Scalar curvature of a Levi-Civita connection on the Cuntz algebra with three generators

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Abstract
A differential calculus on the Cuntz algebra with three generators coming from the action of rotation group in three dimensions is introduced. The differential calculus is shown to satisfy Assumptions I–IV of Bhowmick et al. (A new look at Levi-Civita Connection in NCG. arXiv:1606.08142) so that Levi-Civita connection exists uniquely for any pseudo-Riemannian metric in the sense of Bhowmick et al. [2] Scalar curvature is computed for the Levi-Civita connection corresponding to the canonical bilinear metric.

Keywords Cuntz algebra · Levi-Civita connection · Scalar curvature

Mathematics Subject Classification 46L87 · 58B34

1 Introduction

Notions of connection and curvature are important in any form of geometry, be it classical or noncommutative. In noncommutative geometry, in last few years, various notions of curvature have been introduced. Broadly speaking, there seem to be two main avenues to the definition of curvature of a noncommutative space. One is to define directly Ricci and scalar curvature via an asymptotic expansion of powers of the noncommutative Laplacian (see for example [6–8]), and the other is to prove existence and uniqueness of a Levi-Civita connection on the module of vector fields or one-forms and subsequently compute the curvature operator and scalar curvature (see [1,9,11,13]). The last one is more algebraic and has the advantage of finding the
curvature operator directly. But the difficulty with this approach is to prove the existence and uniqueness of Levi-Civita connection on noncommutative spaces. Recently in Bhowmick et al. [2], the authors gave some sufficient conditions on the differential calculus on noncommutative spaces to ensure existence and uniqueness of a Levi-Civita connection with respect to any pseudo-Riemannian metric in some sense. In a follow-up paper [3], the authors have given examples of noncommutative spaces admitting a unique Levi-Civita connection in the sense of Bhowmick et al. [2].

For any useful concept of a Levi-Civita connection in noncommutative geometry, we need interesting examples where such a connection can be shown to exist and be unique. A formulation also has to perform well in terms of computability of Ricci and scalar curvature. In this paper, we introduce a differential calculus on the Cuntz algebra with three generators coming from the natural action of the rotation group in three dimensions. Then, we show that the differential calculus admits a unique Levi-Civita connection in the sense of Bhowmick et al. [2]. We compute the Ricci curvature and scalar curvature along the lines of Bhowmick et al. [2] with respect to the canonical metric. The scalar curvature turns out to be a negative constant times identity. The result can be generalized to the Cuntz algebra with \( n \)-generators. More precisely, it can be shown that the Cuntz algebra with \( n \)-generators admits a “canonical” metric with scalar curvature \( -\frac{m}{4} \), where \( m = \frac{n(n-1)}{2} \). In this paper, we only give a proof of this result for the particular case when \( n = 3 \). It is worth mentioning that noncommutative geometry of Cuntz algebra has been studied in a more analytic set up (see [4] for example). Absence of any faithful trace on Cuntz algebra poses considerable amount of difficulty there. In a sense, this paper attempts a more algebraic study of the geometry of Cuntz algebra.

2 Preliminaries

2.1 Existence and uniqueness of Levi-Civita connection on a class of modules of one-forms

In this paper, we shall discuss the notions of connection and Riemannian metric on the space of one-forms. Note that, in classical differential geometry, it is more customary to discuss these notions on the space of vector fields. But it is quite standard to discuss these notions on the space of one-forms in the realm of noncommutative geometry. This is so because the space of one-forms is a bimodule over the underlying noncommutative algebra, whereas the space of vector fields fails to be a module. In classical differential geometry, this situation does not arise since the space of vector fields in classical geometry is still a bimodule over the commutative function algebra. Having said that, let us begin by recalling the definition of a differential calculus on a \(*\)-algebra \( \mathcal{A} \) over \( \mathbb{C} \).

**Definition 2.1** A differential calculus on a \(*\)-algebra \( \mathcal{A} \) is a pair \((\Omega(\mathcal{A}), d)\) such that

(i) \( \Omega(\mathcal{A}) \) is an \( \mathcal{A}\)-\( \mathcal{A} \)-bimodule.
(ii) \( \Omega(\mathcal{A}) = \bigoplus_{i \geq 0} \Omega^i(\mathcal{A}) \), where \( \Omega^0(\mathcal{A}) = \mathcal{A} \) and each \( \Omega^i(\mathcal{A}) \) is an \( \mathcal{A}\)-\( \mathcal{A} \)-bimodule.
(iii) There is a bimodule map \( m : \Omega^i(\mathcal{A}) \otimes_\mathcal{A} \Omega^j(\mathcal{A}) \subset \Omega^{i+j}(\mathcal{A}) \) for all \( i, j \).
(iv) \( d : \Omega^i(A) \to \Omega^{i+1}(A) \) satisfies 
\[ d(\omega \eta) = d\omega \eta + (-1)^{\deg(\omega)} \omega d\eta \] 
and \( d^2 = 0 \).

(v) \( \Omega^i(A) \) is spanned by \( da_0 \ldots da_ia_{i+1} \), where \( a_0, a_1, \ldots, a_{i+1} \in A \).

Let \((\Omega(A), d)\) be such a differential calculus on an algebra \( A \).

**Definition 2.2** A \( \mathbb{C} \)-linear map \( \nabla : \Omega^1(A) \to \Omega^1(A) \otimes_A \Omega^1(A) \) is said to be a (right) connection on the \( A \)-\( A \)-bimodule \( \Omega^1(A) \) if

\[ \nabla(\omega \cdot a) = \nabla(\omega)a + \omega \otimes da, \omega, a \in \Omega^1(A), a \in A. \]

**Definition 2.3** A connection \( \nabla \) is said to be torsionless if the right linear map \( T_{\nabla} = m \circ \nabla + d \) is equal to zero.

