COUNTEREXAMPLES AND SUPPLEMENTS TO CHAOS IMPLIED BY FIXED POINTS FOR A CLASS OF DIFFERENTIAL INCLUSIONS THAT ARISE IN ECONOMIC MODELS

BARBORA VOLNÁ

ABSTRACT. In this paper, we deal with continuous multi-valued dynamical systems on $\mathbb{R}^2$ of the form $\dot{x} \in F(x)$ where $F(x)$ is a set-valued function and $F = \{f_1, f_2\}$. We focus on the results from [8] where authors showed that chaos is implied by fixed points for this class of differential inclusions, more precisely they establish the sufficient conditions for Devaney chaos, $\omega$-chaos and infinite topological entropy to be present in such a system. Their formulations of the sufficient conditions are not entirely accurate which might be misleading. We show these problems on counterexamples and we suggest the way how to correct these inaccuracies. At the end, we illustrate these problems on our own macroeconomic model.

1. Introduction and preliminaries

In this article, we closely follow the paper [8] where authors showed that fixed points imply chaos for a class of differential inclusions. A differential inclusion is given by

$$\dot{x} \in F(x)$$

where $F$ is a set-valued map which associates a set $F(x) \subseteq \mathbb{R}^n$ to every point $x \in \mathbb{R}^n$, see e.g. [10]. We can say that such a system defines a multivalued dynamical system. Raines and Stockman researched the particular class of differential inclusion on $\mathbb{R}^2$ for

$$F = \{f_1, f_2\}$$

where $f_{1,2} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are $C^1$ functions. A solution of this differential inclusion is an absolutely continuous function $x : \mathbb{R} \rightarrow \mathbb{R}^2$ such that

$$\dot{x}(t) \in F(x(t)) \text{ a.e.}$$

where $t \in \mathbb{R}$, see [8]. The set of all solutions of this differential inclusion is denoted by $D$. In the set $D$, there can be the solutions of the first branch, i.e. of the differential equation $\dot{x} = f_1(x)$, the solutions of the second branch, i.e. of the differential equation $\dot{x} = f_2(x)$,

2010 Mathematics Subject Classification. 34A60, 54H20, 37D45, 37N40.

Key words and phrases. differential inclusion, chaos, fixed points, economic cycle.
and the solutions constructed by jumping from the integral curves generated by $f_1$ to the integral curves generated by $f_2$, and vice versa, in some points from $R^2$, see [8]. It is closely related to the modelled problem. We say that such a solution 'switches' from the integral curve generated by $f_1$ to the integral curve generated by $f_2$, and vice versa, in these points. Any such solution is connected with the time sequence of switching from the branch $f_1$ to the branch $f_2$, and vice versa. The other way around, we say that the solution $x$ such that $\dot{x}(t) = f_1(x(t))$, or $\dot{x}(t) = f_2(x(t))$, for $t \in [t_a, t_b]$ with $x(t_a) = a$ and $x(t_b) = b$ where $a, b \in R^2$ 'follows' the integral curve generated by $f_1$, or $f_2$, from $a$ to $b$.

Further, we consider the metric on the set $D$ defined by

$$\nu(x, y) = \sup_{t \in R} \frac{\nu_t(x, y)}{2^{|t|}}$$

where $\nu_t(x, y) = \min\{d(x(t), y(t)), 1\}$ and $d(\cdot, \cdot)$ is the usual metric on $R^2$, see [8]. So, $D$ can be considered as a topological space with the topology generated by this metric. Thus, two relevant spaces connected with this dynamical system are considered: the state space $R^2$ and the space of solutions $D$.

Further, there is considered the natural $R$-action on $D$ instead of the action on the differential inclusion. Let

$$T : D \times R \to D$$

be the natural $R$-action on $D$ and $T(x, t) = y = T_t(x)$, $y(s) = x(t + s)$ for all $s \in R$ as in [8]. Naturally, if $T$ is chaotic on $D$ in certain sense then the original multivalued dynamical system $\dot{x} \in \{f_1(x), f_2(x)\}$ is considered chaotic too. In the following, we deal with Devaney chaos, $\omega$-chaos and topological entropy as in [8]. Let $R \subseteq D$ be closed and $T$-invariant. We say that $(R, T)$ has Devaney chaos, if $T$ is topological transitive, has a dense set of periodic points and has sensitive dependence on initial conditions on $R$ as usually considered, see [1]. Let $S \subseteq D$ (having at least two points). We say that $S$ is an $\omega$-scrambled set [8], if for any $x, y \in S$ with $x \neq y$

- $\omega(x) \setminus \omega(y)$ is uncountable,
- $\omega(x) \cap \omega(y)$ is not empty,
- $\omega(x)$ is not included in the set of periodic points

where $\omega(x)$ and $\omega(y)$ is the omega-limit set of $x$ and of $y$ under $T$. The natural $R$-action $T : D \times R \to D$ is called $\omega$-chaotic provided there exists an uncountable $\omega$-scrambled set in $D$, see [8], [9]. Now, for the definition of the topological entropy related to our system
we need the following terms. Let \( s \in \mathbb{R}^+ \) and for each \( x, y \in D \)

\[
\nu_s^T(x, y) = \max_{-s \leq t \leq s} \{ \nu(T_t(x), T_t(y)) \}
\]
gives the metric on the orbit segments, see [8]. The notion of the topological entropy [5] is extended in the usual way for our considered system [11] as we can see below. Let \( B_T(x, \epsilon, s) = \{ y \in D : \nu_s^T(x, y) < \epsilon \} \) denote the open \( \epsilon \)-ball centred at \( x \in D \) in the previous metric. We say that \( E \subseteq D \) is \((s, \epsilon)\)-spanning provided

\[
D \subseteq \bigcup_{x \in E} B_T(x, \epsilon, s).
\]

Let \( S_\nu(T, \epsilon, s) \) be the minimal cardinality of \((s, \epsilon)\)-spanning set. Finally, the topological entropy of \( T \) is defined by

\[
h_{top}(T) = \lim_{\epsilon \to 0} \limsup_{s \to \infty} \frac{1}{s} \log S_\nu(T, \epsilon, s).
\]

Raines and Stockman [8] proved that fixed points imply chaos for our class of differential inclusion in two steps.

Step 1 They formulated the sufficient conditions for a set of solutions to exhibit Devaney chaos, \( \omega \)-chaos and infinite topological entropy in the plane region \( V \) in [8], and Li-Yorke and distributional chaos in [12]. Let these conditions be denoted by (RS1) and (RS2). (RS1) and (RS2) are specified below. But their formulations are not entirely accurate and in this paper we try to specify these problems and suggest a correction of inaccuracies.

