Study of the effects of the curvature term in the density of states on some physical quantities characterizing the deconfining phase transition to a color-singlet QGP

A Ait El Djoudi, R Djida, K Mezouar and N Abbad
Laboratoire de Physique des Particules et Physique Statistique, Ecole Normale Supérieure-Kouba, B.P. 92, 16050, Vieux Kouba, Algiers, Algeria

Abstract. In the present work, we study the effect of including the curvature term, in addition to the volume term, in the density of states used in the calculation of the projected color-singlet partition function of the Quark-Gluon Plasma (QGP), on some features characterizing the phase transition to this new phase of matter. By investigating the behavior of some physical quantities well describing the mixed system with temperature for varying volume, we especially examine the behavior with volume of the shift of the effective transition temperature in a finite volume from the true one, to determine the corresponding scaling exponent. We compare the present results to those obtained in our previous works, where we have used a color-singlet QGP partition function derived with a density of states containing the volume term only.

1. Introduction
In the framework of the study of hadronic matter at extreme conditions of temperature and/or baryonic density, where quarks and gluons are expected to be in a deconfined state called Quark-Gluon Plasma (QGP), we investigate in the present work the deconfining phase transition at high temperature and zero chemical potential.

According to the color confinement property of QCD, this QGP phase may be in a color neutral state, namely a color-singlet state. We account for such a requirement by projecting the partition function of the QGP on the color-singlet SU(3) representation using the projection technique [1, 2]. The density of states used in the calculation is that given by the Multiple Reflection Expansion (MRE) approximation [3], and the curvature and surface terms are considered additionally to the volume term. This is, indeed, a continuation of our previous study of the thermally driven deconfining phase transition from a hadronic gas of pions to a QGP phase with massless u and d quarks, in a finite volume, at zero chemical potential, where we have used a color-singlet partition function (CSPF), for the QGP, derived with a density of states containing the volume term only [4, 5]. Such a CSPF has been used into a simple model of coexisting hadronic and QGP phases in a finite volume [6], additionally to the bag model [7], and finite size effects have been investigated [4, 5].

In the following, let us illustrate first the calculation of the CSPF as well as the model used for the coexistence of the confined-deconfined phases.
2. Partition function of the QGP projected on the color-singlet SU(3) representation

In the aim of implementing the color-singletness constraint for the QGP phase, we use the group theoretical projection technique [1, 2], as done in several works (see for example [8, 9]). The projected partition function for a system of gluons and massless \( u \) and \( d \) quarks, and their antiquarks, in a volume \( V \), at temperature \( T \) and quark chemical potential \( \mu \), can be written after calculation as [10]:

\[
Z_{QGP}(T, V, \mu) = \frac{4}{\sqrt{\pi}} \int_{-\infty}^{+\infty} d\varphi d\psi \ M(\varphi, \psi) e^{\frac{iT}{8 \pi} g_{1}(\varphi, \psi)} e^{-BT/V} \]

where \( B \) is the bag constant, \( R \) the radius of the QGP bubble, \( M(\varphi, \psi) \) the weight function (Haar measure) given by:

\[
M(\varphi, \psi) = \left\{ \sin\left(\frac{1}{2}(\psi + \frac{\varphi}{2})\right) \sin(\varphi) \sin\left(\frac{1}{2}(\psi - \frac{\varphi}{2})\right) \right\}^2
\]

and the functions \( g_{i}(\varphi, \psi, \frac{\mu}{T}) \) and \( g_{2}(\varphi, \psi, \frac{\mu}{T}) \) are given by:

\[
g_{i}(\varphi, \psi, \frac{\mu}{T}) = \frac{\pi}{6} \left[ 4 \sum_{q=r,b,g} \left( \frac{\alpha_{q} - i\frac{\mu}{T}}{\pi} \right)^{2} + \frac{1}{2} \left( \frac{\alpha_{q} - i\frac{\mu}{T}}{\pi} \right)^{4} \right] + \sum_{g=1}^{4} \left[ \frac{7}{30} \left( \frac{\alpha_{g} - \pi}{\pi} \right)^{2} - \frac{1}{2} \left( \frac{\alpha_{g} - \pi}{\pi} \right)^{4} \right]
\]

\[
g_{2}(\varphi, \psi, \frac{\mu}{T}) = \frac{4\pi}{3} \left\{ \sum_{q=r,b,g} \left[ -\frac{1}{3} + \left( \frac{\alpha_{q} - i\frac{\mu}{T}}{\pi} \right)^{2} \right] - \sum_{g=1}^{4} \left[ -\frac{1}{3} + \left( \frac{\alpha_{g} - \pi}{\pi} \right)^{2} \right] \right\}
\]

