ROBUST PORTFOLIO DECISIONS FOR FINANCIAL INSTITUTIONS

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Abstract. The present paper aims to study a robust-entropic optimal control problem arising in the management of financial institutions. More precisely, we consider an economic agent who manages the portfolio of a financial firm. The manager has the possibility to invest part of the firm’s wealth in a classical Black-Scholes type financial market, and also, as the firm is exposed to a stochastic cash flow of liabilities, to proportionally transfer part of its liabilities to a third party as a means of reducing risk. However, model uncertainty aspects are introduced as the manager does not fully trust the model she faces, hence she decides to make her decision robust. By employing robust control and dynamic programming techniques, we provide closed form solutions for the cases of the (i) logarithmic; (ii) exponential and (iii) power utility functions. Moreover, we provide a detailed study of the limiting behavior, of the associated stochastic differential game at hand, which, in a special case, leads to break down of the solution of the resulting Hamilton-Jacobi-Bellman-Isaacs equation. Finally, we present a detailed numerical study that elucidates the effect of robustness on the optimal decisions of both players.

1. Introduction. The use of stochastic optimal control techniques on financial related problems was originated, to the best of our knowledge, by Merton [28], who studied a continuous time investment/consumption problem in a model consisting of two assets: a riskless asset (bond or bank account) and a risky one (stock), whose dynamics are described by the classical Black-Scholes model. For the special case

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of the power utility function, Merton [28], was able to provide closed form solutions for the optimal investment strategy and the optimal consumption rules. Since then, many papers and books that bridge stochastic optimal control techniques and mathematical finance have been written, which in combination with the powerful concept of viscosity solutions, for first and especially for second order partial differential equations, by Crandall et.al [16], established stochastic optimal control theory as a fundamental part of mathematical finance.

The basic philosophy of stochastic optimal control theory, is encapsulated in the following scenario: there exists a decision maker who has control over a system, which is usually subject to noise (at least for the most cases of interest) and by deciding the control process she is able to manipulate the evolution of the system so as to drive it to a desired state in order to achieve some predefined goal. However, it has to be pointed out that while taking her actions, it is silently imposed that the decision maker has complete faith in the model she is given, in the sense that the exact probability law of the stochastic process that introduces stochasticity in the state system is precisely known. As it turns out (see e.g. Anderson et.al [1, 2], Biagini and Pinar [7], Hansen and Sargent [20], Pinar [32], Uppal and Wang [38], Wang and Hou [39] and references therein) this assumption is far from being characterized as realistic.

Model uncertainty aspects are typically introduced when the decision maker does not have complete faith in the model she faces. This idea, although very simple, is quite effective and constitutes the cornerstone of realistic modeling. Typically, according to this form of uncertainty, there exists a “true” reference probability measure (the agent is unaware of) that describes the law of the stochastic process introducing stochasticity in the model and a probability measure which is the agent’s idea about how this “true” law in fact looks like. The decision maker again solves an optimal control problem, but now the problem is solved over the worst possible scenario, that is, by using the model that may create the most unfavorable situation for the problem at hand. In order to mathematically handle such situations, the techniques of stochastic optimal control theory are enhanced with model selection techniques resulting to robust optimal control theory. The most interesting part of robust control theory, is the fact that a robust control problem can always be reformulated as a stochastic differential game with two players (at least in its simplest form). The first, is the decision maker, and the other is an imaginary player called Nature. The decision maker chooses the control process so as to drive the system to a desired state (as was originally planned), while, on the other hand, Nature antagonistically chooses the model (that is, a probability measure, as every stochastic model may be associated with a probability measure) so as to create the most unfavorable scenario for the decision maker.

Once a robust control problem has been reformulated as a stochastic differential game, there typically exist two major ways to proceed with its solution. The first one, relies on the Pontryagin’s maximum principle and backward stochastic differential equations (see e.g. Bachdahn and Li [12], Bayraktar [6] and references therein). The other, is to follow dynamic programming techniques that eventually lead to a second order partial differential equation, known as the Hamilton-Jacobi-Bellman-Isaacs (HJBI). Typically, this is a second order fully non-linear partial differential equation for which the most natural concept of a solution, turns out to be that of viscosity solutions (for more information on this subject, the interested reader is referred to Bachdahn and Li [13], Crandall et.al [16], Fleming and
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Souganidis [17], Nisio [30] and references therein). Even though smooth solutions to the HJBI are extremely rare, under assumption of smoothness it is possible to find some closed form solutions in many interesting problems. This will eventually lead to closed form solutions for the optimal controls of the players which in many situations of interest act as a useful benchmark in the attempt of effective hedging. This method has been heavily used in the relative literature with success by many authors and within a wide range of applications, see e.g. Baltas and Yannacopoulos [5], Branger et.al [8], Cairns [14], Cont [15], Flor and Larsen [18], Korn [22], Rieder and Wopperer [34], Zawisza [40, 41] and references therein.

In the present paper, we study a robust-entropic optimal control problem that arises in the management of financial institutions. To wit, we consider an economic agent who manages the portfolio of a financial firm (e.g. a bank, a hedge fund, e.t.c). The agent (henceforth called risk manager) has the possibility to invest part of the firm’s wealth in a classical Black-Scholes type financial market consisting of a risk-less asset (e.g. bank account, bond, e.t.c) and a risky one (e.g. stock, index, e.t.c). Moreover, we assume that the firm is exposed to a stochastic cash flow of liabilities. The risk manager, as a means of reducing the risk that the firm is exposed to, apart from investing in the financial market, has also the possibility to transfer proportion of its liabilities to a third party. However, model uncertainty aspects are introduced, in the sense that the risk manager does not trust the model she faces. Our aim, is to study an expected utility maximization problem under the worst-case scenario for this model. In this endeavor, we rewrite this robust control problem as a two player, zero sum stochastic differential game, between the risk manager and Nature. The risk manager chooses the optimal investment and the optimal proportional coverage strategies, while, on the other hand, Nature antagonistically chooses the probability measure (that is, the model) in an attempt to mess with the risk manager’s plans.

By employing robust control and dynamic programming techniques, in the special case when the risk manager operates under (i) logarithmic; (ii) exponential and (iii) power-type preferences, we provide closed form solutions for the optimal robust strategies and the robust value function associated with the robust control problem. Especially, in the cases of the exponential and the power utility functions, we employ a useful suggestion proposed by Maenhout [25]. Moreover, we provide a detailed study of the limiting behavior, of the stochastic differential game at hand, which in a special case leads to break down of the solution of the associated Hamilton-Jacobi-Bellman-Isaacs equation. Finally, we present a detailed numerical study that elucidates the effect of robustness on the optimal decisions of both players. The novelty of our work lies in the fact that: (i) we provide a detailed study, from a stochastic differential game point of view, for more than one utility functions (as the vast majority of the relative literature is concerned only with the exponential utility function); (ii) to the best of our knowledge, for the very first time in the relative literature, we solve the HJBI equation for a constant preference for robustness parameter, in the case of the logarithmic utility function; (iii) for the very first time in the relative literature, we provide a detailed study of the limiting behavior (with respect to the preference for robustness parameter) of the stochastic differential game at hand; (iv) we provide a special case where the solution of the HJBI equation breaks down; (v) we provide a detailed verification theorem, and more importantly, we prove that this theorem is satisfied in the cases of the exponential and the power utility functions. This requires to prove uniform integrability of the value function,
in more than one cases; (vi) we provide a detailed numerical study (for the whole trading interval) for the effect of robustness, by employing an Euler-Maruyama scheme and a Monte-Carlo approach, while the existing literature either assumes the optimal wealth process to be a constant or is concerned only with the initial investment behavior (at $t = 0$).

An outline of the paper is as follows. In Section 2, we describe our general model and in Section 3 we introduce model uncertainty aspects to our framework. In Section 4, we provide a general solution and a verification theorem, and in Section 5, we provide closed form solutions in the case of the (i) logarithmic; (ii) exponential and (iii) power-type utility functions. In Section 6, we provide a detailed study, that originates from the structure of the problem at hand, in the limiting cases $\lambda \to 0$ and $\lambda \to +\infty$, where $\lambda$ stands for the preference for the robustness parameter. Finally, in Section 7, we provide a numerical study for the effect of robustness on the optimal controls of both players.

2. The model. We consider the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ that satisfies the usual hypotheses, where $\mathcal{F}_t = \sigma(W(s), B(s), s \leq t)$ is the natural filtration induced by the standard one dimensional independent Brownian motions $\{W(t), t \geq 0\}$ and $\{B(t), t \geq 0\}$.

2.1. The financial market. Suppose that we have a classical Black-Scholes type financial market on the fixed time horizon $[0, T]$, with $T \in (0, \infty)$, and two investment possibilities:

- A risk free asset (bond or bank account) with unit price $S_0(t)$ at time $t$ and dynamics described by the ordinary differential equation
  \[ \frac{dS_0(t)}{S_0(t)} = r dt, \]  
  with initial condition $S_0(0) = 1$.
- A risky asset (stock or index) with unit price $S_1(t)$ at time $t$ which evolves according to the linear stochastic differential equation
  \[ \frac{dS_1(t)}{S_1(t)} = \mu dt + \sigma dW(t), \]  
  with initial condition $S_1(0) > 0$.

Here, the interest rate $r > 0$, the appreciation rate of the stock prices $\mu$ (with $\mu > r$) and the volatility of the stock prices $\sigma > 0$, are given constants.

2.2. The cash flow. In the present paper, we are concerned with the optimal robust behavior of a financial firm in the presence of a stochastic cash flow of liabilities, which is denoted by the stochastic process $\{L(t), 0 \leq t \leq T\}$. Inspired by Browne [11], as this cash flow may be considered as a risk process, it turns out that an appropriate model to describe its dynamics is the following

\[ dL(t) = \alpha dt - \beta dB(t), \]  
where $\alpha$ and $\beta$ are positive constants. In the above model, the drift term can be interpreted as the mean liabilities up to time $t$, while the stochastic term can be interpreted as the fluctuations around the mean liabilities. It has to be pointed out, that this model might give rise to negative values and of course there is also the probability of being of unbounded variation, due to the presence of the Itô integral.
However, as a first simplistic approach, by appropriately choosing the parameters $\alpha$ and $\beta$, these characteristics may be eliminated.

As also pointed out by Browne [11], if the financial firm under consideration is an insurance firm, the cash flow (3) stands for the cumulative claims process the firm faces, see e.g. Bai and Guo [3], Baltas et. al [4], Promislow and Young [33], Schmidli [35] and references therein.

2.3. Stochastic differential equation of the firm’s wealth. We consider a financial firm, that, at time $t = 0$, possesses some wealth $x_0 > 0$ and its actions cannot affect the market prices. At each instant of time $t \in [0, T]$ the risk manager decides the proportion $\pi(t) := \pi(t, \omega) : [0, T] \times \Omega \to [0, 1]$ of the firm’s wealth $X(t)$ to be invested in the risky asset (2). The remaining proportion $(1 - \pi(t))X(t)$ is invested in the risk-less asset, described by (1).

In addition, we assume that the firm is designed to offer some very specific services (e.g. financial investments consultancy, pension fund management, insurance, e.t.c) by entering a contract with its clients. It has been estimated that such a contract generates, from the part of the firm, a stochastic cash flow of liabilities that evolves according to (3). This cash outflow may refer, for example, to short or long term payments, operating costs related to the contract, taxation of income, etc. In exchange for the services it provides, the firm collects a compensation, which for simplicity is assumed to be received continuously at the constant rate $c_0 \mathbb{E}[L(t)] = c_0 \alpha$, where $c_0 \geq 1$ and $\alpha$ denotes the mean appreciation rate of the stochastic process $L$, per unit of time.\(^{1}\) However, by collecting this income, the firm undertakes some additional risk. As a means of reducing this additional exposure, the risk manager of the firm has the ability to transfer a proportion of its liabilities to another party (e.g. external investor, financial fund, reinsurance firm, e.t.c). More precisely, at each time $t \in [0, T]$ the risk manager decides the proportion $q(t) := q(t, \omega) : [0, T] \times \Omega \to [0, 1]$ of its risk process to be covered, by entering a contract with a third party. In exchange, for this coverage, the third party collects a compensation which is assumed to be received continuously at the constant rate $c_1 \mathbb{E}[L(t)] q(t) = c_1 \alpha q(t)$, where $c_1 \geq 1$. A natural requirement at this point would be that $c_0 \leq c_1$, otherwise the firm could exploit arbitrage opportunities.

