambda))).
\]

Let us note that the definition is well posed and that \( \overline{f_{\mathcal{L}}} (\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))) \in B^* \) since, for every \( \varphi \in A^\mathcal{L} \), the function \( f \circ \varphi \in B^\mathcal{L} \). This extension is continuous: let \( \Omega \) be a basis open subset of \( B_{\mathcal{L}} \). If \( \Omega = \{(\lambda, x)\} \) then
\[
\overline{f_{\mathcal{L}}}^{-1} (\Omega) = \bigcup_{y \in f^{-1}(x)} (\lambda, y).
\]
which is open. If $\Omega = N_{\varphi, Q}$ for some $\varphi : L \to R$, $Q \in \mathcal{U}$ then let $\xi \in \overline{f_L^{-1}(\Omega)}$. If $\xi = (\lambda, x)$ for some $x \in A$ then $\{(\lambda, x)\}$ is a neighborhood of $(\lambda, x)$ included in $\overline{f_L^{-1}(\Omega)}$; if $\xi = [\psi]$ then $\overline{f_L([\psi])} = [\varphi]$, therefore there exists $Q_1 \in \mathcal{U}$ such that $f(\psi(\lambda)) = \varphi(\lambda)$ for all $\lambda \in Q_1$, hence if we set $Q_2 = Q \cap Q_1$ we have that $N_{\psi, Q_2}$ is a neighborhood of $\xi$ included in $\overline{f_L^{-1}(\Omega)}$, thus $\overline{f_L^{-1}(\Omega)}$ is open, and this proves that $\overline{f_L}$ is continuous.

Finally, $\overline{f_L}$ restricted to $A$ coincides with $f$ since, for every $a \in A$, by definition

$$\overline{f_L}(a) = \overline{f_L}\left(\lim_{\lambda \to \Lambda} (\lambda, a)\right) = \lim_{\lambda \to \Lambda} (\lambda, f(a)) = f(a).$$

Lemma 22 entails that the following definition is well posed:

**Definition 23.** Given a function

$$f : A \to B$$

the restriction of $\overline{f_L}$ to $A^*$ is called the **natural extension** of $f$ and it will be denoted by

$$f^* : A^* \to B^*.$$  

In particular, $f^*(a) = f(a)$ for every $a \in A$.

### 2.4 The $\Lambda$-limit in $V_\infty(\mathbb{R})$

In this section we want to extend the notion of $\Lambda$-limit to a wider family of functions. To do that, we have to introduce the notion of superstructure on a set (see also [19]):

**Definition 24.** Let $E$ be an infinite set. The **superstructure** on $E$ is the set

$$V_\infty(E) = \bigcup_{n \in \mathbb{N}} V_n(E),$$

where the sets $V_n(E)$ are defined by induction by setting

$$V_0(E) = E$$

and, for every $n \in \mathbb{N}$,

$$V_{n+1}(E) = V_n(E) \cup \mathcal{P}(V_n(E)).$$

Here $\mathcal{P}(E)$ denotes the power set of $E$. Identifying the couples with the Kuratowski pairs and the functions and the relations with their graphs, it follows that $V_\infty(E)$ contains almost every usual mathematical object that can be constructed starting with $E$; in particular, $V_\infty(\mathbb{R})$ contains almost every usual mathematical object of analysis.
Sometimes, following e.g. [19], we will refer to
\[ U := V_\infty(\mathbb{R}) \]
as to the **standard universe**. A mathematical entity (number, set, function or relation) is said to be **standard** if it belongs to \( U \).

Now we want to define the \( \Lambda \)-limit of \((\lambda, \varphi(\lambda))\) where \( \varphi(\lambda) \) is any bounded function of mathematical objects in \( V_\infty(\mathbb{R}) \) (a function \( \varphi : \mathcal{L} \to V_\infty(\mathbb{R}) \) is called bounded if there exists \( n \) such that \( \forall \lambda \in \mathcal{L}, \varphi(\lambda) \in V_n(\mathbb{R}) \)). To this aim, let us consider a function
\[ \varphi : \mathcal{L} \to V_n(\mathbb{R}). \] (4)
We will define \( \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \) by induction on \( n \).