Step 2 They showed that fixed points imply (RS1) and (RS2) [8]. More precisely, the set \( V \subset \mathbb{R}^2 \) with the set of solutions \( D \) fulfilling (RS1) and (RS2) is ensured by conditions:

- \( a^* \in \mathbb{R}^2 \) with \( f_1(a^*) = 0 \) and \( f_2(a^*) \neq 0 \); \( a^* \) is a hyperbolic singular point of the branch \( f_1 \) in the region where the branch \( f_2 \) has no bounded solutions;

- the singular point \( a^* \) is a sink, or a source, or also a saddle point with requirement that \( f_2(a^*) \) is not scalar multiple of an eigenvector of \( Df_1(a^*) \).

Note, the region \( V \) is near the singular point \( a^* \) because of the local character of hyperbolic singular points, generally. In [13], I showed that the last situation where \( a^* \) is a saddle point and \( f_2(a^*) \) is scalar multiple of an eigenvector of \( Df_1(a^*) \) also ensures (RS1) and (RS2).

Now, we present the mentioned Raines-Stockman’s sufficient conditions and for this we need the definition of a simple path between two points in \( \mathbb{R}^2 \) [8], [12]. Let \( a, b \in \mathbb{R}^2 \). There
is a path from \( a \) to \( b \) generated by \( D \) if there exists a solution \( v \in D \) and \( t_0, t_1 \in \mathbb{R} \) such that \( t_0 < t_1 \) with \( v(t_0) = a \) and \( v(t_1) = b \). Moreover, if \( \dot{v} \) has finitely many discontinuities on \([t_0, t_1]\) and \( a \neq v(t) \neq b \) for all \( t_0 < t < t_1 \) than this path is the simple path. The simple path from \( a \) to \( b \) generated by \( D \) is \( \{v(t) : t_0 \leq t \leq t_1\} \subseteq \mathbb{R}^2 \) oriented in the sense of increasing time and is denoted by \( P_{ab} \), or \( P_{ab}(t) \) for \( t_0 \leq t \leq t_1 \), or simply \( P \) if it does not cause any confusion. Note that \( P_{ab} \) and \( P_{ba} \) can be the same subset of \( \mathbb{R}^2 \) with the reverse orientation. In the following, we use also the notation \( P(t_0) = a \) and \( P(t_1) = b \). Let \( V \subseteq \mathbb{R}^2 \) and \( V^* = \{x \in D | x(t) \in V \text{ for all } t \in \mathbb{R}\} \). According to Raines and Stockman [8] the sufficient conditions for \( T|_{V^*} \) to exhibit Devaney chaos are the following.

(RS1) For every \( a, b \in V \) there is a simple path from \( a \) to \( b \) in \( V \) generated by \( D \).

(RS2) There is a solution \( w \in D \) such that \( w(t) \in V \) for all \( t \in \mathbb{R} \) and \( \{w(t) : t \in \mathbb{R}\} \) is not dense in \( V \).

Next to this, only (RS1) is the sufficient condition for \((D, T)\) to exhibit \( \omega \)-chaos and infinite topological entropy according to [8]. Such a statement leads me to find a set of solutions and corresponding \( V \subseteq \mathbb{R}^2 \) where the conditions (RS1) and (RS2) are satisfied, see [13]. But I find also examples where these conditions are fulfilled but \( T|_{V^*} \) is not chaotic in the sense of Devaney, or \((D, T)\) is not \( \omega \)-chaotic and has finite topological entropy. I present some of them in the section 2. From this it follows that Raines-Stockman’s sufficient conditions are not exactly specified. Therefore, in the section 3 we suggest how to correct these inaccuracies and we reformulate the sufficient conditions for a set of solutions to exhibit Devaney chaos, \( \omega \)-chaos and infinite topological entropy. In the section 4, we show that fixed points imply these reformulated sufficient conditions. In this paper, we do not deal with Li-Yorke and distributional chaos but the principle is similar. In conclusion, we illustrate these issues on our own macroeconomic equilibrium model with an economic cycle and we demonstrate the mentioned problems on this application in economics.

2. ILLUSTRATIVE COUNTEREXAMPLES

Let us repeat that according to Raines and Stockman [8] the sufficient conditions for a set of solutions to exhibit Devaney chaos are (RS1) and (RS2), to exhibit \( \omega \)-chaos and infinite topological entropy is (RS1). Essentially, this means that \( V \) can be an arbitrary subset of \( \mathbb{R}^2 \) with restrictions given by (RS1), or by (RS1) and (RS2), to guarantee chaotic behaviour in our multivalued dynamical system. I discover two problems in such a formulation of the sufficient conditions. Firstly, the set \( V \) can not be an arbitrary subset of \( \mathbb{R}^2 \). In Counterexample 2.2 there is the example of the set \( V \subset \mathbb{R}^2 \) where \( \omega \)-chaos is
not present although (RS1) is fulfilled. Secondly, one additional sufficient condition for a set of solutions is frequently used in Raines-Stockman proofs but is not plainly referred, and so this condition should be clearly formulated. In Counterexample 2.1, there is the example of the set of solutions where Devaney chaos is not present although (RS1) and (RS2) are fulfilled, and the example of the set of solutions where $\omega$-chaos and infinite topological entropy are not present although (RS1) is fulfilled.

**Counterexample 2.1.** On Figure 1, there is displayed the considered subset of $\mathbb{R}^2$ denoted by $U$. The set $U$ is depicted as the line segment with endpoints $c_1$ and $c_2$. The appropriate differential inclusion is denoted by $G = \{g_1, g_2\}$, and so $\varphi, \psi$ denote the flows generated by $g_1, g_2$, respectively. The arrows represent the trajectories of the flows $\varphi$ and $\psi$. Further, the subset $U_0$ of the set $U$ is depicted as the line segment with endpoints $c_0$ and $c_2$, see Figure 1. Now, we focus on the specification of two sets of solutions denoted by $\hat{R}$ and by $\bar{R}$. Firstly, let us consider $X$ as the set of solutions representing the periodic solutions passing through the entire set $U$ (cycle). More precise description of the set $X$ is given in the following. Let $x$ denote an element from $X$ and let $X := \{x_{a\varphi}, x_{a\psi}, x_{c_1}, x_{c_2}\}$ where $x_{a\varphi}$ and $x_{a\psi}$ represent the sets of solutions with one solution for every $a \in U \setminus \{c_1, c_2\}$. Each solution $x \in X$ follows the integral curve generated by $g_1$ to the point $c_2$, then $x$ switches to the integral curve generated by $g_2$ and follows this integral curve to the point $c_1$, then in the point $c_1$ the solution $x$ switches to the integral curve generated by $g_1$ and so on for $t \to \infty$ and also for $t \to -\infty$, see Figure 1. So, $x_{a\varphi}$ denotes the solution in $X$ with the initial condition $x_{a\varphi}(0) = a$ for $a \in U \setminus \{c_1, c_2\}$, such that $x_{a\varphi}$ initially follows the integral curve generated by $g_1$ from $a$ to the point $c_2$. Analogously, $x_{a\psi}$ denotes the solution in $X$ with the initial condition $x_{a\psi}(0) = a$ for $a \in U \setminus \{c_1, c_2\}$, such that $x_{a\psi}$ initially follows the integral curve generated by $g_2$ from $a$ to the point $c_1$. $x_{c_1}, x_{c_2}$ denotes the solution with the initial condition $x_{c_1}(0) = c_1$, $x_{c_2}(0) = c_2$, respectively. Obviously, $x_{c_1}$ initially follows the integral curve generated by $g_1$ and $x_{c_2}$ the integral curve generated by $g_2$. Thus, let $\ldots < t^x_{2i-2} < t^x_{2i-1} < t^x_1 < t^x_i < t^x_1 < t^x_2 < \ldots$ be the time sequence of switching from the branch $g_1$ to the branch $g_2$ in times $t^x_{2i+1}$, $i \in \mathbb{Z}$, and from $g_2$ to $g_1$ in times $t^x_{2i}$, $i \in \mathbb{Z}$, for each $x \in X$. Let $t^x_0 < 0 < t^x_1$ for $x_{a\varphi}$,
$t^x_{-1} < 0 < t^x_0$ for $x_{av}$, $t^x_0 = 0$ for $x_{c_1}$ and $t^x_1 = 0$ for $x_{c_2}$. Thus, for each $x \in X$

