COMPACT APPROACH TO THE POSITIVITY OF BROWN-YORK MASS AND RIGIDITY OF MANIFOLDS WITH MEAN-CONVEX BOUNDARIES IN FLAT AND SPHERICAL CONTEXTS

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Abstract. In this article we develop a spinorial proof of the Shi-Tam theorem for the positivity of the Brown-York mass without necessity of building nonsmooth infinite asymptotically flat hypersurfaces in the Euclidean space and use the positivity of the ADM mass proved by Schoen-Yau and Witten. This same compact approach provides an optimal lower bound [HMZ] for the first non null eigenvalue of the Dirac operator of a mean convex boundary for a compact spin manifold with non negative scalar curvature, an a rigidity result for mean-convex bodies in flat spaces. The same machinery provides analogous, but new, results of this type, as far as we know, in spherical contexts, including a version of Min-Oo’s conjecture.

1. Introduction

In a beautiful work by Shi and Tam [ST, Theorem 4.1], the positivity of the ADM mass theorem, proved by Schoen and Yau [SY] and [Wi], independently, was used in a new way to get nice results on the boundary behaviour of a compact Riemannian manifold with non-negative scalar curvature. They proved that if \( \Omega \) is a three-dimensional compact Riemannian manifold with non-negative scalar curvature and strictly convex boundary \( \Sigma \), then

\[
\int_\Sigma H_0 - \int_\Sigma H,
\]

that is, the so called Brown-York mass enclosed in \( \Omega \), is greater than or equal to zero, where \( H \) is the inner mean curvature of \( \Sigma \) in \( \Omega \) and \( H_0 \) is the Euclidean inner mean curvature of \( \Sigma \) in \( \mathbb{R}^3 \) corresponding to the (unique up to rigid motions) Weyl embedding of \( \Sigma \) into \( \mathbb{R}^3 \) obtained by Pogorelov [Po] and Nirenberg [Ni] independently. Moreover, the equality holds for some boundary component \( \Sigma \). The proof uses two fundamental facts. First,
following an idea by Bartnik [Bar], the construction of a suitable infinite asymptotically flat extension of $\Omega$ with non-negative scalar curvature by deforming the exterior of $\Sigma$ in $\mathbb{R}^3$ in order to have the same mean curvature as $\Sigma$ in $\Omega$ with the intention to glue it to manifold $\Omega$ along its boundary $\Sigma$. This construction is equivalent to solve a non-linear parabolic equation in a $C^1$ context, because this is the degree of differentiability after the glueing. The second point is that the difference of integrals

$$\int_{\Sigma_r} H_0 - \int_{\Sigma_r} H_r,$$

where $\Sigma_r$ is an expansion to infinity of the original boundary $\Sigma$ into $\mathbb{R}^3$, where $H_0$ is the mean curvature with respect to the Euclidean metric and $H_r$ is the mean curvature with respect to the Bartnik metric goes in a non-decreasing way to the ADM mass of the built asymptotically flat manifold and, so, one can use the non-negativity of its mass. This approach was successfully used to prove the positivity of other quasi-local masses proposed by Liu and Yau [LiY1, LiY2] and Wang and Yau [WaY]. Along these comments one can see that positivity of quasi-local masses and rigidity of compact manifolds with non-empty boundary are different, although similar, aspects of a quasi same question.

It can be seen, since one starts to study elementary differential geometry, for example, when one begins to prove the Cohn-Vossen rigidity of ovaloids in $\mathbb{R}^3$, that there is a close connexion between rigidity of these manifolds and the fact that the integral its mean curvatures coincide [MR] (pages 218-219). In this paper, we will obtain a compact approach of the positivity of the Brown-York mass for the compact spin manifolds of arbitrary dimension and its corresponding rigidity theorem for mean-convex bodies in the Euclidean spaces. These two type of results are relied with an estimate for the lower eigenvalues of the Dirac operator of these bodies and, in turn, to theorem type Min-Oo. In fact, in the flat case the corresponding Min-Oo conjecture was obtained by Miao [Mi1] (see Remark 1), although it was an easy consequence of the estimate obtained in [HMZ] for this eigenvalue of the Dirac operator.

Also, in this article, for the sake of completeness, we explain the relation between the spin structures of the Euclidean space or the sphere and their hypersurfaces and we will see that their Dirac operator relate basically through the mean curvature of these hypersurfaces and the scalar curvature of the ambient spaces. This fact will allow us, for a $(n + 1)$-dimensional compact spin Riemannian manifold $\Omega$ with non-negative scalar curvature and mean-convex boundary $\Sigma$, to unify a compact proof to obtain a lower estimate for the spectrum of the Dirac of $\Sigma$, the positivity of the Brown-York mass for mean-convex non-necessarily convex domains, the aforementioned resolution of the flat version of Min-Oo’s conjecture and the rigidity of these mean-convex bodies in the Euclidean spaces.
In the final section of the article, we will show that this same scheme, with the important difference of establish the exact value $R = n(n + 1)$ of the scalar curvature of the bulk manifold, works to obtain exactly the same sequence of results. In this case, all of them, as far as we know, are new.

2. **Riemannian spin manifolds and hypersurfaces**

Consider an $(n + 1)$-dimensional spin Riemannian manifold $\Omega$ with non-empty boundary $\partial \Omega = \Sigma$ and denote by $\langle \cdot , \cdot \rangle$ its scalar product and by $\nabla$ its corresponding Levi-Civita connection on the tangent bundle $T\Omega$. We fix a spin structure (and so a corresponding orientation) on the manifold $\Omega$ and denote by $S\Omega$ the associated spinor bundle, which is a complex vector bundle of rank $2^{\lfloor \frac{n+1}{2} \rfloor}$. Then let

$$
\gamma : \mathcal{C}(\Omega) \longrightarrow \text{End}_{\mathbb{C}}(S\Omega)
$$

be the Clifford multiplication, which provides a fibre preserving irreducible representation of the Clifford algebras constructed over the tangent spaces of $\Omega$. When the dimension $n + 1$ is even, we have the standard chirality decomposition

$$
S\Omega = S\Omega^+ \oplus S\Omega^-,
$$

where the two direct summands are respectively the $\pm 1$-eigenspaces of the endomorphism $\gamma(\omega_{n+1})$, with $\omega_{n+1} = i^{\lfloor \frac{n+2}{2} \rfloor}e_1 \cdots e_{n+1}$, the complex volume form. It is well-known (see [LM]) that there are, on the complex spinor bundle $S\Omega$, a natural Hermitian metric $(\cdot , \cdot)$ and a spinorial Levi-Civita connection, denoted also by $\nabla$, which is compatible with both $(\cdot , \cdot)$ and $\gamma$ in the following sense:

\[
\begin{align*}
X(\psi, \phi) &= (\nabla_X \psi, \phi) + (\psi, \nabla_X \phi) \\
\nabla_X (\gamma(Y)\psi) &= \gamma(\nabla_X Y)\psi + \gamma(Y)\nabla_X \psi
\end{align*}
\]

for any tangent vector fields $X, Y \in \Gamma(T\Omega)$ and any spinor fields $\psi, \phi \in \Gamma(S\Omega)$ on $M$. Moreover, with respect to this Hermitian product on $S\Omega$, Clifford multiplication by vector fields is skew-Hermitian or equivalently

\[
(\gamma(X)\psi, \gamma(X)\phi) = |X|^2(\psi, \phi).
\]

Since the complex volume form $\omega_{n+1}$ is parallel with respect to the spinorial Levi-Civita connection, when $n + 1 = \dim \Omega$ is even, the chirality decomposition ([1]) is preserved by $\nabla$. Moreover, from ([4]), one sees that it is an orthogonal decomposition.

