GEOMETRIC PROPERTIES OF DISINTEGRATION OF MEASURES

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Abstract. In this paper we study the connection between disintegration of measures and some geometric properties of probability spaces. We prove a disintegration theorem, and then address disintegration from the perspective of an optimal transport problem. We look at the disintegration of transport plans, which are used to define and study disintegration maps. Using these objects we study the regularity and absolute continuity of disintegration of measures. In particular, we discuss the weak continuity of the disintegration map, we study some conditions for which one can obtain a path of conditional measures in the space of probability measures and we show a rigidity condition for the disintegration of measures to be given into absolutely continuous measures.

1. Introduction

The disintegration of a measure over a partition of the space on which it is defined is a way to rewrite this measure as a combination of probability measures, which are concentrated on the elements of the partition. As an example, consider the probability space \((X, \mathcal{F}, \mu)\) and its partition into a finite number of measurable subsets \(P_1, \ldots, P_n\) with positive measure. A disintegration of \(\mu\) with respect to this partition is a family of probabilities \(\{\mu_1, \ldots \mu_n\}\) on \(X\), such that, for \(i = 1, \ldots, n\), we have \(\mu_i(P_i) = 1\) and, for every measurable set \(E \subset X\), the conditional measures are given by \(\mu_i(E) = \mu(E \cap P_i) / \mu(P_i)\). It is possible to write the original measure as a combination of the conditional ones

\[
\mu(E) = \sum_{i=1}^{n} \mu(P_i) \mu_i(E) = \sum_{i=1}^{n} \mu(P_i) \frac{\mu(E \cap P_i)}{\mu(P_i)}.
\]

More generally, consider a probability space \((X, \mathcal{F}, \mu)\) and a partition \(\mathcal{P}\) of \(X\) into measurable subsets. Let \(\rho\) be the natural projection that associates each point \(x \in X\) to the element \(P \in \mathcal{P}\) which contains \(x\). The measurable function \(\rho\) can be used to induce a probability \(\bar{\mu}\) on \(\mathcal{P}\). A subset \(B \subset \mathcal{P}\) is measurable if, and only if, \(\rho^{-1}(B)\) is a measurable subset of \(X\). Then, the
family $\hat{B}(P)$ of measurable subsets is a $\sigma$-algebra on $P$. Let $\hat{\mu}$ denote the measure given by

$$\hat{\mu}(B) = \varrho_*\mu(B) := \mu(x \in X : \varrho(x) \in B)$$

for every $B \in \hat{B}(P)$. In this case, $\hat{\mu}$ is called the law of $\varrho$, denoted by $\text{law}(\varrho)$.

A disintegration of $\mu$ with respect to $P$ into conditional measures is a family $\{\mu_P : P \in P\}$ of probability measures on $X$, such that, for every $E \in \mathcal{F}$,

1. $\mu_P(P) = 1$ for $\hat{\mu}$-almost every $P \in P$;
2. $P \mapsto \mu_P(E)$ is measurable;
3. $\mu(E) = \int \mu_P(E) \, d\hat{\mu}(P)$.

There are several reasons why one may wish to study such possible combinations of measures. In Ergodic Theory, for example, the disintegration of a measure is directly related to the ergodic decomposition of invariant measures, which are crucial objects encoding the asymptotic behaviour of dynamical systems. The concept of disintegration, however, appears in much broader context in areas, such as Probability [Par67, CP97], Geometry [Stu06], among others [BM17, GL20, Var16, Vil03].

The idea of disintegrating a measure was devised by Von Neumann in 1932 [Von32]. Since then, different versions of disintegration theorems have been presented and used, for example in [AGS05, AP03, DM78, Tue75], to name a few. In particular, the well-known Rokhlin’s Disintegration Theorem shows that there exists a disintegration of $\mu$ relative to $P$ if $X$ is a complete separable metric space and $P$ is a measurable partition. By measurable partition we mean that there exists some measurable set $X_0 \subset X$, such that $\mu(X) = \mu(X_0)$, and $P = \bigvee_{n=1}^{\infty} P_n = \{P_1 \cap P_2 \cap \cdots : P_n \in P_n, \text{ for all } n \geq 1\}$ restricted to $X_0$, for an increasing sequence $P_1 \prec P_2 \prec \cdots \prec P_n \prec \cdots$ of countable partitions [Rok52].

Recently, Simmons has proposed a more general and subtle formulation of Rokhlin’s Disintegration Theorem, where he has considered any universally measurable space $(X, \mathcal{B}, \mu)$ and a measure space $Y$, for which there exists an injective map $Y \to \{0, 1\}^\mathbb{N}$. That is, $Y$ is any subspace of the standard Borel space. He has shown that there is a (unique) system of conditional measures $(\mu_y)_{y \in Y}$, a disintegration of $\mu$ [Sim12]. Then, his formulation is further developed to address $\sigma$-finite measure spaces with absolutely continuous morphisms. One of the facts standing out in Simmons’ formulation is a fibre-wise perspective, which we wish to further explore.

Even though, geometric properties of the space where a measure is defined and statistical properties obtained via disintegration theorems seem to be strictly connected, for instance in foliated manifolds, very little geometric information is taken into account while studying disintegration of measures. In particular, intrinsic geometric properties of probability spaces are very often neglected. The purpose of this paper is to advertise the viewpoint of tackling disintegration of measures taking into consideration intrinsic structures of probability spaces obtained via Optimal Transport Theory. To this
end, we will formulate disintegration of probability measures in terms of a transportation problem to then explore a fibre-wise formulation of a disintegration and its consequences.

**Main results.** As first result in this paper, we prove a disintegration theorem, Theorem A, which allows us to introduce a fibre-wise perspective on disintegration. Using this, we studied some conditions in which one can obtain a path of conditional measures in the space of probability measures: in the propositions 5.2, 5.4 and 5.8, we discuss the weak continuity of a map \( y \mapsto \mu_y \), where \( \{\mu_y\}_{y \in Y} \) is a family of conditional measures, to then proof the first part of Theorem B, which addresses the construction of such path. The second part of Theorem B gives us a sort of rigidity result for disintegration of measures. Namely, we show that (under certain hypotheses) if the disintegration at one of the measures in this path is absolutely continuous, then all measures in the associated path must also be absolutely continuous with respect to the volume measure in a fibre.

The paper is organised as follows. In the next section we present the main concepts from Optimal Transport Theory to be used along the paper. In Section 3, we prove a disintegration theorem, stated as Theorem A. Then, in Section 4, we use Theorem A to define and study what we called disintegration maps. These crucial objects are used in Section 5. There we prove a series of propositions about a disintegration map \( y \mapsto \mu_y \): in Proposition 5.2 we proof that this map is weakly continuous \( \nu \)-almost everywhere, if \( \nu \) is absolutely continuous with respect to volume measure on the set of index points, and in propositions 5.4 and 5.8 we explore certain hypotheses about the disintegration in which this map is weakly continuous. Afterwards we study paths of measures given via these disintegration maps and a rigidity condition establishing absolute continuity of measures in these paths. That is the content of Theorem B, our main result in this section.

## 2. Spaces of probability measures, Wasserstein spaces and Optimal Transport

In this section, we set up some notation to be used throughout the text. We also introduce some basic terminology from Optimal Transport Theory, which is meant for readers not familiar with this area. Those who are skilled on the topic may wish to skip this section. Our main references for this section are [Amb00, AGS05, Vil09].

The cornerstone for the theory of optimal transportation is considered to be a logistic problem addressed by Gaspar Monge in 1781. The main idea is to transport masses from a given location to another one at minimal cost. To state it in a modern formulation, let \( \mathcal{P}(X) \) be the set of all Borel probability measures on \( X \). The problem amounts to:
Monge Transport Problem. Let $X, Y$ be Radon spaces. Given measures $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$, and a fixed Borel cost function $c : X \times Y \to [0, \infty]$, minimise
$$T \mapsto \int_X c(x, T(x)) \, d\mu$$
among all maps $T$, such that, $T_*\mu = \nu$.

The maps $T$ fulfilling $T_*\mu = \nu$ are called transport maps. The Monge Transport Problem actually may be ill-posed and such a map does not need to exist. That is the case, for example, when one of the measures is a Dirac mass and the other one is not. A way around is given by a different formulation as proposed by Kantorovich:

Monge-Kantorovich Transport Problem. Let $X, Y$ be Radon spaces. Given measures $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$, and a fixed Borel cost function $c : X \times Y \to [0, \infty]$, minimise
$$\gamma \mapsto \int_{X \times Y} c(x, y) \, d\gamma(x, y) \tag{1}$$
among all measures $\gamma \in \mathcal{P}(X \times Y)$ with marginals $\mu$ and $\nu$, i.e., $\gamma$ satisfying $(\text{proj}_X)_*\gamma = \mu$ and $(\text{proj}_Y)_*\gamma = \nu$, where $\text{proj}_X$ and $\text{proj}_Y$ are the canonical projections $(x, y) \mapsto x$ and $(x, y) \mapsto y$, respectively.

The measures $\gamma \in \mathcal{P}(X \times Y)$ are called transport plans. We denote the set of all transport plans with marginals $\mu$ and $\nu$ by $\Pi(\mu, \nu)$. The value $C(\mu, \nu) = \inf_{\gamma \in \Pi(\mu, \nu)} \int_{X \times Y} c(x, y) \, d\gamma(x, y)$ is called optimal cost. Note that $(\text{proj}_X)_*\gamma = \mu$ is equivalent to $\gamma[A \times Y] = \mu[A]$, for every $A \in \mathcal{B}(X)$, and $(\text{proj}_Y)_*\gamma = \nu$ is equivalent to $\gamma[X \times B] = \nu[B]$, for every $B \in \mathcal{B}(Y)$. Moreover, it is possible to describe this problem in terms of the coupling of measures, in the following sense:

Definition 2.1. Let $(X, \mu)$ and $(Y, \nu)$ be probability spaces. A coupling of $(\mu, \nu)$ is a pair $(X, Y)$ of measurable functions in a probability space $(\Omega, \mathcal{P})$, such that, law$(X) = \mu$ and law$(Y) = \nu$.

Considering $\Omega = X \times Y$, coupling $\mu$ and $\nu$ means to construct $\gamma \in \mathcal{P}(X \times Y)$ with marginals $\mu$ and $\nu$. Then, the Monge-Kantorovich Transport Problem can be understood as the minimisation of the total cost, over all possible couplings of $(\mu, \nu)$.

Whenever these problems are stated in metric spaces, we may choose the cost function to be the distance function itself, which in turn allows us to introduce a distance function between measures:
Definition 2.2 (Wasserstein distance). Let \((X, d)\) be a separable complete metric space. Consider probability measures \(\mu\) and \(\nu\) on \(X\) and \(p \in [1, \infty)\). The Wasserstein distance of order \(p\) between \(\mu\) and \(\nu\) is given by

\[
W_p(\mu, \nu) := \left( \inf_{\gamma \in \Pi(\mu, \nu)} \int d(x_1, x_2)^p \, d\gamma(x_1, x_2) \right)^{\frac{1}{p}}.
\]

In general, \(W_p\) is not a distance in the strict sense, because it can take the value \(+\infty\). To rule this situation out, it is natural to constrain \(W_p\) to a subset in which it takes finite values.

Definition 2.3 (Wasserstein space). The Wasserstein space of order \(p\) is defined by

\[
\mathcal{P}_p(X) := \left\{ \mu \in \mathcal{P}(X) : \int d(x, \tilde{x})^p \mu(dx) < +\infty \right\}
\]

for \(\tilde{x} \in X\) arbitrary.

Therefore, \(W_p\) sets a (finite) distance on \(\mathcal{P}_p(X)\). It turns out that if \((X, d)\) is a complete separable metric space and \(d\) is bounded, then the \(p\)-Wasserstein distance metrises the weak topology over \(\mathcal{P}(X)\) \cite[Corollary 6.13]{Vil09}. Furthermore, if \(X\) is a complete metric space, then so is \(\mathcal{P}(X)\) with the \(p\)-Wasserstein distance \cite[Theorem 6.18]{Vil09}.

