Prescribed-Time Control and Its Latest Developments
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Abstract—Prescribed-time (PT) control for nonlinear systems, originated from Song et al., has gained increasing attention among the control community. The salient feature of PT control lies in its ability to achieve system stability within a finite settling time user-assignable in advance irrespective of initial conditions. It is such a unique feature that has enticed many follow-up studies on this technically important area, motivating numerous research advancements. In this article, we provide a comprehensive survey on the recent developments in PT control. Through a concise introduction to the concept of PT control, and a unique taxonomy covering: 1) from robust PT control to adaptive PT control; 2) from PT control for single-input–single-output (SISO) systems to multi-input–multioutput (MIMO) systems; and 3) from PT control for an isolated system to multiagent systems; we present an accessible review of this interesting topic. We highlight key techniques, and fundamental assumptions adopted in various developments as well as some new design ideas. We also discuss several possible future research directions toward PT control.

Index Terms—Finite-time (FT) control, prescribed-time (PT) control, state scaling, time scaling, time-varying feedback.

I. INTRODUCTION

THE NOTION of prescribed-time (PT) control for nonlinear systems, pioneered by Song et al. [1], has brought much vitality to finite-time (FT) control, attracting increasing attention from the control community and motivating numerous follow-up studies on this important field over the past few years (see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], [50], [51], [52], [53], [54], [55], [56], [57], [58], [59], [60]).

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The past few decades have witnessed much progress in FT control of dynamic systems [64], [65], [66], [67], [68], [69], [70], [71], [72], [73], [74], [75], [76]. The homogeneous approach, terminal sliding mode, higher-order sliding modes, and adding a power integrator are suggested sequentially in an attempt to achieve FT stability for high-order nonlinear systems [77], [78], [79], [80], [81], [82]. Although convergence may be pursued in a finite time, estimation of the settling time relies explicitly on initial conditions. This may limit the application scope of those existing results when little knowledge of plant initial states are accessible. Later on, a notion termed fixed-time (FxT) control [83], [84], [85], [86], [87], [88], [89], [90], [91], [92] has emerged, which employs odd-order plus fractional-order feedback to provide various closed-loop system dynamics. The upper bound of the settling time can be estimated without using any information on initial conditions.

Despite the benefits of FxT control in the light of settling time estimation, no simple and obvious relationship exists between the control parameters and the intended upper bound of the settling time. The settling time under the FxT control is often overestimated, which may be hundreds or even thousands of times larger than the true settling time, resulting in an inaccurate description of system performance. On the other hand, the settling time is not a directly tunable parameter for either FT control or FxT control, as it also depends on other controller design parameters. To alleviate the problem of overestimation of the settling time while alleviating the dependence of the settling time on design parameters, the predefined-time (PdT) control approach is exploited in [93], [94].
where the least upper bound of the settling time can be preset irrespective of initial conditions and any other design parameter.

Recently, the classical idea that originated in strategic and tactical missile guidance applications [97], [98], [99] has been revisited and further applied to high-order nonlinear systems, namely, PT control, which inherits the advantages of FT control, FxT control, PdT control, and also allows for presetting the settling time precisely. This concept is of great importance in many practical engineering applications where transient processes must occur within a given time (e.g., missile guidance, multiagent rendezvous, emergency braking, and obstacle avoidance in robotic systems, etc.).

More importantly, the PT control is promising since it is robust to external disturbances, the control input is always smooth over the transient process, and there is no need for any information on the upper bound of the nonvanishing perturbations in the control design. The key technical design steps for PT control include: converting the original system to a new system by a time-varying transformation (including state scaling, time scaling, and some other technologies), dealing with matching/mismatching uncertainties and unknown control coefficients to construct appropriate Lyapunov inequalities, and selecting the appropriate control gain $\kappa$ to prove the boundedness of all closed-loop signals, especially the boundedness of the control inputs. Furthermore, because all real PT controllers have infinite gain characteristics as time tends to the window, thereby extending the use of PT control systems. In this article, we perform a complete study on several important theoretical breakthroughs, key technical concerns, and potential research problems in PT control, as well as provide a comprehensive literature survey.

The study will start next in Section II with an overview of some basic propositions of FT/FxT/PdT and PT control. Section III lists some specific literature on PT control for single-input–single-output (SISO) systems and provides some basic design ideas of prescribed robust and/or adaptive controller design, focused on the introduction of state scaling technology and time scaling technology on PT control. Section IV lists some state-of-the-art results on multi-input–multi-output (MIMO) systems and presents a detailed demonstration of PT control for this type of system. It covers square and nonsquare MIMO systems. Section V lists some interesting studies on PT distributed control and addresses some basic issues of PT control for multiagent systems. The organization of Sections II–V is shown in Fig. 1. Section VI provides some connections between FT and PT control and also discusses some possible open areas of research.

II. PRELIMINARIES

A. Definitions

We first consider some basic definitions of infinite-time (asymptotic/exponential) stability.

Definition 1 [108, Ch. 4]: For a nonautonomous system as
\[ \dot{x} = f(x, t) \]  \hspace{1cm} (1)

where $f : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}^n$ is piecewise continuous in $t$ and locally Lipschitz in $x$. The equilibrium point $x = 0$ is as follows.

1) Stable, if there exists a class of $\mathcal{K}$ function $\beta$ such that $\|x(t)\| \leq \beta(\|x(0)\|)$.
2) Asymptotically stable, if it is stable, and $x(t) \rightarrow 0$ as $t \rightarrow \infty$.
3) Exponentially stable, if there exist two positive numbers, $\lambda_1$ and $\lambda_2$, such that for sufficiently small $x(0)$, $\|x(t)\| \leq \lambda_1 \|x(0)\| e^{-\lambda_2 t}$ $\forall t \geq 0$.

The definitions of FT and FxT stability are stated below.

Definition 2 [83, Ch. 4]: For a system as (1), the equilibrium point $x = 0$ is as follow.

1) FT stable, if it is stable and there exists a $x(0)$-dependent settling time function $T(x(0))$ such that $x(t) = 0$ for $t \geq T(x(0))$.
2) FxT stable, if it is stable and the settling time function $T(x(0))$ is upper bounded on $\mathbb{R}$, i.e., $\exists T_{max} > 0$, $x(t) = 0$ for $t \geq T_{max}$.

Obviously, the terminal time always attaches itself to $x(0)$ in FT control, such attachment is however removed in FxT control. An astonishing scenario in FT/FxT stability is the PT stability, where the terminal time has nothing to do with initial condition, and thus can be user-set freely in advance.

B. Propositions on Finite-/Fixed-/Predefined-/Prescribed-Time Stability

Achieving FT stability for dynamic systems is of special theoretical and practical interest. The typical approach for establishing FT stability is to derive Lyapunov differential inequalities. Most of these inequalities can be found in the following works which are summarized as a variety of propositions.

Proposition 1 [78]: For system $\dot{x} = f(x, t)$, if there exists a $C^1$ function $V(x) \geq 0$ such that $\dot{V}(x) \leq -kV^q(x)$, where $k > 0$, $0 < q < 1$, then the closed-loop system is FT stable and the settling time is calculated by

\[ T = \frac{\ln\left(\frac{\|x(0)\|}{\epsilon}\right)}{k} \]
where $0 < \theta < 1$ is a constant.

