A general expression for quasi-local energy flux for spacetime perturbation is derived from covariant Hamiltonian formulation using functional differentiability and symplectic structure invariance, which is independent of the choice of the canonical variables and the possible boundary terms one initially puts into the Lagrangian in the diffeomorphism invariant theories. The energy flux expression depends on a displacement vector field and the 2-surface under consideration. We apply and test the expression in Vaidya spacetime. At null infinity the expression leads to the Bondi type energy flux obtained by Lindquist, Schwartz and Misner. On dynamical horizons with a particular choice of the displacement vector, it gives the area balance law obtained by Ashtekar and Krishnan.

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I. INTRODUCTION

It is well known that in General Relativity, because of equivalence principle, locally we can not detect gravity and therefore renders any unambiguous notion of local density for conserved quantities impossible. On the other hand, the notion of total conserved quantities for spatial and null infinity are well understood. Therefore, there has been hope that one should be able to find an appropriate notion of quasi-local conserved quantities for finite spacetime domains [1].

A systematic way to study conserved quantities is through Hamiltonian. In a series of papers [2, 3, 4, 5], a covariant Hamiltonian formalism was developed to obtain quasi-local energy-momentum expressions and found that the Hamiltonian boundary term also determines the boundary conditions for stationary spacetime.

In general, for a dynamical gravitating system, the gravitational energy-momentum is not conserved, but in such case we would expect a meaningful total or quasi-local energy flux expression. In [6], covariant Noether charge (Hamiltonian) formulation was first used to identify the Bondi energy flux. For trapping and dynamical horizons [7], an energy flux expression was also obtained by Ashtekar and Krishnan [8]. The formalism were then applied to trace the development of trapping horizons in black hole formation and evaporation [9]. For general spacetime regions, definitions of quasi-local energy flux was also developed using covariant Hamiltonian formalism [4]. The Bondi energy flux was derived but the energy flux at dynamical horizons was not addressed there. Much discussions using marginally trapped surfaces and dynamical horizons were given in terms of the Vaidya spacetimes [10, 11, 12].

In this paper, we shall construct explicitly a physically well defined energy flux formula for general dynamical gravitating system using the functional differentiability of the Hamiltonian and symplectic structure invariance of the theory which is independent of the choice of canonical variables and the boundary terms that one initially puts into the Lagrangian. We use spherical symmetric Vaidya spacetime as an example to test our energy flux expression [13]. The Bondi type energy flux at the null infinity and the energy flux at the dynamical horizons are also derived respectively in Schwarzschild and Painlevé-Gullstrand coordinates.

In section II, we derive our energy flux expression within covariant Hamiltonian formulation using functional differentiability and symplectic structure invariance. The important features of this expression are the independence of the choice of the canonical variables and the possible boundary terms that one initially puts into the Lagrangian for diffeomorphism invariant theories. In section III, we apply our expression to General Relativity. Examples for Vaidya spacetime to obtain the Bondi type energy flux at null infinity are then given in the section IV. In there, the area dependence and implications of the first law is also observed. In section V, we calculate the energy flux for dynamical horizons and compare with the results obtained by Ashtekar and Krishnan [8] for a particular choice of the displacement vector. We discuss the results in section VI.

II. NOETHER CONSERVED QUANTITIES AND ENERGY FLUX

In the following, we shall apply the Noether charge approach [3, 6, 14, 15, 16, 17, 18] for dynamical spacetime using self consistence of functionally differentiable Hamiltonian and invariance of the corresponding symplectic structure to define a physical energy flux. Starting with a diffeomorphism invariant first order Lagrangian 4-
form \( L = L(\phi, p) = d\phi \wedge p - \Lambda(\phi, p) \), where \( \phi \) denotes an arbitrary collection of dynamical fields, and \( p \) being the corresponding conjugate momentum. The field equations, \( \frac{\delta L}{\delta \phi} = 0, \frac{\delta L}{\delta p} = 0 \) are obtained by computing the first variation of the Lagrangian,

\[
\delta L = d(\delta \phi \wedge p) + \delta \phi \wedge \frac{\delta L}{\delta \phi} + \frac{\delta L}{\delta p} \wedge \delta p, \tag{1}
\]

where \( \delta \phi \wedge p \) is the symplectic potential 3-form. For any diffeomorphism generated by a smooth vector field \( \xi \), we can replace the variational derivative \( \delta \) by the Lie derivative \( L_\xi \)

\[
L_\xi L = d(L_\xi \phi \wedge p) + L_\xi \phi \wedge \frac{\delta L}{\delta \phi} + \frac{\delta L}{\delta p} \wedge L_\xi p, \tag{2}
\]

Using the identity \( L_\xi = i_\xi d + d i_\xi \) and replacing \( d i_\xi L \) with \( L_\xi L \), one can define a conserved Noether current 3-form (or Hamiltonian 3-form \( H(\xi) \) by

\[
H(\xi) = L_\xi \phi \wedge p - i_\xi L(\phi), \tag{3}
\]

such that by equation (2) the Noether current \( H(\xi) \) is closed \( (dH(\xi) = -L_\xi \phi \wedge \frac{\delta L}{\delta \phi} - \frac{\delta L}{\delta p} \wedge L_\xi p \approx 0) \) when the field equations are satisfied. Locally there exist a 2-form \( Q(\xi) \) (called the Noether charge) such that \( H(\xi) = dQ(\xi) + \) “field equation terms” or in general,

\[
H(\xi) = \xi^\mu H_\mu + dQ(\xi). \tag{4}
\]

