Lamb shift of Unruh detector levels

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Received 29 December 2005
Published 12 May 2006
Online at stacks.iop.org/CQG/23/3917

Abstract

We argue that the energy levels of an Unruh detector experience an effect similar to the Lamb shift in quantum electrodynamics. As a consequence, the spectrum of energy levels in a curved background is different from that in flat space. As examples, we consider a detector in an expanding universe and in Rindler space, and for the latter case we suggest a new expression for the local virtual energy density seen by an accelerated observer. In the ultraviolet domain, that is when the space between the energy levels is larger than the Hubble rate or the acceleration of the detector, the Lamb shift quantitatively dominates over the thermal response rate.

PACS numbers: 98.80.Cq, 04.62.+v, 98.80.−k

1. Unruh detector as a probe for the quantum vacuum

In order to gain an understanding of the physical effects which occur in curved spacetimes, the Unruh detector [1] is often considered. It is an idealized point-like device, solely defined through its energy levels \( E_m \) by

\[
| m, \tau \rangle = e^{-iH_D \tau} | m \rangle = e^{-iE_m \tau} | m \rangle,
\]

where \( H_D \) is the unperturbed detector Hamiltonian and \( \tau \) denotes the proper time along the detector trajectory.

Interactions with a scalar field \( \phi \) take place via the perturbation Hamiltonian

\[
\delta H = \hat{h}\phi(x),
\]

and we denote the matrix elements of the operator \( \hat{h} \) as \( h_{mn} = \langle m | \hat{h} | n \rangle \). Let \( P_{mn}(\tau) \) be the probability for a transition \( m \to n \) after the proper time \( \tau \) has elapsed, and define...
\( \mathcal{F}_{mn} = P_{mn}/|h_{mn}|^2 \). Then, according to the quantum mechanical rules of time-dependent perturbation theory, the response function for \( \tau \to \infty \) is given by [1, 2]

\[
\frac{d\mathcal{F}(\Delta E)}{d\tau} = \int_{-\infty}^{\infty} d\Delta \tau \ e^{i\Delta E \tau} \langle i | \phi(x(-\Delta \tau/2))\phi(x(\Delta \tau/2)) | i \rangle,
\]

where \( \Delta E = E_{\eta} - E_{\mu} \) and \( | i \rangle \) denotes the state of the \( \phi \)-field.

This response function is often discussed for a detector in de Sitter space [3]. We describe the scalar field \( \phi \) by the Lagrangian

\[
\sqrt{-g} \mathcal{L} = \sqrt{-g} \left( \frac{1}{2} g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{2} \xi R \phi^2 \right),
\]

implying the Euler–Lagrange equation of motion

\[
[V^2 + m^2 + \xi R] \phi(x) = 0,
\]

where \( \sqrt{V^2} \) denotes the covariant derivative, such that \( \sqrt{V^2} = (-g)^{-1/2} \partial_{\mu} g^{\mu\nu} \sqrt{-g} \partial_\nu \). The spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) universe has the metric tensor

\[
g_{\mu\nu} = a^2(\eta) \times \text{diag}(1, -1, -1, -1),
\]

where \( \eta \) is the conformal time. For de Sitter space expanding at the Hubble rate \( H \), the scale factor as a function of conformal time is

\[
a(\eta) = -\frac{1}{H \eta},
\]

where \( \eta \) runs in the interval \( -\infty, 0 \) from past to future infinity.

We substitute \( \phi = a \phi \) and decompose \( \phi \) in modes as

\[
\phi(x) = \int \frac{d^3k}{(2\pi)^3} (e^{ikx}\phi(k, \eta)a(k) + e^{-ikx}\phi^*(k, \eta)a^*(k)).
\]

Here, \( a(k) \) and \( a^*(k) \) denote the annihilation and creation operators for the mode with a comoving momentum \( k \), and they are defined by \( a'(k)|0\rangle = |k\rangle \), \( a(k)|k'\rangle = (2\pi)^3 \delta^3(k - k')|0\rangle \), where \( |0\rangle \) denotes the vacuum state and \( |k\rangle \) the one-particle state with momentum \( k \). The physical momentum is related to the conformal momentum by \( k_{\text{phys}} = k/a(\eta) \). From the Klein–Gordon equation (5), one can derive for the mode functions \( \phi(k, \eta) \) the following equation:

\[
\left( \partial_\eta^2 + (k^2 + a^2m^2) + (6\xi - 1) \frac{a''}{a} \right) \phi(k, \eta) = 0,
\]

and spatial homogeneity implies \( \phi(k) = \phi(k) \) \( (k \equiv |k|) \). We define the vacuum by the choice of solutions such that \( \phi(k, \eta) \) reduces to a plane wave of purely negative frequency at infinitely early times. The field \( \phi \) obeys the canonical commutation relation

\[
[\phi(x, \eta), \partial_\eta \phi(x', \eta)] = i \delta^3(x - x'),
\]

which implies the normalization of the mode functions by the Wronskian

\[
\phi(k, \eta)\phi^*(k, \eta) - \phi'(k, \eta)\phi^*(k, \eta) = i,
\]

and for the creation and annihilation operators the commutator

\[
[a_k, a^*_k] = (2\pi)^3 \delta^{(3)}(k - k').
\]

The explicit solution of the mode equation (9) for the minimally coupled massless case, \( m = 0 \) and \( \xi = 0 \), in de Sitter space with the scale factor (7) is

\[
\phi(k, \eta) = \frac{1}{\sqrt{2k}} \left( 1 - \frac{i}{k\eta} \right) e^{-ik\eta},
\]
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from which, when deriving the response function, one finds [2, 4]

\[ \frac{dF(\Delta E)}{d\tau} = \frac{\Delta E}{2\pi} \left( 1 + \frac{H^2}{\Delta E^2} \right) \frac{1}{e^{(2\pi H/\Delta E)} - 1} \quad \text{for} \quad \Delta E \neq 0. \]  

(14)

This result indicates an exponentially falling spectrum of scalar quanta in de Sitter space, often interpreted as the presence of thermal radiation.

On the other hand, from the unrenormalized stress–energy tensor, we obtain the energy density [5]

\[ \varrho = \langle 0| T_{\phi}^\mu(x) |0 \rangle = \frac{1}{a^4} \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{2k^2} \right), \]

suggesting that besides the cosmological constant contribution corresponding to the first term of the integrand, there additionally is a particle spectrum \( \propto 1/k^2 \) produced by the de Sitter background, which is apparently not captured by the response rate (14). A similar power-law behaviour of the stress–energy tensor holds more generally also for massive or nonminimally coupled scalars as well as for adiabatically expanding spacetimes, such as matter and radiation universes [5–7]. Since we are dealing with effects of the quantum vacuum in curved space, in order to understand this discrepancy, it may be useful to recall what possibly related phenomena there are in flat space.

The first experimental result to find an explanation by vacuum fluctuations was the Lamb shift. According to relativistic quantum mechanics, the energy levels \( 2S_{1/2} \) and \( 2P_{1/2} \) of hydrogen are degenerate, despite a tiny correction due to the hyperfine structure, insufficient however to account for the actual shift, which was observed by Lamb and Retherford [8] in 1947. Also in 1947, Bethe has shown in a groundbreaking paper [9] that the split is due to interactions of the electron with the vacuum fluctuations of the electromagnetic field, and a finite answer is obtained when subtracting the self-energy corrections for a free electron, which are infinite, from those of an electron in the Coulomb potential. This is probably the most illustrative, simple and beautiful example for the effects of the quantum vacuum, detected by the hydrogen atom as a probe.

