Induced deformation of the canonical structure and UV/IR
duality in $(1 + 1)D$

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Abstract

The purpose of this work is two fold. Working in the framework of $(1 + 1)D$ Lorentz violating field theories we will investigate in the first place the general claim that fermionic interactions may be equivalent to a deformation of the canonical structure of the theory. Second the deformed theory will be studied using duality reasoning to address the behavior of the Infra-Red and Ultra-Violet regimes.

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I. INTRODUCTION AND MOTIVATIONS

It is known as a general fact that the presence of fermions in a theory can in many instances be effectively accounted for by a deformation in the canonical structure of the theory without fermions [1]. This can be traced to the possible induction of non trivial terms due to quantum fermionic fluctuations [2] which changes the symplectic structure. In this work we want to address this important fact in the context of Lorentz violating theories [3, 4]. In this context our work has a two fold purpose: in the first place, we want to strength the knowledge of the fact that Lorentz violation as described by the deformation of the canonical structure, a sometimes ad hoc imposition, may in fact have its origin in Lorentz violating fermionic interactions. This will be done by working out an explicit example of a scalar field interacting with fermions in $(1 + 1)D$. The effective scalar theory with the quantum fermionic fluctuations accounted for is obtained considering the Goldstone-Wilczek mechanism. This is a new and important result as it provides another non trivial example of generation of deformed structures induced by interactions. As a second objective we will study the resulting deformed theory under the scope of duality. We want to shed some light on the physical mechanism involved in the canonical structure deformation of the dual theory and, in particular, we want to gain some insights on the relations between the IR and UV scales in a Lorentz violating background. This is an important inquire since it is generally demanded that Lorentz violation effects takes place in nearly unobservable scales so that its effects could have managed to remain undetected so far, but if duality could provide us with a map that connects this scale to observable ones the safety of this argument would be spoiled.

It is important to briefly review some results on this subject in order to set the ground on which this work relies. The recent surge in the study of Lorentz violation has its roots in the exploration of physics beyond the standard models of particle physics and cosmology. As a matter of fact it is generally believed that Lorentz invariance cannot hold at such extreme regions as the Planck scale and it is expected that some relics of this violation are translated on to observable effects at accessible scales. Thus a reasonable way to approach the problem is to understand what kind of phenomena signaling a departure of the Lorentz invariance we should expect to see. This is the rationale behind the proposed extended standard model (ESM) [5], for example, which catalogues the possible Lorentz violating terms that can be
added to the conventional standard model lagrangian of particle physics.

Along the same lines, and in fact closely related, is the study of non-commutative theories. Initially proposed by Heisenberg and by Snyder it gained much impulse recently due to its natural appearance in distinct arenas such as the quantum Hall phenomena and in the context of string theory, signaling that such structures might have indeed a fundamental origin. There are two main branches characterizing non-commutative theories: one is the study of spacetime non-commutativity defined by having nonvanishing commutators among the coordinates of spacetime leading to novel field theory models. The other is the noncommutativity of field operators imposed by a deformation in the canonical structure of the field theories. Obviously both approaches may lead to theories in which Lorentz invariance can be broken. In fact there seems to be a remarkable unity in all approaches to Lorentz violation, the ESM Lorentz violating terms may be connected to noncommutativity of spacetime and on the other hand can also be induced by deformation of the canonical structure. Those results builds up the case that consistent theories may be constructed giving up Lorentz invariance without diving in to such unacceptable consequences as causal and unitarity violations.

In this work our interest relies primarily in the study of deformed canonical structures as induced by quantum dynamical effects. This is a very important phenomenon and its occurrence reveals non trivial structures. For example in (2 + 1)$D$ it is known that if the Maxwell theory is written in interaction with massive fermions the Chern-Simons term cannot be disregarded since it will be induced by the fermionic dynamics anyway. The resulting Maxwell-Chern-Simons theory shows non trivial topological properties that are not evident initially. This is a particular case of a fundamental result. There are other examples such as the induced Berry phases represented by Wess-Zumino terms which has been shown recently to have a decisive role in determining the quantum phase transitions of magnetic systems. More akin to our work is the result obtained in which is the generalization to a (3 + 1)$D$ Lorentz violating framework of the (2 + 1)$D$ case discussed above. It was shown that the Carroll-Field-Jackiw model can be written as the conventional Maxwell theory with deformed canonical relations. This result is to be related to that discussed by Kostelecky and Jackiw which investigates the induction of the Lorentz violating Chern-Simons-like term defining the Carroll-Field-Jackiw model by the quantum fluctuations of a Lorentz violating fermionic dynamics. In the present work we will investigate a similar
result in \((1 + 1)D\) concerning scalar fields interacting with fermions and where the fermions will be dealt with by the Goldstone-Wilczek mechanism [12].