Note that here \( H_\mu \) are constraints including matter fields contribution. When integrated on a 3-space \( \Sigma \), it gives a Hamiltonian

\[
H(\xi) = \int_\Sigma H(\xi) = \int_\Sigma \xi^\mu H_\mu + \oint_{\partial \Sigma} Q, \tag{5}
\]

therefore, \( Q \) can also be interpreted as the boundary term \( B = \oint_{\partial \Sigma} Q \) which defines the value of the Hamiltonian. We want to stress here, although \( H \) is called Hamiltonian in the literature, it may not be functionally differentiable to define conserved quantities along the displacement vector field \( \xi \) which generate diffeomorphism invariant transformations. Following the work of Regge and Teitelboim, we justify the functional differentiability of \( H \) as the total Hamiltonian by further varying \( H \),

\[
\delta H(\xi) = \int_\Sigma \delta H(\xi) = \int_\Sigma \delta (L_\xi \phi \wedge p) - i_\xi (\delta L(\phi)). \tag{6}
\]

Using equation (6) and dropping the field equation terms, we arrive at

\[
\delta H(\xi) = \int_\Sigma (L_\xi \phi \wedge \delta p - \delta \phi \wedge L_\xi p) + \oint_{\partial \Sigma} i_\xi (\delta \phi \wedge p), \tag{7}
\]

and the symplectic 3-form \( \Omega \) is defined by

\[
\Omega(\delta_1, \delta_2) = \int_\Sigma (\delta_1 \phi \wedge \delta_2 p - \delta_2 \phi \wedge \delta_1 p). \tag{8}
\]

There can be two possibilities. If \( \oint_{\partial \Sigma} i_\xi (\delta \phi \wedge p) = 0 \) then \( H(\xi) \) is automatically functionally differentiable, conserved along the vector field \( \xi \), i.e. \( L_\xi H(\xi) = 0 \). When one can find boundary conditions (i.e. see reference [4]) to give \( \oint_{\partial \Sigma} i_\xi (\delta \phi \wedge p) = \partial B(\xi) \), to modify \( H \) to \( \tilde{H} = H(\xi) - B(\xi) \) such that \( \tilde{H} \) is still functionally differentiable, conserved along the vector field \( \xi \), i.e. \( L_\xi \tilde{H}(\xi) = 0 \). The function \( H(\xi) \) and \( \tilde{H}(\xi) \) are called the functionally differentiable Hamiltonian conjugate to \( \xi \) [1, 2, 3, 4, 11, 12, 13].

In general dynamical gravitating systems, when space-time is non-stationary, there does not exist boundary conditions to achieve a functionally differentiable Hamiltonian to define conserved quantities. However, the replacement of \( \delta \) by \( L_\xi \) in equation (7) will still lead to the following flux expression,

\[
L_\xi H(\xi) = \oint_{\partial \Sigma} i_\xi (L_\xi \phi \wedge p). \tag{9}
\]

But this flux expression has the ambiguity on the choice of the canonical variables between \( \phi \) and \( p \). Following [6], a prescription was developed by insisting on the self consistency of functionally differentiable Hamiltonian and invariance of symplectic structure (symmetric under \( -\phi \) and \( p \) interchanges) [20].

To proceed, we perform a second variation \( \Delta \) on \( L_\xi H \) in equation (9), and identify the energy flux as in the followings. We have

\[
\Delta L_\xi H(\xi) = \oint_{\partial \Sigma} i_\xi (L_\xi \Delta \phi \wedge p + L_\xi \phi \wedge \Delta p), \tag{10}
\]

where \( \Delta L_\xi = L_\xi \Delta \) is assumed. We subtract a Lie derivative term on both sides,

\[
\Delta L_\xi H(\xi) - L_\xi \oint_{\partial \Sigma} i_\xi (\Delta \phi \wedge p) = \oint_{\partial \Sigma} i_\xi (L_\xi \phi \wedge \Delta p - \Delta \phi \wedge L_\xi p), \tag{11}
\]

and observe that the last term has a symplectic structure and therefore allows us to define a quantity \( E(\xi) \) by

\[
E(\xi) = \Delta H(\xi) - \oint_{\partial \Sigma} i_\xi (\Delta \phi \wedge p). \tag{12}
\]

Note that the above equation (12) bears the same form as equation (7), we arrive at the first main result of this work,

\[
F(\xi) = L_\xi E(\xi) = \oint_{\partial \Sigma} i_\xi (L_\xi \phi \wedge \Delta p - \Delta \phi \wedge L_\xi p). \tag{13}
\]

From the surface integral form of \( F(\xi) \), non-conserving nature and only being functionally differentiable to generate the correct dynamical evolution on the surface
$S = \partial \Sigma$, we can therefore interpret it being the total energy flux across some dynamical surface boundary $S$. To further confirm this energy flux interpretation, we observe another important feature of this flux expression, namely, the symplectic structure invariant character will automatically give the correct boundary terms, boundary conditions and a conserved Hamiltonian as discussed in [4] in the stationary cases. 

