Non-abelian Painlevé systems
with generalized Okamoto integral

I.A. Bobrova,† V.V. Sokolov

Abstract

We study non-abelian systems of Painlevé type. To derive them, we introduce an auxiliary autonomous system with the frozen independent variable and postulate its integrability in the sense of the existence of a non-abelian first integral that generalizes the Okamoto Hamiltonian. All non-abelian $P_6 - P_2$-systems with such integrals are found. A coalescence limiting scheme is constructed for these non-abelian Painlevé systems. This allows us to construct an isomonodromic Lax pair for each of them.

Keywords: non-abelian ODEs, Painlevé equations, isomonodromic Lax pairs

1 Introduction

In Okamoto’s paper [13] all Painlevé equations $P_1 - P_6$ have been written as a polynomial Hamiltonian systems. In particular, the sixth Painlevé equation takes the form

$$
\begin{align*}
\frac{dz}{dz} &= 2u^3v - 2u^2v - \kappa_1 u^2 + \kappa_2 u + z \left( -2u^2v + 2uv + \kappa_4 u + (\kappa_1 - \kappa_2 - \kappa_4) \right), \\
\frac{dz}{dz} &= -3u^2v^2 + 2uv^2 + 2\kappa_1 uv - \kappa_2 v + \kappa_3 + z \left( 2uv^2 - v^2 - \kappa_4 v \right),
\end{align*}
$$

(1)

where $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ are arbitrary constants and $u(z), v(z), z \in \mathbb{C}$. Eliminating $v$, one can obtain the $P_6$-equation for $u(z)$. The Hamiltonian $H$ for (1) is given by

$$
\begin{align*}
z(z-1)H &= u^3v^2 - u^2v^2 - \kappa_1 u^2v + \kappa_2 uv - \kappa_3 u + z \left( -u^2v^2 + uv^2 + \kappa_4 uv + (\kappa_1 - \kappa_2 - \kappa_4)v \right).
\end{align*}
$$

Since system (1) is non-autonomous, the function $H$ is not an integral of motion.

System (1) has the form

$$
\begin{align*}
\frac{df}{dz} &= P_1(u, v) + zQ_1(u, v), \\
\frac{df}{dz} &= P_2(u, v) + zQ_2(u, v),
\end{align*}
$$

(2)

while the Hamiltonian has the following structure: $f(z)H = H_1 + zH_2$, where $P_i, Q_i, H_i$ are polynomials in $u, v$.  

*National Research University Higher School of Economics, Moscow, Russian Federation.
†L.D. Landau Institute for Theoretical Physics, Chernogolovka, Russian Federation. E-mail: vsokolov@landau.ac.ru
Let us consider the system

\[
\begin{align*}
\frac{du}{dt} &= P_1(u, v) + z Q_1(u, v), \\
\frac{dv}{dt} &= P_2(u, v) + z Q_2(u, v),
\end{align*}
\]  

(3)

where we regard \( z \) as a parameter. We call (3) the auxiliary autonomous system for (2). From the fact that (2) is a Hamiltonian system with the Hamiltonian \( H \) it follows that

\[ J = H_1(u, v) + z H_2(u, v) \]  

(4)

is an integral of motion for system (3) i.e. \( \frac{dJ}{dt} = 0 \). We call \( J \) the Okamoto integral.

In Section 2 we consider systems of the form (3), where \( P_i \) and \( Q_i \) are non-commutative polynomials given by

\[
\begin{align*}
P_1(u, v) &= a_1 u^3 v + a_2 u^2 v u + a_3 u v^2 + (2 - a_1 - a_2 - a_3) u v^3 + c_1 u^2 v \\
&\quad + (-2 - c_1 - c_2) u v u + c_2 u v^2 - \kappa_1 u^2 + \kappa_2 u, \\
Q_1(u, v) &= f_1 u^2 v + (-2 - f_1 - f_2) u v + f_2 u v^2 + h_1 u v + (2 - h_1) u v + \kappa_4 u \\
&\quad + (\kappa_1 - \kappa_2 - \kappa_4),
\end{align*}
\]

(5)

\[
\begin{align*}
P_2(u, v) &= b_1 u^2 v^2 + b_2 u v u v + b_3 v^2 u + b_4 u v v + b_5 u v v u + \left(-3 - \sum b_i\right) v^2 u^2 \\
&\quad + d_1 u v^2 + (2 - d_1 - d_2) u v + d_2 u^2 u + e_1 u v + (2 \kappa_1 - e_1) u v - \kappa_2 v + \kappa_3, \\
Q_2(u, v) &= g_1 u v^2 + (2 - g_1 - g_2) u v + g_2 u^2 v - v^2 - \kappa_4 v
\end{align*}
\]

(6)

and \( \kappa_i \) are arbitrary constants. In this paper we assume that all coefficients are complex numbers. If \( f(z) = z(z-1) \), then the corresponding system (2) is a natural non-commutative generalization of the Painlevé-6 system (1). To obtain the ansatz (5), (6), we replace each monomial \( M \) in (1) by a sum of non-commutative monomials such that this sum coincides with \( M \) under the commutative reduction.

The following example of non-abelian Hamiltonian Painlevé-6 system was found in [10]:

\[
\begin{align*}
z(z - 1)u' &= u^2 v u + u v u^2 - 2 u v u - \kappa_1 u^2 + \kappa_2 u \\
&\quad + z \left(-u^2 v - v u^2 + uv + v u + \kappa_4 u + (\kappa_1 - \kappa_2 - \kappa_4)\right), \\
z(z - 1)v' &= -u^2 v - v u v - u v^2 + 2 uv u + \kappa_1 u v + \kappa_1 v u - \kappa_2 v + \kappa_3 \\
&\quad + z \left(u v^2 + v^2 u - v^2 - \kappa_4 v\right).
\end{align*}
\]

In general, the classification problem we have in mind is to find all sets of coefficients in (5), (6) such that the corresponding system (2) is integrable. Consideration of all known non-abelian systems of \( P_1 - P_6 \) type [3, 15, 5, 10, 2, 6] shows that in all cases the corresponding auxiliary system is integrable in one sense or another. This observation led us to the idea of formulating an integrability criterion for non-abelian systems of \( P_1 - P_6 \) in terms of the auxiliary system.

In this paper to find interesting examples of non-abelian systems (2) we postulate for the corresponding system (3) the existence of a non-abelian integral of motion of the form (4).
In the case of $P_6$ systems the polynomials $H_i$ have the form
\begin{equation}
H_1(u,v) = p_1 u^3 v^2 + p_2 u^2 v u + p_3 u^2 v^3 u + p_4 u v u^2 v + p_5 u v u v v + p_6 u v^2 u^2 + p_7 v u^3 v \\
+ p_8 u^2 v u + p_9 v u v u^2 + (1 - \sum p_i) v^2 u^3 + q_1 u^2 v^2 + q_2 u v u v + q_3 v u^2 u \\
+ q_4 v u^2 v + q_5 v u v u + \left( -1 - \sum q_i \right) v^2 u^2 + r_1 u^2 v + r_2 u v v \\
+ (-\kappa_1 - \sum r_i) v u^2 + s_1 u v + (\kappa_2 - s_1) v u - \kappa_3 u,
\end{equation}

\begin{equation}
H_2(u,v) = t_1 u^2 v^2 + t_2 u v v u + t_3 u v^2 u + t_4 v u^2 v + t_5 v u v u + \left( -1 - \sum t_i \right) v^2 u^2 + x_1 u v v \\
+ x_2 v u v + \left( 1 - \sum x_i \right) v^2 u + y_1 u v + (\kappa_4 - y_1) v u + (\kappa_1 - \kappa_2 - \kappa_4) v.
\end{equation}

In the commutative case these polynomials coincide with
\begin{align*}
H_1 &= u^3 v^2 - u^2 v^2 - \kappa_1 u^2 v + \kappa_2 u v - \kappa_3 u, \\
H_2 &= -u^2 v^2 + u v^2 + \kappa_4 u v + (\kappa_1 - \kappa_2 - \kappa_4) v,
\end{align*}

corresponding to (1). Notice that we don’t assume that in the non-abelian case the system (3) is Hamiltonian.

As a result, we found 18 non-abelian systems (2) of Painlevé-6 type (see Appendix A.1). A transformation group acts on the set of these systems. There are 3 orbits of the group action and three non-equivalent systems corresponding to these orbits. Other systems can be derived from these three systems by applying the transformations.

All these systems turn out to be not Hamiltonian and therefore our approach cannot reconstruct the system $P_6^{t'}$.

To justify the integrability of obtained systems, we find the isomonodromic Lax representations of the form
\begin{equation}
A_z - B_\lambda = [B, A]
\end{equation}

for them.

The scalar equations $P_5 - P_1$ are also equivalent to polynomial Hamiltonian systems of the form (2)\(^1\) with $f(z) = z$ or $f(z) = 1$. In Section 3 we find all systems of $P_5$, $P_4$, $P_3'$, and $P_2$ types that have Okamoto integrals and obtain 10, 6, 8, and 2 non-abelian systems, respectively. Such non-abelian systems of type $P_1$ do not exist.

Although the system $P_2^3$ found in [2] as well as six systems of $P_4$ type from [6] are missing in our paper, we find interesting new non-abelian Painlevé type systems.

In Section 4 we extend the scheme

\begin{figure}[h]
\centering
\begin{tikzpicture}
    \node (P6) at (0,0) {$P_6$};
    \node (P5) at (1.5,0) {$P_5$};
    \node (P3) at (1.5,1.5) {$P_3'$};
    \node (P2) at (1.5,-1.5) {$P_2$};
    \node (P1) at (3,0) {$P_1$};
    \node (P4) at (0,-1.5) {$P_4$};
    \draw[->] (P6) -- (P5);
    \draw[->] (P6) -- (P3);
    \draw[->] (P3) -- (P1);
    \draw[->] (P4) -- (P2);
    \draw[->] (P2) -- (P1);
\end{tikzpicture}
\caption{The degeneration scheme of the Painlevé equations [8]}
\end{figure}

of degenerations of the scalar Painlevé equations [8] and their isomonodromic Lax pairs to the non-abelian case and show that all 26 systems of $P_5 - P_2$ type found is Section 3 and their Lax representations can be obtained by limiting transitions from the systems of $P_6$-type

\footnote{Except for some degenerations of the $P_3$ equation. For their non-abelian generalizations see Appendix C.}
described in Section 2 and Appendix A.1. In addition, we somewhat unexpectedly obtain Hamiltonian non-abelian \( P_5 - P_1 \) systems [10, 5] (see Appendix B). These systems do not have the Okamoto integral. The reason for the appearance of these systems is that sometimes the Okamoto integral degenerates into the integral \( I = uv - vu \) that all Hamiltonian systems have (see Lemma 1).

In Appendix A we present explicit list of all \( P_6 - P_2 \) type systems that have the Okamoto integral. In Appendix C we consider degenerate non-abelian \( P'_3 \) systems, which we call systems of \( P'_3(D_7) \) type. Although the corresponding scalar system is not polynomial, its non-abelian counterparts admitting Okamoto integrals were easily found.

1.1 Non-abelian ODE systems

In this paper we consider non-abelian systems of the form

\[
\begin{align*}
\frac{du}{dt} &= F_1(u, v), \\
\frac{dv}{dt} &= F_2(u, v),
\end{align*}
\]

(9)

where \( u \) and \( v \) are generators of the free associative algebra \( A \) over \( \mathbb{C} \) with the unity \( 1^2 \), and \( F_\alpha \in A \). Actually, (9) is a notation for the derivation \( d_t \) of \( A \) such that \( d_t(u) = F_1, \ d_t(v) = F_2 \). The element \( d_t(f) \) is uniquely determined for any element \( f \in A \) by the Leibniz identity.

Usually, the first integrals of a system (9) are some elements of the quotient vector space \( A/[A, A] \). They are a formalization of integrals of the form \( \text{trace}(h(u, v)) \) in the matrix case \( u(t), v(t) \in \text{Mat}_m \). For the Hamiltonian non-abelian systems the Hamiltonians are first integrals of this kind.

The Hamiltonian systems (9) have the form

\[
\begin{align*}
\frac{du}{dt} &= \frac{\partial H}{\partial v}, \\
\frac{dv}{dt} &= -\frac{\partial H}{\partial u},
\end{align*}
\]

(10)

where \( H \in A \) and \( \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \) are non-abelian derivatives (see [11]). For any polynomial \( f(u, v) \in A \) these derivatives are defined by the identity

\[
df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv,
\]

where the additional non-abelian variables \( du \) and \( dv \) are supposed to be moved to the right by the cyclic permutations of generators in monomials. Notice that in the non-abelian case the partial derivatives are not vector fields.

**Remark 1.** It is easy to verify that \( \frac{\partial}{\partial u}(ab - ba) = \frac{\partial}{\partial v}(ab - ba) = 0 \) for any \( a, b \in A \) and therefore the non-abelian partial derivatives are well-defined maps \( A/[A, A] \to A \). For this reason, we can assume that \( H \in A/[A, A] \) in the formula (10).

