Estimation of classical parameters via continuous probing of complementary quantum observables

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Abstract. We discuss how continuous probing of a quantum system allows estimation of unknown classical parameters embodied in the Hamiltonian of the system. We generalize the stochastic master equation associated with continuous observation processes to a Bayesian filter equation for the probability distribution of the desired parameters, and we illustrate its application by estimating the direction of a magnetic field. In our example, the field causes a ground state spin precession in a two-level atom which is detected by the polarization rotation of off-resonant optical probes, interacting with the atomic spin components.
1. Introduction

Quantum control has gained importance in the development of a wide range of quantum-based technology protocols. The application of open-loop control methods to the quantum domain is already successfully employed to design control pulses for chemical reactions [1], to implement robust qubit gate operations (e.g. [2–4]), and to steer the dynamics of degenerate quantum Bose gases [5]. Measurement and feedback based control, however, needs to be significantly revised to gain the same ubiquitous role in the quantum domain as it already has in classical control engineering. Indeed, one of the main lessons of quantum mechanics is that the quantum measurement process is not only a stochastic process but also it intrinsically changes the state and the dynamics of the system under observation. It is unimaginable that future quantum devices will not apply feedback control, but how this will be realized is unclear and the subject of current research investigations. To design efficient and useful protocols it is crucial to understand the interplay between quantum inference and measurement-back action.

Usually, the control of a quantum system refers to its preparation in some (pure) quantum state, like ground state cooling of a mechanical oscillator or an ion. However, the control of a quantum system is also important for estimation purposes, where the goal consists in the assessment, with the highest possible precision, of some (external) parameter like the strength of the Hamiltonian coupling the quantum system to its environment. Knowledge of such parameters is crucial for the ability to control the system with high fidelity and, e.g. for the generation of entanglement and implementation of quantum gates. It is of course also of fundamental relevance to develop estimation strategies for the use of quantum systems as high-sensitivity sensors.

An instance of technological application where estimation is instrumental is high precision metrology, a research field of high relevance to both scientific research and technological applications. Through the quantization of their energy levels, elementary quantum systems provide fundamental time and frequency standards and, due to the highly developed means for preparation, control and detection of these systems, they serve as excellent probes of various perturbations such as applied electric and magnetic fields, and inertial effects associated with rotation, acceleration and classical or relativistic gravitational effects.
The theoretical research proceeds along different directions according to the different measurement schemes. Thus, for experiments where a quantum system is subject to a perturbation for a given short duration of time, the search for the initial quantum states on which different values of the perturbation leads to the most distinguishable outcomes has promoted the use of concepts such as the Fisher information and the Cramer–Rao bound [6], and has identified squeezed states, Schrödinger cat-like states and generalizations hereof as useful resource states in metrology. Along a different path, measurements that occur continuously in time, such as continuous wave laser spectroscopy, are made subject to analyses, that serve to exhaust the information about the desired parameters from the entire sequence of measurement data. While a simple relationship between the average signal, e.g. an absorption profile, and an unknown physical parameter provides a relatively straightforward method for estimation, the systematic extraction of a reliable error-bar on the estimate is more challenging [7–9].

In this work we present such an analysis for the non-trivial case of the continuous probing of a single quantum system. Our analysis has two main goals: (i) we provide a general Bayesian theory along the lines of [10, 11] for the estimation of unknown classical parameters in the framework of continuous quantum measurements; and (ii) we reveal the influence and impact on the estimation process of the (simultaneous) detection of different non-commuting observables. To this end, we establish an augmented stochastic master equation which treats our description of unknown parameter values and the unknown state of the quantum system on an equal footing, and where any piece of measurement data is applied to continuously update our prior estimate of the parameters of interest. When applied to the case of quantum-non-demolition (QND) probing of quantum systems [7, 12], this method is largely equivalent to a Kalman filter [8], while the stochastic master equation formalism presented here is not restricted to QND measurements.

2. Conditional dynamics and estimation of a classical parameter

We are interested in the use of quantum systems to estimate a classical physical parameter or a set of parameters \( \vec{\gamma} \). Here, in order to keep the discussion as general as possible, we treat the unknown parameter \( \vec{\gamma} \) as a vector quantity to indicate that it may be a set of parameters such as damping rates, energy shifts and coupling strengths, or, as in our example below, the directional components of a vector magnetic field. The experiment is sensitive to the value of these parameters, e.g. if they are coefficients in the Hamiltonian \( \hat{H} = \hat{H}(\vec{\gamma}) \) acting on the system, and if this dependence results in a change of the observables probed in the experiment.

We describe the quantum dynamics of the combined system with \( \vec{\gamma} \) belonging to a finite set of values \( \mathcal{V}_f = \{\vec{\gamma}_k : k = 1, \ldots, N\} \).

