Deformations of the Fano scheme of a cubic

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Abstract

We study the deformation theory of the Fano scheme $F = F(X)$ of lines on a cubic $X$ of dimension $d$ with only finitely many singularities. By taking the relative Fano scheme, we define a morphism $\eta : D_X \to D_F$ of the local moduli functors associated to $X$ and $F$, respectively. We show that for $d \geq 5$, $\eta$ yields an isomorphism on first-order deformations; in particular, $\eta$ is an isomorphism whenever $H^0(\Theta_X) = 0$.

1 Introduction

Let $P$ be the complex projective space of dimension $d + 1$, and $X \subset P$ a cubic with a finite number of singularities. For $d \geq 3$, it is well-known that the geometry of $X$ is largely determined by the Hilbert scheme $F = F(X)$ of lines on $X$, which is traditionally called the Fano scheme of $X$. A great deal is known about $F$ for $d = 3$ or $d = 4$ [2, 9, 11, 14, 29], and so our focus is on the $d \geq 5$ case, which has received much less attention. Altman and Kleiman [1] show that $F$ is an irreducible normal local complete intersection of dimension $2d - 4$, and it is known that $X$ can be recovered from $F$ [8].

In this paper, we relate the deformation theory of $X$ to the deformation theory of $F$. It is well-known that every infinitesimal deformation of $X$ is given by a family of cubic hypersurfaces; by taking the relative Hilbert scheme, we define a morphism

$$\eta : D_X \to D_F$$

of local moduli functors. A remarkable result of Beauville and Donagi [2] asserts that if $X$ is smooth of dimension $d = 4$, the scheme $F$ is deformation equivalent to the Hilbert scheme of two points of a K3 surface; in particular, there are deformations of $F$ which are not induced by $X$. In contrast, our main result is:

**Theorem.** Let $X$ be cubic of dimension $d \geq 5$ having only finitely many singularities. The differential

$$d\eta : \Ext^1(\Omega_X^1, \mathcal{O}_X) \to \Ext^1(\Omega_F^1, \mathcal{O}_F)$$

of $\eta$ is an isomorphism. If $H^0(\Theta_X) = 0$, then $\eta$ is an isomorphism.
Our proof relies on the standard description of $F$ as a subscheme of the Grassmannian $G$ of lines in $P$. Parallel to $\eta$, there is a morphism

$$\eta_H : \mathcal{H}_{X/P} \to \mathcal{H}_{F/G}$$

of local Hilbert functors, which is related to $\eta$ by a commutative square

$$\begin{array}{ccc}
\mathcal{H}_{X/P} & \longrightarrow & \mathcal{D}_X \\
\eta_H \downarrow & & \downarrow \eta \\
\mathcal{H}_{F/G} & \longrightarrow & \mathcal{D}_F.
\end{array}$$

where the horizontal morphisms are the forgetful ones. Consider the square

$$\begin{array}{ccc}
\text{H}^0(\mathcal{N}_{X/G}) & \longrightarrow & \text{Ext}^1(\Omega^1_X, \mathcal{O}_X) \\
d\eta_H \downarrow & & \downarrow d\eta \\
\text{H}^0(\mathcal{N}_{F/G}) & \longrightarrow & \text{Ext}^1(\Omega^1_F, \mathcal{O}_F)
\end{array}$$

of differentials. Relying on Borel-Bott-Weil computations and hypercohomology spectral sequences associated to the Koszul resolution of $\mathcal{O}_F$, we show that $\text{H}^0(\mathcal{N}_{F/G}) \to \text{Ext}^1(\Omega^1_F, \mathcal{O}_F)$ and $d\eta_H$ are surjective; we then observe that

$$\dim \text{Ext}^1(\Omega^1_F, \mathcal{O}_F) = \dim \text{Ext}^1(\Omega^1_X, \mathcal{O}_X),$$

using a result of Charles [8] which relates the automorphism groups of $F$ to the one of $X$. The condition $\text{H}^0(\Theta_X) = 0$, which holds for example for Lefschetz cubics, then guarantees that both $\mathcal{D}_X$ and $\mathcal{D}_F$ are pro-representable. Our proof shows that, without assuming $\text{H}^0(\Theta_X) = 0$, $\eta_H$ is an isomorphism and $\eta$ is surjective.

We should discuss the relation of our functorial approach to the work of Borcea [5, 6] and Wehler [29]. Writing $X = Z(f)$ for $f \in \text{H}^0(\mathcal{O}_P(3))$, Borcea [6] considers the deformation of $F$ given by varying $f$ in $\text{H}^0(\mathcal{O}_P(3))$. He checks the conditions

$$H^1(S^3S^\vee \otimes \mathcal{J}_{F/G}) = 0 \quad \text{and} \quad H^1(\Theta_{G|F}) = 0,$$

which guarantee the completeness of the deformation [6, 29], for $d \geq 6$. In contrast to his and other papers [1, 10] using similar methods, we explicitly compute the decomposition of the sheaves $\Lambda^nS^3S$ (which occur in the Koszul resolution of $\mathcal{J}_{F/G}$) into Schur powers. This allows us to check the conditions (1), which play an important role in our proof, for all $d \geq 5$, thus extending Borcea’s result to $d = 5$. In Theorem 3.1, we use the decomposition of $\Lambda^nS^3S$ to express the Hilbert polynomial $\chi(\mathcal{O}_F(n))$ of $F$ in terms of the Pochhammer symbol; this generalises previous results of Altman and Kleiman [1] and Libgober [19].
2 Auxiliary results

2.1 The Borel-Bott-Weil theorem

Let $V$ be a complex vector space of dimension $d + 2$. We write $\mathbb{P} = \mathbb{P}(V)$ for the projective space of one-dimensional linear subspaces of $V$, and $G = \text{Gr}(2, V)$ for the Grassmannian of lines. On $G$ there is a universal exact sequence

$$0 \to S \to \mathcal{O} \otimes V \to Q \to 0$$

of locally free sheaves. The Borel-Bott-Weil theorem, which we will use frequently in this paper, computes the cohomology of sheaves of the form

$$\Sigma^\lambda Q \otimes \Sigma^\mu S,$$

where $\lambda \in \mathbb{Z}^d$ and $\mu \in \mathbb{Z}^2$ are non-increasing. Here $\Sigma^\lambda$ denotes the Schur power corresponding to $\lambda$, generalizing the symmetric power $\Sigma^k = S^k$ and the exterior power $\Sigma^{(1^k)} = \Lambda^k$.

