Increasing of capabilities to manufacture and control small-scale systems together with the discovery of remarkable energy-conversion efficiencies achieved by biomolecular machines has driven a rapidly growing interest in the development of miniature thermal machines. Recent advances for energy harvest in artificial micro- and nanoscale systems undergoing significant thermal fluctuations are being put to the test [1]. For example, realizations of miniature heat engines, featuring a Brownian particle in an optical trap, are reported in Refs. [2, 3]. While these miniature machines require external controls to perform cyclic motions, another promising direction is targeted at autonomous Brownian devices [4–6]. Examples include an on-chip microscale device to convert Joule heating to mechanical oscillation [7] and an on-chip autonomous Maxwell’s demon for refrigeration [8].

As suggested by Feynman [9], an autonomous stochastic heat engine can be facilitated through the simultaneous contacts to two different heat baths although the efficiency cannot reach the Carnot limit due to the built-in nature of irreversibility in such a thought device [10, 11]. The idea has been experimentally realized in a rotational ratchet with an hot bath provided by a granular gas [12].

Other versions of autonomous motors are tested experimentally in mechanical [13, 14] and electrical [15] systems. However, huge thermal noises generated artificially are demanded to show stochastic behaviors. Building an autonomous Brownian ratchet truly relied on real heat baths remains a great challenge.

Brownian gyrators are naturally candidates for the realization of stochastic heat engines for it has been demonstrated in Ref. [16] that autonomous circulating behavior can emerge for stochastic systems in nonequilibrium steady states (NESS). A “minimal” version of Brownian gyration has been proposed in Ref. [17], in which a structureless particle is confined near the center of a harmonic potential in a two-dimensional (2D) space, and is permanently in contact with two different heat baths. The particle undergoing Brownian motion can exhibit an average gyrating motion around the potential minimum if two requirements are met: (1) the potential is not rotationally symmetric, and (2) the directions of the two random forces, derived from their corresponding heat baths, do not coincide with the principal axes of the potential contours. Having gyrating behavior in this linear Brownian system is conceptually simple; however, technically it is tremendously difficult to implement a minuscule in touch with two independent heat baths simultaneously [18].

In this work, we report both experimental and theoretical studies on the stochastic dynamics of a capacitively-coupled resistor-capacitor (RC) circuit as illustrated in Fig. 1a, where the resistors are agitated by thermal noises of two different temperatures [19]. We show that this linear system can be compared exactly to a minimal Brownian gyration [17]. In contrast to the existing studies on this system concerning its entropy fluctuation and the applicability of fluctuation theorems [19, 20], we pay attention to its gyrating behavior concealed in its fluctuating dynamics in the configuration space. While the voltage for each of the electrical element fluctuates due to thermal noises, heat can be conducted from the hot to the cold reservoir via the circuit, leading to an averaged unidirectional gyrating motion.
**Experimental system**

Figure 1a shows the schematic of our study system. Two \( RC \) circuits \((R_1, C_1)\) and \((R_2, C_2)\) are connected through a coupling capacitor \(C_c\). The two resistors \(R_1\) and \(R_2\) are thermalized by heat baths of temperature \(T_1\) and \(T_2\), respectively. In our system of interest, the effects of electromagnetic induction are negligible, and the dynamical change of voltages across the resistors, \(V_1\) and \(V_2\), is governed by the coupled Langevin equation

\[
R_1(C_1 + C_c)V_1 - R_1C_cV_2 = -V_1 + \xi_1, \\
R_2(C_2 + C_c)V_2 - R_2C_cV_1 = -V_2 + \xi_2, \\
\tag{1}
\]

or equivalently,

\[
\hat{R}\hat{C}\hat{V} = -\hat{V} + \hat{\xi}, \\
\tag{2}
\]

where \(\hat{V} \equiv \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}\), \(\hat{\xi} \equiv \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}\), \(\hat{R} \equiv \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}\), and \(\hat{C} \equiv \begin{pmatrix} C_1 + C_c & -C_c \\ -C_c & C_2 + C_c \end{pmatrix}\). The thermal (Johnson-Nyquist) noises \(\xi_1\) and \(\xi_2\) are Gaussian white and uncorrelated, namely \(\langle \xi_i(t)\xi_j(t') \rangle = 2k_BT_1R_i\delta(t - t')\), and \(k_B\) is the Boltzmann constant. Owing to a nonzero \(C_c\), the dynamics of each voltage signal \(V_i\) is influenced explicitly by both thermal noises.