In this setting, the (fundamental) Dirac operator $D$ on the manifold $M$ is the first order elliptic differential operator acting on spinor fields given locally by

$$
D = \sum_{i=1}^{n+1} \gamma(e_i) \nabla_{e_i},
$$
where \(\{e_1, \ldots, e_{n+1}\}\) is a local orthonormal frame in \(T\Omega\). When \(n + 1 = \text{dim}\Omega\) is even, \(D\) interchanges the chirality subbundles \(\mathbb{S}\Omega^\pm\).

The boundary hypersurface \(\Sigma\) is also an oriented Riemannian manifold with the induced orientation and metric. If \(\nabla^\Sigma\) stands for the Levi-Civita connection of this induced metric we have the Gauss and Weingarten equations

\[
\nabla_X Y = \nabla^\Sigma_X Y + \langle AX, Y\rangle N, \quad \nabla_X N = -AX,
\]

for any vector fields \(X, Y\) tangent to \(\Sigma\), where \(A\) is the shape operator or Weingarten endomorphism of the hypersurface \(\Sigma\) corresponding to the unit normal field \(N\) compatible with the given orientation. As the normal bundle of the boundary hypersurface is trivial, the Riemannian manifold \(\Sigma\) is also a spin manifold and so we will have the corresponding spinor bundle \(\mathbb{S}\Sigma\), the Clifford multiplication \(\gamma^\Sigma\), the spinorial Levi-Civita connection \(\nabla^\Sigma\) and the intrinsic Dirac operator \(D^\Sigma\). It is not difficult to show (see [Bä2, BFGK, Bur, Ti, Mo]) that the restricted Hermitian bundle \(S := \mathbb{S}\Omega|\Sigma\)

can be identified with the intrinsic Hermitian spinor bundle \(\mathbb{S}\Sigma\), provided that \(n + 1 = \text{dim}\Omega\) is odd. Instead, if \(n + 1 = \text{dim}\Omega\) is even, the restricted bundle \(S\) could be identified with the sum \(\mathbb{S}\Sigma \oplus \mathbb{S}\Sigma\). With such identifications, for any spinor field \(\psi \in \Gamma(S)\) on the boundary hypersurface \(\Sigma\) and any vector field \(X \in \Gamma(T\Sigma)\), define on the restricted bundle \(S\), the Clifford multiplication \(\gamma^S\) and the connection \(\nabla^S\) by

\[
\gamma^S(X)\psi = \gamma(X)\gamma(N)\psi,
\]

\[
\nabla^S_X \psi = \nabla_X \psi - \frac{1}{2}\gamma^S(AX)\psi = \nabla_X \psi - \frac{1}{2}\gamma(AX)\gamma(N)\psi.
\]

Then it easy to see that \(\gamma^S\) and \(\nabla^S\) correspond respectively to \(\gamma^\Sigma\) and \(\nabla^\Sigma\), for \(n + 1\) odd, and to \(\gamma^\Sigma \oplus -\gamma^\Sigma\) and \(\nabla^\Sigma \oplus \nabla^\Sigma\), for \(n + 1\) even. Then, \(\gamma^S\) and \(\nabla^S\) satisfy the same compatibility relations (2), (3) and (4) and together with the following additional identity

\[
\nabla^S_X (\gamma(N)\psi) = \gamma(N)\nabla^S_X \psi.
\]

As a consequence, the hypersurface Dirac operator \(D\) acts on smooth sections \(\psi \in \Gamma(S)\) as

\[
D\psi := \sum_{j=1}^{n} \gamma^S(u_j)\nabla^S_{u_j} \psi = \frac{n}{2}H\psi - \gamma(N)\sum_{j=1}^{n} \gamma(u_j)\nabla_{u_j} \psi,
\]

where \(\{u_1, \ldots, u_n\}\) is a local orthonormal frame tangent to the boundary \(\Sigma\) and \(H = (1/(n))\text{trace}\ A\) is its mean curvature function, coincides with the intrinsic Dirac operator \(D^\Sigma\) on the boundary, for \(n + 1\) odd, and with the pair \(D^\Sigma \oplus -D^\Sigma\), for \(n + 1\) even. In the particular case where the field \(\psi \in \Gamma(S)\) is the restriction of a spinor field \(\psi \in \Gamma(\Sigma)\) on \(\Omega\), this means that

\[
D\psi = \frac{n}{2}H\psi - \gamma(N)D\psi - \nabla_N \psi.
\]
Note that we always have the anticommutativity property
\begin{equation}
D\gamma(N) = -\gamma(N)D
\end{equation}
and so, when $\Sigma$ is compact, the spectrum of $D$ is symmetric with respect to zero and coincides with the spectrum of $D^\Sigma$, for $n + 1$ odd, and with $\text{Spec}(D^\Sigma) \cup -\text{Spec}(D^\Sigma)$, for $n + 1$ even [see HMR]. In fact, we and other authors also have remarked in several papers that the spectrum $\text{Spec}(D)$ is a $\mathbb{Z}$-symmetric sequence
\[-\infty \vee \cdots \leq -\lambda_k \leq \cdots \leq -\lambda_1 < \lambda_0 = 0 < \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots \nearrow +\infty\]
(each eigenvalue repeated according to its corresponding multiplicity). This is because $D$ is an elliptic operator of order one which is self-adjoint due to the compacity of $\Sigma$ and the fact that $\gamma(N)$ maps the eigenvalue $\lambda_k$ into $-\lambda_k$, being $\gamma$ the Clifford multiplication and $N$ the inner unit normal of $\Sigma$ in $\Omega$. Let’s choose an $L^2(\Sigma\Sigma)$-orthonormal basis $\{\psi_k\}_{k \in \mathbb{Z}}$ of the Hilbert space $L^2(\Sigma\Sigma)$ constituted by eigenspinors of $D$, that is, $D\psi_k = \lambda_k\psi_k$ (and so $\psi_k \in C^\infty(\Sigma)$), for all $k \in \mathbb{Z}$. We have to remark that the presence of the eigenvalue $\lambda_0 = 0$ (repeated with its corresponding multiplicity) is not compulsory. It is a standard fact (Perceval equality) that, if $\phi$ is an $L^2$ spinor field on $\Sigma$, the series
\[\sum_{k \in \mathbb{Z}} \phi_k = \lim_{k \to \infty} \sum_{k \leq 0} \phi_k = \phi,
\]
converges in the strong $L^2(\Sigma\Sigma)$-topology, where each $\phi_k$ is the projection of $\phi$ onto the eigenspace corresponding to the eigenvalue $\lambda_k$. As a consequence of Hölder inequalities and well-known facts, we have $L^1(\Sigma\Sigma)$-convergence as well and moreover, as an easy consequence of the completeness of $L^2(\Sigma\Sigma)$ or $L^1(\Sigma\Sigma)$, pointwise convergence almost everywhere (and so everywhere if $\phi$ is continuous) for a suitable subseries of $\sum_{k = -\infty}^{+\infty} \phi_k$.

