SYMmetry and Nonexistence Results for a Fractional Choquard Equation with Weights

Anh Tuan Duong
Department of Mathematics, Hanoi National University of Education
136 Xuan Thuy, Cau Giay, Hanoi, Vietnam

Phuong Le1,2,*
1Division of Computational Mathematics and Engineering
Institute for Computational Science
Ton Duc Thang University, Ho Chi Minh City, Vietnam
2Faculty of Mathematics and Statistics
Ton Duc Thang University, Ho Chi Minh City, Vietnam

Nhu Thang Nguyen
Department of Mathematics, Hanoi National University of Education
136 Xuan Thuy, Cau Giay, Hanoi, Vietnam

(Communicated by Monica Musso)

Abstract. Let $u$ be a nonnegative solution to the equation

$$(-\Delta)^\alpha u = \left(\frac{1}{|x|^{n-\beta}} * |x|^a u^p\right)|x|^a u^{p-1} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\},$$

where $n \geq 2$, $0 < \alpha < 2$, $0 < \beta < n$, and $\alpha > \max\{-\alpha, -\frac{\alpha + \beta}{2}\}$. By exploiting the method of scaling spheres and moving planes in integral forms, we show that $u$ must be zero if $1 \leq p < n + \beta + 2a \frac{n}{n-\alpha}$ and must be radially symmetric about the origin if $a < 0$ and $\frac{n + \beta + 2a}{n-\alpha} \leq p \leq \frac{n + \beta + a}{n-\alpha}$.

1. Introduction. In this paper, we study the fractional Choquard type equation with weights

$$(-\Delta)^\alpha u = \left(\frac{1}{|x|^{n-\beta}} * |x|^a u^p\right)|x|^a u^{p-1} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\},$$

where $n \geq 2$, $0 < \alpha < 2$, $0 < \beta < n$, and $\alpha > \max\{-\alpha, -\frac{\alpha + \beta}{2}\}$ and $p \geq 1$. Here, the convolution of two functions $f$ and $g$ is defined as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$ 

The fractional Laplacian in $\mathbb{R}^n$ is the following nonlocal pseudo-differential operator

$$(-\Delta)^\alpha u(x) = C_{n,\alpha} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{n+\alpha}}dy,$$

where $n \geq 2$, $0 < \alpha < 2$, $0 < \beta < n$, and $\alpha > \max\{-\alpha, -\frac{\alpha + \beta}{2}\}$.
where $C_{n,\alpha}$ is a normalization constant, $B_{\varepsilon}(x)$ is the ball of radius $\varepsilon$ and center $x \in \mathbb{R}^n$, and $PV$ stands for the Cauchy principle value. This operator is well-defined in the Schwartz space of rapidly decreasing continuously differentiable functions in $\mathbb{R}^n$. In this space, the fractional Laplacian can also be defined via the Fourier transform

$$F[(-\Delta)^{\frac{\alpha}{2}} u](\xi) = |\xi|^\alpha F u(\xi),$$

where $F u$ is the Fourier transform of $u$. One can extend this operator to the distributions $u$ in the space $L^{\alpha}$ by

$$\langle (-\Delta)^{\frac{\alpha}{2}} u, \varphi \rangle = \int_{\mathbb{R}^n} u(-\Delta)^{\frac{\alpha}{2}} \varphi dx, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^n),$$

where

$$L_{\alpha} = \left\{ u \in L^{1,1}_{\text{loc}}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \frac{|u(x)|^2}{1 + |x|^{n+\alpha}} dx < \infty \right\}.$$

It is also clear that, if $u \in L_{\alpha} \cap C^{1,1}_{\text{loc}}(\Omega)$, then $(-\Delta)^{\frac{\alpha}{2}} u(x)$ is well-defined for $x \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is an open domain. In this work, we will study nonnegative solutions of (1) in the strong sense. More precisely, we call $u$ a nonnegative solution of (1) if $u$ is a nonnegative function in $L^{\alpha} \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \cap C(\mathbb{R}^n)$ and $u$ verifies (1) for all $x \in \mathbb{R}^n \setminus \{0\}$.

In the last decades, the fractional Laplacian has been used extensively to model several phenomena in physical sciences, such as water waves, turbulence, quasi-geostrophic flows, flame propagation, anomalous diffusion and phase transitions (see [4,6,9]). It also has several applications in probability, optimization and finance. In particular, the fractional Laplacian can be seen as the infinitesimal generator of a stable Lévy process (see [1,3,5]). Variational methods for nonlocal problems including those with the fractional Laplacian can be found in the excellent monograph [25] and the references therein.

One may observe that the right hand side of (1) is also a nonlocal term. This phenomenon causes some mathematical difficulties which make the study of such problem particularly interesting. Moreover, problem of type (1) has a strong physical motivation. In fact, it is analogous to the nonlinear stationary Choquard-Pekar equation

$$-\Delta u + V(x)u = 2 \left( \frac{1}{|x|} * |u|^2 \right) u \quad \text{in } \mathbb{R}^3.$$

This equation appears in many applications, which include the physics of multiple particle systems, the physics of laser beams, quantum mechanics and the Hartree-Fock theory. We refer to [20,26,28] for interested readers.

Elliptic problems of Choquard type have been studied extensively by several authors in recent years. An introduction to mathematical treatment of Choquard type equations can be found in the review paper [27] by Moroz and Schaftingen. Some recent existence and multiplicity results for Choquard type equations and related problems can be found in [2,12,18,21,22,24,30] and the references therein. In this paper, we focus to problem (1), which is usually called the fractional stationary Choquard or Hartree equation with Hénon-Hardy weights and vanishing potential. This problem was studied by many authors and some optimal Liouville theorems as well as classification results were established recently.
First, we consider the case $a = 0$. One may show that, if $a = 0$ and $0 < \alpha, \beta \leq 2$, then (1) can be transformed into the elliptic system
\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u = u^{p-1} v & \text{in } \mathbb{R}^n \setminus \{0\}, \\
(-\Delta)^{\frac{\beta}{2}} v = C u^p & \text{in } \mathbb{R}^n \setminus \{0\},
\end{cases}
\]
where $C$ is some positive constant. By exploiting the method of moving planes for system (2), Lei [17] proved that equation (1) with $\alpha = \beta = 2$ has no positive solution in the case $1 \leq p < \frac{n+2}{n-2}$ and all positive solutions must assume a specific form in the case $p = \frac{n+2}{n-2}$. The more general cases $0 < \alpha, \beta < 2$ were studied by Ma, Shang and Zhang in [23], where they established the symmetry of positive solutions to (1) in the critical case $p = \frac{n+\beta}{n-\alpha}$ and the nonexistence of such solutions in the subcritical case $\frac{n}{n-\alpha} \leq p < \frac{n+\beta}{n-\alpha}$. The main tool used in [23] is the direct moving plane method, which was developed by Chen, Li and Li [7] for elliptic problems involving the fractional Laplacian. Obviously, the condition $0 < \beta \leq 2$ is required in this approach. Later, Dai, Fang and Qin [10] found a way to apply the method to equation (1) directly, which allowed them to classify all positive solutions of (1) when $\alpha \in (0, \min\{2, \frac{n}{\beta}\})$, $\beta = n - 2\alpha$ and $p = 2$. Very recently, these results were extended to the full range $0 < \alpha < 2$, $0 < \beta < n$ by the second author in [15]. More precisely, it was proved in [15] that, if $0 < \beta < n$ and $u \in L_{2} \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n)$ is a nonnegative solution to equation (1), then $u \equiv 0$ provided that $1 < p \leq \frac{n+\beta}{n-\alpha}$ and $u$ must assume the form $u(x) = c \left( 1 + \frac{4}{n-\alpha} \right)^{-\frac{n-\alpha}{4}}$ provided that $p = \frac{n+\beta}{n-\alpha}$.

