(Learned) Frequency Estimation Algorithms under Zipfian Distribution

Anders Aamand∗ Piotr Indyk† Ali Vakilian†

Abstract

The frequencies of the elements in a data stream are an important statistical measure and the task of estimating them arises in many applications within data analysis and machine learning. Two of the most popular algorithms for this problem, Count-Min and Count-Sketch, are widely used in practice.

In a recent work [Hsu et al., ICLR’19], it was shown empirically that augmenting Count-Min and Count-Sketch with a machine learning algorithm leads to a significant reduction of the estimation error. The experiments were complemented with an analysis of the expected error incurred by Count-Min (both the standard and the augmented version) when the input frequencies follow a Zipfian distribution. Although the authors established that the learned version of Count-Min has lower estimation error than its standard counterpart, their analysis of the standard Count-Min algorithm was not tight. Moreover, they provided no similar analysis for Count-Sketch.

In this paper we resolve these problems. First, we provide a simple tight analysis of the expected error incurred by Count-Min. Second, we provide the first error bounds for both the standard and the augmented version of Count-Sketch. These bounds are nearly tight and again demonstrate an improved performance of the learned version of Count-Sketch.

In addition to demonstrating tight gaps between the aforementioned algorithms, we believe that our bounds for the standard versions of Count-Min and Count-Sketch are of independent interest. In particular, it is a typical practice to set the number of hash functions in those algorithms to $\Theta(\log n)$. In contrast, our results show that to minimise the expected error, the number of hash functions should be a constant, strictly greater than 1.

∗BARC, University of Copenhagen, aa@di.ku.dk
†CSAIL, MIT, {indyk, vakilian}@mit.edu
1 Introduction

The last few years have witnessed a rapid growth in using machine learning methods to solve “classical” algorithmic problems. For example, they have been used to improve the performance of data structures [KBC+18 Mit18], online algorithms [LV18 PSK18 GP19], combinatorial optimization [KDZ+17 BDSV18], similarity search [WLKC16], compressive sensing [MPB15 BJD17] and streaming algorithms [HIKV19]. Multiple frameworks for designing and analyzing such algorithms were proposed [ACC+11 GR17 BDV18 ABL+19]. The rationale behind this line of research is that machine learning makes it possible to adapt the behavior of the algorithms to inputs from a specific data distribution, making them more efficient or more accurate in specific applications.

In this paper we focus on learning-augmented streaming algorithms for frequency estimation. The latter problem is formalized as follows: given a sequence \( S \) of elements from some universe \( U \), construct a data structure that for any element \( i \in U \) computes an estimation \( \tilde{f}_i \) of \( f_i \), the number of times \( i \) occurs in \( S \). Since counting data elements is a very common subroutine, frequency estimation algorithms have found applications in many areas, such as machine learning, network measurements and computer security. Many of the most popular algorithms for this problem, such as Count-Min (CM) [CM05] or Count-Sketch (CS) [CCFC02] are based on hashing. Specifically, these algorithms hash stream elements into \( B \) buckets, count the number of items hashed into each bucket, and use the bucket value as an estimate of item frequency. To improve the accuracy, the algorithms use \( k > 1 \) such hash functions and aggregate the answers. These algorithms have several useful properties: they can handle item deletions (implemented by decrementing the respective counters), and some of them (Count-Min) never underestimate the true frequencies, i.e., \( \tilde{f}_i \geq f_i \).

In a recent work [HIKV19], the authors showed that the aforementioned algorithm can be improved by augmenting them with machine learning. Their approach is as follows. During the training phase, they construct a classifier (neural network) to detect whether an element is “heavy”, i.e., whether \( f_i \) exceeds some threshold. After such a classifier is trained, they scan the input stream, and apply the classifier to each element \( i \). If the element is predicted to be heavy, it is allocated a unique bucket, so that an exact value of \( f_i \) is computed. Otherwise, the element is forwarded to a “standard” hashing data structure \( C \), e.g., CM or CS. To estimate \( \tilde{f}_i \), the algorithm either returns the exact count \( f_i \) (if \( i \) is allocated a unique bucket) or an estimate provided by the data structure \( C \). An empirical evaluation, on networking and query log data sets, shows that this approach can reduce the overall estimation error.

The paper also presents a preliminary analysis of the algorithm. Under the common assumption that the frequencies follow the Zipfian law, i.e., \( f_i \propto 1/i \), and further that item \( i \) is queried with probability proportional to its frequency, the expected error incurred by the learning-augmented version of CM is shown to be asymptotically lower than that of the “standard” CM. However, the magnitude of the gap between the error incurred by the learned and standard CM algorithms has not been established. Specifically, [HIKV19] only shows that the expected error of standard CM with \( k \) hash functions and a total of \( B \) buckets is between \( \frac{k}{B \log(k)} \) and \( \frac{k \log(k+2)/(k-1)(kn/B)}{B} \). Furthermore, no such analysis was presented for CS.

---

1See Figure 1 for a generic implementation of the learning-based algorithms of [HIKV19].
2In fact we will assume that \( f_i = 1/i \). This is just a matter of scaling and is convenient as it removes the dependence of the length of the stream in our bounds
3This assumes that the error rate for the “heaviness” predictor is low enough. Aiming at a theoretical understanding, in this paper we focus on the case where the error rate is zero.
1.1 Our results

In this paper we resolve the aforementioned questions left open in [HIKV19]. Assuming that the frequencies follow a Zipfian law, we show:

- An asymptotically tight bound of $\Theta\left(\frac{k \log (kn/B)}{B}\right)$ for the expected error incurred by the CM algorithm with $k$ hash functions and a total of $B$ buckets. Together with a prior bound for Learned CM (Table 1), this shows that learning-augmentation improves the error of CM by a factor of $\Theta\left(\log (n)/\log (n/B)\right)$ if the heavy hitter oracle is perfect.

- The first error bounds for CS and Learned CS (see Table 1). In particular, we show that for Learned CS, using a single hash function as in [HIKV19] leads to an asymptotically optimal error bound, improving over standard CS by a factor of $\Theta\left(\log (n)/\log (n/B)\right)$ (same as CM).

|                      | $k = 1$                      | $k > 1$                      |
|----------------------|-----------------------------|------------------------------|
| **Count-Min (CM)**   | $\Theta\left(\frac{\log n}{B}\right)$ [HIKV19] | $\Theta\left(\frac{k \log (kn/B)}{B}\right)$ |
| **Learned Count-Min (L-CM)** | $\Theta\left(\frac{\log^2 (\frac{n}{B})}{B \log n}\right)$ [HIKV19] | $\Omega\left(\frac{\log^2 (\frac{n}{B})}{B \log n}\right)$ [HIKV19] |
| **Count-Sketch (CS)** | $\Theta\left(\frac{\log B}{B \log n}\right)$ | $\Omega\left(\frac{k^{1/2}}{B \log k}\right)$ and $O\left(\frac{k^{1/2}}{B}\right)$ |
| **Learned Count-Sketch (L-CS)** | $\Theta\left(\frac{\log \frac{n}{B}}{B \log n}\right)$ | $\Omega\left(\frac{\log \frac{n}{B}}{B \log n}\right)$ |

Table 1: This table summarizes our and previously known results on the expected frequency estimation error of Count-Min (CM), Count-Sketch (CS) and their learned variants (i.e., L-CM and L-CS) that use $k$ functions and overall space $k \times \frac{B}{k}$ under Zipfian distribution. For CS, we assume that $k$ is odd (so that the median of $k$ values is well defined). The lower bounds for L-CM and L-CS are assuming that we use the information from a perfect heavy hitter oracle (i.e., the oracle makes no mistake in its predictions) to place the heavy hitters in separate buckets.