**Theorem 2.1** (Borel-Bott-Weil [4]). Let $\nu = (\lambda, \mu) \in \mathbb{Z}^{d+2}$ and $\rho = (d + 2, d + 1, \ldots, 1)$. If the components of $\nu + \rho$ are pairwise distinct, then the only nonvanishing cohomology group of the sheaf $\Sigma^\lambda Q \otimes \Sigma^\mu S$ is

$$H^l(\sigma)(\Sigma^\lambda Q \otimes \Sigma^\mu S) = \Sigma^\sigma(\nu + \rho) - \rho V,$$

where $\sigma \in \mathcal{S}_{d+2}$ is the unique permutation such that $\sigma(\nu + \rho)$ is non-increasing, and $l(\sigma)$ is its length. If the components of $\nu + \rho$ are not pairwise distinct, then $H^*(\Sigma^\lambda Q \otimes \Sigma^\mu S) = 0$.

We will in particular rely on the following standard applications, where we use tacitly the canonical isomorphism $S^\vee = S \otimes \det(Q)$.

**Example 2.1.** (i) We have $H^0(S^nS^\vee) = S^nV^\vee$ and $H^m(S^nS^\vee) = 0$ for $m \geq 1$.

(ii) Using the decomposition $\mathcal{E}nd(S) = (\Lambda^2S \oplus S^2S) \otimes \det(Q)$, we obtain

$$H^0(\mathcal{E}nd(S)) = \mathbb{C} \quad \text{and} \quad H^m(\mathcal{E}nd(S)) = 0 \quad (m \geq 1).$$

(iii) Tensoring (2) with $S^\vee$, using (ii) and $\Theta_G = \mathcal{H}om(S, Q)$, we get

$$\text{End}(V)/(1) \xrightarrow{\sim} H^0(\Theta_G) \quad \text{and} \quad H^m(\Theta_G) = 0 \quad (m \geq 1).$$

2.2 Fano schemes

Let $S$ be a scheme, and $P_S = P \times S$. For a closed subscheme $X \subset P_S$, we denote by

$$F(X/S) = \text{Hilb}^{T+1}(X/S)$$

the relative Hilbert scheme of lines (Fano scheme). Consider the universal subscheme

$$\mathcal{L}_S \subset P_S \times_S F(P_S/S)$$
and write $q_S$ and $p_S$ for the projections of $P_S \times_S F(P_S/S)$ to $P_S$ and $F(P_S/S)$, respectively. By Theorem 2.17 of [1], the closed subscheme $F(X/S) \subset F(P_S/S)$ is the zero scheme of the canonical morphism

$$q_S^*J_{X/P_S} \to O_{\mathcal{L}_S}$$

(4)

of sheaves on $P_S \times_S F(P_S/S)$. Of course, $F(P_S/S) = G \times S$, where we view $F$ as $F(P)$. Writing $\pi : G \times S \to S$ for the projection, we have

$$L_S = P(\pi^*S) \quad \text{and} \quad O_{\mathcal{L}_S}(1) = q_S^*(O_{P_S}(1))|_{\mathcal{L}_S}. $$

If $X = Z(f)$ for $f \in H^0(O_{P_S}(3))$, then applying $P_S$ to (4) induces a section $\sigma_f$ of

$$p_S^*O_{\mathcal{L}_S}(3) = \pi^*S^3S^\vee,$$

such that, invoking Proposition 2.3 of [1],

$$Z(\sigma_f) = F(X/S).$$

(5)

**Remark 2.1.** The map

$$\sigma : H^0(O_{P_S}(3)) \to H^0(\pi^*S^3S^\vee)$$

is an isomorphism.

If $S = \text{Spec}(\Lambda)$ is affine, we use the abbreviation $F(X/\Lambda) = F(X/\text{Spec}(\Lambda))$. We first consider $F(X) = F(X/C)$ for a cubic $X = Z(f)$, $f \in H^0(O_P(3))$. This scheme is particularly well-behaved when the singular locus of $X$ is finite:

**Theorem 2.2** (Altman-Kleiman [1]). Let $X$ be a cubic with finitely many singularities. The Hilbert scheme $F = F(X)$ is of pure dimension $2d - 4$; moreover, $F$ is reduced for $d \geq 4$.

As the rank of $S^3S^\vee$ is 4, this result in particular implies that the section $\sigma_f$ is regular. Hence $F = Z(\sigma_f)$ is a local complete intersection, the Koszul complex

$$0 \to \Lambda^4S^3S \to \Lambda^3S^3S \to \Lambda^2S^3S \to S^3S \to J_{F/G} \to 0$$

(6)

is exact, $\sigma_f$ induces a canonical isomorphism

$$\mathcal{N}_{F/G} \twoheadrightarrow S^3S|_F^\vee,$$

and the canonical sheaf of $F$ is given by $\omega_F = O_F(4 - d)$, where $O_F(1)$ is given by the Plücker embedding. The proof of our main theorem relies on the following result.

**Lemma 2.1.** We have

$$\Lambda^2S^3S = \Sigma^{5,1}S \oplus \Sigma^{3,3}S, \quad \Lambda^3S^3S = \Sigma^{6,3}S, \quad \Lambda^4S^3S = \Sigma^{6,2}S.$$
Proof. To compute the decomposition of the plethysm $\Lambda^n S^m$ into Schur powers, it suffices to compute the corresponding plethysm of Schur functions

$$s_1 \circ s_m = \sum_\lambda a_{n,m}^\lambda s_\lambda.$$ 

Here the sum is taken over all partitions $\lambda$ of $nm$ with at most $n$ parts, and the numbers $a_{n,m}^\lambda$ can be expressed in terms of generalized Kostka numbers [20]. For small $n$ and $m$, these coefficients are relatively easy to compute; we find

- $s_1 \circ s_3 = s_{5,1} + s_{3,2}$,
- $s_1 \circ s_3 = s_{7,1,2} + s_{6,3} + s_{5,3,1} + s_{3,2}$,
- $s_1 \circ s_3 = s_{9,1,3} + s_{8,3,1} + s_{7,4,1} + s_{7,3,1,2} + s_{6,2} + s_{6,4,2} + s_{6,3,2} + s_{5,2,1,2} + s_{5,3,2,1} + s_{3,4}$.