Just like an atom, Unruh’s detector is a system with discrete energy levels, which by Bethe’s argument should also acquire a Lamb shift correction from the fluctuations of the scalar field \( \phi \). Since quantum field theory in curved space deals with the distortions of the quantum vacuum induced by the gravitational background, it is perhaps more natural to expect that these become manifest in the Lamb shift rather than in the detection rate of scalar quanta. For the case of an accelerated detector coupled to a massless scalar field in four spacetime dimensions, this observation has first been made by Audretsch et al [10–12].

We therefore calculate in the following the self-energy corrections to the energy levels of an Unruh detector in a spacetime \( X \). At second order in perturbation theory, these are given by [9]

\[ \delta E_{nX} = \sum_{m \neq n} \int \frac{d^3k}{(2\pi)^3} \frac{|f(\frac{d\psi}{d\chi})^\mu(m|\hat{h}a^\dagger(k, \eta)\psi(k, \eta)|0, n)|^2}{E_n - E_m - \Omega(k)} \]

\[ = \sum_{m \neq n} \int \frac{d^3k}{(2\pi)^3} \frac{|h^\mu_{mn}|^2}{E_n - E_m - \Omega(k)} \]  

(16)

where \( \Omega(k) \) is the canonical Hamiltonian energy (19) of a \( \phi \) quantum at momentum \( k \). This shift of energy levels has in flat space a square divergence in the ultraviolet. In Minkowski space, one can determine the values of the detector’s energy levels \( E_n \), which are finite, by observation and the infinite shift \( \delta E_{nX} \) is already taken into account for this measurement.
In a curved spacetime $C$, however, the value for the radiative correction differs from the Minkowski space answer; the finite quantity
\[ \delta E_n = \delta E_{nC} - \delta E_{nM} \]  
(17)
can therefore be observed by comparing the spectra of energy levels in flat and in curved background.

To keep notation simple, we drop the summation over energy levels in the following, corresponding to a two-level detector with spacing $\Delta E \equiv E_n - E_m$ and $|h_{mn}|^2 = h^2$. The sum can simply be reinserted into all subsequent results.

2. Lamb shift in the expanding universe

The Hamiltonian for the scalar field in the FLRW background is given by [13]
\[ H(\eta) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left\{ \Omega(k, \eta)(a(k)a\dagger(k) + a\dagger(k)a(k)) + (\Lambda(k, \eta)a(k)a(-k) + \text{h.c.}) \right\}, \]
(18)
where
\[ \Omega(k, \eta) = |\varphi'(k, \eta) - (a'/a)\varphi(k, \eta)|^2 + \bar{\omega}^2(k, \eta)|\varphi(k, \eta)|^2, \]  
(19)
\[ \Lambda(k, \eta) = \left( \varphi'(k, \eta) - \frac{a'}{a}\varphi(k, \eta) \right)^2 + \bar{\omega}^2(k, \eta)\varphi^2(k, \eta), \]  
(20)
and we have defined $\omega^2(k, \eta) = k^2 + a^2m^2$ and $\bar{\omega}^2(k, \eta) = \omega^2(k, \eta) + 6\xi\frac{a^2}{\eta}$.  

The quantity $\Omega(k)$ therefore is the vacuum expectation value for the Hamiltonian energy of the $\phi$-mode at momentum $k$. Note that this canonical energy density equals the covariant energy density as obtained from the stress–energy tensor [2, 6, 7] only in the minimally coupled case $\xi = 0$.

We now make use of these expressions to calculate the shift of detector levels.

2.1. Massless de Sitter case

As a first example, let us consider a minimally coupled massless scalar in de Sitter space, because for this situation exact solutions are available and we do not need to resort to approximation by adiabatic expansion. First, we calculate the shift in Minkowski space,
\[ \delta E^m_{n=0} = \int \frac{d^3k}{(2\pi)^3} \frac{h^2}{2k} \Delta E - k = \frac{h^2}{4\pi^2} \int_0^\infty dk \frac{k}{\Delta E - k} \]
\[ = \frac{h^2}{4\pi^2} \int_0^\infty dk \left\{ -1 + \frac{\Delta E}{\Delta E - k} \right\} = \frac{h^2}{4\pi^2} [-k - \Delta E \log(\Delta E - k)]_0^\infty, \]
(21)
which is divergent and to be subtracted.

The de Sitter mode functions are given by equation (13), such that we find for their squared amplitude
\[ |\varphi(k, \eta)|^2 = \frac{1}{2k} + \frac{1}{2k^3\eta^2}, \]  
(22)
and for the mode energy (19)
\[ \Omega(k, \eta) = k + \frac{1}{2k\eta^2}. \]  
(23)
Let us fix $a = 1$, such that conformal and physical momenta coincide and we have by equation (7) $\eta = -H^{-1}$. The use of equation (16) gives us the unrenormalized Lamb shift in de Sitter space,

$$\delta E_{m=0}^{dS} = \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{2k} + \frac{H^2}{2k^3} \right) \frac{h^2}{\Delta E - (k + \frac{H}{c})}$$

$$= \frac{h^2}{4\pi^2} \int_0^\infty dk \left\{ -1 + \frac{\Delta E}{\Delta E - (k + \frac{H}{c})} \right\}$$

$$= \frac{h^2}{4\pi^2} \left\{ [-k]^0_\infty - \Delta E \int_\Delta E/2^\infty \frac{l + \Delta E/2}{l^2 + H^2 - \Delta E^2/4} \right\}$$

$$= \frac{\Delta E}{2} \log \left( \frac{k + \Delta E/2}{\Delta E^2/4 - H^2 - 1} \right)_0 \quad \text{(24)}$$

We evaluate the boundary terms and subtract the flat space result to find for the finite observable as well as for the scalar field $\phi a(\eta)$, allowing for a general expanding FLRW background given by the scale factor $2.2. The general case

$$\delta E = \delta E_{dS} - \delta E_m$$

$$= \frac{h^2}{4\pi^2} \left\{ \Delta E \log \left| \frac{H}{\Delta E} \right| - \frac{\Delta E^2}{4\sqrt{\Delta E^2/4 - H^2}} \log \left| \frac{\Delta E/2 - \sqrt{\Delta E^2/4 - H^2}}{\Delta E/2 + \sqrt{\Delta E^2/4 - H^2}} \right| \right\} \quad \text{(25)}$$

This expression condenses considerably when expanded in $H/\Delta E$:

$$\delta E = \frac{h^2}{4\pi^2} \Delta E \left( 1 - 2 \log \left| \frac{H}{\Delta E} \right| + O \left( \frac{H}{\Delta E} \right) \right) \quad \text{(26)}$$

and when we reintroduce the sum to treat the case of more than two energy levels, it reads

$$\delta E = \sum_{m \neq n} \frac{|h_m|^2}{4\pi^2} \frac{H^2}{E_n - E_m} \left( -1 - 2 \log \left| \frac{H}{E_n - E_m} \right| + O \left( \frac{H}{E_n - E_m} \right) \right) \quad \text{(27)}$$

When compared to the response function in de Sitter (14), which decays exponentially in $\Delta E$, this power-law behaviour becomes more important in the ultraviolet. Since the mode energy $\Omega$ is contributing, we can consider Lamb shift as a way to observe the energy density (15) produced by the de Sitter background.

