Sharp weak bounds and limiting weak-type behavior for Hardy type operators

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Abstract In this paper, Hardy type operator $H_\beta$ on $\mathbb{R}^n$ and its adjoint operator $H^*_\beta$ are investigated. We use novel methods to obtain two main results. One is that we obtain the operators $H_\beta$ and $H^*_\beta$ being bounded from $L^p(|x|^\alpha)$ to $L^{q,\infty}(|x|^\gamma)$, and the bounds of the operators $H_\beta$ and $H^*_\beta$ are sharp worked out. In particular, when $\alpha = \gamma = 0$, the norm of $H_\beta$ is equal to 1. The other is that we study limiting weak-type behavior for the operator $H_\beta$ and its optimal form was obtained.

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1 Introduction

Let $f$ be a non-negative integrable function on $\mathbb{R}^n$. Christ and Grafakos [11] gave the following definition of $n$-dimensional Hardy operator and its adjoint operator

$$Hf(x) = \frac{1}{|B(0,|x|)|} \int_{|y|<|x|} f(y)dy$$

and

$$H^*f(x) = \int_{|y|\geq|x|} \frac{f(y)}{|B(0,|y|)|}dy,$$

where $x \in \mathbb{R}^n \setminus \{0\}$. In 2012, Zhao, Fu and Lu [12] obtained the weak $(p,p)$ norm of the operator $H$ is 1 with $1 \leq p < \infty$. Furthermore, in 2015, Gao and Zhao showed the sharp constants of the power weight case for the weak $(1,1)$ of $H$ and $H^*$, which can found in the paper [2].

The purpose of this paper is to study the weak bounds property of the $n$-dimensional Hardy type operator along nonnegative locally integrable function $f$ on $\mathbb{R}^n$ which have the form

$$H_\beta f(x) = \frac{1}{|B(0,|x|)|^{1-\frac{\beta}{n}}} \int_{|y|<|x|} f(y)dy$$

(1.1)

and

$$H^*_\beta f(x) = \int_{|y|\geq|x|} \frac{f(y)}{|B(0,|y|)|^{1-\frac{\beta}{n}}}dy,$$

(1.2)
where \( x \in \mathbb{R}^n \setminus \{0\} \) and \( 0 \leq \beta < n \). Here the operator \( H^{*}_{\beta} \) is the adjoint of \( H_{\beta} \), namely the operators \( H_{\beta} \) and \( H^{*}_{\beta} \) satisfy

\[
\int_{\mathbb{R}^n} H_{\beta} f(x) g(x) dx = \int_{\mathbb{R}^n} f(x) H^{*}_{\beta} g(x) dx.
\]

Specifically, when \( \beta = 0 \), we denote \( H_{\beta} \) and \( H^{*}_{\beta} \) by \( H \) and \( H^{*} \), respectively. In 2013, Lu, Yan and Zhao \cite{7} considered the operators defined by (1.1) and obtained weak \((1, \frac{n}{n-\beta})\) bounds is 1.

In \cite{2}, Gao and Zhao studied the Hardy type operator \( H^{*}_{\beta} \), and obtained the following result.

**Theorem A**  
Let \( 0 < \beta < n, 1 \leq p < \infty \) and \( \frac{1}{q} = \frac{1}{p} - \frac{\beta}{n} \), then

\[
\| H^{*}_{\beta} f \|_{L^p(\mathbb{R}^n)} \leq \| f \|_{L^q(\mathbb{R}^n)}.
\]

If \( f \) is a nonnegative locally integrable function on \( \mathbb{R}^n \) and \( 0 < \beta < n \), it easy to obtained weak boundness of the operator \( H_{\beta} \)

\[
\| H_{\beta} f \|_{L^q(\mathbb{R}^n)} \leq 1 \cdot \| f \|_{L^p(\mathbb{R}^n)}.
\]  

(1.3)

But the constant 1 in (1.3) is sharp? For many years, nobody is really sure. Our reslut following will give the affirmative answer. Because it is much hard to apply the method in \cite{12} \cite{2}, we find a new method to prove it.

For the weighted case, in 1972, Muckenhoupt \cite{8} considered the weighted Hardy operator with general weight functions \( u \) and \( v \). However, it it is much hard to apply the method in \cite{8} to solve the weak type estimate of Hardy operator. In \cite{2}, Gao and Zhao studied the Hardy operator with same power weight, and obtained the sharp weak bounds of \( H \) and \( H^{*} \). We main study different power weight and our results following will show that the operator \( H_{\beta} \) and \( H^{*}_{\beta} \) weak bounds are best.

In this paper, we use novel methods and ideas to study Hardy type operators with power weights on \( \mathbb{R}^n \). For generality, we will through make a change of variables to transform the known results, and establish the boundedness of Hardy type with power weights. Moreover, the bounds of operators are sharp worked out. Our proof is very concise and simple.

Throughout this paper, \( 1 < p, p', q < \infty, p \) and \( p' \) are conjugate indices, i.e. \( 1/p + 1/p' = 1 \). Formally, we will also define \( p = 1 \) as conjugate to \( p' = \infty \) and vice versa.

Now, we formulate our main results as follows.

