Covers of stacky curves and limits of plane quintics

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Abstract

We construct a well-behaved compactification of finite covers of a stacky curve using admissible cover degenerations. Using our construction, we compactify the space of tetragonal curves on Hirzebruch surfaces. As an application, we explicitly describe the boundary divisors of the closure in $\overline{M}_g$ of the locus of smooth plane quintic curves.

1 Introduction

Let $X$ be a smooth stacky curve, that is, a connected, proper, one dimensional, smooth Deligne–Mumford stack of finite type over the complex numbers. Consider finite maps $\phi: P \to X$, where $P$ is a smooth orbifold curve. The space of such maps admits a modular compactification as the space of twisted stable maps of Abramovich and Vistoli [AV02]. This compactification is analogous to the space of Kontsevich stable maps in the case of schematic $X$. Although this space is proper, it may have many components, and it is difficult to identify the closure of the locus of finite maps. We construct a compactification which is analogous to the space of admissible covers (Theorem 2.5). This compactification is smooth with a normal crossings boundary divisor, and the locus of finite maps is dense. It admits a morphism to the Abramovich–Vistoli space.

Two classical applications motivate us. First, taking $X = \overline{M}_{0,4}/S_4$ gives us a nice compactification of the space of curves of fiberwise degree 4 on ruled surfaces. Second, taking $X = \overline{M}_{1,1}$ gives us a nice compactification of the space of elliptic fibrations. We work out the first case in detail, leaving the second (and its variants) for the future.

Let us now describe the case of $X = \overline{M}_{0,4}/S_4$ and in particular the application to limits of plane quintics. Let the genus of the domain curve $P$ be 0 and let the map $\phi$ have degree $6d$ on the underlying coarse spaces (the degree is always divisible by 6). A general finite map $\phi: P \to X$ corresponds to the data of $(S \to \mathbb{P}^1, C)$, where $S \to \mathbb{P}^1$ is a $\mathbb{P}^1$-bundle and $C \subset S$ is a divisor of fiberwise degree 4 and genus $3(d−1)$. The space of such maps $\phi$ splits into two connected components, corresponding to the case of $S = \mathbb{F}_n$ for even and odd $n$. In the first case, generically we have $C \subset \mathbb{F}_0$ of class $(4, d)$. In the second case, generically we have $C \subset \mathbb{F}_1$ of class $4\sigma + (d + 2)F$, where $\sigma$ is the class of the exceptional curve and $F$ the class of the fiber. In the second case, taking $d = 3$ means that $C$ is the proper transform of a smooth quintic plane curve. Since our compactification admits a morphism to $\overline{M}_6$, its image in $\overline{M}_6$ is precisely the closure of the locus of plane quintics. We thus get the following explicit description of the limits of plane quintic curves.

Theorem 1.1. Let $Q \subset \overline{M}_6$ be the locus of plane quintic curves and $\overline{Q}$ its closure in $\overline{M}_6$. The generic points of the components of the boundary $\overline{Q} \cap \left(\overline{M}_6 \setminus \overline{M}_g\right)$ of $Q$ represent the following stable curves.
With the dual graph $X \xrightarrow{o} 0$

1. A nodal plane quintic.
2. $X$ hyperelliptic of genus 5.

With the dual graph $X \xrightarrow{p} \xrightarrow{o} Y$

3. $(X, p)$ the normalization of a cuspidal plane quintic, and $Y$ of genus 1.
4. $X$ of genus 2, $Y$ Maroni special of genus 4, $p \in X$ a Weierstrass point, and $p \in Y$ a ramification point of the unique degree 3 map $Y \to \mathbb{P}^1$.
5. $X$ a plane quartic, $Y$ hyperelliptic of genus 3, $p \in X$ a point on a bitangent, and $p \in Y$ a Weierstrass point.
6. $X$ a plane quartic, $Y$ hyperelliptic of genus 3, and $p \in X$ a hyperflex ($K_X = 4p$).
7. $X$ hyperelliptic of genus 4, $Y$ of genus 2, and $p \in X$ a Weierstrass point.
8. $X$ of genus 1, and $Y$ hyperelliptic of genus 5.

With the dual graph $X \xrightarrow{o} \xleftarrow{o} Y$

9. $X$ Maroni special of genus 4, $Y$ of genus 1, and $p, q \in X$ on a fiber of the unique degree 3 map $X \to \mathbb{P}^1$.
10. $X$ hyperelliptic of genus 3, $Y$ of genus 2, and $p \in Y$ a Weierstrass point.
11. $X$ of genus 2, $Y$ a plane quartic, $p, q \in X$ hyperelliptic conjugate, and the line through $p, q$ tangent to $Y$ at a third point.
12. $X$ hyperelliptic of genus 3, $Y$ of genus 2, and $p, q \in X$ hyperelliptic conjugate.

With the dual graph $X \xleftarrow{q} \xrightarrow{o} Y$

13. $X$ hyperelliptic of genus 3, and $Y$ of genus 1.

The closure of $Q$ in $\mathcal{M}_6$ is known to be the union of $Q$ with the locus of hyperelliptic curves $[\text{Gri85}]$. This result also follows from our techniques.

The paper is organized as follows. Section 2 contains the construction of the admissible cover compactification of maps to $\mathcal{X}$. Section 3 deals with the case of $\mathcal{X} = [\mathcal{M}_{0,4}/S_4]$. It generalizes and compactifies the picture of tetragonal curves, trigonal curves, and theta characteristics painted in $[\text{Vak01}]$. Section 4 specializes to the case of plane quintics. Appendix A describes the geometry of $\mathbb{P}^1$ bundles over orbifold curves that we need in Section 4. Theorem 1.1 follows from combining Proposition 4.2 [Proposition 4.3] [Proposition 4.11], [Proposition 4.12], and Proposition 4.14.

We work over the complex numbers $\mathbb{C}$. A stack means a Deligne–Mumford stack. An orbifold means a Deligne–Mumford stack without generic stabilizers. Orbifolds are usually denoted by curly letters $(X, \mathfrak{g})$, and stacks with generic stabilizers by curlier letters $(\mathcal{X}, \mathcal{Y})$. Coarse spaces are denoted by the absolute value sign $|(|X|, |\mathfrak{g}|)|$. A curve is a proper, reduced, connected, one-dimensional scheme/orbifold/stack, of finite type over $\mathbb{C}$. The projectivization of a vector bundle is the space of its one dimensional quotients.
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2 Moduli of branched covers of a stacky curve

In this section, we construct the admissible cover compactification of covers of a stacky curve in three steps. First, we construct the space of branched covers of a family of orbifold curves with a given branch divisor (§ 2.1). Second, we take a fixed orbifold curve and construct the space of branch divisors on it and its degenerations (§ 2.2). Third, we combine these results and accommodate generic stabilizers to arrive at the main construction (§ 2.3).

2.1 Covers of a family of orbifold curves with a given branch divisor

Let $S$ be a scheme of finite type over $\mathbb{C}$ and $\pi : X \to S$ a (balanced) twisted curve as in [AV02]. This means that $\pi$ is a flat family of at worst nodal stacky curves which are isomorphic to their coarse spaces except possibly at finitely many points. The stack structure at these finitely many points is of the following form:

At a node:

$$[\text{Spec } \mathbb{C}[x,y]/xy] / \mu_r$$ where $\zeta \in \mu_r$ acts by $\zeta : (x,y) \to (\zeta x, \zeta^{-1}y)$

At a smooth point:

$$[\text{Spec } \mathbb{C}[x]/\mu_r]$$ where $\zeta \in \mu_r$ acts by $x \to \zeta x$.

Since all our twisted curves will be balanced, we drop this adjective from now on. We call the integer $r$ the order of the corresponding orbifold point.

Let $\Sigma \subset X$ be a divisor that is étale over $S$ and lies in the smooth and representable locus of $\pi$. Define $\text{BrCov}_d(X/S, \Sigma)$ as the category fibered in groupoids over $\text{Schemes}$, whose objects over $T \to S$ are

$$(p : \mathcal{P} \to T, \phi : \mathcal{P} \to X_T),$$

where $p$ is a twisted curve over $T$ and $\phi$ is representable, flat, and finite with branch divisor $\text{br } \phi = \Sigma_T$.

Consider a point of this moduli problem, say $\phi : \mathcal{P} \to X_t$ over a point $t$ of $S$. Then $\mathcal{P}$ is also a twisted curve with nodes over the nodes of $X_t$. Since $\phi$ is representable, the orbifold points of $\mathcal{P}$ are only over the orbifold points of $X_t$.

We can associate some numerical invariants to $\phi$ which will remain constant in families. First, we have global invariants such as the number of connected components of $\mathcal{P}$, their arithmetic genera, and the degree of $\phi$ on them. Second, for every smooth orbifold point $x \in X_t$ we have the local invariant of the cover $\phi : \mathcal{P} \to X_t$ given by its monodromy around $x$. The monodromy is given by the action of the cyclic group $\text{Aut}_x X$ on the $d$ element set $\phi^{-1}(x)$. This data is equivalent to the data of the ramification indices over $x$ of the map $|\phi| : |\mathcal{P}| \to |X_t|$ between the coarse spaces. Notice that the monodromy data at $x$ also determines the number and the orders of the orbifold points of $\mathcal{P}$ over $x$. The moduli problem $\text{BrCov}_d(X/S, \Sigma)$ is thus a disjoint union of the moduli problems with fixed numerical invariants. Note, however, that having fixed $d$, $X/S$, and $\Sigma$, the set of possible numerical invariants is finite.
Proposition 2.1. \( \text{BrCov}_d(\mathcal{X}/S, \Sigma) \to S \) is a separated étale Deligne–Mumford stack of finite type. If the orders of all the orbinodes of all the fibers of \( \mathcal{X} \to S \) are divisible by the orders of all the permutations in the symmetric group \( \Sigma_d \), then \( \text{BrCov}_d(\mathcal{X}/S, \Sigma) \to S \) is also proper.

Proof. Fix non-negative integers \( g, n \). Consider the category fibered in groupoids over \text{Schemes}_S whose objects over \( T \to S \) are \( (p : \mathcal{P} \to T, \phi : \mathcal{P} \to \mathcal{X}) \) where \( \mathcal{P} \to T \) is a twisted curve of genus \( g \) with \( n \) smooth orbifold points and \( \phi \) is a twisted stable map with \( \phi_*([\mathcal{P}]) = d[X_\ast] \). This is simply the stack of twisted stable maps to \( \mathcal{X} \) of Abramovich–Vistoli. By [AV02], this is a proper Deligne–Mumford stack of finite type over \( S \). The conditions that \( \phi : \mathcal{P} \to \mathcal{X}_T \) be finite and unramified away from \( \Sigma \) are open conditions. For \( \phi : \mathcal{P} \to \mathcal{X}_T \) finite and unramified away from \( \Sigma \), the condition that \( \text{br} \phi = \Sigma_T \) is open and closed. Thus, for fixed \( g \) and \( n \) the stack \( \text{BrCov}_d^n(\mathcal{X}/S, \Sigma) \) is a separated Deligne–Mumford stack of finite type over \( S \). But there are only finitely many choices for \( g \) and \( n \), so \( \text{BrCov}_d(\mathcal{X}/S, \Sigma) \) is a separated Deligne–Mumford stack of finite type over \( S \).

To see that \( \text{BrCov}_d(\mathcal{X}/S, \Sigma) \to S \) is étale, consider a point \( t \to S \) and a point of \( \text{BrCov}_d(\mathcal{X}/S, \Sigma) \) over it, say \( \phi : \mathcal{P} \to \mathcal{X}_t \). Since \( \phi \) is a finite flat morphism of curves with a reduced branch divisor \( \Sigma \) lying in the smooth locus of \( \mathcal{X}_t \), it follows that the map on deformation spaces

\[
\text{Def}_\phi \to \text{Def}(\mathcal{X}_t, \Sigma_t)
\]

is an isomorphism. So \( \text{BrCov}_d(\mathcal{X}/S, \Sigma) \to S \) is étale.

Finally, assume that the indices of all the orbinodes of the fibers of \( \mathcal{X} \to S \) are sufficiently divisible as required. Let us check the valuative criterion for properness. For this, we take \( S \) to be a DVR \( \Delta \) with special point 0 and generic point \( \eta \). Let \( \phi : \mathcal{P}_\eta \to \mathcal{X}_\eta \) be a finite cover of degree \( d \) with branch divisor \( \Sigma_\eta \). We want to show that \( \phi \) extends to a finite cover \( \phi : \mathcal{P} \to \mathcal{X} \) with branch divisor \( \Sigma \), possibly after a finite base change on \( \Delta \).

Let \( x \) be the generic point of a component of \( \mathcal{X}_0 \). By Abhyankar’s lemma, \( \phi \) extends to a finite étale cover over \( x \), possibly after a finite base change on \( \Delta \). We then have an extension of \( \phi \) on all of \( \mathcal{X} \) except over finitely many points of \( \mathcal{X}_0 \).

Let \( x \in \mathcal{X}_0 \) be a smooth point. Recall that every finite flat cover of a punctured smooth surface extends to a finite flat cover of the surface. Indeed, the data of a finite flat cover consists of the data of a vector bundle along with the data of an algebra structure on the vector bundle. A vector bundle on a punctured smooth surface extends to a vector bundle on the surface by [Hor64]. The maps defining the algebra structure extend by Hartog’s theorem. Therefore, we get an extension of \( \phi \) over \( x \).

Let \( x \in \mathcal{X}_0 \) be a point that is not a limit of a node in the generic fiber. Then \( \mathcal{X} \) is locally simply connected at \( x \). (That is, \( V \setminus \{x\} \) is simply connected for a sufficiently small étale chart \( V \to \mathcal{X} \) around \( x \)). In this case, \( \phi \) trivially extends to an étale cover locally over \( x \).

Let \( x \in \mathcal{X}_0 \) be a node that is not a limit of a node in the generic fiber. Then \( \mathcal{X} \) has the form \( U = [\text{Spec} \mathbb{C}[x, y, t]/(xy - t^n)/\mu_r] \) near \( x \) where \( r \) is sufficiently divisible. In this case, \( \phi \) extends to an étale cover over \( U \) by [Lemma 2.2] where we interpret an étale cover of degree \( d \) as a map to \( \mathcal{Y} = B \Sigma_d \).

We thus have the required extension \( \phi : \mathcal{P} \to \mathcal{X} \). The equality of divisors \( \text{br} \phi = \Sigma \) holds outside a codimension 2 locus on \( \mathcal{X} \), and hence on all of \( \mathcal{X} \). The proof of the valuative criterion is then complete. \( \square \)

Lemma 2.2. Let \( \mathcal{Y} \) be a Deligne–Mumford stack and let \( \phi \) be the orbinode

\[
U = [\text{Spec} \mathbb{C}[x, y, t]/(xy - t^n)/\mu_r].
\]

Suppose \( \phi : U \to \mathcal{Y} \) is defined away from 0 and the map \( \text{br} \phi \) on the coarse spaces extends to all of \( |U| \).

Suppose for every \( \sigma \in \text{Aut}_{\mathcal{Y}}(0) \) we have \( \sigma^* = 1 \). Then \( \mu \) extends to a morphism \( \phi : U \to \mathcal{Y} \).
Proof. Let $G = \text{Aut}_{\text{ plethora}} Y$. We have an étale local presentation of $Y$ of the form $[Y/G]$ where $Y$ is a scheme. Since it suffices to prove the statement locally around 0 in the étale topology, we may take $\mathcal{Y} = [Y/G]$. Consider the following diagram:

$$
\begin{array}{c}
\text{Spec } \mathbb{C}[u, v, t]/(uv - t) = \tilde{U} \\
\downarrow \\
\text{Spec } \mathbb{C}[x, y, t]/(xy - t^n) = U \\
\downarrow \\
\text{Spec } \mathbb{C}[x, y, t]/(xy - t^n)/\mu_r = U \to [Y/G].
\end{array}
$$

We have an action of $\mu_{nr}$ on $\tilde{U}$ where an element $\zeta \in \mu_{nr}$ acts by $\zeta \cdot (u, v, t) \mapsto (\zeta u, \zeta^{-1} v, t)$. The map $\tilde{U} \to U$ is the geometric quotient by $\mu_n = \langle \zeta \rangle \subset \mu_{nr}$ and the map $U \to Y$ is the stack quotient by $\mu_r = \mu_{nr}/\mu_n$. Since $\tilde{U}$ is simply connected, we get a map $\mu_{nr} \to G$ and a lift $\tilde{U} \to Y$ of $\phi$ which is equivariant with respect to the $\mu_{nr}$ action on $U$ and the $G$ action on $Y$. Since $\sigma^r = 1$ for all $\sigma \in G$, the map $\tilde{U} \to Y$ is equivariant with respect to the $\mu_n$ action on $\tilde{U}$ and the trivial action on $Y$. Hence it gives a map $U \to Y$ and by composition a map $\tilde{U} \to [Y/G]$. Since $\tilde{U} \to U$ is étale, we have extended $\phi$ to a map at 0 étale locally on $U$.

\[ \Box \]

2.2 The Fulton–MacPherson configuration space

The goal of this section is to construct a compactified configuration space of $b$ distinct points on orbifold smooth curves in the style of Fulton and MacPherson. We first recall (a slight generalization of) the notion of a $b$-pointed degeneration from [FM94]. Let $X$ be a smooth schematic curve and let $x_1, \ldots, x_n$ be distinct points of $X$. A $b$-pointed degeneration of $(X, \{x_1, \ldots, x_n\})$ is the data of

$$(\rho : Z \to X, \{\sigma_1, \ldots, \sigma_b\}, \{\bar{x}_1, \ldots, \bar{x}_n\}),$$

where

- $Z$ is a nodal curve and $\{\sigma_j, x_i\}$ are $b + n$ distinct smooth points on $Z$;
- $\rho$ maps $\bar{x}_i$ to $x_i$;
- $\rho$ is an isomorphism on one component of $Z$, called the main component and also denoted by $X$.

The rest of the curve, namely $Z \setminus X$, is a disjoint union of trees of smooth rational curves, each tree meeting $X$ at one point.

We say that the degeneration is stable if each component of $Z$ contracted by $\rho$ has at least three special points (nodes or marked points).

We now define a similar gadget for an orbifold curve.

Definition 2.3. Let $\mathcal{X}$ be a smooth orbifold curve with $n$ orbifold points $x_1, \ldots, x_n$. A $b$-pointed degeneration of $\mathcal{X}$ is the data of

$$(\rho : Z \to \mathcal{X}, \{\sigma_1, \ldots, \sigma_b\}),$$

where

- $Z$ is a twisted nodal curve, schematic away from the nodes and $n$ smooth points, say $\bar{x}_1, \ldots, \bar{x}_n$;
- $\sigma_1, \ldots, \sigma_b$ are $b$ distinct points in the smooth and schematic locus of $Z$;
- $\rho$ maps $\bar{x}_i$ to $x_i$ and induces an isomorphism $\rho : \text{Aut}_x \overset{\rho}{\to} \text{Aut}_x \mathcal{X}$;
- the data on the underlying coarse spaces $(|\rho| : |Z| \to |\mathcal{X}|, \{\sigma_1, \ldots, \sigma_b\}, \{\bar{x}_1, \ldots, \bar{x}_n\})$ is a $b$-pointed degeneration of $(|\mathcal{X}|, \{x_1, \ldots, x_n\})$. 

