A Note on Slice Rank and Matchings in Groups

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Abstract

A multiplicative 3-matching in a group $G$ is a triple of sets $\{a_i\}, \{b_i\}, \{c_i\} \subset G$ such that $a_ib_jc_k = 1$ if and only if $i = j = k$. Here we record the fact that PSL$(2,p)$ has no multiplicative 3-matching of size greater than $O(p^{8/3})$, yet the slice rank of its group algebra’s multiplication tensor is at least $\Omega(p^3)$ over any field. This gives a negative answer to a conjecture of Petrov.

1 Introduction

A multiplicative 3-matching in a finite group $G$, hereon abbreviated to a 3-matching, is a triple of subsets $\{a_i\}_{i=1}^m, \{b_i\}_{i=1}^m, \{c_i\}_{i=1}^m$ of $G$ such that $a_ib_jc_k = 1 \iff i = j = k$. Let $M(G)$ denote the largest size\footnote{By size we mean the parameter $m$.} of a 3-matching in $G$. This quantity is of interest in additive combinatorics, as finite groups provide a model setting for understanding 3-term arithmetic-progression-free sets in the integers. It also has connections to algorithms for fast matrix multiplication \cite{CKSU05}.

The polynomial method of Croot, Lev, and Pach \cite{CLP17} and Ellenberg and Gijswijt \cite{EG17}, and its formulation in terms of slice rank due to Tao \cite{Tao16}, is a powerful tool for establishing upper bounds on $M(G)$. Remarkably, it gives an asymptotically tight bound on $M(G)$ in the case of $\mathbb{F}_p^n$ with $p > 2$ a fixed prime. Specifically, it shows that for a certain $c_p < p$ we have $M(\mathbb{F}_p^n) \leq c_p^n$, and it is also known that $M(\mathbb{F}_p^n) \geq c_p^{(1-o(1))n}$ \cite{KSS16}. This raises the following question, previously asked in \cite{Pet16}: is the slice rank bound on $M(G)$ always tight?

We now make these notions formal. We assume throughout that $k$ is an algebraically closed field\footnote{This is without loss of generality for us, since slice rank can only decrease under field extensions and we will prove a slice rank lower bound.}. Following \cite[Lemma 1(iv)]{TS16}, the slice rank of a trilinear form $T : U_1 \times U_2 \times U_3 \to k$ is

$$SR(T) = \max_{V_i \leq U_i : T(V_1,V_2,V_3) = 0} \text{codim}(V_1) + \text{codim}(V_2) + \text{codim}(V_3).$$

This can be thought of as an analogue of the “codimension of the kernel” definition of matrix rank for trilinear forms. To use slice rank to prove bounds on $M(G)$, we consider the multiplication tensor $T_{k[G]} : (k[G])^3 \to k$ of the group algebra $k[G]$, defined as $T_{k[G]} =$
\[ \sum_{g,h \in G} x_g y_h z_{gh}. \] The key fact is that \( \text{SR}(T_{k[G]}) \) is at least \( M(G) \) for any \( k \) [Tao16, Lemma 1]; this is an analogue of the fact that the rank of a matrix is at least the size of the largest identity-submatrix it contains. Hence upper bounds on \( \text{SR}(T_{k[G]}) \) imply upper bounds on \( M(G) \). Motivated by this, we make the following definition.

**Definition 1.1.** \( \text{SR}(G) = \min_k \text{SR}(T_{k[G]}) \).

Here the minimum is taken over all algebraically closed fields; crucially, the characteristic of \( k \) can be arbitrary. In the case that \( G = \mathbb{P}^n_\mathbb{F} \) for example, \( \text{SR}(T_{\mathbb{C}[\mathbb{F}^n_\mathbb{F}^n]}) = p^n \) [BCC+17, Corollary b.17] but \( \text{SR}(T_{\mathbb{F}_p[\mathbb{F}_p^n]}) \leq c_p^n \). This leads to the following conjecture, which is slightly weaker than one appearing in [Pet16]:

**Conjecture 1.2.** \( \text{SR}(G) \leq M(G) \cdot |G|^{o(1)}. \)

In this work we note that \( \text{PSL}(2, p) \) has no large 3-matchings but has high slice rank over any field, so Conjecture 1.2 is false. Both of these facts are in large part due to the lower bound of [LS74] on the dimensions of nontrivial irreducible representations of \( \text{PSL}(2, p) \). The fact that \( \text{PSL}(2, p) \) has no large 3-matching follows almost immediately from Gowers’s result on quasirandom groups [Gow08]. We now give a quick proof of this.