The work is organized as follows: in the next section our results will be presented. We will start by defining the effective scalar theory originating from taking into account the Lorentz violating fermionic interaction as dictated by the Goldstone-Wilczek mechanism [12]. We will proceed with an analysis of the propagating modes highlighting the effects of the Lorentz violation. After a discussion of the symplectic structure of the effective theory, that sets straight our expectations concerning the canonical relations in a dual picture, we will seek this dual formulation to find that a possible connection can be established between the infra-red and ultra-violet regimes. We will close with the concluding remarks where it is discussed the potential consequences of this IR/UV mapping and our concerns regarding its limitations. An appendix is included to further discuss the derivation of the induced current in the Goldstone-Wilczek mechanism.

II. RESULTS

Before delving into the actual presentation of our results it will pay off to elaborate a bit further on the framework of noncommutative fields. As mentioned above, there are two major trends involving noncommutativity. Without a doubt the great majority of works in this field is dedicated to space-time noncommutativity embodied in relations such as

\[
[x_\mu, x_\nu] = i\theta_{\mu\nu}
\]  

(1)

where \(x_\mu\) are coordinates of space-time and \(\theta_{\mu\nu}\) is an antisymmetric constant tensor with dimension \((\text{length})^2\). This kind of structure shows up in a variety of contexts, most notably in string theory where the coordinates represent longitudinal directions of D-branes, as seen by the ends of open strings, in the presence of a B-field background, and also in the quantum Hall effect where the presence of a strong magnetic field induces a noncommutativity among the coordinates of the particle in a plane. However here the noncommutativity among the coordinates comes from the projection of the operators over the Lowest Landau Level.

But it is the other trend which is the one directly related to our results [1]. The quantum theory of noncommutative fields has been elaborated as a generalization of noncommutative quantum mechanics which are rather different from the usual quantum field theory over a
canonical noncommutative space-time \[13\]. The basic idea of this procedure is to include minimal modifications to the canonical structure of the field theory, or, equivalently, of its symplectic structure \[1\], amounting to adding a tiny violation of the microcausality principle. In these works, the notion of noncommutative fields are introduced by

\[
[\Phi_i(x), \Phi_j(y)] = i\epsilon_{ijk} B^k \delta^{(3)}(x - y)
\]

\[
[\Phi_i(x), \Pi_j(y)] = i\delta_{ij} \delta^{(3)}(x - y)
\]

\[
[\Pi_i(x), \Pi_j(y)] = i\epsilon_{ijk} \Theta^k \delta^{(3)}(x - y)
\]

(2)

where \(i, j = 1, 2, 3\) and \(\Pi_i\) are the conjugate momenta. We mention that, due to the presence of \(\delta^{(3)}(x - y)\) in the right-hand side, the constant vectors \(B_k\) and \(\Theta_k\) have canonical dimension of length and mass, respectively introducing an ultra-violet and an infra-red scale respectively.

It is important to point out that Lorentz symmetry violation due to noncommutativity of fields is not yet established as compared to the usual formulation of field theory over canonical noncommutative spacetime simply because we still lack explicit model realizations. In a recent paper the noncommutative field space formulation was used to analyze the abelian bosonization for a two dimensional system \[14\]. The rationale there was that an analysis in a \(D = 2\) spacetime theory can be useful in disclosing the basic physics underlying this problem. They found that for chiral bosons in a noncommutative field space conformal invariance continues to hold and that the non-commutativity in the field space leads to free fermions when chiral bosons are fermionized.

In a recent report, the connection between Lorentz invariance violation and noncommutativity of fields in a quantum field theory of chiral bosons was resumed with the investigation of a generalized model of non-commutative field space chiral bosons with a real one-parameter deformed symplectic algebra \[15\] which was investigated upon the soldering of the individual chiralities.

