Time-Ordered Products and Exponentials

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Abstract

I discuss a formula decomposing the integral of time-ordered products of operators into sums of products of integrals of time-ordered commutators. The resulting factorization enables summation of an infinite series to be carried out to yield an explicit formula for the time-ordered exponentials. The Campbell-Baker-Hausdorff formula and the nonabelian eikonal formula obtained previously are both special cases of this result.

1 Introduction

It is a great pleasure to dedicate this article to Hiroshi Ezawa on the occasion of his 65th birthday. I am priviledged to have known him for 35 years, and am proud to see that he has such an illustrious career. I wish him and his wife Yoshiko the very best, healthy and a long life that will last for at least another 35 years.

True to the theme of this Workshop I will talk about a mathematical method that has applications in quantum mechanics. This is a method that could simplify computations of (integrals of) time-ordered products and time-ordered exponentials $U$. These quantities are ubiquitous in quantum mechanics. In one form or another they describe the time-evolution operator of quantum systems, the non-integrable phase factor (Wilson line) of Yang-Mills theories, and the scattering amplitudes of perturbation diagrams. The method relies on a decomposition formula, which expresses $U$ in terms of more primitive quantities $C$, the time-ordered commutators.

I shall concentrate in what follows to the description of this and other related formulas. There is certainly no time for the proof and very little to illustrate the applications. For those I refer the readers to the literature [1, 2, 3, 4, 5].
2 Preliminaries

Let $H_i(t)$ be operator-valued functions of time. No attempts will be made to discuss domains and convergence problems of these operators. Let $[s] = [s_1s_2 \cdots s_n]$ be a permutation of the $n$ numbers $[12 \cdots n]$, and $S_n$ the corresponding permutation group. We define the time-ordered product $U[s]$ to be the integral

$$U[s] = U[s_1s_2 \cdots s_n] = \int_{R[s]} dt_1 dt_2 \cdots dt_n H_{s_1}(t_{s_1})H_{s_2}(t_{s_2}) \cdots H_{s_n}(t_{s_n})$$

(1)

taken over the hyper-triangular region $R[s] = \{ T \geq t_{s_1} \geq t_{s_2} \geq \cdots \geq t_{s_n} \geq T' \}$, with operator $H_{s_i}(t_{s_i})$ standing to the left of $H_{s_j}(t_{s_j})$ if $t_{s_i} > t_{s_j}$. The average of $U[s]$ over all permutations $s \in S_n$ will be denoted by $U_n$:

$$U_n = \frac{1}{n!} \sum_{s \in S_n} U[s].$$

(2)

The decomposition formula expresses $U_n$ in terms of the time-ordered commutators $C[s] = C[s_1s_2 \cdots s_n]$. These are defined analogous to $U[s_1s_2 \cdots s_n]$, but with the products of $H_i$’s replaced by their nested multiple commutators:

$$C[s] = \int_{R[s]} dt_1 dt_2 \cdots dt_n$$

$$[H_{s_1}(t_{s_1}), [H_{s_2}(t_{s_2}), [\cdots , [H_{s_n-1}(t_{s_n-1}), H_{s_n}(t_{s_n})] \cdots ]]].$$  (3)

For $n = 1$, we define $C[s_i] = U[s_i]$. Similarly, the operator $C_n$ is defined to be the average of $C[s]$ over all permutations $s \in S_n$:

$$C_n = \frac{1}{n!} \sum_{s \in S_n} C[s].$$  (4)

It is convenient to use a ‘cut’ (a vertical bar) to denote products of $C[\cdots]'s$. For example, $C[312] \equiv C[31]C[2], \text{ and } C[7156423] \equiv C[71]C[564]C[2]C[3]$. Given a sequence $[s]$ of numbers with $s \in S_n$, its cut sequence $[s]_c$ is obtained from $[s]$ by inserting cuts at the appropriate places. A cut is inserted after $s_i$ iff $s_i < s_j$ for all $i < j$. In other words, we should proceed from left to right and put a cut after the smallest number encountered. The first cut is therefore put after the number ‘1’; the second after the smallest number to the right of ‘1’, etc. For example, $[5413267]_c = [5431267]$, and $[1267453]_c = [1267453]$.  