**Proposition 1.3.** \( \text{M}(\text{PSL}(2, p)) \leq O(p^{8/3}). \)

**Proof.** A triple of subsets \( A, B, C \subset G \) is called product-free if \( abc \neq 1 \) for all \( a \in A, b \in B, c \in C \). If \( A, B, C \) is a 3-matching of size \( m \), then there is a product-free triple of sets in \( G \) of size \( m' := [m/3] \) consisting of \( \{a_i\}_{i=1}^{m'}, \{b_i\}_{i=m'+1}^{2m'+1}, \{c_i\}_{i=2m'+1}^{3m'+1} \). In [Gow08] it is shown that \( \text{PSL}(2, p) \) does not contain product-free subsets larger than \( O(p^{8/3}) \), so the proposition follows. \( \square \)

In the next section we will present the following slice rank lower bound.

**Theorem 1.4.** \( \text{SR}(\text{PSL}(2, p)) \geq \Omega(p^3) \).

## 2 Proof of Theorem 1.4

If \( A \) is a finite dimensional algebra over \( k \), we let \( T_A \in A^* \otimes A^* \otimes A \) denote its multiplication tensor. If \( e_1, \ldots, e_n \) is a basis of \( A \) with dual basis \( e_1^*, \ldots, e_n^* \), this tensor is given in coordinates by \( \sum_{1 \leq i, j \leq n} e_i^* \otimes e_j^* \otimes (e_i \cdot e_j) \). We can view this as a trilinear form, for instance by linearly mapping \( e_i^* \otimes e_j^* \otimes e_k \) to the monomial \( x_i y_j z_k \), and define its slice rank as in the introduction. We write \( \text{SR}(A) \) for the slice rank of the multiplication tensor of \( A \).

Now we recall some basic facts about slice rank, namely that it is nonincreasing under linear transformations, and that the slice rank of an algebra is nonincreasing under quotients.

**Lemma 2.1.** [TS16, Lemma 3] Let \( T = \sum c_{ijk} u_i \otimes v_j \otimes w_k \in U \otimes V \otimes W \), and let \( A \in \text{Hom}(U, U'), B \in \text{Hom}(V, V'), C \in \text{Hom}(W, W') \). Then \( \text{SR}(\sum c_{ijk} A(u_i) \otimes B(v_j) \otimes C(w_k)) \leq \text{SR}(T) \).

\(^3\)The conjecture of [Pet16] asked if \( M(G) \) is roughly the sum of codimensions of subspaces multiplying to 0 in \( k[G] \), whereas Conjecture 1.2 is equivalent to asking if \( M(G) \) is roughly the sum of codimensions of subspaces whose product merely vanishes on the coefficient of 1 in \( k[G] \).
Lemma 2.2. If $I$ is a two-sided ideal of $A$, then $\text{SR}(A/I) \leq \text{SR}(A)$.

Proof. Let $\varphi : A \to A/I$ be the quotient map. Let $e_1, \ldots, e_n$ be a basis for $A$. Since $\varphi$ is an onto linear map, there exists $S \subseteq [n]$ so that $(\varphi(e_i))_{i \in S}$ is a basis of $A/I$. Let $P : A \to A/I$ be given by $P(e_i) = \varphi(e_i)$ for $i \in S$, and $P(e_i) = 0$ if $i \notin S$. Similarly define $P' : A^* \to (A/I)^*$ by $P'(e_i^*) = P(e_i)^*$. Applying $P'$ to the first two factors of $T_A$ and $P$ to the third, we obtain $T_{A/I}$. By Lemma 2.1 this proves the claim. \hfill \Box

For an algebra $A$, we denote by $\text{Irr}(A)$ the set of non-isomorphic irreducible representations of $A$ over $k$. Recall that in the ordinary (i.e., characteristic 0) representation theory of finite groups, representations are completely reducible, and in particular the group algebra $k[G]$ is isomorphic to a direct sum of matrix algebras. While this is false when the characteristic of the field divides the order of the group, we still have the following.

Definition 2.3. The radical of $A$, denoted $J(A)$, is the two-sided ideal of all elements of $A$ which act by zero on all irreducible representations of $A$.

