Entanglement Generation of Nearly-Random Operators

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We study the entanglement generation of operators whose statistical properties approach those of random matrices but are restricted in some way. These include interpolating ensemble matrices, where the interval of the independent random parameters are restricted, pseudo-random operators, where there are far fewer random parameters than required for random matrices, and quantum chaotic evolution. Restricting randomness in different ways allows us to probe connections between entanglement and randomness. We comment on which properties affect entanglement generation and discuss ways of efficiently producing random states on a quantum computer.

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Highly entangled, random, quantum states play a central role in many aspects of quantum information processing (QIP). Protocols enabled by random quantum states include superdense coding [1], remote state preparation [2], data hiding schemes [3], and single spin measurement [4]. Random states are produced from computational basis states by applying random unitary operators. However, the implementation of operators randomly drawn from the circular unitary ensemble (CUE), the space of all unitaries, is inefficient.

Independent of its usefulness in QIP, entanglement is a uniquely quantum phenomenon. Entanglement is a conjectured signature of quantum chaos [3, 4, 5, 6, 7, 8, 9] and plays an important role in studies of decoherence [10] and measurement. Understanding entanglement allows us to better exploit it as a QIP resource and provides insight into the working of quantum mechanics.

In this paper we study the entanglement production of operator classes that approach, but do not properly cover, CUE. The purpose of this is two-fold. First, it allows us to explore the relationship between randomness and entanglement and investigate which statistical properties of randomness lead to entanglement production. Second, some of the operators explored here may be implemented efficiently. Thus, this study is an exercise of how best to produce highly entangled random states.

The first class of operators we explore are ensembles of random matrices which interpolate between integrable and CUE [12]. These operators require the same number of random parameters as CUE but the parameters are drawn from restricted intervals. The second class is pseudo-random (PR) operators [11, 13, 14], possibly efficient substitutes for random operators in QIP protocols. These operators fall far short of the requisite number of random parameters when compared to CUE operators. The third class is quantum analogs of classically chaotic systems. These operators are the most restricted in terms of random parameters but are known to exhibit certain statistical properties of random matrices [15] and can be efficiently implemented on a quantum computer.

As a practical measure of multi-partite entanglement, we use the average bipartite entanglement between each qubit and the rest of the system [14, 17],

$$Q(|\psi\rangle) = \frac{4}{n} \sum_{j=1}^{n} D(|\tilde{u}_j\rangle, |\tilde{v}_j\rangle) = 2 - 2 \sum_{j=1}^{n} Tr[\rho_j^2],$$

where $|\psi\rangle = |0\rangle_j \otimes |\tilde{u}_j\rangle + |1\rangle_j \otimes |\tilde{v}_j\rangle$, $D(|\tilde{u}_j\rangle, |\tilde{v}_j\rangle)$ is the norm-squared of the wedge product between $|\tilde{u}_j\rangle$ and $|\tilde{v}_j\rangle$, and $\rho_j$ is the reduced density matrix of qubit $j$. We apply matrices from the above classes to computational basis states and the average entanglement produced as a function of time (number of iterations), $\langle Q(t)\rangle$, is compared to the CUE average entanglement $\langle Q_{\text{CUE}}(t)\rangle = (N - 2) / (N + 1)$, where $N$ is the Hilbert space dimension. Other entanglement measures, specifically the concurrence between the two most significant qubits and linear entropy between the two $N/2$ dimensional subspaces, exhibit behavior similar to $Q$ for the operators explored here.

The statistical properties we examine are the level or number variance, the randomness of the eigenvectors, and the matrix element distribution. The number variance measures a two-point eigenvalue correlation function which, for many dynamical systems, is known to deviate from CUE at long range due to short periodic orbits [10]. Thus, the number variance provides insight into entanglement generation as a function of time since periodic orbits will cause deviations from CUE. In the limit of large $N$, the CUE number variance is \cite{Euler}

$$\Sigma_{\text{CUE}}^2(L) = \frac{1}{2L} (\ln(2\pi L) + 1 + \gamma),$$

where $\gamma \approx .577$ is the Euler constant.

