Stability as a Whole of a Family of Fibers Maps and \(\Omega\)-Stability of \(C^1\)-Smooth Skew Products of Maps of an Interval

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Abstract. Properties of \(C^1\)-smooth skew products of maps of an interval with stable as a whole family of fibers maps are established. These results are applied to the proof of the criterion of \(\Omega\)-stability of \(C^1\)-smooth skew products of maps of an interval (with respect to homeomorphisms of skew products class). The proper subspace of the space of \(C^1\)-smooth skew products of maps of an interval is distinguished, where \(\Omega\)-stable \(C^1\)-smooth skew products are contained. It is proved that \(\Omega\)-stable skew products are not dense in the distinguished subspace of \(C^1\)-smooth maps.

1. Introduction
Different aspects of \(C^1\)-structural stability and \(C^1\)-\(\Omega\)-stability are investigated in works [1 – 9] for diffeomorphisms and flows and in works [10 – 12] for endomorphisms. Some properties of fibers maps of \(C^1\)-smooth \(\Omega\)-stable skew products of maps of an interval (with respect to homeomorphisms of skew products class) are investigated in papers [13 – 16]. The concept of stability as a whole of a family of fibers maps for a \(C^1\)-smooth skew product of maps of an interval is introduced in [15].

This work is the direct continuation of papers [13 – 18]. We solve here the nonlocal problem of description of the set of \(C^1\)-smooth \(\Omega\)-stable skew products of maps of an interval. First, with the use of the concept of stability as a whole in \(C^1\)-norm of a family of fibers maps we prove the criterion of \(C^1\)-\(\Omega\)-stability (with respect to homeomorphisms of skew products class). Second, using the decomposition theorem for the space of \(C^1\)-smooth skew products of maps of an interval with a complicated dynamics of a quotient map (see [17, 18]), we distinguish the proper subspace of this space, where \(C^1\)-smooth \(\Omega\)-stable skew products of maps of an interval with a complicated dynamics of a quotient map are contained. Third, we prove that \(\Omega\)-stable \(C^1\)-smooth skew products are not dense in the mentioned above proper subspace. This last result strengthens the main result of paper [16], where nondensity of \(\Omega\)-stable \(C^1\)-smooth skew products of maps of an interval is proved in the space of all \(C^1\)-smooth skew products of maps of an interval.

Let \(I = I_1 \times I_2\) be a closed rectangle in the plane (\(I_1, I_2\) are closed intervals). We consider a skew product of maps of an interval, i. e. a dynamical system \(F : I \rightarrow I\), where

\[
F(x, y) = (f(x), g_x(y)), \quad \text{and} \quad g_x(y) = g(x, y), \quad (x; y) \in I.
\]
The map \( f : I_1 \to I_1 \) is called the \textit{quotient map} of skew product (1), and the map \( g_x : I_2 \to I_2 \) is called the map \textit{acting in the fiber over an arbitrary point} \( x \in I_1 \).

By formula (1) the equality

\[
F^n(x, y) = (f^n(x), g_{x,n}(y)), \quad \text{where} \quad g_{x,n} = g_{f^{n-1}(x)} \circ \ldots \circ g_x,
\]

is valid for every natural number \( n \) and every point \((x, y) \in I_i\).

Let, as usually, \( T^0(I) \) \( (T^1(I)) \) be the space of all continuous \((\text{all} \ C^1\!\text{-smooth})\) skew products of maps of an interval with the standard \( C^0\!\text{-norm} \) \((\text{the standard} \ C^1\!\text{-norm})\).

Denote by \( C^1_\partial(I_k) \) \((k = 1, 2)\) the subspace of the space \( C^1(I_k) \) \((\text{of} \ C^1\!\text{-smooth maps of the segment} \ I_k \ \text{into itself with the standard} \ C^1\!\text{-norm})\), which consists of all maps \( \psi \in C^1(I_k) \) satisfying the condition of \( \psi\)-invariance of the boundary \( \partial I_k \) of the segment \( I_k \):

\[
\psi(\partial I_k) \subseteq \partial I_k.
\]

Let us remind that a map \( \xi \in C^1_\partial(I_k) \) \((k = 1, 2)\) is \( \Omega \)-stable in \( C^1\!\text{-norm} \) \((\text{i. e. in the space} \ C^1_\partial(I_k))\) if for every \( \delta > 0 \) there exists \( \varepsilon > 0 \) such that for every map \( \varphi \in B^1_{\partial k,\varepsilon}(\xi) \) one can find \( \delta \)-closed in \( C^0\!\text{-norm} \) to the identity map homeomorphism \( h : \Omega(\xi) \to \Omega(\varphi) \) satisfying the equality

\[
h \circ \xi|_{\Omega(\xi)} = \varphi|_{\Omega(\varphi)} \circ h,
\]

where \( B^1_{\partial k,\varepsilon}(\xi) \) is \( \varepsilon \)-neighborhood of a map in the space \( C^1_\partial(I_k) \) \((\text{with respect to} \ C^1\!\text{-norm}); \ \Omega(\xi) \) is the nonwandering set of a map \( \xi \).

Denote by \( C^1_\omega(I_k) \) the space of all \( \Omega \)-stable in \( C^1_\partial(I_k) \) maps of the closed interval \( I_k \) into itself \((k = 1, 2)\).

