MULTIPLICATIVE REDUCTION AND THE CYCLOTOMIC MAIN CONJECTURE FOR GL₂

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ABSTRACT. We show that the cyclotomic Iwasawa–Greenberg Main Conjecture holds for a large class of modular forms with multiplicative reduction at $p$, extending previous results for the good ordinary case. In fact, the multiplicative case is deduced from the good case through the use of Hida families and a simple Fitting ideal argument.

1. Introduction

The cyclotomic Iwasawa–Greenberg Main Conjecture was established in [18], in combination with work of Kato [13], for a large class of newforms $f \in S_k(\Gamma_0(N))$ that are ordinary at an odd prime $p \nmid N$, subject to $k \equiv 2 \pmod{p-1}$ and certain conditions on the mod $p$ Galois representation associated with $f$. The purpose of this note is to extend this result to the case where $p | N$ (in which case $k$ is necessarily equal to 2).

Recall that the coefficients $a_n$ of the $q$-expansion $f = \sum_{n=1}^{\infty} a_n q^n$ of $f$ at the cusp at infinity (equivalently, the Hecke eigenvalues of $f$) are algebraic integers that generate a finite extension $\mathbb{Q}(f) \subset \mathbb{C}$ of $\mathbb{Q}$. Let $p$ be an odd prime and let $L$ be a finite extension of $\mathbb{Q}(f)$ at a chosen prime above $p$ (equivalently, let $L$ be a finite extension of $\mathbb{Q}_p$ in a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ that contains the image of a chosen embedding $\mathbb{Q}(f) \hookrightarrow \mathbb{Q}_p$). Suppose that $f$ is ordinary at $p$ with respect to $L$ in the sense that $a_p$ is a unit in the ring of integers $\mathcal{O}$ of $L$. Then the $p$-adic $L$-function $L_f$ of $f$ is an element of the Iwasawa algebra $\Lambda_{\mathcal{O}} = \mathcal{O}[\Gamma]$, where $\Gamma = \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ is the Galois group of the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$. A defining property of $L_f$ is that it interpolates normalized special values of the $L$-function of $f$ twisted by Dirichlet characters associated with finite-order characters of $\Gamma$. The Iwasawa–Greenberg Selmer group $\text{Sel}_{\mathbb{Q}_\infty,L}(f)$, defined with respect to the $p$-adic Galois representation $V_f$ of $f$ over $L$ - a two-dimensional $\mathcal{O}$-vector space - and a Galois-stable $\mathcal{O}$-lattice $T_f \subset V_f$, is a discrete, cofinite $\Lambda_{\mathcal{O}}$-module, and the Iwasawa–Greenberg characteristic ideal $Ch_L(f) \subset \Lambda_{\mathcal{O}}$ is the characteristic $\Lambda_{\mathcal{O}}$-ideal of the Pontryagin dual $X_{\mathbb{Q}_\infty,L}(f)$ of $\text{Sel}_{\mathbb{Q}_\infty,L}(f)$. The Iwasawa–Greenberg Main Conjecture for $f$ then asserts that there is an equality of ideals $Ch_L(f) = (L_f)$ in $\Lambda_{\mathcal{O}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and even in $\Lambda_{\mathcal{O}}$ if $T_f$ is residually irreducible.

Theorem A. Let $p \geq 3$ be a prime. Let $f \in S_k(\Gamma_0(N))$ be a newform and let $L$ and $\mathcal{O}$ be as above and suppose $f$ is ordinary at $p$ with respect to $L$. If

(i) $k \equiv 2 \pmod{p-1}$;
(ii) the reduction $\bar{\rho}_f$ of the representation $\rho_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}_\mathcal{O}(T_f)$ modulo the maximal ideal of $\mathcal{O}$ is irreducible;

(iii) there exists a prime $q \neq p$ such that $q \mid N$ and $\bar{\rho}_f$ is ramified at $q$,

then $\text{Ch}_L(f) = (L_f)$ in $\Lambda_\mathcal{O}$. That is, the Iwasawa–Greenberg Main Conjecture is true.

When $p \nmid N$ this is just Theorem 1 of [18]. When $p \mid N$, in which case the ordinary hypothesis forces $p \mid\mid N$ and $k = 2$, this is not an immediate consequence of the results in [18], as this case is excluded from Kato’s divisibility theorem [13, Thm. 17.4], which is a crucial ingredient in the deduction of the main conjecture from the main results in [18]. However, as we explain in this note, the main conjecture in the case $p \mid N$ can be deduced from knowing it when $p \nmid N$.

Having the cyclotomic main conjecture in hand, one obtains results toward special value formulas. For example:

**Theorem B.** Let $p \geq 3$ be a prime. Let $f \in S_2(\Gamma_0(N))$ be a newform and let $L$ and $\mathcal{O}$ be as above and suppose $f$ is ordinary. Suppose also that

(i) the reduction $\bar{\rho}_f$ of the representation $\rho_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}_\mathcal{O}(T_f)$ modulo the maximal ideal of $\mathcal{O}$ is irreducible;

(ii) there exists a prime $q \neq p$ such that $q \mid\mid N$ and $\bar{\rho}_f$ is ramified at $q$;

(iii) if $p \mid N$ and $a_p = 1$, then the $L$-invariant $\mathcal{L}(V_f) \in L$ is nonzero.

Let

$$L^\text{alg}(f,1) = \frac{L(f,1)}{-2\pi i \Omega_f^+}.$$  

Then

$$\#\mathcal{O}/(L^\text{alg}(f,1)) = \#\text{Sel}_L(f) \cdot \prod_\ell c_\ell(T_f).$$

In particular, if $L(f,1) = 0$, then $\text{Sel}_L(f)$ has $\mathcal{O}$-corank at least one.

Here $\Omega_f^+$ is one of two canonical periods associated with $f$ as in [18, §3.3.3] (and well-defined up to an element of $\mathcal{O}_x^* \cap \mathbb{Q}(f)$), $\text{Sel}_L(f)$ is the Selmer group associated by Bloch-Kato to the Galois lattice $T_f$, $c_\ell(T_f)$ is the Tamagawa factor at $\ell$ of $T_f$ (and equals 1 unless $\ell \mid N$), and $\mathcal{L}(V_f)$ is the $\mathcal{L}$-invariant of a modular form $f$ (or of $V_f$) with split multiplicative reduction at $p$ introduced by Mazur, Tate, and Teitelbaum in [14] (see also [11, §3]). It is conjectured that $\mathcal{L}(V_f)$ is always nonzero; this is known if $f$ is the modular form associated to an elliptic curve, but in general it is an open question.

As a special case of Theorem B, obtained by taking $f$ to be the newform associated with an elliptic curve $E$ over $\mathbb{Q}$, we have:

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1In order to conclude that the equality holds in $\Lambda_\mathcal{O}$ and not just $\Lambda_\mathcal{O} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, Theorem 1 in [18] requires that $\rho_f$ have an $\mathcal{O}$-basis with respect to which the image contains $\text{SL}_2(\mathbb{Z}_p)$. But as we explain in Section 2.5, hypotheses (ii) and (iii) of Theorem A are enough for the arguments. We also explain that the reference to [20] in [18] should have been augmented with a reference to [4].
**Theorem C.** Let $E$ be an elliptic curve over $\mathbb{Q}$ with good ordinary or multiplicative reduction at a prime $p \geq 3$. Suppose that

(i) $E[p]$ is an irreducible $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation;

(ii) there exists a prime $q \neq p$ at which $E$ has multiplicative reduction and $E[p]$ is ramified.

If $L(E, 1) \neq 0$ then

$$\left| \frac{L(E, 1)}{\Omega_E} \right|_p^{-1} = \left| \# \Sha(E) \prod_{\ell} c_{\ell}(E) \right|_p^{-1},$$

and if $L(E, 1) = 0$ then $\text{Sel}_{\infty}(E)$ has $\mathbb{Z}_p$-corank at least one.

Here, $\Omega_E$ is the Néron period of $E$, $\Sha(E)$ is the Tate-Shafarevich group of $E/\mathbb{Q}$, and the $c_{\ell}(E)$ are the Tamagawa numbers of $E$. In particular, $c_{\ell}(E)$ is the order of the group of irreducible components of the special fiber of the Néron model of $E$ over $\mathbb{Z}_\ell$.

Our proof of Theorem A is relatively simple. Let $N = pM$. We first make two reductions: (1) it suffices to prove the theorem with the field $L$ replaced by any finite extension, and (2) it suffices to prove the equality $Ch^c_E(f) = (\mathcal{L}^c_f)$, where $\Sigma$ is any finite set of primes containing all $\ell \mid N$, $\mathcal{L}^c_f$ being the incomplete $p$-adic $L$-function with the Euler factors at primes in $\Sigma$ different from $p$ removed and $Ch^c_E(f)$ being the characteristic ideal of the Pontryagin dual $X^\Sigma_{\ell, \infty}(f)$ of the Iwasawa–Greenberg Selmer group $\text{Sel}_{\infty, \ell}(f)$ with all conditions at primes in $\Sigma$ different from $p$ relaxed. Then we exploit Hida theory to deduce that one can choose $L$ so that for each integer $m > 0$ there exists a newform $f_m \in S_{k_m}(\Gamma_0(M))$ with $k_m \equiv k \pmod{(p-1)p^m}$, $\mathbb{Q}(f_m) \subset L$ and $f_m$ ordinary at $p$ with respect to $L$, and the ordinary $p$-stabilization $f_m^*$ of $f_m$ satisfies $f_m^* \equiv f \pmod{p^m}$ in the sense that the $q$-expansions (which have coefficients in $\mathcal{O}$) are congruent modulo $p^m$. Furthermore, as a consequence of the existence of the ‘two-variable’ $p$-adic $L$-function associated to a Hida family we also have $\mathcal{L}^c_{f_m} \equiv \mathcal{L}^c_f \pmod{p^m \Lambda_\ell}$. An argument of Greenberg shows that since $X^\Sigma_{\ell, \infty}(f_m)$ is a torsion $\Lambda_\ell$-module, it has no non-zero finite-order $\Lambda_\ell$-submodules. From this it follows that $Ch^c_{E}(f_m)$ equals the $\Lambda_\ell$-Fitting ideal $F^c_E(f_m)$ of $X^\Sigma_{\ell}(f_m)$. The congruence $f_m^* \equiv f \pmod{p^m}$ implies that $\text{Sel}^c_{\infty, \ell}(f_m[p^m]) \cong \text{Sel}^c_{\infty, \ell}(f_m[p^m])$, so comparing Fitting ideals yields

$$F^c_E(f, p^m) = (F^c_E(f_m, p^m), (Ch^c_E(f_m, p^m) \subset \Lambda_\ell).$$

From the main conjecture for $f_m$ (the congruence $f_m^* \equiv f \pmod{p}$ ensures that the hypotheses of Theorem A also hold for $f_m$) and the congruence modulo $p^m$ of $p$-adic $L$-functions we then have

$$F^c_E(f, p^m) = (Ch^c_E(f_m, p^m) = (\mathcal{L}^c_{f_m, p^m} = (\mathcal{L}^c_f, p^m) \subset \Lambda_\ell$$

for all integers $m > 0$. This, together with the non-vanishing of the $p$-adic $L$-function $\mathcal{L}^c_f$, implies that $F^c_E(f) \neq 0$ and hence that $X^\Sigma_{\ell, \infty}(f)$ is a torsion $\Lambda_\ell$-module. The earlier argument of Greenberg then gives $Ch^c_E(f) = F^c_E(f)$, and so $(Ch^c_E(f), p^m) = \Lambda_\ell$. 


\((L^\Sigma(f), p^m) \subset \Lambda_\O\) for all \(m > 0\). As \(Ch^\Sigma_L(f) \subset \Lambda_\O\) is a principal ideal, it then easily follows that \(Ch^\Sigma_L(f) = (L^\Sigma_f)\), proving Theorem A.

The deduction of Theorem B from Theorem A follows an argument of Greenberg from [8].

In addition to extending the main conjecture to the case of multiplicative reduction, our motivation for writing this note was in part to provide an explicit reference for the expression for the special value \(L^{\text{alg}}(f, 1)\) in terms of the size of Selmer groups that is required for the arguments in [22] and, by including the multiplicative reduction case, also provide an important ingredient for the extension of the main results of [22] to cases of multiplicative reduction. Additional motivation for the latter stems from the author’s collaboration with Manjul Bhargava and Wei Zhang to provide lower bounds on the proportion of elliptic curves that satisfy the rank part of the Birch–Swinnerton-Dyer conjecture.

While preparing this note the author learned of Olivier Fouquet’s work [7] on the equivariant Tamagawa number conjecture for motives of modular forms. That work should provide another means for deducing Theorem A in the case \(p \nmid N\) from the main results\(^2\) in [18] as well as some additional weakening of the conditions on primes away from \(p\). The deduction of Theorem A for \(p \mid N\) in this paper uses no more machinery than already developed in [18] or than is required for our deduction of Theorem B.

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2. Gathering the pieces

In this section we recall the various objects that go into the Iwasawa–Greenberg Main Conjecture for modular forms, some of their properties, and some useful relations. Throughout \(p\) is a fixed odd prime.

Let \(\overline{\Q} \subset \C\) be the algebraic closure of \(\Q\) and let \(G_\Q = \text{Gal}(\overline{\Q}/\Q)\). For each prime \(\ell\), let \(\overline{\Q}_\ell\) be a fixed algebraic closure of \(\overline{\Q}_\ell\). For each \(\ell\) we also fix an embedding \(\overline{\Q} \rightarrow \overline{\Q}_\ell\), which identifies \(G_{\Q_\ell} = \text{Gal}(\overline{\Q}_\ell/\Q)\) with a decomposition subgroup in \(G_\Q\); let \(I_\ell \subset G_{\Q_\ell}\) be the inertia subgroup. Let \(\text{frob}_\ell \in G_{\Q_\ell}\) be (a lift of) an arithmetic Frobenius element.

Let \(\epsilon : G_\Q \rightarrow \Z_p^\times\) be the \(p\)-adic cyclotomic character. This is just the projection to \(\text{Gal}(\Q[\mu_p^\infty]/\Q)\), the latter being canonically isomorphic to \(\Z_p^\times\). Similarly, let \(\omega : G_\Q \rightarrow \Z_p^\times\) be the mod \(p\) Teichmüller character. This is just the composition of the reduction of \(\epsilon\) mod \(p\) and the multiplicative homomorphism \((\Z/p\Z)^\times \rightarrow \Z_p^\times\) defined by the Teichmüller lifts.