3. A spinorial Reilly inequality

A basic tool to relate the eigenvalues of the Dirac operator and the geometry of the manifold $\Omega$ and those of its boundary $\Sigma$ will be, as in the closed case (see [Fr1]), the integral version of the Schrödinger-Lichnerowicz formula

\begin{equation}
D^2 = \nabla^*\nabla + \frac{1}{4}R,
\end{equation}
where $R$ is the scalar curvature of $\Omega$. In fact, given a spinor field $\psi$ on $\Omega$, taking into account the formula above, if we compute the divergence of the one-form $\alpha$ defined by

\[\alpha(X) = \langle \gamma(X)D\psi + \nabla_X\psi, \psi \rangle, \quad \forall X \in T\Omega\]
and integrate, one gets

\[-\int_{\Sigma} \langle \gamma(N)D\psi + \nabla_{N}\psi, \psi \rangle = \int_{\Omega} \left( |\nabla\psi|^2 - |D\psi|^2 + \frac{1}{4}R|\psi|^2 \right),\]

which by (7), could be written as

\[\int_{\Sigma} \left( (D\psi, \psi) - \frac{n}{2}H|\psi|^2 \right) = \int_{\Omega} \left( |\nabla\psi|^2 - |D\psi|^2 + \frac{1}{4}R|\psi|^2 \right).\]

Finally, we will use the pointwise spinorial Schwarz inequality

\[|D\psi|^2 \leq (n + 1)|\nabla\psi|^2, \quad \forall \psi \in \Gamma(S\Omega),\]

where the equality is achieved only by the so-called twistor spinors, that is, those satisfying the following over-determined first order equation

\[\nabla_X \psi = -\frac{1}{n + 1} \gamma(X)D\psi, \quad \forall X \in T\Omega.\]

Then we get the following integral inequality, called Reilly inequality [see HMZ, for example], because of its similarity with the corresponding one obtained in [Re] for the Laplace operator,

\[\int_{\Sigma} \left( (D\psi, \psi) - \frac{n}{2}H|\psi|^2 \right) \geq \int_{\Omega} \left( \frac{1}{4}R|\psi|^2 - \frac{n}{n + 1}|D\psi|^2 \right),\]

with equality only for twistor spinors on Ω.

4. The APS Boundary Condition

It is a well-known fact that the Dirac operator $D$ on a compact spin Riemannian manifold $\Omega$ with boundary, $D : \Gamma(S\Omega) \to \Gamma(S\Omega)$ has an infinite dimensional kernel and a closed image with finite codimension. People has looked for conditions $B$ to be imposed on the restrictions to the boundary $\Sigma$ of the spinor fields on $\Omega$ so that this kernel becomes finite dimensional and then the boundary problem

\[(BP) \quad \begin{cases} D\psi = \Phi & \text{on } \Omega \\ B\psi_{|\Sigma} = \chi & \text{along } \Sigma, \end{cases}\]

for $\Phi \in \Gamma(S\Omega)$ and $\chi \in \Gamma(S)$, is of Fredholm type. In this case, we will have smooth solutions for any data $\Phi$ and $\chi$ belonging to a certain subspace with finite codimension and these solutions will be unique up to a finite dimensional kernel.

To our knowledge, the study of boundary conditions suitable for an elliptic operator $D$ (of any order, although for simplicity, we only consider first order operators) acting on smooth sections of a Hermitian vector bundle $F \to \Omega$ has been first done in the fifties of past century by Lopatinsky and Shapiro ([HÖ, Lo]), but the main tool was discovered by Calderón in the sixties: the so-called Calderón projector

\[P_{+}(D) : H^{\frac{d}{2}}(F_{|\Sigma}) \longrightarrow \{ \psi_{|\Sigma} \ | \ \psi \in H^{1}(F), D\psi = 0 \}.\]
This is a pseudo-differential operator of order zero (see [BW, Se]) with principal symbol \( p_+(D) : T\Sigma \rightarrow \text{End}_C(F) \) depending only on the principal symbol \( \sigma_D \) of the operator \( D \) and can be calculated as follows

\[
(11) \quad p_+(D)(X) = -\frac{1}{2\pi i} \int_{\Gamma} \left[ (\sigma_D(N))^{-1} \sigma_D(X) - \zeta I \right]^{-1} d\zeta,
\]

for any \( p \in \Sigma \) and \( X \in T_p\Sigma \), where \( N \) is the inner unit normal along the boundary \( \Sigma \) and \( \Gamma \) is a positively oriented cycle in the complex plane enclosing the poles of the integrand with negative imaginary part. Although the Calderón projector is not unique for a given elliptic operator \( D \), its principal symbol is uniquely determined by \( \sigma_D \). One of the important features of the Calderón projector is that its principal symbol detects the ellipticity of a boundary condition, or in other words, if the corresponding boundary problem (BP) is a well-posed problem (according to Seeley in [Se]). In fact (cfr. [Se] or [BW, Chap. 18]), a pseudo-differential operator

\[
B : L^2(F|\Sigma) \rightarrow L^2(V),
\]

where \( V \rightarrow \Sigma \) is a complex vector bundle over the boundary, is called a (global) elliptic boundary condition when its principal symbol \( b : T\Sigma \rightarrow \text{Hom}_C(F|\Sigma, V) \) satisfies that, for any non-trivial \( X \in T_p\Sigma \), \( p \in \Sigma \), the restriction

\[
b(X)|_{\text{image} p_+(D)(X)} : \text{image} p_+(D)(X) \subset F_p \rightarrow V_p
\]

is an isomorphism onto \( \text{image} b(X) \subset V_p \). Moreover, if \( \text{rank} V = \dim \text{image} p_+(D)(X) \), we say that \( B \) is a local elliptic boundary condition. When \( B \) is a local operator this definition yields the so-called Lopatinsky-Shapiro conditions for ellipticity (see for example [Hö]).

When these definitions and the subsequent theorems are applied to the case where the vector bundle \( F \) is the spinor bundle \( S\Omega \) and the elliptic operator \( D \) is the Dirac operator \( D \) on the spin Riemannian manifold \( \Omega \), we obtain the following well-known facts in the setting of the general theory of boundary problems for elliptic operators (see for example [BrL, BW, GLP, Hö, Se]):

It is easy to see that the principal symbol \( \sigma_D \) of the Dirac operator \( D \) on \( \Omega \) is given by

\[
\sigma_D(X) = i\gamma(X), \quad \forall X \in T\Omega.
\]

Then by (11), the principal symbol of the Calderón projector of the Dirac operator is given by

\[
p_+(D)(X) = -\frac{1}{2|X|} (i\gamma(N)\gamma(X) - |X|I) = \frac{1}{2|X|} (i\gamma^S(X) + |X|I),
\]

for each non-trivial \( X \in T\Sigma \) and where \( \gamma^S \) is identified in (53) as the intrinsic Clifford product on the boundary. As the endomorphism \( i\gamma(N)\gamma(u) = -i\gamma^S(X) \) is self-adjoint and its square is \( |X|^2 \) times the identity map, then it has exactly two eigenvalues, say \( |X| \) and \( -|X| \), whose eigenspaces are of the same dimension \( \frac{1}{2} \dim S\Omega_p = 2^{[\frac{n+1}{2}]} - 1 \), since they are interchanged by \( \gamma(N) \).
Hence the symbol $p_+(D)(X)$ is, up to a constant, the orthogonal projection onto the eigenspace corresponding to the eigenvalue $-|X|$ and so
\[
\text{image } p_+(D)(X) = \{ \eta \in S_{\Omega_p} \mid i\gamma(N)\gamma(X)\eta = -|X|\eta \},
\]
\[
\dim \text{image } p_+(D)(X) = \frac{1}{2} \dim S_{\Omega_p} = 2^{\left[\frac{n+1}{2}\right]} - 1.
\]
From these equalities and from the definition of ellipticity for the boundary condition represented by the pseudo-differential operator $B$, we have that the first equation in the statement above is equivalent to the injectivity of the map $b(X) |\text{image } p_+(D)(X)$. The second one implies that $\dim \text{image } b(X) = \dim \text{image } p_+(D)(X)$ and so, together with the injectivity, this means that $b(X) |\text{image } p_+(D)(X)$ is surjective. So we have proved that the two claimed conditions are equivalent to the ellipticity of a given boundary condition $B$ for the Dirac operator $D$ on $\Omega$. Now, from this ellipticity, one may deduce that the problem (BP) and the spectral corresponding one are of Fredholm type and the remaining assertions on eigenvalues and eigenspaces follow in a standard way (see [BW, Hö]).