**Example 1.** The auxiliary system for \( P^H_6 \) is Hamiltonian with

\[
H = H_1(u, v) + z H_2(u, v),
\]

\[
H_1 = u^2 v w - u v w - \kappa_1 u^2 v + \kappa_2 u w - \kappa_3 u, \quad H_2 = -u^2 v^2 + u v^2 + \kappa_4 u v + (\kappa_1 - \kappa_2 - \kappa_4) v.
\]

\(^{2}\)For any \( k \in \mathbb{C} \) we often write \( k \) instead of \( k1 \).
Indeed, \(dH = dH_1 + z dH_2\), where
\[
dH_1 = du uvuv + u du vuv + u^2 du vv + u^2 v du v + u^2 v uvv - du vv vuv - uv du v \quad \text{and}
\]
\[
dH_2 = -du uu^2 - u du v^2 - u^2 dv v + u^2 v du v + u dv vv - uv du v - uvu dv - \kappa_1 (du uvv + u du v + u^2 dv) + \kappa_2 (du v + u dv) - \kappa_3 du,
\]

Making cyclic permutations in all monomials to bring \(du\) and \(dv\) to the right, we obtain
\[
dH_1 = (uvuv + vuuv + vu^2 v - 2 uvv - \kappa_1 uvv - \kappa_1 vu + \kappa_2 v - \kappa_3) du
\]
\[
+ (uvu^2 + u^2 vv - 2 uvv - \kappa_1 u^2 + \kappa_2 u) dv,
\]
\[
dH_2 = (-uv^2 - v^2 u + v^2 + \kappa_4 v) du + (-uv^2 - u^2 v + uu + uv + \kappa_4 u
\]
\[
+ (\kappa_1 - \kappa_2 - \kappa_4)) dv.
\]

Therefore,
\[
\frac{\partial H}{\partial v} = u^2 vu + uvu^2 - 2 uvu - \kappa_1 u^2 + \kappa_2 u + z (-u^2 v - vu^2 + uv + vu + \kappa_4 u + (\kappa_1 - \kappa_2 - \kappa_4))
\]
\[
\frac{\partial H}{\partial u} = -uvuv - vuuv - vu^2 v + 2 uvv + \kappa_1 uv + \kappa_1 uv + \kappa_2 v + \kappa_3 + z (uv^2 + v^2 u - v^2 - \kappa_4 v)
\]

and (10) coincides with the auxiliary system for \(P^H_6\).

In this paper we are dealing with non-abelian first integrals, which are elements of \(A\) but not traces from \(A/[A, A]\).

**Definition 1.** An element \(h \in A\) are called a non-abelian first integral for system (9) if \(\partial_t (h) = 0\).

For non-abelian Hamiltonian systems with two variables \(u\) and \(v\) a special integral \(I = uv - vu\) appears in the following statement:

**Lemma 1.** Any system of the form (10) has the non-abelian first integral \(I = uv - vu\).

**Proof.** It follows from the following well-known identity [11] for partial derivatives:
\[
\left[ u, \frac{\partial f}{\partial u} \right] + \left[ v, \frac{\partial f}{\partial v} \right] = 0, \quad f \in A.
\]

In our paper we assume that the auxiliary system (3) for Painlevé type system (2) has a non-abelian Okamoto first integral (see Introduction) of the form (4). Both in the auxiliary system and in the Okamoto integral, the variable \(z\) plays the role of an arbitrary parameter. Using the terminology of the bi-Hamiltonian formalism, we have pencils of two non-abelian dynamical systems and two non-abelian first integrals.

---

^3 Notice that the Hamiltonian \(H\) of a system (10) is not a first integral in the sense of Definition 1.
Instead of algebra $\mathcal{A}$ with multiplication $xy$ one can consider the associative algebra with the opposite product $x \star y = yx$. The transition to the opposite multiplication is represented by the involution $\tau: \mathcal{A} \to \mathcal{A}$ defined by
\[
\tau(u) = u, \quad \tau(v) = v, \quad \tau(ax + by) = a\tau(x) + b\tau(y), \quad \tau(xy) = \tau(y)\tau(x),
\]
where $x, y \in \mathcal{A}$, $a, b \in \mathbb{C}$. This involution is called transposition.

All key properties of integrable systems, such as the existence of first integrals, infinitesimal symmetries, Lax representations, etc., are invariant under $\tau$.

2 Painlevé-6 systems

Consider the non-abelian systems (3), where the non-commutative polynomials $P_i(u, v), Q_i(u, v)$ are given by the formulas (5), (6).

**Proposition 1.** Such a system possesses a non-abelian Okamoto integral of the form (4), (7), where $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ are arbitrary parameters iff the corresponding $P_6$ system belongs to the list $1P_6 - 18P_6$ from Appendix A.1.

**Proof.** Differentiating the integral (4) with respect to the system defined by (3), (5), (6), we obtain a polynomial $Y(u, v, z)$ of degree 8 in $u, v$. Equating to zero the coefficients of different monomials in $Y(u, v, z)$, we arrive at a system of nonlinear algebraic equations. The simplest equations from this system are:

\[
p_1 = p_3 = p_5 = p_7 = 0, \quad p_2 = 1 - p_4 - p_5 - p_8 - p_9.
\]

It turns out that all coefficients of polynomials $P_i, Q_i$ can be expressed in terms of the Okamoto integral in the following way:

\[
a_1 = 1 - p_4 - p_5 - p_8 - p_9, \quad a_2 = 1 + p_4 - p_8 - p_9, \quad a_3 = p_5 + 2p_8 + p_9, \\
b_1 = -1 + p_4 + p_5 + p_8 + p_9, \quad b_2 = -2 + p_5 + 2p_8 + 2p_9, \\
b_3 = 0, \quad b_4 = -p_4 - p_5 - p_8, \quad b_5 = -p_5 - 2p_8 - 2p_9, \\
c_1 = -d_1 = 2q_1 + q_2, \quad c_2 = -d_2 = -2 - 2q_1 - 2q_2 - 2q_3 - 2q_4 - q_5, \quad h_1 = 2x_1 + x_2, \\
e_1 = 2\kappa_1 + 2r_1 + r_2, \quad f_2 = 1 - t_3 + x_1 + x_2 - p_5 - p_8 - 2p_9 + 2q_1 + 2q_2 + q_3 + 2q_4 + q_5, \\
f_1 = -2 - t_3 - x_1 + p_4 + p_5 + 2p_8 + 2p_9 - 2q_1 - q_2 - q_3.
\]

Equating to zero the coefficients of different monomials of degree 8 in $Y(u, v, z)$, we obtain a system of nonlinear algebraic equations for the variables $a_i, i = 1, 2, 3, b_i, i = 1, \ldots, 5$ and $p_i, i = 1, \ldots, 9$. Using the above formulas, we can eliminate $a_i$ and $b_i$ and obtain a system for $p_4, p_5, p_8, p_9$, which is equivalent to

\[
(p_4 - 1)p_4 = (p_5 - 1)p_5 = (p_8 - 1)p_8 = (p_9 - 1)p_9 = 0; \quad p_4 p_5 = p_4 p_8 = p_4 p_9 = p_5 p_8 = p_5 p_9 = p_8 p_9 = 0.
\]

This system have 5 solutions which leads to the following cases:

**Case 1:** $a_1 = 0, a_2 = 0, a_3 = 2, b_1 = 0, b_2 = 0, b_3 = 0, b_4 = -1, b_5 = -2$,
\[
p_1 = 0, p_2 = 0, p_3 = 0, p_4 = 0, p_5 = 0, p_6 = 0, p_7 = 0, p_8 = 1, p_9 = 0;
\]

**Case 2:** $a_1 = 0, a_2 = 0, a_3 = 1, b_1 = 0, b_2 = 0, b_3 = 0, b_4 = 0, b_5 = -2$,
\[
p_1 = 0, p_2 = 0, p_3 = 0, p_4 = 0, p_5 = 0, p_6 = 0, p_7 = 0, p_8 = 0, p_9 = 1;
Case 3: \(a_1 = 0, a_2 = 1, a_3 = 1, b_1 = 0, b_2 = -1, b_3 = 0, b_4 = -1, b_5 = -1,\)
\(p_1 = 0, p_2 = 0, p_3 = 0, p_4 = 0, p_5 = 1, p_6 = 0, p_7 = 0, p_8 = 0, p_9 = 0;\)

Case 4: \(a_1 = 0, a_2 = 2, a_3 = 0, b_1 = 0, b_2 = -2, b_3 = 0, b_4 = -1, b_5 = 0,\)
\(p_1 = 0, p_2 = 0, p_3 = 0, p_4 = 1, p_5 = 0, p_6 = 0, p_7 = 0, p_8 = 0, p_9 = 0;\)

Case 5: \(a_1 = 1, a_2 = 1, a_3 = 0, b_1 = -1, b_2 = -2, b_3 = 0, b_4 = 0, b_5 = 0,\)
\(p_1 = 0, p_2 = 1, p_3 = 0, p_4 = 0, p_5 = 0, p_6 = 0, p_7 = 0, p_8 = 0, p_9 = 0.\)

In each case, equating to zero the remaining coefficients in the polynomial \(Y(u, v, z),\) we
obtain a large but rather simple algebraic system for \(c_i, d_i, f_i, g_i, q_i, r_i, s_i, t_i, x_i, y_i.\) This
system contains \(\kappa_1, \kappa_2, \kappa_3, \kappa_4\) as parameters. Solving the algebraic system in Case 1, we
obtain systems \(1P_6 - 3P_6,\) Case 2 leads to \(4P_6 - 6P_6,\) in Case 3 we have \(7P_6 - 12P_6.\) In Cases
4 and 5 the systems are given by \(13P_6 - 15P_6\) and \(16P_6 - 18P_6\) respectively. All systems
contain four arbitrary parameters \(\kappa_1 - \kappa_4.\) Notice that additional systems that correspond
to particular values of the parameters do not exist.

\[\square\]

### 2.1 Transformation group

The transformations
\[
\begin{align*}
  r_1 : (z, u, v) &\mapsto (1 - z, 1 - u, -v), \\
  r_2 : (z, u, v) &\mapsto (z^{-1}, z^{-1}u, zv),
\end{align*}
\]
act on the set of eighteen systems from Proposition 1. They change the parameters in the
following way
\[
\begin{align*}
  r_1 : (\kappa_1, \kappa_2, \kappa_3, \kappa_4) &\mapsto (\kappa_1, 2\kappa_1 - \kappa_2 - \kappa_4, \kappa_3, \kappa_4), \\
  r_2 : (\kappa_1, \kappa_2, \kappa_3, \kappa_4) &\mapsto (\kappa_1, \kappa_4 - 1, \kappa_3, \kappa_2 + 1).
\end{align*}
\]
These transformations generate a group isomorphic to \(S_3.\) The involution \(\tau\) defined by (11)
commutes with \(r_i\) and also acts on the set of systems. We denote the action of \(\tau\) by the
superscript \(T.\) The whole transformation group is isomorphic to \(S_3 \times \mathbb{Z}_2\) and acts on the \(P_6\)
type systems as follows:

**Cases 1 and 4:**
\[
\begin{align*}
  r_1(P_6) &\mapsto 3P_6, & r_2(P_6) &\mapsto 1P_6, & (P_6)^T &\mapsto 13P_6, \\
  r_1(2P_6) &\mapsto 2P_6, & r_2(2P_6) &\mapsto 3P_6, & (2P_6)^T &\mapsto 15P_6, \\
  r_1(3P_6) &\mapsto 1P_6, & r_2(3P_6) &\mapsto 2P_6, & (3P_6)^T &\mapsto 14P_6.
\end{align*}
\]

**Cases 2 and 5:**
\[
\begin{align*}
  r_1(4P_6) &\mapsto 6P_6, & r_2(4P_6) &\mapsto 5P_6, & (4P_6)^T &\mapsto 16P_6, \\
  r_1(5P_6) &\mapsto 3P_6, & r_2(5P_6) &\mapsto 4P_6, & (5P_6)^T &\mapsto 17P_6, \\
  r_1(6P_6) &\mapsto 4P_6, & r_2(6P_6) &\mapsto 6P_6, & (6P_6)^T &\mapsto 18P_6.
\end{align*}
\]

**Case 3:**
\[
\begin{align*}
  r_1(7P_6) &\mapsto 8P_6, & r_2(7P_6) &\mapsto (7P_6)^T, & (7P_6)^T &\mapsto 10P_6, \\
  r_1(8P_6) &\mapsto 7P_6, & r_2(8P_6) &\mapsto (8P_6)^T, & (8P_6)^T &\mapsto 11P_6, \\
  r_1(9P_6) &\mapsto (9P_6)^T, & r_2(9P_6) &\mapsto (9P_6)^T, & (9P_6)^T &\mapsto 12P_6.
\end{align*}
\]
It follows from these relations that there are three orbits of the action of the transformation group:

Orbit 1 = \{3P_6, 4P_6, 5P_6, 13P_6, 14P_6, 15P_6\},
Orbit 2 = \{4P_6, 5P_6, 6P_6, 16P_6, 17P_6, 18P_6\},
Orbit 3 = \{7P_6, 8P_6, 9P_6, 10P_6, 11P_6, 12P_6\}.