Proposition 4 [76]: For system $\dot{x}(t, x)$, if there exists a $C^1$ function $V(x) \geq 0$ such that $\dot{V}(t) \leq -\alpha V^p(x) + \beta V^q(x)$ where $\alpha > 0$, $\beta > 0$, $p > 0$, $q > 0$, $k > 0$, and $pk < 1$, $qk > 1$, then the closed-loop system is FxT stable and the settling time is bounded by

$$T := \frac{\ln \left( \frac{k \theta V^q(0)}{k_1} \right)}{k_2 \theta (1 - q)}$$

where $0 < \theta < 1$ is a constant.

Proposition 5 [84]: For system $\dot{x}(x, t)$, if there exists a $C^1$ function $V(x) \geq 0$ such that $\dot{V}(x) \leq -\alpha V^p(x) + \beta V^q(x)$ where $\alpha > 0$, $\beta > 0$, $p > 0$, $q > 0$, $k > 0$, and $pk < 1$, $qk > 1$, then the closed-loop system is FxT stable and the settling time is bounded by

$$T := \frac{\ln \left( \frac{k \theta V^q(0) + k_1}{k_1} \right) \ln \left( \frac{k \theta V^q(0) + k_1}{k_2} \right)}{k_2 \theta (1 - q)}$$

Proposition 6 [83]: For system $\dot{x} = f(x, t)$, if there exists a $C^1$ function $V(x) \geq 0$ such that $\dot{V}(x) \leq -\alpha V^p(x) - \beta V^q(x)$ where $\alpha > 0$, $\beta > 0$, $p > 0$, $q > 0$, $k > 0$, and $pk < 1$, $qk > 1$, then the closed-loop system is FxT stable and the settling time is bounded by

$$T := \frac{1}{\alpha k (1 - pk)} + \frac{1}{\beta k (qk - 1)}$$

Proposition 7 [86]: For system $\dot{x} = f(x, t)$, if there exists a $C^1$ function $V(x) \geq 0$ such that $\dot{V}(x) \leq -\alpha V^p(x) - \beta V^q(x)$ where $\alpha > 0$, $\beta > 0$, $p > 0$, $q > 0$, $k > 0$, and $pk < 1$, $qk > 1$, then the closed-loop system is FxT stable and the settling time is bounded by

$$T := \frac{\pi \gamma}{\sqrt{\alpha \beta}}$$

where $\alpha > 0$, $\beta > 0$, $p > 0$, $q > 0$, $k > 0$, and $pk < 1$, $qk > 1$.
TABLE 1
RELATED PROPOSITIONS ON FT, FXT, PDT, AND PT CONTROL (THE MEANING OF THE RELATED PARAMETERS SEE PROPOSITIONS 1–11)

| Proposition | Expression of $\dot{V}(t)$ | Setting time function $T$ |
|-------------|-----------------------------|--------------------------|
| 1           | $\dot{V}(x) \leq -kV^q(x)$ | $T := \frac{1}{k(1-q)}V^{1-q}(x(0))$ |
| 2           | $\dot{V}(x) \leq -k_1V^q(x) + k_2V(x)$ | $T := \frac{1}{k_1(1-q)}\ln\left(1 - \frac{k_1V^{1-q}(x(0))}{k_2}\right)$ |
| 3           | $\dot{V}(x) \leq -kV^q(x) + \eta$ | $T := \frac{1}{k_1(1-q)}\left(V^{1-q}(x(0)) - \left(\frac{n}{k_2(1-q)}\right)^{\frac{1}{1-q}}\right)$ |
| 4           | $\dot{V}(x) \leq -k_1V^q(x) - k_2V(x) + \eta$ | $T := \max\left\{\frac{1}{k_1(1-q)}\ln\left(\frac{k_2V^{1-q}(x(0)) + k_3}{k_2}\right), \frac{1}{k_2(1-q)}\ln\left(\frac{k_2V^{1-q}(x(0)) + k_3}{k_2}\right)\right\}$ |
| 5           | $\dot{V}(x) \leq -aV^{\beta}(x) + \beta V^q(x)\frac{d}{dt}$ | $T := \frac{1}{\alpha(1-p)}V^{1-p}(x)$ |
| 6           | $\dot{V}(x) \leq -aV^{1-\frac{1}{2}}(x) - \beta V^{\gamma}(x)$ | $T := \frac{\lambda}{\sqrt{\gamma}}$ |
| 7           | $\dot{V}(x) \leq -k_1V^q(x) - k_2V^{\gamma}(x)$ | $T := \frac{1}{\alpha(1-p)}(1 + \frac{1}{k_2})$ |
| 8           | $\dot{V}(x) \leq -k_2V^q(x)$ | $T := T_p$ |
| 9           | $\dot{V}(x) \leq -k_2\mu(t)V + |d(t)|$ | $T := T_{user}$ |
| 10          | $\dot{V}(x) \leq -k_2\mu(t)V + |d(t)|^2$ | $T := T_{user}$ |
| 11          | $\dot{V}(x) \leq -k_2\mu(t)V + |d(t)|^2$ | $T := T_{user}$ |

**Definition 4 [1]**: The system $\dot{x} = f(x, t, d)$ ($d$ represents nonvanishing perturbations) is PT globally uniformly asymptotically stable in time $T$ if there exist class $\mathcal{K}\mathcal{L}$ functions $\beta$ and $\beta_1$, a class $\mathcal{K}$ function $\gamma$, and a time-varying function $\mu(t) : [0, T] \rightarrow \mathbb{R}^+$ with $\mu(t)$ approaching to $\infty$ as $t \rightarrow T$ such that

$$\|x(t)\| \leq \beta(\|x(0)\|, \mu(t)) \forall t \in [0, T).$$

(7)

Clearly, Definition 4 is a generalization of Definition 3 in the presence of nonvanishing perturbations in the system.

To achieve PT convergence, two general PT controller design approaches, namely, the state scaling-based approach and the time scaling-based approach, are used in the literature.

1) **State Scaling**: Using a monotonically increasing function $\mu(t)$ that grows to infinity in finite time $T$ to scale the state, thereby constructing a new variable $z = \mu x$. A control design that keeps $z$ bounded will implicitly make $x$ go to zero as $t \rightarrow T$.

2) **Time Scaling**: Using a nonlinear temporal transformation $\tau = a(t)$ with $a(0)$ being a function defined such that $a(0) = 0$ and $\lim_{t \rightarrow T} a(t) = \infty$. Since this time scale transformation maps $t \in [0, T]$ to $\tau \in [0, \infty)$. A control design that achieves asymptotic convergence in the light of the time variable $\tau$ explicitly achieves PT convergence in terms of the time variable $t$.

In the seminal work for nonlinear system PT stabilizing design [1], a novel yet simple time-varying scaling of the form

$$\mu(t) = \frac{T}{T - t}, \quad t \in [0, T)$$

(8)

or its extended version is proposed, where $T > 0$ is the settling time specified by the designer.

The basis of time scaling-based design, originated from [54], is a temporal axis mapping $\tau = a(t)$, with the properties defined as follows. Let $a'(t) = (da/dt)$ and $a(\tau) = \frac{1}{T - \tau}$.

