AR-COMPONENTS FOR GENERALIZED BEILINSON ALGEBRAS

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Abstract. We show that the generalized $W$-modules defined in 2013 determine $\mathbb{Z}A_\infty$-components in the Auslander-Reiten quiver $\Gamma(n, r)$ of the generalized Beilinson algebra $B(n, r), n \geq 3$. These components entirely consist of modules with the constant Jordan type property. We arrive at this result by interpreting $B(n, r)$ as an iterated one-point extension of the $r$-Kronecker algebra $K_r$, which enables us to generalize findings concerning the Auslander-Reiten quiver $\Gamma(K_r)$ presented in 2013 to $\Gamma(n, r)$.

INTRODUCTION

Motivated by work of Carlson, Friedlander, Pevtsova and Suslin on modular representations of elementary abelian $p$-groups (cf. [4], [5]), we introduced in a foregoing paper [12] modules with the equal images property, the equal kernels property and modules of constant Jordan type for the generalized Beilinson algebra $B(n, r)$, the path algebra of the quiver

\[
\begin{align*}
0 & \xrightarrow{\gamma_1} 1 \\
& \vdots \\
& \gamma_{r-1} \\
& \gamma_r \\
& \gamma_1 \\
& \gamma_{r-1} \\
& \gamma_r \\
& \vdots \\
& \gamma_{r-1} \\
& \gamma_r \\
& \vdots
\end{align*}
\]

modulo commutativity relations $\gamma_i \gamma_j = \gamma_j \gamma_i$. These classes are defined such that a faithful exact functor $\mathfrak{F}: \text{mod } B(n, r) \to \text{mod } kE_r$ maps a $B(n, r)$-module with one of the above properties to a module that satisfies the respective property over an elementary abelian $p$-group $E_r$ of rank $r \geq 2$ [12 2.3]. We moreover gave a generalization of the so-called $W$-modules, a special class of $kE_2$-modules with the equal images property defined in [5] via generators and relations, to elementary abelian $p$-groups of arbitrary rank and showed that these modules might as well be considered modules with the equal images property over $B(n, r)$ via $\mathfrak{F}$.

The Auslander-Reiten quiver constitutes an important invariant of the Morita equivalence class of an algebra. While in general it is hard to compute the components of this quiver, one has a good knowledge of the Auslander-Reiten components for group algebras of finite groups and hereditary algebras. In particular, by work of Ringel [2] and Erdmann [6], the regular Auslander-Reiten components of wild hereditary algebras and of $p$-elementary abelian groups are of tree class $A_\infty$. 

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On the contrary, not much is known about the Auslander-Reiten theory of generalized Beilinson algebras. These algebras, however, constitute an interesting class of algebras of global dimension \( \geq 2 \) for \( n \geq 3 \) [13, 3.7].

In the present paper, we interpret \( B(n, r) \) as an iterated one-point extension of the hereditary path algebra \( B(2, r) \) of the \( r \)-Kronecker by duals of generalized \( W \)-modules. As a consequence, we can make use of certain lifting properties of Auslander-Reiten sequences [10, 2.5] in combination with torsion theoretic arguments to generalize our findings from [12] as follows:

**Theorem.** Let \( r \geq 2, m > n \geq 2 \).

(i) If \( (n, r) \neq (2, 2) \), then the generalized \( W \)-module \( W_{m,n}^{(r)} \) is a quasi-simple module in a \( \mathbb{Z}A_\infty \)-component \( C_m \) of \( \Gamma(n, r) \) which contains two disjoint cones: one consisting of modules with the equal images property and one consisting of modules with the equal kernels property. Moreover, all modules in \( C_m \) have constant Jordan type.

(ii) If \( r > 2 \), then the two cones in (i) are adjacent. In case \( r = 2 \), there is exactly one quasi-simple module that has neither the equal images nor the equal kernels property.

For \( r > 2 \), these components can be visualized as follows with \( W = W_{m,n}^{(r)} \) being the uniquely determined generalized \( W \)-module of the component.

![Figure 1](image-url)

The gray and black bullets refer to modules with the equal images property and the equal kernels property, respectively.