We choose the systems 3P_6, 6P_6, and 7P_6 as representatives of these orbits. Other systems can be obtained from them by the formulas:

\begin{align*}
1P_6 &= r_1(3P_6), \quad 2P_6 = r_2(3P_6), \quad 13P_6 = (r_1(3P_6))^T, \quad 14P_6 = (3P_6)^T, \quad 15P_6 = (r_2(3P_6))^T, \\
4P_6 &= r_1(6P_6), \quad 5P_6 = r_1r_2(6P_6), \quad 16P_6 = (r_1(6P_6))^T, \quad 17P_6 = (r_1r_2(6P_6))^T, \quad 18P_6 = (6P_6)^T, \\
8P_6 &= r_1(7P_6), \quad 9P_6 = (r_1r_2(7P_6))^T, \quad 10P_6 = (7P_6)^T, \quad 11P_6 = (r_1(7P_6))^T, \quad 12P_6 = r_1r_2(7P_6).
\end{align*}

2.2 Isomonodromic representations

It is well-known [9] that the scalar system (1) has the isomonodromic representation (8), where matrices \(A(z, \lambda)\) and \(B(z, \lambda)\) have the form

\[
A(z, \lambda) = \frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1} + \frac{A_2}{\lambda - z}, \quad B(z, \lambda) = B_0 - \frac{A_2}{\lambda - z}
\]

with the following matrices \(A_0, A_1, A_2,\) and \(B_0:\)

\[
A_0 = \begin{pmatrix} -1 - \kappa_1 + \kappa_4 & uz^{-1} - 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -uv + \kappa_1 \\ -u^2v^2 + \kappa_1uv + \kappa_3 \\ uv \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{uv + (\kappa_1 - \kappa_2 - \kappa_4)}{zuw^2 + (\kappa_1 - \kappa_2 - \kappa_4)zw} & -uz^{-1} \\ zuv^2 + (\kappa_1 - \kappa_2 - \kappa_4)zv & -uv \end{pmatrix},
\]

\[
B_0 = \begin{pmatrix} (z(z - 1))^{-1} \left(2u^2v - \kappa_1u - z(2uv + (\kappa_1 - \kappa_2 - \kappa_4)) \right) & 0 \\ -uv^2 - (\kappa_1 - \kappa_2 - \kappa_4)v & 0 \end{pmatrix}.
\]

It is clear that the shifts

\[
A \mapsto A + p(\lambda)I, \quad B \mapsto B + q(z, u, v)I
\]

are allowed by the relation (13). Note that in the non-abelian case the second shift is admissible only if \(q\) does not depend on \(u\) and \(v\).

We generalize this representation to the case of non-abelian systems \(3P_6, 6P_6, 7P_6\) by means of the non-abelinizing procedure described in [7].

2.2.1 System 3P_6:

\[
A_0 = \begin{pmatrix} -1 - \kappa_1 + \kappa_4 & uz^{-1} - 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -uv + \kappa_1 \\ -vuv^2 + \frac{1}{2}(\kappa_1 + \kappa_3)uv \\ + \frac{1}{2}(\kappa_1 - \kappa_3)vu + \frac{3}{4}(\kappa_3^2 - \kappa_1^2) \\ 0 \end{pmatrix}.
\]
\[
A_2 = \begin{pmatrix}
uv + (\kappa_1 - \kappa_2 - \kappa_4) & -uz^{-1} \\
zuv + (\kappa_1 - \kappa_2 - \kappa_4)zv & -vu
\end{pmatrix},
\]
\[
B_0 = \begin{pmatrix}
(z(z-1))^{-1} (2vu - \kappa_1 u - z(uv + vu + (\kappa_1 - \kappa_2 - \kappa_4))) & 0 \\
-vu - (\kappa_1 - \kappa_2 - \kappa_4)v & (z-1)^{-1} (vu - v)
\end{pmatrix}.
\]

2.2.2 System \textbf{6P}_6:

\[
A_0 = \begin{pmatrix}
-1 - \kappa_1 + \kappa_4 & uz^{-1} - 1 \\
0 & 0
\end{pmatrix}, \quad A_1 = \begin{pmatrix}
-vu + \kappa_1 & 1 \\
-vu + \kappa_1 uv + \kappa_3 & uv
\end{pmatrix},
\]
\[
A_2 = \begin{pmatrix}
vu + (\kappa_1 - \kappa_2 - \kappa_4) & -uz^{-1} \\
-zv^2 u + (\kappa_1 - \kappa_2 - \kappa_4) zv & -vu
\end{pmatrix},
\]
\[
B_0 = \begin{pmatrix}
(z(z-1))^{-1} (uv + vu^2 - \kappa_1 u - z(vu + v + (\kappa_1 - \kappa_2 - \kappa_4))) & 0 \\
-vu^2 - (\kappa_1 - \kappa_2 - \kappa_4)v & (z-1)^{-1} (-vu + v)
\end{pmatrix}.
\]

2.2.3 System \textbf{7P}_6:

\[
A_0 = \begin{pmatrix}
-1 - \kappa_1 + \kappa_4 & uz^{-1} - 1 \\
0 & 0
\end{pmatrix}, \quad A_1 = \begin{pmatrix}
-vu + \kappa_1 & 1 \\
-vu + \kappa_1 uv + \kappa_3 & uv
\end{pmatrix},
\]
\[
A_2 = \begin{pmatrix}
vu + (\kappa_1 - \kappa_2 - \kappa_4) & -uz^{-1} \\
-zv^2 u + (\kappa_1 - \kappa_2 - \kappa_4) zv & -vu
\end{pmatrix}, \quad (16)
\]
\[
B_0 = \begin{pmatrix}
(z(z-1))^{-1} (u^2 v + uvu - \kappa_1 u - z(2uv + vu - v + (\kappa_1 - \kappa_2 - \kappa_4))) & 0 \\
-uv^2 - (\kappa_1 - \kappa_2 - \kappa_4)v & (z-1)^{-1} (-vu + v)
\end{pmatrix}.
\]

The isomonodromic representations for remaining 15 systems from Proposition 1 can be obtained using the transformation group from Section 2.1. Since the involutions \( r_1 \) and \( r_2 \) do not preserve the structure of the Lax pair, we should supplement each of them by a proper gauge transformation

\[
A \mapsto g (A \ g^{-1} + g_z \ g^{-1}), \quad B \mapsto g (B \ g^{-1} + g_z \ g^{-1}) \quad (17)
\]

followed by a change of variables \( z, u, v, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \lambda \).

The involution \( r_1 \) (12) we extend to the Lax pair by the formulas

\[
\lambda \mapsto (1 - z) \lambda (\lambda - z)^{-1}, \quad g = \begin{pmatrix}
(z-1)^{-1} (\lambda - z) & 0 \\
\lambda v & -1
\end{pmatrix}. \quad (18)
\]

For the involution \( r_2 \) a similar transformation is defined by

\[
\lambda \mapsto \lambda^{-1}, \quad g = \begin{pmatrix}
\lambda & 0 \\
0 & 1
\end{pmatrix}.
\]

As a result, in both cases we arrive at a Lax pair, which coincides with (14) the under the commutative reduction.
Example 2. Let us show how to get a Lax pair for the $\mathfrak{sp}_6$ system starting with the pair \((16)\) for the $\mathfrak{sp}_6$ system.

As it was mentioned in Section 2.1, the $\mathfrak{sp}_6$ system is related to the system $\mathfrak{sp}_6$ via the involution $r_1$. This connection on the level of Lax pairs is described above. Applying to the pair \((16)\) the gauge transformation \((17)\) with the matrix $g$ defined in \((18)\) and making the change of the variables and parameters

$$z \mapsto 1-z, \ u \mapsto 1-u, \ v \mapsto -v, \ \kappa_2 \mapsto 2\kappa_1 - \kappa_2 - \kappa_4, \ \lambda \mapsto (1-z)\lambda(\lambda-z)^{-1},$$

we obtain a pair of the form \((13)\), where matrices $A_0$, $A_1$, $A_2$, and $B_0$ are

$$A_0 = \begin{pmatrix} -1 - \kappa_1 + \kappa_4 & uz^{-1} - 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -uv + \kappa_1 & 1 \\ -uvu + \kappa_1uv + \kappa_3 & uv \end{pmatrix},$$

$$A_2 = \begin{pmatrix} uv + (\kappa_1 - \kappa_2 - \kappa_4) & -uz^{-1} \\ zvu + (\kappa_1 - \kappa_2 - \kappa_4)zv & -vu \end{pmatrix},$$

$$B_0 = \begin{pmatrix} (z(z-1))^{-1}(uvu-uvu - \kappa_1 u - z(2uv + vu - v + (\kappa_1 - \kappa_2 - \kappa_4))) & 0 \\ -vu - (\kappa_1 - \kappa_2 - \kappa_4)v & (z-1)^{-1}(-uv + v) \end{pmatrix}.$$  \(19\)

The zero-curvature condition for this pair leads to the $\mathfrak{g}_6$ system.

In the case of transposition \((11)\) the situation is slightly different. The matrices of a pair transform as

$$A \mapsto -A^t, \quad B \mapsto -B^t, \quad (20)$$

where each matrix entry should be transformed by $\tau$. Here the subscript $t$ means the matrix transpose. After that, using the map \((17)\) with

$$g = \lambda^{-1-\kappa_1+\kappa_4}(\lambda - 1)^{\kappa_1}(\lambda - z)^{\kappa_1-\kappa_2-\kappa_4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (21)$$

we get matrices $A_0$, $A_1$, $A_2$ with the same dependence on the variables and parameters. The diagonal entries of the matrix $B_0$ have a different structure. However, in the scalar case the resulting matrix $B_0$ can be brought to the form \((14)\) by a shift \((15)\). In the non-abelian case it is impossible.

Example 3. The $\mathfrak{sp}_6$ system can be obtained from the system $\mathfrak{sp}_6$ by the transposition \((11)\). Let us extend this transformation to the Lax pairs. According to \((20)\), the pair \((16)\) turns into

$$A_0 = -\begin{pmatrix} -1 - \kappa_1 + \kappa_4 & 0 \\ uz^{-1} - 1 & 0 \end{pmatrix}, \quad A_1 = -\begin{pmatrix} -uv + \kappa_1 & -uvu + \kappa_1uv + \kappa_3 \\ 1 & uv \end{pmatrix},$$

$$A_2 = -\begin{pmatrix} vu + (\kappa_1 - \kappa_2 - \kappa_4) & zvu + (\kappa_1 - \kappa_2 - \kappa_4)zv \\ -uv - (\kappa_1 - \kappa_2 - \kappa_4)v & -vu \end{pmatrix},$$

$$B_0 = -\begin{pmatrix} (z(z-1))^{-1}(vu^2 + vu - \kappa_1u - z(2vu + uv - v + (\kappa_1 - \kappa_2 - \kappa_4))) & 0 \\ -v^2u - (\kappa_1 - \kappa_2 - \kappa_4)v & (z-1)^{-1}(-uv + v) \end{pmatrix}.$$
Now, using the gauge transformation with \((21)\), we get

\[
    A_0 = \begin{pmatrix} -1 - \kappa_1 + \kappa_4 & uz^{-1} - 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -v\kappa_1 + 1 \\ -v\kappa_1 + \kappa_1 \kappa_3 \end{pmatrix},
\]

\[
    A_2 = \begin{pmatrix} \frac{v}{v} + (\kappa_1 - \kappa_2 - \kappa_4) \\ zv^2 u + (\kappa_1 - \kappa_2 - \kappa_4) z v \end{pmatrix}, \quad B_0 = \begin{pmatrix} (z - 1)^{-1} (uv - v) \\ -v^2 u - (\kappa_1 - \kappa_2 - \kappa_4) v (z(z - 1))^{-1} (-v^2 - u v + \kappa_1 u + z (2 v u + v - v + (\kappa_1 - \kappa_2 - \kappa_4))) \end{pmatrix},
\]

(22)

3 Systems of \(P_5 - P_2\) type

3.1 Painlevé-5 systems

The scalar \(P_5\) system has the following structure:

\[
\begin{align*}
    \frac{d}{dz} (u) &= P_1(u, v) + \kappa_4 z u, \\
    \frac{d}{dz} (v) &= P_2(u, v) - \kappa_4 z v.
\end{align*}
\]

(23)

The non-abelian ansatz for the polynomials \(P_1(u, v)\) and \(P_2(u, v)\) are given by

\[
    P_1(u, v) = a_1 u^3 v + a_2 u^2 v u + a_3 u v u^2 + \left( 2 - \sum a_i \right) v u^3 + c_1 u^2 v + (-4 - c_1 - c_2) u v u + c_2 v u^2 - \kappa_1 u^2 + e_1 u v + (2 - e_1) v u + (\kappa_1 + \kappa_2) u - \kappa_2,
\]

\[
    P_2(u, v) = b_1 u^2 v^2 + b_2 u v u^2 + b_3 u v u^2 + b_4 v^2 u^2 + b_5 v v u u + \left( 3 - \sum b_i \right) v^2 u^2 + d_1 u w^2 + (4 - d_1 - d_2) v u v - d_2 v^2 u - v^2 + f_1 u v + (2 f_1 - f_1) v u - (\kappa_1 + \kappa_2) v + \kappa_3,
\]

