HIGHER DEGENERATE HARMONIC MEAN CURVATURE FLOW

M.C. CAPUTO* AND P. DASKALOPOULOS**

ABSTRACT. We study the evolution of a weakly convex surface Σ₀ in R³ with flat sides by the Harmonic Mean Curvature flow. We establish the short time existence as well as the optimal regularity of the surface and we show that the boundaries of the flat sides evolve by the curve shortening flow. It follows from our results that a weakly convex surface with flat sides of class Cᵏ,γ, for some k ∈ N and 0 < γ ≤ 1, remains in the same class under the flow. This distinguishes this flow from other, previously studied, degenerate parabolic equations, including the porous medium equation and the Gauss curvature flow with flat sides, where the regularity of the solution for t > 0 does not depend on the regularity of the initial data.

1. INTRODUCTION

We consider the motion of a compact, weakly convex two-dimensional surface Σ₀ in space R³ under the harmonic mean curvature flow (HMCF)

\[ \frac{\partial P}{\partial t} = \frac{K}{H} N \]

where each point P of Σ₀ moves in the inward normal direction N with velocity equal to the harmonic mean curvature of the surface, namely the harmonic mean

\[ \frac{K}{H} = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} \]

of the two principal curvatures \( \lambda_1, \lambda_2 \) of the surface.

The existence of solutions to the HMCF with strictly convex smooth initial data was first shown by Andrews in [2]. He also showed that, under the HMCF, strictly convex, smooth surfaces converge to round points in finite time. In [10], Dieter established the short time existence of solutions to the HMCF with weakly convex smooth initial data and mean curvature H > 0. More precisely, Dieter showed that if at time t = 0 the surface Σ₀ satisfies K ≥ 0 and H > 0, then there exists a unique strictly convex smooth solution Σ₁ of the HMCF defined on 0 < t < τ, for some τ > 0. By the results of Andrews, this solution exists up to the time where its enclosed volume becomes zero. However, the highly degenerate case where the initial data is weakly convex and both K and H vanish in a region is not studied in [10].

We will consider in this work the evolution of a surface Σ₀ with flat sides by the HMCF. The parabolic equation describing the motion of the surface becomes degenerate at points where both curvatures K and H become zero. Our main

*: Partially supported by the NSF grant DMS-03-54639.
**: Partially supported by the NSF grants DMS-01-02252, DMS-03-54639 and the EPSRC in the UK.
objective is to study the solvability and optimal regularity of the evolving surface for \( t > 0 \), by viewing the flow as a free-boundary problem. It will be shown that a surface \( \Sigma_0 \) of class \( C^{k,\gamma} \) with \( k \in \mathbb{N} \) and \( 0 < \gamma \leq 1 \) at \( t = 0 \), will remain in the same class for \( t > 0 \). In addition, we will show that the strictly convex parts of the surface become instantly \( C^\infty \) smooth up to the flat sides and the boundaries of the flat sides evolve by the curve shortening flow.

For simplicity we will assume that the surface \( \Sigma_0 \) has only one flat side, namely \( \Sigma = \Sigma_1 \cup \Sigma_2 \), with \( \Sigma_1 \) flat and \( \Sigma_2 \) strictly convex (both principal curvatures are strictly positive). We may also assume that \( \Sigma_1 \) lies on the \( z = 0 \) plane and that \( \Sigma_2 \) lies above this plane since the equation is invariant under rotation and translation. Therefore, the lower part of the surface \( \Sigma_0 \) can be written as the graph of a function

\[
z = h(x, y)
\]

over a compact domain \( \Omega \subset \mathbb{R}^2 \) containing the initial flat side \( \Sigma_1 \). Let \( \Gamma \) denote the boundary of the flat side \( \Sigma_1 \). We define \( g = h^p \), for some \( 0 < p < 1 \). Our main assumption on the initial surface \( \Sigma_0 \) is that it satisfies the following non-degeneracy condition \((\ast)\):

\[
(\ast) \quad |Dg(P)| \geq \lambda \quad \text{and} \quad g_{\tau \tau}(P) \geq \lambda, \quad \text{for all } P \in \Gamma
\]

for some number \( \lambda > 0 \). Here \( \tau \) denotes the tangential direction to the level sets of \( g \) and \( g_{\tau \tau} \) denotes the second order derivative in this direction.

Under the above conditions, our main results show that for \( t \in (0, T) \):

1. The HMCF admits a solution \( \Sigma_t = (\Sigma_1)_t \cup (\Sigma_2)_t \) of class \( C^{k,\gamma} \), for some \( k \in \mathbb{N} \) and \( 0 < \gamma \leq 1 \) depending on \( p \), which is smooth up to \( \Gamma_t = \partial(\Sigma_1)_t \).

2. \( (\Sigma_1)_t \) is flat and its boundary \( \Gamma_t \) evolves by the curve shortening flow.

The fact that the solution \( \Sigma_t \) remains in the class \( C^{k,\gamma} \) distinguishes this flow from other, previously studied, degenerate free-boundary problems (such as the Gauss curvature flow with flat sides, the porous medium equation and the evolution p-laplacian equation) in which the regularity of the solution for \( t > 0 \) does not depend on the regularity of the initial data.

We define \( \mathcal{S} \) to be the class of weakly convex compact surfaces \( \Sigma_0 \) in \( \mathbb{R}^3 \) so that \( \Sigma = \Sigma_1 \cup \Sigma_2 \), where \( \Sigma_1 \) is a surface contained in the plane \( z = 0 \) and \( \Sigma_2 \) is a strictly convex and smooth surface contained in the half-space \( z \geq 0 \). The main result states as follows:

**Main Theorem.** Assume that at time \( t = 0 \), \( \Sigma_0 \) is a weakly convex compact surface in \( \mathbb{R}^3 \) which belongs to the class \( \mathcal{S} \) so that the function \( g = h^p \) defined as above is smooth up to the interface \( \Gamma \) and satisfies \((\ast)\). Then, there exists a time \( T > 0 \) such that the HMCF admits a solution \( \Sigma_t \in \mathcal{S} \) on \([0, T)\). Moreover, the function \( g = h^p \), defined as above for \( \Sigma_t \), is smooth up to the interface \( z = 0 \) and satisfies \((\ast)\) on \([0, T)\). In particular, the interface \( \Gamma_t \) between the flat side and the strictly convex side is a smooth curve for all \( t \) in \( 0 < t \leq T \) and it evolves by the curve shortening flow.

**Sketch of the proof.** A standard computation shows that when \( \Sigma_t \) solves the HMCF, the function \( h \) evolves by the equation

\[
h_t = \frac{h_{zz}h_{yy} - h_{zy}^2}{(1 + h_y^2)h_{zz} - 2h_zh_yh_{zy} - (1 + h_z^2)h_{yy}} \quad \text{on } \quad z > 0.
\]
The HMCF can be seen as a free boundary problem arising from the degeneracy near the flat side of the fully nonlinear parabolic PDE which describes the flow. We will show in section 5 that, via a global change of coordinates, this free boundary problem is equivalent to an initial value problem on $D \times [0, T]$, with $D = \{(u, v); u^2 + v^2 \leq 1\}$, namely
\[
\begin{cases}
Mw = 0 & \text{on } D \times [0, T] \\
w = w_0 & \text{at } t = 0
\end{cases}
\] (1.2)

The operator $Mw = w_t - F(t, u, v, Dw, D^2w)$, is a fully non-linear operator which becomes degenerate at $\partial D$, the boundary of $D$. We will apply the Inverse Function Theorem between appropriately defined Banach spaces to show that this problem admits a solution.

The linearization of the operator $M$ at a point $\bar{w}$ close to the initial data $w_0$, can be modeled, after straightening the boundary, on the degenerate equation
\[
f_t = z^2 a_{11} f_{zz} + 2 z a_{12} f_{zy} + a_{22} f_{yy} + b_1 z f_z + b_2 f_y
\] (1.3)
on $z > 0$ with no extra conditions on $f$ along the boundary $z = 0$. We observe that the diffusion in the above equation is governed by the Riemannian metric $ds^2 = d\bar{s}^2 + |dt|$ where
\[
d\bar{s}^2 = \frac{dz^2}{z^2} + dy^2.
\]
The distance (with respect to the singular metric $\bar{s}$) of an interior point ($z > 0$) from the boundary ($z = 0$) is hence infinite. This fact distinguishes our problem from other, previously studied, degenerate free-boundary problems such as the degenerate Gauss curvature flow and the porous medium equation.

The plan of the paper is the following: in section 2 we will introduce a local change of coordinates that fixes the free-boundary $\Gamma$ in equation (1.1). We will compute the linearization of equation (1.3) in this new change of coordinates, and motivate the use of the appropriate Banach spaces $C^{2+\alpha,p}_s$ for our problem. The detailed definition of these Banach spaces will be given in section 3 where we will also present the appropriate Schauder estimates for our problem. In section 4 we will study the fully-nonlinear degenerate equations (1.2) and establish the short time existence for such equations in the Banach spaces $C^{2+\alpha,p}_s$. The global change of coordinates and the proof of the Main Theorem will be given in sections 5 and 6 respectively. In the last section we will establish the comparison principle for equation (1.1) and characterize our solutions as viscosity solutions.

