Symmetry and short interval mean-squares

GIOVANNI COPOLLA - MAURIZIO LAPORTA

Abstract. The so-called weighted Selberg integral is a discrete mean-square, that generalizes the classical Selberg integral of primes to an arithmetic function $f$, whose values in a short interval are suitably attached to a weight function. We give conditions on $f$ and select a particular class of weights in order to investigate non-trivial bounds of weighted Selberg integrals of both $f$ and $f \ast \mu$. In particular, we discuss the cases of the symmetry integral and the modified Selberg integral, that involves the Cesaro weight, and prove some side results when $f$ is a divisor function.

1. Introduction and statement of the results.

The symmetry integral of a complex-valued arithmetic function $f$ is a short interval mean-square of the type

$$J_{\text{sgn}, f}(N, H) \overset{\text{def}}{=} \sum_{x \sim N} \left| \sum_{x-H \leq n \leq x+H} f(n) \text{sgn}(n-x) \right|^2,$$

where $N, H$ are positive integers such that $H = o(N)$ as $N \to \infty$, $\text{sgn}(0) \overset{\text{def}}{=} 0$, $\text{sgn}(t) \overset{\text{def}}{=} |t|/t$ for $t \neq 0$, and $x \sim N$ in sums means that $x$ is an integer belonging to $[N, 2N]$.

These kinds of mean-squares have been intensively studied by the first author ([C1],[C2],[C3],[C5]) for different instances of the arithmetic function $f$ with a particular attention to the divisor function $d_k$. Symmetry integrals are so called because of the Kaczorowski and Perelli discover [K-P], during the 1990s, about a remarkable link between the symmetry integral of the primes and the classical Selberg integral [Se], respectively

$$\int_1^{2N} \left| \sum_{x < n \leq x+H} \Lambda(n) - \sum_{x-H < n < x} \Lambda(n) \right|^2 \, dx,$$

where $\Lambda$ is the well-known von Mangoldt function, defined as $\Lambda(n) \overset{\text{def}}{=} \log p$ if $n = p^r$ for some prime number $p$ and for some positive integer $r$, otherwise $\Lambda(n) \overset{\text{def}}{=} 0$. In fact, even the discrete version of the last integral can be generalized to define the Selberg integral of any arithmetic function $f$,

$$J_f(N, H) \overset{\text{def}}{=} \sum_{x \sim N} \left| \sum_{x-H \leq n \leq x+H} f(n) - M_f(x, H) \right|^2,$$

where $M_f(x, H)$ is the so-called short interval mean-value of $f$ and $H = o(N)$ as $N \to \infty$ hereafter.

As in the prototype case of the function $d_k$, non-trivial estimates of $J_{\text{sgn}, f}(N, H)$ are obtained for essentially bounded $f$, that is $f(n) \ll_{\varepsilon} n^{\varepsilon} \forall \varepsilon > 0$. Plainly, the wider is the range of the width $\theta$ of the short interval $[x-H, x+H]$ with $H \asymp N^\theta$ (i.e $N^\theta \ll H \ll N^\theta$) for which a non-trivial bound holds, the finer is the result.

Our first theorem gives a link between non-trivial estimates of the symmetry integrals of both $g$ and the Dirichlet convolution product $g \ast 1$, where $1$ denotes the constantly 1 function and $g$ is a real-valued and essentially bounded function. Note that $g \ast 1$ is essentially bounded as well.

**Theorem 1.** Let $g$ be an essentially bounded real arithmetic function and let $H \asymp N^\theta$ for a fixed $\theta \in (1/3, 1)$. If there exists $G \in (0, 1)$ such that

$$J_{\text{sgn}, g}(N, h) \ll Nh^{2-G} \text{ for every integer } h \asymp N^\theta \text{ with } \theta \in \left(\frac{3 \theta - 1}{(1-G)\theta + G + 1}, \theta\right),$$

then

$$J_{\text{sgn}, g\ast 1}(N, H) \ll NH^{2-G'} \text{ for every } G' \in \left(0, \min\left(\frac{3 - \vartheta^{-1}}{1 + 2\vartheta^{-1}}, \vartheta^{-1} - 1\right)\right).$$

Mathematics Subject Classification (2010) : 11N37, 11N36.
Moreover, where the mean value $M$ with respect to $G$ and both cases imply an exponent gain $G' \to 0$, i.e. a trivial bound for $J_{\text{sgn},g=1}(N,H)$.

Moreover,

$$\vartheta \in \left( \frac{1}{3}, \frac{G + 1}{2G + 1} \right) \iff \min \left( \frac{3 - \vartheta^{-1}}{1 + 2G^{-1}}, \vartheta^{-1} - 1 \right) = \frac{3 - \vartheta^{-1}}{1 + 2G^{-1}}$$

and in particular for $\vartheta = \frac{G + 1}{2G + 1} \in \left( \frac{1}{3}, 1 \right)$ one gets the largest possible range for $G' \in \left( 0, \frac{G}{G+1} \right)$.

While we postpone the proof of Theorem 1 until §4 after the necessary Lemmata, here we state and prove a first noteworthy consequence, in the special cases of both the divisor function $d_3(n) \overset{\text{def}}{=} \sum_{ab=1} n$ and the function $\omega(n) \overset{\text{def}}{=} \sum_{p|n} 1$ that counts the number of distinct prime divisors of $n$.

**Corollary.** For $H \asymp N^\vartheta$ non-trivial bounds hold for both

$$J_{\text{sgn},d_3}(N,H) \text{ if } \vartheta \in (1/3, 1/2 - \varepsilon) \quad \text{and} \quad J_{\text{sgn},\omega}(N,H) \text{ if } \vartheta \in (7/17 + \varepsilon, 1 - \varepsilon) \quad \forall \varepsilon > 0.$$

**Proof.** Since $d_3 = d \ast 1$ with $d(n) \overset{\text{def}}{=} \sum_{d|n} 1$, we can use the bound given in [C-S],

$$J_{\text{sgn},d}(N,h) \ll NH^{2-G}$$

for every integer $h \asymp N^\vartheta$ with $\vartheta \in (0, 1/2)$, that holds for any $G \in (0, 1)$. Hence, from Theorem 1 we get an exponent gain $G' = G'(\vartheta, G) > 0$ so that

$$J_{\text{sgn},d_3}(N,H) \ll NH^{2-G'}$$

for every integer $H \asymp N^\vartheta$ with $\vartheta \in (1/3, 1/2 - \varepsilon)$.

In order to exhibit a non-trivial bound for $J_{\text{sgn},\omega}(N,H)$, note that $\omega = 1_p \ast 1$, where $1_p$ is the characteristic function of the set $\mathcal{P}$ of prime numbers. Thus, we can apply Theorem 1 with aid of the bound

$$J_{\text{sgn},1_p}(N,h) \ll NH^{2-G}$$

for every integer $h \asymp N^\vartheta$ with $\vartheta \in (1/6, 1)$, that is a consequence of Huxley's zero density estimate [H]. Indeed, it is well known that within the same range of width $\vartheta \in (1/6, 1)$ such an estimate implies a non-trivial bound for the classical Selberg integral,

$$\int_{2N}^{2N} \left| \sum_{x<n \leq x+h} \Lambda(n) - h \right|^2 \, dx,$$

whose difference from $J_{\Lambda}(N,h)$ is negligible. Thus, one obtains the above non-trivial bound for $J_{\text{sgn},1_p}(N,h)$ by using the inequalities

$$J_{\text{sgn},1_p}(N,h) \ll J_{1_p}(N-h-1,h) + J_{1_p}(N,h), \quad J_{1_p}(N,h) \ll J_{\Lambda}(N,h) \log^{-2} N + Nh \log^{2} N,$$

the former valid for any $f$, here $f = 1_p$, and the second applying partial summation (to $p$ powers).

After the introduction by the first author [C0] of the modified Selberg integral,

$$\tilde{J}_f(N,H) \overset{\text{def}}{=} \sum_{x \sim N} \left| \frac{1}{H} \sum_{h \leq H, x-h < n < x+h} f(n) - M_f(x,H) \right|^2,$$

where $M_f(x,H)$ is the same mean-value that appears in $J_f(N,H)$, we have further generalized mean-squares in short intervals for arithmetic functions gauged by a general weight $w$ (see [C-L]). Indeed, $J_{\text{sgn},f}(N,H)$, $J_f(N,H)$ and $\tilde{J}_f(N,H)$ are particular instances of the so-called weighted Selberg integral of $f$, i.e.

$$J_{w,f}(N,H) \overset{\text{def}}{=} \sum_{x \sim N} \left| \sum_n w_H(n-x)f(n) - M_f(x,w_H) \right|^2,$$

where $w_H$ is the product of the weight $w : \mathbb{R} \to \mathbb{C}$ and the characteristic function $1_H$ of the set $[-H,H] \cap \mathbb{Z}$, whereas the mean value $M_f(x,w_H)$ has to be determined in agreement with the choice of $w$. Namely, the weights involved in $J_f(N,H)$ and $\tilde{J}_f(N,H)$ are respectively

- the **unit step** function $u(a) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } a > 0 \\ 0 & \text{otherwise} \end{cases}$ and
- the Cesaro weight $C_H(a) \overset{\text{def}}{=} \max \left( 0 - \left| \frac{a}{H} \right|, 0 \right)$.  

