Filtering information in a connected network

Andrea Capocci\textsuperscript{a}, František Slanina\textsuperscript{b}, and Yi-Cheng Zhang\textsuperscript{a}

\textsuperscript{a}Institut de Physique Théorique, Université de Fribourg, Perolles CH-1700, Switzerland

\textsuperscript{b}Institute of Physics, Academy of Sciences of the Czech Republic, Na Slovance 2, CZ-18221 Praha, Czech Republic
e-mail: slanina@fzu.cz

Abstract

We introduce a new kind of Information Theory. From a finite number of local, noisy comparisons, we want to design a robust filter such that the outcome is a high ranking number. Both analytical and numerical results are encouraging and we believe our toy model has wide ranging implications in the future Internet-based information selection mechanism.

Key words:

Introduction

Claude Shannon, more than half a century ago, has introduced the modern Information Theory [1]. This theory was a timely contribution to the explosive growth of long distance communications during, or just after, war time. He studied, in a general framework, the coding strategies that allow one to send a message through a noisy channel, such that the received signal contains no error. Actually, it is an easy task, if one is allowed unlimited information capacity, that is, infinite number of bits transmitted through the channel in a time unit. For instance, one could repeat the same binary signal many times. On the receiving end, taking the average or using the majority rule [2], the original message can be restored to any desired precision.

But, unlike in academic economics literature, unlimited information resources are not available. Therefore, one usually deals with the problem of minimizing the information capacity needed to transmit messages with a required degree of precision. Indeed, one can code a message in an efficient way such that only a very limited amount of redundancy is needed. For this purpose, Shannon
proved two fundamental theorems and found a lower bound on the necessary redundancy. However, his theory does not provide a constructive method to code signals. In other words, no methods were introduced by Shannon to solve the problem, except in some very special cases. Since then, a number of coding methods have been built, that reach the optimal efficiency computed by Shannon. Indeed, much of the state-of-art of the information theory is dedicated to sophisticated coding/decoding business [2].

The current Internet era poses new challenges. Nowadays, the problem does not lie in the lack of information, but in the fact that too much of it is available. One therefore has to set up methods to sort the specific information out of a highly disordered environment according to any given relevance criteria. Having infinite resources, in principle one could examine in depth each available information source, selecting the relevant ones. Unfortunately, one does not have such resources. Many recent researches emphasize that scarcity does not affect available information any longer, but rather the capability to process it. In other words, the most valuable good of the new economic era is attention [3]. Thus, in a finite time one can obtain at most an approximate estimate of the true relevance of all available information sources.

In addition, in many common cases the quality of an information source cannot be observed directly, but only through comparison with peers. For example, one cannot measure the intrinsic relevance of a web page by simply observing it: most of the times, one needs a collection of other pages to compare their content, in order to choose the best. Unfortunately, the results of such comparisons are often fuzzy, since a clear-cut assessment would need prohibitive amount of time and attention resources. Nevertheless, a certain number of matches allows one to have an approximate estimate of the intrinsic relevance. Clearly, as the number of matches is increased, the accuracy in the observation grows.

Finally, the results of a search have to be presented in a shape which takes into account the human interaction. For example, new generation WWW search engines put a strong effort in establishing reliable rankings of relevant web pages matching any given query [4]. Instead of showing a whole rankings, some search engine provide a restricted number of matches to a query, or even a single one, as in Google’s “I’m feeling lucky” option. Again, this option implies a neat gain in ease at use, though it rarely corresponds to the best possible answer.

In this paper we propose a toy model of information filtering. Our model deals with the problem of finding the most relevant element from a large set, in presence of a stochastic noise which prevents a perfect perception of the intrinsic quality of each item. For the sake of simplicity, we represent the quality of each item by a real number randomly drawn from a uniform
distribution in the range \([0, 1]\).

Knowing all the qualities, it would be trivial to sort the items and build a ranking: in this case, the higher the quality of an item, the higher its rank. But, as explained above, we assume that these numbers cannot be accessed directly: any information can be obtained by a noisy comparisons among the items. In particular, we choose a pairwise filtering architecture, i.e. information can be gathered only by comparing pairs of items.

