A Conjectural Formula for Genus One Gromov-Witten Invariants of a Class of Local Calabi-Yau $n$-folds

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Abstract

We conjecture a formula for the generating function of genus one Gromov-Witten invariants of the local Calabi-Yau manifolds which are the total spaces of splitting bundles over projective spaces. We prove this conjecture in several special cases, and assuming the validity of our conjecture we check the integrality of genus one BPS numbers of local Calabi-Yau 5-folds defined by A. Klemm and R. Pandharipande.

1 Introduction

After a series of splendid works with Jun Li and R. Vakil (see [10] the references therein), A. Zinger finally explicitly computed the genus one Gromov-Witten invariants of Calabi-Yau hypersurfaces in projective spaces. This result is generalized to complete intersections in projective spaces by A. Popa in [8]. Our object is to find a similar formula for the genus one Gromov-Witten invariants of the local Calabi-Yau $n$-fold

$$X = \text{Tot}(\mathcal{O}(-c_1) \oplus \cdots \mathcal{O}(-c_m) \rightarrow \mathbb{P}^{n-m}),$$

where $c_i \in \mathbb{Z}_{>0}$ for $1 \leq i \leq m$ and $\sum_{i=1}^{m} c_i = n - m + 1$.

Let us first recall Zinger’s formula. Let the target space $Y$ be a degree $n$ hypersurface in $\mathbb{P}^{n-1}$. For $q = 0, 1, \ldots$, define $I_{0,q}$ by

$$\sum_{q=0}^{\infty} I_{0,q}(t) w^q = e^{wt} \sum_{d=0}^{\infty} e^{dt} \prod_{r=1}^{d} (nw + r) \prod_{r=1}^{d} (w + r)^n.$$  

(2)

It is easy to see that for $0 \leq q \leq n - 2$, $I_{0,q}$ are solutions of the Picard-Fuchs operator

$$\mathcal{L} = \left( \frac{d}{dt} \right)^{n-1} - nc' \prod_{r=1}^{n-1} (\frac{d}{dt} + r).$$

(3)

For $q \geq p \geq 0$, we inductively define

$$I_{p,q}(t) = \frac{d}{dt} \left( \frac{I_{p-1,q}(t)}{I_{p-1,0}(t)} \right),$$

(4)

and the mirror map is given by

$$T = \frac{I_{0,1}(t)}{I_{0,0}(t)}.$$  

(5)
Thus $T - t$ and $I_{p, p}(t)$ are series of $e^t$ for $p \geq 0$. The genus one degree $d$ Gromov-Witten invariants $N_{1, d}^X$ are given by

$$
\sum_{d=1}^{\infty} N_{1, d}^X e^{dT} = \left( \frac{(n - 2)(n + 1)}{48} + \frac{1 - (1 - n)^n}{24n^2} \right) (T - t) + \frac{n^2 - 1 + (1 - n)^n}{24n} \ln I_{0, 0}(t)
- \left\{ \begin{array}{l}
\frac{n-1}{48} \ln(1 - n^n e^t) + \frac{n-1}{8n} \sum_{p=0}^{n-3} \frac{(n-1-2p)^2}{8} \ln I_{p, p}(t), \quad \text{if} \quad 2 \nmid n;
\frac{n-1}{48} \ln(1 - n^n e^t) + \frac{n-1}{8n} \sum_{p=0}^{n-3} \frac{(n-1-2p)^2}{8} \ln I_{p, p}(t), \quad \text{if} \quad 2 \nmid n.
\end{array} \right.
$$

(6)

Before Zinger’s work, the formula for $n = 5$ or 6 ($Y$ is a quintic 3-fold or a sextic 4-fold, resp.) had been conjectured via mirror symmetry and physical arguments on the B-side, see [2] and [6]. For $n \geq 7$, the B-side interpretation is still absent, at least to the best knowledge of the author.

The Gromov-Witten invariants of local Calabi-Yau manifolds which are total spaces of vector bundles over toric varieties are in principle less difficult to compute, because we can directly apply the virtual localization method. But in dimension greater than 3, it seems not easy to get a closed formula due to the complicated combinatorics. Thus to get a formula for local Calabi-Yau spaces, a possible approach is just to adapt Zinger’s method to the local case, i.e., we need to

1. Find a standard vs reduced comparison formula for relevant Hodge integrals on $\mathcal{M}_{1, k}(\mathbb{P}^{n-m+1}, d)$ and $\mathfrak{M}_{1, k}(\mathbb{P}^{n-m+1}, d)$.
2. Find a formula for Hodge integrals on $\mathcal{M}_{1, k}$.
3. Write the Hodge integrals on $\mathfrak{M}_{1, k}(\mathbb{P}^{n-m+1}, d)$ as contributions of graphs by localization.
4. Generalize the combinatorial arguments in [10] to the local cases.

In principle also, the above procedure should be less difficult than that of the compact cases, since in the latter cases the involved sheaves $R^0\pi_* f^* \mathcal{O}(n - m + 2)$ is not locally free. We have made some progress on this and hope to address it in the future. In this article, however, we get a formula by a mixture of physical arguments and mathematical observations on Zinger’s proof, and we check the formula by proving it in several most simple cases, and also by checking the integrality of the BPS numbers of local Calabi-Yau 5-folds.

Now let us take a closer look at [6]. For the first term, the coefficient of $T - t$ physically (see [2]) comes from the integral

$$
\frac{1}{24} \int_{Y} k \wedge c_{n-3}(Y),
$$

(7)

where $k$ is the Kähler class of $Y$ associated with the variable $T$, and is $H$ here, the class induced by the hyperplane class in the ambient space $\mathbb{P}^{n-1}$. The Chern class is easily computed

$$
c_{n-3}(Y) = \left( \frac{(n - 2)(n + 1)}{2n} + \frac{1 - (1 - n)^n}{n^3} \right) H^{n-3}.
$$

(8)

For the local case, for the target space $X$ of the form [11], the series corresponding to [2] is

$$
\sum_{q=0}^{\infty} I_{0, q}(t) w^q = e^{wt} \sum_{d=0}^{\infty} e^{dt} \prod_{i=1}^{m} \prod_{s=0}^{d-1} \frac{(-c_i w - s)}{(w + s)^{n-m+1}},
$$

(9)

which encodes the genus zero one-point and two-point Gromov-Witten invariants of $X$ by [9]. It is easy to see that, when $m > 1$ the mirror map is the identity map $T = t$, so the first term of [9] has

\footnotesize

The potential $F_1$ differs from the nowadays usual choice of potential by a factor 2, so the coefficient $\frac{1}{12}$ is taken as $\frac{1}{24}$ here.

\normalsize
no counterpart in these cases. When \( m = 1 \), \( X \) is the total space of the canonical bundle of \( \mathbb{P}^{n-1} \), and \( c_{n-1}(X) = -\frac{n(n+1)(n-2)}{4} H^{n-1} \). The Kähler class is still \( H \), but the integral of \( c_{n-1}(X) \wedge H \) over the local space \( X \) should be taken as the integral of the (formal) quotient of \( c_{n-1}(X) \wedge H \) by the Euler class of \( \mathcal{O}(-n) \) over the compact part \( \mathbb{P}^{n-1} \), as a general principle.

We can also get the same result in another way. In the mathematical proof of Zinger, the coefficient of \( T - t \) comes from a computation of residues. In fact, the first term of the coefficient comes from a residue at 0, and the second term from a residue at \( -n \). In the local case, by a speculation on Zinger’s proof, there should be no residues at \( -n \) and the residue at 0 is the same as the global case.

So the counterpart of the first term in the formula for \( K_{p_{n-1}} \) should be

\[
\frac{(n+1)(n-2)}{48} (T - t).
\]  

(10)

For the second term of (6), since in the local case we always have \( I_{0,0}(t) = 1 \) from (9), it has no counterpart in the local case.

For the third term of (6), we follow the arguments in [6]. By some physical argument, this term comes from the behavior of the potential at the conifold point of the moduli space on the B-side, and the coefficient \( -\frac{n-1}{48} \) (or \( -\frac{n-4}{8} \) (if \( n \) is odd or even, resp.) should be universal. The \( 1 - n^e t^e \) comes from the discriminant of the Picard-Fuchs operator (3). In the local case, the Picard-Fuchs operator is

\[
\mathcal{L} = \left( \frac{d}{dt} \right)^{n-m+1} - e^t \prod_{i=1}^{m} (-c_i \frac{d}{dt} - s),
\]

and the discriminant is

\[
\Delta = 1 - \prod_{i=1}^{m} (-c_i)^e t^e.
\]  

(11)

So the counterpart of the third term in the local case should be

\[
- \left\{ \begin{array}{ll}
\frac{n+1}{48} \ln(1 - \prod_{i=1}^{m} (-c_i)^e t^e), & \text{if } 2 \nmid n; \\
\frac{n-4}{8} \ln(1 - \prod_{i=1}^{m} (-c_i)^e t^e), & \text{if } 2 \mid n.
\end{array} \right.
\]  

(12)

The fourth group of terms of (6) seems the most mysterious. On one hand, I believe that, to get a series of \( e^t \) (not a mixture of \( t \) and \( e^t \), or equivalently, without log \( q \) terms, where \( q = e^t \)) from the solutions of the corresponding Picard-Fuchs equation, and to encode enough data from these solutions to get the genus one invariants, the inductive procedure [4] is somewhat ubiquitous, and thus in the same way we obtain \( I_{p,p}(t) \) in the local case. On the other hand, by a speculation on the argument in [6], I believe that if one could find a B-side interpretation of (6), the coefficient of \( I_{p,p}(t) \) would come from the fact \( h^{p,p} = 1 \) for \( 0 \leq p \leq n - 2 \) (corresponding to the Ramond-Ramond sector on the B-side) and the elementary identities

\[
\frac{1}{2} + \frac{3}{2} + \cdots + \frac{n-2}{2} - 2p = \frac{(n-1-2p)^2}{8}
\]

or

\[
\frac{2}{2} + \frac{4}{2} + \cdots + \frac{n-2}{2} - 2p = \frac{(n-2p)(n-2-2p)}{8}
\]

for \( n \) is odd or even, resp.. So the counterpart of the fourth group of terms in the local cases should be

\[
- \left\{ \begin{array}{ll}
\sum_{p=1}^{(n-1)/2} \frac{(n+1-2p)^2}{8} \ln I_{p,p}(t), & \text{if } 2 \nmid n; \\
\sum_{p=1}^{(n-2)/2} \frac{(n+2-2p)(n-2-2p)}{8} \ln I_{p,p}(t), & \text{if } 2 \mid n.
\end{array} \right.
\]  