In the present work we want to study the possibility of having the low-energy sector of an effective real scalar field model having its canonical structure deformed by quantum fluctuations of a fermionic field coupled to this scalar field, such that

\[
[\Pi(x), \Pi(y)] \rightarrow l(x - y)
\]

(3)

with \(l(x - y)\) an anti-symmetry form to be determined below, while the other brackets remain
A. The Goldstone-Wilczek mechanism

Here we shall follow the strategy of [5] and consider bosonic/fermionic model with Lorentz violating interaction. Consider the following Lorentz violating action:

\[
S = \int d^2x \left( \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + i \bar{\psi} \gamma_{\mu} \partial_{\mu} \psi - \theta \phi \bar{\psi} \gamma^1 \psi - g \bar{\psi} e^{g \phi} \psi \right)
\]  (4)

it describes a scalar field \( \phi \) coupled with a massless fermion field \( \psi \). Here \( \gamma^0 = \sigma^1 \), \( \gamma^1 = i\sigma^3 \) and \( \gamma_5 = i\sigma^2 \). The derivative interaction explicitly violates Lorentz invariance as it defines a constant tensor which selects a preferred Lorentz frame:

\[
\phi \bar{\psi} \gamma^1 \psi = P_{\mu \nu} (\partial_{\mu} \phi) \bar{\psi} \gamma_{\nu} \psi; \quad P_{\mu \nu} = \begin{pmatrix} 0 & 0 \\ 0 & \theta \end{pmatrix}.
\]  (5)

We want to consider a framework in which the space-time variations of the scalar field may be neglected in a first approximation. Ultimately we are searching for an effective theory describing the low momenta excitations of the scalar field. We also demand the Lorentz violating effects to be small thus retaining only first order terms in \( \theta \). To construct such an effective theory we must take into account the contribution of the fermionic fluctuations defining the effective action by the expression:

\[
e^{iS_{eff}} = \int D\bar{\psi} D\psi e^{iS}.
\]  (6)

Further, neglecting higher order scalar derivatives we can write

\[
e^{iS_{eff}} = e^{\int d^2x \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi} e^{-\int d^2x \theta \phi \bar{\psi} \gamma^1 \psi} = e^{\int d^2x \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \int d^2x \theta \phi \bar{\psi} \gamma^1 \psi} \]

where

\[
<\bar{\psi} \gamma^{\mu} \psi> = \int D\bar{\psi} D\psi (\bar{\psi} \gamma^{\mu} \psi) e^{\int d^2x \left( \bar{\psi} \gamma^{\mu} \partial_{\mu} \psi - g \bar{\psi} e^{g \phi} \psi \right)}
\]  (8)

is the induced fermion current due to the non-derivative interaction given by the last term in (4). This expression neglects higher \( \theta \) order contributions and higher \( \phi \) derivatives also. The physical meaning is that we are supposing a classical behavior for the fermionic current
at the scale of variations of $\phi$, that is: $<(j^\mu - <j^\mu>)^2> = 0$. It is straightforward to calculate this induced current \[12, 16\] imposing the condition $|\partial \phi| \ll |g|$, it is given by

$$
<\psi \gamma^\mu \psi> = -\frac{1}{2\pi} \varepsilon^{\mu\nu} \partial_\nu \phi. \tag{9}
$$

For a somewhat more detailed discussion of these matters and a bosonized view see the appendix. Then the effective Lorentz violating action is given by

$$
S_{\text{eff}} = \int d^2 x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\theta}{2\pi} \phi' \dot{\phi} \right). \tag{10}
$$

where $\phi' \equiv \partial_t \phi$ and $\dot{\phi} \equiv \partial_\phi \phi$. Observe that it does not depend on the coupling $g$, nevertheless the scale of validity of our approximation, and therefore of the effective theory itself, is defined by $g$, which has mass dimension 1 setting an ultraviolet cutoff for the scalar momentum in the effective theory. This observation will be very important in the discussion of the dual formulation and the UV/IR dual map.

**B. Modes decomposition**

The violation of the Lorentz invariance has a very interesting manifestation here. The effective action \[10\] describes two independent propagating modes with different velocities. This can be explicitly seen by making a chiral decomposition as follows. Let us write \[10\] as:

$$
S_{\text{eff}} = \int d^2 x \left( \Pi \dot{\phi} - \frac{1}{2} \Pi^2 - \frac{1}{2} \phi'^2 - \frac{\theta}{2\pi} \phi' \dot{\phi} \right). \tag{11}
$$

where $\Pi$ is an auxiliary field. We can further redefine $\Pi \rightarrow \eta + \frac{\theta}{2\pi} \phi'$ giving

$$
S_{\text{eff}} = \int d^2 x \left( \eta \dot{\phi} - \frac{1}{2} \left( \eta + \frac{\theta}{2\pi} \phi' \right)^2 - \frac{1}{2} \phi'^2 \right). \tag{12}
$$

