A New Ansatz for Quark and Lepton Mass Matrices

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ABSTRACT

A new ansatz for quark and lepton mass matrices is proposed in the context of supersymmetric grand unified theories. The 13 parameters describing fermion masses and mixings are determined in terms of only 6 free parameters, allowing 7 testable predictions. The values of $V_{us}$, $V_{cb}$, $V_{ub}$, $m_u$, $m_d$, $m_s$, and $m_b$ are then predicted as a function of the 3 charged lepton masses, $m_c$, $m_t$, and $\tan \beta$, the ratio of Higgs vacuum expectation values. In particular the Cabibbo angle and $m_s/m_d$ are determined in terms of only lepton masses. All predictions are in very good agreement with experiments.

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One of the main open problems in particle physics is the understanding of the quark and lepton mass matrices. In the Standard Model, the fermion masses and mixings are described by 13 free parameters. Attempts to compute these 13 phenomenological parameters within the framework of extensions of the Standard Model so far have not proven very successful. A less ambitious approach is to assume a particular form of the fermion mass matrices, which allows a description of the physical observables in terms of fewer parameters and which therefore makes some testable predictions. It is hoped that this ansatz hints at the necessary symmetries of the interactions responsible for fermion mass generation and then leads to the construction of a successful theory beyond the Standard Model.

In this paper, I will present a new ansatz for quark and lepton mass matrices which involves only 6 free parameters, leading to 7 testable predictions. I begin by defining the energy scale at which the ansatz holds. If the ansatz has anything to do with the symmetries of an underlying theory of fermion mass generation, then it should hold at the energy scale where the extended theory breaks down into the Standard Model. The subsequent running of the mass matrices from this energy scale to the weak scale, according to the renormalization group equations, may spoil the simplicity of the ansatz and hide its symmetry relations, but, it is hoped, provides the correct values of the fermion masses and mixings.

I will assume here that the underlying theory is some kind of Grand Unified Theory (GUT). Given the accurate measurements of $\alpha_s$ and $\sin^2 \theta_W$ at LEP, it is now clear that the simplest way to achieve grand unification is to consider a supersymmetric particle spectrum [1]. I will therefore assume the minimal supersymmetric model below the GUT scale $M_X$, and an unspecified GUT theory above $M_X$. Here I closely follow the strategy proposed in ref. [2], where a different ansatz, that of Georgi-Jarlskog [3], is investigated.

Let us write the fermion mass terms, after spontaneous breaking of the electroweak symmetry, as:

$$\mathcal{L}_{\text{mass}} = \bar{q}_L^i U_{ij} u^j_R \frac{v}{\sqrt{2}} \sin \beta + \bar{q}_L^i D_{ij} \nu_R \frac{v}{\sqrt{2}} \cos \beta + \bar{l}_L^i E_{ij} \tau_R \frac{v}{\sqrt{2}} \cos \beta + h.c., \quad (1)$$

where $i, j = 1, 2, 3$ are generation indices, $\tan \beta$ is the ratio of Higgs vacuum expectation values, and $v = 246$ GeV. The ansatz proposed here for the Yukawa couplings is that, at the scale $M_X$, $U$, $D$, and $E$ are Hermitian.
matrices of the form:

\[
U = \begin{pmatrix} 0 & 0 & b \\ 0 & b & 0 \\ b & 0 & a \end{pmatrix}, \quad D = \begin{pmatrix} 0 & f & 0 \\ f^* & d & d' \\ 0 & d^* & c \end{pmatrix}, \quad E = \begin{pmatrix} 0 & f & 0 \\ f^* & -3d & d' \\ 0 & d'^* & c \end{pmatrix}.
\] (2)

