The Topological Structure of the Space-Time Disclination

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Abstract

The space-time disclination is studied by making use of the decomposition theory of gauge potential in terms of antisymmetric tensor field and ϕ-mapping method. It is shown that the self-dual and anti-self-dual parts of the curvature compose the space-time disclinations which are classified in terms of topological invariants—winding number. The projection of space-time disclination density along an antisymmetric tensor field is quantized topologically and characterized by Brouwer degree and Hopf index.

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I. INTRODUCTION

The topological space-time defects, if they exist, may help to explain some of the largest-scale structure seen in the universe today. They may have been found at phase transitions in the early history of the universe, analogous to those found in some condensed matter systems—vortex line in liquid helium, flux tubes in type-II superconductors, or disclination lines in liquid crystals.

As a kind of space-time defect, torsion plays an important role in modern physics. Recently, a lot of works on the spin and torsion have been done by many physicists. Many of them are focused on the so-called space-time dislocation by considering the effect of topology of torsion in the Riemann-Cartan manifold. The consideration partially comes from the gauge field theory of dislocation and disclination continuum about deformable material media. These works is trying to clarify the quantization of the gravitation (i.e. to quantize the Riemann-Cartan space-time itself). Moreover, they hope to find the dynamical relation between the stress-energy-momentum tensor and curvature is expressed in general relativity by Einstein equations.

On the other hand, there exist another kind of space-time defect called space-time disclination. This kind of defect has the same important meaning as that of the torsion. Space-time disclination reflects the intrinsic property of the curved space-time. It is caused by inserting solid angles into the flat space-time. In Riemann-Cartan geometry, this effect is showed by the integral of the affine curvature along a closed surface. Duan, Duan and Zhang had discussed the disclinations in deformable material media by applying the gauge field theory and decomposition theory of gauge potential. In their works, the projection of disclination density along the gauge parallel vector was found corresponding to a set of isolated disclinations in the three dimensional sense and being topologically quantized. The decomposition theory of gauge potential and $\phi$-mapping method they used are good tools to study the problems of gauge field.

In this paper, we discuss the space-time disclinations in the similar way as that was used
in Duan et al’s works on the condensed matter disclination.\textsuperscript{12} The decomposition formulas of $SO(4)$ and $SU(2)$ gauge potential are given in terms of antisymmetric tensor field and unit vector field respectively. Using this decomposition theory the space-time disclinations are classified into two classes which are marked by the self-dual and anti-self-dual parts of the curvature. The projection of space-time disclination density along the gauge parallel tensor can be written in terms of winding numbers. Moreover we show that the projection of space-time disclination density, which is quantized topologically, corresponds to two groups of isolated disclinations. The positions of the disclination vertices are determined by the zeroes of the self-dual or anti-self-dual part of an antisymmetric tensor. And the Hopf index and Brouwer degree classify the disclinations and characterize the local nature of the space-time disclinations. For this quantization of the space-time disclinations is related to the space-time curvature directly as that will be shown in this paper, it perhaps relates to the quantization of space-time.

This paper is arranged as follows: In section 2, we discuss the representation of the space-time disclination. In section 3, the decomposition of $SO(4)$ gauge potential is given by considering the relationship between $SO(4)$ gauge theory and $SU(2)$ gauge theory. At last, we discuss the topological quantization and the local topological property of the space-time disclination in section 4.

\section*{II. THE REPRESENTATION OF SPACE-TIME DISCLINATION}

Let $M$ be a compact, oriented 4-dimensional Riemannian-Cartan manifold and $P(M, G, \pi)$ be a principal bundle with structure group $G = SO(4)$. As it was shown in Ref. 12, the dislocation and disclination continuum can be described by the reference, deformed and natural states. For the natural state there is only an anholonomic rectangular coordinate $Z^a (a = 1, 2, 3, 4)$ and

\[ \delta Z^a = e^a_{\mu} dx^\mu, \]  

\[ (1) \]
where $e^a_\mu$ is vielbein. The metric tensor of the Riemann-Cartan manifold of natural state is defined by

$$g_{\mu\nu} = e^a_\mu e^a_\nu.$$  \hfill (2)

