EXOTIC AUTOMORPHISMS OF THE
SCHOUTEN ALGEBRA OF POLYVECTOR FIELDS

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Abstract. Using a new compactification of the (braid) configuration space of \( n \) points in the upper half plane we construct a family of exotic \( \mathcal{L}ie_\infty \) automorphisms of the Schouten algebra of polyvector fields on an affine space depending on a Kontsevich type propagator.

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1. Introduction

1.1. Statement of the result. This paper gives explicit formulae for a family of exotic \( \mathcal{L}ie_\infty \)-automorphisms,

\[
F^{\mathcal{L}ie} = \{ F^{\mathcal{L}ie}_n : \wedge^n T_{\text{poly}}(\mathbb{R}^d) \to T_{\text{poly}}(\mathbb{R}^d)[2 - 2n] \}_{n \geq 1}
\]

of the Lie algebra, \( T_{\text{poly}}(\mathbb{R}^d) \), of polyvector fields on \( \mathbb{R}^d \) (equipped with the grading in which the Schouten brackets have degree \(-1\)). The formulae are universal, i.e. independent of the dimension \( d < \infty \), have the first component \( F^{\mathcal{L}ie}_1 \) equal to the identity map, and all the other components
are given by sums,

\[ F_{n}^{\text{Lie}} = \sum_{\Gamma \in \mathfrak{G}_{n,2n-2}} C_{\Gamma} \Phi_{\Gamma}, \quad n \geq 2, \]

running over a family of graphs, \( \mathfrak{G}_{n,2n-2} \), with \( n \) vertices and \( 2n-2 \) edges, where

- \( \Phi_{\Gamma} : \otimes^{n} T_{\text{poly}}(\mathbb{R}^{d}) \to T_{\text{poly}}(\mathbb{R}^{d})[2-2n] \) is a linear map constructed from the graph \( \Gamma \) via a certain simple procedure explained in §4.1,
- the numerical coefficient, \( C_{\Gamma} \), is given by an integral,

\[ C_{\Gamma} = \int_{C_{n,0}}^{\hat{c}_{n,0}} \Phi_{\Gamma}(\omega) \frac{2}{2\pi} \]

over a compactified configuration space, \( \hat{C}_{n,0} \), of certain equivalence classes of \( n \) pairwise distinct points in the upper half plane

\[ \mathbb{H} := \{ x + iy \in \mathbb{C} \mid y \geq 0 \}. \]

The big open cell, \( C_{n,0} \), of \( \hat{C}_{n,0} \) is exactly the same as in [Ko2],

\[ C_{n,0} := \{ z_{1}, \ldots, z_{n} \in \mathbb{H} \mid z_{i} \neq z_{j} \text{ for } i \neq j \}/G^{(1)}, \]

\[ G^{(1)} := \{ z \to az + b \mid a, b \in \mathbb{R}, a > 0 \}, \]

but our compactification, \( \hat{C}_{n,0} \), of \( C_{n,0} \) is different from Kontsevich’s one, \( \overline{\mathcal{T}}_{n,0} \). In the above formula the symbol \( p_{e} \) stands for a surjection (not equal to the ordinary forgetful map, see §3.6) \( \hat{C}_{n,0} \to \hat{C}_{2,0} \) associated with an edge \( e \) of a graph \( \Gamma \in \mathfrak{G}_{n,2n-2} \), and \( \omega \) stands for an arbitrary closed differential form on \( \hat{C}_{2,0} \) whose restriction to the boundary \( \partial \hat{C}_{2,0} \simeq S^{1} \cup S^{1} \) coincides with the standard homogeneous volume form on each of the two boundary topological circles. If we drop the requirement of homogeneity, then our formula describes a \( \text{Lie}_{\infty} \)-quasi-isomorphism between certain \( \text{Lie}_{\infty} \)-extensions of the Schouten bracket canonically associated with the values of \( \omega \) on the first and, respectively, the second boundary circle of \( \partial \hat{C}_{2,0} \); we show in §3 an explicit formula for such a \( \text{Lie}_{\infty} \)-extension which looks as the one above except that involves a family of graphs, \( \mathfrak{G}_{n,2n-3} \), with \( n \) vertices and \( 2n-3 \) edges and a different compactified configuration space; the resulting family of \( \text{Lie}_{\infty} \)-extensions of the Schouten bracket is parameterized by semialgebraic functions on \( S^{1} \) and includes, for example, the one constructed by Shoikhet in [Sh2].

The family of new compactifications, \( \{ \hat{C}_{n,0} \}_{n \geq 1} \), as well as their braid version, \( \{ \hat{B}_{n,0} \}_{n \geq 1} \), discussed in §5, have nice operadic interpretations: the cell complex of the first one is naturally the 2-coloured dg operad of \( \text{Leib}_{\infty} \)-morphisms of \( \text{Leib}_{\infty} \)-algebras while the cell complex of the second has a natural structure of the 2-coloured dg operad of \( \text{Lie}_{\infty} \)-morphisms of \( \text{Lie}_{\infty} \)-algebras. Here \( \text{Leib} \) stands for the operad of Leibniz algebras introduced by J.-L. Loday in [Lo], and \( \text{Leib}_{\infty} \) for its minimal resolution. Thus the face structure underlying the compactification \( \hat{C}_{n,0} \) suggests that there might exist a generalization of the above construction producing more general \( \text{Leib}_{\infty} \)-automorphisms,

\[ F_{n}^{\text{Leib}} = \{ F_{n}^{\text{Leib}} : \otimes^{n} T_{\text{poly}}(\mathbb{R}^{d}) \to T_{\text{poly}}(\mathbb{R}^{d})[2-2n] \}_{n \geq 1}, \]

of the Schouten algebra. Any \( \text{Lie}_{\infty} \)-automorphism is, of course, a \( \text{Leib}_{\infty} \)-automorphism but not vice versa. Though the symmetrization of a generic \( \text{Leib}_{\infty} \) automorphism does not give a \( \text{Lie}_{\infty} \) automorphism in general, they both induce automorphisms,

\[ \alpha \to F_{n}^{\text{Leib}}(\alpha) := \sum_{n \geq 1} \frac{h^{n-1}}{n!} F_{n}^{\text{Leib}}(\alpha, \ldots, \alpha), \quad \alpha \to F_{n}^{\text{Lie}}(\alpha) := \sum_{n \geq 1} \frac{h^{n-1}}{n!} F_{n}^{\text{Lie}}(\alpha, \ldots, \alpha), \]

of one and the same set

\[ \mathcal{MC}(T_{\text{poly}}(\mathbb{R}^{d})[[h]]) := \{ \alpha \in T_{\text{poly}}(\mathbb{R}^{d}) \otimes \mathbb{C}[[h]] : |\alpha| = 2 \text{ and } |\alpha, \alpha|_{\text{Schouten}} = 0 \}, \]

of Poisson structures on \( \mathbb{R}^{d} \) depending on a formal parameter \( h \). The space \( \mathbb{R}^{d} \) can, in general, be equipped with a non-trivial \( \mathbb{Z} \)-grading, and \( |\alpha| \) stands above for the total degree of a polyvector field \( \alpha \) (so that \( |\alpha| = 2 \) does not necessarily imply that \( \alpha \) is a bi-vector field). Therefore, in the
context of Poisson geometry, one can skip distinguishing the two notions, \( \Lie b_\infty \) and \( \Lie c_\infty \), and talk simply about exotic automorphisms of finite-dimensional Poisson structures, or, even better, about exotic automorphisms,

\[
F : (\Poly, d) \rightarrow (\Poly, d),
\]
of a certain very simple dg free wheeled prop \( \Poly \), controlling finite-dimensional Poisson geometry (see, e.g., [Me2, Me3] for an elementary introduction into the language of wheeled operads and props in the context of Poisson geometry).

It is worth emphasizing that our formulae for exotic automorphisms of Poisson structures depend on the choice of a Kontsevich type propagator, \( \omega(z_1, z_2) \). Propagators introduced by Kontsevich in his theory of formality maps [Ko2, Ko3] give suitable propagators for our model, but there are other ones. In fact, the original Kontsevich propagator gives via our formula a highly non-trivial quasi-isomorphism from the original Schouten algebra to its \( \Lie c_\infty \)-extension constructed by Shoikhet in [Sh2].

The set of non-trivial exotic \( \Lie c_\infty \) automorphisms of \( \calT_{\Poly}(\mathbb{R}^d) \) is non-empty. We show in §4 several non-trivial examples. We also give a de Rham field theory interpretation of famous Duflo’s strange automorphism and of its generalization by Kontsevich. It is shown in §5 that the exotic automorphism corresponding to the propagator

\[
\omega(z_1, z_2) = \frac{1}{2i} \left( d\log \frac{z_1 - z_2}{z_1 - z_2} + d\log \frac{z_2 - z_1}{z_2 - z_1} \right)
\]
gives at the infinitesimal level the differential equation for an exotic flow of Poisson structures which was introduced by Kontsevich long ago (see §4.6.3 in [Ko2]) and which describes a homotopy non-trivial \( \Lie c_\infty \)-automorphism of the Schouten algebra.

1.2. A motivation. Let \( \calD_{\Poly}^\bullet(\mathbb{R}^d) \) be the Hochschild dg Lie algebra of polydifferential operators on smooth (formal) functions on \( \mathbb{R}^d \). Tamarkin proved [Ta1] existence of a family of \( \Lie c_\infty \)-quasi-isomorphisms,

\[
\{ F_a : \calD_{\Poly}^\bullet(\mathbb{R}^d) \rightarrow \Lambda^\bullet \calT_{\Poly}(\mathbb{R}^d) \}_{a \in \calM},
\]

parameterized by the set, \( \calM \), of all possible Drinfeld’s Lie associators (see the original paper [Dr] or the book [ES] for a definition of \( \calM \)). The Grothendieck–Teichmüller group, \( \GT \), acts on \( \calM \) [Dr] and hence on the above family, \( \{ F_a \} \), of formality maps. This in turn defines a map,

\[
\rho : \GT \rightarrow \text{Aut}(\calT_{\Poly}(\mathbb{R}^d)),
\]

\[
G \rightarrow F_{G(a)} \circ F_a^{-1},
\]

where \( F_a^{-1} : \calT_{\Poly}(\mathbb{R}^d) \rightarrow \calD_{\Poly}(\mathbb{R}^d) \) is a \( \Lie c_\infty \)-morphism which is homotopy inverse to \( F_a \) (it exists but, in general, is not uniquely defined).

Conjecture. There exists a non-trivial representation, \( \GT \rightarrow \text{Aut}(\calT_{\Poly}(\mathbb{R}^d))/\sim \), where \( \sim \) stands for the homotopy equivalence relation.

It is shown in §4-5 that some exotic \( \Lie c_\infty \) transformations involve an infinite sequence of numbers,

\[
\left\{ \frac{\zeta(n)}{n(2\pi \sqrt{-1})^n} \right\}_{n \in \mathbb{N}+1},
\]

which one might interpret as a result of the \( \GT \) action (cf. [Dr, Ko3]) and hence as an evidence in support of the above conjecture.

Another motivation — which might lead to a new kind of \( \GT \) twisted differential and algebraic geometry — is outlined in §6.

1.3. Some notation. The set \( \{1, 2, \ldots, n\} \) is abbreviated to \( [n] \); its group of automorphisms is denoted by \( S_n \). The cardinality of a finite set \( A \) is denoted by \( \#A \). If \( V = \bigoplus_{i \in \mathbb{Z}} V^i \) is a graded vector space, then \( V[k] \) stands for the graded vector space with \( V[k]^i := V^{i+k} \), for \( v \in V^i \) we

\[\footnote{In the same vein the Kontsevich formality map [Ko2] can be understood as a morphism of dg wheeled props \( D_{\Eff} \rightarrow Poly \) (see [Me2]).} \]
set \( |v| := i \). If \( \omega_1 \) and \( \omega_2 \) are differential forms on manifolds \( X_1 \) and, respectively, \( X_2 \), then the form 
\[ p^*_1(\omega_1) \wedge p^*_2(\omega_2) \] 
on manifolds \( X_1 \times X_2 \), where \( p_1 : X_1 \times X_2 \to X_1 \) and \( p_2 : X_1 \times X_2 \to X_2 \) are natural projections, is often abbreviated to \( \omega_1 \wedge \omega_2 \).

The algebra, \( T_{poly}(\mathbb{R}^d) \), of smooth polyvector fields on a finite-dimensional \( \mathbb{Z} \)-graded vector space \( V \simeq \mathbb{R}^d \) is understood in this paper as a \( \mathbb{Z} \)-graded commutative algebra of smooth functions on the \( \mathbb{Z} \)-graded manifold, \( T_{gZ}[1] \), which is isomorphic to the tangent bundle on \( \mathbb{R}^d \) with degrees of the fibers shifted by 1. If \( x^a \) are homogeneous coordinates on \( \mathbb{R}^d \) and \( \psi_a := \partial/\partial x^a[1] \), then a polyvector field \( \gamma \in T_{poly}(\mathbb{R}^d) \) is just a smooth function, \( \gamma(x, \psi) \), of these coordinates, and the Schouten brackets are given by,
\[
[\gamma_1 \bullet \gamma_2] := \Delta(\gamma_1 \gamma_2) - \Delta(\gamma_1) \gamma_2 - (-1)^{\gamma_1} \gamma_1 \Delta(\gamma_2),
\]
where \( \Delta = \sum_{a=1}^{d} (-1)^{|x^a|} \frac{\partial^2}{\partial x^a \partial \psi_a} \). As \( |\psi_a| = 1 - |x^a| \), the operator \( \Delta \) and, therefore, the Schouten brackets have degree \(-1\).

We work throughout in the category of smooth manifolds with corners. However, all the main theorems of this paper hold true in the category of semialgebraic manifolds introduced in \([KS]\) and further developed in \([HLTV]\) so that in applications one can employ not only ordinary smooth differential forms but also \( PA \)-forms, where \( PA \) stands for “piecewise semi-algebraic” as defined in the above mentioned papers.

2. Configuration space \( C_n \)

2.1. A Fulton-MacPherson type compactification of \( C_n \) \([Ko2]\). Let
\[
Conf_n := \{ z_1, \ldots, z_n \in \mathbb{C} \mid z_i \neq z_j \text{ for } i \neq j \}
\]
be the configuration space of \( n \) pairwise distinct points in the complex plane \( \mathbb{C} \). The space \( C_n \) is a smooth \((2n - 3)\)-dimensional real manifold defined as the orbit space \([Ko2]\),
\[
C_n := Conf_n / G^{(2)},
\]
with respect to the following action of a real 3-dimensional Lie group,
\[
G^{(2)} = \{ z \to az + b \mid a \in \mathbb{R}^+, b \in \mathbb{C} \}.
\]
Its compactification, \( \overline{C}_n \), was defined in \([Ko2]\) (see also \([Ga]\)) as the closure of an embedding,
\[
\begin{align*}
(z_1, \ldots, z_n) &\quad \mapsto (\mathbb{R}/2\pi \mathbb{Z})^{n(n-1)} \times [0, +\infty]^{n(n-1)^2} \\
&\quad \times \prod_{i \neq j} Arg(z_i - z_j) \times \prod_{k, l \neq i, j} \frac{|z_l - z_i|}{|z_k - z_l|}.
\end{align*}
\]
The space \( \overline{C}_n \) is a smooth (naturally oriented) manifold with corners. Its codimension 1 strata is given by
\[
\partial \overline{C}_n = \bigsqcup_{A \subseteq [n] \atop |A| \geq 2} C_n - \#A + 1 \times C_\#A
\]
where the summation runs over all possible proper subsets of \([n]\) with cardinality of at least two. Geometrically, each such a strata corresponds to the \( A \)-labeled elements of the set \( \{z_1, \ldots, z_n\} \) moving very close to each other.

The natural action of the permutation groups on the standard face complex of \( \overline{C}_\bullet \) is trivial as permutations preserve all the big cells together with their natural orientations (if we were working with the configurations of points in the odd dimensional affine space \( \mathbb{R}^{2k+1} \) rather than in \( \mathbb{R}^2 \), then orientations might be changed by a sign); it was noticed in \([GeJa]\) that this face complex has a natural structure of an operad of \( Lie_{\infty} \)-algebras. However, in the applications below the points of all configuration spaces considered in this paper always come decorated with vertices of certain graphs (and such decorations extend naturally to the compactifications!). The natural action of the permutation groups on the cell complexes of such decorated compactified configuration spaces is non-trivial already at the level of big cells and hence induces a non-trivial \( S_n \)-action on the associated de Rham field theories (see §4 below). To keep this subtlety under control we assume from now on that our configuration spaces consist of not only distinct but also distinctly decorated
points \((z_1, \ldots, z_n)\); equivalently, one may think of a choice of a total order on the set \((z_1, \ldots, z_n)\) (which may not coincide with the natural order) because such a structure on \(C_n\) extends obviously to its compactification \(\overline{C}_n\); our final formulae involve a summation over all possible decorations (in particular, over all possible orderings) so that eventually nothing depends on such a choice. In this \textit{decorated} case the face complex of \(\overline{C}_\bullet\) has again a natural structure of a dg operad which is different from the operad \(\text{Lie}_\infty\).