For an observer who knows the true value of \( \vec{\gamma} = \vec{\gamma}_0 \), we assume that the system is described by the following stochastic master equation [13]:

\[
d\hat{\rho}_0 = \frac{i}{\hbar} [\hat{\rho}_0, \hat{H}(\vec{\gamma}_0)] dt + \sum_j \Gamma_j \mathcal{D}[\hat{O}_j] \hat{\rho}_0 dt \\
+ \sum_{n=1}^{M} M_n \mathcal{D}[\hat{M}_n] \hat{\rho}_0 dt + \sqrt{\eta_n} M_n \sqrt{\mathcal{H}[\hat{M}_n]} \hat{\rho}_0 dW_n(t), \tag{1}
\]

conditioned by the measurement signals

\[
dY_n(t) = \sqrt{\eta_n} dW_n(t) + \eta_n \sqrt{M_n} (\hat{M}_n + \hat{M}_n^\dagger) \hat{\rho}_0 dt, \tag{2}
\]

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where
\[
\mathcal{D}(\hat{f}) \hat{\rho}_n^k = \hat{f} \hat{\rho}_n^k \hat{f}^\dagger - \frac{\hat{f}^\dagger \hat{f} \hat{\rho}_n^k + \hat{\rho}_n^k \hat{f}^\dagger \hat{f}}{2},
\]
and
\[
\mathcal{H}(\hat{f}) \hat{\rho}_n^k = \hat{f} \hat{\rho}_n^k + \hat{\rho}_n^k \hat{f}^\dagger - (\hat{f} + \hat{f}^\dagger) \hat{\rho}_n^k,
\]
\[
\langle \hat{M}_n + \hat{M}_n^\dagger \rangle = \text{Tr}_S(\langle \hat{M}_n + \hat{M}_n^\dagger \rangle \hat{\rho}_0^k),
\]
\(M_n\) are effective interaction parameters, \(dW_n(t)\) are standard Wiener processes and \(\text{Tr}_S(\cdot)\) is the trace on the system Hilbert space. We note that equation (1) follows from a Markovian and perturbative treatment of the interaction of the system with its environment. For every concrete physical example one has to validate such a treatment and to explicitly analyze the appropriate interactions and field measurement schemes to obtain the operators and parameters in equation (1), as we do for our specific example in the appendix. Furthermore, in the equations (1) and (2) the parameters \(\eta_n\) account for the detector efficiencies and transmission losses of the probe beams between the system and the detector. If \(\eta_n = 0\), the corresponding probe field merely contributes extra damping and decoherence to the system, while if \(\eta_n = 1\), and in the absence of other damping or ineffective probing terms, the system may be described by a stochastic wave function rather than a density matrix [14–17].

2.1. Augmented quantum filter equation

In our formulation of the estimation problem we will treat the classical parameter \(\gamma\) as unknown with an assigned probability distribution \(P(\gamma)\). The measurements then cause an update of the probability distribution, which is governed by Bayes rule for conditional probabilities. Until the actual value of \(\gamma\) is known, we thus have to treat each candidate value with a probability factor, and for each possible value of \(\gamma\), the corresponding state of the quantum system evolves under the quantum filtering equation with the corresponding dependence of \(\gamma\).

To this end, it is convenient [7, 9–11, 18] to consider the augmented Hilbert space \(\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_\gamma\), where \(\mathcal{H}_s\) and \(\mathcal{H}_\gamma\) refer to the quantum system Hilbert space and the space of classical states for the variable \(\gamma\), respectively. The latter space describes states with definite values of the parameters, and superposition states are not populated. The quantum mechanical notation, however, still applies and, e.g. describes a probability distribution for a set of values \(\gamma_k\) as a diagonal density matrix \(\hat{\rho}_\gamma = \sum_k P_k |\gamma_k\rangle \langle \gamma_k|\). The classical variables \(\gamma_k\) are equivalent to QND variables of an ancillary quantum system that interacts with our probe system, and for which the Bayesian probability update is fully equivalent to the quantum measurement back-action. When we incorporate the parameters \(\gamma\) in this way we can directly apply the filtering equation on the augmented space. We note that the equivalence of classical unknown parameters and QND variables of ancillary quantum systems goes both ways. This permits certain types of quantum evolution to be described exactly by classical probability distributions, and has recently been applied, e.g. in the context of decoherence of two-level quantum systems driven by coherent laser light [19–21].

The observer who does not know the value of \(\gamma\) describes the combined quantum and classical system by the augmented density matrix
\[
\hat{\rho} = \sum_{k=1}^N P_k |\gamma_k\rangle \langle \gamma_k| \otimes \hat{\rho}_k^k,
\]
where \(\hat{\rho}_k^k\) is the normalized system state density matrix associated with the specific value \(\gamma = \gamma_k\).
The combined system evolves according to the quantum filter equation
\[
\frac{\partial}{\partial t} \hat{\varrho} = \left( \frac{i}{\hbar} [\hat{\varrho}, \hat{H}] + \sum_j \Gamma_j D(\hat{\mathcal{O}}_j)\hat{\varrho} + \sum_{n=1}^M M_n D(\hat{\mathcal{M}}_n)\hat{\varrho} \right) dt + \sum_{n=1}^M \sqrt{M_n} \mathcal{H}[\hat{\mathcal{M}}_n] \hat{\varrho} \left( dY^D_n(t) - \eta_n \sqrt{M_n} \langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}_n^\dagger \rangle_E dt \right),
\]
(5)
where the operator \( \mathcal{H} \) is defined as
\[
\mathcal{H}[\hat{f}]\hat{\varrho} = \hat{f} \hat{\varrho} + \hat{\varrho} \hat{f}^\dagger - \langle \hat{f} + \hat{f}^\dagger \rangle_E \hat{\varrho}.
\]
(6)
Here we use the notation \( \langle \hat{f} \rangle_E = \text{Tr} \{ \hat{f} \hat{\varrho} \} = \sum_{k=1}^N P_k \text{Tr}_S \{ \hat{f} \hat{\varrho}^k \} \) to explicitly recall that the expectation value of the signal should be determined by the full augmented quantum state, equivalent to a weighted average over the ensemble of states of the quantum system governed by the different values of \( \vec{\gamma} \).