Now the following field redefinition can be made

$$
\phi = \phi_+ + \phi_- \quad \eta = u \left( \phi'_+ - \phi'_- \right) \tag{13}
$$

where $u = \sqrt{1 + \left( \frac{\theta}{2\pi} \right)^2}$. With this we finally obtain the result

$$
S_{\text{eff}} = S_+ + S_-, \tag{14}
$$
where
\begin{align*}
S_+ &= \int d^2x \left( u\dot{\phi}_+ \phi'_+ - \frac{1}{2} \left[ 1 + \left( u + \frac{\theta}{2\pi} \right)^2 \right] \phi'^2_+ \right)
S_- &= \int d^2x \left( -u\dot{\phi}_- \phi'_- - \frac{1}{2} \left[ 1 + \left( u - \frac{\theta}{2\pi} \right)^2 \right] \phi'^2_- \right).
\end{align*}

(15)

We immediately see that the velocities of each mode are different. In fact, for the sensible case \( \theta \ll 1 \), we have:
\begin{align*}
v_+ &= \left( 1 + \frac{\theta}{2\pi} \right) \\
v_- &= \left( 1 - \frac{\theta}{2\pi} \right).
\end{align*}

(16)

Of course, this could be inferred directly from the dispersion relation following from the original effective action (10), but it is instructive to reveal the possibility of factorization of the modes as depicted in (15). This is to be compared to the well known factorization of the Proca model in \((2 + 1)D\) in its self-dual components [17].

C. Symplectic structure

From the symplectic matrix one can read the Poisson brackets of the model. These are not the canonical ones. But the action (10) and the free scalar theory with non-canonical brackets both lead to the same equations of motion.

More quantitatively, we can use the reduced order form (11) again from which follows immediately the inverse symplectic matrix
\begin{align*}
f &= \begin{pmatrix} 0 & -1 \\ 1 & -\frac{\theta}{\pi} \partial_x \end{pmatrix} \delta(x - y)
\end{align*}

(17)

which shows that the bracket \{\(\Pi(x),\Pi(y)\)\} is deformed. It is then easy to verify that the free scalar Hamiltonian
\begin{align*}
H &= \int dx \frac{1}{2} [\Pi^2 + (\phi')^2]
\end{align*}

with the brackets given by
\begin{align*}
\{\phi(x),\phi(y)\} &= 0 \\
\{\phi(x),\Pi(y)\} &= \delta(x - y) \\
\{\Pi(x),\Pi(y)\} &= \frac{\theta}{\pi} \partial_x \delta(x - y)
\end{align*}

(19)
lead to the same equations of motion obtained by minimizing the action (10). Thus we may interpret the Lorentz violating induced term due to fermionic interaction as a modification of the canonical Poisson brackets of the free field theory.

Observe that the deformation showed up in the momentum sector. The parameter $\theta$ is dimensionless but the deformation is proportional to the derivative and thus has mass dimension 1, fitting the general discussion below (2). Incidentally this is interpreted as an infra-red deformation (see the comments on [18]). Since duality has as a general property the interchange of potential and kinetic contributions, could it be possible to transfer this deformation to the ultra-violet sector by duality transformations? In fact we will show in the next section that an exact dual representation exists with brackets given by

$$\{\Sigma(x), \Sigma(y)\} = \frac{\theta}{2\pi} \epsilon(x - y)$$
$$\{\Sigma(x), P(y)\} = \delta(x - y)$$
$$\{P(x), P(y)\} = 0$$ (20)

where the escalar field $\Sigma$ is the dual representation of the $\phi$ field and $\epsilon(x - y)$ is the skew symmetric step function with the property $\partial_x \epsilon(x - y) = 2\delta(x - y)$. This would stand for a ultra-violet deformation. We will discuss these remarks more precisely in the following section.

D. The dual formulation

We will now seek the dual formulation of the theory discussed so far. Observe that the action (10) can be cast in the form:

$$S_{\text{eff}} = \int d^2x \frac{1}{2} \partial_\mu \phi M^{\mu\nu} \partial_\nu \phi$$ (21)

where

$$M^{\mu\nu} = \begin{pmatrix} \frac{\theta}{2\pi} & 1 \\ \frac{\theta}{2\pi} & -1 \end{pmatrix} = \left( 1 + \left( \frac{\theta}{2\pi} \right)^2 \right) M_{\mu\nu}^{-1},$$ (22)

the physics described by (21) is not altered if we introduce an auxiliary field $\Pi^\mu$.