Acknowledgments. The authors wish to thank G. Huisken for suggesting the problem and R. Hamilton for many stimulating discussions.

2. Local Change of Coordinates

We will assume throughout this section that the surface $\Sigma_0$ belongs to the class $\mathcal{S}$. Let $\Sigma_t$ be a solution to the HMCF on $[0, T)$, for some $T > 0$ in the sense that $\Sigma_t = (\Sigma_1)_t \cup (\Sigma_2)_t$, with $(\Sigma_1)_t$ flat and $(\Sigma_2)_t$ strictly convex. Let $P_0(x_0, y_0, t_0)$ be a point on the interface $\Gamma_{t_0}$, for $t_0 > 0$ sufficiently small. Then, the strictly convex part of surface $(\Sigma_2)_{t_0}$, $t < t_0$ can be expressed locally around $P_0$ as the graph of a function $z = h(x, y, t)$. Let $g$ be defined by $g = h^p$, for $0 < p < 1$, such that $g$ is smooth up to the interface and satisfies condition $(\star)$. We can assume, by rotating
the coordinates, that at the point \( P_0 \) the normal vector to \( \Gamma_{t_0} \) facing outwards the flat side \( \Sigma(t_0) \) is parallel to the \( x \)-axis, so that at \( P_0 \) we have 
\[ g_x(P_0) > 0 \quad \text{and} \quad g_y(P_0) = 0. \]

Then we solve locally around the point \( P_0 \) the equation \( z = h(x, y, t) \) with respect to \( x \). This yields to a map \( x = f(z, y, t) \). The condition on \( g \) expressed in terms of \( f \) gives the following non-degeneracy condition \((\star \star)\) :
\[ \left( -z^{2-p} f_{zz} \quad z^{1-p} f_{zy} \right) \right) \geq \bar{\lambda} \]
in the sense that both eigenvalues of the above matrix are bounded from below by a number \( \bar{\lambda} > 0 \).

Since \( f \) is the inverse of \( h \) and the HMCF is invariant under rotation, the function \( f \) satisfies the same equation as \( h \) on \( z > 0 \):
\[
f_t = \frac{f_{zz}f_{yy} - f_y^2}{(1 + f_y^2)f_{zz} - 2 f_z f_y f_{zy} - (1 + f_z^2)f_{yy}}.
\]

We will construct a smooth solution to this equation by using the Inverse Function Theorem. To do so, we will define the Banach space \( C^{2+\alpha,p}_s \) in the next section.

According to our notation, the constants \( \alpha \) and \( p \) indicate “how the surface becomes flat”, while \( s \) refers to the hyperbolic metric which governs the problem.

We will prove in the next sections that when \( f \in C^{2+\alpha,p}_s \) and satisfies condition \((\star \star)\), then the equation \((2.1)\) becomes degenerate at \( z = 0 \) implying that:
\[
f_t = \frac{f_{yy}}{1 + f_y^2} \quad \text{at the interface} \quad z = 0;
\]

This is equivalent to say that the free boundary \( \Gamma_t \) evolves by the curve shortening flow.

3. The \( C^{2+\alpha,p}_s \) space and Schauder estimates

Let \( \mathcal{A} \) be a compact subset of the half space \( \{ (z, y) \in \mathbb{R}^2 : z \geq 0 \} \) such that \((0, 0) \in \mathcal{A}\). Then, we define:
\[
\mathcal{A}^o := \{ y \in \mathbb{R} : (0, y) \in \mathcal{A} \} \\
\mathcal{A} := \{ (w, y) \in \mathbb{R}^2 : w = \ln z, (z, y) \in \mathcal{A}, z \neq 0 \} \\
Q_T := \mathcal{A} \times [0, T], \ T > 0 \\
Q^o_T := \mathcal{A} \times [0, T] \\
Q_T := \mathcal{A} \times [0, T].
\]

Let \( 0 < p < 1 \). Given a function \( f \) on \( \mathcal{A} \) we define:
\[
f^o(y) := f(0, y) \\
f(w, y) := e^{-pw} \left( f(z, y) - f^o(y) \right)
\]

with \( w = \ln z \), for \( z > 0 \).

Analogously, given a function \( f \) on \( Q_T \) we define:
\[
f^o(y, t) := f(0, y, t) \\
f(w, y, t) := e^{-pw} \left( f(z, y, t) - f^o(y, t) \right).
\]

Given a subspace \( \mathcal{A} \) as above, we define the hyperbolic distance \( \tilde{s}(P_1, P_2) \) between two points \( P_1 = (z_1, y_1) \) and \( P_2 = (z_2, y_2) \) in \( \mathcal{A} \) \( z_i > 0 \), \( i = 1, 2 \) to be:
\[
\tilde{s}(P_1, P_2) := \sqrt{|\ln z_1 - \ln z_2|^2 + |y_1 - y_2|^2}, \quad \text{if } 0 < z_1, z_2 \leq 1
\]
otherwise it is defined to be equivalent to the standard euclidean metric.

We define the parabolic hyperbolic distance between two points \( \hat{P}_i = (z_i, y_i, t_i) \) and \( \hat{P}_2 = (z_2, y_2, t_2) \) with \( z_i > 0, i = 1, 2 \) to be:
\[
s(\hat{P}_1, \hat{P}_2) := \tilde{s}(P_1, P_2) + \sqrt{|t_1 - t_2|}
\]
where \( P_1 = (z_1, y_1) \), \( P_2 = (z_2, y_2) \).

Let \( 0 < \alpha \leq 1 \). We define the Hölder space \( C^{\alpha, \beta}_s(A) \) in terms of the above distance. We start defining the Hölder semi-norm:
\[
\| f \|_{H^s_\alpha(A)} := \sup_{P_1 \neq P_2 \in A \cap \{(x, y) \in \mathbb{R}^2 : z > 0\}} \frac{|f(P_1) - f(P_2)|}{s(P_1, P_2)^{\alpha}}
\]
and the norm
\[
\| f \|_{C^\beta(A)} := \| f \|_{C^0(A)} + \| f \|_{H^s_\alpha(A)}
\]
where \( ||f||_{C^0(A)} := \sup_{P \in A} |f(P)| \).

We say that a function \( f \) belongs to \( C^{\alpha, \beta}_s(A) \) if \( f^o \in C^\alpha(A^o) \) and \( \tilde{f} \in C^\beta(A) \). The norm of \( f \) in the space \( C^{\alpha, \beta}_s(A) \) is defined as:
\[
\| f \|_{C^{\alpha, \beta}_s(A)} := \| f^o \|_{C^\alpha(A^o)} + \| \tilde{f} \|_{C^\beta(A)}
\]
Moreover, we define:
\[
\| f \|_{C^\alpha^{\beta, \gamma}(A)} := \| f^o \|_{C^\alpha(A^o)} + \| \tilde{f} \|_{C^\beta(A)}
\]

Remark 1. We observe that \( f(w, y) \in C^\alpha(\hat{A}) \) if and only if \( f(z, y) \in C^\alpha_s(A) \), where \( w = \ln z \).

We say that a continuous function \( f \) on \( A \) belongs to \( C^{2+\beta}_s(A) \) if \( f^o \in C^2(A^o) \) and \( f \) has continuous derivatives
\[
f_{z}, f_y, f_{zz}, f_{zy}, f_{yy}
\]
in the interior of \( A \), such that
\[
z^{-p}(f - f^o), z^{1-p}f_z, z^{-p}(f_y - f^o_y), z^{2-p}f_{zz}, z^{1-p}f_{zy}, z^{-p}(f_{yy} - f^o_{yy})
\]
extend continuously up to the boundary. The norm of \( f \) in the space \( C^{2+\beta}_s(A) \) is defined as follows:
\[
\| f \|_{C^{2+\beta}_s(A)} := \| f^o \|_{C^2(A^o)} + \sum_{m=0}^{2} \sum_{n=0}^{2} \| D_z^m D_y^n f^o \|_{C^\beta(A)}
\]
Given \( f \in C^{2+\beta}_s(A) \), we say that \( f \) belongs to \( C^{2+\alpha, \gamma}_s(A) \) if 
\[
f^o \in C^{2+\alpha}(A^o), \quad f_z, f_y, z^2 f_{zz}, z f_{zy}, f_{yy}
\]
extend continuously up to the boundary, and the extensions are Hölder continuous on \( A \) of class \( C^{2+\alpha, \beta}_s(A) \). The norm of \( f \) in the space \( C^{2+\alpha, \beta}_s(A) \) is defined as:
\[
\| f \|_{C^{2+\alpha, \beta}_s(A)} := \| f^o \|_{C^{2+\alpha}(A^o)} + \sum_{m=0}^{2} \sum_{n=0}^{2} \| z^m D_z^m D_y^n f^o \|_{C^{2+\gamma, \beta}_s(A)}
\]
Remark 2. It follows by definition that \( \tilde{f}_w = -p \tilde{f} + z^{1-p} f_z \) and \( \tilde{f}_{ww} = -p \tilde{f}_z + (1-p) z^{2-p} f_{zz} \), which implies that:

\[
\sum_{m+n=0}^2 \| z^m D_z^m D_y^n f \|_{C^{2+\alpha}((A))} \sim \| \tilde{f} \|_{C^{2+\alpha}((\tilde{A}))}.
\]

