On Non-Equilibrium Thermodynamics of Space-Time and Quantum Gravity

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Abstract

Based on recent results from general relativistic statistical mechanics and black hole information transfer limits a space-time entropy-action equivalence is proposed as a generalization of the holographic principle. With this conjecture, the action principle can be replaced by the second law of thermodynamics, and for the Einstein-Hilbert action the Einstein field equations are conceptually the result of thermodynamic equilibrium. For non-equilibrium situations Jaynes’ information-theoretic approach to maximum entropy production is adopted instead of the second law of thermodynamics. As it turns out, for appropriate choices of constants quantum gravity is obtained. For the special case of a free particle the Bekenstein-Verlinde entropy-to-displacement relation of holographic gravity, and thus the traditional holographic principle, emerges. Although Jacobson’s original thermodynamic equilibrium approach proposed that gravity might not necessarily be quantized, this particular non-equilibrium treatment might require it.

1 Introduction

Gravity, thermodynamics and quantum mechanics are deeply connected.

The partition function for a grand canonical ensemble and the quantum partition function derived from Feynman’s path integral formulation are analogous with the substitution of time and inverse temperature \( \frac{it\hbar}{\hbar} \leftrightarrow \frac{1}{k_b T} \). While such a substitution — formally corresponding to a Wick rotation — might be useful for computations, it is also a key to connect both theories via Shannon information theory [1, 2, 3, 4].

As regards thermodynamics and gravity, Jacobson showed that the Einstein field equations could be derived based on space-time thermodynamic assumptions of Unruh temperature and Rindler coordinates [5]. This connection was deepened with the emergence of the holographic principle from string theory, loop quantum gravity and entropic gravity theories [6, 7, 8, 9, 10, 11, 12, 13, 14].

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This suggests that entropy — and thus Shannon information theory — might potentially be an important aspect in the connection between thermodynamics and gravity as well.

However, as regards entropy, a complete covariant theory of thermodynamics and statistical mechanics in a full general relativistic context is yet to be established [15]. Initial investigations have shown that for a constant "thermal time" there seems to be a direct relation between temperature and inverse proper time [15, 16, 17]. With the assumption that entropy is negative information $I$ there is also a direct connection between entropy, energy $E$ and time $t$ in upper limits of the bound on the information $I$ transfer in black hole thermodynamics [18]:

$$I < \gamma Et$$

(1)

for the constant $\gamma = \pi/\hbar \ln 2$. Although this is in line with the holographic principle, it is perhaps worthwhile to consider an alternative holographic principle based on these results. A principle which connects entropy dynamically to energy and time, such that an equilibrium configuration is a special case, like in [19]. Non-equilibrium is after all believed to be the most common state of a thermodynamics system [19, 20].

2 Entropy-action equivalence conjecture

If there is a potential proportionality between inverse temperature and time the thermodynamic relation $dH = \delta Q/T$ proposes a direct proportionality between entropy $H$, energy $E$ and time $t$:

$$H \propto E \cdot t,$$

(2)

which is up to a constant the the upper information transfer limit for black holes [1] derived by Bekenstein [18]. This general entropy is dependent on coordinate variables of space and time. For most purposes the second law of thermodynamics is enforced to constrain the dynamics. In effect this means a maximization of the entropy. Keep in mind that the form of energy $E$ is not constrained here, and that basically it is set up to match units of the dynamical components of (1) and that of thermal time.

Thus consider, for the time being, the possibility that $E$ is the Lagrangian $L$ of a system (in classical terms) and that the Lagrangian is integrated over time $t$ to bring the entropy $H$:

$$H \propto \int L(q, \dot{q}, t) \, dt,$$

(3)

with a generalized coordinate $q$. This is dramatically different from Bekenstein’s information transfer bound, but the units of the dynamical components remain. And for the special case of $L = E$ and that $E$ is independent of time $t$ brings $H \propto E \cdot t$ which is the upper limit of Bekenstein’s information transfer bound. Also, if one applies the second law of thermodynamics to (3), with respect to
the generalized coordinates \( q \) and \( \dot{q} \), it would be equivalent to the variational principle. This is worthwhile investigating deeper, since if applicable it could potentially dispense with the use of certain action principles in physics.