(24)

while the non-abelian Okamoto integral has the form

\[
    J = p_1 u^3 v^2 + p_2 u^2 v u + p_3 u^2 v^2 + p_4 u v u^2 + p_5 v u u v + p_6 u^2 v^2 + p_7 v u^3 v + p_8 u^2 v v u + p_9 v u u^2 + \left( 1 - \sum p_i \right) v^2 u^3 + q_1 u^2 v^2 + q_2 u u v + q_3 v^2 u + q_4 u^2 v + q_5 v u u + \left( -\kappa_1 - \sum r_i \right) v u v + s_1 u v^2 + s_2 v u v + (1 - \sum s_i) v^2 u + t_1 u v + (\kappa_1 + \kappa_2 - t_1) v u - \kappa_3 u - \kappa_2 v + \kappa_3 + z (w_1 u v + (\kappa_4 - w_1) v u).
\]

(25)

Proposition 2. The auxiliary system

\[
\begin{align*}
    \frac{d}{dt} (u) &= P_1(u, v) + \kappa_4 z u, \\
    \frac{d}{dt} (v) &= P_2(u, v) - \kappa_4 z v
\end{align*}
\]

for (23), (24) has a non-abelian integral of the form (25) iff the corresponding \(P_5\) system belongs to the list \(P_5 - P_2\) from Appendix A.2.
There are five orbits,

Orbit 1 = \{1P_5, 7P_5\}, Orbit 2 = \{2P_5, 8P_5\}, Orbit 3 = \{3P_5, 9P_5\},
Orbit 4 = \{4P_5, 10P_5\}, Orbit 5 = \{5P_5, 6P_5\},

with respect to the transposition (11).

3.2 The Painlevé-4 systems

The anzats [6] for non-abelian system of P_4 type can be written as

\[
\begin{align*}
\frac{du}{dz} &= -u^2 + 2uv + \alpha [u,v] - 2zv + \kappa_2, \\
\frac{dv}{dz} &= -v^2 + 2vu + \beta [v,u] + 2zv + \kappa_3
\end{align*}
\]  

(26)

with the parameters \(\alpha, \beta \in \mathbb{C}\) to be defined. The non-abelian Okamoto integral has the following structure:

\[
J = a_1 uv^2 + (1 - a_1 - a_2) uvu + a_2 u^2 v + b_1 u^2 v + (-1 - b_1 - b_2) uvu + b_2 vu^2 - \kappa_3 u + \kappa_2 v + z (c_1 uv + (-2 - c_1)vu).
\]  

(27)

**Proposition 3.** There are six systems 1P_4 – 6P_4 of the form (26) whose auxiliary system have the Okamoto integral of the form (27). They are listed in Appendix A.3.

**Remark 2.** In the papers [6, 7] all systems (26) that satisfy the matrix Painlevé test have been found. The corresponding pairs \((\alpha, \beta)\) are shown in the following figure

![Diagram of orbits and parameters](image)

The systems from Appendix A.3 are marked by blue dots without orange rims and belong to the same orbit with respect to the transformation group used in these papers. The central red dot corresponds to the Hamiltonian P_4 system (see Appendix B) which will appear in Section 4 devoted to limiting transitions.

The second order non-abelian Painlevé equation corresponding to the system with \((\alpha, \beta) = (0, -1)\) was first obtained in the paper [1].
3.3 Painlevé-3′ systems

In the scalar case, the Hamiltonian $h$ for $P_3'(D_6)$-system [14]

$$\begin{cases}
    z u' = 2u^2v + \kappa_1 u + \kappa_2 u^2 + \kappa_4 z, \\
    z v' = -2uv^2 - \kappa_1 v - 2\kappa_2 uv - \kappa_3
\end{cases} \quad (28)$$

is given by

$$z h = u^2v^2 + \kappa_1 uv + \kappa_2 u^2v + \kappa_3 u + \kappa_4 z.$$  \quad (29)

Proposition 4. Let

$$P_1(u, v) = a_1 u^2 v + (2 - a_1 - a_2) uv u + a_2 vu^2 + \kappa_1 u + \kappa_2 u^2,$$

$$P_2(u, v) = b_1 uv^2 - (2 + b_1 + b_2) uv v + b_2 u^2 v - \kappa_1 v + c_1 uv + (-2\kappa_2 - c_1) v u - \kappa_3,$$

and

$$J = d_1 u^2 v^2 + d_2 uv^2 u + d_3 uv u v + d_4 vu^2 v + d_5 v u v + \left(1 - \sum d_i\right) v^2 u^2 + e_1 uv$$

$$+ (\kappa_1 - e_1) v u + h_1 u^2 v + (\kappa_2 - h_1 - h_2) u v u + h_2 v u^2 + \kappa_3 u + \kappa_4 z v.$$ \quad (30)

Then a non-abelian system (29) has an Okamoto integral $J$ of the form (30) iff the corresponding $P_3'$ system belongs to the list $1P_3' - 8P_3'$ from Appendix A.4.

There are four orbits,

**Orbit 1** $= \{1P_3', 2P_3'\}$, **Orbit 2** $= \{3P_3', 8P_3'\}$,

**Orbit 3** $= \{4P_3', 6P_3'\}$, **Orbit 4** $= \{5P_3', 7P_3'\}$,

with respect to the transposition action.

3.4 Painlevé-2 systems

The scalar $P_2$-system

$$\begin{cases}
    u' = -u^2 + v - \frac{1}{2} z, \\
    v' = 2uv + \kappa_3
\end{cases} \quad (31)$$

has the following Hamiltonian:

$$h = \frac{1}{2} v^2 - u^2 v - \kappa_3 u - \frac{1}{2} z v.$$ \quad (32)

A non-abelian generalization of (31) can be written as

$$\begin{cases}
    u' = -u^2 + v - \frac{1}{2} z, \\
    v' = 2vu + \beta [v, u] + \kappa_3.
\end{cases} \quad (33)$$

The ansatz for a non-abelian analog of (32) is given by

$$J = a_1 u^2 v + (-1 - a_1 - a_2) uv u + a_2 vu^2 + \frac{1}{2} v^2 - \kappa_3 u - \frac{1}{2} z v.$$ \quad (34)
**Proposition 5.** There are two systems of the form (33) that have the Okamoto integral (34) (see Appendix A.5).

**Remark 3.** These systems are related by transposition (11). There exist two more non-equivalent integrable systems of \( P_2 \) type (see [2]).

## 4 Tree of degenerations

Schematically, the degenerations of non-abelian \( P_6 \) systems into systems of types \( P_5 - P_1 \) can be represented as follows. The red arrows correspond to the representatives of the \( P_6 \) orbits (see Section 2.1) and their degenerations.

![Figure 2: The degeneration scheme of the systems from the Orbit 1](image)

Two schemes of Figure 2 show separately chains of degenerations of the types \( P_6 \rightarrow P_5 \rightarrow P_4 \rightarrow P_2 \rightarrow P_1 \) and \( P_6 \rightarrow P_5 \rightarrow P'_3 \rightarrow P_2 \rightarrow P_1 \) for systems from Orbit 1.

The degenerations of \( P_6 \) systems from Orbits 2 and 3 are shown in Figure 3 below.

![Figure 3: The degeneration scheme of the systems from the Orbits 2 and 3](image)

### 4.1 Description of degenerations

In this section, we present explicit formulas corresponding to the arrows in Figures 2 and 3. In addition to all systems that possess the Okamoto integrals, the non-abelian Hamiltonian
Painlevé systems of types $P_5 - P_1$ appear here (see Appendix B). They are denoted by $P^H_i$.

We also describe limiting transitions for isomonodromic Lax pairs. The Lax representations for all systems shown in Figures 2 and 3 can be obtained from ones for the $P_6$ systems (see Section 2.2).

**Remark 4.** In the explicit formulas for the Lax pair degenerations $P_4 \rightarrow P_2$, $P'_6 \rightarrow P_2$ and $P_2 \rightarrow P_1$ the function $g$ contains a constant different for different systems. If we take this constant equal to zero, then in some cases we get a Lax pair that is not polynomial on $\varepsilon$ but has a pole at $\varepsilon = 0$. In such a situation, we choose a constant so that the pole disappears, and after that we set $\varepsilon = 0$.

Notice that for branches starting with the three representatives and marked in red, all constants are zero. The same is true for all branches starting with 6 systems connected to the representatives via $r_1, r_2$, but not via the transposition (11).

### 4.1.1 $P_6 \rightarrow P_5$

The systems of $P_6$ type can be reduced to $P_5$ systems by the following map with the small parameter $\varepsilon$:

$$
\begin{align*}
    z & \mapsto \varepsilon^{-1}(z - 1), \\
    \kappa_2 & \mapsto -\kappa_1 + \kappa_2 + \kappa_4, \\
    \kappa_4 & \mapsto -\varepsilon \kappa_1 + \varepsilon \kappa_4.
\end{align*}
$$

To obtain the corresponding Lax pair, we supplement the transformation (35) by

$$
\lambda \mapsto (z - 1)^{-1}(\lambda - 1).
$$

As a result, a pair of the form (13) degenerates to a Lax pair of the following structure:

$$
A(\lambda, z) = A_0 + \frac{A_1}{\lambda} + \frac{A_2}{\lambda - 1}, \quad B(\lambda, z) = B_1 \lambda + B_0
$$

for a $P_5$ system.

**Remark 5.** In the formulas (13), (36), (40), (44), (47), (52) we indicate only the dependence of $A$ and $B$ on $\lambda$. We hope that the use of the same notation for different coefficients $A_i$ and $B_i$ in these formulas will not lead to misunderstanding.

In the following example we demonstrate the mechanism for the appearance of Hamiltonian systems in the process of limiting transitions.

**Example 4 ($7P_6 \rightarrow P_5^H$).** After changing the variables and parameters given by (35), system $7P_6$ and the corresponding Okamoto integral turn into

$$
\begin{align*}
    z(1 + \varepsilon z) u' & = u^2 u v + u v u^2 - u^2 v - 2 u v u - v u^2 - \kappa_1 u^2 + u v + v u + (\kappa_1 + \kappa_2) u - \kappa_2 + \kappa_4 z u + \varepsilon \ v (-u v u - v u^2 + u v + v u + \kappa_1 u - \kappa_2), \\
    z(1 + \varepsilon z) v' & = -u v w - u v^2 v - v w u + u v^2 + 2 u v u + v^2 u - v^2 + \kappa_1 v u + \kappa_1 v u - (\kappa_1 + \kappa_2) v + \kappa_3 - \kappa_4 z v + \varepsilon \ z (v u v + v^2 u - v^2 - \kappa_1 v), \\
    J & = -\kappa_4 \varepsilon^{-1} (u v - v u) + u v v u - u w u - v w u - \kappa_1 u v u + v u + \kappa_1 v u + \kappa_2 u w - \kappa_2 v - \kappa_3 u + \kappa_4 z v u + \varepsilon \ z (-u v u + v u + \kappa_1 v u - \kappa_2 v).
\end{align*}
$$
Under the limit $\varepsilon \to 0$, the system becomes the Hamiltonian $P_5^H$ system

\[
\begin{align*}
z u' &= u^2v + uuv^2 - u^2v - 2uvu - v + uv + vu \\
z v' &= -uvu - vu^2v - v - u + 2vuv + v^2u - v^2 + \kappa_1uv + \kappa_1uv - (\kappa_1 + \kappa_2)v + \kappa_3 - \kappa_4 z \nu,
\end{align*}
\]

and the integral degenerates to (cf. Lemma 1)

\[I = uv - vu.\]

To get a Lax pair for the resulting system, in addition to the above transformation, we should replace the spectral parameter $\lambda$ by $1 + \varepsilon z \lambda$. As a result, the Lax pair (10) takes the form (36), where matrices $A_0$, $A_1$, $A_2$, $B_1$, and $B_0$ are given by

\[
A_0 = \begin{pmatrix} \kappa_4 z & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -uv + \kappa_1 \\ -uvu + \kappa_1 uv + \kappa_3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} uv - \kappa_2 \\ uv^2 - \kappa_2v - uv \end{pmatrix},
\]

\[
B_1 = \begin{pmatrix} \kappa_4 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_0 = z^{-1} \begin{pmatrix} u^2v + uvu - 2uv - v - \kappa_1u + \kappa_4 & -u + 1 \\ -uvu + uv^2 + \kappa_1uv - \kappa_2v + \kappa_3 & -uv + v \end{pmatrix}.
\]