Remark 3. The function \( f \in C^{2+\alpha}_s(A) \) if and only if \( f^\circ \in C^{2+\alpha}(A^\circ) \) and \( \tilde{f} \in C^{2+\alpha}(\tilde{A}) \), therefore, the following norms are equivalent:

\[
\| f \|_{C^{2+\alpha}_s(A)} \sim \| f^\circ \|_{C^{2+\alpha}(A^\circ)} + \| \tilde{f} \|_{C^{2+\alpha}(\tilde{A})}.
\]

Let \( T > 0 \). The definitions above can be naturally extended on the space-time domain \( Q_T \) by using the parabolic distance \( ds^2 = d\bar{s}^2 + |dt| \). We define the space \( C^\alpha_\circ(Q_T) \) to be the standard Hölder space with respect to the metric \( d\bar{s}^2 \). We say that a continuous function \( f \) on \( Q_T \) belongs to \( C^{2+\alpha}(Q_T) \) if \( f \) has continuous derivatives

\[
f_t, f_z, f_y, f_{zz}, f_{zy}, f_{yy}
\]

in the interior of \( Q_T \) and \( f^\circ \) has continuous derivatives that extend continuously up to the boundary and

\[
z^{-p}(f - f^\circ), z^{-p}(f_t - f^\circ_t), z^{1-p}f_z, z^{-p}f_y, z^{2-p}f_{zz}, z^{1-p}f_{zy}, z^{-p}(f_{yy} - f_y^\circ)
\]

extend continuously up to the boundary. The norm of \( f \) in the space \( C^{2+\alpha}(Q_T) \) is defined as follows:

\[
\| f \|_{C^{2+\alpha}(Q_T)} : = \| f^\circ \|_{C^2} + \sum_{l+m-n=0}^2 \| D_z^l D_y^m D_t^n f \|_{C^{2+\alpha}}.
\]

The function \( f \) belongs to \( C^{2+\alpha}_s(Q_T) \) if \( f \in C^{2+\alpha}(Q_T) \),

\[
f, f_t, z f_z, f_y \quad \text{and} \quad z^2 f_{zz}, z f_{zy}, f_{yy}
\]

belong to \( C^\alpha_\circ(Q_T) \).

Throughout the paper \( k \) will denote a positive integer. We can extend these definitions to spaces of higher order derivatives. We denote by \( C^{k,\alpha}(Q_T) \) the space of all functions \( f \) whose \( k \)-th order derivatives \( D_z^i D_y^j D_t^l f \), \( i+j+2l = k \) in the interior of \( Q_T \) and \( z^i D_z^i D_y^j D_t^l (f - f^\circ) \), \( i+j+2l = k \) exist and belong to the space \( C^\alpha_\circ(Q_T) \). We define \( C^{\infty,\alpha}(Q_T) = \cap_k C^{k,\alpha}(Q_T) \).

We denote by \( C^{k+\alpha}_s(Q_T) \) the space of all functions \( f \in C^{k,\alpha}(Q_T) \) such that \( z^i D_z^i D_y^j D_t^l f \), for \( i+j+2l = k \) belong to the space \( C^\alpha_\circ(Q_T) \). The space \( C^{k+\alpha}_s(Q_T) \) is equipped with the norm:

\[
\| f \|_{C^{k+\alpha}_s(Q_T)} : = \sum_{l+j+2l\leq k} \| z^i D_z^i D_y^j D_t^l f \|_{C^{\alpha}_\circ(Q_T)}.
\]

Remark 4. A function \( f \in C^{k+\alpha}_s(Q_T) \) if \( f^\circ \in C^{k+\alpha}(Q^\circ_T) \) and \( \tilde{f} \in C^{k+\alpha}(\tilde{Q}_T) \). Moreover,

\[
\| f \|_{C^{k+\alpha}_s(Q_T)} \simeq \| f^\circ \|_{C^{k+\alpha,\alpha/2}(Q^\circ_T)} + \| \tilde{f} \|_{C^{k+\alpha,\alpha/2}(\tilde{Q}_T)}.
\]
In the next paragraph we denote by $S_0$ the half space $x \geq 0$ in $\mathbb{R}^2$, by $S$ the space $S = S_0 \times [0, \infty)$, and by $S_T$ the space $S \times [0, T]$, for $T > 0$. The operator $L_k : C^{k+2+\alpha,p}_s(Q_T) \to C^{k+\alpha,p}_s(Q_T)$ is defined as:

\[
L_k[f] := f_t - (z^2 a_{11} f_{zz} + 2 z a_{12} f_{zy} + a_{22} f_{yy} + b_1 z f_z + b_2 f_y + c f)
\]

where the coefficients $\{a_{ij}\}_{i,j}$ are uniformly elliptic and $\{a_{ij}, b_i, c\} \subseteq C^{k+2+\alpha}_s(q_T)$, $i, j = 1, 2$.

**Theorem 1.** (Existence and Uniqueness) Let $L_k$ be defined as above. Assume that $\phi \in C^{k+\alpha,p}_s(S)$ and $f_0 \in C^{k+2+\alpha,p}_s(S_0)$, and that $\phi, f_0$ are compactly supported in $S$ and $S_0$, respectively. Then, for any $T > 0$, the initial value problem:

\[
\begin{cases}
L_k[f] = \phi & \text{in } S_T \\
f(\cdot, 0) = f_0 & \text{on } S_0
\end{cases}
\]

admits a unique solution $f \in C^{k+2+\alpha,p}_s(S_T)$. Moreover

\[
\|f\|_{C^{k+2+\alpha,p}_s(S_T)} \leq C(T) \left(\|f_0\|_{C^{k+2+\alpha,p}_s(S_0)} + \|\phi\|_{C^{k+\alpha,p}_s(S)}\right)
\]

for some constant $C(T)$, depending on $\alpha, p, k$ and $T$.

**Proof.** To solve the above Cauchy problem is equivalent to solve the following Cauchy problems \((3.3)\) and \((3.5)\).

The problem \((3.3)\) is obtained by evaluating \((3.2)\) at $z = 0$ and the problem \((3.5)\) is obtained by solving the corresponding one for $\tilde{f}$.

\[
\begin{cases}
(L_k)_0[f^0] = \phi^0 & \text{in } \mathbb{R} \times [0, T] \\
\tilde{f}^0(\cdot, 0) = (f_0)^0 & \text{on } \mathbb{R}
\end{cases}
\]

\[
\begin{cases}
\tilde{L}_k[\tilde{f}] = \tilde{\phi} & \text{in } S_T \\
\tilde{f}(\cdot, 0) = \tilde{f}_0 & \text{on } S_0
\end{cases}
\]

where the operators $(L_k)_0$ and $\tilde{L}_k$ are defined respectively as follows:

\[
(L_k)_0[f^0] = a_{11}^0 f_{ww}^0 + 2 a_{12}^0 f_{wy}^0 + a_{22}^0 f_{yy}^0 + b_1^0 \tilde{f}_w + b_2^0 \tilde{f}_y + c^0 \tilde{f} + \tilde{G}
\]

\[
\tilde{L}_k[\tilde{f}] = \tilde{f}_t - (a_{11} \tilde{f}_{ww} + 2 a_{12} \tilde{f}_{wy} + a_{22} \tilde{f}_{yy} + b_1 \tilde{f}_w + b_2 \tilde{f}_y + c \tilde{f} + \tilde{G})
\]

with

\[
\begin{align*}
\hat{a}_{ij}(w, y, t) & := a_{ij}(x, y, t) \\
\hat{b}_1(w, y, t) & := (2p - 1) a_{11}(w, y, t) + b_1(x, y, t) \\
\hat{b}_2(w, y, t) & := b_2(x, y, t) \\
\hat{c}(w, y, t) & := e^{-p \lambda} \left[ f^2 \hat{a}_{11}(x, y, t) - 2 p \hat{a}_{12}(w, y, t) + p b_1(x, y, t) \right] \\
\hat{G}(w, y, t) & := \hat{b}_2(w, y, t) g^0_{yy}(y, t) + \hat{a}_{22}(w, y, t) g^0_{yy}(y, t)
\end{align*}
\]

By the assumptions on the operator $L_k$ it is clear that the coefficients of the two operators $(L_k)_0$ and $\tilde{L}_k$ satisfy classical conditions. We first find the solution $f^0$ to \((3.4)\), then we solve \((3.5)\). By classical theory both problems have a unique solution. The function $\tilde{f}$ defined by $f(w, y, t) := f^0(y, t) + z^p \tilde{f}(w, y, t)$ is a solution to \((3.2)\).
Let $C^{k+\alpha}$ and $C^{k+2+\alpha}$ denote classical parabolic H"older spaces. Then the following inequalities hold:
\[
\begin{align*}
\|f^a\|_{C^{k+2+\alpha}(\mathbb{R}^+ \times [0,T])} & \leq C(T) \left( \|f_0^a\|_{C^{k+2+\alpha}(\mathbb{R}^+)} + \|g^a\|_{C^{k+\alpha}(\mathbb{R}^+)} \right) \\
\|\tilde{f}\|_{C^{k+2+\alpha}(\mathbb{S}_T)} & \leq C(T) \left( \|\tilde{f}_0\|_{C^{k+2+\alpha}(\mathbb{S}_0)} + \|\tilde{\phi}\|_{C^{k+\alpha}(\mathbb{S})} \right)
\end{align*}
\]
It follows that the solution to (3.2) is unique and satisfies the inequality (3.3).

