Nonparametric estimation in functional linear models with second order stationary regressors.

Jan Johannes*

January 27, 2009

Abstract

We consider the problem of estimating the slope parameter in functional linear regression, where scalar responses $Y_1, \ldots, Y_n$ are modeled in dependence of second order stationary random functions $X_1, \ldots, X_n$. An orthogonal series estimator of the functional slope parameter with additional thresholding in the Fourier domain is proposed and its performance is measured with respect to a wide range of weighted risks covering as examples the mean squared prediction error and the mean integrated squared error for derivative estimation. In this paper the minimax optimal rate of convergence of the estimator is derived over a large class of different regularity spaces for the slope parameter and of different link conditions for the covariance operator. These general results are illustrated by the particular example of the well-known Sobolev space of periodic functions as regularity space for the slope parameter and the case of finitely or infinitely smoothing covariance operator.

Keywords: Orthogonal series estimation, Spectral cut-off, Derivatives estimation, Mean squared error of prediction, Minimax theory, Sobolev space.

AMS 2000 subject classifications: Primary 62J05; secondary 62G20, 62G08.

1 Introduction

Functional linear models have become very important in a diverse range of disciplines, including medicine, linguistics, chemometrics as well as econometrics (see for instance Ramsay and Silverman [2005] and Ferraty and Vieu [2006], for several case studies, or more specific, Forni and Reichlin [1998] and Preda and Saporta [2005] for applications in economics). Roughly speaking, in all these applications the dependence of a response variable $Y$ on the variation of an explanatory random function $X$ is modeled by

$$Y = \int_0^1 \beta(t)X(t)dt + \sigma \varepsilon, \quad \sigma > 0,$$

for some error term $\varepsilon$. One objective is then to estimate nonparametrically the slope function $\beta$ based on an independent and identically distributed (i.i.d.) sample of $(Y, X)$.

In this paper we suppose that the random function $X$ is taking its values in $L^2[0,1]$, which is endowed with the usual inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, and that $X$ has

*Universität Heidelberg, Institut für Angewandte Mathematik, Im Neuenheimer Feld, 294, D-69120 Heidelberg, Germany, e-mail: johannes@statlab.uni-heidelberg.de
a finite second moment, i.e., \( \mathbb{E}\|X\|^2 < \infty \). In order to simplify notations we assume that the mean function of \( X \) is zero. Moreover, the random function \( X \) and the error term \( \varepsilon \) are uncorrelated, where \( \varepsilon \) is assumed to have mean zero and variance one. This situation has been considered, for example, in Cardot et al. [2003] or Müller and Stadtmüller [2005]. Then multiplying both sides in (1.1) by \( X \) and taking the expectation leads to

\[
g(s) := \mathbb{E}[YX(s)] = \int_0^1 \beta(t) \text{cov}(X(t), X(s))dt =: [T_{\text{cov}} \beta](s), \quad s \in [0, 1],
\]

where \( g \) belongs to \( L^2[0, 1] \) and \( T_{\text{cov}} \) denotes the covariance operator associated to the random function \( X \). Estimation of \( \beta \) is thus linked with the inversion of the covariance operator \( T_{\text{cov}} \) of \( X \) and, hence called an inverse problem. We assume that there exists a unique solution \( \beta \in L^2[0, 1] \) of equation (1.2), i.e., \( g \) belongs to the range \( \mathcal{R}(T_{\text{cov}}) \) of \( T_{\text{cov}} \), and \( T_{\text{cov}} \) is injective. However, as usual in the context of inverse problems all the results below could also be obtained straightforward for the unique least-square solution with minimal norm, which exists if and only if \( g \) is contained in the direct sum of \( \mathcal{R}(T_{\text{cov}}) \) and its orthogonal complement \( \mathcal{R}(T_{\text{cov}})^{\perp} \) (for a definition and detailed discussion in the context of inverse problems see chapter 2.1 in Engl et al. [2000], while in the special case of a functional linear model we refer to Cardot et al. [2003]).

The normal equation (1.2) is the continuous equivalent of a normal equation \( \mathbb{E}XY = \mathbb{E}XX^t \beta^* \) in a linear model \( \mathbb{E}Y = X^t \beta + \varepsilon \), where the covariance matrix \( \mathbb{E}XX^t \) has always a continuous generalized inverse. However, due to the finite second moment of \( X \) the covariance operator \( T_{\text{cov}} \) of \( X \) defined in (1.2) is nuclear (c.f. Dauxois et al. [1982]). Thereby, unlike in the linear model, a continuous generalized inverse of \( T_{\text{cov}} \) does not exist if the range of the operator \( T_{\text{cov}} \) is an infinite dimensional subspace of \( L^2[0, 1] \). This corresponds to the setup of ill-posed inverse problems (with the additional difficulty that \( T_{\text{cov}} \) in (1.2) is unknown and hence, has to be estimated).

In the literature several approaches are proposed in order to circumvent the instability issue due to an inversion of \( T_{\text{cov}} \). Essentially, all of them replace the operator \( T_{\text{cov}} \) in equation (1.2) by a regularized version having a continuous generalized inverse. A popular example is based on a functional principal components regression (c.f. Bosq [2000], Cardot et al. [2007] or Müller and Stadtmüller [2005]), which corresponds to a method called spectral cut-off in the numerical analysis literature (c.f. Tautenhahn [1996]). An other example is the Tikhonov regularization (c.f. Hall and Horowitz [2007]), where the regularized solution \( \beta_\alpha \) is defined as unique minimizer of the Tikhonov functional \( F_\alpha(\beta) = \|T_{\text{cov}} \beta - g\|^2 + \alpha \|\beta\|^2 \) for some strictly positive \( \alpha \). A regularization through a penalized least squares approach after projection onto some basis (such as splines) is also considered in Ramsay and Dalzell [1991], Eilers and Marx [1996] or Cardot et al. [2003].

In opposite to the model assumptions considered until now in the literature in this paper we suppose that the regressor \( X \) is second order stationary. Over relatively short periods of time, the assumption of second order stationarity is in many situations realistic and can be checked from the data by estimating the covariance function using the multiple realizations of \( X \). Moreover, assuming second order stationarity allows us to generalize the known results in essentially two directions. First, we can unify the measures of performances for the estimator as considered in the literature and second it is possible to present a simple estimation strategy which is optimal in a minimax sense over a wide range of possible regularity spaces for the slope functions \( \beta \) as well as various forms of link conditions for the covariance operators \( T_{\text{cov}} \). To be more detailed:
In this paper we show that in case of second order stationary regressor \( X \) the associated covariance operator \( T_{\text{cov}} \) admits a spectral decomposition \( \{\lambda_j, \psi_j, j \geq 1\} \) given by the trigonometric basis \( \{\psi_j\} \) (defined below) as eigenfunctions and a strictly positive, possibly not ordered, zero-sequence \( \lambda := (\lambda_j)_{j \geq 1} \) of corresponding eigenvalues. Then the normal equation can be rewritten as follows
\[
\beta = \sum_{j=1}^{\infty} \frac{g_j}{\lambda_j} \cdot \psi_j \quad \text{with} \quad g_j := (g, \psi_j), \; j \geq 1. \tag{1.3}
\]

It is well-known that even in case of an a-priori known sequence \( \lambda \) of eigenvalues replacing in (1.3) the unknown function \( g \) by a consistent estimator \( \hat{g} \) does in general not lead to a \( L^2 \)-consistent estimator of \( \beta \). To be more precise, since \( \lambda \) is a zero-sequence, \( \mathbb{E}\|\hat{g} - g\|^2 = o(1) \) does generally not imply \( \sum_{j=1}^{\infty} \lambda_j^{-2} \cdot \mathbb{E}|(\hat{g} - g, \psi_j)|^2 = o(1) \), i.e., the inverse operation of the covariance operator \( T_{\text{cov}} \) is not continuous. Essentially, all of the approaches mentioned above circumvent this instability issue by replacing equation (1.3) by a regularized version which avoids that the denominator becomes too small. For instance, in case of a Tikhonov regularization (c.f. Hall and Horowitz [2007]) in (1.3) the factor \( 1/\lambda_j \) is replaced by \( \lambda_j/(\alpha + \lambda_j^2) \).

In the literature so far the performance of an estimator of \( \beta \) has been measured either by considering a squared prediction error or an integrated squared error. We show in this paper that these approaches can be unified by considering a loss given by a weighted norm. To be more precise for \( f \in L^2[0,1], \) we define
\[
\|f\|_\omega^2 := \sum_{j=1}^{\infty} \omega_j |(f, \psi_j)|^2 \tag{1.4}
\]
for some strictly positive sequence of weights \( \omega := (\omega_j)_{j \geq 1} \). Then, the performance of an estimator \( \hat{\beta} \) of \( \beta \) is measured by the \( \mathcal{F}_\omega \)-risk, that is \( \mathbb{E}\|\hat{\beta} - \beta\|_\omega^2 \). This general framework allows us with an appropriate choice of the weight sequence \( \omega \) to cover both, the risk in terms of mean integrated squared error, i.e., \( \omega \equiv 1 \), as well as the mean squared prediction error. Indeed, the squared prediction error of a new value of \( Y \) given any random function \( X_{n+1} \) possessing the same distribution as \( X \) and being independent of \( X_1, \ldots, X_n \) can be evaluated as follows (see for example Cardot et al. [2003] or Crambes et al. [2009] for similar setups)
\[
\mathbb{E}\left[ \|\hat{\beta} - \beta\|_{\omega}^2 \right] = \sum_{j \geq 1} \lambda_j |(\hat{\beta} - \beta, \psi_j)|^2,
\]
where we have used for the last identity that the regressor is second order stationary, i.e, \( T_{\text{cov}} \) admits \( \{\lambda_j, \psi_j, j \geq 1\} \) as spectral decomposition. Consequently, choosing \( \omega \equiv \lambda \) the \( \mathcal{F}_\omega \)-risk is equivalent to the mean squared prediction error. We present this specific situation in Section 4 below. It is worth to note, that the \( L^2 \)-norm \( \|f^{(s)}\| \) of the \( s \)-th weak derivative \( f^{(s)} \) of a function \( f \), if it exists, is also equivalently given by a specific weighted norm \( \|\cdot\|_\omega \) with an appropriate choice of weights \( \omega \) (c.f. Neubauer [1988a]). Thus, by considering the corresponding \( \mathcal{F}_\omega \)-risk we also cover the estimation of derivatives of the slope function. This question is also discussed in detail in Section 4.

