Uniform semantic treatment of default and autoepistemic logics

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Abstract

We revisit the issue of connections between two leading formalisms in nonmonotonic reasoning: autoepistemic logic and default logic. For each logic we develop a comprehensive semantic framework based on the notion of a belief pair. The set of all belief pairs together with the so called knowledge ordering forms a complete lattice. For each logic, we introduce several semantics by means of fixpoints of operators on the lattice of belief pairs. Our results elucidate an underlying isomorphism of the respective semantic constructions. In particular, we show that the interpretation of defaults as modal formulas proposed by Konolige allows us to represent all semantics for default logic in terms of the corresponding semantics for autoepistemic logic. Thus, our results conclusively establish that default logic can indeed be viewed as a fragment of autoepistemic logic. However, as we also demonstrate, the semantics of Moore and Reiter are given by different operators and occupy different locations in their corresponding families of semantics! This result explains the source of the longstanding difficulty to formally relate these two semantics. In the paper, we also discuss approximating skeptical reasoning with autoepistemic and default logics and establish constructive principles behind such approximations.

1 INTRODUCTION

Due to their applications in knowledge representation and, more specifically, in commonsense reasoning, abduction, diagnosis, belief revision, planning and reasoning about action, default and autoepistemic logics are among the most extensively studied nonmonotonic formalisms. Still, after almost two decades of research in the area several key questions remain open.

The first of them is the question of the relationship between default and autoepistemic logics. Default logic was introduced by Reiter [Rei80] to formalize reasoning about defaults, that is, statements that describe what normally is the case, in the absence of contradicting information. Autoepistemic logic was proposed by Moore [Moo84] to describe the belief states of rational agents reflecting upon their own beliefs and disbeliefs. Although the motivation and syntax of both logics are different, it has been clear for a long time that they are closely related. However, despite much work [Kon88, MT89a, Tru91, MT93, Got95] no truly satisfactory account of the relationship was found. Konolige [Kon88] related default logic to a version of autoepistemic logic based on the notion of a strongly grounded expansion — a concept that depends on a syntactic representation of a theory. Marek and Truszczynski related default logic to two modal nonmonotonic logics related but different from autoepistemic logic: the nonmonotonic modal logic N [MT89a] and nonmonotonic modal logic S4F [Tru91]. Finally, Gottlob [Got95] found a relationship between default and autoepistemic logics but the translation he used was not modular. In fact, he proved that a modular translation of default logic into autoepistemic logic does not exist. These results seem to point to some misalignment between extensions of default theories and expansions of modal theories. Our results in this paper finally clarify the picture.

Another problem is related to the fixpoint definitions of extensions (in the case of default logic) and expansions (in the case of autoepistemic logic).
provide no insights into constructive processes agents might use to build their belief sets based on default or modal theories describing base facts. Finally, there is a problem of high computational complexity of reasoning with extensions and expansions. The problems to decide the existence of an extension (expansion) is \( \Sigma_2^P \)-complete, the problem to compute the intersection of extensions (expansions) — it is needed for skeptical reasoning — is \( \Pi_2^P \)-hard.

In this paper we develop a unifying semantic treatment of default and autoepistemic logics and use it to address the three issues discussed above. For each logic we define a family of 2-, 3- and 4-valued semantics and show that they include all major semantics for default and autoepistemic logics. Within our framework we define semantics that generalize Kripke-Kleene and well-founded semantics for logic programs. These semantics allow us to approximate skeptical default and autoepistemic reasoning and they can be computed faster than extensions and expansions (assuming that the polynomial hierarchy does not collapse). Most importantly, we show that the unified semantic picture of default and autoepistemic logics described here allows us to pinpoint the exact nature of how they are related.

Our approach is motivated by the algebraic approach proposed by Fitting in his analysis of semantics for logic programs with negation [Fit99], and extends our earlier work on 3-valued semantics for autoepistemic logics [DMT98]. It relies on the concept of a belief pair, a pair \((P, S)\), where \(P\) and \(S\) are sets of 2-valued interpretations. The concept of a belief pair generalizes the notion of a possible-world structure, that is, a set of 2-valued interpretations, often used to define the semantics of the modal logic S5 and also used by Moore and Levesque [Moo84, Lev90] in their work on autoepistemic logic. Specifically, each possible-world structure \(Q\) can be identified with a belief pair \((Q, Q)\). Belief pairs allow us to approximate the state of beliefs of an agent whose beliefs are represented by a possible-world structure.

Given a possible-world structure \(Q\) or a belief pair \(B\), it is often possible to describe a way in which \(Q\) (or \(B\)) could be revised to more accurately reflect the agent’s beliefs. Such a revision procedure can be formally described by an operator. Fixpoints of such operators are often used to specify semantics of nonmonotonic formalisms as they represent those belief states of the agent that cannot be revised away. Sometimes fixpoints that satisfy some minimality conditions are additionally distinguished.

These intuitions underlie the original definition of an expansion of a modal theory \(T\) as a fixpoint of a certain operator \(D_T\) on the set of all possible-world structures [Moo84]. In [DMT98], it is shown that the operator \(D_T\) can be extended to the set of belief pairs. The resulting operator, \(D_T\), yields a multi-valued generalization of extensions and a semantics that approximates skeptical autoepistemic reasoning. We refer to it as the Kripke-Kleene semantics as it generalizes Kripke-Kleene semantics for logic programs. In this paper, we derive from the operator \(D_T\) two new operators and show that their fixpoints give rise to semantics for autoepistemic theories under which circular dependence of beliefs upon themselves, present in autoepistemic logic of Moore, is eliminated. One of these semantics is shown to correspond to Reiter’s semantics of default logic. The other one can be viewed as a generalization of the well-founded semantics.