We have known that the metric tensor $g_{\mu\nu}$ is invariant under the local $SO(4)$ transformation of vielbein. The corresponding gauge covariant derivative 1-form of an antisymmetric tensor field $\phi^{ab} = -\phi^{ba}$ on $\mathcal{M}$ is given as

$$D\phi^{ab} = d\phi^{ab} - \omega^{ac} \phi^{cb} - \omega^{bc} \phi^{ac},$$  \hfill (3)

where $\omega^{ab}$ is $SO(4)$ spin connection 1-form

$$\omega^{ab} = -\omega^{ba} \quad \omega^{ab} = \omega^{ab}_{\mu} dx^\mu.$$  \hfill (4)

The affine connection of the Riemann–Cartan space is determined by

$$\Gamma^\lambda_{\mu\nu} = e^\lambda_a D_\mu e^a_\nu.$$  \hfill (5)

The torsion tensor is the antisymmetric part of $\Gamma^\lambda_{\mu\nu}$ and is expressed as

$$T^\lambda_{\mu\nu} = e^\lambda_a T^a_{\mu\nu},$$  \hfill (6)

where

$$T^a_{\mu\nu} = \frac{1}{2} (D_\mu e^a_\nu - D_\nu e^a_\mu).$$  \hfill (7)

The Riemannian curvature tensor is equivalent to the $SO(4)$ gauge field strength tensor 2-form $F^{ab}$, which is given by

$$F^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega^{cb} \quad F^{ab} = \frac{1}{2} F_{\mu\nu}^{ab} dx^\mu \wedge dx^\nu$$  \hfill (8)

and relates with the Riemann curvature tensor by

$$F^{ab}_{\mu\nu} = -R^\lambda_{\mu\nu\sigma} e^a_\lambda e^{\sigma b}.$$  \hfill (9)

The dislocation density is defined by
\[ \alpha^a = T^a \quad T^a = \frac{1}{2} T^a_{\mu\nu} dx^\mu \wedge dx^\nu. \]  

(10)

Analogous to the definition of the 3-dimensional disclination density in the gauge field theory of condensed matter, we define the space-time disclination density as

\[ \theta^{ab} = \frac{1}{2} R_{\lambda\mu\nu\sigma}^{\lambda} e^{a}_\lambda e^{b}_\lambda d x^\mu \wedge d x^\nu = - F^{ab}. \]  

(11)

The size of the space-time disclination can be represented by the means of the surface integral of the projection of the space-time disclination density along an antisymmetric tensor field \( n^{ab} \)

\[ \Omega = \oint_{\Sigma} \theta^{ab} n^{ab} = - \oint_{\Sigma} F^{ab} n^{ab} \]  

(12)

where \( \Sigma \) is a closed surface including the disclinations. The new quantity \( \Omega \) defined by (12) is dimensionless. Using the so-called \( \phi \)-mapping method and the decomposition of gauge potential Y. S. Duan et al have proved that the dislocation flux is quantized in units of the Planck length.\(^\text{11}\) In this paper we will show the size of the space-time disclination (12) is topologically quantized and is the sum of two groups of solid angles representing the size of disclinations.

**III. DECOMPOSITION THEORY OF \( \text{SO}(4) \) SPIN CONNECTION**

In this section we will give the decomposition theory of \( \text{SO}(4) \) gauge potential in terms of an antisymmetric tensor field on a compact and oriented 4-dimensional manifold, which is the foundation of topological gauge field theory. It will be seen in the following discussion that this decomposition of \( \text{SO}(4) \) gauge potential is expressed by the combination of \( \text{SU}(2) \) gauge potential. Hence we must study the decomposition of \( \text{SU}(2) \) gauge potential in terms of the sphere bundle on a compact and oriented 4-dimensional manifold firstly.