To sum up, the wealth process of the firm corresponding to the strategy $\eta_1 = (\pi(t), q(t))$, is denoted as $X^{\eta_1}(t)$ and is defined as the solution of the following linear stochastic differential equation

$$dX^{\eta_1}(t) = \pi(t)X^{\eta_1}(t) \frac{dS_1(t)}{S_1(t)} + (1 - \pi(t))X^{\eta_1}(t) \frac{dS_0(t)}{S_0(t)} + dR(t), \quad (4)$$

where

$$dR(t) = \left[\alpha c_0 - \alpha c_1 q(t)\right]dt - dL(t) + q(t)dL(t)$$

$$= \left[-(1 - c_0)\alpha + (1 - c_1)\alpha q(t)\right]dt + \beta(1 - q(t))dB(t). \quad (5)$$

\(^{1}\)The idea behind this is the following. The firm has estimated that such a contract is expected to generate liabilities equal to $\alpha$. As a result, it seems quite logical for the firm to demand a proportional amount of the estimated expected liabilities. To wit, when $c_0 = 1$, the firm asks for a compensation equal to the estimated expected liabilities and in this case the firm is expected to make no profit. On the other hand, if $c_0 > 1$, the firm is expected not only to cover the expected liabilities but also to make some profit. Of course, in the case $c_0 < 1$, the firm is expected to lose money.
Therefore, in view of Equations (1), (2), (3) and (5), it follows that the wealth of the financial firm, evolves according to the stochastic differential equation

\[ dX^n(t) = \left[ X^n(t)(r + (\mu - r)\pi(t)) - (1 - c_0)\alpha + (1 - c_1)\alpha q(t) \right] dt \\
+ \beta(1 - q(t))dB(t) + \sigma\pi(t)X^n(t)dW(t), \]

(6)

with initial condition \( X^n(0) = x_0 > 0. \)

**Definition 2.1.** Let \( \mathbb{F} \) be a general filtration. We denote by \( \mathcal{A}^\mathbb{F} \) the class of admissible strategies \( (\pi(t), q(t)) \) that satisfy the following conditions:

(i). \( \pi(t), q(t) \) are \( \mathbb{F} \)-progressively measurable.

(ii). \( 0 \leq \pi(t) \leq 1 \) and \( 0 \leq q(t) \leq 1. \)

(iii). \( \mathbb{E}\left[ \int_0^T (\sigma\pi(t))^2 dt \right] < \infty \) and \( \mathbb{E}\left[ \int_0^T \beta^2(1 - q(t))^2 dt \right] < \infty. \)

(iv). The stochastic differential equation (6) admits a unique strong solution.

Even though the framework adopted in the current paper is quite general, it is designed so as to find applications in many areas of interest. In this vein, we provide the following two characteristic examples. Of course, a modification of the dynamics of the cash flow process (3) including jumps would be much more interesting from a mathematical point of view but this comes with a cost concerning the analytical tractability of our model. As in the present paper we focus on closed form solutions, such an extension will be the focus of a future work.

**Example 1.** Consider a financial institution (e.g. a bank) that issues mortgage loans. All the loans that have been issued are part of the bank’s loan portfolio. This portfolio generates both income and claims. Income steams from the loan repayments, which are assumed to be made continuously and at a constant interest rate. However, as already mentioned, this portfolio also generates claims. These claims are generated when for example a loan goes default or when the bank faces liquidity issues. As these claims constitute a collection of random variables it is reasonable to state that they form a stochastic process. One of the possible ways to describe the evolution of this process is the model (3). Furthermore, the bank, in order to reduce the risk associated with the claims generated by this portfolio, decides to sell part of it to some external investor (e.g. a hedging fund, a foreign investor, an insurance company etc.). Finally, the bank also has the opportunity to invest part of its assets/reserves in a financial market, like the one described by equations (1) and (2), as a means of transferring risk and wealth to different agents and different future states of the world. This is a characteristic example that may be considered as a special case of our general model.

**Example 2.** Another interesting example which falls in the above mentioned general framework, is the classical insurance/reinsurance setting. Let us for example consider an insurance firm who has the opportunity to invest part of its reserves in the financial market described by Equations (1) and (2). In addition, the insurance firm collects premia continuously and at a constant rate from its clients. In this case, the stochastic process (3) is considered as the claims process the insurance firm faces. As a means of reducing the underlying risk involved with this claims process, the insurance firm faces the possibility of entering a reinsurance contract.
and purchase coverage (not necessary proportional). For this coverage, the reinsurance firm also receives premia at some constant rate. This is another example that may be considered as a special case of our general model.

3. Misspecification concerns. We further assume that the risk manager of the firm is uncertain as to the true nature of the stochastic processes $W$ and $B$ that introduce uncertainty into the state equation (6), in the sense that the exact law for $W$ and $B$ is not known. To specify this a bit more, we assume that there exists an unknown drift $y_1$ related to the Brownian motion $W$ and an unknown drift $y_2$ related to the Brownian motion $B$. Furthermore, we assume that these two drifts are stochastic processes. While this is not the most general form of uncertainty concerning the stochastic processes $W$ and $B$, it provides a simplistic, well-known and heavily studied framework to effectively study the problem at hand. Stated differently, there exists a “true” probability measure related to the true law of the processes $W$ and $B$, the risk manager is unaware of, and a probability measure $Q$, which is her idea of what the exact law of $W$ and $B$ looks like. The manager is going to solve an optimal control problem related to the maximization of an expected utility under the probability measure $Q$, which is her idea of the future states of the work. In other words, the manager seeks to solve the optimal control problem

$$
\sup_{\pi, q \in A} \mathbb{E}_Q \left[ U(X^{\pi}(T)) \bigg| \mathcal{F}_t \right],
$$

where $U$ denotes the utility function (to be specified later). At this point, observe that the risk manager seeks to maximize the expected utility under the probability measure $Q$ she adopts concerning the future states of the world.

However, the manager is uncertain about validity of $Q$ as the appropriate way to describe the future evolution of (6). In this vein, she seeks to adopt a more careful approach, that of seeking to minimize the worst possible scenario concerning the true description of the noise terms. In other words, she seeks to maximize the minimum possible value of the expected utility over all possible scenarios concerning the true state of the system, which is quantified as

$$
\inf_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[ U(X^{y_1}(T)) \bigg| \mathcal{F}_t \right],
$$

where $\mathcal{Q}$ is an appropriate class of probability measures to be specified shortly. Putting all these together, the risk manager faces the robust control problem

$$
\sup_{\pi, q \in A} \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[ U(X^{\pi}(T)) \bigg| \mathcal{F}_t \right],
$$

subject to the state process (6).

In what follows, we are going to restrict the class $\mathcal{Q}$ in the following two ways.

**Definition 3.1.** [The set $\mathcal{Q}$] The set of acceptable probability measures $Q$ for the agent is a set enjoying the following two properties:

(i). We will only consider the class of measures $\mathcal{Q}$, such that considering the stochastic process $W$ under the reference probability measure $\mathbb{P}$ and under the probability measure $Q$ results to a change of drift to the Brownian motion $W$.

(ii). There is a maximum allowed deviation of the managers measure $Q$ from the reference measure $\mathbb{P}$. In other words, the manager is not allowed to freely choose between various probability models as every departure will be penalized.
by an appropriately defined penalty function, a special case of which is the Kullback-Leibler relative entropy $\mathcal{H}(\mathbb{P} \| \mathbb{Q})$.

The above two conditions specify the set $\mathbb{Q}$. In what follows we provide a more concrete characterization of the set $\mathbb{Q}$.

3.1. Characterization of $\mathbb{Q}$. In this section we explicitly characterize the set of measures $\mathbb{Q}$ as given in Definition 3.1.

3.1.1. Change of measure. We first analyze restriction (i) in Definition 3.1. This is related to the celebrated Girsanov theorem in stochastic analysis.

To be more precise, we consider the progressively measurable stochastic processes $(y_1, y_2) := (y_1(t), y_2(t), t \in [0, T])$ taking values in some compact and convex set $Y \subset \mathbb{R}^2$ and satisfying the integrability conditions

$$\mathbb{P} \left[ \int_0^T y_1^2(s) ds < \infty \right] = 1 \quad \text{and} \quad \mathbb{P} \left[ \int_0^T y_2^2(s) ds < \infty \right] = 1. \quad (8)$$

Additionally, we define the stochastic process

$$\Xi(t) = \exp \left( \int_0^t y_1(s) dW(s) + \int_0^t y_2(s) dB(s) - \frac{1}{2} \int_0^t y_1^2(s) + y_2^2(s) ds \right). \quad (9)$$

Remark 1. Given the compactness of $\mathcal{Y}$, it follows that the stochastic processes $(y_1(t), y_2(t), t \in [0, T])$ satisfy the integrability condition

$$\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^T y_1^2(s) + y_2^2(s) ds \right) \right] < \infty, \quad (10)$$

which is known as the Novikov condition [31].

Note that, the stochastic process $(y_1, y_2)$ will serve as possible allowed drifts the agent may assign to the Wiener processes $W$ and $B$, respectively.

Theorem 3.2 (Girsanov [19]). Assume that $(y_1, y_2) \in \mathcal{Y}$ satisfy condition (10) and define the probability measure $\mathbb{Q}$ on $\mathcal{F}_T$, as $d\mathbb{Q} / d\mathbb{P} := \Xi(T)$. Additionally, assume that the stochastic process $\Xi$ is an $(\mathbb{F}, \mathbb{P})$ martingale. Then, the stochastic processes $\tilde{W}$ and $\tilde{B}$ with decomposition given by

$$\tilde{W}(t) = W(t) - \int_0^t y_1(s) ds, \quad (11)$$

and

$$\tilde{B}(t) = B(t) - \int_0^t y_2(s) ds, \quad (12)$$

are $(\mathbb{P}, \mathbb{Q})$ Brownian motions.

Proposition 1. The wealth process under the equivalent probability measure $\mathbb{Q} \in \mathbb{Q}$ is denoted as $\tilde{X}^{\eta_1, \eta_2}(t)$, where $\eta_1 = (\pi(t), q(t))$ and $\eta_2 = (y_1(t), y_2(t))$ and is defined as the solution of the following stochastic differential equation

$$d\tilde{X}^{\eta_1, \eta_2}(t) = \left[ \r \tilde{X}^{\eta_1, \eta_2}(t) + (\mu - r) \pi(t) \tilde{X}^{\eta_1, \eta_2}(t) - (1 - c_0) \alpha \right.$$

$$\left. + \frac{1}{2} \sigma \alpha \pi(t) \tilde{X}^{\eta_1, \eta_2}(t) + \sigma \pi(t) y_1(t) \tilde{X}^{\eta_1, \eta_2}(t) + \beta (1 - q(t)) y_2(t) \right] dt \quad (13)$$

$$+ \sigma \pi(t) \tilde{X}^{\eta_1, \eta_2}(t) d\tilde{W}(t) + \beta (1 - q(t)) d\tilde{B}(t),$$

with initial condition $\tilde{X}^{\eta_1, \eta_2}(0) = x > 0$. 


Proof. The proof follows immediately by substituting the semimartingale decompositions (11) and (12) in the SDE (6) that describes the wealth process under the reference probability measure $P$.