$$S_{\text{eff}} \rightarrow \int d^2x \left( \Pi^\mu \partial_\mu \phi - \frac{1}{2} \Pi_\mu (M^{-1})^{\mu\nu} \Pi_\nu \right)$$ (23)
Πμ can be integrated out (in a path integral sense) leading us back to (21). On the other hand φ can be viewed as a Lagrange multiplier forcing the constraint

\[ \partial_\mu \Pi^\mu = 0 \Rightarrow \Pi^\mu = \varepsilon^{\mu\nu} \partial_\nu \Sigma, \]  

(24)

where \( \varepsilon_{01} = 1 \Rightarrow \varepsilon^{01} = -1 \). Σ is the dual field and the action in terms of it is the dual action, which in this particular case is just the original one, that is, the theory is self-dual after a trivial scaling.

\[ S_{eff} \rightarrow *S_{eff} = \int d^2 x \frac{1}{2 \left( 1 + \left( \frac{\theta}{2\pi} \right)^2 \right)} \partial_\mu \Sigma M^{\mu\nu} \partial_\nu \Sigma \]  

(25)

From (23) a map between φ and Σ follows

\[ \begin{pmatrix} \dot{\Sigma} \\ \Sigma' \end{pmatrix} = R(\theta) \begin{pmatrix} \dot{\phi} \\ \phi' \end{pmatrix} = \begin{pmatrix} \frac{\theta}{2\pi} & -1 \\ -1 & \frac{\theta}{2\pi} \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \phi' \end{pmatrix}. \]  

(26)

\( R(\theta) \) is the dual map, but it is not the most general one that preserves the equations of motion. Linearity allows us to consider a linear combination including the trivial map (the identity map)

\[ \tilde{R}(\theta) = a \mathbf{1} + b R(\theta) \]  

(27)

where \( a \) and \( b \) are real constants.

This is a good place to comment on what seems to be a general feature of the kind of duality discussed here. Under a direct application of the map (27) the action (21) transforms as

\[ S_{eff} \rightarrow \left( \frac{1}{a^2 - b^2 \left( 1 + \left( \frac{\theta}{2\pi} \right)^2 \right)} \right) S_{eff} \]  

(28)

For \( a = 0, b = 1 \) the map reduces to the duality map (26), but there is a sign difference with respect to the dual obtained in (25) through the duality procedure. This will not affect the equations of motion of course, but it is nevertheless disturbing and begs for an explanation. The reason for this difference has its roots in a misuse of the map. For the map is constructed using the equations of motion of both \( \Pi^\mu \) and \( \phi \) that follows from (23), so it is not rigourously licit to substitute this on-shell information on the action. Nevertheless since we are dealing with quadratic actions this sign change is the only effect of the direct
use of the map on the action level, it is only a reflection of a general property of duality in these cases: it interchanges kinetic and potential contributions. But care should be taken in more general cases. This in fact should be all very familiar from Maxwell theory in \((3 + 1)D\): the analogous duality amounts to an interchange of electric and magnetic fields \((\vec{E} \rightarrow \vec{B}, \vec{B} \rightarrow -\vec{E})\) which of course changes the sign of the action when naively applied to it even though the Maxwell theory is self-dual.

With these warnings in mind we proceed now to the study of the Hamiltonian structure of this theory under the duality map. We are seeking for a dual description in which the deformed momentum brackets \((19)\) are mapped to the field space deformed ones \((20)\). The relevant relation is then

\[
\{\Sigma'(x), \Sigma'(y)\} = \{b\dot{\phi} + (a + b \left( \frac{\theta}{2\pi} \right)) \phi'(x), b\dot{\phi} + (a + b \left( \frac{\theta}{2\pi} \right)) \phi'(x)\} = -b^2 \left( \frac{\theta}{\pi} \right) \partial_x \delta(x - y) + 2b \left( a + b \left( \frac{\theta}{2\pi} \right) \right) \partial_x \delta(x - y) \tag{29}\]

where the map \((27)\) has been used as well as the brackets satisfied by \(\phi\). We immediately see that we can obtain what we intended for if we make \(a = -b \left( \frac{\theta}{2\pi} \right)\) and \(b = 1\). This particular map takes the Hamiltonian \((18)\) to

\[
H \rightarrow \int dx \left[ \frac{\Sigma'^2}{2} + \frac{P^2}{2} \right] \tag{30}\]

where \(P = \dot{\Sigma} + \frac{\theta}{\pi} \Sigma'\) is the canonical momentum as can be explicitly verified calculating the remaining bracket structure:

\[
\{\Sigma'(x), P(y)\} = \partial_x \delta(x - y) \tag{31}\]

\[
\{P(x), P(y)\} = 0 \tag{31}\]

which along with the Hamiltonian \((30)\) and the brackets \((29)\)

\[
\{\Sigma'(x), \Sigma'(y)\} = -\left( \frac{\theta}{\pi} \right) \partial_x \delta(x - y) \tag{32}\]

defines the dual formulation of \((18), (19)\).

