Recursion Relations in Semirigid Topological Gravity

Eugene Wong
Physics Department
University of Pennsylvania
Philadelphia, PA 19104 USA

A field theoretical realization of topological gravity is discussed in the semirigid geometry context. In particular, its topological nature is given by the relation between deRham cohomology and equivariant BRST cohomology and the fact that all but one of the physical operators are BRST-exact. The puncture equation and the dilaton equation of pure topological gravity are reproduced, following reference [1].
1. Introduction

In four dimensions, field theories of quantum gravity break down due to non-renormalizability, signaling the need for a more fundamental theory. This is however not the case in two dimensions. It is adequate to describe 2-d quantum gravity within the context of a field theory. Moreover, if 2-d gravity is exactly solvable then it gives us a handle to search for qualitative features that may persist in 4-d. However, a renormalizable theory does not imply that it is solvable, and here the difficulty lies in the Liouville mode of 2-d quantum gravity. The Liouville mode decouples in a modified version of 2-d quantum gravity called topological gravity, much like string theory in the critical dimension.

In the path integral context, quantizing gravity amounts to integrating over all different metrics \( g_{\mu\nu}(x) \). This integration possesses two subtleties, namely what we mean by “all” and what we mean by “different”. Certainly, we must at least include all of the moduli space: the space of all metrics modulo coordinate transformations and conformal transformations when scaling is also a true symmetry. This moduli space is equivalent to the space of all conformal structures modulo coordinate transformations (see eg. ). There exists yet another definition of this moduli space, namely by the universal family of stable algebraic curves . In rough correspondence with these three constructions of moduli space, three main classes of modified 2-d gravity have emerged, each simpler to deal with than pure Liouville gravity. Corresponding to the sum over all metrics, we have discretizations of spacetime at some critical point, yielding various matrix models; to the sum over conformal structures we associate critical field theoretic realizations of topological gravity (see e.g. ). Finally, corresponding to the space of stable curves, one is led to consider the topological invariants of this moduli space .

These theories are all related to the 2-d quantum gravity Polyakov first wrote down but the relations between the elementary variables among the different approaches are not so clear. Moreover, there is an added complication of coupling matter to gravity which may naturally appear in the above approaches. However, the end results of correlation functions of observables in all cases seem to reproduce the same physics, in particular, when pure topological gravity is concerned. In this paper we will concentrate on pure topological gravity.

Let us briefly recall each of the three modifications to pure gravity mentioned above, starting with a description of the different field theoretical realizations of topological gravity. By now, there are numerous topological quantum field theories of 2-d gravity obtained
in different ways. What these theories lack in summary is an underlying geometrical principle to define and obtain a topological field theory of gravity and compute correlation functions.

In this paper, we follow the approach taken by Distler and Nelson which is best described in the following interpretation. Field theories live on spacetime manifolds. Integrating over inequivalent spacetime manifolds then gives quantum gravity. In Polyakov’s theory, conformal symmetry is not exact. Therefore, quantum gravity is given by integrating over all inequivalent complex spacetime manifolds and also the Liouville mode. To obtain topological gravity from 2-d quantum gravity, Distler and Nelson proposed a new spacetime geometry which again possesses conformal symmetry. However, besides having complex structures on a 2-d manifold, they suggested that this manifold also possesses a “semirigid structure”.

One constructs semirigid geometry starting with a geometry associated to local $N = 2$ supersymmetric 2-d gravity. One then constrains and twists the two supersymmetry transformations to leave only the semirigid symmetry transformations. Since supersymmetry is a spacetime symmetry, so is the semirigid symmetry. The moduli space now becomes the space of all semirigid complex manifolds modulo isomorphisms due to coordinate transformations. This semirigid moduli space is well understood. On these semirigid manifolds, a topological field theory of gravity can be defined. From this point of view, we can imagine a phase transition at the Planck scale to the new semirigid symmetry as opposed to zero vacuum expectation of the metric. The stress energy tensor, with its topological character, and the BRST charge come out naturally in superfields. One can then apply the operator formalism and construct the correlation functions of non-trivial observables given by.

There are two approaches other than field theoretical mentioned above which also simplify Polyakov’s quantum gravity and one of them is the matrix models. We will show in this paper that in two respects, correlation functions obtained in the semirigid formulation reproduce those of a particular matrix model. Matrix models provide an alternate way to sum all the metrics on spacetime lattices. In these discrete models, the volume of diffeomorphisms gets replaced by a finite factor much as in lattice gauge theories, so one needs not fix a gauge. One then takes the continuum limit and obtains theories of

---

1 A partial list includes [10][15][16][17][11][18].
quantum gravity (and sometimes coupled to matter). To establish the link between topological gravity and matrix models, we recall some results of the relevant matrix model. The one matrix model with an \( N \times N \) hermitian matrix corresponds to a dynamically triangulated random surface. This matrix model is generalized \[7\] to correspond also to surfaces generated by squares, pentagons, etc. Such a one matrix model exhibits multi-critical behavior indexed by the integer \( k \). For example, in the large \( N \) and at the \( k = 2 \) critical point double scaling limit, it gives the same scaling relations as the Liouville theory of pure quantum gravity \[7\]. At the \( k = 1 \) critical point in the double scaling limit, the one matrix model reproduces topological gravity of various other formulations \[13\][11][18]. In principle, this \( 1/N \) expansion in the double scaling limit of the matrix model provides a non-perturbative definition of 2-d quantum gravity. However, the results can of course be expanded in the string coupling to get an expansion in the number of handles \( g \) of individual surfaces. Correlation functions in this perturbative expansion obey recursion relations \[22\]. In particular, at the \( k = 1 \) critical point, one has the puncture \( \mathcal{O}_0 \) and dilaton \( \mathcal{O}_1 \) equations,

\[
\langle \mathcal{O}_0 \mathcal{O}_{n_1} \ldots \mathcal{O}_{n_N} \rangle_g = \sum_{i=1}^{N} n_i \langle \mathcal{O}_{n_i - 1} \prod_{j \neq i} \mathcal{O}_{n_j} \rangle_g \quad \text{and} \quad \langle \mathcal{O}_1 \mathcal{O}_{n_1} \ldots \mathcal{O}_{n_N} \rangle_g = (2g - 2 + N) \langle \mathcal{O}_{n_1} \ldots \mathcal{O}_{n_N} \rangle_g. \tag{1.1}
\]

In \( \langle 1.2 \rangle \), we recognize the prefactor as a topological invariant, the Euler number of an \( N \) punctured Riemann surface. In this paper, as a sequel to \[1\], we use the field theoretical method in the operator formalism \[20\][21] to show how the boundary of moduli space of \( N = 0 \) gravity can contribute to give topological coupling \( \langle 1.1 \rangle \) and how the bulk of the moduli space contributes the factor of \( 2g - 2 \) in \( \langle 1.2 \rangle \). The contact bit of \( \langle 1.2 \rangle \) was studied in \[1\]. Hence we will show that two recursion relations in the one matrix model at the \( k = 1 \) critical point are reproduced in the semirigid topological field theory. Other recursion relations involve Riemann surfaces with different genera \[22\]. To recover non-perturbative physics, one turns these recursion relations into differential equations and shows that their solutions will contain the same non-perturbative information as in the one matrix model \[23\].

Finally, the third approach to simplifying 2-d gravity mentioned above was intersection theory. In such theory, one computes the topologically invariant intersection numbers of certain subspaces of the space of stable algebraic curves. There exist established intersection theories to compute these topological invariants \[13\]. In \[14\], the puncture equation
(1.1) is derived for arbitrary genus from the intersection theory following Deligne. More recently, Witten has shown [24] indirectly that the other recursion relations are true as well by establishing a link between Kontsevich’s formulation of intersection theory and the one matrix model.

These intersection numbers obey the axioms of a topological quantum field theory [25] [26]. Furthermore, they are independent of any specific field theoretical implementation. This generality however is also a reason why the intersection theories are more abstract and hence more difficult to compute than in a specific field theoretic realization of topological gravity. Here, we trade this abstractness with a field theoretic calculation that is simple and is valid for all genera.

The paper is organized as follows. In section 2, we review the semirigid geometry and describe its moduli space. In section 3, we introduce an operator formalism which is used to construct correlation functions. The relation between BRST cohomology on the Hilbert space and the deRham cohomology on the moduli space is discussed. We end the section by giving the observables in topological gravity first obtained by E. Verlinde and H. Verlinde. In section 4, the dilaton equation (1.2) is computed ignoring contact terms and in section 5, the puncture equation (1.1) is derived.

2. Semirigid geometry

In this section, we will review semirigid geometry [12] [1] [27] [19]. In particular, we will show how to impose a semirigid structure on an \( N = 2 \) super Riemann surface, obtain the stress energy tensor and the BRST charge. The semirigid moduli space is then described and the sewing prescription is given. The sewing prescription is needed for the contact term contributions to (1.1) and (1.2).

2.1. Semirigid gravity from \( N=2 \) superconformal geometry

An \( N = 2 \) super Riemann surface is patched from pieces of \( \mathbb{C}^{1|2} \) with coordinate \((z, \theta, \xi)\). The transition function on an overlap is given by the superconformal coordinate transformation [12],

\[
\begin{align*}
z' &= f + \theta t \psi + \xi s \tau + \theta \xi \partial (\tau \psi) \\
\theta' &= \tau + \theta t + \theta \xi \partial \tau, \quad \xi' = \psi + \xi s - \theta \xi \partial \psi,
\end{align*}
\] (2.1)
where $f, t, s, \tau, \psi$ are even (odd) functions of $z$ and restricted to transformations with $\partial f = ts - \tau \partial \psi - \psi \partial \tau$.

The $N = 2$ super Riemann geometry only dictates that $\theta \xi$ be of spin one, like the coordinate $z$. The spin of $\theta$ is not defined a priori. To obtain a spin zero $\theta$ geometrically, we restrict to transformations with $\tau = 0$ and $t = 1$ so that $\theta$ does not change under coordinate transformation. We are left with

$$z' = f + \theta \psi, \quad \theta' = \theta$$

$$\xi' = \psi + \xi \partial f - \theta \xi \partial \psi.$$  \hspace{1cm} (2.2)

On a super manifold with patching functions of this form, $D_\theta = \partial / \partial \theta + \xi \partial / \partial z$ becomes a global vector field and $\{ \tilde{D}_\xi = \partial / \partial \xi + \theta \partial / \partial z \}$ span a line bundle $D_-$. We can thus mod out the flow of $D_-$ and obtain semirigid coordinate transformations in terms of $(z, \theta)$, $\theta$ being spin zero.

An infinitesimal $N = 2$ coordinate transformation is generated by

$$V_{\tilde{v}^z} = \tilde{v}^z \partial_z + \frac{1}{2} (D\tilde{v}^z) \tilde{D} + \frac{1}{2} (\tilde{D} \tilde{v}^z) D$$

where $\tilde{v}^z = \tilde{v}^z(z, \theta, \xi)$ is an even tensor field. Since $D$ does not transform on a semirigid surface, we impose that $\tilde{v}^z$ satisfies $\tilde{D} \tilde{v}^z = 0$, hence

$$\tilde{v}^z = v^z_0 + \theta \omega^z + \theta \xi v^z_0'.$$ \hspace{1cm} (2.3)

Substituting (2.3) back into $V_{\tilde{v}^z}$, it generates the infinitesimal version of semirigid coordinate transformation (2.2).

