Incoherence induced sign change in noise cross-correlations: A case study in the full counting statistics of a pure spin pump

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The full counting statistics of a non adiabatic pure spin pump are calculated with particular emphasis on the second and third moments. We show that incoherence can change the sign of spin shot noise cross-correlations from negative to positive, implying entanglement for spin-singlet electronic sources, a truly counterintuitive result. The third moment on the other hand is shown to be much more resilient to incoherence.

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Introduction: Charge or spin transport is a statistical process involving electrons carrying definite amounts of spin or charge, since charge or spin current fluctuates in time. Therefore, in addition to knowing the mean charge or spin current passing through a normal conductor one needs to know the noise as well as the other transport moments in order to fully characterize charge or spin electron motion. To do this one takes recourse to the full counting statistics (FCS), which gives us the complete knowledge about all the moments of the distribution of the number of transferred charges or spins. The full counting statistics for a non-adiabatic pure spin pump is analyzed in both the completely coherent and incoherent regimes.

Shot noise cross-correlations, the second moment, in solid state devices have been studied for a long time. Some of these studies include normal metal-superconductor hybrid structures, coulomb blockaded quantum dots, exploiting the Rashba scattering, etc. However, an experimental demonstration has thus far been lacking. This is mainly due to the difficulty in controlling environmental effects like incoherence. It begs the question how to deal with incoherence and reduce it. In this work a novel scheme is proposed in which the incoherence present in such systems can be used as a resource. We particularly concentrate on the electronic spin. The reason for dwelling on the spin instead of charge is because there have been many works on the charge counting statistics however works on the full counting statistics for spin are less visible. However, they have been attempted in different context to that which is the topic of this rapid communication. For example, in the FCS of spin currents was first attempted, the FCS of spin transfer through ultrasmall quantum dots in context of Kondo effect was attempted in while in a study of FCS in interacting quantum dots attached to ferromagnetic leads revealed super-poissonian transport. Many works revolve around the spin shot-noise cross-correlations. Among the notable works on spin shot-noise cross-correlations mention may be made of: (i) a realistic superconductor-quantum dot entangler, and (iii) a spin transistor. Positive spin shot noise cross-correlations for spin-singlet electronic sources could be a signature of entanglement too. In this letter the properties of the third moment are also calculated. The reason for looking at the third moment is because the third moment is predicted to be much more resilient to incoherence. In our work we prove this statement by an exact analytical calculation.

In this letter we find that in the coherent transport regime the current and spin shot noise cross-correlations are similar to that in Ref. The effect of incoherence on odd moments is negligible. The current and the third moment do not change much with incoherence. In contrast the second moment, i.e., spin shot noise becomes completely positive. An extremely counter-intuitive result. For the third moment spin auto or cross-correlations do not change much from the coherent and incoherent transport regimes. This shows the resilience of the third moment to incoherence. The main body of this letter starts with an explanation of the model. The coherent density matrix equation is then analyzed separate from the incoherent density matrix equation to bring out the differences. Lastly we bring out a perspective on future endeavors.

FIG. 1: The model system.

Model: The model of Ref. is the starting point. It is depicted in Fig.1. The single electron levels in the dot are split by an external magnetic field . , where is effective electron g-factor
in z-direction and $\mu_B$ is Bohr magneton. No bias voltage is applied across the leads. An additional oscillating magnetic field $B_{rt}(t) = (B_{rf} \cos(\omega t), B_{rf} \sin(\omega t))$ applied perpendicularly to constant field $B$ with frequency $\omega$ nearly equal to $\Delta$ can pump the electron to higher level where its spin is flipped, then the spin down electron can tunnel out of the leads. Coulomb interaction in the quantum dot is considered to be strong enough to prohibit double occupation. No extra electrons can enter the quantum dot before the spin-down electron exits. The Hamiltonian of ESR induced spin battery under consideration is written as:

$$H = \sum_{\eta,k,\sigma} \epsilon_{\eta,k,\sigma} c_{\eta,k,\sigma}^\dagger c_{\eta,k,\sigma} + \sum_{\sigma} \epsilon_{\sigma} c_{d\sigma}^\dagger c_{d\sigma} + U n_{d\uparrow} n_{d\downarrow}$$

$$+ \sum_{\eta,k,\sigma} (V_{\eta} c_{\eta,k,\sigma}^\dagger c_{d\sigma} + h.c.) + H_{rf}(t)$$