**Proposition 1** \([10, 20]\). Let \( f \in C^1_\partial(I_1) \). Then

(1.1) \( \text{either} \ f \ \text{is a map of type} \ < 2^{\infty} \ (\text{i. e. the set of the (least) periods of} \ f\)-periodic points \( \text{coincides with the set} \ \{1, 2, \ldots, 2^n\} \ \text{for some} \ 0 \leq \mu < +\infty\)), \ \text{and in this case the nonwandering set} \ \Omega(f) \ \text{is finite and consists of hyperbolic periodic points;}

(1.2) \( \text{or} \ f \ \text{is a map of type} \ > 2^{\infty} \ (\text{i. e. there exists an} \ f\)-periodic point \( x \in \text{Per}(f) \) \( \text{with the (least) period} \ n(x) \leq 2^n \) \( \text{for some} \ n \geq 0 \)), \ \text{and in this case the nonwandering set} \ \Omega(f) \ \text{is the union of finitely many hyperbolic periodic points and finitely many locally maximal quasiminimal sets}^2, \ \text{which are hyperbolic, perfect and nowhere dense. ("Locally maximal" means "maximal in a neighborhood of itself").}

The set \( C^1_\omega(I_1) \) is open and everywhere dense in \( C^1_\partial(I_1) \).

Define the space \( T^1(I) \) of \( C^1\!\text{-smooth skew products of maps of an interval (with the standard} \ C^1\!\text{-norm}) \) as the subspace of the space \( T^1(I) \), which consists of skew products of maps of an interval with quotients from the space \( C^1_\partial(I_1) \). As it follows from Proposition 1, the set \( T^1(I) \) is open and everywhere dense in the subspace of the space \( T^1(I) \) consisting of skew products with quotient maps from \( C^1_\partial(I_1) \).

Results of this paper are obtained with the use of the special multifunctions related to an arbitrary continuous skew product of maps of an interval.

**Definition 1** \([13]\). The \( \Omega\)-function of a map \( F \in T^0(I) \) is the multifunction \( \xi^F : \Omega(f) \to 2^{I_2} \) satisfying the equality

\[
\xi^F(x) = (\Omega(F))(x)
\]

for any \( x \in \Omega(f) \), where \( (\Omega(F))(x) = \{ y \in I_2 : (x, y) \in \Omega(F) \} \) is the slice of the nonwandering set \( \Omega(F) \) by the vertical fiber over a point \( x \), \( 2^{I_2} \) is the topological space of closed subsets of \( I_2 \) with the exponential topology \([22, 23]\).

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1 Definitions of wandering, nonwandering points and sets one can find in [19].

2 A quasiminimal set of a map is the closure of an infinite recurrent trajectory [21].
The following definition will be given here for a skew product $F \in T^1_s(I)$. Let $n$ be a natural number. By Proposition 1 the equality $\Omega(f^n) = \Omega(f)$ holds in considering case. Let us use the skew product

$$F_n(x, y) = (id(x), g_{x,n}(y))$$

and the direct product

$$F_{n,1}(x, y) = (f^n(x), id(y)),$$

where $id(x)$ and $id(y)$ are the identity maps of the closed intervals $I_1$ and $I_2$, respectively, and $F_n, F_{n,1} : I \to I$. Then the following formula is valid:

$$F^n = F_{n,1} \circ F_n.$$  \hspace{1cm} (4)

Formula (4) allows to define new multifunctions for each iteration of $F$. The graphs of these multifunctions are using to form the nonwandering set of the map $F \in T^1_s(I)$ (or, equivalently, to form the graph of the $\Omega$-function of $F$).

**Definition 2.** An auxiliary multifunction of a map $F \in T^1_s(I)$ is a multifunction $\eta_n : \Omega(f) \to 2^{I^2}$ satisfying

$$\eta_n(x) = \Omega(g_{x,n})$$

for any $x \in \Omega(f)$ (see [13]), where $\Omega(g_{x,n})$ is the nonwandering set of a map $g_{x,n} : I_2 \to I_2$.

A function $\pi_n : \Omega(f) \to 2^{I^2}$ is said to be a multifunction suitable to the $\Omega$-function of a map $F \in T^1_s(I)$ if the graph of $\pi_n$ in $I$ is the closure of the graph of the auxiliary function $\eta_n$. We have

$$\pi_n(x) = (\pi_n)(x) \text{ for any } x \in \Omega(f)$$

(see [17, 18]). Here $(\pi_n)(x)$ denotes the slice of the graph of $\pi_n$ by the fibre over $x$ (or, equivalently, the slice of the closure of the graph of $\eta_n$).

Having defined auxiliary functions $\eta_n$ (suitable functions $\pi_n$) for all $n > 1$, we must move each point $(x; y)$ on the graph of $\eta_n$ (on the graph of $\pi_n$, respectively) to the point $(f^n(x) ; y)$ using the direct product $F_{n,1}$ (see equality (4)). We can therefore define multifunctions $\eta_{n,1} : \Omega(f) \to 2^{I^2}$ ($\pi_{n,1} : \Omega(f) \to 2^{I^2}$), $n > 1$, by the equalities

$$\eta_{n,1}(x) = (F_{n,1}(\eta_n))(x) \quad (\pi_{n,1}(x) = (F_{n,1}(\pi_n))(x))$$

for any $x \in \Omega(f)$. Here $\eta_n$ ($\pi_n$) is the graph of the corresponding multifunction in $I$, and $(F_{n,1}(\eta_n))(x)$ ($F_{n,1}(\pi_n))(x)$ is the slice of the set $F_{n,1}(\eta_n)$ (of the set $F_{n,1}(\pi_n)$) by the fibre over $x \in \Omega(f)$. Since any point $(x; y)$ on the graph of $\eta_{n,1}$ (on the graph of $\pi_{n,1}$) can be reached from any point $(x; y)$ on the graph of $\eta_n$ (on the graph of $\pi_n$) using $F_{n,1}$, where $x$ is a point in the $n$-th complete preimage of $x$ under the map $f|\Omega(f)$. it follows that

$$\eta_{n,1}(x) = \bigcup_{x \in (f^{-n}(x))} \eta_{n}(x) \quad (\eta_{n,1}(x) = \bigcup_{x \in (f^{-n}(x))} \pi_{n}(x)).$$

We consider the subspace of the space $T^1_s(I)$, which consists of skew products with quotient maps of type $\succ 2^\infty$. Following [17, 18] we distinguish four main subspaces of this space.