Although the Wasserstein distance is defined for every \(p \geq 1\), in this paper we will chose either \(p = 1\) or \(p = 2\), as stated later on. The reason is that for \(W_1\) we can explicitly compute distance bounds, while for \(W_2\) the space \(\mathcal{P}_2(X)\) inherits geometric properties of the space \(X\). The choice is always indicated throughout the text.

We finish this section recalling two well-known results for later use. The first one is regarding measurable and continuous functions, the Lusin’s Theorem in a specific form. The other one gives us a “recipe” to glue different couplings.

Theorem 2.4. \cite[Theorem 2.3.5]{Fed69}. Let \(M\) be a locally compact metric space and \(N\) a separable metric space. Consider \(\mu\) a Borel measure on \(M\), \(A \subset M\) a measurable set with finite measure and \(f : M \to N\) a measurable map. Then, for each \(\delta > 0\) there is a closed set \(K \subset A\), with \(\mu(A \setminus K) < \delta\), such that the restriction of \(f\) to \(K\) is continuous.

Lemma 2.5 (Gluing Lemma). \cite[Chapter 1]{Vil09} Let \((X_i, \mu_i), i = 1, 2, 3,\) be complete separable metric probability spaces. If \((X_1, X_2)\) is a coupling of \((\mu_1, \mu_2)\) and \((Y_2, Y_3)\) is a coupling of \((\mu_2, \mu_3)\), then one can construct a triple of random variables \((Z_1, Z_2, Z_3)\) such that \((Z_1, Z_2)\) has the same law as \((X_1, X_2)\) and \((Z_2, Z_3)\) has the same law as \((Y_2, Y_3)\). If \(\mu_{12}\) stands for the law of \((X_1, X_2)\) on \(X_1 \times X_2\) and \(\mu_{23}\) stands for the law of \((X_2, X_3)\) on
$X_2 \times X_3$, then to construct the joint law $\mu_{123}$ of $(Z_1, Z_2, Z_3)$ one just has to glue $\mu_{12}$ and $\mu_{23}$ along their common marginal $\mu_2$.

3. Disintegration of measures

In order to grasp some of the properties of the probability spaces while studying disintegration of measures, we would like to associate the latter with the Optimal Transport Theory. Before doing so, we prove the following disintegration theorem. Our proof is based on the idea of choosing a dense subset of a vector space using separability. Then, we extend a tailored made linear functional to the whole space, which is implicitly used in the proof of Dellacherie and Meyer [DM78, III-70], although our conditions are different.

**Theorem A.** Let $X$ and $Y$ be locally compact and separable metric spaces. Let $\pi : X \to Y$ be a Borel map, and take $\mu \in \mathcal{M}_+(X)$, where $\mathcal{M}_+(X)$ is the set of all positive and finite Radon measures on $X$. Define $\nu = \pi_*\mu$ in $\mathcal{M}_+(Y)$. Then, there exist measures $\mu_y \in \mathcal{M}_+(X)$, such that,

1. $y \mapsto \mu_y$ is a Borel map and $\mu_y \in \mathcal{P}(X)$ for $\nu$-almost every $y \in Y$;
2. $\mu = \nu \otimes \mu_y$, that is, $\mu(A) = \int_Y \mu_y(A) \, d\nu(y)$ for every $A \in \mathcal{B}(X)$;
3. $\mu_y$ is concentrated on $\pi^{-1}(y)$ for $\nu$-almost every $y \in Y$.

**Proof.** We shall first consider the disintegration of measures on compact metric spaces. Then, we tackle the general case as it is stated.

**Step 1:** To get started, consider $X$ to be a compact metric space with its Borel $\sigma$-algebra $\mathcal{B}(X)$, and let $\mu$ be a Radon measure on $X$. If $(Y, \mathcal{E})$ is a measurable space, we define a measurable map $q : \mathcal{B}(X) \to \mathcal{E}$, so that we set $\nu = q_*\mu$. Let $C(X)$ be the set of all continuous real functions $\omega : X \to \mathbb{R}$. Then, for each $\omega \in C(X)$, we associate a measure $\lambda$ given by

\[
\lambda(A) = \int_{q^{-1}(A)} \omega(x) \, d\mu(x)
\]

for every $A \in \mathcal{E}$. The measure $\lambda$ is absolutely continuous with respect to $\nu$. Indeed, for every $A \in \mathcal{E}$ we have that $\nu(A) = \mu(q^{-1}(A)) = 0$ implies $\lambda(A) = 0$. Therefore, since $\nu$ and $\lambda$ are positive measures and $\lambda \ll \nu$, by the Radon-Nikodym Theorem, there exists $h : Y \to [0, \infty]$, the density of $\lambda$ with respect to $\nu$, such that, $\lambda(A) = \int_A h \, d\nu$ for $A \in \mathcal{E}$. Thus,

\[
\int_A h(y) \, d\nu(y) = \int_{q^{-1}(A)} \omega(x) \, d\mu(x).
\]

Recall that $C(X)$ is a separable space with the supremum norm. Let $\mathcal{H} = \{\omega_1 \equiv 1, \omega_2, \omega_3, \ldots\}$ be a dense subset of $C(X)$. Suppose, without loss of generality, that $\mathcal{H}$ is a vector space over $\mathbb{Q}$. Then, for each $n \in \mathbb{N}$ we
consider the Radon-Nikodym density \( h_n \) associated with \( \omega_n \) given by (2), so that, for each \( A \in \mathcal{E} \),

\[
\int_A h_n(y) \, d\nu(y) = \int_{q^{-1}(A)} \omega_n(x) \, d\mu(x). \tag{3}
\]

Note that \( h_n \geq 0 \) almost always, if \( \omega_n \geq 0 \), since \( \nu = q_* \mu \).

**Step 2:** We will use the associated densities to construct a linear functional which will be extended using the Hahn-Banach Theorem. In order to do so, we denote by \( A \) the set of all \( y \in Y \), such that, if \( \omega_i = \alpha \omega_j + \beta \omega_k \), then we have for their associated densities that the relation \( h_i(y) = \alpha h_j(y) + \beta h_k(y) \) holds true, where \( \alpha, \beta \in \mathbb{Q} \), and the associated density to \( \omega_1 \) is set to \( h_1(y) = 1 \). The set \( A \) is measurable and \( \nu(A) = 1 \). Indeed,

\[
\int_{q^{-1}(Y)} \omega_i(x) \, d\mu = \int_{q^{-1}(Y)} (\alpha \omega_j + \beta \omega_k)(x) \, d\mu = \alpha \left( \int_{q^{-1}(Y)} \omega_j(x) \, d\mu \right) + \beta \left( \int_{q^{-1}(Y)} \omega_k(x) \, d\mu \right), \text{ which by (3),}
\]

\[
= \alpha \left( \int_Y h_j(y) \, d\nu \right) + \beta \left( \int_Y h_k(y) \, d\nu \right)
\]

so \( \int h_i(y) \, d\nu = \int \alpha h_j(y) + \beta h_k(y) \, d\nu \). Consequently, \( h_i(y) = \alpha h_j(y) + \beta h_k(y) \) \( \nu \)-almost always. Therefore, whenever \( \omega_n \) is a linear combination of elements of \( \mathcal{H} \), then the associated densities defined by (3) can also be written point-wise as linear combinations of Radon-Nikodym densities.

**Step 3:** For each \( y \in \mathcal{A} \), we define the functional \( \varphi_y : \mathcal{H} \to \mathbb{R} \), given by \( \varphi_y(\omega) := h_n(y) \). Note that \( \varphi_y : \mathcal{H} \to \mathbb{R} \) is \( \mathbb{Q} \)-linear with \( \|\varphi_y\| \leq 1 \) and, since \( \varphi_y(1) = 1 \), we have actually that \( \|\varphi_y\| = 1 \). Therefore, by the Hahn-Banach Theorem, \( \varphi_y \) can be extended to a continuous positive linear functional \( \varphi_y : C(X) \to \mathbb{R} \), with \( \|\varphi_y\| = 1 \). Furthermore, the Riesz-Markov-Kakutani Representation Theorem assures that there exists a unique Radon measure \( \mu_y \) on \( X \), such that, \( \varphi_y(\omega) = \int \omega \, d\mu_y \) for every \( \omega \in C(X) \) and \( \varphi_y(1) = 1 \). Note that \( \int \omega \, d\mu_y \leq 1 \), since \( \|\varphi_y\| = 1 \). Thus, we conclude that \( \mu_y \) is a probability measure. Note also that \( \mu_y \) is supported on \( q^{-1}\{y \in \mathcal{A}\} \). For \( y \notin \mathcal{A} \), consider \( \mu_y = 0 \).

**Step 4:** Observe that \( y \mapsto \int_X \omega_n \, d\mu_y \) is \( \mathcal{E} \)-measurable for every \( \omega_n \in \mathcal{H} \) and

\[
\int_Y \int_{q^{-1}(y)} \omega_n(x) \, d\mu_y \, d\nu = \int_Y \varphi_y(\omega_n) \, d\nu = \int_Y h_n(y) \, d\nu = \int_X \omega_n(x) \, d\mu.
\]
By the definition of $H$, we have that for each $\omega \in C(X)$, there exists a sequence $(\omega_i)_i$, with $\omega_i \to \omega$ uniformly. So, by uniform convergence, we have that $y \mapsto \int_X \omega \, d\mu_y$ is $E$-measurable for every $\omega \in C(X)$. Furthermore,

$$\int_Y \int_{q^{-1}(y)} \omega(x) \, d\mu_y \, dv = \int_X \omega(x) \, d\mu.$$ 

The same holds true for any bounded and $B(X)$-measurable function $\omega$. Indeed, denote by $\mathcal{C}$ the class of functions such that $y \mapsto \int_X \omega \, d\mu_y$ is $E$-measurable and $\int_Y \int_{q^{-1}(y)} \omega(x) \, d\mu_y \, dv = \int_X \omega(x) \, d\mu$. Note that $C(X) \subset \mathcal{C}$, from what was shown before. If $A \subset X$ is an open set, then $1_A \in \mathcal{C}$. So, let $D$ be the set of all characteristic functions in $\mathcal{C}$. If $1_A$ is $\mathcal{D}$, then $1_A \in D$. Thus, the class of measurable sets whose characteristic functions are in $\mathcal{D}$ is $B(X)$. Therefore, the result follows by monotone convergence. This completes the proof of Theorem A for compact spaces.

**Step 5:** Let $X$ be a locally compact and separable metric space. Then, the set of continuous real-valued functions with compact support on $X$, denoted by $C_c(X)$, is a vector space. Such a vector space can be seen as the union of the spaces $C_c(K_i)$ of continuous functions with support on compact sets $K_i$. Since $\mu$ is a Radon measure on $X$, the map $\varphi : C_c(X) \to \mathbb{R}$, such that, $\omega \mapsto \int_X \omega(x) \, d\mu$, is a continuous positive linear map. Note also that $\mu$ is supported on a set $\tilde{K}$, which is a countable union of compact subsets $K_i \subset X$. So, we can imbed $X$ into a compact metric space $\mathcal{K}$ and identify $\mu$ with a measure on $\mathcal{K}$ with support on $\tilde{K}$, and construct the measures $\mu_y$ as above. Consider $\mu_y = 0$ for $y$ such that $\pi^{-1}(y) \notin \tilde{K}$. Thus, there exist probability measures $\mu_y$ on $X$, such that, each $\mu_y$ is supported on $\pi^{-1}(y)$ and, for every $\omega \in C_c(X),$

$$\int_Y \int_{\pi^{-1}(y)} \omega(x) \, d\mu_y \, dv = \int_X \omega(x) \, d\mu. \quad (4)$$

Since $Y$ is a locally compact and separable metric space and $\pi : X \to Y$ is a Borel map, then $y \mapsto \mu_y$ is a Borel map for $\nu$-almost every $y \in Y$. Furthermore, note that (4) is equivalent to say that $\mu(A) = \int_Y \mu_y(A) \, dv(y)$ for every $A \in B(X)$ and $\mu_y$ is concentrated on $\pi^{-1}(y)$ for $\nu$-almost every $y \in Y$. This concludes the proof. \qed

Many interesting examples arise when we consider Theorem A for the case of product spaces with the Borel map $\pi$ as the canonical projection on the first component, as follows:

**Corollary 3.1.** Let $X$ and $Y$ be locally compact and separable metric spaces. Let $\text{proj}_X : X \times Y \to X$ be the canonical projection on the first component,
take \( \gamma \in \mathcal{M}_+(X \times Y) \) and set \( \mu = \text{proj}_X \gamma \in \mathcal{M}_+(X) \). Then, there exist measures \( \gamma_x \in \mathcal{M}_+(X \times Y) \), such that,

1. \( x \mapsto \gamma_x \) is a Borel map and \( \gamma_x \in \mathcal{P}(X \times Y) \) for \( \mu \)-a.e. \( x \in X \);
2. \( \gamma = \mu \otimes \gamma_x \), i.e., \( \gamma(A) = \int_X \gamma_x(A) \, d\mu(x) \) for every \( A \in \mathcal{B}(X \times Y) \);
3. \( \gamma_x \) is concentrated on \( \text{proj}_{X}^{-1}(x) \) for \( \mu \)-almost every \( x \in X \).

