Ground state energy of the polarized diluted gas of interacting spin 1/2 fermions

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Abstract

The effective field theory approach simplifies the perturbative computation of the ground state energy of the diluted gas of repulsive fermions allowing in the case of the unpolarized system to easily rederive the classic results up to the \((k_Fa_0)^2\) order (where \(k_F\) is the system’s Fermi momentum and \(a_0\) the s-wave scattering length) and (with more labour) to extend it up to the order \((k_Fa_0)^4\). The analogous expansion of the ground state energy of the polarized gas of spin 1/2 fermions is known only up to the \(k_Fa_0\) order (where \(k_F\) stands for \(k_F^\uparrow\) or \(k_F^\downarrow\)); the order \((k_Fa_0)^2\) contribution has been computed (analytically) only using a specific (hard-core type) interaction potential. Here we show that the effective field theory method also allows to easily obtain the order \((k_Fa_0)^2\) correction to the ground state of the polarized gas in a way applicable to all repulsive interactions.

Keywords: Diluted gas of interacting fermions, nonzero polarization, effective field theory, scattering length

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1 Introduction

Effective field theories are used in high energy physics since already half a century. They allow for example to quantitatively capture, by exploiting essentially only the information about exact and broken symmetries of the underlying theory, characteristic features of the low energy dynamics of light mesons in the regime in which quantum chromodynamics is genuinely nonperturbative and, on the other hand, routinely serve to parametrize potential effects of yet unknown new physics in processes involving well known particles. These applications rely on the separation of energy scales involved which makes reliable the expansion of the computed quantities in powers of their ratio and on the possibility of fixing values of the parameters, which cannot be obtained by matching onto the underlying more fundamental theory, by directly extracting them from low energy data.

Relatively more recent are applications of the effective field theory methods to non-relativistic many body problems [1]–[5]. A particularly instructive is the application of this technique [2] to the classic problem of computing the energy \( E_\Omega \) of the ground state of the system of \( N \) fermions (enclosed in the volume \( V \)) interacting through a two-body spin independent potential which may not be specified explicitly but is, instead, characterized by the (in principle infinite) set of the scattering lengths \( a_\ell \) and the effective radii \( r_\ell \) (\( \ell = 0,1,\ldots \)) parametrizing the expansion of the resulting partial amplitudes of elastic scattering of two fermions in powers of their relative momentum. Stated in this form the problem is ideally suited for handling it in the framework of an effective theory, because the information on the fundamental dynamics (the two-body potential) is traded from the beginning for the (infinite) set of low energy data. The crucial observation making possible the application of the effective theory technique to it is that if the underlying interaction potential is natural in the sense that the magnitudes of the scattering lengths \( a_\ell \) and the effective radii \( r_\ell \) it gives rise to are set by some common scale \( 1/\Lambda \) (this excludes from the considerations attractive potentials which can lead to bound states and formation of resonances), this scale is, if the gas of fermions is sufficiently diluted, well separated from its characteristic momentum scale set by the Fermi momentum (wave vector) \( k_F \) of the gas. The expansion of the ground state energy \( E_\Omega/N \) or \( E_\Omega/V \) in powers of \( k_F a_\ell \propto k_F/\Lambda \) and \( k_F r_\ell \propto k_F/\Lambda \) naturally provided by the effective theory methods is then reliable. In the textbook treatment of this problem [6] which summarizes results obtained in the past within the conventional approaches (mean field approach [7], ordinary perturbative expansion [8], Goldstone diagrams, [9] or Green’s functions methods [10]–[12]) computing already the term of order \( O(k_F^2 a_0^2) \) in the expansion of the energy \( E_\Omega \) of the gas of unpolarized fermions requires a sophisticated procedure consisting of summation of an infinite subset of Goldstone diagrams and re-expanding the resulting expressions. In contrast, in the effective field theory determination of the term of order \( O(k_F^2 a_0^2) \) reduces to a computation of a single (divergent) diagram and removing its divergence with the help the standard renormalization procedure. Owing to its simplicity this method allowed to obtain [13] recently the complete fourth \( O(k_F^4/\Lambda^4) \) and \( O((k_F^4/\Lambda^4) \ln(k_F/\Lambda)) \) order terms which complement the third, \( O(k_F^3/\Lambda^3) \), order result obtained earlier [14] by more conventional (semi-analytic) methods.

The interest in properties of a diluted gas of fermions stemmed originally from the
study of nuclear matter, although this model obviously cannot capture all realistic features of systems of nucleons interacting through (mostly) attractive potentials. More recently models of diluted gases (of fermions and bosons) find their more natural application as continuum models of interacting cold atomic gases bound in optical or harmonic traps, complementing more traditional ways of investigating properties of such systems based on lattice models known generally as Hubbard models (paradigmatic for condensed matter physics) which, despite of more than 60 years of development, still leave many problems without clear answers. One of them is the mechanism of emergence of ferromagnetism in systems of mutually repelling atoms which has been observed experimentally [15]. Theoretical investigations of many questions of interest related to this result, like the problem of itinerant ferromagnetism on lattice models of mutually repelling spin 1/2 fermions as well as the possibility of spontaneous separation of magnetic and nonmagnetic phases begun already in the sixtieth of the XX century, but a successful explanation of these phenomena with the help of the so-called dynamical mean field theory (DMFT) approach [16] has been achieved only some twenty years ago.

Clearly, any study of the emergence of magnetism in systems of \( N \) mutually repelling spin 1/2 fermions within the continuum model must start with the computation of the ground state energy of such a system for \( N_\uparrow \neq N_\downarrow \), where \( N_\uparrow \) and \( N_\downarrow \) (\( N_\uparrow + N_\downarrow = N \)) are the (conserved by the assumed interaction) numbers of spin up and spin down fermions. Using the ordinary Rayleigh-Schrödinger perturbative expansion it is easy to obtain the relevant expression in the first order in the \( s \)-wave scattering length \( a_0 \) (this result can be given also a mathematically more rigorous foundation [17], [18]). The second order term has been computed analytically using the traditional approach only within the hard spheres model interaction [19]. This approach does not allow, however, to easily recognize the universality of this result (in the class of natural spin-independent repulsive potentials). Apart from these result, there are also Monte Carlo simulations [20], [21] which – while providing quite reliable numerical estimates of the exact (nonperturbative) ground state energy – must necessarily employ concrete model potentials and therefore suffer from the lack of universality.

In this paper we compute the second order correction to the ground state energy of the polarized gas of spin 1/2 fermions with the help of the effective field theory method. It makes it clear from the outset that this correction can only depend on the \( s \)-wave scattering length \( a_0 \) and, therefore, that the result of [19] is universal. Nevertheless, it is interesting to recover it using this new method as this may pave the way to extend the computation to yet higher orders. From the conceptual point of view, this task reduces to only a minor modification of the computation performed in [2] for \( N_\uparrow = N_\downarrow \), but it is more involved from technical point of view. While the order \( a_0^2 \) correction to the ground state energy is in [2] given in a completely analytic form, and, as the result of [19] shows, the same is possible also in the case of \( N_\uparrow \neq N_\downarrow \), we do not attempt to perform the resulting integrals analytically and content ourselves with providing the formulae which involve integrals which can be easily evaluated numerically with the help of a three-line Mathematica code.

Since our computation parallels that of [2], we take the opportunity to give here more details (and to make some general comments on the renormalization procedure) which, although probably obvious to experts, may be of some pedagogical value. There-
fore in Section 2 we state first the problem in the formalism of second quantization and compute by the standard perturbative method the first (order $a_0$) correction to the ground state energy of the system of interacting fermions. The purpose of this section is also to fix the notation and possible ways of parametrizing the result. In Section 3 we recall the effective field theory approach and give some details of the procedure allowing to relate the couplings of the effective Lagrangian (Hamiltonian) to the “low energy data”, that is, to the scattering lengths $a_\ell$ and effective radii $r_\ell$. Instead of using as in [2] the dimensional reduction as the regularization method, we cut off divergent integrals over the wave vectors at the scale $\Lambda$. While being technically more troublesome (but only in higher orders) this regularization prescription seems more in line with the main idea of the effective theory method and moreover it allows to partly control the correctness of the calculation, which is not possible with the dimensional regularization which automatically sets to zero all power-like divergences. In Section 4 we compute the second-order correction to the ground state energy using the effective theory method, demonstrate explicitly cancellation of the dependence on the cutoff $\Lambda$ and give the result in the form dependent on a single function of the ratio of Fermi momenta of spin up and spin down fermions which is evaluated numerically. We also compare our perturbative result with the existing nonperturbative estimates based on Monte Carlo simulations. We summarize the results in Section 5 and speculate about perspectives of generalizing them.

2  Zeroth and first order results

We would like to compute energy $E_{\Omega}$ of the ground state of $N$ identical (nonrelativistic) fermions of mass $m_f$ and spin $s = 1/2$ ($g = 2s + 1 = 2$ spin states per fermion) interacting through a spin independent two-body potential which, instead of being specified explicitly, is characterized in terms of the scattering lengths and effective ranges of the elastic scattering amplitude it gives rise to. We will not assume equal densities $\rho_\uparrow = N_\uparrow/V$ of fermions having spin up and $\rho_\downarrow = N_\downarrow/V$ ($N_\uparrow + N_\downarrow = N$) of fermions having spin down.

