One-loop $\beta$-functions in 4-derivative gauge theory in 6 dimensions

Lorenzo Casarin$^a$ and Arkady A. Tseytlin$^{b,c,1}$

$^a$Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut), Am Mühlenberg 1, DE-14476 Potsdam, Germany
$^b$Blackett Laboratory, Imperial College, London SW7 2AZ, U.K.
$^c$Institute of Theoretical and Mathematical Physics, Moscow State University, Moscow 119991, Russia

E-mail: lorenzo.casarin@aei.mpg.de, tseytlin@imperial.ac.uk

ABSTRACT: A classically scale-invariant 6d analog of the 4d Yang-Mills theory is the 4-derivative $(\nabla F)^2 + F^3$ gauge theory with two independent couplings. Motivated by a search for a perturbatively conformal but possibly non-unitary 6d models we compute the one-loop $\beta$-functions in this theory. A systematic way of doing this using the background field method requires the (previously unknown) expression for the $b_6$ Seeley-DeWitt coefficient for a generic 4-derivative operator; we derive it here. As an application, we also compute the one-loop $\beta$-function in the $(1,0)$ supersymmetric $(\nabla F)^2$ 6d gauge theory constructed in hep-th/0505082.

KEYWORDS: Field Theories in Higher Dimensions, Supersymmetric Gauge Theory

ArXiv ePrint: 1907.02501

1Also at the Lebedev Institute, Moscow.
1 Introduction

Like Einstein theory in 4 dimensions, the 6d Yang-Mills theory with the standard $F^2$ action has dimensional coupling and is not power-counting renormalizable. A 6d analog of the classically scale invariant and renormalizable $R^2 + C^2$ 4d gravity is the 4-derivative $(\nabla F)^2 + F^3$ gauge theory. Such 4-derivative terms are induced as counterterms when considering the standard scalars, fermions or YM vectors coupled to a background gauge field in 6d [1]. While non-unitary, this model may serve as a building block of possible higher-derivative (super)conformal theories in 6 dimensions.\footnote{Similar 4-derivative 6d gauge theories were discussed, e.g., in [5–13].}

The aim of the present paper is to compute the one-loop $\beta$-functions in the Euclidean 6d theory with the action\footnote{In 4 dimensions the $F^2 + (\nabla F)^2 + F^3$ theory was studied in [2] and later in [3]. The result of [2] for the one-loop divergences in this 4d theory was corrected in [4] making it in agreement with that of [3].}

\begin{equation}
S = \frac{1}{g^2} \int d^6x \left[ (\nabla_m F_{mn})^2 + 2\gamma F_{mn} F_{nk} F_{km} \right] \equiv \frac{1}{2g^2} \int d^6x \left[ (\nabla_m F^a_{mn})^2 + \gamma f^{abc} F^a_{mn} F^b_{nk} F^c_{km} \right].
\end{equation}

Here $g$ and $\gamma$ are the two independent dimensionless coupling parameters.\footnote{We use $m, n, k, \ldots = 1, \ldots, 6$ for coordinate indices and flat Euclidean 6d metric so that the position of contracted indices is irrelevant. The gauge group generators are normalized as $\text{tr}(t^a t^b) = -T_R \delta^{ab}$, $[t^a, t^b] = f^{abc} t^c$, where $T_R = \frac{1}{2}$ in the fundamental representation of $SU(N)$ (we denote the trace in this case as $\text{Tr}$) and $T_R = C_2 = N$ in the adjoint representation.}

Two other possible 4-derivative $\nabla F \nabla F$ invariants are related to the above two by the Bianchi identity, e.g., $F_{mn} \nabla^2 F_{mn} = -2 (\nabla_m F_{mn})^2 + 4 F_{mn} F_{nk} F_{km} + \text{total derivative}$.\footnote{Two other possible 4-derivative $\nabla F \nabla F$ invariants are related to the above two by the Bianchi identity, e.g., $F_{mn} \nabla^2 F_{mn} = -2 (\nabla_m F_{mn})^2 + 4 F_{mn} F_{nk} F_{km} + \text{total derivative}$.}
In general, the UV logarithmically divergent part of the 6d one-loop effective action \( \Gamma_1 \) in a gauge field background may be written as

\[
\Gamma_{1\text{c}} = -\frac{\log \Lambda}{(4\pi)^3} \int d^6x \text{ tr} \left[ -\frac{1}{60} \beta_2 (\nabla_m F_{mn})^2 + \frac{1}{90} \beta_3 F_{mn} F_{nk} F_{km} \right],
\]

where the 1-loop \( \beta \)-function coefficients \( \beta_2, \beta_3 \) depend on the field content of the theory. As we shall find below, their values in the case of the 4-derivative theory (1.1) are given by the following functions of the coupling \( \gamma \) (1-loop coefficients do not depend on the overall \( g^2 \) coupling)

\[ \beta_{2A} = 249, \quad \beta_{3A} = 9 - 900 \gamma + \frac{405}{2} \gamma^3. \]

Somewhat surprisingly, the coefficient \( \beta_{2A} \) of the \((\nabla F)^2\) divergence turns out to be independent of the coupling \( \gamma \).

The total values of \( \beta_2, \beta_3 \) in a 6d renormalizable model containing the gauge theory (1.1) minimally coupled to the ordinary-derivative “matter” fields — \( N_0 \) real scalars, \( N_1 \) Weyl fermions, \( N_1 \) YM vectors and \( N_T \) self-dual tensors (interacting with \( A_m \) as in [12]) are then [1, 12]

\[ \beta_2 = \beta_{2A} - 27N_T - 36N_1 + N_0 + 16N_{1/2}, \]
\[ \beta_3 = \beta_{3A} - 57N_T + 4N_1 + N_0 - 4N_{1/2}. \]

Note that for the ordinary spin 0, 1/2, 1 fields their contributions to \( \beta_3 \) are proportional to the number of dynamical degrees of freedom. The same is true also for the 4-derivative gauge theory (1.1) with \( \gamma = 0 \): \( \beta_{3A} = 9 \) is the number of d.o.f. of a 4-derivative gauge vector in 6d. \(^5\) As a consequence one should get \( \beta_3 = 0 \) in a supersymmetric theory; this is consistent with the non-existence of a super-invariant containing \( \text{tr}(F_{mn} F_{nk} F_{km}) \). Indeed, for the standard 2-derivative 6d \((1,0)\) SYM theory \((N_1 = 1, N_{1/2} = 1)\) and for the scalar (hyper) multiplet \((N_0 = 4, N_{1/2} = 1)\) one finds

\[ \beta_{2(1,0)\text{SYM}} = -20, \quad \beta_{2\text{scal}} = 20, \quad \beta_{3(1,0)\text{SYM}} = \beta_{3\text{scal}} = 0. \]

Since \( \nabla_m F_{mn} = 0 \) on the standard YM equations of motion the \((1,0)\) SYM theory is 1-loop finite on shell. The sum of the contributions of the two multiplets in (1.5) corresponds to the \((1,1)\) SYM theory in 6d (and thus to \( N = 4 \) SYM in 4d) which is 1-loop finite even off-shell [1]

\[ \beta_{2(1,1)\text{SYM}} = \beta_{3(1,1)\text{SYM}} = 0. \]

\(^4\)Here \( \text{tr} \) is the trace over the matrix indices of a particular representation to which the quantum field belongs; for example, in the gauge theory case it is in the adjoint representation \( A^a_{ab} = f^{ac} A^c_{ab} \), \( f_{ab} f_{cd} = C_{abc} \delta_{bd} \).

\(^5\)Here all the fields are taken for simplicity in the adjoint representation; in the case of other representations one is to rescale the numbers \( N_s \) by \( T_R C_2 \). We corrected misprints in [1] mentioned in [12]. Note that the vector \( N_1 \) terms here are formal: they indicate the 6d YM contribution in the absence of higher-derivative terms in (1.1). In the combined \( F^2 + (\nabla F)^2 + F^3 \) theory discussed below in appendix B the values of \( \beta_2 \) and \( \beta_3 \) are the same as in the theory (1.1) without the YM \( F^2 \) term.

\(^6\)While the 2-derivative YM vector in \( d \) dimensions has \( \frac{1}{2}(d-1)(d-2) = d(d-2) \) dynamical d.o.f., for the 4-derivative gauge vector in (1.1) one finds \( \frac{1}{2}(4d-1)(4d-2) = 2d(d-3) \), i.e. 5 in \( d = 4 \) and 9 in \( d = 6 \).
In the (1,0) supersymmetric 4-derivative gauge theory with the action given by the super-extension \[5\] of \(\text{tr} (\nabla_m F_{mn})^2\) (containing also interacting \(\nabla^3\) Weyl fermion and three \(\nabla^2\) scalars) we will find below that

\[ \beta_2(1,0) = 220, \quad \beta_3(1,0) = 0. \]  

(1.7)

This result is in agreement (modulo notation change) with the one given in the recently revised version of \[5\]. This theory is non-unitary and is also formally inconsistent having a chiral anomaly \[6\] (the same as in the (1,0) 6d SYM theory containing Weyl fermion). One may still hope to cancel all of its anomalies by adding some higher derivative 6d “matter” multiplets (cf. \[14–16\]).

