Martingale Problem under Nonlinear Expectations

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May 5, 2014

Abstract

We formulate and solve the martingale problem in a nonlinear expectation space. Unlike the classical work of Stroock and Varadhan (1969) where the linear operator in the associated PDE is naturally defined from the corresponding diffusion process, the main difficulty in the nonlinear setting is to identify an appropriate class of nonlinear operators for the associated fully nonlinear PDEs.

Based on the analysis of the martingale problem, we introduce the notion of weak solution for stochastic differential equations under nonlinear expectations and obtain an existence theorem under the Hölder continuity condition of the coefficients. The approach to establish the existence of weak solutions generalizes the classical Girsanov transformation method in that it no longer requires the two (probability) measures to be absolutely continuous.

MSC2000 subject classification: 60G40, 60H30, 49J10, 49K10, 93E20.

Key words and phrases: fully nonlinear PDE, nonlinear martingale problem, (conditional) nonlinear expectation, weak solution to $G$-SDE.

1 Introduction

1.1 Background

A probability measure and its associated linear expectation is a special case of nonlinear expectations. A particular nonlinear expectation is the sublinear or $G$-expectation introduced in [P07a], defined as the following. Let $\Omega = C[0, \infty)$ be the space of all real valued continuous functions $(\omega(t))_{t \geq 0}$, and let $\mathcal{H}$ be the linear space of random variables on $\Omega$. A sublinear expectation $\hat{E}$ is a functional on $\mathcal{H}$ satisfying, for all $X, Y \in \mathcal{H}$,

1. Monotonicity: $\hat{E}[X] \geq \hat{E}[Y]$, if $X \geq Y$;
2. Constant preserving: $\hat{E}[c] = c$, for $c \in \mathbb{R}$;
3. Sub-additivity: $\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y]$;
4. Positive homogeneity: $\hat{E}[aX] = a\hat{E}[X]$, for $a \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space. In a sublinear expectation space, there

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is no longer one-to-one correspondence between the nonlinear expectation and its induced capacity, unlike the linear expectation and its induced probability measure.

One motivation for developing the G-theory is the theory of risk measure. A coherent risk measure $\rho$ is first introduced in [ADEH99], which can be associated with a sublinear expectation $\mathbb{E}$ via $\rho(X) = \mathbb{E}[-X]$ for any random variable $X$. G-theory provides a rigorous mathematical framework for time-consistent risk measures, which were previously restricted to be static. Sublinear expectation is also related to model uncertainty. An insightful result in [DHP11] shows that a sublinear expectation is connected to classical expectation through a class of probability measures that measure the “size” of uncertainty in the following way: there exists a weakly compact family $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{B}(\Omega))$ such that $\mathbb{E}[X] = \max_{P \in \mathcal{P}} E^P[X]$ for a proper class of $X$. Here $\mathcal{B}(\Omega)$ is the Borel $\sigma$-algebra on $\Omega$, and $E^P[X]$ is the linear expectation with respect to $P$. Consequently, the notion of “quasi-sure” in a sublinear expectation replaces that of “almost-sure” in a probability space. From this perspective, a sublinear expectation “measures” the model uncertainty: the bigger the expectation $\mathbb{E}$, the more the uncertainty.

The very first building block of G-theory is the G-normal distribution, i.e., a normal distribution with an uncertain variance written as $N(0 \times [\sigma^2, \sigma^2])$. It is characterized by the G-heat equation

$$\partial_t u - G(D^2 u) = 0, \quad u|_{t=0} = \varphi. \quad (1)$$

Here, $G : \mathbb{R} \to \mathbb{R}$ is a monotonic sublinear function given by $G(\gamma) = \frac{1}{2} \sup_{\alpha \in [\sigma^2, \sigma^2]} \gamma \alpha$, where $\sigma^2 = -\mathbb{E}[-X^2]$ and $\sigma^2 = \mathbb{E}[X^2]$. The G-theory is then developed in a way similar to the classical probability theory: the notion of G-(in)dependence and a G-central limit theorem are developed. Especially, in order to define the conditional expectation, a backward recursive procedure is adopted to first define a pre-expectation, starting from the solution of the G-heat equation with $\varphi$. This idea is analogous to defining stochastic processes from a finite-dimensional distribution. Such a procedure is well-defined once Kolmogorov’s time-consistency theorem or the semi-group property is established, as shown in [P05]. From here, the G-Brownian motion, G-Itô’s calculus, G-SDEs, and G-martingale are developed similarly as the classical Itô’s calculus. This is the G-theory in the spirit of Kolmogorov and Itô.

1.2 The martingale problem with $\tilde{G}$

In this paper, we consider the martingale problem in the spirit of Stroock and Varadhan [SV69], albeit in a nonlinear expectation space.

Problem formulation The classical martingale problem studies a diffusion process and its distributions with a parabolic PDE with a linear differential operator $L_\theta$ and their semi-group properties, and shows the equivalence of solving the martingale problem to the unique weak solution of an associated stochastic differential equation with given drift and diffusion coefficients. Moreover, the probability measure is built along with the underlying random processes and its uniqueness is established. Naturally, under a nonlinear expectation, the corresponding martingale problem is to find a family of nonlinear operators $\{\tilde{E}_t\}_{t \geq 0}$ on a nonlinear expectation space $(\Omega, \mathcal{H})$ such that

$$\varphi(X_t) - \int_0^t \tilde{G}(X_\theta, D\varphi(X_\theta), D^2\varphi(X_\theta)) d\theta, \quad t \geq 0 \quad (2)$$
is an $\tilde{E}$-martingale for all $\varphi \in C_c^\infty(\mathbb{R}^d)$. Here $X_t(\omega) = \omega(t), \omega \in \Omega$, $\tilde{G} : \mathbb{R}^d \times \mathbb{R}^d \times S^d \to \mathbb{R}$ is a given continuous function satisfying certain properties to be specified later, where $S^d$ is the collection of $d \times d$ symmetric matrices with usual order.

**Appropriate class for $\tilde{G}$** However, there is a major issue. Unlike the classical martingale problem where the linear differential operator $L_\theta$ is defined naturally as the infinitesimal generator of a diffusion process with given drift terms $b(\cdot, \cdot)$ and diffusion terms $a^{ij}(\cdot, \cdot)$, the specification of the continuous function $\tilde{G}$ is not so obvious in a nonlinear setting. Given the nonlinear nature of the PDE, identifying the appropriate class of $\tilde{G}$ is critical for the scope and feasibility of our study.

To study the martingale problem in the nonlinear expectation space, one would need to analyze the class of fully nonlinear PDEs of the following form

$$
\partial_t u - \tilde{G}(x, Du, D^2 u) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d.
$$

(3)

Assuming that $\tilde{G}$ were chosen so that this fully nonlinear PDE had a unique solution in some sense, then one could proceed to construct a sequence of conditional expectations $\tilde{E}_t$. Now, in order for such a nonlinear expectation to be consistent with the existing literature in both linear and sublinear spaces, it would be natural to require that $\tilde{G}$ be monotonic, subadditive, and positive homogeneous. However, this class of $\tilde{G}$ would appear too restrictive. In this paper, we will show that an appropriate class of $\tilde{G}$ needs not be sublinear itself. Instead $\tilde{G}$ should be “dominated” by some sublinear and continuous function $G$. We call such a condition the “DOM” condition and define a class $D$ for $\tilde{G}$ (see Section 2.1).

**Fully nonlinear PDEs and weak solutions of $G$-SDEs** Once such a class of $\tilde{G}$ is fixed, then one needs to analyze the associated fully nonlinear PDE. For the pair of $G$ and $\tilde{G}$, there is a pair of the associated PDEs $(\mathcal{P})$ and $(\tilde{\mathcal{P}})$ respectively (see Section 2.2). The specification of the class $D$ as outlined in Section 2.1 has several implications. First, it ensures the uniqueness of the solutions for the PDEs. Second, it guarantees that a conditional expectation, i.e., a family of operators $\{\tilde{E}_t\}_{t \geq 0}$ can be constructed from the viscosity solution of this PDE, and the constructed conditional expectation has reasonable properties such as time consistency. Finally, the condition “DOM” not only allows one to build a piecewise Brownian motion in the $\tilde{G}$ space based on the $G$-Brownian motion embedded in the sub-linear expectation space, but also ensures a much simpler stochastic calculation within the framework of Brownian motion.

Once the martingale problem is solved, it is natural to introduce the notion of weak solution of $G$-SDE under the nonlinear expectation and discuss its existence as in the classical probability theory.

**Our work vs. related work** The literature on sublinear expectation is growing rapidly (see [P10a] and references therein). In contrast to the bottom-up approach in $G$-theory, where the $G$-martingale and $G$-Itô’s calculus are developed on a sublinear expectation space from the basic $G$-heat equation, our approach starts with a general class of fully nonlinear PDEs which includes the $G$-heat equation as a very special case. These PDEs are state dependent. Consequently, our analysis on the existence and uniqueness of their solutions not only generalizes existing results including [P10a] and [FS92], but also leads to the construction of nonlinear expectations that goes
beyond the sublinear ones in [P10a], [STZ12] and [DHP11]. As a result, random processes, especially martingales, and the stochastic calculus are all developed under a more general framework.

It is worth mentioning that there are other approaches in addition to ours of constructing non-linear expectations from PDEs. For example, [STZ12] uses the classical stochastic control approach of building regular conditional probability when considering a family of backward stochastic differential equations; and [N13] takes a control approach, where random G-expectations are constructed based on an optimal control formulation with path-dependent control sets, hence a path-dependent family of probabilities. All these approaches, however, leads to sublinear expectation spaces (see also [DHP11]).

Finally, our approach to establish the existence of weak solutions generalizes the classical Girsanov transformation method in that it no longer requires the two (probability) measures to be absolutely continuous. Instead it is critical that probability measures singular from each other should be “dominated” by a certain sublinear expectation.

### Outline of the paper
The paper starts with the discussion of G and $\tilde{G}$ in the class $\mathcal{D}$ (Section 2.1) and states some of the key properties of the solutions to the associated PDEs (Section 2.2). Following the Kolmogorov’s idea, a family of nonlinear operators $\{\tilde{E}_t\}_{t \geq 0}$ is constructed via solutions of the PDEs (P) and (P) with analysis of their properties (Section 2.3); Section 2.4 finishes the proof of the martingale problems. Finally, the weak solution of G-SDE is introduced and analyzed in Section 3. Appendix contains some technical details for stochastic calculus under nonlinear expectations and proofs on the existence and uniqueness for the PDEs that are associated with the martingale problem.

## 2 Martingale problems

### 2.1 G and $\tilde{G}$

Let us start by defining a class $\mathcal{D}$ of functions $\tilde{G}$. $\tilde{G}$ is continuous and “dominated” by a continuous and sublinear function $G$.

**Definition 1 (Class $\mathcal{D}$)** A continuous function $\tilde{G} : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}^d \rightarrow \mathbb{R}$ is of class $\mathcal{D}$, if there exists a continuous function $G : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}^d \rightarrow \mathbb{R}$ such that

- $\tilde{G}(x, \alpha p, \alpha A) = \alpha \tilde{G}(x, p, A)$ for all $x \in \mathbb{R}^d$, $\alpha \geq 0$,
- for each $x \in \mathbb{R}^d$, $(p, A), (p', A') \in \mathbb{R}^d \times \mathcal{S}^d$,

\[ \tilde{G}(x, p, A) - \tilde{G}(x, p', A') \leq G(x, p - p', A - A'), \]  

(DOM)

with $G$ satisfying

(A). (Subadditivity) $G(x, p + \bar{p}, A + \bar{A}) \leq G(x, p, A) + G(x, \bar{p}, \bar{A})$,

(B). (Positive Homogeneity) $G(x, \beta p, \beta A) = \beta G(x, p, A)$, $\beta \geq 0$,

(C). (Monotonicity) $G(x, p, A) \leq G(x, p, A + \bar{A})$,  

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(D). (Uniform Lipschitz Continuity) \[ |G(x, p, A) - G(x, \bar{p}, \bar{A})| \leq L(|p - \bar{p}| + |A - \bar{A}|), \] for some \( L > 0 \), for any \( x, p, \bar{p} \in \mathbb{R}^d \), \( A, \bar{A}, \bar{A} \in \mathbb{S}^d \), and \( \bar{A} \geq 0 \).

**Example 1** When \( G \) satisfies conditions (A) and (B), then by Theorem I.2.1 of [P10a], for each given \( x \in \mathbb{R}^d \), there exists a bounded, closed, and convex subset \( U(x) \subset \mathbb{S}^d \times \mathbb{R}^d \), such that

\[
G(x, p, A) = \sup_{(a, b) \in U(x)} \left\{ \frac{1}{2} \text{tr}[aA] + \langle b, p \rangle \right\}.
\]

Since for each given \( x \), there exists a dense subset \( U_0(x) \subset U(x) \), which is countable, also denoted by \( \{\{\frac{1}{2}a(x, i), b(x, j)\}\}_{i,j} \in \mathbb{N} \), one can rewrite the above expression in the following sublinear form

\[
G(x, p, A) = \sup_{(\gamma, \beta) \in \Gamma} \left\{ \frac{1}{2} \text{tr}[a(x, \gamma)A] + \langle b(x, \gamma), p \rangle \right\},
\]

where \( \Gamma \) is an index set, and \( a(x, \gamma) \in \mathbb{S}^d, b(x, \gamma) \in \mathbb{R}^d \) are bounded. Moreover, when \( G \) also satisfies (C), then \( a(x, \gamma) \geq 0 \) for any \( x \in \mathbb{R}^d \) and \( \gamma \in \Gamma \). And there exists \( \sigma(x, \gamma) \in \mathbb{R}^{d \times d} \), such that \( a(x, \gamma) = \sigma(x, \gamma)\sigma^T(x, \gamma) \). With additional condition (D), one can simply write

\[
G(p, A) = \sup_{(\gamma, \beta) \in \Gamma} \left\{ \frac{1}{2} \text{tr}[\gamma A] + \langle \beta, p \rangle \right\}
\]

with \( \Gamma \subset \mathbb{S}^d_+ \times \mathbb{R}^d \) being bounded, convex, and closed.