Let $F \in T^1_s(I)$ be a skew product with a quotient map of type $\succ 2^\infty$. Then by Proposition 1 the perfect part of the nonwandering set of its quotient map $\Omega_p(f)$ is not empty. Let $K(f) \subset \Omega_p(f)$ be a locally maximal quasiminimal set of $f$, and $\tau(f|_{K(f)})$ be the set of the (least) periods of
periodic points of $f_{\mid K(f)}$. There exist natural numbers $m_0 = m_0(\mid K(f))$, $i_0 = i_0(\mid K(f))$ and a finite subset $N_\ast = N_\ast(\mid K(f))$ of the set $\mathbb{N}$ of natural numbers (possibly, empty) such that

$$\tau(f_{\mid K(f)}) = \{m_0i\}_{i \geq i_0} \cup N_\ast$$

(see [25, 26]).

We need the following natural numbers:

$$m_\ast = \min \{m_0(\mid K(f))\},$$

$$n_\ast = \min \{n \in N_\ast(\mid K(f))\},$$

$$i_\ast = \max \{i_0(\mid K(f))\},$$

where l.c.m. is the least common multiple of a finite set of natural numbers.

**Definition 3** [17, 18]. We say that a skew product $F \in T_4^4(I)$ with a quotient map of type $\succ 2^\infty$ satisfies condition $\mathbf{H}$ (strong condition $\mathbf{H}$) if for any sequence of natural numbers $\{i_t^*\}_{t \geq i}$, with

$$l_t^* = m_an_t^i,$$  \hspace{1cm} (5)

the multifunctions $\eta_{t^*}^\ast$ (the multifunctions $\eta_{t^*}^\ast$ respectively) are continuous for all $i \geq i^*$, where $i^* \geq i_\ast$.

As it follows from the definition of multifunctions $\eta_{t^*}^\ast$ (and $\eta_{t^*}^{i,i}$) continuity of a multifunction $\eta_{t^*}^{i,i}$ implies continuity of $\eta_{t^*}^{i,i}$ (and $\eta_{t^*}^{i,i}$ respectively).

Let us also note that if a skew product $F \in T_4^4(I)$ with a quotient map of type $\succ 2^\infty$ satisfies condition $\mathbf{H}$ (strong condition $\mathbf{H}$) then the sequence $\{\eta_{t^*}^\ast\}_{i \geq i}$ (the sequence $\{\eta_{t^*}^\ast\}_{i \geq i}$) can only contain a finitely many discontinuous functions.

We denote the subspace of $T_4^4(I)$ consisting of skew products whose quotient maps have type $\succ 2^\infty$ and satisfy strong condition $\mathbf{H}$ by $T_{4,1}^4(I)$, and the subspace of $T_4^4(I)$ consisting of skew products whose quotients have type $\succ 2^\infty$ and satisfy condition $\mathbf{H}$, but do not satisfy strong condition $\mathbf{H}$, by $T_{4,2}^4(I)$. We let $T_{4,3}^4(I)$ denote the subspace of $T_4^4(I)$ consisting of skew products whose quotients of type $\succ 2^\infty$, each of which has a sequence of suitable functions $\{\eta_{t^*}^\ast\}_{i \geq i}$ containing infinitely many discontinuous functions and has a continuous $\Omega$-function. Finally, we let $T_{4,4}^4(I)$ denote the subspace of $T_4^4(I)$ consisting of maps with quotients of type $\succ 2^\infty$, each of which has a sequence of suitable functions $\{\eta_{t^*}^\ast\}_{i \geq i}$ containing infinitely many discontinuous functions and has a discontinuous $\Omega$-function. The subspaces $T_{4,1}^4(I) - T_{4,4}^4(I)$ are pairwise disjoint.

**Decomposition Theorem** [17, 18]. Each of the subspaces $T_{4,1}^4(I)$ for $1 \leq i \leq 4$ is nonempty, and their union $\bigcup_{i=1}^4 T_{4,i}^4(I)$ coincides with the part of the space $T_4^4(I)$ consisting of skew products with quotients of type $\succ 2^\infty$.

Definition of subspaces $T_{4,i}^4(I)$ ($1 \leq i \leq 4$) is based on the use of special multifunctions. Therefore, these subspaces require an explicit functional description. We begin such description in this paper.

2. Properties of Skew Products with Stable as a Whole in $C^1$-norm Family of Fibers Maps
Let $\tilde{T}_4^4(I)$ be the subspace of maps $F \in T_4^4(I)$ satisfying the inclusion

$$F(\partial I) \subseteq \partial I,$$
where $\partial I$ is the boundary of $I$.

**Definition 4** [15]. We say that a family of fibers maps of a skew product $F \in \hat{T}_1^1(I)$ with a quotient map of type $\succ 2^{\infty}$ is stable as a whole in $C^1$-norm if for any $\delta > 0$ there is a neighborhood $B^1_\delta(F)$ of $F$ in $\hat{T}_1^1(I)$ such that for any map $\Phi \in B^1_\delta(F)$, where $\Phi(x, y) = (\varphi(x), \psi_\varphi(y))$, and any $l^*_i (i \geq i^* \text{ for some } i^* \geq i_\delta)$ one can find $\delta$-close to the identity map in $C^0$-norm homeomorphism

$$H^{<l^*_i>} : \eta^F_{l^*_i} \to \eta^\Phi_{l^*_i} \quad (H^{<l^*_i>}(x, y) = (h_1(x), h_{2,x}^{<l^*_i>}(y)))$$

satisfying the equality

$$h_{2,x}^{<l^*_i>} \psi_{\eta^F_{l^*_i}}(x, y) = \psi_{h_1(x), l^*_i}(h_1(x)) \circ h_{2,x}^{<l^*_i>}(y), \quad (6)$$

where $(x; y)$ is a point of the graph of a function $\eta^F_{l^*_i}$ in $I$.