Throughout, we denote by \( k \) an algebraically closed field and we let \( n, r \geq 2 \). We assume the reader to be familiar with the concept of an algebra given by a quiver with relations and refer to [1] and [2] for basic notions of Auslander-Reiten theory.
1. Module categories for $B(n, r)$ and the case $n = 2$

In this section, we recall the module categories we defined in [12] and provide the reader with the relevant information we have on the Auslander-Reiten quiver of $B(2, r)$.

Let $E(n, r)$ be the path algebra of the quiver $Q(n, r)$ with $n$ vertices and $r$ arrows between the vertices $i$ and $i + 1$ for all $0 \leq i \leq n - 2$:

$\begin{align*}
0 & \quad 1 & \quad 2 & \quad \cdots & \quad n - 2 & \quad n - 1 \\
\gamma_0^{(0)} & \quad \gamma_1^{(0)} & \quad \gamma_1^{(1)} & \quad \gamma_1^{(n-2)} & \quad \gamma_r^{(n-2)}
\end{align*}$

The generalized Beilinson algebra $B(n, r)$ is the factor algebra $E(n, r)/I$ where $I$ is the ideal generated by the commutativity relations $\gamma_s^{(i+1)} \gamma_t^{(i)} - \gamma_t^{(i+1)} \gamma_s^{(i)}$ for all $s, t \in \{1, \ldots, r\}$ and $i \in \{0, \ldots, n - 1\}$. These algebras generalize the algebras of the form $B(n) = B(n, n)$ introduced by Beilinson in [3]. We denote by $S(i)$, $P(i)$, $I(i)$ and $e_i$ the simple, the projective and the injective indecomposable $B(n, r)$-module and the primitive orthogonal idempotent corresponding to the vertex $i \in \{0, \ldots, n - 1\}$. For $\alpha \in k^r \setminus 0$ and $M \in \text{mod} B(n, r)$, we consider the linear operator $\alpha_M : M \rightarrow M$ given by left-multiplication with the element $\sum_{i=0}^{n-2} (\alpha_1 \gamma_1^{(i)} + \cdots + \alpha_r \gamma_r^{(i)}) \in B(n, r)$.

**Definition 1.1** (cf. [12]). We denote by

(i) $\text{EIP}(n, r) := \left\{ M \in \text{mod} B(n, r) \mid \forall \alpha \in k^r \setminus 0 : \text{im}(\alpha_M) = \bigoplus_{i=1}^{n-1} (e_i \cdot M) \right\}$,

(ii) $\text{EKP}(n, r) := \left\{ M \in \text{mod} B(n, r) \mid \forall \alpha \in k^r \setminus 0 : \text{ker}(\alpha_M) = e_{n-1} \cdot M \right\}$,

(iii) $\text{CR}^J(n, r) := \left\{ M \in \text{mod} B(n, r) \mid \exists c_j \in \mathbb{N}_0 \forall \alpha \in k^r \setminus 0 : \text{rk}(\alpha_M)^j = c_j \right\}$,

(iv) $\text{CJT}(n, r) := \bigcap_{j=1}^n \text{CR}^j(n, r)$

the categories of modules with the equal images property and the equal kernels property, and the categories of modules of constant $j$-rank and of constant Jordan type, respectively.

Due to the fact that $B(n, r) \cong B(n, r)^\text{op}$, there is a duality $D : \text{mod} B(n, r) \rightarrow \text{mod} B(n, r)$ induced by taking the linear dual and relabelling the vertices of the quiver $Q(n, r)$ in the reversed order. The functor $D$ is compatible with the Auslander-Reiten translation $\tau$ in the sense that $D \tau \cong \tau^{-1} D$ [13 3.3] and restricts to a duality between the categories $\text{EIP}(n, r)$ and $\text{EKP}(n, r)$. Note, moreover, that $\text{EIP}(n, r) \cup \text{EKP}(n, r) \subseteq \text{CJT}(n, r)$. The following is a direct consequence of Definition 1.1.

**Remark 1.2.** Let $M \in \text{EIP}(n, r)$, $N \in \text{EKP}(n, r)$.

(i) If $e_0 \cdot M = 0$, then $M = 0$.

(ii) If $e_{n-1} \cdot N = 0$, then $N = 0$.