In addition to clarifying the gap between the learned and standard variants of popular frequency estimation algorithms, our results provide interesting insights about the algorithms themselves. For example, for both CM and CS, the number of hash functions $k$ is often selected to be $\Theta(\log n)$, in order to guarantee that every frequency is estimated up to a certain error bound. In contrast, we show that if instead the goal is to bound the expected error, then setting $k$ to a constant (strictly greater than 1) leads to the asymptotic optimal performance. We remark that the same phenomenon holds not only for a Zipfian query distribution but in fact for an arbitrary distribution on the queries, e.g. the uniform (see Remark 2.2).

1.2 Related work

In addition to the aforementioned hashing-based algorithms [CM05, CCFC02], multiple non-hashing algorithms were also proposed, e.g., [MG82, MM02, MAEA05]. These algorithms often exhibit better accuracy/space tradeoffs, but do not posses many of the properties of hashing-based methods, such as the ability to handle deletions as well as insertions.

Zipf law is a common modeling tool used to evaluate the performance of frequency estimation algorithms, and has been used in many papers in this area, including [MM02, MAEA05, CCFC02].
In its general form it postulates that $f_i$ is proportional to $1/i^s$ for some exponent parameter $s > 0$. In this paper we focus on the “original” Zipf law where $s = 1$. However, the techniques introduced in this paper can be applied to other values of the exponent $s$ as well.

1.3 Our techniques

Our main contribution is our analysis of the standard Count-Min and Count-Sketch algorithms for Zipfians with $k > 1$ hash functions. Showing the improvement for the learned counterparts is relatively simple (for Count-Min it was already done in [HIKVD19]). In both of these analyses we consider a fixed item $i$ and bound $\mathbb{E}[|f_i - \tilde{f}_i|]$ whereupon linearity of expectation leads to the desired results. In the following we assume that $f_j = 1/j$ for each $j \in [n]$ and describe our techniques for bounding $\mathbb{E}[|f_i - \tilde{f}_i|]$ for each of the two algorithms.

**Count-Min.** With a single hash function and $B$ buckets the head of the Zipfian distribution, namely the items of frequencies $(f_i)_{i \in [B]}$, contribute with $\log B/B$ to the expected error. Our main observation is that the fast decay of $(f_i)_{i \in [B]}$ reduces this to $1/B$ for $k \geq 2$ and that in fact the main contribution to the error comes from the light items $(f_i)_{i \in [n] \setminus [B]}$. The expected contribution of these items is easily upper bounded and can be lower bounded using Bennett’s concentration inequality. In contrast to the analysis from [HIKV19] which is technical and leads to suboptimal bounds, our analysis is short, simple, and yields completely tight bounds in terms of all of the parameters $k, n$ and $B$.

**Count-Sketch.** Simply put, our main contribution is an improved understanding of the distribution of random variables of the form $S = \sum_{i=1}^{n} f_i \eta_i \sigma_i$. Here the $\eta_i \in \{0, 1\}$ are i.i.d Bernoulli random variables and the $\sigma_i \in \{-1, 1\}$ are independent Rademachers, that is, $\Pr[\eta_i = 1] = \Pr[\eta_i = -1] = 1/2$. Note that the counters used in CS are random variables having precisely this form. Usually such random variables are studied for the purpose of obtaining large deviation results. In contrast, in order to analyze CS, we are interested in a fine-grained picture of the distribution within a “small” interval $I$ around zero, say with $\Pr[S \in I] = 1/2$. For example when proving a lower bound on $\mathbb{E}[|f_i - \tilde{f}_i|]$ we must prove a certain anti-concentration of $S$ around $0$. More precisely we find an interval $J \subset I$ centered at zero such that $\Pr[S \in J] = O(1/\sqrt{k})$. Combined with the fact that we use $k$ independent hash functions as well as properties of the median and the binomial distribution, this gives that $\mathbb{E}[|f_i - \tilde{f}_i|] = \Omega(|J|)$. Anti-concentration inequalities of this type are in general notoriously hard to obtain but it turns out that we can leverage the properties of the Zipfian distribution, specifically its heavy head. For our upper bounds on $\mathbb{E}[|f_i - \tilde{f}_i|]$ we need strong lower bounds on $\Pr[S \in J]$ for intervals $J \subset I$ centered at zero. Then using concentration inequalities we can bound the probability that half of the $k$ relevant counters are smaller (larger) than the lower (higher) endpoint of $J$, i.e., that the median does not lie in $J$. Again this requires a precise understanding of the distribution of $S$ within $I$.

1.4 Structure of the paper

In Section 2 we describe the algorithms Count-Min and Count-Sketch. We also formally define the estimation error that we will study as well as the Zipfian distribution. In Sections 3 and 4 we provide our analyses of the expected error of Count-Min and Count-Sketch. In Section 5 we analyze the performance of learned Count-Sketch.
2 Preliminaries

We start out by recapping the sketching algorithms Count-Min and Count-Sketch. Common to both of these algorithms is that we sketch a stream $S$ of elements coming from some universe $U$ of size $n$. For notational convenience we will assume that $U = [n] := \{1, \ldots, n\}$. If item $i$ occurs $f_i$ times then either algorithm outputs an estimate $\hat{f}_i$ of $f_i$.

**Count-Min.** We use $k$ independent and uniformly random hash functions $h_1, \ldots, h_k : [n] \rightarrow [B]$. Letting $C$ be an array of size $[k] \times [B]$ we let $C[\ell, b] = \sum_{j \in [n]} [h_\ell(j) = b]f_j$. When querying $i \in [n]$ the algorithm returns $\hat{f}_i = \min_{\ell \in [k]} C[\ell, h_\ell(i)]$. Note that we always have that $\hat{f}_i \geq f_i$.

**Count-Sketch.** We pick independent and uniformly random hash functions $h_1, \ldots, h_k : [n] \rightarrow [-1, 1]$ and $s_1, \ldots, s_k : [n] \rightarrow \{-1, 1\}$. Again we initialize an array $C$ of size $[k] \times [B]$ but now we let $C[\ell, b] = \sum_{j \in [n]} [h_\ell(j) = b]s_\ell(j)f_j$. When querying $i \in [n]$ the algorithm returns the estimate $\hat{f}_i = \text{median}_{\ell \in [k]} s_\ell(i) \cdot C[\ell, h_\ell(i)]$.

**Remark 2.1.** The bounds presented in Table 1 assumes that the hash functions have codomain $[B/k]$ and not $[B]$, i.e., that the total number of buckets is $B$. In the proofs to follows we assume for notational ease that the hash functions take value in $[B]$ and the claimed bounds follows immediately by replacing $B$ by $B/k$.

**Estimation Error.** To measure and compare the overall accuracy of different frequency estimation algorithms, we will use the expected estimation error which is defined as follows: let $F = \{f_1, \cdots, f_n\}$ and $\tilde{F}_A = \{\tilde{f}_1, \cdots, \tilde{f}_n\}$ respectively denote the actual frequencies and the estimated frequencies obtained from algorithm $A$ of items in the input stream. We remark that when $A$ is clear from the context we denote $\tilde{F}_A$ as $\tilde{F}$. Then we define

$$\text{Err}(F, \tilde{F}_A) := \mathbb{E}_{i \sim D}[f_i - \tilde{f}_i], \quad (1)$$

where $D$ denotes the query distribution of the items. Here, similar to previous work (e.g., [RKA16, HIKV19]), we assume that the query distribution $D$ is the same as the frequency distribution of items in the stream, i.e., for any $i^* \in [n]$, $Pr_{i \sim D}[i = i^*] \propto f_{i^*}$ (more precisely, for any $i^* \in [n]$, $Pr_{i \sim D}[i = i^*] = f_{i^*}/N$ where $N = \sum_{i \in [n]} f_i$ denotes the total sum of all frequencies in the stream).

**Remark 2.2.** As all upper/lower bounds in this paper are proved by bounding the expected error of estimating the frequency a single item, $E[|\hat{f}_i - f_i|]$, then using linearity of expectation, in fact we obtain analogous bounds for any query distribution $(p_i)_{i \in [n]}$. In particular this means that the bounds of Table 1 for CM and CS hold for any query distribution. For L-CM and L-CS the factor of $\log(n/B)/\log n$ gets replaced by $\sum_{i=B_{h+1}}^{n} p_i$ where $B_h = \Theta(B)$ is the number of buckets reserved for heavy hitters.