In this paper we characterize the a-priori information on the slope parameter such as smoothness by considering ellipsoids (see definition below) in \( L^2[0,1] \) with respect to a
weighted norm $\|\cdot\|_\gamma$ for a pre-specified weight sequence $\gamma$. Again an appropriate choice of the sequence $\gamma$ enables us not only to restrict the slope parameter to a class of differentiable functions (considered, e.g. in Crambes et al. [2009]) but, for instance, also to a class of analytic functions. Moreover, it is usually assumed that the sequence $\lambda$ of eigenvalues of $T_{cov}$ has a polynomial decay (c.f. Hall and Horowitz [2007] or Crambes et al. [2009]). However, it is well-known that this restriction may exclude several interesting cases, such as an exponential decay. Therefore, we do not impose a specific form of a decay, but consider a third sequence of weights $\upsilon$ characterizing the decay of $\lambda$. Then we show that the three sequences $\gamma$ (regularity of $\beta$), $\upsilon$ (regularity of $T_{cov}$) and $\omega$ (measure of the performance of the estimator) determine together the obtainable accuracy of any estimator. In other words, in Section 3 we derive a lower bound under minimal regularity conditions on these sequences. It is remarkable, that a simple orthogonal series estimator attains this lower bound up to a constant under very mild moment assumptions on the regressor and the error term.

To be more precise, we replace the unknown quantities $g_j$ and $\lambda_j$ in equation (1.3) by their empirical counterparts. That is, if $(Y_1, X_1), \ldots, (Y_n, X_n)$ denotes an i.i.d. sample of $(Y, X)$, then for each $j \geq 1$, we consider the unbiased estimator

$$\hat{g}_j := \frac{1}{n} \sum_{i=1}^{n} Y_i \langle X_i, \psi_j \rangle, \quad \text{and} \quad \hat{\lambda}_j := \frac{1}{n} \sum_{i=1}^{n} \langle X_i, \psi_j \rangle^2 \quad (1.5)$$

for $g_j$ and $\lambda_j$ respectively. The orthogonal series estimator $\hat{\beta}$ of $\beta$ is then defined by

$$\hat{\beta} := \sum_{j=1}^{m} \frac{\hat{g}_j}{\hat{\lambda}_j} \cdot 1\{\hat{\lambda}_j \geq \alpha\} \cdot \psi_j, \quad (1.6)$$

where the dimension parameter $m = m(n)$ and the threshold $\alpha = \alpha(n)$ has to tend to infinite and zero respectively as the sample size $n$ increases. Note that we introduce an additional threshold $\alpha$ on each estimated eigenvalue $\hat{\lambda}_j$, since it could be arbitrarily close to zero even in case that the true eigenvalue $\lambda_j$ is sufficiently far away from zero. Thresholding in the Fourier domain has been used, for example, in a deconvolution problem in Mair and Ruymgaart [1996], Neumann [1997] or Johannes [2009] and coincides with an approach called spectral cut-off in the numerical analysis literature (c.f. Tautenhahn [1996]).

The paper is organized in the following way. In Section 2 we formalize the regularity conditions on the slope parameter $\beta$ and the covariance operator $T_{cov}$ characterized through different weight sequences. Moreover, we state the minimal conditions on these weight sequences as well as the moments of the random function $X$ and the error term $\varepsilon$ used throughout the paper. In Section 3 we show consistency in the $F_\omega$-risk of the proposed orthogonal series estimator under very mild assumptions. For example, considering the $L^2$-risk, i.e., $\omega \equiv 1$, there are no additional regularity conditions on the slope parameter needed. Furthermore, we derive a lower and an upper bound for the $F_\omega$-risk only supposing the minimal conditions on the sequences $\gamma$, $\omega$ and $\upsilon$. These results are illustrated in Section 4 by considering the mean squared prediction error as well as the optimal estimation of derivatives of $\beta$ in case that the slope function belongs to a Sobolev space of periodic functions and that the covariance operator $T_{cov}$ is finitely or infinitely smoothing. All proofs can be found in the Appendix.
2 Notations and basic assumptions

**Second order stationarity.** In this paper we suppose that the regressor \( X \) is second order stationary, i.e., there exists a positive definite function \( c : [-1, 1] \to \mathbb{R} \) such that \( \text{cov}(X(t), X(s)) = c(t-s), \ s, t \in [0, 1] \). Thereby we show in Proposition A.1 in the Appendix that the eigenfunctions of the covariance operator \( T_{\text{cov}} \) associated to \( X \) are given by the trigonometric basis

\[
\psi_1 := 1, \ \psi_{2j}(s) := \sqrt{2} \cos(2\pi js), \ \psi_{2j+1}(s) := \sqrt{2} \sin(2\pi js), \ s \in [0, 1], \ j \in \mathbb{N} \quad (2.1)
\]

and the corresponding eigenvalues satisfy

\[
\lambda_1 = \int_{-1}^{1} c(s)ds, \ \lambda_{2j} = \lambda_{2j+1} = \int_{-1}^{1} \cos(2\pi js)c(s)ds, \ j \in \mathbb{N}. \quad (2.2)
\]

Notice that the eigenfunctions are known to the statistician and only the eigenvalues depend on the unknown covariance function \( c(\cdot) \), i.e., have to be estimated.

**Minimal regularity conditions.** It is well-known that the obtainable accuracy of any estimator of the slope parameter \( \beta \) is essentially determined by additional regularity conditions imposed on both the slope parameter \( \beta \) and the sequence of eigenvalues \( (\lambda_j) \) of the covariance operator. In this paper these conditions are characterized through different weighted norms in \( L^2[0, 1] \), which we formalize now. Given a strictly positive sequence of weights \( w := (w_j)_{j \geq 1} \) and a constant \( c > 0 \) denote for all \( r \in \mathbb{R} \) by \( \mathcal{F}_{w,r}^c \) the ellipsoid given by

\[
\mathcal{F}_{w,r}^c := \left\{ f \in L^2[0, 1] : \sum_{j=1}^{\infty} w_j^2 |\langle f, \psi_j \rangle|^2 =: \| f \|_{w,r}^2 \leq c \right\}.
\]

Furthermore, let \( \mathcal{F}_{w,r} := \{ f \in L^2[0, 1] : \| f \|_{w,r}^2 < \infty \} \). Here and subsequently, we suppose that given a strictly positive sequence of weights \( \gamma := (\gamma_j)_{j \geq 1} \) the slope function \( \beta \) belongs to the ellipsoid \( \mathcal{F}_{\gamma}^\rho \) for some \( \rho > 0 \). The ellipsoid \( \mathcal{F}_{\gamma}^\rho \) captures then all the prior information (such as smoothness) about the unknown slope function \( \beta \). It is worth to note, that in case \( \gamma \equiv 1 \) the set \( \mathcal{F}^\rho \) denotes an ellipsoid in \( L^2[0, 1] \) and hence does not imposes additional restrictions on \( \beta \). Furthermore, given a strictly positive sequence of weights \( v := (v_j)_{j \geq 1} \) we assume that the sequence of eigenvalues \( (\lambda_j)_j \) of the covariance operator \( T_{\text{cov}} \) is an element of the set \( S^d_v \) defined for \( d \geq 1 \) by

\[
S^d_v := \left\{ (\lambda_j)_{j \geq 1} : 1/d \leq \lambda_j/v_j \leq d, \ \forall j \in \mathbb{N} \right\}. \quad (2.3)
\]

Notice that the sequence of eigenvalues \( (\lambda_j)_{j \geq 1} \) is summable, since \( \sum_{j \in \mathbb{N}} \lambda_j = \mathbb{E}\|X\|^2 < \infty \). Therefore, the sequence \( v \) has also to be summable. We consider this quite general class of eigenvalues first. However, we illustrate condition (2.3) in Section 4 below by assuming a “regular decay” of the eigenvalues. Moreover, consider a strictly positive sequence of weights \( \omega := (\omega_j)_{j \geq 1} \). Then we shall measure the performance of an estimator \( \hat{\beta} \) of \( \beta \) by the \( \mathcal{F}_{\omega}-\text{risk} \), that is \( \mathbb{E}\| \hat{\beta} - \beta \|_2^2 \). In Section 4 this approach is illustrated by considering different weight sequences \( \omega \). Roughly speaking, an appropriate choice of \( \omega \) enables us to cover both the estimation of derivatives of \( \beta \) as well as the optimal estimation in terms of the mean prediction error. Finally, all the results below are derived under the following minimal regularity conditions.
Assumption 2.1. Let \( \omega := (\omega_j)_{j \geq 1} \), \( \gamma := (\gamma_j)_{j \geq 1} \) and \( \nu := (\nu_j)_{j \geq 1} \) be strictly positive sequences of weights with \( \omega_1 = 1 \), \( \gamma_1 = 1 \) and \( \nu_1 = 1 \) such that \( \gamma \) and \( (\gamma_j / \omega_j)_{j \geq 1} \) are nondecreasing and \( \nu \) is nonincreasing with \( \Lambda := \sum_j \nu_j < \infty \).