4.1.2 $P_5 \to P_4$

In the case of $P_5 \to P_4$, one can consider the transformation

\[
z \mapsto \frac{1}{\sqrt{z}} \varepsilon^{-1}(z - 1), \quad u \mapsto \sqrt{z} \varepsilon^{-1}u, \quad v \mapsto \sqrt{z} \varepsilon v,
\]

taking together with the following changes of the parameters $\kappa_i$:

\[
\kappa_1 = -\kappa_3 + 2\varepsilon^{-1}, \quad \kappa_2 \mapsto -2\kappa_2, \quad \kappa_3 \mapsto 2\kappa_3 - 2\varepsilon^{-2}, \quad \kappa_4 = -\varepsilon^{-2};
\]

\[
\kappa_1 = \kappa_3 + 2\varepsilon^{-1}, \quad \kappa_2 \mapsto -2\kappa_2, \quad \kappa_3 \mapsto -2\kappa_3 - 2\varepsilon^{-2}, \quad \kappa_4 = -\varepsilon^{-2}
\]

for Cases 1 and 4; and

\[
\kappa_1 = \varepsilon^{-1}, \quad \kappa_2 \mapsto -2\kappa_2, \quad \kappa_3 \mapsto 2\varepsilon^2 \kappa_3, \quad \kappa_4 = -\varepsilon^{-2}
\]

for Cases 2, 3, 5. For the degeneration of Lax pairs, in addition to the above formulas, consider the following mapping:

\[
\lambda \mapsto \frac{1}{\sqrt{z}} \varepsilon^{-1}(\lambda - 1), \quad A \mapsto g Ag^{-1}, \quad B \mapsto gBg^{-1}, \quad g = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{z} \varepsilon \end{pmatrix},
\]

which brings pair (36) to the form

\[
A(\lambda, z) = A_1 \lambda + A_0 + A_{-1} \lambda^{-1}, \quad B(\lambda, z) = B_1 \lambda + B_0.
\]

4.1.3 $P_5 \to P_3'$

The mapping

\[
u \mapsto \varepsilon^{-1}(u - 1), \quad v \mapsto \varepsilon v,
\]

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together with the following change of the parameters:

\[ \begin{align*}
\kappa_1 & \mapsto -\kappa_1 + \kappa_2, & \kappa_2 & \mapsto -\frac{1}{2}\varepsilon(\kappa_1 - \kappa_3), & \kappa_3 & \mapsto \frac{1}{4}\varepsilon(\kappa_1^2 - \kappa_3^2), & \kappa_4 & \mapsto \varepsilon^{-1}\kappa_4; \\
\kappa_1 & \mapsto -\kappa_1 + \kappa_2, & \kappa_2 & \mapsto -\frac{1}{2}\varepsilon(\kappa_1 + \kappa_3), & \kappa_3 & \mapsto \frac{1}{4}\varepsilon(\kappa_1^2 - \kappa_3^2), & \kappa_4 & \mapsto \varepsilon^{-1}\kappa_4
\end{align*} \]

for Cases 1 and 4; and

\[ \begin{align*}
\kappa_1 & \mapsto -\kappa_1 + \kappa_2, & \kappa_2 & \mapsto -\varepsilon\kappa_2, & \kappa_3 & \mapsto -\varepsilon\kappa_3, & \kappa_4 & \mapsto \varepsilon^{-1}\kappa_4
\end{align*} \tag{42} \]

for Cases 2, 3, 5 describes the degeneration \( P_5 \to P_3' \). Supplementing the latter formulas with

\[ \begin{align*}
\lambda & \mapsto \varepsilon\lambda, & A & \mapsto gA^{-1}, & B & \mapsto gB^{-1}, & g & = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}, \tag{43}
\end{align*} \]

and taking the limit \( \varepsilon \to 0 \), we bring matrices (36) to the form

\[ \begin{align*}
A(\lambda, z) &= A_0 + A_{-1}\lambda^{-1} + A_{-2}\lambda^{-2}, & B(\lambda, z) &= B_1\lambda + B_0. \tag{44}
\end{align*} \]

**4.1.4 \( P_4 \to P_2 \)**

Putting

\[ \begin{align*}
z & \mapsto \frac{1}{4}\varepsilon^{-4} - \varepsilon^{-1}z, & u & \mapsto -\frac{1}{4}\varepsilon^{-2} - \varepsilon u, & v & \mapsto -\frac{1}{2}\varepsilon^{-1}v, & \kappa_2 & = -\frac{1}{16}\varepsilon^{-6}, & \kappa_3 & \mapsto \frac{1}{2}\kappa_3 \tag{45}
\end{align*} \]

and taking the limit \( \varepsilon \to 0 \), one can obtain the degeneration \( P_4 \to P_2 \) for \( P_4 \) systems. The corresponding degeneration for a pair (40) is given by

\[ \begin{align*}
\lambda & \mapsto \frac{1}{4}\varepsilon^{-2} + 2\varepsilon\lambda, & A & \mapsto gA^{-1}, & B & \mapsto gB^{-1} + g'g^{-1}, & g & = e^{\text{const}}\varepsilon^{-3}z \begin{pmatrix} 1 & 0 \\ -\varepsilon & \varepsilon \end{pmatrix}. \tag{46}
\end{align*} \]

As a result, we obtain a pair

\[ \begin{align*}
A(\lambda, z) &= A_2\lambda^2 + A_1\lambda + A_0, & B(\lambda, z) &= B_1\lambda + B_0 \tag{47}
\end{align*} \]

of the Jimbo-Miwa type for the corresponding non-abelian \( P_2 \) system.

**4.1.5 \( P_3' \to P_2 \)**

The following map

\[ \begin{align*}
z & \mapsto -\frac{1}{2}\varepsilon^{-2} - \frac{1}{2}\varepsilon z, & u & \mapsto \frac{1}{4}\varepsilon^{-1}(u - 1), & v & \mapsto 2\varepsilon v, & \kappa_1 & = \frac{1}{2}\varepsilon^{-3}, & \kappa_2 & = -\frac{1}{4}\varepsilon^{-3}, & \kappa_3 & = -4\varepsilon^3\kappa_3, & \kappa_4 & = \frac{1}{4} \tag{48}
\end{align*} \]

reduces \( P_3' \) type systems to systems of \( P_2 \) type. To get the corresponding Lax pair, we consider the transformation

\[ \begin{align*}
\lambda & \mapsto -\frac{1}{2}\varepsilon^{-1}(\lambda + 1), & A & \mapsto gA^{-1}, & B & \mapsto gB^{-1} + g'g^{-1}, & g & = e^{\text{const}}z \begin{pmatrix} 1 & 0 \\ 0 & 2\varepsilon^2 \end{pmatrix}, \tag{49}
\end{align*} \]

that, after taking the limit \( \varepsilon \to 0 \), brings (44) to the form (47).
The degeneration of $P_2$ systems is given by

$$z \mapsto \varepsilon^{-2}z + 6\varepsilon^{-12}, \quad u \mapsto -\varepsilon^{-1}u + \varepsilon^4v + 3\varepsilon^{-6}, \quad v \mapsto -2\varepsilon^{-4}u + \varepsilon v + 4\varepsilon^{-9},$$

$$\kappa_3 = 4\varepsilon^{-15}. \quad (50)$$

Combining this mapping with the following degeneration formulas

$$\lambda \mapsto \varepsilon^{-6} + \varepsilon^{-1}\lambda,$$

$$A \mapsto g A g^{-1} + g' \lambda g^{-1}, \quad B \mapsto g B g^{-1} + g' \lambda g^{-1},$$

$$g = e^{4\varepsilon^{-10} + \varepsilon^{-5}\lambda^2 + \text{const} \varepsilon^{-5}z} \begin{pmatrix} 1 & 0 \\ \varepsilon^2 u & \varepsilon^2 \end{pmatrix}, \quad (51)$$

we obtain from (47) a Lax pair of the form

$$A(\lambda, z) = A_2 \lambda^2 + A_1 \lambda + A_0, \quad B(\lambda, z) = B_1 \lambda + B_0$$

(52)

for the corresponding $P_1$ type system.

**Example 5.** Below we list, as an example, the degenerations of Lax pairs for system $8P_6$ (see Example 2). The scheme of the degenerations is given in Figure 4:

![Diagram](image)

Figure 4: Degenerations of $8P_6$

- $8P_6 \rightarrow 6P_5$. The limiting transition given by formula (35) transforms Lax pair (19) to a pair of the form (36) with

$$A_0 = \begin{pmatrix} \kappa_4 z & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -uv + \kappa_1 & 1 \\ -uvw + \kappa_1 uv + \kappa_3 & uv \end{pmatrix}, \quad A_2 = \begin{pmatrix} uv - \kappa_2 & -u \\ vuv - \kappa_2 v & -vu \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \kappa_4 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_0 = (2uv - uv) \begin{pmatrix} u^2v + uv - 2uv - vu - \kappa_1 u + v + \kappa_1 & -u + 1 \\ -uvuv + vuv + \kappa_1 uv - \kappa_2 v + \kappa_3 & -vu + v \end{pmatrix} \quad (53)$$

for the $6P_5$ system.

- $6P_5 \rightarrow 1P_4$. Using the map (37), (38), (39), we reduce Lax pair (53) to (40), where

$$A_1 = \begin{pmatrix} -2 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} -2z & 1 \\ uv + \kappa_3 & 0 \end{pmatrix}, \quad A_{-1} = \begin{pmatrix} uv + \kappa_2 & -u \\ vuv + \kappa_2 v & -vu \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -2 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} -u + v - 2z & 1 \\ uv + \kappa_3 & v \end{pmatrix}. \quad (54)$$
- $6P_5 \to 6P'_3$. The substitution of the formulas (41), (42), (43) into (53) gives a pair (44) with

$$A_0 = \begin{pmatrix} \kappa_4 z & 0 \\ 0 & 0 \end{pmatrix}, \quad A_{-1} = - \begin{pmatrix} \kappa_1 & u \\ \kappa_2 u + \kappa_3 & \kappa_2 \end{pmatrix} \begin{pmatrix} 1 & \kappa_1 \\ 0 & \kappa_2 \end{pmatrix}, \quad A_{-2} = \begin{pmatrix} \kappa_2 + \kappa_3 & -1 \\ \kappa_1 & \kappa_2 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \kappa_4 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_0 = z^{-1} \begin{pmatrix} \kappa_2 u + \kappa_3 & -u \\ \kappa_2 u + \kappa_3 & -v \end{pmatrix}$$

for the $6P'_3$ system.

- $1P_4, 6P'_3 \to P'_2^H$. The degeneration data (45), (46) with $\text{const} = 0$ and (48), (49) with $\text{const} = 0$ for the $1P_4$ and $6P'_3$ systems, respectively, lead to the $P'_2^H$ system with a pair (47), where

$$A_2 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & -1 \\ -v & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} -v + z & -2v \\ \kappa_2 u & \kappa_2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} -u & 1 \\ \kappa_2 u & 0 \end{pmatrix}.$$ (54)

- $P'_2 \to P'_1^H$. The limiting transition defined by (50), (51) with $\text{const} = 0$ reduces Lax pair (52) to a pair (54) with

$$A_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & -2 \\ -2u & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} -v & -2v \\ 2u^2 + \kappa_2 u & \kappa_2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & -1 \\ -2u & 0 \end{pmatrix}.$$ (55)

for the $P'_1^H$ system.

**Example 6.** To illustrate Remark 4, we present the degenerations of Lax pairs for the branch starting from system $10P_6$ (see Example 3). Degenerations are shown in Figure 5:

![Figure 5: Degenerations of $10P_6$](image)

- $10P_6 \to P'_5^H$. Substituting (35) into Lax pair (22), we obtain a pair of the form (36) for the $P'_5^H$ system. The matrices read as

$$A_0 = \begin{pmatrix} \kappa_4 z & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} \kappa_1 & 1 \\ -v \kappa_2 u + \kappa_3 v u + \kappa_4 & v u \\ \kappa_2 & \kappa_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \kappa_2 & -u \\ \kappa_2 u + \kappa_3 & \kappa_2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} \kappa_4 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_0 = z^{-1} \begin{pmatrix} \kappa_2 u + \kappa_3 & -u + 1 \\ -v \kappa_2 u + \kappa_3 v u - \kappa_2 v + \kappa_4 & -v \kappa_2 u + \kappa_3 v u - \kappa_2 v + \kappa_4 \end{pmatrix}.$$ (56)
• \( P_5^H \to P_4^H \). Using the map (37), (38), (39), we reduce Lax pair (56) to (40), with

\[
A_1 = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} -2z & 1 \\ vu + \kappa_3 & 0 \end{pmatrix}, \quad A_{-1} = \frac{1}{2} \begin{pmatrix} vu + \kappa_2 & -u \\ v^2u + \kappa_2v & -vu \end{pmatrix}, \quad B_1 = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} v - 2z & 1 \\ vu + \kappa_3 & u - v \end{pmatrix}.
\] (57)

• \( P_5^H \to P_3^H \). The limiting transition defined by the formulas (41), (42), (43) transforms the pair (56) to (44) with

\[
A_0 = \begin{pmatrix} \kappa_4z & 0 \\ 0 & 0 \end{pmatrix}, \quad A_{-1} = \begin{pmatrix} \kappa_1 \\ vu + \kappa_1v + \kappa_2vu + \kappa_3 \\ 0 \end{pmatrix}, \quad A_{-2} = \begin{pmatrix} v + \kappa_2 \\ v^2 + \kappa_2v \\ -v \end{pmatrix}, \quad B_1 = \begin{pmatrix} \kappa_4 \\ 0 \\ 0 \end{pmatrix}, \quad B_0 = z^{-1} \begin{pmatrix} u - v - \kappa_1 \\ -v\alpha + \kappa_1v + \kappa_2vu + \kappa_3 \\ -vu - \kappa_2u - \kappa_1 \end{pmatrix}.
\] (58)

• \( P_4^H, P_3^H \to P_2^H \). The substitution of (45), (46) with \( \text{const} = \frac{1}{4} \) into (57) and (48), (49) with \( \text{const} = -\frac{1}{4} \) into (58) leads to the pair of the form (47) for the \( P_2^H \) system, where

\[
A_2 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & -2 \\ -v & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} -v + z \\ vu + \kappa_3 \\ -2u \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 \\ -\frac{1}{2}v \\ -1 \end{pmatrix}.
\] (59)

• \( P_2^H \to P_1^H \). The mapping (50), (51) with \( \text{const} = 0 \) reduces Lax pair (59) to pair (55) for the \( P_1^H \) system.