2.2. The general case

Now, we allow for a general expanding FLRW background given by the scale factor $a(\eta)$, as well as for the scalar field $\phi$, a curvature coupling $\xi$ and a constant mass $m$. Adiabatic expansion gives up to second order in $d/d\eta$ [5] (here we keep the scale factor $a$ explicitly)

$$|\phi|^2 = \frac{1}{2\omega} - \frac{1}{4\omega^3} \left\{ (6\xi - 1) \frac{a'' + 1}{a} - \frac{m^2(aa'' + a'^2)}{2 \omega^2} + \frac{5 m^4 a^2 a'^2}{4 \omega^4} \right\} \quad \text{(28)}$$

and

$$\Omega = \omega + \frac{1}{2\omega} \left( \frac{a'^2}{a^2} + 6\xi \frac{a''}{a} \right) + \frac{1}{2} \frac{a'^2 a^2 m^2}{\omega^4} + \frac{1}{8} \frac{a'^2 a^2 m^4}{\omega^6} \quad \text{(29)}$$
\[
\Lambda = \left\{ \frac{1}{2\omega} \left( \frac{a^2 + a''}{a} \right) + \frac{1}{4} \frac{a'' + 3 \frac{a^2}{a^2} - \frac{a^2}{2a^2} \omega^2 - \frac{a^4}{2a^2} \omega^5 + \frac{a^6}{a} \left( 1 + \frac{a^2}{2} \omega^2 \right) }\right\} \\
\times \exp \left( -2i \int_0^\infty W(\eta') \, d\eta' \right),
\]
where \( \varphi(\eta) = (2W(\eta))^{-1/2} \exp \left( -i \int_0^\infty W(\eta') \, d\eta' \right) \). We therefore define
\[
\Delta_{\Lambda'} = \frac{1}{\omega} \left\{ \frac{1 - 6\xi}{2} - \frac{1}{4} \frac{m^2(a'' + a^2)}{\omega^2} - \frac{5}{8} \frac{m^4 a^2 a'^2}{\omega^5} \right\},
\]
\[
\Delta_\Omega = \frac{1}{2\omega} \left( \frac{a'' + 6\xi}{a^2} \right) + \frac{1}{2} \frac{a^2}{a^2} \omega^2 - \frac{1}{8} \frac{a^2 a^2 m^2}{\omega^3} + \frac{1}{8} \frac{a^2 a^4 m^4}{\omega^5},
\]
such that \( |\psi|^2 = 1/(2\omega) + \Delta_{\Lambda'}/(2\omega^2) \) and \( \Omega = \omega + \Delta_\Omega \).

The Lamb shift in FLRW universe with respect to flat space is then
\[
\delta E = \delta E_{\text{FLRW}} - \delta E_{\text{M}} = h^2 \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{2}\omega \frac{\Delta_{\Lambda'}}{\Delta E - \omega - \Delta_\Omega} - \frac{1}{2}\omega \frac{\Delta E - \omega}{\Delta E - \omega} \right\}
\approx \frac{h^2}{4\pi^3} \int_0^\infty \frac{dk}{\omega^2} \left\{ \frac{\Delta_{\Lambda'}}{\omega \Delta E - \omega} \right\} + \frac{h^2}{8\pi^2 \Delta E} \left\{ -\frac{5}{6} \frac{a^2}{a^2} + (1 - 6\xi) \log \left( 2\Delta E - \omega \right) \right\} + O \left( \frac{m a''}{\Delta E^2}, \frac{ma'^2}{\Delta E^2} \right),
\]
where the relevant integrals are given in the appendix. For \( m \to 0 \), there occurs a logarithmic infrared divergence. This is however an artefact of adiabatic expansion, which breaks down in this limit. Note, in particular, that the exact expression (25) for the massless de Sitter case is infrared finite.

Let us comment in more detail on the sensitivity of the detector to the energy density produced by the expanding background. From expression (16) we immediately see the contribution of the mode energies \( \Omega(\mathbf{k}, \eta) \) to the Lamb shift, but additionally there also enters the mode amplitude through \( |\varphi(\mathbf{k}, \eta)|^2 \), which is also influenced by the background. For this quantity, the relation
\[
|\varphi(\mathbf{k}, \eta)|^2 = \frac{1}{2} \left( \Omega(\mathbf{k}, \eta) - \text{Re} \left( \Lambda(\mathbf{k}, \eta) \exp \left( 2i \int_0^\infty W(\eta') \, d\eta' \right) \right) \right)^{-1}
\]
generally holds. We hence found that although we cannot write the result for the Lamb shift solely in terms of \( \Omega(\mathbf{k}, \eta) \) or \( T_0^0 \), we can express it in terms of contributions to the canonical Hamiltonian (18).

### 3. Lamb shift in Rindler space

It was suggested by Unruh [1] that an accelerated observer should perceive particles even in the vacuum, which is due to the fact that quantization in a coordinate system suitable for the observer, referred to as Rindler space, is inequivalent to quantization in Minkowski space. Therefore, accelerated observer vacuum and inertial Minkowski vacuum do not coincide [14]. The quantum state in the accelerated system that is equivalent to the inertial state in Minkowski vacuum can be constructed through a Bogolyubov transformation, which corresponds to mode mixing and is known as the Unruh effect [1].

Just as in de Sitter space, the response function of Unruh’s detector falls off exponentially [1, 15], therefore resembling to a thermal spectrum. As we have observed for expanding universes, this effect is quantitatively dominated by the Lamb shift of energy levels. In the following, we shall demonstrate that the same holds also true for an accelerated detector.
3.1. Scalar field in Rindler coordinates

In flat two-dimensional space with the line element
\[ ds^2 \equiv g_{\mu\nu} \, dx^\mu \, dx^\nu = -dt^2 + dx^2, \quad g_{\mu\nu} = \text{diag}(-1, 1), \] (35)
we consider an observer of mass \( m_O \), who is constantly accelerated by the force \( f \), for example an ion in a homogeneous electric field. Let us determine his trajectory \( y(\tau) = (t(\tau), x(\tau))^T \), where \( \tau \) is his proper time, defined by \( d\tau^2 = -ds^2 \).

The Minkowski vector describing the force in the inertial system where the observer is instantaneously at rest is
\[ \vec{f} = \begin{pmatrix} 0 \\ f \end{pmatrix}. \] (36)
When we see the observer moving at the instantaneous velocity \( v \), the force vector \( f \) in our coordinate system is obtained from
\[ f = \Lambda(-v) \vec{f} = f \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} \\ \frac{v}{\sqrt{1-v^2}} \end{pmatrix} = f \begin{pmatrix} \sinh \psi \\ \cosh \psi \end{pmatrix}, \] (37)
where \( \Lambda(-v) \) denotes the Lorentz boost transformation, \( \psi \) is the rapidity parameter, \( \tanh \psi = v \), and the velocity vector is of the standard form
\[ u = \frac{dy}{d\tau} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} \\ \frac{v}{\sqrt{1-v^2}} \end{pmatrix} = \begin{pmatrix} \cosh \psi \\ \sinh \psi \end{pmatrix}. \] (38)

With \( p \) being his momentum, the observer then follows a trajectory which is solution to the relativistic equation of motion
\[ \frac{dp}{d\tau} = m_O \frac{d^2y}{d\tau^2} = f. \] (39)
A solution for \( dy/d\tau \) is easily found when setting \( \psi = \alpha \tau \) and \( \alpha = f/m_O \), and we can interpret the parameter \( \alpha \) as a constant proper acceleration
\[ \alpha = \left[ \left( \frac{d^2y}{d\tau^2} \right)^2 \right]^{\frac{1}{2}} = \left[ -\left( \frac{d^2t}{d\tau^2} \right)^2 + \left( \frac{d^2x}{d\tau^2} \right)^2 \right]^{\frac{1}{2}}. \] (40)
A special \( y(\tau) \) is given by
\[ y(\tau) = \begin{pmatrix} \alpha^{-1} \sinh \alpha \tau \\ \alpha^{-1} \cosh \alpha \tau \end{pmatrix}, \] (41)
implying the trajectory
\[ x(t) = (t^2 + \alpha^{-2})^{1/2}, \] (42)
on which we shall consider Unruh’s detector in the following.

