Quantization of dynamical symplectic reduction

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Abstract

A long-standing problem in quantum gravity and cosmology is the quantization of systems in which evolution is generated by a constraint that must vanish on solutions. Here, an algebraic formulation of this problem is presented, together with new structures and results that prove the existence of specific conditions for well-defined evolution to be possible.

1 Introduction

Given a symplectic manifold \((M, \Omega)\) and \(C \in C^\infty(M)\), the \textit{symplectic reduction} \(M/C\) of \(M\) by \(C\) is the orbit space of \(M \supset M_C\) with respect to the gauge flow \(F_C(\epsilon) = \exp(\epsilon X_C)\) in \(M_C\) generated by the Hamiltonian vector field \(X_C\) of \(C\), \(dC = \Omega(X_C, \cdot)\). Because \(L_{X_C}C = \Omega(X_C, X_C) = 0\), the flow preserves \(M_C\), and the orbit space inherits a unique symplectic form \(\Omega_{M/C}\) from the presymplectic form \(i^*\Omega\) on \(M_C\), where \(i: M_C \rightarrow M\) is the inclusion of \(M_C\) in \(M\). The set of \textit{observables} of the constrained system, which solve the constraint equation \(C = 0\) and are invariant under the gauge flow, is given by \(C^\infty(M/C)\).

In addition to implementing a constraint \(C = 0\) by symplectic reduction, physical systems usually require the definition of a \textit{dynamical flow}. The canonical way is to select a Hamiltonian function \(H \in C^\infty(M)\) which generates the dynamical flow \(F_H(t) = \exp(tX_H)\) with the Hamiltonian vector field \(X_H\) of \(H\). A dynamical flow in the presence of a constraint \(C = 0\) is consistent if it preserves the constraint surface, that is, \(X_HC = \Omega(X_C, X_H) = -\{C, H\} = 0\) on \(M_C\) with the Poisson bracket \(\{\cdot, \cdot\}\) defined by \(\Omega\). The same condition ensures that the dynamical flow is well-defined on the reduced phase space \(M/C\) because it is compatible with the gauge flow: By the Jacobi identity of \(\{\cdot, \cdot\}\), a gauge transformation (that is, the diffeomorphism induced by a gauge flow) commutes with the dynamical flow up to a gauge transformation. Since \(\{C, H\} = 0\) on \(M_C\), there is a \(\lambda \in C^\infty(M)\) such that \(\{C, H\} = \lambda C\) on \(M\), and

\[ [X_C, X_H] = \{\cdot, H\}, C\} - \{\cdot, C\}, H\} = -\{\cdot, \{C, H\}\} = -X_{\{C, H\}} = -X_{\lambda C}. \]
In systems typically encountered in general relativity or its cosmological models, the dynamical flow is simultaneously a gauge flow. A system is time-reparameterization invariant if, given a solution $f(t)$ of its dynamical flow such that $df/dt = \{f, H\}$ for all $t \in \mathbb{R}$, $f(T(t))$ is also a solution for any monotonic $T \in C^\infty(\mathbb{R})$. Any such $f(T(t))$ can be obtained from $f(t)$ by following the flow generated by the Hamiltonian itself together with a suitable non-zero multiplier $N \in C^\infty(\mathbb{R})$ via

$$\lim_{\epsilon \to 0} \frac{f(t + \epsilon N(t)) - f(t)}{\epsilon} = N(t) \frac{df}{dt} = \{f, NH\}.$$ 

Therefore the Hamiltonian function is itself the generator of a gauge flow. Observables are functions on the orbit space of the gauge flow. This orbit space inherits a Poisson structure from $M$, with symplectic leaves given by the level surfaces of $H$ \cite{2}. Adding a constant to $H$ does not change the dynamical flow. Therefore, without loss of generality, we can assume the relevant symplectic leaf to be given by $H = 0$, such that the dynamical generator $H$ is also a constraint.

The Hamiltonian of a time-reparameterization invariant system is therefore a constraint, called the Hamiltonian constraint. In order to emphasize its nature as a constraint, we will slightly change notation and refer to a Hamiltonian constraint as $C$. We refer to symplectic reduction with a Hamiltonian constraint as dynamical symplectic reduction. Associated with this process is the following long-standing problem \cite{3,4}: Any observable $O \in C^\infty(M/C)$ on the reduced phase space can be pulled back to a function on $M_C: C = 0$ using the projection $p: M_C \to M/C$ to the orbit space. By definition, $p^*O$ is constant on the orbits, or time independent if $C$ is a Hamiltonian constraint. In the reduced phase space, therefore, there is no recognizable time evolution in a time-reparameterization invariant theory.

Classically, the problem of identifying time evolution in a time-reparameterization invariant system is usually solved by fixing the gauge flow generated by a Hamiltonian constraint. This construction to determine observables and their evolution does not use the reduced phase space. Given a symplectic manifold $(M, \Omega)$ and a Hamiltonian constraint $C \in C^\infty(M)$, a gauge fixing of the gauge flow is accomplished by a global incisive section.

**Definition 1** A section $(L, \Omega_L, \iota)$ of the gauge flow generated by a constraint $C$ on $(M, \Omega)$ is a symplectic manifold $(L, \Omega_L)$ (called the gauge-fixed phase space) together with an embedding $\iota: L \to M_C$ such that $\Omega_L = \iota^*\iota^*\Omega$.

A section $(L, \Omega_L, \iota)$ of the gauge flow generated by a constraint $C$ on $(M, \Omega)$ is global if for every $y \in M_C$ there is an $x \in L$ such that $y = F_C(\epsilon)\iota(x)$ for some $\epsilon$.

A section $(L, \Omega_L, \iota)$ of the gauge flow generated by a constraint $C$ on $(M, \Omega)$ is incisive if, for all $x_1, x_2 \in L$, $\iota(x_1) = F_C(\epsilon)\iota(x_2)$ for some $\epsilon$ implies $x_1 = x_2$.

The pull-back $\iota^*: C^\infty(M_C) \to C^\infty(L)$ maps functions on the constraint surface $M_C$ to gauge-fixed observables on $L$.

**Proposition 1** If $(L, \Omega_L, \iota)$ is a global incisive section of the gauge flow of $C$ on $(M, \Omega)$, the gauge-fixed phase space $(L, \Omega_L)$ is symplectomorphic to the reduced phase space $(M/C, \Omega_{M/C})$.
Proof: Since a global incisive section intersects each gauge orbit exactly once, there is a bijection between $L$ and the reduced phase space. The symplectomorphism property can then be shown in local coordinates: Locally, $C$ can be used as a coordinate in a neighborhood around a given point $x \in M_C \subset M$. We use the gauge flow $F_C(\epsilon): x \mapsto x_\epsilon \in M_C$ to introduce a second coordinate $z$ such that $z(x) = 0$ and $z(x_\epsilon) = \epsilon$. The two functions $C$ and $z$ are canonically conjugate: $\{z, C\} = X_C z = d z / d \epsilon = 1$. By Darboux' theorem, there are $\dim M - 2$ additional local coordinates $q_j$ and $p_k$, such that

$$
\Omega_M = dz \wedge dC + \sum_{j=1}^{\frac{1}{2}\dim M - 1} dq_j \wedge dp_j.
$$

Since $0 = \{q_j, z\} = \partial q_j / \partial C$ and $0 = \{p_j, z\} = \partial p_j / \partial C$, $q_j$ and $p_k$ together with $z$ define a local coordinate system on $M_C$.

On $M_C$, $i^* \Omega = \sum_{j=1}^{\frac{1}{2}\dim M - 1} dq_j \wedge dp_j$ is a presymplectic form. Local intervals of gauge orbits of $C$ are the coordinate lines of $z$. Therefore, $q_j$ and $p_k$ are local coordinates on the reduced phase space, with symplectic form $\Omega_{M/C} = \sum_{j=1}^{\frac{1}{2}\dim M - 1} dq_j \wedge dp_j$. In order for $i^* i^* \Omega$ to be symplectic, any section of the gauge flow must locally be of the form $t: y \mapsto (s(y), z(s(y)))$ with a canonical transformation $s: y \mapsto (q_j, p_k)$ and a smooth function $z(q_j, p_k)$. Therefore, $\Omega_L = i^* i^* \Omega = s^* \sum_{j=1}^{\frac{1}{2}\dim M - 1} dq_j \wedge dp_j = s^* \Omega_{M/C}$. 

An incisive section $(L, \Omega_L, i)$ evolves in $M$ if there is a $1$-parameter family of incisive sections $(L, \Omega_L, i_t), t \in (t_1, t_2) \subset \mathbb{R}$, such that $i = i_{t_0}$ for some $t_0 \in (t_1, t_2)$, and $L \times (t_1, t_2) \to U, (y, t) \mapsto i_t(y)$ is a diffeomorphism to an open submanifold $U \subset M_C$. For each value of $t \in (t_1, t_2)$ the hypersurface $i_t(L) \subset M_C$ plays the role of a surface of a fixed value of time. With this structure in place, any function $f \in C^\infty(M_C)$ can be viewed as evolving in time along any given gauge orbit by tracing it along the intersection between the orbit and the constant-time surfaces.

$$
f_{[x]}(t) = f([x] \cap i_t(L)),
$$

where $[x]$ is the gauge orbit passing through some $x \in M_C$.

This time evolution takes place on the pre-symplectic manifold $M_C$ which is not the usual setting for describing a dynamical physical system. Moreover, it is not a good starting point for standard quantization as functions on $M_C$ do not have a well-defined Poisson bracket due to the degeneracy of $i^* \Omega$. However, an evolving incisive section defines a family of functions in $C^\infty(U)$, namely those that are constant along the curves traced by points in $L$ from one section to the next $x(t) = i_t(y)$ for a fixed $y \in L$. This family of functions can be arbitrarily extended to the entirety of $M_C$:

**Definition 2** A subset $\mathcal{F} \subset C^\infty(M_C)$ is fashionable with respect to an evolving incisive section $(L, \Omega_L, i_t)$ if for all $t, t' \in (t_1, t_2)$ the map $i_t^*: \mathcal{F} \to C^\infty(L)$ is a bijection and $i_{t'}^* f = i_t^* f$ for all $f \in \mathcal{F}$. 

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Given a choice of fashionables, each function on the symplectic manifold $L$ corresponds to an evolving observable on $M_C$ and conversely, the set of evolving observables $\mathcal{F}$ inherits a Poisson bracket from the symplectic structure on $L$.

This notion of evolution has no known analog in the reduced phase space. In practice it is usually implemented through deparameterization $[5, 6, 7, 8, 9]$, provided the constraint surface admits a factorization of the form $M_C \cong \iota(L) \times \mathbb{R} \ni (\iota(x), Z)$ with a global coordinate $Z \in \mathbb{R}$ such that $\{Z, C\} \neq 0$. Then the map $\iota_t : L \to M_C, x \mapsto (x, t)$ defines a family of global incisive sections. Evolution defined by this family of sections and $\mathcal{F}$, the $Z$-independent functions on $M_C$, is called global relational evolution with respect to $Z$.

\textbf{Example:} Let $M = \mathbb{R}^{2(n+1)} \ni (Z, E, q_1, p_1, \ldots q_n, p_n)$ with
\[
\Omega = dZ \wedge dE + \sum_{i=1}^{n} dq_i \wedge dp_i
\]
and a constraint $C = E + h(Z, q_i, p_i)$ linear in $E$. The constraint surface here consists of points with coordinates $(Z, -h(q_i, p_i), q_i, p_i)$, so that $(Z, q_i, p_i)$ serve as coordinates on $M_C$. The choice $L = \mathbb{R}^{2n} \ni (Q_1, P_1, \ldots Q_n, P_n)$ then leads to global incisive deparameterized sections via $\iota_t : (Q_i, P_i) \mapsto (t, Q_i, P_i) \in M_C$. Since for $C^\infty(M_C) \ni f = f(Z, q_i, p_i)$ under this family of embeddings $(\iota_t^* f)(Q_i, P_i) = f(t, Q_i, P_i)$, the corresponding fashionables consist precisely of the functions that do not depend on $Z$. Since the Hamiltonian vector field of $E$ generates translations in $Z$ and hence shifts from $\iota_t(L)$ to $\iota_{t'}(L)$, the set of fashionables correspond to the Poisson commutant $E' = \{ f \in C^\infty(M) : \{ f, E \} = 0 \}$ of $E$ pulled back to $M_C$. Relational evolution with respect to $Z$ is identical with Hamiltonian evolution in $L$ generated by $H_t(Q_i, P_i) = h(t, Q_i, P_i)$: The gauge flow $F_C(\epsilon)$ on $M$ maps a function $g \in C^\infty(M)$ to $g_{\epsilon} = F_C(\epsilon)^* g$. In an infinitesimal version, $\delta g / \delta \epsilon := \lim_{\epsilon \to 0} (g_{\epsilon} - g) / \epsilon$ is given by
\[
\frac{\delta g}{\delta \epsilon} = \{ g, C \} = \frac{\partial g}{\partial Z} + \{ g, h \}.
\]
Specifically,
\[
\frac{\delta q_i}{\delta \epsilon} = \frac{\partial h}{\partial p_i}, \quad \frac{\delta p_i}{\delta \epsilon} = -\frac{\partial h}{\partial q_i}, \quad \frac{\delta Z}{\delta \epsilon} = 1.
\]
This pulls back to $L$ as
\[
\frac{\delta Q_i}{\delta \epsilon} = \frac{\partial H_t}{\partial P_i}, \quad \frac{\delta P_i}{\delta \epsilon} = -\frac{\partial H_t}{\partial Q_i}.
\]
For a function on $L$, we have
\[
\frac{\delta f}{\delta \epsilon} = \lim_{\epsilon \to 0} \frac{f(Q_i + \epsilon \delta Q_i / \delta \epsilon, P_i + \epsilon \delta P_i / \delta \epsilon) - f(Q_i, P_i)}{\epsilon}
\]
\[
= \sum_{i=1}^{n} \left( \frac{\partial f}{\partial Q_i} \frac{\delta Q_i}{\delta \epsilon} + \frac{\partial f}{\partial P_i} \frac{\delta P_i}{\delta \epsilon} \right) = \{ f, H \}_L
\]
computed precisely according to Hamilton’s equations on $L$. 

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The quantization of a reduced phase space exists in the sense of deformation quantization \[10,11\] à la Fedosov or Kontsevich \[12\]. On the other hand, dynamical symplectic reduction is usually quantized only for deparameterized systems as in the immediately preceding example, using a standard Hilbert-space quantization of \(L\) on which the reduced Hamiltonian \(H_t(Q_i, P_i)\) is represented as an operator. In such examples, quantum evolution exists and is unitary, but there are long-standing problems when one tries to extend this notion to more complicated constrained systems in which no global analog of \(Z\) exists \[3,4\]. For instance, given a constraint quadratic in \(E\) on the same phase space as in the example, \(\{Z, C\} \propto E\) may become zero along a gauge orbit such that \(Z = \text{const}\) no longer defines a gauge section. In Section 2 we define algebraic quantization of dynamical symplectic reduction and prove several useful properties of the resulting quantum evolution on an algebra of observables. For deparameterizable systems, which can be quantized by well-established means as representations on a fixed Hilbert space, our algebraic results provide a more general treatment because they apply to all possible choices of the Hilbert space. Moreover, our construction applies to non-deparameterizable systems, even though the results in that case are less specific than for deparameterizable systems. Several results and examples in Section 3 will demonstrate the non-trivial nature of our constructions.