(1)

In the above equation, $c_{\eta,k,\sigma}^\dagger (c_{\eta,k,\sigma})$ and $c_{d\sigma}^\dagger (c_{d\sigma})$ are the creation and annihilation operators for electrons with momentum $k$, spin $\sigma$ and energy $\epsilon_{\eta,k,\sigma}$ in lead $\eta$ ($L, R$) and for spin $\sigma$ electron on the quantum dot. The third term describes coulomb interaction among electrons on the quantum dot. The fourth term describes tunnel coupling between quantum dot and reservoirs. $H_{rf}(t)$ describes the coupling between the spin states due to the rotating field $B_{rf}(t)$ and can be written in rotating wave approximation as:

$$H_{rf}(t) = R_{rf}(c_{d\uparrow}^\dagger c_{d\uparrow} e^{i\omega t} + c_{d\downarrow}^\dagger c_{d\downarrow} e^{-i\omega t})$$

(2)

with, ESR rabi frequency $R_{rf} = g_{\perp} \mu_B B_{rf}/2$, with $g_{\perp}$ factor of rf field $B_{rf}$.

The quantum rate equations for the density matrix can be easily derived as in Ref.[12]. $\rho_{00}$ and $\rho_{\sigma\sigma}$ describe occupation probability in QD being respectively unoccupied and spin-$\sigma$ states and off-diagonal term $\rho_{\uparrow \downarrow}(t)$ denotes coherent superposition of two coupled spin states in quantum dot. The doubly occupied is prohibited due to infinite coulomb interaction $U \rightarrow \infty$. To derive the density matrix, we proceed as follows. The time dependence can be removed from Eqs. [12], by using the following unitary transformation:

$$U = e^{-\frac{i}{\hbar}[\sum_{(k,n)}(c_{d\uparrow}^\dagger c_{d\uparrow} - c_{d\downarrow}^\dagger c_{d\downarrow}) + (c_{\eta,k,\sigma}^\dagger c_{\eta,k,\sigma} - c_{\eta,k,\sigma}^\dagger c_{\eta,k,\sigma})]}$$

(3)

The Hamiltonian is then redefined in the rotating reference frame as follows:

$$H_{RF} = U^{-1} H U + i \frac{dU^{-1}}{dt} U$$

$$= \sum_{\eta,k,\sigma} \bar{\epsilon}_{\eta,k,\sigma} c_{\eta,k,\sigma}^\dagger c_{\eta,k,\sigma} + \sum_{\sigma} \bar{\epsilon}_{\sigma} c_{d\sigma}^\dagger c_{d\sigma} + U n_{d\uparrow} n_{d\downarrow}$$

$$+ \sum_{\eta,k,\sigma} (V_{\eta} c_{\eta,k,\sigma}^\dagger c_{d\sigma} + h.c.) + R_{rf}(c_{d\uparrow}^\dagger c_{d\uparrow} + c_{d\downarrow}^\dagger c_{d\downarrow})$$

(4)

In the above equation, $\bar{\epsilon}_{\eta,k,\sigma} = \epsilon_{\eta,k,\sigma} - \frac{\Delta}{2} + \frac{\Delta}{2}$, and $\bar{\epsilon}_{\sigma} = \epsilon_{\sigma} + \frac{\Delta}{2} - \frac{\Delta}{2}$, while $\bar{\epsilon}_{\eta,k,\sigma} = \epsilon_{\eta,k,\sigma} + \frac{\Delta}{2} - \frac{\Delta}{2}$.