Since we describe the time evolution of the detector in terms of its proper time \( \tau \), we also use \( \tau \) as the time variable for canonical quantization of the scalar field, which then manifestly separates into modes which the observer perceives as of positive and of negative frequency, respectively. Let us therefore transform the system to the Rindler coordinates as [16]
\[ t = \alpha^{-1} e^\xi \sinh \alpha \tau, \quad x = \alpha^{-1} e^\xi \cosh \alpha \tau, \] (43)
such that the metric becomes
\[ ds^2 = -e^{2\xi} \, dt^2 + \alpha^{-2} \, e^{2\xi} \, d\xi^2, \] (44)
where the detector’s site is at $\xi = 0$. The dependence of the metric (44) on $\xi$ indicates that Minkowski space appears inhomogeneous to an accelerated observer.

According to the Lagrangian

$$\sqrt{-g} L = \sqrt{-g} \left(-\frac{1}{2} g_{\mu\nu} \partial^\mu \varphi \partial^\nu \varphi - \frac{1}{2} m^2 \varphi^2\right),$$

the Klein–Gordon equation for a scalar field with mass $m$ is

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - m^2\right) \varphi(x, t) = 0.$$  (46)

In the Rindler coordinate system, this transforms to

$$\left(-\frac{\partial^2}{\partial \tau^2} + \alpha^2 \frac{\partial^2}{\partial \xi^2} - e^{2\xi} m^2\right) \varphi(\xi, \tau) = 0.$$  (47)

We shall take the scalar field to be in the Rindler vacuum $|0\rangle$, with the field operator

$$\hat{\varphi}(\xi, \tau) = \int_0^\infty \frac{d\lambda}{2\pi} \left\{c^\dagger_{\lambda} \varphi^\dagger_{\lambda}(\xi, \tau) + c_{\lambda} \varphi_{\lambda}(\xi, \tau)\right\},$$

where the creation and annihilation operators act as $c^\dagger_{\lambda}|0\rangle = |\lambda\rangle$ and $c_{\lambda}|\lambda\rangle = 2\pi \delta(\lambda - \lambda')|0\rangle$.

Making use of the Jacobian of the coordinate transformation (43), $J = \alpha^{-1} \exp(2\xi)$, and the identity $\int dt \int dx H^\varphi(x, t) = \int d\tau \int d\xi H^\varphi(\xi, \tau)$, one arrives at the following Hamiltonian:

$$H^\varphi = \int_{-\infty}^{\infty} d\xi \mathcal{H}^\varphi, \quad \mathcal{H}^\varphi = \frac{1}{2\alpha} \left\{\left(\frac{\partial \hat{\varphi}}{\partial \tau}\right)^2 + \alpha^2 \left(\frac{\partial \hat{\varphi}}{\partial \xi}\right)^2 + e^{2\xi} m^2 \hat{\varphi}^2\right\}.$$  (49)

The normalizable negative-frequency mode functions in (48) can be expressed in terms of Hankel functions as (cf [14, 17, 18])

$$\varphi_{\lambda}(\xi, \tau) = e^{-i\lambda \tau} e^{-\frac{\pi}{2\alpha} \frac{\lambda}{\alpha} \sinh \left(\frac{\pi \lambda}{\alpha}\right)} iH_{\frac{\lambda}{i\alpha}} \left(\frac{m}{\alpha} e^{\xi}\right).$$

In order to fix the normalization, we define the scalar product as implied by Green’s theorem [19]

$$(\varphi_{\lambda}, \varphi'_{\lambda}) = i \int_{\Sigma} d^{D} \hat{S} \sqrt{-g} \left(\varphi_{\lambda}^* \partial_n \varphi_{\lambda'}\right),$$

where $\Sigma$ is a spatial hypersurface, for example the surface defined by $\tau = 0$, which we consider in the following. When introducing

$$\rho = \alpha^{-1} e^{\xi}$$

and an infinitesimal regulator $\varepsilon$, the scalar product turns out to be the desired $\delta$-function representation

$$(\varphi_{\lambda}, \varphi'_{\lambda}) = -i \int_{-\infty}^{\infty} d\xi \frac{1}{\alpha} (\varphi_{\lambda}^* \partial_\xi \varphi_{\lambda'})$$

$$= N_{\lambda} N_{\lambda'} \int_0^{\infty} \frac{d\varrho}{\varrho \alpha^{1+\varrho}} \left|H^{(1)}_{\frac{\lambda}{i\alpha}}(i\varrho)\right|^2 \left|H^{(1)}_{\frac{\lambda'}{i\alpha}}(i\varrho)\right|^2$$

$$= N_{\lambda} N_{\lambda'} \frac{1}{8\alpha} \left|\Gamma(\varepsilon + i\frac{\lambda + \lambda'}{2\alpha})\right|^2 \left|\Gamma(\varepsilon + i\frac{\lambda - \lambda'}{2\alpha})\right|^2$$

$$\approx N_{\lambda} N_{\lambda'} \frac{1}{2\alpha} \left|\Gamma \left(\frac{\lambda + \lambda'}{2\alpha}\right)\right|^2 \left|\frac{\varepsilon}{\varepsilon^2 + \left(\frac{\lambda - \lambda'}{\alpha}\right)^2}\right| = 2\pi \delta(\lambda - \lambda').$$  (53)
For solving the integral, we made use of the formula GR 6.576.4 (we denote the equalities taken from Gradshteyn and Ryzhik [20] by GR).

It is also of interest to compute the Bogolyubov coefficients for the matching to the Minkowski modes

\[ \psi_k(x, t) = \frac{1}{\sqrt{2\omega}} e^{-i\omega t + ikx}, \]

(54)

where \( \omega = \sqrt{k^2 + m^2} \). When we use the formula GR 6.621.3 for the occurring integrals and GR 9.121.19, GR 9.121.21 in order to express hypergeometric functions in terms of elementary functions, we obtain (cf [14])

\[ \alpha_{\lambda, k} = \braket{\psi_k | \phi_\lambda} = \frac{(2m)^{\frac{1}{2}}}{\sqrt{\omega \alpha}} (1 - e^{-2\pi \xi})^{-\frac{1}{2}} \left( \frac{\omega + k}{m} \right)^{\frac{1}{2}}, \]

(55)

and likewise

\[ \beta_{\lambda, k} = \braket{\psi_k^* | \phi_\lambda} = (2m)^{\frac{1}{2}} \frac{\pi}{\sqrt{\omega \alpha}} (e^{2\pi \xi} - 1)^{-\frac{1}{2}} \left( \frac{\omega - k}{m} \right)^{-\frac{1}{2}}. \]

(56)

The \( k \)-dependent phase is missed when matching instead of along \( \tau = t = 0 \) along \( U = t - x \) and \( V = t + x \) [1], because the Rindler modes reduce to massless modes along these lightlike surfaces and Green’s theorem is strictly speaking only applicable along proper spacelike surfaces.

Note that the mode mixing is exponentially suppressed, since \( \beta_{\lambda, k} \propto e^{-\pi \lambda / \alpha} \) for large \( \lambda \).