To define a field theory on the semirigid manifold, we begin with the (twisted) $N = 2$ superconformal ghosts and their stress energy tensor,

$$C^z = c^z + \theta \gamma^z + \xi \gamma^z + \theta \xi \tilde{c}, \quad B_z = \tilde{b}_z + \theta \tilde{\beta}_z - \xi \beta_z + \theta \xi (b_z + \partial_z \tilde{b}_z),$$ \hspace{1cm} (2.4)

and

$$T_z = J_z + \theta \tilde{G}_{\theta z} - \xi G_{\xi z} + \theta \xi ((T_B)_{zz} + \partial_z J_z).$$ \hspace{1cm} (2.5)

Keep in mind that $\theta$ and $\xi$ in (2.4) and (2.5) are spin zero and one respectively. We have the desired supersymmetric partners of the same spin, namely $(b_{zz}, \beta_{\xi z})$ and $(c^z, \gamma^z)$, but we also have the unwanted fields $(\tilde{b}_z, \tilde{\beta}_{\theta z})$ and $(\tilde{c}, \tilde{\gamma}^{} \theta)$. Therefore we will constrain the theory and eliminate half of the degrees of freedom. Since $\tilde{\gamma}^{} \theta$ is a scalar field, it make
sense to constrain it to be a constant. To be consistent, this constraint must follow from a superfield constraint. This leads us to impose

\[ \tilde{D}_\xi C^z = q \]  

(2.6)

where \( q \) is a constant. This superfield constraint breaks the full \( N = 2 \) symmetry group down to the subgroup (2.2) since the latter preserves \( D_\theta \) and the lhs of (2.6) (\( \tilde{D}_\xi C^{\xi\theta} \)) transforms dually to \( D_\theta \). Equation (2.6) implies that \( \tilde{\gamma} \) is the constant \( q \) and \( \tilde{c} = \partial c \). The definition of the components of \( B_z \) in (2.4) is twisted so that \( (\tilde{b}, \tilde{\beta}) \) are conjugate to \( (c, \tilde{\gamma}) \) which are eliminated by (2.6). Moreover, the zero mode insertions which we will use in section 3 to define correlation functions are of the form

\[ \oint [dz d\theta d\xi] B_z \tilde{v}^z = - \oint dz (\beta \xi \omega + b_z \tilde{v}_0^z) \]

since \( \tilde{v}^z \) that generates infinitesimal semirigid coordinate transformation is given by (2.3). Hence if we consider inserting only operators independent of \( (\tilde{b}_z, \tilde{\beta}_\theta z) \) and \( (\tilde{c}, \tilde{\gamma}^\theta) \), then these fields will decouple altogether from the theory.

To generate coordinate transformation (2.2), only part of the stress energy tensor (2.5) is needed. The definition of the bosonic energy tensor \( T_B \) in (2.5) is twisted so that the unbroken generators of (2.2) are modes of \( G_{\xi z} \) and \( T_B \). To see that we again use \( \tilde{v}^z \) of (2.3) which generates infinitesimal semirigid coordinate transformation and obtain the corresponding generators of the stress energy tensor (2.5) given by

\[ - \oint [dz d\theta d\xi] T_z \tilde{v}^z = \oint dz (T_B \tilde{v}_0^z + G_{\xi z} \omega^\xi). \]

This twisting as in [28] will lead to an anomaly free theory, as we will see.

The full \( N = 2 \) stress energy tensor is given by [12]

\[ T_z = \partial (CB) - \frac{1}{2} (DB \tilde{D}C + \tilde{D}BDC) \]  

(2.7)

and the BRST charge

\[ Q = - \frac{1}{2} \oint [dz d\theta d\xi] C^z T_z. \]  

(2.8)

Imposing the constraint (2.6), we obtain in components the unbroken generators by substituting (2.4) and (2.5) into the constrained (2.7) and (2.8),

\[ T_B = - 2b \partial c - (\partial b)c - 2\beta \partial \gamma - (\partial \beta)\gamma, \]

(2.9)
\[ G_{\xi z} = -2\beta \partial c - (\partial \beta) c + \frac{q}{2} b, \quad (2.10) \]

and
\[
Q_T = -\frac{1}{2} \oint dz (-c\bar{z} T_{zz} + \gamma^z G_{\xi z} + q \bar{G}_{\theta z}) \\
= \oint dz [-c\bar{b} \partial c + \beta \gamma \partial c - \beta c \partial \gamma - \frac{q}{2} b \gamma]. \quad (2.11)
\]

Since \((\beta, \gamma)\) have the same spin as \((b, c)\) but the opposite statistics, the central charges from them cancel. Moreover, note that in (2.10) and (2.11), \(G\) and \(Q_T\) differ from their \(N = 1\) counterparts. Using the \(N = 1\) superfield expressions and replacing the spin one half \(\theta\) by spin zero \(\theta\) give incorrect answers.

Next, we expand in modes \(L_n = \oint T_B z^{n+1} dz\) and \(G_n = \oint G_{\xi z} z^{n+1} dz\) and similarly for the ghosts \((b, c)\) and \((\beta, \gamma)\). Imposing the usual commutation relations \([b_m, c_n] = \delta_{m+n,0}\) and \([\gamma_m, \beta_n] = \delta_{m+n,0}\) where \([,\] is the graded bracket, we obtain the following algebra

\[
L_m, L_n = (m - n) L_{m+n}, \quad [L_m, G_n] = (m - n) G_{m+n}, \quad [G_m, G_n] = 0, \\
G_m, b_n = (m - n) \beta_{m+n}, \quad [G_m, c_n] = \frac{q}{2} \delta_{m+n,0}, \\
G_m, \gamma_n = -(2m + n) c_{m+n}, \quad [G_m, \beta_n] = 0, \\
L_m, b_n = (m - n) b_{m+n}, \quad [L_m, c_n] = -(2m + n) c_{m+n}, \\
L_m, \gamma_n = -(2m + n) \gamma_{m+n}, \quad [L_m, \beta_n] = (m - n) \beta_{m+n}, \\
Q_T, b_n = L_n, \quad [Q_T, c_n] = \sum_m (m - n + 1) c_m c_{n-m} - \frac{q}{2} \gamma_n, \\
Q_T, \beta_n = -G_n, \quad \text{and} \quad [Q_T, \gamma_n] = \sum_m (2m - n) c_m \gamma_{n-m}. \quad (2.12)
\]

We set \(q = -2\) as in [1] and [11]. From the commutator of the BRST charge with the mode \(b_n\), we obtain \([Q_T, b(z)] = T_B(z)\). The stress energy tensor being BRST-exact is the signature of a topological theory, implying that the metric dependence of the action decouples [13].

**2.2. Semirigid moduli space**

We will now discuss the semirigid moduli space following [12] [1] [27] [19]. To understand the semirigid moduli space, we introduce a family of augmented surfaces. To build an augmented surface, we start with an ordinary Riemann surface given by patching maps \(z' = f(z)\). We then introduce a new global spinless anticommuting coordinate \(\theta\) and
promote all patching functions to superfields in $\theta$. Thus, an augmented surface is obtained with patching maps

$$z' = f(z, \theta) \equiv f(z) + \theta \phi(z); \quad \theta' = \theta.$$ (2.13)

This is the same as (2.2) after modding out $D_-$ when we identify $\phi(z)$ with $\psi(z)$ of (2.2). Hence, surfaces patched together by the augmented maps have a one to one correspondence with the semirigid surfaces since given either patching function, we can recover the other.

Suppose now we have a family of Riemann surfaces parametrized by $\vec{m}$, that is $z' = f(z; \vec{m})$ are the patching maps. To obtain a family of semirigid surfaces, we merely need a family of augmented Riemann surfaces. We let the family of augmented Riemann surfaces have patching functions $\theta' = \theta$ and

$$z' = f(z; \vec{m} + \theta \vec{\zeta}) = f(z; \vec{m}) + \theta \zeta^i h_i(z; \vec{m})$$ (2.14)

where $h_i(z; \vec{m}) = \partial m_i f(z; \vec{m})$. The original moduli $\vec{m}$ are augmented by $\theta \vec{\zeta}$ giving an equal number of odd moduli $\vec{\zeta}$. One can now show using (2.14) that another parametrization $(\vec{m}', \vec{\zeta}')$ of the same family of surfaces induces $(\vec{m}', \vec{\zeta}')$ which are related to the original ones by a split coordinate transformation. That is, $\vec{m}' = \vec{m}'(\vec{m})$ and $\zeta^i = (\partial m^j m'^j) \zeta^j$.

We will use this property later on.

Consider the moduli space of genus $g$ semirigid surfaces with one puncture at $P$, $\hat{M}_{g,1} = \frac{[\text{all semirigid complex manifolds with puncture}]}{[\text{isomorphisms preserving puncture}]}$. It has a natural projection to the unpunctured moduli space $\hat{M}_{g,0} = \frac{[\text{all semirigid complex manifolds}]}{[\text{isomorphisms}]}$ simply by forgetting $P$. More generally, we can have a moduli space of genus $g$ semirigid surfaces with $N$ punctures $\hat{M}_{g,N}$. The integration density on $\hat{M}_{g,0}$ is interpreted to be the integrand in the path integral without source terms. Hence, the integral over $\hat{M}_{g,0}$ of the volume density gives the partition function. The one point correlation function is then the integral of a certain integration density defined on $\hat{M}_{g,1}$ as we will recall below. One of the consequences of having a split coordinate transformation on semirigid moduli space $\hat{M}_{g,N}$ is that there also exists a natural projection $\Pi : \hat{M}_{g,N} \to M_{g,N}$ to the ordinary moduli space of Riemann surfaces with $N$ punctures $\Pi$. This means that if we have a
measures on $\hat{M}_{g,N}$, then we can without further obstruction integrate along the fibers $\Pi^{-1}$ (ie. integrate out all the odd moduli $\vec{\zeta}$ ) leaving a measure on the ordinary moduli space $\hat{M}_{g,1}$.

A bundle $\hat{P}_{g,1}$ with base space $\hat{M}_{g,1}$ can be constructed [20][29], where the fiber over each point $(\hat{\Sigma}, P) \in \hat{M}_{g,1}$ consists of the germs of coordinate systems $z_P(\cdot)$ on $\hat{\Sigma}$ defined near $P$ with their origins at $P$. (We always take $\theta_P(\cdot) = \theta(\cdot)$, the global odd coordinate on $\hat{\Sigma}$.) Thus, we have $\hat{\pi} : \hat{P}_{g,1} \to \hat{M}_{g,1}$. The analog of the Virasoro action on ordinary $P_{g,1}$ is defined by the infinitesimal form of coordinate transformation (2.13) on $z_P(\cdot)$. If we let $f(z) = z - \epsilon z^{n+1}$ and $\phi(z) = \alpha z^{m+1}$ in (2.13), where $\epsilon$ and $\alpha$ are commuting and anticommuting infinitesimal parameters respectively, the corresponding generators are defined as $l_n = -z^{n+1}\partial_z$ and $g_m = -\theta z^{m+1}\partial_z$ so that $z' = (1 + \epsilon l_n + \alpha g_n)z$. $l_n$ and $g_m$ are the generators of the augmented version of the Virasoro group. In the following, we denote a semirigid surface $\hat{\Sigma}$ with a unit disk $|z_P(\cdot)| \leq 1$ centered at $P$ removed by $(\hat{\Sigma}, z_P(\cdot))$, and similarly for a Riemann surface with a unit disk removed $(\Sigma, z_P(\cdot))$.

We will review some facts about a vector field $\tilde{v}_i \in TP_{g,1}$ for ordinary geometry $\pi : P_{g,1} \to M_{g,0}$ [20].