A homological characterization of these categories [12 2.5] yields that the category $\text{EIP}(n, r)$ is the torsion class $\mathcal{T}$ of a torsion pair $(\mathcal{F}, \mathcal{F})$ in $\text{mod} B(n, r)$ with $\text{EKP}(n, r) \subseteq \mathcal{F}$ that is closed under the Auslander-Reiten translation $\tau$ and
contains all preinjective modules [12, 2.7]. In particular, there are no non-trivial maps $EIP(n, r) \to EKP(n, r)$. This yields [12, 2.9]:

**Theorem 1.3.** Let $C$ be a regular $\mathbb{Z}A_\infty$-component of $\Gamma(n, r)$. If $EIP(n, r) \cap C \neq \emptyset$, then either $C \subseteq EIP(n, r)$ or there exists a quasi-simple module $W_C$ such that $(\to W_C) = C \cap EIP(n, r)$. Dually, if $EKP(n, r) \cap C \neq \emptyset$, then either $C \subseteq EKP(n, r)$ or there exists a quasi-simple module $M_C$ such that $(M_C \to) = C \cap EKP(n, r)$.

Here, for $M \in \text{mod } B(n, r)$ indecomposable, we denote by $(M \to)$ and $(\to M)$ the sets consisting of $M$ and all successors and all predecessors of $M$ in $\Gamma(n, r)$, respectively. In case $C$ contains both modules with the equal images and the equal kernels property, we define $W(C)$ by the property

$$\tau^{W(C)+1}M_C = W_C;$$

i.e. $W(C)$ is the number of quasi-simple modules in $C$ that satisfy neither the equal images nor the equal kernels property. In [12, 3.4] it is shown that the width $W(C)$ can in general be arbitrarily large.

We furthermore denote by $R = k[X_1, \ldots, X_r]$ the polynomial ring in $r$ variables and by $I = (X_1, \ldots, X_r)$ the ideal generated by $X_1, \ldots, X_r$. There is an equivalence of categories $B(n, r) \cong C_{[0, n-1]}$; where $C_{[0, n-1]}$ denotes the category of $\mathbb{Z}$-graded $R$-modules with support contained in $\{0, \ldots, n-1\}$. We denote by $[-]$ the shift to the right in the category of $\mathbb{Z}$-graded $R$-modules. Via this identification the $\mathbb{Z}$-graded $R$-module

$$M^{(r)}_{m,n} = (I^{m-n}/I^n)[n-m],$$

$m \geq n$, is an indecomposable object in $EKP(n, r)$ with $W^{(r)}_{m,n} := D M^{(r)}_{m,n} \in EIP(n, r)$ [13, 3.6]. In case $m < n$, we define $M^{(r)}_{m,n} := M^{(r)}_{n,n}$. We call modules of the form $W^{(r)}_{m,n}$ generalized $W$-modules, since for $n \leq p$ and $r = 2$, these modules correspond to the $kE_2$-modules defined by Carlson, Friedlander and Suslin in [5] via generators and relations. The module $M^{(3)}_{3,2} \in EKP(2, 3)$, for example, can be depicted as follows:

The dots represent the canonical basis elements given by the monomials in degree one and two and $\to$, $\longrightarrow$ and $\leadsto$ denote the action of $\gamma^{(0)}_1$, $\gamma^{(0)}_2$ and $\gamma^{(0)}_3$, respectively.

The algebra $B(2, r)$ is the path algebra $K_r$ of the $r$-Kronecker quiver. Whenever $r > 2$, $B(2, r)$ is wild, and due to a result by Ringel [9] all regular components are of type $\mathbb{Z}A_\infty$. In [12, §3], we have shown that the module $W^{(r)}_{m,2}$ is quasi-simple in a $\mathbb{Z}A_\infty$-component $C_m$ of $\Gamma(2, r)$ with $W_{c_m} = W^{(r)}_{m,2}$ and $W(C_m) = 0$. Thus the module $W^{(r)}_{m,2}$ is in the rightmost position in the equal images cone of $C_m \subseteq \Gamma(2, r)$; i.e. $C_m$ can be visualized as in Figure 1 with $W = W^{(r)}_{m,2}$. The dual statement holds for modules of the form $M^{(r)}_{m,2}$. 