**Theorem 1.1** Suppose that \( f \) is a nonnegative locally integrable function on \( \mathbb{R}^n \). Let \( 1 \leq p < \infty, 1 < q < \infty, 0 \leq \beta < n, \gamma > -n, \alpha \leq \beta(p - 1) \) and \( \frac{n+\beta}{q} + \beta = \frac{\alpha+n}{p} \), then the inequality

\[
\| H_{\beta} f \|_{L^q(|x|^\gamma)} \leq C_{\text{sharp}} \| f \|_{L^p(|x|^\alpha)}
\]

holds with sharp constant

\[
C_{\text{sharp}} = \left[ \frac{n(p-1)}{n(p-1)-\alpha} \right]^{\frac{1}{p'}} \left( \frac{n}{n+\gamma} \right)^{\frac{\beta}{q}} v_n^{\frac{\alpha+n}{q} - \frac{1}{p'}}.
\]

**Theorem 1.2** Suppose that \( f \) is a nonnegative locally integrable function on \( \mathbb{R}^n \). Let \( 1 \leq p < \infty, 1 < q < \infty, 0 \leq \beta < n, \gamma > -n \) and \( \frac{n+\beta}{q} + \beta = \frac{\alpha+n}{p} \), then the inequality

\[
\| H^{*}_{\beta} f \|_{L^q(|x|^\gamma)} \leq C_{\text{sharp}}^* \| f \|_{L^p(|x|^\alpha)}
\]

holds with sharp constant

\[
C_{\text{sharp}}^* = \left( \frac{q}{p'} \right)^{\frac{1}{p'}} \left( \frac{n}{n+\gamma} \right)^{\frac{\beta+p}{q}} v_n^{\frac{\alpha+n}{q} - \frac{1}{p'}}.
\]
Remark 1.3 When $\alpha = \gamma = 0$, the conclusion of Theorem 1.1 will coincide with that of inequality (1.3). This confirms that constant in (1.3) is sharp.

Remark 1.4 We note that our results contain all possible case of weak type estimate for Hardy type operator with power weight. This means that we completely solve problem about weak bounds for Hardy type operators.

2 Preliminaries

To reduce the dimension of function space, we need the following lemma which was obtained in [7, 10].

Lemma 2.1 For a function $f \in L^p(\mathbb{R}^n)$, let

$$
g_f(y) = \frac{1}{\omega_n} \int_{|\xi|=1} |f(|y|\xi)|d\xi, \quad y \in \mathbb{R}^n,
$$

where $\omega_n = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$. Then

$$
H_\beta(|f|)(x) = H_\beta(g_f)(x), \quad H_\beta^*(|f|)(x) = H_\beta^*(g_f)(x)
$$

and choose $\alpha$ satisfy appropriate condition

$$
\|g_f\|_{L^p(|x|^\alpha)} \leq \|f\|_{L^p(|x|^\alpha)}.
$$

Remark 2.2 It follows from the lemma 2.1 that for $\gamma > -n$

$$
\frac{\|H_\beta(f)\|_{L^{q,\infty}(|x|^\gamma)}}{\|f\|_{L^p(|x|^\alpha)}} \leq \frac{\|H_\beta(g_f)\|_{L^{q,\infty}(|x|^\gamma)}}{\|g_f\|_{L^p(|x|^\alpha)}} \quad \text{and} \quad \frac{\|H_\beta^*(f)\|_{L^{q,\infty}(|x|^\gamma)}}{\|f\|_{L^p(|x|^\alpha)}} \leq \frac{\|H_\beta^*(g_f)\|_{L^{q,\infty}(|x|^\gamma)}}{\|g_f\|_{L^p(|x|^\alpha)}}.
$$

Therefore, the norm of the operators $H_\beta$ and $H_\beta^*$ from $L^p(|x|^\alpha)$ to $L^q(|x|^\gamma)$ are equal to that $H_\beta$ and $H_\beta^*$ restrict to radial functions.

With the help of previous consequences, we shall prove our main statements.

3 Proof of main results

We present proof of our main results in this section.

Proof of Theorem 1.1 We will divide into two cases to prove this theorem: $\alpha = 0$, $\alpha \neq 0$.

Case 1: $\alpha = 0$. Since $\frac{n+\gamma}{q} + \beta = \frac{n}{p}$ and using Hölder’s inequality, we have

$$
H_\beta f(x) = \frac{1}{|B(0,|x|)|^{\frac{n}{p}} - \frac{n+\gamma}{n}} \int_{|y|<|x|} f(y)dy \leq \frac{1}{\nu_n}\|f\|_{L^p(\mathbb{R}^n)}.
$$

Hence,

$$
\lambda \left( \int_{\{x \in \mathbb{R}^n : |H_\beta f(x)| > \lambda \}} |x|^\gamma dx \right)^{\frac{1}{\gamma}} \leq \nu_n^{\frac{\beta}{\gamma}} \|f\|_{L^p(\mathbb{R}^n)} \lambda \left( \int_{\{x \in \mathbb{R}^n : |x|^{\beta - \frac{\beta}{\gamma}} > \lambda \}} |x|^\gamma dx \right)^{\frac{1}{\gamma}}
$$

$$
= \nu_n^{\frac{\beta}{\gamma}} \left( \frac{n}{n+\gamma} \right)^{\frac{1}{\gamma}} \|f\|_{L^p(\mathbb{R}^n)}, \quad (3.6)
$$
where we use the condition $\gamma > -n$ of the last step.