In fact, since \( \gamma_x \) is concentrated on \( \text{proj}_{X}^{-1}(x) = \{x\} \times Y \), we can consider each \( \gamma_x \) as a measure on \( Y \), writing \( \gamma_x(B) = \int_X \gamma_x(\{y : (x, y) \in B\}) \, d\mu \) for every \( B \in \mathcal{B}(X \times Y) \), adding a different point of view to disintegration. The following example illustrates this case of disintegration of measures.

**Example 3.2.** Consider a Solid Torus \( S^1 \times D^2 \). Let \( \mathcal{F}^s = \{\{x\} \times D^2\}_{x \in S^1} \) be a foliation of \( S^1 \times D^2 \), as represented in Figure 1. Given a measure \( \gamma \in \mathcal{P}(S^1 \times D^2) \), let \( \text{proj}_{S^1} : S^1 \times D^2 \to S^1 \) be the canonical projection on the first component and set \( \mu = \text{proj}_{S^1} \gamma \). The Theorem A, with \( \pi := \text{proj}_{S^1} \), gives us a disintegration \( \{\gamma_x : x \in S^1\} \) of \( \gamma \) along the leaves. Since the measures \( \gamma_x \) are concentrated on \( \text{proj}_{S^1}^{-1}(x) = \{x\} \times D^2 \) for \( \mu \)-a.e. \( x \in S^1 \), we can consider each \( \gamma_x \) as a measure on \( D^2 \). That is, we can define a probability on \( D^2 \) for each \( x \in S^1 \).

![Figure 1. Representation of \( \mathcal{F}^s = \{\{x\} \times D^2\}_{x \in S^1} \).](image)

The point of view from Corollary 3.1 is, somehow, a generalisation of cases of disintegration of a probability measure along leaves in a foliated compact Riemannian manifold, as in Example 3.2. In this sense, we remark that different versions of disintegration theorems can be related by taking suitable hypotheses, as we do in the following example.

**Example 3.3.** Let \( M_1 \) and \( M_2 \) be compact Riemannian manifolds and set the product space \( \Sigma = M_1 \times M_2 \). Let \( \mathcal{F}^s = \{\{x\} \times M_2\}_{x \in M_1} \) be a foliation of \( \Sigma \). Given \( \gamma \in \mathcal{P}(M_1 \times M_2) \), let \( \text{proj}_{M_1} : \Sigma \to M_1 \) be the canonical projection on \( M_1 \) and set \( \mu = \text{proj}_{M_1} \gamma \). By Theorem A, there exists a family \( \{\gamma_x : x \in M_1\} \subset \mathcal{P}(M_1 \times M_2) \), such that,

1. \( x \mapsto \gamma_x \) is a Borel map and \( \gamma_x \in \mathcal{P}(M_1 \times M_2) \) for \( \mu \)-a.e. \( x \in M_1 \);
(2) $\gamma(A) = \int_{M_1} \gamma_x(A) \, d\mu(x)$ for every $A \in \mathcal{B}(M_1 \times M_2)$;

(3) $\gamma_x$ is concentrated on $\operatorname{proj}_{M_1}^{-1}(x)$ for $\mu$-a. e. $x \in M_1$.

Furthermore, we can consider each $\gamma_x$ as a probability on $M_2$, as stated above. Note that this result agrees with Rokhlin’s Disintegration Theorem. In fact, let $\varrho : \Sigma \to \mathcal{F}^s$ be a map that associates each point $(x, y) \in \Sigma$ to the $\zeta$ element of $\mathcal{F}^s$ that contains $(x, y)$. Consider $\hat{\gamma} = \varrho_* \gamma$. Note that $\mathcal{F}^s$ is a measurable partition of $\Sigma$ and $\Sigma$ is a complete separable metric space. So, Rokhlin’s Disintegration Theorem describes a disintegration of $\gamma$ relative to $\mathcal{F}^s$ by a family $\{\gamma_\zeta : \zeta \in \mathcal{F}^s\}$, such that, for $E \subset \Sigma$ measurable set,

1. $\gamma_\zeta(\zeta) = 1$, for $\hat{\gamma}$-almost every $\zeta \in \mathcal{F}^s$;

2. $\zeta \mapsto \gamma_\zeta(E)$ is measurable;

3. $\gamma(E) = \int \gamma_\zeta(E) \, d\hat{\gamma}(\zeta)$.

Rewrite $\mathcal{F}^s$ by $\{\zeta_x\}_{x \in M_1}$, where $\zeta_x = \{x\} \times M_2$ for each $x \in M_1$, and consider $\gamma'_x = \operatorname{proj}_{M_1} \gamma$. For each $x \in M_1$, let $\gamma_{\zeta_x}$ be the restriction of $\gamma_\zeta$ to $\zeta = \{x\} \times M_2$. Note that $\varrho^{-1}(\zeta) = \{x\} \times M_2 = \zeta_x = \operatorname{proj}_{M_1}^{-1}(x)$, so

$$
\int \gamma_\zeta(E) \, d\hat{\gamma}(\zeta) = \int \gamma_\zeta(E) \gamma(\varrho^{-1}(d\zeta))
= \int_{M_1} \gamma_{\zeta_x}(E \cap \zeta_x) \gamma(\operatorname{proj}_{M_1}^{-1}(dx))
= \int_{M_1} \gamma_{\zeta_x}(E \cap \zeta_x) \, d\gamma'_x
$$

and then

$$
\gamma(E) = \int_{M_1} \gamma_{\zeta_x}(E \cap \zeta_x) \, d\gamma'_x.
$$

Moreover, note that $\gamma_{\zeta_x}$ is supported in $\operatorname{proj}_{M_1}^{-1}(x) = \zeta_x$. Hence, we have a disintegration $\{\gamma_{\zeta_x} : \zeta_x \in \mathcal{F}^s\}$ along the leaves of $\Sigma$ associated to $\gamma'_x$. In addition, it is possible to identify $\operatorname{proj}_{M_1}^{-1}(x)$ with $M_2$, and write this disintegration by $\{\gamma_x : x \in M_1\} \subset \mathcal{P}(M_2)$, as desired.

Such example is one of the possible roles of disintegration of measures. In Dynamics, for instance, this kind of disintegration appears in several contexts. To name a few, in [BM17] the regularity of this kind of disintegration is investigated considering invariant measures for hyperbolic skew products. Specifically, for this purpose, a function that associates each $x \in X$ with a probability measure $\gamma_x$ obtained via a disintegration of $\gamma$ is analysed. In the next section, we will see that this type of application can be thought in a more general framework and it has important properties. We can also cite [GL20] and [Gal17], where the disintegration of Example 3.2 is used to study the behaviour of the transfer operator in a solenoidal map.

We can actually obtain uniqueness and absolute continuity of the disintegration in the context of Theorem A. If $\lambda \in \mathcal{M}_+(Y)$ is another measure, such that, there exists a Borel map $y \mapsto \eta_y$ for which $\mu = \lambda \otimes \eta_y$, with $\eta_y$
Proposition 3.4. Let $\mu$ be a Borel measure concentrated on $\pi^{-1}(y)$ for $\nu$-almost every $y \in Y$, then $\lambda|_{C} \ll \nu$, where $C = \{y \in Y : \eta_y(X) > 0\}$, and $\eta_y \ll \mu_y$ for $\nu$-almost every $y \in Y$. See the details in [AP03]. In the following proposition, we focus on the case of product spaces. Taking $\gamma \in \mathcal{M}_{+}(X \times Y)$ with $\mu = \text{proj}_{X*}\gamma$, we obtain uniqueness of $\gamma_x$ and $\mu$ in $\gamma = \mu \otimes \gamma_x$.

**Proposition 3.4.** Let $X \times Y$, $X$ be locally compact and separable metric spaces. Let $\text{proj}_X : X \times Y \to X$ be the projection on the first component and consider $\gamma \in \mathcal{M}_{+}(X \times Y)$, $\nu \in \mathcal{M}_{+}(X)$ and $\mu = \text{proj}_{X*}\gamma$. Let $x \mapsto \eta_x$ be a Borel $\mathcal{M}_{+}$-valued map defined on $X$ such that

1. $\gamma = \nu \otimes \eta_x$;
2. $\eta_x$ is concentrated on $\text{proj}_{X}^{-1}(x)$ for $\nu$-almost every $x \in X$.

Then, the $\eta_x$ are uniquely defined $\nu$-almost always by (1) and (2). Moreover, $\nu_{C}$ is absolutely continuous with respect to $\mu$. In particular, $\frac{d\nu}{d\mu}\eta_x = \gamma_x$ for $\mu$-almost every $x \in X$, where $\gamma_x$ are the conditional probabilities as in Corollary 3.1.

**Proof.** Let $\eta_x$ and $\eta'_x$ be measures satisfying the conditions (1) and (2). Let $(A_n)_n$ be a sequence of open sets such that the finite intersection is also an open set which generates $\mathcal{B}(X \times Y)$, the Borel $\sigma$-algebra of $X \times Y$. Consider $B \in \mathcal{B}(X \times Y)$ and $A = A_n \cap \text{proj}_{X}^{-1}(B)$, for any $n \in \mathbb{N}$. By the condition (1), we have that

$$\gamma(A) = \int_X \eta_x(A) d\nu = \int_B \eta_x(A_n) d\nu$$

and

$$\gamma(A) = \int_X \eta'_x(A) d\nu = \int_B \eta'_x(A_n) d\nu.$$ 

Therefore,

$$\int_B \eta_x(A_n) d\nu = \int_B \eta'_x(A_n) d\nu.$$ 

Given that $B$ is arbitrary, then $\eta_x(A_n) = \eta'_x(A_n)$ for $\nu$-almost every $x \in X$ and for any $n \in \mathbb{N}$. So, there exists a set $N \subset X$, with $\nu(N) = 0$, such that, $\eta_x(A_n) = \eta'_x(A_n)$ for any $n \in \mathbb{N}$, $x \in (X - N)$. Hence, for $\nu$-almost every $x \in X$, $\eta_x = \eta'_x$.

Let us denote $C = \{x \in X : \eta_x(X \times Y) > 0\}$. Let $\mathcal{G} \subset C$ be such that $\mu(\mathcal{G}) = 0$. So, $\text{proj}_{X}^{-1}(\mathcal{G})$ is such that $\gamma(\text{proj}_{X}^{-1}(\mathcal{G})) = 0$. Therefore, the condition (2) implies

$$0 = \int_X \eta_x(\text{proj}_{X}^{-1}(\mathcal{G})) d\nu = \int_{\mathcal{G}} \eta_x(X \times Y) d\nu.$$ 

Since $\eta_x(X \times Y) > 0$ in $C \supset \mathcal{G}$, we have that $\nu(\mathcal{G}) = 0$. That is, $\nu|_{C} \ll \mu$. Moreover, writing $\nu|_{C} = f\mu$ implies that $\gamma = f \mu \otimes \eta_x$. But, by the Corollary 3.1, $\gamma = \mu \otimes \gamma_x$ and then $\frac{d\nu}{d\mu}\eta_x = \gamma_x$. 

$\square$
4. Disintegration Maps

From the Optimal Transport perspective, Theorem A, in fact, deals with the disintegration of transport plans. In this sense, it is possible to define a function from $Y$ to $\mathcal{P}(X)$ with certain properties which actually establishes the link between disintegration of measures and the geometric properties of the measure spaces. We will call such an object the disintegration map.