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2. One-point extensions

We will now provide the necessary theoretical framework and notation for the theory of one-point extensions. For a general introduction, the reader is referred to [10] or [11, XV.1].

**Definition 2.1** (Ringel [10]). Let $A$ be an algebra, $M$ in $\text{mod} \ A$. The algebra $A[M] = \begin{pmatrix} A & M \\ 0 & k \end{pmatrix}$ with usual matrix addition and multiplication is referred to as the **one-point extension** of $A$ by $M$.

If $A = kQ_A/I$ is a basic algebra, we obtain the quiver $Q_{A[M]}$ of $A[M]$ by adding a source vertex together with some arrows to $Q_A$. A module over $A[M]$ so fits the form $(N V) \varphi$, where $N \in \text{mod} \ A$, $V \in \text{mod} \ k$ and $\varphi \in \text{Hom}_k(V, \text{Hom}_A(M, N))$. The $A[M]$-module structure is then given via $(a m 0 \lambda)(n v) = (a.n + \varphi(v)(m) \lambda v)$.

Given $\tilde{X} = \begin{pmatrix} X \\ V \end{pmatrix} \varphi$, $\tilde{Y} = \begin{pmatrix} Y \\ W \end{pmatrix} \psi \in \text{mod} A[M]$, a morphism $\tilde{X} \to \tilde{Y}$ in mod $A[M]$ corresponds to a pair $(f_0, f_1)$, where $f_0 \in \text{Hom}_A(X, Y)$, $f_1 \in \text{Hom}_k(V, W)$ and $\text{Hom}_A(M, f_0) \circ \varphi = \psi \circ f_1$. Since $A$ is a factor algebra of $A[M]$, there is a full exact embedding $\iota_A : \text{mod} A \to \text{mod} A[M]$, sending $N \in \text{mod} A$ to the $A[M]$-module $\begin{pmatrix} N \\ 0 \end{pmatrix}$.

There is a simple injective module $\tilde{S} = \begin{pmatrix} 0 \\ k \end{pmatrix}_0 \in \text{mod} A[M]$. The indecomposable projective $A[M]$-modules are exactly the images of the projective indecomposables of $A$ under $\iota_A$ together with the module $P(\tilde{S}) = \begin{pmatrix} M \\ k \end{pmatrix}_{\lambda \to \lambda \text{id}_M}$.

The following lemma due to Ringel gives information on how almost split sequences in mod $A$ “lift” to mod $A[M]$ [10, 2.5].

**Lemma 2.2.** Let $A$ be an algebra, $M$ an $A$-module. Furthermore let $0 \to \tau N \xrightarrow{f} E \xrightarrow{g} N \to 0$ be an Auslander-Reiten sequence in mod $A$. Then

$$0 \to \begin{pmatrix} \tau N \\ \text{Hom}_A(M, \tau N) \end{pmatrix} \xrightarrow{(f, \text{id})} \begin{pmatrix} E \\ \text{Hom}_A(M, \tau N) \end{pmatrix} \xrightarrow{g} \begin{pmatrix} N \\ 0 \end{pmatrix} \to 0$$

is an Auslander-Reiten sequence in mod $A[M]$. 

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3. **BEILINSON ALGEBRAS AS ITERATED ONE-POINT EXTENSIONS**

The simple module \( S(0) \in \text{mod}\, B(n, r) \) is injective. In view of [11 XV.1], we hence obtain

\[
B(n, r) \cong (1 - e_0)B(n, r)(1 - e_0)[\text{rad}\, P(0)].
\]

Since \( B(n - 1, r) \) is isomorphic to the algebra \( (1 - e_0)B(n, r)(1 - e_0) \), we can thus regard \( B(n, r) \) as a one-point extension of \( B(n - 1, r) \) and consider \( \text{rad}\, P(0) \) an object in \( \text{mod}\, B(n - 1, r) \). By [13 3.23], we have isomorphisms \( M^{(r)}_{n,n} \cong P(0) \) in \( \text{mod}\, B(n, r) \) and \( \text{rad}\, P(0) \cong M^{(r)}_{n,n-1} \) in \( \text{mod}\, B(n - 1, r) \). Inductively, we obtain

\[
B(n, r) \cong B(n - 1, r)[M^{(r)}_{n,n-1}] \cong \mathcal{K}_r[M_{3,2}^{(r)} \cdots M^{(r)}_{n,n-1}].
\]