2.2. The face complex of \(\{C_n\}\) as an operad of Leibniz\(_\infty\) algebras. The faces of \(\overline{C}_n\) are isomorphic to the products of the form \(C_{k_1} \times \ldots \times C_{k_m}\). The stratification of the (decorated) configuration space \(\overline{C}_n\) is best coded by its face complex, \((\bigoplus_{n \geq 2} C_\bullet(\overline{C}_n), \partial)\), which is a dg free operad, \(\text{Free}(E_o)\), generated by an \(\mathbb{S}\)-module \(E_o = \{E_o(n)\}\) with

\[
E_o(n) = \begin{cases} 
\mathbb{C}[\Sigma_n][2n - 3] = \text{span} \left( \begin{array}{c} \sigma(1) \sigma(2) \cdots \sigma(n) \end{array} \right)_{\sigma \in \mathbb{S}_n} & \text{for } n \geq 2 \\
0 & \text{otherwise.}
\end{cases}
\]

Each plain corolla with \(n\) legs corresponds to \(\overline{C}_n\). As we prefer working with cochain complexes, we assign to this corolla degree \(-(2n - 3) = 3 - 2n\). The boundary differential is given on the generators by

\[
\partial = \sum_{k=0}^{n-2} \sum_{\substack{\{A_i\} \subseteq [n] \setminus \{k+1\} \setminus \{k+1+i_1+i_2\} \setminus \# I_1 \geq 1}} \sum_{\substack{\{I_1\} \subseteq [n-A_{\inf A-1}]} \setminus \{k+1+i_1+i_2\}} \sum_{\substack{\{I_2\} \subseteq [n-A_{\inf A-1}]} \setminus \{k+1+i_1+i_2\}} \sum_{\substack{\{I_3\} \subseteq [n-A_{\inf A-1}]} \setminus \{k+1+i_1+i_2\}} \cdots
\]

A \textit{smooth local coordinate system} on \(\overline{C}_n\) near the face corresponding to a graph \(G \in \text{Free}(E_o)\) can be described, as it is explained in detail in §5.2 of [Ko2], by an associated \textit{metric} graph in which every internal edge, \(e\), of \(G\) is assigned a small positive number \(\varepsilon_e\). For example, the face in \(\overline{C}_7\) corresponding to a graph

![Graph](image)

has associated the following metric graph,

![Graph with edges labeled](image)
describing an open subset of $\overline{C}_7$ consisting of all possible configurations of 7 points obtained in the following way:

(i) take first a standardly positioned configuration of 3 points labeled by 1 and, say, $a$ and $b$,
(ii) replace point $a$ (respectively, $b$) by an $\varepsilon(1)$-magnified standardly positioned configuration of two points labeled by $c$ and 6 (respectively, by an $\varepsilon(2)$-magnified standard configuration of three points labeled by 2, 4 and 7)
(iii) finally, replace the point $c$ by an $\varepsilon(3)$-magnified standard configuration of two points labeled by 3 and 5, and project the resulting configuration in $Conf_7$ into $C_7$.

The embedding of the boundary faces into this smooth coordinate neighborhood is given by the equation $\varepsilon(1)\varepsilon(2)\varepsilon(3) = 0$.

We conclude this subsection with a curious observation. The values of differential (3) on 2- and 3-corollas are given by

\[ \partial \left( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) = 0, \quad \partial \left( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) = \left( \begin{array}{c} 1 \\ 3 \\ 2 \end{array} \right) + \left( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right). \]

Hence the map

\[ \alpha : (Free(E_0), \partial) \rightarrow \mathcal{F}ree \left( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \]

which sends to zero all generating $n$-corollas except the ones with $n = 2$ is a morphism of dg operads. The operad on the r.h.s. is a quadratic operad generated by $\alpha$ the following way:

As a dg operad, the face complex, $C_\bullet(\mathcal{C})$, of the family of compactified configurations spaces, $\{\overline{C}_n\}_{n \geq 2}$, is canonically isomorphic to the minimal resolution, $\text{Leib}_\infty$, of the operad, $\text{Leib}$, of Leibniz algebras.2

Thus a structure of $\text{Leib}_\infty$-algebra on a dg vector space $(\mathfrak{g}, d)$ is given by a collection of linear maps

\[ \left\{ \mu_n : \mathfrak{g}^{\otimes n} \rightarrow \mathfrak{g}[3 - 2n], \gamma_1 \otimes \cdots \otimes \gamma_n \rightarrow \mu_n(\gamma_1, \ldots, \gamma_n) \right\}_{n \geq 2} \]

satisfying the equations,

\[ (d\mu_n)(\gamma_1, \ldots, \gamma_n) = \sum_{A \subseteq \{1, \ldots, n\}} (-1)^{\#A} \mu_n(\gamma_1, \ldots, \gamma_{\#A}, \gamma_{\#A+1}, \ldots, \gamma_n). \]

Here and elsewhere we use a notation,

\[ (\gamma S) := \gamma_{i_1} \otimes \cdots \otimes \gamma_{i_\ell}, \]

2 The projection $Conf_n \rightarrow C_n$ has a natural section $s : C_n \rightarrow Conf_n$ representing every point $p \in C_n$ as a collection of $n$ pairwise distinct points $(z_1, \ldots, z_n) \in Conf_n$ such that the minimal Euclidean circle enclosing $(z_1, \ldots, z_n)$ has radius 1 and center at $0 \in \mathbb{C}$. The point $s(p)$ is called the standard position of $p$.\[3\]

The author is grateful to Michel van den Bergh for this observation.
for a naturally ordered subset $S = \{i_1, \ldots, i_l\} \subseteq [n]$. If the tensors $\mu_n$ happen to be graded symmetric, $\mu_n : \otimes^n g \to g[3 - 2n]$, then the above equation is precisely the equation for a $\Lie_{\infty}$ structure on $g$, i.e. there exists a canonical morphism of dg operads,

$$\Lie_{\infty} \to \Lie_{\infty}.$$

Thus any Lie algebra, $g$, is also a Leibniz algebra. However, the groups of automorphisms of $g$ in the categories of $\Lie_{\infty}$-algebras and of $\Lieib_{\infty}$-algebras may be different.

It is worth noting that the symmetrization of a generic $\Lieib_{\infty}$-algebra structure, $\{\mu_n\}$, does not give, in general, a $\Lie_{\infty}$-structure, i.e. there is no simple morphism of dg operads of the form, $\Lie_{\infty} \to \Lieib_{\infty}$.

### 2.3. Hyperbolic standard position.

For future reference we shall describe a new atlas on $C_n$ which uses only configurations of points in the upper half plane. To an arbitrary collection of points in the complex plane,

$$p = \{z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n \in \mathbb{C}\}$$

we associate a point $p^{\min} = x^{\min} + iy^{\min} \in \mathbb{C}$ as follows,

$$y^{\min} := \inf_{i \in [n]} y_i, \quad x^{\min} := \sum_{i=1}^n \frac{1}{n} x_i.$$

Given a point $z_0 \in \mathbb{H}$ and a positive real number $\varepsilon$. Any point $p$ in the orbit space $C_n$ can be uniquely represented by a configuration, $p(z_0, \varepsilon) = (z_1, \ldots, z_n)$, in $\Conf_n$ satisfying the following two conditions

(i) $z_1, \ldots, z_n \in \mathbb{H}$;

(ii) the Euclidean diameter of $\{z_1, \ldots, z_n\}$ is equal to $\varepsilon$;

(iii) $p^{\min} = z_0$.

The image of the resulting section,

$$C_n \longrightarrow \Conf_n$$

$$p \longrightarrow p(z_0, \varepsilon),$$

of the projection $\Conf_n \to C_n$ is denoted by $C_n(z_0, \varepsilon)$. If $z_0 = i$ and $\varepsilon = 1$ we use a simpler notation, $C^h_n := C_n(i, 1)$, e.g.

The upper half space representation, $p_{(i, 1)} \in C^h_n$, of a point, $p \in C_n$, is called its **standard hyperbolic position** (or, shortly, hyperposition). The space $C^h_n$ is the union of subspaces in $\mathbb{R}^{2n}$ given by polynomial (of degree 1) equalities and inequalities and hence has a natural structure of a semialgebraic manifold or, if one prefers, of a smooth oriented manifold with corners. For example, $C^h_2$ is naturally a smooth 1-dimensional manifold with corners isomorphic to the following union of two interval.

---

4We show sometimes in our pictures dotted minimal Euclidean circles enclosing groups of points as a way to indicate their (relative) “sizes.”
with the lower interval corresponding to the configurations given in the left picture above and the upper interval to the similar configurations but with \( z_1 \) and \( z_2 \) swapped; the left and right \( \bullet \)-points represent, respectively, the following two configurations,

\[
\begin{align*}
\bullet_{z_1} & \quad \leftrightarrow \quad i_{z_2} \\
\bullet_{z_2} & \quad \leftrightarrow \quad i_{z_1}
\end{align*}
\]

It is clear from the construction that the configuration spaces \( C_n(z_0, \varepsilon) \subset Conf_n \) for different values of the parameters \( z_0 \) and \( \varepsilon \) are canonically isomorphic to each other,

\[
\Phi^{(z_2, \varepsilon_2)}_{(z_1, \varepsilon_1)} : C_n(z_1, \varepsilon_1) \longrightarrow C_n(z_2, \varepsilon_2),
\]

for some uniquely determined element \( \Phi^{(z_2, \varepsilon_2)}_{(z_1, \varepsilon_1)} \in G^{(2)} \).

The map,

\[
\varepsilon : = \Phi^{(i, \varepsilon)}_{(i, 1)} : \begin{array}{c}
Ch_n \longrightarrow C_n(i, \varepsilon) \\
p_{(i, 1)} \longrightarrow p_{(i, \varepsilon)}
\end{array},
\]

is called \( \varepsilon \)-magnification, e.g., for \( \varepsilon < 1 \),

\[
\begin{array}{c}
\Phi^{(i, \varepsilon)}_{(i, 1)} : \begin{array}{c}
C_n^{i} \longrightarrow C_n(i, \varepsilon) \\
p_{(i, 1)} \longrightarrow p_{(i, \varepsilon)}
\end{array}
\end{array}
\]

Finally, let \( p \in C_n(i, \varepsilon) \) and \( p' \in C_n(i, \varepsilon') \) be magnified standard configurations with the ratio \( \varepsilon/\varepsilon' \gg 0 \). Then with any point \( z \) in the configuration \( p \) we associate a kind of translation map,

\[
T_z : = \Phi^{(i, \varepsilon'/\varepsilon)}_{(i, \varepsilon)} : \begin{array}{c}
C_n(i, \varepsilon) \longrightarrow C_n(z, \varepsilon'/\varepsilon) \\
p' \longrightarrow T_z(p')
\end{array},
\]

which preserves all the relative Euclidean angles of the points in \( p' \). The image of \( T_z(p') \) under the projection \( Conf_n \rightarrow C_n \) is called the configuration \( p' \) placed at the point \( z \) of the configuration \( p \).

With these new notions of a standard position, a magnification and placing of a magnified standard configuration at a given point of another magnified standard configuration one can apply the same idea of metric graphs as in the previous subsection to define a semialgebraic (or smooth with corners) atlas on \( C_n \). It is worth noting that this atlas makes sense not only for the case of small values of \( \varepsilon \)-parameters but also for the case when they are large. Indeed, the boundary strata are given, in the above notations, by equations of the form \( \varepsilon'/\varepsilon = 0 \), and this can be achieved, for example, for \( \varepsilon \) finite and \( \varepsilon' \rightarrow 0 \) and also for \( \varepsilon' \) finite and \( \varepsilon \rightarrow +\infty \) (or, better, when both \( \varepsilon \) and \( \varepsilon' \) tend to \( +\infty \) in such a way that \( \varepsilon/\varepsilon' \gg 0 \)).

Our modification of Kontsevich's smooth atlas on \( C_n \) makes no big difference when one studies limit configurations of points moving too close to each other. However, that difference becomes important when we study below compactified configurations of points moving too far away from each other.
2.3.1. Remark. The spaces $C^h_n$ can be given also an equivariant structure of a smooth manifold as in [Ko2]; for example, the space $C^h_2$ can be parameterized by $\text{Arg}(z_1 - z_2)$ and hence identified with the smooth circle,

In this case one gets a smooth atlas on $C_n$, which is equivalent to Kontsevich’s one but uses only points in the upper half plane. Which atlas to use on $C_n$, a semialgebraic atlas or a smooth (with corners) one, depends on what kind of propagators one wishes to work with, i.e. it depends on the choice of a de Rham algebra on $\mathcal{C}_n$. We shall use below both versions so that sometimes $C^h_n$ is an ordinary smooth circle and sometimes is a topological circle with two zero dimensional corners.

3. Configuration space $C_{n,0}$ and its new compactification

3.1. $C_{n,0}$ as a magnified $C_n$. Let $\text{Conf}_{n,0} := \{z_1, \ldots, z_n \in \mathbb{H} \mid z_i \neq z_j \text{ for } i \neq j\}$ be the configuration space of $n$ pairwise distinct points in the upper half plane $\mathbb{H}$. The space $C_{n,0}$ is a smooth (naturally) oriented real $(2n - 2)$-dimensional manifold defined as the orbit space 

$C_{n,0} := \text{Conf}_{n,0}/G^{(1)},$

with respect to the following free action of a 2-dimensional real Lie group,

$G^{(1)} = \{z \to az + b \mid a, b \in \mathbb{R}, a > 0\}.$

Any point in the orbit space $C_{n,0}$ can be uniquely represented by a configuration, $p = (z_1, \ldots, z_n)$, in $\text{Conf}_{n,0}$ such that $p^{\min} = 0$ (see §2.3 for a definition of the point $p^{\min}$). This particular configuration is called the \textit{standard hyperbolic position}, or \textit{hyperposition}, of $p$. The image of the resulting section of the projection $\text{Conf}_{n,0} \to C_{n,0}$ is denoted by $C^h_{n,0}$.

The topological space $C^h_{n,0}$ is homeomorphic to the product $C^h_n \times \mathbb{R}^+$,

$\Psi_n : C^h_{n,0} \to C^h_n \times \mathbb{R}^+$,

$p = (z_1, \ldots, z_n) \mapsto (p_0, \lambda)$

where $\lambda$ is the Euclidean diameter of the configuration $p$ and $p_0$ is the unique point in $C^h_n$ whose $\lambda$-magnification $\Psi$ gives $p$. It is worthwhile pointing out that in geometric terms the inverse isomorphism,

$C^h_n \times \mathbb{R}^+ \to C^h_{n,0}$

$(p_0, \lambda) \mapsto p := \lambda \cdot p_0,$

is given by “exploding” a diameter 1 configuration $p_0 \in C^h_n$ by the factor $\lambda$. The magnification map $\Psi$ extends naturally to $C_{n,0}$,

$C_{n,0} \times \mathbb{R}^+ \to C_{n,0}$

$(p, \varepsilon) \mapsto \varepsilon \cdot p$

as the composition,

(7) $\varepsilon : C_{n,0} \simeq C^h_{n,0} \xrightarrow{\Psi_n^{-1}} C^h_n \times \mathbb{R}^+ \to C^h_n \times \mathbb{R}^+ \xrightarrow{\Psi_n} C_{n,0}$

It is worth noting that this magnification map preserves all the Euclidean angles $\text{Arg}(z_i - z_j)$ and all the relative Euclidean distances, $|z_i - z_j|$, of points in a configuration $p \in C_{n,0}$ as it can be represented by an action of an element of the group $G^{(2)}$ on $\text{Conf}_{n,0}$.

---

5We write here “exploding” instead of “expanding” in the anticipation of the main trick of the new compactification which formally allows $\lambda$ to be equal not only to zero (as in §2) but also to $+\infty$. 
3.2. A new compactification of \( C_{n,0} \). The set \( \text{Conf}_{n,0} \) can be understood as the set of all injections \([n] \hookrightarrow \mathbb{H}\). For any finite set \( A \) one can define the spaces \( \text{Conf}_{A,0} := \{ A \hookrightarrow \mathbb{H} \} \) and \( \text{Conf}_A := \{ A \hookrightarrow \mathbb{C} \} \), and, in a full analogy to the case \( A = [n] \), the associated orbit spaces \( C_{A,0}, C_A \), and their hyperbolic models \( C^h_{A,0} \) and, respectively, \( C^h_A \). Denote the composition of isomorphisms,

\[
C_{A,0} \longrightarrow C^h_{A,0} \longrightarrow C^h_A \times (0, +\infty)
\]

by \( \Psi_A \). For any non-empty subset \( A \subseteq [n] \) there is a natural projection,

\[
\pi_A : C_{n,0} \longrightarrow C_{A,0}
\]

which forgets the points which are parameterized by the complementary set \([n] \setminus A\).

A compactification, \( \hat{C}_{n,0} \), of \( C_{n,0} \) can be defined as the closure of a composition,

\[
\prod_{A \subseteq [n], \#A \geq 2} \bigg[ \prod_{A \subseteq [n], \#A = \#B, \#A = \#B} \prod_{A \subseteq [n], \#A \neq \#B} \bigg] C_{n,0} \times \prod_{A \subseteq [n], \#A \geq 2} \bigg[ \prod_{A \subseteq [n], \#A = \#B, \#A = \#B} \prod_{A \subseteq [n], \#A \neq \#B} \bigg] C^h_{A,0} \times (0, +\infty) \longrightarrow \prod_{A \subseteq [n], \#A \geq 2} C^h_A \times [0, +\infty].
\]

Thus all the limiting points come from configurations when a group (or groups) of points move too close to each other within each group and/or a group or groups of points are moving too far (with respect to the relative distances inside each group) away from each other.