On the other hand, we have following estimate for $H_{\beta}$

$$\int_{\{x \in \mathbb{R}^n: H_{\beta}f(x) > \lambda\}} |x|^\gamma dx \geq \int_{\{x \in B(0, (1/v_n)^{1/\beta}^n): H_{\beta}f(x) > \lambda\}} |x|^\gamma dx \geq \int_{\{x \in B(0, (1/v_n)^{1/\beta}^n): H_{\beta,p}f(x) > \lambda\}} |x|^\gamma dx,$$

where

$$H_{\beta,p}f(x) := \frac{1}{|B(0, |x|)^{\frac{1}{\beta}}|^{\frac{n}{\beta}}} \int_{|y| < |x|} f(y)dy, \quad x \in B(0, (1/v_n)^{\frac{1}{\beta}}^n).$$

Therefore,

$$\|H_{\beta}\|_{L^p(\mathbb{R}^n) \to L^{q,\infty}(\mathbb{R}^n)} = \sup_{\|f\|_{L^p(\mathbb{R}^n)} \neq 0} \frac{\|H_{\beta}f\|_{L^{q,\infty}(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)}} \geq \frac{\|H_{\beta,p}f\|_{L^{q,\infty}(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)}}. \quad (3.8)$$

Now taking

$$f_0(x) = \chi_{B(0, (1/v_n)^{\frac{1}{\beta}} - \delta)}(x),$$

for any $0 < \delta < (1/v_n)^{\frac{1}{\beta}}$, we have

$$f_0 \in L^p(\mathbb{R}^n) \quad \text{and} \quad \|f_0\|_{L^p(\mathbb{R}^n)} = v_n^{\frac{1}{p}} ((1/v_n)^{\frac{1}{\beta}} - \delta)^{\frac{n}{\beta}}.$$

If $0 < |x| \leq (1/v_n)^{\frac{1}{\beta}} - \delta$, we obtain

$$H_{\beta,p}f(x) = \frac{1}{(0, |x|)^{\frac{1}{\beta}}|^{\frac{n}{\beta}}} \int_{|y| < |x|} \chi_{B(0, (1/v_n)^{\frac{1}{\beta}} - \delta)}(y)dy = v_n^{\frac{1}{p}} |x|^{\frac{\beta}{p}}.$$  

If $(1/v_n)^{\frac{1}{\beta}} - \delta \leq |x| \leq (1/v_n)^{\frac{1}{\beta}}$, we get $H_{\beta,p}f(x) = v_n^{\frac{1}{p}} ((1/v_n)^{\frac{1}{\beta}} - \delta)^n |x|^{\frac{n}{\beta}}$.  

For any $\lambda > 0$, we conclude that

$$\int_{\{x \in B(0, (1/v_n)^{\frac{1}{\beta}}^n): H_{\beta,p}^*f_0(x) > \lambda\}} |x|^\gamma dx \geq \int_{\{x \in B(0, (1/v_n)^{\frac{1}{\beta}}^n): H_{\beta,p}f_0(x) > \lambda\}} |x|^\gamma dx + \int_{\{x \in B(0, (1/v_n)^{\frac{1}{\beta}}^n \setminus B(0, (1/v_n)^{\frac{1}{\beta} - \delta})^n): H_{\beta,p}^*f_0(x) > \lambda\}} |x|^\gamma dx$$

$$= \left[\frac{\lambda^{\frac{\beta}{p}} + \frac{\lambda^{\frac{\beta}{p}}}{v_n} v_n^{\frac{\beta}{p}} ((1/v_n)^{\frac{1}{\beta}} - \delta)^n}{\lambda} \right]^{\frac{n+\gamma}{n+\frac{n}{\beta}}} v_n^{\frac{n}{\beta}} \left[\frac{(1/v_n)^{\frac{1}{\beta}} - \delta)^n}{v_n^{\frac{\beta}{p}} + \frac{\lambda^{\frac{\beta}{p}}}{v_n}} \right]^{\frac{n+\gamma}{n+\frac{n}{\beta}}} v_n^{\frac{n}{\beta}}.$$  

If there exists a constant $C$ such that

$$\int_{\{x \in B(0, (1/v_n)^{\frac{1}{\beta}}^n): H_{\beta}^*(f_0(x) > \lambda\}} |x|^\gamma dx \leq \left(\frac{C\|f\|_{L^p(\mathbb{R}^n)}}{\lambda}\right)^q$$

holds for all $f \in L^p(\mathbb{R}^n)$. Then we can choose that

$$f(x) = f_0(x) = \chi_{B(0, (1/v_n)^{\frac{1}{\beta}} - \delta)}(x).$$
It follows from equality (3.9) that
\[
\left[ \frac{v_{n}^{\frac{1}{p}+\frac{1}{q}} t}{(1/v_{n}^{\frac{1}{p}} - \delta)^{n}} \right]^{q} v_{n}^{n+1} \frac{n v_{n}}{n+\gamma} \left[ \frac{\lambda}{v_{n}^{\frac{1}{p}+\frac{1}{q}}} \right]^{\frac{n+\gamma}{n+\gamma}} n v_{n}^{n+1} \frac{n v_{n}}{n+\gamma} \leq \left( \frac{C ||f_{0}||_{L^{p}(\mathbb{R}^{n})}}{\lambda} \right)^{q}
\]
always holds for every \( \lambda > 0 \). Letting \( \lambda \to 0^{+} \) and \( \delta \to 0^{+} \), this forces that
\[
C \geq v_{n}^{\frac{-q}{nq}} \left( \frac{n}{n+\gamma} \right)^{\frac{1}{q}}.
\] (3.10)

Combining (3.6), (3.8) together (3.10), we obtain that
\[
\|H_{\beta}\|_{L^{p}(\mathbb{R}^{n})} \to L^{q,\infty}(|x|^{\alpha}) = v_{n}^{\frac{-q}{nq}} \left( \frac{n}{n+\gamma} \right)^{\frac{1}{q}}.
\] (3.11)

Case 2: \( \alpha \neq 0 \). By using main result and idea for case \( \alpha = 0 \), we derive the sharp result for the case \( \alpha \neq 0 \). It follows from Lemma 2.1 that the norm of the norm of operator \( H_{\beta} \) from \( L^{p}(|x|^{\alpha}) \) to \( L^{q,\infty}(|x|^{\gamma}) \) is equal to the norm \( H_{\beta} \) restricts to radial functions. Consequently, without loss of generality, it suffices to carry out the proof of the case by assuming that \( f \) is a nonnegative, radial, smooth function with compact support on \( \mathbb{R}^{n} \).