The circuit parameters are \(C_1 = 488\ \text{pF}\), \(R_1 = 9.01\ \text{M}\Omega\), \(C_2 = 420\ \text{pF}\), \(R_2 = 9.51\ \text{M}\Omega\), and \(T_2 = 296\ \text{K}\). The coupling capacitance \(C_c\) varies from 100 pF to 10 nF and \(T_1\) varies from 120 K to 296 K. The measurement observables, the voltages \(V_i\), are amplified and sampled with a rate of 2048 Hz. (For experimental details, see Methods.) Fig. 1b shows a snapshot of the concurrent voltage time traces \(V_1(t)\) and \(V_2(t)\) with \(C_c = 1.0\ \text{nF}\) and \(T_1 = 120\ \text{K}\). \(V_1(t)\) and \(V_2(t)\) resemble each other owing to the large \(C_c\).

One can compare the electric circuit system to a Brownian particle in two dimensions, as depicted in Fig. 1c. The vector \(\vec{V}(t)\) can serve as the position of this virtual Brownian particle at time \(t\). A small segment of its trajectory is shown by the orange lines. In the thermal equilibrated case, \(T_1 = T_2\), the virtual particle does not exhibit any net movement besides thermal fluctuations; when \(T_1 \neq T_2\), the particle is unequally agitated by the two heat baths, causing a persistent, unidirectional movement on average. In the latter case, we set \(T_1\) to be the colder heat bath throughout our study.

To compare our system to the Brownian gyrator described in Ref. [17], we introduce the linear transformation

\[
\vec{q} \equiv \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \hat{C}\hat{V}, \\
\tag{3}
\]

where \(q_1\) and \(q_2\) represent the total capacitor charges in the neighbors of nodes 1 and 2, respectively. The potential energy of the system (depicted by equipotential elliptical contours in Fig. 1c), as stored in the capacitors, is \(U = \frac{1}{2}C_1V_1^2 + \frac{1}{2}C_2V_2^2 + \frac{1}{2}C_c(V_1 - V_2)^2 = \frac{1}{2}\hat{V}^T\hat{C}\hat{V} = \frac{1}{2}\vec{q}^T\hat{C}^{-1}\vec{q}\). With this transformation of variables, the coupled Langevin equation (equation(2)) now reads

\[
\hat{R}\vec{q} = -\hat{C}^{-1}\vec{q} + \hat{\xi} = -\nabla_q U + \hat{\xi}. \\
\tag{4}
\]

FIG. 1: **Electrical autonomous Brownian gyrator.**

a. Schematics of the experimental system, featuring a capacitively-coupled \(RC\) circuit agitated by two heat baths. b. A snapshot of concurrent \(V_1(t)\) and \(V_2(t)\) over 30 ms with \(C_c = 1.0\ \text{nF}\) and \(T_1 = 120\ \text{K}\). c. A virtual particle evolving in the 2D phase space formed by \(V_1\) and \(V_2\). A small segment of its trajectory (corresponding to the data in b) is shown by the orange lines. The dashed lines indicate the \(q_1\) and \(q_2\) axes; see text. The virtual particle is thermalized by two heat baths and experiences two random noises from directions parallel to \(q_1\) and \(q_2\) axes, respectively, as noises are depicted by two sets of wavy arrows. The tilt ellipses designate potential contours with a minimum in the origin.

Along the direction of the \(q_1\) axis, \(q_2\) stays fixed, and the system is subject to the thermal bath \(T_1\) only (and vice versa). The virtual particle simultaneously experiences two thermal noises from directions parallel to the \(q_1\) and \(q_2\) axes, respectively, as noises are depicted by the wavy arrows in Fig. 1c. In this work, most results are presented on the \(V_1 - V_2\) plane, and the dynamics of each \(V_i\) is influenced explicitly by both thermal noises.
Behavior in nonequilibrium steady state: probability flux circulation

The main results of this work are visualized in Fig. 2. First, we present the equipotential lines of $U$ as white concentric elliptical contours in Fig. 2. Due to the presence of a nonzero $C_c$, the potential possesses no rotational symmetry, and the principle axes of the contours are tilted in the $V_1-V_2$ frame and do not coincide with $T_1-T_2$ axes. Therefore, this setup meets the two aforementioned requirements and can serve as an electrical version of Brownian gyrator [17].

Our measured steady state distribution $P_{ss}(V)$ is presented as a colormap plot in Figs. 2a and 2b. At thermal equilibrium ($T_1 = T_2 = T$), $P_{ss}$ follows the Boltzmann distribution. Thus $P_{ss}$ is constant on equipotential contours, as shown in Fig. 2a. The system relaxes into a NESS when $T_1 < T_2$ with a nonzero heat flow on average going steadily from the $T_2$ heat bath to the $T_1$ heat bath through the circuit (see Methods for evaluating heat conduction). In a NESS, however, $P_{ss}$ does not stay in accordance with the potential landscape, as demonstrated in Fig. 2b for the case of $T_1 = 120$ K. While $P_{ss}$ in Fig. 2b still has an elliptic shape, its principle axes rotate counterclockwise slightly when compared with that in Fig. 2a. This behavior is attributed to the narrower distribution in $V_1$ in our NESS case due to the lower $T_1$.