The calculation of the \(\beta\)-functions (1.3) is most straightforward in the background field method and using the heat kernel expansion to extract the log divergences of the determinants. This requires the knowledge of the corresponding \(b_6\) Seeley-DeWitt coefficient for the 4-derivative operator \(\Delta_4 = \nabla^4 + \ldots\) in a gauge field background. While \(b_6\) is available for the 2-derivative \(\Delta_2\) operators \[17\], its expression for \(\Delta_4\) was not known so far. The main new technical result of this paper is the computation of \(b_6(\Delta_4)\). We shall use the same strategy as employed previously in \[2\] to obtain \(b_4(\Delta_4)\) from the known expression for \(b_4(\Delta_2)\) by considering special factorized cases of the operator \(\Delta_4\).

It would be interesting to extend the computation of the \(b_6\) coefficient for the 4-derivative operators to the case of a curved metric background (finding the analog of the corresponding expression for \(b_4\) in \[2\]). This would allow, in particular, to compute the one-loop UV divergences in \(d = 6\) conformal supergravity and verify the expectation \[7, 8\] that the higher derivative (2,0) 6d conformal supergravity coupled to exactly 26 (2,0) tensor multiplets has vanishing conformal anomaly.\(^7\) Another important step would be to extend the background field approach to the computation of UV divergences in 4-derivative gauge theories to the two-loop level generalizing the methods of \[20–22\].

The rest of the paper is organized as follows. In section 2 we present the general form of the one-loop effective action of the theory (1.1). In section 3 the result for the heat kernel coefficient \(b_6\) that controls the logarithmic divergence of the determinant of a generic 4-derivative operator is given. In section 4 this expression is applied to compute the one-loop divergences in the bosonic gauge theory (1.1) and its (1,0) supersymmetric extension (with \(\gamma = 0\)). Details of the derivation of \(b_6(\Delta_4)\) are described in appendix A. In appendix B we discuss divergences of the combined 2- and 4-derivative \(\frac{1}{g^2}[\kappa^2 F^2 + (\nabla F)^2 + \gamma F^3]\) gauge theory and its (1,0) supersymmetric version: adding \(F^2\) does not change the \(\beta\)-functions (1.3) for \(g\) and \(\gamma\) but leads to the \(\gamma\)-dependent \(\beta\)-function for \(\kappa\).

2 One-loop effective action

The derivation of the one-loop effective action in the 4-derivative theory (1.1) in 6d follows the same steps as in the 4d case discussed in appendix C of \[2\] (for a review, see also \[4\]).

\(^7\)This is the 6d counterpart of the known fact of cancellation of the conformal anomaly in the 4d system of \(\mathcal{N}=4\) conformal supergravity coupled to 4 vector \(\mathcal{N}=4\) multiplets \[18, 19\].
Expanding the invariants in (1.1) near a classical background $A_m^a \to A_m^a + \tilde{A}_m^a$ we get
\[
\text{Tr}(\nabla_m F_{mn})^2 \to -\frac{1}{2} \tilde{A}_m^a \left[ \delta_{mn} \nabla^4 + 4 F_{mn} \nabla^2 - 2 \left( \nabla_k F_{km} \delta_{nr} + 2 \nabla_k F_{[m} \delta_{r]n} \right) \nabla_r \right] - 2 \nabla_n \nabla_k F_{km} + 4 F_{mn} F_{kn} \right] \tilde{A}_n^b - \frac{1}{2} (\nabla_m \tilde{A}_m^a) \nabla^2 (\nabla_n \tilde{A}_n^a),
\]
(2.1)

where $F_{mn}$ and $\nabla_m$ depend on the background $A_m$ and $a, b$ are indices in the adjoint representation. Then the quadratic part of the fluctuation Lagrangian in (1.1) may be written as
\[
\mathcal{L}^{(2)} = \frac{1}{2 g^2} \tilde{A}_m^a (\Delta_{4A})_{mn}^{ab} \tilde{A}_n^b + \frac{1}{2 g^2} (\nabla_m \tilde{A}_m^a) (-\nabla^2) (\nabla_n \tilde{A}_n^a).
\]
(2.3)

The second term here can be cancelled by adding a gauge-fixing $(\nabla_m \tilde{A}_m = f(x))$ term averaged with the operator $-\nabla^2$. The 4-derivative operator $\Delta_{4A}$ acting on $\tilde{A}_m^a$ can be written in the following “symmetric” form
\[
\Delta_4 = \nabla^4 + \nabla_r \tilde{V}_{rk} \nabla_k + \tilde{N}_k \nabla_k + \nabla_k \tilde{N}_k + \tilde{U}, \quad \tilde{V}_{rk} = \tilde{V}_{kr},
\]
(2.4)

where $\tilde{V}_{rk}, \tilde{N}_k, \tilde{U}$ are local covariant matrices in the internal $(a, m), (b, n)$ indices reading
\[
(\tilde{V}_{rk})_{mn} = (4 + 3 \gamma) F_{mn}^{\delta^r_k} - 6 \gamma F_{mn}^{(r \delta^k_n)},
\]
(2.5)

\[
(\tilde{N}_k)_{mn} = \frac{1}{2} (2 + 3 \gamma) \nabla_r F_{rk} \delta_{mn} - \frac{1}{2} (4 + 3 \gamma) \nabla_r F_{(m} \delta_{nk)} - \frac{3}{2} \gamma \nabla_r F_{(m} \delta_{nk)} - \frac{3}{2} \gamma \nabla_r F_{(m} \delta_{nk)} + \frac{3}{2} \gamma \nabla_r F_{(m} \delta_{nk)} + \frac{3}{2} \gamma \nabla_r F_{(m} \delta_{nk)} + 3 \nabla^2 F_{mn}.
\]
(2.6)

The operator that appears in the effective action after path-integral is performed (i.e. $\Delta_{4A}$ in (2.3)) should be self-adjoint and this is so for (2.4) with (2.5).\(^8\)

The 1-loop effective action is then given by
\[
\Gamma_1 = \frac{1}{2} \log \frac{\det \Delta_{4A}}{(\det \Delta_{gh})^2} \log H = \frac{1}{2} \log \det \Delta_{4A} - \frac{3}{2} \log \det \Delta_{2,0}, \quad \Delta_{2,0} = -\nabla^2,
\]
(2.7)

where $\Delta_{gh} = -\nabla^2$ is the ghost operator and $H = -\nabla^2$ is the gauge-condition averaging operator required to cancel the last term in (2.3). Using the proper-time cutoff, the log divergent part of a determinant can be expressed (in general dimension $d$) in terms of the corresponding Seeley-DeWitt coefficient $B_d$,\(^9\)
\[
\Gamma_1(\Delta) = \frac{1}{2} (\log \det \Delta)_{\infty} = -\frac{\log \Lambda}{(4\pi)^{d/2}} B_d(\Delta), \quad B_d = \int d^d x b_d(\Delta).
\]

\(^8\)Note that (2.4) is a completely general form for a fourth-order elliptic differential operator without the three-derivative term. The self-adjointness can be imposed via the following additional conditions on the coefficients $\tilde{V}_{mn}^\dagger = \tilde{V}_{mn}, \tilde{N}_m^\dagger = -\tilde{N}_m, \tilde{U}^\dagger = \tilde{U}$ where $\dagger$ is transposition if the field is real, and hermitian conjugation if the field is complex.

\(^9\)Here we ignore boundary terms. Note also that in the dimensional regularization one is to replace $\log \Lambda \to -\frac{1}{4\pi^2}$ where $d$ is integer and $d < d$ is its analytic continuation.
The values of $b_p$ for 2-derivative Laplacian $\Delta_2$ (in general curved space and gauge field background) are known up to $p = 10$ (see, e.g., [17, 23–26]) while for the 4-derivative operator $\Delta_4$ only $b_2$ and $b_4$ were found so far [2, 27–29]. Thus to compute the divergent part of (2.6) we need first to determine the coefficient $b_6$ for $\Delta_4$ in (2.4). This will be the subject of the next section and appendix A.