**Definition 4.1 (Disintegration map).** Let $X$ and $Y$ be locally compact and separable metric spaces. Consider a measure $\mu \in \mathcal{M}_+(X)$, a Borel map $\pi : X \to Y$, and a disintegration of $\mu$ given by Theorem A, so that $\mu = \nu \otimes \mu_y$. We define the disintegration map:

$$f : Y \to (\mathcal{P}(X), W_p)$$

$$y \mapsto \mu_y,$$

such that $\mu = \nu \otimes f(y)$, where $W_p$ is the $p$-Wasserstein distance.

**Remark 4.2.** In order to make clear which measures are associated with the disintegration map, we will say that "$f$ is a disintegration map of $\mu$ w.r.t. $\nu$".

Although we may define such map to a general $W_p$, our main interest will be when either $p = 1$ or $p = 2$. On the one hand, for $p = 1$ there is an explicit formula for the Wasserstein distance which we can apply to study the disintegration of measures in product spaces. On the other hand, when $p = 2$, the theory of Optimal Transport allows for a geometric characterisation of $\mathcal{P}_2(X)$ in terms of the geometric properties of $X$. More precisely, the study of geodesics defined in $\mathcal{P}_2(X)$ and the convexity properties of certain functionals along these geodesics play a crucial role in the Metric Theory of Gradient Flows, which allows us to infer geometric properties of $X$ itself. We shall address the case when $p = 2$ in the next section. Before that, we use the disintegration maps to further characterise the disintegration of measures in product spaces. In this section, consider the following definition of a disintegration map:

**Definition 4.3 (Disintegration map - product spaces).** Let $X$ and $Y$ be locally compact and separable metric spaces. Consider $\gamma \in \mathcal{P}(X \times Y)$ and a disintegration of $\gamma$ given by Corollary 3.1, so that $\gamma = \mu \otimes \gamma_x$. In this case, the disintegration map reads as

$$f : X \to (\mathcal{P}(Y), W_1)$$

$$x \mapsto \gamma_x$$

such that $\gamma = \mu \otimes f(x)$. 

In the lines of [AP03] and [GM13], we start by showing the following:

**Proposition 4.4.** A map \( f : X \to (\mathcal{P}(Y), W_1) \) is a disintegration map if, and only if, it is Borel.

**Proof.** Denote by \( \text{Lip}_1(Y) \) the set of Lipschitz functions whose Lipschitz constants are less or equal 1. By Ascoli-Arzelà Theorem, the space \( \text{Lip}_1(Y) \) is compact with respect to the uniform convergence [AGS05, Proposition 3.3.1]. Let \( D \subset \text{Lip}_1(Y) \) be a countable dense subset and take \( \varphi \in \text{Lip}_1(Y) \).

If the measures \( \nu_1, \nu_2 \in \mathcal{P}(Y) \) have bounded support, we can use the duality formula to obtain

\[
W_1(\nu_1, \nu_2) = \sup_{\varphi \in \text{Lip}_1(Y)} \left\{ \int_Y \varphi \ d(\nu_1 - \nu_2) \right\} = \sup_{\varphi \in D} \int_Y \varphi \ d(\nu_1 - \nu_2);
\]

see [AGS05, Section 7.1] for details. For all \( \varphi \in \text{Lip}_1(Y) \),

\[
\psi_\varphi : X \to \mathbb{R} \quad x \mapsto \int_Y \varphi \ d(\nu - f(x))
\]
is Borel. For \( f \) to be a Borel map it suffices that

\[
f^{-1}(B(\nu, r)) = \bigcap_{\varphi \in D} \psi_\varphi^{-1}((-r, r)) := A,
\]

where \( B(\nu, r) \) is an open ball of radius \( r \) centred at \( \nu \) in \( (\mathcal{P}(Y), W_1) \). In fact, if \( x \in A \), then \( |\psi_\varphi(x)| < r \) for all \( \varphi \in D \) and \( W_1(\nu, f(x)) < r \) (by the definition of \( \psi_\varphi \)), so that \( f(x) \in B(\nu, r) \). In the same way, if \( f(x) \in B(\nu, r) \), then \( W_1(\nu, f(x)) < r \) and, by the duality formula,

\[
|\psi_\varphi(x)| := |\int_Y \varphi \ d(\nu - f(x))| < r
\]
for every \( \varphi \in D \), so that \( x \in A \). Thus, one way is proven. Conversely, let \( A \subset Y \) be an open subset. Note that

\[
f(x)(A) = \int_Y 1_A(y) \ df(x).
\]

Let \( I_\varphi \) be a function given by:

\[
I_\varphi : (\mathcal{P}(Y), W_1) \to \mathbb{R} \quad \lambda \mapsto \int_Y \varphi(y) \ d\lambda,
\]

where \( \varphi \) is a lower semicontinuous function over \( Y \). Since \( W_1 \) metrises the weak* topology of \( \mathcal{P}(Y) \) [AGS05, Chapter 7], the function \( I_\varphi \) is lower semicontinuous. For every \( x \in X \), one obtains \( \int_Y \varphi(y) \ df(x) = I_\varphi(f(x)) \). By assumption, \( f \) is Borel and then \( f(\cdot)(A) : X \to \mathbb{R} \) is a composition of a lower semicontinuous function and a Borel map, so it is a Borel map. Therefore, \( f \) is a disintegration map, which concludes the proof. \( \square \)
The disintegration map in fact can be written in terms of the Monge Problem 2. Given a transport map \( T : X \to Y \) and the measures \( \mu \in \mathcal{P}(X) \), and \( \nu = T_*\mu \in \mathcal{P}(Y) \), the disintegration map is given by \( x \mapsto \delta_{T(x)} \), where \( \delta_{T(x)} \) is a Dirac measure at \( T(x) \). It is possible show that exists a relationship among measures in \( (\mathcal{P}(Y), W_1) \), via push forward of \( \mu \) by disintegration maps, and the second marginal of transport plans induced by distinct transport maps.

**Lemma 4.5.** Let \( T : X \to Y \) and \( S : X \to Y \) be transport maps. Consider a measure \( \mu \in \mathcal{P}(X) \) and the applications \( f \), and \( g \) given by:

\[
\begin{align*}
 f : X &\to (\mathcal{P}(Y), W_1) \\
 x &\mapsto \delta_{T(x)} \\
 g : X &\to (\mathcal{P}(Y), W_1) \\
 x &\mapsto \delta_{S(x)}.
\end{align*}
\]

Then, \( T_*\mu = S_*\mu \) if, and only if, \( f_*\mu = g_*\mu \).

**Proof.** Define \( \varphi(y) = \psi(\delta_y) \), where \( \psi \in C((\mathcal{P}(Y), W_1)) \) is arbitrarily chosen. Then,

\[
\begin{align*}
 \int_Y \psi d(f_*\mu) &= \int_X \psi(f(x))d\mu \\
 &= \int_X \psi(\delta_{T(x)})d\mu \\
 &= \int_X \varphi(T(x))d\mu \\
 &= \int_X \varphi(S(x))d\mu \\
 &= \int_Y \psi d(g_*\mu).
\end{align*}
\]

Given the arbitrary choice of \( \psi \), it follows that \( f_*\mu = g_*\mu \). Conversely, consider \( \varphi \in C(Y) \) and the application \( I_\varphi \) defined by equation (5). Note that

\[
\int_{\mathcal{P}(Y)} I_\varphi(\lambda)d(f_*\mu) = \int_{\mathcal{P}(Y)} I_\varphi(\lambda)d(g_*\mu)
\]

if, and only if,

\[
\int_X I_\varphi(f(x))d\mu = \int_X I_\varphi(g(x))d\mu,
\]

which in turn occurs if, and only if,

\[
\int_X \left( \int_Y \varphi(y)df(x) \right)d\mu = \int_X \left( \int_Y \varphi(y)dg(x) \right)d\mu.
\]
Since
\[ \int_Y \varphi(y) \, df(x) = \varphi(T(x)) \]
and
\[ \int_Y \varphi(y) \, dg(x) = \varphi(S(x)), \]
the last equation can be written as
\[ \int_X \varphi(T(x)) \, d\mu = \int_X \varphi(S(x)) \, d\mu. \]
Due to the arbitrary choice of \( \varphi \), it follows that \( T^*\mu = S^*\mu \).

\[ \square \]

Corollary 4.6. Let \( T : X \rightarrow Y \) and \( S : X \rightarrow Y \) be transport maps. Consider a measure \( \mu \in \mathcal{P}(X) \), the disintegration maps \( f \) and \( g \) defined as in the Lemma 4.5, and the measures \( \gamma = \mu \otimes f(x) \) and \( \eta = \mu \otimes g(x) \). Then, \( f_*\mu = g_*\mu \) if, and only if, \( \text{proj}_Y \gamma = \text{proj}_Y \eta \).

Given \( \gamma, \eta \in \Pi(\mu, \nu) \), as defined in Section 2, with \( \gamma = \mu \otimes f(x) \) and \( \eta = \mu \otimes g(x) \), we say that \( \gamma \) is equivalent by disintegration to \( \eta \) (and we denote \( \gamma \approx \eta \)) if \( f_*\mu = g_*\mu \). With this equivalence in mind, it is possible to define an equivalent class among the transport plans as follows:

Definition 4.7. Given \( \gamma \in \Pi(\mu, \nu) \) with \( \gamma = \mu \otimes f(x) \) the transport class of \( \gamma \) is defined as the equivalence class \( [\gamma] = \{ \eta = \mu \otimes g(x) : g_*\mu = f_*\mu \} \).

Thus, we have that all transport plans induced by transport maps belong to the same transport class. Moreover, in the next proposition, we prove that it is possible to use equivalence by disintegration to assure the existence of a transport map.

Proposition 4.8. Consider a transport map \( T : X \rightarrow Y \), such that \( T^*\mu = \nu \) and \( \gamma = \mu \otimes \delta_{T(x)} \), for a non-atomic measure \( \mu \in \mathcal{P}(X) \). If \( \eta \in [\gamma] \), then there exists a transport map \( S : X \rightarrow Y \), such that, \( \eta = \mu \otimes \delta_{S(x)} \).

Proof. By the Approximation Theorem [Amb00, Theorem 9.3] and by the definition of equivalent by disintegration, there exist a sequence of Borel functions \( S_n : X \rightarrow Y \), such that, \( \eta = \lim_{n \to \infty} \mu \otimes \delta_{S_n(x)} \) and \( (S_n)_*\mu = \nu \) for every \( n \in \mathbb{N} \). So \( \mu \otimes \delta_{S_n(x)} \in [\gamma] \), for every \( n \in \mathbb{N} \). Consider \( \varphi(y) = |y|^2 \) and observe that
\[ \int_X |S_n(x)|^2 \, d\mu = \int_X \varphi(S_n(x)) \, d\mu = \int_Y \varphi(y) \, d\nu = \int_Y |y|^2 \, d\nu < \infty. \]
Let \( \psi \in C((\mathcal{P}(Y), W_1)) \) be a function given by \( \psi(\delta_y) = |y|^2 \), and let \( \Delta \subset \mathcal{P}(Y) \) be the set of Dirac measures. Observe that \( \psi \) is Lipschitz over
\[ \frac{\Delta}{\partial} \text{, with respect to } W_1. \] So, take \( \psi \) any Lipschitz extension over \( \mathcal{P}(Y) \). Since \( (\delta_{S_n})* \mu = (\delta_T)* \mu \), we have that

\[ \int_X |S_n(x)|^2 \, d\mu = \int_X \psi(\delta_{S_n(x)}) \, d\mu = \int_X \psi(\delta_{T(x)}) \, d\mu = \int_X |T(x)|^2 \, d\mu \]

for every \( n \in \mathbb{N} \). Therefore, moving on to a subsequence, we can assume that \( S_n \) is weakly convergent to \( S \). By [Amb00, Lemma 9.1] \( \mu \otimes \delta_{S_n(x)} \) converges weakly to \( \mu \otimes \delta_{S(x)} \). This concludes the proof.