Of course, such a \( G \) is dominated by itself in the sense of (DOM), thus in the class \( \mathcal{D} \).

**Example 2** Assume \( G \) of the form

\[
\tilde{G}(x, p, A) = \sup_{\gamma \in \Gamma} \inf_{\lambda \in \Lambda} \left\{ \frac{1}{2} \text{tr}[\sigma(x, \gamma, \lambda)\sigma^T(x, \gamma, \lambda)A] + \langle b(x, \gamma, \lambda), p \rangle \right\}
\]

or

\[
\tilde{G}(x, p, A) = \inf_{\gamma \in \Gamma} \sup_{\lambda \in \Lambda} \left\{ \frac{1}{2} \text{tr}[\sigma(x, \gamma, \lambda)\sigma^T(x, \gamma, \lambda)A] + \langle b(x, \gamma, \lambda), p \rangle \right\}.
\]

Here \( \Gamma \) and \( \Lambda \) are compact metric spaces, \( \sigma, b \in C_b(\mathbb{R}^d \times \Gamma \times \Lambda) \), and \( \sigma(\cdot, \gamma, \lambda) \) and \( b(\cdot, \gamma, \lambda) \) are uniformly Lipschitz continuous in the following sense

\[
|\sigma(x, \gamma, \lambda) - \sigma(y, \gamma, \lambda)| + |b(x, \gamma, \lambda) - b(y, \gamma, \lambda)| \leq \tilde{L}|x - y|, \text{ for all } \gamma \in \Gamma, \lambda \in \Lambda,
\]

with \( \tilde{L} > 0 \) a constant.

Such form of \( \tilde{G} \) is important for stochastic games [BBP97] and is in class \( \mathcal{D} \): it is clearly dominated by \( G \) specified by

\[
G(x, p, A) = \sup_{\gamma \in \Gamma, \lambda \in \Lambda} \left\{ \frac{1}{2} \text{tr}[\sigma(x, \gamma, \lambda)\sigma^T(x, \gamma, \lambda)A] + \langle b(x, \gamma, \lambda), p \rangle \right\}.
\]

Throughout the paper unless otherwise specified, \( G \) and \( \tilde{G} \) satisfy the conditions specified in the definition of class \( \mathcal{D} \). And without loss of generality, we assume \( G \) is of the form (5) as discussed in Example 1.
2.2 PDEs associated with $G$ and $\tilde{G}$

Now we introduce two classes of fully nonlinear PDEs associated with $G$ and $\tilde{G}$ in class $D$.

State-dependent parabolic PDEs associated with $G$

\[
\begin{cases}
\partial_t u(t, x) - G(x, Du(t, x), D^2 u(t, x)) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d, \\
u(0, x) = \varphi(x), \quad x \in \mathbb{R}^d.
\end{cases}
\] (P)

Fully nonlinear PDEs associated with $\tilde{G}$

\[
\begin{cases}
\partial_t u(t, x) - \tilde{G}(x, Du(t, x), D^2 u(t, x)) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d, \\
u(0, x) = \varphi(x), \quad x \in \mathbb{R}^d.
\end{cases}
\] (P̃)

Clearly the PDE (P) is a special case of the PDE (P̃).

The conditions specified in class $D$ for $G$ and $\tilde{G}$ are essential to ensure the existence of the viscosity solutions for PDE (P) and PDE (P̃) and to guarantee that such solutions have nice properties.

In the following, we will discuss the existence, the uniqueness, and the properties of the solutions associated with PDEs (P) and (P̃). Note that some results hold under more general conditions for $G$ and $\tilde{G}$ than those specified in class $D$. To avoid confusions, all conditions in the theorems and lemmas are specified for $G$ and $\tilde{G}$.

First, recall the definition of the viscosity solutions for the associated PDEs in (P̃) and (P).

**Definition 2** Given a constant $T > 0$. A viscosity subsolution of the PDE in (P̃) on $(0, T) \times \mathbb{R}^d$ is an upper semicontinuous (USC) function $u$ in $(0, T) \times \mathbb{R}^d$ such that for all $(t, x) \in (0, T) \times \mathbb{R}^d, \phi \in C^2((0, T) \times \mathbb{R}^d)$ such that $u(t, x) = \phi(t, x)$ and $u < \phi$ on $(0, T) \times \mathbb{R}^d \setminus \{(t, x)\}$, we have

$$\partial_t \phi(t, x) - \tilde{G}(x, D\phi(t, x), D^2 \phi(t, x)) \leq 0;$$

likewise, a viscosity supersolution of the PDE in (P̃) on $(0, T) \times \mathbb{R}^d$ is a lower semicontinuous (LSC) function $v$ in $(0, T) \times \mathbb{R}^d$ such that for all $(t, x) \in (0, T) \times \mathbb{R}^d, \psi \in C^2((0, T) \times \mathbb{R}^d)$ such that $v(t, x) = \psi(t, x)$ and $v > \psi$ on $(0, T) \times \mathbb{R}^d \setminus \{(t, x)\}$, we have

$$\partial_t \psi(t, x) - \tilde{G}(x, D\psi(t, x), D^2 \psi(t, x)) \geq 0.$$

And a viscosity solution of the PDE in (P̃) on $(0, T) \times \mathbb{R}^d$ is a function that is both a viscosity subsolution and a viscosity supersolution of the PDE in (P̃) on $(0, T) \times \mathbb{R}^d$.

The definition of the viscosity solution to PDE in (P) is similar, with $\tilde{G}$ replaced by $G$.

Now, note that PDE (P) has been extensively studied, for example, in the literature of portfolio selections (see for instance [Ph09]). And its comparison theorem can be established with slightly modified techniques from [FS92].
Theorem 1 (Comparison theorem for PDE in (P)) Given a continuous function \( G : \mathbb{R}^d \times \mathbb{R}^d \times S^d \rightarrow \mathbb{R} \), which satisfies conditions (A), (B), (C). Suppose \( \sigma, b \) are uniformly Lipschitz continuous with respect to \( x \). Let \( u \in \text{USC}([0, T] \times \mathbb{R}^d) \) be a viscosity subsolution of the PDE in (P) and \( \overline{u} \in \text{LSC}([0, T] \times \mathbb{R}^d) \) be a viscosity supersolution of the PDE in (P) on \([0, T] \times \mathbb{R}^d\) with polynomial growth. Then \( u \leq \overline{u} \) when \( u|_{t=0} \leq \overline{u}|_{t=0} \).

Perron’s existence result of the solution of (P) follows from Appendix C.3 of [P10a].

Theorem 2 (Existence for PDE (P)) Assuming a comparison theorem holds for (P). Moreover, suppose that there is a viscosity subsolution of (P) \( \underline{u} \) and a viscosity supersolution \( \overline{u} \) of (P) such that \( \underline{u}|_{t=0} = \overline{u}|_{t=0} = \varphi \in C(\mathbb{R}^d) \) with polynomial growth. Here \( u^* \) is the upper semicontinuous envelope of \( u \) and \( u^* \) is lower semicontinuous envelope of \( u \). Then

\[
w(t, x) = \sup\{W(t, x); \underline{u} \leq W \leq \overline{u} \text{ and } W \text{ is a viscosity subsolution of (P)}\},
\]

is a viscosity solution of (P).

In particular, if problem (P) satisfies conditions (A), (B), and (C), and \( \sigma, b \) are bounded and uniformly Lipschitz continuous, then it has a unique solution.

We next state the comparison theorem for PDE (\( \tilde{P} \)) which relies on a technical condition (27) as detailed in the Appendix. We also outline its proof in the Appendix.

Theorem 3 (Comparison theorem for PDE in (\( \tilde{P} \))) Suppose both \( G \) and \( \tilde{G} \) satisfy condition (27), with their respective corresponding continuous decomposition functions satisfying condition (G). Suppose \( \tilde{G} \) satisfies condition (DOM) and \( G \) satisfies conditions (C) and (D). Let \( \underline{u} \in \text{USC}([0, T] \times \mathbb{R}^d) \) be a subsolution of the PDE in (\( \tilde{P} \)) and \( \overline{u} \in \text{LSC}([0, T] \times \mathbb{R}^d) \) be a supersolution of the PDE in (\( \tilde{P} \)) on \((0, T) \times \mathbb{R}^d\) and \( \overline{w} \) is a supersolution of the PDE in (P). They all satisfy the polynomial growth condition. If \( (\overline{u} - \underline{u})|_{t=0} = \overline{u}|_{t=0} = \varphi \in C(\mathbb{R}^d) \) with polynomial growth. Here \( \underline{u} \leq \overline{u} \) on \([0, T] \times \mathbb{R}^d\) provided that \( \underline{u}|_{t=0} \leq \overline{u}|_{t=0} \).

Moreover, the same proof of Theorem C.3.1 of [P10a] leads to

Theorem 4 (Existence of the solution of (\( \tilde{P} \))) Suppose \( G \) satisfies conditions (A), (B), (C), and (D). Assume that both \( G \) and \( \tilde{G} \) satisfy condition (27), with their respective corresponding continuous decomposition functions satisfying condition (G). If \( \tilde{G} \) satisfies the (DOM) condition, the relation

\[
\tilde{G}(x, \alpha p, \alpha A) - \tilde{G}(x, 0, 0) = \alpha [\tilde{G}(x, p, A) - \tilde{G}(x, 0, 0)] \text{ for all } \alpha \geq 0,
\]

and \( \tilde{G}(x, 0, 0) \) has polynomial growth, then there exists a unique solution for PDE (\( \tilde{P} \)).

Remark 1 In fact, the positive homogeneity condition and condition (27) are not necessary for the uniqueness of the solution for PDE (\( \tilde{P} \)). For instance, take

\[
\tilde{G}(x, p, A) = \sup_{\gamma \in \Gamma} \{a(\gamma)A + g(x, p, \gamma)\}, (x, p, A) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R},
\]

(8)
where $\Gamma$ is an index set such that $a(\gamma) \geq 0$ is uniformly bounded, and the continuous function $g$ is dominated by a continuous function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in the sense of

$$g(x, p, \gamma) - g(x, \tilde{p}, \gamma) \leq h(x, p - \tilde{p}), \text{ for every } \gamma \in \Gamma.$$ 

Suppose $g$ satisfies $|g(x, p, \gamma) - g(y, p, \gamma)| \leq L_g (1 + |x| + |y|)|x - y|(1 + |p|), x, y, p \in \mathbb{R}$ uniformly in $\gamma \in \Gamma$ and $L_g > 0$ is a constant, and $|h(x, p) - h(y, p)| \leq L_h (1 + |x| + |y|)|x - y|(1 + |p|).$

Then this $\tilde{G}$ is dominated by $G(x, p, A) = \sup_{\gamma \in \Gamma} \{a(\gamma) A\} + h(x, p)$, yet $\tilde{G}$ does not satisfy the positive homogeneity condition and condition (27). Nevertheless, the same approach shows that the comparison still holds.

For $\tilde{G}$ of the forms as in Example 2, we have the following results about the associated PDEs.

**Theorem 5 [BBP97]** Suppose $\tilde{G}$ is of the form (6). Let $u_1, -u_2 \in \text{USC}([0, T] \times \mathbb{R}^d)$, $u_1$ and $u_2$ are subsolution and supersolution to problem (P) with initial conditions $u_1(0, x) = \varphi_1(x)$ and $u_2(0, x) = \varphi_2(x)$, respectively. Set $w = u_1 - u_2$. Then $w$ is a subsolution to

$$
\begin{cases}
\partial_t w(t, x) - G(x, Dw(t, x), D^2 w(t, x)) = 0, & (t, x) \in (0, T] \times \mathbb{R}^d, \\
w(0, x) = \varphi_2(x) - \varphi_1(x), & x \in \mathbb{R}^d,
\end{cases}
$$

with $G(x, p, A) = \sup_{\gamma \in \Gamma, \lambda \in \Lambda} \left\{ \frac{1}{2} \text{tr} \left[ \sigma(x, \gamma, \lambda) \sigma^T(x, \gamma, \lambda) A \right] + \langle b(x, \gamma, \lambda), p \rangle \right\}$. Consequently, let $u \in \text{USC}([0, T] \times \mathbb{R}^d), v \in \text{LSC}([0, T] \times \mathbb{R}^d)$ be the subsolution and supersolution to the PDE (P), respectively. If both $u$ and $v$ have at most polynomial growth and $u(0, x) \leq v(0, x)$, then $u(t, x) \leq v(t, x)$ in $(0, T] \times \mathbb{R}^d$.