A major difference between thermal equilibrium and NESS lies in time reversibility. Theoretically, the former is achieved through the detailed balance condition, which is itself a signature of time reversibility. On the other hand, the detailed balance condition can fail in a nonequilibrium process, leading to persistent probability flows even in its steady state. Here we evaluate the probability flux $\vec{J}_{ss}(\vec{V}) = P_{ss}(\vec{V})\vec{v}_{\text{flow}}(\vec{V})$ from the experimental trajectory of the virtual particle, where $\vec{v}_{\text{flow}}(\vec{V})$ represents the steady-state flow velocity at $\vec{V}$ [17]. (See Methods for the definition of $\vec{v}_{\text{flow}}$.) The experimental result of $\vec{J}_{ss}$ are presented as vector fields in Figs. 2(a) and 2b. There is clearly a circulating probability flux field in the NESS case (Fig. 2b), while no significant flow occurs in the thermal equilibrium case (Fig. 2a). Therefore, in a NESS, the motion of the virtual particle can be depicted by Brownian dynamics with an average counterclockwise circulation, i.e., it manifests as a Brownian gyrator in an electrical system.

The circulation of probability flux in the NESS results from the unbalanced competition between the conservative and diffusive driving forces. Naively speaking, on the $V_1-V_2$ plane, the conservative force pulls the virtual particle inward. Meanwhile, the diffusive force results from gradient changes of $P_{ss}$ and tends to push the virtual particle outward. At thermal equilibrium (see Fig. 2a), the two sets of contour lines have identical shape, and their representative forces cancel out exactly. Thus the net flux is zero everywhere, a signature of the detailed balance. In the NESS case, however, due to the temperature difference, the principle axes for the contours of $P_{ss}$ and $U$ are different, and the two driving forces are mostly not balanced. As a result, the net force contributes to a non-vanishing flux. Circulating motion therefore emerges naturally since the flux at the steady state must be divergence-free (the curl of a nonzero field must exist somewhere for the divergence-free case). Note that this directional circulation motion requires no non-linear rectification like the Feynman ratchet; this system is completely linear. Moreover, owing to the conservation of probability,

$$\frac{dP_{ss}}{dt} = \nabla P_{ss} \cdot \vec{v}_{\text{flow}} + \frac{\partial P_{ss}}{\partial t} = 0 \quad (5)$$

holds along the steady-state circulation trajectories. Since $\frac{\partial P_{ss}}{\partial t} = 0$, $\vec{v}_{\text{flow}}$ and thus $\vec{J}_{ss}$ must be perpendicular.
to $\nabla P_{ss}$ (a rigorous proof is provided in Theoretical analysis). This feature is also confirmed in our experiment, as demonstrated in Fig. 2b.

The results for the curl of the steady-state flux, $\nabla \times \vec{J}_{ss}$, are shown in the insets of Fig. 2, where $\nabla \times \vec{J}_{ss}$ points out of the $V_1 - V_2$ plane. In the NESS case (see the inset of b and c), the large positive curling trend near the origin causes the virtual particle to gyrate counterclockwise on average. Note that away from the origin, small regions with a negative curling direction (shown as dark blue) can be observed from both our experimental and theoretical analysis. Negative curl exists in regions of approximately parallel field lines whose magnitude decreases as the virtual particle marches outward. Note that even in the negative curl regions, the flux field lines still follow the counterclockwise gyration trend with respect to the origin.

**Energy flow in a cycle**

The above observation shows that on average, the virtual particle rotates about the origin in the $V_1 - V_2$ phase space, while its Brownian-motion signature is revealed within short-time intervals. The average circulating behavior shows periodic oscillations in $V_1$ and $V_2$ with an identical frequency and a constant phase difference. The system therefore acts like a mini electricity generator powered solely by the temperature difference, and thus can be compared to a Brownian ratchet or a mini heat engine. The ac voltage could be conceivably used to power up devices or rectified to store electric energy.