We first assume that the interaction potential is given explicitly and the Hamiltonian of the system (enclosed in the box of volume $V$ and with periodic boundary conditions imposed) in the standard formalism of second quantization (see e.g. [6, 22]) has the usual form $H = H_0 + V_{\text{int}}$ with

$$H_0 = \sum_{p,\sigma=\uparrow,\downarrow} \frac{\hbar^2 p^2}{2m_f} a_{p,\sigma}^\dagger a_{p,\sigma},$$

$$V_{\text{int}} = \frac{1}{2V} \sum_{q} \tilde{V}_{\text{pot}}(q) \sum_{p_1, p_2, \sigma_1, \sigma_2=\uparrow,\downarrow} a_{p_1+q,\sigma_1}^\dagger a_{p_2,\sigma_2}^\dagger a_{p_2,\sigma_2} a_{p_1,\sigma_1},$$

where $\tilde{V}_{\text{pot}}(q)$ is the Fourier transform of the assumed spin-independent two-body interaction potential $V_{\text{pot}}(r_2 - r_1)$. The Hamiltonian $H$ is ordered normally with respect to the lowest vector $|\text{void}\rangle$ of the Fock space (in which act the creation $a_{p,\sigma}^\dagger$ and annihilation $a_{p,\sigma}$ operators); this ensures equivalence of the second quantization formalism
with the one based on the \( N \)-body Schrödinger wave equation. The fixed numbers \( N_\uparrow \) and \( N_\downarrow \) determine the Fermi momenta \( p_{F\uparrow} \) and \( p_{F\downarrow} \) through the relation

\[
N_{\uparrow/\downarrow} = V \int \frac{d^3k}{(2\pi)^3} \theta(p_{F\uparrow/\downarrow} - |k|) = \frac{V}{6\pi^2} p_{F\uparrow/\downarrow}^3.
\] (3)

We seek the ground state energy \( E_\Omega \) of \( H \) in the form of the series \( E_\Omega = E_{\Omega_0} + E_{\Omega_1} + \ldots \). The first term \( E_{\Omega_0} = E_{\Omega_0} \) is the energy of the ground state \( |\Omega_0\rangle \) of the system of \( N \) noninteracting fermions

\[
E_{\Omega_0} = V \sum_{s=\uparrow,\downarrow} \int \frac{d^3k}{(2\pi)^3} \frac{h^2 p_s^2}{2m_f} \theta(p_{Fs} - |k|) = \frac{V}{6\pi^2} \frac{3}{5} \frac{h^2}{2m_f} (p_{F\uparrow}^5 + p_{F\downarrow}^5)
\]

\[
= \frac{3}{5} \left( N_\uparrow \frac{h^2 p_{F\uparrow}^2}{2m_f} + N_\downarrow \frac{h^2 p_{F\downarrow}^2}{2m_f} \right) = \frac{3}{5} \frac{h^2}{2m_f} (6\pi^2)^{2/3} \left( \rho_{\uparrow}^{5/3} + \rho_{\downarrow}^{5/3} \right),
\]

where \( \rho_{\uparrow/\downarrow} = N_{\uparrow/\downarrow}/V \).

The first order correction \( E_{\Omega_1} \) can be computed in different ways. Application of the standard Rayleigh-Schrödinger perturbative expansion gives it in the form of the matrix element

\[
E_{\Omega_1}^{(1)} = \frac{1}{2V} \sum_q \sum_{p_1,p_2} \tilde{\mathcal{V}}_{\text{pot}}(q) \langle \Omega_0 | a_{p_1+q,\sigma_1}^\dagger a_{p_2-q,\sigma_2}^\dagger a_{p_2,\sigma_2} a_{p_1,\sigma_1} | \Omega_0 \rangle,
\]

which can be straightforwardly evaluated by applying the Wick theorem to the string of two creation and two annihilation operators

\[
a_{p_1+q,\sigma_1}^\dagger a_{p_2-q,\sigma_2}^\dagger a_{p_2,\sigma_2} a_{p_1,\sigma_1} = [a_{p_1+q,\sigma_1}^\dagger a_{p_2-q,\sigma_2}] [a_{p_2-q,\sigma_2} a_{p_1,\sigma_1}] - [a_{p_1+q,\sigma_1}^\dagger a_{p_2,\sigma_2}] [a_{p_2,\sigma_2} a_{p_1,\sigma_1}] + [a_{p_1+q,\sigma_1}^\dagger a_{p_2-q,\sigma_2}] [a_{p_2-q,\sigma_2} a_{p_1,\sigma_1}] + \ldots,
\]

where the ellipsis stands for operator terms ordered normally with respect to the state \( |\Omega_0\rangle \) (characterized by \( p_{F\uparrow} \) and \( p_{F\downarrow} \)) of \( N \) noninteracting fermions, and the pairings \( [a_{p_1+q,\sigma_1}^\dagger a_{p_1,\sigma_1}] \) etc. are given by (pairings of \( aa \) and \( aa^\dagger \) vanish)

\[
[a_{p,\sigma}^\dagger a_{p',\sigma'}^\dagger] = \langle \Omega_0 | a_{p,\sigma}^\dagger a_{p',\sigma'} | \Omega_0 \rangle = \delta_{\sigma\sigma'} \delta_{p,p'} \theta(p_{F\sigma} - |p|) \theta(p_{F\sigma} - |p'|).
\]

The contributions to \( E_{\Omega_1}^{(1)} \) of the terms in which all operators have spin up or all operators have spin down are proportional to

\[
\sum_{p_1,p_2} \left( \tilde{\mathcal{V}}_{\text{pot}}(0) - \tilde{\mathcal{V}}_{\text{pot}}(p_2 - p_1) \right) \theta(p_{F\uparrow} - |p_1|) \theta(p_{F\downarrow} - |p_2|),
\]

and vanish in the approximation \( p_2 - p_1 \sim 0 \) which is valid if the gas is diluted so that both \( p_{F\uparrow} \) and \( p_{F\downarrow} \) are small. The two remaining terms give equal contributions and one obtains

\[
E_{\Omega_1}^{(1)} = \frac{1}{V} \tilde{\mathcal{V}}_{\text{pot}}(0) \sum_{p_1} \theta(p_{F\uparrow} - |p_1|) \sum_{p_2} \theta(p_{F\downarrow} - |p_1|) = V \tilde{\mathcal{V}}_{\text{pot}}(0) \frac{p_{F\uparrow}^3 + p_{F\downarrow}^3}{6\pi^2}.
\] (4)

\footnote{The standard replacement \( \sum_k \to V \int d^3k/(2\pi)^3 \) is assumed.}
Even if fermions interact via a hardcore-type potential $V_{pot}(|x_1 - x_2|)$ the Fourier transform of which may be ill defined, the correct result, as argued e.g. in [6], is obtained by trading $\tilde{V}_{pot}(0)$ for the $s$-wave scattering length $a_0$ (which is defined as the $k \to 0$ limit of the always well defined scattering amplitude) according to the formula

$$\tilde{V}_{pot}(0) = \int d^3x V_{pot}(x) = \lim_{k \to 0} \left(-\frac{2\pi \hbar^2}{m_{red}} f_{\text{Born}}(k, \theta)\right) = \frac{2\pi \hbar^2}{m_{red}} a_0,$$

in which $m_{red} = m_f/2$, because the scattering fermions are identical. In this way one arrives at

$$\frac{E^{(1)}_{\Omega}}{V} = \frac{4\pi \hbar^2}{m_f} a_0 \frac{p_p^2 + p_{\bar{p}}^2}{6\pi^2} = \frac{\hbar^2}{2m_f} 8\pi a_0 \rho_\uparrow \rho_\downarrow,$$

the result obtained in [17], [18] by much more sophisticated methods as a mathematically rigorous estimate valid also for arbitrary strengths of the potential provided the densities $\rho_\uparrow/\rho_\downarrow$ are small.

3 Effective Field theory approach

Another method of computing the ground state energy, proposed in [2], is based on the effective theory. The most general Lagrangian density consistent with the Galileo, parity and time-reversal symmetries the dynamics of the spinor field is assumed to be subjected to, has the form (spinor indices of field in the brackets are implicitly contracted)

$$L = \psi^\dagger \left(i\hbar \partial_t + \frac{\hbar^2 \nabla^2}{2m_f}\right) \psi - \frac{C_0}{2} : (\psi^\dagger \psi)^2 : + \frac{C_2}{16} : [ (\psi^\dagger \psi^\dagger) (\psi \nabla^2 \psi) + \text{H.c.}] : + \ldots$$

It consists of infinitely many local operator structures of increasing dimensions. Their coefficients $C_0, C_2$, etc. have to be determined by comparing the scattering amplitude of two fermions computed using this effective theory with the one known from the potential scattering (i.e. the one parametrized by the scattering lengths etc.), or - see below - by matching onto the “fundamental” theory [1].

The first term of the effective interaction $V_{\text{int}}$ (obtained as minus the interaction term of the above Lagrangian integrated over the space) of the effective Hamiltonian

and, in the finite volume $V$, setting

$$\psi_\sigma(x) = \frac{1}{\sqrt{V}} \sum_p a_{p,\sigma} e^{ip\cdot x},$$

differs from $V_{\text{int}}$ of [1] only by the replacement of $\tilde{V}_{pot}(q)$ by $C_0$ and obviously reproduces the first order correction $E^{(1)}_{\Omega}$ to the ground state energy of the system of $N$ fermions, provided $C_0$ is (in the lowest order) identified with $(4\pi \hbar^2/m_f)a_0$.