A simple time scaling is $t = T(1 - e^{-\tau}) \Leftrightarrow \tau = a(t) = \ln T - \ln(T - t)$. In this case, $a'(t) = 1/(T - t)$, which becomes a particular case of the above relation (8).
The well-known proportional navigation law in tactical and strategic missile guidance (see, for instance, [97], [98]) is the primary motivation for PT control. The early work addressing optimal PT control of linear time-invariant (LTI) systems was reported in [99], [100], [101], which however, is difficult to be extended to high-order nonlinear systems. The first work on PT control applicable to nonlinear systems is documented in [1], that is based on state scaling. Later on, various PT control methods and extensions are proposed, among which the most typical ones include: PT control via time base generator [106], [107]; super-exponential and PT precise tracking control for normal-form systems [3]; PT observer [4] and output feedback design for linear time-invariant (LTI) systems based upon the separation principle [5]; PT observer for LTI systems with measurement delay [6]; predictor-feedback PT stabilization of LTI systems with input delay [7]; arbitrary time stabilization of integrators via contraction analysis and an special exponential-like scaling [8]; PT stabilizing control for a full-scale 4-degrees-of-freedom permanent-magnet synchronous motor system via a modified exponential-like scaling [9]; PT stabilizing control for LTI systems via a new modified exponential-like scaling [10]; time scaling-based output feedback design for strict-feedback-like systems [11], [12]; PT stabilizing/tracking control for strict-feedback-like systems via a dynamic gain feedback design [13], [14]; PT stabilization via adding a power integrator technique [15]; PT estimation and output regulation of the linearized schrödinger equation [16]; PT stabilization for stochastic nonlinear systems, where a nonscaling method is used [17], [18], [19]; PT control for nonlinear systems within a liner decay rate [22]; PT control for normal-form systems, where Faà di Bruno’s formula and Bell polynomials are used [23]; frozen-time eigenvalues for PT-stabilized linear time-varying systems [20]; PT control via bounded time-varying feedback and parametric Lyapunov equation [24]; parametric Lyapunov equation-based output feedback PT control [25]; bounded time-varying feedback-based PT control for normal-form systems and satellite formation flying [26], [27]; PT control for p-normal nonlinear systems [28]; PT sliding mode control [102], [103], [104]; a general time transformation for PT control [29]; adaptive PT control for strict-feedback systems [30], [31]; PT differentiator and switched feedback-based PT controller [32], [33]; PT stabilization of a perturbed chain of integrators within the framework of time-varying homogeneity [34]; PT control with bounded time-varying control gain [35], [36]; PT control for affine systems and rigid bodies [37]; PT and prescribed performance tracking control for certain nonlinear systems [38]; practical PT control, namely, the output state/tracking error converges to a certain set within a prescribed time [39], [40], [41].

The representative results of the PT control for SISO systems via time-varying feedback are summarized in Table II. Most of them are based on state feedback. Due to the difficulties of designing complex uncertain systems, most results assume that the control coefficients (including the control direction) of the system model are precisely known without nonvanishing perturbations in the system. In addition, most results consider only robust control schemes and do not consider adaptive control schemes. Because in adaptive control, it is necessary to guarantee the boundedness of parameter estimation (it seems to be difficult to do this with the state scaling-based PT control approach) in addition to the boundedness of the control signal, which usually poses a challenge for the controller design. The following about robust and adaptive PT control will be addressed.

3Since both \(x(t)\) and \(\tilde{x}(\tau)\) relate to the value of the same signal at the same physical time point represented as \(t\) in the original time axis and \(\tau\) in the converted time axis, we use the notation \(\tilde{x}(\tau)\) to express a signal \(x(t)\) as a function of the transformed time variable \(\tau\), i.e., \(x(t) = \tilde{x}(\tau)\). Hence, \(\tilde{x}(\tau) = (d\tilde{x}/dt) = (dt/\tau)(dx/dt) = (1/\alpha(t))(dx/dt) = (1/\alpha(\tau))\tilde{x}(\tau)\).
\section{B. Robust Prescribed-Time Control}

In this section, we adopt the state scaling method to design a control \( u(t) \) to stabilize a scalar system with unknown control coefficient and nonvanishing perturbation in prescribed time \( T \). Consider
\[
\dot{x} = b(x, t)u + f(x, t) \tag{9}
\]
where \( x \) and \( u \) are the state and the control input, respectively, \( b(x, t) \) and \( f(x, t) \) are nonlinear time-varying functions and satisfy the following assumptions.

\textbf{Assumption 1 [11]}: The function \( f(x, t) \) is smooth and satisfies \( |f(x, t)| \leq d(t)|x| \), where \( d(t) \) is a bounded but unknown perturbation, and \( \psi(x) \geq 0 \) is a known computable function.

\textbf{Assumption 2}: The time-varying function \( b(t) \) is away from zero, without losing generality, we assume that \( b(t) > 0 \) and there exists an unknown \( b \) such that \( 0 < b \leq |b(x, t)| < \infty \) for all \( x \in \mathbb{R}, t \in [0, +\infty) \).

\textbf{Remark 1}: Assumption 2 is more general than the one used in [1] since the latter requires that \( b \) be known.

\textbf{Theorem 1}: Under Assumptions 1 and 2, the closed-loop system consisting of (9) and the control law (10) is PT stable in the sense of Definition 4 and all internal signals are bounded over \([0, T]\)
\[
u = -k(\mu x) - \theta(\mu x)\left(\psi + |\mu^{-2}(\mu x)|/2\right)^2 \tag{10}
\]
where \( k \) and \( \theta > 0 \).

\textbf{Proof}: Denote \( \mu x \) by \( z \), and denote \( \psi + |\mu^{-2}(\mu x)| \) by \( \Phi(x, t) \). Choose a Lyapunov function as \( V = 1/(2\beta z^2) \), then
\[
\dot{V} = \frac{b}{\mu z}u + \frac{1}{\mu z}(\mu^{-2}z + f) \leq \frac{b}{\mu z}u + \max\{1, ||d(t)||/b\} \mu z \Phi(x, t).
\]
Let \( \Delta = \max\{1, \sup(|d(t)|)/b\} \), applying Young’s inequality with \( \theta > 0 \), we get
\[
\Delta \mu z \Phi(x, t) \leq \theta \mu z^2 \frac{\mu^2}{\Delta^2} + \frac{\mu^2}{4\theta}.
\]
Note that \( b/\beta \geq 1 \), then substituting (10) and (12) into (11), we have
\[
\dot{V} \leq -k\mu z^2 + \frac{\mu^2}{4\theta} \leq -2b\mu V + \frac{\mu^2}{4\theta}.
\]
According to Proposition 10, (13) results in \( V \in \mathcal{L}_\infty[0, T] \), and hence \( z \in \mathcal{L}_\infty[0, T] \). Furthermore, the state \( x = \frac{1}{\mu^{-2}}V^{1/2} \) is bounded and converges to zero as \( t \to T \). Since \( \mu^{-2} \equiv 1/T \) is bounded, then \( \Phi(x, t) \) is bounded, establishing the same for \( u(t) \). Therefore, all signals are bounded, and the closed-loop system is PT stable in the sense of Definition 4.

\textbf{Remark 2}: It is worth mentioning that, for system (9), as long as \( b(x, t) \) satisfies \( b(x, t) \geq b > 0 \) with \( b \) being some unknown constant, the proposed PT control does not need a priori information on \( b(x, t) \), such simple control algorithm without involving \( b \) can be readily extended to higher-order systems in normal form [1]. Other robust PT control results can be found in [13], [17], [29], and [60] and the references therein.

\section{C. Adaptive Prescribed-Time Control}

It is interesting yet challenging to develop adaptive control schemes to regulate the system state to zero in a prescribed time. So far, the related results in this area are very limited. The following sections present three basic frameworks of adaptive PT control through a scalar system: (Section III-C1) adaptive design for systems with time-invariant parameters; (Section III-C2) adaptive design for systems with time-varying parameters; and (Section III-C3) adaptive Nussbaum gain design for systems with time-varying parameters.

