Sufficient spectral conditions on Hamiltonian and traceable graphs

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Abstract

In this paper, we give sufficient conditions on the spectral radius for a bipartite graph to Hamiltonian and traceable, which expand the results of Lu, Liu and Tian (2012) \cite{10}. Furthermore, we also present tight sufficient conditions on the signless Laplacian spectral radius for a graph to Hamiltonian and traceable, which improve the results of Yu and Fan (2012) \cite{12}.

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Key words: Spectral radius; Hamiltonian bipartite graph; Traceable bipartite graph; Signless Laplacian spectral radius; Hamiltonian graph; Traceable graph

1 Introduction

All graphs considered here are simple and undirected. Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. For $v_i \in V(G)$, we denote by $d(v_i)$ or $d_i$ the degree of $v_i$. Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of $G$, where $d_1 \leq d_2 \leq \cdots \leq d_n$. Denote by $\delta(G)$ or simply $\delta$ the minimum degree of $G$, i.e., $\delta = d_1$. The disjoint union of $k$ copies of a graph $G$ is denoted by $kG$. The join of $G$ and $H$, denoted by $G \vee H$, is the graph obtained from disjoint union of $G$ and $H$ by adding all possible edges between them. Write $K_{n-1} + e$ for the complete graph on $n - 1$ vertices with a pendant edge, and $K_{n-1} + v$ for the complete graph on $n - 1$ vertices together with an isolated vertex.

A cycle passing through all the vertices of a graph is called a Hamiltonian cycle. A graph containing a Hamiltonian cycle is called a Hamiltonian graph. A path passing

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through all the vertices of a graph is called a *Hamiltonian path* and a graph containing a Hamiltonian path is said to be *traceable*.

The *adjacency matrix* $A(G) = (a_{ij})_{n \times n}$ of a simple graph $G$ is the matrix indexed by the vertices of $G$, where $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$, and $a_{ij} = 0$ otherwise. The largest eigenvalue of $A(G)$, denoted by $\rho(G)$, is called the *spectral radius* of $G$. Let $D(G)$ be the degree diagonal matrix of $G$. The matrix $Q(G) = D(G) + A(G)$ is the *signless Laplacian matrix* of $G$. Denote by $q(G)$ the *signless Laplacian spectral radius*.

The problem of deciding whether a given graph is Hamiltonian or traceable is NP-complete. Many reasonable sufficient or necessary conditions were given for a graph to be Hamiltonian or traceable. Recently, spectral theory of graphs has been applied to the problem. Fiedler and Nikiforov [7] gave sufficient conditions for a graph to be Hamiltonian and traceable in terms of the spectral radius of the graph or its complement. Lu, Liu and Tian [10] showed a sufficient condition for a graph to be traceable in terms of the spectral radius of the graph. Subsequently, Zhou [14] investigated the signless Laplacian spectral radius of the complement of a graph, and provided tight conditions for the existence of Hamiltonian cycles or paths. Using Laplacian of graphs, Butler and Chung [3] established a sufficient condition for a graph to be Hamiltonian. Fan and Yu [8] gave a sufficient condition for a graph to be Hamiltonian with respect to normalized Laplacian.

For a bipartite graph, Lu, Liu and Tian [10] gave a sufficient condition for a bipartite graph being Hamiltonian in terms of the spectral radius of the quasi-complement of a bipartite graph. In Section 2, we give sufficient conditions for a bipartite graph to Hamiltonian and traceable in terms of spectral radius of the bipartite graph.

Yu and Fan [12] mentioned the signless Laplacian spectral conditions for a graph to be Hamiltonian and traceable, while investigating spectral conditions for a graph to Hamilton-connected. However, there is a flaw in their result which left out two exceptional graphs. Hence, it needs further investigate on the sufficient conditions for a graph to be Hamiltonian and traceable in terms of the signless Laplacian spectral radius of a graph. In Section 3, we provide tight sufficient conditions on the signless Laplacian spectral radius for a graph to Hamiltonian and traceable, which improve the results of Yu and Fan [12].

Note that $\delta \geq 2$ and $\delta \geq 1$ are trivial necessary conditions for a graph to be Hamiltonian and traceable, respectively. Hence we always make the assumption while finding spectral conditions for Hamiltonian and traceable graphs or bipartite graphs throughout this paper.

## 2 Hamiltonian and traceable bipartite graphs

In this section, we consider the bipartite graphs. Let $G[X, Y]$ be a bipartite graph. The bipartite graph $G^*[X, Y]$ is called *quasi-complement* of $G$, which is constructed as follows: $V(G^*) = V(G)$ and $xy \in E(G^*)$ if and only if $xy \notin E(G)$ for $x \in X$, $y \in Y$. Let $K_{n, n-1}$ be a complete bipartite graph with bipartition $(X, Y)$, where $|X| = n$ and $|Y| = n - 1$. Denote $K_{n, n-1} + e$ the bipartite graph obtained from $K_{n, n-1}$ by adding a
dependent edge to one of vertices in $X$. The following result was given by Lu et al. in [10].