Since our PDE has a simpler form than that in [BBP97] without the jump term, their proof can be greatly simplified, as illustrated in the Appendix. Similar results and proof hold for PDEs with $\tilde{G}$ of form (7).

Finally, we discuss the properties of the solutions for the PDEs, when exist. Clearly, from the definition of class $\mathcal{D}$, one sees

**Theorem 6 (Properties of the solutions of PDEs)** Let $u^\phi, \tilde{u}^\phi \in C([0, T] \times \mathbb{R}^d)$ denote the unique solutions of (P) and (P) with polynomial growth, with the boundary conditions $\phi$ and $\varphi$ respectively. Then we have

$$
\begin{align*}
\tilde{u}^\phi + c &= \tilde{u}^\phi + c, \\
\tilde{u}^\phi - \tilde{u}^\phi &\leq u^{\varphi - \phi}, \\
\tilde{u}^{\alpha \varphi} &\leq \alpha \tilde{u}^\varphi, \quad \alpha \geq 0,
\end{align*}
$$

where $c \in \mathbb{R}$ is a constant, and $\varphi, \phi$ are continuous functions with polynomial growth.
2.3 Nonlinear expectations $\tilde{E}$ and $E$

2.3.1 Construction of $\tilde{E}$ and $E$ from the associated PDEs

Assuming the unique solution $\tilde{u}^\varphi$ to PDE $(\overline{P})$, one can define the ‘conditional expectation’ $\tilde{E}_t$ for $t \in [0, T], T < \infty$.

The construction starts from the “pre-expectation”. Let $\Omega = C_x([0, \infty); \mathbb{R}^d) = \{\omega(\cdot); \omega \text{ is a continuous } \mathbb{R}^d\text{-valued function on } [0, \infty) \text{ and } \omega(0) = x\}$. Fix $N > 0$, take $0 = t_0 \leq t_1 \leq \cdots \leq t_N \leq T$. Take $\varphi_0$ from a proper function space on $(\mathbb{R}^d)^N$ denoted by $\mathcal{C}((\mathbb{R}^d)^N)$, and set $\xi(\omega) = \varphi_0(X_{t_1}, \cdots, X_{t_N})$. Let $\mathcal{T}_t[\varphi(\cdot)](x) := u(t, x)$, and for $0 \leq j \leq N$, define

\[
\varphi_1(x_1, \cdots, x_{N-1}) = \mathcal{T}_{t_{N} - t_{N-1}}[\varphi_0(x_1, \cdots, x_{N-1}, \cdot)](x_{N-1}), \\
\vdots \\
\varphi_{N-j}(x_1, \cdots, x_j) = \mathcal{T}_{t_{N-j} - t_{j}}[\varphi_{N-j-1}(x_1, \cdots, x_j, \cdot)](x_j), \\
\vdots \\
\varphi_{N-1}(x_1) = \mathcal{T}_{t_2 - t_1}[\varphi_{N-2}(x_1, \cdot)](x_1), \\
\varphi_N = \mathcal{T}_{t_1}[\varphi_{N-1}(\cdot)](x),
\]

where $\varphi_k \in \mathcal{C}((\mathbb{R}^d)^{N-k}), 0 \leq k \leq N - 1$ and $\varphi_N \in \mathbb{R}$. Then define

\[
\tilde{E}_t[\xi] = \varphi_{N-j}(X_{t_1}, \cdots, X_{t_j}), \text{ if } t = t_j, 0 \leq j \leq N.
\]

This construction approach is in the spirit of Nisio’s semigroup theory (see [N76a], [N76b], and [P05]), where nonlinear expectations are constructed from nonlinear Markov chains after establishing a generalized Kolmogorov’s consistency theorem and pre-expectations.

In our case, we denote such an ‘expectation’ by $\tilde{E}$. Since PDE $(P)$ is a special case of PDE $(\overline{P})$, a sublinear expectation $E$ can be similarly defined from the its solution. It is a special case of $\tilde{E}$.

Remark 2 1) One can set $\mathcal{C}((\mathbb{R}^d)^N) := \mathcal{C}_{l,\text{Lip}}((\mathbb{R}^d)^N)$, where $\mathcal{C}_{l,\text{Lip}}(\mathbb{R}^n)$ is the space of real valued continuous functions defined on $\mathbb{R}^n$ such that

\[
|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \forall x, y \in \mathbb{R}^n,
\]

for some $C > 0$, and $m \in \mathbb{N}$ depending on $\varphi$.

Now let $\Omega_T := \{\omega(\cdot \land T); \omega \in \Omega\}$ and $\mathcal{H} := L_{ip}(\Omega_T) = \{\varphi(X_{t_1}, \cdots, X_{t_N}); \varphi \in \mathcal{C}_{l,\text{Lip}}((\mathbb{R}^d)^N) \text{ for some } N \in \mathbb{N} \text{ and } 0 \leq t_1 \leq \cdots \leq t_N \leq T\}$, and call $(\Omega, \mathcal{H}, \tilde{E})$ a nonlinear expectation space. If $\xi = \varphi(X_{t_1}, \cdots, X_{t_N})$ with $t_N \leq t \in [0, T]$, we say $\xi \in L_{ip}(\Omega_t)$. It is clear that $L_{ip}(\Omega_t) \subset L_{ip}(\Omega_T), t \leq T$.

2) With a sublinear $E$, for each $t \in [0, T]$, one can extend the space $L_{ip}(\Omega_t)$ to a Banach space $L^1_T(\Omega_t)$ under the norm $\| \cdot \| := E[\| \cdot \|]$ as in [P10a], or see the Appendix, since the nonlinear expectation $E$ is sublinear. And from now on, we take $\mathcal{H} = L^1_T(\Omega) := \bigcup_{T > 0} L^1_T(\Omega_T)$. 

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Remark 3  We point out here the above construction procedure for \( \tilde{E} \) depends critically on the homogeneity of the PDE \((\tilde{P})\). Otherwise, the resulting \( \tilde{E} \) may not be well defined as seen from the following example. Consider the following linear nonhomogeneous PDE

\[
\begin{aligned}
\partial_t u - \partial_x^2 u &= x, \quad (t, x) \in (0, T] \times \mathbb{R}, \\
u(0, x) &= c, \quad x \in \mathbb{R}.
\end{aligned}
\]

The solution is \( u(t, x) = tx + c \). If we were to define \( \tilde{E} \) as suggested in the above procedure, and consider the constant as a function

\[\phi_0 : \mathbb{R}^2 \to \{c\}; (x_1, x_2) \mapsto \phi_0(x_1, x_2).\]

Then

\[c = \phi_0(X_{t_1}, X_{t_2}), \quad \forall t_1 \leq t_2 \in (0, T].\]

Now clearly

\[\tilde{E}_{t_1}[c] = T_{t_2 - t_1}[\phi_0(x_1, \cdot)|_{x_1 = X_{t_1}} = [(t_2 - t_1)x_1 + c]|_{x_1 = X_{t_1}} =: \phi_1(x_1)|_{x_1 = X_{t_1}},\]

\[\tilde{E}_0[\phi_1(X_{t_1})] = t_1X_0 + (t_2 - t_1)X_0 + c = t_2X_0 + c.\]

We would have \( \tilde{E}[c] = t_2X_0 + c \) for any arbitrary \( t_2 \). Thus the nonlinear expectation is not well defined.

2.3.2 Properties of nonlinear expectations \( \tilde{E} \) and \( E \)

Given a nonlinear expectation space \((\Omega, \mathcal{H}, \tilde{E})\), a stochastic process \((\xi_t)_{t \geq 0}\) is a collection of random variables on \((\Omega, \mathcal{H})\). That is, for each \( t \geq 0 \), \( \xi_t \in \mathcal{H} \). Moreover,

Definition 3 (\( \tilde{E} \)-Martingale) A stochastic process \((M_t)_{t \geq 0}\) is called an \( \tilde{E} \)-martingale if for each \( t \in [0, \infty) \), \( M_t \in L^1_\mathcal{E}(\Omega_t) \), and for each \( s \in [0, t] \),

\[\tilde{E}_s[M_t] = M_s.\]

Remark 4 In this paper, since \( \tilde{E} \) is constructed from the PDEs associated with \( \tilde{G} \), sometimes the \( \tilde{E} \)-martingale is also referred to as \( \tilde{G} \)-martingale when there is no risk of confusion.

Moreover, the \( \tilde{E} \) and \( E \) constructed in Section 2.3.1 have the following properties.

Proposition 7 Given a nonlinear expectation space \((\Omega, \mathcal{H}, \tilde{E})\), let \( \xi, \eta \in \mathcal{H} \).

(I) For \( \varphi \in C_{1, \text{Lip}}(\mathbb{R}) \) and \( s \leq t \)

\[\tilde{E}_s[\varphi(X_t)] = u^\varphi(t - s, X_s).\]

(II) (Monotonicity) If \( \xi \leq \eta \),

\[\tilde{E}_t[\xi] \leq \tilde{E}_t[\eta].\]
(III) (Constant preserving) If \( \xi \in L_{ip}(\Omega_s), \eta \in L_{ip}(\Omega_{s+h}), s, h \geq 0, \)

\[
\tilde{E}_s[\xi + \eta] = \xi + \tilde{E}_s[\eta].
\]

In particular, \( \tilde{E}[\xi + c] = \tilde{E}[\xi] + c, \) with \( c \in \mathbb{R} \) a constant.

(IV) (Tower property) For any \( s, h > 0, \)

\[
\tilde{E}_s \circ \tilde{E}_{s+h} = \tilde{E}_s.
\]

(V) (Domination)

\[
\tilde{E}_t[\xi] - \tilde{E}_t[\eta] \leq \tilde{E}_t[\xi - \eta].
\]

In addition, we have for the sublinear expectation \( \mathcal{E}, \)

(VI) (Subadditivity)

\[
\mathcal{E}_t[\xi + \eta] \leq \mathcal{E}_t[\xi] + \mathcal{E}_t[\eta].
\]

(VII) (Positive homogeneity)

If \( \xi \in L_{ip}(\Omega_s), \eta \in L_{ip}(\Omega_{s+h}), s, h \geq 0, \)

\[
\mathcal{E}_s[\xi \eta] = \xi^+ \mathcal{E}_s[\eta] + \xi^- \mathcal{E}_s[-\eta].
\]

In particular, for any constant \( \lambda \geq 0, \)

\[
\mathcal{E}_t[\lambda \xi] = \lambda \mathcal{E}_t[\xi].
\]

**Proof.** For (I), note that \( \varphi(X_t) = \varphi(X_t + X_s - X_s) =: \psi(X_s, X_t), \)

\[
\tilde{E}_s[\varphi(X_t)] = \tilde{E}_s[\psi(X_s, X_t)] = \mathcal{T}_{t-s}[\psi(x_1, \cdot)](x_1)|_{x_1=x_s} = \varphi(t-s, x_1)|_{x_1=x_s}.
\]

(II) is an implication of the comparison theorem for the PDE in (P).

For (III), without loss of generality, assume \( \xi = \varphi(X_s), \eta = \phi(X_{s+h}), \varphi, \phi \in \mathcal{C}(\mathbb{R}^d). \) Then

\[
\tilde{E}_s[\xi + \eta] = \tilde{E}_s[\varphi(X_s) + \phi(X_{s+h})]
\]

\[
= \mathcal{T}_h[\varphi(y) + \phi(\cdot)](y)|_{y=x_s}
\]

\[
= \varphi(X_s) + \mathcal{T}_h[\phi(\cdot)](y)|_{y=x_s}
\]

\[
= \xi + \tilde{E}_s[\eta],
\]

where the third equality follows from Theorem 6.

To show (IV), without loss of generality, we assume \( \xi = \varphi_0(X_{t_1}, X_{t_2}, X_{t_3}), t_1 = s, t_2 = s+h, t_2 \leq t_3 \leq T, \varphi_0 \in \mathcal{C}((\mathbb{R}^d)^3). \) Then

\[
\tilde{E}_{t_2}[\xi] = \varphi_1(X_{t_1}, X_{t_2}) \text{ and } \tilde{E}_{t_1}[\xi] = \varphi_2(X_{t_1}), \varphi_1 \in \mathcal{C}((\mathbb{R}^d)^2), \varphi_2 \in \mathcal{C}(\mathbb{R}^d),
\]

from the construction procedure for \( \tilde{E}. \) Meanwhile,

\[
\tilde{E}_{t_1}[\tilde{E}_{t_2}[\xi]] = \tilde{E}_{t_1}[\varphi_1(X_{t_1}, X_{t_2})] = \varphi_2(X_{t_1}),
\]

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with \( \varphi_2(x) = \mathcal{T}_{t_0 - t_1} \varphi_1(x, \cdot))(x) = \varphi_2(x) \), since the PDE \((\tilde{P})\) has a unique solution for a given appropriate initial condition.

\( (V) \) can be derived directly from Theorem 6 and the construction of \( \tilde{E} \), while \( (VI) \) is a special case of \( (V) \).