2 Sections in quantum symplectic reduction

The set of observables of a quantum system is given by the \(*\)-invariant elements of a complex, unital \(*\)-algebra \(\mathcal{A}\). In this paper, we assume that \(\mathcal{A}\) is associative. (This assumption rules out some physical systems, such as magnetic monopole densities \[13,14\], which however are usually considered exotic.) Physical states of the quantum system defined by \(\mathcal{A}\) are normalized positive linear functionals \(\omega: \mathcal{A} \to \mathbb{C}\), such that \(\omega(1) = 1\) and

\[
\omega(AA^*) \geq 0 \quad \text{for all } A \in \mathcal{A}.
\]

The condition that \((AA^*)\) is real for all \(A \in \mathcal{A}\) on its own implies that a physical state is real — \(\omega(A) = \overline{\omega(A^*)}\). In addition, the stronger inequality condition leads to the Cauchy–Schwarz inequality

\[
|\omega(AB^*)|^2 \leq |\omega(AA^*)||\omega(BB^*)| \quad \text{for all } A, B \in \mathcal{A};
\]

see for instance \[15\].

As we will see, intermediate stages of quantum symplectic reduction require a weaker notion of states which are not completely positive. We begin with

**Definition 3** The set of kinematical states \(\Gamma\) on a unital \(*\)-algebra \(\mathcal{A}\) is the set of normalized linear functionals \(\omega: \mathcal{A} \to \mathbb{C}\), such that \(\omega(1) = 1\).

Given the normalization condition, \(\Gamma\) is not a vector space, but it is closed with respect to normalized sums: for any integer \(N \geq 1\), states \(\omega_1, \ldots, \omega_N \in \Gamma\) and complex numbers \(a_1, \ldots, a_N\), \(\sum_{j=1}^{N} a_j \omega_j \in \Gamma\) if \(\sum_{j=1}^{N} a_j = 1\).
Definition 4 A dynamical flow on \( \mathcal{A} \) is a one-parameter family of derivations \( \bar{D}_t : (a, b) \times \mathcal{A} \rightarrow \mathcal{A} \), where \( (a, b) \subset \mathbb{R} \), which is compatible with the \( * \)-structure on \( \mathcal{A} \) — (\( \bar{D}_t A \))\(^*\) = \( \bar{D}_t A^* \) for all \( A \in \mathcal{A} \) — and such that \( \omega(\bar{D}_t A) \) is continuously differentiable with respect to \( t \) for all \( \omega \in \Gamma \).

Given a dynamical flow \( \bar{D}_t \) on \( \mathcal{A} \), the time evolution of a kinematical state \( \omega \in \Gamma \) is a map \( (a, b) \times \mathcal{A} \rightarrow \mathbb{C}, \{(t, A) \mapsto \omega_t(A)\} \) such that \( \omega_t \) is a kinematical state and

\[
\frac{d}{dt} \omega_t(A) = \omega_t \left( \bar{D}_t A \right)
\]

for all \( t \in (a, b) \), with initial conditions \( \omega_{t_0} = \omega \) for some \( t_0 \in (a, b) \).

In order to make sure that a state has a unique time evolution (or a unique gauge flow in what follows), we will assume that, for all algebras we consider, a differential equation of the form (1) has a unique solution with the specified initial condition. Standard results do not necessarily apply because our differential equations, though linear, are, in general, formulated on an infinite-dimensional space and may have time-dependent coefficients. (Although we will not pursue a formal proof of existence and uniqueness of solutions, we note that time evolution in systems of interest in physics is usually obtained as a unique Dyson series on a Hilbert space; see for instance [10]. Such solutions may be constructed within the algebraic setting after using the GNS construction based on the initial state \( \omega \).)

Lemma 1 If \( \omega \in \Gamma \) is a kinematical state, its time evolution with respect to \( \bar{D}_t \), \( t \in (a, b) \), returns a kinematical state for any \( t \in (a, b) \).

Proof: By definition, a derivation satisfies

\[
\bar{D}_t (AB) = \bar{D}_t (A) B + A \bar{D}_t (B)
\]

for all \( A, B \in \mathcal{A} \). Choosing \( B = 1 \), we have \( \bar{D}_t (A) = \bar{D}_t (A) + A \bar{D}_t (1) \) for all \( A \in \mathcal{A} \). It follows that \( \bar{D}_t (1) = 0 \), whence \( d\omega_t (1) / dt = 0 \) for all \( t \). Therefore, \( \omega_t (1) = 1 \) for all \( t \). \( \square \)

Lemma 2 If \( \omega \in \Gamma \) is positive, its time evolution is positive.

Proof: To prove that \( \omega_t(AA^*) \geq 0 \) continues to hold along the flow, it is sufficient to show that (i) \( \omega_t(AA^*) \) is real for all \( t \) and (ii) \( d\omega_t(AA^*)/dt \) is non-negative whenever \( \omega_t(AA^*) = 0 \).

To prove (i), for each \( A \in \mathcal{A} \) define a function of \( t \) via \( f_A(t) = \omega_t(AA^*) - \omega_t(\bar{D}_t(A)A^*) \) on \( \Gamma \), so that \( \omega_t(AA^*) \) is real iff \( f_A(t) = 0 \). Suppose all of the functions \( f_A(t') = 0 \) for some \( t' \in (a, b) \), then \( \omega_{t'}(AA^*) \) is real for all \( A \in \mathcal{A} \), which implies \( \omega_{t'}(A) = \omega_{t'}(A^*) \), and we get

\[
\frac{d}{dt} \omega_t (AA^*) \bigg|_{t=t'} = \left. \omega_t \left( \bar{D}_t (AA^*) \right) \right|_{t=t'} = \omega_{t'} \left( \left( \bar{D}_{t'} A \right) A^* \right) + \omega_{t'} \left( A \left( \bar{D}_{t'} A^* \right) \right)
\]

\[
= \omega_{t'} \left( \left( \bar{D}_{t'} A \right) A^* \right) + \omega_{t'} \left( \left( \bar{D}_{t'} A^* \right) A \right)
\]

\[
= 2 \text{Re} \left[ \omega_{t'} \left( \left( \bar{D}_{t'} A \right) A^* \right) \right] = \text{Re} \left[ \frac{d}{dt} \omega_t (AA^*) \right] \bigg|_{t=t'},
\]
which means that \( df_A(t)/dt = 0 \) at \( t = t' \). Since \( \omega_{t_0} = \omega \) is positive, we have the initial conditions \( f_A(t_0) = 0 \) for all \( A \in \mathcal{A} \). We see that \( \{ f_A(t) = 0, \forall t \in (a,b) \}_{A \in \mathcal{A}} \) satisfies the first-order ordinary differential equation system induced by the dynamical flow and matches the given set of initial conditions. As previously discussed, here we assume such solutions to the dynamical flow to be unique. Therefore \( \omega_t(AA^*) \) is real for all \( t \in (a,b) \).

To prove (ii) we use the above result and assume that the inequality holds at \( t = t' \).

\[
\left| \frac{d}{dt} \omega_t(AA^*) \right|_{t=t'}^2 = 4 \left| \text{Re} \left[ \omega_{t'} \left( \left( \tilde{D}_{t'}A \right) A^* \right) \right] \right|^2 \\
\leq 4 \left| \omega_{t'} \left( \left( \tilde{D}_{t'}A \right) A^* \right) \right|^2 \\
\leq 4 \omega_{t'}(A^*A) \omega_{t'} \left( \left( \tilde{D}_{t'}A \right) \left( \tilde{D}_{t'}A \right)^* \right).
\]

Since \( \omega_{t'}(A^*A) \in \mathbb{R} \), \( \omega_{t'}(A^*A) = \omega_{t'}(AA^*) \), and the expression on the right is zero if \( \omega_{t'}(AA^*) = 0 \).

\[\square\]

2.1 Constrained quantization

A singly constrained quantum system is a complex, unital \(*\)-algebra \( \mathcal{A} \) together with a constraint \( C \in \mathcal{A} \) such that \( C^* = C \) and \( C \) does not have a left-inverse in \( \mathcal{A} \).

**Definition 5** The algebra of Dirac observables of a singly constrained quantum system \((\mathcal{A},C)\) is the commutant of \( C \) in \( \mathcal{A} \):

\[
\mathcal{A}_{\text{obs}} = C' = \{ A \in \mathcal{A} : [A,C] = 0 \}.
\]

**Lemma 3** \( \mathcal{A}_{\text{obs}} \) is a unital \(*\)-subalgebra of \( \mathcal{A} \).

**Proof:** Defined as the commutant of \( C \), \( \mathcal{A}_{\text{obs}} \) is a subalgebra. Since \( [1,C] = 0 \) and \( [A^*,C] = -[A,C^*] = -[A,C]^* = 0 \) if \( A \in \mathcal{A}_{\text{obs}} \), using \( C^* = C \), it is a unital \(*\)-subalgebra. \( \square \)

**Definition 6** A kinematical state \( \omega \in \Gamma \) is a solution of the constraint \( C \) if \( \omega(AC) = 0 \) for all \( A \in \mathcal{A} \). The constraint surface \( \Gamma_C \subset \Gamma \) is the subset of all solutions of \( C \), closed with respect to normalized sums.

**Remark:** Since we have assumed that \( C \) is without left-inverse in \( \mathcal{A} \), \( AC \subset \mathcal{A} \) is a strict subalgebra without unit. The condition \( \omega(AC) = 0 \) is therefore consistent with normalization of kinematical states.

The constraint \( C \) in a singly constrained quantum system induces a gauge flow:
**Definition 7** Two kinematical states $\psi, \omega \in \Gamma$ are $C$-equivalent, $\omega \sim_C \psi$, if there exist a positive integer $M$, $A_1, A_2, \ldots, A_M \in \mathcal{A}$ and $\lambda_1, \lambda_2, \ldots, \lambda_M \in \mathbb{R}$, such that

$$\psi = S_{A_1C}(\lambda_1)S_{A_2C}(\lambda_2) \ldots S_{A_MC}(\lambda_M)\omega$$

where for $A \in \mathcal{A}$ and $\lambda \in \mathbb{R}$, the flow $S_A(\lambda) : \Gamma \to \Gamma$ is defined by $S_A(0) = \text{id}$ and

$$i\hbar \frac{d}{d\lambda} (S_A(\lambda)\omega(B)) = S_A(\lambda)\omega([B, A]) .$$

Since $B \mapsto [B, A]$ is a derivation, $S_A(\lambda)$ is well-defined by Lemma 1. By analogy with classical reduction, we refer to flows generated by elements of $\mathcal{AC}$ as gauge.

**Lemma 4** The constraint surface $\Gamma_C$ is preserved by the flow induced by any algebra element $AC$.

**Proof:** For any fixed $A \in \mathcal{A}$ and $\omega \in \Gamma_C$, following the same argument as in Lemma 2, define functions $F f_B(\lambda) = S_{AC}(\lambda)\omega(BC)$ on $\Gamma$, for $B \in \mathcal{A}$. Suppose all $f_B(\lambda) = 0$ for some $\lambda'$, then

$$i\hbar \frac{d}{d\lambda} \bigg|_{\lambda=\lambda'} = i\hbar \frac{d}{d\lambda} (S_{AC}(\lambda)\omega(BC)) \bigg|_{\lambda=\lambda'}$$

$$= S_{AC}(\lambda')\omega(\lambda')$$

$$= S_{AC}(\lambda)\omega((BC, AC)_{\lambda=\lambda'}) = f_{\{BC, A\} + A[B, C]}(\lambda') = 0$$

for all $B \in \mathcal{A}$. Moreover, we have initial conditions $f_B(0) = \omega(BC) = 0$ for all $B$. It follows that $\{f_B(\lambda) = 0, \forall \lambda \} B \in \mathcal{A}$ is the solution to the flow induced by any algebra element of the form $AC$ that satisfies our initial conditions. Therefore, $S_{AC}(\lambda)\omega(BC) = 0$ for all $\lambda$, and $S_{AC}(\lambda)\omega \in \Gamma_C$.

Any two $C$-equivalent states on $\Gamma_C$ are indistinguishable by their evaluation in Dirac observables:

**Lemma 5** For any $\omega, \psi \in \Gamma_C$, if $\omega \sim_C \psi$, then $\omega(O) = \psi(O)$ for any $O \in \mathcal{A}_{\text{obs}}$.

**Proof:** The two states $\omega$ and $\psi$ are related by a succession of gauge flows $S_{AC}(\lambda)$. By Lemma 4, each of these flows preservers $\Gamma_C$. Therefore, for any $A \in \mathcal{A}$ and $B \in \mathcal{A}_{\text{obs}},$

$$i\hbar \frac{d}{d\lambda} (S_{AC}(\lambda)\omega(B)) = S_{AC}(\lambda)\omega([B, AC])$$

$$= S_{AC}(\lambda)\omega(A[BC] + [B, A]C)$$

$$= S_{AC}(\lambda)\omega([B, A]C) = 0 ,$$

since $S_{AC}(\lambda)\omega \in \Gamma_C$. Therefore, $S_{AC}(\lambda)\omega(B)$ is constant along any gauge flow $S_{AC}(\lambda)$.

Equivalence classes $[\omega]_C \in \Gamma_C / \sim_C$ therefore define states on $\mathcal{A}_{\text{obs}}$.

**Definition 8** The space of physical states $\Gamma_{\text{phys}}$ is the convex subset of $\Gamma_C / \sim_C$ containing all $[\omega]_C$ with $\omega$ positive on $\mathcal{A}_{\text{obs}}$.

As in the classical case, there is no evolution for physical states in a dynamical constrained system. Solving this problem requires the introduction of gauge sections.
2.2 Gauge sections

**Definition 9** A section \((A_\mathcal{O}, \mathcal{O}, \eta)\) of the gauge flow in a quantum system with a single constraint \(C \in \mathcal{A}\) is a surjective \(*\)-algebra homomorphism \(\eta: A_\mathcal{O} \to \mathcal{O}\) such that \(\eta(AC \cap A_\mathcal{O}) = \{0\}\), where \(A_\mathcal{O} \subset \mathcal{A}\) is a unital \(*\)-subalgebra and \(\mathcal{O}\) a unital \(*\)-algebra.

A section \((A_\mathcal{O}, \mathcal{O}, \eta)\) is expansive if \(A_\mathcal{O} + AC = \mathcal{A}\).

The space of gauge-fixed states with respect to a section \((A_\mathcal{O}, \mathcal{O}, \eta)\) is

\[
\Gamma_{A_\mathcal{O}}|_\eta = \{\bar{\omega} \in \Gamma_{A_\mathcal{O}} : \bar{\omega}(B) = 0 \text{ for all } B \in \ker \eta\}.
\]

The space of reduced states is given by \(\Gamma_{\mathcal{O}}\).

**Remark:** The subalgebra \(AC \subset \mathcal{A}\) on which constrained kinematical states vanish is, in general, a left-ideal but not two-sided, and it is not \(*\)-invariant. Therefore, the factor space \(\mathcal{A}/AC\) is not a \(*\)-algebra and cannot be used to define observables. This property is different from the classical case, in which \(\iota^*: C^\infty(M) \to C^\infty(M_C)\) defines a natural algebra of observables on the constraint surface. Our definition therefore introduces a new \(*\)-algebra \(A_\mathcal{O}\). The roles of \(\mathcal{O}\) and \(\eta\), respectively, are comparable to those of \(L\) and \(\iota\) classically.

Given a section \((A_\mathcal{O}, \mathcal{O}, \eta)\) of the gauge flow of \(C \in \mathcal{A}\), any kinematical state on \(\mathcal{A}\) can be restricted to a state on \(A_\mathcal{O} \subset \mathcal{A}\) using \(\phi: \Gamma \to \Gamma_{A_\mathcal{O}}\) with \(\phi(\omega)(B) = \omega(B)\) for \(B \in A_\mathcal{O}\). Each reduced state \(\bar{\omega} \in \Gamma_\mathcal{O}\), pulled back to \(\Gamma_{A_\mathcal{O}}\) through \(\eta\), has a corresponding fiber \(\phi^{-1}(\bar{\omega} \circ \eta)\) in \(\Gamma\), where

\[
\phi^{-1}(\bar{\omega}) := \{\omega \in \Gamma : \omega(B) = \bar{\omega}(B), \forall B \in A_\mathcal{O}\}
\]

for \(\bar{\omega} \in \Gamma_{A_\mathcal{O}}\).