For \( n = 3, r = 2 \), we can visualize this as follows: Extending the path algebra \( \mathcal{K}_2 \) of the quiver

![Quiver Diagram]

by the module \( M^{(2)}_{3,2} \)

yields the path algebra of

![Extended Quiver Diagram]

with relations \( \gamma_2 \cdot x_1 = \gamma_1 \cdot x_2 \), which is easily seen to be isomorphic to \( B(3, 2) \).

From now on, we will identify the algebras \( B(n, r) \) and \( B(n - 1, r)[M^{(r)}_{n,n-1}] \). Note that when writing \( \tilde{M} \in \text{mod}\, B(n, r) \) in the form

\[
\tilde{M} = \left( \begin{array}{c} M \\ V \end{array} \right), \quad \varphi \in \text{mod}\, B(n - 1, r)[M^{(r)}_{n,n-1}],
\]

the dimension vector \( \dim \tilde{M} \) coincides with the vector \((\dim_k V, \dim M)\).

We want to study the Auslander-Reiten quiver \( \Gamma(n, r) \) by making use of the information on \( \Gamma(2, r) \) presented in Section 2. An application of Lemma 2.2 yields [13 5.6]:

**Proposition 3.1.** Let \( m \geq n \geq 3 \). We have

(i) \( t_{B(n-1,r)}^{-1} M^{(r)}_{m,n-1} M^{(r)}_{n,n-1} \cong \tau_{B(n,r)}^{-1} M^{(r)}_{m,n} \),

(ii) \( t_{B(n-1,r)}^{-1} W^{(r)}_{m,n-1} W^{(r)}_{n,n-1} \cong \tau_{B(n,r)}^{-1} W^{(r)}_{m+1,n} \).

Hence the Auslander-Reiten sequences in \( \text{mod}\, B(n - 1, r) \) starting in \( M^{(r)}_{m,n-1} \) and \( W^{(r)}_{m,n-1} \) lift to Auslander-Reiten sequences in \( \text{mod}\, B(n, r) \) that start in \( M^{(r)}_{m,n} \) and \( W^{(r)}_{m+1,n} \), respectively.
4. Occurrence of Generalized $W$-modules in $\Gamma(n, r)$

We now show that generalized $W$-modules determine $ZA_\infty$-components in $\Gamma(n, r)$, $n \geq 3$, $r \geq 2$ that entirely consist of modules with the constant Jordan type property.

Let us consider the case $r = 2$. On the level of the Auslander-Reiten quiver $\Gamma(2, 2)$ of the tame algebra $K_2$, we do not have any $ZA_\infty$-components to start out with, and due to [7, 4.2.2], the $W$-modules correspond to the preinjective $K_2$-modules. With the use of tilting theory one can show that all regular components of $\Gamma(3, 2)$ are of type $ZA_\infty$, as has been communicated to me by Otto Kernser [5]: There exists a preprojective tilting module $T$ over the path algebra of the extended Kronecker quiver

$$
\begin{array}{ccc}
0 & \rightarrow & 1 \\
\uparrow & & \downarrow \\
& \circlearrowright & 2
\end{array}
$$

such that $\text{End}(T)$ is isomorphic to $B(3, 2)$. The regular components of $\Gamma(\text{End}(T))$ are of type $ZA_\infty$, while there is a preprojective and preinjective component consisting of the $\tau$- and $\tau^{-1}$-shifts of all projective indecomposables and injective indecomposables, respectively.

**Proposition 4.1.** For $m > 3$, the module $W^{(2)}_{m, 3}$ is quasi-simple in a $ZA_\infty$-component $C_m$ of $\Gamma(3, 2)$ with $W(C_m) = 1$. Dually, $M^{(2)}_{m, 3}$ is quasi-simple in a $ZA_\infty$-component of $\Gamma(3, 2)$.