5 Conclusion

We have constructed a special class of Painlevé type systems and presented isomonodromic Lax pairs for them. Our systems from Appendix A, taking together with Hamiltonian systems from Appendix B found by H. Kawakami [10], form a collection of systems that are closed with respect to degenerations.

The following unsolved problems appear in connection with the obtained results:

• Extend this class of systems to a complete list of non-abelian analogs of Painlevé equations, including all known examples;

• Find the corresponding Painlevé - Calogero systems [4];

• Interpret obtained systems as symmetry reductions of 1 + 1 or 2 + 1 dimensional integrable systems;

• Prove that their solutions are meromorphic;

• Find Bäcklund transformations for the resulting systems.
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Appendices

A Lists of non-abelian systems of Painlevé type that have Okamoto integral

A.1 Systems of $P_6$ type

A.1.1 Case 1

$$\begin{align*}
z(z - 1)u' &= 2\nu v^2 - \nu v - \nu^2 - \kappa_1 u^2 + \kappa_2 u \\
&\quad + z (-\nu v u + \nu v^2 + 2\nu u + \kappa_4 u + (\kappa_1 - \kappa_2 - \kappa_4)), \\
z(z - 1)v' &= -\nu v^2 v - 2\nu u v^2 + \nu v + \nu^2 u + \frac{1}{2}(\kappa_1 + \kappa_3)uv + \frac{1}{2}(3\kappa_1 - \kappa_3)uv \\
&\quad - \kappa_2 v + \frac{1}{4}(\kappa_3^2 - \kappa_1^2) + z (\nu v v + \nu^2 u - \nu^2 - \kappa_4 v), \\
J &= \nu^2 v u - \nu v u - \frac{1}{2}(\kappa_3 + \kappa_1)uv + \frac{1}{2}(\kappa_3 - \kappa_1)\nu u^2 + \kappa_2 \nu u - \frac{1}{4}(\kappa_3^2 - \kappa_1^2)u + z (-\nu v u + \nu^2 u + \kappa_4 u + (\kappa_1 - \kappa_2 - \kappa_4)v).
\end{align*}$$

$$\begin{align*}
z(z - 1)u' &= 2\nu^2 u^2 - \nu u v - \nu^2 - \kappa_1 u^2 + \kappa_2 u \\
&\quad + z (-2\nu u v + uv + \nu u + \kappa_4 u + (\kappa_1 - \kappa_2 - \kappa_4)), \\
z(z - 1)v' &= -\nu^2 v^2 - 2\nu u v^2 + \nu v + \nu^2 u + \frac{1}{2}(\kappa_1 + \kappa_3)uv + \frac{1}{2}(3\kappa_1 - \kappa_3)uv \\
&\quad - \kappa_2 v + \frac{1}{4}(\kappa_3^2 - \kappa_1^2) + z (2\nu v u - \nu^2 - \kappa_4 v), \\
J &= v^2 u v - v u v - \frac{1}{2}(\kappa_3 + \kappa_1)uv + \frac{1}{2}(\kappa_3 - \kappa_1)\nu u^2 + \kappa_2 \nu u - \frac{1}{4}(\kappa_3^2 - \kappa_1^2)u + z (-\nu v u + \nu^2 u + \kappa_4 u + (\kappa_1 - \kappa_2 - \kappa_4)v).
\end{align*}$$
A.1.2 Case 2

\[
\begin{align*}
\begin{cases}
z(z-1)u &= uu^2 + vu^3 - 2vu^2 - \kappa_1 u^2 + \kappa_2 u \\
z(z-1)v &= -2vuvu - v^2 u^2 + 2v^2 u + 2\kappa_1 vu - \kappa_2 v + \kappa_3 \\
&+ z (vuv + v^2 u - v^2 - \kappa_4)
\end{cases}
\end{align*}
\]

\[J = vuvu^2 - v^2 u^2 - \kappa_1 vu^2 + \kappa_2 vu - \kappa_3 u + z (-vuvu + v^2 u + \kappa_4 v u + (\kappa_1 - \kappa_2 - \kappa_4) v).
\]

\[
\begin{align*}
\begin{cases}
z(z-1)u &= uu^2 + vu^3 - uu - v^2 - \kappa_1 u^2 + \kappa_2 u \\
z(z-1)v &= -2vuvu - v^2 u^2 + vuv + v^2 u + 2\kappa_1 vu - \kappa_2 v + \kappa_3 \\
&+ z (2v^2 u - v^2 - \kappa_4)
\end{cases}
\end{align*}
\]

\[J = vuvu^2 - vuvu - \kappa_1 vu^2 + \kappa_2 vu - \kappa_3 u + z (-vuvu + vuv + \kappa_4 v u + (\kappa_1 - \kappa_2 - \kappa_4) v).
\]

A.1.3 Case 3

\[
\begin{align*}
\begin{cases}
z(z-1)u &= u^2 vu + uu^2 - u^2 v - uu - \kappa_1 u^2 + \kappa_2 u \\
z(z-1)v &= -uvuv - uv^2 v - uvuv + u^2 v + uu + \kappa_1 uv + \kappa_2 v + \kappa_3 \\
&+ z (vuv + v^2 u - v^2 - \kappa_4)
\end{cases}
\end{align*}
\]

\[J = uvuv - uvuv - \kappa_1 uv + \kappa_2 uv - \kappa_3 u + z (-uvuv + uv + \kappa_4 v u + (\kappa_1 - \kappa_2 - \kappa_4) v).
\]

\[
\begin{align*}
\begin{cases}
z(z-1)u &= u^2 vu + uu^2 - 2uu - \kappa_1 u^2 + \kappa_2 u \\
z(z-1)v &= -uvuv - uv^2 v - uvuv + 2uv + \kappa_1 uv + \kappa_2 v + \kappa_3 \\
&+ z (uv + v^2 u - v^2 - \kappa_4)
\end{cases}
\end{align*}
\]

\[J = uvuv - uv^2 u - \kappa_1 uv + (\kappa_2 - \kappa_1 + \kappa_4) uv + (\kappa_1 - \kappa_4) vu - \kappa_3 u + z (-uvuv + v^2 u + \kappa_4 v u + (\kappa_1 - \kappa_2 - \kappa_4) v).
\]
\[
\begin{align*}
\{ z(z-1)u' &= u^2vu + uvu^2 - uv^2 - uvu - \kappa_1u^2 + \kappa_2u \\
&\quad + z(-2uv + 2uv + \kappa_4u + (\kappa_1 - \kappa_2 - \kappa_4)), \\
\{ z(z-1)v' &= -uvv - vu^2v - vuv + uv + u\kappa v + \kappa_1uv - \kappa_2v + \kappa_2 \\
&\quad + z(2uv - v^2 - \kappa_4v), \\
J &= uvvu - uvv - \kappa_1uv + \kappa_2uv - \kappa_3u + z(-uv + uv + \kappa_4uv + (\kappa_1 - \kappa_2 - \kappa_4)v). 
\end{align*}
\]

\[
\begin{align*}
\{ z(z-1)u' &= u^2vu + uvu^2 - uv^2 - \kappa_1u^2 + \kappa_2u \\
&\quad + z(-u^2v - uv + uv + \kappa_4u + (\kappa_1 - \kappa_2 - \kappa_4)), \\
\{ z(z-1)v' &= -uvv - vu^2v - vuv + uv + u\kappa v + \kappa_1uv - \kappa_2v + \kappa_2 \\
&\quad + z(2uv - v^2 - \kappa_4v), \\
J &= uvvu - uv^2 - \kappa_1uv + (\kappa_1 - \kappa_4)uv + (\kappa_2 - \kappa_1 + \kappa_4)uv - \kappa_3u \\
&\quad + z(-uv + uv^2 + \kappa_4uv + (\kappa_1 - \kappa_2 - \kappa_4)v). 
\end{align*}
\]

\[
\begin{align*}
\{ z(z-1)u' &= u^2vu + uvu^2 - uv^2 - \kappa_1u^2 + \kappa_2u \\
&\quad + z(-2uv + 2uv + \kappa_4u + (\kappa_1 - \kappa_2 - \kappa_4)), \\
\{ z(z-1)v' &= -uvv - vu^2v - vuv + uv + v^2u + \kappa_1uv + \kappa_1vu - \kappa_2v + \kappa_3 \\
&\quad + z(2uv - v^2 - \kappa_4v), \\
J &= uvvu - uvu - \kappa_1uv + \kappa_2uv - \kappa_3u + z(-uv^2u + v^2u + (\kappa_4 - \kappa_1 + \kappa_2)uv \\
&\quad + (\kappa_1 - \kappa_2)vuv + (\kappa_1 - \kappa_2 - \kappa_4)v). 
\end{align*}
\]

A.1.4 Case 4

\[
\begin{align*}
\{ z(z-1)u' &= 2u^2vu - u^2v - uv - \kappa_1u^2 + \kappa_2u \\
&\quad + z(-u^2v - uv + 2uv + \kappa_4u + (\kappa_1 - \kappa_2 - \kappa_4)), \\
\{ z(z-1)v' &= -2uvv - vu^2v + uv + \frac{1}{2}(\kappa_1 - \kappa_3)uv + \frac{1}{2}(\kappa_1 + \kappa_3)vu \\
&\quad - \kappa_2v + \frac{1}{2}(\kappa_3^2 - \kappa_2^2) + z(2uv - v^2 - \kappa_4v), \\
J &= uv^2v - uv^2 + \frac{1}{2}(\kappa_3 - \kappa_1)u^2v - \frac{1}{2}(\kappa_3 + \kappa_1)uv + \kappa_2uv - \frac{1}{2}(\kappa_3 - \kappa_1)u \\
&\quad + z(-uvu + uv + \kappa_4uv + (\kappa_1 - \kappa_2 - \kappa_4)v). 
\end{align*}
\]
\[
\begin{align*}
\{\begin{array}{l}
z(z-1)u' &= 2u^2vu - 2uvu - \kappa_1u^2 + \kappa_2u + z \left( -u^2v - uvu + uv + vu + \kappa_4u + (\kappa_1 - \kappa_2 - \kappa_4) \right), \\
z(z-1)v' &= -2uvuv - uv^2v + 2uv + \frac{1}{2}(3\kappa_1 - \kappa_3)uv + \frac{1}{2}(\kappa_1 + \kappa_3)vu - \kappa_2v + \frac{1}{4}(\kappa_3^2 - \kappa_2^2) + z(\kappa_3^2 - \kappa_2^2) + z(\kappa_3^2 - \kappa_2^2), \\
J &= uvuv^2 - vuv^2 + \frac{1}{2}(\kappa_3 - \kappa_1)u^2v - \frac{1}{2}(\kappa_3 - \kappa_1)uvuv + \frac{1}{2}(2\kappa_2 - \kappa_1 - \kappa_3)uv + \frac{1}{2}\left( (\kappa_1 + \kappa_3)u - \frac{1}{4}(\kappa_3^2 - \kappa_1^2) \right) + z(-uvuv + uvuv + \kappa_4uv + (\kappa_1 - \kappa_2 - \kappa_4)v).
\end{array}\}
\end{align*}
\]

\[
\begin{align*}
\{\begin{array}{l}
z(z-1)u' &= 2u^2vu - u^2v - uvu - \kappa_1u^2 + \kappa_2u + \frac{1}{2}(3\kappa_1 - \kappa_3)uv + \frac{1}{2}(\kappa_1 + \kappa_3)vu - \kappa_2v + \frac{1}{4}(\kappa_3^2 - \kappa_2^2) + z(2uvuv - v^2 - \kappa_4v), \\
z(z-1)v' &= -2uvuv - uv^2v + uv + vu + \frac{1}{2}(3\kappa_1 - \kappa_3)uv + \frac{1}{2}(\kappa_1 + \kappa_3)vu - \kappa_2v + \frac{1}{4}(\kappa_3^2 - \kappa_2^2) + z(2uvuv - v^2 - \kappa_4v), \\
J &= uvuv^2 - uvuv + \frac{1}{2}(\kappa_3 - \kappa_1)u^2v - \frac{1}{2}(\kappa_3 + \kappa_1)uvuv + \frac{1}{2}(2\kappa_2 - \kappa_1 - \kappa_3)uv + \frac{1}{2}(\kappa_1 + \kappa_3)uv + (\kappa_1 - \kappa_2 - \kappa_4)v).
\end{array}\}
\end{align*}
\]