5
We say that the degeneration is \textit{stable} if the degeneration of the underlying coarse spaces is stable.

Let $\mathcal{X}$ be an orbifold curve with $n$ orbifold points $x_1, \ldots, x_n$. Let $U \subset \mathcal{X}^b$ be the complement of all the diagonals and orbifold points. Let $\pi: \mathcal{X} \times U \rightarrow U$ be the second projection and $\sigma_j: U \rightarrow \mathcal{X} \times U$ the section of $\pi$ corresponding to the $j$th factor, namely

$$\sigma_j(u_1, \ldots, u_b) = (u_j, u_1, \ldots, u_b).$$

Let $\rho: \mathcal{X} \times U \rightarrow \mathcal{X}$ be the first projection.

\begin{proposition}
There exists a smooth Deligne–Mumford stack $\mathcal{X}[b]$ along with a family of twisted curves $\pi: \mathcal{Z} \rightarrow \mathcal{X}[b]$ such that the following hold:
1. $\mathcal{X}[b]$ contains $U$ as a dense open substack and $\pi: \mathcal{Z} \rightarrow \mathcal{X}[b]$ restricts over $U$ to $\mathcal{X} \times U \rightarrow U$.
2. $\mathcal{X}[b] \setminus U$ is a divisor with simple normal crossings and the total space $\mathcal{Z}$ is smooth.
3. The sections $\sigma_j: U \rightarrow \mathcal{X} \times U$ and the map $\rho: \mathcal{X} \times U \rightarrow \mathcal{X}$ extend to sections $\sigma_j: \mathcal{X}[b] \rightarrow \mathcal{Z}$ and a map $\rho: \mathcal{Z} \rightarrow \mathcal{X}$.
4. For every point $t$ in $\mathcal{X}[b]$, the datum $(\rho: \mathcal{Z}, \pi, \{\sigma_j\})$ is a $b$-pointed stable degeneration of $\mathcal{X}$.
5. We may arrange $\mathcal{X}[b]$ and $\pi: \mathcal{Z} \rightarrow \mathcal{X}[b]$ so that the indices of the orbinodes of the fibers of $\pi$ are sufficiently divisible.
\end{proposition}

\textbf{Proof.} Let $X$ be the coarse space of $\mathcal{X}$. Let $\mathcal{X}[b]$ be the Fulton–MacPherson moduli space of $b$ points on $X$ where the points are required to remain distinct and also distinct from the $x_i$. It is a smooth projective variety containing $U$ as a dense subset with a normal crossings complement. It admits a universal family of nodal curves $\pi: Z \rightarrow \mathcal{X}[b]$ with smooth total space $Z$ along with $b + n$ sections $\sigma_j: X[b] \rightarrow Z$ and $\tilde{x}_i: X[b] \rightarrow Z$, and a map $\rho: Z \rightarrow X$. The universal family extends the constant family $X \times U \rightarrow U$; the sections $\sigma_j$ extend the tautological sections; the sections $\tilde{x}_i$ extend the constant sections $x_i$; and the map $\rho$ extends the projection $X \times U \rightarrow X$. The fibers of $Z \rightarrow \mathcal{X}[b]$ along with the points $\sigma_j$, $\tilde{x}_i$, and the map $\rho$ form a stable $b$-pointed degeneration of $(X, x_1, \ldots, x_n)$.

We can construct $\mathcal{X}[b]$ following the method of [FM94]—start with $X^b$ and the constant family $X \times X^b \rightarrow X$ and one-by-one blow up the proper transforms of the strata where the sections $\sigma_j$ coincide among themselves or coincide with the points $x_i$. We summarize the picture so far in the following diagram:

\[
\begin{array}{c}
\mathcal{X} \times U & \xrightarrow{\sigma_j, x_i} & Z & \xrightarrow{\rho} & X \\
\downarrow & & \downarrow & & \\
U & \xrightarrow{\sigma_j, \tilde{x}_i} & \mathcal{X}[b] & \xrightarrow{\sigma_j} & \mathcal{X} \\
\end{array}
\]

We now use the results of Olsson [Ols07] to modify $\mathcal{X}[b]$ and $Z \rightarrow \mathcal{X}[b]$ in a stacky way to obtain the claimed $\mathcal{X}[b]$ and $Z \rightarrow \mathcal{X}[b]$. Our argument follows the proof of [Ols07] Theorem 1.9. Fix a positive integer $d$ that is divisible by $a_i = |\text{Aut}_{X[i]} X|$ for all $i$. The simple normal crossings divisor $X[b] \setminus U$ gives a canonical log structure $\mathcal{M}$ on $X[b]$. This log structure agrees with the log structure that $X[b]$ gets from the family of nodal curves $Z \rightarrow \mathcal{X}[b]$ as explained in [Ols07] § 3. Denote by $r$ the number of irreducible components of $X[b] \setminus U$. Let $\alpha$ be the vector $(d, \ldots, d)$ of length $r$. Let $\mathcal{X}[b]$ be the stack $\mathcal{F}(\alpha)$ constructed in [Ols07] Lemma 5.3. The defining property of $\mathcal{F}(\alpha)$ is the following: a map $T \rightarrow \mathcal{F}(\alpha)$ corresponds to a map $T \rightarrow X[b]$ along with an extension of log structures $\mathcal{M}_T \rightarrow \mathcal{M}'_T$ which locally has the form $\mathbb{N}^r \xrightarrow{d \cdot} \mathbb{N}^r$. Thus, $\mathcal{X}[b]$ maps to $X[b]$ and comes equipped with a tautological extension $\mathcal{M} \rightarrow \mathcal{M}'$, where we have used the same symbol $\mathcal{M}$ to denote the pullback to $X[b]$ of the log structure $\mathcal{M}$ on $X[b]$. By [Ols07] Theorem 1.8, the data $(Z \times_{\mathcal{X}[b]} X[b], \{x_i, a_i\}, \mathcal{M} \rightarrow \mathcal{M}')$ defines a
twisted curve $Z \to X[b]$ with a map $Z \to Z$ which is a purely stacky modification (an isomorphism on coarse spaces).

Before we proceed, let us describe the modification $X[b] \to X[b]$ and $Z \to Z$ explicitly in local coordinates. Let $0 \in X[b]$ be a point such that $Z_0$ is an $l$-nodal curve. Étale locally around $0$, the pair $(X[b], X[b] \setminus U)$ is isomorphic to

$$\text{Spec} \mathbb{C}[x_1, \ldots, x_b, x_1 \ldots x_l].$$

(1)

Étale locally around a node of $Z_0$, the map $Z_0 \to X[b]$ is isomorphic to

$$\text{Spec} \mathbb{C}[x_1, \ldots, x_b, y, z]/(yz - x_1) \to \text{Spec} \mathbb{C}[x_1, \ldots, x_b].$$

(2)

In the coordinates of (1), the map $X[b] \to X[b]$ is given by

$$\left[ \text{Spec} \mathbb{C}[u_1, \ldots, u_l, x_{i+1}, \ldots, x_b]/\mu_d^l \right] \to \text{Spec} \mathbb{C}[x_1, \ldots, x_b]$$

$$(u_1, \ldots, u_l, x_{i+1}, \ldots, x_b) \mapsto (u_1^l, \ldots, u_l^l, x_{i+1}, \ldots, x_b).$$

(3)

Here $\mu_d^l$ acts by multiplication on $(u_1, \ldots, u_l)$ and trivially on the $x_i$. Having described $X[b] \to X[b]$, we turn to $Z \to Z$. Let

$$V = \text{Spec} \mathbb{C}[u_1, \ldots, u_l, x_{i+1}, \ldots, x_b]$$

be the étale local chart of $X[b]$ from (3). The map $Z_V \to Z_V$ is an isomorphism except over the points $\bar{x}_i$ and the nodes of $Z_0$. Around the point $\bar{x}_i$ of $Z_0$, the map $Z_V \to Z_V$ is given by

$$[\text{Spec} \mathbb{C}[s]/\mu_{a_i}] \to \text{Spec} \mathbb{C}[t],$$

$$s \mapsto t^{a_i},$$

(4)

where $\zeta \in \mu_{a_i}$ acts by $\zeta \cdot s = \zeta s$. Around the node of $Z_0$ from (2), the map $Z_V \to Z_V$ is given by

$$[\text{Spec} \mathbb{C}[a, b]/(ab - u_l)/\mu_d] \to \text{Spec} \mathbb{C}[y, z]/(yz - u_1^d)$$

$$(a, b) \mapsto (a^d, b^d),$$

(5)

where $\zeta \in \mu_d$ acts by $\zeta \cdot (a, b) = (\zeta a, \zeta^{-1} b)$.

We now check that our construction has the claimed properties. From (3), it follows that $X[b] \to X[b]$ is an isomorphism over $U$. Therefore, $X[b]$ contains $U$ as a dense open. From (3), we also see that the complement is simple normal crossings. From (4), we see that the map $Z_U \to Z_U$ is the root stack of order $a_i$ at $x_i \times U$. Therefore, $Z_U \to U$ is isomorphic to $X \times U \to U$. From the local description, we see that $Z$ is smooth. The sections $\sigma_j: X[b] \to Z$ give sections $\sigma_j: X[b] \to Z \times_{X[b]} X[b]$. But $Z \to Z \times X[b]$ is an isomorphism around $\sigma_j$. So we get sections $\sigma_j: X[b] \to Z$. To get $\rho: Z \to \bar{X}$, we start with the map $|\rho|: \bar{Z} \to X$ obtained by composing $\bar{Z} \to Z$ with $\rho: Z \to X$. To lift $|\rho|$ to $\rho: Z \to \bar{X}$, we must show that the divisor $|\rho|^{-1}(x_i) \subset \bar{Z}$ is $a_i$ times a Cartier divisor. The divisor $|\rho|^{-1}(x_i) \subset Z$ consists of multiple components: a main component $\bar{x}_i(X[b])$ and several other components that lie in the exceptional locus of $Z \to X \times X[b]$. From (4), we see that the multiplicity of the preimage in $\bar{Z}$ of $\bar{x}_i(X[b])$ is precisely $a_i$. In the coordinates of (2), the exceptional components are cut out by powers of $y$, $z$, or $x_i$. In any case, we see from (5) that their preimage in $\bar{Z}$ is divisible by $d$, which is in turn divisible by $a_i$. Therefore, $|\rho|: \bar{Z} \to X$ lifts to $\rho: Z \to \bar{X}$. For a point $t$ in $X[b]$, the datum $(\rho: Z_t \to X, \{\sigma_j\}, \{\bar{x}_i\})$ is a $b$-pointed stable degeneration of $(X, \{\bar{x}_i\})$. From the description of $Z \to Z$, it follows that $(\rho: Z_t \to X, \{\sigma_j\})$ is a $b$-pointed stable degeneration of $X$. Finally, we see from (5) that the order of the orbinodes of the fibers of $Z \to X[b]$ is $d$, which we can take to be sufficiently divisible.
2.3 Moduli of branched covers of a stacky curve

The goal of this section to combine the results of the previous two sections and accommodate generic stabilizers.

Let \( \mathcal{X} \) be a smooth stacky curve. We can express \( \mathcal{X} \) as an étale gerbe \( \mathcal{X} \to \mathcal{X} \), where \( \mathcal{X} \) is an orbifold curve. Fix a positive integer \( b \) and let \( \mathcal{X}[b] \) be a Fulton–MacPherson space of \( b \) distinct points constructed in [Proposition 2.4]. Let \( \pi : Z \to \mathcal{X}[b] \), and \( \sigma_j : \mathcal{X}[b] \to Z \), and \( \rho : Z \to \mathcal{X} \) be as in [Proposition 2.4]. We think of \( \mathcal{X}[b] \) as the space of \( b \)-pointed stable degenerations of \( \mathcal{X} \), and the data \( (Z, \rho : Z \to \mathcal{X}, \sigma_j) \) as the universal object.

Let the divisor \( \Sigma \subset Z \) be the union of the images of the sections \( \sigma_j \). Set \( \mathcal{X} = Z \times_\rho \mathcal{X} \). Define \( \text{BrCov}_d(\mathcal{X}, b) \) as the category fibered in groupoids over \( \text{Schemes} \) whose objects over a scheme \( T \) are

\[
(T \to \mathcal{X}[b], \phi : P \to \mathcal{X}_T),
\]
such that \( \psi : P \to \mathcal{X}_T \) induced by \( \phi \) is representable, flat, and finite of degree \( d \) with \( \text{br} \psi = \Sigma_T \).

**Theorem 2.5.** \( \text{BrCov}_d(\mathcal{X}, b) \) is a Deligne–Mumford stack smooth and separated over \( \text{C} \) of dimension \( b \). If the indices of the orbinodes of the fibers of \( Z \to \mathcal{X}[b] \) are sufficiently divisible, then it is also proper.

**Remark 2.6.** Strictly speaking, \( \text{BrCov}_d(\mathcal{X}, b) \) is an abuse of notation since this object depends on the choice of \( \mathcal{X}[b] \). However, as long as the nodes of \( Z \to \mathcal{X}[b] \) are sufficiently divisible, this choice will not play any role.

**Proof.** We have a natural transformation \( \text{BrCov}_d(\mathcal{X}, b) \to \text{BrCov}_d(Z/\mathcal{X}[b], \Sigma) \) defined by \( \phi \to \psi \). Let \( S \) be a scheme with a map \( \mu : S \to \text{BrCov}_d(Z/\mathcal{X}[b], \Sigma) \). Such \( \mu \) is equivalent to \((\pi : P \to S, \psi : P \to Z_S)\). Then \( S \times_\mu \text{BrCov}_d(\mathcal{X}, b) \) is just the stack of lifts of \( \psi \) to \( Z_S \). Equivalently, setting \( \mathcal{P} = P \times_\phi \mathcal{X} \), the objects of \( S \times_\mu \text{BrCov}_d(\mathcal{X}, b) \) over \( T/S \) are sections \( P_T \to \mathcal{P}_T \) of \( \mathcal{P}_T \to \mathcal{P} \). We denote the latter by \( \text{Sect}_S(\mathcal{P} \to P) \).

That \( S \times_\mu \text{BrCov}_d(\mathcal{X}, b) = \text{Sect}_S(\mathcal{P} \to P) \) is a separated Deligne–Mumford stack of finite type over \( S \) follows from [AV02]. That \( \text{Sect}_S(\mathcal{P} \to P) \to S \) is étale follows from the unique infinitesimal extension property for sections of étale morphisms (This is standard for étale morphisms of schemes and not too hard to check for étale morphisms of Deligne–Mumford stacks.) Assume that the orders of the nodes of \( Z_S \to S \) are sufficiently divisible. The nodes of \( P \) are étale over the nodes of \( Z \) via the map \( \psi \) of degree \( d \). Therefore, the orders of the nodes of \( P \to S \) are at worst \( 1/d \) times the orders of the nodes of \( Z_S \to S \). So we may assume that the orders of the nodes of \( P \to S \) are also sufficiently divisible. To check that \( \text{Sect}_S(\mathcal{P} \to P) \to S \) is proper, let \( S = \Delta \) be a DVR and let a section \( s : P \to \mathcal{P} \) be given over the generic point of \( \Delta \). First, \( s \) extends to a section over the generic points of \( P_0 \) after replacing \( \Delta \) by a finite cover. Second, \( s \) extends to a section over the smooth points and the generic nodes of \( P_0 \) since \( P \) is locally simply connected at these points. Finally, \( s \) extends over the non-generic nodes by [Lemma 2.2] (the extension of the section on the coarse spaces follows from normality).

We have thus proved that \( \text{BrCov}_d(\mathcal{X}, b) \to \text{BrCov}_d(Z/\mathcal{X}[b], \Sigma) \) is representable by a separated étale morphism of Deligne–Mumford stacks which is also proper if the orders of the nodes of \( Z \to \mathcal{X}[b] \) are sufficiently divisible. By combining this with [Proposition 2.4], we complete the proof. \( \square \)

Let \( K(\mathcal{X}, d) \) be the Abramovich–Vistoli space of twisted stable maps to \( \mathcal{X} \). This is the moduli space of \( \phi : P \to \mathcal{X} \), where \( P \) is a twisted curve and \( \phi \) is a representable morphism such that the map on the underlying coarse spaces is a Kontsevich stable map that maps the fundamental class of \( |P| \) to \( d \) times the fundamental class of \( |\mathcal{X}| \).
Proposition 2.7. $\text{BrCov}_d(\mathcal{X}, b)$ admits a morphism to $K(\mathcal{X}, d)$. 

Proof. On $\text{BrCov}_d(\mathcal{X}, b)$ we have a universal twisted curve $\mathcal{P} \to \text{BrCov}_d(\mathcal{X}, b)$ with a morphism $\mathcal{P} \to \mathcal{X}$. This morphism is obtained by composing the universal $\mathcal{P} \to \mathcal{Z}$ with $\rho: \mathcal{Z} \to \mathcal{X}$. By [AV02, Proposition 9.11], there exists a factorization $\mathcal{P} \to \mathcal{P}' \to \mathcal{X}$, where $\mathcal{P}' \to \text{BrCov}_d(\mathcal{X}, b)$ is a twisted curve and $\mathcal{P}' \to \mathcal{X}$ is a twisted stable map. On coarse spaces, this factorization is the contraction of unstable rational components. The twisted stable map $\mathcal{P}' \to \mathcal{X}$ gives the morphism $\text{BrCov}_d(\mathcal{X}, b) \to K(\mathcal{X}, d)$.

3 Moduli of tetragonal curves on Hirzebruch surfaces

In this section, we apply our results to $\mathcal{X} = [M_{0,4}/S_4]$. Set

$$\tilde{M}_{0,4} := [M_{0,4}/S_4].$$

We interpret the quotient as the moduli space of stable marked rational curves where the marking consists of a divisor of degree 4. Let

$$\pi: (\tilde{S}, \tilde{c}) \to \tilde{M}_{0,4}$$

be the universal family where $\tilde{S} \to \tilde{M}_{0,4}$ is a nodal curve of genus 0 and $\tilde{c} \subset \tilde{S}$ a divisor of relative degree 4.

The action of $S_4$ on $\tilde{M}_{0,4}$ has a kernel: the Klein four group

$$K = \{\text{id}, (12)(34), (13)(24), (14)(23)\},$$

acts trivially. Therefore, a generic $t \to \tilde{M}_{0,4}$ has automorphism group $K$. The action of $K$ on $\tilde{S}_t$ and $\tilde{c}_t$ is faithful. There are three special points $t$ at which Aut$_t\tilde{M}_{0,4}$ jumps. The first, which we label $t = 0$, is specified by

$$\tilde{S}_t, \tilde{c}_t \cong (\mathbb{P}^1, \{1, i, -1, -i\}),$$

with Aut$_0\tilde{M}_{0,4} = D_4 \subset S_4$. The second, which we label $t = 1$, is specified by

$$\tilde{S}_t, \tilde{c}_t \cong (\mathbb{P}^1, \{0, 1, e^{2\pi i/3}, e^{-2\pi i/3}\}),$$

with Aut$_1\tilde{M}_{0,4} = A_4 \subset S_4$. The third, which we label $t = \infty$, is specified by

$$\tilde{S}_t, \tilde{c}_t \cong (\mathbb{P}^1 \cup \mathbb{P}^1, \{0, 1, 0, 1\})$$

where the two $\mathbb{P}^1$s are attached at a node (labeled $\infty$ on both). We have Aut$_\infty\tilde{M}_{0,4} = D_4 \subset S_4$.