\(^2\)But see also footnote 1, especially as the main results in [7] rely on Theorem A as stated, at least for the \(p \nmid N\) case.
Let $\mathbb{Q}_\infty \subset \mathbb{Q}[\mu^{p^\infty}] \subset \overline{\mathbb{Q}}$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. That is, $\mathbb{Q}_\infty$ is the unique abelian extension of $\mathbb{Q}$ such that $\Gamma = \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \cong \mathbb{Z}_p$. Let $\gamma \in \Gamma$ be a fixed topological generator. As $\text{Gal}(\mathbb{Q}[\mu^{p^\infty}]/\mathbb{Q}) \cong \mathbb{Z}_p \times \Gamma$, there is a lift $\bar{\gamma}$ of $\gamma$ to $\text{Gal}(\mathbb{Q}[\mu^{p^\infty}]/\mathbb{Q})$ identified with $(1, \gamma)$, and we let $u = \epsilon(\bar{\gamma}) \in \mathbb{Z}_p^\times$.

2.1. Galois representations and (ordinary) newforms. Let $f \in S_k(\Gamma_0(N))$ be a newform. Let $\mathbb{Q}(f) \subset \mathbb{C}$ be the finite extension of $\mathbb{Q}$ generated by the Fourier coefficients $a_n(f)$ of the $q$-expansion $f = \sum_{n=0}^{\infty} a_n(f)q^n$ of $f$ at the cusp at infinity (equivalently, the field obtained by adjoining the eigenvalues of the action of the usual Hecke operators on $f$). Fix an embedding $\mathbb{Q}(f) \hookrightarrow \overline{\mathbb{Q}}_p$ and let $L \subset \overline{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$ containing the image of $\mathbb{Q}(f)$. Let $\mathcal{O}$ be the ring of integers of $L$ (the valuation ring), let $m$ be its maximal ideal, and let $\kappa = \mathcal{O}/m$ be its residue field.

Associated with $f$ and $L$ (and the embedding $\mathbb{Q}(f) \hookrightarrow L$) is a two-dimensional $L$-space $V_f$ and an absolutely irreducible continuous $G_{\mathbb{Q}}$-representation $\rho_f : G_{\mathbb{Q}} \to \text{Aut}_L(V_f)$ such that $\rho_f$ is unramified at all primes $\ell \nmid Np$ and $\det(1 - X \cdot \rho_f(\text{frob}_\ell)) = 1 - a_\ell(f)X + \ell^{k-1}X^2$ for such $\ell$. In particular, trace $\rho_f(\text{frob}_\ell) = a_\ell(f)$ if $\ell \nmid pN$, and det $\rho_f = \epsilon^{k-1}$.

Let $T, T' \subset V_f$ be two $G_{\mathbb{Q}}$-stable $\mathcal{O}$-lattices. Let $\bar{\rho}$ and $\bar{\rho}'$ denote, respectively, the two-dimensional $\kappa$-representations $T/mT$ and $T'/mT'$. The following lemma is well-known, but we include it for later reference.

**Lemma 2.1.1.**

(a) If $\bar{\rho}$ or $\bar{\rho}'$ is irreducible, then $\bar{\rho}$ and $\bar{\rho}'$ are equivalent as $\kappa$-representations. In particular, $\bar{\rho}$ is irreducible if and only if $\bar{\rho}'$ is irreducible.

(b) If $\bar{\rho}$ or $\bar{\rho}'$ is irreducible, then there exists $a \in L^\times$ such that $T = aT'$.

**Proof.** Replacing $T'$ with some $\mathcal{O}$-multiple, we may assume that $T'$ is a sublattice of $T$. Then $T/T' \cong \mathcal{O}/m^n \times \mathcal{O}/m^n$ with $n \leq m$. Let $\varpi$ be a uniformizer of $\mathcal{O}$ (a generator of $m$). Then $\varpi^n T/(T' + \varpi^{n+1}T) \cong \mathcal{O}/m^{\min(1, m-n)}$ is a $G_{\mathbb{Q}}$-stable quotient of $T/mT \cong \varpi^n T/\varpi^{n+1}T$ of at most one-dimension over $k$. If $\bar{\rho}$ is irreducible, then this quotient must be trivial and so $m-n = 0$ and $T' = \varpi^n T$, in which case $T'/mT' \cong \varpi^n T/\varpi^{n+1}T \cong T/mT$ as $G_{\mathbb{Q}}$-representations over $\kappa$. Reversing the roles of $T$ and $T'$ in this argument then yields the lemma. \hfill $\square$

We then define $\bar{\rho}_f$ to be the $\kappa$-representation $T/mT$ of $G_{\mathbb{Q}}$ for a Galois-stable $\mathcal{O}$-lattice $T \subset V_f$. By the above lemma, if $\bar{\rho}_f$ is irreducible for some choice of $T$, then it is irreducible for any choice of $T$, and the equivalence class of $\bar{\rho}_f$ is independent of $T$. Of course, it is not difficult to show that the semisimplification of $\bar{\rho}_f$ is independent of $T$ even when $\bar{\rho}_f$ is not irreducible, but will not need this.

Suppose $k \geq 2$ and $f$ is ordinary with respect to the embedding $\mathbb{Q}(f) \hookrightarrow L$. That is, $a_p(f) \in \mathcal{O}^\times$. Then $V_f$ has a unique $G_{\mathbb{Q}_p}$-stable $L$-line $V_f^+ \subset V_f$ such that $G_{\mathbb{Q}_p}$ acts on $V_f^+$ via the character $\alpha_f^{-1} \ell^{k-1}$, where $\alpha_f : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ is the unique unramified character such that $\alpha_f(\text{frob}_p)$ equals the (unit) root $\alpha_p$ in $\mathcal{O}^\times$ of the polynomial $x^2 - a_p(f)x + p^{k-1}$.
if \( p \nmid N \) and \( \alpha_f(f \text{rob}_p) = a_p(f) \) if \( p \mid N \). (Note that the reduction of the polynomial \( x^2 - a_p(f)x + p^{k-1} \) modulo \( m \) is \( x(x - \alpha_p(f)) \) and so, by Hensel’s lemma, \( a_p(f) \) lifts to a root in \( \mathcal{O}^\times \).) The action of \( G_{\mathbb{Q}_p} \) on the quotient \( V_f^- = V_f/V_f^+ \) is via \( \alpha_f \). Given any \( G_{\mathbb{Q}} \)-stable \( \mathcal{O} \)-lattice \( T \subset V_f \) we let \( T^+ = T \cap V_f^+ \) and \( T^- = T/T^+ \). Then \( T^+ \) is the unique \( G_{\mathbb{Q}_p} \)-stable free \( \mathcal{O} \)-summand of rank one on which \( G_{\mathbb{Q}_p} \) acts via \( \alpha_f^{-1}e^{k-1} \), and \( T^- \) is the unique \( G_{\mathbb{Q}_p} \)-stable free \( \mathcal{O} \)-module quotient of rank one on which \( G_{\mathbb{Q}_p} \) acts via \( \alpha_f \).

The following lemma is also well-known, but we also include it for completeness.

**Lemma 2.1.2.** Suppose \( a_p(f) \in \mathcal{O}^\times \). If \( p \mid N \), then \( p \mid N \), \( k = 2 \), and \( a_p(f) = \pm 1 \).

**Proof.** If \( f \in S_k(\Gamma_0(N)) \) is a newform with trivial Nebentypus such that \( p \mid N \), then \( a_p(f) \neq 0 \) if and only if \( p \mid N \), in which case \( a_p(f)^2 = p^{k-2} \). If \( a_p(f) \in \mathcal{O}^\times \), then it follows that \( k = 2 \) and \( a_p(f)^2 = 1 \), so \( a_p(f) = \pm 1 \).

Note that if \( f \) is a newform with \( p \mid N \) that is ordinary with respect to some embedding \( \mathbb{Q}(f) \to \overline{\mathbb{Q}_p} \), then, since \( a_p(f) = \pm 1 \) by the lemma, it is ordinary with respect to all such embeddings. Also, as noted in the proof of the lemma, if \( f \in S_2(\Gamma_0(N)) \) is a newform with \( p \mid N \) then \( a_p(f) = \pm 1 \) and so \( f \) is ordinary with respect to any embedding \( \mathbb{Q}(f) \to \overline{\mathbb{Q}_p} \).

In keeping with the terminology for elliptic curves, we say that a newform \( f \in S_2(\Gamma_0(N)) \) has multiplicative reduction at \( p \) if \( p \mid N \) and that it has good reduction at \( p \) if \( p \nmid N \). Additionally, we say \( f \) has split (resp. non-split) multiplicative reduction at \( p \) if \( p \mid N \) and \( a_p(f) = 1 \) (resp. \( a_p = -1 \)).

### 2.2. \( \mathfrak{L} \)-invariants

Suppose \( f \in S_2(\Gamma_0(N)) \) is a newform with split multiplicative reduction at \( p \). The Galois representation \( V_f \) restricted to \( G_{\mathbb{Q}_p} \) is an extension

\[
0 \to V_f^+ \cong L(1) \to V_f \to V_f^- \cong L \to 0.
\]

This extension is known to be non-split and semistable but not crystalline\(^3\). Let \( \pi_{V_f} : H^1(\mathbb{Q}_p, V_f) \to H^1(\mathbb{Q}_p, L) \) be the induced map in cohomology. As the extension is non-split, the image of \( \pi_{V_f} \) is a one-dimensional \( L \)-space. The \( \mathfrak{L} \)-invariant \( \mathfrak{L}(V_f) \) of \( V_f \) is the negative of the ‘slope’ of the line \( \text{im}(\pi_{V_f}) \) with respect to a particular basis of the two-dimensional \( L \)-space \( H^1(\mathbb{Q}_p, L) \).

We have

\[
H^1(\mathbb{Q}_p, L) = \text{Hom}_{\text{cts}}(G_{\mathbb{Q}_p}, L) = \text{Hom}_{\text{cts}}(G_{\mathbb{Q}_p}^{ab}, L),
\]

\(^3\)This is reflected in the compatibility, proved in [17], of the Weil-Deligne representation attached to the dual \( V_f^* \) by Fontaine with the Weil-Deligne representation attached by the local Langlands correspondence to the \( p \)-component \( \pi_p \) of the automorphic representation \( \pi = \otimes_v \pi_v \) of \( GL_2(A_f) \) corresponding to the newform \( f \). If \( f \) has split (resp. non-split) multiplicative reduction at \( p \) then \( \pi_p \) is the special representation (resp. the twist of the special representation by the unramified quadratic character).
where $G^\text{ab,p}_\mathbb{Q}_p$ is the maximal abelian pro-$p$ quotient of $G_{\mathbb{Q}_p}$. Local class field theory gives an identification\(^4\)
\[
\lim_{n \to \infty} \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^{p^n} \cong G^\text{ab,p}.
\]
From the decomposition $\mathbb{Q}_p^\times = p^\mathbb{Z} \times \mathbb{Z}_p^\times$ we obtain an $L$-basis $\{\psi_{ur}, \psi_{\text{cyc}}\}$ of $H^1(\mathbb{Q}_p, L) = \text{Hom}_{cts}(G^\text{ab,p}_\mathbb{Q}_p, \mathbb{L})$, with
\[
\psi_{ur}(p) = 1 = (\log_p u)^{-1} \cdot \psi_{\text{cyc}}(u) \quad \text{and} \quad \psi_{ur}(u) = 0 = \psi_{\text{cyc}}(p).
\]
Recall that $u = e(\tilde{\gamma})$ is a topological generator of $1 + p\mathbb{Z}_p$. The condition that $V_f$ is not crystalline is equivalent to $\text{im}(\pi_{V_f}) \not\subset L \cdot \psi_{ur}$. Let $0 \neq \lambda \in \text{im}(\pi_{V_f})$ and write $\lambda = x \cdot \psi_{\text{cyc}} + y \cdot \psi_{ur}$. Then $x \neq 0$, and the $\mathcal{L}$-invariant $\mathcal{L}(V_f)$ of the extension $V_f$ is defined to be
\[
\mathcal{L}(V_f) = -x^{-1}y \in L.
\]
This is independent of the choice of $\lambda$.

The non-split extension $V_f$ also defines a line $\ell_{V_f} = H^1(\mathbb{Q}_p, L(1))$ (the image of the boundary map $L = H^0(\mathbb{Q}_p, L) \to H^1(\mathbb{Q}_p, L(1))$). Under the perfect pairing $\langle \cdot, \cdot \rangle : H^1(\mathbb{Q}_p, L) \times H^1(\mathbb{Q}_p, L(1)) \to H^2(\mathbb{Q}_p, L(1)) = L$ of Tate local duality, the lines $\text{im}(\pi_{V_f})$ and $\ell_{V_f}$ are mutual annihilators. So $\mathcal{L}(V_f)$ can also be expressed in terms of $\langle \psi_{ur}, c \rangle$ and $\langle \psi_{\text{cyc}}, c \rangle$ for $0 \neq c \in \ell_{V_f}$.

The Kummer isomorphism yields an identification
\[
(\lim_{n} \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^{p^n}) \otimes_{\mathbb{Z}_p} L \cong H^1(\mathbb{Q}_p, L(1)).
\]
Then, together with the above identification of $H^1(\mathbb{Q}_p, L)$, the pairing $\langle \cdot, \cdot \rangle$ of local Tate duality is identified with the usual $L$-linear pairing
\[
\text{Hom}_{\mathbb{Z}_p}((\lim_{n} \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^{p^n}), L) \times (\lim_{n} \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^{p^n}) \otimes_{\mathbb{Z}_p} L \to L.
\]
So if $0 \neq c \in \ell_{V_f}$, then
\[
\mathcal{L}(V_f) = \psi_{ur}(c)^{-1} \psi_{\text{cyc}}(c).
\]
Note that the condition that $V_f$ not be crystalline is equivalent to $\ell_{V_f} \not\subset H^1_f(\mathbb{Q}_p, L(1))$, so $\psi_{ur}(c) \neq 0$ as $H^1_f(\mathbb{Q}_p, L(1))$ is identified with $(\lim_{n} \mathbb{Z}_p^\times / (\mathbb{Z}_p^\times)^{p^n}) \otimes_{\mathbb{Z}_p} L$.