**Proof.** Consider $W^{(2)}_{m-1, 2} \in \text{mod } K_2$. Due to Proposition [3.1] (ii), we have an isomorphism $\tau_{B(3,2)}^{-1} W^{(2)}_{m, 3} \cong \iota_{B(2,2)} \tau_{K_2}^{-1} W^{(2)}_{m-1, 2}$. Note that for $k \geq 1$, the module $W^{(2)}_{k, 2}$ is the preinjective $K_2$-module with dimension vector $(k - 1, k)$ [7, 4.2.2]. It is well-known that there is an Auslander-Reiten sequence

$$
0 \rightarrow W^{(2)}_{m-1, 2} \rightarrow W^{(2)}_{m-2, 2} \oplus W^{(2)}_{m-2, 2} \rightarrow W^{(2)}_{m-3, 2} \rightarrow 0
$$

in $\text{mod } K_2$. We hence obtain

$$
\tau_{B(3,2)}^{-1} W^{(2)}_{m, 3} \cong \iota_{B(2,2)} W^{(2)}_{m-3, 2}.
$$

Remark [12] yields that $\iota_{B(2,2)} W^{(2)}_{m-3, 2} \notin \text{EIP}(3, 2)$ since $e_0 \cdot (\iota_{B(2,2)} W^{(2)}_{m-3, 2}) = 0$. Hence we have $\tau_{B(3,2)}^{-1} W^{(2)}_{m, 3} \notin \text{EIP}(3, 2)$ and in view of [12, 2.7], this implies that $W^{(2)}_{m, 3}$ is neither preinjective nor preprojective. Thus $W^{(2)}_{m, 3}$ is a regular module with the equal images property and therefore contained in a $ZA_\infty$-component $C_m$ of $\Gamma(3, 2)$. Due to the fact that $e_0 \cdot \tau_{B(3,2)}^{-1} W^{(2)}_{m, 3} = 0$, the module $\tau_{B(3,2)}^{-1} W^{(2)}_{m, 3}$ cannot contain a non-trivial module with the equal images property and is hence torsion-free with respect to the torsion pair $(\text{EIP}(3, 2), F)$, where $\text{EKP}(3, 2) \subseteq F$; cf. [12, 2.7]. This yields that there is no non-trivial map $W^{(2)}_{m, 3} \rightarrow \tau_{B(3,2)}^{-1} W^{(2)}_{m, 3}$, and hence $W^{(2)}_{m, 3}$ is quasi-simple in $C_m$. Since $W^{(2)}_{m, 3} \in \text{EIP}(2, 2)$, we obtain $W^{(2)}_{m-3, 2} \notin \text{EKP}(2, 2)$ by [12, 2.7], which yields that $\iota_{B(2,2)} W^{(2)}_{m-3, 2} \notin \text{EKP}(3, 2)$ by [13, 3.32]. Furthermore, due to the fact that $e_0 \cdot (\tau_{B(3,2)}^{-1} W^{(2)}_{m, 3}) = 0$, the dual of [12, 2.10] yields that $\tau_{B(3,2)}^{-2} W^{(2)}_{m, 3} \in \text{EKP}(3, 2)$ and hence $V(C_m) = 1$. The dual statement holds in view of the duality $D$ on mod $B(n, r)$, which is compatible with $\tau$. \[\square\]
In the proof, we made use of the fact that for $m > 3$, we have $\tau_{B(3,2)}^{-1} W_{m,3}^{(2)} \cong \tau_{B(2,2)} W_{m-3,2}^{(2)}$. Proposition 3.1 (ii) now inductively yields for $m > n \geq 3$,

$$\tau_{B(n,2)}^{-1} W_{m,n}^{(2)} \cong \tau_{B(n-1,2)} \cdots \tau_{B(2,2)} W_{m-n,2}^{(2)}$$

and dually

$$\tau_{B(n,2)} M_{m,n}^{(2)} \cong D \tau_{B(n-1,2)} \cdots \tau_{B(2,2)} W_{m-n,2}^{(2)}.$$  

In particular, we have

$$\tau_{B(n,2)} M_{n+1,n}^{(2)} \cong S(1).$$

An alternative proof for the fact that generalized $W$-modules determine $ZA_{\infty}$-components in $\Gamma(n, 2), n \geq 3$, can be found in [13] 5.7.