2
While this is readily seen for \( u \), the relation between \( \tilde{J}_f(N, H) \) and \( C_H \) relies on a well-known observation going back to the Italian mathematician Cesaro, i.e.

\[
\sum_{0 \leq |n-x| \leq H} \left( 1 - \frac{|n-x|}{H} \right) f(n) = \frac{1}{H} \sum_{h \leq H} \sum_{|n-x| < h} f(n).
\]

Concerning the mean value terms, whereas it is plain that \( M_f(x, \text{sgn}_H) \) vanishes identically for any \( f \), according to Ivč [Iv] if \( f \) has Dirichlet series \( F(s) \) that is meromorphic in \( \mathbb{C} \) and absolutely convergent in the half-plane \( \Re(s) > 1 \) at least, then the mean value appearing in \( J_f(N, H) \) and \( \tilde{J}_f(N, H) \) has the analytic form

\[
M_f(x, H) = H p_f(\log x),
\]

and more in general in \( J_w,f(N, H) \) one has (compare [C-L], §1)

\[
M_f(x, w_H) = \tilde{w}_H(0) p_f(\log x),
\]

where \( p_f(\log x) \equiv \text{Res}_{s=1} F(s)x^{s-1} \) is the logarithmic polynomial of \( f \) and \( \tilde{w}_H(0) \equiv \sum_n w_H(a) \) is the so-called mass of \( w \) in \([-H, H]\) (see §2). From our study (see [C-L], §2-3) it turns out that, if \( f = g * 1 \), then under suitable conditions on \( g \) and \( w \) one might expect \( M_f(x, w_H) \) to be close, at least in the mean-square, to its arithmetic form

\[
\tilde{w}_H(0) \sum_{q \leq Q} \frac{g(q)}{q},
\]

for some \( Q = Q(x) \ll N \).

This is a new feature exploited by the first author ([C1], [C2], [C3]) for a sieve function \( f \), namely

\[
f(n) \equiv \sum_{q | n, q \leq Q} g(q),
\]

where \( g : \mathbb{N} \to \mathbb{R} \) is essentially bounded and \( Q \) does not depend on \( x \sim N \).

If \( Q \ll N^\lambda_f \), uniformly \( \forall n \ll N \) and for some \( \lambda_f \in [0,1] \), then we say that \( f \) has LEVEL (at most) \( \lambda_f \). Trivially, any essentially bounded arithmetic function is a sieve function of level at most 1, while in the sequel a particular attention is given to arithmetic functions of level (strictly) less than 1.

A typical case with an effective dependence of \( Q \) on \( x \) is the divisor function \( d_k = d_{k-1} * 1 \), to which the last section §5 is devoted, in order to accomplish a discussion commenced in [C-L], on the problem of showing sufficient proximity of the analytic and the arithmetic forms of the short interval mean value.

Here we are going to explore further the relation between weighted Selberg integrals of \( g * 1 \) and \( g \). Our next result yields that (roughly speaking) if \( J_g(N, H) \) is close enough to \( \tilde{J}_g(N, H) \), then the same happens with \( J_{g*1}(N, H) \) and \( \tilde{J}_{g*1}(N, H) \).

**Theorem 2.** Let \( g \) be an essentially bounded and real arithmetic function and let \( w' \) be a weight such that \( w'_H = C_H - u_H \) with \( H \asymp N^\vartheta \) for a fixed \( \vartheta \in (1/3, 1) \). If there exists \( G \in (0, 1) \) such that

\[
J_{w',g}(N, h) \ll Nh^{2-G} \quad \text{for every integer } h \asymp N^\vartheta \text{ with } \vartheta \in \left( \frac{3\vartheta - 1}{1-G}\vartheta + G + 1, \vartheta \right],
\]

then \( J_{g*1}(N, H) \ll NH^{2-3G'} \) for some \( G' = G'(\vartheta, G) > 0 \).

Since the proof goes as for Theorem 1 (in analogy with \( \text{sgn} \), the weight \( w' \) has zero mass in \([-H, H]\)), we just show that from the assertion of Theorem 2 one has

\[
J_{g*1}(N, H) - \tilde{J}_{g*1}(N, H) \ll NH^{2-G'}.
\]

To this end, set \( w''_H = C_H + u_H \) and apply Cauchy’s inequality to write

\[
\tilde{J}_f(N, H) - J_f(N, H) = \sum_{x \sim N} \left( \sum_n C_H(n-x) f(n) - M_f(x, H) \right)^2 - \left( \sum_n u_H(n-x) f(n) - M_f(x, H) \right)^2 =
\]

3
\[ \sum_{x<H} \left( \sum_{n} w'_H(n-x)f(n) \right) \left( \sum_{n} w''_H(n-x)f(n) - 2M_f(x,H) \right) \leq \sqrt{J_{w',f}(N,H)(\tilde{J}_f(N,H) + J_f(N,H))} \ll \varepsilon \sqrt{J_{w',f}(N,H)N^{1+\varepsilon}H^2} \ll_{\varepsilon} NH^{2-3G'/2}N^{\varepsilon} \ll NH^2-G', \]

by using the trivial bound for \( \tilde{J}_f(N,H) + J_f(N,H) \) and the non-trivial one yielded by Theorem 2.

Next theorem is a generalization to \( J_{w,f}(N,H) \) of results in [C1] and [C3], when \( f \) is a sieve function and \( w \) belongs to a particular class of weights, that we describe as follows. If the correlation of \( w_H \), i.e.

\[ \mathcal{C}_{w_H}(a) \overset{def}{=} \sum_{m-n=a} w_H(m)w_H(n), \]

satisfies the formula

\[ (A) \quad \sum_{a \equiv 0(\text{mod } \ell)} \mathcal{C}_{w_H}(a) = \frac{1}{\ell} \sum_a c_{w_H}(a) + O(H) \quad \forall \ell \leq 2H, \]

then \( w_H \) is said to be arithmetic and, if this is the case for every \( H \), then we say so for \( w \). In particular, \( w \) is a good weight if it is arithmetic and absolutely bounded on the integers, i.e. \( |w(n)| \leq K \quad \forall n \in \mathbb{Z} \) for some \( K \in (0, +\infty) \). In §2 we show that the weights \( u, \text{sgn and } C_H \) are arithmetic (consequently good weights, it being plain that they are absolutely bounded by 1).

In order to state next Theorem 3, we need some further notation and convention. First, a modified Vinogradov notation (introduced by Kolesnik) is useful to hide arbitrarily small powers (of the main variable):

\[ A \ll B \overset{def}{\iff} A \ll_{\varepsilon} N^{\varepsilon}B, \quad \forall \varepsilon > 0 \]

Then, by \( \text{supp } g \subset [1, Q] \) we implicitly mean that the arithmetic function under consideration is \( g \cdot 1_{[1, Q]} \), where \( 1_{[1, Q]} \) is the characteristic function of \([1, Q] \cap \mathbb{N} \), so that \( g \cdot 1 \) is a sieve function.

**Theorem 3.** Let \( g \) be a real and essentially bounded arithmetic function such that \( \text{supp } g \subseteq [1, Q] \) with \( Q = Q(N,H) \to \infty \). For every good and real weight \( w \) one has

\[ J_{w,g \cdot 1}(N,H) \ll NH + Q^2H + QH^2 + H^3. \]

The proof is postponed until §4. Here we point out that, under the same hypotheses for \( g \) and \( w \), if \( H \asymp N^\vartheta \) with \( \vartheta \in (0, 1/2) \), then one can establish

\[ J_{w,g \cdot 1}(N,H) \ll NH + N^{4\vartheta}Q^{5/48}H^2 + N^{1-2\vartheta/3}H^2 + QH^2 \quad \forall \vartheta > 0 \]

by means of the Main Lemma from a recent work of the first author [C3] (a very technical result coming from averages of Kloosterman sums [D-F-I]). We will return to this in a future paper.

The last result of this section exhibits a length-inertia property for the Selberg integral \( J_f(N,H) \), that allows to preserve non-trivial bounds, as the length of the short interval increases. While the symmetry integral has no such features, a rather cumbersome amount of calculations prevents us to give a definitive answer to the analogous question for the modified Selberg integral \( \tilde{J}_f(N,H) \).

Let \([x]\) denote the integer part of \( x \in \mathbb{R} \) and let \( L \overset{def}{=} \log N \). In §4 we prove also the following theorem.

**Theorem 4.** For every \( H > h \) one has

\[ J_f(N,H) \ll \left( \frac{H}{h} \right)^2 J_f(N,h) + J_f \left( N, H - h \left[ \frac{H}{h} \right] \right) + H^3 \left( \|f\|_{\infty}^2 + L^2c \right), \]

where \( \|f\|_{\infty} \overset{def}{=} \max_{[N-H, 2N+H]} |f| \) and \( c \) is the degree of the logarithmic polynomial of \( f \).  

4
Plan of the paper

After a brief paragraph on some further notation and definitions, in §2 we introduce the sporadic functions and discuss some properties on the arithmetic weights. The necessary Lemmata for Theorems 1 and 3 constitute the third section of the paper, whereas in §4 one finds the proofs of Theorems 1, 3 and 4. The last section complements our study [C-L] regarding the case of the divisor function \( d_k \).