For a real life instance, one could think to football teams in a national league. To choose the best team, we cannot restrict ourselves to examine each team and draw a judgment: many people do so, without ever reaching an ultimate answer. On the other hand, we can guess an approximate answers, using a finite number of pairwise matches. Actually, the outcome of a match does not signal precisely a better intrinsic quality of one team over the other. But if a team is intrinsically better than another one, it has a greater chance of winning in a real game. If an infinite number of games was allowed, we would be able to discern the intrinsic superiority by pitting two teams against each other.

In reality, we must be content with approximate answers, using rather a limited number of comparisons (in the soccer language, number of games). Clearly, this introduces a design problem. The aim is to get as good an approximation using a given number of comparisons; or, equivalently, to achieve a given level of approximation using minimalizing necessary resources. In this sense, we say our approach is a generalization of Shannon’s Information Theory.

From sport events, we learn that a different structure gives rise to a different quality of filtering. In the final round of the Soccer World Cup, for example, teams are disqualified after a single defeat, up to the final match. This architecture does not often yield a very good approximation of intrinsic quality ranking, because a good team defeated by a worse team will be put out of the competition. But this tree structure is the most economical one, needing a minimal number of games.

On the other hand, in european national leagues each team plays each other. This structure yields in general a more reliable approximation of the intrinsic ranking, since more redundancy is built into the scheme, as teams are not menaced from chancy elimination. But, as in Shannon Information theory, more precision requires more resources, and a trade-off has to be made between the two considerations. In our metaphor, the World Cup need less time resources, usually one month, whereas many national leagues take almost a year. This example illustrates the dilemma.

We study a very simple design structure which allows both elimination of low quality items, which decreases the time needed to the final selection, and a
certain degree of redundancy, which provides reliability to our mechanism.

The model

The underlying structure of our filtering model is a one-dimensional lattice of \( L \) nodes with periodic boundary conditions. Time is assumed to be an integer variable \( t = 0, 1, 2, \ldots \). Every node \( i = 1, \ldots, L \) is attached a value \( x_i(t) \). The starting configuration is an array \( x_i(t = 0)_{i=1,\ldots,L} \) of random variables drawn from the range \([0, 1]\) with uniform probability: these values represent the intrinsic qualities we introduced above. At each time step, every couple of neighbors values \((x_i(t), x_{i+1}(t + 1))\) for \( i = 1, \ldots, L \) gives rise to a new value \( x_i(t + 1) \), according to the following rule:

\[
x_i(t + 1) = \begin{cases} 
  x_i(t) \text{ with probability } \frac{x_i(t)}{x_i(t)+x_{i+1}(t)}, \\
  x_{i+1}(t) \text{ with probability } \frac{x_{i+1}(t)}{x_i(t)+x_{i+1}(t)} 
\end{cases}
\]

Therefore, in a time step the whole array is updated. The periodic boundary conditions ensure that \( x_{L+1} = x_1 \) for all values of \( t \), and thus consistency of the evolution law.

As the chain evolves, connected domains, i.e. regions of the lattice composed of sites occupied by the same value, emerge: high values have a greater probability to spread over neighbor sites, and low values are more likely to vanish, as described in figure. Eventually, after a time \( t^* \), all sites are occupied by a single domain associated to the value \( x^* \): the process has reached a stationary state. Accordingly, we call this system a filter, since it exerts a selection on the initial \( L \) values, favoring by its dynamics the propagation of domains associated with high values.

The value \( x^* \) is attached to a given site \( s \) in the starting configuration, i.e. \( x^* \in x_i(0)_{i=1,\ldots,L} \). We say that the site \( s^* \) and the value \( x^* = x_{s^*}(0) \) associated with it at the beginning have been selected by the filter. We define the search time \( t^*(L) \) needed to reach the stationary state, and the inefficiency of this filter, i.e. the rank \( R(L) \) of the selected value in the starting configuration. An ideal filter would select the site associated to the highest value in the starting configuration \( (R(L) = 1) \). Due to the randomness in the initial condition and the stochastic dynamics, our filter may select a different site. We will investigate these two quantities, \( t^*(L) \) and \( R(L) \), as a function of the total number of sites \( L \).