Example. Suppose $f$ is associated with an elliptic curve $E/\mathbb{Q}$ with split multiplicative reduction at $p$ and let $q_E \in \mathbb{Q}_p^\times$ be the Tate period of $E$. Then $V_f = T_p E \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is the $G_{\mathbb{Q}_p}$-extension associated to the image of $q_E$ in $H^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$ under the Kummer map. That is, $\ell_{V_f} = \mathbb{Q}_p \cdot q_E \subseteq (\lim_{n} \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^{p^n}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and so $\mathcal{L}(V_f) = \log_{\mathbb{Z}_p} q_E / \text{ord}_{\mathbb{Z}_p}(q_E)$. As the $j$-invariant $j(q_E) = j(E) \in \mathbb{Q}$ of $E$ is algebraic, $q_E$ is transcendental by a theorem of Barré-Sirieix, Diaz, Gramain, and Philibert [2], and so $\log_{\mathbb{Z}_p} q_E \neq 0$. Therefore, $\mathcal{L}(V_f) \neq 0$.

\(^4\)To be precise, we normalize the reciprocity law so that uniformizers are taken to arithmetic Frobenius elements.
2.3. Iwasawa–Greenberg Selmer groups. Let \( f \in S_{k}(\Gamma_0(N)) \) be a newform that is ordinary with respect to an embedding \( \mathbb{Q}(f) \rightarrow \mathbb{Q} \). Let \( L \subset \bar{\mathbb{Q}}_p \) be any finite extension of \( \mathbb{Q}_p \) containing the image of \( \mathbb{Q}(f) \) and let \( \mathcal{O} \) be the ring of integers of \( L \). Let \( T_f \subset V_f \) be a fixed \( \mathbb{G}_m \)-stable \( \mathcal{O} \)-lattice.

Let \( \Lambda_\mathcal{O} = \mathcal{O}[\Gamma] \). Let \( \Psi : G_\mathcal{Q} \rightarrow \Gamma \subset \Lambda_{\mathcal{O}}^* \) be the natural projection. This is a continuous \( \Lambda_\mathcal{O} \)-valued character that is unramified away from \( p \) and totally ramified at \( p \). Let \( \Lambda_{\mathcal{O}}^* = \text{Hom}_{cts}(\Lambda_\mathcal{O}, \mathbb{Q}_p/\mathbb{Z}_p) \) be the Pontryagin dual of \( \Lambda_\mathcal{O} \). This is a discrete \( \Lambda_\mathcal{O} \)-module via \( r \cdot \varphi(x) = \varphi(rx) \), for \( r, x \in \Lambda_\mathcal{O} \) and \( \varphi \in \Lambda_{\mathcal{O}}^* \). We similarly define a \( \Lambda_\mathcal{O} \)-module structure on the Pontryagin dual of any \( \Lambda_\mathcal{O} \)-module.

Put \( \mathcal{M} = T_f \otimes \mathcal{O} \Lambda_{\mathcal{O}}^* \), with \( G_\mathcal{Q} \)-action given by \( \rho_f \otimes \Psi^{-1} \). Let \( \mathcal{M}^+ = T_f^+ \otimes \mathcal{O} \Lambda_{\mathcal{O}}^* \) and \( \mathcal{M}^- = \mathcal{M}/\mathcal{M}^+ \). Let \( \Sigma \) be any finite set of primes containing \( p \). Let \( S = \Sigma \cup \{ \ell \mid N \} \). Let \( Q_S \) be the maximal extension of \( \mathbb{Q} \) unramified outside \( S \) and \( \infty \), and let \( G_S = \text{Gal}(Q_S/\mathbb{Q}) \). Following Greenberg, we define a Selmer group \( \text{Sel}_Q^{\Sigma, L}(f) \) by

\[
\text{Sel}_Q^{\Sigma, L}(f) = \ker \left\{ H^1(G_S, \mathcal{M}) \rightarrow H^1(I_\ell, \mathcal{M}^-)^{G_{Q_\ell}} \times \prod_{\ell \in S \setminus \Sigma} H^1(I_\ell, \mathcal{M})^{G_{Q_\ell}} \right\}.
\]

This is a discrete, cofinite \( \Lambda_\mathcal{O} \)-module. Its Pontryagin dual \( X^{\Sigma, L}_Q(f) \) is a finite \( \Lambda_\mathcal{O} \)-module. We denote by \( Ch^\Sigma_L(f) \) the \( \Lambda_\mathcal{O} \)-characteristic ideal of \( X^{\Sigma, L}_Q(f) \); this is a principal ideal. In general, these all depend on the choice of \( T_f \), but if \( \rho_f \) is irreducible, then Lemma 2.1.1 shows that \( \text{Sel}_Q^{\Sigma, L}(f) \) is independent of \( T_f \) up to isomorphism, and hence so is \( X^{\Sigma, L}_Q(f) \). In particular, if \( \rho_f \) is irreducible, then the ideal \( Ch^\Sigma_L(f) \) does not depend on the choice of \( T_f \).

Furthermore, if \( L_1 \supset L \) is a finite extension with ring of integers \( \mathcal{O}_1 \supset \mathcal{O} \), then \( T_{f, 1} = T_f \otimes \mathcal{O} \mathcal{O}_1 \) is a \( G_{Q_\ell} \)-stable \( \mathcal{O}_1 \)-lattice in \( V_1 = V_f \otimes L L_1 \) and \( T_{f, 1}^+ = T_f^+ \otimes \mathcal{O}_1 \mathcal{O}_1 \). Hence the Selmer group \( \text{Sel}_Q^{\Sigma, L_1}(f) \), defined with respect to the lattice \( T_{f, 1} \), is canonically isomorphic to \( \text{Sel}_Q^{\Sigma, L}(f) \otimes \mathcal{O} \mathcal{O}_1 \) as a \( \Lambda_\mathcal{O}_1 \) \( \Lambda_\mathcal{O} \otimes \mathcal{O} \mathcal{O}_1 \)-module, from which it follows that its Pontryagin dual \( X^{\Sigma, L_1}_Q(f) \) is isomorphic to \( X^{\Sigma, L}_Q \otimes \mathcal{O} \mathcal{O}_1 \) as a \( \Lambda_\mathcal{O}_1 \)-module and therefore

\[
(2.3.1) \quad Ch^\Sigma_{L_1}(f) = Ch^\Sigma_L(f) \cdot \Lambda_\mathcal{O}_1.
\]

The relation between the Selmer groups \( \text{Sel}_Q^{\Sigma_1, L}(f) \) and \( \text{Sel}_Q^{\Sigma_2, L}(f) \) with \( \Sigma_1 \subset \Sigma_2 \) is clear:

\[
\text{Sel}_Q^{\Sigma_1, L}(f) = \ker \left\{ \text{Sel}_Q^{\Sigma_2, L}(f) \xrightarrow{\rho_{f, \Sigma_1}} \prod_{\ell \in S_2 \setminus S_1} H^1(I_\ell, \mathcal{M})^{G_{Q_\ell}} \right\}.
\]

Each \( H^1(I_\ell, \mathcal{M})^{G_{Q_\ell}}, \ell \neq p \), is a cotorsion \( \Lambda_\mathcal{O} \)-module, and the \( \Lambda_\mathcal{O} \)-characteristic ideal of its Pontryagin dual is generated by \( P_\ell(\Psi^{-1}\epsilon^{-1}(\text{frob}_\ell)) \), where

\[
P_\ell(X) = \det(1 - \rho_f(\text{frob}_\ell) | V_{f, I_\ell})
\]
with \( V_{f,t} \) being the space of \( I_{c}\)-coinvariants of the representation \( V_f \). In particular, \( X_{\Q\infty,L}^{\Sigma}(f) \) is a torsion \( \Lambda_\Q \)-module if and only if \( X_{\Q\infty,L}^{\Sigma_1}(f) \) is, and
\[
Ch_{L}^{\Sigma_2}(f) \supseteq Ch_{L}^{\Sigma_1}(f) \cdot \prod_{\ell \in \Sigma_2 \setminus \Sigma_1} (P_{\ell}(\Psi^{-1}(\text{frob}_\ell))).
\]
Later, we shall see that this last inclusion is often an equality.

If \( \Sigma = \{p\} \) then we will omit it from our notation, writing \( \Sel_{\Q\infty,L}(f), X_{\Q\infty,L}(f), \) and \( Ch_L \) instead.

The following lemma shows that if \( \Sigma \) is large enough and that if \( \tilde{\rho}_f \) is irreducible, then \( \Sel_{\Q\infty,L}^{\Sigma}(p^m) \) and \( X_{\Q\infty,L}^{\Sigma}/p^m X_{\Q\infty,L}^{\Sigma}(f) \) depend only on the pair \( (T_f/p^m T_f, T_f^+/p^m T_f^+) \) (up to isomorphism).

**Lemma 2.3.1.** Suppose \( \Sigma \supset \{\ell \mid N\} \) and that \( \tilde{\rho}_f \) is irreducible. Then the inclusion \( \mathcal{M}[p^m] \subset \mathcal{M} \) induces an identification
\[
\Sel_{\Q\infty,L}^{\Sigma}(f)[p^m] = \ker \left\{ H^1(G_S, \mathcal{M}[p^m]) \xrightarrow{\text{res}} H^1(I_p, \mathcal{M}^{-}[p^m]) \right\}.
\]
That the dependence is only on the pair \( (T_f/p^m T_f, T_f^+/p^m T_f^+) \) follows since \( \mathcal{M}[p^m] = T_f/p^m T_f \otimes \Lambda_{\Q}^*[p^m], \mathcal{M}^+[p^m] = T_f^+/p^m T_f^+ \otimes \Lambda_{\Q}^*[p^m], \) and \( \mathcal{M}^{-}[p^m] = \mathcal{M}[p^m]/\mathcal{M}^+[p^m] \).

**Proof.** Since \( \tilde{\rho}_f \) is irreducible, the inclusion \( \mathcal{M}[p^m] \hookrightarrow \mathcal{M} \) induces an identification \( H^1(G_S, \mathcal{M}[p^m]) \cong H^1(G_S, \mathcal{M})[p^m] \). So \( \Sel_{\Q\infty,L}^{\Sigma}(f)[p^m] \) is the kernel of the restriction map \( H^1(G_S, \mathcal{M}[p^m]) \to H^1(I_p, \mathcal{M}^{-}[p^m]) \), which factors through the restriction map \( H^1(G_S, \mathcal{M}[p]) \to H^1(I_p, \mathcal{M}^{-}[p^m]) \). The kernel of the natural map \( H^1(I_p, \mathcal{M}^{-}[p^m]) \to H^1(I_p, \mathcal{M}^{-}) \) is the image of \( (\mathcal{M}^{-})^{1/p}/(\mathcal{M}^{-})^{1/p} \) via the boundary map. But \( (\mathcal{M}^{-})^{1/p} \cong \text{Hom}_{cts}(\mathcal{O}, \Q_p/Z_p) \) since \( I_p \) acts via \( \Psi^{-1} \) on \( \mathcal{M}^{-} \cong \Lambda_{\Q}^* \), and so \( (\mathcal{M}^{-})^{1/p}/(\mathcal{M}^{-})^{1/p} = 0 \) as \( \text{Hom}_{cts}(\mathcal{O}, \Q_p/Z_p) \) is \( p \)-divisible.

The key to our proofs of both Theorems A and B is an understanding of the images of the restriction maps
\[
(2.3.2) \quad H^1(G_S, \mathcal{M}) \xrightarrow{\text{res}} H^1(\Q_p, \mathcal{M}^{-}) \times \prod_{\ell \not\in S, \ell \neq p} H^1(I_{\ell}, \mathcal{M})^{G_{\Q_{\ell}}}
\]
and
\[
(2.3.3) \quad H^1(G_S, \mathcal{M}) \xrightarrow{\text{res}} H^1(I_p, \mathcal{M}^{-})^{G_{\Q_p}} \times \prod_{\ell \not\in S, \ell \neq p} H^1(I_{\ell}, \mathcal{M})^{G_{\Q_{\ell}}},
\]
where \( S \supset \{\ell \mid Np\} \) is any finite set of primes. The kernel of (2.3.3) is, of course, just \( \Sel_{\Q\infty,L}(f) \). We denote the kernel of (2.3.2) by \( \mathcal{S} \) (it is independent of \( S \)) and let \( \mathcal{X} \) be its Pontryagin dual. As \( \mathcal{S} \) is a submodule of each \( \Sel_{\Q\infty,L}^{\Sigma}(f), \mathcal{X} \) is a quotient of each \( X_{\Q\infty,L}^{\Sigma}(f) \).

The next two propositions record some properties of the above restriction maps. The ideas behind the proofs of these propositions are due to Greenberg (see especially [8],...
§§3.4, [9], and [10]). As there is not a convenient reference for the exact case considered here, we have included the details of the arguments.

**Proposition 2.3.2.** Suppose \( k \equiv 2 \pmod{p-1} \), \( \bar{\rho}_f \) is irreducible, and \( X_{\mathbb{Q},\ell}(f) \) is a torsion \( \Lambda_{\mathcal{O}} \)-module. The restriction maps (2.3.2) and (2.3.3) are surjective.

**Proof.** As \( H^1(\mathbb{Q}_p, \mathcal{M}^-) \to H^1(I_p, \mathcal{M}^-)^G_{\mathbb{Q}_p} \), (2.3.3) is surjective if (2.3.2) is. That is, to prove the proposition it suffices to prove surjectivity of (2.3.2). To establish this surjectivity we introduce some auxiliary Selmer groups.