**Zipfian Distribution** In our analysis we assume that the frequency distribution of items follows Zipf’s law. That is, if we sort the items according to their frequencies with no loss of generality assuming that $f_1 \geq f_2 \geq \cdots \geq f_n$, then for any $j \in [n]$, $f_j \propto 1/j$. Given that the frequencies of items follow Zipf’s law and assuming that the query distribution is the same as the distribution of
the frequency of items in the input stream (i.e., \( \Pr_{i \sim D}[i^*] = f_{i^*} / N = 1/(i^* \cdot H_n) \)) where \( H_n \) denotes the \( n \)-th harmonic number), we can write the expected error defined in eq. (1) as follows:

\[
\text{Err}(F, \tilde{F}_A) = \mathbb{E}_{i \sim D}[|f_i - \tilde{f}_i|] = \frac{1}{N} \cdot \sum_{i \in [n]} |\tilde{f}_i - f_i| \cdot f_i = \frac{1}{H_n} \cdot \sum_{i \in [n]} |\tilde{f}_i - f_i| \cdot \frac{1}{i} \quad (2)
\]

Throughout this paper, we present our results with respect to the objective function in the right hand side of eq. (2), i.e., \((1/H_n) \cdot \sum_{i=1}^n |\tilde{f}_i - f_i| \cdot f_i\). We further assume that in fact \( f_i = 1/i \). At first sight this assumption may seem strange since it says that item \( i \) appears a non-integral number of times in the stream. This is however just a matter of scaling and the assumption is convenient as it removes the dependence on the length of the stream in our bounds.

**Algorithm 1** Learning-Based Frequency Estimation

```
1: procedure LearnSketch(B, B_h, HH-Oracle, Sketch-Alg)
2:     for each stream element \( i \) do
3:         if HH-Oracle(\( i \)) = 1 then  \triangleright the oracle predicts \( i \) as heavy (one of \( B_h \) most frequent items)
4:             if a unique bucket is already assigned to item \( i \) then
5:                 counter_\( i \) \leftarrow counter_\( i \) + 1
6:             else
7:                 allocate a new unique bucket to item \( i \) and counter_\( i \) \leftarrow 1
8:         end if
9:     else   \triangleright an instance of Sketch-Alg with \( B - B_h \) many buckets
10:         feed \( i \) to Sketch-Alg(\( B - B_h \))
11:     end if
12: end for
13: end procedure
```

Figure 1: A generic learning augmented algorithm for the frequency estimation problem. HH-Oracle denotes a given learned oracle for detecting whether the item is among the top \( B_h \) frequent items of the stream and Sketch-Alg is a given (sketching) algorithm (e.g., CM or CS) for the frequency estimation problem.

**Learning Augmented Sketching Algorithms for Frequency Estimation.** In this paper, following the approach of [HIKV19], the learned variants of CM and CS are algorithms augmented with a machine learning based heavy hitters oracle. More precisely, we assume that the algorithm has access to an oracle HH-Oracle that predicts whether an item is “heavy” (i.e., is one of the \( B_h \) most frequent items) or not. Then, the algorithm treats heavy and non-heavy items differently: (a) a unique bucket is allocated to each heavy item and their frequencies are computed with no error, (b) the rest of items are fed to the given (sketching) algorithm Sketch-Alg using the remaining \( B - B_h \) buckets and their frequency estimates are computed via Sketch-Alg. See Figure 1.

Note that, in general the oracle HH-Oracle can make errors. Aiming at a theoretical understanding, in this paper we focus on the case where the oracle is perfect, i.e., the error rate is zero. We also assume that \( B_h = \Theta(B - B_h) = \Theta(B) \), that is, we use approximately the same number of buckets for the heavy items as for the sketching of the light items. One justification for this assumption is that in any case we can increase both the number of buckets for heavy and light items to \( B \) without affecting the overall asymptotic space usage.
3 Tight Bounds for Count-Min with Zipfians

For both Count-Min and Count-Sketch we aim at analyzing the expected value of the variable \( \sum_{i \in [n]} f_i \cdot |\tilde{f}_i - f_i| \) where \( f_i = 1/i \) and \( \tilde{f}_i \) is the estimate of \( f_i \) output by the relevant sketching algorithm. Throughout this paper we use the following notation: For an event \( E \) we denote by \( [E] \) the random variable in \( \{0, 1\} \) which is 1 if and only if \( E \) occurs. We begin by presenting our improved analysis of Count-Min with Zipfians. The main theorem is the following.

**Theorem 3.1.** Let \( n, B, k \in \mathbb{N} \) with \( k \geq 2 \) and \( B \leq n/k \). Let further \( h_1, \ldots, h_k : [n] \rightarrow [B] \) be independent and truly random hash functions. For \( i \in [n] \) define the random variable \( \tilde{f}_i = \min_{\ell \in [k]} \left( \sum_{j \in [n]} [h_\ell(j) = h_\ell(i)] f_j \right) \). For any \( i \in [n] \) it holds that

\[
E[|\tilde{f}_i - f_i|] = \Theta \left( \frac{\log \left( \frac{n}{B} \right)}{B} \right)
\]

Replacing \( B \) by \( B/k \) in Theorem 3.1 and using linearity of expectation we obtain the desired bound for Count-Min in the upper right hand side of Table 1. The natural assumption that \( B \leq n/k \) simply says that the total number of buckets is upper bounded by the number of items.

To prove Theorem 3.1 we start with the following lemma which is a special case of the theorem.

**Lemma 3.2.** Suppose that we are in the setting of Theorem 3.1 and further that \( n = B \). Then

\[
E[|\tilde{f}_i - f_i|] = O \left( \frac{1}{n} \right).
\]

**Proof.** It suffices to show the result when \( k = 2 \) since adding more hash functions and corresponding tables only decreases the value of \( |\tilde{f}_i - f_i| \). Define \( Z_\ell = \sum_{j \in [n] \setminus \{i\}} [h_\ell(j) = h_\ell(i)] f_j \) for \( \ell \in [2] \) and note that these variables are independent. For a given \( t \geq 3/n \) we wish to upper bound \( \Pr[Z_\ell \geq t] \).

Let \( s < t \) and note that if \( Z_\ell \geq t \) then either of the following two events must hold:

- \( E_1 \): There exists a \( j \in [n] \setminus \{i\} \) with \( f_j > s \) and \( h_\ell(j) = h_\ell(i) \).
- \( E_2 \): The set \( \{ j \in [n] \setminus \{i\} : h_\ell(j) = h_\ell(i) \} \) contains at least \( t/s \) elements.

Union bounding we find that

\[
\Pr[Z_\ell \geq t] \leq \Pr[E_1] + \Pr[E_2] \leq \frac{1}{ns} + \left( \frac{n}{t/s} \right) n^{-t/s} \leq \frac{1}{ns} + \left( \frac{es}{t} \right)^{t/s}.
\]

Choosing \( s = \frac{t}{\log(t/n)} \), a simple calculation yields that \( \Pr[Z_\ell \geq t] = O \left( \frac{\log(t/n)}{tn} \right) \). As \( Z_1 \) and \( Z_2 \) are independent, \( \Pr[Z \geq t] = O \left( \left( \frac{\log(t/n)}{tn} \right)^2 \right) \), so

\[
E[Z] = \int_0^\infty \Pr[Z \geq t] \, dt \leq \frac{3}{n} + O \left( \int_{3/n}^\infty \left( \frac{\log(t/n)}{tn} \right)^2 \, dt \right) = O \left( \frac{1}{n} \right).
\]

\[\square\]

\(^4\)In particular we dispose with the assumption that \( B \leq n/k \).
Before proving the full statement of Theorem 3.1, we recall Bennett’s inequality.