$$\dot{x}(t) = g_1(x) \text{ for } t \in \bigcup_{i \in \mathbb{Z}}(t^x_{2i}; t^x_{2i+1})$$

$$\dot{x}(t) = g_2(x) \text{ for } t \in \bigcup_{i \in \mathbb{Z}}(t^x_{2i-1}; t^x_{2i})$$

with $x(t^x_{2i}) = c_1$ and $x(t^x_{2i+1}) = c_2$ where $i \in \mathbb{Z}$, see Figure 1. Let $\tau_1 := t^x_{2i+1} - t^x_{2i}$ and $\tau_2 := t^x_{2i} - t^x_{2i-1}$ for each $x \in X$. Note that $\tau_1$ and $\tau_2$ are the same values for all $x$. We see that each solution $x$ is the periodic solution of the length $\tau = \tau_1 + \tau_2$, i.e. $x(t+\tau) = x(t)$ for every $t \in \mathbb{R}$. Secondly, let us consider $X_0$ as the set of solutions representing the periodic solutions passing through the entire subset $U_0 \subset U$ (cycle). More precise description of the set $X_0$ is the analogous description as previous with the difference that the end points of the set $U_0$ are $c_0$ and $c_2$, so we write $c_0$ instead of $c_1$ in the previous description. Finally, define $\hat{\mathcal{R}} := X \cup X_0$ and $\hat{R} := X$. Let $\hat{T}$ denote the natural $\mathbb{R}$-action on $\hat{R}$ and let $\hat{T}$ denote the natural $\mathbb{R}$-action on $\hat{R}$. Clearly, $\hat{R}$, $\hat{R}$ are both closed and $\hat{T}$-, $\hat{T}$-invariant, respectively. Obviously, for every $a, b \in U$ there is a simple path from $a$ to $b$ in $U$ generated by $X$ and there is a solution $w \in X_0$ such that $w(t) \in U$ for all $t \in \mathbb{R}$ and $\{w(t) : t \in \mathbb{R}\}$ is not dense in $U$. So, if we consider $U$ with $\hat{R}$ the conditions (RS1) and (RS2) are fulfilled, and if we consider $U$ with $\hat{R}$ the condition (RS1) is fulfilled.

**Lemma 2.1.** Let $U$ be the subset of $\mathbb{R}^2$ specified above and $\hat{R}$ be the set of solutions specified above. Let $U^* = \{y \in \hat{R} | y(t) \in U \text{ for all } t \in \mathbb{R}\}$. Then $\hat{T}|_{U^*}$ is not chaotic in the sense of Devaney.

**Proof.** Obviously, $U^* = \hat{R}$. We use the proof by contradiction to show that $\hat{T}|_{U^*}$ has not sensitive dependence on initial conditions, and therefore $\hat{T}|_{U^*}$ is not chaotic in the sense of Devaney. So, let the sensitive dependence on initial conditions be assumed in this system. Let $0 < \epsilon_0 < 1$. Pick $x_{c_1} \in U^*$ with $x_{c_1}(0) = c_1$ specified above. Let $B_{\epsilon_0}(x_{c_1})$ be the open $\epsilon_0$-neighbourhood around $x_{c_1}$ with respect to the $\nu$ metric. Let $\delta > 0$ be the sensitivity constant, i.e. for any $x \in U^*$ and $\epsilon > 0$ there is a solution $z$ and $s \in \mathbb{R}$ such that $\nu(\hat{T}(x,s), \hat{T}(z,s)) > \delta$ and $z \in B_{\epsilon}(x)$. Pick $z_0 \in U^*$ such that $z_0 \in B_{\epsilon_0}(x_{c_1})$ and $\nu(\hat{T}(x_{c_1}, s_0), \hat{T}(z_0, s_0)) > \delta$ for some $s_0 \in \mathbb{R}$. Let $\delta_0 := \nu(x_{c_1}, z_0)$. Since $z_0 \in B_{\epsilon_0}(x_{c_1}) \subseteq U^*$, $x_{c_1} \neq z_0$, $x_{c_1}(0) \neq z_0(0)$ and $\epsilon_0 < 1$ we see that $d(x_{c_1}(t), z_0(t)) \leq d(x_{c_1}(0), z_0(0)) < 1$ for every $t$ and $0 < \delta_0 < \epsilon_0$. Thus, we have

$$0 < \delta < \nu(\hat{T}(x_{c_1}, s_0), \hat{T}(z_0, s_0)) \leq \delta_0 < \epsilon_0$$
for some $s_0 \in \mathbb{R}$. Now, let $0 < \epsilon_1 < \delta$. Let $B_{\epsilon_1}(x_{c_1})$ be the open $\epsilon_1$-neighbourhood around $x_{c_1}$ with respect to the $\nu$ metric. Pick $z_1 \in U^*$ such that $z_1 \in B_{\epsilon_1}(x_{c_1})$ and $\nu(\hat{T}(x_{c_1}, s_1), \hat{T}(z_1, s_1)) > \delta$ for some $s_1 \in \mathbb{R}$. Let $\delta_1 := \nu(x_{c_1}, z_1)$. Finally, there is the following contradiction:

$$\delta < \nu(\hat{T}(x_{c_1}, s_1), \hat{T}(z_1, s_1)) \leq \delta_1 < \epsilon_1 < \delta$$

for some $s_1 \in \mathbb{R}$.

Lemma 2.2. Let $U$ be the subset of $\mathbb{R}^2$ specified above and $\bar{R}$ be the set of solutions specified above. Then $(\bar{R}, \bar{T})$ is not $\omega$-chaotic.