**Theorem 2.1** ([10]) Let $G[X, Y]$ be a bipartite graph and $G^*$ the quasi-complement of $G$, where $|X| = |Y| = n \geq 2$. If

$$\rho(G^*) \leq \sqrt{n - 1},$$

then $G$ is Hamiltonian unless $G \cong K_{n,n-1} + e$.

Theorem 2.1 provided a sufficient condition on Hamiltonian bipartite graph in terms of spectral radius of the quasi-complement of the bipartite graph. But there is a minor error in its proof. It should be:

**Theorem 2.2** ([9]) Let $G[X, Y]$ be a bipartite graph and $G^*$ the quasi-complement of $G$, where $|X| = |Y| = n \geq 2$. If

$$\rho(G^*) \leq \sqrt{\frac{n-2}{2}},$$

then $G$ is Hamiltonian.

Next we will show sufficient conditions on Hamiltonian and traceable bipartite graphs in terms of the spectral radius of bipartite graphs, respectively. First we state a sharp upper bound on the spectral radius of a bipartite graph.

**Lemma 2.3** ([1]) If $G$ is a bipartite graph with $m \geq 1$ edges and $n$ vertices, then

$$\rho(G) \leq \sqrt{m},$$

and equality holds if and only if $G \cong K_{p,q} \cup (n-p-q)K_1$, where $pq = m$.

A sufficient condition for a bipartite graph to be Hamiltonian was given in [2, Ex. 18.3.9].

**Lemma 2.4** ([2]) Let $G[X, Y]$ be a bipartite graph, where $|X| = |Y| = n \geq 2$, with degree sequence $(d_1, d_2, \ldots, d_{2n})$, where $d_1 \leq d_2 \leq \cdots \leq d_{2n}$. If there is no integer $k \leq n/2$ such that $d_k \leq k$ and $d_n \leq n - k$. Then $G$ is Hamiltonian.

**Lemma 2.5** ([10]) Let $G[X, Y]$ be a bipartite graph with $\delta \geq 1$ and $m$ edges, where $|X| = |Y| = n \geq 2$. If

$$m \geq n^2 - n + 1,$$

then $G$ is Hamiltonian unless $G \cong K_{n,n-1} + e$.

Let $K_{p,n-2} + 4e$ be a bipartite graph obtained from $K_{p,n-2}$ by adding two vertices which are adjacent to two common vertices with degree $n-2$ in $K_{p,n-2}$, respectively, where $p \geq n - 1$. Next we obtain a result which is similar to Lemma 2.5.
Lemma 2.6 Let $G[X, Y]$ be a bipartite graph with $\delta \geq 2$ and $m$ edges, where $|X| = |Y| = n \geq 4$. If
\[ m \geq n^2 - 2n + 4, \]
then $G$ is Hamiltonian unless $G \cong K_{n,n-2} + 4e$.

Proof. Suppose that $G$ is a non-Hamiltonian bipartite graph with $\delta \geq 2$ and degree sequence $(d_1, d_2, \ldots, d_{2n})$, where $d_1 \leq d_2 \leq \cdots \leq d_{2n}$. By Lemma 2.4 there exists an integer $k \leq n/2$ such that $d_k \leq k$ and $d_n \leq n-k$. Then
\[
m = \frac{1}{2} \sum_{i=1}^{2n} d_i \leq \frac{k^2 + (n-k)^2 + n^2}{2} = n^2 - 2n + 4 + (k-2)(k-n+2).
\]
Since $m \geq n^2 - 2n + 4$, $(k-2)(k-n+2) \geq 0$. Note that since $k \geq d_k \geq \delta \geq 2$ and $n-k \geq d_n \geq 2$, $(k-2)(k-n+2) \leq 0$. Hence $(k-2)(k-n+2) = 0$ and $m = n^2 - 2n + 4$, and all inequalities in the above argument should be equalities. Then $k = 2$ or $k = n - 2$. If $k = 2$, then $G$ is a bipartite graph with $n^2 - 2n + 4$ edges and $d_1 = d_2 = 2$, $d_3 = \cdots = d_n = n-2$ and $d_{n+1} = \cdots = d_{2n} = n$. This implies $G \cong K_{n,n-2} + 4e$. If $k = n-2$, then $2 \leq n - 2 \leq n/2$ and hence $n = 4$. Then $G$ is a bipartite graph with 12 edges and degree sequence $(2, 2, 2, 2, 4, 4, 4, 4)$. Thus $G \cong K_{4,2} + 4e$. We complete the proof. □

Theorem 2.7 Let $G[X, Y]$ be a bipartite graph with $\delta \geq 2$, where $|X| = |Y| = n \geq 4$. If
\[ \rho(G) \geq \sqrt{n^2 - 2n + 4}, \]
then $G$ is Hamiltonian unless $G \cong K_{n,n-2} + 4e$.