How does energy transfer from the hot heat reservoir to the cold one in a directed cycle? First we note that if the virtual particle circulates along some closed loop on the $V_1 - V_2$ diagram, the amount of energy flowing into the circuit through resistor $R_i$ (the same as the amount of heat flowing out of the heat bath coupled to $R_i$) during one cycle is $Q_1 = \int V_i i_{R,i} dt = \int V_i dq_i$, where $i_{R,i}$ is the current through $R_i$. One can find $Q_1 = -C_c \int V_1 dV_2$ and $Q_2 = -C_c \int V_2 dV_1 = C_c \int V_1 dV_2 = -Q_1$, and their magnitude is proportional to the loop area $|\int V_2 dV_1|$ on the $V_1 - V_2$ diagram. Thus over each counterclockwise cycle $Q_2 = -Q_1 > 0$, and a net energy is flowing from the hot reservoir towards the cold one.

The energy transfer can be better understood using Fig. 3a as a schematic example. The cyclic diagram is constituted by four simple paths (I, II, III, IV), which form a parallelogram with endpoints A, B, C, and D. The four paths are chosen such that $q_1$ stays fixed during I and III while $q_2$ stays fixed during II and IV. As a consequence, for processes I and III, no currents are flowing through $R_1$, and the system can be considered adiabatic to the cold reservoir $T_1$, while it can exchange energy with the hot reservoir $T_2$. Similarly, for processes II and IV, $i_{R,2} = 0$, and the system can be considered adiabatic to the $T_2$ bath, while it can exchange energy with the $T_1$ reservoir.

One can show that during processes I and III, the resistor $R_2$ is exerting positive work on capacitors $C_1$ and $C_2$, while the bridging capacitor $C_c$ is discharging and also releasing energy into the other capacitors. And during processes II and IV, the capacitors $C_1$ and $C_2$ are discharging and releasing energy into $R_1$ and $C_c$. The directions of net energy flows and electric currents are shown in Fig. 3b. Other than the reversed polarity in charges and currents, the processes III and IV simply repeat I and II, respectively. After a full cycle, the system resumes its original state, and a net energy is transferred from the $T_2$ to the $T_1$ heat bath through the circuit elements. The amount of transferred energy can be characterized via the enclosed area of the cycle, as larger cycles and faster gyration rates signify higher heat conduction rates.

Note that for a parallelogram of the aforementioned semi-adiabatic processes without centering at the origin, the magnitude and even the sign of transported energy to and from the capacitors for each individual process may vary. Yet the total amount of energy transported in the two processes I and III remains unchanged (and so on for II and IV). As a result, the energy transfer can be characterized in terms of area on the $V_1 - V_2$ plot. Furthermore, one can dissect any closed cycle (e.g. the elliptical contour in our experimental observation in a NESS) into infinite pavements of parallelograms (see Fig. 3c for an illustrated example). Thus any closed cycle can be treated as a composite of semi-adiabatic processes.

The linear coupled circuit described here does not
convert any heat into work. Therefore, although the gyrating behavior is observed in our system, currently it remains meaningless to discuss about its efficiency and output power. Nonetheless, we can briefly remark on the possibility of extracting work from the system. For all our discussed cases, the average entropy of this stochastic system (up to addition by some constant owing to $I_{ss}$ is not a dimensionless quantity) is $\langle S \rangle/k_B = -\int P_{ss}(\vec{V}) \ln P_{ss}(\vec{V}) d\vec{V} = 1 + \ln(2k_B\pi) - \frac{1}{2} \ln \det(\mathbf{C}) + \frac{1}{2} \ln \left\{ T_1T_2 + \frac{C^2_R}{\det(\mathbf{C})} \left[ \frac{1}{2} + \frac{1}{8} \frac{C^2_R}{\det(\mathbf{C})} \right] \right\}$ (see Methods). Furthermore, the average internal energy is $\langle U \rangle = \frac{1}{2} k_B(T_1 + T_2)$. Therefore, for the NESS case we consider, where the circuit is thermalized by two different heat baths at $T_1$ and $T_2$, its average energy is identical with that in thermal equilibrium with the mean temperature $T = (T_1 + T_2)/2$. On the other hand, one can easily show that the NESS average entropy is less than the equilibrium result at the average temperature, suggesting that the circuit in a NESS is more ordered. Since in thermal equilibrium, entropy is a monotonic function of energy, thus in principle, the circuit in a NESS should be capable of providing work via some relaxation process towards the equilibrium where the system entropy is preserved. Note that with proper external driving, this linear system can function as a heat engine or a refrigerator [21].

Rotation speed of the gyror

The gyrating motion can also be well visualized through the time trace of $\phi \equiv \tan^{-1}(V_2/V_1)$, the angle between $\vec{V}$ and the horizontal axis in the $V_1 - V_2$ plane. Fig. 4a presents the gross behavior of $\phi$ for various $T_1$, while its stochastic behavior is exemplified in the inset. In the NESS cases ($T_1 < T_2$), the overall trend exhibits a linear growth in time, while such feature is absent in the thermal-equilibrium case ($T_1 = T_2 = 296$ K).