**Theorem 3.3** (Bennett’s inequality [Ben62]). Let $X_1, \ldots, X_n$ be independent, mean zero random variables. Let $S = \sum_{i=1}^n X_i$, and $\sigma^2, M > 0$ be such that $\text{Var}[S] \leq \sigma^2$ and $|X_i| \leq M$ for all $i \in [n]$. For any $t \geq 0$,

$$\Pr[S \geq t] \leq \exp\left(-\frac{\sigma^2}{M^2} h\left(\frac{tM}{\sigma^2}\right)\right),$$

where $h : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is defined by $h(x) = (x+1) \log(x+1) - x$. The same tail bound holds on the probability $\Pr[S \leq -t]$.

**Remark 3.4.** It is well known and easy to check that for $x \geq 0$,

$$\frac{1}{2}x \log(x+1) \leq h(x) \leq x \log(x+1).$$

We will use these asymptotic bounds repeatedly in this paper.

**Proof of Theorem 3.1.** We start out by proving the upper bound. Let $N_1 = [B] \setminus \{i\}$ and $N_2 = [n] \setminus ([B] \cup \{i\})$. Let $b \in [k]$ be such that $\sum_{j \in N_1} f_j \cdot [h_b(j) = h_b(i)]$ is minimized. Note that $b$ is itself a random variable. We also define

$$Y_1 = \sum_{j \in N_1} f_j \cdot [h_b(j) = h_b(i)],$$

$$Y_2 = \sum_{j \in N_2} f_j \cdot [h_b(j) = h_b(i)].$$

Clearly $|\tilde{f}_i - f_i| \leq Y_1 + Y_2$. Using Lemma 3.2, we obtain that $\mathbb{E}[Y_1] = O\left(\frac{n}{B}\right)$. For $Y_2$ we observe that

$$\mathbb{E}[Y_2 | b] = \sum_{j \in N_2} \frac{f_j}{B} = O\left(\frac{\log\left(\frac{n}{B}\right)}{B}\right).$$

We conclude that

$$\mathbb{E}[|\tilde{f}_i - f_i|] \leq \mathbb{E}[Y_1] + \mathbb{E}[Y_2] = \mathbb{E}[Y_1] + \mathbb{E}[\mathbb{E}[Y_2 | b]] = O\left(\frac{\log\left(\frac{n}{B}\right)}{B}\right),$$

as desired.

Next we show the lower bound. For $j \in [n]$ and $\ell \in [k]$ we define $X^{(j)}_\ell = f_j \cdot ([h_\ell(j) = h_\ell(i)] - \frac{1}{B})$. Note that the variables $(X^{(j)}_\ell)_{j \in [n]}$ are independent. We also define $S_\ell = \sum_{j \in N_2} X^{(j)}_\ell$ for $\ell \in [k]$. Observe that $|X^{(j)}_\ell| \leq f_j \leq \frac{1}{B}$ for $j \geq B$, $\mathbb{E}[X^{(j)}_\ell] = 0$, and that

$$\text{Var}[S_\ell] = \sum_{j \in N_2} f_j^2 \left(\frac{1}{B} - \frac{1}{B^2}\right) \leq \frac{2}{B^2}.$$ 

Applying Bennett’s inequality with $\sigma^2 = \frac{1}{B^2}$ and $M = 1/B$ thus gives that

$$\Pr[S_\ell \leq -t] \leq \exp\left(-2h\left(\frac{tB}{2}\right)\right).$$
Defining \( W_\ell = \sum_{j \in N_2} f_j \cdot [h_\ell(j) = h_\ell(i)] \) it holds that \( \mathbb{E}[W_\ell] = \Theta \left( \frac{\log(\frac{1}{\ell})}{B} \right) \) and \( S_\ell = W_\ell - \mathbb{E}[W_\ell] \), so putting \( t = \mathbb{E}[W_\ell]/2 \) in the inequality above we obtain that

\[
\Pr[W_\ell \leq \mathbb{E}[W_\ell]/2] = \Pr[S_\ell \leq -\mathbb{E}[W_\ell]/2] \leq \exp\left(-2h\left(\Omega \left( \log \frac{n}{B} \right) \right)\right).
\]

Appealing to Remark 3.4 and using that \( B \leq n/k \) the above bound becomes

\[
\Pr[W_\ell \leq \mathbb{E}[W_\ell]/2] \leq \exp\left(-\Omega \left( \log k \cdot \log(\log k + 1) \right) \right) = \exp\left(-\Omega(\log(\log k + 1))\right), \tag{3}
\]

By the independence of the events \( W_\ell > \mathbb{E}[W_\ell]/2 \), we have that

\[
\Pr[|\tilde{f}_i - f_i| \geq \mathbb{E}[W_\ell]/2] \geq (1 - k^{-\Omega(\log(\log k + 1))})^k = \Omega(1),
\]

and so \( \mathbb{E}[|\tilde{f}_i - f_i|] = \Omega(\mathbb{E}[W_\ell]) = \Omega \left( \frac{\log(\frac{n}{B})}{B} \right) \), as desired.

\[\square\]

### 4. (Nearly) tight Bounds for Count-Sketch with Zipfians

In this section we proceed to analyze Count-Sketch for Zipfians either using a single or more hash functions. We start with two simple lemmas which for certain frequencies \( (f_i)_{i \in [n]} \) of the items in the stream can be used to obtain respectively good upper and lower bounds on \( \mathbb{E}[|\tilde{f}_i - f_i|] \) in Count-Sketch with a single hash function. We will use these two lemmas both in our analysis of standard and learned Count-Sketch for Zipfians.

**Lemma 4.1.** Let \( w = (w_1, \ldots, w_n) \in \mathbb{R}^n \), \( \eta_1, \ldots, \eta_n \) independent Bernoulli variables taking value 1 with probability \( p \), and \( \sigma_1, \ldots, \sigma_n \in \{-1, 1\} \) independent Rademachers, i.e., \( \Pr[\eta_i = 1] = \Pr[\eta_i = -1] = 1/2 \). Let \( S = \sum_{i=1}^n w_i \eta_i \sigma_i \). Then

\[
\mathbb{E}[|S|] = O(\sqrt{p} ||w||_2).
\]

**Proof.** Using that \( \mathbb{E}[\sigma_i \sigma_j] = 0 \) for \( i \neq j \) and Jensen’s inequality

\[
\mathbb{E}[|S|^2] \leq \mathbb{E}[S^2] = \mathbb{E} \left[ \sum_{i=1}^n w_i^2 \eta_i \right] = p ||w||_2^2,
\]

from which the result follows. \( \square \)

**Lemma 4.2.** Suppose that we are in the setting of Lemma 4.1. Let \( I \subset [n] \) and let \( w_I \in \mathbb{R}^n \) be defined by \((w_I)_i = |i \in I| \cdot w_i \). Then

\[
\mathbb{E}[|S|] \geq \frac{1}{2} p (1 - p)^{|I|-1} \|w_I\|_1.
\]
Proof. Let $J = [n] \setminus I$, $S_1 = \sum_{i \in J} w_i \eta_i \sigma_i$, and $S_2 = \sum_{i \in J} w_i \hat{\eta}_i \sigma_i$. Let $E$ denote the event that $S_1$ and $S_2$ have the same sign or $S_2 = 0$. Then $Pr[E] \geq 1/2$ by symmetry. For $i \in I$ we denote by $A_i$ the event that $\{j \in I : n_j \neq 0\} = \{i\}$. Then $Pr[A_i] = p(1 - p)|I|^{-1}$ and furthermore $A_i$ and $E$ are independent. If $A_i \cap E$ occurs, then $|S| \geq |w_i|$ and as the events $(A_i \cap E)_{i \in I}$ are disjoint it thus follows that
\[ \mathbb{E}[|S|] \geq \sum_{i \in I} Pr[A_i \cap E] \cdot |w_i| = \frac{1}{2} p(1 - p)|I|^{-1} \|w_I\|_1. \]

\[ \square \]

4.1 One hash-function

We are now ready to commence our analysis of Count-Sketch for Zipfians. As in the discussion succeeding Theorem 3.1 the following theorem yields the desired result for a single hash function as presented in Table 1.