**Definition 25.** For \( n = 0 \), \( \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \) exists by Thm 4; so by induction we may assume that the limit is defined for \( n-1 \) and we define it for the function (4) as follows:
\[ \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) = \left\{ \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)) \mid \psi : \mathcal{L} \to V_{n-1}(\mathbb{R}) \text{ and } \forall \lambda \in \mathcal{L}, \psi(\lambda) \in \varphi(\lambda) \right\}. \]
Clearly \( \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \) is a well defined set in \( V_\infty(\mathbb{R}^*) \).

**Definition 26.** A mathematical entity (number, set, function or relation) which is the \( \Lambda \)-limit of a function is called **internal**.

Notice that \( V_\infty(\mathbb{R}^*) \) contains sets which are not internal.

**Example 27.** Each real number is standard and internal. However the set of real numbers \( \mathbb{R} \in V_\infty(\mathbb{R}^*) \) is standard, but not internal. In order to see this let us suppose that there is a function \( \varphi : \mathcal{L} \to V_1(\mathbb{R}) \) such that \( \mathbb{R} = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \). Therefore, by definition, we would have
\[ \mathbb{R} = \left\{ \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)) \mid \psi : \mathcal{L} \to \mathbb{R} \text{ and } \forall \lambda \in \mathcal{L}, \psi(\lambda) \in \varphi(\lambda) \right\}. \]
In particular, for every constant \( c \in \mathbb{R} \) we have that \( c \in \varphi(\lambda) \); therefore, \( \varphi(\lambda) = \mathbb{R} \) for every \( \lambda \in \mathcal{L} \), and this is absurd because
\[ \lim_{\lambda \to \Lambda} (\lambda, \mathbb{R}) = \mathbb{R}^*, \]
and (except trivial cases) \( \mathbb{R}^* \) properly includes \( \mathbb{R} \). Let us explicitly observe that (except trivial cases), while for every \( c \in \mathbb{R} \) the function \( \lambda \to (\lambda, c) \) converges to \( c \), given \( A \in V_n(\mathbb{R}) \), for \( n \geq 1 \) the function \( \lambda \to (\lambda, A) \) converges to a proper superset of \( A \).

**Definition 28.** A mathematical entity (number, set, function or relation) which is not internal is called **external**.
The definition of limit given by Def 25 is purely insiemistic and it is not related to any topology. Thus a question arises naturally: is there a topological Hausdorff space such that the limit given by Def 25 is the topological limit of a function?

The answer is positive. Let

$$U_L = [L \times V_\infty(\mathbb{R})] \cup V_\infty(\mathbb{R}^*)$$

and let $\tau_U$ be the topology on $U_L$ defined as follows:

- every subset of $L \times V_\infty(\mathbb{R})$ is open;
- if $x \in V_\infty(\mathbb{R}^*)$ is external then \(\{x\}\) is open;
- if $x$ is internal and $\varphi$ is a bounded function such that
  
  $$x = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))$$

  then
  
  $$N_{\varphi, Q} := \{(\lambda, \varphi(\lambda)) | \lambda \in Q\} \cup \{x\}$$

  is open.

It is clear that this topology is Hausdorff and that the $\Lambda$-limit is a limit in this topology.

The set

$$U_L = [L \times V_\infty(\mathbb{R})] \cup V_\infty(\mathbb{R}^*)$$

will be called the expanded universe. Let us note that, by construction, $U_L \subseteq V_\infty(\mathbb{R}_E)$.