A lower bound for the asymptotic bipartite entanglement production with respect to time, $S_{\text{asy}}$, is the bipartite entanglement of the systems’ eigenvectors, $S_{\text{eig}}$ minus one \cite{Bennett}. This result can be extended to $Q$ since it is an average of bipartite entanglements, $Q_{\text{asy}} \geq Q_{\text{eig}} - 1$. Let $c_L^k$ denote the $k$th component of the $l$th system eigen...
vector. The distribution of amplitudes, \( \eta = |c_k|^2 \), for CUE eigenvectors in the limit \( N \to \infty \) and after rescaling to unit mean is \( P_{\text{CUE}}(y) = e^{-y} \), where \( y = N \eta \).

When applying an operator to a computational basis state the resulting state is a column of the applied operator. Repeated applications are simply powers of the column. Thus, an operator’s matrix elements, the resulting state elements, play a central role in the amount of entanglement generated. This is seen by writing the average \( Q \) over all states in terms of \( |c_i|^2 \), the state element amplitudes

\[
\langle Q \rangle = 4 \left( \sum_{m=1}^{N/2} \sum_{n=\frac{N}{2}+1}^{N} \langle |c_m|^2 |c_n|^2 \rangle - \sum_{q=1}^{N/2} \langle |c_q|^2 |c_q+\frac{N}{2}|^2 \rangle \right).
\]

CUE matrices can be generated by multiplying eigenvectors of a Gaussian unitary ensemble (GUE) Hermitian matrix by random phases and using the resulting vectors as matrix columns \[23\]. Since the eigenvector distribution of CUE and GUE are the same \[13\] and multiplication by a phase does not change the amplitude of the elements, \( P_{\text{CUE}}(x) \), the distribution of the rescaled amplitude of CUE matrix elements \( x \), is equal to \( P_{\text{CUE}}(y) \). The closeness of an operator’s matrix element amplitude distribution to that of CUE indicates of how much entanglement the operator can generate.

The interpolating ensembles are a one-parameter interpolation between diagonal matrices with uniform, independently distributed elements, and CUE \[12\]. They are constructed based on the Hurwitz parameterization of CUE matrices. The CUE construction starts with elementary unitary transformations, \( E^{(i,j)}(\phi, \psi, \chi) \), with non-zero elements \[23\], \[24\]

\[
E_{kk}^{(i,j)} = 1, \quad k = 1, \ldots, N, \quad k \neq i, j
\]

\[
E_{ij}^{(i,j)} = e^{i\psi} \cos \phi, \quad E_{ij}^{(i,j)} = e^{i\chi} \sin \phi
\]

\[
E_{ji}^{(i,j)} = -e^{-i\chi} \sin \phi, \quad E_{jj}^{(i,j)} = e^{-i\psi} \cos \phi
\]

which are used to form \( N - 1 \) composite rotations

\[
E_1 = E^{(N-1,N)}(\phi_01, \psi_01, \chi_1)
\]

\[
E_2 = E^{(N-2,N-1)}(\phi_12, \psi_12, 0) E^{(N-1,N)}(\phi_02, \psi_02, \chi_2)
\]

\[
\vdots
\]

\[
E_{N-1} = E^{(1,2)}(\phi_{N-2,N-1}, \psi_{N-2,N-1}, 0) \times E^{(2,3)}(\phi_{N-3,N-2}, \psi_{N-3,N-2}, 0) \times \cdots \times E^{(N-1,N)}(\phi_{0,N-1}, \psi_{0,N-1}, \chi_{N-1})
\]

and, finally, \( U_{\text{CUE}} = e^{\alpha \cdot E_1 E_2 \ldots E_{N-1}} \). Angles \( \psi, \chi, \) and \( \alpha \) are drawn uniformly from the intervals