\footnote{The effective theory free Hamiltonian $H_0$ is constructed in the standard way as the Legendre transform of $L_0$ consisting of the terms of $L$ bilinear in $\psi^\dagger$ and $\psi$. This obviously gives the same $H_0$ as in [1].}
In higher orders the local (i.e. singular) nature of the interaction terms of the Lagrangian density results in short-distance (i.e. ultraviolet) divergences (evidently absent in the original theory defined by the Hamiltonian) in various quantities computed with the help of it. These should be regularized and removed by applying the standard renormalization procedure. Since we are interested here only in a directly measurable quantity (the ground state energy $E_\Omega$), renormalization can be straightforwardly carried out by simply computing (using the same regularization) an appropriate set of observables other than $E_\Omega$ itself (in this case the scattering lengths $a_\ell$ and effective radii $r_\ell$ in the expansion of the elastic scattering amplitude of two fermions) and expressing the computed quantity, $E_\Omega$, in terms of them. Any consistent regularization can be used for this purpose because when the computed physical quantities (such as $E_\Omega$) are expressed in terms of other observables ($a_\ell$’s and $r_\ell$’s) they become independent of it (in the limit of removed regularization). The most popular in high energy physics computations (in which preservation of gauge invariance is the main concern) is the dimensional regularization and it is this one which was used in the seminal paper [2]. It however can hardly be given any physical interpretation. The regularization implemented by cutting off all integrals over the wave vectors $k$ at the scale $\Lambda$ is better in line with the entire effective theory approach which should be viewed in the wilsonian spirit as follows: Suppose physical observables are computed both in the effective theory with the cut-off $\Lambda$ imposed and directly in the “fundamental” theory. The (“bare”) coefficients $C_i$ of the Lagrangian could be then adjusted to reproduce the results of the “fundamental” finite theory - this is the mentioned matching of the effective theory onto the fundamental one. This would make these coefficients explicitly dependent on the cut-off scale $\Lambda$ and they, of course, would diverge in the (unphysical) limit $\Lambda \to \infty$. But obviously all physical quantities computed in terms of cutoff dependent bare coefficients would be finite and independent of the scale $\Lambda$, if the dependence of the bare coefficients $C_i$ on this scale, which can be encoded in the wilsonian renormalization group equations were taken into account. Of course, once observables like $E_\Omega$ and $a_\ell$, $r_\ell$ are computed in terms of $C_i(\Lambda)$ (obtained as sketched above), the latter can be eliminated altogether by expressing

\[ Z_\psi = 1 + \sum_{k=1}^{\infty} \delta Z^{(k)}_\psi, \]

in case of requiring finiteness of Green’s functions; the coefficients $C_i$ and the Heisenberg picture operators $\psi$ are usually called “bare” in this context) and imposing appropriate “renormalization conditions”. This is not necessary to the order we will be working.

The counterterm technique through the splitting $C_i(\Lambda) = C_i^{(R)}(\mu) + \delta C_i(\Lambda, \mu)$ allows to introduce an auxiliary scale $\mu$ (introduced automatically also in the dimensional regularization) on which computed physical quantities do not depend if the explicit dependence of renormalized couplings $C_i^{(R)}(\mu)$ on this scale (encoded in the standard renormalization group equations satisfied by the renormalized coefficients $C_i^{(R)}(\mu)$) is taken into account.

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say $E_0$ through $a_\ell$, $r_\ell$’s. The dependence of $E_0$ on $\Lambda$ (originating from cutting off integrals over the wave vectors) remaining after this operation is already harmless, and can be removed by formally taking the limit $\Lambda \to \infty$ which simplifies the expressions and is also justified from the practical point of view, if the (arbitrarily chosen) scale $\Lambda$ of matching of the fundamental and effective theories is high compared to the typical physical scale involved in the computed quantity of interest (in the case of $E_0$ this physical scale is $k_F$). In passing, it is interesting to note that the nonrelativistic theories: the fundamental one (1), and the effective one (6) provide explicit illustration of the role of the cutoff scale $\Lambda$ in effective theories on which an attempt, proposed in [27], to resolve the so-called hierarchy problem of high energy physics was based.

We now proceed to the determination of $C_0$ and other coefficients in terms of the scattering lengths $a_\ell$ and effective radii $r_\ell$. Given (in principle) the spin independent two-body interaction potential $V_{\text{pot}}(r_1 - r_2)$, the amplitude of elastic scattering of two nonrelativistic (not necessarily identical at this point) particles of spins $s_1$ and $s_2$ and masses $m_1$ and $m_2$ is obtained by rewriting the two-body Schrödinger equation in terms of the relative $r = r_1 - r_2$ and the center of mass $R = (m_1 r_1 + m_2 r_2)/(m_1 + m_2)$ variables, separating it by setting $\Psi(R, r) = \psi(r) \exp(i P \cdot R / \hbar)$ and solving the resulting equation (which corresponds to the motion of a fictitious particle of mass $m_{\text{rel}}$ with energy $E'$ in the potential $V_{\text{pot}}$)

$$\left( -\frac{\hbar^2}{2m_{\text{rel}}} \nabla_r^2 + V_{\text{pot}}(r) \right) \psi(r) = E' \Psi(r),$$

in which $m_{\text{rel}} = m_1 m_2 / (m_1 + m_2)$ and $E' = E - h^2 P^2 / 2M$, in the center of mass system (CMS) that is, setting $P = 0$. In this frame the momenta of the two particles are $h k$ and $-h k$, respectively, and the energy $E' = E$ of the fictitious particle is the total energy of the two colliding particles:

$$E = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{\hbar^2}{2} \left( m_1 \frac{k^2}{m_1^2} + m_2 \frac{k^2}{m_2^2} \right) = \frac{\hbar^2 k^2}{2m_{\text{rel}}}.$$ 

It follows that the parameter $k$ playing the role of the wave vector of the fictitious particle should be identified with the wave vector of one of the scattering particles (the true relative momentum of the two scattering particles is $2k$). As long as particles are distinguishable (and the interaction potential is spin independent), individual spins of the two particles and their total spin $S$ are preserved and do not play any role: the scattering amplitude $f(k, \theta)$ is defined in terms of the asymptotic form ($\hbar \equiv |k|$)

$$\psi_k^{(+)}(r) = e^{i k r} + \frac{f(k, \theta)}{r} e^{i k r},$$

of the (in-state) solution of the Schrödinger equation (7) with $E' = \hbar^2 k^2 / 2m_{\text{rel}}$ and the differential scattering cross section $d\sigma/d\Omega$ is simply given by $|f(k, \theta)|^2$.

If the two particles are identical and both have spin $s$ (integer or half-integer), the complete wave function of the system written as a product $\Psi(r_1, r_2)\chi(\sigma_1, \sigma_2)$, where

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6The proliferation of possible operator structures of higher dimensions has the effect that also amplitudes of three-body scattering must be used as the input to determine all independent coefficients of the effective Lagrangian (6).
\(\sigma_{1,2} = -s, \ldots, +s\) must be either totally symmetric or totally antisymmetric depending on whether \(s\) is integer or half-integer. If the interaction is spin-independent, one can choose \(\chi(\sigma_1, \sigma_2)\) to be symmetric or antisymmetric. \(\psi(\mathbf{r})\) must, therefore, also be constructed as symmetric \(\psi(-\mathbf{r}) = \psi(\mathbf{r})\) or antisymmetric and the simple rule \([28]\) is then that if the total spin \(S\) of the two particles is even, \(\psi(\mathbf{r})\) must be symmetric, and antisymmetric, when the total spin \(S\) is odd. Thus, if \(s = \frac{1}{2}\) but \(S = 0\), the scattering amplitude of the two indistinguishable spin 1/2 fermions is given by \(f(k, \theta) + f(k, \pi - \theta)\), while if \(S = 1\), it is \(f(k, \theta) - f(k, \pi - \theta)\).

In the formalism of second quantization one computes the \(S\)-matrix element\(^7\) \([29]\)

\[
S_{\beta\alpha} = \langle \beta_0 | T \exp \left( -\frac{i}{\hbar} \int_{-\infty}^{\infty} dt \, V_{\text{int}}^f(t) \right) | \alpha_0 \rangle ,
\]

in which as \(|\alpha_0\rangle\) and \(|\beta_0\rangle\) are the states of two fermions:

\[
|\alpha_0\rangle = a_{\sigma_2}^\dagger(k_2) a_{\sigma_1}^\dagger(k_1)|\text{void}\rangle ,
|\beta_0\rangle = a_{\sigma_2}^\dagger(k'_2) a_{\sigma_1}^\dagger(k'_1)|\text{void}\rangle ,
\]

and \(V_{\text{int}}^f(t)\) is the interaction picture counterpart of the interaction term of the Hamiltonian \([1]\) written in the position representation (in the normalization in the continuum, as requires the application to the scattering theory)

\[
V_{\text{int}} = \frac{1}{2} \int d^3x \int d^3y \, \psi^\dagger_\alpha(x) \psi^\dagger_\beta(y) V_{\text{pot}}(x - y) \psi_\beta(y) \psi_\alpha(x).
\]

that is, expressed in terms of the interaction picture field operators

\[
\psi^f_\sigma(t, \mathbf{x}) = e^{iH_0 t/\hbar} \psi_\sigma(x) e^{-iH_0 t/\hbar} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x} + i\mathbf{k}\cdot\mathbf{a}_\sigma(\mathbf{k})},
\]

in which \(\omega_\mathbf{k} \equiv \hbar k^2/2m_f\) (the notation \(k^0 = \omega_\mathbf{k}\) will also be employed below).