Next, we define the boxes in which we prove the Schauder estimates. Let $0 < r \leq 1$. We denote by $B_r(P)$ the box
\[
B_r(P) = \left\{ \begin{pmatrix} z \\ y \\ t \end{pmatrix} : \begin{array}{l} z \geq 0, |x - x_0| \leq e^r \\ |y - y_0| \leq r \\ t_0 - r^2 \leq t \leq t_0 \end{array} \right\}
\]
around the point $P = \begin{pmatrix} x_0 \\ y_0 \\ t_0 \end{pmatrix}$. We set $B_r$ to be the box around the point $P = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

**Remark 5.** The choice of the box $B_r$ is made so that it has the right rescaling. The operators $L_{k_0}$ and $L_k$ are well understood on the corresponding boxes $B_r^c$ and $B_r$.

**Theorem 2. (Schauder Estimate)** Assume that all the coefficients of the operator
\[
L f = f_t - (z^2 a_{11} f_{zz} + 2z a_{12} f_{zy} + a_{22} f_{yy} + z b_1 f_x + b_2 f_y + cf)
\]
belong to the space $C^{k+\alpha}(B_1)$ and that the coefficients $a_{22}, b_2$ and $c$ belong to $C^{k+\alpha,p}(Q_T)$ for some numbers $\alpha$, $p$ in $0 < p < 1$, $0 < \alpha \leq 1$ and satisfy
\[
a_{ij} \xi^i \xi^j \geq \lambda |\xi|^2, \ \forall \xi \in \mathbb{R}^2 \setminus \{0\}, \ \lambda > 0
\]
with $|a_{ij}|_{C^{k+\alpha}(Q_T)}$, $|b_i|_{C^{k+\alpha}(Q_T)}$, $|a_{22}|_{C^{k+\alpha,p}(Q_T)}$, $|b_2|_{C^{k+\alpha,p}(Q_T)}$, $|c|_{C^{k+\alpha,p}(Q_T)} \leq \frac{1}{\lambda}$.
Then, there exists a constant $C$ depending only on $\alpha$, $\lambda$ and $p$ such that
\[
\|f\|_{C^{2+k+\alpha,p}(B_{1/2})} \leq C \left( \|f\|_{C^{\alpha,p}(B_1)} + \|L[f]\|_{C^{\alpha,p}(B_1)} \right)
\]
for all functions $f \in C^{2+k+\alpha,p}(B_1)$.

**Proof.** The proof follows by the same argument as in Theorem 1 and classical Schauder estimates for strictly parabolic operators.

4. The Degenerate Equation on the Disc

We will extend in this section the existence and uniqueness the Theorem to the following class of linear degenerate equations:
\[
L_w := w_t - (a^{ij} w_{ij} + b^i w_i + c w)
\]
on the cylinder $D \times [0,T)$, $T > 0$, where $D$ denotes the unit disk in $\mathbb{R}^2$. The sub-indices $i, j \in \{x, y\}$ denote differentiation with respect to the space variables $x, y$. The matrix $\{a^{ij}\}$ is assumed to be symmetric. Certain assumptions on the coefficients will be made so that this class of equations includes, under appropriate change of coordinates, the operator $L$ given by (3.1).
We define the distance function $s$ in $D$ as follows: in the interior of $D$, $\bar{s}$ it is equivalent to the standard euclidean distance, while around any boundary point $P \in \partial D$, $\bar{s}$ is defined as the pull back of the distance function induced by the metric

$$ds^2 = \frac{dz^2}{z^2} + dy^2$$

on the half space $S_0 = \{(z,y) : z \geq 0\}$, via a map $\varphi : S_0 \cap D \to D$ that flattens the boundary of the disk $D$ near $P$.

The parabolic distance is defined by

$$s \left[ \left( \begin{array}{c} P_1 \\ t_1 \end{array} \right), \left( \begin{array}{c} P_2 \\ t_2 \end{array} \right) \right] = \bar{s}(P_1, P_2) + \sqrt{|t_1 - t_2|}, \quad P_1, P_2 \in D, \quad 0 < t_1 \leq t_2$$

We define the spaces $C^{k,\alpha,p}(D)$ and $C^{k+2,\alpha,p}(D)$. For a fixed small number $\delta$ in $0 < \delta < 1$, we write

$$D = \mathcal{D}_{1-\delta/2} \cup \bigcup_l (\mathcal{D}_{\delta}(P_l) \cap D)$$

for finite many points $P_l \in \partial D$, $l \in I$, with $\mathcal{D}_{1-\delta/2}$ denoting the disk centered at the origin of radius $1 - \delta/2$ and $\mathcal{D}_{\delta}(P_l)$ denoting the disk of radius $\delta$ centered at $P_l$.

We denote by $D_+$ the half disk

$$D_+ = \{(x,y) \in D : x \geq 0\}.$$  

We can choose charts $\Upsilon_l : D_+ \to \mathcal{D}_{\delta}(x_l) \cap D$ which flatten the boundary of $D$ and such that $\Upsilon_l(0) = P_l$, $l \in I$. Let $\psi, \psi_l$ be a partition of unity subordinated to the cover

$$\{ \mathcal{D}_{1-\delta/2}, (\mathcal{D}_{\delta}(P_l) \cap D) \}$$

of $D$, with $l \in I$.

We define $C^{k,\alpha,p}(D)$ to be the space of all functions $w$ on $D$ such that $w \in C^{k,\alpha}(\mathcal{D}_{1-\delta/2})$ and $w \circ \Upsilon_l \in C^{k,\alpha,p}(D_+)$ for all $l \in I$.

Also, we define $C^{k+2,\alpha,p}(D)$ to be the space of all functions $w$ on $D$ such that $w \in C^{k+2,\alpha}(\mathcal{D}_{1-\delta/2})$ and $w \circ \Upsilon_l \in C^{k+2,\alpha,p}(D_+)$ for all $l \in I$. Here $C^{k,\alpha}$ and $C^{k+2,\alpha}$ denote the regular Hölder Spaces, while $C^{k,\alpha,p}(D_+)$ and $C^{k+2,\alpha,p}(D_+)$ denote the Hölder Spaces defined in section 3. One can show that both spaces $C^{k,\alpha,p}(D)$ and $C^{k+2,\alpha,p}(D)$ are Banach Spaces under the norms

$$\|w\|_{C^{k,\alpha,p}(D)} = \|w\|_{C^{k,\alpha}(\mathcal{D}_{1-\delta/2})} + \sum_l \|\psi_l(w \circ \Upsilon_l)\|_{C^{k,\alpha,p}(D_+)}$$

and

$$\|w\|_{C^{k+2,\alpha,p}(D)} = \|w\|_{C^{k+2,\alpha}(\mathcal{D}_{1-\delta/2})} + \sum_l \|\psi_l(w \circ \Upsilon_l)\|_{C^{k+2,\alpha,p}(D_+)}.$$ 

The above definitions can be extended in a straight forward manner to the parabolic spaces $C^{\alpha,p}(Q)$ and $C^{2+\alpha,p}(Q)$ where $Q$ is the cylinder $Q = D \times [0,T]$, for some $T > 0$. Before we state the main result in this section, we will give the assumptions on the coefficients of the equation

$$w_t = a^{ij} w_{ij} + b^i w_i + c w$$

on the cylinder $Q = D \times [0,T)$, $i, j = 1, 2$.

We first assume that for any $\delta$ in $0 < \delta < 1$, the coefficients $\{a^{ij}\}$, $\{b^i\}$ and $c$ belong to the Hölder class $C^\alpha(\mathcal{D}_{1-\delta/2} \times [0,T])$, which means that the coefficients are of the class $C^\alpha$ in the interior of $D$. For a number $\delta$ in $0 < \delta < 1$, let $\Upsilon_l :$
$D_+ \to D_\delta(P_l) \cap D$ be the collection of charts which flatten the boundary of $D$, considered above. We assume that there exists a number $\delta$ so that for every $l \in I$, the coordinate change introduced by each of the $\Upsilon_l$ transforms the operator 

\[(4.1) \quad L[w] = w_t - (a^{ij} w_{ij} + b^i w_i + c w)\]
on $D_\delta(P_l) \cap D$, into an operator $\tilde{L}_l$ on $D_+$ of the form

\[\tilde{L}_l [\tilde{w}] = \tilde{w}_t - (x^2 a_{11} \tilde{w}_{xx} + 2x a_{12} \tilde{w}_{xy} + a_{22} \tilde{w}_{yy} + x \tilde{b}_1 \tilde{w}_x + \tilde{b}_2 \tilde{w}_y + \tilde{c} \tilde{w})\]

with the coefficients $\tilde{a}_{ij}$, $\tilde{b}_i$ and $\tilde{c}$ belonging to the class $C^{k+\alpha}_s(D_+)$, with $a_{22}$, $b_2$ and $c \in C^{k+\alpha,p}_s$ such that:

\[\tilde{a}_{ij} \xi^i \xi^j \geq \lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}\]

for some number $\lambda > 0$.

