Converse Sturm-Hurwitz-Kellogg theorem and related results

Serge Tabachnikov*

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Abstract

We prove that if $V^n$ is a Chebyshev system on the circle and $f(x)$ is a continuous function with at least $n+1$ sign changes then there exists an orientation preserving diffeomorphism of $S^1$ that takes $f$ to a function $L^2$-orthogonal to $V$. We also prove that if $f(x)$ is a function on the real projective line with at least four sign changes then there exists an orientation preserving diffeomorphism of $\mathbb{RP}^1$ that takes $f$ to the Schwarzian derivative of a function on $\mathbb{RP}^1$. We show that the space of piece-wise constant functions on an interval with values $\pm 1$ and at most $n+1$ intervals of constant sign is homeomorphic to an $n$-dimensional sphere.

To V. I. Arnold for his 70th birthday

1 Introduction and formulation of results

The classic four vertex theorem asserts that the curvature of a plane oval (strictly convex smooth closed curve) has at least four extrema. Discovered about 100 years ago by S. Mukhopadhyaya, this theorem and its numerous

*Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA; e-mail: tabachni@math.psu.edu
generalizations and refinements continue to attract attention up to this day; see [7] for a sampler.

One such result is the converse four vertex theorem proved by Gluck for strictly convex, and by Dahlberg for general curves [3, 1]: a periodic function having at least two local minima and two local maxima is the curvature function of a simple closed plane curve. See [2] for a very well written survey.

The radius of curvature \( \rho(\alpha) \) of an oval, considered as a function of the direction of the tangent line to the curve, is \( L^2 \)-orthogonal to the first harmonics:

\[
\int_0^{2\pi} \rho(\alpha) \cos \alpha \, d\alpha = \int_0^{2\pi} \rho(\alpha) \sin \alpha \, d\alpha = 0.
\]

Such a function must have at least four critical points. The converse four vertex theorem can be restated as follows: if a function \( \rho(\alpha) \) has at least two local minima and two local maxima then there is a diffeomorphism \( \varphi \) of the circle such that the function \( \rho(\varphi(\alpha)) \) is \( L^2 \)-orthogonal to the first harmonics.

Our first result is the following generalization.

A Chebyshev system is an \( n \)-dimensional space \( V \) of functions on the circle \( S^1 = \mathbb{R}/2\pi \mathbb{Z} \) such that every non-zero function from \( V \) has at most \( n - 1 \) zeros (counted with multiplicities). According to the Sturm-Hurwitz-Kellogg theorem, if a smooth function on \( S^1 \) is \( L^2 \)-orthogonal to a Chebyshev system \( V^n \) then this function has at least \( n + 1 \) sign changes; see, e.g., [7]. In particular, a function orthogonal to \( \{1, \cos \alpha, \sin \alpha\} \) has at least four zeros; applied to the derivative of the radius of curvature of an oval, this implies the four vertex theorem.

We prove the next converse Sturm-Hurwitz-Kellogg theorem.

**Theorem 1** Let \( V^n \) be a Chebyshev system on \( S^1 \). If \( f(x) \) is a continuous function on \( S^1 \) with at least \( n + 1 \) sign changes then there exists an orientation preserving diffeomorphism \( \varphi : S^1 \to S^1 \) such that \( f(\varphi(x)) \) is \( L^2 \)-orthogonal to \( V \).

Our strategy of proof is that of Gluck [3, 2] which we illustrate by the following simplest case of the above theorem.

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1Where \( n \) is odd. One can define a Chebyshev system on a segment as well, and then there is no restriction on the parity of its dimension.
2Smoothness is not needed; one can work with finitely differentiable or continuous functions.
Example 1.1 Let $f(x)$ be a continuous function on $S^1$ that has both positive and negative values. One claims that there exists an orientation preserving diffeomorphism $\varphi : S^1 \to S^1$ such that $f(\varphi(x))$ has zero average value:

$$\int_0^{2\pi} f(\varphi(x)) \, dx = 0.$$ 

Of course, this is obvious, but we shall describe an argument that exemplifies the method of proof of Theorem 1 and other results of this paper.