The relation between \(D\) and \(E\) is natural in GUT models where the down quarks and the charged leptons lie in the same multiplet. For instance, the coupling of a Higgs boson in the 5 of SU$_5$ (or 10 of SO$_{10}$) gives certain entries in the Yukawa coupling matrices of the form \(D_{ij} = E_{ij}\), while a Higgs boson in the 45 of SU$_5$ (or 126 of SO$_{10}$) gives \(-3D_{ij} = E_{ij}\). I will also assume \(d' = 2d\). This is taken here as a purely phenomenological assumption, but it is hoped that it will find a group theoretical explanation, analogous to the idea proposed in ref.\[^3\], in a complete GUT involving some generation symmetry. I want to stress once again that my goal here is to find simple relations that give successful phenomenological predictions, which can be used as a guide for constructing realistic models, rather than to find an explicit realization of the \textit{ansatz}. Using the freedom to redefine the fermion phases, the \textit{ansatz} of eq.(2) now becomes:

\[
U = \begin{pmatrix} 0 & 0 & b \\ 0 & b & 0 \\ b & 0 & a \end{pmatrix}, \quad D = \begin{pmatrix} 0 & f e^{i\phi} & 0 \\ f e^{-i\phi} & d & 2d \\ 0 & 2d & c \end{pmatrix}, \quad E = \begin{pmatrix} 0 & f & 0 \\ f & -3d & 2d \\ 0 & 2d & c \end{pmatrix}.
\] (3)

In eq.(3), \(a, b, c, d, f, \phi\) are 6 real parameters which will be used to compute the fermion masses and mixings, thus providing us with 7 predictions. Notice that without the requirement that \(U, D,\) and \(E\) are Hermitian, one would find two independent phases in eq.(3). As an alternative to the hermiticity condition, I could have imposed different relations among the phases in order to get rid of the extra parameter.

The \textit{ansatz} in eq.(3) has to be run down to the weak scale. The Yukawa coupling constants satisfy the one-loop renormalization group equations:

\[
16\pi^2 \frac{d}{dt} U = (3UU^\dagger + DD^\dagger + 3TrUU^\dagger - G_U)U, \tag{4}
\]

\[
16\pi^2 \frac{d}{dt} D = (3DD^\dagger + UU^\dagger + 3TrDD^\dagger + TrEE^\dagger - G_D)D, \tag{5}
\]

\[
16\pi^2 \frac{d}{dt} E = (3EE^\dagger + 3TrDD^\dagger + TrEE^\dagger - G_E)E, \tag{6}
\]
where \( t = \log \mu \), and \( \mu \) is the renormalization scale, and

\[
G_a = \sum_{i=1}^{3} c_a^i g_i^2, \quad a = U, D, E.
\]  

(7)

Finally, the \( g_i \) are the gauge coupling constants which satisfy the one-loop renormalization group equations:

\[
16\pi^2 \frac{d}{dt} g_i = b_i g_i^3,
\]

and the coefficients \( c_a^i \) and \( b_i \) for the minimal supersymmetric model are given in table 1.

In order to solve eqs.(4-6), I first redefine the quark fields so that the Yukawa matrices transform as:

\[
U \rightarrow K^\dagger U K \equiv U', \quad D \rightarrow K^\dagger D K \equiv D',
\]

(9)

with \( K \) chosen such that \( U' \) is diagonal. Neglecting in eqs.(4-6) all non-leading terms in Yukawa couplings different from that of the top quark, I find:

\[
16\pi^2 \frac{d}{dt} U' = \left[ 3U'^2_{33} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 3U'^2_{33} - G_U \right] U',
\]

(10)

\[
16\pi^2 \frac{d}{dt} D' = \left[ U'^2_{33} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - G_D \right] D',
\]

(11)

\[
16\pi^2 \frac{d}{dt} E = -G_E E.
\]

(12)

The solutions of eqs.(10-12) are:

\[
U'(t) = \gamma_U \begin{pmatrix} \xi^3 \\ \xi^3 \\ \xi^6 \end{pmatrix} U'(t_0),
\]

(13)

\[
D'(t) = \gamma_D \begin{pmatrix} 1 \\ 1 \\ \xi \end{pmatrix} D'(t_0),
\]

(14)

\[
E(t) = \gamma_E E(t_0).
\]

(15)
where
\[
\gamma_a = \exp \left( -\frac{1}{16\pi^2} \int_{t_0}^t dt G_a \right), \quad \xi = \exp \left( \frac{1}{16\pi^2} \int_{t_0}^t dt U'^2_{33} \right),
\]
(16)
and the initial condition is \( t_0 = \log M_X \). These integrals can be performed with the help of eq.(8) to obtain:
\[
\gamma_a = \prod_{i=1}^3 \left[ \frac{g_i(t_0)}{g_i(t)} \right]^{c_i/b_i},
\]
(17)
\[
\xi = \left[ 1 + \frac{3}{4\pi^2} I U'^2_{33}(t_0) \right]^{-1/12} = \left[ 1 - \frac{3}{4\pi^2} \frac{I}{\gamma_U^2} U'^2_{33}(t) \right]^{1/12},
\]
(18)
where \( I = -\int_{t_0}^t dt \gamma_U^2 \).