Let \( n \) be an unit \( \text{SU}(2) \) Lie algebraic vector

\[ n = n^A I_A , \quad A = 1, 2, 3 ; \]  

(13)
and

\[ n^A n^A = 1, \quad (14) \]

in which \( I_A \) is the generator of the group \( SU(2) \), which satisfies the commutation relation

\[ [I_A, I_B] = -\epsilon^{ABC} I_C. \quad (15) \]

The covariant derivative 1-form of \( n \) is given by

\[ D_{SU(2)} n = dn - [\omega_{SU(2)}, n], \quad (16) \]

where \( \omega_{SU(2)} \) is the \( SU(2) \) gauge potential 1-form:

\[ \omega_{SU(2)} = \omega^A_{SU(2)} I_A, \quad (17) \]

and

\[ \omega^A_{SU(2)} = \omega^A_{SU(2)\mu} dx^\mu \quad \mu = 0, 1, 2, 3. \quad (18) \]

The curvature 2-form is

\[ F_{SU(2)} = F^A_{SU(2)} I_A = d\omega_{SU(2)} - \omega_{SU(2)} \wedge \omega_{SU(2)}, \quad (19) \]

and

\[ F^A_{SU(2)} = \frac{1}{2} F^A_{SU(2)\mu\nu} dx^\mu \wedge dx^\nu \\
= d\omega^A_{SU(2)} + \frac{1}{2} \epsilon^{ABC} \omega^B_{SU(2)} \wedge \omega^C_{SU(2)}. \quad (20) \]

Let \( m = m^A I_A, l = l^A I_A \) be another two unit \( SU(2) \) algebraic vectors orthonormal to \( n \) and to each other, satisfying

\[ n^A = \epsilon^{ABC} m^B l^C, \quad (21) \]

It can be proved that the \( SU(2) \) gauge potential can be decomposed by \( n, m, l \) as

\[ \omega^A_{SU(2)} = \epsilon^{ABC} (dn^B n^C - D_{SU(2)} n^B n^C) - n^A n, \quad (22) \]
where

\[ A = dm^A t^A - D_{SU(2)} m^A t^A \]  \hspace{1cm} (23)

is a \( U(1) \)-like gauge potential.

If \( n \) is taken as a gauge parallel vector, i.e.

\[ D_{SU(2)} n = 0, \]  \hspace{1cm} (24)

the \( SU(2) \) gauge potential becomes

\[ \omega^A_{SU(2)} = \epsilon^{ABC} n^B n^C - n^A A. \]  \hspace{1cm} (25)

Then we can easily get the curvature 2-form \( F^A_{SU(2)} \) as

\[ F^A_{SU(2)} = n^A dA - \frac{1}{2} \epsilon^{ABC} dn^B \wedge dn^C. \]  \hspace{1cm} (26)

Now let us consider the \( SO(4) \) gauge theory. Let the 4-dimensional Dirac matrix \( \gamma_a \) \((a = 1, 2, 3, 4) \) be the basis of the Clifford algebra which satisfies

\[ \gamma_a \gamma_b + \gamma_b \gamma_a = 2 \delta_{ab}. \]  \hspace{1cm} (27)

The antisymmetric tensor field \( \phi^{ab} \) on \( M \) can be expressed in the following matrix form

\[ \phi = \frac{1}{2} \phi^{ab} I_{ab}, \]  \hspace{1cm} (28)

in which \( I_{ab} \) is the generator of the group \( SO(4) \)

\[ I_{ab} = \frac{1}{4} [\gamma_a, \gamma_b]. \]  \hspace{1cm} (29)

Similarly, the spin connection 1-form and curvature 2-form can be expressed as

\[ \omega = \frac{1}{2} \omega^{ab} I_{ab}, \quad F = \frac{1}{2} F^{ab} I_{ab}. \]  \hspace{1cm} (30)