Notation and definitions

If the implicit constants in the symbols \( O \) and \( \ll \) depend on some parameters like \( \varepsilon > 0 \), then mostly we specify it by introducing subscripts like \( O_\varepsilon \) and \( \ll_\varepsilon \), whereas we omit subscripts for \( \ll \) defined above. Notice that the value of \( \varepsilon \) may change from statement to statement, since \( \varepsilon > 0 \) is arbitrarily small.

For the sake of clarity, let us remark that throughout the paper \( H, N \) denote positive integers such that \( H = o(N) \) as \( N \to \infty \), i.e. \( H/N \to 0 \) as \( N \to \infty \). The notation \( H = o(N) \) is used synonymously with \( N = \infty (H) \). Typically we write \( H \asymp N^\theta \) for \( N^\theta \ll H \ll N^\theta \) with \( \theta \in (0,1) \), that we call the \emph{width} of the short interval \([x-H,x+H]\). As already mentioned, we use to abbreviate \( L = \log N \).

The symbol \( 1 \) denotes the constantly 1 function, while \( 1_U \) is the characteristic function of the set \( U \cap \mathbb{Z} \) for every \( U \subseteq \mathbb{R} \). In particular, we abbreviate \( 1_H = 1_{[-H,H]} \), so that for \( w : \mathbb{R} \to \mathbb{C} \) we write \( w_H = w \cdot 1_H \). Given the arithmetic functions \( f, g : \mathbb{N} \to \mathbb{C} \), if \( f \) is the convolution product of \( g \) and \( 1 \), i.e. \( f = g * 1 \), then \( g \) is called the \emph{Eratosthenes transform} of \( f \). Thus, \( 1 \) is the Eratosthenes transform of the divisor function \( d = 1 * 1 \). More in general, \( d_k = 1 * \cdots * 1 = d_{k-1} * 1 \) for \( k \geq 3 \).

In sums like \( \sum_{n \leq x} \) it is implicit that \( a \geq 1 \). As usual, \( e(\alpha) \overset{\text{def}}{=} e^{2\pi i \alpha} \forall \alpha \in \mathbb{R} \) and \( e_q(\alpha) \overset{\text{def}}{=} e(\alpha/q) \forall q \in \mathbb{N}, \forall \alpha \in \mathbb{Z} \). Specially within formulae, sometimes we abbreviate \( n \equiv 0 (q) \) for \( n \equiv 0 (\mathbb{Z}) \). The symbol \( \sum^* \) indicates that the sum is taken over the reduced residues. The distance of \( \alpha \in \mathbb{R} \) from the nearest integer is \( \| \alpha \| \overset{\text{def}}{=} \min(|\alpha|,1-|\alpha|) \), where \( \{ \alpha \} \overset{\text{def}}{=} \alpha - \lfloor \alpha \rfloor \) is the fractional part of \( \alpha \). Throughout the paper we apply standard formulae without further references. For example, we use the asymptotic equation \( \gamma \) is the \emph{Euler-Mascheroni constant}:

\[
\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O(1/x).
\]

2. Sporadic functions and arithmetic weights.

As already said, any function of the type \( w : \mathbb{R} \to \mathbb{C} \) can take over the role of a weight here, though de facto we deal with the weighted characteristic function \( w_H = w \cdot 1_H \) of the integers in the short interval \([-H,H]\).

By the \emph{discrete Fourier transform} (DFT) of \( w_H \) we mean the exponential sum

\[
\widehat{w_H}(\alpha) \overset{\text{def}}{=} \sum_a w_H(a)e(a\alpha) \quad \forall \alpha \in [0,1),
\]

whose value for \( \alpha = 0 \) is the \emph{mass} of \( w \) in \([-H,H]\), i.e.

\[
\widehat{w_H}(0) = \sum_a w_H(a).
\]

Since \( \#\{n \in [x-H,x+H] : n \equiv 0 (\mathbb{Z})\} \leq 1 \) when \( q > 2H \), then we call

\[
W_H(x;q) \overset{\text{def}}{=} \sum_{n \equiv 0 (q)} w_H(n-x) = \sum_{a \equiv -x (q)} w_H(a),
\]

the (weighted) \emph{sporadic sum}, while the (weighted) \emph{sporadic function} is

\[
\chi_{q}(x,w_H) \overset{\text{def}}{=} W_H(x;q) - \frac{\widehat{w_H}(0)}{q} = \sum_{a \equiv -x (q)} w_H(a) - \frac{1}{q} \sum_a w_H(a)
\]

where \( \widehat{w_H}(0)/q \) exhibits a behavior of a \emph{local} mean-value for \( W_H(x;q) \).
In the next proposition we give the so-called Fourier-Ramanujan expansion of the sporadic function.

**Proposition 1.** For every \( w : \mathbb{R} \to \mathbb{C} \) and all positive integers \( q, H \) one has

\[
\chi_q(x, w_H) = \frac{1}{q} \sum_{d \mid q, (d,q)=1} \sum_{j \leq d} \hat{w}_H\left(\frac{j}{d}\right) e_d(jx)
\]

(assume that for \( q = 1 \) the sum vanishes).

**Proof.** From the orthogonality of the additive characters,

\[
\frac{1}{q} \sum_{r \leq q} e_q(ar) = \begin{cases} 
1 & \text{if } q \mid a, \\
0 & \text{otherwise},
\end{cases}
\]

one gets

\[
\sum_{n=0(q)} w_H(n-x) = \sum_{\alpha+x=0(q)} w_H(\alpha) = \frac{1}{q} \sum_{r \leq q} e_q(rx) \sum_{a} w_H(a) e_q(ar) = \frac{1}{q} \sum_{r \leq q} \hat{w}_H\left(\frac{r}{q}\right) e_q(rx).
\]

Thus, since \( \hat{w}_H(0) = \hat{w}_H(1) \), we can write

\[
\chi_q(x, w_H) = \sum_{n=0(q)} w_H(n-x) - \frac{\hat{w}_H(0)}{q} = \frac{1}{q} \sum_{r \leq q} \hat{w}_H\left(\frac{r}{q}\right) e_q(rx) =
\]

\[
= \frac{1}{q} \sum_{d \mid q} \sum_{j \leq d} \hat{w}_H\left(\frac{j}{q}\right) e_d(jx) = \frac{1}{q} \sum_{d \mid q} \sum_{j \leq d} \hat{w}_H\left(\frac{j}{d}\right) e_d(jx)
\]

and the Proposition is proved.

Recall that \( w_H \) is said to be arithmetic when the correlation \( \mathcal{C}_{w_H}(a) \) satisfies the formula (A) in §1. Next propositions express some properties of such weights.

**Proposition 2.**
1) \( w_H \) is arithmetic if and only if

\[
(B) \quad \frac{1}{q} \sum_{j < q} \left| \hat{w}_H\left(\frac{j}{q}\right) \right|^2 \ll H \quad \forall q \leq 2H.
\]

2) If \( w \) is a good weight, then the inequality (B) holds for all \( q \geq 1 \).
3) If \( w_H \) is arithmetic, so is its normalized correlation \( \mathcal{C}_{w_H}/H \).

**Proof.** 1) Through an application of the orthogonality of additive characters similar to that of Proposition 1, it easily seen that (A) is equivalent to

\[
\frac{1}{q} \sum_{j < q} \mathcal{C}_{w_H}\left(\frac{j}{q}\right) \ll H \quad \forall q \leq 2H.
\]

Therefore, (B) follows immediately from the identity

\[
\mathcal{C}_{w_H}(\alpha) = \sum_{a} \mathcal{C}_{w_H}(a) e(a\alpha) = \sum_{a} \sum_{m-n=a} w_H(m) \hat{w}_H(n) e(a\alpha) = \left| \sum_{r} w_H(r) e(r\alpha) \right|^2 = \left| \hat{w}_H(\alpha) \right|^2 \forall \alpha \in [0,1).
\]

2) Note that if \( w \) is absolutely bounded, then the inequality (B) holds for high divisors \( q > 2H \) because

\[
\frac{1}{q} \sum_{j < q} \left| \hat{w}_H\left(\frac{j}{q}\right) \right|^2 \leq \frac{1}{q} \sum_{j < q} \left| \hat{w}_H\left(\frac{j}{q}\right) \right|^2 = \frac{1}{q} \sum_{j < q} \sum_{h_1} \sum_{h_2} w_H(h_1) \hat{w}_H(h_2) e_q(j(h_1 - h_2)) =
\]
= \sum_{0 \leq |h_1| \leq H} \sum_{0 \leq |h_2| \leq H} \sum_{h_2 \equiv h_1 \pmod{q}} w_H(h_1) w_h(h_2) = \sum_{0 \leq |h_1| \leq H} w(h_1) \sum_{0 \leq |h_2| \leq H} w(h_2) \ll \frac{H^2}{q} + H.

3) Let us show that (B) holds for \( C_{w_H}/H \). Indeed, since trivially \( \tilde{w}_H(\alpha) \ll H, \forall \alpha \in [0,1) \), then

\[
\frac{1}{q} \sum_{j < q} \left| \frac{\tilde{w}_H(\frac{j}{q})}{H} \right|^2 = \frac{1}{qH^2} \sum_{j < q} \left| \frac{\tilde{w}_H(\frac{j}{q})}{H} \right|^2 = \frac{1}{qH^2} \sum_{j < q} \left| w_H(\frac{j}{q}) \right|^2 \ll \frac{1}{q} \sum_{j < q} \left| w_H(\frac{j}{q}) \right|^2 \ll H \quad \forall q \leq 2H.
\]

The Proposition is completely proved.