It remains to observe that since $S$ has rank 2, we have $\Sigma^\lambda S = 0$ if $\lambda$ has more than two parts. (Note that since $\Lambda^4 S^3 S = \det(S^3 S)$, it is easy to show $\Lambda^4 S^3 S = \det(S)^{\otimes 6}$ directly.)

**Proposition 2.1.** Consider the sheaves $\Lambda^n S^3 S \otimes \Theta_G$ and $\Lambda^m S^3 S \otimes S^3 S^\vee$ on the Grassmannian $G$. For $d \geq 6$, $1 \leq n \leq 4$, and $2 \leq m \leq 4$ the cohomology of these sheaves is zero. For $d = 5$, the only non-vanishing cohomology groups of these sheaves are

$$H^4(\Lambda^2 S^3 S \otimes \Theta_G) = V^* \quad \text{and} \quad H^5(\Lambda^2 S^3 S \otimes S^3 S^\vee) = V^*.$$ 

For any $d \geq 5$, the only non-vanishing cohomology group of $S^3 S \otimes S^3 S^\vee$ is

$$H^0(S^3 S \otimes S^3 S^\vee) = \det(V)^{\otimes 3}.$$ 

**Proof.** By applying Lemma 2.1, $S^\vee = S \otimes \det(Q)$, and the Pieri rule, we obtain

- $S^3 S \otimes \Theta_G = \Sigma^{2,1,d-1} Q \otimes (S^4 S \oplus \Sigma^{3,1} S)$,
- $\Lambda^2 S^3 S \otimes \Theta_G = Q \otimes (S^5 S \oplus \Sigma^{4,1} S \oplus \Sigma^{3,2} S)$,
- $\Lambda^3 S^3 S \otimes \Theta_G = Q \otimes (S^6,2 S \oplus \Sigma^{5,3} S)$,
- $\Lambda^4 S^3 S \otimes \Theta_G = Q \otimes \Sigma^{6,5} S$,

for all $d \geq 3$. Similarly, we have the decompositions

- $S^3 S \otimes S^4 S^\vee = \Sigma^{3,d} Q \otimes (S^6 S \oplus \Sigma^{5,1} S \oplus \Sigma^{4,2} S \oplus \Sigma^{3,3} S)$,
- $\Lambda^2 S^3 S \otimes S^4 S^\vee = \Sigma^{3,d} Q \otimes (\Sigma^{8,1} S \oplus \Sigma^{7,2} S \oplus \Sigma^{6,3} S^{\otimes 2} \oplus \Sigma^{5,4} S)$,
- $\Lambda^3 S^3 S \otimes S^3 S^\vee = S^6 S \oplus \Sigma^{5,1} S \oplus \Sigma^{4,2} S \oplus \Sigma^{3,3} S$,
- $\Lambda^4 S^3 S \otimes S^3 S^\vee = S^{6,3} S$,

for all $d \geq 3$. It remains to apply the Borel-Bott-Weil theorem.

The Koszul resolution (6) induces a hypercohomology spectral sequence

$$E_1^{pq} = H^q(\Lambda^{-p+1} S^3 S \otimes F) \Rightarrow H^{p+q}(J_{F/G} \otimes F)$$

for any locally free sheaf $F$ on $G$. 

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Corollary 2.1. For \( d \geq 5 \), the hypercohomology spectral sequences

\[
E_1^{pq} = H^q(\Lambda^{-p+1}S^3 \otimes \Theta_G) \Rightarrow H^{p+q}(J_{F/G} \otimes \Theta_G) \\
E_1^{pq} = H^q(\Lambda^{-p+1}S^3 \otimes S^3 S^\vee) \Rightarrow H^{p+q}(J_{F/G} \otimes S^3 S^\vee)
\]
degenerate at the \( E_1 \)-page.

For certain classes of complete intersections (including cubics of dimension \( d \geq 6 \)), the latter result was obtained by Borcea [5]; our approach is similar to his, but Borcea does not explicitly compute the plethysms of Lemma 2.1 — by employing weight considerations, he instead proves a vanishing theorem (which, by Proposition 2.1, does not hold for \( d = 5 \)).

2.3 Deformation theory

We recall now some well-known general facts about functors of Artin rings [22, 24], and explain our notation; for us an Artin ring is a local \( \mathbb{C} \)-algebra which is finite over \( \mathbb{C} \). For a functor of Artin rings \( F \), we denote by \( t_F = F(C[\varepsilon]) \) the tangent space of \( F \), and if \( \varphi : F \to G \) is a functorial morphism, we refer to

\[
d\varphi = \varphi(C[\varepsilon]) : t_F \to t_G
\]
as the differential of \( \varphi \). For future reference, we recall the following general result [24]:

Lemma 2.2. Let \( \varphi : F \to G \) be a morphism of functors of Artin rings.

(i) If \( F \) and \( G \) have a pro-representable hull, \( F \) is smooth and \( d\varphi \) surjective, then \( \varphi \) is smooth.

(ii) If \( F \) and \( G \) are pro-representable, \( F \) is smooth and \( d\varphi \) bijective, then \( \varphi \) is an isomorphism.

The local moduli functor \( D_S \) of a projective scheme \( S \) takes an Artin ring \( \Lambda \) to the set \( D_S(\Lambda) \) of isomorphism classes of deformations of \( S \) over \( \Lambda \). A basic result is:

Theorem 2.3. (i) The functor \( D_S \) has a pro-representable hull.

(ii) If \( H^0(\Theta_S) = 0 \), then \( D_S \) is pro-representable.

(iii) If \( S \) is reduced, then \( t_{D_S} = \text{Ext}^1(\Omega^1_S, \mathcal{O}_S) \); if \( S \) also a local complete intersection, then \( \text{Ext}^2(\Omega^1_S, \mathcal{O}_S) \) is an obstruction space for \( D_S \).