While possible-world semantics were used in the study of autoepistemic logic, they had only a marginal effect on the development of default logic. In this paper, we show that possible-world semantics approach can be extended to the case of default logic. Namely, in a close analogy with autoepistemic logic, for every default theory \(\Delta\), we introduce an operator \(E_\Delta\) defined on the set of belief pairs. We show that the operator \(E_\Delta\) gives rise to three other operators and that the fixpoints of these operators yield several semantics for default theories. Among these semantics are the semantics of extensions by Reiter, the stationary semantics [PP94] of the well-founded semantics for default logic [BS91] (it approximates skeptical reasoning under extensions), the semantics of weak extensions [MT89a] and the Kripke-Kleene semantics (it approximates skeptical reasoning under weak extensions).

Our results settle the issue of the relationship between autoepistemic and default logics. We show that the operators \(E_\Delta\) and \(D_T\) are closely related if a default theory \(\Delta\) is interpreted as a modal theory \(T\) given by the translation proposed by Konolige [Kon88]. Our results show that under the Konolige’s translation, the families of semantics for default and autoepistemic logics are isomorphic and default logic can be viewed, as has long been expected, as a fragment of autoepistemic logic. But this correspondence does not relate extensions and expansions! The semantics corresponding to these concepts occupy different locations in their respective families of semantics and have different properties.

We also point out that the Kripke-Kleene and well-founded semantics studied in the paper have better computational properties than the semantics of expansions and extensions. We conclude with comments...
about further generalizations and open problems.

2 PRELIMINARIES

The formal language for the semantic study developed in this paper is that of lattices, operators and fixpoints. The key result is that by Tarski and Knaster [Tar55] stating that a monotone operator on a complete lattice has a least fixpoint.

By $At$ we denote the set of atoms of the propositional language under consideration. The set of all 2-valued interpretations of $At$ will be denoted by $\mathcal{A}$. Any set $\mathcal{Q} \subseteq \mathcal{A}$ is called a possible-world structure and can be viewed as a universal Kripke model [Che80]. Possible-world structures are a basic tool in semantic studies of modal logics. They were also used in the context of autoepistemic logic [Moo84, Lev90, MT93].

A collection of all possible-world structures will be denoted by $\mathcal{W}$. This set can be ordered by the reverse set inclusion: for $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{W}$, $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$ if $\mathcal{Q}_2 \subseteq \mathcal{Q}_1$ (the smaller the set of possible worlds, the bigger the theory it determines). Clearly, $(\mathcal{W}, \subseteq)$ is a complete lattice. To study formalisms based on the modal language, we define the truth function $\mathcal{H}_{Q,I}$ inductively as follows ($Q$ is a possible-world structure, $I \in \mathcal{A}$ is an interpretation):

1. $\mathcal{H}_{Q,I}(p) = I(p)$, if $p$ is an atom
2. Boolean connectives are handled in the standard Tarskian way
3. $\mathcal{H}_{Q,I}(K\varphi) = t$, if for every interpretation $J \in \mathcal{Q}$,
   $\mathcal{H}_{Q,J}(\varphi) = t$, and $\mathcal{H}_{Q,I}(K\varphi) = f$, otherwise.

The value of a modal atom $K\varphi$ given by $\mathcal{H}_{Q,I}$ does not depend on $I$. Thus, it is determined only by the possible-world structure in question. It is clear that the meta-knowledge specified by a possible-world structure $Q$ is complete: all modal atoms $K\varphi$ are either true or false with respect to $Q$.

For every modal theory $T$, Moore [Moo84] defined an operator $D_T$ on $\mathcal{W}$ by:

$$D_T(Q) = \{ I : \mathcal{H}_{Q,I}(\varphi) = t, \text{ for every } \varphi \in T \}.$$  

Moore called the theory of a fixpoint of $D_T$ an expansion$^3$.

In [DMT98], Moore’s approach was extended to the 3-valued case, in which a possibility of incomplete meta-knowledge is admitted. A key concept is that of a belief

$\text{pair}$, that is, a pair $(P,S)$ of possible-world structures. In a belief pair $(P,S)$, $P$ can be viewed as a representation of a conservative (pessimistic) view on what is believed while $S$ can be regarded as a representation of a liberal (gullible) view. If $S \subseteq P$, formulas believed in according to the conservative view captured by $P$, are believed in according to the liberal view represented by $S$. Consequently, a belief pair $(P,S)$ such that $S \subseteq P$ is called consistent. However, the ways in which the agent establishes the estimates $P$ and $S$ may be independent of each other and, thus, we allow belief pairs $(P,S)$ such that $S$ is not a subset of $P$. We refer to them as inconsistent$^4$. In applications we are mostly interested in consistent belief pairs. Constructive techniques for building belief pairs given a base theory, which are described in the paper, result in consistent belief pairs. However, admitting inconsistent belief pairs completes the picture, leads to simple intuitions behind mathematical arguments and results in more elegant algebraic structures.