3 Heat kernel coefficient $b_6(\Delta_4)$

In general, given an elliptic differential operator $\Delta_\ell$ of an even order $\ell$ in $d$ dimensions one has

$$\log \det \Delta_\ell = - \int d^dx \int_\epsilon^\infty \frac{dt}{t} \tr \langle x | e^{-t \Delta_\ell} | x \rangle,$$  

(3.1)

where $\tr$ is the trace over internal indices of the operator. The heat kernel has an asymptotic expansion for $t \to 0$ so that (see, e.g., [25, 28, 29])

$$\tr \langle x | e^{-t \Delta_\ell} | x \rangle \equiv \tr K(t; x, x; \Delta_\ell) \simeq \sum_{p \geq 0} \frac{2}{(4\pi)^{d/2} \ell} t^{(p-d)/\ell} b_p(\Delta_\ell).$$  

(3.2)

The Seeley-DeWitt coefficients $b_p$ are local invariant expressions of dimension $p$ constructed out of the background metric and gauge field (below we shall consider them up to total derivative terms). Using the proper-time cutoff $\epsilon = \Lambda^{-\ell}$ we obtain for the divergent part of (3.1)\textsuperscript{10}

$$(\log \det \Delta_\ell)_{\infty} = - \frac{2}{(4\pi)^{d/2}} \left[ \sum_{p=0}^{d-1} \frac{B_p(\Delta_\ell)}{d-p} \Lambda^{d-p} + B_d(\Delta_\ell) \log \frac{\Lambda}{\mu} \right],$$  

(3.3)

$$B_p(\Delta_\ell) = \int d^dx \ b_p(\Delta_\ell).$$

The renormalization scale $\mu$ in log will be sometimes left implicit below. For example, for the 2-derivative operator defined on a vector bundle with the covariant derivative $\nabla_m$ and the curvature $F_{mn} = [\nabla_m, \nabla_n]$ one has\textsuperscript{11}

$$\Delta_2 = -\nabla^2 + X,$$

$$b_6(\Delta_2) = \tr \left[ -\frac{1}{60} (\nabla_m F_{mn})^2 + \frac{1}{90} F_{mn} F_{nk} F_{km} - \frac{1}{12} X F_{mn} F_{mn} + \frac{1}{12} X \nabla^2 X - \frac{1}{6} X^3 \right].$$  

(3.5)

To find $b_6(\Delta_4)$ for the operator in (2.4) we will use the same idea as in [2] and consider several special cases of factorized operators $\Delta_4$ for which

$$\Delta_4 = \Delta_2 \Delta_2', \quad \det \Delta_4 = \det \Delta_2 \det \Delta_2', \quad b_p(\Delta_2 \Delta_2') = b_p(\Delta_2) + b_p(\Delta_2').$$  

(3.6)
The 4-derivative operator that we are interested in is given in (2.4). As explained in appendix A, a general expression for its $b_6$ coefficient is ($\hat{V} \equiv \hat{V}_{mn}$)

$$b_6(\Delta_4) = \text{tr} \left[ \hat{k}_1 (\nabla_m F_{mn})^2 + \hat{k}_2 F_{mn} F_{nk} F_{km} + \hat{k}_3 \hat{V}_{mn} \hat{V}_{km} + \hat{k}_4 \hat{V}_{mn} \hat{V}_{mm} \hat{V} + \hat{k}_5 \hat{V} \hat{V} \hat{V} + \hat{k}_6 \hat{V}_{mn} \nabla (\nabla k) \hat{V}_{km} + \hat{k}_7 \hat{V}_{mn} \nabla^2 \hat{V}_{mn} + \hat{k}_8 \hat{V}_{mn} \nabla_m \nabla_n \hat{V} + \hat{k}_9 \hat{V} \nabla^2 \hat{V} + \hat{k}_{10} \hat{V}_{mn} \hat{V}_{nk} F_{mk} + \hat{k}_{11} \hat{V}_{mn} \nabla (m \nabla k) \hat{V}_{kn} + \hat{k}_{12} \hat{V}_F F_{mn} F_{mk} + \hat{k}_{13} \hat{V}_{mn} F_{mk} F_{nk} + \hat{k}_{14} \hat{V}_{mn} \hat{N}_n + \hat{k}_{15} \hat{V}_m \nabla_m \hat{N}_n + \hat{k}_{16} \hat{V} \nabla_m \hat{N}_m + \hat{k}_{17} \hat{N}_m \hat{N}_m + \hat{k}_{18} \hat{U} \hat{V} \right].$$

In contrast to what happens in the case of $\Delta_2$ in (3.5), some of the coefficients in (3.7) in general depend on the number of dimensions $d$. In the case of $d = 6$ we are interested in here one finds

$$\hat{k}_1 = -\frac{1}{30}, \quad \hat{k}_2 = \frac{1}{45}, \quad \hat{k}_3 = \frac{1}{360}, \quad \hat{k}_4 = \frac{1}{480}, \quad \hat{k}_5 = \frac{1}{2880}, \quad \hat{k}_6 = -\frac{1}{120},$$

$$\hat{k}_7 = \frac{1}{120}, \quad \hat{k}_8 = \frac{1}{60}, \quad \hat{k}_9 = \frac{1}{240}, \quad \hat{k}_{10} = -\frac{1}{24}, \quad \hat{k}_{11} = 0, \quad \hat{k}_{12} = \frac{1}{24},$$

$$\hat{k}_{13} = -\frac{1}{6}, \quad \hat{k}_{14} = -\frac{1}{3}, \quad \hat{k}_{15} = 0, \quad \hat{k}_{16} = 0, \quad \hat{k}_{17} = -\frac{1}{6}, \quad \hat{k}_{18} = -\frac{1}{12} .$$

### 4 Divergences of 4-derivative 6d gauge theories

Let us now apply the above general expression (3.7), (3.8) for $b_6(\Delta_4)$ to the gauge theories of interest.

#### 4.1 Bosonic theory

Starting with the explicit form of the coefficient functions (2.4), (2.5) in the operator $\Delta_{4A}$ and applying (3.7), (3.8) as well as (3.5), we can compute the coefficient $b_6$ in the divergent part of the effective action (2.6), (2.7) of the 4-derivative bosonic 6d gauge theory (1.1)

$$b_6 = b_6(\Delta_{4A}) - 3b_6(\Delta_{2,0}) ,$$

$$b_6(\Delta_{4A}) = \text{tr} \left[ -\frac{21}{5} (\nabla_m F_{mn})^2 + \left( \frac{2}{15} - 10\gamma + \frac{9}{4}\gamma^3 \right) F_{mn} F_{nk} F_{km} \right] ,$$

$$b_6(\Delta_{2,0}) = \text{tr} \left[ -\frac{1}{60} (\nabla_m F_{mn})^2 + \frac{1}{90} F_{mn} F_{nk} F_{km} \right] .$$

Thus finally

$$b_6 = \text{tr} \left[ -\frac{83}{20} (\nabla_m F_{mn})^2 + \left( \frac{1}{10} - 10\gamma + \frac{9}{4}\gamma^3 \right) F_{mn} F_{nk} F_{km} \right] .$$

Comparing to (1.2) we end up with the values of the one-loop $\beta$-function coefficients $\beta_{2A}$, $\beta_{4A}$ quoted in (1.3). It is remarkable that the divergence proportional to $(\nabla F)^2$ turned

---

In applying (3.7) to the gauge field case, the trace there is acting on the full internal index structure of the operator $\Delta_{4A}$, i.e. involving both spacetime and gauge indices (cf. footnote 11).
out to be independent of the parameter $\gamma$: various terms in $b_6$ in (3.7) generically do give $\gamma$-dependent $(\nabla F)^2$ contributions and they cancel out only when combined together weighted with the $k_i$ coefficients in (3.8).

The corresponding RG equations for the renormalized couplings $g(\mu)$ and $\gamma(\mu)$ in (1.1) may be written as ($t = (\frac{1}{4\pi^2}) \log \mu^2$, $C_2(\SU(N)) = N$)

$$\frac{dg^{-2}}{dt} = \beta_2 A C, \quad \frac{d\gamma}{dt} = \beta\gamma g^2, \quad C \equiv \frac{1}{60} C_2,$$  

$$\beta_2 A \equiv 249, \quad \beta_\gamma = -\gamma \beta_2 A - \frac{1}{3} \beta_{3A} = \frac{3}{2} ( -2 + 34 \gamma - 45 \gamma^3 ).$$  

The flow of $g$ is independent of the parameter $\gamma$ and the sign of $\beta_2 A$ corresponds to asymptotic freedom. The fixed points of the flow of $\gamma$ are the solutions of $\beta_\gamma = 0$, i.e. $\gamma_1 \simeq -0.897$, $\gamma_2 \simeq 0.059$, $\gamma_3 \simeq 0.838$. Since $\beta_\gamma > 0$ for $\gamma \gamma_1$ or $\gamma_2 \gamma_2 < \gamma < \gamma_3$, we have that $\gamma_1$ and $\gamma_3$ are attractive fixed points of the flow.

As the sign of the $F^3$ term in (1.1) is not a priori constrained by the requirement of positivity of the Euclidean action we formally define a second coupling $h_2 \equiv \frac{1}{g_2}$ that may assume positive as well as negative values. Then near the fixed points $h_2$ also goes to zero in the UV, i.e. like $g^2$ the second coupling is also asymptotically free.

In appendix B we shall present also the one-loop $\beta$-functions for the combined YM plus 4-derivative gauge theory with $\mathcal{L} = \frac{1}{g^2} \left[ \kappa^2 F^2 + (\nabla F)^2 + \gamma F^3 \right]$.  

### 4.2 (1,0) supersymmetric theory

Let us now consider the 6d supersymmetric version of the theory (1.1) constructed in [5]. In this case $\gamma = 0$ since, in general, there is no supersymmetric extension of the $F^3$ term.\footnote{This can be easily understood using, e.g., the standard $N = 1$ 4d superspace formulation: the YM field strength $F_{mn}$ is part of the spinor superfield strength $W_\alpha$ and thus constructing an invariant cubic in $W_\alpha$ is not possible.} The field content includes the 4-derivative gauge field $A_m$, the 3-derivative 6d Weyl spinor $\Psi$, and the three 2-derivative real scalars $\Phi_I$ ($I = 1, 2, 3$). In total, one has $9 + 3$ bosonic and $3 \times 4$ fermionic on-shell degrees of freedom (for each value of the internal index).