To get the density matrix from the above Hamiltonian, the following procedure is used. An electron operator affecting only the electron on the dot can be written in terms of $|p >, |q >, |r >, |s >$. Writing, for the annihilation operator of the dot $c_{d\sigma} = |0 > < \sigma |$, and for the creation operator for the dot $c_{d\sigma}^\dagger = | \sigma > < 0 |$, the Hamiltonian is rewritten in terms of the three states: $|0 >, | \uparrow >, | \downarrow >$, corresponding to empty state, a single electron with spin-up and single electron with spin-down. The doubly occupied state in the dot is prohibited by the fact that $U$ is taken to be extremely large. Thus the Hamiltonian reduces to:

$$H = \sum_{\eta,k,\sigma} \bar{\epsilon}_{\eta,k,\sigma} c_{\eta,k,\sigma}^\dagger c_{\eta,k,\sigma} + \sum_{\sigma} \bar{\epsilon}_{\sigma} | \sigma > < \sigma | + U n_{d\uparrow} n_{d\downarrow}$$

$$+ \sum_{\eta,k,\sigma} (V_{\eta} c_{\eta,k,\sigma}^\dagger |0 > < \sigma | + h.c.) + R_{rf}(| \uparrow > < \downarrow |$$

$$+ | \downarrow > < \uparrow |)$$

(5)

The elements of the density matrix $\rho_{mn}$ in dot spin basis are expectation values of operators $|n > < m |$, with $n, m = 0, \uparrow, \downarrow$, so we can write $\rho_{00} = |0 > < 0 |$, $\rho_{\uparrow\uparrow} = | \uparrow > < \uparrow |$, $\rho_{\downarrow\downarrow} = | \downarrow > < \downarrow |$, $\rho_{\uparrow\downarrow} = | \uparrow > < \downarrow |$. The time evolution of the density matrix elements can be expressed in terms of expectation values for new operators [12]. For instance,

$$i \frac{d}{dt} \rho_{00} = i \left[ H |0 > < 0 | - |0 > < 0 | H \right]$$

$$= i \left[ V_{\eta}^* |\sigma > < 0 | c_{\eta,k,\sigma} - V_{\eta} |0 > < \sigma | c_{\eta,k,\sigma}^\dagger \right]$$

$$= [V_{\eta}^* G_{\eta\eta\sigma}(t, t) - V_{\eta} G_{\eta\sigma\eta}(t, t)]$$

(6)

The approximated current Green’s functions are (using Ref. [17]) as a guide we have:

$$G_{\sigma\eta\sigma}(t, t') = \int dt_{1} [G_{\sigma\sigma}^{R}(t_{1}, t_{1}) V_{\eta\sigma}^* g_{\eta\sigma}(t_{1}, t')]$$

$$+ G_{\sigma\sigma}^{A}(t_{1}, t_{1}) V_{\eta\sigma}^* g_{\eta\sigma}(t_{1}, t')]$$

$$G_{\eta\sigma\eta}(t, t') = \int dt_{1} [G_{\eta\sigma}^{R}(t_{1}, t_{1}) V_{\eta\sigma}^* G_{\sigma\sigma}(t_{1}, t')]$$

$$+ g_{\eta\sigma}(t_{1}, t') V_{\eta\sigma} G_{\sigma\sigma}^{A}(t_{1}, t')]$$

(7)

The $G_{\sigma\eta\sigma}$’s are the green functions for the dot, while $g_{\eta\sigma}$ is the Green’s function for the $\eta$-lead in absence of tunnelling.

From the convolution theorem for Fourier transforms,

$$\int dt_{1} A(t_{1} - t) B(t_{1} - t) = \int du A(u) B(-u) = \int \frac{du}{2\pi} A(w) B(w)$$

(8)

Inserting the approximated current Green’s functions from Eqs[17] into Eq[16] and Fourier transforming one gets:

$$\rho_{00} = [V_{\eta}^2 G_{\eta\sigma\sigma}(w) (g_{\eta\sigma}^{R}(w) - g_{\eta\sigma}^{A}(w)) + g_{\eta\sigma}(w) (G_{\sigma\sigma}^{A}(w) - G_{\sigma\sigma}^{R}(w))]$$

(9)
The lesser Green’s function then becomes-

\[ \rho_{00} = |V_{\eta}|^2[G_{0\eta\sigma}^<(w)(g_{\eta\kappa\sigma}^<(w) - g_{\eta\kappa\sigma}^>(w)) + g_{\eta\kappa\sigma}^<(w)(G_{0\eta\sigma}^< - G_{0\eta\sigma}^>)] \] (10)