The number of different values in the lattice monotonically decreases with time. We denote by \( n(t, L) \) the number of these domains. A relation between
t, L and n can be derived by simple reasoning. Let us assume that at time $t(n, L)$ only $n$ domains remain in a chain of length $L$.

Each of them occupies, on average, a region of size $\frac{L}{n}$. Therefore, $t(n, L)$ approximately corresponds to the time needed to reach the stationary state for a filter acting over this sub-region of size $\frac{L}{n}$. This reads

$$ t(n, L) = t(1, \frac{L}{n}). \quad (1) $$

By assuming $n(t, L) \sim t^{-\alpha} L^\beta$, with $\alpha > 0$ and $\beta > 0$, we write $t(n, L) \sim n^{-1/\alpha} L^{\beta/\alpha}$, and replace this expression in eq. 1. This way, we obtain

$$ n^{-\beta/\alpha} L^{\beta/\alpha} = n^{-1/\alpha} L^{\beta/\alpha}, \quad (2) $$

which implies $\beta = 1$ and $t^*(L) \sim L^\frac{\alpha}{\beta}$. This rough estimate of the scaling behavior with respect to $L$ is confirmed by the analysis of the population dynamics of the coalescing domain walls.

**Search time**

At time 0, there are $L$ domain walls, since each domain is made of a single site. As time passes by, domains vanishes. A domain vanishes when the two surrounding domain walls coalesce. By tracing all the domain walls as a function of time, following the framework of 1 + 1 dimensional directed polymers, we observe a tree-like structure, whose source is in the end point of the filter process. At time $t$ the lattice is occupied by $n(t, L)$ domains $\Gamma_k$, corresponding to the values $y_k = x_{i_k}(0)$, with $k = 1, \ldots, n(t, L)$. Let us denote by $h_k(t)$ the position of the $k$-th wall between domains $\Gamma_k$ and $\Gamma_{k+1}$. The wall between these domains performs a random walk of unitary steps whose drift $v(n)$ is equal to $\frac{y_{k+1} - y_k}{y_{k+1} + y_k}$. To evaluate the time $\Delta(n)$ between two subsequent coalescing along the same domain wall, when the surviving domain walls are $n$, we write

$$ \Delta(n) \sim \frac{L}{n \langle |v(n)| \rangle}, \quad (3) $$

where $L/n$ is the average length of a domain, and $\langle |v(n)| \rangle$ is the typical speed with which a wall moves towards its neighbor, and the average is performed over the distribution of the remaining values $\{y_k\}_{k=1,\ldots,n}$ at time $t$. Therefore $\Delta(n)^{-1}$ is the probability per time step that a domain wall encounters a neighbor. Since there are $n$ walls, the total probability of a coalescing anywhere in
the lattice is \( n/\Delta(n) \). At each intersection, the number of walls decreases by one, therefore we can write a differential equation for \( n(t,L) \),

\[
\dot{n} = -\frac{n}{\Delta(n)}.
\]  

(4)

We can give an estimate of \( \langle |v(n)| \rangle \) by assuming that the values \( y_{k=1,...,n} \) are uniformly distributed in the range \([1-n/L,1]\). Under this assumption,

\[
\langle |v(n)| \rangle = \frac{L^2}{n^2} \int_{1-n/L}^{1} dx \int_{1-n/L}^{1} dy \frac{|x-y|}{x+y} = \\
= \frac{1}{2} \left[ -\frac{2n}{L} + \frac{n^2}{L^2} + 2 \ln 2 + 2 \left( \frac{n^2}{L^2} - 2 \right) \ln 2 - \frac{n^2}{L^2} - 2 \left( \frac{n}{L} - 1 \right)^2 \ln 2 \left( 1 - \frac{n}{L} \right) \right].
\]  

(5)

(6)

(7)