The most general classical description of physics is given by general relativity, whose dynamics is traditionally derived from the Einstein-Hilbert action. As the main conjecture of this paper, let’s assume, for the time being, that the space-time thermodynamic entropy \( H \) for a space-time region \( D \) can be identified with the Einstein-Hilbert action \( S_{EH} \) with a proportionality constant \( \kappa \):

\[
H(g_{\mu\nu}, L_M, D) \equiv \kappa S_{EH}(g_{\mu\nu}, L_M, D) = \kappa \int_D \left( \frac{c^4}{16\pi G} R + L_M \right) \sqrt{-g} \, d^4x. \tag{4}
\]

where \( R = R(g_{\mu\nu}) \) is the Ricci-scalar based on the metric tensor \( g_{\mu\nu} \) and \( L_M \) is the matter-Lagrangian. The entropy is then a function of the metric \( g_{\mu\nu} \) (as a set of generalized coordinates), the classical matter-Lagrangian \( L_M \) and space-time region \( D \). This means that the entropy is dependent on both space-time configuration and matter content within a domain of space-time. Entropy, as defined here, is still in practice in proportionality with the dynamical variables of energy and time.

If entropy \( H \) is varied with respect to the generalized coordinates, that is the metric \( g_{\mu\nu} \), then a "thermodynamic equilibrium state" is conceptually obtained. Formally the Einstein field equations are given, since that variation corresponds to the traditional variation of the Einstein-Hilbert action. These derivations are formally equivalent, but there is a conceptual difference for the variation: maximization of entropy instead of traditional variation of the action, which will be of importance as the paper progresses.

There is no formal novelty in varying the Einstein-Hilbert action with respect to the metric, but there is a conceptual gain via the connection to entropy and thermodynamics. In thermodynamics, equilibrium is not always achieved. In fact a state of maximum entropy is rarely achieved, and most systems are likely in a non-equilibrium state \([19, 20]\). Theoretically, for thermodynamic applications, in non-equilibrium situations a maximum entropy production approach is adopted instead of the second law of thermodynamics which postulates a maximum entropy approach \([19, 20]\).

In Jacobson’s original thermodynamic approach to gravity the Einstein field equations were derived on the assumption of thermodynamic equilibrium \([5]\). This ansatz was extended to non-equilibrium space-time thermodynamics by Eling, Guedens and Jacobson in \([19]\). This study assumed that the entropy was a function of the Ricci scalar:

\[
H \propto \sigma + f(R). \tag{5}
\]

As a means to estimate the rate of entropy change semi-classical components such as the Unruh temperature \( T \) and "internally developed entropy" \( dH_i \) were used in the entropy balance equation \( dH = \delta Q/T + d_i H \). Generally, see \([20]\) for an overview of entropy production methodologies. Albeit a semi-classical
approach the study in [19] revealed interesting connections between shear viscosity of the horizon in the Einstein field equations and entropy. While the entropy-action equivalence (4) is a function of space and time, it is also — in similarity with (5) a function of the Ricci scalar. Eling, Guedens and Jacobson’s approach regarded a general setup, while the approach here is more specified and by onset not based on semi-classical concepts. In their study gravity was based on semi-classical concepts, but remained, however, classic.

An alternative approach for describing non-equilibrium thermodynamics is to utilize Jaynes’ information theoretic setup [21, 22, 23, 24]. As a macroscopic approach — similar to that of thermodynamics itself — that method devises a partition function based on all possible states of entropy and uses a maximum entropy production principle to constrain probabilities for different states. If applied to Eling et al.’s study it would imply setting up a partition function for the unknown function of the Ricci scalar. Formally in this approach, we have the following.

Assume the proposed space-time entropy of the classical Einstein-Hilbert action (4), but resist the hesitation to apply variation to obtain the thermodynamic equilibrium. Instead, setup all possible states of configuration for $H$ in terms of $g_{\mu\nu}$. Based on these states a new form of entropy $H_2$ can be identified. In traditional non-equilibrium thermodynamics this is labeled “the second entropy” [21, 25] which is the pure information-theoretic entropy of all states of entropy $H[g_{\mu\nu}]$:

\[ H_2 = \sum_{\text{All } g_{\mu\nu}} p[g_{\mu\nu}] \log p[g_{\mu\nu}] = -\int Dg_{\mu\nu} p[g_{\mu\nu}] \log p[g_{\mu\nu}] \] \hspace{1cm} (6)