A.1.5 Case 5

\[
\begin{align*}
\{\begin{array}{l}
z(z-1)u' &= u^3v + u^2vu - 2u^2v - \kappa_1u^2 + \kappa_2u + z \left( -2uvu + uv + vu + \kappa_4u + (\kappa_1 - \kappa_2 - \kappa_4) \right), \\
z(z-1)v' &= -u^2v^2 - uvuv + 2uv^2 + 2\kappa_1uv - \kappa_2v + \kappa_3 + z(\kappa_2v - \kappa_2v - \kappa_4v), \\
J &= u^2uv - u^2v^2 - \kappa_1u^2v + \kappa_2uv - \kappa_3u + z(-uvuv + uv^2 + \kappa_4uv + (\kappa_1 - \kappa_2 - \kappa_4)v).
\end{array}\}
\end{align*}
\]

\[
\begin{align*}
\{\begin{array}{l}
z(z-1)u' &= u^3v + u^2vu - u^2v - uvu - \kappa_1u^2 + \kappa_2u + z \left( -2u^2v + 2uv + \kappa_4u + (\kappa_1 - \kappa_2 - \kappa_4) \right), \\
z(z-1)v' &= -u^2v^2 - 2uvuv + uv^2 + vu + 2\kappa_1uv - \kappa_2v + \kappa_3 + z(2uvuv - v^2 - \kappa_4v), \\
J &= u^2uv - uvuv - \kappa_1u^2v + \kappa_2uv - \kappa_3u + z(-u^2v^2 + uv^2 + \kappa_4uv + (\kappa_1 - \kappa_2 - \kappa_4)v).
\end{array}\}
\end{align*}
\]

\[
\begin{align*}
\{\begin{array}{l}
z(z-1)u' &= u^3v + u^2vu - u^2v - uv - \kappa_1u^2 + \kappa_2u + z \left( -u^2v - uvu + vu + \kappa_4u + (\kappa_1 - \kappa_2 - \kappa_4) \right), \\
z(z-1)v' &= -u^2v^2 - 2uvuv + uv^2 + vu + 2\kappa_1uv - \kappa_2v + \kappa_3 + z(2uvuv - v^2 - \kappa_4v), \\
J &= u^2uv - uvuv - \kappa_1u^2v + \kappa_2uv - \kappa_3u + z(-uvuv + uvuv + \kappa_4uv + (\kappa_1 - \kappa_2 - \kappa_4)v).
\end{array}\}
\end{align*}
\]
A.2 Systems of $P_5$ type

A.2.1 Case 1

\[ z' \begin{align*} u &= 2uv^2 - 2uv - 2v^2 - \kappa_1 u^2 + 2vu + (\kappa_1 + \kappa_2)u - \kappa_2 + \kappa_4 z, \\
v &= -v^2 - (\kappa_1 + \kappa_2)v + \frac{1}{4}(\kappa_3 - \kappa_1)^2 - \kappa_4 z, \\
J &= vu^2 - 2vuv - \frac{1}{4}(\kappa_3 + \kappa_1)uv + \frac{1}{4}(\kappa_3 - \kappa_1)u^2 + v^2 + (\kappa_1 + \kappa_2)\nu \\
&\qquad - \frac{1}{4}(\kappa_3 - \kappa_1^2)u - \kappa_2 v + \kappa_4 zu. \end{align*} \]

A.2.2 Case 2

\[ z' \begin{align*} u &= 2uv^2 - 3uv - 2v - \kappa_1 u^2 + uv + vu + (\kappa_1 + \kappa_2)u - \kappa_2 + \kappa_4 z, \\
v &= -2vuv - v^2 u^2 + 2v^2 u + 2v^2 u + 2v^2 u - \kappa_2 + \kappa_4 z, \\
J &= vu^2 - 2vuv - \kappa_1 u^2 + uv + (\kappa_1 + \kappa_2)nu - \kappa_2 v + \kappa_4 zu. \end{align*} \]

A.2.3 Case 3

\[ z' \begin{align*} u &= u^2 + 3u^3 - 3u^2 - \kappa_1 u^2 + 2uv + (\kappa_1 + \kappa_2)u - \kappa_2 + \kappa_4 z, \\
v &= -2vuv - v^2 u^2 + 2v^2 u + 3v^2 u + 2v^2 u - \kappa_2 + \kappa_4 z, \\
J &= u^2 + 3u^3 - u^2 + v^2 + (\kappa_1 + \kappa_2)nu - \kappa_2 v + \kappa_4 zu. \end{align*} \]
A.2.4 Case 4

\[
\begin{align*}
    z u' &= 2u^2vu - 2u^2v - 2uvu - \kappa_1 u^2 + 2uv + (\kappa_1 + \kappa_2)u - \kappa_2 + \kappa_4zu, \\
    z v' &= -2uvu - vu^2v + 2uv^2 + 2vuv + \frac{1}{2}(3\kappa_1 - \kappa_3)uv + \frac{1}{2}(\kappa_1 + \kappa_3)uv - v^2 - (\kappa_1 + \kappa_2)v + \frac{1}{2}(\kappa_3 - \kappa_1^2) - \kappa_4zv, \\
    J &= uvu^2v - 2uuv + \frac{1}{2}(\kappa_3 - \kappa_1)u^2v - \frac{1}{2}(\kappa_3 + \kappa_1)uvu + uv^2 + (\kappa_1 + \kappa_2)uv - \frac{1}{2}(\kappa_3 - \kappa_1^2)u - \kappa_2v + \kappa_4zuv. 
\end{align*}
\]

A.2.5 Case 5

\[
\begin{align*}
    z u' &= u^3v + u^2vu - 2u^2v - 2uvu - \kappa_1 u^2 + uv + vu + (\kappa_1 + \kappa_2)u - \kappa_2 + \kappa_4zu, \\
    z v' &= -u^2v^2 - 2uvuv + 2u^2v + 2vuv + 2\kappa_1 uv - v^2 - (\kappa_1 + \kappa_2)v + \kappa_3 - \kappa_4zv, \\
    J &= u^2vuv - 2uvuv - \kappa_1 u^2v + uvu + (\kappa_1 + \kappa_2)uv - \kappa_3u - \kappa_2v + \kappa_4zuv. 
\end{align*}
\]

A.3 Systems of $P_4$ type

\[
\begin{align*}
    u' &= -u^2 + 2vu - 2zu + \kappa_2, \\
    v' &= -v^2 + vu + uv + 2zu + \kappa_3, \\
    J &= v^2u - uvu - \kappa_3u + \kappa_2v - 2zvu. 
\end{align*}
\]
\[ \begin{align*}
\{ u' & = -u^2 + 2uv - 2zu + \kappa_2, \\
v' & = -v^2 + 2uv + 2zv + \kappa_3, \quad 5P_4 \\
J & = uv^2 - u^2v - \kappa_3u + \kappa_2v - 2zuv. \end{align*} \]

A.4 Systems of P'_3 type

A.4.1 Case 1

\[ \begin{align*}
\{ z u' & = 2uu + \kappa_1 u + \kappa_2 u^2 + \kappa_4z, \\
z v' & = -2uv - \kappa_1 v - 2\kappa_2vu - \kappa_3, \quad 1P'_3 \\
J & = vu^2v + \kappa_1vu + \kappa_3 \kappa_2^{-1}[u, v] + \kappa_2vu^2 + \kappa_3u + \kappa_4zv. \end{align*} \]

A.4.2 Case 2

\[ \begin{align*}
\{ z u' & = 2vu^2 + \kappa_1 u + \kappa_2 u^2 + \kappa_4z, \\
z v' & = -2v^2u - \kappa_1 v - 2\kappa_2vu - \kappa_3, \quad 2P'_3 \\
J & = v^2u^2 + \kappa_1vu + \kappa_2vu^2 + \kappa_3u + \kappa_4zv. \end{align*} \]

A.4.3 Case 3

\[ \begin{align*}
\{ z u' & = uvu + vu^2 + \kappa_1 u + \kappa_2 u^2 + \kappa_4z, \\
z v' & = -uvu - v^2u - \kappa_1 v - \kappa_2uv - \kappa_2vu - \kappa_3, \quad 4P'_3 \\
J & = uvvu + \kappa_1vu + \kappa_2vu + \kappa_3u + \kappa_4zv. \end{align*} \]

A.4.4 Case 4

\[ \begin{align*}
\{ z u' & = u^2v + uvu + \kappa_1 u + \kappa_2 u^2 + \kappa_4z, \\
z v' & = -uv^2 - v^2u - \kappa_1 v - \kappa_2uv - \kappa_2vu - \kappa_3, \quad 6P'_3 \\
J & = uvuv + \kappa_1uv + \kappa_2uv + \kappa_3u + \kappa_4zv. \end{align*} \]
\[ \begin{align*}
\{ u' &= u^2v + uvu + \kappa_1u + \kappa_2u^2 + \kappa_4z, \\
       v' &= -uv^2 - vu - \kappa_1v - 2\kappa_2uv - \kappa_3, \}
\end{align*} \tag{A.4.5} \]

\[ \begin{align*}
    J &= uvuv + \kappa_1uv + \kappa_2u^2v + \kappa_3u + \kappa_4zv. 
\end{align*} \tag{P_3'} \]

A.4.5 Case 5

\[ \begin{align*}
\{ u' &= 2u^2v + \kappa_1u + \kappa_2u^2 + \kappa_4z, \\
       v' &= -2uv^2 - \kappa_1v - 2\kappa_2uv - \kappa_3, \}
\end{align*} \tag{A.5} \]

\[ \begin{align*}
    J &= u^2v^2 + \kappa_1uv + \kappa_2u^2v + \kappa_3u + \kappa_4zv. 
\end{align*} \tag{sP_3'} \]

A.5 Systems of P_2 type

\[ \begin{align*}
\{ u' &= -u^2 + v - \frac{1}{2}z, \\
       v' &= 2uv + \kappa_3, \}
\end{align*} \tag{1P_2} \]

\[ \begin{align*}
\{ u' &= -u^2 + v - \frac{1}{2}z, \\
       v' &= 2uv + \kappa_3, \}
\end{align*} \tag{2P_2} \]

\[ \begin{align*}
    J &= -u^2v + \frac{1}{2}v^2 - \kappa_3u - \frac{1}{2}zv. 
\end{align*} \]

\[ \begin{align*}
    J &= -vu^2 + \frac{1}{2}v^2 - \kappa_3u - \frac{1}{2}zv. 
\end{align*} \]

B List of Hamiltonian non-abelian systems of Painlevé type

\[ \begin{align*}
\{ u' &= u^2vu + uvu^2 - 2uvu - \kappa_1u^2 + \kappa_2u \\
       &\quad + z \left(-u^2v - vu^2 + uv + vu + \kappa_4u + (\kappa_1 - \kappa_2 - \kappa_4)\right), \\
       v' &= -uvuv - vu - \kappa_1v - 2\kappa_2uv + \kappa_3
\end{align*} \tag{P_6H} \]

\[ \begin{align*}
\{ u' &= u^2vu + uvu^2 - u^2v - 2uvu - vu^2 - \kappa_1u^2 + uv + vu + (\kappa_1 + \kappa_2)u \\
       &\quad - \kappa_2 + \kappa_4zv, \\
       v' &= -uvuv - vu^2v - vuuv + uv^2 + 2vu + v^2u + \kappa_1uv + \kappa_1vu - v^2 \\
       &\quad - (\kappa_1 + \kappa_2)v + \kappa_3 - \kappa_4zv. \}
\end{align*} \tag{P_5H} \]

\[ \begin{align*}
\{ u' &= -u^2 + uv + v^2 - 2zu + \kappa_2, \\
       v' &= -v^2 + vu + uv + 2zv + \kappa_3. \}
\end{align*} \tag{P_4H} \]

\[ \begin{align*}
\{ u' &= 2uvu + \kappa_1u + \kappa_2u^2 + \kappa_4z, \\
       v' &= -2vu - 2uv - \kappa_1v - \kappa_2uv - \kappa_3 
\end{align*} \tag{P_3'H} \]

\[ \begin{align*}
\{ u' &= -u^2 + v - \frac{1}{2}z, \\
       v' &= vu + uv + \kappa_3. \}
\end{align*} \tag{P_2'H} \]
\[
\begin{cases}
  u' = v, \\
  v' = 6u^2 + z.
\end{cases}
\]