We use the time scaling method to develop our adaptive control design and hence it is necessary to restate some basic concepts: 1) \( \tilde{x}(\tau) = x(t) \); 2) \( \alpha(t)\tilde{x}(\tau) = \dot{x}(t) \); and 3) \( \alpha(t) > 0 \).

\textbf{1) Design for Systems With Time-Invariant Parameters:}

\textbf{Assumption 3 [30]}: The nonlinearity \( f(x, t) \) can be parameterized as \( f(x, t) = \theta \psi(x) \) with \( \psi(x) \) being a known smooth function and \( \psi(0) = 0 \), and \( \theta \) being an unknown constant.

\textbf{Assumption 4 [110]}: The function \( b(x, t) \), called control coefficient, satisfies \( b(x, t) \equiv b \) with \( b \) being an unknown nonzero constant. The sign of \( b \) is available for control design. Furthermore, we assume that there exists a known constant \( \tilde{b} \) satisfies \( b < \tilde{b} \).

Since \( \psi \in C^1 \) and \( \psi(0) = 0 \), then by Hadamard’s Lemma, we know that there exists a known smooth mapping \( \psi \) such that \( \psi = \psi(x) \).

\textbf{Theorem 2}: Under Assumptions 3 and 4, the closed-loop system consisting of (9) and the adaptive control law (14) is PT stable in the sense of Definition 3 and all internal signals are bounded over \([0, T]\)
\[
f(t) = \tilde{\rho}(t)\tilde{u}(t)
\]
\[
\tilde{u}(t) = -\left(ka(t)x(t) + 2\tilde{\rho}^2x(t) + \psi^2(x(t))\right)
\]
\[
\tilde{\rho}(t) = -\gamma_\rho \text{sgn}(b(t))\tilde{u}(t)\tilde{u}(t)
\]
where \( k > 1/(\tilde{\rho}(0)b) \), \( \gamma_\rho > 0 \), \( \gamma_\rho > 0 \) and \( a'(t) = (da/dt) \) is a time-varying function as defined in footnote 2, the initial value of \( \rho \) is chosen as \( \rho(0) > 0 \) for \( b > 0 \) (or \( \rho(0) < 0 \) for \( b < 0 \)).

\textbf{Proof}: According to footnote 3, we know that \( \dot{x}(\tau) = x(t) \) and \( \alpha(t)\tilde{x}(\tau) = \dot{x}(\tau) \). By using Assumption 3, we rewrite (9) as
\[
\tilde{x} = \frac{1}{\alpha(t)}(bu + \theta\psi) = \frac{1}{\alpha(t)}\left(b\tilde{u}(t) + \theta\psi(\tilde{x})\right).
\]

Let \( \tilde{u} = \tilde{\rho}(t)\tilde{u}(t) \), and choose a Lyapunov function \( V(t) : [0, +\infty) \to [0, +\infty) \) as
\[
V(t) = \frac{1}{2}\tilde{x}^2 + \frac{1}{2\gamma_\rho}\left(\theta - \tilde{\rho}(t)\right)^2 + \frac{1}{2\gamma_\rho}\left(\frac{1}{b} - \tilde{\rho}(t)^2\right)^2.
\]
The derivative of \( V(t) \) along the trajectory of the system (15) is
\[
\frac{dV(t)}{dt} = \frac{\dot{x}}{\alpha(t)}(b\tilde{u}(t) + \theta\psi(\tilde{x})) + \left(\theta - \tilde{\rho}(t)\right)\left(\gamma_\rho\tilde{x}\psi(\tilde{x}) - \frac{d\theta}{\alpha(t)}\right) - \left(\frac{1}{b} - \tilde{\rho}(t)^2\right)^2\tilde{x}^2.
\]
Note that the control law and update laws designed in Theorem 2 are equivalent to
\begin{equation}
\tilde{u}(t) = -k\alpha(t)x - \frac{1}{2}b^2(\tau)x + \frac{1}{2}\psi^2x + \frac{1}{2}\psi^2x.
\end{equation}
It follows that from (19) that
\begin{equation}
dV(t) = -k\chi^2(t) \leq 0.
\end{equation}
It is straightforward to prove that
\begin{equation}
\lim_{t \to T} e^{A(t)}(T - t) = 0
\end{equation}
and
\begin{equation}
\lim_{t \to T} e^{-\frac{1}{2} \frac{k(t)}{T - t} dt} \leq \lim_{t \to T} e^{-\frac{1}{2} \frac{k(t)}{T - t} dt} = \lim_{t \to T} \frac{e^{\min\ln(T - t)}}{T - t} = \lim_{t \to T} \frac{e^{\min\ln(T - t)}}{T - t} = \lim_{t \to T} \frac{e^{\min\ln(T - t)}}{T - t}.
\end{equation}
According to Squeeze Theorem, we obtain lim_{t \to T} e^{A(t)}(T - t) = 0. Next, we continue to prove lim_{t \to T} x/(T - t) = 0.

To proceed, we rewrite the closed-loop dynamics as
\begin{equation}
\dot{x} = -kb\hat{\rho}a'(t)x - \frac{1}{2}b\hat{\rho}(\hat{\theta}^2x + \hat{\psi}^2x) + \theta \psi.
\end{equation}
Recall footnote 2, we know that a(t) = ln(T/(T - t)), and
\begin{equation}
as(t) = 1, a(T) = +\infty, a'(t) = \frac{1}{T - t}.
\end{equation}
Then, (20) can be simplified as
\begin{equation}
\dot{x} = -k(T - t)x + f(x, t).
\end{equation}
where k(t) = \frac{kb\hat{\rho}(t)}{T - t} is a positive function and f(x, t) = -(1/2)b\hat{\rho}(\hat{\theta}^2x + \hat{\psi}^2x) + \theta \psi is a bounded function. Solving the differential inequality (21) gives
\begin{equation}
x(t) = e^{A(t)}\int_0^t f(x, s) e^{-A(s)} ds + e^{A(t)}x(0), A(t) = \int_0^t \frac{k(s)}{T - s} ds.
\end{equation}
To show the boundedness of u(t), we state the following Lemma.

**Lemma 1:** For (22), if lim_{t \to T} f(x, t) = 0 and a constant k_{\min} = \inf\{k(t)\} satisfies k_{\min} > 1, then the following equations hold:
\begin{equation}
\lim_{t \to T} e^{A(t)}(T - t) = 0, \lim_{t \to T} \frac{x}{T - t} = 0.
\end{equation}
It is important to ensure that \tilde{u} \leq 0 since this guarantees the monotonicity of \hat{\rho}(t) and thus allows to explicitly pick a suitable control gain k.

**Proof:** It is straightforward to prove that
\begin{equation}
\lim_{t \to T} e^{A(t)}(T - t) = \lim_{t \to T} e^{-\frac{1}{2} \frac{k(t)}{T - t} dt} \geq 0
\end{equation}
and
\begin{equation}
\lim_{t \to T} e^{-\frac{1}{2} \frac{k(t)}{T - t} dt} \leq \lim_{t \to T} e^{-\frac{1}{2} \frac{k(t)}{T - t} dt} = \lim_{t \to T} \frac{e^{\min\ln(T - t)}}{T - t} = \lim_{t \to T} \frac{e^{\min\ln(T - t)}}{T - t} = \lim_{t \to T} \frac{e^{\min\ln(T - t)}}{T - t}.
\end{equation}
Since k(t) \geq k_{\min} > 1 and lim_{t \to T} f(x, t) = 0, then lim_{t \to T} e^{A(t)}(T - t) \int_0^t f(x, s) e^{-A(s)} ds = 0, implying lim_{t \to T} x/(T - t) = 0. The proof of Lemma 1 is completed.