Using an off-shell harmonic superspace formulation ref. [5] found the following (1,0) supersymmetric 6d action\footnote{In the case of the standard (1,0) SYM theory (corresponding to $N = 2$ SYM theory in 4d) the latter correspond to the auxiliary scalars.}

$$S_{(1,0)} = \frac{\alpha}{2} \int d^6 x \text{Tr} \left[ (\nabla_m F_{mn})^2 - i \Psi \nabla^2 \Psi - (\nabla_m \Phi_I)^2 - i \frac{1}{2} \bar{\Psi} \Gamma_k \Gamma_{mn} \nabla_k [F_{mn}, \Psi] + 2i \nabla_m F_{mn} \bar{\Psi} \Gamma_n \Phi + \mathcal{O}(\Phi \Phi^2, \Phi^3) \right].$$

We suppressed interactions that are more than second order in the scalars and fermions, as they will not contribute to the one-loop divergences in a gauge-field background. Note

$$\Gamma_{(m} \Gamma_{n)} = \frac{1}{4} \{ \Gamma_m, \Gamma_n \} = \delta_{mn} \text{ and } \Gamma_{mn} \equiv \Gamma_{[m} \Gamma_{n]}.$$

JHEP08(2019)159
that with our definition of the coupling constant \( g \) (i.e. the choice of the overall sign of the action) the gauge field term in (4.7) is positive definite (cf. (1.1)) but the scalar term is not, and this is one indication of the non-unitarity of the theory.\(^{16}\)

The 4-derivative operator for the fluctuations of the gauge field is given by (2.4), (2.5) with \( \gamma = 0 \), i.e. it is \( \Delta_{4A}^{(0)} \equiv \Delta_{4A} \big|_{\gamma=0} \), while the 3-derivative fermion and the 2-derivative scalar operators in gauge field background may be written as\(^{17}\)

\[
\Delta_3 \Psi = i \nabla^2 + \frac{i}{2} \nabla \Gamma_{mn} F_{mn} + i \Gamma_n (\nabla_m F_{mn}) = i \Psi^3 + i \Gamma_n (\nabla_m F_{mn}),
\]

\[
\Delta_2 \Phi = - \nabla^2 = \Delta_{2,0}.
\]

Here \( i \Psi^3 \) is the cube of the Dirac operator \( \Delta_1 \Psi = -i \nabla = -i \Gamma^m \nabla_m \) whose square is

\[
\Delta_2 \Phi = - \nabla^2 = - \nabla^2 - \frac{1}{2} \Gamma_{mn} F_{mn}.
\]

As a result, the one-loop effective action of the supersymmetric theory (4.7) is the following generalization of the bosonic case (2.6)

\[
\Gamma_{1(1,0)} = \frac{1}{2} \log \det \Delta_{4A}^{(0)} \left[ \det \Delta_{2,0} \right]^3 \det \Delta_{3 \Phi} = \frac{1}{2} \log \det \Delta_{4A}^{(0)} - \frac{1}{2} \log \det \Delta_{3 \Phi}.
\]

Here the contributions of the ghost and gauge-averaging operators in (2.6) got canceled against the contribution of the three scalars \( \Phi_f \). We also used that \( \det \Delta_{\Phi} \) is defined for the Dirac 6d spinors so that the factor \( \frac{1}{2} \) accounts for the fact that the fermion \( \Psi \) is a Weyl spinor. As a result, the coefficient of the log divergent part of the effective action (2.7) is given by (cf. (4.1))

\[
b_6(1,0) = b_6(\Delta_{4A}^{(0)}) - b_6(\Delta_{3 \Phi}).
\]

Setting \( \gamma = 0 \) in (4.2) gives

\[
b_6(\Delta_{4A}^{(0)}) = \text{tr} \left[ - \frac{21}{5} (\nabla_m F_{mn})^2 + \frac{2}{15} F_{mn} F_{nk} F_{km} \right].
\]

To compute the fermionic contribution, let us first construct a 4-derivative operator by taking the product of \( \Delta_3 \Phi \) in (4.8) with the standard Dirac operator

\[
\Delta_{4 \Psi} \equiv \Delta_{1 \Psi} \Delta_{3 \Phi} = \nabla^2 + \nabla \Gamma_n (\nabla_m F_{mn}) , \quad b_6(\Delta_{3 \Phi}) = b_6(\Delta_{4 \Psi}) - b_6(\Delta_{1 \Psi}).
\]

\( \Delta_{4 \Psi} \) is then a 4-order operator of the form (2.4) with the coefficients\(^{18}\)

\[
\hat{V}_{rk} = \Gamma_{mn} F_{mn} \delta_{rk}, \quad \hat{N}_k = \frac{1}{2} \Gamma_k \Gamma_n \nabla_m F_{mn},
\]

\[
\hat{U} = \frac{1}{2} \Gamma_{mn} \nabla^2 F_{mn} + \frac{1}{4} \Gamma_{mn} \Gamma_{rk} F_{mn} F_{rk} + \frac{1}{2} \Gamma_k \Gamma_n \nabla_k \nabla_m F_{mn}.
\]

\(^{16}\)In [5] the opposite overall sign was chosen so that their coupling is related to ours by \( g^2 \rightarrow -g^2 \). This translates into the opposite sign of the \( \beta \)-function for \( g \) in (4.17). Note that here there is thus no "preferred" choice of the sign of the action (redefining the scalars \( \Phi_f \rightarrow i \Phi_f \) leads to imaginary \( \Phi^4 \) interaction, i.e. to non-hermiticity of the action). For a review of related issues in higher-derivative theories see [30].

\(^{17}\)In the first form of \( \Delta_{3 \Phi} \) the derivative in the second term acts all the way to the right while in the third term it acts only on \( F_{mn} \).

\(^{18}\)Notice that this operator is not self-adjoint, i.e. the symmetry requirements in footnote 8 are not satisfied.
Applying the general expression for $b_6(\Delta_4)$ that we found in (3.7), (3.8) (where now the connection and its curvature are understood to include also the internal spinor indices, see footnote 11) and also using that squaring $\Delta_1\Phi$ one obtains (4.9), for which $b_6$ can then found from (3.5), we end up with

$$b_6(\Delta_3\Phi) = b_6(\Delta_4\Phi) - \frac{1}{2} b_6(\Delta_2\Phi) = \text{tr} \left[ -\frac{8}{15} (\nabla_m F_{mn})^2 + \frac{2}{15} F_{mn} F_{nk} F_{km} \right]. \quad (4.15)$$

Combining the bosonic (4.12) and the fermionic (4.15) contributions to (4.11) we conclude that the $F^3$ terms cancel as expected and finally

$$b_6(1,0) = -\frac{11}{3} \text{tr} (\nabla_m F_{mn})^2. \quad (4.16)$$

This is the same result as quoted in (1.2), (1.7). The resulting renormalized coupling in (4.7) is thus (cf. (2.7), (4.7))

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(\Lambda)} - \frac{22}{3} \frac{C_2}{(4\pi)^3} \log \frac{\Lambda}{\mu}, \quad (4.17)$$

corresponding to an asymptotically free behaviour. This agrees with the (recently revised) result of [5] (cf. footnote 16). Note that the computation of the $\beta$-function in [5] was done in the scalar field $\Phi_I$ background while here we used the gauge field background, thus providing an independent check of the result.

For comparison, let us recall the result [1] of a similar computation in the ordinary-derivative (1,0) 6d SYM theory

$$S_{(1,0)\text{SYM}} = -\frac{\kappa^2}{g^2} \int d^6x \text{Tr} \left( \frac{1}{2} F_{mn} F_{mn} + i \bar{\Psi} \nabla \Psi - \Phi_I \Phi_I \right), \quad (4.18)$$

where $\Psi$ is a Weyl spinor, $\Phi_I$ are 3 auxiliary fields (cf. (4.7)) and $\kappa$ is a mass scale. The analog of the one-loop effective action in a gauge field background (4.10) here is

$$\Gamma_{1,(1,0)\text{SYM}} = \frac{1}{2} \log \frac{\Delta_{2A}}{[\det \Delta_{2,0}]^2 [\det \Delta_{1\Psi}]}, \quad (\Delta_{2A})_{mn} = -\delta_{mn} \nabla^2 - 2 F_{mn}. \quad (4.19)$$

Using (3.5) we get

$$b_6(\Delta_{2A}) = \text{tr} \left[ \frac{17}{30} (\nabla_m F_{mn})^2 + \frac{1}{15} F_{mn} F_{nk} F_{km} \right],$$

$$b_6(\Delta_{2,0}) = \text{tr} \left[ -\frac{1}{60} (\nabla_m F_{mn})^2 + \frac{1}{90} F_{mn} F_{nc} F_{cm} \right], \quad (4.20)$$

$$b_6(\Delta_{1\Psi}) = \frac{1}{2} b_6(\Delta_{2\Psi}) = \text{tr} \left[ \frac{4}{15} (\nabla_m F_{mn})^2 + \frac{2}{45} F_{mn} F_{nc} F_{cm} \right].$$

As a result, the one-loop logarithmic divergence is given by (2.7) with

$$b_{6,(1,0)\text{SYM}} = b_6(\Delta_{2A}) - 2 b_6(\Delta_{2,0}) - b_6(\Delta_{1\Psi}) = \frac{1}{3} \text{tr} (\nabla_m F_{mn})^2. \quad (4.21)$$
Once again, the $F^3$ divergence cancels, and (4.21) implies the value of $\beta_2 = -20$ in (1.2), (1.5). Since here $\nabla_m F_{mn} = 0$ is an equation of motion, the divergence (4.21) vanishes on-shell, i.e. the (1,0) 6d SYM theory is finite on-shell\(^\text{19}\) though is not renormalizable off-shell.

