Article

Hybrid Nil Radical of a Ring

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Abstract: The nature of universe problems is ambiguous due to the presence of asymmetric data in almost all disciplines, including engineering, mathematics, medical sciences, physics, computer science, operations research, artificial intelligence, and management sciences, and they involve various types of uncertainties when dealing with them on various occasions. To deal with the challenges of uncertainty and asymmetric information, different theories have been developed, including probability, fuzzy sets, rough sets, soft ideals, etc. The strategies of hybrid ideals, hybrid nil radicals, hybrid semiprime ideals, and hybrid products of rings are introduced in this paper and hybrid structures are used to examine the structural properties of rings.

Keywords: rings; hybrid structure; hybrid ideals; hybrid nil radical; hybrid product

1. Introduction

In [1], Zadeh developed and investigated fuzzy set theory and its features in relation to set theory, which gives a suitable structure for generalising fundamental algebraic ideas, including set theory, group theory, ring theory, etc. In group theory, Rosenfeld [2] explored the perception of a fuzzy set. Kuroki [3–5] has proposed fuzzy ideal structures and bi-ideals in the theory of semigroups. Liu [6] explored the fuzzy ideal of a ring and obtained significant results. Mukherjee et al. [7] defined and discussed the fuzzy prime ideal notion of a ring. Later, fuzzy prime ideals were further discussed by Swamy et al. [8]. Malik [9] studied the fuzzy maximal and radical of a ring. Kumbhojkar et al. [10] studied fuzzy nil radicals of a ring. A fuzzy nil radical in different algebraic structures was explored by several researchers [11–14]. The fuzzy subnear-ring theory concept was presented by Abou-Zaid [15]. Since then, many mathematicians have pursued their research in fuzzy algebraic structures in different directions and obtained significant results in semigroups, groups, rings, etc.

Molodtsov [16] introduced and examined the soft set theory idea to deal with uncertainty concepts. Later on, different researchers looked at these ideas. The hybrid structure idea, which is parallel to fuzzy set theory and soft set theory, was initiated by Jun et al. [17] and obtained various results, and then applied to the theory of BCI/BCK-algebras. Anis et al., in [18], presented hybrid structures in semigroups, including hybrid subsemigroups, and hybrid ideals, and analysed their significant properties.

Elavarasan et al. [19], investigated the concept of hybrid ideals and hybrid bi-ideals in semigroups and established equivalent conditions in terms of hybrid ideals and hybrid bi-ideals for a semigroup to be regular and intra-regular. They also used hybrid ideals and hybrid bi-ideals to characterise the left and right simple semigroups, as well as the completely regular semigroups. In [20], Porselvi et al. defined hybrid interior ideals and hybrid characteristic interior ideals of a semigroup and found some equivalent conditions for a hybrid structure to be a hybrid interior ideal of a semigroup. For a regular semigroup...
and an intra-regular semigroup, hybrid interior ideals and hybrid ideals are also shown to be the same.

In [21], Porselvi et al. introduced the concepts of hybrid interior ideals and hybrid simple in an ordered semigroup, discussed the characteristic hybrid structures using ideals and interior ideals, and defined the ordered semigroup in terms of various hybrid ideal structures. The equivalent condition for an ordered semigroup was also established to be simple. In [22], Porselvi et al. presented the ideas of hybrid AG-groupoid as well as equivalent claims for a semigroup to be regular, and explored several properties of hybrid ideal structures.

In [23], Elavarasan et al. hooked up the idea of hybrid ideals in a near-ring, established various properties, and discussed hybrid structure images using homomorphism. Muhiuddin et al. in [24] proposed and investigated a hybrid subsemimodule over semirings, investigated the representations of hybrid subsemimodules and hybrid ideals using hybrid products, and obtained some intriguing results on t-pure ideal structures.

In a ternary semiring, Muhiuddin et al. [25] introduced hybrid ideals and k-hybrid ideals, proved various properties of k-hybrid ideals, and gave characterisations of hybrid intersections with respect to these k-hybrid ideals. There were also results based on the homomorphic hybrid preimage of a k-hybrid ideal. More concepts related to this study have been studied in [26–34].

The study of symmetry is one of the most important and beautiful themes uniting various areas of contemporary arithmetic. Rings are useful structures in pure mathematics and for learning a geometrical object’s symmetries. The most important functions in ring theory are those that preserve the ring operation, which are referred to as homomorphism. Showing a link from ring theory to real life is a simple way to see a possible connection between homomorphism and reality.

In this expository article, we discuss hybrid structures in ring theory, namely, hybrid ideals, hybrid semi-prime ideals, hybrid nil radicals, hybrid products, and characteristic functions in rings, and present some results regarding these concepts. We also prove that the hybrid intersection of a hybrid radical will coincide with the hybrid radical of a hybrid intersection, the hybrid image of a hybrid radical will coincide with the hybrid radical of a hybrid image under certain conditions, and the hybrid preimage of a hybrid radical will coincide with the hybrid radical of a hybrid preimage. Further, we obtain an equivalent assertion for the hybrid semi-prime ideal in the ring.

2. Definitions and Results

We will present the glossary of definitions that we will use in this sequel.

**Definition 1.** A set \( \mathcal{R}(\neq \emptyset) \) together with two binary operations \('+\' and '\.'\) is said to be a ring if it fulfills the following assertions:

(i) \( \mathcal{R} \) is an abelian group under \('+\',
(ii) \( \mathcal{R} \) is associative under '\.'\,'
(iii) \( c.(u + k) = c.u + c.k \) and \( (c + u).k = c.k + u.k \) for all \( c, u, k \in \mathcal{R} \).

Throughout this paper, unless stated otherwise, \( \mathcal{R} \) denotes a ring and \( \mathcal{P}(\mathcal{R}) \), the power set of a set \( \mathcal{R} \).

**Definition 2.** Let \( I \) be the unit interval and \( \mathcal{U} \) be an initial universal set. Consider a mapping \( \bar{f}_\mu := (\bar{f}, \mu) : \mathcal{R} \rightarrow \mathcal{P}(\mathcal{U}) \times I, x_1 \mapsto (\bar{f}(x_1), \mu(\bar{f}(x_1))) \), where \( \bar{f} : \mathcal{R} \rightarrow \mathcal{P}(\mathcal{U}) \) and \( \mu : \mathcal{R} \rightarrow I \). Then, \( \bar{f}_\mu \) is described as a hybrid structure in \( \mathcal{R} \) over \( \mathcal{U} \).

Let all the hybrid structures collected in \( \mathcal{R} \) over \( \mathcal{U} \) be described by \( \mathcal{H}(\mathcal{R}) \). An order \( \ll \) in \( \mathcal{H}(\mathcal{R}) \) is outlined as follows: For every \( \bar{f}_\mu, \bar{f}_\gamma \in \mathcal{H}(\mathcal{R}) \), \( \bar{f}_\mu \ll \bar{f}_\gamma \) if and only if \( \bar{f}(w) \subseteq \bar{f}(\gamma(w)) \) and \( \mu(w) \geq \gamma(w) \) for all \( w \in \mathcal{R} \). For any \( x_1, x_2 \in \mathcal{R} \), \( \bar{f}_\mu(x_1) = \bar{f}_\gamma(x_2) \) if and only if \( \mu(x_1) \ll \gamma(x_2) \) and \( \bar{f}_\mu(x_2) \ll \gamma(x_1) \). Additionally, \( \bar{f}_\mu \ll \bar{f}_\gamma \) and \( \bar{f}_\gamma \ll \bar{f}_\mu \) if and only if \( \bar{f}_\mu = \bar{f}_\gamma \). It is noted that \( (\mathcal{H}(\mathcal{R}), \ll) \) is a poset.
Definition 3. Suppose \((\mathcal{G}, .)\) is a group. A hybrid structure \(\tilde{i}_\Omega \in \mathcal{H}(\mathcal{R})\) is a hybrid subgroup in \(\mathcal{G}\) if it fulfills the following assertions: \(\forall a_1, a_2 \in \mathcal{G}\),

(i) \(\tilde{i}(a_1a_2) \supseteq \tilde{i}(a_1) \cap \tilde{i}(a_2)\),

(ii) \(\tilde{i}(a_1^{-1}) = \tilde{i}(a_1)\),

\(\Omega(\mathcal{G})\)

\(\Omega(\mathcal{G})\)

Equivalently, \(\tilde{i}(a_1a_2)^{-1} \supseteq \tilde{i}(a_1) \cap \tilde{i}(a_2)\) and \(\Omega(\mathcal{G}) \leq \tilde{i}(a_1) \cap \tilde{i}(a_2)\).