Given $\Sigma$, we can deform it to a neighboring $\Sigma'$ by some $\tilde{v}_i \in TP_{g,1}$. We will classify into three categories the action of a Virasoro generator $v = -\epsilon \sigma^{n+1}\partial_\sigma$, where $\sigma = z_P(\cdot)$ is some local coordinate. We denote the Virasoro action on $P_{g,1}$ by $i_\sigma(v) = \tilde{v}_i$, a tangent to $P_{g,1}$ at $\sigma$. To construct $\Sigma'$, we begin with $(\Sigma, \sigma)$. We then identify points on the boundary of $(\Sigma, \sigma)$ with those of a unit disk $D$ via a composition of $\sigma$ with the map $1 + v$, yielding $\sigma' = \sigma - \epsilon \sigma^{n+1}$, a “Schiffer variation” of $(\Sigma, \sigma)$. If $v$ extends analytically to $\Sigma \setminus P$ (eg. $n \leq 1 - 3g$), then $\Sigma'$ is identical to $\Sigma$ because the variation can be undone by a coordinate transformation generated by $v$ on the rest of the surface $\Sigma$. Thus, $i_\sigma(v) = 0$. If $v$ extends holomorphically to $D$ and vanishes at $\sigma = 0$ $(n \geq 0)$, then we merely have a coordinate transformation on $\sigma$. $i_\sigma(v)$ is then vertical along the fiber in $P_{g,1}$ and $\pi_*$ kills it. The Weierstrass gap theorem [20] states that on an unpunctured Riemann surface with $g > 1$, every meromorphic vector field $v$ on the disk $D$ can be written as the sum of a holomorphic vector on $D$ and a vector field that extends to the rest of $\Sigma$, except for a $(3g - 3)$ dimensional subspace. The $3g - 3$ dimensional subspace of vector fields have simultaneously a pole in $D$ and in the rest of $\Sigma$ (eg. $-2 \geq n \geq 2 - 3g$). This $3g - 3$ dimensional vector space when projected down to $M_{g,0}$ by $\pi_*$ yields the full $TM_{g,0}$, hence the ordinary moduli space $M_{g,0}$ is $3g - 3$ dimensional. By the augmented construction
of semirigid moduli space, we similarly see that $\hat{M}_{g,0}$ is $3g - 3|3g - 3$ dimensional [12]. Finally, $\tilde{v} = \epsilon \partial_{\sigma} \in TP_{g,1}$ gives a vector field that moves the puncture $P$.

Consider a family of once-punctured semirigid surfaces with a family of semirigid coordinates $\sigma$ centered at the puncture $P$, constructed by augmenting a similar family of ordinary Riemann surfaces as in (2.14). Thus, as shown in figure 1, we have $\sigma = f(z; \bar{m}) + \theta \zeta^i h_i(z; \bar{m})$ where $h_i = \partial_{m^i} f$ and $(\bar{m}, \bar{\zeta})$ the coordinates for the $3g - 2|3g - 2$ dimensional $\hat{M}_{g,1}$. We will see that a puncture in semirigid geometry has one even and one odd modulus associated with it and hence the dimension of $\hat{M}_{g,1}$ is 1/1 bigger than $\hat{M}_{g,0}$. Keep in mind that $\sigma$ has to be holomorphic in $z$ and $\theta$ but not necessary in $\bar{m}$. Then for $k = 1, \ldots, 3g - 2$,

$$\sigma_{\ast}(\partial \xi_k) = \frac{\partial \sigma}{\partial m^k} \frac{\partial}{\partial \sigma} = \frac{\partial \sigma}{\partial m^k} \frac{\partial}{\partial \sigma} = (\partial \xi_k f + \theta \zeta^i \partial_{m^i} h_i) \partial_{\sigma} + (\partial \xi_k \bar{f} + \bar{\theta} \bar{\zeta}^i \partial_{m^i} \bar{h}_i) \partial_{\sigma} \equiv \tilde{v}_k$$

(2.15)

is the push forward of an even vector $\partial \xi_k \in T\hat{M}_{g,1}$ to $\tilde{v}_k \in T\hat{P}_{g,1}$ and

$$\sigma_{\ast}(\partial \xi_k) = -\theta h_k \partial_{\sigma} \equiv \tilde{v}_k$$

(2.16)

is the push forward of an odd vector $\partial \xi_k \in T\hat{M}_{g,1}$ to $\tilde{v}_k \in T\hat{P}_{g,1}$. Note that in $\tilde{v}_k = \tilde{v}_k^z \partial_z + \tilde{v}_k^\xi \partial_\xi$ of (2.15), $\tilde{v}_k^\xi$ is not necessary the complex conjugate of $\tilde{v}_k^\xi$; also, $\tilde{v}_k$ of (2.16) is proportional to $\theta$, a special property of the augmented coordinates. These vectors $(\tilde{v}_k, \bar{\tilde{v}}_k)$ when projected down by $\hat{\pi}_\ast$ span the $3g - 2|3g - 2$ dimensional holomorphic tangent space $T\hat{M}_{g,1}$ analogous to the vectors $\pi_{\ast}\tilde{v}_k$ with $\tilde{v}_k \in TP_{g,1}$ in the above discussion of ordinary geometry.

We now give the prescription for sewing two semirigid surfaces that is compatible with the compactification of moduli space by stable curves. In superspace, a “point” $P$ is defined by the vanishing of some functions. In particular, if we have some even function $f(z, \theta)$ on a semirigid surface, then $P$ can be defined by where $f = \theta = 0$ [12]. Note that with any invertible function $g$, $g \cdot f$ defines the same $P$. A divisor in the semirigid superspace is thus given by a semirigid coordinate $z_P(\cdot) = z - z_0 - \theta \theta_0$ centered at $P$. Higher Taylor coefficients in $z$ do not matter since they can be introduced or removed freely by $g$. $(z_0, \theta_0)$ are the 1/1 parameters associated to the position of $P$. Consider a semirigid surface $\hat{\Sigma}$ with genus $g$ without punctures degenerating into two pieces $\Sigma_L$ and $\Sigma_R$ of genera $g_L$ and $g_R$. Let $P_L$ and $P_R$ be the double points at the node on $\hat{\Sigma}_L$ and $\hat{\Sigma}_R$. Counting the number of moduli of $\hat{\Sigma}$ ($3g - 3|3g - 3$) and the sum of that of $(\hat{\Sigma}_L, P_L)$ ($3g_L - 2|3g_L - 2$) and $(\hat{\Sigma}_R, P_R)$
(3g_R − 2|3g_R − 2), we have an excess of 1|1 dimension in \( \hat{\Sigma} \)'s moduli space over that of \( \hat{\Sigma}_L \) and \( \hat{\Sigma}_R \). Hence the plumbing fixture joining the two semirigid surfaces \( \hat{\Sigma}_L \) and \( \hat{\Sigma}_R \) must depend on a 1|1 sewing moduli. The augmented sewing prescription tells us to join the two surfaces by relating the coordinates \( z_L|\theta_L \) and \( z_R|\theta_R \) centered at the sewing points \( P_L \) and \( P_R \) in the following way [12][4],

\[
z_L = \frac{q + \theta_R \delta}{z_R}, \quad \theta_L = \theta_R. \tag{2.17}
\]

\((q, \delta)\) are the 1|1 sewing moduli because changing \((q, \delta)\) alter the resultant surface \( \hat{\Sigma} \). Since (2.17) is of the form (2.2), we get a semirigid surface after sewing.

We introduce the plumbing fixture because we want to complete the moduli space to a compact space. For this, we need a precise description of how semirigid surfaces may degenerate. The stable curve compactification of the moduli space requires that \( \hat{\Sigma}_L \) and \( \hat{\Sigma}_R \) be independent of \((q, \delta)\) and that this dependence come solely from sewing in the limit \( q \to 0 \). Away from \( q \to 0 \), it does not matter whether \( \hat{\Sigma}_{L,R} \) depend on \( q \).

Another rule from the stable curve compactification is that no two nodes of \( \hat{\Sigma} \) can collide and neither can two punctures. Two colliding points \( P_1 \) and \( P_2 \) on \( \hat{\Sigma} \) is replaced by a conformally equivalent degenerating surface. The conformally equivalent surface consists of the same surface \( \hat{\Sigma} \) with one puncture \( P_L \) at the would be colliding point identified with the puncture \( P_R \) of a fixed three punctured sphere \( \mathbf{P}^1 (P_R, P_1, P_2) \). Here, we see that we can apply the sewing prescription with \( \hat{\Sigma}_L = \hat{\Sigma}_{P_L} \) and \( \hat{\Sigma}_R = \mathbf{P}^1 (P_R, P_1, P_2) \) via the plumbing fixture (2.17). The three punctured sphere is rigid and has no moduli associated with it [12]. The moduli from the plumbing fixture are associated to the distance between \( P_1 \) and \( P_2 \).

The most general coordinates near the three punctures \((P_R, P_1, P_2)\) centered at \((\infty, E, \tilde{E})\) on the sphere can be given by [30][4]

\[
(\mathbf{P}^1; z^{-1}, (z - E) + a_1(z - E)^2 + a_2(z - E)^3 + \ldots, (z - \tilde{E}) + \tilde{a}_1(z - \tilde{E})^2 + \ldots). \tag{2.18}
\]

Let the coordinate centered at \( P_L \) be \((\sigma, \theta)\). Then by (2.17) with \( z_L = \sigma \) and \( z_R = z^{-1} \), we can express \( z \) in terms of \( \sigma \), \( z = q^{-1}\sigma(1 - q^{-1}\theta\delta) \). Substituting this back into (2.18), we obtain the most general coordinates centered at the colliding \( P_1 \) and \( P_2 \) using the local coordinates \((\sigma, \theta)\). They are

\[
z_{P_1}(\cdot) = q^{-1}\sigma(1 - q^{-1}\theta\delta) - E + a_i[q^{-1}\sigma(1 - q^{-1}\theta\delta) - E]^2 + \ldots, \tag{2.19}
\]

and similarly for \( z_{P_2} \) with \( a_i \to \tilde{a}_i \) and \( E \to \tilde{E} \). These two coordinate slices will be used in calculating the contact terms in section 5.
3. Operator Formalism

Following G. Segal’s construction [31], a conformal field theory associates a state in the Hilbert space $\ket{\hat{\Sigma}, z}$ to a one-punctured semirigid Riemann surface with local coordinate $z$ denoted by $(\hat{\Sigma}, z)$. Recall that $z$ is a coordinate we put on $\hat{\Sigma}$ at the point where $z = 0$ and a unit disk $|z| \leq 1$ has been removed from the semirigid surface. Since a state in the Hilbert space associated to $(\hat{\Sigma}, z)$ depends on $z$, the Virasoro action previously defined on $\hat{\mathcal{P}}$ also acts on the Hilbert space. Following [24], we define using mode expansions $L_n, G_m$ of $T_B, G_{\xi z}$ and their conjugates

$$\langle \hat{\Sigma}, z - \epsilon z^{n+1} - \alpha \theta z^{m+1} \rangle = \langle \hat{\Sigma}, z \rangle [(1 + \epsilon L_n + \alpha G_m)(1 + i\bar{L}_n + i\bar{G}_m) + \ldots]$$

where the ellipsis refer to terms of order $\epsilon^2$, etc..