Proof. Pick arbitrary $x \in \bar{R}$. Clearly, by construction $\bar{T}(x, \tau) = x$ and for every $y \in \bar{R}$ with $y \neq x$ there exists $r \in (0, \tau)$ such that $y = \bar{T}(x, r)$. And, $\bar{T}(x, r) = \bar{T}(x, r + n\tau) = y$ for $n \in \mathbb{N}$. So, $\lim_{n \to \infty} \bar{T}(x, r + n\tau) = y$ and thus $y \in \omega(x)$. We see that $\omega(x) = \bar{R}$ for each $x \in \bar{R}$ and $\bar{R}$ contains only periodic points. Thus, there is no $\omega$-scrambled set. ☐

Lemma 2.3. Let $U$ be the subset of $\mathbb{R}^2$ specified above and $\bar{R}$ be the set of solutions specified above. Then $\bar{T}$ has zero topological entropy.

Proof. Initially, we show that $S_{\nu}(\bar{T}, \epsilon, s) = 1$ for $s \geq \frac{\tau}{2}$ and for arbitrary $\epsilon > 0$. The orbit of $\bar{T}$ through $x \in \bar{R}$ is $\{\bar{T}_t(x) | t \in \mathbb{R}\}$. Note that $\bar{T}_t(x) \in \bar{R}$ for every $t$. We know that each solution $x \in \bar{R}$ is periodic solution of the length $\tau$. Pick arbitrary $v, w$ from $\bar{R}$. So, we have

$$\{\bar{T}_t(v) | t \in \mathbb{R}\} = \{\bar{T}_t(v) | t \in [0, \tau)\} = \{\bar{T}_t(w) | t \in [0, \tau)\} = \{\bar{T}_t(w) | t \in \mathbb{R}\}$$

with $\bar{T}_0(v) = \bar{T}_\tau(v)$ and $\bar{T}_0(w) = \bar{T}_\tau(w)$. Let $\{v_i\}_{i \in [0, \tau)}$ and $\{w_j\}_{j \in [0, \tau)}$ denote elements of the orbit of $\bar{T}$ through $v$ and through $w$, respectively. By construction, for every $x, y \in \bar{R}$ there exists $r \in [0, \tau)$ such that $\bar{T}(x, r) = y$. Let $z \in \bar{R}$ and let $\{z_k\}_{k \in [-\frac{\tau}{2}, \frac{\tau}{2})}$ be elements from the orbit of $\bar{T}$ through $z$. Thus, for all $v_i$ we find $r_i \in [0, \tau)$ such that $\bar{T}(v_i, r_i) = z_i$ for some $l \in [-\frac{\tau}{2}, \frac{\tau}{2})$ and $\{z_l\}_{l \in [-\frac{\tau}{2}, \frac{\tau}{2})}$ contains all elements of the orbit of $\bar{T}$ through $z$. Analogously, for all $w_j$ we find $r_j \in [0, \tau)$ such that $\bar{T}(w_j, r_j) = z_m$ for some $m \in [-\frac{\tau}{2}, \frac{\tau}{2})$ and $\{z_m\}_{m \in [-\frac{\tau}{2}, \frac{\tau}{2})}$ contains all elements of the orbit of $\bar{T}$ through $z$. Obviously, $\{z_l\}_{l \in [-\frac{\tau}{2}, \frac{\tau}{2})} = \{z_m\}_{m \in [-\frac{\tau}{2}, \frac{\tau}{2})}$. So, we see that

$$\nu_s^\tau(v, w) = \max_{-s \leq t \leq s} \{\nu(\bar{T}_t(v), \bar{T}_t(w))\} = 0$$

for $s \geq \frac{\tau}{2}$ and for arbitrary $v, w \in \bar{R}$. From this it follows that $(s, \epsilon)$-spanning $\bar{E} = \{x\}$ where $x \in \bar{R}$ arbitrary, and $B_T(x, \epsilon, s) = \bar{R}$ for $s \geq \frac{\tau}{2}$ and for arbitrary $\epsilon > 0$. Thus,
the minimal cardinality of \((s, \epsilon)-\text{spanning } \mathcal{E}\) is \(S_\nu(\bar{T}, \epsilon, s) = 1\) for \(s \geq \frac{\tau}{2}\) and for arbitrary \(\epsilon > 0\). Finally,

\[
h_{\text{top}}(\bar{T}) = \lim_{\epsilon \to 0} \limsup_{s \to \infty} \frac{1}{s} \log S_\nu(\bar{T}, \epsilon, s) = \lim_{\epsilon \to 0} \limsup_{s \to \infty} \frac{1}{s} \log 1 = 0.
\]

\[\square\]

**Counterexample 2.2.** On Figure 2, there is displayed the considered subset of \(\mathbb{R}^2\) denoted by \(W\). The set \(W\) is depicted as two curved lines connected in the point \(c_1\) and in the point \(c_2\). The appropriate differential inclusion is denoted by \(H = \{h_1, h_2\}\), and \(\varphi\), \(\psi\) denote the flows generated by \(h_1\), \(h_2\), respectively. The arrows represent trajectories of the flows \(\varphi\) and \(\psi\). The set of solutions denoted by \(\tilde{S}\) is specified in the following. For each \(a \in W\) there is just one solution \(x_a \in \tilde{S}\) such that \(x_a(0) = a\). This solution passes through the entire set \(W\) and alternately follows the integral curve generated by \(h_1\) and the integral curve generated by \(h_2\). In the point \(c_1\) the solution \(x_a\) switches from the integral curve generated by \(h_2\) to the integral curve generated by \(h_1\), and in the point \(c_2\) from the integral curve generated by \(h_1\) to the integral curve generated by \(h_2\). Obviously, each \(x_a\) is the periodic solution. Let \(\bar{T}\) denote the natural \(\mathbb{R}\)-action on \(\tilde{S}\). \(\tilde{S}\) is closed and \(\bar{T}\)-invariant. Evidently, for every \(a, b \in W\) there is a simple path from \(a\) to \(b\) in \(W\) generated by \(\tilde{S}\), so the conditions (RS1) is fulfilled. But we can see that \(\tilde{S}\) contains only periodic solutions and so \((\tilde{S}, \bar{T})\) is not \(\omega\)-chaotic.

### 3. Reformulated sufficient conditions

For now, we say that the subset \(V\) of the state space \(\mathbb{R}^2\) of our system is named the *chaotic set* if there are fulfilled the sufficient conditions for a set of solutions to exhibit the chaos in all mentioned senses and the infinite topological entropy. The name of chaotic set was used in [12]. According to [5], these sufficient conditions are (RS1) and (RS2) for \(V \subset \mathbb{R}^2\). As we show in the previous section, the chaotic set \(V \subset \mathbb{R}^2\) can not be an arbitrary subset of \(\mathbb{R}^2\) with restrictions given only by (RS1) and (RS2), and the
sufficient conditions for a set of solutions are not all highlighted. So, we firstly point out the limitation of $V$ as a subset of $\mathbb{R}^2$. Secondly, we formulate the additional sufficient condition for a set of solutions. Finally, we reformulate the sufficient conditions for $T|_V$ to be Devaney chaotic, for $(D,T)$ to be $\omega$-chaotic and for $T$ to have infinite topological entropy. These reformulated sufficient conditions also ensure Li-Yorke and distributional chaos in our system and the principles are quite similar but in this paper we do not focus on these types of chaoses.