Definition 4. Let \(\tilde{i}_\Omega \in \mathcal{H}(\mathcal{R})\). Then, \(\forall k_1, w_2 \in \mathcal{R}\),

(i) \(\tilde{i}(k_1 - w_2) \supseteq \tilde{i}(k_1) \cap \tilde{i}(w_2)\),

(ii) \(\tilde{i}(k_1w_2) \supseteq \tilde{i}(w_2)\),

(iii) \(\tilde{i}(k_1w_2) \supseteq \tilde{i}(k_1)\).

\(\tilde{i}_\Omega\) is a hybrid left ideal in \(\mathcal{R}\) if it fulfills assertions (i) and (ii), \(\tilde{i}_\Omega\) is a hybrid right ideal in \(\mathcal{R}\) if it fulfills assertions (i) and (iii), and \(\tilde{i}_\Omega\) is a hybrid ideal if it fulfills assertions (i), (ii), and (iii).

Example 1. Consider the set \(S_0 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}\). Then, \((S_0, +, \cdot, \cdot, \cdot)\) is a ring.

Let \(\tilde{i}_\Omega\) in \(\mathcal{H}(\mathcal{R})\) over \(\mathcal{U} = \{0, 1\}\) be described as \(\tilde{i}(k) = \begin{cases} [0, 0.9] & \text{if } k = 0 \\ [0, 0.6] & \text{if } k = 3 \text{ or } 6 \text{ and } \text{the mapping } \Omega : \mathcal{R} \to I \text{ is constant} \end{cases}\).

Then, \(\tilde{i}_\Omega\) is a hybrid ideal in \(S_0\).

Definition 5. Let \(\tilde{j}_\nu, \tilde{i}_\Omega \in \mathcal{H}(\mathcal{R})\). Then, the hybrid structure \(\tilde{j}_\nu \circ \tilde{i}_\Omega := (\tilde{j} \circ \tilde{i}_\nu \circ \nu \tilde{\Omega})\) is the hybrid product of \(\tilde{j}_\nu\) and \(\tilde{i}_\Omega\), where

\(\tilde{j} \circ \tilde{i}_\nu(t) = \begin{cases} \bigcup\{\tilde{i}(w) \cap \tilde{j}(k)\} & \text{if } \exists w, k \in \mathcal{R} : t = wk \\ \emptyset & \text{otherwise} \end{cases}\)

and

\(\nu \tilde{\Omega}(t) = \begin{cases} \bigwedge\{\nu(w) \cap \tilde{\Omega}(k)\} & \text{if } \exists w, k \in \mathcal{R} : t = wk \\ 1 & \text{otherwise} \end{cases}\)

\(\forall t \in \mathcal{R}\).

Definition 6. Let \(\tilde{i}_\Omega, \tilde{j}_\gamma \in \mathcal{H}(\mathcal{R})\). Then, the hybrid intersection of \(\tilde{i}_\Omega\) and \(\tilde{j}_\gamma\) is defined as \(\tilde{i}_\Omega \cap \tilde{j}_\gamma : \mathcal{R} \to \mathcal{H}(\mathcal{U}) \times I, w \mapsto ((\tilde{i}_\Omega \cap \tilde{j}_\gamma)(w), (\tilde{\Omega} \cap \tilde{\gamma})(w))\), where \(\tilde{i}_\Omega \cap \tilde{j}_\gamma : \mathcal{R} \to \mathcal{H}(\mathcal{U}), w \mapsto \tilde{i}(w) \cap \tilde{j}(w)\) and \(\tilde{\Omega} \cap \tilde{\gamma} : \mathcal{R} \to I, w \mapsto \tilde{\Omega}(w) \cap \tilde{\gamma}(w)\).

Definition 7 ([11]). Let \(R\) be a commutative ring. An ideal \(K \subseteq \mathcal{R}\) is termed as semiprime if, whenever \(d^n \in K\) for some positive integer \(n\) and \(d \in \mathcal{R}\), then \(d \in K\).

Definition 8 ([11]). A ring \(\mathcal{R}\) is referred to as regular if for each \(d \in \mathcal{R}\), \(\exists s \in \mathcal{R} \ni d = dsd\).

Theorem 1 ([11]). Suppose \(\mathcal{R}\) is commutative. Then, every ideal of \(\mathcal{R}\) is semiprime if and only if \(\mathcal{R}\) is regular.

Definition 9. Consider a map \(\Psi : \mathcal{F} \to \mathcal{G}\), where \(\mathcal{F}\) and \(\mathcal{G}\) are sets. For any hybrid structure \(\tilde{i}_\Omega \in \mathcal{H}(\mathcal{F})\) over \(\mathcal{U}\), consider \(\Psi^{-1}(\tilde{i}_\Omega) := (\Psi^{-1}(\tilde{i}), \Psi^{-1}(\tilde{\Omega})) \in \mathcal{H}(\mathcal{F})\) where \(\Psi^{-1}(\tilde{i}(w)) = \)
The hybrid preimage of \( \tilde{t} \Omega \) under the mapping \( \Psi \) is represented by \( \Psi^{-1}(\tilde{t} \Omega) \).

For any \( \tilde{t} \Omega \in \mathcal{H}(\mathcal{R}) \) over \( \mathcal{U} \), under the mapping \( \Psi \), the hybrid image of \( \tilde{t} \Omega \) is considered as a hybrid structure \( \Psi(\tilde{t} \Omega) := (\Psi(\tilde{t}), \Psi(\Omega)) \) in \( \mathcal{F} \), where

\[
\Psi(\tilde{t}(c)) = \begin{cases} 
\bigcup_{i \in \Psi^{-1}(c)} \tilde{i}(i) & \text{if } \Psi^{-1}(c) \neq \phi, \\
\phi & \text{otherwise,}
\end{cases}
\]

\[
\Psi(\Omega(\Omega)) = \begin{cases} 
\bigwedge_{i \in \Psi^{-1}(c)} \Omega(i) & \text{if } \Psi^{-1}(c) \neq \phi, \\
1 & \text{otherwise}
\end{cases}
\]

\[\forall c \in \mathcal{F}.
\]

**Definition 10.** For \( A \in \mathcal{P}(\mathcal{R}) \), define \( \chi_A(\tilde{t} \Omega) \in \mathcal{H}(\mathcal{R}) \) as follows:

\[\chi_A(\tilde{t} \Omega) = (\chi_A(\tilde{t}), \chi_A(\Omega)), \text{ where } \chi_A(\tilde{t}) : \mathcal{R} \rightarrow \mathcal{P}(\mathcal{U}), a \mapsto \begin{cases} 
\mathcal{U} & \text{if } a \in A \\
\phi & \text{otherwise}
\end{cases}
\]

and \( \chi_A(\Omega) : \mathcal{R} \rightarrow I, a \mapsto \begin{cases} 
0 & \text{if } a \in A \\
1 & \text{otherwise}
\end{cases}.
\]

This is called characteristic hybrid structure of \( A \).

### 3. Hybrid Structures in Rings

We define the hybrid radical notion in a ring and explore its properties. We also show that for any two hybrid ideals, the hybrid radical of the hybrid intersection is equivalent to the hybrid intersection of the hybrid radical, and the hybrid radical of the hybrid sum is equivalent to the hybrid sum of the hybrid radical.

**Definition 11.** Suppose \( \tilde{t} \Omega \in \mathcal{H}(\mathcal{R}) \) is a hybrid ideal in \( \mathcal{R} \). Then, the hybrid nil radical of \( \tilde{t} \Omega \) is the hybrid structure in \( \mathcal{R} \) over \( \mathcal{U} \), represented by \( \sqrt{\tilde{t} \Omega} := (\sqrt{\tilde{t}}, \sqrt{\Omega}) \) where \( \sqrt{\tilde{t}}(x) = \bigcup_{n \geq 1} \tilde{t}(x^n) \)
and \( \sqrt{\Omega}(x) = \bigwedge_{n \geq 1} \Omega(x^n) \), for \( x \in \mathcal{R} \).