According to Theorem 1, we know that
\begin{equation}
u(t) = -\hat{\rho}(t)\left(\frac{kx(t)}{T - t} + \frac{1}{2}\hat{\psi}^2x + \frac{1}{2}\hat{\psi}^2x\right).
\end{equation}
In terms of Lemma 1, we obtain that the control input u(t) is bounded over [0, T] and lim_{t \to T} u(t) = 0. Therefore, the closed-loop system is PT stable in the sense of Definition 3.

**Remark 3:** Since |\hat{\rho}(t)| is a monotone increasing function and b\hat{\rho} > 0, then k_{\min} = \hat{\rho}(0)b. Therefore, we only need to pick k > 1/|\hat{\rho}(0)b| to ensure that k_{\min} > 1. Particularly, when the control coefficient b is known, as assumed in [30], there is no need for using Lemma 1, we just need to choose k > 1.

2) **Design for Systems With Time-Varying Parameters:**

**Assumption 5:** The nonlinearity f(x, t) satisfies that f(x, t) = \theta(t)\psi(x), where \psi(x) is a known smooth function, \psi(0) = 0, and the time-varying parameter \theta(t) takes values in an unknown compact set, i.e., there exists an unknown constant \delta_\theta such that |\theta(t)| \leq \delta_\theta.

**Remark 4:** Such an Assumption is more general than the one used in [30], since the latter requires that \theta(t) be time-invariant. It is also more general than the Assumption used in [109] because the latter requires that \delta_\theta be known.

**Theorem 3:** Under Assumptions 4 and 5, the closed-loop system consisting of (9) and the adaptive control law (25) is
PT stable in the sense of Definition 3 and all internal signals are bounded over \([0, T)\)

\[
\begin{align*}
    u(t) &= \hat{\rho}(t)\hat{\psi}(t) \\
    \hat{\psi}(t) &= -\left(ka'(t)x(t) + \frac{1}{2}d^2(t)x(t) + \frac{1}{2}\bar{\theta}^2x(t) + d(t)\right) \\
    \hat{\theta}(t) &= \hat{\gamma}(t)x(t) \\
    \hat{\delta}(t) &= -\gamma(t)\hat{\Psi}(t)\hat{\theta}(t)
\end{align*}
\]  

(25)

where \(k > 1/(\hat{\rho}(0)\bar{\theta})\), \(\gamma > 0\), \(\gamma > 0\), and \(d'(t) = (da/dt)\) is a time-varying function as defined in Section III-A. The initial value of \(\rho\) is chosen as \(\rho(0) > 0\) for \(b > 0\) (or \(\rho(0) < 0\) for \(b < 0\)).

**Proof:** Similar to the proof of Theorem 2, we first rewrite (9) as

\[
\dot{x} = \frac{1}{a(t)}\left(\hat{b}(t)\hat{\psi} + (\ell_0 - \hat{\rho})\hat{\psi} + \Delta_0 \hat{\psi}\right)
\]

(26)

with \(\ell_0\) being some constant and \(\Delta_0 = \theta(\tau) - \ell_0\). We then choose a Lyapunov function \(V(\tau) : [0, +\infty) \rightarrow [0, +\infty)\)

\[
V(\tau) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\gamma(\hat{\theta}(\tau) - \hat{\rho}(\tau))^2 + \frac{1}{\gamma}(\hat{\theta}(\tau) - \hat{\rho}(\tau))^2
\]

(27)

With (25), it follows that:

\[
\frac{dV(\tau)}{\tau} \leq -k\dot{x}^2(\tau) - \frac{1}{a(t)}(\hat{\psi}(t) - \hat{\psi}(t)) - \frac{1}{a(t)}(\Delta_0 - \hat{\Delta}_0)\dot{x}^2(1 + \hat{\psi}^2(\tau)).
\]

(28)

Since

\[
\Delta_0 \hat{\psi} = \hat{\Delta}_0 \hat{\psi}(\tau)x^2 \leq \frac{\Delta_0}{2}\hat{\Delta}_0 \hat{\psi}^2(\tau) + \frac{\hat{\Delta}_0}{2}x^2
\]

(29)

then substituting \(v\) into (29) yields \(dV/d\tau \leq -k\dot{x}^2 \leq 0\). Thus, according to an analysis similar to that in the proof of Theorem 2, it can be concluded that all signals are bounded and the closed-loop system is PT stable in the sense of Definition 3.

3) Adaptive Nussbaum Gain Design:

**Assumption 6:** The function \(b(x, t)\), called control coefficient, is away from zero and takes values in a compact set. However, its magnitude and sign are unknown. There exists a known positive constant \(b\) satisfies \(b \leq |b(x, t)|\).

**Lemma 2 [111]:** Consider two \(C^\infty\) positive functions \(V(t) : [0, \infty) \rightarrow \mathbb{R}^+\) and \(\dot{N}(t) : [0, \infty) \rightarrow \mathbb{R}^+\). Let \(b(t) : [0, \infty) \rightarrow [b, \tilde{b}]\) for two constants \(b\) and \(\tilde{b}\) satisfying \(b \leq \tilde{b} > 0\). If, for \(\forall \tau \geq 0\)

\[
\dot{V}(\tau) \leq (b(t), V(\tau)) \dot{N}(\tau) + 1|\xi(\tau)| \xi(\tau) \geq 0
\]

(30)

for an enhanced Nussbaum function \(V(\tau)\), then \(\dot{N}(\tau) \dot{V}(\tau)\) are bounded over the whole time interval \([0, \infty)\).

**Theorem 4:** Under Assumptions 5 and 6, all internal signals are bounded over \([0, T)\) and system state \(x(t)\) converges to a compact set within preset time \(T\), if the control law and update laws are designed as

\[
\begin{align*}
    u(t) &= N(\xi)\dot{\psi}(t) \\
    \dot{\psi}(t) &= ka'(t)x(t) + \frac{1}{2}(1 + \hat{\theta}^2\hat{\psi}^2 + \hat{\Delta}_0(1 + \hat{\psi}^2))x(t) \\
    \dot{\theta}(t) &= \gamma x(t)\dot{\psi}(t), \\
    \dot{\delta}(t) &= \hat{\delta}(t) + \hat{\delta}(t)
\end{align*}
\]  

(31)

where \(k > 0\), \(\gamma > 0\), and \(a'(t) = (da/dt)\) is a time-varying function as defined in Section III-A and \(\dot{N}(\xi)\) is an enhanced type B-L Nussbaum function as defined in [111, Definition 4.2].