The dependence of average growth rate, $\langle \dot{\phi} \rangle$, on the temperature difference, $\Delta T \equiv T_2 - T_1$, is shown in Fig. 4b. Note that experimentally we obtain $\langle \dot{\phi} \rangle$ via two methods: the first method is finding the slopes of the fitted straight lines in Fig. 4a (solid square), while the second is evaluating the average rotating speed from the probability flux: $\langle \dot{\phi} \rangle = \int \frac{\vec{V} \times \vec{J}_{ss}}{V^2} d\vec{V}$ (open circle). Both experimental evaluations agree well and indicate an approximately proportional relation between $\langle \dot{\phi} \rangle$ and $\Delta T$. The approximately linear dependence is also justified in our theoretical analysis when $\Delta T$ is small (see the dashed line in Fig. 4b).

We further study the dependence of rotating speed on the coupling strength, as is shown in Fig. 4c. Remarkably, $\langle \dot{\phi} \rangle$ does not increase monotonically with $C_c$. Our theoretical result (dashed curve in Fig. 4c) predicts a broad peak near $C_c \approx 700$ pF, while the peak circulating speed is about 5 rev/s. And the evaluation of $\langle \dot{\phi} \rangle$ from the experimental data (open circles) follow well with the theoretical curve, proving the existence of an optimal coupling for gyration. The decreasing trend of $\langle \dot{\phi} \rangle$ can be understood by recognizing that at large $C_c$ it takes a long time for the system to charge/discharge (a more elaborated discussion can be found in the Methods section).

In Fig. 4d we present the phase difference $\langle \alpha \rangle$ between $V_1$ and $V_2$ along the elliptical contours of constant $P_{ss}(\vec{V})$ (positive if $V_1$ leads $V_2$). For the special case that $C_c$ vanishes, the elliptical contours are nontilted, and thus $\langle \alpha \rangle$ is equal to 90 degrees. As $C_c$ increases, the ellipses start to tilt due to the coupling between the signals $V_1$ and $V_2$, and as a result $\langle \alpha \rangle$ decreases. Again the experimental results are well confirmed by theoretical analysis.

Conclusion

We demonstrate in this work that the linear, coupled $RC$ circuit system, under the agitation of two different thermal baths near room temperature scale, can serve as a non-mechanical realization of an autonomous Brownian motor. The incomplete cancellation between the diffusive drive and the dragging force from the potential gradient results in a net circulating probability flux in the steady state. For such an arrangement, the system acts as a mini electricity generator, while the possibility for the usage of this generated power is currently under exploration.

The observation that heat is conducted from the hot
to the cold reservoir is simply consistent with the second law of thermodynamics. Yet the heat-transfer mechanism through the gyration in the configuration space is plausible, noting that the conducting element possesses two thermal degrees of freedom only. The direction of heat flow, the gyrating dynamics, and the total entropy production, are all representations of the second law which states the time irreversibility in a nonequilibrium steady state.

Methods

Experimental setup. The measured RC circuits in metal shielding boxes are placed in a Faraday cage on an optical table. The resistor $R_1$ in a metal shielding box is cooled in a semiclosed liquid nitrogen dewar by liquid nitrogen vapor. We use voltage amplifiers with gain of $10^4$ to magnify the thermal voltages before sampling. The amplified signals are filtered by a 160-kHz antialiasing filter, digitized at 262.1 kHz, and averaged over 128 digitized points for a sample to achieve sampling rate of 2048 Hz. Typically $10^6$ pairs of $(V_1, V_2)$ are recorded during each run. The value of $C_1$, $C_2$, $R_1$ and $R_2$ are determined from the measured noise power spectrums of $V_1$ and $V_2$ at $C_c = 0$ when both circuits are at room temperature. The value of $C_c$ in the circuit is independently measured by a LCR meter. The value of $T_1$ below room temperature is measured by a K-type thermocouple and confirmed by the variance of probability distribution of $V_1$ with known $C_1$, $C_2$, $R_1$, $R_2$, $C_c$ and $T_2$.