Proof. By Lemma 2.3 we have
\[ \sqrt{n^2 - 2n + 4} \leq \rho(G) \leq \sqrt{m}, \]
where $m$ is the number of edges in $G$. Then $m \geq n^2 - 2n + 4$. By Lemma 2.6 the result follows. □

Let $G[X, Y]$ be a traceable bipartite graph. Then $|X| = |Y|$ or $|X| = |Y| + 1$. These two types will be discussed separately.

Lemma 2.8 (\cite{5}) Let $G[X, Y]$ be a bipartite graph, where $|X| = |Y| = n \geq 2$. If
\[ d(x) + d(y) \geq n + 1 \]
for every pair of nonadjacent vertices $x \in X$ and $y \in Y$, then $G$ is Hamiltonian.
**Lemma 2.9** Let $G[X, Y]$ be a bipartite graph with $\delta \geq 1$ and $m$ edges, where $|X| = |Y| = n \geq 3$. If

$$m \geq n^2 - 2n + 3,$$

then $G$ is traceable.

**Proof.** Let $d(x_0) + d(y_0)$ be the minimum of $d(x) + d(y)$ for every pair of nonadjacent vertices $x \in X$ and $y \in Y$. If $d(x_0) + d(y_0) \geq n + 1$, then by Lemma 2.8, $G$ is Hamiltonian, and thus $G$ is traceable. Suppose that $d(x_0) + d(y_0) \leq n$. Then

$$m(G - \{x_0, y_0\}) \geq n^2 - 2n + 3 - n = (n - 1)^2 - (n - 1) + 1.$$

By Lemma 2.5, we have $G - \{x_0, y_0\}$ is Hamiltonian or $G - \{x_0, y_0\} \cong K_{n-1, n-2} + e$.

**Case 1.** $G - \{x_0, y_0\}$ is Hamiltonian.

Assume that $C = x_1y_1x_2y_2 \cdots x_{n-1}y_{n-1}x_1$ is a Hamiltonian cycle in $G - \{x_0, y_0\}$.

Denote by $N(x_0)$ and $N(y_0)$ the neighborhoods of vertices $x_0$ and $y_0$, respectively. Let $A = \{y_i \mid x_i \in N(y_0), 1 \leq i \leq n - 1\}$ and $B = \{x_j+1 \mid y_j \in N(x_0), 1 \leq j \leq n - 1\}$. Note that by convention we let $x_n = x_1$.

Claim: There exist $y_s \in A$ and $x_t \in B$ such that $x_t y_s \in E(G)$.

Suppose not. Note that $d(x_0) = |B|$ and $d(y_0) = |A|$. Then

$$m \leq (n - 1)^2 - |A||B| + |A| + |B|.$$

Since

$$m \geq n^2 - 2n + 3,$$

then $(|A| - 1)(|B| - 1) \leq -1$ which yields a contradiction. Thus Claim holds.

Let $y_s \in A$ and $x_t \in B$ such that $x_t y_s \in E(G)$. Then $x_s y_0 \in E(G)$ and $x_0 y_{t-1} \in E(G)$. Without loss of generality, we can assume that $s \geq t$. Then we find a Hamiltonian path

$$P = x_0y_{t-1}x_{t-1}y_{t-2} \cdots x_{s+1}y_s x_t y_s x_t \cdots y_s - 1 x_s y_0.$$

Then $G$ is traceable.

**Case 2.** $G - \{x_0, y_0\} \cong K_{n-1, n-2} + e$.

Let $e = uv$ with $u \in X$ and $v \in Y$, where $v$ is the pendant vertex in $G - \{x_0, y_0\}$. Hence $d(v) = 1$ or 2 in $G$.

Suppose $d(y_0) \geq 2$. If $d(v) = 1$, then $x_0 v \notin E(G)$. Thus $d(x_0) + d(v) = d(x_0) + d(y_0)$ contradicts the minimality of $d(x_0) + d(y_0)$.

If $d(v) = 2$. In this case, $x_0 v \in E(G)$. Let $u_1 y_0 \in E(G)$, where $u_1 \neq u$. Then $x_0 uvu$ is the path starting from $x_0$ and goes into the graph isomorphic to $K_{n-1, n-2}$. We can travel all vertices of this complete bipartite graph once and end at $u_1$. Then go to $y_0$. Hence $G$ is traceable.
Now, we suppose \( d(y_0) = 1 \). Since \( n^2 - 2n + 3 \leq m \leq n(n - 2) + d(y_0) + d(v) \leq n^2 - 2n + 1 + d(v) \), \( d(v) = 2 \). Hence \( x_0v \in E(G) \). Since \( |E(K_{n-1,n-2})| = n^2 - 3n + 2 \leq n^2 - 2n - 1 \leq m - 4 \), there are at least 4 edges being removed when we remove \( x_0, y_0 \) and \( v \) from \( G \). Since \( d(v) = 2 \), \( d(y_0) = 1 \) and \( x_0y_0 \notin E(G) \), \( d(x_0) \geq 2 \). Without loss of generality, we let \( x_1y_0 \in E(G) \) and let \( x_0y_1 \in E(G) \).