If the dependence on \( \vec{\gamma} \) only enters via the Hamiltonian \( \hat{H}(\vec{\gamma}) \), the different terms in equation (5) are implemented as the following product operators on \( \mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_Y \):
\[
\hat{H} = \sum_{k=1}^N |\vec{\gamma}_k\rangle \langle \vec{\gamma}_k| \otimes \hat{H}(\vec{\gamma}_k),
\]
\[
\hat{\mathcal{O}}_j = \sum_{k=1}^N |\vec{\gamma}_k\rangle \langle \vec{\gamma}_k| \otimes \hat{\mathcal{O}}_j,
\]
\[
\hat{\mathcal{M}}_n = \sum_{k=1}^N |\vec{\gamma}_k\rangle \langle \vec{\gamma}_k| \otimes \hat{\mathcal{M}}_n.
\]
(7)
The quantum filter equation (5) is established as an augmented version of equation (1) to efficiently include the probabilistic representation of the unknown variables and their correlations with the quantum system. As the main objective of our analysis is the estimation of the parameter \( \vec{\gamma} \), we shall in the following section reformulate equation (5) to explicitly extract the conditioned dynamics of the probability distribution \( P_k \) in equation (4).

2.2. Detection signal properties

The stochastic process appearing in equation (5),
\[
\sqrt{\eta_n} dV_n(t) := dY^D_n(t) - \eta_n \sqrt{M_n} \langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}_n^\dagger \rangle_E dt.
\]
(8)
is not a standard Wiener process. This is because we subtract from the measured signal a weighted average based on our probabilistic description of \( \vec{\gamma} \), while the measured photocurrent \( dY^D_n \) in the experiment is governed by the actual value of the unknown parameter \( \vec{\gamma} = \vec{\gamma}_k \).

Equation (2) characterizes the properties of such a realistic detection record, and when inserted in equation (5), we find that the stochastic process in equation (8) can be rewritten as
\[
dV_n = dW_n - \sqrt{\eta_n} M_n [\langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}_n^\dagger \rangle_E - \langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}_n^\dagger \rangle_0] dt.
\]
(9)
Hence, $dV_n(t)$ is a stochastic Gaussian process with variance $dr$, and with (statistical) mean value $\langle dV_n(t) \rangle = \sqrt{\eta_n M_n}(\langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}^1_n \rangle_0 - \langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}^1_n \rangle_E)dt$, reflecting precisely the difference in the expectation value assumed by the weighted average over different $\tilde{\gamma}_k$ and by the correct value $\tilde{\gamma}_k$.

As shown in detail in appendix A.3, equation (5) leads to two separate equations for $\hat{\gamma}_k$ and $P_k$ that permit a full simulation of the detection process

$$d\hat{\gamma}_k = \frac{i}{\hbar}[\hat{\gamma}_k, \hat{H}(\tilde{\gamma}_k)]dt + \sum_j \Gamma_j [\hat{D}_j(\hat{\gamma}_k) d\hat{\gamma}_k^j dt + \frac{\sum M_n D[\hat{\mathcal{M}}_n] \hat{\gamma}_k^j dt + \sqrt{M_n} \mathcal{H}[\hat{\mathcal{M}}_n] \hat{\gamma}_k^j d\tilde{V}_n^{(k)}(t)]}{(10)}$$

with $d\tilde{V}_n^{(k)}(t) = dY_n^{(D)}(t) - \eta_n \sqrt{M_n}(\langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}^1_n \rangle_0 - \langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}^1_n \rangle_E)dt$ and

$$dP_k = \text{Tr}_S(d\hat{\gamma}_k) = P_k \sum_{n=1}^M \eta_n M_n(\langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}^1_n \rangle_0 - \langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}^1_n \rangle_E) dV_n(t).$$

In both equations the photocurrent $dY_n^{(D)}(t)$, observed or simulated according to equation (1), appears, and equation (11) agrees with the expression given in [10]. We note, however, that the stochastic term in our equation (10) contains the expectation value $\langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}^1_n \rangle_k$ corresponding to the parameter value $\tilde{\gamma}_k$, while in [10] the ensemble average $\langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}^1_n \rangle_E$ has been used.

By inserting the expression (9) for $dV_n$ in equation (11), we see that the change of $dP_k$ due to the measurements is given by

$$\langle (\hat{\mathcal{M}}_n + \hat{\mathcal{M}}^1_n)_k - (\hat{\mathcal{M}}_n + \hat{\mathcal{M}}^1_n)_E \rangle [dW_n(t) + \sqrt{\eta_n M_n}(\langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}^1_n \rangle_0 - \langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}^1_n \rangle_E)dr].$$

This equation has a natural interpretation: for parameter values $\tilde{\gamma}_k$ where the expected mean current $\propto \langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}^1_n \rangle_k$ differs in the same (opposite) direction from the ensemble mean as the one expected for the actual value $\langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}^1_n \rangle_0$, the two parentheses will typically have the same (opposite) sign, and $P_k$ will increase (decrease). Due to the random contribution $dW_n(t)$, however, the probabilities will show fluctuations, and their increase (decrease) with time will appear only as an average trend, leading, in particular, to a typically increasing value for the probability of the correct value $P_k$.

3. Vector magnetometry with a two-level atom

We now apply the formalism to the two-level atom coupled to a magnetic field $\vec{B}$ with known magnitude $|\vec{B}|$, but unknown direction in space, and we will assume that we probe all three spin components ($\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$) with measurement strengths ($M_1$, $M_2$, $M_3$) and efficiencies ($\eta_1$, $\eta_2$, $\eta_3$). Since such a field will cause a spin precession around the magnetic field axis, we expect that optical probing of a single spin component will not be sensitive to the magnetic field projection along the spin direction probed, while the simultaneous probing of all three spin components is bound to reveal any motion of the mean spin vector due to the magnetic precession.

