Economic relationships evolve: If the fish you bought on the fish market is spoiled you will probably switch the dealer next time. On larger scales, markets may develop into complex interaction structures solely from agents using their freedom of choosing their partners.

Statistical physics has a tradition of studying systems of interacting agents motivated by economic or social interactions. One example is evolutionary game theory on the lattice where agents can be thought of as particles with generalized interactions and internal memory. These properties give rise to complex spatiotemporal dynamics when agents try to locally optimize their payoffs \([1, 2, 3]\). The goal of this paper is to relax a further constraint in this class of models, namely the fixed interaction topology. Agents may use the choice of interaction partners as another means to locally optimize their payoff and, thereby, shape the interaction matrix and the topology of the system.

An interesting aspect of agent systems with evolving neighborhoods is the structure of the emerging interaction networks. The theoretical physics community has focused on complex network structures in diverse natural systems recently \([4, 5]\). Game theoretic models on evolving networks may provide a fresh view on the emergence of complex networks where strategic interactions occur. One example are economic systems where complex global structures and dynamics are observed \([6, 7]\).

The dynamics of games on static networks has been studied recently, namely the question of stability of cooperative behavior in the iterated Prisoner’s Dilemma (IPD) \([8, 9, 10]\), also under the influence of noise in the spatial relationships of agents \([11]\). The question how the game itself may affect network topology was asked by Zimmermann et al. in a simple setting of cooperating and defecting players, leading to a network that self-organizes into a steady state with highly connected cooperators \([12]\).

In the following, let us consider the IPD with memory on an evolving network. Details of this game on static networks can be found in Ref. \([13]\). The Prisoner’s Dilemma \([3, 14]\) is a two-person game defined by the payoff matrix

\[
A = \begin{pmatrix}
3 & 0 \\
5 & 1
\end{pmatrix}
\]

with the entries \(a_{ij}\) being the payoffs of agent 1 playing strategy \(s_i\) against agent 2 using strategy \(s_j\) (\(s_1\), cooperating; \(s_2\), defecting). In general, the entries \(a_{ij}\) have to fulfill \(a_{12} < a_{22} < a_{11} < a_{21}\) and \(a_{12} + a_{21} < 2a_{11}\).

In the IPD, a strategy is interpreted as a map of an agent’s knowledge to an action. Here, knowledge is defined by the number of the \(m\) previous moves the agent can remember. We will confine the strategy space to strategies with \(m = 1\), i.e., one agent knows only about the latest move of its opponent. At the beginning of an encounter, there is no previous move. Therefore, the first action is also part of the strategy (Table 1). It can be viewed as a lookup table where each history is mapped to an action. In the case of \(m = 1\), there are \(2^2 \times 1 = 8\) possible strategies. The strategies “always defect” (000) and “tit for tat” (011) are always Nash equilibria of this finite normal form game. Initially, a random network of a given average degree (i.e., number of a node’s next neighbors) \(\langle k \rangle\) is generated resulting in a Poissonian degree distribution. Strategies are assigned to the players at random. The coevolutionary dynamics consists of one part for the evolution of the strategies and another for network evolution. In each iteration cycle the following steps take place. Strategy evolution:

(i) One agent \(i\) is chosen randomly and its strategy is mutated...
to a strategy picked at random. (ii) The mutated agent plays against its neighbors and its payoff is compared to its payoff before the mutation. In case of a payoff increase the mutation is accepted and the payoffs of all neighbors are updated. This strategy update has been first used in Ref. [10]. Step (ii) corresponds to the assumptions that changing the strategy may bear some costs (risk) for the player and that mutations occur before the mutation. In case player

Network evolution: (i) With probability $\alpha$, one randomly chosen agent $i$ is connected to a new neighbor taken at random. (ii) If the new connection leads to an increase in average payoff, player $i$ will accept the new connection and remove the link to the neighbor it scores worst against. In case player $i$ had no neighbor before, it accepts the connection, and a link between a random pair of players will be removed to keep the average degree $\langle k \rangle$ constant [20]. Thereafter, all payoffs are updated. Note that the information used by an agent (its own payoff) is fully local.