Finally we would like to discuss the existence of a duality between the IR and UV scales as defined in a general way by the quantities \(B^k\) and \(\Theta^k\) in the introductory remarks of our results \((2)\). In the model we studied in this work there are two relevant constants: \(g\) with mass dimension 1 and \(\theta\) which is dimensionless. As discussed previously, the role of
$g$ is to set the scale of our approximations related to the Goldstone-Wilczek mechanism thus defining the region of validity of the effective scalar theory. $\theta$ on the other hand defines the deformation of the canonical structure but being dimensionless it seems to lack the significance that was carried by its analogs $B^k$ and $\Theta^k$. But we should be careful here because the interplay of $g$ and $\theta$ in the definition of scale and deformation makes this analysis non-trivial.

To make a proper analysis it is interesting to make a Fourier decomposition of the effective scalar theory and study the duality mode by mode. We shall adopt the $O(2)$ decomposition, instead of the usual textbook $U(1)$, given by (for details see [19, 20]).

$$\phi(x,t) = \int dk \, q_a(t,k) \hat{e}_a(k,x); \quad a, b = 1, 2$$

(33)

where the $O(2)$ basis is such that

$$\int dx \, \hat{e}_a(k,x) \hat{e}_b(k',x) = \delta_{ab} \delta(k-k').$$

(34)

Using this and the property: $\partial_x \hat{e}_a(k,x') = \varepsilon_{ab} k \hat{e}_b(k,x')$, we find that each mode is controlled by the Lagrangean

$$L = \frac{1}{2} q_a^2 - \frac{k^2}{2} q_a^2 + \theta k q_a \varepsilon_{ab} \dot{q}_b$$

(35)

representing a two dimensional harmonic oscillator under the influence of an external magnetic field $B \sim \theta k$.

In this illuminating mechanical language the duality discussed above becomes expressed in terms of a two-dimensional harmonic oscillator hamiltonian

$$H(p,q) = \frac{1}{2} p_a^2 + \frac{k^2}{2} q_a^2$$

(36)

with deformed canonical brackets

$$\{q_a, q_b\} = 0$$

$$\{q_a, p_b\} = \delta_{ab} \delta(k-k')$$

$$\{p_a, p_b\} = k \theta \varepsilon_{ab} \delta(k-k')$$

(37)

and a dual formulation given by

$$*H = H(*p, *q)$$

(38)
\[
\{*q_a,*q_b\} = \theta \frac{k}{k} \varepsilon_{ab} \delta(k - k')
\]
\[
\{*q_a,*p_b\} = \delta_{ab} \delta(k - k')
\]
\[
\{*p_a,*p_b\} = 0.
\]

(39)

We may therefore define the effective deformation parameters

\[
\Theta = k \theta
\]

(40)

and

\[
B = \frac{\theta}{k}
\]

(41)

such that \(\Theta\) has mass dimension 1 and \(B\) has mass dimension \(-1\) as expected. We are then led to the conclusion that duality may provide a map between the infra-red scale characterized by \(\Theta\) and the ultra-violet scale characterized by \(B\). This self-duality would then tells us that those scales sustain the same physics in opposition to expected arguments [1]. On the other hand this may give us a hint on the connections of the results found in [8] and [21] regarding the existence of a Lorentz violation spectrum.

We should be careful though. We are working with a Fourier slice and in the field theory we must sum over all momentum contributions. In fact, after we sum up all modes, there should remain no difference between the infra-red and ultra-violet regions as this scalar theory is scale invariant. But there is a catch. We are working with an effective theory with a built in scale defined by the coupling \(g\). Even though it does not appear directly in the action it will appear in the summation of the modes. The scale invariance is lost and \(g\) is the defining scale. The effective theory is valid for \(k \ll g\) and the original theory has a Lorentz violating deformation given by the effective parameter \(k \theta\). In order for this to be a tiny Lorentz violation \(\theta\) must be a small quantity such that \(k \theta \ll k\), that is, the scale associated with the Lorentz violation should be much lower then the scale of the relevant phenomenon, this is why it is considered an IR deformation. This just amounts to demand that \(\theta \ll 1\). Observe that this demand is sufficient to guarantee that Lorentz violations effects are small even in the complete theory were fermionic excitations must be considered, as depicted in
the energy scale diagram below.