3.1.2. The entropic constraint. In this section we express restriction (ii) in Definition 2.1 in a more convenient form, using the stochastic processes $(y_1, y_2) \in \mathcal{Y}$ used to characterize the measure $Q$.

Proposition 2. The Kullback-Leibler relative entropy between the equivalent probability measures $P$ and $Q$, is denoted as $H(P | Q)$ and is given by

$$H(P | Q) = \mathbb{E}_Q \left[ \frac{1}{2} \int_0^T y_1^2(s) + y_2^2(s) ds \right]. \quad (14)$$

Proof. The proof is standard but is repeated here for the sake of completeness. The relative entropy is defined as

$$H(P | Q) = \mathbb{E}_Q \left[ \log \Xi(T) \right], \quad (15)$$

where the stochastic process $\Xi(t)$ is defined in (9). Hence, by taking into account the semimartingale decompositions (11) and (12), it follows that

$$H(P | Q) = \mathbb{E}_Q \left[ \int_0^T y_1(s) dW(s) + \int_0^T y_2(s) dB(s) - \frac{1}{2} \int_0^T y_1^2(s) + y_2^2(s) ds \right]$$

$$= \mathbb{E}_Q \left[ \int_0^T y_1(s) d\tilde{W}(s) + \int_0^T y_2(s) d\tilde{B}(s) + \frac{1}{2} \int_0^T y_1^2(s) + y_2^2(s) ds \right],$$

and the proof follows by noticing that the stochastic processes $\tilde{W}$ and $\tilde{B}$ are $(\mathbb{F}, Q)$ Brownian motions.

3.1.3. A modified cost functional. Motivated by the above observations and the pioneering work of Anderson et. al [1, 2], in the present paper we decide to penalize deviations from the reference probability model by employing the Kullback-Leibler relative entropic function (14), weighted by the term $1/\lambda$. The non-negative constant $\lambda$ is a measure to quantify the preference for robustness. More specifically, when $\lambda \to 0$, the risk manager has complete faith in her model and seeks no robustness. In this case, the robust control problem (7) reduces to the expected utility maximization problem

$$\sup_{\pi, q \in \mathcal{A}^q} \mathbb{E} \left[ U(X^{\eta_1}(T)) \bigg| \mathcal{F}_t \right], \quad (16)$$

subject to the state dynamics (13), which can be solved by following the classical dynamic programming techniques. On the other hand, a value of $\lambda = +\infty$ dictates that the controller does not believe at all in her model and is willing to consider alternative models with larger entropy (for information on the limiting behavior of our model cf. Section 6).

To sum up, the payoff functional associated with the robust control problem (7) can be expressed in a much simpler form by using the pseudo-entropic functional (14), as follows

$$J(t, x) := \mathbb{E}_Q \left[ U(\hat{X}^{\eta_1, \eta_2}(T)) \right] + \frac{1}{\lambda} H(P | Q)$$
\[
\begin{align*}
V(t, x) &= \sup_{\pi, q \in A^\ell} \inf_{y_1, y_2 \in Y} J(t, x),
\end{align*}
\]
subject to the dynamics \( (19) \).

**Remark 2.** The robust control problem \( (18) \) is in the form of a zero sum stochastic differential game with two players: the risk manager (player I) and a fictitious adversarial agent, who is commonly referred to, as Nature (player II). The risk manager of the firm decides the optimal investment strategy as well as the proportion of the firm’s claims to be transferred. On the other hand, Nature, antagonistically chooses the probability measure \( Q \in \mathcal{Q} \), equivalently the stochastic processes \( (y_1, y_2) \in Y \), so as to create the most unfavorable scenario for the risk manager.

**Definition 3.3.** The upper value for the stochastic differential game \( (18) \) with initial data \( (t, x) \in [0, T] \times \mathbb{R}_+ \) is defined as
\[
V(t, x) = \sup_{\pi, q \in A^\ell} \inf_{Q \in \mathcal{Q}} J(t, x)
\]
subject to the state dynamics \( (19) \).

The lower value for the stochastic differential game \( (18) \) with initial data \( (t, x) \in [0, T] \times \mathbb{R}_+ \) is defined as
\[
\underline{V}(t, x) = \inf_{Q \in \mathcal{Q}} \sup_{\pi, q \in A^\ell} J(t, x)
\]
subject to the state dynamics \( (19) \).

where both problems are considered subject to the state dynamics \( (19) \).
If \( V(t, x) = V(t, x) \) then we say that the stochastic differential game (18) has a value given by (20). However, this is a point that requires special treatment and constitutes a matter of research on its own, as this condition (which has its origins to the famous Isaacs minimax condition) does not hold in general. Within the framework of viscosity solutions, this condition has been proven by Fleming and Souganidis [17]. For more information on this matter, the interested reader is referred, among others, to Nikaidó [29] or Sion [36] along with the classical work of Isaacs [21].

4. The Hamilton-Jacobi-Bellman-Isaacs equation and general solution.

The present section focuses on the derivation of the Hamilton-Jacobi-Bellman-Isaacs equation which is associated with the stochastic differential game (18) subject to the state dynamics (19). Furthermore, we provide a general solution for the problem at hand. In this endeavor, we resort to dynamic programming techniques. Before proceeding any further, we let \( S = [0, T] \times \mathbb{R}_+ \). We will also denote by \( C^{1,2} \) the space of functions which are once continuously differentiable with respect to the time variable and twice continuously differentiable with respect to the space variable.

**Assumption 1.** The utility function \( U : \mathbb{R}_+ \to \mathbb{R}_+ \) is a function that is strictly increasing, strictly concave, \( C^1 \) and satisfies the usual Inada conditions.

\[
\lim_{x \to +\infty} U'(x) = 0 \\
\lim_{x \to 0} U'(x) = +\infty.
\]

**Assumption 2.** The value function \( V(t, x) \) is smooth enough, that is, \( V \in C^{1,2}(S) \) and moreover it satisfies the conditions \( V_x > 0 \) and \( V_{xx} < 0 \).

The generator for the state process (19) subject to the choice \((\pi, q)\) and \((y_1, y_2)\) is denoted by \( \tilde{L}^{\eta_1, \eta_2} \) and admits the form

\[
\tilde{L}^{\eta_1, \eta_2} h(t, x) = h_t + [r x + (\mu - r) \pi x - (1 - c_0) \alpha + (1 - c_1) \alpha q + \sigma \pi y_1 x + \beta (1 - q) y_2] h_x + \frac{1}{2} \left( \sigma^2 \pi^2 x^2 + \beta^2 (1 - q)^2 \right) h_{xx},
\]

where \( h \in C^{1,2}(S) \). Thus, under Assumption 2, one can show that the value function \( V \) satisfies the following Hamilton-Jacobi-Bellman-Isaacs equation

\[
\sup_{\pi, q \in \mathcal{A}} \inf_{y_1, y_2 \in \mathcal{Y}} \tilde{L}^{\eta_1, \eta_2} V(t, x) + F(y_1, y_2; \lambda) = 0 \\
V(T, x) = U(x),
\]

where we have defined for simplicity

\[
F(y_1, y_2; \lambda) = \frac{1}{2\lambda} (y_1^2 + y_2^2).
\]

We are now in a position to present the first result of the paper.

**Theorem 4.1.** Suppose that the risk manager has preference for robustness as described by the non-negative constant \( \lambda \). The optimal robust strategy is to invest in the risky asset proportion of the firm’s wealth equal to

\[\text{...}\]

\[\text{...}\]

\[\text{...}\]

\[\text{...}\]

\[\text{...}\]

\[\text{...}\]
\[ \pi^*(t, x) = -\frac{\mu - r}{\sigma^2 x} \frac{V_x}{V_{xx} - \lambda V_x^2}, \]  

(26)

and also, to purchase proportional coverage for the firm’s liabilities, equal to

\[ q^*(t, x) = 1 - \frac{\alpha(1 - c_1)}{\beta^2} \frac{V_x}{V_{xx} - \lambda V_x^2}. \]  

(27)

On the other hand, Nature chooses the worst-case scenario defined by

\[ y_1^*(t, x) = \frac{\mu - r}{\sigma} \frac{\lambda V_x^2}{V_{xx} - \lambda V_x^2}, \]  

(28)

and

\[ y_2^*(t, x) = -\frac{\alpha(1 - c_1)}{\beta} \frac{\lambda V_x^2}{V_{xx} - \lambda V_x^2}. \]  

(29)

In this case, the optimal robust value function is a smooth solution of the following non-linear partial differential equation

\[ V_t + (rx + \alpha(c_0 - c_1))V_x - \frac{1}{2} \left[ \frac{(\mu - r)^2}{\sigma^2} + \frac{\alpha^2(1 - c_1)^2}{\beta^2} \right] \frac{V_x^2}{V_{xx} - \lambda V_x^2} = 0, \]  

(30)

with boundary condition \( V(T, x) = U(x) \), assuming that such a solution exists and it satisfies the conditions \( V_x > 0 \) and \( V_{xx} < 0 \).

**Proof.** Assume that the HJBI Equation (24) admits a classical solution \( V \in C^{1,2}(S) \) and moreover that it satisfies the concavity conditions \( V_x > 0 \) and \( V_{xx} < 0 \). For each pair \((t, x)\) we must solve the static max-min problem

\[ V_t + rx + \max_{\pi, q} \left[ \left( (\mu - r)\pi x - (1 - c_0)\alpha + (1 - c_1)\alpha q + \sigma \pi y_1 x \right. \right. \]

\[ \left. \left. + \beta(1 - q)y_2 \right) V_x + \frac{1}{2} \left( \sigma^2 \pi^2 x^2 + \beta^2(1 - q)^2 \right) V_{xx} + \frac{1}{2\lambda}(y_1^2 + y_2^2) \right] = 0. \]  

(31)

Concerning the inner minimization problem, the first order conditions, yield to the following candidate optimal controls for Nature

\[ \dot{y}_1 = -\pi x \lambda V_x, \]  

(32)

and

\[ \dot{y}_2 = -\beta(1 - q)\lambda V_x. \]  

(33)

Substituting (32) and (33) back in (31), yields to the maximization problem

\[ V_t + rx + \max_{\pi, q} \left[ (\mu - r)\pi x - (1 - c_0)\alpha + (1 - c_1)\alpha q \right] V_x \]

\[ + \frac{1}{2} \sigma^2 \pi^2 x^2 \left[ V_{xx} - \lambda V_x^2 \right] + \frac{1}{2} \beta^2 (1 - q)^2 \left[ V_{xx} - \lambda V_x^2 \right] = 0. \]  

(34)

Differentiating the above equation with respect to \((\pi, q)\) and setting the derivatives equal to zero, gives the optimal robust strategies (26) and (27). Moreover, by substituting (26) and (27) in (32) and (33) gives the worst-case scenario strategies (28) and (29). Finally, by substituting (26) and (27) back in (34) yields to the partial differential equation (30)

According to our program, it remains to justify that the control processes (26), (27), (28) and (29) we derived in Theorem 4.1 are indeed optimal. In this direction, we modify a similar result of Mataramwura and Øksendal [26] and Zawisza [40], and get the following result. Before proceeding any further, we define \( \bar{S} := [0, T) \times \mathbb{R}_+ \).
Theorem 4.2 (Verification). Suppose that there exists a function $\varphi \in C^{1,2}(\bar{S}) \cap C(S)$ and a Markovian control pair $(\eta_1^n, \eta_2^n) = ((\pi^*, q^*), (y_1^n, y_2^n)) \in A^F \times Y$, such that

(i). $\bar{L}^{n_1, n_2} \varphi(t, x) + F(y_2^n; \lambda) \geq 0$, for all $\eta_1 \in A^F$, $(t, x) \in \bar{S}$.