Next, we turn our attention to the weighted case, i.e., $a \neq 0$. When $a < 0$, the study of equation (1) is motivated by the doubly weighted Hardy-Littlewood-Sobolev type inequality
\[
\left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)|^{n+\beta+2a} |u(y)|^{n+\beta+2a}}{|x-y|^{\frac{n\alpha+\beta}{2}}-\frac{\alpha}{2}} dxdy \right)^{\frac{2}{2+n+\beta+2a}} \leq C \|(-\Delta)^{\frac{\alpha}{2}} u\|_{L^2(\mathbb{R}^n)},
\]
see [19, 29] for more details. In [13], Du, Gao and Yang proved that, if $\alpha = 2$, $\frac{n-\beta-\min\{4,\alpha\}}{2} \leq a < 0$ and $p = \frac{n+\beta+2a}{n-2}$, then every positive $D^{1,2}_{\text{loc}}(\mathbb{R}^n)$ solution of (1) must be radially symmetric about the origin. There is also a radial symmetry result for positive solutions $u$ of the Choquard type equation involving fractional $p$-Laplacian
\[
(-\Delta)^{\frac{\alpha}{p}} u = \left( \frac{1}{|x|^{n-\beta}} * |x|^a u^p \right) |x|^a u^{p-1} \quad \text{in } \mathbb{R}^n \setminus \{0\},
\]
where $a < 0$ and $p > 2$, see Le [16]. However, a priori asymptotic assumptions on $u$ is required in [16].

The main purpose of this paper is to extend the result in [15] to the case $a \neq 0$. Our first result is a Liouville type theorem for (1).

**Theorem 1.1.** Assume $n \geq 2$, $0 < \alpha < 2$, $0 < \beta < n$, $a > \max\{-\alpha, -\frac{\alpha+\beta}{2}\}$ and $1 \leq p < \frac{n+\beta+2a}{n-\alpha}$. If $u$ is a nonnegative solution of equation (1), then $u \equiv 0$.

**Remark 1.** The case $a = 0$ was already established in [15] via the direct method of moving planes. However, that method cannot be applied to the case $a \neq 0$ to derive Liouville theorems. If one uses that method on equation (1) with $a \neq 0$, one can only obtain the radial symmetry of nonnegative solutions in the range $1 \leq p < \frac{n+\beta+2a}{n-\alpha}$.
To obtain Liouville theorem in the full range $1 \leq p < \frac{n+\beta+2a}{n-\alpha}$, we will take another approach: the method of scaling spheres. This method was introduced by Dai and Qin recently in [11] and was successfully used to establish the optimal Liouville theorem for nonnegative solutions of the fractional Hénon-Hardy equation

$$(-\Delta)^{\frac{\alpha}{2}} u = |x|^{a} |u|^{p-1}$$

in $\mathbb{R}^n \setminus \{0\}$.

To prove Theorem 1.1, we will turn equation (1) into an equivalent integral system, then we extend the scaling sphere method to that system to obtain the desired result.

Our second result is concerned with the radial symmetry of nonnegative solutions when $p \geq \frac{n+\beta+2a}{n-\alpha}$.

**Theorem 1.2.** Assume $n \geq 2$, $0 < \alpha < 2$, $0 < \beta < n$, $\max\{-\alpha, -\frac{\alpha+\beta}{2}\} < a < 0$ and $\frac{n+\beta+2a}{n-\alpha} \leq p \leq \frac{n+\beta+a}{n-\alpha}$. If $u$ is a nonnegative solution of equation (1), then $u$ must be radially symmetric about the origin.

**Remark 2.** Clearly, if $\frac{n+\beta+2a}{n-\alpha} \leq p \leq \frac{n+\beta+a}{n-\alpha}$, then $a \leq 0$. Furthermore, when $a = 0$, the assumption $\frac{n+\beta+2a}{n-\alpha} \leq p \leq \frac{n+\beta+a}{n-\alpha}$ reduces to $p = \frac{n+\beta}{n-\alpha}$. This case was already studied in [15], where it was proved that every nonnegative solution must be radially symmetric about some point in $\mathbb{R}^n$ and therefore assumes an explicit form. Such an explicit form is, however, not available for equation (1) in the singular case $a < 0$.

Note that Theorem 1.2 can be proved by the direct method of moving planes as in [15]. However, in this paper, we will utilize the method of moving planes in integral forms instead, see [8]. This gives us a new proof which is simpler than that in [15] and more consistent with the proof of Theorem 1.1.

The remainder of this paper is organized as follows. In the next section, we use the maximum principle and Liouville theorem for $\alpha$-harmonic functions to transform equation (1) into an equivalent integral system. Then we exploit the method of scaling spheres for this system to prove Theorem 1.1 in Section 3. Section 4 is devoted to the proof of the radial symmetry of solutions, namely, Theorem 1.2.

Throughout this paper, we use $C$ to denote some positive constant which may vary from place to place. Moreover, we may append subscripts to $C$ to emphasize its dependence on the subscript parameters. The Lebesgue measure of a subset $\Omega \subset \mathbb{R}^n$ will be denoted by $|\Omega|$. For the sake of simplicity, we also denote by $B_R$ the ball of center 0 with radius $R > 0$.

2. Preliminaries. It is not easy to investigate the qualitative properties of solutions to (1) directly due to the presence of the convolution term in the right hand side of the equation. To overcome this difficulty, throughout this paper, we denote

$$v = \frac{1}{|x|^{n-\beta}} * |x|^{a} |u|^p,$$

then we transform (1) into an equivalent integral system. More precisely, we have

**Theorem 2.1.** If $u$ is a nonnegative solution of (1), then $(u, v)$ is a solution of the integral system

$$
\begin{align*}
  u(x) &= R_{n,\alpha} \int_{\mathbb{R}^n} |y|^{a} |u|^{p-1} \frac{v(y)}{|x-y|^{n-\alpha}} \, dy, \quad x \in \mathbb{R}^n, \\
  v(x) &= \int_{\mathbb{R}^n} \frac{|y|^{a} |u|^p(y)}{|x-y|^{n-\beta}} \, dy, \quad x \in \mathbb{R}^n,
\end{align*}
$$

(3)
where $R_{\alpha,\alpha} = \frac{\Gamma(\frac{\alpha-n}{2})}{\pi^{\frac{n-\alpha}{2}} \Gamma(\frac{\alpha}{2})}$ is the Riesz potential’s constant and $\Gamma$ denotes the gamma function.

**Lemma 2.2** (Lemma 2.2 in [11]). Assume $n \geq 2$ and $0 < \alpha < 2$. If $w$ is $\alpha$-harmonic in $B_R \setminus \{0\}$ and satisfies $w(x) = \alpha(|x|^{\alpha-n})$ as $|x| \to 0$, then $w$ can be defined at 0 so that it is $\alpha$-harmonic in $B_R$.

**Lemma 2.3** (Maximum principle [7]). Let $n \geq 2$, $0 < \alpha < 2$ and $\Omega$ be a bounded domain in $\mathbb{R}^n$. Assume that $w \in L_\alpha \cap C^{1,1}_{\text{loc}}(\Omega)$ is lower semi-continuous on $\overline{\Omega}$. If
\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} w(x) \geq 0, & x \in \Omega, \\
w(x) \geq 0, & x \in \mathbb{R}^n \setminus \Omega,
\end{cases}
\]
then $w(x) \geq 0$ for all $x \in \mathbb{R}^n$. Moreover, if $w = 0$ at some point in $\Omega$, then $w(x) = 0$ almost everywhere in $\mathbb{R}^n$. The same conclusions also hold for the unbounded domain $\Omega$ if we further assume $\liminf_{|x| \to \infty} w(x) \geq 0$.

**Lemma 2.4** (Liouville theorem [31]). Assume $n \geq 2$, $0 < \alpha < 2$ and $w \in L_\alpha \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n)$ is a solution of
\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} w(x) = 0, & x \in \mathbb{R}^n, \\
w(x) \geq 0, & x \in \mathbb{R}^n.
\end{cases}
\]
Then $u \equiv C$ for some nonnegative constant $C$.