Evaluated using the standard methods (Wick theorem, etc.) the \(S\)-matrix element takes, in the case of spin independent interaction, the general form

\[
S_{\beta\alpha} = \langle \beta_0 | \alpha_0 \rangle - \frac{i}{\hbar} \langle \beta_0 | (2\pi)^4 \delta(\omega_{k'_1} + \omega_{k'_2} - \omega_{k_1} - \omega_{k_2}) \delta^{(3)}(k'_1 + k'_2 - k_1 - k_2) \nonumber \\
\times \left[ A((k_1 - k'_1)^2) \delta_{\sigma_1\sigma_1} \delta_{\sigma_2\sigma_2} - A((k_1 - k'_2)^2) \delta_{\sigma_1\sigma_2} \delta_{\sigma_2\sigma_1} \right] .
\]

\(^7\)The formula \([3]\) is based on the very strong assumption that there is a well defined, strict one-to-one correspondence between the \(\text{in}\) and \(\text{out}\) eigenstates of the complete Hamiltonian and the eigenstates of \(H_0\). In most relativistic field theories, except for the simplest ones, like \(\varphi^4\) models and quantum electrodynamics (which probably explains its ubiquity in elementary textbooks without any qualifications) these assumptions cannot be satisfied. Even in cases the required one-to-one correspondence can be enforced, the use of the formula \([3]\) necessarily entails working in the so-called On-Shell renormalization scheme, that is with physical mass \(m_{\text{phys}}\) in the free propagator and with physically normalized Heisenberg picture field operators. In more complicated cases \(S\)-matrix elements can only be obtained by applying the LSZ prescription to appropriate Green’s functions. In the nonrelativistic theory considered here the assumptions are satisfied precisely owing to the form of the interaction (its normal ordering with respect to the vector \(|\text{void}\rangle\)) and to the absence of antiparticles which together preclude any genuine self-interaction of the particles.
Since \((k_1 - k'_1)^2 = 4|k|^2 \sin^2(\theta/2)\), \((k_1 - k'_2)^2 = 4|k|^2 \sin((\pi - \theta)/2)\), the two terms in the bracket must be proportional to the nonrelativistic amplitudes \(f(k, \theta)\) and \(f(k, \pi - \theta)\). It is then easy to check that the proper identification reads

$$\frac{-m_f}{4\pi\hbar^2} A(k, \theta) = f(k, \theta). \quad (10)$$

In general, the scattering amplitude \(f(k, \theta)\) can be expressed through the partial wave shifts as

$$f(k, \theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l + 1) e^{i\delta_l(k)} \sin \delta_l(k) P_l(\cos \theta) = \sum_{l=0}^{\infty} (2l + 1) \frac{1}{k \cot \delta_l - ik} P_l(\cos \theta),$$

and the scattering lengths \(a_l\) and the effective ranges \(r_l\) are defined as the coefficients in the small \(k\) expansions

$$k \cot \delta_0 = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \ldots, \quad k \cot \delta_1 = -\frac{3}{k^2 a_1^2} + \ldots.$$

Therefore, the small \(k\) expansion of the scattering amplitude reads

$$f(k, \theta) = -a_0 \left[ 1 - ia_0 k + \left( \frac{1}{2} a_0 r_0 - a_0^2 \right) k^2 + \ldots \right] - a_1^2 k^2 \cos \theta + \ldots. \quad (11)$$

Evaluating the formula (8) for

$$|\alpha_0\rangle = a_1^\dagger(k_2) a_1^\dagger(k_1)|\text{void}\rangle,$$

$$|\beta_0\rangle = a_1^\dagger(k_2) a_1^\dagger(k_1)|\text{void}\rangle,$$

(in order to extract directly one part of the scattering amplitude) one easily gets

$$S_{\beta\alpha} = \langle \beta_0 | \alpha_0 \rangle = \frac{i}{\hbar} C_0 \times (2\pi)^4 \delta^{(4)}(k'_1 + k'_2 - k_1 - k_2),$$

(the wave vectors \(k_i\)'s are treated as four-vectors: \(k^0 = \omega_k\)). Thus \(A = C_0\) simply. Using the rule (10) one obtains that in the first order \(C_0 = (4\pi \hbar^2 / m_f) a_0\).

The next order contribution to \(S_{\beta\alpha}\) is (we omit the superscript \(I\) - interaction picture - on operators; \(dt \times dx \equiv dt \, dx\))

$$\frac{1}{2!} \left( \frac{C_0}{2\hbar} \right)^2 \int d^4 x \int d^4 y \langle a_1^\dagger a_2^\dagger T \left[ (\psi_1^\dagger \psi_0)^2(x) : (\psi_1^\dagger \psi_\sigma)^2(y) : \right] a_2^\dagger a_1^\dagger \rangle.$$
Figure 1: Two one-loop (second order) diagrams potentially contributing to the elastic scattering amplitude of two fermions: diagram a) is nonzero, diagram b) vanishes. The “sausage”-type diagrams c) arising in higher orders. Time flows from the left to the right.

and final states). We will consider first the term which corresponds to the diagram of Fig. 1a. Its contribution is

\[
\left( \frac{C_0}{i\hbar} \right)^2 \int d^4x \int d^4y e^{ik_2'x} e^{ik_1'x} iG^{(0)}(x-y) iG^{(0)}(x-y) e^{-ik_2y} e^{-ik_1y},
\]

The propagator \((iG_{\alpha\beta}^{(0)}(x-y) = \delta_{\alpha\beta} iG^{(0)}(x-y))\) is given by

\[
iG^{(0)}_{\alpha\beta}(x-y) = \langle \text{void} | T[\bar{\psi}_\alpha(x) \psi_\beta^\dagger(y)] | \text{void} \rangle = \delta_{\alpha\beta} \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{i}{\omega - \omega_q + i0} e^{-i\omega(x^0-y^0)+i\mathbf{q}(x-y)}.
\]

Using this form of the propagator, the integrals over \(d^4x\) and \(d^4y\) can be taken and one gets \((2\pi)^4\delta^{(4)}(k_1' + k_2' - k_1 - k_2)\) times

\[
\left( \frac{C_0}{i\hbar} \right)^2 \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{i^2}{[\omega - \omega_q + i0][-\omega + k_1^0 + k_2^0 - \omega_{k_1+k_2} - q + i0]}.
\]

The integral over \(d\omega\) is nonzero because the two simple poles of the integrand are located one above and one below the real \(\omega\) axis. It should be remarked here that the right diagram shown in Fig. 1b gives zero because it leads to a similar integral in which, however, both poles are below the real \(\omega\) axis; the integral over \(d\omega\) then gives zero.\(^9\)

This simplification arises because of the absence of antiparticle in the nonrelativistic theory. Picking up the pole below the real axis to perform the integral over \(d\omega\) and then working out the denominator \((\omega_q = \hbar q^2/2m_f, k_1^0 = \hbar k_1^2/2m_f, \text{etc.})\) one obtains

\[
\left( \frac{C_0}{i\hbar} \right)^2 \int \frac{d^3q}{(2\pi)^3} \frac{-i m_f/\hbar}{\mathbf{q}^2 - \mathbf{q} \cdot (\mathbf{k}_1 + \mathbf{k}_2) + \mathbf{k}_1 \cdot \mathbf{k}_2 - i0}.
\]

Instead of evaluating this integral keeping the momenta \(\mathbf{k}_1\) and \(\mathbf{k}_2\) arbitrary, one can take advantage of the fact that the scattering amplitude is computed in the CMS in which \(\mathbf{k}_1 + \mathbf{k}_2 = 0\) and \(\mathbf{k}_1 \cdot \mathbf{k}_2 = -\mathbf{k}^2\). This reduces the integral to the form (7) in [2]:

\[
I_0 = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\mathbf{q}^2 - \mathbf{k}^2 - i0}.
\]

\(^9\)Other diagrams which one could draw having two incoming, two outgoing lines and two vertices are in fact absent because the interaction \(V_{\text{int}}\) is normally ordered with respect to the \(|\text{void}\rangle\) state.
It is not difficult to see that there is a whole class of diagrams shown in Fig. 1c representing the elastic scattering which is generated by the interaction term proportional to $C_0$. Their contribution to $S_{\beta\alpha}$ can immediately be written down:

\[
(2\pi)^4 \delta^{(4)}(k'_1 + k'_2 - k_1 - k_2) \left\{ \frac{C_0}{i\hbar} + \left( \frac{C_0}{i\hbar} \right)^2 \frac{m_f}{i\hbar} I_0 + \left( \frac{C_0}{i\hbar} \right)^3 \left( \frac{m_f}{i\hbar} I_0 \right)^2 + \ldots \right\}.
\]

This gives the scattering amplitude $A = i\hbar \{ \ldots \}$ and, according to the rule (10), the corresponding contribution to the scattering amplitude $f(k, \theta)$ is

\[
f(k, \theta) = -\frac{m_f}{4\pi\hbar^2} A = -\frac{m_f}{4\pi\hbar^2} C_0 \left\{ 1 + \left( \frac{C_0}{i\hbar} \right) \left( \frac{m_f}{i\hbar} I_0 \right) + \left( \frac{C_0}{i\hbar} \right)^2 \left( \frac{m_f}{i\hbar} I_0 \right)^2 + \ldots \right\}. \tag{13}
\]

This amplitude, supplemented with terms which come from tree level diagrams generated by the interactions proportional to the coefficients $C_2$ and $C'_2$, and with yet higher orders from loop diagrams involving all other vertices present in the effective Lagrangian should be matched onto the expansion (11).