The quotient $S_4/K$ is isomorphic to $S_3$. Therefore, the orbifold curve underlying $\tilde{M}_{0,4}$ is $[M_{0,4}/S_3]$. Consider the inclusion $S_3 \subset S_4$ as permutations acting only on the first three elements. The inclusion $S_3 \to S_4$ is a section of the projection $S_4 \to S_3$. We can thus think of $S_3$ as acting on $\tilde{M}_{0,4}$ by permuting three of the four points and leaving the fourth fixed. Set $\tilde{M}_{0,1+3} := [M_{0,4}/S_3]$. We can interpret this quotient as the moduli space of stable marked rational curves, where the marking consists of a point and a divisor of degree 3 (hence the notation “1+3”).
We have the fiber product diagram

\[
\begin{array}{ccc}
\widetilde{M}_{0,4} & \longrightarrow & B \mathcal{S}_4 \\
\downarrow & & \downarrow \\
\widetilde{M}_{0,1+3} & \longrightarrow & B \mathcal{S}_3.
\end{array}
\]

Since \( \mathcal{S}_4 = K \times \mathcal{S}_3 \), the map \( B \mathcal{S}_4 \to B \mathcal{S}_3 \) is the trivial \( \mathcal{K} \) gerbe \( B \mathcal{K} \), where \( \mathcal{K} \) is the sheaf of groups on \( B \mathcal{S}_3 \) obtained by the action of \( \mathcal{S}_3 \) on \( K \) by conjugation. Therefore, we get that \( \widetilde{M}_{0,4} = B \mathcal{K} \times_{B \mathcal{S}_3} \widetilde{M}_{0,1+3} \).

The relation \( \mathcal{S}_4 = K \times \mathcal{S}_3 \) indicates a relation between quadruple and triple covers. This is the classic correspondence due to Recillas.

**Proposition 3.1.** Let \( \mathcal{P} \) be a Deligne-Mumford stack. We have a natural bijection between \( \{ \phi : \mathcal{C} \to \mathcal{P} \} \), where \( \phi \) is a finite étale cover of degree 4 and \( \{ (\psi : \mathcal{D} \to \mathcal{P}, \mathcal{L}) \} \), where \( \psi \) is a finite étale cover of degree 3 and \( \mathcal{L} \) a line bundle on \( \mathcal{D} \) with \( \mathcal{L}^2 = O_{\mathcal{D}} \) and \( \text{Norm}_\psi \mathcal{L} = O_{\mathcal{P}} \). Furthermore, under this correspondence we have \( \phi_* O_{\mathcal{C}} = O_{\mathcal{P}} \oplus \psi_* \mathcal{L} \).

**Proof.** This is essentially the content of [Rec73]. We sketch a proof in our setup.

An étale cover of degree \( d \) is equivalent to a map to \( B \mathcal{S}_d \). From \( \mathcal{S}_4 = K \times \mathcal{S}_3 \), we get that an étale cover \( \mathcal{C} \to \mathcal{P} \) of degree 4 is equivalent to a map \( \mu : \mathcal{P} \to B \mathcal{S}_3 \) and a section of \( \mathcal{K} \times_{\mu} \mathcal{P} \). Such a section is in turn equivalent to an element of \( H^1(\mathcal{P}, \mathcal{K}) \). Let \( \psi : \mathcal{D} \to \mathcal{P} \) be the étale cover of degree 3 corresponding to \( \mu \). Denote \( \mathcal{K} \times_{\mu} \mathcal{P} \) by \( \mathcal{K} \) for brevity. We have the following exact sequence on \( \mathcal{P} \) (pulled back from an analogous exact sequence on \( B \mathcal{S}_3 \)):

\[
0 \to \mathcal{K} \to \psi_* (\mathcal{Z}_2)_{\mathcal{P}} \to \psi_* (\mathcal{Z}_2)_{\mathcal{P}} \to 0.
\]

The associated long exact sequence gives

\[
H^1(\mathcal{P}, \mathcal{K}) = \ker \left( H^1(\mathcal{D}, \mathcal{Z}_2) \to H^1(\mathcal{P}, \mathcal{Z}_2) \right).
\]

If we interpret \( H^1(\mathcal{P}, \mathcal{K}) \) as two-torsion line bundles, then the map \( H^1(\mathcal{D}, \mathcal{Z}_2) \to H^1(\mathcal{P}, \mathcal{Z}_2) \) is the norm map. The bijection follows.

In the rest of the proof, we view the data of the line bundle \( \mathcal{L} \) as the data of an étale double cover \( \tau : \tilde{\mathcal{D}} \to \mathcal{D} \). The double cover and the line bundle are related by \( \tau_* O_{\tilde{\mathcal{D}}} = O_{\mathcal{D}} \oplus \mathcal{L} \).

It suffices to prove the last statement universally on \( B \mathcal{S}_4 \)—it will then follow by pullback. A cover of \( B \mathcal{S}_4 \) is just a set with an \( \mathcal{S}_4 \) action and a sheaf on \( B \mathcal{S}_4 \) is just an \( \mathcal{S}_4 \)-representation. Consider the 4-element \( \mathcal{S}_4 \)-set \( \mathcal{C} = \{1, 2, 3, 4\} \). The corresponding 3-element \( \mathcal{S}_3 \)-set \( \mathcal{D} \) with an étale double cover \( \tau : \tilde{\mathcal{D}} \to \mathcal{D} \) is given by

\[
\tilde{\mathcal{D}} = \{(12), (13), (14), (23), (24), (34)\} \to \mathcal{D} = \{(12)(34), (13)(23), (14)(23)\}.
\]

It is easy to check that we have an isomorphism of \( \mathcal{S}_4 \)-representations

\[
\mathbb{C} \oplus \mathbb{C}[\tilde{\mathcal{D}}] = \mathbb{C}[\mathcal{D}] \oplus \mathbb{C}[\mathcal{C}].
\]

In terms of \( \phi : \mathcal{C} \to B \mathcal{S}_4 \), \( \psi : \mathcal{D} \to B \mathcal{S}_4 \), and \( \tau : \tilde{\mathcal{D}} \to \mathcal{D} \), this isomorphism can be written as an isomorphism of sheaves on \( B \mathcal{S}_4 \):

\[
O \oplus \psi_* O_{\mathcal{D}} \oplus \psi_* \mathcal{L} = \phi_* O_{\mathcal{C}} \oplus \phi_* O_{\mathcal{C}}.
\]

Canceling \( \psi_* O_{\mathcal{D}} \) gives the statement we want. \( \square \)
Let $f : S \to \mathbb{P}^1$ be a $\mathbb{P}^1$-bundle and $C \subset S$ a smooth curve such that $f : C \to \mathbb{P}^1$ is a finite simply branched map of degree 4. Away from the $b = 2g(C) + 6$ branch points $p_1, \ldots, p_b$ of $C \to \mathbb{P}^1$, we get a morphism

$$\phi : \mathbb{P}^1 \setminus \{p_1, \ldots, p_b\} \to \overline{\mathcal{M}}_{0,4} \setminus \{\infty\}.$$ 

Set $\mathcal{P} = \mathbb{P}^1(\sqrt{p_1}, \ldots, \sqrt{p_b})$. Abusing notation, denote the point of $\mathcal{P}$ over $p_i$ by the same letter. Then $\phi$ extends to a morphism $\tilde{\phi} : \mathcal{P} \to \overline{\mathcal{M}}_{0,4}$, which maps $p_1, \ldots, p_b$ to $\infty$, is étale over $\infty$, and the underlying map of orbifolds $\mathcal{P} \to \overline{\mathcal{M}}_{0,1+3}$ is representable of degree $b$. We can construct the family of 4-pointed rational curves that gives $\phi$ as follows. Consider the $\mathbb{P}^1$-bundle $S \times_{\phi_i} \mathcal{P}$ and the curve $C \times_{\phi_i} \mathcal{P} \subset S \times_{\phi_i} \mathcal{P}$. Since $C \to \mathbb{P}^1$ was simply branched, $C \times_{\phi_i} \mathcal{P}$ has a simple node over each $p_i$. Let $\widetilde{S} = S \times_{\phi_i} \mathcal{P}$ be the blow up at these nodes and $C$ the proper transform of $C$. The pair $(\widetilde{S}, C)$ over $\mathcal{P}$ gives the map $\phi : \mathcal{P} \to \overline{\mathcal{M}}_{0,4}$. The geometric fiber of $(\widetilde{S}, C)$ over $p_i$ is isomorphic to $(\mathbb{P}^1 \cup \mathbb{P}^1, \{0, 1; 0, 1\})$ where we think of the $\mathbb{P}^1$'s as joined at $\infty$. The action of $\text{Aut}_p \mathcal{P}$ is trivial on one component and is given by $x \mapsto 1 - x$ on the other component.

Conversely, let $\phi : \mathcal{P} \to \overline{\mathcal{M}}_{0,4}$ be a morphism that maps $p_1, \ldots, p_b$ to $\infty$, is étale over $\infty$, and the underlying map of orbifolds $\mathcal{P} \to \overline{\mathcal{M}}_{0,1+3}$ is representable of degree $b$. Let $f : (\widetilde{S}, C) \to \mathcal{P}$ be the corresponding family of 4-pointed rational curves. Away from $p_1, \ldots, p_b$, the map $f : \widetilde{S} \to \mathcal{P}$ is a $\mathbb{P}^1$-bundle. Locally near $p_i$, we have

$$\mathcal{P} = [\text{Spec } \mathbb{C}[t]/\mathbb{Z}_2].$$

Set $U = \text{Spec } \mathbb{C}[t]$. Since $\phi$ is étale at $t = 0$, the total space $\widetilde{S}_U$ is smooth. Since $t = 0$ maps to $\infty$, the fiber of $f$ over 0 is isomorphic to $(\mathbb{P}^1 \cup \mathbb{P}^1, \{0, 1, 0, 1\})$ where we think of the $\mathbb{P}^1$'s as joined at $\infty$. Consider the map $\mathbb{Z}_2 = \text{Aut}_p \mathcal{P} \to D_4 = \text{Aut}_{\overline{\mathcal{M}}_{0,4}}$. Since the map induced by $\phi$ on the underlying orbifolds is representable, the image of the generator of $\mathbb{Z}_2$ is an element of order 2 in $D_4$ not contained in the Klein four subgroup. The only possibility is a 2-cycle, whose action on the fiber is trivial on one component and $x \mapsto 1 - x$ on the other. Let $S_U = S'_U$ be obtained by blowing down the component on which the action is non-trivial and let $C'_U \subset S_U$ be the image of $C_U$. Note that the $\mathbb{Z}_2$ action on $(S_U, C_U)$ descends to an action on $(S'_U, C'_U)$ which is trivial on the central fiber. Thus $S'_U/\mathbb{Z}_2 \to U/\mathbb{Z}_2$ is a $\mathbb{P}^1$ bundle and $C'_U/\mathbb{Z}_2 \to U/\mathbb{Z}_2$ is simply branched (the quotients here are geometric quotients, not stack quotients). Let $(S, C)$ be obtained from $(\widetilde{S}, C)$ by performing this blow down and quotient operation around every $p_i$. Then $f : S \to \mathbb{P}^1$ is a $\mathbb{P}^1$-bundle and $C \subset S$ is a smooth curve such that $C \to \mathbb{P}^1$ is a finite, simply branched cover of degree 4. We call the construction of $(S, C)$ from $(\widetilde{S}, C)$ the blow-down construction.

We thus have a natural bijection

$$\{f : (S, C) \to \mathbb{P}^1\} \leftrightarrow \{\phi : \mathcal{P} \to \overline{\mathcal{M}}_{0,4}\},$$

where on the left we have a $\mathbb{P}^1$-bundle $S \to \mathbb{P}^1$ and a smooth curve $C \subset S$ such that $f : C \to \mathbb{P}^1$ is a finite, simply branched cover of degree 4 and on the right we have $\mathcal{P} = \mathbb{P}^1(\sqrt{p_1}, \ldots, \sqrt{p_b})$ and $\phi$ that maps $p_i$ to $\infty$, is étale over $\infty$, and induces a representable finite map of degree $b$ to $\overline{\mathcal{M}}_{0,1+3}$.

Assume that $C \subset S$ on the left is general so that the induced map $\mathcal{P} \to \overline{\mathcal{M}}_{0,1+3}$ is simply branched over distinct points away from 0, 1, or $\infty$. Then the monodromy of $\mathcal{P} \to \overline{\mathcal{M}}_{0,1+3}$ over 0, 1, and $\infty$ is given by a product of 2-cycles, a product of 3-cycles, and identity, respectively. Said differently, the map $|\phi| : \mathbb{P}^1 \to \mathbb{P}^1$ has ramification $(2, 2, \ldots)$ over 0; $(3, 3, \ldots)$ over 1, and $(1, 1, \ldots)$ over $\infty$. In particular, the degree $b$ of $|\phi|$ is divisible by 6. Taking $b = 6d$, the map $\phi : \mathcal{P} \to \overline{\mathcal{M}}_{0,1+3}$ has $5d - 2$ branch points.

Denote by $\Omega_d$ the open and closed substack of $\text{BrCov}_{6d}(\overline{\mathcal{M}}_{0,4}, 5d - 2)$ that parametrizes covers with connected domain and ramification $(2, 2, \ldots)$ over 0, $(3, 3, \ldots)$ over 1, and $(1, 1, \ldots)$ over $\infty$. Denote
by $\mathcal{U}_d$ the open and closed substack of $\text{BrCov}_{6d}(\overline{\mathcal{M}}_{0,1+3}, 5d - 2)$ defined by the same two conditions. Let $\overline{\mathcal{U}}_{d,g}$ be the space of admissible covers of degree $d$ of genus $0$ curves by genus $g$ curves as in [ACV03].

**Proposition 3.2.** $\Omega_d$ is a smooth and proper Deligne–Mumford stack of dimension $5d - 2$. It has three connected (= irreducible) components $\Omega^0_d, \Omega^{odd}_d, \Omega^{even}_d$. Via (6), general points of these components correspond to the following $f: (S, C) \rightarrow \mathbb{P}^1$:

- $\Omega^0_d$: $S = F_d$ and $C$ is a disjoint union of the directrix $\sigma$ and a general curve of class $3(\sigma + dF)$.
- $\Omega^{odd}_d$: $S = F_1$ and $C$ is a general curve of class $4\sigma + (d + 2)F$.
- $\Omega^{even}_d$: $S = F_0$ and $C$ is a general curve of class $(4, d)$.

The components of $\Omega_d$ admit morphisms to the corresponding spaces of admissible covers, namely $\Omega^0_d \rightarrow \overline{\mathcal{H}}_{3,3d-2}$, $\Omega^{odd}_d \rightarrow \overline{\mathcal{H}}_{4,3d-3}$, and $\Omega^{even}_d \rightarrow \overline{\mathcal{H}}_{4,3d-3}$.

**Proof.** That $\Omega_d$ is a smooth and proper Deligne–Mumford stack of dimension $5d - 2$ follows from [Theorem 2.5]. That the components admit morphisms to the spaces of admissible covers follows from the same argument as in [Proposition 2.7].

Recall that $\Omega_d$ parametrizes covers of $\overline{\mathcal{M}}_{0,4}$ and its degenerations. Let $U \subset \Omega_d$ be the dense open subset of non-degenerate covers. It suffices to show that $U$ has three connected components. Via (6), the points of $U$ parametrize $f: (S, C) \rightarrow \mathbb{P}^1$, where $S \rightarrow \mathbb{P}^1$ is a $\mathbb{P}^1$-bundle and $C \subset S$ is a smooth curve such that $C \rightarrow \mathbb{P}^1$ is simply branched of degree $4$. Say $S = F_n$. Since $C \rightarrow \mathbb{P}^1$ is degree 4 and ramified at $6d$ points, we get

$$[C] = 4\sigma + (d + 2n)F.$$

Let $U^0 \subset U$ be the open and closed subset where $C$ is disconnected. Since $C$ is smooth, the only possibility is $n = d$ and $C$ is the disjoint union of $\sigma$ and a curve in the class $3(\sigma + dF)$. As a result, $U^0$ is irreducible and hence a connected component of $U$.

Let $U^{even} \subset U$ be the open and closed subset where $n$ is even. Since $C$ is smooth, we must have $d + 2n \geq 4d$. In particular, $H^1(S, \mathcal{O}_S(C)) = H^2(S, \mathcal{O}_S(C)) = 0$, and hence $(S, C)$ is the limit of $(F_0, C_{gen})$, where $C_{gen} \subset F_0$ is a curve of type $(4, d)$. Therefore, $U^{even}$ is irreducible, and hence a connected component of $U$.

By the same reasoning, the open and closed subset $U^{odd} \subset U$ where $n$ is odd is the third connected component of $U$. 

There is a second explanation for the connected components of $\Omega_d$, which involves the theta characteristics of the trigonal curve $D \rightarrow \mathbb{P}^1$ associated to the tetragonal curve $C \rightarrow \mathbb{P}^1$ via the Recillas correspondence (see [Vak01]). Let $V \subset T_d$ be the open set parametrizing non-degenerate covers of $\overline{\mathcal{M}}_{0,1+3}$. It is easy to check that $V$ is irreducible and the map $\Omega_d \rightarrow T_d$ is representable, finite, and étale over $V$. Therefore, the connected components of $\Omega_d$ correspond to the orbits of the monodromy of $\Omega_d \rightarrow T_d$ over $V$. Let $v$ be a point of $V$, $\psi_v: \mathcal{P} \rightarrow \overline{\mathcal{M}}_{0,1+3}$ the corresponding map, $(\overline{\mathcal{T}}, \overline{\sigma} \cup \overline{D}_v) \rightarrow \mathcal{P}$ the corresponding $(1+3)$-pointed family of rational curves, and $f: (S, \sigma \cup D_i) \rightarrow \mathbb{P}^1$ the family obtained by the blow-down construction as in (6). Note that $D_v$ is the coarse space of $\overline{D}_v$.

By [Proposition 3.1] the points of $\Omega_d$ over $v$ are in natural bijection with the norm-trivial two-torsion line bundles $\mathcal{L}$ on $\overline{D}_v$. Since $[\mathcal{P}]$ has genus 0, a line bundle on $\mathcal{P}$ is trivial if and only if it has degree 0 and the automorphism groups at the orbifold points act trivially on its fibers. Let $p_1, \ldots, p_{6d}$ be the orbifold points of $\mathcal{P}$. Note that $\overline{D}_v$ also has the same number of orbifold points, say $q_1, \ldots, q_{6d}$, with $q_i$ lying over $p_i$. All the orbifold points, $\{p_i\}$ and $\{q_i\}$, have order 2. Since $q_i$ is the only orbifold point over $p_i$, the action of $\text{Aut}_{q_i} \mathcal{P}$ on the fiber of Norm $\mathcal{L}$ is trivial if and only if the action of $\text{Aut}_{q_i} \overline{D}_v$ on the fiber of $\mathcal{L}$ is trivial. If this is the case for all $i$, then $\mathcal{L}$ is a pullback from the coarse space $\overline{D}_v$. Thus,
norm-trivial two-torsion line bundles on \( \mathcal{D}_v \) are just pullbacks of two-torsion line bundles on \( D_v \). The component \( \mathcal{O}_v^0 \) corresponds to the trivial line bundle. The non-trivial ones split into two orbits because of the natural theta characteristic \( \theta = f^*\mathcal{O}_\mathfrak{P}(d-1) \) on \( D_v \).