It is well known that the spin representation of \( SO(4) \) group is hormorphic to the direct product of the representations of two \( SU(2) \) group\(^{17}\)
The generators of $SO(4)$ group can be divided into two terms. Each term is a generator of some $SU(2)$ group. Define

\begin{align*}
I^1_\pm &= \frac{1}{2} I_{23}(1 \mp \gamma_5) = \frac{1}{4}(\gamma_2 \gamma_3 \pm \gamma_1 \gamma_4); \\
I^2_\pm &= \frac{1}{2} I_{31}(1 \mp \gamma_5) = \frac{1}{4}(\gamma_3 \gamma_1 \pm \gamma_2 \gamma_4); \\
I^3_\pm &= \frac{1}{2} I_{12}(1 \mp \gamma_5) = \frac{1}{4}(\gamma_1 \gamma_2 \pm \gamma_3 \gamma_4),
\end{align*}

in which $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$. It can be proved $I^A_\pm (A = 1, 2, 3)$ satisfy the commutation relation of group $SU(2)_\pm$

\begin{equation}
I^A_\pm = -\epsilon^{ABC} [I^B_\pm, I^C_\pm],
\end{equation}

and

\begin{equation}
[I^A_\pm, I^B_\mp] = 0.
\end{equation}

Therefore $I^A_\pm$ are the generators of two different groups $SU(2)_\pm$, and they are the basis of $SU(2)_\pm$ Lie algebraic spaces.

Arbitrary antisymmetric tensor field $\phi^{ab}$ can be decomposed as

\begin{equation}
\phi^{ab} = \phi^+ + \phi^-,
\end{equation}

where

\begin{equation}
\phi^+ = \frac{1}{2}(\phi^{ab} + \frac{1}{2} \epsilon^{abcd} \phi^{cd}), \quad \phi^- = \frac{1}{2}(\phi^{ab} - \frac{1}{2} \epsilon^{abcd} \phi^{cd})
\end{equation}

are the self-dual and anti-self-dual parts of $\phi^{ab}$. Define

\begin{equation}
\phi^1_\pm = \phi^{23} \pm \phi^{14}, \quad \phi^2_\pm = \phi^{31} \pm \phi^{24}, \quad \phi^3_\pm = \phi^{12} \pm \phi^{34}.
\end{equation}

We can rewrite $\phi$ as

\begin{equation}
\phi = \frac{1}{2} \phi^{ab} I_{ab} = \phi^+ + \phi^-,
\end{equation}
and

\[ ||\phi_\pm||^2 = \phi_\pm^A \phi_\pm^A = \phi_\pm^{ab} \phi_\pm^{ab}, \]  

(41)

in which

\[ \phi_\pm = \frac{1}{2} \phi(1 \mp \gamma_5) = \phi_\pm^A I_\pm^A. \]  

(42)

From above discussion, we see that \( \phi_\pm \) are the \( SU(2)_\pm \) algebraic vectors. With the similar decomposition as above it can be verified that

\[ D\phi_+ + D\phi_- = d\phi_+ - [\omega_+, \phi_+] + d\phi_- - [\omega_-, \phi_-]. \]  

(43)

For the independent of \( I_+ \) and \( I_- \) we have

\[ D\phi_\pm = d\phi_\pm - [\omega_\pm, \phi_\pm] \quad \text{or} \quad D\phi_\pm^A = d\phi_\pm^A - \epsilon^{ABC} \omega_\pm^C \phi_\pm^B, \]  

(44)

in which

\[ \omega_\pm = \omega_\pm^A I_\pm^A \]  

(45)

is the gauge potential of \( SU(2)_\pm \) gauge field. Similarly curvature 2-form can be decomposed as

\[ F = F_+ + F_-; \]  

(46)

and

\[ F_\pm = d\omega_\pm - \omega_\pm \wedge \omega_\pm. \]  

(47)

\( F_\pm = F_\pm^A I_\pm^A \) are the curvature 2-forms of \( SU(2)_\pm \) gauge field.