At the risk of being too long, we will explain what is the Atiyah, Patodi and Singer condition in this context. This well-known boundary condition was introduced in [APS] in order to establish index theorems for compact manifolds with boundary. Later, this condition has been used to study the positive mass and the Penrose inequalities (see [He2, Wi]). Such a condition does not allow to model confined particle fields since, from the physical point of view, its global nature is interpreted as a causality violation. Although it is a well-known fact that the APS condition is an elliptic boundary condition, we are going to sketch the proof in the setting established above, for three reasons: first, for completeness, second for pointing out that the APS condition for an achiral Dirac operator covers both cases of odd and even dimension, although the latter case is not referred to the spectral resolution of the intrinsic Dirac operator $D^\Sigma$ but to the system $D^\Sigma \oplus -D^\Sigma$ and third, because we will use it with some little modifications which may not familiar to some readers.

Precisely, this condition can be described as follows. Choose the aforementioned Hermitian bundle $V$ over the boundary hypersurface $\Sigma$ as the restricted spinor bundle $S$ defined in Section 2, and define, for each $a \in \mathbb{Z}$, as $B^a_{\text{APS}} := B_{\geq a} : L^2(S) \to L^2(S)$ as the orthogonal projection onto the subspace spanned by the eigenvalues of the self-adjoint intrinsic operator $D$ greater or equal to $a$. Atiyah, Patodi and Singer showed in [APS] (see also [BW, Prop. 14.2]) that $B^a_{\text{APS}}$ is a zero order pseudo-differential operator whose principal symbol $b_{\text{APS}}$ (independent of $a$) satisfies the following fact: for each $p \in \Sigma$ and $X \in T_p\Sigma - \{0\}$, the map $b_{\text{APS}}(X)$ is the orthogonal projection onto the eigenspace of $\sigma_D(X) = i\gamma^S(X)$ corresponding to the positive eigenvalue $|X|$. That is
\[
b_{\text{APS}}(X) = \frac{1}{2} (i\gamma^S(X) + |X|I) = \frac{1}{2} (-i\gamma(N)\gamma(X) + |X|I),
\]
and so the principal symbol $b_{\text{APS}}$ of the APS operator coincides, up to a constant, with the principal symbol $p_+(D)$ of the Calderón projector of $D$, for all $a \in \mathbb{Z}$. From this, it is immediate to see that the two ellipticity required conditions are satisfied.

Analytically, this APS boundary condition can be formulated as follows. If $\phi$ is a spinor on $\Sigma$ and $a \in \mathbb{Z}$ is an integer number, we will denote by $b_{\text{APS}}^a(\phi) = \phi_{\geq a}$ the orthonormal projection referred to the diagonalization of $D$ given above just after (8)

$$b_{\text{APS}}^a = \phi_{\geq a} = \sum_{k \geq a} \phi_k = \lim_{l \to \infty} \sum_{m=a}^{l} \phi_m,$$

where the above subseries convergences remain to be valid, as we had already commented for this truncated series, and moreover

$$\int_{\Sigma} |\phi_{\geq a}|^2 \leq \int_{\Sigma} |\phi|^2 \quad \forall a \in \mathbb{Z}$$

(Bessel inequality). It was long ago known the fact [see APS, Se, BB, HMR, BäBa1,BäBa2] that this spectral projection $b_{\text{APS}}^a$ trough $D$ provided an elliptic self-adjoint boundary condition on $\Omega$ for $D$ and for all $a \in \mathbb{R}$, if we take the spectral projection on eigenspaces corresponding to its non negative eigenvalues. (Be careful, in some papers, as [APS and BäBa1,BäBa2], the adapted Dirac operator $D$ on the boundary is taken to be $-D$ and, so, it is necessary to reverse positive and negative). But, we strongly advise to study all those questions about first order differential operators, existence of solutions and their regularity in the two recent works by Ballmann and Bär [BäBa1,BäBa2]. They are much more well clearly written and in a much more modern language. We have procured attain at the maximum number of readers.

Anyway, we conclude that, for each integer number $a \leq 0$, we can pose the elliptic boundary problem

$$\begin{cases} D\psi_a = \xi \\
\psi_{\geq a} = \phi_{\geq a}, \end{cases}$$

where $\psi$ and $\xi$ are spinor fields on the bulk manifold $\Omega$, $\phi$ is a spinor field on the boundary $\Sigma$ and the meaning of $\psi_{\geq a}$ should be already evident. This problem will be a Fredholm type problem. Analogously, the corresponding eigenvalue problem

$$\begin{cases} D\psi_a = \lambda \psi_a \\
\psi_{\geq a} = 0, \end{cases}$$

[see HMR, BaBä1,BaBä2], is also well posed and has nice existence and regularity properties.
5. A compact approach spinor proof of the Shi-Tam theorem valid for mean-convex domains

We assume that the scalar curvature $R$ of the compact spin Riemannian manifold $\Omega$ is non-negative and consider a solution to the following homogeneous problem on $\Omega$

$$\begin{cases} D\psi_a = 0 \\
\psi_{\ge a} = 0, \end{cases}$$

for any $a \in \mathbb{Z}$. Then, if we put it in the inequality (10) and use that $R \ge 0$. It only remains the terms on the boundary $\Sigma$ of $\Omega$. We get

$$0 \le \int_\Sigma \left( \langle D\psi_a, \psi_a \rangle - \frac{n}{2} H|\psi_a|^2 \right),$$

and the equality is attained if and only if $\psi_a$ is a parallel (twistor plus harmonic) spinor on $\Omega$. Now, choose $a \le 0$ in the problem above and substitute the corresponding solution in the integral inequality above. Since $\langle D\psi_{<a}, \psi_{<a} \rangle$ and $\langle D\psi_{\ge a}, \psi_{\ge a} \rangle$ are clearly $L^2$-orthogonal, we have

$$\int_\Sigma \langle D\psi_a, \psi_a \rangle = \int_\Sigma \langle D\psi_{<a}, \psi_{<a} \rangle = \sum_{k < a \le 0} \lambda_k \int_\Sigma |\psi_k|^2 \le 0,$$

for each integer $a \le 0$. If, moreover, we add the hypothesis that the inner mean curvature $H$ is non-negative along the boundary $\Sigma$, (16) implies

$$0 \le \int_\Sigma \left( \langle D\psi_a, \psi_a \rangle - \frac{n}{2} H|\psi_a|^2 \right) \le 0.$$ 