In this context, the Monge Problem 2 can be interpreted as: minimise \( \int_{X \times Y} c(x, y) \, d\gamma \) in a fixed transport class of \( \Pi(\mu, \nu) \), i.e., obtain

\[ \min \left\{ \int_{X \times Y} c(x, y) \, d\gamma : \gamma \in [\mu \otimes \delta_T] \right\} \]

for a given transport map \( T \). Regarding the Monge-Kantorovich Problem, note that the second part of the proof of Lemma 4.5 applies to general transport plans, so that the second marginal can be fixed by the disintegration maps, as follows:

**Lemma 4.9.** Consider \( \mu \in \mathcal{P}(X) \), the disintegration maps \( f : X \to \mathcal{P}(Y) \) and \( g : X \to \mathcal{P}(Y) \), and the transport plans \( \gamma = \mu \otimes f(x) \) and \( \eta = \mu \otimes g(x) \). Then, \( f_* \mu = g_* \mu \) implies \( \text{proj}_Y^* \gamma = \text{proj}_Y^* \eta \).

The reciprocal of Lemma 4.9, however, is not true. Consider, for instance, the following transport problem: Three factories, \( x_1, x_2, \) and \( x_3 \), with the same production, supply 100% of their products to two stores, \( y_1 \) and \( y_2 \). Suppose store \( y_2 \) has a demand five times greater than store \( y_1 \). Let \( \mu \) be the measure related to the production of the factories and \( \nu \) be the measure related to the amount of delivered products to the stores. These measures are given by

\[ \mu = \frac{1}{3} \delta_{x_1} + \frac{1}{3} \delta_{x_2} + \frac{1}{3} \delta_{x_3} \]

\[ \nu = \frac{1}{6} \delta_{y_1} + \frac{5}{6} \delta_{y_2}. \]

Consider the transport plan illustrated in Figure 2, which divides the products which are produced in the factory \( x_1 \) between the two stores. The corresponding disintegration map \( f \) is given by \( f(x_1) = \frac{2}{3} \delta_{y_1} + \frac{1}{3} \delta_{y_2} \), \( f(x_2) = \delta_{y_2} \) and \( f(x_3) = \delta_{y_2} \).
Note that, since $\mu$ is of the type $\sum_i \alpha_i \delta_{x_i}$, we have that $f_*\mu = \sum_i \alpha_i \delta_{f(x_i)}$.

Suppose, however, that there is a change in logistics and in the new transport plan, Figure 3, the factory $x_1$ delivers its production only to the store $y_2$ and the factory $x_2$ divides its production between the two stores. In this new situation, the disintegration map is given by $g(x_1) = \delta_{y_2}$, $g(x_2) = \frac{1}{2}\delta_{y_1} + \frac{1}{2}\delta_{y_2}$ and $g(x_3) = \delta_{y_2}$. Nevertheless, on the one hand $f_*\mu = g_*\mu$, i.e., the transport class does not change. On the other hand, if in this new transport plan the factories $x_1$ and $x_2$ deliver to both stores, Figure 4, in such a way that $x_1$ sends 30% of its production to $y_1$ and 70% to $y_2$, and $x_2$ sends 20% of its production to $y_1$ and 80% to $y_2$, the disintegration map is given by $h(x_1) = \frac{3}{10}\delta_{y_1} + \frac{7}{10}\delta_{y_2}$, $h(x_2) = \frac{2}{10}\delta_{y_1} + \frac{8}{10}\delta_{y_2}$ and $h(x_3) = \delta_{y_2}$. In this case, $f_*\mu \neq h_*\mu$. So, the transport plans 1 and 3 do not belong to the same transport class.

Furthermore, if there is a small change in transport plan 3, so that $x_1$ sends 10% of its production to $y_1$ and 90% to $y_2$, and $x_2$ sends 40% of its production to $y_1$ and 60% to $y_2$, the new disintegration map is given by $k(x_1) = \frac{1}{10}\delta_{y_1} + \frac{9}{10}\delta_{y_2}$, $k(x_2) = \frac{4}{10}\delta_{y_1} + \frac{6}{10}\delta_{y_2}$ and $k(x_3) = \delta_{y_2}$. Therefore, $h_*\mu \neq k_*\mu$. Thus, with the previous definition for transport class, the transport class changes when either the number of factories that deliver products to more than one store is changed or even if the fraction of delivered production is changed.

Therefore, in order to be compatible with the Monge-Kantorovich Problem, we need another definition of transport class. Take $\mu \otimes f \in \Pi(\mu, \nu)$ and
\[ \Lambda = f_*\mu, \text{ and recall that } (\text{proj}_Y)_*(\mu \otimes f) = \nu, \] then for every \( \varphi \in C(Y) \),
\[
\int_Y \varphi(y) \, d\nu = \int_X \left( \int_Y \varphi(y) \, df(x) \right) \, d\mu \\
= \int_X I_\varphi(f(x)) \, d\mu \\
= \int_{\mathcal{P}(Y)} I_\varphi(\lambda) \, d\Lambda(\lambda) \\
= \int_{\mathcal{P}(Y)} \left( \int_Y \varphi(y) \, d\lambda \right) \, d\Lambda,
\]
where \( I_\varphi \) is given by equation (5) and \( \lambda \in \mathcal{P}(Y) \). Hence, every probability \( \Lambda \) in \((\mathcal{P}(Y), W_1)\) satisfying
\[
\int_{\mathcal{P}(Y)} \lambda \, d\Lambda = \nu
\]
defines a transport class \([\gamma] = \{\mu \otimes f : f_*\mu = \Lambda\}\). As an example, a transport class \([\mu \times \nu]\) corresponds to the measure \( \Lambda = \delta_\nu \). From this point of view, the Monge-Kantorovich Problem in the class \( \Lambda \) can be thought as:
\[
MK_\Lambda(c, \mu, \nu) = \inf_{\gamma} \left\{ \int_{X \times Y} c(x, y) \, d\gamma : \gamma = \mu \otimes f, f_*\mu = \Lambda \right\}.
\]
This notion of transport class also allows us to see the Monge-Kantorovich Problem as an abstract Monge Problem between the spaces \( X \) and \( \mathcal{P}(Y) \), considering the cost
\[
\tilde{c}(x, \lambda) = \int_Y c(x, y) \, d\lambda.
\]
In fact, for every disintegration map \( f : X \to \mathcal{P}(Y) \), such that \( f_*\mu = \Lambda \),
\[
\int_X \tilde{c}(x, f(x)) \, d\mu = \int_X \left( \int_Y c(x, y) \, df(x) \right) \, d\mu \\
= \int_{X \times Y} c(x, y) \, d(\mu \otimes f)
\]
and the problem \( MK_\Lambda(c, \mu, \nu) \) is equivalent to the Monge Problem with parameters \((\tilde{c}, \mu, \Lambda)\). Thus, the disintegration of transport plans introduces another perspective for the optimisation problems proposed by Monge and Kantorovich.

5. Absolute continuity from disintegration maps

In order to connect what we just developed with the geometric properties of the measure spaces and the space of measures, we shall study how paths of measures are given by disintegration maps. Our interest hereafter is to use the 2-Wasserstein distance.

To construct such paths in 2-Wasserstein space, it is essential to analyse the weak continuity of the disintegration map. For instance, considering \( f \) a
disintegration map of $\mu$ w.r.t. $\nu$, if $\nu$ is absolutely continuous with respect to the volume measure of $Y$, we can apply Lusin’s Theorem 2.4 to show that $f$ is weakly continuous at almost every point. We also need the following lemma, which is actually a known fact within the Optimal Transport community. For completeness, we add it here in our context.

**Lemma 5.1.** Let $(X, d)$ be a separable metric space. The $2$-Wasserstein space $(\mathcal{P}(X), W_2)$ is a separable metric space.

**Proof.** Let $D \subset X$ be a countable and dense subset. Consider the space $\mathcal{M}$ defined by $\mathcal{M} := \{\nu \in \mathcal{P}(X) : \nu = \sum_j a_j \delta_{x_j}, \text{ with } a_j \in \mathbb{Q} \text{ and } x_j \in D\}$. We want to show that $\mathcal{M}$ is dense in $(\mathcal{P}(X), W_2)$.

Given $\mu \in (\mathcal{P}(X), W_2)$, then for any $\varepsilon > 0$ and $x_0 \in D$, there exists a compact set $K \subset X$, such that

$$\int_{X \setminus K} d(x_0, x)^2 \, d\mu \leq \varepsilon^2.$$

Since $K$ is compact, we may cover it with a family $\{B(x_k, \frac{\varepsilon}{2}) : 1 \leq k \leq N, x_k \in D\}$ and define

$$B'_k := B(x_k, \varepsilon) \setminus \bigcup_{j < k} B(x_j, \varepsilon)$$

so that $\{B'_k\}$ are disjoint and still cover $K$. Define $\varphi$ on $X$ by

$$\begin{align*}
\varphi(B'_k \cap K) &= \{x_k\} \\
\varphi(X - K) &= \{x_0\}.
\end{align*}$$

So $d(x, x_k) \leq \varepsilon$ for every $x \in K$. Therefore,

$$\int_X d(x, \varphi(x))^2 \, d\mu \leq \varepsilon^2 \int_K d\mu + \int_{X \setminus K} d(x, x_0)^2 \, d\mu \leq 2\varepsilon^2$$

and then $W_2(\mu, \varphi_*\mu) \leq 2\varepsilon^2$. Note that $\varphi_*\mu$ can be written as $\sum_{0 \leq j \leq N} a_j \delta_{x_j}$, that is, $\mu$ might be approximated by a finite combination of Dirac masses. Moreover, the coefficients $a_j$ might be replaced by rational coefficients (up to a small error in Wasserstein distance): since Wasserstein distances are controlled by weighted total variation [Vil09, Theorem 6.15],

$$W_2\left( \sum_{0 \leq j \leq N} a_j \delta_{x_j}, \sum_{0 \leq j \leq N} \beta_j \delta_{x_j} \right) \leq 2^{\frac{1}{2}} \left[ \max_{k,l} d(x_k, x_l) \right] \sum_{0 \leq j \leq N} |a_j - \beta_j|^{\frac{1}{2}},$$

which can become small, taking suitable coefficients $\beta_j \in \mathbb{Q}$. Thus, it follows that $\mathcal{M}$ is dense in $\mathcal{P}(X)$. Consequently, $(\mathcal{P}(X), W_2)$ is separable. \qed

**Proposition 5.2.** Let $X$, $Y$ be locally compact, complete, and separable metric spaces. Consider $\pi : X \to Y$ a Borel map, $\mu \in \mathcal{M}_+(X)$ and $\nu := \pi_*\mu$. If $\nu$ is absolutely continuous with respect to the volume measure on $Y$, then
The disintegration map of $\mu$ w.r.t. $\nu$, is weakly continuous for $\nu$-almost every $y \in Y$.

Proof. From Theorem A, the map $y \mapsto \mu_y \in \mathcal{P}(X)$ is measurable. We endow $\mathcal{P}(X)$ with the 2-Wasserstein distance $W_2$ on $\mathcal{P}(X)$, considering $d$ a complete bounded metric for $X$. Therefore, $W_2$ metrises the weak convergence of $\mathcal{P}(X)$ [Vil09, Theorem 6.9]. Furthermore, from Lemma 5.1, $(\mathcal{P}(X), W_2)$ is a separable space. Then, by Lusin’s Theorem 2.4, for a subset $A \subset Y$ with finite measure, and each $\varepsilon > 0$, there exists $K \subset A$ with $\nu(A \setminus K) < \varepsilon$ such that the disintegration map restricted to $K$ is continuous. The sets such that this property holds generate a $\sigma$-algebra. Thus, there is a unique extension to the completion of the generated $\sigma$-algebra of $A$. □

Although Proposition 5.2 is relevant by itself, we would like to have conditions for which the disintegration map is weakly continuous at every point, so that, given any two points $y, y'$ in $Y$, we can construct a path of conditional measures connecting $\mu_y$ and $\mu_{y'}$. Note that the 2-Wasserstein space is actually connected, and therefore we can always find a path connecting two measures. Nevertheless, we require this path to be specifically given by the disintegration map. This is not trivial indeed, and we can easily construct examples in which this map is not weakly continuous.

Example 5.3. Consider $X = Y = [0, 1]$. Let $\mu$ be the Lebesgue measure on $X$ and take the map $\pi : X \to Y$ given by

$$\pi(x) = \begin{cases} 2x, & \text{if } x < \frac{1}{2} \\ 1, & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Note that $\pi$ is Borel measurable, since it is continuous. Define

$$\nu := \pi_* \mu = \frac{1}{2} \lambda + \frac{1}{2} \delta_1$$

where $\lambda$ is the Lebesgue measure on $Y$ and $\delta_1$ is the Dirac measure at $y = 1$.