(ii). $\bar{L}^{n_1, n_2} \varphi(t, x) + F(y_2^n; \lambda) \leq 0$, for all $\eta_2 \in Y$, $(t, x) \in \bar{S}$.

(iii). $\bar{L}^{n_1, n_2} \varphi(t, x) + F(\eta_2^n; \lambda) = 0$, for all $(t, x) \in \bar{S}$.

(iv). $\varphi(T, x) = U(x)$, for all $x \in \mathbb{R}_+$.

(v). The family \( \{ \varphi(\tau, \bar{X}^{n_1, n_2}(\tau)) \}_{\tau \in \mathcal{T}_{[0, T]}} \) is uniformly integrable, where by $\mathcal{T}_{[0, T]}$ we denote the set of $\mathbb{F}$-stopping times on $[0, T]$.

Then, \( J^{n_1, n_2}(t, x) \leq \varphi(t, x) \leq J^{n_1, n_2}(t, x) \), \( J^{n_1, n_2}(t, x) \leq \varphi(t, x) \leq J^{n_1, n_2}(t, x) \), \( (35) \) and

\[ \varphi(t, x) = J^{n_1, n_2}(t, x) = V(t, x). \] \( (36) \)

Proof. First of all, we recall that $\bar{X}^{n_1, n_2}(t)$ denotes the state system at time $t$ under the Markovian control pair $(\eta_1, \eta_2) = ((\pi, q), (y_1, y_2)) \in A^F \times Y$. We define the localizing sequence of stopping times $\tau_n$, as \[ \tau_n := T \land n \land \inf \{ t > 0 : \left| \bar{X}^{n_1, n_2}(t) \right| \geq n \}, \quad n \in \mathbb{N}. \]

It is clear that as $n \to \infty$ we have that $\tau_n \to T$. A direct application of the Dynkin formula for the stopping time $\tau_n$ yields

\[ \mathbb{E}_Q \left[ \varphi(\bar{X}^{n_1, n_2}(\tau_n), \tau_n) \right] = \varphi(t, x) + \mathbb{E}_Q \left[ \int_t^{\tau_n} \bar{L}^{n_1, n_2} \varphi(s, \bar{X}^{n_1, n_2}(s)) ds \right]. \] \( (37) \)

**Step 1.** We apply (37) to the Markovian pair $(\eta_1, \eta_2)$ and take into account condition (ii). In this vein, we have

\[ \mathbb{E}_Q \left[ \varphi(\bar{X}^{n_1, n_2}(\tau_n), \tau_n) \right] \leq \varphi(t, x) - \frac{1}{2\lambda} \mathbb{E}_Q \left[ \int_t^{\tau_n} (y_1^n(s))^2 + (y_2^n(s))^2 ds \right], \]

which yields \[ \varphi(t, x) \geq \mathbb{E}_Q \left[ \varphi(\bar{X}^{n_1, n_2}(\tau_n), \tau_n) \right] + \frac{1}{2\lambda} \mathbb{E}_Q \left[ \int_t^{\tau_n} (y_1^n(s))^2 + (y_2^n(s))^2 ds \right]. \]

By applying the dominated convergence theorem (this is possible due to condition (v)) we let $n \to \infty$ (and also we take into account (iv)), we arrive at \[ \varphi(t, x) \geq J^{n_1, n_2}(t, x). \] \( (38) \)

**Step 2.** We apply (37) to the Markovian pair $(\eta_1^n, \eta_2^n)$ and take into account (i). In this vein, we have

\[ \mathbb{E}_Q \left[ \varphi(\bar{X}^{n_1, n_2}(\tau_n), \tau_n) \right] \geq \varphi(t, x) - \frac{1}{2\lambda} \mathbb{E}_Q \left[ \int_t^{\tau_n} y_1^n(s) + y_2^n(s) ds \right]. \]
which yields

\[ \varphi(t, x) \leq \mathbb{E}_Q \left[ \varphi(\tilde{X}^{\eta_1^*, \eta_2^*}(\tau_n), \tau_n) + \frac{1}{2\lambda} \mathbb{E}_Q \left[ \int_t^{\tau_n} \int_s^y \left( q_1^2(s) + q_2^2(s) \right) ds \right] \right]. \]

Again, by applying the dominated convergence theorem (this is possible due to condition (v)) we let \( n \to \infty \) (and also take into account (iv)) and arrive at

\[ J^{\eta_1^*, \eta_2^*}(t, x) \geq \varphi(t, x). \]  
(39)

**Step 3.** We apply (37) to the Markovian pair \((\eta_1^*, \eta_2^*)\) and take into account (iii). The proof is concluded by taking into account (38) and (39)

5. **Closed form solutions.** In this section, our aim is to provide solutions, in feedback form, for the optimal robust policies and the robust value function for a variety of utility functions: (i) logarithmic; (ii) exponential (iii) power. However, due to the non-linear character of the partial differential equation (30) this is an extremely difficult task and the difficulty stems from the presence of the preference for robustness parameter \( \lambda \). To the best of our knowledge, a closed form solution can only be derived in the case of the logarithmic utility function. In order to provide solutions in the cases of the exponential and the power utility functions, we appropriately adopt in our framework an interesting modification for the preference for robustness parameter proposed by Maenhout [25].

5.1. **The case of the logarithmic utility function.** In this section, we provide solutions in feedback form for the optimal robust policies and the robust value function in the case of the logarithmic utility function, that is, for a utility function of the form

\[ U(x) = \log(x). \]  
(40)

Before proceeding any further, we make the assumption \( c_0 = c_1 \) which is needed for the derivation of a closed form solution. This means that the firm pays to the third party, for the coverage of its liabilities, the whole amount it has collected from its client.

**Theorem 5.1.** Assume logarithmic preferences of the form (40) and also that the preference for robustness is described by the positive constant \( \lambda > 0 \). Furthermore let \( c_0 = c_1 \). The optimal robust value function, which is also the solution of the stochastic differential game (18), admits the form:

\[ V(t, x) = \log(x) + \mathcal{K}(t), \]  
(41)

where

\[ \mathcal{K}(t) = \left( r + \frac{1}{2(1 + \lambda)} \left[ \frac{(\mu - r)^2}{\sigma^2} + \frac{\alpha^2(1 - c_1)^2}{\beta^2} \right] \right)(T - t). \]  
(42)

In this case, the optimal robust strategy for the risk manager is to invest in the risky asset the constant amount

\[ \pi^*(t, x) = \frac{\mu - r}{\sigma^2(1 + \lambda)}, \]  
(43)

and also, to purchase proportional coverage for the firm’s liabilities, equal to

\[ q^*(t, x) = 1 + \frac{\alpha(1 - c_1)}{\beta^2(1 + \lambda)} x. \]  
(44)
On the other hand, Nature chooses the worst-case scenario defined by

\[ y_1^*(t, x) = -\frac{\mu - r}{\sigma} \frac{\lambda}{1 + \lambda}, \]  

(45)

and

\[ y_2^*(t, x) = \frac{\alpha(1 - c_1)}{\beta} \frac{\lambda}{1 + \lambda}. \]  

(46)

Proof. Suppose that the partial differential equation (30) admits a classical solution \( V \in C^{1,2}(S) \). We look for a solution using the following guess

\[ V(t, x) = \log(x) + K(t), \]

where \( K(t) \) is an appropriate function (to be determined later) with boundary condition \( K(T) = 0 \) (this follows from the boundary condition \( V(T, x) = U(x) \)).

Differentiating the above expression with respect to \((t, x)\), yields

\[ V_t(t, x) = K'(t) \]
\[ V_x(t, x) = 1/x \]
\[ V_{xx}(t, x) = -1/x^2. \]

Substituting the above expressions back in the partial differential equation (30), results to the following ordinary differential equation

\[ K'(t) + r + \frac{1}{2(1 + \lambda)} \left[ \frac{(\mu - r)^2}{\sigma^2} + \frac{\alpha^2(1 - c_1)^2}{\beta^2} \right] = 0. \]  

(47)

with boundary condition \( K(T) = 0 \). The solution of this equation gives the robust value function (41). Furthermore, substituting the above partial derivatives to equations (26), (27), (28) and (29) gives the robust optimal controls (43), (44), (45) and (46). The proof is concluded by an application of Proposition 4.2, as it is straightforward to check that all conditions are satisfied.

5.2. State dependent preference for robustness. In the stochastic differential game (7), departures from the reference model are penalized by the entropic penalty function (14). This penalization term is weighted by a constant \( \lambda > 0 \) which aims to capture the preference for robustness of the risk manager. To wit, as \( \lambda \) decreases the faith the risk manager has in her model increases, while, as \( \lambda \) increases she becomes more and more robust and seeks the transfer to another probability model by paying the penalty. This is the standard approach to robust modeling as pioneered by Anderson et. al [1, 2].

However, the existence of this non-negative constant \( \lambda \) yields to problems regarding the analytical tractability of our problem. These problems stem from the term \(-\lambda V_x^2\) appearing in the partial differential equation (30), which is the result of the robustness in our model. In fact, to the best of our knowledge, closed form solutions for the stochastic differential game (7) are only available when the risk manager of the firm operates under logarithmic preferences. In order to tackle this obstacle, there are two major suggested ways to adopt. The first one is to follow a viscosity solution approach for second order partial differential equations, see e.g. Crandall et. al. [16], Fleming and Souganidis [17] and references therein. However, in this case one is able to prove (under some additional structural assumptions on the model) that the stochastic differential game has a value and that this value is the unique viscosity solution of the associated HJBI equation. As in this framework
smoothness is no longer an issue, closed form solutions for the optimal controls are no longer possible.

In the present paper, we focus more on the existence of closed form solutions than on the existence of a weak solution for the HJBI equation (30). Closed form solutions, if possible, provide a useful benchmark that is translated to effective portfolio risk management. In this vein, we adopt a modification for the preference for robustness, initially proposed by Maenhout [25] and subsequently adopted by many authors, see for example Anderson et al [2], Branger et al [8], Flor and Larsen [18], Lioni and Poncet [23], Liu [24], Skiadas [37] and references therein. Maenhout [25] proposed replacing the constant preference for robustness parameter $\lambda$ by its state dependent version, denoted by the function $\psi(t, x) > 0$. This function is supposed to play the role of the performance parameter $\lambda$ and measures the preference for robustness. In fact, the behavior of the risk manager according to the values of the parameter $\lambda$, discussed earlier, still holds. That is, the larger the $\psi$ is, the less the faith in the model. As it turns out, this choice is driven by a natural economic interpretation (for more information on this we refer the interested reader to Maenhout [25]).

To sum up, from now on we consider the modified stochastic differential game

$$
\sup_{\pi, q \in A_F} \inf_{Q \in Q} J(t, x) = \sup_{\pi, q \in A_F} \inf_{y_1, y_2 \in Y} E_Q \left[ U(\tilde{X}^{n_1, n_2}(T)) + \frac{1}{2\psi(t, x)} \int_t^T (y_1^2(s) + y_2^2(s)) ds \right],
$$

subject to the state dynamics (19). Comparing with the stochastic differential game (7), the only thing that has changed is $\lambda$ which has now been replaced by the function $\psi(t, x)$. From now on, we work with the stochastic differential game (48) subject to the system dynamics (19).