Step 1. Let $h(x)$ be the step function that takes value 1 on $[0, \pi)$ and $-1$ on $[\pi, 2\pi)$. This step function has zero average value.

Step 2. Since $f(x)$ changes sign, there is a number $c \neq 0$ such that $f$ assumes both values $\pm c$. Scaling $f$, assume that $c = 1$ and that $f(x_1) = 1, f(x_2) = -1$. For every $\varepsilon > 0$, there exists a diffeomorphism $\varphi \in \text{Diff}^+(S^1)$ which stretches neighborhoods of the points $x_1$ and $x_2$ so that $\varphi^*(f)$ is $\varepsilon$-close in measure to $h$.

Step 3. For a sufficiently small real $\alpha$, consider an orientation preserving diffeomorphism $\psi_\alpha \in \text{Diff}^+(S^1)$ that fixes 0 and stretches the interval $[0, \pi]$ to $[0, \pi + \alpha]$. We assume that the dependence of $\psi_\alpha$ on $\alpha$ is smooth. The correspondence $\alpha \mapsto \psi_\alpha$ is a map of an interval $I$ to the group $\text{Diff}^+(S^1)$. Consider the function

$$F(\alpha) = \int_0^{2\pi} (\psi_\alpha^*(h))(x) \, dx.$$ 

One has: $F(0) = 0$ and $F'(0) \neq 0$. In particular, making the interval $I$ smaller, if needed, $F$ has opposite signs at the end points of $I$.

Step 4. Finally, replace $h$ in the definition of $F$ by the function $\varphi^*(f)$ from Step 2. If $\varepsilon$ is small enough, the resulting function $\tilde{F} : I \to \mathbb{R}$ still has opposite signs at the end points of $I$, hence there exists $\alpha$ such that $\tilde{F}(\alpha) = 0$. Thus the function $\psi_\alpha^*(\varphi^*(f))$ has zero average.
Remark 1.2 An object invariantly related to a function is its differential \( df = f'(x)dx \) (rather than the derivative). If \( \lambda \) is a differential 1-form on \( S^1 \) and
\[
\int_0^{2\pi} \lambda = 0
\]
then \( \lambda \) has sign changes, but the converse does not hold since
\[
\int_0^{2\pi} \varphi^*(\lambda) = \int_0^{2\pi} \lambda
\]
for every \( \varphi \in \text{Diff}_+(S^1) \). This explains why we deal with a function, rather than a differential 1-form.

Another, rather recent, four vertex-type theorem is due to E. Ghys: the Schwarzian derivative of a diffeomorphism of the real projective line has at least four zeros. Choose an affine coordinate \( x \) on \( \mathbb{RP}^1 \) and let \( f(x) \) be a diffeomorphism. Then the Schwarzian derivative \( S(f) \) is given by the formula
\[
S(f) = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2;
\]
it measures the failure of \( f \) to preserve the projective structure; see [7].

We prove a converse theorem.

**Theorem 2** If \( f(x) \) is a smooth function on \( \mathbb{RP}^1 \) with at least four sign changes then there exists an orientation preserving diffeomorphisms of the projective line \( \varphi \) and \( g(x) \) such that \( \varphi^*(f) = S(g) \).

**Remark 1.3** The invariant meaning of the Schwarzian is not a function but rather a quadratic differential, see, e.g., [7] for a detailed discussion:
\[
S(f) = \left( \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 \right) dx^2.
\]

Similarly to Remark 1.2 the property of a quadratic differential on \( \mathbb{RP}^1 \) to be the Schwarzian derivative of a diffeomorphism is invariant under the action of the group \( \text{Diff}(\mathbb{RP}^1) \).
2 Proof of the converse Sturm-Hurwitz-Kellogg theorem

The proof consists of the same four steps as in Example 1.1.

Step 1.

Lemma 2.1 There exists a piece-wise constant function on $S^1$ with values $\pm 1$ and exactly $n+1$ intervals of constant sign which is $L^2$-orthogonal to $V$.