With a belief pair $(P,S)$ and a 2-valued interpretation $I$ we associate a 2-valued truth function $\mathcal{H}^2_{(P,S),I}$ defined on a modal language. The idea is to define $\mathcal{H}^2_{(P,S),I}(\varphi)$ so that it provides a conservative estimate to the truth value of $\varphi$ with respect to a belief pair $(P,S)$. Since $P$ represents a conservative point of view and $S$ a liberal one, to get a conservative estimate we use $P$ to evaluate positive occurrences of modal atoms and $S$ to evaluate negative ones. The definition is inductive:

1. $\mathcal{H}^2_{(P,S),I}(p) = I(p)$, for every atom $p$
2. Conjunction and disjunction are treated in the standard Tarskian way
3. $\mathcal{H}^2_{(P,S),I}(\neg \varphi) = \neg \mathcal{H}^2_{(P,S),I}(\varphi)$
4. $\mathcal{H}^2_{(P,S),I}(K\varphi) = t$ if $\mathcal{H}^2_{(P,S),I}(\varphi) = t$ for all $J \in P$.
   $\mathcal{H}^2_{(P,S),I}(K\varphi) = f$, otherwise.

We stress that when evaluating the negation of a formula the roles of $P$ and $S$ are switched, which ensures that modal literals appearing positively in a formula are evaluated with respect to $P$ while those appearing negatively are evaluated with respect to $S$.

Clearly, to construct a liberal estimate for the truth value of $\varphi$ with respect to a belief pair $(P,S)$ we can proceed similarly and use $S$ $(P)$ to evaluate modal literals appearing positively (negatively) in $\varphi$. It is easy to see, however, that the resulting truth function

$^3$In [DMT98] only consistent belief pairs were considered.

$^4$In [DMT98]
can be expressed as $\mathcal{H}_{(S,P),I}^2$ (we reverse the roles of $P$ and $S$).

Conservative and liberal estimates of truth values of formulas can be combined into a single 4-valued estimate. We say that the logical value of a formula $\varphi$ is true, $t_4$ (false, $f_4$), if both conservative and liberal estimates of its truth value are equal to $t$ ($f$). We say that the logical value $\varphi$ is unknown, $u$, if the conservative estimate is $f$ and the liberal estimate is $t$. Finally, the logical value of $\varphi$ is inconsistent, $i$, if the conservative estimate is $t$ and the liberal estimate is $f$. Thus, the estimates given by $\mathcal{H}_{(P,S),I}^4(\varphi)$ and $\mathcal{H}_{(S,P),I}^2(\varphi)$ yield a 4-valued truth function

$$\mathcal{H}_{(P,S),I}^4(\varphi) = (\mathcal{H}_{(P,S),I}^2(\varphi), \mathcal{H}_{(S,P),I}^2(\varphi)),$$

where $t_4 = (t, t)$, $f_4 = (f, f)$, $u = (f, t)$ and $i = (t, f)$.

We define the meta-knowledge of a belief pair $(P, S)$ as the set of all formulas $\varphi$ such that both conservative and liberal estimates of the truth value of the epistemic atom $K\varphi$ coincide (are both true or are both false). Clearly, the meta-knowledge of a belief pair $(P, S)$ need not be complete. For some modal atoms $K\varphi$ the two estimates may disagree ($K\varphi$ may be assigned value $u$ or $i$). However, it is easy to see, that for a complete belief pair $(P, P)$, $\mathcal{H}_{(P,P),I}^4 = \mathcal{H}_{P,I}$, for every interpretation $I \in \mathcal{A}$. In other words, belief pairs and the truth function $\mathcal{H}_{(P,S),I}^4$ generalize possible-world structures and the truth function $\mathcal{H}_{P,I}$. In addition, for a consistent belief pair $(P, S)$, $\mathcal{H}_{(P,S),I}^4$ never assigns value $i$ and coincides with the 3-valued truth function defined in [DMT98].

The set of all belief pairs is denoted by $B$. It can be ordered by the knowledge ordering $\leq_{kn}$: $(P_1, S_1) \leq_{kn} (P_2, S_2)$ if $S_1 \subseteq S_2$ and $P_2 \subseteq P_1$. The set $B$ with the ordering $\leq_{kn}$ forms a complete lattice and, consequently, a $\leq_{kn}$-monotone operator on $B$ is guaranteed to have a least fixpoint by the result of Tarski and Knaster [Tars]. The ordering $\leq_{kn}$ coincides with the ordering of increasing meta-knowledge (decreasing meta-ignorance): $(P_1, S_1) \leq_{kn} (P_2, S_2)$ if and only if the set of modal atoms with the same conservative and liberal estimates with respect to $(P_1, S_1)$ is contained in the set of modal atoms for which conservative and liberal estimates with respect to $(P_2, S_2)$ are the same.

Let $(P, S)$ be a belief pair. Following [DMT98], we set

$$D_T(P, S) = (D_T^P(P, S), D_T^S(P, S)),$$

where

$$D_T^P(P, S) = \{ I : \mathcal{H}_{(S,P),I}^2(T) = t \}$$

and

$$D_T^S(P, S) = \{ I : \mathcal{H}_{(P,S),I}^2(T) = t \}.$$

Fixpoints of the operator $D_T$ will be called partial expansions.

Speaking intuitively, the operator $D_T$ describes how an agent might revise a belief pair $(P, S)$. The possible-world structure $P$ is replaced by the structure $P'$ consisting of those interpretations $I$ for which all formulas from $T$ are true according to the liberal estimates of truth values (given $(P, S)$). A liberal criterion for selecting possible worlds (interpretations) to $P'$ results in a possible-world structure capturing a conservative point of view. By duality between conservative and liberal approaches, $S$ can be replaced by $S'$ consisting of all interpretations $I$ such that all formulas from $T$ are true according to the conservative estimates of truth values (given $(P, S)$).