**Lemma 1.** Let \( \tilde{t} \Omega, \tilde{t}_1 \in \mathcal{H}(\mathcal{R}) \) be hybrid ideals in \( \mathcal{R} \). Then, the following assertions hold:

(i) \( \tilde{t} \Omega \ll \sqrt{\tilde{t}} \Omega \)
(ii) \( \tilde{t} \Omega \ll \tilde{t}_1 \Rightarrow \sqrt{\tilde{t}} \Omega \ll \sqrt{\tilde{t}_1} \)
(iii) \( \sqrt{\sqrt{\tilde{t} \Omega}} = \sqrt{\tilde{t}_1} \)

**Proof.**

(i) Let \( s \in \mathcal{R} \). Then, \( \sqrt{\tilde{t}}(s) = \bigcup_{k \geq 1} \tilde{t}(s^k) \supseteq \tilde{t}(s) \supseteq \tilde{t}(s) \), and \( \sqrt{\Omega}(s) = \bigwedge_{k \geq 1} \Omega(s^k) \leq \Omega(s) \). So \( \sqrt{\tilde{t} \Omega} \ll \sqrt{\tilde{t}} \Omega \).

(ii) Let \( s \in \mathcal{R} \). Then, \( \sqrt{\tilde{t}}(s) = \bigcup_{k \geq 1} \tilde{t}(s^k) \subseteq \bigcup_{k \geq 1} \tilde{t}_1(s^k) = \sqrt{\tilde{t}_1}(s) \), and \( \sqrt{\Omega}(s) = \bigwedge_{k \geq 1} \Omega(s^k) \geq \bigwedge_{k \geq 1} \gamma(s^k) = \sqrt{\tilde{t}_1}(s) \). So \( \sqrt{\tilde{t} \Omega} \ll \sqrt{\tilde{t}_1} \).

(iii) Let \( s \in \mathcal{R} \). Then,

\[
\sqrt{\tilde{t}}(s) = \bigcup_{r \geq 1} \sqrt{\tilde{t}}(s^r) = \bigcup_{r \geq 1} \bigcup_{m \geq 1} \tilde{t}(s^m) = \bigcup_{r \geq 1} \tilde{t}(s^r) = \sqrt{\tilde{t}}(s) \text{ and }
\]

\[
\sqrt{\Omega}(s) = \bigwedge_{r \geq 1} \sqrt{\Omega}(s^r) = \bigwedge_{r \geq 1} \bigwedge_{m \geq 1} \Omega(s^m) = \bigwedge_{r \geq 1} \Omega(s^r) = \sqrt{\Omega}(s).
\]

So, \( \sqrt{\sqrt{\tilde{t} \Omega}} = \sqrt{\tilde{t}_1} \). □
Theorem 2. Let \( \mathcal{R} \) be a commutative ring and \( I_\Omega \in \mathcal{H}(\mathcal{R}) \) a hybrid ideal in \( \mathcal{R} \). Then, \( \sqrt{I_\Omega} \) is a hybrid ideal in \( \mathcal{R} \).

Proof. Let \( v, u \in \mathcal{R} \). Then, for positive integers \( r, s \), we have

\[
\bigcap_{r \geq 1} \left\{ \sqrt{I(v)}, \sqrt{I(u)} \right\} = \bigcap_{r \geq 1} \left\{ \bigcap_{s \geq 1} \left\{ I(v), I(u) \right\} \right\},
\]

and

\[
\bigvee_{r \geq 1} \left\{ \sqrt{I(v)}, \sqrt{I(u)} \right\} = \bigvee_{r \geq 1} \left\{ \bigvee_{s \geq 1} \left\{ \Omega(v), \Omega(u) \right\} \right\}.
\]

Since \( \mathcal{R} \) is commutative, all the terms in \( (v + u)^{r+s} \) contain either \( v^r \) or \( u^s \) as a factor. Therefore, \( (v + u)^{r+s} = xv^r + yu^s \) for some \( x, y \in \mathcal{R} \). Now,

\[
\bigcap_{r \geq 1} \left\{ I(v^r), I(u^s) \right\} \subseteq \bigcap_{r \geq 1} \left\{ I(v^r) \cup I(x), I(u^s) \cup I(y) \right\} \\
\subseteq \bigcap_{r \geq 1} \left\{ I(xv^r), I(yu^s) \right\} \\
\subseteq \bigcap_{r \geq 1} \left\{ I(xv^r + yu^s) \right\} \\
\subseteq \bigcap_{r \geq 1} \left\{ I(v^r) \cup I(u^s) \right\} \\
= \sqrt{I(v^r) + I(u^s)}.
\]

Therefore, we have \( \bigcap \{ \sqrt{I(v)}, \sqrt{I(u)} \} \subseteq \sqrt{I(v^r + I(u^s))} \). Now, for a positive integer \( r \),

\[
\bigcup_{r \geq 1} \left\{ \sqrt{I(v)}, \sqrt{I(u)} \right\} = \bigcup_{r \geq 1} \left\{ \bigcup_{s \geq 1} \left\{ I(v), I(u) \right\} \right\} = \bigcup_{r \geq 1} \left\{ I(v^r) \cup I(u^s) \right\},
\]

and

\[
\bigwedge_{r \geq 1} \left\{ \sqrt{I(v)}, \sqrt{I(u)} \right\} = \bigwedge_{r \geq 1} \left\{ \bigwedge_{s \geq 1} \left\{ \Omega(v), \Omega(u) \right\} \right\} = \bigwedge_{r \geq 1} \left\{ \Omega(v^r) \cup \Omega(u^s) \right\}.
\]

Then,

\[
\bigcup_{r \geq 1} \left\{ I(v^r), I(u^s) \right\} \subseteq I(v^r) \cup I(u^s) \\
= \sqrt{I(v^r)} + \sqrt{I(u^s)} \\
\subseteq \bigcup_{k \geq 1} \left\{ I(v^r + yu^s) \right\} \\
= \sqrt{I(v^r + I(u^s))}.
\]

Therefore, \( \bigwedge_{r \geq 1} \left\{ \Omega(v^r), \Omega(u^s) \right\} \subseteq \Omega(v^r) \cup \Omega(u^s) \).

\[
\bigwedge_{r \geq 1} \left\{ \Omega(v^r), \Omega(u^s) \right\} \geq \Omega(v^r) \cup \Omega(u^s) \\
= \Omega((vu)^k) \\
\geq \bigwedge_{k \geq 1} \Omega((vu)^k) \\
= \sqrt{I(v^r + I(u^s))}.
\]
Thus, \( \bigcup \{ \sqrt[|i|]{v}, \sqrt[|i|]{u} \} \subseteq \sqrt[|i|]{vu} \) and \( \bigwedge \{ \sqrt[|i|]{v}, \sqrt[|i|]{u} \} \geq \sqrt[|i|]{vu} \) and hence \( \sqrt[i]{\Omega} \) is a hybrid ideal in \( \mathcal{R} \). \( \square \\

**Definition 12.** Let \( i_\Omega, j_\gamma \in \mathcal{H}(\mathcal{R}) \) be hybrid ideals in \( \mathcal{R} \). Then, the hybrid intrinsic product \( i_\Omega \otimes j_\gamma \) is the hybrid structure in \( \mathcal{R} \) stated as below: For \( x \in \mathcal{R} \), define

\[
(i_\Omega \otimes j_\gamma)(x) = \begin{cases} 
\bigcup_{x=\sum_{i=1}^{\infty} a_i w_i} \{ i(a_1), i(a_2), ..., i(a_m), j(w_1), j(w_2), ..., j(w_m) \} & \text{if } x = a_1 w_1 + a_2 w_2 + ... + a_m w_m \text{ for some } a_i, w_i \in \mathcal{R}, \\
\phi, & \text{otherwise}
\end{cases}
\]

\[
(\Omega \otimes \gamma)(x) = \begin{cases} 
\bigwedge_{x=\sum_{i=1}^{\infty} a_i w_i} \{ \Omega(a_1), \Omega(a_2), ..., \Omega(a_m), \gamma(w_1), \gamma(w_2), ..., \gamma(w_m) \} & \text{if } x = a_1 w_1 + a_2 w_2 + ... + a_m w_m \text{ for some } a_i, w_i \in \mathcal{R}, \\
1, & \text{otherwise}
\end{cases}
\]

It is evident that if \( \mathcal{R} \) is commutative, then \( i_\Omega \otimes j_\gamma \) is commutative.

Let \( i_\Omega, j_\gamma \in \mathcal{H}(\mathcal{R}) \) be hybrid ideals in \( \mathcal{R} \). Then, the hybrid intrinsic product \( i_\Omega \otimes j_\gamma \) is also a hybrid ideal, and whenever \( i_\Omega \) is a hybrid ideal, \( \forall \ n \geq 1, (i_\Omega)^n = i_\Omega \otimes i_\Omega \otimes ... \otimes i_\Omega(n \ \text{times}) \) is also a hybrid ideal.