It is easy to see from the above definition that the codimension 1 boundary strata in \( \hat{C}_{n,0} \) are given by

\[
\partial \hat{C}_{n,0} = \bigcup_{A \subseteq [n], \#A \geq 2} \bigg( \prod_{A \subseteq [n], \#A = \#B, \#A = \#B} \prod_{A \subseteq [n], \#A \neq \#B} \bigg) (C_{n,0} - \#A+1 \times C_{\#A}) \bigcup_{[n] = B_1 \cup \ldots \cup B_k} \big[ \prod_{2 \leq k \leq n} (C_{#B_1,0} \times \ldots \times C_{#B_k,0}) \big],
\]

where the first summation runs over all possible subsets, \( A \), of \([n]\) with cardinality at least two, and the second summation runs over all possible decompositions of \([n]\) into (at least two) disjoint non-empty subsets \( B_1, \ldots, B_k \). Geometrically, a strata in the first group of summands corresponds to \( A \)-labeled elements of the set \( \{ z_1, \ldots, z_n \} \) moving close to each other, while a strata in the second group of summands corresponds to \( k \) clusters of points (labeled, respectively, by disjoint ordered subsets \( B_1, \ldots, B_k \) of \([n]\)) moving far from each other in the sense that, for any \( a, b, c \in [k] \) with \( a \neq b \), the ratio \( \rho_{B_a, B_b} / \rho_{B_b} \) tends to +\( \infty \), where \( \rho_{B_a, B_b} \) is the Euclidean (or, equivalently, hyperbolic) distance between clusters \( B_a \) and \( B_b \) and \( \rho_{B_b} \) is the Euclidean (or, equivalently, hyperbolic) diameter of the cluster \( B_c \).

By analogy to \( C_n \), the face stratification of \( \hat{C}_{n,0} \) can be nicely described in terms of graphs (in fact, in terms of a dg operad describing strongly homotopic morphisms of \( \text{Leib}_{\infty} \)-algebras) while the structure of a \( PA \)-manifold on \( \hat{C}_{n,0} \) is best described in terms of the associated \textit{metric} graphs. These are the main themes of the rest of this section.

3.3. The face complex of \( \hat{C}_{n,0} \) as an operad \( \text{Mor}(\text{Leib}_{\infty}) \). Note that all the boundary faces of the compactification \( \hat{C}_{n,0} \) are products of the form

\[
\overline{C}_{p_1} \times \ldots \times \overline{C}_{p_k} \times \hat{C}_{n_1,0} \times \ldots \times \hat{C}_{n_k,0} \times \overline{C}_{q_1} \times \ldots \times \overline{C}_{q_l}
\]

where factors \( \overline{C}_{q_m} \) correspond to the first group of boundary terms in \((8)\) and the factors \( \overline{C}_{p_n} \) to the second one. Let us distinguish these two types of strata by interpreting them as operads in \textit{two} different colours, that is, by representing the associated generators, \( \overline{C}_p \) and, respectively, \( \overline{C}_q \), as, say, \textit{dashed} white corollas and, respectively, \textit{solid} white corollas,

\[
\overline{C}_p \simeq \begin{array}{*{5}{c}} \ldots \rightarrow \ldots & 1 & 2 & 3 & \ldots & p-1 & p \end{array}, \quad \overline{C}_q \simeq \begin{array}{*{5}{c}} \ldots \rightarrow \ldots & 1 & 2 & 3 & \ldots & q-1 & q \end{array}
\]

\footnote{In fact, the description of \( \hat{C}_{n,0} \) in terms of metric graphs given below in \S\S\ 3.3, 3.4 and Appendix 2 can be taken as an independent (of the embedding formula) definition of the compactification. It is the metric graph description of \( C_{n,0} \) which we use in applications.}
Next we represent a boundary factor of the type $\hat{C}_{n,0}$ as a degree $2 - 2n$ black vertex corolla,

$$\hat{C}_{n,0} \simeq \cdots$$

After these notational preparations one can describe the face complex of the compactification $\hat{C}_{\bullet,0}$ as follows.

3.3.1. Proposition. The face complex of the disjoint union $\hat{C}_{\bullet} \sqcup \hat{C}_{\bullet,0} \sqcup \hat{C}_{\bullet}$ has naturally a structure of a free 2-coloured operad,

$${\mathcal{M}}\text{or}(\text{Leib}_\infty) := \mathcal{F}ree\left(\begin{array}{c}
\text{black corollas} \\
\text{white corollas of both colours}
\end{array}\right)$$

equipped with a differential which is given on white corollas of both colours by formula (3) and on black corollas by the following formula

$$\partial = -\sum_{k=0}^{n-2} \sum_{\begin{array}{c}
\text{white corollas with } k \text{ colours in }
\text{black corollas with } \geq 1 \text{ colours}
\end{array}}$$

(9)

Representations of this operad in a pair of dg vector spaces, $V_{in}$ and $V_{out}$, is the same as a triple, $(\mu_{in}, \mu_{out}, F)$, consisting of a $\text{Leib}_\infty$ structure, $\mu_{in}$, on $V_{in}$, a $\text{Leib}_\infty$ structure, $\mu_{out}$, on $V_{out}$, and of a morphism, $F : (V_{in}, \mu_{in}) \rightarrow (V_{out}, \mu_{out})$, of $\text{Leib}_\infty$ algebras.

This first half of the above claim follows from formula (5). The second half (more precisely, the last three rows) follows from Proposition 2.2.1 and some standard manipulations with the bar-cobar constructions of the operad of Leinbiz algebras. We omit these manipulations so that the reader can interpret the sentence in the last row as a definition of the notion of $\text{Leib}_\infty$ morphism. The only property of such a morphism which we use in this paper is stated and proven in Appendix 3.

3.4. A smooth structure on $\hat{C}_{n,0}$. Let $z_0$ be any point of a configuration $p_0 \in C_{k,0}$. A placed at $z_0$ configuration $p$ in $C_n$ (or, in $C_{n,0}$) means a configuration in $\text{Conf}_{n+k-1,0}$ (and also its image under the projection $\text{Conf}_{n+k-1,0} \rightarrow C_{n+k-1,0}$) obtained from $z_0$, $p_0$ and $p$ in the following two steps

(i) put $p$ into its hyperposition $p^h$ in $C_n^h$ (respectively, hyperposition $p^h$ in $C_{n,0}^h$);

(ii) apply the transformation $T_{z_0}$ to $p^h$, see (4).

Note that this operation of placing a configuration of points, $(z_1, \ldots, z_n)$, in the upper half plane at a position $z_0 \in \mathbb{H}$ preserves all their relative angles, $\text{Arg}(z_i - z_j)$. Note also that it depends on the size of the configuration $p_0$.

The boundary strata of $\hat{C}_{n,0}$ are given by graphs $G \in {\mathcal{M}}\text{or}(\text{Leib}_\infty)$ containing at least one black corolla. A smooth structure in the neighborhood of a boundary face corresponding to a graph $G$ is best defined in terms of the associated metric graph, $G_{\text{metric}}$, by the following procedure:

(a) every internal edge of the form $\begin{array}{c}
\text{black corollas}
\end{array}$ or $\begin{array}{c}
\text{white corollas of both colours}
\end{array}$, is assigned a small positive real number $\varepsilon \ll +\infty$,
(b) every white vertex of a dashed corolla is assigned a large positive real number \( \tau \gg 0 \),

![diagram](image)

(c) for every (if any) two vertex subgraph of \( G_{\text{metric}} \) of the form \( \tau_1 \tau_2 \), there is associated a relation, \( \tau_2 = \varepsilon \tau_1 \), between the parameters (which essentially says that \( \tau_1 \gg \tau_2 \gg 0 \)).

Such a metric graph defines a smooth local coordinate chart, \( U_G \), on \( \hat{C}_{n,0} \) in which the face \( G \) is given by the equations: all \( \varepsilon = 0 \) and all \( \tau = +\infty \) (or, better, all \( \tau' := \tanh \tau = 1 \)). The construction of \( U_G \) should be clear from a pair of explicit examples one of which we show here and another (illustrating a relation of the type \( \tau_2 = \varepsilon \tau_1 \)) in Appendix 2.

Let \( G \) be the face of \( \hat{C}_{n,0} \) corresponding to a graph

![diagram](image)

The associated metric graph is given by

![diagram](image)

and the smooth coordinate chart \( U(G) \cap C_{8,0} \) is, by definition, an open subset of \( C_{8,0} \) consisting of all those configurations, \( p \), of 8 points in \( \mathbb{H} \) which result from the following four step construction:

**Step 1:** take an arbitrary hyperpositioned configuration, \( p^{(1)} = (z', z'', z''') \in C_3 \), and magnify it, \( p^{(1)} \rightarrow \tau \cdot p^{(1)} \) (see §2.3);

**Step 2:** take arbitrary hyperpositioned configurations of points, \( p_1^{(2)} \in C_{2,0} \), \( p_2^{(2)} \in C_{2,0} \) and \( p_3^{(2)} \in C_{3,0} \), labelled, respectively, by sets, \{1, 8\}, \{z''', 6\} and \{2, 4, 7\}, and place them at the positions \( z', z'' \) and, respectively, \( z''' \);

**Step 3:** take an arbitrary hyperpositioned configuration, \( p^{(4)} \in C_2 \), of two points labelled by 3 and 5, \( \varepsilon \)-shrink it as explained in §2.3, and finally place the result at the point \( z''' \).
The final result is a hyperpositioned point $p = (z_1, z_3, z_5, z_6, z_2, z_7, z_8)$ in $C_{8,0}$ of the form

$$z_4 \times z_7 \times z_3 \times z_5 \times z_6 \times z_1 \times z_8 \times z_2.$$ 

Thus $\mathcal{U}_G \cap C_{8,0} \simeq C_3 \times C_{2,0} \times C_{2,0} \times C_{3,0} \times C_2 \times \mathbb{R} \times \mathbb{R}$. The boundary strata are given by setting formally $\tau = \infty$ and/or $\varepsilon = 0$.

### 3.4.1. Remark
The smooth structure on $\hat{C}_{2,0}$ depends on the choice of a smooth structure on $C_\bullet$. For our applications the natural semialgebraic structure on $C_\bullet$ is the most appropriate one to work with but this choice is by no means canonical. In fact, the boundary factors of $\hat{C}_{2,0}$ of the type $C_\bullet \simeq C_\bullet^n$ corresponding to collapsing points and to the exploded points never intersect in the compactification $\hat{C}_{2,0}$ so that for collapsing strata one can choose an ordinary smooth structure on $C_\bullet^n$ (see Remark 2.3.1) while for exploded strata one can choose the smooth structure with corners introduced in §2.3. With such a choice the space $\hat{C}_{2,0}$, for example, becomes identical as a manifold with corners to the Kontsevich eye [Ko].

### 3.5. Angle functions on $\hat{C}_{2,0}$

The space $\hat{C}_{2,0}$ is the closure of an embedding,

$$C_{2,0} \longrightarrow C_2^h \times [0, +\infty]$$

and hence is diffeomorphic to the following manifold with corners (see, however, Remark 3.4.1 above),

whose inner topological circle describes the first boundary component, $C_{1,0} \times C_2^h$ (two point moving very close to each other), in the face decomposition

$$\partial \hat{C}_{2,0} = C_{1,0} \times C_2 \bigcup C_2 \times C_{1,0} \times C_{1,0},$$

while the outer topological circle describes the second boundary component (two points moving very far — in the Euclidean or Poincaré metric — from each other). Therefore, $\hat{C}_{2,0}$ is homeomorphic to the Kontsevich’s eye $\overline{C}_{2,0}$. It also has a natural structure of a semialgebraic manifold so we can work both with smooth and $PA$ (see [KS] and [HIV]) differential forms on $\hat{C}_{2,0}$. In fact, taking into account the natural action of the semigroup $\mathbb{R}^+$ on the big cell, $C_{2,0}$, of $\hat{C}_{2,0}$, the
latter space is better visualized geometrically as a closed infinitely long cylinder,

![Image of a closed infinitely long cylinder]

rather than a topological disk.

3.5.1. Definition. An angle function on \( \hat{C}_{2,0} \) is a smooth (or PA) map,

\[
\phi : \hat{C}_{2,0} \longrightarrow \mathbb{R}/2\pi\mathbb{Z}
\]

such that the boundary restrictions, \( d\phi \mid_{\text{outer circle}} \) and \( d\phi \mid_{\text{inner circle}} \), represent normalized cohomology classes in \( H^1(S^1) \), i.e.

\[
\int_{\text{outer circle}} d\phi = 2\pi \quad \text{and} \quad \int_{\text{inner circle}} d\phi = 2\pi.
\]

The associated differential form, \( d\phi \), is called a propagator on \( \hat{C}_{2,0} \).

3.5.2. Example: Kontsevich’s propagator. It is not hard to check that the function

\[
\phi_h : C_{2,0} \rightarrow \mathbb{R}/2\pi\mathbb{Z}
\]

\[
p = (z_1, z_2) \rightarrow \phi_h(p) := \text{Arg} \frac{z_2 - z_1}{z_2 - z_1}
\]

extends to an angle function on \( \hat{C}_{2,0} \) whose restriction to the inner circle gives the angle measured in the anti-clockwise direction from the vertical line, and which maps the outer circle to \( \mathbb{R}/2\pi\mathbb{Z} \) while taking its lower lid into the point 0. The associated propagator is denoted by \( \omega_K \),

\[
\omega_K(z_1, z_2) := d\phi_h(z_1, z_2).
\]

The geometric meaning of \( \phi_h \) is explained by the picture,

![Image of geometric meaning of \( \phi_h \)]

It measures (in the anticlockwise direction) the angle between two hyperbolic geodesics passing through \( z_i \), the first one in the vertical direction (towards the point \( \infty \)) and the second one towards \( z_j \).

3.5.3. Example: Kontsevich’s antipropagator. The function

\[
\phi_h^{\text{op}} : C_{2,0} \rightarrow \mathbb{R}/2\pi\mathbb{Z}
\]

\[
p = (z_1, z_2) \rightarrow \phi_h(p) := \text{Arg} \frac{z_1 - z_2}{z_1 - z_2}
\]

extends to an angle function on \( \hat{C}_{2,0} \) whose restriction to the inner circle gives the angle measured in the anti-clockwise direction from the vertical line, and which maps the outer circle to \( \mathbb{R}/2\pi\mathbb{Z} \) while taking its upper lid into the point 0. The associated propagator is denoted by \( \omega_K^{\text{op}} \),

\[
\omega_K^{\text{op}}(z_1, z_2) := \omega_K(z_2, z_1).
\]
3.5.4. Example: symmetrized Kontsevich’s propagator. The function
\[ C_{2,0} \longrightarrow \mathbb{R}/2\pi \mathbb{Z} \]
\[ (z_1, z_2) \longrightarrow \text{Arg}(z_1 - z_2) \]
extends to an angle function on \( \hat{C}_{2,0} \) whose restriction to the outer circle equals the angle measurers anticlockwise from the horizontal line. It is not hard to check that
\[ d\text{Arg}(z_1 - z_2) = \frac{1}{2} (\omega_K(z_1, z_2) + \omega_K(z_2, z_1)) \]
The associated propagator is denoted by \( \omega_{K}^{\text{sym}} \).

3.5.5. Example: a one-parameter family of propagators. For any \( t \in [0, 1] \) the function
\[ \phi_h^t : C_{2,0} \longrightarrow \mathbb{R}/2\pi \mathbb{Z} \]
\[ p = (z_1, z_2) \longrightarrow t\phi_h(z_1, z_2) + (1-t)\phi_h(z_2, z_1) \]
extends to an angle function on \( \hat{C}_{2,0} \). The associated propagator is denoted by \( \omega_K^t \),
\[ \omega_K^t = t\omega_K + (1-t)\omega_{K}. \]

3.5.6. Lemma For any \( t \in [0, 1] \) one has
\[ \omega_K^t(z_1, z_2)|_{\text{inner circle}} = d\text{Arg}(z_1 - z_2). \]

Proof. Setting \( z_1 - z_2 = re^{i\text{Arg}(z_1 - z_2)} \) one easily finds,
\[ \lim_{r \to 0} \omega_K(z_1, z_2) = \lim_{r \to 0} \omega_{K}(z_1, z_2) = d\text{Arg}(z_1 - z_2). \]

\[ \square \]

3.5.7. A special function on \( \hat{C}_{2,0} \). Note that the closed PA-differential form \( \omega_K - \omega_{K} \) on \( \hat{C}_{2,0} \) satisfies obviously the conditions,
\[ \int_{S^1} (\omega_K - \omega_{K})|_{\text{inner circle}} = \int_{S^1} (\omega_K - \omega_{K})|_{\text{outer circle}} = 0, \]
and hence defines a trivial cohomology class on \( \hat{C}_{2,0} \). Therefore, there exists a smooth (or PA) function, \( \Phi \), on \( \hat{C}_{2,0} \) (defined up to addition of a constant) such that
\[ \omega_K = \omega_{K} + d\Phi. \]
This fact will imply (see below) that exotic automorphisms of the Schouten algebra of polyvector fields defined by any propagator from the one-parameter family \( \{t\omega_K + (1-t)\omega_{K}\} \) are homotopy equivalent to each other as \( \text{Lie}_\infty \)-maps.

By Lemma 3.5.6,
\[ d\Phi|_{\text{inner circle}} = 0. \]
so that there exists a unique choice of \( \Phi \) satisfying the condition,
\[ \Phi|_{\text{inner circle}} = 0. \]
However, the PA-function on \( S^1 \) defined by
\[ \Phi_{\text{out}} := \Phi|_{\text{outer circle}} \]
is non-trivial (and its symbol is chosen to reflect somehow the opposite angle-like behavior of \( \omega_K \) and \( \omega_{K} \) on the outer circle). The constructed function \( \Phi_{\text{out}} : S^1 \to \mathbb{R} \) can be visualized as a natural continuous lift,
of the following $PA$-map of circles,

$$
\phi'_{\text{out}} : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}/2\pi\mathbb{Z},
\phi \rightarrow \phi'_{\text{out}}(\phi) := \begin{cases} 
2\phi & \text{for } \phi \in [0,\pi] \\
-2\phi & \text{for } \phi \in [\pi,2\pi].
\end{cases}
$$

3.6. Propagators on $\hat{C}_{n,0}$. Note that for any pair of integers $i,j \in [n], i \neq j$, there is an associated forgetting map,

$$
p_{ij} : C_{n,0} \rightarrow C_{2,0},
(z_1,\ldots,z_n) \rightarrow (z_i,z_j),
$$

which extends to a smooth map of their compactifications,

$$
p_{ij} : \hat{C}_{n,0} \rightarrow \hat{C}_{2,0}.
$$

Hence, for any propagator $\omega$ on $\hat{C}_{2,0}$ the pull-back $p^*_ij(\omega)$ is a well-defined one-form on $\hat{C}_{n,0}$.