The (1,1) 6d SYM found by combining the (1,0) SYM with a scalar multiplet (cf. (1.5)) is one-loop finite even off-shell [1] (cf. also [32]). Let us also note that it is easy to check the cancellation of $F^3$ divergences in the (1,0) supersymmetric gauge theory (4.7) by restricting the background to satisfy $\nabla_m F_{mn} = 0$ (which is a special on-shell background also in this theory). Then $\Delta_{1\Psi}$ in (4.8) becomes simply $\left(\frac{1}{2}\right)$ and also the vector field operator in (2.4), (2.5) (with $\gamma = 0$) becomes a square of the standard YM operator in (4.19), i.e. $\Delta_{4A} = (\Delta_{2A})^2$. As a result, the effective action (4.10) reduces to

$$\Gamma_{1(1,0)} = \frac{1}{2} \log \det (\Delta_{2A})^2 - \frac{1}{2} \log \det (\Delta_{1\Psi})^3$$

$$= 2 \cdot \frac{1}{2} \left[ \log \det \Delta_{2A} - 2 \log \det \Delta_{2,0} - \log \det \Delta_{1\Psi} \right] + \frac{1}{2} \left[ 4 \log \det \Delta_{2,0} - \log \det \Delta_{1\Psi} \right]$$

$$= 2 \Gamma_{1(1,0)\text{SYM}} + \Gamma_{1\text{scal}},$$

(4.22)

i.e. equal to the sum of twice the effective action of the standard (1,0) SYM in (4.19) with the effective action of the scalar (hyper) multiplet (containing 4 real scalars and one Weyl fermion). Each of these do not contribute to the $F^3$ divergent terms according to (1.5).

**Acknowledgments**

We are grateful to E. Ivanov and A. Smilga for discussions related to the value of the $\beta$-function in [5]. LC wishes to thank T. Bertolini for useful discussions. AAT acknowledges K.-W. Huang and R. Roiban for discussions related to [12]. LC is supported by the International Max Planck Research School for Mathematical and Physical Aspects of Gravitation, Cosmology and Quantum Field Theory. AAT was supported by the STFC grant ST/P000762/1.

**Note added.** After this paper was submitted to the arXiv we learned about the earlier work [33] (see also [34]) in which a diagrammatic computation of the two-loop $\beta$-functions in the 6d gauge theory (1.1) coupled to standard fermions was performed.\(^\text{20}\) After correcting a mistake in the original version of this paper we found that our result (1.3), (1.4) for the $\beta$-functions of the theory (1.1) coupled to fermions is in full agreement with the one-loop $\beta$-functions in [33].\(^\text{21}\)

\(^\text{19}\)The coefficient in (4.21) here is, in fact, gauge-dependent, see also [31].

\(^\text{20}\)We are grateful to I. Klebanov for drawing our attention to this paper.

\(^\text{21}\)The translation between the notation in [33] and ours is as follows. Instead of $(\nabla_m F_{mn})^2$ in (1.1) the action in [33] contained $(\nabla_k F_{mn})^2$ with the two invariants related as in footnote 3. As a result, the couplings $g_1$ and $g_2$ in [33] are related to ours as $g_1 = g$, $g_2 = 3g(1 + \gamma)$ (using also that $g_2 \rightarrow -g_2$ due to apparent sign difference in notation for $F_{mn}$). For the gauge theory (1.1) coupled to Weyl fermions in generic representation our result (1.3), (1.4) for the $\beta$-functions reads (cf. (4.5), (4.6)): $\beta_\gamma \equiv \frac{d g_1}{d t} = -\frac{1}{120} C_2 \beta_2$, $\beta_2 \equiv \frac{d g_2}{d t} = -\frac{1}{720} C_2 (2\gamma + 3\beta_2 - \beta_1)g^2$, $\beta_2 = 249 + N_3 + 9 - 900\gamma + \frac{900}{2} \gamma^2 - 4N_4$, $N_1 = \frac{720}{C_2} N_1 + \frac{540}{2} N_2$. Then the $\beta$-functions for the above $g_1$ and $g_2$, i.e. $\beta_{g_1} = \frac{d g_1}{d t} = \beta_\gamma$, $\beta_{g_2} = \frac{d g_2}{d t} = 3\beta_\gamma (1 + \gamma) + 3g_2$, match the expressions in [33].
A Derivation of the expression for $b_6(\Delta_4)$

The operator that we shall consider is

$$\Delta_4 = \nabla^4 + V_{mn}\nabla_m \nabla_n + 2N_m \nabla_m + U,$$

which is the most general fourth-order elliptic differential operator without 3-derivative term. It is related to the “symmetrized” operator in (2.4) by

$$V_{mn} = \dot{V}_{mn}, \quad N_m = \dot{N}_m + \frac{1}{2} \nabla_m V_{mn}, \quad U = \dot{U} + \nabla_m \dot{N}_m. \quad (A.2)$$

The general expression for its coefficient $b_6$ including only independent invariants may be written as ($V \equiv V_{mn}$)

$$b_6(\Delta_4) = \text{tr} \left[ k_1 (\nabla_m F_{mn})^2 + k_2 F_{mn} F_{nk} F_{km} \right. \right.
\left. + k_3 V_{mn} V_{nk} V_{km} + k_4 V_{mn} V_{nn} V + k_5 V_{V} V + k_6 V_{mn} \nabla_m \nabla_n \nabla_k V_{km} \right. \left. + k_7 V_{mn} \nabla^2 V_{mn} + k_8 V_{mn} \nabla_n \nabla_m V + k_9 V \nabla^2 V + k_{10} V_{mn} V_{nk} F_{km} 
\left. + k_{11} F_{mn} \nabla_{(m} \nabla_k) V_{kn} + k_{12} V F_{mn} F_{mn} + k_{13} V_{mn} F_{nk} F_{km} \right. \left. + k_{14} F_{mn} \nabla_m N_n + k_{15} V_{mn} \nabla_m N_n + k_{16} V \nabla_m N_m + k_{17} N_m N_m + k_{18} U V \right],$$

where the trace is over internal indices and $k_i$ are real coefficients.\(^{22}\) Their values in $d = 6$ found below are

$$k_1 = -\frac{1}{30}, \quad k_2 = \frac{1}{45}, \quad k_3 = \frac{1}{360}, \quad k_4 = \frac{1}{480}, \quad k_5 = \frac{1}{2880}, \quad k_6 = \frac{1}{30}, \quad k_7 = \frac{1}{120}, \quad k_8 = -\frac{1}{40}, \quad k_9 = \frac{1}{240}, \quad k_{10} = -\frac{1}{12}, \quad k_{11} = \frac{1}{6}, \quad k_{12} = \frac{1}{24}, \quad (A.4)$$

$$k_{13} = -\frac{1}{6}, \quad k_{14} = \frac{1}{3}, \quad k_{15} = -\frac{1}{6}, \quad k_{16} = \frac{1}{12}, \quad k_{17} = -\frac{1}{6}, \quad k_{18} = -\frac{1}{12}.$$

To determine $k_i$ we shall exploit the factorization property (3.6), i.e.

$$b_6(\Delta_4) = b_6(\Delta_2) + b_6(\Delta_2'), \quad \Delta_4 = \Delta_2 \Delta_2', \quad (A.5)$$

where $b_6(\Delta_2)$ is given by (3.5). One needs to identify enough special cases and consistency conditions to fix all $k_i$. When comparing the two sides of the $b_6$-relation in (A.5) it is important to take into account (i) that they are defined up to total derivatives (which we drop in discussing UV divergences), (ii) that the terms can be cyclically permuted because they appear under an overall trace, and (iii) relations between the invariants (implied, e.g., by the Bianchi identity).