Here (i) is a theorem of Schlessinger [22], while (ii) goes back to Kodaira and Spencer [17] (see also [12]). For a closed subscheme \( Z \) of \( S \), the Hilbert functor of \( S \) induces a functor of Artin rings (local Hilbert functor) \( \mathcal{H}_{Z/S} \), which takes an Artin ring \( \Lambda \) to the set \( \mathcal{H}_{Z/S}(\Lambda) \) of deformations of \( Z \) in \( S \) over \( \Lambda \). By the existence of the Hilbert scheme of \( S \), \( \mathcal{H}_{Z/S} \) is pro-representable and \( t_{\mathcal{H}_{Z/S}} = H^0(\mathcal{N}_{Z/S}) \). It is related to \( D_Z \) by a forgetful morphism

\[
\mathcal{H}_{Z/S} \to D_Z.
\]

We now assume that \( S \) is smooth, and \( F \) a locally free sheaf on \( S \).
Lemma 2.3. Let $Z = Z(\sigma)$ be the zero scheme of a regular section $\sigma \in H^0(F)$.  
(i) The differential of $H_{Z/S} \to D_Z$ can be identified with the connecting morphism 
$$H^0(N_{Z/S}) \to \text{Ext}^1(\Omega^1_Z, \mathcal{O}_Z)$$
associated to conormal sequence of $Z \subset S$.  
(ii) Under the canonical identification $F|_Z \simeq N_{Z/S}$, the restriction map 
$$H^0(F) \to H^0(N_{Z/S})$$
takes $\tau \in H^0(F)$ to the first-order deformation of $Z$ in $S$ given by $Z(\sigma + \epsilon \tau)$.

Finally, we consider the closed subscheme $Z$ of $H^0(F) \times S$, which parametrises pairs $(\sigma, s)$ with $\sigma(s) = 0$. The fibre of the projection 
$$\pi: Z \to H^0(F)$$
over $\sigma \in H^0(F)$ is the zero scheme $Z = Z(\sigma)$. If $\sigma$ is regular, then $\pi$ is flat in a neighbourhood of $\sigma$, thus inducing a deformation of $Z$. The following result gives a criterion for the completeness of the latter deformation.

Lemma 2.4. If $H^1(F \otimes J_{Z/S}) = 0$ and $H^1(\Theta_S|_Z) = 0$, then the Kodaira-Spencer map 
$$\kappa_{\pi, \sigma}: H^0(F) \to \text{Ext}^1(\Omega^1_Z, \mathcal{O}_Z)$$
is surjective.

3 The Hilbert polynomial of $F$

3.1 Related results

Using Schubert calculus, Altman and Kleiman [1] prove that the Plücker degree of $F$ is 
$$\int_F c_1(\mathcal{O}_F(1))^{2d-4} = \frac{27}{d!(d-1)!} (3d^2 - 7d + 4).$$
In the special case $d = 3$, this is a theorem of Fano [11]. It is thus a natural question to determine, more generally, the Hilbert polynomial 
$$\chi(\mathcal{O}_F(n)) = \sum_{k=0}^{2d-4} \frac{n^k}{k!} \int_F c_1(\mathcal{O}_F(1))^k \cap \text{Td}(F)$$
of $F$. Altman and Kleiman (and, independently, Libgober [19]) show that for $d = 3$, we have 
$$\chi(\mathcal{O}_F(n)) = \frac{45}{2} n^2 - \frac{45}{2} n + 6.$$ 
In this section, we use ?? to express the Hilbert polynomial $\chi(\mathcal{O}_F(n))$, for any dimension $d$, in terms of the Pochhammer symbol.
3.2 $\chi(\mathcal{O}_F(n))$ via the Pochhammer symbol

Recall that the Pochhammer symbol $(x)_d$ is defined by

$$(x)_d = \prod_{j=0}^{d-1} (x+j).$$

**Theorem 3.1.** The Hilbert polynomial of $F$ is given by

$$\chi(\mathcal{O}_F(n)) = \frac{1}{d!(d+1)!} (n+1)_d(n+2)_d - 4(n-2)_d(n+2)_d + (n-2)_d(n-1)_d$$

$$+ 5(n-4)_d(n+1)_d - 4(n-5)_d(n-1)_d + (n-5)_d(n-4)_d.$$

**Proof.** By the Koszul resolution, we obtain

$$\chi(\mathcal{O}_F(n)) = \chi(\mathcal{O}_G(n)) - \chi(S^3\mathcal{S}(n)) - \chi(\Lambda^2S^3\mathcal{S}(n)) + \chi(\Lambda^3S^3\mathcal{S}(n)).$$

Using Lemma 2.1, it suffices to describe the Hilbert polynomial of $\Sigma^{\mu_1,\mu_2}S$ for any $\mu_1 \geq \mu_2$. We now establish the equality

$$\chi(\Sigma^{\mu_1,\mu_2}S(n)) = \frac{(\mu_1 - \mu_2 + 1)}{(d+1)!} (n - \mu_1 + 1)_d(n - \mu_2 + 2)_d.$$  (7)

To prove (7), we may assume $n \geq \mu_1$. Since $\mathcal{O}_G(n) = \Sigma^{(n^d)}Q$,

$$\chi(\Sigma^{n^d}Q \otimes \Sigma^{\mu_1,\mu_2}S) = \dim \Sigma^{n^d,\mu_1,\mu_2}V,$$

by the Borel-Bott-Weil theorem. Hence

$$\dim \Sigma^{n^d,\mu_1,\mu_2}V = (\mu_1 - \mu_2 + 1) \prod_{j=1}^{d} \frac{(n - \mu_1 + j)(n - \mu_2 + j + 1)}{j(j+1)}$$

by the Weyl dimension formula. In particular, combining (7) with Lemma 2.1, we obtain

$$\chi(\mathcal{O}_G(n)) = \frac{1}{d!(d+1)!} (n+1)_d(n+2)_d,$$

$$\chi(S^3\mathcal{S}(n)) = \frac{4}{d!(d+1)!} (n-2)_d(n+2)_d,$$

$$\chi(\Lambda^2S^3\mathcal{S}(n)) = \frac{1}{d!(d+1)!} (n-2)_d(n-1)_d + \frac{5}{d!(d+1)!} (n-4)_d(n+1)_d,$$