We shall need below a “renormalized” version of the forgetting map,

$$
p_{ij} : C_{n,0} \rightarrow C_{2,0},
p = (z_1,\ldots,z_n) \rightarrow p_{ij}(p) := \varepsilon \cdot (z_i,z_j),
$$

where $\varepsilon$ is the Euclidean diameter of the standard hyperbolic position of the configuration $(z_1,\ldots,z_n)$ and $\varepsilon$ is the magnification map defined in (7). For any any propagator $\omega$ on $\hat{C}_{2,0}$ the pull-back $p^*_ij(\omega)$ is a well-defined (semialgebraic) one-form on $\hat{C}_{n,0}$. It is clear that for any $i$ and $j$ one has

$$
\lim_{\varepsilon \to \infty} p^*_ij(\omega) = q^*_ij(\omega|_{\text{outer circle}}),
$$

where $q_{ij} : C_{n,0} \rightarrow C_2$ is the natural forgetful map.

4. De Rham field theories on configuration spaces

4.1. Families of graphs. Let $\mathcal{G}_{n,l}$ stand for a family of graphs, $\{\Gamma\}$, with $n$ vertices and $l$ edges such that

- the edges of $\Gamma$ are directed, beginning and ending at different vertices;
- the set of vertices, $V(\Gamma)$, is labeled by the set $[n]$;
- the set of edges, $E(\Gamma)$, is totally ordered.

We identify two total orderings on the set $E(\Gamma)$ (that is, isomorphisms $E(\Gamma) \simeq [\#E(\Gamma)]$), if they differ by an even permutation of $[\#E(\Gamma)]$. Thus there are precisely two possible orderings on the set $E(\Gamma)$ and the group $\mathbb{Z}_2$ acts freely on $\mathcal{G}_{n,l}$ by ordering changes; its orbit space, $(\Gamma,\Gamma_{\text{opp}})$, is denoted by $\mathfrak{S}_{n,l}$.

With every graph $\Gamma \in \mathcal{G}_{n,l}$ one can associate a linear map,

$$
\Phi_\Gamma : \otimes^n T_{\text{poly}}(\mathbb{R}^d) \rightarrow T_{\text{poly}}(\mathbb{R}^d)[-l],
\gamma_1 \otimes \cdots \otimes \gamma_n \rightarrow \Phi_\Gamma(\gamma_1,\ldots,\gamma_n)
$$

where

$$
\Phi_\Gamma(\gamma_1,\ldots,\gamma_n) = \left[ \prod_{e \in E(\Gamma)} \Delta_e \right] \gamma_1(\psi_{(1)},x_{(1)}) \cdots \gamma_n(\psi_{(n)},x_{(n)})
$$

and, for an edge $e$ beginning at a vertex labelled by $i$ and ending at a vertex labelled by $j$,

$$
\Delta_e := \sum_{a=1}^d \left( \frac{\partial^2}{\partial x^a_{(j)} \partial \psi_{(i)}^a} \right)
$$

It is useful sometimes to identify an orientation of $\Gamma \in \mathcal{G}_{n,l}$ with a vector $O_\Gamma := \wedge_\epsilon \in E(\Gamma) e$ in the real one dimensional vector space $\wedge \mathbb{R}[E(\Gamma)]$, where $\mathbb{R}[E(\Gamma)]$ is the $l$-dimensional vector space spanned over $\mathbb{R}$ by the set $E(\Gamma)$.
4.1.1. Complete subgraphs. For any subset $A \subset [n]$ and any graph $\Gamma$ in $G_{n,l}$ (or in $G_{n,l}$), there is an associated complete subgraph $\Gamma_A$ of $\Gamma$ whose vertices are, by definition, those vertices of $\Gamma$ which are labelled by elements of $A$, and whose edges are all the edges of $\Gamma$ which connect these $A$-labelled vertices. If we shrink all the $A$-labelled vertices of $\Gamma$ (together with all the edges connecting these $A$-labelled vertices) into a single vertex, then we obtain from $\Gamma$ a new graph which we denote by $\Gamma/\Gamma_A$.

Similarly, for any family of disjoint subsets $A_1, \ldots, A_k$ of $[n]$ and any graph $\Gamma$ in $G_{n,l}$ (or in $G_{n,l}$) one can define complete subgraphs $\Gamma_{A_1}, \ldots, \Gamma_{A_k}$ in $\Gamma$ as well as the quotient graph $\Gamma/\{\Gamma_{A_1}, \ldots, \Gamma_{A_k}\}$.

4.1.2. Lemma. Let $A$ be a (naturally ordered) proper subset of $[n]$, $\Gamma_1 \in G_{n-\#A+1,l_1}$ and $\Gamma_2 \in G_{\#A,l_2}$. Then

$$\Phi_{\Gamma_1}(\gamma_1, \ldots, \gamma_{n-\inf A-1}) \Phi_{\Gamma_2}(\gamma_A), \gamma_{[n-\inf A+1]\setminus A}) = \sum_{\Gamma \in G_{n,l_1+l_2}(A,\Gamma_{1,2})} \Phi_{\Gamma}(\gamma_1, \ldots, \gamma_n),$$

where $G_{n,l_1+l_2}(A,\Gamma_{1,2})$ is a subset of $G_{n,l_1+l_2}$ consisting of all those graphs $\Gamma$ whose complete subgraph $\Gamma_A$ is isomorphic to $\Gamma_2$ and the quotient graph $\Gamma/\Gamma_A$ is isomorphic to $\Gamma_1$.

Proof uses only definition (15) and the Leibniz rule for partial derivatives. The above formula holds literally true under the assumption that all elements $\gamma_i$ are evenly graded (this is the only interesting case for us in this paper as MC elements of the Schouten algebra have grading 2 in our conventions); in general, there is a standard Koszul sign on the r.h.s.

4.1.3. Definitions. (i) Given a graph $\Gamma \in G_{n,2n-4}$. A subset $A \subset Vert(\Gamma) \simeq [n]$ is called admissible if $2 \leq \#A \leq n - 1$ and the associated subgraph $\Gamma_A$ belongs to $G_{\#A,2\#A-3}$. Note that in this case $\Gamma/\Gamma_A \in G_{n-\#A+1,2(\#A-1)}$.

(ii) Given a graph $\Gamma \in G_{n,2n-3}$. A subset $A \subset Vert(\Gamma) \simeq [n]$ is called admissible if the associated subgraph $\Gamma_A$ belongs to $G_{\#A,2\#A-3}$. Note that in this case $\Gamma/\Gamma_A \in G_{n-\#A+1,2(\#A-1)-2}$.

(iii) Given a graph $\Gamma \in G_{n,2n-3}$. A decomposition $Vert(\Gamma) = A_1 \cup \ldots \cup A_k$, $k \geq 2$, is called admissible if $\Gamma_{A_i} \in G_{\#A_i,2\#A_i-2}$, $i = 1, \ldots, k$. In this case $\Gamma/\{\Gamma_{A_1}, \ldots, \Gamma_{A_k}\} \in G_{k,2k-3}$.

Analogous definitions can be made for graphs from the sets $G_{n,2n-4}$ and $G_{n,2n-3}$.

4.2. De Rham field theory on $\mathcal{C}$. For any proper subset $A \subset [n]$ of cardinality at least two there is an associated embedding,

$$i_A : \mathcal{C}_{n-\#A+1} \times \mathcal{C}_{\#A} \to \mathcal{C}_n,$$

of the corresponding boundary component into $\mathcal{C}_n$. For example, for $A = \{3, 5, 6, 7\} \subset \{7\}$ the image of $i_A$ is a face of $\mathcal{C}_8$ represented by the following graph

```
     1
    /|
   / \
  2---3---4
 /       |
5        6
       /|
        7
```

Let $\Omega^*(\mathcal{C}_n) = \oplus_{p\geq 0} \Omega^p(\mathcal{C}_n)$ stand for the de Rham algebra of smooth (or $PA$) differential forms on the space $\mathcal{C}_n$. A de Rham field theory on the family of compactified configuration spaces $\{\mathcal{C}_n\}_{n \geq 2}$ is, by definition, a collections of maps,

$$\{ \Omega : G_{n,l} \to \Omega^l(\mathcal{C}_n) \}_{n \geq 2, l \geq 1},$$

such that $d\Omega = 0$, $\Omega_{\Gamma_{opp}} = -\Omega_{\Gamma}$, and, for any $\Gamma \in G_{n,2n-4}$ and any proper subset $A \subset Vert(\Gamma)$ with $\#A \geq 2$, one has

$$\tag{16} i_A^*(\Omega_{\Gamma}) \simeq (-1)^{\#A} \Omega_{\Gamma/\Gamma_A} \wedge \Omega_{\Gamma_A},$$

where the sign $(-1)^{\#A}$ is defined by the equality $\Omega_{\Gamma} = (-1)^{\#A} \Omega_{\Gamma/\Gamma_A} \wedge \Omega_{\Gamma_A}$ (see footnote 7). The symbol $\simeq$ means here and below equality modulo differential forms whose integral over the corresponding boundary component is zero.
4.2.1. Theorem. Given a de Rham field theory on $\overline{\mathcal{C}}$, then, for any $d \in \mathbb{N}$, there is an associated $\text{Lie}_{\infty}$-algebra structure,

$$
\mu_n : \otimes^n \mathcal{T}_{\text{poly}}(\mathbb{R}^d) \rightarrow \mathcal{T}_{\text{poly}}(\mathbb{R}^d)[3 - 2n],
$$

on $\mathcal{T}_{\text{poly}}(\mathbb{R}^d)$ given by

$$
\mu_n(\gamma_1, \ldots, \gamma_n) := \begin{cases} 
0 & \text{for } n = 1, \\
\sum_{\Gamma \in \mathfrak{S}_n, 2n - 3} c_\Gamma \Phi_\Gamma(\gamma_1, \ldots, \gamma_n) & \text{for } n \geq 2
\end{cases}
$$

with

$$
c_\Gamma := \int_{C_\Gamma} \Omega_\Gamma.
$$

Proof. By the Stokes theorem, for any $\Gamma \in \mathfrak{S}_n, 2n - 4$,

$$
0 = \int_{C_\Gamma} d\Omega_\Gamma = \int_{\partial C_\Gamma} \Omega_\Gamma = \sum_{A \in \mathfrak{S}_n} (-1)^{\#A} \int_{C_{\Gamma - A}} \Omega_{\Gamma - A} = \sum_{\Gamma \in \mathfrak{S}_n, 2n - 4} (\sum_{\Gamma \in \mathfrak{S}_n, 2n - 4} (-1)^{\#A} c_\Gamma \Phi_\Gamma(\gamma_1, \ldots, \gamma_n)).
$$

Then, using Lemma 4.1.2, one obtains

$$
\sum_{A \in \mathfrak{S}_n, 2n - 4} (-1)^{\#A} \sum_{\Gamma \in \mathfrak{S}_n, 2n - 4} (-1)^{\#A} c_\Gamma \Phi_\Gamma(\gamma_1, \ldots, \gamma_n)
$$

which proves the claim.

For any pair of different integers, $i, j \in [n]$, there is an associated map,

$$
\pi_{ij} : \overline{\mathcal{C}}_n \rightarrow \overline{\mathcal{C}}_2 = S^1
$$

which extends to the compactifications, $\pi_{ij} : \overline{\mathcal{C}}_n \rightarrow \overline{\mathcal{C}}_2$.

4.2.2. Proposition. For any differential 1-form, $\omega_0$, on the circle $C_2 = S^1$ satisfying the condition

$$
\int_{S^1} \omega_0 = 2\pi,
$$

the associated map

$$
\Omega : \mathfrak{G}_{n, d} \rightarrow \Omega(\overline{\mathcal{C}}_n),
$$

with $\pi_e := \pi_{ij}$ for an edge $e$ beginning at a vertex labelled by $i \in [n]$ and ending at a vertex labelled by $j \in [n]$, defines a non-trivial de Rham field theory on $\overline{\mathcal{C}}$ (and hence an associated non-trivial $\text{Lie}_{\infty}$-structure on $\mathcal{T}_{\text{poly}}(\mathbb{R}^d)$).
Proof. Basic condition (19) can be easily checked in the coordinate chart near the stratum \( \text{Im} i_A \) (see §2.2). The weights, \( c_\Gamma = \int_{C_n} \Omega_\Gamma \), are independent of the labelling maps, \( V(\Gamma) \to [n] \), so that the resulting \( \text{Lie}_{\infty} \) operations \( \mu_n \) are graded symmetric and give, therefore, a \( \text{Lie}_{\infty} \)-structure on \( T_{\text{poly}}(\mathbb{R}^d) \). Its non-triviality follows from a particular example studied in the next subsection. \( \Box \)

4.2.3. Example: Schouten brackets from a propagator. The first natural choice for a differential 1-form on \( C_2 \) satisfying condition (19) is, of course, the following one

\[
\omega_0(z_1, z_2) = d\text{Arg}(z_1 - z_2).
\]

By Kontsevich’s “vanishing” Lemma 6.4 in [Ko2], the associated weights \( c_\Gamma \) are zero for all graphs \( \Gamma \in \sqcup_{n \geq 2} G_{n, 2n-3} \) except for \( \Gamma_1 = \begin{array}{c} \bullet \\ 2 \end{array} \) and \( \Gamma_2 = \begin{array}{c} \bullet \\ 2 \end{array} \) which both have weight 1. Hence all \( \text{Lie}_{\infty} \)-operations (17) are zero for \( n \geq 3 \), and

\[
\mu_2(\gamma_1, \gamma_2) = c_{\Gamma_1} \Phi_{\Gamma_1}(\gamma_1, \gamma_2) + c_{\Gamma_2} \Phi_{\Gamma_2}(\gamma_1, \gamma_2) = (-1)^{\gamma_1} s^{[\gamma_1, \gamma_2]}.
\]

Thus the propagator \( d\text{Arg}(z_1 - z_2) \) on \( C_2 \) is responsible for the existence in nature of the Schouten bracket on polyvector fields.

4.2.4. From semialgebraic functions on \( S^1 \) to \( \text{Lie}_{\infty} \)-structures on \( T_{\text{poly}}(\mathbb{R}^d) \). An arbitrary differential 1-form \( \omega \) on \( C_2 \) satisfying normalization condition (19) is given by,

\[
\omega(z_1, z_2) = d\text{Arg}(z_1 - z_2) + df \left( \frac{z_i - z_j}{|z_i - z_j|} \right),
\]

for some \( PA \)-function \( f \) on \( S^1 \) (which we can always choose to satisfy an extra condition \( \int_{C_2} fd\text{Arg}(z_1 - z_2) = 0 \)). The \( \text{Lie}_{\infty} \) structure (17) associated to such a propagator has \( \mu_2 \) always equal to the Schouten bracket, but has in general higher homotopy compositions \( \mu_n \) non-trivial. For example, the \( \text{Lie}_{\infty} \) structure generated on \( T_{\text{poly}}(\mathbb{R}^d) \) by the propagator

\[
\omega_0(z_1, z_2) = d\text{Arg}(z_1 - z_2) + \frac{1}{2} df_{\text{out}} \left( \frac{z_i - z_j}{|z_i - z_j|} \right) = \omega_{\text{K}}|_{\text{outer circle}},
\]

where \( f_{\text{out}} \) is the \( PA \) function on \( S^1 \) defined in (11), has operation \( \mu_4 \) non-zero. For example, the weight of the graph

\[
\Gamma = \begin{array}{c} \bullet \\ 2 \end{array} \in G_{4,5}
\]

with respect to this propagator equals \( \frac{13}{12} \) so that the associated differential operator,

\[
c_{\Gamma} \Phi_{\Gamma} : \otimes^4 T_{\text{poly}}(\mathbb{R}^d) \to T_{\text{poly}}(\mathbb{R}^d)[-5],
\]

gives a non-zero contribution to \( \mu_4 \). It is easy to see that the \( \text{Lie}_{\infty} \)-structure on \( T_{\text{poly}}(\mathbb{R}^d) \) associated by Proposition 4.2.2 to the propagator \( \omega_{\text{K}}|_{\text{outer circle}} \) coincides precisely with the one constructed in [Sh2].

4.2.5. On homotopy equivalence of \( \text{Lie}_{\infty} \) algebras. Let \( \mathbb{R}[t, dt] \) stand for the polynomial de Rham algebra on \( \mathbb{R} \) and \( d \) for the de Rham differential. The tensor product \( T_{\text{poly}}(\mathbb{R}^d)[t, dt] := T_{\text{poly}}(\mathbb{R}^d) \otimes_{\mathbb{R}} \mathbb{R}[t, dt] \) is naturally a dg module over \( \mathbb{R}[t, dt] \). A \( \text{Lie}_{\infty} \)-structure, \( \mu_\bullet(t, dt) \), on \( T_{\text{poly}}(\mathbb{R}^d)[t, dt] \) such that

\[
\mu_1(t, dt) = d
\]

and all the higher operations, \( \mu_n(t, dt) \), are morphisms of \( \mathbb{R}[t, dt] \)-modules is uniquely determined by two families of operations,

\[
\left\{ \begin{array}{l}
\mu_n(t) : \otimes^n T_{\text{poly}}(\mathbb{R}^d) \to T_{\text{poly}}(\mathbb{R}^d)[3-2n] \otimes \mathbb{R}[t], \\
\mu_n(t) : \otimes^n T_{\text{poly}}(\mathbb{R}^d) \to T_{\text{poly}}(\mathbb{R}^d)[2-2n] \otimes \mathbb{R}[t]
\end{array} \right\}_{n \geq 2}
\]
such that \( \mu_n(t, dt) = \mu'_n(t) + dt \mu''_n(t) \) for \( n \geq 2 \). We call such a \( \text{Lie}_\infty \)-structure on \( T_{\text{poly}}(\mathbb{R}^d)[t, dt] \) a path one. Two minimal \( \text{Lie}_\infty \)-structures, say \( \mu_\bullet \) and \( \dot{\mu}_\bullet \), on \( T_{\text{poly}}(\mathbb{R}^d) \) are called gauge or homotopy equivalent [21] if there exists a path \( \text{Lie}_\infty \)-structure \( \mu_\bullet(t, dt) \) such that \( \mu_\bullet(t)|_{t=0} = \mu_\bullet \) and \( \mu_\bullet(t)|_{t=1} = \dot{\mu}_\bullet \).