**Theorem 3.** Let \( i_\Omega, j_\gamma \in \mathcal{H}(\mathcal{R}) \) be hybrid ideals in \( \mathcal{R} \). Then, \( \sqrt[|i|]{i_\Omega \otimes j_\gamma} = \sqrt[|i|]{i_\Omega} \otimes \sqrt[|i|]{j_\gamma} \).

**Proof.** Let \( s \in \mathcal{R} \) with \( s = a_1 w_1 + a_2 w_2 + ... + a_m w_m \), and \( a_i w_i \neq 0 \) for each \( i \in \{1, 2, ..., m\} \).

Then, \( i(a_1) \cap j(w_1) \subseteq i(a_1) \subseteq i(a_1 w_1) \) and \( \Omega(a_i) \geq \Omega(a_i) \geq \Omega(a_i w_i) \). Now,

\[
\bigcap \{ i(a_1), i(a_2), ..., i(a_m), j(w_1), j(w_2), ..., j(w_m) \} \subseteq \bigcap \{ i(a_1 w_1), i(a_2 w_2), ..., i(a_m w_m) \} \\
= \{ i(s) \}. \\
\]

\[
\bigvee \{ \Omega(a_1), \Omega(a_2), ..., \Omega(a_m), \gamma(w_1), \gamma(w_2), ..., \gamma(w_m) \} \geq \bigvee \{ \Omega(a_1 w_1), \Omega(a_2 w_2), ..., \Omega(a_m w_m) \} \\
= \Omega(s).
\]

So, \( (i_\Omega \otimes j_\gamma)(s) \subseteq (i_\Omega)(s) \) and \( (\Omega \otimes \gamma)(s) \geq (\Omega)(s) \). Hence \( i_\Omega \otimes j_\gamma \ll i_\Omega \). Similarly, we can show that \( i_\Omega \otimes j_\gamma \ll j_\gamma \). So, \( (i_\Omega \otimes j_\gamma)(s) \subseteq (i_\Omega)(s) \cap (j_\gamma)(s) = (i_\Omega \otimes j_\gamma)(s) \) and \( (\Omega \otimes \gamma)(s) \geq (\Omega)(s) \lor (\gamma)(s) = (\Omega \lor \gamma)(s) \) and hence \( i_\Omega \otimes j_\gamma \ll i_\Omega \otimes j_\gamma \).

Therefore, by Lemma 1(ii), \( \sqrt[i]{i_\Omega \otimes j_\gamma} \ll (\sqrt[i]{i_\Omega} \otimes \sqrt[i]{j_\gamma}) \).

Let \( s \in \mathcal{R} \). Then,

\[
\sqrt[i]{i_\Omega \otimes j_\gamma}(s) = \bigcup_{k \geq 1} (i_\Omega \otimes j_\gamma)(s^k) \\
\supseteq (i_\Omega \otimes j_\gamma)(s^{2n}) \\
= (i_\Omega \otimes j_\gamma)(s^n) \ \forall \ n \geq 1.
\]
\[
\sqrt{\Omega} \ast \gamma(s) = \bigwedge_{k \geq 1} (\Omega \ast \gamma)(s^k)
\]
\[
\leq (\Omega \ast \gamma)(s^{2n})
\]
\[
\leq \Omega(s^n) \lor \gamma(s^n)
\]
\[
= (\Omega \lor \gamma)(s^n) \quad \forall n \geq 1.
\]
Therefore, \( \sqrt[\Omega \ast \gamma]{\bigodot} \ll \sqrt[\Omega \ast \gamma]{\bigodot} \) which implies \( \sqrt[\Omega \ast \gamma]{\bigodot} \ll \sqrt[\Omega \ast \gamma]{\bigodot} \), by Lemma 1(ii). Hence,
\[
\sqrt[\Omega \ast \gamma]{\bigodot} = \sqrt[\Omega \ast \gamma]{\bigodot}.
\]
Now, by Lemma 1(iii), we have
\( \sqrt[\Omega \ast \gamma]{\bigodot} \ll \sqrt[\Omega \ast \gamma]{\bigodot} \) and \( \sqrt[\Omega \ast \gamma]{\bigodot} \ll \sqrt[\Omega \ast \gamma]{\bigodot} \) which imply \( \sqrt[\Omega \ast \gamma]{\bigodot} \ll \sqrt[\Omega \ast \gamma]{\bigodot} \).

Conversely, let \( v \in \mathcal{K} \). Then, for any two positive integers \( t, r \), we have
\[
(\sqrt[\Omega \ast \gamma]{\bigodot})(v) = \bigcap_{t \geq 1} \bigcup_{r \geq 1} (\hat{i}(v), \cup_{r \geq 1} (\hat{j}(v'))
\]
\[
= \bigcup_{t \geq 1} (\hat{i}(v) \cap \hat{j}(v'))
\]
\[
= \hat{i}(v) \lor \gamma(v')
\]
which implies \( (\sqrt[\Omega \ast \gamma]{\bigodot})(v) \leq (\sqrt[\Omega \ast \gamma]{\bigodot})(v) \).

Additionally,
\[
\Omega(v') \lor \gamma(v') \geq \Omega(v') \lor \gamma(v') = (\Omega \lor \gamma)(v')
\]
\[
geq \bigwedge_{k \geq 1} (\Omega \lor \gamma)(v^k)
\]
\[
= \sqrt{\Omega \lor \gamma}(v)
\]
which implies \( (\sqrt[\Omega \lor \gamma]{\bigodot})(v) \geq (\sqrt[\Omega \lor \gamma]{\bigodot})(v) \).

So, \( \sqrt[\Omega \ast \gamma]{\bigodot} \ll \sqrt[\Omega \ast \gamma]{\bigodot} \). Hence, \( \sqrt[\Omega \ast \gamma]{\bigodot} = \sqrt[\Omega \ast \gamma]{\bigodot} \).

**Corollary 1.** Let \( \sqrt[\Omega \ast \gamma]{\bigodot} \in \mathcal{H}(\mathcal{R}) \) be a hybrid ideal. Then, \( \sqrt[\Omega \ast \gamma]{\bigodot} = \sqrt[\Omega \ast \gamma]{\bigodot} \) for all \( n \geq 1 \), where \( \sqrt[\Omega \ast \gamma]{\bigodot} = \sqrt[\Omega \ast \gamma]{\bigodot} \) (n - times).

**Proof.** Taking \( \sqrt[\Omega \ast \gamma]{\bigodot} \) in Theorem 3, we obtain \( \sqrt[\Omega \ast \gamma]{\bigodot} \). So, by induction principle, \( \sqrt[\Omega \ast \gamma]{\bigodot} = \sqrt[\Omega \ast \gamma]{\bigodot} \) for all \( n \geq 1 \).

**Corollary 2.** Let \( \sqrt[\Omega \ast \gamma]{\bigodot} \in \mathcal{H}(\mathcal{R}) \) be hybrid ideals in \( \mathcal{R} \). If \( \sqrt[\Omega \ast \gamma]{\bigodot} \ll \sqrt[\Omega \ast \gamma]{\bigodot} \) for some \( k \geq 1 \), then \( \sqrt[\Omega \ast \gamma]{\bigodot} \ll \sqrt[\Omega \ast \gamma]{\bigodot} \).
Proof. By applying Corollary 1, $\sqrt{R} = \sqrt{\Omega}$ for all $k \geq 1$. Then, by Lemma 1(ii), $\sqrt{R} \gg \sqrt{\Omega} = \sqrt{\Omega}$. □

Definition 13. For any $\bar{i}, \bar{j} \in H(\mathcal{R})$, the hybrid sum of $\bar{i}$ and $\bar{j}$, represented by $\bar{i} \oplus \bar{j}$, is defined as $\bar{i} \oplus \bar{j} = (\bar{i} \cup \bar{j}, \Omega + \gamma) \in H(\mathcal{R})$, where

$$(\bar{i} \cup \bar{j})(a') = \bigcup_{\bar{a}' = s' + s''} \{\bar{i}(s') \cap \bar{j}(s'')\} \quad \text{if } \exists s', s'' \in \mathcal{R} \exists a' = s' + s''$$

and

$$(\Omega + \gamma)(a') = \bigcup_{\bar{a}' = s' + s''} \{\Omega(s') \cup \gamma(s'')\} \quad \text{if } \exists s', s'' \in \mathcal{R} \exists a' = s' + s''$$

for all $a' \in \mathcal{R}$.