(13)

Combining the above discussions, we obtain the following

\[\text{2Writing the local Gromov-Witten invariants as Hodge integrals over the moduli space of stable maps to the compact part, to make the WDVV equation still hold, we need to cancel one of the two copies of contributions of the Euler class at the node, in the usual derivation of the WDVV equation.}\]
Conjecture 1. Let $X$ be of the form $[1]$. For $m = 1$, we have

$$\sum_{d=1}^{\infty} N_{1,d} x^d e^{dT} = \frac{(n-2)(n+1)}{48} (T-t) - \left\{ \begin{array}{ll} \frac{n+1}{48} \ln(1+n^ne^t) + \frac{\sum_{p=1}^{n-1} (n+1-2p)^2}{8} \ln I_{p,p}(t), & \text{if } 2 \nmid n; \\ \frac{n^2}{48} \ln(1-n^ne^t) + \frac{\sum_{p=1}^{n-2} (n+2-2p)^2}{8} \ln I_{p,p}(t), & \text{if } 2 \mid n. \end{array} \right. \quad (14)$$

For $m \geq 2$, we have (in these cases $T = t$)

$$\sum_{d=1}^{\infty} N_{1,d} x^d e^{dt} = -\left\{ \begin{array}{ll} \frac{n+1}{48} \ln(1 - \prod_{i=1}^{m} (-c_i)^e e^t) + \sum_{p=1}^{(n-1)/2} \frac{(n+1-2p)^2}{8} \ln I_{p,p}(t), & \text{if } 2 \nmid n; \\ \frac{n^2}{48} \ln(1 - \prod_{i=1}^{m} (-c_i) e^t) + \sum_{p=1}^{(n-2)/2} \frac{(n+2-2p)^2}{8} \ln I_{p,p}(t), & \text{if } 2 \mid n. \end{array} \right. \quad (15)$$

In fact, the above discussions suggest a recipe to get genus one Gromov-Witten invariants from genus zero invariants for Calabi-Yau $n$-folds with $h^{1,1} = 1$. Thus one can try to make similar conjectures for, e.g., Calabi-Yau complete intersections in Grassmannians. It is very desirable to give a B-side interpretation of these formulae, e.g., by solving the $tt^*$-equations.

The $n = 3$ and $n = 4$ cases of the conjecture $[1]$ has been given in $[1]$ and $[6]$. The main theorem of this article is

**Theorem 1.** The conjecture $[1]$ holds for degree one invariants, and holds for $X = \text{Tot}(\mathcal{O}(1)^{\oplus(l+1)} \to \mathbb{P}^l)$ and $X = \text{Tot}(\mathcal{O}(1)^{\oplus(-1)} \oplus \mathcal{O}(-2) \to \mathbb{P}^l)$ in all degrees, for $l \geq 1$.

We prove this theorem by virtual localization ($[3]$). Finally, we check the integrality of $n_{1,d}$ defined for Calabi-Yau 5-folds in $[2]$, from our conjectural formulae $[14]$ and $[15]$.

**Conventions:**
- We use $[x^k] f(x)$ to represent the coefficient of $x^k$ in the Laurent expansion of $f(x)$ at $x = 0$. In this article $x$ may be $q$, $e^t$, $Q$ or $w$.
- Since the compact part of the target spaces that we consider in this article are always projective spaces, we use $H$ to denote the hyperplane class throughout. Also, $N_{1,q}$ always denotes the genus one Gromov-Witten invariants of the Calabi-Yau space $X$ with no insertion.
- We always understand $Q = e^T$ and $q = e^t$. In the first three sections we usually use $e^t$ and $e^T$. In the section 4 we use $Q$ and $q$, and understand that $I_{p,p}(q)$ means replacing $e^t$ by $q$ in the expansion of $I_{p,p}(t)$.
- In the graphs that represent the fixed loci in the moduli spaces of genus one stable maps, $\circ$ represents a genus one component, and $\bullet$ represents a genus zero component.
- The formal integrals over $\mathcal{M}_{0,1}$ and $\mathcal{M}_{0,2}$ are understood as extending the range of $n$ in the following identity to $n \geq 1$:

$$\int_{\mathcal{M}_{0,n}} \prod_{i=1}^{n} (w_i - \psi_i) = \prod_{i=1}^{n} w_i \left( \sum_{i=1}^{n} w_i \right)^{n-3}.$$

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## 2 Degree one invariants

The genus one degree one invariants of local Calabi-Yau $n$-folds of the form of $[1]$ can be easily computed by virtual localization. Let the torus $(\mathbb{C}^*)^{n-m+1}$ acts on $\mathbb{P}^{n-m}$ with fixed point $P_i$, $1 \leq i \leq n - m + 1$, such that the $n - m$ weights at $P_i$ is $\alpha_i - \alpha_k$, for $k \in \{1, \ldots, n - m + 1\} \setminus \{i\}$. We choose the linearizations of $\mathcal{O}(-c_i)$ with weight $-c_i \alpha_k$ at $P_k$, for $1 \leq i \leq m$, $1 \leq k \leq n - m + 1$. The
torus action naturally induces an action on $\mathcal{M}_{1,0}(\mathbb{P}^{n-1}, 1)$, whose fixed loci are corresponding to the graphs of the form

$$
\Gamma_{ij} = \begin{array}{c}
\bullet \\
\circ \\
\bullet
\end{array},
$$

where $1 \leq i \neq j \leq n - m + 1$. Let us first assume $m = 1$. Then the contribution of $\Gamma_{ij}$ is

$$
\int_{\mathcal{M}_{1,1}} \frac{(\alpha_j - \alpha_i) \prod_{k \neq i} \Lambda_1^i(\alpha_i - \alpha_k) \cdot \Lambda_1^j(-\nu \alpha_i) \prod_{a=1}^{n-1} (-n \alpha_j + a(\alpha_j - \alpha_i))}{(\alpha_i - \alpha_j - \psi)(\alpha_i - \alpha_j)(\alpha_j - \alpha_i) \prod_{k \neq i,j} \prod_{a=0}^{1} (\alpha_i - \alpha_k + a(\alpha_j - \alpha_i))} = \frac{(-1)^{n-1} \alpha_i \prod_{a=1}^{n-1} ((n-a)\alpha_j + a\alpha_i)}{24 \prod_{j \neq i,j} (\alpha_j - \alpha_k)} \left( \sum_{k \neq i,j} \frac{1}{\alpha_i - \alpha_k} - \frac{1}{n \alpha_i} \right).
$$

Note that

$$
\sum_{j \neq i} \frac{\prod_{a=1}^{n-1} ((n-a)\alpha_j + a\alpha_i)}{\prod_{k \neq i,j} (\alpha_j - \alpha_k)} = n^{n-1} \alpha_i^{n-1} \sum_{j \neq i} \frac{1}{(\alpha_j - \alpha_i) \prod_{k \neq i,j} (\alpha_j - \alpha_k)} + (n-1)!
$$

$$
= -\frac{n^{n-1} \alpha_i^{n-1}}{\prod_{j \neq i} (\alpha_i - \alpha_j)} + (n-1)!,
$$

which are easily to show by the residue theorem on $\mathbb{P}^1$. Thus we have

$$
\sum_{j \neq i} \frac{n \alpha_i \prod_{a=1}^{n-1} ((n-a)\alpha_j + a\alpha_i)}{\prod_{k \neq i,j} (\alpha_j - \alpha_k)} \left( \sum_{k \neq i,j} \frac{1}{\alpha_i - \alpha_k} - \frac{1}{n \alpha_i} \right)
$$

$$
= n \alpha_i \left( \sum_{k \neq i} \frac{1}{\alpha_i - \alpha_k} \right) \sum_{j \neq i} \frac{\prod_{a=1}^{n-1} ((n-a)\alpha_j + a\alpha_i)}{\prod_{k \neq i,j} (\alpha_j - \alpha_k)} - \sum_{j \neq i} \frac{\prod_{a=1}^{n-1} ((n-a)\alpha_j + a\alpha_i)}{\prod_{k \neq i,j} (\alpha_j - \alpha_k)}
$$

$$
+ n \alpha_i \sum_{j \neq i} \frac{\prod_{a=1}^{n-1} ((n-a)\alpha_j + a\alpha_i)}{\prod_{k \neq i,j} (\alpha_j - \alpha_k)}
$$

$$
= \left[ n \alpha_i \left( \sum_{k \neq i} \frac{1}{\alpha_i - \alpha_k} \right) - 1 \right] \left( -\frac{n^{n-1} \alpha_i^{n-1}}{\prod_{j \neq i} (\alpha_i - \alpha_j)} + (n-1)! \right)
$$

$$
+ \prod_{j \neq i} (\alpha_i - \alpha_j) \sum_{j \neq i} \frac{1}{\alpha_i - \alpha_j} - \frac{n^{n-1} \alpha_i^{n-1}}{2 \prod_{j \neq i} (\alpha_i - \alpha_j)}
$$

$$
= n! \alpha_i \left( \sum_{k \neq i} \frac{1}{\alpha_i - \alpha_k} \right) - (n-1)! - \frac{n^{n-1} \alpha_i^{n-1}}{2 \prod_{j \neq i} (\alpha_i - \alpha_j)}
$$

and thus

$$
\sum_{i=1}^{n} \sum_{j \neq i} \frac{n \alpha_i \prod_{a=1}^{n-1} ((n-a)\alpha_j + a\alpha_i)}{\prod_{k \neq i,j} (\alpha_j - \alpha_k)} \left( \sum_{k \neq i,j} \frac{1}{\alpha_i - \alpha_k} - \frac{1}{n \alpha_i} \right)
$$

$$
= n! \cdot \frac{n(n-1)}{2} - n! - \frac{n^{n-1} \alpha_i^{n-1}}{2} = \frac{(n! - n^{n-1} \alpha_i^{n-1})(n-2)(n+1)}{2}.
$$