Let us note that by Definition 4 and by the choice of numbers $l^*_i$ (see equality (5)), iterations of a skew product $F \in \hat{T}_1^1(I)$ with a quotient map of type $\succ 2^{\infty}$ and with stable as a whole in $C^1$-norm fibers maps have also stable as a whole in $C^1$-norm families of fibers maps.

To give the criterion of stability as a whole in $C^1$-norm of a family of fibers maps of a skew product $F \in \hat{T}_1^1(I)$ with a quotient map of type $\succ 2^{\infty}$ we use auxiliary skew products $F_{l^*_i}(x, y) = (x, g_x, l^*_i(y))$ and $\Phi_{l^*_i}(x, y) = (x, \psi_x, l^*_i(y))$ for corresponding iterations of $F$ and $\Phi$ respectively. As it follows from [27] the graphs of multifunctions $\eta^F_{l^*_i}$ and $\eta^\Phi_{l^*_i}$ in $I$ coincide with wandering sets of the maps $F_{l^*_i} : \Omega(f) \times I_2$ and $\Phi_{l^*_i} : \Omega(\varphi) \times I_2$ respectively.

Therefore, the following result is the direct corollary of Definition 4.

**Theorem 1.** A family of fibers maps of a skew product $F \in \hat{T}_1^1(I)$ with a quotient map of type $\succ 2^{\infty}$ is stable as a whole in $C^1$-norm iff for any $\delta > 0$ there exists a neighborhood $B^1_\delta(F)$ of $F$ in $\hat{T}_1^1(I)$ such that for any map $\Phi \in B^1_\delta(F)$ and any $l^*_i (i \geq i^* \text{ for some } i^* \geq i_\delta)$ one can find $\delta$-close to the identity map in $C^0$-norm homeomorphism $H^{<l^*_i>} : \eta^F_{l^*_i} \to \eta^\Phi_{l^*_i}$ of skew products class possessing the property:

maps $F_{l^*_i} : \Omega(f) \times I_2$ and $\Phi_{l^*_i} : \Omega(\varphi) \times I_2$ are $\Omega$-conjugate under homeomorphism $H^{<l^*_i>}$.  

Let us use Theorem 1 and the above definition of maps $F_{l^*_i}$ and $\Phi_{l^*_i}$ for skew products $F \in \hat{T}_1^1(I)$ and $\Phi \in B^1_\delta(F)$ respectively (with quotients of type $\succ 2^{\infty}$ and stable as a whole in $C^1$-norm families of fibers maps). Then we have

$$H^{<l^*_i>}(\{x\} \times (\eta^F_{l^*_i})(x)) = \{h_1(x)\} \times (\eta^\Phi_{l^*_i})(h_1(x)).$$

By Definition 2 the equalities $\eta^F_{l^*_i}(x) = \Omega(g_x, l^*_i)$ and $\eta^\Phi_{l^*_i}(x) = \Omega(\psi_x, l^*_i)$ hold. Since $H^{<l^*_i>}$ is homeomorphism, then for any $x \in \Omega(p(f))$ the equality

$$H^{<l^*_i>}(\{x\} \times (\eta^F_{l^*_i})(x)) = \{h_1(x)\} \times (\eta^\Phi_{l^*_i})(h_1(x))$$

is valid. Hence, the equality $H^{<l^*_i>} (\eta^F_{l^*_i}) = \eta^\Phi_{l^*_i}$ is fulfilled simultaneously with the equality

$$H^{<l^*_i>} (\eta^F_{l^*_i}) = \eta^\Phi_{l^*_i}$$

and inequality

$$\text{dist}_I(\eta^F_{l^*_i}, \eta^\Phi_{l^*_i}) < \delta,$$

where dist$_I$ is Hausdorff metric in the space $2^I$ of all closed subsets of the phase space $I$ (for details, see [22, 23]).
Completing these preliminary considerations, one can prove the following statement.

**Theorem 2.** Let \( F \in \tilde{T}_1^I(I) \) have a quotient map of type \( \geq 2\infty \) and have a stable as a whole in \( C^1 \)-norm family of fibers maps. Then \( F \) satisfies strong condition \( H \) and has a continuous \( \Omega \)-function.

**Corollary 1.** Let a skew product \( F \in \tilde{T}_1^I(I) \) have a quotient map of type \( \geq 2\infty \) and have a stable as a whole in \( C^1 \)-norm family of fibers maps. Then there exists a neighborhood \( U_{\delta}(\Omega(f)) \) in \( I_1 \) of the set \( \Omega(f) \) such that for every \( x \in U_{\delta}(\Omega(f)) \) and \( t_i^* \) for \( i \geq i^* \) a map \( g_{x, t_i^*} \) is \( \Omega \)-stable in \( C^1 \)-norm with respect to the family of fibers maps \( \{g_{x, t_i^*}\}_{x \in U_{\delta}(\Omega(f))} \).

The above Corollary 1 shows that \( \eta_i^F \) of auxiliary functions \( \eta_i^F \) on a neighborhood \( U_{\delta}(\Omega(f)) \) of the set \( \Omega(f) \) is continuous for every \( i \geq i^* \). Therefore, fibers maps over points of the set \( U_{\delta}(\Omega(f)) \) do not influence on the structure of the set \( \Omega(F) \).