Considering $\Delta_2 = -\nabla^2 + X$ and $\Delta_2' = -\nabla^2 + X'$ their product is given by (A.1) with

$$V_{mn} = -\delta_{mn}(X + X'), \quad N_m = -\nabla_m X', \quad U = XX' - \nabla^2 X', \quad V = -6(X + X'). \quad (A.6)$$

\(^{22}\)The relations between the $k_i$ and $\bar{k}_i$ in (3.8) are, using (A.2), $\bar{k}_6 = k_6 + \frac{1}{2} k_{15} - \frac{1}{4} k_{17}$, $\bar{k}_8 = k_8 + \frac{1}{2} k_{16}$, $\bar{k}_{10} = k_{10} - \frac{1}{2} k_{15} + \frac{1}{4} k_{17}$, $\bar{k}_{13} = k_{11} + \frac{1}{2} k_{14}$, $k_{15} = k_{15} - k_{17}$, $\bar{k}_{16} = k_{16} + k_{18}$ with $\bar{k}_i = k_i$ otherwise.
Using (3.5) and comparing with (A.3) gives
\[
\begin{align*}
    k_1 &= \frac{1}{30}, & k_3 + 6k_4 + 36k_5 &= \frac{1}{30}, & k_{13} + 6k_{12} &= \frac{1}{12}, & k_{17} &= -\frac{1}{6}, \\
    k_2 &= \frac{1}{45}, & k_6 + 6k_7 + 6k_8 + 36k_9 &= \frac{1}{12}, & k_{15} + 6k_{16} &= \frac{1}{3}, & k_{18} &= -\frac{1}{12}.
\end{align*}
\]

Next, let us assume that
\[
\Delta_4 = \Delta_+ \Delta_-, \quad \nabla_m^\pm \equiv \nabla_m \pm K_m, \tag{A.8}
\]
\[
\Delta_+ = -(\nabla_m^\pm)^2 = -\nabla^2 + 2K_m \nabla_m + (\nabla_m K_m - K_m K_m). \tag{A.9}
\]

Here \(\nabla_m K_n = \partial_m K_n + [A_m, K_n] (K_m \text{ is in the adjoint representation of the gauge group}).\)

The coefficient functions in the corresponding operator \(\Delta_4 = \Delta_+ \Delta_-\) in (A.1) read
\[
\begin{align*}
    V_{mn} &= -4\nabla_{(m} K_{n)} + 2K^2 \delta_{mn} - 4K_{(m} K_{n)}, \quad V = -4\nabla_n K_n + 8K^2, \\
    N_m &= -\nabla^2 K_m - \nabla_m \nabla_m K_m + \nabla_m K^2 \\
    &+ K_m K^2 - K^2 K_m - 2K_m \nabla_m K_m - K_m \nabla_n K_n + 2K_n F_{nm}, \quad (A.10) \\
    U &= -\nabla^2 \nabla_m K_n + \nabla_m K^2 - 2K_m \nabla_m \nabla_n K_n + 2K_m \nabla_n K^2 - (\nabla_n K_n)^2 + K^4 \\
    &+ (\nabla_n K_n) K^2 - K^2 \nabla_m K_m - 2\nabla_m K_n F_{mn} - 2K_m K_n F_{mn} + 2K_m \nabla_n F_{mn}.
\end{align*}
\]

Using (3.5) and the relations
\[
\begin{align*}
    F_{mn}^\pm &= \left[\nabla_m^\pm, \nabla_n^\pm \right] = F_{mn} + [K_m, K_n] \pm (\nabla_m K_n - \nabla_n K_m), \tag{A.11} \\
    \nabla_m^\pm F_{mn}^\pm &= \nabla_m \left[ F_{mn} + [K_m, K_n] \pm (\nabla_m K_n - \nabla_n K_m) \right] \\
    &\pm \left[ K_m, F_{mn} + [K_m, K_n] \pm (\nabla_m K_n - \nabla_n K_m) \right], \tag{A.12}
\end{align*}
\]

one can compute \(b_6(\Delta_\pm)\) and then compare to \(b_6(\Delta_4)\) in (A.5).

It is enough to consider the following special cases:

1. Abelian gauge group, \(\nabla_n K_m = \partial_n K_m, [F_{mn}, K_k] = 0\). In (A.5) we consider the terms with \(\partial^r K_m, \ r = 0, 1, 4\) that can always be uniquely cast into the form
\[
K^6, \quad K^4 \partial_n K_m, \quad (\partial_m K_m) \partial^2 (\partial_n K_n), \quad K_m \partial^4 K_m. \tag{A.13}
\]

Then comparing also the coefficients of \(F_{mn} K^2 \partial_n K_m\) and \(F_{nm} K_m \partial_n K^2\) (the latter does not actually appear) one obtains
\[
\begin{align*}
    32k_3 + 16 \cdot 12k_4 + 512k_5 + 8k_8 &= 0, & k_{15} + 4k_{16} - k_{17} + 4k_{18} &= 0, \\
    4k_6 + 8k_7 + 2k_{15} - k_{17} &= \frac{1}{30}, & 12k_{18} + 256 \cdot 3k_5 + 16 \cdot 18k_4 + 48k_3 &= 0, \\
    12k_6 + 8k_7 + 16k_8 + 16k_9 + 6k_{15} + 8k_{16} - 3k_{17} + 4k_{18} &= -\frac{1}{30}.
\end{align*}
\]

2. \(K_n\) constrained by \(\nabla_m K_n = 0\), implying \(2\nabla_{[k} \nabla_{m]} K_n = [F_{km}, K_n] = 0\). This leads to a number of nontrivial relations, e.g., \(\text{tr}([K_m, K_n] F_{nk} F_{mn}) = 0\). All the remaining invariants can be uniquely written as a combination of
\[
\begin{align*}
    K^6, & \quad K_m K_n K_k K_m K_n K_k, \quad K^2 K_m K^2 K_m, \quad K_m K_n K_m K_k K_n K_k, \\
    K^2 F_{mn} F_{mn}, & \quad K^2 K_m K_n K_n K_n, \quad K_m K_n F_{nk} F_{nk}, \quad F_{mn} K_m K_n K^2. \tag{A.15}
\end{align*}
\]
Their coefficients can then be compared to get \(((K_m K_n K_p)^2\) and \((K_m K_p K_m)^2\) give the same equation)

\[
\begin{align*}
24k_3 &= \frac{1}{15}, & -8k_3 + 64k_4 + 512k_3 + 8k_{18} - 2k_{17} &= -\frac{2}{45}, \\
-k_{17} + 2k_{18} &= 0, & 64k_4 + 48k_3 + 2k_{17} &= -\frac{1}{15}, & 24k_3 + 64k_4 &= \frac{1}{5}, \\
-k_{13} - k_{17} &= 0, & 8k_{12} + 2k_{13} &= 0.
\end{align*}
\] (A.16)

3. General unconstrained \(K_n\), comparing the terms with one \(K_m\) or two of them contracted together. A basis of such tensors contains

\[
\begin{align*}
K_m \nabla^2 \nabla_n F_{mn}, & \quad K_m F_{nk} \nabla_n F_{km}, & \quad K_m F_{mn} \nabla_k F_{kn}, \\
K_m \nabla_n F_{km} F_{nk}, & \quad K_m \nabla_k F_{kn} F_{mn}, & \quad K_m \nabla^4 K_m, \\
K_m \nabla_k F_{kn} \nabla_n K_m, & \quad K^2 F_{kn} F_{km}, & \quad K_m F_{kn} K_m F_{kn}.
\end{align*}
\] (A.17)

In this case we obtain (the two \(KKFF\) terms give the same equation)

\[
\begin{align*}
2k_{11} + k_{14} &= 0, & 8k_{12} + 2k_{13} &= 0, & 2k_{11} - 2k_{13} + 2k_{14} &= 0, \\
4k_6 + 8k_7 + 2k_{15} - k_{17} &= \frac{1}{30}, & 2k_6 + 16k_7 + 2k_{10} + 8k_{12} + 2k_{13} &= \frac{1}{30}, \\
4k_6 + 16k_7 + 4k_{10} &= -\frac{1}{15}, & 2k_6 + 16k_7 + 2k_{10} &= \frac{1}{30}, & k_{11} + k_{13} &= 0.
\end{align*}
\] (A.18)

The final system of equations is given by (A.7), (A.14), (A.16) and (A.18). This system is over-determined, with the unique solution for \(k_i\) given by (A.4). That some of the equations are actually redundant gives a non-trivial consistency check of the calculation. We also checked some of the coefficients \(k_i\) by explicit diagrammatic calculations of the corresponding UV divergences.