Note that our general expression $F(\xi)$ is precisely the sum of the two flux expressions of [4],

$$F(\xi) = F_{\text{Dirichlet}}(\xi) + F_{\text{Neumann}}(\xi) = F_{\text{dynamic}}(\xi) + F_{\text{constraint}}(\xi),$$

where $F_{\text{Dirichlet}}(\xi)$, $F_{\text{Neumann}}(\xi)$, $F_{\text{dynamic}}(\xi)$ and $F_{\text{constraint}}(\xi)$ are the flux expressions with certain variables being fixed on the boundary (see Appendix). For a general dynamic spacetime, we want to allow our symplectic variables to be completely dynamical without any variable being fixed on the boundary. An example is dynamical black holes which we shall discuss in section V.

We would also like to point out that it is the functional differentiability and the symplectic structure invariance which frees the energy flux expression from the ambiguities in determining the boundary terms one initially put into the Lagrangian and therefore allows us to define the energy flux expression uniquely.

III. ENERGY FLUX FROM SPACETIME PERTURBATIONS

We now apply the prescription described in the previous section to General Relativity,

$$S = \int_{\Sigma} \mathcal{L} = \frac{1}{16\pi} \int_{\Sigma} R^{ab} \wedge (\ast(\partial_a \wedge \partial_b)) + \mathcal{L}_{\text{matter}},$$

(15)

where $R^{ab} = \delta\omega^{ab} + \omega^a_c \wedge \omega^{cb}$ is the curvature 2-form constructed from the connection 1-form $\omega^{ab}$, $\ast(\partial_a \wedge \partial_b) = \frac{1}{2} \epsilon_{abcd} \partial^c \wedge \partial^d$, and $g = \eta_{ab} \partial^a \otimes \partial^b$ is the metric, where $\eta_{ab} = \text{diag}(-1,1,1,1)$ and $\partial^a$ is the orthonormal frame 1-form. It is important to note that one can add a boundary term to the above action, however, adding such a boundary term will not change the result using our symplectic structure invariant prescription described in the previous section.

Denote $\eta_{ab} = \ast(\partial_a \wedge \partial_b) = \frac{1}{2} \epsilon_{abcd} \partial^c \wedge \partial^d$, $\omega^{ab}$ being the spin connection that solves the equation of motion. The expression of Noether current 3-form in equation (5) is given by

$$H(\xi) = \int_{\Sigma} \mathcal{H}(\xi) = \int_{\Sigma} dQ(\xi) = \frac{1}{16\pi} \int_{\partial \Sigma} i\xi \omega^{ab} \wedge \eta_{ab},$$

(16)

where $Q(\xi)$ is the Noether charge 2-form appears as a total derivative therefore can also be interpreted as the boundary term, here, we assume the field equations are satisfied. For stationary spacetime but rather generic situation ($\oint_{\partial \Sigma} i\xi \phi \wedge p \neq 0$), we require

$$\delta \tilde{H} = \delta (H - B)$$

$$= \frac{1}{16\pi} \left[ \int_{\partial \Sigma} i\xi \omega^{ab} \wedge \eta_{ab} - \int_{\partial \Sigma} i\xi (\delta \omega^{ab} \wedge \eta_{ab}) \right]$$

$$= \frac{1}{16\pi} \left[ \int_{\partial \Sigma} i\xi \omega^{ab} \wedge \delta \eta_{ab} + \delta \omega^{ab} \wedge i\xi \eta_{ab} \right] = 0.$$

(17)

This can be satisfied by the boundary conditions on a bifurcate Killing horizons for stationary black holes [14] such that $\mathcal{L}_{\xi} \phi^a = 0$, $\mathcal{L}_{\xi} \omega^{ab} = 0$. In this case the energy flux is zero, $\mathcal{L}_{\xi} \tilde{H} = 0$ and from equation (16) we can obtain the Noether charge for stationary case.

For dynamical cases, following equation (14) we obtain the corresponding energy flux formula using the previous described invariant symplectic structure prescription,

$$F(\xi) = \frac{1}{16\pi} \int_{\partial \Sigma} i\xi \left( \mathcal{L}_{\xi} \omega^{ab} \wedge \Delta \eta_{ab} - \Delta \omega^{ab} \wedge \mathcal{L}_{\xi} \eta_{ab} \right).$$

(18)

For small perturbation away from stationary spacetime, we can define $\Delta \phi = \omega_{\text{dynamic}} - \omega_{\text{stationary}}$, and $\Delta \eta = \eta_{\text{dynamic}} - \eta_{\text{stationary}}$. This is a consistent definition, as when in the stationary limit, the expression will give zero flux, correct boundary conditions and correct value of the Noether charges for stationary black holes.