where $p[g_{\mu\nu}]$ is an unknown probability function defined for each configuration of $g_{\mu\nu}$. This produces expected values, for example for the (“first”) entropy $H:

\[ \langle H \rangle = \sum_{\text{All } g_{\mu\nu}} p[g_{\mu\nu}] H[g_{\mu\nu}] = \int Dg_{\mu\nu} p[g_{\mu\nu}] H[g_{\mu\nu}], \] \hspace{1cm} (7)

which is proportional to the expected action via (4). The probability function $p[g_{\mu\nu}]$ is found by using Lagrange multipliers, as was done for non-equilibrium thermodynamics in [21, 22, 23, 24] and analogously for information-theoretic interpretations of quantum mechanics in [1, 3, 4, 26]. By employing complex Lagrange multipliers, $\lambda$ and $\alpha$, the second entropy $H_2$ is maximized in terms of ”first” entropy $H$ by:

\[ H_2 = -\int Dg_{\mu\nu} p[g_{\mu\nu}] \log p[g_{\mu\nu}] + \lambda \left(1 - \int Dg_{\mu\nu} p[g_{\mu\nu}] \right) + \] \hspace{1cm}

\[ +\alpha \left( \langle H[g_{\mu\nu}] \rangle - \int Dg_{\mu\nu} p[g_{\mu\nu}] H[g_{\mu\nu}] \right), \] \hspace{1cm} (8)

which simplified becomes:

\[ H_2 = \lambda + \alpha \langle H \rangle - \int Dg_{\mu\nu} \left(p[g_{\mu\nu}] \log p[g_{\mu\nu}] + \lambda p[g_{\mu\nu}] + \alpha p[g_{\mu\nu}] H[g_{\mu\nu}] \right). \] \hspace{1cm} (9)
If we perform variation on the probability distribution we get:

$$\delta H_2 = -\int Dg_{\mu\nu}(\delta p[g_{\mu\nu}])(\log p[g_{\mu\nu}] + 1 + \lambda + \alpha H[g_{\mu\nu}])$$

(10)

which is extremized when $$\delta H_2 = 0$$, which corresponds to the probability distribution:

$$p[g_{\mu\nu}] = e^{-1-\lambda \alpha H[g_{\mu\nu}] + \frac{1}{Z}e^{-\alpha H[g_{\mu\nu}]}}.$$  

(11)

By varying the Lagrange multipliers one enforces the two constraints, giving $$\lambda$$ and its connection to $$\alpha$$. Especially one gets: $$e^{-1-\lambda} = \frac{1}{Z}$$ where $$Z$$ is a form of partition function:

$$Z = \sum_{g_{\mu\nu}} e^{-\alpha H[g_{\mu\nu}]} = \int Dg_{\mu\nu}e^{-\alpha H[g_{\mu\nu}]}.$$  

(12)

This yields the expected entropy:

$$\langle H \rangle = \sum_{g_{\mu\nu}} p[g_{\mu\nu}]H[g_{\mu\nu}] = \frac{1}{Z}e^{-\alpha H[g_{\mu\nu}]}H[g_{\mu\nu}] = \int Dg_{\mu\nu}\frac{1}{Z}e^{-\alpha H[g_{\mu\nu}]}H[g_{\mu\nu}],$$

(13)

which by (4) is proportional to the expected action with proportionality constant $$\kappa$$:

$$\langle S \rangle = \kappa \sum_{g_{\mu\nu}} p[g_{\mu\nu}]S_{EH}[g_{\mu\nu}] = \kappa \sum_{g_{\mu\nu}} \frac{1}{Z}e^{-\alpha \kappa S_{EH}[g_{\mu\nu}]}S_{EH}[g_{\mu\nu}] =$$

$$= \kappa \int Dg_{\mu\nu}\frac{1}{Z}e^{-\alpha \kappa S_{EH}[g_{\mu\nu}]}S_{EH}[g_{\mu\nu}].$$

(14)

The second entropy — also called the ”entropy of entropy” arises in Jaynes’ non-equilibrium formulation here and is defined as:

$$H_2 = -\sum_{g_{\mu\nu}} p[g_{\mu\nu}] \log(p[g_{\mu\nu}]) = -\int Dg_{\mu\nu}p[g_{\mu\nu}] \log([g_{\mu\nu}]),$$

(15)

which by the criterion $$\sum_{g_{\mu\nu}} p[g_{\mu\nu}] 1_{\mathcal{A}(g_{\mu\nu})} = 1$$ and algebraic manipulations become (see derivation in [3, 4]):

$$H_2 = -\alpha \langle H \rangle + \log Z.$$  

(16)

This characterizes the complete description of possible states of entropy and second entropy in this setup, with undefined constants $$\kappa$$ connecting action and entropy, and $$\alpha$$ related to entropy production maximization. Sections 3 and 4 will outline the pursuit of these constants.