A disintegration $\{\mu_y\}_{y \in Y}$ of $\mu$ with respect to $\pi$ is given by

$$\mu_y = \begin{cases} \delta_{\pi^{-1}(y)}, & \text{if } y < 1 \\ 2\lambda \mid \left[\frac{1}{2}, 1\right], & \text{if } y = 1. \end{cases}$$

In this case, the disintegration map is not weakly continuous at $y = 1$. In fact, consider a sequence $(y_n)_n$ in $Y$ such that $y_n \to y = 1$. Note that

$$W_2^2(\mu_{y_n}, \mu_y) = \inf_{\gamma \in \Pi(\mu_{y_n}, \mu_y)} \int_{X \times X} d(x_1, x_2)^2 \, d\gamma$$

$$= \int_{X \times X} d(x_1, x_2)^2 \, d(\delta_{\pi^{-1}(y_n)} \times \mu_y)$$

$$= \int_X d(\pi^{-1}(y_n), x_2)^2 \, d\mu_y.$$
Then, at the limit $y_n \to 1$, we have

$$W_2^2(\mu_{y_n}, \mu_y) \to \int_X d\left(\frac{1}{2}, x_2\right)^2 d\mu_y \neq 0.$$ 

Therefore, $f$ is not weakly continuous at $y = 1$. Furthermore, note that $\nu(\{1\}) = \frac{1}{2}$. This example also shows that we cannot consider a weak continuity at almost every point when $\nu$ is not absolutely continuous with respect to the volume measure on $Y$.

We can actually ask for some additional hypotheses so that the disintegration map is weakly continuous. One possibility is to take $\pi$ as a bijective and continuous map.

**Proposition 5.4.** Let $X$ and $Y$ be locally compact and separable metric spaces. Consider $\mu \in \mathcal{M}_+(X)$, $\pi : X \to Y$ a Borel map, $\nu := \pi_* \mu$, $\{\mu_y\}_{y \in Y}$ a disintegration of $\mu$ given by Theorem A, and $f$ the disintegration map of $\mu$ w.r.t. $\nu$. If $\pi$ is bijective and continuous, then $f$ is weakly continuous.

**Proof.** On the one hand,

$$\mu(B) = \int_Y \mu_y(B) \ d\nu$$

$$= \int_Y \mu_y(B \cap \pi^{-1}(y)) \ d\nu.$$ 

On the other hand,

$$\mu(B) = \int_X 1_B \ d\mu$$

$$= \int_X 1_B \ d(\nu \circ \pi)$$

$$= \int_Y 1_B(\pi^{-1}(y)) \ d\nu$$

$$= \int_Y 1_B(\pi^{-1}(y)) \delta_{\pi^{-1}(y)}(\pi^{-1}(y)) \ d\nu$$

$$= \int_Y \delta_{\pi^{-1}(y)}(B \cap \pi^{-1}(y)) \ d\nu.$$ 

Then,

$$\int_Y \mu_y(B \cap \pi^{-1}(y)) \ d\nu = \int_Y \delta_{\pi^{-1}(y)}(B \cap \pi^{-1}(y)) \ d\nu.$$ 

Moreover, since $\mu_y$ is a probability and $B \cap \pi^{-1}(y)$ is either a singleton or an empty set, we have that

$$\delta_{\pi^{-1}(y)}(B \cap \pi^{-1}(y)) = \delta_{\pi^{-1}(y)}(B \cap \pi^{-1}(y)) \geq \mu_y(B \cap \pi^{-1}(y)).$$

Then $\mu_y(B \cap \pi^{-1}(y)) = \delta_{\pi^{-1}(y)}(B \cap \pi^{-1}(y))$, and it follows that $\mu_y = \delta_{\pi^{-1}(y)}$. 


Suppose $y_n \rightarrow y$. Since $\pi$ is continuous, $\pi^{-1}$ is continuous, and then
\[
\int g \, d\mu_{y_n} \rightarrow \int g \, d\mu_y
\]
for every bounded uniformly continuous function $g$. Therefore, $f$ is weakly continuous. □

Asking for the bijectivity of $\pi$ is, in a way, quite strong. We want to explore ways to ease this restriction. In some examples which $\pi$ is a quotient map, we have the continuity of the disintegration map.

**Example 5.5.** $\ell_q$-product space from metric measures spaces. Let $(Y, d_Y, m_Y)$, $(Z, d_Z, m_Z)$ denote the metric spaces $(Y, d_Y)$ and $(Z, d_Z)$ which are endowed with probability measures $m_Y$ and $m_Z$, respectively. We shall call these metric measure spaces. For $q \in [0, \infty]$, define the $\ell_q$-product space $X := Y \times \ell_q Z$ as the product space $Y \times Z$ equipped with the measure $\mu = m_Y \times m_Z$ and the distance $d_{\ell_q}$ given by
\[
d_{\ell_q}((y, z), (y', z')) = \begin{cases} \left( d_Y(y, y')^q + d_Z(z, z')^q \right)^{\frac{1}{q}}, & \text{if } 1 \leq q < \infty \\ \max\{d_Y(y, y'), d_Z(z, z')\}, & \text{if } q = \infty. \end{cases}
\]
Consider the projection $p : X \rightarrow Y$ and $\{\mu_y\}_{y \in Y}$ the disintegration of $\mu$ with respect to $\nu := p_*\mu$. Note that
\[
d_{\ell_q}(p^{-1}(y), p^{-1}(y')) = d_Y(y, y')
\]
for every $q \in [1, \infty]$ and, for every conditional measures $\mu_y, \mu_{y'}$,
\[
W_2^2(\mu_y, \mu_{y'}) = \inf_{\gamma \in \Pi(\mu_y, \mu_{y'})} \int d_{\ell_q}((y_1, z_1), (y_2, z_2))^2 \, d\gamma
\]
\[
= \inf_{\gamma \in \Pi(\delta_y, \delta_{y'})} \int d_Y(y_1, y_2)^2 \, d\gamma
\]
\[
= d_Y(y, y')^2.
\]
Then $W_2(\mu_y, \mu_{y'}) = d_Y(y, y')$ and, therefore, the disintegration map is weakly continuous.

In fact, this is one example of a disintegration of measures associated with a special kind of foliation, called **metric measure foliation**, covered in [GKMS18]. For a precise definition, we need to introduce some concepts. Let $(X, d)$ be a metric space. A foliation $\mathcal{F}$ of $X$ is called a **metric foliation** if for every $F, F' \in \mathcal{F}$ and every $x \in F$, $d(F, F') = d(x, F')$. Given a metric foliation $\mathcal{F}$ of $X$, define the equivalence relation $x \sim x'$ if, and only if, there exists $F \in \mathcal{F}$ such that $x, x' \in F$. Therefore, consider the quotient space $X^* := X/\sim$ and the **quotient map** $p : X \rightarrow X^*$. Define a distance function $d^*$ on $X^*$ as $d^*(y, y') := d(p^{-1}(y), p^{-1}(y'))$, for $y, y' \in X^*$. 
Definition 5.6. Let \((X, d, \mu)\) be a metric measure space and \(\mathcal{F}\) a metric foliation. We call \(\mathcal{F}\) a metric measure foliation if \(p_*\mu \in \mathcal{M}_{\text{loc}}(X^*)\), where \(\mathcal{M}_{\text{loc}}(X^*)\) is the set of all locally finite Borel measure on \(X^*\), and there exists a Borel subset \(\Omega \subset X^*\) with \(p_*\mu(X^* \setminus \Omega) = 0\) such that

\[
W_2(\mu_y, \mu_{y'}) = d^*(y, y') = d(p^{-1}(y), p^{-1}(y'))
\]

(7)

for any \(y, y' \in \Omega\), where \(p\) is the quotient map and \(\{\mu_y\}_{y \in Y}\) is a disintegration of \(\mu\) with respect to \(p_*\mu\).

A very important example of metric measure foliation is related to the action of isometry group.

Example 5.7. Let \((X, d, \mu)\) be a metric measure space and \(G\) a compact topological group. Let

\[G \times X \ni (g, x) \mapsto gx \in X\]

be an isometric action of \(G\) on \(X\). Suppose this action is metric measure isomorphic, that is, for every \(g \in G\) the map \(X \ni x \mapsto gx \in X\) is an isometry preserving the measure \(\mu\). Consider \([x]\) the \(G\)-orbit of a point \(x \in X\) and the quotient space \(X/G\) endowed with the distance

\[d_{X/G}([x], [x']) = \inf_{g, g' \in G} d(gx, g'x').\]

Consider \(p : X \to X/G\) the projection map, that is, \(p\) is given by \(x \mapsto [x]\). The family \(\mathcal{F} := \{p^{-1}(y) : y \in X/G\}\) is a metric measure foliation on \(X\).

With the Definition 5.6 we have a direct association of the 2-Wasserstein distance between two conditional measures (given by Theorem A) and the distance between points to which these measures were indexed. It is clear that in this case we have the weak continuity of the disintegration map. Furthermore, each conditional measure \(\mu_y\) is supported on a leaf, this leaf is indexed by an element of the quotient space, and the distance between leaves is related to the distance between elements of the leaves. Thus, we can relate the 2-Wasserstein distance between conditional measures to the distance between two points of the quotient space looking at a direction perpendicular to the leaves, passing along the leaves. Roughly speaking, we can think of a kind of space fibration, where one of the directions has been “collapsed”, and it becomes a parameter for the disintegration family, so that each conditional measure is supported on the underlying fibre (see Figure 5). That is, the measures \(\mu_y\) are of the type \(\delta_y \times \lambda\), where \(\lambda\) is a measure on the fibre, in a similar way to the Example 3.2.
With what we have seen so far we have been able to classify some situation in which the disintegration map is weakly continuous:

**Proposition 5.8.** Let $X$ be locally compact and separable metric space. Consider $\mu \in M_+(X)$, a metric foliation $\mathcal{F}$ of $X$, the quotient space $X^*$, and the quotient map $p : X \to X^*$. If there exists a metric measure foliation of $X$, with $\Omega = X^*$, then the disintegration map of $\mu$ w.r.t. $\nu$ is weakly continuous.

**Proof.** When there exists such metric measure foliation of $X$, then for every $y, y' \in X^*$, $W_2(\mu_y, \mu_{y'}) = d^*(y, y')$, where $\mu_y = f(y)$ and $\mu_{y'} = f(y')$. Now, consider a sequence $(y_n)_n$ in $X^*$ such that $y_n \to y$. Note that $W_2(\mu_{y_n}, \mu_y) \to 0$, since $d^*(y_n, y) \to 0$. Therefore we have the weak continuity of $f$.

□

**Remark 5.9.** If in Proposition 5.8 we do not ask for $\Omega = X^*$ (see Definition 5.6), the weakly continuity of the disintegration map is given for $p^*\mu$-almost every $y \in X^*$ due the definition of metric measure foliation. Although it seems to be a strong condition, in general cases of interest we have it satisfied, as in Example 5.5 and Example 5.7, for instance.

**Remark 5.10.** One can associate hypotheses about the map $\pi$ and the type of disintegration obtained. In light of what we have seen, we obtain the following dichotomy:

1. Under the hypotheses of Proposition 5.4, that is, when the map $\pi$ is bijective and continuous, the conditional measures given by Theorem A are Dirac deltas.
2. Under the hypotheses of Proposition 5.8, that is, when we have a metric measure foliation of $X$, the conditional measures given by Theorem A are of the type $\delta_y \times \lambda$, where $\lambda$ is a measure on the fibre.
The entire study carried out on the disintegration map makes clear a fundamental condition for it to be weakly continuous: the supports of the conditional measures \( \{ \mu_y \} \) must be disjoint. Moreover, if we want some kind of absolute continuity with respect to \( \mu \), the supports must have \( \mu \)-positive measure. In fact, both, in the example of disintegration of a probability with respect to a finite measurable partition of space (see introduction), and in discrete cases, these conditions are satisfied. However, in the cases of propositions 5.4 and 5.8, we do not obtain supports with a \( \mu \)-positive measure. Another way is to think about the absolute continuity of measures with respect to a reference measure on the fibres. That is what is given in the next theorem.

