Abstract

The peakons discussed here are singular solutions of the dispersionless Camassa-Holm (CH) shallow water wave equation in one spatial dimension. These are reviewed in the context of asymptotic expansions and Euler-Poincaré variational principles. The dispersionless CH equation generalizes to the EPDiff equation, whose singular solutions are peakon wave fronts in higher dimensions. The reduction of these singular solutions of CH and EPDiff to canonical Hamiltonian dynamics on lower dimensional sets may be understood, by realizing that their solution ansatz is a momentum map, and momentum maps are Poisson.

1 Introduction

Camassa and Holm [7] discovered the “peakon” solitary traveling wave solution for a shallow water wave,

$$u(x,t) = ce^{-|x-ct|/\alpha},$$

(1)
whose fluid velocity \( u \) is a function of position \( x \) on the real line and time \( t \). The peakon traveling wave moves at a speed equal to its maximum height, at which it has a sharp peak (jump in derivative). Peakons are an emergent phenomenon, solving the initial value problem for a partial differential equation derived by an asymptotic expansion of Euler’s equations using the small parameters of shallow water dynamics. Peakons are nonanalytic solitons, which superpose as

\[
  u(x, t) = \sum_{a=1}^{N} p_a(t) e^{-|x-q_a(t)|/\alpha},
\]

for sets \( \{p\} \) and \( \{q\} \) satisfying canonical Hamiltonian dynamics. Peakons arise for shallow water waves in the limit of zero linear dispersion in one dimension. Peakons satisfy a partial differential equation (PDE) arising from Hamilton’s principle for geodesic motion on the smooth invertible maps (diffeomorphisms) with respect to the \( H^1 \) Sobolev norm of the fluid velocity. Peakons generalize to higher dimensions, as well. We explain how peakons were derived in the context of shallow water asymptotics and describe some of their remarkable mathematical properties.

2 Shallow water background for peakons

Euler’s equations for irrotational incompressible ideal fluid motion under gravity with a free surface have an asymptotic expansion for shallow water waves that contains two small parameters, \( \epsilon \) and \( \delta^2 \), with ordering \( \epsilon \geq \delta^2 \). These small parameters are \( \epsilon = a/h_0 \) (the ratio of wave amplitude to mean depth) and \( \delta^2 = (h_0/l_x)^2 \) (the squared ratio of mean depth to horizontal length, or wavelength). Euler’s equations are made non-dimensional by introducing \( x = l_xx' \) for horizontal position, \( z = h_0z' \) for vertical position, \( t = (l_x/c_0)t' \) for time, \( \eta = a\eta' \) for surface elevation and \( \varphi = (gl_xa/c_0)\varphi' \) for velocity potential, where \( c_0 = \sqrt{gh_0} \) is the mean wave speed and \( g \) is the constant gravity. The quantity \( \sigma = \sigma'/(h_0\rho c_0^2) \) is the dimensionless Bond number, in which \( \rho \) is the mass density of the fluid and \( \sigma' \) is its surface tension, both of which are taken to be constants. After dropping primes, this asymptotic expansion yields the nondimensional Korteweg-de Vries (KdV) equation for the horizontal velocity variable \( u = \varphi_x(x,t) \) at linear order in the small dimensionless ratios \( \epsilon \) and \( \delta^2 \), as the left hand side of

\[
  u_t + u_x + \frac{3\epsilon}{2} uu_x + \frac{\delta^2}{6}(1-3\sigma)u_{xxx} = O(\epsilon\delta^2).
\]
Here, partial derivatives are denoted using subscripts, and boundary conditions are 
\( u = 0 \) and 
\( u_x = 0 \) at spatial infinity on the real line. The famous \( \text{sech}^2(x-t) \) 
traveling wave solutions (the solitons) for KdV (3) arise in a balance between 
its (weakly) nonlinear steepening and its third-order linear dispersion, when the 
quadratic terms in \( \epsilon \) and \( \delta^2 \) on its right hand side are neglected.