**Theorem 4.3.** Suppose that $B \leq n$ and let $h : [n] \rightarrow [B]$ and $s : [n] \rightarrow \{-1, 1\}$ be truly random hash functions. Define the random variable $\tilde{f}_i = \sum_{j \in [n]} h(j) = h(i)s(j)f_j$ for $i \in [n]$. Then
\[ \mathbb{E}[|\tilde{f}_i - s(i)f_i|] = O\left(\frac{\log B}{B}\right). \]

**Proof.** Let $i \in [n]$ be fixed. We start by defining $N_1 = [B] \setminus \{i\}$ and $N_2 = [n] \setminus ([B] \cup \{i\})$ and note that
\[ |\tilde{f}_i - s(i)f_i| \leq \sum_{j \in N_1} |h(j) = h(i)|s(j)f_j| + \sum_{j \in N_2} |h(j) = h(i)|s(j)f_j| := X_1 + X_2. \]

Using the triangle inequality
\[ \mathbb{E}[X_1] \leq \frac{1}{B} \sum_{j \in N_1} f_j = O\left(\frac{\log B}{B}\right). \]

Also, by Lemma 4.1, $\mathbb{E}[X_2] = O\left(\frac{1}{B}\right)$ and combining the two bounds we obtain the desired upper bound.

For the lower bound we apply Lemma 4.2 with $I = N_1$ concluding that
\[ \mathbb{E}[|\tilde{f}_i - s(i)f_i|] \geq \frac{1}{2B} \left(1 - \frac{1}{B}\right)^{|N_1|^{-1}} \sum_{i \in N_1} f_i = \Omega\left(\frac{\log B}{B}\right). \]

\[ \square \]

4.2 Multiple hash functions

Let $k \in \mathbb{N}$ be odd. For a tuple $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ we denote by $\text{median} x$ the median of the entries of $x$. The following theorem immediately leads to the result on CS with $k \geq 3$ hash functions claimed in Table 1.
Theorem 4.4. Let \( k \geq 3 \) be odd, \( n \geq kB \), and \( h_1, \ldots, h_k : [n] \to [B] \) and \( s_1, \ldots, s_k : [n] \to \{-1, 1\} \) be truly random hash functions. Define the random variable \( \tilde{f}_i = \text{median}_{\ell \in [k]} \left( \sum_{j \in [n]} [h_\ell(j) = h_\ell(i)] s_\ell(j) f_j \right) \) for \( i \in [n] \). Assume that \( k \leq B \). Then

\[
\mathbb{E}[|\tilde{f}_i - s(i) f_i|] = \Omega \left( \frac{1}{B \sqrt{k \log k}} \right), \quad \text{and} \quad \mathbb{E}[Z] = O \left( \frac{1}{B \sqrt{k}} \right)
\]

The assumption \( n \geq kB \) simply says that the total number of buckets is upper bounded by the number of items. Again using linearity of expectation for the summation over \( i \in [n] \) and replacing \( B \) by \( B/k \) we obtain the claimed upper and lower bounds of \( \frac{\sqrt{k}}{B \log k} \) and \( \frac{\sqrt{k}}{B} \) respectively. We note that even if the bounds above are only tight up to a factor of \( \log k \) they still imply that it is asymptotically optimal to choose \( k = O(1) \), e.g. \( k = 3 \). To settle the correct asymptotic growth is thus of merely theoretical interest.

Proof. If \( B \) (and hence \( k \)) is a constant the results follows easily from Lemma 4.1 so in what follows we may assume that \( B \) is larger than a sufficiently large constant.

We first prove the upper bound. Define \( N_1 = [B] \setminus \{i\} \) and \( N_2 = [n] \setminus ([B] \cup \{i\}) \). Let for \( \ell \in [k] \), \( X_1^{(\ell)} = \sum_{j \in N_1} [h_\ell(j) = h_\ell(i)] s_\ell(j) f_j \) and \( X_2^{(\ell)} = \sum_{j \in N_2} [h_\ell(j) = h_\ell(i)] s_\ell(j) f_j \). Finally write \( X^{(\ell)} = X_1^{(\ell)} + X_2^{(\ell)} \).

As the absolute error in Count-Sketch with one pair of hash functions \((h, s)\) is always upper bounded by the corresponding error in Count-Min with the single hash function \( h \), we can use the bound in the proof of Lemma 3.2 to conclude that

\[
\Pr[|X_1^{(\ell)}| \geq t] = O \left( \frac{\log(tB)}{tB} \right),
\]

when \( t \geq 3/B \). Also

\[
\text{Var}[X_2^{(\ell)}] = \left( \frac{1}{B} - \frac{1}{B^2} \right) \sum_{j \in N_2} f_j^2 \leq \frac{2}{B^2},
\]

so by Bennett’s inequality (with \( M = 1/B \) and \( \sigma^2 = 2/B^2 \)) and Remark 3.4

\[
\Pr[|X_2^{(\ell)}| \geq t] \leq 2 \exp \left( -2h(tB/2) \right) \leq 2 \exp \left( -\frac{1}{2} tB \log \left( \frac{tB}{2} + 1 \right) \right) = O \left( \frac{\log(tB)}{tB} \right),
\]

for \( t \geq \frac{3}{B} \). It follows that for \( t \geq 3/B \),

\[
\Pr[|X^{(\ell)}| \geq 2t] \leq \Pr[|X_1^{(\ell)}| \geq t] + \Pr[|X_2^{(\ell)}| \geq t] = O \left( \frac{\log(tB)}{tB} \right).
\]

Let \( C \) be the implicit constant in the \( O \)-notation above. If \( |\tilde{f}_i - s(i) f_i| \geq 2t \), at least half of the values \((|X^{(\ell)}|)_{\ell \in [k]}\) are at least \( 2t \). For \( t \geq 3/B \) it thus follows by a union bound that

\[
\Pr[|\tilde{f}_i - s(i) f_i| \geq 2t] \leq \frac{k}{[k/2]} C \left( \frac{\log(tB)}{tB} \right)^{[k/2]} \leq \left( 4C \frac{\log(tB)}{tB} \right)^{[k/2]}. \tag{4}
\]

This very mild assumption can probably be removed at the cost of a more technical proof. In our proof it can even be replaced by \( k \leq B^{2-\varepsilon} \) for any \( \varepsilon = \Omega(1) \).
If $\alpha = O(1)$ is chosen sufficiently large it thus holds that

\[
\int_{\alpha/B}^{\infty} \Pr[|\tilde{f}_i - s(i)f_i| \geq t] dt = 2 \int_{\alpha/(2B)}^{\infty} \Pr[|\tilde{f}_i - s(i)f_i| \geq 2t] dt \\
\leq \frac{4}{B} \int_{\alpha/2}^{\infty} \left( 4C^2 \log(t) \right)^{[k/2]} dt \\
\leq \frac{1}{B^{2k}} \leq \frac{1}{B\sqrt{k}}.
\]

Here the first inequality uses eq. (11) and a change of variable. The second inequality uses that \(4C^2 \log(t)^{[k/2]} \leq (C'/t)^{2k/5}\) for some constant \(C'\) followed by a calculation of the integral. For our upper bound it therefore suffices to show that \(\int_0^{\alpha/B} \Pr[|\tilde{f}_i - s(i)f_i| \geq t] dt = O\left( \frac{1}{B\sqrt{k}} \right)\). For this we need the following claim:

**Claim 4.5.** Let \(I \subset \mathbb{R}\) be the closed interval centered at the origin of length \(2t\), i.e., \(I = [-t, t]\). Suppose that \(\frac{1}{\sqrt{Bk}} \leq t \leq \frac{1}{B^2}\). For \(\ell \in [k]\), \(\Pr[X^{(\ell)} \in I] = \Omega(tB)\).