**Lemma 6** For every \(\bar{\omega} \in \Gamma_\mathcal{O}\) of a section \((A_\mathcal{O}, \mathcal{O}, \eta)\), the fiber \(\phi^{-1}(\bar{\omega} \circ \eta)\) has a non-empty intersection with \(\Gamma_C\). If the section is expansive, there is a unique constrained state in the intersection. If \(\bar{\omega} \neq \bar{\psi} \in \Gamma_\mathcal{O}\), \(\phi^{-1}(\bar{\omega} \circ \eta) \neq \phi^{-1}(\bar{\psi} \circ \eta)\).

**Proof:** The spaces \(AC\) and \(A_\mathcal{O}\) are linear subspaces of \(\mathcal{A}\). There is therefore a linear map \(\omega: \mathcal{A} \to \mathbb{C}\) which extends a given linear map \(\bar{\omega} \in \Gamma_{A_\mathcal{O}}\) and assigns zero to all elements of \(AC\), provided that \(\omega = \bar{\omega}\) on \(AC \cap A_\mathcal{O}\). Since \(\eta(AC \cap A_\mathcal{O}) = \{0\}\) for a section \((A_\mathcal{O}, \mathcal{O}, \eta)\), the condition is fulfilled for \(\bar{\omega} = \bar{\omega} \circ \eta\). Since \(1 \in A_\mathcal{O}\) and \(\eta\) is a homomorphism, any such extension \(\omega\) is normalized and therefore a state in \(\Gamma\). For an expansive section, the extension of the state to \(\mathcal{A}\) is unique.

If \(\phi^{-1}(\bar{\omega} \circ \eta) = \phi^{-1}(\bar{\psi} \circ \eta)\), \(\bar{\omega} \circ \eta = \bar{\psi} \circ \eta\). Since \(\eta\) is onto, this implies \(\bar{\omega} = \bar{\psi}\).

Thanks to the Lemma, an expansive section makes it possible to identify any state in \(\Gamma_\mathcal{O}\) with a unique constrained state in \(\Gamma_C\). However, without further conditions, states
in $\Gamma_O$ are not guaranteed to be physical because the gauge flow on $\Gamma_C$ induces a residual gauge flow on $\Gamma_{\mathcal{A}_O}$ which in general is non-trivial.

Any 1-parameter family $\omega_\lambda$ of states in $\Gamma_C$ (or $\Gamma$) can be restricted to a 1-parameter family $\tilde{\omega}_\lambda$ in $\Gamma_{\mathcal{A}_O}$, via $\tilde{\omega}_\lambda(B) = \omega_\lambda(B)$ for all $B \in \mathcal{A}_O$. A flow uniquely projects from $\Gamma_C$ to $\Gamma_{\mathcal{A}_O}$ if it preserves the $\phi$-fiber structure introduced in $\Gamma_C$ by a section. Specifically, let $\tilde{D}$ be a derivation that preserves $\mathcal{A}C$ and, therefore, generates a flow tangent to $\Gamma_C$, and let us pick a state $\tilde{\omega} \in \Gamma_{\mathcal{A}_O}$. Each element of the fiber $\omega \in \phi^{-1}(\tilde{\omega})$ is mapped by the flow to a 1-parameter family $\omega_\lambda$ with $\omega_0 = \omega$ and $d\omega_\lambda(A)/d\lambda = \omega_\lambda(\tilde{D}A)$ for all $\lambda \in \mathbb{R}$. Starting at elements of $\phi^{-1}(\tilde{\omega}) \cap \Gamma_C$, the flow projects to a unique 1-parameter family $\tilde{\omega}_\lambda$ if for any pair $\omega, \omega' \in \phi^{-1}(\tilde{\omega}) \cap \Gamma_C$ we have $\omega_\lambda(B) = \omega'_\lambda(B)$ for all $B \in \mathcal{A}_O$ and all $\lambda \in \mathbb{R}$. This condition is satisfied if

$$\frac{d}{d\lambda} \omega_\lambda(B) = \frac{d}{d\lambda} \omega'_\lambda(B)$$

for all $\lambda$, or $\omega_\lambda(\tilde{D}B) = \omega'_\lambda(\tilde{D}B)$. Two elements $\omega, \omega' \in \phi^{-1}(\tilde{\omega}) \cap \Gamma_C$ have identical restrictions both to $\mathcal{A}_O$ and to $\mathcal{A}C$. Therefore, a flow can be projected to $\Gamma_{\mathcal{A}_O}$ if it is generated by a derivation $\tilde{D}$ such that $\tilde{D}B \in \mathcal{A}C + \mathcal{A}_O$ for all $B \in \mathcal{A}_O$. Hence, $\tilde{D}$ preserves the fiber structure if it preserves the algebra $\mathcal{A}_O$ up to adding elements of $\mathcal{A}C$. If $\mathcal{A}_O + \mathcal{A}C = \mathcal{A}$, every flow satisfies this condition.

In general, it is possible that for a given section there are physical states $[\omega]_C$ such that $\omega'(A) \neq 0$ for all $\omega' \in [\omega]_C$ and some $A \in \ker \eta$, while a pullback $\tilde{\omega} \circ \eta$ vanishes on $\ker \eta$. If such a state exists, it is not sampled by the section: there is no $\tilde{\omega} \in \Gamma_O$ and $\omega' \in \Gamma_C$ such that $\omega \sim_C \omega' \in \phi^{-1}(\tilde{\omega} \circ \eta)$. A section samples all physical states if it is global, according to

**Definition 10** A section $(\mathcal{A}_O, \mathcal{O}, \eta)$ of the gauge flow given by $C \in \mathcal{A}$ is **global** if for every $\omega \in \Gamma_C$ there exist $\tilde{\omega} \in \Gamma_O$ and $\omega_0 \in \Gamma_C$, such that $\omega \sim_C \omega_0$ and $\omega_0(A) = \tilde{\omega}(\eta(A))$ for all $A \in \mathcal{A}_O$.

If $(\mathcal{A}_O, \mathcal{O}, \eta)$ is a global section of the gauge flow of $C \in \mathcal{A}$, every equivalence class $[\omega]_C$ of a constrained state $\omega \in \Gamma_C$ has a non-trivial intersection with some fiber $\phi^{-1}(\tilde{\omega} \circ \eta)$. In this paper, we are more concerned about making sure that every physical state $[\omega]_C$ which does have a non-trivial intersection with some $\phi^{-1}(\tilde{\omega} \circ \eta)$ intersects in an appropriate way, in the sense of the following two definitions:

**Definition 11** A section $(\mathcal{A}_O, \mathcal{O}, \eta)$ of the gauge flow of $C \in \mathcal{A}$ is **transversal** if for all $A \in \mathcal{A}$ and $B \in \ker \eta$, $[B, \mathcal{A}C] \notin \ker \eta$.

**Definition 12** A section $(\mathcal{A}_O, \mathcal{O}, \eta)$ of the gauge flow of $C \in \mathcal{A}$ is **incisive** if, given $\tilde{\omega}_1, \tilde{\omega}_2 \in \Gamma_O$ such that $\omega_1 \in \phi^{-1}(\tilde{\omega}_1 \circ \eta)$ and $\omega_2 \in \phi^{-1}(\tilde{\omega}_2 \circ \eta)$, $\omega_1 \sim_C \omega_2$ implies $\tilde{\omega}_1 = \tilde{\omega}_2$.

For a transversal section, there is no gauge flow that preserves the space $[\mathcal{B}]$ of gauge-fixed states. In general, this condition does not imply that the section is incisive because the intersection of a gauge orbit with the gauge-fixed states may contain a countable number greater than one of states. Such a section may suffer from the analog of a Gribov problem
in gauge theories, unlike an incisive section where each physical state is sampled at most once.

For an incisive section, gauge relations between constrained states in \( \Gamma_C \) occur only within a \( \phi \)-fiber. In order for \( \Gamma_C \) to represent physical states, it is also desirable that all constrained states within a given \( \phi \)-fiber are, in fact, gauge related:

**Definition 13** A section \( (A_O, O, \eta) \) of the gauge flow of \( C \in A \) is maximally resolved if for every \( \bar{\omega} \in \Gamma_{A_O} \) and \( \omega_1, \omega_2 \in \phi^{-1}(\bar{\omega}) \cap \Gamma_C \) we have \( \omega_1 \sim_C \omega_2 \).

**Remark:** This property depends only on the choice of \( A_O \), and not on the \( * \)-homomorphism \( \eta \). We have not introduced a classical analog of this condition in Section \( \square \) because \( C^\infty(M_C) \) is then a natural choice. However, in specific cases it may be convenient to work with a different set of functions on \( M \) as a classical analog of \( A_O \).

For example, any expansive section is maximally resolved: according to Lemma \( \square \) for any \( \bar{\omega} \in \Gamma_{A_O} \) there is at most one state in the intersection \( \phi^{-1}(\bar{\omega}) \cap \Gamma_C \). On the other hand, a section is not maximally resolved if, for example, there is a non-trivial flow on \( \Gamma_C \) that translates states along the fibers (and therefore vanishes on \( \Gamma_{A_O} \)), but is not generated by a combination of the residual gauge flows on \( \Gamma_{A_O} \). In some cases the resolution of a gauge section may be increased by “refining” the constraint conditions, factorizing the constraint. In Section \( \square \) we will come back to this issue which is important for dynamical flows.

**Example:** Suppose that we know some unital \( * \)-subalgebra of the algebra of Dirac observables, \( D \subseteq A_{\text{obs}} = C' \subset A \). We choose \( A_O = D \). Since \( DC_C \subseteq D \) is a two-sided \( * \)-ideal, we may choose \( O = D/DC_C \). The choice \( \eta = \pi \) with the canonical projection map \( \pi: D \to D/DC \) completes the definition of a section \( (D, D/DC, \pi) \). For any state \( \omega \in \Gamma_C \), the restricted state \( \phi(\omega) \in \Gamma_D \) annihilates the kernel of the quotient map \( \pi \). Therefore, it corresponds to the pull-back of some \( \bar{\omega} \in \Gamma_{D/DC} \). Thus, this section is global. Since any two \( C \)-equivalent states have identical restrictions to \( A_{\text{obs}} \) thanks to Lemma \( \square \) and thus to \( D \), all \( C \)-equivalent states belong to the same fiber and the section is incisive. However, in general it is possible that two distinct physical states have identical restrictions to \( D \). Therefore, the section is not maximally resolved.

This example motivates

**Definition 14** A section \( (A_O, O, \eta) \) of the gauge flow generated by \( C \in A \) is Dirac if \( A_O = D \subseteq A_{\text{obs}} \) is a subset of Dirac observables and \( O = D/DC \), \( \eta = \pi: D \to D/DC \).

**Definition 15** A set \( D \subseteq A_{\text{obs}} \) of Dirac observables with respect to a constraint \( C \in A \) is complete if the Dirac section \( (D, D/DC, \pi) \) is maximally resolved.

In general, it is difficult to find a complete set of Dirac observables. An alternative section then requires an \( O \) that is large enough to represent all physical degrees of freedom.
Example: For any given $*$-invariant element $Z = Z* \in \mathcal{A}$, the commutant $Z' = \{ A \in \mathcal{A} : [A, Z] = 0 \}$ is a unital $*$-subalgebra of $\mathcal{A}$. We choose $\mathcal{A}_C = Z'$. Since $Z$ is central in $Z'$, $(Z - t \mathbf{1})Z'$ is an ideal in $Z'$ for any given $t \in \mathbb{R}$. We choose $\mathcal{O} = Z'/(Z - t \mathbf{1})Z'$ and $\eta$ the canonical projection $\pi_t : Z' \to Z'/(Z - t \mathbf{1})Z'$. We have a family of sections $(Z, Z'/(Z - t \mathbf{1})Z', \pi_t)$ if $Z' \cap \mathcal{A}C = \{0\}$. (In particular, we need $Z \neq a \mathbf{1}$ and $Z \neq bC$ with real numbers $a, b$.) Heuristically, we can interpret $\mathcal{O}$ as the algebra of degrees of freedom the system has when the value of $Z$ is set equal to $t$.

Expansive sections of this kind appear in a large class of physical examples:

Definition 16 A section $(\mathcal{A}_C, \mathcal{O}, \eta)$ of the gauge flow generated by $C \in \mathcal{A}$ is relational if it is expansive and there are $t \in \mathbb{R}$ and $Z = Z* \in \mathcal{A}$ with $Z' \cap \mathcal{A}C = \{0\}$, such that $\mathcal{A}_C = Z'$, $\mathcal{O} = Z'/(Z - t \mathbf{1})Z'$ and $\eta = \pi_t : Z' \to Z'/(Z - t \mathbf{1})Z'$.

2.3 Gauge flows on relational sections

Evolution of states with respect to a Hamiltonian $H \in \mathcal{A}$,

$$\frac{d \omega_t(A)}{dt} = \frac{1}{i\hbar} \omega_t ([A, H]) + \omega_t \left( \frac{d}{dt} A \right), \quad (4)$$

can be formulated as constrained dynamics if we extend the kinematical algebra $\mathcal{A}$ by two new generators, “time” $T = T*$ and “energy” $E = E*$, such that $[T, E] = i\hbar \mathbf{1}$ and $[T, A] = 0 = [E, A]$ for all $A \in \mathcal{A}$. On this extended algebra $\mathcal{A}_{\text{ext}}$, the constraint $C := E + H \in \mathcal{A}_{\text{ext}}$ has a gauge flow

$$\frac{d}{d\lambda} \omega_\lambda = \frac{1}{i\hbar} \omega_\lambda ([A, C]) + \frac{1}{i\hbar} \omega_\lambda ([A, H] + [A, E]) = \omega_\lambda ([A, H]) + \omega_\lambda \left( \frac{d}{dT} A \right) \quad (5)$$

resembling the original dynamical flow on $\mathcal{A}$. Explicit time dependence of $A(t)$ in the Hamiltonian case corresponds to $T$-dependence of $A \in \mathcal{A}_{\text{ext}}$ in the constrained case. This process is called parameterization of the dynamical flow.

The gauge flow (5) is equivalent to the dynamical flow (4) if $T \in \mathcal{A}_{\text{ext}}$ can be “deconstituted to a real number” $t$. The deparameterization process of passing from $\mathcal{A}_{\text{ext}}$ back to the smaller algebra $\mathcal{A}$ can therefore be interpreted as finding the physical states on $\mathcal{A}_{\text{ext}}$ when “the value of $T$ is fixed” to equal $t$. A relational gauge section with $Z = T$ provides a precise formulation of this heuristic idea. We demonstrate this claim in two steps. First, in this subsection, we find specific conditions under which a relational section accurately captures the reduced degrees of freedom. In the subsection that follows, we show how physical states give rise to a dynamical evolution via a suitable family of relational sections.

Throughout the remainder of this section we will denote the constraint element by $C_H$, rather than $C$, anticipating that a constraint should, in general, be factorized to make it deparameterizable, in the sense that $C = NC_H$ with suitable $C_H = C_H*, N \in \mathcal{A}$; see Section [3]. We begin with the following useful result, which quickly follows from Definition [16].

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Lemma 7 In a relational section with respect to $Z$ of the gauge flow of $C_H \in A$, each $\phi$-fiber in $\Gamma$ defined by $Z' \subset A$ contains exactly one $\omega \in \Gamma_{C_H}$. In particular, a relational section is maximally resolved.