Average heat conduction. The heat productions in both heat reservoirs can be experimentally determined from the measured voltage time traces by analyzing the currents flowing through both resistors with the method described in Ref. [19]. In a NESS, a nonzero heat flow steadily going from the $T_2$ bath to the $T_1$ bath through the circuit is found (for example 8.86 × 10$^{-20}$ W for $T_1 = 120$ K and $C_c = 1.0$ mF), and is consistent with the theoretical value (8.18 × 10$^{-20}$ W) computed by the expression [19]

$$\langle Q \rangle = \frac{C^2 k_B (T_2 - T_1)}{\det(\mathbf{C}) \cdot \text{Tr}(\mathbf{R} \mathbf{C})}.$$  

(6)

Steady-state flow velocity. The operational definition of the steady-state flow velocity is $\vec{v}_{\text{flow}}(\vec{V}) = (\langle \vec{V} (t + \Delta t) - \vec{V} (t) \rangle / \Delta t) - \langle \vec{V} (t) \rangle$. As a result, $\Delta t = 4.88$ ms is the sampling interval (corresponding to the sampling rate of 2048 Hz, and the phase space is divided into grids with resolution $\Delta V = 0.67 \mu V$ in order to accumulate decent statistics.

Leading angle of gyrating motion. The leading angle ($\alpha$) in Fig. 4d is experimentally evaluated by the average of the instantaneous angle difference $\alpha = \tan^{-1} \left( \frac{\omega V_y}{\omega V_x} \right) - \tan^{-1} \left( \frac{\omega V_y}{\omega V_x} \right)$, where $\omega = |\vec{V} \times \vec{V}|$ is the instantaneous angular velocity of the virtual particle in the $V_1 - V_2$ phase space.

Theoretical analysis. The Fokker-Planck equation corresponding to the coupled Langevin equation, equation (1), is

$$\frac{\partial P(\vec{V}, t)}{\partial t} = \nabla \cdot [\hat{\mathbf{M}}^{-1} \bar{V} P(\vec{V}, t)] + \frac{1}{2} \nabla \cdot \hat{\mathbf{M}}^{-1} \bar{\nabla} P(\vec{V}, t),$$  

(7)

where $\hat{\mathbf{M}} = \hat{\mathbf{R}} \hat{\mathbf{C}}$, $\hat{\Gamma} = \left( \begin{array}{cc} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{array} \right)$ and $\Gamma_m \equiv 2 R_m k_B T_m$ for $m = 1, 2$. Its steady-state distribution is Gaussian:

$$P_{\text{ss}}(\vec{V}) = \frac{\sqrt{\text{det}(\hat{\mathbf{M}}^T \hat{\mathbf{X}} \hat{\mathbf{M}})}}{\pi^2} \exp(-\vec{V}^T \hat{\mathbf{M}}^{-1} \hat{\mathbf{X}} \vec{V}),$$  

(8)

where

$$\hat{X} = \frac{(\hat{\mathbf{M}}^{-1})^T \hat{\mathbf{X}}^{-1} \hat{\mathbf{M}}^{-1} + \hat{\mathbf{F}}^{-1}}{\text{det}(\hat{\mathbf{M}})}$$  

(9)

is symmetric, and $\{M_{ij}\}$ represent the elements of the matrix $\hat{\mathbf{M}}$. With a little algebra one can show that

$$\hat{\mathbf{X}} \hat{\mathbf{M}} = \frac{\text{Tr}(\hat{\mathbf{M}})}{B} (\hat{\mathbf{A}} \hat{\mathbf{X}}^{-1} - \epsilon \hat{\mathbf{Y}}),$$  

(10)

where $A \equiv \text{Tr}(\hat{\mathbf{M}}) \text{det}(\hat{\Gamma})$, $\hat{\mathbf{Y}} = \left( \begin{array}{c} 0 \\ 1 \end{array} \right)$, $B \equiv \text{det}(\hat{\mathbf{A}} \hat{\mathbf{X}}^{-1} - \epsilon \hat{\mathbf{Y}}) = A^2 / \text{det}(\hat{\Gamma}) + \epsilon^2$, and $\epsilon \equiv \Gamma_1 M_{21} - \Gamma_2 M_{12} = 2 k_B R_1 R_2 C_c (T_2 - T_1)$.

The probability flux of the system is

$$\vec{J} = -\hat{\mathbf{M}}^{-1} \vec{V} P - \frac{1}{2} \hat{\mathbf{M}}^{-1} \hat{\mathbf{F}} \vec{V} P, \quad \text{in which the first term results from the restoring force towards the origin, while the second term can be attributed to the diffusive driving force. At the steady state, one has}$$

$$\vec{J}_{\text{ss}} = -\hat{\mathbf{M}}^{-1}[\vec{V} - \hat{\Gamma} \vec{X} P_{\text{ss}}],$$  

(12)

In the case of thermal equilibrium, $T_1 = T_2$, $P_{\text{ss}}$ exhibits a Boltzmann distribution, and $\vec{X} = \Gamma^{-1} \mathbf{M}$. As a result, $\vec{J}_{\text{ss}} = 0$, which is a signature of the detailed balance. On the other hand, in a NESS case, the two representative forces in Eq. 11 do not cancel out, causing a persistent net flux in a NESS.