Now, let us show that (B) holds in particular for \( u, \text{sgn} \) and \( C_H \), that is to say, such weights are good. To this end, we set

\[
\mathcal{L}_q^2(\tilde{w}_H) \overset{\text{def}}{=} \frac{1}{q} \sum_{j < q} \left| \tilde{w}_H(\frac{j}{q}) \right|^2
\]

and prove the following property that plainly implies (B) for \( u_H, \text{sgn}_H \) and \( C_H \).

**Proposition 3.** If \( w_H \) is one of our weights \( u_H, \text{sgn}_H, C_H \), then

\[
\mathcal{L}_q^2(\tilde{w}_H) \ll \min\left(1, \frac{H}{q}\right).
\]

**Proof.** Let us start with \( w_H = u_H \), whose DFT satisfies the well known inequality

\[
|\tilde{w}_H(\alpha)| = \left| \sum_{n \leq H} e(n\alpha) \right|^2 = \left| \frac{\sin(\pi Hao)}{\sin(\pi \alpha)} \right| \ll \min\left(H, \frac{1}{|\alpha|}\right).
\]

Thus, for \( H_q \overset{\text{def}}{=} q\{H/q\} \ll \min(q, H) \) we get, assuming \( H_q > 0 \) (otherwise the trivial \( \mathcal{L}_q^2(\tilde{w}_H) = 0 \) gives the inequality),

\[
\mathcal{L}_q^2(u_H) = \frac{1}{H} \sum_{j < H} \frac{\sin(\pi Hao)}{\sin(\pi j/q)} \ll \frac{1}{H} \sum_{0 < |j| \leq \frac{H}{q}} H_q^2 + \sum_{\frac{H}{q} < |j| \leq H} \frac{1}{j^2} \ll H_q \ll \min\left(1, \frac{H}{q}\right).
\]

Since the Cesaro weight is the normalized correlation of \( u \), i.e.

\[
C_H(\alpha) = \frac{1}{H} \sum_{t \leq H-|\alpha|} 1 = \frac{1}{H} \sum_{m,n \leq H} 1 = \frac{\tilde{e}_{u_H}(\alpha)}{H},
\]

then \( C_H \) is arithmetic because of the third assertion in Proposition 2. However, it turns out that the stated inequality holds also in this case:

\[
\mathcal{L}_q^2(C_H) = \mathcal{L}_q^2(\tilde{e}_{u_H}/H) = \frac{1}{H} \mathcal{L}_q^2(|\tilde{w}_H|^2) \ll \mathcal{L}_q^2(u_H) \ll \min\left(1, \frac{H}{q}\right).
\]

Now, let us consider the case \( w_H = \text{sgn}_H \) and recall (see [Da, Ch.25]) that for every \( \alpha \in \mathbb{R} \setminus \mathbb{Z} \) we can write

\[
|\tilde{\text{sgn}}_H(\alpha)| = 2 \left| \sum_{h \leq H} \sin(2\pi h\alpha) \right| \ll |\cot(\pi \alpha)| \sin^2(\pi Hao) + |\sin(2\pi Hao)| \ll \frac{\sin^2(\pi Hao)}{|\sin(\pi \alpha)|} \ll \frac{H_q^2}{|\alpha|^2},
\]

while \( \tilde{\text{sgn}}_H(\alpha) = 0 \) when \( \alpha \in \mathbb{Z} \). Thus, as above one gets (compare [C-S])

\[
\mathcal{L}_q^2(\tilde{\text{sgn}}_H) \ll \frac{1}{q^2} \sum_{0 < |j| \leq \frac{H_q}{q}} \frac{|j|H_q^2}{q} + \sum_{\frac{H_q}{q} < |j| \leq \frac{H}{2}} \frac{H_q}{q^2} \ll \min\left(1, \frac{H}{q}\right). \quad \square
\]
Remark. Similarly to the unit step function $u$, it is not difficult to see that any piecewise-constant weight $w$ is arithmetic. Moreover, for $u$, it is plain that for the normalized correlation of $	ext{sgn}$ we get

$$L_q^2\left(\frac{\hat{\text{sgn}}_H}{H}\right) = \frac{1}{H^2}L_q^2(|\hat{\text{sgn}}_H|^2) \ll \min\left(1, \frac{H}{q}\right).$$

We close this section with an example of an absolutely bounded weight that is not arithmetic. Let us take $w_H(n) = e(2\pi n\alpha)u_H(n)$ for a fixed $\alpha \in [0, 1)$ to be chosen later. Then, its correlation is

$$\mathcal{C}_{w_H}(a) = \sum_{0 < m \leq H \atop |a-m| < H} e(\alpha m) e(-m\alpha) = e(\alpha a) \sum_{\max(0, a) < m \leq \min(H, H+a)} 1 = e(\alpha a) \max(H - |a|, 0),$$

that satisfies the formula

$$\sum_{a \equiv 0 (\ell)} \mathcal{C}_{w_H}(a) = \sum_{0 \leq |b| \leq |H/\ell|} (H - \ell|b|) e(\ell b \alpha) = \ell \sum_{0 \leq |b| \leq |H/\ell|} (|H/\ell| - |b|) e(\ell b \alpha) + O(H) = \ell \left| \sum_{h \leq |H/H|} e(h \ell \alpha) \right|^2 + O(H).$$

In particular, by taking $a = m/\ell$ with $1 \leq m < \ell$ one has

$$\sum_{a \equiv 0 (\ell)} \mathcal{C}_{w_H}(a) = \ell \left| \sum_{h \leq |H/\ell|} e(h \ell) \right|^2 + O(H) = \frac{H^2\ell}{\ell} + O(H),$$

so that

$$\sum_{a \equiv 0 (\ell)} \mathcal{C}_{w_H}(a) = \mathcal{C}_{\hat{w_H}}(0) = |\hat{w_H}(0)|^2 = |\sum_{h \leq H} e(h \ell)|^2,$$

since (see the proof of 1) in Proposition 2 above

$$\sum_{a} \mathcal{C}_{w_H}(a) = \mathcal{C}_{\hat{w_H}}(0) = |\hat{w_H}(0)|^2 = \left| \sum_{h \leq H} e(h \ell) \right|^2,$$

then for $a = m/\ell$ we get

$$\sum_{a} \mathcal{C}_{w_H}(a) = \left| \sum_{h \leq H} e(\ell mh) \right|^2 = \left| \sum_{h \leq \ell(H/\ell)} e(\ell mh) \right|^2 \ll \ell^2.$$ 

Hence, the formula (A) cannot hold for any choice of $\ell = o(H)$.

3. Lemmata for Theorems 1 and 3.

Next Lemmata 1-3 are applied within the proof of Theorem 1, while Lemma 4 is of use for Theorem 3.

Lemma 1. Let $\kappa : x \in (N, 2N] \rightarrow \kappa(x) \in [0, +\infty)$ be strictly increasing and such that $\kappa(2N) \ll Q \ll N$, where $Q$ may depend on $N, H$. For every essentially bounded $g : N \rightarrow C$ and every good weight $w$ one has

$$\sum_{x \sim N} \sum_{q \leq \kappa(x)} |g(q)\chi_q(x, w_H)|^2 \ll (N + Q^2)H.$$ 

Proof. By applying Proposition 1 of §2 we write

$$\sum_{x \sim N} \left| \sum_{q \leq \kappa(x)} g(q)\chi_q(x, w) \right|^2 = \sum_{x \sim N} \sum_{q \leq \kappa(x)} |g(q)\chi_q(x, w)|^2.$$
and the Lemma is proved.

**Remark.** We explicitly note that under the same hypothesis for $g$ and $w$ through a similar proof one gets

$$\sum_{x \sim N} \left| \sum_{q \leq Q(x, w)} g(q) \chi_q(x, w) \right|^2 \ll (N + Q^2) H \sum_{1 < d \leq Q} \frac{1}{d} \ll (N + Q^2) H$$

and the Lemma is proved.

**Lemma 2.** Let $A, B, Q, H, N$ be positive real numbers such that, as $N \to \infty$, $H = o(N)$, $H \to \infty$, and $Q \ll A < B \ll Q \ll N$ with $Q = \infty(N/H)$.