Let us recall that $D_T$ stands for the operator introduced by Moore. The operator $D_T$ allows us to reconstruct the semantic approach to autoepistemic logic proposed by Moore.

**Theorem 2.1** Let $T$ be a modal theory. Then, for every possible-world structure $P$, $D_T(P, P) = (D_T(P), D_T(P))$. Consequently, a belief pair $(P, P)$ is a fixpoint of $D_T$ if and only if $P$ is a fixpoint of $D_T$.

**Theorem 2.1** implies that there is a natural one-to-one correspondence between the complete partial expansions of $T$ and the expansions of $T$. Thus, partial expansions (consistent partial expansions) can be viewed as 4-valued (3-valued) generalizations of Moore’s expansions.

The key property of the operator $D_T$ is its $\leq_{kn}$-monotonicity. It follows that $D_T$ has a unique $\leq_{kn}$-least fixpoint. We denote it by $KK(T)$ and refer to it as the Kripke-Kleene fixpoint (or semantics) for $T$. Kripke-Kleene fixpoint has a clear constructive flavor (it can be obtained by iterating the operator $D_T$, starting at the least informative belief pair, $(A, \emptyset)$). It approximates all partial expansions and, in particular, approximates the skeptical reasoning with expansions.

**Theorem 2.2** Let $T$ be a modal theory.

1. The fixpoint $KK(T)$ is consistent.

2. For every partial expansion $B$ of $T$, $KK(T) \leq_{kn} B$.

3. If $K\varphi$ is true with respect to $KK(T)$ then $\varphi$ belongs to every expansion of $T$. If $K\varphi$ is false with

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4There is a close analogy between the least fixpoint of the operator $D_T$ and the Kripke-Kleene semantics for logic programs.
Deciding the truth value of a modal atom $K\varphi$ with respect to the Kripke-Kleene fixpoint is in the class $\Delta^P_2$. Thus, unless the polynomial hierarchy collapses, it is a simpler problem than the problem of computing expansions or their intersection.

We can also use the Kripke-Kleene semantics as a test for the uniqueness of an expansion. Namely, we can first compute $KK(T)$ and check if $KK(T)$ is complete. If it is, $T$ has a unique expansion. This method is computationally better (again, if the polynomial hierarchy does not collapse) than the straightforward one which computes all expansions of $T$. However, it is incomplete — there are theories $T$ with a unique expansion and such that $KK(T)$ is not complete.

3 AUTOEPISTEMIC LOGIC

The operator $D_T$ allows us to define two additional operators: the operator $D^f_T$ defined on the lattice $W$, and the operator $D^s_T$ defined on the lattice $B$. They give rise to new semantics for autoepistemic logic that are closely related to the semantics of extensions for default logic. One of them is a perfect match to Reiter’s semantics of extensions for default logic, an object long sought after in the autoepistemic logic.

Let us recall that the operator $D_T$ associates with each belief pair $(P, S)$ its revised variant $(P’, S’) = D_T(P, S)$. The way in which $P’$ is obtained is described by the operator $D^s_T$. Namely, $P’ = D^s_T(P, S)$. If we fix $S$, this operator becomes a monotone operator on $W$ (it follows from the fact that $D_T$ is $\preceq_{\text{kn}}$-monotone). Hence, its least fixpoint can be viewed as the preferred revision of $P$, given a fixed $S$. Let us define, then,

$$D^f_T(S) = \text{lfp}(D^f_T(\cdot, S)).$$

Similarly, we can argue that $\text{lfp}(D^s_T(P, \cdot))$ can be regarded as a preferred revision of $S$, given $P$. It turns out that $\text{lfp}(D^s_T(P, \cdot)) = D^s_T(P)$. Thus, we define an operator $D^s_T$ on belief pairs as follows:

$$D^s_T(P, S) = (D^f_T(S), D^s_T(P)).$$

Clearly, $D^s_T$ is an operator on $W$. The fixpoints of the operator $D^s_T$ (and also their theories) will be referred to as extensions. The choice of the term is not arbitrary. We show in Section 4 that extensions of modal theories can be regarded as generalizations of extensions of default theories. We have the following property relating fixpoints of the operators $D^f_T$ and $D^s_T$.

**Theorem 3.1** For every modal theory $T$, a possible-world structure $P$ is a fixpoint of $D^s_T$ if and only if a belief pair $(P, P)$ is a fixpoint of $D^f_T$.

It follows that the semantics of fixpoints of $D^s_T$ can be viewed as a 4-valued version of the semantics of extensions. Similarly, consistent fixpoints can be thought of as a 3-valued generalizations of extensions. Consequently, we refer to the fixpoints of the operator $D^s_T$ as partial extensions.

The circular dependence allowing the agent to accept $p$ to the belief set just on the basis of this agent believing in $p$, allowed under the semantics of extensions, is eliminated in the case of extensions. For instance, the theory $\{Kp \Rightarrow p\}$ has two expansions. One of them is determined by the possible-world structure consisting of all interpretations, the other one — by the possible-world structure consisting of all interpretations in which $p$ is true. It is this second expansion that suffers from circular-argument problem: the belief in $p$ is the only justification for having $p$ in this expansion. In the same time, the theory $\{Kp \Rightarrow p\}$ has exactly one extension, the one given by the possible-world structure consisting of all interpretations. The atom $p$ is not true in it and, hence, circular arguments are not used in the construction of this expansion.