The acquisition of data from a real or a simulated experiment causes a continuous update of the probability distribution $P_k$ for the magnetic field, represented in the following by the dimensionless vector, $\vec{b}_k = \mu_B \vec{B}_k/(\hbar M)$ (see also appendix A.4 for definitions). The angular
measure, \( \cos \theta(t) = |\vec{b}_u|^{-2} \sum_k P_k(t) (\vec{b}_k \cdot \vec{b}_u) \), quantifies the scattering of the magnetic field directions \( \vec{b}_k \) inferred from a single experimental run around the actual value \( \vec{b}_u \). By carrying out a large number of simulations, we thus quantify the average performance of the method by the (average) scalar product

\[
\langle \langle \cos \theta \rangle \rangle(t) = |\vec{b}_u|^{-2} \sum_k \langle \langle P_k(t) (\vec{b}_k \cdot \vec{b}_u) \rangle \rangle
\]  

as a function of the measurement time \( t \).

3.1. Direction of a magnetic field along a given axis

Following [10], we have first investigated the case in which the initial state of the atom is represented by the Bloch vector \( \vec{r}_0 = (0, 1, 0) \) with \( M_{1,2} = 0 \) but \( M_3 \neq 0 \) (i.e. we probe only the component of the spin along \( z \) and \( M \equiv M_3 \)), and where the magnetic field has a known strength while it has equal prior probabilities to point in the positive and in the negative \( x \)-axis directions. In figure 1 (left) we show the behavior of \( \langle \langle \cos \theta \rangle \rangle(t) \) as function of time in the two cases of a weak \( |\vec{b}_u| = 0.1 \) (lower, black curve) and a strong \( |\vec{b}_u| = 1.5 \) field (upper, red curve). The curves are obtained by averaging over 104 simulated detection records. The stronger the field amplitude, the better is the directional estimate, but, as also observed in [10], the quantum filter does not unambiguously identify the direction \( \vec{b}_u \) of the unknown field.

We emphasize that the inability to determine the direction of the field unambiguously is not due to the method of analysis of the simulated measurement data. The filter equation, indeed, provides a full Bayesian analysis, but the method of detection restricts the information content of the measurement data and hence the sensitivity of the quantum system to the Hamiltonian.
unknown parameter(s). For example, the measured \( z \)-component of the spin is sensitive to a precession around the \( y \)-axis, but only the first transient evolution of the Bloch vector away from the equatorial plane yields information about the sense of rotation. This is reflected by the curves in figure 1 (left), which reach constant levels after finite time, and hereafter the detection scheme cannot teach us more about the direction of the magnetic field. In the case of the weak field, we learn only very little, because the measurement processes forces the \( z \)-component of the spin to acquire definite values on a time scale too short for the result to be sensitive to the small rotation angle toward the positive or negative \( z \)-axis caused by the magnetic field. Hereafter, further evolution of the \( z \)-component may be suppressed by a quantum Zeno-like effect, and the sense of rotation away from the \( z \)-axis is anyway not discernible by measurements of \( \sigma_z \), only.

This situation changes if the experiment is changed and all the three spin components are detected. As illustrated in figure 1 (right), the accumulation of results from all three detectors causes an effective update of the probability distribution that proceeds much longer in time than when only one detector is applied. For high strength of the field we obtain a probability distribution which is well converged to the correct \( \vec{b}_u \) within the measurement time in the figure, and for the weaker field, we show a more slowly, but steadily, growing probability of the correct value. In these numerical experiments 104 trials have been performed in order to accumulate sufficient data for the statistical averages within a reasonable computational time. A small fraction (~1%) of the simulated trajectories are unstable in the case of the three detectors and they have been rejected from the statistical averages. The probing of several components on the one hand allows discrimination between left and right circular precession, and, on the other hand prevents (Zeno-) locking of the system to eigenstates of the probed quantities as they do not commute and do not have common eigenstates.

3.2. Estimation of a spherically random direction of a magnetic field

Finally, we have analyzed the scenario in which the magnetic field has an isotropic prior probability distribution, represented by an ensemble \( V_B \) with \( N = 98 \) directions on the unit sphere. Here, we assume the stronger field \( |\vec{b}_u| = 1.5 \), and we assume equal strength probing of all three spin components.

The initial condition for the probabilities with all \( P_k = 1/N \) is illustrated by the (green) sphere displayed in figure 2 at time \( Mt = 0 \). We study the convergence of the probability distribution as a function of the probing time, and for long times (\( MT = 25 \)), the filter converges steadily toward a single direction. We note that the spheres are cut on the top because in the simulation we have considered an interval \( \theta = (0, \pi) \) equally spaced with \( N_\theta = 7 \) grid points and an interval \( \phi = [0, 2\pi) \) with \( N_\phi = 14 \) grid points for the azimuthal angle.

The equal strength probing of the three cartesian spin components \( \sigma_x \), \( \sigma_y \) and \( \sigma_z \) is equivalent [10, 22] to probing of any other cartesian set, including for example one parallel component and two perpendicular components to the applied magnetic field, and we find that the monitoring of all three spin components lead to unambiguous identification of the direction of the applied field.