Suppose \( x_1 = u \). Then \( y_0x_1y_0y_1 \) is a path starting from \( y_0 \) and goes into the graph isomorphic to \( K_{n-2,n-2} \). Hence \( G \) is traceable.

Suppose \( x_1 \neq u \). Then \( y_0x_1 \) is a path starting from \( y_0 \) and goes into a graph isomorphic to \( K_{n-1,n-2} \). We can travel all vertices of this complete bipartite graph once and end at \( u \). Then pass \( v \) and go to \( x_0 \). Hence \( G \) is traceable.

This completes the proof. □

**Theorem 2.10** Let \( G[X,Y] \) be a bipartite graph with \( \delta \geq 1 \), where \( |X| = |Y| = n \geq 3 \). If

\[
\rho(G) \geq \sqrt{n^2 - 2n + 3},
\]

then \( G \) is traceable.

**Proof.** By Lemma 2.3 we have

\[
\sqrt{n^2 - 2n + 3} \leq \rho(G) \leq \sqrt{m},
\]

then \( m \geq n^2 - 2n + 3 \). By Lemma 2.9 the theorem holds. □

Next, we consider the other type \( |X| = |Y| + 1 \). Let \( G[X,Y] \) be a bipartite graph, where \( |X| = n + 1 \) and \( |Y| = n \geq 2 \). Denote by \( \delta_X \) and \( \delta_Y \) the minimum degrees of vertices in \( X \) and \( Y \), respectively. Note that \( \delta_X \geq 1 \) and \( \delta_Y \geq 2 \) are the trivial necessary conditions for \( G \) to be traceable. Let \( G[X,Y + v] \) be the bipartite graph obtained from \( G[X,Y] \) by adding a vertex \( v \) which is adjacent to every vertex in \( X \). It is easy to see that \( G[X,Y] \) is traceable if and only if \( G[X,Y + v] \) is Hamiltonian.

Let \( K_{n,n-1} + 2e \) be a graph obtained from \( K_{n,n-1} \) by adding two vertices which are adjacent to a common vertex with degree \( n - 1 \), respectively.

**Theorem 2.11** Let \( G[X,Y] \) be a bipartite graph with \( \delta_X \geq 1 \) and \( \delta_Y \geq 2 \), where \( |X| = n + 1 \) and \( |Y| = n \geq 3 \). If

\[
\rho(G) \geq \sqrt{n^2 - n + 2},
\]

then \( G \) is traceable unless \( G \in \{K_{n+1,n-2} + 4e, K_{n,n-1} + 2e\} \).

**Proof.** Let \( G[X,Y] \) be a bipartite graph with \( m \) edges. Let \( G[X,Y] \) be a bipartite graph with \( m \) edges. By Lemma 2.3 we have

\[
\sqrt{n^2 - n + 2} \leq \rho(G) \leq \sqrt{m}.
\]

Hence \( m \geq n^2 - n + 2 \). Note that \( d(v) = n + 1 \) in \( G[X,Y + v] \), hence

\[
m(G[X,Y + v]) = m + (n + 1) \geq n^2 + 3 = (n + 1)^2 - 2(n + 1) + 4.
\]

By Lemma 2.6 \( G[X,Y + v] \) is Hamiltonian or \( G[X,Y + v] \cong K_{n+1,n-2} + 4e \). Hence \( G[X,Y] \) is traceable or \( G \cong K_{n+1,n-2} + 4e \) or \( K_{n,n-1} + 2e \). So we have the theorem. □
3 Hamiltonian and traceable graphs

In [14], Zhou gave a sufficient condition for a graph to be Hamiltonian and traceable in terms of the signless Laplacian spectral radius of the complement of a graph.

Let $EC_n$ be the set of graphs of the following two types of graphs on $n$ vertices: (a) the join of a trivial graph and a graph consisting of two complete components, and (b) the join of a regular graph of degree $\frac{n-1}{2} - r$ and a graph on $r$ vertices, where $1 \leq r \leq \frac{n-1}{2}$.

Let $EP_n$ be the set of graphs of the following three types of graphs on $n$ vertices: (a) a regular graph of degree $\frac{n}{2} - 1$, (b) a graph consisting of two complete components, and (c) the join of a regular graph of degree $\frac{n}{2} - 1 - r$ and a graph on $r$ vertices, where $1 \leq r \leq \frac{n}{2} - 1$.