Proof: Since $Z' + AC_H = A$ for a relational section, any linear functional on $A$ is completely defined by its restrictions to $AC_H$ and $Z'$. This implies that there is a one-to-one correspondence between $\Gamma_{C_H}$ and $\Gamma_{Z'}$. \qed

States in $\Gamma_{C_H}$ can now be identified with states on the unital $*$-algebra $Z'$. We further note that for a relational section the two properties $Z' \cap AC_H = \{0\}$ and $Z' + AC_H = A$ imply that every $A$ can be written as $A = B + GC_H$, where $B \in Z'$ is unique and $G \in A$ is unique up to adding terms that are annihilated by $C_H$ multiplied on their right. We can also express $G$ as a sum of elements from $Z'$ and $AC_H$ to get a second-order expression $A = B + B'C_H + G'C_H^2$ with $B, B' \in Z'$ and $G \in A$. Depending on the structure of $A$ and the nature of the constraint element, iterating this process may converge either by terminating at a finite order or by leading to an infinite converging sum. In order to prove several of our stronger results without having to deal with a notion of convergence on $A$ we are going to assume that every $A \in A$ can be written as

$$A = B_0 + B_1C_H + B_2C_H^2 + \ldots B_MC_H^M$$

(6)

with $B_i \in Z'$ and $M \in \mathbb{Z}$. In other words, we assume that the set $Z' \cup \{C_H\}$ algebraically generates the whole of $A$. (This situation is common in physical examples, in which both $C_H$ and relevant $A$ are polynomial in an “energy” $E \in A$; see our examples below.)

Definition 17 A relational section with respect to $Z$ of the gauge flow generated by $C_H \in A$ is linear if $[B, C_H] \in Z'$ for all $B \in Z'$.

For a linear section with respect to $Z$,

$$[[Z, C_H], B] = [[Z, B], C_H] + [[B, C_H], Z] = 0$$

for $B \in Z'$. Therefore, $\tilde{D}_HZ \in \mathcal{C}(Z')$ where $\mathcal{C}(Z') := \{B \in Z' : [B, G] = 0, \forall G \in Z'\}$ is the center of $Z'$. In this sense, $C_H$ can be considered a constraint linear in a momentum canonically conjugate to $Z$. Since $Z' \cap AC_H = \{0\}$, $[Z, AC_H] = 0$ implies $AC_H = 0$ for any $A \in A$. Moreover, any linear functional on $Z'$ can be extended to the whole of $A$, such that it annihilates all elements of $AC_H$. Therefore, every $\phi$-fiber defined by $Z'$ on $\Gamma$ intersects $\Gamma_{C_H}$.

With these assumptions in place, every flow induced on $\Gamma_{C_H}$ by the elements of $AC_H$ can be restricted to a flow on $\Gamma_{Z'}$.

Lemma 8 For a linear relational gauge section with respect to $Z$ such that $Z' \cup \{C_H\}$ algebraically generates $A$, any residual gauge flow on $\Gamma_{Z'}$ is uniquely determined by the derivation $\tilde{D}_H$ induced by $C_H$ on $Z'$, via $\tilde{D}_HF = \frac{1}{m}[F, C_H]$. 

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Proof: First, we note that for $F, B \in A$

$$[F, BC_H] = B(ih\bar{D}_H F) + [F, B]C_H$$

Iterating by replacing $B$ with $BC_H$, we get for any integer $n \geq 1$,

$$[F, BC_H^n] = \sum_{i=1}^{n} \binom{n}{i} (-1)^{i-1} B((ih)^i \bar{D}_H^i F)C_H^{n-i} + [F, B]C_H^n = (-1)^{n-1} B((ih)^n \bar{D}_H^n F) + GC_H.$$  

We have combined the terms proportional to $C_H$ using

$$G = \sum_{i=1}^{n-1} \binom{n}{i} (-1)^{i-1} B((ih)^i \bar{D}_H^i F)C_H^{n-i-1} + [F, B]C_H^{n-1}.$$

They all vanish when evaluated on states in $\Gamma_{C_H}$. Using this and writing $A \in A$ as a polynomial in $C_H$ as in equation 6, we have

$$[F, AC_H] = \sum_{n=1}^{M+1} (-1)^{n-1} B_{n-1}((ih)^n \bar{D}_H^n F) + GC_H$$

for some $G \in A$ and $B_i \in Z'$. For any state $\omega \in \Gamma_{C_H}$, we have $\omega(GC_H) = 0$, and hence the flows induced by constraint elements $AC_H$ satisfy

$$i\hbar \frac{d}{d\lambda} \omega_\lambda(F) = \omega_\lambda([F, AC_H]) = \sum_{n=1}^{M+1} (-1)^{n-1} \omega_\lambda \left( B_{n-1}((ih)^n \bar{D}_H^n F) \right),$$

Since for a linear section if $F \in Z'$ then $\bar{D}_H^n F \in Z'$, the above expression can be computed entirely through states restricted to $Z'$, and the action of $\bar{D}_H$ on $Z'$.

Linear sections are not always incisive or transversal. Consider the space $\Gamma_{Z'|_{\pi_t}}$ of gauge-fixed states in $\Gamma_{A_0} = \Gamma_{Z'}$, which, according to Definition 9, contains precisely those states on $Z'$ that vanish on ker $\eta = \ker \pi_t$. A gauge flow that preserves the values assigned by a state to ker $\pi_t$ is tangent to $\Gamma_{Z'|_{\pi_t}}$ and therefore remains unfixed by this gauge section, which is then not transversal.

Lemma 9 If a linear relational section with respect to $Z$ of the gauge flow of $C_H \in A$, where $Z' \cup \{C_H\}$ generates $A$, is transversal, then $\bar{D}_H Z \notin (Z - t1)Z'$.

Proof: Suppose a constraint element $AC_H$, where $A \in A$, generates a gauge flow on $\Gamma_{C_H}$ that is restricted to $\bar{\omega}_\lambda \in \Gamma_{Z'}$. Suppose further that this gauge flow intersects $\Gamma_{Z'|_{\pi_t}}$ at $\bar{\omega} = \bar{\omega}_{\lambda_0}$. We determine whether the flow generated by $AC_H$ preserves the zero values that the elements of $\Gamma_{Z'|_{\pi_t}}$ assign to ker $\pi_t$. This is the case if $d\bar{\omega}_\lambda((Z - t1)F)/d\lambda|_{\lambda=\lambda_0} = 0$ holds for all $F \in Z'$. We can evaluate this derivative by using equation 7

$$i\hbar \frac{d}{d\lambda} \bar{\omega}_\lambda((Z - t1)F)\bigg|_{\lambda=\lambda_0} = \bar{\omega} \left( \sum_{n=1}^{M+1} (-1)^{n-1}(ih)^n B_{n-1} \bar{D}_H^n ((Z - t1)F) \right).$$

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If \( \bar{D}_H^n Z \in (Z - t1)Z' \) for \( 0 < n \leq M \), then also \( \bar{D}_H^{M+1} Z \in (Z - t1)Z' \). Since, for \( n \leq M \), we can write \( \bar{D}_H^n Z = (Z - t1)F_n \), for some \( F_n \in Z' \) it follows that

\[
\bar{D}_H^{M+1} Z = \bar{D}_H(\bar{D}_H^M Z) = \bar{D}_H ((Z - t1)F_M) = (\bar{D}_H Z)F_M + (Z - t1)\bar{D}_H F_M
\]

\[
= (Z - t1) (F_1F_M + \bar{D}_H F_M) .
\]

Thus \( \bar{D}_H^n Z \in (Z - t1)Z' \) for all \( n > 0 \). This further implies that \( \bar{D}_H ((Z - t1)G) \in (Z - t1)Z' \) for all \( G \in Z' \), since

\[
\bar{D}_H^n ((Z - t1)G) = \sum_{i=0}^{n} \binom{n}{i} (\bar{D}_H^i (Z - t1)) (\bar{D}_H^{n-i} G)
\]

\[
= (Z - t1) (\bar{D}_H^n G) + \sum_{i=1}^{n} \binom{n}{i} (\bar{D}_H^i Z) (\bar{D}_H^{n-i} G)
\]

\[
= (Z - t1) \left[ (\bar{D}_H^n G) + \sum_{i=1}^{n} \binom{n}{i} F_i (\bar{D}_H^{n-i} G) \right] .
\]

Substituting this into \( \emptyset \), shows that every term in the sum on the right-hand side is an element of \( \ker \pi_t \). Therefore \( d\omega (\emptyset (Z - t1)F) /d\lambda = 0 \) for all \( F \in Z' \) everywhere on \( \Gamma_{Z'|\pi_t} \).

\( \square \)

Let us now examine the flow generated by \( C_H \) itself, so that \( B_0 = 1 \) and \( B_n = 0 \) for \( n > 1 \) in \( \emptyset \), and consider \( F = 1 \). In this situation equation \( \emptyset \) implies \( d\omega (\emptyset (Z - t1)) /d\lambda = 0 \) if \( \bar{\omega}(\bar{D}_H Z) = 0 \). So long as \( \bar{D}_H Z \notin (Z - t1)Z' \), this condition cannot hold for all states on \( \Gamma_{Z'|\pi_t} \), and the flow generated by \( C_H \) is indeed fixed on some subset of the gauge-fixed states. However, in order to ensure that this gauge section fixes the flow throughout \( \Gamma_{Z'|\pi_t} \), we need a stronger condition. Clearly \( \bar{\omega}(\bar{D}_H Z) \neq 0 \) for all \( \bar{\omega} \in \Gamma_{Z'|\pi_t} \) is a sufficient condition to fix the one flow generated by \( C_H \) everywhere on \( \Gamma_{Z'|\pi_t} \). This condition is satisfied if \( \bar{D}_H Z \) is in the coset of the identity element, defined by the ideal \( (Z - t1)Z' \), since we can write

\[
\bar{D}_H Z = a1 + (Z - t1)G \tag{9}
\]

for some non-zero \( a \in \mathbb{C} \) and \( G \in Z' \). From \( \emptyset \) it immediately follows that for any \( \bar{\omega} \in \Gamma_{Z'|\pi_t} \), and for any \( F \in Z' \)

\[
\bar{\omega} (\bar{D}_H Z F) = \bar{\omega} ((a1 + (Z - t1)G) F) = \bar{\omega} (aF) + \bar{\omega} ((Z - t1)GF) = a \bar{\omega} (F) . \tag{10}
\]

In fact, demanding that \( \bar{\omega} (\bar{D}_H Z F) = a \bar{\omega} (F) \) for some non-zero \( a \in \mathbb{C} \) and all \( F \in Z' \) is a sufficiently strong condition to ensure the existence of a transversal section of the gauge flow generated by any \( AC_H \).
In order to show this, we start by establishing a useful result for a pair of integers $i \leq n$
\[
\bar{D}_H^i (Z - t1)^n = \bar{D}_H^{i-1} \left( n(Z - t1)^{n-1}(\bar{D}_H Z) \right) = \bar{D}_H^{i-2} \left( n(n-1)(Z - t1)^{n-2}(\bar{D}_H Z)^2 + n(Z - t1)^{n-1}(\bar{D}_H^2 Z) \right) = \ldots = (Z - t1)^{n-i} \binom{n}{i} (\bar{D}_H Z)^i + (Z - t1)^{n-i+1} \left( \ldots \right).
\]

Thus, for $\bar{\omega} \in \Gamma_{Z'|\pi_t}$, and for any $F_1, F_2 \in Z'$
\[
\bar{\omega} \left[ F_1 \left( \bar{D}_H^i (Z - t1)^n \right) F_2 \right] = \begin{cases} 0 & \text{for } i < n \\ \bar{\omega} \left( (\bar{D}_H Z)^n F_1 F_2 \right) & \text{for } i = n \end{cases}
\]

(11)

if we have a linear section, such that $\bar{D}_H Z$ commutes with elements of $Z'$.

**Definition 18** A linear section $(Z', Z'/(Z - t1)Z', \pi_t)$ is of constant rate $a \in \mathbb{R}$ if $\bar{\omega} \left( (\bar{D}_H Z) F \right) = a \bar{\omega}(F)$ for all $F \in Z'$ and $\bar{\omega} \in \Gamma_{Z'|\pi_t}$.

**Lemma 10** A linear section of constant rate, where $Z' \cup \{C_H\}$ generates $A$, is transversal.

**Proof:** Consider the flow generated by $AC_H$. Using (6)
\[
A = B_0 + B_1 C_H + B_2 C_H^2 + \ldots B_M C_H^M
\]

with $B_i \in Z'$. Suppose this flow is not fixed at $\bar{\omega}$. Then, according to (8), for any $F \in Z'$
\[
0 = i\hbar \frac{d}{d\lambda} \bar{\omega}_{\lambda} \left( (Z - t1)^{M+1} F \right) \bigg|_{\lambda = \lambda_0} = \bar{\omega} \left( \sum_{n=1}^{M+1} (-1)^{n-1}(i\hbar)^n B_{n-1} \bar{D}_H^n ((Z - t1)^{M+1} F) \right)
\]
\[
= \sum_{n=1}^{M+1} (-1)^{n-1}(i\hbar)^n B_{n-1} \sum_{m=1}^{n} \binom{n}{m} \left( \bar{D}_H^m (Z - t1)^{M+1} \right) \left( \bar{D}_H^{n-m} F \right)
\]
\[
= \sum_{n=1}^{M+1} (-1)^{n-1}(i\hbar)^n \sum_{m=1}^{n} \binom{n}{m} \delta_{m,M+1} \bar{\omega} \left( B_{n-1} \left( \bar{D}_H^m (Z - t1)^{M+1} \right) (\bar{D}_H^{n-m} F) \right)
\]
\[
= \sum_{n=1}^{M+1} (-1)^{n-1}(i\hbar)^n \sum_{m=1}^{n} \binom{n}{m} \delta_{m,M+1} \bar{\omega} \left( B_{n-1} (\bar{D}_H Z)^{M+1} \left( \bar{D}_H^{n-m} F \right) \right)
\]
\[
= (-1)^M (i\hbar)^{M+1} \bar{\omega} \left( B_M (\bar{D}_H Z)^{M+1} F \right) = (-1)^M (i\hbar a)^{M+1} \bar{\omega} (B_M F)
\]

(12)

where the Kronecker delta comes directly from (11). This implies that $\bar{\omega} (B_M F) = 0$ for all $F \in Z'$. Iterating the argument, (8) also implies
\[ 0 = \left. i\hbar \frac{d}{d\lambda} \bar{\omega}_\lambda \right|_{\lambda=\lambda_0} \]

\[ = \bar{\omega} \left( \sum_{n=1}^{M+1} (-1)^n (i\hbar)^n \mathcal{B}_{n-1} \bar{D}_H^n \left((Z-t\mathbf{1})^M F\right) \right) \]

\[ = \bar{\omega} \left( \sum_{n=1}^{M} (-1)^n (i\hbar)^n \mathcal{B}_{n-1} \bar{D}_H^n \left((Z-t\mathbf{1})^M F\right) \right) \]

\[ + (-1)^M (i\hbar)^M \bar{\omega} \left( \mathcal{B}_M \left[ \bar{D}_H^{M+1} \left((Z-t\mathbf{1})^M F\right) \right] \right). \]

By (12), the second term in the final expression is zero for any \( F \in Z' \), giving

\[ 0 = \bar{\omega} \left( \sum_{n=1}^{M} (-1)^n (i\hbar)^n \mathcal{B}_{n-1} \bar{D}_H^n \left((Z-t\mathbf{1})^M F\right) \right) \]

\[ = \sum_{n=1}^{M} (-1)^n (i\hbar)^{n-1} \sum_{m=1}^{n} \binom{n}{m} \bar{\omega} \left( \mathcal{B}_{n-1} \left( \bar{D}_H^m (Z-t\mathbf{1})^M \left( \bar{D}_H^{n-m} F \right) \right) \right) \]

\[ = (-1)^{M-1} (i\hbar)^M \bar{\omega} \left( \mathcal{B}_{M-1} \left( \bar{D}_H Z \right)^M F \right) = (-1)^{M-1} (i\hbar)^M \bar{\omega} \left( \mathcal{B}_{M-1} F \right) , \]

which implies \( \bar{\omega} \left( \mathcal{B}_{M-1} F \right) = 0 \) for all \( F \in Z' \). Continuing in this way, we establish that \( \bar{\omega} \left( \mathcal{B}_{n} F \right) = 0 \) for all \( n \). Therefore the flow must completely vanish at \( \bar{\omega} \) since

\[ i\hbar \left. \frac{d}{d\lambda} \bar{\omega}_\lambda \right|_{\lambda=\lambda_0} = \sum_{n=1}^{M+1} (-1)^n (i\hbar)^n \bar{\omega} \left( \mathcal{B}_{n-1} \left( \bar{D}_H^n F \right) \right) = 0 . \]

□

We see that relational sections of this type fix all of the gauge degrees of freedom at least locally. Furthermore, a relational section possesses a natural relational interpretation. Any kinematical observable \( A \) that commutes with the gauge-fixing variable \( Z \) is in \( Z' \) and can be projected to the quotient \( Z'/(Z-t\mathbf{1})Z' \). The resulting element \( \pi_t(A) \) (which is \(*\)-invariant as long as \( A^* = A \)) can be interpreted as “\( A \) when \( Z = t \).