In short, this approach represents the complete non-equilibrium dynamics of general relativity based on the action-entropy equivalence conjecture and Jayne’s second entropy approach for maximum entropy production.
3 $\alpha\kappa$ and quantum gravity

In this setup two constants are undefined so far: $\kappa$ which connects entropy $H$ to the action $S$, and $\alpha$ which connects entropy $H$ to second entropy $H_2$. By replacing $H$ with $\kappa S$ in the partition function, according to (4), and assuming that $\alpha\kappa \equiv i/\hbar$ then one gets:

$$Z = \sum_{g_{\mu\nu}} e^{i\frac{\kappa}{\hbar} S[g_{\mu\nu}]} = \int Dg_{\mu\nu} e^{i\frac{\kappa}{\hbar} S[g_{\mu\nu}]},$$

(17)

which is the quantum partition function for quantum gravity [27]. This also renders the second entropy equivalent to an information-theoretic entropy in quantum mechanics, also called “quantropy”, as the term was coined by Baez and Pollard [3, 4].

Since this connects to information theory, and previous studies on information-theoretic approaches to quantum mechanics, an explicit probability for each configuration of $g_{\mu\nu}$ is provided (seen for general configurations in [1, 3, 4]):

$$p[g_{\mu\nu}] = \frac{1}{Z} e^{-\frac{i}{\hbar} S[g_{\mu\nu}]}.$$

(18)

Because $\alpha\kappa = i/\hbar$ is complex the probability for each state is complex, which is problematic from probability theory perspective. However, that can be mended with the following setup, which was devised by Lisi for general quantum systems [1]. The probability for the system to be on a specific path in a set of possible paths in configuration space is:

$$p(set) = \sum_{\text{paths}} \delta^{set}_{\text{path}} p[\text{path}] = \int Dg_{\mu\nu} \delta(set - g_{\mu\nu}) p[g_{\mu\nu}].$$

(19)

Typically the system reverses sign under inversion of parameter integration limits [1]:

$$S_{t'} = \int_{t'}^{t} dt L(g_{\mu\nu}) = -\int_{t}^{t'} dt L(g_{\mu\nu}) = -S_{t'}.$$

(20)

This implies that the probability for the system to pass through configuration $g_{\mu\nu}'$ at parameter value $t'$ is:

$$p(g_{\mu\nu}', t') = \int Dg_{\mu\nu} \delta(g_{\mu\nu}(t') - g_{\mu\nu}) p[g_{\mu\nu}] =$$

$$= \left( \int_{g_{\mu\nu}(t') = g_{\mu\nu}'} Dg_{\mu\nu} \right) \left( \int_{g_{\mu\nu}(t') = g_{\mu\nu}'} Dg_{\mu\nu} p'[g_{\mu\nu}] \right) =$$

$$= \psi(g_{\mu\nu}', t') \psi^\dagger(g_{\mu\nu}', t'),$$

(21)

in which we can identify the quantum wave function:

$$\psi = \int_{g_{\mu\nu}(t') = g_{\mu\nu}'} Dg_{\mu\nu} p'[g_{\mu\nu}] = \frac{1}{\sqrt{Z}} \int_{g_{\mu\nu}(t') = g_{\mu\nu}'} Dg_{\mu\nu} e^{-\alpha S_{t'}} =$$
This gives the probability amplitude for the probability of a system to pass through metric $g'_{\mu\nu}$ at time $t'$. See the similarity with quantum mechanics in [1].

It should be noted that the quantum wave function in quantum mechanics is subordinate to the partition function since it only works when $t'$ is a physical parameter of the system and that the system is $t'$ symmetric, which provides a real partition function $Z$.