In equation (3), a normal form transformation due to Kodama [22] has been 
used to remove the other possible quadratic terms of order \( O(\epsilon^2) \) and \( O(\delta^4) \). The 
remaining quadratic correction terms in the KdV equation (3) may be collected 
at order \( O(\epsilon \delta^2) \). These terms may be expressed, after introducing a “momentum 
variable,”

\[
m = u - \nu \delta^2 u_{xx},
\]

and neglecting terms of cubic order in \( \epsilon \) and \( \delta^2 \), as

\[
m_t + m_x + \frac{\epsilon}{2}(um_x + bmu_x) + \frac{\delta^2}{6}(1 - 3\sigma)u_{xxx} = 0.
\]

In the momentum variable \( m = u - \nu \delta^2 u_{xx} \), the parameter \( \nu \) is given by [11]

\[
\nu = \frac{19 - 30\sigma - 45\sigma^2}{60(1 - 3\sigma)}.
\]

Thus, the effects of \( \delta^2 \)-dispersion also enter the nonlinear terms. After restor-
ing dimensions in equation (5) and rescaling velocity \( u \) by \( (b + 1) \), the following 
“\( b \)-equation” emerges,

\[
m_t + c_0 m_x + um_x + bmu_x + \Gamma u_{xxx} = 0,
\]

where \( m = u - \alpha^2 u_{xx} \) is the dimensional momentum variable, and the constants \( \alpha^2 \) 
and \( \Gamma/c_0 \) are squares of length scales. When \( \alpha^2 \to 0 \), one recovers KdV from the 
\( b \)-equation (7), up to a rescaling of velocity. Any value of the parameter \( b \neq -1 \) may be achieved in equation (7) by an appropriate Kodama transformation [11].

As we have emphasized, the values of the coefficients in the asymptotic analysis 
of shallow water waves at quadratic order in their two small parameters only hold, 
modulo the Kodama normal-form transformations. Hence, these transformations 
may be used to advance the analysis and thereby gain insight, by optimizing the 
choices of these coefficients. The freedom introduced by the Kodama transforma-
tions among asymptotically equivalent equations at quadratic order in \( \epsilon \) and \( \delta^2 \) also helps to answer the perennial question, “Why are integrable equations so 
ubiquitous when one uses asymptotics in modeling?”
Integrable cases of the $b$–equation \((7)\). The cases $b = 2$ and $b = 3$ are special values, for which the $b$–equation becomes a completely integrable Hamiltonian system. For $b = 2$, equation \((7)\) specializes to the integrable CH equation of Camassa and Holm \[7\]. The case $b = 3$ in \((7)\) recovers the integrable DP equation of Degasperis and Procesi \[9\]. These two cases exhaust the integrable candidates for \((7)\), as was shown using Painlevé analysis. The $b$–family of equations \((7)\) was also shown in \[26\] to admit the symmetry conditions necessary for integrability, only in the cases $b = 2$ for CH and $b = 3$ for DP.

The $b$–equation \((7)\) with $b = 2$ was first derived in Camassa and Holm \[7\] by using asymptotic expansions directly in the Hamiltonian for Euler’s equations governing inviscid incompressible flow in the shallow water regime. In this analysis, the CH equation was shown to be bi-Hamiltonian and thereby was found to be completely integrable by the inverse scattering transform (IST) on the real line. Reviews of IST may be found, for example, in Ablowitz et al. \[1\], Dubrovin et al. \[10\], Novikov et al. \[27\]. For discussions of other related bi-Hamiltonian equations, see \[9\].

Camassa and Holm \[7\] also discovered the remarkable peaked soliton (peakon) solutions of \((1,2)\) for the CH equation on the real line, given by \((7)\) in the case $b = 2$. The peakons arise as solutions of \((7)\), when $c_0 = 0$ and $\Gamma = 0$ in the absence of linear dispersion. Peakons move at a speed equal to their maximum height, at which they have a sharp peak (jump in derivative). Unlike the KdV soliton, the peakon speed is independent of its width ($\alpha$). Periodic peakon solutions of CH were treated in Alber et al. \[2\]. There, the sharp peaks of periodic peakons were associated with billiards reflecting at the boundary of an elliptical domain. These billiard solutions for the periodic peakons arise from geodesic motion on a tri-axial ellipsoid, in the limit that one of its axes shrinks to zero length.