Proof (suggested by D. Khavinson). Extend $V^n$ to a larger Chebyshev system $W^{n+2}$ and pick $f \in W - V$. Consider $g$, the best $L^1$ approximation of $f$ by a function in $V$. The function $g$ exists since $V$ is finite dimensional.

Since $W$ is a Chebyshev system, $f - g$ has at most $n+1$ intervals of constant sign (obviously, $f - g \neq 0$). Let $I_k$ be these intervals, and let $h$ be the function that has alternating values $\pm 1$ on the intervals $I_k$. Since $g$ is best approximation of $f$, one has the Lagrange multipliers condition:

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_0^{2\pi} |(f - g)(x) + \varepsilon v(x)| \, dx = 0 \quad (1)$$

for every $v(x) \in V$. It follows from (1) that

$$0 = \sum_k (-1)^k \int_{I_k} v(x) \, dx = \int_0^{2\pi} h(x)v(x) \, dx,$$

that is, $h$ is orthogonal to $V$.

By the Sturm-Hurwitz-Kellog theorem, $h$ has at least $n+1$ sign changes (Proof, for completeness: if not, one can find a function from $V$ with the same intervals of constant sign as $h$; such a function cannot be orthogonal to $h$).

Step 2. Since $f(x)$ changes sign at least $n+1$ times, there is a non-zero constant $c$ such that $f$ takes the alternating values $\pm c$ at points, say, $x_0, \ldots, x_n$. Multiplying $f$ by a constant, assume that $c = 1$.

Let $h(x)$ be the function from Lemma 2.1. For every $\varepsilon > 0$, there exists a diffeomorphism $\varphi \in \text{Diff}_+(S^1)$ which stretches neighborhoods of the points $x_0, \ldots, x_n$ so that the function $\varphi^*(f)$ is $\varepsilon$-close in measure to $h$.
Step 3. Consider the function $h(x)$ and let $[0, x_1], [x_1, x_2], \ldots, [x_n, 2\pi]$ be its intervals of constant sign. For $\alpha = (\alpha_1, \ldots, \alpha_n)$, consider an orientation preserving diffeomorphism $\psi_\alpha \in \text{Diff}_+(S^1)$ that stretches the intervals $[x_i, x_{i+1}]$ so that point $x_i$ goes to $x_i + \alpha_i$ and which fixes 0. We assume that each $|\alpha_i|$ is sufficiently small and that the dependence of $\psi_\alpha$ on $\alpha$ is smooth. The correspondence $\alpha \mapsto \psi_\alpha$ is a map of an $n$-dimensional disc $D^n$ to $\text{Diff}_+(S^1)$.

The formula $F(\alpha)(g) = \langle \psi_\alpha^*(h), g \rangle$ defines a smooth map $D \to V^*$ that takes the origin to the origin (the scalar product is understood in the $L^2$ sense).

Lemma 2.2 The differential $dF$ is non-degenerate at the origin.

Proof. Let $g_1, \ldots, g_n$ be a basis of $V$. We want to prove that the matrix

$$c_{ij} = \left. \frac{\partial F(\alpha)(g_i)}{\partial \alpha_j} \right|_{\alpha=0}, \quad i, j = 1, \ldots, n$$

is non-degenerate. One has:

$$F(\alpha)(g) = \sum_{k=0}^{n} (-1)^k \int_{x_k+\alpha_k}^{x_{k+1}+\alpha_{k+1}} g(x) \, dx$$

where we assume that $x_0 = 0, x_{n+1} = 2\pi, \alpha_0 = \alpha_{n+1} = 0$. It follows that $c_{ij} = 2(-1)^{j+1} g_i(x_j)$, and it suffices to show that the matrix $g_i(x_j)$ is non-degenerate. This is indeed a fundamental property of Chebyshev systems, see [6] (Proof, for completeness: if $c = (c_1, \ldots, c_n)$ is a non-zero vector such that $\sum c_i g_i(x_j) = 0$ for each $j$ then the function $\sum c_i g_i(x)$ has $n$ zeros, which contradicts the definition of Chebyshev systems). \(\square\)

Step 4. It follows from Lemma 2.2 that there exists $\delta > 0$ such that the map $F$, restricted to the cube $D^n$ given by the conditions $|\alpha_i| < \delta$, $i = 1, \ldots, n$, has degree one, and the hypersurface $F(\partial D)$ has the rotation number one with respect to the origin in $V^*$.