Lemma 2. Let $\bar{i} \Omega, \bar{j} \gamma \in H(\mathcal{R})$ be hybrid ideals in $\mathcal{R}$. Then, $\bar{i} \Omega \oplus \bar{j} \gamma$ is a hybrid ideal in $\mathcal{R}$.

Proof. As $\bar{i} \Omega, \bar{j} \gamma \in H(\mathcal{R})$ are hybrid subgroups of commutative group $(\mathcal{R}, +)$, $\bar{i} \Omega \oplus \bar{j} \gamma$ is also a hybrid subgroup of commutative group $(\mathcal{R}, +)$. Let $v, u \in \mathcal{R}$. If $v = a + b$ for $a, b \in \mathcal{R}$, then

$$(\bar{i} \cup \bar{j})(vu) = \bigcup_{vu = (a + b)u} \{\bar{i}(au) \cap \bar{j}(bu)\}$$

$$\supseteq \bigcup_{vu = (a + b)u} \{\bar{i}(a) \cap \bar{j}(b)\}$$

$$= \bigcup_{u = a + b} \{\bar{i}(a) \cap \bar{j}(b)\}$$

$$= (\bar{i} \cup \bar{j})(v),$$

$$(\Omega + \gamma)(vu) = \bigcup_{vu = (a + b)u} \{\Omega(au) \cup \gamma(bu)\}$$

$$\supseteq \bigcup_{vu = (a + b)u} \{\Omega(a) \cup \gamma(b)\}$$

$$= \bigcup_{u = a + b} \{\Omega(a) \cup \gamma(b)\}$$

$$= (\Omega + \gamma)(v).$$

In addition,

$$(\bar{i} \cup \bar{j})(vu) = \bigcup_{vu = (r + s)u} \{\bar{i}(ru) \cap \bar{j}(sv)\}$$

$$\supseteq \bigcup_{vu = (r + s)u} \{\bar{i}(r) \cap \bar{j}(s)\}$$

$$= \bigcup_{u = r + s} \{\bar{i}(r) \cap \bar{j}(s)\}$$

$$= (\bar{i} \cup \bar{j})(u),$$

$$(\Omega + \gamma)(vu) = \bigcup_{vu = (r + s)u} \{\Omega(ri) \cup \gamma(vs)\}$$

$$\supseteq \bigcup_{vu = (r + s)u} \{\Omega(r) \cup \gamma(s)\}$$

$$= \bigcup_{u = r + s} \{\Omega(r) \cup \gamma(s)\}$$

$$= (\Omega + \gamma)(u).$$
Thus, $i_\Omega \oplus j_\gamma$ is a hybrid ideal in $\mathcal{R}$. \qed

**Theorem 4.** Suppose $i_\Omega, j_\gamma, f_\nu, p_\mu \in \mathcal{H}(\mathcal{R})$. If $f_\nu \ll j_\gamma$ and $i_\Omega \ll p_\mu$, then $i_\Omega \oplus j_\gamma \ll f_\nu \oplus p_\mu$.

**Proof.** Suppose $w \in \mathcal{R}$. If $w \neq s + r$ for $r, s \in \mathcal{R}$, then clearly $i_\Omega \oplus j_\gamma \ll f_\nu \oplus p_\mu$.

Assume $w = s + r$ for some $s, r \in \mathcal{R}$. Then,

$$(j + i)(w) = \bigcup_{w = s + r} \{j(s) \cap i(r)\}$$

$$\subseteq \bigcup_{w = s + r} \{f(s) \cap p(r)\}$$

$$= (j + p)(w).$$

Additionally,

$$(\Omega + \gamma)(w) = \bigwedge_{w = s + r} \Omega(s) \lor \gamma(r)$$

$$\geq \bigwedge_{w = s + r} \nu(s) \lor \mu(r)$$

$$= (\nu + \mu)(w).$$

Therefore, $i_\Omega \oplus j_\gamma \ll f_\nu \oplus p_\mu$. \qed

**Theorem 5.** Let $\mathcal{R}$ be a commutative ring and $i_\Omega, j_\gamma \in \mathcal{H}(\mathcal{R})$ be hybrid ideals in $\mathcal{R}$. Then,

$$\sqrt{i_\Omega \oplus j_\gamma} = \sqrt{\sqrt{i_\Omega} \oplus \sqrt{j_\gamma}} \gg \sqrt{i_\Omega} \oplus \sqrt{j_\gamma}.$$

**Proof.** By Lemma 1(i), we have $i_\Omega \ll \sqrt{i_\Omega}$ and $j_\gamma \ll \sqrt{j_\gamma}$. Again, by Lemma 1(i) and Theorem 4, we have $\sqrt{i_\Omega \oplus j_\gamma} \ll \sqrt{\sqrt{i_\Omega} \oplus \sqrt{j_\gamma}}$.

On the other hand, let $x \in \mathcal{R}$. Then,

$$(\sqrt{i} + \sqrt{j})(x) = \bigcup_{x = v + u} \{\sqrt{i}(v) \cap \sqrt{j}(u)\}$$

$$= \bigcup_{x = v + u} \{\bigcup_{m \geq 1} i(v^m) \cap \bigcup_{n \geq 1} j(u^n)\}$$

$$= \bigcup_{x = v + u} \{\bigcup_{m \geq 1, n \geq 1} \{i(v^m) \cap j(u^n)\}\},$$

and

$$(\sqrt{\Omega} + \sqrt{\gamma})(x) = \bigwedge_{x = v + u} \{\sqrt{\Omega}(v) \lor \sqrt{\gamma}(u)\}$$

$$= \bigwedge_{x = v + u} \{\bigwedge_{m \geq 1} \Omega(v^m) \lor \bigwedge_{n \geq 1} \gamma(u^n)\}$$

$$= \bigwedge_{x = v + u} \{\bigwedge_{m \geq 1, n \geq 1} \{\Omega(v^m) \lor \gamma(u^n)\}\}.$$
\[
\Omega(u^m) \vee \gamma(v^n) \geq \Omega(tu^m) \vee \gamma(rv^n) \\
\geq (\Omega + \gamma)(tu^m + rv^n) \\
= (\Omega + \gamma)(\kappa + v) \\
\geq \bigwedge_{\alpha \geq 1} (\Omega + \gamma)(\kappa^\alpha) \\
= \sqrt{\Omega + \gamma}(k).
\]

Thus, \((\sqrt{i} + \sqrt{j})(k) \subseteq \sqrt{i} + \sqrt{j}(k)\) and \((\sqrt{\Omega} + \sqrt{\gamma})(k) \geq \sqrt{\Omega + \gamma}(k)\).

Hence, \(\sqrt{i} \oplus \sqrt{j} \ll \sqrt{i} \oplus \sqrt{j}\). By Lemma 1, \(\sqrt{i} \oplus \sqrt{j} \ll \sqrt{i} \oplus \sqrt{j}\). Therefore, \(\sqrt{i} \oplus \sqrt{j} = \sqrt{i} \oplus \sqrt{j}\). \(\square\)

**Definition 14.** Consider the function \(\Psi : \mathcal{S} \to \mathcal{T}\). A hybrid structure \(\tilde{i}_\Omega\) in \(\mathcal{S}\) is stated as \(\Psi\)-invariant whenever \(\Psi(u) = \Psi(v) \Rightarrow \tilde{i}_\Omega(u) = \tilde{i}_\Omega(v)\) and \(\Omega(u) = \Omega(v)\) for any \(u, v \in \mathcal{S}\).

**Theorem 6.** Consider the ring homomorphism \(\Psi : \mathcal{R} \to \mathcal{S}\). If \(\tilde{i}_\Omega\) and \(\tilde{j}_\gamma\) are hybrid ideals in \(\mathcal{R}\) and \(\mathcal{S}\), respectively, then the following assertions hold:

(i) \(\sqrt{\Psi^{-1}(\tilde{j}_\gamma)} = \Psi^{-1}(\sqrt{\tilde{j}_\gamma})\).

(ii) \(\sqrt{\Psi(\tilde{i}_\Omega)} = \Psi(\sqrt{\tilde{i}_\Omega})\), provided \(\Psi\) is onto and \(\tilde{i}_\Omega\) is \(\Psi\)-invariant.