We need the next Lemma to prove the invertibility of the operator $L$:

**Lemma 1.** (Hölder Interpolation). For every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ depending on $\varepsilon$, $p$, $k$ and $\alpha$ such that for any $g \in C^{k+2+\alpha,p}_s(Q_\delta)$, the following inequality holds:

\[(4.2) \quad || \partial D g ||_{C^{k+2+\alpha,p}_s(Q_\delta)} \leq \varepsilon || g ||_{C^{k+2+\alpha,p}_s(Q_\delta)} + C(\varepsilon) || g ||_{C^{k+\alpha}_s(Q_\delta)} \]

where $\partial$ behaves like distance to the boundary.

**Proof.** It follows by standard arguments. \qed

The following existence result readily follows from Theorem 1 and the above discussion.

**Theorem 3.** Assume that the operator $L$ satisfies all the above conditions on the cylinder $Q = D \times [0,T]$. Then, given any function $w_0 \in C^{k+2+\alpha,p}_s(D)$ and any function $g \in C^{k+\alpha}_s(Q)$ there exists a unique solution $w \in C^{k+2+\alpha,p}_s(Q_T)$ of the initial value problem

\[
\begin{cases}
Lw = g & \text{in } Q \\
w(\cdot,0) = w_0 & \text{on } D
\end{cases}
\]

satisfying

\[(4.3) \quad ||w||_{C^{k+2+\alpha,p}_s(Q_T)} \leq C(T) \left(||w_0||_{C^{k+2+\alpha,p}_s(D)} + ||g||_{C^{k+\alpha}_s(Q)}\right)\]

The constant $C(T)$ depends only on the numbers $\alpha$, $k$, $\lambda$ and $T$.

**Proof.** We can assume, without loss of generality, that $w_0 \equiv 0$ and that $g$ is a function in $C^{k+\alpha,p}_s(Q_T)$, which vanishes at $t = 0$.

For $\delta > 0$, set $Q_\delta = D \times [0,\delta]$ and denote by $C^{k+2+\alpha,p}_s(Q_\delta)$ and $C^{k+\alpha,p}_s(Q_\delta)$ the subspaces of $C^{k+2+\alpha}_s(Q)$ and $C^{k+\alpha}_s(Q)$ respectively, consisting out of all functions which vanish identically at $t = 0$. Also, we denote by $I$ the identity operator on $C^{k+\alpha,p}_s(Q_\delta)$. We will show that, if $\delta$ is sufficiently small, there exists an operator $M: C^{k+\alpha,p}_s(Q_\delta) \to C^{k+2+\alpha,p}_s(Q_\delta)$ such that

\[||LM - I|| \leq \frac{1}{2}\]

This implies that the operator $LM: C^{k+\alpha,p}_s(Q_\delta) \to C^{k+\alpha,p}_s(Q_\delta)$ is invertible and therefore $L: C^{k+2+\alpha,p}_s(Q_\delta) \to C^{k+\alpha,p}_s(Q_\delta)$ is onto, as desired.
We begin by expressing the compact domain $D$ as the finite union

$$D = D_0 \cup \bigcup_{l \geq 1} D_l$$

of compact domains in such a way that

$$\text{dist}(D_0, \partial D) \geq \frac{\rho}{2} > 0$$

and for all $l \geq 1$

$$D_l = B_\rho(x_l) \cap D$$

with $B_\rho(x_l)$ denoting the ball centered at $x_l \in \partial D$ of radius $\rho > 0$. The number $\rho > 0$ will be determined later.

The operator $L$ is non-degenerate when restricted on the interior domain $D_0$. Therefore, the classical Schauder theory for linear parabolic equations implies that $L$ is invertible when restricted on functions which vanish outside $D_0$, $\rho > 0$, nonnegative, smooth bump functions $\psi_l$ with $\psi_l \equiv 1$ on the support of $\phi_l$. Then $\sum_{l \geq 0} \phi_l = 1$ and $\psi_l \phi_l = \phi_l$ for all $l$. We denote by $M_0 : C^{k,\alpha,p}_s(D_0 \times [0,\delta]) \rightarrow C^{k,\alpha,p}_s(D_0 \times [0,\delta])$ the inverse of the operator $L$ restricted on $D_0$. Next, we consider the domains $D_l$, $l \geq 1$, close to the boundary of $D$, which can be chosen in such a way that the sets $B_\rho/4(x_l) \cap D$ are disjoint. Denoting by $\overline{B}$ the half unit ball

$$\overline{B} = \{(x, y) \in B_1(0) : x \geq 0\}$$

and by $\overline{Q}_\delta$ the cylinder

$$\overline{Q}_\delta = \overline{B} \times [0,\delta]$$

we select smooth charts $\Upsilon_l : \overline{Q}_\delta \rightarrow D_l$, which flatten the boundary of $D$, i.e., they map $\overline{B} \cap \{x = 0\}$ onto $D_l \cap \partial D$ and have $\Upsilon_l(0, x) = x_l$. This is possible if the number $\rho$ is chosen sufficiently small. Under the change of coordinates induced by the charts $\Upsilon_l$, the operator $L$, restricted on each $D_l \times [0,\delta]$, is transformed to an operator $\tilde{L}_l$ of the form

$$\tilde{L}_l[\tilde{w}] = \tilde{w}_1 - \left( x^2 a^{11}_1 \tilde{w}_1 + 2 x a^{12}_1 \tilde{w}_2 + a^{22}_1 \tilde{w}_2 + x a^{11}_i \tilde{w}_i + \tilde{b}_1^2 \tilde{w}_2 + \tilde{b}_2^2 \tilde{w}_1 + \tilde{c}_1 \tilde{w}\right)$$

defined on $\overline{B} \times [0,\delta]$. Moreover, the charts $\Upsilon_l$ can be chosen appropriately so that the coefficients of $\tilde{L}_l$ satisfy

$$a^{ij}_l \xi_i \xi_j \geq \lambda |\xi|^2 > 0 \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}$$

and

$$||\tilde{a}^{ij}_l||_{C^{k,\alpha,p}(Q_\delta)} \quad ||\tilde{b}_1^2||_{C^{k,\alpha,p}(Q_\delta)} \quad ||\tilde{c}_1||_{C^{k,\alpha,p}(Q_\delta)} \leq 1/\tilde{\lambda}$$

for some positive constant $\tilde{\lambda}$.

Each of the operators $\tilde{L}_l$ has the form of the model operators previously studied. Denote by $S_0$ the half space $x \geq 0$ in $\mathbb{R}^2$ and by $S_\delta$ the space $S_0 \times [0,\delta]$. Also, consider the subspace $\overline{C}^{k,\alpha,p}_s(S_\delta)$ of $C^{k,\alpha,p}_s(S_\delta)$, consisting out of functions which are compactly supported on $S_\delta$. Then, Theorem[I] implies that for every $l = 1, 2, ...$ there is an operator $\overline{M}_l : \overline{C}^{k,\alpha,p}_s(S_\delta) \rightarrow C^{k+2,\alpha,p}_s(S_\delta)$ such that

$$\overline{L}_l \overline{M}_l = I$$

with $I$ denoting the identity operator on $\overline{C}^{k,\alpha,p}_s(S_\delta)$. Let $M_l$ be the pull back of the operator $\overline{M}_l$ via the chart $\Upsilon_l$. Next, choose a nonnegative partition of unity $\phi_l$ with $l = 0, 1, ...$ subordinated to the cover $D_l$; $l = 0, 1, ...$ of $D$ and also choose, for each $l \geq 0$, nonnegative, smooth bump functions $\psi_l$, $0 \leq \psi_l \leq 1$, supported in $D_l$ with $\psi_l \equiv 1$ on the support of $\phi_l$. Then $\sum_{l \geq 0} \phi_l = 1$ and $\psi_l \phi_l = \phi_l$ for all $l$. 

