A new recursion procedure for deriving renormalized perturbation expansions for the one-dimensional anharmonic oscillator is offered. Based upon the $\hbar$-expansions and suitable quantization conditions, the recursion formulae obtained have the same simple form both for ground and excited states and can be easily applied to any renormalization scheme. As an example, the renormalized expansions for the sextic anharmonic oscillator are considered.

1. Introduction

In the past few decades intensive investigations have been carried out on the one-dimensional anharmonic oscillator because of both its role in the modeling of quantum field theory and its usefulness in atomic, and molecular physics $^{1-4}$. The conventional way to study the energy eigenvalues and eigenfunctions of this bound-state problem is the practical application of perturbation expansions in a coupling constant. But the nonconvergence of obtained expansions $^{5,6}$ compels us to resort to modern procedures of summation of divergent series. Ones of the most common among them are various versions of the renormalization technique $^{7-9}$. The inclusion of one or more free parameters corrects for the above mentioned deficiency, by controlling and accelerating the convergence of the expansions not only for eigenvalues but for eigenfunctions as well $^{10,11}$.

However, to provide a reasonable accuracy this method needs to know high orders of the perturbation series. Unfortunately we can easily obtain the high-order corrections only in the case of ground states when the simple recursion relations of the logarithmic perturbation theory $^{12-18}$ or the Bender-Wu method $^5$ for performing the Rayleigh-Schrödinger perturbation expansion are applied. The description of radially excited states involves including the nodes of wave function in consideration that makes these approaches extremely cumbersome and, practically, inapplicable.

Very recently, a new semiclassical technique for deriving results of logarithmic perturbation theory within the framework of the one-dimensional Schrödinger equation has been proposed $^{19}$. Based upon an $\hbar$-expansion this technique leads to recursion formulae having the same simple form both for the ground and excited states.

The object of this paper is to extend the above mentioned formalism and to adapt it to the treatment of any renormalization scheme in terms of handy recursion relations within the framework of the united approach. To this end the paper is presented as follows. Section 2 contains a general discussion and the necessary assumptions in the semiclassical treatment of logarithmic perturbation theory. In Sect.3 quantization conditions obtained are used for deriving the recursion formulae for perturbation expansions. Sect.4 focuses on a derivation of the recursion formulae for renormalized perturbation expansions. Sect.5 gives the example of explicit application of proposed technique. Finally, Sect.6 contains concluding remarks.

2. Semiclassical treatment of logarithmic perturbation theory

In this section, our goal is to describe a straightforward semiclassical approach to obtaining results of the logarithmic perturbation theory.

We shall confine our interest to bound-state problem for the Schrödinger equation in one degree of freedom

$$-\frac{\hbar^2}{2m}U''(x) + V(x)U(x) = EU(x).$$

(1)

We shall study the discrete piece of the spectrum related to the potential function, $V(x)$, having a single simple minimum and can be given by expression

$$V(x) = \frac{1}{2}m\omega^2x^2 + \sum_{i\geq1} f_ix^{i+2}.$$  

(2)
However, if the potential has a finite number of local minima, the proposed below technique also can be applied. In this case the obtained eigenvalues are the energies of quasidiscrete states in a vicinity of each local minimum when the tunneling can be neglected.

We want to restore the results of logarithmic perturbation theory for the \( n \)th eigenfunction and corresponding energy eigenvalue by means of the explicit semiclassical expansions in powers of the Planck constant, \( \hbar \).

Indeed, after the scale transformation, \( x \to \sqrt{\hbar}x \), the coupling constants, \( f_i \), appear in common with powers of Planck’s constant. Therefore, the perturbation series must be semiclassical \( \hbar \)-expansion, too.

It has been proved \(^20\) that the energy eigenvalues under consideration should be concentrated near the minimum of the potential and should behave as

\[
E = \hbar \omega(n + 1/2) + \sum_{k \geq 2} E_k(\omega, n)\hbar^k.
\] (3)

A few procedures are elaborated for computing the coefficients \( E_i(\omega, n) \). They involve, in particular, applying the methods of the comparison equation \(^21\) and complex ‘sprout’ \(^22\); an analytic continuation in the \( \hbar \)-plane \(^23\); various approaches within the framework of the WKB-approximation \(^24\)–\(^28\); quantization using the methods of classical mechanics \(^26,29\); and, lastly, expansions in the \( \hbar^{1/2} \)-series \(^30\).

