Fifth order of non-polynomial spline functions to find a second type integral numerical solution linear Volterra with weakly singular kernel

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Abstract. In this work, the fifth order non-polynomial spline functions are used to solve linear Volterra integral equations with weakly singular kernel. Numerical example is presented to illustrate the applications of this method and to compare the computed results with other numerical methods. We take integral equation when it has the kernel and it is non-linear. The results are the approximations that depended on Volterra integral equation. Our work calculated the results for the Volterra integral equation, which are approximate that the error is found.

Keywords: non-polynomial spline functions, linear Volterra integral equations with weakly singular kernel.

1. Introduction
In this paper, we take integral equation when it has a kernel, and it is non-linear and the results are approximations since its use depended on Volterra integral equation. In the last few years, there has been an increase in interest in integral equations, especially the Volterra equation and several fields such as probability theory and particle transport problems in astrophysics. In the mid-sixties of the last century, the researchers began to solve the integral equations numerically when the kernel was weak. In the early eighties, numerical methods began to impose on it when it was not possible to find analytical solutions, especially when the kernel was singular.

In 1997, Diogo and T-Lima presented an inductive solution method for finding the numerical solution to Volterra integral equation with weakly singular kernel [1]. In 2012, Hosseinpour solved integral differential equations using non-polynomial spline function [2]. In 2015, S-Murad, M-Majeced and Harbi presented the numerical solution to linear Volterra integral equation using the third-order non-polynomial spline functions [3]. In our work, we will study the numerical solution for linear Volterra integral equation with the weakly singular kernel using the fifth-order non-polynomial spline functions.

The definitions of the integral equations will be discussed hereafter. The following equation is called an integral equation, which is given on the form:

\[ h(x) + u(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t) u(x) \, dt \]

where \( a(x), b(x) \) are bounded of integration and \( \lambda \) constant coefficient, \( k(x,t) \) is a function known by two variables \( (x,t) \) and called the kernel of the integral equation [4]. The previous equation is non-linear if its kernel is written in the form [5]:

\[ K(x,t) = k(x, t, u(t)) \]

The previous equation is homogenous if \( f(x) = 0 \). Otherwise, they are non-homogeneous [1].
The previous equation is linear of the first type if $h(x) = 0$ and linear of the second type if $h(x) = 1$ [6]. The previous equation is called Volterra integral equation when $(x) = a$, $b(x) = x$ where $a$ is a constant. The equation becomes as the form given below [7]:

$$h(x) + u(x) = f(x) + \lambda \int_{a}^{x} k(x, t) u(t) \, dt \quad x \in [a, b]$$

(3)

The kernel of a linear integral equation is adjective if written in the form below [5]:

$$k(x, t) = \sum_{k=1}^{n} a_k (x) b_k(t)$$

(4)

The kernel of integral equation is weakly singular if written in the form below [7]:

$$k(x, t) = \frac{H(x, t)}{(x-e)^a} , \quad a \leq t \leq b , \quad a \leq x \leq b$$

(5)

In the sequential approximations method, linear Volterra integral equation is given in the form [8,4]:

$$u(x) = f(x) + \lambda \int_{a}^{b} k(x, t) u(t) \, dt$$

(6)

where $f(x)$ is a continous function on $[0, a]$ and $k(x, t)$ is continous function where, $0 \leq t \leq x$, $0 \leq x \leq a$.

Let $u_0(x)$ is continuous function on $[0, a]$ and substitute $u(x)$ in $u_0(x)$, we get on approximation from first order:

$$u_1(x) = f(x) + \lambda \int_{a}^{x} k(x, t) u_0(x) \, dt$$

(7)

Furthermore, we get a sequence of dependencies $u_0, u_1, u_2, \ldots, u_n$.

$$u_n(x) = f(x) + \lambda \int_{a}^{x} k(x, t) u_{n-1}(x) \, dt$$

(8)

For the discussion on theorem, let Volterra integral equation with weakly singular kernel to take the form [9]:

$$u(x) - \int_{a}^{x} t^{\mu-1} u(t) \, dt = f(x) \quad x \in [0, T]$$

(9)

$$i f \quad 0 < \mu \leq 1 , \; f \in C^1[0, T] , \text{where } f(0) = 0 \text{ if } \mu = 1$$