Theorem 3.1 ([14]) Let $G$ be a graph on $n$ vertices with complement $\bar{G}$.

(i) If $n \geq 3$, $q(\bar{G}) \leq n - 1$ and $G \notin EC_n$, then $G$ is Hamiltonian.

(ii) If $q(\bar{G}) \leq n$ and $G \notin EP_n$, then $G$ is traceable.

Yu and Fan [12] mentioned a sufficient condition for a graph to be Hamiltonian and traceable in terms of the signless Laplacian spectral radius of the graph. However, there is a flaw in their result which left out two exceptional graphs $K_2 \vee 3K_1$ and $K_{1,3}$. The complete result is as follows.

Theorem 3.2 ([12]) Let $G$ be a graph of order $n \geq 3$.

(i) If $q(G) > 2n - 4$ and $G$ is neither $K_2 \vee 3K_1$ nor $K_{n-1} + e$, then $G$ is Hamiltonian.

(ii) If $q(G) \geq 2n - 4$ and $G$ is neither $K_{1,3}$ nor $K_{n-1} + v$, then $G$ is traceable.

In this section, we present new spectral conditions for a graph to be Hamiltonian and traceable in terms of the signless Laplacian spectral radius of the graph, which improve the results in Theorem 3.2.

The following better sharp upper bound on the signless Laplacian spectral radius for a connected graph $G$ was given in [6], also see [4].

Lemma 3.3 ([6]) Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then

$$q(G) \leq \frac{2m}{n - 1} + n - 2,$$

with equality if and only if $G$ is $K_{1,n-1}$ or $K_n$.

If $G$ is disconnected, by considering a connected component of $G$, Yu and Fan [12] obtained the following sharp upper bound on the signless Laplacian spectral radius for a general graph $G$.  

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Lemma 3.4 (\cite{12}) Let \( G \) be a graph of order \( n \) with \( m \) edges. Then
\[ q(G) \leq \frac{2m}{n-1} + n - 2. \]

If \( G \) is connected, then the equality holds if and only if \( G \) is \( K_{1,n-1} \) or \( K_n \). Otherwise, the equality holds if and only if \( G \) is \( K_{n-1} + v \).

A sufficient condition for a graph to be Hamiltonian was given by Chvátal in 1972.

Lemma 3.5 (\cite{5}) Let \( G \) be a simple graph with the degree sequence \( (d_1, d_2, \ldots, d_n) \), where \( d_1 \leq d_2 \leq \cdots \leq d_n \) and \( n \geq 3 \). Suppose that there is no integer \( k < n/2 \) such that \( d_k \leq k \) and \( d_{n-k} \leq n-k-1 \). Then \( G \) is Hamiltonian.

Let \( \text{NC} = \{ K_1 \lor 5K_1, K_2 \lor (K_3 + 2K_1), K_3 \lor 4K_1, K_{1,2} \lor 4K_1, K_2 \lor (K_1 + K_{1,3}), K_2 \lor (K_2 + 2K_1), K_1 \lor 2K_3, K_2 \lor 3K_1 \} \) be the set of some graphs. Obviously, the graphs in \( \text{NC} \) are non-Hamiltonian.

A stronger version of Lemma 3.6 occurs in \cite{11}. In order to keep this paper complete, independent and self-contained, we provide a detailed proof again.

Lemma 3.6 \footnote{See \cite{11} for a stronger result.} Let \( G \) be a graph with \( \delta \geq 2 \). If
\[ m > \frac{n^2 - 4n + 6}{2}, \]
then \( G \) is Hamiltonian unless \( G \in \text{NC} \).

Proof. Let \( G \) be a graph on \( n \) vertices and \( m \) edges with \( \delta \geq 2 \). Let \( d_1 \leq d_2 \leq \cdots \leq d_n \) be its degree sequence. Suppose that \( G \) is a non-Hamiltonian graph. By Lemma 3.5 there exists an integer \( k < \frac{n}{2} \) such that \( d_k \leq k \) and \( d_{n-k} \leq n-k-1 \). Then
\[
2m = \sum_{i=1}^{n} d_i = \sum_{i=1}^{k} d_i + \sum_{i=k+1}^{n-k} d_i + \sum_{i=n-k+1}^{n} d_i \leq k^2 + (n-2k)(n-k-1) + k(n-1) \tag{1}
\]
where \( f(k) = 3k^2 + (1-2n)k + 3n - 6 = n^2 - 4n + 6 + f(k), \) and \( k \geq 2 \). Since \( m > \frac{n^2 - 4n + 6}{2}, \) \( f(k) > 0. \)

The roots of \( f(k) \) are \( r_1 = \frac{1}{6}(2n - 1 - \sqrt{4n^2 - 40n + 73}) \) and \( r_2 = \frac{1}{6}(2n - 1 + \sqrt{4n^2 - 40n + 73}) \). Since \( f(k) > 0, \) either \( 2 \leq k < r_1 \) or \( \frac{n}{2} > k > r_2. \)