Note that under Assumption 2.1 the ellipsoid \( \mathcal{F}_\omega \) is a subset of \( \mathcal{F}_\omega^2 \), and hence the \( \mathcal{F}_\omega \)-risk a well-defined risk for \( \beta \). Roughly speaking, if \( \mathcal{F}_\omega^2 \) describes \( p \)-times differentiable functions, then the Assumption 2.1 ensures that the \( \mathcal{F}_\omega \)-risk involves maximal \( s \leq p \) derivatives.

Moment assumptions. The results derived below involve additional conditions on the moments of the random function \( X \) and the error term \( \varepsilon \), which we formalize now. Let \( \mathcal{X} \) be the set of all centered second order stationary random functions \( X \) with finite second moment, i.e., \( \mathbb{E}[|X|^2] < \infty \), and strictly positive covariance operator. Then given \( X \in \mathcal{X} \) the random variables \( \{\langle X, \psi_j \rangle / \sqrt{\lambda_j}, j \in \mathbb{N}\} \) are centered with variance one and moreover pairwise uncorrelated. Here and subsequently, \( \mathcal{X}^m \), \( m \in \mathbb{N} \), \( \eta \geq 1 \), denotes the subset of \( \mathcal{X} \) containing all random functions \( X \) such that the \( m \)-th moment of the corresponding standardized random variables \( \{\langle X, \psi_j \rangle / \sqrt{\lambda_j}, j \in \mathbb{N}\} \) are uniformly bounded, that is

\[
\mathcal{X}^m := \left\{ X \in \mathcal{X} \; \text{such that} \; \sup_{j \in \mathbb{N}} \mathbb{E}\left| \frac{\langle X, \psi_j \rangle}{\sqrt{\lambda_j}} \right|^m \leq \eta \right\},
\]

(2.4)

It is worth noting that in case \( X \in \mathcal{X} \) is a Gaussian random function the corresponding random variables \( \{\langle X, \psi_j \rangle / \sqrt{\lambda_j}, j \in \mathbb{N}\} \) form an i.i.d. sample of Gaussian random variables with mean zero and variance one. Hence, for each \( k \in \mathbb{N} \) there exists \( \eta \) such that any Gaussian random function \( X \in \mathcal{X} \) belongs also to \( \mathcal{X}^k \). In what follows, \( \mathcal{C}^m \) stands for the set of all centered error terms \( \varepsilon \) with variance one and finite \( m \)-th moment, i.e., \( \mathbb{E}[|\varepsilon|^m] \leq \eta \).

3 Optimality in the general case

Consistency. The \( \mathcal{F}_\omega \)-risk of the estimator \( \hat{\beta} \) given in (1.6) is essentially determined by the deviation of the estimators of \( (g_j)_j \) and \( (\lambda_j)_j \) and by the regularization error due to the threshold. The next assertion summarizes minimal conditions to ensure consistency of the estimator defined in (1.6).

Proposition 3.1 (Consistency). Assume an \( n \)-sample of \( (Y, X) \) satisfying (1.1) with \( \sigma > 0 \). Let \( \beta \in \mathcal{F}_\gamma \), \( X \in \mathcal{X}^2 \) and \( \varepsilon \in \mathcal{C}^4, \eta \geq 1 \). Consider the estimator \( \hat{\beta} \) with threshold \( m := m(n) \) and parameter \( \alpha := \alpha(n) \) satisfying \( m \to \infty \), \( \alpha = o(1) \) and \( \sup_{j \leq m} |\omega_j|/(\sigma^2)^{-1} = o(1) \) as \( n \to \infty \). If in addition \( \gamma \) and \( \omega \) satisfy Assumption 2.1, then \( \mathbb{E}[\|\hat{\beta} - \beta\|_2^2] = o(1) \) as \( n \to \infty \).

Remark 3.1. Since the last result covers the case \( \gamma \equiv \omega \equiv 1 \) it follows that the estimator \( \hat{\beta} \) is consistent without any additional restriction on \( \beta \in L^2[0,1] \) provided \( m \to \infty \), \( \alpha = o(1) \) and \( n\alpha^2 \to \infty \) as \( n \to \infty \). \( \square \)

The lower bound. It is well-known that in general the hardest one-dimensional subproblem does not capture the full difficulty in estimating the solution of an inverse problem even in case of a known operator (for details see e.g. the proof in Mair and Ruymgaart [1996]). In other words, there does not exist two sequences of slope functions \( \beta_{1,n}, \beta_{2,n} \in \mathcal{F}_\gamma^2 \), which are statistically not consistently distinguishable and satisfy \( \|\beta_{1,n} - \beta_{2,n}\|_2^2 \geq C\delta_n^* \), where \( \delta_n^* \) is the optimal rate of convergence. Therefore we need to consider subsets of \( \mathcal{F}_\gamma^2 \) with growing number of elements in order to get the optimal lower bound. More specific, we obtain
the following lower bound by applying Assouad’s cube technique (see e.g. Korostolev and Tsybakov [1993] or Chen and Reiß [2008]) under the additional assumption that the error term $\varepsilon$ is standard normal distributed, i.e., $\varepsilon \sim \mathcal{N}(0, 1)$, and independent of the regressor.

**Theorem 3.2.** Assume an $n$-sample of $(Y, X)$ obeying (1.1) with $\sigma > 0$. Suppose that the error term $\varepsilon \sim \mathcal{N}(0, 1)$ is independent of the second order stationary regressor $X$ with associated sequence of eigenvalues $(\lambda_j) \in S^d_{\eta}$. Consider $\mathcal{F}^\triangle$, $\rho > 0$, as set of slope functions. Let $m_\ast := m_\ast(n) \in \mathbb{N}$ and $\delta_n^\ast := \delta_n^\ast(m_\ast) \in \mathbb{R}^+$ for some $\Delta \geq 1$ be chosen such that

$$1/\Delta \leq \frac{\gamma_{m_\ast}}{\sqrt{n} \omega_{m_\ast}} \sum_{j=1}^{m_\ast} \frac{\omega_j}{v_j} \leq \Delta \quad \text{and} \quad \delta_n^\ast := \omega_{m_\ast}/\gamma_{m_\ast}. \quad (3.1)$$

If in addition the Assumption 2.1 is satisfied then for any estimator $\hat{\beta}$ we have

$$\sup_{\beta \in \mathcal{F}^\triangle} \left\{ \mathbb{E}\|\hat{\beta} - \beta\|_2^2 \right\} \geq \frac{1}{4\Delta} \min \left( \frac{\sigma^2}{2d}, \frac{\rho}{\Delta} \right) \max(\delta_n^\ast, 1/n).$$

**Remark 3.2.** The normality assumption in the last theorem is only used to simplify the calculation of the distance between distributions corresponding to different slope functions. Obviously the derived lower bound is still valid if we consider the less restrictive assumption that the error term $\varepsilon$ belongs to $\mathcal{E}_{\eta}^m$ for some $m \in \mathbb{N}$ and sufficiently large $\eta$. Furthermore, it is worth to note that the lower bound tends only to zero if $(\omega_j/\gamma_j)$ is a zero sequence. In other words, in case $\gamma \equiv 1$, i.e., without any additional restriction on $\beta \in L^2[0, 1]$, uniform consistency over $L^2[0, 1]$ in the $\mathcal{F}_\omega$-risk is only possible if the weighted norm $\|\cdot\|_\omega$ is weaker than the usual $L^2$-norm, that is, $\omega$ is a zero sequence. This obviously reflects the ill-posedness of the underlying inverse problem.

**The upper bound.** The next theorem states that the rate $\max(\delta_n^\ast, 1/n)$ of the lower bound given in Theorem 3.2 provides also an upper bound of the proposed estimator $\hat{\beta}$. Therefore the rate $\max(\delta_n^\ast, 1/n)$ is optimal and hence the estimator $\hat{\beta}$ is minimax-optimal.

**Theorem 3.3.** Assume an $n$-sample of $(Y, X)$ satisfying (1.1) with $\sigma > 0$. Suppose that the regressor $X$ is second order stationary with associated sequence of eigenvalues $(\lambda_j) \in S^d_{\eta}$. Consider $m_\ast := m_\ast(n)$ and $\delta_n^\ast := \delta_n^\ast(n)$ given in (3.1) for some $\Delta \geq 1$. Let $\hat{\beta}$ be the estimator defined in (1.6) with $m := m_\ast$ and $\alpha := (1/n) \min(1, \gamma_{m_\ast}/(2d\Delta))$. If in addition $X \in \mathcal{X}_{\eta}^{4k}$ and $\mathcal{E}_{\eta}^{4k}$, $k \geq 4$, then for some generic constant $C > 0$ we have

$$\sup_{\beta \in \mathcal{F}^\triangle} \left\{ \mathbb{E}\|\hat{\beta} - \beta\|_2^2 \right\} \leq C d^3 \Delta^3 \eta \left[ \sigma^2 d \Lambda + \sigma^2 \right] \max(\delta_n^\ast, 1/n),$$

for all sequences $\gamma$, $\omega$ and $\upsilon$ satisfying Assumption 2.1.