**Proof of Theorem 2.1.** Assume that $u$ is a nonnegative solution of (1). For arbitrary $R > 0$, let
\[
uR(x) = \int_{B_R} G_R^\alpha(x,y)|y|^\alpha u^{p-1}(y)v(y)dy,
\]
where $G_R^\alpha$ is the Green’s function for $(-\Delta)^{\frac{\alpha}{2}}$ on $B_R$. That is, $G_R^\alpha$ is given by
\[
G_R^\alpha(x,y) = \begin{cases}
\frac{R_{\alpha,\alpha}}{|x-y|^\alpha} \int_0^{|x-y|^\alpha} t^{\frac{\alpha-1}{2}} \left(1 + t\right)^{-\frac{\alpha}{2}} dt, & \text{if } x, y \in B_R, \\
0, & \text{if } x \text{ or } y \in \mathbb{R}^n \setminus B_R,
\end{cases}
\]
where $s = \frac{|x-y|^2}{R^2}$ and $t = \left(1 - \frac{|x|^2}{R^2}\right)\left(1 - \frac{|y|^2}{R^2}\right)$ (see [14]). Then $u_R \in L_\alpha \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \cap C(\mathbb{R}^n)$ and satisfies
\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u_R(x) = |x|^\alpha u^{p-1}(x)v(x), & x \in B_R \setminus \{0\}, \\
uR(x) = 0, & x \in \mathbb{R}^n \setminus B_R.
\end{cases}
\]
Let $\piR(x) = u(x) - u_R(x)$, then $(-\Delta)^{\frac{\alpha}{2}} \piR = 0$ in $B_R \setminus \{0\}$. By Lemma 2.2, we have $\piR \in L_\alpha \cap C^{1,1}_{\text{loc}}(B_R)$ and
\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} \piR(x) = 0, & x \in B_R, \\
\piR(x) \geq 0, & x \in \mathbb{R}^n \setminus B_R.
\end{cases}
\]
By Lemma 2.3, we derive
\[
\piR(x) \geq 0, \quad x \in \mathbb{R}^n.
\]
Therefore, letting $R \to \infty$, we obtain
\[
uu(x) = R_{\alpha,\alpha} \int_{\mathbb{R}^n} |y|^\alpha u^{p-1}(y)v(y) |x-y|^{2-\alpha} dy.
\]
and \( u_\infty \in L_\alpha \cap C^{1,1}_{\text{loc}}(\mathbb{R}\setminus \{0\}) \cap C(\mathbb{R}^n) \) is a solution of
\[
(-\Delta)^{\frac{s}{2}} u_\infty (x) = |x|^\alpha u^{p-1}(x)v(x), \quad x \in \mathbb{R}^n \setminus \{0\}.
\]

Now let \( \pi_\infty (x) = u(x) - u_\infty (x) \), then by Lemma 2.2, we have \( \pi_\infty \in L_\alpha \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n) \) and satisfies
\[
\begin{cases}
(-\Delta)^{\frac{s}{2}} \pi_\infty (x) = 0, & x \in \mathbb{R}^n, \\
\pi_\infty (x) \geq 0, & x \in \mathbb{R}^n.
\end{cases}
\]

Applying Lemma 2.4, we obtain \( \pi_\infty \equiv C_1 \geq 0 \), which indicates \( u \geq C_1 \). Thus,
\[
v(x) = \int_{\mathbb{R}^n} \frac{|y|^a u^p(y)}{|x - y|^{n-\beta}} dy \geq C_1^p \int_{\mathbb{R}^n} \frac{|y|^a}{|x - y|^{n-\beta}} dy = C_1^p C_2
\]
and
\[
u(x) = C_1 + R_{n,\alpha} \int_{\mathbb{R}^n} \frac{|y|^a u^{p-1}(y)v(y)}{|x - y|^{n-\alpha}} dy \geq C_1 + R_{n,\alpha} C_1^{2p-1} C_2 \int_{\mathbb{R}^n} \frac{|y|^a}{|x - y|^{n-\alpha}} dy.
\]

If we choose \( x = 0 \), then the above inequality and the fact \( a > -\alpha \) imply \( C_1 = 0 \). Therefore, \((u, v)\) satisfies integral system (3).

Conversely, assume that \((u, v)\) is a nonnegative solution of integral system (3), then
\[
(-\Delta)^{\frac{s}{2}} u(x) = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} \left( \frac{R_{n,\alpha}}{|x - y|^{n-\alpha}} \right) \frac{|y|^a u^{p-1}(y)v(y)}{|x - y|^{n-\alpha}} dy
\]
\[= \int_{\mathbb{R}^n} \delta_x(y) \frac{|y|^a u^{p-1}(y)v(y)}{|x - y|^{n-\alpha}} dy = |x|^a u^{p-1}(x)v(x).
\]

That is, \( u \) satisfies equation (1). This completes the proof of Theorem 2.1. \( \square \)

For any \( \lambda > 0 \), we denote by \( S_\lambda = \partial B_\lambda \) the sphere of center \( 0 \) and radius \( \lambda \). We also denote by
\[
x^\lambda = \frac{\lambda^2 x}{|x|^2}
\]
the inversion of \( x \in \mathbb{R}^n \setminus \{0\} \) about the sphere \( S_\lambda \). We then define
\[
u_{\lambda}(x) = \left( \frac{\lambda}{|x|} \right)^{n-\alpha} u(x^\lambda) \quad \text{and} \quad v_{\lambda}(x) = \left( \frac{\lambda}{|x|} \right)^{n-\beta} v(x^\lambda),
\]
which are called the Kelvin transform of \( u \) and \( v \), respectively, with respect to \( S_\lambda \).

One may easily check that if \((u, v)\) is a solution of (3), then \((u_{\lambda}, v_{\lambda})\) satisfies the integral system
\[
\begin{cases}
u_{\lambda}(x) = R_{n,\alpha} \int_{\mathbb{R}^n} \frac{|y|^a}{|x - y|^{n-\alpha}} \left( \frac{\lambda}{|y|} \right)^\tau u_{\lambda}^{p-1}(y)v_{\lambda}(y) dy, & x \in \mathbb{R}^n \setminus \{0\}, \\
u_{\lambda}(x) = \int_{\mathbb{R}^n} \frac{|y|^a}{|x - y|^{n-\beta}} \left( \frac{\lambda}{|y|} \right)^p u_{\lambda}^{p}(y) dy, & x \in \mathbb{R}^n \setminus \{0\},
\end{cases}
\]
where
\[
\tau = n + \beta + 2a - p(n-\alpha).
\]

From Theorem 2.1, we deduce that, if \( u \) is a nonnegative solution of (1), then \( u, v \) are both positive or \((u, v) \equiv (0, 0)\). Therefore, to prove Theorems 1.1 and 1.2, we may assume that \((u, v)\) is a positive solution of (3) in the rest of this paper.

Then we aim to obtain a contradiction in the case \( 1 \leq p < \frac{n+\beta+2a}{n-\alpha} \) and the radial symmetry of \( u \) in the case \( \frac{n+\beta+2a}{n-\alpha} \leq p \leq \frac{n+\beta+2a}{n-\alpha} \).
3. **Nonexistence of positive solutions.** In this section, we utilize the direct method of scaling spheres to prove Theorem 1.1. To this end, let \((u,v)\) be a positive solution of \((3)\) and define

\[
U_\lambda(x) = u_\lambda(x) - u(x), \quad V_\lambda(x) = v_\lambda(x) - v(x).
\]

From the first equation in \((3)\), we have

\[
u(x) = R_{n,\alpha} \int_{B_\lambda} \frac{|y|^a}{|x - y|^{n-\alpha}} u^{p-1}(y) v(y) dy + R_{n,\alpha} \int_{B_\lambda} \frac{|y|^a}{|x - y|^{n-\alpha}} \left( \frac{\lambda}{|y|} \right)^\tau u_\lambda^{p-1}(y) v_\lambda(y) dy.
\]

(5)

Similarly, from the first equation in \((4)\), we deduce

\[
u_\lambda(x) = R_{n,\alpha} \int_{B_\lambda} \frac{|y|^a}{|x - y|^{n-\alpha}} u^{p-1}(y) v(y) dy + R_{n,\alpha} \int_{B_\lambda} \frac{|y|^a}{|x - y|^{n-\alpha}} \left( \frac{\lambda}{|y|} \right)^\tau u_\lambda^{p-1}(y) v_\lambda(y) dy.
\]

(6)

By combining \((5)\) and \((6)\), we derive that, for any \(x \in B_\lambda \setminus \{0\}\),

\[
U_\lambda(x) = R_{n,\alpha} \int_{B_\lambda} \left( \frac{|y|^a}{|x - y|^{n-\alpha}} - \frac{|y|^a}{|x - y|^{n-\alpha}} \right) u^{p-1}(y) v(y) dy + \left( \left( \frac{\lambda}{|y|} \right)^\tau u_\lambda^{p-1}(y) v_\lambda(y) - u^{p-1}(y) v(y) \right) dy
\]

(7)

where we have used the fact \(\tau > 0\) and the following formulae in the last inequality

\[
\left| \frac{|y|^a}{|x - y|^{n-\alpha}} \right|^2 = \frac{|x|^2 - \lambda^2}{\lambda^2} > 0 \quad \text{for } x, y \in B_\lambda \setminus \{0\}.
\]

Notice that, for any \(0 < a \leq b \) and \(q \geq 0\), we have the elementary inequality

\[
a^q - b^q \geq \max\{q,1\} b^{q-1}(a - b).
\]

(8)

Using \((8)\), for each \(y \in B_\lambda\), we consider the following cases:

- If \(u_\lambda(y) \geq u(y)\) and \(v_\lambda(y) \geq v(y)\), then

  \[
  u_\lambda^{p-1}(y) v_\lambda(y) - u^{p-1}(y) v(y) \geq 0.
  \]

- If \(u_\lambda(y) \geq u(y)\) and \(v_\lambda(y) < v(y)\), then

  \[
  u_\lambda^{p-1}(y) v_\lambda(y) - u^{p-1}(y) v(y) \geq u^{p-1}(y) V_\lambda(y).
  \]

- If \(u_\lambda(y) < u(y)\) and \(v_\lambda(y) \geq v(y)\), then

  \[
  u_\lambda^{p-1}(y) v_\lambda(y) - u^{p-1}(y) v(y) \geq \left( u_\lambda^{p-1}(y) - u^{p-1}(y) \right) v(y) \geq \max\{p - 1,1\} u^{p-2}(y) v(y) U_\lambda(y).
  \]
If \( u_\lambda(y) < u(y) \) and \( v_\lambda(y) < v(y) \), then
\[
u_\lambda^{p-1}(y)v_\lambda(y) - u^{p-1}(y)v(y) = (u_\lambda^{p-1}(y) - u^{p-1}(y))v_\lambda(y) + u^{p-1}(y)(v_\lambda(y) - v(y)) \geq \max\{p-1,1\}u^{p-2}(y)v(y)U_\lambda(y) + u^{p-1}(y)V_\lambda(y).
\]
Consequently, we always have
\[
u_\lambda^{p-1}(y)v_\lambda(y) - u^{p-1}(y)v(y) \geq \max\{p-1,1\}u^{p-2}(y)v(y)U_\lambda^-(y) + u^{p-1}(y)V_\lambda^-(y), \quad (9)
\]
where \( t^- = \min\{t,0\} \) for \( t \in \mathbb{R} \).

From (7) and (9), we derive
\[
U_\lambda(x) > R_{n,\alpha} \int_{B_\lambda} \left( \frac{|y|^a}{|x-y|^{n-\beta}} - \frac{|y|^a}{|x-\lambda y|^{n-\beta}} \right) \left( \left( \frac{\lambda}{|y|} \right)^\tau u_\lambda^{p-1}(y) - u^p(y) \right) \, dy 
\]
\[
\geq C \int_{B_\lambda^+} \frac{|y|^a u^{p-2}(y)v(y)U_\lambda(y)}{|x-y|^{n-\alpha}} \, dy + C \int_{B_\lambda^-} \frac{|y|^a u^{p-1}(y)V_\lambda(y)}{|x-y|^{n-\alpha}} \, dy,
\]
where
\[
B_\lambda^+ = \{x \in B_\lambda \setminus \{0\} \mid U_\lambda(x) < 0\} \quad \text{and} \quad B_\lambda^- = \{x \in B_\lambda \setminus \{0\} \mid V_\lambda(x) < 0\}.
\]

By a similar argument,
\[
V_\lambda(x) = \int_{B_\lambda} \left( \frac{|y|^a}{|x-y|^{n-\beta}} - \frac{|y|^a}{|x-\lambda y|^{n-\beta}} \right) \left( \left( \frac{\lambda}{|y|} \right)^\tau u_\lambda^{p-1}(y) - u^p(y) \right) \, dy 
\]
\[
> \int_{B_\lambda} \left( \frac{|y|^a}{|x-y|^{n-\beta}} - \frac{|y|^a}{|x-\lambda y|^{n-\beta}} \right) \left( u_\lambda^{p-1}(y) - u^p(y) \right) \, dy, \quad (11)
\]
which leads to
\[
V_\lambda(x) > p \int_{B_\lambda^+} \frac{|y|^a u^{p-1}(y)U_\lambda(y)}{|x-y|^{n-\beta}} \, dy. \quad (12)
\]

To continue, we recall the following Hardy-Littlewood-Sobolev inequality.

**Lemma 3.1** (Hardy-Littlewood-Sobolev inequality [19, 29]). Let \( 0 < \alpha < n \) and \( p,q > 1 \) be such that \( \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n} \). Then
\[
\left\| \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy \right\|_{L^q(\mathbb{R}^n)} \leq C_{n,\alpha,p} \|f\|_{L^p(\mathbb{R}^n)}
\]
for any \( f \in L^p(\mathbb{R}^n) \).

Let \( s > \frac{n}{n-\alpha} \) and \( t > \frac{n}{n-\beta} \) be such that
\[
\frac{1}{s} - \frac{1}{t} = \frac{\beta - \alpha}{2n}.
\]

Applying Hardy-Littlewood-Sobolev and Hölder’s inequality, from (10) we have
\[
\|U_\lambda\|_{L^s(B_\lambda)} \leq C \left( \|x|^a u^{p-2} vU_\lambda\|_{L^{\frac{n}{n-\alpha}}(B_\lambda)} + C \|x|^a u^{p-1} V_\lambda\|_{L^{\frac{n}{n+\beta}}(B_\lambda)} \right) 
\]
\[
\leq C \left( \|x|^a u^{p-2} v\|_{L^{\frac{n}{n-\beta}}(B_\lambda)} \|U_\lambda\|_{L^s(B_\lambda)} + C \|x|^a u^{p-1}\|_{L^{\frac{2n}{n+\beta}}(B_\lambda)} \|V_\lambda\|_{L^t(B_\lambda)} \right), \quad (13)
\]
Similarly, from (12), we have
\[
\|V_\lambda\|_{L^p(B_\lambda^0)} \leq C \|x|^{\alpha}u^{p-1}U_\lambda\|_{L^{\frac{n+\alpha}{n-\alpha}}(B_\lambda^0)} \\
\leq C \|x|^{\alpha}u^{p-1}\|_{L^{\frac{n+\alpha}{n-\alpha}}(B_\lambda^0)} \|U_\lambda\|_{L^p(B_\lambda^0)}.
\] (14)

Combining (13) and (14), we deduce
\[
\|U_\lambda\|_{L^p(B_\lambda^0)} \leq C \left(\|x|^{\alpha}u^{p-2}v\|_{L^\frac{n+\alpha}{n-\alpha}(B_\lambda^0)} + \|x|^{\alpha}u^{p-1}\|_{L^{\frac{n+\alpha}{n-\alpha}}(B_\lambda^0)}\right) \|U_\lambda\|_{L^p(B_\lambda^0)},
\] (15)
where \(C\) is independent of \(\lambda\). Inequality (15) is the key ingredient in the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Assume that (3) has a positive solution \((u, v)\).

**Step 1.** (Start dilating the sphere from near \(\lambda = 0\))
We prove that
\[
U_\lambda \geq 0 \text{ in } B_\lambda \setminus \{0\}
\] (16)
for \(\lambda > 0\) sufficiently small. Indeed, since \(a > \max\{-\alpha, -\frac{\alpha+\beta}{\alpha}\}\) and \(u, v\) are continuous, there exists \(\varepsilon_0 > 0\) small enough, such that
\[
\left\|x|^{\alpha}u^{p-2}v\right\|_{L^\frac{n+\alpha}{n-\alpha}(B_\lambda)} + \|x|^{\alpha}u^{p-1}\|_{L^{\frac{n+\alpha}{n-\alpha}}(B_\lambda)} \leq \frac{1}{2C}
\]
for all \(0 < \lambda < \varepsilon_0\), where the constant \(C\) is the same as in (15). Hence, (15) indicates \(\|U_\lambda\|_{L^p(B_\lambda^0)} = 0\), which means \(B_\lambda^0 = \{0\}\). Therefore, (16) holds for all \(0 < \lambda < \varepsilon_0\).

**Step 2.** (Dilate the sphere \(S_\lambda\) outward until \(\lambda = \infty\))
Step 1 provides us a starting point to dilate the sphere \(S_\lambda\) from near \(\lambda = 0\). Now we dilate the sphere \(S_\lambda\) outward as long as (16) holds. Let
\[
\lambda_0 = \sup\{\lambda > 0 \mid U_\mu \geq 0 \text{ in } B_\mu \setminus \{0\} \text{ for all } \mu \in (0, \lambda]\}.
\]
We will point out that
\[
\lambda_0 = \infty.
\] (17)

By contradiction, let us assume \(0 < \lambda_0 < \infty\). Since \(U_\lambda\) is continuous with respect to \(\lambda\), we already have \(U_{\lambda_0} \geq 0\) in \(B_{\lambda_0} \setminus \{0\}\). From (11), we deduce \(V_{\lambda_0} > 0\) in \(B_{\lambda_0} \setminus \{0\}\). Then (7) yields \(U_{\lambda_0} \geq 0\) in \(B_{\lambda_0} \setminus \{0\}\).