The arrow is pointing to increasing energies

Through duality however the Lorentz violation shows up in the effective parameter $\frac{\theta}{k}$, this can still be a tiny violation, an UV one due to its mass dimension, and in this sense there is a map UV/IR. But the condition discussed above ($\theta \ll 1$) does not guarantee anymore that the Lorentz violation is a small effect. The condition in the dual picture reads $\frac{\theta}{k} \ll \frac{1}{k}$ but since we also have $\frac{1}{g} \ll \frac{1}{k}$ the only way to be certain that the Lorentz violating effects are small (even for the complete theory) is to demand that $\frac{\theta}{k} \ll \frac{1}{g}$ (see the distance scale diagram below) but this does not follow from the original theory.

The arrow is pointing to increasing distances

III. CONCLUSIONS AND PERSPECTIVES

In this work we have put forward another example of a Lorentz violating fermionic interaction that can be written as an effective theory where all the information about the Lorentz violation is carried by the deformed canonical relations (or equivalently, a deformed symplectic sector in the lagrangean). This is an important result because it helps to corroborate a general expectation that the physical meaning of these, otherwise ad hoc, deformations seems to be an underlying Lorentz violating dynamics.

We further analyzed this canonical structure making use of duality relations. This revealed that the deformation appearing in the field momentum sector of the symplectic matrix can be mapped to the field configuration sector. In both of the deformations the breaking of Lorentz invariance manifests itself: for the field momentum sector deformation it shows up as an infra-red effect, for the dual field configuration deformation it is an ultra-violet effect. But there is also the possibility that the mapping connects small, yet inaccessible, scales with possibly observable ones. This may rise an interesting conundrum as it suggests the possibility that even if we try to hide the Lorentz violation in some yet unreachable scale it
may come to haunt us by duality showing up in the measuring of some dual observable. We have drawn this conclusion from a particular \((1+1)D\) example but since it was based on general duality properties we think that there are grounds to believe that it may be a more general feature.

IV. APPENDIX

Here we will sketch how the mean value of the fermionic current can be obtained following the method presented in \cite{16}. We will also comment on how the effective action \cite{11} can be seen to arise from the bosonized representation of \cite{11} under the appropriate limits.

The general expression for the mean value of the current is given by the equation 3.12 in \cite{16}:

\[
< j^\mu(x) > = -i \left[ Tr \frac{1}{\not{p} - M_0} \gamma^\mu \delta(x - y) + Tr \frac{1}{\not{p} - \tilde{M}_0} \frac{1}{\not{p} - M_0} \gamma^\mu \delta(x - y) + \ldots \right]. \tag{42}
\]

In the present case

\[
M(\phi) = \theta \phi' \gamma^1 + g e^{\gamma^5 \phi} \tag{43}
\]
\[
M_0 = M(\phi_0) = M(\phi(x_0)) \tag{44}
\]
\[
\tilde{M}(\phi) = M(\phi) - M(\phi_0), \tag{45}
\]

where \(\phi(x_0)\) is the value of the field \(\phi\) evaluated at an arbitrary point \(x_0\). It is convenient to write \cite{43} as:

\[
M(\phi) = \theta \phi' \gamma^1 + g (\cos \phi + \gamma^5 \sin \phi) = \theta \phi' \gamma^1 + g (\phi_1 + \gamma^5 \phi_2) \tag{46}
\]

In \cite{42} the \(Tr\) symbol stands for \(\gamma\)-matrices traces, momentum integration and space-time integration as well. The omitted higher order terms refers to higher derivatives. The heart of the matter, as explained in \cite{16}, is that \(\tilde{M}\) contains functions of \(x\) so it does not commute with the momenta. Because of this we must order each term in this expression isolating the \(x\)’s from the \(p\)’s ending with something like

\[
< j^\mu(x) > = f^\mu(x) \int d^2 p g(p) \tag{47}
\]
This is accomplished using relations like
\[
\phi_\mu \frac{1}{p^2 - g^2} = \frac{1}{p^2 - g^2} \phi + \frac{1}{(p^2 - g^2)^2} [p^2, \phi] + \text{higher derivatives}
\]
\[
[p^2, \phi(x)] = \square \phi + 2i\gamma^\mu \partial_\mu \phi
\]
\[
[p^\mu, \phi] = i\partial^\mu \phi
\] (48)

This ordering procedure constitutes the core of the calculation, once it is done the momenta integrations gives only a constant factor.