Suppose now that for general $r \geq 2$, we have a $ZA_{\infty}$-component $C$ in $\Gamma(n - 1, r)$ with $EKP(n - 1, r) \cap C \neq \emptyset$ which is not completely contained in the category $EKP(n - 1, r)$. It is not known whether, in general, there exist $ZA_{\infty}$-components that entirely consist of modules with the equal kernels property. At the level of the Kronecker quiver, however, this cannot happen [12 3.3]. By Theorem 1.3 the component $C$ contains an equal kernels cone $(M_C \rightarrow) = C \cap EKP(n,r)$ consisting of a distinct quasi-simple module $M = M_C \in EKP(n,r)$ and all its successors:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{}
\end{figure}

**Proposition 4.2.** Let $C$ be a component of $\Gamma(n - 1, r)$ as above and let

$$0 \rightarrow \tau N \rightarrow E \rightarrow N \rightarrow 0$$

be an Auslander-Reiten sequence in the subcone $(\tau^{-1} M_C \rightarrow)$ of the equal kernels cone $(M_C \rightarrow)$. Then

$$0 \rightarrow \tau_{B(n-1,r)} N \rightarrow \tau_{B(n-1,r)} E \rightarrow \tau_{B(n-1,r)} N \rightarrow 0$$

is an Auslander-Reiten sequence in $\Gamma(n,r)$. 
Proof. In order to determine the lift of (3) to mod $B(n, r)$, we need to compute

$$\text{Hom}_{B(n-1, r)}(M_{n, n-1}^{(r)}, \tau N)$$

in view of Lemma 2.2. We have $\tau^2 N \in \text{EKP}(n-1, r)$, since $\tau N \in (\tau^{-1} M_n \to)$. For $r > 2$, we have $\tau M_{n, n-1}^{(r)} \in \text{EIP}(n-1, r)$ according to 12.11, whereas in view of (2) we have $\tau M_{n, n-1}^{(2)} \cong S(1) \in \text{mod } B(n-1, 2)$. In either case, $\tau M_{n, n-1}^{(r)}$ does not contain a non-trivial factor module with the equal kernels property, and in view of (2), we obtain

$$0 = \text{Hom}_{B(n-1, r)}(\tau M_{n, n-1}^{(r)}, \tau^2 N).$$

The Auslander-Reiten formula yields an isomorphism of vector spaces

$$(4) \quad 0 = \text{Hom}_{B(n-1, r)}(\tau M_{n, n-1}^{(r)}, \tau^2 N) \cong \text{Hom}_{B(n-1, r)}(M_{n, n-1}^{(r)}, \tau N).$$

The module $M_{n, n-1}^{(r)}$ is indecomposable, non-projective and generated by $e_0 \cdot M_{n, n-1}^{(r)}$, while $e_0 \cdot P(i) = 0$ for $0 < i \leq n - 1$ and furthermore $\dim_k (e_0 \cdot P(0)) = 1$. Hence we have $\text{Hom}_{B(n-1, r)}(M_{n, n-1}^{(r)}, P(i)) = 0$ for all $0 \leq i \leq n - 1$. In view of (4), we thus obtain $\text{Hom}_{B(n-1, r)}(M_{n, n-1}^{(r)}, \tau N) = 0$, and due to Lemma 2.2, the sequence (3) lifts to the Auslander-Reiten sequence

$$0 \to \left( \frac{\tau N}{0} \right)_0 \to \left( \frac{E}{0} \right)_0 \to \left( \frac{N}{0} \right)_0 \to 0$$

in mod $B(n, r)$. \hfill \Box

Note that Proposition 3.1 implies that the component $\hat{D}_n$ of $\Gamma(n-1, r)$ that contains the module $M_{n, n-1}^{(r)}$ gives rise to a component $D_n$ of $\Gamma(n, r)$ containing the projective module $P(0) \cong M_{n, n}^{(r)} \in \text{mod } B(n, r)$. We now prove our main result:

**Theorem 4.3.** If $(n, r) \neq (2, 2)$, then the module $W_{m, n}^{(r)}$ is quasi-simple in a $ZA_\infty$-component $C_m$ of $\Gamma(n, r)$ which contains two non-empty disjoint cones $\text{EIP}(n, r) \cap C_m$ and $\text{EKP}(n, r) \cap C_m$. Moreover, all modules in $C_m$ have constant Jordan type while $W(C_m) = 1$ for $r = 2$ and $W(C_m) = 0$ in case $r > 2$.