The limitation on the set $V$ as a subset of $\mathbb{R}^2$ is the following. 

$$\mathbb{R}^2 \setminus V$$

is one unbounded subset of $\mathbb{R}^2$. Such restriction excludes the situation from Counterexample 2.2. As we said before, Raines and Stockman [8], [12] used one more sufficient condition for the set of solutions in their proofs and this condition is not adequately highlighted. So, we formulate the new additional condition denoted (BV3) using the definition of a concatenation of paths.

**Definition 3.1.** Let $a_1, a_2, a_3, \ldots \in \mathbb{R}^2$. Let $P_{a_1 a_2}$ be the path from $a_1$ to $a_2$ generated by $D$, let $P_{a_2 a_3}$ be the path from $a_2$ to $a_3$ generated by $D$, etc. Let $t_{a_1}, t_{a_2}, t_{a_3} \ldots \in \mathbb{R}$ be such that $t_{a_1} < t_{a_2} < t_{a_3} < \ldots$ with $P_{a_1 a_2}(t_{a_1}) = a_1, P_{a_1 a_2}(t_{a_2}) = a_2 = P_{a_2 a_3}(t_{a_2}), P_{a_2 a_3}(t_{a_3}) = a_3$, etc. We say that there is a concatenation of paths $P_{a_1 a_2}, P_{a_2 a_3}$, etc. generated by $D$ provided there exists a solution $\gamma \in D$ such that the path $Q$ fulfilled

$$Q(t_{a_1}) = a_1;$$

$$Q(t) = P_{a_1 a_2}(t) \text{ for } t_{a_1} \leq t \leq t_{a_2};$$

$$Q(t_{a_2}) = a_2;$$

$$Q(t) = P_{a_2 a_3}(t) \text{ for } t_{a_2} \leq t \leq t_{a_3};$$

$$Q(t_{a_3}) = a_3;$$

etc.

is generated by $\{\gamma\}$. The concatenation of paths $P_{a_1 a_2}, P_{a_2 a_3}$, etc. generated by $D$ is the path $Q := \{\gamma(t) : t_{a_1} \leq t \leq \ldots\}$.

**BV3** There are all concatenations of simple paths specified in (RS1) generated by $D$. The cases in Counterexample 2.1 show the sets of solutions not fulfilling this condition and so not being chaotic in the mentioned sense. But if there exist another solutions fulfilling the condition (BV3) in these cases then this system is chaotic in the mentioned senses.

In all Raines-Stockman’s proofs [8], [12] there was automatically used the condition (BV3) besides (RS1) and (RS2), i.e. authors did concatenations of simple paths in their
So, (BV3) is one of the sufficient conditions for $T|_V$ to exhibit Devaney chaos, for $(D,T)$ to exhibit $\omega$-chaos and for $T$ to have infinite topological entropy. Additionally, Raines and Stockman [8], [12] considered only the set $V$ with the property that $\mathbb{R}^2 \setminus V$ is one unbounded subset of $\mathbb{R}^2$. So, it seems to be suitable to add (BV3) and our limitation of the set $V \subset \mathbb{R}^2$ to the assumptions of their lemmas and theorems.

Now, we show that (RS2) follows from (BV3) and (RS1) for the set $V$ with the property that $\mathbb{R}^2 \setminus V$ is one unbounded subset of $\mathbb{R}^2$.

Lemma 3.1. (RS1) & (BV3) $\Rightarrow$ (RS2)

Proof. We assume that for every $a, b \in V$ there is a simple path from $a$ to $b$ in $V$ generated by $D$ and that there are all concatenations of these simple paths generated by $D$. We can see that there exist $\bar{a}, \bar{b}, \bar{c} \in V$ and $\epsilon > 0$ such that $P_{\bar{a}\bar{b}}$ and $P_{\bar{b}\bar{a}}$ do not intersect $D_\epsilon(\bar{c})$ where $P_{\bar{a}\bar{b}}$ is the simple path from $\bar{a}$ to $\bar{b}$ in $V$ generated by $D$, $P_{\bar{b}\bar{a}}$ is the simple path from $\bar{b}$ to $\bar{a}$ in $V$ generated by $D$ and $D_\epsilon(\bar{c})$ is the open $\epsilon$-neighbourhood around $\bar{c}$ with respect to the usual metric on $\mathbb{R}^2$. In fact, in each set $V$ complying with our assumptions we can find two points $\bar{a}$ and $\bar{b}$ sufficiently close together such that $P_{\bar{a}\bar{b}}$ and $P_{\bar{b}\bar{a}}$ are represented by the same subset of $\mathbb{R}^2$ with the reverse orientation (see Figure 3), or are represented by the different subset of $\mathbb{R}^2$ with $P_{\bar{a}\bar{b}} \cap P_{\bar{b}\bar{a}} = \{\bar{a}, \bar{b}\}$ (see Figure 4), and the point $\bar{c}$ with $D_\epsilon(\bar{c})$ such that $P_{\bar{a}\bar{b}} \cap D_\epsilon(\bar{c}) = \emptyset$ and $P_{\bar{b}\bar{a}} \cap D_\epsilon(\bar{c}) = \emptyset$. On Figure 3, there are depicted the subset of $V$ with empty interior sketched by the line segment (on the left scheme) and the set $V$ with non-empty interior (on the right scheme) where points $\bar{a}, \bar{b}$ and $\bar{c}$ with $D_\epsilon(\bar{c})$ are located, and there are depicted $P_{\bar{a}\bar{b}}$ and $P_{\bar{b}\bar{a}}$ by the same curve segment with arrows representing the orientation of $P_{\bar{a}\bar{b}}$ and $P_{\bar{b}\bar{a}}$ in the sense of increasing time. On Figure 4, there are depicted the set $V$ with non-empty interior, points $\bar{a}, \bar{b}$, paths $P_{\bar{a}\bar{b}}$ and $P_{\bar{b}\bar{a}}$ and...
the point \( \bar{c} \) with \( \epsilon \)-neighbourhood \( D_\epsilon(\bar{c}) \) located outside (displayed on the left scheme) or inside (displayed on the right scheme) the region bordered by \( P_{\bar{a}\bar{b}} \) and \( P_{\bar{b}\bar{a}} \). In each case, we construct the solution \( w \in D \) so that \( w \) alternately follows the simple path \( P_{\bar{a}\bar{b}} \) and the simple path \( P_{\bar{b}\bar{a}} \). Such a solution exists in \( D \) because of our assumptions (simple paths and concatenations). Let \( t_{\bar{a}\bar{b}} > 0 \) and \( t_{\bar{b}\bar{a}} > 0 \) be defined such that

\[
P_{\bar{a}\bar{b}}(t_{\bar{a}\bar{b}}) = \bar{b}
\]

and

\[
P_{\bar{b}\bar{a}}(t_{\bar{b}\bar{a}}) = \bar{a}.
\]