Using the polar coordinate transformation, we can rewrite (1.4) as
\[
\sup_{\lambda > 0} \lambda \left( \int_{E_{n,\beta}} x^{\gamma+n-1} dx \right)^{\frac{1}{q}} \leq v_{n}^{\frac{1}{p}+\frac{1}{q}} n^{\frac{1}{q}} \left( \frac{n}{p-1} \right)^{-\frac{1}{q}} C_{\text{sharp}} \left( \int_{0}^{\infty} f^{p}(x) x^{\alpha+n-1} dx \right)^{\frac{1}{p}}
\] (3.12)
where
\[
E_{n,\beta} := \left\{ x \in (0, \infty) : \frac{1}{x^{n-\beta}} \int_{0}^{x} f(t) t^{n-1} dt > \lambda \right\}.
\]

First, we make a change of variables in (3.12) by putting
\[
s = s(x) = \int_{0}^{x} t^{\frac{\alpha}{p-1}} t^{n-1} dt = \frac{p-1}{n(p-1) - \alpha} x^{\frac{\alpha(p-1)-\alpha}{p-1}}
\] (3.13)
and define
\[
g(s) = g(s(x)) = f(x) x^{\frac{\alpha}{p-1}}.
\] (3.14)

Then
\[
\int_{0}^{\infty} f^{p}(x) x^{\alpha+n-1} dx = \int_{0}^{\infty} f^{p}(x) x^{\frac{\alpha}{p-1}} x^{-\frac{\alpha}{p-1}} x^{n-1} dx = \int_{0}^{\infty} g(s) ds
\] (3.15)
and
\[
H_{0} := \lambda \left( \int_{E_{n,\beta}} x^{\gamma+n-1} dx \right)^{\frac{1}{q}} = \lambda \left( \int_{E_{n,\beta}} x^{\frac{(\alpha+n)q}{p}-\beta q-1} dx \right)^{\frac{1}{q}}
\]
\[
= \lambda \left( \int_{E_{n,\beta}} x^{\frac{(\alpha+n)q}{p}-\beta q-n+\frac{\alpha}{p-1}+n-1} dx \right)^{\frac{1}{q}}.
\]
Hence, since $f(t)t_1^{n-1}dt = f(t)t^{n-1}t^{-\alpha/n} + n-1 dt = g(r)dr$ and $x = \left(\frac{n(p-1) - \alpha}{p-1}\right)^{\frac{1}{q}}$, we have

$$II_0 = \left[\frac{p-1}{n(p-1) - \alpha}\right]^{\frac{1}{q} + \frac{1}{q}} \lambda \left(\int_{E_{n,p,\alpha,\beta}} \frac{(\alpha + n - \beta p)q(p-1)}{p(n(p-1) - \alpha^{\frac{1}{q}})} ds \right)^{\frac{1}{q}}$$

(3.16)

where

$$E_{n,p,\alpha,\beta} = \left\{ s \in (0, \infty) : \frac{1}{s^{\frac{1}{1} - \frac{n(p-1) - \alpha}{n}} - 1} \int_0^s g(r)dr > \lambda \right\}.$$ 

Now let

$$\gamma' = \frac{(\alpha + n - \beta p)q(p-1)}{p(n(p-1) - \alpha)} - 1, \quad \alpha' = 0 \quad \text{and} \quad \beta' = \frac{\beta(p-1) - \alpha}{n(p-1) - \alpha},$$

then $\gamma', \alpha', \beta'$ satisfy condition $\frac{\gamma' + 1}{q} + \beta' = \frac{\alpha' + 1}{q}$ and $0 \leq \beta' < 1$. Combining (3.15), (3.16) and applying result of the case 1 with $n = 1$, we find that

$$\sup_{\lambda > 0} II_0 \leq \left[\frac{p-1}{n(p-1) - \alpha}\right]^{\frac{1}{q} + \frac{1}{q}} \left(1 + \gamma'\right)^{\frac{1}{q}} \|g\|_{L^p(\mathbb{R})}. \quad (3.17)$$

Comparing (3.12) with (3.17) implies that

$$\frac{1}{v_n} \frac{q-\beta}{n} \left[\frac{p-1}{n(p-1) - \alpha}\right]^{\frac{1}{q} + \frac{1}{q}} \left[\frac{p(n(p-1) - \alpha)}{(\alpha + n - \beta p)q(p-1)}\right]^{\frac{1}{q}} C_{\text{sharp}} = \left[\frac{n(p-1)}{n(p-1) - \alpha}\right]^{\frac{1}{q} + \frac{1}{q}} \left(\frac{n}{n + \gamma}\right)^{\frac{1}{q}} v_n^{\frac{q}{q} + \frac{1}{p} - \frac{1}{q}} C_{\text{sharp}}.$$ 

Therefore,

$$C_{\text{sharp}} = \left[\frac{n(p-1)}{n(p-1) - \alpha}\right]^{\frac{1}{q} + \frac{1}{q}} \left(\frac{n}{n + \gamma}\right)^{\frac{1}{q}} v_n^{\frac{q}{q} + \frac{1}{p} - \frac{1}{q}} C_{\text{sharp}}.$$ 

This completes the proof of Theorem 1.4

\[\Box\]

**Proof of Theorem 1.2** The proof of theorem 1.2 is similar to prove theorem 1.1. We will consider two case to prove theorem 1.2 for $\alpha = 0$ and $\alpha \neq 0$.