**Corollary 2.** Let a skew product \( F \in \tilde{T}_1^I(I) \) have a quotient map of type \( \geq 2\infty \) and stable as a whole in \( C^1 \)-norm family of fibers maps. Then, for any different points \( x', x'' \in \text{Per}(f) \cap K(f) \) with the (least) periods \( m(x') \) and \( m(x'') \) respectively \( (K(f) \) is a locally maximal quasiminimal set of \( f \) \), maps \( \gamma_{x'}^{m(x')} \) and \( \gamma_{x''}^{m(x'')} \) are \( \Omega \)-conjugate, where \( m \) is the least common multiple of numbers \( m(x') \) and \( m(x'') \); the notation \( g_x \) for every \( f \)-periodic point \( x \) means the composition \( g_{f_{m-1}(x)} \circ \ldots \circ g_x \) of fibers maps over the points of periodic orbit \( \{x, f(x), \ldots, f^{m-1}(x)\} \) (see equality (2)).

**Corollary 3.** Let a map \( F \in T_1^I(I) \) satisfy conditions of Theorem 2. Then the equality

\[
\Omega(F|\Omega_p(F)) = \bigcup_{x \in \text{Per}_p^*(f)} \{x\} \times \Omega(g_x)
\]

holds for an arbitrary, everywhere dense invariant subset \( \text{Per}_p^*(f) \) of the set of all periodic points of \( \Omega_p(f) \), where \( \Omega_p(F) = \Omega_p(f) \times I_2 \).

3. **Criterion of \( C^1 \)-\( \Omega \)-stability**

We begin this part of the paper giving the standard definition of \( C^1 \)-\( \Omega \)-stability with respect to homeomorphisms of skew products class.

**Definition 5.** We say that a map \( F \in \tilde{T}_1^I(I) \) is \( \Omega \)-stable in \( C^1 \)-norm (in the space \( \tilde{T}_1^I(I) \)) if for any \( \delta > 0 \) there exists \( \varepsilon > 0 \) such that for any map \( \Phi \in B_1^I(F) \), \( \Phi(x, y) = (\varphi(x), \psi_x(y)) \), one can find \( \delta \)-close in \( C^0 \)-norm to the identity map homeomorphism

\[
H : \Omega(F) \to \Omega(\Phi), \quad H(x, y) = (h_1(x), h_2, x(y)),
\]

such that for every point \((x, y) \in \Omega(F)\) the equalities hold:

\[
h_1 \circ f_{\Omega(f)}(x) = \varphi \circ h_1_{\Omega(f)}(x);
\]

\[
h_2, f(x)_{\Omega(f)} \circ g_x \circ (\Omega(f))(x) = \psi \circ h_1_{\Omega(f)}(x) \circ h_2, x(\Omega(f))(x) \circ y.
\]

**Theorem 3.** A skew product \( F \in \tilde{T}_1^I(I) \) with a quotient map of type \( \geq 2\infty \) is \( \Omega \)-stable in \( C^1 \)-norm if its family of fibers maps is stable as a whole in \( C^1 \)-norm.

We give here only the proof of sufficiency of the property of stability as a whole in \( C^1 \)-norm of a family of fibers maps of a skew product \( F \in \tilde{T}_1^I(I) \) for its \( C^1 - \Omega \)-stability.

1. In fact, let a family of fibers maps of a skew product \( F \in \tilde{T}_1^I(I) \) be stable as a whole in \( C^1 \)-norm. Then by Definition 4 for every \( \delta > 0 \) there exists a neighborhood \( B_\delta^I(F) \subset \tilde{T}_1^I(I) \) such that for every map \( \Phi \in B_\delta^I(F) \), \( \Phi(x, y) = (\varphi(x), \psi_x(y)) \), and every \( t_i^* \) (\( i \geq i^* \) for \( i^* \geq i_\delta \)), one can find \( \delta \)-close to the identity map in \( C^0 \)-norm homeomorphism \( H < t_i^* : \eta_i^F \to \eta_i^F \).
(H_{1}^{\infty}(x, y) = (h_{1}(x), h_{2, x}^{\infty}(y))) satisfying equalities (6).

2. Let $\Phi \in B_{L}^{1}(F)$. We construct a homeomorphism $H : \Omega(F) \to \Omega(\Phi)$ of skew products class such that

$$H \circ F_{\mid \Omega(F)} = \Phi_{\mid \Omega(\Phi)} \circ H,$$

where first coordinate function $h_{1}$ of $H$ is defined for $\varphi$ with the use of condition of $\Omega$-stability of $f$ in $C^{1}$-norm (see equality (3)).

Get over construction of fibers maps for $H$ over points of the set $\text{Per}(f)$ of $f$-periodic points. Let $x$ be an arbitrary point of the set $\text{Per}(f)$ with the (least) period $m(x)$. Let a natural number $i(m(x))$, $i(m(x)) \geq i^{*}$, be so that $m(x)$ divides $l_{i(m(x))}$. Set

$$h_{2, x|\Omega(\tilde{g}_{x})} = h_{2, x|\Omega(\tilde{g}_{x})}^{l_{i(m(x))}},$$

where

$$l_{i(m(x))}^{\eta_{i(m(x))}}(x) = \Omega(\tilde{g}_{x}^{l_{i(m(x))}/m(x)}).$$

By Corollary 1 we have

$$\Omega(\tilde{g}_{x}^{l_{i(m(x))}/m(x)}) = \Omega(\tilde{g}_{x}).$$

As it follows from paper [27], equality (7) and Corollary 3, a map $h_{2, x}(y)$ is defined on the completely invariant set $\bigcup_{x \in \text{Per}(f)} \{x\} \times \Omega(\tilde{g}_{x})$, which is everywhere dense in $\Omega(F)$; moreover, $h_{2, x}(y)$ is one to one map on $\bigcup_{x \in \text{Per}(f)} \{x\} \times \Omega(\tilde{g}_{x})$. Continuity of $h_{2, x}(y)$ on $\bigcup_{x \in \text{Per}(f)} \{x\} \times \Omega(\tilde{g}_{x})$ follows from Theorem 2, Corollaries 2 and 3.