HIGHLY DEGENERATE HARMONIC MEAN CURVATURE FLOW 11
We aim to show that the operator $M : C^{k+\alpha,p}_s(Q_\delta) \to C^{k+2+\alpha,p}_s(Q_\delta)$ defined as

$$Mg = \sum \psi_l M_l \phi_l g$$

satisfies

$$\| LMg - g \|_{C^{k+\alpha,p}_s(Q_\delta)} < \frac{1}{2} \| g \|_{C^{k+\alpha,p}_s(Q_\delta)} \quad \forall g \in C^{k+\alpha,p}_s(Q_\delta)$$

if the cover $\{D_l\}$ and $\delta$ are chosen appropriately. Indeed, we can write

$$LMg - g = \sum_l L \psi_l M_l \phi_l g - \sum \phi_l g = \sum \psi_l (LMl - I) \phi_l g + \sum [L, \psi_l] M_l \phi_l g$$

with $[L, \psi_l]$ denoting the commutator of $L$ and $\psi_l$. The commutator $[L, \psi_l]$ is only of first order and it can be estimated as

$$\|[L, \psi_l] M_l \phi_l g\|_{C^{k+\alpha,p}_s(Q_\delta)} \leq C \left( \|\partial D(M_l \phi_l g)\|_{C^{k+\alpha,p}_s(Q_\delta)} + \|M_l \phi_l g\|_{C^{k+\alpha,p}_s(Q_\delta)} \right).$$

Let $\epsilon > 0$. It follows via the Hölder spaces interpolation from Lemma[1] that

$$\|\partial D(M_l \phi_l g)\|_{C^{k+\alpha,p}_s(Q_\delta)} \leq \epsilon \|M_l \phi_l g\|_{C^{k+2+\alpha,p}_s(Q_\delta)} + C(\epsilon) \|M_l \phi_l g\|_{C^{k,p}_s(Q_\delta)}.$$

However, for each $k$ we have

$$\|M_l \phi_l g\|_{C^{k+2+\alpha,p}_s(Q_\delta)} \leq C \|g\|_{C^{k+\alpha,p}_s(Q_\delta)}$$

and therefore, since $M_l \phi_l g \equiv 0$ at $t = 0$,

$$\|M_l \phi_l g\|_{C^{k,p}_s(Q_\delta)} \leq C \delta \|g\|_{C^{k+\alpha,p}_s(Q_\delta)}.$$

It follows that if we choose $\delta$ sufficiently small:

$$\sum_l \| [L, \psi_l] M_l \phi_l g \|_{C^{k+\alpha,p}_s(Q_\delta)} \leq \frac{1}{4} \|g\|_{C^{k+\alpha,p}_s(Q_\delta)}.$$

On the other hand we have $(LM_0 - I)\varphi_0 g = 0$, while for $l \geq 1$, we can make the norm of each of the operators $LM_l - I$ arbitrarily close to zero by choosing the diameters of the domains $D_l$ sufficiently small:

$$\| \sum_l \psi_l (LM_l - I) \phi_l g \|_{C^{k+\alpha,p}_s(Q_\delta)} < \frac{1}{4} \|g\|_{C^{k+\alpha,p}_s(Q_\delta)}$$

for all $g \in C^{k+\alpha,p}_s(Q_\delta)$, if $\rho$ and $\delta$ are both sufficiently small. The above inequalities give

$$\| LMg - g \|_{C^{k+\alpha,p}_s(Q_\delta)} \leq \frac{1}{2} \|g\|_{C^{k+\alpha,p}_s(Q_\delta)}$$

for all $g \in C^{k+\alpha,p}_s(Q_\delta)$. We conclude that for every $g \in C^{k+\alpha,p}_s(Q_\delta)$ there exists a function $w \in C^{k,2+\alpha}_s(Q_\delta)$ such that $Lw = g$. In addition

$$\|w\|_{C^{k+2+\alpha,p}_s(Q_\delta)} \leq C \|g\|_{C^{k+\alpha,p}_s(Q_\delta)}$$

with $C$ depending only on $D$ and the constants $\alpha$, $k$, $\lambda$ and $T$.

The last inequality implies we can extend the solution on a bigger interval. Hence, one can show that

$$\|w\|_{C^{k+2+\alpha,p}_s(Q)} \leq C(T) \left( \|w_0\|_{C^{k+2+\alpha,p}_s(D)} + \|g\|_{C^{k+\alpha,p}_s(Q)} \right)$$

where the constant $C(T)$ depends only on the numbers $\alpha$, $k$, $\lambda$ and $T$. This last inequality implies uniqueness. \qed
Finally, the following existence result follows from Theorem 3 and the Inverse Function Theorem between Banach spaces along the line of the proof of Theorem 7.3 in [9]:

**Theorem 4.** Let $w_0$ be a function in $C^{k+2+\alpha,p}(\mathcal{D})$. Assume that the linearization $DM(\bar{w})$ of the fully-nonlinear operator

$$Mw = wt - F(t, u, v, w, Dw, D^2w)$$

defined on the cylinder $Q = \mathcal{D} \times [0, T]$, satisfies the hypotheses of Theorem 3 at all points $\bar{w} \in C^{k+2+\alpha,p}(Q)$, with $||\bar{w} - w_0||_{C^{k+2+\alpha,p}(Q)} \leq \mu$, for some $\mu > 0$. Then, there exists a number $\tau_0$ in $0 < \tau_0 \leq T$ depending on the constants $\alpha, p, k, \lambda$ and $\mu$, for which the initial value problem

$$\begin{cases}
w_t = F(t, u, v, w, Dw, D^2w) & \text{in } \mathcal{D} \times [0, \tau_0] \\
w(\cdot, 0) = w_0 & \text{on } \mathcal{D}
\end{cases}$$

admits a solution $w$ in the space $C^{k+2+\alpha,p}(\mathcal{D} \times [0, \tau_0])$. Moreover,

$$||w||_{C^{k+2+\alpha,p}(\mathcal{D} \times [0, \tau_0])} \leq C \||w_0||_{C^{k+2+\alpha,p}(\mathcal{D})}$$

for some positive constant $C$ which depends only on $\alpha, p, k, \lambda$ and $\mu$.

5. **Global change of coordinates and existence in $C^{k+2+\alpha,p}$**

In this section we introduce a global change of coordinates which transforms the HMCF for a surface $\Sigma_0$ into a fully-nonlinear degenerate parabolic PDE on $\mathcal{D}$.

Let $S$ be a smooth surface close to $\Sigma_2$. Let $S : \mathcal{D} \to \mathbb{R}^3$, indicate a parameterization of $S$ on the unit disk $\mathcal{D}$. We denote by $x_u, x_v, x_{uu}, x_{uv}, x_{vv}$ the partial derivatives of $x$ with respect to $u$ and $v$. The same notation will be used for the function $y$.

Let $\eta > 0$ be sufficiently small. Let $T = (T_1, T_2, T_3)$ be a smooth vector field transverse to $S$. We define the global change of coordinates $\Phi : \mathcal{D} \times [-\eta, \eta] \to \mathbb{R}^3$ by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Phi \begin{pmatrix} u \\ v \\ w \end{pmatrix} = S(u, v) + wT(u, v)$$

or more explicitly

$$\begin{align*}
x &= S_1(u, v) + wT_1(u, v) \\
y &= S_2(u, v) + wT_2(u, v) \\
z &= S_3(u, v) + wT_3(u, v)
\end{align*}$$

Let $\delta > 0$ be small number, such that

$$T_3(u, v) = 0 \quad \text{on } \mathcal{D} \setminus \mathcal{D}_{1-\delta}$$

denoting, as above, the transverse vector field to the surface $\mathcal{S}$. Notice that by choosing the smooth surface sufficiently close to the surface $z = h(x, y)$, we can make $\delta$ to depend only on the constant $\lambda$ which depends on the initial non-degeneracy conditions on the surface $\Sigma_0$. 


We write the first and second derivatives of $z$ with respect to $x, y$ and $t$ in terms of the first and second derivatives of $w$ with respect to $u, v$ and $t$.

If $z = h_0(x, y)$ then we compute the first and second partial derivatives of $z$ with respect to $x$ and $y$ in terms of $w = l(u, v)$ as functions of $u$ and $v$.

Let $A$ be the Jacobian matrix relative to the transformation of coordinates:

$$A = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let $\nabla z, \nabla u, \nabla v$ be, respectively the gradients of $z, u$ and $v$:

$$\nabla z = \begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \nabla u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \nabla v = \begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

We denote by $D^2 u$ and $D^2 v$ the following matrices:

$$D^2 u = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{pmatrix} = \begin{pmatrix} a_1 & c_1 \\ c_1 & c_2 \end{pmatrix}$$

$$D^2 v = \begin{pmatrix} \frac{\partial^2 v}{\partial x^2} & \frac{\partial^2 v}{\partial x \partial y} \\ \frac{\partial^2 v}{\partial y \partial x} & \frac{\partial^2 v}{\partial y^2} \end{pmatrix} = \begin{pmatrix} b_1 & d_1 \\ d_1 & d_2 \end{pmatrix}$$

Based on the above, we define $a_2 := c_1, b_2 := d_1$ and denote $e_1 = (1, 0), e_2 = (0, 1)$ to be the basis vectors. Let $A^{-1}$ be the inverse matrix of $A$:

$$A^{-1} := \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

Next, we introduce the matrices $B_1$ and $B_2$ which denote, respectively, the derivative of the inverse matrix of $A$ and $A^{-1}$ with respect to $x$ and $y$; $B_1 = \frac{\partial A^{-1}}{\partial x}$, $B_2 = \frac{\partial A^{-1}}{\partial y}$ which can be computed as:

$$B_1 = \begin{pmatrix} a x_{11} + b x_{12} & a y_{11} + b y_{12} \\ a x_{12} + b x_{22} & a y_{12} + b y_{22} \end{pmatrix}$$

$$B_2 = \begin{pmatrix} c x_{11} + d x_{12} & c y_{11} + d y_{12} \\ c x_{12} + d x_{22} & c y_{12} + d y_{22} \end{pmatrix}$$