To prove \( (VII) \), assume \( \xi = \varphi(X_s), \eta = \phi(X_{s+h}) \), \( \varphi, \phi \in \mathcal{C}(\mathbb{R}^d) \). Then, by Theorem 6,

\[
\mathcal{E}_s[\xi] = \mathcal{E}_s[\xi] = \mathcal{E}_s[\varphi(X_s)\phi(X_{s+h})] = \mathcal{T}_h[\varphi(y)\phi(\cdot)](y)|_{y=X_s} = \mathcal{T}_h[\varphi(y)^+\phi(\cdot) - \varphi(y)^-\phi(\cdot)](y)|_{y=X_s} = \xi^+\mathcal{T}_h[\phi(\cdot)]|_{y=X_s} + \xi^-\mathcal{T}_h[-\phi(\cdot)]|_{y=X_s} = \xi^+\mathcal{E}_s[\eta] + \xi^-\mathcal{E}_s[-\eta].
\]

\( \Box \)

In addition, both \( \tilde{E} \) and \( E \) enjoy the following property.

**Lemma 8** Let \( \xi, \eta \) be two random variables in the sublinear expectation space \( (\Omega, \mathcal{H}, \mathcal{E}) \) such that \( \mathcal{E}[\xi] = -\mathcal{E}[-\xi] \), and \( \tilde{E} \) be a nonlinear expectation dominated by \( \mathcal{E} \). Then

\[
\tilde{E}[\alpha \xi + \eta] = \alpha \tilde{E}[\xi] + \tilde{E}[\eta], \quad \text{for } \forall \alpha \in \mathbb{R}.
\]

The following lemma follows easily with the (DOM) condition.

**Lemma 9** Given a nonlinear expectation space \( (\Omega, \mathcal{H}, \tilde{E}) \) and \( \xi \in \mathcal{H} \). If \( \{\varphi_n\}_{n=1}^\infty \subset \mathcal{C}(\mathbb{R}^d) \) satisfying \( \varphi_n \downarrow 0 \), then \( \tilde{E}[\varphi_n(\xi)] \downarrow 0 \).

### 2.4 Martingale problem and the solution

**Definition 4** (\( \tilde{E} \)-martingale problem) Given the sample space \( \Omega = \mathcal{C}([0, \infty); \mathbb{R}^d) \) with the canonical process \( Z \), the \( \tilde{E} \)-martingale problem is to find a time-consistent nonlinear expectation \( \tilde{E} \) defined on \( (\Omega, \mathcal{H}) \) such that, for each \( \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d) \),

\[
\varphi(Z_t) - \varphi(Z_0) - \int_0^t \tilde{G}(Z_\theta, D_z\varphi(Z_\theta), D_z^2\varphi(Z_\theta)) \, d\theta
\]

is an \( \tilde{E} \)-martingale on \([0, \infty)\).

Now, we will solve the martingale problem with \( Z_t \) in the canonical space \( \Omega = C_z([0, \infty); \mathbb{R}^{2d}) \) being the generalized \( G \)-Brownian motion in \([P10a]\). To this end, consider the following Cauchy problem, which is a special form of PDE \((\tilde{P})\).

\[
\begin{cases}
\partial_t u(t, x, y) - \tilde{G}(x, y, D_y u(t, x, y), D_y^2 u(t, x, y)) = 0, (t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d, \\
u(0, x, y) = \varphi(x, y), (x, y) \in \mathbb{R}^d \times \mathbb{R}^d,
\end{cases}
\]

\((\tilde{P}2)\)
where \( \varphi \in C(\mathbb{R}^d \times \mathbb{R}^d) \) with polynomial growth and \( \tilde{G} \) satisfies the continuity condition:

\[
|\tilde{G}(z, p, A) - \tilde{G}(\tilde{z}, p, A)| \leq C(1 + |A|)(1 + |z|^l + |\tilde{z}|^l)[(1 + |p|)|z - \tilde{z}|^\alpha], \quad z \in \mathbb{R}^d,
\]

for some constants \( C > 0, l \in \mathbb{N}, \) and \( \alpha \in (0, 1] \).

Clearly the existence and uniqueness of PDE (\( \tilde{P}_1 \)) imply the existence and uniqueness of PDE (\( \tilde{P}_2 \)), and both Example 1 and Example 2 satisfy condition (10). Note also the \( G \) for the \( G \)-Brownian motion in [P10a] is a special case of the \( G \) in the PDE (\( \tilde{P} \)).

**Theorem 10 (Martingale problem)** Take the canonical process \( (X_t, y_t) \) as the generalized \( G \)-Brownian motion. Then there exists a time consistent nonlinear expectation \( \tilde{E} \), together with its conditional expectations \( \{\tilde{E}_t\}_{t \in \mathbb{R}^d} \), defined on the sublinear expectation space \( (\Omega, \mathcal{H}) \) such that for \( 0 \leq s \leq t \leq T \), and \( \varphi \in C^\infty_0(\mathbb{R}^d \times \mathbb{R}^d) \),

\[
\tilde{E}_s[\varphi(X_t, y_t) - \varphi(X_s, y_s)] - \int_s^t \tilde{G}(X_\theta, y_\theta, D_y \varphi(X_\theta, y_\theta), D^2_x \varphi(X_\theta, y_\theta)) \, d\theta = 0. \tag{11}
\]

That is,

\[
\varphi(X_t, y_t) - \varphi(X_0, y_0) - \int_0^t \tilde{G}(X_\theta, y_\theta, D_y \varphi(X_\theta, y_\theta), D^2_x \varphi(X_\theta, y_\theta)) \, d\theta
\]

is an \( \tilde{E} \)-martingale on \([0, T] \).

To prove Theorem 10, it suffices to prove the following Proposition 11. Indeed the following identity can be easily established by taking the Itô’s formula for the generalized \( G \)-Brownian motion:

\[
\varphi(X_t, y_t) - \varphi(X_s, y_s) = \int_s^t \left\{ \langle D_x \varphi(X_\theta, y_\theta), dX_\theta \rangle + \langle D_y \varphi(X_\theta, y_\theta), dy_\theta \rangle + \frac{1}{2} \text{tr}[D^2_x \varphi(X_\theta, y_\theta) d\langle X \rangle_\theta] \right\}. 
\]

**Proposition 11** Let \( M_0 \in \mathbb{R}, \zeta, q \in M^2_\mathbb{E}(0, T; \mathbb{R}^d) \), and \( \eta \in M^2(0, T; \mathbb{S}^d) \) be given continuous processes, and let

\[
M_t = M_0 + \int_0^t \left( \zeta_\theta^T \, dX_\theta + q_\theta^T \, dy_\theta + \text{tr}[\eta_\theta \, d\langle X \rangle_\theta] \right) - \int_0^t \tilde{G}(X_\theta, y_\theta, q_\theta, 2\eta_\theta) \, d\theta, \quad 0 \leq t \leq T.
\]

Then \( M \) is a \( \tilde{G} \)-martingale.

We will prove Proposition 11 in several steps.

**Step 1.** Since \( X \) is a symmetric \( G \)-Brownian motion, by Lemma 8 and Proposition III.9.1-(iii) of [P10a], it is also a symmetric \( \tilde{G} \)-martingale. Thus it suffices to prove that

\[
\tilde{M}_t = M_0 + \int_0^t \left( q_\theta^T \, dy_\theta + \text{tr}[\eta_\theta \, d\langle X \rangle_\theta] \right) - \int_0^t \tilde{G}(X_\theta, y_\theta, q_\theta, 2\eta_\theta) \, d\theta, \quad 0 \leq t \leq T
\]

is a \( \tilde{G} \)-martingale.

To this end, we need the following lemma.
Lemma 12 Let $\varphi \in C_{l, \text{Lip}}(\mathbb{R}^d)$ be given and assume that $f : \mathbb{R}^d \to \mathbb{R}$ satisfies
\[ |f(z) - f(\tilde{z})| \leq C(1 + |z|^l + |\tilde{z}|^l)|z - \tilde{z}|^\alpha, \]
for some constants $C > 0, l \in \mathbb{N}$, and $\alpha \in (0, 1]$. We have
\[ u(t, Z_t) = \tilde{E}_t[\varphi(Z_T) + \int_t^T f(Z_s)ds], \]
where $u \in C([0, T] \times \mathbb{R}^d)$ with polynomial growth is the unique viscosity solution of the problem
\[ \partial_t u + \overline{G}(z, D_y u, D_x^2 u) + f(x, y) = 0, \quad t \in [0, T), \quad z = (x, y) \in \mathbb{R}^d, \]
\[ u(T, x, y) = \varphi(x, y). \]

Proof. For a fixed $\bar{t} \in [0, T)$, we set $t^n_i = i(T - \bar{t})/n$, for $i = 0, 1, \ldots, n$, and $f_n(s, \omega) = \sum_{i=0}^{n-1} f(\omega(t^n_i))1_{[t^n_i, t^n_{i+1})}(s)$, then denote
\[ u^n_i(t, Z_{t_i}; Z) := \tilde{E}_t[\varphi(Z_T) + \int_t^T f_n(s, Z.)ds], \quad t \in [t^n_i, t^n_{i+1}). \]

According to the definition of the conditional expectation $\tilde{E}_t$, it is not hard to see that $u^n_i(t, z; \omega)$ solves the following PDEs parameterized by $\omega$:
\[
\partial_t u^n_i(t, z; \omega) + \overline{G}(z, D_y u^n_i, D_x^2 u^n_i) + f(\omega(t^n_i)) = 0, \quad t \in [t^n_i, t^n_{i+1}), \quad z \in \mathbb{R}^d, \\
u^n_i(t^n_{i+1}, z; \omega) = u^n_{i+1}(t^n_{i+1}, z; \omega),
\]
for $i = n - 1, n - 2, \ldots, 1, 0$. The terminal condition for $u^n_k$, $k = n - 1$, is at $t^n_n = T$, $u^n_{n-1}(t^n_n, z; \omega) = \varphi(z)$. By the comparison theorem of PDE, backwardly and successively, we can check that
\[ u^n_i(t, z; \omega) - u(t, z) \leq \hat{u}^n_i(t, z; \omega), \quad t \in [t^n_i, t^n_{i+1}), \quad z \in \mathbb{R}^d, \]
where $\hat{u}^n_i(t, z; \omega)$, $i = n - 1, n - 2, \ldots, 1, 0$, solves the PDEs:
\[
\partial_t \hat{u}^n_i(t, z; \omega) + G(D_z \hat{u}^n_i, D_x^2 \hat{u}^n_i) + f(\omega(t^n_i)) - f(z) = 0, \quad t \in [t^n_i, t^n_{i+1}), \quad z \in \mathbb{R}^d, \\
\hat{u}^n_i(t^n_{i+1}, z; \omega) = \hat{u}^n_{i+1}(t^n_{i+1}, z; \omega).
\]
Now, since $G(p, A)$ is a sublinear function which does not depend on $z$, we claim that
\[ |\hat{u}^n_0(\bar{t}, Z_{\bar{t}}; Z.)| \leq \mathcal{E}_\bar{t}[\int_\bar{t}^T |f(Z_s) - f_n(s, Z.)|ds] \to 0 \text{ in } L^1_{\mathbb{E}}(\Omega_T) \text{ as } n \to \infty, \]
from which we have
\[ |u(\bar{t}, Z_{\bar{t}}) - u^n_0(\bar{t}, Z_{\bar{t}}; Z.)| \leq |\hat{u}^n_0(\bar{t}, Z_{\bar{t}}; Z.)| \to 0 \text{ as } n \to \infty, \]
and thus
\[
u^n_0(\bar{t}, Z_{\bar{t}}; Z.) := \tilde{E}_\bar{t}[\varphi(Z_T) + \int_\bar{t}^T f_n(s, Z.)ds] \\
\to u(\bar{t}, Z_{\bar{t}}) = \tilde{E}_\bar{t}[\varphi(Z_T) + \int_\bar{t}^T f(s, Z_s)ds].
\]

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Now let us prove the claim. Set
\[ \Delta = \mathcal{E}_{t} \left[ \int_{t}^{T} |f(Z_s) - f_n(s, Z_s)| \, ds \right], \]
then
\[ \Delta \leq \mathcal{E}_{t} \left[ \int_{t}^{T} \sum_{i=0}^{n-1} C(1 + |Z_s|^l + |Z^n_t|^l)|Z_s - Z^n_t|^\alpha \mathbf{1}_{[t^n_i, t^n_{i+1})}(s) \, ds \right], \]
and
\[ \mathcal{E}[\Delta] \leq \mathcal{E} \left[ \int_{t}^{T} \sum_{i=0}^{n-1} C(1 + |Z_s|^l + |Z^n_t|^l)|Z_s - Z^n_t|^\alpha \mathbf{1}_{[t^n_i, t^n_{i+1})}(s) \, ds \right] \]
\[ \leq C \sum_{i=0}^{n-1} \int_{t^n_i}^{t^n_{i+1}} \mathcal{E}[(1 + |Z_s|^l + |Z^n_t|^l)|Z_s - Z^n_t|^\alpha] \, ds \]
\[ \leq C \sum_{i=0}^{n-1} \int_{t^n_i}^{t^n_{i+1}} \sqrt{\mathcal{E}[(1 + |Z_s|^l + |Z^n_t|^l)^2]|Z_s - Z^n_t|^{2\alpha}} \, ds \]
\[ \leq C \sum_{i=0}^{n-1} \int_{t^n_i}^{t^n_{i+1}} (s - t^n_i)^\alpha \, ds \]
\[ \leq C \sum_{i=0}^{n-1} (t^n_{i+1} - t^n_i)^{1+\alpha} \to 0, \text{ as } n \to \infty. \]
And a slight modification of the approach in Chapter V.3 of [P10a] yields the inequality in the claim. \[ \square \]