Hence, the first term in the integral above implies $\psi_{<a} = 0$ for all $a \le 0$. Since, from the boundary condition in (15), $\psi_{\ge a} = 0$, for all $a \in \mathbb{Z}$, $a \le 0$, we conclude

$$\psi_a = \psi_{<a} + \psi_{\ge a} = 0,$$

for all $a \in \mathbb{Z}$, $a \le 0$ along $\Sigma$. So, the spinor $\psi_a$ is identically zero along $\Sigma$. As $\psi_a$ was harmonic on $\Omega$, its length must be identically zero because the Hausdorff measure of $\Sigma$ is greater than two in the bulk manifold as was shown by Bär in [33]. Then, the unique solution to the homogeneous equation (15) is $\psi_a = 0$. Using that this equation is of type Fredholm, we need to study its cokernel, that is, that the adjoint problem to (15) has also trivial kernel. Thus, consider the homogeneous problem

$$\begin{cases} D\psi_a = 0 \\
\psi_{>a} = 0, \end{cases}$$

where it is easy to imagine what $\psi_{>a}$ denotes. Working as in (15), we also obtain the same inequality (17) for all $a \le 0$. But, in this case, when the equality is attained, we are not sure that $\psi_{<a} = 0$ for all $a \le 0$, because $\psi_0$ could be vanish, that is, $\psi$ could contain a non trivial harmonic component $\psi_0$. So, the homogeneous problem (18) could have non trivial harmonic solutions, unless $H > 0$, that is, the boundary $\Sigma$ be strictly mean convex.
In any of these two cases, the unique solutions to (15) and its adjoint (18) would be the null ones. Then, the non homogeneous problems would always unique solutions.

**Proposition 1.** Let $\Omega$ be a compact spin manifold with non negative scalar curvature and non-empty boundary $\Sigma$ such that its inner mean curvature is non-negative. Then, if we suppose that $\Sigma$ either does not admit harmonic spinors or is strictly mean convex ($H > 0$) in $\Omega$, both homogeneous problems (15) and (18) have as unique solution the zero spinor. As a consequence of this, the inhomogeneous problem

\[
\begin{align*}
    D\psi_a &= \xi \\
    \psi_{\geq a} &= \phi_{\geq a},
\end{align*}
\]

has a unique solution $\psi_a \in H^1(\Omega) \cap H^{\frac{1}{2}}(\Sigma)$ for prescribed spinor fields $\xi \in L^2(\Omega)$ and $\phi \in L^2(\Sigma)$ for each integer number $a \leq 0$.

We will apply this result à la Reilly, thought for solving equations on the bulk manifold, to obtain results on its boundary.

**Theorem 2.** Let $\Omega$ be a compact spin Riemannian manifold of dimension $n + 1$ with non-negative scalar curvature $R \geq 0$ and having a non-empty boundary $\Sigma$ whose inner mean curvature $H \geq 0$ is also non-negative (mean-convex). Suppose also that $\Sigma$ either does not carry harmonic spinors or $H > 0$. Then, for every spinor $\phi \in H^1(\Sigma)$, we have

\[
\int_\Sigma |D\phi||\phi| \geq \frac{n}{2} \int_\Sigma H|\phi|^2.
\]

The equality occurs if and only if $\phi$ is the restriction to $\Sigma$ of a parallel spinor on $\Omega$ and so

\[
D\phi = \frac{n}{2} H\phi.
\]

**Proof.** Using the Proposition above, for all integer $a \leq 0$, solve the type (19) equation

\[
\begin{align*}
    D\psi_a &= 0 \\
    \psi_{\geq a} &= \phi_{\geq a},
\end{align*}
\]

By working in same manner as in the introduction of this section, we can arrive to the inequality (16) because, until this moment, we did not use the boundary conditions. On the other hand, by using the orthogonal projections defined above, we know that

\[
\int_\Sigma \langle D\psi_a, \psi_a \rangle \leq \int_\Sigma \langle D\psi_{\geq a}, \psi_{\geq a} \rangle,
\]

because the summand

\[
\int_\Sigma \langle D\psi_{< a}, \psi_{< a} \rangle
\]
only contains non positive eigenvalues of $D$. So, since $\psi_{\geq a} = \phi_{\geq a}$, using the pointwise Cauchy-Schwarz inequality, the unique solution $\psi_a$ of (20) satisfies
\begin{equation}
\left(\int_{\Sigma} \langle D \psi_a, \psi_a \rangle \leq \int_{\Sigma} \langle D \phi_{\geq a}, \phi_{\geq a} \rangle \right) \leq \int_{\Sigma} |D \phi_{\geq a}| |\phi_{\geq a}|,
\end{equation}
for all $a \in \mathbb{Z}, a \leq 0$. Analogously, we have the $L^2$ decomposition
\begin{equation}
0 \leq \int_{\Sigma} |\psi_a|^2 = \int_{\Sigma} |\psi_{\geq a}|^2 + \int_{\Sigma} |\psi_{< a}|^2.
\end{equation}
So, in a similar way as above,
\begin{equation}
\int_{\Sigma} |\psi_a|^2 \leq \int_{\Sigma} |\psi_{\geq a}|^2 = \int_{\Sigma} |\phi_{\geq a}|^2,
\end{equation}
for each non positive integer $a$. From the fact that $\psi_{\geq a} = \phi_{\geq a}$ lies in $H^{1/2}(\Sigma)$, we know that this sequence with index $a$ converges strongly in the $L^2(\Sigma)$ topology and, so, in the $L^1$ topology as well. As the non negative mean curvature function $H$ is smooth on the compact manifold $\Sigma$, $\sqrt{H} \in L^2(\Sigma)$. Then,
\begin{equation}
\lim_{a \to -\infty} \int_{\Sigma} H|\phi_{\geq a}|^2 \, d\Sigma = \int_{\Sigma} H|\phi|^2
\end{equation}
strongly in $L^2(\Sigma)$. From (16), (20) and this last limit, we have
\begin{equation}
0 \leq \lim_{a \to -\infty} \int_{\Sigma} \left( |D \phi_{\geq a}| |\phi_{\geq a}| - \frac{n}{2} H|\phi_{\geq a}|^2 \right) = \int_{\Sigma} \left( |D \phi||\phi| - \frac{n}{2} H|\phi|^2 \right).
\end{equation}
Hence, the inequality is true.

Let us see that the solution $\psi_a$ to the boundary problem (10) is smooth on $\bar{\Omega}$ for all integer $a \leq 0$. In fact, the classical theory for the Dirac operator [see BäBa1,BäBa2] assures the smoothness of solutions in the interior $\Omega$. With respect to the regularity at $\Sigma$, we can apply Theorems 6.11 and 7.17 and Corollary 7.18 in the first on the papers cited above (using a bootstrapping process) and obtain smoothness on $\Sigma$ as well. Then $\psi_a \in \cap_{k=1}^{\infty} H^k(\Omega) = C^\infty(\bar{\Omega})$. For all non positive integer number $a$, the limit function $\psi$ will also be smooth on the compact manifold $\Omega$. This implies that the limit function
\begin{equation}
\psi = \lim_{a \to -\infty} \psi_a
\end{equation}
is also strongly harmonic on $\Omega$. Taking limit in (13) and taking into account that we also have the equality in the right side of (10) we see that $\psi$ is also a parallel spinor on the whole of $\Omega$. And it is clear that
\begin{equation}
\phi = \lim_{a \to -\infty} \phi_{\geq a} = \psi|\Sigma
\end{equation}
as well. But the relation between the operators $D$ and its restriction $D$ is
\begin{equation}
D\phi = \frac{n}{2} H\phi - \gamma(N)D\psi - \nabla_N \psi.
\end{equation}
Then, [see Bä1,BFGK,Bur] the spinor \( \phi \) has to satisfy on the whole \( \Sigma \) the equality

\[
D\phi = \frac{n}{2} H\phi.
\]

The converse is obvious.