3 General Decomposition Formula

The main formula $\triangleright$ states that

$$n!U_n = \sum_{s \in S_n} U[s] = \sum_{s \in S_n} C[s]_c.$$  (5)
Great simplification occurs when all $H_i(t) = H(t)$ are identical, for then $U[s]$ and $C[s]$ depend only on $n$ but not on the particular $s \in S_n$. In that case all $U[s] = U_n$ and all $C[s] = C_n$, and the decomposition theorem becomes

$$U_n = \sum m_1 m_2 \cdots m_k C_{m_1} C_{m_2} \cdots C_{m_k},$$

$$\xi(m_1 m_2 \cdots m_k) = \prod_{i=1}^{k} \left( \sum_{j=m_i}^{m_{i+1}} m_j \right)^{-1}.$$ (7)

The sum in the first equation is taken over all $k$, and all $m_i > 0$ such that $\sum_{i=1}^{k} m_i = n$. The quantity $\xi(m_1 \cdots m_k)^{-1}$ is just the product of the number of numbers to the right of every cut (times $n$). Note that it is not symmetric under the interchange of the $m_i$'s. It is this asymmetry that makes the formulas for $K_n$ in (2) rather complicated.

We list below this special decompositions up to $n = 5$:

$$1! U_1 = C_1,$$

$$2! U_2 = C_1^2 + C_2,$$

$$3! U_3 = C_1^3 + 2C_2C_1 + C_1C_2 + 2C_3,$$

$$4! U_4 = C_1^4 + 6C_3C_1 + 2C_1C_3 + 3C_2^2 + 3C_2C_1^2 + 2C_1C_2C_1 + C_1^2C_2 + 6C_4,$$

$$5! U_5 = C_1^5 + 24C_4C_1 + 6C_3C_1 + 12C_2C_3 + 8C_2C_2 + 12C_3C_2^2 + 6C_1C_3C_1 + 2C_2^2C_3 + 8C_2C_1^2 + 4C_2C_1C_2 + 3C_1C_2^2 + 4C_2C_1^3 + 3C_1C_2C_1^2 + 2C_1^2C_2C_1 + C_1^3C_2.$$ (8)
The graphical expression for $U_3$ is given in Fig. 2.

5 Exponential Formula for Time-Ordered Exponentials

The time-ordered exponential

$$U = T \left( \exp \left( \int_T^T H(t) dt \right) \right) = \sum_{n=0}^{\infty} U_n$$

(9)

can be computed from the time-ordered products $U_n$. The factorization character in (9) and (8) suggests that it may be possible to sum up the power series $U_n$ to yield an explicit exponential function of the $C_m$’s. This is indeed the case.

5.1 Commutative $C_i$’s

First assume all the $C_m$ in (9) commute with one another. Then it is possible to show that

$$U = \sum_{n=0}^{\infty} U_n$$

$$= \prod_{j=1}^{\infty} \sum_{m=0}^{\infty} \frac{1}{j^{m+1}} C_m^j$$

$$= \exp \left[ \sum_{j=1}^{\infty} \frac{C_j}{j} \right].$$

(10)

5.2 General Exponential Formula

In general the $C_j$’s do not commute with one another so the exponent in (10) must be corrected by terms involving commutators of the $C_j$’s. The exponent $K$ can be computed by taking the logarithm of $U$ (9):

$$U = 1 + \sum_{n=1}^{\infty} U_n = \exp[K] \equiv \exp \left[ \sum_{i=1}^{\infty} K_i \right],$$

$$K = \ln \left[ 1 + \sum_{n=1}^{\infty} U_n \right] = \sum_{\ell=1}^{\infty} (-1)^{\ell-1} \frac{\ell}{\ell} \left[ \sum_{n=1}^{\infty} U_n \right]^\ell.$$

(11)
The resulting expression must be expressible as (multiple-)commutators of the $C$’s. In other words, only commutators of $H(t)$, in the form of $C_m$ and their commutators, may enter into $K$. This is so because in the special case when $H(t)$ is a member of a Lie algebra, $\mathcal{U}$ is a member of the corresponding Lie group and so $K$ must also be a member of a Lie algebra.