First we consider case for $\alpha = 0$. Since $\beta, \gamma$ satisfy condition $\frac{\gamma + n}{q} + \beta = \frac{\alpha}{p}$ and

$$H_\beta f(x) \leq \left(\int_{|t|<|x|} \frac{1}{|B(0,|x|)|p'(1-\beta/n)}} dt\right)^{\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}^n)} = v_n^{\frac{q}{p} + \frac{1}{q} - \frac{1}{q}} \left[\frac{nq}{p'(\gamma + n)}\right]^{\frac{1}{p'}} |x|^{-\frac{\gamma + n}{q}} ||f||_{L^p(\mathbb{R}^n)},$$

we have that

$$\lambda \left(\int_{\{x \in \mathbb{R}^n : |H_\beta f(x)| > \lambda\}} |x|^\gamma dx\right)^{\frac{1}{\gamma}} = v_n^{\frac{q}{p} + \frac{1}{q} - \frac{1}{q}} \left[\frac{nq}{p'(\gamma + n)}\right]^{\frac{1}{p'}} ||f||_{L^p(\mathbb{R}^n)} \lambda \left(\int_{\{x \in \mathbb{R}^n : |x|^\gamma dx > \lambda\}} |x|^\gamma dx\right)^{\frac{1}{\gamma}} = v_n^{\frac{q}{p} + \frac{1}{q} - \frac{1}{q}} \left(\frac{n}{\gamma + n}\right)^{\frac{1}{p'}} \left(\frac{q}{p'}\right)^{\frac{1}{p'}} ||f||_{L^p(\mathbb{R}^n)}.$$ 

(3.18)

On the other hand, we consider test function

$$f_0^\delta(x) = |x|^{\frac{\beta - n}{n}} \chi_{\{x \in \mathbb{R}^n : |x| > 1\}^o}(x).$$
A simple calculation shows that
\[ f_0^* \in L^p(\mathbb{R}^n) \quad \text{and} \quad \|f_0^*\|_{L^p(\mathbb{R}^n)} = \left(\frac{p-1}{n-p\beta} |S^{n-1}|\right)^{\frac{1}{p}}. \]

Note that \( \frac{n+\gamma}{q} = \frac{n}{p} - \beta \) and \( v_n = \frac{|S^{n-1}|}{n} \). Hence,
\[ \|f_0^*\|_{L^p(\mathbb{R}^n)} = \left[ \frac{qnv_n}{p'(\gamma + n)} \right]^{\frac{1}{p}}. \]

By the definition of \( H_{\beta}^* \), we obtain
\[ H_{\beta}^* f_0^*(x) = \frac{1}{v_n^{1-\beta/n}} \int_{|t|\geq|x|} |t|^\frac{\beta-n}{p-1} \chi_{\{t \in \mathbb{R}^n : |t| > 1\}}(t) dt. \]
If \(|x| \leq 1\), we have
\[ H_{\beta}^* f_0^*(x) = \frac{1}{v_n^{1-\beta/n}} \int_{|t|\geq|x|} |t|^\frac{\beta-n}{p-1} dt = \frac{n(p-1)}{n-p\beta} v_n^\frac{\beta}{n} = \frac{qn}{p'(n+\gamma)} v_n^\frac{\beta}{n}. \]
If \(|x| > 1\), we have
\[ H_{\beta}^* f_0^*(x) = \frac{1}{v_n^{1-\beta/n}} \int_{|t|\geq|x|} |t|^\frac{\beta-n}{p-1} dt = \frac{qn}{p'(n+\gamma)} v_n^\frac{\beta}{n} |x|^{-\frac{p'(n+\gamma)}{q}} < \frac{qn}{p'(n+\gamma)} v_n^\frac{\beta}{n}. \]

Therefore, we obtain
\[ \left\{ x \in \mathbb{R}^n : |H_{\beta}^* f_0^*(x)| > \lambda \right\} = \left\{ |x| \leq 1 : \frac{qn}{p'(n+\gamma)} v_n^\frac{\beta}{n} > \lambda \right\} \cup \left\{ |x| > 1 : \frac{qn}{p'(n+\gamma)} v_n^\frac{\beta}{n} |x|^{-\frac{p'(n+\gamma)}{q}} > \lambda \right\}. \]

In fact, we can easily verify that if \( \lambda > \frac{qn}{p'(n+\gamma)} v_n^\frac{\beta}{n} \), then
\[ E := \left\{ x \in \mathbb{R}^n : |H_{\beta}^* f_0^*(x)| > \lambda \right\} = \emptyset. \]

Let \( L = \frac{qn}{p'(n+\gamma)} v_n^\frac{\beta}{n} \). For \( 0 < \lambda < L \), we obtain
\[ E = \left\{ x \in \mathbb{R}^n : |x| < \left(\frac{L}{\lambda}\right)^{\frac{q}{p'n}} \right\}. \]