For every essentially bounded $g : \mathbb{N} \to \mathbb{C}$ and every absolutely bounded weight $w$ with $\overline{w}(0) = 0$, one has

$$\sum_{x \sim N} \left| \sum_{A < q \leq B} g(q) \chi_q(x, w_H) \right|^2 \ll \sum_{x \sim N} \left| \sum_{q < m \leq \frac{N}{Q}} \sum_{q} g(q) w_H(mq - x) \right|^2 + (H^4 + H^2 Q^2) N^{-1}.$$
Proof. First, notice that \( \hat{w}_H(0) = 0 \) implies
\[
\chi_q(x, w_H) = W_H(x; q) = \sum_{n \equiv 0 (\text{mod } q)} w_H(n - x).
\]
Then, let us apply Dirichlet’s hyperbola method to get
\[
\sum_{A < q \leq B} g(q)\chi_q(x, w_H) = \sum_{A < q \leq B} g(q) \sum_{n \equiv 0 (\text{mod } q)} w_H(n - x) = \sum_{\frac{x-H}{m} \leq m < \frac{x+H}{m}} \sum_{A < q \leq B} g(q)w(mq - x) =
\]
\[
= \sum_{\frac{x-H}{m} \leq m < \frac{x+H}{m}} \sum_{\frac{e-H}{m} \leq q \leq \frac{e+H}{m}} g(q)w(mq - x) + O(\varepsilon \sum_{\frac{x-H}{m} \leq m < \frac{x+H}{m}} I_H(x; m) + N^e \sum_{\frac{x-H}{m} \leq m < \frac{x+H}{m}} I_H(x; m)) =
\]
\[
= \sum_{\frac{x-H}{m} \leq m < \frac{x+H}{m}} \sum_{q \equiv n \text{ (mod } q)} g(q)w(mq - x) + O(\varepsilon \sum_{\frac{x-H}{m} \leq m < \frac{x+H}{m}} I_H(x; m) + N^e \sum_{\frac{x-H}{m} \leq m < \frac{x+H}{m}} I_H(x; m)),
\]
where \( Q = \infty(N/H) \) implies that
\[
I_H(x; m) \overset{\text{def}}{=} \sum_{a \equiv -x (\text{mod } m)} 1_{H}(a) = \sum_{\frac{x-H}{m} \leq q \leq \frac{x+H}{m}} 1 \leq \frac{2H}{m} + 1 \ll \frac{H}{m}.
\]
Thus, the Lemma is proved once we show that the two \( m \)-sums in the \( O(\varepsilon) \)-terms give a mean-square contribution
\[
\ll (H^4 + H^2Q^2)N^{-1}.
\]
To this end, we note that
\[
\# \left[ \frac{x-H}{A}, \frac{x+H}{A} \right] \cap N \ll \frac{H}{Q} + 1
\]
and apply the Cauchy inequality to get
\[
\sum_{x \sim N} \left| \sum_{\frac{x-H}{m} \leq m \leq \frac{x+H}{m}} I_H(m) \right|^2 \ll \left( \frac{H}{Q} + 1 \right) \sum_{x \sim N} \sum_{\frac{x-H}{m} \leq m \leq \frac{x+H}{m}} |I_H(m)|^2 \ll
\]
\[
\ll H^2Q^2 \left( \frac{H}{Q} + 1 \right)^2 \sum_{x \sim N} \frac{1}{(x-H)^2} \ll H^2Q^2 \left( \frac{H^2}{Q^2} + 1 \right) \frac{H^4}{N} \ll \frac{H^4}{N} + \frac{H^2Q^2}{N}.
\]
Analogously we proceed for the other \( m \)-sum in the \( O(\varepsilon) \)-term. The Lemma is completely proved. 

Lemma 3. Let \( A, B, Q, H, N \) be as in Lemma 2. For every essentially bounded function \( g : \mathbb{N} \to \mathbb{C} \), we get
\[
\sum_{x \sim N} \left| \sum_{\frac{e}{q} \leq m \leq \frac{e+H}{q}} \sum_{q} g(q) \text{sgn}_H(mq - x) \right|^2 \ll \frac{N}{Q} \sum_{m \approx \frac{H}{q}} m \sum_{n \approx \frac{H}{q}} \left| \sum_{q} g(q) \text{sgn}_H(mq - x) \right|^2 + \frac{N^3}{Q^2} + H^3.
\]
Proof. By the Cauchy inequality we get
\[
\sum_{x \sim N} \left| \sum_{\frac{e}{q} \leq m \leq \frac{e+H}{q}} \sum_{q} g(q) \text{sgn}_H(mq - x) \right|^2 =
\]
\[
= \sum_{\frac{e}{m_1, m_2 \leq \frac{e}{H}} m_1 A \leq x < m_1 B} \sum_{q_1} g(q_1) \text{sgn}_H(m_1q_1 - x) \sum_{q_2} g(q_2) \text{sgn}_H(m_2q_2 - x) \ll
\]
\[
\ll \sum_{\frac{e}{m_1, m_2 \leq \frac{e}{H}}} \left( \sum_{x \sim N} \left| \sum_{q_1} g(q_1) \text{sgn}_H(m_1q_1 - x) \right|^2 \right)^{1/2} \sum_{x \sim N} \left| \sum_{q_2} g(q_2) \text{sgn}_H(m_2q_2 - x) \right|^2 \ll
\]

10
\[ N \sum_{m \ll \frac{Q}{x}} \sum_{q \sim x} \sum_{q} g(q) \text{sgn}(mq - x)^2. \]

Thus, by writing \( x = mn + r \) with \( 0 \leq r \leq m - 1 \) (compare [C6], Lemma 2.4), we see that
\[ \sum_{x \sim N} \left| \sum_{q} g(q) \text{sgn}(H/m)(q - [x/m]) \right|^2 + m[H/m]^2, \]
where \( m[H/m]^2 \ll H^2/m \) gives clearly a contribution \( \ll H^3 \). The Lemma is completely proved.

**Lemma 4.** Let \( g : \mathbb{N} \to \mathbb{R} \) be a real and essentially bounded function such that \( \text{supp } g \subseteq [1, Q] \) with \( Q = Q(N, H) \ll N \) and let \( f = g \ast 1 \).

1) For every real and absolutely bounded weight \( w \) one has
\[ J_{w, f}(N, H) - \sum_{0 \leq |a| \leq 2H} c_{w, H}(a) c_f(a) + N \overline{w}_H(0)^2 \left( \sum_{q \leq Q} \frac{g(q)}{q} \right)^2 \ll H^3 + QH^2. \]

2) For every integer \( a \neq 0 \) one has
\[ c_f(a) = \sum_{\ell|d} \sum_{(d, q)=1} g(\ell d) \frac{g(\ell q)}{q} \left( \left\lfloor \frac{2N}{\ell d} \right\rfloor - \left\lceil \frac{N}{\ell d} \right\rceil \right) + R_f(a) \]
with
\[ R_f(a) \overset{\text{def}}{=} \sum_{\ell|a} \sum_{(d, q)=1} g(\ell d) \frac{g(\ell q)}{q} \sum_{j(q)} e_q(-ja/\ell) \sum_{m \sim \frac{N}{\ell d}} e_q(jdm), \]
where in the dashed sum with \( j(q) \) we take all the non zero residue classes \( j \) mod \( q \).

Furthermore, if \( K : [-2H, 2H] \to \mathbb{C} \) is an even function, then
\[ \sum_{a \neq 0} K(a) R_f(a) = \sum_{\ell \leq 2H} \sum_{(d, q)=1} g(\ell d) \frac{g(\ell q)}{q} \sum_{j(q)} \sum_{m \sim \frac{N}{\ell d}} \cos \frac{2\pi jdm}{q} a \sum_{a \neq 0} K(\ell a) e_q(ja). \]

3) For every real and absolutely bounded weight \( w \) one has
\[ \sum_{a \neq 0} c_{w, H}(a) R_f(a) \ll NH + QH + QH^2. \]

**Proof.** 1) is a consequence of Lemma 1 in [C-L], that precisely yields the formula
\[ J_{w, f}(N, H) = \sum_{0 \leq |a| \leq 2H} c_{w, H}(a) c_f(a) - 2 \sum_n f(n) \sum_{x \sim N} w_H(n-x)M_f(x, w_H) + \sum_{x \sim N} M_f(x, w_H)^2 + O(H^3 \|f\|_\infty^2), \]
where

\[ \|f\|_\infty \equiv \max_{N-H \leq n \leq 2N+H} |f(n)| \]

and, since \( f \) is a sieve function, then we can take the \( x \)-independent mean-value

\[ M_f(x, w_H) = \bar{w}_H(0) \sum_{q \leq Q} \frac{g(q)}{q}. \]

Then, it suffices to observe that \( M_f(x, w_H) \ll H \) and

\[
\sum_{x \sim N} \sum_{n} f(n) w_H(n - x) = \sum_{x \sim N} \sum_{n} g(q) \sum_{x-H \leq n \leq x+H} w(n - x) = \sum_{q \leq Q} g(q) \sum_{n \sim \frac{N}{x}} \sum_{n \leq n \leq n+H} w(n - x) =
\]

\[
= \sum_{q \leq Q} g(q) \sum_{n \sim \frac{N}{x}} \sum_{n-H \leq n \leq x+H} w(n - x) + O_x \left( N^\varepsilon H^2 \right) = N \bar{w}_H(0) \sum_{q \leq Q} \frac{g(q)}{q} + O_x \left( N^\varepsilon \left( H^2 + QH \right) \right).
\]

While 2) is a straightforward adaptation of Lemma 2.3 in [C1], in order to prove 3) we closely follow the proof of Theorem 1.1 in [C1].