We can summarize the above discussion in the following sequence of bijections:

\[
\begin{align*}
\{ \text{Points in } \Omega_d \text{ over } a \} & \leftrightarrow \{ \text{Two torsion line bundles on } D_v \} \\
\{ \text{general } v \in \mathcal{I}_d \} & \leftrightarrow \{ \text{Theta characteristics on } D_v \}.
\end{align*}
\]

(7)

**Proposition 3.3.** Under the bijection in (7), the points of \( \Omega_d^{\text{even}} \) correspond to even theta characteristics and the points of \( \Omega_d^{\text{odd}} \) correspond to odd theta characteristics.

**Proof.** Let \( u \in \Omega_d \) be a point over \( v \). Let \( f : (S, \mathcal{C}) \to \mathbb{P}^1 \) be the corresponding 4-pointed curve on a Hirzebruch surface and \( L \) the corresponding two-torsion line bundle on \( D_v \). By Proposition 3.1, we get

\[
\mathcal{O}_\mathfrak{P} \oplus f_* L = f_* \mathcal{O}_\mathcal{C}.
\]

Tensoring by \( \mathcal{O}_\mathfrak{P} (d-1) \) gives

\[
\mathcal{O}_\mathfrak{P} (d-1) \oplus f_* \theta = f_* \mathcal{O}_\mathcal{C} \otimes \mathcal{O}_\mathfrak{P} (d-1).
\]

Thus the parity of \( \theta \) is the parity of \( h^0(C, f^* \mathcal{O}_\mathfrak{P} (d-1)) - d \). It is easy to calculate that for \( C \) on \( F_0 \) of class \( (4, d) \), this quantity is 0 and for \( C \) on \( F_1 \) of class \( (4\sigma + (d+2)F) \), this quantity is 1. \( \square \)

It will be useful to understand the theta characteristic \( \theta \) on \( D_v \) in terms of the map to \( \overline{\mathcal{M}}_{0,1+3} \). Let \( (\mathcal{I}, \sigma \cup \mathcal{D}) \to \overline{\mathcal{M}}_{0,1+3} \) be the universal \((1+3)\)-pointed curve. The curve \( \mathcal{D} \) has genus 0 and has two orbifold points, both of order 2, one over 0 and one over \( \infty \). Let \( \overline{\mathcal{M}}_{0,1+3} \to \overline{\mathcal{M}}'_{0,1+3} \) be the coarse space around \( \infty \) and \( \mathcal{D} \to \mathcal{D}' \) the coarse space around the orbifold point over \( \infty \). Then \( \mathcal{D}' \) is a genus 0 orbifold curve with a unique orbifold point of order 2. Furthermore, \( \mathcal{D}' \to \overline{\mathcal{M}}'_{0,1+3} \) is simply branched over \( \infty \) and the line bundle \( \mathcal{O}(1/2) \) on \( \mathcal{D}' \) is the square root of the relative canonical bundle of \( \mathcal{D}' \to \overline{\mathcal{M}}'_{0,1+3} \). We have the fiber diagram

\[
\begin{array}{ccc}
\mathcal{D}_v & \overset{\mu}{\longrightarrow} & \mathcal{D}' \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & \longrightarrow & \overline{\mathcal{M}}'_{0,1+3}.
\end{array}
\]

Thus \( \theta_{\text{rel}} = \mu^* \mathcal{O}(1/2) \) is a natural relative theta characteristic on \( D_v \). With the unique theta characteristic \( \mathcal{O}(-1) \) on \( \mathbb{P}^1 \), we get the theta characteristic \( \theta = \theta_{\text{rel}} \otimes \mathcal{O}(-1) \).

## 4 Limits of plane quintics

In this section, we fix \( d = 3 \) and write \( \mathcal{Q} \) for \( \Omega_d \). By Proposition 3.2, a general point of \( \mathcal{Q}^{\text{odd}} \) corresponds to a curve of class \( (4\sigma + 5F) \) on \( F_1 \). Such a curve is the proper transform of a plane quintic under a blow up \( F_1 \to \mathbb{P}^2 \) at a point on the quintic. Therefore, the image in \( \overline{\mathcal{M}}_6 \) of \( \mathcal{Q}^{\text{odd}} \) is the closure of the locus \( Q \) of plane quintic curves. The goal of this section is to describe the elements in the closure. More specifically, we will determine the stable curves corresponding to the generic points of the irreducible components of \( \Delta \cap \mathcal{Q} \), where \( \Delta \subset \overline{\mathcal{M}}_6 \) is the boundary divisor.
Figure 1: An admissible cover, and its dual graph with and without the redundant components

We have the sequence of morphisms

$$\mathcal{Q}^{\text{odd}} \xrightarrow{\alpha} \overline{\mathcal{H}}_{4,6} \xrightarrow{\beta} \overline{\mathcal{M}}_5.$$ 

Set $\overline{Q} = \alpha(\mathcal{Q}^{\text{odd}})$. We then get the sequence of surjections

$$\mathcal{Q}^{\text{odd}} \xrightarrow{\alpha} \overline{Q} \xrightarrow{\beta} \overline{Q}.$$ 

Let $U \subset \mathcal{Q}$ be the locus of non-degenerate maps. Call the irreducible components of $\mathcal{Q} \setminus U$ the boundary divisors of $\mathcal{Q}$.

**Proposition 4.1.** Let $B$ be an irreducible component of $\overline{Q} \cap \Delta$. Then $B$ is the image of a boundary divisor $\overline{B}$ of $\mathcal{Q}^{\text{odd}}$ such that $\dim \alpha(B) = \dim B = 12$, $\dim \beta \circ \alpha(B) = 11$, and $\beta \circ \alpha(B) \subset \Delta$.

**Proof.** Note that $\dim \mathcal{Q}^{\text{odd}} = \dim \overline{Q} = 13$ and $\dim \overline{Q} = 12$. Since $\Delta$ is a Cartier divisor, we have $\text{codim}(B, \overline{Q}) = 1$. Let $\overline{B} \subset \overline{Q}$ be an irreducible component of $\beta^{-1}(B)$ that surjects onto $B$. Then $\text{codim}(\overline{B}, \overline{Q}) = 1$. Let $\mathcal{B} \subset \mathcal{Q}^{\text{odd}}$ be an irreducible component of $\alpha^{-1}(\overline{B})$ that surjects onto $\overline{B}$. Then $\mathcal{B}$ is the required boundary divisor of $\mathcal{Q}$. $\square$

Recall that points of $\mathcal{Q}$ correspond to certain finite maps $\phi: \mathcal{P} \to \mathcal{Z}$, where $\mathcal{Z} \to \overline{\mathcal{M}}_{0,4}$ is a pointed degeneration. Set $P = |\mathcal{P}|$ and $Z = |\mathcal{Z}|$. The map $P \to Z$ is an admissible cover with $(2, 2, \ldots)$ ramification over $0$, $(3, 3, \ldots)$ ramification over $1$, and $(1, 1, \ldots)$ ramification over $\infty$. We encode the topological type of the admissible cover by its dual graph $(\Gamma_Z: \Gamma_P \to \Gamma_Z)$. Here $\Gamma_Z$ is the dual graph of $(Z, \{0, 1, \infty\})$, $\Gamma_P$ is the dual graph of $P$, and $\Gamma_Z$ is a map that sends the vertices and edges of $\Gamma_P$ to the corresponding vertices and edges of $\Gamma_Z$. We decorate each vertex of $\Gamma_P$ by the degree of $\phi$ on that component and each edge of $\Gamma_P$ by the local degree of $\phi$ at that node. We indicate the main component of $Z$ by a doubled circle. For the generic points of divisors, $Z$ has two components, the main component and a ‘tail’. In this case, we will omit writing the vertices of $\Gamma_P$ corresponding to the ‘redundant components’—these are the components over the tail that are unramified except possibly over the node and the marked point. These can be filled in uniquely. Figure 1 shows an example of an admissible cover and its dual graph, with and without the redundant components.

**Proposition 4.2.** Let $\mathcal{B} \subset \mathcal{Q}$ be a boundary divisor such that $\dim \alpha(\mathcal{B}) = \dim \mathcal{B}$ and $\alpha(\mathcal{B}) \subset \overline{\mathcal{H}}_{4,6} \setminus \mathcal{K}_{4,6}$. Then the generic point of $\mathcal{B}$ has one of the following dual graphs (drawn without the redundant components).
Consider the finite map $br : \overline{H}_{4,6} \to \overline{M}_{0,18}$ that sends a branched cover to the branch points. Under this map, the preimage of $\mathcal{M}_{0,18}$ is $\mathcal{H}_{4,6}$. So it suffices to prove the statement with $\gamma = br \circ \alpha$ instead of $\alpha$ and $\overline{M}_{0,18}$ instead of $\overline{H}_{4,6}$. Notice that $\gamma$ sends $(\phi : \mathcal{P} \to \mathbb{Z})$ to the stabilization of $(P, \phi^{-1}(\infty))$.

Assume that $\mathcal{B} \subset \mathcal{Q}$ is a boundary divisor satisfying the two conditions. Let $\phi : \mathcal{P} \to \mathbb{Z}$ be a generic point of $\mathcal{B}$. The dual graph of $Z$ has the following possibilities:

1. \[ \begin{array}{c}
1 \\
0 \\
\infty \\
0 \\
2 \\
6 \\
\end{array} \]

2. \[ \begin{array}{c}
1 \\
0 \\
\infty \\
0 \\
2 \\
9 \\
\end{array} \]

3. \[ \begin{array}{c}
1 \\
0 \\
\infty \\
0 \\
3 \\
15 \\
\end{array} \]

4. \[ \begin{array}{c}
1 \\
0 \\
\infty \\
0 \\
2 \\
10 \\
\end{array} \]

5. \[ \begin{array}{c}
1 \\
0 \\
\infty \\
0 \\
2 \\
16 \\
\end{array} \]

6. \[ \begin{array}{c}
1 \\
0 \\
\infty \\
0 \\
1 \\
18 \\
\end{array} \]

7. \[ \begin{array}{c}
1 \\
0 \\
\infty \\
0 \\
1 + j \\
12 \\
\end{array} \]

8. \[ \begin{array}{c}
1 \\
0 \\
\infty \\
0 \\
1 + j + k \\
6 \\
\end{array} \]

Let $M \subset Z$ be the main component, $T \subset Z$ the tail and set $t = M \cap T$.

Suppose $Z$ has the form (I), (II), or (III). Let $E \subset P$ be a component over $T$ that has at $s$ points over $t$ where $s \geq 2$. Since $\gamma(\phi)$ does not lie in $\mathcal{M}_{0,18}$, such a component must exist. The contribution of $E$ towards the moduli of $\gamma(\phi)$ is due to $(E, \phi^{-1}(t))$, whose dimension is bounded above by $\max(0, s - 3)$. The contribution of $E$ towards the moduli of $\phi$ is due to the branch points of $E \to T$. Let $e$ be the degree and $b$ the number of branch points of $E \to T$ away from $t$ (counted without multiplicity). Then $b$ equals $e + s - 2$ in case (I), $e/2 + s - 1$ in case (II), and $e/3 + s - 1$ in case (III). Since $\gamma$ is generically finite on $\mathcal{B}$, we must have $b - \dim \text{Aut}(T,t) = b - 2 \leq \max(0, s - 3)$. The last inequality implies that $s = 2$, $e = 2$ in cases (I) and (II), and $s = 2$, $e = 3$ in case (III). We now show that all other components of $P$ over $T$ are redundant. Suppose $E' \subset P$ is a non-redundant component over $T$ different from $E$. This means that $E' \to T$ has a branch point away from $t$ and the marked point (which is present only in cases (II) and (III)). Composing $E' \to T$ with an automorphism of $T$ that fixes $t$ and the marked point (if any) gives another $\phi$ with the same $\gamma(\phi)$. Since there is a positive dimensional choice of such automorphisms and $\alpha$ is generically finite on $\mathcal{B}$, such $E'$ cannot exist. We now turn to the picture of $P$ over $M$. Since $s = 2$, $P$ has two components over $M$. We also know the ramification profile over 0, 1, $\infty$, and $t$. This information restricts the degrees of the two components modulo 6: in case (I), they must both be 0 (mod 6); in case (II), they must both be 3 (mod 6); and in case (III), they must be 4 and 2 (mod 6). Taking these possibilities gives the pictures (1)–(5).

Suppose $Z$ has the form (IV). By the same argument as above, $P$ can have at most one non-redundant component over $T$. On the other side, we see from the ramification profile over 0 and 1 that the components of $P$ over $M$ have degree divisible by 6. We get the three possibilities (6), (7), or (8) corresponding to whether $P$ has 1, 2, or 3 components over $M$. \qed

The next step is to identify the images in $\overline{\mathcal{M}}_g$ of the boundary divisors of the form listed in Proposition 4.2. Recall that the map $\mathcal{Q} \to \overline{\mathcal{M}}_g$ factors through the stabilization map $\mathcal{Q} \to K(\overline{\mathcal{M}}_{0,4})$, where
$K(\widetilde{M}_{0,4})$ is the Abramovich–Vistoli space of twisted stable maps. The flavor of the analysis in cases (1)–(5) versus cases (6)–(8) is quite different. So we treat them in two separate sections that follow.

### 4.1 Divisors of type (1)–(5)

**Proposition 4.3.** There are 5 irreducible components of $\bar{Q} \cap \Delta$ which are the images of the divisors of $\bar{Q}^\text{odd}$ of type (1)–(5). Their generic points correspond to one of the following stable curves:

- With the dual graph $\circ \rightarrow$
  
  1. A nodal plane quintic.

- With the dual graph $x \circ \rightarrow \sigma \circ \rightarrow y$
  
  2. $X$ hyperelliptic of genus 3, $Y$ a plane quartic, and $p \in Y$ a hyperflex ($K_Y = 4p$).

- With the dual graph $x \circ \rightarrow \sigma \circ \rightarrow y$
  
  3. $X$ Maroni special of genus 4, $Y$ of genus 1, and $p, q \in X$ in a fiber of the degree 3 map $X \rightarrow \mathbb{P}^1$.

- With the dual graph $x \circ \rightarrow \sigma \circ \rightarrow y$
  
  4. $X$ hyperelliptic of genus 3, $Y$ of genus 2, and $p \in Y$ a Weierstrass point.

- With the dual graph $x \circ \rightarrow \sigma \circ \rightarrow y$
  
  5. $X$ hyperelliptic of genus 3, and $Y$ of genus 1.

The rest of § 4.1 is devoted to the proof of Proposition 4.3.

The map $\Omega \rightarrow \tilde{M}_6$ factors via the space $K(\widetilde{M}_{0,4})$ of twisted stable maps. Let $(\tilde{\phi} : \tilde{P} \rightarrow \tilde{S}, \tilde{Z} \rightarrow \widetilde{M}_{0,4})$ correspond to a generic point of type (1)–(5). Under the morphism to $K(\widetilde{M}_{0,4})$, all the components of $\bar{P}$ over the tail of $\mathbb{Z}$ are contracted. The resulting twisted stable map $\phi : \mathcal{M} \rightarrow \mathcal{M}_{0,4}$ has the following form: $\mathcal{M}$ is a twisted curve with two components joined at one node; $\phi$ maps the node to a general point in case (1), to 0 in cases (2) and (3), and to 1 in cases (4) and (5). In all the cases, $\phi$ is étale over $\infty$. Let $(\breve{S}, \breve{C}) \rightarrow \mathcal{M}$ be the pullback of the universal family of 4-pointed rational curves. Let $\mathcal{M} \rightarrow \mathcal{P}$ be the coarse space at the 18 points $\phi^{-1}(\infty)$ and let $f : (S, C) \rightarrow \mathcal{P}$ be the family obtained from $(\breve{S}, \breve{C})$ by the blow-down construction as in (6) on page 11. Then $S \rightarrow \mathcal{P}$ is a $\mathbb{P}^1$ bundle and $C \rightarrow \mathcal{P}$ is simply branched over 18 points.

Every $\mathbb{P}^1$-bundle over $\mathcal{P}$ is the projectivization of a vector bundle (see, for example, [Pom13]). It is easy to check that vector bundles on $\mathcal{P}$ split as direct sums of line bundles and line bundles on $\mathcal{P}$ have integral degree. Therefore, $S = PV$ for some vector bundle $V$ on $\mathcal{P}$ of rank two. The degree of $V$ (modulo 2) is well-defined and determines whether $(S, C)$ comes from $\bar{Q}^\text{odd}$ or $\bar{Q}^\text{even}$. The normalization of $\mathcal{P}$ is the disjoint union of two orbi-curves $P_1$ and $P_2$, both isomorphic to $\mathbb{P}^1(\sqrt{0})$. The number $r$ is the order of the orbinode of $\mathcal{P}$. Since $\phi : \mathcal{P} \rightarrow \mathcal{M}_{0,4}$ is representable, the possible values for $r$ are 1 and 2 in case (1), 2 and 4 in cases (2) and (3), and 3 in cases (4) and (5). Set $V_i = V |_{P_i}$ and $C_i = f^{-1}(P_i) \subset PV_i$. The number of branch points of $C_i \rightarrow P_i$ is $\deg \phi |_{P_i}$ and $C_i \rightarrow P_i$ is étale over 0. Let $[C_i] = 4\sigma_i + m_i F$, where $\sigma_i \subset PV_i$ is the class of the directrix. Using the description of curves in $\mathbb{P}^1$-bundles over $\mathbb{P}^1(\sqrt{0})$ from Appendix A [Proposition A.3 and Corollary A.4], we can list the possibilities for $V_i$ and $m_i$. These are enumerated in Table 1. An asterisk in front of the (arithmetic) genus means that the curve is disconnected. In these disconnected cases, it is the disjoint union $\sigma \sqcup D$, where $D$ is in the linear system.
Table 1: Possibilities for the divisors of type (1)–(5)

| Number | Dual graph | $r$ | $V$ | $m_1$ | $m_2$ | $g(C_1)$ | $g(C_2)$ |
|--------|------------|----|-----|-------|-------|-----------|-----------|
| 1      | (0, 0), (0, 1) | 2  | 3   | 3     | 3     | 0         |           |
| 2      | (0, 1), (0, 0) | 4  | 1   | 3     | 0     |           |           |
| 3      | (0, 2), (0, -1) | 6  | 3   | 3$^*$ | 0$^*$ |           |           |
| 4      | (0, 1/2), (0, 1/2) | 3  | 2   | 4     | 1     |           |           |
| 5      | (0, 1/2), (0, -3/2) | 5/2 | 9/2 | 2     | 2$^*$ |           |           |
| 6      | (0, 1/2), (0, 1/2) | 5/2 | 5/2 | 2     | 2     |           |           |
| 7      | (0, 1/4), (0, 3/4) | 2  | 3   | 3     | 3     |           |           |
| 8      | (0, 1/2), (0, 1/2) | 7/2 | 3/2 | 5     | -1$^*$ |           |           |
| 9      | (0, 3/4), (0, 1/4) | 4  | 1   | 6     | 0     |           |           |
| 10     | (0, 5/4), (0, -1/4) | 5  | 1   | 6     | 0     |           |           |
| 11     | (0, 1/3), (0, -4/3) | 7/3 | 4   | 3     | 2$^*$ |           |           |
| 12     | (0, 1/3), (0, 2/3) | 7/3 | 8/3 | 3     | 2     |           |           |
| 13     | (0, 2/3), (0, 1/3) | 3  | 2   | 3     | 2     |           |           |
| 14     | (0, 5/3), (0, -2/3) | 5  | 8/3 | 3$^*$ | 2     |           |           |
| 15     | (0, 2/3), (0, 1/3) | 4  | 1   | 6     | -1$^*$ |           |           |
| 16     | (0, 4/3), (0, -1/3) | 16/3 | 1   | 6     | -1$^*$ |           |           |

3$\sigma + mF$. The notation $(0, a_1), (0, a_2)$ represents the vector bundle $\mathcal{O} \oplus L$, where $L$ is the line bundle on $P$ whose restriction to $P_i$ is $\mathcal{O}(a_i)$.