**Step 2:** Now, we prove that, for each fixed \((\eta, q) \in L_{ip}(\Omega_s, \mathbb{S}^d \times \mathbb{R}^d)\), we have the relation, for \(0 \leq s < t \leq T\),
\[ \frac{1}{2} \langle \eta X_s, X_s \rangle + \langle q, y_s \rangle = \mathcal{E}_{s} \left[ \frac{1}{2} \langle \eta X_t, X_t \rangle + \langle q, y_t \rangle - \int_{s}^{t} \tilde{G}(X_\theta, y_\theta, q, \eta) \, d\theta \right] \] (14)
or
\[ \mathcal{E}_{s} \left[ \frac{1}{2} \text{tr}[\eta(\langle X \rangle_t - \langle X \rangle_s)] + \langle q, y_t - y_s \rangle - \int_{s}^{t} \tilde{G}(X_\theta, y_\theta, q, \eta) \, d\theta \right] = 0. \] (15)

**Proof of (15).** We can fix \((\eta, q)\) as constants. According to Lemma 12, the right hand side of (14) equals \(u(s, X_s, y_s)\), where \(u(t, x, y)\) is the viscosity solution of the following PDE:
\[ \partial_t u + \tilde{G}(x, y, D_y u, D^2_x u) - \tilde{G}(x, y, q, \eta) = 0, \quad t \in [t_1, t_2], \quad x, y \in \mathbb{R}^d, \]
\[ u(t_2, x, y) = \frac{1}{2} \langle q x, x \rangle + \langle q, y \rangle. \]
But it is easy to check that \(u(t, x, y) \equiv \frac{1}{2} \langle q x, x \rangle + \langle q, y \rangle\) is the unique solution of this PDE, from which we prove the first relation (14). For relation (15), we just need to move the terms of the right hand side to the left, inside the \(\mathcal{E}_{s}\). Note by Itô's formula,
\[ \frac{1}{2} \langle \eta X_t, X_t \rangle - \frac{1}{2} \langle \eta X_s, X_s \rangle = \int_{s}^{t} \langle \eta X_\theta, dX_\theta \rangle + \frac{1}{2} \text{tr}[\eta(\langle X \rangle_t - \langle X \rangle_s)] \]
and, for each $\xi \in L^1(\Omega_T)$, we have $\tilde{E}_s [\int_s^t 2(\eta X_\theta, dX_\theta) + \xi] = \tilde{E}_s [\int_s^t 2(\eta X_\theta, dX_\theta) + \xi] = \tilde{E}_s [\xi]$, by Lemma 8.

**Step 3:** Now we are ready to finish proving Proposition 11.

By the domination inequality, the equality (15) can be extended to the case $(\zeta, \eta, q) \in L^2(\Omega_t, \mathbb{R} \times \mathbb{S}^d \times \mathbb{R}^d)$. Now for step processes:

$$\eta^K_t = \sum_{j=0}^{K-1} \eta_j 1_{[s_j, s_{j+1})}(t), \quad q^K_t = \sum_{j=0}^{K-1} q_j 1_{[s_j, s_{j+1})}(t), \quad s = s_1 < \cdots < s_K = T,$$

$$(\eta_j, q_j) \in L^2(\Omega_t, \mathbb{S}^d \times \mathbb{R}^d), \quad \zeta \in L^2(\Omega_t, \mathbb{R})$$

We can repeat the equality (15) to prove

$$\tilde{E}_s \left[ \zeta + \int_s^t \text{tr}[\eta^K_t d(X)_r] + \int_s^t \langle q^K_t, dy_r \rangle - \int_s^t \tilde{G}(t, \omega, q^K_t, 2\eta^K_t) dr \right] = \zeta, \quad s \leq t \leq T.$$

From the domination of $\tilde{E}$ by $E$, we then can prove, for $(\eta, q) \in M^2(0, T; \mathbb{S}^d \times \mathbb{R}^d)$, and

$$\tilde{E}_s \left[ \int_s^t \text{tr}[\eta_r d(X)_r] + \int_s^t \langle q_r, dy_r \rangle - \int_s^t \tilde{G}(s, q_r, 2\eta_r) dr \right] = 0,$$

from which the proof is complete.

As a corollary of Theorem 10, we have the following result.

**Corollary 13** *The martingale Theorem 10 holds when $\varphi$ is a polynomial.*

**Proof.** For each given polynomial $\varphi$, one can find a sequence of functions $\varphi_n \in C^0(\mathbb{R}^d \times \mathbb{R}^d)$ such that $|\varphi_n - \varphi| \downarrow 0, |D_y \varphi_n - D_y \varphi| \downarrow 0$, and $|D^2_x \varphi_n - D^2_x \varphi| \downarrow 0$, then we have

$$\tilde{E}_s [\varphi(Z_t) - \varphi(Z_s) - \int_s^t \tilde{G}(Z_\theta, D_y \varphi(Z_\theta), D^2_x \varphi(Z_\theta)) d\theta]$$

$$\leq \tilde{E}_s [\varphi(Z_t) - \varphi(Z_s) - \int_s^t \tilde{G}(Z_\theta, D_y \varphi(Z_\theta), D^2_x \varphi(Z_\theta)) d\theta]$$

$$- \tilde{E}_s [\varphi(Z_t) - \varphi_n(Z_t) - \int_s^t \tilde{G}(Z_\theta, D_y \varphi_n(Z_\theta), D^2_x \varphi_n(Z_\theta)) d\theta]$$

$$\leq \tilde{E}_s [(\varphi(Z_t) - \varphi_n(Z_t)) - (\varphi(Z_s) - \varphi_n(Z_s))]$$

$$- \int_s^t [\tilde{G}(Z_\theta, D_y \varphi(Z_\theta), D^2_x \varphi(Z_\theta)) - \tilde{G}(Z_\theta, D_y \varphi_n(Z_\theta), D^2_x \varphi_n(Z_\theta))] d\theta]$$

$$\leq \tilde{E}_s [(\varphi(Z_t) - \varphi_n(Z_t))] + \tilde{E}_s [(\varphi(Z_s) - \varphi_n(Z_s))]$$

$$+ \int_s^t \tilde{E}_s [(\varphi(Z_\theta, D_y \varphi(Z_\theta), D^2_x \varphi_n(Z_\theta)) - \tilde{G}(Z_\theta, D_y \varphi_n(Z_\theta), D^2_x \varphi_n(Z_\theta))] d\theta$$

$$\leq \tilde{E}_s [(\varphi(Z_t) - \varphi_n(Z_t))] + \tilde{E}_s [(\varphi(Z_s) - \varphi_n(Z_s))]$$

$$+ L \int_s^t \tilde{E}_s [(D_y \varphi(Z_\theta) - D_y \varphi_n(Z_\theta)) + (D^2_x \varphi(Z_\theta) - D^2_x \varphi_n(Z_\theta))] d\theta.$$
Letting $n \to \infty$, according to Lemma 9, we see

$$\tilde{E}_s[\varphi(Z_t) - \varphi(Z_s) - \int_s^t \tilde{G}(Z_\theta, D_y \varphi(Z_\theta), D^2_x \varphi(Z_\theta)) \, d\theta] = 0.$$ 

3 Weak solution of $G$-SDE

In this section, we will develop a notion of weak solution of general $G$-SDE, and show the existence of such weak solutions in comparison with the strong solutions within the existing $G$-framework. To this end, the $G$ in this section is from [P10a] which is a special case of the $G$ (with the form (5)) in class $\mathcal{D}$. We will rely on the analysis and results for the martingale problem in the previous section.

3.1 Weak solution of SDE in $G$-framework

Consider a $d$-dimensional $G$-SDE

$$\begin{cases}
    dz_t = b(z_t)dt + r(z_t)d\langle B \rangle_t + \sigma(z_t)dB_t, & 0 \leq t \leq T, \\
    z_0 = z, & z \in \mathbb{R}^d,
\end{cases}$$

(16)

where $b = (b^i)_{1 \leq i \leq d}$, $r = (r^i_{jk})_{1 \leq i,j,k \leq d}$, and $\sigma = (\sigma_{ij})_{1 \leq i,j \leq d}$, and $b^i$, $r^i_{jk}$, and $\sigma_{ij}$ are continuous functions on $\mathbb{R}^d$, and $\sigma \geq \sigma_0 I$ for some constant $\sigma_0 > 0$. Here $B$ is a $d$-dimensional generalized $\tilde{E}$-Brownian motion introduced in [P10a] (for additional background material, please consult Appendix).

**Definition 5 (Weak solution)** A weak solution of $G$-SDE (16) is a triple $((\Omega, \mathcal{H}, \tilde{E}), z, B)$, where

i) $(\Omega, \mathcal{H}, \tilde{E})$ is a nonlinear expectation space,

ii) $z$ is a $d$-dimensional continuous process on the nonlinear expectation space $(\Omega, \mathcal{H}, \tilde{E})$, and $B$ is a $d$-dimensional $\tilde{E}$-Brownian motion in the sense of [P10a],

iii) the identity

$$z_t = z_0 + \int_0^t b(z_\theta) \, d\theta + \int_0^t r(z_\theta) \, d\langle B \rangle_\theta + \int_0^t \sigma(z_\theta) \, dB_\theta$$

(17)

holds in the nonlinear expectation space.

3.2 Existence of weak solutions for $G$-SDE

We will establish the existence result for a random process $(z_t)_{t \geq 0}$ which is a weak solution of a $G$-SDE. For comparison with the existing $G$-framework, we assume that a random process $z_t$ in a nonlinear expectation space can be decomposed into two parts $X_t$ and $y_t$, with $X_t$ being a
symmetric martingale and \( y_t \) with finite variation. This is a generalization of the generalized \( G \)-Brownian motion \((X_t, y_t)\). For more about this martingale representation in a nonlinear expectation space, see for instance [PSZ12]. More specifically, we consider the following 1-dimensional \( G \)-SDE:

\[
\begin{aligned}
    dz^i_t &= b^i(z_t) \, dt + r^{ij}_k(z_t) \, d\langle B \rangle^{jk}_t + \sigma_{ij}(z_t) \, dB^j_t, \\
    z^i_0 &= z \in \mathbb{R}^d,
\end{aligned}
\]

where \( b : \mathbb{R}^d \to \mathbb{R}^d, \quad r : \mathbb{R}^d \to L(\mathbb{R}^{d \times d}; \mathbb{R}^d), \) and \( \sigma : \mathbb{R}^d \to L(\mathbb{R}^d; \mathbb{R}^d) \) are bounded and continuous functions such that the inversed matrix \( \sigma^{-1}(z) \) is also bounded, and they satisfy the Hölder continuity condition

\[ |b(z) - b(\tilde{z})| + |r(z) - r(\tilde{z})| + |\sigma(z) - \sigma(\tilde{z})| \leq L_0 |z - \tilde{z}|^\alpha. \]

We also assume that there exists a sublinear monotone function \( \bar{G} : \mathbb{S}^d \to \mathbb{R} \), satisfying

\[ \bar{G}(A) \geq \lambda \text{tr}[A], \quad \forall A \in \mathbb{S}^d \quad (\lambda > 0), \]

define

\[ \tilde{G}(x, y, p, A) = \bar{G}\left( [2r^k_{ij}(x + y)p_k + \sigma_{i'j'}(x + y)A^{i'j'}]_{i',j'=1} \right) + b^i(x + y)p_i, \]

here we use Einstein convention, namely the repeated indices \( i, j \) implies taking sum from 1 to \( d \). One can see that \( \tilde{G} \) satisfies the continuity condition (10).

**Lemma 14** For the case

\[ \tilde{G}(x, y, p, A) = \bar{G}\left( [2r^k_{ij}(x + y)p_k + \sigma_{i'j'}(x + y)A^{i'j'}]_{i',j'=1} \right) + b^i(x + y)p_i, \]

we denote \( z_t = X_t + y_t \), where \((X, y) = \omega(\cdot) \in \Omega\) is the canonical process. Then, for each \( p \in M^2_E(0,T;\mathbb{R}^d) \) and \( \eta \in M^1_E(0,T;\mathbb{S}^d) \), the process

\[
N_t^{p,\eta} = \int_0^t p_s^T d\zeta_s + \int_0^t \frac{1}{2} \text{tr}[\eta_s d(\zeta)_s] - \int_0^t \tilde{G}(2r(z_s)p_s + \sigma^T(\zeta_s)\eta_s\sigma(z_s)) + p_s^T b(z_s)) ds
\]

is a martingale under \( \tilde{E} \).

**Proof.** Since, for each \( \xi, p, \) and \( \eta \)

\[
\int_0^t \zeta_s^T \, dX_s + \int_0^t p_s^T \, dy_s + \int_0^t \frac{1}{2} \text{tr}[\eta_s d(\zeta)_s] - \int_0^t \tilde{G}(X_s, y_s, p_s, \eta_s) \, ds
\]

is a martingale under \( \tilde{E} \) and \((X)_s \equiv (\zeta)_s\), by taking \( \zeta_s \equiv p_s \) we obtain that for each \( p \in M^2_E(0,T;\mathbb{R}^d) \) and \( \eta \in M^1_E(0,T;\mathbb{S}^d) \), \( N_t^{p,\eta} \) is an \( \tilde{E} \)-martingale.