Now replace $h$ in the definition of the map $F$ by the function $\varphi^*(f)$ from Step 2, and denote the new map by $\tilde{F} : D^n \to V^*$. We shall be done if we show that there exists $\alpha$ such that $\tilde{F}(\alpha) = 0$. Indeed, if $\varepsilon$ is small enough then $\tilde{F}(\partial D)$ still has rotation number one with respect to the origin in $V^*$, and therefore $\tilde{F}(D)$ contains the origin. \(\square\)
3 Digression: the space of step functions with values ±1 on an interval

An extension of Lemma 2.1 to the case when \( V \) is not assumed to be a Chebyshev system is the following Hobby–Rice theorem [4], see also [10, 12].

**Theorem 3** Let \( V \) be an \( n \)-dimensional subspace in \( L^1([0,1]) \). Then there exists a piece-wise constant function on \( I \) with values ±1 and at most \( n+1 \) intervals of constant sign which is \( L^2 \)-orthogonal to \( V \).

**Proof** ([4, 10]). Let \( x = (x_0, x_1, \ldots, x_n) \), \( \sum_i x_i^2 = 1 \), be a point of the sphere \( S^n \). Assign to \( x \) the partition of \([0,1]\) on the intervals of consecutive lengths \( x_0^2, x_1^2, \ldots, x_n^2 \) and the piece-wise constant function \( h_x \) with value equal to sign \( x_i \) on the respective interval. We obtain a map \( F : S_n \to V^* \) given by the formula:

\[
\langle F(x), g \rangle = \int_0^1 h_x(t)g(t) \, dt.
\]

This map is odd: \( F(-x) = -F(x) \), and it follows from the Borsuk-Ulam theorem (see e.g., [5]) that \( F(x) = 0 \) for some \( x \in S_n \). Thus \( h_x \) is orthogonal to \( V \). \( \square \)

From the point of view of topology, it is interesting to consider the space \( S_n \subset L^1([0,1]) \) of piece-wise constant function on \([0,1]\) with values ±1 and at most \( n+1 \) intervals of constant sign. We complement the proof of Theorem 3 with the following result.

**Theorem 4** \( S_n \) is homeomorphic to \( n \)-dimensional sphere.

**Proof.** We give \( S_n \) the structure of a finite cell complex with two cells in every dimension \( 0, 1, \ldots, n \) and prove, by induction on \( n \), that \( S_n \) is homeomorphic to \( S^n \). For \( n = 0 \), the set \( S_0 \) consists of two constant functions with values +1 or −1 and is homeomorphic to \( S^0 \).

Let \( \Delta^n = \{ x = (x_0, \ldots, x_n) | x_i \geq 0, \sum x_i = 1 \} \) be the standard simplex. Consider the subset \( C \subset S_n \) consisting of functions with exactly \( n+1 \) intervals of constant sign. The lengths of these intervals are positive numbers \( x_0, x_1, \ldots, x_n \) satisfying \( \sum x_i = 1 \), and a function from \( C \) is determined by \( x = (x_0, \ldots, x_n) \) and the sign ± that the function has on the first interval.
Thus we obtain two embeddings $\psi_n^\pm : \text{Int} \Delta^n \to C$, and $C$ is the disjoint union of the images of $\psi_n^+$ and $\psi_n^-$.