The operator $D^s_T$ is antimonotone. It follows that the operator $D^f_T(P, S)$ is $\preceq_{\text{kn}}$-monotone. Thus, by the result of Tarski and Knaster, it has the least fixpoint. We will denote this fixpoint by $WF(T)$ and refer to it as the well-founded fixpoint (or semantics) of $T$. Our choice of the term is again not accidental. This semantics is closely related to the well-founded semantics of default logic [BS91] and logic programming [VRS91]. We have the following result indicating that well-founded semantics can be used to approximate all partial extensions and, in particular, all extensions of a modal theory. It also shows that the well-founded semantics provides a sufficient condition for the uniqueness of an extension.

**Theorem 3.2** Let $T$ be a modal theory and let $WF(T) = (P, S)$. Then:

1. The fixpoint $WF(T)$ is consistent.
2. For every partial extension $B$, $WF(T) \preceq_{\text{kn}} B$.
3. If $\mathcal{H}_T(K\varphi) = t$, then $\varphi$ belongs to every extension of $T$. Similarly, if $\mathcal{H}_T(K\varphi) = f$, then $\varphi$ does not belong to any extension.
4. If $WF(T)$ is complete (that is, $P = S$) then $T$ has a unique extension corresponding to the possible-world structure $P$. 

Based on the approach developed in [DMT98] for computing the Kripke-Kleene semantics, we can establish computational properties of the well-founded semantics. We have the following result.

**Theorem 3.3** The problem of computing the well-founded semantics is the class $\Delta_\delta^p$.

Thus, assuming that the polynomial hierarchy does not collapse, computing the well-founded semantics of a theory $T$ is easier than computing the intersection of extensions of $T$ (that is, the set of skeptical consequences of $T$). Since the well-founded semantics of a modal theory $T$ approximates all extensions, it can be used to speed up the computation of their intersection.

The next result connects expansions and the Kripke-Kleene semantics with extensions and the well-founded semantics. It shows that the well-founded semantics is stronger than the Kripke-Kleene semantics and that (partial) extensions of $T$ are (partial) expansions of $T$ satisfying some minimality condition.

**Theorem 3.4** Let $T$ be a modal theory. Then:

1. $KK(T) \preceq_{kn} WF(T)$.
2. Every extension of $T$ is a $\subseteq$-minimal expansion of $T$.
3. Every partial extension $(P, S)$ of $T$ is a minimal partial expansion of $T$ in the following sense: for every partial expansion $(P', S')$, if $P' \subseteq P$ and $S' \subseteq S$, then $P = P'$ and $S = S'$.

We conclude this section with a schematic illustration of the panorama of semantics for autoepistemic logic. The central position is occupied by the operator $D_T$. Its fixpoints yield the semantics of partial expansions and its least fixpoint yields the Kripke-Kleene semantics. Restriction of the operator $D_T$ to complete belief pairs leads to the operator $D_{\text{st}}$, originally introduced by Moore, and results in the semantics of expansions. The operator $D_T$ also gives rise to the operators $D_T^{\square}$ and $D_T^{\Box}$ that yield new semantics for autoepistemic logic: the semantics of extensions, the semantics of partial extensions and the well-founded semantics.

## 4 DEFAULT LOGIC

While possible-world semantics played a prominent role in the study of autoepistemic logics [Moo84, Lev90, DMT98] they have not, up to now, had a similar impact on default logic. In this section we will introduce a comprehensive semantic treatment of default logic in terms of possible-world structures and belief pairs. Our approach will follow closely that used in the preceding sections.

We observed earlier that autoepistemic logic can be viewed as the logic of the operator $D_T$. Its fixpoints, and fixpoints of the operators that can be derived from $D_T$, determine all major semantics for autoepistemic logic. We will now develop a similar treatment of default logic.

As before, we start with a 2-valued truth function that gives a conservative estimate of the logical value of a formula or a default with respect to a belief pair $(P, S)$ and an interpretation $I$. For a propositional formula $\varphi$, we define $H^{dl}_{(P, S), I}(\varphi) = I(\varphi)$. For a default $d = \alpha; \beta_1, \ldots, \beta_k$, we set $H^{dl}_{(P, S), I}(d) = t$ if at least one of the following conditions holds:

1. there is $J \in S$ such that $J(\alpha) = f$
2. there is $i$, $1 \leq i \leq k$ such that for every $J \in P$, $J(\beta_i) = f$
3. $I(\gamma) = t$.

We set $H^{dl}_{(P, S), I}(d) = f$, otherwise. Clearly, the definition of $H^{dl}_{(P, S), I}(d)$ agrees with the intuitive reading of a default $d$: it is true, according to a conservative point of view, if its prerequisite is false (even with respect to a liberal view captured by $S$) or if at least one of its justifications is definitely impossible (it is false according to a conservative point of view captured by $P$) or if its consequent is true (in $I$). As before, we can also argue that $H^{dl}_{(S, P), I}(d)$ provides a liberal estimate for a truth value of $d$ with respect to $(P, S)$ (the roles of $P$ and $S$ are reversed).

Let $\Delta = (D, W)$ be a default theory. We use the truth function $H^{dl}_{(P, S), I}$ to define an operator $E_\Delta$ on
the lattice $\mathcal{B}$ of belief pairs:

$$\mathcal{E}_\Delta(P, S) = (\mathcal{E}'_\Delta(P, S), \mathcal{E}''_\Delta(P, S)),$$

where

$$\mathcal{E}'_\Delta(P, S) = \{I : \mathcal{H}_{(S,P)}I(\Delta) = t\}$$

and

$$\mathcal{E}''_\Delta(P, S) = \{I : \mathcal{H}_{(S,P)}I(\Delta) = t\}.$$

This definition can be justified similarly as that of the operator $\mathcal{D}$.