(16)
Now assume \( m \geq 2 \). The contribution of \( \Gamma_{ij} \) is

\[
\int_{M_{ij}} \frac{(\alpha_j - \alpha_i) \prod_{k \neq i} \Lambda^j_k(\alpha_i - \alpha_k) \cdot \prod_{i=1}^{m} \left( \Lambda^j_i(-c_i\alpha_i) \prod_{a=1}^{n-1}(\alpha_j - \alpha_a) + a(\alpha_j - \alpha_a) \right)}{(\alpha_i - \alpha_j - \psi)(\alpha_i - \alpha_j)(\alpha_j - \alpha_k) \prod_{k \neq i,j}(\alpha_i - \alpha_k)(\alpha_j - \alpha_k)}
\]

\[
= \frac{(-1)^{n-m+1} \alpha_i^m \prod_{i=1}^{m} \left( c_i \prod_{a=1}^{n-1}(\alpha_j - \alpha_a) + a(\alpha_j - \alpha_a) \right)}{24 \prod_{k \neq i,j}(\alpha_i - \alpha_k)(\alpha_j - \alpha_k)} \left( \sum_{k \neq i,j} \frac{1}{\alpha_i - \alpha_k} - \sum_{l=1}^{m} \frac{1}{c_l \alpha_i} \right).
\]

Similar to the \( m = 1 \) case, we have

\[
\sum_{j \neq i} \prod_{k=i}^{m} \prod_{k \neq j}(\alpha_i - \alpha_j) = \frac{\prod_{i=1}^{m} c_i^j \alpha_i^{n-2m+1}}{\prod_{j \neq i}(\alpha_i - \alpha_j)} \sum_{j \neq i} \frac{1}{\alpha_i - \alpha_j} - \frac{(n-2m+1) \prod_{i=1}^{m} c_i^j \alpha_i^{n-2m}}{2 \prod_{j \neq i}(\alpha_i - \alpha_j)}.
\]

So

\[
\sum_{j \neq i} \frac{\alpha_i^m \prod_{i=1}^{m} \left( c_i \prod_{a=1}^{n-1}(\alpha_j - \alpha_a) + a(\alpha_j - \alpha_a) \right)}{(\alpha_i - \alpha_j) \prod_{k \neq i,j}(\alpha_j - \alpha_k)} \left( \sum_{k \neq i,j} \frac{1}{\alpha_i - \alpha_k} - \sum_{l=1}^{m} \frac{1}{c_l \alpha_i} \right)
\]

\[
\quad = \sum_{j \neq i} \left[ \sum_{k \neq i} \frac{1}{\alpha_i - \alpha_k} \cdot \frac{\alpha_i^m \prod_{i=1}^{m} \left( c_i \prod_{a=1}^{n-1}(\alpha_j - \alpha_a) + a(\alpha_j - \alpha_a) \right)}{(\alpha_i - \alpha_j) \prod_{k \neq i,j}(\alpha_j - \alpha_k)} \right] + \alpha_i^m \prod_{i=1}^{m} \left( c_i \prod_{a=1}^{n-1}(\alpha_j - \alpha_a) + a(\alpha_j - \alpha_a) \right)
\]

\[
\quad = -\sum_{k \neq i} \frac{1}{\alpha_i - \alpha_k} \cdot \frac{\prod_{i=1}^{m} c_i^k \alpha_i^{n-m+1}}{\prod_{j \neq i}(\alpha_i - \alpha_j)} + \frac{\prod_{i=1}^{m} c_i^j \alpha_i^{n-m+1}}{\prod_{j \neq i}(\alpha_i - \alpha_j)} \sum_{j \neq i} \frac{1}{\alpha_i - \alpha_j}
\]

\[
\quad = \left( \sum_{l=1}^{m} \frac{1}{c_l} - \frac{n-2m+1}{2} \right) \prod_{i=1}^{m} c_i^j \alpha_i^{n-m} \prod_{j \neq i}(\alpha_i - \alpha_j).
\]

Therefore for \( m \geq 2 \) we obtain

\[
N_{1,d}^X = \frac{(-1)^{n-m+1}}{24} \left( \sum_{l=1}^{m} \frac{1}{c_l} - \frac{n-2m+1}{2} \right) \prod_{i=1}^{m} c_i^j \alpha_i^{n-m}.
\] (17)

We need to check that our conjectural formulae \([14] \) and \([15] \) match \([10] \) and \([17] \). First we give a lemma.

**Lemma 2.1.** If \( n \) is odd, suppose \( n = 2r + 1 \), we have

\[
\text{Res}_{w=0} \frac{(2w+1) \prod_{i=1}^{n} c_i \alpha_i^{n-1}}{(w+1)^{r-m+1} \prod_{i=1}^{m} c_i \alpha_i^{n-1}} = -\frac{1}{12} \left( \frac{1}{c_i} - \sum_{l=1}^{m} c_l \right) \prod_{i=1}^{m} c_i^j.
\] (18)
If \( n \) is even, suppose \( n = 2r \), we have

\[
\text{Res}_{w=0} \frac{\prod_{i=1}^{m} \Gamma_s^{-1} (c_i w + s)}{(w + 1)^{-m+1} w^{r-m}} = -\frac{1}{24} \left( \frac{1}{c_i} - \sum_{i=1}^{m} c_i + \frac{3}{2} \right) \prod_{i=1}^{m} c_i^{-1}.
\] (19)

Proof: The crucial point is to notice that

\[
\text{Res}_{w=0} \frac{(2w + 1) \prod_{i=1}^{m} \Gamma_s^{-1} (c_i w + s)}{(w + 1)^{-m+1} w^{r-m+1}} = \text{Res}_{w=-1} \frac{(2w + 1) \prod_{i=1}^{m} \Gamma_s^{-1} (c_i w + s)}{(w + 1)^{-m+1} w^{r-m+1}}
\]

and

\[
\text{Res}_{w=0} \frac{\prod_{i=1}^{m} \Gamma_s^{-1} (c_i w + s)}{(w + 1)^{-m+1} w^{r-m+1}} = \text{Res}_{w=-1} \frac{\prod_{i=1}^{m} \Gamma_s^{-1} (c_i w + s)}{(w + 1)^{-m+1} w^{r-m+1}}
\]

by substitution of variables. Thus by the residue theorem on \( \mathbb{P}^1 \), it suffices to compute

\[
\text{Res}_{w=\infty} \frac{(2w + 1) \prod_{i=1}^{m} \Gamma_s^{-1} (c_i w + s)}{(w + 1)^{-m+1} w^{r-m+1}}
\]

and

\[
\text{Res}_{w=\infty} \frac{\prod_{i=1}^{m} \Gamma_s^{-1} (c_i w + s)}{(w + 1)^{-m+1} w^{r-m+1}}.
\]

We leave the details to the reader. \( \square \)

The functions \( I_{0,q}(t) \) for \( X \) are defined by

\[
\sum_{q=0}^{\infty} I_{0,q}(t) w^q = e^t \sum_{d=0}^{\infty} \prod_{i=1}^{m} \Gamma_s^{d-1} (-c_i w - s) \prod_{i=1}^{m} (w + s)^{n-m+1} = e^t \left[ 1 + \sum_{d=1}^{\infty} \prod_{i=1}^{m} \Gamma_s^{d-1} (c_i w + s) \prod_{i=1}^{m} (w + s)^{n-m+1} \right].
\] (20)

For \( q \geq p \), define

\[
I_{p,q}(t) = \frac{d}{dt} \left( \frac{I_{p-1,q}(t)}{I_{p-1,p-1}(t)} \right).
\] (21)

Proposition 2.1. For \( 2 \nmid n \),

\[
\sum_{p=1}^{(n-1)/2} \frac{(n + 2p - 2p)^2}{8} [c^t] (\ln I_{p,p}(t)) = -\frac{(-1)^{\sum_{i=1}^{m} c_i}}{24} \left( \sum_{i=1}^{m} \frac{1}{c_i} - \sum_{i=1}^{m} c_i \right) \prod_{i=1}^{m} c_i^t,
\] (22)

and for \( 2 \mid n \),

\[
\sum_{p=1}^{(n-2)/2} \frac{(n + 2p - 2p)(n - 2p)}{8} [c^t] (\ln I_{p,p}(t)) = -\frac{(-1)^{\sum_{i=1}^{m} c_i}}{24} \left( \sum_{i=1}^{m} \frac{1}{c_i} - \sum_{i=1}^{m} c_i + \frac{3}{2} \right) \prod_{i=1}^{m} c_i^t.
\] (23)

Proof: For a fixed \( n \), suppose

\[
\frac{(-1)^{n-m+1} \prod_{i=1}^{m} \Gamma_s^{c_i-1} (c_i w + s)}{(w + 1)^{n-m+1}} = a_1 w + a_2 w^2 + \cdots,
\] (24)

then a straightforward induction shows

\[
I_{p,p}(t) = 1 + e^t \sum_{k=1}^{p} a_k \binom{p-1}{k-1} + O(e^t).
\] (25)
Now we treat the cases that \( n \) is odd or even separately.