Here we describe a new, simpler and more straightforward semiclassical technique \(^19\). It involves two successive stages: first, we imply that the eigenvalue selection is effectively achieved by enforcing not the square integrability of the eigenfunction, but its analyticity, with applying the quantization condition; and second, we state such a classical analog of the system, in the limit \( \hbar \to 0 \), which is appropriate for consideration of the low-lying levels.

To begin with, we use the substitution, \( C(x) = \hbar U'(x)/U(x) \), accepted in logarithmic perturbation theory, and rewrite Eq. (1) in the Riccati form

\[
\hbar C'(x) + C^2(x) = 2m[V(x) - E].
\] (4)

It will be recalled that we attempt to solve this equation in a semiclassical manner with the following asymptotic series expansions

\[
E = \sum_{k=0}^{\infty} E_k\hbar^k, \quad C(x) = \sum_{k=0}^{\infty} C_k(x)\hbar^k.
\] (5)

On substituting these expansions into the Riccati equation (4) and collecting coefficients of the same powers of \( \hbar \), one obtains the system

\[
C_0^2(x) = 2mV(x),
C_0'(x) + 2C_0(x)C_1(x) = -2mE_1,
\ldots
C_{k-1}'(x) + \sum_{i=0}^{k} C_i(x)C_{k-1}(x) = -2mE_k.
\] (6)

For nodeless states this system coincides with those derived by means of the standard perturbation approach and can be solved straightforwardly. However, when radial excitations are described within the framework of logarithmic perturbation theory the nodes of wave functions are included in some separate factor and consideration becomes extremely cumbersome. We intend to circumvent this difficulty by making use of the quantization condition, inspired by the WKB-approximation \(^31,32\).

Generally speaking, such the quantization condition simple means monitoring whether eigenfunction \( U(x) \) is analytic inside a closed contour. We would remind that a necessary condition for analyticity is

\[
\frac{1}{2\pi i} \oint U'(x)/U(x) \, dx = n,
\] (7)

which must be a non-negative integer, giving the number of zeros of \( U(x) \) which lie inside the contour of integration. Hence, the transition to the logarithmic derivative, \( C(x) \), allows the nodes of wave functions to be easily involved in consideration.
Taking into account that due to the Sturm-Liouville theory the eigenfunction of the \( n \)th radially excited state has precisely \( n \) zeros on the real axis, we have

\[
\frac{1}{2\pi i} \oint C(x) \, dx = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \hbar^k \oint C_k(x) \, dx = \hbar n, \quad n = 0, 1, 2, \ldots
\]  

where a contour of integration encloses the nodes of wave function and no other singularities.

The above quantization condition must be supplemented with the rule of achieving a classical limit for its right-hand side. Unfortunately, under the influence of standard textbooks on quantum mechanics, the semiclassical \( \hbar \)-expansions are usually associated solely with the WKB-approach, for which the rule of achieving a classical limit is

\[
\hbar \to 0, \quad n \to \infty, \quad \hbar n = \text{const.}
\]  

That implies that the WKB method is more suitable for obtaining energy eigenvalues in the limiting case of large quantum numbers.

The semiclassical treatment of the low lying levels demands the use of the different, as compared with the WKB-approach, rule of achieving a classical limit, namely

\[
\hbar \to 0, \quad n \sim O(1), \quad \hbar n \to 0.
\]  

This rule does provide lowering a particle to the bottom of a potential well as \( \hbar \to 0 \) and was formerly applied in deriving coefficients of the \( 1/N \)-expansions. As it will be shown below, in one dimension case this way lead us to restoring the logarithmic perturbation series.

Thus, in view of the new rule of passing to the classical limit, the quantization conditions are rewritten in the final form as

\[
\frac{1}{2\pi i} \oint C_1(x) \, dx = n, \quad \frac{1}{2\pi i} \oint C_k(x) \, dx = 0, \quad k \neq 1.
\]  

A further application of the theorem of residues to the explicit form of functions \( C_k(x) \) easily solves the problem of taking into account nodes of the wave functions for excited states.

### 3. Recursion formulae for perturbation expansions

We proceed now to deriving the recursion relations for obtaining the \( n \)th eigenfunction and corresponding energy eigenvalue.