**Remark 3.3.** It is worth to note that the bound derived in the last theorem is non asymptotic. Furthermore, as in case of the lower bound (see Remark 3.2) also the upper bound tends only to zero, if $(\omega_j/\gamma_j)$ is a zero sequence. Therefore the estimator $\hat{\beta}$ is consistent even without any additional restriction on $\beta \in L^2[0, 1]$, i.e., $\gamma \equiv 1$, as long as $\omega$ is a zero sequence. We shall stress that from Theorem 3.3 follows that for all sequences $\gamma$, $\omega$ and $\upsilon$ satisfying the minimal regularity Assumption 2.1 the orthogonal series estimator $\hat{\beta}$ attains the optimal rate $\max(\delta_n^\ast, 1/n)$ and hence is minimax-optimal. In particular, it is easily seen that the optimal rate $\max(\delta_n^\ast, 1/n)$ is parametric if and only if $\sum_{j=1}^{\infty} \omega_j/v_j < \infty$. Hence,
in this case the rate of the orthogonal series estimator \( \hat{\beta} \) is parametric again without any additional restriction on \( \beta \in L^2[0,1] \), i.e., \( \gamma \equiv 1 \). Finally as long as the sequence \( \gamma \) is unbounded in Theorem 3.3 the threshold parameter \( \alpha \) satisfies \( \alpha = 1/n \) for all sufficiently large \( n \). Thus in this situation as open problem remains only how to choose the dimension parameter \( m \) adaptively from the data. We are currently exploring this issue. \( \square \)

4 Mean prediction error and derivative estimation

In this section we suppose the slope function \( \beta \) is an element of the Sobolev space of periodic functions \( \mathcal{W}_p \) given for \( p > 0 \) by

\[
\mathcal{W}_p = \left\{ f \in H_s : f^{(j)}(0) = f^{(j)}(1), \quad j = 0, 1, \ldots, p - 1 \right\},
\]

where \( H_p := \{ f \in L^2[0,1] : f^{(p-1)} \text{ absolutely continuous }, f^{(p)} \in L^2[0,1] \} \) is a Sobolev space (c.f. Neubauer [1988a,b], Mair and Ruymgaart [1996] or Tsybakov [2004]). However, if we consider the sequence of weights \( \{ w_j \} \in \mathbb{N} \) given by

\[
w_0^p = 1 \quad \text{and} \quad w_{2j}^p = w_{2j+1}^p = j^{2p}, \quad j \in \mathbb{N}.
\] (4.1)

Then the Sobolev space \( \mathcal{W}_p \) of periodic functions is equivalently given by \( \mathcal{F}_w \). Therefore, let us denote by \( \mathcal{W}_p^0 := \mathcal{F}_w^0, \rho > 0 \), an ellipsoid in the Sobolev space \( \mathcal{W}_p \). We use in case \( p = 0 \) again the convention that \( \mathcal{W}_p^0 \) denotes an ellipsoid in \( L^2[0,1] \).

Mean prediction error. We shall first measure the performance of an estimator \( \hat{\beta} \) by the mean prediction error (MPE), i.e., \( \mathbb{E}(T_{\text{cov}}(\hat{\beta} - \beta), (\hat{\beta} - \beta)) \). Consequently, if the sequence of eigenvalues \( (\lambda_j) \) associated to the covariance operator \( T_{\text{cov}} \) satisfies a link condition, that is \( (\lambda_j) \in \mathcal{S}_p^d \) for some weight sequence \( v \) (see definition (2.3)). Then the MPE is equivalent to the \( \mathcal{F}_w \)-risk with \( \omega \equiv v \), that is \( \mathbb{E}||\hat{\beta} - \beta||^2_v \approx_d \mathbb{E} \langle T_{\text{cov}}(\hat{\beta} - \beta), (\hat{\beta} - \beta) \rangle \). To illustrate the previous results we assume in the following the sequence \( v \) to be either polynomially decreasing, i.e., \( v_1 = 1 \) and \( v_j = |j|^{-2a}, j \geq 2 \), for some \( a > 1/2 \), or exponentially decreasing, i.e., \( v_1 = 1 \) and \( v_j = \exp(-|j|^{2a}), j \geq 2 \), for some \( a > 0 \). In the polynomial case easy calculus shows that a covariance operator \( T_{\text{cov}} \) with eigenvalues \( (\lambda_j) \in \mathcal{S}_v^d \), i.e., \( \lambda_j \approx_d |j|^{-2a} \), acts like integrating \( (2a)\)-times and hence it is called finitely smoothing (c.f. Natterer [1984]). This is the case considered, for example, in Crambes et al. [2009]. On the other hand in the exponential case it can easily be seen that the link condition \( (\lambda_j) \in \mathcal{S}_v^d \), i.e., \( \lambda_j \approx_d \exp(-|j|^{2a}) \), implies \( \mathcal{R}(T_{\text{cov}}) \subset \mathcal{W}_s \) for all \( s > 0 \), therefore the operator \( T_{\text{cov}} \) is called infinitely smoothing (c.f. Mair [1994]). Since in both cases the minimal regularity conditions given in Assumption 2.1 are satisfied, the lower bounds presented in the next assertion follow directly from Theorem 3.2. Here and subsequently, we write \( a_n \lesssim b_n \) when there exists \( C > 0 \) such that \( a_n \leq C b_n \) for all sufficiently large \( n \in \mathbb{N} \) and \( a_n \sim b_n \) when \( a_n \leq c b_n \) and \( b_n \lesssim a_n \) simultaneously.

**Proposition 4.1.** Under the assumptions of Theorem 3.2 we have for any estimator \( \hat{\beta} \)

(i) in the polynomial case, i.e. \( v_1 = 1 \) and \( v_j = |j|^{-2a}, j \geq 2 \), for some \( a > 1/2 \), that

\[
\sup_{\beta \in \mathcal{W}_p^0} \left\{ \mathbb{E} \left< T_{\text{cov}}(\hat{\beta} - \beta), (\hat{\beta} - \beta) \right> \right\} \gtrsim n^{-(2p+2a)/(2p+2a+1)},
\]

(ii) in the exponential case, i.e. \( v_1 = 1 \) and \( v_j = \exp(-|j|^{2a}), j \geq 2 \), for some \( a > 0 \), that

\[
\sup_{\beta \in \mathcal{W}_p^0} \left\{ \mathbb{E} \left< T_{\text{cov}}(\hat{\beta} - \beta), (\hat{\beta} - \beta) \right> \right\} \gtrsim n^{-1}(\log n)^{1/2a}.
\]
On the other hand, if the dimension parameter $m$ and the threshold $\alpha$ in the definition of the estimator $\beta$ given in (1.6) are chosen appropriate, then by applying Theorem 3.3 the rates of the lower bound given in the last assertion provide up to a constant also the upper bound of the risk of the estimator $\hat{\beta}$, which is summarized in the next proposition. We have thus proved that these rates are optimal and the proposed estimator $\hat{\beta}$ is minimax optimal in both cases.

**Proposition 4.2.** Under the assumptions of Theorem 3.3 consider the estimator $\hat{\beta}$

(i) in the polynomial case, i.e. $v_1 = 1$ and $v_j = |j|^{-2a}$, $j \geq 2$, for some $a > 1/2$, with dimension $m \sim n^{1/(2p+2a+1)}$ and threshold $\alpha \sim 1/n$. Then we have

$$\sup_{\beta \in \mathcal{W}_p} \left\{ E \left( T_{\text{cov}}(\hat{\beta} - \beta), (\hat{\beta} - \beta) \right) \right\} \lesssim n^{-1} (2p+2a)(2p+2a+1),$$

(ii) in the exponential case, i.e. $v_1 = 1$ and $v_j = \exp(-|j|^{2a})$, $j \geq 2$, for some $a > 0$, with dimension $m \sim (\log n)^{1/(2a)}$ and threshold $\alpha \sim 1/n$. Then

$$\sup_{\beta \in \mathcal{W}_p} \left\{ E \left( T_{\text{cov}}(\hat{\beta} - \beta), (\hat{\beta} - \beta) \right) \right\} \lesssim n^{-1}(\log n)^{1/2a}.$$  

**Remark 4.1.** It is of interest to compare our results with those of Crambes et al. [2009] who measure the performance of their estimator in terms of the prediction error. In their notations the decrease of the eigenvalues of $T_{\text{cov}}$ is assumed to be of order $(|j|^{-2q-1})$, i.e., $q = a - 1/2$. Furthermore they suppose the slope function to be $m$-times continuously differentiable, i.e., $m = p$. By using this reparametrization we see that our results in the polynomial case imply the same rate of convergence in probability of the prediction error as it is presented in Crambes et al. [2009]. However, from our general results follows a lower and an upper bound of the MPE not only in the polynomial case but also in the exponential case.

Furthermore, we shall emphasize the interesting influence of the parameters $p$ and $a$ characterizing the smoothness of $\beta$ and the decay of the eigenvalues of $T_{\text{cov}}$, respectively. As we see from Propositions 4.1 and 4.2, in the polynomial case an increasing value of $p$ leads to a faster optimal rate. In other words, as expected, a smoother regression function can be faster estimated. The situation in the exponential case is extremely different. It seems rather surprising that, contrary to the polynomial case, in the exponential case the optimal rate of convergence does not depend on the value of $p$, however this dependence is clearly hidden in the constant. Furthermore, the dimension parameter $m$ does not even depend on the value of $p$. Thereby, the proposed estimator is automatically adaptive, i.e., it does not involve an a-priori knowledge of the degree of smoothness of the slope function $\beta$. However, the choice of the dimension parameter depends on the value $a$ specifying the decay of the eigenvalues of $T_{\text{cov}}$. Note further that in both cases an increasing value of $a$ leads to a faster optimal rate of convergence, i.e., we may call $1/a$ a degree of ill-posedness (c.f. Natterer [1984]). Finally, we shall stress that Proposition 4.2 covers the case $p = 0$, i.e., $\hat{\beta}$ is consistent with optimal MPE-rate without additional restrictions on $\beta \in L^2[0,1]$. □