5.3. The case of exponential utility functions. In this section, we derive closed form solutions for the special case of the exponential utility function, that is, the utility function of the form

$$
u_0 - \frac{\nu_1}{\gamma} e^{-\gamma x},
$$

where $\nu_1, \gamma > 0$ and $\nu_0 > \nu_1/\gamma$. Inspired by Maenhout [25], we propose a specific form for the preference for robustness function $\psi$, as given by

$$
\psi(t, x) = - \frac{\lambda}{\gamma [V(t, x) - \nu_0]} > 0.
$$

It has to be pointed out, that with the above choice for the preference for robustness function, robustness is independent of the current level of wealth.
Theorem 5.2. Assume Exponential preferences of the form (50) and also that the preference for robustness is described by the function \( \psi(t, x) > 0 \) of the form (51). The optimal robust value function, which is also the solution of the stochastic differential game (48), admits the form:

\[
V(t, x) = \nu_0 - \frac{\nu_1}{\gamma} \exp \left[ -\gamma xe^{r(T-t)} + g(t) \right],
\]

where

\[
g(t) = \alpha \gamma (c_0 - c_1) \frac{1 - e^{r(T-t)}}{r} - \frac{\gamma}{2(\lambda + \gamma)} \left[ \frac{(\mu - r)^2}{\sigma^2} + \frac{\alpha^2(1 - c_1)^2}{\beta^2} \right] (T - t). \tag{53}
\]

In this case, the optimal robust strategy for the risk manager is to invest in the risky asset the constant amount

\[
\pi^*(t, x) = \frac{\mu - r e^{-r(T-t)}}{\sigma^2 x} \frac{1}{\lambda + \gamma}, \tag{54}
\]

and also, to purchase proportional coverage for the firm’s liabilities, equal to

\[
g^*(t, x) = 1 + \frac{\alpha(1 - c_1)}{\beta^2} \frac{e^{-r(T-t)}}{\lambda + \gamma}. \tag{55}
\]

On the other hand, Nature chooses the worst-case scenario defined by

\[
y_1^*(t, x) = -\frac{\mu - r}{\sigma} \frac{\lambda}{\lambda + \gamma}, \tag{56}
\]

and

\[
y_2^*(t, x) = \frac{\alpha(1 - c_1)}{\beta} \frac{\lambda}{\lambda + \gamma}. \tag{57}
\]

Proof. Suppose that the partial differential equation (30) admits a classical solution \( V \in C^{1,2}(S) \). We look for a solution using the following guess

\[
V(t, x) = \nu_0 - \frac{\nu_1}{\gamma} \exp \left[ -\gamma xe^{r(T-t)} + g(t) \right],
\]

where \( g(t) \) is an appropriate function (to be determined later) with boundary condition \( g(T) = 0 \) (this follows from the boundary condition \( V(T, x) = U(x) \)). Differentiating the above expression with respect to \( (t, x) \), yields

\[
V_t(t, x) = [V(t, x, y) - \nu_0] \gamma xe^{r(T-t)} + g_t
\]

\[
V_x(t, x) = [V(t, x, y) - \nu_0] \gamma e^{r(T-t)}
\]

\[
V_{xx}(t, x) = [V(t, x, y) - \nu_0] \gamma^2 e^{2r(T-t)}.
\]

Substituting the above expressions back in the partial differential equation (30), results to the robust value function (52). Furthermore, substituting the above partial derivatives to equations (26), (27), (28) and (29) gives the robust optimal controls (54), (55), (56) and (57). The other half of the proof relies on Proposition 4.2. More precisely, As conditions (i)-(iv) are obviously satisfied, it remains to prove condition (v) of Proposition 4.2 (as augmented with Remark 3). This will be carried out in three steps.
**Step 1.** At first, we note that under Exponential preferences (Equation (52)), the optimal wealth process, denoted as $X^*(\cdot) := X^\eta;\gamma^2(\cdot)$, is the solution of the following stochastic differential equation

$$dX^*(t) = \left\{ rX^*(t) + \alpha(c_0 - c_1) + \left[ \frac{(\mu - r)^2}{\sigma^2} + \frac{\alpha^2(1-c_1)^2}{\beta^2} \right] \frac{\gamma e^{-r(T-t)}}{(\lambda + \gamma)^2} \right\} dt$$

$$+ \frac{\mu - r e^{-r(T-t)}}{\lambda + \gamma} dW(t) - \frac{\alpha(1-c_1) e^{-r(T-t)}}{\beta} d\tilde{B}(t),$$

with initial condition $X^*(0) = x_0 > 0$. This follows immediately by substituting the optimal robust strategies (54), (55), (56) and (57) back in the stochastic differential equation (19) that describes the wealth of the firm’s portfolio under the equivalent probability measure $Q$.

**Step 2.** We will prove that for the function (52) it holds that

$$\mathbb{E}_Q \left[ \sup_{t \in [0,T]} |V(t, X^*(t))|^2 \right] < \infty. \quad (59)$$

Indeed, from Step 1, we have that the optimal wealth process is given, in integral form, as

$$X^*(t) = x_0 e^r t + \int_0^t \alpha(c_0 - c_1) ds + \int_0^t \left[ \frac{(\mu - r)^2}{\sigma^2} + \frac{\alpha^2(1-c_1)^2}{\beta^2} \right] \frac{\gamma e^{-r(T-s)}}{(\lambda + \gamma)^2} ds$$

$$+ \int_0^t \frac{\mu - r e^{-r(T-s)}}{\lambda + \gamma} dW(s) - \int_0^t \frac{\alpha(1-c_1) e^{-r(T-s)}}{\beta} d\tilde{B}(s).$$

Hence, by taking into account Equation (52), we have that (we let for simplicity $\nu_0 = 0$)

$$|V(t, X^*(t))|^2 = \left( \frac{\nu_1}{\gamma} \right)^2 \exp \left[ 2 \left( -\gamma X^*(t)e^{r(T-t)} + g(t) \right) \right]$$

$$\leq A_1 \exp \left[ -2\gamma X^*(t)e^{r(T-t)} \right]$$

$$= A_1 \exp \left[ -2\gamma e^{r(T-t)} \left( x_0 e^r t + \int_0^t \alpha(c_0 - c_1) ds + \int_0^t e^{r(T-s)} M_0 ds \right. \right.$$

$$+ \left. \int_0^t \frac{\mu - r e^{-r(T-s)}}{\lambda + \gamma} dW(s) - \int_0^t \frac{\alpha(1-c_1) e^{-r(T-s)}}{\beta} d\tilde{B}(s) \right),$$

for some appropriate constant $A_1$, where the second line follows from the boundedness of the deterministic function $g(t)$, as given in (53), on $[0,T]$ and the positive constant $M_0$ is defined as

$$M_0 = \left[ \frac{(\mu - r)^2}{\sigma^2} + \frac{\alpha^2(1-c_1)^2}{\beta^2} \right] \frac{\gamma}{(\lambda + \gamma)^2}. \quad (60)$$

As a result, from the above inequality, it follows that (for some appropriate chosen constant $A_2$)

$$|V(t, X^*(t))|^2 \leq A_2 \exp(I_1(t)) \exp(I_2(t)) < \infty,$$

where

$$I_1(t) = -2\gamma \int_0^t \frac{\mu - r e^{-r(T-s)}}{\lambda + \gamma} dW(s),$$

$$I_2(t) = 2\gamma X^*(t)e^{r(T-t)}.$$
and
\[ I_2(t) = 2\gamma \int_0^t \frac{\alpha(1-c_1)}{\beta} \frac{e^{-r(T-s)}}{s+\gamma} dB(s), \]
which follows immediately by realizing that the stochastic processes \( \exp(I_1(t)) \) and \( \exp(I_2(t)) \) are exponential martingales. Finally, we have
\[
\mathbb{E}_Q \left[ |V(t, X^*(t))|^2 \right] \leq A_2 \mathbb{E}_Q \left[ \exp(I_1(t)) \exp(I_2(t)) \right] \\
\leq A_2 \left( \mathbb{E}_Q \left[ \exp(2I_1(t)) \right] \right) \left( \mathbb{E}_Q \left[ \exp(2I_2(t)) \right] \right)^{\frac{1}{2}} < \infty,
\]
where the last line follows from the Cauchy-Schwartz inequality.

**Step 3.** We will prove that
\[
\mathbb{E}_Q \left[ \sup_{t \in [0, T]} |\Gamma(t, X^*(t), y_1^*(t), y_2^*(t))|^2 \right] < \infty. \tag{61}
\]
Indeed, by taking into account (49) and (51), it follows that
\[
\mathbb{E}_Q \left[ |\Gamma(t, X^*(t), y_1^*(t), y_2^*(t))|^2 \right] = \mathbb{E}_Q \left[ |V(t, X^*(t))|^2 \lambda M_6 \right]^2 < \infty,
\]
where \( M_6 \) is defined in (74) and the last inequality follows from Step 2. As a result, Proposition 4.2 can be applied and this concludes the proof of the theorem.

**5.4. The case of power utility functions.** In this section, we derive closed form solutions for the special case of the power utility function, that is, the utility function of the form
\[ u(x) = x^{1-\rho}, \tag{62} \]
where \( 0 < \rho < 1 \) is the risk aversion parameter. With the same reasoning as before, we assume that the preference function \( \psi \) admits the form
\[ \psi(t, x) = \frac{\lambda}{(1-\rho)V(t, x)} > 0. \tag{63} \]

As in the case of the Logarithmic utility function, we again make the assumption \( c_0 = c_1 \) which is needed for the derivation of a closed form solution.

**Theorem 5.3.** Assume power-type preferences of the form (62) and also that the preference for robustness is described by the function \( \psi(t, x) > 0 \) of the form (63). Furthermore assume that \( c_0 = c_1 \). The optimal robust value function, which is also the solution of the stochastic differential game (48), admits the form:
\[ V(t, x) = x^{1-\rho} \exp \left[ \xi(t) \right], \tag{64} \]
where
\[ \xi(t) = r(1-\rho) + \frac{1-\rho}{\lambda + \rho} \left[ \frac{(\mu - r)^2}{\sigma^2} + \frac{\alpha^2(1-c_1)^2}{\beta^2} \right] (T-t). \tag{65} \]

In this case, the optimal robust strategy for the risk manager is to invest in the risky asset the constant amount
\[ \pi^*(t, x) = \frac{\mu - r}{\sigma^2} \frac{1}{\lambda + \rho}, \tag{66} \]
and also, to purchase proportional coverage for the firm’s liabilities, equal to...
\[ q^*(t,x) = 1 + \frac{\alpha(1 - c_1)}{\beta^2} \frac{x}{\lambda + \rho}. \]  

On the other hand, Nature chooses the worst-case scenario defined by

\[ y_1^*(t,x) = -\frac{\mu - r}{\sigma} \frac{\lambda}{\lambda + \rho}, \]  

and

\[ y_2^*(t,x) = \frac{\alpha(1 - c_1)}{\beta} \frac{\lambda}{\lambda + \rho}. \]  

**Proof.** Suppose that the partial differential equation (30) admits a classical solution \( V \in C^{1,2}(S) \). We look for a solution using the following guess

\[ V(t,x) = \frac{x^{1-\rho}}{1-\rho} h(t), \]

where \( h(t) \) is an appropriate function (to be determined later) with boundary condition \( h(T) = 1 \) (this follows from the boundary condition \( V(T, x) = U(x) \)). Differentiating the above expression with respect to \((t,x)\), yields

\[ V_t(t,x) = \frac{x^{1-\rho}}{1-\rho} h_t, \]
\[ V_x(t,x) = x^{-\rho} h, \]
\[ V_{xx}(t,x) = -\rho x^{-\rho-1} h. \]

Substituting the above expressions back in the partial differential equation (30), results to the following ordinary differential equation

\[ h'(t) + \left( r(1-\rho) + \frac{1-\rho}{\lambda+\rho} \left[ \frac{(\mu-r)^2}{\sigma^2} + \frac{\alpha^2(1-c_1)^2}{\beta^2} \right] \right) h(t) = 0, \]  

with boundary condition \( h(T) = 0 \). The solution of this equation gives the robust value function (64). Furthermore, substituting the above partial derivatives to equations (26), (27), (28) and (29) gives the robust optimal controls (66), (67), (68) and (69). Moreover, as conditions (i)-(iv) are obviously satisfied, it remains to prove condition (v) of Proposition 4.2 (as augmented with Remark 3). The proof, as in the case of Theorem 5.2, will be served in three steps.