We will now discuss how to obtain a measure on $\hat{\mathcal{M}}_{g,1}$ from $\hat{\mathcal{P}}_{g,1}$ following [32]. $N$-point correlation functions can be easily generalized. We insert a state $\ket{\Psi}$ at $P$ on $\hat{\Sigma}$ as shown in figure 2. Instead of using the projection $\hat{\pi}$ as in [20], we choose a section $\sigma : \hat{\mathcal{M}}_{g,1} \to \hat{\mathcal{P}}_{g,1}$. A volume density $\Omega_\sigma$ on $\hat{\mathcal{M}}_{g,1}$ is related to an integration density $\tilde{\Omega}$ on $\hat{\mathcal{P}}_{g,1}$ by the pullback, $\Omega_\sigma = \sigma^*\tilde{\Omega}$ or

$$\Omega_\sigma (V_1, \ldots, V_{3g-2}, \Upsilon_1, \ldots, \Upsilon_{3g-2}) = \tilde{\Omega}(\tilde{v}_1, \ldots, \tilde{v}_{3g-2}, \tilde{\nu}_1, \ldots, \tilde{\nu}_{3g-2}) \quad (3.2)$$

where $\tilde{v}_i = \sigma_* V_i$ and $\tilde{\nu}_i = \sigma_* \Upsilon_i$, $i = 1, \ldots, 3g - 2$ are linearly independent even and odd vectors respectively and $(\tilde{V}, \tilde{\Upsilon})$ span the full $T\hat{\mathcal{M}}_{g,1}$. All complex conjugates are suppressed.

Let $z_P(\cdot)$ be the coordinate centered at the chosen point $P$. We define $\tilde{\Omega}$ on $\hat{\mathcal{P}}_{g,1}$ by [21] [10]

$$\tilde{\Omega}(\tilde{v}_1, \ldots, \tilde{v}_{3g-2}, \tilde{\nu}_1, \ldots, \tilde{\nu}_{3g-2}) \equiv \langle \hat{\Sigma}, z_P(\cdot) | B[\tilde{v}_1] \ldots B[\tilde{v}_{3g-2}] \delta(B[\tilde{v}_1]) \ldots \delta(B[\tilde{v}_{3g-2}]) | \Psi \rangle^P \quad (3.3)$$

where $B[\tilde{v}] = \tilde{v}^z \partial_z + \tilde{v}^\bar{z} \partial_{\bar{z}} \equiv \oint [dz d\theta d\xi] B_z \tilde{v}^z + \oint [dz d\theta d\bar{\xi}] B_{\bar{z}} \tilde{v}^\bar{z}$, $B[\tilde{\nu}] = \tilde{\nu}^z \partial_z$ and similarly its conjugate. $B_z$ is the antighost superfield in [2.4]. These definitions can be simplified by integrating out $\theta$ and $\xi$, but before doing so, we need to know the following. When we mod out $\mathcal{D}_-$, a holomorphic even tensor field [23] becomes $\tilde{v}^z = v^z_0 + \theta \omega^\xi$, but given the latter, we can uniquely lift it back. Hence, lifting the augmented $\tilde{v} \in T\hat{\mathcal{P}}_{g,1}$ to $\tilde{\nu} = (v^z_0 + \theta \omega^\xi + \theta \xi \partial_z v^z_0) \partial_z + (v^\bar{z} + \bar{\theta} \omega^\bar{\xi} + \bar{\theta} \bar{\xi} \partial_z v^\bar{z}) \partial_{\bar{z}}$, we obtain

$$B[\tilde{v}] = -\oint dz (\beta \omega^\xi + b v^\bar{z}_0) - \oint d\bar{z} (\bar{\beta} \omega^\bar{\xi} + b \bar{v}^z_0). \quad (3.4)$$
Since an odd vector \( \bar{\nu} = \theta \bar{w} \xi \partial_z \in T\hat{\mathcal{M}}_{g,1} \) is always proportional to \( \theta \) in the augmented construction (see (2.16)), we get the very simple form

\[
B[\bar{\nu}] = -\int dz \beta \bar{w} \xi
\]

and similarly its conjugate.

The \( B \) insertions are required also to absorb the fermionic and bosonic zero modes of \( b's \) and \( \beta's \) respectively. The state \( \langle \hat{\Sigma}, z_P(\cdot) \rangle \) has an anomaly associated with ghost charges \( (U_{bc}, U_{\beta, \gamma}) = (3g - 3, 3g - 3) \), and the antighosts \( b \) and \( \beta \) have ghost charges \((-1, 0)\) and \((0, -1)\), the inserted state \( \Psi \)'s (and in general \( N \) inserted states') ghost charges must add to yield a net zero ghost number for \( \bar{\Omega} \) in order to have a nonzero answer.

We can generalize the above prescription slightly \[20\]. Suppose we have a family of surfaces with more than one puncture. We then can (and will later) consider families of local coordinates where several \( \sigma_{P_i} \) all depend on the same modulus \( m^a \). Then \( \sigma^*(\partial_{m^a}) \) will involve Schiffer variations of \((\Sigma, \sigma_{P_1}, \ldots, \sigma_{P_N})\) at several points, that is,

\[
\sigma^*(\partial_{m^a}) = i_{\sigma_{P_1}} (v^{(1)}_a) + \ldots + i_{\sigma_{P_N}} (v^{(N)}_a)
\]

where \( v^{(i)}_a \) acts at \( P_i \). In this case we replace insertions like (3.4) and (3.5) with the sum of the corresponding insertions at \( P_i \) using \( v^{(i)}_a \).

Since we have chosen a section \( \sigma : \hat{\mathcal{M}}_{g,1} \to \hat{\mathcal{P}}_{g,1} \), there is no ambiguity in pushing forward the given vectors \((\bar{V}, \bar{\bar{Y}})\) from \( T\hat{\mathcal{M}}_{g,1} \) to \((\bar{v}_i, \bar{\bar{v}}_i) \in T\hat{\mathcal{P}}_{g,1} \) by \( \sigma^* \). However, the correlation function \( \int_{\hat{\mathcal{M}}_{g,1}} \Omega_\sigma \) now has an apparent dependence on what section is being chosen. Consider a nearby slice \( \sigma' \). Then to eliminate this dependence, we impose the condition \( \int_{\hat{\mathcal{M}}_{g,1}} \Omega_\sigma - \Omega_{\sigma'} = 0 \). Substituting \( \Omega_\sigma = \sigma^* \bar{\Omega} \), we have

\[
0 = \int_{\hat{\mathcal{M}}_{g,1}} (\sigma - \sigma')^* \bar{\Omega} = \int_{\partial K} \bar{\Omega} = \int_K d\Omega,
\]

where \( K \) is the enclosed volume between \( \sigma \) and \( \sigma' \), and the superspace form of Stokes’ theorem \[33\] is used in the last step. This analysis is correct for the case when \( K \) stays away from the boundaries of \( \hat{\mathcal{M}}_{g,1} \), otherwise, see below. Since the space \( K \) is arbitrary, we therefore must demand that \( \bar{\Omega} \) to be a closed form \( d\bar{\Omega} = 0 \). We have a closed \( \bar{\Omega} \) only if the state \( |\Psi\rangle \) inserted is BRST closed, that is \( Q_T |\Psi\rangle = 0 \) \[20\]. Thus for \( \int \Omega_\sigma \) to be
\(\sigma\) independent, we must at least impose the condition that all inserted states be BRST closed. Since \(d\tilde{\Omega} = 0\), it follows that \(d\Omega_\sigma = \sigma^*d\tilde{\Omega} = 0\). Therefore, \(\Omega_\sigma\) is a closed form on \(\hat{M}_{g,1}\).

It seems that the only condition on the states is that they are BRST closed. However, the above analysis assumes the existence of a global slice \(\sigma\). Recall that a point in \(\sigma\) is given by \((\hat{\Sigma}, P, z_P(\cdot))\). We will discuss the situation with an ordinary Riemann surface \(\Sigma\) and then obtain the results on a semirigid surface \(\hat{\Sigma}\) by the method of augmentation. Certainly, we can have a coordinate \(z_P(\cdot)\) centered at \(P\) when the point \(P\) is chosen on any Riemann surface. However, for \(g \neq 1\) Riemann surface, there is a topological obstruction to have a smoothly varying \(\text{family}\) of local coordinate systems \(z_P(\cdot)\) when \(P\) spans the entire surface. This topological obstruction to having a global coordinate on a compact \(\Sigma\) is given by the Euler number of the manifold.

Let us elaborate on this point. We cover the ordinary moduli space \(M_{g,1}\) with a family of local coordinate patches. On a patch \(\alpha\), recall that the coordinate \(z_\alpha^\alpha(Q)\) centered at \(P\) is a holomorphic function of \(Q\), but not necessary of \(P\). On an overlap of two patches with transition given by \(z_\alpha^\alpha(Q) = M^{\alpha\beta}(P)z_\beta^\beta(Q)\), \(M^{\alpha\beta}\) is a \(P\)-dependent element of the complexified Virasoro semigroup [31]. But the latter is topologically equivalent to the \(U(1)\) group generated by \(l_0 - \bar{l}_0\). So we may deform these families of coordinates until they agree up to functions with values in \(U(1)\). An explicit construction of a global family of coordinates modulo \(U(1)\) phases was given by Polchinski [34].

A similar analysis for the family of semirigid coordinates can be carried out. We begin with a family of ordinary Riemann surfaces with local holomorphic coordinates. As argued above, we can choose families of coordinates so that the transition functions are \(U(1)\)-valued. We can then augment the coordinate families as well as families of Riemann surfaces by the procedure given in (2.14). We must choose however to augment \(\vec{m}\) (which include the \(3g - 3\) moduli and the modulus associated to \(P\)) to \(\vec{m} + \theta\vec{\zeta}\) but leave \(\vec{\bar{m}}\) alone so that we get

\[
z_\alpha^\alpha(Q) = \exp[2\pi if_{\alpha\beta}(\vec{m} + \theta\vec{\zeta}, \vec{\bar{m}})]z_\beta^\beta(Q).
\]

We have to leave \(\vec{\bar{m}}\) alone because augmenting them yield \(\bar{\theta} = \bar{\theta}(Q)\) which is not a holomorphic function of \(Q\). This transition function between the two semirigid coordinates is not a pure phase nor is it \(Q\)-independent. However, it differs from a pure phase by a factor of \([1 + \theta(Q)\zeta^a\partial_m^a f_{\alpha\beta}(\vec{m}, \vec{\bar{m}})]\). The one forms \(\{dm^a\partial_m^a f_{\alpha\beta}\}\) amount to a 1-cocycle of smooth sections of the holomorphic cotangent to the moduli space. Since the associated
Čech cohomology vanishes, we may finally modify the coordinates $z_P^\alpha(Q)$ on each patch to eliminate these factors. Hence one can choose a family of local semirigid coordinates so that on each overlap, they differ by a pure phase,

$$z_P^\alpha(\cdot) = e^{2\pi i f_{\alpha\beta}(P)} z_P^\beta(\cdot). \quad (3.8)$$

Thus, we have only to ensure that $\tilde{\Omega}$ is insensitive to this phase so that $\Omega_\sigma = \sigma^* \tilde{\Omega}$ is a well defined measure on $\hat{\mathcal{M}}_{g,1}$.

Locally on the overlap of two coordinate patches differing by an infinitesimal phase, $\sigma' = \sigma + i\epsilon \sigma$ where $\sigma = z_P(\cdot)$ of the above discussion (note that it is not $\sigma' = \sigma + i\theta \rho \sigma$), there is thus a remaining ambiguity in lifting the vectors $(V_i, \Upsilon_i) \in T\hat{\mathcal{M}}_{g,1}$ to $(\sigma_* V_i, \sigma_* \Upsilon_i)$ or to $(\sigma'_* V_i, \sigma'_* \Upsilon_i)$. Imposing in (3.3) that $\tilde{\Omega}$ be insensitive to this lifting, $\delta \tilde{\Omega}(\tilde{v}_1, \ldots, \tilde{v}_{3g-2}, \tilde{v}_1, \ldots, \tilde{v}_{3g-2}) = 0$, we obtain the condition $(b_0 - \bar{b}_0)|\Psi\rangle = 0$. Moreover, there are coordinate dependences on the state via (3.1). Imposing the condition that this dependence also drop out and by using (2.12), it yields $(L_0 - \bar{L}_0)|\Psi\rangle = 0$. Stronger conditions on the state that save algebra without sacrificing any interesting observables are

$$L_0|\Psi\rangle = b_0|\Psi\rangle = Q_T|\Psi\rangle = 0 \quad \text{and their conjugates.} \quad (3.9)$$

These are known as the equivariance or weak physical state conditions (WPSC) on $|\Psi\rangle$ [30]. Note that there is no such condition imposed by $G$ or $\beta$ on $|\Psi\rangle$.