Let \( \mathcal{N} = \text{Hom}_\mathcal{O}(T_f, \mathcal{O}(1)) \otimes_{\mathcal{O}} \Lambda_{\mathcal{O}} \), with \( G_{\mathbb{Q}} \)-action given by \( \epsilon \rho_j^* \otimes \Psi \), and let \( \mathcal{N}^+ = \text{Hom}_\mathcal{O}(T_f/T_f^+, \mathcal{O}(1)) \otimes_{\mathcal{O}} \Lambda_{\mathcal{O}} \), which is \( G_{\mathbb{Q}_p} \)-stable with \( G_{\mathbb{Q}_p} \) acting via \( \alpha_f^{-1} \epsilon \otimes \Psi \). These are free \( \Lambda_{\mathcal{O}} \)-modules, and \( \mathcal{N}^+ \) is a \( \Lambda_{\mathcal{O}} \)-direct summand of \( \mathcal{N} \). Let \( \mathcal{N}^- = \mathcal{N}/\mathcal{N}^+ \). The pairing

\[
(\cdot, \cdot) : \mathcal{M} \times \mathcal{N} \to \mathbb{Q}_p/\mathbb{Z}_p, \quad (t \otimes \varphi, \phi \otimes r) = \varphi(\phi(t) \cdot r),
\]

is a \( \mathcal{O} \)-equivariant perfect pairing under which \( \mathcal{M}^+ \) and \( \mathcal{N}^+ \) are mutual annihilators. Under the induced (perfect) local Tate pairing

\[
H^1(\mathbb{Q}_p, \mathcal{M}) \otimes H^2(\mathbb{Q}_p, \mathcal{N}) \to \mathbb{Q}_p/\mathbb{Z}_p,
\]

\[
L_p(\mathcal{M}) = \text{im}\{H^1(\mathbb{Q}_p, \mathcal{M}^+) \to H^1(\mathbb{Q}_p, \mathcal{M})\} \quad \text{and} \quad L_p(\mathcal{N}) = \text{im}\{H^1(\mathbb{Q}_p, \mathcal{N}^+) \to H^1(\mathbb{Q}_p, \mathcal{N})\}
\]

are also mutual annihilators. Let

\[
\text{Sel}^S(\mathcal{N}) = \ker \left\{ H^1(G_S, \mathcal{N}) \stackrel{\text{res}}{\to} H^1(\mathbb{Q}_p, \mathcal{N})/L_p(\mathcal{N}) \hookrightarrow H^1(\mathbb{Q}_p, \mathcal{N}^-) \right\}.
\]

Let \( \text{III}^1(\mathbb{Q}, S, \mathcal{N}) \subseteq \text{Sel}^S(\mathcal{N}) \) consist of those classes that are trivial at all places in \( S \).

For \( \ell \neq p \), \( H^1(\mathbb{F}_\ell, \mathcal{M}^\ell) = 0 \) and so \( H^1(\mathbb{F}_\ell, \mathcal{M}) \cong H^1(\mathbb{Q}_p, \mathcal{M}) \). Also, \( H^2(\mathbb{Q}_p, \mathcal{M}^+) = 0 \) as its dual is \( H^0(\mathbb{Q}_p, \mathcal{N}^-) = 0 \), so \( H^1(\mathbb{Q}_p, \mathcal{M})/L_p(\mathcal{M}) \to H^1(\mathbb{Q}_p, \mathcal{M}^-) \). Global Tate duality then identifies the dual of the cokernel of (2.3.2) with \( \text{Sel}^S(\mathcal{N})/\text{III}^1(\mathbb{Q}, S, \mathcal{N}) \) (cf. [9, Prop. 3.1]). As \( \bar{\rho}_f \) is irreducible, \( H^1(G_S, \mathcal{N}) \) is \( \Lambda_{\mathcal{O}} \)-torsion-free, and hence so are \( \text{Sel}^S(\mathcal{N}) \) and \( \text{III}^1(\mathbb{Q}, S, \mathcal{N}) \). Therefore, to prove the desired surjectivity it suffices to show that \( \text{Sel}^S(\mathcal{N}) \) is a torsion \( \Lambda_{\mathcal{O}} \)-module (and so trivial). We prove that \( \text{Sel}^S(\mathcal{N}) \) is torsion by exhibiting elements in \( x \) in the maximal ideal of \( \Lambda_{\mathcal{O}} \) such that \( \text{Sel}^S(\mathcal{N})/x\text{Sel}^S(\mathcal{N}) \) has finite order.

Let \( x = \gamma - u^m \in \Lambda_{\mathcal{O}} \) with \( m \) an integer. Let \( N_x = \mathcal{N}/x\mathcal{N}, \ N_x^+ = \mathcal{N}^+/x\mathcal{N}^+ \), and \( N_x^- = N_x/N_x^+ \). These are free \( \mathcal{O} \)-modules. If \( p \nmid N \) or \( m \neq 0 \), then the natural injection

\[
H^1(G_S, \mathcal{N})/xH^1(G_S, \mathcal{N}) \hookrightarrow H^1(G_S, N_x)
\]

induces an injection

\[
(2.3.4) \quad \text{Sel}^S(\mathcal{N})/x\text{Sel}^S(\mathcal{N}) \hookrightarrow \text{Sel}^S(N_x) = \ker \left\{ H^1(G_S, N_x) \to H^1(\mathbb{Q}_p, N_x^-) \right\}.
\]

For this, we first note that the image of the induced map from \( \text{Sel}^S(\mathcal{N})/x\text{Sel}^S(\mathcal{N}) \) to \( H^1(G_S, N_x) \) lies in \( \text{Sel}^S(N_x) \). It remains to prove injectivity. Let \( c \in \text{Sel}^S(\mathcal{N}) \) be such that it has trivial image in \( \text{Sel}^S(N_x) \). Then \( c = xd \) for some \( d \in H^1(G_S, \mathcal{N}) \) such that \( xd = 0 \) in \( H^1(\mathbb{Q}_p, \mathcal{N}^-) \). The kernel of multiplication by \( x \) on \( H^1(\mathbb{Q}_p, \mathcal{N}^-) \) is the
image of $H^0(\mathbb{Q}_p, N_x^-)$. But $N_x^-$ is a free $\mathcal{O}$-module with $G_{\mathbb{Q}_p}$ acting via the character $\alpha_f^{e^2 - k + m\omega - m}$, and so $H^0(\mathbb{Q}_p, N_x^+) = 0$ unless $m = k - 2$ and $\alpha_f = 1$. But $\alpha_f = 1$ only if $p \nmid N$ and $k = 2$. It follows that that if $p \nmid N$ or $m \neq 0$, then multiplication by $x$ is injective on $H^1(\mathbb{Q}_p, N^-)$ and, therefore, $d \in \text{Sel}^S(N)$, proving the injectivity in (2.3.4).

From (2.3.4) it follows that to prove $\text{Sel}^S(N)$ is torsion it suffices to show that there is some $m \neq 0$ such that $\text{Sel}^S(N_x)$ has finite order. As $\text{Sel}^S(N_x)$ has finite order if and only if $\text{Sel}^S(N_x) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ has finite order - in which case it must be trivial - it suffices to prove the latter. Furthermore, as $\tilde{\rho}_f$ is irreducible and so $H^1(G_S, N_x)$ and hence also $\text{Sel}^S(N_x)$ - is a torsion-free $\mathcal{O}$-module and therefore free, it would then follow that $\text{Sel}^S(N_x) = 0$.

Let $M_x = N_x \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ and $M_x = N_x \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. From the injections $H^1(G_S, N_x) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^1(G_SM_x)$ and $H^1(\mathbb{Q}_p, N_x^-) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^1(\mathbb{Q}_p, M_x^-)$ we deduce an injection

$$\text{Sel}^S(N_x) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \text{Sel}^S(M_x) = \ker \left\{ H^1(G_S, M_x) \overset{\gamma \circ \rho}{\rightarrow} H^1(\mathbb{Q}_p, M_x^-) \right\}. \tag{2.3.5}$$

So to prove that there is an $m \neq 0$ such that $\text{Sel}^S(N_x)$ has finite order, it suffices to find such an $m$ for which $\text{Sel}^S(M_x)$ has finite order.

Let $m \neq 0$ be an integer such that $m \equiv 0 \pmod{p - 1}$. Let $y = \gamma - u^{k-2-m}$. Then, as $k \equiv 2 \pmod{p - 1}$, $M_x \cong M[y]$ as $\mathcal{O}[G]\text{-modules}$, and the isomorphism can be chosen so that $M_x$ is identified with $M^-[y]$. It follows that

$$\text{Sel}^S(M_x) = \text{Sel}^S(M[y]) \hookrightarrow \text{Sel}^S_{\mathbb{Q}_\infty, L}(f)[y], \tag{2.3.6}$$

where $\text{Sel}^S(M[y])$ is defined just as $\text{Sel}^S(M_x)$, and where the injection is induced by the natural identification $H^1(G_S, M[y]) \cong H^1(G_S, M)[y]$ (which is injective as $\tilde{\rho}_f$ is irreducible).

As $X_{\mathbb{Q}_\infty, L}(f)$ is a torsion $\Lambda_{\mathcal{O}}$-module so is $X_{\mathbb{Q}_\infty, L}(f)$. Therefore, for all but finitely many integers $m$, $X_{\mathbb{Q}_\infty, L}(f)/y X_{\mathbb{Q}_\infty, L}(f)$ has finite order. As the latter is dual to $\text{Sel}^S_{\mathbb{Q}_\infty, L}(f)[y]$, it follows from (2.3.6) that there is an $m \neq 0$ with $m \equiv 0 \pmod{p - 1}$ such that $\text{Sel}^S(M_x)$ has finite order. As explained above, the existence of such an $x$ implies the desired surjectivity of (2.3.2).

**Proposition 2.3.3.** Suppose $k \equiv 2 \pmod{p - 1}$, $\tilde{\rho}_f$ is irreducible, and $X_{\mathbb{Q}_\infty, L}(f)$ is a torsion $\Lambda_{\mathcal{O}}$-module.

(i) $X$ has no non-zero finite-order $\Lambda_{\mathcal{O}}$-submodules.

(ii) Let $\Sigma$ be any finite set of primes containing $p$. Then $X_{\mathbb{Q}_\infty, L}(f)$ has no non-zero finite-order $\Lambda_{\mathcal{O}}$-submodules.

**Proof.** To prove part (i), let $S \supset \{\ell \mid Np\}$ be any finite set of primes and let

$$\mathcal{P}_S = H^1(\mathbb{Q}_p, M^-) \times \prod_{\ell \in S, \ell \neq p} H^1(\mathbb{Q}_\ell, M).$$

For \( x = \gamma - u^m \in \Lambda_{\mathcal{O}} \), \( \mathcal{P}_S[x] \) is a quotient of

\[
P_{S,x} = H^1(\mathbb{Q}_p, \mathcal{M}[x]) / L_p(\mathcal{M}[x]) \times \prod_{\ell \in S, \ell \neq p} H^1(\mathbb{Q}_\ell, \mathcal{M}[x]),
\]

where \( L_p(\mathcal{M}[x]) = \text{im}\{H^1(\mathbb{Q}_p, \mathcal{M}^+)[x] \to H^1(\mathbb{Q}_p, \mathcal{M}[x])\} \). Therefore the cokernel of the restriction map \( H^1(G_S, \mathcal{M}[x]) = H^1(G_S, \mathcal{M})[x] \to \mathcal{P}_S[x] \) is a quotient of the cokernel of the restriction map \( H^1(G_S, \mathcal{M})[x] \to \mathcal{P}_{S,x} \). By global Tate duality, the Pontryagin dual of the latter is a subquotient of \( \text{Sel}^S(N_x) \), where \( N_x \) and \( \text{Sel}^S(N_x) \) as in (2.3.4). But, as shown in the proof of Proposition 2.3.2, \( m \) can be chosen so that \( \text{Sel}^S(N_x) = 0 \) and hence so that \( H^1(G_S, \mathcal{M})[x] \to \mathcal{P}_S[x] \). It then follows from an application of the snake lemma to multiplication by \( x \) of the short exact sequence

\[
0 \to S \to H^1(G_S, \mathcal{M}) \to \mathcal{P}_S \to 0
\]

that, for such a choice of \( m \),

\[
(2.3.7) \quad S / xS \hookrightarrow H^1(G_S, \mathcal{M}) / xH^1(G_S, \mathcal{M}).
\]

However, as shown in both [18, Lem. 3.3.18] and [10, Prop. 2.6.1], the right-hand side of (2.3.7) is trivial for all but finitely many \( m \), so the \( m \) can also be chosen so that \( S / xS = 0 \). Let \( X \subseteq \mathcal{X} \) be a sub-\( \Lambda_{\mathcal{O}} \)-module of finite order, and let \( X^* \) be its Pontryagin dual. Then \( X^* / xX^* \) is a quotient of \( S / xS \) and so is 0. By Nakayama’s lemma \( X^* = 0 \), hence \( X = 0 \). This proves (i).

To prove part (ii), let \( S \supseteq \Sigma \cup \{ \ell \mid Np \} \) and let

\[
\mathcal{P}_{S,\Sigma} = H^1(I_p, \mathcal{M}^-)^{G_{\mathbb{Q}_p}} \times \prod_{\ell \in S \setminus \Sigma} H^1(\mathbb{Q}_\ell, \mathcal{M})
\]

and

\[
\mathcal{P}_{S,\Sigma,x} = H^1(\mathbb{Q}_p, \mathcal{M}[x]) / L_p(\mathcal{M}[x]) \times \prod_{\ell \in S \setminus \Sigma} H^1(\mathbb{Q}_\ell, \mathcal{M}[x]).
\]

We may then argue as in the proof of part (i) but with \( \mathcal{P}_S \) replaced by \( \mathcal{P}_{S,\Sigma} \). Then \( S \) is replaced by \( \text{Sel}_{\mathbb{Q}_{\infty,L}}^\Sigma(f) \). Furthermore, as \( \mathcal{P}_{S,\Sigma,x} \) is a quotient of \( \mathcal{P}_{S,x} \), the surjectivity of the restriction map \( H^1(G_S, \mathcal{M}[x]) \to \mathcal{P}_{S,\Sigma,x} \), and hence of the restriction map \( H^1(G_S, \mathcal{M}) \to \mathcal{P}_{S,\Sigma,x} \), follows for a suitable \( x = \gamma - u^m \in \Lambda_{\mathcal{O}} \) from the surjectivity of the restriction map onto \( \mathcal{P}_{S,x} \) established in the proof of part (i).

Let \( F_L^\Sigma(f) \) be the \( \Lambda_{\mathcal{O}} \)-Fitting ideal of \( X_{\mathbb{Q}_{\infty,L}}^\Sigma(f) \). The following is a straight-forward consequence of the preceding propositions.

**Lemma 2.3.4.** Suppose \( k \equiv 2 \mod{p - 1} \) and \( \tilde{\rho}_f \) is irreducible.

(i) \( \text{Ch}^\Sigma_L(f) = \text{Ch}_L(f) \cdot (\prod_{\ell \in \Sigma, \ell \neq p} P_\ell(\Psi^{-1} \epsilon^{-1}(\text{frob}_\ell))) \).

(ii) \( F_L^\Sigma(f) = \text{Ch}^\Sigma_L(f) \).
Proof. If \( X_{Q_{\infty},L}^\Sigma(f) \) is not a torsion \( \Lambda_\mathcal{O} \)-module (equivalently, \( X_{Q_{\infty},L}^\Sigma(f) \) is not a torsion \( \Lambda_\mathcal{O} \)-module), then \( \text{Ch}_L(f), \text{Ch}_L^f(f) \), and \( F_L^\Sigma(f) \) are all zero, so there is nothing to prove. We suppose then that \( X_{Q_{\infty},L}^\Sigma(f) \) is a torsion \( \Lambda_\mathcal{O} \)-module.