**Definition 19** For any \( A \in \mathcal{A} \) that commutes with the gauge-fixing element \( Z \), the relational observable for \( A \) when the value of \( Z \) is \( t \in \mathbb{R} \) is the image of \( A \) under the canonical projection, \( A_{Z=t} := \pi_t(A) \).

**Lemma 11** Each state on the full collection of relational observables associated with a relational section \( (Z', Z'/(Z-t\mathbf{1})Z', \pi_t) \), specifies a unique gauge orbit on \( \Gamma_{C_V} \).

**Proof:** A state on the full set of relational observables evaluated at a fixed value of \( Z \) is a state on the quotient algebra \( Z'/(Z-t)Z' \cong \{ A_{Z=t} : A \in Z' \} \). Such a state pulls back to a unique state \( \bar{\omega} \in \Gamma_{Z'}/\pi_t \subset \Gamma_{Z'} \), which in turn defines a fiber \( \phi^{-1}(\bar{\omega}) \subset \Gamma \) within
the space of kinematical states. By Lemma 7, \( \phi^{-1}(\bar{\omega}) \) intersects \( \Gamma_{CH} \) at exactly one state, which belongs to exactly one gauge orbit, as claimed.

Within a transversal section, unless it is also incisive, a gauge orbit on \( \Gamma_C \) sampled by the relational section may be represented by a countable number of states on relational observables rather than just one. This could lead to the analogue of the Gribov problem, as already mentioned earlier. However, it constitutes an ambiguity of physical states only if there is more than one state that is positive on \( \mathcal{O} = Z'(Z - t_1)Z' \) in the intersection of the relational section and a gauge orbit. In Section 2.4 under an additional assumption that \( CH \) is not a divisor of zero in \( \mathcal{A} \) we prove that the gauge orbit intersecting the fiber of a state \( \bar{\omega} \in \Gamma_{Z'|\pi_t} \) that is positive on \( Z' \), is also positive on \( \mathcal{A}_{obs} \) (see Proposition 4), and therefore corresponds to a physical state. Any two such states are related by the flow of a \( * \)-compatible derivation on \( Z' \). From equation (8) (together with (6)), we see that the only gauge flow that, when projected onto \( \Gamma_{Z'} \), is generated by a \( * \)-compatible derivation, and therefore preserves positivity, is the one generated by \( CH \) itself. For this particular flow we note the following result.

**Lemma 12** A linear section of constant rate is incisive with respect to the flow generated by \( CH \).

**Proof:** If the section \( (Z', Z'/((Z - t_1)Z', \pi_t) \) is linear of constant rate \( a \), we have \( \bar{\omega}(\bar{D}_HZ) = d\bar{\omega}(Z)/d\lambda = a \neq 0 \) for any \( \bar{\omega}_\lambda \in \Gamma_{Z'|\pi_t} \). Therefore, \( \bar{\omega}_\lambda(Z) \) is monotonic in \( \lambda \) along the gauge orbit of \( \bar{\omega} \), and any two gauge-related states have different values \( \bar{\omega}(Z - t_1) \). Since \( Z - t_1 \in \text{ker} \pi_t \), it is not possible that two states along the flow of \( CH \) are contained in the same \( \Gamma_{Z'|\pi_t} \).

**Remark:** This Lemma is our equivalent to the consistency of standard deparameterized quantization on a fixed Hilbert-space representation of \( \mathcal{O} \). Because our treatment is algebraic, its results apply to all possible representations of the deparameterized system, while the standard treatment is restricted to a single representation and has no complete analogue to our definition of an incisive section. For a given deparameterizable system, the existence of a standard quantization on a single Hilbert space therefore does not imply the existence of an incisive section on the corresponding algebra.

As an analog to Proposition 1 we summarize our results as

**Proposition 2** Given \( CH \) that is not a divisor of zero in \( \mathcal{A} \) and a linear relational section of the corresponding gauge \( (Z', Z'/((Z - t_1)Z', \pi_t) \) of constant rate, such that \( Z' \cup \{CH\} \) generates \( \mathcal{A} \), the full collection of relational observables, taken at a fixed value of \( Z \), uniquely specifies a physical state.

**Example:** Let \( \mathcal{A} \) be the polynomial algebra, generated by complex polynomials in the
basic elements $Z, E, A_i$, with $i = 1, 2, \ldots, M$, and $1$ ($=: A_0$). The generating elements are star-invariant, $Z = Z^*$, $E = E^*$ and $A_i = A_i^*$, and are subject to commutation relations $[Z, E] = i\hbar 1, [Z, A_i] = 0 = [E, A_i]$ and $[A_i, A_j] = i\hbar \sum_{k=0}^{M} \alpha_{ijk} A_k$ for some $\alpha_{ijk} \in \mathbb{C}$. For any $M$-tuple of integers $\vec{n} = (n_1, n_2, \ldots, n_M)$, we define $A_{\vec{n}} = A_1^{n_1} A_2^{n_2} \ldots A_M^{n_M}$, with $A_0 := 1$. The set of monomials $\{A_{\vec{n}} Z^m E^l\}$ is linearly independent and forms a linear basis on $A$.

Let this system be subject to a single constraint of the form $C_H = E + h(Z, A_i)$, where $h$ is a polynomial with an ordering such that $h = h^*$ and therefore $C_H = C_H^*$. Consider the gauge section $(Z', Z'/ (Z - t1) Z', \pi_t)$. Any element of $Z'$ can be written as a linear combination of monomials of the form $A_{\vec{n}} Z^m$, while the expression for any non-zero element of $AC_H$ in terms of the basis monomials includes at least one term of the form $A_{\vec{n}} Z^m E^l$ with $l \neq 0$. Therefore $Z' \cap AC_H = \{0\}$. By substituting $E = C_H - h(Z, A_i)$, we can write any element of $A$ as a polynomial in $Z$, $C_H$, and $A_i$. Using the commutation relations, a factor of $C_H$ can be iteratively moved to the right whenever present, so that any $A \in A$ can be written as

$$A = p_0(Z, A_i) + p(Z, C_H, A_i) C_H,$$

for some polynomials $p_0$ and $p$. The first term is in $Z'$, while the second is in $AC_H$, the two sets therefore linearly generate the whole of $A$, and the section is expansive. It is also immediately clear here that $Z' \cup \{C_H\}$ algebraically generates $A$. Iteratively moving every factor of $C_H$ to the right we can write

$$A = p_1(Z, A_i) + p_2(Z, A_i) C_H + \ldots + p_N(Z, A_i) C_H^N.$$

Furthermore, any $B \in Z'$ can be written as a polynomial $p(Z, A_i)$, so that

$$[B, C_H] = [B, E] + [B, h] = i\hbar \frac{\partial p(s, A_i)}{\partial s} \bigg|_{s=Z} + [p(Z, A_i), h(Z, A_i)],$$

both terms in the final expression are in $Z'$. Therefore, for every $t \in \mathbb{R}$, $(Z', Z'/ (Z - t1) Z', \pi_t)$ with the canonical projection $\pi_t$ is linear. For any $\omega \in \Gamma_{Z'}|_{\pi_t}$ and any $F \in Z'$,

$$\omega \left( (\tilde{D}_H Z) F \right) = \frac{1}{i\hbar} \omega ([Z, C_H] F) = \frac{1}{i\hbar} \omega ([Z, E] F) = \omega (F)$$

and the section is of constant rate. According to Lemma [10], the section is transversal. The relational observables here are given by the projection $\pi_t(A) \in O$ for any $A \in Z'$, interpreted as “$A$ when $Z = t$.” For a basis monomial,

$$A_{\vec{n}} Z^m = A_{\vec{n}} ((Z - t1) + t1)^m = A_{\vec{n}} \sum_{k=0}^{m} \binom{m}{k} (Z - t1)^k t^{m-k} = A_{\vec{n}} t^m + A_{\vec{n}} \sum_{k=1}^{m} \binom{m}{k} (Z - t1)^k t^{m-k}.$$
The last sum lies in the ideal \((Z - t1)Z'\), and, therefore, in the coset of the zero element; hence,

\[
[A_{\tilde{n}}Z^m] = [A_{\tilde{n}}t^m] = t^m[A_{\tilde{n}}]
\]

Therefore, “\(A_{\tilde{n}}Z^m\) when \(Z = t\)” is \(t^m[A_{\tilde{n}}]\).

In this example, \(A_{\text{obs}} = C_H'\) contains \(C_H\) itself and the identity element \(1\). Any element of \(A_{\text{obs}}\) which is not a linear combination of a power of \(C_H\) and \(1\) is a constant of motion of the (possibly time-dependent) Hamiltonian \(h\). For most choices of a classical polynomial \(h_{\text{class}}\), constants of motion which are \(E\)-independent and fulfill \(\{O, h_{\text{class}}\} = 0\), are generically non-polynomial, if they even exist in closed form \([17, 18]\). (The system may be non-integrable.) No quantization of such an observable can exist in our \(A\), and the available \(A_{\text{obs}}\) is incomplete in the sense of Definition 15. Even if one extends \(A\), for instance by using deformation quantization, in most cases of physical interest it is impossible to find a complete set of Dirac observables. Nevertheless, we have shown that it is possible to fix the gauge relative to \(Z\) in any such system and uniquely specify physical states by relational observables, with the only requirement on \(h\) that \(h \in Z'\) and \(h^* = h\).

### 2.4 The dynamical relational flow

Suppose that \((Z', Z'/\{Z - t1\}Z', \pi_t\)) is a linear relational section for some \(t_1 \in \mathbb{R}\). The conditions of a linear relational section do not depend on \(t\), and therefore \((Z', Z'/\{Z - t1\}Z', \pi_t\), \(t \in \mathbb{R}\), is a 1-parameter family of linear relational sections.

**Definition 20** Given a physical state \([\omega]_{C_H} \in \Gamma_{\text{phys}}\), its history in \(\Gamma_{Z'}\) relative to a relational section \((Z', Z'/\{Z - t1\}Z', \pi_t)\) is given by the 1-parameter family

\[
I_t([\omega]_{C_H}, Z) = \phi([\omega]_{C_H}) \cap \Gamma_{Z'}|_{\pi_t} \subset \Gamma_{Z'}
\]

of intersections between the gauge orbit corresponding to \([\omega]_{C_H}\) on \(\Gamma_{Z'}\) and the subspaces \(\Gamma_{Z'}|_{\pi_t}\) of gauge-fixed states.

**Definition 21** A one-parameter family of states \(\bar{\omega}_t \in \Gamma_{Z'}\), is a time evolution relative to \(Z\) if there exists a physical state \([\omega]_{C_H} \in \Gamma_{\text{phys}}\), such that \(\bar{\omega}_t \in I_t([\omega]_{C_H}, Z)\) for each \(t\).

**Lemma 13** If \((Z', Z'/\{Z - t1\}Z', \pi_t)\) is a relational section with respect to \(C_H\), such that \([Z, C_H] = iha1\) for some \(a \in \mathbb{R}\), then the flow generated by \(C_H\) on any state, projected to \(\Gamma_{Z'}\), is a time evolution.

**Proof:** If \([Z, C_H] = iha1\), then \(\tilde{Z} = Z/a\) defines a linear relational section at constant rate 1, such that \([\tilde{Z}, C_H] = ih1\). Hence, without loss of generality we will use the standard commutation relation in what follows. Let \(S_{C_H}(\lambda)\) denote the flow induced on \(\Gamma\) by the adjoint action of \(C_H\) on \(A\), which by Lemma 8 preserves the constraint surface \(\Gamma_{C_H}\). This
flow maps to a flow on $\Gamma_{Z'}$ via the pullback map $\bar{S}_{C_H}(\lambda)\bar{\phi}(\omega) = \phi(S_{C_H}(\lambda)\omega)$. Using this relation and Definition 17 it follows that for any $\bar{\omega} \in \Gamma_{Z'}$ and any $A \in Z'$

$$i\hbar \frac{d}{d\lambda}(\bar{S}_{C_H}(\lambda)\bar{\omega}(A)) = \bar{S}_{C_H}(\lambda)\bar{\omega}([A,C_H]) \ . \quad (13)$$

Therefore, the projected flow is generated by the adjoint action of $C_H$, this time on the subalgebra $Z'$. Since $S_{C_H}(\lambda)$ flows along gauge orbits, it follows that $\bar{S}_{C_H}(\lambda)$ flows along the projections of the gauge orbits on $\Gamma_{Z'}$. Thus, if $\bar{\omega}$ is in the orbit of a physical state $[\omega]_{C_H}$, then so is $\bar{S}_{C_H}(\lambda)\bar{\omega}$; they correspond to two points on the history $I_t([\omega]_{C_H}, Z)$ defined by this physical state on $\Gamma_{Z'}$. All that remains to be shown is that the gauge flow $\bar{S}_{C_H}(\lambda)$ maps states from $\Gamma_{Z'|_{\pi_t}}$ to $\Gamma_{Z'|_{\pi_{t+\lambda}}}$.

For convenience let us denote the one-parameter family of states $\bar{\omega}_\lambda := \bar{S}_{C_H}(\lambda)\bar{\omega}$, where $\bar{\omega} \in \Gamma_{Z'|_{\pi_t}}$ for some fixed $t$. Following the method of Lemma 2, for each $A \in Z'$ we define a function that varies along the flow $f_A(\lambda) = \bar{\omega}_\lambda((Z - (t + \lambda)1)A)$. The state $\bar{\omega}_\lambda$ belongs to $\Gamma_{Z'|_{\pi_{t+\lambda}}}$ if and only if $f_A(\lambda) = \bar{\omega}_\lambda((Z - (t + \lambda)1)A) = 0$ for all $A \in Z'$. Suppose all of the functions $f_A(\lambda') = 0$ for some $\lambda'$. We can compute their derivatives along the flow using equation (13):

$$\left.\frac{df_A}{d\lambda}\right|_{\lambda=\lambda'} = \frac{d}{d\lambda}(\bar{\omega}_\lambda((Z - (t + \lambda)1)A)) = \frac{d}{d\lambda}(\bar{\omega}_\lambda((Z - t1)A) - \lambda'\bar{\omega}_\lambda(A))$$

$$= \frac{1}{i\hbar}\bar{\omega}_\lambda'([Z-t1,A,C_H]) - \frac{\lambda'}{i\hbar}\bar{\omega}_\lambda([A,C_H]) - \bar{\omega}_\lambda(A)$$

$$= \frac{1}{i\hbar}\bar{\omega}_\lambda'([Z-t1,A,C_H]) - \frac{\lambda'}{i\hbar}\bar{\omega}_\lambda([A,C_H])$$

$$= \frac{1}{i\hbar}\bar{\omega}_\lambda'((Z - (t + \lambda')1)A) = \frac{1}{i\hbar}f_A([A,C_H])(\lambda')$$

The last equality follows since $[A,C_H] \in Z'$ by Definition 17. Furthermore, by our initial conditions $f_A(0) = 0$ for all $A \in \mathcal{A}$ since $\bar{\omega}_0 = \bar{\omega} \in \Gamma_{Z'|_{\pi_{t+0}}}$. It follows that $\{f_A(\lambda) = 0, \forall\lambda\}_{A \in \mathcal{A}}$ is the solution to the flow given by equation (13) with $\bar{\omega} \in \Gamma_{Z'|_{\pi_t}}$. Thus at any point along the flow

$$\bar{\omega}_\lambda((Z - (t + \lambda)1)A) = 0, \ \text{for all} \ A \in Z' \ .$$

Therefore $\bar{S}_{C_H}(\lambda)\bar{\omega} \in \Gamma_{Z'|_{\pi_{t+\lambda}}}$, as required.