Now define an antisymmetric tensor

\[ n^{ab} = \frac{\phi_+^{ab}}{||\phi_+||} + \frac{\phi_-^{ab}}{||\phi_-||}, \]  

(48)

i.e.
\[ n^a_b = \frac{\phi^a_b}{||\phi^a||}, \quad n^a_- = \frac{\phi^a_-}{||\phi^-||}. \] (49)

Then

\[ n = \frac{1}{2} n^{ab} I_{ab} = n_+ + n_- \] (50)

and

\[ n_\pm = \frac{1}{2||\phi_\pm||} \phi(1 \mp \gamma_5) = n^A_\pm I^A_\pm. \] (51)

It naturally guarantees the constraint

\[ n^A_\pm n^A_\pm = 1. \] (52)

i.e. \( n^A_\pm \) are the \( SU(2)_\pm \) lie algebraic unit vectors. If \( n \) is taken as a gauge parallel tensor field

\[ Dn = 0, \] (53)

we have

\[ Dn_\pm = dn_\pm - [\omega_\pm, n_\pm] = 0. \] (54)

From the decomposition formula of \( SU(2) \) gauge potential \((52)\) we get

\[ F^A_\pm = n^A_\pm dA_\pm - \frac{1}{2} \epsilon^{ABC} n^B_\pm \wedge d n^C_\pm. \] (55)

This formula is very useful in the following discussing of space-time disclinations.

**IV. THE TOPOLOGICAL QUANTIZATION AND THE LOCAL TOPOLOGICAL STRUCTURE OF THE SPACE-TIME DISCLINATION**

It is easy to see that

\[ F^{ab} n^{ab} = F^a_+ n^a_+ + F^a_- n^a_- \] (56)
and

\[ F_{\pm}^A n_{\pm}^A = F_{\pm}^{ab} n_{\pm}^{ab}. \]  

(57)

By considering (55) we have

\[ F_{\pm}^A n_{\pm}^A = dA_{\pm} - \frac{1}{2} \epsilon^{ABC} n_{\pm}^A dn_{\pm}^B \wedge dn_{\pm}^C. \]  

(58)

Define

\[ \Omega_{\pm} = \oint_\Sigma (\frac{1}{2} \epsilon^{ABC} n_{\pm}^A dn_{\pm}^B \wedge dn_{\pm}^C - dA_{\pm}). \]  

(59)

\( \Omega_{\pm} \) represent the components of space-time disclination corresponding to different gauge group \( SU(2)_\pm \). The space-time disclination can be expressed as

\[ \Omega = \Omega_+ + \Omega_- \]  

(60)

If the surface \( \Sigma \) is closed, the second term of the right side of (59) contribute nothing to \( \Omega_{\pm} \), i.e.

\[ \Omega_{\pm} = \oint_\Sigma (\frac{1}{2} \epsilon^{ABC} n_{\pm}^A dn_{\pm}^B \wedge dn_{\pm}^C). \]  

(61)

To study the relationship between the solid angle \( \Omega_{\pm} \) and the local properties of the disclinations inside the closed surface \( \Sigma \), using the Stoke’s formula, \( \Omega_{\pm} \) can be expressed as

\[ \Omega_{\pm} = \int_V \frac{1}{2} \epsilon^{ABC} d n_{\pm}^A \wedge d n_{\pm}^B \wedge d n_{\pm}^C, \]  

(62)

in which \( \partial V = \Sigma \). Let us choose coordinates \( y = (u^1, u^2, u^3, v) \) on \( M \) such that \( u = (u^1, u^2, u^3) \) are intrinsic coordinate on \( V \). For the coordinate component \( v \) does not belong to \( V \). Then

\[ \Omega_{\pm} = \int_V \frac{1}{2} \epsilon^{ABC} \partial_i n_{\pm}^A \partial_j n_{\pm}^B \partial_k n_{\pm}^C d u^i \wedge d u^j \wedge d u^k \]

\[ = \int_V \frac{1}{2} \epsilon^{ijk} \epsilon^{ABC} \partial_i n_{\pm}^A \partial_j n_{\pm}^B \partial_k n_{\pm}^C d^3 u \]  