So, we define \( w \) by

\[
w(t) = a \quad \text{for } t = kt_{\bar{a}\bar{b}} + kt_{\bar{b}\bar{a}};
\]

\[
w(t) = P_{\bar{a}\bar{b}}(t) \quad \text{for } kt_{\bar{a}\bar{b}} + kt_{\bar{b}\bar{a}} \leq t \leq (k + 1)t_{\bar{a}\bar{b}} + kt_{\bar{b}\bar{a}};
\]

\[
w(t) = b \quad \text{for } t = (k + 1)t_{\bar{a}\bar{b}} + kt_{\bar{b}\bar{a}};
\]

\[
w(t) = P_{\bar{b}\bar{a}}(t) \quad \text{for } (k + 1)t_{\bar{a}\bar{b}} + kt_{\bar{b}\bar{a}} \leq t \leq (k + 1)t_{\bar{a}\bar{b}} + (k + 1)t_{\bar{b}\bar{a}};
\]

for \( k \in \mathbb{N}_0 \). We know that \( \{w(t) : t \in \mathbb{R}\} \cap D_\epsilon(\bar{c}) = \emptyset \) and \( D_\epsilon(\bar{c}) \) is open. Since \( \{w(t) : t \in \mathbb{R}\} \subseteq V \setminus D_\epsilon(\bar{c}) \) the solution \( w \) is not dense in \( V \). \( \square \)

Finally, we reformulate the definition of the chaotic set \( V \subset \mathbb{R}^2 \). From the other point of view, we can say that in this definition there are formulated the sufficient conditions for the chaos in the mentioned senses and for the infinite topological entropy to be present in our multi-valued dynamical system.

**Definition 3.2.** Let \( V \) be the subset of \( \mathbb{R}^2 \) such that \( \mathbb{R}^2 \setminus V \) is one unbounded subset of \( \mathbb{R}^2 \). Then \( V \) is called the **chaotic set** provided

(RS1) for every \( a, b \in V \) there is a simple path from \( a \) to \( b \) in \( V \) generated by \( D \);

(BV3) there are all concatenations of these simple paths generated by \( D \).

4. **Fixed points imply reformulated sufficient conditions**

Raines and Stockman [8], [12] showed that for differential inclusion \( F = \{f_1, f_2\} \) with the property that

- \( f_1 \) has a hyperbolic singular point \( a^* \) in a region where \( f_2 \) has no bounded solutions,
- and \( a^* \) is a sink, or a source, or also a saddle point with requirement that \( f_2(a^*) \) is not a scalar multiple of an eigenvector of \( Df_1(a^*) \),

there can be constructed a set \( V \) satisfying (RS1) and (RS2). In [13], we showed that in the last case where \( f_2(a^*) \) is a scalar multiple of an eigenvector of \( Df_1(a^*) \) there can be also constructed a set \( V \) satisfying (RS1) and (RS2). Furthermore in [13], we provided an overview of sets \( V \) satisfying (RS1) and (RS2) for hyperbolic singular points corresponding to both branches not lying in the same point in \( \mathbb{R}^2 \). Naturally, in this section, we show that for our differential inclusion with the property that
• \( f_1 \) has a hyperbolic singular point \( a^* \) in a region where \( f_2 \) has no bounded solutions, we can construct a set \( V \) fulfilling

• \( \mathbb{R}^2 \setminus V \) is one unbounded subset of \( \mathbb{R}^2 \);
• (RS1) and (BV3);

i.e. we can construct the chaotic set \( V \) in the new sense (see Definition 3.2). And then, according to the section 3, we see that the corresponding differential inclusion evinces the chaotic behaviour and the infinite topological entropy.

The construction of the set \( V \) is based on the construction method from [8] but we add supplements and better specifications of some problems, and it is outlined below. Let \( K \subset \mathbb{R}^2 \) denote the non-empty compact set, where \( f_2 \) has no bounded solutions. Let \( a \in K \). Let \( P \) denote the simple path from \( a \) to \( a \) generated by \( D \) such that \( P \subset K \). Thus, \( P \) is a finite union of arcs and is compact. \( \mathbb{R}^2 \setminus P \) has only one unbounded component denoted by \( C_0 \). The set \( \mathbb{R}^2 \setminus P \) can be written as \( \left( \bigcup_{\alpha \in A} C_\alpha \right) \cup C_0 \) where \( C_\alpha \) are bounded components. So, the set \( V \) is given by

\[
\left( \bigcup_{\alpha \in A} C_\alpha \right) \cup P, \tag{1}
\]

is closed, and meets the condition (RS1) and the condition (BV3), see the following paragraphs I, II, III. By construction, \( \mathbb{R}^2 \setminus V \) is one unbounded subset of \( \mathbb{R}^2 \).

I Remind that the solution of our differential inclusion is an absolutely continuous function \( x : \mathbb{R} \to \mathbb{R}^2 \) such that

\[
\dot{x}(t) \in \{ f_1(x(t)), f_2(x(t)) \} \text{ a.e.} \tag{2}
\]

where \( t \in \mathbb{R} \). In the set \( D \), there is each solution fulfilling \( (2) \) which is constructed by jumping from the integral curve generated by \( f_1 \) to the integral curve generated by \( f_2 \), and vice versa, in arbitrary times. So, there exist any path (and corresponding solution) connecting the path belonging to the branch \( f_1 \) and the path belonging to the branch \( f_2 \) with one equal endpoint. More precisely, let \( P_{b_1b_2} = \{ v_1(t) : t_{b_1} \leq t \leq t_{b_2} \} \) be the simple path from \( b_1 \) to \( b_2 \) generated \( D \) such that \( v_1(t) = f_1(v_1(t)) \) for \( t_{b_1} \leq t \leq t_{b_2} \) and \( P_{b_2b_3} = \{ v_2(t) : t_{b_2} \leq t \leq t_{b_3} \} \) be the simple path from \( b_2 \) to \( b_3 \) generated \( D \) such that \( v_2(t) = f_2(v_2(t)) \) for \( t_{b_2} \leq t \leq t_{b_3} \) then there exist \( v \in D \) such that \( P_{b_1b_3} = \{ v(t) : t_{b_1} \leq t \leq t_{b_3} \} \) and \( P_{b_1b_3} = P_{b_1b_2} \cup P_{b_2b_3} \). Analogously for \( v_1, v_2 \) such that \( v_1(t) = f_2(v_1(t)) \) for \( t_{b_1} \leq t \leq t_{b_2} \) and \( v_2(t) = f_1(v_2(t)) \) for \( t_{b_2} \leq t \leq t_{b_3} \).