We have ignored the boundaries of moduli space in the above discussion. However, they are important because they contribute to the correlation function as contact terms. If there are boundaries in the moduli space, then (3.7) is not valid, but if we specify boundary conditions on $\sigma$, then we may recover a well defined integral $\int \Omega_\sigma$ over the moduli space. To get such boundary conditions, note that $\hat{\mathcal{M}}_{g,N}$ is non-compact, but near a boundary, there exists a notion of good coordinates, those which are compatible with the stable-curve compactification of moduli space [12]. Hence, we can specify what the asymptotic slices are at the relevant boundaries and thus get a well defined measure over the entire $\hat{\mathcal{M}}_{g,N}$. They are given by (2.19) near a boundary of $\hat{\mathcal{M}}_{g,N}$ when two punctures approach each other. Moreover, we will see in section 5 that the choices $(a_i, E)$ and $(\bar{a}_i, \bar{E})$ of the asymptotic slices (2.19) drop out at the end of the calculation.

2 In other words, since a coordinate centered at $P$ is equivalent to another if they differ by an even invertible function, we can redefine the local coordinate at each $P$ to absorb the unwanted invertible factor.
We will now give the relation between equivariant BRST cohomology and deRham cohomology. Here, we are not interested in the moduli space’s boundary contributions, hence we will assume that \( \hat{\mathcal{M}}_{g,N} \) is a compact manifold without boundaries. First, consider a closed \( n \)-form \( \Omega \) defined on a compact Riemann surface \( M \). If \( \Omega = d\mu \), where \( \mu \) is an \( (n-1) \)-form, then we call \( \Omega \) exact. The deRham cohomology is defined as the vector space \( H^n(M) \), where \( H^n(M) \equiv \text{closed } n \text{ forms/exact } n \text{ forms} \). That is, if \( H^n(M) \neq 0 \), then a closed \( n \)-form need not be exact. In fact, \( \int_M \Omega \) depends only on the cohomology class of \( \Omega \) in \( H^n(M) \). For a BRST-exact state, \( |\Psi\rangle = (Q_T + \bar{Q}_T)|\lambda\rangle \), one can show that the closed form \( \Omega_\Psi = d\mu_\lambda \) is \( d \)-exact \([20],[29]\); then \( \int_M d\mu_\lambda = 0 \) because \( M \) is compact. In other words, BRST-exact states decouple. This discussion can be generalized to the top density \( \Omega_\sigma \) on \( \hat{\mathcal{M}}_{g,1} \) we obtained from (3.2). Then, integrating over the fiber of \( \Pi : \hat{\mathcal{M}}_{g,1} \to \mathcal{M}_{g,1} \), \( \int_{\Pi^{-1}} \Omega_\sigma \) yields the closed form \( \Omega \) of the above discussion where the manifold \( M \) becomes \( \mathcal{M}_{g,1} \).

In the above argument, we need to keep in mind the equivariance condition or WPSC (3.9). In fact in topological gravity all BRST-closed states are also BRST-exact \([11]\), except for the states created by the puncture operator and the vacuum state. The other non-trivial observables in topological gravity can therefore all be written as \([11],[30]\)

\[ |\Psi\rangle = (Q_T + \bar{Q}_T)|\lambda\rangle, \quad \text{where } |\lambda\rangle \text{ fails the WPSC.} \quad (3.10) \]

It is convenient to break the Hilbert space of BRST-closed states into two disjoint sectors, \( \mathcal{H}^{\text{BRST-closed}} = \mathcal{H}^{\text{WPS}} \oplus \mathcal{H}^{\text{rest}} \), so that \( |\Psi\rangle \in \mathcal{H}^{\text{WPS}} \) and \( |\lambda\rangle \in \mathcal{H}^{\text{rest}} \). However, this does not mean that the theory is empty. A state like (3.10) need not decouple because it is not really a BRST-exact state in the equivariant sense, so that \( \mu_\lambda \) in the previous paragraph is not globally defined on \( \mathcal{M}_{g,1} \).

For an \( n \)-form \( \Omega \) on a compact manifold \( M \) where the \( n \)-th-cohomology is non-trivial \( H^n(M) \neq 0 \), by the Poincaré lemma we can always write \( \Omega = d\mu_i \) on a local patch \( U_i, \cup U_i = M \). Then

\[ \int_M \Omega = \sum_i \int_{U_i} d\mu_i = \sum_i \int_{\partial U_i} \mu_i. \quad (3.11) \]

If the \( \mu_i \) agree along the boundaries \( \partial U_i \), then \( \Omega \) is an exact form globally on \( M \) and \( \int_M \Omega = 0 \). Generalizing to our semirigid situation, we insert the state (3.10) into \( \tilde{\Omega} \) of (3.3) and obtain \( \tilde{\Omega}_\Psi = d\tilde{\mu}_\lambda \), where \( d \) is the ordinary exterior derivative and \( \tilde{\mu}_\lambda \) is an integration density of degree \( (3g-3,3g-2) \). Total derivatives in the Grassmann variables are ignored because they vanish when integrated. Observe that \( \tilde{\mu}_\lambda \) will not be invariant

---

16
under the change in phase in the coordinates across patches because the state $|\lambda\rangle$ does not satisfy the WPSC. What this means is that on a local patch on the moduli space, we have by (3.2) $\Omega_{\sigma}|_{\alpha} = d(\sigma^*|_{\alpha}\tilde{\mu}_{\lambda})$, and on an overlapping patch, $\Omega_{\sigma}|_{\beta} = d(\sigma^*|_{\beta}\tilde{\mu}_{\lambda})$, but $\sigma^*|_{\alpha}\tilde{\mu}_{\lambda}$ and $\sigma^*|_{\beta}\tilde{\mu}_{\lambda}$ do not agree on the overlap. Just like the discussion above, since $\mu_i = \int_{\Pi^{-1}} \sigma^*|_{i}\tilde{\mu}_{\lambda}$ (the integral over the fiber of $\Pi: \hat{M}_{g,1} \to M_{g,1}$) do not agree on the overlap, the top form $\int_{\Pi^{-1}} \Omega_{\sigma}$ can give a nontrivial element in $H^{6g-4}(M_{g,1})$. Therefore the correlation function $\int_{\Omega_{\sigma}} = \int \sigma^*\tilde{\Omega}_{\Psi}$ probes non-trivial topology of the moduli space $M_{g,1}$ of the family of Riemann surfaces.

The non-trivial observables were given by E. and H. Verlinde \[1\]. We will in this paper use the modified version of the non-trivial observables given by Distler and Nelson in \[1\], and further, we normalize them by the factor $1/2\pi i$. They are for $n \geq 0$,

$$|O_n\rangle = \frac{1}{2\pi i} \gamma_0^n c_1 \gamma_1^{-1} | -1\rangle, \quad | -1\rangle \equiv \delta(\gamma_1)\delta(\bar{\gamma}_1)|0\rangle,$$

where $| -1\rangle$ denotes the vacuum state at Bose sea level $-1$. These observables $|O_n\rangle$ satisfy the WPSC \[3.9\]. However, these states are non-trivial because when written as BRST-exact states using \[2.12\], for $n \geq 1$,

$$|O_n\rangle = (Q_T + \bar{Q}_T)|\lambda_n\rangle, \quad |\lambda_n\rangle = c_0|O_{n-1}\rangle,$$

$|\lambda_n\rangle$ fails the WPSC since $b_0|\lambda_n\rangle \neq 0$ although it satisfies the rest of the WPSC. This fits the category of states \[3.10\]. That is, these states are BRST-closed but nonetheless not BRST-exact in the equivariant sense. In particular, $O_0$ is called the puncture operator and $O_1$ the dilaton. The puncture operator is the unique operator that satisfies stronger conditions than WPSC, namely the strong physical state condition \[1][30\]: $Q_T|O_0\rangle = L_n|O_0\rangle = G_n|O_0\rangle = b_n|O_0\rangle = \beta_n|O_0\rangle = 0$ for $n \geq 0$ and their complex conjugates. Note that $|O_{n \geq 1}\rangle$ would have been a strong physical state if it were not for $\beta_0|O_n\rangle = 0$. Although $O_0$ is a BRST-closed state it cannot be written as BRST-exact state, hence it need not decouple.

By conservation of ghost charges ($U_{bc}, U_{\beta\gamma}$), one can show that for a correlation of $N$ observables $O_n\rangle$ to be nonzero, the number $N$ and type $n_i$ of observables \[3.12\] are constrained by $\sum_{i=1}^{N} n_i = 3g - 3 + N \[1\]$. In the remaining sections we will assume that this constraint is satisfied by having the correct amount and type of observables. (Sometimes we will not show all of them explicitly.)
4. The dilaton equation

In this section, we will integrate over the location of the dilaton first and stay away from the boundaries of the moduli space. The boundary contributions giving contact terms are considered in [1]. We will suppress all moduli from the formulas except the 1|1 moduli associated to the chosen puncture \( P \). We will also suppress other inserted states needed for ghost charge conservation and display only the dilaton. In particular, the zero mode insertions for the \( 3g - 3|3g - 3 \) moduli associated to the unpunctured surface are inserted at punctures other than \( P \). This is allowed by the operator formalism [20].

To begin, we take an ordinary Riemann surface and choose a covering by coordinate patches and a corresponding tiling by polygons with edges along two-fold patch overlaps and vertices at three-fold overlaps. For example, we can choose in such a way that each polygon is a triangle. We then augment this Riemann surface to a semirigid surface with semirigid coordinate patches. On the overlap of two patches, recall that their respective coordinates can be chosen to differ by a \( U(1) \) phase as in (3.8). This prompts us to define a semirigid coordinate family to include a possible \( U(1) \) phase \( M(P) \), \( z_P(\cdot) \equiv M(P)\sigma_P(\cdot) \) on a local patch where \( \sigma_P(\cdot) \) is a general coordinate. When \( P \) is on the overlap with another patch, \( \sigma_P(\cdot) \) is chosen to be common to the two overlapping patches whereas the phase \( M(P) \) jumps across patches. To keep our calculations simple, instead of the general coordinate \( \sigma_P(\cdot) \), we will illustrate with the “conformal normal ordered” [36][29] coordinate \( z - (r + \theta \rho) \) where \( (r, \rho) \) are the moduli associated to \( P \). As we will see, it is only the phase difference across patches that matters and hence we will obtain the same answer if we begin with a general coordinate \( \sigma_P(\cdot) \). Thus, on a local patch \( U_\alpha \) the modified conformal normal ordered coordinate

\[
z_P(\cdot) = M(P)(z - r - \theta \rho) \tag{4.1}
\]

will be used. Recall that the phase \( M(P) \) by construction depends on the even moduli \( r \) and \( \bar{r} \) but not their odd partners associated to the location of \( P \) and jumps across coordinate patches.