Let us consider the system and investigate the behaviour of the functions \( C_k(x) \). From the first equation it is seen that \( C_0(x) \) can be written in the form

\[
C_0(x) = -[2mV(x)]^{1/2} = -m\omega x(1 + \frac{2}{m\omega^2} \sum_{i \geq 1} f_i x^i)^{1/2} = x \sum_{i=0}^{\infty} C_0^i x^i,
\]  

where the minus sign is chosen from boundary conditions, and coefficients \( C_0^i \) are defined by parameters of the potential through the relations

\[
C_0^0 = -m\omega, \quad C_i^0 = \frac{1}{2m\omega} \left( \sum_{p=1}^{i-1} C_p^0 C_{i-p}^0 - 2mf_i \right), \quad i \geq 1.
\]  

From we recognize that \( C_0(0) = 0 \) and, consequently, the function \( C_1(x) \) has a simple pole at the origin, while \( C_k(x) \) has a pole of the order of \( (2k-1) \). Thus, \( C_k(x) \) can be represented by the Laurent series

\[
C_k(x) = x^{1-2k} \sum_{i=0}^{\infty} C_i^k x^i, \quad k \geq 1.
\]  

With definition of residues, this expansion permits us to express the quantization conditions explicitly in terms of the coefficients \( C_i^k \) as
\[ C_{2k-2}^k = n \delta_{1,k}, \] (15)

where the Kronecker delta is used. Thereby the common consideration of the ground and excited states has indeed proved to be possible.

The substitution of the series \((13)\) and \((14)\) into \((6)\) in the case \(i \neq 2k - 2\) yields the recursion relation for obtaining the coefficients \(C_{2k}^i\):

\[
C_i^k = -\frac{1}{12C_0} [(3 - 2k + i)C_i^{k-1} + \sum_{j=1}^{k-1} \sum_{p=0}^i C_p^j C_i^{k-j} + 2 \sum_{p=1}^i C_p^0 C_i^{k-1}] , \tag{16}
\]

whereas if \(i = 2k - 2\) we would find the recursion formula for the energy eigenvalues

\[
2mE_k = -C_{2k-2}^{k-1} - \sum_{j=0}^{k} \sum_{p=0}^{2k-2} C_p^j C_{2k-2-j}^{k-1}. \tag{17}
\]

It is easy to verify that proposed technique does indeed restore the results of the logarithmic perturbation theory. Upon calculating first corrections to the energy eigenvalues we arrive at the form familiar from standard textbooks

\[
E = \hbar \omega \left( n + \frac{1}{2} \right) - \hbar^2 \frac{15f_2^2}{4m^3 \omega^4} (n^2 + n + \frac{11}{30}) + \hbar^2 \frac{3f_2}{2m^2 \omega^2} (n^2 + n + \frac{1}{2}) + O(\hbar^3), \tag{18}
\]

with the obvious constraints

\[
n + \frac{1}{2} \ll \frac{2m^2 \omega^3}{3hf_2}, \frac{4m^3 \omega^5}{15hf_1^2}. \tag{19}
\]

The equations \((16)\) and \((13)\) have the same simple form both for ground and excited states and provide, in principle, the calculation of the perturbation corrections up to an arbitrary order in the analytical or numerical form.

Another salient feature of this technique is the fact that for obtaining corrections to the energy eigenvalues and eigenfunctions we need not have initially the explicit form of the exact solution for the unperturbed potential. This solution is found automatically in the process of solving recursion formulae.

Moreover, in this way the quasi-exact solutions may be obtained as well. Indeed, to be more concrete, let us consider the ground state for the anharmonic potential

\[
V(x) = V_2 x^2 + V_4 x^4 + V_6 x^6 \] (20)