IV. VAIDYA SPACETIME EXAMPLE, BONDI TYPE ENERGY FLUX AND FIRST LAW

As an example for calculations of our energy flux expression, we consider the Vaidya spacetime which describes a spherically symmetric collapse of null dust (radiation). The metric is given by

$$ds^2 = -e^{2\psi} dt^2 + e^{-2\psi} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(19)

where $\psi = \psi(t,r)$ and $e^{2\psi} = 1 - 2m(t,r)/r$. In this coordinate, the marginally trapped surfaces are given by $r = 2m(t,r)$. For constant $m(t,r)$, this is just the standard Schwarzschild metric. Now consider a perturbation $\Delta m(t,r)$ away from the stationary solution, $m(t,r) = m_0 + \Delta m(t,r)$,

$$m(t,r) = m_0 + \Delta m(t,r),$$

(20)

because $m_0$ is a constant, this implies $m' = \partial_r (\Delta m)$ and $\dot{m} = \partial_t (\Delta m)$. In terms of the orthonormal frames, the natural choice is,

$$\partial^0 = e^\psi dt, \quad \partial^1 = e^{-\psi} dr,$n

$$\partial^2 = rd\theta, \quad \partial^3 = r \sin \theta d\phi,$$

(21)

with corresponding basis vectors,

$$e_0 = e^{-\psi} \partial_t, \quad e_1 = e^\psi \partial_r, \quad e_2 = \frac{1}{r} \partial_{\theta}, \quad e_3 = \frac{1}{r \sin \theta} \partial_\phi.$$
For $\xi = c_1 \partial_t + c_2 \partial_r$, the Lie derivatives of $\vartheta^a$ are

$$\mathcal{L}_\xi \vartheta^0 = -\frac{1}{re^\psi} \left[ c_1 \dot{m} + c_2 \left( m' - \frac{m}{r} \right) \right] dt,$$

$$\mathcal{L}_\xi \vartheta^1 = \frac{1}{re^{3\psi}} \left[ c_1 \dot{m} + c_2 \left( m' - \frac{m}{r} \right) \right] dr,$$

$$\mathcal{L}_\xi \vartheta^2 = c_2 d\vartheta,$$

$$\mathcal{L}_\xi \vartheta^3 = c_2 \sin \theta d\phi.$$ (23)

The spin-connection $\omega^{ab}$ has the following nonvanishing terms:

$$\omega^{01} = \frac{1}{re^4\psi} m dr - \frac{1}{r} \left( m' - \frac{m}{r} \right) dt = -\omega^{10},$$

$$\omega^{12} = -e^\psi d\theta = -\omega^{21},$$

$$\omega^{13} = -e^\psi \sin \theta d\phi = -\omega^{31},$$

$$\omega^{23} = -\cos \theta d\phi = -\omega^{32}. $$ (24)

The corresponding Lie derivatives of $\omega^{ab}$ have the following nonvanishing terms,

$$\mathcal{L}_\xi \omega^{01} = \frac{1}{e^4\psi} \left[ \frac{c_1}{r} \left( \dot{m} + \frac{4\dot{m}^2}{re^2\psi} \right) \right] dr$$

$$+ \frac{1}{e^4\psi} \left[ \frac{c_2}{r} \left( \dot{m} + \frac{4\dot{m}m' - \dot{m} - 4m\dot{m}}{r^2e^2\psi} \right) \right] dr$$

$$- \left[ \frac{c_1}{r} \left( \dot{m} - \frac{\dot{m}}{r} \right) \right] dt$$

$$- \left[ \frac{c_2}{r} \left( m' - \frac{2m'}{r^2} + \frac{2m}{r^2} \right) \right] dt$$

$$= -\mathcal{L}_\xi \omega^{10},$$

$$\mathcal{L}_\xi \omega^{12} = \frac{1}{re^\psi} \left[ c_1 \dot{m} + c_2 \left( m' - \frac{m}{r} \right) \right] d\theta$$

$$= -\mathcal{L}_\xi \omega^{21},$$

$$\mathcal{L}_\xi \omega^{13} = \frac{1}{re^{3\psi}} \left[ c_1 \dot{m} + c_2 \left( m' - \frac{m}{r} \right) \right] \sin \theta d\phi$$

$$= -\mathcal{L}_\xi \omega^{31},$$

$$\mathcal{L}_\xi \omega^{23} = 0 = -\mathcal{L}_\xi \omega^{32}. $$ (25)

The perturbation of the orthonormal tetrad the spin-connection have the following forms respectively,

$$\Delta \vartheta^0 = -\frac{1}{re^\psi} \Delta m dt,$$

$$\Delta \vartheta^1 = \frac{1}{re^{3\psi}} \Delta m dr,$$

$$\Delta \vartheta^2 = 0,$$

$$\Delta \vartheta^3 = 0,$$

$$\Delta \omega^{01} = \left( \frac{4\dot{m}\Delta m}{re^{3\psi}} + \frac{1}{re^\psi} \Delta \dot{m} \right) dr$$

$$- \frac{1}{r} \left( \Delta m' - \frac{\Delta m}{r} \right) dt$$

$$= -\Delta \omega^{10},$$

$$\Delta \omega^{12} = \frac{1}{re^\psi} \Delta m d\theta = -\Delta \omega^{21},$$

$$\Delta \omega^{13} = \frac{1}{re^{3\psi}} \Delta m \sin \theta d\phi = -\Delta \omega^{31},$$

$$\Delta \omega^{23} = 0 = -\Delta \omega^{32}. $$ (27)