Thus, the equation has a set of solutions that are given on the form:

$$u(t) = c_0 t^{1-\mu} + f(t) + \gamma + t^{1-\mu} \int_{0}^{t} s^{\mu-2} \left( f(s) - f'(0) \right) ds$$

(10)

$$\gamma = \begin{cases} 1 & \text{if } \mu < 1 \\ 0 & \text{if } \mu = 1 \end{cases}$$

(11)

if $\mu > 1$, $f \in C^m[0, T], m \geq 0$, $f(0) = 0$ if $\mu = 1$

Then the only solution $u \in C^{m}[0, T]$, for the equation given by:

$$u(t) = f(t) + t^{1-\mu} \int_{0}^{t} s^{\mu-2} f(s) ds$$

(12)

The fifth order non-polynomial spline functions method is used to find the approximate solution of the second type Volterra Integral Equation (VIE) [3].

Let group of points $t_i$ on the interval $[a, b]$:

$$a = t_0 < t_1 < \cdots < t_n = b , h = \frac{b-a}{n} , t_i = t_0 + ih$$

(13)

where $u(t)$ is exact solution of the VIE, $p_i(t)$ is exact solution of the VIE. We take the fifth order non-polynomial spline functions as:

$$p_i(t) = a_i \cos (t - t_i) + b_i \sin (t - t_i) + c_i (t - t_i) + d_i (t - t_i)^2 + e_i (t - t_i)^3 + r_i (t - t_i)^4$$

(14)

where $a_i, b_i, c_i, d_i, e_i, r_i, z_i, g_i$ are constants.

We differentiate the previous equations seven times with respect to $t$ and the following equations were obtained:

$$p_i(t) = -k_1a_i \cos (t - t_i) + k_2b_i \cos (t - t_i)$$

$$+ c_i + 2d_i (t - t_i) + 3e_i (t - t_i)^2 + 4r_i (t - t_i)^3 + 5z_i (t - t_i)^4$$

(15)

$$p_i(t) = -k_2a_i \cos (t - t_i) - k_2^2b_i \sin (t - t_i) + 2d_i + 6e_i (t - t_i)^2 + 12r_i (t - t_i)^2 + 20z_i (t - t_i)^3$$

(16)
\[ p_i^{(2)}(t) = k^3 a_i \sin k(t - t_i) - k^3 b_i \cos k(t - t_i) + 6e_i + 24r_i(t - t_i) + 20 z_i(t - t_i)^2 \]
\[ p_i^{(4)}(t) = k^4 a_i \cos k(t - t_i) + k^4 b_i \sin k(t - t_i) + 24r_i + 120 z_i(t - t_i) \]
\[ p_i^{(5)}(t) = -k^5 a_i \sin k(t - t_i) + k^5 b_i \cos k(t - t_i) + 24r_i + 120 z_i \]
\[ p_i^{(6)}(t) = -k^6 a_i \cos k(t - t_i) - k^6 b_i \sin k(t - t_i) \]
\[ p_i^{(7)}(t) = k^7 a_i \sin k(t - t_i) - k^7 b_i \cos k(t - t_i) \]

The \( t \) was replaced with \( t_0 \) and the following equations were obtained:
\[
\begin{align*}
    p_i^{(1)}(t_0) &= a_i + g_i \\
    p_i^{(1)}(t_0) &= b_i + c_i \\
    p_i^{(2)}(t_0) &= -k^2 a_i + 2d_i \\
    p_i^{(3)}(t_0) &= -k^3 b_i + 6e_i \\
    p_i^{(4)}(t_0) &= k^4 a_i + 24r_i \\
    p_i^{(5)}(t_0) &= k^5 b_i + 120 z_i \\
    p_i^{(6)}(t_0) &= -k^6 a_i \\
    p_i^{(7)}(t_0) &= -k^7 b_i
\end{align*}
\]