**Proof:** First, we rewrite (9) as

\[
\dot{x} = \frac{1}{a(t)}\left(\bar{b}(t)\hat{N}(\xi)\hat{\psi}(t) + \hat{\theta}(t)\hat{\psi}(t) + (\ell_0 - \hat{\theta})\hat{\psi}(t) + \Delta_0 \hat{\psi}(t)\right)
\]

(32)

with \(\ell_0\) being an unknown constant and \(\Delta_0 = \theta(\tau) - \ell_0\). Then, choosing a Lyapunov function \(V(\tau)\) candidate as

\[
V(\tau) = \frac{1}{2}\bar{b}^2(\tau) + \frac{1}{2}\gamma(\hat{\theta}(\tau) - \hat{\rho}(\tau))^2 + \frac{1}{\gamma}(\hat{\theta}(\tau) - \hat{\rho}(\tau))^2
\]

(33)

Taking derivative of \(V(\tau)\) along the trajectory of (32), we get

\[
\frac{dV(\tau)}{\tau} = (\bar{b}(\tau)\hat{N}(\xi) + 1)\frac{d\bar{\psi}}{\tau} - \frac{1}{a(t)}\hat{\psi}(\tau) + \frac{1}{\gamma}(\hat{\theta}(\tau) - \hat{\rho}(\tau))\hat{\psi}(\tau) - \frac{d\bar{\psi}}{\tau}
\]

(34)

Inserting the control law designed in (31) into (34), yields

\[
\frac{dV(\tau)}{\tau} \leq (\bar{b}(\tau)\hat{N}(\xi) + 1)\frac{d\bar{\psi}}{\tau}
\]

(35)

where \(d\bar{\psi}/d\tau = \hat{\psi}(\tau)x(t) + \hat{\psi}(t)(1/2)(1 + \hat{\theta}^2\hat{\psi}^2)\hat{\psi}(t) + (1/2)(\hat{\Delta}_0(1 + \hat{\psi}^2))\hat{\psi}(t) \geq 0 \forall \tau \geq 0\). Thus, it follows from Lemma 2 that \(V(\tau) \in L_\infty\) and \(\hat{\psi} \in L_\infty\). Note that the boundedness of \(\hat{N}(\xi)\) is guaranteed by the boundedness of \(\xi\). Therefore, it can be concluded that \(\hat{x} \in L_\infty\), which further indicates that \(\lim_{t \rightarrow \infty} \hat{x}(t) = 0\) via Barbait’s Lemma. In addition, in terms of the analysis similar to that in the proof of Theorem 2, the boundedness of all closed-loop signals can be guaranteed and hence the closed-loop system is PT stable in the sense of Definition 3.

**Remark 5:** It is noted that the adaptive PT control is developed for the system with unknown yet time-varying parameters in both feedback and input channels. These parameters are not slowly time-varying, but rather, are fast time-varying or even involve abrupt changes, thereby making the controller design quite challenging. Although the control algorithm is based on the first-order system, the fundamental idea and the key design steps are worth extending to more general systems. Some attempts about PT control for high-order systems with unknown directions can be found in [42]. In addition, since neural network (NN) can be combined with robust adaptive to deal with modeling uncertainties, how to compensate the NN reconstruction error to get PT stability represents an increasing topic for future study.
IV. PRESCRIBED-TIME CONTROL FOR MIMO SYSTEMS

PT control for MIMO nonlinear systems is an open area of research that is both theoretically and practically important and urgent, especially with new problems arising from emerging applications, such as missile guidance, accurate and timely weather forecasting, aircraft and spacecraft flight control, and obstacle avoidance in robotic systems, all of which require new control technologies for time optimization. There have been few research on PT control for MIMO nonlinear systems, particularly when the control gain matrix is unknown, and essentially no findings that can provide PT stabilization, regulation, or tracking. In [107], by using time-based generators, a PT control algorithm is applied to a 7-DoF robot manipulator with a precondition that all information in the control gain matrix is available. In [24], a parametric Lyapunov function-based PT controller is applied to a spacecraft rendezvous system with unknown control gain matrix and nonvanishing performance control theory [114], [115], and finally to obtain in the sense of Definition 4 and all internal signals are bounded over the time interval [0, T] of coordinate transformation similar to that in the prescribed constrained system into an unconstrained one by using the idea given value at the prescribed time, and to convert the original in the prescribed performance control theory [114], [115], and finally to obtain the tracking error of the original system that can converge to a given accuracy at the prescribed time by proving the boundedness of the converted system. In the following sections, we introduce a powerful design approach for MIMO system that applies not only to square systems but also to nonsquare systems.

A. Square System

Consider an MIMO nonlinear system as follows:

$$\dot{X} = B(X, t)U + F(X, t)$$

where $U \in \mathbb{R}^n$ and $X \in \mathbb{R}^n$ are the input and the state vector, respectively. $F(X, t) = [f_1, \ldots, f_n]^T \in \mathbb{R}^n$ denotes the modeling uncertainties and external perturbations and each $f_i$ satisfies Assumption 1, i.e., $\|F\| \leq d(t)\Psi$ with $\Psi = [\psi_1, \ldots, \psi_n]^T \in \mathbb{R}^n$.

Assumption 7 [60]: The matrix $B(X, t) \in \mathbb{R}^{n \times n}$ is square and unknown. The only information available for control design is that $(B + B^T)$ is positive definite and symmetric.

Theorem 5: Under Assumptions 1 and 7, the closed-loop system consisting of (36) and the control law (37) is PT stable in the sense of Definition 4 and all internal signals are bounded over the time interval [0, T]

$$U = -kZ - \theta Z\|\Phi\|^2$$

where $k > 0$, $\theta > 0$, $Z = (TX/(T - t))$, and $\Phi = \Psi + \mu\mu^{-2}\|Z\|$.

Proof: Consider $V = 1/(2w_B)Z^TZ$ with $w_B$ being an unknown positive constant, then

$$\dot{V} = \frac{1}{w_B}\mu Z^T(BU + F + \mu\mu^{-2}Z)$$

$$= \mu Z^T\left(\frac{B + B^T}{2w_B} - \frac{B - B^T}{2w_B}\right)(-kZ - \theta Z\|\Phi\|^2)$$

$$+ \frac{1}{w_B}\mu Z^T(F + \mu\mu^{-2}Z).$$

In light of Assumption 7, there exists some unknown constant $w_B > 0$, such that $0 < w_B \leq (1/2)\lambda_{\min}(B + B^T)$. Therefore

$$\frac{1}{2w_B}Z^T(B + B^T)Z \geq \|Z\|^2.$$ 

In addition, $(B - B^T)$ is skew symmetric and hence $Z^T(B - B^T)Z = 0 \forall Z \in \mathbb{R}^n$. Now, it follows from (38) that

$$\dot{V} \leq -k\mu\|Z\|^2 - \theta\|Z\|^2\|\Phi\|^2 + \frac{1}{w_B}\mu Z^T(F + \mu\mu^{-2}Z).$$

With Young’s inequality, we get

$$\frac{1}{w_B}\mu Z^T(F + \mu\mu^{-2}Z) \leq \frac{1}{w_B}\mu\mu\|Z\|^2(d(t)\Psi + \mu\mu^{-2}\|Z\|) \leq \mu\|Z\|\Delta\Phi \leq \theta\mu\|Z\|^2\|\Phi\|^2 + (\mu\Delta^2/4\theta)$$

where $\Delta = (1/w_B) \times \max[1, \|d(t)\|]\) and $\Phi = \mu\mu^{-2}\|Z\| + \Psi$. Therefore, we have

$$\dot{V} \leq -k\mu\|Z\|^2 + \frac{\mu\Delta^2}{4\theta} = -2w_Bk\mu(t)U + \frac{\mu\Delta^2}{4\theta}. \quad (41)$$

It follows from Proposition 10 that $V \in L_{\infty}[0, T)$. Using the analysis similar to that below (13), one can conclude that all signals are bounded over $[0, T)$ and $x(t) \to 0$ as $t \to T$.

Therefore, (36) is PT stable in the sense of Definition 3.

B. Nonsquare System

Now, we consider a nonsquare MIMO system $\dot{X} = BU + F$ satisfying the following Assumption:

Assumption 8 [60]: The high-frequency gain matrix $B(X, t) \in \mathbb{R}^{n \times m}$ can be characterized as $B(X, t) = A(X, t)M(X, t)$, where $M \in \mathbb{R}^{m \times n}$ is uncertain yet possibly asymmetric and $A \in \mathbb{R}^{n \times n}$ is a known matrix with full row rank. The message usable for synthesis is that $A(M + M^T)A^T$ is symmetric and positive definite.