Note that we are using the following notation:

$$x_{11} = x_{uu}, x_{12} = x_{uv}, x_{22} = x_{vv}, y_{11} = y_{uu}, y_{12} = y_{uv}, y_{22} = y_{vv}.$$  

The coefficients of the matrices $D^2 u$ and $D^2 v$ are evaluated as follows:

$$\begin{pmatrix} a_1 \\ c_1 \end{pmatrix} = -A \cdot B_1 \cdot \nabla u; \quad \begin{pmatrix} b_1 \\ d_1 \end{pmatrix} = -A \cdot B_1 \cdot \nabla v$$

$$\begin{pmatrix} a_2 \\ c_2 \end{pmatrix} = -A \cdot B_2 \cdot \nabla u; \quad \begin{pmatrix} b_2 \\ c_2 \end{pmatrix} = -A \cdot B_2 \cdot \nabla v$$
Since: \( \nabla z = A \cdot \left( \frac{\partial w}{\partial x} \right) + z_w \left( \frac{\partial w}{\partial y} \right) \), we obtain that:

\[
\frac{\partial}{\partial x} \nabla z = A_x \left( \frac{\partial w}{\partial x} \right) + A \left( a z_{uu} + b z_{uv} + z_{uuw} \frac{\partial w}{\partial x} \right) + (a z_{uw} + b z_{vw}) \left( \frac{\partial w}{\partial y} \right) + z_w \left( \frac{\partial^2 w}{\partial x \partial y} \right).
\]

\[
\frac{\partial}{\partial y} \nabla z = A_y \left( \frac{\partial w}{\partial y} \right) + A \left( c z_{uv} + d z_{vv} + z_{uvw} \frac{\partial w}{\partial y} \right) + (c z_{uw} + d z_{vw}) \left( \frac{\partial w}{\partial y} \right) + z_w \left( \frac{\partial^2 w}{\partial x \partial y} \right).
\]

The gradient of the function \( w \), as well as its partial derivatives, can be expressed by using the matrix \( A \):

\[
\left( \frac{\partial w}{\partial x} \right) = \left( \begin{array}{c} a w_u + b w_v \\ c w_u + d w_v \end{array} \right)
\]

\[
\left( \frac{\partial^2 w}{\partial x \partial y} \right) = \left( \begin{array}{c} a_1 w_u + a (a w_{uu} + b w_{uv}) + b_1 w_v + b (a w_{uu} + b w_{uv}) \\ c_1 w_u + c (a w_{uu} + b w_{uv}) + d_1 w_v + d (a w_{uu} + b w_{uv}) \end{array} \right)
\]

Using the substitutions above we get:

\[
\frac{\partial^2 z}{\partial x^2} = A_{11}^1 w_{11} + A_{12}^1 w_{12} + A_{22}^1 w_{22} + B_{11}^1 w_1 + B_{12}^1 w_2 + B_{11}^2 w_{11} w_2 + B_{22}^2 w_{22}^2 + C_1
\]

\[
\frac{\partial^2 z}{\partial y^2} = A_{11}^2 w_{11} + A_{12}^2 w_{12} + A_{22}^2 w_{22} + B_{11}^2 w_1 + B_{12}^2 w_2 + B_{22}^2 w_{11} w_2 + B_{22}^2 w_{22}^2 + C_2
\]

\[
\frac{\partial^2 z}{\partial x \partial y} = A_{11}^3 w_{11} + A_{12}^3 w_{12} + A_{22}^3 w_{22} + B_{11}^3 w_1 + B_{12}^3 w_2 + B_{12}^3 w_{11} w_2 + B_{22}^3 w_{22}^2 + C_3
\]

where the coefficients \( A_{i,j}^k, B_{i,j}^k, C^k \) with \( k = 0, 1, 2, i, j = 1, 2 \) are defined as follows:

\[
A_{11}^1 := -a^2 (z_w + b y_w(z_u + w_1 z_w)) + d y_w(z_v + w_2 z_w)
\]

\[
A_{12}^1 := -b^2 (z_w + b y_w(z_u + w_1 z_w)) + d y_w(z_v + w_2 z_w)
\]

\[
A_{11}^2 := -2 a b (z_w + b y_w(z_u + w_1 z_w)) + d y_w(z_v + w_2 z_w)
\]

\[
A_{12}^2 := c^2 (z_w - d y_w(z_v + w_2 z_w))
\]

\[
A_{12}^2 := d^2 (z_w + d y_w(z_v + w_2 z_w))
\]

\[
A_{12}^2 := -2 c d (z_w + d y_w(z_v + w_2 z_w))
\]
\[ A_1^0 : = -ac(-z_w + by_w(z_u + w_1 z_w) + dy_w(z_v + w_2 z_w)) \]
\[ A_2^0 : = -bd(-z_w + by_w(z_u + w_1 z_w) + dy_w(z_v + w_2 z_w)) \]
\[ A_1^1 : = -b(2ac y_{uw} + bcy_{uv} + ady_{vw}) z_v \]
\[ A_2^1 : = -b(2acy_{uw} + bcy_{uv} + ady_{vw}) z_w \]
\[ B_1^1 : = -d(by_{uw} + ady_{uw} + 2bdy_{uv}) z_w \]
\[ B_2^1 : = -((b^2 c + a(b + 2c)) y_{uw} + d(b(2b + c) + ad) y_{uv}) z_w \]
\[ B_1^2 : = -a^2 cx_{uv} + dx_{uv}) z_w - b(by_{uw} z_u + cdy_{uv} z_v - cz_{uv} + bcy_{uv} z_w + bd y_{uv} z_w) - a(-2cz_{uw} + 2cdy_{uw} z_v + d^2 y_{uw} z_v - dz_{uw} + b(2cy_{uw} z_u + dy_{uv} z_v + c(x_{uv} + y_{uv}) z_v + d(x_{uv} + y_{uv}) z_w)) \]
\[ B_2^2 : = -b^2 (cy_{uw} + 2dy_{uv}) z_w - b(-cz_{uw} + d(ay_{uw} z_u + cy_{uv} z_v + 2dy_{uw} z_v - 2z_{uw}) + (c + d) x_{uv} + c dy_{uv} + d^2 y_{uv}) z_v - a(c^2 x_{uw} z_v + d(-z_{uw} + c(x_{uw} + y_{wu}) z_w + d(y_{uw} z_v + y_{uv} z_w))) \]
\[ C_1 : = -2ab(ay_{uw} + by_{vw}) \]
\[ C_2 : = -2bd(by_{uw} + by_{vw}) \]
\[ C_3 : = -2(b^2 + ad)(ay_{uw} + by_{uv}) z_v \]
\[ C_4 : = 0 \]
\[ C_5 : = -2 b^2 (cy_{uw} + dy_{uv}) z_w \]
\[ C_6 : = -2c d^2 (y_{uv} + dy_{uw}) z_w \]
\[ C_7 : = -2c d^2 (y_{uv} + dy_{uv}) z_w \]
\[ C_8 : = -2c d^2 (y_{uv} + dy_{uw}) z_w \]
\[ C_9 : = 0 \]
Evolution of $w_t$. The evolution equation of $w$ is:

\begin{equation}
w_t = \frac{1}{zy_y w - zw_z} z_t
\end{equation}

where the function $z$ satisfies the non linear PDE:

\begin{equation}
z_t = \frac{z_{xx} z_{yy} - z_{xy}^2}{(1 + z_y^2) z_{xx} - 2 z_x z_y z_{xy} - (1 + z_x^2) z_{yy}}
\end{equation}

By replacing the partial derivatives of $z$ in terms of the derivatives of $w$, we find the linearization $L$ to the fully non linear operator given by the Equation (5.5).

Linearization. Let $\tilde{w}$ be a point close to the initial data $w_0$. Then, the linearization of the operator given by the Equation (5.5) is given by:

\[ L(\tilde{w}) = a_{11} \tilde{w}_{11} + 2a_{12} \tilde{w}_{12} + a_{22} \tilde{w}_{22} + b_1 \tilde{w}_1 + b_2 \tilde{w}_2 + c \tilde{w} + d \]

where its coefficients behave as follows:

\[
\begin{align*}
a_{11} &\approx \frac{w_2^2}{w_{12}} g_{11}; \quad a_{12} \approx \frac{w_{11} w_{12}}{w_2^2} g_{12}; \\
a_{11} &\approx \frac{w_2^2}{w_{12}} g_{22}; \quad b_1 \approx \frac{w_{11} w_1}{w_2^2} h_1; \\
b_2 &\approx \frac{w_2}{w_{11}} h_2
\end{align*}
\]

where $\{g_{ij}\}, h_i, c$ and $d$ with $i, j = 1, 2$ are functions which belong to the space $C^\alpha_s$. In particular $g_{22}$ and $b_2$ belong to $C^{\alpha,p}_s$. It follows that the coefficients of the operator $L$ satisfy the same conditions of the operator of Theorem 3.