3. Complete a definition of $h_{2, x}(y)$ on $\Omega(F)$ setting

$$h_{2, x}(y) = \lim_{n \to +\infty} h_{2, x_{n}}(y_{n})$$

for any point $(x, y) \in \Omega(F)$ and for any convergent to $(x, y)$ sequence $\{(x_{n}, y_{n})\}_{n \geq 1} \subset \bigcup_{x \in \text{Per}(f)} \{x\} \times \Omega(\tilde{g}_{x}).$

Equality (8) defines correctly second coordinate function $h_{2, x}(y)$ of homeomorphism $H$ on the set $\Omega(F)$. In fact, take a point $(x, y) \in \Omega(F)$. Let $\{(x_{n}, y_{n})\}_{n \geq 1}$ and $\{(x_{n}', y_{n}')\}_{n \geq 1}$ be any different, convergent to $(x, y)$ sequences of points of the set $\bigcup_{x \in \text{Per}(f)} \{x\} \times \Omega(\tilde{g}_{x})$. Using these two sequences, we form a new, convergent to $(x, y)$ sequence $\{(\tau_{n}, \varphi_{n})\}_{n \geq 1}$, for example, setting

$$(\tau_{2n-1}, \varphi_{2n-1}) = (x_{n}, y_{n}); \quad (\tau_{2n}, \varphi_{2n}) = (x_{n}', y_{n}').$$

Then, by continuity of $h_{2, x}(y)$ on the set $\bigcup_{x \in \text{Per}(f)} \{x\} \times \Omega(\tilde{g}_{x})$ sequence $\{h_{2, \tau}(\varphi)\}_{n \geq 1}$ is fundamental and, hence, it converges. Using definition of a sequence $\{(\tau_{n}, \varphi_{n})\}_{n \geq 1}$ and equalities (8) – (9) we obtain

$$\lim_{n \to +\infty} h_{2, \tau_{n}}(\varphi_{n}) = \lim_{n \to +\infty} h_{2, \tau_{n}}(\varphi_{n}') = \lim_{n \to +\infty} h_{2, x_{n}}(y_{n}) = h_{2, x}(y).$$

Thus, the value of the function $h_{2, x}(y)$ in any point $(x, y) \in \Omega(F)$ does not depend on the choice of a sequence $\{(x_{n}, y_{n})\}_{n \geq 1} \subset \bigcup_{x \in \text{Per}(f)} \{x\} \times \Omega(\tilde{g}_{x})$ convergent to a point $(x, y)$.

4. Verify that the above map $H : \Omega(F) \to \Omega(\Phi)$ is homeomorphism. In fact, as it follows from its definition, $H$ is surjection of the set $\Omega(F)$ on the set $\Omega(\Phi)$. By compactness of $I$ the map $H$
is closed. Therefore, $H$ is mutually continuous (see [22, 23]).
Prove that $H$ is one to one map. Suppose the contrary. Since $h_1 : \Omega(f) \rightarrow \Omega(\varphi)$ is homeomorphism, then there exist different points $(x, y_1)$ and $(x, y_2)$, where $x \notin \operatorname{Per}(f)$, satisfying
\[ h_{2,x}(y_1) = h_{2,x}(y_2). \tag{10} \]
Let $\{(x_n, y_{1n})\}_{n \geq 1}$ and $\{(x_n, y_{2n})\}_{n \geq 1}$ be any convergent to $(x, y_1)$ and $(x, y_2)$ respectively sequences of points of the set $\bigcup_{x \in \operatorname{Per}(f)} \{x\} \times \Omega(\tilde{g}_x)$. Then by equality (10) corresponding sequence $\{H(x_n, y_{1n}), H(x_n, y_{2n})\}_{n \geq 1}$ is fundamental, in addition,
\[ \{H(x_n, y_{1n}), H(x_n, y_{2n})\}_{n \geq 1} \subseteq \bigcup_{x \in \operatorname{Per}(\varphi)} \{x\} \times \Omega(\tilde{\psi}_x). \]
Since restriction of $H$ on the set $\bigcup_{x \in \operatorname{Per}(f)} \{x\} \times \Omega(\tilde{g}_x)$ is homeomorphism and
\[ H(\bigcup_{x \in \operatorname{Per}(f)} \{x\} \times \Omega(\tilde{g}_x)) = \bigcup_{x \in \operatorname{Per}(\varphi)} \{x\} \times \Omega(\tilde{\psi}_x), \]
then a sequence $\{(x_n, y_{1n}), (x_n, y_{2n})\}_{n \geq 1}$ is also fundamental. Hence, $y_1 = y_2$, and the map $H$ is one to one. This property with mutual continuity of $H$ means that $H$ is homeomorphism (see [22, 23]). Hence, $F \in \tilde{T}^1_{\ast,1}(I)$ is $\Omega$-stable in $C^1$-norm.

**Corollary 4.** Let a skew product $F \in \tilde{T}^1_{\ast,1}(I)$ with a quotient map of type $> 2^\infty$ be $\Omega$-stable in $C^1$-norm. Then $F \in \tilde{T}^1_{\ast,1}(I)$, and the $\Omega$-function of $F$ is continuous.