$$\chi(\Lambda^3S^3\mathcal{S}(n)) = \frac{4}{d!(d+1)!} (n-5)_d(n-1)_d,$$

$$\chi(\Lambda^4S^3\mathcal{S}(n)) = \frac{1}{d!(d+1)!} (n-5)_d(n-4)_d.$$

A result of Schömlich [23] explicitly describes the coefficients of

$$(x)_d = \sum_{k=0}^{d} \left[ \frac{d}{k} \right] x^k,$$
which are the (unsigned) Stirling numbers of the first kind, in terms of binomial coefficients:

\[
\left[ \begin{array}{c} d \\ k \end{array} \right] = (-1)^{d-k} \sum_{m=0}^{d-k} \binom{d-1+m}{k-1} \binom{2d-k}{d+m} \sum_{n=0}^{m} \frac{(-1)^n n^{d-k+m}}{n!(m-n)!}.
\]

Applying this to Theorem 3.1, we can express the coefficients of \( \chi(O_F(n)) \) as sums of binomial coefficients; this shows in particular that our expression for \( \chi(O_F(n)) \) is a polynomial of degree \( 2d-4 \), as it should.

**Corollary 3.1.** We have the expansion

\[
\chi(O_F(n)) = 27 (3d-4) n^{2d-4} + 27 (3d-4)(d-4) n^{2d-5} + \ldots.
\]

Theorem 3.1 and Kodaira vanishing

\[
\chi(O_F(n)) = h^0(O_F(n)) \quad (n \geq 5 - d)
\]

allow one to compute the dimension of the space \( H^0(J_F/G(n)) \) of global sections of \( O_G(n) \) vanishing on \( F \). Indeed, for \( d \geq 4 \), Debarre and Manivel [10] prove that \( H^1(J_F/G(n)) = 0 \) for \( n \geq 0 \). There is thus an exact sequence of the form.

\[
0 \rightarrow H^0(J_F/G(n)) \rightarrow H^0(O_G(n)) \rightarrow H^0(O_F(n)) \rightarrow 0.
\]

**3.3 Examples**

Writing \( X_d \) to indicate the dimension \( d \) of the cubic \( X \), combining Theorem 3.1 with Schlömilch’s formula gives

\[
\begin{align*}
\chi(O_{F(X_4)}(n)) &= \frac{9}{2} n^4 + \frac{15}{2} n^2 + 3, \\
\chi(O_{F(X_5)}(n)) &= \frac{33}{80} n^6 + \frac{99}{80} n^5 + \frac{57}{16} n^4 + \frac{81}{16} n^3 + \frac{241}{40} n^2 + \frac{37}{10} n + 1, \\
\chi(O_{F(X_6)}(n)) &= \frac{7}{320} n^8 + \frac{7}{40} n^7 + \frac{391}{480} n^6 + \frac{39}{16} n^5 + \frac{4889}{960} n^4 + \frac{591}{80} n^3 + \frac{1697}{240} n^2 + 4n + 1, \\
\chi(O_{F(X_7)}(n)) &= \frac{17}{22400} n^{10} + \frac{51}{4480} n^9 + \frac{589}{6720} n^8 + \frac{979}{2240} n^7 + \frac{4903}{3200} n^6 + \frac{2493}{640} n^5 + \frac{4023}{560} n^4 + \frac{10503}{420} n^3 + \frac{34421}{140} n^2 + \frac{599}{140} n + 1.
\end{align*}
\]

**4 Deformations of X**

**4.1 Generalities**

Consider a cubic \( X \subset P \) of dimension \( d \geq 3 \), having only finitely many singularities, and defined by \( f \in H^0(O_P(3)) \).
Lemma 4.1. (i) The restriction map
\[ H^0(\mathcal{O}_P(3)) \to H^0(\mathcal{O}_X(3)) \]
is surjective with kernel \((f)\).

(ii) The restriction map
\[ H^0(\Theta_P) \to H^0(\Theta_P|_X) \]
is an isomorphism, and \(H^1(\Theta_P|_X) = 0\).

(iii) We have \(\text{Ext}^2(\Omega^1_X, \mathcal{O}_X) = 0\).

The proof is straightforward. Part (iii) implies that \(D_X\) is smooth, and a consequence of (ii) is that the forgetful morphism
\[ \mathcal{H}_{X/P} \to D_X \]
is smooth, in particular surjective.

Remark 4.1. In fact, any deformation \(\mathcal{X} \subset P_\Lambda\) of \(X\) in \(P\) over an Artin ring \(\Lambda\) is a cubic: there exists a section \(f_\Lambda\) of \(\mathcal{O}_{P_\Lambda}(3)\) extending \(f\), such that \(\mathcal{X} = Z(f_\Lambda)\) [17, 30].

4.2 Automorphisms

As the vanishing of \(H^0(\Theta_X)\) guarantees the pro-representability of \(D_X\), we are led to study \(H^0(\Theta_X)\). It is well-known that \(H^0(\Theta_X) = 0\) when \(X\) is smooth [15]. We observe here that this holds for the simplest class of singular cubics: Lefschetz cubics in the sense of [9], i.e. those with at most one node. Since \(H^0(\Theta_X)\) is the kernel of the derivative
\[ df : H^0(\Theta_P|_X) \to H^0(\mathcal{O}_X(3)), \]
which under the identification
\[
\begin{array}{ccc}
H^0(\Theta_P|_X) & \xrightarrow{df} & H^0(\mathcal{O}_X(3)) \\
\uparrow & & \uparrow \iota \\
H^0(\Theta_P) & \to & H^0(\mathcal{O}_P(3))/(f)
\end{array}
\]
is given by \(df(\sum L_i \partial_i) = \sum L_i \partial_i f \mod (f)\). We can thus view \(H^0(\Theta_X)\) as the subspace of \(H^0(\Theta_P)\) consisting of all \(\sum L_i \partial_i\) such that
\[ \sum L_i \partial_i f = \lambda f \]
for some constant \(\lambda\).