**Theorem B.** Let \( X \) and \( Y \) be locally compact, complete, separable metric spaces. Consider \( \pi : X \to Y \) a Borel map, \( \mu \) in \( \mathcal{M}_+(X) \) and \( \nu := \pi_*\mu \). If the disintegration map of \( \mu \) w.r.t. \( \nu \) is weakly continuous, then:

(i) given two points \( y, y' \in Y \), there exists a path on \((\mathcal{P}(X), W_2)\), given by \( f \), connecting \( \mu_y \) and \( \mu_{y'} \) (the respective conditional measures given by Theorem A).

(ii) if, in addition, \( \pi \) is such that \( \{ \pi^{-1}(y) \}_{y \in Y} \) is a metric measure foliation of \( X \), the fibres are smooth compact Riemannian manifolds, and either \( \mu_y \) or \( \mu_{y'} \) is absolutely continuous with respect to the volume measure on the respective support fibre, then all the measures \( \mu_{y_t} \) on this path are absolutely continuous with respect to the volume measure on the fibre.

**Proof.**

(i) This is a direct consequence of the weakly continuity of the disintegration map \( f \). Consider \( y, y' \in Y \). Let \( \psi \) be a continuous curve in \( Y \) connecting \( y \) and \( y' \), that is, \( \psi = \{ y_t : t \in [0,1], \psi(0) = y, \psi(1) = y' \} \). Taking \( y_t \to y \in Y \), we have \( f(y_t) \xrightarrow{w} f(y) \), that is, \( W_2(\mu_{y_t}, \mu_{y}) \to 0 \). Then, \( \zeta = \{ \mu_{y_t} : t \in [0,1] \} \), where \( \mu_{y_t} \) is the conditional measure associated with \( y_t \) via \( f \), for every \( t \in [0,1] \), is a weakly continuous curve in \((\mathcal{P}(X), W_2)\) connecting \( \mu_y \) and \( \mu_{y'} \). This proves the first part of our theorem.

The proof of (ii) will be done in a few steps. The idea is to use a sort of “time-dependent” version of optimal transport. See [Vil09, Chapter 7], for example. In short, we will consider the curve \( \zeta \) given by the disintegration map as an interpolation between probability measures, called displacement interpolation. To this end, we consider a transport problem such that the initial and final probability measures are defined on the same space \( F \), and we associate \( \zeta \) with a random curve \( \xi \) in \( F \). In our case, the space \( F \) is the space equivalent to each fibre.
Step 1: A minimising curve in the space of measures.

Consider the path $\zeta = \{\mu_{yt} : t \in [0, 1]\}$ as constructed in item (i). Note that, as in this case we have disintegration associated with a metric measure foliation of $X$, we can think of it as a fibration of the space where each conditional measure is supported on the underlying fibre. Therefore, we can think of each $\mu_{yt}$ as a measure on the fibre $F$ (such as in Example 3.2 and Example 3.3).

Furthermore, taking $\psi$ as a minimising curve in $Y$, $\zeta$ is a minimising curve in $P(F)$, since we have the equality (7) between the Wasserstein distance and the distance in $Y$. In fact, take a sequence $(y_t)$ in $Y$ converging to $y$ along $\psi$. Since the disintegration map is weakly continuous, there is a sequence $(\mu_{yt})_{y_t}$ in $P(F)$ weakly converging to $\mu_y$, and then $W_2(\mu_y, \mu_{yt}) \to 0$. Moving $y_t$ along a minimising curve in $Y$, implies that $\mu_{yt}$ moves along a curve in $P(F)$ which minimises $W_2$. By abuse of notation we use the weak continuous path $\zeta: [0, 1] \to (P(F), W_2)$ while referring to this path in $(P(F), W_2)$ given by the disintegration map $f$ evaluated at the minimal curve $\psi$ joining $[y, y']$ in $Y$. More precisely, we consider the composition $f \circ \psi$ to describe $\zeta$.

Step 2: A random curve in $F$.

We want to associate $\zeta$ with a random curve $\xi: [0, 1] \to F$. In order to set some notation, let $e_t$ be the evaluation map, given by $e_t(\xi) = \xi(t) := \xi_t$, meaning the evaluation of $\xi$ at $t$. We will also use some usual concepts related to geodesics in Riemannian manifolds throughout the demonstration. We suggest [Jost11] for a comprehensive reading.

Consider the curve $\zeta: [0, 1] \to \mathcal{P}(F)$ joining $\mu_0$ and $\mu_1$ (from step 1). Suppose that there is a transport problem associated with this curve, whose respective spatial distributions are modelled by these probability measures. Assume that the cost function for the transport between the initial point $x_0 \in F$ (at time 0) and the final point $x_1 \in F$ (at time 1), denoted by $c^{0,1}(x_0, x_1)$, is associated with a family of functionals parametrised by the initial and the final times. Denote by $A^{0,1}$ the functional on the set of curves $[0, 1] \to F$, such that,

$$c^{0,1}(x_0, x_1) = \inf \{ A^{0,1}(\xi) : \xi_0 = x_0, x_1 = x_1, \xi \in C([0, 1]; F) \}.$$

In other words, $c^{0,1}(x_0, x_1)$ is the minimal cost needed to go from point $x_0$ at initial time 0, to point $x_1$ at final time 1. Moreover, let $C^{0,1}(\mu_0, \mu_1)$ be the optimal transport cost between $\mu_0$ and $\mu_1$ for the cost $c^{0,1}(x_0, x_1)$. For $t_1, t_2 \in [0, 1]$, define

$$A^{t_1, t_2}(\xi) = \frac{L(\xi)^2}{t_2 - t_1}$$
so that
\[ c^{t_1,t_2}(x_0, x_1) = \frac{d(x_0, x_1)^2}{t_2 - t_1} \]
and
\[ C^{t_1,t_2} (\mu_0, \mu_1) = \frac{W_2^2(\mu_0, \mu_1)^2}{t_2 - t_1}. \] (8)

We want to show that there exists a random minimiser \( \xi : [0, 1] \to F \), such that, \( \text{law} (\xi_t) = \mu_t \), for every \( t \in [0, 1] \). In other words, we want to show that \( \zeta \) is a curve in the space of measures which interpolates all possible measures along the minimising path joining \( \mu_0 \) and \( \mu_1 \). Such a curve is called displacement interpolation.

**Step 3:** \( \zeta \) is a displacement interpolation.

In this step, \( \xi \) will be constructed by dyadic approximation, according to couplings of measures in \( \zeta \) associated with times \( t = \frac{k}{2^n} \). In order to achieve this, we will use the iterative construction in lines with [Vil09, Theorem 7.21].

Let \( \Gamma \) be the set of minimising curves in \( F \). It will be necessary throughout the text, to consider subsets of \( \Gamma \) in which the geodesics are defined for certain time intervals and endpoints (or endpoints regions). So, for \( s, t \in [0, 1], x_s, x_t \in F \), let \( \Gamma_{x_s \rightarrow x_t}^{s,t} \) be the set of minimising curves in \( F \) starting at \( x_s \) at time \( s \), and ending at \( x_t \) at time \( t \). Similarly, for any two compact sets \( K_s, K_t \subset F \), let \( \Gamma_{K_s \rightarrow K_t}^{s,t} \) be the set of minimising curves starting in \( K_s \) at time \( s \), and ending in \( K_t \) at time \( t \).

Considering the measures along \( \zeta \), for \( t_1, t_2, t_3 \in [0, 1] \), let \( \gamma_{t_1 \rightarrow t_2} \) be an optimal transference plan between \( \mu_{t_1} \) and \( \mu_{t_2} \) for \( c^{t_1,t_2}(x_{t_1}, x_{t_2}) \), and let \( \gamma_{t_2 \rightarrow t_3} \) be an optimal transference plan between \( \mu_{t_2} \) and \( \mu_{t_3} \) for \( c^{t_2,t_3}(x_{t_2}, x_{t_3}) \).

By Lemma 2.5, it is possible take random variables \((\xi_{t_1}, \xi_{t_2}, \xi_{t_3})\), such that, 

\[
\text{law}(\xi_{t_1}, \xi_{t_2}, \xi_{t_3}) = \gamma_{t_1 \rightarrow t_2}, \text{law}(\xi_{t_2}, \xi_{t_3}) = \gamma_{t_2 \rightarrow t_3} \text{ and law}(\xi_{t_i}) = \mu_{t_i} \text{ for } i = 1, 2, 3.
\]

Since \( \zeta \) is minimising in \( \mathcal{P}_2(F) \) (see step 1) and \( \mathcal{P}_2(F) \) is a geodesic space, it follows from equation (8), that

\[ C^{t_1,t_2} (\mu_{t_1}, \mu_{t_2}) + C^{t_2,t_3} (\mu_{t_2}, \mu_{t_3}) = C^{t_1,t_3} (\mu_{t_1}, \mu_{t_3}). \]

This, in particular, implies

(a) \((\xi_{t_1}, \xi_{t_2})\) is an optimal coupling of \((\mu_{t_1}, \mu_{t_2})\) for \( c^{t_1,t_3}(\xi_{t_1}, \xi_{t_3})\);

(b) \(c^{t_1,t_3}(\xi_{t_1}, \xi_{t_3}) = c^{t_1,t_2}(\xi_{t_1}, \xi_{t_2}) + c^{t_2,t_3}(\xi_{t_2}, \xi_{t_3})\) almost surely.

Let \((\xi_0, \xi_1)\) be an optimal coupling of \((\mu_0, \mu_1)\). Consider optimal transference plans \(\gamma_{0 \rightarrow \frac{1}{2}}, \gamma_{\frac{1}{2} \rightarrow 1}\), as above, and construct random variables \((\xi^{(1)}_{0}, \xi^{(1)}_{\frac{1}{2}}, \xi^{(1)}_{0})\),
such that \( (\xi_0^{(1)}, \xi_{1/2}^{(1)}) \) is an optimal coupling of \( (\mu_0, \mu_{1/2}) \), for \( c^{0,1/2} \left( \xi_0^{(1)}, \xi_{1/2}^{(1)} \right) \), \\
\left( \xi_{1/2}^{(1)}, \xi_1^{(1)} \right) \) is an optimal coupling of \( (\mu_{1/2}, \mu_1) \), for \( c^{1,1/2} \left( \xi_{1/2}^{(1)}, \xi_1^{(1)} \right) \), and \law(\xi_i^{(1)}) = \mu_i \) for \( i = 0, \frac{1}{2}, 1 \). Moreover, item (a) implies that \( (\xi_0^{(1)}, \xi_1^{(1)}) \) is an optimal coupling of \( (\mu_0, \mu_1) \), and item (b) implies \\
\[ c^{0,1}(\xi_0^{(1)}, \xi_1^{(1)}) = c^{0,1/2} \left( \xi_0^{(1)}, \xi_{1/2}^{(1)} \right) + c^{1,1/2} \left( \xi_{1/2}^{(1)}, \xi_1^{(1)} \right) \]
almost surely. Iterating this process, at the step \( k \), we have random variables \\
\left( \xi_0^{(k)}, \xi_{1/2^k}^{(k)}, \xi_{1/2^k}^{(k)}, \ldots, \xi_{1/2^k}^{(k)} \right), \]
so that for any two \( i, j \leq 2^k \), \( (\xi_i^{(k)}, \xi_j^{(k)}) \) is an optimal coupling of \( (\mu_{1/2^k}, \mu_{1/2^k}) \). Furthermore, for \( i_1, i_2, i_3 \leq 2^k \), \\
\[ \frac{i_1}{2^k} \frac{i_2}{2^k} \frac{i_3}{2^k} \left( \xi_{i_1/2^k}^{(k)}, \xi_{i_2/2^k}^{(k)}, \xi_{i_3/2^k}^{(k)} \right) = c^{2^k, 2^k, 2^k} \left( \xi_{i_1/2^k}^{(k)}, \xi_{i_2/2^k}^{(k)}, \xi_{i_3/2^k}^{(k)} \right) \]
almost always.