Before Camassa and Holm derived their shallow water equation in \[7\], a class of integrable equations existed, which was later found to contain equation \((7)\) with $b = 2$. This class of integrable equations was derived using hereditary symmetries in Fokas and Fuchssteiner \[13\]. However, equation \((7)\) was not written explicitly, nor was it derived physically as a shallow water equation and its solution properties for $b = 2$ were not studied before Camassa and Holm \[7\]. See Fuchssteiner \[15\] for an insightful history of how the shallow water equation \((7)\) in the integrable case with $b = 2$ relates to the mathematical theory of hereditary symmetries.

Equation \((7)\) with $b = 2$ was recently re-derived as a shallow water equation by using asymptotic methods in three different approaches in Dullin et al. \[11\], in Fokas and Liu \[14\] and also in Johnson \[21\]. These three derivations all used different variants of the method of asymptotic expansions for shallow water waves.
in the absence of surface tension. Only the derivation in Dullin et al. [11] used the Kodama normal-form transformations to take advantage of the non-uniqueness of the asymptotic expansion results at quadratic order.

The effects of the parameter $b$ on the solutions of equation (7) were investigated in Holm and Staley [19], where $b$ was treated as a bifurcation parameter, in the limiting case when the linear dispersion coefficients are set to $c_0 = 0$ and $\Gamma = 0$. This limiting case allows several special solutions, including the peakons, in which the two nonlinear terms in equation (7) balance each other in the absence of linear dispersion.

3 Peakons: Singular solutions without linear dispersion in one spatial dimension

Peakons were first found as singular soliton solutions of the completely integrable CH equation. This is equation (7) with $b = 2$, now rewritten in terms of the velocity, as

$$u_t + c_0 u_x + 3u u_x + \Gamma u_{xxx} = \alpha^2 (u_{xxt} + 2u_x u_{xx} + uu_{xxx}).$$

(8)

Peakons were found in [7] to arise in the absence of linear dispersion. That is, they arise when $c_0 = 0$ and $\Gamma = 0$ in CH (8). Specifically, peakons are the individual terms in the peaked $N$–soliton solution of CH (8) for its velocity,

$$u(x, t) = \sum_{b=1}^{N} p_b(t) e^{-|x-q_b(t)|/\alpha},$$

(9)

in the absence of linear dispersion. Each term in the sum is a soliton with a sharp peak at its maximum. Hence, the name “peakon.” Expressed using its momentum, $m = (1 - \alpha^2 \partial_x^2)u$, the peakon velocity solution (9) of dispersionless CH becomes a sum over a delta functions, supported on a set of points moving on the real line. Namely, the peakon velocity solution (9) implies

$$m(x, t) = 2\alpha \sum_{b=1}^{N} p_b(t) \delta(x - q_v(t)), \quad (10)$$

because of the relation $(1 - \alpha^2 \partial_x^2)e^{-|x|/\alpha} = 2\alpha \delta(x)$. These solutions satisfy the $b$–equation (7) for any value of $b$, provided $c_0 = 0$ and $\Gamma = 0$. 


Thus, peakons are singular momentum solutions of the dispersionless $b$-equation, although they are not stable for every value of $b$. From numerical simulations [19], peakons are conjecture to be stable for $b > 1$. In the integrable cases $b = 2$ for CH and $b = 3$ for DP, peakons are stable singular soliton solutions. The spatial velocity profile $e^{-|x|/\alpha}/(2\alpha)$ of each separate peakon in (9) is the Green’s function for the Helmholtz operator on the real line, with vanishing boundary conditions at spatial infinity. Unlike the KdV soliton, whose speed and width are related, the width of the peakon profile is set by its Green’s function, independently of its speed.