Assuming \( m \gg \alpha \), as required by the expansions we shall use in the following, \( \beta_{\lambda, k} \ll 1 \), and one does not need to account for the mode mixing. In order not to distract from the main line of argument, we therefore neglect it here.

The spatial parts of the Rindler modes (50) are real-valued functions of \( \xi \), because the operator (47) corresponds to a quantum mechanical particle which is reflected by a potential \( e^{2\alpha \xi} \) [14]. For \( \xi \to -\infty \), the solutions therefore reduce to standing plane waves. This picture also explains the exponential decay of the mode functions for \( m \approx \lambda / \alpha \), which is by GR 8.451.3

\[ H_{\frac{1}{2}}^{(1)}(\im \varpi) \sim -i \sqrt{-\frac{2}{\pi m \varpi}} \exp \left( -m \varpi + \frac{\pi \lambda}{2 \alpha} \right). \]

(57)

As a consequence, the condition \( \lambda \gg \alpha \) effectively holds whenever \( m \gg \alpha \), due to the exponential vanishing of \( \varphi \) at the detector’s site \( \varpi = \alpha^{-1} \) when \( \lambda \ll m \). For calculational purposes and in order to keep notation simple, it proves useful to complexify the expansion. First, note the identity [21] (we introduce an infinitesimal real part \( \varepsilon > 0 \) of the argument of the Bessel function for later use)

\[ J_{\frac{1}{2} \pm \varepsilon} \left( i \frac{m}{\alpha} e^\frac{\varepsilon}{\alpha} - \varepsilon \right) = \left[ J_{-\frac{1}{2} \pm \varepsilon} \left( i \frac{m}{\alpha} e^{-i \varepsilon} + e^{-i \varepsilon} \right) \right]^*, \]

\[ = e^{-\frac{i m}{\alpha} e^\frac{\varepsilon}{\alpha}} \left[ J_{-\frac{1}{2} \pm \varepsilon} \left( i \frac{m}{\alpha} e^{\frac{\varepsilon}{\alpha}} \right) \right]^*. \]

(58)

which, together with the definitions GR 8.403.1 and GR 8.405.1, allows us to express

\[ H_{\frac{1}{2}}^{(1)} \left( i \frac{m}{\alpha} e^\varepsilon \right) = \frac{4}{1 - e^{-2 \pi \xi}} \text{Im} \left[ J_{\frac{1}{2}} \left( i \frac{m}{\alpha} e^\varepsilon \right) \right]. \]

(59)
since the argument fulfills arg(c)| < π/2 of the approximation by tangents. Indeed, when the parameter λ becomes large compared to the acceleration α, the modes (50), (61) asymptotically reduce to plane waves. We provide here a systematic expansion of the mode functions (61) in Rindler space in the ultraviolet domain, that is where λ ≫ α, m. All terms involving powers up to α² and m⁴ are displayed. Since |λ|/α ≫ 1, we need an asymptotic expansion of Bessel functions of large order, which is given by the approximation by tangents [21]:

\[
J_{\pm \lambda} \left( \frac{m}{\alpha} e^{\frac{\pi}{2}} + \epsilon \right) \sim \frac{\exp \left( \frac{\lambda}{2} \left( \tanh \beta - \beta \right) - \frac{1}{4} \pi i \right)}{(2\pi \frac{\lambda}{2} \tanh \beta)^{1/2}} \left\{ 1 - \frac{\alpha}{\lambda} \left( \frac{1}{8} \coth \beta - \frac{5}{24} \coth^3 \beta \right) \right. \\
\left. - \frac{\alpha^2}{\lambda^2} \left( \frac{9}{128} \coth^2 \beta - \frac{231}{576} \coth^4 \beta + \frac{1155}{3456} \coth^6 \beta \right) + O \left( \frac{\alpha^3}{\lambda^3} \right) \right\},
\]

for \( \lambda > 0 \),

\[\text{(63)}\]

The expansion (63) corresponds to the region 2 of figure 22 in [21].

When \( \lambda < 0 \), we make use of the identity (58) to bring the argument of the Bessel function in (61) into the region of validity (\( \text{arg}(c) \) < π/2) of the approximation by tangents. Indeed, since the argument fulfills \( \text{arg}(c(m/\alpha) e^{\frac{\pi}{2}} + \epsilon) < \pi/2 \), we can use the approximation by tangents (cosh(β) = \( \nu/\zeta \) lies again in the region 2 of figure 22 in [21])

\[
J_{\pm \lambda} \left( \frac{m}{\alpha} e^{\frac{\pi}{2}} - \epsilon \right) \sim e^{-\pi \lambda/\alpha} \exp \left( \frac{\lambda}{2} \left( \tanh \beta - \beta \right) + \frac{1}{4} \pi i \right) \left\{ 1 - \frac{\alpha}{\lambda} \left( \frac{1}{8} \coth \beta - \frac{5}{24} \coth^3 \beta \right) \right. \\
\left. - \frac{\alpha^2}{\lambda^2} \left( \frac{9}{128} \coth^2 \beta - \frac{231}{576} \coth^4 \beta + \frac{1155}{3456} \coth^6 \beta \right) + O \left( \frac{\alpha^3}{\lambda^3} \right) \right\},
\]

for \( \lambda < 0 \).

\[\text{(64)}\]

The following expressions, which are valid for both \( \lambda < 0 \) and \( \lambda > 0 \), completely specify β,

\[
\coth \beta = \left( 1 - \frac{\alpha^2}{\lambda^2} e^{2\epsilon} \right)^{-\frac{1}{2}},
\]

\[\text{(65)}\]

\[
\cosh \beta = \frac{\left| \lambda \right|}{m} e^{\epsilon} - i \epsilon \text{sign}(\lambda).
\]

\[\text{(66)}\]
We can now write a general approximation by tangents for the Rindler modes (50) (valid for any $|\lambda| \gg \alpha, m$):

$$
\hat{\varphi}_\lambda \sim \frac{\exp(-i|\lambda|\tau + \frac{\pi}{2} \frac{|\lambda|}{\alpha})}{\sqrt{2|\lambda|}} \left| \Gamma\left(1 + \frac{\lambda}{\alpha}\right) \right| \frac{\exp(i\frac{\pi}{2} \left(\tanh\beta - \beta - \frac{\alpha}{\lambda} \cdot \text{sign}(\lambda)\right))}{[2\pi(|\lambda|/\alpha) \tanh \beta]^{1/2}} \times \left\{ 1 - i\frac{\alpha}{\lambda} \left(1 + \frac{\pi}{8} \coth \beta - \frac{5}{24} \coth^3 \beta \right) - \frac{\alpha^2}{\lambda^2} \left(9 \frac{\alpha}{128} \coth^5 \beta \right) \right. \\
- \frac{231}{576} \coth^4 \beta + \frac{1155}{3456} \coth^6 \beta \right\} \cdot (68)
$$