Part (i) is immediate from Proposition 2.3.2 and the definition of characteristic ideals. For part (ii), we first note that \( F_L^\Sigma(f) \subset \text{Ch}_L^f(f) \). Let \( a \) be the kernel of the quotient \( \Lambda_\mathcal{O}/F_L^\Sigma(f) \to \Lambda_\mathcal{O}/\text{Ch}_L^f(f) \). Since \( X_{Q_{\infty},L}^\Sigma(f) \) is a torsion \( \Lambda_\mathcal{O} \)-module and \( \text{Ch}_L^f(f) \) is a principal ideal, there exists \( \lambda = \gamma - u^m \in \Lambda_\mathcal{O} \) such that \( \lambda \) is not a zero-divisor in \( \Lambda_\mathcal{O}/\text{Ch}_L^f(f) \) and \( X_{Q_{\infty},L}^\Sigma(f)/\lambda X_{Q_{\infty},L}^\Sigma(f) \) is a torsion \( \Lambda_\mathcal{O}/\Lambda_\mathcal{O} = \mathcal{O} \)-module. The size of this module is then equal to the size of both \( \Lambda_\mathcal{O}/(\lambda, F_L^\Sigma(f)) \) and \( \Lambda_\mathcal{O}/(\lambda, \text{Ch}_L^f(f)) \) (which are necessarily finite), the first by basic properties of Fitting ideals and the second by Proposition 2.3.3(ii) and a standard argument\(^5\) from Iwasawa theory. If follows that the natural projection \( \Lambda_\mathcal{O}/(\lambda, F_L^\Sigma(f)) \to \Lambda_\mathcal{O}/(\lambda, \text{Ch}_L^f(f)) \) is an isomorphism. Applying the snake lemma to the diagram obtained by multiplying the short exact sequence

\[
0 \to a \to \Lambda_\mathcal{O}/F_L^\Sigma(f) \to \Lambda_\mathcal{O}/\text{Ch}_L^f(f) \to 0
\]

by \( \lambda \) then yields an exact sequence

\[
0 \to a/\lambda a \to \Lambda_\mathcal{O}/(\lambda, F_L^\Sigma(f)) \to \Lambda_\mathcal{O}/(\lambda, \text{Ch}_L^f(f)) \to 0.
\]

Therefore \( a/\lambda a \), and hence \( a \), is 0. \( \square \)

2.4. \( p \)-adic \( L \)-functions. Let \( f, L, \mathcal{O} \), and \( \Lambda_\mathcal{O} \) be as in the preceding section, with the assumption that \( k \geq 2 \) and \( f \) is ordinary with respect to \( L \). Amice and Vélu [1] and Vishik [21] (see also [14]) constructed a \( p \)-adic \( L \)-function for \( f \). This is a power series \( L_f \in \Lambda_\mathcal{O} \) with the property that if \( \phi: \Lambda_\mathcal{O} \to \overline{\mathbb{Q}}_p \) is a continuous \( \mathcal{O} \)-homomorphism such that \( \phi(\gamma) = \zeta u^m \) with \( \zeta \) a primitive \( p^{k-1} \)-th root of unity and \( 0 \leq m \leq k-2 \) an integer,

\(^5\)The argument: A finitely-generated torsion \( \Lambda_\mathcal{O} \)-algebra \( X \) admits a \( \Lambda_\mathcal{O} \)-homomorphism \( X \to Y = \prod_{i=1}^r \Lambda_\mathcal{O}/(f_i) \) with finite-order kernel \( a \) and cokernel \( b \) and such that the \( \Lambda_\mathcal{O} \)-characteristic ideal of \( X \) is \( (f_1 \cdots f_r) \). Let \( f = f_1 \cdots f_r \). If \( X \) has no finite-order \( \Lambda_\mathcal{O} \)-submodules, then the map to \( Y \) is an injection. Multiplying the short exact sequence \( 0 \to X \to Y \to b \to 0 \) by \( \lambda = \gamma - u^m \) and applying the snake lemma is easily seen to give

\[
\#X/\lambda X = \#Y/\lambda Y = \prod \#\Lambda_\mathcal{O}/(\lambda, f_i) = \prod \#\mathcal{O}/(f_i(u^m - 1)) = \#\mathcal{O}/(f(u^m - 1)) = \#\mathcal{O}/(\lambda, f),
\]

where we have written \( f_i(u^m - 1) \) and \( f(u^m - 1) \) for the respective images of \( f_i \) and \( f \) under the continuous \( \mathcal{O} \)-algebra homomorphism \( \Lambda_\mathcal{O} \to \mathcal{O} \) sending \( \gamma \) to \( u^m \).
then\(^6\)
\[
\mathcal{L}_f(\phi) := \phi(\mathcal{L}_f) = e(\phi) \frac{p^t_{\phi}(m+1) m! L(f, \chi_\phi^{-1} \omega^{-m}, m + 1)}{(-2\pi i)^{m+1} G(\chi_\phi^{-1} \omega^{-m}) \Omega_f^{\text{sgn}((-1)^m)}},
\]
\[(2.4.1)\]
\[
e(\phi) = \alpha_p^{-t_\phi} \left(1 - \frac{\omega^{-m} \chi_\phi^{-1} p^{k-2-m}}{\alpha_p} \right) \left(1 - \frac{\omega^m \chi_\phi(p) p^m}{\alpha_p} \right),
\]
where \(\alpha_p\) is the unique (unit) root in \(O^\times\) of \(x^2 - a_p(f)x + p^{k-1}\) if \(p \nmid N\) and \(\alpha_p = a_p(f)\) if \(p \mid N\), \(t_\phi = 0\) if \(t_\phi = 1\) and \(p - 1 \mid m\) and otherwise \(t_\phi = t_\phi\), \(\chi_\phi\) is the primitive Dirichlet character of \(p\)-power order and conductor (which can be viewed as a finite-order character of \(\mathbb{Z}_p^\times\)) such that \(\chi_\phi(u) = \zeta^{-1}\), \(G(\chi_\phi^{-1} \omega^{-m})\) is the usual Gauss sum (and so equals 1 if \(t_\phi = 0\)), and \(\Omega_f^\pm\) are the canonical periods of \(f\) (these are well-defined up to a unit in \(O\); see \([18, \S3.3.3]\)).

Let \(\Sigma\) be a finite set of primes. We define an incomplete \(p\)-adic \(L\)-function \(\mathcal{L}_f^\Sigma \in \Lambda_O\) by
\[
\mathcal{L}_f^\Sigma = \mathcal{L}_f \cdot \prod_{\ell \in \Sigma, \ell \neq p} P_{\ell}(\Psi^{-1} \epsilon^{-1}(\text{frob}_\ell)).
\]
Note that
\[
P_{\ell}(\Psi^{-1} \epsilon^{-1}(\text{frob}_\ell)) = \begin{cases} 
1 - a_\ell(f) \ell^{-1} \Psi^{-1}(\text{frob}_\ell) + \ell^{k-3} \Psi^{-2}(\text{frob}_\ell) & \ell \nmid N \\
1 - a_\ell(f) \ell^{-1} \Psi^{-1}(\text{frob}_\ell) & \ell \mid N.
\end{cases}
\]
In particular, the value of \(\mathcal{L}_f^\Sigma\) under a continuous \(O\)-algebra homomorphism \(\phi : \Lambda_O \to \overline{\mathbb{Q}}_p\) such that \(\phi(\gamma) = \zeta u^m, 0 \leq m \leq k - 2\), can be expressed in terms of a special value of an incomplete \(L\)-function:
\[
\mathcal{L}_f^\Sigma(\phi) = e(\phi) \frac{p^t_{\phi}(m+1) m! L^{\Sigma \setminus \{p\}}(f, \chi_\phi^{-1} \omega^{-m}, m + 1)}{(-2\pi i)^{m+1} G(\chi_\phi^{-1} \omega^{-m}) \Omega_f^{\text{sgn}((-1)^m)}}.
\]

Remark 2.4.1. Let \(Z(f)\) be the ring of integers of \(\mathbb{Q}(f)\) and let \(p\) be the prime of \(Z(f)\) determined by the chosen embedding \(\mathbb{Q}(f) \hookrightarrow \overline{\mathbb{Q}}_p\). Then \(\Omega_f^\pm\) is well-defined up to a unit in the localization \(Z(f)(p)\) of \(Z(f)\), and the value of the \(p\)-adic \(L\)-function under a homomorphism \(\phi\) as above lies in a finite extension of \(Z(f)(p)\). It is in this way that period-normalized values of the \(L\)-function \(L(f, s)\) and its twists, which \textit{a priori} are complex values, can be viewed as being in \(\overline{\mathbb{Q}}_p\) without fixing an isomorphism \(\overline{\mathbb{Q}}_p \cong \mathbb{C}\).

Suppose \(f\) has split multiplicative reduction at \(p\). Then it follows easily from (2.4.1) that if \(\phi_0 : \Lambda_O \to \overline{\mathbb{Q}}_p\) is the \(O\)-algebra homomorphism such that \(\phi_0(\gamma) = 1\), then \(\mathcal{L}_f(\phi_0) = 0\). In particular, \(\mathcal{L}_f = (\gamma - 1) \cdot \mathcal{L}_f^'\) for some \(\mathcal{L}_f^' \in \Lambda_O\). Greenberg and Stevens

\(^6\)The power of \(-2\pi i\) in the denominator of this formula is incorrectly given as \((-2\pi i)^m\) in some of the formulas in [18], namely in the introduction, in \(\S3.4.4\), and in Theorem 3.26 of \textit{loc. cit.} In these cases the correct factor is \((-2\pi i)^{m+1}\). This error originates in the difference between \(\Omega_f^\pm\) as defined in [18, \S3.3.3] and the \(\Omega_f^\pm\) in [14, I.9]: \(\Omega_f^\pm = -2\pi i \Omega_f^\pm\). The exponents of \(-2\pi i\) are correct in the formulas in [18] for the \(L\)-function of \(f\) twisted by a Hecke character of the imaginary quadratic field \(\mathcal{K}\).
[11, Thm. 7.1] proved that \( L'_f(\phi_0) = \phi_0(L'_f) \) is related to the \( L \)-invariant of \( V_f \) by the formula

\[
(2.4.3) \quad L'_f(\phi_0) = (\log p u)^{-1} \mathcal{L}(V_f) \frac{L(f,1)}{-2\pi i \Omega^f_f}.
\]

More precisely, if we identify \( \Lambda_\mathcal{O} \) with the power-series ring \( \mathcal{O}[T] \) by sending \( \gamma \) to \( 1 + T \), and if we let \( L_p(f,s) = L_f(u^{s-1} - 1), s \in \mathbb{Z}_p \), then Greenberg and Stevens proved that

\[
\frac{d}{ds} L_p(f,s) \bigg|_{s=1} = \mathcal{L}(V_f) \frac{L(f,1)}{-2\pi i \Omega^f_f}.
\]

This is easily seen to be equivalent to (2.4.3). This formula was conjectured by Mazur, Tate, and Teitelbaum [14, §13].

2.5. The Iwasawa–Greenberg Main Conjecture. Let \( f, L, \mathcal{O}, \Lambda_\mathcal{O}, \mathcal{L}_f \), etc., be as in the preceding sections. Along the lines of Iwasawa’s original Main Conjecture for totally real number fields, Mazur and Swinnerton-Dyer (for modular elliptic curves) and Greenberg (more generally) made the following conjecture.

**Conjecture 2.5.1.** If \( \Sigma \) is any finite set of primes containing \( p \), then \( X_{\mathcal{O},L}^\Sigma(f) \) is a torsion \( \Lambda_\mathcal{O} \)-module and \( Ch_{\Sigma}^\mathcal{L}(f) = (\mathcal{L}_f^\Sigma) \) in \( \Lambda_\mathcal{O} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) and even in \( \Lambda_\mathcal{O} \) if \( \bar{\rho}_f \) is irreducible.

It follows easily from Lemma 2.3.4(i) and (2.4.2) that if this conjecture holds for one set \( \Sigma \) then it holds for all sets \( \Sigma \). Also, the conjecture with \( L \) replaced by any finite extension implies the conjecture for \( L \), as can be seen by the observations in Section 2.3 on the relation (2.3.1) between \( Ch_{\mathcal{L}_L}(f) \) and \( Ch_{\mathcal{L}_{L_1}}(f) \) for a finite extension \( L_1 \supset L \).

In [18] the following theorem was proved, in combination with results of Kato [13], which established this conjecture for a large class of modular forms.

**Theorem 2.5.2.** Suppose

(i) \( k \equiv 2 \pmod{p-1} \);
(ii) \( \bar{\rho}_f \) is irreducible;
(iii) there exists a prime \( q \neq p \) such that \( q \parallel N \) and \( \bar{\rho}_f \) is ramified at \( q \);
(iv) \( p \nmid N \) (this is automatic if \( k \neq 2 \)).

Then for any finite set of primes \( \Sigma \), \( X_{\mathcal{O},L}^\Sigma(f) \) is a torsion \( \Lambda_\mathcal{O} \)-module and \( Ch_{\mathcal{L}}^\Sigma(f) = (\mathcal{L}_f^\Sigma) \) in \( \Lambda_\mathcal{O} \).

In [18] an additional hypothesis is required to conclude equality in \( \Lambda_\mathcal{O} \) and not just in \( \Lambda_\mathcal{O} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \):

(*) there exists an \( \mathcal{O} \)-basis of \( T_f \) such that the image of \( \rho_f \) contains \( SL_2(\mathbb{Z}_p) \).

This hypothesis was included because it is part of the statement of [13, Thm. 17.4]. However, a closer reading of the proof of loc. cit. shows that all that is necessary is that (a) \( \bar{\rho}_f \) be irreducible and (b) there exist an element \( g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\mu_{p^\infty}]) \) such
that $T_f/(\rho_f(g) - 1)T_f$ is a free $\mathcal{O}$-module of rank one, as we explain in the following paragraph. All references to theorems or sections in the following paragraph are to [13] unless otherwise indicated.