C Special cases of \( P'_3 \) type systems

Let us consider the scalar \( P'_3 \) system (28). Eliminating \( v \) from the system, one arrives at the Painlevé-3'(\( D_6 \)) equation for \( y(z) = u(z) \) of the form

\[
y'' = \frac{1}{y}(y')^2 - \frac{1}{z}y' + \frac{1}{z^2}y^2(\gamma y + \alpha) + \frac{\beta}{z} + \frac{\delta}{y},
\]

\[
\alpha = \kappa_1\kappa_2 - 2\kappa_3, \quad \beta = \kappa_4(1 - \kappa_1), \quad \gamma = \kappa_2^2, \quad \delta = -\kappa_4^2.
\]

It follows from these relations that the equation

\[
y'' = \frac{1}{y}(y')^2 - \frac{1}{z}y' + \frac{1}{z^2}y^2(\gamma y + \alpha) + \frac{\beta}{z}
\]

of \( P'_3(\text{D}_7) \) type with \( \delta = 0 \), \( \beta \neq 0 \) cannot be obtained from (28). However, this equation can be represented [12, p. 12142, Painlevé III (\( D_7^{(1)} \))-2] as the Hamiltonian system

\[
\begin{cases}
  z u' = 2u^2v + \kappa_1 u + \kappa_2 u^2, \\
  z v' = -2uv^2 - \kappa_1 v - 2\kappa_2 uv - \kappa_3 + \kappa_4 z u^{-2},
\end{cases}
\]

where

\[
\alpha = \kappa_1\kappa_2 - 2\kappa_3, \quad \beta = 2\kappa_4, \quad \gamma = \kappa_2^2.
\]

The Hamiltonian for this system is given by

\[zh = u^2v^2 + \kappa_1 uv + \kappa_2 u^2 v + \kappa_3 u + \kappa_4 z u^{-1}.\]

Notice that system (60) has the structure

\[
\begin{cases}
  z u' = P_1(u, v), \\
  z v' = P_2(u, v) + \kappa_4 u^{-2}.
\end{cases}
\]

C.1 Non-abelian systems of \( P'_3(\text{D}_7) \) type

**Proposition 6.** There are six non-abelian systems (61) with components \( P_1(u, v) \) and \( P_2(u, v) \) given by the formulas

\[
P_1(u, v) = a_1u^2v + (2 - a_1 - a_2)uvu + a_2vu^2 + \kappa_1 u + \kappa_2 u^2,
\]

\[
P_2(u, v) = b_1uv^2 - (2 + b_1 + b_2)uvu + b_2v^2u - \kappa_1 vc_1uv + (-2\kappa_2 - c_1)uv - \kappa_3,
\]

whose auxiliary system

\[
\begin{cases}
  \frac{du}{dt} = P_1(u, v), \\
  \frac{dv}{dt} = P_2(u, v) + \kappa_4 u^{-2}.
\end{cases}
\]
have an Okamoto integral of the form

\[ J = d_1 u^2 v^2 + d_2 uv^2 u + d_3 uvu + d_4 vu^2 v + d_5 vvu + \left( 1 - \sum d_i \right) v^2 u^2 + e_1 uv + (\kappa_1 - e_1) vu + h_1 u^2 v + (\kappa_2 - h_1 - h_2) uu + h_2 vu^2 + \kappa_4 z u^{-1}. \]

These systems are given in the list below:

- **Case 1**
  \[
  \begin{align*}
  z u' &= 2 uv + \kappa_1 u + \kappa_2 u^2, \\
  z v' &= -2 uv - \kappa_1 v - 2 \kappa_2 uv - \kappa_3 + \kappa_4 z u^{-2},
  \end{align*}
  \]
  \[ J = vu^2 v + \kappa_1 uv + \kappa_3 \kappa_2^{-1} [u, v] + \kappa_2 v u^2 + \kappa_3 u + \kappa_4 z u^{-1}; \]

- **Case 2**
  \[
  \begin{align*}
  z u' &= uv + v^2 u + \kappa_1 u + \kappa_2 u^2, \\
  z v' &= -uv - v^2 u - \kappa_1 v - \kappa_2 uv - \kappa_3 + \kappa_4 z u^{-2},
  \end{align*}
  \]
  \[ J = vuuv + \kappa_1 uv + \kappa_2 uvu + \kappa_3 u + \kappa_4 z u^{-1}; \]

- **Case 3**
  \[
  \begin{align*}
  z u' &= u^2 v + uv + \kappa_1 u + \kappa_2 u^2, \\
  z v' &= -uv^2 - uv - \kappa_1 v - \kappa_2 uv - \kappa_3 + \kappa_4 z u^{-2},
  \end{align*}
  \]
  \[ J = uvuv + \kappa_1 uv + \kappa_2 uvu + \kappa_3 u + \kappa_4 z u^{-1}; \]

The action of the transposition (11) on this list of systems defines three non-equivalent orbits:

- **Orbit 1** = \{1P_3'(D_7), 2P_3'(D_7)\},
- **Orbit 2** = \{3P_3'(D_7), 5P_3'(D_7)\},
- **Orbit 3** = \{4P_3'(D_7), 6P_3'(D_7)\}.
C.2 Limiting transitions

The degenerations of non-abelian special cases of $P'_3$ type systems are given in Figure 6. As in Section 4.1, the red arrows correspond to the representatives of the $P_0$ orbits (see Section 2.1) and their degenerations.

$$
\begin{align*}
1P'_3 & \rightarrow 1P'_3(D_7) \\
2P'_3 & \rightarrow 2P'_3(D_7) \\
4P'_3 & \rightarrow 3P'_3(D_7) \\
3P'_3 & \rightarrow P'_3(D_7) \\
8P'_3 & \rightarrow 4P'_3(D_7) \\
6P'_3 & \rightarrow 5P'_3(D_7) \\
7P'_3 & \rightarrow 6P'_3(D_7) \\
P'_3 & \rightarrow P'_3(D_7) \\
31P'_3 & \rightarrow P'_3(D_7)
\end{align*}
$$

Figure 6: Degeneration scheme for special cases of $P'_3$ type systems

C.2.1 $P'_3 \rightarrow P'_3(D_7)$

Systems of $P'_3$ type from Appendix A.4 can be reduced to $P'_3(D_7)$ systems by the following limiting transition with the small parameter $\varepsilon$:

$$
v \mapsto v - \varepsilon^{-1}u^{-1}, \quad \kappa_1 \mapsto \kappa_1 + 2\varepsilon^{-1}, \quad \kappa_3 \mapsto \kappa_3 + \varepsilon^{-1}\kappa_2, \quad \kappa_4 \mapsto \varepsilon^{-1}\kappa_4. \quad (62)
$$

Using these formulas, we obtain the following degenerations:

$$
\begin{align*}
1P'_3 & \rightarrow 1P'_3(D_7), \quad 2P'_3 \rightarrow 2P'_3(D_7), \quad 3P'_3 \rightarrow (63), \quad 4P'_3 \rightarrow 3P'_3(D_7), \\
5P'_3 & \rightarrow 4P'_3(D_7), \quad 6P'_3 \rightarrow 5P'_3(D_7), \quad 7P'_3 \rightarrow 6P'_3(D_7), \quad 8P'_3 \rightarrow (64), \\
P'_3 & \rightarrow P'_3(D_7).
\end{align*}
$$

Here $P'_3(D_7)$ is the Hamiltonian system

$$
\begin{align*}
z u' & = 2uvu + \kappa_1u + \kappa_2u^2, \\
z v' & = -2vuv - \kappa_1v - \kappa_2uv - \kappa_3 - \kappa_4zu^{-2}.
\end{align*}
$$

Remark 6. Systems
\[
\begin{aligned}
    z_u' &= 2vu^2 + \kappa_1 u + \kappa_2 u^2, \\
    z_v' &= -2u^2 v - \kappa_1 v - 2\kappa_2 uv \\
    \quad &\quad - \kappa_3 + \kappa_4 z v^{-2}, \\
    z' &= 2v^2 u + \kappa_1 u + \kappa_2 u^2, \\
    v' &= -2uv^2 - \kappa_2 v - 2\kappa_1 uv \\
    \quad &\quad - \kappa_3 + \kappa_4 z v^{-2},
\end{aligned}
\]

(63) \quad (64)

\[J = u^{-1}(uv - vu) u, \quad J = u(vw - vu) u^{-1},\]

are equivalent to the \(P'_{3}^H(D_7)\) system. This equivalence is defined by the Laurent mappings \((u, v) \mapsto (u, u^{-1}vu)\) and \((u, v) \mapsto (u, uv u^{-1})\), respectively.

Supplementing formulas (62) by the following mapping

\[\lambda \mapsto -\lambda, \quad A \mapsto g A g^{-1} + g' A g^{-1}, \quad B \mapsto g B g^{-1} + g' B g^{-1}, \quad g = \lambda^{-\varepsilon^{-1}} z^{\text{const} \varepsilon^{-1}} \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix},\]

we degenerate a pair of the form (44) for any system of \(P'_{3}^H(D_6)\) type to a pair of the same structure for the corresponding \(P'_{3}(D_7)\) system.

**C.2.2 \( P'_{3}(D_7) \to P_1 \)**

The \(P'_{3}(D_7)\)-systems listed in Appendix C.1 can be degenerated to the Painlevé-1 system \(P_1^H\) by the map

\[z \mapsto \varepsilon z - 2\varepsilon^{-4}, \quad u \mapsto \varepsilon^{-2} (u - 1), \quad v \mapsto \varepsilon^2 v - 2\varepsilon^{-3}, \quad \kappa_1 = \varepsilon, \quad \kappa_2 = -4\varepsilon^{-5}, \quad \kappa_3 = 12\varepsilon^{-10}, \quad \kappa_4 = 2.\]

To get a Lax pair of the form (52) for \(P_1^H\), one may consider the degeneration data

\[\lambda \mapsto \varepsilon^{-2} (\lambda - 1), \quad A \mapsto g A g^{-1} + g' A g^{-1}, \quad B \mapsto g B g^{-1} + g' B g^{-1}, \quad g = \varepsilon^{2\varepsilon^{-1} \lambda^{-1} + \text{const} \varepsilon z} \begin{pmatrix} 2 & 0 \\ -\varepsilon^4 v & \varepsilon^4 \end{pmatrix}.\]

**References**

[1] V. E. Adler. Painlevé type reductions for the non-Abelian Volterra lattices. *Journal of Physics A: Mathematical and Theoretical*, 54(3):035204, 2021. [arXiv:2010.09021](https://arxiv.org/abs/2010.09021).

[2] V. E. Adler and V. V. Sokolov. On matrix Painlevé II equations. *Theoret. and Math. Phys.*, 207(2):188–201, 2021. [arXiv:2012.05639](https://arxiv.org/abs/2012.05639).

[3] S. P. Balandin and V. V. Sokolov. On the Painlevé test for non-Abelian equations. *Physics letters A*, 246(3-4):267–272, 1998.

[4] M. Bertola, M. Cafasso, and V. Rubtsov. Noncommutative Painlevé equations and systems of Calogero type. *Communications in Mathematical Physics*, 363(2):503–530, 2018. [arXiv:1710.00736](https://arxiv.org/abs/1710.00736).

[5] P. Boalch. Simply-laced isomonodromy systems. *Publications mathématiques de l’IHÉS*, 116:1–68, 2012. [arXiv:1107.0874](https://arxiv.org/abs/1107.0874).

[6] I. A. Bobrova and V. V. Sokolov. On matrix Painlevé-4 equations. Part 1: Painlevé-Kovalevskaya test. *arXiv preprint arXiv:2107.11680*, 2021.
[7] I. A. Bobrova and V. V. Sokolov. On matrix Painlevé-4 equations. Part 2: Isomonodromic Lax pairs. arXiv preprint arXiv:2110.12159, 2021.

[8] B. Gambier. Sur les équations différentielles du second ordre et du premier degré dont l’intégrale générale est à points critiques fixes. Acta Mathematica, 33(1):1–55, 1910.

[9] M. Jimbo and T. Miwa. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II. Physica D: Nonlinear Phenomena, 2(3):407–448, 1981.

[10] H. Kawakami. Matrix Painlevé systems. Journal of Mathematical Physics, 56(3):033503, 2015.

[11] M. Kontsevich. Formal (non)-commutative symplectic geometry, The Gelfand Mathematical Seminars, 1990–1992. Fields Institute Communications, Birkhäuser Boston, pages 173–187, 1993.

[12] Y. Ohyama and S. Okumura. A coalescent diagram of the Painlevé equations from the viewpoint of isomonodromic deformations. Journal of Physics A: Mathematical and General, 39(39):12129, 2006. arXiv:math/0601614v1.

[13] K. Okamoto. Polynomial Hamiltonians associated with Painlevé equations, I. Proceedings of the Japan Academy, Series A, Mathematical Sciences, 56(6):264–268, 1980.

[14] K. Okamoto. Studies on the Painlevé equations IV. Third Painlevé equation PIII. Funkcial. Ekvac, 30:305–332, 1987.

[15] V. S. Retakh and V. N. Rubtsov. Noncommutative Toda Chains, Hankel Quasideterminants and Painlevé II Equation. Journal of Physics. A, Mathematical and Theoretical, 43(50):505204, 2010. arXiv:1007.4168.