Assumption 8 [60]: The high-frequency gain matrix $B(X, t) \in \mathbb{R}^{n \times m}$ can be characterized as $B(X, t) = A(X, t)M(X, t)$, where $M \in \mathbb{R}^{m \times n}$ is uncertain yet possibly asymmetric and $A \in \mathbb{R}^{n \times n}$ is a known matrix with full row rank. The message usable for synthesis is that $A(M + M^T)A^T$ is symmetric and positive definite.

Under Assumption 8, we get a new MIMO system as follows:

$$\dot{X} = AMU + F(X, t)$$

where $U \in \mathbb{R}^m$ and $X \in \mathbb{R}^n$ are the input and the state vector, respectively.

According to Assumption 7, we known that the positive definiteness of $(B + B^T)$ ensures that $\lambda_{\min}(B + B^T)$ is always positive and there exists some positive unknown constant $w_A$, such that $0 < w_A \leq (1/\|A\|)\lambda_{\min}(A(M + M^T)A^T)$.

Theorem 6: Under Assumptions 1 and 8, the closed-loop system consisting of (36) and the control law (43) is PT stable in the sense of Definition 4 and all internal signals are bounded over $[0, T)$

$$U = -\frac{A^T}{\|A\|}(kZ + \theta Z\|\Phi\|^2)$$

(43)
where \( k > 0, \theta > 0, Z = \mu X \) with \( \mu(t) = (T/(T-t)) \) and \( \Phi = \Psi + \dot{\mu} \mu^{-2}\|Z\| \).

**Proof:** This proof is omitted as it is straightforward by taking the analysis in the proof of Theorem 5. The difference is that we need to replace the inequality \( Z^2(B + B^T/2)Z \geq w_B \|Z\|^2 \) in (39) with \( Z^2(A(M + M^T)A^T/[2\|A\|])Z \geq w_A \|Z\|^2 \).

**Remark 6:** The main challenges in designing a PT controller for a high-order MIMO system are how to cope with the unknown nonlinear perturbations due to the unknown control matrix and how to relax the assumptions on the control matrix in order to make more general control algorithms.

V. LATEST DEVELOPMENTS IN PRESCRIBED-TIME CONTROL

In this section, we aim to present a literature survey of the foundations of PT decentralized control theory. Knowledge of graph theory can be found in any of the papers about multiagents, which we have omitted here due to space constraints.

The idea of using time-varying feedback to obtain PT stability has already appeared in early distributed control and has accomplished a large diffusion in recent years. For example, PT consensus on single and double integrator dynamics cases [43], [105]; PT consensus under undirected/directed graph and PT containment under multiple leaders of first-order networked multiagent systems [44]; leader-following control of high-order multiagent systems without/mismatched uncertainties [45], [46]; PT consensus via time base generator [47]; cluster synchronization of complex networks [48]; lag consensus of second-order leader-following multiagents [49]; PT consensus observer for high-order multiagents [50]; PT bipartite consensus tracking [51], [52], [53]; PT consensus over time-varying graph via time scaling [54], and then generalized in [55], [56], [57], and [58], in which, PT formation tracking, leader-following control, uncertain multiagent dynamics, multiagent rigid body system, are considered.

**A. Prescribed-Time Consensus Protocol**

Consider a multiagent system where the dynamics of each subagent is a single integrator

\[
\dot{x}_i = u_i, \quad i = 1, 2, \ldots, n. \tag{44}
\]

The general consensus protocol is

\[
\dot{u}_i = -k \sum_{j=1}^{n} a_{ij} \text{sgn}(x_j - x_i)|x_j - x_i|^{\alpha_{ij}}, \quad 1 \leq i \leq n \tag{45}
\]

where \( 0 \leq \alpha_{ij} \leq 1 \) and \( k > 0 \). Obviously, protocol (45) covers several common cases.

1) When \( \alpha_{ij} = 1 \), it simplifies to the classical asymptotic consensus protocol studied in [116]; then the original system can be abbreviated as \( \dot{x} = -Lx \), where \( x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n \) and \( L \) is the Laplacian matrix of the system, and \( L = [l_{ik}]_{n \times n} \)

\[
l_{ik} = \begin{cases} \sum_{j \in N_i} a_{ij}, & k = i \\ -a_{ik}, & k \neq i. \end{cases} \tag{46}
\]

Meanwhile, the Laplace matrix \( L \) has only one zero eigenvalue and all other eigenvalues with positive real parts if and only if the corresponding directed graph \( G \) contains a spanning tree [117].

2) When \( \alpha_{ij} = 0 \), it corresponds to the discontinuous FT consensus protocol outlined in [118].

3) When \( 0 < \alpha_{ij} < 1 \), it reduces to the continuous however nonsmooth FT consensus protocol established in [119].

It is important to note that with \( 0 < \alpha_{ij} < 1 \), the finite settling time \( T \) is determined by Proposition 1 as \( T = (V^1-\alpha(0)/[c(1-\alpha)]) \) with \( c > 0 \) being some constant associated with the design parameters \( k, \alpha_{ij}, \) and \( \lambda_2(L)^6 \) (which relies on the structure of \( G \)). There are several issues associated with the settling time \( T \):

1) The settling time \( T \) is affected by design parameters \( k \) and \( \alpha_{ij} \), the initial state \( V(0) \), as well as the topological structure.
2) To produce a lower \( T \), one can increase \( k \) or decrease \( \alpha_{ij} \) (creating a larger \( c \) or a smaller \( \alpha \)), but the control effort increases with a smaller \( \alpha_{ij} \).
3) If a settling time \( T \) is imposed, it is necessary to try to find the relevant parameters \( c \) and \( \alpha \) based upon \( V(0) \) from Proposition 1, which cannot be explicitly preset because \( \alpha_{ij} \) is implicitly involved in the function and the initial condition may be unknown.

The following PT consensus protocol circumvents all the aforementioned shortcomings [44]:

\[
u_i = -\left(k + c \frac{\dot{\mu}}{\mu}\right) e_i, \quad i = 1, 2, \ldots, n \tag{47}
\]

where \( k > 0, c \) is a parameter will be designed later, \( e_i = \sum_{j \in N_i} a_{ij}(x_j - x_i) \) is the local neighborhood error, and \( \mu(t) \) is well defined on \([0, T]\) as in (8).

**Theorem 7:** Consider system (44) in conjunction with the protocol (47). If the graph \( G \) is undirected and connected, and the design parameter \( c \) is selected as \( c \geq 1/\lambda_2(L) \), then the consensus is attained in PT, namely

\[
\|x(t)\| \leq \frac{1}{\mu(t)} \|\delta(0)\|e^{-\lambda_2(L)t} \quad \forall t \in [0, T] \tag{48}
\]

where \( x = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n \) and \( x_i = x_i - (1/n) \sum_{i=1}^{n} x_i \). Furthermore, the control input remain \( C^1 \) smooth and bounded over \([0, T]\).

**B. Prescribed-Time Containment Protocol**

When the communication topology structure involves multiple leaders, the containment control can be naturally evolved from the consensus control. In this section, the achieved consensus result is extended to the scenario of containment.