Upon expanding $\varphi_\lambda$ in powers of $\alpha/\lambda, (m/\lambda)^2$ and $\xi$, we get (up to corrections of order $O((\alpha/\lambda)^3, \xi^2, (m/\lambda)^6$)

$$
\hat{\varphi}_\lambda \simeq e^{-|\lambda|\tau} \frac{\alpha^{1/2}}{(4\pi)^{1/2}|\lambda|} \exp \left(\frac{\pi |\lambda|}{2 \alpha} - i\frac{\pi}{4} \text{sign}(\lambda)\right) |\Gamma\left(1 + \frac{\lambda}{\alpha}\right)| \times \exp \left[\frac{\lambda}{\alpha}\left(1 - \log\left(\frac{2|\lambda|}{m}\right) - \frac{1}{2} \frac{m^2}{\lambda^2} - \frac{1}{8} \frac{m^4}{4 \lambda^4}\right)\right] \exp \left[i\frac{\lambda}{\alpha}\left(1 - \frac{m^2}{2\lambda^2} - \frac{m^4}{8\lambda^4}\right)\right] \times \left\{ 1 + \frac{1}{4} \frac{m^2}{\lambda^2} + \frac{5}{32} \frac{m^4}{\lambda^4} + \frac{5}{2} \frac{m^2}{\lambda^2} + \frac{5}{8} \frac{m^4}{\lambda^4} \right\} \\
\times \left\{ 1 + \frac{i}{12} \frac{\alpha}{\lambda} \left[1 + \frac{3}{8} \frac{m^2}{\lambda^2} + \frac{33}{8} \frac{m^4}{\lambda^4} + 6\frac{m^4}{\lambda^4} \right] \right. \\
- \frac{1}{288} \frac{\alpha^2}{\lambda^2} \left[1 + \frac{78}{4} \frac{m^4}{\lambda^4} + \frac{1005}{8} \frac{m^4}{\lambda^4} + 156\frac{m^4}{\lambda^4} + 1005\frac{m^4}{\lambda^4}\right] \right\} \right. \\
\left. (68) \right.
$$

For $\lambda \to \infty$, this reduces to a plane-wave solution as it should, since the acceleration parameter $\alpha$ becomes irrelevant. Investigating the $\xi$-dependent part of this expansion, we note

$$
\exp \left[i\frac{\lambda}{\alpha}\left(1 - \frac{m^2}{2\lambda^2} - \frac{m^4}{8\lambda^4} + O\left(\frac{m^6}{\lambda^6}\right)\right)\right] = \exp \left[i\frac{\lambda}{\alpha}\sqrt{\lambda^2 - m^2}\right]. (69)
$$

Therefore, we can interpret the modes locally as particles of energy $\lambda$ and momentum $\sqrt{\lambda^2 - m^2}$.

### 3.2. Lamb shift

In order to calculate the Lamb shift, we need the amplitude squared of the expanded mode functions (68), which is at the site $\xi = 0$,

$$
|\hat{\varphi}_\lambda(\xi = 0, \tau)|^2 = \frac{1}{2\lambda} \left[ 1 - e^{-2|\lambda|\tau/\alpha} \left(1 + \frac{1}{2} \frac{m^2}{\lambda^2} + \frac{3}{8} \frac{m^4}{4 \lambda^4} + \cdots \right) \left(1 + \frac{1}{2} \frac{\alpha^2 m^2}{\lambda^4} + \cdots \right) \right]. (70)
$$

The Hamiltonian (49), which is quadratic in the field operators (48), can be recast into the following quadratic form in terms of the creation and annihilation operators of the Rindler vacuum:

$$
H^\varphi = \frac{1}{2} \int \frac{d\lambda}{2\pi} \Omega_\lambda (\hat{c}_\lambda \hat{c}_\lambda + \hat{c}_\lambda^\dagger \hat{c}_\lambda^\dagger) + (\Lambda_\lambda \hat{c}_\lambda \hat{c}_\lambda - \text{h.c.}). (71)
$$

where $\Omega_\lambda$ denotes the virtual energy of a Rindler quasiparticle excitation of momentum $\lambda$ and $\Lambda_\lambda$ is the amplitude for annihilation of a Rindler pair, with the momenta $\lambda$ and $-\lambda$, respectively. Here, we are interested in the ultraviolet domain, i.e. in the portion of (71), where $|\lambda| \gg \alpha$. Thus, it is possible to choose $1 \gg \Delta \xi \gg \alpha/|\lambda|$, which is what we assume in the following. From the analytic behaviour of the Rindler modes (50), it then follows that in the detector's...
neighbourhood at \( \xi = 0 \) the ultraviolet contributions to the Hamiltonian (71) are dominated by the local contribution from \( \xi \in (-\Delta \xi / 2, \Delta \xi / 2) \).

\[
H^{\xi} \approx \int_{-\Delta \xi / 2}^{\Delta \xi / 2} d\xi \mathcal{H}^{\alpha}.
\]

Then, by using the following approximate relation,

\[
\int_{-\Delta \xi / 2}^{\Delta \xi / 2} d\xi \exp \left( i\xi \frac{\lambda}{\alpha} \left( 1 - \frac{m^2}{2 \lambda^2} - \frac{1}{8 \lambda^4} \right) \pm i\xi \frac{\lambda'}{\alpha} \left( 1 - \frac{m^2}{2 \lambda^2} - \frac{1}{8 \lambda^4} \right) \right)
\approx 2\pi \alpha \delta(\lambda \pm \lambda') \left( 1 - \frac{m^2}{2 \lambda^2} - \frac{1}{8 \lambda^4} \right),
\]

we find

\[
\Omega_{\lambda} = \left\{ (\lambda^2 + m^2)|\psi_{\lambda}|^2 + \alpha^2|\partial_\xi \psi_{\lambda}|^2 \right\} \left( 1 - \frac{m^2}{2 \lambda^2} - \frac{1}{8 \lambda^4} \right),
\]

\[
\Lambda_{\lambda} = \left\{ (m^2 - \lambda^2)|\psi_{\lambda}|^2 + \alpha^2|\partial_\xi \psi_{\lambda}|^2 \right\} \left( 1 - \frac{m^2}{2 \lambda^2} - \frac{1}{8 \lambda^4} \right) e^{-2i|\lambda| \tau}.
\]

Assuming \( m \gg \alpha \), by the same token as for neglecting the mode mixing, we have dropped here the exponentially falling prefactor occurring in expression (70).

Upon substituting the expanded mode functions (68), one obtains after some algebra

\[
\Omega_{\lambda} = \frac{1}{|\lambda|} \left( \lambda^2 - \frac{3}{8} \frac{\alpha^2 m^4}{\lambda^4} + \cdots \right),
\]

\[
\Lambda_{\lambda} = -\frac{\alpha^2 m^2}{2|\lambda|^3} e^{-2i|\lambda| \tau} + \cdots.
\]

We can now assemble the difference between Lamb shift in Rindler space \( \delta E_R \) and flat space \( \delta E_M \). For the lower limit of the self-energy integral we take \( \lambda = m \), because for \( \lambda < m \) the modes are exponentially suppressed. According to the two-dimensional case of equation (16), we find

\[
\delta E = \delta E_R - \delta E_M = 2\hbar^2 \int_{m}^{\infty} \frac{dk}{2\pi} \frac{1}{2\lambda} \left( 1 + \frac{m^2}{2 \lambda^2} + \frac{3}{8} \frac{m^4}{\lambda^4} \right) \left\{ \frac{1 + \frac{m^2}{2 \lambda^2}}{\Delta E - \lambda} + \frac{\frac{3}{8} \frac{m^4}{\lambda^4}}{\lambda} \right\} = \frac{\hbar^2}{4\pi} \alpha^2 \frac{m^2}{\Delta E m^2} (1 + \mathcal{O}(m / \Delta E))
\]

where we have identified the momentum according to the identity (69) as \( k = \sqrt{\lambda^2 - m^2} \) and substituted \( d\lambda \rightarrow dk \). The integrals are evaluated according to (A.15), (A.17), and the final result is displayed up to leading order in \( 1 / \Delta E \), assuming that \( \Delta E \gg m \). When compared with the exponentially falling particle number by mode mixing, we see that in the ultraviolet the Unruh effect gets a boost.