**Integrable peakon dynamics of CH.** Substituting the peakon solution ansatz (9) and (10) into the dispersionless CH equation,

$$m_t + um_x + 2mu_x = 0, \quad \text{with} \quad m = u - \alpha^2 u_{xx},$$

yields Hamilton’s canonical equations for the dynamics of the discrete set of peakon parameters $p_a(t)$ and $q_a(t)$,

$$\dot{q}_a(t) = \frac{\partial h_N}{\partial p_a} \quad \text{and} \quad \dot{p}_a(t) = - \frac{\partial h_N}{\partial q_a},$$

for $a = 1, 2, \ldots, N$, with Hamiltonian given by [7],

$$h_N = \frac{1}{2} \sum_{a,b=1}^{N} p_a p_b e^{-|q_a - q_b|/\alpha}.$$  (13)

Thus, one finds that the points $x = q^a(t)$ in the peakon solution (9) move with the flow of the fluid velocity $u$ at those points, since $u(q^a(t), t) = \dot{q}^a(t)$. This means the $q^a(t)$ are Lagrangian coordinates. Moreover, the singular momentum solution ansatz (10) is the Lagrange-to-Euler map for an invariant manifold of the dispersionless CH equation (11). On this finite-dimensional invariant manifold for the partial differential equation (11), the dynamics is canonically Hamiltonian.

With Hamiltonian (13), the canonical equations (12) for the $2N$ canonically conjugate peakon parameters $p_a(t)$ and $q_a(t)$ were interpreted in [7] as describing geodesic motion on the $N$-dimensional Riemannian manifold whose co-metric is $g^{ij}(\{q\}) = e^{-|q_i - q_j|/\alpha}$. Moreover, the canonical geodesic equations arising from Hamiltonian (13) comprise an integrable system for any number of peakons $N$. This integrable system was studied in [7] for solutions on the real line, and in [2, 25] and references therein, for spatially periodic solutions.
Being a completely integrable Hamiltonian soliton equation, the continuum CH equation (8) has an associated isospectral eigenvalue problem, discovered in [7] for any values of its dispersion parameters \( c_0 \) and \( \Gamma \). Remarkably, when \( c_0 = 0 \) and \( \Gamma = 0 \), this isospectral eigenvalue problem has a purely discrete spectrum. Moreover, in this case, each discrete eigenvalue corresponds precisely to the time-asymptotic velocity of a peakon. This discreteness of the CH isospectrum in the absence of linear dispersion implies that only the singular peakon solutions (10) emerge asymptotically in time, in the solution of the initial value problem for the dispersionless CH equation (11). This is borne out in numerical simulations of the dispersionless CH equation (11), starting from a smooth initial distribution of velocity [16, 19].

Figure 1 shows the emergence of peakons from an initially Gaussian velocity distribution and their subsequent elastic collisions in a periodic one-dimensional domain. This figure demonstrates that singular solutions dominate the initial value problem and, thus, that it is imperative to go beyond smooth solutions for the CH equation; the situation is similar for the EPDiff equation.

**Peakons as mechanical systems.** Being governed by canonical Hamiltonian equations, each \( N \)-peakon solution can be associated with a mechanical system of moving particles. Calogero et al. [6] further extended the class of mechanical systems of this type. The \( r \)-matrix approach was applied to the Lax pair formulation of the \( N \)-peakon system for CH by Ragnisco and Bruschi [28], who also pointed out the connection of this system with the classical Toda lattice. A discrete version of the Adler-Kostant-Symes factorization method was used by Suris [29] to study a discretization of the peakon lattice, realized as a discrete integrable system on a certain Poisson submanifold of \( gl(N) \) equipped with an \( r \)-matrix Poisson bracket. Beals et al. [5] used the Stieltjes theorem on continued fractions and the classical moment problem for studying multi-peakon solutions of the CH equation. Generalized peakon systems are described for any simple Lie algebra by Alber et al. [2].