**Corollary 5.** Let a skew product $F \in \tilde{T}^1_{\ast,1}(I)$ satisfy conditions of Corollary 4. Then for any different periodic points $x', x''$ ($m(x')$ and $m(x'')$ are the (least) periods of $x'$ and $x''$ respectively) of every locally maximal quasiminimal set $K(f)$ of the quotient map $f$, fibers maps $\tilde{g}_{x'}^{m'/m(x')}$ and $\tilde{g}_{x''}^{m'/m(x'')}$ are $\Omega$-conjugate, where $m'$ is the least common multiple of numbers $m(x')$ and $m(x'')$.

4. **$\Omega$-Stable Skew Products of Maps of an Interval Are Not Dense in $\tilde{T}^1_{\ast,1}(I)$**

Nongenericity of $\Omega$-stability for $C^r$-diffeomorphisms ($r \geq 2$) in dimensions $\geq 3$ is proved in [1] and in dimension 2 follows from [7].

In this section we prove that $C^1$-smooth $\Omega$-stable skew products of maps of an interval do not form everywhere dense subset in the subspace $\tilde{T}^1_{\ast,1}(I) = T^1_{\ast,1}(I) \cap \tilde{T}^1_{\ast,1}(I)$.

**Theorem 4.** There exists a skew product $F \in \tilde{T}^1_{\ast,1}(I)$ such that some its neighborhood $B^1_{\ast,1}(F)$ in the space $\tilde{T}^1_{\ast,1}(I)$ does not contain $\Omega$-stable skew products of maps of an interval.

**Proof.** 1. In fact, let the skew product $F(x, y) = (f(x), \lambda(x)y(1- y))$ be defined on the unit square $[0, 1]^2$. Here the quotient map is given by the equality
\[ f(x) = \begin{cases} \tilde{h}(x), & \text{if } x \in [0, \frac{1}{4}); \\ 9\left(\frac{1}{4} - x\right)(x - \frac{3}{4}) + \frac{1}{4}, & \text{if } x \in \left[\frac{1}{4}, \frac{3}{4}\right); \\ \tilde{h}(1 - x), & \text{if } x \in \left[\frac{3}{4}, 1\right]. \end{cases} \tag{11} \]
We use the function $\tilde{h}(x) = \alpha \sin^2 \pi \beta x$ for $x \in [0, 1/4]$; moreover, $\alpha > 0$ and $\beta \in (4, 6)$ are defined uniquely from the system of equations
\[ \begin{cases} \alpha \sin^2 \pi \beta / 4 = 1/4; \\ 2\alpha \pi \beta \sin \pi \beta / 4 \cos \pi \beta / 4 = 9/2. \end{cases} \]
This system is equivalent to the condition of \( C^1 \)-smoothness of \( f \) in the point \( x = 1/4 \). Then \( f \in C^0_b(I_1) \), and \( \Omega(f) = \{0\} \cup K(f) \), where \( K(f) \) is the unique locally maximal quasiminimal set of the quotient map \( f \), \( K(f) = \Omega_p(f) \subset \{\frac{1}{4}, \frac{2}{3}\} \), \( \tau(f|_{K(f)}) = N \).

Note that the quotient \( f \) has two fixed points \( x_1 = 1/4, x_2 = 23/36 \) on the closed interval \([1/4; 3/4]\).

Let \( \lambda = \lambda(x) \) be \( C^1 \)-smooth function \([0, 1]^2 \) given by formula

\[
\lambda(x) = \begin{cases} 
\frac{9r}{2}(\lambda^* - 2)(x - \frac{1}{4}) + 2, & \text{if } x \in [0, \frac{1}{4}); \\
(\lambda^* - 2)\sin \frac{2\pi r}{2}(x - \frac{1}{4}) + 2, & \text{if } x \in [\frac{1}{4}, \frac{2}{3}); \\
\lambda^*, & \text{if } x \in [\frac{2}{3}, 1],
\end{cases}
\]

where the number \( \lambda^* \) is such that the map \( \bar{\gamma} = \lambda^* y(1 - y) \) has type \( 2^n \) (i.e. \( f \) contains periodic points of periods \( 1, 2, 2^2, \ldots, 2^n, \ldots \) and does not contain periodic points of other periods), \( \lambda^* \approx 3.569 \).

Call attention on the properties of fibers maps over \( f \)-fixed points \( x_1 \) and \( x_2 \).

\( (i_f) \) The fiber map \( \bar{g}_{x_1} \) is defined by the equality \( \bar{g}_{x_1} = 2y(1 - y) \). Hence, \( \bar{g}_{x_1} \) is \( \Omega \)-stable in the space \( C^0_b(I_2) \) Morse-Smale endomorphism with two hyperbolic fixed points (\( \bar{g}_{x_1} \) does not contain other periodic points). It means that there is a \( \varepsilon \)-neighborhood \( B^1_{\varepsilon, x_1}(\bar{g}_{x_1}) \) of the map \( \bar{g}_{x_1} \) in the space \( C^0_b(I_2) \) such that every map from this neighborhood is \( \Omega \)-conjugate with \( \bar{g}_{x_1} \).

\( (ii_f) \) The fiber map \( \bar{g}_{x_2} \) is defined by the equality \( \bar{g}_{x_2} = \lambda^* y(1 - y) \). In this case there exists strictly decreasing sequence of positive numbers \( \{\varepsilon_r\}_{r \geq 2} \), \( \lim_{r \rightarrow +\infty} \varepsilon_r = 0 \), such that every fiber map \( g_{x_2} \in B^1_{\varepsilon_r, \bar{g}_{x_2}}(\bar{g}_{x_2}) \) for \( x \in (1/4; 23/36) \) (\( x \) belongs also to some left-side neighborhood of the point \( x_2 \) defined for \( \varepsilon_r \) contains periodic points with the (least) periods \( \{1, 2, \ldots, 2^{\mu_r}\} 0 < \mu_r < \infty \). Here \( B^1_{\varepsilon_r, \bar{g}_{x_2}}(\bar{g}_{x_2}) \) is an \( \varepsilon \)-neighborhood of the map \( \bar{g}_{x_2} \) in the space \( C^0_b(I_2) \); a sequence \( \{\mu_r\}_{r \geq 2} \) strictly increases\(^5 \) [28].