Therefore, we have
\[
\|H_{\beta}^* f_0^*\|_{L^{q,\infty}(\mathbb{R}^n)} = \sup_{0<\lambda<L} \lambda \left( \frac{1}{v_n^{\frac{\beta}{n}}} \int_E |x|^\gamma dx \right)^{\frac{q}{\gamma}} = \sup_{0<\lambda<L} \lambda v_n^{\frac{1}{p'} \left(\frac{L}{\lambda}\right)^{\frac{1}{p'}}} \left( \frac{n}{n+\gamma} \right)^{\frac{1}{q}} = \frac{q}{p'} \left( \frac{n}{n+\gamma} \right)^{\frac{1}{q}+1} v_n^{\frac{\beta}{n} + \frac{1}{q}}.
\]
Then we make a change of variables in (3.19) by taking

\[ H^*_\beta f_0 \|_{L^q(\mathbb{R}^n)} = v_n^{\frac{1}{q} - \frac{1}{p} + \frac{2}{n}} \left( \frac{n}{n + \gamma} \right) \frac{1}{p^\frac{1}{q}} \| f_0 \|_{L^p(\mathbb{R}^n)}. \]

Hence, since

\[ \| H^*_\beta f \|_{L^q(\mathbb{R}^n)} = v_n^{-\gamma} \left( \frac{n}{n + \gamma} \right) \frac{1}{p^\frac{1}{q}} \| f \|_{L^p(\mathbb{R}^n)}. \]

Thus we obtain

\[ \| H^*_\beta f \|_{L^q(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)} = v_n^{-\gamma} \left( \frac{n}{n + \gamma} \right) \frac{1}{p^\frac{1}{q}}. \]

Next we consider case for \( \alpha \neq 0 \). It is similar to theorem 1.1 we only need consider the norm that \( H^*_\beta \) restricts to radical functions.

Using the polar coordinate transformation, we can rewrite (1.5) as

\[ f_n(t^\beta) = \int_0^{\infty} f(t) t^{n-1} dt = \int_0^{\infty} g(t) t^{n-1} dt = \int_0^{\infty} g(s) ds \]

where

\[ E^*_n(x) := \left\{ x \in (0, \infty) : \int_0^x \frac{f(t)}{t^{n-1}} dt \right\}. \]

Now we make a change of variables in (3.19) by taking

\[ s = s(x) = \int_0^x t^{\frac{n(\alpha - p\beta + \beta)}{n - \beta}} t^{n-1} dt = \frac{n - \beta}{n(\alpha + n - p\beta)} \]

and define

\[ g(s) = f(s(x)) = f(x)^{\frac{np\beta - (\alpha + n)\beta}{(n - \beta)p}}. \]

Then

\[ \int_0^{\infty} f(t^\beta) t^{n-1} dt = \int_0^{\infty} f(t^\beta) t^{n-1} dt = \int_0^{\infty} g(s) ds \]

and

\[ II^*_\alpha := \lambda \left( \int_{E^*_n} x^{\gamma} x^{n-1} dx \right)^{\frac{1}{q}} = \lambda \left( \int_{E^*_n} x^{\frac{(\alpha + n - p\beta)q}{p - (n - \beta)}} - n x^{n-1} dx \right)^{\frac{1}{q}}. \]

Hence, since \( x = \left( \frac{n(\alpha + n - p\beta)}{n - \beta} \right) s \) and

\[ \frac{f(t)}{t^{n-1}} dt = \frac{f(t^\beta)}{t^{n-1} \frac{np\beta - (\alpha + n)\beta}{(n - \beta)p} + \frac{n(\alpha - p\beta + \beta)}{n - \beta}} t^{n-1} dt = \left[ \frac{n - \beta}{n(\alpha + n - p\beta)} \right]^{1 - \frac{\beta}{pn}} g(r) r^{-\frac{\beta}{pn}} dr, \]

we have that

\[ II^*_\alpha = \left[ \frac{n - \beta}{n(\alpha + n - p\beta)} \right]^{1 + \frac{1}{q}} \lambda \left( \int_{E^*_n} \left( \frac{q}{pn} - 1 \right) ds \right)^{\frac{1}{q}}, \]

(3.21)
where
\[ E_{n,p,\beta}^* = \left\{ s \in (0, \infty) : \int_0^s \frac{g(r)}{r^{1-\frac{\beta}{pm}}} dr > \lambda \right\} . \]

Now we write
\[ \gamma^* = \frac{q(n-\beta)}{pm} - 1, \quad \alpha^* = 0 \quad \text{and} \quad \beta^* = \frac{\beta}{pm} , \]
then \( \alpha^*, \beta^*, \gamma^* \) satisfy condition \( \frac{\gamma^*+1}{q} + \beta^* = \frac{1}{p} \) and \( 0 \leq \beta^* < 1 \). Combining (3.20), (3.21) and using the result for the case \( \alpha = 0 \) with \( n = 1 \), we obtain that
\[ \sup_{\lambda > 0} II_0^* \leq \left( \frac{1}{\gamma + n} \right)^{1+q} \left( \frac{q}{p'} \right)^{\frac{1}{p'}} \|g\|_{L^p(\mathbb{R})} . \] (3.22)
Comparing (3.19) with (3.22), we have that
\[ v_n^{1+\frac{\beta}{n} - \frac{\beta}{p}} C_{\text{sharp}}^* = \left( \frac{1}{\gamma + n} \right)^{1+q} \left( \frac{q}{p'} \right)^{\frac{1}{p'}} . \]
Therefore,
\[ C_{\text{sharp}}^* = v_n^{\frac{n}{\gamma + n} \frac{1+\beta}{n} - \frac{1}{p}} \left( \frac{q}{p'} \right)^{\frac{1}{p'}} . \]
This completes the proof of Theorem 1.2.

4 Limiting weak type behavior for Hardy type operator \( H_\beta \)

In 2006, Janakiraman [5] first studied the limiting weak type estimate for the Hardy-Littlewood maximal function and some class of singular integrals, and many researchers were focused on limiting weak type for some operators, see [3, 4, 6, 11]. Based on this intuition, we consider limiting weak type behavior for Hardy type operator \( H_\beta \).