**Step 1.** At first, we note that under power-type preferences (Equation (52)), the optimal wealth process, denoted as \( X^*(\cdot) := X^{\eta^{\cdot,\tilde{\omega}}(\cdot)} \), is the solution of the following stochastic differential equation

\[ dX^*(t) = X^*(t) \left\{ r + \left[ \frac{(\mu-r)^2}{\sigma^2} + \frac{\alpha^2(1-c_1)^2}{\beta^2} \right] \frac{\rho}{(\lambda+\rho)^2} \right\} dt \]
\[ + \frac{\mu-r}{\sigma(\lambda+\rho)} X^*(t) d\tilde{W}(t) - \frac{\alpha(1-c_1)}{\beta(\lambda+\rho)} X^*(t) d\tilde{B}(t), \]  

with initial condition \( X^*(0) = x_0 > 0 \). This follows immediately by substituting the optimal robust strategies (66), (67), (68) and (69) back in the stochastic differential equation (19) that describes the wealth of the firm’s portfolio under the equivalent probability measure \( Q \).

**Step 2.** We will prove that for the function (64) it holds that

\[ \mathbb{E}_Q \left[ \sup_{t \in [0,T]} |V(t, X^*(t))|^2 \right] < \infty. \]
Indeed, from Step 1, we have that the stochastic differential equation for the optimal wealth process, admits the following unique strong solution

$$X^*(t) = x_0 \exp \left\{ \left[ r + \left( \frac{(\mu - r)^2}{\sigma^2} + \frac{\alpha^2(1 - c_1)^2}{\beta^2} \right) \frac{2\rho - 1}{2(\lambda + \rho)^2} \right] t \right\} \times \exp \left[ \frac{\mu - r}{\sigma(\lambda + \rho)} \bar{W}(t) - \frac{\alpha(1 - c_1)}{\beta(\lambda + \rho)} \bar{B}(t) \right] = x_0 \exp(\bar{\mu}(t)) \exp(I_3(t)),$$

where

$$\bar{\mu}(t) = \left[ r + \left( \frac{(\mu - r)^2}{\sigma^2} + \frac{\alpha^2(1 - c_1)^2}{\beta^2} \right) \frac{2\rho - 1}{2(\lambda + \rho)^2} \right] t,$$

and

$$I_3(t) = \frac{\mu - r}{\sigma(\lambda + \rho)} \bar{W}(t) - \frac{\alpha(1 - c_1)}{\beta(\lambda + \rho)} \bar{B}(t).$$

Hence, by taking into account Equation (64), we have that

$$|V(t, X^*(t))|^2 = \frac{(X^*(t))^{2(1 - \rho)}}{(1 - \rho)^2} \exp[2\xi(t)] \leq A_4(X^*(t))^2 = A_4x_0^2 \exp(2\bar{\mu}(t)) \exp(2I_3(t)) < \infty,$$

where $A_4$ is some appropriately chosen constant. The first inequality follows from the fact that the function $\xi(t)$, as defined in Equation (65), is deterministic and bounded on $[0, T]$ and the last inequality follows from the fact that the stochastic process $\exp(2I_3(t))$ is a martingale. Finally, it follows that

$$E_Q \left[ |V(t, X^*(t))|^2 \right] \leq A_4x_0^2E_Q \left[ \exp(2\bar{\mu}(t)) \exp(2I_3(t)) \right]$$

$$\leq A_4x_0^2 \left( E_Q \left[ \exp(4\bar{\mu}(t)) \right] E_Q \left[ \exp (4I_3(t)) \right] \right)^{\frac{1}{2}} < \infty,$$

where the last line follows from the Cauchy-Schwartz inequality.

**Step 3.** We will prove that

$$E_Q \left[ \sup_{t \in [0, T]} |\Gamma(t, X^*(t), y_1^*(t), y_2^*(t))|^2 \right] < \infty. \quad (73)$$

Indeed, by taking into account (49) and (63), it follows that

$$E_Q \left[ |\Gamma(t, X^*(t), y_1^*(t), y_2^*(t))|^2 \right] = E_Q \left[ |V(t, X^*(t))|^2 M_7^2 \right] \leq A_5E_Q \left[ |V(t, X^*(t))|^2 \right] < \infty,$$

for some appropriate constant $A_5$, where $M_7$ is given by

$$M_7 = \left[ \frac{(\mu - r)^2}{\sigma^2} + \frac{\alpha^2(1 - c_1)^2}{\beta^2} \right] \frac{\lambda(1 - \rho)}{(\lambda + \rho)^2}, \quad (74)$$

and the last inequality follows from Step 2. As a result, Proposition 4.2 can be applied and this concludes the proof of the theorem.
6. Limiting behavior and special solutions. The present section is devoted to the study of the limiting behavior of our model as $\lambda \to 0$ and $\lambda \to +\infty$. These two cases are of great interest and importance and will have to be examined separately, as each one of them requires special treatment. Roughly speaking, it is well known (see e.g. Anderson et.al [1, 2], Hansen and Sargent [20] and references therein) that as $\lambda \to 0$ the decision maker fully trusts her model and exhibits no preference for robustness, while, as $\lambda \to +\infty$, the decision maker has no faith in the model she is offered and is willing to consider alternative models with larger relative entropy. Even though this is a common (and also very natural) remark in the relative literature, to the best of our knowledge, the vast majority of the available works examine the limiting behavior of the optimal robust strategies, after the problem has been solved, and are not concerned with the structural behavior of the robust control problem itself in these limiting cases, something that raises serious questions concerning the well-posedness of the problem in the first place, especially in the limit $\lambda \to +\infty$, as this is the problematic case. In the current section, we will attempt to give a rigorous answer to this crucial question.

In the present paper, we studied the stochastic differential games (18) and (48) subject to the state dynamics (19). Substantially, we studied one problem from two different point of views, namely, when the preference for robustness is (i) constant and (ii) state dependent. The payoff function associated with both cases is of the form $G(\eta_1, \eta_2)$, where $\eta_1$ and $\eta_2$ are the control of the players $^4$ and the function $G$ is concave with respect to $\eta_1$ and convex with respect to $\eta_2$. As the control spaces are defined as compact and convex sets, it follows, by a direct application of the Von Neumann min-max theorem, that the stochastic differential game has a value. However, the limiting behavior of this problem remains an open question. In this section, we will examine each limiting case separately. In what follows, unless stated otherwise, we consider the stochastic differential game (18) subject to the state dynamics (19).

6.1. The case $\lambda \to 0$. Let us consider the payoff functional (17). In this case, it is straightforward to see that the convex term dominates the concave term. However, the concave term is still present, something that allows us to proceed as usual. The stochastic differential game (18) has a value, as the Sion’s min-max theorem is still applicable. More precisely, we have the following result.

**Theorem 6.1** (Limiting behavior as $\lambda \to 0$). The optimal robust strategy for the risk manager is to invest in the risky asset proportion of the firm’s wealth equal to

$$\pi^*(t, x) = -\frac{\mu - r}{\sigma^2 x} \frac{V_x}{V_{xx}},$$

and also, to purchase proportional coverage for the firm’s liabilities, equal to

$$q^*(t, x) = 1 - \frac{\alpha(1 - c_1)}{\beta^2} \frac{V_x}{V_{xx}}.$$  

On the other hand, Nature chooses the myopic worst-case scenario defined by

$$y_1^*(t, x) = y_1^*(t, x) = 0.$$  

$^4$the structure of the game is, maximize with respect to $\eta_1$ and minimize with respect to $\eta_2$
In this case, the optimal robust value function is a smooth solution of the following non-linear partial differential equation

$$V_t + [rx + \alpha(\theta - \eta)]V_x - \frac{1}{2} \left[ \frac{(\mu - r)^2}{\sigma^2} + \frac{\alpha^2(1 - c_1)^2}{\beta^2} \right] V_{xx} = 0,$$

with boundary condition $V(T, x) = U(x)$, assuming that such a solution exists and it satisfies the conditions $V_x > 0$ and $V_{xx} < 0$.

**Proof.** Assume that the HJBI Equation (24) admits a classical solution $V \in C^{1,2}(\mathbb{S})$ and moreover that it satisfies the concavity conditions $V_x > 0$ and $V_{xx} < 0$. We consider the following static max-min problem

$$V_t + rxV_x + \max_{\pi, q} \min_{y_1, y_2} \left[ \left( (\mu - r)\pi x - (1 - c_0)\alpha + (1 - c_1)\alpha q + \sigma\pi y_1 x \right. \right.

$$

$$\left. + \beta(1 - q)y_2 \right] V_x + \frac{1}{2} \left( \sigma^2 \pi^2 x^2 + \beta^2 (1 - q)^2 \right) V_{xx} + \frac{1}{2\lambda} (y_1^2 + y_2^2) = 0. \quad (79)$$

Concerning the minimization problem, as $\lambda \to 0$, the only accepted policy for Nature is $y_1^*(t, x) = y_2^*(t, x) = 0$. This yields the maximization problem

$$V_t + rxV_x + \max_{\pi, q} \left[ \left( (\mu - r)\pi x - (1 - c_0)\alpha + (1 - c_1)\alpha q \right) V_x \right. \left. + \frac{1}{2} \left( \sigma^2 \pi^2 x^2 + \beta^2 (1 - q)^2 \right) V_{xx} = 0. \right. \quad (79)$$

Differentiating the above equation with respect to $(\pi, q)$ and setting the derivatives equal to zero, gives the optimal robust strategies (75) and (76). Moreover, by substituting (75) and (76) back in (79) yields to the partial differential equation (78).

**Remark 4.** In this limit $\lambda \to 0$, the risk manager has complete faith in the model described by Equations (2) and (3) and as a result operates under the probability measure $\mathbb{P}$. Note that, in this case, the optimal investment strategy (75), the optimal proportional coverage strategy (76) and the partial partial differential equation (78), are the optimal Markovian control laws and partial differential equation associated with the stochastic optimal control problem

$$\sup_{\pi, q \in A^F} \mathbb{E}_P \left[ U(X^{n_1}(T)) \right],$$

subject to the state dynamics (13). As a result, when $\lambda \to 0$, the stochastic differential game (18) is equivalent to a stochastic optimal control problem.

**Remark 5.** In the limit $\lambda \to 0$ (no robustness), we have the following results.

(i). When the risk manager operates under logarithmic preferences (Equation (40)), the optimal robust strategy is to invest in the risky asset proportion of the firm’s wealth equal to

$$\pi^*(t, x) = \frac{\mu - r}{\sigma^2}, \quad (80)$$

and to purchase proportional coverage for the firm’s liabilities, equal to

$$q^*(t, x) = 1 + \frac{\alpha(1 - c_1)}{\beta^2} x. \quad (81)$$

This follows by substituting Equation (41) in Equations (75) and (76).
(ii). When the risk manager operates under exponential preferences (Equation (50)), the optimal robust strategy is to invest in the risky asset proportion of the firm’s wealth equal to
\[
\pi^*(t,x) = \frac{\mu - r e^{r(T-t)}}{\sigma^2 x \gamma},
\]
and to purchase proportional coverage for the firm’s liabilities, equal to
\[
q^*(t,x) = 1 + \frac{\alpha(1 - c_1) e^{-r(T-t)}}{\beta^2 x \gamma}.
\]
This follows by substituting Equation (52) in Equations (75) and (76).

(iii). When the risk manager operates under power preferences (Equation (62)), the optimal robust strategy is to invest in the risky asset proportion of the firm’s wealth equal to
\[
\pi^*(t,x) = \frac{\mu - r}{\sigma^2 \rho},
\]
and to purchase proportional coverage for the firm’s liabilities, equal to
\[
q^*(t,x) = 1 + \frac{\alpha(1 - c_1)}{\beta^2 \rho x}.
\]
This follows by substituting Equation (64) in Equations (75) and (76).

**Remark 6.** Comparing the optimal robust strategies of the risk manager, with robustness and without robustness (Remark 5), we get the following results.