Thus the Yukawa coupling matrices renormalized at the weak scale \( \mu \) are given by:
\[
U_R = \gamma_U \begin{pmatrix} \xi^3 & \xi^3 \\ \xi^6 & \xi \end{pmatrix} K^\dagger U K,
\]
(19)
\[
D_R = \gamma_D \begin{pmatrix} 1 & \\ 1 & \xi \end{pmatrix} K^\dagger D K,
\]
(20)
\[
E_R = \gamma_E E,
\]
(21)
where \( U, D, E \) are the matrices at \( M_X \) given by the ansatz in eq.(3). The matrix \( K \), defined in eq.(9), is given by:
\[
K = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \quad \tan \theta = \frac{2b}{a}.
\]
(22)

Since \( U_R \) is already diagonal, the Cabibbo-Kobayashi-Maskawa matrix is simply given by the matrix \( V \), such that \( V^\dagger D_R D_R^\dagger V \) is diagonal.

In order to diagonalize eqs.(20-21), I use the fact that, because of the observed hierarchy of fermion masses, the parameters in eq.(3) must satisfy \( a >> b, c >> d >> f \). Then I express \( a, b, c, d, f \) in terms of \( m_e, m_\mu, m_\tau, \)
$m_c, m_t$. After diagonalization, I find that the Cabibbo-Kobayashi-Maskawa matrix becomes

$$V = \begin{pmatrix}
  c_1c_3e^{i\phi} - s_1s_2s_3 & s_1c_3e^{i\phi} + c_1s_2s_3 & -c_2s_3 \\
  -s_1c_2 & c_1c_2 & s_2 \\
  c_1s_3e^{i\phi} + s_1s_2c_3 & s_1s_3e^{i\phi} - c_1s_2c_3 & c_2c_3
\end{pmatrix},$$

(23)

where $c_1 \equiv \cos \theta_1, s_1 \equiv \sin \theta_1$, etc. The 7 predictions of the ansatz in eq.(3) are:

$$V_{us} = |s_1| = 3 \sqrt{\frac{m_e}{m_\mu}} \left(1 - \frac{25}{2} \frac{m_e}{m_\mu} + 16 \frac{m_e}{9 m_\tau}\right)$$

(24)

$$V_{cb} = |s_2| = \frac{2}{3} \frac{m_\mu}{m_\tau} \left(1 - \frac{m_e}{m_\mu} - \frac{m_\mu}{9 m_\tau}\right)$$

(25)

$$V_{ub} = |s_3| = \frac{\xi^2 m_e}{\eta_c m_t}$$

(26)

$$m_u = \frac{\xi^3 \eta_u m_\tau}{\eta_c^2 m_t}$$

(27)

$$m_d = \frac{\gamma_D}{\gamma_E} \eta_d 3m_e \left(1 - 8 \frac{m_e}{m_\mu} + 16 \frac{m_\mu}{9 m_\tau}\right)$$

(28)

$$m_s = \frac{\gamma_D}{\gamma_E} \eta_s \frac{m_\mu}{3} \left(1 + 8 \frac{m_e}{m_\mu} - 16 \frac{m_\mu}{9 m_\tau}\right)$$

(29)

$$m_b = \frac{\gamma_D}{\gamma_E} \xi \eta_b m_\tau$$

(30)

where $\xi = [1 - (3m_\mu^2T)/(2\pi^2\eta_\mu a^2 \sin^2 \beta)]^{1/12}$. Eqs.(24-30) have been obtained by diagonalizing eqs.(20-21), keeping the first order corrections in $m_e/m_\mu$ and $m_\mu/m_\tau$.