From the previous relationships, we obtain the values of each of the parameters as below:
\[
\begin{align*}
    a_i &= -\frac{1}{k^6} u^{(6)}(t_i) \\
    b_i &= -\frac{1}{k^7} u^{(7)}(t_i) \\
    c_i &= u(t_i) - k b_i \\
    d_i &= \frac{1}{2} [u'(t_i) + k^2 a_i] \\
    e_i &= \frac{1}{6} [u''(t_i) + k^3 b_i] \\
    r_i &= \frac{1}{24} [u^{(4)}(t_i) - k^4 a_i] \\
    z_i &= \frac{1}{120} [u^{(5)}(t_i) - k^5 b_i] \\
    g_i &= u(t_i) - a_i
\end{align*}
\]

2. Volterra integral equation of the second type with weakly singular kernel

The VIE with weakly singular kernel is given as [5, 6]:
\[
u(x) - \int_0^x \frac{t^{\mu-1}}{x^{\mu}} u(t) dt = f(x) \quad x \in [0, T]
\]
where \( f \) is a known function and \( 0 < \mu < 1 \).

To solve the equation (23), we multiply both sides with \( x^\mu \) and we obtained:
\[
x^\mu u(x) - \int_0^x t^{\mu-1} u(t) dt = f(x). x^\mu
\]

The equation (24) was derived as follows:
\[ x^\mu u'(x) + \mu x^{\mu-1} u(x) - \frac{1}{x^{1-\mu}} u(x) = \mu x^{\mu-1}. f(x) + f'(x). x^\mu \]

Both sides were multiplied by \( x^{1-\mu} \) and following equation was obtained:
\[
x x u(x) + (\mu - 1). u(x) = \mu x f(x) + x f'(x)
\]
\[
u(0) = \frac{\mu}{\mu - 1} f(0)
\]

The equation (26) was differentiated seven times with respect to \( x \) and following equations were the results:
\[ x u^{(7)}(x) + \mu u^{(6)}(x) = (\mu + 1) f'(x) + x f''(x) \]
\[ xu''(x) + (\mu + 1) u'(x) = (\mu + 2)f''(x) + xf'''(x) \]
\[ xu^{(4)}(x) + (\mu + 2) u''(x) = (\mu + 3)f''(x) + xf^{(4)}(x) \]
\[ xu^{(5)}(x) + (\mu + 3) u'(x) = (\mu + 4)f^{(4)}(x) + xf^{(5)}(x) \]
\[ xu^{(6)}(x) + (\mu + 4) u(x) = (\mu + 5)f^{(5)}(x) + xf^{(6)}(x) \]
\[ xu^{(7)}(x) + (\mu + 5) u(x) = (\mu + 6)f^{(6)}(x) + xf^{(7)}(x) \]
\[ xu^{(8)}(x) + (\mu + 6) u(x) = (\mu + 7)f^{(7)}(x) + xf^{(8)}(x) \]  

(28)

The \( x \) terms were replaced with \( a \) (the minimum bound of integration) in the previous equations and then, we get the following equations:

\[ u_0' = \frac{\mu + 1}{\mu} f'(a) \]
\[ u_0'' = \frac{\mu + 2}{\mu + 1} f''(a) \]
\[ u_0''' = \frac{\mu + 3}{\mu + 2} f'''(a) \]
\[ u_0^{(4)} = \frac{\mu + 4}{\mu + 3} f^{(4)}(a) \]
\[ u_0^{(5)} = \frac{\mu + 5}{\mu + 4} f^{(5)}(a) \]
\[ u_0^{(6)} = \frac{\mu + 6}{\mu + 5} f^{(6)}(a) \]
\[ u_0^{(7)} = \frac{\mu + 7}{\mu + 6} f^{(7)}(a) \]  

(29)

Next, we will present an algorithm for finding the numerical solution to the Linear Volterra Integral Equation of the second type with a weakly singular kernel using the fifth order non-polynomial Spline functions. The steps to acquire the solution are listed below:

1) Values of parameters used:

\[ u_0 = f(a) \quad , \quad t_i = t_0 + i h \quad , \quad h = \frac{b - a}{n} \]
\[ a = t_0 \quad , \quad t_n = b \quad , \quad \text{where} \; i = 0, 1, 2, ..., n \]

2) Parameters calculated:

\[ a_0, b_0, c_0, d_0, e_0, r_0, z_0, g_0 \]

Substitutes equations (27), (29) into the equations (15) \( \rightarrow \) (22).