First, since \( \mathcal{E}_{w_H}(a) \) is an even function, in 2) we can take \( K(a) = \mathcal{E}_{w_H}(a) \) \( \forall a \in [-2H, 2H] \) and write

\[
\sum_{a \neq 0} \mathcal{E}_{w_H}(a) R_f(a) = \sum_{\ell \leq 2H} \sum_{(d, q) = 1} g(\ell d) \frac{g(\ell q)}{q} \sum_{j(q) \sim \frac{N}{\ell}} \frac{2\pi j dm}{q} \sum_{a \sim \frac{N}{\ell}} \mathcal{E}_{w_H}(a) e_q(j a) + O_x(N^{1+\varepsilon} H),
\]

where we have completed the last sum with the term \( a = 0 \) at the cost \( O_x(N^{1+\varepsilon} H) \) yielded by

\[
\mathcal{E}_{w_H}(0) \sum_{\ell \leq 2H} \sum_{(d, q) = 1} g(\ell d) \frac{g(\ell q)}{q} \sum_{j(q) \sim \frac{N}{\ell}} \frac{2\pi j dm}{q} = \]

\[
= \mathcal{E}_{w_H}(0) \sum_{\ell \leq 2H} \sum_{(d, q) = 1} g(\ell d) g(\ell q) \left( \sum_{m \sim \frac{N}{\ell}} 1 - \frac{1}{q} \sum_{m \sim \frac{N}{\ell}} 1 \right) \ll \]

\[
\ll H \sum_{\ell \leq 2H} \left( \sum_{d \leq \frac{Q}{\ell}} \sum_{m \sim \frac{N}{\ell}} d(m) + \frac{N}{\ell} \right) \ll HN \sum_{\ell \leq 2H} \frac{1}{\ell} \ll HN.
\]

Since \( \widehat{\mathcal{E}_{w_H}}(\alpha) \geq 0 \) \( \forall \alpha \in \mathbb{R} \), then from the orthogonality of the additive characters it follows that

\[
\sum_{a} \mathcal{E}_{w_H}(a) e(a \beta) = \sum_{h \equiv 0(\ell)} \mathcal{E}_{w_H}(h) e(h \beta) = \frac{1}{\ell} \sum_{j \leq \ell} \sum_{h} \mathcal{E}_{w_H}(h) e(h(j + \beta)) \geq 0 \quad \forall \beta \in \mathbb{R}.
\]

In particular, one has \( \sum_{a} \mathcal{E}_{w_H}(a) e_q(j a) \geq 0 \) \( \forall j (\mod q) \), that together with the usual bound [Da, Ch.25] implies

\[
\sum_{a \neq 0} \mathcal{E}_{w_H}(a) R_f(a) \ll \sum_{\ell \leq 2H} \sum_{(d, q) = 1} \sum_{d \leq \frac{Q}{\ell}} \sum_{j(q) \sim \frac{N}{\ell}} \sum_{a \sim \frac{N}{\ell}} \frac{1}{q} \mathcal{E}_{w_H}(a) e_q(j a) + NH \ll \]

\[
\ll \sum_{\ell \leq 2H} \sum_{q \leq \frac{Q}{\ell}} \sum_{d \leq \frac{Q}{2q}} \frac{1}{q} \sum_{j(q) \sim \frac{N}{\ell}} \sum_{a \sim \frac{N}{\ell}} \mathcal{E}_{w_H}(a) e_q(j a) + NH.
\]
Now, let us set $j' = jd$, so that $j = 7j'$ with $7d \equiv 1 \pmod{q}$, and write (recalling correlations are even)

\[
\frac{1}{q} \sum_{j(q)}' \frac{1}{q} \sum_{a} e_{\chi_H}(\alpha t)e_{\chi}(ja) = \frac{1}{q} \sum_{j'(q)}' \frac{1}{q} \sum_{a} e_{\chi_H}(\alpha t)e_{\chi}(j'a7) \ll \\
\ll \sum_{j' \leq q/2} \frac{1}{q} \sum_{a} e_{\chi_H}(\alpha t)e_{\chi}(j'a7) \ll \sum_{j \leq q/2} \frac{1}{q} \sum_{a} e_{\chi_H}(\alpha t)e_{\chi}(jan)
\]

as the variable $7 = n$ ranges over a complete set of reduced residue classes, then

\[
\sum_{d \leq 2q} \frac{1}{q} \sum_{j(q)}' \frac{1}{d} \sum_{a} e_{\chi_H}(\alpha t)e_{\chi}(ja) \ll \\
\ll \sum_{n \leq q} \sum_{j \leq q/2} \frac{1}{q} \sum_{a} e_{\chi_H}(\alpha t)e_{\chi}(jan) \ll \sum_{n \leq q} \sum_{j \leq q/2} \frac{1}{q} \sum_{a} e_{\chi_H}(\alpha t)e_{\chi}(jan).
\]

Again by orthogonality of characters we get

\[
\sum_{n \leq q} \sum_{a} e_{\chi_H}(\alpha t)e_{\chi}(jan) = qe_{\chi_H}(0) + q \sum_{\alpha \neq 0, j\equiv 0(q)} e_{\chi_H}(\alpha t),
\]

whence

\[
\sum_{n \leq q} \sum_{j \leq q/2} \frac{1}{q} \sum_{a} e_{\chi_H}(\alpha t)e_{\chi}(jan) \ll qH + \sum_{j \leq q/2} \frac{q}{j} \sum_{\alpha \neq 0, j\equiv 0(q)} e_{\chi_H}(\alpha t) \ll qH + H^2,
\]

where we have seen that

\[
\sum_{j \leq q/2} \frac{q}{j} \sum_{\alpha \neq 0, j\equiv 0(q)} e_{\chi_H}(\alpha t) \ll H \sum_{\ell | q, t \leq q/2} \frac{1}{\ell} \sum_{a | t, a \equiv 0(t)} e_{\chi_H}(\alpha t) \ll H \sum_{\ell | q, t \leq q/2} \frac{H}{\ell} + 1 \ll H \sum_{\ell | q, t \leq q/2} \frac{H}{\ell} \ll H^2.
\]

Finally, we have that

\[
\sum_{a \neq 0} e_{\chi_H}(a)R_f(a) \ll NH + \sum_{\ell \leq 2H} \sum_{q \leq \frac{Q}{2}} (qH + H^2) \ll \\
\ll NH + Q^2H \sum_{\ell \leq 2H} \ell^2 + QH^2 \sum_{\ell \leq 2H} \ell^{-1} \ll NH + Q^2H + QH^2. \quad \square
\]
4. Proofs of the Theorems 1, 3 and 4.

Proof of Theorem 1. First, let us note that
\[ \theta \in \left( \frac{3\vartheta - 1}{(1 - G)\vartheta + G + 1}, \vartheta \right) = \left( 1 + \frac{\vartheta - 1}{\lambda_0}, \vartheta \right) \] if and only if \( \theta = 1 + \frac{\vartheta - 1}{\lambda} \) for some \( \lambda \in (\lambda_0, 1] \)
where \( \lambda_0 \triangleq 1 - \vartheta + \frac{3\vartheta - 1}{G + 2} > \max(1 - \vartheta, 1/2) \). Indeed, \( \lambda_0 > 1/2 \iff G + 2\theta(1 - G) > 0. \)

Then, recalling that \( \chi_q(x, \text{sgn}_H) = \sum_{m \equiv q \pmod{H}} \text{sgn}(qm - x) \), let us apply a dyadic argument to write
\[ J_{\text{sgn}, g^*1}(N, H) = \sum_{x \sim N} \left| \sum_{q \sim Q} g(q) \chi_q(x, \text{sgn}_H) \right|^2 \ll \max_{Q \leq N} \sum_{x \sim N} \left| \sum_{q \sim Q} g(q) \chi_q(x, \text{sgn}_H) \right|^2. \]

From Lemma 1 (see its remark) one has
\[ \sum_{x \sim N} \left| \sum_{q \sim Q} g(q) \chi_q(x, \text{sgn}_H) \right|^2 \ll NH + Q^2H, \quad \forall Q \ll N. \]

By taking \( A = Q, B = 2Q \) and \( Q \triangleq N^\lambda = \infty(N/H) \) for \( \lambda \in (\lambda_0, 1) \) in Lemmata 2 and 3
\[ \sum_{x \sim N} \left| \sum_{q \sim Q} g(q) \chi_q(x, \text{sgn}_H) \right|^2 \ll \sum_{x \sim N} \left| \sum_{m \geq \frac{N}{Q}} \sum_{q \sim Q} g(q) \text{sgn}_H(mq - x) \right|^2 + (H^4 + H^2Q^2)N^{-1} \ll \]
\[ \ll \frac{N}{Q} \sum_{m \geq \frac{N}{Q}} mJ_{\text{sgn}, g}(\frac{N}{m}, \frac{H}{m}) + \frac{N^3}{Q^2} + \frac{H^2Q^2}{N} + H^3. \]

Note that \( QHN^{-1} = Q^\theta \) for \( \theta = 1 + (\vartheta - 1)\lambda^{-1} \). Therefore, \( |H/m| \asymp (N/m)^\theta \quad \forall m \asymp N/Q \) and by hypothesis
\[ J_{\text{sgn}, g}(\frac{N}{m}, \frac{H}{m}) \ll \frac{NH^{2-G}}{m^{3-G}} \quad \forall m \asymp \frac{N}{Q} \]

Hence, we obtain (here, \( Q = N^\lambda \) as above is implicit)
\[ \sum_{x \sim N} \left| \sum_{q \sim Q} g(q) \chi_q(x, \text{sgn}_H) \right|^2 \ll NH^2 \left( \frac{N}{QH} \right)^G + \frac{N^3}{Q^2} + \frac{H^2Q^2}{N} + H^3. \]

In particular, for \( Q_0 \triangleq N^{\lambda_0} = \infty(N/H) \) it turns out that
\[ Q_0^2H = NH^2 \left( \frac{N}{Q_0H} \right)^G \quad \text{and} \quad \frac{H^2Q_0^2}{N} \ll H^3. \]