The maps $\psi_n^\pm$ extend continuously to the boundary $\partial \Delta^n$: when some $x_i$’s shrink to zero, the respective segments of constant sign of a function disappear, and if the function has the same signs in the neighboring segments, they merge together. For example, let $n = 2$. Then $\psi_2^+(0, x_1, x_2)$ has two intervals of constant sign and equals $\psi_1^+(x_1, x_2)$, whereas $\psi_2^+(x_0, 0, x_2)$ is constant function with value $+1$, i.e., equals $\psi_0^+(1)$.

We have: $S_n - C = S_{n-1}$, and the latter is homeomorphic to $S^{n-1}$ by the induction assumption. Each map $\psi_n^\pm$ sends $\partial \Delta^n$ to $S_{n-1}$, and we claim that the degree of $\psi_n^\pm$ is one. Indeed, the faces of $\partial \Delta^n$ are given by one of the conditions: $x_0 = 0, x_1 = 0, \ldots, x_n = 0$. Since $\psi_n^+ (0, x_1, \ldots, x_n) = \psi_n^{-1}(x_1, \ldots, x_n)$ and $\psi_n^-(x_0, x_1, \ldots, x_n) = \psi_n^{-1}(x_0, \ldots, x_n)$, the map $\psi_n^\pm$ sends the faces $x_0 = 0$ and $x_n = 0$ to the two $n-1$-dimensional cells of $S_{n-1}$, and the other faces are sent to the $n-2$-skeleton of $S_{n-1}$. Therefore $\deg \psi_n^\pm = 1$.

Since the attaching maps of two $n$-dimensional discs $\Delta^n$ to $S^{n-1}$ have degree one, $S_n$ is $n$-dimensional sphere.

One can also consider the space of piece-wise constant function on the circle with values $\pm 1$ and at most $n$ intervals of constant sign ($n$ even). Such a space is also homeomorphic to $S^n$. cut the circle at, say, point 0 to obtain a piece-wise constant function on an interval with at most $n + 1$ intervals of constant sign, and apply Theorem 4.

4 Proof of the converse Ghys theorem

Let us start with a reformulation described in [3].

A diffeomorphism $f : \mathbb{RP}^1 \to \mathbb{RP}^1$ has a unique lifting to a homogeneous of degree one area preserving diffeomorphism $F$ of the punctured plane. If $f$ is a projective transformation then $F \in SL(2, \mathbb{R})$. Let $x$ be the angular parameter on $\mathbb{RP}^1$ so that $x$ and $x + \pi$ describe the same point. Then $(x, r)$ are the polar coordinates in the plane and

$$F(x, r) = (f(x), rf'^{-1/2}(x)).$$

Let $\gamma(x)$ be the image of the unit circle under $F$, this is a centrally symmetric
curve that bounds area $\pi$. The curve $\gamma$ satisfies the differential equation

$$\gamma''(x) = -k(x)\gamma(x)$$

(2)

where $k(x)$ is a $\pi$-periodic function called the potential. The relation of the potential with the Schwarzian is as follows:

$$k = \frac{1}{2} S(f) + 1.$$ 

In particular, the zeros of the Schwarzian corresponds to the values 1 of the function $k(x)$ (indeed, if $k(x) \equiv 1$ then $\gamma$ is a central ellipse, $F \in SL(2, \mathbb{R})$ and $f$ is a projective transformation).

Thus we arrive at the following reformulation of Theorem 2: if a function $k(x) - 1$ on $\mathbb{RP}^1$ changes sign at least four times then there exists an orientation preserving diffeomorphism $\varphi$ of the projective line such that the function $\tilde{k} = \varphi^*(k)$ is the potential of a centrally symmetric closed parametric curve $\gamma(x)$ in the punctured plane bounding area $\pi$, that is, a curve satisfying the differential equation $\gamma''(x) = -\tilde{k}(x)\gamma(x)$.

The proof consists of the same four steps as in Example 1.1.

**Step 1.** Let $k_1, k_2$ be two positive numbers satisfying $k_1 > 1, k_1 + k_2 = 2$ and both sufficiently close to 1. We claim that there exists a $\pi$-periodic step function $h(x)$ with four intervals of constant values $k_1, k_2, k_1, k_2$ on $[0, \pi]$ such that the respective solution of the differential equation (2) is a closed curve.