The results about extensions of subsets of $\mathbb{R}$ and of functions $f : A \to B$, $A, B \subseteq \mathbb{R}$, can be generalized to our new general setting. Since a function $f$ can be identified with its graph then the natural extension of a function is defined by the above definition. Moreover we have the following result, that can be proved as Lem 22:

**Theorem 29.** For every sets $E, F \in V_\infty(\mathbb{R})$ and for every function $f : E \to F$ the natural extension of $f$ is a function

$$f^* : E^* \to F^*,$$

and for every function $\varphi : \mathcal{L} \to E$ we have that

$$\lim_{\lambda \to \Lambda} f(\lambda, \varphi(\lambda)) = f^* \left( \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \right).$$
3 Comparison between Λ-theory and NSA

3.1 Λ-theory and nonstandard universes

In this section we want to show the relationship between Λ-theory and NSA. To be more precise, we will show that \( U_L \) contains a nonstandard universe in the sense of Keisler [19]. We recall the main definitions of [19].

**Definition 30.** A superstructure embedding is a one to one mapping \( * \) of \( V_\infty(\mathbb{R}) \) into another superstructure \( V_\infty(\mathbb{S}) \) such that

1. \( \mathbb{R} \) is a proper subset of \( \mathbb{S} \), \( r^* = r \) for all \( r \in \mathbb{R} \), and \( \mathbb{R}^* = \mathbb{S} \);
2. for \( x, y \in V_\infty(\mathbb{R}) \), \( x \in y \) if and only if \( x^* \in y^* \).

To avoid confusion, in this section we will use the letter \( K \) to denote the non-Archimedean field constructed in section 2.2, while \( \mathbb{R}^* \) will be used as in Def 30.

Let us denote by \( L \) a formal language relative to a first order predicate logic with the equality symbol, a binary relation symbol \( \in \), and a constant symbol for each element in \( V_\infty(\mathbb{R}) \). We recall that a sentence \( p \in L \) is bounded if every quantifier in \( p \) is bounded (see e.g. [19]). The notion of bounded sequence allows to define the notion of nonstandard universe.

**Definition 31.** A nonstandard universe is a superstructure embedding \( * : V_\infty(\mathbb{R}) \to V_\infty(\mathbb{R}^*) \) which satisfies Leibniz’ Principle, which is the property that for each bounded sentence \( p \in L \), \( p \) is true in \( V_\infty(\mathbb{R}) \) if and only if \( p^* \) is true\(^3\) in \( V_\infty(\mathbb{R}^*) \).

**Definition 32.** We let \( * : V_\infty(\mathbb{R}) \to V_\infty(\mathbb{K}) \) be the map defined as follows: for every element \( x \in V_\infty(\mathbb{R}) \) we set

\[
x^* = \lim_{\lambda \to \Lambda} (\lambda, x).
\]

**Remark 33.** Following Keisler (see [19]), in the Def 31 we have called nonstandard universe just the superstructure embedding; however, in our approach, probably, it would be more appropriate to call nonstandard universe the set \( V_\infty(\mathbb{K}) \); in this case the global picture would be the following one: the extended universe

\[
U_\Sigma = [L \times V_\infty(\mathbb{R})] \sqcup V_\infty(\mathbb{K})
\]

contains couples \((\lambda, x)\) and elements of the nonstandard universe \( V_\infty(\mathbb{K}) \); the latter contains the following objects:

- standard elements, namely objects \( x \in V_\infty(\mathbb{R}) \subseteq V_\infty(\mathbb{K}) \);
- nonstandard elements, namely objects \( x \in V_\infty(\mathbb{K}) \setminus V_\infty(\mathbb{R}) \);

\(^3p^*\) is the bounded sentence obtained by changing every constant symbol \( c \in V_\infty(\mathbb{R}) \) that appears in \( p \) with \( c^* \).
• hyperimages, namely objects $x$ such that there exists $y \in V_\infty(\mathbb{R})$ with $x = y^*$;

• internal objects, namely $\Lambda$-limits of bounded functions;

• external objects.