The flow $\bar{S}_{C_H}(\lambda)$ provides a one-to-one invertible map from $I_t([\omega]_{C_H}, Z)$ to $I_{t+\lambda}([\omega]_{C_H}, Z)$ via time evolution curves. In the case where $I_t([\omega]_{C_H}, Z)$ contains a finite or a countable number of points this leads to a history consisting of a finite or countable number of distinct time evolution curves.
Corollary 1 If \((Z', Z'/(Z - t1)Z'), \pi_t\) is a relational section with respect to \(C_H\), such that 
\([Z, C_H] = \text{iha1} \) for some \(a \in \mathbb{R}\) and \(Z' \cup \{C_H\}\) algebraically generates \(\mathcal{A}\), then each history

\(I_t(\omega|_{C_H}, Z)\) consists of a countable number of time evolution curves.

In particular, the number of points in \(I_t(\omega|_{C_H}, Z)\) remains the same for all values of \(t\). For example, if the section \((Z', Z'/(Z - t1)Z'), \pi_t\) is incisive at one value of \(t\), then each history consists of precisely one time evolution curve, and the section remains incisive for all \(t\).

Not every state \(\tilde{\omega}\) on the algebra \(Z'\) has a relational interpretation since this would require \(\tilde{\omega}_t((Z - t1)A) = 0\) for any \(A \in Z'\) for some \(t \in \mathbb{R}\). On the other hand, while any state on a quotient \(Z'/(Z - t1)Z'\) does have a relational interpretation, these quotients give distinct algebras for different values of \(t\). In order to define time-dependent expectation values \(\tilde{\omega}_t(F)\) that can be freely specified for a fixed quantity \(F\), we introduce an additional structure.

Definition 22 A \(Z\)-fashionable algebra of a family \((Z', Z'/(Z - t1)Z'), \pi_t\) of relational sections in \(\mathcal{A}\) is a unital *-subalgebra \(\mathcal{F} \subset Z'\), such that for all \(t \in \mathbb{R}\), we have \(\mathcal{F} \cap \ker \pi_t = \{0\}\) and \(\pi_t(\mathcal{F}) = Z'/(Z - t1)Z'\).

The two conditions guarantee that \(\pi_t\) restricted to \(\mathcal{F}\) is a *-algebra isomorphism. The algebra of fashionables is therefore a realization of a family of quotient algebras \(Z'/(Z - t1)Z'\) as a single subalgebra of \(Z'\) (and hence of \(\mathcal{A}\)). We denote the *-isomorphism \(\nu_t : Z'/(Z - t1)Z' \to \mathcal{F}\), where for any \(X \in Z'/(Z - t1)Z'\),

\[\nu_t(X) := \pi_t^{-1}(X) \cap \mathcal{F}\]

yields a single element of \(\mathcal{F}\). This isomorphism inverts \(\pi_t\) when the latter is restricted to \(\mathcal{F}\), so that \(\pi_t \circ \nu_t = \text{id}\). As a direct consequence, we note

Lemma 14 For every \(t \in \mathbb{R}\), \(\mathcal{F} + (Z - t1)Z' = Z'\).

Additionally, for each value of \(t\) we have a projection from \(Z'\) to its subalgebra \(\mathcal{F}\) via the composition of *-homomorphisms \(\nu_t \circ \pi_t\), which can be used to drag the *-compatible derivation generated by the adjoint action of \(C_H\) from \(Z'\) to \(\mathcal{F}\). Defining \(\bar{D}_H := \frac{1}{\hbar}[A, C_H]\), \(\bar{D}_H\) is a derivation on \(Z'\) if the relational section is linear. We define

\[\bar{D}'_H(t)F := \nu_t \circ \pi_t\left(\bar{D}_H F\right), \quad (14)\]

which is a *-compatible derivation on \(\mathcal{F}\) thanks to the fact that \(\nu_t\) and \(\pi_t\) are *-homomorphisms.

The fashionable algebra gives us the structure necessary to define time translation of a state from \(Z = t_1\) to \(Z = t_2\): \(\tilde{\omega}_1 \in \Gamma_{Z'}|_{\pi_{t_1}}\) and \(\tilde{\omega}_2 \in \Gamma_{Z'}|_{\pi_{t_2}}\) represent the same unevolved state at two different times \(t_1\) and \(t_2\) if \(\tilde{\omega}_1(F) = \tilde{\omega}_2(F)\) for all \(F \in \mathcal{F}\).

Lemma 15 If \(\omega|_{C_H}\) is a physical state with respect to \(C_H \in \mathcal{A}\) and, for all \(t\) in some range \((a, b) \subset \mathbb{R}\), \((Z', Z'/(Z - t1)Z'), \pi_t\) is an incisive linear relational section of the flow of \(C_H\) with a \(Z\)-fashionable algebra \(\mathcal{F}\), such that the history \(I_t(\omega|_{C_H}, Z)\) is non-zero, then \(I_t(\omega|_{C_H}, Z)\) is equivalent to a state \(\tilde{\omega}_t\) on \(\mathcal{F}\) that evolves according to

\[\frac{d}{dt} \tilde{\omega}_t(F) = \tilde{\omega}_t(\bar{D}'_H(t)F)\]. \quad (15)
Proof: For each \( t \in (a, b) \), there is a \( \bar{\omega}_t \in \Gamma_{Z'} \) such that \( I_t([\omega]_{C_H}, Z) = \{ \bar{\omega}_t \} \) because the section at \( t \) is incisive. The 1-parameter family \( \bar{\omega}_t \), by inclusion \( \iota: \mathcal{F} \hookrightarrow Z' \), is also a 1-parameter family of states \( \bar{\omega}_t \) on \( \mathcal{F} \), where \( \bar{\omega}_t(F) := \bar{\omega}_t(F) \) for any \( F \in \mathcal{F} \). In fact, the 1-parameter family \( \bar{\omega}_t \) is generated precisely by the flow associated with the derivation \( \vec{D}'_H(t)F \) defined above since, using equation (13), for any \( F \in \mathcal{F} \)

\[
\frac{d}{dt} \bar{\omega}_t(F) = \frac{d}{dt} \bar{\omega}_t(F) = \bar{\omega}_t(\vec{D}_H F) .
\]

Because \( \bar{\omega}_t \in \Gamma_{Z'|\pi_t} \), it assigns the same value to all elements that belong to a given coset generated by the ideal \( (Z - t1)Z' \). By definition in equation (14), for any value of \( t \), \( \vec{D}'_H(t)F \) and \( \vec{D}_H F \) are in the same coset relative to the ideal \( (Z - t1)Z' \). Hence \( \bar{\omega}_t(\vec{D}_H F) = \bar{\omega}_t(\vec{D}'_H(t)F) \), from which (15) follows. \( \square \)

Remark: If the section is not incisive, but merely transversal, there is a countable number of evolving states, \( \bar{\omega}_t^{(i)} \) with \( i \in \mathcal{I} \subset \mathbb{Z} \).

**Definition 23** A relational section of \( C_H \) with respect to \( Z \) is a deparameterization of \( C_H \) if \([Z, C_H] = i\hbar 1, Z' \cup \{C_H\} \) generates \( \mathcal{A} \), and \( Z' \) possesses a fashionable subalgebra \( \mathcal{F} \).

**Example:** In our previous example of a linear relational section, the algebra generated by the basis elements \( \{A_{\vec{n}}\} \) is a fashionable algebra \( \mathcal{F} \). Moreover, since \( [A_{\vec{n}}, E] = 0 \), for any \( F \in \mathcal{F} \) we have

\[
\vec{D}_H F = \frac{1}{i\hbar} [F, C_H] = \frac{1}{i\hbar} [F, h(\mathbb{Z}, A_{\vec{n}})] .
\]

The projection from \( Z' \) to \( \mathcal{F} \) here has the form

\[\nu_t \circ \pi_t (A_{\vec{n}} Z^m) = \nu_t ([A_{\vec{n}} Z^m]) = A_{\vec{n}} t^m ,\]

so that the commutator of two basis elements projects as

\[\nu_t \circ \pi_t ([A_{\vec{n}_1}, A_{\vec{n}_2} Z^m]) = [A_{\vec{n}_1}, A_{\vec{n}_2}] t^m .\]

It follows that

\[
\vec{D}'_H(t) F = \frac{1}{i\hbar} [F, h(t, A_{\vec{n}})] .
\]

In this example, therefore, any history with respect to the relational section associated with \( Z \) can be cast as time evolution of a quantum system with degrees of freedom generated by the basis \( \{A_{\vec{n}}\} \), driven by the time-dependent Hamiltonian \( h(t, A_{\vec{n}}) \).

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Combining Definition 17 and Lemma 14, deparameterization is accomplished by finding a local clock \(Z\) such that \([Z, C_H] = i\hbar \mathbf{1}\), and by splitting the kinematical algebra into subalgebras that share only the null element,

\[
A = AC_H + (Z - t\mathbf{1})Z' + \mathcal{F},
\]

where \(\mathcal{F}\) is a \(\ast\)-subalgebra of \(Z'\) isomorphic to \(Z'/((Z - t\mathbf{1})Z')\) at each \(t\). If these conditions are satisfied, the section \(\pi_t: Z' \rightarrow Z'/((Z - t\mathbf{1})Z')\) is transversal and maximally resolved at each \(t\).

Specifically, for each \(t\) the orbit \([\omega]_{C_H}\) of a physical state contains a countable number of representative states \(\omega_{i}^{(t)}\), \(i \in \mathcal{I} \subset \mathbb{Z}\), each of which agrees with a state \(\bar{\omega}_{i}^{(t)}\) on \(Z'\) such that \(I_t([\omega]_{C_H}, Z) = \{\bar{\omega}_{i}^{(t)}\}_{i \in \mathcal{I}}\). Therefore, for any \(B \in Z'\)

\[
\omega_{i}^{(t)}((Z - t\mathbf{1})B) = \bar{\omega}_{i}^{(t)}((Z - t\mathbf{1})B) = 0
\]

because \(\bar{\omega}_{i}^{(t)} \in \Gamma_{Z'}|_{\pi_t}\). This, in turn, implies that

\[
\omega_{i}^{(t)}(Z) = t \quad \text{and} \quad \omega_{i}^{(t)}(ZB) = t \omega_{i}^{(t)}(B) = \omega_{i}^{(t)}(Z) \omega_{i}^{(t)}(B).
\]

We find that this relation extends beyond \(Z'\) to the whole of \(A\): Given any \(A \in A\), and any state \(\omega \in \Gamma_{C_H}\), that satisfies \(\omega(ZB) = \omega(Z) \omega(B)\) for all \(B \in Z'\), there are \(B \in Z'\) and \(G \in A\) such that

\[
\omega(ZA) = \omega(ZB + ZGC_H) = \omega(ZB) = \omega(Z)\omega(B) = \omega(Z)\omega(B + GC_H) = \omega(Z)\omega(A).
\]

In order to interpret each \(\omega_{i}^{(t)}\) as a physical state on the fashionable algebra \(\mathcal{F}\) at the instant \(t\) of internal clock \(Z\), we need \(\omega_{i}^{(t)}\) to be positive when restricted to \(\mathcal{F}\). Thanks to Lemma 2 and the fact that \(C_H\) defines a \(\ast\)-compatible derivation on \(\mathcal{F}\), the whole history \(\omega_{i}^{(t)}\) is positive on \(\mathcal{F}\) if it is positive at one time.

**Proposition 3** Given a deparameterization with respect to \(Z\), a gauge-fixed state which is positive on the fashionable algebra \(\mathcal{F}\) is positive on \(Z'\).

**Proof:** According to Lemma 14, for any element \(B \in Z'\) there are \(F \in \mathcal{F} \subset Z'\) and \(B_0 \in Z'\) such that \(B = F + (Z - t\mathbf{1})B_0\). Now consider a state \(\omega\) that is positive on \(\mathcal{F}\), satisfying \(\omega(Z) = t\) and \(\omega(ZB) = \omega(Z)\omega(B)\). We have

\[
\omega(BB^*) = \omega\left(FF^* + F(Z - t\mathbf{1})B_0 + (Z - t\mathbf{1})B_0F^* + (Z - t\mathbf{1})B_0(Z - t\mathbf{1})B_0^*\right)
= \omega(FF^*) + (\omega(Z) - t)\left(\omega(FB_0^*) + \omega(B_0F^*) + \omega(B_0(Z - t\mathbf{1})B_0^*)\right)
= \omega(FF^*) \geq 0.
\]

The following definition is therefore meaningful:
Definition 24 A state $\omega \in \Gamma$ is almost-positive with respect to a deparameterization of $C_H$ by $Z$ if

1. it annihilates the left ideal generated by $C_H$: $\omega(AC_H) = 0$ for all $A \in \mathcal{A}$;
2. it is positive on the commutant of $Z$: $\omega(BB^*) \geq 0$ for all $B \in Z'$;
3. it parameterizes left multiplication by $Z$: for all $A \in \mathcal{A}$, $\omega(ZA) = \omega(Z)\omega(A)$.

If $\omega \in \Gamma$ is almost-positive, the positive state $\omega|_F$ is called a relational physical state.

The first condition ensures that $\omega \in \Gamma_{C_H}$ solves the constraint. The second condition ensures that $\omega$ is in the $\phi$-fiber induced by some positive state on $\mathcal{F}$. The third condition ensures that the pullback of $\omega$ to $\Gamma_{Z'}$ belongs to the subspace $\Gamma_{Z'}|_{\pi_\omega(Z)}$ of gauge-fixed states.

In other words, $\phi(\omega)$ represents a positive relational physical state on $\mathcal{F}$ at time $\omega(Z)$.

Corollary 2 Every relational physical state has a unique extension to an almost-positive state.

Remark: One way to interpret the last condition in Definition 24 is to notice that it requires fluctuations of $Z$ to vanish. For example, we have $(\Delta_\omega Z)^2 := \omega(Z^2) - \omega(Z)^2 = 0$. Just like a time parameter in ordinary quantum mechanics, $Z$ is sharply defined in such a state, but it does not correspond to an evolving observable since $Z$ is not an element of $\mathcal{F}$. Note that the combination of almost-positivity and conditions required for deparameterization prevent $\omega$ from being positive on the whole of $\mathcal{A}$. For example, full positivity would require $\omega(ZC_H + C_HZ) \in \mathbb{R}$ because $C_H^* = C_H$ and $Z^* = Z$, but our constraint and gauge conditions stipulate that

$$\omega(ZC_H + C_HZ) = \omega(2ZC_H - i\hbar 1) = 2\omega(ZC_H) - i\hbar \omega(1) = -i\hbar ,$$

which is purely imaginary. The new notion of an almost-positive state, introduced here, may therefore be considered a maximal implementation of positivity in an internal-time setting, in which evolution is defined with respect to an algebra element. According to Proposition 3, positivity of states can be extended from observables (realized here by $\mathcal{F}$) to time ($Z \in Z'$), but not to the full algebra $\mathcal{A}$.

According to Lemma 13, the gauge flow $S_{C_H}(\lambda)$ generated by $C_H$ and projected to $\Gamma_{Z'}$ drags states from the subspace $\Gamma_{Z'}|_{\pi_\omega}$ to the subspace $\Gamma_{Z'}|_{\pi_{\omega+\lambda}}$ of gauge-fixed states.