Thus, with appropriate constants it is perhaps unexpectedly possible to obtain quantum gravity from this non-equilibrium ansatz. However, only the product of constants $\alpha \kappa$ have been determined so far. In order to obtain the equivalence proportionality constant $\kappa$ a connection to the holographic principle is made in the following section.

4 $\alpha$ and holographic gravity

The entropy-action equivalence conjecture with the Einstein-Hilbert action here proposes that the action principle is equivalent to the second law of thermodynamics. That is, classical dynamics of a system is obtained for thermodynamic equilibrium situations and quantum dynamics arise in non-equilibrium situations. However all of this requires that the entropy-action equivalence conjecture in fact is valid and indeed supersedes the holographic principle.

A first step on that path is to show equivalence under certain conditions. According to the entropy-action equivalence conjecture [1] the entropy of a stationary particle with mass $m$ is:

$$H = \kappa \int mc^2 dt = \kappa mc^2 t.$$  \hspace{1cm} (23)

If we assume that time can be expressed as $\Delta x/c$ for a "time-like" distance $\Delta x$ then we get:

$$\Delta S = \kappa mc \Delta x,$$  \hspace{1cm} (24)

and if we set $\kappa = -2\pi k_B/\hbar$, where $k_B$ is Boltzmann’s constant, then we get the Bekenstein-Verlinde expression from holographic gravity [14]:

$$\Delta S = -2\pi k_B \frac{mc}{\hbar} \Delta x.$$  \hspace{1cm} (25)

This expression was originally derived by Bekenstein (up to a constant) [18] for particles falling into black holes, and it was more recently slightly altered by Verlinde to develop his approach to entropic gravity [14]. If this special case of the entropy can be assumed equivalent to the Bekenstein-Verlinde expression it is possible to identify $\alpha$ as well since $\alpha \kappa = \frac{i}{\pi}$:

$$\alpha = \frac{i}{\hbar \kappa} = -\frac{i}{2\pi k_B}.$$  \hspace{1cm} (26)
This defines both quantum dynamics in [17] and the entropy-action proportionality for the Einstein-Hilbert action $S_{EH}$:

$$H \equiv -\frac{2\pi k_B}{\hbar}S_{EH},$$  

which completes the investigation on the generalization of the holographic principle by means of an entropy-action equivalence conjecture in this paper.

5 Discussion and conclusions

Based on recent developments in general relativistic statistical mechanics and black hole thermodynamics an entropy-action equivalence is conjectured. With it, Einsteinian gravity is obtained for equilibrium situations — analogous to Jacobson’s results — and a non-equilibrium theory of gravity is also developed based on Jaynes’ information-theoretic approach to maximum entropy production.

This approach to maximum entropy production, in thermodynamics and other fields, is one among several different approaches to characterize non-equilibrium dynamics, and the non-equilibrium concept is not without controversy [20]. But the fact that for a specific choice of the coupling constant and a Lagrange multiplier the result is quantum gravity is perhaps interesting enough to consider this possibility. And if true, it also presents a form of information-theoretic quantization principle: maximum entropy production. Another interesting feature is also that the correspondence to classical physics is obtained by enforcing the second law of thermodynamics, which dispenses with the use of a classical action principle.

One advantage of this approach is the derivation of an explicit probability for each potential metric configuration, which could perhaps be useful for certain calculations [1]. The entropy-action equivalence is conjectured as a generalization of the holographic principle and shown to correspond to the Bekenstein-Verlinde entropy-to-displacement relation of holographic gravity for the special case of a stationary mass. That being said, this conjecture likely violates the holographic principle, even under classical considerations, for non-stationary situations.

The entropy of any object, even a black hole, is universal for all observers in this approach (since the classical Lagrangian is universal) and based on the energy and matter content of space-time. This information-theoretic quantization of gravity suggests that space-time is encoded with the fundamental stochastic nature of quantum mechanics. Even though entropy is invariant for all observers, the second entropy: quantropy, is not. Since observers may or may not gain information regarding the state of any object, quantropy is by necessity observer dependent [1, 4].

In a foundational sense, since the theory is based on accessible information and observer dependence, it seems to demand some form of principle of information covariance: the laws of physics can only be defined on the basis of the
information accessible to each observer [4]. This suggests that perhaps Rovelli’s relational interpretation for quantum mechanics is favorable for this theory [28].

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