We must identify in classical terms (as in Proposition 4.3) the curves $C_i$ appearing in Table 1. Let $C$ be a general curve in the linear system $4\sigma + mF$ on $\mathbf{F}_a$ for a fractional $a$. Let $X = |L|$ and let $\hat{X} \to X$ be the minimal resolution of singularities. Denote also by $C$ the proper transform of $C \subset X$ in $\hat{X}$. From Proposition A.2 we can explicitly describe the pair $(\hat{X}, C)$. By successively contracting exceptional curves on $X$, we then transform $(\hat{X}, C)$ into a pair where the surface is a minimal rational surface. We describe these modifications using the dual graph of the curves involved in these modifications, namely the components of the fiber of $\hat{X} \to \mathbf{P}^1$ over 0, and the proper transforms in $\hat{X}$ of the directrix $\sigma$ and the original curve $C$. The components of $\hat{X} \to \mathbf{P}^1$ over 0 are drawn in the top row whereas $\sigma$ and $C$ are drawn in the bottom row. A vertex is labeled by the self-intersection of the corresponding curve and an edge by the intersection multiplicity of the corresponding intersection. A 2-cell represents coincident intersections. The edges emanating from $C$ are in the same order before and after.

We can read-off the classical descriptions in Proposition 4.3 from the resulting diagrams. For example, diagram 2 implies that a curve of type $4\sigma + (5/2)F$ on $\mathbf{F}_{1/2}$ is of genus 2; it has three points on the fiber over 0, namely $\sigma(0)$ (the leftmost edge), $\tau(0)$ (the rightmost edge), and $x$ (the middle edge), of which $\sigma(0)$ and $x$ are hyperelliptic conjugates. Likewise, diagram 3 implies that a curve of type $4\sigma + 3F$ on $\mathbf{F}_{1/2}$ is Maroni special of genus 4 and its two points over 0 lie on a fiber of the unique map $C \to \mathbf{P}^1$ of degree 3. We leave the remaining such interpretations to the reader.

1. $4\sigma + 2F$ on $\mathbf{F}_{1/2}$:

![Diagram 1](image1)

2. $4\sigma + (5/2)F$ on $\mathbf{F}_{1/2}$:

![Diagram 2](image2)
We similarly analyze the curves \( \sigma \cup C \) where \( C \) is of type \( 3\sigma + mF \).

The proof of Proposition 4.3 now follows from Table 1 and the diagrams above. We discard rows 9, 10, 15, and 16 of Table 1 since they map to the interior of \( \mathcal{M}_6 \). For the remaining ones, we read off the description of \( C_1 \) and \( C_2 \) from the corresponding diagram and get \( C = C_1 \cup C_2 \). While attaching \( (C_1, S_1) \) to \( (C_2, S_2) \), we must take into account whether the directrices of the \( S_i \) meet each other or whether the directrix of one meets a co-directrix of the other. From \( S = PV \) and \( V = \emptyset \oplus L \), we get that if the
restrictions of \( L \) to \( \mathcal{P}_i \) have the same sign, then the two directrices meet and if they have the opposite sign, then a directrix meets a co-directrix. Proceeding in this way, we get that rows 2, 5, 6, 13, and 14 map to loci in \( \overline{\mathcal{M}}_{4} \) of dimension at most 10, and hence do not give divisors of \( \mathcal{O} \). Row 1 gives divisor 5; rows 3 and 4 give divisor 3; rows 7 and 11 give divisor 2; row 8 gives divisor 1; and row 12 gives divisor 4. The proof of Proposition 4.3 is thus complete.

### 4.2 Divisors of type (6)–(8)

To handle boundary divisors of type (6)–(8), we need to do some preparatory work. First, we need to understand the tetragonal curves arising from finite maps to \( \overline{\mathcal{M}}_{4,4} \). Away from the points mapping to \( \infty \), a map to \( \overline{\mathcal{M}}_{4,4} \) gives a fiberwise degree 4 curve in a \( \mathbf{P}^1 \)-bundle. The question that remains is then local around the points that map to \( \infty \). Let \( D \) be a disk, set \( D = D(\sqrt{0}) \), and let \( \phi : D \to \overline{\mathcal{M}}_{4,4} \) be a representable finite map that sends 0 to \( \infty \). Let \( n \) be the local degree of the map \( D \to \mathbf{P}^1 \) of the underlying coarse spaces. Let \( f : (S, e) \to D \) be the pullback of the universal family of 4-pointed rational curves. Then \( (S_0, e_0) \cong (\mathbf{P}^1 \cup \mathbf{P}^1, \{1, 2, 3, 4\}) \), where the \( \mathbf{P}^1 \)'s meet in a node and 1, 2 lie on one component and 3, 4 lie on the other. Let \( \pi \in \mathcal{S}_4 \) be the image in \( \text{Aut}_{\infty} \overline{\mathcal{M}}_{4,4} \) of a generator of \( \text{Aut}_0 D \).

In the following proposition, an \( A_n \) singularity over a disk with uniformizer \( t \) is the singularity with the formal local equation \( x^2 - t^{n+1} \). Thus an \( A_0 \) singularity is to be interpreted as a smooth ramified double cover and an \( A_{-1} \) singularity as a smooth unramified double cover.

**Proposition 4.4.** With the notation above, the curve \( |C| \) is the normalization of \( |C'| \), where \( C' \) is a curve of fiberwise degree 4 on a \( \mathbf{P}^1 \)-bundle \( S' \) over an orbifold disk \( D' \) of one of the following forms.

**Case 1 :** \( \pi \) preserves the two components of \( S_0 \). Then \( D' = D \) and \( S' = \mathbf{P}^1 \times D' \). On the central fiber of \( S' \to D' \), the curve \( C' \) has an \( A_i \) and an \( A_j \) singularity over \( D' \) with \( i + j = n - 2 \). If \( n \) is even and \( i \) is even, then \( \pi \) is trivial; if \( n \) is even and \( i \) is odd, then \( \pi \) has the cycle type \( (2, 2) \); and if \( n \) is odd then \( \pi \) has the cycle type \( (2) \).

**Case 2 :** \( \pi \) switches the two components of \( S_0 \). Then \( D' = D(\sqrt{0}) \) and \( S' = \mathbf{P}^1 \oplus \mathbf{P}^1(1/2) \). Let \( u : \tilde{D}' \to D' \) be the universal cover. On the central fiber of \( S' \times_u \tilde{D}' \to \tilde{D}' \), the curve \( C' \times_u \tilde{D}' \) has two \( A_{n-1} \) singularities over \( \tilde{D}' \) that are conjugate under the action of \( \mathbf{Z}_2 \). If \( n \) is even then \( \pi \) has the cycle type \( (2, 2) \), and if \( n \) is odd then \( \pi \) has the cycle type \( (4) \).

**Remark 4.5.** The apparent choice of \( i \) and \( j \) in the first case is not a real ambiguity. By an elementary transformation centered on the \( A_i \) singularity, we can transform \( (i, j) \) to \( (i-2, j+2) \).

**Proof.** Since \( \phi : D \to \overline{\mathcal{M}}_{0,4} \) is representable, the map \( \text{Aut}_0 D \to \text{Aut}_{\infty} \overline{\mathcal{M}}_{0,4} = D_4 \) is injective. So the order \( r \) of \( 0 \in D \) is 1, 2, or 4. Let \( D \to \overline{\mathcal{M}}_{0,4} \). Then \( \tilde{S} \) is a surface with an action of \( \mathbf{Z}_r \) compatible with the action of \( \mathbf{Z}_r \) on \( D \). The action of the generator of \( \mathbf{Z}_r \) on the central fiber of \( \tilde{S} \to \overline{\mathcal{M}}_{0,4} \) is given by \( \pi \). Note that \( r n \) is the local degree of the map \( D \to \overline{\mathcal{M}}_{0,4} \). Therefore, the surface \( \tilde{S} \) has an \( A_{m-1} \) singularity at the node of the central fiber \( S_0 \), where \( m = r n / 2 \). We take the minimal desingularization of \( \tilde{S} \), successively blow down the \(-1\) curves on the central fiber compatibly with the action of \( \mathbf{Z}_r \) until we arrive at a \( \mathbf{P}^1 \)-bundle, and then take the quotient by the induced \( \mathbf{Z}_r \) action. The resulting surface \( S' \) and curve \( C' \) are as claimed in the proposition.
We illustrate the process for Case 2 and odd \( n \), in which case \( r = 4 \). Let \( t \) be a uniformizer on \( \mathcal{D} \). In suitable coordinates, \( \mathcal{S} \rightarrow \mathcal{D} \) has the form
\[
\mathcal{C}[x, y, t]/(xy - t^m) \leftarrow \mathcal{C}[t],
\]
where \( m = 2n \). A generator \( \zeta \in \mathbb{Z}_4 \) acts by
\[
\zeta \cdot t = it, \quad \zeta \cdot x = y, \quad \zeta \cdot y = -x.
\]
Let \( \mathcal{S} \rightarrow \mathcal{S} \) be the minimal desingularization. Then \( \mathcal{S}_0 \) is a chain of \( \mathbb{P}^1 \)s, say
\[
\mathcal{S}_0 = P_0 \cup P_1 \cup \cdots \cup P_n \cup \cdots \cup P_{m-1} \cup P_m,
\]
where \( P_i \) meets \( P_{i+1} \) nodally at a point. Under the induced action of \( \mathbb{Z}_4 \), \( \zeta \) sends \( P_i \) to \( P_{m-i} \). Contract \( P_0, \ldots , P_{n-1} \) and \( P_{m}, \ldots , P_{n+1} \) successively, leaving only \( P_n \). Let \( \mathcal{S} \) (resp. \( \mathcal{C} \)) be the image of \( \mathcal{S} \) (resp. \( \mathcal{C} \)) under the contraction. Then \( \mathcal{C} \) has two \( A_{m-1} \) singularities on \( P_n \), say at 0 and \( \infty \). The \( \mathbb{Z}_4 \) action descends to an action on \( \mathcal{S} \) and the generator exchanges 0 and \( \infty \). Note that the \( \mathbb{Z}_2 \subset \mathbb{Z}_4 \) acts trivially on the central fiber. Let us replace \( \mathcal{S} \) (resp. \( \mathcal{C} \)) by its geometric quotient under the \( \mathbb{Z}_2 \) action. Then \( \mathcal{S} \rightarrow \mathcal{D} \) is a \( \mathbb{P}^1 \) bundle and \( \mathcal{C} \subset \mathcal{D} \) has two \( A_{m-1} \) singularities on the central fiber. The group \( \mathbb{Z}_4 \) acts compatibly on \( (\mathcal{S}, \mathcal{C}) \) and \( \mathcal{D} \) and exchanges the two singularities of \( \mathcal{C} \). Setting \( \mathcal{S}' = [\mathcal{S}/\mathbb{Z}_2] \), \( \mathcal{C}' = [\mathcal{C}/\mathbb{Z}_2] \), and \( \mathcal{D}' = [\mathcal{D}/\mathbb{Z}_2] \) gives the desired claim.

The other cases are analogous. \( \square \)

Second, we consider maps that contract the domain to \( \infty \in \mathcal{M}_{0,4} \). Let us denote the geometric fiber of the universal family \( (\mathcal{S}, \mathcal{C}) \rightarrow \mathcal{M}_{0,4} \) over \( \infty \) by
\[
(\mathcal{S}, \mathcal{C})_{\infty} = (P_A \cup P_B, \{1, 2, 3, 4\}) \text{ where } 1, 2 \in P_A \cong \mathbb{P}^1 \text{ and } 3, 4 \in P_B \cong \mathbb{P}^1.
\]
We have
\[
D_A \cong \text{Aut}_\infty \mathcal{M}_{0,4} = \text{Stab}(\{1, 2\}, \{3, 4\}) \subset \mathcal{S}_4.
\]
Over the \( BD_4 \subset \mathcal{M}_{0,4} \) based at \( \infty \), the universal family is given by
\[
(\mathcal{S}, \mathcal{C})_{BD_4} = [(P_A \cup P_B, \{1, 2, 3, 4\})/D_A].
\]
We have the natural map
\[
[\{1, 2, 3, 4\}/D_A] \rightarrow [\{A, B\}/\mathbb{Z}_2]. \quad (9)
\]
Let \( \mathcal{P} \) be a smooth connected orbifold curve and let \( \phi : \mathcal{P} \rightarrow BD_4 \subset \mathcal{M}_{0,4} \) be a representable morphism, where the \( BD_4 \) is based at \( \infty \). Set \( \mathcal{C}_\phi = \mathcal{C} \times_\phi \mathcal{M}_{0,4} \). From (9), we see that the degree 4 cover \( \mathcal{C}_\phi \rightarrow \mathcal{P} \) factors as a sequence of two degree 2 covers
\[
\mathcal{C}_\phi \rightarrow \mathcal{G}_\phi \rightarrow \mathcal{P}. \quad (10)
\]
The two points of \( \mathcal{G}_\phi \) over a point of \( \mathcal{P} \) are identified with the two components of \( \mathcal{S} \) over that point.

Let us analyze this factorization from the point of view of the tetragonal-trigonal correspondence (Proposition 3.1). Consider the induced map \( \psi : \mathcal{P} \rightarrow B \mathbb{Z}_2 \subset \mathcal{M}_{0,1+3} \), where the \( B \mathbb{Z}_2 \) is based at \( \infty \). It is important to distinguish between the 4 numbered points for \( \mathcal{M}_{0,4} \) and those for \( \mathcal{M}_{0,1+3} \). We denote the
latter by $I, II, III, IV$ with the convention that the $G_d = \mathfrak{S}_d / K$ action is given by conjugation via the identification

$I \leftrightarrow (13)(24), \quad II \leftrightarrow (14)(32), \quad III \leftrightarrow (12)(34), \quad IV \leftrightarrow \text{id}$.

Then the $Z_2 = D_4 / K$ action switches $I$ and $II$ and leaves $III$ and $IV$ fixed. As a result, the trigonal curve $D = D_\psi$ of Proposition 3.1 is the disjoint union

$$D_\psi = \mathcal{P} \sqcup \mathcal{P}$$

where $\mathcal{P} \to \mathcal{P}$ is a double cover. (Caution: the double cover $\mathcal{Q}_\phi \to \mathcal{P}$ of (10) is different from the double cover $\mathcal{Q} \to \mathcal{P}$ of (11)). Let $\mathcal{L}$ be the norm-trivial two-torsion line bundle on $\mathcal{D}_\psi$ corresponding to the lift $\phi: \mathcal{P} \to \mathcal{M}_{0,4}$ of $\psi: \mathcal{P} \to \mathcal{M}_{0,1+3}$. Since $\mathcal{L}^2$ is trivial, the action of the automorphism groups of points of $\mathcal{D}_\psi$ on the fibers of $\mathcal{L}$ is either trivial or by multiplication by $-1$. The following proposition relates the ramification of $|\mathcal{D}_\phi| \to |\mathcal{P}|$ with this action.

**Proposition 4.6.** Identify $\mathcal{P}$ with its namesake connected component in $\mathcal{D}_\psi = \mathcal{P} \sqcup \mathcal{P}$. Let $p \in \mathcal{P}$ be a point. Then $|\mathcal{D}_\phi| \to |\mathcal{P}|$ is ramified over $p$ if and only if the action of $\text{Aut}_p \mathcal{P}$ on $\mathcal{D}_p$ is nontrivial.

**Proof.** Write $\mathcal{Q}_\phi = \mathcal{Q}$, $\mathcal{Q}_\phi = \mathcal{Q}$ and so on. The map $|\mathcal{Q}| \to |\mathcal{P}|$ is ramified over $p$ if and only if the action of $\text{Aut}_p \mathcal{P}$ on the fiber $\mathcal{Q}_p$ is non-trivial. Denote the fiber of $\mathcal{Q} \to \mathcal{P}$ over $p$ by $\{1,2,3,4\}$, considered as a set with the action of $\text{Aut}_p \mathcal{P}$. Then the fiber of $\mathcal{D} \to \mathcal{P}$ over $p$ is

$$\{I, II, III, IV\} = \{(13)(24), (14)(32), (12)(34)\},$$

among which $\{(13)(24), (14)(32)\}$ comprise points of $\mathcal{E}$ and $\{(12)(34)\}$ the point of $\mathcal{P}$. From the proof of Proposition 3.1, we know that the two-torsion line bundle $\mathcal{L}$ on $\mathcal{D}$ corresponds to the étale double cover $\tau: \mathcal{D} \to \mathcal{D}$ where the fiber of $\mathcal{D}$ over $p$ is $\{(12), (13), (14), (23), (24), (34)\}$. The action of $\text{Aut}_p \mathcal{P}$ on $\mathcal{D}_p$ is non-trivial if and only if the action of $\text{Aut}_p \mathcal{P}$ on $\{(12), (34)\}$ is non-trivial. But we can identify the $\text{Aut}_p \mathcal{P}$ set $\{(12), (34)\}$ with the $\text{Aut}_p \mathcal{P}$ set $\{A, B\}$, which is precisely the fiber of $\mathcal{Q} \to \mathcal{P}$ over $p$. $\square$

We have all the tools to determine the stable images of the divisors of $\mathcal{Q}$ of type (6)–(8), but to be able to separate $\mathcal{Q}$ from $\mathcal{Q}$, we need some further work.

We need to extend the blow-down construction in (9), which we recall. Let $\mathcal{P}$ be a smooth orbifold curve of with $b$ orbifold points of order 2 and let $\phi: \mathcal{P} \to \mathcal{M}_{0,4}$ be a finite map of degree $b$ such that the underlying map $\mathcal{P} \to \mathcal{M}_{0,1+3}$ is representable and has ramification type $(1,1,\ldots)$ over $\infty$. Let $f: (\mathcal{S}, \mathcal{C}) \to \mathcal{P}$ be the pullback of the universal family of 4-pointed rational curves. Let $p \in \mathcal{P}$ be an orbifold point. Then we have $\mathcal{S}_p \cong \mathbb{P}^1 \sqcup \mathbb{P}^1$ and the $Z_2 = \text{Aut}_p \mathcal{P}$ acts trivially on one $\mathbb{P}^1$ and by an involution on the other. By blowing down the component with the non-trivial action and taking the coarse space, we get a family $f: (\mathcal{S}, \mathcal{C}) \to \mathcal{P}$, where $\mathcal{P} = |\mathcal{P}|, \mathcal{S} \to \mathcal{P}$ is a $\mathbb{P}^1$ bundle, and $\mathcal{C} \subset \mathcal{S}$ is simply branched over $\mathcal{P}$.