(i) \( n = 2r + 1 \), and \( r \geq 0 \). By (25) we have

\[
\sum_{p=1}^{(n-1)/2} \frac{(n+1-2p)^2}{8} |e^r| (\ln I_{p,p}(t)) = \sum_{p=1}^{r} \frac{(r+1-p)^2}{2} \sum_{k=1}^{p} a_k \left( \frac{p-1}{k-1} \right)
\]

\[
= \sum_{k=1}^{r} a_k \sum_{p=1}^{r} \frac{(r+1-p)^2}{2} \left( \frac{p-1}{k-1} \right).
\]

Since

\[
(m + 1 - p)^2 = (p + 1)p - p(2m + 3) + (m + 1)^2,
\]

we have

\[
\sum_{p=1}^{r} (r+1-p)^2 \left( \frac{p-1}{k-1} \right)
\]

\[
= \sum_{p=1}^{r} \left( (p+1)p \left( \frac{p-1}{k-1} \right) - (2r+3)p \left( \frac{p-1}{k-1} \right) + (r+1)^2 \left( \frac{p-1}{k-1} \right) \right)
\]

\[
= \sum_{p=1}^{r} \left( (k+1)k \left( \frac{p+1}{k+1} \right) - (2r+3)k \left( \frac{p}{k} \right) + (r+1)^2 \left( \frac{r}{k} \right) \right)
\]

\[
= (k+1)k \left( \frac{r+2}{k+2} \right) - (2r+3)k \left( \frac{r+1}{k+1} \right) + (r+1)^2 \left( \frac{r}{k} \right)
\]

\[
= (k+2)(k+1) - 2(k+2) + 2 \left( \frac{r+2}{k+2} \right) - (2r+3)(k+1) - (r+1) + (r+1)^2 \left( \frac{r}{k} \right)
\]

\[
= 2 \left( \frac{r+2}{k+2} \right) - \left( \frac{r+1}{k+1} \right).
\]

Thus by (26) and (27) we have

\[
\sum_{p=1}^{(n-1)/2} \frac{(n+1-2p)^2}{8} |e^r| (\ln I_{p,p}(t)) = \frac{1}{2} \sum_{k=1}^{r} a_k \left( \frac{r+2}{k+2} - \left( \frac{r+1}{k+1} \right) \right)
\]

\[
= \frac{1}{2} \|w^{-2}\| \left( \frac{2(-1)^{n-m+1} \prod_{i=1}^{m} c_i \cdot (c_i w + s)}{(w+1)^{n-m+1}} \cdot \left( 1 + \frac{1}{w} \right)^{r+2} \right)
\]

\[- \frac{1}{2} \|w^{-1}\| \left( \frac{(-1)^{n-m+1} \prod_{i=1}^{m} c_i \cdot (c_i w + s)}{(w+1)^{n-m+1}} \cdot \left( 1 + \frac{1}{w} \right)^{r+1} \right)
\]

\[
= \frac{(-1)^{n-m+1} \prod_{i=1}^{m} c_i \cdot (c_i w + s)}{2 \|w^{-m}\|} \left( \frac{2w+1}{w+1} \prod_{i=1}^{m} c_i \cdot \left( (w+1)^{w+1} \right) \right).
\]

Thus by (18) we obtain (22).

(ii) \( n = 2r \), and \( r \geq 1 \). By (25) we have

\[
\sum_{p=1}^{(n-2)/2} \frac{(n+2-2p)(n-2p)}{8} |e^r| (\ln I_{p,p}(t)) = \sum_{p=1}^{r-1} \frac{(r+1-p)(r-p)}{2} \sum_{k=1}^{p} a_k \left( \frac{p-1}{k-1} \right)
\]

\[
= \sum_{k=1}^{r-1} a_k \sum_{p=1}^{r-1} \frac{(r+1-p)(r-p)}{2} \left( \frac{p-1}{k-1} \right).
\]
A similar computation as in the $n$ odd case shows
\[
\sum_{p=1}^{r-1} (r + 1 - p)(r - p) \binom{p - 1}{k - 1} = 2 \binom{r + 1}{k + 2}.
\] (29)

Thus by (28) and (29), we see
\[
\sum_{p=1}^{(n-2)/2} \frac{(n-2p)(n-2-2p)}{8} \left[ \ln I_{p,p}(t) \right] = \sum_{k=1}^{r} a_k \binom{r + 1}{k + 2}
\]
\[
= \left[ w^{-2} \right] \left[ -1 \right]^{n-m+1} \prod_{i=1}^{m} \left( \frac{c_i - 1}{w + 1} \right) \cdot \left( 1 + \frac{1}{w} \right)^{r+1}
\]
\[
= (-1)^{n-m+1} \prod_{i=1}^{m} c_i \cdot \left[ w^{-m-1} \right] \left( \frac{2w + 1}{w + 1} \right)^{r-m} \cdot \left( \frac{\prod_{i=1}^{m} \left( c_i w + s \right)}{(w + 1)^{n-m+1}} \right).
\]

Then (23) follows from (19). □

When $m \geq 2$, Prop.2.1 together with the contribution from $-\frac{n+1}{48} \ln(1 - \prod_{i=1}^{m} (-c_i)^{e_i})$ or $-\frac{n-1}{48} \ln(1 - \prod_{i=1}^{m} (-c_i)^{e_i})$ (n is odd or even, resp.) gives (17). When $m = 1$, from (20) it is easy to see
\[
T = t + \sum_{d=1}^{\infty} e^{dt} \frac{(-1)^{nd}}{d} \frac{(nd)!}{(d!)^n}
\]
Take this into account, we also recover (10). So we have proved

**Theorem 2.1.** The conjecture (7) holds for all degree one invariants. □

**Remark 2.1.** The same method shows that for the Calabi-Yau hypersurface $Y$ in $\mathbb{P}^{n-1}$ we have
\[
N_{1,1}^Y = n! \left[ \frac{(n-2)(n+1)}{48} + \frac{1 - (1-n)^n}{24n^2} \sum_{s=2}^{n} \frac{s^n - s + 1}{s} \right] - \frac{n^{n-1}(n-1)(n+2)}{48}.
\]

## 3 Two extremal cases

In general as the degree $d$ increase, the graphs and their contributions corresponding to the fixed loci will become more and more complicated, and thus a direct computation through virtual torus localization seems very difficult. But for some special target spaces we can make a good choice of the linearization so that a lot of graphs give zero contributions (see, e.g., [4]). In principle, the larger $m$ is, the more flexible the choice of the linearizations is. We shall consider the two extremal cases: $X = \text{Tot}(O(-1)^{\oplus(l+1)} \rightarrow \mathbb{P}^l)$ and $X = \text{Tot}(O(-1)^{\oplus(l-1)} \oplus O(-2) \rightarrow \mathbb{P}^l)$. In these two cases it is easy to see from (20) and (21) that $I_{p,p}(t) = 1$ for $p$ in the ranges that appear in (14) and (15). So to prove conjecture (1) in these two cases is equivalent to show

**Theorem 3.1.** For $X = \text{Tot}(O(-1)^{\oplus(l+1)} \rightarrow \mathbb{P}^l)$ we have
\[
N_{1,d} = \frac{(-1)^{(l+1)d}(l+1)}{24d}. \quad (30)
\]

For $X = \text{Tot}(O(-1)^{\oplus(l-1)} \oplus O(-2) \rightarrow \mathbb{P}^l)$ we have
\[
N_{1,d}^{X} = \frac{(-1)^{(l-1)d}(l-1)^4d}{24d}. \quad (31)
\]

In the following we treat the two cases separately. The choice of linearizations are following those of the similar cases in [6] and [7]. In the following computations we shall make repeatedly use of $\lambda^2 = 0$ on $\mathcal{M}_{1,m}$ for $m \geq 1$, for example from this we have $\Lambda_Y^2(x)\Lambda_Y^2(-x) = -x^2$.
3.0.1 $\mathcal{O}(-1)^{(l+1)} \rightarrow \mathbb{P}^l$

Write $\mathcal{O}(-1)^{(l+1)} = \bigoplus_{i=1}^{l+1} L_i$, and choose torus linearizations on $L_i$ with weight $\alpha_i - \alpha_k$ at $P_k$, for $1 \leq i, k \leq l + 1$. In particular, $L_i$ has weight zero at $P_i$. The fixed loci with nonzero contributions are of the form

$$\Gamma_{ij} = i \overset{\alpha}{\underset{d}{\rightarrow}} j,$$

where $1 \leq i \neq j \leq l + 1$. The contribution of $\Gamma_{ij}$ is

$$\frac{1}{d} \int_{\mathcal{X}_{1,1}} \frac{\alpha_i - \alpha_k}{d} \prod_{k \neq i,j} \Lambda_1^j (\alpha_i - \alpha_k) \prod_{k=1}^{l+1} \left(\Lambda_1^j (\alpha_k - \alpha_i) \prod_{a=1}^{d-1} (\alpha_k - \alpha_j + a \frac{\alpha_i - \alpha_k}{d})\right)$$

$$= - \frac{1}{24d} \left(\alpha_j - \alpha_i\right) \prod_{k \neq i,j} (\alpha_i - \alpha_k)(\alpha_k - \alpha_i) \prod_{k=1}^{l+1} \prod_{a=0}^{d-1} (\alpha_k - \alpha_j + a \frac{\alpha_j - \alpha_i}{d}),$$

Note that

$$\alpha_i - \alpha_k + (d - a) \frac{\alpha_j - \alpha_i}{d} = - \left(\alpha_k - \alpha_j + a \frac{\alpha_j - \alpha_i}{d}\right),$$

so the contribution is

$$\frac{(-1)^{(l-1)d} \prod_{k \neq i,j} (\alpha_i - \alpha_k)}{24d} \prod_{k \neq i,j} (\alpha_j - \alpha_k)$$

Since for any fixed $i$ we have

$$\sum_{j \neq i} \prod_{k \neq i,j} (\alpha_i - \alpha_k) = 1,$$

we obtain

$$N_{1,d} = \frac{(-1)^{(l-1)d}(l+1)}{24d}.$$  

3.0.2 $\mathcal{O}(-1)^{(l-1)} \oplus \mathcal{O}(-2) \rightarrow \mathbb{P}^l$

Choose the linearizations on $L_i$ such that for $1 \leq i \leq l - 1$, $L_i$ has weight $\alpha_i - \alpha_k$ at $P_k$, and $L_l$ has weight $\alpha_l + \alpha_{l+1} - 2\alpha_k$ at $P_k$, $1 \leq k \leq l + 1$. The fixed loci which may have nonzero contributions are of three types.