Based on the above definition, we state the following theorem:

**Theorem 5.** Assume that the initial surface $\Sigma_0$ belongs to the class $C^{k+2+\alpha,p}_s$ and satisfies the non-degeneracy condition ($\star$). Then, under the coordinate change $\Phi$, the HMCF with initial data the surface $\Sigma_0$ converts into the initial value problem:

\[
\begin{align*}
M w &= 0 \quad \text{on } D \times [0,T] \\
w &= w_0 \quad \text{at} \quad t = 0
\end{align*}
\]

with $w_0 \in C^{k+2+\alpha,p}_s(D)$ and

\[ Mw = w_t - F(t,u,v,w,Dw,D^2w) \]

satisfying the hypotheses of Theorem 4.

As an immediate Corollary of Theorem 4 and Theorem 5 we obtain the following existence result:

**Theorem 5.1.** Under the same assumptions as in Theorem 4, there exists a number $\tau_k > 0$ for which the HMCF with initial data the surface $\Sigma_0$ admits a solution $\Sigma_t$ on $0 \leq t \leq \tau_k$. Moreover, under the coordinate change $\Phi$ the strictly convex part $\Sigma_2(t)$ of $\Sigma_t$ is converted to a function $w(t)$ which belongs to the Hölder class $C^{k+2+\alpha,p}_s(Q_k)$, on $Q_k = D \times [0,\tau_k]$.

**Theorem 6.** Assume that the initial surface $\Sigma_0$ satisfies the assumptions of Theorem 4. Then, the solution $\Sigma_t$ of the HMCF is converted, via the coordinate change studied in Section 4, to a function $w$ which belongs, for any positive integer $k$, to the
H"older class $C^{k+2+\alpha,p}(Q)$, on $Q = \mathcal{D} \times (0,T]$. Moreover, for any $\tau$ in $0 < \tau < T$ we have
\begin{equation}
||w||_{C^{k+2+\alpha,p}(\mathcal{D} \times [\tau,T])} \leq C_k(\tau, ||w_0||_{C^{2+\alpha,p}(\mathcal{D})})
\end{equation}

Proof. We omit the proof as it is similar to the one done in [9]. Also, for more details we invite the reader to look at the Ph.D. thesis [5] of the first author. □

6. THE PROOF OF THE MAIN THEOREM

In this section we will give the proof of the Main Theorem stated in the Introduction. We will actually prove the following stronger result, where we relax the regularity assumptions on the initial surface.

Theorem 7. Assume that the strictly convex part $\Sigma_2$ of the initial surface $\Sigma_0$ belongs to the class $C^{2+\alpha,p}$ and satisfies the non-degeneracy conditions ($\star$). Then, the HMCF
\[
\frac{\partial \mathcal{S}}{\partial t} = \frac{K}{H} \mathcal{N} \quad t \in [0,T]
\]
with initial data the surface $\Sigma_0$ admits a solution $\Sigma_t$ which is smooth up to the interface, for $0 < t \leq T$. In particular, the interface $\Gamma_t$ is a smooth curve for every $0 < t \leq T$ which moves by the curve shortening Flow.

Remark. It can be easily checked that if the initial surface satisfies the conditions of the Main Theorem, then it will satisfy the weaker conditions of Theorem 7.

Proof. Assume that the strictly convex part $\Sigma_2$ of the initial surface $\Sigma_0$ belongs to the class $C^{2+\alpha,p}$ and satisfies the non-degeneracy condition ($\star$), then we have proven existence for the HMCF in Theorem 5.

From Theorem 5 we have that $w \in C^{k+2+\alpha,p}(\mathcal{D} \times (0,T])$, for all nonnegative integers $k$. In particular this implies that for all integers $k$ we have $w(t) \in C^{k+\alpha,p}(\mathcal{D})$, for all $\tau$ in $0 < t < T$. It follows that $w$ is $C^{\infty,p}$ smooth up to the boundary of $\mathcal{D}$. Going back to the original coordinates, we conclude that the strictly convex part of the surface $\Sigma_t$, $0 < t \leq T$ is smooth up to $z = 0$ and that the interface $\Gamma_t$ is smooth. □

7. COMPARISON PRINCIPLE

In this final section we will give the proof of the comparison principle for the HMCF and we will show that the solution given in the Main Theorem is a viscosity solution.

Proposition 7.1. (Comparison principle) Let $\Sigma_0$ be a surface of class $C^{2+\alpha,p}$ that satisfies condition ($\star$), and let $\Sigma^+$ be a smooth, strictly convex surface containing $\Sigma_0$ at time $t = 0$, then the surface $\Sigma_t$ obtained by evolving $\Sigma_0$ by the HMCF, is contained in the surface $\Sigma^+_t$ obtained by evolving $\Sigma^+_0$ by the HMCF up to the time of existence of $\Sigma_t$. Analogously, if $\Sigma_0$ contains a smooth, strictly convex surface $\Sigma^-$ at time $t = 0$, then the surface $\Sigma_t$ contains the surface $\Sigma^-_t$ obtained by evolving $\Sigma^-_0$ by the HMCF up to the time of existence of $\Sigma_t$.

Proof. We observe that by the classical maximum principle the surfaces $\Sigma_t$ and $\Sigma^+_t$ cannot touch were they are both strictly convex. Next, we suppose that there exists a time $\bar{t}$ where they first touch at a point $\bar{P}$, then this cannot happen in the interior of the flat side. Hence $\bar{P}$ belongs to the boundary of the flat side.
Suppose $\Sigma_t$ has the flat side on the $x = 0$ plane, then the tangent plane to the surface at the point $\bar{P}$ would be contained in the $x = 0$ plane. Now if the two regions touch, then, because they are of class $C^1$, the tangent to $\Sigma^+_t$ at $\bar{P}$ would be contained in the $x = 0$ plane. But $\Sigma^+_t$ is strictly convex, hence this would imply that a part of $\Sigma^+_t$ is inside $\Sigma_t$ which leads to a contradiction. The second part of the proof is straightforward since if $\Sigma$ contains a smooth, strictly convex surface $\Sigma^-$ at time $t = 0$, then the two surfaces cannot touch at the flat side of $\Sigma_t$ because the flat side does not move in its normal direction. Once again, by the classical maximum principle for parabolic equations the two surfaces cannot touch where they are strictly convex either.

\[ \Box \]

**Corollary 1. (Viscosity solutions)** Let $h \in \mathbb{N}$, $0 < \gamma \leq 1$. Let $\Sigma_0$ be a convex surface of class $C^{h,\gamma}$ that satisfies the hypothesis of the Main Theorem. Then the solution $\Sigma_t$ given by Theorem 7 is a viscosity solution of class $C^{h,\gamma}$. Moreover, this solution is unique upon satisfying the non-degeneracy condition (⋆).

**Proof.** It is a consequence of the comparison principle. \[ \Box \]

**References**

[1] B. Andrews. Contraction of convex hypersurfaces in Riemannian spaces J. Diff. Geometry. 39, no. 2, 407-431, 1994

[2] B. Andrews. Motion of hypersurfaces by Gauss curvature Pacific J. Math. 195, no. 1, 1-34, 2000

[3] B. Andrews. Pinching estimates and motion of hypersurfaces by curvature functions ArXiv: Math. DG /0402311

[4] S. Altschuler. A geometric heat flow for one-forms on three dimensional manifolds Illinois J. Math 39, 98-118, 1995

[5] M.C. Caputo. Highly Degenerate Harmonic Mean Curvature Flow PhD Thesis, Columbia University, May 2006

[6] I. Chavel. Riemannian Geometry: A Modern Introduction. New York: Cambridge University Press, 1994

[7] P. Daskalopoulos, R. Hamilton. Harmonic mean curvature flow on Surfaces of Negative Gaussian Curvature. Comm. Anal. Geom, Vol 14, N. 5, 907-943, 2006

[8] P. Daskalopoulos, R. Hamilton. The free boundary for the n-dimensional porous medium equation. Internat. Math. Res. Notices, 17: 817–831, 1997

[9] P. Daskalopoulos, R. Hamilton, The Free Boundary on the Gauss curvature Flow with Flat Sides, J. Reine Angew. Math., no 510, (1999) pp 187-227.

[10] S. Dieter. Nonlinear degenerate Curvature flow for weakly convex hypersurfaces Calculus of Variations and Partial Differential Equations, 22, 2: 229 - 251, 2005

[11] M.E. Gage, R. Hamilton. The heat equation shrinking convex plane curves. J. Differential Geometry no. 23, 69–96, 1986

[12] O. A. Ladyzhenskaya, V.A. Solonnikov and N.N. Ural’ceva. Linear and quasilinear equations of parabolic type Amer. Math. Soc. Providence, RI 1968

**Department of Mathematics, University of Texas at Austin, TX**

**E-mail address:** caputo@math.utexas.edu

**Department of Mathematics, Columbia University, NY**

**E-mail address:** pdaskalo@math.columbia.edu