Proposition 4.1. If \(X\) is a Lefschetz cubic, then \(H^0(\Theta_X) = 0\).
Proof. Consider a cubic \( X \) with a single node \( x_0 \). After a linear change of coordinates, we may assume \( x_0 = [0 : \cdots : 0 : 1] \). Then the equation defining \( X \) can be written as
\[
f(x_0, \ldots, x_{d+1}) = g(x_0, \ldots, x_d) + x_{d+1}h(x_0, \ldots, x_d),
\] (10)
where \( g \) is a cubic and \( h \) a non-degenerate quadric. Inserting (10) into (9), we have to show that if
\[
\sum_{i=0}^{d} L_i \partial_i g + x_{d+1} \sum_{i=0}^{d} L_i \partial_i h + L_{d+1} h = \lambda (g + x_{d+1} h)
\] (11)
for some constant \( \lambda \), then \( L_i = \mu x_i \) for some constant \( \mu \). Write
\[
L_i(x_0, \ldots, x_{d+1}) = \lambda_i x_{d+1} + l_i(x_0, \ldots, x_d).
\]
Taking the coefficient of \( x_{d+1}^2 \) in (11), we obtain
\[
\sum_{i=0}^{d} \lambda_i \partial_i h = 0,
\]
and in particular, since \( h \) is non-degenerate, \( \lambda_i = 0 \) for \( 0 \leq i \leq d \). On the other hand, taking the coefficient of \( x_{d+1} \) in (11) gives
\[
\sum_{i=0}^{d} l_i \partial_i h + \lambda_{d+1} h = \lambda h, \quad \sum_{i=0}^{d} l_i \partial_i g + l_{d+1} h = \lambda g.
\] (12)
Consider now the linear subspace \( \mathbb{P}' = Z(x_{d+1}) \subset \mathbb{P} \), and the smooth complete intersection \( Z = Z(g, h) \subset \mathbb{P}' \). The restriction maps induce isomorphisms
\[
H^0(\Theta_{\mathbb{P}')} \cong H^0(\Theta_{\mathbb{P}'|Z}), \quad H^0(\mathcal{O}_{\mathbb{P}'(2)}/(h)) \cong H^0(\mathcal{O}_{Z}(2)),
\]
\[
H^0(\mathcal{O}_{\mathbb{P}'(3)}/(g, h\mathcal{O}_{\mathbb{P}'(1)})) \cong H^0(\mathcal{O}_{Z}(3)).
\]
Using these isomorphisms, one can, parallel to our description of \( H^0(\Theta_X) \), explicitly describe \( H^0(\Theta_Z) \) as a subspace of \( H^0(\Theta_{\mathbb{P}'}) \). Then (12) precisely means that
\[
\sum_{i=0}^{d} l_i \partial_i h \in H^0(\Theta_Z).
\]
Since \( Z \) is smooth, we have \( H^0(\Theta_Z) = 0 \); in particular, \( l_i(x_0, \ldots, x_d) = \mu x_i \) for a constant \( \mu \). Inserting this into (12) gives
\[
2\mu h + \lambda_{d+1} h = \lambda h, \quad 3\mu g + l_{d+1} h = \lambda g.
\]
Since \( X \) is irreducible, the second equation implies \( l_{d+1} = 0 \) and \( \lambda = 3\mu \), while the first one gives \( 2\mu + \lambda_{d+1} = \lambda \). \( \square \)

More generally, we expect that
\[
H^0(\Theta_X) = 0
\]
for any nodal cubic. Low-dimensional (\( 2 \leq d \leq 4 \)) nodal cubics are in fact known to be stable in the sense of geometric invariant theory [18], and so this holds for \( 2 \leq d \leq 4 \).
4.3 Locally trivial deformations

Instead of $D_X$, one could consider the subfunctor $D'_X$ of $D_X$ given by the locally trivial deformations of $X$; here $H^2(\Theta_X)$ is an obstruction space of $D'_X$ [24]. While it is known that if $d = 2$ or $d = 3$, then $H^2(\Theta_X) = 0$ [21], this vanishing need not hold when $d$ is large. In fact, the following holds:

**Proposition 4.2.** Let $X$ be a nodal cubic with $H^0(\Theta_X) = 0$. If $X$ has $\delta > \left(\frac{d+2}{3}\right)$ nodes, then $H^2(\Theta_X) \neq 0$.

Indeed, $H^0(\Theta_X) = 0$ and Lemma 4.1 imply that

$$\dim \text{Ext}^1(\Omega^1_X, \mathcal{O}_X) = \left(\frac{d+2}{3}\right),$$

and there is an exact sequence of the form

$$\text{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to H^0(\text{Ext}^1(\Omega^1_X, \mathcal{O}_X)) \to H^2(\Theta_X) \to 0,$$

coming from the local-to-global spectral sequence; here $\text{Ext}^1(\Omega^1_X, \mathcal{O}_X)$ is the structure sheaf of the singular locus. As a special case of a result of Varchenko [27],

$$\delta \leq \left(\frac{d+2}{3}\right)$$

which turns out to be optimal; hence $\delta > \left(\frac{d+2}{3}\right)$ is possible only for $d \geq 7$.

**Remark 4.2.** The space $H^2(\Theta_X)$ is canonically isomorphic to $H^1(N'_{X/P})$, where $N'_{X/P}$ is the equisingular normal sheaf of $X \subset P$. We can view $X$ as a point of the Hilbert scheme $V^\delta_d$ of cubic hypersurfaces in $P$ with $\delta$ nodes (Severi scheme); $H^0(N'_{X/P})$ and $H^1(N'_{X/P})$ are then the tangent and obstruction spaces of $V^\delta_d$ at $[X]$ [13]. Proposition 4.2 naturally leads to an extension of Theorem 111 of [7].