For \( 2 \leq k < r_1, \) we have \( 2n - 13 > \sqrt{4n^2 - 40n + 73} \). It is easily to get \( n < 8. \)

For \( \frac{n}{2} > k > r_2, \) we have \( \frac{2n - 1}{2} > r_2 \) for odd \( n \). Then we will get that \( n^2 - 12n + 23 < 0. \) So we have \( 2 < n < 10. \) Similarly, for even \( n, \) we will get that \( 2 < n < 8. \) Combining with the lower bound on \( n \) obtained above, finally we have \( n = 9, 7, 6, 5. \)
Case 1. \( n = 9 \). We have \( k \in \{2, 3, 4\} \). Since \( f(2) = -1 \), \( f(3) = -2 \) and \( f(4) = 1 \), we have \( k = 4 \). That is, \( d_4 \leq 4 \) and \( d_5 \leq 4 \). Note that \( 51 < \sum_{i=1}^{9} d_i = 2m \leq 52 \). So \( \sum_{i=1}^{9} d_i = 52 \) and the equality in Eq. (1) holds. That is, the degree sequence of \( G \) is \( (4, 4, 4, 4, 4, 8, 8, 8, 8) \). Hence \( G \cong K_4 \lor 5K_1 \).

Case 2. \( n = 7 \). We have \( k \in \{2, 3\} \). We have \( f(2) = 1 \) and \( f(3) = 3 \). If \( k = 2 \), then \( 27 < \sum_{i=1}^{7} d_i = 2m \leq 28 \). So \( \sum_{i=1}^{7} d_i = 28 \) and the equality in Eq. (1) holds. That is, the degree sequence of \( G \) is \( (2, 2, 4, 4, 4, 6, 6) \). Hence \( G \cong K_2 \lor (K_3 + 2K_1) \).

If \( k = 3 \), then \( d_3 \leq 3 \) and \( d_4 \leq 3 \). Since \( 27 < \sum_{i=1}^{7} d_i = 2m \leq 30 \), \( \sum_{i=1}^{7} d_i = 30 \) or 28. When \( \sum_{i=1}^{7} d_i = 30 \). Similar to the above case we obtain that the degree sequence of \( G \) is \( (3, 3, 3, 3, 6, 6, 6) \). Thus \( G \cong K_3 \lor 4K_1 \). If \( \sum_{i=1}^{7} d_i = 28 \), then the degree sequences of \( G \) and \( G \) are as follows:

\[
\begin{cases}
(3, 3, 3, 5, 5, 6), & \text{then } G \cong K_{1, 2} \lor 4K_1; \\
(3, 3, 3, 3, 4, 6, 6), & \text{then } G \cong K_2 \lor (K_2 + K_{1, 2}); \\
(2, 3, 3, 3, 5, 6, 6), & \text{then } G \cong K_2 \lor (K_1 + K_{1, 3}).
\end{cases}
\]

Case 3. \( n = 6 \). We have \( k = 2 \). Thus \( 18 < \sum_{i=1}^{6} d_i = 2m \leq 20 \). So \( \sum_{i=1}^{6} d_i = 20 \) and the equality in Eq. (1) holds. Then the degree sequence of \( G \) is \( (2, 2, 3, 3, 5, 5) \), and hence \( G \cong K_2 \lor (K_2 + K_{1, 2}) \).

Case 4. \( n = 5 \). We have \( k = 2 \). Thus \( 11 < \sum_{i=1}^{5} d_i = 2m \leq 14 \), that is, \( \sum_{i=1}^{5} d_i = 12 \) or 14. Since \( d_2 \leq 2 \) and \( d_3 \leq 2 \), we obtain the degree sequences of \( G \) and \( G \) are:

\[
\begin{cases}
(2, 2, 2, 2, 4), & \text{then } G \cong K_1 \lor 2K_2; \\
(2, 2, 2, 3, 3), & \text{then } G \cong K_{2, 3}; \\
(2, 2, 2, 4, 4), & \text{then } G \cong K_2 \lor 3K_1.
\end{cases}
\]

Note that \( K_2 \lor (K_2 + K_{1, 2}) \) contains a Hamiltonian cycle and the others obtained graphs are nonhamiltonian. Thus the proof is completed. \( \square \)

By Lemma 3.6 we present one of the main results.

**Theorem 3.7** Let \( G \) be a graph on \( n \geq 4 \) vertices with \( \delta \geq 2 \). If

\[
q(G) \geq 2n - 5 + \frac{3}{n-1},
\]

then \( G \) is Hamiltonian unless \( G \in \{K_3 \lor 4K_1, K_2 \lor 3K_1\} \).