**Theorem 15** Under the nonlinear expectation \( \tilde{E} \) derived from the PDE

\[ \partial_t u(t, x, y) - \tilde{G}(x, y, Du, D^2 u) = 0, \quad (t, x, y) \in (0,T] \times \mathbb{R}^{2d}, \]

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together with the canonical space $\Omega = C_{x,y}([0,T];\mathbb{R}^{2d})$, the linear space of $d$-dimensional random variables $\mathcal{H}$, the process $z = X + y$ for $(X, y) \in \Omega$, and

$$B_i^t = \int_0^t \sigma_{ij}^{-1}(z_s)dz_s^j - \int_0^t \sigma_{ij}^{-1}(z_s)b^j(z_s)ds - \int_0^t \sigma_{ik}^{-1}(z_s)r^k_{jl}(z_s)d\langle B \rangle_{s}^{jl},$$

$$i = 1, \ldots, d, \quad 0 \leq t \leq T,$$

is a weak solution of the $G$-SDE (18).

Several steps are needed for proving the theorem. **Step 1.** For the canonical process $(z_s)_{s \geq 0}$, we construct the following Itô process:

$$B_i^t = \int_0^t \sigma_{ij}^{-1}(z_s)dz_s^j - \int_0^t \sigma_{ij}^{-1}(z_s)b^j(z_s)ds - \int_0^t \sigma_{ik}^{-1}(z_s)r^k_{jl}(z_s)d\langle B \rangle_{s}^{jl}. \tag{20}$$

By Proposition 21, the quadratic variation process of this $G$-Itô process $B$ is given by

$$\langle B \rangle_{s}^{ii'} = \int_0^t \sigma_{ij}^{-1}(z_s)\sigma_{i'j'}^{-1}(z_s)d\langle z \rangle_{s}^{jj'}. \tag{21}$$

Thus it is clear that

$$dz_s = b(z_s)ds + \sigma(z_s)dB_s + r(z_s)d\langle B \rangle_s, \quad z_0 = z(= x + y).$$

It remains to prove that, under $\tilde{\mathcal{E}}$, $B$ is a $d$-dimensional $\tilde{G}$-Brownian motion. From (21), we rewrite (20) as

$$B_i^t = \int_0^t \sigma_{ij}^{-1}(z_s)dz_s^j - \int_0^t \sigma_{ij}^{-1}(z_s)b^j(z_s)ds \tag{22}$$

$$- \sum_{i,i'} \int_0^t \sigma_{ik}^{-1}(z_s)r^k_{ii'}(z_s)\sigma_{i'j'}^{-1}(z_s)d\langle z \rangle_{s}^{jj'}.$$

We need to prove that the $G$-Itô process defined by

$$M_t := \sum_k \int_0^t \zeta_s^k dB_s^k + \frac{1}{2} \sum_{i,i'} \int_0^t \eta_{i'i'}^s d\langle B \rangle_{s}^{ii'} - \int_0^t \tilde{G}(\eta_s)ds \tag{23}$$

is an $\tilde{\mathcal{E}}$-martingale. Indeed, we have

$$M_t = \int_0^t \zeta_s^i \sigma_{ij}^{-1}(z_s)dz_s^j$$

$$+ \int_0^t \left[ \frac{1}{2} \eta_{ii'}^s \sigma_{ij}^{-1}(z_s)\sigma_{i'j'}^{-1}(z_s) - \sum_{i,i'} \zeta_s^i \sigma_{ik}^{-1}(z_s)r^k_{ii'}(z_s)\sigma_{i'j'}^{-1}(z_s) \right] d\langle z \rangle_{s}^{jj'}$$

$$- \int_0^t [\tilde{G}(\eta_s) + \zeta_s^i \sigma_{ij}^{-1}(z_s)b^j(z_s)]ds = N_t^{\tilde{\mathcal{E}}}.$$
where we set
\[
(\bar{p}_s)_j = \zeta^1_i \sigma^1_{ij}(z_s), \quad \text{and}
(\bar{\eta}_s)_{jj'} = \sigma^1_{ij}(z_s) \eta^{ij'}(z_s) \sigma^1_{jj'}(z_s) - 2 \sum_{k,i'} \zeta^1_k \sigma^1_{ik}(z_s) \eta^{i'k}(z_s) \sigma^1_{jj'}(z_s).
\]

This, with Lemma 14, shows that \(\{M_t\}_{0 \leq t \leq T}\) defined in (23) is an \(\bar{E}\)-martingale.

**Step 2.** The proof can be completed by applying the following proposition.

**Proposition 16** Let \((B_t)_{t \geq 0}\) be a \(d\)-dimensional \(G\)-Itô process defined on sublinear expectation space \((\Omega, L^1(\Omega), \mathcal{E})\) and let \(\bar{G}\) be dominated by \(G\) such that for each bounded \(\zeta \in M^2_{\bar{E}}(0, T; \mathbb{R}^d)\) and \(\eta \in M^2_{\bar{E}}(0, T; \mathbb{R}^d)\),
\[
\int_0^t \zeta^T_s dB_s + \frac{1}{2} \int_0^t \text{tr}[\eta_0 d(B)_s] - \int_0^t \bar{G}(\eta_s) ds,
\]
is an \(\bar{E}\)-martingale, where \(\bar{G}\) is given as before. Then \(B\) is a \(\bar{G}\)-Brownian motion under \(\bar{E}\).

**Corollary 17** We assume the same condition for \(B_t\) and \(\bar{G}\) as given in Proposition 16. If for each \(\varphi \in C^2_b(\mathbb{R}^d)\),
\[
\int_0^t \langle D_\varphi(B_s), dB_s \rangle + \frac{1}{2} \int_0^t \text{tr}[D^2_\varphi(B_s) d(B)_s] - \int_0^t \bar{G}(D^2_\varphi(B_s)) ds,
\]
is an \(\bar{E}\)-martingale, then \(B\) is also a \(\bar{G}\)-Brownian motion under \(\bar{E}\).

**Corollary 18** We set \(d = 1\) and assume the same condition for \(B\) as in Proposition 16. If there exist two constants \(\bar{\sigma} > \sigma > 0\), such that

(1). \(B\) is a symmetric \(\bar{E}\)-martingale,
(2). the process \(\{B^2_t - \bar{\sigma}^2 t\}\) is an \(\bar{E}\)-martingale,
(3). the process \(\{\sigma^2 t - B^2_t\}\) is an \(\bar{E}\)-martingale.

Then \(B\) is a \(\bar{G}\)-Brownian motion under \(\bar{E}\), where \(\bar{G}\) is a sublinear monotone function of the form
\[
\bar{G}(a) := \frac{1}{2}(\bar{\sigma}^2 a^+ - \sigma^2 a^-), \quad a \in \mathbb{R}.
\]

**Proof.** For simple processes \(\zeta, \eta \in M^2_{\bar{E}}(0, T)\) of the form
\[
\zeta_t = \sum_{i=0}^{n-1} \zeta^i_{1[t_i, t_{i+1})}(t), \quad \eta_t = \sum_{i=0}^{n-1} \eta^i_{1[t_i, t_{i+1})}(t), \quad \zeta^i, \eta^i \in L_{ip}(\Omega_{t_i}),
\]
we can check that the process defined by \(M_t := \int_0^t \zeta_s dB_s + \frac{1}{2} \int_0^t \eta_s d(B)_s - \int_0^t \bar{G}(\eta_s) ds\) is an \(\bar{E}\)-martingale. It is also easy to extend this property to the case of bounded \(\zeta, \eta \in M^2_{\bar{E}}(0, T)\). Thus Proposition 16 can be applied.
Proof of Proposition 16. For each \( \varphi(x, \bar{x}) \in C^2_0(\mathbb{R}^d) \), we solve the following PDE, parameterized by \( \bar{x} \in \mathbb{R}^d \),

\[
\partial_t u_\varepsilon(t, x; \bar{x}) + G(D_x^2 u_\varepsilon(t, x; \bar{x})) = 0,
\]

defined on \( t \in [0, T + \varepsilon) \times \mathbb{R}^d \) with terminal condition \( u_\varepsilon(T + \varepsilon, x; \bar{x}) = \varphi(x, \bar{x}) \). Since \( G \) is convex and \( G(A) \geq \lambda \text{tr}[A] \), by Krylov [K87], the internal regularity

\[
\|u_\varepsilon\|_{C^{1+\alpha/2,2+\alpha}([0,T] \times \mathbb{R}^d)} < \infty.
\]

We then can apply G-Itô’s formula to get

\[
M^\varepsilon_t := u_\varepsilon(t, B_t^0; t) - u_\varepsilon(t, 0; B_t)
\]

\[
= \int_t^T \partial_t u_\varepsilon(s, B_s^t; B_t) \, ds + \int_t^T \langle D_x u_\varepsilon(s, B_s^t; B_t), dB_s \rangle + \frac{1}{2} \int_t^T \text{tr}[D_x^2 u_\varepsilon(s, B_s^t; B_t) \, dB_s]^2 \, ds
\]

\[
= \int_t^T \langle D_x u_\varepsilon(s, B_s^t; B_t), dB_s \rangle + \frac{1}{2} \int_t^T \text{tr}[D_x^2 u_\varepsilon(s, B_s^t; B_t) \, dB_s]^2 \, ds
\]

\[
- \int_t^T G(D_x^2 u_\varepsilon(s, B_s^t; B_t)) \, ds,
\]

where \( B_s^t = B_s - B_t, t \leq s \leq T \).

But, as a condition of Proposition 16, \( M^\varepsilon_t \) is an \( \tilde{E} \)-martingale. It then follows that

\[
u_\varepsilon(t, 0; B_t) = \tilde{E}_t[u_\varepsilon(T, B_T - B_t; B_t)].
\]

Let \( u \) be the viscosity solution of the same PDE (24) defined on \( [0, T) \times \mathbb{R}^d \) with terminal value \( u(T, x; \bar{x}) = \varphi(x, \bar{x}) \). By using the stability of viscosity solution (Lemma II.6.2 of [FS92]) and the internal regularity of \( u \), letting \( \varepsilon \to 0 \) in the above identity, we obtain \( u(t, 0; B_t) = \tilde{E}_t[u(T, B_T - B_t; B_t)] = \tilde{E}_t[\varphi(B_T - B_t; B_t)] \). It follows that

\[
\tilde{E}_t[\varphi(B_T - B_t; B_t)] = \tilde{E}_t[\varphi(\sqrt{T - t} \xi, \bar{x})]_{\xi = B_t},
\]

where \( \xi \) is a \( G \)-normal distributed random variable. It follows that \( B_T - B_t \overset{d}{=} \sqrt{T - t} \xi \) and \( B_T - B_t \) is independent of \( B_t \). In fact, we can applying the above method to the case \( \varphi = \varphi(B_T - B_t, B_{t_1}, \ldots, B_{t_N}) \), for \( t_1 \leq \cdots \leq t_N \leq t \), to prove that, for \( \varphi(x_1, \cdots, x_N, x) \in C^2_0(\mathbb{R}^{d \times (N + 1)}) \), we have

\[
\tilde{E}[\varphi(B_{t_1}, \cdots, B_{t_N}, B_T - B_t)] = \tilde{E}[\tilde{E}[\varphi(x_1, \cdots, x_N, B_T - B_t)|x_1 = B_{t_1}, \cdots, x_N = B_{t_N}]]
\]

\[
= \tilde{E}[\tilde{E}[\varphi(x_1, \cdots, x_N, \sqrt{T - t} \xi)|x_1 = B_{t_1}, \cdots, x_N = B_{t_N}]]
\]

This implies that \( B_T - B_t \) is also independent of \( B_{t_1}, \cdots, B_{t_N} \). It then follows that \( (B_t)_t \geq 0 \) is a \( G \)-Brownian motion. The proof is complete.

Remark 5 The method to establish the existence of weak solutions of SDE (16) is by and large a generalization of the classical Girsanov transformation for change of measures. However, the Girsanov transformation is limited to the transform of two measures that are absolutely continuous, and even a small change of the diffusion coefficient may cause the singularity between two measures. In this regard, our method is new and the key is to have a sublinear expectation of \( \mathcal{E} \) that dominates a class of probability measures singular from each other.
4 Appendix

4.1 Related stochastic calculus under nonlinear expectations

We first recall some notions under $G$-framework mainly from [P10a]. We then develop some new results under general sublinear expectations.