4.2.6. Proposition. For any smooth (or \( PA \)) function \( f \) on \( S^1 \), the operators \( \mu'_n(t) := \sum_{\Gamma \in \mathfrak{S}_{n, 2n-3}} c'_\Gamma(t) \Phi_\Gamma \) and \( \mu''_n(t) := \sum_{\Gamma \in \mathfrak{S}_{n, 2n-2}} c''_\Gamma(t) \Phi_\Gamma \) with

\[
c'_\Gamma(t) := \int_{E_n} \bigwedge_{e \in E(\Gamma)} \frac{\pi^*_e(\omega_0 + tdf)}{2\pi}, \quad \text{and} \quad dt c'_\Gamma(t) := \int_{E_n} \bigwedge_{e \in E(\Gamma)} \frac{\pi^*_e(\omega_0 + tdf + f dt)}{2\pi},
\]

define a path \( \text{Lie}_\infty \)-structure,

\[
\mu'_n(t, dt) = \mu'_n(t) + dt \mu''_n(t), \quad n \geq 2,
\]

on \( T_{\text{poly}}(\mathbb{R}^d)[t, dt] \) such that \( \mu'_n(t)|_{t=0} \) equals the standard Schouten algebra structure on \( T_{\text{poly}}(\mathbb{R}^d) \) and \( \mu'_n(t)|_{t=1} \) equals the \( \text{Lie}_\infty \)-structure on \( T_{\text{poly}}(\mathbb{R}^d) \) associated by Proposition 4.2.2 to the propagator [21].

Proof. First we note that the definition of the weight \( c''_\Gamma(t) \) makes sense as it involves integration of an \((2n-2)\)-form over an \((2n-3)\)-dimensional manifold. In fact, one has,

\[
c''_\Gamma(t) := \sum_{e \in E(\Gamma)} (-1)^{|e|} \int_{E_n} \bigwedge_{e' \in E(\Gamma)} \frac{\pi^*_{e'}(f)}{2\pi} \wedge \frac{\pi^*_{e'}(\omega_0 + tdf)}{2\pi},
\]

where \(|e|\) counts the number of edges of \( \Gamma \) staying before the edge \( e \) in the chosen total ordering, \( o : E(\Gamma) \to \{\#E(\Gamma)\} \), of edges, i.e. \(|e| := o(e) - 1\).

It is obvious that the required conditions on the boundary values \( \mu'_\bullet(t)|_{t=0} \) and \( \mu_\bullet(t)|_{t=1} \) are satisfied.

Next it is a straightforward calculation (which is fully analogous to the one made on the proof of Theorem 4.2.1) to check that maps \([23]\) define a \( \text{Lie}_\infty \)-algebra structure on \( T_{\text{poly}}(\mathbb{R}^d) \otimes \mathbb{K}[t, dt] \) if and only if one has, for any \( \Gamma \in \mathcal{G}_{n, 2n-4} \),

\[
\sum_{A \in V(\Gamma)} (-1)^{\sigma \cdot A} c'_{\Gamma A}(t)c'_{\Gamma A}(t) = 0
\]

and, for any \( \Gamma \in \mathcal{G}_{n, 2n-3} \),

\[
dc'_\Gamma(t) = \sum_{A \in V(\Gamma) \text{ such that } \gamma_A \in \mathcal{S}_{2n, 2n-2}} (-1)^{\sigma \cdot A} c'_{\Gamma A}(t)c''_{\Gamma A}(t) + \sum_{A \in V(\Gamma) \text{ such that } \gamma_A \in \mathcal{S}_{2n, 2n-2}} (-1)^{\sigma \cdot A} c''_{\Gamma A}(t)c'_{\Gamma A}(t).
\]
The first condition is obvious (see again the proof of Theorem 4.2.1 above) while the second one follows from the following calculation,

\[
\frac{dc_t^e(t)}{dt} = \sum_{e \in E(t)} (-1)^{|e|} \left( \sum_{e' \in E(t) : e' 
eq e} \int_{\bar{c}_n} \frac{\pi_{e'}^*(df)}{2\pi} \bigwedge_{e'' \in E(t) : e'' 
eq e'} \frac{\pi_{e''}^*(\omega_0 + tdf)}{2\pi} \right) 
\]

\[= \sum_{e \in E(t)} (-1)^{|e|} \left( \sum_{e' \in E(t) : e' 
eq e} (-1)^{|e'|} \int_{\bar{c}_n} \frac{\pi_{e'}^*(\omega_0 + tdf)}{2\pi} \bigwedge_{e'' \in E(t) : e'' 
eq e'} \frac{\pi_{e''}^*(\omega_0 + tdf)}{2\pi} \right) 
\]

\[+ \sum_{\bar{C}_n \in \bar{E}_A} (-1)^{|e|} \int_{\bar{c}_n} \frac{\pi_{e'}^*(\omega_0 + tdf)}{2\pi} \bigwedge_{e'' \in E(t) : e'' 
eq e'} \frac{\pi_{e''}^*(\omega_0 + tdf)}{2\pi} \]

\[= \sum_{\bar{C}_n \in \bar{E}_A} (-1)^{|e|} c_t^e(t)c_t^{\bar{e}}(t) + \sum_{\bar{C}_n \in \bar{E}_A} (-1)^{|e|} c_t^{\bar{e}}(t)c_t^e(t).
\]

\[\square\]

4.2.7. De Rham field theory of path \(Lie_{\infty}\)-structures. One can generalize the notion of a de Rham field theory on \(\bar{C}\) to a map

\[\{\Omega(t, dt) : G_n \rightarrow \Omega^1(C_n \times \mathbb{R})\}_{n \geq 2, t \geq 1},\]

taking values in \(\Omega^1(C_n \times \mathbb{R}) := \Omega^1(C_n)[t, dt]\) and satisfying the same closeness and factorization condition \([16]\) as the map \(\Omega\) in the beginning of §4.2. An analogue of Theorem 4.2.1 claiming that every such a de Rham field theory \(\Omega(t, dt)\) defines a path \(Lie_{\infty}\)-algebra structure on \(T_{\text{poly}}(\mathbb{R}^d)[t, dt]\) holds true with the only difference that the summation in formula \([17]\) goes over graphs \(\Gamma \in \mathcal{G}_{n, 2n - 3} \cup \mathcal{G}_{n, 2n - 2}\); weights \([18]\) take now values in \(\mathbb{R}[t, dt]\) rather than in \(\mathbb{R}\). Proposition 4.2.6 gives us an explicit example of such a generalized de Rham field theory and of the corresponding path \(Lie_{\infty}\)-algebra associated with the propagator

\[
\omega^f(z_1, z_2) := d\text{Arg}(z_1 - z_2) + tdf \left( \frac{z_1 - z_j}{|z_i - z_j|} \right) + f \left( \frac{z_1 - z_i}{|z_i - z_j|} \right) dt \in \Omega^1(C_2 \times \mathbb{R}).
\]

4.2.8. Homotopy equivalence theorem. Let \(f\) be an arbitrary smooth (or PA) function on \(S^1\) and let \(\mu^f(t, dt)\) be the associated path \(Lie_{\infty}\) structure \([23]\) on \(T_{\text{poly}}(\mathbb{R}^d)\). There exists a morphism of \(Lie_{\infty}\)-algebras,

\[H(t, dt) : (T_{\text{poly}}(\mathbb{R}^d), [\cdot, \cdot]) \rightarrow (T_{\text{poly}}(\mathbb{R}^d)[t, dt], \mu^f(t, dt))\]

whose composition with the evaluation map at \(t = 0\),

\[\left( T_{\text{poly}}(\mathbb{R}^d)[t, dt], \mu^f(t, dt) \right) \xrightarrow{ev = 0} \left( T_{\text{poly}}(\mathbb{R}^d), [\cdot, \cdot] \right),\]

equals the identity map.

This theorem can be proven by a direct but very tedious inspection of the polynomial dependence of the weights \(c^e_t(t)\) on \(t\) along the lines of the proof of Proposition 4.2.6. We shall show below in §4.5 a short and more elegant proof (as well as explicit formulae for \(H(t, dt)\)) using the compactified configuration space \(C_{n, 0}\). Note that the composition of the morphism \(H(t, dt)\) with the evaluation morphism at \(t = 1\) gives a \(Lie_{\infty}\) morphism from \(T_{\text{poly}}(\mathbb{R}^d)\) equipped with the standard Lie algebra structure to itself but equipped with the \(Lie_{\infty}\)-structure associated to the propagator \([21]\).

4.2.9. Remark. Let \(Aut(T_{\text{poly}}(\mathbb{R}^d))\) be the group of \(Lie_{\infty}\) automorphisms of the Schouten algebra. Two elements, \(F_0, F_1 \in Aut(T_{\text{poly}}(\mathbb{R}^d))\) are called homotopy equivalent \([24]\) if there
exists a $\mathcal{L}ie_{\infty}$ morphism, $F(t, dt)$, from the Lie algebra $(T_{\text{poly}}(\mathbb{R}^d), [\cdot, \cdot])$ to the dg Lie algebra $(T_{\text{poly}}(\mathbb{R}^d)[t, dt], [\cdot, \cdot], d)$ such that the compositions of $F(t, dt)$ with the evaluations at $t = 0$ and, respectively, at $t = 1$ give $F_0$ and, respectively, $F_1$. An element $F = \{F_n\}_{n \geq 1} \in \text{Aut}(T_{\text{poly}}(\mathbb{R}^d))$ is called homotopy trivial if it is homotopy equivalent to the identity map, and exotic if it is homotopy non-trivial. If $F_1 = \text{Id}$, then the first non-zero higher composition, $F_{\text{min}} \geq 2$, of an automorphism $F$ gives a cycle in the Chevalley-Eilenberg complex $C_*^{CE}(T_{\text{poly}}(\mathbb{R}^d), T_{\text{poly}}(\mathbb{R}^d))$ of the Schouten algebra. It is easy to check that if the automorphism $F$ is homotopy trivial, then $F_{\text{min}}$ is a coboundary. Thus if $F_{\text{min}}$ gives a non-trivial cohomology class in $H^*(T_{\text{poly}}(\mathbb{R}^d), T_{\text{poly}}(\mathbb{R}^d))$, then the automorphism is exotic.

4.3. De Rham field theory on $\overline{C} \sqcup \hat{C} \sqcup \overline{C}$. For any proper subset $A \subset [n]$ of cardinality at least two and for any decomposition, $[n] = A_1 \sqcup \ldots \sqcup A_k$, of $[n]$ into disjoint non-empty subsets, there are associated embeddings,

\begin{equation}
(25) \quad j_A : \hat{C}_{n-\#A+1,0} \times \overline{C}_{\#A} \hookrightarrow \hat{C}_{n,0}, \quad j_{A_1, \ldots, A_k} : \overline{C}_k \times \hat{C}_{n-\#A_1+1,0} \times \ldots \times \hat{C}_{n-\#A_k+1,0} \hookrightarrow \hat{C}_{n,0},
\end{equation}

of the corresponding boundary components into $\hat{C}_{n,0}$ (see (9)).

A de Rham field theory on $\overline{C} \sqcup \hat{C} \sqcup \overline{C}$ is, by definition, a pair, $\Omega^\text{in}$ and $\Omega^\text{out}$, of de Rham field theories on $\overline{C}$ together with a family of maps,

\begin{equation}
\{ \Xi : G_{n,l} \rightarrow \Omega^l(\hat{C}_{n,0}) \}_{n \geq 2}
\end{equation}

such that $d\Xi = 0$, $\Xi_{\text{opp}} = -\Xi$, and, for any $\Gamma \in G_{n,2n-3}$, and any boundary embedding (25) one has

\begin{equation}
(26) \quad j_A^*(\Xi_\Gamma) \simeq (-1)^{\sigma_A} \Xi_\Gamma / \Gamma_A \wedge \Omega^{\text{in}}_{\Gamma A},
\end{equation}

\begin{equation}
(27) \quad j_{A_1, \ldots, A_k}^*(\Xi_\Gamma) \simeq (-1)^{\sigma_{A_1, \ldots, A_k}} \Omega^{\text{out}}_{\Gamma / (\Gamma_{A_1}, \ldots, \Gamma_{A_k})} \wedge \Xi_{\Gamma_{A_1}} \wedge \ldots \wedge \Xi_{\Gamma_{A_k}},
\end{equation}

where the sign $(-1)^{\sigma_{A_1, \ldots, A_k}}$ is defined by the equality

\begin{equation}
\sigma_{\Gamma} = (-1)^{\sigma_{A_1, \ldots, A_k}} \sigma_{\Gamma / (\Gamma_{A_1}, \ldots, \Gamma_{A_k})} \wedge \sigma_{\Gamma_{A_1}} \wedge \ldots \wedge \sigma_{\Gamma_{A_k}},
\end{equation}

i.e. it is given just by a rearrangement of the wedge product of edges of $\Gamma$.

4.3.1. Theorem. Given a de Rham field theory, $(\Omega^\text{in}, \Omega^\text{out}, \Xi)$, on $\overline{C} \sqcup \hat{C} \sqcup \overline{C}$, then, for any $d \in \mathbb{N}$, there are associated

(i) two $\mathcal{L}ie_{\infty}$-algebra structures, $\mu^\text{in}$ and $\mu^\text{out}$, on $T_{\text{poly}}(\mathbb{R}^d)$ given by formulae (17)-(18) for $\Omega = \Omega^\text{in}$ and, respectively, $\Omega = \Omega^\text{out}$, and

(ii) a $\mathcal{L}ie_{\infty}$ morphism,

\begin{equation}
(28) \quad F_n^{\text{Leib}}(\gamma_1, \ldots, \gamma_n) := \begin{cases} \text{Id} & \text{for } n = 1, \\ \sum_{\gamma \in G_{n,2n-2}} C_\gamma \Phi_\gamma(\gamma_1, \ldots, \gamma_n) & \text{for } n \geq 2 \end{cases},
\end{equation}

with

\begin{equation}
(29) \quad C_\gamma := \int_{\hat{C}_{n,0}} \Xi_\gamma.
\end{equation}
where, as before,

\[ \pi(31) \]

and except the two ones which are the boundary vertices of the edge 4.4.1. Theorem

Equation (26) is equivalent to the following one,

\[ \Omega(\Gamma) = \sum_{c \in E(\Gamma)} \frac{\pi^*(\omega_{\text{in}})}{2\pi} = \sum_{c \in E(\Gamma)} \frac{\pi^*(\omega_{\text{out}})}{2\pi} \]

Proof is completely analogous to the proof of Theorem 4.2.1 above. The required claim follows immediately from the definitions and the Stokes theorem,

\[ 0 = \int_{\mathring{C}_{n,0}} d\Xi = \int_{\partial \mathring{C}_{n,0}} \Xi = \]

\[ = - \sum_{A \subseteq [n]} (-1)^{\sigma_A} \int_{C_{n,A}} \frac{\Omega_{\Gamma,\Lambda}}{\Lambda} + \sum_{2 \leq k \leq n} (-1)^{\sigma_{A_{1} \ldots A_{k}}} \int_{C_{k}} \frac{\Omega_{\Gamma,\Lambda}}{\Lambda} \int_{C_{\Lambda}} \Xi_{\Gamma_{A_{1} \ldots A_{k}}} \]

\[ = - \sum_{A \subseteq V(\Gamma)} (-1)^{\sigma_{A}} \int_{C_{\Lambda}} \Xi_{\Gamma_{A_{0}}} + \sum_{k=2}^{n} \sum_{V(\Gamma) = A_{1} \cup \ldots \cup A_{k}} (-1)^{\sigma_{A_{1} \ldots A_{k}}} \int_{C_{\Lambda}} \Xi_{\Gamma_{A_{1} \ldots A_{k}}} \]

\[ \hat{4.4}. \text{ De Rham field theories from angular functions on } \mathring{C}_{2,0}. \text{ Let } \phi \text{ be an angular function on } \mathring{C}_{2,0}, \omega(z_{1}, z_{2}) = d\phi(z_{1}, z_{2}) \text{ the associated propagator on } \mathring{C}_{2,0}, \text{ and let } \]

\[ \omega_{\text{in}} := \omega|_{\text{inner circle}} \quad \text{and} \quad \omega_{\text{out}} := \omega|_{\text{outer circle}} \]

be the 1-forms on S1 obtained by restricting \( \omega \) to the inner and, respectively, outer circles of \( \mathring{C}_{2,0} \).