To give some examples: $7, \mathbb{R}, \mathcal{P}(\mathbb{R} \times \mathcal{P}(\mathbb{R}))$ are all standard elements; $7$ is also an hyperimage, while $\mathbb{R}, \mathcal{P}(\mathbb{R} \times \mathcal{P}(\mathbb{R}))$ are not; $\mathbb{K}, \mathcal{P}(\mathbb{R})^*$ and $\lim_{\lambda \to \Lambda}(\lambda, \varphi(\lambda))$ for every $\varphi : \mathcal{L} \to \mathbb{R}$ which is not eventually constant are nonstandard elements, and they are all internal; $\mathbb{R}$ and $\mathbb{K} \setminus \mathbb{R}$ are external objects.

An interesting class of internal objects is that of hyperfinite elements:

**Definition 34.** An object $\xi \in V_\infty(\mathbb{K})$ is hyperfinite if there exists a natural number $n$ and a bounded function $\varphi : \mathcal{L} \to \mathcal{P}_{\text{fin}}(V_n(\mathbb{R}))$ such that $\xi = \lim_{\lambda \to \Lambda}(\lambda, \varphi(\lambda))$.

Hyperfinite elements are the analogue, in the universe $V_\infty(\mathbb{K})$, of finite elements in $V_\infty(\mathbb{R})$. The notion of hyperfinite object will be used in section 4 to show some applications of $\Lambda$-theory.

To precise the relationship between $\Lambda$-theory and nonstandard universes in the sense of Keisler we need to precise how we interpret formulas in $V_\infty(\mathbb{K})$:

**Definition 35.** Let $p(x_1, \ldots, x_n) \in \mathcal{L}$ be a bounded formula having $x_1, \ldots, x_n$ as its only free variables. Let $\xi_1 = \lim_{\lambda \to \Lambda}(\lambda, \varphi_1(\lambda)), \ldots, \xi_n = \lim_{\lambda \to \Lambda}(\lambda, \varphi_n(\lambda))$. We say that $p^*(\xi_1, \ldots, \xi_n)$ holds in $V_\infty(\mathbb{K})$ iff $p(\varphi_1(\lambda), \ldots, \varphi_n(\lambda))$ is eventually true in $V_\infty(\mathbb{R})$, namely iff

$$\{\lambda \in \mathcal{L} \mid p(\varphi_1(\lambda), \ldots, \varphi_n(\lambda)) \text{ holds in } V_\infty(\mathbb{R})\} \in \mathcal{U}.$$

**Theorem 36.** Let $*$ be defined as in Def 32 then

$$(V_\infty(\mathbb{R}), V_\infty(\mathbb{K}), *)$$

is a nonstandard universe.

**Proof.** That $* : V_\infty(\mathbb{R}) \to V_\infty(\mathbb{K})$ is a superstructure embedding follows immediately by the definitions. Moreover, for every bounded formula $p(x_1, \ldots, x_n) \in \mathcal{L}$ having $x_1, \ldots, x_n$ as its only free variables, for every $\xi_1 = \lim_{\lambda \to \Lambda}(\lambda, \varphi_1(\lambda)), \ldots, \xi_n = \lim_{\lambda \to \Lambda}(\lambda, \varphi_n(\lambda))$, we have that

$$p(\xi_1, \ldots, \xi_n) \text{ holds in } V_\infty(\mathbb{K}) \iff \{\lambda \in \mathcal{L} \mid p(\varphi_1(\lambda), \ldots, \varphi_n(\lambda)) \text{ holds in } V_\infty(\mathbb{R})\} \in \mathcal{U} \iff \mathcal{P}(p([\varphi_1], \ldots, [\varphi_n])) \text{ holds in } \mathbb{R}_U^\mathcal{U}.$$ 

This equivalence can be used to easily prove the transfer property for $* : V_\infty(\mathbb{R}) \to V_\infty(\mathbb{K})$ by induction on the complexity of formulas. □
3.2 General remarks

We want to compare our topological approach to Λ-theory with nonstandard analysis as presented by Keisler [19]. We point out the main differences:

1. in Λ-theory we assume the existence of a unique mathematical universe $\mathcal{U}_L \subset V_\infty(\mathcal{L} \cup \mathcal{K})$. Inside this universe there are entities that do not appear in traditional mathematics but that can be obtained as limits of traditional objects, namely the internal elements. Moreover, there are also external objects, and some of them are objects of traditional mathematics (e.g., $\mathbb{R}$);

2. in nonstandard analysis the primitive concept is that of hyperimage, the other concepts (e.g., the concept of internal object) are derived by that one; in Λ-theory, the primitive concept is that of Λ-limit, while the concept of hyperimage is derived by the limit. So, within Λ-theory the notion of internal object (being defined as a Λ-limit) is more primitive than that of hyperimage;

3. while (almost) all traditional approaches to nonstandard analysis are based on the idea of nonstandard image and require an explicit use of logic notions, the Λ-theory based on the notion of a topological limit is, in our opinion, more intuitive especially for the "working mathematician" in the sense of [18]. In fact, Λ-limits are defined in terms of functions, which are a natural generalization of the notion of sequence. Moreover within our approach it is possible to prove directly many instances of the transfer property for elementary properties, by showing that those elementary properties "go to the limit". Of course Leibniz' principle holds also in our approach (as we proved in Thm 36);

4. the construction of the hyperreal field in our approach has a topological "flavour" which is similar to other constructions in traditional mathematics. In fact, e.g. within our approach the construction of $\mathbb{R}^*$ as "set of limits of functions with values in $\mathcal{L} \times \mathbb{R}$" has some similarities with the construction of $\mathbb{R}$ as set of limits of Cauchy sequences with values in $\mathbb{Q}$.

4 Generalized Solutions

In many circumstances, the notion of function is not sufficient to the needs of a theory and it is necessary to extend it. In this section we want to apply Λ-theory to construct spaces of generalized functions called ultrafunctions (see also [3, 5, 9, 10, 11, 12, 13]), and to use them to study a simple class of problems in the calculus of variations. As we are going to show, ultrafunctions are constructed by means of a particular version of the hyperfinite approach which can be naturally introduced by means of Λ-theory.
In this section we will adopt the following shorthand notation: for every bounded function $\phi : \mathcal{L} \rightarrow V_\infty(\mathbb{R})$ we let
\[
\lim_{\lambda \uparrow \Lambda} \phi(\lambda) := \lim_{\lambda \rightarrow \Lambda} (\lambda, \phi(\lambda)).
\]

### 4.1 Ultrafunctions

Let $N$ be a natural number, let $\Omega$ be a set in $\mathbb{R}^N$ and let $V(\Omega)$ be a function vector space. We want to define the space of ultrafunctions generated by $V(\Omega)$. We assume that
\[
\mathcal{L} = \mathcal{P}_{\text{fin}}(V_\infty(\mathbb{R})).
\]
For any $\lambda \in \mathcal{L}$, we set
\[
V_\lambda(\Omega) = \text{Span}\{\lambda \cap V(\Omega)\}.
\]
Let us note that, by construction, $V_\lambda(\Omega)$ is a finite dimensional vector subspace of $V(\Omega)$.

**Definition 37.** Given the function space $V(\Omega)$ we set
\[
V_\Lambda(\Omega) := \lim_{\lambda \uparrow \Lambda} V_\lambda(\Omega) = \left\{ \lim_{\lambda \uparrow \Lambda} u_\lambda \mid u_\lambda \in V_\lambda(\Omega) \right\}.
\]
$V_\Lambda(\Omega)$ will be called the space of **ultrafunctions** generated by $V(\Omega)$.