### B One-loop divergences in \(F^2 + (\nabla F)^2 + F^3\) theory

It is straightforward to generalize the expression for the effective action (2.6) to the case when one adds to the action (1.1) the standard YM term, i.e. the first term in (4.18)

\[
\Gamma_1 = \frac{1}{2} \log \frac{\det \Delta'_{4A}}{\det (\nabla^2)^2 \det (-\nabla^2 + \kappa^2)}, \quad \Delta'_{4A} = \Delta_{4A} + \kappa^2 \Delta_{2A}.
\] (B.1)

Here \(\Delta_{2A}\) is given in (4.19). The quadratic and logarithmic divergences of (B.1) are determined by the total \(b_4\) and \(b_6\) coefficients (cf. (3.3))

\[
\Gamma_\infty = -\frac{1}{(4\pi)^3} \left( \frac{1}{2} B_4 \Lambda^2 + B_6 \log \Lambda \right),
\] (B.2)

\[
B_p = \int d^6x \, b_p, \quad b_p = b_p(\Delta'_{4A}) - 2b_p(-\nabla^2) - b_p(-\nabla^2 + \kappa^2).
\] (B.3)
The expression for $b_4$ is known for both for $\Delta_2$ (3.4) and $\Delta_4$ (A.1) operators\footnote{Here $\text{tr}$ and $F_{mn}$ are the general trace and the curvature on the bundle, cf. footnote 11.}

\begin{equation}
   b_4(\Delta_2) = \text{tr} \left[ \frac{1}{12} F_{mn} F_{mn} + \frac{1}{2} X^2 \right], \quad (B.4)
\end{equation}

\begin{equation}
   b_4(\Delta_4) = \text{tr} \left[ \frac{1}{6} F_{mn} F_{mn} + p_1 V_{mn} V_{mn} + p_2 V V - U \right]. \quad (B.5)
\end{equation}

While the coefficients in (B.4) are universal, i.e. the same in any dimension $d$ [17], the coefficients $p_1$ and $p_2$ in (B.5) are dimension-dependent. In $d = 4$ their values are [2]
\begin{equation}
   p_1 = \frac{1}{21}, \quad p_2 = \frac{1}{48} \quad \text{while for general $d$ we found}
\end{equation}

\begin{equation}
   p_1 = -\frac{8 - 8d + d^2}{16d(d-1)}, \quad p_2 = \frac{1}{16(d-1)}, \quad p_1\big|_{d=6} = \frac{1}{120}, \quad p_2\big|_{d=6} = \frac{1}{80}. \quad (B.6)
\end{equation}

The coefficient $b_4$ controls the logarithmic divergences in the corresponding 4d theory where their computation was done in [2] (see also [4]). For the operators in (B.1) we get in $d = 6$ (here $\text{tr}$ is in the adjoint representation and $F_{mn}$ is the gauge field strength)\footnote{The gauge fixing contributions are independent of $d$. As a result, in 4d theory the coefficient $\beta_{1A}$ in (B.13) below is given by $\beta_{1A} = -2(43 + 108\gamma + 27\gamma^2)$ (cf. [2, 4]).}

\begin{equation}
   b_4(-\nabla^2 + \kappa^2) = \frac{1}{12} \text{tr} F_{mn} F_{mn} + \frac{1}{2} \kappa^4 C_2, \quad (B.7)
\end{equation}

\begin{equation}
   b_4(\Delta_4') = -\left( 3 + 14\gamma + \frac{12}{5} \gamma^2 \right) \text{tr} F_{mn} F_{mn} + 3 \kappa^4 C_2. \quad (B.8)
\end{equation}

Similarly, using (3.5) and (3.7), (3.8) we find

\begin{equation}
   b_6(\Delta_4') = -\frac{21}{5} \text{tr}(\nabla_m F_{mn})^2 + \left( \frac{1}{10} - 10\gamma + \frac{9}{4} \gamma^3 \right) \text{tr} F_{mn} F_{nk} F_{km} + \left( \frac{3}{2} + 9\gamma + 3\gamma^2 \right) \kappa^2 \text{tr} F_{mn} F_{mn} - \kappa^6 C_2, \quad (B.9)
\end{equation}

\begin{equation}
   b_6(-\nabla^2 + \kappa^2) = -\frac{1}{60} \text{tr}(\nabla_m F_{mn})^2 + \frac{1}{90} \text{tr} F_{mn} F_{nk} F_{km} - \frac{1}{12} \kappa^2 \text{tr} F_{mn} F_{mn} - \frac{1}{6} \kappa^6 C_2. \quad (B.10)
\end{equation}

As a result, the total values of the coefficients of the quadratic and logarithmic divergences in (B.2) in $d = 6$ are (omitting field-independent terms)

\begin{equation}
   b_4 = \frac{1}{12} \beta_1 \text{tr} F_{mn} F_{mn}, \quad (B.11)
\end{equation}

\begin{equation}
   b_6 = \kappa^2 \beta_2 \text{tr} F_{mn} F_{mn} - \frac{1}{60} \beta_{2A} \text{tr}(\nabla_m F_{mn})^2 + \frac{1}{90} \beta_{3A} \text{tr} F_{mn} F_{nk} F_{km}, \quad (B.12)
\end{equation}

\begin{equation}
   \beta_{1A} = -39 - 168\gamma - \frac{144}{5} \gamma^2, \quad \beta_{\kappa A} = \frac{19}{12} + 9\gamma + 3\gamma^2, \quad (B.13)
\end{equation}

where $\beta_{2A}$ and $\beta_{3A}$ in (B.12) are the same as in (1.3). Ignoring non-universal quadratic divergence (absent in dimensional regularization), the logarithmic renormalization of $\kappa$ is controlled by $\beta_{\kappa A}$ with the RG equation (cf. (4.5), (4.6))\footnote{Recall that the coefficient of the YM term is chosen as $\frac{\kappa^2}{g^2}$, cf. (4.18).}

\begin{equation}
   \frac{d\kappa^2}{dt} = -\left( \beta_{\kappa A} + \frac{1}{60} \beta_{2A} \right) 2\kappa^2 g^2 C_2 = \left( \frac{175}{12} - 18\gamma - 6\gamma^2 \right) \kappa^2 g^2 C_2. \quad (B.14)
\end{equation}
Near both the attractive fixed points $\gamma_1 \simeq -0.897$ and $\gamma_3 \simeq 0.838$ of $\beta_\gamma$ in (4.6), the r.h.s of (B.14) is negative and thus $\kappa^2 \to 0$ in the UV.

Let us now consider the log divergence in the $(1,0)$ supersymmetric extension of this bosonic model, i.e. the $(1,0)$ SYM combined with the $(1,0)$ theory (4.7). Here the operators in the 1-loop effective action (10.10) get $\kappa$-dependent terms as in (B.1) (with $\gamma = 0$)

$$\Delta_{4A}^{(0)} = \Delta_{4A}^{(0)} + \kappa^2 \Delta_{2A}, \quad \Delta_{3\Psi} = \Delta_{3\Psi} + \kappa^2 \Delta_{1\Psi}, \quad \Delta_{2\Phi} = \Delta_{2\Phi} + \kappa^2,$$  

(B.15)

where $\Delta_{1\Psi} = -i \nabla$ and $\Delta_{2\Phi} = -\nabla^2$. Explicitly, we get (cf. (4.10), (B.1))

$$\Gamma'_{(1,0)} = \frac{1}{2} \log \left[ \frac{\det \Delta_{4A}^{(0)}}{[\det(-\nabla^2)]^2 \det(-\nabla^2 + \kappa^2) \det \Delta_{3\Psi}} \right]$$

(B.16)

$$= \frac{1}{2} \log \left[ \frac{\det \Delta_{4A}^{(0)}}{[\det(-\nabla^2 + \kappa^2)]^2 \det \Delta_{3\Psi}} \right].$$

For the gauge field and scalar determinants the expressions for $b_4$ and $b_6$ are given by (B.7)–(B.10) with $\gamma = 0$ while for the fermion contribution we get as in (4.13),

$$b_6(\Delta_{3\Psi}) = b_6(\Delta_{1\Psi} \Delta_{3\Psi}) - b_6(\Delta_{1\Psi}) = b_6(\Delta_{3\Psi}) + \frac{14}{3} \kappa^2 \text{tr} F_{mn} F_{mn} - \frac{4}{3} \kappa^8 C_2.$$  

(B.17)

As a result, the analog of (B.13) is

$$b_6 = \kappa^2 \beta_{\kappa(1,0)} \text{tr} F_{mn} F_{mn} - \frac{1}{60} \beta_{2(1,0)} \text{tr} F_{mn} F_{mn}^2,$$

(B.18)

$$\beta_{\kappa(1,0)} = -\frac{29}{6}, \quad \beta_{2(1,0)} = 220.$$  

where $\beta_{2(1,0)}$ is the same as in (4.16), (1.7). Since the combination $\beta_{\kappa(1,0)} + \frac{1}{60} \beta_{2A}$ is negative, as a result of (B.14) we do not have asymptotic freedom in the supersymmetric case.

Let us note also that on $\nabla_m F_{mn} = 0$ background (B.16) becomes the following generalization of (4.22)

$$\Gamma'_{(1,0)} = \frac{1}{2} \left[ \log \det \Delta_{2A} - 2 \log \det \Delta_{2,0} - \det \Delta_{1\Psi} \right]$$

$$+ \frac{1}{2} \left[ \log \det(\Delta_{2A} + \kappa^2) - 2 \log \det(\Delta_{2,0} + \kappa^2) - \det(\Delta_{1\Psi} + \kappa^2) \right]$$

$$+ \frac{1}{2} \left[ \log \det(\Delta_{2,0} + \kappa^2) - \det(\Delta_{1\Psi} + \kappa^2) \right],$$  

(B.19)

i.e. the sum of contributions of massless $(1,0)$ SYM, its massive analog, and massive analog of scalar multiplet. From (B.19) it is easy to compute the quadratic divergence coefficient $b_4$ (B.4) or $\beta_1$ coefficient in (B.11) in the effective action (B.16) or (B.19)$^{26}$

$$\beta_{1(1,0)\text{SYM}} = -12, \quad \beta_{1(1,0)\text{scal}} = 12, \quad \beta_{1(1,0)} = \beta_{1(1,0)}' = -12 + 12 = -12.$$  

(B.20)

The coefficient of the field independent $\kappa^4$ quadratic divergence is proportional to the number of degrees of freedom and thus vanishes in supersymmetric cases.