Many of the terms vanish in the energy flux $F(\xi)$, the remaining nonvanishing terms that will contribute are,

$$\frac{1}{8\pi} \int_{\partial \Sigma} i\xi (\Delta \omega^{12} \wedge \Delta \eta_2 + \Delta \omega^{13} \wedge \Delta \eta_3)$$

$$= -c_1 \Delta \frac{m}{re^{2\psi}} \left[ c_1 \dot{m} + c_2 \left( m' - \frac{m}{r} \right) \right],$$

$$\frac{1}{8\pi} \int_{\partial \Sigma} i\xi (\Delta \omega^{12} \wedge \Delta \xi \eta_2 + \Delta \omega^{13} \wedge \Delta \xi \eta_3)$$

$$= c_1 \left[ \frac{\Delta m}{re^{2\psi}} \left[ c_1 \dot{m} + c_2 \left( m' - \frac{m}{r} \right) \right] - c_2 \frac{\Delta m}{r} \right],$$

and

$$\frac{1}{8\pi} \int_{\partial \Sigma} i\xi (\Delta \omega^{01} \wedge \Delta \xi \eta_0)$$

$$= c_2 \left[ \left( \Delta m' - \frac{\Delta m}{r} \right) c_1 - \left( 4\dot{m} \frac{\Delta m}{e^{3\psi}} + \frac{\Delta \dot{m}}{e^{4\psi}} \right) c_2 \right].$$ (30)

Finally, the total energy flux is

$$F(\xi) = c_2 \left[ \left( \Delta m' - \frac{\Delta m}{r} \right) c_1 - \left( 4\dot{m} \frac{\Delta m}{e^{3\psi}} + \frac{\Delta \dot{m}}{e^{4\psi}} \right) c_2 \right].$$ (31)

Taking $u = t - r = const$ and $t, r \to \infty$ to approach the null infinity and dropping the term contains $\dot{m} \Delta m$ which is of higher order in $\Delta$, we arrive at the Bondi type energy flux

$$F(\xi) = c_2 (c_1 m' - c_2 \dot{m}) = -\partial_u m(u),$$

where we have put $c_1 = 1$ and $c_2 = 1$ by requiring that $\xi$ and $\Delta$ defines the same direction of mass changes for the consistency of interchanging $\Delta$ and $\mathcal{L}_\xi$. The same energy flux result, equation (29), was also obtained long ago by Lindquist, Schwartz and Misner using Landau-Lifshitz stress-energy pseudotensor. Such energy flux $-\partial_u m(u)$ has the interpretation as the luminosity of the star as seen by an observer at null infinity.

It is interesting to notice that the nonvanishing term come only from the following equations

$$F(\xi) = -\frac{1}{16\pi} \int_{\partial \Sigma} i\xi [\Delta \omega^{01} \wedge \mathcal{L}_\xi (\vartheta^2 \wedge \vartheta^3)]$$

$$- \frac{1}{16\pi} \int_{\partial \Sigma} i\xi [\mathcal{L}_\xi \vartheta^3 \wedge \Delta \omega^{12} - \Delta \omega^{13} \wedge \mathcal{L}_\xi \vartheta^2]$$

$$= \frac{1}{16\pi} \int_{\partial \Sigma} i\xi [\Delta \omega^{01} + \Delta \omega^{12} \wedge \mathcal{L}_\xi (\vartheta^2 \wedge \vartheta^3)],$$

(33)
where \( \vartheta^2 \land \vartheta^3 \) is the area element. This indicates the first law for general spacetime regions. For stationary spacetimes where \( \mathcal{L}_\xi (\vartheta^2 \land \vartheta^3) = 0 \), the energy flux vanishes. The appearance of the first law provides a nontrivial consistent check of our energy flux expression.

V. ENERGY FLUX FROM DYNAMICAL HORIZONS: PAINELEVÉ-GULLSTRAND COORDINATES

On dynamical black-hole horizons, \( r = 2m \), which implies \( m' = 1/2 \), \( \Delta m' = 1/2 \), \( m/r = 1/2 \). The first term of the above total energy flux becomes

\[
\mathcal{F}(\xi) = \left[ \frac{c_1 c_2}{2} \left( 1 - 4\Delta m \right) \right]^{r_2}_{r_1}. \tag{31}
\]

Note that unlike in the Bondi type energy flux case which is defined at the null infinity 2-sphere boundary, here for dynamical horizons, \( \Sigma \) is bounded by two cross sections, \( \partial \Sigma = \partial \Sigma_1 + \partial \Sigma_2 \), with radius changes from \( r_1 \) to \( r_2 \) dynamically because of the outgoing energy flux. However, the second term in \( \mathcal{F}(\xi) \) is singular on dynamical horizons where \( e^\psi = 0 \) (although \( \dot{m} = 0 \)) because of the coordinate singularity on the horizons.