Define a series of maps,

\[ \Omega_{\Gamma}^{in} : \mathcal{G}_{n,l} \to \Omega_{\Gamma}(\mathring{C}_{n}) \quad \Omega_{\Gamma}^{out} : \mathcal{G}_{n,l} \to \Omega_{\Gamma}(\mathring{C}_{n}) \]

\[ \Omega(\Gamma) := \sum_{c \in E(\Gamma)} \frac{\pi^*(\omega_{\text{in}})}{2\pi} \quad \Omega(\Gamma) := \sum_{c \in E(\Gamma)} \frac{\pi^*(\omega_{\text{out}})}{2\pi} \]

and

\[ \Xi : \mathcal{G}_{n,l} \to \Omega_{\Gamma}(\mathring{C}_{n,0}) \]

\[ \Xi(\Gamma) := \sum_{c \in E(\Gamma)} \frac{\pi^*(\omega)}{2\pi} \]

where, as before, \( \pi_{e} : C_{n} \to C_{2} \) is the map which forgets all the points in the configurations except the two ones which are the boundary vertices of the edge \( e \), and \( p_{e} : C_{n,0} \to C_{2,0} \) is a “renormalized” version (see [130]) of the forgetful map \( p_{e} : C_{n,0} \to C_{2,0} \).

4.4.1. Theorem. For any angular function on \( \mathring{C}_{2,0} \) the associated data \([50], [71]\) define a de Rham field theory on \( \mathring{C} \cup \mathring{C} \).

Proof. Equation (26) is equivalent to the following one,

\[ \int_{\mathring{C}_{n-\#A_{1}+1,0} \times \mathring{C}_{\#A} \times \mathring{C}_{n-\#A_{1}+1,0}} \frac{j_{A}^{\star}(\Xi_{\Gamma})}{2\pi} = (-1)^{\sigma_{A}} \int_{\mathring{C}_{n-\#A_{1}+1,0} \times \mathring{C}_{\#A}} \frac{\Xi_{\Gamma}}{2\pi} = \frac{\Omega_{\Gamma}^{in}}{2\pi} \]

By definition of the boundary strata \( \mathring{C}_{n-\#A_{1}+1,0} \times \mathring{C}_{\#A} \hookrightarrow \mathring{C}_{n,0} \), both sides of the above equation are zero unless \( \Gamma_{A} \) is an admissible subgraph of \( \Gamma \) in which case the equality is obvious when one uses local coordinates defined in §3.

Consider next an arbitrary strata of the form,

\[ j_{A_{1}, \ldots, A_{k}} : C_{k} \times C_{n-\#A_{1}+1,0} \times \ldots \times C_{n-\#A_{k}+1,0} \hookrightarrow \mathring{C}_{n,0} \]

By definition, this is a subset of \( \mathring{C}_{n,0} \) obtained in the limit \( \varepsilon \to \infty \) from a class of configurations in \( C_{n,0} \) determined by the data:

(a) a configuration \( p = (z_{1}, \ldots, z_{k}) \in C_{n}(i, \varepsilon) \),

(b) a collection of configurations, \( p_{1} \in C_{n-\#A_{1}+1,0}, \ldots, p_{k} \in C_{n-\#A_{k}+1,0}, 1 \leq i \leq k \), placed, respectively, at the positions \( z_{1}, \ldots, z_{k} \) in \( \mathbb{H} \).
Note that all the configurations \{p_j\}_{j \in [k]} are shrunk by the factor of \(\varepsilon\) when they are placed at the respective positions so that in the limit \(\varepsilon \to \infty\) they are eventually shrunk to the points \(\lim_{\varepsilon \to \infty}(z_1, \ldots, z_k) \in C_{n}(i, \infty)\). Therefore, the differential form

\[
j^*_{A_1, \ldots, A_k}(\Xi_\Gamma) = (-1)^{\sigma_A, \ldots, A_k} \bigwedge_{e \in E(\Gamma/(\Gamma_{A_1} \ldots \Gamma_{A_k}))} \lim_{\varepsilon \to \infty} \frac{p^*_\varepsilon(\omega)}{2\pi} \prod_{i=1}^{k} \lim_{\varepsilon \to \infty} \frac{p^*_\varepsilon(\omega)}{2\pi}
\]

vanishes unless the decomposition \([n] = A_1 \cup \ldots \cup A_k\) is admissible. Let us now assume that the decomposition is admissible. It follows from (14) that

\[
\bigwedge_{e \in E(\Gamma/(\Gamma_{A_1} \ldots \Gamma_{A_k}))} \lim_{\varepsilon \to \infty} \frac{p^*_\varepsilon(\omega)}{2\pi} = \Omega^{\text{out}}_{\Gamma/(\Gamma_{A_1} \ldots \Gamma_{A_k})}
\]

Note in this connection that although the configuration \((z_1, \ldots, z_k) \in C_{n}(i, \infty)\) has infinite size, it is not true that all the points in such a limit configuration are necessarily at infinite distance from each other. Hence the fact that we used the renormalized forgetful map \(p_{\varepsilon}\) is crucial for the above equality to hold. It is also clear that

\[
\bigwedge_{e \in E(\Gamma_{A_i})} \lim_{\varepsilon \to \infty} \frac{p^*_\varepsilon(\omega)}{2\pi} = \Xi_{\Gamma_{A_i}}.
\]

Indeed, the two rescaling effects — the first one associated with shrinking the configuration of points labelled by \(A_i\) by the factor \(\varepsilon\) and the second one associated via \(p_{\varepsilon}\) with “expanding” the propagator by the same factor \(\varepsilon\) — cancel each other so that we are left with the differential form \(\Xi_{\Gamma_{A_i}}\). This proves equality (27) and hence completes proof of the Theorem.

\[\square\]

\subsection*{4.4.2. Corollary.} For any angular function \(\phi\) on \(\hat{C}_{2,0}\) there is an associated \(\text{Lie}_{\infty}\) automorphism of the Schouten algebra of polyvector fields.

\[\text{Proof.}\] As weights (29) are \(S_n\)-invariant, Proposition 4.2.2 implies that the map \(F\) given by formula (28) describes a \(\text{Lie}_{\infty}\) morphism between the \(\text{Lie}_{\infty}\) structures \(\mu^{in}_\bullet, \mu^{out}_\bullet\) corresponding to the 1-forms \(\omega^{in}\) and \(\omega^{out}\) respectively. By Theorem 4.2.8, both these \(\text{Lie}_{\infty}\) structures are homotopy equivalent to the Schouten Lie algebra via certain homotopy equivalence maps, \(H_{\text{in}}\) and \(H_{\text{out}}\) (see the next section §4.5 for their explicit formulae). Then the composition

\[\mathcal{F} : (\mathcal{T}_{\text{poly}}(\mathbb{R}^d), [\bullet]) \xrightarrow{H_{\text{in}}} (\mathcal{T}_{\text{poly}}(\mathbb{R}^d), \mu^{in}_\bullet) \xrightarrow{F} (\mathcal{T}_{\text{poly}}(\mathbb{R}^d), \mu^{out}_\bullet) \xrightarrow{H_{\text{out}}^{-1}} (\mathcal{T}_{\text{poly}}(\mathbb{R}^d), [\bullet])\]

is an automorphism of the Schouten algebra.

\[\square\]

\subsection*{4.4.3. One parameter family of de Rham field theories.} In fact, any angular function on \(\hat{C}_{2,0}\) defines a one-parameter family, \((\Omega^{in}, \Omega^{out}, \Xi(s)), s > 0,\) of de Rham field theories on \(\bar{C}_* \cup \bar{C}_{*,0} \cup \bar{C}_*\) where the map \(\Xi(s)\) is given by

\[
\Xi(s) : G_{n,l} \rightarrow \Omega^l(\hat{C}_{0,n,0})
\]

\[
\Gamma \rightarrow \Xi_\Gamma(s) := \bigwedge_{e \in E(\Gamma)} \frac{p_{e}(s)^*(\omega)}{2\pi}
\]

Here, for an edge \(e \in V(\Gamma)\) beginning at a vertex labelled by \(i\) and ending at a vertex labelled by \(j\), one sets

\[
p_{e}(s) : C_{n,0} \rightarrow C_{2,0}
\]

\[
p = (z_1, \ldots, z_n) \rightarrow p_{e}(s)(p) := \varepsilon^s \cdot (z_i, z_j),
\]

with \(\varepsilon\) being the Euclidean diameter of the standard hyperbolic position of the configuration \((z_1, \ldots, z_n)\) (cf. (13)). For any any propagator \(\omega\) on \(\hat{C}_{2,0}\) the pull-back \(p_{e}(s)^*(\omega)\) is a well-defined one-form on \(\hat{C}_{n,0}\) so that for arbitrary graph \(\Gamma \in \mathcal{S}_{n,2n-2}\) its weight

\[
C_\Gamma(s) := \int_{\hat{C}_{n,0}} \Xi_\Gamma(s)
\]
is a well-defined continuous function of \( s \in (0, +\infty) \) as the form \( \Xi_{\Gamma}(s) \) extends, for any \( s > 0 \), to a well-defined form on the compactification \( \hat{C}_{n,0} \). Moreover, this function extends to a continuous function on \([0, +\infty)\) as for \( s = 0 \) the map \( p_\varepsilon(s) \) becomes the ordinary forgetful map \( C_{n,0} \to C_{2,0} \) and the form \( \Xi_{\Gamma}(s) \big|_{s=0} \) extends to the Kontsevich compactification, \( \overline{C}_{n,0} \), of \( C_{n,0} \). We shall use this observation below when estimating (homotopy) non-triviality of our exotic \( \mathcal{L}ie_{\infty}(\text{auto}) \) morphisms: it is often much easier to compute a weight \( C_{\Gamma}(s) \) for \( s = 0 \) rather than for generic non-negative \( s \), and if \( C_{\Gamma}(0) \neq 0 \) for some propagator \( \Phi \) on \( \hat{C}_{2,0} \simeq \overline{C}_{2,0} \), then, by continuity, we can be sure that \( C_{\Gamma}(s) \neq 0 \) at least for small positive values of the parameter \( s \).

### 4.5. Gauge equivalence propagators and a proof of the homotopy equivalence theorem.

A function,

\[
I : \quad C_{2,0} \quad \longrightarrow \quad \mathbb{R}^2
\]

\[
I(z_1, z_2) \quad \longrightarrow \quad I(z_1, z_2) := \tanh b(z_1, z_2)
\]

where \( b(z_1, z_2) \) is the hyperbolic distance between the points \( z_1 \) and \( z_2 \), extends to a smooth function on its compactification, \( I : \hat{C}_{2,0} \to [0, 1] \), which takes values 0 at the inner circle and value 1 at the outer one. For an arbitrary piecewise smooth function \( f \) on \( S^1 \) we consider a propagator

\[
\omega(z_1, z_2, t, dt) = d\text{Arg}(z_1 - z_2) + td\left( I(z_1, z_2) f\left( \frac{z_i - z_j}{|z_i - z_j|} \right) \right) + I(z_1, z_2) f\left( \frac{z_i - z_j}{|z_i - z_j|} \right) dt \in \Omega^1(\hat{C}_{2,0} \times \mathbb{R})
\]

which satisfies the boundary conditions,

\[
\omega(z_1, z_2, t, dt)|_{\text{inner circle}} = d\text{Arg}(z_1 - z_2), \quad \text{and} \quad \omega(z_1, z_2, t, dt)|_{\text{outer circle}} = (24).
\]

Hence formulae (23) with summation \( \sum_{\Gamma \in \mathcal{G}_{n,2n-2}} \) extended to \( \sum_{\Gamma \in \mathcal{G}_{n,2n-2} \sqcup \mathcal{G}_{n,2n-1}} \) give us a \( \mathcal{L}ie_{\infty} \) morphism

\[
H(t, dt) : (T_{\text{poly}}(\mathbb{R}^d), [\bullet, \bullet]) \longrightarrow (T_{\text{poly}}(\mathbb{R}^d)[t, dt], \mu^t(t, dt))
\]

which obviously has the property stated at the end of Theorem 4.2.8.

The map \( H^\text{out} \) used in the proof of Corollary 4.4.2 is equal to \( H(t, dt)|_{t=1} \). The map \( H^\text{in} \) is constructed similarly.

### 4.6. Exotic transformations of Poisson structures.

Any \( \mathcal{L}ie_{\infty} \)-automorphism \( F \) of the algebra \( T_{\text{poly}}(\mathbb{R}^d)[[\hbar]] \) acts on its set of Maurer-Cartan elements,

\[
\gamma \to F(\gamma) = \sum_{n \geq 1} \frac{\hbar^{n-1}}{n!} F_n(\gamma, \ldots, \gamma).
\]

If \( F \) is a \( \mathcal{L}ie_{\infty} \)-automorphism given by a de Rham field theory on \( \hat{C}_\bullet \sqcup \hat{C}_{\ast,0} \sqcup \overline{C}_\bullet \), then

\[
F(\gamma) := \gamma + \sum_{n \geq 2} \frac{\hbar^{n-1}}{n!} \sum_{\Gamma \in \mathcal{G}_{n,2n-2}} \frac{C_\Gamma(s)}{\# Aut(\Gamma)} \Phi_{\Gamma}(\gamma^{\otimes n})
\]

where \( \# Aut(\Gamma) \) is the cardinality of the group of automorphisms of the graph \( \Gamma \), \( C_\Gamma(s) = \int_{C_{n,0}} \Xi_{\Gamma}(s) \) and \( \mathcal{G}_{n,2n-2} \) means the family of graphs \( \mathcal{G}_{n,2n-2} \) with labeling of vertices forgotten. In particular, \( F \) acts on the set of ordinary Poisson structures on \( \mathbb{R}^d \), that is, on the set of bivector fields, \( \gamma = \sum_{i,j} \gamma_{ij}(x) \psi_i \psi_j \) satisfying the equation \( [\gamma, \gamma] = 0 \). In this case only those graphs \( \Gamma \in \mathcal{G}_{n,2n-2} \) can give a non-trivial contribution to \( F(\gamma) \) which have at most two output edges at each vertex. The wheels,

\[
w_n = \begin{cases} n & n \geq 2, \\ n-1 & n = 1, \end{cases}
\]
and their unions do have this property and hence give a non-trivial contribution into $F(\gamma)$ provided their weights, $C_{w_n}(s)$, are non-zero with respect to a propagator $\omega$ and some value of $s \in (0, +\infty)$.

An easy calculation based on definition \[\text{(13)}\] gives (with the total ordering of $E(w_n)$ chosen to be \{(1, 2), (2, 3), \ldots, (n - 1, n), (1, n + 1), \ldots, (n, n + 1)\})

$$\Phi_{w_n}(\gamma \otimes n + 1) = \left(-1\right)^{n(n-1)/2} \frac{1}{2} \sum \frac{\partial^n \alpha_{ij}}{\partial x^{k_1} \cdots \partial x^{k_n}} \frac{\partial \gamma_{k_1 l_1}}{\partial x^{l_1}} \frac{\partial \gamma_{k_2 l_2}}{\partial x^{l_2}} \cdots \frac{\partial \gamma_{k_n l_n}}{\partial x^{l_n}} (\psi_i \psi_j).$$

As $\text{Aut}(w_n) = \mathbb{Z}/n\mathbb{Z}$, the contribution of graphs $w_n$ into $F$ is given by

$$F(\gamma) = \gamma + \sum_{n \geq 2} \frac{C_{w_n}(s)}{\#\text{Aut}(w_n)} \Phi_{w_n}(\gamma \otimes n + 1) + \ldots$$

\[\text{(34)}\]

We shall study such contributions to $F$ for several concrete propagators below.

4.7. Examples: Kontsevich’s symmetrized propagators. The symmetrized propagator (see §3.5.5),

$$\omega_{\text{sym}} = \frac{1}{2} (\omega_K(z_1, z_2) + \omega_K(z_2, z_1)) = \text{dArg}(z_1 - z_2),$$

on $\hat{C}_{2,0}$ restricts to its inner and outer boundary circles as $\text{dArg}(z_1 - z_2)$ and hence defines, by Example 4.2.3 and Theorem 4.3.1, an automorphism, $F_{\text{sym}}^K$, of the Schouten algebra $\mathcal{T}_{\text{pol}(\mathbb{R}^d)}$. This is, however, a trivial automorphism (i.e. the one with \[\text{(34)}\]) and hence defines, by Example 4.2.3 and Theorem 4.3.1, an automorphism, $F_{\text{sym}}^K$, of the Schouten algebra $\mathcal{T}_{\text{pol}(\mathbb{R}^d)}$. This is, however, a trivial automorphism (i.e. the one with $F_{n \geq 2} = 0$) as the differential $(2n - 2)$-forms $\prod_{i \in E(V)} p(l) \left(\omega_{\text{sym}}^K\right) = \prod_{i \in E(V)} p(l) \left(\omega_{\text{sym}}^K\right)$ are invariant under the action of the semigroup, $z \rightarrow z + n, \nu \in \mathbb{R}^+$, and hence vanish identically on $C_{0,0}$ for dimensional reasons.