The effects of QCD renormalization of the quark mass $m_q$ from the energy scale $\mu$ to the energy scale $\Lambda_q$, at which the input quark mass is given, is contained in $\eta_q \equiv m_q(\Lambda_q)/m_q(\mu)$. This means that each quark mass appearing in eqs.(24-30) is defined at the energy scale $\Lambda_q$; I choose $\Lambda_q = 1$ GeV for the three light quarks, and $\Lambda_q = m_q$ for the three heavy quarks. The renormalization scale $\mu$ is taken to be equal to $m_t$; also the

\footnote{I have checked that higher order terms give corrections in eqs.(24-30) always smaller than $10^{-2}$. Notice that terms of order $m_\mu^2/m_\tau^2$ always contribute much less than terms of order $m_e/m_\mu$ and $m_\mu/3m_\tau$.}
supersymmetry breaking scale is taken to be equal to $\mu (= m_t)$, and threshold effects due to the supersymmetric particle spectrum have been neglected. The numerical predictions following from eqs.(24-30), as a function of $m_t$ and $\xi$ (or, equivalently, $m_t$ and $\tan \beta$) are contained in table 2, with the details of the input values illustrated in the table caption.

A correct prediction of the bottom quark mass constrains $\xi = 0.81 \pm 0.02$, and therefore the ratio of Higgs vacuum expectation values must satisfy:

$$\sin \beta \simeq \frac{m_t}{180 \text{ GeV}}.$$  \hfill (31)

The condition $\tan \beta > 1$, usually required by the electroweak breaking mechanism in supersymmetric models, provides a lower bound for the top quark mass of about 125 GeV. An upper bound of about 170 GeV is obtained by requiring that the prediction for $m_u$ is consistent with the result from chiral perturbation theory and QCD sum rules ($m_u = 5.1 \pm 1.5$ MeV \cite{4}), after errors on $m_c$ and $\xi$ have been taken into account. Note that because of the upper bound on $m_t$, eq.(31) gives $\tan \beta < 3$ and therefore the approximation of neglecting the Yukawa coupling for the bottom quark in the solution of the renormalization group equations is justified.

As it is apparent from table 2, all the predictions are in good agreement with the experimental results, for $\xi = 0.81$ and $m_t$ in the range 125–170 GeV. In particular, the Cabibbo angle, which has been measured to one part in a hundred, and $m_s/m_d$, which is precisely determined from second order chiral perturbation theory \cite{3}, are successfully predicted in terms of only lepton masses.

Notice that the phase $\phi$ does not appear in eqs.(24-30). The CP violation effects of the Cabibbo-Kobayashi-Maskawa matrix are then determined in terms of a new free parameter, $\phi$. The parametrization-invariant CP violating quantity

$$|J| = |\text{Im}(V_{ij}V_{ik}V_{kj}^* V_{ij}^*)| \quad \text{for any} \quad i \neq l, \quad j \neq k$$  \hfill (32)

is given by

$$|J| = c_1 c_2 c_3 s_1 s_2 s_3 |\sin \phi| = \xi \frac{130 \text{ GeV}}{m_t} \cdot 3.7 \cdot 10^{-5} |\sin \phi|,$$  \hfill (33)

\footnote{Here and in table 2 I have neglected the uncertainties in the determination of the quark mass due to the $\mu (= m_t)$ dependence of the $\eta$'s and $\gamma$'s. Such uncertainties, which amount to a few percent, can be eliminated by computing the explicit dependence on $m_t$, but nevertheless they cancel in the predictions for the ratios of quark masses.}
where the numerical values from table 2 have been used and $\xi \equiv \zeta / 0.81$.

In conclusion, I have proposed a new ansatz for quark and lepton mass matrices, remnant of some unspecified supersymmetric GUT, much in the same spirit of ref.[2]. This ansatz, eq.(3), involves 6 free parameters and therefore leads to 7 predictions. These are listed in eqs.(24-30) and compared with experimental results in table 2. The striking agreement between predictions and experiment suggests that eq.(3) may have something to do with the theory of fermion mass generation.

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References

[1] U. Amaldi, W. de Boer, and H. Fürstenau, Phys. Lett. B260 (1991) 447; P. Langacker and M. Luo, Phys. Rev. D44 (1991) 817.

[2] S. Dimopoulos, L.J. Hall, and S. Raby, OSU preprint-DOE-ER-01545-567 (1991).

[3] H. Georgi and C. Jarlskog, Phys. Lett. 86B (1979) 297.

[4] J. Gasser and H. Leutwyler, Phys. Rep. 87 (1982) 77.