Type I:

$$\Gamma_{s;k_1,d_1;\ldots;k_m,d_m} = \begin{array}{c}
\bullet \\
\vdots \\
\bullet \\
\bullet \\
\end{array}$$

where $1 \leq k_1, \ldots, k_m \leq l - 1$, $m \geq 1$, with edges of degree $d_1, \ldots, d_m$ respectively, and $s = l$ or
The crucial observation is that, in these contributions the factor $\alpha$ and we shall see that $\alpha$ of the other types. So we are able to set $\alpha$, where $1 \leq l + 1$. When $s = l + 1$, the contribution is

\[
\frac{1}{|\text{Aut}(\Gamma_{x:ki_{1},\ldots,ki_{m}})|} \prod_{i=1}^{m} \frac{d_{i}}{d_{i}} \int_{\mathcal{M}_{x:ki_{1},\ldots,ki_{m}}} \prod_{i=1}^{m} (\alpha_{l+1} - \alpha_{j})^{m-1} \Lambda\gamma (\alpha_{l+1} - \alpha_{j}) \prod_{j=1}^{l} \left( (\alpha_{l+1} - \alpha_{j})^{m-1} \Lambda\gamma (\alpha_{l+1} - \alpha_{j}) \right)
\]

\[
\prod_{j=1}^{l-1} \left( (\alpha_{j} - \alpha_{l+1})^{m-1} \Lambda\gamma (\alpha_{j} - \alpha_{l+1}) \prod_{i=1}^{m} d_{i-1} \prod_{i=1}^{m} (\alpha_{l+1} - \alpha_{j}) + a_{\alpha_{j+1}-\alpha_{k_{i}}} \right)
\]

\[
\prod_{i=1}^{m} \left( (\alpha_{l+1} - \alpha_{1})^{m-1} \Lambda\gamma (\alpha_{l+1} - \alpha_{1}) \prod_{i=1}^{m} d_{i-1} \prod_{i=1}^{m} (\alpha_{l+1} - \alpha_{j}) + a_{\alpha_{l+1}-\alpha_{k_{i}}} \right)
\]

\[
\prod_{i=1}^{m} \left( \frac{d_{i}}{d_{i}} \prod_{j=1}^{l} (\alpha_{l+1} - \alpha_{k_{j}})^{d_{i}-1} (\alpha_{k_{j}} - \alpha_{i}) \prod_{r \neq l+1,l,k_{j}} (\alpha_{k_{j}} - \alpha_{i}) \right)
\]

Similarly, when $s = l + 1$, the contribution is

\[
\frac{1}{|\text{Aut}(\Gamma_{k_{1},\ldots,k_{m}})|} \prod_{i=1}^{m} d_{i}^{2} \int_{\mathcal{M}_{x:ki_{1},\ldots,ki_{m}}} \prod_{i=1}^{m} \left( \frac{d_{i}}{d_{i}} \prod_{j=1}^{l} (\alpha_{l+1} - \alpha_{k_{j}})^{d_{i}-1} (\alpha_{k_{j}} - \alpha_{i}) \prod_{r \neq l+1,l,k_{j}} (\alpha_{k_{j}} - \alpha_{i}) \right)
\]

The crucial observation is that, in these contributions the factor $\alpha_{l+1} - \alpha_{l}$ appears at least once, and we shall see that $\alpha_{l+1} - \alpha_{l}$ does not appear in the denominator of the sums of the contributions of the other types. So we are able to set $\alpha_{l+1} = \alpha_{l}$ and thus the type I graphs contribute nothing.

Type II:

$$\Gamma_{x:k_{0},d_{0};k_{1},d_{1};\ldots;k_{m},d_{m}} = \begin{array}{c}
\includegraphics[scale=0.5]{diagram}
\end{array}$$

where $1 \leq k_{0}, k_{1}, \ldots, k_{m} \leq l - 1$, $m \geq 0$, with edges of degree $d_{0}, d_{1}, \ldots, d_{m}$ respectively, and $s = l$.
or \( l + 1 \). When \( s = l + 1 \), the contribution is

\[
\frac{1}{|\text{Aut}(\Gamma'_{s;k_0,k_1,\ldots,k_m})|} \prod_{i=0}^{n} d_i \int_{\mathcal{M}_{l,1}} \prod_{j \neq k_0} \Lambda_{l}^{\gamma} (\alpha_{k_0} - \alpha_j) \prod_{i=1}^{l-1} \Lambda_{l-1}^{\gamma} (\alpha_i - \alpha_{k_0}) \cdot \Lambda_{l}^{\gamma} (\alpha_l + \alpha_{l+1} - 2\alpha_{k_0})
\]

\[
= \frac{1}{|\text{Aut}(\Gamma'_{s;k_0,k_1,\ldots,k_m})|} \prod_{i=0}^{m} \alpha_i \frac{1}{\alpha_{l} - \alpha_{l+1}} \prod_{j=1}^{l-1} \frac{1}{\alpha_{l} - \alpha_{l+1}} \prod_{j=1}^{l-1} (\alpha_j - \alpha_{l+1} + a \frac{\alpha_{l+1} - \alpha_{k_0}}{d_i})
\]

When \( m > 0 \), the power of \( \alpha_{l+1} - \alpha_l \) in the numerator is not less than that in the denominator. To show that the sums of contributions of the type II graphs has no factor of \( \alpha_{l+1} - \alpha_l \) in its denominator, we only need to consider the \( m = 0 \) case. When \( m = 0 \), the above contribution is

\[
\frac{(-1)^{(l-1)d_l+1} d_0^{d_l-1} \prod_{j=1}^{l-1, j \neq k_0} (\alpha_{k_0} - \alpha_j) \cdot \prod_{k=0}^{d_0} (\alpha_l - \alpha_{k_0} + a \frac{\alpha_{l+1} - \alpha_{k_0}}{d_i})}{24d_0!} \prod_{r=1}^{d_0} (\alpha_{l+1} - \alpha_{k_0})^{d_0} \prod_{r \neq 1, l, k_0} (\alpha_{l+1} - \alpha_r) \cdot (\alpha_{l+1} - \alpha_l).
\]

Thus the sum of the contributions of \( \Gamma'_{l;k_0} \) and \( \Gamma'_{l+1;k_0} \) is

\[
\frac{(-1)^{(l-1)d_l+1} d_0^{d_l-1} \prod_{j=1}^{l-1} (\alpha_{k_0} - \alpha_j)}{24d_l!} \prod_{r=1}^{d_0} (\alpha_{l+1} - \alpha_{k_0})^{d_0} \prod_{r \neq 1, l, k_0} (\alpha_{l+1} - \alpha_r) (\alpha_{l+1} - \alpha_l)
\]

The sum of the group of terms in the square brackets of the last expression is divisible by \( \alpha_{l+1} - \alpha_l \). Therefore we have shown that the sum of the contributions of type II graphs has no factor \( \alpha_{l+1} - \alpha_l \) in its denominator. We shall see the type III contribution also has no factor \( \alpha_{l+1} - \alpha_l \) in the denominator. So we are able to set \( \alpha_{l+1} = \alpha_l \). Then we see that a type II graph has no contribution unless \( m = 0 \) or \( m = 1 \). Now we compute the contributions of \( m = 0 \) and \( m = 1 \) cases separately.
\[
\begin{align*}
\prod_{a=0}^{d} (\alpha_l - \alpha_k + a, \frac{\alpha_{l+1} - \alpha_k}{d}) &= (\alpha_l - \alpha_k) \frac{d!}{d^d} \prod_{a=1}^{d} \left( \frac{d(\alpha_l - \alpha_k)}{a} + \alpha_{l+1} - \alpha_k \right) \\
&= (\alpha_l - \alpha_k) \frac{d!}{d^d} \left( (2d)! (\alpha_l - \alpha_k)^{d} + (\alpha_{l+1} - \alpha_l) \frac{(2d)!}{(d)!^2} (\alpha_l - \alpha_k)^{d-1} \sum_{a=1}^{d} \frac{a}{d} + O[(\alpha_{l+1} - \alpha_l)^2] \right) \\
&= \frac{(2d)!}{d^d} (\alpha_l - \alpha_k)^{d+1} + (\alpha_{l+1} - \alpha_l) \frac{(2d)!}{d^d} (\alpha_l - \alpha_k)^{d} \sum_{a=0}^{d} \frac{d}{d+a} + O[(\alpha_{l+1} - \alpha_l)^2], \\
(\alpha_{l+1} - \alpha_k)^{d} &= (\alpha_l - \alpha_k)^{d} + d(\alpha_{l+1} - \alpha_l)(\alpha_l - \alpha_k)^{d-1} + O[(\alpha_{l+1} - \alpha_l)^2], \\
\prod_{r \neq l, l, k_0} (\alpha_{l+1} - \alpha_r) &= \prod_{r \neq l, l, k_0} (\alpha_l - \alpha_l) \prod_{r \neq l, l, k_0} (\alpha_l - \alpha_l) \sum_{r \neq l, l, k_0} \frac{1}{\alpha_l - \alpha_l} + O[(\alpha_{l+1} - \alpha_l)^2], \\
\text{we have} \\
\prod_{a=0}^{d} (\alpha_l - \alpha_k + a, \frac{\alpha_{l+1} - \alpha_k}{d}) &= (\alpha_l - \alpha_k) \frac{d!}{d^d} \prod_{r \neq l, l, k_0} (\alpha_l - \alpha_l) \\
&- \prod_{a=0}^{d} (\alpha_{l+1} - \alpha_k + a, \frac{\alpha_{l+1} - \alpha_k}{d}) \prod_{r \neq l, l, k_0} (\alpha_{l+1} - \alpha_l) \\
&= (\alpha_{l+1} - \alpha_l) \frac{(2d)!}{d^d} (\alpha_l - \alpha_k)^{d} \sum_{a=1}^{d} \frac{a}{d+a} \cdot (\alpha_l - \alpha_k)^{d} \prod_{r \neq l, l, k_0} (\alpha_l - \alpha_l) \\
&- (\alpha_{l+1} - \alpha_l) \frac{(2d)!}{d^d} (\alpha_l - \alpha_k)^{d} \sum_{a=0}^{d} \frac{d}{d+a} \cdot (\alpha_l - \alpha_k)^{d} \prod_{r \neq l, l, k_0} (\alpha_l - \alpha_l) \\
&- d(\alpha_{l+1} - \alpha_l)(\alpha_l - \alpha_k)^{d-1} \cdot \frac{(2d)!}{d^d} (\alpha_l - \alpha_k)^{d+1} \prod_{r \neq l, l, k_0} (\alpha_l - \alpha_l) \\
&- (\alpha_{l+1} - \alpha_l) \prod_{r \neq l, l, k_0} (\alpha_l - \alpha_l) \sum_{r \neq l, l, k_0} \frac{1}{\alpha_l - \alpha_l} \cdot \frac{(2d)!}{d^d} (\alpha_l - \alpha_k)^{d+1}(\alpha_l - \alpha_k)^{d} + O[(\alpha_{l+1} - \alpha_l)^2] \\
&= (\alpha_{l+1} - \alpha_l) \frac{(2d)!}{d^d} (\alpha_l - \alpha_k)^{2d} \prod_{r \neq l, l, k_0} (\alpha_l - \alpha_l) \\
&\cdot \left( \sum_{a=1}^{d} \frac{a}{d+a} - 2d - 1 - (\alpha_l - \alpha_k) \sum_{r \neq l, l, k_0} \frac{1}{\alpha_l - \alpha_l} \right) + O[(\alpha_{l+1} - \alpha_l)^2].
\end{align*}
\]
Thus setting $\alpha_{t+1} = \alpha_t = \alpha$, the sum of the contributions of $\Gamma'_{t;\alpha_{t0}}$ and $\Gamma'_{t+1;\alpha_{t1}}$ is