The iteration of these steps eventually leads to stationary states. The choice of the value of $\alpha$ that determines the speed of network evolution is not critical and convergence has been verified for values between $\alpha = 0.001$ and $1$. The stationary states, where no further increase in payoff is possible for any agent, are cooperative and dominated by the strategy “tit for tat” (011). With regard to strategies, the stationary states correspond to game theoretic Nash equilibria [15]. Network evolution here does not interfere with the notion of Nash equilibria. In a stationary state no player can improve its payoff, neither by changing its strategy nor by choosing a different neighborhood. In the remainder of this paper, such an equilibrium will be called a network Nash equilibrium.

One essential property of this spatially extended coevolutionary game is the stability of the network Nash equilibria. Exploring the IPD on static networks [13], perturbations of the game’s Nash equilibria have been observed to cause avalanches of $M$ mutation events until stationary states are reestablished. Depending on the crucial parameter of the payoff matrix, the temptation to defect $a_{21}$, three different regimes of avalanche dynamics were found. First, for low temptations, the distribution of avalanche size $P(M)$ is supercritical with few mutation events necessary to reach again an equilibrium. In an intermediate range of $a_{21}$, scale-free behavior occurs as the distribution of avalanche size exhibits a power law with exponent $\gamma = -1.39 \pm 0.10$. For high values of $a_{21}$, the game becomes supercritical with a probability for very large avalanches much higher compared to the critical regime.

Let us similarly perturb the network Nash equilibrium by randomly changing the strategy of an arbitrary player [21]. The deviating strategy offers new opportunities for strategy changes to the neighboring players. The perturbation then can spread which leads to an avalanche of mutation events. Additionally, payoff can be improved by changing one’s neighborhood which results in a network evolution closely connected to the strategy evolution of the individual agents.

One finds that the model with network dynamics considered here results in a similar avalanche dynamics as observed for the static network case. For temptations to defect $a_{21} \leq 4.8$ and $a_{21} \geq 5.2$, the subcritical and supercritical regimes emerge (inset in Fig. 1). The critical regime exists in the same range $4.9 \leq a_{21} \leq 5.1$ as for static networks with same mean degree (Fig. 1). In addition, the scaling exponent $\gamma = -1.35 \pm 0.04$ agrees with the static case. Surprisingly, however, evolution of network structure is profoundly affected by the avalanche dynamics.

After an avalanche, network evolution comes to an end when a network Nash equilibrium is reached. The equilibrium network, which emerges in the course of subsequent perturbations and avalanches, strongly differs from the initial random graph. For practical purposes we here speak of an equilibrium network when the first three cumulants of the degree distribution do not change over a period of time much larger than the time needed to reach these values. The first three cumulants are given by the mean degree $\langle k \rangle$ (being constant because of

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**FIG. 1:** Distribution $P(M)$ of avalanche size $M$ in the critical regime. $P(M)$ exhibits scale-free behavior with a scaling exponent of $\gamma = -1.35 \pm 0.04$. The inset shows the sub- and supercritical regimes ($N = 500$, $\langle k \rangle = 5, a_{21} = 3.5, 5.0, 5.1$).

**FIG. 2:** Degree distributions $P(k)$ for critical (solid line, $a_{21} = 5.0$) and subcritical (dotted line, $a_{21} = 3.5$) avalanche dynamics. The dashed plot shows the initial Poissonian degree distribution. ($N = 500$, $\langle k \rangle = 20$).
errors are given by standard deviations). This reflects in the emerging link distribution. It appears to be more or less attractive depending on its strategy agent who decides. However, the agent who receives the link neighbors as the number of neighbors is kept constant for the payoffs an agent receives, it does not select for high number of degree distribution is independent of the number of agents. Note that although the quantity selected for is the sum of the quantity is always the number of agents N. Similarly to the treatment of the clustering coefficient, shortest path lengths for Poissonian random networks (ℓ0) and for random networks with identical degree distributions (ℓ′) are calculated. The observed values of ℓ are slightly larger than the average shortest paths in a Poissonian random network resulting in still very short paths between any two nodes in the network (Table II). The estimate ℓ′ does not yield results as good as C′ in the case of the clustering coefficient. Obviously, for the shortest path length ℓ, the assumption of randomly assigned links is not sufficient for an accurate estimate.