Now assume that $\mathcal{P}$ is reducible and $p \in \mathcal{P}$ is a smooth orbifold point of order 2 lying on a component that is contracted to $\infty$ by $\phi$. (We still require that the underlying map $\phi: \mathcal{P} \to \mathcal{M}_{0,1+3}$ be representable at $p$.) Locally near $p$, the curve $\mathcal{P}$ has the form $\text{Spec} \mathcal{C}[x]/Z_2$. Set $U = \text{Spec} \mathcal{C}[x]$. Then $\mathcal{S}_U \cong \mathcal{P} \sqcup \mathcal{P}_2$, where $\mathcal{P}_1 \cong \mathbb{P}^1 \times U$ and the $\mathcal{P}_1$ are joined along sections $s_1: U \to \mathcal{P}_1$ (see Figure 2). Like the case before, the action of $Z_2$ on the fiber $\mathcal{P}_1|_0 \sqcup \mathcal{P}_2|_0$ must be trivial on one component, say the first, and an involution on the other. Unlike the case before, we cannot simply blow down the $\mathbb{P}^1$ with the non-trivial action. Let $\mathcal{S}_U$ be the blow up of $\mathcal{S}_U$ along the (non-Cartier) divisor $\mathcal{P}_1|_0$ (see Figure 2). Then $\mathcal{S}_U = \mathcal{P}_1 \sqcup \mathcal{P}_2$, where $\mathcal{P}_1$ is the blow up of $\mathcal{P}_1$ at the point $s_1(0) = \mathcal{P}_1 \cap \mathcal{P}_2|_0$, and $\mathcal{P}_3$ and $\mathcal{P}_2$ are joined along the proper
transform \( \hat{s}_i \) of \( s_i \) and \( s_2 \). The proper transform of \( P_1|_0 \) is a \(-1\) curve on the \( \hat{P}_1 \) component of \( \hat{S}_U \). Let \( \hat{S}_U \to S'_U \) be the blow-down along this \(-1\) curve. Then \( S'_U \to U \) is a \( \mathbb{P}^1 \cup \mathbb{P}^1 \)-bundle with a trivial \( \mathbb{Z}_2 \) action on the central fiber \( S'_U \). Therefore the quotient \( S = S'_U/\mathbb{Z}_2 \) is a \( \mathbb{P}^1 \cup \mathbb{P}^1 \) bundle over the coarse space \( P \) of \( \mathcal{P} \) at \( p \). The image \( C \subset S \) of \( \tilde{C} \subset \tilde{S} \) is simply branched over \( p \in P \) and is disjoint from the singular locus of \( S \). We call the construction of \((S, C)\) from \((\tilde{S}, \tilde{C})\) the blow-up blow-down construction.

![Diagram](image)

**Figure 2:** The blow-up blow-down construction

Let us verify that the blow-up blow-down construction is compatible in a one-parameter family with the blow-down construction. This verification is local around the point \( p \). Let \( \Delta \) be a DVR and \( P \to \Delta \) a smooth (not necessarily proper) curve with a section \( p: \Delta \to P \). Set \( \mathcal{P} = P(\sqrt{P}) \). Let \( \phi: \mathcal{P} \to \mathcal{M}_{0,4} \) be a map such that the underlying map \( \mathcal{P} \to \mathcal{M}_{0,1+3} \) is representable. Assume that for a generic point \( t \in \Delta \), the map \( \phi_t \) maps \( p \) to \( \infty \) and is étale around \( p \) but \( \phi_0 \) contracts \( \mathcal{P}_0 \) to \( \infty \). Let \( f: (S, \tilde{C}) \to \mathcal{P} \) be the pullback by \( \phi \) of the universal family of 4-pointed rational curves.

**Proposition 4.7.** There exists a (flat) family \( S \to P \) over \( \Delta \) such that the generic fiber \( S_t \to P_t \) is the \( \mathbb{P}^1 \) bundle obtained from \( \tilde{S}_t \to \hat{P}_t \) by the blow-down construction and the special fiber \( S_0 \to P_0 \) is the \( \mathbb{P}^1 \cup \mathbb{P}^1 \) bundle obtained from \( \tilde{S}_0 \to \tilde{P}_0 \) by the blow-up blow-down construction.

**Proof.** We may take \( \mathcal{P} = [U/\mathbb{Z}_2] \), where \( U \to \Delta \) is a smooth curve and \( \mathbb{Z}_2 \) acts freely except along a section \( p: \Delta \to U \). Say \( \tilde{S}_p = P_1 \cup P_2 \), where \( P_i \to \Delta \) are \( \mathbb{P}^1 \)-bundles meeting along a section and the \( \mathbb{Z}_2 \) acts trivially on \( P_2 \) and by an involution on \( P_1 \). Note that \( \tilde{S}_U|_t \) is a smooth surface for a generic \( t \) and \( P_1|_t \subset \tilde{S}_U|_t \) is a \(-1\) curve. Let \( \beta: \hat{S}_U \to S_U \) be the blow-up along \( P_2 \). Then \( \beta \) is an isomorphism for a generic \( t \in \Delta \). We claim that \( \beta_0 \) is is the blow up of \( P_2|_0 \) in \( \tilde{S}_U|_0 \). To check the claim, we do a local computation. Locally around \( p(0) \), we can write \( U \) as

\[
\text{Spec } \mathbb{C}[x, t],
\]

where \( p \) is cut out by \( x \). Now \( \tilde{S}_U \to U \) is a family of curves whose generic fiber is \( \mathbb{P}^1 \), and whose discriminant locus (where the fiber is singular) is supported on \( xt = 0 \). Furthermore, we know that the multiplicity of the discriminant along \((x = 0)\) is 1. Therefore, around the node of \( \tilde{S}_U|_0 \), we can write \( \tilde{S}_U \) as

\[
\text{Spec } \mathbb{C}[x, y, z, t]/(yz - xt^n).
\]

In these coordinates, say \( P_2 \subset \tilde{S}_U \) is cut out by the ideal \((x, y)\). Direct calculation shows that the specialization of \( \text{Bl}_{(x,y)} \text{Spec } \mathbb{C}[x, y, z]|/(yz - xt^n) \) at \( t = 0 \) is \( \text{Bl}_{(x,y)} \text{Spec } \mathbb{C}[x, y, z]/(yz) \), as claimed. Let \( P_1 \subset \tilde{S}_U \) be the proper transform of \( P_1 \). Then \( P_1|_t \subset \tilde{S}_U|_t \) is a \(-1\) curve for all \( t \). Let \( \hat{S}_U \to S'_U \) be the blow-down. Then the action of \( \mathbb{Z}_2 \) on \( S'_U|_p \) is trivial. The quotient \( S = S'/\mathbb{Z}_2 \) with the map \( S \to P \) is the required family. \( \square \)
Let \( \phi : \mathcal{P} \to \widetilde{\mathcal{M}}_{0,4} \) be an Abramovich–Vistoli stable map arising from a generic point of a divisor in \( \mathcal{O} \) of type \((6)–(8)\). Then \( \mathcal{P} \) has 18 smooth orbifold points of order 2. Let \( \mathcal{P} \to P \) be the coarse space at these 18 points. Let \( f : (\tilde{S}, \tilde{C}) \to \mathcal{P} \) be the pullback of the universal family of 4-pointed rational curves and let \( (S, C) \to P \) be the family obtained by the blow-up blow-down construction. Then the surface \( S \) is a degeneration of \( F_0 \) or \( F_1 \). The following observation lets us distinguish the two cases.

**Proposition 4.8.** Suppose \( s : P \to S \) is a section lying in the smooth locus of \( S \to P \). Then the self-intersection \( s^2 \) is an integer. If it is even (resp. odd), then \( S \) is a degeneration of \( F_0 \) (resp. \( F_1 \)).

**Proof.** Note that \( s(P) \subset S \) is a Cartier divisor. Let \( \mathcal{L} \) be the associated line bundle. Then \( s^2 = \deg(s^* \mathcal{L}) \).

Since degrees of line bundles on \( P \) are integers, \( s^2 \) is an integer. Suppose \( S \) is a degeneration of \( F_i \). Then a smoothing of \( \mathcal{L} \) is a line bundle on \( F_i \) of fiberwise degree 1. Its self-intersection determines the parity of \( i \).

Such a section \( s \) does not always exist, however. For example, the \( \mathbb{P}^1 \cup \mathbb{P}^1 \) bundle over a contracted component of \( P \) may have non-trivial monodromy that exchanges the two components. To distinguish odd and even in these cases, we must understand the parity of the limiting theta characteristic on the associated trigonal curve.

We quickly recall the theory of limiting theta characteristic from [Chi08]. Consider a one-parameter family of smooth curves degenerating to a nodal curve \( C \). Suppose we have a theta-characteristic on this family away from the central fiber. Then, after possibly making a base change and replacing the nodes by orbifold nodes of order two, the theta-characteristic extends uniquely to a (locally free) theta-characteristic on the central fiber. Note that the limit theta-characteristic may not be a line bundle on \( C \) itself, but on \( \tilde{C} \), where \( \tilde{C} \to C \) is an orbinal modification. By a *limiting theta characteristic* on \( C \), we mean a theta characteristic on an orbinal modification of \( C \). Suppose \( \mathcal{L} \) is a theta characteristic on \( \tilde{C} \) and \( x \in \tilde{C} \) is an orbinode. Then \( Aut_x \tilde{C} \) acts on \( \mathcal{L}_x \) by \( \pm 1 \). Suppose the action is non-trivial. Let \( \nu : \tilde{C} \to \tilde{C} \) be the normalization at \( x \) and \( c : \tilde{C} \to \tilde{C}' \) the coarse space at the two points of \( \tilde{C} \) over \( x \). Then \( c_x \nu^* \mathcal{L} \) is a theta characteristic on \( \tilde{C}' \) and

\[
\chi(C, \mathcal{L}) = \chi(C', c_x \nu^* \mathcal{L}).
\] (12)

Suppose the action is trivial. Then \( \mathcal{L} \) is a pullback from the coarse space around \( x \), so we may assume that \( Aut_x \tilde{C} \) is trivial. Let \( \nu : \tilde{C} \to \tilde{C} \) be the normalization at \( x \), as before, and \( x_1, x_2 \) the two points of \( \tilde{C} \) over \( x \). Let \( \epsilon_x \) be the two-torsion line bundle on \( \tilde{C} \) obtained by taking the trivial line bundle on \( \tilde{C} \) and gluing the fibers over \( x_1 \) and \( x_2 \) by \( -1 \). Then \( \mathcal{L} \otimes \epsilon_x \) is another theta characteristic on \( \tilde{C} \), and by [Har82, Theorem 2.14] we have

\[
\chi(C, \mathcal{L} \otimes \epsilon_x) = \chi(C, \mathcal{L}) \pm 1.
\] (13)

Let \( \mathcal{Z} \to \widetilde{\mathcal{M}}_{0,1+3} \) be a pointed degeneration and \( \psi : \mathcal{P} \to \mathcal{Z} \) a finite cover corresponding to a generic point of a divisor of type \((6), (7), \) or \((8)\). Let \( f : \mathcal{D}_\psi \to \mathcal{P} \) the corresponding étale triple cover. We assume that the orders of the orbnodes of \( \mathcal{Z} \) and therefore \( \mathcal{D}_\psi \) are sufficiently divisible. Therefore, we have a limiting theta characteristic \( \theta \) on \( \mathcal{D}|_\psi \). Denote by the same symbol its pullback to \( \mathcal{D}_\psi \). Note that the action on \( \theta_x \) of \( Aut_x \mathcal{D}_\psi \) is trivial for all \( x \) except possibly the nodes.

Let \( \mathcal{P}^{\text{main}} \subset \mathcal{P} \) (resp. \( \mathcal{P}^{\text{tail}} \)) be the union of the components that lie over the main (resp. tail) component of \( \mathcal{Z} \). Denote by \( \mathcal{D}_\psi^{\text{main}} \) (resp. \( \mathcal{D}_\psi^{\text{tail}} \)) the pullback of \( \mathcal{D}_\psi \) to \( \mathcal{P}^{\text{main}} \) (resp. \( \mathcal{P}^{\text{tail}} \)). Then \( \mathcal{D}_\psi^{\text{tail}} \) is the disjoint union \( \mathcal{P}^{\text{tail}} \cup \mathcal{E}_\psi \), where \( \mathcal{E}_\psi \to \mathcal{P}^{\text{tail}} \) is a double cover.
Proposition 4.9. Let \( x \) be a node of the \( \mathcal{P}_{\text{tail}} \) component of \( \mathcal{D}_{\text{tail}} \). Then the action of \( \text{Aut}_x \mathcal{D}_\psi \) on \( \theta_x \) is non-trivial.

Proof. We look at the limiting relative theta characteristic on the universal family. Let \( \mathcal{D} \to \mathcal{Z} \) be the pullback along \( \mathcal{Z} \to \overline{M}_{0,1+3} \) of the universal triple cover on \( \overline{M}_{0,1+3} \). Note that \( \mathcal{D} \) has three components, say \( \mathcal{D}_1, \mathcal{E}_1, \) and \( \mathcal{E}_2, \) and the dual graph of \( \mathcal{D} \to \mathcal{Z} \) is as follows:

\[
\begin{array}{c}
\bullet & \bullet & \bullet \\
1 & 2 & 3 \\
\end{array}
\]

Let \( \mathcal{Z}' \) be obtained from \( \mathcal{Z} \) by taking the coarse space at \( \infty \) and \( \mathcal{D}' \) (resp. \( \mathcal{E}_1', \mathcal{E}_2' \)) from \( \mathcal{D} \) (resp. \( \mathcal{E}_1, \mathcal{E}_2 \)) by taking the coarse space at the points over \( \infty \). Then the cover \( \mathcal{D}' \to \mathcal{Z}' \) is simply branched over \( \infty \), and it is a degeneration of the cover \( \mathcal{D}' \to \mathcal{M}_{0,1+3}' \) in (8). The relative dualizing sheaf of \( \mathcal{D}' \to \mathcal{Z}' \) has degree 0 on \( \mathcal{E}_2' \). Let \( \theta_{\text{rel}} \) be the limiting theta characteristic on \( |\mathcal{D}'| \). Since \( x_2 = \mathcal{E}_2' \cap \mathcal{D}_1 \) is the unique orbifold point on \( \mathcal{E}_2' \), the action of \( \text{Aut}_x \mathcal{D}_\psi \) on \( \theta_{\text{rel}} \) at \( x_2 \) must be trivial.

The map \( \psi: \mathcal{D}_\psi \to \mathcal{D} \) maps a node \( x \) on the \( \mathcal{P}_{\text{tail}} \) component to \( x_2 \). Therefore, the action of \( \text{Aut}_x \mathcal{D}_\psi \) on \( \psi^* \theta_{\text{rel}}|_x \) is trivial. Let \( \theta_{\text{rel}} \) be the unique limiting theta characteristic on \( |\mathcal{P}| \). Then the action of \( \text{Aut}_{f(x)} \mathcal{P} \) on \( \theta_{\text{rel}} \) at \( f(x) \) is by \( -1 \) and \( f: \text{Aut}_x \mathcal{D}_\psi \to \text{Aut}_{f(x)} \mathcal{P} \) is an isomorphism. Since \( \theta = \psi^* \theta_{\text{rel}} \otimes f^* \theta_{\text{rel}} \), we get the assertion. \( \square \)

Remark 4.10. Let \( \psi': \mathcal{P}' \to \overline{M}_{0,1+3} \) be the Abramovich–Vistoli stable map obtained from \( \psi: \mathcal{P} \to \mathcal{Z} \) by contracting the unstable (= redundant) components of \( \mathcal{P}_{\text{tail}} \). Let \( \mathcal{D}' \to \mathcal{P}' \) be the corresponding triple cover and let \( \theta' \) be the limiting theta characteristic. The statement of Proposition 4.9 holds also for \( \mathcal{D}' \to \mathcal{P}' \) and \( \theta' \). Indeed, in a neighborhood of the node \( x \), the pairs \( (\mathcal{D}_\psi, \theta) \) and \( (\mathcal{D}', \theta') \) are isomorphic.

We now have all the tools to determine the images in \( \overline{M}_6 \) of the boundary components of \( \mathcal{Q}_{\text{odd}} \) of type (6), (7), and (8).

4.2.1 Type (6)

Proposition 4.11. There are 10 irreducible components of \( \mathcal{Q} \cap \Delta \) which are images of divisors of type (6) in \( \mathcal{Q}_{\text{odd}} \). Their generic points correspond to the following stable curves:

- With the dual graph \( x \circ \infty \)

  1. A nodal plane quintic.

- With the dual graph \( x \circ n \to \circ v \)

  2. \( (X, p) \) the normalization of a cuspidal plane quintic and \( Y \) of genus 1.

  3. \( X \) Maroni special of genus 4, \( Y \) of genus 2, \( p \in X \) a ramification point of the unique degree 3 map \( X \to \mathbb{P}^1 \), and \( p \in Y \) a Weierstrass point.
4. $X$ a plane quartic, $Y$ hyperelliptic of genus 3, $p \in X$ a point on a bitangent, and $p \in Y$ a Weierstrass point.

5. $X$ of genus 2, $Y$ hyperelliptic of genus 4, $p \in Y$ a Weierstrass point.

6. $X$ a plane quartic, $Y$ hyperelliptic of genus 3, and $p \in X$ a hyperflex ($K_X = 4p$).

7. $X$ of genus 1, $Y$ hyperelliptic of genus 5.

- With the dual graph $\xymatrix{ X \ar@{.>}[r]^{p} & Y }$

8. $X$ Maroni special of genus 4, $Y$ of genus 1, and $p, q \in X$ on a fiber of the unique degree 3 map $X \to \mathbb{P}^1$.

9. $X$ a curve of genus 2, $Y$ hyperelliptic of genus 3, and $p, q \in Y$ hyperelliptic conjugate.

The rest of § 4.2.1 is devoted to the proof.

Recall that type (6) corresponds to $(\phi : \mathcal{P} \to \mathbb{Z}, \mathbb{Z} \to \overline{\mathcal{M}}_{0,4})$ where $\phi$ has the following dual graph:

![Dual Graph](image)

Let $(\bar{S}, \bar{C}) \to \mathcal{P}$ be the pullback of the universal family of 4-pointed rational curves. Let $\mathcal{P} \to P$ be the coarse space away from the node. Let $f : (S, C) \to P$ be obtained from $(\bar{S}, \bar{C})$ by the blow-up blow-down construction. Define $C_1 = f^{-1}(P_1)$, $C_{\text{tail}} = f^{-1}(P_{\text{tail}})$, and similarly for $S_1$ and $S_{\text{tail}}$. Set $x = P_1 \cap P_{\text{tail}}$.