**Proof.**

(i) Let \(v \in \mathcal{R}\). Then,

\[
\sqrt{\Psi^{-1}(\tilde{j}_\gamma)(v)} = \bigcup_{r \geq 1} (\Psi^{-1}(\tilde{j}_\gamma))(v') \\
= \bigcup_{r \geq 1} (\tilde{j}(\Psi(v')) \\
= \bigcup_{r \geq 1} (\tilde{j}(\Psi(v)))^r \\
= \sqrt{\tilde{j}(\Psi(v))} \\
= (\Psi^{-1}(\sqrt{\tilde{j}_\gamma}))(v),
\]

\[
\sqrt{\Psi^{-1}(\tilde{j}_\gamma)(v)} = \bigcup_{r \geq 1} (\Psi^{-1}(\tilde{j}_\gamma))(v') \\
= \bigcup_{r \geq 1} (\gamma(\Psi(v')) \\
= \bigcup_{r \geq 1} (\gamma(\Psi(v)))^r \\
= \sqrt{\gamma(\Psi(v))} \\
= (\Psi^{-1}(\sqrt{\gamma}))\Psi(v).
\]

So, \(\sqrt{\Psi^{-1}(\tilde{j}_\gamma)} = \Psi^{-1}(\sqrt{\tilde{j}_\gamma})\).

(ii) Since \(\Psi\) is onto, for \(v \in \mathcal{S}\), \(\Psi(u) = v\) for some \(u \in \mathcal{R}\). In addition, \(\Psi^{-1}(\tilde{i}_\Omega) = \tilde{i}_\Omega\), as \(\tilde{i}_\Omega\) is \(\Psi\)-invariant. Now, for some integer \(m\), we have
\[
\sqrt{\Psi(l)}(v) \subseteq (\Psi(l))(v^m) \\
= (\Psi(l))(\Psi(u^m)) \\
= \Psi^{-1}(\Psi(l))(u^m) \\
= i(u^m) \\
\subseteq \sqrt{i}(u) \\
\subseteq \bigcup_{t \in \Psi^{-1}(v)} \sqrt{i}(t) \\
= (\Psi(\sqrt{i}))(v),
\]

\[
(\sqrt{\Psi(\Omega)})(v) > (\Psi(\Omega))(v^m) \\
= (\Psi(\Omega))(\Psi(u^m)) \\
= \Psi^{-1}(\Psi(\Omega))(u^m) \\
= \Omega(u^m) \\
\geq \sqrt{\Omega}(u) \\
\geq \bigwedge_{t \in \Psi^{-1}(v)} \sqrt{\Omega}(t) \\
= (\Psi(\sqrt{\Omega}))(v).
\]

So, \( \sqrt{\Psi(l)}(v) \ll \sqrt{\Psi(l)} \).

On the other hand, \( \exists u_0 \in \mathcal{R} \ni \Psi(u_0) = v \), and \( (\Psi(\sqrt{i}))(v) \subset \sqrt{i}(u_0) = \bigcup_{n \geq 1} i(u_0^n) \) and

\[
(\Psi(\sqrt{\Omega}))(v) > \sqrt{\Omega}(u_0) = \bigwedge_{n \geq 1} \Omega(u_0^n). \]

Then, \( \exists \) a positive integer \( k \ni \)

\[
(\Psi(\sqrt{i}))(v) \subseteq \bigcup_{w \in \Psi^{-1}(v^k)} \sqrt{i}(w) \\
= (\Psi(\sqrt{i}))(v^k) \\
\subseteq \bigcup_{n \geq 1} (\Psi(\sqrt{i}))(v^n) \\
= \sqrt{\Psi(i)}(v)
\]

and

\[
(\Psi(\sqrt{\Omega}))(v) \geq \bigwedge_{w \in \Psi^{-1}(v^k)} \Omega(w) \\
\geq \bigwedge_{n \geq 1} (\Psi(\Omega))(v^n) \\
= \sqrt{\Psi(\Omega)}(v)
\]

Thus, \( \Psi(\sqrt{l}) \ll \sqrt{\Psi(l)} \) and hence \( \sqrt{\Psi(l)} = \Psi(\sqrt{\Omega}) \). \( \square \)

**Theorem 7.** Consider \( \Psi : \mathcal{R} \to \mathcal{S} \) as the ring homomorphism. If \( \sqrt{i} \) is a hybrid left (resp. right, ideal) ideal in \( \mathcal{S} \), then \( \Psi^{-1}(\sqrt{i}) \) is a hybrid left (resp. right, ideal) in \( \mathcal{R} \).

**Proof.** Let \( c, v \in \mathcal{R} \) and \( i \) be a hybrid left ideal in \( \mathcal{S} \). Then,
Consider \( R \) hybrid left (resp., right) ideal in \( \Psi^{-1}(\Psi(c - v)) \).

\[
\Psi^{-1}(\tilde{\iota}(c - v)) = \tilde{\iota}(\Psi(c - v))
\geq \tilde{\iota}(\Psi(c)) \cap \tilde{\iota}(\Psi(v))
= \Psi^{-1}(\tilde{\iota}(c)) \cap \Psi^{-1}(\tilde{\iota}(v)).
\]

\[
\Psi^{-1}(\Omega(c - v)) = \Omega(\Psi(c - v))
\geq \Omega(\Psi(c)) \cap \Omega(\Psi(v))
= \Psi^{-1}(\Omega(c)) \cap \Psi^{-1}(\Omega(v)).
\]

Additionally,

\[
\Psi^{-1}(\tilde{\iota}(cv)) = \tilde{\iota}(\Psi(cv))
\geq \tilde{\iota}(\Psi(c)) \cap \tilde{\iota}(\Psi(v))
= \Psi^{-1}(\tilde{\iota}(v)).
\]

So, \( \Psi^{-1}(\tilde{\iota}_\Omega) \in \mathcal{H}(\mathcal{R}) \) is hybrid left ideal. \( \Box \)

**Theorem 8.** Consider \( \Psi : \mathcal{R} \to \mathcal{J} \) as onto ring homomorphism. If \( \tilde{j}_\Omega \in \mathcal{H}(\mathcal{R}) \) is a \( \Psi \)-invariant hybrid left (resp., right) ideal in \( \mathcal{R} \), then \( \Psi(\tilde{j}_\Omega) \) is a hybrid left (resp., right) ideal in \( \mathcal{J} \).

**Proof.** Let \( w_1, w_2 \in \mathcal{J} \). Then, \( \{ k : k \in \Psi^{-1}(w_1 - w_2) \} \supseteq \{ u_1 - r_2 : u_1 \in \Psi^{-1}(w_1), r_2 \in \Psi^{-1}(w_2) \} \).

Now,

\[
\Psi(\tilde{j})(w_1 - w_2) = \bigcup_{k \in \Psi^{-1}(w_1 - w_2)} \tilde{j}(k)
\supseteq \bigcup_{u_1 \in \Psi^{-1}(w_1), r_2 \in \Psi^{-1}(w_2)} \tilde{j}(u_1 - r_2)
\supseteq \bigcup_{u_1 \in \Psi^{-1}(w_1), r_2 \in \Psi^{-1}(w_2)} \{ j(u_1) \cap \tilde{j}(r_2) \}
= \{ \bigcup_{u_1 \in \Psi^{-1}(w_1)} j(u_1) \} \cap \{ \bigcup_{r_2 \in \Psi^{-1}(w_2)} \tilde{j}(r_2) \}
= \Psi(\tilde{j})(w_1) \cap \Psi(\tilde{j})(w_2),
\]

\[
\Psi(\gamma)(w_1 - w_2) = \bigwedge_{k \in \Psi^{-1}(w_1 - w_2)} \gamma(k)
\leq \bigwedge_{u_1 \in \Psi^{-1}(w_1), r_2 \in \Psi^{-1}(w_2)} \gamma(u_1 - r_2)
\leq \bigwedge_{u_1 \in \Psi^{-1}(w_1), r_2 \in \Psi^{-1}(w_2)} \{ \gamma(u_1) \cap \gamma(r_2) \}
= \{ \bigwedge_{u_1 \in \Psi^{-1}(w_1)} \gamma(u_1) \} \cap \{ \bigwedge_{r_2 \in \Psi^{-1}(w_2)} \gamma(r_2) \}
= \Psi(\gamma)(w_1) \cap \Psi(\gamma)(w_2),
\]
Theorem 9. Let $\chi$ only if semiprime ideal. Proof. By Theorem 3, hybrid semiprime ideal. Let Lemma 3. $R$ semiprime ideal in Definition 15. A hybrid ideal. Throughout this section, unless otherwise stated, $R$ rings in terms of hybrid semiprime ideals. We also obtain an equivalent assertion for the hybrid semiprime ideal. 4. Hybrid Semiprime Ideals in Commutative Rings We discuss some properties of hybrid semiprime ideals of a ring and we characterise rings in terms of hybrid semiprime ideals. We also obtain an equivalent assertion for the hybrid semiprime ideal. Throughout this section, unless otherwise stated, $R$ denotes a commutative ring.