The integral $I_0$ is divergent and requires regularization. Instead of using for this purpose the dimensional regularization as in [2], we regularize it by imposing the UV cut-off $\Lambda$ on $q = |q|$.

\[
I_0(k, \Lambda) = \frac{1}{2\pi^2} \int_0^{\Lambda} dq \frac{q^2}{q^2 - k^2 - i0} = \frac{1}{2\pi^2} \int_0^{\Lambda} dq \left[ \frac{1}{q - k - i0} + \frac{1}{q + k + i0} \right].
\]

Since the integral runs from zero, the pole is only in the first part of the integrand, and, upon using the Sochocki formula ($P$ stands for principal value), one obtains

\[
I_0(k, \Lambda) = \frac{i}{4\pi} k + \frac{1}{2\pi^2} \Lambda - \frac{1}{2\pi^2} \frac{k^2}{\Lambda} + \ldots \tag{14}
\]

Inserting this into the formula (13) matched onto the expansion (11) and solving for (real) $C_0$ setting $C_0 = (4\pi\hbar^2/m_f) a_0 (1 + \Delta/a_0)$ one obtains

\[
C_0 = \frac{4\pi\hbar^2}{m_f} a_0 \left( 1 + \frac{2}{\pi} a_0 \Lambda + \ldots \right). \tag{15}
\]

The procedure leading to $C_0$ is more complicated when the cut-off regularization is used and the resulting $C_0$ is given in the form of an infinite power series in $a_0 \Lambda$ (this is why the dimensional regularization is more convenient) but, as will be seen, allows to better control cancellation of divergences in physical quantities.

4 Corrections to the ground state energy

We now compute the correction $E^{(2)}_{\Omega}$ to the ground state energy diagrammatically, treating the terms of the Lagrangian (6) as the interaction vertices. The calculation closely parallels that of [2] except that we do not assume that the numbers $N_\uparrow$ and
$N_i$ of spin up and spin down fermions are equal. The basic formula employed for this purpose reads\(^{10}\)

$$
\lim_{T \to \infty} \exp(-i T (E_\Omega - E_{\Omega_0})/\hbar) = \lim_{T \to \infty} \langle \Omega_0 | T \exp \left( -\frac{i}{\hbar} \int_{-T/2}^{T/2} dt V_{\text{int}}(t) \right) | \Omega_0 \rangle. \tag{16}
$$

In other words, $-i T (E_\Omega - E_{\Omega_0})/\hbar$ is (in the limit $T \to \infty$, $V \to \infty$) given by $(2\pi)^4 \delta^{(4)}(0)$ times the sum of the momentum space connected vacuum diagrams (diagrams without external lines). The factor $(2\pi)^4 \delta^{(4)}(0)$ arising in evaluating diagrams in position space (expressing the overall four-momentum conservation) is interpreted as $VT$. It follows that $i\hbar$ times the expression arising from summing the momentum space connected vacuum diagrams is just $(E_\Omega - E_{\Omega_0})/V$.

As explained in \(^{2}\), to compute the order $(k_F a_0)^2$ correction to the ground state energy only the interaction term proportional to $C_0$ of \(^{6}\) is needed. It simplifies considerably if there are only two possible spin projections, because $\psi_\uparrow \psi_\uparrow \dagger = \psi_\alpha \psi_\alpha = 0$ (one can treat fields as anticommuting), and reads\(^{11}\)

$$
V_{\text{int}} = C_0 \int d^3x : (\psi_\uparrow \psi_\uparrow ) (\psi_\downarrow \psi_\downarrow ) : = C_0 \int d^3x : (\psi_\downarrow \psi_\downarrow ) (\psi_\uparrow \psi_\uparrow ) : . \tag{17}
$$

In evaluating the right hand side of the formula \(^{16}\) expanding the exponents and using the Wick theorem needed are the propagators (see e.g. \(^{6}\))

$$
i G_{\pm}^{(0)}(t', x', t, x) \equiv \langle \Omega_0 | T \psi_{\pm}^{\dagger}(t', x') \psi_{\pm}(t, x) | \Omega_0 \rangle
= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} e^{-i\omega(t'-t) + ik(x'-x)} i G_{\pm}(\omega, k),
$$

(all other Wick contractions vanish) in which

$$
i G_{\pm}(\omega, k) = i \left[ \frac{\theta(|k| - p_{F\pm})}{\omega - \omega_k + i0} + \frac{\theta(p_{F\pm} - |k|)}{\omega - \omega_k - i0} \right], \tag{18}
$$

where $\omega_k = \hbar k^2/2m_f$. Analytical expressions are ascribed to individual diagrams according to the standard rules of quantum field theory; to account for normal ordered form of the interaction one has only to add the rule \(^{6}\) that if a line originates from and ends up in one and the same vertex, the propagator \(^{18}\) corresponding to this line has to be multiplied by $e^{\eta \omega}$ with the limit $\eta \to 0^+$ taken at the end.

In the first order in $C_0$ there is only one connected vacuum graph shown in Figure \(^2\) Applying the Feynman rules one easily obtains

$$
TE_{\Omega}^{(1)} = C_0 VT i G_+(0) i G_-(0) = C_0 VT \frac{p_{F-}^3}{6\pi^2} \frac{p_{F+}^3}{6\pi^2},
$$

and recovers, after using (up to the first order in $a_0$) the result \(^{15}\), the formula \(^{5}\).

\(^{10}\)The symbol T of the chronological ordering should not be confused with $T$ denoting time.

\(^{11}\)In what follows we will denote by $+$ ($-$) quantities and operators pertaining to the spin projection of larger (smaller) density; thus we will use the Fermi momenta $p_{F\uparrow}$ and $p_{F\downarrow}$ ($p_{F\uparrow} \geq p_{F\downarrow}$) understanding that $p_{F\uparrow} = p_{F\dagger}$ when $p_{F\dagger} \geq p_{F\downarrow}$. 

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Figure 2: The effective theory connected vacuum diagram of order $C_0$ reproducing the first order correction $E^{(1)}_m$. Solid and dashed lines represent propagators of fermions with opposite spin projections.

$$k + q \quad p - q \quad p \quad k$$

Figure 3: The only nonvanishing three loop connected vacuum diagram contributing to the ground state energy of the diluted gas of spin $\frac{1}{2}$ fermions. The solid and dashed lines correspond to propagators of fermions having opposite spin projections. The two kinds of propagators differ by the values of the Fermi momenta: we assume that $p_{F+} \geq p_{F-}$.

Several connected vacuum diagrams of the next order can be drawn but, as explained in [2], nonzero is only the one shown in Figure 3. The contribution of this diagram is given by

$$\frac{E^{(2)}_Q}{V} = -\frac{i}{2!} \frac{C_0^2}{\hbar} \int \frac{d^4 q}{(2\pi)^4} \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} \bar{\mathcal{G}}_-(p) i\bar{\mathcal{G}}_-(p - q) i\mathcal{G}_+(k) i\mathcal{G}_+(k + q)$$

$$= -\frac{i}{2!} \frac{C_0^2}{\hbar} \int \frac{d^3 q}{(2\pi)^3} \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d q_0}{2\pi} \left[ \ldots \right],$$

where $[\ldots]$ stands for the product of two integrals over frequencies

$$\int_{-\infty}^{\infty} \frac{d k_0}{2\pi} \left[ \frac{\theta(|p| - p_{F+}) + \theta(p_{F+} - |p|)}{k_0 - \omega_k + i0} \right] \left[ \frac{\theta(|k + q| - p_{F+}) + \theta(p_{F+} - |k + q|)}{k_0 + q_0 - \omega_{k+q} + i0} \right] \left[ \frac{\theta(|p - q| - p_{F-}) + \theta(p_{F-} - |p - q|)}{p_0 - q_0 - \omega_{p-q} + i0} \right].$$

Each of these integrals gives only two terms (not four) because the two poles of the integrands must lie one above and the other one below the real axis - the integrands with both poles above or both poles below the axis give zero. Performing next the integrations over $dk_0$, $dp_0$ and then over $dq_0$ one gets

$$\int_{-\infty}^{\infty} \frac{d q_0}{2\pi} \left[ \ldots \right] = \frac{i}{\omega_k + \omega_p - \omega_{k+q} - \omega_{p-q} + i0} \frac{\theta(|k| - p_{F+}) \theta(|p - p_{F-}) \theta(|k + q| - p_{F+}) \theta(|p - q| - p_{F-})}{\omega_k + \omega_p - \omega_{k+q} - \omega_{p-q} + i0} \frac{\theta(|k| - p_{F+}) \theta(|p - p_{F-}) \theta(|p - |k + q|) \theta(|p - |p - q|)}{\omega_k + \omega_p - \omega_{k+q} - \omega_{p-q} + i0}.$$

Making then the substitutions $k + q = -k'$, $p - q = -p'$ in the second integral one
finds that the two terms above give equal contributions. Thus,

\[
\frac{E_{11}^{(2)}}{V} = \frac{C_0^2}{\hbar} \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{\theta(pF_- - |k|) \theta(pF_- - |p|) \theta(|k + q - pF_+|) \theta(|p - q| - pF_-)}{\omega_k + \omega_p - \omega_{k+q} - \omega_{p-q} + i0}.
\]

The last step \[2\] is to pass to the integrations over the variables \(s\), \(t\), and \(u\) defined by the relations (the Jacobian \(J = 8\))

\[
k = (s - t), \quad p = (s + t), \quad q = (t - u).
\]

The denominator of the integrand then becomes equal \((t^2 - u^2)/m_f\). Defining

\[
I = \int d^3s \int d^3t \int d^3u \frac{\theta(pF_- - |t + s|) \theta(pF_+ - |t - s|) \theta(|u + s| - pF_-) \theta(|u - s| - pF_+)}{t^2 - u^2 + i0},
\]

one can write the combined first and second order contributions in the form

\[
\frac{E_{11}^{(1)} + E_{11}^{(2)}}{V} = C_0 \frac{p_F^3 - p_{F+}^3}{36\pi^3} + \frac{8C_0^2}{\hbar^2} m_f \frac{I}{(2\pi)^9}.
\]