[5] H. Leutwyler, Nucl. Phys. B337 (1990) 108.

[6] Review of Particle Physics, Phys. Lett. B239 (1990) 1.

[7] G. Altarelli, Cern preprint CERN-TH.6245/91 (1991).

[8] J. Ellis, G.L. Fogli, and E. Lisi, Phys. Lett. B274 (1992) 456.
Table 1: The coefficient $c^i_a$ and $b_i$ for the minimal supersymmetric model.

| $i$ = | 1   | 2   | 3   |
|------|-----|-----|-----|
| $c^i_U$ | $13/15$ | 3 | $16/3$ |
| $c^i_D$ | $7/15$ | 3 | $16/3$ |
| $c^i_E$ | $9/5$ | 3 | 0   |
| $b_i$ | $33/5$ | 1 | $-3$ |
Table 2: The predictions of the ansatz of eq.(3) compared with experimental results. I have taken $M_X = 1 \cdot 10^{16}$ GeV and $\alpha^{-1} = 25.1$, as predicted by supersymmetric grand unification. For a range of $\mu$ between 125 and 170 GeV, I find $\gamma_U = 3.2$, $\gamma_D/\gamma_E = 2.1$, $I = 113$, $\eta_b = 1.4$, $\eta_c = 1.8$, $\eta_{s,d,u} = 2.0$. The masses of the 3 light quarks are defined at 1 GeV, and the masses of the 3 heavy quarks are defined at an energy scale equal to their masses. As input, I have taken $m_c = 1.27 \pm 0.05$ and the charged lepton masses given in ref.[6]. The errors shown for the predictions take into account only the uncertainty on $m_c$. Finally $\bar{\xi} \equiv \xi/0.81.$

|               | prediction | experiment               | reference                |
|---------------|------------|--------------------------|--------------------------|
| $V_{us}$      | 0.218      | $0.221 \pm 0.003$        | @ 90% CL [6]             |
| $V_{cb}$      | $\bar{\xi}^{-1}$ $4.8 \cdot 10^{-2}$ | $(4.4 \pm 0.9) \, 10^{-2}$ | [6]                      |
| $V_{ub}$      | $\bar{\xi}^{-1} \frac{130 \text{ GeV}}{m_t}$ $(3.6 \pm 0.1) \, 10^{-3}$ | $(4 \pm 3) \, 10^{-3}$ | @ 90% CL [6]             |
| $V_{ub}/V_{cb}$ | $\bar{\xi}^{-3} \frac{130 \text{ GeV}}{m_t}$ $(7.5 \pm 0.3) \, 10^{-2}$ | $(9 \pm 4) \, 10^{-2}$ | [6]                      |
| $m_u$         | $\bar{\xi}^{-3} \frac{130 \text{ GeV}}{m_c}$ $(4.1 \pm 0.3) \, \text{MeV}$ | $5.1 \pm 1.5 \, \text{MeV}$ | QCD sum rules [4]          |
|               |            |                          | SU(4) mass relations [4]  |
| $m_d$         | 6.9 MeV    | $8.9 \pm 2.6 \, \text{MeV}$ | QCD sum rules [4]          |
|               |            | $7.9 \pm 2.4 \, \text{MeV}$ | SU(4) mass relations [4]  |
| $m_s$         | 138 MeV    | $175 \pm 55 \, \text{MeV}$ | QCD sum rules [4]          |
|               |            | $155 \pm 50 \, \text{MeV}$ | SU(4) mass relations [4]  |
| $m_b$         | $\bar{\xi} \frac{130 \text{ GeV}}{m_t}$ $4.25 \, \text{GeV}$ | $4.25 \pm 0.10 \, \text{GeV}$ | [4]                      |
| $m_u/m_d$     | $\bar{\xi}^{-3} \frac{130 \text{ GeV}}{m_t}$ $(0.59 \pm 0.05)$ | $0.56 \pm 0.08$ | chiral pert. theory [5]   |
| $m_s/m_d$     | 20         | 20 $\pm 2$              | chiral pert. theory [5]   |
| $m_t$         | 125 $-$ 170 GeV | 140 $\pm 35 \, \text{GeV}$ | fit of LEP data [7]        |
|               |            | $120 \pm 27_{28} \, \text{GeV}$ | fit of LEP data [8]        |