5 Deformations of $F$.

5.1 The functorial morphism $\eta$.

Consider a cubic $X$ with finitely many singularities, and an infinitesimal deformation $\mathcal{X}$ of $X$ over an Artin ring $\Lambda$. Then $\mathcal{X}$ is induced by a deformation $\mathcal{X} \subset P_\Lambda$ of $X$ in $P$, and $\mathcal{X} \subset P_\Lambda$ is a cubic (Remark 4.1). Using the induced polarisation $\mathcal{O}_X(1)$ of $\mathcal{X}$ over $\Lambda$, we can consider the relative Hilbert scheme of lines $F(\mathcal{X}/\Lambda)$, which is naturally a closed subscheme of $G_\Lambda$. The morphism

$$F(\mathcal{X}/\Lambda) \to \text{Spec}(\Lambda)$$
is flat, because the fibre over the closed point $F = F(X)$ is a local complete intersection (Theorem 2.2). In particular, $F(\mathfrak{X}/\Lambda)$ can be thought of as an infinitesimal deformation of $F$ in $G$ over $\Lambda$. For any morphism of local Artin rings $\Lambda \to \Lambda'$, we have

$$F(\mathfrak{X}/\Lambda) \times_\Lambda \Lambda' = F(\mathfrak{X}'_\Lambda/\Lambda')$$

as a subscheme of $G_{\Lambda'} = G_\Lambda \times_\Lambda \Lambda'$. The relative Hilbert scheme thus defines a morphism

$$\eta_H : \mathcal{H}_{X/P} \to \mathcal{H}_{F/G}.$$

of local Hilbert functors. Since $\text{Pic}(\mathfrak{X}) = \mathbb{Z}$ by the Grothendieck-Lefschetz theorem and $\omega_{\mathfrak{X}/\Lambda} = \mathcal{O}_\mathfrak{X}(1 - d)$, the isomorphism class of the deformation $F(\mathfrak{X}/\Lambda)$ of $F$ over $X$ depends only on the isomorphism class of the deformation $\mathfrak{X}$ of $X$ over $\Lambda$, and so we get a morphism

$$\eta : \mathcal{D}_X \to \mathcal{D}_F,$$ (13)

related to $\eta_H$ by a commutative diagram

$$\begin{array}{ccc}
\mathcal{H}_{X/P} & \longrightarrow & \mathcal{D}_X \\
\eta_H \downarrow & & \downarrow \eta \\
\mathcal{H}_{F/G} & \longrightarrow & \mathcal{D}_F.
\end{array}$$ (14)

The proof of our main theorem requires an analogue of Lemma 4.1 for $F \subset G$.

**Lemma 5.1.** Let $d \geq 5$. (i) The restriction map

$$H^0(S^3S^\vee) \to H^0(S^3S_{|F}^\vee)$$

is surjective with kernel $(\sigma_f)$.

(ii) The restriction map

$$H^0(\Theta_G) \to H^0(\Theta_G|_F)$$

is an isomorphism, and $H^1(\Theta_G|_F) = 0$.

**Proof.** (i) By Corollary 2.1, the spectral sequence

$$E_1^{pq} = H^q(\Lambda^{-p+1}S^3S \otimes S^3S^\vee) \Rightarrow H^{p+q}(\mathcal{J}_{F/G} \otimes S^3S^\vee)$$

degenerates at the $E_1$-page. In particular,

$$H^0(\mathcal{J}_{F/G} \otimes S^3S^\vee) \simeq H^0(S^3S \otimes S^3S^\vee) \quad \text{and} \quad H^1(\mathcal{J}_{F/G} \otimes S^3S^\vee) = 0.$$

Here $H^0(S^3S \otimes S^3S^\vee)$ is one-dimensional (Proposition 2.1), and it remains to combine this with the exact sequence in cohomology associated to

$$0 \to \mathcal{J}_{F/G} \otimes S^3S^\vee \to S^3S^\vee \to S^3S_{|F}^\vee \to 0.$$
(ii) Similarly, by Corollary 2.1 the spectral sequence
\[ E_1^{pq} = H^q(\Lambda^{-p+1}S^3S \otimes \Theta_G) \Rightarrow H^{p+q}(\mathcal{J}_{F/G} \otimes \Theta_G) \]
degenerates at the \( E_1 \)-page, and we obtain
\[ H^0(\mathcal{J}_{F/G} \otimes \Theta_G) = H^1(\mathcal{J}_{F/G} \otimes \Theta_G) = H^2(\mathcal{J}_{F/G} \otimes \Theta_G) = 0. \]
The result follows from this vanishing, and the exact sequence
\[ 0 \to \mathcal{J}_{F/G} \otimes \Theta_G \to \Theta_G \to \Theta_G|_F \to 0. \]

**Corollary 5.1.** The forgetful morphism \( \mathcal{H}_{F/G} \to D_F \) is smooth.

We now apply Lemma 2.4 to \( F = S^3S^\vee \) on \( S = G \). Let \( \pi : Z \to H^0(S^3S^\vee) \) be as in Lemma 2.4, and put \( \phi = \sigma^{-1} \circ \pi \), where \( \sigma \) is the isomorphism of Remark 2.1.

**Corollary 5.2.** The Kodaira-Spencer map \( \kappa_{\phi,f} : H^0(\mathcal{O}_F(3)) \to \text{Ext}^1(\Omega^1_{F}, \mathcal{O}_F) \) is surjective.

In other words, the deformation of \( F \) induced by \( \phi \) is complete at \( f \in H^0(\mathcal{O}_F(3)) \). For \( d \geq 6 \), this is a theorem of Borcea [5].

**Lemma 5.2.** There is a canonical isomorphism
\[ H^0(\Theta_X) \cong H^0(\Theta_F). \]

**Proof.** Consider the canonical morphism of automorphism groups
\[ \text{Aut}(X) \to \text{Aut}(F). \] (15)

Since \( X \) can be covered by lines, this morphism is injective; by [8] the image of (15) is the subgroup \( \text{Aut}(F, \mathcal{O}_F(1)) \) of automorphisms preserving the Plücker polarization. As \( H^1(\mathcal{O}_F) = 0, H^0(\Theta_F) \) the tangent space of \( \text{Aut}(F, \mathcal{O}_F(1)) \) at the identity. \( \square \)

### 5.2 Proof of the main theorem

**Theorem 5.1.** Let \( d \geq 5 \). Then the differential
\[ d\eta : \text{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to \text{Ext}^1(\Omega^1_F, \mathcal{O}_F) \]
of \( \eta \) is an isomorphism. If \( H^0(\Theta_X) = 0 \), then \( \eta \) is an isomorphism.