Now we claim that, there exist \(C > 0\) and \(\eta > 0\) such that
\[
U_{\lambda_0}, V_{\lambda_0} \geq C \text{ in } B_{\eta} \setminus \{0\}.
\] (18)

Indeed, from (7), we can derive that, for any \(x \in B_{\lambda_0} \setminus \{0\}\),
\[
U_{\lambda_0}(x) \geq R_{n,\alpha} \int_{B_{\lambda_0}} \left( \frac{|y|^\alpha}{|x-y|^{n-\alpha}} - \frac{|y|^\alpha}{|x-y|^{\frac{n-\alpha}{\alpha}}} \right) \left( u_{\lambda_0}^{p-1}(y)v_{\lambda_0}(y) - u^{p-1}(y)v(y) \right) dy
\]
\[
\geq R_{n,\alpha} \int_{B_{\lambda_0}} \left( \frac{|y|^\alpha}{|x-y|^{n-\alpha}} - \frac{|y|^\alpha}{|x-y|^{\frac{n-\alpha}{\alpha}}} \right) u^{p-1}(y)V_{\lambda_0}(y) dy.
\] (19)
From (19) and Fatou’s lemma, we have
\[
\liminf_{|x| \to 0} U_{\lambda_0}(x) \geq R_{n, \alpha} \int_{B_{\lambda_0}} \left( \frac{1}{|y|^n} - \frac{1}{\lambda_0^n} \right) |y|^a u^{p-1}(y) V_{\lambda_0}(y) dy > 0.
\]
Hence for \( x \in B_\eta \setminus \{0\} \), where \( \eta \) is sufficiently small, we have \( U_{\lambda_0}(x) > C \).

Similarly, from (11), we can derive
\[
V_{\lambda_0}(x) > \int_{B_{\lambda_0}} \left( \frac{|y|^a}{|x-y|^{n-\beta}} - \frac{|y|^a}{|x-y|^n} \right) (u^p_{\lambda_0}(y) - u^p(y)) dy \geq C > 0
\]
for all \( x \in B_\eta \setminus \{0\} \), where \( \eta \) is chosen smaller if necessary. This proves (18).

Now we fix \( 0 < r_0 < \frac{\lambda_0}{2} \) small enough, such that
\[
\|x|^a u^{p-2}v\|_{L^\infty(B_{\lambda_0+r_0} \setminus B_{\lambda_0-r_0})} + \|x|^a u^{p-1}\|_{L^{\frac{2n}{n-\beta}}(B_{\lambda_0+r_0} \setminus B_{\lambda_0-r_0})} \leq \frac{1}{2C}, \quad (20)
\]
where the constant \( C \) is the same as in (15).

Combining (18) with the fact that \( U_{\lambda_0} \) and \( V_{\lambda_0} \) are continuous and positive in \( B_{\lambda_0} \setminus \{0\} \), we can find a constant \( C > 0 \) such that
\[
U_{\lambda_0}, V_{\lambda_0} \geq C \quad \text{in} \quad B_{\lambda_0-r_0} \setminus \{0\}.
\]

Because \( u \) and \( v \) are uniformly continuous on arbitrary compact set, there exists \( \rho_0 \in (0, r_0) \) such that, for any \( \lambda \in (\lambda_0, \lambda_0 + \rho_0) \),
\[
U_\lambda, V_\lambda \geq \frac{C}{2} > 0 \quad \text{in} \quad B_{\lambda_0-r_0} \setminus \{0\}.
\]

Therefore, for any \( \lambda \in (\lambda_0, \lambda_0 + \rho_0) \),
\[
B^x_{\lambda} \subset B_{\lambda_0+r_0} \setminus B_{\lambda_0-r_0}.
\]

Hence, estimates (15) and (20) yield \( \|U_\lambda\|_{L^\infty(B^x_{\lambda})} = 0 \), which means \( |B^x_{\lambda}| = 0 \).

Thus, for any \( \lambda \in (\lambda_0, \lambda_0 + \rho_0) \),
\[
U_\lambda \geq 0 \quad \text{in} \quad B_\lambda \setminus \{0\}.
\]

However, this contradicts the definition of \( \lambda_0 \) and (17) is proved.

**Step 3.** (Derive lower bound estimates on \( u \) and \( v \))

Since \( \lambda_0 = \infty \), we have \( U_\lambda \geq 0 \) in \( B_\lambda \setminus \{0\} \) for all \( \lambda > 0 \). That is,
\[
u(x) \geq \left( \frac{\lambda}{|x|} \right)^{n-\alpha} \frac{\lambda^2 x}{|x|^2} \quad \text{for all} \quad \lambda > 0 \quad \text{and} \quad |x| \geq \lambda.
\]
Choosing \( \lambda = \sqrt{|x|} \), we have
\[
u(x) \geq \frac{1}{|x|^\frac{n-\alpha}{2}} u \left( \frac{x}{|x|} \right) \quad \text{for all} \quad |x| \geq 1.
\]

Therefore,
\[
u(x) \geq \frac{\min_{S_{\lambda_0}} u}{|x|^{\frac{n-\alpha}{2}}} = \frac{C_0}{|x|^{\frac{n-\alpha}{2}}} \quad \text{for all} \quad |x| \geq 1.
\]

That is,
\[
u(x) \geq \frac{C_0}{|x|^{\tau_0}} \quad \text{for all} \quad |x| \geq 1,
\]
where \( \tau_0 = \frac{n-\alpha}{2} \).
Now we have for $|x| \geq 1$,

$$v(x) = \int_{\mathbb{R}^n} \frac{|y|^a u^p(y)}{|x-y|^{n-\beta}} \, dy \geq \frac{C}{|x|^{n-\beta}} \int_{|2|x| \leq |y| \leq 3|x|} \frac{dy}{|y|^{p\tau_0-a}} = \frac{C}{|x|^{p\tau_0-(a+\beta)}}$$

and hence

$$u(x) = R_{n,\alpha} \int_{\mathbb{R}^n} \frac{|y|^a u^{p-1}(y)v(y)}{|x-y|^{n-\alpha}} \, dy \geq \frac{C}{|x|^{n-\alpha}} \int_{|2|x| \leq |y| \leq 3|x|} \frac{dy}{|y|^{(p-1)\tau_0-a+p\tau_0-(a+\beta)}}$$

$$= \frac{C_1}{|x|^{(2p-1)\tau_0-(2a+\alpha+\beta)}}.$$

That is,

$$u(x) \geq \frac{C_1}{|x|^\tau_1} \quad \text{for all} \quad |x| \geq 1,$$

where $\tau_1 = (2p-1)\tau_0 - (2a + \alpha + \beta)$.

Continuing the above process, we obtain the following lower bounds of $u$ for any $k \in \mathbb{N}$,

$$u(x) \geq \frac{C_k}{|x|^\tau_k} \quad \text{for all} \quad |x| \geq 1,$$

where

$$\tau_{k+1} = (2p-1)\tau_k - (2a + \alpha + \beta).$$

If $p = 1$, then

$$\tau_k = \tau_0 - k(2a + \alpha + \beta) \to -\infty \quad \text{as} \quad k \to \infty.$$

If $1 < p < \frac{n+\beta+2a}{n-\alpha}$, then

$$\tau_k = (2p-1)^k \left( \frac{n-\alpha}{2} - \frac{2a + \alpha + \beta}{2(p-1)} \right) + \frac{2a + \alpha + \beta}{2(p-1)} \to -\infty \quad \text{as} \quad k \to \infty.$$

Therefore, in both cases, $\tau_k \to -\infty$ as $k \to \infty$. Combining this fact with (21), we arrive at

$$u(x) \geq C \quad \text{for} \quad |x| \geq 1.$$

Hence we have

$$v(x) = \int_{\mathbb{R}^n} \frac{|y|^a u^p(y)}{|x-y|^{n-\beta}} \, dy \geq \frac{C}{|x|^{n-\beta}} \int_{|2|x| \leq |y| \leq 3|x|} |y|^a \, dy = C|x|^{\alpha+\beta} \quad \text{for all} \quad |x| \geq 1.$$

Then

$$\infty > u(0) = R_{n,\alpha} \int_{\mathbb{R}^n} \frac{|y|^a u^{p-1}(y)v(y)}{|y|^{n-\alpha}} \, dy \geq C \int_{\mathbb{R}^n \setminus B_1} \frac{dy}{|y|^{n-2a-\alpha-\beta}}.$$