**Remark 1.** Suppose that, in Theorem 2 above, we choose the spinor \( \phi \) on the boundary \( \Sigma \) as an eigenvalue corresponding to the first non-negative eigenvalue \( \lambda_1(D) \) of the intrinsic Dirac operator \( D \) of the boundary. A direct application of Theorem 2 gives

\[
\int_{\Sigma} \left( \lambda_1(D) - \frac{n}{2} H \right) |\phi|^2 \geq 0.
\]

As a consequence, we obtain the following lower bound which we firstly found in [HM]:

\[
\lambda_1(D) \geq \frac{n}{2} \max_{\Sigma} H
\]

and that improves, in the immersed case, the well-known lower bound by Fiedrich [F] for boundary manifolds in \( \mathbb{R}^{n+1} \).

**Remark 2.** It is also worthy to note that, as a consequence of this estimate, if \( \Sigma \) is a compact boundary in an Euclidean space and we know that \( \lambda_1(D) \leq n/2 \) and \( H \geq 1 \), then have the equality and *a fortiori* the mean curvature \( H \) of \( \Sigma \) must be constant. Hence, from the Alexandrov Theorem, \( \Omega \) is a round disc. This is an Euclidean analogue to the Min-Oo conjecture and was remarked to be true by Miao in the paper [Mi2], by supposing that \( \Sigma \) was isometric to a unit \( n \)-sphere.

**Theorem 3** (Brown-York mass for mean-convex surfaces). *Let \( \Omega \) be a compact spin Riemannian manifold of dimension \( n+1 \) with non-negative scalar curvature \( R \geq 0 \) and having a non-empty boundary \( \Sigma \) whose inner mean curvature \( H \geq 0 \) is strictly (strictly mean-convex). Suppose that there is an isometric and isospin immersion from \( \Sigma \) into another spin manifold \( \Omega_0 \) carrying on a non-trivial parallel spinor field and let \( H_0 \) its mean curvature with respect to any of its orientations. Then, we have*

\[
\int_{\Sigma} H \leq \int_{\Sigma} |H_0|.
\]

*The equality implies that \( H = |H_0| = H_0 \). Then, if \( n = 2 \), \( \Omega_0 \) is a domain in \( \mathbb{R}^3 \) and the two embeddings differ by a direct rigid motion.*

**Remark 3.** Note that, in the original Shi-Tam original result, the authors assume that the boundary \( \Sigma \) is strictly convex. Then a well-known by Pogorelov [P] and Nirenberg [Ni] guarantees the existence of an isometric embedding into the Euclidean space. Here, we need suppose the existence of this second isometric immersion. Moreover we have \( H_0 > 0 \) *a fortiori.*
Proof. Denote by $\psi$ the parallel spinor on $\Omega_0$ and let $\phi = \psi|_\Sigma$ its restriction onto $\Sigma$ through the existent immersion. Let’s recall that the parallelism of $\psi$ (see (22)) gives
\[ D\phi = \frac{n}{2}H_0\phi \quad \text{and} \quad |\phi| = 1. \]
Now, we apply Theorem 2 and have the desired inequality
\[ \int_{\Sigma} H \leq \int_{\Sigma} |H_0| d\Sigma. \]
If the equality is attained, then
\[ \frac{n}{2}H_0 = D\phi = \frac{n}{2}H \]
and so $H = H_0 > 0$. Then, the immersion of $\Sigma$ into the second ambient space $\Omega_0$ is strictly mean-convex as well (with respect to the choice of inner normal to $\Omega$).

When $n = 2$, from this equality and the fact that $K = K_\phi$ (because the two embeddings are isometric and preserve the Gauss curvatures), we deduce that the two second fundamental forms coincide. The Fundamental Theorem of the Local Theory of Surfaces allows us to conclude that the two boundaries differ by a direct rigid motion of the Euclidean space.

Remark 4. It is obvious that the result above is a generalization of the positivity theorem for the Brown-York mass previously proved for strictly convex surfaces by Shi and Tam. In their proof, the solution of difficult boundary equations and the positivity of the ADM-mass obtained by Shoen-Yau and Witten \cite{SY, Wi} in the context of asymptotically flat manifolds are essential components. Here, these difficulties are avoided and, as we have already remarked somewhere above, this compact version of the theorem implies the asymptotically flat version for the ADM mass (see \cite{HMRa}).

Corollary 4. Let $\Omega$ be a compact spinor manifold of dimension 3 with non-negative scalar curvature $R \geq 0$ and having a mean-convex boundary $\Sigma$ isometric to a sphere of any radius. Then, we have
\[ \int_{\Sigma} H \leq \sqrt{\pi A(\Sigma)} \]
where $A(\Sigma)$ is the area of $\Sigma$. If the equality holds, then the two boundaries are spheres of the same radius.

Proof. It is clear that the boundary $S^2$ of $\Omega$ admits and isometric and isospin (the sphere supports a unique spin structure) embedding into the Euclidean space $\mathbb{R}^3$ with $|H_0| = 1/r$ and area $A(\Sigma) = \pi r^2$, where $r > 0$ is the radius of the sphere. The fact that the two embeddings of $S^2$ are isometric allows us to finish.

Corollary 5 (Cohn-Vossen rigidity theorem for mean-convex domains). Two isometric and isospin strictly mean-convex compact surfaces in the Euclidean space $\mathbb{R}^3$ must be congruent.
Proof. Let $\Omega$ and $\Omega_0$ be the two domains determined in $\mathbb{R}^3$ by two corresponding surfaces identified by means of an isometry. Then, we can apply Theorem 3 interchanging the roles of $\Omega$ and $\Omega_0$ and applying the case of the equality.

Remark 5. The integral inequality in Corollary 4, for strictly convex surfaces of $\mathbb{R}^3$, is attributed to Minkowski (1901), although its very probable that it were previously known to Alexandrov and Fenchel. Recently have proved that the Minkowski inequality is not valid for any compact surface, although they proved it is for the axisymmetric ones. Its is also worthy to remark the following conjecture by Gromov: If $\Sigma$ is the boundary of a compact Riemannian manifold $\Omega$, then, if $R \geq \sigma$, for a certain constant $\sigma$, where $R$ is the scalar function of $\Omega$, then there exists a constant $\Lambda(\Sigma, \sigma)$ such that

$$\int_\Sigma H d\Sigma \leq \Lambda(\Sigma, \sigma).$$

Remark 6. Much more recently, in the context of fill in problems posed firstly by Bartnik, proved that, if $\Omega$ is the hemisphere $B^{n+1}$ and $\gamma$ is a metric on the boundary $S^n$ isotopic to the standard one with mean curvature $H > 0$, then there is a constant $h_0 = h_0(\gamma)$ such that

$$\int_\Sigma H \leq h_0.$$

It is clear that this result and our Corollary 4 belong to a same family.