$$\frac{(-1)^{(l-1)d_1+d_1}d_1^{d_1-1}}{24d!} \cdot \frac{\prod_{j=1, \neq \alpha_{t0}}^{l-1} (\alpha_{t0} - \alpha_j)}{(\alpha_{t+1} - \alpha_{t0})^d (\alpha_t - \alpha_{t0})^d \cdot \prod_{r \neq l+1, l, k_0} (\alpha_{t+1} - \alpha_r) (\alpha_t - \alpha_r)} \cdot \frac{(2d)!}{d!} (\alpha_t - \alpha_{t0})^{2d} \prod_{r \neq l+1, l, k_0} (\alpha_t - \alpha_r) \cdot \left(2 \sum_{a=1}^{d} \frac{a}{d + a} - 2d - 1 - (\alpha_t - \alpha_{t0}) \sum_{r \neq l+1, l, k_0} \frac{1}{\alpha_t - \alpha_r}\right).$$

The contribution of $\Gamma'_{t+1;\alpha_{t1},k_1}$ is

$$\frac{1}{d_0!d_1!} \left( - \frac{1}{24} (-1)^{(l-1)d_1} d_0 (\alpha_{t0} - \alpha_t) \prod_{j=1, \neq \alpha_{t0}}^{l-1} (\alpha_{t0} - \alpha_j)^2 \cdot (\alpha_t + \alpha_{t+1} - 2\alpha_{t0}) \right)$$

$$\frac{\int_{\mathbb{R}^{d_1}} \frac{1}{\prod_{i=0}^{l-1} (\alpha_t + \alpha_{t+1} - 2\alpha_{t0})} \cdot \prod_{\alpha_{t0} - \alpha_{t+1} + (\alpha_{t+1} - \alpha_{t0}) \cdot (\alpha_{t+1} - \alpha_t) (\alpha_{t+1} - \alpha_t)} (-1)^{(l-1)(d_1-1)} \prod_{i=0}^{d_1-1} (\alpha_t - \alpha_{t+1} + a \frac{\alpha_{t+1} - \alpha_{t0}}{d_1}) \prod_{i=0}^{l-1} (\alpha_{t0} - \alpha_{t+1}) \cdot (\alpha_{t0} - \alpha_t) \cdot (\alpha_{t0} - \alpha_t) \cdot (\alpha_{t0} - \alpha_t)} {24d_0!d_1!} \left( \prod_{j=1, \neq \alpha_{t0}}^{l-1} (\alpha_{t0} - \alpha_j) \cdot (\alpha_t + \alpha_{t+1} - 2\alpha_{t0}) \right)$$

$$\frac{1}{d_0!d_1!} \cdot \left( \prod_{i=0}^{d_1-1} (\alpha_t - \alpha_{t+1} + a \frac{\alpha_{t+1} - \alpha_{t0}}{d_1}) \prod_{i=0}^{l-1} (\alpha_{t0} - \alpha_{t+1}) \cdot (\alpha_{t0} - \alpha_t) \cdot (\alpha_{t0} - \alpha_t) \cdot (\alpha_{t0} - \alpha_t) \right)$$

setting $\alpha_{t+1} = \alpha_t = \alpha$, the above contribution becomes

$$\frac{(-1)^{(l-1)d_1+d_1}d_0(2d_0)!/2(d_1)!}{48(d_0)!^2(d_1)!^2d_1} \prod_{j=1, \neq \alpha_{t0}}^{l-1} (\alpha_{t0} - \alpha_j) \cdot \prod_{j=1, \neq \alpha_{t0}}^{l-1} (\alpha_{t1} - \alpha_j) \cdot \frac{\alpha - \alpha_{t0}}{d\alpha - d_1 \alpha_{t0} - d_0 \alpha_{t1}}.$$

Therefore the sum of the contributions of $\Gamma'_{t+1;\alpha_{t0},k_1}$ and $\Gamma'_{t;\alpha_{t1},k_1}$ is

$$\frac{(-1)^{(l-1)d_1+d_1}d_0(2d_0)!/2(d_1)!}{24(d_0)!^2(d_1)!^2d_1} \prod_{j=1, \neq \alpha_{t0}}^{l-1} (\alpha_{t0} - \alpha_j) \cdot \prod_{j=1, \neq \alpha_{t0}}^{l-1} (\alpha_{t1} - \alpha_j) \cdot \frac{\alpha - \alpha_{t0}}{d\alpha - d_1 \alpha_{t0} - d_0 \alpha_{t1}}.$$

Type III:

$$\Gamma_{ij} = \frac{i d j}{\alpha - \alpha_{t0}}.$$
where \(1 \leq i, j \leq l - 1, i \neq j\). The contribution of \(\Gamma_{ij}\) is

\[
\frac{1}{d} \int_{X_{1,1}} \prod_{k=1, k \neq i}^{l-1} \Lambda_{i}^{\gamma}(\alpha_{i} - \alpha_{k}) \cdot \prod_{k=1}^{l-1} \Lambda_{k}^{\gamma}(\alpha_{k} - \alpha_{i}) \cdot \Lambda_{i}^{\gamma}(\alpha_{l} + \alpha_{l+1} - 2\alpha_{i})
\]

Combining the three type of contributions, we obtain

\[
\begin{align*}
\frac{(-1)^{(l-1)}d}{24d} & \prod_{k=1, k \neq i, j}^{l-1} (\alpha_{i} - \alpha_{k}) \cdot (\alpha - \alpha_{j})^{3} \\
& \cdot \prod_{\alpha=1}^{l-1} (2\alpha - 2\alpha_{j} + a^{2}\alpha_{j} - a_{i}) \\
& \prod_{\alpha=1}^{l-1} (\alpha_{i} - \alpha_{k}) \prod_{\alpha=0}^{d}(\alpha_{i} - \alpha + a^{2}\alpha_{j} - a_{i})^{2}.
\end{align*}
\]

Combining the three type of contributions, we obtain

\[
N_{1,d}^{X} = \sum_{k_{0}=1}^{l-1} \frac{(-1)^{(l-1)}d(2d)!}{24(d_{0})^{2}d} \prod_{j=1, j \neq k_{0}}^{l-1} (\alpha_{k_{0}} - \alpha_{j}) \left(2d \sum_{a=1}^{d} \frac{1}{d + a} + 1 + (\alpha - \alpha_{k_{0}}) \sum_{j=1, \neq k_{0}}^{l-1} \frac{1}{\alpha - \alpha_{j}}\right)
\]

\[
\sum_{d_{0}+d_{1}+d_{k}=d} \sum_{k=1, k \neq 1, k \neq 1}^{l-1} \frac{(-1)^{(l-1)}d}{24(d_{0})^{2}d_{1}d_{1}} \prod_{j=1, \neq k}^{l-1} (\alpha_{k} - \alpha_{j}) \cdot \prod_{\alpha=1}^{l-1} (\alpha - \alpha_{j})
\]

\[
\sum_{i=1}^{l-1} \sum_{j=1, \neq i}^{l-1} \frac{(-1)^{(l-1)}d}{12d} \prod_{k=1, \neq i, j}^{l-1} (\alpha_{i} - \alpha_{k}) \cdot \prod_{\alpha=1}^{l-1} (\alpha - \alpha_{i})^{3} \prod_{\alpha=1}^{l-1} (2\alpha - 2\alpha_{j} + a^{2}\alpha_{j} - a_{i})
\]

\[
\prod_{\alpha=1}^{l-1} (\alpha_{i} - \alpha_{j}) \prod_{\alpha=0}^{d}(\alpha_{i} - \alpha + a^{2}\alpha_{j} - a_{i})^{2}.
\]

Let us first assume \(l \geq 3\). Note that \(N_{1,d}^{X}\) is a priori a rational number. So it is straightforward to see that, for fixed \(1 \leq i, j \leq l - 1, i \neq j\), to cancel the denominators of the form \(d\alpha - d_{1}\alpha_{i} - d_{0}\alpha_{j}\), it
forces that there exist \( b \in \mathbb{Q}, \beta_{ij} \in \mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_{l-1}, \) such that

\[
\begin{align*}
\beta_{ij} + \beta_{ij} = \sum_{d_0, d_1 \geq 1} \frac{(-1)^{l-1} d_0 (2d_0)! (2d_1)!}{24(d_0)!^2 (d_1)!^2 d_1} \cdot \frac{(\alpha - \alpha_j)^2}{d_0 - d_1 \alpha_i - d_0 \alpha_j} + \frac{(-1)^{l-1} d_0}{24d} \cdot \frac{d_0 (2d_0)! (2d_1)!}{24(d_0)!^2 (d_1)!^2 d_1}.
\end{align*}
\]

Dividing both side by \( \alpha \) and let \( \alpha \to \infty \), we obtain

\[
\begin{align*}
b &= \sum_{d_0, d_1 \geq 1} \frac{(-1)^{l-1} d_0 (2d_0)! (2d_1)!}{24(d_0)!^2 (d_1)!^2 d_1 d} + \frac{(-1)^{l-1} d_0 2^{2d-1}}{12d} \\
&= \frac{(-1)^{l-1} d_0}{24d} \left( \sum_{d_0, d_1 \geq 1} \frac{d_0 (2d_0)! (2d_1)!}{(d_0)!^2 (d_1)!^2 d_1} + 4^d \right).
\end{align*}
\]