Since $2a + \alpha + \beta > 0$, the last integral is infinite and we have a contradiction. Therefore, system (3) has no positive solution and the proof of Theorem 1.1 is completed. \qed
4. Symmetry of positive solutions. Let \((u, v)\) be a positive solution of (3) and denote by \(\varpi\) and \(\varpi\) the Kelvin transform of \(u\) and \(v\), respectively, with respect to \(S_1\). That is,

\[
\varpi(x) = \frac{1}{|x|^{n-\alpha}} u \left( \frac{x}{|x|^2} \right) \quad \text{and} \quad \varpi(x) = \frac{1}{|x|^{n-\beta}} v \left( \frac{x}{|x|^2} \right) \quad \text{for} \ x \in \mathbb{R}^n \setminus \{0\}.
\]

From (4), we see that \((\varpi, \varpi)\) satisfies the integral system

\[
\begin{split}
\varpi(x) &= R_{n,\alpha} \int_{\mathbb{R}^n} \frac{\varpi^{p-1}(y)\varpi(y)}{|x-y|^{n-\alpha}|y|^\gamma} \, dy, \quad x \in \mathbb{R}^n \setminus \{0\}, \\
\varpi(x) &= \int_{\mathbb{R}^n} \frac{\varpi^{p}(y)}{|x-y|^{n-\beta}|y|^\gamma} \, dy, \quad x \in \mathbb{R}^n \setminus \{0\},
\end{split} \tag{22}
\]

where

\[ \gamma = n + \beta + a - p(n - \alpha) \geq 0. \]

Moreover, we have

\[
\varpi(x) \sim \frac{1}{|x|^{n-\alpha}} \quad \text{and} \quad \varpi(x) \sim \frac{1}{|x|^{n-\beta}} \quad \text{as} \ |x| \to \infty. \tag{23}
\]

To prove Theorem 1.2, we exploit the method of moving planes in integral forms. For arbitrary \(\lambda \in \mathbb{R}\), we denote

\[ T_\lambda = \{ x \in \mathbb{R}^n \mid x_1 = \lambda \} \quad \text{and} \quad \Sigma_\lambda = \{ x \in \mathbb{R}^n \mid x_1 < \lambda \}, \]

which are the moving plane and the left region of the plane, respectively.

In this section, we redefine \(x^\lambda, U_\lambda, V_\lambda\) as follows

- \(x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)\) is the reflection of the point \(x = (x_1, x_2, \ldots, x_n)\) about the plane \(T_\lambda\),
- \(\varpi_\lambda(x) = \varpi(x^\lambda)\) and \(\varpi_\lambda(x) = \varpi(x^\lambda)\),
- \(U_\lambda(x) = \varpi_\lambda(x) - \varpi(x)\) and \(V_\lambda(x) = \varpi_\lambda(x) - \varpi(x)\).

For \(\lambda < 0\), we denote

\[ \Sigma^*_\lambda = \Sigma_\lambda \setminus \{0^\lambda\}, \]

\[ \Sigma^\lambda_\lambda = \{ x \in \Sigma^*_\lambda \mid U_\lambda(x) < 0 \}, \]

\[ \Sigma^*\lambda = \{ x \in \Sigma^*_\lambda \mid V_\lambda(x) < 0 \}. \]

For any \(x \in \Sigma^*_\lambda\), we may use (22) to write

\[
\varpi(x) = R_{n,\alpha} \int_{\Sigma_\lambda} \frac{\varpi^{p-1}(y)\varpi(y)}{|x-y|^{n-\alpha}|y|^\gamma} \, dy + R_{n,\alpha} \int_{\Sigma_\lambda} \frac{\varpi^{p-1}(y)\varpi(y)}{|x-y|^{n-\alpha}|y|^\gamma} \, dy \tag{24}
\]

and

\[
\varpi_\lambda(x) = R_{n,\alpha} \int_{\Sigma_\lambda} \frac{\varpi^{p-1}(y)\varpi(y)}{|x-y|^{n-\alpha}|y|^\gamma} \, dy + R_{n,\alpha} \int_{\Sigma_\lambda} \frac{\varpi^{p-1}(y)\varpi(y)}{|x-y|^{n-\alpha}|y|^\gamma} \, dy. \tag{25}
\]

Since \(|x-y| = |x^\lambda - y|\) and \(|x-y| = |x^\lambda - y^\lambda|\), from (24) and (25), we obtain

\[
U_\lambda(x) = R_{n,\alpha} \int_{\Sigma_\lambda} \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \right) \left( \frac{\varpi^{p-1}(y)\varpi(y)}{|y|^\gamma} - \frac{\varpi^{p-1}(y)\varpi(y)}{|y|^\gamma} \right) \, dy \
\geq R_{n,\alpha} \int_{\Sigma_\lambda} \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda - y|^{n-\alpha}} \right) \frac{\varpi^{p-1}(y)\varpi(y)}{|y|^\gamma} \, dy. \tag{26}
\]
Arguing as in (9), we have
\[
\pi^{-1}_\lambda(y)\pi(y) - \pi^{-1}(y)\pi(y) \geq \max\{p - 1, 1\} \pi^{p-2}(y)\pi(y)U_\lambda^-(y) + \pi^{-1}(y)V_\lambda^-(y).
\] (27)

From (26) and (27), we deduce
\[
U_\lambda(x) \geq C \int_{\Sigma_\lambda} \frac{\pi^{p-2}(y)\pi(y)U_\lambda(y)}{|x - y|^{n-\alpha}|y|^{\gamma}} dy + C \int_{\Sigma_\lambda} \frac{\pi^{-1}(y)V_\lambda(y)}{|x - y|^{n-\alpha}|y|^{\gamma}} dy.
\] (28)

Similarly,
\[
V_\lambda(x) = \int_{\Sigma_\lambda} \left( \frac{1}{|x - y|^{n-\beta}} - \frac{1}{|x^{\lambda} - y|^{n-\beta}} \right) \left( \frac{\pi^{p}(y)}{|y|^{\gamma}} - \frac{\pi^{p}(y)}{|y|^{\gamma}} \right) dy
\]
\[
\geq \int_{\Sigma_\lambda} \left( \frac{1}{|x - y|^{n-\beta}} - \frac{1}{|x^{\lambda} - y|^{n-\beta}} \right) \frac{\pi^{p}(y)}{|y|^{\gamma}} dy,
\] (29)

which leads to
\[
V_\lambda(x) \geq p \int_{B_x^+} \frac{\pi^{-1}(y)U_\lambda(y)}{|x - y|^{n-\beta}|y|^{\gamma}} dy.
\] (30)

Similar to the previous section, we let \( s > \frac{n}{n-\alpha} \) and \( t > \frac{n}{n-\beta} \) be such that
\[
\frac{1}{s} - \frac{1}{t} = \frac{\beta - \alpha}{2n}.
\]

Then using (28), (30) and arguing as in Section 3, we derive
\[
\|U_\lambda\|_{L^s(\Sigma_\lambda)} \leq C \left( \left\| \pi^{p-2}\pi \right\|_{L^\frac{n}{2}(\Sigma_\lambda)} + \left\| \pi^{p-1}\pi \right\|_{L^{\frac{2n}{\alpha}}(\Sigma_\lambda)} \left\| \pi^{p-1}\pi \right\|_{L^{\frac{2n}{\beta}}(\Sigma_\lambda)} \right) \times \|U_\lambda\|_{L^s(\Sigma_\lambda)},
\] (31)

where \( C \) is independent of \( \lambda \).

Using (23) and the assumption \( a > \max\{-\alpha, -\frac{\alpha + \beta}{2}\} \), it is easy to check that
\[
\pi^{p-2}\pi \in L^{\frac{n}{2}}(\mathbb{R}^n \setminus B_\epsilon) \quad \text{and} \quad \pi^{p-1}\pi \in L^{\frac{2n}{\alpha}}(\mathbb{R}^n \setminus B_\epsilon) \quad \text{for all} \quad \epsilon > 0.
\] (32)

**Proof of Theorem 1.2.** There are two cases to be considered.

**Case 1:** The subcritical case \( p < \frac{n+\beta}{n-\alpha} \). In this case, \( \gamma > 0 \).

We move \( T_\lambda \) from near \( \lambda = -\infty \) along the \( x_1 \)-direction to the right until it reaches the limiting position. The proof in this case has two steps.