(63)

where \( i, j, k = 1, 2, 3 \) and \( \partial_i = \partial / \partial u^i \). Define solid angle densities as
\[
\rho_{\pm} = \frac{1}{2} \varepsilon^{ijk} \varepsilon^{ABC} \partial_i n^A_{\pm} \partial_j n^B_{\pm} \partial_k n^C_{\pm}.
\]  
(64)

Then we get

\[
\Omega_{\pm} = \int_V \rho_{\pm} d^3 u.
\]
(65)

For the equation (51), the unit vectors \(n^A_{\pm}(x)\) can expressed as follows:

\[
n^A_{\pm} = \frac{\phi^A_{\pm}}{||\phi_{\pm}||}
\]
(66)

Hence

\[
dn^A_{\pm} = \frac{1}{||\phi_{\pm}||} d\phi^A_{\pm} + \phi^A_{\pm} d\left(\frac{1}{||\phi_{\pm}||}\right),
\]
(67)

and

\[
\frac{\partial}{\partial \phi^A_{\pm}} \left(\frac{1}{||\phi_{\pm}||}\right) = -\frac{\phi^A_{\pm}}{||\phi_{\pm}||^3}
\]
(68)

Substituting above equations into the solid angle density (64) we obtain

\[
\rho = \frac{1}{2} \varepsilon^{ABC} \varepsilon^{ijk} \partial_i (n^A_{\pm} \partial_j n^B_{\pm} \partial_k n^C_{\pm})
\]
\[= \frac{1}{2} \varepsilon^{ABC} \varepsilon^{ijk} \partial_i \frac{\phi^A_{\pm}}{||\phi_{\pm}||^3} \partial_j \phi^B_{\pm} \partial_k \phi^C_{\pm}
\]
\[= -\frac{1}{2} \varepsilon^{ABC} \varepsilon^{ijk} \frac{\partial}{\partial \phi^D_{\pm}} \frac{\partial}{\partial \phi^A_{\pm}} \left(\frac{1}{||\phi_{\pm}||}\right) \partial_i \phi^D_{\pm} \partial_j \phi^B_{\pm} \partial_k \phi^C_{\pm}.
\]
(69)

Define the Jacobian \(J(\frac{\phi_{\pm}}{u})\) as

\[
\varepsilon^{ABC} J(\frac{\phi_{\pm}}{u}) = \varepsilon^{ijk} \partial_i \phi^A_{\pm} \partial_j \phi^B_{\pm} \partial_k \phi^C_{\pm}.
\]
(70)

By making use of the Laplacian relation in \(\phi\)-space

\[
\partial_A \partial_A \frac{1}{||\phi_{\pm}||} = -4\pi \delta^3(\phi_{\pm}), \quad \partial_A = \frac{\partial}{\partial \phi^A_{\pm}},
\]
(71)

we can write the density of the solid angle as the \(\delta\)-like expression

\[
\rho_{\pm} = 4\pi \delta^3(\phi_{\pm}) J(\frac{\phi_{\pm}}{u})
\]
(72)
and\[\Omega_\pm = \int_V 4\pi \delta^3(\phi_\pm) J(\frac{\phi_\pm}{u}) d^3 u.\tag{73}\]

It obvious that $\rho_\pm$ are non-zero only when $\phi_\pm = 0$.  

Suppose that $\phi_A^\pm(x) (A = 1, 2, 3)$ possess $K_\pm$ isolated zeros, according to the deduction of Ref. 18 and the implicit function theorem,\textsuperscript{19} the solutions of $\phi_\pm(u^1, u^2, u^3, v) = 0$ can be expressed in terms of $u = (u^1, u^2, u^3)$ as
\[u^i = z^i_\pm(v), \quad i = 1, 2, 3\tag{74}\]

and
\[\phi_A^\pm(z^1_\pm(v), z^2_\pm(v), z^3_\pm(v), v) \equiv 0,\tag{75}\]

where the subscript $l = 1, 2, \cdots, K_\pm$ represents the $l$th zero of $\phi_A^\pm$, i.e.
\[\phi_A^\pm(z^i_{\pm l}) = 0, \quad l = 1, 2, \cdots, K_\pm; \quad A = 1, 2, 3.\tag{76}\]