Now, we formulate our main results as follows.

**Theorem 4.1** Let \( 0 \leq \beta < n, 1 \leq p, q < \infty \) and \( 1/q + \beta/n = 1/p \), for \( f \in L^p(\mathbb{R}^n) \)

(a) If \( p = 1 \), then
\[ \lim_{\lambda \to 0} \{ x \in \mathbb{R}^n : |H_\beta f(x)| > \lambda \}^{1/q} = \|f\|_1 . \] (4.23)

(b) If \( 1 < p < \infty \), then
\[ \lim_{\lambda \to 0} \{ x \in \mathbb{R}^n : |H_\beta f(x)| > \lambda \}^{1/q} = 0 . \]

**Remark 4.2** A proof of a more general statement than Theorem 1.1 (b) may found in the recent paper [7]. The limiting behavior (4.23) is very interesting, it gives some information of the best constant for the weak type \((q,1)\) estimate of the Hardy type operator \( H_\beta \) in some sense.

**Proof of Theorem 4.1** (a) First, without loss of generality, we suppose that \( f \geq 0 \). For any \( t > 0 \), let \( f_t(x) = \frac{1}{t^n} f\left( \frac{x}{t} \right) \), we obtain
\[ \|f_t\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f_t(x) dx = \int_{\mathbb{R}^n} \frac{1}{t^n} f(x/t) dt = \int_{\mathbb{R}^n} f(x) dy = \|f\|_{L^1(\mathbb{R}^n)} . \]
By variables transform, we know
\[
\frac{1}{t^{n-\beta}} H_\beta f(x/t) = \frac{t^{\beta-n}}{|B(0, |x|)|^{1-n}} \int_{|y|<|x|} f(y) dy = \frac{1}{|B(0, |x|)|^{1-n}} \int_{|y|<|x|} \frac{1}{t^n} f(y/t) dy = H_\beta f(x).
\] (4.24)

Applying (4.24), we show that
\[
|\{x \in \mathbb{R}^n : |H_\beta f_t(x)| > \lambda\}| = \left| \left\{ x \in \mathbb{R}^n : \int_{\mathbb{R}^n} f_t(y) dy > \lambda \right\} \right| = t^n |\{x \in \mathbb{R}^n : |H_\beta f(x)| > t^{n-\beta} \lambda\}|.
\]

Hence, for any \( \lambda > 0 \), we have that
\[
\lambda |\{x \in \mathbb{R}^n : |H_\beta f_t(x)| > \lambda\}|^{\frac{n-\beta}{n}} = t^n \lambda \left| \{x \in \mathbb{R}^n : |H_\beta f(x)| > t^{n-\beta} \lambda\} \right|^{\frac{n-\beta}{n}}.
\] (4.25)

Using the equality (4.23) and (4.25), we only need to prove that
\[
\lim_{\epsilon \to 0} \lambda |\{x \in \mathbb{R}^n : |H_\beta f_t(x)| > \lambda\}|^{\frac{n-\beta}{n}} = \|f\|_{L^1(\mathbb{R}^n)}.
\]

Now taking \( 0 < \epsilon_1 < 1, 2\epsilon_t < \epsilon_1 \) and satisfy: if \( t \to 0 \), then \( \epsilon_t \to 0 \). And for any \( \epsilon > 0 \), when \( t \) enough small
\[
\int_{B(0,\epsilon_t/t)} f(y) dy > \|f\|_{L^1(\mathbb{R}^n)} - \epsilon
\]
holds, where \( B(0, \epsilon_t/t) \) denote center at origin and radial \( \epsilon_t/t \) ball.

When \( |x| > \epsilon_1 > 2\epsilon_t \), we now take part two for \( H_\beta f_t(x) \)
\[
H_\beta f_t(x) = \int_{B(0,\epsilon_t)} \frac{1}{|B(0, |x|)|^{1-n}} f_t(y) dy + \int_{B(0,|x|) \setminus B(0,\epsilon_t)} \frac{1}{|B(0, |x|)|^{1-n}} f_t(y) dy := I + II.
\]

Let \( \delta \ll 1 \), we have
\[
\lambda |\{x \in \mathbb{R}^n : |H_\beta f_t(x)| > \lambda\}|^{\frac{n-\beta}{n}} \leq \lambda (|\{x \in \mathbb{R}^n : |I| > (1-\delta)\lambda\}| + |\{x \in \mathbb{R}^n : |II| > \delta\lambda\}|^{\frac{n-\beta}{n}}) \leq \lambda |\{x \in \mathbb{R}^n : |I| > (1-\delta)\lambda\}|^{\frac{n-\beta}{n}} + \lambda |\{x \in \mathbb{R}^n : |II| > \delta\lambda\}|^{\frac{n-\beta}{n}}.
\] (4.26)