**Estimation of the derivatives.** Let us consider now the estimation of derivatives of the slope function $\beta$. It is well-known, that for any function $g$ belonging to a Sobolev-ellipsoid $\mathcal{W}_p^s = \mathcal{F}_{uw}^s$ with weights $w^p$ given in (4.1) the weighted norm $\|g\|_{w^s}$ for each $0 \leq s \leq p$ is equivalent to the $L^2$-norm of the $s$-th weak derivative $g^{(s)}$, that is, $\|g^{(s)}\|_{L^2} \lesssim \|g\|_{w^s}$. Thereby, the results in the Section 3 imply again a lower bound as well as an upper bound of the $L^2$-risk for the estimation of the $s$-th weak derivative of $\beta$. In the following we
In their notations the decrease of the eigenvalues of $T$ we see from Propositions 4.3 and 4.4, in both cases an decreasing of the value of $\beta = \alpha$. Furthermore the Fourier coefficients of the slope function decay at least with rate $s$. Horowitz [2007] in the polynomial case with the MPE by considering the $L$-hat by the relation exp($2\pi kt = 2^{-1/2}(\psi_{2k}(t) + \epsilon \psi_{2k+1}(t))$, for $k \in \mathbb{Z}$ and $t \in [0,1]$, with $\epsilon^2 = -1$, then for $0 \leq s < p$ the $s$-th derivative $\hat{\beta}(s)$ of $\beta$ in a weak sense is

$$\hat{\beta}(s)(t) = \sum_{k \in \mathbb{Z}} (2\pi k)^s \left( \int_0^1 \hat{\beta}(u) \exp(-2\pi ku) \, du \right) \exp(2\pi kt), \quad t \in [0,1]. \tag{4.2}$$

Note, that the sum in (4.2) contains only a finite number of nonzero summands and hence its numerical implementation is straightforward. Furthermore, if the dimension parameter $m$ and the threshold $\alpha$ in the definition of the estimator $\hat{\beta}$ given in (1.6) are chosen appropriate, then by applying Theorem 3.3 the rates of the lower bound given in the last assertion provide up to a constant again the upper bound of the $L^2$-risk of the estimator $\hat{\beta}(s)$, which is summarized in the next proposition. We have thus proved that these rates are optimal and the proposed estimator $\hat{\beta}(s)$ is minimax optimal in both cases.

**Proposition 4.4.** Under the assumptions of Theorem 3.3 consider the estimator $\hat{\beta}(s)$

(i) in the polynomial case, i.e. $v_1 = 1$ and $v_j = |j|^{-2a}$, $j \geq 2$, for some $a > 1/2$, with $m \sim n^{1/(2p+2a+1)}$ and threshold $\alpha \sim n$. Then

$$\sup_{\beta \in W_p^n} \{ \mathbb{E}[\hat{\beta}(s) - \beta(s)]^2 \} \lesssim n^{-(2p-2s)/(2p+2a+1)},$$

(ii) in the exponential case, i.e. $v_1 = 1$ and $v_j = \exp(-|j|^{2a})$, $j \geq 2$, for some $a > 0$, with $m \sim (\log n)^{1/(2a)}$ and threshold $\alpha \sim n$. Then

$$\sup_{\beta \in W_p^n} \{ \mathbb{E}[\hat{\beta}(s) - \beta(s)]^2 \} \lesssim (\log n)^{-(p-s)/a}.$$

**Remark 4.2.** It is worth noting that the $L^2$-risk in estimating the slope function $\beta$ itself, i.e., $s = 0$, has been considered in Hall and Horowitz [2007] only in the polynomial case. In their notations the decrease of the eigenvalues of $T_{cov}$ is of order $(\log n)^{-\alpha}$, i.e., $\alpha = 2a$. Furthermore the Fourier coefficients of the slope function decay at least with rate $j^{-\beta}$, i.e., $\beta = p + 1/2$. By using this reparametrization we see that we recover the result of Hall and Horowitz [2007] in the polynomial case with $s = 0$, but without the additional assumption $\beta > \alpha/2 + 1$ or $\beta > \alpha - 1/2$.

Furthermore, we shall discuss again the influence of the parameters $p$, $s$ and $a$. As we see from Propositions 4.3 and 4.4, in both cases an decreasing of the value of $a$ or an increasing of the value $p$ leads to a faster optimal rate of convergence. Hence, in opposite to the MPE by considering the $L^2$-risk the parameter $a$ describes in both cases the degree of ill-posedness. Furthermore, the estimation of higher derivatives of the slope function, i.e. by
considering a larger value of $s$, is as usual only possible with a slower optimal rate. Finally, as for the MPE in the exponential case the dimension parameter $m$ does not depend on the values of $p$ or $s$, hence the proposed estimator is automatically adaptive.

**Remark 4.3.** There is an interesting issue hidden in the parametrization we have chosen. Consider a classical indirect regression model with known operator given by $T_{\text{cov}}$, i.e., $Y = [T_{\text{cov}}\beta](U) + \varepsilon$ where $U$ has a uniform distribution on $[0, 1]$ and $\varepsilon$ is white noise (for details see e.g. Mair and Ruymgaart [1996]). If in addition the operator $T_{\text{cov}}$ is finitely smoothing, i.e., $(v_j)$ is polynomially decreasing with $v_j = j^{-2a}$, $j \geq 2$. Then given an $n$-sample of $Y$ the optimal rate of convergence of the $L^2$-risk of any estimator of $\beta(s)$ is of order $n^{-2(p-s)/(2(p+2a)+1)}$, since $R(T_{\text{cov}}) = \mathcal{W}_{2a}$ (c.f. Mair and Ruymgaart [1996] or Chen and Reiß [2008]). However, we have shown that in a functional linear model even with estimated operator the optimal rate is of order $n^{-2(p-s)/(2(p+a)+1)}$. Thus comparing both rates we see that in a functional linear model the covariance operator $T_{\text{cov}}$ has the degree of ill-posedness a while the same operator has in the indirect regression model a degree of ill-posedness (2a).

In other words in a functional linear model we do not face the complexity of an inversion of $T_{\text{cov}}$ but only of its square root $T_{\text{cov}}^{1/2}$. This, roughly speaking, may be seen as a multiplication of the normal equation $YX = \langle \beta, X \rangle X + X\varepsilon$ by the inverse of $T_{\text{cov}}^{1/2}$.

Notice that $T_{\text{cov}}$ is also the covariance operator associated to the error term $\varepsilon X$. Thus the multiplication by the inverse of $T_{\text{cov}}^{1/2}$ leads, roughly speaking, to white noise and hence to an indirect regression model rather defined by $T_{\text{cov}}^{1/2}$ than $T_{\text{cov}}$. The same finding holds true in case of an infinitely smoothing operator $T_{\text{cov}}$. However, in this situation $(\log n)^{-(p-s)/a}$ is the optimal rate in an indirect regression model given by $T_{\text{cov}}$ as well as $T_{\text{cov}}^{1/2}$. Thus, the above described effect is not visible formally, but is actually hidden in the order symbol. $\square$

## A Appendix

**Proposition A.1.** Let $X$ be second order stationary with $E[X(t)X(s)] = c(t - s)$, $t, s \in [0, 1]$, for some positive definite function $c : [-1, 1] \to \mathbb{R}$. Then the associated covariance operator $T_{\text{cov}}$ admits an eigenvalue decomposition with eigenfunctions given by the trigonometric basis defined in (2.1) and corresponding eigenvalues given by (2.2).

**Proof.** Let $f \in L^2[0, 1]$ and consider $g = T_{\text{cov}} f = \int_0^1 f(t)c(t - s)dt$. Since $c$ is even, it is straightforward to show that $\int_0^1 g(s)e^{-i\lambda}ds = \int_0^1 f(s)e^{-i\lambda}ds \int_{-1}^1 c(s)\cos(s\lambda)ds$ and $\int_0^1 g(s)e^{i\lambda}ds = \int_0^1 f(s)e^{i\lambda}ds \int_{-1}^1 c(s)\cos(s\lambda)ds$ for all $\lambda \in \mathbb{R}$. Due to this we obtain for all $\lambda \in \mathbb{R}$ the following identities

$$\int_0^1 g(s)\cos(s\lambda)ds = \int_0^1 f(s)\cos(s\lambda)ds \int_{-1}^1 c(s)\cos(s\lambda)ds,$$

$$\int_0^1 g(s)\sin(s\lambda)ds = \int_0^1 f(s)\sin(s\lambda)ds \int_{-1}^1 c(s)\cos(s\lambda)ds.$$ 

Consider the trigonometric basis $\{\psi_n\}$ and the values $\{\lambda_n\}$ given in (2.1) and (2.2), respectively, then we have just shown, that $\langle T_{\text{cov}} f, \psi_n \rangle = \langle f, \psi_n \rangle \lambda_n$ for all $f \in L^2[0, 1]$ and $n \in \mathbb{N}$, which proves the result. $\square$
A.1 Proofs of Section 3

We begin by defining and recalling notations to be used in the proofs:

\[ X_{ij} := \langle X_i, \psi_j \rangle, \quad \beta_j = \langle \beta, \psi_j \rangle, \quad T_{n,j} := \frac{1}{n} \sum_{i=1}^{n} (Y_i X_{ij} - X_{ij}^2 \beta_j), \quad \lambda_j = \mathbb{E}X_{ij}^2, \]

\[ \tilde{\beta}_m := \sum_{j=1}^{m} \beta_j \cdot 1 \{ \hat{\lambda}_j \geq \alpha \} \cdot \psi_j, \quad \beta_m := \sum_{j=1}^{m} \beta_j \cdot \psi_j. \]  

(A.1)

We shall prove in the end of this section two technical Lemma (A.2 - A.3) which are used in the following proofs.

Proof of consistency.

Proof of Proposition 3.1. The proof is based on the decomposition

\[ \mathbb{E}\|\hat{\beta} - \beta\|_\omega^2 \leq 2 \{ \mathbb{E}\|\hat{\beta} - \tilde{\beta}_m\|_\omega^2 + \mathbb{E}\|\tilde{\beta}_m - \beta\|_\omega^2 \}. \]  

(A.2)

We show below under the moment condition \( X \in \mathcal{X}_\eta \) defined in (2.4) and \( \varepsilon \in \mathcal{E}_\eta \) for some universal constant \( C > 0 \) the following bound

\[ \mathbb{E}\|\hat{\beta} - \tilde{\beta}_m\|_\omega^2 \leq C \left( \sup_{j \leq m} \omega_j \right) (n\alpha^2)^{-1} \mathbb{E}\|X\|_2^2 \{ \sigma^2 + \|\beta\|_2^2 \mathbb{E}\|X\|_2^2 \} \eta, \]  

(A.3)

while given \( \|\beta\|_\omega < \infty \) we conclude from Lebesgue’s dominated convergence theorem

\[ \mathbb{E}\|\tilde{\beta}_m - \beta\|_\omega^2 = o(1) \]  

in case that \( 1/m = o(1), \alpha = o(1) \) as \( n \to \infty \). (A.4)

Thereby, the conditions on \( m \) and \( \alpha \) ensure the convergence to zero of the two terms on the right hand side in (A.2) as \( n \to \infty \), which gives the result.