**Theorem 8:** Consider system (44) in conjunction with the protocol (47). If the graph \( G \) has a directed spanning tree leaded by the root node \( x_i \) and the design parameter \( c \) is selected as \( c \geq 2\lambda_{\text{max}}(\overline{P})/\lambda_1(\overline{Q}) \), then, for \( \forall t \in [0, T] \), the containment is attained in PT, namely

\[6\lambda_1(L) \] denotes the \( i \)-th minimum eigenvalue of the Laplace matrix \( L \).
\[
\left\| \dot{Z}(t) \right\| \leq \frac{1}{\mu(t)} \lambda_{\max}(\tilde{P}) \left\| L_1^{-1} \otimes I_n \right\| \left\| \tilde{E}(0) \right\| e^{\lambda_{\min}(\tilde{P}) t}
\]

where \( L_1 \in \mathbb{R}^{(n-1) \times (n-1)} \) is a nonsingular matrix with all eigenvalues satisfying \( \lambda_i(L_1) > 0 \), \( i = 1, \ldots, n-1 \), whose specific expression can be obtained according to the Laplace matrix \( L \), namely

\[
L = \begin{bmatrix} 0 & 0_{1 \times (n-1)} \\ L_2 & L_1 \end{bmatrix}
\]

and \( \tilde{Z} = [z_2, z_3, \ldots, z_n]^T \in \mathbb{R}^{n-1} \) with \( z_i = x_i - x_1 \), \( \tilde{E} = [e_2, e_3, \ldots, e_n]^T \in \mathbb{R}^{n-1} \) and \( \tilde{Q} = PL_1 + L_1^T \tilde{P} \) with \( \tilde{P} = \text{diag}(p_2, \ldots, p_n) \) and \( [p_2, \ldots, p_n]^T = (L_1)^{-1}1_{n-1} \). Furthermore, the control \( u_i \) remains \( C^1 \) smooth and bounded over \([0, T]\).

**Proof:** The proofs of Theorems 7 and 8 are omitted as they can be found in [44].

**VI. COMPARISON AND CONCLUSION**

To close this article, we recap the connection between the PT results to FT results and discuss the related numerical implementation issues. In addition, we compare the differences between typical FT/FxT/PdT and PT controllers via simulation on a double integrator. Finally, we conclude by giving some future research challenges.

**A. Controller Structure**

Consider the first-order integrator as follows:

\[
\dot{x} = u, \quad x(0) = x_0.
\]

From [24], one can immediately obtain a time-varying feedback-based PT controller as

\[
u_{\text{prescribed}} = -\frac{kx}{T - t}, \quad k \geq 1.
\]

Also, from [65], we get the classical FT autonomous controller as

\[
u_{\text{finite}} = -k \text{sgn}(x)|x|^\alpha, \quad 0 < \alpha < 1, \quad k > 0
\]

then the solution of (49) with (51) is

\[
\begin{align*}
\{ x(t) &= \left( |x_0|^{1-\alpha} - k(1-\alpha)t \right)^{\frac{1}{1-\alpha}} \text{sgn}(x_0), \quad t \in [0, T_f] \\
&= 0, \quad t \in [T_f, +\infty)
\end{align*}
\]

with

\[
T_f = \frac{1}{k} \left( 1 - \frac{1}{\alpha} \right) |x_0|^{1-\alpha}.
\]

Therefore, we have \( |x_0|^{1-\alpha} = kT_f(1-\alpha) \), and the control law (51) can be rewritten as

\[
u_{\text{finite}} = -k \text{sgn}(x)|x|^\alpha |x| - k(1-\alpha)t}^{\frac{1}{1-\alpha}} \text{sgn}(x)
\]

\[
= -k \left( |x_0|^{1-\alpha} - k(1-\alpha)t \right)^{\frac{1}{1-\alpha}} x
\]

\[
= \frac{x}{(1-\alpha)(T_f - t)}.
\]

Note that if we choose \( k \equiv 1/(1-\alpha) \), the PT controller (50) becomes the FT controller (51), which means that the PT controller (51) is indeed a special case of the PT controller (50). In fact, they share the same property that the control gain tends to \( \infty \) as \( t \to T \). As a matter of fact, all FT controllers (including FxT controllers, PT controllers, and PdT controllers) share this property. Also, note that the magnitude of the PT control input (50) (consisting of a high-gain function \( -k/(T - t) \) and a feedback signal \( x \)) does not become large when the feedback signal decays faster than the high-gain function grows.

**B. Discussion on Implementation**

In the implementation of FT control algorithms, it is necessary to introduce sign function \( \text{sgn}(x) \) to avoid singularity when \( x(t) = 0 \). For example, the control law \( u = -x^{1/3} \) is programmed to be replaced by \( u = -\text{sgn}(x)|x|^{1/3} \). Two effective ways of implementing PTC are as follows.

1. Letting \( T = T_s \) (scheduled time) \( + \) (small constant) so that the controller works for the scheduled time.
2. Setting an upper bound on the scaling function \( \mu(t) \) before the time variable approaches the desired preset time \( T \).

Anyhow, unbounded control gain will not cause unbounded control input, and many simulation results show that the PT regulation is achieved with a suitable control effort, without an exorbitant price. Both of the above implement methods slightly sacrifice the control precision in favor of promoting practical implementation by avoiding unbounded gains. The major concern for time-varying feedback control is its robustness against measurement noise. In [59], a robust approach has been established for second-order systems where bounded disturbances corrupt the measurements. For the problem of PT stabilization of high-order systems in the presence of measurement disturbances, further research is required.

To show the characteristics and the differences of FT, FxT, PdT, and PT control schemes, we consider a double integrator for numerical simulation.

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = u.
\]

The FT controller [65, Example 1], the FxT controller [83, Example 5.11], the PdT controller [93, Example 4.2], and the PT controller [30] are shown below.

1. \( u_{\text{finite}} = -x_2^{1/3} - (x_1 + (3/5)x_2^2)^{1/5} \)
2. \( u_{\text{fixed}} = -(1 + 3x_2^2/2) \text{sgn}(s) - (s + x_1^2)^{1/2} \) with \( s = x_2 + (x_2^2 + x_1 + x_2)^{1/3} \) and \( |v| \approx |v|^{3/5} \) for \( v \neq 0 \)
3. \( u_{\text{predifined}} = -[(\sigma \Phi_1 + (1/T_s)^{1/2})x_2 - \Phi_2(x, T_s)] \) with \( \sigma = \Phi_1 + 3x_2 \) and \( \Phi_i(v, T_s) = (5/2T_s) \exp(|v|^{1/2} |v|^{3/5}) \text{sgn}(v) \)
4. \( u_{\text{prescribed}} = -k(x_2 + 3x_2) - 3x_2^2 \text{sgn}(x_2) \) \( k \) and \( \mu = 1/(T - t) \)

Two scenarios are considered for simulation: \( x_1(0) = 0.2 \) and \( x_2(0) = -0.2 \) and \( 0 \). Figs. 3–6 illustrate the simulation results, from which, it can be seen that the FT controller achieves FT regulation in \( T = 1 \) s, whereas the settling time of the PT controller depends on the initial conditions, the FxT controller depends on the design parameters and the upper limitations.

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As for PT control for multiagent systems, it is interesting to generalize the simple framework on first or second-order integrators to agents having high-order uncertain nonlinear dynamics and to investigate PT decentralized control algorithms under complex communication topologies, as well as to study how to achieve consensus with as little information interaction between agents as possible without losing controllability.

5) The study of more types of system models, more low-conservative control algorithms, or the pursuing for better control performance of closed-loop systems are all interesting future research topics in the field of PT control.

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