Definition 15. A hybrid ideal $i_\Omega \in \mathcal{H}(R)$ in $R$ is hybrid semiprime whenever $\sqrt{i_\Omega} = i_\Omega$.

Example 2. Let $\mathcal{H}$ be a semiprime ideal of $R$. Let $i_\Omega$ in $\mathcal{H}(R)$ over $\mathcal{W} = [0,1]$ be given by $i(k) = \begin{cases} [0,0.5] & \text{if } k \in \mathcal{H} \\ [0,0.1] & \text{if } k \notin \mathcal{H} \end{cases}$ and a mapping $\Omega : R \to I$ be constant. Then, $i_\Omega$ is a hybrid semiprime ideal in $R$.

Lemma 3. Let $i_\Omega, i_\gamma \in \mathcal{H}(R)$ be hybrid semiprime ideals in $R$. Then, $i_\Omega \cap i_\gamma \in \mathcal{H}(R)$ is a hybrid semiprime ideal.

Proof. By Theorem 3, $\sqrt{i_\Omega \cap i_\gamma} = \sqrt{i_\Omega} \cap \sqrt{i_\gamma} = i_\Omega \cap i_\gamma$. Therefore, $i_\Omega \cap i_\gamma$ is a hybrid semiprime ideal.

Theorem 9. For any $Q \in \mathcal{P}(R)$ and $i_\Omega \in \mathcal{H}(R)$, we have that $Q$ is a semiprime ideal if and only if $\chi_Q(i_\Omega)$ is a hybrid semiprime ideal.

Proof. Assume $Q$ is a semiprime ideal. Let $u \in R$. If $u^2 \in Q$, then $u \in Q$. Thus, $\chi_Q(i_\Omega)(u) = \mathcal{W} = \chi_Q(i_\Omega)(u^2)$ and $\chi_Q(\Omega)(u) = \chi_Q(\Omega)(u_2)$ which imply $\sqrt{\chi_Q(i_\Omega)} = \chi_Q(i_\Omega)$. If $u^2 \notin Q$, then $\chi_Q(i_\Omega)(u_2) = \phi \subseteq \chi_Q(i_\Omega)(u)$ and $\chi_Q(\Omega)(u_2) = 1 \geq \chi_Q(\Omega)(u)$ which imply $\sqrt{\chi_Q(i_\Omega)} = \chi_Q(i_\Omega)$. So, $\chi_Q(i_\Omega)$ is a hybrid semiprime ideal.
Conversely, suppose \( \chi_{Q}(\tilde{i}_Ω) \) is a hybrid semiprime ideal and \( u^2 \in Q \). Then, \( \chi_{Q}(\tilde{i})(u) \supseteq \chi_{Q}(\tilde{i})(u^2) = \psi \) and \( \chi_{Q}(\tilde{i})(u) \subseteq \chi_{Q}(\Omega)(u) = 0 \) which imply \( u \in Q \). Hence \( Q \) is a semiprime ideal. \( \square \)

**Theorem 10.** Let \( \tilde{i}_Ω \in \mathcal{H}(\mathcal{R}) \) be a hybrid ideal in \( \mathcal{R} \). Then, the following assertions are equivalent:

(i) \( \tilde{i}_Ω \) is a hybrid semiprime ideal,

(ii) \( \tilde{i}(u^n) = \tilde{i}(u) \) and \( \Omega(u^n) = \Omega(u) \forall u \in \mathcal{R}, \forall n \geq 1 \),

(iii) \( \tilde{i}(u^n) = \tilde{i}(u) \) and \( \Omega(u^n) = \Omega(u) \forall u \in \mathcal{R} \) with some \( m \geq 2 \).

**Proof.** (i) \( \Rightarrow \) (ii) Let \( n_1 \) be any positive integer and \( u \in \mathcal{R} \). Then,

\[
\tilde{i}(u) \subseteq \tilde{i}(u^{n_1}) \subseteq \bigcup_{n_1 \geq 1} \tilde{i}(u^{n_1}) = \sqrt{\tilde{i}(u)} = \tilde{i}(u)
\]

and

\[
\Omega(u) \supseteq \Omega(u^{n_1}) \supseteq \bigwedge_{n_1 \geq 1} \Omega(u^{n_1}) = \sqrt{\Omega(u)} = \Omega(u)
\]

\[
\Rightarrow \Omega(u^{n_1}) = \Omega(u).
\]

(ii) \( \Rightarrow \) (iii) Evident.

(iii) \( \Rightarrow \) (i) Suppose \( w \in \mathcal{R} \) with \( m < s \). Then, for some positive integer \( k, m^k > s \). Now, for all \( s > m \)

\[
i(w) = i(w^m) = i(w^{m^k}) = \ldots = i(w^s) \supseteq i(w^s) = i(w),
\]

\[
\Omega(w) = \Omega(w^m) = \Omega(w^{m^k}) = \ldots = \Omega(w^s) \supseteq \Omega(w^s) = \Omega(w)
\]

\[
\Rightarrow i(w) = i(w^s), \Omega(w) = \Omega(w^s).
\]

Thus, \( \sqrt{i}(w) = \bigcup_{s \geq 1} i(w^s) = i(w) \) and \( \sqrt{\Omega}(w) = \bigwedge_{s \geq 1} \Omega(w^s) = \Omega(w) \), and hence \( \sqrt{\tilde{i}_Ω} = \tilde{i}_Ω \). \( \square \)

**Theorem 11.** If \( \tilde{i}_Ω \in \mathcal{H}(\mathcal{R}) \) is a hybrid ideal in \( \mathcal{R} \), then \( \sqrt{\tilde{i}_Ω} = \bigcap \{ j_γ : i_Ω \ll j_γ \} \), \( j_γ \) is a hybrid semiprime ideal in \( \mathcal{R} \).

**Proof.** Let \( \{ j_γ \}_i \in \mathcal{H}(\mathcal{R}) \) be hybrid semiprime ideals in \( \mathcal{R} \) containing \( \tilde{i}_Ω \). Let \( \sigma = \bigcap j_γ \). By Lemma 1(i), \( \sqrt{\tilde{i}_Ω} \ll \sqrt{j_γ} \ll \tilde{i}_Ω \forall \gamma \), and hence \( \sqrt{\tilde{i}_Ω} \ll \bigcap j_γ = \sigma \). So, \( \sqrt{\tilde{i}_Ω} \) is hybrid semiprime, by Theorem 2. In other words, \( \sqrt{\tilde{i}_Ω} \) is one of the \( j_γ \). So, \( \sigma \ll \sqrt{\tilde{i}_Ω} \) and hence \( \sigma = \sqrt{\tilde{i}_Ω} \). \( \square \)

**Theorem 12.** Let \( \tilde{i}_Ω \in \mathcal{H}(\mathcal{R}) \). Then, the equivalent assertions are as follows:

(i) Every hybrid ideal \( \tilde{i}_Ω \) in \( \mathcal{R} \) is a hybrid semiprime ideal,

(ii) \( \mathcal{R} \) is regular.

**Proof.** (i) \( \Rightarrow \) (ii) Suppose \( X \) is an ideal of \( \mathcal{R} \). Then, \( \chi_X(\tilde{i}_Ω) \) is a hybrid semiprime ideal in \( \mathcal{R} \). By Theorem 9, \( \mathcal{R} \) is semiprime. Thus, \( \mathcal{R} \) is regular by Theorem 1.

(ii) \( \Rightarrow \) (i) Assume \( \mathcal{R} \) is regular with \( x \in \mathcal{R} \). Then, \( \exists r \in \mathcal{R} \ni x = xrx = x^2r \) which gives \( x = x^n + 1 \forall n \geq 2 \). Let \( i_Ω \in \mathcal{H}(\mathcal{R}) \) be a hybrid ideal. Then, \( i(x) = i(x^m + 1) \supseteq i(x^m) \supseteq i(x) \) and \( \Omega(x) = \Omega(x^m + 1) \supseteq \Omega(x^m) \leq \Omega(x) \) which imply \( i(x^m) = \tilde{i}(x) \) and \( \Omega(x^m) = \Omega(x) \forall x \in \mathcal{R} \), \( n \geq 1 \). Therefore, by Theorem 10, \( i_Ω \) is hybrid semiprime. \( \square \)

**Theorem 13.** Consider the ring homomorphism \( \Psi : \mathcal{R} \rightarrow \mathcal{S} \). If \( \tilde{i}_Ω \in \mathcal{H}(\mathcal{R}) \) and \( j_γ \in H(\mathcal{S}) \) are hybrid semiprime in \( \mathcal{R} \) and \( \mathcal{S} \), respectively, then we have
(i) $Ψ^{-1}(j_γ)$ is hybrid semiprime in $R$.
(ii) $Ψ(i_Ω)$ is hybrid semiprime in $S$, provided $Ψ$ is onto and $i_Ω$ is $Ψ$-invariant.