3) \( p_0(t) \) calculated using the second step and using equations (13).

4) Approximated \( u_1 \approx p_0(t_1) \).

5) To acquire the \( i = 1 \) to \( n-1 \) terms, following steps are taken.

6) Terms \( a_i, b_i, c_i, d_i, e_i, r_i, z_i, g_i \) calculated using the equations (15) \( \rightarrow \) (22). Also, all \( u(t_i) \) and its derivatives were substituted with \( p_i(t_i) \) and its derivatives.

7) \( p_i(t_i) \) was calculated using the sixth step and equation (13).

8) Approximated \( u_{i+1} \approx p_i(t_{i+1}) \).

A program was written in Matlab language to find a numerical solution to linear Volterra integral equation of the second type with weakly singular kernel using the fifth order non-polynomial spline functions as follows:

```matlab
function [err, u] = singular (f,ex,a,b,n,m)
syms
h= (b-a)/n;
x=a:h:b;
```


\% x = x + h;
u(1) = (m / (m-1)) * subs(f, 0);
du0 = ((m+1)/m) * subs(diff(f, 1), 0);
d2u0 = ((m+2)/m+1) * subs(diff(f, 2), 0);
d3u0 = ((m+3)/m+2) * subs(diff(f, 3), 0);
d4u0 = ((m+4)/m+3) * subs(diff(f, 4), 0);
d5u0 = ((m+5)/m+4) * subs(diff(f, 5), 0);
d6u0 = ((m+6)/m+5) * subs(diff(f, 6), 0);
d7u0 = ((m+7)/m+6) * subs(diff(f, 7), 0);
a(1) = -d6u0;
b(1) = -d7u0;
c(1) = du0 + d7u0;
a(i+1) = a(i) * cos(h) + b(i) * sin(h) + c(i) + d(i) * h^2 + e(i) * h^3 + r(i) * h^4 + z(i) * h^5 + g(i);
d2u(i+1) = a(i) * cos(h) + b(i) * sin(h) + c(i) + 2 * d(i) * h + e(i) * h^2 + r(i) * h^3 + z(i) * h^4 + 12 * r(i) * h^2 + 20 * z(i) * h^3;
d3u(i+1) = a(i) * sin(h) - b(i) * cos(h) + 6 * e(i) + 24 * r(i) * h + 60 * z(i) * h^2;
d4u(i+1) = a(i) * cos(h) + b(i) * sin(h) + 24 * r(i) * h + 120 * z(i) * h;
d5u(i+1) = a(i) * sin(h) + b(i) * cos(h) + 120 * z(i) * h;
d6u(i+1) = a(i) * cos(h) - b(i) * sin(h);
d7u(i+1) = a(i) * sin(h) - b(i) * cos(h);
a(i+1) = -d6u(i+1);
b(i+1) = d7u(i+1);
c(i+1) = du0(i+1) + d7u(i+1);
a(i+1) = (1/2) * (d2u(i+1) + du0(i+1));
c(i+1) = (1/6) * (d3u(i+1) + du0(i+1));
z(i+1) = (1/120) * (d5u(i+1) + du0(i+1));
r(i+1) = (1/24) * (d4u(i+1) + du0(i+1));
g(i+1) = u(i+1) + d6u(i+1);
for i = 1:n
erri = abs(u(i) - subs(ex, x(i)));
end

The program was applied for a sample calculation that will be shown in this section. Let a VEI equation of the second type with weakly singular kernel as below:

\[ u(x) - \int_0^x t^{\mu-1} x^\omega u(t) dt = f(x) \quad 0 \leq t \leq 1 \]
\[ f(x) = 0.71428571 \ x^3, \ h = 0.1, \ \mu = 0.5, \ u(x) = x^3. \]