For a bimodule \( E \) over \( A \), the center of the bimodule will be denoted by \( Z(E) \), i.e.,

\[ Z(E) = \{ e \in E : a.e = e.a \text{ for all } a \in A \}. \]

Similarly, \( Z(A) \) will denote the center of an algebra \( A \). Now, we are going to state a few assumptions on a given differential calculus over a \( * \)-algebra \( A \) as given in [2]. These assumptions enable one to prove the existence and uniqueness of a Levi-Civita connection on the space of one-forms. In the following, we shall denote the bimodule of one-forms by \( E \).

**Assumption I** The \( A \)-\( A \) bimodule \( E \) is a finitely generated projective right \( A \)-module. Moreover, the map \( u^E : Z(E) \otimes Z(A) \to E \) given by \( u^E(\sum e_i \otimes a_i) = \sum e_i a_i \), is an isomorphism of vector spaces.

**Assumption II** The right \( A \)-module \( E \otimes A E \) admits a splitting \( E \otimes A E = \ker(m) \oplus F \), where \( F \cong \text{Im}(m) \).

**Assumption III** If we denote the idempotent projecting onto \( \ker(m) \) by \( P_{\text{sym}} \), then the map \( \sigma := (2P_{\text{sym}} - 1) \) should satisfy \( \sigma(\omega \otimes \eta) = \eta \otimes \omega \) for \( \omega, \eta \in Z(E) \).

Before stating the IVth and final assumption, we recall the definition of a pseudo-Riemannian metric on a differential calculus over a \( * \)-algebra \( A \) satisfying Assumptions I–III. In this paper, for a right \( A \)-module \( E \), the right \( A \)-module \( \text{Hom}_A(E, A) \) will be denoted by \( E^* \).

**Definition 2.4** A pseudo-Riemannian metric \( g \) on \( E \) is an element of \( \text{Hom}_A(E \otimes_A E, A) \) such that

(i) \( g \circ \sigma = g \).

(ii) \( g \) is non-degenerate in the sense that the map \( V_g : E \to E^* \) defined by \( V_g(\omega)(\eta) = g(\omega \otimes \eta) \) is an isomorphism of right \( A \)-modules.

Now we are ready to state the final assumption.

**Assumption IV** There is a bilinear pseudo-Riemannian metric \( g \in \text{Hom}_A(E \otimes_A E, A) \).
Definition 2.5 (see Proposition 2.12 and Definition 2.13 of [2]) Suppose for a \( \ast \)-algebra \( \mathcal{A} \), the differential calculus satisfies Assumptions I–IV. Then, a connection \( \nabla \) on the space of one-forms is said to be unitary with respect to a pseudo-Riemannian metric \( g \) if \( dg = \Pi_g(\nabla) \), where \( \Pi_g(\nabla) : \mathcal{E} \otimes_\mathcal{A} \mathcal{E} \to \mathcal{E} \) is defined as

\[
\Pi_g(\nabla)(\omega \otimes \eta) = (g \otimes \text{id})\sigma_{23}(\nabla(\omega) \otimes \eta + \nabla(\eta) \otimes \omega), \quad \sigma_{23} = (\text{id} \otimes \sigma).
\]

Recall the definition of the right \( \mathcal{A} \)-linear map \( \Phi_g : \text{Hom}(\mathcal{E}, \mathcal{E} \otimes^\text{sym}_\mathcal{A} \mathcal{E}(:= P_{\text{sym}}(\mathcal{E} \otimes_\mathcal{A} \mathcal{E}))) \to \text{Hom}(\mathcal{E} \otimes^\text{sym}_\mathcal{A} \mathcal{E}, \mathcal{E}) \) (see Theorem 2.14 of [2]). We state the following theorem from [2, Theorem 3.12 and 2.14]:

Theorem 2.6 For a \( \ast \)-algebra \( \mathcal{A} \) such that the differential calculus over \( \mathcal{A} \) satisfies assumptions I–IV, a unique Levi-Civita connection (a connection which is torsion-less and unitary) exists. Moreover, for any torsionless connection \( \nabla_0 \), the Levi-Civita connection \( \nabla \) is given by

\[
\nabla = \nabla_0 + L,
\]

where \( L = \Phi_g^{-1}(dg - \Pi_g(\nabla_0)) \).

Let us recall some definitions which will be used later in the paper. For two bimodules \( \mathcal{E}, \mathcal{F} \), the map \( \zeta_{\mathcal{E},\mathcal{F}} : \mathcal{E} \otimes_\mathcal{A} \mathcal{F}^* \to \text{Hom}_\mathcal{A}(\mathcal{F}, \mathcal{E}) \) is defined as

\[
\zeta_{\mathcal{E},\mathcal{F}} \left( \sum_i e_i \otimes \phi_i \right)(f) = \sum_i e_i(\phi_i(f)).
\]

(1)

When both \( \mathcal{E} \) and \( \mathcal{F} \) are finitely generated and projective, \( \zeta_{\mathcal{E},\mathcal{F}} \) is an isomorphism. Let \( \mathcal{E} \) be an \( \mathcal{A} \)-\( \mathcal{A} \) bimodule satisfying Assumptions I–IV. Then, for any \( \mathcal{A} \)-\( \mathcal{A} \) bimodule \( \mathcal{F} \), the map \( T_{\mathcal{E},\mathcal{F}}^L : \mathcal{E} \otimes_\mathcal{A} \mathcal{F} \to \mathcal{Z}(\mathcal{E}) \otimes \mathcal{Z}(\mathcal{A}) \mathcal{F} \) given by

\[
T_{\mathcal{E},\mathcal{F}}^L := ((\mu^\mathcal{E})^{-1} \otimes \text{id}_\mathcal{F}),
\]