4.1.1 Review: Itô’s integral with $G$-Brownian motion in $L^2_G(\Omega)$

We briefly present some useful results of stochastic calculus under $G$-expectation. Recall that, since $E$ is a sublinear expectation defined on $(\Omega, L_{lip}(\Omega_T))$, thus, for each $p \geq 1$, $T > 0$, we can define a Banach norm

$$\|\xi\|_{L^p_E} = (E[|\xi|^p])^{1/p}, \quad \xi \in L_{lip}(\Omega_T).$$

The completion of $L_{lip}(\Omega_T)$ under this norm is denoted by $L^p_E(\Omega_T)$. Both $E$-expectation and $\tilde{E}$-expectation, as well as their conditional expectations $E_t, \tilde{E}_t$ are extended in $L^1_E(\Omega_T), T \geq 0$ and the properties obtained in Proposition 7 still hold true for $L^p_E(\Omega_T)$ in the place of $L_{lip}(\Omega_T)$. Moreover, it is proved in [DHP11] that there exists a weakly compact subset $P_G$ of probability measures on the Borel measurable space $(\Omega, \mathcal{B}(\Omega))$ such that

$$E[\xi] = \sup_{P \in P_G} \int_{\Omega} \xi(\omega) dP, \quad \xi \in L^1_E(\Omega_T),$$

and, in fact $L^p_E(\Omega_T)$ belongs to the space of $\mathcal{B}(\Omega)$-measurable functions

$$\sup_{P \in P_G} \int_{\Omega} |\xi(\omega)|^p dP < \infty. \quad (25)$$

The usual language of $P$-almost surely is replaced by $c_G$-quasi surely with

$$c_G(A) := \sup_{P \in P_G} P(A), \quad A \in \mathcal{B}(\Omega).$$

In fact $\xi \in L^p_E(\Omega)$ iff $\xi \in L^0(\Omega)$ has $c_G$-quasi continuous modification such that (25) and

$$\lim_{N \to \infty} \sup_{P \in P_G} \int_{\Omega} |\xi(\omega)|^p 1_{\{|\xi| > N\}} dP = 0.$$

We give the definition of $\tilde{G}$-Brownian motion here.

**Definition 6 (\(\tilde{G}\)-Brownian Motion)** A $d$-dimensional process $(B_t)_{t \geq 0}$ defined on a sublinear expectation space $(\Omega, \mathcal{H}, \mathcal{E})$ is called a $\tilde{G}$-Brownian motion under a given nonlinear expectation $\mathcal{E}$ dominated by $\mathcal{E}$, if the following conditions are satisfied:

(i). $B_0(\omega) = 0$.

(ii). For each $t, s \geq 0$, $B_{t+s} - B_t$ and $B_s$ are identically distributed and $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$. 

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Indeed, for each $\phi$ where $\Gamma_1$ is a convex subset of $\mathbb{R}^d$ such that $\max_{x \in \Gamma_1} \langle p, x \rangle = g_0(p) := G(p, 0)$, for all $p \in \mathbb{R}^d$. A typical situation of such kind of Brownian motion is the quadratic variation process $\langle X \rangle_t$ of the above symmetric Brownian motion.

Definition 7 For $T \in [0, \infty)$, a partition $\pi_T$ of $[0, T]$ is a finite ordered subset $\pi_T = \{t_0, t_1, \ldots, t_N \}$ such that $0 = t_0 < t_1 < \cdots < t_N = T$,

$$\mu(\pi_T) := \max\{|t_{i+1} - t_i|; i = 0, 1, \ldots, N - 1\}.$$ 

We use $\pi_T^N = \{t_0^N, t_1^N, \ldots, t_N^N \}$ to denote a sequence of partitions of $[0, T]$ such that $\lim_{N \to \infty} \mu(\pi_T^N) = 0$.

Let $p \geq 1$ be fixed. We consider the following type of simple processes: for a given partition $\pi_T = \{t_0, \ldots, t_N \}$ of $[0, T]$ we set

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) 1_{[t_k, t_{k+1})}(t),$$

where $\xi_k \in L^p(\Omega, \mathcal{F}, \mathbb{P})$, $k = 0, 1, 2, \ldots, N - 1$ are given. The collection of these processes is denoted by $M^{p,0}_\mathcal{E}(0, T)$.

Definition 8 For an $\eta \in M^{p,0}_\mathcal{E}(0, T)$ with $\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) 1_{[t_k, t_{k+1})}(t)$, the related integrals are

$$\int_0^T \eta_t(\omega) dt := \sum_{k=0}^{N-1} \xi_k(\omega)(t_{k+1} - t_k),$$

$$\int_0^T \eta_t(\omega) dX_t := \sum_{k=0}^{N-1} \xi_k(\omega)(X_{t_{k+1}} - X_{t_k}),$$

$$\int_0^T \eta_t(\omega) dy_t := \sum_{k=0}^{N-1} \xi_k(\omega)(y_{t_{k+1}} - y_{t_k}).$$

Definition 9 For each $p \geq 1$, we denote by $M^{p}_\mathcal{E}(0, T)$ the completion of $M^{p,0}_\mathcal{E}(0, T)$ under the norm

$$\|\eta\|_{M^p_\mathcal{E}(0, T)} := \left\{ \mathcal{E}\left[ \int_0^T |\eta_t|^p dt \right] \right\}^{1/p}.$$ 

It is clear that $M^p_\mathcal{E}(0, T) \supset M^q_\mathcal{E}(0, T)$ for $1 \leq p \leq q$. We also use $M^p_\mathcal{E}(0, T; \mathbb{R}^d)$ for all $d$-dimensional stochastic processes $\eta_t = (\eta^1_t, \ldots, \eta^d_t)^T$, $t \geq 0$ with $\eta^i \in M^p_\mathcal{E}(0, T)$, $i = 1, 2, \ldots, d$. 

(iii). $\lim_{t \to 0} \mathcal{E}[|B_t|^3] t^{-1} = 0$.

$B$ is called a symmetric Brownian motion if $\mathcal{E}[B_t] = \mathcal{E}[-B_t] = 0$. In the finite dimensional case nonlinear distribution of $B$ is fully determined by the function: $G(A) = \frac{1}{2}\mathcal{E}[(AB_1, B_1)]$, defined on $\mathcal{S}^d$. We often call it a $\tilde{G}$-Brownian motion.

In fact in the main text of the paper we have introduced 2$d$-dimensional stochastic process $(X_t - X_0, y_t - y_0)$. They are Brownian motion under $\mathcal{E}$, but in general not under $\tilde{\mathcal{E}}$. Furthermore, under $\mathcal{E}$, $X_t$ is a symmetric $G_0$-Brownian motion with $G_0(A) = G(0, A)$, while $y_t$ is not symmetric. Indeed, for each $\varphi \in C_b(\mathbb{R}^d)$

$$\mathcal{E}[\varphi(y_t)] = \max_{v \in \Gamma_1} \varphi(v t),$$

where $\Gamma_1$ is a convex subset of $\mathbb{R}^d$ such that $\max_{p \in \Gamma_1} \langle p, v \rangle = g_0(p) := G(p, 0)$, for all $p \in \mathbb{R}^d$. A typical situation of such kind of Brownian motion is the quadratic variation process $\langle X \rangle_t$ of the above symmetric Brownian motion.
4.1.2 \textit{G-Itô’s calculus}

In the above space, it is easy to check that, for fixed $p_i \in \mathbb{R}^d$, $i = 1, \ldots, n$, the $n$-dimensional process defined by $(\langle Z_t(\omega), p_1 \rangle, \cdots, \langle Z_t(\omega), p_n \rangle)_{t \geq 0}$ is also a Brownian motion, particularly $X_t + y_t$ is a $G$-Brownian motion under $\mathcal{E}$.

Therefore, the process $z_t(\omega) = X_t(\omega) + y_t(\omega) = \omega(t)$, $\omega \in \Omega = C([0, \infty), \mathbb{R}^d)$ is a $G$-Brownian motion under $\mathcal{E}$, namely $(z_t)_{t \geq 0}$ is $\alpha_G$-quasi surely continuous process such that the nonlinear distribution of $z_t - z_s$ is that of $z_{t-s}$, and $z_t - z_s$ is independent from $(z_{s1}, \ldots, z_{sN})$ for each $t \geq s \geq t_{si}$, $i = 1, 2, \ldots, N$.

It is worth noticing that $(z_t)_{t \geq 0}$ is also a nonlinear diffusion process under $\mathcal{E}$: for each $t_i \geq t \geq 0$, $i = 1, \ldots, N$, $\mathcal{E}_t[\varphi(z_{t1}, \cdots, z_{tn})]$ depends only on $z_t$. Such nonlinear Markovian property plays an important role in this paper.

\textbf{Proposition 19} If the function $\tilde{G}$ is of the form $\tilde{G}(x + y, p, A)$, then $z_t = X_t + y_t$ still satisfies a martingale problem with nonlinear expectation derived from the PDE

$$
\partial_t u(t, z) + \tilde{G}(z, D_z u(t, z), D_z^2 u(t, z)) = 0 \quad (t, x) \in (0, \infty) \times \mathbb{R}^d.
$$

\textbf{Proof.} In this case the solution of the PDE

$$
\partial_t u(t, x, y) - \tilde{G}(x + y, D_y u(t, x, y), D_y^2 u(t, x, y)) = 0, \quad u(0, x, y) = \varphi(x + y)
$$

coincides with $\tilde{u}(t, x + y)$, where $\tilde{u}(t, z)$ is the solution to the PDE

$$
\partial_t \tilde{u}(t, z) - \tilde{G}(z, D_z \tilde{u}(t, z), D_z^2 \tilde{u}(t, z)) = 0, \quad \tilde{u}(0, z) = \varphi(z),
$$

$(t, z) \in (0, \infty) \times \mathbb{R}^d$.

Notice that nonlinear expectation $\mathcal{E}$ is dominated by the sublinear expectation $\mathcal{E}$, the nonlinear expectation $\mathcal{E}$ can still be defined on the Banach space $L_p^\alpha$, $p \geq 1$. We give Itô’s formula for a “$G$-Itô process”. For simplicity, we first consider the case of the function $\Phi$ being sufficiently regular and consider the general $n$-dimensional $G$-Itô’s process

$$
\xi_t = \xi_0 + \int_0^t \alpha_s ds + \int_0^t \beta_s dX_s + \int_0^t \eta_s d\langle X \rangle_s + \int_0^t \kappa_s dy_s,
$$

where $\xi_0 \in \mathbb{R}^n, \alpha_s \in \mathbb{R}^n, \beta_s, \kappa_s \in L(\mathbb{R}^d; \mathbb{R}^n)$, and $\eta_s \in L(\mathbb{R}^{d \times d}; \mathbb{R}^n)$.

\textbf{Theorem 20 (Itô’s formula)} Let $\Phi$ be a $C^2$-function on $\mathbb{R}^n$ such that $\partial_{x^{\nu}x^{\nu}} \Phi$ satisfies the polynomial growth condition for $\mu, \nu = 1, \cdots, n$. Let $\alpha^{\nu}, \beta^{\mu\nu}$, and $\eta^{\nu}_{ij}, \nu = 1, \ldots, n, i, j = 1, \ldots, d$ be bounded processes in $M_2^2(0, T)$. Then for each $t \geq s \geq 0$ we have in $L_2^\alpha(\Omega_t)$

$$
\Phi(\xi_t) - \Phi(\xi_s) = \int_s^t \partial_{x^{\nu}} \Phi(\xi_t)d\xi^{\nu}_t + \frac{1}{2} \sum_{i,j} \int_s^t \partial_{x^{\nu}x^{\nu}} \Phi(\xi_t) \beta^{\nu\mu}_i \beta^{\nu\mu}_j d\langle X \rangle^{ij}_t
$$

$$
= \int_s^t \langle D_\gamma \Phi(\xi_t), d\xi_\gamma \rangle + \frac{1}{2} \int_s^t \text{tr}[\beta^T D_2^2 \Phi(\xi_t) \beta d\langle X \rangle_\theta],
$$

where $\langle X \rangle^{ij} = \langle X^i, X^j \rangle$. 

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We have the following lemma.

**Proposition 21**

\[
\langle \xi_t \rangle_1 \triangleq \left( \langle \xi^i_t \rangle \right) = \left( \int_0^t \sum_{\mu, \nu} \beta^\mu_s \beta^\nu_s \, d\langle X^\mu_s, X^\nu_s \rangle_s \right) = \int_0^t \beta_s \, d\langle X \rangle_s \beta_s^T.
\]

**Proof.** Let \( \pi^N_t, N \in \mathbb{N} \), be a sequence of partitions of \([0, t]\). We denote \( \xi^{(N)}_t = \sum_{j=0}^{N-1} \xi^N_{t_j} 1_{[t_j, t_{j+1})}(t) \), then

\[
\xi_t \xi^T_t - \xi_0 \xi^T_0 = \sum_{j=0}^{N-1} (\xi^N_{t_j+1} - \xi^N_{t_j}) (\xi^N_{t_j+1} - \xi^N_{t_j})^T
\]

\[
= \int_0^t \left[ (\xi^{(N)}_t)^T \, d\xi^T_t + d\xi_t (\xi^{(N)}_t)^T \right] + \sum_{j=0}^{N-1} (\xi^N_{t_{j+1}} - \xi^N_{t_j}) (\xi^N_{t_{j+1}} - \xi^N_{t_j})^T.
\]

As \( \mu(\pi^N_t) \to 0 \), the first term on the right hand side converges to \( \int_0^t [\xi_s \, d\xi^T_s + d\xi_s \, \xi^T_s] \) in \( L^2_\xi(\Omega_t) \), the second one must be convergent, and we denote its limit by \( \langle \xi_t \rangle \),

\[
\langle \xi_t \rangle = \xi_t \xi^T_t - \xi_0 \xi^T_0 - \int_0^t [\xi_s \, d\xi^T_s + d\xi_s \, \xi^T_s]
\]

But by Theorem 20

\[
\xi_t \xi^T_t - \xi_0 \xi^T_0 = \int_0^t (\xi^i_t \, d\xi^j_s + d\xi^i_s \, \xi^j_0) + \sum_{\mu, \nu} \int_0^t \beta^\mu_s \beta^\nu_s \, d\langle X \rangle^\mu_s^\nu_s, \quad 1 \leq i, j \leq n.
\]

\[\square\]