Given any vector space of functions $V(\Omega)$, we have the following three properties:

1. the ultrafunctions in $V_\Lambda(\Omega)$ are $\Lambda$-limits of functions valued in $V(\Omega)$, so they are all internal functions;
2. the space of ultrafunctions $V_\Lambda(\Omega)$ is a vector space of hyperfinite dimension;
3. if we identify a function $f$ with its natural extension $f^*$ then $V_\Lambda(\Omega)$ includes $V(\Omega)$, hence we have that
\[
V(\Omega) \subset V_\Lambda(\Omega) \subset V(\Omega)^*.
\]

**Remark 38.** Notice that the natural extension $f^*$ of a function $f$ is an ultrafunction if and only if $f \in V(\Omega)$.

**Proof** See [9] for a proof. □

Ultrafunctions give generalized solutions to some problems in the calculus of variations (see e.g. [10]). Usually this kind of problems have a "natural space" where to look for solutions: the appropriate function space has to be a space in which the problem is well posed and (relatively) easy to be solved. For a very
large class of problems the natural space is a Sobolev space. However, many
times even the best candidates to be natural spaces are inadequate to study the
problem, since there is no solution in them. So the choice of the appropriate
function space is part of the problem itself; this choice is somewhat arbitrary
and it might depend on the final goals. In the framework of ultrafunctions this
situation persists. The general rule is: choose the "natural space" $V(\Omega)$ and
look for a generalized solution in $V_\lambda(\Omega)$. For many applications, an hypothesis
that we need to assume is that $D(\Omega) \subseteq V(\Omega) \subseteq L^2(\Omega)$. In this case, since
$V_\lambda(\Omega) \subset [L^2(\mathbb{R})]^*$, we can equip $V_\lambda(\Omega)$ with the following scalar product:

$$(u, v) = \int^* u(x)v(x) \, dx,$$

where $\int^*$ is the natural extension of the Lebesgue integral considered as a func-
tional

$$\int : L^1(\Omega) \to \mathbb{R}.$$ 

The norm of an ultrafunction will be given by

$$\|u\| = \left(\int^* |u(x)|^2 \, dx\right)^{\frac{1}{2}}.$$

Moreover, using the inner product (5), we can identify $L^2(\Omega)$ with a subset
of $V'(\Omega)$ and hence $[L^2(\Omega)]^*$ with a subset of $[V'(\Omega)]^*$; in this case, $\forall f \in
[L^2(\Omega)]^*$, we let $\tilde{f}$ be the unique ultrafunction such that, $\forall v \in V_\lambda(\Omega),$

$$\int \tilde{f}(x)v(x) \, dx = \int f(x)v(x) \, dx,$$

namely we associate to every $f \in L^2(\Omega)^*$ the function $\tilde{f} = P_\lambda(f)$, where

$$P_\lambda : [L^2(\Omega)]^* \to V_\lambda(\Omega)$$

is the orthogonal projection. Let us note that the key property to introduce
this association is that $[L^2(\Omega)]^*$ can be identified with a subset of $[V'(\Omega)]^*$. Therefore, using a similar idea, it is also possible to extend a large class of
operators:

**Definition 39.** Given an operator

$$\mathcal{A} : V(\Omega) \to V'(\Omega),$$

we can extend it to an operator

$$\tilde{\mathcal{A}} : V_\lambda(\Omega) \to V_\lambda(\Omega)$$

---

E.g., in [11] a (slightly modified) version of this hypothesis is used to construct an embed-
dding of the space of distributions in a particular algebra of functions constructed by means of
ultrafunctions.
in the following way: given an ultrafunction $u$, $\mathcal{A}_\Lambda(u)$ is the unique ultrafunction such that
\[
\forall v \in V_\Lambda(\Omega), \quad \int \tilde{\mathcal{A}}(u)v dx = \int \mathcal{A}^*(u)v dx;
\]
amely
\[
\tilde{\mathcal{A}} = P_\Lambda \circ \mathcal{A}^*,
\]
where $P_\Lambda$ is the canonical projection.

This association can be used, e.g., to define the derivative of an ultrafunction, by setting
\[
Du := \tilde{\partial}u = P_\Lambda(\partial^* u)
\]
for every ultrafunction $u \in V_\Lambda(\Omega) \cap C^1(\Omega)^*$.  