We will first find the vectors \( (\tilde{v}_i = \sigma_* V_i, \tilde{\nu}_i = \sigma_* \Upsilon_i) \in T_P \) that deform the moduli \( (r, \rho) \) associated to \( P \) on the coordinate patch \( U_\alpha \) and the corresponding ghost insertions for the measure. Here, we have \( (V_{3g-2}, \Upsilon_{3g-2}) = (\partial_r, \partial_\rho) \) and their complex conjugates
(\partial_r, \partial_\rho). Using (4.1), we get

\[ \sigma_s \frac{\partial}{\partial r} = \frac{\partial z_P(\cdot)}{\partial r} \frac{\partial}{\partial z_P(\cdot)} + \frac{\partial z_P(\cdot)}{\partial r} \frac{\partial}{\partial z_P(\cdot)} \]

\[ = -M \frac{\partial}{\partial z_P(\cdot)} + A z_P(\cdot) \frac{\partial}{\partial z_P(\cdot)} + B z_P(\cdot) \frac{\partial}{\partial z_P(\cdot)} \]

\[ = M i_\sigma(l_{-1}) - A i_\sigma(l_0) - B i_\sigma(l_0) \]  

(4.2)

where \( A \equiv M^{-1} \partial_r M \), \( B \equiv M^{-1} \partial_\rho M \) and \( \sigma = z_P(\cdot) \). Similarly,

\[ \sigma_s \left( \frac{\partial}{\partial \bar{r}} \right) = \bar{M} i_\sigma(\bar{l}_{-1}) - \bar{A} i_\sigma(\bar{l}_0) - \bar{B} i_\sigma(l_0) \]

\[ \sigma_s \left( \frac{\partial}{\partial r} \right) = -M i_\sigma(g_{-1}), \quad \text{and} \quad \sigma_s \left( \frac{\partial}{\partial \rho} \right) = -\bar{M} i_\sigma(g_{-1}). \]  

(4.3)

According to (3.4) and (3.5), therefore

\[ B \left[ \frac{\partial}{\partial r} \right] = Mb_{-1} - Ab_0 - \bar{B} b_0 \equiv \hat{b}_{-1}, \quad B \left[ \frac{\partial}{\partial \rho} \right] = \bar{M} b_{-1} - \bar{A} \bar{b}_0 - B b_0 \equiv \hat{b}_{-1}, \]

\[ B \left[ \frac{\partial}{\partial \rho} \right] = -M \beta_{-1} \equiv -\hat{\beta}_{-1}, \quad \text{and} \quad B \left[ \frac{\partial}{\partial \rho} \right] = -\bar{M} \bar{\beta}_{-1} \equiv -\hat{\bar{\beta}}_{-1}. \]  

(4.4)

On another overlapping patch \( U_\beta \) in which a coordinate with the phase \( \tilde{M} \) is used, we let

\[ M = e^{2\pi i f} \tilde{M} \]  

(4.5)

to be the transition map, where \( f \) is real and depends only on \( r \) and \( \bar{r} \). Thus on the overlap between these two patches, we have

\[ \tilde{A} - A = -2\pi i \partial_r f \quad \text{and} \quad \tilde{B} - B = -2\pi i \partial_\rho f. \]  

(4.6)

From the discussion in section 4, if we have a measure \( \tilde{\Omega}_{\psi = Q \lambda} \), then

\[ \int \tilde{\Omega}_{Q \lambda} = \sum_i \int_{U_i} d\tilde{\mu}_\lambda|_i = \sum_i \int_{\partial U_i} \tilde{\mu}_\lambda|_i, \]  

(4.7)

where \( \tilde{\mu}_\lambda|_i \) means \( \tilde{\mu}_\lambda \) evaluated in the coordinate patch \( U_i \). On the patch \( U_\alpha \), using (4.4), the boundary contributions give

\[ \int_{\partial U_\alpha} \tilde{\mu}_\lambda|_\alpha = \int_{\partial U_\alpha} d\rho(\Sigma, z_P(\cdot)) |\hat{b}_{-1}\rangle \delta(\hat{\beta}_{-1}) \delta(\hat{\bar{\beta}}_{-1}) |\lambda\rangle \]

\[ + \int_{\partial U_\alpha} d\bar{\rho}(\Sigma, z_P(\cdot)) |\hat{\bar{b}}_{-1}\rangle \delta(\hat{\bar{\beta}}_{-1}) \delta(\hat{\beta}_{-1}) |\lambda\rangle \]  

(4.8)

19
and on the patch $U_\beta$, we let $M$ in (4.8) go to $\tilde{M}$. The sum in (4.7) can be rearranged, replacing integrals along $\partial U_i$ by integrals along the common edges $W_{\alpha\beta}$ on the overlaps $U_\alpha \cap U_\beta$ as follows.

$$
\int \tilde{\Omega}_{Q\lambda} = \sum_{(\alpha,\beta)} \int_{\partial U_\alpha \cap W_{\alpha\beta}} \tilde{\mu}_\lambda|_\alpha + \int_{\partial U_\beta \cap W_{\alpha\beta}} \tilde{\mu}_\lambda|_\beta
$$

$$
= \sum_{(\alpha,\beta)} \int_{W_{\alpha\beta}} \tilde{\mu}_\lambda|_\alpha - \tilde{\mu}_\lambda|_\beta
$$

(4.9)

where $(\alpha, \beta)$ denote that the sum is over each overlap $W_{\alpha\beta}$ once. The sign change in the last step is due to reversing the orientation of the line integral.

To evaluate (4.8), we first integrate over the odd moduli $\rho$, so we need to extract the $\rho$ dependence in $\langle \Sigma, z_p(\cdot) \rangle$. That is from (3.1),

$$
\langle \Sigma, z_p(\cdot) \rangle = M(z - r - \theta \rho)|
$$

$$
= \langle \Sigma, z_r(\cdot) = M(z - r)| \ (1 - \rho M G_{-1})(1 - \bar{\rho} \bar{M} \bar{G}_{-1}) \rangle.
$$

(4.10)

Since there are no other $\rho$ dependencies in the rest of the measure, integrating over $\rho$ gives us the picture changing operators, and we are left with

$$
\int_{\partial U_\alpha} \tilde{\mu}_\lambda|_\alpha = \int_{\partial U_\alpha} \mu(\Sigma, z_r(\cdot)) \hat{b}_{-1} G_{-1} \delta(\beta_{-1}) G_{-1} \delta(\bar{\beta}_{-1})|\lambda\rangle
$$

$$
+ \int_{\partial U_\alpha} \bar{\rho}(\Sigma, z_r(\cdot)) \hat{b}_{-1} G_{-1} \delta(\beta_{-1}) G_{-1} \delta(\bar{\beta}_{-1})|\lambda\rangle
$$

(4.11)

where $\delta(\hat{\beta}_{-1}) = |M|^{-1} \delta(\beta_{-1})$ and its complex conjugate are used. Since $|O_1\rangle = \langle Q_T + \bar{Q}_T|\lambda\rangle$ where $|\lambda\rangle = (2\pi i)^{-1} c_0 c_1 \bar{c}_1| - 1\rangle$ and $G_{-1} \delta(\beta_{-1}) G_{-1} \delta(\bar{\beta}_{-1}) c_1 \bar{c}_1| - 1\rangle = -|0\rangle$, we obtain

$$
(2\pi i) \int_{\partial U_\alpha} \tilde{\mu}_\lambda|_\alpha = - \int_{\partial U_\alpha} \mu(\Sigma, z_r(\cdot)) \hat{b}_{-1} c_0|0\rangle - \int_{\partial U_\alpha} \bar{\rho}(\Sigma, z_r(\cdot)) \hat{b}_{-1} c_0|0\rangle.
$$

(4.12)

Using (4.4), the only non-vanishing term is

$$
(2\pi i) \int_{\partial U_\alpha} \tilde{\mu}_\lambda|_\alpha = \left( \int_{\partial U_\alpha} \mu(A - \tilde{A}) + \int_{\partial U_\alpha} \bar{\rho} B \right) Z
$$

(4.13)

where $Z = \langle \Sigma, z_p(\cdot)|0\rangle$ along with the suppressed $(3g - 3, 3g - 3)$ ghost insertions. Thus by (4.9),

$$
\langle O_1 \ldots \rangle = \sum_{(\alpha,\beta)} (2\pi i)^{-1} \left( \int_{W_{\alpha\beta}} [\mu(A - \tilde{A}) + \bar{\rho} B]|\lambda\rangle \right) Z
$$

$$
= \sum_{(\alpha,\beta)} \left( \int_{W_{\alpha\beta}} \mu(A - \tilde{A}) + \bar{\rho} B]|\lambda\rangle \right) Z
$$

$$
= \chi Z, \quad \text{where} \quad \chi = 2g - 2.
$$

20
The last step is given by the standard definition of the Euler number. As mentioned we can tie on the calculation of (1.1) to get the full dilaton equation (1.2).

5. The puncture equation

The expression (1.1) can be derived by integrating out the puncture operator $O_0$ from the lhs of (1.1). We will first show that the puncture operator, unlike the dilaton, gives no contribution when integrated over the bulk of the moduli space. The contributions come only near the boundaries, when the puncture runs into other inserted states. We will then analyze a contact term and show that

$$
\langle O_0 O_1 \cdots \rangle = \frac{n}{n-1} \langle O_{n-1} \cdots \rangle
$$

which can be generalized to the case when there are $N$ contact terms, $\langle O_0 O_1 \cdots O_{n_N} \rangle$.

To begin, we ignore the boundaries of the moduli space and use the family of modified conformal normal ordered coordinates (3.8) on the bulk of moduli space $\hat{M}_{g,1}$. The correlation function is given by

$$
\langle O_0 \prod_{i=1}^N O_{n_i} \rangle = \int d^2 r d^2 \rho \langle \hat{\Sigma}, \sigma_P, \ldots | B[\sigma_{P*}(\partial r)]B[\sigma_{P*}(\partial \rho)]\delta(B[\sigma_{P*}(\partial r)])\delta(B[\sigma_{P*}(\partial \rho)])\rangle|O_0\rangle^P \otimes_{i=1}^N |O_{n_i}\rangle,
$$

where $(r, \rho)$ are the parameters (moduli) describing the position $P$ of the puncture operator. $|B[\sigma_{P*}(\partial r)]\delta(B[\sigma_{P*}(\partial \rho)])|^2$ is the zero mode insertion associated to the point $P$. The $(3g-3+N, 3g-3+N)$ other zero mode insertions are suppressed from notation and inserted elsewhere away from $P$. Thus, we only display terms that depend on $(r, \rho)$ which are going to be integrated out.

On a coordinate patch $U_\alpha$, we use (4.1) for $\sigma_P$ and by substituting its push forward (4.4) into (5.1), we obtain

$$
\langle O_0 \prod_{i=1}^N O_{n_i} \rangle = \int d^2 r d^2 \rho \langle \hat{\Sigma}, M(z-r-\theta \rho), \ldots | \hat{b}_{-1}^P \hat{b}_{-1}^P \delta(\hat{\beta}_{-1}^P)\delta(\hat{\bar{\beta}}_{-1}^P)\rangle|O_0\rangle^P \otimes_{i=1}^N |O_{n_i}\rangle,
$$

\[3\] Since the puncture operator satisfies the strong physical state conditions, it does not matter what local coordinate slice we choose to evaluate its contribution in the bulk of moduli space. We may as well use the family of conformal normal ordered coordinates. However, we will use the family of modified conformal normal ordered coordinates and see explicitly that the phase $M$ is irrelevant.
where the integral over other moduli are again implicit. Substituting (4.4) and (3.12), we have \( b_1^P \delta_p^P \delta_p^P |O_0\rangle^P = -|0\rangle^P \). Next, integrating over the fibers \( \Pi : \tilde{\mathcal{M}}_{g,1} \rightarrow \mathcal{M}_{g,1} \) implies integrating over \( \rho \). Since there are no dependences on \( \rho \) other than the state \( \langle \tilde{\Sigma}, M(z - r - \theta \rho), \ldots | \rangle \), this integration brings out the factor \(|M|^2 \tilde{G}_P^0 \tilde{G}_P^0 \) using (1.10).