(2)
is a right \( \mathcal{A} \)-linear isomorphism. Similarly, we have the map \( T_{\mathcal{E},\mathcal{F}}^R \) (see [2, Proposition 4.3]) which is a left \( \mathcal{A} \)-linear isomorphism between \( \mathcal{F} \otimes_\mathcal{A} \mathcal{E} \) and \( \mathcal{F} \otimes \mathcal{Z}(\mathcal{A}) \mathcal{Z}(\mathcal{E}) \). Let us also recall the following maps for future purpose:

(i) \( \text{flip} : \mathcal{Z}(\mathcal{E}) \otimes \mathcal{Z}(\mathcal{A}) \mathcal{E}^* \to \mathcal{E}^* \otimes \mathcal{Z}(\mathcal{A}) \mathcal{E} \) is defined as \( \text{flip}(e' \otimes \phi) = \phi \otimes e' \).

(ii) \( \rho : \mathcal{E} \otimes_\mathcal{A} \mathcal{E}^* \to \mathcal{E}^* \otimes_\mathcal{A} \mathcal{E} \) is defined as \( \rho := (T_{\mathcal{E},\mathcal{E}^*}^R)^{-1} \text{flip} T_{\mathcal{E},\mathcal{E}^*}^L \).

(iii) \( \text{ev} : \mathcal{E}^* \otimes \mathcal{E} \to \mathcal{A} \) is defined as \( \text{ev}(\sum_i \phi_i \otimes e_i) = \sum_i \phi_i(e_i) \).

We also recall the map \( H : \mathcal{E} \otimes_\mathcal{A} \mathcal{E} \to \mathcal{E} \otimes_\mathcal{A} \mathcal{E} \otimes_\mathcal{A} \mathcal{E} \) (see [2, Lemma 4.1]) for a connection \( \nabla \).

\[
H(\omega \otimes \eta) = (1 - P_{\text{sym}})_{23}(\nabla(\omega) \otimes \eta) + \omega \otimes Q^{-1}(d\eta),
\]

(3)

where \( Q : \text{Im}(1 - P_{\text{sym}}) \to \text{Im}(m) \) is the isomorphism from Assumption II.
3 Connes’ space of forms for the Cuntz algebra with three generators

Let $\mathcal{A}$ be a $\ast$-algebra. We call a triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where $\mathcal{A}$ is faithfully represented on $\mathcal{B}(\mathcal{H})$ and $\mathcal{D}$ is apriori an unbounded operator, a Dirac triple if $[\mathcal{D}, a] \in \mathcal{B}(\mathcal{H})$ for all $a \in \mathcal{A}$. Note that a Dirac triple in our sense is called spectral triple in general. But a spectral triple comes with some kind of summability or compactness criterion. We do not need this in the paper and that is why we are content with a Dirac triple. For a $\ast$-algebra $\mathcal{A}$, recall the reduced universal differential algebra $(\Omega^\bullet(\mathcal{A}) := \bigoplus_k \Omega^k(\mathcal{A}), \delta)$ from Connes [5]. Given a Dirac triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ over $\mathcal{A}$, there is a well defined $\ast$-representation $\Pi$ of $\Omega^\bullet(\mathcal{A})$ on $\mathcal{B}(\mathcal{H})$ given by the following (see [10]):

$$\Pi(a_0 \delta a_1 \cdots \delta a_k) = a_0 [\mathcal{D}, a_1] \cdots [\mathcal{D}, a_k],$$

where $a_0, \ldots, a_k \in \mathcal{A}$.

Let $J_0^k = \{ \omega \in \Omega^k(\mathcal{A}) : \Pi(\omega) = 0 \}$. The Connes’ space of $k$-forms is defined to be

$$\Omega^k_D(\mathcal{A}) = \Pi(\Omega^k(\mathcal{A}))/\Pi(\delta J_0^{k-1}).$$

$\Pi(\delta J_0^{k-1})$ is a two-sided ideal of $\Pi(\Omega^k(\mathcal{A}))$ and is called the space of junk forms. For any element $\omega \in \Omega^k(\mathcal{A})$, we denote the image of $\Pi(\omega)$ in $\Omega^k_D(\mathcal{A})$ by $\overline{\Pi(\omega)}$. It can be shown that $(\Omega^\bullet_D(\mathcal{A}) := \bigoplus_k \Omega^k_D(\mathcal{A}), d)$ where $d$ is defined as $d \overline{\Pi(\omega)} := \overline{\Pi(\delta \omega)}$, satisfies the conditions of Definition 2.1.