It is easy to get the following formula from the ordinary theory of $\delta$-function that
\[\delta^3(\phi_\pm) J(\frac{\phi_\pm}{u}) = \sum_{i=1}^{K_\pm} \beta_\pm \eta_{\pm l} \delta^3(u - z_{\pm l})\tag{77}\]

in which
\[\eta_{\pm l} = \text{sign} J(\frac{\phi_\pm}{u})|_{x = z_{\pm l}} = \pm 1,\tag{78}\]

is the Brouwer degree of $\phi$-mapping and $\beta_{\pm l}$ are positive integers called the Hopf index of map $\phi_{\pm}$ which means while the point $x$ covers the region neighboring the zero $x = z_{\pm l}$ once, $\phi_\pm$ covers the corresponding region $\beta_{\pm l}$ times. Therefore the slid angle density becomes
\[\rho_\pm = 4\pi \sum_{l=1}^{K_\pm} \beta_\pm \eta_{\pm l} \delta^3(u - z_{\pm l})\tag{79}\]

and
\[\Omega_\pm = 4\pi \sum_{l=1}^{K_\pm} \beta_\pm \eta_{\pm l} \int_V \delta^3(u - z_{\pm l}) d^3 u = 4\pi \sum_{l=1}^{K_\pm} \beta_{\pm l} \eta_{\pm l}\tag{80}\]
Therefore the space-time disclination is

\[ \Omega = 4\pi \sum_{l=1}^{K_+} \beta_+ \eta_+ l + 4\pi \sum_{l=1}^{K_-} \beta_- \eta_- l \quad (81) \]

We find that (79) is the exact density of a system of \( K_+ \) and \( K_- \) classical point-like objects with “charge” \( \beta_+ \eta_+ l \) and \( \beta_- \eta_- l \) in space-time, i.e. the topological structure of disclinations formally corresponds to a point-like system. These point objects may be called disclination points as in nematic crystals.\(^{20}\) In Ref. 21, it was shown that the existence of disclination points is related to a kind of broken symmetries. The dislocations and disclinations appear as singularities of distortions of an order parameter.\(^{22}\) In our paper, the disclination points are identified with the isolated zero points of vector field \( \phi_\pm_A(x) \). From (72) we know that these singularities are those of the disclination density as well.

On another hand, the winding number \( W_\pm \) of the surface \( \Sigma \) and of the mapping \( \phi_\pm \) is defined as\(^{23}\)

\[ W_\pm = \oint_{\Sigma} \frac{1}{8\pi} \epsilon^{ABC} \frac{\phi_\pm^A}{||\phi_\pm||^3} d\phi_\pm^B \wedge d\phi_\pm^C \quad (82) \]

which is equal to the number of times \( \Sigma \) encloses (or, wraps around) the point \( \phi_\pm = 0 \). Hence, the space-time disclinations is quantized by the winding numbers

\[ \Omega_\pm = 4\pi W_\pm. \quad (83) \]

The winding number \( W_\pm \) of the surface \( \Sigma \) can be interpreted or, indeed, defined as the degree of the mappings \( \phi_\pm \) onto \( \Sigma \). By (62) and (72) we have

\[ \Omega_\pm = 4\pi \int_V \delta(\phi_\pm) J_\pm \left( \frac{\phi_\pm}{u} \right) d^3 u \]

\[ = 4\pi \deg \phi_\pm \int_{\phi_\pm(V)} \delta(\phi_\pm) d^3 \phi_\pm \]

\[ = 4\pi \deg \phi_\pm \quad (84) \]

where \( \deg \phi_\pm \) are the degrees of map \( \phi_\pm : V \rightarrow \phi_\pm(V) \). Compared above equation with (83), it shows the degrees of map \( \phi_\pm : V \rightarrow \phi_\pm(V) \) is just the winding number \( W_\pm \) of surface \( \Sigma \) and map \( \phi_\pm \), i.e.
\[ \deg \phi_\pm = W_\pm(\Sigma, \phi_\pm). \] (85)