Denote the fiber of $S \to P$ over $x$ by $(P_1 \cup P_1, \{1, 2, 3, 4\})$, where 1, 2 lie on one component and 3, 4 on the other.

![Figure 3](image)

First, we look at $P_{\text{tail}}$. The map $S_{\text{tail}} \to P_{\text{tail}}$ is a $\mathbb{P}^1 \cup \mathbb{P}^1$ bundle. Recall the étale double cover $\mathcal{S} \to \mathcal{P}_{\text{tail}}$ in (10) on page (10), whose fibers correspond to the components of $\bar{S}$. Since the action of $\text{Aut}_x \mathcal{P}_{\text{tail}}$ on the two components is trivial for all $x$ except possibly the node, $\mathcal{S} \to \mathcal{P}_{\text{tail}}$ descends to an étale double cover $G \to P_{\text{tail}}$. Since $P_{\text{tail}}$ has only one orbifold point, it is simply connected, and hence $G$ is the trivial cover $P_{\text{tail}} \sqcup P_{\text{tail}}$. The degree 4 cover $C_{\text{tail}} \to P_{\text{tail}}$ factors as $C_{\text{tail}} \to G \to P_{\text{tail}}$. Hence, it is a disjoint union $C_{\text{tail}}^{1,2} = C_{\text{tail}}^{1} \sqcup C_{\text{tail}}^{2}$. Both $C_{\text{tail}}^{1,2}$ are hyperelliptic curves, each contained in a component of $S_{\text{tail}}$ and lying away from the singular locus (see a sketch in Figure 3). We claim that if both $C_{\text{tail}}^{1} \to P_{\text{tail}}$ and $C_{\text{tail}}^{2} \to P_{\text{tail}}$ are nontrivial covers, then the boundary divisor maps to a locus of codimension at least 2 in $\overline{Q}$. Indeed, compose $C_{\text{tail}}^{2} \to P_{\text{tail}}$ by an automorphism of $P_{\text{tail}}$ that fixes $x$. The resulting cover also represents an element of the same boundary divisor and has the same image if $\overline{Q}$. The claim follows
from the fact that there is a 2-dimensional choice of moduli for \( \text{Aut}(P_{\text{tail}}, x) \). We may thus assume that \( C_{\text{tail}}^2 = P_{\text{tail}} \cup P_{\text{tail}} \). Without loss of generality, take \( C_{\text{tail}}^2 \big|_{x} = \{3, 4\} \). Then the monodromy of \( \{1, 2, 3, 4\} \) at \( x \) is either trivial or \( \langle 12 \rangle \). The map \( c_{\text{tail}}^1 \to P_{\text{tail}} \) is ramified at \( i \) points. The component of \( S_{\text{tail}} \) containing \( C_{\text{tail}}^1 \) is the bundle \( P(\mathcal{O} \oplus \mathcal{O}(i/2)) \). The component of \( S_{\text{tail}} \) containing \( C_{\text{tail}}^2 \) is the trivial bundle \( \mathbb{P}^1 \times P_{\text{tail}} \) (See Figure 3).

Second, we look at \( P_i \). The map \( S_1 \to P_i \) is a \( \mathbb{P}^1 \)-bundle away from \( x \); over \( x \) the fiber is isomorphic to \( \mathbb{P}^1 \cup \mathbb{P}^1 \) (See Figure 3). By blowing down the component containing 1 and 2 as in the proof of Proposition 4.4, we see that \( |C_{1,1}| \) is the normalization of a curve \( C_{1,1} \) on a Hirzebruch surface \( F_i \) which has an \( A_{i-1} \) singularity over the fiber over \( x \) along with two smooth unramified points, namely \( \{3, 4\} \). Note that \( S \to P \) admits a section of self-intersection \( l \) (mod 2), which consists of a horizontal section of the component of \( S_{\text{tail}} \) containing \( C_{\text{tail}}^2 \) and a section of \( S_1 \to P_1 \) that only intersects the component of \( S_1 \big|_x \) containing \( \{3, 4\} \). Also note that \( C_{1,1} \subset F_i \) is of class \( 4\sigma + (3 + 2l)F_i \). Since \( C \) and hence \( C_{1,1} \) is connected, the only possible odd choice of \( l \) is \( l = 1 \).

In conclusion, the (pre-)stable images of generic points of divisors of type (6) are of the following two forms: First, for odd \( i \) we get \( C_1 \cup \cup C_{\text{tail}} \), where \( (C_1, p) \) is the normalization of a curve of class \( 4\sigma + 5F \) on \( F_i \) with an \( A_{i-1} \) singularity, \( C_{\text{tail}} \) is a hyperelliptic curve of genus \( (i - 1)/2 \), and \( p \in C_{\text{tail}} \) is a Weierstrass point. Second, for even \( i \) we get \( C_1 \cup p, q C_{\text{tail}} \) where \( (C_1, \{p, q\}) \) is the normalization of a curve of class \( 4\sigma + 5F \) on \( F_i \) with an \( A_{i-1} \) singularity, \( C_{\text{tail}} \) is a hyperelliptic curve of genus \( i/2 \) and \( p, q \in C_{\text{tail}} \) are hyperelliptic conjugate. In both cases, we have \( 1 \leq i \leq 14 \). The case of \( i = 1 \) gives a smooth stable curve so we discard it. The cases \( i = 3, 5, 7, 9 \) give the divisors \( 2, 3, 4, 5 \) of Proposition 4.11. The case of \( i = 11 \) yields a codimension 2 locus. The case of irreducible \( C_i \) and \( i = 2, 4, 6, 8 \) give the divisors \( 1, 2, 3, 4, 5 \), respectively. The cases of \( i = 10, 12 \) yield codimension 2 loci. We also have cases with reducible \( C_i \) for \( i = 2, 4, 6 \). For \( i = 2 \), we have \( C_{2,1} \) be the union of \( \sigma \) with a tangent curve of class \( 3\sigma + 5F \), which again gives divisor \( 2 \). For \( i = 4 \), we can have \( C_{4,1} \) be the union of \( \sigma + F \) with a 4-fold tangent curve of class \( 3\sigma + 4F \), which gives divisor \( 2 \). For \( i = 6 \), we can have \( C_{6,1} \) be the union of \( \sigma + 2F \) with a 6-fold tangent curve of class \( 3\sigma + 3F \) or the union of \( 2\sigma + 2F \) with a 6-fold tangent curve of class \( 2\sigma + 3F \), both of which give divisor \( 7 \). The proof of Proposition 4.11 is thus complete.

4.2.2 Type (7)

**Proposition 4.12.** There are 8 irreducible components of \( \overline{Q} \cap \Delta \) which are images of divisors of type (7) in \( \Omega_6^{\text{odd}} \). Their generic points correspond to the following stable curves:

- With the dual graph \( X \circ \circ \)
  1. \( X \) hyperelliptic of genus 5.
- With the dual graph \( X \circ \circ \longrightarrow \bullet \bullet \bullet \)
  2. \( X \) of genus 2, \( Y \) Maroni special of genus 4, \( p \in X \) a Weierstrass point, \( p \in Y \) a ramification point of the unique degree 3 map \( Y \to \mathbb{P}^1 \).
  3. \( X \) hyperelliptic of genus 3, \( Y \) of genus 3, \( p \in X \) a Weierstrass point, \( p \in Y \) a point on a bitangent.

- With the dual graph \( X \circ \circ \bullet \bullet \bullet \)
  4. \( X \) hyperelliptic of genus 4, \( Y \) of genus 2, \( p \in X \) a Weierstrass point.
5. \(X\) hyperelliptic of genus 3, \(Y\) of genus 2, and \(p \in Y\) a Weierstrass point.
6. \(X\) of genus 2, \(Y\) a plane quartic, \(p, q \in X\) hyperelliptic conjugate, the line through \(p, q\) tangent to \(Y\) at a third point.
7. \(X\) hyperelliptic of genus 3, \(Y\) of genus 2, \(p, q \in X\) hyperelliptic conjugate.

- With the dual graph \(\xrightarrow{\cdots} Y\)

8. \(X\) hyperelliptic of genus 3, \(Y\) of genus 1.

The rest of § 4.2.2 is devoted to the proof. Recall that type (7) corresponds to \((\phi : \mathcal{P} \to \mathbb{Z}, \mathbb{Z} \to \mathcal{M}_{0,4})\) where \(\phi\) has the following dual graph:

\[
\begin{array}{c}
\{1, 2\} \xrightarrow{\phi} \{5\} \\
\{5\} \xrightarrow{\phi} \{12, 11\} \\
\{11, 12\} \xrightarrow{\phi} \{1, 2\}
\end{array}
\]

Let \((S, C) \to \mathcal{P}\) be the pullback of the universal family of 4-pointed rational curves. Let \(\mathcal{P} \to \mathcal{P}\) be the coarse space away from the nodes. Let \(f : (S, C) \to \mathcal{P}\) the family obtained by the blow-up blow-down construction. Define \(C_1 = f^{-1}(P_1)\), and similarly for \(C_2, C_{\text{tail}}, S_1, S_2, \text{ and } S_{\text{tail}}\). Set \(x_1 = P_{\text{tail}} \cap P_1\) and \(x_2 = P_{\text{tail}} \cap P_2\).

First, we look at \(P_{\text{tail}}\). The map \(S_{\text{tail}} \to P_{\text{tail}}\) is a \(\mathbb{P}^1 \cup \mathbb{P}^1\) bundle. The étale double cover given by the two components is \(\mathcal{S} \to P_{\text{tail}}\) of \((10)\) which induces an étale double cover \(G \to P_{\text{tail}}\) as in type (6). We have the factorization \(C_{\text{tail}} \to G \to P_{\text{tail}}\). Since \(P_{\text{tail}}\) has two orbifold points \(x_1\) and \(x_2\), this cover may be nontrivial. If \(G \to P_{\text{tail}}\) is trivial, then \(C_{\text{tail}}\) is the disjoint union \(C_{\text{tail}}^1 \cup C_{\text{tail}}^2\) of two double covers of \(P_{\text{tail}}\). If \(G \to P_{\text{tail}}\) is nontrivial, then \(|G|\) is a rational curve and \(|C_{\text{tail}}|\) is its double cover.

Second, we look at \(P_1\). Denote the fiber of \(S_1 \to P_1\) over \(x_1\) by \((P_A \cup P_B, \{1, 2, 3, 4\})\), where \(P_A \cong P_1 \cong \mathbb{P}^1\), \(1, 2 \in P_A\), and \(3, 4 \in P_B\). Let \(\pi \in \mathcal{G}_4\) be a generator of the monodromy of \(C_1 \to P_1\) at \(x_1\).

By Proposition 4.4, \(|C_1|\) is the normalization of \(|C'_1|\), where \(C'_1\) is a fiberwise degree 4 curve on a \(\mathbb{P}^1\) bundle \(S'_1 = F_1\) over \(P'_1\) where \(P'_1 = \mathbb{P}^1\) or \(P'_1 = \mathbb{P}^1(\sqrt{5})\). In either case, \(C'_1\) is of class \(4\sigma + (1 + 21)F\).

From Corollary 4.4 we see that the only possibilities for \(l\) are \(l = 0\) and \(l = 1\) if \(P'_1 = \mathbb{P}^1\) and \(l = 1/2\) if \(P'_1 = \mathbb{P}^1(\sqrt{5})\). Also, if \(l = 1\) then \(C'_1\) is the disjoint union of \(\sigma\) with a curve in the class \(3\sigma + 3F\).

The case \(P'_1 = \mathbb{P}^1\) occurs if \(\pi\) preserves the two components \(P_A\) and \(P_B\). By Proposition 4.4, \(C'_1\) has an \(A_{i-1}\) and an \(A_{i+1}\) singularity over 0 for some \(k = 1, \ldots, i + 1\). By Remark 4.3, we may assume that the singularities are \(A_{i-1}\) and \(A_{i+1}\) or \(A_0\) (if \(k\) is even).

The case \(P'_1 = \mathbb{P}^1(\sqrt{5})\) occurs if \(\pi\) switches the two components \(P_A\) and \(P_B\). By Proposition 4.4, over an étale chart around 0 \(\in \mathbb{P}^1(\sqrt{5})\), the pullback of \(C'_1\) has two \(A_{i-1}\) singularities over 0 that are conjugate under the \(\mathbb{Z}_2\) action. To identify such a curve in more classical terms, we use the strategy of § 4.1. Indeed, by diagram \((1)\) on page \(17\) we get that \(|C'_1|\) is a curve of class \(2\sigma + 4F\) on \(F_2\) disjoint from the directrix and with an \(A_{i-1}\) singularity on the fiber of \(F_2 \to \mathbb{P}^1\) over 0.

We now simply enumerate the possibilities for \(|C_1|\) along with its attaching data with the rest of \(C\), namely \(|D_1| = f^{-1}(x_1)\). We list the possible dual graphs for \((|C_1|, |D_1|)\), where the vertices represent connected components of \(|C_1|\) labeled by their genus, and the half-edges represent points of \(|D_1|\), labeled by their multiplicity in \(|D_1|\). In the case where \(\pi\) preserves \(A\) and \(B\), we record some additional data as follows. We make the convention that the half edges depicted on top (resp. bottom)
are images of the points which lie on $P_A \subset S_1$ (resp. $P_B \subset S_1$). We then record the self-intersection (modulo 2) of a section $\sigma_A$ (resp. $\sigma_B$) of $S_1 \to P_1$ that lies in the smooth locus of $S_1 \to P_1$ and meets $P_A$ (resp. $P_B$). In the case where $\pi$ switches $A$ and $B$, there is no such additional information. Here we make the convention that the half-edges are depicted on the sides.

For example, let us take $i = 1$. For $l = 0$, we get $C'_1 \subset S'_1 = F_0$ of class $4\sigma + F$ with an $A_0$ singularity (that is, a point of simple ramification) over $0 \in P^1$. This gives us the dual graph in 1.1. To get the additional data, we must reconstruct $S_1$ from $S'_1$, which we do by a stable reduction of the 4-pointed family $(S'_1, C'_1) \to P^1$ of rational curves around 0. To do so, set $P_1 = P^1(\sqrt{0})$. We first pass to the base change $S'_1 \times_{P^1} P_1$, on which the curve $C'_1 \times_{P^1} P_1$ has a node. The blow up of $S'_1 \times_{P^1} P_1$ at the node and the proper transform of $C'_1 \times_{P^1} P_1$ gives the required family $(S_1, C_1)$. The central fiber of $S_1 \to P_1$ is $P_A \cup P_B$, where $P_A$ is the exceptional curve of the blow up and $P_B$ is the proper transform of the original fiber.

The self-intersection of a section meeting $P_A$ (resp. $P_B$) is $-1/2$ (resp. 0). This leads to the complete picture 1.1. For $l = 1$, the same procedure gives 1.2. For $l = 1/2$, we get that $|C'_1|$ is a curve of class $2\sigma + 4F$ on $F_2$ disjoint from $\sigma$ and with an $A_0$ singularity (that is, a point of simple ramification) over the fiber $F$ of $F_2 \to P^1$ over 0. The divisor $|D'_1|$ is $|C'_1| \cap 2F$. This leads to the picture 1.3. We get the pictures for $i = 2, 3, 4$ analogously.

• $i = 1$

1.1. $\sigma_A^2 = -1/2, \sigma_B^2 = 0$

1.2. $\sigma_A^2 = 1/2, \sigma_B^2 = -1$

1.3. $\sigma^2 - 4$

• $i = 2$

1.4. $\sigma_A^2 = -1, \sigma_B^2 = 0$

1.5. $\sigma_A^2 = -1/2, \sigma_B^2 = -1/2$

1.6. $\sigma^2 - 2$

• $i = 3$

1.7. $\sigma_A^2 = -1/2, \sigma_B^2 = 1$

1.8. $\sigma - 4$

• $i = 4$
Third, we look at $P_2$. The story here is entirely analogous to that of $P_1$, except that the curve $C_2' \subset F_l$ is of class $4\sigma + (2 + 2l)F$, and the allowed values of $l$ are $l = 0, 1/2, 1, \text{and } 2$. The case of $l = 2$ corresponds to a disjoint union of $\sigma$ and $3\sigma + (3 + 2l)F$. The case of $l = 1/2$ corresponds to diagram 3 on page 17, which shows that $|C''_2|$ is a curve of class $3\sigma + 6F$ on $F_2$ disjoint from $\sigma$ and with an $A_{j-1}$ singularity on the fiber over 0. We enumerate the possibilities with the same conventions as before.

1. $\sigma^2_A \equiv 1, \sigma^2_B \equiv 1$

2. $\sigma^2_A \equiv p + 1/2, \sigma^2_B \equiv 1$

3. $\sigma^2_A \equiv p - 1/2, \sigma^2_B \equiv 0$

4. $\sigma^2_A \equiv p - 1/2, \sigma^2_B \equiv 0$

5. $\sigma^2_A \equiv p + 1, \sigma^2_B \equiv 1$

6. $\sigma^2_A \equiv p, \sigma^2_B \equiv 0$

7. $\sigma^2_A \equiv p, \sigma^2_B \equiv 0$

8. $\sigma^2_A \equiv 1, \sigma^2_B \equiv 1$

9. $\sigma^2_A \equiv 0, \sigma^2_B \equiv 0$

10. $\sigma^2_A \equiv 0, \sigma^2_B \equiv 0$

11. $\sigma^2_A \equiv p - 3/2, \sigma^2_B \equiv 3/2$

12. $\sigma^2_A \equiv p - 1/2, \sigma^2_B \equiv 1/2$

13. $\sigma^2_A \equiv p - 1/2, \sigma^2_B \equiv 0$

14. $\sigma^2_A \equiv 0, \sigma^2_B \equiv 0$

15. $\sigma^2_A \equiv 0, \sigma^2_B \equiv 0$

16. $\sigma^2_A \equiv 0, \sigma^2_B \equiv 0$
For \( p = 4 \).