3.1 A Dirac triple on $\mathcal{O}_3$: Connes’ space of forms

We begin this subsection by recalling the definition of the Cuntz algebra with $n$-generators. The Cuntz algebra $\mathcal{O}_n$ with $n$-generators is defined to be the universal $C^\ast$-algebra generated by $n$ elements $\{S_i\}_{i=1,\ldots,n}$ satisfying

$$S_i^\ast S_i = 1, \quad i = 1, \ldots, n$$

$$\sum_{j=1}^n S_j S_j^\ast = 1.$$

We shall limit ourselves to the Cuntz algebra $\mathcal{O}_3$ with three generators in this paper. For $A = ((a_{ij}))_{i,j=1,2,3} \in SO(3)$, define $\alpha_A(S_i) = \sum_{j=1}^3 a_{ij} S_j$ for $i = 1, 2, 3$. Then,

$$\alpha_A(S_i)^\ast \alpha_A(S_i) = \sum_{j,k} a_{ij} a_{ik} S_j^\ast S_k$$

$$= \sum_{j} a_{ij} a_{ij}$$

$$= 1.$$
system \((\mathcal{O}_3, \alpha, SO(3))\). Now let us describe a Dirac triple arising from the above \(\mathcal{C}^*\)-dynamical system. To that end recall that \(SO(3)\) has three-dimensional Lie algebra \(so(3)\). Let us choose three basis elements

\[
X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

which define three derivations on \(\mathcal{O}_3\). They are given by

\[
\partial_i(a) = d\frac{d}{d\theta} \exp(\theta X_i)(a)|_{\theta=0}, \quad a \in \mathcal{O}_3, \ i = 1, 2, 3.
\]

Note the following:

\[
\exp(\theta X_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad \exp(\theta X_2) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix},
\]

\[
\exp(\theta X_3) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Then, it is easy to see the actions of the derivations on the generators of \(\mathcal{O}_3\). They are as follows:

\[
\partial_1(S_1) = 0, \quad \partial_1(S_2) = -S_3, \quad \partial_1(S_3) = S_2
\]

\[
\partial_2(S_1) = -S_3, \quad \partial_2(S_2) = 0, \quad \partial_2(S_3) = S_1
\]

\[
\partial_3(S_1) = S_2, \quad \partial_3(S_2) = -S_1, \quad \partial_3(S_3) = 0.
\]

It is clear that the derivations are actually \(*\)-derivations. The commutation relations are given by

\[
[\partial_1, \partial_2] = -\partial_3, \quad [\partial_2, \partial_3] = \partial_1, \quad [\partial_1, \partial_3] = -\partial_2. \quad (4)
\]

We denote the unique faithful KMS state of \(\mathcal{O}_3\) by \(\tau\). Let \(\mathcal{H}\) be the Hilbert space \(L^2(\mathcal{O}_3, \tau) \otimes \mathbb{C}^3\). Then, we define a Dirac triple \((\mathcal{O}_3, \mathcal{H}, \mathcal{D})\) where \(\pi : \mathcal{O}_3 \to B(\mathcal{H})\) is given by \(\pi(a) := a \otimes \mathbb{1}\) and \(\mathcal{D}\) is the densely defined operator \(\sum_{i=1}^3 \partial_i \otimes \sigma_i\), where \(\sigma_i\)’s are the \(3 \times 3\) Pauli Spin matrices satisfying \(\sigma_i^2 = \mathbb{1}\) and \(\sigma_i \sigma_j = -\sigma_j \sigma_i\) for \(i \neq j\). Using the derivation properties of \(\partial_i\)’s, it can be shown that for all \(a \in \mathcal{O}_3\),

\[
[\mathcal{D}, \pi(a)] = \sum_{i=1}^3 (\partial_i(a)) \otimes \sigma_i.
\]

Hence, \((\mathcal{O}_3, \mathcal{H}, \mathcal{D})\) is in deed a Dirac triple.
Proposition 3.1  Connes’ space of one-forms $\Omega^1_D(\mathcal{O}_3)$ is a free module of rank 3.

Proof Recall the differential $\delta$ on the reduced universal differential algebra $\Omega^*(\mathcal{O}_3)$. Also recall the well defined representation $\Pi : \Omega^*(\mathcal{O}_3) \to B(\mathcal{H})$. Then by definition,

$$\Omega^1_D(\mathcal{O}_3) = \{ \Pi \left( \sum a_i \delta b_i \right) \subset B(\mathcal{H}) : a_i, b_i \in \mathcal{O}_3 \}.$$ 

For $a_i, b_i \in \mathcal{O}_3$,

$$\Pi \left( \sum_i a_i \delta b_i \right) = \sum_i (a_i \otimes I)[D, \pi(b_i)] = \sum_j 3 \left( \sum_i a_i \partial_j (b_i) \otimes \sigma_j \right).$$

The above calculation proves that $\Omega^1_D(\mathcal{O}_3) \subset \mathcal{O}_3 \oplus \mathcal{O}_3 \oplus \mathcal{O}_3$. We shall show that $\Omega^1_D(\mathcal{O}_3) = \mathcal{O}_3 \oplus \mathcal{O}_3 \oplus \mathcal{O}_3$. To that end observe that

$$\Pi(S_1^* \delta S_2) = (S_1^* \otimes I)[D, S_2] = -S_1^* S_1 \otimes \sigma_3 = -1 \otimes \sigma_3.$$ 

Similarly, it can be shown that $\Pi(S_1^* \delta S_3) = 1 \otimes \sigma_2$ and $\Pi(S_2^* \delta S_3) = 1 \otimes \sigma_1$. Hence,

$$\Omega^1_D(\mathcal{O}_3) = \mathcal{O}_3 \oplus \mathcal{O}_3 \oplus \mathcal{O}_3.$$ 

So $\Omega^1_D(\mathcal{O}_3)$ is a free module of rank 3. The freeness follows from the linear independence of $\sigma_1, \sigma_2, \sigma_3$. $\square$

Proposition 3.2 $\Omega^2_D(\mathcal{O}_3)$ is also a free module of rank 3.