In $\text{Ko3}$ Kontsevich introduced a new $\frac{\omega_{\text{sym}}}{2}$-propagator,

$$\omega_{1/2}^K(z_i, z_j) := \frac{1}{i} \text{dlog} \frac{z_i - z_j}{z_i - z_j},$$

and claimed that “all identities proven in $\text{Ko2}$ remain true”. In particular, all the integrals $\int_{C_{0,0}} \wedge_{i \in E(V)} p(l) \left(\omega\right), \Gamma \in \mathcal{G}_{0,2n-2}$, are finite. This is by no means an obvious claim as the differential forms $p(l) \left(\omega_{\text{sym}}^K\right), i, j \in [n]$, extend neither to the compactification $\hat{C}_{0,0}$ nor to $\hat{C}_{n,0}$. In fact such forms extend nicely to all boundary components of both compactifications except those of the form $C_{n-k+1,0} \times \mathbb{R}$ which describe a group of $k$ points moving too close to each other in $\mathbb{H}$ (and which are the only ones which are common to both compactifications for all $n$). We refer to $\text{AiTa}$ for a discussion of why the Kontsevich $\frac{1}{2}$-propagator works. As the symmetrized version of this propagator,

$$\omega_{\text{sym}}^{1/2}(z_i, z_j) := \frac{1}{2} \left(\omega_{1/2}^K(z_i, z_j) + \omega_{1/2}^K(z_j, z_i)\right),$$

restricts to the outer circle of $\hat{C}_{2,0}$ as $\text{dArg}(z_1 - z_2)$ and tends towards the inner circle as $\text{dArg}(z_1 - z_2) + \text{dln} \epsilon, \epsilon \rightarrow 0$, we infer from Example 4.2.3 and Theorem 4.3.1 that the associated universal map $F_{\text{sym}}^{1/2}$ given by formulæ $\text{(28)}$–$\text{(29)}$ gives an exotic automorphism of the Schouten algebra without any homotopy adjustments. The automorphism $F_{\text{sym}}^{1/2}$ is non-trivial (even homotopy non-trivial!) as its lowest in $n$ non-trivial component is given by the graph $w_3$ whose weight $C_{w_3}(0)$ is equal to $\frac{\zeta(3)}{(4\pi)^{4}}$ $\text{Gr2}$.

It is worth noting that for any symmetrized propagator the choice of arrows on a graph $\Gamma \in \mathcal{G}_{0,2n-2}$ does not affect its weight $C_{\Gamma}(s)$ (but does affect the associated operator $\Phi_{\Gamma}$). Such theories are better understood as de Rham field theories on braid configuration spaces, see §5 below.

4.8. Example: Kontsevich’s (anti)propagator. In the case of the Kontsevich propagator

$$\omega_K(z_1, z_2) = \text{dArg} \frac{z_1 - z_2}{z_1 - z_2}$$
or the antipropagator,

$$\omega_K(z_1, z_2) = d\text{Arg} \frac{z_2 - z_1}{z_2 - z_1}$$

formulae \([28], [29]\) define a \(\mathcal{L}\text{ie}_{\infty}\) morphism from the Schouten algebra to its \(\mathcal{L}\text{ie}_{\infty}\)-extension constructed by Shoikhet in \([Sh2]\) (see §4.2.4). Both such morphisms, \(F_K\) and, respectively, \(F_K^{-}\), must be highly non-trivial as they encode all the obstructions to existence of universal Kontsevich type formality morphism for \(\text{infinite-dimensional}\) Schouten algebras (non-existence of such a formality morphism, i.e. non-emptiness of the set of obstructions, was proven in \([Me2]\)). All the weights \(C_{w_n}(0)\) vanish in the case of the propagator \(\omega_K\) \([Ko2, Sh1]\) and, in the case of the antipropagator one has \(C_{w_{2n+1}}(0) = 0\) and, for even \(n\) and the total ordering of \(E(w_n)\) chosen to be \(\{(1, 2), (2, 3), \ldots, (n - 1, n), (1, n + 1), \ldots, (n, n + 1)\}\), one has \([VdB]\)

$$C_{w_n}(0) = (-1)^{n(n-1)/2} n B_n,$$

where \(B_n\) are modified Bernoulli numbers.\(^9\)

The associated automorphisms, \(H^{-1}_\omega \circ F_K\) and \(H^{-1}_\omega \circ F_K^{-}\), of the Schouten algebra are, perhaps, homotopy equivalent to each other and, moreover, are homotopy trivial. Therefore it is more interesting to consider \(\mathcal{L}\text{ie}_{\infty}\) morphisms, \(F_\omega^{(1)}\) and \(F_\omega^{(1)}\) associated with Kontsevich’s \(\frac{1}{2}\)- propagator \(\omega_\mathbb{H}^{(1)}\) and, respectively, antipropagator \(\omega_{\mathbb{R}}^{(1)}(z_1, z_2) := \omega_{\mathbb{H}}^{(1)}(z_2, z_1)\). The associated automorphisms, \(H^{-1}_\omega \circ F_\omega^{(1)}\) and \(H^{-1}_\omega \circ F_\omega^{(1)}\), must be homotopy non-trivial (at least for some \(s \in (0, +\infty)\)) as the weights \(C_{w_n}(0)\) with respect to \(\omega_\mathbb{H}^{(1)}\) are non-zero for all \(n\); it is proven in Appendix 1 that

$$C_{w_n}(0) = (-1)^{n(n-1)/2} \frac{\zeta(n)}{(2\pi i)^n},$$

where \(\zeta(n)\) is the Riemann zeta function.

### 4.9. De Rham field theory of Duflo’s strange automorphism

Let \(\omega\) be a propagator on \(\hat{C}_{2,0}\). Had we defined the map \(\Xi\) in \([31]\) with help of the ordinary forgetful map \(p_e : C_{n,0} \to C_{2,0}\) (rather than with \(p_e\)),

$$\Xi' : G_{n,t} \to \Omega(\hat{C}_{n,0})$$

then we would not get in general a de Rham field theory \((\Omega^{in}, \Xi', \Omega^{out})\) on \(\hat{C}_{n,0}\) as for generic graphs \(\Gamma\) the factorization \([27]\) might fail. For a fixed propagator \(\omega\), let us denote by \(G_{sing}(\omega)\) that set of graphs \(\Gamma\) for which it fails indeed.

The boundary values of the propagator \(\omega\) determine the associated \(\mu^{in}\) and \(\mu^{in}\) \(\text{Lieb}_{\infty}\)-structures on \(\mathbb{R}^d\) as it is explained in §3. It is clear that if \(\gamma\) is a Maurer-Cartan element of the \(\mu^{in}\) structure such that for any \(\Gamma \in G_{sing}(\omega)\) the value, \(\Phi_{\Gamma}(\alpha, \ldots, \alpha)\), of the associated operator \(\Phi_{\Gamma}\) vanishes, then the transformation

$$F(\gamma) := \gamma + \sum_{n \geq 2} h^{n-1} \sum_{\Gamma \in G_{n,2n-2}} \frac{C_{\Gamma'}}{\# Aut(\Gamma)} \Phi_{\Gamma}(\gamma^{\otimes n})$$

defines a Maurer-Cartan element, \(F(\alpha)\), of the the \(\mu^{out}\)-structure on \(\mathbb{R}^d\), i.e. in such a case our machinery “works”.

As an illustration of how it works, let consider a family, \(\{\gamma^{P.D.}\}\), of polyvector fields on \(\mathbb{R}^d\) of the form \(\gamma^{P.D.} = \sum_{i \geq 0} \gamma^i, \gamma^i \in \wedge^i T_{\mathbb{R}^d}\), with all \(\gamma^i\) vanishing except for \(i = 0\) and \(2\), and with

$$\gamma_2^2 = \frac{1}{2} \sum_{i,j} \alpha_{i+k}^j x^i \psi_j \psi_j$$

being a linear Poisson structure. Equation \([\gamma^i, \gamma^j]_{\text{Schouten}} = 0\) implies then that \(\gamma^0 = \gamma^0(x)\) is an invariant polynomial on \(\mathbb{R}^d\), that is, an element of \((\mathcal{C}^{0,0})^\infty\), where \(\mathcal{C}\) is the space

\(^9\)For \(n \geq 2\), \(B_n = \frac{B_{2n}}{2n2n}\), where \(B_n\) are the ordinary Bernoulli numbers \(B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}\) etc.
The conclusion is that the exotic transformation can define a compactification, action of \( G \) and the ordinary projection product of their weights giving rise above to the summation over \( C_1 \) associated with Kontsevich's

Here we used the fact that the weight, \( C_{w_n} \) of wheel with respect to Kontsevich's antipropagator (and the ordinary projection \( \Gamma \)) is given by

Analogously, the transformation of Poisson-Duflo structures,

associated with Kontsevich's \( \frac{1}{2} \)-antipropagator,

is given by

and at \( \hbar = 1 \) coincides with Kontsevich's modification (see §4.6 in [Ko3]) of Duflo's strange transformation.

5. Braid configuration spaces

5.1. Compactified braid configuration spaces \( \overline{B}_n \) as an operad \( \text{Lie}_\infty \). For a topological space \( X \) the symmetric cartesian product \( X^p / S_p \) is denoted by \( X^\circ p \).

Let \( B_n \cong C_n / S_n \) be the configuration space of braid pairwise distinct points in \( \mathbb{C} \) modulo the group action of \( G(2) \). Choosing an ordering, that is a section \( s : B_n \to C_n \) of the natural projection, one can define a compactification, \( \overline{B}_n \), of \( B_n \) as the closure of the following map

and the symmetric cartesian product \( \rightarrow \)

for \( n \) even (see [Ko2]) and equals to \( C_{w_n} = (-1)^{n(n-1)/2} n B_n \) for \( n \) even [VdB]. The weight of a union of \( m \) wheels is equal to the product of their weights giving rise above to the summation over \( m \geq 1 \).

The conclusion is that the exotic transformation \( F^{\overline{R}} \) associated with Kontsevich’s antipropagator preserves the class of Poisson-Duflo structures, and, at \( \hbar = 1 \), coincides precisely with the famous strange Duflo automorphism (see, e.g., [Ko2, ChRo] and references cited there).

Analogously, the transformation of Poisson-Duflo structures,

associated with Kontsevich’s \( \frac{1}{2} \)-antipropagator,

is given by

and at \( \hbar = 1 \) coincides with Kontsevich’s modification (see §4.6 in [Ko3]) of Duflo’s strange transformation.

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which does not depend on the chosen section $s$. One has

$$
\partial \mathcal{B}_n = \bigsqcup_A \mathcal{B}_{n-\#A+1} \times \mathcal{B}_{\#A}
$$

where the summation runs over all proper unordered subsets, $A$, of the set $[n]$ with cardinality at least 2. Identifying space $\mathcal{B}_n$ with a degree $2n - 3$ symmetric $n$-corolla

$$
\begin{array}{ccc}
1 & 2 & 3 \\
\cdots & n-1 & n \\
\end{array}
= \begin{array}{ccc}
\sigma(1) & \sigma(2) & \cdots \\
\sigma(n) & & \\
\end{array}, \quad \forall \sigma \in S_n, \ n \geq 2
$$

we can rewrite the boundary differential as

$$
\partial = \sum_{A \subseteq [n], \#A \geq 2} A_{\subseteq [n]} \nabla_{[n]\setminus A}
$$

and hence establish the following

5.1.1. Proposition[13]. The face complex, $C^\bullet(\mathcal{B})$, of the family of compactified braid configurations spaces, $\{\mathcal{B}_n\}_{n \geq 2}$, has a structure of a dg operad canonically isomorphic to the operad, $\mathcal{L}ie_\infty$, of strong homotopy Lie algebras.

Introducing a metric structure on the graphs from $C^\bullet(\mathcal{B})$ as in §2 one can make each $\mathcal{B}_n$ into a smooth manifold with corners.

5.2. Compactified braid configuration spaces $\hat{\mathcal{B}}_{n,0}$ as an operad $\text{Mor}(\mathcal{L}ie_\infty)$. Let $\hat{\mathcal{B}}_{n,0} \simeq C_{n,0}/S_n$ be the configuration space of braid, or unordered, pairwise distinct points in $\mathbb{H}$ modulo the group action of $G^{(1)}$. Its compactification can be defined either using metric trees or as the closure of the natural embedding (cf. §3),

$$
\mathcal{B}_n \longrightarrow \prod_{\text{subsets } A, \text{ of the set of } n \text{ points}} \mathcal{B}_A \times [0, +\infty]
$$

One has

$$
\partial \hat{\mathcal{B}}_{n,0} = \bigsqcup_{A \subseteq [n], \#A \geq 2} (\mathcal{B}_{n-\#A+1,0} \times \mathcal{B}_{\#A}) \bigsqcup_{\{n\} = B_1 \sqcup \ldots \sqcup B_k \quad 2 \leq A \leq n} (\mathcal{B}_k \times \mathcal{B}_{\#B_1,0} \times \ldots \times \mathcal{B}_{\#B_k,0})
$$

and, as in §3.3, makes the following observation.

5.2.1. Proposition. The face complex of $\mathcal{B} \sqcup \hat{\mathcal{B}} \sqcup \mathcal{B}$ has structure of a dg 2-coloured operad which is canonically isomorphic to the dg 2-coloured operad $\text{Mor}(\mathcal{L}ie_\infty)$ describing pairs of $\mathcal{L}ie_\infty$-algebras and $\mathcal{L}ie_\infty$-morphisms between them.

5.3. De Rham field theories on braid configuration spaces. Let $\mathcal{B}_{n,l}$ stand for a family of graphs, $\{\Gamma\}$, such that (i) $\#V(\Gamma) = n$, (ii) $\#E(\Gamma) = l$, (iii) $\Gamma$ has no loop type edges, and (iv) the set $E(\Gamma)$ is totally ordered (up to an even permutation). Let $\mathfrak{B}_{n,l}$ be the version of $\mathcal{B}_{n,l}$ with data (iv) forgotten. Note that edges of these graphs are not directed, and their vertices are not ordered.

With any graph $\Gamma \in \mathfrak{B}_{n,l}$ one can associate a linear map

$$
\Phi_\Gamma^f : \text{poly}(\mathbb{R}^d) \longrightarrow \text{poly}(\mathbb{R}^d)[-l]
$$

$$
\gamma_1 \otimes \ldots \otimes \gamma_n \longrightarrow \Phi_\Gamma^f(\gamma_1, \ldots, \gamma_n),
$$

$$
\Phi_\Gamma^f(\gamma_1, \ldots, \gamma_n) := \frac{1}{n!} \sum_{f: \text{Vert}(\Gamma) \to [n]} \left[ \prod_{e \in \text{Edges}(\Gamma)} \Delta_e \prod_{v \in \text{Vert}(\Gamma)} \gamma f(v)(\psi f(v), x f(v)) \right]
$$

where $\Delta_e := \sum_{j=1}^{\#(\Gamma)} \sum_{x \in \text{Edges}(\Gamma)} 1 \prod_{v \in \text{Vert}(\Gamma)} \psi_{v(x)} x_{v(x)}$.
where the operator $\Delta_e$ corresponding to an edge $e$ beginning at a vertex labelled by $i \in [n]$ and ending at a vertex labelled by $j \in [n]$ is given by

$$\Delta_e := \sum_{a=1}^d \left( \frac{\partial^2}{\partial x^a_{(i)} \partial \psi_{(j)} a} + \frac{\partial^2}{\partial x^a_{(j)} \partial \psi_{(i)} a} \right).$$

**De Rham field theories** on $\overline{B}$ and $\hat{B}$ can be defined in a full analogy with the ones on $\overline{C}$ and $\hat{C}$. Moreover, Theorems 4.2.1 and 4.3.1 hold true with symbols $\overline{C}$, $\hat{C}$ and $\mathcal{L}_{\text{Lie}}\infty$ replaced, respectively, by $\overline{B}$, $\hat{B}$ and $\mathcal{L}_{\text{Lie}}\infty$.

A class of de Rham theories on $\overline{C} \sqcup \hat{C} \sqcup \overline{C}$ determined by a propagator $\omega(z_i, z_j)$ on $\hat{C}_2 \times_0$, satisfying the symmetry condition (cf. §4.5),

$$\omega(z_i, z_j) = \omega(z_j, z_i),$$

comes in fact from a class of de Rham field theories on $\overline{B} \sqcup \hat{B} \sqcup \overline{B}$. It is easy to see that for any choice of such a symmetric propagator the weight, $C_{w_n}(s)$, of the following graph

$$w_n = \begin{array}{c}
\includegraphics{graph.png}
\end{array} \in G_{n,2n-2}, \ n \geq 2,$$

is equal to zero for even $n$. For odd $n$ its weight is, in general, non-zero. For example, $C_{w_3}(0) = \zeta(3)/(4\pi)^3i$ with respect to to the symmetrized Kontsevich’s $\frac{1}{2}$-propagator and, therefore, the infinitesimal part, $\delta\alpha$, of the associated exotic transformation of an ordinary Poisson structure,

$$F_{\text{sym}}^{\mathcal{K}}(\alpha) = \alpha + \frac{C_{w_3}(s)}{\#\text{Aut}(w_3)} \Phi^s_1(\alpha, \alpha, \alpha) + \text{higher order (in } \alpha \text{) terms},$$

is controlled by the graph $w_3$ (which is the same as the tetrahedron graph $\includegraphics{tetrahedron.png}$) so that we get, up to a non-zero numerical factor, the following infinitesimal change of the Poisson structure,

$$\delta\alpha \sim \sum_{i,j,k,l,m,k',l',m'} \left( \frac{\partial^3 \alpha_{ij}}{\partial x^k \partial x^l \partial x^m} \frac{\partial \alpha_{kk'}}{\partial x^l} \frac{\partial \alpha_{ll'}}{\partial x^m} \frac{\partial \alpha_{mm'}}{\partial x^k} + \frac{4}{3} \frac{\partial^3 \alpha_{im}}{\partial x^k \partial x^l \partial x^m} \frac{\partial \alpha_{kk'}}{\partial x^l} \frac{\partial \alpha_{ll'}}{\partial x^m} \frac{\partial \alpha_{mm'}}{\partial x^k} \right) (\partial_i \wedge \partial_j).$$

According to Kontsevich [Ko1], the second term vanishes identically for any Poisson structure $\alpha$ so that the above relation simplifies further,

$$\delta\alpha \sim \sum_{i,j,k,l,m,k',l',m'} \frac{\partial^3 \alpha_{ij}}{\partial x^k \partial x^l \partial x^m} \frac{\partial \alpha_{kk'}}{\partial x^l} \frac{\partial \alpha_{ll'}}{\partial x^m} \frac{\partial \alpha_{mm'}}{\partial x^k} (\partial_i \wedge \partial_j).$$

Thus the flow of Poisson structures,

$$\frac{d\alpha}{dt} := \delta\alpha,$$

associated with the infinitesimal part of the exotic $\mathcal{L}_{\text{Lie}}\infty$ morphism $F_{\text{sym}}^{\mathcal{K}}$ is precisely the one which was found by Kontsevich long ago in §4.6.3 of [Ko2] as an example of an exotic (i.e. homotopy non-trivial, see Remark 4.2.9) infinitesimal $\mathcal{L}_{\text{Lie}}\infty$ automorphism of the Schouten algebra.
6. TOWARDS A NEW DIFFERENTIAL GEOMETRY

The classical architecture of geometry and theoretical physics can be described as follows: a geometric structure is a function (a “field” or an “observable”) on a manifold (“space-time”) satisfying some differential equations.