It is straightforward to see that the first term in (42) is zero. For the second term we have
\[
Tr \frac{1}{p^2 - g^2} \tilde{M} \frac{1}{p^2 - g^2} \gamma^\mu \delta(x - y) = Tr \frac{\tilde{\psi} + M_0^\dagger}{p^2 - g^2} M_0 \frac{\tilde{\psi} + M_0^\dagger}{p^2 - g^2} \gamma^\mu \delta(x - y)
\]
\[
= Tr y Tr_p \left[ \frac{1}{p^2 - g^2} Tr [\tilde{\psi} - M_0] \tilde{M} (\tilde{\psi} - M_0) \gamma^\mu \right] \frac{1}{p^2 - g^2} \delta(x - y) \] (49)

We can first perform the $\gamma$-trace. Further neglecting the $\theta$-terms, which would give a high order $\theta^2$ contribution upon substituting back in the action, we obtain
\[
<j^\mu(x)> = -i Tr y Tr_p \left[ \frac{1}{p^2 - g^2} \left( 2g^2 p^\mu (\tilde{\phi}_1 \phi_0^0 + \tilde{\phi}_2 \phi_0^2) + 2ig^2 \varepsilon^{\mu\nu} p_\nu (\tilde{\phi}_2 \phi_0^0 - \tilde{\phi}_1 \phi_0^2) \right) \right] \frac{1}{p^2 - g^2} \delta(x - y). \] (50)

Before performing the momenta integrals we must order this expression by bringing all momenta to the left, say. After doing that the space-time integral is trivially evaluated and most momenta integrals results to be null by symmetry. We are left with
\[
<j^\mu(x)> = -i \left[ \int d^2p \frac{1}{(p^2 - g^2)^2} \left( -2ig^2(\phi_1^0 \partial^\mu \tilde{\phi}_1 + \phi_2^0 \partial^\mu \tilde{\phi}_2) \right) \right] 
+ \int d^2p \frac{p^\mu p^\nu}{(p^2 - g^2)^3} \left( 8ig^2(\phi_1^0 \partial_\nu \tilde{\phi}_1 + \phi_2^0 \partial_\nu \tilde{\phi}_2) \right) 
+ \int d^2p \frac{1}{(p^2 - g^2)^2} \left( -2g^2 \varepsilon^{\mu\nu} \partial_\nu \phi \right) \] (51)

The first two terms can be seen to cancel each other after doing the momenta integrals. So the final answer in the approximations we are considering is the same as the one obtained by Goldstone and Wilczek [12].
\[
<j^\mu(x)> = -\frac{1}{2\pi} \varepsilon^{\mu\nu} \partial_\nu \phi(x). \] (52)
There is also a nice, heuristic, way to reach the effective theory (10) by reasoning with the bosonized version of (4) which is given by [12]:

$$S = \int d^2x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - \frac{\theta}{\sqrt{\pi}} \phi' \chi - g \mu \cos(2\sqrt{\pi} \chi - \phi) \right)$$

(53)

where $\chi$ is the bosonized version of the fermionic field and $\mu$ is an arbitrary energy scale introduced for dimensional reasons. In the approximations we are considering the fermionic current is viewed as having its origin totally determined by the original scalar field $\phi$. Furthermore as discussed after (8) it has a classical character resembling a given external input. This prompt us, since we are interested only in the scalar $\phi$ field dynamics, to neglect the bosonized fermionic kinetic term $\frac{1}{2} \partial_\mu \chi \partial^\mu \chi$ in (53). This is in tune with considering a large energy gap between the characteristic energies of the scalar $\phi$ field and of the fermions, that is, $\partial \phi \ll g$. In this limit we have a situation analogous to the London limit forcing the cosine potential to a minimum value, fixing the condition $\chi \to \frac{1}{2\sqrt{\pi}} \phi$, after which we obtain the effective theory (10)

$$S = \int d^2x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\theta}{2\pi} \phi' \phi \right)$$

(54)

V. NOTE ADDED

It came to our attention, after finishing the first version of this paper, an interesting work by Passos and Petrov, [22]. Working also in $(1 + 1)D$ they have obtained another example of the kind of phenomenon studied here: a Lorentz violating fermionic interaction inducing a deformed scalar theory. Their results differ from ours since they work with a model containing two scalar fields. Nevertheless it helps to corroborate our general claim that deformed canonical structures might be described by fermionic effects.

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17
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