Proof By definition, $\Omega^2_D = \Pi(\Omega^2)/\Pi(\delta J^1_0)$. We claim that

$$\Pi(\Omega^2(\mathcal{O}_3)) = \mathcal{O}_3 \oplus \mathcal{O}_3 \oplus \mathcal{O}_3 \oplus \mathcal{O}_3.$$ 

$\square$
To prove the above claim, let $\sum_i a_i \delta b_i \delta c_i \in \Omega^2(\mathcal{O}_3)$.

$$
\Pi\left(\sum_i a_i \delta b_i \delta c_i\right) = \sum_i (a_i \otimes \mathbb{I})[\mathcal{D}, b_i][\mathcal{D}, c_i]
$$

$$
= \sum_i (a_i \otimes \mathbb{I}) \left( \sum_{j=1}^3 \partial_j(b_i) \otimes \sigma_j \right) \left( \sum_{k=1}^3 \partial_k(c_i) \otimes \sigma_k \right)
$$

$$
= \sum_i \left( a_i \sum_{j=1}^3 \partial_j(b_i) \partial_j(c_i) \otimes \mathbb{I} + \sum_{j<k} a_i (\partial_j(b_i) \partial_k(c_i) - \partial_k(b_i) \partial_j(c_i)) \otimes \sigma_j \sigma_k \right),
$$

which proves $\Pi(\Omega^2(\mathcal{O}_3)) \subset \mathcal{O}_3 \oplus \mathcal{O}_3 \oplus \mathcal{O}_3$. To prove the equality, note that $\Pi(\delta(S^*_1)\delta(S_1)) = \sum_{j=1}^3 \partial_j(S^*_1)\partial_j(S_1) \otimes \mathbb{I} = 2 \otimes \mathbb{I}$. Also,

$$
\Pi(\delta S^*_2 \delta S_1) = \sum_{j<k} (\partial_j(S^*_2)\partial_k(S_1) - \partial_k(S^*_2)\partial_j(S_1)) = 1 \otimes \sigma_1 \sigma_2.
$$

Similarly, choosing appropriate elements from $\mathcal{O}_3$, we can establish the equality. We are left with finding the junk forms. First, we shall show that $\Pi(\delta J^1_1) \subset \mathcal{O}_3 \otimes \mathbb{I}$. Choose $\omega = \sum_i a_i \delta b_i$ so that $\Pi(\omega) = 0$. But $\Pi(\omega) = 0$ implies the following:

$$
\sum_i (a_i \otimes \mathbb{I}) \left( \sum_{j=1}^3 \partial_j(b_i) \otimes \sigma_j \right) = 0
$$

$$
\Rightarrow \sum_j \left( \sum_i a_i \partial_j(b_i) \right) \otimes \sigma_j = 0.
$$

Using the linear independence of $\sigma_j's$, we get $\sum_i a_i \partial_j(b_i) = 0$ for $j = 1, 2, 3$. Note that $\delta \omega = \sum_i \delta a_i \delta b_i$. Therefore,

$$
\Pi(\delta \omega) = \sum_i \left( \sum_{j=1}^3 \partial_j(a_i) \partial_j(b_i) \otimes \mathbb{I} + \sum_{j<k} (\partial_j(a_i) \partial_k(b_i) - \partial_k(a_i) \partial_j(b_i)) \otimes \sigma_j \sigma_k \right).
$$

$\sum_i a_i \partial_j(b_i) = 0$ implies that for all $j, k = 1, 2, 3$, we have $\sum_i \partial_k(a_i) \partial_j(b_i) = -a_i \partial_k \partial_j(b_i)$. Hence,

$$
\Pi(\delta \omega) = \sum_i \left( \sum_{j=1}^3 \partial_j(a_i) \partial_j(b_i) \otimes \mathbb{I} - \sum_{j<k} (a_i [\partial_j, \partial_k](b_i) \otimes \sigma_j \sigma_k) \right).
$$
Using the commutation relations (4), $\Pi(\delta \omega)$ reduces to
\[
\left( \sum_{i}^{3} \sum_{j=1}^{3} \partial_{j}(a_{i})\partial_{j}(b_{i}) \otimes I - \sum_{i} a_{i} \partial_{1}(b_{i}) \otimes \sigma_{2}\sigma_{3} + \sum_{i} a_{i} \partial_{2}(b_{i}) \otimes \sigma_{1}\sigma_{3} + \sum_{i} a_{i} \partial_{3}(b_{i}) \otimes \sigma_{1}\sigma_{2} \right).
\]

But $\sum_{i} a_{i} \partial_{j}(b_{i}) = 0$ for $j = 1, 2, 3$. So
\[
\Pi(\delta J_{0}^{1}) \subset O_{3} \otimes I.
\]

We shall prove that the above inclusion is actually an equality. Pick $\omega = S_{1}^{s}\delta S_{1}$. Then $S_{1}^{s} \partial_{j}(S_{1}) = 0$ for all $j$ so that $\Pi(\omega) = 0$. But $\Pi(\delta \omega) = \sum_{j=1}^{3} \partial_{j}(S_{1}^{s}) \partial_{j}(S_{1}) \otimes I = 2 \otimes I$. Therefore,
\[
\Pi(\delta J_{0}^{1}) = O_{3} \otimes I.
\]

Hence, we conclude that
\[
\Omega_{D}^{2}(O_{3}) = O_{3} \oplus O_{3} \oplus O_{3}.
\]

From the proof of Proposition 3.1, it follows that $\{1 \otimes \sigma_{i} : i = 1, 2, 3\}$ is an $O_{3}$-basis for $\Omega_{D}^{1}(O_{3})$. We denote them by $\{e_{i}\}_{i=1,2,3}$. It is clear from the module structure that $e_{i} \in Z(\Omega_{D}^{1}(O_{3}))$. Similarly $\{1 \otimes \sigma_{ij}\}_{i<j}$ is an $O_{3}$-basis for $\Omega_{D}^{2}(O_{3})$. We denote the basis elements by $\{e_{ij}\}_{i<j}$. With these notations we have the following