The theory of (wheeled) props offers a different picture \[\text{Me1, Me2}\] in which “space-time” equipped with a geometric structure is itself a function (a representation) on a more fundamental object — a certain dg free prop, a kind of a graph complex.

Here \(V\) stands for a vector space modelling a local coordinate chart of some (say, real analytic) manifold \(M\). A real analytic gluing map, \(V \to W\), of two coordinate charts on \(M\) can be understood (via a power series decomposition) as a morphism of props \(\text{End}_V \to \text{End}_W\). Then a consistent gluing of local geometric structures, \(\{\mathcal{P} \to \text{End}_V\}\), controlled by some prop \(\mathcal{P}\) into a global one on the manifold \(M\) can be understood as commutativity of diagrams of the form,

\[
\begin{array}{ccc}
\text{End}_V & \longrightarrow & \text{End}_W \\
\mathcal{P} & \downarrow & \\
\end{array}
\]

If, however, the dg prop \(\mathcal{P}\) admits a non-trivial group of automorphisms, then the above gluing pattern can be replaced by the following one,

\[
\begin{array}{ccc}
\text{End}_V & \longrightarrow & \text{End}_W \\
\mathcal{P} & \downarrow & \\
\end{array}
\]

i.e. the group \(\text{Aut}(\mathcal{P})\) to allowed to twist the standard local coordinate gluing mappings. Note that the group \(\text{Aut}(\mathcal{P})\) is universal and does not depend on a particular manifold \(M\), i.e. one modifies in this way the whole category of geometric structures of type \(\mathcal{P}\).

Poisson structures on any finite-dimensional manifold are controlled by a dg wheeled prop \(\textit{Poly}\) (see Appendix 4) whose automorphism group is very non-trivial. It is worth noting that another important class of geometric structures — the so called Nijenhuis structures (coming from the famous Nijenhuis integrability condition for an almost complex structure) — are also controlled by a certain dg (wheeled) prop (see \[\text{Me1, St}\]).
Theorem A. The weight,

\[ C_{W_n} := \frac{1}{(2\pi)^{2n}} \int_{C_{n+1,0}} \omega_{\frac{1}{2}}(z_{n+1}, z_1) \wedge \ldots \wedge \omega_{\frac{1}{2}}(z_{n+1}, z_n) \wedge \omega_{\frac{1}{2}}(z_1, z_2) \wedge \omega_{\frac{1}{2}}(z_2, z_3) \wedge \ldots \wedge \omega_{\frac{1}{2}}(z_n, z_1) \]

of the following graph with \( n + 1 \) vertices,

\[ W_n = \begin{array}{c}
1 \\
2 \\
n \\
n+1 \\
n-1
\end{array} \]

with respect to Kontsevich’s 1/2-propagator \( \omega_{\frac{1}{2}}(z_i, z_j) := \frac{1}{i} d \log \frac{z_i - z_j}{z_i + z_j} \) is given by

\[ C_{W_n} = (-1)^{n(n-1)/2} \frac{\zeta(n)}{(2\pi)^n} = (-1)^{n(n-1)/2} \frac{\sum_{p=1}^{\infty} \frac{1}{p^n}}{(2\pi)^n} \]

Proof. We identify \( C_{n+1,0} \) with a subspace of \( \text{Conf}_{n+1,0} \) consisting of all configurations, \( \{z_1, \ldots, z_n, z_{n+1}\} \), with \( z_{n+1} = i \), and introduce in \( C_{n+1,0} \) a system of coordinates, \( \{\rho_i, \phi_i \mid 0 < \rho_i < 1, 0 < \phi_i \leq 2\pi\}_{1 \leq i \leq n} \), as follows

\[ \frac{z_i - i}{z_i + 1} =: \rho_i e^{i\phi_i}. \]

Thus \( \rho_i = \tanh(\frac{1}{2} h(i, z_i)) \), where \( h(i, z_i) \) is the hyperbolic distance from \( i \) to \( z_i \in \mathbb{H} \) and \( \phi_i \) is the angle between the vertical line and the hyperbolic geodesic from \( i \) to \( z_i \).

Let \( I \) be the ideal in the de Rham algebra on \( C_{n,0} \) generated by 1-forms \( d(\rho_i e^{i\phi_i}) \), \( 1 \leq i \leq n \). As

\[ z_i = \frac{1 + \rho_i e^{i\phi_i}}{1 - \rho_i e^{i\phi_i}} \]

we have

\[ \frac{1}{i} d \log \frac{z_i - z_j}{z_i + z_j} = \frac{1}{i} d \log \frac{1 - \rho_i e^{-i\phi_i}}{(1 - \rho_i e^{i\phi_i})(1 - \rho_i e^{i(\phi_j - \phi_i)})} = \frac{1}{i} \left( \frac{\rho_j e^{i\phi_j}}{1 - \rho_i e^{i(\phi_j - \phi_i)}} - \frac{1}{1 - \rho_i e^{-i\phi_i}} \right) d (\rho_i e^{-i\phi_i}) \mod I. \]

Hence, modulo \( I \),

\[ \frac{1}{i} d (\rho_i e^{i\phi_i}) \wedge \frac{1}{i} d \log \frac{z_i - z_j}{z_i + z_j} = \frac{2}{i} d\phi_i \wedge d\rho_i \left( \frac{\rho_j e^{i(\phi_j - \phi_i)}}{1 - \rho_i \rho_j e^{i(\phi_j - \phi_i)}} - \frac{e^{-i\phi_i}}{1 - \rho_i e^{-i\phi_i}} \right) \]

As

\[ \int e^{ik\phi} d\phi = \begin{cases} 2\pi & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \]
we finally get (identifying $k_{-1}$ with $k_n$)

$$C_{W_n} = \frac{2^n(-1)^{n(n-1)/2}}{(i)^n(2\pi)^{2n}} \int d\phi_1 \ldots d\phi_n d\rho_1 \ldots d\rho_n \sum_{k=0}^{\infty} \prod_{i=1}^{n} \rho_i^{k_{i-1}+k_{i+1}+1} e^{i(k_{i-1}-k_i)\phi_i} \rho_1 \ldots \rho_n \sum_{k=0}^{\infty} (\rho_1 \ldots \rho_n)^{2k+1}$$

$$= \frac{2^n(-1)^{n(n-1)/2}}{(i)^n(2\pi)^{2n}} \int_0^1 \ldots \int_0^1 \rho_1 \ldots \rho_n \rho_1 \ldots \rho_n \sum_{k=0}^{\infty} (\rho_1 \ldots \rho_n)^{2k+1}$$

$$= \frac{2^n(-1)^{n(n-1)/2}}{(i)^n(2\pi)^{2n}} \int_0^1 \ldots \int_0^1 \rho_1 \ldots \rho_n \rho_1 \ldots \rho_n 1 - (\rho_1 \ldots \rho_n)^2$$

$$= \frac{(-1)^{n(n-1)/2}}{(i)^n(2\pi)^{2n}} \int_0^1 \ldots \int_0^1 dx_1 \ldots dx_n$$

$$= \frac{(-1)^{n(n-1)/2}}{(i)^n(2\pi)^{2n}} \zeta(n).$$

Note that $\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n} B_{2n}}{2(2n)!}$ so that for $n$ even

$$C_{W_n} = -(-1)^{n(n-1)/2} \frac{1}{2n!} B_n = -(-1)^{n(n-1)/2} n B_n.$$

**Appendix 2: Example of a boundary strata**

Consider a pair

$$G = \begin{array}{c}
1 \\
3 & 5 & 6
\end{array}, \quad G_{\text{metric}} = \begin{array}{c}
1 & 2 & 4 & 7
\end{array}$$

consisting of a graph $G$ from the face complex $\mathcal{C}_* (\hat{C}_{\tau,0})$ and an associated metric graph. The latter defines a smooth coordinate chart,

$$U(G) \simeq C_3 \times C_2 \times \hat{C}_{3,0} \times \hat{C}_{2,0} \times \mathbb{R} \times \mathbb{R},$$

near the face $G \subset \hat{C}_{\tau,0}$ whose intersection with $C_{\tau,0}$ consists, by definition, of all those configurations, $p$, of 7 points in $\mathbb{H}$ which result from the following four step construction:

**Step 1**: take an arbitrary hyperpositioned configuration, $p_0^{(1)} \in C_3$, of 3 points labelled by 1, and, say, $z'$ and $z''$, and magnify it, $p_0^{(1)} \rightarrow \tau_1 \cdot p_0^{(1)}$;

**Step 2**: take an arbitrary hyperpositioned configuration, $p_0^{(2)} \in C_2$, of 2 points labelled by 6, and, say, $z'''$, magnify it, $p_0^{(2)} \rightarrow \tau_2 \cdot p_0^{(2)}$, and place the result at the position $z'$;

**Step 3**: take an arbitrary hyperpositioned configuration of 3 points, $p^{(3)} \in C_{3,0}$, labelled by $\{2, 4, 7\}$ and place it at the position $z'''$;

**Step 4**: take an arbitrary hyperpositioned configuration, $p^{(4)} \in C_{2,0}$, of 2 points labelled by 3 and 5 and place it at the position $z'''$. 


The final result is a hyperpositioned point \( p = (z_1, z_3, z_5, z_6, z_2, z_4, z_7) \) in \( C_{7,0} \) of the form

\[
\begin{align*}
&= z_4 \cdot z_7 \cdot z_3 \cdot z_5 \cdot z_6 \cdot z_1 \cdot z_2 \\
&\rightarrow \text{ } \rightarrow \\
&\downarrow \downarrow \downarrow \\
&\rightarrow \rightarrow \rightarrow \\
&\uparrow \uparrow \uparrow \\
&z_1
\end{align*}
\]

An equivalent coordinate chart results from an alternative 5-step construction which uses instead of the parameters \((\tau_1, \tau_2)\) another pair of independent parameters, \(\tau_1\) and \(\varepsilon\):

**Step 1:** take an arbitrary hyperpositioned configuration, \( p^{(1)}_0 \in C_3, \) of 3 points labelled by 1, and, say, \( z' \) and \( z'' \);

**Step 2:** take an arbitrary hyperpositioned configuration, \( p^{(2)}_0 \in C_2, \) of 2 points labelled by 6, and, say, \( z''', \varepsilon\)-magnify it, \( p^{(2)}_0 \rightarrow \varepsilon \cdot p^{(2)}_0, \) and place the result at the position \( z' \);

**Step 3:** magnify the resulting configuration of points \((1, 6, z'', z''')\) by the factor \(\tau_1\);

**Step 4:** take an arbitrary hyperpositioned configuration of 3 points, \( p^{(3)} \in C_{3,0}, \) labelled by \(\{2, 4, 7\}\) and place it at the position \( z'' \);

**Step 5:** take an arbitrary hyperpositioned configuration, \( p^{(4)} \in C_{2,0}, \) of 2 points labelled by 3 and 5 and place it at the position \( z''' \).

The 10-dimensional face \( G \) lies in the intersection of two 11-dimensional faces described by the following graphs,

\[
G_1 = \begin{array}{c}
1 \\
3 \\
5 \\
6 \\
2 \\
4 \\
7
\end{array}
\]

\[
G_2 = \begin{array}{c}
1 \\
3 \\
5 \\
6 \\
4 \\
2 \\
7
\end{array}
\]

and the above two constructions give us explicit descriptions of the associated embeddings, \( G \hookrightarrow G_1 \) and \( G \hookrightarrow G_2 \).

**APPENDIX 3: \( \text{Leib}_\infty \) AUTOMORPHISMS OF MAURER-CARTAN SETS**

**Proposition.** Any \( \text{Leib}_\infty \)-automorphism,

\[
\{F_n : \otimes g \rightarrow g[2 - 2n]\}_{n \geq 1},
\]

of a Lie algebra \((g, [\ , ] : \otimes^2 g \rightarrow g[-1])\) induces an automorphism of its set,

\[
\{\alpha \in g[[h]] : [\alpha, \alpha] = 0 \& |\alpha| = 2\},
\]

of Maurer-Cartan elements by the formula

\[
\alpha \rightarrow F^{\text{Leib}}(\alpha) := \sum_{n \geq 1} \frac{h^{n-1}}{n!} F_n^{\text{Leib}}(\alpha \otimes^n)
\]
Proof. It follows from \([\text{Me3}]\) that, for \(n \geq 2\),
\[
0 = \sum_{\substack{|n| = |B_1| \leq |B_2| \\
\text{int } B_1 \leq \text{int } B_2}} [F_{\#B_1}(\alpha \otimes \#B_1), F_{\#B_2}(\alpha \otimes \#B_2)]
\]
\[
= \sum_{\substack{|n| = |S_1 + 1| \leq |S_2| \\
\text{int } S_1 \leq \text{int } S_2 \geq 1}} [F_{\#S_1 + 1}(\alpha \otimes (\#S_1 + 1)), F_{\#S_2}(\alpha \otimes \#S_2)]
\]
\[
= \sum_{k=0}^{n-2} \frac{(n-1)!}{k!(n-k-1)!} [F_{k+1}((\alpha \otimes (k+1)), F_{n-k-1}(\alpha \otimes (n-k-1))]
\]
\[
= \frac{1}{2} \sum_{k=0}^{n-2} \left( \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(n-k-2)!(k+1)!} \right) [F_{k+1}((\alpha \otimes (k+1)), F_{n-k-1}(\alpha \otimes (n-k-1))]
\]
\[
= \frac{1}{2} \sum_{k=0}^{n-2} \frac{n!}{(k+1)!(n-k-1)!} [F_{k+1}((\alpha \otimes (k+1)), F_{n-k-1}(\alpha \otimes (n-k-1))].
\]
\[
= \frac{n!}{2} \sum_{\substack{p+q=1 \\
p,q \geq 1}} \frac{\text{sgn}(\alpha)}{p!q!} [F_p(\alpha^\otimes p), F_q(\alpha^\otimes q)].
\]

Then
\[
h^2[F^{\text{Leib}}(\alpha), F^{\text{Leib}}(\alpha)] = \sum_{n \geq 2} h^n \sum_{\substack{p+q=1 \\
p,q \geq 1}} \frac{1}{p!q!} [F_p(\alpha^\otimes p), F_q(\alpha^\otimes q)]
\]
\[
= 0.
\]

\[\square\]

APPENDIX 4: WHEELED PROP OF POLYVECTOR FIELDS

We refer to \([\text{Me3}]\) for an elementary introduction into the language of (wheeled) operads and props.

Definition. The wheeled prop of polyvector fields, Poly, is a dg free wheeled prop, \((\text{Free} \langle E \rangle, \delta)\), generated by an \(S\)-bimodule \(E = \{E(m,n)\}_{m,n \geq 0}\),
\[
E(m,n) = \text{sgn}_m \otimes 1_n [m-2] = \text{span}\langle \cdot \rangle_\text{span}
\]

and equipped with the differential \(\delta\) given on the generators by the formula
\[
\delta = \sum_{\substack{|m| = |I_1 \cup I_2| \\
\text{int } I_1 \leq \text{int } I_2 \geq 1 \ \\
|I_1| \geq |I_2| \geq 0}} (-1)^{\sigma(I_1 \cup I_2) + |I_1||I_2| + 1} \ circ \ (I_1 \cup I_2)
\]

where \(\sigma(I_1 \cup I_2)\) is the sign of the permutation \([n] \rightarrow I_1 \cup I_2\).
Here \(\text{sgn}_m\) (resp. \(1_n\)) is the 1-dimensional sign (resp. trivial) representation of \(S_n\). Representations, \(f : \text{Poly} \rightarrow \text{End}_\mathbb{C}\), of this prop in a finite-dimensional \(\mathbb{C}\)-graded vector space \(V\) are in one-to-one correspondence with formal \(\mathbb{C}\)-graded Poisson structures on \(V\) \([\text{Me2}]\). The group, \(\text{Aut}(\text{Poly})\), of automorphisms of this prop consists of all automorphisms, \(F : \text{Free} \langle E \rangle \rightarrow \text{Free} \langle E \rangle\), of the free
prop which respect the differential, \( F \circ \delta = \delta \circ F \). Every such an automorphism is uniquely determined by its values on the generators,

\[
F \left( \begin{array}{cccc}
1 & 2 & \cdots & m \\
1 & 2 & \cdots & n
\end{array} \right) = \sum_{k \geq 2} \sum_{\Gamma \in G_{k,2k-2}^{\text{cor}}(m,n)} c_{\Gamma} \Gamma,
\]

which, for purely degree reasons, must be a sum over a family \( G_{k,2k-2}^{\text{cor}}(m,n) \) of graphs \( \Gamma \) which are built from the generating corollas by taking their disjoint unions and then gluing some output legs with with the same number of input legs and which satisfy three conditions: \( \Gamma \) has \( k \) vertices, \( 2k - 2 \) edges, \( n \) input legs and \( m \) output legs (cf. [Me3]). The main result of our paper can be restated as follows: any de Rham field theory on \( \mathcal{C} \sqcup \hat{\mathcal{C}} \) defines an exotic automorphism of \( (\text{Poly}, \delta) \) with weights \( c_{\Gamma} \) given by (29).

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