4. Lamb shift versus response rate

While the response rate of an Unruh detector falls off exponentially with the level spacing \( \Delta E \), which holds true for de Sitter as well as for Rindler space, we have shown that Lamb shift exhibits a power-law behaviour. From the response function \( dF(\Delta E) / d\tau \) (14) for positive and negative \( \Delta E \), one can derive according to the principle of detailed balance the probability...
to find the detector on an excited level when being in equilibrium with the background, see e.g. [2]. This probability turns out to be exponentially falling with \( \Delta E \) too. Since Lamb shift also corresponds to the mixing of energy levels both effects are therefore quantitatively comparable, and Lamb shift is clearly more important in the ultraviolet. Yet, the difference is of course that the response of the detector is a time-dependent while Lamb shift a time-independent effect\(^3\).

For the expanding universe, the power-law behaviour is expected when considering the Hamiltonian or the local covariant energy density. For the accelerated observer, we have derived a new expression for a local virtual energy density, which also corresponds to a power law of the mode energy. It is however interesting to note that for the case of the detector exposed to a massless field in four spacetime dimensions, using a different formalism\(^4\), Audretsch and Müller also derive a Lamb shift correction \( \propto 1/\Delta E \) from an intermediate expression with manifestly thermal features [10]. Note that for the calculation of Lamb shift we had to perform renormalizations, though at a rather crude technical level. Therefore, we would have to verify whether the response function (3) for the unrenormalized, bare detector correctly reproduces the response rate for its renormalized, dressed counterpart. In QFT, this question is answered positively by the LSZ reduction formula for scattering amplitudes, the proof of which in particular requires that the external states of the matrix element correspond to well-separated wave packets. It is not clear whether this condition can be met for the state being the product of detector in the ground state and curved spacetime vacuum. In particular, our discussion of boundary effects in [2] indicates possible problems, since a huge period of interaction with the vacuum and a tremendous coherence is required, while scattering in flat space is a resonance phenomenon on rather short timescales. However, we do not decide this question in this paper.

Since the Lamb shift for the Unruh detector can be interpreted as a self-energy, it is interesting to compare with the effects of self-energies for quantum fields. In [22–24], the vacuum polarization of a photon coupled to minimally and nearly minimally coupled scalars in de Sitter space is calculated, and in [25, 26] the fermion self-energy for the case of Yukawa theory in de Sitter space is discussed. As a main result, a one-loop effective equation of motion is derived, which indicates mass terms for the photon and for the fermion, but no damping term corresponding to scattering from a thermal scalar background.

Therefore, self-energy effects known for fields in curved spacetimes are also of relevance for the Unruh detector. In fact, it is not via the response rate\(^5\) but through the Lamb shift how a bound state probes the quantum vacuum and the energy produced by the background.

### Appendix. Integrals for Lamb shift calculation

For the calculation of Lamb shift in FLRW background, we need the following integrals:

\[
I_1 = \int \frac{k^2 \, dk}{(\Delta E - \omega)\omega^3} = -\frac{k}{\Delta E \omega} - \frac{m}{\Delta E^2} \arctan \frac{k}{m} + \frac{\sqrt{\Delta E^2 - m^2}}{\Delta E^2} \frac{1}{2} \left\{ \log \left[ \frac{\sqrt{\Delta E^2 - m^2} + k}{\sqrt{\Delta E^2 - m^2} - k} \right] + \log \left[ \frac{\sqrt{\Delta E^2 - m^2 \omega + \Delta E k}}{\sqrt{\Delta E^2 - m^2 \omega - \Delta E k}} \right] \right\},
\]

\( I_1 \)  

\(^3\) Strictly speaking, the Lamb shift in the expanding universe (33) varies as the Hubble rate changes with time. Time independence means here that we use time-independent perturbation theory.

\(^4\) We were unaware of [10–12] before completion of the first version of the manuscript.

\(^5\) The conclusion for the response rate made in [5] is incorrect.
\[ I_2 = \int \frac{k^2 \, dk}{(\Delta E - \omega) \omega^3} \]
\[ = \frac{1}{E} \frac{k^3}{3 m^2 \omega^3} + \frac{1}{\Delta E^2} \left[ \frac{1}{2m} \arctan \left( \frac{k}{m} \right) - \frac{k}{2 \omega^2} \right] - \frac{1}{\Delta E^3} \frac{k}{\omega} - \frac{m}{\Delta E^4} \arctan \left( \frac{k}{m} \right) \]
\[ + \frac{\sqrt{\Delta E^2 - m^2}}{2 \Delta E^4} \int \log \frac{\sqrt{\Delta E^2 - m^2} + k}{\sqrt{\Delta E^2 - m^2} - k} + \log \frac{\sqrt{\Delta E^2 - m^2 \omega + \Delta E k}}{\sqrt{\Delta E^2 - m^2 \omega - \Delta E k}} \].

\[ I_3 = \int \frac{k^2 \, dk}{(\Delta E - \omega) \omega^3} \]
\[ = \frac{1}{\Delta E} \frac{k^3}{15 m^2 \omega + \frac{1}{5 m^2 \omega^3} - \frac{1}{5 \omega^3}} + \frac{1}{\Delta E^2} \left[ \frac{1}{8 m^3} \arctan \left( \frac{k}{m} \right) + \frac{k}{8 \omega^2} - \frac{1}{4 \omega^3} \right] \]
\[ + \frac{1}{\Delta E^3} \frac{k^3}{5 m^2 \omega^3} + \frac{1}{\Delta E^4} \left[ \frac{1}{2m} \arctan \left( \frac{k}{m} \right) - \frac{k}{2 \omega^2} \right] - \frac{1}{\Delta E^5} \frac{k}{\omega} - \frac{m}{\Delta E^6} \arctan \left( \frac{k}{m} \right) \]
\[ + \frac{\sqrt{\Delta E^2 - m^2}}{2 \Delta E^6} \int \log \frac{\sqrt{\Delta E^2 - m^2} + k}{\sqrt{\Delta E^2 - m^2} - k} + \log \frac{\sqrt{\Delta E^2 - m^2 \omega + \Delta E k}}{\sqrt{\Delta E^2 - m^2 \omega - \Delta E k}} \].

\[ J_1 = \int \frac{k^2 \, dk}{(\Delta E - \omega) \omega^2} \]
\[ = \frac{k}{\Delta E (\Delta E - \omega)} - \frac{m}{\Delta E^2} \arctan \frac{k}{m} \]
\[ - \frac{m^2}{\Delta E^2 \sqrt{\Delta E^2 - m^2}} \int \log \frac{\sqrt{\Delta E^2 - m^2} + k}{\sqrt{\Delta E^2 - m^2} - k} + \log \frac{\sqrt{\Delta E^2 - m^2 \omega + \Delta E k}}{\sqrt{\Delta E^2 - m^2 \omega - \Delta E k}} \].

\[ J_2 = \int \frac{k^2 \, dk}{(\Delta E - \omega) \omega^4} \]
\[ = \frac{1}{\Delta E^2} \left[ \frac{1}{2m} \arctan \left( \frac{k}{m} \right) - \frac{k}{2 \omega^2} \right] + \frac{k}{\Delta E^4 (\Delta E - \omega)} - \frac{2}{\Delta E^3} \frac{k}{\omega} - \frac{3m}{\Delta E^4} \arctan \left( \frac{k}{m} \right) \]
\[ + \frac{2 \Delta E^2 - 3 m^2}{\Delta E^4 \sqrt{\Delta E^2 - m^2}} \int \log \frac{\sqrt{\Delta E^2 - m^2} + k}{\sqrt{\Delta E^2 - m^2} - k} + \log \frac{\sqrt{\Delta E^2 - m^2 \omega + \Delta E k}}{\sqrt{\Delta E^2 - m^2 \omega - \Delta E k}} \].