To prove this, consider the frame $F(x) = (\gamma(x), \gamma'(x))$. The differential equation (2) rewrites as

$$F'(x) = F(x)A(x)$$

(3)

where

$$A(x) = \begin{pmatrix} 0 & -k(x) \\ 1 & 0 \end{pmatrix}.$$ 

Equation (3) defines a curve on the group $SL(2, \mathbb{R})$: the curve $\gamma$ is centrally symmetric and closed iff $F(\pi) = -F(0)$. Let us refer to the last equality as the monodromy condition.

Let the desired step function $h(x)$ have intervals of constant values of lengths $t_1, t_2, t_3, t_4$ with $t_1 + t_2 + t_3 + t_4 = \pi$. For a constant potential $k$, equation (3) is easily solved:

$$F(x) = F(0)e^{xA} = F(0) \begin{pmatrix} \cos(\sqrt{k}x) & -\sqrt{k}\sin(\sqrt{k}x) \\ \frac{1}{\sqrt{k}}\sin(\sqrt{k}x) & \cos(\sqrt{k}x) \end{pmatrix}.$$
It follows that the monodromy condition is

\[ e^{t_1A}e^{t_2B}e^{t_3A}e^{t_4B} = -E \]  

(4)

where

\[ A = \begin{pmatrix} 0 & -k_1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -k_2 \\ 1 & 0 \end{pmatrix} \]

and \( E \) is the unit matrix.

Let us look for a solution satisfying \( t_3 = t_1, t_4 = t_2 \); then \( t_1 + t_2 = \pi/2 \). Set: \( \alpha = t_1\sqrt{k_1}, \beta = t_2\sqrt{k_2} \). A direct computation shows that (4) is satisfied once

\[ \tan \alpha \tan \beta = \sqrt{k_1 k_2}. \]  

(5)

The constraint on \( \alpha \) and \( \beta \) is

\[ \frac{\alpha}{\sqrt{k_1}} + \frac{\beta}{\sqrt{k_2}} = \frac{\pi}{2}. \]

If \( \alpha \) is close to \( \pi/2 \) then the left hand side of (5) is greater, and if \( \alpha \) is close to 0 then it is smaller than the right hand side. It follows that (5) has a solution.

**Step 2.** Since \( k(x) - 1 \) changes sign at least four times, there is a constant \( c > 0 \) such that \( k \) takes the values \( 1 + c, 1 - c, 1 + c, 1 - c \) at points, say, \( x_1, x_2, x_3, x_4 \). Let \( k_1 = 1 + c, k_2 = 1 - c \), and let \( h(x) \) be the step function from Step 1. For every \( \varepsilon > 0 \), there exists a diffeomorphism \( \varphi \in \text{Diff}_+(\mathbb{RP}^1) \) which stretches neighborhoods of the points \( x_1, \ldots, x_4 \) so that the function \( \varphi^*(k) \) is \( \varepsilon \)-close in measure to \( h \).

**Step 3.** Similarly to Step 3 in Section 2, consider a 3-parameter family of diffeomorphisms \( \psi_\alpha \in \text{Diff}(\mathbb{RP}^1) \) that change the intervals of constant values of the step function \( h(x) \). Given \( \alpha \), consider the function \( \psi_\alpha^*(h) \) as the potential of equation (3) with the initial conditions \( F(0) = E \). The formula \( G(\alpha) = F(\pi) \) defines a smooth map \( D^3 \to SL(2, \mathbb{R}) \) that takes the origin to the matrix \( -E \).