To gain further insight into the structure of the evolved networks, the Pearson coefficient r is calculated. It provides a measure of the degree correlation between neighboring nodes

\[ r = \frac{1}{\sigma_Q} \sum_{\mu, \nu} \mu \nu (Q(\mu, \nu) - Q(\mu)Q(\nu)) \]  

(4)

Q(μ) is the distribution of the remaining degree μ (i.e., μ = k - 1) and Q(μ, ν) the joint probability distribution that the two nodes at the ends of a randomly chosen link have the remaining degrees μ and ν. A positive Pearson coefficient indicates that nodes with a high degree prefer linking to other highly connected nodes (assortative mixing). For r > 0 high degree nodes tend to link to poorly connected agents (disassortative mixing). No preference is given when such correlations are absent, r = 0. The evolved networks studied here are neutral in the subcritical avalanche regime, as expected for the initial Poissonian random network (Table II). However, for critical and supercritical behavior, assortative mixing is observed (comparable in size to the assortative mixing of real-world graphs as, for example, co-authorship networks [18]).

To summarize we have studied a model for a coevolutionary Prisoner’s Dilemma game on an evolving network where...
each agent is allowed to alter its neighborhood in order to enhance its payoff, thereby shaping the topology of the network. One observes that the network, after an initial perturbation and a subsequent avalanche of rearrangements, relaxes into a stationary state which is stable w.r.t. strategy variations (Nash equilibrium), as well as against local topology changes (network Nash equilibrium). In this equilibrium, even a change of topology cannot enhance any player’s payoff. While the dynamics of the relaxation process is similar to the corresponding model on static networks without topology updates, the equilibrium structure of the network, emerged in the course of avalanches, strongly deviates from the initial random graph. The network evolves to a statistically stationary state with a broad degree distribution, suggesting scale-free behavior in the critical avalanche regime. It exhibits small-world behavior and in the critical and supercritical regimes also shows assortative mixing, a property of social networks not shared by many standard models.

The spatial coevolutionary game studied here is a minimal non-trivial model of network evolution driven by game theoretic interactions. It may help in understanding the emergence of global organization from local interactions as observed in social and economic systems. It also provides an alternative mechanism for the evolution of complex networks possibly relevant for real-world systems where simple growth models as, for example, preferential linking, are not easily applicable.

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[1] M. A. Nowak and R. M. May, Nature (London) 359, 826 (1992).
[2] A. V. M. Herz, J. Theor. Biol. 169, 65 (1994).
[3] K. Lindgren and M. G. Nordahl, Physica D 75, 292 (1994).
[4] S. H. Strogatz, Nature (London) 410, 268 (2001).
[5] R. Albert and A.-L. Barabási, Rev. Mod. Phys. 74, 47 (2002).
[6] W. B. Arthur, Science 284, 107 (1999).
[7] A. Kirman, in Handbook of Graphs and Networks, edited by S. Bornholdt and H. G. Schuster (Wiley-VCH, Weinheim, 2002), 273-293, 12.
[8] G. Abramson and M. Kuperman, Phys. Rev. E 63, 030901(R) (2001).
[9] B. J. Kim, A. Trusina, P. Holme, P. Minnhagen, J. S. Chung, and M. Y. Choi, Phys. Rev. E 66, 021907 (2002).
[10] H. Ebel and S. Bornholdt, Phys. Rev. E 66, 056118 (2002).
[11] M. D. Cohen, R. L. Riolo, and R. Axelrod, Rationality and Society 13, 5 (2001).
[12] M. G. Zimmermann, V. M. Eguíluz, and M. S. Miguel, in Economics with Heterogeneous Interacting Agents, edited by A. Kirman and J.-B. Zimmermann (Springer, Berlin, 2001), pp. 73–86.
[13] R. Axelrod and W. D. Hamilton, Science 211, 1390 (1981).
[14] R. K. Axelrod, The Evolution of Cooperation (Basic Books, New York, 1984).
[15] J. F. Nash, Proc. Natl. Acad. Sci. U.S.A. 36, 48 (1950).
[16] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. E 64, 026118 (2001).
[17] J. Davidsen, H. Ebel, and S. Bornholdt, Phys. Rev. Lett. 88, 128701 (2002).
[18] M. E. J. Newman, Phys. Rev. Lett. 89, 208701 (2002).
[19] D. J. Watts and S. H. Strogatz, Nature (London) 393, 440 (1998).
[20] The case of growing $\langle k \rangle$ only adds a drift $\langle k \rangle_t \propto \sqrt{t}$ to the dynamics. This is caused by the occurrence of disconnected nodes with probability $P_t(k = 0) \propto 1/\sqrt{t}$. The resulting clustering $C_{\Delta,t} \propto \sqrt{t}$ is fully reproduced by $\Delta$.
[21] One can additionally change a link which, however, does not significantly alter the system’s response to the perturbation.