\( \alpha_{q}(q = r, b, g) \) and \( \alpha_{g}(g = 1, \ldots, 4) \) being the angles determined by:

\[
\alpha_{r} = \frac{\varphi}{2} + \frac{\psi}{3}, \quad \alpha_{g} = -\frac{\varphi}{2} + \frac{\psi}{3}, \quad \alpha_{b} = -\frac{2\psi}{3}
\]

\[
\alpha_{1} = \alpha_{r} - \alpha_{g}, \quad \alpha_{2} = \alpha_{g} - \alpha_{b}, \quad \alpha_{3} = \alpha_{b} - \alpha_{r}, \quad \alpha_{4} = 0
\]

Let us note that the density of states used in this work is that determined by the Multiple Reflection Expansion (MRE) approximation developed by Balian and Bloch [3], given for a spherical system of quarks and gluons, described by the bag model, by:

\[
\rho_{i}(k, \nu_{QGP}) = \frac{V_{QGP} k^{2}}{2\pi^{2}} + f_{A,i} k^{4} + f_{C,i} 8\pi R + ...
\]

where \( i = Q, G \) for quarks and gluons respectively, and \( f_{A,i}, f_{C,i} \) are the area and curvature coefficients, respectively. The area and curvature coefficients \( f_{A,Q}, f_{C,G} \) for the gluons are given by [11]:

\[
f_{A,G} = 0, \quad f_{C,G} = -\frac{1}{6\pi^{2}}
\]

The area coefficient for quarks \( f_{A,Q} \) as given in [12] is:
The curvature coefficient $f_{c,0}$ for massive quarks is given by [13]:

$$f_{c,0} = \frac{1}{12\pi^2} \left[ 1 - \frac{3k}{2m} \arctan \left( \frac{k}{m} \right) \right]$$

and it takes the value $-1/24\pi^2$ for massless quarks [14]. It is then clear that in the present case of massless $u$ and $d$ quarks, the area term does not contribute, and it is just the curvature term which is involved in the density of states, in addition to the usual volume term.

3. Deconfinement phase transition in a finite volume

It seems that the only experimental way to create the QGP phase in laboratory is to collide ultra-relativistic heavy ions in particle colliders. Thus, the volume in which this eventual formation would take place is certainly finite.

In what follows, we study the advent of the deconfinement phase transition in a finite volume, by means of a simple model [6] assuming the coexistence of the hadronic gas (HG) and QGP phases in a finite volume $V$, with fractional volumes $V_{HG}$ and $V_{QGP}$ such that: $V = V_{HG} + V_{QGP}$. A parameter $h$ labeling the fraction of volume occupied by the HG phase, i.e., $h = V_{HG}/V$, is then defined which lies between 0 and 1. Thus, the value $h=1$ corresponds to a pure HG system while the value $h=0$ corresponds to a pure QGP system. The mean value of such a parameter seems to be suitable to play the role of an order parameter for the occurring DPT.

We use then the probability of finding the system in a state $h$, defined by [6]:

$$p(h,T,\mu,V) = \frac{Z(h,T,\mu,V)}{\int_0^1 Z(h,T,\mu,V)dh}$$

(11)

to calculate the mean value of any physical quantity $A$ characterizing the system, by the relation:

$$\langle A(T,\mu,V) \rangle = \int_0^1 A(h,T,\mu,V) p(h,T,\mu,V) dh$$

(12)

where $A(h,T,\mu,V)$ is the total thermodynamic quantity in the state $h$, and $Z(h,T,\mu,V)$ in equation (11) is the total partition function of the system in the state $h$, which, assuming non-interacting phases, can be written on the form:

$$Z(h,T,\mu,V) = Z_{HG}(h,T,\mu,V)Z_{QGP}(h,T,\mu,V)$$

(13)

with $Z_{HG}(h,T,\mu,V)$ and $Z_{QGP}(h,T,\mu,V)$ the partition functions of the individual HG and QGP phases, respectively.