In order to study the energy flux in dynamical horizons, we make a coordinate transformation to the Painlevé-Gullstrand time coordinate \( T \) which is related to the Schwarzschild coordinate \( t \) by

\[
T = t + 4m \left[ \frac{r}{2m} + \frac{1}{2} \ln \left( \frac{\sqrt{2m} - 1}{\sqrt{2m} + 1} \right) \right]. \tag{35}
\]

In terms of the Painlevé-Gullstrand coordinates, the metric can be written as

\[
ds^2 = -dT^2 + (dr + \sqrt{\frac{2m(T,r)}{r}} dT)^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\phi^2).
\]

with the choice of the orthonormal frames,

\[
\vartheta^0 = dT, \quad \vartheta^1 = dr + \sqrt{\frac{2m(T,r)}{r}} dT, \quad \vartheta^2 = rd\vartheta, \quad \vartheta^3 = r \sin \vartheta d\phi,
\]

and the corresponding basis vectors,

\[
e_0 = \partial_T - \sqrt{\frac{2m(T,r)}{r}} \partial_r, \quad e_1 = \partial_r, \quad e_2 = \frac{1}{r} \partial_\vartheta, \quad e_3 = \frac{1}{r \sin \vartheta} \partial_\varphi. \tag{38}
\]

In order to compare results for dynamical horizons obtained by Ashtekar and Krishnan previously, we choose an \( r \) dependent \( \tilde{c}_2 \) in the displacement vector, \( \xi = \tilde{c}_1 \partial_T + \tilde{c}_2(r) \partial_r \) instead of a constant \( \tilde{c}_2 \). We can now calculate the Lie derivatives of \( \vartheta^a \) and obtain

\[
\mathcal{L}_\xi \vartheta^0 = 0, \quad \mathcal{L}_\xi \vartheta^1 = \frac{1}{\sqrt{2mr}} \left[ \tilde{c}_1 \dot{m} + \tilde{c}_2(r) \left( m' - \frac{m}{r} \right) \right] dT + \tilde{c}_2'(r) dr, \quad \mathcal{L}_\xi \vartheta^2 = \tilde{c}_2(r) d\vartheta, \quad \mathcal{L}_\xi \vartheta^3 = \tilde{c}_2(r) \sin \theta d\phi. \tag{39}
\]

The corresponding spin-connection \( \omega^{ab} \) will have the following nonvanishing terms:

\[
\omega^{01} = -(m' - \frac{m}{r}) \left( \frac{1}{\sqrt{2mr}} dr + \frac{1}{r} dT \right) = -\omega^{10}, \quad \omega^{02} = -\frac{2m}{r} d\theta = -\omega^{20}, \quad \omega^{03} = -\frac{2m}{r} \sin \theta d\phi = -\omega^{30},
\]

\[
\omega^{12} = -d\theta = -\omega^{21}, \quad \omega^{13} = -\sin \theta d\phi = -\omega^{31}, \quad \omega^{23} = -\cos \theta d\phi = -\omega^{32}, \tag{40}
\]

and the Lie derivatives of \( \omega^{ab} \) have the following nonvanishing terms:

\[
\mathcal{L}_\xi \omega^{01} = \frac{\tilde{c}_1}{\sqrt{2mr}} \left( \tilde{m}' m' + \frac{\tilde{m} m'}{2m} + \frac{\dot{m} m'}{2r} \right) dr + \tilde{c}_2(r) \left( m' m'' + \frac{m'^2}{2m} + \frac{m'}{r} \right) dr - \frac{3m}{2r^2} dr + \frac{\tilde{c}_2(r)}{\sqrt{2mr}} \left( \frac{m}{r} - m'\right) dT + \tilde{c}_2'(r) \left( \frac{2m}{r^2} - 2m/r^3 - m'' \right) dT = -\mathcal{L}_\xi \omega^{10},
\]

\[
\mathcal{L}_\xi \omega^{02} = -\frac{1}{\sqrt{2mr}} \left[ \tilde{c}_1 \dot{m} + \tilde{c}_2(r) \left( m' - \frac{m}{r} \right) \right] d\theta = -\mathcal{L}_\xi \omega^{20}, \quad \mathcal{L}_\xi \omega^{03} = -\frac{1}{\sqrt{2mr}} \left[ \tilde{c}_1 \dot{m} + \tilde{c}_2(r) \left( m' - \frac{m}{r} \right) \right] \sin \theta d\phi = -\mathcal{L}_\xi \omega^{30},
\]

\[
\mathcal{L}_\xi \omega^{12} = 0 = -\mathcal{L}_\xi \omega^{21}, \quad \mathcal{L}_\xi \omega^{13} = 0 = -\mathcal{L}_\xi \omega^{31}, \quad \mathcal{L}_\xi \omega^{23} = 0 = -\mathcal{L}_\xi \omega^{32}. \tag{41}
\]