Then the space-time disclination is
\[ \Omega = 4\pi(\deg \phi_+ + \deg \phi_-) = 4\pi(W_+ + W_-). \] (86)

Divide \( V \) by
\[ V = \sum_{l=1}^{K_\pm} V_{\pm l} \] (87)
and \( V_{\pm l} \) includes only one zero \( z_{\pm l} \) of \( \phi_\pm \), i.e. \( z_{\pm l} \in V_{\pm l} \). The winding number \( W_\pm \) of the surface \( \Sigma_{\pm l} = \partial V_{\pm l} \) and the mapping \( \phi_\pm \) is defined as
\[ W_{\pm l} = \oint_{\Sigma_{\pm l}} \frac{1}{8\pi} \epsilon^{ABC} \frac{\phi^A_\pm}{||\phi_\pm||^3} d\phi^B_\pm \land d\phi^C_\pm, \] (88)
which is equal to the number of times \( \Sigma_{\pm l} \) encloses (or, wraps around) the point \( \phi_\pm = 0 \). It is easy to see that
\[ W_\pm = \sum_{l=1}^{K_\pm} W_{\pm l} \] (89)
and
\[ |W_{\pm l}| = \beta_{\pm l}. \] (90)

Then
\[ \Omega = 4\pi \sum_{l=1}^{K_+} W_{+ l} + 4\pi \sum_{l=1}^{K_-} W_{- l}. \] (91)

According that is proved in Ref. 24, the increment
\[ \frac{1}{2} \epsilon^{ABC} \frac{\phi^A_\pm}{||\phi_\pm||^3} d\phi^B_\pm \land d\phi^C_\pm = \epsilon^{ABC} \epsilon_{ijk} \frac{\phi^A_\pm}{||\phi_\pm||^3} \partial_i \phi^B_\pm \partial_j \phi^C_\pm dS^k \] (92)
may be regarded as the solid angle on \( \phi_\pm(\Sigma_{\pm l}) \) subtended by the surface element \( dS \) on \( \Sigma_{\pm l} \).

Now it is easy to see why the space-time disclination is caused by inserting solid angles into the flat space-time.

Also we can write
\[ \Omega = 4\pi(N_+^+ + N_+^-) - 4\pi(N_-^+ + N_-^-) \] (93)

in which \(N_\pm^\pm\) are the sums of the Hopf indexes with respect to \(\eta_+ = \pm 1\) and \(N_\pm^-\) are the sums of the Hopf indexes with respect to \(\eta_- = \pm 1\). We see that while \(x\) covers \(V\) once, \(\phi_+^A\) must cover \(\phi_+(V) N_+^+\) times with \(\eta = 1\) or \(N_+^-\) times with \(\eta = -1\) while \(\phi_-^A\) must cover \(\phi_-(V) N_-^+\) times with \(\eta = 1\) or \(N_-^-\) times with \(\eta = -1\). Therefore the topological disclinations are distinguished by the sign of the Jacobian \(J(\phi_+/u)\) at the zero point \(x^\mu = z_\pm^\mu\). Consequently, the global topological property of disclinations is characterized by the Brouwer degrees and Hopf indices of each disclination point, which label the local structure of disclinations.

In this paper, we have studied the topological structure, global and local properties of space-time disclinations. The disclinations we discussed are the general 4-dimensional case, which is characterized by the so-called topological points. The decomposition of gauge potential, introduction of the gauge parallel tensor field \(n^{ab}(x)\) \((a, b = 1, 2, 3, 4)\) and the \(\phi\)-mapping method play important roles in establishing our theory. The disclination points are the zeros of the self-dual or anti-self-dual parts of the tensor field \(\phi^{ab}(x)\) and the topological characteristics of the space-time disclinations are determined by the winding numbers which can be represented by the Brouwer degrees and Hopf indices at the disclination points. From the definition of the space-time disclination (11), we know that the space-time disclination is expressed exactly by the curvature of space-time. Therefore the quantization of the space-time disclination has close relationship with the quantization of space-time.

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