Hypothesis (*) intervenes in the proof of Theorem 17.4 through Theorem 15.5(4), which is proved in §13.14. Hypothesis (a) together with Lemma 2.1.1 of this paper implies that, in the notation of [13], the conclusion in §13.14 that $T_f = a \cdot V_{O,\lambda}(f)$ for some $a \in F_N$ holds; Lemma 2.1.1 of this paper can replace the reference to Lemma 14.7 in §13.14, which is the only explicit use of a basis with an image containing $\text{SL}_2(\mathbb{Z}_p)$ in the proof of Theorem 15.5(4). Hypothesis (a) also, of course, ensures that the hypotheses of Theorem 12.4(3) hold, as needed in §13.14. Hypothesis (b) ensures that the hypotheses of Theorem 13.4(3) hold. The proof of Theorem 15.5(4) in §13.14 then holds with (*) replaced by the hypotheses (a) and (b) above.

We now check that (a) and (b) hold under the hypotheses of Theorem 2.5.2. Hypothesis (a) is just hypothesis (ii) of the theorem. Hypothesis (b) is satisfied in light of hypothesis (iii) of the theorem: As $q || N$, the action of $I_q$ on $V_f$ is nontrivial and unipotent and in particular factors through the tame quotient (this is a consequence of the ‘local-global’ compatibility of the Galois representation $\rho_f$ [3, Thm. A]). It follows that $\tilde{\rho}_f(\tau)$ is unipotent for any $\tau \in I_q$ projecting to a topological generator of the tame quotient and, since $\tilde{\rho}_f$ is ramified at $q$, $\tilde{\rho}_f(\tau) \neq 1$, hence $T_f/(\rho_f(\tau) - 1)T_f$ is a free $\mathcal{O}$-module of rank one. As $\tau \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\mu_{p^\infty}])$, condition (b) holds for $g = \tau$.

We also take this opportunity to note that the reference to [20] in the proof of [18, Prop. 12.3.6] is not sufficient. It may be that the weight two specialization of the Hida family in loc. cit. that has trivial character also has multiplicative reduction at $p$. This case is excluded in [20], though the ideas in that paper can be extended to this case, as is explained in [4]. The reference to [20, Thm. 1.1] must be augmented by a reference to [4, Thm. C].

The purpose of this paper is, of course, to show that hypothesis (iv) of Theorem 2.5.2 can be removed.

The main results of [18] show that for a suitable imaginary quadratic field $K$ and a large enough set $\Sigma$, the equality $\text{Ch}_L^\Sigma(f)\text{Ch}_L^\Sigma(f \otimes \chi_K) = (L_f^\Sigma\ell_f^\Sigma \otimes \chi_K)$ holds, where $f \otimes \chi_K$ is the newform associated with the twist of $f$ by the primitive quadratic Dirichlet character corresponding to $K$. When $p \nmid N$, this equality can be refined to an equality of the individual factors via the inclusions $L_f^\Sigma \subset \text{Ch}_L^\Sigma(f)$ and $L_f^\Sigma \otimes \chi_K \subset \text{Ch}_L^\Sigma(f \otimes \chi_K)$, which are proved in [13]. When $p \mid N$, these inclusions do not follow directly from [13]; additional arguments are required.

2.6. Hida families. Let $f \in S_k(\Gamma_0(N))$ be a newform that is ordinary with respect to an embedding $\mathbb{Q}(f) \hookrightarrow \overline{\mathbb{Q}}_p$. Write $N = p^r M$ with $p \nmid M$ (so $r = 0$ or 1 by Lemma 2.1.2). Let $L \subset \overline{\mathbb{Q}}_p$ be any finite extension of $\mathbb{Q}_p$ containing the image of $\mathbb{Q}(f)$ and let $\mathcal{O}$ be the ring of integers of $L$. Let $R_0 = \mathcal{O}[X]$. Hida proved that there is a finite, local $R_0$-domain
Then, as explained by Greenberg and Stevens [11] (see also [15, (1.4.7)]),

\[
R = R_0[\{a_\ell : \ell = \text{prime}\}];
\]

- if \(\phi : R \to \mathbb{Q}_p\) is a continuous \(\mathcal{O}\)-algebra homomorphism such that \(\phi(1+X) = (1+p)^k\) with \(k' > 2\) and \(k' \equiv k \pmod{p-1}\), then \(\sum_{n=1}^{\infty} \phi(a_n)q^n\) is the \(q\)-expansion of a \(p\)-stabilized newform, in the sense that there is a newform \(f_\phi \in S_k(\Gamma_0(M))\) and an embedding \(\mathbb{Q}(f_\phi) \hookrightarrow \mathbb{Q}_p\) such that \(\phi(a_\ell) = a_\ell(f_\phi)\) for all primes \(\ell \neq p\) and \(\phi(a_p)\) is the unit root of the polynomial \(x^2 - a_p(f) + p^{k' - 1}\);

- there is a continuous \(\mathcal{O}\)-algebra homomorphism \(\phi_0 : R \to \mathcal{O}\) such that \(\phi_0(1+X) = (1+p)^k\) and \(\phi_0(a_\ell) = a_\ell(f), \ell \neq p\), and \(\phi_0(a_p)\) is the unit root of \(x^2 - a_p(f) + p^{k' - 1}\) if \(r = 0\) and \(\phi_0(a_p) = a_p(f)\) if \(r = 1\).

Furthermore, after possibly replacing \(L\) with a finite extension, we may assume

- \(\mathcal{O}\) is integrally closed in \(R\).

Then, as explained by Greenberg and Stevens [11] (see also [15, (1.4.7)]),

- there is an integer \(c\) and an \(\mathcal{O}\)-algebra embedding

\[
R \hookrightarrow R_c = \left\{ \sum_{i=0}^{\infty} u_i(x - k)^i : u_i \in L, \lim_{i \to \infty} \text{ord}_p(u_i) + ci = +\infty \right\} \subset L[x]
\]

such that the induced embedding of \(R_0\) sends \(1+X\) to the power series expansion of \((1+p)^x\) about \(x = k\) and \(\phi_0\) is the homomorphism induced by evaluating at \(x = k\).

Then evaluating at \(x = k'\) for an integer \(k' > 2\) with \(k' \equiv k \pmod{(p-1)p^c}\) defines a continuous \(\mathcal{O}\)-algebra homomorphism \(\phi_{k'} : R \to L\) such that \(\phi_{k'}(1+X) = (1+p)^{k'}\) with corresponding newform \(f_{\phi_{k'}} \in S_{k'}(\Gamma_0(M))\). Furthermore, it is clear that given any integer \(m > 0\), there is an integer \(r_m > 0\) such that if \(k' \equiv k \pmod{(p-1)p^m}\), then \(\phi_{k'} \equiv \phi_0 \pmod{p^m\mathcal{O}}\); in particular, for all primes \(\ell \neq p\)

\[
a_\ell(f_{\phi_{k'}}) \equiv a_\ell(f) \pmod{p^m\mathcal{O}}.
\]

For each integer \(m\) we choose such a \(k' = k_m\) and write \(f_m\) for the corresponding \(f_{\phi_{k_m}}\). Note that we have chosen \(k_m > 2\) so that \(f_m\) is a newform of level not divisible by \(p\), though \(p\) might divide the level of \(f\).

Suppose that \(\bar{\rho}_f\) is irreducible. Then there is a free rank two \(R\)-module \(T\) and a continuous Galois representation

\[
\rho_R : G_\mathbb{Q} \to \text{Aut}_R(T)
\]
that is unramified at each $\ell \nmid pN$ and such that for any such prime trace $\rho_R(\text{frob}_\ell) = a_\ell \in R$. In particular for $\phi : R \to O$ being $\phi_0$ or one of the homomorphisms $\phi_k$, $T_{f_\phi} = \mathbb{T} \otimes_{R,\phi} O$ is a $G_{\mathbb{Q}}$-stable $O$-lattice in $\mathbb{T} \otimes_{R,\phi} L \cong V_{f_\phi}$. Let $T_f = T_{f_{\phi_0}}$ and $T_m = T_{f_{\phi_m}}$. Since $\phi_0$ and $\phi_m$ agree modulo $p^m$, reduction modulo $p^m$ induces identifications
\begin{equation}
T_f/p^mT_f = \mathbb{T} \otimes_{R,\phi_0} O/p^mO = \mathbb{T} \otimes_{R,\phi_m} O/p^mO = T_f/p^mT_m
\end{equation}
as $O[G_{\mathbb{Q}}]$-modules.

Suppose also that
\[ \alpha_f^{-1} \epsilon_{k-1} \neq \alpha_f \pmod{m}. \]
Then there is a free rank-one $G_{\mathbb{Q}_p}$-stable $R$-summand $\mathbb{T}^+ \subset \mathbb{T}$ such that for any of the $\phi$ as before, $\mathbb{T}^+ \otimes_{R,\phi} O = T_{f_{\phi}}^+$. The identification $T_f/p^mT_f = T_f/p^mT_f$ induces an identification
\begin{equation}
T_f^+/p^mT_f^+ = T_m^+/p^mT_m^+.
\end{equation}

Greenberg and Stevens [11] and others have shown that the $p$-adic $L$-functions $L_{f_\phi}$ for the forms $f_\phi$ arising from a Hida family fit into a `two-variable’ $p$-adic $L$-function. In particular, following Emerton, Pollack, and Weston, we have the following.

**Proposition 2.6.1.** ([6, §3 esp. Prop. 3.4.3]) Let $\Sigma$ be a finite set of primes containing $p$. If $\hat{\rho}_f$ is irreducible, then there exists $L_{f_\phi}^\Sigma \in R[\Gamma]$ such that for each continuous $O$-algebra homomorphism $\phi : R \to \overline{\mathbb{Q}}_p$ as above, the image of $L_{f_\phi}^\Sigma$ in $R[\Gamma] \otimes_{R,\phi} \phi(R)' = \Lambda_{\phi(R)'}$ is a multiple of the $p$-adic $L$-function $L_{f_\phi}^\Sigma$ by a unit in $\phi(R)'$.

Here $\phi(R)'$ is the integral closure of $\phi(R)$ in its field of fractions (which is a finite extension of $L$). In particular, as $\phi_k(R) = O$, the image of $L_{f_\phi}^\Sigma$ in $R[\Gamma] \otimes_{R,\phi_k} O = \Lambda_O$ is just $u_mL_{f_m}^{\Sigma}$, for some $u_m \in O^\times$. Assuming that $\hat{\rho}_f$ is irreducible, for each $m$ we then have an equality of $\Lambda_O$-ideals
\begin{equation}
(L_{f_m}^{\Sigma}, p^m) = (L_{f_m}^{\Sigma}, p^m) \subseteq \Lambda_O.
\end{equation}

3. **Assembling the pieces**

We can now put together the various objects and results from Section 2 to prove Theorems A and B as indicated in the introduction. We will freely use the notation introduced in Section 2.

### 3.1. Proof of Theorem A

Let $f$, $L$, $O$ be as in the statement of Theorem A. In particular, $f \in S_k(N)$ is a newform of some weight $k \geq 2$ that is congruent to 2 modulo $p - 1$ and some level $N$. Furthermore, if $f = \sum_{n=1}^{\infty} a_n(f)q^n$ is the $q$-expansion of $f$, then $a_p(f) \in O^\times$. If $p \nmid N$, then by Theorem 2.5.2 the Iwasawa–Greenberg Main Conjecture is true: for any finite set of primes $\Sigma$ containing $p$, $\text{Ch}_{\Sigma,f}^\Sigma(f) = (L_{f,\Sigma}^\Sigma)$ in $\Lambda_O$. So we assume that $p \mid N$. By Lemma 2.1.2 we then have $N = pM$ with $p \nmid M$ and $k = 2$. Let $T_f \subset V_f$
be a $G_{\mathbb{Q}}$-stable $\mathcal{O}$-lattice. By Lemma 2.1.1 this lattice is unique up to $L^\times$-multiple since $\bar{\rho}_f$ is assumed irreducible.

Let $\Sigma \supset \{\ell \mid N\}$ be a finite set of primes. After possibly replacing $L$ with a finite extension, for each integer $m > 0$ there exists

(a) a newform $f_m \in S_{k_m}(\Gamma_0(M))$ with $\mathbb{Q}(f_m) \subset L$, $k_m > 2$, and $k_m \equiv 2 \pmod{p - 1}$ and such that $a_p(f_m) \in \mathcal{O}^\times$;
(b) a $G_{\mathbb{Q}}$-stable $\mathcal{O}$-lattice $T_{f_m} \subset V_{f_m}$ and an isomorphism $T_f/p^mT_f \cong T_{f_m}/p^mT_{f_m}$ as $\mathcal{O}[G_{\mathbb{Q}}]$-modules that identifies $T_f^+/p^mT_f^+$ with $T_{f_m}^+/p^mT_{f_m}^+$ as $\mathcal{O}[G_{\mathbb{Q}_p}]$-modules;
(c) an equality of ideals $(\mathcal{L}_f^\Sigma, p^m) = (\mathcal{L}_{f_m}^\Sigma, p^m) \subset \Lambda_{\mathcal{O}}$.

The forms $f_m$ in (a) are just those defined in the discussion of Hida families in Section 2.6. Then (b) is just (2.6.1) and (2.6.2), and (c) is (2.6.3). Furthermore, we also have

(d) $\bar{\rho}_{f_m} \cong \bar{\rho}_f$ is irreducible and ramified at some $q \neq p$ such that $q \mid M$;
(e) $X_{Q\infty, L}^\Sigma(f_m)$ is a torsion $\Lambda_{\mathcal{O}}$-module and $Ch_{\mathcal{O}}^\Sigma(f_m) = (\mathcal{L}_{f_m}^\Sigma) \subset \Lambda_{\mathcal{O}}$.
(f) $X_{Q\infty, L}^\Sigma(f_m)$ has no nonzero finite-order $\Lambda_{\mathcal{O}}$-submodules, so $F_{\mathcal{O}}^\Sigma(f_m) = Ch_{\mathcal{O}}^\Sigma(f_m)$.

Note that (d) follows from (b) and the hypotheses on $N$ and $\bar{\rho}_f$ in Theorem A, while (e) and (f) follow from the Iwasawa–Greenberg Main Conjecture for $f_m$ (which holds by (a), (d), and Theorem 2.5.2 since $f_m$ is of level $M$ and $p \mid M$) together with Proposition 2.3.3 and Lemma 2.3.4.