$^{26}$Explicitly, one finds from (B.4) in general dimension $d$ [1]: $\beta_1(\Delta_{2,0}) = 1$, $\beta_1(\Delta_{2A}) = d - 24$, $\Delta_{1\Psi} = -\nu$, $\nu = 2^{d/2}$, so that in $d = 6$ we get $\beta_{1(1,0)\text{SYM}} = -12$, $\beta_{1(1,0)\text{scal}} = 12$, $\beta_{1(1,0)} = 0$. 

– 15 –
Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY 4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

[1] E.S. Fradkin and A.A. Tseytlin, Quantum Properties of Higher Dimensional and Dimensionally Reduced Supersymmetric Theories, Nucl. Phys. B 227 (1983) 252 [inSPIRE].

[2] E.S. Fradkin and A.A. Tseytlin, Renormalizable asymptotically free quantum theory of gravity, Nucl. Phys. B 201 (1982) 469 [inSPIRE].

[3] B. Grinstein and D. O’Connell, One-Loop Renormalization of Lee-Wick Gauge Theory, Phys. Rev. D 78 (2008) 105005 [arXiv:0801.4034] [inSPIRE].

[4] B. Grinstein and D. O’Connell, One-Loop Renormalization of Lee-Wick Gauge Theory, Phys. Rev. D 78 (2008) 105005 [arXiv:0801.4034] [inSPIRE].

[5] L. Casarin, On higher-derivative gauge theories, arXiv:1710.08021 [inSPIRE].

[6] L. Casarin, On higher-derivative gauge theories, arXiv:1710.08021 [inSPIRE].

[7] E.A. Ivanov and A.V. Smilga, Chiral anomalies in higher-derivative supersymmetric 6D theories, Phys. Lett. B 647 (2007) 298 [hep-th/0606139] [inSPIRE].

[8] E.A. Ivanov and A.V. Smilga, Chiral anomalies in higher-derivative supersymmetric 6D theories, Phys. Lett. B 647 (2007) 298 [hep-th/0606139] [inSPIRE].

[9] M. Beccaria and A.A. Tseytlin, Conformal a-anomaly of some non-unitary 6d superconformal theories, JHEP 09 (2015) 017 [arXiv:1506.08727] [inSPIRE].

[10] M. Beccaria and A.A. Tseytlin, Conformal anomaly c-coefficients of superconformal 6d theories, JHEP 01 (2016) 001 [arXiv:1510.02685] [inSPIRE].

[11] M. Beccaria and A.A. Tseytlin, Conformal anomaly c-coefficients of superconformal 6d theories, JHEP 01 (2016) 001 [arXiv:1510.02685] [inSPIRE].

[12] S. Giombi, I.R. Klebanov and G. Tarnopolsky, Conformal QED, F-Theorem and the c Expansion, J. Phys. A 49 (2016) 135403 [arXiv:1508.06354] [inSPIRE].

[13] S. Giombi, I.R. Klebanov and G. Tarnopolsky, On C_J and C_T in Conformal QED, JHEP 08 (2016) 156 [arXiv:1602.01076] [inSPIRE].

[14] S. Giombi, I.R. Klebanov and G. Tarnopolsky, On C_J and C_T in Conformal QED, JHEP 08 (2016) 156 [arXiv:1602.01076] [inSPIRE].

[15] H. Osborn and A. Stergiou, C_T for non-unitary CFTs in higher dimensions, JHEP 06 (2016) 079 [arXiv:1603.07307] [inSPIRE].

[16] H. Osborn and A. Stergiou, C_T for non-unitary CFTs in higher dimensions, JHEP 06 (2016) 079 [arXiv:1603.07307] [inSPIRE].

[17] K.-W. Huang, R. Roiban and A.A. Tseytlin, Self-dual 6d 2-form fields coupled to non-abelian gauge field: quantum corrections, JHEP 06 (2018) 134 [arXiv:1804.05059] [inSPIRE].

[18] K.-W. Huang, R. Roiban and A.A. Tseytlin, Self-dual 6d 2-form fields coupled to non-abelian gauge field: quantum corrections, JHEP 06 (2018) 134 [arXiv:1804.05059] [inSPIRE].

[19] H. Johansson, G. Mogull and F. Teng, Unraveling conformal gravity amplitudes, JHEP 09 (2018) 080 [arXiv:1806.05124] [inSPIRE].

[20] E.A. Ivanov and A.V. Smilga, Conformal properties of hypermultiplet actions in six dimensions, Phys. Lett. B 637 (2006) 374 [hep-th/0510273] [inSPIRE].

[21] E.A. Ivanov and A.V. Smilga, Conformal properties of hypermultiplet actions in six dimensions, Phys. Lett. B 637 (2006) 374 [hep-th/0510273] [inSPIRE].

[22] S.M. Kuzenko, J. Novak and S. Theisen, New superconformal multiplets and higher derivative invariants in six dimensions, Nucl. Phys. B 925 (2017) 348 [arXiv:1707.04445] [inSPIRE].

[23] S.M. Kuzenko, J. Novak and I.B. Samsonov, Chiral anomalies in six dimensions from harmonic superspace, JHEP 11 (2017) 145 [arXiv:1708.08238] [inSPIRE].

[24] P.B. Gilkey, The Spectral geometry of a Riemannian manifold, J. Diff. Geom. 10 (1975) 601 [inSPIRE].

[25] E.S. Fradkin and A.A. Tseytlin, Conformal Anomaly in Weyl Theory and Anomaly Free Superconformal Theories, Phys. Lett. 134B (1984) 187 [inSPIRE].
[19] E.S. Fradkin and A.A. Tseytlin, Conformal supergravity, Phys. Rept. 119 (1985) 233 [arXiv:hep-th/0306138] [SPIRE].

[20] I. Jack and H. Osborn, Two Loop Background Field Calculations for Arbitrary Background Fields, Nucl. Phys. B 207 (1982) 474 [arXiv:hep-th/0306138] [SPIRE].

[21] I. Jack and H. Osborn, Background Field Calculations in Curved Space-time. I. General Formalism and Application to Scalar Fields, Nucl. Phys. B 234 (1984) 331 [arXiv:hep-th/0306138] [SPIRE].

[22] B. Grinstein, A. Stergiou, D. Stone and M. Zhong, Two-loop renormalization of multiflavor $\phi^4$ theory in six dimensions and the trace anomaly, Phys. Rev. D 92 (2015) 045013 [arXiv:1504.05959] [SPIRE].

[23] A.E.M. van de Ven, Explicit Counter Action Algorithms in Higher Dimensions, Nucl. Phys. B 250 (1985) 593 [arXiv:hep-th/0306138] [SPIRE].

[24] R.R. Metsaev and A.A. Tseytlin, On loop corrections to string theory effective actions, Nucl. Phys. B 298 (1988) 109 [arXiv:hep-th/0306138] [SPIRE].

[25] D.V. Vassilevich, Heat kernel expansion: User’s manual, Phys. Rept. 388 (2003) 279 [arXiv:hep-th/0306138] [SPIRE].

[26] A.E.M. van de Ven, Index free heat kernel coefficients, Class. Quant. Grav. 15 (1998) 2311 [arXiv:hep-th/9708152] [SPIRE].

[27] P.B. Gilkey, The spectral geometry of the higher order Laplacian, Duke Math. J. 47 (1980) 511 [Erratum ibid. 48 (1981) 887].

[28] A.O. Barvinsky and G.A. Vilkovisky, The Generalized Schwinger-Dewitt Technique in Gauge Theories and Quantum Gravity, Phys. Rept. 119 (1985) 1 [SPIRE].

[29] I.G. Avramidi, Heat kernel and quantum gravity, Lect. Notes Phys. Monogr. 64 (2000) 1 [SPIRE].

[30] A. Smilga, Classical and quantum dynamics of higher-derivative systems, Int. J. Mod. Phys. A 32 (2017) 1730025 [arXiv:1710.11538] [SPIRE].

[31] I.L. Buchbinder, E.A. Ivanov, B.S. Merzlikin and K.V. Stepanyantz, Gauge dependence of the one-loop divergences in 6D, $N = (1,0)$ abelian theory, Nucl. Phys. B 936 (2018) 638 [arXiv:1808.08446] [SPIRE].

[32] G. Bossard, E. Ivanov and A. Smilga, Ultraviolet behavior of 6D supersymmetric Yang-Mills theories and harmonic superspace, JHEP 12 (2015) 085 [arXiv:1509.08027] [SPIRE].

[33] J.A. Gracey, Six dimensional QCD at two loops, Phys. Rev. D 93 (2016) 025025 [arXiv:1512.04443] [SPIRE].

[34] J.A. Gracey, $\beta$-functions in higher dimensional field theories, PoS(LL2016)063 (2016) [arXiv:1610.04447] [SPIRE].