We will now define a 2-valued version of the operator $\mathcal{E}_\Delta$. To this end, we use the following result.

**Theorem 4.1** Let $\Delta$ be a default theory. If $B \in \mathcal{B}$ is complete then $\mathcal{E}_\Delta(B)$ is also complete.

Let $Q$ be a possible-world structure. We define

$$E_\Delta(Q) = Q',$$

where $Q'$ is a possible-world structure such that $\mathcal{E}_\Delta(Q, Q) = (Q', Q')$ (its existence is guaranteed by Theorem 4.1).

To the best of our knowledge, the operator $E_\Delta$ has not appeared explicitly in the literature before. Its fixpoints, however, did. In [MT89a], the concept of a weak extension of a default theory was introduced and studied (the approach used there was proof-theoretic).

It turns out that fixpoints of the operator $E_\Delta$ correspond precisely to weak extensions of $\Delta$. Thus, the semantics given by the operator $E_\Delta$ is precisely the semantics of weak extensions.

**Theorem 4.2** Let $\Delta$ be a default theory. Then:

1. A propositional theory $T$ is a weak extension of a default theory $\Delta$ according to [MT89a] if and only if $T = \{\varphi : I(\varphi) = t, \text{ for every } I \in Q\}$ for a fixpoint $Q$ of $E_\Delta$.

2. A possible-world structure $Q$ is a fixpoint of $E_\Delta$ if and only if a belief pair $(Q, Q)$ is a fixpoint of $\mathcal{E}_\Delta$.

In view of Theorem 4.2(1), we call fixpoints of the operator $E_\Delta$ weak extensions. Theorem 1.2(2) implies that complete fixpoints of the operator $\mathcal{E}_\Delta$ are in one-to-one correspondence with the fixpoints of the operator $E_\Delta$. Thus, we call fixpoints of the operator $\mathcal{E}_\Delta$ — partial weak extensions (they can be regarded as a 4-valued generalization of weak extensions, consistent fixpoints can be regarded as 3-valued generalizations).

The key property of the operator $\mathcal{E}_\Delta$ is its $\preceq_{kn}$-monotonicity. Thus, $\mathcal{E}_\Delta$ has a least fixpoint. We call it the Kripke-Kleene fixpoint and denote it by $KK(\Delta)$. We refer to the corresponding semantics as the Kripke-Kleene semantics for $\Delta$.

The Kripke-Kleene semantics can be obtained by iterating the operator $\mathcal{E}_\Delta$ starting with the least informative belief pair $(\mathcal{A}, \emptyset)$. Thus, it has a constructive flavor. Second, it approximates the skeptical reasoning with weak extensions and provides a test for uniqueness of a weak extension.

**Theorem 4.3** Let $\Delta$ be a default theory and let $KK(\Delta) = (P, S)$.

1. The fixpoint $KK(\Delta)$ is consistent, that is, $S \subseteq P$.

2. If $I(\varphi) = t$ for every $I \in P$, then $\varphi$ belongs to every weak extension. If $J(\varphi) = f$ for some $J \in S$, $\varphi$ does not belong to any weak extension.

3. If $P = S$ (that is, if $KK(\Delta)$ is complete) then $\Delta$ has a unique weak extension corresponding to the possible-world structure $P$.

The Kripke-Kleene semantics is computationally attractive. Deciding whether a formula is in the intersection of the weak extensions of a default theory $\Delta$ is $\Pi^1_2$-complete. In contrast, by adapting the methods developed in [DMT98] to the case of default logic we can show that the problem of computing $KK(\Delta)$ (specifically, assuming $KK(\Delta) = (P, S)$), deciding whether $I(\varphi) = t$ for every $I \in P$, or whether $J(\varphi) = f$ for some $J \in S$) is in the class $\Delta^P_f$.

So far we have not yet reconstructed the concept of an extension. In order to do so, we will now derive from $\mathcal{E}_\Delta$ two other operators related to default logic. Let us consider a belief pair $(P, S)$. We want to revise it to a belief pair $(P', S')$. We might do it by fixing $S$ and taking for $P'$ a preferred revision of $P$, and by fixing $P$ and taking for $S'$ a preferred revision of $S$.

It is easy to see that $\preceq_{kn}$-monotonicity of $\mathcal{E}_\Delta$ implies that the operator $\mathcal{E}'_\Delta(\cdot, S)$ is $\sqsubseteq$-monotone operator on $W$. Consequently, it has a least fixpoint. This fixpoint can be taken as the preferred way to revise $P$ given $S$.

Thus, we define

$$E^u_\Delta(S) = \text{lfp}(\mathcal{E}'_\Delta)^f(\cdot, S))$$

As in the case of autoepistemic logic, one can see that $E^u_\Delta$ also specifies the preferred way to revise $S$ given $P$, that is $E^u_\Delta(P) = \text{lfp}(\mathcal{E}'_\Delta(P, \cdot))$. Thus, we define the operator on $\mathcal{B}$ as follows:

$$\mathcal{E}'_\Delta(P, S) = (E^u_\Delta(S), E^u_\Delta(P)).$$
It turns out that the concept of extension as defined by Reiter can be obtained from the operator $E^t_{\Delta}$. It is known that Reiter’s extensions are theories of fixpoints of the operator $\Sigma_{\Delta}$ introduced by Guerreiro and Casanova (see also [Li90, MT93]). One can show that the operator $E^t_{\Delta}$ coincides with the well-founded semantics it implies a well-founded semantics $WF$ fixpoint. We will denote it by $WF$.