**Lemma 3.3** The multiplication map $m : \Omega_{D}^{1}(O_{3}) \otimes_{O_{3}} \Omega_{D}^{1}(O_{3}) \to \Omega_{D}^{2}(O_{3})$ is surjective and is given by
\[
m \left( \sum_{i=1}^{3} e_{i}a_{i}, \sum_{i=1}^{3} e_{i}b_{i} \right) = \sum_{i<j} e_{ij}(a_{i}b_{j} - a_{j}b_{i}). \tag{5}
\]

**Proof** The surjectiveness of $m$ follows immediately once we prove (5). But (5) is a consequence of the following:
\[
m(e_{i} \otimes e_{i}) = 0, m(e_{i} \otimes e_{j}) = e_{i}e_{j} \text{ for } i < j, m(e_{i} \otimes e_{j}) = -e_{j}e_{i} \text{ for } i > j.
\]

From now on we shall denote the $O_{3}$-bimodule $\Omega_{D}^{1}(O_{3})$ by $E$. 

\[\text{ Springer}\]
Theorem 3.4 For the Connes’ differential calculus coming from the $C^*$-dynamical system $(O_3, \alpha, SO(3))$, a unique Levi-Civita connection exists for any pseudo-Riemannian metric.

Proof We shall show that the Connes’ differential calculus satisfies Assumptions I–IV of Sect. 2.1 and hence by Theorem 2.6, the conclusion of the above theorem will follow. So let us check the assumptions one by one. □

Assumption I $\mathcal{E}$ is a free module of rank 3. Therefore, it is finitely generated and projective as both right and left module. It is easy to see that the map $u: \mathcal{E} \otimes \mathcal{Z}(O_3) \to \mathcal{E}$ given by $u(e_i \otimes a_i) = \sum e_i a_i$ is an isomorphism with the inverse $u^{-1}(\sum_{i=1}^{3} e_i a_i) = \sum_{i=1}^{3} e_i \otimes a_i$.

Assumption II–III Recall the multiplication map $m$. From Lemma 3.3, we have the following short exact sequence:

$$0 \to \ker(m) \to \mathcal{E} \otimes O_3 \to \Omega^2_D(O_3) \to 0$$

$\Omega^2_D(O_3)$ being a free module, the above short exact sequence splits and consequently we have

$$\mathcal{E} \otimes O_3 = \ker(m) \oplus \mathcal{F},$$

where $\mathcal{F} \cong \Omega^2_D(O_3)$. As in [2], this splitting implies that the map $P_{sym}$ on the elements $e_i \otimes e_j$ is given by $P_{sym}(e_i \otimes e_j) = \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i)$. As a consequence, $\sigma(e_i \otimes e_j) = e_j \otimes e_i$ for all $i, j = 1, 2, 3$. Thus, Connes’ differential calculus satisfies Assumptions II–III.

Assumption IV Consider the metric $g: \mathcal{E} \otimes O_3 \to O_3$ defined by

$$g(e_i \otimes e_j a) = \delta_{ij} a, \text{ where } a \in O_3.$$  \hfill (6)

Then, it is easy to see that $g$ is a pseudo-Riemannian bilinear non-degenerate metric such that $g \circ \sigma = g$. Hence, Assumption IV is satisfied. □

Henceforth, we shall call the metric $g$ the canonical metric.

4 Scalar curvature for the canonical metric

In this section, we shall compute the scalar curvature of $O_3$ for the canonical metric $g$ given by (6).

4.1 Computation of Christoffel symbols

Let us recall from Theorem 2.6 that the Levi-Civita connection for any pseudo-Riemannian metric $g$ is given by $\nabla = \nabla_0 + L$, where $\nabla_0$ is any torsionless connection.
(see Definition 2.3) and \( L = \Phi_g^{-1}(dg - \Pi_g(\nabla_0)) \). We prove the following lemma which will enable us to choose a torsionless connection.

**Lemma 4.1**

\[
de_1 = m(e_2 \otimes e_3) \tag{7}
de_2 = m(e_3 \otimes e_1) \tag{8}
de_3 = m(e_1 \otimes e_2) \tag{9}
\]

**Proof** Since \( e_1 = \sigma^* S^2(S_3) \), we have

\[
de_1 = \Pi(\delta S^*_2 \delta S_3)
= (-S^*_3 \otimes \sigma_1 - S^*_1 \otimes \sigma_3)(S_2 \otimes \sigma_1 + S_1 \otimes \sigma_2)
= 1 \otimes \sigma_2 \sigma_3
= m(e_2 \otimes e_3)
\]

The other equalities follow from similar calculations. \( \square \)

**Proposition 4.2** A choice of torsionless connection \( \nabla_0 \) is given on the \( O_3 \)-basis of the free module \( \Omega^1(O_3) \) by

\[
\nabla_0(e_1) = (e_3 \otimes e_2), \quad \nabla_0(e_2) = (e_1 \otimes e_3), \quad \nabla_0(e_3) = (e_2 \otimes e_1). \tag{10}
\]

**Proof** Consequence of Lemma 4.1 and the fact that \( T_{\nabla_0} \) is right \( O_3 \)-linear. \( \square \)

**Lemma 4.3** For the canonical metric \( g \) and the connection \( \nabla_0 \), let \( \Pi_g(\nabla_0) \) be given by

\[
\Pi_g(\nabla_0)(e_i \otimes e_j) = -\sum_{m=1}^3 T^m_{ij} e_m.
\]