By definition, $K_i$ contains $i$ factors of $H(t)$. Calculation for the first five gives

$$
\begin{align*}
K_1 &= C_1 \\
K_2 &= \frac{1}{2}C_2 \\
K_3 &= \frac{1}{3}C_3 + \frac{1}{12}[C_2, C_1] \\
K_4 &= \frac{1}{4}C_4 + \frac{1}{12}[C_3, C_1] \\
K_5 &= \frac{1}{5}C_5 + \frac{3}{40}[C_4, C_1] + \frac{1}{60}[C_3, C_2] + \frac{1}{360}[C_1, [C_1, C_3]] + \\
&\quad + \frac{1}{240}[C_2, [C_2, C_1]] + \frac{1}{720}[C_1, [C_1, [C_1, C_2]]]
\end{align*}
$$

(12)

$K_n$ consists of $C_n/n$, plus the compensating terms in the form of commutators of the $C$’s. By counting powers of $H(t)$ it is clear that the subscripts of these $C$’s must add up to $n$, but beyond that all independent commutators and multiple commutators may appear. For that reason it is rather difficult to obtain an explicit formula valid for all $K_n$, if for no other reason than the fact that new commutator structures appear at every new $n$. It is however very easy to compute $K_n$ using (11) with the help of a computer. This is actually how $K_5$ was obtained.

Moreover, when we stick to commutators of a definite structure, their coefficients in $K$ can be computed. For example, the coefficient of the commutator term $[C_m, C_n]$ in any $K_{m+n}$ can be shown to be

$$
\eta_2 = \frac{n-m}{2mn(m+n)}
$$

(13)

See Ref. [5] for similar formulas for multiple commutators.

6 Applications

These formulas can be applied to mathematics and physics in various ways, depending on our choice of $H_i(t)$ and the integration interval $[T', T]$. If we choose the interval to be $[T', T] = [0, 2]$, and the operator $H(t)$ to be $P$ for $t \in [1, 2]$ and $Q$ for $t \in [0, 1]$, then $\mathcal{U} = \exp(P) \exp(Q)$, $C_1 = P + Q$, $C_{m+1} = (ad P)^m Q/m!$, and
and eqs. (11) and (12) lead to the Baker-Campbell-Hausdorff formula

\[
\exp(P)\exp(Q) = \exp[K_1 + K_2 + K_3 + K_4 + K_5 + \cdots]
\]

\[
K_1 = P + Q
\]

\[
K_2 = \frac{1}{2}[P, Q]
\]

\[
K_3 = \frac{1}{12}[P, [P, Q]] + \frac{1}{12}[Q, [Q, P]]
\]

\[
K_4 = -\frac{1}{24}[P, [Q, [P, Q]]]
\]

\[
K_5 = -\frac{1}{720}[P, [P, [P, [P, Q]]]] - \frac{1}{720}[Q, [Q, [Q, [Q, P]]]]
\]

\[
+ \frac{1}{360}[P, [Q, [Q, [Q, P]]]] + \frac{1}{360}[Q, [P, [P, Q]]]
\]

\[
+ \frac{1}{120}[P, [Q, [Q, [Q, P]]]] + \frac{1}{120}[Q, [P, [P, Q]]],
\] \hspace{1cm} (14)

The case when \([P, Q]\) commutes with \(P\) and \(Q\) is well known. In that case all \(K_n\) for \(n \geq 3\) are zero. Otherwise, up to and including \(K_4\) this formula can be found in eq. (15), §6.4, Chapter II of Ref. [6].

By choosing the interval to be \([T', T] = [-\infty, \infty]\), and the operators to be \(H(t) = \exp(ip \cdot k, t)V_i\), we obtain a nonabelian eikonal formula useful in physics. In that case \(U_n\) is the \(n\)th order tree amplitude for an energetic particle with four-momentum \(p^\mu\) to emit \(n\) bosons with momenta \(k^\mu_i \ll p^0\) via vertex factors \(V_i\). The decomposition formula (14) can then be interpreted as a repackaging of the tree amplitude into terms in which interference patterns in the spacetime and the internal quantum number variables are explicitly displayed. Such (destructive) interferences lead to cancellations, and the formula can be conveniently used to demonstrate the cancellations necessary for the self consistency of baryonic amplitudes in large-\(N_c\) QCD. It can also be used to obtain a simple understanding as to why gluons reggeize in QCD but photons do not reggeize in QED.

References

[**] Contribution to the Second Jagna International Workshop on Mathematical Methods of Quantum Physics, January 4-8, 1998, at Jagna, Bohol, Philippines, in honour of Prof. Hiroshi Ezawa on the occasion of his 65th birthday.

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