Before proving this claim we will first show how to use it to establish the desired result. For this let \(\frac{1}{\sqrt{Bk}} \leq t \leq \frac{1}{B^2}\) be fixed. If \(|\tilde{f}_i - s(i)f_i| \geq t\), at least half of the values \((X^{(\ell)})_{\ell \in [k]}\) are at least \(t\) or at most \(-t\). Let us focus on bounding the probability that at least half are at least \(t\), the other bound being symmetric giving an extra factor of 2 in the probability bound. By symmetry and the claim, \(\Pr[X^{(\ell)} \geq t] = \frac{1}{2} - \Omega(tB)\). For \(\ell \in [k]\) we define \(Y_\ell = [X^{(\ell)} \geq t]\), and we put \(S = \sum_{\ell \in [k]} Y_\ell\). Then \(E[S] = k \left( \frac{1}{2} - \Omega(tB) \right)\). If at least half of the values \((X^{(\ell)})_{\ell \in [k]}\) are at least \(t\) then \(S \geq k/2\). By Hoeffding’s inequality we can bound the probability of this event by

\[
\Pr[S \geq k/2] = \Pr[S - E[S] = \Omega(ktB)] = \exp(-\Omega(kt^2B^2)).
\]

It follows that \(\Pr[|\tilde{f}_i - s(i)f_i| \geq t] \leq 2 \exp(-\Omega(kt^2B^2))\). Thus

\[
\int_0^{\alpha/B} \Pr[|\tilde{f}_i - s(i)f_i| \geq t] dt \leq \frac{1}{B\sqrt{k}} + \int_{\alpha/(2B)}^{\frac{\alpha}{B}} 2 \exp(-\Omega(kt^2B^2)) dt + \int_{\alpha/(2B)}^{\alpha/B} 2 \exp(-\Omega(k)) dt \\
= O\left( \frac{1}{B\sqrt{k}} \right) + \frac{1}{B\sqrt{k}} \int_1^{\alpha/B} \exp(-t^2) dt = O\left( \frac{1}{B\sqrt{k}} \right).
\]

It thus suffices to prove the claim.

**Proof of Claim 4.5.** We first show that with probability \(\Omega(1)\), \(X^{(\ell)}_2\) lies in the interval \([1/B, \gamma/B]\) for some constant \(\gamma\). To see this we note that by Lemma 4.4, \(E[|X^{(\ell)}_2|] = O\left( \frac{1}{B} \right)\), so it follows by Markov’s inequality that if \(\gamma = O(1)\) is large enough, the probability that \(|X^{(\ell)}_2| \geq \gamma/B\) is at most \(\frac{1}{200}\). For a constant probability lower bound on \(|X^{(\ell)}_2|\) we write

\[
X^{(\ell)}_2 = \sum_{j \in \mathbb{N} \cap \{B+1,\ldots,2B\}} [h_{\ell}(j) = h_{\ell}(i)] s_\ell(j)f_j + \sum_{j \in \mathbb{N} \cap \{2B+1,\ldots,\infty\}} [h_{\ell}(j) = h_{\ell}(i)] s_\ell(j)f_j := S_1 + S_2.
\]
Condition on $S_2$. If $B \geq 4$ the probability that there exist exactly two $j \in N_2 \cap \{B+1, \ldots, 2B\}$ with $h_\ell(j) = h_\ell(i)$ is at least
\[
\left(\left|N_2 \cap \{B+1, \ldots, 2B\}\right| \cdot \frac{1}{B^2} \left(1 - \frac{1}{B}\right)^{B-2}\right) \geq \left(\frac{B-1}{2}\right)^{\frac{1}{eB^2}} \geq \frac{1}{8e}.
\]
With probability $1/4$ the corresponding signs $s_\ell(j)$ are both the same as that of $S_2$. By independence of $s_\ell$ and $h_\ell$ the probability that this occurs is at least $\frac{1}{32e}$ and if it does, $|X_2^{(\ell)}| \geq 1/B$. Combining these two bounds it follows that $|X_2^{(\ell)}| \in [1/B, \gamma/B]$ with probability at least $\frac{1}{32e} - \frac{1}{200} \geq \frac{1}{200}$. By symmetry, $\Pr[X_2^{(\ell)} \in [1/B, \gamma/B]] \geq \frac{1}{400} = \Omega(1)$. Denote this event by $E$. Also let $F$ be the event that $\{|j \in N_1 : h_\ell(j) = h_\ell(i)\}| = 1$. Then $\Pr[F] = \Omega(1)$ and as $E$ and $F$ are independent $\Pr[E \cap F] = \Omega(1)$. Conditioned on $E \cap F$ we now lower bound the probability that $X_2^{(\ell)} \in I$. For this it suffices to fix $X_2^{(\ell)} = \frac{\eta}{B}$ for some $1 \leq \eta \leq \gamma$ and lower bound the probability that $\frac{B}{2} + \sigma f \in [-t, t]$ where $\sigma$ is a Rademacher and $f \in \{1/j : j \in N_1\}$ is chosen uniformly at random. Let $m_1, m_2 \in \mathbb{R}_{>0}$ be such that
\[
\eta \cdot \frac{1}{B} - \frac{1}{m_1} = -t \quad \text{and} \quad \eta \cdot \frac{1}{B} - \frac{1}{m_2} = t.
\]
Then $m_2 - m_1 = B \cdot \frac{2B}{\eta B^2} = \Omega(tB^2)$. Using that $B$ is larger than a big enough constant and the mild assumption $k \leq B$, we have that $m_2 - m_1 = \Omega(B/\sqrt{k}) = \Omega(\sqrt{B}) \geq 1$, and so $|m_2 - m_1| = \Omega(m_2 - m_1) = \Omega(tB^2)$ as well. As we have $|N_2| \geq B - 1$ options for $f$ it follows that
\[
\Pr\left[\frac{\eta}{B} + \sigma f \in [-t, t]\right] \geq \frac{|m_2 - m_1|}{B - 1} = \Omega(tB),
\]
as desired. \hfill \square

This completes the proof of the upper bound and we proceed with the lower bound. Fix $\ell \in [k]$ and let $M_1 = [B \log k] \setminus \{i\}$ and $M_2 = [n] \setminus ([B \log k] \cup \{i\})$. Write
\[
S := \sum_{j \in M_1} [h_\ell(j) = h_\ell(i)] s_\ell(j) f_j + \sum_{j \in M_2} [h_\ell(j) = h_\ell(i)] s_\ell(j) f_j := S_1 + S_2.
\]
We also define $J := \{j \in M_1 : h_\ell(j) = h_\ell(i)\}$. Let $I \subseteq \mathbb{R}$ be the closed interval around $s_\ell(i) f_i$ of length $\frac{1}{B \sqrt{k \log k}}$. We now upper bound the probability that $S \in I$ conditioned on the value of $S_2$. To ease the notation the conditioning on $S_2$ has been left out in the notation to follow. Note first that
\[
\Pr[S \in I] = \sum_{r=0}^{\lceil M_1 \rceil} \Pr[S \in I \mid |J| = r] \cdot \Pr[|J| = r].
\]
For a given $r \geq 1$ we now proceed to bound $\Pr[S \in I \mid |J| = r]$. This probability is the same as the probability that $S_2 + \sum_{j \in R} \sigma_j f_j \in I$, where $R \subseteq M_1$ is a uniformly random $r$-subset and the $\sigma_j$’s are independent Rademachers. Suppose that we sample the elements from $R$ as well as the corresponding signs $(\sigma_j)_{j \in R}$ sequentially, and let us condition on the values and signs of the first $r - 1$ sampled elements. At this point at most $\frac{B \log k}{\sqrt{k}} + 1$ possible samples for the last element in $R$
brings $S$ into $I$. Indeed, the minimum distance between consecutive points in $\{f_j : j \in M_1\}$ is at most $1/(B \log k)^2$ so at most

$$\frac{1}{B\sqrt{k \log k}} \cdot (B \log k)^2 + 1 = \frac{B \log k}{\sqrt{k}} + 1$$

samples brings $S$ into $I$. For $1 \leq r \leq (B \log k)/2$ we can thus can upper bound

$$\Pr[S \in I \mid |J| = r] \leq \frac{B \log k}{|M_1| - r + 1} \leq \frac{2}{\sqrt{k}} + \frac{2}{B \log k} \leq \frac{3}{\sqrt{k}}.$$