Then the lowest order corrections derived by means of formulae \((16)\) and \((17)\) are written as

\[
\begin{align*}
E_1 &= \sqrt{V_2}, \\
E_2 &= \frac{3V_4}{2V_2}, \\
E_3 &= \frac{3(-7V_4^2 + 5V_2 V_6)}{4 V_2^{5/2}}, \\
E_4 &= \frac{-9(-37V_4^3 + 40V_2V_4 V_6)}{8 V_2^4}, \\
E_5 &= \frac{-15(2059V_4^4 - 2992 V_2 V_4^2 V_6 + 466 V_2^2 V_6^2)}{64 V_2^{11/2}}, \\
E_6 &= \frac{9(101859V_4^5 - 186380 V_2 V_4^3 V_6 + 61420 V_2^2 V_4 V_6^2)}{128 V_2^7},
\end{align*}\tag{21}
\]

setting

\[
V_2 = V_4^2/(4 V_6) - 3\sqrt{V_6} \tag{22}
\]
immediately leads to the analytical expression for the ground state energy and the wave function
\[ E = V_4/(2\sqrt{V_6}) \] (23)
and
\[ \Psi(x) = \exp(-V_4 x^2/(4\sqrt{V_6}) - \sqrt{V_6} x^4/4) \] (24)
where for the simplicity we put \( \hbar = 2m = 1 \).

4. Renormalized perturbation expansions

The renormalization of perturbation expansions is based on a redistribution of values of series items for a partial summation of them and improving the convergence of expansions. From mathematical point of view it means the transition from the asymptotical expansion according to Poincare to the asymptotical expansion according to Erdélyi. Characteristic of this method is that the zeroth approximation involves a set of artificial parameters, which are not contained in the original Hamiltonian and are determined order by order after calculating the physical quantity perturbatively.

For the discussed one-dimensional anharmonic oscillator, as a trial parameters we can choose a frequency and a mass. Though proposed technique is easily applied to the mass renormalization we do consider only the case of renormalization of a frequency as being more physically motivated. For this purpose it is enough to think of the harmonic oscillator frequency incoming in the potential as a function of Plank’s constant variable, with subsequent its expansion in an \( \hbar \)-series. However, for later use it is more convenient to take
\[ \omega^2 = \sum_{k=0}^{\infty} \omega^2_k \hbar^k \] (25)
The recursion system (25) then reads
\[ C_0^2(x) = m^2 \omega_0^2 x^2 + 2m \sum_{i \geq 1} f_i x^{i+2}, \]
\[ C_0'(x) + 2C_0(x)C_1(x) = m^2 \omega_1^2 x^2 - 2mE_1, \]
\[ \ldots \]
\[ C_{k-1}^2(x) + \sum_{i=0}^{k} C_i(x)C_{k-i}(x) = m^2 \omega_k^2 x^2 - 2mE_k, \] (26)

with
\[ C_0^0 = -m\omega_0, \quad C_1^0 = \frac{1}{2m\omega_0} \left( \sum_{p=1}^{i-1} C_p^0 C_{i-p}^0 - 2mf_i \right), \quad i \geq 1. \] (27)

In the case \( k \geq 1 \), when \( i \neq 2k - 2 \) and \( i = 2k - 2 \) that results in
\[ C_i^k = -\frac{1}{2C_0^2} \left[ -m^2 \omega_i^2 \delta_{i,2k} + (3 - 2k + i)C_i^{k-1} \right. \]
\[ + \sum_{j=1}^{k-1} \sum_{p=0}^{i-j} C_p^j C_{i-p}^{k-j} + 2 \sum_{p=1}^{i} C_p^0 C_{i-p}^k \left], \right. \]
\[ 2mE_k = -C_{2k-2}^{k-1} - \sum_{j=0}^{k-2} \sum_{p=0}^{2k-2-j} C_j^p C_{2k-2-p}^j, \] (28)
respectively. Here the coefficients \( \omega^2_k \) are defined by the chosen version of renormalization.

In practice, the one-parameter schemes are usually used. They are obtained with the truncation of series \( \omega^2 = \omega_0^2 + \omega_k^2 \hbar^k \) as
\[ \omega^2 = \omega_0^2 + \omega_k^2 \hbar^k, \] (29)
where \( \omega_0 \) is a trial frequency and an order in \( \hbar \) of the remainder is determined by the anharmonicity of a potential \( V \).

A free parameter, \( \omega_0 \), is order-dependent. At any finite order of approximation, \( N \), its optimal value is picked up by the condition of "minimal difference" \( E_N(\omega_0) = 0 \) \(^{39,40} \), and "minimal sensitivity", \( \langle d/d\omega_0 \rangle \sum_{i=0}^{N} E_i(\omega_0) = 0 \)^ \( ^{10,41} \), because perturbatively calculated quantity should not depend on the artificial parameter at all.