Theorem 4.4 A theory $T$ is an extension of a default theory $\Delta$ if and only if $T = \{ \varphi: I(\varphi) = t, \text{ for every } I \in Q \}$ for some fixpoint $Q$ of $E^t_{\Delta}$.

In view of Theorem 4.4, we refer to the fixpoints of $E^t_{\Delta}$ as extensions. We have the following result relating fixpoints of the operators $E^t_{\Delta}$ and $E^s_{\Delta}$.

Theorem 4.5 Let $\Delta$ be a default theory. For every possible-world structure $P$, $P$ is a fixpoint of $E^s_{\Delta}$ if and only if $(P, P)$ is a fixpoint of $E^t_{\Delta}$.

It follows that the fixpoints of $E^t_{\Delta}$ can be regarded as 4-valued (3-valued, in the case of consistent fixpoints) generalizations of an extension of a default theory. We will therefore call them partial extensions. It turns out that partial extensions coincide with stationary extensions defined in [PP94].

Our next result describes monotonicity properties of the operators $E^t_{\Delta}$ and $E^s_{\Delta}$.

Theorem 4.6 Let $\Delta$ be a default theory. Then, the operator $E^t_{\Delta}$ is $\subseteq$-antimonotone and the operator $E^s_{\Delta}$ is $\subseteq$-kn-monotone.

Theorem 4.6 implies that the operator $E^t_{\Delta}$ has a least fixpoint. We will denote it by $WF(\Delta)$ and refer to it as the well-founded fixpoint of $\Delta$. We will call the semantics it implies a well-founded semantics of $\Delta$. The well-founded semantics of $\Delta$ coincides with the well-founded semantics of default logic introduced by Baral and Subrahmanian [BS91]. The well-founded semantics allows us to approximate skeptical reasoning with extensions and yields a sufficient condition for the uniqueness of an extension.

Theorem 4.7 Let $\Delta$ be a default theory and let $WF(\Delta) = (P, S)$. Then:

1. The fixpoint $WF(\Delta)$ is consistent.
2. For every partial extensions $B$ of $E^t_{\Delta}$, $WF(\Delta) \subseteq B$.
3. If $J(\varphi) = t$ for every $I \in P$, then $\varphi$ belongs to every extension. If $J(\varphi) = f$ for some $J \in S$, $\varphi$ does not belong to any extension.

4. If $P = S$ (that is, if $WF(\Delta)$ is complete) then $\Delta$ has a unique extension corresponding to the possible-world structure $P$.

Well-founded semantics has a constructive flavor. It can be obtained by iterating the operator $E^t_{\Delta}$ over the belief pair $(A, \emptyset)$. In addition, by extending the approach described in [MT93], one can show that the problem of computing the well-founded semantics is in the class $\Delta^2$.

Finally, let us note connections between (partial) weak extensions and (partial) extensions, and between the Kripke-Kleene and well-founded semantics for default logic.

Theorem 4.8 Let $\Delta$ be a default theory. Then:

1. $KK(\Delta) \subseteq WF(\Delta)$.
2. Every extension of $\Delta$ is a $\subseteq$-minimal weak extension of $\Delta$.
3. Every partial extension $(P, S)$ of $\Delta$ is a minimal partial weak extension of $\Delta$ in the following sense: for every partial weak extension $(P', S')$, if $P' \subseteq P$ and $S' \subseteq S$, then $P = P'$ and $S = S'$.

In summary, default logic can be viewed as the logic of the operator $E_{\Delta}$. Its fixpoints define the semantics of partial weak extensions. The least fixpoint of $E_{\Delta}$ defines the Kripke-Kleene semantics. The operator $E_{\Delta}$ gives rise to the operator $E_{\Delta}$, which yields the semantics of weak extensions. Kripke-Kleene semantics provides an approximation for the skeptical reasoning under the semantics of weak extensions. The operator $E_{\Delta}$ also leads to the operator $E^t_{\Delta}$. Consistent fixpoints of this operator yield stationary extensions. Fixpoints of a related operator $E^t_{\Delta}$, defined on the lattice $\mathcal{W}$, correspond to extensions by Reiter. The least fixpoint of the operator $E^t_{\Delta}$ results in the well-founded semantics for default logic and approximates the skeptical reasoning under the semantics of extensions. The relationships between the operators of default logic are illustrated in Figure 3.

5 DEFAULT LOGIC VERSUS AUTOEPISTEMIC LOGIC

The results of the paper shed new light on the relationship between default and autoepistemic logics. The nature of this relationship was the subject of extensive investigations since the time both systems were introduced in early 80s. Konolige [Kon88] proposed to encode a default $d = (\alpha, \beta)$ by the modal formula
A default theory \( \Delta = (D, W) \) by a modal theory \( m(\Delta) = W \cup \{ m(d) : d \in D \} \). Despite the fact that the encoding is intuitive it does not provide a correspondence between default logic as defined by Reiter and autoepistemic logic as defined by Moore. Consider a default theory \( \Delta \) with \( \Delta \) yields the theory \( m(\Delta) = \{ m(d) : d \in D \} \). The theory \( m(\Delta) \) has two extensions. One of them is generated by the theory \( Cn(\emptyset) \) and corresponds to the only extension of \( \Delta \). The other expansion is generated by the theory \( Cn(\{ p \}) \). Thus, the Konolige’s translation does not give a one-to-one correspondence between extensions of default theories and expansions of their modal encodings.