After using \[15\], i.e. replacing \(C_0\) by \((4\pi \hbar^2/m_f)a_0(1 + 2a_0\Lambda/\pi)\), one gets

\[
\frac{E_{11}^{(1)} + E_{11}^{(2)}}{V} = \frac{p_F^3 - p_{F+}^3}{9\pi^3} m_f a_0 \frac{\hbar^2}{a_0^2 \Lambda + \frac{\hbar^2}{m_f} 32a_0^2 \frac{I}{(2\pi)^7}}.
\]

The regions of integrations over \(d^3u\) and over \(d^3t\) in \(I\) are determined by the intersections of two Fermi spheres of unequal radii, \(pF_-\) and \(pF_+\), the centers of which are displaced from the origin of the \(u\) (of the \(t\)) space by the vectors \(-s\) (\(s\) will be taken to determine the \(z\)-axes of the \(u\) and \(t\) spaces in the integrals over \(d^3u\) and \(d^3t\)) and \(s\), respectively. The integral over \(u\) is over the infinite exterior of both spheres and is, therefore, divergent; the integration over \(t\) covers the intersection of the two spheres. For this reason the outermost integration over \(s \equiv |s|\) is restricted to \(s \leq s_{\text{max}} = \frac{1}{2}(pF_+ + pF_-)\) because if \(s > \frac{1}{2}(pF_+ + pF_-)\), the two spheres which determine the region of integration over \(t\) become disjoint. It will be convenient to write \(I = 8(2\pi)^3 J(pF_-, pF_+)\) with

\[
J(pF_-, pF_+) = \int_{s_0}^{s_{\text{max}}} ds s^2 \frac{1}{4\pi} \int d^3t \frac{\theta(pF_- - |t + s|) \theta(pF_+ - |t - s|)}{g(t, s)} g(t, s) \equiv g(|t|, s) = \frac{1}{4\pi} \int d^3u \frac{\theta(|u + s| - pF_-) \theta(|u - s| - pF_+)}{t^2 - u^2 + i0}.
\]

As far as the integral \(g(t, s)\) is concerned, the range of the variable \(s\) splits into two domains: if \(0 \leq s \leq s_0 = \frac{1}{2}(pF_+ - pF_-)\), the sphere of radius \(pF_-\) is entirely contained inside the one of radius \(pF_+\) (see Fig. \[4\]) and plays no role in determining the domain of integration over \(u\); this domain is then just the (infinite) exterior of the sphere of radius \(pF_+\), the center of which is at \(u_z = 0\), when \(s = 0\) and moves to the right as \(s\) increases but the origin of the \(u\) space \((u_x = u_y = u_z = 0)\) always remains inside this sphere. Thus, in the spherical variables \(u_x = u\sin \vartheta \cos \varphi\), \(u_x = u\sin \vartheta \sin \varphi\),
Figure 4: Configurations of the Fermi spheres. Dots mark their centers shifted by ±s from the origin of the space. 

a) $p_{F-}/p_{F+} < 1/3$; in this case, if $p_{F-} < s < (p_{F+} - p_{F-})/2$, the smaller sphere is entirely in the left half of the space.

b) $p_{F-}/p_{F+} > 1/3$; in this case part of the smaller sphere is always in the right half of the space.

c) The spheres intersect for $(p_{F+} - p_{F-})/2 < s < (p_{F+} + p_{F-})$. Marked is the “critical” polar angle $\vartheta_0$.

$u_z = u \cos \vartheta \equiv u \xi$, the integration over $du$ is bounded from below by the condition $u^2 - 2us \xi - (p_{F+}^2 - s^2) = 0$ and one can write

$$g(t, s) = \frac{1}{2} \int_{-1}^{+1} d\xi \int_{0}^{\infty} du \frac{u^2}{t^2 - u^2 + i0}$$

$$= \int_{0}^{\infty} du \frac{u^2}{t^2 - u^2 + i0} + \frac{1}{2} \int_{-1}^{+1} d\xi \int_{0}^{\infty} du \frac{u^2}{u^2 - t^2 - i0},$$

where $u(\xi, s) = s\xi + \sqrt{p_{F+}^2 - s^2(1 - \xi^2)}$. The first, divergent, integral is proportional to the integral (12). The second one can be worked out using the trick given in Appendix C of [4], that is introducing under the integral over $\xi$ the factor $1 = d\xi/d\eta$, taking this integral by parts and then trading the remaining integration over $\xi$ for the integration over $u(\xi, s)$. This, upon using the result (14), leads to $(v = u^2(\xi, s);$ terms of (14) vanishing in the limit $\Lambda \to \infty$ are omitted)

$$g(t, s) = -i\pi \frac{\Lambda}{2} t - \Lambda + \frac{1}{2} \left\{ \int_{0}^{u(1,s)} du \frac{u^2}{u^2 - t^2 - i0} ight.$$

$$+ \int_{0}^{u(-1,s)} du \frac{u^2}{u^2 - t^2 - i0} - \frac{1}{4\pi} \int_{0}^{u^2(1,s)} dv \frac{v - (p_{F+}^2 - s^2)}{v - t^2 - i0} \right\}.$$
only the first two do develop an imaginary part\textsuperscript{12} We therefore compute these integrals using the standard tricks (the Sochocki formula)
\[
\int_0^{u_{\text{max}}} du \frac{u^2}{u^2 - t^2 - i0} = i \frac{\pi}{2} t + u_{\text{max}} + \frac{t}{2} \int_0^{u_{\text{max}}} du \left( \frac{1}{u - t} - \frac{1}{u + t} \right) = i \frac{\pi}{2} t + u_{\text{max}} + \frac{t}{2} \ln \frac{u_{\text{max}} - t}{u_{\text{max}} + t}.
\] (21)

Since \( u(1, s) = p_{F+} + s \), \( u(-1, s) = p_{F+} - s \) one obtains (the imaginary part of \( g(t, s) \) indeed disappears)
\[
g(t, s) = -\Lambda + p_{F+} + \frac{t}{4} \ln \frac{p_{F+} - s - t}{p_{F+} - s + t} + \frac{t}{4} \ln \frac{p_{F+} - s - t}{p_{F+} - s + t} - \frac{1}{8s} \int_{u_{\text{max}}(-1, s)}^{u_{\text{max}}(1, s)} dv \frac{v - (p_{F+}^2 - s^2)}{v - t^2 - i0}.
\]

Working out the last integral (ignoring \(-i0\), as the pole of the integrand is outside the range of the integration) one gets the formula for the function \( g(t, s) \) in the range \( 0 \leq s \leq s_0 \)
\[
g(t, s) = -\Lambda + \frac{1}{2} p_{F+} + \frac{t}{4} \ln \frac{(p_{F+} - t)^2 - s^2}{(p_{F+} + t)^2 - s^2} + \frac{p_{F+}^2 - s^2 - t^2}{8s} \ln \frac{(p_{F+} + s)^2 - t^2}{(p_{F+} - s)^2 - t^2}. \] (22)

Integrated over \( t \), \( \eta \) and then over \( s \) from 0 to \( s_0 \), it gives the corresponding contribution to the ground state energy density. The integration over \( t \) and \( \eta \) is in this case \( 0 < s < s_0 \); the smaller sphere entirely inside the larger one) over the interior of the (smaller) sphere of radius \( p_{F-} \). If \( p_{F-}/p_{F+} > 1/3 \) the origin \( t = 0 \) of the \( t \)-space is, as long as \( 0 < s < s_0 \), always inside this sphere (see Fig. 4b) and the integrations over \( t \) and \( \eta \) are given by the expression
\[
\int_{-1}^{1} d\eta \int_{0}^{t(\eta, s)} dt t^2 g(t, s),
\]

with \( t(\eta, s) = -s \eta + \sqrt{p_{F-}^2 - s^2(1 - \eta^2)} \). If, instead \( p_{F-}/p_{F+} < 1/3 \), the integration domain \( 0 \leq s \leq s_0 \) has to be further split into \( 0 \leq s \leq p_{F-} \) and \( p_{F-} \leq s \leq s_0 \). In the first case the integrations over \( t \) and \( \eta \) are given by the expression given above. However, when \( p_{F-} \leq s \leq s_0 \), the sphere of radius \( p_{F-} \) is entirely in the left half of the \( t \)-space (see Fig. 4b) and the integrations over \( t \) and \( \eta \) is given by\textsuperscript{13}
\[
\int_{-1}^{-\sqrt{1-p_{F-}^2/s^2}} d\eta \int_{t_-(\eta, s)}^{t_+(\eta, s)} dt t^2 g(t, s),
\]

with \( t_-(\eta, s) = -s \eta \pm \sqrt{p_{F-}^2 - s^2(1 - \eta^2)} \) (the upper limit of the integral over \( \eta \) is determined by the equality \( t_- = t_+ \).