Lemma 16 The gauge flow $S_{C_H}$ on $\Gamma$ drags an almost-positive state $\omega$ to another almost-positive state $S_{C_H}(\lambda)\omega$, such that $S_{C_H}(\lambda)\omega(Z) = \omega(Z) + \lambda$.

Proof: Lemma 4 guarantees that the flow remains on $\Gamma_{C_H}$. Since the adjoint action of $C_H$ is a $*$-compatible derivation on $Z'$, according to Lemma 2 the corresponding flow maps
states that are positive on \( Z' \) to other states that are positive on \( Z' \). For any almost-positive \( \omega \), along the flow \( \omega_\lambda := S_{CH}(\lambda)\omega \) we have
\[
\frac{d}{d\lambda} \omega_\lambda(Z) = \omega_\lambda(D_H Z) = 1.
\]
Therefore, \( \omega_\lambda(Z) = \omega(Z) + \lambda \).

To prove that parameterization of \( Z \) is preserved along the flow, we follow the method of Lemma \([2]\) and define a function \( f_A(\lambda) = \omega_\lambda(Z\bar{\lambda}A) - \omega_\lambda(Z)\omega_\lambda(A) \) for each \( A \in \mathcal{A} \). Condition 3 of Definition \([24]\) holds for \( \omega \) if and only if \( f_A(\lambda) = 0 \) for all \( A \in \mathcal{A} \). Suppose all of the functions \( f_A(\lambda') = 0 \) for some \( \lambda' \). Taking an arbitrary \( A \in \mathcal{A} \)
\[
\frac{d}{d\lambda} \omega_\lambda(ZA) \bigg|_{\lambda=\lambda'} = \left( \omega_\lambda \left( Z\bar{\lambda}A \right) + \omega_\lambda(A) \right) \bigg|_{\lambda=\lambda'} - \omega_\lambda(A) \Bigg[ \omega_\lambda(Z) + \lambda \Bigg] = \omega_\lambda(Z) \frac{d}{d\lambda} \omega_\lambda(A) \bigg|_{\lambda=\lambda'} \cdot
\]
Consequently, \( df_A(\lambda)/d\lambda = 0 \) at \( \lambda = \lambda' \) for all \( f_A(\lambda) \). Since \( f_A(0) = \omega(ZA) - \omega(Z)\omega(A) = 0 \) for all \( A \in \mathcal{A} \), it follows that \( \{ f_A(\lambda) = 0, \forall \lambda \} \in \mathcal{A} \) is the solution to the flow generated by \( C_H \). Hence \( \omega_\lambda(ZA) = \omega_\lambda(Z)\omega_\lambda(A) \) remains true everywhere along the flow. \( \square \)

Since \( \mathcal{A}_{\text{obs}} \) is not available in general, there is no full quantum analog of Proposition \([1]\). But any available Dirac observable \( O \in \mathcal{A}_{\text{obs}} \) is a valid observable with respect to any almost-positive state, under a mild condition on \( C_H \):

**Proposition 4** If \( O \in \mathcal{A}_{\text{obs}} \) and \( C_H \) is not a divisor of zero in \( \mathcal{A} \), \( \omega(OO^*) \geq 0 \) for any almost-positive functional \( \omega \) with respect to a deparameterization of \( C_H \) by some \( Z \in \mathcal{A} \).

**Proof:** Since \( O \in \mathcal{A}_{\text{obs}} \subset \mathcal{A} \) is also an element of \( \mathcal{A} \), the decomposition of Lemma \([14]\) induced by deparameterization implies that we can write it as \( O = AC_H + B \) for some \( A \in \mathcal{A} \) and \( B \in Z' \). The fact that \( O \) is in the commutant of \( C_H \) implies \( [O,C_H] = [B,C_H] + [A,C_H]C_H = 0 \). The first term on the left-hand side is in \( Z' \) by Definition \([17]\) the second term is in \( AC_H \). Since the two subalgebras are linearly independent, the two terms must vanish separately, implying that \( [B,C_H] = 0 \). Since \( C_H \) is not a divisor of zero, for \( A \in \mathcal{A} \) we have that \( AC_H = 0 \) implies \( A = 0 \), and therefore \( [A,C_H] = 0 \). These results also imply that \( A^* \) and \( B^* \) commute with \( C_H \). Now suppose that \( \omega \) is almost-positive with respect to deparameterization of \( C_H \) by \( Z \), then
\[
\omega(OO^*) = \omega((AC_HC_HA^* + BC_HA^* + AC_HB^* + BB^*)) = \omega((AA^*C_H + BA^* + AB^*)C_H) + \omega(BB^*) = \omega(BB^*) \geq 0
\]
because \( B \in Z' \), using \( \omega(AC) = 0 \). \( \square \)
3 General polynomial constraints

A general constraint element $C$ is not immediately of the form required for a linear section at constant rate, or a corresponding deparameterization, to exist. For instance, most Hamiltonian constraints that are of interest to quantum gravity and quantum cosmology are quadratic in momenta, resulting in $[Z, [Z, C_H]] \neq 0$ for a clock $Z$ which violates the condition of a linear section in Definition 17. There are also examples of constraints for which $[Z, C_H] \in Z'$ allows a linear section but not at constant rate because $[Z, C_H] \neq i\hbar a\mathbf{1}$ is not a multiple of the unit.

3.1 Linearization and cancellation

In some of these cases, it may be possible to “linearize” the constraint by finding a suitable $C_H \in A$ which satisfies all four criteria of a deparameterization with respect to $Z$ and has a gauge flow and a constraint surface related to those of $C$: If $N \in A$ is such that $C = NC_H$ and $C_H$ as in Definition 17, we have $A_C \subset A_{C_H}$ and therefore $\Gamma_{C_H} \subset \Gamma_C$. Moreover, $\omega \sim C \psi$ if $\omega \sim C_H \psi$. If $N$ is invertible in $A$, $\Gamma_{C_H} = \Gamma_C$ and $\sim C_H = \sim C$, but this case is too restrictive for most practical purposes.

**Example:** The constraint $C = E^2 - h(A_i)^2$ with $Z$-independent $h = h^*$ can be factorized as $C = (E-h)(E+h) = NC_H$ with $N = E-h$ and $C_H = E+h$. We have $[N, C_H] = 0$, but $N$ does not have an inverse. However, if $\omega \in \Gamma_{C_H}$ and $\omega(E) \neq 0$ it follows that $\omega(N) \neq 0$, in which case it may be of interest to study evolution of the state with respect to $C_H$. More generally, we define $C_\pm = E \pm h(A_i)$ such that

$$C = C_+ C_- = C_- C_+ .$$

Since either $\omega(AC_+) = 0$ or $\omega(AC_-) = 0$ also imply $\omega(AC) = 0$, every left solution of $C_\pm$ is also a left solution of $C$. Therefore both constraint surfaces $\Gamma_{C_\pm}$ are contained within the constraint surface $\Gamma_C$. Furthermore, normalized combinations of states from $\Gamma_{C_+}$ and $\Gamma_{C_-}$ also give us solutions to $C$. In particular, for any $a_i^{(+)}$, $a_i^{(-)} \in \mathbb{C}$, $\omega_i^{(+)} \in \Gamma_{C+}$, and $\omega_i^{(-)} \in \Gamma_{C-}$

$$\omega = \sum_i a_i^{(+)} \omega_i^{(+)} + \sum_i a_i^{(-)} \omega_i^{(-)}$$

is a left solution of $C$, which is normalized provided that $\left( \sum_i a_i^{(+)} + \sum_i a_i^{(-)} \right) = 1$.

In this example the two subsets $\Gamma_{C_\pm}$ are not disjoint. A solution to both constraint factors must satisfy $\omega(AC_+) = 0$ and $\omega(AC_-) = 0$ for any $A \in A$. These conditions are equivalent to requiring that both $\omega(AE) = 0$ and $\omega(Ah) = 0$ for all $A \in A$, since

$$\omega(AE) = \omega\left( A\frac{1}{2}(C_+ + C_-) \right) = \frac{1}{2} (\omega(AC_+) + \omega(AC_-)) = 0 ,$$

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and similarly
\[ \omega(Ah) = \omega \left( A \frac{1}{2} (C_+ - C_-) \right) = \frac{1}{2} \left( \omega(AC_+) - \omega(AC_-) \right) = 0. \]

Conversely, \( \omega(AE) = 0 \) and \( \omega(Ah) = 0 \) imply \( \omega(AC_+) = 0 \) and \( \omega(AC_-) = 0 \). The only restriction on the values assigned by a kinematical state \( \omega \in \Gamma \) is normalization \( \omega(1) = 1 \). It is therefore possible to satisfy both \( \omega(AE) = 0 \) and \( \omega(Ah) = 0 \) for all \( A \), unless \( AE + Bh = 1 \) for some \( A, B \in A \). No such \( A \) and \( B \) exist within our polynomial \( A \), hence the intersection \( \Gamma_{C_+} \cap \Gamma_{C_-} \) is non-empty. However if we consider only the states that are positive on \( Z' \), as required by Definition 24, there may be additional restrictions: suppose \( h = FF^* + \epsilon_01 \) for some \( F \in F \) and real \( \epsilon_0 > 0 \). Then, for any normalized state that is positive on \( F \)
\[ \omega(h) = \omega(FF^*) + \epsilon_0 \geq \epsilon_0 > 0, \]
which means \( \omega(h) = 0 \) cannot be satisfied. Hence depending on \( h \) the sets of almost-positive states with respect to internal clock \( Z \) defined by the two constraint factors may be disjoint.

Using the original constraint \( C \), the orbits are generated by the subalgebra \( AC \), as opposed to \( AC_\pm \) if we use one of the factors instead. Neither of the two factors has an inverse already contained within \( A \) (the only element with an explicit inverse in \( A \) here is \( 1 \)). Thus \( AC \) is a proper subset of \( AC_\pm \), and hence the original orbits of \( C \) are contained within the larger orbits of \( C_\pm \). This guarantees, via Lemma 4, that a physical state of the original constraint \( C \) is either entirely inside \( \Gamma_{C_+} \) or entirely outside of it. However, some gauge flows generated by the factor constraints \( C_\pm \) are not gauge orbits of \( C \) and can potentially link distinct gauge orbits of the original constraint \( C \). Therefore, a physical state with respect to \( C_\pm \) generally corresponds to a region of the space of physical states with respect to the original constraint \( C \).

This complication would not arise if \( N \) had an inverse in \( A \). However, even if \( N \) is non-invertible there are in general some states on which its action can be “reversed” in the following sense.

**Definition 25** Left multiplication of \( A \in A \) can be canceled in \( \omega \in \Gamma \) if for any \( B \in A \), \( \omega(GAB) = 0 \) for all \( G \in A \) implies \( \omega(GB) = 0 \) for all \( G \in A \).

This state-by-state condition differs from the algebraic cancellation property. In our concrete example, only the zero element is a divisor of zero in \( A \). However, even though \( CB = 0 \) implies \( B = 0 \), left multiplication by \( C \) cannot be canceled in any of its left solutions \( \omega \in \Gamma_C \). Setting \( B = 1 \), we get \( \omega(GC1) = \omega(GC) = 0 \) for all \( G \in A \), which is not equivalent to \( \omega(G1) = 0 \) for all \( G \in A \), since setting \( G = 1 \) would violate normalization.

**Definition 26** A constraint \( C \) is deparameterized by factorization with respect to an internal clock \( Z \), if there are \( N, C_H \in A \), such that (i) \( C = NC_H \), (ii) there is at least one state \( \omega \in \Gamma_{C_H} \) in which left multiplication by \( N \) can be canceled, and (iii) \( C_H = C_H^* \) is deparameterized by \( Z \).
In our concrete example, if we deparameterize our system using $C_+$ as the constraint, we consider only the states $\omega \in \Gamma_{C_+}$ in which the left multiplication of $C_-$ can be canceled. (In particular, this means that $\omega \notin \Gamma_{C_-}$.)

**Lemma 17** For a constraint that is deparameterized by factorization as $C = NC_H$, for any $A \in A_{\text{obs}}$ of $C$ and $\omega \in \Gamma_{C_H}$ such that left multiplication by $N$ can be canceled in $\omega$, the value $\omega(A)$ is invariant along all of the gauge flows generated by $C_H$.

**Proof:** We first observe that, since $[A, C] = 0$, in particular $\omega(B[A, C]) = 0$ for any $B \in A$. Which means

$$0 = \omega(B[A, NC_H]) = \omega(BN[A, C_H]) + \omega(B[A, N]C_H) = \omega(BN[A, C_H]) .$$

Since this holds for any $B$, cancellation of left multiplication by $N$ in $\omega$ implies that $\omega(B[A, C_H]) = 0$, for all $B \in A$.

The above property holds along all of the gauge flows generated by $C_H$. To see this let us fix an arbitrary $G \in A$ and, following the method of Lemma 12, define functions $f_B(\lambda) = S_{GC_H}(\lambda)\omega(B[A, C_H])$ for each $B \in A$. Clearly, $f_B(0) = 0$ for all $B \in A$. Suppose all functions $f_B(\lambda') = 0$ for some $\lambda'$, then

$$i\hbar \frac{df_B}{d\lambda} \bigg|_{\lambda=\lambda'} \quad = \quad i\hbar \frac{d}{d\lambda} (S_{GC_H}(\lambda)\omega(B[A, C_H])) \bigg|_{\lambda=\lambda'}$$

$$\quad = \quad S_{GC_H}(\lambda')\omega(B[A, C_H]GC_H - GC_HB[A, C_H])$$

$$\quad = \quad S_{GC_H}(\lambda')\omega(B[A, C_H]GC_H) - f_{GC_H,B}(\lambda^\prime) = 0 ,$$

where we used the fact that $S_{GC_H}(\lambda')\omega \in \Gamma_{C_H}$ according to Lemma 11. We see that \{\(f_B(\lambda) = 0, \forall \lambda\}\} is the solution of the dynamical flow generated by $GC_H$ that agrees with our initial conditions. Since $G$ was arbitrary, $S_{GC_H}(\lambda)\omega(B[A, C_H]) = 0$ for all $B, G \in A$ and $\lambda \in \mathbb{R}$.

Using the above result, the value of $\omega(A)$ along the gauge flow generated by $BC_H$, with arbitrary $B \in A$, varies according to

$$i\hbar \frac{d}{d\lambda} (S_{BC_H}(\lambda)\omega(A)) \quad = \quad S_{BC_H}(\lambda)\omega([A, BC_H])$$

$$\quad = \quad S_{BC_H}(\lambda)\omega(B[A, C_H]) + S_{BC_H}(\lambda)\omega([A, B]C_H) = 0 , \text{ for all } \lambda .$$

Therefore, using gauge flows generated by $C_H$ does not affect the values assigned to the set of Dirac observables of the original constraint $C$, so long as we use states on which left multiplication of the factor $N$ can be canceled. In this section’s example, the roles of $C_+$ and $C_-$ can be reversed since the two factors commute.
In principle this construction also applies to a constraint that can be written as a product of non-commuting factors, as one would expect in the case of time-dependent Hamiltonians $h(A_i, Z)$. However, factorizing such a constraint is much more complicated.

Example: If we factorize a constraint of the form $C = E^2 - H^2$ with $[E, H] \neq 0$, we have $C \neq (E - H)(E + H)$, but we can try to find $X \in Z'$ such that $C = (E - H + X)(E + H - X)$. Multiplying the two factors, we have

$$C = E^2 - H^2 - X^2 + [E, H] - [E, X] + [X, H] + 2HX$$

provided that

$$2HX = [H, E] + [H, X] + [E, X] + X^2.$$

This equation has a formal power-series solution $X = \sum_{n=1}^{\infty} (ih)^n X_n$ with

$$2HX_1 = \frac{[H, E]}{ih}$$

and

$$2HX_n = \frac{[H, X_{n-1}]}{ih} + \frac{[E, X_{n-1}]}{ih} + \sum_{a=1}^{n-1} X_{n-a} X_a.$$

We can split $X = \frac{1}{2}(X_+ + X_-)$ into its $*$-invariant and anti-$*$-invariant contributions, $X_+ = \frac{1}{2}(X + X^*)$ and $X_- = \frac{1}{2}(X - X^*)$, and define

$$H' = H - X_+ \quad \text{and} \quad E' = E - X_-. $$

As in the example with commuting factors, $H'^* = H'$ and $[Z, E'] = [Z, E] = i\hbar \mathbf{1}$ but $E'^* \neq E'$. There are therefore almost-positive states, but the gauge flow of $C_H = E' + H'$ does not induce a $*$-compatible derivation, unless it so happens that $X_- = 0$.