Then

\[
T^m_{ij} = \begin{cases} -1, & i, j, m \text{ are distinct} \\ 0, & \text{otherwise}. \end{cases} \tag{11}
\]

**Proof** Using Eq. (10), we get the following:

\[
\Pi_g(\nabla_0)(e_1 \otimes e_2) = (g \otimes \text{id})\sigma_{23}(\nabla_0(e_1) \otimes e_2 + \nabla_0(e_2) \otimes e_1)
= (g \otimes \text{id})\sigma_{23}(e_3 \otimes e_2 \otimes e_2 + e_1 \otimes e_3 \otimes e_1)
= e_3
\]

With similar computations, one can show that \( \Pi_g(\nabla_0)(e_2 \otimes e_3) = e_1 \) and \( \Pi_g(\nabla_0)(e_1 \otimes e_3) = e_2 \). Using the easy to see fact that \( T^m_{ij} = T^m_{ji} \) for all \( i, j, m \), we have for distinct
$i, j, m, T^m_{ij} = -1$ for all $i, j, m = 1, 2, 3$ and $T^m_{ij} = 0$ for $i \neq j$ whenever either $m = i$ or $m = j$. So we are left with proving that $T^m_{ii} = 0$ for all $i, m$. To that end, note the following:

\[ \Pi_g(\nabla_0)(e_1 \otimes e_1) = (g \otimes \text{id})\sigma_{23}2(\nabla_0(e_1) \otimes e_1) = 2(g \otimes \text{id})(e_3 \otimes e_2 \otimes e_1) = 0. \]

Similarly, we can show that $\Pi_g(\nabla_0)(e_i \otimes e_i) = 0$ for $i = 2, 3$, finishing the proof of the lemma. \hfill \Box

**Theorem 4.4** Let $\Gamma^i_{jk}$ be the Christoffel symbols of the Levi-Civita connection $\nabla$ i.e. $\nabla(e_i) = \sum_{j,k=1}^3 e_j \otimes e_k \Gamma^i_{jk}$ for $i = 1, 2, 3$. Then, we have the following:

\[ \Gamma^1_{32} = \Gamma^2_{13} = \Gamma^3_{21} = \frac{1}{2}, \Gamma^1_{23} = \Gamma^2_{31} = \Gamma^3_{12} = -\frac{1}{2}. \] (12)

Rest of the Christoffel symbols are zero.

**Proof** $\nabla - \nabla_0 = L$ for some $L \in \text{Hom}(\mathcal{E}, \mathcal{E} \otimes_{O_3}^{\text{sym}} \mathcal{E})$ (see [2, Theorem 2.14]). If we write $L(e_j) = \sum_{i,m=1}^3 e_i \otimes e_m L^j_{im}$ for $j = 1, 2, 3$, then adapting the same arguments used in [2, Theorem 4.24], we get

\[ L^j_{im} = \frac{1}{2}(T^m_{ij} + T^i_{jm} - T^j_{im}). \]

Hence by Equation (11),

\[ L^j_{im} = \begin{cases} -\frac{1}{2}, & i, j, m \text{ are distinct} \\ 0, & \text{otherwise.} \end{cases} \] (13)

\[ \nabla(e_1) = \nabla_0(e_1) + L(e_1) = e_3 \otimes e_2 - \frac{1}{2}e_3 \otimes e_2 - \frac{1}{2}e_2 \otimes e_3 \text{ (by (13))} = \frac{1}{2}e_3 \otimes e_2 - \frac{1}{2}e_2 \otimes e_3. \]

Therefore, $\Gamma^1_{23} = -\frac{1}{2}, \Gamma^1_{32} = \frac{1}{2}, \Gamma^1_{11} = \Gamma^1_{22} = \Gamma^1_{12} = \Gamma^1_{21} = 0$. With similar computations, we can prove the rest of the equalities of (12). \hfill \Box

**4.2 Computation of scalar curvature**

Scalar curvature is obtained by contracting the Ricci tensor by the metric. Recall the maps $\rho$ and $\text{ev}$ from Sect. 2.1. The Ricci tensor $\text{Ric} \in \mathcal{E} \otimes_{O_3} \mathcal{E}$ is given by

\[ \text{Ric} = (\text{id} \otimes \text{ev} \circ \rho)(\Theta), \]
Lemma 4.5 \( R(\nabla)(e_i) = \frac{1}{8} \sum_{k \neq i} (e_k \otimes e_k \otimes e_i - e_k \otimes e_i \otimes e_k), \) for \( i = 1, 2, 3. \)

**Proof** By definition of \( R(\nabla) \), we have

\[
R(\nabla)(e_1) = H \circ \nabla(e_1) \quad \text{(for the definition of } H, \text{ see (3))}
\]

\[
= \frac{1}{2} H(e_3 \otimes e_2 - e_2 \otimes e_3)
\]

\[
= \frac{1}{2} [(1 - P_{sym})_{23}(\nabla(e_3) \otimes e_2 - \nabla(e_2) \otimes e_3) + e_3 \otimes Q^{-1}(de_2) - e_2 \otimes Q^{-1}(de_3)] \text{ (by (3))}
\]

We calculate each terms individually.

\[
(1 - P_{sym})_{23}(\nabla(e_3) \otimes e_2) = \frac{1}{2} (1 - P_{sym})_{23}(e_2 \otimes e_1 \otimes e_2 - e_1 \otimes e_2 \otimes e_2)
\]

\[
= \frac{1}{4} (e_2 \otimes e_1 \otimes e_2 - e_2 \otimes e_2 \otimes e_1)
\]