In order to probe the system at the transition, we may examine some characteristic quantities, such that the order parameter, which is nothing but the mean value of the hadronic volume fraction, as yet mentioned. For this, we consider for the hadronic phase, a gas consisting of massless pions $(\pi^0, \pi^+, \pi^-)$, for which the relevant partition function is simply given by:

$$Z_{HG}(T,V) = e^{\frac{x^2}{2}V_{HG}}$$

(14)
The order parameter according to the previous definitions is then given by:

\[
\langle h(T,V) \rangle = \frac{\int_0^1 h Z(h,T,V) dh}{\int_0^1 Z(h,T,V) dh}
\] (15)

Thus, the analytical calculation of this mean value is done, using expression (1) at \( \mu = 0 \), since we work at zero chemical potential, for the CSPF calculated with the contribution of the curvature term additionally to the volume term in the density of states, and equation (14) for the HG partition function, and the obtained integral expressions are calculated in a suitable numerical way. A similar calculus can be carried out with the CSPF obtained with the volume contribution only in the density of states [4, 10].

In the following, figure 1 illustrates the variations of the order parameter, \( \langle h(T,V) \rangle \), with temperature for various volumes of the system, in these two cases, with a bag constant value \( B^{1/4} = 145 MeV \).

Figure 1. Plot of the order parameter versus temperature for different system volumes, in the two cases of a QGP partition function calculated with the contribution of (a) the volume term only in the density of states and (b) the curvature and volume terms, at \( \mu = 0 \) and with \( B^{1/4} = 145 MeV \).

Detailed analysis of the variations of this quantity, in addition to other quantities representing response functions characterizing the DPT, with temperature for various volumes, can be found in [10], where the volume term only was contributing in the quarks and gluons density of states, as in part (a) of figure 1. The first feature to notice from the graphs in figure 1, is the similarity of the general variations of \( \langle h(T,V) \rangle \) in the two cases. The main results which can be drawn are that in a finite volume, no singularity occurs and the variations of the order parameter are continuous. The finite sharp jump appearing in \( \langle h(T,V) \rangle \) at the thermodynamic limit, at a transition temperature noted \( T_c(\infty) \) [10], is rounded off in a finite volume, with a shift of the curve on the \( T \) axis. At the level of its first derivative with respect to \( T \), called thermal susceptibility \( \chi = \partial \langle h \rangle / \partial T \), as illustrated on figure 2 for the volume \( V = 150 fm^3 \), this jump transforms to a \( \delta \)-peak observed at the thermodynamic limit, which is, in a finite volume, smeared out into a finite peak over a range of temperature \( \delta T(V) \), with a maximum occurring at an effective transition temperature \( T_c(V) \) shifted away from the transition temperature for infinite volume \( T_c(\infty) \).
Figure 2. Plot of the order parameter (solid line), its first thermal derivative \( \chi = \partial \langle h \rangle / \partial T \) (dashed line) and its second thermal derivative \( \langle h'' \rangle = \partial^2 \langle h \rangle / \partial T^2 \) (dash dot dot) versus temperature for the system volume \( V = 150 \text{fm}^3 \), in the case of a QGP partition function calculated with the contribution of the volume term only in the density of states.

This shift is a consequence of the color-singletness requirement since the effective number of internal degrees of freedom for a color-singlet QGP is drastically reduced with decreasing volume, as found in [10] and as it has been shown in [8, 9]. The second derivative of the order parameter \( h(T,V) = \partial^2 \langle h(T,V) \rangle / \partial T^2 \) is also illustrated on figure 2 as function of temperature, and it can be noted that this quantity reaches its extrema at temperatures \( T_1(V) \) and \( T_2(V) \). The width of the transition region \( \delta T(V) \) can simply be defined by the gap between these two temperatures, i.e., \( \delta T(V) = T_2(V) - T_1(V) \), and we have found that the gap between these two extrema decreases with increasing volume [10]. Therein, finite size effects have been analyzed, and summarized in four points which are the rounding of the discontinuities, the smearing of the singularities, the shifting of the transition point and the broadening of the transition region. To these effects, useful characteristic quantities can be associated, among which are the maxima of the peaks of the susceptibility \( \chi_V^{\text{max}}(V) \), the shift of the transition temperature \( \tau_T(V) = T_c(V) - T_c(\infty) \) and the width of the transition region \( \delta T(V) \). Each of these quantities can be considered as a signature which may anticipate the behavior in the thermodynamic limit, and is expected to exhibit a scaling behavior described by a power law of the volume \( V \), characterized by a \textit{scaling critical exponent}. For a first order phase transition, the set of power laws is:

\[
\begin{align*}
\chi_V^{\text{max}}(V) & \propto V^\gamma \\
\delta T(V) & \propto V^{-\theta} \\
\tau_T(V) & \propto V^{-\lambda}
\end{align*}
\]
and it has been shown in the FSS theory [15-17] that in this case of a first order phase transition, the scaling exponents \( \gamma, \theta \) and \( \lambda \) are all equal to unity. The values of the scaling critical exponents may then be used as indicators of the order of a phase transition [18-21].