**Step 1.** We show that
\[
U_\lambda \geq 0 \quad \text{in} \quad \Sigma_\lambda
\] (33)

for all \( \lambda \) sufficiently negative.

Indeed, using (32), we can find \( R_0 > 0 \) sufficiently large such that for \( \lambda \leq -R_0 \), we have
\[
\left\| \pi^{p-2}\pi \right\|_{L^{\frac{n}{2}}(\Sigma_\lambda)} + \left\| \pi^{p-1}\pi \right\|_{L^{\frac{2n}{\alpha}}(\Sigma_\lambda)} \left\| \pi^{p-1}\pi \right\|_{L^{\frac{2n}{\beta}}(\Sigma_\lambda)} \leq \frac{1}{2C},
\] (34)

where the constant \( C \) is the same as in (31).

Now (31) and (34) imply \( \|U_\lambda\|_{L^s(\Sigma_\lambda)} = 0 \) and hence \( |\Sigma_\lambda|^s = 0 \) for \( \lambda \leq -R_0 \). Thus, (33) holds for \( \lambda \leq -R_0 \).

**Step 2.** Let
\[
\lambda_0 = \sup\{\lambda \leq 0 \mid U_\mu \geq 0 \quad \text{in} \quad \Sigma_\mu \quad \text{for all} \quad \mu \leq \lambda\}.
\] (35)
We point out in this step that
\[ \lambda_0 = 0. \quad (36) \]

By contradiction, assume that \( \lambda_0 < 0 \). By continuity, we already have \( U_{\lambda_0} \geq 0 \) in \( \Sigma_{\lambda_0}^* \). Therefore, (29) yields that
\[
V_{\lambda_0}(x) \geq \int_{\Sigma_{\lambda_0}} \left( \frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x_{\lambda_0} - y|^{n-\beta}} \right) \left( \frac{1}{|y|^{\gamma}} - \frac{1}{|y|^{\gamma}} \right) \pi^{p}(y) dy > 0
\]
and hence (26) implies
\[
U_{\lambda_0}(x) \geq \int_{\Sigma_{\lambda_0}} \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_{\lambda_0} - y|^{n-\alpha}} \right) \left( \frac{1}{|y|^{\gamma}} - \frac{1}{|y|^{\gamma}} \right) \pi^{p-1}(y) \pi(y) dy > 0.
\]
That means that \( U_{\lambda_0} \) and \( V_{\lambda_0} \) are positive in \( \Sigma_{\lambda_0}^* \).

To obtain a contradiction, we will show that \( T_\lambda \) can be move a little bit to the right such that (33) still holds. That is, \( U_{\lambda} \geq 0 \) in \( \Sigma_{\lambda}^* \) for all \( \lambda \in [\lambda_0, \lambda_0 + \varepsilon) \), where \( \varepsilon > 0 \) is sufficiently small.

To this end, we aim to find some \( \varepsilon > 0 \) sufficiently small such that
\[
\left\| \frac{\pi^{p-2}\pi}{|x|^{\gamma}} \right\|_{L^{\frac{n}{\pi}}(\Sigma_{\lambda}^*)} + \left\| \frac{\pi^{p-1}}{|x|^{\gamma}} \right\|_{L^{\frac{n}{\pi+\pi}}(\Sigma_{\lambda}^*)} \leq \frac{1}{2C}
\]
for all \( \lambda \in [\lambda_0, \lambda_0 + \varepsilon) \), where the constant \( C \) is the same as in (31).

Using (32), we can find some \( R > 0 \) sufficiently large such that
\[
\left\| \frac{\pi^{p-2}\pi}{|x|^{\gamma}} \right\|_{L^{\frac{n}{\pi}}(\Sigma_{\lambda}^* \setminus B_R)} + \left\| \frac{\pi^{p-1}}{|x|^{\gamma}} \right\|_{L^{\frac{n}{\pi+\pi}}(\Sigma_{\lambda}^* \setminus B_R)} \leq \frac{1}{4C},
\]
Now fix this \( R \), in order to derive (37), we only need to point out that
\[ \lim_{\lambda \to \lambda_0^+} |\Sigma_{\lambda}^* \cap B_R| = \lim_{\lambda \to \lambda_0^+} |\Sigma_{\lambda}^* \cap B_R| = 0. \quad (39) \]

Let us define \( D_\delta = \{ x \in \Sigma_{\lambda_0}^* \cap B_R \mid U_{\lambda_0}(x) > \delta \} \) and \( E_\delta = (\Sigma_{\lambda_0}^* \cap B_R) \setminus D_\delta \) for \( \delta > 0 \). Then we denote \( F_\lambda = (\Sigma_{\lambda}^* \setminus \Sigma_{\lambda_0}^*) \cap B_R \) for \( \lambda > \lambda_0 \). Clearly,
\[ \lim_{\delta \to 0^+} |E_\delta| = \lim_{\lambda \to \lambda_0^+} |F_\lambda| = 0 \]
and
\[ \Sigma_{\lambda}^* \cap B_R \subset \Sigma_{\lambda}^* \cap (D_\delta \cup E_\delta \cup F_\lambda) \subset \Sigma_{\lambda}^* \cap D_\delta \cup E_\delta \cup F_\lambda \]

Fix arbitrary \( \delta > 0 \). We will derive
\[ \lim_{\lambda \to \lambda_0^+} |\Sigma_{\lambda}^* \cap D_\delta| = 0. \quad (42) \]

In fact, \( \pi(x_{\lambda_0}) - \pi(x_\lambda) = U_{\lambda_0}(x) - U_\lambda(x) > \delta \) for \( x \in \Sigma_{\lambda}^* \cap D_\delta \). Hence \( \Sigma_{\lambda}^* \cap D_\delta \subset H_\delta := \{ x \in B_R \mid \pi(x_{\lambda_0}) - \pi(x_\lambda) > \delta \} \). Let \( e_1 = (1, 0, \ldots, 0) \). For any \( \varepsilon > 0 \) sufficiently small, we have
\[
|H_\delta \setminus B_\varepsilon(2\lambda_0 e_1)| \leq \frac{1}{\delta} \int_{H_\delta \setminus B_\varepsilon(2\lambda_0 e_1)} |\pi(x_{\lambda_0}) - \pi(x_\lambda)| \ dx
\]
\[
\leq \frac{1}{\delta} \int_{B_R(2\lambda_0 e_1) \setminus B_\varepsilon} |\pi(x) - \pi(x + 2(\lambda_0 - \lambda)e_1)| \ dx,
\]
Since \( \pi \) is continuous in \( \mathbb{R}^n \setminus \{0\} \) and \( \varepsilon \) is arbitrarily small, the above inequality indicates \( \lim_{\lambda \to \lambda_0^+} |H_\delta| = 0 \), which yields (42).
By combining (40), (41) and (42), we deduce
\[
\lim_{\lambda \to \lambda_0^+} |\Sigma_\lambda^u \cap B_R| \leq |E_\delta|.
\]
Letting \(\delta \to 0^+\) and using (40), we arrive at the first claim in (39). The second limit in (39) can be obtained by a similar reasoning. Combining (38) and (39), we arrive at (37).

From (31) and (37), we can find \(\varepsilon > 0\) sufficiently small such that \(|\Sigma_\lambda^u| = 0\) for any \(\lambda \in [\lambda_0, \lambda_0 + \varepsilon]\). Hence \(U_\lambda \geq 0\) in \(\Sigma_\lambda^u\) for any \(\lambda \in [\lambda_0, \lambda_0 + \varepsilon]\). Since this contradicts the definition of \(\lambda_0\) in (35), we must have (36) and hence \(U_0(x) \geq 0\) in \(\Sigma_0^u\).

Similarly, we can move \(T_\lambda\) from near \(\lambda = +\infty\) to the left in order to deduce \(U_0(x) \leq 0\) in \(\Sigma_0^u\). Therefore, \(U_0 \equiv 0\), which means that \(\pi\) is symmetric about \(T_0\). By repeating the previous arguments to arbitrary direction, we derive that \(u\) is radially symmetric about the origin. So is \(u\).

**Case 2:** The critical case \(p = \frac{n+\beta+a}{n-\alpha}\). In this case, \(\gamma = 0\).

Suppose on the contrary that (3) admits a positive solution \((u,v)\) such that \(u\) is not radially symmetric about the origin. Then there exists a hyperplane \(H\) passing through the origin such that \(u\) is not symmetric about \(H\). For simplicity, we may assume \(H = T_0\).