II We show that the set \( V \) given by \( (1) \) meets the condition (RS1), similarly as in [8]. But Raines and Stockman [8] constructed a path in \( V \) between two arbitrary points from
V which may not be the simple path. Here, we make this construction more accurate, i.e. we construct the simple path in V between two arbitrary points c, d ∈ V. Remind that K ⊂ \( \mathbb{R}^2 \) is a region where the branch \( f_2 \) has no bounded solutions, V ⊂ K and P is the simple path from a to a and represents the boundary of V. So, we can find a simple path in V from c to some \( c_P \in P \) and a simple path in V from some \( d_P \in P \) to d. Thus, the main idea from [8] relies on the reasoning that we can connect four simple paths - from c to \( c_P \), from \( c_P \) to a, from a to \( d_P \), from \( d_P \) to d - and the resulting path is the simple path in V. The meaning of the simple paths connection is described in the previous paragraph. But this new path may not be the simple path if \( c \in P \& d \notin P \), or \( c \notin P \& d \in P \), or \( c, d \in P \). We can illustrate this using Figure 5. There are depicted the set \( V \subset K \) with non-empty interior (on the left scheme) and with empty interior (on the right scheme), the simple path P from a to a with arrows displaying the orientation of P in the sense of increasing time and the trajectories of the flow \( \psi \) generated by \( f_2 \) represented by arrows passing through K. If we consider \( V \) with non-empty interior then according to [8] the path from \( c_1 \notin P \) to \( d_2 \in P \) connects the simple path from \( c_1 \) to \( c_P = c_2 \), the simple path from \( c_P \) to a, the simple path from a to \( d_2 \), and the resulting path is not the simple path (see the left scheme on Figure 5). Or if we consider \( V \) with non-empty interior then according to [8] the path from \( c_2 \in P \) to \( d_2 \in P \) connects the simple path from \( c_2 \) to a, the simple path from a to \( d_2 \), and the resulting path is not the simple path (see the left scheme on Figure 5). Analogously for the V with empty interior, according to [8] the path from \( c_1 = c_P \) to \( d_1 = d_P \) connecting simple paths from \( c_1 \) to a and from a to \( d_1 \) is not simple (see the right scheme on Figure 5). Next to this our main idea relies on the reasoning that we can use the subset of P from \( c_P \) to \( d_P \) which does not have to go through the point a but such that the resulting path is simple. And so, for our previous example for \( V \) with non-empty interior, the simple path from \( c_1 \) to \( d_2 \) connects the simple path from \( c_1 \) to \( c_P = c_2 \) and from \( c_P \) to \( d_2 \), the simple path from \( c_2 \) to \( d_2 \) is the subset of P not containing the point a (see the left scheme on Figure 5). For our previous example for \( V \) with empty interior, the simple path from \( c_1 \) to \( d_1 \) is the subset of P not containing the point a (see the right scheme on Figure 5). And so, using this way we can construct the simple path in V between two arbitrary points from \( V \) given by (1).

III We show that the set \( V \) given by (1) meets the condition (BV3). Note that all solutions from D have the following character - the solution passes through any point
from state space $\mathbb{R}^2$ in one of two possible ways: this solution follows the same integral curve through this point or this solution switches from the original integral curve generated by one branch to the integral curve generated by the other branch of $F = \{f_1, f_2\}$ in this point. Such a character of solutions guarantees the existence of all concatenations of simple paths constructed in the way described in the previous paragraph. We can demonstrate this using Figure 5. For $V$ with non-empty interior, the solution corresponding to the concatenation of the simple paths $P_{c_1c_2}$ and $P_{c_2d_2}$ switches from the integral curve generated by $f_2$ to the integral curve generated by $f_1$ in $c_2$, or the solution corresponding to the concatenation of the simple paths $P_{c_1d_1}$ and $P_{d_1c_2}$ follows the integral curve generated by $f_2$ through $d_1$ (see the left scheme on Figure 5). For $V$ with empty interior, the solution corresponding to the concatenation of the simple paths $P_{c_1c_2}$ and $P_{c_2d_1}$ switches from the integral curve generated by $f_2$ to the integral curve generated by $f_1$ in $c_2$, or the solution corresponding to the concatenation of the simple paths $P_{c_1d_1}$ and $P_{d_1d_2}$ follows the integral curve generated by $f_2$ through $d_1$ (see the right scheme on Figure 5). Using the schemes on Figure 5 we can see that the concatenations of the simple paths for $n \geq 3$ is similar and is done in the same way.

Finally, in [8], [12] and [13], there was shown that for differential inclusion $F = \{f_1, f_2\}$ with the property that $f_1$ has a hyperbolic singular point $a^*$ in a region where $f_2$ has no bounded solutions we can construct a set $V$ given by (1). Such a set $V$ is so-called chaotic set and is located near $a^*$, and so our differential inclusion evinces Devaney chaos, $\omega$-chaos and infinite topological entropy on the set $V$.

**Example 4.1.** Let us consider the differential inclusion $\dot{x} = \{g_1(x), g_2(x)\}$ on $\mathbb{R}^2$ where $g_{1,2} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are linear. Let $\varphi, \psi$ denote the flows generated by $g_1, g_2$, respectively. Let $a_1^*$ and $a_2^*$ be the singular point of the branch $\dot{x} = g_1(x)$ and of the branch $\dot{x} = g_2(x)$ with
\( a_1^* \neq a_2^* \) and let these singular points be sources (i.e. unstable nodes). Thus, we have the similar situation as in Counterexample 2.1 and this situation is depicted on Figure 6. On this figure, we can see the stable nodes \( a_1^*, a_2^* \) not lying in the same point in \( \mathbb{R}^2 \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{Figure 6.}
\end{figure}

illustrated by the appropriate trajectories of \( \varphi, \psi \), and the set \( U \) with empty interior in the \( \epsilon \)-neighbourhood around \( a_1^* \). Thus, the branch \( g_1 \) has the hyperbolic singular point \( a_1^* \) in the region where the branch \( g_2 \) has no bounded solutions and this singular point is a source. Unlike the counterexample, there are also solutions which switch from the integral curve generated by one branch to the integral curve generated by the second branch of this differential inclusion in other points from \( U \) not only in the points \( c_1 \) and \( c_2 \), or \( c_0 \) and \( c_2 \). Thus, there are all concatenations of simple paths generated by \( D \). So, this set \( U \) is so-called chaotic set in the new sense.