The lesser Green’s function then becomes-

\[ g_{\eta\kappa\sigma}^< (t) \equiv \langle e_{\eta\kappa\sigma}^t c_{\eta\kappa\sigma}^t (t) \rangle = i e^{-i\eta\kappa\sigma t} < e_{\eta\kappa\sigma}^t c_{\eta\kappa\sigma}^t > = i e^{-i\eta\kappa\sigma t} f_\eta (\epsilon_{\eta\kappa\sigma}) \] (11)

where, \( f(\epsilon) \) is the Fermi function. Performing a fourier transformation yields

\[ g_{\eta\kappa\sigma}^< (w) = 2\pi i f_\eta (\epsilon_{\eta\kappa\sigma}) \delta (w - \epsilon_{\eta\kappa\sigma}), \text{ and similarly} \]
\[ g_{\eta\kappa\sigma}^> (w) = -2\pi i (1 - f_\eta (\epsilon_{\eta\kappa\sigma})) \delta (w - \epsilon_{\eta\kappa\sigma}) \] (12)

Substituting the above expressions in Eqs (6) and using the coupling parameter \( \Gamma_\eta^\sigma (\epsilon) = 2\pi \sum_k |V_{\eta k}|^2 \delta (\epsilon - \epsilon_{\eta\kappa\sigma}) \) gives-

\[ \rho_{00} = -\frac{i}{2\pi} \int dw \sum_{\eta\sigma} \{ \Gamma_\eta^\sigma (1 - f_\eta (w)) G_{0\eta\sigma\sigma}^> (w) + \Gamma_\eta^\sigma f_\eta (w) G_{0\eta\sigma\sigma}^< (w) \} \] (13)

The lesser and greater Greens functions for the dot can be derived using the same formalism as in Ref. [17].

\[ M = \begin{pmatrix}
-\left( \Gamma_1^L + \Gamma_1^R \right) & 0 & 0 & 0 & 0 & 0 \\
\Gamma_1^L e^{i\chi_1^L} + \Gamma_1^R e^{-i\chi_1^R} & 0 & 0 & -2R_{\eta f} & \delta_{\text{ESR}} & \left( \Gamma_1^L + \Gamma_1^R \right) \\
0 & 0 & 0 & 2R_{\eta f} & \delta_{\text{ESR}} & \left( \Gamma_1^L + \Gamma_1^R \right) \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & R_{\eta f} & \delta_{\text{ESR}} & \left( \Gamma_1^L + \Gamma_1^R \right) & 0 & 0 \\
R_{\eta f} & \delta_{\text{ESR}} & \left( \Gamma_1^L + \Gamma_1^R \right) & 0 & 0 & 0
\end{pmatrix} \] (16)

and \( \delta_{\text{ESR}} = \Delta - \omega \). The normalization relation \( \rho_{00} + \sum_{\sigma\sigma'} \rho_{\sigma\sigma'} = 1 \) holds for the conservation and \( \Gamma_{\eta\sigma} = 2\pi \sum_k |V_{\eta k}|^2 \delta (w - \epsilon_{\eta\kappa\sigma}) \). We assume the spin relaxation time of an excited spin state into the thermal equilibrium to be very large.

We calculate the eigenvalues of Eq (16). The minimal of these eigenvalues defines the full counting statistics (as, \( \chi_{\eta\sigma} \to 0, \eta = L, R; \sigma = \uparrow, \downarrow \)). After finding this eigenvalue \( \epsilon_{\eta\sigma} \), we then by using the approach pioneered in Ref. [14], we calculate the first, second and higher cumulants. Note that the approach of Ref. [14] has been generalized in Refs. [13, 16] to include both coherent and incoherent transport regimes.

The first cumulant is defined as the current, we calculate the individual spin polarized currents as follows:

\[ I_{\eta\sigma} = \frac{\partial E_{\text{En}}}{\partial x_{\eta\sigma}} |_{x_{\eta\sigma} = 0}. \]

The spin current is thus \( I_\eta^\uparrow = I_{\eta\uparrow} - I_{\eta\downarrow} \), while the charge current is \( I_\eta^\prime = I_{\eta\uparrow} + I_{\eta\downarrow} \). The second cumulant defines the shot-noise. The shot noise auto and cross-correlations can be calculated as follows. The spin shot noise auto and cross-correlation is what we concentrate on. \( S_{\eta L} = S_{\eta L}^\uparrow + S_{\eta L}^\downarrow - S_{\eta L}^{\uparrow\downarrow} \) and \( S_{\eta R} = S_{\eta R}^\uparrow + S_{\eta R}^\downarrow - S_{\eta R}^{\uparrow\downarrow} \) wherein, \( S_{\eta\sigma\sigma'} = \frac{\partial^2 E_{\text{En}}}{\partial x_{\eta\sigma} \partial x_{\eta\sigma'}} |_{x_{\eta\sigma} = 0, x_{\eta\sigma'} = 0} \).