If the remaining domains are \( n \ll L \), we can expand this expression in a series of powers up to the first order in \( \frac{n}{L} \), getting

\[
\langle |v(n)| \rangle \simeq \frac{n}{12L} + \ldots.
\]  

(8)

By replacing eq. (8) in eq. (3), we obtain \( \Delta \sim \frac{L^2}{n^2} \). Then, for \( n \ll L \), the time evolution of \( n(t,L) \) reads \( \dot{n} \sim n^3L^{-2} \). Given the assumption made on the scaling behavior of \( n \) with respect to \( t \) and \( L \), the previous equation provides us with two relations for \( \alpha \) and \( \beta \), which yield \( \alpha = \frac{1}{2} \) and \( \beta = 1 \). By replacing the steady state condition \( n(t^*) = 1 \) in the scaling relation of \( n \) as a function of \( t \) and \( L \), we obtain \( (t^*)^{-\frac{3}{2}}L \sim L^{-2} \). This gives the scaling relation

\[
t^* \sim L^2;
\]  

(9)

which is confirmed by numerical simulation, as shown in fig. .

**Inefficiency**

To compute the inefficiency of the filter, we have to estimate the rank of the finally selected value in the starting configuration. Let us make some strong (but reasonable) approximation: (i) we assume that the distribution \( \psi(y) \) of the remaining \( n(t) \) numbers on the lattice is uniform for all times, with mean value \( \bar{y} \) and support \([2\bar{y} - 1,1]\). (ii) we make a mean field approximation about the time evolution of the domains. In order to explain better hypothesis (ii),
let us focus on a single domain. Let $y_k$ be the value occupying a domain, $l_k(t)$ the length of this domain, $y_{k-1}$ and $y_{k+1}$ the values occupying the neighboring domains. The time evolution of $l_k$ reads

$$l_k(t+1) = l_k(t) - \frac{1}{2} [\eta(t, y_{k-1}, y_k) - \eta(t, y_k, y_{k+1})]$$  \hspace{1cm} (10)$$

where

$$\eta(t, x, y) = \begin{cases} 1 & \text{with probability } \frac{x}{x+y} \\ -1 & \text{with probability } \frac{y}{x+y} \end{cases}$$

We now replace in eq. (10) the neighboring values $y_{k+1}$ by the mean value of the distribution of remaining numbers, $\bar{y}$. This assumption is equivalent to considering each domain size as a biased random walk starting at position $l_k(0)$ with an absorbing boundary at the origin (that corresponds to the coalescence of two neighboring domain walls). The bias of the random walk is given by the interaction between the domain occupied by the value $y_k$ and the effective medium, which is assumed to be occupied by the value $\bar{y}$. In the same mean field approach, we assume that $l_k(0) = L/n(t)$, which is the typical distance between two adjacent walls. Such a random walk [5] is absorbed by the boundary with probability 1 if $y_k < \bar{y}$ and probability $q(y_k) = \left( \frac{\bar{y}}{y_k} \right)^{2L/n(t)}$ otherwise. In both cases, the absorption occurs after the same typical time $t_0$.

After $m$ typical times (not all equal!), the mean value of the distribution is $\bar{y}(m) = 1 - 2^{-m-1}$. In fact, in our approximation, during a typical time all domains occupied by values lower than $\bar{y}$ have vanished, as does a fraction of the domains occupied by values higher than $\bar{y}$. This fraction is estimated by

$$q(m) = \int_\bar{y}^1 \psi(y) \left( \frac{\bar{y}}{y} \right)^{2L/n(m)} dy =$$

$$= \frac{\bar{y}^{2L/n(m)}}{1 - \bar{y}} \int_\bar{y}^1 \frac{1}{y^{2L/n(m)}} dy =$$

$$= (1 - 2^{-m-1})^{2L/n(m)} + \ldots =$$

$$\approx e^{-2^{-m-1}(m)L},$$  \hspace{1cm} (13)$$

for large $m$ and $L$. Thus, we can write the time evolution (more precisely than in the previous section) for the number of domains after $m$ typical times, $n(m)$, which reads
\[ n(m+1) = \frac{1}{2} n(m)[1 - q(m)] \]
\[ = \frac{1}{2} n(m)[1 - e^{-2^{-m} L}] . \quad (15) \]