2. Since \( \tau(f|_{K(f)}) = N \) then \( \lambda_r = i \) (\( i \geq 1 \)). Using properties of period doubling bifurcations in one-dimensional maps one can prove that the constructed above skew product \( F \) satisfies strong condition \( \mathbf{H} \). This property (with equalities (11), (12) and formulas for fibers maps) means that \( F \in \mathbb{T}_{\varepsilon, 1}^1(I) \).

3. Point out a neighborhood \( B^1_{\varepsilon_r, \bar{g}_{x_1}}(\bar{g}_{x_1}) \) in \( C^0_b(I_2) \) which does not contain \( \Omega \)-stable skew products. In fact, let a natural number \( r \geq 2 \) be such that each map from the neighborhood \( B^1_{\varepsilon_r, \bar{g}_{x_2}}(\bar{g}_{x_2}) \) contains periodic points with (least) periods \( \{1, 2, \ldots, 2^{\mu_r}\} 3 \leq \mu_r < \infty \). Let \( \varepsilon \)-neighborhood \( B^1_{\varepsilon_r, \bar{g}_{x_2}}(\bar{g}_{x_2}) \) of the map \( \bar{g}_{x_1} \) in \( C^0_b(I_2) \) be chosen accordingly item 1 of the proof such that property \( (i_f) \) is fulfilled.

Set \( \varpi = \min\{\varepsilon_1, \varepsilon_r\} \).

4. Unique uniform continuity of \( C^1 \)-presentation \( \rho_1 \) on the compact \( K(f) \). For number \( \varpi > 0 \) we find \( 0 < \delta_1 < 1/16 \) such that for any \( x', x'' \in K(f) \) satisfying the inequality \( |x' - x''| < \delta_1 \), we have \( g_{x''} \in B^1_{\varpi}(g_{x'}) \). Here \( \rho_1(x') = g_{x'}, \rho_1(x'') = g_{x''}. \)

Since \( f \in C^1(\omega(I)) \) then for \( \delta_1 > 0 \), there exists \( \varpi \)-neighborhood \( B^1_{\varpi, \bar{g}_{x_2}}(\bar{g}_{x_2}) \) of the quotient map \( g_{x_1} \) consisting of maps each of them is \( \Omega \)-conjugate with \( f \) over homeomorphism \( \delta_1 \)-close to the identity map. Point out that \( \delta_1 \)-neighborhoods of \( f \)-fixed points \( U_{1, \delta_1}(0), U_{1, \delta_1}(1/4), U_{1, \delta_1}(23/36) \) for \( 0 < \delta_1 < 1/16 \) are pairwise disjoint, and each of them contains unique fixed point of any map from \( B^1_{\varpi, \bar{g}_{x_2}}(\bar{g}_{x_2}) \).

Set \( \varepsilon = \min\{\varpi, \varpi'\} \).

5. Prove that a neighborhood \( B^1_{\varepsilon_r, \bar{g}_{x_2}}(\bar{g}_{x_2}) \) of the skew product \( F \in \mathbb{T}_{\varepsilon, 1}^1(I) \) does

\[^5\] Maps \( \bar{g}_{x_2} \) and \( \bar{g}_{x_2}^{n} \) for all \( n \geq 1 \) possess properties analogous properties \((ii_f) \). But a sequence \( \{\mu_n(n)\}_{n \geq 1} \) for \( n > 1 \) can be nonmonotone. One can explain it using definition of fibers maps \( g_{x_2}^{n} \) as compositions of logistic maps with different values of function \( \lambda(x) \) in the points \( x, f(x), \ldots, f^{n-1}(x) \) (see equality (2)).
not contain $\Omega$-stable maps. Suppose the contrary. Then there exists $\Omega$-stable skew product $\Phi \in T^*_f(I)$ (see Corollary 4) such that $\Phi \in B^1(\varphi_\varepsilon)$, where $\Phi(x, y) = (\varphi(x), \psi_\delta(y))$. A map $\Phi$ possesses the following properties:

(i) $\varphi \in B^1_{\varepsilon, \delta}(f)$;
(ii) $\varphi$ has unique locally maximal quasiminimal set $K(\varphi)$ containing two fixed points $x_1(\varphi) \in U_{1, \delta_1}(I)$ and $x_2(\varphi) \in U_{1, \delta_1}(23/36)$;
(iii) $\tilde{\psi}_{x_1(\varphi)} \in B^1_{2, \varepsilon_1}(\tilde{g}_{x_1})$; (iv) $\tilde{\psi}_{x_2(\varphi)} \in B^1_{2, \varepsilon_2}(\tilde{g}_{x_2})$.

By properties (i) and (ii) the map $\tilde{\psi}_{x_1(\varphi)}$ is Morse-Smale endomorphism with two fixed points. At the same time, by properties (iii) and (iv) the map $\tilde{\psi}_{x_2(\varphi)}$ contains periodic orbits with (least) periods $\{1, 2, \ldots, 2^n\}$ for $3 \leq \mu_r < \infty$. Therefore, there are no iterations of maps $\tilde{\psi}_{x_1(\varphi)}$ and $\tilde{\psi}_{x_2(\varphi)}$ which are $\Omega$-conjugate. Contradiction with Corollary 5 is obtained. Theorem 4 is proved.

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