\[
0 \leq \psi_{rs} \leq 2\pi, \quad 0 \leq \chi_s \leq 2\pi, \quad 0 \leq \alpha \leq 2\pi,
\]

and \( \phi_{rs} = \sin^{-1}(\xi_{rs}^{1/2r+2}) \), with \( \xi_{rs} \) drawn uniformly from 0 to 1. The 2 \( \times \) 2 block \( E^{(i,j)}_{m,n} \) with \( m, n = i, j \) and \( r = 0 \) is a random SU(2) rotation with respect to the Haar measure. Interpolating ensemble construction is the same with the angles drawn from constricted intervals

\[
0 \leq \psi_{rs} \leq 2\pi \delta, \quad 0 \leq \chi_s \leq 2\pi \delta, \quad 0 \leq \alpha \leq 2\pi \delta,
\]

with \( \phi_{rs} = \sin^{-1}(\delta \xi_{rs}^{1/(2r+2)}) \) and \( \xi_{rs} \) drawn from 0 to 1. The whole is multiplied by a diagonal matrix of random phases drawn uniformly from 0 to 2\( \pi \). The parameter \( \delta \) ranges from 0 to 1 and provides a smooth transition of certain statistical properties between the diagonal circular Poisson ensemble and CUE \[12\].

For our purposes the interpolating ensembles have the same number of random parameters as CUE matrices, drawn, however, from restricted intervals. We stress that \( \delta \) restricts all \( N^2 \) independent variables, such that even ensembles of the highest \( \delta \) used here cover only an extremely small fraction of CUE space. Figure\[11\] shows the difference between \( \langle Q \rangle_\delta \) and \( \langle Q \rangle_{\text{CUE}} \) as a function of time and \( \delta \). For \( \delta \gtrsim .96 \), \( \langle Q \rangle_\delta \) approaches \( \langle Q \rangle_{\text{CUE}} \) as an exponential whose rate decreases with decreasing \( \delta \). For \( \delta \lesssim .5 \), the approach is a power-law. For constant \( \delta \), \( \langle Q \rangle_\delta \) approaches its asymptotic value faster for lower \( N \) while the difference between the asymptotic value and \( \langle Q \rangle_{\text{CUE}} \) increases for lower \( N \).

The next class of operators we investigate is PR operators \[11\], \[13\], \[14\], potentially efficient replacements of CUE operators for QIP. To implement a PR operator apply \( m \) iterations of the \( n \) qubit gate: random SU(2) rotation, Eqs. \[3\] and \[5\], to each qubit, evolve the system via all nearest neighbor couplings \[11\]. The coupling operator is \( U_{\text{nic}} = e^{i(n/4) \sum_{j=1}^{n} \sigma_z^j \otimes \sigma_z^{j+1}} \), where \( \sigma_z^j \) is the \( j \)th qubit z-direction Pauli spin operator. The random rotations are different for each qubit and each iteration.

**FIG. 1:** (Color online) Absolute value of difference between the CUE average entanglement, \( \langle Q \rangle_{\text{CUE}} \) and the average \( Q \) for interpolating ensemble operators, \( \langle Q \rangle_\delta \), applied \( t \) times to each computational basis state. (left) \( N = 256 \), \( \langle Q \rangle_{\text{CUE}} = .9883 \) and \( \delta = .5 \) (△), .8 (×), .9 (○), .94 (□), .96 (γ), .98 (o), .99 (left-triangles), .999 (six-pointed stars), and .9999 (⋆). The dotted lines show \( Q_{\text{avg}} \) determined by the eigenvectors of each set of operators. The dashed line shows the lower-bound for CUE eigenvectors. (right) \( N = 256 \) (□), 128 (×), 64 (○), and 32 (△). As \( N \) decreases the difference between \( \langle Q \rangle_\delta \) and \( \langle Q \rangle_{\text{CUE}} \) as a function of time goes from power-law to exponential (for \( \delta = .8 \)) and the exponential rate increases. This is due to the increased randomness of the matrix elements.
The above shows that one can create states with CUE levels of multi-partite entanglement though only a small portion of CUE is covered. An $m = 8$ PR operator, for example, has only 193/256 = 0.3% of the random parameters needed for CUE operators but can generate CUE levels of entanglement by iterating the operator 5-6 times.