Similarly, \( (1 - P_{sym})_{23}(\nabla(e_2) \otimes e_3) = \frac{1}{4} (e_3 \otimes e_3 \otimes e_1 - e_3 \otimes e_1 \otimes e_3). \) Also using the fact that \( Q^{-1}(e_i \otimes e_j) = \frac{1}{2} (e_i \otimes e_j - e_j \otimes e_i) \) for all \( i, j, \) it is easy to see the following:

\[
e_3 \otimes Q^{-1}(de_2) = \frac{1}{2} (e_3 \otimes e_3 \otimes e_1 - e_3 \otimes e_1 \otimes e_3), e_2 \otimes Q^{-1}(de_3)
\]

\[
= \frac{1}{2} (e_2 \otimes e_1 \otimes e_2 - e_2 \otimes e_2 \otimes e_1).
\]

Hence,

\[
R(\nabla)(e_1) = \frac{1}{8} (e_2 \otimes e_1 \otimes e_2 - e_2 \otimes e_2 \otimes e_1 - e_3 \otimes e_3 \otimes e_1 + e_3 \otimes e_1 \otimes e_3
\]

\[
+ 2e_2 \otimes e_3 \otimes e_1 - 2e_2 \otimes e_1 \otimes e_3 - 2e_2 \otimes e_1 \otimes e_2 + 2e_2 \otimes e_2 \otimes e_1)
\]

\[
= \frac{1}{8} (e_2 \otimes e_2 \otimes e_1 + e_3 \otimes e_3 \otimes e_1 - e_2 \otimes e_1 \otimes e_2 - e_3 \otimes e_1 \otimes e_3).
\]

With similar computations we have

\[
R(\nabla)(e_2) = \frac{1}{8} (e_1 \otimes e_1 \otimes e_2 + e_3 \otimes e_3 \otimes e_2 - e_1 \otimes e_2 \otimes e_1 - e_3 \otimes e_2 \otimes e_3).
\]
\[
R(\nabla)(e_3) = \frac{1}{8}(e_1 \otimes e_1 \otimes e_3 + e_2 \otimes e_2 \otimes e_3 - e_1 \otimes e_3 \otimes e_1 - e_2 \otimes e_3 \otimes e_2).
\]

Combining the above expressions, we get the result of the lemma. \(\square\)

**Lemma 4.6** The Ricci tensor \(\text{Ric}\) is given by

\[
\text{Ric} = \sum_{j=1}^{3} \left( -\frac{1}{4} \right) e_j \otimes e_j.
\]

**Proof** Let us compute the curvature operator \(\Theta\). Denoting the element of \(\mathcal{E}^*\) which takes the value 1 at \(e_i\) and 0 on the other basis elements by \(e^*_i\), we do the following calculation:

\[
\Theta = (\sigma_{23} \otimes \text{id}) \zeta \xi (\xi_{\mathcal{E} \otimes \mathcal{O}_3} \xi_{\mathcal{O}_3}) R(\nabla)
\]

\[
= (\sigma_{23} \otimes \text{id}) \frac{1}{8} \sum_{i \neq j} (e_j \otimes e_j \otimes e_i \otimes e^*_i - e_j \otimes e_i \otimes e_j \otimes e^*_i)
\]

\[
= \frac{1}{8} \sum_{i \neq j} (e_j \otimes e_i \otimes e_j \otimes e^*_i - e_j \otimes e_j \otimes e_i \otimes e^*_i)
\]

Now recall the map \(\rho : \mathcal{E} \otimes \mathcal{E}^* \to \mathcal{E}^* \otimes \mathcal{E}\). It is easy to see that \(\rho(e_i \otimes e^*_i) = e^*_i \otimes e_i\) for all \(i, j = 1, 2, 3\). Using this, we get

\[
\text{Ric} = (\text{id}_{\mathcal{E} \otimes \mathcal{E}} \otimes \text{ev} \circ \rho)(\Theta)
\]

\[
= \frac{1}{8} \sum_{i \neq j} (\text{id}_{\mathcal{E} \otimes \mathcal{E}} \otimes \text{ev} \circ \rho)(e_j \otimes e_i \otimes e_j \otimes e^*_i - e_j \otimes e_j \otimes e_i \otimes e^*_i)
\]

\[
= \frac{1}{8} \sum_{i \neq j} (e_j \otimes e_i \text{ev}(e^*_i \otimes e_j) - e_j \otimes e_j \text{ev}(e^*_i \otimes e_i)).
\]

Using the fact that \(\text{ev}(e^*_i \otimes e_j) = \delta_{ij}\), we get

\[
\text{Ric} = -\frac{1}{8} \sum_{i \neq j} e_j \otimes e_j = \sum_{j=1}^{3} \left( -\frac{1}{4} \right) e_j \otimes e_j.
\]

This proves the lemma. \(\square\)

**Theorem 4.7** The Cuntz algebra with three generators with the differential calculus coming from the action of \(SO(3)\) has a constant negative scalar curvature \(-\frac{3}{4}\) for the canonical metric.
\textbf{Proof} By the definition of the scalar curvature (see after Definition 4.4 in [2]),
\[
\text{Scal} = \text{ev} \circ (V_g \otimes \text{id})(\text{Ric})
\]
\[
= \left( -\frac{1}{4} \right) \sum_{j=1}^{3} g(e_j \otimes e_j)
\]
\[
= -\frac{3}{4}.
\]
\[
\square
\]

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