### 4.2 Proof of Theorems 3 and 5

**Proof of Theorem 3.** The proof is based on a key lemma.

**Lemma 22** Suppose that each continuous function \( G_i : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R}, i = 0, 1, \ldots, k \) satisfies

\[
\lambda G_i(t, x, v, p, A) = G_i^{(1)}(t, x, v, \lambda p, \lambda A) + \lambda G_i^{(2)}(t, x, v, p),
\]

for all \( (t, x, v, p, A) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d, \lambda \geq 0, i = 0, 1, \ldots, k, \) where its respective decomposition functions \( G_i^{(1)} \) and \( G_i^{(2)} \) being continuous and satisfying the following condition

\[\text{(G). (Condition (G))}\]

\[
|G(t, x, v, p, A) - G(t, y, v, p, A)| \leq \rho_G(1 + (T - t)^{-1} + |x| + |y| + |v|)\rho_G(|x - y| + |p| \cdot |x - y|),
\]

for each \( t \in [0, \infty), v \in \mathbb{R}, x, y, p \in \mathbb{R}^d, \) and \( A \in \mathbb{S}^d \), where \( \rho_G, \bar{\rho}_G : [0, \infty) \to [0, \infty) \) are continuous functions that satisfy \( \rho_G(0) = 0, \bar{\rho}_G(0) = 0. \)
Moreover, assume that

\[ \sum_{i=1}^{k} G_i(t, x, v_i, p_i, A_i) \leq G_0(t, x, \sum_{i=1}^{k} v_i, \sum_{i=1}^{k} p_i, \sum_{i=1}^{k} A_i), \]

for each \((t, x, v_i, p_i, A_i) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d\), and that

\[ G_0(t, x, v, p, A) \leq G_0(t, x, v, p, A + \bar{A}), \]

\[ |G_0(t, x, v, p, A) - G_0(t, x, v, p, \bar{A})| \leq L_0(|u - v| + |A - \bar{A}|), \]

where \((t, x, p) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, u, v \in \mathbb{R}, A, \bar{A} \in \mathbb{S}^d\) with \(\bar{A} \geq 0\), and \(L_0 > 0\) is a constant. For each \(i = 1, \ldots, k\), let \(u_i \in \text{USC}([0, T] \times \mathbb{R}^d)\) be a viscosity subsolution of PDE

\[ \partial_t u(t, x) - G_i(t, x, u(t, x), Du(t, x), D^2 u(t, x)) = 0, (t, x) \in (0, T) \times \mathbb{R}^d, \]

and let \(u_0 \in \text{LSC}([0, T] \times \mathbb{R}^d)\) be a viscosity supersolution of PDE

\[ \partial_t u(t, x) - G_0(t, x, u(t, x), Du(t, x), D^2 u(t, x)) = 0, (t, x) \in (0, T) \times \mathbb{R}^d, \]

such that each \(u_i, i = 0, 1, \ldots, k\) is with polynomial growth. Then \(\sum_{i=1}^{k} u_i \leq u_0\) on \((0, T) \times \mathbb{R}^d\) provided

\[ \sum_{i=1}^{k} u_i(0, x) \leq u_0(0, x), x \in \mathbb{R}^d. \]

Note that the above lemma corrects Theorem C.2.3 of [P10a] where a condition of type (27) was missing. Now, take \(G_0 = G, G_1 = \bar{G}\), and define \(G_2(t, x, p, A) = -\bar{G}(t, x, -p, -A)\). Since \(\bar{G}\) satisfies condition (DOM), we have

\[
G_1(t, x, p_1, A_1) + G_2(t, x, p_2, A_2) = \bar{G}(t, x, p_1, A_1) - \bar{G}(t, x, -p_2, -A_2) \\
\leq G(t, x, p_1 - (-p_2), A_1 - (-A_2)) \\
= G_0(t, x, p_1 + p_2, A_1 + A_2).
\]

Applying Lemma 22 yields Theorem 3.

\[ \square \]

**Proof of Theorem 5** Let \(\Phi \in C^2([0, T] \times \mathbb{R}^d)\), and \(w - \Phi\) achieves its global maximum at \((t_0, x_0) \in (0, T) \times \mathbb{R}^d\). Set

\[ \Psi_{\varepsilon, \delta}(t, x, s, y) = u_1(t, x) - u_2(s, y) - \frac{|x - y|^2}{\varepsilon^2} - \frac{|t - s|^2}{\delta^2} - \Phi(t, x), \]

where \(\varepsilon, \delta > 0\). Since \((t_0, x_0)\) is a strict global maximum point of \(w - \Phi\), as the proof for Lemma 3.1 in [CIL92], there exists a sequence \((\bar{t}, \bar{x}, \bar{s}, \bar{y}) = (\bar{t}(\varepsilon, \delta), \bar{x}(\varepsilon, \delta), \bar{s}(\varepsilon, \delta), \bar{y}(\varepsilon, \delta))\) such that

- \((\bar{t}, \bar{x}, \bar{s}, \bar{y})\) is a global maximum point of \(\Psi_{\varepsilon, \delta}\) in \([0, T] \times B_R^2\);
\( (\tilde{t}, \tilde{x}), (s, \tilde{y}) \to (t_0, x_0) \) as \( (\varepsilon, \delta) \to 0; \)

\( \frac{|\tilde{x} - \tilde{y}|}{\varepsilon^2} \) and \( \frac{|\tilde{t} - s|}{\delta^2} \) are bounded and tend to zero as \( (\varepsilon, \delta) \to 0. \)

According to Theorem 8.3 in [CIL92], there exist \( X, Y \in S^d \) such that

\[
\left( \frac{2(\tilde{t} - \tilde{s})}{\delta^2} + \frac{\partial \Phi}{\partial t}(\tilde{t}, \tilde{x}), \frac{2(\tilde{x} - \tilde{y})}{\varepsilon^2} + D\Phi(\tilde{t}, \tilde{x}), X \right) \in \mathcal{P}^2_{B_R} u_1(\tilde{t}, \tilde{x}),
\]

\[
\left( \frac{2(\tilde{t} - \tilde{s})}{\delta^2}, \frac{2(\tilde{x} - \tilde{y})}{\varepsilon^2}, Y \right) \in \mathcal{P}^2_{B_R} u_2(s, \tilde{y}),
\]

\[
\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{4}{\varepsilon^2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} D^2\Phi(\tilde{t}, \tilde{x}) & 0 \\ 0 & 0 \end{pmatrix}.
\]

Without loss of generality, assume that \((\tilde{t}, \tilde{x}, s, \tilde{y})\) is a global maximum point of \( \Psi_{\varepsilon, \delta} \) in \([0, T] \times \mathbb{R}^d \). Since \( u_1 \) and \( u_2 \) are subsolution and supersolution to the PDE \((\tilde{P})\), we have

\[
-\frac{2(\tilde{t} - \tilde{s})}{\delta^2} - \frac{\partial \Phi}{\partial t}(\tilde{t}, \tilde{x}) - \bar{G}(\tilde{x}, \frac{2(\tilde{x} - \tilde{y})}{\varepsilon^2}) + D\Phi(\tilde{t}, \tilde{x}), X \leq 0
\]

and

\[
-\frac{2(\tilde{t} - \tilde{s})}{\delta^2} - \bar{G}(\tilde{y}, \frac{2(\tilde{x} - \tilde{y})}{\varepsilon^2}), Y \geq 0.
\]

Therefore

\[
-\frac{\partial \Phi}{\partial t}(\tilde{t}, \tilde{x}) - \left[ \bar{G}(\tilde{x}, \frac{2(\tilde{x} - \tilde{y})}{\varepsilon^2}) + D\Phi(\tilde{t}, \tilde{x}), X \right] - \bar{G}(\tilde{y}, \frac{2(\tilde{x} - \tilde{y})}{\varepsilon^2}), Y \leq 0. \tag{28}
\]

Note that

\[
\bar{G}(\tilde{x}, \frac{2(\tilde{x} - \tilde{y})}{\varepsilon^2}) + D\Phi(\tilde{t}, \tilde{x}), X - \bar{G}(\tilde{y}, \frac{2(\tilde{x} - \tilde{y})}{\varepsilon^2}), Y
\]

\[
= \sup_{\Gamma} \inf_{\Lambda} \left\{ \frac{1}{2} \text{tr} [\sigma(\tilde{x}, \gamma, \lambda)\sigma^T(\tilde{x}, \gamma, \lambda)X] + \left\langle b(\tilde{x}, \gamma, \lambda), \frac{2(\tilde{x} - \tilde{y})}{\varepsilon^2} + D\Phi(\tilde{t}, \tilde{x}) \right\rangle \right\}
\]

\[
- \sup_{\Gamma} \inf_{\Lambda} \left\{ \frac{1}{2} \text{tr} [\sigma(\tilde{y}, \gamma, \lambda)\sigma^T(\tilde{y}, \gamma, \lambda)Y] + \left\langle b(\tilde{x}, \gamma, \lambda), \frac{2(\tilde{x} - \tilde{y})}{\varepsilon^2} \right\rangle \right\}
\]

\[
\leq \sup_{\Gamma, \Lambda} \left\{ \frac{1}{2} \text{tr} [\sigma(\tilde{x}, \gamma, \lambda)\sigma^T(\tilde{x}, \gamma, \lambda)X - \sigma(\tilde{y}, \gamma, \lambda)\sigma^T(\tilde{y}, \gamma, \lambda)Y] + \left\langle b(\tilde{x}, \gamma, \lambda) - b(\tilde{y}, \gamma, \lambda), \frac{2(\tilde{x} - \tilde{y})}{\varepsilon^2} \right\rangle 
\right.
\]

\[
+ \langle D\Phi(\tilde{t}, \tilde{x}), b(\tilde{x}, \gamma, \lambda) \rangle \right\},
\]

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\[
\begin{align*}
&\text{tr}[\sigma(\tilde{x}, \gamma, \lambda)\sigma^T(\tilde{x}, \gamma, \lambda)X - \sigma(\tilde{y}, \gamma, \lambda)\sigma^T(\tilde{y}, \gamma, \lambda)Y] \\
&= \text{tr} \left[ \begin{pmatrix} \sigma(\tilde{x}, \gamma, \lambda)\sigma^T(\tilde{x}, \gamma, \lambda) & \sigma(\tilde{x}, \gamma, \lambda)\sigma^T(\tilde{y}, \gamma, \lambda) \\ \sigma(\tilde{y}, \gamma, \lambda)\sigma^T(\tilde{x}, \gamma, \lambda) & \sigma(\tilde{y}, \gamma, \lambda)\sigma^T(\tilde{y}, \gamma, \lambda) \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \right] \\
&\leq \frac{4}{\varepsilon^2} \text{tr} \left[ \begin{pmatrix} \sigma(\tilde{x}, \gamma, \lambda)\sigma^T(\tilde{x}, \gamma, \lambda) & \sigma(\tilde{x}, \gamma, \lambda)\sigma^T(\tilde{y}, \gamma, \lambda) \\ \sigma(\tilde{y}, \gamma, \lambda)\sigma^T(\tilde{x}, \gamma, \lambda) & \sigma(\tilde{y}, \gamma, \lambda)\sigma^T(\tilde{y}, \gamma, \lambda) \end{pmatrix} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \right] \\
&\quad + \text{tr}[\sigma(\tilde{x}, \gamma, \lambda)\sigma^T(\tilde{x}, \gamma, \lambda)D^2\Phi(\tilde{t}, \tilde{x})] \\
&= \frac{4}{\varepsilon^2} \text{tr}((\sigma(\tilde{x}, \gamma, \lambda)\sigma^T(\tilde{x}, \gamma, \lambda) - \sigma(\tilde{x}, \gamma, \lambda)\sigma^T(\tilde{y}, \gamma, \lambda)))(\sigma(\tilde{y}, \gamma, \lambda)\sigma^T(\tilde{x}, \gamma, \lambda) - \sigma(\tilde{y}, \gamma, \lambda)\sigma^T(\tilde{y}, \gamma, \lambda))] \\
&\quad + \text{tr}[\sigma(\tilde{x}, \gamma, \lambda)\sigma^T(\tilde{x}, \gamma, \lambda)D^2\Phi(\tilde{t}, \tilde{x})] \\
&\leq \frac{4L^2}{\varepsilon^2}|\tilde{x} - \tilde{y}|^2 + \text{tr}[\sigma(\tilde{x}, \gamma, \lambda)\sigma^T(\tilde{x}, \gamma, \lambda)D^2\Phi(\tilde{t}, \tilde{x})],
\end{align*}
\]

and
\[
\langle b(\tilde{x}, \gamma, \lambda) - b(\tilde{y}, \gamma, \lambda), \frac{2(\tilde{x} - \tilde{y})}{\varepsilon^2} \rangle \leq \tilde{L}\frac{|\tilde{x} - \tilde{y}|^2}{\varepsilon^2}.
\]

We have
\[
-\frac{\partial\Phi}{\partial t}(\tilde{t}, \tilde{x}) - G(\tilde{x}, D\Phi(\tilde{t}, \tilde{x}), D^2\Phi(\tilde{t}, \tilde{x})) \leq 5\tilde{L}\frac{|\tilde{x} - \tilde{y}|^2}{\varepsilon^2},
\]

and the right hand side tends to 0 as \((\varepsilon, \delta) \rightarrow 0\).

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