4. Influence of the curvature term on some features of the deconfinement phase transition

4.1. Influence of the curvature term on the transition point

In the following, we pursue our analysis of the graphs of the order parameter obtained in figure 1, by examining the effect of including the curvature term in the density of states in the present study. For this, we compare the behavior of the order parameter and the thermal susceptibility in the two cases, with and without the contribution of the curvature term.

We illustrate the order parameter on figure 3 as well as the thermal susceptibility on figure 4, versus temperature for the volumes \( V = 300 \text{ fm}^3 \) (left) and \( V = 2000 \text{ fm}^3 \) (right), in the two cases of a QGP CSPF calculated with the contribution of the volume term only in the density of states (solid line) and with the contribution of the curvature and volume terms (dashed line). The striking difference is the shift of the curves to higher temperatures when the curvature term is included, or in more accurate terms the shift of the effective transition temperature in a finite volume, since this latter can be defined at \( \langle \hat{h} \rangle = 0.5 \), or equivalently at the susceptibility maximum (see figure 2).

By examining the curvature coefficients in equations (8) and (10), we see that both are negative and this is exactly what causes this shift of the transition temperature. It turns out that the pressure calculated in the presence of the curvature contribution in the density of states, at a given temperature, has a lower value. Thus, the equilibrium between the two phases, according to the Gibbs criterion, is reached at temperatures greater than those in the absence of the curvature term. Also, it is clear that this shift is more considerable in small volumes, while it is reduced with increasing volume. This is well expected, and agrees very well with the results of prior works [8, 22, 23], indicating that the finite size corrections between a color unprojected QGP and a color projected one disappear as the size and/or temperature of the system increase.

**Figure 3.** Variations of the order parameter with temperature for the volumes (a) \( V = 300 \text{ fm}^3 \) and (b) \( V = 2000 \text{ fm}^3 \), in the two cases of a QGP partition function calculated with the contribution (solid line) of the volume term only in the density of states and (dashed line) of the curvature and volume terms.
just the curvature contrib

In the present work, we have calculated the partition function of a QGP in the color-singlet representation of the SU(3) symmetry group, using a density of states given by the MRE approximation for the density of states, where the curvature, area and volume terms contribute. In the present case of a QGP consisting of gluons, massless up and down quarks and their antiquarks, it is just the curvature contribution which is accounted for, additionally to the volume one.

4.2. Influence of the curvature term on the shift scaling exponent

In our previous work [10], we have used a finite size scaling (FSS) analysis to recover the scaling exponents γ, θ and λ for the thermally driven deconfinement phase transition. For this purpose, we proceeded by studying the behavior of the susceptibility maxima, its smearing as well as the shift of the effective transition temperature with varying volume. Numerical parametrizations of the obtained data with the power-law forms set in (16) gave the following results for the two first corresponding scaling exponents: γ = 1.01 ± 0.03 and θ = 1.03 ± 0.03 , where the associated errors are systematic ones.

For the study of the shift of the transition temperature, τf(V) = Tc(V) − Tc(∞) , we need to locate the effective transition temperature in a finite volume Tc(V). A way to define Tc(V) , is at the value (h) = 0.5 , as yet mentioned, meaning that both HG and QGP phases contribute with equal probability to the state of the total system [6], or equivalently at the location of the maximum of the rounded peak of the susceptibility. Results of the shift of the transition temperature obtained in this way were fitted in [10], where only the volume term contributed to the density of states, to the form: τf(V) ∝ V−λ3 , and the obtained shift scaling exponent was: λ3 = 0.876 ± 0.041 . Such a value ≠ 1 was interpreted as suggesting the contribution of non-leading terms in the expression of the volume variation of the shift.