\textsuperscript{12}Indeed, it is easy to see geometrically that the maximal value of \( t \) reached in the outer integral is \( s + p_{F-} \), whereas the lower limit of the third integral is \( (p_{F-} - s)^2 \); since the function \( g(t, s) \) computed here is valid only up to \( s \leq s_0 = (p_{F+} - p_{F-})/2 \), the variable \( t^2 \) never exceeds the lower limit of integration over \( v \).
\textsuperscript{13}Again it can be checked that if \( g(t, s) = 1 \), this expression gives \((2/3)p_{F-}^3\).
Thus if we write the integral \[20\) as \(J = J_1 + J_2\) where \(J_1\) is the contribution of the range of \(s\) for which the sphere of radius \(p_{F-}\) is entirely inside the one of radius \(p_{F+}\), we get

\[
J_1 = \int_0^{s_0} ds \frac{s^2}{2} \int_{-1}^1 d\eta \int_{t_s(\eta,s)}^{t_s(\eta,s)} dt \ t^2 g(t,s), \quad \text{if} \quad p_{F-} > \frac{1}{3} p_{F+}, \quad \text{and}
\]

\[
J_1 = \int_0^{p_{F-}} ds \frac{s^2}{2} \int_{-1}^1 d\eta \int_{t_s(\eta,s)}^{t_s(\eta,s)} dt \ t^2 g(t,s) + \int_{p_F}^{s_0} ds \frac{s^2}{2} \int_{-1}^1 \sqrt{1-p_{F-}^2/s^2} d\eta \int_{t_s(\eta,s)}^{t_s(\eta,s)} dt \ t^2 g(t,s),
\]

if \(p_{F-} < \frac{1}{3} p_{F+}\). Since when \(g(t,s) \equiv 1\) both integrations over \(\eta\) and \(t\) give \((2/3)p_{F-}^3\), the divergent part of \(J_1\) is

\[
J_1^{\text{div}} = \frac{1}{3} s_0^2 \frac{1}{3} p_{F-}^3 (-\Lambda) = -\frac{1}{72} (p_{F+} - p_{F-})^3 p_{F-}^3 \Lambda. \tag{23}
\]

To simplify the evaluation of the (finite part of the) integral \(J_1\) one can shift the variable \(t\) by writing \(t = t' - s\) and introducing the spherical coordinate system in the \(t'\) space with the \(t'_z\) axis taken in the direction of the vector \(s\). Then

\[
J_1 = \frac{1}{2} \int_0^{s_0} ds \frac{s^2}{2} \int_{-1}^1 d\eta \int_0^{p_{F-}} dt' t^2 g\left(\sqrt{t'^2 - 2t's\eta + s^2}, s\right),
\]

without the need to distinguish the cases \(p_{F-} > \frac{1}{3} p_{F+}\) and \(p_{F-} < \frac{1}{3} p_{F+}\). Numerical evaluation shows that both ways of computing \(J_1\) yield the same results. \[14\]

We now compute the function \(g(t,s)\) for \(s_0 \leq s \leq s_{\text{max}}\) and the corresponding contribution \(J_2\) to the integral \(20\). In this regime the two Fermi spheres which determine the ranges of integrations over \(u\) and over \(t\) intersect one another. In the \(u\) space the \(z\) coordinate \(u_z\) of the intersection and its distance \(u_0\) from the origin are determined by solving the equations

\[
\begin{align*}
  u_{z}^2 + (u_z - s)^2 &= p_{F+}^2, \\
  u_{z}^2 + (u_z + s)^2 &= p_{F-}^2,
\end{align*}
\]

which give

\[
\begin{align*}
  u_0^2 &= \frac{-p_{F+}^2 - p_{F-}^2}{4s}, & u_0^2 &= \frac{1}{2} (p_{F+}^2 + p_{F-}^2) - s^2.
\end{align*}
\]

In the spherical system the “critical” angle \(\vartheta_0\) corresponding to the intersection of the spheres (marked in Fig. 4c) is given by

\[
\cos \vartheta_0 = \xi_0 = \frac{u_0}{u_0} = -\frac{p_{F+}^2 - p_{F-}^2}{4s \sqrt{\frac{1}{2} (p_{F+}^2 + p_{F-}^2) - s^2}}.
\]

\[14\)Yet another, the simplest way, of evaluating this contribution (without distinguishing the cases \(p_{F-} > \frac{1}{3} p_{F+}\) and \(p_{F-} < \frac{1}{3} p_{F+}\)) is to use the Mathematica package instruction

0.5NIntegrate[s^2 t^2 g[t, s] Boole[t^2 + 2 t s x + s^2 < p_{F-}^2], \{s, 0, s_0\}, \{x, -1, 1\}, \{t, 0, \infty\}].

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Therefore, if \( s_0 \leq s \leq s_{\text{max}} \) (i.e. when the two Fermi spheres intersect), the function \( g(t, s) \) is given by

\[
g(t, s) = \frac{1}{2} \int_{\xi_0}^{\infty} d\xi \int_{u_+(\xi, s)}^{\infty} du \frac{u^2}{t^2 - u^2 + i0} + \frac{1}{2} \int_{-1}^{\xi_0} d\xi \int_{u_-^{\prime}(\xi, s)}^{\infty} du \frac{u^2}{t^2 - u^2 + i0},
\]

where now \( u_+(\xi, s) = s \xi + \sqrt{p_{F+}^2 - s^2(1 - \xi^2)} \), \( u_-^{\prime}(\xi, s) = -s \xi + \sqrt{p_{F-}^2 - s^2(1 - \xi^2)} \);

of course \( u_+(\xi_0, s) = u_-^{\prime}(\xi_0, s) \equiv u_0 \). (Because the integrals are over the exterior of the spheres, the fact that the origin of the \( u \) space may not, for some ranges of the parameters, be inside the smaller sphere of radius \( p_{F-} \) is irrelevant here). After extracting the divergent part of these two integrals as previously, they combine giving to the integral \( -2\pi^2 I_0 \) and one obtains

\[
g(t, s) = -\Lambda - \frac{\pi}{2} t + \frac{1}{2} \int_{\xi_0}^{1} d\xi \int_{0}^{u_+(\xi, s)} du \frac{u^2}{u^2 - t^2 - i0} + \frac{1}{2} \int_{-1}^{\xi_0} d\xi \int_{0}^{u_-^{\prime}(\xi, s)} du \frac{u^2}{u^2 - t^2 - i0}.
\]

It is now straightforward to compute \( J_2^{\text{div}} \) and to check the cancellation of \( \Lambda \). Indeed, the integral

\[
\frac{1}{4\pi} \int d^3t \theta(p_{F-} - |t + s|) \theta(p_{F+} - |t - s|)(-\Lambda),
\]

can be done using the already computed integrals: shifting the origin of the coordinate system of the \( t \) space so that the intersection of the two Fermi spheres occurs at \( t_0' = 0 \) one readily obtains the result

\[
-\frac{\Lambda}{2} \left\{ \left[\frac{1}{3}p_{F-}^3 - \frac{1}{2}p_{F-}^2(s - \varepsilon) + \frac{1}{6}(s - \varepsilon)^3 \right] + \left[\frac{1}{3}p_{F+}^3 - \frac{1}{2}p_{F+}^2(s + \varepsilon) + \frac{1}{6}(s + \varepsilon)^3 \right] \right\},
\]

with \( \varepsilon = (p_{F+}^2 - p_{F-}^2)/2s \equiv -u_0^4 \). This should be integrated from \( s_0 = \frac{1}{2}(p_{F+} - p_{F-}) \) to \( s_{\text{max}} = \frac{1}{2}(p_{F+} + p_{F-}) \) with the weight \( s^2 \). Mathematica does it readily and the result is

\[
J_2^{\text{div}} = -\Lambda \left( \frac{p_{F+}^4 + p_{F-}^4}{24} - \frac{p_{F+}^5 + p_{F-}^5}{24} + \frac{p_{F-}^6}{72} \right).
\]

Combining this with the divergent part (23) of \( J_1 \) one gets

\[
J_1^{\text{div}} + J_2^{\text{div}} = -\Lambda \frac{p_{F+}^3 p_{F-}^3}{72},
\]

which is precisely what is needed to cancel in (19) the term explicitly proportional to \( \Lambda \) which comes from expressing \( C_0 \) in terms of the scattering length in the first order result. Thus, as expected, the divergences disappear when observable quantities (the ground state energy) computed within the effective theory are expressed in terms of other observable quantities (the scattering lengths).

To work out the imaginary parts of the two remaining integrals we again use the trick with taking the integral over \( \xi \) by parts after inserting into it \( 1 = d\xi/d\xi \). This
chocki formula that is, using the result (21); their sum (the above expression enters
the first two integrals do have imaginary parts which can be extracted using the So-
c cancel out, when the two integrals are added, and the sum reads
\[ g(t, s) = \Lambda + \frac{1}{2} (p_{F^+} + p_{F^-} + 2s) + \frac{t}{4} \ln \frac{p_{F^+} + s - t}{p_{F^+} + s + t} + \frac{t}{4} \ln \frac{p_{F^-} + s - t}{p_{F^-} + s + t} \]
\[ - \frac{1}{8s} \int_{u_0^2}^{(p_{F^+} + s)^2} dv \frac{v - (p_{F^+}^2 - s^2)}{v - t^2 - i0} - \frac{1}{8s} \int_{u_0^2}^{(p_{F^-} + s)^2} dv \frac{v - (p_{F^-}^2 - s^2)}{v - t^2 - i0}. \]

The first two integrals do have imaginary parts which can be extracted using the So-
chocki formula that is, using the result (21); their sum (the above expression enters
the last integral are interchanged)

Since \( u_+ (\xi_0, s) = u_- (\xi_0, s) = u_0 \), the terms explicitly proportional to \( \xi_0 \) mutually
cancel out, when the two integrals are added, and the sum reads

\[ \int_{-1}^{\xi_0} d\xi \int_{0}^{u_+ (\xi, s)} du \frac{u^2}{u^2 - t^2 - i0} = \int_{0}^{u_+ (1, s)} du \frac{u^2}{u^2 - t^2 - i0} - \frac{1}{4s} \int_{u_0^2}^{u_+^2 (1, s)} dv \frac{v - (p_{F^+}^2 - s^2)}{v - t^2 - i0}. \]

and, similarly,

\[ \int_{-1}^{\xi_0} d\xi \int_{0}^{u_- (\xi, s)} du \frac{u^2}{u^2 - t^2 - i0} = \frac{1}{4s} \int_{u_0^2}^{u_-^2 (\xi_0, s)} dv \frac{v - (p_{F^-}^2 - s^2)}{v - t^2 - i0}. \]