Proof of (A.3). By making use of the notations given in (A.1) it follows that

\[ \mathbb{E}\|\hat{\beta} - \tilde{\beta}_m\|_\omega^2 = \sum_{j=1}^{m} \omega_j \mathbb{E}\frac{\hat{g}_j - \beta_j \hat{\lambda}_j}{\lambda_j^2} 1 \{ \hat{\lambda}_j \geq \alpha \} \leq \frac{1}{\alpha^2} \sum_{j=1}^{m} \omega_j \mathbb{E}|T_{n,j}|^2 \]

and hence by using (A.10) in Lemma A.2 we obtain (A.3).

The proof of (A.4) is based on the decomposition

\[ \mathbb{E}\|\tilde{\beta}_m - \beta\|_\omega^2 \leq 2 \left\{ \sum_{j=1}^{\infty} \omega_j \beta_j^2 1 \{ j > m \} + \sum_{j=1}^{m} \omega_j \beta_j^2 P(\hat{\lambda}_j < \alpha) \right\} \leq 2 \sum_{j=1}^{\infty} \omega_j \beta_j^2 = \|\beta\|_\omega^2 < \infty. \]

Thus Lebesgue’s dominated convergence theorem implies the result since in case \( 1/m = o(1) \) and \( \alpha = o(1) \) as \( n \to \infty \) for each \( j \in \mathbb{N} \) \( 1 \{ j > m \} = 0 \) and \( P(\hat{\lambda}_j < \alpha) = o(1) \), which can be realized as follows. By using that \( \alpha = o(1) \) as \( n \to \infty \) there exists \( n_j > 0 \) such that for all \( n \geq n_j \) it holds \( \lambda_j \geq 2n \) and hence \( P(\hat{\lambda}_j < \alpha) \leq P(\hat{\lambda}_j/\lambda_j < 1/2) \) together with (A.12) in Lemma A.2 implies the assertion, which completes the proof.  

\[ \square \]
Proof of Theorem 3.2. Let $X_i$, $i \in \mathbb{N}$, be i.i.d. copies of $X$ which is second order
stationary with associated sequence of eigenvalues $(\lambda_j)_{j \geq 1} \in \mathcal{S}^d$. Consider independent
error terms $\varepsilon_i \sim \mathcal{N}(0, 1)$, $i \in \mathbb{N}$, which are independent of the random functions \{X_i\}. Let
$\theta \in \{-1, 1\}^{m_*}$, where $m_* := m_* (n) \in \mathbb{N}$ satisfies (3.1) for some $\delta \geq 1$. Consider the $m*$
vector $b$ of coefficients $b_j$ given in (A.15) in Lemma A.3. For each $\theta$ define a slope function
$\beta_\theta := \sum_{j=1}^{m_*} \theta_j \beta_j \psi_j$ which belongs to $\mathcal{F}^d$ due to (A.16) in Lemma A.3. Consequently, for
each $\theta$ the random variables $(Y_i, X_i)$ with $Y_i := \int_0^1 \beta_\theta(s)X_i(s)ds + \sigma \varepsilon_i$, $i = 1, \ldots, n$, form
a sample of the model (1.1) and we denote its joint distribution by $P_{\theta}$. Furthermore, for
$j = 1, \ldots, m_*$ and each $\theta$ we introduce $\theta^{(j)}$ by $\theta^{(j)} = \theta_j$ for $j \neq l$ and $\theta^{(j)} = -\theta_j$. As in case
of $P_{\theta}$ the conditional distribution of $Y_i$ given $X_i$ is Gaussian with mean $\sum_{j=1}^{m_*} \theta_j \beta_j \psi_j$
and variance $\sigma^2$ it is easily seen that the log-likelihood of $P_{\theta}$ w.r.t. $P_{\theta}$ is given by
\[
\log \left( \frac{dP_{\theta}(\theta)}{dP_{\theta}} \right) = -\frac{1}{\sigma^2} \sum_{i=1}^{n} \left( Y_i - \sum_{l=1}^{m_*} \theta_l \beta_l \psi_l \right) \theta_l \beta_l \psi_l \psi_l - \frac{2}{\sigma^2} \sum_{i=1}^{n} b_j^2 X_{ij}^2
\]
and its expectation w.r.t. $P_{\theta}$ satisfies $\mathbb{E}_{P_{\theta}}[\log(dP_{\theta}(\theta)/dP_{\theta})] = -(2n/\sigma^2) b_j^2 \mathbb{E}X_{ij}^2$. In terms of
Kullback-Leibler divergence this means $KL(P_{\theta}(\theta), P_{\theta}) = (2n/\sigma^2) b_j^2 \mathbb{E}X_{ij}^2 \leq (2dn/\sigma^2) b_j^2 v_j$
by using that $(\lambda_j)_{j \geq 1} \in \mathcal{S}^d$. Since the Hellinger distance $H(P_{\theta}(\theta), P_{\theta})$ satisfies
$H^2(P_{\theta}(\theta), P_{\theta}) \leq KL(P_{\theta}(\theta), P_{\theta})$ it follows from (A.16) in Lemma A.3 that
\[
H^2(P_{\theta}(\theta), P_{\theta}) \leq \frac{2dn}{\sigma^2} \cdot b_j^2 \cdot v_j \leq 1, \quad j = 1, \ldots, m_*.
\] (A.5)
Consider the Hellinger affinity $\rho(P_{\theta}(\theta), P_{\theta}) = \int dP_{\theta}(\theta)dP_{\theta}$, then for any estimator $\tilde{\beta}$ follows
\[
\rho(P_{\theta}(\theta), P_{\theta}) \leq \int \frac{|\langle \tilde{\beta} - \beta_{\theta}(\theta) , \psi_j \rangle|}{|\langle \beta_{\theta} - \beta_{\theta}(\theta) , \psi_j \rangle|} \sqrt{dP_{\theta}(\theta)dP_{\theta}} + \int \frac{|\langle \tilde{\beta} - \beta_{\theta}(\theta) , \psi_j \rangle|}{|\langle \beta_{\theta} - \beta_{\theta}(\theta) , \psi_j \rangle|} \sqrt{dP_{\theta}(\theta)dP_{\theta}}
\leq \left( \int \frac{|\langle \tilde{\beta} - \beta_{\theta}(\theta) , \psi_j \rangle|^2}{|\langle \beta_{\theta} - \beta_{\theta}(\theta) , \psi_j \rangle|^2} dP_{\theta}(\theta) \right)^{1/2} + \left( \int \frac{|\langle \tilde{\beta} - \beta_{\theta}(\theta) , \psi_j \rangle|^2}{|\langle \beta_{\theta} - \beta_{\theta}(\theta) , \psi_j \rangle|^2} dP_{\theta}(\theta) \right)^{1/2}.
\] (A.6)
Due to the identity $\rho(P_{\theta}(\theta), P_{\theta}) = 1 - \frac{1}{2}H^2(P_{\theta}(\theta), P_{\theta})$ combining (A.5) with (A.6) yields
\[
\left\{ \mathbb{E}_{\theta}(\langle \beta - \beta_{\theta}(\theta) , \psi_j \rangle)^2 + \mathbb{E}_{\theta}(\langle \beta - \beta_{\theta}(\theta) , \psi_j \rangle)^2 \right\} \geq \frac{1}{2} b_j^2, \quad j = 1, \ldots, m_*
\]
From this we conclude for each estimator $\tilde{\beta}$ that
\[
\sup_{\tilde{\beta} \in \mathcal{F}^d} \mathbb{E}_{\tilde{\beta}}[\tilde{\beta}^2] \geq \sup_{\theta \in \{-1, 1\}^{m_*}} \mathbb{E}_{\theta}[\|\tilde{\beta} - \beta_{\theta}\|^2] \geq \frac{1}{2m_*} \sum_{\theta \in \{-1, 1\}^{m_*}} \sum_{j=1}^{m_*} \omega_j \mathbb{E}_{\theta}(|\tilde{\beta} - \beta_{\theta}, \psi_j|^2)
\geq \frac{1}{2m_*} \sum_{\theta \in \{-1, 1\}^{m_*}} \frac{1}{2} \sum_{j=1}^{m_*} \omega_j \left\{ \mathbb{E}_{\theta}(|\tilde{\beta} - \beta_{\theta}, \psi_j|^2) + \mathbb{E}_{\theta}(\langle \tilde{\beta} - \beta_{\theta}, \psi_j \rangle)^2 \right\}
\geq \frac{1}{4} \sum_{j=1}^{m_*} b_j^2 \omega_j \geq \frac{1}{4} \min \left( \frac{\sigma^2}{2d^2 \Delta}, \frac{\rho d^2}{\Delta} \right)
\]
where the last inequality follows again from (A.16) in Lemma A.3, which completes the proof. \hfill \Box
Proof of the upper bound.