**Proof.** Suppose that \( G \) is a non-Hamiltonian graph with \( m \) edges. Obviously, \( K_n \) is Hamiltonian, \( \delta(K_{n-1}) = 1 \) and \( \delta(K_{n-1} + v) = 0 \). By Lemma 3.4 we have \( q(G) < \frac{2m}{n-1} + n - 2 \). Since \( q(G) \geq 2n - 5 + \frac{3}{n-1} \), \( m > \frac{n^2 - 4n + 6}{2} \). By Lemma 3.6 \( G \in \mathcal{NC} \). By directed calculation (see Table 1 at the next page of this paper), we have \( q(G) < 2n - 5 + \frac{3}{n-1} \) for the graphs in \( \mathcal{NC} \) except \( K_3 \lor 4K_1 \) and \( K_2 \lor 3K_1 \). This completes the proof. \( \square \)

The following equivalent condition for a graph to be Hamiltonian and traceable is an exercise in [2] (see Ex. 18.1.6).
Lemma 3.8 (\cite{2}) Let $G$ be a graph. Then $G$ is traceable if and only if $G \lor K_1$ is Hamiltonian.

Let $\mathbb{NP} = \{K_3 \lor 5K_1, K_1 \lor (K_3 + 2K_1), K_2 \lor 4K_1, K_2, K_1 \lor (K_1 + K_{1,3}), K_1 \lor (K_2 + 2K_1), 2K_2, K_{1,3}\}$ be the set of some graphs. Obviously, the graphs in $\mathbb{NP}$ are nontraceable. By Lemmas 3.6 and 3.8, we obtain a sufficient condition for a graph to be traceable.

Table 1: The signless Laplacian spectral radius of some graphs

| $G$                      | $q(G)$ | $G$                      | $q(G)$ |
|--------------------------|--------|--------------------------|--------|
| $K_4 \lor 5K_1$          | 13.1789| $K_3 \lor 5K_1$          | 10.8990|
| $K_2 \lor (K_3 + 2K_1)$  | 9.3408 | $K_1 \lor (K_3 + 2K_1)$  | 6.9095 |
| $K_3 \lor 4K_1$          | 9.7720 | $K_2 \lor 4K_1$          | 7.4641 |
| $K_{1,2} \lor 4K_1$      | 8.8965 | $K_{2,4}$                | 6.0000 |
| $K_2 \lor (K_1 + K_{1,3})$ | 9.3408 | $K_1 \lor (K_1 + K_{1,3})$ | 6.9095 |
| $K_2 \lor (K_2 + 2K_1)$  | 7.7588 | $K_1 \lor (K_2 + 2K_1)$  | 5.3234 |
| $K_1 \lor 2K_2$          | 5.5616 | $2K_2$                   | 2.0000 |
| $K_{2,3}$                | 5.0000 | $K_{1,4}$                | 5.0000 |
| $K_2 \lor 3K_1$          | 6.3723 | $K_{1,3}$                | 4.0000 |

A stronger result of Lemma 3.9 can be found in \cite{3}.

Lemma 3.9 Let $G$ be a graph with $\delta \geq 1$ and $m$ edges. If $m > \frac{n^2 - 4n + 3}{2}$, then $G$ is traceable unless $G \in \mathbb{NP}$.

Proof. Since $|V(G \lor K_1)| = n + 1$ and $|E(G \lor K_1)| = m + n > \frac{n^2 - 4n + 3}{2} + n = \frac{(n+1)^2 - 4(n+1) + 6}{2}$, by Lemma 3.6 $G \lor K_1$ is Hamiltonian unless $G \lor K_1 \in \mathbb{NC}$. According to Lemma 3.8, G is traceable unless $G \in \mathbb{NP}$. $\square$

By Lemma 3.9 we easily obtain the following result.

Theorem 3.10 Let $G$ be a graph on $n \geq 4$ vertices with $\delta \geq 1$. If

$$q(G) \geq 2n - 5,$$

then $G$ is traceable unless $G \in \{K_2 \lor 4K_1, K_1 \lor (K_2 + 2K_1), K_{1,3}, K_{1,4}\}$.

Proof. By Lemma 3.4 and the hypothesis, we have $2n - 5 \leq q(G) \leq \frac{2m}{n-1} + n - 2$, where $m$ is the size of $G$.

Suppose the last equality holds. By Lemma 3.4, $G \in \{K_{1,n-1}, K_n, K_{n-1} + v\}$. Clearly, $K_n$ is traceable. Since $\delta(K_{n-1} + v) = 0$, it is not a case. If $G = K_{1,n-1}$, then $q(G) = n$. Under our assumption, we have $n \leq 5$. So $G \in \{K_{1,3}, K_{1,4}\}$.