By defining $\Delta \mathbf{M} \equiv \hat{\mathbf{F}} \vec{X} P_{\text{ss}} - \hat{\mathbf{I}}$, where $\hat{\mathbf{I}}$ is the 2 × 2 identity matrix, one can show that

$$\hat{\Delta} \hat{\mathbf{M}} = -\frac{\epsilon^2 \hat{\mathbf{I}}}{B} - \frac{\epsilon \hat{\mathbf{X}} \hat{\mathbf{M}}}{\text{det}(\hat{\mathbf{M}})},$$  

(13)

Therefore, $\hat{\Phi} = \hat{\mathbf{M}}^{-1} \hat{\Delta} \hat{\mathbf{M}}$, and both $\hat{\Phi}$ and $\vec{J}_{\text{ss}}$ are approximately proportional to $\epsilon$ and hence $\Delta T \equiv T_2 - T_1$ when the temperature difference is small.

To prove that $\vec{J}_{\text{ss}}$ and $\nabla P_{\text{ss}}$ are perpendicular, we note that $\nabla P_{\text{ss}} = -2 \hat{\mathbf{M}}^T \hat{\mathbf{X}} \hat{\mathbf{M}} \vec{V} P_{\text{ss}}$. According to equation (12) and (13),

$$\nabla P_{\text{ss}} \cdot \vec{J}_{\text{ss}} = -2 \vec{V}^T (\hat{\mathbf{M}}^T \hat{\mathbf{X}} \hat{\mathbf{M}})(\hat{\mathbf{M}}^{-1} \hat{\Delta} \hat{\mathbf{M}}) \vec{V} P_{\text{ss}}^2 = \frac{2 \epsilon}{\text{Tr}(\hat{\mathbf{M}})} \vec{V}^T (\hat{\mathbf{X}} \hat{\mathbf{M}})^T \hat{\Phi} \hat{\Phi} P_{\text{ss}}^2 = 0,$$  

(14)
for all $\vec{V}$, since the matrix $(\hat{X} M)^T \hat{Y} (\hat{X} M)$ is antisymmetric.

The gyration direction of the virtual particle can be identified by comparing the directions of $\vec{J}_{ss}$ and $\nabla P_{ss} \times \vec{z}$, where $\vec{z}$ is the unit vector point out of the $V_1 - V_2$ plane. The virtual particle is gyroting counterclockwise if both vectors are parallel and clockwise if they are antiparallel.

Note that the first term in equation (16) is positive while $\vec{J}_{ss}$ is gyrating counterclockwise if $\epsilon > 0$, i.e., $T_2 > T_1$, and vice versa.

The curl of the steady-state flux is

$$\nabla \times \vec{J}_{ss} = P_{ss} \nabla \times (\hat{Y} \vec{V}) + (\nabla P_{ss}) \times (\hat{Y} \vec{V}) = P_{ss} [\text{Tr}(\hat{Y} \vec{V}) \vec{z} - 2(\hat{X} \hat{X}^T \hat{Y} \vec{V}) \times (\hat{Y} \vec{V})].$$

Note that the first term in equation (16) is positive while the second term is negative in the $\vec{z}$ direction. The opposing behavior explains the possible existence of negative curl in Fig. 2. Following the feature in $\vec{Y}$, $\nabla \times \vec{J}_{ss}$ is approximately proportional to $\Delta T$ when the temperature difference is small.

To study the rotation speed of the virtual particle, we first find from the average heat transfer rate equation (6) that $\langle \dot{Q} \rangle$ increases monotonically over $C_c$, as $\langle \dot{Q} \rangle \sim O(C_c^2)$ in the weak-coupling regime (when $C_c$ is small) and $\langle \dot{Q} \rangle \sim O(1)$ in the strong coupling regime (large $C_c$). Furthermore, the average heat transferred from $T_2$ to $T_1$ reservoir during one gyration cycle, $Q_{cycle}$, is equal to the product of $C_c$ and the average area of gyration on the $V_1 - V_2$ diagram. Therefore, $Q_{cycle}$ is proportional to $C_c \pi \sqrt{\det(M^T X M)}$. One can show that $Q_{cycle}$ is also increasing monotonically over $C_c$, as $Q_{cycle} \sim O(\sqrt{C_c})$ for large $C_c$ and $Q_{cycle} \sim O(C_c)$ for small $C_c$.