Notice that our approach can be also applied to more general scheme such as, for instance, the renormgroup conditions \(^{40} \) or the simple equating to zero some corrections:

\[
E_i(\omega_0, ..., \omega_k) = 0, \quad i = 2, 3, ..., N. \tag{30}
\]

5. Example of application

As a rule, various schemes of renormalization are illustrated with describing ground states of the quartic anharmonic oscillator. Here we consider the less traditional example of obtaining energy eigenvalues for both ground and excited states of the sextic anharmonic oscillator with a potential

\[
V(x) = \frac{1}{2} (x^2 + \lambda x^6), \tag{31}
\]

where we put \( \omega = m = 1 \) for simplicity.

The analytical representation of perturbation expansion for its energy obtained by using formulae \(^{11} \) and \(^{17} \) has the form

\[
E_1 = \frac{1}{2} + n,
E_3 = 2^{-4}5 \lambda (1 + 2n) (3 + 2n + 2n^2),
E_5 = -2^{-8} \lambda^2 (1 + 2n) (3495 + 4538 n + 5324 n^2 + 1572 n^3 + 786 n^4),
E_7 = 2^{-11}5 \lambda^3 (1 + 2n) (247935 + 444014 n + 600050 n^2 + 323868 n^3
+ 191424 n^4 + 35388 n^5 + 11796 n^6).
\tag{32}
\]

This series diverges for any finite value of \( \lambda \) and requires the use of some renormalization procedure. Since for discussed potential the above series is, in fact, the expansion in powers of the Planck constant squared, the trial frequency within any one-parameter scheme of renormalization is given by \(^{24} \) as

\[
\omega^2 = \omega_0^2 + \omega_2^2 \hbar^2. \tag{33}
\]

By taking use of relations \(^{28} \), the first terms of the renormalized perturbation series for the energy eigenvalues then read

\[
E_1 = \omega_0 \left( \frac{1}{2} + n \right)
E_3 = 2^{-4} \omega_0^{-3} (1 + 2n) \left[ 5\lambda (3 + 2n + 2n^2) + 4 \omega_0^2 (1 - \omega_0^2) \right]
E_5 = -2^{-8} \omega_0^{-7} (1 + 2n) \times
\left[ \lambda^2 (3495 + 4538 n + 5324 n^2 + 1572 n^3 + 786 n^4)
+ 120 \lambda \omega_0^2 (1 - \omega_0^2) (3 + 2n + 2n^2) + 16 \omega_0^4 (1 - \omega_0^2)^2 \right]
E_7 = 2^{-11} \omega_0^{-11} (1 + 2n) \times
\left[ 5 \lambda^3 (247935 + 444014 n + 600050 n^2 + 323868 n^3
+ 191424 n^4 + 35388 n^5 + 11796 n^6)
+ 28 \lambda^2 \omega_0^2 (1 - \omega_0^2) \times
\left[ 3495 + 4538 n + 5324 n^2 + 1572 n^3 + 786 n^4 \right]
+ 1200 \lambda \omega_0^4 (1 - \omega_0^2)^2 (3 + 2n + 2n^2) + 64 \omega_0^6 (1 - \omega_0^2)^3 \right] \tag{34}
\]

where \( \omega_2 \) is excluded due to equation \(^{33} \).
The obtained recursion formulae enable us to calculate high orders of the renormalized perturbation series for wide ranges of the coupling constant $\lambda$ and the radial quantum number, $n$.

We have computed the partial sums of $N$ corrections to the energy eigenvalues with defining the free parameter $\omega_0$, order by order under the conditions of the minimal difference, $E_N(\omega_0) = 0$, and the minimal sensitivity, $(d/d\omega_0)E_N(\omega_0) = 0$ and $(d/d\omega_0)\sum_{i=0}^{N} E_i(\omega_0) = 0$.

Contrary to the current view $^{42}$, we recognize that in the case of sextic anharmonic oscillator all these schemes appear to give approximately an equal accuracy.

It has been observed that in the range of values $0 < \lambda \leq 1000$ and $0 \leq n \leq 1000$ the best accuracy can be achieved in the limiting case of the weak coupling, for small $\lambda$, $n$. The obtained energy eigenvalues are most inaccurate in the intermediate region ($\lambda = O(1)$ and $0 \leq n \leq 2$). Further, the $N$th-order renormalized approximation becomes improve as $n$ and $\lambda$ increase.