(i). When the risk manager operates under logarithmic preferences (Equation (40)), comparing equations (43) and (44) with (80) and (81), we can see that the relative risk aversion, which is equal to 1 in this case, is replaced with \(1 + \lambda > 1\).

(ii). When the risk manager operates under exponential preferences (Equation (50)), comparing equations (54) and (55) with (82) and (83), we can see that the relative risk aversion, which is equal to \(\gamma x\) in this case, is replaced with \((\gamma + \lambda)x > \gamma x\).

(iii). When the risk manager operates under power preferences (Equation (62)), comparing equations (66) and (67) with (84) and (85), we can see that the relative risk aversion, which is equal to \(\rho\) in this case, is replaced with \(\rho + \lambda > \rho\).

As a result, we can say that the effect of the robustness on the optimal investment and proportional coverage strategies is an increase in relative risk aversion. That is, the less the faith in the model, the more risk averse the risk manager is. This result coincides with a similar result of Baltas and Yannacopoulos [5] and Maenhout [25], who however adopt a robust control setting in different frameworks than the one studied in the present paper.

6.2. **The case** \(\lambda \to +\infty\). This case is much more challenging that the limit \(\lambda \to 0\), as now the convex term in (17) vanishes and the payoff functional is only concave (a property that is inherited from the utility function). As a result, the game lacks the necessary structure in order to guarantee the existence of a value. At first, we seek to solve the problem
\[
\sup_{\pi, q \in \mathcal{A}^p} \inf_{Q \in \mathcal{Q}} J^0(t,x)
\]
\[
= \sup_{\pi, q \in \mathcal{A}^p} \inf_{\eta_1, \eta_2 \in \mathcal{Y}} \mathbb{E}_Q \left[ U(\tilde{X}^{\eta_1, \eta_2}(T)) \right],
\]

where
subject to the state dynamics (19). In order to proceed even further in our analysis, we recall that \((\pi, q) \in [0, 1]^2\) and assume that \(\mathcal{Y}\) is the rectangle \([y_1, y_1] \times [y_2, y_2] := \mathcal{Y}_1 \times \mathcal{Y}_2\). In this direction, we have the following result.

**Theorem 6.2** (Limiting behavior as \(\lambda \to +\infty\)). The optimal robust strategy for the risk manager is to invest in the risky asset proportion of the firm’s wealth equal to

\[
\pi^*(t, x) = -\left(\frac{\mu - r}{\sigma} + \frac{y_1}{\sigma x V_{xx}}\right),
\]

and to purchase proportional coverage for the firm’s liabilities, equal to

\[
q^*(t, x) = 1 + \left(\frac{\alpha(c_1 - 1)}{\beta} + \frac{y_2}{\beta V_{xx}}\right).
\]

On the other hand, Nature chooses the myopic worst-case scenario defined by

\[
y_1^*(t, x) = y_1,
\]

and

\[
y_2^*(t, x) = y_2.
\]

In this case, the optimal robust value function is a smooth solution of the following non-linear partial differential equation

\[
V_t + rxV_x + \max_{\pi, q} \min_{y_1 \in \mathcal{Y}_1, y_2 \in \mathcal{Y}_2} \left[\left(\mu - r\right)\pi x - \left(1 - c_0\right)\alpha + \left(1 - c_1\right)\alpha q + \sigma \pi y_1 x \right.
\]

\[
+ \beta(1 - q)y_2 \right]V_x + \frac{1}{2} \left(\sigma^2 \pi^2 x^2 + \beta^2 (1 - q)^2\right)V_{xx} = 0.
\]

Concerning the minimization problem, it follows that the minimum in the expressions

\[
\min_{y_1 \in \mathcal{Y}_1} \left\{\sigma \pi y_1 x V_x\right\},
\]

\[
\min_{y_2 \in \mathcal{Y}_2} \left\{\beta(1 - q)y_2 V_{xx}\right\},
\]

is attained at the values \(y_1\) and \(y_2\), respectively. Substituting these solutions back in (92), yields the maximization problem

\[
V_t + rxV_x + \max_{\pi, q} \left[\left(\mu - r\right)\pi x - \left(1 - c_0\right)\alpha + \left(1 - c_1\right)\alpha q + \sigma \pi y_1 x \right.
\]

\[
+ \beta(1 - q)y_2 \right]V_x + \frac{1}{2} \left(\sigma^2 \pi^2 x^2 + \beta^2 (1 - q)^2\right)V_{xx} = 0.
\]

Differentiating the above equation with respect to \((\pi, q)\) and setting the derivatives equal to zero, gives the optimal robust strategies (87) and (88). Moreover, by
substituting (87) and (88) back in (93) yields to the partial differential equation (91)

6.2.1. Solution breakdown. Up to this point, it should have been made clear that the solution of the HJBI equation plays a central role in the solution of the associated stochastic differential game and as a consequence to the derivation of the optimal robust controls of the players. In the present paper, we have focused on the existence of smooth solutions for the HJBI equation, and in fact, we provided three different cases when such a solution exists, to wit, we examined the case of the (i) exponential; (ii) power and (iii) logarithmic utility functions. However, in the relative literature, there exists only a few papers that focus on the non-existence rather than on existence of a solution to the HJBI equation, like for example, the papers of Brock et.al [9] and McMillan and Triggiani [27], who studied a robust control problem in an infinite horizon. Motivated by these works, in what follows, we construct a case where the loss of convexity, due to the limit \( \lambda \to \infty \), results in the breakdown of the solution of the associated HJBI equation. First of all, for simplicity we assume that \( c_0 = c_1 \) and we rewrite the partial differential equation (91), as

\[
V_t + rxV_x - A(y_1, y_2) \frac{V^2}{V_{xx}} = 0,
\]

where

\[
A(y_1, y_2) := \frac{1}{2} \left[ \left( \frac{\mu - r}{\sigma} + y_1 \right)^2 + \left( \frac{\alpha(c_1 - 1)}{\beta} + y_2 \right)^2 \right] \geq 0.
\]

Furthermore, we modify our boundary condition as \( V(T, x) = e^{-\delta T} U(x) \), where \( \delta > 0 \) is a discounting factor. This does not create any inconsistency problems with the rest of the paper, as this condition is equivalent to the one we have imposed throughout this work. Furthermore, in what follows, we assume that the risk manager operates under quadratic preferences, that is a utility function of the form

\[
U(x) = \kappa \frac{x^\rho}{\rho},
\]

for some \( \kappa > 0 \) and \( 0 < \rho < 1 \).

**Proposition 3.** Assume power-type preferences of the form (94) and moreover that \( \delta \geq r \rho \). In this case, the solution of the partial differential equation

\[
V_t + rxV_x - A(y_1, y_2) \frac{V^2}{V_{xx}} = 0,
\]

along with the boundary condition \( V(T, x) = e^{-\delta T} U(x) \), may break down.

**Proof.** Assume that the partial differential equation (95) admits a classical solution \( V \in C^{1,2}(\mathbb{S}) \) and moreover that it satisfies the concavity conditions \( V_x > 0 \) and \( V_{xx} < 0 \). We look for a solution using the guess

\[
V(t, x) = e^{-\delta t} \tilde{V}(x),
\]

where \( \tilde{V} \in C^{1,2}(\mathbb{S}) \). Differentiating the above expression with respect to \((t, x)\), yields

\[
V_t = -\delta e^{-\delta t} \tilde{V}(x)
\]

\[
V_x = e^{-\delta t} \tilde{V}_x
\]

\[
V_{xx} = e^{-\delta t} \tilde{V}_{xx}.
\]
Substituting these expressions back in the partial differential equation (95), results to the elliptic partial differential equation
\[ \delta V - rx \tilde{V}_x + A(y_1, y_2) \frac{\tilde{V}_x^2}{V_{xx}} = 0. \] (96)

We propose a solution to the partial differential equation of the form
\[ \tilde{V}(x) = \kappa \frac{x^\rho}{\rho}. \]

Inserting this trial solution in (96), yields to the following condition for the discounting factor
\[ \delta = r \rho - A(y_1, y_2) \frac{\rho}{\rho - 1}, \]
or equivalently
\[ A(y_1, y_2) = \frac{1 - \rho}{\rho} (\delta - r \rho). \] (97)

Equation (97) serves an indication of a solution breakdown. To be more precise, both sides of the equality are always greater than or equal to zero (thanks to the assumption \( \delta \geq r \rho \)). In general, if the values of \( y_1 \) and \( y_2 \) are such that the above equality holds, then a solution exists. However, we distinguish the following cases:

(i). If \( A(y_1, y_2) = 0 \) and \( \delta = r \rho \), a solution exists.

(ii). If \( A(y_1, y_2) > 0 \) and \( \delta = r \rho \), the solution breaks down.

(iii). If \( A(y_1, y_2) = 0 \) and \( \delta > r \rho \), the solution breaks down.

(iv). If \( A(y_1, y_2) > 0 \) and \( \delta - r \rho > 0 \), as \( y_1 \) and \( y_2 \) increase in absolute value, the solution breaks down.

This completes the proof

**Remark 7.** In the proof of Proposition 3, we saw that the term \( A(y_1, y_2) \) determines if a solution to the partial differential equation (96) exists or not. As this term depends on the optimal worst-cases strategies of Nature, namely \( y_1 \) and \( y_2 \), this is equivalent to saying, that, in the case at hand, Nature is the one who determines if the solution breaks down. In fact, a safe scenario for the existence of a solution, is scenario (i) in the proof of Proposition 3. In order to have \( A(y_1, y_2) = 0 \), Nature must pick the worst-case choice
\[ y_1 = -\frac{\mu - r}{\sigma} \quad \text{and} \quad y_2 = -\frac{\alpha(c_1 - 1)}{\beta}. \]

In any other case, it is clear that as the term \( A(y_1, y_2) \) increases, the probability of a solution break down also increases.

**Remark 8.** Proposition 3 leads to the relation (97) that determines if a solution to the partial differential equation (96) exists. As already stated in Remark 7, this relation heavily depends on the choice of the worst-case strategies \( y_1 \) and \( y_2 \) of Nature. Loosely speaking, this is equivalent to saying that \( \delta \) and \( \rho \) create an “uncertainty rectangle”. The size of this rectangle plays a crucial role in the existence of a solution for the partial differential equation (96). To be more precise, if the “uncertainty rectangle” is too large, then the solution breaks down. In other words, this is a clear manifestation that in the case of full robustness, we may have...
a solution breakdown. This case corresponds to a hot spot of type I in the paper of Brock et.al [9].

6.2.2. The min-max problem. As in the limiting case $\lambda \to +\infty$ it is evident that the game does not have a value, that is, the Isaacs min-max condition does not hold (and as a result the upper and the lower value of the game do not necessarily coincide), it is of great interest to also study the min-max problem. That is, the problem

$$
\inf_{Q \in \mathcal{Q}} \sup_{\pi, q \in \mathcal{A}} J_0(t, x) = \inf_{y_1, y_2 \in \mathcal{Y}} \sup_{\pi, q \in \mathcal{A}} \mathbb{E}_Q \left[ U(\tilde{X}_{\eta_1 \eta_2}(T)) \right],
$$

subject to the state dynamics (19). In this direction, we have the following theorem.

**Theorem 6.3** (The min-max problem). The optimal robust strategy for the risk manager is to turn her attention in the risk-less asset, that is,

$$
\pi^*(t, x) = 0 \quad \text{(99)}
$$

and to purchase full proportional coverage for the firm’s liabilities,

$$
q^*(t, x) = 1 \quad \text{(100)}.
$$

On the other hand, Nature chooses the worst-case scenario defined by

$$
y_1^*(t, x) = -\frac{\mu - r}{\sigma}, \quad \text{(101)}
$$

and

$$
y_2^*(t, x) = -\alpha\frac{(c_1 - 1)}{\beta}. \quad \text{(102)}
$$

In this case, the optimal robust value function is a smooth solution of the following linear partial differential equation

$$
V_t + \left[ rx + \alpha(c_0 - c_1) \right] V_x = 0, \quad \text{(103)}
$$

with boundary condition $V(T, x) = U(x)$, assuming that such a solution exists.