As before, we express the perturbation of mass as,

\[
m(T,r) = m_0 + \Delta m(T,r), \quad \Delta \sqrt{m} = \frac{\Delta m}{2\sqrt{m}}, \quad \Delta \left( \frac{1}{\sqrt{m}} \right) = -\frac{\Delta m}{2m\sqrt{m}}. \tag{42}
\]
and obtain the following perturbation of the orthonormal tetrad and spin-connection respectively,

\[
\begin{align*}
\Delta \theta^0 &= 0, \\
\Delta \theta^1 &= \frac{1}{\sqrt{2mr}} (\Delta m) dT, \\
\Delta \theta^2 &= 0, \\
\Delta \theta^3 &= 0, \\
\Delta \omega^{01} &= -\Delta \omega^{10} \\
&= - (\Delta m' - \frac{\Delta m}{r}) \left( \frac{1}{\sqrt{2mr}} dr + \frac{1}{r} dT \right) \\
&+ (m' - m) \frac{\Delta m}{2mr} dr, \\
\Delta \omega^{02} &= - \frac{1}{\sqrt{2mr}} (\Delta m) d\theta = - \Delta \omega^{20}, \\
\Delta \omega^{03} &= - \frac{1}{\sqrt{2mr}} (\Delta m) \sin \theta d\phi = - \Delta \omega^{30}, \\
\Delta \omega^{12} &= 0 = - \Delta \omega^{21}, \\
\Delta \omega^{13} &= 0 = - \Delta \omega^{31}, \\
\Delta \omega^{23} &= 0 = - \Delta \omega^{32}.
\end{align*}
\]  

(45)

Put all the above results into the energy flux expression \( F(\xi) \), we obtain the following nonvanishing contributions, which includes:

\[
\begin{align*}
\frac{1}{8\pi} \oint_{\partial \Sigma} i_{\xi} (\mathcal{L}_\xi \omega^{02} \wedge \Delta \eta_{02} + \mathcal{L}_\xi \omega^{03} \wedge \Delta \eta_{03}) \\
= - \frac{1}{2} \Delta m \left( \frac{\tilde{c}_1 \tilde{m}}{m} + \frac{\tilde{c}_1 \tilde{c}_2 (r) m'}{m} - \frac{\tilde{c}_1 \tilde{c}_2 (r)}{r} \right),
\end{align*}
\]

(47)

\[
\begin{align*}
- \frac{1}{8\pi} \oint_{\partial \Sigma} i_{\xi} (\Delta \omega^{01} \wedge \mathcal{L}_\xi \eta_{01}) \\
= \tilde{c}_2 (r) \left( - \left( \Delta m' - \frac{\Delta m}{r} \right) \left( \tilde{c}_2 (r) \sqrt{\frac{r}{2m}} + \tilde{c}_1 \right) \\
+ \tilde{c}_2 (r) \left( 2m' - m \right) \left( \tilde{c}_2 (r) \frac{\Delta m}{2m} \sqrt{\frac{r}{2m}} \right) \right),
\end{align*}
\]

(48)

and

\[
\begin{align*}
- \frac{1}{8\pi} \oint_{\partial \Sigma} i_{\xi} (\Delta \omega^{02} \wedge \Delta \eta_{02}) - \frac{1}{8\pi} \oint_{\partial \Sigma} i_{\xi} (\Delta \omega^{03} \wedge \Delta \eta_{03}) \\
= \tilde{c}_2 (r) \frac{\Delta m}{\sqrt{2mr}} \left( 2m' - m \right) \sqrt{\frac{2m}{r}} + \tilde{c}_2 (r) \right) \\
+ \frac{1}{2} \Delta m \left( \frac{\tilde{c}_1 \tilde{m}}{m} + \frac{\tilde{c}_1 \tilde{c}_2 (r) m'}{m} - \frac{\tilde{c}_1 \tilde{c}_2 (r)}{r} \right) \\
+ \frac{1}{2} \Delta m \tilde{c}_2 (r) \frac{\tilde{c}_2 (r)}{r} \sqrt{\frac{2m}{r}}.
\end{align*}
\]

(49)

Finally, the total energy flux becomes,

\[
F(\xi) = \left[ - \tilde{c}_2 (r) \left( \Delta m' - \frac{\Delta m}{r} \right) \left( \tilde{c}_2 (r) \sqrt{\frac{2m}{r}} + \tilde{c}_1 \right) \\
+ \frac{1}{2} \tilde{c}_2 (r) \Delta m \left( \tilde{c}_2 (r) \sqrt{\frac{2m}{r}} + \tilde{c}_1 \right) + \tilde{c}_2 (r) \left( \tilde{c}_2 (r) \sqrt{\frac{2m}{r}} + \tilde{c}_1 \right) \right],
\]

(50)

Note that an expression on the dynamical horizon was obtained by Ashtekar and Krishnan \[8\] previously,

\[
\left( \frac{r^2}{2} - \frac{r_1}{2} \right) = \int_{\Delta H} T_{ab} r^a e^b_r d^3V \\
+ \frac{1}{16\pi} \int_{\Delta H} N_r \left\{ |\sigma|^2 + 2|\xi|^2 \right\} d^3V.
\]

(51)

At the dynamical horizon, \( r = 2m \), therefore \( m' = 1/2, \Delta m' = 1/2, m/r = 1/2 \), our flux expression \[50\] reduced to

\[
F(\xi) = \frac{1}{2} \left[ \tilde{c}_2 (r) (\tilde{c}_1 + \tilde{c}_2 (r)) (1 - 4 \frac{\Delta m}{r} \right] r^2_1 \\
+ \frac{1}{2} \left( \Delta m \tilde{c}_2 (r) \tilde{c}_2 (r) \right) r^2_1.
\]

(52)

Therefore, on the dynamical horizon when we choose \( \tilde{c}_1 = 0 \) and \( \tilde{c}_2 (r) = \sqrt{r} \) for the displacement vector \( \xi \), contribution from the second term vanishes and our result reduced to an “area balance law”,

\[
F(\xi) = \left( \frac{r^2}{2} - \frac{r_1}{2} \right),
\]

(53)

which agrees with the Ashtekar-Krishnan energy flux formula \[51\], with the shear \( |\sigma| \) and the twist \( |\xi| \) becomes zero in Vaidya spacetime.