Next, we will show above two parts integral. For second \( II \), by weak \( (1, \frac{n}{n-\beta}) \) boundedness of Hardy type operator \( H_\beta \), we have for \( t \) enough small
\[
\lambda |\{x \in \mathbb{R}^n : |II| > \delta\lambda\}|^{\frac{n-\beta}{n}} \leq \lambda |\{x \in \mathbb{R}^n : |H_\beta f_t \chi_{B(0,|x|) \setminus B(0,\epsilon_t)}(x)| > \delta\lambda\}|^{\frac{n-\beta}{n}} \leq \|f_t \chi_{B(0,|x|) \setminus B(0,\epsilon_t)}\|_{L^1(\mathbb{R}^n)} \leq \frac{C_{n,\beta}}{\delta} \int_{B(0,\epsilon_t/t)^c} f(y) dy < \frac{C_{n,\beta} \epsilon}{\delta}.
\] (4.27)
For firstly $I$, when $x \in B(0, \varepsilon_1)^c$ and $t$ enough small, we have

$$
\lambda\{x \in \mathbb{R}^n : |I| > (1 - \delta)\lambda\} \frac{n-\beta}{n} \\
\leq \lambda \left( \{x \in B(0, \varepsilon_1) : |I| > (1 - \delta)\lambda\} + \{x \in B(0, \varepsilon_1) \setminus B(0, \varepsilon_t) : |I| > (1 - \delta)\lambda\} \right) \frac{n-\beta}{n} \\
\leq \lambda |B(0, \varepsilon_1)| \frac{n-\beta}{n} + \lambda \left( \{x \in B(0, \varepsilon_1) \setminus B(0, \varepsilon_t) : |I| > (1 - \delta)\lambda\} \right) \frac{n-\beta}{n} \\
\leq \lambda |B(0, \varepsilon_1)| \frac{n-\beta}{n} + \lambda \left\{ x \in B(0, \varepsilon_1) \setminus B(0, \varepsilon_t) : v_n^{-1+\beta/n} |x|^{-n+\beta} \|f\|_{L^1(\mathbb{R}^n)} > (1 - \delta)\lambda \right\} \frac{n-\beta}{n} \\
= \lambda v_n^{-\beta} \varepsilon_1^{-n+\beta} + \frac{\|f\|_{L^1(\mathbb{R}^n)}}{1 - \delta}.
$$

(4.28)

On the other hand

$$
\lambda\{x \in \mathbb{R}^n : |I| > \lambda\} \frac{n-\beta}{n} \geq \lambda\{x \in B(0, \varepsilon_1)^c : |I| > \lambda\} \frac{n-\beta}{n} \\
\geq \lambda \left\{ x \in B(0, \varepsilon_1)^c : v_n^{-1+\beta/n} |x|^{-n+\beta} \|f\|_{L^1(\mathbb{R}^n)} - \varepsilon > \lambda \right\} \frac{n-\beta}{n} \\
= \lambda \left( \frac{\|f\|_{L^1(\mathbb{R}^n)} - \varepsilon}{\lambda} \right)^{\frac{n-\beta}{n}} - v_n \varepsilon_1^n \right) \frac{n-\beta}{n}.
$$

(4.29)

Hence, using (4.26), (4.27) and (4.28), we obtain

$$
\lambda\{x \in \mathbb{R}^n : |H_\beta f_t(x)| > \lambda\} \frac{n-\beta}{n} \leq \frac{C_{n,\beta,\varepsilon}}{\delta} + \lambda v_n^{-\beta} \varepsilon_1^{-n+\beta} + \frac{\|f\|_{L^1(\mathbb{R}^n)}}{1 - \delta}.
$$

(4.30)

And applying (4.29), we show that

$$
\lambda\{x \in \mathbb{R}^n : |H_\beta f_t(x)| > \lambda\} \frac{n-\beta}{n} \geq \lambda\{x \in \mathbb{R}^n : |I| > \lambda\} \frac{n-\beta}{n} \\
\geq \lambda \left( \frac{\|f\|_{L^1(\mathbb{R}^n)} - \varepsilon}{\lambda} \right)^{\frac{n-\beta}{n}} - v_n \varepsilon_1^n \right) \frac{n-\beta}{n}.
$$

(4.31)

Combining (4.30) with (4.31), we conclude that

$$
\lambda \left( \left( \frac{\|f\|_{L^1(\mathbb{R}^n)} - \varepsilon}{\lambda} \right)^{\frac{n-\beta}{n}} - v_n \varepsilon_1^n \right) \frac{n-\beta}{n} \leq \lambda\{x \in \mathbb{R}^n : |H_\beta f_t(x)| > \lambda\} \frac{n-\beta}{n} \\
\leq \frac{C_{n,\beta,\varepsilon}}{\delta} + \lambda v_n^{-\beta} \varepsilon_1^{-n+\beta} + \frac{\|f\|_{L^1(\mathbb{R}^n)}}{1 - \delta}.
$$

(4.32)

Therefore, taking limiting two side of inequality (4.32), we obtain

$$
\lambda \left( \left( \frac{\|f\|_{L^1(\mathbb{R}^n)} - \varepsilon}{\lambda} \right)^{\frac{n-\beta}{n}} - v_n \varepsilon_1^n \right) \frac{n-\beta}{n} \leq \lim_{t \to 0} \lambda\{x \in \mathbb{R}^n : |H_\beta f_t(x)| > \lambda\} \frac{n-\beta}{n} \\
\leq \frac{C_{n,\beta,\varepsilon}}{\delta} + \lambda v_n^{-\beta} \varepsilon_1^{-n+\beta} + \frac{\|f\|_{L^1(\mathbb{R}^n)}}{1 - \delta}.
$$
As \( \varepsilon \) arbitrary, we first let \( \varepsilon \to 0 \), next let \( \varepsilon_1 \to 0 \) and \( \delta \to 0 \). We obtain the desired result
\[
\lim_{\lambda \to 0} \lambda \| \{ x \in \mathbb{R}^n : |H_{\beta} f(x)| > \lambda \} \|_{\mathbb{R}^n}^{\frac{n-\beta}{n}} = \| f \|_{L^1(\mathbb{R}^n)}.
\]

This completes the proof of Theorem 4.1. \( \square \)

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