**Proof.** The proof follows along the same lines with the proof of Theorem 6.1 and hence is omitted.

**Remark 9.** Theorem 6.3 provides a very interesting result. In Remark 7 we said that the safest scenario that guarantees the existence of a solution to the partial differential equation (95) is the choice

$$
y_1 = -\frac{\mu - r}{\sigma} \quad \text{and} \quad y_2 = -\alpha\frac{(c_1 - 1)}{\beta}.
$$

From Theorem 6.3, we know that, for the lower value of the game, the worst-case decision for Nature are indeed those values, which furthermore lead the risk manager to abandon the risky asset, turn her attention in the safe hands of the interest rate and to purchase full coverage for the firm’s potential claims. Combining Theorems 6.2 and 6.3, concerning the limiting behavior of our game when $\lambda \to +\infty$, we have the following

(i). If the Isaacs min-max condition holds, then the non-linear partial differential equation (91) reduces to the linear partial differential equation (103), which can be solved. In this case, the optimal robust controls for the risk manager are (99) and (100) and for Nature, are (101) and (102), as long as $\delta = \rho \lambda$. 


(ii). If the Isaacs min-max condition does not hold, the max-min (Theorem 6.2) and min-max (Theorem 6.3) problems do not coincide. In this case, the min-max problem is always solvable, while, according to Proposition 3, the value of the max-min problem may break down.

**Remark 10.** Let us assume that the Isaacs min-max condition holds. In this case, the optimal robust strategy for the risk manager is to turn her full attention in the risk-free interest rate and purchase full proportional coverage for the firm’s claims. On the other hand, Nature chooses the worst-case scenarios described by (101) and (102). The quantities \((\mu - r)/\sigma\) and \(-\alpha(c_1 - 1)/\beta\) may be recognized as the Sharpe ratios of the financial market and of the “coverage market”, respectively. In fact, these are the average returns earned per unit of the volatilities \(\sigma\) and \(\beta\). If Nature plays against these average returns, it is like it absorbs all the potential profits of the risk manager, this is why the manager turns her attention in the bank account and purchases full coverage.

7. **A numerical study for the effect of robustness.** In the present section, we provide a detailed numerical simulation of the optimal robust strategies to be followed by both the risk manager and Nature, in the case of the exponential utility function (50). Our aim is to study the effect that robustness has on the optimal decisions of the players. However, as the optimal wealth process \(X^*\) appears within the control functions, as given by Equations (54), (55), (68) and (69), it is apparent that special treatment is needed in order to effectively capture the behavior of the risk manager (as the optimal decisions of Nature are independent of the optimal wealth process), with the major obstacle being the simulation of \(X^*(t)\). In this direction, we follow the next steps:

(i) First of all, we begin with an appropriate Euler-Maruyama approximation scheme for the stochastic differential equation for \(X^*(t)\) (as described by (58)). In order to implement the method, for a time step of size \(\Delta t = T/N\) with \(N = 2^{11}\) points, we define the step size in the Euler-Maruyama scheme as \(\delta t = \Delta t\).

(ii) After we have approximated the solution of \(X^*\), we follow a Monte-Carlo approach to calculate the mean path over a large number of realizations of the optimal robust strategies. In order to implement the Monte Carlo method, we simulate a large number \(M\) of paths of \(\pi^*\) in the time interval \([0,T]\) and at each time point we plot the average of \(M\) different values. We also use for each path \(N = 2^{11}\) number of points (here \(N = 2^{11}\) and \(M = 6000\) paths).

In what follows, unless stated otherwise, we let \(M = 6000\), \(c = 11\), \(T = 10\) months, \(X(0) = 1\), \(\gamma = 0.5\) and \(\lambda = 0.2\). The parameters of the financial market are chosen as \(\mu = 12\%\), \(r = 6\%\), \(\sigma = 40\%\). The parameters for the insurance market are chosen as \(\alpha = 1\), \(\beta = 0.5\) and \(c_1 = 1.1\).

7.1. **The case of the exponential utility function.** When the risk manager operates under exponential preferences (Equation (50)), her optimal investment decision is given by (54) and the optimal proportional coverage strategy by (55). In this case, Nature chooses the worst-case scenario described by Equations (56) and (57).

First of all, concerning the effect of robustness on the optimal investment and proportional coverage strategies, from Figures 1 and 3, we have the following results...
As the level of the preference for the robustness parameter $\lambda$ increases, the risk manager is expected to cover a larger proportion of the firm’s claims. In fact, this result is in accordance with her investment behavior in this case, as explained above. The faith in the model decreases and the manager becomes more and more ambiguous about the model she faces. As a result, she decides that the best strategy is to enjoy the returns of the bank account and purchase a full cover for the firm’s claims.

Concerning the effect of the initial wealth level on the optimal robust investment decisions, from Figure 2 we have that as the initial wealth level increases, the risk manager is expected to also turn her attention to the bond market. Again, this is natural and in accordance with our findings. In fact, this behavior is a consequence of the exponential utility function, and has a mathematical explanation. In Equation (54) the $X^*(t)$ term appears in the denominator. From Equation (71) we can see that the stochastic differential equation that describes the optimal wealth in a linear one. If we simulate a large number of such paths and plot their mean path, this will be monotonically increasing. Hence as the optimal mean path increases, the optimal robust investment strategy decreases. Of course, this attitude towards risk has also an economic explanation. As the portfolio of the firm becomes more and more wealthy, there is no need to take any additional risks associated with the stock market. The firm is satisfied with its current wealth and probably has reached its goals. As a result, the risk manager takes the decision to sit and do nothing.
Finally, from Figure 4, we have the following findings.

- As $\lambda \to 0$, Nature is expected to follow the myopic strategy $y^*_1 = y^*_2 = 0$. In fact this numerical finding is in accordance with Theorem 6.1. In this case, as already stated, the risk manager has total faith in the model described by Equations (2) and (3) and operates under the reference probability measure $\mathbb{P}$. Thus, the stochastic differential game reduces to a stochastic optimal control problem and Nature vanishes.
On the other hand, as $\lambda$ increases, we can see that the worst-case decisions for Nature seem to converge to some value. This result is in accordance to the findings of Theorems 6.2 and 6.3. These values are respectively $y_1$ and $y_2$.

8. Conclusions. In the present paper we studied a robust control problem arising in the management of financial institutions. More precisely, we envisioned a risk manager who is responsible for optimizing the expected returns of a firm’s portfolio. In this vein, the manager has the possibility to invest part of the firm’s wealth in a classical Black-Scholes type financial market and also purchase proportional coverage for the liabilities the firm faces. However, we assume that the manager does not fully trust the model she is given, hence she decides to make her decision robust. By employing robust control and dynamic programming techniques, we provide closed form solutions for the cases of the (i) logarithmic; (ii) exponential and (iii) power utility functions. Moreover, we provide a detailed study of the limiting behavior, of the stochastic differential game at hand, which in a special case leads to break down of the solution of the associated HJBI equation. Finally, we conclude with a detailed numerical study that elucidates the effect of robustness on the optimal investment decisions of both players.

REFERENCES

[1] E. Anderson, L. Hansen and T. Sargent, A quartet of semigroups for model specification, robustness, prices of risk, and model detection, Journal of the European Economic Association, 1 (2003), 68–123.

[2] E. Anderson, E. Ghysels and J. Juergens, The impact of risk and uncertainty on expected returns, Journal of Financial Economics, 94 (2009), 233–263.

[3] L. Bai and G. Guo, Optimal proportional reinsurance and investment with multiple risky assets and no-shorting constraint, Insurance: Mathematics and Economics, 42 (2008), 968–975.

[4] I. D. Baltas, N. E. Frangos and A. N. Yannacopoulos, Optimal investment and reinsurance policies in insurance markets under the effect of inside information, Applied Stochastic Models in Business and Industry, 28 (2012), 506–528.

[5] I. D. Baltas and A. N. Yannacopoulos, Uncertainty and inside information, Journal of Dynamics and Games, 3 (2016), 1–24.

[6] E. Bayraktar and S. Yao, Doubly reflected BSDEs with integrable parameters and related Dynkin games, Stochastic Processes and their Applications, 125 (2015), 4489–4542.
[7] S. Biagini and M. Pinar, The Robust Merton Problem of an Ambiguity-Averse Investor, *Mathematics and Financial Economics*, 11 (2017), 1–24.

[8] N. Branger, L. Larsen and C. Munk, Robust portfolio choice with ambiguity and learning predictability, *Journal of Banking and Finance*, 37 (2013), 1397–1411.

[9] W. A. Brock, A. Xepapadeas and A. N. Yannacopoulos, Robust control and hot spots in spatiotemporal economic systems, *Dyn. Games Appl.*, 4 (2014), 257–289.

[10] W. A. Brock, A. Xepapadeas and A. N. Yannacopoulos, Robust control of a spatially distributed commercial fishery, in *Dynamic Optimization in Environmental Economics*, (eds. E. Moser, W. Semmler, G. Tragler, V. Veliov), Springer-Verlag, Heidelberg, 15 (2014), 215–241.

[11] S. Browne, Optimal investment policies for a firm with a random risk process: Exponential utility and minimizing the probability of ruin, *Mathematics of Operations Research*, 20 (1995), 937–958.

[12] R. Buckdahn and J. Li, Stochastic differential games with reflection and related obstacle problems for Isaacs equations, *Acta Mathematicae Applicatae Sinica, English Series*, 27 (2011), 647–678.

[13] A. Cairns, A discussion of parameter and model uncertainty in insurance, *Insurance: Mathematics and Economics*, 27 (2000), 313–330.

[14] R. Cont, Model uncertainty and its impact on the pricing of derivative instruments, *Mathematical Finance*, 16, (2004), 519–547.

[15] I. Girsanov, On transforming a certain class of stochastic processes by absolutely continuous substitution of measures, (Russian) *Teor. Verojatnost. i Primenen.*, 5 (1960), 314–330.

[16] L. Hansen and T. Sargent, Robust control and model uncertainty, *Uncertainty within Economic Models*, 6 (2014), 145–154. Available from: http://www.jstor.org/stable/2677734

[17] H. Nikaido, On Von Neumann’s minimax theorem, *Pacific Journal of Mathematics*, 4 (1954), 65–72.

[18] M. Nisio, Stochastic differential games and viscosity solutions of Isaacs equations, *Stochastics: An International Journal of Probability and Stochastic Processes*, 80 (2008), 317–337.
[34] U. Rieder and C. Wupperer, Robust consumption-investment problems with random market coefficients, Math Finan Econ, 6 (2012), 295–311.

[35] H. Schmidli, Diffusion approximations for a risk process with the possibility of borrowing and investment, Communications in Statistics, Stochastic Models, 10 (1994), 365–388.

[36] M. Sion, On general minimax theorems, Pacific Journal of Mathematics, 8 (1958), 171–176.

[37] C. Skiadas, Robust control and recursive utility, Finance and Stochastics, 7 (2003), 475–489.

[38] R. Uppal and T. Wang, Model misspecification and underdiversification, The Journal of Finance, 58 (2003), 2465–2486.

[39] H. Wang and S. Hou, Robust consumption and portfolio choice with habit formation, the spirit of capitalism and recursive utility, Annals of Economics and Finance, 16 (2015), 393–416.

[40] D. Zawisza, Robust portfolio selection under exponential preferences, Applicationes Mathematicae, 37 (2010), 215–230.

[41] D. Zawisza, Robust consumption-investment problem on infinite horizon, Appl. Math. Optim, 72 (2015), 469–491.

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