Note that similar to the previous section, in this example the nonvanishing term come from

\[
F(\xi) = - \frac{1}{16\pi} \oint_{\partial \Sigma} i_{\xi} [\Delta \omega^{01} \wedge \mathcal{L}_\xi (\vartheta^2 \wedge \vartheta^3) \\
- \frac{1}{16\pi} \oint_{\partial \Sigma} i_{\xi} [\vartheta^1 \Delta \omega^{02} \wedge \mathcal{L}_\xi (\vartheta^3) \wedge \Delta \omega^{03} \wedge \mathcal{L}_\xi (\vartheta^3),
\]

(54)

where \( \vartheta^2 \wedge \vartheta^3 \) is the area element and again hints at an area dependent first law. For stationary spacetimes where \( \mathcal{L}_\xi (\vartheta^2 \wedge \vartheta^3) = 0 \), the energy flux vanishes.

VI. DISCUSSIONS

A general expression for quasi-local energy flux expression is derived from covariant Hamiltonian formulation.
using functionally differentiability and symplectic structure invariance, which is coordinate independent. The energy flux expression is given by the boundary term. The benefits of using symplectic structure invariance is to avoid the ambiguity in choosing the correct boundary terms in the Lagrangian that one begins with. This was a core problem for many other formalisms for diffeomorphism invariant theories in the literatures. Another important features of this expression are the independence of the choice of the canonical variables. The expression $F(\xi)$ depends on the vector field $\xi$ and the choice of the boundary surfaces which depends on the type of the physics under investigation.

For the boundary surface taking to be the null infinity, the expression leads to the Bondi type energy flux obtained by Lindquist, Schwartz and Misner [13] for Vaidya spacetime, where the energy flux has the interpretation as the luminosity of the star as seen by an observer at infinity. If the boundary surface is taken to be the dynamical horizons, the expression gives rise to the energy flux obtained by Ashtekar and Krishnan for Vaidya spacetime.

Note that for Painlevé-Gullstrand coordinates our expression fails to give the correct Bondi type energy flux at null infinity because of the $1/\sqrt{\ell}$ fall off of the metric. Also for Schwarzschild coordinates the expression fails on the interesting dynamical horizon. This only indicates a trivial fact that one requires to use at least two coordinate patches to cover the whole spacetime range of interest except for the uninteresting flat spacetime. However, this is a good news to our energy flux expression which derived from a coordinate independent covariant formalism and detects the limits of the applicability of the coordinate system employed in the calculations.

Another interesting observation is that the expression gives an area dependence which hints at the first law for general Vaidya spacetime.

Appendix

In comparison to the flux expressions in [4], our flux expression [13] is simply the sum of the Dirichlet and Neumann boundary flux expressions given in [4]. In the special case when $\phi$ is a fixed variable, $\Delta \phi = 0$, our flux expression [13] reduced to the “Dirichlet boundary flux expression” (equation (30) of [4]),

$$F_{\text{Dirichlet}}(\xi) = \oint_{\partial \Sigma} \iota_\xi (\ell \xi \phi \wedge \Delta p).$$  \hfill (55)

If $p$ is a fixed variable, $\Delta \phi = 0$, our flux expression [13] reduced to the “Neumann boundary flux expression” (equation (31) of [4]),

$$F_{\text{Neumann}}(\xi) = \oint_{\partial \Sigma} \iota_\xi (-\Delta \phi \wedge \ell \xi p).$$  \hfill (56)

Similarly, when the spatial projections of the variables $\phi$ and $p$ are fixed, we obtain the “dynamic boundary flux expression” (equation (32) of [4]),

$$F_{\text{dynamic}}(\xi) = \oint_{\partial \Sigma} (\zeta \ell \xi \phi \wedge \iota_\xi \Delta p - \iota_\xi \Delta \xi \phi \wedge \ell \xi p),$$  \hfill (57)

where $\zeta = (-1)^f$ for $\phi$ being an $f$-form field. If the time projections of the variables $\phi$ and $p$ are fixed, we obtain the “constraint boundary flux expression” (equation (33) of [4]),

$$F_{\text{constraint}}(\xi) = \oint_{\partial \Sigma} (\iota_\xi \ell \xi \phi \wedge \Delta p - \iota_\xi \Delta \xi \phi \wedge \ell \xi p).$$  \hfill (58)

Our flux expression [13] is the general symplectic invariant expression without any variable being fixed.

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