Now we assume $2n - 5 \leq q(G) < \frac{2m}{m} + n - 2$. This implies that $m > \frac{n^2 - 4n + 3}{2}$. By Lemma 3.9, $G$ is traceable unless $G \in \mathbb{NP}$. For $G \in \mathbb{NP}$, it is easy to check that only
graphs $K_2 \lor 4K_1, K_1 \lor (K_2 + 2K_1)$, and $K_{1,3}$ satisfying the condition $q(G) \geq 2n - 5$ (see Table 1).

Combining both cases, we have the theorem. □

For $n \geq 7$, Theorem 3.10 can be considered as a corollary of the following recent result:

**Theorem 3.11 (13)** Let $G$ be a connected graph of order $n \geq 4$. If

$$q(G) \geq \frac{2(n-2)^2 + 4}{n-1} \left( = 2n - 6 + \frac{6}{n-1} \right),$$

then $G$ is traceable.

Theorem 3.10 improves the result in Theorem 3.2 (ii). As a corollary of our result, we present the proof of Theorem 3.2 (ii).

**Proof of Theorem 3.2 (ii).** If $n = 3$, then the result obviously holds.

Now we assume that $n \geq 4$. Suppose $\delta \geq 1$. Since $q(G) \geq 2n - 4 > 2n - 5$, by Theorem 3.10 $G$ is traceable unless $G \in \{K_2 \lor 4K_1, K_1 \lor (K_2 + 2K_1), K_{1,3}, K_{1,4}\}$. By Table 1 and $q(G) \geq 2n - 4$, we have that $G = K_{1,3}$ is the only exceptional case.

Suppose $\delta = 0$ and let $d(v) = 0$ in $G$. If $G \neq K_{n-1} + v$, then $G$ is a proper spanning subgraph of $K_{n-1} + v$. Then $q(G) < q(K_{n-1} + v) = 2n - 4$. It is not a case. Thus $G = K_{n-1} + v$, which is not traceable. This completes the proof. □

**Remark 3.12** In fact, by much more complicated analysis and excluding much more exceptional graphs, similar to the proofs of Lemmas 3.6 and 3.9, we can reduce the number of edges to $m \geq \binom{n-2}{2} + 2$ and $m \geq \binom{n-2}{2}$ for a graph to be Hamiltonian and traceable, respectively. Using the two results, we can obtain the signless Laplacian spectral conditions $q(G) \geq 2n - 6 + \frac{1}{n-1}$ and $q(G) \geq 2n - 6$ for a graph to be Hamiltonian and traceable except several specific graphs, respectively.

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**References**

[1] A. Bhattacharya, S. Friedland, U.N. Peled, On the first eigenvalue of bipartite Graphs, The Electronic Journal of Combinatorics 15 (2008) R144.

[2] J.A. Bondy, U.S.R. Murty, Graph Theory, GTM 244, Springer, New York, 2008.

[3] S. Butler, F. Chung, Small spectral gap in the combinatorial Laplacian implies Hamiltonian, Annals Comb. 13 (2010) 403-412.
[4] D. Cvetković, S.K. Simić, Towards a spectral theory of graphs based on the signless Laplacian, III, Appl. Anal. Discrete Math. 4 (2010) 156-166.

[5] V. Chvátal, On Hamilton’s ideals, J. Combin. Theory Ser. B 12 (1972) 163-168.

[6] L. Feng, G. Yu, On three conjectures involving the signless Laplacian spectral radius of graphs, Publ. Inst. Math. (Beograd) 85 (2009) 35-38.

[7] M. Fiedler, V. Nikiforov, Spectral radius and Hamiltonicity of graphs, Linear Algebra Appl. 432 (2010) 2170-2173.

[8] Y.-Z. Fan, G.-D. Yu, Spectral conditions for a graph to be Hamiltonian with respect to normalized Laplacian, arXiv:1207.6824v1 [math.CO] 30 July 2012.

[9] R. Li, Eigenvalues, Laplacian eigenvalues and some Hamiltonian properties of graphs, Utilitas Math. 88 (2012), 247-257.

[10] M. Lu, H. Liu, F. Tian, Spectral radius and Hamiltonian graphs, Linear Algebra Appl. 437 (2012) 1670-1674.

[11] B. Ning, J. Ge, Spectral radius and Hamiltonian properties of graphs, Linear and Multilinear Algebra (2014), http://dx.doi.org/10.1080/03081087.2014.947984.

[12] G.-D. Yu, Y.-Z. Fan, Spectral conditions for a graph to be Hamilton-connected, arXiv:1207.6447v1 [math.CO] 27 July 2012.

[13] G.-D. Yu, M. Ye, G. Cai, J. Cao, Signless Laplacian spectral conditions for Hamiltonicity of graphs, J. Appl. Math. (2014), article ID 282053, http://www.hindawi.com/journals/jam/aip/282053/

[14] B. Zhou, Signless Laplacian spectral radius and Hamiltonicity, Linear Algebra Appl. 432 (2010) 566-570.