We want to extend the random variables \( \xi^{(k)} \), defined for times \( \frac{i}{2^k} \), \( i \leq 2^k \), to continuous curves \( (\xi^{(k)})_{0 \leq t \leq 1} \). For this, note that for all times \( s, t \in [0, 1] \), \( s < t \), there exists a Borel map \( S_{s \rightarrow t} : F \times F \rightarrow \mathcal{C}([s, t] ; X) \), such that for all \( x, z \in F \), \( S(x, z) \) belongs to \( \Gamma_{x \rightarrow z}^{s,t} \) [Vil09, Proposition 7.16]. Indeed, let \( E_{s,t} \) be the function given by \( E_{s,t}(\xi) := (\xi_s, \xi_t) \), for \( \xi : [s, t] \rightarrow F \) minimising curve. Since \( F \) is a geodesic space, any two points of \( F \) can be joined by at least one minimising curve, so \( E_{s,t} \) is onto \( F \times F \). Moreover, \( E_{s,t} \) is a continuous map between complete separable metric spaces, and \( E_{s,t}^{-1}(x, z) \) is compact for every \( x, z \). Therefore, \( E_{s,t} \) admits a measurable right-inverse \( S_{s \rightarrow t} \) [Del75], that is, \( E_{s,t} \circ S_{s \rightarrow t} = Id \). Thus \( S_{s \rightarrow t} \) is a measurable recipe to join two points \( x, z \) by a minimising curve, which was to be proved.

For \( t \in (\frac{i}{2^k}, \frac{i+1}{2^k}) \), define \( \xi_t^{(k)} \) by \( e_t \left( S_{1/2^k, 1/2^k} \left( \xi_{1/2^k}^{(k)}, \xi_{1/2^k}^{(k)} \right) \right) \). Then, the law of \( (\xi_t^{(k)})_{0 \leq t \leq 1} \) is a probability on \( C(F) \). Let us denote it by \( \Theta^{(k)} \). Note that \( (e_t)_* \Theta^{(k)} = \mu_t \) for every \( t = \frac{i}{2^k}, i \leq 2^k \), and \( \Theta^{(k)} \) is concentrated on \( \Gamma \).

In short, from \( \zeta \) we iteratively constructed probability measures \( \Theta^{(k)} \) on \( \Gamma \) up to a step \( k \). We want to pass to the limit as \( k \rightarrow \infty \). Given \( \varepsilon > 0 \), since \( \mu_0 \) and \( \mu_1 \) are Radon measures, there exist compact sets \( K_0 \) and \( K_1 \) such that \( \mu_0(X \setminus K_0) \leq \varepsilon, \mu_1(X \setminus K_1) \leq \varepsilon \). Also, the set \( \Gamma_{K_0 \rightarrow K_1}^{0,1} \) is compact.
[Vil09, Definition 7.13 and Example 7.15] and
to show that $(k \to \infty)$ bounded continuous function on $F$ bounded continuous function on $t$ and $\mu$ of $t$ and $\xi$ of $t$ and $\mu$.

By [Vil09, Theorem 8.5], if $(\Theta_{t})$ is a displacement interpolation function on $F$, $U(t) = E \xi(t)$ is a continuous function of $t$ if $\xi$ is random geodesic with law $\Theta$. In fact, since $t \to \xi(t)$ is continuous and the composition of continuous functions is also continuous, $t \to u(\xi(t))$ is continuous. Moreover, let $\{u_{n}\}$ be Lebesgue integrable functions, such that, $u_{n} \to u$. Since $u$ is bounded, $|u_{n}| \leq g$ for some integrable function $g$; by Lebesgue’s Dominated Convergence Theorem $E u_{n} \to E u$. From these results the continuity of $U(t)$ follows, as we wanted.

Accordingly, we constructed $\xi$, such that, for each $t \in [0, 1]$, $\mu_{t}$ is the law of $\xi_{t}$, where $(\xi_{t})_{0 \leq t \leq 1}$ is a dynamical optimal coupling of $(\mu_{0}, \mu_{1})$. In other terms, we say that $\xi$ is displacement interpolation.

**Step 4:** An important observation about displacement interpolation.

By [Vil09, Theorem 8.5], if $\{\mu_{t}\}$ is a displacement interpolation between two compactly supported probability measures on $F$, and $t_{0} \in (0, 1)$ is given, then, for every $t \in [0, 1]$, the transport map $T_{t_{0} \to t}$ between the points $\xi(t_{0})$ and $\xi(t)$ it is well-defined $\mu_{t_{0}}$-almost everywhere and it is Lipschitz continuous. In other words, $T_{t_{0} \to t}$ is a solution of the Monge Problem between $\mu_{t_{0}}$ and $\mu_{t}$. For the completeness of the text, we will comment briefly on the proof.

Note that $(e_{0}, e_{1})_{*}(\Theta \otimes \Theta) = \gamma_{0 \to 1} \otimes \gamma_{0 \to 1}$. So, if one property holds true $\gamma_{0 \to 1} \otimes \gamma_{0 \to 1}$-almost always for quadruples, this property, for the endpoints of pairs of curves, holds true $\Theta \otimes \Theta$-almost always. Since $\gamma_{0 \to 1}$ is optimal, it has a property named c-cyclical monotonicity [Vil09, Theorem 5.10], so that $c(x, y) + c(\tilde{x}, \tilde{y}) \leq c(x, \tilde{y}) + c(\tilde{x}, y)$, $\Theta \otimes \Theta(dx, dy, d\tilde{x}, d\tilde{y})$-a. a. Thus, $c(\xi(0), \xi(1)) + c(\tilde{\xi}(0), \tilde{\xi}(1)) \leq c(\xi(0), \tilde{\xi}(1)) + c(\tilde{\xi}(0), \xi(1))$, $\Theta \otimes \Theta(d\xi, d\tilde{\xi})$-a. a.
Moreover, by Mather’s Shortening Lemma [Vil09, Theorem 8.1 and Corollary 8.2]
\[
\sup_{0 \leq t \leq 1} d(\xi_t, \tilde{\xi}_t) \leq C_K d(\xi_{t_0}, \tilde{\xi}_{t_0}).
\]
where $C_K$ is a constant. Suppose that $\Theta$ is supported on a compact set $S$. The equation (9) defines a closed set for all pairs of curves $\xi, \tilde{\xi} \in S \otimes S$. Let $e_{t_0}(S)$ be the union of all $\xi(t_0)$, when $\xi$ varies over $S$, and $T_{t_0 \rightarrow t}$ by $T_{t_0 \rightarrow t}(\xi(t_0)) = \xi(t)$. Note that if $\xi, \tilde{\xi}$ in $S$ are such that $\xi(t_0) = \tilde{\xi}(t_0)$, then (9) implies $\xi = \tilde{\xi}$. Moreover, $T_{t_0 \rightarrow t}$ is Lipschitz-continuous. So, $(\xi(t_0), T_{t_0 \rightarrow t}(\xi(t_0)))$ is a Monge coupling of $(\mu_{t_0}, \mu_t)$.

**Step 5:** Absolute continuity of measures $\mu_t$ on $\zeta$ with respect to the volume measure on $F$, when $\mu_0$ and $\mu_1$ are compactly supported.

Without loss of generality, let us suppose that $\mu_1$ is absolutely continuous with respect to the volume measure on $F$, vol. If $\mu_0$ and $\mu_1$ are compactly supported, then $\zeta$ has a compact support. Indeed, let $A_0, A_1 \subset F$ be the compact supports of $\mu_0, \mu_1$ and $\gamma_{0 \rightarrow 1}$ be the transference plan with marginals $\mu_0$ and $\mu_1$. Consider the canonical projections $(\text{proj}_1), (\text{proj}_2)$ on the first and second components, respectively. Since $(\text{proj}_1)_* \gamma_{0 \rightarrow 1} = \mu_0$ and $(\text{proj}_2)_* \gamma_{0 \rightarrow 1} = \mu_1$,
\[
(\text{proj}_1)_* \gamma_{0 \rightarrow 1}(F \times F) = \mu_0(F) = \mu_0(A_0) = \gamma_{0 \rightarrow 1}(A_0 \times F)
\]
\[
(\text{proj}_2)_* \gamma_{0 \rightarrow 1}(F \times F) = \mu_1(F) = \mu_1(A_1) = \gamma_{0 \rightarrow 1}(F \times A_1).
\]
Therefore $\gamma_{0 \rightarrow 1}$ is concentrated in a compact set $A_0 \times A_1$. Moreover, since $\gamma_{0 \rightarrow 1} = (e_0, e_1)_* \Theta$ and the evaluation map is continuous, $\Theta$ is concentrated in a compact set. The compactness of the $\zeta$ support follows from $(e_t)_* \Theta = \mu_t$.

We can use Step 4, and there is a Lipschitz map $T$ solving the Monge problem between $\mu_t$ and $\mu_1$, $t \in (0, 1)$. Let $N$ be a set such that the volume measure is zero and consider $T(N)$. If $T(N)$ is not Borel measurable, consider a negligible Borel set that contains $T(N)$ (which by abuse of notation, we will continue denoting $T(N)$). Note that $N \subset T^{-1}(T(N))$, so
\[
\mu_1(N) \leq \mu_1(T^{-1}(T(N))) = (T_* \mu_t)(T(N)) = \mu_1(T(N))
\]
and then $\mu_t(T(N)) = 0$, since vol($T(N)$) $\leq$ $\|T\|_{\text{Lip}}$ vol($N$) = 0 (this inequality occurs since $T$ is Lipschitz and $\|T\|_{\text{Lip}}$ stands for the Lipschitz constant) and $\mu_1 \ll \text{vol}$ by hypothesis. So, $\mu_t(N) = 0$ for every Borel set $N$ such that vol($N$) = 0, that is, $\mu_t \ll \text{vol}$.

**Step 6:** Absolute continuity of measures $\mu_t$ of $\zeta$ with respect to the volume measure on $F$: the general case.

Without loss of generality, suppose $\mu_1 \ll \text{vol}$. Let us assume that there is some case in which neither $\mu_0$ nor $\mu_1$ is compactly supported. We will
prove our statement \((ii)\) by contradiction. Suppose that \(\mu_\tau\), for \(\tau \in (0, 1)\), is not absolutely continuous with respect to the volume measure on \(F\). Then, there exists a set \(Z_\tau \subset F\), such that \(\text{vol}(Z_\tau) = 0\) and \(\mu_\tau(Z_\tau) > 0\). Consider \(Z := \{\xi \in \Gamma : \xi_\tau \in Z_\tau\}\) so that \(\Theta(Z) = \mathbb{P}(\xi_\tau \in Z_\tau) = \mu_\tau(Z_\tau) > 0\). Since \(\Theta\) is a regular measure, there exists \(K \subset Z\) compact such that \(\Theta(K) > 0\). So, if we set

\[
\Theta' := \frac{\Theta}{\Theta(K)}
\]

and consider \(\gamma'_{0 \to 1} := (e_0, e_1)_* \Theta'\) and \(\mu'_t = (e_t)_* \Theta'\), we have

\[
\mu'_t \leq \frac{(e_t)_* \Theta}{\Theta(K)} = \frac{\mu_t}{\Theta(K)}.
\]

In this way, \((\mu'_t)\) is a displacement interpolation and, considering the previous equation for \(t = 1\), \(\mu'_1 \ll \mu_1 \ll \text{vol}\). Note that, now \(\mu'_t\) is concentrated on \(e_\tau(K) \subset e_\tau(Z) \subset Z_\tau\) and then \(\mu'_t\) is singular. Although \(\mu'_0\) is supported in \(e_0(K)\) and \(\mu'_1\) is supported in \(e_1(K)\), which are compact. This is the case of Step 5. Then, \(\mu'_\tau \ll \text{vol}\), which is a contradiction. \(\square\)

**Remark 5.11.** Part \((ii)\) of Theorem \(B\) holds, for instance, in the case of the disintegration of the volume measure in the Solid Torus (Example \(3.2\)), or in spaces like those in the examples \(3.3, 5.5,\) and \(5.7\).  

**Remark 5.12.** Since \(\zeta\) is a displacement interpolation, it is a constant speed geodesic, by [AG13, Theorem 2.10]. That is, the path in \((\mathcal{P}(F), W_2)\) given by the disintegration map \(f\) evaluated at the minimal curve \(\psi\) in \(Y\) is a constant speed geodesic in \(\mathcal{P}_2(F)\).  

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