Since \( |0\rangle \) is an augmented-\( SL(2, \mathbb{C}) \) invariant vacuum, \( G_P^0 |0\rangle^P = 0 \). Thus on each coordinate patch \( \langle O_0 \prod_{i=1}^N O_{n_i} \rangle \) vanishes; this correlation function does not get contributions from the bulk of moduli space.

At a boundary of the moduli space when the puncture runs into an inserted state, the modified conformal normal ordered coordinate near \( Q \) and similarly for the coordinate centered at \( P \) is given by

\[
\langle O_0 O_n \rangle = \int d^2 q d^2 \delta \langle \tilde{\Sigma}, \sigma_P, \sigma_Q | B[\sigma_P^{PQ}(\partial_q)] B[\sigma_Q^{PQ}(\partial_q)] \rangle 
\times \delta(B[\sigma_P^{PQ}(\partial_q)]) \delta(B[\sigma_Q^{PQ}(\partial_q)]) |O_0\rangle^P \otimes |O_n\rangle^Q.
\]

(5.3)

(\( q, \delta \)) is the position of the point \( P \) relative to \( Q \), and \( \sigma_P^{PQ} = \sigma_P \oplus \sigma_Q \). The push forward of a vector by \( \sigma_P^{PQ} \) lives in the vector space \( T\tilde{\mathcal{P}}_{g,2} \) which is the generalization of \( T\tilde{\mathcal{P}}_{g,1} \). \( \langle \tilde{\Sigma}, \sigma_P, \sigma_Q \rangle \) is a state in the dual of the tensor product of two copies of Hilbert space. The two copies of Hilbert space consist of states labelled by \( P \) and \( Q \) respectively.

When \( q \approx \epsilon \) is small and given that \( \sigma \) is the coordinate on \( \tilde{\Sigma} \) centered at the attachment point to the standard three punctured sphere, the sewing prescription yields the general coordinate (2.19) centered at \( P \)

\[
\bar{\sigma}_P = \Sigma_P + a_1 \Sigma_P^2 + a_2 \Sigma_P^3 + \ldots \quad \text{where} \quad \Sigma_P = \frac{\sigma}{q} - E - \frac{\theta \delta}{q^2} \sigma,
\]

and similarly for the coordinate centered at \( Q \) with \( E \rightarrow \tilde{E} \) and \( a_i \rightarrow \tilde{a}_i \). When \( P \) and \( Q \) are far apart, \( q \gg \epsilon \), we should interpolate to the conformal normal ordered slice \( \sigma_P^{c.n.o.} = \sigma - qE - \theta \delta E \) up to a phase. However, we have imposed a stronger condition WPSC (3.9) than the necessary phase independent condition; the density \( \tilde{\Omega} \) is insensitive to changes in the section \( \sigma \) by complex multiplicative factors. Hence, the section is globally defined modulo a complex multiplicative factor. This saves algebra because we can smoothly interpolate between \( \tilde{\sigma} \) for \(|q| < \epsilon \) and \( q^{-1} \sigma_P^{c.n.o.} \) for \(|q| > 2\epsilon \) by an interpolating function \( f(|q|) \) as in [30], where \( f = 0 \) for \(|q| < \epsilon \) and \( f \rightarrow 1 \) as \(|q| \rightarrow \infty \). Hence the interpolated slice is given by

\[
\bar{\sigma}_P = \Sigma_P + A_1 \Sigma_P^2 + \frac{f \theta \delta}{q} \Sigma_P + \ldots \quad (5.4)
\]
where the ellipses are terms involving \(a_{n \geq 2}\) and \(A_1 = a_1(1-f)\). We have a similar expression for the interpolated slice at \(Q\) by letting the parameters become their tildes. The integrand has two parts, the state \(\langle \hat{\Sigma}, \hat{\sigma}_P, \hat{\sigma}_Q \rangle\) and the zero mode insertions \(|B[(\partial_q)]\delta(B[\partial_\delta])|^2\). Their \((q,\delta)\) dependences will be extracted independently, combined and integrated over, \(\delta\) first and then \(q\).

By using (3.1), we extract those possibly contributing \((q,\delta)\) moduli dependences from \(\langle \hat{\Sigma}, \hat{\sigma}_P, \hat{\sigma}_Q \rangle\). To do this, we turn \(\hat{\sigma}_P\) into composition of maps,

\[
\hat{\sigma}_P = (s + A_1 s^2) + \frac{2A_1 f \delta \theta}{q} (s + A_1 s^2)^2 + \ldots \text{ where } s = (1 + q^{-1} f \theta \delta) \Sigma_P.
\]

Hence we have

\[
\langle \hat{\sigma}_P \rangle = \langle s \rangle (1 - A_1 L_1)(1 - 2A_1 f q^{-1} \delta G_1)
\]

(5.5)

where we have left out terms involving \(L_{n \geq 2}\) and \(G_{n \geq 2}\) and along with them are \(a_{n \geq 2}\). If the \(a_1\) terms do not contribute then neither will the higher modes as we will see. Complex conjugates are again suppressed until needed. Furthermore, we have

\[
\langle s \rangle = \langle \Sigma_P \rangle (1 + \delta f q^{-1} G_0)
\]

\[
= \langle z_E \rangle (1 - \delta E q^{-1} G_{-1})(1 + \delta f q^{-1} G_0)
\]

\[
= \langle \sigma \rangle q^{L_0} e^{EL} (1 - \delta q^{-1} G_0)(1 - \delta E q^{-1} G_{-1})(1 + \delta f q^{-1} G_0)
\]

(5.6)

where \(z_E = (q^{-1} \sigma - E) - \theta \delta q^{-1}(q^{-1} \sigma - E)\). \(\langle \hat{\sigma}_Q \rangle\) has the same expansion as \(\langle \hat{\sigma}_P \rangle\) with \(A_1 \rightarrow A_1\) and \(E \rightarrow \tilde{E}\). We will however set \(\tilde{E} = 0\) because at the end of the calculation, we wish that whatever is inserted at \(Q\) will be inserted with the coordinate \(\sigma\). Putting together the expansion in \(\hat{\sigma}_P\) and \(\hat{\sigma}_Q\), we end up with

\[
\langle \hat{\Sigma}, \hat{\sigma}_P, \hat{\sigma}_Q \rangle
\]

\[
= \langle \hat{\Sigma}, z_E, \sigma \rangle \{(1 - \delta E q^{-1} G_{-1}^P)(1 + \delta f q^{-1} G_0^P)(1 - A_1 L_1^P)(1 - 2A_1 f q^{-1} \delta G_1^P)\}
\]

\[
\times \{q^{L_0^Q} (1 - \delta q^{-1} G_0^Q)(1 + \delta f q^{-1} G_0^Q)(1 - A_1 L_1^Q)(1 - 2A_1 f q^{-1} \delta G_1^Q)\}.
\]

(5.7)

We are left with giving the zero mode insertions. They are gotten from the push forwards of \(\partial_q\) and \(\partial_\delta\) and their complex conjugates by the interpolated slices \(\hat{\sigma}_P\) and \(\hat{\sigma}_Q\) in (3.4). Setting \(\tilde{E} = 0\) at \(Q\) and keeping only to \(a_1\) terms, we obtain

\[
qB[\partial_q] = E b_{-1}^P + \tilde{\partial}(2q)^{-1} f'|q|(\tilde{\beta}_{0}^P + \tilde{\beta}_{0}^Q) + \delta q^{-1} F^{PQ}(E, |q|, a_1),
\]

(5.8)

23
where \( F^{PQ}(E, |q|, a_1) \) is linear in the operators \( \beta^{P,Q} \) and a function of \( E, |q| \) and \( a_1 \). We will later see that this term does not contribute because it has a \( \delta \) coefficient. Similarly, we have the conjugation of (5.8) giving us the push forward of \( \partial_q \). We also have

\[
qB[\partial_\delta] = -E\beta_{P-1} - (1-f)(\beta_0^P + \beta_0^Q) + 2a_1E\beta_0^P,
\]

thus giving

\[
\delta(qB[\partial_\delta]) = |qE|^{-1}\delta(\beta_{P-1}^P) + \delta'(E\beta_{P-1}^P)\{(1-f)(\beta_0^P + \beta_0^Q) - 2a_1E\beta_0^P\} \tag{5.9}
\]

and similarly its conjugate. We will first evaluate (5.3) with only the first term of (5.9) and with \( a_1 = \bar{a}_1 = 0 \) which imply \( A_1 = \bar{A}_1 = 0 \). It will be shown later that the terms ignored now do not contribute when turned on.

Substitute (5.8), the first term of (5.9) and their conjugates into \( B[\partial_\delta]B[\partial_\delta]\delta(B[\partial_\delta])\delta(B[\partial_\delta])|\mathcal{O}_n\rangle^P \otimes |\mathcal{O}_n\rangle^Q \) of (5.3), we get

\[
\begin{align*}
&\{E\beta_{P-1}^P + \delta(2q)^{-1}f'|q|(\beta_0^P + \beta_0^Q) + \delta q^{-1}F^{PQ}\} \\
&\times \{E\beta_{P-1}^P + \delta(2q)^{-1}f'|q|(\beta_0^P + \beta_0^Q) + \delta q^{-1}F^{PQ}\}(2\pi i)^{-1}c_1^Pc_1^PE^{-2}|0\rangle^P \otimes |\mathcal{O}_n\rangle^Q, \tag{5.10}
\end{align*}
\]

where \([\delta(\gamma_n), \delta(\beta_m)] = \delta_{n+m,0}\) is used. What we want is to integrate out the puncture operator completely and in (5.10) we have to remove its remnant \( c_1^Pc_1^P \). First we observe from (2.12) that only the operators \( b_{-1} \) and \( G_{-1} \) are conjugate to \( c_1 \). Without sandwiching with the state \( |\hat{\Sigma}, \hat{\sigma}_P, \hat{\sigma}_Q\rangle \), we have to take \( E^2b_{-1}^Pb_{-1}^P \) to wipe out \( c_1^Pc_1^P \). However, integrating over \( d^2\delta \) kills it. Hence we need at least one \( G_{-1}^P \) and/or \( \bar{G}_{-1}^P \) from the state since every \( G \) (\( \bar{G} \)) comes with a \( \delta \) (\( \bar{\delta} \)). On the other hand if we choose both \( \delta G_{-1}^P \) \( \bar{\delta}G_{-1}^P \) from the state, then picking any terms in \{ \} of (5.10) will give zero because \( \delta^2 = \bar{\delta}^2 = 0 \) and \( G_{-1}|0\rangle = \bar{G}_{-1}|0\rangle = 0 \). Hence the only possible non-vanishing term has either \( \delta G_{-1} \) or \( \bar{\delta}G_{-1} \) from the state. Since we have \( \gamma_0^n \) but not its complex conjugate in \(|\mathcal{O}_n\rangle \) in (5.12), the only contributing combination is \( -\delta\bar{q}^{-1}E\bar{G}_{-1}^P \) from the state along with \( E\beta_{P-1}^P \) and \( (2q)^{-1}\delta f'|q|(\beta_0^P + \beta_0^Q) \) from the zero mode insertions in (5.10). Substituting (5.7) with \( a_1 = \bar{a}_1 = 0 \), we finally arrive at

\[
\langle \mathcal{O}_0|\mathcal{O}_n\rangle = (2\pi i)^{-1} \int d^2qd^2r\delta(\hat{\Sigma}, z_E, \sigma)|q|^{\beta_0^Q}q^{\beta_0^Q}\delta\delta(2|q|)^{-1}f'\beta_0^Qb_{-1}^PB_{-1}^Pc_1^Pc_1^P|0\rangle^P \otimes |\mathcal{O}_n\rangle^Q \tag{5.11}
\]
since $\beta_0^P |0\rangle^P = 0$. Using $[L_0, \beta_0] = 0, L_0^Q |O_n\rangle^Q = \tilde{L}_0^Q |O_n\rangle^Q = 0 \text{ (WPSC)},$ and $\beta_0^Q |O_n\rangle = -n |O_{n-1}\rangle$ for $n \geq 0$, we have

$$
\langle O_0 O_n \ldots \rangle = (2\pi i)^{-1} (-n) \int d^2 q (2|q|)^{-1} f' (\hat{\Sigma}, z_E, \sigma) |0\rangle^P \otimes |O_{n-1}\rangle^Q \\
= n \langle \hat{\Sigma}, \sigma | O_{n-1} \rangle^Q \int d|q| f'(|q|) \\
= n \langle O_{n-1} \ldots \rangle.
$$