Then since

\[
\begin{align*}
\sum_{j=1}^{l-1} \sum_{i=1, i \neq j}^{l-1} \sum_{k=1, k \neq i, j}^{l-1} \frac{\alpha_i - \alpha_k}{\alpha_j - \alpha_k} &= \sum_{i=1}^{l-1} \left( \sum_{j=1, j \neq i}^{l-1} (\alpha_i - \alpha_j) \cdot \prod_{j=1, j \neq i}^{l-1} \frac{1}{(\alpha_j - \alpha_i) \prod_{k=1, k \neq i, j}^{l-1} (\alpha_j - \alpha_k)} \right) \\
&= \sum_{i=1}^{l-1} \left( \sum_{j=1, j \neq i}^{l-1} (\alpha_i - \alpha_j) \cdot \frac{1}{\prod_{j=1, j \neq i}^{l-1} (\alpha_i - \alpha_j)} \right) \\
&= l - 1,
\end{align*}
\]

we have

\[
\begin{align*}
N_{1,d}^X &= (l-1)b + (l-1) \sum_{d_0, d_1 = d} \frac{(-1)^{l-1} d_0 (2d_0)! (2d_1)!}{24(d_0)!^2 (d_1)!^2 d_1} \\
&= \frac{(-1)^{l-1} (l-1)4^d}{24d}.
\end{align*}
\]

For \( l = 2 \), (32) still holds, and has been proved in [6] without giving the details. Here we give another proof for this, which is interesting itself since we make use of the proof of the \( l = 3 \) case to prove a combinatorial identity. It suffices to prove the following lemma.

**Lemma 3.1.**

\[
\begin{align*}
\frac{(2d)!}{(d!)^2} \left( 2d \sum_{a=1}^{d} \frac{1}{d + a} + 1 \right) - \sum_{d_0, d_1 = d} d_0 (2d_0)! (2d_1)! (d_0)!^2 (d_1)!^2 d_1 = 4^d.
\end{align*}
\]

\( ^3\text{Thanks Si-Qi Liu for telling the author that (33) can also be proved using Mathematica.} \)}
Proof: Consider the case $l = 3$. We have

$$N_{1,d}^X = -\frac{(2d)!}{24(d!)^2d} \left( 2d \sum_{a=1}^{d} \frac{1}{d+a} + 1 \right) \frac{(\alpha - \alpha_2)^2}{(\alpha - \alpha_1)(\alpha - \alpha_2)}$$

$$+ \frac{(2d)!}{24(d!)^2d} \left( \frac{(\alpha - \alpha_2)(\alpha - \alpha_1)}{(\alpha - \alpha_2)^2} + \frac{(\alpha_2 - \alpha_1)(\alpha - \alpha_2)}{(\alpha - \alpha_1)^2} \right)$$

$$- 2 \sum_{d_0 + d_1 = d} \frac{d_0(2d_0)!}{24(d_0!)^2(d_1!)^2d_1d}$$

$$+ \sum_{d_0 + d_1 = d} \frac{d_0(2d_0)!}{24(d_0!)^2(d_1!)^2d_1d} \left( \frac{\alpha - \alpha_1}{d_0a - d_1\alpha_1 - d_0\alpha_2} + \frac{\alpha - \alpha_2}{d_0\alpha - d_1\alpha_2 - d_0\alpha_1} \right)$$

$$+ \sum_{i=1}^{2} \sum_{j=1, \neq i}^{2} \frac{(-1)^{(i-1)d}}{12d} \cdot \frac{(\alpha - \alpha_i) \prod_{a=1}^{2d-1} (2\alpha - 2\alpha_j + \alpha \cdot \alpha_i)}{(\alpha - \alpha_j) \prod_{a=1}^{2d-1} (\alpha - \alpha_i + \alpha \cdot \alpha_i - \alpha_j)^2}$$

$$= -\frac{(2d)!}{24(d!)^2d} \left( 2d \sum_{a=1}^{d} \frac{1}{d+a} + 1 \right) \frac{(\alpha - \alpha_2)^2}{(\alpha - \alpha_1)(\alpha - \alpha_2)}$$

$$- 2 \sum_{d_0 + d_1 = d} \frac{d_0(2d_0)!}{24(d_0!)^2(d_1!)^2d_1d}$$

$$+ \frac{1}{(\alpha - \alpha_2)^2(\alpha - \alpha_1)^2} \left( 1 + (\alpha - \alpha_2)^3(b\alpha + \beta_12 + \frac{(2d)!}{24(d!)^2d}(\alpha - \alpha_2)) \right)$$

$$+(\alpha - \alpha_2)^3(b\alpha + \beta_21 + \frac{(2d)!}{24(d!)^2d}(\alpha_2 - \alpha_1)).$$

It forces that $\alpha - \alpha_2$ divides $b\alpha + \beta_12 + \frac{(2d)!}{24(d!)^2d}(\alpha - \alpha_2)$, and also $\alpha - \alpha_1$ divides $b\alpha + \beta_21 + \frac{(2d)!}{24(d!)^2d}(\alpha_2 - \alpha_1)$. Thus

$$\beta_12 = \frac{(2d)!}{24(d!)^2d} - b\alpha - \frac{(2d)!}{24(d!)^2d} \alpha_1,$$

$$\beta_21 = \frac{(2d)!}{24(d!)^2d} - b\alpha - \frac{(2d)!}{24(d!)^2d} \alpha_2,$$

and

$$N_{1,d}^X = -\frac{(2d)!}{24(d!)^2d} \left( 2d \sum_{a=1}^{d} \frac{1}{d+a} + 1 \right) \frac{(\alpha - \alpha_2)^2}{(\alpha - \alpha_1)(\alpha - \alpha_2)}$$

$$+ \frac{b(\alpha - \alpha_1)^3(\alpha - \alpha_2) + b(\alpha - \alpha_2)^3(\alpha - \alpha_1)}{(\alpha - \alpha_1)^2(\alpha - \alpha_2)^2}$$

$$= -\frac{(2d)!}{24(d!)^2d} \left( 2d \sum_{a=1}^{d} \frac{1}{d+a} + 1 \right) \frac{(\alpha - \alpha_2)^2}{(\alpha - \alpha_1)(\alpha - \alpha_2)}$$

$$+ \frac{2b + \frac{b(\alpha - \alpha_2)^2}{(\alpha - \alpha_1)(\alpha - \alpha_2)}}{2}$$

Therefore it forces that

$$b = \frac{(2d)!}{24(d!)^2d} \left( 2d \sum_{a=1}^{d} \frac{1}{d+a} + 1 \right).$$
4 Integality of \( n_{1,d} \) for local Calabi-Yau 5-folds

The Gopokumar-Vafa invariants \( n_{0,d}(\gamma_1, \cdots, \gamma_k) \) for a Calabi-Yau \( n \)-fold \( X \), where \( \gamma_1, \cdots, \gamma_k \in H^*(X) \) are defined by (see, e.g., [6], [7])

\[
\sum_{\beta \neq 0} \langle \gamma_1, \cdots, \gamma_k \rangle^X_{0,k,d} q^\beta = \sum_{\beta \neq 0} n_{0,\beta}(\gamma_1, \cdots, \gamma_k) \sum_{d=1}^{\infty} \frac{1}{d^{1-k}} q^{d\beta}. \tag{34}
\]

When \( n \geq 6 \), the definition of Gopokumar-Vafa invariants in genus one is still absent. For \( n = 4 \), the invariants \( n_{1,d} \) are defined in [6], and for \( n = 5 \) in [7]. The integality of \( n_{1,d} \) has been verified in low degrees in [6] for \( X \) of the form \( \mathbb{P}^1 \) when \( n = 4 \), and in [7] the case \( X = \text{Tot}(\mathcal{O}(-1)^{\mathbb{P}^3} \to \mathbb{P}^2) \) when \( n = 5 \). The remaining three cases for \( n = 5 \) are \( \mathcal{O}(-1) \oplus \mathcal{O}(-3) \to \mathbb{P}^3 \), \( \mathcal{O}(-2) \oplus \mathcal{O}(-2) \to \mathbb{P}^3 \), \( \mathcal{O}(-5) \to \mathbb{P}^4 \).

For Calabi-Yau 5-folds, once we have \( N_{1,d}, n_{0,1}(\gamma_1) \) and \( n_{0,i}(\gamma_2, \gamma_3) \) for \( 1 \leq i \leq d \), all \( \gamma_1 \in H^6(X) \) and all \( \gamma_2, \gamma_3 \in H^4(X) \) as inputs, the invariants \( n_{1,d} \) are defined through a complicated simultaneous recursion of many invariants. For the details we refer the reader to [7]. The invariants \( n_{0,i}(\gamma_1) \) and \( n_{0,i}(\gamma_2, \gamma_3) \) are defined by [34], and the one-point and two-point genus zero Gromov-Witten invariants on the left of [34] can be extracted from the formulae in [6] (see also [35]). Assuming the validity of our conjectural formulae [13] and [15] for \( n = 5 \), we have checked the integality of \( n_{1,d} \) in for \( 1 \leq d \leq 100 \) for these three cases using a Maple programme, and for \( 1 \leq d \leq 20 \) we list them in the following.