The spin vector may, inadvertently, align, parallel or anti-parallel with the applied magnetic field axis, but the random back-action of the probing of the non-commuting orthogonal spin observables will kick the system away from these directions and give rise to renewed observable precession.
**Figure 2.** Upper panels: time evolution of the probability distribution $P_k$ for an ensemble $V_B$ of 98 elements initially equally distributed over the unit sphere. The unknown magnetic field that has to be determined is the strong field $\vec{b}_u = (1.5, 0, 0)$. Lower panel: evolution of $\langle \langle \cos \theta \rangle \rangle(t)$ for the single shot experiment simulated in the upper panels.

### 4. Conclusion

In this work, we have demonstrated a Bayesian filter for classical parameters, which affect the dynamics of a quantum system. Previous studies along the same lines have focused on QND measurements of typically a single variable, but as shown by our analysis, a non-QND setting may be analyzed by the very same assumptions and methods. Non-QND probing may
have specific advantages and provide more decisive results, when the parameters affect different observables of the quantum probe, as illustrated explicitly by our numerical simulations.

The Bayesian filter is derived from a standard quantum filter formulation of conditioned quantum dynamics. In this mapping, we model the classical parameters as QND observables of auxiliary quantum systems, and their classical probability distribution thus coincides with the conventional reduced density matrix elements for a quantum system. Since the quantum state description cannot be completed by further knowledge in the form of hidden variables, our formulation of the parameter estimation problem, indeed, provides the tightest and most precise probability distribution for the variable of interest conditioned on the measurement outcome and on the prior probability distribution.

The discretization of the parameter space and solution of a quantum system master equation associated with each potential parameter value naturally puts limit on the precision of the method and the number of variables that can be realistically determined. A natural next step would be to apply methods that gradually refine the parameter space around the most likely values and suppress the most unlikely ones from the calculation. Such weighted stochastic differential equations are known in statistics [23], and we imagine that they may be used to decide objective means to suppress and to breed new parameter values without enlarging the memory and computational demands of the method. Another interesting approach, put forward in [24], involves projection of the complete system on a nonlinear lower-dimensional manifold on which the integration of the stochastic differential equations of motion is faster. Alternatively, maximum likelihood methods and random searches through the parameter space, e.g. by Markov chain Monte Carlo methods [25], may be effectively applied to even very large search spaces. We imagine that our simulations may serve as useful reference data for testing such alternative estimation techniques [26].

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Appendix

A.1. Dynamics of a two-state atom and an off-resonant laser field

We consider an atomic quantum system with degenerate ground states \( |g_m\rangle \) and excited states \( |e_m\rangle \), where \( m = \pm 1/2 \) denotes the azimuthal quantum number with respect to the quantization \( z \)-axis. The atom interacts through its magnetic moment \( \hat{\mu} \) with a classical magnetic field \( \vec{B} = (B_x, B_y, B_z) \), which drives a Larmor precession of the ground state atomic spin. The atom is probed by an off-resonant laser field, which is linearly polarized along the \( x \)-axis. Due to the dipole selection rules, the two circularly polarized field components with annihilation operators \( \hat{a}_\pm = (\mp \hat{a}_x + i\hat{a}_y) / \sqrt{2} \) couple individually to the two different ground state populations.
During off-resonant probing (i.e. $\Delta \gg g$), the atomic excited state can be adiabatically eliminated, and the quantized field–atom Hamiltonian is given by [27]
\[
\hat{H} = \frac{\hbar g^2}{\Delta} \sum_{\ell = \pm 1} \hat{a}_\ell^\dagger \hat{a}_\ell |g_{\ell/2}\rangle \langle g_{\ell/2}| + \hat{\mu} \cdot \hat{B},
\]  
(A.1)
where $\Delta = \omega_L - \omega_A$ is the laser atom detuning, $g = \hat{a} \cdot \hat{E}_0 / \hbar$, where $\hat{a}$ is the atomic electric dipole moment and $|\hat{E}_0| = \sqrt{\hbar \omega_L / (V \epsilon_0)}$ is the electric field per photon in the quantization volume $V$ and $\epsilon_0$ is the (electric) vacuum permeability. As result of equation (A.1), the two circularly polarized field components experience phase shifts that depend on the atomic occupation of the two ground states. This implies a (Faraday) rotation of the field polarization, which is proportional to the population difference between the ground states $|g_{\pm 1/2}\rangle$.

Because of the strong linearly polarized probe field with photon number $N_{\text{ph}} \gg 1$, the Stokes operators of the field can be written as
\[
\begin{align*}
\hat{J}_x &= \frac{\hat{a}_-^\dagger \hat{a}_+ - \hat{a}_+^\dagger \hat{a}_-}{2} \approx \frac{N_{\text{ph}}}{2}, \\
\hat{J}_y &= \frac{\hat{a}_-^\dagger \hat{a}_+ + \hat{a}_+^\dagger \hat{a}_-}{2} \approx \sqrt{\frac{N_{\text{ph}}}{2}} (\hat{a}_+ + \hat{a}_-^\dagger) = \sqrt{N_{\text{ph}}} \hat{y}, \\
\hat{J}_z &= \frac{\hat{a}_-^\dagger \hat{a}_+ - \hat{a}_+^\dagger \hat{a}_-}{2i} \approx \frac{\sqrt{N_{\text{ph}}}}{2i} (\hat{a}_+ - \hat{a}_-) = \sqrt{N_{\text{ph}}} \hat{p}_y,
\end{align*}
\]  
(A.2)
defining the canonical conjugate operators $\hat{y}$ and $\hat{p}_y$. The polarization rotation of the field is conveniently measured by subtracting the intensities of polarization components linearly polarized at $\pm 45^\circ$ with respect to the incident field, and when this quantity is expressed in terms of the Stokes observables we recover an expression proportional to the operator $\hat{y}$.