**Lemma 4.1** The differential \( dG \) is non-degenerate at the origin.
Proof  Stretch the intervals of constant values of the potential function to \( t_i + \varepsilon s_i, \ i = 1, 2, 3, 4; \) the vector \( s = (s_1, s_2, s_3, s_4), \ s_1 + s_2 + s_3 + s_4 = 0 \) is interpreted as a tangent vector to \( D^3 \) at the origin. Using the formula for monodromy (4), we compute:

\[
-dG(s) = s_1 A + s_4 B + s_2 e^{t_1 A} B e^{-t_1 A} + s_3 e^{t_2 B} A e^{-t_2 B}
\]

where \( A, B, t_1, t_2 \) are as in Step 1. We need to check that the linear map \( dG : \mathbb{R}^4 \to \mathfrak{sl}_2 \), given by (6), is surjective and that its kernel is transverse to the hyperplane \( s_1 + s_2 + s_3 + s_4 = 0. \) Both claims follow, by a direct computation, from the explicit formulas for the matrices \( A, B \) and their exponents given in Step 1. \( \Box \)

Step 4. This last step is identical to Step 4 in Section 2; replace the potential \( h \) in the definition of the map \( G \) in Step 3 by \( \varphi^*(k) \). We obtain a new monodromy map \( \bar{G} : D^3 \to SL(2, \mathbb{R}) \) whose image contains the matrix \( -E \). The respective curve closes up, and we are done.

Remark 4.2  The Ghys theorem is closely related to the four vertex theorem in the hyperbolic plane [11]. Let \( \gamma \) be an oval in \( H^2. \) Each tangent line to \( \gamma \) intersects the circle at infinity at two points, and this defines a circle diffeomorphism \( f_\gamma. \) In the projective model of hyperbolic geometry, the circle at infinity is represented by a conic in \( \mathbb{R}P^2. \) A conic has a canonical projective structure, hence \( f_\gamma \) can be viewed as a diffeomorphism of \( \mathbb{R}P^1. \) Singer’s theorem asserts that the zeros of the Schwarzian \( S(f_\gamma) \) correspond to the vertices of \( \gamma \) (in the hyperbolic metric, of course), see [7] for a discussion.

Note however that a converse four vertex theorem for the hyperbolic plane does not hold in the same way as in the Euclidean plane: if the positive curvature function is too small then the respective curve in the hyperbolic plane does not close up.

5 Problems and conjectures

There are many other results extending the four vertex theorem. In each case, it is interesting to find the converse theorem; we mention but a few.
**Problem 1.** Another classic theorem of Mukhopadhyaya is that a plane oval has at least six affine vertices (also known as sextactic points). An affine vertex is a point at which the curve is abnormally well approximated by a conic: at a generic point, a conic passes through five infinitesimally close points of the curve, whereas at an affine point, this number equals six. Every oval $\gamma$ can be given an affine parameterization such that $\det(\gamma'(x), \gamma''(x))$ is constant. Then $\gamma'''(x) = -k(x)\gamma'(x)$ where the function $k(x)$ is called the affine curvature. The affine vertices are the critical points of the affine curvature, see, e.g., [7].

A conjectural converse theorem asserts that *if a periodic function $k(x)$ has at least six extrema then there exists a plane oval $\gamma(x)$ whose affine curvature at point $\gamma(x)$ is $k(x)$* (of course, here $x$ is not necessarily an affine parameter).

**Problem 2.** The four vertex theorem has numerous discrete versions, see, e.g., [7, 9] for surveys and references. For example, let $P$ be a convex $n$-gon with vertices $x_1, \ldots, x_n$. Assume that $n \geq 4$ and that no four consecutive vertices lie on a circle. Consider the circles circumscribing triples of consecutive vertices $x_{i-1}x_ix_{i+1}$, and assume that the center of this circle lies inside the cone of the vertex $x_i$ (such a polygon is called coherent). Let $r_1, \ldots, r_n$ be the cyclic sequence of the radii of the circles. Then the sequence $r_1, \ldots, r_n$ has at least two local maxima and two local minima.

A conjectural converse theorem asserts that *if a cyclic sequence $r_1, \ldots, r_n$ has at least two local maxima and two local minima then it corresponds, as described above, to a coherent convex polygon.*

Another version of discrete four vertex theorem concerns the circles tangent to the triples of consecutive sides of a polygon: the radii of such inscribed circles also form a cyclic sequence with at least two local maxima and two local minima. One conjectures that a converse theorem holds as well.

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