This mismatch can be explained within the semantic framework introduced in the paper. Konolige’s translation does not establish correspondence between extensions and expansions because they are associated with different operators. Extensions are associated with fixpoints of the operator \( D_T \). Its counterpart on the side of default logic is the operator \( E_\Delta \). Fixpoints of this operator are not extensions but weak extensions of \( \Delta \). Extensions turn out to be associated with the operator \( E^s_\Delta \). Its counterpart on the side of autoepistemic logic is the operator \( D^s_T \), introduced in Section 3. This operator, to the best of our knowledge, has not appeared in the literature and properties of its fixpoints and the relationship to McDermott-Doyle style logics \[ MD80, McD82, MT93 \] are not known.

Once we properly align concepts from default logic with those from autoepistemic logic, Konolige’s translation works! This alignment is illustrated in Figure 3 and is formally described in the following theorem.

**Theorem 5.1** Let \( \Delta \) be a default theory and let \( T = m(\Delta) \). Then the following pairs of operators coincide and, thus, have the same fixpoints:

1. \( E_\Delta = D_T \) (that is, weak extensions correspond to expansions).
2. \( E^s_\Delta = D^s_T \) (that is, partial weak extensions correspond to partial expansions; Kripke-Kleene semantics for \( \Delta \) and Kripke-Kleene semantics for \( T \) coincide).
3. \( E^s_\Delta \) and \( D^s_T \) (that is, extensions correspond to strong expansions).
4. \( E^s_\Delta \) and \( D^s_T \) (that is, partial extensions correspond to partial strong expansions, well-founded semantics for \( \Delta \) and well-founded semantics for \( T \) coincide).

**6 DISCUSSION AND FUTURE WORK**

We presented results uncovering the semantic properties of default and autoepistemic logics. In each case, a whole family of semantics can be derived from a single operator by purely algebraic transformations. Most importantly, the translation of Konolige establishes a perfect correspondence between the families of semantics of default and autoepistemic logics. This elegant picture can be further extended to the case of logic programming. As discovered by Fitting, all key semantics for logic programs can be similarly obtained from a single operator, the 4-valued van Emden-Kowalski one-step provability operator \[ EK70 \]. The resulting semantic structure for logic programming is shown in Figure 4.

In addition, the translation of logic program clauses into default rules proposed in \[ BF91, MT89b \] establishes an embedding of logic programming into default
Kripke-Kleene fixpoint

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[Lev90] H. J. Levesque. All I know: a study in autoepistemic logic. *Artificial Intelligence*, 42(2-3):263–309, 1990.

[Lif90] V. Lifschitz. On open defaults. In J. Lloyd, editor, *Proceedings of the symposium on computational logic*, pages 80–95. Berlin: Springer-Verlag, 1990. ESPRIT Basic Research Series.

[McD82] D. McDermott. Nonmonotonic logic II: nonmonotonic modal theories. *Journal of the ACM*, 29(1):33–57, 1982.

[MD80] D. McDermott and J. Doyle. Nonmonotonic logic I. *Artificial Intelligence*, 13(1-2):41–72, 1980.

[Moo84] R.C. Moore. Possible-world semantics for autoepistemic logic. In *Proceedings of the Workshop on Non-Monotonic Reasoning*, pages 344–354, 1984. Reprinted in: M. Ginsberg, ed., *Readings on nonmonotonic reasoning*, pp. 137–142, Morgan Kaufmann, 1990.

[MT89a] W. Marek and M. Truszczynski. Relating autoepistemic and default logics. In *Proceedings of the First International Conference on Principles of Knowledge Representation and Reasoning (Toronto, ON, 1989)*, Morgan Kaufmann Series in Representation and Reasoning, pages 276–288, San Mateo, CA, 1989. Morgan Kaufmann.

[MT89b] W. Marek and M. Truszczynski. Stable semantics for logic programs and default theories. In E. Lusk and R. Overbeek, editors, *Proceedings of the North American Conference on Logic Programming*, pages 243–256. MIT Press, 1989.

[MT93] W. Marek and M. Truszczynski. *Nonmonotonic logics: context-dependent reasoning*. Springer-Verlag, Berlin, 1993.

[PP94] H. Przymusinska and T. Przymusinski. Stationary default extensions. *Fundamenta Informaticae*, 21(1-2):67–87, 1994.

[Rei80] R. Reiter. A logic for default reasoning. *Artificial Intelligence*, 13(1-2):81–132, 1980.

[Sch95] J. Schlipf. The expressive powers of the logic programming semantics. *Journal of the Computer Systems and Science*, 51(1):64–86, 1995.

[Tar55] A. Tarski. Lattice-theoretic fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5:285–309, 1955.

[Tru91] M. Truszczynski. Modal interpretations of default logic. In *Proceedings of IJCAI-91*, pages 393–398, San Mateo, CA, 1991. Morgan Kaufmann.

[vEK76] M.H. van Emden and R.A. Kowalski. The semantics of predicate logic as a programming language. *Journal of the ACM*, 23(4):733–742, 1976.

[VRS91] A. Van Gelder, K.A. Ross, and J.S. Schlipf. The well-founded semantics for general logic programs. *Journal of the ACM*, 38(3):620–650, 1991.