6. Ambients with positive scalar curvature

Until now we have suppose that the scalar curvature of our compact spin Riemannian manifold $\Omega$ satisfied $R \geq 0$ (Euclidean context). Let’s enhance this positivity assumption to $R \geq n(n+1)$ (spherical context). This lower bound is just the constant value of the scalar curvature of the $(n+1)$-dimensional unit sphere. Then, by putting this assumption and the Schwarz inequality

$$|D\psi|^2 \leq (n+1)|\nabla\psi|$$

(already used in Section 3) into the right side of the Weitzenbök-Lichnerowicz formula, we obtain

$$\int_\Sigma \left( (D\psi, \psi) - \frac{n}{2} H |\psi|^2 \right) \geq \int_\Omega \left( -\frac{1}{n+1} |D\phi|^2 + \frac{n+1}{4} |\psi|^2 \right),$$

for all compact spin manifold $\Omega$, with equality only for the twistor spinor fields on $\Omega$. From now on, by using this integral inequality, we will work in a similar, but a few more elaborated, way as in Theorem 2, and will get the following result.
Theorem 6. Let $\Omega$ be a $(n+1)$-dimensional compact spin Riemannian manifold whose scalar curvature is constant $R = n(n+1)$ and having a non-empty boundary $\Sigma$ whose inner mean curvature $H \geq 0$ is non negative (mean-convex). Suppose also that $\Sigma$ does not admit harmonic spinors. Then, for every spinor $\phi \in H^1(\Sigma)$, we have
\[
\int_\Sigma |D\phi||\phi| \geq \frac{n}{2} \int_\Sigma |\phi|^2 \sqrt{H^2 + 1}.
\]
The equality holds if and only if $\phi$ is a spinor field coming from a real Killing spinor $\psi$ defined on $\Omega$, and so
\[
D\phi = \frac{n}{2} H\phi - \frac{n}{2} \gamma(N)\phi.
\]

Remark 7. Note that our hypotheses impose the equality $R = n(n+1)$ and not the maybe expected inequality $R \geq n(n+1)$, conjectured by Min-Oo and based on the flat case. Obviously the condition $R \geq n(n+1)$ is necessary, because one can otherwise perturb the hemisphere at an interior point so that $R \geq n(n+1) - \varepsilon$, for some small $\varepsilon > 0$, without changing the assumptions on the boundary [HW1]. Moreover the other possibility $R > n(n+1)$ had to be invalidated by the counterexamples built by Brendle, Marques and Neves [BMN], since all of them require at least one point with this strict inequality in the bulk manifold. Hence, we are brought to suppose the equality.

Proof. Given $a \in \mathbb{Z}$, $a \leq 0$, consider now the self-adjoint eigenvalue problem for the Dirac operator $D$ on the bulk manifold $\Omega$
\[
\begin{cases}
D\psi_a = \mu_a \psi_a \\
\psi_{\geq a} = 0,
\end{cases}
\] corresponding to the Dirac operator on $\Omega$ subjected to the usual APS elliptic boundary condition as in the section above. Since the boundary does not carry harmonic spinors, then the corresponding $D$ is symmetric in $\Omega$ and its eigenvalues are real numbers. If $\psi_a$ is a solution to (25), from Schwarz inequality (23), we get from the spinorial Reilly inequality
\[
0 \geq \int_\Sigma \left( \langle D\psi_a, \psi_a \rangle - \frac{n}{2} H|\psi_a|^2 \right) \geq \int_\Omega \left( - \frac{\mu_a^2}{n+1} + \frac{n+1}{4} \right) |\psi_a|^2,
\]
Since we have the $L^2$-orthogonal decomposition $\psi|_{\Sigma} = \psi_{\geq a} + \psi_{< a} = 0$ on $\Sigma$, $\psi_{\geq a} = 0$, from (20) and $H \geq 0$ in the hypotheses, the integral on the right side is non positive. Then
\[
\mu_a^2 \geq \frac{(n+1)^2}{4},
\]
with equality if and only $\psi_a$ is a twistor spinor and, in this case, a real Killing spinor on $\Omega$ with
\[
\nabla_X \psi_a = -\mu_a \gamma(X)\psi_a, \quad \forall X \in T\Omega.
\]
The equality would imply as well
\[ \int_{\Sigma} \left( \langle D \psi_{<a}, \psi_{<a} \rangle - \frac{n}{2} H |\psi|^2 \right) = 0, \]
and then \( \psi_{a}|_{\Sigma} = 0 \). But a real Killing spinor has constant length and so \( \psi_{a} \)
would identically null on the whole of \( \Omega \). As a consequence of this, the first eigenvalue \( \mu_{1}(a) \) of (25) satisfies
\[ \mu_{1}(a) > \frac{n + 1}{2}, \]
for all \( a \) integer non positive. Henceforth, there exists a unique solution with \( \psi_{a} \in H^{1}(\Omega) \cap H^{\frac{3}{2}}(\Sigma) \) for the problem
(27)
\[
\begin{cases}
D \psi_{a} = \frac{n + 1}{2} \psi_{a} \\
\psi_{\geq a} = \phi_{\geq a},
\end{cases}
\]
for each \( a \in \mathbb{Z}, a \leq 0 \) and for all \( \phi \in L^{2}(\Sigma) \). Take a such solution to (26) and put it in (24). Then
(28)
\[ 0 \leq \int_{\Sigma} \left( \langle D \psi_{a}, \psi_{a} \rangle - \frac{n}{2} H |\psi_{a}|^2 \right), \quad \forall a \in \mathbb{Z}, a \leq 0, \]
which is exactly the same expression that we obtained in the flat ambient case, but here the equality would attain only in the case where \( \psi_{a} \) is a real Killing spinor (and not a parallel one)
\[ \nabla_X \psi_{a} = -\frac{1}{2} \gamma(X) \psi_{a}, \quad \forall X \in T\Omega. \]
From this point, by working in the very exact way as in the proof of Theorem 2, we arrive at
(29)
\[ 0 \leq \lim_{a \to -\infty} \int_{\Sigma} \left( |D \phi_{\geq a}| |\phi_{\geq a}| - \frac{n}{2} H |\phi_{\geq a}|^2 \right). \]
But, with this choice of \( \psi_{a} \) as a solution to (26), the left side in the integral Reilly inequality after (25) vanishes. Hence, \( \psi_{a} \) is a twistor spinor and also
\[ D \psi_{a} = \frac{n + 1}{2} \psi_{a} \]
on the whole \( \Omega \). That is, \( \psi_{a} \) is a real Killing spinor. More precisely, we arrive just to the expression which had already rejected some times to avoid several reductionis ad absurdum, more precisely,
(30)
\[ \nabla_X \psi_{a} = -\frac{1}{2} \gamma(X) \psi_{a}, \quad \forall X \in T\Omega. \]
By using (7) and the equality above, we have
(31)
\[ D \psi_{a} = \frac{n}{2} H \psi_{a} - \frac{n}{2} \gamma(N) \psi_{a}, \]
along the boundary \( \Sigma \). Taking into account the orthogonal \( L^{2} \) decomposition \( \psi_{a} = \psi_{\geq a} + \psi_{<a} \), the fact that \( D \) preserves this decomposition and that
\( \gamma(N) \) reverses it pointwise, if we multiply scalarly by the spinor \( \gamma(N)\psi_a \) and integrate on \( \Sigma \), we get
\[
\int_{\Sigma} \langle D\psi_a, \gamma(N)\psi_a \rangle = -\frac{n}{2} \int_{\Sigma} |\psi_a|^2.
\]
By working in an analogous, but not exactly, equal way as in (20), we obtain
\[
(32) \quad \lim_{a \to -\infty} \int_{\Sigma} |D\phi \geq a||\phi \geq a| \geq \lim_{a \to -\infty} \frac{n}{2} \int_{\Sigma} |\phi \geq a|^2,
\]
for all \( a \in \mathbb{Z}, a \leq 0 \) and the convergence is in \( H^1_\Omega(\Sigma) \) and, so, strongly in \( H^2(\Sigma) \).