**Proof of Theorem 3.3.** The proof is based on the decomposition (A.2), where we show below under the condition $X \in \mathcal{X}_\eta^8$, $\varepsilon \in \mathcal{E}_\eta^8$ and $\lambda_j \geq 2\alpha$, $1 \leq j \leq m$, for some generic constant $C > 0$ the following two bounds

\[
\mathbb{E}\|\tilde{\beta} - \tilde{\beta}_m\|_\omega^2 \leq C \frac{m}{\eta} \sum_{j=1}^m \frac{\omega_j}{\lambda_j} \left\{ \|\beta\|^2 \mathbb{E}\|X\|^2 + \sigma^2 \right\} \eta \left\{ \lambda_j^2/(n\alpha)^2 + (1/n) + 1 \right\}, \tag{A.7}
\]

\[
\mathbb{E}\|\tilde{\beta}_m - \beta\|_\omega^2 \leq C \{\omega_m/\gamma_m + \eta/n\} \|\beta\|_\omega^2. \tag{A.8}
\]

Consequently, for all $\beta \in \mathcal{F}_1^d$ and $(\lambda_j)_{j \geq 1} \in \mathcal{S}_d^\eta$, i.e., $\lambda_j \leq d v_j \leq d$ and $\mathbb{E}\|X\|^2 \leq d \Delta$, follows

\[
\mathbb{E}\|\tilde{\beta} - \beta\|_\omega^2 \leq C \left\{ d (d^2/(n\alpha)^2 + 1/n + 1) \sum_{j=1}^m \frac{\omega_j}{n v_j} + \omega_m/\gamma_m + 1/n \right\} \eta [\rho d \Delta + \sigma^2].
\]

Let $m_*$ and $\delta_n^\alpha$ be given by (3.1) for some $\Delta \geq 1$ then the condition on $m$ and $\alpha$, i.e., $m = m_*$ and $\alpha = (1/n) \min(1, \gamma_m/(2d\Delta))$, implies

\[
\mathbb{E}\|\tilde{\beta} - \beta\|_\omega^2 \leq C d (d^2/(n\alpha)^2 + 1/n + 1) \Delta \max(\delta_n^\alpha, 1/n) \eta [\rho d \Delta + \sigma^2].
\]

because $\omega_m/\gamma_m = \delta_n^\alpha$, $\sum_{j=1}^m \omega_j/(n v_j) \leq \Delta \delta_n^\alpha$ and $\lambda_j \geq 2\alpha$, $1 \leq j \leq m$ by using that $\gamma_m \geq \gamma_m/(n\Delta)$ and $(\lambda_j)_{j \geq 1} \in \mathcal{S}_d^\eta$. Hence, from $n\alpha \geq 1/(2d\Delta)$ follows the result.

Proof of (A.7). By using $T_{n,j}$ introduced in (A.1) we obtain the identity

\[
\mathbb{E}\|\tilde{\beta}_\omega - \tilde{\beta}_\omega^{n}\|_\omega^2 = \sum_{j=1}^m \frac{\omega_j}{\lambda_j} \cdot \mathbb{E} \left[ \frac{T_{n,j}^2}{\lambda_j} \cdot \left( \lambda_j/\hat{\lambda}_j \right) \right]^{2} \left( \hat{\lambda}_j \geq \alpha \right) \tag{A.9}
\]

By using the elementary inequality $1/2 \leq |\hat{\lambda}_j/\lambda_j - 1|^2 + |\hat{\lambda}_j/\lambda_j|^2$ it follows that

\[
|\lambda_j/\hat{\lambda}_j|^{2} \left( \hat{\lambda}_j \geq \alpha \right) \leq 2 \left\{ 2(\lambda_j/\alpha)^2 |\hat{\lambda}_j/\lambda_j - 1|^4 + 2|\hat{\lambda}_j/\lambda_j - 1|^2 + 1 \right\}.
\]

Therefore, by combination of the last estimate and (A.9) we have

\[
\mathbb{E}\|\tilde{\beta} - \tilde{\beta}_\omega^{n}\|_\omega^2 \leq 4 \sum_{j=1}^m \frac{\omega_j}{\lambda_j} \left( \mathbb{E} |T_{n,j}|^4 \right)^{1/2} \left( \lambda_j/\alpha \right)^2 \left( \mathbb{E} |\hat{\lambda}_j/\lambda_j - 1|^8 \right)^{1/2} \left( \mathbb{E} |\hat{\lambda}_j/\lambda_j - 1|^4 \right)^{1/2} + 1
\]

The estimate (A.7) follows now from (A.10) and (A.12) in Lemma A.2.

Proof of (A.8). Following along the lines of the proof of (A.4) we obtain

\[
\mathbb{E}\|\tilde{\beta}_m - \beta\|_\omega^2 \leq 2\{\|\beta_m - \beta\|_\omega^2 + C(n/\eta) \|\beta_m\|_\omega^2 \}
\]

where under the condition $\lambda_j \geq 2\alpha$ for each $1 \leq j \leq m$ we have used that $P(\hat{\lambda}_j < \alpha) \leq C \eta/n$. Then, under Assumption 2.1, i.e., $(\omega_j/\gamma_j)$ is non-increasing, the usual estimate $\|\beta_m - \beta\|_\omega^2 \leq \omega_m/\gamma_m \|\beta\|_\omega^2$ implies (A.8), which completes the proof. \qed
Technical assertions.

The following two lemmas gather technical results used in the proof of Proposition 3.1, Theorem 3.2 and Theorem 3.3.

**Lemma A.2.** Suppose \( X \in \mathcal{X}_n^{4m} \) and \( \varepsilon \in \mathcal{E}_n^{4m}, \ m \in \mathbb{N} \). Then for some constant \( C > 0 \) only depending on \( m \) we have

\[
\sup_{j \in \mathbb{N}} \left\{ \frac{\lambda_j}{\varepsilon_j} \cdot \mathbb{E}[T_{n,j}]^{2m} \right\} \leq C \cdot n^{-m} \cdot \left\{ \| \beta \|^{2m} \cdot (\mathbb{E}[\| X \|^2])^m + \sigma^{2m} \right\} \cdot \eta, \tag{A.10}
\]

\[
\sup_{j \in \mathbb{N}} \mathbb{E}[\lambda_j / \varepsilon_j - 1]^{2m} \leq C \cdot n^{-m} \cdot \eta. \tag{A.11}
\]

If in addition \( w_1 \geq 2 \) and \( w_2 \leq 1/2 \), then we obtain

\[
\sup_{j \in \mathbb{N}} P(\lambda_j / \varepsilon_j \geq w_1) \leq C \cdot n^{-m} \cdot \eta \ \text{and} \ \sup_{j \in \mathbb{N}} P(\lambda_j / \varepsilon_j < w_2) \leq C \cdot n^{-m} \cdot \eta. \tag{A.12}
\]

**Proof.** Let \( \zeta_{ij} := \sum_{i \neq j} \beta_i X_{il} \), \( i = 1, \ldots, n \) and \( j \in \mathbb{N} \). Then we have

\[
T_{n,j} = \frac{1}{n} \sum_{i=1}^{n} \{ \zeta_{ij} + \sigma \varepsilon_i \} X_{ij} =: T_1 + T_2,
\]

where we bound below each summand separately, that is

\[
\mathbb{E}[T_1]^{2m} \leq C \cdot \lambda_j^{2m} / \varepsilon_j^{2m} \cdot \| \beta \|^{2m} \cdot (\mathbb{E}[\| X \|^2])^m \cdot \eta, \tag{A.13}
\]

\[
\mathbb{E}[T_2]^{2m} \leq C \cdot \lambda_j^{2m} / \varepsilon_j^{2m} \cdot \sigma^{2m} \cdot \eta \tag{A.14}
\]

for some \( C > 0 \) only depending on \( m \). Consequently, the inequality (A.10) follows from (A.13) and (A.14). Consider \( T_1 \). For each \( j \in \mathbb{N} \) the random variables \( (\zeta_{ij} \cdot X_{ij}), i = 1, \ldots, n, \) are independent and identically distributed with mean zero. From Theorem 2.10 in Petrov [1995] we conclude \( \mathbb{E}[T_1]^{2m} \leq C n^{-m} \mathbb{E}[\zeta_{ij} X_{ij}]^{2m} \) for some constant \( C > 0 \) only depending on \( m \). Then we claim that (A.13) follows in case of \( T_1 \) from the Cauchy-Schwarz inequality together with \( X_1 \in \mathcal{X}_n^{4m}, \ i.e., \sup_j \mathbb{E}[X_{ij} / \sqrt{\lambda_j}]^{4m} \leq \eta \). Indeed, we have

\[
\mathbb{E}[\zeta_{ij} X_{ij}]^{2m} \leq (\sum_{l \neq j} \beta_l^2)^m \sum_{l_1 \neq j} \cdots \sum_{l_m \neq j} \mathbb{E}[X_{ij}]^{2m} \prod_{k=1}^{m} |X_{il_k}|^2 \leq \| \beta \|^{2m} \cdot \lambda_j^m \cdot (\sum_{l \neq j} \lambda_l)^m \cdot \eta.
\]

Consider \( T_2 \). (A.14) follows in analogy to the case of \( T_1 \), because \( \{ \sigma \varepsilon_i X_{ij} \} \) are independent and identically distributed with mean zero, and \( \mathbb{E}[\sigma \varepsilon_i X_{ij}]^{2m} \leq \sigma^{2m} \cdot \lambda_j^m \cdot \eta \).

Proof of (A.11). Since \( \{ |X_{ij}|^2 / \lambda_j - 1 \} \) are independent and identically distributed with mean zero, and \( \mathbb{E}[X_{ij}^2 / \lambda_j]^{2m} \leq \eta \), the result follows by applying Theorem 2.10 in Petrov [1995].

Proof of (A.12). If \( w \geq 2 \) then \( P(\lambda_j / \varepsilon_j \geq w) \leq P(|\lambda_j / \varepsilon_j - 1| \geq 1) \). Thus applying Markov’s inequality together with (A.11) implies the first bound in (A.12), while the second follows in analogy, which proves the lemma.