By writing $\hat{\mu} = \mu \hat{\sigma}$, where $\hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ are the Pauli matrices, and using the definitions in equation (A.2), the total Hamiltonian can be written
\[
\hat{H} = \frac{\hbar g^2}{\Delta} \sqrt{N_{\text{ph}}} \hat{p}_y \hat{\sigma}_z + \mu \sum_{a = x, y, z} \hat{\sigma}_a B_a.
\]  
(A.3)
A term $\frac{\hbar g^2}{\Delta} (\hat{a}_-^\dagger \hat{a}_+ + \hat{a}_+^\dagger \hat{a}_-)$ has been omitted from equation (A.3) as it gives rise to a common phase shift, but no polarization rotation of the probe laser field.

The continuous interaction between the atom and the light beam can then be represented as a sequence of interactions of the atom with one beam segment of length $L = c \tau$ after the other. Each segment, in turn, is described as a single harmonic oscillator mode described by the operators in equation (A.2). In the time interval $\tau$ the atom interacts with $N_{\text{ph}} = \Phi \tau$ photons, where $\Phi$ is the photon flux, and the dynamics is well described by the coarse-grained propagator
\[
\hat{U} = e^{-\frac{i}{\hbar} \hat{H} \tau} \simeq e^{-\frac{i}{\hbar} \kappa \tau \hat{p}_y \hat{\sigma}_z} e^{-\frac{i}{\hbar} \mu \tau |\tilde{B}| \hat{\sigma}_B},
\]  
(A.4)
where
\[
\kappa = \frac{\hbar g^2 \tau}{\Delta} \sqrt{N_{\text{ph}}}, \quad \mu = \mu \tau,
\]  
(A.5)
and $\tilde{B} \equiv (n_B^x, n_B^y, n_B^z)$ is the unit vector pointing in the magnetic field direction. We assume that with probe beams of appropriate polarization and propagation direction we can probe the

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atomic ground state spin projection \( \hat{\sigma}_m = \hat{\sigma} \cdot \vec{m} \) along any direction \( \vec{m} \), and we have introduced \( \hat{\sigma}_{m_B} = \hat{\sigma} \cdot \vec{n}_B \). Since \( N_{ph} \propto \tau \) and \( g \propto \tau^{-1/2} \) (through the volume \( V = A \tau c \)), the dimensionless coupling constant \( \kappa_z \) is proportional to \( \tau^{1/2} \), while \( \mu_{y} \) is linear in \( \tau \), and we have thus neglected terms of order \( \tau^{-3/2} \) and higher in equation (A.4).

The incident laser beam is in a coherent state of linearly polarized light described by a Gaussian wave function \( \pi^{-1/4} e^{-p_y^2/2} \), associated with the observable \( \hat{p}_y \) introduced in equation (A.2). The joint state of an atomic ground superposition state \( |\psi_s(t)\rangle = \sum_{\ell=\pm 1} c_{\ell/2} |g_{\ell/2}\rangle \) and the incident \( y \)-polarization component of the quantized probe field can thus be written in the product basis \( |p_y, g_{\ell/2}\rangle \equiv |p_y\rangle \otimes |g_{\ell/2}\rangle \),

\[
|\Psi_{ps}(t)\rangle = \frac{1}{\pi^{1/4}} \sum_{\ell=\pm 1} c_{\ell/2} \int dp_y e^{-p_y^2/2} |p_y, g_{\ell/2}\rangle. \tag{A.6}
\]

Under the action of (A.4), this state evolves into the entangled state

\[
|\Psi_{ps}(t + \tau)\rangle = e^{-\frac{1}{2} \mu_{y} |\vec{B}|} e^{-\frac{1}{2} \kappa_{y} \hat{p}_y \hat{\sigma}_m} |\Psi_{ps}(t)\rangle = \frac{1}{\pi^{1/4}} \sum_{\ell=\pm 1} \int dp_y c'_{\ell/2}(p_y) e^{-p_y^2/2} |p_y, g_{\ell/2}\rangle, \tag{A.7}
\]

where, to first order in \( \tau \) (second order in \( \sqrt{\tau} \)), the new expansion coefficients \( c'_{\ell/2}(p_y) \) are

\[
c'_{\ell/2} = c_{\ell/2} \left[ 1 - \frac{1}{2} \left( \frac{\kappa_{y}}{\hbar} \right)^2 p_y^2 - i \ell \left( m_{\ell} \frac{\kappa_{y}}{\hbar} p_y + n_{\ell} \mu_{y} \frac{|\vec{B}|}{\hbar} \right) \right]
- i c_{-\ell/2} \left[ \kappa_{y} \frac{\mu_{y} |\vec{B}|}{\hbar} (n_{\ell y} - i \ell m_y) + \frac{\mu_{y} |\vec{B}|}{\hbar} (n_{\ell} - i \ell n_{\ell y}) \right]. \tag{A.8}
\]

A.2. A quantum filtering equation for the two-state atom

To describe the back-action due to the field measurement, it is convenient to transform the entangled state (A.7) to the \( y \) rather than \( p_y \) representation of the field, using the relation

\[
|p_y\rangle = \frac{1}{\sqrt{2\pi}} \int dy e^{-ip_y y} |y\rangle. \tag{A.9}
\]