The result then is (it is clear that \( u_0 < p_{F^\pm} + s \), so the limits of the
the two integrals are interchanged)

\[ g(t, s) = -\Lambda + \frac{1}{2} (p_{F^+} + p_{F^-} + 2s) + \frac{t}{4} \ln \frac{p_{F^+} + s - t}{p_{F^+} + s + t} + \frac{t}{4} \ln \frac{p_{F^-} + s - t}{p_{F^-} + s + t} \]
\[ - \frac{1}{8s} \int_{u_0^2}^{(p_{F^+} + s)^2} dv \frac{v - (p_{F^+}^2 - s^2)}{v - t^2 - i0} - \frac{1}{8s} \int_{u_0^2}^{(p_{F^-} + s)^2} dv \frac{v - (p_{F^-}^2 - s^2)}{v - t^2 - i0}. \]

The final step is to work out the remaining integrals. They are similar to the one
already computed and one gets

\[ g(t, s) = -\Lambda + \frac{1}{4} (p_{F^+} + p_{F^-} + 2s) + \frac{t}{4} \ln \frac{p_{F^+} + s - t}{p_{F^+} + s + t} + \frac{t}{4} \ln \frac{p_{F^-} + s - t}{p_{F^-} + s + t} \]
\[ + \frac{p_{F^+}^2 - t^2 - s^2}{8s} \ln \frac{(p_{F^+} + s)^2 - t^2}{u_0^2 - t^2} + \frac{p_{F^-}^2 - t^2 - s^2}{8s} \ln \frac{(p_{F^-} + s)^2 - t^2}{u_0^2 - t^2}. \] (24)

In the limit \( p_{F^-} = p_{F^+} = k_F \) this goes over into the function

\[ g(t, s) = -\Lambda + \frac{1}{2} (k_F + s) + \frac{t}{2} \ln \frac{k_F + s - t}{k_F + s + t} + \frac{k_F^2 - s^2 - t^2}{4s} \ln \frac{(k_F + s)^2 - t^2}{k_F^2 - s^2 - t^2}. \]
Figure 5: Plot of the function $J(r, 1)$. The value $J(1, 1) = 0.0114449 = (11 - 2 \ln 2)/840$ is the result of [2].

arising in the case of equal densities of spin up and spin down fermions. In this case, in which the integral $J_1$ is zero, the remaining integrals over $s, \eta$ and $t$ which give $J_2 = J$ can be even worked out explicitly [2] with the result

$$J(k_F, k_F) = -\frac{1}{72} \Lambda k_F^6 + k_F^7 \frac{11 - 2 \ln 2}{24 \cdot 35}.$$

If $p_{F-} < p_{F+}$, the integrals $s, \eta$ and $t$ can be easily evaluated numerically. It is convenient to write the complete function $J(p_{F-}, p_{F+})$ defined in (20), setting $\Lambda = 0$ as $p_{F+}^7 J(r, 1)$ with $0 \leq r \equiv p_{F-}/p_{F+} \leq 1$. The function $J(r, 1)$ is shown in Fig. 5. The complete result can be therefore written as

$$E_\Omega V = \frac{\hbar^2 p_{F+}^3 p_{F-}^3}{m_f 6 \pi^2} \left\{ \frac{3}{10} (1 + r^5) + \frac{2}{3 \pi} r^3 (p_{F+} a_0) + \frac{96}{\pi^2} (p_{F+} a_0)^2 J(r, 1) + \ldots \right\}. \quad (25)$$

It is, however better to express it in terms of $k_F = (3\pi^2 \rho)^{1/3}$, where $\rho = N/V$ - the Fermi wave vector in the case $N_+ = N_- = N/2$, which does not change when the ratio $r = p_{F-}/p_{F+}$ (i.e. the system’s polarization) is varied. Since $p_{F+} = k_F (2/(1 + r^3))^{1/3}$,

$$E_\Omega V = \frac{\hbar^2 k_F^6}{m_f 6 \pi^2} \left( \frac{2}{1 + r^3} \right)^{5/3} \left\{ \frac{3}{10} (1 + r^5) + \frac{2}{3 \pi} r^3 \left( \frac{2}{1 + r^3} \right)^{1/3} (k_F a_0) \right. \left. + \frac{96}{\pi^2} \left( \frac{2}{1 + r^3} \right)^{2/3} (k_F a_0)^2 J(r, 1) + \ldots \right\}. \quad (26)$$

This energy density is plotted in Fig. 6 as a function of the system’s polarization $P = (N_+ - N_-)/N$ (0 \leq P \leq 1; it is related to the variable $r$ by $r = [(1 - P)/(1 + P)]^{1/3}$) for five different values of the expansion parameter $k_F a_0$. We have checked that our

\[15\] The simplest way is to use the Mathematica numerical integration routine to integrate over the domain \(s_0 \leq s \leq s_{\text{max}}, -1 \leq x \leq 1, 0 \leq t \leq \infty\), imposing the conditions \(t^2 + 2txs + s^2 < p_{F-}^2\) and \(t^2 - 2txs + s^2 < p_{F+}^2\), but we have also evaluated it using other methods always with the same results.
Figure 6: Energy density $E_0/V$ in units $(3/5)\hbar^2 k_F^5/6\pi^2 m_f = (N/V)(\hbar^2 k_F^5/2m_f)(3/5)$ of the polarized gas of spin $1/2$ fermions as a function of its polarization $P = (N_+ - N_-)/N$ for different values (from below) 0.1 (blue), 0.2 (yellow), 0.3 (green), 0.4 (red) and 0.6 (blue) of the expansion parameter $k_F a_0$.

result agrees with that of [19]. The computed correction of order $(k_F a_0)^2$ is rather small, $\sim 1\%$ for $k_F a_0 = 0.2$ and $r = 1$ and decreases with decreasing $r$. All curves assume at $P = 1$ the same value $2^{2/3} = 1.5874$ - due to the Pauli exclusion principle interactions do not induce any corrections to the ground state of a fully polarized $(P = 1, r = 0)$ system of fermions.\footnote{The precise relation, checked numerically, of the function $I(P)$ used in [19] (the variable $r$ used there is our polarization $P$) to the function $J$ defined by (20) evaluated with the cutoff $\Lambda = 0$ and $k_F = 1$ is

$$I(P) = 160(1 + P)^{7/3}J(r(P), 1), \quad r(P) = ((1 - P)/(1 + P))^{1/3}.\footnote{This readily follows from the form of the effective interaction written in terms of the field operators $\phi_{\pm}$, $\psi_{\pm}^\dagger$ introduced in [17] and the absence of the “sea” of oppositely polarized fermions; the same can be also inferred by writing explicitly the original interaction Hamiltonian $\mathcal{H}$ in terms of creation and annihilation operators associated with different spin projections.}$$}

Note also that the prefactor $\hbar^2 k_F^5/6\pi^2 m_f$ in (26) can be written in the form $(N/V)(\hbar^2 k_F^5/2m_f)$. Therefore, our Fig. 6 can be directly compared to Fig. 3 of ref. [20]: our curve for $k_F a_0 = 0.6$ corresponds to the lowest curve in this plot obtained for a model repulsive potential by a numerical estimate of the exact ground state energy. It is seen that the result of the second order expansion is somewhat lower than the numerical estimate. This agrees with the comparison of the ground state energies of the unpolarized system $(P = 0, r = 1)$ performed in Fig. 2 of ref. [20] which shows that the perturbative expansion of the ground state energy in powers of $k_F a_0$ remains reliable up to $k_F a_0 \lesssim 0.5$ but is systematically below the numerical estimates of the exact value.
5 Summary

We have recomputed the order \((k_F a_0)^2\), where \(a_0\) is the s-wave scattering length and \(k_F = (3 \pi^2 N/V)^{1/3}\), correction to the ground state energy of a polarized gas of (non-relativistic) fermions of spin \(1/2\) using the effective theory approach proposed in [2] which does not require specifying explicitly the (spin independent) interaction potential. We have demonstrated the cancellation of ultraviolet divergences when the result is expressed in terms of the scattering length. Our result obtained by the method applicable to arbitrary repulsive interaction potentials is identical with that of [19] obtained with the help of traditional methods within the specific model of hard spheres. That it should be so is almost obvious in the effective theory approach but wasn’t such in the old framework.

Since the main technical problem of this approach is only isolating ultraviolet divergences and working out cancellation of imaginary contributions, it seems that with some more labour the computations presented here could be extended to yet higher orders of the expansion, similarly as was done in the case of unpolarized system in [2] and [13]. A more challenging task would be obtaining a rigorous estimate of the high order terms of the perturbative expansion which could allow to assess the range of its convergence.

In this paper we have considered the polarized diluted gas of (nonrelativistic) interacting spin \(1/2\) fermions, working in the continuum version of the theory. Our results can be most naturally applied to atomic gases bound in traps. An analogous problem can of course be also formulated using the lattice version, that is within the paradigmatic Hubbard model, with obvious applications to atomic gases bound in periodic laser traps and to the solid state systems. As far as we know, there are no second order results similar to ours in this other version (rigorous first order results have been given in [26] and [30]) and it would be interesting to try to obtain them.

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