The interpolating ensemble operators and PR operators lead to similar average entanglement generation behavior as a function of time. For both the entanglement approaches $\langle Q \rangle_{CUE}$ as a power-law and, as the operators cover more of CUE, an exponential. A priori, there is no reason that different restrictions on CUE should give rise to similar average entanglement behavior, especially since the distributions of $Q$ after one iteration are very different for the two types of operators (not shown).

In light of these results Fig. 3 shows how various statistical properties relate to entanglement production with the aim: to explain why the entanglement generation approaches $\langle Q \rangle_{CUE}$ as a power-law, or exponential, and why $\langle Q \rangle_{PR}$ is dependent on $N$ while $\langle Q \rangle_{PR}$ is not.

Based on our numerical investigations, the number variance determines the approach of $\langle Q \rangle$ to $\langle Q \rangle_{CUE}$. For the interpolating ensembles with $\delta < 0.9$ and low $m$ PR operators the number variance diverges almost immediately from $\Sigma^2_{CUE}(L)$. For values of $\delta$ and $m$ where $\langle Q \rangle$ approaches $\langle Q \rangle_{CUE}$ exponentially as a function of time, $\Sigma^2_{CUE}(L)$ follows CUE faithfully even for large $L$. In other words, operators for which $\langle Q \rangle$ approaches $\langle Q \rangle_{CUE}$ as a power-law do not follow $\Sigma^2_{CUE}(L)$, while those that approach $\langle Q \rangle_{CUE}$ as an exponential follow $\Sigma^2_{CUE}(L)$ up to long range correlations, corresponding to short time.

As expressed in Eq. 2 the matrix element distribution (Fig. 3) is key in determining entanglement generation. The matrix element distribution approaches the CUE distribution at a rate similar to that of the one iteration entanglement distribution, $P(Q)$ (not shown). This rate of convergence is slower than that of other explored statistics. Second, the matrix element distributions explains the behavior of the entanglement generation as a function of $N$. For the interpolating ensemble operators, the eigenvector and eigenvalue statistics are practically unchanged with $N$ [12]. The matrix element distribution, however, is strongly dependent on $N$ as is the entanglement generation. For $\delta = 0.9$ operators, the element distribution is fully random for $N = 8$ but deviates as $N$ increases. For the PR operators the eigenvector and eigenvalue statistics depend only slightly on $N$, as with the matrix element distribution. The entanglement generation is also essentially unchanged.

The final class of operators we explore are quantum chaotic operators. While a full analysis is beyond the scope of this paper we discuss them in comparison to the other operators. Figure 3 shows entanglement generation as a function of time for the quantum baker’s map [25] and an ensemble of chaotic sawtooth [20], and Harper
The chaotic maps require on order $n^2$ gates. The PR operators require $n - 1$ coupling terms and $n$ rotations per iteration, approximately $2mn$ gates per operator. These are comparable as long as $m$ is less than quadratic in $n$. However, unlike the quantum chaotic operators, the coupling terms at every iteration of the PR algorithm can be applied simultaneously. The PR operators thus require only $2m + 1$ gates which appears to be less then that required of chaotic operators.

In conclusion, we have studied the entanglement generation of interpolating ensemble, PR, and quantum chaotic operators as a function of time. These operators restrict the full space of CUE in different ways and the effect of these restrictions can be seen in the entanglement generation. The statistical properties which influence the entanglement generation include the number variance, which effects the entanglement generation as a function of time, and the matrix element distribution, which determines other aspects of the entanglement. Finally, we note that the PR operators may be efficiently implemented on a quantum computer and provide a way to create highly entangled, random states.

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