Hence, the state (A.7) can be rewritten as

\[
|\Psi_{ps}(t + \tau)\rangle = \frac{1}{\pi^{1/4}} \sum_{\ell=\pm 1} \int dy \tilde{c}_{\ell/2}(y) e^{-y^2/2} |y, g_{\ell/2}\rangle \tag{A.10}
\]

with the new coefficient \( \tilde{c}_{\ell/2}(y) \) given by

\[
\tilde{c}_{\ell/2} = i c_{-\ell/2} \left[ \frac{\mu_{y} |\vec{B}|}{\hbar} (i \ell n_{\ell y} - n_{\ell y}) + \frac{\kappa_{y}}{\hbar} y (\ell m_y + im_x) \right]
+ c_{\ell/2} \left[ 1 - \frac{1}{2} \left( \frac{\kappa_{y}}{\hbar} \right)^2 (1 - y^2) - i \ell \left( n_{\ell} \frac{\mu_{y} |\vec{B}|}{\hbar} - im_{\ell y} \frac{\kappa_{y}}{\hbar} y \right) \right]. \tag{A.11}
\]

A measurement of the light probe observable \( \hat{y} \) with outcome \( y^D \) projects the state of the system (A.10) onto the state component with that definite value, i.e. the atomic part of the
In order to derive equations (A.3), Quantum filtering equations for the parameter estimation problem, components, and, to lowest order in $\tau^2$, the quantum filtering equation for the state of the atomic system expresses for the state amplitudes to lowest order in $\tau$.

Given the quantum state (A.10), the probability to measure a given value $y^D$ is

$$P(y^D) = \frac{e^{-(y^D)^2}}{\sqrt{\pi}} \sum_{l=\pm 1} |\tilde{c}_{l/2}(y^D)|^2 \sim \pi^{-1/2} e^{-(y^D-y_0)^2},$$

(A.13)

where $y_0 = \frac{\kappa}{\hbar} \langle \hat{\sigma}_m \rangle$. This explicitly shows how the optical probing yields a signal proportional to the desired mean value $\langle \hat{\sigma}_m \rangle$, and it enables us to model the measurement outcome $y^D$ as a stochastic variable

$$y^D = \frac{\kappa}{\hbar} \langle \hat{\sigma}_m \rangle + \frac{\Delta W}{\sqrt{2}\tau},$$

(A.14)

where $\Delta W$ is a (finite) Gaussian Wiener increment with mean zero and variance $\tau$.

By replacing $y$ with the expression (A.14) for $y^D$ in equation (A.12), and by expanding the expressions for the state amplitudes to lowest order in $\tau$, we obtain, in the continuous limit, the quantum filtering equation for the state of the atomic system

$$d\hat{\rho}_k = -i\frac{\hbar}{\rho} [\hat{\sigma}_B, \hat{\rho}_k] dt + M D[n_{\hat{\rho}}] \hat{\rho}_k dt + \sqrt{M} H[\hat{\sigma}_m] \hat{\rho}_k dW(t).$$

(A.15)

Here the interaction parameter $M$ is given explicitly by

$$M = \frac{g^4 \tau^2}{4(\omega_k - \omega_\Lambda)^2} \Phi,$$

(A.16)

and an infinitesimal Wiener process models the noise in the detected signal, $dY^D(t) = 2\sqrt{M} \langle \hat{\sigma}_m \rangle dt + dW$

We note that several probe fields may be used to simultaneously probe different spin components, and, to lowest order in $\tau$, their effects on the quantum state commute. They may hence be included as separate terms with independent Wiener processes $dW_n$.

A.3. Quantum filtering equations for the parameter estimation problem

In order to derive equations (10) and (11) we first need the stochastic master equation for the (unnormalized) density operator $\hat{\rho}_k$, which is given by

$$d\hat{\rho}_k \over \rho_k = \left( i \frac{\hbar}{\rho} [\hat{\rho}_k, \hat{H}(\hat{\gamma}_k)] + \sum_j \gamma j D[\hat{\rho}_j] \hat{\rho}_k + \sum_{n=1}^M M_n D[\hat{M}_n] \hat{\rho}_k \right) dt$$

$$+ \sum_{n=1}^M \sqrt{\eta_n M_n} \left( \hat{M}_n \hat{\rho}_k^+ + \hat{\rho}_k^+ \hat{M}_n^\dagger - \langle \hat{M}_n + \hat{M}_n^\dagger \rangle \right) dV_n(t),$$

(A.17)

where we note that $d\hat{\rho}_k \equiv d(\langle \hat{\rho}_k | \hat{\gamma}_k \rangle)$. Given this, the equation of motion for the probability $P_k$ is easily obtained by computing the trace of equation (A.17), which provides precisely

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equation (11). Now, since $\hat{\gamma}_k^s = \hat{\rho}_k^s / P_k$ we have

$$d\hat{\gamma}_k^s = \frac{1}{P_k} \cdot d\hat{\rho}_k^s + \hat{\rho}_k^s \cdot d \left( \frac{1}{P_k} \right),$$

(A.18)

where (classical Itô formula [28])

$$d \left( \frac{1}{P_k} \right) = -\frac{1}{P_k^2} dP_k + \frac{1}{P_k^3} (dP_k)^2.$$  

(A.19)

Thus, we have

$$\frac{(dP_k)^2}{P_k^2} = \sum_{n=1}^{M} \eta_n M_n \left( \langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}_n^\dagger \rangle_k - \langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}_n^\dagger \rangle_k \right) dt,$$  

(A.20)

where we used the fact that $dW_n(t) dt = 0$ and $dW_n(t) dW_n(t) = \delta_{n,n'} dt$. Consequently, we obtain

$$d \left( \frac{1}{P_k} \right) = \frac{1}{P_k} \sum_{n=1}^{M} \left[ \eta_n M_n \left( \langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}_n^\dagger \rangle_k - \langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}_n^\dagger \rangle_k \right) dt - \sqrt{\eta_n M_n} \right] \times \left( \langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}_n^\dagger \rangle_k - \langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}_n^\dagger \rangle_k \right) dW_n(t),$$