Consequently
\[ J_{\text{sgn}, g^*1}(N, H) \ll NH^2 \left( \frac{N}{Q_0H} \right)^G + \frac{N^3}{Q_0^2} + H^3 = NH^2 \left( \frac{G}{G + 2} \right)^{1/(G + 2)} + H^{-2(G + 1)/(G + 2)} \Rightarrow H^3 \ll \]
\[ \ll NH^2 \cdot H^{- \frac{G(3\vartheta - 1)}{(G + 2)^2}} + H^3, \]
\[ J_{\text{sgn}, g*1}(N, H) \ll NH^2 \cdot H^{-G'} \]

when
\[ 0 < G' < \min \left( \frac{3 - \theta^{-1}}{1 + 2G^{-1}}, \theta^{-1} - 1 \right). \]

**Proof of Theorem 3.** Recalling that \( \mathcal{C}_f(0) \mathcal{C}_{wH}(0) \ll NH \), by Lemma 4 we easily infer
\[ J_{w, f}(N, H) = \Delta + O_\varepsilon(N^\varepsilon(NH + Q^2H + QH^2 + H^3)), \]
where
\[ \Delta \overset{\text{def}}{=} \sum_{a \neq 0} \mathcal{C}_{wH}(a) \sum_{\ell | a} \sum_{(d, q) = 1} g(\ell \delta) \frac{g(\ell q)}{q} \left( \left\lfloor \frac{2N}{\ell d} \right\rfloor - \left\lfloor \frac{N}{\ell d} \right\rfloor \right) - N \bar{w}_{H}(0)^2 \left( \sum_{q \leq Q} \frac{g(q)}{q} \right)^2. \]

Clearly, we may confine to prove
\[ \Delta \ll NH + QH^2. \]

To this end, observe that
\[ \sum_{a \neq 0} \mathcal{C}_{wH}(a) \sum_{\ell | a} \sum_{(d, q) = 1} g(\ell \delta) \frac{g(\ell q)}{q} \left( \left\lfloor \frac{2N}{\ell d} \right\rfloor - \left\lfloor \frac{N}{\ell d} \right\rfloor \right) = \]
\[ = N \sum_{\ell | a, \ell \neq 0} \sum_{(d, q) = 1} g(\ell \delta) \frac{g(\ell q)}{q} - \sum_{a \neq 0} \mathcal{C}_{wH}(a) \sum_{\ell | a} \sum_{(d, q) = 1} g(\ell \delta) \frac{g(\ell q)}{q} \left( \left\lfloor \frac{2N}{\ell d} \right\rfloor - \left\lfloor \frac{N}{\ell d} \right\rfloor \right) = \]
\[ = N \sum_{\ell = 1}^{\infty} \sum_{a \neq 0} \mathcal{C}_{wH}(a) \sum_{(d, q) = 1} g(\ell \delta) \frac{g(\ell q)}{q} + O_\varepsilon(N^\varepsilon QH^2), \]
because
\[ \sum_{a \neq 0} \mathcal{C}_{wH}(a) \sum_{\ell | a} \sum_{(d, q) = 1} g(\ell \delta) \frac{g(\ell q)}{q} \left( \left\lfloor \frac{2N}{\ell d} \right\rfloor - \left\lfloor \frac{N}{\ell d} \right\rfloor \right) \ll N^\varepsilon H \sum_{\ell | a, \ell \neq 0} \sum_{(d, q) \neq 0} \sum_{\ell \neq 0} \frac{1}{q} \ll N^\varepsilon QH^2. \]

Now we apply the hypothesis that \( w \) is arithmetic, i.e.
\[ \sum_{a \neq 0} \mathcal{C}_{wH}(a) = \sum_{a \equiv 0 \pmod{\ell}} \mathcal{C}_{wH}(a) + O(H) = \frac{\bar{w}_{H}(0)^2}{\ell} + O(H), \]
to get the conclusion
\[ N \sum_{\ell = 1}^{\infty} \sum_{a \neq 0} \mathcal{C}_{wH}(a) \sum_{(d, q) = 1} g(\ell \delta) \frac{g(\ell q)}{q} = N \bar{w}_{H}(0)^2 \sum_{\ell = 1}^{\infty} \sum_{a \equiv 0 \pmod{\ell}} \frac{g(\ell \delta) g(\ell q)}{q} + O_\varepsilon (N^{1+\varepsilon} H) = \]
\[ = N \bar{w}_{H}(0)^2 \sum_{\ell = 1}^{\infty} \sum_{d', q'} \frac{g(d') g(q')}{q'} + O_\varepsilon (N^{1+\varepsilon} H) = N \bar{w}_{H}(0)^2 \left( \sum_{n} \frac{g(n)}{n} \right)^2 + O_\varepsilon (N^{1+\varepsilon} H). \]

**Proof of Theorem 4.** Without further references, in what follows we will appeal to the formula
\[ p_f \left( \log(x + m) \right) = p_f \left( \log x + \log(1 + m/x) \right) = p_f \left( \log x \right) + O(m(\log x)^{\varepsilon - 1}/x) \text{ for } m = o(x) \text{ as } x \to \infty. \]
Recall that \( M_f(x, H) \overset{\text{def}}{=} Hp_f(\log x) \) and note that for any \( H > h \) we can write

\[
\sum_{x < n \leq x + H} f(n) - M_f(x, H) = \sum_{x < n \leq x + h} f(n) - hp_f(\log x) + \sum_{x + h < n \leq x + H} f(n) - (H - h)p_f(\log(x + h)) + Hp_f(\log(x + h)) - p_f(\log x) + h(p_f(\log x) - p_f(\log(x + h))).
\]

Therefore, since \( h = O(N) \) from hypothesis \( H = o(N) \), then by assuming also that \( H < 2h \) one has

\[
J_f(N, H) \ll J_f(N, h) + \sum_{N+h < x \leq 2N+h} \left| \sum_{x < n \leq x+H-h} f(n) - (H-h)p_f(\log x) \right|^2 + N^{-1}H^2h^2L^{2c-2} \ll
\]

\[
\ll J_f(N, h) + J_f(N, H - h) + h(H-h)^2\left(\|f\|_{\infty}^2 + L^{2c}\right) + N^{-1}H^2h^2L^{2c-2} \ll
\]

\[
\ll J_f(N, h) + J_f(N, H - h) + hH^2\left(\|f\|_{\infty}^2 + L^{2c}\right),
\]

that gives the desired conclusion for \( h < H < 2h \).

Now let us assume that \( H \geq 2h \) and write

\[
\sum_{x < n \leq x + H} f(n) = \sum_{j \leq H/h} \sum_{x + h_{j-1} < n \leq x + h_j} f(n) + \sum_{x + [H/h]h < n \leq x + H} f(n)
\]

and

\[
Hp_f(\log x) = \left[\frac{H}{h}\right]h_p(\log x) + \left\{\frac{H}{h}\right\}hp_f(\log x) = \sum_{j \leq H/h} hp_f(\log x) + \left\{\frac{H}{h}\right\}hp_f(\log x),
\]

where we set \( h_j \overset{\text{def}}{=} jh \). Consequently,

\[
\sum_{x < n \leq x + H} f(n) - Hp_f(\log x) =
\]

\[
= \sum_{j \leq H/h} \left( \sum_{x + h_{j-1} < n \leq x + h_j} f(n) - hp_f(\log(x + h_{j-1})) \right) + \sum_{x + [H/h]h < n \leq x + H} f(n) - \left\{\frac{H}{h}\right\}hp_f(\log(x + [H/h]h)) + O(x^{-1}H^2L^{c-1}).
\]

Hence, by the Cauchy inequality

\[
J_f(N, H) \ll \frac{H}{h} \sum_{j \leq H/h} \left( J_f(N, h) + H\left(\|f\|_{\infty}^2 + L^{2c}\right)h^2 \right) + J_f\left(N, h\left\{\frac{H}{h}\right\}\right) + Hh^2\left(\|f\|_{\infty}^2 + L^{2c}\right) + N^{-1}H^4L^{2c-2} \ll \left(\frac{H}{h}\right)^2 J_f(N, h) + J_f\left(N, h\left\{\frac{H}{h}\right\}\right) + H^3\left(\|f\|_{\infty}^2 + L^{2c}\right),
\]

that gives the desired conclusion also for any \( H \geq 2h \). □
5. Further comments and properties.

The study of the Selberg integral $J_k(N, H)$ associated to the divisor function $d_k$ has an enormous attraction, because of the deep implication with the $2k$–th moments of the Riemann zeta function $\zeta$ (see [C4]). In the present section we add some further properties for these particular functions, that complement the results of [C-L]. Let us recall that in [C-L] we find the following explicit expression of the short interval mean value in the Selberg integral $J_3(N, H)$ of $d_3$, i.e.

$$M_3(x, H) \overset{\text{def}}{=} H \left( \frac{\log^2 x}{2} + 3\gamma \log x + 3\gamma^2 + 3\gamma_1 \right),$$

where $\gamma$ is the Euler-Mascheroni constant and $\gamma_1 \overset{\text{def}}{=} \lim_{m \to \infty} \left( \frac{\log^2 m}{2} - \sum_{j=1}^{m} \frac{\log j}{j} \right)$ is a Stieltjes constant.