By a standard Chernoff bound

$$\Pr[|J| \geq B \log k/2] = \exp(-\Omega(B \log k)) = k^{-\Omega(B)}.$$

If we assume that $B$ is larger than a constant, then $\Pr[|J| \geq B \log k/2] \leq k^{-1}$. Finally, $\Pr[|J| = 0] = (1 - 1/B)^{B \log k} \leq k^{-1}$. Combining these three bounds,

$$\Pr[S \in I] \leq \Pr[|J| = 0] + \sum_{r=1}^{(B \log k)/2} \Pr[S \in I \mid |J| = r] \cdot \Pr[|J| = r] + \sum_{r=(B \log k)/2}^{|M_1|} \Pr[|J| = r] = O\left(\frac{1}{\sqrt{k}}\right),$$

which holds even after removing the conditioning on $S_2$. We now show that with probability $\Omega(1)$ at least half the values $(X^{(t)})_{t \in [k]}$ are at least $\frac{1}{2B\sqrt{k \log k}}$. Let $p_0$ be the probability that $X^{(t)} \geq \frac{1}{2B\sqrt{k \log k}}$. This probability does not depend on $t \in [k]$ and by symmetry and what we showed above, $p_0 = 1/2 - O(1/\sqrt{k})$. Define the function $f : \{0, \ldots, k\} \to \mathbb{R}$ by

$$f(t) = \binom{k}{t} p_0^t (1 - p_0)^{k - t}.$$

Then $p(t)$ is the probability that exactly $t$ of the values $(X^{(t)})_{t \in [k]}$ are at least $\frac{1}{2B\sqrt{k \log k}}$. Using that $p_0 = 1/2 - O(1/\sqrt{k})$, a simple application of Stirling’s formula gives that $f(t) = \Theta\left(\frac{1}{\sqrt{k}}\right)$ for $t = [k/2], \ldots, [k/2 + \sqrt{k}]$ when $k$ is larger than some constant $C$. It follows that with probability $\Omega(1)$ at least half the $(X^{(t)})_{t \in [k]}$ are at least $\frac{1}{2B\sqrt{k \log k}}$ and in particular

$$\mathbb{E}[\hat{f}_t - f_t] = \Omega\left(\frac{1}{B \sqrt{k \log k}}\right).$$

Finally we handle the case where $k \leq C$. It is easy to check (e.g. with Lemma 4.2) that $X^{(t)} = \Omega(1/B)$ with probability $\Omega(1)$. Thus this happens for all $t \in [k]$ with probability $\Omega(1)$ and in particular $\mathbb{E}[\hat{f}_t - f_t] = \Omega(1/B)$, which is the desired for constant $k$. \hfill \square

## 5 Learned Count-Sketch for Zipfians

We now proceed to analyze the learned Count-Sketch algorithm. In Section 5.1 we estimate the expected error when using a single hash function and in Section 5.2 we show that the expected error only increases when using more hash functions. Recall that we assume on the number of buckets $B_h$ used to store the heavy hitters that $B_h = \Theta(B - B_h) = \Theta(B)$. 13
5.1 One hash function

By taking $B_1 = B_h = \Theta(B)$ and $B_2 = B - B_h = \Theta(B)$ in the theorem below the result on L-CS for $k = 1$ claimed in Table 4 follows immediately.

**Theorem 5.1.** Let $h : [n] \setminus [B_1] \to [B_2]$ and $s : [n] \to \{-1, 1\}$ be truly random hash functions where $n, B_1, B_2 \in \mathbb{N}$ and $n - B_1 \geq B_2 \geq B_1$. Define the random variable $\tilde{f}_i = \sum_{j=B_1+1}^n [h(j) = h(i)]s(j)f_j$ for $i \in [n] \setminus [B_1]$. Then

$$\mathbb{E}[|\tilde{f}_i - s(i)f_i|] = \Theta\left(\frac{\log \frac{B_2+B_h}{B_2}}{B_2}\right).$$

**Proof.** Let $N_1 = [B_1 + B_2] \setminus ([B_1] \cup \{i\})$ and $N_2 = [n] \setminus ([B_1 + B_2] \cup \{i\})$. Let $X_1 = \sum_{j \in N_1} [h(j) = h(i)]s(j)f_j$ and $X_2 = \sum_{j \in N_2} [h(j) = h(i)]s(j)f_j$. By the triangle inequality and linearity of expectation,

$$\mathbb{E}[|X_1|] = O\left(\frac{\log \frac{B_2+B_1}{B_1}}{B_2}\right).$$

Moreover, it follows directly from Lemma 4.4 that $\mathbb{E}[|X_2|] = O\left(\frac{1}{B_2}\right)$. Thus

$$\mathbb{E}[|\tilde{f}_i - s(i)f_i|] \leq \mathbb{E}[|X_1|] + \mathbb{E}[|X_2|] = O\left(\frac{\log \frac{B_2+B_1}{B_2}}{B_2}\right),$$

as desired.

For the lower bound on $\mathbb{E}\left[|\tilde{f}_i - s(i)f_i|\right]$ we apply Lemma 4.2 to obtain that,

$$\mathbb{E}\left[|\tilde{f}_i - s(i)f_i|\right] \geq \frac{1}{2B_2} \left(1 - \frac{1}{B_2}\right)^{|N_1| - 1} \sum_{i \in N_1} f_i = \Omega\left(\frac{\log \frac{B_2+B_1}{B_2}}{B_2}\right).$$

\[\square\]

**Corollary 5.2.** Let $h : [n] \setminus [B_h] \to [B - B_h]$ and $s : [n] \to \{-1, 1\}$ be truly random hash functions where $n, B_1, B_2 \in \mathbb{N}$ and $B_h = \Theta(B) \leq B/2$. Define the random variable $\tilde{f}_i = \sum_{j=B_h+1}^n [h(j) = h(i)]s(j)f_j$ for $i \in [n] \setminus [B_h]$. Then

$$\mathbb{E}[|\tilde{f}_i - s(i)f_i|] = \Theta\left(\frac{1}{B}\right).$$

5.2 More hash functions

We now show that, like for Count-Min, using more hash functions does not decrease the expected error. We first state the Littlewood-Offord lemma as strengthened by Erdős.

\[\text{The first inequality is the standard assumption that we have at least as many items as buckets. The second inequality says that we use at least as many buckets for non-heavy items as for heavy items (which doesn’t change the asymptotic space usage).}\]
Theorem 5.3 (Littlewood-Offord [LO39], Erdős [Erd45]). Let $a_1, \ldots, a_n \in \mathbb{R}$ with $|a_i| \geq 1$ for $i \in [n]$. Let further $\sigma_1, \ldots, \sigma_n \in \{-1, 1\}$ be random variables with $\Pr[\sigma_i = 1] = \Pr[\sigma_i = -1] = 1/2$ and define $S = \sum_{i=1}^n \sigma_i a_i$. For any $v \in \mathbb{R}$ it holds that

$$\Pr[|S - v| \leq 1] \leq \left(\frac{n}{|n/2|}\right) \cdot \frac{1}{2^n} = O\left(\frac{1}{\sqrt{n}}\right).$$

Setting $B_1 = B_h = \Theta(B)$ and $B_2 = B - B_h = \Theta(B)$ in the theorem below gives the final bound from Table I on L-CS with $k \geq 3$.