If we define \( \phi(m) = 2^{-m} n^{-1} L \) and if we consider \( m \) as a continuous variable, the last equation can be written as
\[
\frac{d\phi}{dm} = \phi e^{-\phi} , \quad (17)
\]

whose solution can be written in an implicit form by means of the exponential integral function \( Ei(x) \)

\[
Ei(\phi(m)) - Ei(\phi(m_0)) = m - m_0 \quad (18)
\]

There is a problem of treating the initial conditions properly. It is tempting to take the initial condition \( m_0 = 1 \) and therefore have formally \( \phi(m_0) = 1/2 \). However, it is not a prudent approximation to assume the variable \( m \) as continuous for very small values. Taking the the initial condition for \( m_0 > 1 \) we do not know how to compute reliably the value \( \phi(m_0) \). However, we found that with \( m_0 = 3 \) and \( n(3) = L/4 \), i.e. \( \phi(3) = 1/2 \) we match well the numerical simulations data. Thus, we have the equation \( Ei(\phi(m)) = m - 3 + Ei(1/2) \), where \( Ei(1/2) = 0.454... \). For the estimated rank \( R(L) \), we have \( R(L) \simeq 1 + 2^{-m^*} L \), for the assumptions (i). Let us define the reduced variable \( \rho(L) = R(L) - 1, \rho(L) \in [0, L - 1] \) so that we can write \( \rho(L) = \phi(m^*) \). For \( \phi(m^*) \) we have the equation \( Ei(\phi(m^*)) = m^* - 3 + Ei(1/2) \), thus
\[
Ei(\rho(L)) + \frac{\ln \rho(L)}{\ln 2} = \ln L / \ln 2 - 3 + Ei(1/2) \quad (19)
\]

For \( \rho(L) \to \infty \) we have \( \ln(Ei(\rho(L)) + \ln \rho(L) / \ln 2) \simeq \rho(L) \) (we checked by plotting the functions that it holds well for \( \rho(L) \) larger than \( \simeq 30 \), so we end up with the conclusion that asymptotically for \( L \to \infty \) the rank behaves like
\[
\rho(L) \simeq \ln \ln L . \quad (20)
\]

However, this asymptotic regime is reached only for extremely large systems. Rank \( \rho(L) \) larger than about 30 means \( L \) larger than about \( 10^{10^{12}} \): a number beyond any imaginable application. For smaller sizes, up to about \( L = 1000 \), we found approximately, by expanding the LHS of equation (19) in Taylor series around \( \rho(L) = 1 \) and by replacing \( \rho(L) \) by \( R(L) - 1, \)
\[
R(L) \simeq 0.347 \ln L + 0.933... \quad (21)
\]
We may compare this result with numerical results and find an excellent agreement, as shows fig. .

**Improving the filter**

In order to improve the performance of the filter, one could follow the suggestions coming from the Shannon information theory [1], where the addition of redundant information (e.g., by repeating the transmission of the message) helps in recovering the source information. In the same approach, we could build a number $\alpha$ of replicas of the initial chain, in which each site corresponds to a different value. Then, by linking the $\alpha$ chains together, keeping periodic boundary condition, we obtain a chain of length $L' = \alpha L$. In the same Shannon’s spirit, we could wonder if there exists a finite value $\alpha_c$ of $\alpha$ such that the inefficiency of the filter decreases to its minimal value, and the final selected value is (almost) always the highest one. In the new $L'$-chain, the first $\alpha$ places in the ranking of the values are occupied by the $\alpha$ replicas. Therefore, if $R(L') < \alpha$, the selected value is the best one attached to a replica of the original site. We have the following condition for $\alpha_c$,