In the present study taking the curvature term contribution into account, additionally to the volume term, data of the shift of the transition temperature are fitted to the power law form: τf(V=Cur) ∝ V−λ2 , and the obtained shift scaling exponent is: λ2 = 0.732 ± 0.002 . When contrasting this result with that found in [24], where the same model of coexisting HG and QGP phases in a finite volume was used and the shift scaling exponent at zero chemical potential obtained was: λ ≈ 0.75 , we note a large agreement, and we can speak as in [24], about a universal scaling exponent in the range 0.75 < λ < 1 .

5. Conclusion

In the present work, we have calculated the partition function of a QGP in the color-singlet representation of the SU(3) symmetry group, using a density of states given by the MRE approximation for the density of states, where the curvature, area and volume terms contribute. In the present case of a QGP consisting of gluons, massless up and down quarks and their antiquarks, it is just the curvature contribution which is accounted for, additionally to the volume one.
Probing the behavior of the mixed HG-QGP system in a finite volume, at the thermally driven phase transition, at zero chemical potential, we examined some response functions, namely the order parameter and its first thermal derivative representing the susceptibility. Through a comparison of the behavior of the studied quantities with and without the contribution of the curvature term in the density of states, the influence of this latter appeared, since it turned out that the effective transition temperature in a finite volume is shifted to higher values when the curvature is accounted for. This is probably due to the fact that the curvature coefficients are negative and this has as a consequence that the pressure in the presence of the curvature contribution, at a given temperature, has a lower value and the equilibrium between the two phases according to the Gibbs criterion is then reached at temperatures greater than those in the absence of the curvature contribution. Also, this shift is more considerable in small volumes, while it reduces with increasing volume, and this result agrees very well with those of prior works [8, 22, 23], indicating that the finite size corrections between a color unprojected QGP and a color projected one disappear as the size and/or temperature of the system increase, since the color projection no longer plays any role at this limit.

We have also investigated the influence of including the curvature term in the density of states on the shift scaling exponent $\lambda$, and we have found a value of $\lambda$ close to 0.75, which seems to agree with the result found in [24].

**Acknowledgments**

This research work is supported by the Algerian Ministry of Higher Education and Scientific Research. The authors acknowledge financial support from the Direction Générale de la Recherche Scientifique et du Développement Technologique (DGRSDT).

**References**

[1] Redlich K and Turko L 1980 Z. Phys. C 5 201.
[2] Turko L 1981 Phys. Lett. B 104 153.
[3] Balian R and Bloch C 1970 Ann. Phys. (N.Y.) 60 401.
[4] Ladrem M and Ait El Djoudi A 2005 Eur. Phys. J. C 44 257.
[5] Ait El Djoudi A 2013 Canad. J. Phys. 91 793.
[6] Spieles C, Stöcker H and Greiner C 1998 Phys. Rev. C 57 908.
[7] Chodos A and al. 1974 Phys. Rev. D 9 3471.
[8] Elze H -Th, Greiner W and Rafelski J 1983 Phys. Lett. B 124 515.
[9] Elze H -Th and Greiner W 1986 Phys. Lett. B 179 385.
[10] Ait El Djoudi A, Djida R and Mezouar K 2019 SSRN: http://dx.doi.org/10.2139/ssrn.3372375.
[11] Balian R and Bloch C 1971 Ann. Phys. (N.Y.) 64 271; 1974 ibid 84 559 (Erratum).
[12] Berger M S and Jaffe R L 1987 Phys. Rev. C 35 213; 1991 ibid 44 566 (Erratum).
[13] Madsen J 1994 Phys. Rev. D 50 3328.
[14] Farhi E and Jaffe R L 1984 Phys. Rev. D 30 2379.
[15] Fisher M E and Berker A N 1982 Phys. Rev. B 26 2507.
[16] Privman V and M E Fischer 1983 J. Stat. Phys. Vol 33 N°2 385.
[17] Challa M S et al. 1986 Phys. Rev. B 34 1841.
[18] Binder K and Heermann D W 1988 Monte Carlo Simulations in Statistical Physics (Springer-Verlag).
[19] Meyer-Ortmanns H 1996 Rev. Mod. Phys. 68 473.
[20] Fukugita M et al. 1987 Phys. Rev. Lett. 58 2515; 1988 Phys. Rev. Lett. 60 178.
[21] Fukugita M 1988 Nucl. Phys. Proc. Suppl. 4 105.
[22] Gorenstein M I, Lipskikh S I, Petrov V K and Zinov'jev G M 1983 Phys. Lett. B 123 437.

References
[24] Spieles C, Bleicher M and Greiner C 2020 arXiv: 1908.05927 [hep-ph].