**Pulsons: Generalizing the peakon solutions of the dispersionless \( b \)-equation for other Green’s functions.** The Hamiltonian \( h_N \) in equation (13) depends on the Green’s function for the relation between velocity \( u \) and momentum \( m \). However, the singular momentum solution ansatz (10) is independent of this Green’s function. Thus, as discovered in Fringer and Holm [16],

\footnote{The figures in this article were kindly supplied by Martin Staley}
Figure 1: A smooth localized (Gaussian) initial condition for the CH equation breaks up into an ordered train of peakons as time evolves (the time direction being vertical). The peakon train eventually wraps around the periodic domain, thereby allowing the leading peakons to overtake the slower emergent peakons from behind in collisions that cause phase shifts, as discussed in [7].

The singular momentum solution ansatz \([10]\) for the dispersionless equation,

\[
m_t + um_x + 2mu_x = 0, \quad \text{with} \quad u = g \ast m,
\]

provides an invariant manifold on which canonical Hamiltonian dynamics occurs, for any choice of the Green’s function \(g\) relating velocity \(u\) and momentum \(m\) by the convolution \(u = g \ast m\).

The fluid velocity solutions corresponding to the singular momentum ansatz \([10]\) for equation \([14]\) are the pulsons. Pulsons are given by the sum over \(N\) velocity profiles determined by the Green’s function \(g\), as

\[
u(x, t) = \sum_{a=1}^{N} p_a(t) g(x, q_a(t)).
\]

Again for \([14]\), the singular momentum ansatz \([10]\) results in a finite-dimensional invariant manifold of solutions, whose dynamics is canonically Hamiltonian. The
Hamiltonian for the canonical dynamics of the $2N$ parameters $p_a(t)$ and $q_a(t)$ in
the “pulson” solutions (15) of equation (14) is

$$h_N = \frac{1}{2} \sum_{a,b=1}^N p_a p_b g(q_a, q_b).$$

Again for the pulsons, the canonical equations for the invariant manifold of singular momentum solutions provide a phase-space description of geodesic motion, this time with respect to the co-metric given by the Green’s function $g$. Mathematical analysis and numerical results for the dynamics of these pulson solutions are given in [16]. These results describe how the collisions of pulsons (15) depend upon their shape.

**Compactons in the $1/\alpha^2 \to 0$ limit of CH.** As mentioned earlier, in the limit that $\alpha^2 \to 0$, the CH equation (8) becomes the KdV equation. In the opposite limit that $1/\alpha^2 \to 0$ CH becomes the Hunter-Zheng equation [20]

$$\left( u_t + uu_x \right)_x = \frac{1}{2} (u_x^2)_x \quad \text{(Hunter-Zheng)}$$

This equation has “compacton” solutions, whose collision dynamics was studied numerically and put into the present context in [16]. The corresponding Green’s function satisfies $-\partial_x^2 g(x) = 2\delta(x)$, so it has the triangular shape, $g(x) = 1 - |x|$ for $|x| < 1$, and vanishes otherwise, for $|x| \geq 1$. That is, the Green’s function in this case has compact support; hence, the name “compactons” for these pulson solutions, which as a limit of the integrable CH equations are true solitons, solvable by IST.

**Pulson solutions of the dispersionless $b$–equation.** Holm and Staley [19] give the pulson solutions of the traveling wave problem and their elastic collision properties for the dispersionless $b$–equation,

$$m_t + um_x + bmu_x = 0, \quad \text{with} \quad u = g \ast m, \quad (17)$$

with any (symmetric) Green’s function $g$ and for any value of the parameter $b$. Numerically, pulsons and peakons are both found to be stable for $b > 1$, [19]. The reduction to noncanonical Hamiltonian dynamics for the invariant manifold of singular momentum solutions (10) of the other integrable case $b = 3$ with peakon Green’s function $g(x, y) = e^{-|x-y|/\alpha}$ is found in [9].
4 Euler-Poincaré theory in more dimensions