In order to obtain the inequality in the Theorem, note that (28) and (31) provide us integral sequences which converge strongly in \( L^2(\Sigma) \). Then, we can extract from each one of them a subsequence converging a.e. (and, so, since \( \phi \) is continuous converging on the whole) to the corresponding function limit. These two subsequences will be (probably different), but have been extracted from a same converging sequence, so, we will label them with the same indices as the original ones, and have
\[
\lim_{a \to -\infty} |D\phi \geq a||\phi \geq a| \geq \frac{n}{2} H|\phi|^2 \quad \text{and} \quad \lim_{a \to -\infty} |D\phi \geq a||\phi \geq a| \geq \frac{n}{2} |\phi|^2.
\]
As a consequence, we obtain
\[
\lim_{a \to -\infty} |D\phi \geq a||\phi \geq a| \geq \frac{n}{2} \sqrt{1 + H^2}|\phi \geq a|^2,
\]
which was the inequality we are looking for.

For the case of the equality, as in the proof of Theorem 2, we can use the standard and well written results of regularity results in [BaBa1, BaBa2] to show that the solution \( \psi_a \) to the boundary problem (26) is smooth on \( \Omega \) for all integer \( a \leq 0 \). As before, with respect to the boundary \( \Sigma \), we can apply Theorem 7.17 and Corollary 7.18 in [BaBa] repeatedly in a bootstrapping process and obtain smoothness on \( \Sigma \) as well. Then \( \psi_a \in \cap_{k=1}^{\infty} H^k(\Omega) = C^\infty(\Omega) \). For all non positive integer number \( a \), the limit function \( \psi \) is also smooth on the compact manifold \( \bar{\Omega} \). This implies that the limit function
\[
\psi = \lim_{a \to -\infty} \psi_a
\]
satisfies (15) on \( \Omega \) as well. That is, \( D\psi = (n + 1)/\psi \). Taking limit in (29) and substituting in the integral Reilly inequality we have, from the left side, that \( \psi \) is a parallel spinor on the whole of \( \Omega \). Also it is clear that
\[
\phi = \lim_{a \to -\infty} \phi \geq a = \psi|_\Sigma.
\]
But the relation between the operators \( D \) and its restriction \( D \) is
\[
(33) \quad D\phi = \frac{n}{2} H\phi - \gamma(N)D\psi - \nabla_N\psi
\]
and the spinor $\phi$ comes from a real Killing spinor on $\Omega$ with constant $-1/2$. Hence, it has to satisfy on the whole $\Sigma$ the equality

$$D\phi = \frac{n}{2} H\phi - \frac{n}{2}\phi.$$ 

The converse is obvious.

As far as we know, we obtain from Theorem 6 a lower estimate for the first eigenvalue of the Dirac operator in a hypersurface in a context on scalar curvature positive.

**Corollary 7.** Consider a compact spin Riemannian manifold $\Omega$ of dimension $n+1$ and scalar curvature $R = n(n+1)$ and mean-convex boundary. Suppose that $\Sigma$ does not support harmonic spinors and let $\phi$ be the eigen-spinor corresponding to the first eigenvalue $\lambda_1(D)$ positive of the intrinsic Dirac operator $D$ of the boundary. A direct application of Theorem 6 gives

$$\int_{\Sigma} \left( \lambda_1(D) - \frac{n}{2} \sqrt{1 + H^2} \right) |\phi|^2 \geq 0.$$ 

As a consequence, we obtain the following lower bound:

$$\lambda_1(D) \geq \frac{n}{2} \max_{\Sigma} \sqrt{1 + H^2}$$

that improves the well-known lower bound by Friedrich [F] for non immersed manifolds boundary manifolds (the Gauss for the curvature in the unit $(n+1)$-dimensional sphere gives

$$\sqrt{\frac{nR}{4(n-1)}} \leq \frac{n}{2} \sqrt{1 + H^2}$$

and the equality is attained only by the umbilical hypersurfaces).

**Remark 8.** It is also worthy to note that, as a consequence of this estimate, if $\Sigma$ is a compact boundary in an Euclidean space and we know that $\lambda_1(D) \leq n/2$ and $H \geq 0$, then we have the equality and *a fortiori* the mean curvature $H$ of $\Sigma$ must be identically null. Moreover, $\Omega$ supports the existence of a non trivial real Killing spinor (see [Bä]). Hence, $\Omega$ is close to being a spherical domain bounded by a minimal hypersurface. This would be the most similar to the solution to the spherical analogue to the Min-Oo conjecture.

As in the flat case, we can obtain a kind of positivity for a possible quasi-local mass in this new context.

**Theorem 8** (Brown-York in mean-convex and spherical case). Let $\Omega$ be a spin Riemannian manifold of dimension $n+1$ with scalar curvature $R = n(n+1)$ and having a non-empty boundary $\Sigma$ whose inner mean curvature $H \geq 0$ is also non-negative (mean-convex). Suppose that there is an isometric and isospin immersion from $\Sigma$ into another spin manifold $\Omega_0$ carrying
on a non-trivial real Killing spinor field and let $H_0$ its mean curvature with respect to any of its orientations. Then, we have
\[ \int_{\Sigma} \sqrt{1 + H^2} \leq \int_{\Sigma} \sqrt{1 + H_0^2}, \]
provided that the boundaries do not admit harmonic spinors. The equality implies that $H = H_0$. Then, if $n = 2$, $\Omega_0$ is a domain in $S^3$ and the two embeddings differ by a direct rigid motion.

Proof. Denote by $\psi$ the spinor on $\Omega_0$ and let $\phi = \psi|_{\Sigma}$ its restriction onto $\Sigma$ through the existent immersion. Let’s recall that the parallelism of $\psi$ gives
\[ D\phi = \frac{n}{2} \sqrt{1 + H_0^2} \phi \quad \text{and} \quad |\phi| = 1. \]
Now, we apply Theorem 2 and have the desired inequality
\[ \int_{\Sigma} \sqrt{1 + H^2} \leq \int_{\Sigma} \sqrt{1 + H_0^2}. \]
If the equality is attained, then
\[ \frac{n}{2} \sqrt{1 + H_0^2} = D\phi = \frac{n}{2} \sqrt{1 + H^2} \]
and so $H = H_0 \geq 0$. Then, the immersion of $\Sigma$ into the second ambient space $\Omega_0$ is mean-convex as well (with respect to the choice of inner normal to $\Omega$).

When $n = 2$, from this equality and the fact that $K = K_0$ (because the two embeddings are isometric and preserve the Gauss curvatures), we deduce that the two second fundamental forms coincide. The Fundamental Theorem of the Local Theory of Surfaces allows us to conclude that the two boundaries differ by a direct rigid motion of the Euclidean space.

Corollary 9 (Cohn-Vossen rigidity theorem in the sphere). Two isometric and isospin mean-convex compact surfaces embedded in a sphere $S^3$ must be congruent, provided they do not admit harmonic spinors.

Proof. Let $\Omega$ and $\Omega_0$ be the two domains determined in $S^3$ by two corresponding surfaces identified by means of an isometry. Then, we can apply Theorem 8 interchanging the roles of $\Omega$ and $\Omega_0$ and applying the case of the equality.

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