Table (1) gives a comparison between the exact solution presented by Diogo, T-Lima, P in 2008 and the numerical solutions achieved in this study. The numerical solutions include the solutions achieved using the non-polynomial spline functions of the first order (linear), the second order (quadratic), and the non-singular spline functions of the fifth order [12,13].
Table (1). Comparison between the exact and the numerical solution

| x   | Exact solution | Linear non-polynomial spline functions | Quadratic non-polynomial spline functions | Fifth order non-polynomial spline functions |
|-----|----------------|----------------------------------------|------------------------------------------|-------------------------------------------|
| 0   | 0              | 0                                      | 0                                        | 0                                         |
| 0.1 | 0.001000000000000 | 0.000999500113034 | 0.000999500113034 | 0.000999999940000 |
| 0.2 | 0.000800000000000 | 0.007984015118729 | 0.007984015118729 | 0.007999999520000 |
| 0.3 | 0.027000000000000 | 0.026878598706900 | 0.026878598706900 | 0.026999998380000 |
| 0.4 | 0.064000000000000 | 0.063348994576715 | 0.063489945767158 | 0.063999999616000 |
| 0.5 | 0.125000000000000 | 0.123446767634102 | 0.123446767634102 | 0.124999999250000 |
| 0.6 | 0.216000000000000 | 0.212145158356918 | 0.212145158356918 | 0.215999998704000 |
| 0.7 | 0.343000000000000 | 0.334693874565692 | 0.334693874565692 | 0.342999997942000 |
| 0.8 | 0.512000000000000 | 0.495163451627685 | 0.495163451627685 | 0.511999996928000 |
| 0.9 | 0.729000000000000 | 0.700385803487272 | 0.700385803487272 | 0.728999995626000 |
| 1   | 1.000000000000000 | 0.951174085445580 | 0.951174085445580 | 0.999999940000000 |

Table (2) gives the calculated errors the solutions in the Table (1) and it were calculated with following equations:

\[
Error = |\text{exact value} - \text{numerical value}|
\]

\[
\|err\|_\infty = \max |\text{Error}|
\]

Table (2). Comparison between linear, quadratic and the fifth order error

| x   | Error In Linear | Error In Quadratic | Error In The Fifth order |
|-----|-----------------|--------------------|--------------------------|
| 0.1 | 0               | 0                  | 0                        |
| 0.2 | 4.998869659316330 e -07 | 4.998869659316330 e -07 | 5.99999997603 e -12 |
| 0.3 | 1.598418271247777 e -05 | 1.598418271247777 e -05 | 4.79999980820e -11 |
| 0.4 | 1.212401293096826 e -04 | 1.212401293096826 e -04 | 1.620000036949 e -10 |
| 0.5 | 5.100542328419638 e -04 | 5.100542328419638 e -04 | 3.839999846561e -10 |
| 0.6 | 1.553232365897550 e -03 | 1.553232365897550 e -03 | 7.99999788855e -10 |
| 0.7 | 3.854841643081697 e -03 | 3.854841643081697 e -03 | 1.2959991849361 e -9 |
| 0.8 | 8.306125434307476 e -03 | 8.306125434307476 e -03 | 2.0579982611055e -9 |
| 0.9 | 1.613658437231471 e -02 | 1.613658437231471 e -02 | 3.0719998772489e -9 |
| 1   | 2.896146196512817 e -02 | 2.896146196512817 e -02 | 4.3739982899933 e -9 |

\[
\|err\|_\infty = 2.896146196512817 e -02
\]

\[
2.896146196512817 e -02
\]

\[
4.3739982899933 e -9
\]
3. Conclusion
Our work calculates the results for the Volterra integral equation with errors. Computational method for finding the numerical solution to the linear Volterra integral equation of the second type with weakly singular kernel using the fifth order non-polynomial Spline functions was shown. This method is characterized by being efficient, easy, fast, and has high accuracy compared to other numerical methods. As for the future work, we will investigate a numerical solution to the nonlinear Volterra integral equation using non-boundary Spline functions.

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