**Lemma A.3.** Let \( m_s \in \mathbb{N} \) and \( \delta_n^s \) be chosen such that (3.1) is satisfied for some \( \Delta \geq 1 \). Consider a (infinite) vector \( b \) with components \( b_j \) satisfying

\[
b_j^2 = \frac{\zeta}{n \cdot v_j}, \quad j \in \mathbb{N}, \quad \text{with} \quad \zeta := \min \left( \frac{\sigma^2}{2d}, \frac{\rho}{\Delta} \right), \tag{A.15}
\]
then we have for all $j \in \mathbb{N}$

\[
\frac{2dn}{\sigma^2} b_j^2 v_j \leq 1, \quad \sum_{j=1}^{m_*} b_j^2 \gamma_j \leq \rho, \quad \text{and} \quad \sum_{j=1}^{m_*} b_j^2 \omega_j \geq \min \left( \frac{\sigma^2}{2d} \frac{\rho}{\Delta} \right) \max \left( \delta_n^*, 1/n \right).
\]

(A.16)

**Proof.** The first inequality in (A.16) follows trivially by using the definition of $\zeta$. Since by Assumption 2.1 the sequence $\gamma_j/\omega_j$ is nondecreasing the definition of $m_*$ given in (3.1) implies the second estimate in (A.16), i.e.,

\[
\sum_{j=1}^{m_*} b_j^2 \gamma_j \leq \zeta(\gamma_m/\omega_m) \sum_{j=1}^{m_*} \omega_j/(nv_j) \leq \zeta \Delta \leq \rho.
\]

To deduce the third inequality in (A.16) from the definition of $m_*$ and $\delta_n^*$ observe that

\[
\sum_{j=1}^{m_*} b_j^2 \omega_j = \delta_n^* \zeta(\gamma_m/\omega_m) \sum_{j=1}^{m_*} \omega_j/(nv_j) \geq \delta_n^* \zeta/\Delta \quad \text{and} \quad \sum_{j=1}^{m_*} b_j^2 \omega_j \geq \zeta/n \quad \text{since} \quad \omega_1/v_1 = 1,
\]

which proves the lemma. \hfill \Box

### A.2 Proofs of Section 4

#### The mean prediction error.

**Proof of Proposition 4.1.** Given the eigenvalues $(\lambda_j)$ of $T_{\text{cov}}$ satisfy a link condition, that is $(\lambda_j) \in S_d^1, d \geq 1$. It follows that $E[\|\beta - \beta^*\|_v^2] \geq d E(T_{\text{cov}}(\beta - \beta^*), \beta - \beta)$. Therefore, we can apply the general results by considering the $F_\omega$-risk with $\omega \equiv v$. Furthermore, in case (i) the definition of $\gamma \equiv w^p$ and $v$ imply together $(\gamma_m/\omega_m) \sum_{j=1}^{m_*} \omega_j/v_j = m_*^{2a+2p+1}$. It follows that the condition on $m_*$ and $\delta_n^*$ given in (3.1) of Theorem 3.2 can be rewritten as $m_* \sim n^{1/(2p+2a+1)}$ and $\delta_n^* \sim n^{-(2p+2a)/(2p+2a+1)}$, respectively. On the other hand, in case (ii) $(\gamma_m/\omega_m) \sum_{j=1}^{m_*} \omega_j/v_j = m_*^{2p+1} \exp(m_*^{2a})$ implies that the condition on $m_*$ and $\delta_n^*$ writes $m_* \sim (\log n)^{1/(2a)}$ and $\delta_n^* \sim n^{-1}(\log n)^{1/(2a)}$, respectively. Consequently, the lower bounds in Proposition 4.1 follow by applying Theorem 3.2. \hfill \Box

**Proof of Proposition 4.2.** Since in both cases the condition on the dimension parameter $m$ and the threshold $\alpha$ ensures that $m \sim m_*$ and $\alpha \sim 1/n$ (see the proof of Proposition 4.1) the result follows from Theorem 3.3. \hfill \Box

#### The estimation of derivatives.

**Proof of Proposition 4.3.** Due to $E\|\hat{\beta}^{(s)} - \beta^{(s)}\|^2 \asymp (2n)^{2s} E\|\hat{\beta} - \beta\|^2_{\text{diag}}, 0 \leq s \leq p$, we can apply again the general results by considering the $F_\omega$-risk with $\omega \equiv w^s$. In case (i) the well-known approximation $\sum_{j=1}^{m_*} j^r \sim m_*^{r+1}$ for $r > 0$ together with the definition of $\gamma \equiv w^p$ and $v$ implies $(\gamma_m/\omega_m) \sum_{j=1}^{m_*} \omega_j/v_j \sim m_*^{2a+2p+1}$. It follows that the condition on $m_*$ and $\delta_n^*$ given in (3.1) of Theorem 3.2 writes $m_* \sim n^{(2p+2a+1)}$ and $\delta_n^* \sim n^{-(2p-2a)/(2p+2a+1)}$, respectively. On the other hand, in case (ii) by applying Laplace’s Method (c.f. chapter 3.7 in Olver [1974]) the definition of $\gamma \equiv w^p$ and $v$ imply $(\gamma_m/\omega_m) \sum_{j=1}^{m_*} \omega_j/v_j \sim m_*^{2p} \exp(m_*^{2a})$. Therefore, the condition on $m_*$ and $\delta_n^*$ can be rewritten as $m_* \sim (\log n)^{1/(2a)}$ and $\delta_n^* \sim n^{-1}(\log n)^{1/(2a)}$, respectively. Consequently, the lower bounds in Proposition 4.1 follow by applying Theorem 3.2. \hfill \Box

**Proof of Proposition 4.4.** Since in both cases the condition on the dimension parameter $m$ and the threshold $\alpha$ ensures that $m \sim m_*$ and $\alpha \sim 1/n$ (see the proof of Proposition 4.3) the result follows from Theorem 3.3. \hfill \Box
References

D. Bosq. *Linear Processes in Function Spaces.*, volume 149 of *Lecture Notes in Statistics*. Springer-Verlag, 2000.

H. Cardot, F. Ferraty, and P. Sarda. Spline estimators for the functional linear model. *Statistica Sinica*, 13:571–591, 2003.

H. Cardot, A. Mas, and P. Sarda. CLT in functional linear regression models. *Prob. Theory and Rel. Fields*, to appear, 2007.

X. Chen and M. Reiß. On rate optimality for ill-posed inverse problems in econometrics. Technical report, Yale University, 2008.

C. Crambes, A. Kneip, and P. Sarda. Smoothing splines estimators for functional linear regression. *Annals of Statistics*, 37(1):35–72, 2009.

J. Dauxois, A. Pousse, and Y. Romain. Asymptotic theory for principal components analysis of a random vector function: some applications to statistical inference. *Journal of Multivariate Analysis*, 12:136–154, 1982.

P. H. Eilers and B. D. Marx. Flexible smoothing with b-splines and penalties. *Statistical Science*, 11:89–102, 1996.

H. W. Engl, M. Hanke, and A. Neubauer. *Regularization of inverse problems*. Kluwer Academic, Dordrecht, 2000.

F. Ferraty and P. Vieu. *Nonparametric Functional Data Analysis: Methods, Theory, Applications and Implementations*. Springer-Verlag, London, 2006.

M. Forni and L. Reichlin. Let’s get real: A factor analytical approach to disaggregated business cycle dynamics. *Review of Economic Studies*, 65:453–473, 1998.

P. Hall and J. L. Horowitz. Methodology and convergence rates for functional linear regression. *Annals of Statistics*, 35(1):70–91, 2007.

J. Johannes. Deconvolution with unknown error distribution. Forthcoming in Annals of Statistics, 2009.

A. P. Korostolev and A. B. Tsybakov. *Minimax Theory for Image Reconstruction.*, volume 82 of *Lecture Notes in Statistics*. Springer-Verlag, 1993.

B. A. Mair. Tikhonov regularization for finitely and infinitely smoothing operators. *SIAM Journal on Mathematical Analysis*, 25:135–147, 1994.

B. A. Mair and F. H. Ruymgaart. Statistical inverse estimation in Hilbert scales. *SIAM Journal on Applied Mathematics*, 56(5):1424–1444, 1996.

H.-G. Müller and U. Stadtmuller. Generalized functional linear models. *Ann. Stat.*, 33:774–805, 2005.

F. Natterer. Error bounds for Tikhonov regularization in Hilbert scales. *Applicable Analysis*, 18:29–37, 1984.
A. Neubauer. When do Sobolev spaces form a Hilbert scale? *Proc. Amer. Math. Soc.*, 103 (2):557–562, 1988a.

A. Neubauer. An a posteriori parameter choice for Tikhonov regularization in Hilbert scales leading to optimal convergence rates. *SIAM J. Numer. Anal.*, 25(6):1313–1326, 1988b.

M. H. Neumann. On the effect of estimating the error density in nonparametric deconvolution. *Journal of Nonparametric Statistics*, 7:307–330, 1997.

F. Olver. *Asymptotics and special functions*. Academic Press, 1974.

V. V. Petrov. *Limit theorems of probability theory. Sequences of independent random variables*. Oxford Studies in Probability. Clarendon Press., Oxford, 4. edition, 1995.

C. Preda and G. Saporta. PLS regression on a stochastic process. *Computational Statistics & Data Analysis*, 48:149–158, 2005.

J. Ramsay and B. Silverman. *Functional Data Analysis*. Springer, New York, second ed. edition, 2005.

J. O. Ramsay and C. J. Dalzell. Some tools for functional data analysis. *Journal of the Royal Statistical Society, Series B*, 53:539–572, 1991.

U. Tautenhahn. Error estimates for regularization methods in Hilbert scales. *SIAM Journal on Numerical Analysis*, 33(6):2120–2130, 1996.

A. B. Tsybakov. *Introduction à l’estimation non-paramétrique (Introduction to nonparametric estimation)*. Mathématiques & Applications (Paris). 41. Springer: Paris, 2004.