Similarly the third moment spin correlations are calculated as follows:

\[ C_{\eta\eta'\eta''} = C_{\eta\eta'\eta''}^\uparrow + C_{\eta\eta'\eta''}^\downarrow - C_{\eta\eta'\eta''}^{\uparrow\downarrow} \]

wherein \( C_{\eta\eta'\eta''} = \frac{\partial^3 E_{\text{En}}}{\partial x_{\eta\sigma} \partial x_{\eta\sigma'} \partial x_{\eta\sigma''}} |_{x_{\eta\sigma} = 0, x_{\eta\sigma'} = 0, x_{\eta\sigma''} = 0} \). The existence of a pure spin current is a signature of a spin-singlet electronic source. Since, in a pure spin current electrons of opposite spin move in exactly opposite directions.

**Incoherent regime:** To go into the incoherent or sequential transport regime as exemplified in Refs. [13], we use the complete coherent matrix, Eq (16). The coefficient matrix for incoherent transport can be obtained from Eq (16) via setting \( \Re \{ \rho_{\eta\eta'} \} = 0 \) and \( \Im \{ \rho_{\eta\eta'} \} = 0 \) and then solving the two simultaneous equations for \( \Re \{ \rho_{\eta\eta'} \} \)
The elements of the matrix: $\dot{\mathcal{M}}(t) = (\dot{\rho}_{00}, \dot{\rho}_{11}, \dot{\rho}_{11}) = \mathcal{M}(t)$ with

$$\mathcal{M} = \begin{pmatrix}
- (\Gamma_{L1} + \Gamma_{R1}) & 0 & \Gamma_{L1} e^{i\chi_{L1}} + \Gamma_{R1} e^{i\chi_{R1}} \\
\Gamma_{L1} e^{-i\chi_{L1}} + \Gamma_{R1} e^{-i\chi_{R1}} & -z & z \\
0 & z & - (\Gamma_{L1} + \Gamma_{R1})
\end{pmatrix},$$

and $z = \frac{R_{LR}^2 (\Gamma_{L1} + \Gamma_{R1})}{\delta_{\rho_{R1}}^2 (\Gamma_{L1} + \Gamma_{R1})}$. The minimal eigenvalue of this equation is again what we require. $E_{V0} = \frac{1}{6a} [K - 2b - \frac{4(3c^2 - b^2)}{K}]$, here $K = 36cba - 108 da^2 - 8b^3 + 12\sqrt{3} \sqrt{4c^2 - a^2 - c^2 b^2 - 18cba + 27d^2 a^2 + 4db^2 a}$, and the elements $a, b, c, d$ are as follows (with $\Gamma_{L1} = \Gamma_{L1} = \Gamma_{R1} = \Gamma_{R1} = \Gamma / 2$):

$$a = 4\delta_{ESR}^2 - \Gamma^2, \quad b = 8\delta_{ESR}^2 - 8R_{rf}^2 \Gamma - 2\Gamma^2$$
$$c = 12R_{rf}^2 \Gamma^2 - 4\Gamma^2 \delta_{ESR}^2, \quad X = R_{rf}^2 \Gamma^3.$$
TABLE I: Comparing first three moments in coherent and incoherent regimes

| Moment | Coherent | Incoherent |
|--------|----------|------------|
| 1st    | Pure spin current | Pure spin current |
| 2nd    | Shot-noise cross-correlations positive for certain range of parameters | Shot-noise cross-correlations always positive |
| 3rd    | Third moment finite | Third moment finite, No qualitative change |

of incoherence is shown. Future endeavors on effects of incoherence on different geometries especially including superconductors are contemplated.

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