Generalizing the peakon solutions of the CH equation to higher dimensions. In [19], weakly nonlinear analysis and the assumption of columnar motion in the variational principle for Euler’s equations are found to produce the two-dimensional generalization of the dispersionless CH equation (11). This generalization is the Euler-Poincaré (EP) equation [18] for the Lagrangian consisting of the kinetic energy,

\[
\ell = \frac{1}{2} \int \left[ |u|^2 + \alpha^2 (\text{div } u)^2 \right] dxdy ,
\]

in which the fluid velocity \( u \) is a two-dimensional vector. Evolution generated by kinetic energy in Hamilton’s principle results in geodesic motion, with respect to the velocity norm \( \| u \| \), which is provided by the kinetic energy Lagrangian. For ideal incompressible fluids governed by Euler’s equations, the importance of geodesic flow was recognized by Arnold [3] for the \( L^2 \) norm of the fluid velocity. The EP equation generated by any choice of kinetic energy norm without imposing incompressibility is called “EPDiff,” for “Euler-Poincaré equation for geodesic motion on the diffeomorphisms.” EPDiff is given by [18]

\[
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) m + \nabla u^T \cdot m + m (\text{div } u) = 0 ,
\]

with momentum density \( m = \delta \ell / \delta u \), where \( \ell = \frac{1}{2} \| u \|^2 \) is given by the kinetic energy, which defines a norm in the fluid velocity \( \| u \| \), yet to be determined. By design, this equation has no contribution from either potential energy, or pressure. It conserves the velocity norm \( \| u \| \) given by the kinetic energy. Its evolution describes geodesic motion on the diffeomorphisms with respect to this norm [18]. An alternative way of writing the EPDiff equation (19) in either two, or three dimensions is,

\[
\frac{\partial}{\partial t} m - u \times \text{curl } m + \nabla (u \cdot m) + m (\text{div } u) = 0 .
\]

This form of EPDiff involves all three differential operators, curl, gradient and divergence. For the kinetic energy Lagrangian \( \ell \) given in (18), which is a norm for irrotational flow (with \( \text{curl } u = 0 \)), we have the EPDiff equation (19) with momentum \( m = \delta \ell / \delta u = u - \alpha^2 \nabla (\text{div } u) \).

EPDiff (19) may also be written intrinsically as

\[
\frac{\partial}{\partial t} \frac{\delta \ell}{\delta u} = - \text{ad}^*_{\delta u} \frac{\delta \ell}{\delta u} ,
\]
where \( \text{ad}^* \) is the \( L^2 \) dual of the ad-operation (commutator) for vector fields. See [4, 24] for additional discussions of the beautiful geometry underlying this equation.

**Reduction to the dispersionless CH equation in 1D.** In one dimension, the EPDiff equation (19-21) with Lagrangian \( \ell \) given in (18) simplifies to the dispersionless CH equation (11). The dispersionless limit of the CH equation appears, because we have ignored potential energy and pressure.

**Strengthening the kinetic energy norm to allow for circulation.** The kinetic energy Lagrangian (18) is a norm for irrotational flow, with \( \text{curl} \ u = 0 \). However, inclusion of rotational flow requires the kinetic energy norm to be strengthened to the \( H^1_\alpha \) norm of the velocity, defined as

\[
\ell = \frac{1}{2} \int \left[ |u|^2 + \alpha^2 (\text{div} \ u)^2 + \alpha^2 (\text{curl} \ u)^2 \right] dxdy
\]

\[
= \frac{1}{2} \int \left[ |u|^2 + \alpha^2 |\nabla u|^2 \right] dxdy = \frac{1}{2} \|u\|_{H^1_\alpha}^2.
\]

(22)

Here we assume boundary conditions that give no contributions upon integrating by parts. The corresponding EPDiff equation is (19) with \( m = \delta \ell / \delta u = u - \alpha^2 \Delta u \). This expression involves inversion of the familiar Helmholtz operator in the (nonlocal) relation between fluid velocity and momentum density. The \( H^1_\alpha \) norm \( \|u\|_{H^1_\alpha}^2 \) for the kinetic energy (22) also arises in three dimensions for turbulence modeling based on Lagrangian averaging and using Taylor’s hypothesis that the turbulent fluctuations are “frozen” into the Lagrangian mean flow [12].