For a systematic analysis of suitable factorizations, we need to carefully consider the adjointness conditions imposed on the factors of the constraint.

### 3.2 Adjointness relations

The adjointness relation $C^* = C$ imposed on constraints guarantees that $A_{\text{obs}} \subset A$ inherits a $*$-relation, which in turn makes it possible to define physical states as positive linear functionals on $A_{\text{obs}}$. This condition also restricts possible factorization choices that could be applied to linearize constraints. Suppose a constraint $C = C^*$ can be written as $C = NC_H$, where $N$ can be algebraically canceled within $A$ and $C_H = C_H^*$ allows a deparameterization with respect to $Z^* = Z \in A$. Then $C$ can be deparameterized with respect to $Z$ by factorization, using the same method as we applied to $C = C_- C_+$ to cast a subset of its physical states as dynamical evolution in $Z$. Under these conditions, $C_H$ uniformizes the flow generated by $C$: Since $[Z, C] = i\hbar N$, we may consider $N$ as the “non-constant rate” of evolution determined by $C$, while evolution with respect to $C_H$ has constant rate.
In order to satisfy $C = C^*$ we need $NC_H = C_H N^*$, which can be rewritten as

$$[N, C_H] = C_H (N^* - N)$$.

If the non-constant rate is required to be real when evaluated in a positive state $\omega$, we need $N^* = N$. In this case, (16) implies $[N, C_H] = 0$, such that the rate is, in fact, constant on solutions of the constraint because $N$ is a constant of motion with respect to $C_H$. Conversely, if $[N, C_H] = 0$ we obtain $(N - N^*)C_H = 0$, and if $C_H$ can be algebraically canceled within $\mathcal{A}$, we get $N = N^*$. These cases constitute two sufficient conditions for factorization to result in a deparameterization.

Provided that the clock is part of a canonical pair, $[Z, E] = i\hbar 1$, as in the example from the previous subsection, the most general form of a factorizable constraint is $C = N(E + H)$, where $H = H^*$ commutes with $Z$, and condition (16) holds for $C_H = E + H$. Further properties depend on the $E$-dependence of $C$.

### 3.2.1 Non-relativistic constraints

**Definition 27** A constraint $C \in \mathcal{A}$ is non-relativistic of rate $N \in \mathcal{A}$ if there is a canonical generator $E \in \mathcal{A}$ conjugate to $Z \in \mathcal{A}$, $[Z, E] = i\hbar 1$, such that $[Z, C] = i\hbar N \in Z'$.

**Definition 28** A non-relativistic constraint $C \in \mathcal{A}$ is of constant flow rate $N \in \mathcal{A}$ if there is a $C_H \in \mathcal{A}$ such that $C = NC_H$ and $[N, C_H] = 0$.

**Lemma 18** Every deparameterizable non-relativistic constraint is of constant flow rate.

**Proof:** Since $C^* = C$ and $Z^* = Z$ imply $[Z, C]^* = -[Z, C]$, we immediately obtain $N^* = N$ from $N = [Z, C]/(i\hbar)$. Using this in (16), we have $[N, C_H] = 0$. 

**Remark:** The condition $[N, C_H] = 0$ of constant flow rate shows the restrictive nature of adjointness conditions: Only constants of motion with respect to $C_H$ are allowed as factors of $E$ in non-relativistic constraints. Written as $[N, E] = -[N, H]$ if $C_H = E + H$, the condition amounts to a partial differential equation for $N$ as a function of $Z$ and the remaining canonical variables.

**Lemma 19** If a non-relativistic constraint $C$ is deparameterizable, it is of the form $C = A_1 E + A_0$ such that $A_1 = A_1^*$ and $[A_0, A_1] = A_1[A_1, E]$.

**Proof:** Since the constraint is non-relativistic, it is linear in $E$ and can be written as $C = A_1 E + A_0$ with $A_1$ and $A_0$ such that $[Z, A_1] = [Z, A_0] = 0$. The conditions $C = C^*$ and $Z = Z^*$ imply that $[Z, C]^* = -[Z, C]$, and thus $A_1 = A_1^*$. Since $A_1$ plays the role of the factor $N$, it must be a left factor of $A_0$: There must exist $H \in \mathcal{A}$ such that $A_0 = A_1 H$ and

$$C = A_1 (E + H)$$.
In order for $C$ to be deparameterizable, according to Lemma \[18\] the flow rate $A_1 = N$ must be constant with respect to $C_H = E + H$. Therefore, $[A_1, E] = -[A_1, H]$, which, upon left multiplication with $A_1$, implies $A_1[A_1, E] = -[A_1, A_0]$ because $A_0 = A_1H$.

**Remark:** The condition $H = H^*$, obtained from $C_H = C_H$ for a deparameterizable constraint, implies

$$A_0^* = HA_1 = A_0 + [A_1, H].$$

Therefore, $A_0$ in $C = A_1E + A_0$ is not self-adjoint unless $A_1$ commutes with $H$.

**Remark:** If, in spite of Lemma \[18\], we try to factorize a constraint of non-constant flow rate, we end up with a non-self-adjoint $C_H$. To see this, consider a non-relativistic constraint of the form $C = \frac{1}{2}(B_1E + EB_1) + B_0$ with $B_0 = B_0^*$ and invertible $B_1 = B_1^*$, we can write

$$C = B_1E + B_0 - \frac{1}{2}[B_1, E] = B_1 \left( E + \frac{1}{2}(B_1^{-1}B_0 + B_0B_1^{-1}) + \frac{1}{2}[B_1^{-1}, B_0] - \frac{1}{2}B_1^{-1}[B_1, E] \right).$$

Defining $N = B_1$,

$$H = \frac{1}{2}(B_1^{-1}B_0 + B_0B_1^{-1})$$

and

$$E' = E + \frac{1}{2}[B_1^{-1}, B_0] - \frac{1}{2}B_1^{-1}[B_1, E],$$

we can write $C = NC_H$ with $C_H = E' + H$. It follows that $H = H^*$, and $[Z, E'] = [Z, E] = i\hbar$ since $[Z, B_1] = 0$ and $[Z, B_0] = 0$ for a non-relativistic constraint. However,

$$E'' = E - \frac{1}{2}[B_1^{-1}, B_0] + \frac{1}{2}[B_1, E]B_1^{-1} \neq E'$$

and therefore $C_H^* \neq C_H$. For $\omega \in \Gamma_{C_H}$, we have $\omega(E') = -\omega(H) \in \mathbb{R}$. If $\omega$ is almost-positive, this equation is consistent even though $E' \neq E''$ while $H^* = H$: because $E' \notin Z'$, an almost-positive state may take on a real value in a non-self-adjoint $E'$. However, the gauge flow of $C_H \neq C_H^*$ does not induce a $*$-preserving derivation on any fashionable algebra $F \subset Z'$ because, in general, $[f, E'] \neq 0$ for $f \in F$ unless $B_0$ and $B_1$ are multiples of the unit.

### 3.3 Relativistic constraints

**Definition 29** A constraint $C$ is relativistic if there is a canonical generator $E$ conjugate to $Z$, $[Z, E] = i\hbar 1$, such that $0 \neq [Z, [Z, C]] \in Z'$.

A relativistic constraint that is deparameterizable by factorization has the form

$$C = (N_1E + N_0)(E + H) = N_1E^2 + (N_0 + N_1H)E + N_1[E, H] + N_0H$$
where $H^* = H$, and $[Z, N_1] = [Z, N_0] = [Z, H] = 0$. Using $C^* = C$ and $Z = Z^*$, we have

$$(\frac{1}{i\hbar} [Z, \frac{1}{i\hbar} [Z, C]])^* = (\frac{1}{i\hbar} [Z, \frac{1}{i\hbar} [Z, C]])^*,$$

which quickly yields $N_1 = N_1^*$. The flow rate of $C$ with respect to $C_H = E + H$ is given by $N_1 E + N_0$, such that $C = NC_H$. In contrast to linear or relativistic constraints, the flow rate depends on $E$.

**Lemma 20** If a relativistic constraint $C$ that is deparameterizable by factorization is of constant real flow rate $N$, it is of the form $C = NC_H$ with $N = N_1 E + N_0$ and $C_H = E + H$ such that

$$N_0^* = N_0 + [N_1, E],$$

$$[N_1, E] + [N_1, H] = 0,$$  

and

$$N_1 [H, E] = [N_0, E] + [N_0, H].$$

**Proof:** For real flow rate, $N = N^*$ implies $N^* = EN_1 + N_0^* = N$ and therefore (17). Constant flow rate, $[N, C_H] = 0$, results in

$$0 = ([N_1, E] + [N_1, H]) E + N_1 [E, H] + [N_0, E] + [N_0, H].$$

(20)

Taking a commutator with $Z$ on both sides, only the term proportional to $E$ survives giving us (18). Substituting this back into (20) results in (19).

The three conditions of Lemma 20 together are sufficient to make the quadratic constraint deparameterizable by factorization.

**Lemma 21** A relativistic constraint with real constant flow rate $N = E + N_0$ is of the form $C = \tilde{E}^2 - h$ such that $\tilde{E}^* = \tilde{E}$ and $h^* = h$ as well as $[Z, \tilde{E}] = i\hbar 1$ and $[Z, h] = 0$.

**Proof:** A relativistic constraint with flow rate $N = E + N_0$, using $N_1 = 1$ in terms Lemma 20, can be written as $C = E^2 + A_1 E + A_0$, where $[A_i, Z] = 0$. Using the notation of Lemma 20

$$A_1 = N_0 + H$$ and $$A_0 = [E, H] + N_0 H.$$

(21)

We have $A_1^* = A_1$ because $N_0^* = N_0$ from equation (17). Equation (18) is trivially satisfied, while (19) becomes

$$[H, E] = [N_0, E] + [N_0, H].$$

(22)

We rewrite

$$C = (E^2 + \frac{1}{2} (A_1 E + EA_1) + \frac{1}{4} A_1^2) - \frac{1}{4} A_1^2 + \frac{1}{2} [A_1, E] + A_0$$

$$= (E + \frac{1}{2} A_1)^2 - \left(\frac{1}{4} A_1^2 - \frac{1}{2} [A_1, E] - A_0\right)$$

$$= \tilde{E}^2 - h.$$
setting \( h = \frac{1}{4}A_1^2 - \frac{1}{2}[A_1, E] - A_0 \) and \( \tilde{E} = E + \frac{1}{2}A_1 \). Using (21) and (22), we compute

\[
  h = \frac{1}{4}A_1^2 - \frac{1}{2}[N_0, E] + \frac{1}{2}[H, E] - N_0 H
\]

\[
  = \frac{1}{4}A_1^2 - \frac{1}{2}[N_0, E] + \frac{1}{2}([N_0, E] + [N_0, H]) - N_0 H
\]

\[
  = \frac{1}{4}A_1^2 - \frac{1}{2}(N_0 H + H N_0)
\]

such that \([Z, h] = 0\). By inspection, \( h^* = h \) as well as \( \tilde{E}^* = \tilde{E} \). Moreover, since \([Z, A_1] = 0\), we have \([Z, \tilde{E}] = [Z, E] = i\hbar 1\).

\[\Box\]

**Lemma 22** A relativistic constraint of the form \( C = (E + g)^2 - h \), such that \( g^* = g \), \( h^* = h \), and \([Z, g] = [Z, h] = 0\), is deparameterizable by factorization only if \([E + g, h] = 0\).

**Proof:** The factorized version of such a constraint must be of the form

\[
  C = (E + N_0)(E + H) = E^2 + (N_0 + H)E + [E, H] + N_0 H ,
\]

which we compare with

\[
  (E + g)^2 - h = E^2 + g^2 + 2gE + [E, g] - h .
\]

Taking a commutator with \( Z \) and equating the two expressions yields \( g = \frac{1}{2}(N_0 + H) \).

Using this result to eliminate \( g \) and setting the two expressions equal gives

\[
  [E, H] + N_0 H = \frac{1}{4}(N_0 + H)^2 + \frac{1}{2}[E, N_0] + \frac{1}{2}[E, H] - h .
\]

This expression can be rearranged to solve for \( h \) in terms of \( H, N_0 \), and their commutators with \( E \)

\[
  h = \frac{1}{2}([H, E] + [E, N_0]) + \frac{1}{2} (N_0^2 + H^2 + 2N_0H - [N_0, H]) - N_0 H .
\]

We combine the first two terms using equation (19) (with \( N_1 = 1 \)):

\[
  h = \frac{1}{2}N_0 H + \frac{1}{4}N_0^2 + \frac{1}{4}H^2 - \frac{1}{2}N_0 H - \frac{1}{4}[N_0, H]
\]

\[
  = \frac{1}{4} (N_0^2 + H^2 - 2N_0H + [N_0, H]) = \frac{1}{4} (N_0 - H)^2 .
\]

Now consider the commutator

\[
  [E + g, N_0 - H] = [E + \frac{1}{2}(N_0 + H), N_0 - H]
\]

\[
  = [H, E] + [E, N_0] - \frac{1}{2}[N_0, H] + \frac{1}{2}[H, N_0]
\]

\[
  = [N_0, H] + [H, N_0] = 0 ,
\]

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where in the final equality we once again used (19). This result immediately implies
\[ [E + g, h] = [\tilde{E}, \frac{1}{4}(N_0 - H)^2] = 0 \]
as a necessary condition for our constraint to be deparameterizable by factorization.

Example: We assume that \( h = \sqrt{h^2} \) has a square root \( \sqrt{h} = \sqrt{h}^* \) in \( \mathcal{A} \). Comparison of the two constraint forms results in
\[ g = \frac{1}{2}(N_0 + H) , \quad \text{and} \quad \sqrt{h} = \frac{1}{2}(N_0 - H) . \]
The factorizability condition (19) now gives
\[ 2[E, \sqrt{h}] = [\sqrt{h}, g] - [g, \sqrt{h}] , \]
or
\[ i\hbar \frac{\partial \sqrt{h}}{\partial Z} = [\sqrt{h}, g] . \tag{23} \]

For example, in a two-component system with canonical generators \([Z, E] = [q, p] = i\hbar 1\), setting
\[ \sqrt{h} = p + \frac{1}{2}(q^2 - Z^2) , \quad \text{and} \quad g = Z(q - Z) , \]
satisfies equation (23) and leads to the factorization
\[ C = (E + Z(q - Z))^2 - \left( p + \frac{1}{2}(q^2 - Z^2) \right)^2 \]
\[ = \left( E + p + \frac{1}{2}q^2 - \frac{3}{2}Z^2 + qZ \right) \left( E - \left( p + \frac{1}{2}(q - Z)^2 \right) \right) . \]

As this example demonstrates, in general a constraint \( C \in \mathcal{A} \) has to be of a specific form in order for a deparameterization and therefore evolution with respect to a gauge section to exist. This result showcases the power of our general approach to quantum dynamical reduction. The restrictions of the type found in Lemmas 20–22 have not been anticipated by the standard method of deparameterization on a fixed Hilbert space, which treats each specific scenario individually and has mainly been applied to time-independent systems in which \( C = NC_H \), where \( N \) and \( C_H \) commute. The additional restrictions derived here are the consequence of the inclusion of time dependence from the outset, as well as the general algebraic treatment that is not tied to a specific Hilbert-space representation.

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