(A.21)

and therefore

$$d\hat{\rho}_k^s \cdot d \left( \frac{1}{P_k} \right) = \sum_{n=1}^{M} \eta_n M_n \left( \langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}_n^\dagger \rangle_k - \langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}_n^\dagger \rangle_k \right)$$

$$\times \left( \langle \hat{\mathcal{M}}_n \hat{\gamma}_k^s + \hat{\gamma}_k^s \hat{\mathcal{M}}_n^\dagger - \langle \hat{\mathcal{M}}_n + \hat{\mathcal{M}}_n^\dagger \rangle_k \right) \right) dt.$$  

(A.22)

Putting all together into equation (A.18) and by using the definition (2) we derive (10).

A.4. Numerical simulations

We can represent the (normalized) density matrices associated with each value $\tilde{y}_k$ as a Bloch vector, and propagate the collection of Bloch vectors as subject to the noisy detection signals—governed by equation (2). These equations have the form

$$d\bar{r}_k = 2[(\bar{\mathbf{b}}_k \times \bar{r}_k) - (\alpha_1 + \alpha_2 + \alpha_3) \bar{r}_k + \bar{\alpha} \times \bar{r}_k + (1 - \delta_{k\alpha,\kappa}) (\bar{I}_k (1 - r_k^2) - \bar{r}_k \times (\bar{r}_k \times \tilde{\bar{r}}_k))] dt$$

$$+ d\hat{\Omega}(1 - r_k^2) \bar{r}_k \times (\bar{r}_k \times d\hat{\Omega}),$$

(A.23)

where we have passed to dimensionless units by performing the replacement $t \to M t$ with $M = \max\{M_1, M_2, M_3\}$. Besides this, we have defined the vectors $\bar{\alpha}$, with components $\alpha_n = M_n / M$, $\bar{\eta} = (\eta_1, \eta_2, \eta_3)$, $b_k = \mu_b \bar{B}_k / (\hbar M)$ and $d\hat{\Omega}$, with components $d\Omega_n = \sqrt{\eta_n \alpha_n} dW_n$. The symbol $\ast$ in equation (A.23) indicates the pointwise product, $(\bar{\mathbf{p}} \ast \bar{\mathbf{q}})_n \equiv p_n q_n$.

Instead, the probabilities represented as the column vector $\bar{\mathbf{P}} = (P_1, P_2, \ldots, P_N)^T$ obey the following matrix equation:

$$d\bar{\mathbf{P}} = 4(\bar{\mathbf{P}}^\dagger \cdot (\tilde{\mathbf{C}}^T \cdot \tilde{\mathbf{C}})^T \mathbf{P}^\dagger - \bar{\mathbf{P}}^\dagger \cdot (\tilde{\mathbf{C}}^T \cdot \tilde{\mathbf{C}})^T)$$

$$- \bar{\mathbf{P}}^\dagger [(\tilde{\mathbf{C}} \mathbf{P}^\dagger \tilde{\mathbf{C}}^T + \bar{\mathbf{P}}^\dagger [(\tilde{\mathbf{C}} \mathbf{P})^T \cdot (\tilde{\mathbf{C}} \mathbf{P})]) dt + G \ d\hat{\Omega}.$$  

(A.24)
Here, $C$ and $\tilde{C}$ are $3 \times N$ matrices and $G$ is an $N \times 3$ matrix containing the Bloch vector solutions of equation (A.23), $C_{n,j} = x_j^{(n)}$, $\tilde{C}_{n,j} = \eta_n \alpha_n C_{n,j}$ and $G_{j,n} = 2P_j (x_j^{(n)} - \tilde{P}^T \cdot \tilde{X}_n)$ where $\tilde{X}_n = (x_1^{(n)}, x_2^{(n)}, \ldots, x_N^{(n)})^T$, and $\tilde{v} = \tilde{\eta} * a * \tilde{r}_0$.

For the numerical simulation of both (A.23) and (A.24) we employed an Itô–Euler integrator [28] with a time step $\Delta t$ ranging from $2 \times 10^{-7}$ to $10^{-5} \, M^{-1}$ depending on the size of the set $\mathcal{V}_H$ and on the number of switched off detectors. Such a choice enabled us to have an efficient integrator, even though some of the quantum trajectories might have been unstable. To solve the instability problem, we have first tried to apply an implicit Milstein method [28, 29], but since both (A.23) and (A.24) are nonlinear, one has to solve numerically at each time step (e.g. by means of the Nelder–Mead method [25]) implicit equations like $\tilde{r}_1(t + \Delta t) = \tilde{r}_1(t) + f(\tilde{r}_1(t + \Delta t))$, where $f$ is the rhs of equation (A.23) plus some additional term due to the Milstein routine. While such a strategy might solve the problem, we have numerically observed that such an approach is significantly more time consuming than the Itô–Euler integrator. Thus, we also applied a derivative free order 2.0 weak predictor corrector method [29], which turns out to be quite efficient in the case of a single Wiener noise process, but in the case of three detectors we noticed, as for other predictor–corrector methods, that the instability could not be fixed. Hence, we employed the simple Itô–Euler integrator with (rather) small time steps (up to $\Delta t = 2 \times 10^{-7} \, M^{-1}$). We noticed that with such a simple strategy the unstable quantum trajectories could have been reduced or even suppressed, but at the expenses of a very long numerical computation.

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