The typical behaviour of the speed and accuracy of these one-parameter schemes is illustrated by Table 1. Here the partial sums of $N$ renormalized corrections to energy eigenvalues derived under the condition of minimal sensitivity, $(d/d\omega_0)\sum_{i=0}^{N} E_i(\omega_0) = 0$, are compared with the energy eigenvalues obtained by the numerical integration of the Schrödinger equation. As the numerical integration procedure it was used the improved shooting method with the Noumerov finite difference scheme, described in literature $^{43}$, which ensures the necessary exactness of calculation for the Sturm-Liouville problems. Notice that the condition of minimal sensitivity yields a large number of the trial $\omega_0$-values from which we choose the one with the flattest extremum $^{42,44}$.

6. Conclusion

A new simple recursion procedure, suitable to applying any renormalization scheme of improving the perturbation expansions for the bound-state problem of anharmonic oscillator within the framework of the one-dimensional Schrödinger equation, has been proposed.

Based upon the semiclassical treatment, including $\hbar$-expansions and proper quantization conditions, the main disadvantage of logarithmic perturbation theory, namely, the cumbersome description of excited states when nodes of wave functions are taking into account, has been avoided.

The derived formulae have been adapted for the use of any renormalization scheme. As a result, the recursion formulae, obtained for renormalized perturbation expansions, have the same simple form both for ground and excited states and provide, in principle, the calculation of renormalization corrections up to an arbitrary order.

Moreover, the advantage of the method is such flexibility that within the united approach one can apply various schemes of renormalization and choose different values of the trial parameter depending on the order of perturbation.

The proposed technique has been developed for investigation of the bound-state problem within the one-dimensional Schrödinger equation with any potential described by an analytic function having a simple global minimum. However, taking into account that the potential having a finite number of local minima becomes, as $\hbar \rightarrow 0$, a sum of infinite deep wells, our procedure may be applied as well for investigation of bound states in a vicinity of a local minimum, when tunneling can be neglected.

Acknowledgments

This work was supported in part by the International Soros Science Education Program (ISSEP) under Grant APU052102.

References

[1] R. F. Bishop and M. F. Flynn, Phys. Rev. A38, 2211 (1988).
[2] F. T. Hioe, D. MacMillen and E. W. Montroll, Phys. Rep. 43, 305 (1978).
[3] C. M. Bender and T. T. Wu, Phys. Rev. Lett. 37, 117 (1976).
[4] C. E. Reid, J. Mol. Spectrosc. 36, 183 (1970).
[5] C. M. Bender and T. T. Wu, Phys. Rev. 184, 1231 (1969).
[6] B. Simon, Ann. Phys. (N.Y.) 58, 76 (1970).
[7] Large Order Behaviour of Perturbation Theory, Current Physics - Sources and Comments vol. 7, ed. J. C. Le Guillon and J. Zinn-Justin (North-Holland, Amsterdam, 1990).

[8] G. A. Arteca, F. M. Fernandez and E. A. Castro, Large Order Perturbation Theory and Summation Methods in Quantum Mechanics (Springer-Verlag, Berlin, 1990).

[9] H. Kleinert, Path Integrals in Quantum Mechanics, Statistical and Polymer Physics (World Scientific, Singapore, 1995).

[10] J. Killingbeck, J. Phys. A14, 1005 (1981).

[11] F. M. Fernandez and E. A. Castro, Phys. Rev. A27, 663 (1983).

[12] V. S. Polikanov, it Zh. Eks. Teor. Fiz. 52, 1326 (1967).

[13] V. S. Polikanov, Teor. Mat. Fiz. 24, 230 (1975).

[14] A. D. Dolgov and V. S. Popov, Phys. Lett. B79, 403 (1978).

[15] Y. Aharonov and C. K. Au, Phys. Rev. A20, 2245 (1979).

[16] A. V. Turbiner, Usp. Fiz. Nauk 144, 35 (1984).

[17] T. Imbo and U. Sukhatme, Am. J. Phys. 52, 140 (1984).

[18] G. M. Rogers, J. Math. Phys. 26, 567 (1985).

[19] I. V. Dobrovolska and R. S. Tutik, J. Phys. A32, 563 (1999).

[20] B. Simon, Ann. Inst. H. Poincaré 38, 295 (1983).

[21] S. Yu. Slav’yakov, Differ. Equ. 5, 313 (1969).