(5.12)

Recall that we left out the term with $\delta' (E \beta_{-1}^P)$ in (5.9). It requires a $E \beta_{-1}^P$ term so that using $x\delta' (x) = -\delta(x)$ we get $\delta(E \beta_{-1}^P)$ to raise the Bose see level back to 0 in $|O_0\rangle^P$. However, we did not pick up any $\beta_{-1}^P$ term in integrating out $d^2 \delta$. Thus including $\delta' (E \beta_{-1}^P)$ will not yield new contribution to integrating out the punchure operator.

Finally, we put back the $a_1$ dependences. First note that all $a_1$ dependences in the state $\langle \hat{\Sigma}, \hat{\sigma}_P, \hat{\sigma}_Q \rangle$ come with $L_1^P$ or $G_1^P$ (5.7). Since all the operators in $|B[\partial_q] \delta(B[\partial_q])|^2$ have mode expansion greater than $-1$ and the puncture operator is inserted at $P$, hence by (2.12), $L_1^P$ and $G_1^P$ will annihilate the state at $P$. The other $a_1$ dependences come from the zero mode insertions. $a_1$ comes in via $F^{PQ}$ in $B[\partial_q]$ (5.8) which did not contribute. It also comes into $\delta(B[\partial_q])$, appearing with the the term $\delta' (E \beta_{-1}^P)$ in (5.9) which we argued will not contribute. Thus, turning on $a_1$ does not affect the result we obtained. Similarly, $a_{i \geq 2}$ will drop out since they are associated with higher modes $L_{n \geq 2}, G_{n \geq 2}$ and $\beta_{n \geq 1}$. As for $\tilde{a}_i$, the dependences come only from the state since we have set $\tilde{E} = 0$. Just like before, they come with $L_{n \geq 1}^Q$ and $G_{n \geq 1}^Q$. Although these operators do not appear in the WPSC, they vanish when applied to the state $|O_n\rangle^Q$ in (5.10). Hence, no $\tilde{a}_i$ terms contribute either. Thus, the answer is completely independent of the choice of coordinates around $P$ and $Q$. More generally, we get a contribution similar to (5.12) in integrating out the puncture operator each time it comes in contact with an operator in $\langle O_0 O_{n_1} \ldots O_{n_N} \rangle$. Hence we finally have the desired recursion relation (1.1).

6. Conclusions

In two dimensions, field theory, quantum mechanics and gravity are compatible. By imposing the semirigid symmetry, we simplify the situation enough to see what a theory of quantum gravity predicts. In particular, we have derived two recursion relations involving $N$ point and $N-1$ point correlation functions in a topological quantum field theory with the semirigid geometry. These same relations partially characterize amplitudes of the one
matrix model at its topological critical point. Thus, there is hope for one to continue to show the rest of the recursion relations involving Riemann surfaces with different genera are reproduced in the topological semirigid gravity. It is only then that one can say topological semirigid gravity and the one matrix model are equivalent.

Since intersection theory on moduli space and the one matrix model at its topological point are equivalent in the Kontsevich model and since these intersection numbers satisfy the axioms extracted from topological quantum field theories, a field theory of topological gravity such as the semirigid formulation may be equivalent to the one matrix model. The results of [1] and this paper give a concrete example of how a field theory of topological gravity and the one matrix model can be equivalent as suggested by the intersection theory.

One can also imagine coupling topological matter to semirigid gravity and computing correlation functions including the matter fields. We can then compare to the higher matrix models or the one matrix model at a different critical point and see if any of the recursion relations in the matrix models are reproduced. This will help us sort out what these matrix models correspond to.

Finally, a comment is in order on a string theory interpretation of this 2-d semirigid gravity. Since the Liouville mode decouples in the semirigid pure gravity, there is no constraint analogous to the critical dimension $c = 26$ on the type of topological matter we may couple. Hence, we may be able to construct a semirigid string theory that lives in a four dimensional spacetime.

I would like to thank Gabriel Cardoso, Mark Doyle, Suresh Govindarajan, Steve Griffies, and especially Philip Nelson for valuable discussions. I would also like to thank Philip Nelson for making numerous suggestions to this manuscript. This work was supported in part by DOE grant DOE-AC02-76-ERO-3071 and NSF grant PHY88-57200.
References

[1] J. Distler and P. Nelson, “The Dilaton Equation in Semirigid String Theory,” PUPT-1232 = UPR0428T (1991), to appear in Nucl. Phys. B.
[2] A. M. Polyakov, Mod. Phys. Lett. A2 (1987) 893.
[3] V. G. Knizhnik, A. M. Polyakov and A. B. Zamoldchikov, Mod. Phys. Lett. A3 (1988) 819.
[4] J. Distler and H. Kawai, Nucl. Phys. B321 (1989) 509.
[5] N. Seiberg, “Notes on quantum Liouville theory and quantum gravtiy,” RU-90-29.
[6] P. Nelson, “Lecture on strings and moduli space,” Phys. Reports 149 (1987) 304.
[7] E. Brezin and V. A. Kazakov, “Exactly solvable field theories of closed strings,” Phys. Lett. B236 No. 2 (1990) 144.
[8] D. J. Gross and A. A. Migdal, “A nonperturbative treatment of two dimensional quantum gravity,” Phys. Rev. Lett. 64 (1990) 127.
[9] M. Douglas and S. Shenker, “Strings in less than one dimension,” Nucl. Phys. B335 (1990) 635.
[10] J. Labastida, M. Pernici, and E. Witten, “Topological gravity in two dimensions,” Nucl. Phys. B310 (1988) 611.
[11] E. Verlinde and H. Verlinde, “A solution of two-dimensional topological gravity,” Nucl. Phys. B348 (1991) 457.
[12] J. Distler and P. Nelson, “Semirigid supergravity,” Phys. Rev. Lett. 66 (1991) 1955.
[13] E. Witten, “On the structure of the topological phase of two-dimensional gravity,” Nucl. Phys. B340 (1990) 281.
[14] R. Dijkgraaf and E. Witten, “Mean field theory, topological field theory, and multi-matrix models,” Nucl. Phys. B342 (1990) 486.
[15] D. Montano and J. Sonnenschein, “Topological strings,” Nucl. Phys. B333 (1989) 258; “The topology of moduli space and quantum field theory,” Nucl. Phys. B324 (1989) 348.
[16] R. Myers and V. Periwal, Nucl. Phys. B333 (1990) 536.
[17] L. Baulieu and I. Singer, “Conformally invariant gauge fixed actions for 2D topological gravity,” Commun. Math. Phys. 135 (1991) 253.
[18] J. Distler, “2-d quantum gravity, topological field theory, and the multircritical matrix models,” Nucl. Phys. B342 (1990) 523.
[19] S. Govindarajan, P. Nelson, and E. Wong, “Semirigid Geometry,” UPR-0477T, to appear in Commun. Math. Phys.
[20] L. Alvarez-Gaumé, C. Gomez, G. Moore and C. Vafa, Nucl. Phys. B303 (1988) 455.
[21] L. Alvarez-Gaumé, C. Gomez, P. Nelson, G. Sierra and C. Vafa, Nucl. Phys. B311 (1988) 333.
[22] R. Dijkgraaf, Herman Verlinde, and Erik Verlinde, “Loop equations and Virasoro constraints in nonperturbative 2-d quantum gravity,” Nucl. Phys. B348 (1991) 435.

[23] R. Dijkgraaf, H. Verlinde, and E. Verlinde, in String theory and Quantum Gravity, eds. M. Green et al. (World Scientific, 1990).

[24] E. Witten, “On the Kontsevich model and other models of two dimensional gravity,” IASSNS-HEP-91/24.

[25] M. Atiyah, “Topological quantum field theories,” IHES Publications Mathematiques 68 (1988) 175.

[26] E. Witten, “Two dimensional gravity and intersection theory on moduli space,” IASSNS-HEP-90/45, to appear in J. Diff. Geometry.

[27] S. Govindarajan, P. Nelson, and S. J. Rey, “Semirigid construction of topological supergravities,” UPR-0472T, to appear in Nucl. Phys. B.

[28] T. Eguchi and S.-K. Yang, “N=2 superconformal models as topological field theories,” Mod. Phys. Lett. A5 (1990) 1693.

[29] H. S. La and P. Nelson, “Effective field equations for fermionic strings,” Nucl. Phys. B332 (1990) 83.

[30] J. Distler and P. Nelson, “Topological couplings and contact terms in 2d field theory,” Commun. Math. Phys. 138 (1991) 273.

[31] G. B. Segal, “The definition of conformal field theory,” Differential geometrical methods in theoretical physics (1987) 165.

[32] P. Nelson, “Covariant insertions of general vertex operators,” Phys. Rev. Lett. 62 (1989) 993.

[33] Yu. Manin, Gauge field theory and complex geometry, (Springer-Verlag, 1988).

[34] J. Polchinski, “Factorization of bosonic string amplitudes,” Nucl. Phys. B307 (1988) 61.

[35] Philip Nelson, private communication

[36] J. Polchinski, “Vertex operators in the Polyakov path integral,” Nucl. Phys. B289 (1987) 465.
Figure Captions

Fig. 1. A coordinate patch $U$ with coordinates $(z, \theta)$ and origin at $O$ on the semirigid surface $\hat{\Sigma}$ is shown. All odd coordinates are suppressed. A local coordinate $\sigma = z_P(\cdot)$ centered at $P = (r, \rho)$ is given by $\sigma = f(z - r - \theta \rho)$ where $f$ is some holomorphic function in the coordinates $(z, \theta)$. A family of local coordinates at $P$ can be obtained if we now let $f$ be parametrized by the moduli $(\vec{m}, \vec{\zeta})$ of the once-punctured surface at $P$.

Fig. 2. A semirigid surface $\hat{\Sigma}$ with a unit disk $|z_P(\cdot)| < 1$ removed $(\hat{\Sigma}, z_P(\cdot))$ is shown. A change of the local coordinate $z_P(\cdot)$ to $z'_P(\cdot) = (1 + v)z_P(\cdot)$ is performed and then the boundaries of $(\hat{\Sigma}, z'_P(\cdot))$ and the unit disk $D$ are identified. A state $|\Psi\rangle$ is inserted at the puncture $P$ on the disk.