The marked curves appearing as \( C_2 \) above are not arbitrary in moduli. But it is easy to find which marked covers appear by using that they are normalizations of a singular curve \( C'_2 \) on a known surface of a known class and a known singularity. We now write down these descriptions. We denote by \( a \) or \( a_1, a_2 \) (resp. \( b \) or \( b_1, b_2 \)) the point(s) represented by the half-edges on top (resp. bottom). The numbering goes from the left to the right.

| Dual graph | \( p \) | Description |
|------------|--------|-------------|
| 2.1        | 0      | Plane quartic with \( 2a + b_1 + b_2 \) a canonical divisor |
| 2.1        | 1      | Genus 2 with \( b_1 \) and \( b_2 \) hyperelliptic conjugate |
| 2.1        | 2, 3   | Any moduli |
| 2.2        | 0      | Hyperelliptic genus 3 with 3 marked points |
| 2.2        | 1      | Genus 2 with \( a \) a Weierstrass point |
| 2.2        | 2, 3   | Any moduli |
| 2.3        | 0      | \( \mathbb{P}^1 \sqcup \) Maroni special of genus 3 with \( 2a + b_2 \) the \( g_3^1 \) |
| 2.3        | 1      | \( \mathbb{P}^1 \sqcup \) plane quartic with \( 2a + 2b_2 \) a canonical divisor |
| 2.3        | 2      | \( \mathbb{P}^1 \sqcup \) genus 2 with \( b_2 \) a Weierstrass point |
| 2.3        | 3      | \( \mathbb{P}^1 \sqcup \) genus 1 with \( a - b_2 \) two-torsion |
| 2.3        | 4      | Any moduli |
| 2.4        | 0      | Maroni special genus 4 with a ramification point of the \( g_3^1 \) |
| 2.4        | 1      | Plane quartic with a point on a bitangent |
| 2.4        | 2, 3, 4| Any moduli |
| 2.5        | 1      | Genus 2 with \( b_1 + b_2 \) hyperelliptic conjugate |
| 2.5        | 2, 3   | Any moduli |
| 2.6        | 1      | Genus 2 with \( a_1 + a_2 \) hyperelliptic conjugate |
| 2.6        | 2, 3   | Any moduli |
| 2.7        | 1      | \( \mathbb{P}^1 \sqcup \) plane quartic with \( a_1 + a_2 + 2b_2 \) a canonical divisor |
| 2.7        | 2      | \( \mathbb{P}^1 \sqcup \) genus 2 with \( b_2 \) a Weierstrass point |
| 2.7        | 3      | \( \mathbb{P}^1 \sqcup \) genus 1 with \( a_1 + a_2 = 2b_2 \) |
| 2.7        | 4      | Any moduli |
| 2.8        | 2, 9   | Any moduli |
| 2.10       | –      | Genus 1 with \( a - b \) two-torsion |
| 2.11       | 1      | Hyperelliptic genus 3 with any 2 points |
| 2.11       | 2      | Genus 2 with \( a \) a Weierstrass point |
| 2.11       | 3, 4   | Any moduli |
| 2.12       | 1      | Plane quartic with \( 2a_1 + 2a_2 \) a canonical divisor |
| 2.12       | 2      | Genus 2 with \( b \) a Weierstrass point |
| 2.12       | 3, 4   | Any moduli |
| 2.13       | 1      | Plane quartic with the line joining the two points tangent at a third |
| 2.13       | 2, 3, 4| Any moduli |
| 2.14       | –      | Any moduli |

Having described \( C_{\text{tail}} \to P_{\text{tail}}, C_1 \to P_1, \) and \( C_2 \to P_2 \) individually, we now put them together. Let us first consider the case where \( G \to P_{\text{tail}} \) is trivial. Recall that in this case \( C_{\text{tail}} \) is a disjoint union of two
double covers $C^1_{\text{tail}}$ and $C^2_{\text{tail}}$ of $P_{\text{tail}}$. The dual graph of the coarse space of $C = C_1 \cup C^1_{\text{tail}} \cup C^2_{\text{tail}} \cup C_2$ has the following form

Here a dashed line represents one or two nodes with the following admissibility criterion: In the case of one node, the node point is a ramification point of the map to $|P|$ on both curves. In the case of two nodes, the two node points are unramified points in a fiber of the map to $|P|$ on both curves. The convention for drawing points of $A$ (resp. $B$) on top (resp. bottom) for $C_1$ and $C_2$ still applies, except that the $A$ and $B$ for $C^1_{\text{tail}}$ and $C^2_{\text{tail}}$ may be switched. Note that $C$ comes embedded in a surface $S$ fibered over $P$ obtained by gluing $S_1 \rightarrow P_1$, $S_2 \rightarrow P_2$, and $S_{\text{tail}} \rightarrow P_{\text{tail}}$ (see Figure 4). We can determine the parity of $f : S \rightarrow P$ using Proposition 4.8. We produce a section of $S \rightarrow P$ by gluing sections of $S_i \rightarrow P_i$ and of $S_{\text{tail}} \rightarrow P_{\text{tail}}$. We have recorded the self-intersections of the sections $\sigma_i$ of $S_i \rightarrow P_i$ (modulo 2). Consider a section $\sigma_{\text{tail}}$ of $S_{\text{tail}} \rightarrow P_{\text{tail}}$ that matches with $\sigma_i$ over $x_i$ and lies in the smooth locus of $S_{\text{tail}} \rightarrow P_{\text{tail}}$. Such a section is a section of the $\mathbf{P}^1$ bundle $S^1_{\text{tail}} \rightarrow P_{\text{tail}}$ or $S^2_{\text{tail}} \rightarrow P_{\text{tail}}$, say the first. Then the self-intersection of $\sigma_{\text{tail}}$ (modulo 2) is $b_1/2$, where $b_1$ is the number of ramification points of $C^1_{\text{tail}} \rightarrow P_{\text{tail}}$. We then get

$$\sigma^2 = \sigma^2_1 + \sigma^2_2 + \sigma^2_{\text{tail}}.$$  

The parity of $\sigma^2$ determines the parity of $f : S \rightarrow P$ by Proposition 4.8.

![Figure 4: A sketch of $C_1 \subset S_1$, $C_{\text{tail}} \subset S_{\text{tail}}$, and $C_2 \subset S_2$ as in type (7)](image)

For example, taking $C_1$ as in 1.9, $C_2$ as in 2.2, $C^1_{\text{tail}}$ of genus 0, and $C^2_{\text{tail}}$ of genus $p + 1$ gives the following instance of (14).

![Diagram of the resulting $S \rightarrow P$]  

The resulting $S \rightarrow P$ admits a section with self-intersection $p + 1 \pmod{2}$ and hence represents a divisor of $Q_{\text{odd}}$ for even $p$. For $p = 0$, we get the divisor $8$ in Proposition 4.12. For $p = 2$, we get a codimension 2 locus.

We similarly take all possible combinations of $C_1$, $C_2$, and $C_{\text{tail}}$, compute the stable images (see Table 2), and do a dimension count to see which ones give divisors. The combinations not shown in the Table 2 correspond to boundary divisors of $Q_{\text{odd}}$ whose images in $Q$ have codimension higher than one. A prime (') denotes the dual graph obtained by a vertical flip (that is, by switching $A$ and $B$).
The combinations not shown in Table 3 give loci of codimension higher than one. The proof of Proposition 4.12 is thus complete.
4.2.3 Type (8)

**Proposition 4.14.** There are 2 irreducible components of $\bar{Q} \cap \Delta$ which are images of divisors of type (8) in $Q_{6}$ odd. Their generic points correspond to the following stable curves:

- With the dual graph $\xymatrix{ & x \ar@(dl,ul)[] \ar[r] & y}$
  1. $X$ hyperelliptic of genus 5.
- With the dual graph $\xymatrix{ & i \ar@(dl,ul)[] \ar[r] & j + k \ar[r] & P_{1} \ar[r] & Z_{1}}$

2. $X$ hyperelliptic of genus 3, $Y$ of genus 1.

The rest of § 4.2.3 is devoted to the proof.

Recall that type (8) corresponds to $\phi : P \to \mathbb{Z}$ with the following dual graph.

$$
\xymatrix{ & P_{2} \\
& P_{1} \ar[r] & P_{tail} \ar@(dl,ul)[] \ar[r] & i + j + k \\
& \ar[r] & P_{3} \ar[r] & Z_{1} \ar[r] & \mathbb{Z} \ar[r] & \mathbb{Z} \ar[r] & \infty}
$$

We have already developed all the tools to analyze this case in § 4.2.2. The cover $C_{tail} \to P_{tail}$ factors as $C_{tail} \to G \to P_{tail}$. Since $P_{tail}$ has at most 3 orbifold points, the cover $G \to P_{tail}$ is either trivial or non-trivial. In the trivial case, $C_{tail}$ is the disjoint union $C_{tail}^{1} \sqcup C_{tail}^{2}$ of two double covers of $P_{tail}$. In the non-trivial case, $|G|$ is a rational curve and $|C_{tail}|$ is its double cover. The curves $C_{i}$ given by $P_{i} \to Z_{1}$ for $i = 1, 2, 3$ are enumerated as $1.7$ or $1.9$. The first has $|C_{tail}^{1}|$ of genus $-1$ and $|C_{tail}^{2}|$ of genus 5 and it gives divisor 1. The second has $|C_{tail}^{1}|$ of genus 1 and $|C_{tail}^{2}|$ of genus 3 and it gives divisor 2. All other combinations give loci of codimension higher than one.

The case of non-trivial $G \to P_{tail}$ gives one divisor. By renumbering the subscripts if necessary, say that the non-trivial monodromy of $G \to P_{tail}$ is at the node $x_{1} = P_{tail} \cap P_{1}$ and $x_{2} = P_{tail} \cap P_{2}$. Then taking $C_{1}$ and $C_{2}$ from $1.8, 1.10$ and $C_{3} = 1.9$ gives divisor 1. All other combinations give loci of codimension higher than one. Note that the question of parity is moot in this case by the same argument as in Proposition 4.13. The proof of Proposition 4.14 is thus complete.

33
A  Linear series on orbifold scrolls

Let \( \mathcal{P} \) be the orbifold curve \( \mathbb{P}^1(\sqrt{5}) \), which has one orbifold point with stabilizer \( \mathbb{Z}_5 \) over 0. The goal of this section is to describe \( \mathbb{P}^1 \) bundles over \( \mathcal{P} \), their coarse spaces, and linear series on them. We recall the following standard facts about \( \mathcal{P} \).

**Proposition A.1.** Let \( \mathcal{P} = \mathbb{P}^1(\sqrt{5}) \).
1. Every \( \mathbb{P}^1 \)-bundle over \( \mathcal{P} \) is the projectivization of a rank two vector bundle.
2. Every vector bundle on \( \mathcal{P} \) is the direct sum of line bundles.
3. The line bundles on \( \mathcal{P} \) are of the form \( \mathcal{O}_\mathcal{P}(a) \) for \( a \in \frac{1}{5}\mathbb{Z} \), where \( \mathcal{O}_\mathcal{P}(1/r) \) refers to the dual of the ideal sheaf of the unique (reduced) orbifold point on \( \mathcal{P} \).

Let \( c : \mathcal{P} \to \mathbb{P}^1 \) be the coarse space map. Note that \( c^*\mathcal{O}_\mathcal{P}(a) = \mathcal{O}_\mathbb{P}(a) \) for \( a \in \mathbb{Z} \) and \( c_\ast\mathcal{O}_\mathcal{P}(a) = \mathcal{O}_\mathbb{P}([a]) \) for \( a \in \frac{1}{5}\mathbb{Z} \). Set \( \mathcal{F}_a = \text{Proj}(\mathcal{O}_\mathcal{P} \oplus \mathcal{O}_\mathcal{P}(-a)) \). The tautological line bundle \( \mathcal{O}_{\mathcal{F}_a}(1) \) on \( \mathcal{F}_a \) has a unique section. We denote its zero locus by \( \sigma \) and call it the directrix. It is the unique section of \( \mathcal{F}_a \to \mathcal{P} \) with negative self intersection \( \sigma^2 = -a \). It corresponds to the projection \( \mathcal{O}_\mathcal{P} \oplus \mathcal{O}_\mathcal{P}(-a) \to \mathcal{O}_\mathcal{P}(-a) \). There are sections \( \tau \) disjoint from \( \sigma \) corresponding to projections \( \mathcal{O}_\mathcal{P} \oplus \mathcal{O}_\mathcal{P}(-a) \to \mathcal{O}_\mathcal{P} \). These \( \tau \) lie in the divisor class \( \sigma + aF \), where \( F \) is the pullback of \( \mathcal{O}_\mathbb{P}(1) \). Observe that if \( a \) is not an integer, then \( \tau(0) \) is independent of the choice of \( \tau \). We call \( \tau \) a co-directrix.

**Proposition A.2.** Retain the notation introduced above. If \( a \in \mathbb{Z} \), then \( |\mathcal{F}_a| \) is smooth and \( |\mathcal{F}_a| \to \mathbb{P}^1 \) is the \( \mathbb{P}^1 \)-bundle \( \text{Proj}(\mathcal{O}_\mathcal{P} \oplus \mathcal{O}_\mathcal{P}(-a)) \). If \( a \not\in \mathbb{Z} \), then \( |\mathcal{F}_a| \) is smooth except at the two points \( \sigma(0) \) and \( \tau(0) \). At \( \sigma(0) \), it has the singularity \( \frac{1}{r}(1, r-a) \). At \( \tau(0) \), it has the singularity \( \frac{1}{r}(1, r-r-a) \). Furthermore, the scheme theoretic fiber of \( |\mathcal{F}_a| \to \mathbb{P}^1 \) over 0 has multiplicity \( n/ \text{gcd}(r, ra) \).

**Proof.** Fix a generator \( \zeta \in \mu_r \). In local coordinates around 0, we can write \( \mathcal{P} \) as

\[
[\text{Spec } \mathbb{C}[x]/\mu_r],
\]

where \( \zeta \) acts by \( x \to \zeta x \). In these coordinates, we can trivialize \( \mathcal{O}_\mathcal{P} \oplus \mathcal{O}_\mathcal{P}(-a) \) as a \( \mu_r \)-equivariant vector bundle with basis \( \langle X, Y \rangle \) on which \( \zeta \) acts by \( X \to X \) and \( Y \to \zeta^{-a}Y \). We think of \( X \) and \( Y \) as homogeneous coordinates on the projectivization. Then \( \tau \) corresponds to \( X = 0 \) and \( \sigma \) to \( Y = 0 \). Locally around \( \tau(0) \) we can write \( \mathcal{F}_a \) as

\[
[\text{Spec } \mathbb{C}[x, X/Y]/\mu_r], \quad \text{where } \zeta \cdot (x, X/Y) = (\zeta x, \zeta^{-a}X/Y).
\]

Similarly, around \( \sigma(0) \) we can write \( \mathcal{F}_a \) as

\[
[\text{Spec } \mathbb{C}[x, Y/X]/\mu_r], \quad \text{where } \zeta \cdot (x, Y/X) = (\zeta x, \zeta^{ra}Y/X).
\]

The claims about the singularities follow from these presentations.

In either chart, if we invert the second coordinate, we see that the invariant ring is generated by \( u = x^{\text{gcd}(r, ra)}(X/Y)^{-\text{gcd}(r, ra)} \). On the other hand, the invariant ring in \( \mathbb{C}[x] \) is generated by \( v = x^t \). Up to an invertible function, the preimage of \( v \) is \( u^{\ell/\text{gcd}(r, ra)} \). The claim about the multiplicity follows.

We now turn to linear systems on \( \mathcal{F}_a \). Let \( \pi : \mathcal{F}_a \to \mathcal{P} \) be the projection.

**Proposition A.3.** Let \( \mathcal{C} \subset \mathcal{F}_a \) be a member of \( |n\sigma + mF| \). Then \( \text{deg } \omega_{\mathcal{C}/\mathcal{P}} = (n-1)(2m-an) \). If \( \mathcal{C} \) does not pass through \( \sigma(0) \), then \( m - na \) is a non-negative integer. If \( \mathcal{C} \) is étale over 0, then at least one of \( m-na \) or \( m-(n-1)a \) is a non-negative integer. If \( \mathcal{C} \) is smooth, then \( m-na \geq 0 \) or \( m-na = -a \). In the former case, \( \mathcal{C} \) is connected. In the latter case, \( \mathcal{C} \) is the disjoint union of \( \sigma \) and a curve in \( |(n-1)\tau| \).
Proof. We have $\omega_{F_i/F} = -2\sigma - aF$. By adjunction, $\omega_{E/F} = (n-2)\sigma + (m+a)F$. Hence

$$\deg \omega_{E/F} = ((n-2)\sigma + (m-a)F)(na + mF) = (n-1)(2m-an).$$

For the next two statements, expand a global section $s$ of $\pi_*\mathcal{O}(n\sigma + mF)$ locally around $0$ as a homogeneous polynomial of degree $n$ in local coordinates $X \oplus Y$ for $\mathcal{O} \oplus \mathcal{O}(-a)$. Say

$$s = p_0X^n + p_1X^{n-1}Y + \cdots + p_{n-1}XY^{n-1} + p_nY^n,$$

where $p_i$ is the restriction of a global section of $\mathcal{O}(m-ia)$. For $\mathcal{C}$ to not pass through $\sigma(0)$, $p_n$ must not vanish at $0$. For the zero locus of $s$ to be étale over $0$, at least one of $p_n$ or $p_{n-1}$ must not vanish at $0$. But $\mathcal{O}(m-ia)$ has a section not vanishing at $0$ if and only if $m-ia$ is a nonnegative integer.

For the next statements, note that $\mathcal{C} \cdot \sigma = m-na$. If $\mathcal{C}$ is smooth and $m-na < 0$, then $\mathcal{C}$ must contain $\sigma$ and have $\sigma \cdot (\mathcal{C} \setminus \sigma) = 0$. This forces $\mathcal{C}$ to be the disjoint union of $\sigma$ and a curve in $|(n-1)\tau|$. If $m-na \geq 0$, then we see that $h^0(\mathcal{C}, \mathcal{O}_\mathcal{C}) = 1$, which implies that $\mathcal{C}$ is connected.

**Corollary A.4.** Let $\mathcal{C} \subset F_1$ be a curve in the linear system $4\sigma + mF$ such that the degree of the ramification divisor of $\mathcal{C} \to \mathcal{P}$ is $b$. Then $m = b/6 + 2a$. If $\mathcal{C}$ does not pass through $\sigma(0)$, then $a \leq b/12$. If $\mathcal{C}$ is étale over $0$, then $a \leq b/6$. If $\mathcal{C}$ is smooth, then either $a \leq b/12$ or $a = b/6$.

**References**

[ACV03] Dan Abramovich, Alessio Corti, and Angelo Vistoli, *Twisted bundles and admissible covers*, Comm. Algebra 31 (2003), no. 8, 3547–3618.

[AV02] Dan Abramovich and Angelo Vistoli, *Compactifying the space of stable maps*, J. Amer. Math. Soc. 15 (2002), no. 1, 27–75 (electronic).

[Chi08] Alessandro Chiodo, *Stable twisted curves and their r-spin structures*, Ann. Inst. Fourier (Grenoble) 58 (2008), no. 5, 1635–1689.

[FM94] William Fulton and Robert MacPherson, *A compactification of configuration spaces*, Ann. of Math. (2) 139 (1994), no. 1, 183–225.

[Gri85] Edmond E. Griffin, II, *Families of quintic surfaces and curves*, Compositio Math. 55 (1985), no. 1, 33–62.

[Har82] Joe Harris, *Theta-characteristics on algebraic curves*, Trans. Amer. Math. Soc. 271 (1982), no. 2, 611–638.

[Hor64] Geoffrey Horrocks, *Vector bundles on the punctured spectrum of a local ring*, Proc. London Math. Soc. (3) 14 (1964), 689–713.

[Ols07] Martin C. Olsson, *(Log) twisted curves*, Compos. Math. 143 (2007), no. 2, 476–494.

[Pom13] Flavia Poma, *Étale cohomology of a DM curve-stack with coefficients in $\mathcal{G}_m$*, Monatshefte für Mathematik 169 (2013), no. 1, 33–50 (English).

[Rec73] Sevin Recillas, *Maps between Hurwitz spaces*, Bol. Soc. Mat. Mexicana (2) 18 (1973), 59–63.

[Vak01] Ravi Vakil, *Twelve points on the projective line, branched covers, and rational elliptic fibrations*, Math. Ann. 320 (2001), no. 1, 33–54.