\[ J_3 = \int \frac{k^2 \, dk}{(\Delta E - \omega) \omega^6} \]
\[ = \frac{1}{\Delta E^2} \left[ \frac{1}{8 m^3} \arctan \left( \frac{k}{m} \right) + \frac{k}{8 m^2 \omega^2} - \frac{1}{8 \omega^3} \right] + \frac{1}{\Delta E^3} \frac{k^3}{3 m^2 \omega^3} \]
\[ + \frac{1}{\Delta E^4} \left[ \frac{3}{2m} \arctan \left( \frac{k}{m} \right) - \frac{3 k}{2 \omega^2} \right] - \frac{1}{\Delta E^5} \frac{4k}{\omega} - \frac{5m}{\Delta E^6} \arctan \left( \frac{k}{m} \right) \]
\[ + \frac{1}{\Delta E^5} \frac{k}{\omega} + \frac{4 \Delta E^2 - 5 m^2}{\Delta E^6 \sqrt{\Delta E^2 - m^2}} \times \int \frac{1}{2} \left\{ \log \frac{\sqrt{\Delta E^2 - m^2} + k}{\sqrt{\Delta E^2 - m^2} - k} + \log \frac{\sqrt{\Delta E^2 - m^2 \omega + \Delta E k}}{\sqrt{\Delta E^2 - m^2 \omega - \Delta E k}} \right\} .
\]
We evaluate the above integrals at their boundaries and obtain

\[ [I_1]_0^\infty = -\frac{1}{\Delta E} - \frac{\pi}{2} \frac{m}{\Delta E} + \frac{\sqrt{\Delta E^2 - m^2}}{\Delta E} \frac{1}{2} \log \frac{\sqrt{\Delta E^2 - m^2} + \Delta E}{\sqrt{\Delta E^2 - m^2} - \Delta E}, \]

\[ [I_2]_0^\infty = \frac{1}{\Delta E} \frac{\pi}{3} \left( \frac{3}{m^2} + \frac{1}{4} \frac{k}{m} \right) + \frac{1}{\Delta E^2} \frac{\pi}{2} \frac{m}{\Delta E^2} + \frac{\sqrt{\Delta E^2 - m^2}}{\Delta E^2} \frac{1}{2} \log \frac{\sqrt{\Delta E^2 - m^2} + \Delta E}{\sqrt{\Delta E^2 - m^2} - \Delta E}, \]

\[ [I_3]_0^\infty = \frac{2}{15} \frac{\Delta E}{m^3} + \frac{1}{6} \frac{\Delta E^2 m^3}{m^3} + \frac{1}{16} \frac{\Delta E}{m^3} + \frac{1}{4} \frac{\Delta E^4}{m^3} \frac{1}{2} \frac{1}{\Delta E^2} + \frac{\sqrt{\Delta E^2 - m^2} + \Delta E}{\sqrt{\Delta E^2 - m^2} - \Delta E}. \]

When expanded up to second order in \(1/\Delta E\), the above expressions read

\[ [I_1]_0^\infty \approx -\frac{1}{\Delta E} - \frac{\pi}{2} \frac{m}{\Delta E} + \frac{1}{2} \frac{\Delta E}{m}, \]

\[ [I_2]_0^\infty \approx \frac{2}{3} \frac{\Delta E}{m^2} + \frac{1}{4} \frac{\Delta E^2 m^2}{m^2}, \]

\[ [I_3]_0^\infty \approx \frac{2}{15} \frac{\Delta E}{m^2} + \frac{1}{16} \frac{\Delta E^2}{m^2}. \]

The integrals which we need for obtaining the Lamb shift in Rindler space are

\[ (k = \sqrt{\lambda^2 - m^2}) \]

\[ R_1 = \int \frac{dk}{\lambda^2 - \Delta E - \lambda} \]

\[ = \frac{1}{\Delta E} \left( \frac{2}{3} m^2 \lambda + \frac{1}{3} k \right) + \frac{1}{\Delta E^2} \left( \frac{1}{2} \arctan \frac{k}{m} + \frac{1}{2} \frac{k}{m^2} \lambda^2 \right) \]

\[ + \frac{1}{\Delta E^3} \frac{1}{m^2} \lambda + \frac{1}{\Delta E^4} \frac{1}{m} \arctan \frac{k}{m} + \frac{1}{\Delta E^4} \frac{\sqrt{\Delta E^2 - m^2}}{\Delta E^2} \]

\[ \times \frac{1}{2} \left\{ \log \frac{\sqrt{\Delta E^2 - m^2} + k}{\sqrt{\Delta E^2 - m^2} - k} + \log \frac{\sqrt{\Delta E^2 - m^2} + \Delta E k}{\sqrt{\Delta E^2 - m^2} - \Delta E k} \right\}. \]
\[ |R_1|_0^\infty = \frac{1}{\Delta E} \left( \frac{2}{3m^2} + \frac{1}{m^2} \frac{\pi}{\Delta E^3} \right) + \frac{1}{\Delta E^4} \left( \frac{1}{\sqrt{\Delta E^2 - m^2}} \right) + \log \left( \frac{\sqrt{\Delta E^2 - m^2} + \Delta E}{\sqrt{\Delta E^2 - m^2} - \Delta E} \right) \quad (A.16) \]

\[ R_2 = \int \frac{dk}{\lambda^6} \frac{1}{(\Delta E - \lambda)^2} \]
\[ = \frac{1}{\Delta E^2} \left( \frac{2}{3m^2} \text{arctan} \left( \frac{k}{m} \right) + \frac{3}{8m^2} \lambda^2 + \frac{1}{4m^2} \lambda^2 \right) + \frac{2}{\Delta E^3} \left( \frac{2}{3m^2} \lambda^2 + \frac{1}{3m^2} \lambda^3 \right) \]
\[ + \frac{3}{\Delta E^4} \left( \frac{1}{2\pi} \text{arctan} \left( \frac{k}{m} \right) + \frac{4}{m^2} \lambda^2 + \frac{1}{5m} \text{arctan} \left( \frac{k}{m} \right) \right) \]
\[ + \frac{1}{\Delta E^5} \left( \frac{6\Delta E^2 - 5m^2}{\Delta E^2 - m^2} \right) \frac{\Delta E - \lambda}{\Delta E^6} - \frac{\lambda}{\Delta E^6} (\Delta E^2 - m^2)^{3/2} \]
\[ \times \frac{1}{2} \left\{ \log \left( \frac{\sqrt{\Delta E^2 - m^2} + k}{\sqrt{\Delta E^2 - m^2} - k} \right) + \log \left( \frac{\Delta E^2 - m^2 + \Delta E k}{\Delta E^2 - m^2 - \Delta E k} \right) \right\} \quad (A.17) \]

\[ |R_2|_0^\infty = \frac{1}{\Delta E^2} \left( \frac{16}{16m^2} + \frac{\Delta E^3}{3m^2} \right) + \frac{\Delta E^4}{4m^2} + \frac{\Delta E^5}{m^2} + \frac{\Delta E^6}{2m} + \frac{\lambda}{\Delta E^6} \left( \frac{6\Delta E^2 - 5m^2}{\Delta E^2 - m^2} \right) \frac{1}{2} \log \left( \frac{\sqrt{\Delta E^2 - m^2} + \Delta E}{\sqrt{\Delta E^2 - m^2} - \Delta E} \right) \quad (A.18) \]

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