**Example 4.2.** Let us consider the differential inclusion \( \dot{x} = \{h_1(x), h_2(x)\} \) on \( \mathbb{R}^2 \) and let the singular point \( b_1^* \) of the branch \( \dot{x} = h_1(x) \) be the saddle point and the singular point \( b_2^* \) of the branch \( \dot{x} = h_2(x) \) be the unstable node with \( b_1^* \neq b_2^* \). Let \( \varphi, \psi \) denote the flows generated by \( h_1, h_2 \), respectively. Thus, we have the similar situation as in Counterexample 2.2 with the difference that we consider the set \( W \) with interior, not only the boundary. On Figure 6 we can see the set \( W \) with non-empty interior depicted by grey area (with the boundary) in the \( \epsilon \)-neighbourhood around \( b_1^* \), the saddle point \( b_1^* \) and the unstable node \( b_2^* \) not lying in the same point in \( \mathbb{R}^2 \) illustrated by the appropriate trajectories of \( \varphi \) and \( \psi \). Thus, the branch \( h_1 \) has the hyperbolic singular point \( b_1^* \) in the region where the branch \( h_2 \) has no bounded solutions and this singular point is a saddle.
point. Unlike the counterexample, this set $W$ has non-empty interior, i.e. $\mathbb{R}^2 \setminus W$ is one unbounded subset of $\mathbb{R}^2$. So, this set $W$ is so-called chaotic set in the new sense.

5. Conclusion and chaotic sets in economics

In this paper, we try to better clarify the problem of chaos existence for a class of differential inclusion in $\mathbb{R}^2$ which is implied by fixed points. We present several counterexamples and supplements to [8] and we suggest the corrections of inaccuracies from [8]. Finally, fixed points imply chaos for this class of differential inclusion but Raines-Stockman’s two steps of the proof are modified and made more precise.

Here, we provide our economic application and illustrations of the mentioned problems. A macroeconomic model with an economic cycle can be an example of our differential inclusion $\dot{x} \in \{f_1(x), f_2(x)\}$ with two branches $f_1$ and $f_2$. The economic cycle (or the business cycle) consists of the expansion phases and of the recession phases which alternate, see [3], [6], [7]. In the peak points the recession replaces the expansion and in the trough points the expansion replaces the recession, see [3], [6], [7]. The first branch $\dot{x} = f_1(x)$ represents the description of macroeconomic situation in a recession and the second branch $\dot{x} = f_2(x)$ represents the description of macroeconomic situation in an expansion. The solution fulfilling $\dot{x}(t) \in \{f_1(x(t)), f_2(x(t))\}$ a.e. and switching from the integral curve generated by $f_1$ to the integral curve generated by $f_2$ and vice versa represents the alternation of these phases in an economy. The trough points are represented
by points where this solution switches from the integral curve generated by $f_1$ to the integral curve generated by $f_2$, and vice versa for the peak points. In [13], we created such a model - the branch belonging to the recession phase is the well-known macroeconomic equilibrium model called IS-LM model (see for example [2], [4]), and the branch belonging to the expansion phase is the newly created model called QY-ML model. This model is named IS-LM/QY-ML model and can be briefly described by

$$
\begin{pmatrix}
\dot{Y} \\
\dot{R}
\end{pmatrix} \in \left\{ \begin{pmatrix}
\alpha_1[I(Y, R) - S(Y, R)] \\
\beta_1[L(Y, R) - M(Y, R)]
\end{pmatrix}, \begin{pmatrix}
\alpha_2[Q(Y, R) - Y] \\
\beta_2[M(Y, R) - L(Y, R)]
\end{pmatrix} \right\}
$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and

- $Y$ is an aggregate income (GDP, GNP),
- $R$ is an interest rate,
- $I$ is an investment,
- $S$ is a saving,
- $Q$ is a production,
- $L$ is a demand for money,
- $M$ is a supply of money.

Firstly, let us consider the set $U \subset \mathbb{R}^2$ and the sets of solutions $\hat{R}$ and $\hat{R}$ described in Counterexample 2.1 occurring in the system given by this model. The set of solutions $\hat{R}$, or $\hat{R}$, can be interpreted so that there exists one regular economic cycle in an economy, or there exist two regular economic cycles with different period. This situation requires an additional restriction on the set of solutions of this system and this restriction follows from the modelled economic problem. Secondly, let us consider the set $W \subset \mathbb{R}^2$ and the set of solutions $\tilde{S}$ described in Counterexample 2.2 occurring in the system given by this model. The interpretation is such that the set $W$ can occur in an economy with some statutory limitations on the levels of the aggregate income $Y$ and of the interest rate $R$ (levels belonging to the 'interior' and 'exterior' are forbidden). This situation requires an additional restriction on the range of solutions, i.e. solutions are absolutely continuous functions $x : \mathbb{R} \rightarrow K_1 \subset \mathbb{R}^2$ with $W \subseteq K_1$, and this restriction follows from the modelled economic problem. And finally, let us consider that properties of relevant economic quantities are such that there are chaotic sets $V$ in the system given by this model (see more in [13]). This case can be interpreted so that there exist economic cycles with all possible periods and lengths of the recession and expansion phases and there are no statutory limitations on the levels of the aggregate income $Y$ and of the interest rate $R$. 

Acknowledgements

The research was supported by Mathematical Institute of Silesian University in Opava, Czech Republic.

References

[1] R.L. Devaney, An introduction to chaotic dynamical systems, Westview Press, Boulder, Colorado, 2nd ed., 2003.
[2] G. Gandolfo, Economic Dynamics, 4th ed., Springer-Verlag, Berlin-Heidelberg, 2009.
[3] J.D. Hamilton, A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle, Econometrica 57 (2) (1989), 357–384.
[4] J.R. Hicks, Mr. Keynes and the classics - a suggested interpretation, Econometrica 5 (2) (1937), 147–159.
[5] A. Katok, B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, New York, NY, 1995.
[6] Motohiro Yogo, Measuring business cycle: A wavelet analysis of economic time series, Econ. Letters 100 (2008), 208–2012.
[7] G. Orlando, A discrete mathematica model for chaotic dynamics in economics: Kaldor’s model on business cycle, Math. and Comput. in Simul., 125 (2016), 83–98.
[8] B.R. Raines, D.R. Stockman, Fixed points imply chaos for a class of differential inclusions that arise in economic models, Trans. American Math. Society 364 (5) (2012), 2479–2492.
[9] Shi Hai Li, ω-chaos and topological entropy, Trans. American Math. Society 339 (1) (1993), 243–249.
[10] G.V. Smirnov, Introduction to the Theory of Differential Inclusions, Graduate Studies in Mathematics, volume 41, American Math. Society, Providence, Rhode Island, 2002.
[11] D.R. Stockman, Chaos and capacity utilization under increasing returns to scale, J. of Econ. Behaviour & Organization, 77 (2011), 147–162.
[12] D.R. Stockman, B.R. Raines, Chaotic sets and Euler equation branching, J. of Math. Econ. 46 (2010), 1173–1193.
[13] B. Volná, Existence of chaos in the plane $\mathbb{R}^2$ and its application in macroeconomics, Appl. Math. Comput. 258 (2015), 237–266.