4.2 Applications to calculus of variations

To give an example of application of ultrafunctions to calculus of variations, we will see the ultrafunction interpretation of the Lavrentiev phenomenon. Let us consider the following problem: minimize the functional
\[
J_0(u) = \int_0^1 \left[ (|\nabla u|^2 - 1)^2 + |u|^2 \right] dx
\]
in the function space $C^1_0(\Omega) = C^1(\Omega) \cap C_0(\overline{\Omega})$. We assume $\Omega$ to be bounded to avoid problems of summability.

It is not difficult to realize that any minimizing sequence $u_n$ converges uniformly to 0 and that $J_0(u_n) \to 0$, but $J_0(0) > 0$ for any $u \in C^1_0(0,1)$.

On the contrary, it is possible to show that this problem has a minimizer in the space of ultrafunctions
\[
V^1_0(\Omega) = [C^1(\Omega) \cap C_0(\overline{\Omega})]_\Lambda.
\]
So our problem becomes
\[
\text{find } v \in V^1_0(\Omega) \text{ s.t. } \tilde{J}_0(v) = \min_{u \in V^1_0(\Omega)} \tilde{J}_0(u). \quad (P)
\]

To solve (P), let us prove the following ”ultrafunction version” of an existence result for minimizers of coercive continuous operators; the proof is based on a variant of Faedo-Galerkin method.

**Theorem 40.** Let $V(\Omega) \subseteq L^2(\Omega)$ be a vector space and let
\[
J : V(\Omega) \to \mathbb{R}
\]
be an operator continuous and coercive on finite dimensional spaces. Then the operator
\[
\tilde{J} : V_\Lambda(\Omega) \to \mathbb{R}^*
\]
has a minimum point. If $J$ itself has a minimizer $u$, then $u^*$ is a minimizer of $\tilde{J}$.  

---

5 This example has already been studied in greater detail in [10].
Proof Take $\lambda \in \mathcal{L}$; since the operator
$$J_{|_{V_{\lambda}}} : V_{\lambda}(\Omega) \rightarrow \mathbb{R}$$
is continuous and coercive, it has a minimizer; namely
$$\exists u_{\lambda} \in V_{\lambda} \forall v \in V_{\lambda} \ J(u_{\lambda}) \leq J(v).$$
We set
$$u_{\Lambda} = \lim_{\lambda \uparrow \Lambda} u_{\lambda}.$$ 
We show that $u_{\Lambda}$ is a minimizer of $\tilde{J}$. Let $v \in V_{\Lambda}(\Omega)$. Let us suppose that
$$v = \lim_{\lambda \uparrow \Lambda} v_{\lambda};$$
then by construction
$$\forall \lambda \in \mathcal{L} \ J(u_{\lambda}) \leq J(v_{\lambda}),$$therefore
$$\tilde{J}(u_{\Lambda}) \leq \tilde{J}(v).$$
If $J$ itself has a minimizer $\overline{\pi}$, then $u_{\lambda}$ is eventually equal to $\overline{\pi}$ and hence
$$u_{\Lambda} = \overline{\pi}.$$ 
□

As a consequence, problem (P) has a solution, since the functional $J_0$ satisfies the hypothesis of Thm \[40\] So there exists an ultrafunction $u \in V_{0}^{1}(\Omega)$ that minimizes $\tilde{J}_0$. Moreover, it can be represented as the $\Lambda$-limit of a function of minimizers of the approximate problems on the spaces $[C^1(\Omega) \cap C_0(\Omega)]_{\lambda}$. By using this characterization, it is also possible to derive some qualitative properties of $u$, e.g. it is not difficult to show that, $\forall x \in (0,1)^*$, the minimizer $u_{\Lambda}(x) \sim 0$ and that $\tilde{J}_0(u_{\Lambda})$ is a positive infinitesimal.

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