**Proof.** Consider the diagram
\[
\begin{array}{ccc}
H^0(N_{X/G}) & \longrightarrow & \text{Ext}^1(\Omega^1_X, \mathcal{O}_X) \\
\downarrow d\eta & & \downarrow d\eta \\
H^0(N_{F/G}) & \longrightarrow & \text{Ext}^1(\Omega^1_F, \mathcal{O}_F)
\end{array}
\]

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of differentials induced by (14). By Lemma 2.3 (i) and Lemma 5.1 (ii), the differential
\[ H^0(N_{F/G}) \to \text{Ext}^1(\Omega^1_F, \mathcal{O}_F) \]
of the forgetful morphism is surjective. To show that \( d\eta \) is surjective, it remains to observe that \( d\eta_H \) is surjective. The diagram
\[
\begin{array}{ccc}
H^0(\mathcal{O}_P(3)) & \longrightarrow & H^0(N_{X/P}) \\
\sigma \downarrow & & \downarrow d\eta_H \\
H^0(S^3S^\vee) & \longrightarrow & H^0(N_{F/G}),
\end{array}
\]
where the horizontal maps are given by restriction, is commutative; indeed, we have
\[ F(Z(f + \varepsilon g)/C[\varepsilon]) = Z(\sigma_f + \varepsilon\sigma_g) \]
by the description of section 2.2. Since \( \sigma \) is an isomorphism and the restriction map
\[ H^0(S^3S^\vee) \to H^0(N_{F/G}) \]
is surjective by Lemma 5.1 (i), it follows that \( d\eta_H \) is surjective. It now suffices to show that
\[ \dim \text{Ext}^1(\Omega^1_X, \mathcal{O}_X) = \dim \text{Ext}^1(\Omega^1_F, \mathcal{O}_F). \]
Consider the pair of exact sequences
\[
0 \to H^0(\Theta_X) \to H^0(\Theta_P|_X) \to H^0(N_{X/P}) \to \text{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to 0 \\
0 \to H^0(\Theta_F) \to H^0(\Theta_G|_F) \to H^0(N_{F/G}) \to \text{Ext}^1(\Omega^1_F, \mathcal{O}_F) \to 0
\]
associated to the conormal sequences of \( X \subset P \) and \( F \subset G \), respectively. By Lemma 5.2 we have
\[ h^0(\Theta_X) = h^0(\Theta_F), \]
while
\[ h^0(\Theta_P|_X) = h^0(\Theta_G|_F), \quad \text{and} \quad h^0(N_{X/P}) = h^0(N_{F/G}) \]
result from Lemma 4.1, Lemma 5.1, and Example 2.1.

If \( H^0(\Theta_X) = 0 \), then \( H^0(\Theta_F) = 0 \) by Lemma 5.2. Hence both \( \mathcal{D}_X \) and \( \mathcal{D}_F \) are pro-representable; since \( \mathcal{D}_X \) is smooth and \( d\eta \) bijective, it remains to apply Lemma 2.2 (ii).

**Corollary 5.3.** The morphism \( \eta_H \) is an isomorphism, and \( \eta \) is surjective.

**Proof.** This is a consequence of the proof of Theorem 5.1 rather than Theorem 5.1 itself. The proof shows that \( d\eta_H \) can be identified with the isomorphism
\[ H^0(\mathcal{O}_P(3))/(f) \cong H^0(S^3S^\vee)/(\sigma_f) \]
induced by \( \sigma \). As \( \mathcal{H}_{X/P} \) and \( \mathcal{H}_{F/G} \) are pro-representable, and \( \mathcal{H}_{X/P} \) smooth, \( \eta_H \) is an isomorphism by Lemma 2.2 (ii). Finally, \( \eta \) is surjective by Lemma 2.2 (i), as both \( \mathcal{D}_X \) and \( \mathcal{D}_F \) have a pro-representable hull by Schlessinger’s theorem, Theorem 2.3 (i). \[ \square \]
Remark 5.1. The proof of Theorem 5.1 depends on [8]. One could get rid of this dependence by establishing a commutative diagram

\[ \begin{array}{ccc}
H^0(\Theta_P|_X) & \xrightarrow{df} & H^0(\mathcal{N}_X/P) \\
\downarrow & & \downarrow d_{\mathcal{R}h} \\
H^0(\Theta_G|_F) & \xrightarrow{d\sigma_f} & H^0(\mathcal{N}_F/G),
\end{array} \]

where the isomorphism on the left is induced by Chow’s isomorphism \( \text{Aut}(P) \rightarrow \text{Aut}(G) \), and Lemma 4.1 (ii), Lemma 5.1 (ii). We expect \( \eta \) to be an isomorphism without assuming the condition \( H^0(\Theta_X) = 0 \).

5.3 Further questions

There are number of follow-up questions. If \( X \) is a Lefschetz cubic with a node at \( x_0 \), then the singular locus of \( F \) can be identified with a smooth complete intersection \( \Sigma \subset P_d \) of type \((2, 3)\). The scheme \( F \) has rational singularities, and the blow up

\[ \tilde{F} \rightarrow F \]

of \( F \) along \( \Sigma \) provides a resolution of singularities of \( F \) [9]. In such a situation, a general construction of Wahl [28] yields a blow-down morphism

\[ \beta : D_{\tilde{F}} \rightarrow D_F. \]

Here \( \tilde{F} \) is closely related to the Hilbert scheme of points \( \Sigma^{[2]} \). By [3], one has a canonical isomorphism \( H^1(\Theta_{\Sigma}) \xrightarrow{\sim} H^1(\Theta_{\Sigma^{[2]}}) \), which shows in particular that \( H^1(\Theta_{\Sigma^{[2]}}) \) has dimension \( \binom{d+2}{3} \); since this is also the dimension of \( \text{Ext}^1(\Omega^1_{\tilde{F}}, \mathcal{O}_F) \), the morphism \( \beta \) might be an isomorphism.

On the other hand, for smooth \( X \) it would be interesting to relate the non-commutative deformation theory (in the sense of [25]) of \( X \) to the one of \( F \). A crucial role is played by the Hochschild cohomology

\[ \text{HH}^2(F) = H^0(\Lambda^2\Theta_F) \oplus H^1(\Theta_F), \]

and the first step in this direction would be to compute the space \( H^0(\Lambda^2\Theta_F) \) of bivector fields on \( F \), and to exhibit Poisson structures on \( F \).

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