$$\frac{1}{3} \log (\alpha_c L) \leq \alpha_c,$$

which has a finite solution $\alpha(L)$, for all values of $L$, which increases with $L$. As a consequence, however, the number of matches that have to be done to reach a steady state is increased as the number of sites grows from $L$ to $L' = \alpha L$. We denote by $M(l)$ the total number of matches needed to reach a steady state in a chain of $l$ sites. If we assume that each match costs a unit time, the quantity $C = \frac{M(L)}{M(\alpha_c L) L}$, is analogous to the inverse of the information capacity in Shannon’s theory, which decrease with increasing redundancy. In the original chain of length $L$, at each time step $L$ matches are made. Then, from 9, we have $M(l) \approx l^3$. From the definition of $C$, we get the relation

$$C \simeq \alpha_c(L).$$

**Conclusions**

We have introduced a toy model of search engine, i.e. an algorithm which elects a best element in a large set, according to some relevance criterion. Such a device is gaining a growing importance in the current information-based economy: nowadays, gathering large amounts of informations is a widely affordable task; on the other hand, selecting, examining and judging information
requires a huge (and increasing) processing capacity, though it is necessary to exploit such available information.

These algorithms face two main problems. First, the exact relevance of an item (an information, a webpage, a people) in the most common situations can be measured only within a certain degree of uncertainty. Only an infinite available time would allow a deep knowledge about the items. Second, the results have to be presented in a user-friendly manner, e.g. in ranking order. In the extreme case, the algorithm will even yield only the best selected item.

This puzzle recalls the one solved by pioneers of Information Theory, who dealt with the challenge of recovering the original message transmitted through a noisy channel, by knowing only the corrupted received message. They established that, even with a finite information capacity, one is able to achieve error-free communication, though they rarely constructed such algorithms. Analogously, nowadays one looks for methods that are able to order large sets of items with respect to their intrinsic relevance, without having a full knowledge about them.

Accordingly, we assumed that the quality of each element can not be measured directly, and one can only compare elements pairwise. Each element is put on a site of a linear chain with periodic boundary conditions, and can spread over neighboring sites, thus creating domains. The "search engine" stops when a domain occupies the whole lattice and the value attached to the domain is selected as the best one.

The model is approached by applying methods issued from a directed polymers field, since its properties can be investigated by focusing on the domain walls dynamics. We analytically computed the search time and the inefficiency of the "filter", and verified our results by numerical simulations.

Most interestingly, we found that the error made by the filter (the intrinsic ranking of the elected item) grows only logarithmically with respect to the number of items. Moreover, we determined the minimal redundancy to be added to the filter in order to achieve full efficiency, i.e. to always select the intrinsically most relevant item in the set. In the analogy with classical Information Theory, this would correspond to the well-known Shannon limit.

Acknowledgments

The authors thank P. Laureti, P. De Los Rios, Y. Manida and Y. Pismak for useful discussions. This work was supported by the Grant Agency of the Czech Republic, grant project No. 202/01/1091, and by the Swiss National...
Fig. 1. A realization of the model. The vertical direction represents the time evolution.

Fig. 2. Search time as a function of the total number of sites $L$. Circles correspond to numerical simulations. The solid line has slope 1.94 in log-log scale.

Fund, Grant No. 20-61470.00, and by the Grant Agency of the Czech Republic, Grant No. 202/01/1091. F. Slanina acknowledges the financial support from the University of Fribourg, Switzerland.
Fig. 3. Inefficiency of the filter as a function of the total number of sites $L$. Circles correspond to numerical simulation; the solid line corresponds to eq. 21

References

[1] D. Slepian, editor, “Key Papers in the Development of Information Theory”, IEEE Press, New York 1974.

[2] D. McKay, “Information Theory, Inference and Learning Algorithms”, available at [http://www.inference.phy.cam.ac.uk/mackay/itprnn/book.html](http://www.inference.phy.cam.ac.uk/mackay/itprnn/book.html)

[3] T.H. Davenport, J.C. Beck, “The Attention Economy”, Harvard Business School Press, Cambridge, 2001.

[4] See documentation at the address [http://www.google.com](http://www.google.com)

[5] W. Feller, “An introduction to probability theory and its applications”, J. Wiley & sons, 1950.