**Proof.** We consider the dual component $D_m = D C_m$ containing the module $D W_{m, n}^{(r)} = M_{m, n}^{(r)}$. In view of the duality between $\text{EIP}(n, r)$ and $\text{EKP}(n, r)$, the compatibility between $\tau$ and $D$ and in view of the fact that the notion of constant Jordan type is self-dual 13.15, it suffices to prove the assertion for $D_m$.

As mentioned in Section 1, the statement holds for $n = 2$ and $r > 2$ and due to Proposition 4.1 for $(n, r) = (3, 2)$. Now let $n \geq n(r)$ with $n(2) = 4$ and $n(3) = 3$ if $r > 2$ and assume that the statement is true for $n - 1$. In view of Proposition 3.1 the regular $ZA_\infty$-component $\hat{D}_m$ of $\Gamma(n-1, r)$ containing the quasi-simple module $M_{m, n}^{(r)}$ lifts to the component $D_m$ in which $M_{m, n}^{(r)}$ is quasi-simple and the cone $(\tau^{-1} M_{m, n}^{(r)} \to)$ coincides via $\iota_{B(n-1, r)}$ with the cone $(\tau^{-1} M_{m, n-1}^{(r)} \to) \subseteq \hat{D}_m$ by Proposition 4.2.

Let $\mathcal{D} = \{ \iota^k X \mid X \in (M_{m, n}^{(r)} \to), \ k \in \mathbb{Z} \} \subseteq D_m$. Due to the fact that $m > n$, $\mathcal{D}$ does not contain $P(0)$, and hence $\mathcal{D}$ does not contain any projective vertices since all modules in $(\tau^{-1} M_{m, n-1}^{(r)} \to)$ are regular $B(n-1, r)$-modules. Furthermore, $\mathcal{D}$ is $\tau$-stable as well as $\tau^{-1}$-stable, and for $X \in \mathcal{D}$, we have $Y \in \mathcal{D}$ if there is an
irreducible map $X \to Y$ or $Y \to X$. Since $\mathcal{D}_m$ is connected, we have $\mathcal{D}_m = \mathcal{D}$ while $\mathcal{D}_m$ is of type $Z\mathcal{A}_\infty$ since it is induced by the cone $(\tau^{-1}M_{m,n}^{(r)})$.

According to [12, 2.11], we have $\tau M_{m,n}^{(r)} \in \text{EIP}(n, r)$ in case $r > 2$. In view of (4), we have $\tau_{B(n,2)} M_{m,n}^{(2)} \cong D t_{B(n-1,2)} \cdots t_{B(2,2)} W_{m-n,2}^{(2)}$. This implies that $\tau_{B(n,2)} M_{m,n}^{(2)} \notin \text{EIP}(n, 2)$ due to the fact that $D W_{m-n,2}^{(2)} \in \text{EKP}(2, 2)$. However, since $D W_{m-n,2}^{(2)} \in \text{CJT}(2, 2)$, we obtain that $D t_{B(n-1,2)} \cdots t_{B(2,2)} W_{m-n,2}^{(2)} \in \text{CJT}(n, 2)$. Moreover, we have $e_{n-1} \cdot (\tau_{B(n,2)} M_{m,n}^{(2)}) = 0$, which implies that $\tau_{B(n,2)} M_{m,n}^{(2)} \notin \text{EKP}(n, 2)$ by Remark [12, 2.10]. Since in either case all quasi-simple modules in $\mathcal{D}_m$ are of constant Jordan type, we have $\mathcal{D}_m \subset \text{CJT}(n, r)$ by [13, 3.27]. The foregoing observations yield that $\mathcal{W}(\mathcal{D}_m) = 0$ if $r > 2$ and $\mathcal{W}(\mathcal{D}_m) = 1$ if $r = 2$.

The distribution of equal images and equal kernels modules in $\mathcal{C}_m$ for $r > 2$ is hence as in Figure 1.

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