From (b) together with Lemma 2.3.1 we conclude that there is a $\Lambda_{\mathcal{O}}$-isomorphism

$$\text{Sel}_{\mathcal{O}}^\Sigma_{\infty, L}(f)[p^m] \cong \text{Sel}_{\mathcal{O}}^\Sigma_{\infty, L}(f)[p^m]$$

of $\Lambda_{\mathcal{O}}$-modules, and hence, upon taking Pontryagin duals, also a $\Lambda_{\mathcal{O}}$-isomorphism

$$X_{Q\infty, L}^\Sigma(f)/p^m X_{Q\infty, L}^\Sigma(f) \cong X_{Q\infty, L}(f_m)/p^m X_{Q\infty, L}(f_m).$$

From basic properties of Fitting ideals we then conclude that there as an equality of $\Lambda_{\mathcal{O}}$-ideals

$$(F_{\mathcal{O}}^\Sigma(f), p^m) = (F_{\mathcal{O}}^\Sigma(f_m), p^m).$$

Together with (c), (e), and (f) we then have

(3.1.1) $$(F_{\mathcal{O}}^\Sigma(f), p^m) = (\mathcal{L}_f^\Sigma, p^m) \subset \Lambda_{\mathcal{O}}.$$ \hspace{1cm}

As $\mathcal{L}_f$, and hence $\mathcal{L}_f^\Sigma$, is non-zero by a well-known theorem of Rohrlich [16, Thm. 1], if $m$ is large enough then $(\mathcal{L}_f^\Sigma, p^m) \neq p^m \Lambda_{\mathcal{O}}$. From this and (3.1.1) it then follows that if $m$ is large enough, then $(F_{\mathcal{O}}^\Sigma(f), p^m) \neq p^m \Lambda_{\mathcal{O}}$ and hence $F_{\mathcal{O}}^\Sigma(f) \neq 0$. As $F_{\mathcal{O}}^\Sigma(f) \neq 0$, $X_{Q\infty, L}^\Sigma(f)$ must be a torsion $\Lambda_{\mathcal{O}}$-module. It then follows from Proposition 2.3.3(ii) and Lemma 2.3.4(ii) that $Ch_{\mathcal{O}}^\Sigma(f) = F_{\mathcal{O}}^\Sigma(f)$. Combining this with (3.1.1) we then conclude that for all integers $m$

(3.1.2) $$(Ch_{\mathcal{O}}^\Sigma(f), p^m) = (\mathcal{L}_f^\Sigma, p^m) \subset \Lambda_{\mathcal{O}}.$$
The characteristic ideal \(Ch^\Sigma_L(f)\) is a principal ideal. Let \(C^\Sigma_f\) be a generator. From (3.1.2) it follows that for each integer \(m\) there is an \(u_m \in \Lambda_\mathcal{O}\) such that

(3.1.3) \[ C^\Sigma_f - u_m L^\Sigma_f \in p^m \Lambda_\mathcal{O}. \]

Let \(\varpi\) be a uniformizer of \(\mathcal{O}\) and let \(\epsilon\) be such that \((\varpi) = (\varpi^\epsilon)\). As \(L^\Sigma_f \neq 0\), there exists an integer \(m_0 \geq 0\) such that \(L^\Sigma_f(f) \in \varpi^{m_0} \Lambda_\mathcal{O}\) but \(L^\Sigma_f(f) \notin \varpi^{m_0 + 1} \Lambda_\mathcal{O}\). It then follows from (3.1.3) that

\[ u_{m'} - u_m \in \varpi^{me-m_0} \Lambda_\mathcal{O}, \quad m' \geq m. \]

Therefore the sequence \(\{u_m\}\) converges in \(\Lambda_\mathcal{O}\) to an element \(u \in \Lambda_\mathcal{O}\) such that for all \(m\) \(u - u_m \in \varpi^{me-m_0} \Lambda_\mathcal{O}\). From this and (3.1.3) it follows that

\[ C^\Sigma_f - u L^\Sigma_f \in \varpi^{me-m_0} \quad \forall m \geq 0, \]

whence \(C^\Sigma_f = u L^\Sigma_f\). That is \(C^\Sigma_f \in (L^\Sigma_f)\).

Since \(X^\Sigma_{Q_\infty,L}(f)\) is a torsion \(\Lambda_\mathcal{O}\)-module, \(Ch^\Sigma_L(f)\) is non-zero, and so \(C^\Sigma_f \neq 0\). We may then reverse the roles of \(C^\Sigma_f\) and \(L^\Sigma_f\) in the above argument to show that \(L^\Sigma_f \in (C^\Sigma_f)\). From the two inclusions we then conclude

\[ (L^\Sigma_f) = (C^\Sigma_f) = Ch^\Sigma_L \subseteq \Lambda_\mathcal{O}. \]

This proves the desired equality, at least for the chosen \(L\) and for \(\Sigma\) containing all primes \(\ell \mid N\). But, as observed in Section 2.5, this implies the desired equality for all sets \(\Sigma\) and all possible \(L\). That is, the Iwasawa–Greenberg Main Conjecture holds for \(f\): Theorem 2.5.2 holds without hypothesis (iv).

3.2. Proof of Theorem B. Let \(f, L, \mathcal{O}\) be as in the statement of Theorem B. As these also satisfy the hypotheses of Theorem A, \(X_{Q_\infty,L}(f)\) is a torsion \(\Lambda_\mathcal{O}\)-module and its \(\Lambda_\mathcal{O}\)-characteristic ideal \(Ch_L(f)\) is generated by the \(p\)-adic \(L\)-function \(L_f\). Furthermore, by Proposition 2.3.3, neither \(X_{Q_\infty,L}(f)\) nor \(\mathcal{X}\) have a nonzero finite-order \(\Lambda_\mathcal{O}\)-submodule. To deduce the conclusions of Theorem B from this, we make a close study of \(Sel_{Q_\infty}(f)[\gamma - 1]\) and \(S[\gamma - 1]\), following the methods of Greenberg [8].

By Proposition 2.3.2 there is an exact sequence

\[ 0 \to S \to Sel_{Q_\infty,L}(f) \to H^1(F_p, (\mathcal{M}^-)^{1_p}) \to 0. \]

As \(G_{\mathbb{Q}_p}\) acts on \(\mathcal{M}^- \cong \Lambda^*\) through the character \(\alpha_f \Psi^{-1}\), \((\mathcal{M}^-)^{1_p} \cong \Lambda^*[\gamma - 1] = Hom_{\mathbb{Z}_p}(\mathcal{O}, \mathbb{Q}_p/\mathbb{Z}_p) \cong L/\mathcal{O}\) with \(G_{\mathbb{Q}_p}\) acting through the unramified character \(\alpha_f\). Let

\[ \alpha_p = \alpha_f(f^{1_p}). \]

Then \(H^1(F_p, (\mathcal{M}^-)^{1_p}) = 0\) unless \(\alpha_p = 1\) (i.e., unless \(f\) has split multiplicative reduction at \(p\)), in which case it is isomorphic to \(L/\mathcal{O}\). Letting \(Ch_L(f)'\) be the \(\Lambda_\mathcal{O}\)-characteristic ideal of \(\mathcal{X}\), it follows that

\[ Ch_L(f) = Ch_L(f)' \cdot \begin{cases} (\gamma - 1) & \text{if } f \text{ has split multiplicative reduction at } p \\ 1 & \text{otherwise}. \end{cases} \]
This reflects the ‘extra zero’ phenomenon in the split multiplicative case observed at the end of Section 2.4. In fact, we then have

\[ Ch_L(f)' = \begin{cases} (L'_f) & \text{if } f \text{ has split multiplicative reduction at } p \\ (L_f) & \text{otherwise.} \end{cases} \]

As \( X \) has no non-zero finite-order \( \Lambda_O \)-submodules, a standard result\(^7\) in Iwasawa theory gives \( \#X/(\gamma - 1)X = \#\Lambda_O/(\gamma - 1, Ch_L(f)') \). As \( \#S[\gamma - 1] = \#X/(\gamma - 1)X \), we then find

\[ (3.2.1) \quad \#S[\gamma - 1] = \begin{cases} \#O/(L'_f(\phi_0)) & \text{if } f \text{ has split multiplicative reduction at } p \\ \#O/(L_f(\phi_0)) & \text{otherwise,} \end{cases} \]

where \( \phi_0 : \Lambda_O \to O \) is the continuous \( O \)-algebra homomorphism sending \( \gamma \) to 1.

Let \( \Sigma = \{ \ell \mid Np \} \). Let

\[ W = M[\gamma - 1] \cong T_f \otimes \mathbb{Z}_p \mathbb{Q}_p / \mathbb{Z}_p \text{ and } W^\pm = M[\gamma - 1]^\pm \cong T_f^\pm \otimes \mathbb{Z}_p \mathbb{Q}_p / \mathbb{Z}_p. \]

Let

\[ P_S = H^1(\mathbb{Q}_p, M^-) \times \prod_{\ell \in \Sigma, \ell \neq p} H^1(\mathbb{Q}_\ell, M) \]

and

\[ P = H^1(\mathbb{Q}_p, W)/L_p(W) \times \prod_{\ell \in \Sigma, \ell \neq p} H^1(\mathbb{Q}_\ell, W), \]

where \( L_p(W) = \text{im}\{H^1(\mathbb{Q}_p, W^+) \to H^1(\mathbb{Q}_p, W)\} \). Let \( P^{\text{div}}_S \) be defined just as \( P_S \) but with \( L_p(W) \) replaced by its maximal divisible subgroup \( L_p(W)^{\text{div}} \). The usual (torsion) Bloch-Kato Selmer group for \( T_f \) is just

\[ \text{Sel}_L(f) = \ker\{H^1(G_S, W) \to P^{\text{div}}_S\}. \]

As the the restriction map \( H^1(G_S, M) \to P_S \) is surjective by Proposition 2.3.2, we conclude that there is a short exact sequence

\[ 0 \to \text{Sel}_L(f) \to S[\gamma - 1] \to \text{im}\{H^1(G_S, W) \to P^{\text{div}}_S\} \cap \ker\{P^{\text{div}}_S \to P_S[\gamma - 1]\} \to 0. \]

Let \( K = \ker\{P^{\text{div}}_S \to P_S[\gamma - 1]\} \). We claim that

\[ (3.2.2) \quad \#S[\gamma - 1] = \#\text{Sel}_L(f) \cdot \#K. \]

If \( \text{Sel}_L(f) \) is infinite, then there is nothing to prove since \( \text{Sel}_L(f) \subset S[\gamma - 1] \). Suppose then that \( \text{Sel}_L(f) \) is finite. We will show that the restriction map \( H^1(G_S, W) \to P^{\text{div}}_S \) is surjective, from which the claim follows.

By global duality, the cokernel of the restriction map \( H^1(G_S, W) \to P^{\text{div}}_S \) is dual to a subquotient of

\[ \text{Sel}^\Sigma(T_f)^{\text{sat}} = \ker\{H^1(G_S, T_f) \to H^1(\mathbb{Q}_p, T_f) / L_p(T_f)^{\text{sat}}\}, \]

\(^7\)See footnote 5.
where \( L_p(T_f) = \text{im}\{H^1(\mathbb{Q}_p, T_f^+) \to H^1(\mathbb{Q}_p, T_f)\} \) and
\[
L_p(T_f)^{\text{sat}} = \{ x \in H^1(\mathbb{Q}_p, T_f) : p^n x \in L_p(T_f) \text{ some } n \geq 0 \}.
\]
Here we have used that \( T_f \cong \text{Hom}_{\mathbb{Z}_p}(W, \mathbb{Q}_p/\mathbb{Z}_p(1)) \) as an \( \mathcal{O}[G_{\mathbb{Q}}] \)-module and that such an isomorphism identifies \( L_p(T_f)^{\text{sat}} \) and \( L_p(W)^{\text{div}} \) as mutual annihilators under local Tate duality. Then \( \text{Sel}^\Sigma(T_f)^{\text{sat}} \) is a torsion-free \( \mathcal{O} \)-module (as \( \tilde{\rho}_f \) is irreducible) and its \( \mathcal{O} \)-rank equals the \( \mathcal{O} \)-corank of \( \text{Sel}_L(f) \). In fact, \( \text{Sel}^\Sigma(T_f)^{\text{sat}} = H^1(G_{\Sigma}, T_f) \cap H^1_f(\mathbb{Q}, V_f) \), where
\[
H^1_f(\mathbb{Q}, V_f) = \ker\{H^1(G_{\Sigma}, V_f) \to H^1(\mathbb{Q}_p, V_f)/L_p(V_f) \times \prod_{\ell \in \Sigma, \ell \neq p} H^1(\mathbb{Q}_\ell, V_f)\}
\]
and \( L_p(V_f) = \text{im}\{H^1(\mathbb{Q}_p, V_f^+) \to H^1(\mathbb{Q}_p, V_f)\} \). (So \( H^1_f(\mathbb{Q}, V_f) \) is just the usual characteristic zero Bloch-Kato Selmer group of \( V_f \).) In particular, the \( \mathcal{O} \)-rank of \( \text{Sel}^\Sigma(T_f)^{\text{sat}} \) is the \( L \)-dimension of \( H^1_f(\mathbb{Q}, V_f) \). The image of \( H^1_f(\mathbb{Q}, V_f) \) in \( H^1(G_{\Sigma}, T_f \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p) \cong H^1(G_{\Sigma}, W) \) is the maximal divisible submodule of \( \text{Sel}_L(f) \). However, the latter is assumed to be of finite-order, so its maximal divisible subgroup is trivial. This proves that \( \text{Sel}^\Sigma(T_f)^{\text{sat}} = 0 \) and hence that the restriction map \( H^1(G_{\Sigma}, W) \to \mathcal{P}^{\text{div}}_{\Sigma,x} \) is a surjection. The equality (3.2.2) follows.

Put
\[
L^{\text{alg}}(f, 1) = \frac{L(f, 1)}{-2\pi i \Omega_f}.
\]
Combining (3.2.1) with (3.2.2), the specialization formula for \( \mathcal{L}_f \), and the Greenberg-Stevens formula (2.4.3) yields
\[
\#\text{Sel}_L(f) \cdot \#K = \begin{cases} 
\#\mathcal{O}/(\mathcal{O}_{\text{alg}} \cdot \mathcal{L}(V_f) \cdot L^{\text{alg}}(f, 1)) & \alpha_p = 1 \\
\#\mathcal{O}/((1 - \alpha_p)^2 \cdot L^{\text{alg}}(f, 1)) & \text{otherwise}
\end{cases}
\]
Therefore, to complete the proof Theorem B it remains to express \( \#K \) in terms of Tamagawa factors and the \( L \)-invariant \( \mathcal{L}(V_f) \).