Theorem 5.4. Let $n \geq B_1 + B_2 \geq 2B_1$, $k \geq 3$ odd, and $h_1, \ldots, h_k : [n] \setminus [B_1] \to [B_2/k]$ and $s_1, \ldots, s_k : [n] \setminus [B_1] \to [B_2/k]$ be independent and truly random. Define the random variable $\tilde{f}_i = \text{median}_{\ell \in [k]} \left(\sum_{j \in [n]\setminus[B_1]} [h_\ell(j) = h_\ell(i)] s_\ell(j) f_j\right)$ for $i \in [n] \setminus [B_1]$. Then

$$\mathbb{E}[|\tilde{f}_i - s(i) f_i|] = \Omega\left(\frac{1}{B_2}\right).$$

Proof. Like in the proof of the lower bound of Theorem 4.4 it suffices to show that for each $i$ the probability that the sum $S_\ell := \sum_{j \in [n]\setminus([B_1] \cup \{i\})} [h_\ell(j) = h_\ell(i)] s_\ell(j) f_j$ lies in the interval $I = [-1/(2B_2), 1/(2B_2)]$ is $O(1/\sqrt{k})$. Then at least half the $(S_\ell)_{\ell \in [k]}$ are at least $1/(2B_2)$ with probability $\Omega(1)$ by an application of Stirling’s formula, and it follows that $\mathbb{E}[|\tilde{f}_i - s(i) f_i|] = \Omega(1/B_2)$.

Let $\ell \in [k]$ be fixed, $N_1 = [2B_2] \setminus ([B_2] \cup \{i\})$, and $N_2 = [n] \setminus (N_1 \cup \{i\})$, and write

$$S_\ell = \sum_{j \in N_1} [h_\ell(j) = h_\ell(i)] s_\ell(j) f_j + \sum_{j \in N_2} [h_\ell(j) = h_\ell(i)] s_\ell(j) f_j := X_1 + X_2.$$

Now condition on the value of $X_2 = x_2$ of $X_2$. Letting $J = \{j \in N_1 : h_\ell(j) = h_\ell(i)\}$ it follows by Theorem 5.3 that

$$\Pr[S_\ell \in I \mid X_2 = x_2] = O\left(\frac{\sum_{J' \subseteq N_1} \Pr[J = J']}{\sqrt{|J'| + 1}}\right) = O\left(\Pr[|J| < k/2] + 1/\sqrt{k}\right).$$

An application of Chebyshev’s inequality gives that $\Pr[|J| < k/2] = O(1/k)$, so $\Pr[S_\ell \in I] = O(1/\sqrt{k})$. Since this bound holds for any possible value of $x_2$ we may remove the conditioning and the desired result follows. \hfill $\square$

Remark 5.5. The bound above is probably only tight for $B_1 = \Theta(B_2)$. Indeed, we know that it cannot be tight for all $B_1 \leq B_2$ since when $B_1$ becomes very small, the bound from the standard Count-Sketch with $k \geq 3$ takes over — and this is certainly worse than the bound in the theorem. It is an interesting open problem (that requires a better anti-concentration inequality than the Littlewood-Offord lemma) to settle the correct bound when $B_1 \ll B_2$. And more generally to obtain an understanding for more general values of the frequencies $(f_i)_{i \in [n]}$.

References

[ACC+11] Nir Ailon, Bernard Chazelle, Kenneth L Clarkson, Ding Liu, Wolfgang Mulzer, and C Seshadhri. Self-improving algorithms. *SIAM Journal on Computing*, 40(2):350–375, 2011.
[AKL+19] Daniel Alabi, Adam Tauman Kalai, Katrina Ligett, Cameron Musco, Christos Tzamos, and Ellen Vitercik. Learning to prune: Speeding up repeated computations. In Conference on Learning Theory, 2019.

[BDSV18] Maria-Florina Balcan, Travis Dick, Tuomas Sandholm, and Ellen Vitercik. Learning to branch. In International Conference on Machine Learning, pages 353–362, 2018.

[BDV18] Maria-Florina Balcan, Travis Dick, and Ellen Vitercik. Dispersion for data-driven algorithm design, online learning, and private optimization. In 2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS), pages 603–614. IEEE, 2018.

[Ben62] George Bennett. Probability inequalities for the sum of independent random variables. Journal of the American Statistical Association, 57(297):33–45, 1962.

[BJPD17] Ashish Bora, Ajil Jalal, Eric Price, and Alexandros G Dimakis. Compressed sensing using generative models. In International Conference on Machine Learning, pages 537–546, 2017.

[CCFC02] Moses Charikar, Kevin Chen, and Martin Farach-Colton. Finding frequent items in data streams. In International Colloquium on Automata, Languages, and Programming, pages 693–703. Springer, 2002.

[CM05] Graham Cormode and Shan Muthukrishnan. An improved data stream summary: the count-min sketch and its applications. Journal of Algorithms, 55(1):58–75, 2005.

[Erd45] Paul Erdös. On a lemma of littlewood and offord. Bulletin of the American Mathematical Society, 51(12):898–902, 1945.

[GP19] Sreenivas Gollapudi and Debmalya Panigrahi. Online algorithms for rent-or-buy with expert advice. In Proceedings of the 36th International Conference on Machine Learning, pages 2319–2327, 2019.

[GR17] Rishi Gupta and Tim Roughgarden. A pac approach to application-specific algorithm selection. SIAM Journal on Computing, 46(3):992–1017, 2017.

[HIKV19] Chen-Yu Hsu, Piotr Indyk, Dina Katabi, and Ali Vakilian. Learning-based frequency estimation algorithms. In International Conference on Learning Representations, 2019.

[KBC+18] Tim Kraska, Alex Beutel, Ed H Chi, Jeffrey Dean, and Neoklis Polyzotis. The case for learned index structures. In Proceedings of the 2018 International Conference on Management of Data, pages 489–504, 2018.

[KDZ+17] Elias Khalil, Hanjun Dai, Yuyu Zhang, Bistra Dilkina, and Le Song. Learning combinatorial optimization algorithms over graphs. In Advances in Neural Information Processing Systems, pages 6348–6358, 2017.

[LO39] John Edensor Littlewood and Albert C Offord. On the number of real roots of a random algebraic equation. ii. In Mathematical Proceedings of the Cambridge Philosophical Society, volume 35, pages 133–148. Cambridge University Press, 1939.
[LV18] Thodoris Lykouris and Sergei Vassilvitskii. Competitive caching with machine learned advice. In *International Conference on Machine Learning*, pages 3302–3311, 2018.

[MAEA05] Ahmed Metwally, Divyakant Agrawal, and Amr El Abbadi. Efficient computation of frequent and top-k elements in data streams. In *International Conference on Database Theory*, pages 398–412. Springer, 2005.

[MG82] Jayadev Misra and David Gries. Finding repeated elements. *Science of computer programming*, 2(2):143–152, 1982.

[Mit18] Michael Mitzenmacher. A model for learned bloom filters and optimizing by sandwiching. In *Advances in Neural Information Processing Systems*, pages 464–473, 2018.

[MM02] Gurmeet Singh Manku and Rajeev Motwani. Approximate frequency counts over data streams. In *VLDB’02: Proceedings of the 28th International Conference on Very Large Databases*, pages 346–357. Elsevier, 2002.

[MPB15] Ali Mousavi, Ankit B Patel, and Richard G Baraniuk. A deep learning approach to structured signal recovery. In *Communication, Control, and Computing (Allerton), 2015 53rd Annual Allerton Conference on*, pages 1336–1343. IEEE, 2015.

[PSK18] Manish Purohit, Zoya Svitkina, and Ravi Kumar. Improving online algorithms via ml predictions. In *Advances in Neural Information Processing Systems*, pages 9661–9670, 2018.

[RKA16] Pratanu Roy, Arijit Khan, and Gustavo Alonso. Augmented sketch: Faster and more accurate stream processing. In *Proceedings of the 2016 International Conference on Management of Data*, pages 1449–1463, 2016.

[WLKC16] Jun Wang, Wei Liu, Sanjiv Kumar, and Shih-Fu Chang. Learning to hash for indexing big data - a survey. *Proceedings of the IEEE*, 104(1):34–57, 2016.