**Generalizing the CH peakon solutions to \( n \) dimensions.** Building on the peakon solutions (9) for the CH equation and the pulsons (15) for its generalization to other traveling-wave shapes in [16], Holm and Staley [19] introduced the following measure-valued singular momentum solution ansatz for the \( n \)–dimensional solutions of the EPDiff equation (19):

\[
m(x,t) = \sum_{a=1}^N \int P^a(s,t) \delta \left( x - Q^a(s,t) \right) ds.
\]

(23)

These singular momentum solutions, called “diffeons,” are vector density functions supported in \( \mathbb{R}^n \) on a set of \( N \) surfaces (or curves) of codimension \( (n - k) \) for \( s \in \mathbb{R}^k \) with \( k < n \). They may, for example, be supported on sets of points (vector
peakons, \( k = 0 \), one-dimensional filaments (strings, \( k = 1 \)), or two-dimensional surfaces (sheets, \( k = 2 \)) in three dimensions.

Figure 2 shows the results for the EPDiff equation when a straight peakon segment of finite length is created initially moving rightward (East). Because of propagation along the segment in adjusting to the condition of zero speed at its ends and finite speed in its interior, the initially straight segment expands outward as it propagates and curves into a peakon “bubble.”

![Figure 2: A peakon segment of finite length is initially moving rightward (East). Because its speed vanishes at its ends and it has fully two-dimensional spatial dependence, it expands into a peakon “bubble” as it propagates. (The colors indicate speed: red is highest, yellow is less, blue low, grey zero.)](image)

Figure 3 shows an initially straight segment whose velocity distribution is exponential in the transverse direction, but is wider than \( \alpha \) for the peakon solution. This initial velocity distribution evolves under EPDiff to separate into a train of curved peakon “bubbles,” each of width \( \alpha \). This example illustrates the emergent property of the peakon solutions in two dimensions. This phenomenon is observed in nature, for example, as trains of internal wave fronts in the south China Sea [23].
Figure 3: An initially straight segment of velocity distribution whose exponential profile is wider than the width $\alpha$ for the peakon solution will break up into a train of curved peakon “bubbles,” each of width $\alpha$. This example illustrates the emergent property of the peakon solutions in two dimensions.

Substitution of the singular momentum solution ansatz (23) into the EPDiff equation (19) implies the following integro-partial-differential equations (IPDEs) for the evolution of the parameters $\{P\}$ and $\{Q\}$,

$$
\frac{\partial}{\partial t} Q^a(s,t) = \sum_{b=1}^{N} \int P^b(s',t) G(Q^a(s,t) - Q^b(s',t)) ds',
$$

$$
\frac{\partial}{\partial t} P^a(s,t) = -\sum_{b=1}^{N} \int (P^a(s,t) \cdot P^b(s',t)) \frac{\partial}{\partial Q^a(s,t)} G(Q^a(s,t) - Q^b(s',t)) ds'.
$$

(24)

Importantly for the interpretation of these solutions, the coordinates $s \in \mathbb{R}^k$ turn out to be Lagrangian coordinates. The velocity field corresponding to the
Figure 4: A single collision is shown involving reconnection as the faster peakon segment initially moving Southeast along the diagonal expands, curves and obliquely overtakes the slower peakon segment initially moving rightward (East). This reconnection illustrates one of the collision rules for the strongly two-dimensional EPDiff flow.