Lemma 2.4. [EGH+11, Theorem 2.12] Let $A$ be a finite-dimensional algebra. Then

$$A/J(A) \cong \bigoplus_{V \in \text{Irr}(A)} \text{End}(V).$$

We will use the following fact, which says that the slice rank of direct sums of matrix multiplication tensors is maximal.

Lemma 2.5. [BCC+17, Proposition B.6] For any field $k$, $\text{SR}(\bigoplus_{i=1}^m \text{End}(k^{d_i})) = \sum_{i=1}^m d_i^2$.

The proof of Theorem 1.4 will go as follows. By Lemma 2.4, $k[G]/J(k[G])$ is a direct sum of matrix algebras, one for each irreducible representation of $k[G]$. So by Lemma 2.5, $k[G]/J(k[G])$ has full slice rank, and by Lemma 2.2 this is a lower bound on the slice rank of $k[G]$. So if we can show that there are many sufficiently large irreps of $G$, we conclude that $\text{SR}(k[G])$ is large. To give an example of when this fails dramatically, when $G$ is any $p$-group, the only irrep of $G$ when $k$ has characteristic $p$ is the trivial one [S+77, Corollary of Proposition 26], so this argument only says that $\text{SR}(k[G]) \geq 1$.

Proof of Theorem 1.4. Let $G = \text{PSL}(2,p)$ and let $k$ be a field of characteristic $\ell$. First, if $\ell = 0$ or $\ell$ is coprime to $|G| = (p-1)p(p+1)/2$, then $k[G]$ is semisimple and so by Lemma 2.5 the slice rank of $k[G]$ equals $|G| = \Omega(p^3)$. Next, if $\ell = p$, then the irreps of $\text{SL}(2,p)$ are given by the action of $G$ on homogeneous polynomials in two variables of degree up to $p-1$ with coefficients in $k$ [Alp93, p. 15]; since the center of $\text{SL}(2,p)$ acts trivially on even degree polynomials, these are also irreps of $\text{PSL}(2,p)$ (in fact, all of them). By Lemma 2.4 the dimension of $k[G]/J(k[G])$ is then $\sum_{i=0}^{(p-1)/2} (2i+1)^2 \geq \Omega(p^3)$, so the claim holds.

So suppose $\ell \neq p$ divides $|G| = (p-1)p(p+1)/2$. By [Alp93, I.3 Theorem 2], the number of irreducible representations of $G$ equals the number of conjugacy classes having order coprime to $\ell$. Next we show that there are $\Omega(p)$ such conjugacy classes of $G$. This follows from the more general bound of [HM22, Theorem 6.1]; here we sketch a proof for the special case of $\text{PSL}(2,p)$. See [FH13, p. 71] for a reference on conjugacy classes of $\text{SL}(2,p)$, which we adapt to $\text{PSL}(2,p)$. Most elements in $G$ are either conjugate to an element of the split torus, a
cyclic subgroup of order \((p-1)/2\), or the non-split torus, a cyclic subgroup of order \((p+1)/2\). The number of non-conjugate elements in the split torus is at least \((p-3)/4\), and the number of non-conjugate elements in the non-split torus is at least \((p-5)/4\) (with the exact values depending on \(p \mod 4\)). Because the orders of these tori are coprime, all conjugacy classes of elements in at least one of the subgroups have order coprime to \(\ell\). So there are at least \((p-5)/4\) such conjugacy classes. Finally, since the minimum dimension of a nontrivial irrep of \(G\) is at least \((p-1)/2\) in characteristic \(\ell \neq p\) [LS74], we conclude by Lemma 2.4 that \(\dim k[G]/J(k[G]) \geq \Omega(p^3)\), and thus by Lemma 2.5 \(\text{SR}(k[G]) \geq \Omega(p^3)\). }

One might wonder if all sufficiently quasirandom groups (groups with no small nontrivial irreps over \(\mathbb{C}\)) have high slice rank. Here is a conjecture towards this question.

**Conjecture 2.6.** For a fixed \(\varepsilon > 0\), let \(G\) be a group of order \(n\) that is \(n^\varepsilon\)-quasirandom. Then for all fields \(k\), we have the uniform bound of \(\dim k[G]/J(k[G]) \geq \Omega(n)\).

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