From the definition of \( K \),
\[
K = \prod_{\ell \in \Sigma} K_\ell
\]
with
\[
K_\ell = \begin{cases} 
\ker\{H^1(\mathbb{Q}_\ell, W) \to H^1(\mathbb{Q}_\ell, \mathcal{M})\} & \ell \neq p \\
\ker\{H^1(\mathbb{Q}_p, W)/L_p(W)^{\text{div}} \to H^1(\mathbb{Q}_p, \mathcal{M}^-)\} & \ell = p.
\end{cases}
\]
If \( \ell \neq p \), then \( \mathcal{M}^{\ell \ell} \) is \( \gamma - 1 \)-divisible and so \( H^1(I_{\ell \ell}, W) \hookrightarrow H^1(I_{\ell \ell}, \mathcal{M}) \) and
\[
K_\ell = \ker\{H^1(\mathbb{F}_\ell, W^{I_{\ell \ell}}) \to H^1(\mathbb{F}_\ell, \mathcal{M}^{I_{\ell \ell}}) = 0\} = H^1(\mathbb{F}_\ell, W^{I_{\ell \ell}}).
\]
Therefore
\[
\#K_\ell = \#H^1(\mathbb{F}_\ell, W^{I_{\ell \ell}}) = c_\ell(T_f),
\]
where \( c_\ell(T_f) = \# H^1(F_\ell, W^\ell) \) is just the Tamagawa number at \( \ell \neq p \) defined by Bloch and Kato for the \( p \)-adic representation \( T_f \). Note that \( c_\ell(T_f) = 1 \) if \( \ell \nmid N \) (i.e., if \( T_f \) is unramified at \( \ell \)). Hence to complete the proof of Theorem B it remains to express \( #K_p \) in terms of \( \alpha_p \), if \( f \) does not have split multiplicative reduction at \( p \) (equivalently \( \alpha_p \neq 1 \)) and in terms of \( L(V_f) \) and the Tamagawa number at \( p \) of \( T_f \) otherwise.

Let
\[
c'_p = \# \ker\{H^1(\mathbb{Q}_p, W)/L_p(W)_{\text{div}} \rightarrow H^1(\mathbb{Q}_p, W)/L_p(W)\}
\]
and
\[
c''_p = \# \ker\{H^1(\mathbb{Q}_p, W)/L_p(W) \rightarrow H^1(\mathbb{Q}_p, M)/L_p(M)\}.
\]
Then
\[
#K_p = c'_p c''_p.
\]

By Tate local duality, \( L_p(W) \) is dual to \( H^1(\mathbb{Q}_p, T_f)/L_p(T_f) \) and \( L_p(W)_{\text{div}} \) is dual to \( H^1(\mathbb{Q}_p, T_f)/L_p(T_f)^{\text{sat}} \). Therefore
\[
c'_p = \#(L_p(W)/L_p(W)_{\text{div}}) = \#(L_p(T_f)^{\text{sat}}/L_p(T_f)).
\]
Since \( H^1(\mathbb{Q}_p, T_f)/L_p(T_f) \hookrightarrow H^1(\mathbb{Q}_p, T_f^-) \) and \( H^1(\mathbb{Q}_p, T_f)/L_p(T_f)^{\text{sat}} \hookrightarrow H^1(\mathbb{Q}_p, V_f^-) \), we find that the (injective) image of \( L_p(T_f)^{\text{sat}}/L_p(T_f) \) in \( H^1(\mathbb{Q}_p, T_f^-) \) is just
\[
\text{im}(H^1(\mathbb{Q}_p, T_f)/L_p(T_f) \hookrightarrow H^1(\mathbb{Q}_p, T_f^-)) \cap \ker\{H^1(\mathbb{Q}_p, T_f^-) \rightarrow H^1(\mathbb{Q}_p, V_f^-)\}.
\]
But \( H^1(I_p, T_f^-) \hookrightarrow H^1(I_p, V_f^-) \), so \( \ker\{H^1(\mathbb{Q}_p, T_f^-) \rightarrow H^1(\mathbb{Q}_p, V_f^-)\} = H^1(\mathbb{F}_p, T_f^-) \). On the other hand, the boundary map injects the cokernel of \( H^1(\mathbb{Q}_p, T_f)/L_p(T_f) \hookrightarrow H^1(\mathbb{Q}_p, T_f^-) \) into \( H^2(\mathbb{Q}_p, T_f^+) \) but sends the the subgroup \( H^1(\mathbb{F}_p, T_f^-) \) to zero (since \( \text{Gal}(\mathbb{F}_p/\mathbb{F}_p) \) has cohomological dimension one). Hence the image of \( H^1(\mathbb{Q}_p, T_f)/L_p(T_f) \hookrightarrow H^1(\mathbb{Q}_p, T_f^-) \) contains \( H^1(\mathbb{F}_p, T_f^-) \). It then follows that
\[
L_p(T_f)^{\text{sat}}/L_p(T_f) \simeq H^1(\mathbb{F}_p, T_f^-) \cong \begin{cases} 0 & \alpha_p = 1 \\ \mathcal{O}/(\alpha_p - 1) & \text{otherwise.} \end{cases}
\]
In particular,
\[
c'_p = \begin{cases} 1 & \alpha_p = 1 \\ \#(\mathcal{O}/(\alpha_p - 1)) & \text{otherwise.} \end{cases}
\]

It remains to deduce the desired expression for \( c''_p \). By definition \( c''_p \) equals
\[
\#(\text{im}(H^1(\mathbb{Q}_p, W)/L_p(W) \hookrightarrow H^1(\mathbb{Q}_p, W^-)) \cap \ker\{H^1(\mathbb{Q}_p, W^-) \rightarrow H^1(\mathbb{Q}_p, M^-)\}) .
\]
Since \( H^2(\mathbb{Q}_p, W^+) \) is dual to \( H^0(\mathbb{Q}_p, T_f^-) \) and the latter is 0 if \( \alpha_p \neq 1 \), we have \( H^1(\mathbb{Q}_p, W)/L_p(W) \simeq H^1(\mathbb{Q}_p, W^-) \) if \( \alpha_p = 1 \). It follows that in this case
\[
c''_p = \# \ker\{H^1(\mathbb{Q}_p, W^-) \rightarrow H^1(\mathbb{Q}_p, M^-)\} = \#(M^-)^{G_\mathbb{Q}_p}/(\gamma - 1) \cdot (M^-)^{G_\mathbb{Q}_p}.
\]
As \( I_p \) acts on \( M^- \) through the character \( \Psi^{-1} \) and \( \text{frob}_p \) acts on \( (M^-)^{I_p} = M^-[\gamma - 1] \cong L/\mathcal{O} \) as multiplication by \( \alpha_p \), we find
\[
c''_p = \#L/\mathcal{O}[\alpha_p - 1] = \#\mathcal{O}/(\alpha_p - 1), \quad \alpha_p \neq 1.
\]
Suppose then that \( \alpha_p = 1 \). It follows from local duality that \( c_p \) equals the index of the \( \mathcal{O} \)-submodule of \( H^1(\mathbb{Q}_p, T_f^+) \) generated by \( \ker\{H^1(\mathbb{Q}_p, T_f^+) \to H^1(\mathbb{Q}_p, T_f)\} \) and the annihilator of \( \ker\{H^1(\mathbb{Q}_p, W^-) \to H^1(\mathbb{Q}_p, \mathcal{M}^-)\} \). The first is just the image of \( \mathcal{O} \cong H^0(\mathbb{Q}_p, T_f^-) \to H^1(\mathbb{Q}_p, T_f^+) \) determined by the \( G_{\mathbb{Q}_p} \)-extension \( T_f \). Let \( c_{T_f} \) be an \( \mathcal{O} \)-generator; this is a non-zero element in \( \ell_{T_f} \) in the notation of Section 2.2. On the other hand, as \( H^1(\mathbb{Q}_p, W^-) \cong \text{Hom}_{\text{cts}}(G_{\mathbb{Q}_p}, L/\mathcal{O}) \), the kernel \( \ker\{H^1(\mathbb{Q}_p, W^-) \to H^1(\mathbb{Q}_p, \mathcal{M}^-)\} \) is readily seen to be \( \text{Hom}_{\text{cts}}(\Gamma, L/\mathcal{O}) \) - those homomorphisms that factor through \( \Gamma \).

Then, under the identification
\[
H^1(\mathbb{Q}_p, T_f^+) = H^1(\mathbb{Q}_p, \mathcal{O}(1)) = (\varprojlim_n \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^{p^\alpha}) \otimes_{\mathbb{Z}_p} \mathcal{O},
\]
the annihilator of \( \text{Hom}_{\text{cts}}(\Gamma, L/\mathcal{O}) \) is identified with the \( \mathcal{O} \)-module \( p \otimes \mathcal{O} \) generated by the image of \( p^{\mathbb{Z}} \). The index of \( \mathcal{O} \cdot c_{T_f} + p \otimes \mathcal{O} \) is just the index of the projection of \( c_{T_f} \) to \((\varprojlim_n \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^{p^\alpha}) \otimes_{\mathbb{Z}_p} \mathcal{O} \). From the definition of \( \psi_{\text{cyc}} \) in Section 2.2, this index is just \( \#\mathcal{O}/(\ell_{T_f} \cdot \psi_{\text{cyc}}(c_{T_f})) \). So by the definition of \( \psi_{\text{ur}} \) (which is non-zero on \( c_{T_f} \) as \( 0 \neq c_{T_f} \in \ell_{T_f} \)) and the definition of \( \mathfrak{L}(V_f) \),
\[
c_p'' = \#\mathcal{O}/(1/\log_p u \cdot \psi_{\text{cyc}}(c_{T_f})) = \#\mathcal{O}/(1/\log_p u \cdot \mathfrak{L}(V_f) \cdot \psi_{\text{ur}}(c_{T_f})), \quad \alpha_p = 1.
\]

Combining the formulas for \( c_p'' \) in the two cases with those for \( c_p \) we find
\[
\#K_p = \begin{cases} \#\mathcal{O}/(1/\log_p u \cdot \mathfrak{L}(V_f) \cdot \psi_{\text{ur}}(c_{T_f})) & \alpha_p = 1 \\ \#\mathcal{O}/(\alpha_p - 1)^2 & \alpha_p \neq 1. \end{cases}
\]

Suppose \( \mathfrak{L}(V_f) \neq 0 \) if \( \alpha_p = 1 \). Then combining (3.2.3) with (3.2.4), (3.2.5), and (3.2.6) yields
\[
\#\mathcal{O}/(L^\text{alg}(f, 1)) = \#\text{Sel}_L(f) \cdot \prod_{\ell \neq p} c_{\ell}(T_f) \cdot \begin{cases} \#\mathcal{O}/(\psi_{\text{ur}}(c_{T_f})) & \alpha_p = 1 \\ 1 & \alpha_p \neq 1. \end{cases}
\]

That the final term is just the Bloch-Kato Tamagawa number at \( p \) of the representation \( T_f \), which we denote \( \text{c}_p(T_f) \), can be shown as in [5]; in loc. cit. \( \text{c}_p(T_f) \) is denoted \( \text{Tam}_M^0(T_f) \). The only significant change is the need to include the \( \mathcal{O} \)-action, but this is a straightforward modification. In the \( p \nmid N \) case - that is, the case where \( V_f \) is a crystalline representation of \( G_{\mathbb{Q}_p} \) - that \( \text{c}_p(T_f) = 1 \) follows by the arguments used to prove [5, Thm. 5.1]. The \( p \mid N \) case - in which case \( V_f \) is a semistable representation of \( G_{\mathbb{Q}_p} \) follows as in [5, §7] from the arguments used to prove [5, Thm. 6.1]. We therefore have the formula asserted in Theorem B:
\[
\#\mathcal{O}/(L^\text{alg}(f, 1)) = \#\text{Sel}_L(f) \cdot \prod_{\ell} c_{\ell}(T_f).
\]

This completes the proof of Theorem B.
3.3. **Proof of Theorem C.** Theorem C is just a special case of Theorem B. To see this, let $E$ be as in Theorem C and let $f \in S_2(\Gamma_0(N))$ be the newform associated with $E$, so $N$ is the conductor of $E$ and $L(E, s) = L(f, s)$. For Theorem C to follow from Theorem B, if suffices to have that under the hypotheses of Theorem C, hypotheses (i), (ii), and (iii) of Theorem B hold for $f$ and $\Omega_E$ is a $\ZZ_p$-multiple of $-2\pi \Omega_f^+$. That hypotheses (i) and (ii) of Theorem C imply hypotheses (i) and (ii) of Theorem B is immediate. Furthermore, as noted in the example at the end of 2.2, if $E$ has split multiplicative reduction at $p$ then the $L$-invariant $L(V_f)$ of $f$ is non-zero, hence hypothesis (iii) of Theorem C also holds.

To compare periods, we first recall that if $\omega_E$ is a Néron differential of $E$ then

$$\Omega_E = \int c^+ \omega_E \in \CC^\times,$$

where $c^+$ is a generator of the submodule $H_1(E(\CC), \ZZ)^+ \subset H_1(E(\CC), \ZZ)$ that is fixed by the action of Gal($\CC/\RR$); this is well-defined up to multiplication by $\pm 1$. Now let

$$\phi : X_1(N) \to E^{\text{opt}}$$

be an optimal parameterization for the $\QQ$-isogeny class of $E$ as in [19, Prop. (1.4)]. Then, as demonstrated in the proof of [12, Prop. (3.1)], $\Omega_{E^{\text{opt}}}$ equals $-2\pi i \Omega_f^+$ up to a $\ZZ_p^\times$-multiple. Let

$$\beta : E^{\text{opt}} \to E$$

be a $\QQ$-isogeny. Since $E[p]$ is an irreducible $G_{\QQ}$-representation, $\beta$ can be chosen so that its degree is prime to $p$. Then $\beta^* \omega_E$ is a $\ZZ_p^\times$-multiple of $\omega_{E^{\text{opt}}}$, and so $\Omega_E$ is a $\ZZ_p^\times$-multiple of $\Omega_{E^{\text{opt}}}$ and hence also of $-2\pi i \Omega_f^+$.

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8The key points are [12, Prop. (3.3)], which shows that if $\text{ord}_p(N) \leq 1$ then $\phi^* \omega_{E^{\text{opt}}} = c \cdot 2\pi i f(z)dz$ for some integer $c \in \ZZ$ such that $p \nmid c$, and the fact - by the definition of an optimal parameterization - that $\phi$ induces a surjection $H_1(X_1(N)(\CC), \ZZ) \to H_1(E^{\text{opt}}(\CC), \ZZ)$. 

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