On the other side, Proposition 3 in [C-L] suggests that the expected mean value in the modified Selberg integral $\tilde{J}_3(N, H)$, where $d_3$ is weighted with $C_H$, is given by

$$\tilde{M}_3(x, H) \overset{\text{def}}{=} H \left( \sum_{q \leq Q} \frac{d(q)}{q} + \sum_{d_1 < x/Q} \frac{1}{d_1} \sum_{d_2 \leq Q/d_1} \frac{1}{d_2} + \left( \sum_{d < x/Q} \frac{1}{d} \right)^2 \right) \text{ with } Q = \frac{x}{[(N - H)^1/3]}.$$

Indeed, more in general, the aforementioned Proposition 3 implies the following inequality, for every divisor function $d_k$ with $k > 2$ and every good weight $w$:

$$\sum_{x \sim N} \left| \sum_n d_k(n)w_H(n - x) - \tilde{w}_H(0) \sum_{q \leq Q_k} \frac{g_k(q)}{q} \right|^2 \ll N^{2-2/k} H + N^{-1} H^k,$$

where $Q_k \overset{\text{def}}{=} x/[(N - H)^1/k]$ and

$$g_k(q) \overset{\text{def}}{=} d_{k-1}(q) + \sum_{j \leq k-1} \sum_{n_1 \cdots n_{k-1} \leq q} \sum_{n_1, \ldots, n_{k-1} < x/Q_k} 1$$

is the short Eratosthenes transform of $d_k$ coming out from the $k$–folding method (see [C-L], Proposition 2).

Since Proposition 1 in [C-L] yields

$$M_3(x, H) - \tilde{M}_3(x, H) \ll HN^{-1/3}$$

uniformly for $x \sim N$,

then this justifies the presence of the analytic mean value $M_3(x, H)$ in place of the arithmetic $\tilde{M}_3(x, H)$, within the definition of $J_3(N, H)$.

Here we take the opportunity to prove next proposition, that yields for any $k > 2$ the proximity in the mean square between the arithmetic form of the mean value

$$M_k(x, w_H) \overset{\text{def}}{=} \tilde{w}_H(0) \sum_{q \leq Q_k} \frac{g_k(q)}{q}$$

and its analytic counterpart

$$\tilde{w}_H(0)p_{k-1}(\log x) = \tilde{w}_H(0) \text{Res}_{s=1} \zeta^k(s)x^{s-1},$$

where $p_{k-1}$ is the logarithmic polynomial of $d_k$.

To this end, first let us recall that Ivić’s bounds [Iv] yield an exponent gain for the Selberg integral of $d_k$,

$$J_k(N, H) \overset{\text{def}}{=} \sum_{x \sim N} \left| \sum_{x < n \leq x + H} d_k(n) - Hp_{k-1}(\log x) \right|^2,$$
whenever $H \asymp N^\vartheta$ for $\vartheta > \theta_k \triangleq 2\sigma_k - 1$, where $\sigma_k$ is the Carleson abscissa.

Then we prove the following property.

**Proposition 4.** For every integer $k > 2$ and every integer $H \asymp N^\vartheta$ with $\max(\theta_k, 1 - 2/k) < \vartheta < 1$ there exists $G = G(k, \vartheta) > 0$ such that

$$\sum_{x \sim N} \left| \sum_{q \leq Q_k} \frac{g_k(q)}{q} - \sigma_{k-1}(\log x) \right|^2 \ll N^{1-G}.$$ 

**Proof.** Since $\hat{\mu}_H(0) = H$, from the above-cited inequality yielded by Proposition 3 in [C-L] we write

$$H^2 \sum_{x \sim N} \left| \sum_{q \leq Q_k} \frac{g_k(q)}{q} - \sigma_{k-1}(\log x) \right|^2 \ll J_k(N, H) + \sum_{x \sim N} \sum_n \frac{d_k(n)u_H(n-x) - \hat{u}_H(0)}{q \leq Q_k} \sum_{q \leq Q_k} \frac{g_k(q)}{q} \ll$$

$$\ll NH^2 (N^{-G_1} + N^{1-2/k-\vartheta} + N^{-2(1-\vartheta)})$$

where the exponent gain $G_1$ for the Selberg integral of $d_k$,

$$J_k(N, H) \overset{def}{=} \sum_{x \sim N} \left| \sum_{x < n \leq x + H} d_k(n) - Hp_{k-1}(\log x) \right|^2,$$

follows from the aforementioned Ivić’s bounds [Iv] for $\vartheta > \theta_k = 2\sigma_k - 1$.

Hence, the conclusion follows by taking $G < \min(G_1, 2/k + \vartheta - 1, 2(1-\vartheta))$.

**Remark.** Implying in particular that $f = g * 1$ is a sieve function, the hypotheses of Theorem 3 legitimate the assumption that the analytic form

$$M_f(x, w) \overset{def}{=} \hat{w}_H(0) \operatorname{Res}_{s=1} F(s)x^{s-1} \quad \text{with} \quad F(s) \overset{def}{=} \sum_{n=1}^\infty f(n)n^{-s}$$

is sufficiently close to its arithmetic form, so that we can take

$$M_f(x, w_H) = \hat{w}_H(0) \sum_{q \leq Q} \frac{g(q)}{q},$$

that does not depend both on $x$ and on Dirichlet series.

From this point of view, from what we have seen before it turns out that the divisor functions are very close to be sieve functions. Roughly speaking, they are well approximated by some arithmetic functions, which are Dirichlet convolutions of the functions $g_k$ as the $k$-folding method reveals.

**Acknowledgements.** We wish to thank Professor Martin N. Huxley for two reasons. First, for his leading contributes in Analytic Number Theory, a source of inspiration for many colleagues and us. Then, for his patience in (electronic) correspondence with the first author, who took the idea, for the title of the present paper, from the subject of one of his email messages.

**References**

[C0] Coppola, G. - *On the modified Selberg integral* - http://arxiv.org/abs/1006.1229v1

[C1] Coppola, G. - *On the Correlations, Selberg integral and symmetry of sieve functions in short intervals* - http://arxiv.org/abs/0709.3648v3 - extended version on J. Combinatorics and Number Theory 2.2 (2010), Article 1.
[C2] Coppola, G. - *On the Correlations, Selberg integral and symmetry of sieve functions in short intervals, II* - Int. J. Pure Appl. Math. **58.3**(2010), 281–298. - available online

[C3] Coppola, G. - *On the Correlations, Selberg integral and symmetry of sieve functions in short intervals, III* - http://arxiv.org/abs/1003.0302v1

[C4] Coppola, G. - *On the Selberg integral of the $k$-divisor function and the $2k$-th moment of the Riemann zeta-function* - Publ. Inst. Math. (Beograd) (N.S.) **88**(102) (2010), 99–110. MR **2011m:11173** - available online

[C5] Coppola, G. - *On the symmetry of arithmetical functions in almost all short intervals, V* - (electronic) http://arxiv.org/abs/0901.4738v2

[C6] Coppola, G. - *On the symmetry of square-free supported arithmetical functions in short intervals* - JIPAM. J. Inequal. Pure Appl. Math. **5**(2004), no. **2**, Article 33, 11 pp. (electronic). MR **2005f:11210** - available online

[C-L] Coppola, G. and Laporta, M. - *Generations of correlation averages* - (submitted) - previous version available online at http://arxiv.org/abs/1205.1706v3

[C-S] Coppola, G. and Salerno, S. - *On the symmetry of the divisor function in almost all short intervals* - Acta Arith. **113** (2004), no. **2**, 189–201. MR **2005a:11144**

[Da] Davenport, H. - *Multiplicative Number Theory* - Third Edition, GTM 74, Springer, New York, 2000. MR **2001f:11001**

[D-F-I] Duke, W., Friedlander, J. and Iwaniec, H. - *Bilinear forms with Kloosterman fractions* - Invent. Math. **128** (1997), no. **1**, 23–43. MR **97m:11109**

[H] Huxley, M.N. - *On the difference between consecutive primes* - Invent. Math. **15** (1972), no. **2**, 164–170. MR **45#1856**

[Iv] Ivić, A. - *On the mean square of the divisor function in short intervals* - J. Théor. Nombres Bordeaux **21** (2009), no. **2**, 251–261. MR **2010k:11151**

[K-P] Kaczorowski, J. and Perelli, A. - *On the distribution of primes in short intervals* - J. Math. Soc. Japan **45** (1993), no. **3**, 447–458. MR **94e:11100**

[Se] Selberg, A. - *On the normal density of primes in small intervals, and the difference between consecutive primes* - Arch. Math. Naturvid. **47** (1943), no. **6**, 87–105. MR **7,48e**

[V] Vinogradov, I.M. - *The Method of Trigonometrical Sums in the Theory of Numbers* - Interscience Publishers LTD, London, 1954. MR **15,941b**

---

Giovanni Coppola
Università degli Studi di Salerno  
Home address: Via Partenio 12 - 83100, Avellino(AV), ITALY  
e-page: www.giovannicoppola.name  
e-mail: gcoppola@diima.unisa.it

Maurizio Laporta
Università degli Studi di Napoli  
Dipartimento di Matematica e Appl. Compl. Monte S. Angelo  
Via Cinthia - 80126, Napoli, ITALY  
e-mail: mlaporta@unina.it