Finally, the average entropy of the system in a NESS, in multiples of $k_B$, is

$$\langle S \rangle/k_B = \langle \vec{V}^T (\hat{M}^T \hat{X} M) \vec{V} \rangle + \frac{1}{2} \ln \frac{\pi^2}{\det(M^T X M)} = 1 + \ln(2k_B \tau) - \frac{1}{2} \ln \det(C) + \frac{1}{2} \ln \left\{ T_1 T_2 + \frac{C_c^2 R_1 R_2 (T_2 - T_1)^2}{\text{Tr}(R C)} \right\} ,$$

where $T_1$ and $T_2$ are reservoir temperatures.

---

[1] Martinez, I. A., Roldán, É., Dinis, L., & Rica, R. A. Colloidal heat engines: a review. Soft Matter 13, 22 (2017).
[2] Blickle, V. & Bechinger, C. Realization of a micrometre-sized stochastic heat engine. Nat. Phys. 8, 143 (2012).
[3] Martinez, I. A. et al. Brownian Carnot engine. Nat. Phys. 12, 67 (2016).
[4] Reimann, P. Brownian motors: noisy transport far from equilibrium. Phys. Rep. 361, 57 (2002).
[5] Hanggi, P. & Marchesoni, F. Artificial Brownian motors: Controlling transport on the nanoscale. Rev. Mod. Phys. 81, 387 (2009).
[6] Van den Broeck, C., Kawai, R. & Meurs, P. Microscopic analysis of a thermal Brownian motor. Phys. Rev. Lett. 93, 090601 (2004).
[7] Steeneken, P. G. et al. Piezoresistive heat engine and refrigerator. Nat. Phys. 7, 354 (2011).
[8] Koski, J. V. et al. On-chip Maxwell’s demon as an information-powered refrigerator. Phys. Rev. Lett. 115, 260602 (2016).
[9] Feynman, R. P., Leighton, R. B. & Sands, M. The Feynman Lectures on Physics, Vol. 1 (Addison-Wesley, Boston, 1963).
[10] Parrondo, J. M. R. & Espanol, P. Am. J. Phys. 64, 1125 (1996).
[11] Roldan, E. Irreversibility and Dissipation in Microscopic Systems (Springer, Switzerland, 2014).
[12] Eshuis, P. et al. Experimental realization of a rotational ratchet in a granular gas. Phys. Rev. Lett. 104, 248001 (2010).
[13] Peng, Z. & To, K. Biased Brownian motion in narrow channels with asymmetry and anisotropy. Phys. Rev. E 94, 022902 (2016).
[14] Serra-Garcia, M. et al. Mechanical autonomous stochastic heat engine. Phys. Rev. Lett. 117, 010602 (2016).
[15] Hartmann, F., Pfeffer, P. , Höfling, S., Kamp, M. & Worschech, L. Voltage fluctuation to current converter with Coulomb-coupled quantum dots, Phys. Rev. Lett. 114, 146805 (2015).
[16] Kwon, C., Ao, P. & Thouless, D. J. Structure of stochastic dynamics near fixed points. Proc. Natl. Acad. Sci. 102, 13029 (2005).
[17] Filliger, R. & Reimann, P. Brownian Gyrator: A minimal heat engine on the nanoscale. Phys. Rev. Lett. 99, 230602 (2007).
[18] Seifert, U. Stochastic thermodynamics, fluctuation theorems and molecular machines. Rep. Prog. Phys. 75, 126001 (2012).
[19] Ciliberto, S., Imparato, A., Naert, A. & Tanase, M. Heat flux and entropy produced by thermal fluctuations. Phys. Rev. Lett. 110, 180601 (2013).
[20] Chiang, K.-H., Lee, C.-L., Lai, P.-Y. & Chen, Y.-F. Entropy production and irreversibility of dissipative trajectories in electric circuits. Phys. Rev. E 95, 012158 (2017).
[21] Park, J.-M., Chun, H.-M. & Noh, J. D. Efficiency at maximum power and efficiency fluctuations in a linear Brownian heat-engine model. Phys. Rev. E 94, 012127 (2016).
Acknowledgments

The authors wish to thank Prof. Yonggun Jun and Prof. Jae Dong Noh for fruitful discussions. This work has been supported by Ministry of Science and Technology in Taiwan. C.-L.L. acknowledges the support from NCTS thematic group Complex Systems.

Author contributions

K.-H.C. developed the experimental system, performed the experiments, and analyzed the experimental data. C.-L.L. developed the theoretical aspects. P.-Y.L. provided theoretical supports. C.-L.L. and Y.-F.C. established and supervised the project. All authors discussed the results and wrote the manuscript.

Additional information

The authors declare no competing financial interests.