4.1 \( X = \mathbb{P}^4 \)

\[
\langle H^3 \rangle_{0,1,d} = -\frac{1}{5} [x^2 Q^d] \left( e^{-x f(q)} \sum_{d=0}^{\infty} q^d \prod_{s=0}^{d-1} (-5x-s)^{5} \right),
\]

where \( Q = q e^{f(q)} \) and the mirror map

\[
f(q) = \sum_{d=1}^{\infty} q^d \frac{(-1)^d (5d)!}{d!(d!)^5}. \]

For \( \langle H^2, H^2 \rangle_{0,2,d} \), we follow the notations in the remark 3.4 in [9] and define \( F(w, q) \) and \( F_i(q) \) by

\[
F(w, q) = \sum_{d=0}^{\infty} q^d \prod_{r=1}^{5d} (-5w-r)^{5} = F(0, q) + \sum_{i=1}^{\infty} F_i(q) w^i,
\]

and let

\[
I_1(q) = 1 + q \frac{d}{dq} \frac{F_1(q)}{F(0, q)}.
\]

Then

\[
\langle H^2, H^2 \rangle_{0,2,d} = -\frac{1}{5} [Q^d] \left( -f(q) + \frac{F_1(q)}{F(0, q)} + q \frac{d}{dq} \frac{F_2(q)}{F(0, q)} I_1(q) \right).
\]

\footnote{When \( n \geq 4 \), the Gromov-Witten invariants in genus at least two are trivall, due to the dimension constraint and the string equation.}

\footnote{We need also the Poincaré pairing on \( H^4(X) \oplus H^6(X) \), which in the local cases are defined via the general principle mentioned in the footnote in Page 3. For example, for \( X = \mathbb{P}^4 \), we have \( \langle H^2, H^2 \rangle_X = -\frac{1}{2} \).}
The conjectural formula (14) in this case reads
\[
\sum_{d=1}^{\infty} N_{1,d}Q^d = \frac{3}{8} f(q) - \frac{1}{8} \ln(1 + 5^5q) - 2 \ln I_{1,1}(q) - \frac{1}{2} \ln I_{2,2}(q),
\]
where
\[
I_{1,1}(q) = 1 + \sum_{d=1}^{\infty} \frac{(-1)^d(5d)!}{(d!)^5} q^d,
\]
and
\[
I_{2,2}(q) = 1 + \frac{1}{I_{1,1}(q)} \sum_{d=1}^{\infty} (-1)^d \frac{(nd)!}{(d!)^n} q^d + \frac{1}{I_{1,1}(q)} \sum_{d=1}^{\infty} \left( \frac{(-1)^n nd(nd)!}{(d!)^n} \sum_{s=d+1}^{nd-1} \frac{1}{s} \right) q^d.
\]

\begin{tabular}{|c|c|}
\hline
\(d\) & \(n_{0,d}(H^3)\) of \(K_{P^4}\) \\
\hline
1 & 130 \\
2 & -58345 \\
3 & 55837430 \\
4 & -73589158000 \\
5 & 115854201969950 \\
6 & -20434235412313875 \\
7 & 39005119173987697630 \\
8 & -78913606642194095804000 \\
9 & 1669447288789130694033224250 \\
10 & -365889343261650276399759555175 \\
11 & 825627129183279407802045607394310 \\
12 & -190615095874156816116638558317767574480 \\
13 & 4491714794958888771450771829378033670230 \\
14 & -107667316864820156273192312584585440698457095 \\
15 & 261915168370711178492182001044618321338813469450 \\
16 & -645393917552138476376093839553201039666790189529280 \\
17 & 160844564437001168916957634727089346407225594867091080 \\
18 & -4049011495564074654404411325327805800339427963862185528005 \\
19 & 10284566695008271699589128589728350347114600022600731093548670 \\
20 & -263343050244486861033964360994375819798940753071425109074393898000 \\
\hline
\end{tabular}
| $d$ | $n_{0,d}(H^2, H^2)$ of $K_{P^4}$ | $n_{1,d}$ of $K_{P^4}$ |
|-----|--------------------------------|-----------------|
| 1   | 245                            | 0               |
| 2   | -289035                        | 0               |
| 3   | 499858460                      | 138263175125    |
| 4   | -101355889195                 | -502345733521805|
| 5   | 224234151509675               | 1625739014586631100|
| 6   | -524191823614046300            | -4991836999879872628150|
| 7   | 1272851040234464504790         | 14290114955810054172550700|
| 8   | -31777727076990402350118750    | -439386009096882061090032617300|
| 9   | 81033105451821118038400330625  | 128301145686957983686779831220|
| 10  | -21010809962234326675476798422750 | -37279008029268264120510592773314550|
| 11  | 5521684097530427421557084925003565 | 108007778797127346376877874887763587150 |
| 12  | -1467310058144521736953946444230597767540 | -31243380949758383754174103026588005609926750 |
| 13  | 393563544488399018105036315876566792311135 | 903072807874710632569913387841027502777326800 |
| 14  | -106400976807608701622296726078463500726511377961970 | -260959628664299991445716478810388636452207299316310 |
| 15  | 289624478492348885737946426072924327337201627396250 | 7541452197993604753879936408390248835523907361074200 |
| 16  | -79304558509583208268501575306282770036005404547746270 | -21800287431262666464163674964378026073821556734372213400 |
| 17  | 1867580405051157938147098147931632353311673301482974031124005645 | 630457897353330241059812044159053398376532549249949897687300 |
| 18  | -603583838621757243435085701726681467968037123021501795632035 | -182421186456815829506157885517426326597137430136994284773704950 |
| 19  | 1675850405051157938147098147931632353311673301482974031124005645 | 528133007550254253143927733347423831897591633865113159271700486035 |

It is interesting to note that they are all multiples of 5, and when $5 \nmid d$, $n_{1,d}$ is a multiple of 25.

4.2 $X = \text{Tot}(\mathcal{O}(–1) \oplus \mathcal{O}(–3) \to \mathbb{P}^3)$

\[
\langle H^3 \rangle_{0,1,d} = \frac{(d-1)!(3d-1)!}{(d!)^4}.
\]

\[
\langle H^2, H^2 \rangle_{0,2,d} = [q^d] \left( \frac{\sum_{d=1}^{\infty} q^d (3d)!}{1 + \sum_{d=1}^{\infty} q^d (3d)!} \right)^{\frac{1}{d}}.
\]
$$
\sum_{d=1}^{\infty} N_{1,d} q^d = -\frac{1}{8} \ln(1 - 27q) - \frac{1}{2} \ln \left(1 + \sum_{d=1}^{\infty} q^d \frac{(3d)!}{(d!)^3}\right).
$$

Table 1: Low degree genus 0 and genus 1 BPS numbers of $\text{Tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-3) \to \mathbb{P}^3)$

| $d$ | $n_{0,d}(H^3)$ | $n_{0,d}(H^2,H^2)$ | $n_{1,d}$ |
|-----|----------------|---------------------|----------|
| 1   | 2              | 5                   | 0        |
| 2   | 7              | 53                  | 0        |
| 3   | 62             | 888                 | 135      |
| 4   | 720            | 16578               | 4069     |
| 5   | 10090          | 336968              | 102497   |
| 6   | 158809         | 7208592             | 2529330  |
| 7   | 2714782        | 159953128           | 62485370 |
| 8   | 49299360       | 3644804226          | 1549538856 |
| 9   | 937750740      | 8475783392          | 38632050468 |
| 10  | 1850320115     | 200278261068        | 968230418446 |
| 11  | 376107425518   | 47940402636848      | 24386703246083 |
| 12  | 7835027188272  | 1159841269631844    | 616987529756004 |
| 13  | 166623467599342 | 28312447677391792 | 1567308566208659 |
| 14  | 3606416097808937 | 696398907175066480 | 39958344201671692 |
| 15  | 79251821904257590 | 17241740125645491096 | 1022055487533281200 |
| 16  | 1764772740099673920 | 429315366375232815762 | 262188626394087701664 |
| 17  | 39757622487694555282 | 10743399666271987545848 | 6743753349276509395348 |
| 18  | 904958567371990915302 | 270039166920941445186084 | 173872012409851929166786 |
| 19  | 2078888672249855553518 | 6814313281153255310131216 | 4492655791971935260396097 |
| 20  | 481526012065391894029200 | 172564210354543917847594608 | 116315885319017767137751283 |

4.3 $X = \text{Tot}(\mathcal{O}(-2) \oplus \mathcal{O}(-2) \to \mathbb{P}^3)$

$$
\langle H^3 \rangle_{0,1,d}^X = \frac{(2d-1)!(2d-1)!}{(d!)^4}.
$$

$$
\langle H^2, H^2 \rangle_{0,2,d}^X = [q^d] \left( \frac{\sum_{d=1}^{\infty} q^d (\frac{(2d)!}{(d!)^2})^2 \sum_{r=d+1}^{2d+1} r}{1 + \sum_{d=1}^{\infty} q^d (\frac{(2d)!}{(d!)^4})^2} \right).
$$

$$
\sum_{d=1}^{\infty} N_{1,d} q^d = -\frac{1}{8} \ln(1 - 16q) - \frac{1}{2} \ln \left(1 + \sum_{d=1}^{\infty} q^d \frac{(2d)!^2}{(d!)^4}\right).
$$
Table 2: Low degree genus 0 and genus 1 BPS numbers of $\text{Tot}(O(-2) \oplus O(-2) \rightarrow \mathbb{P}^3)$

| $d$ | $n_{0,d}(H^3)$ | $n_{0,d}(H^2, H^2)$ | $n_{1,d}$ |
|-----|----------------|---------------------|-----------|
| 1   | 1              | 2                   | 0         |
| 2   | 2              | 12                  | 0         |
| 3   | 11             | 122                 | 20        |
| 4   | 76             | 1344                | 411       |
| 5   | 635            | 16182               | 6228      |
| 6   | 5926           | 204508              | 92696     |
| 7   | 60095          | 2683410             | 1372416   |
| 8   | 647000         | 36160512            | 20351408  |
| 9   | 7296000        | 49743288            | 303008660 |
| 10  | 8536790        | 6954446148          | 4529630140|
| 11  | 1028170055     | 98509313850         | 6798636924|
| 12  | 12695240996    | 1410519352384       | 1024271346252|
| 13  | 160018462071   | 20380347529206      | 15484823717804|
| 14  | 2052731611966  | 29674754566052      | 234834989626688|
| 15  | 26734938900985  | 439510828254174     | 357157291880416|
| 16  | 328292721754800 | 64120438449904656   | 54460625124782072|
| 17  | 4710828711092291 | 950056145934862062 | 83296434040238536 |
| 18  | 6354790178313374 | 14139866390015314240 | 12750049354231063044 |
| 19  | 865157668345976759 | 211286868769225452618 | 195680390778912132364 |
| 20  | 11870040942305597380 | 31684875775889623680 | 300860642249496135414 |

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