[22] V. P. Maslov, Complex Method WKB in Nonlinear Equations (Nauka, Moscow, 1977) (in Russian).

[23] F. H. Stillinger, Phys. Rev. A43, 3317 (1991).

[24] A. Rubinovicz, Quantum Mechanics (Elsevier, Warsaw, 1968).

[25] G. H. Walker, Phys. Lett. A74, 170 (1974).

[26] S. Yu. Slav’yakov, Asymptotic of Solutions for the One-Dimensional Schrödinger Equation (Leningrad University Press, Leningrad, 1990) (in Russian).

[27] A. Voros, Phys. Rev. A40, 6814 (1989).

[28] E. Delabaere, H. Dillinger and F. Pham, J. Math. Phys. 38, 6126 (1997).

[29] S. S. Stepanov, R. S. Tutik, A. P. Yaroshenko and W. von Schlippe, Nuovo Cimento B106, 329 (1991).

[30] R. Zucchini, Ann. Phys. (N. Y.) 159, 199 (1985).

[31] A. Zwaan, Arch. Neerland. Sci. Exact. Natur. Ser. 3A12, 1 (1929).

[32] J. L. Dunham, Phys. Rev. 41, 713 (1932).

[33] S. S. Stepanov, R. S. Tutik, J. Phys. A24, L469 (1991).

[34] S. S. Stepanov, R. S. Tutik, Zh. Eksp. Teor. Fiz. 100, 415 (1991).

[35] S. S. Stepanov, R. S. Tutik, Zh. Eksp. Teor. Fiz. 101, 18 (1992).

[36] S. S. Stepanov, R. S. Tutik, Phys. Lett. A163, 26 (1992).

[37] L. Skála, J. Dvořák, J. Čížek and V. Špirko, Phys. Rev. A53, 2009 (1996).

[38] M. V. Fedoryuk, Asymptotical methods for liner differential equations (Moscow, Nauka, 1983).
Table 1: The sequences of the partial sums of $N$ renormalized corrections to the energy eigenvalues of the sextic anharmonic oscillator with the potential $V(x) = \frac{1}{2} (x^2 + \lambda x^6)$, computed within the one-parameter scheme of renormalization under the condition of minimal sensitivity, $(d/d\omega_0) \sum_{i=0}^{N} E_i(\omega_0) = 0$. The values of $E_{num}$ were obtained by numerical integration of the Schrödinger equation.

| N   | $n = 0$       | $n = 1$       | $n = 5$       |
|-----|---------------|---------------|---------------|
|     | $\lambda = 0.01$ | $\lambda = 10$ | $\lambda = 0.01$ | $\lambda = 10$ |
| 1   | 0.508693705   | 1.161458     | 1.55611747   | 4.210051     | 6.59434725 | 25.95659 |
| 3   | 0.508378396   | 1.110292     | 1.55399477   | 4.054344     | 6.57502024 | 25.54466 |
| 5   | 0.508371342   | 1.104354     | 1.55397174   | 4.047270     | 6.61788050 | 26.4265  |
| 10  | 0.508370692   | 1.102706     | 1.55398991   | 4.056856     | 6.61764448 | 26.4234  |
| 15  | 0.508370674   | 1.102541     | 1.55398998   | 4.057586     | 6.61764001 | 26.4256  |
| 20  | 0.508370673   | 1.102651     | 1.55398999   | 4.057838     | 6.61764261 | 26.4280  |
| 25  | 0.508370689   | 1.102586     | 1.55398995   | 4.057637     | 6.61763908 | 26.4259  |
| 30  | 0.508370674   | 1.102729     | 1.55398996   | 4.057495     | 6.61763913 | 26.4249  |
| 35  | 0.508370726   | 1.102819     | 1.55398997   | 4.057749     | 6.61764005 | 26.4250  |
| 40  | 0.508370676   | 1.102768     | 1.55398996   | 4.057442     | 6.61763907 | 26.4247  |
| 45  | 0.508370682   | 1.102796     | 1.55398996   | 4.057401     | 6.61763907 | 26.4248  |
| 50  | 0.508370681   | 1.102829     | 1.55398996   | 4.057410     | 6.61763906 | 26.4247  |

$E_{num}$ 0.508370682 1.102862 1.55398996 4.057422 6.61763908 26.42476