Proof.
(i) By Theorem 6, we can have $\sqrt{Ψ^{-1}(j_γ)} = Ψ^{-1}(\sqrt{j_γ}) = Ψ^{-1}(j_γ)$, and so $Ψ^{-1}(j_γ)$ is hybrid semiprime ideal in $R$.
(ii) By Theorem 6, we can have $\sqrt{Ψ(i_Ω)} = Ψ(\sqrt{i_Ω}) = Ψ(i_Ω)$, and so $Ψ(i_Ω)$ is hybrid semiprime ideal in $S$.

5. Conclusions

We developed an understanding of how hybrid nil radicals, hybrid ideals, hybrid products, and hybrid semiprime ideals operate in rings and discussed several of their properties. We have shown that for any hybrid ideal, the hybrid radical of the hybrid intersection is equivalent to the hybrid intersection of the hybrid radical. We also obtained an equivalent assertion for the hybrid radical of the hybrid intrinsic product, as well as a hybrid semiprime ideal. A future research project explores hybrid primes and hybrid quasi-primes in algebraic structures and establishes their structural properties.

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References
1. Zadeh, L.A. Fuzzy sets. Inform. Control 1965, 8, 338–353. [CrossRef]
2. Rosenfeld, A. Fuzzy groups. J. Math. Anal. Appl. 1971, 35, 512–517. [CrossRef]
3. Kuroki, N. Fuzzy bi-ideals in semigroups. Comment. Math. Univ. St. Paul. 1980, 28, 17–21.
4. Kuroki, N. On fuzzy ideals and fuzzy bi-ideals in semigroups. Fuzzy Sets Syst. 1981, 5, 203–215. [CrossRef]
5. Kuroki, N. Fuzzy semiprime ideals in semigroups. Fuzzy Sets Syst. 1982, 8, 71–79. [CrossRef]
6. Liu, W.J. Fuzzy invariant subgroups and fuzzy ideals. Fuzzy Sets Syst. 1982, 8, 133–139. [CrossRef]
7. Mukherjee, T.K.; Sen, M.K. On fuzzy ideals of a ring1. Fuzzy Sets Syst. 1987, 21, 99–104. [CrossRef]
8. Swamy, U.M.; Swamy, K.L.N. Fuzzy prime ideals of rings. J. Math. Anal. 1988, 134, 94–103. [CrossRef]
9. Malik, D.S.; Mordeson, J.N. Fuzzy Maximal, Radical, and Primary ideals of a ring. Inform. Sci. 1991, 55, 151–165. [CrossRef]
10. Kumbhojkar, H.V.; Bapat, M.S. On prime and primary fuzzy ideals and their radicals. Fuzzy Sets Syst. 1993, 53, 203–216. [CrossRef]
11. Gupta, K.C; Manoj, K.K. The nil radical of a fuzzy ideal. Fuzzy Sets Syst. 1993, 59, 87–93. [CrossRef]
12. Kumar, R. Fuzzy nil radicals and fuzzy primary ideals. Fuzzy Sets Syst. 1991, 43, 81–93. [CrossRef]
13. Kumar, R. Fuzzy cosets and fuzzy radicals. Fuzzy Sets Syst. 1992, 46, 261–265. [CrossRef]
14. Sidky, F.I.; Khatab, S.A. Nil radical of fuzzy ideal. Fuzzy Sets Syst. 1992, 47, 117–120. [CrossRef]
15. Abou-Zaid, S. On fuzzy subnear-rings and ideals. Fuzzy Sets Syst. 1991, 44, 139–146. [CrossRef]
16. Molodtsov, D. Soft set theory-First results. Comput. Math. Appl. 1999, 37, 19–31. [CrossRef]
17. Jun, Y.B.; Song, S.Z.; Muthuudi, G. Hybrid structures and applications. Ann. Commun. Math. 2018, 11, 11–25.
18. Anis, S.; Khan, M.; Jun, Y.B. Hybrid ideals in semigroups. Cogent Math. 2017, 4, 1352117. [CrossRef]
19. Elavarasan, B.; Jun, Y.B. Regularity of semigroups in terms of hybrid ideals and hybrid bi-ideals. *Kragujev. J. Math.* 2022, 46, 857–864.

20. Porselvi, K.; Elavarasan, B. On hybrid interior ideals in semigroups. *Probl. Anal. Issues Anal.* 2019, 8, 137–146. [CrossRef]

21. Porselvi, K.; Elavarasan, B.; Jun, Y.B. On hybrid interior ideals in ordered semigroups. *New Math. Nat. Comput.* 2022, 18, 1–8. [CrossRef]

22. Porselvi, K.; Muhiuddin, G.; Elavarasan, B.; Jun, Y.B.; Catherine Grace John, J. Hybrid ideals in an AG-groupoid. *New Math. Nat. Comput.* 2022, preprint. [CrossRef]

23. Elavarasan, B.; Muhiuddin, G.; Porselvi, K.; Jun, Y.B. Hybrid structures applied to ideals in near-rings. *Complex Intell. Syst.* 2021, 7, 1489–1498. [CrossRef]

24. Muhiuddin, G.; Catherine Grace John, J.; Elavarasan, B.; Jun, Y.B.; Porselvi, K. Hybrid structures applied to modules over semirings. *J. Intell. Fuzzy Syst.* 2022, 42, 2521–2531. [CrossRef]

25. Muhiuddin, G.; Catherine Grace John, J.; Elavarasan, B.; Porselvi, K.; Al-Kadi, D. Properties of k-hybrid ideals in ternary semiring. *J. Intell. Fuzzy Syst.* 2022, preprint. [CrossRef]

26. Jan, N.; Ullah, K.; Mahmood, T.; Garg, H.; Davvaz, B.; Saeid, A.B.; Broumi, S. Some Root Level Modifications in Interval Valued Fuzzy Graphs and Their Generalizations Including Neutrosophic Graphs. *Mathematics* 2019, 7, 72. [CrossRef]

27. Muhiuddin, G.; Al-Kadi, D. Interval valued m-polar fuzzy BCK/BCI-algebras. *Int. J. Comput. Intell. Syst.* 2021, 14, 1014–1021. [CrossRef]

28. Al-Kadi, D.; Muhiuddin, G. Bipolar fuzzy BCI-implicative ideals of BCI-algebras. *Ann. Commun. Math.* 2020, 3, 88–96.

29. Said, B.; Ullah, K.; Bakali, A.; Talea, M.; Singh, P.K.; Mahmood, T.; Smarandache, F.; Bahnasse, A.; Kumar, S.; de Oliveira, A. A Novel System and Method for Telephone Network Planning Based on Neutrosophic Graph; Version 1.0; Global Journal of Computer Science and Technology, E Network, Web & Security: Framingham, MA, USA, 2018; Volume 18.

30. Mahmood, T. A Novel Approach toward Bipolar Soft Sets and Their Applications. *J. Math.* 2020, 2020, 4690808. [CrossRef]

31. Mahmood T.; Ali, Z. Entropy Measure and TOPSIS Method Based on Correlation Coefficient using Complex q-Rung Orthopair Fuzzy Information and Its Application to Multiple Attribute Decision Making. *Soft Comput.* 2021, 25, 1249–1275. [CrossRef]

32. Mahmood, T.; Mehmoond, F.; Khan, Q. Some Generalized Aggregation Operators for Cubic Hesitant Fuzzy Sets and Their Applications to Multi Criteria Decision Making. *Punjab Univ. J. Math.* 2017, 49, 31–49.

33. Ullah, K.; Mahmood, T.; Jan, N.; Said, B.; Khan, Q. On Bipolar-Valued Hesitant Fuzzy Sets and its Applications in Multi-Attribute Decision Making. *Nucleus* 2018, 55, 93–101.

34. Ullah, K.; Mahmood, T.; Ali, Z.; Jan, N. On some distance measures of complex Pythagorean fuzzy sets and their applications in pattern recognition. *Complex Intell. Syst.* 2020, 6, 15–27. [CrossRef]