The momentum solution ansatz (23) is given by

\[ u(x, t) = G \ast m = \sum_{b=1}^{N} \int P^b(s', t) G(x - Q^b(s', t)) ds', \quad (25) \]

for \( u \in \mathbb{R}^n \). When evaluated along the curve \( \mathbf{x} = Q^a(s, t) \), this velocity satisfies,

\[ u(Q^a(s, t), t) = \sum_{b=1}^{N} \int P^b(s', t) G(Q^a(s, t) - Q^b(s', t)) ds' \]

\[ = \frac{\partial Q^a(s, t)}{\partial t}. \quad (26) \]
Consequently, the lower-dimensional support sets defined on $x = Q^a(s,t)$ and parameterized by coordinates $s \in \mathbb{R}^k$ move with the fluid velocity. This means the $s \in \mathbb{R}^k$ are Lagrangian coordinates. Moreover, equations (24) for the evolution of these support sets are canonical Hamiltonian equations,

$$\frac{\partial}{\partial t} Q^a(s,t) = \frac{\delta H_N}{\delta P^a}, \quad \frac{\partial}{\partial t} P^a(s,t) = -\frac{\delta H_N}{\delta Q^a}.$$  \hspace{1cm} (27)

The corresponding Hamiltonian function $H_N : (\mathbb{R}^n \times \mathbb{R}^n)^N \rightarrow \mathbb{R}$ is,

$$H_N = \frac{1}{2} \int \int \sum_{a,b=1}^N (P^a(s,t) \cdot P^b(s',t)) G(Q^a(s,t), Q^b(s',t)) \, ds \, ds'.$$ \hspace{1cm} (28)

This is the Hamiltonian for geodesic motion on the cotangent bundle of a set of curves $Q^a(s,t)$ with respect to the metric given by $G$. This dynamics was investigated numerically in [19] to which we refer for more details of the solution properties. One important result found numerically in [19] is that only codimension-one singular momentum solutions appear to be stable under the evolution of the EPDiff equation. Thus, we have

**Stability for codimension-one: the singular momentum solutions of EPDiff are stable, as points on the line (peakons), as curves in the plane (filaments, or wave fronts), or as surfaces in space (sheets).**

Proving this stability result analytically remains an outstanding problem. The stability of peakons on the real line is proven in [8].

**Reconnections in oblique overtaking collisions of peakon wave fronts.** Figures [4] and [5] show results of oblique wave front collisions producing reconnections for the EPDiff equation in two dimensions. Figure [4] shows a single oblique overtaking collision, as a faster expanding peakon wave front overtakes a slower one and reconnects with it at the collision point. Figure [5] shows a series of reconnections involving the oblique overtaking collisions of two trains of curved peakon filaments, or wave fronts.

**The peakon reduction is a momentum map.** As shown in [17], the singular solution ansatz (23) is a momentum map from the cotangent bundle of the smooth embeddings of lower dimensional sets $\mathbb{R}^s \in \mathbb{R}^n$, to the dual of the Lie algebra of vector fields defined on these sets. (Momentum maps for Hamiltonian dynamics
Figure 5: A series of multiple collisions is shown involving reconnections as the faster wider peakon segment initially moving Northeast along the diagonal expands, breaks up into a wave train of peakons, each of which propagates, curves and obliquely overtakes the slower wide peakon segment initially moving rightward (East), which is also breaking up into a train of wavefronts. In this series of oblique collision, the now-curved peakon filaments exchange momentum and reconnect several times.

are reviewed in [24], for example.) This geometric feature underlies the remarkable reduction properties of the EPDiff equation, and it also explains why the reduced equations must be Hamiltonian on the invariant manifolds of the singular solutions; namely, because momentum maps are Poisson maps. This geometric feature also underlies the singular momentum solution (23) and its associated velocity (25) which generalize the peakon solutions, both to higher dimensions and to arbitrary kinetic energy metrics. The result that the singular solution ansatz (23) is a momentum map helps to organize the theory, to explain previous results and to suggest new avenues of exploration.
Acknowledgments. I am grateful to R. Camassa, J. E. Marsden, T. S. Ratiu and A. Weinstein for their collaboration, help and inspiring discussions over the years. I also thank M. F. Staley for providing the figures obtained from his numerical simulations in our collaborations. US DOE provided partial support, under contract W-7405-ENG-36 for Los Alamos National Laboratory, and Office of Science ASCAR/AMS/MICS.

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