Similar to Case 1, we can show that
\[
U_\lambda \geq 0 \quad \text{in } \Sigma_\lambda^u
\]
for \(\lambda\) sufficiently negative. Let
\[
\lambda_0 = \sup\{\lambda \leq 0 \mid U_\mu \geq 0 \text{ in } \Sigma_\mu^u \text{ for all } \mu \leq \lambda\}.
\]
We will show that
\[
\lambda_0 = 0.
\]

By contradiction, assume \(\lambda_0 < 0\). We have the following possibilities.

**Possibility (i):** \(U_{\lambda_0} = 0\) in \(\Sigma_{\lambda_0}^u\). From (29), we also have \(V_{\lambda_0} = 0\) in \(\Sigma_{\lambda_0}^u\). Therefore, 0 is not a singular point of \(\pi\) and \(\pi\). Furthermore,
\[
\lim_{x \to 0} \pi(x) = \pi(0^{\lambda_0}) > 0 \quad \text{and} \quad \lim_{x \to 0} \pi(x) = \pi(0^{\lambda_0}) > 0.
\]

Hence
\[
u(x) \sim \frac{1}{|x|^{n-\beta}} \quad \text{as } |x| \to \infty.
\]
Therefore,
\[
|x|^\alpha u^{p-2}v \in L^{\frac{n}{n-\alpha}}(\mathbb{R}^n \setminus B_\varepsilon) \quad \text{and} \quad |x|^\alpha u^{p-1} \in L^{\frac{n}{n-\beta}}(\mathbb{R}^n \setminus B_\varepsilon) \quad \text{for all } \varepsilon > 0.
\]

Using this fact, we may apply the moving plane method to integral system (3) directly to get that \(u\) is symmetric about the origin, which is a contradiction. The proof is very similar to that of Case 1. The only difference is that we now deal with \(u, v, -\alpha\) instead of \(\pi, \pi, \gamma\).

**Possibility (ii):** \(U_{\lambda_0} \geq 0\), but \(U_{\lambda_0} \neq 0\) in \(\Sigma_{\lambda_0}^u\). It follows from (29) that
\[
V_{\lambda_0}(x) = \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x_{\lambda_0} - y|^{n-\beta}}\right) (\pi_{\lambda_0}^p(y) - \pi_{\lambda_0}^p(y)) \, dy > 0.
\]
and hence (26) implies
\[ U_{\lambda_0}(x) = R_{n,\alpha} \int_{\Sigma_{\lambda_0}} \left( \frac{1}{|x - y|^{n-\alpha}} - \frac{1}{|x - \lambda_0 y|^{n-\alpha}} \right) \times \left( \frac{u^{p-1}}{\varpi_{\lambda_0}(y)} \varpi_{\lambda_0}(y) - \frac{1}{\varpi^{p-1}(y)} \varpi(y) \right) dy > 0. \]

Similar to Case 1, we can show that \( T_{\lambda} \) can be moved a little bit to the right such that (43) still holds. However, this contradicts the definition of \( \lambda_0 \).

Therefore, \( \lambda_0 = 0 \). Similarly, one can move the plane \( T_{\lambda} \) from \( \lambda = +\infty \) to the left to finally deduce that \( u \) is symmetric about \( T_0 \), which is a contradiction.

Hence \( u \) must be radially symmetric about the origin.

\begin{proof}

\end{proof}

Acknowledgments. This work was finished while the authors were visiting Vietnam Institute for Advanced Study in Mathematics (VIASM) in 2019. They are grateful to the institute for financial support and hospitality. The research is funded by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2017.307.

REFERENCES

[1] D. Applebaum, \textit{Lévy Processes and Stochastic Calculus}, vol. 116 of Cambridge Studies in Advanced Mathematics, 2nd edition, Cambridge University Press, Cambridge, 2009.
[2] P. Belchior, H. Bueno, O. H. Miyagaki and G. A. Pereira, Remarks about a fractional Choquard equation: Ground state, regularity and polynomial decay, \textit{Nonlinear Anal.}, 164 (2017), 38–53.
[3] J. Bertoin, \textit{Lévy Processes}, vol. 121 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1996.
[4] J.-P. Bouchaud and A. Georges, Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications, \textit{Phys. Rep.}, 195 (1990), 127–293.
[5] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, \textit{Comm. Partial Differential Equations}, 32 (2007), 1245–1260.
[6] L. A. Caffarelli and A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, \textit{Ann. of Math. (2)}, 171 (2010), 1903–1930.
[7] W. Chen, C. Li and Y. Li, A direct method of moving planes for the fractional Laplacian, \textit{Adv. Math.}, 308 (2017), 404–437.
[8] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, \textit{Comm. Pure Appl. Math.}, 59 (2006), 330–343.
[9] P. Constantin, Euler equations, Navier-Stokes equations and turbulence, in \textit{Mathematical Foundation of Turbulent Viscous Flows}, vol. 1871 of Lecture Notes in Math., Springer, Berlin, 2006, 1–43.
[10] W. Dai, Y. Fang and G. Qin, Classification of positive solutions to fractional order Hartree equations via a direct method of moving planes, \textit{J. Differential Equations}, 265 (2018), 2044–2063.
[11] W. Dai and G. Qin, Liouville type theorems for fractional and higher order Hénon-Hardy type equations via the method of scaling spheres, Preprint, \texttt{arXiv:1810.02752}.
[12] P. d’Avenia, G. Siciliano and M. Squassina, On fractional Choquard equations, \textit{Math. Models Appl. Sci.}, 25 (2015), 1447–1476.
[13] L. Du, F. Gao and M. Yang, Existence and qualitative analysis for nonlinear weighted Choquard equations, Preprint, \texttt{arXiv:1810.11759}.
[14] T. Kulczycki, Properties of Green function of symmetric stable processes, \textit{Probab. Math. Statist.}, 17 (1997), 339–364.
[15] P. Le, Liouville theorem and classification of positive solutions for a fractional Choquard type equation, \textit{Nonlinear Anal.}, 185 (2019), 123–141.
[16] P. Le, Symmetry of singular solutions for a weighted Choquard equation involving the fractional \( p \)-Laplacian, \textit{Commun. Pure Appl. Anal.}, 19 (2020), 527–539.
[17] Y. Lei, Liouville theorems and classification results for a nonlocal Schrödinger equation, \textit{Discrete Contin. Dyn. Syst.}, 38 (2018), 5351–5377.
[18] E. H. Lieb, Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation, Studies in Appl. Math., 57 (1976/77), 93–105.
[19] E. H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. of Math. (2), 118 (1983), 349–374.
[20] E. H. Lieb and B. Simon, The Hartree-Fock theory for Coulomb systems, Comm. Math. Phys., 53 (1977), 185–194.
[21] P.-L. Lions, The Choquard equation and related questions, Nonlinear Anal., 4 (1980), 1063–1072.
[22] L. Ma and L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, Arch. Ration. Mech. Anal., 195 (2010), 455–467.
[23] P. Ma, X. Shang and J. Zhang, Symmetry and nonexistence of positive solutions for fractional Choquard equations, Pacific J. Math., 304 (2020), 143–167.
[24] P. Ma and J. Zhang, Existence and multiplicity of solutions for fractional Choquard equations, Nonlinear Anal., 164 (2017), 100–117.
[25] G. Molica Bisci, V. D. Radulescu and R. Servadei, Variational Methods for Nonlocal Fractional Problems, vol. 162 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2016, With a foreword by Jean Mawhin.
[26] I. M. Moroz, R. Penrose and P. Tod, Spherically-symmetric solutions of the Schrödinger-Newton equations, Classical Quantum Gravity, 15 (1998), 2733–2742, Topology of the Universe Conference (Cleveland, OH, 1997).
[27] V. Moroz and J. Van Schaftingen, A guide to the Choquard equation, J. Fixed Point Theory Appl., 19 (2017), 773–813.
[28] G. I. Nazin, Limit distribution functions of systems with many-particle interactions in classical statistical physics, Teoret. Mat. Fiz., 25 (1975), 132–140.
[29] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.
[30] W. Zhang and X. Wu, Nodal solutions for a fractional Choquard equation, J. Math. Anal. Appl., 464 (2018), 1167–1183.
[31] R. Zhuo, W. Chen, X. Cui and Z. Yuan, Symmetry and non-existence of solutions for a nonlinear system involving the fractional Laplacian, Discrete Contin. Dyn. Syst., 36 (2016), 1125–1141.

Received for publication August 2019.

E-mail address: tuanda@hnue.edu.vn
E-mail address: lephuong@tdtu.edu.vn
E-mail address: thangnn@hnue.edu.vn