Optimization of Flat to Round Transformers Using Adjoint Techniques

L. Dovlatyan*, B.L. Beaudoin, S. Bernal, I. Haber, D. Sutter, and T.M. Antonsen Jr.
(Institute For Research in Electronics and Applied Physics, University of Maryland, College Park, Maryland 20742, USA)
(Dated: February 16, 2021)

A continuous system of moment equations is introduced that models the transverse dynamics of a beam of charged particles as it passes through an arbitrary lattice of quadrupoles and solenoids in the presence of self-fields. Then, figures of merit are introduced specifying system characteristics to be optimized. The resulting model is used to optimize the parameters of the lattice elements of a flat to round transformer, as could be applied in relativistic electron cooling. Results are shown for a case of no space-charge and two cases with space-charge. The optimization is based on a gradient descent algorithm in which the gradient is calculated using adjoint methods that prove to be very computationally efficient. In the optimized configurations for each of the three cases, the beam in the solenoid is round and with uniform second moments.

I. INTRODUCTION

Beams of charged particles in accelerators are guided and manipulated by complex systems of magnets whose design optimization requires tracking the trajectories of the particles through the fields of the magnets as well as through the self-fields of the beam. Since the properties of the beam after passing through a lattice of magnetic focusing elements depend in a complicated way on the many parameters describing the lattice, optimizing the lattice is a computationally intensive task, and historically, a large body of knowledge and techniques for addressing the optimization problem have been developed.

Traditionally, the design of these magnet systems is carried out using computer codes that calculate the beam particle phase space trajectories in the prescribed lattice of magnets. The process often reduces to the optimization of figures of merit (FoMs) in the high dimensional parameter space characterizing the lattice. Because of the large number of parameters, the efficiency of the optimization algorithm is critical. One class of optimization algorithms, based on calculating the gradient of the FoM in parameter space, becomes computationally prohibitive if the gradient is to be calculated directly (by individually varying each of the parameters) in the high dimensional parameter space.

The number of computations needed to calculate the gradient can be reduced via the introduction of adjoint techniques [1–5]. In this approach an alternate, but related, mathematical problem is introduced in which in a single (or in several) computation(s) the linear dependence of the FoM on all the parameters can be determined. The adjoint approach has previously been applied in circuit theory [1], electromagnetics [2], aerodynamics [3], and accelerator physics [4, 5], as well as in other fields.

In this paper we will illustrate the application of the adjoint approach to the design of Flat-to-Round (FTR) or Round-to-Flat (RTF) transformers as have been proposed for use in relativistic electron cooling [6–9]. As the names suggest these transformers are systems of magnets that will convert an unmagnetized beam that has a high aspect ratio, elliptical spatial cross section, to a round beam in a solenoidal magnetic field, or vice versa. In its simplest form this conversion is accomplished with a triplet of quadrupoles, and a solenoid. See Fig. 1. Parameters that can be varied to optimize this conversion are the positions and strengths of the four magnet elements, including the orientations of the quadrupoles (11 parameters including the location of the first quadrupole). The method we will introduce also allows for more detailed optimization in terms of the spatial profiles of the magnet elements. In addition, one might have the option to vary the parameters of the incoming beam, but we will take these to be given. The description of the beam that we will use is based on the solution of second moment equations describing the four-dimensional transverse phase space of the beam. There are 10 such moments corresponding to the ten independent elements of the matrix of second moments, the so-called sigma matrix [10]. The effect of self-fields is included in the calculations under the assumption that the beam maintains an elliptical spatial cross section of uniform density. A future publication will extend the approach to a kinetic or phase space description. The FoM

* levondov@umd.edu

FIG. 1. General geometry of quadrupole triplet for FTR beam transformations.
will be a weighted sum of squares of nine combinations of the moments at the exit of the transformer. Each of these nine combinations ideally should vanish in a successful transformation. We will find that combining the adjoint calculation with a gradient descent algorithm allows us to effectively satisfy all nine conditions.

We chose FTR and RTF transformers as an example because of their multiple applications. The original motivation for the development of circular mode adapters, introduced by Derbenev in 1998 [6], was to improve the rates and efficiency of relativistic electron cooling [6–9] of hadron beams. Here a high energy electron beam and a hadron beam are made to co-propagate during which the beams interact and the hadrons are cooled. High energy electron beam sources are generally electron storage devices, in which the beams tend to become flat through radiation effects and intra-beam scattering. The cooling of the hadron beam is optimized when the electron beam interacting with the hadron beam is magnetized and matches the hadron beam size and shape, which is approximately round, and has a low effective transverse temperature [11]. The adapters provide the means for converting an unmagnetized beam, to the “magnetized” state in the cooling solenoid and then converting back to the unmagnetized state at the exit of the solenoid for reinjection into the storage device.

Almost immediately after Derbenev’s initial proposal, it was realized that an RTF adapter following an electron source immersed in a solenoid field could be used to produce a flat, uncoupled beam of high enough emittance ratio and charge to potentially replace the high energy electron damping rings required in proposed designs for high energy linear colliders [12]. Experiments at Fermilab and DESY confirmed the feasibility [13–15]. Since then, the possible use of circular mode adapters for phase space manipulations have been expanded to include generating round beams at select points in circular colliders to compensate for beam-beam effects at high energy and to offset the effects of space-charge and the cyclotron component of the ion beam in low energy cooler rings [9]. Additionally, their use has been explored for injection into high intensity X-ray Sources [16, 17] or into dielectric laser driven accelerators [18, 19].

The FTR and RTF beam transformations typically involve a solenoid and three, 45-degree skew quadrupoles, as depicted in Fig. 1. The overall length of the triplet, excluding the solenoid, scales inversely with the strength of the solenoidal field and linearly with the momentum of the beam, as reflected in the reference length

$$\beta_s = \frac{2p}{|qB|}$$ (1)

where $p$ is the momentum, $q$ the charge, and $B$ the maximum solenoidal field. In symmetrical triplets, the simplest configuration, the two outer quadrupoles have the same strength and polarity and are equidistant from the central one with opposite polarity. In asymmetric triplets [8, 9], the two outer quadrupoles have differing strengths and different separations from the inner quadrupole.

Using a thin lens model for the symmetric triplet, the distance $d$ between the outer thin quadrupoles $Q_1$, $Q_3$ and the central one $Q_2$, and the inverse focal lengths, $q_1 = q_3$, $q_2$, are, following Wolske, all given in terms of $\beta_s$ [20]:

$$d = \frac{\beta_s}{2\sqrt{1 + \sqrt{2}}}, q_1 = q_3 = -\frac{\sqrt{2} + 1}{\beta_s}, q_2 = \frac{2\sqrt{2}}{\beta_s}$$ (2)

In this design, the flat beam (for FTR transformer) is incident directly on the first quadrupole which is located at $z = 0$, i.e., no drift space is assumed between the input beam and the first quadrupole and between the last quadrupole and the solenoid ($s_1 = s_4 = 0$ in Fig 1). More general equations, ascribed to Edwards [21, 22], still apply to thin lenses and no initial/final drift spaces, but relax the condition of symmetry, i.e., $q_1 \neq q_3$, $s_2 \neq s_3$. For the triplet described by Eq. (2) an initial horizontal flat beam is transformed into a round beam with a canonical angular momentum of opposite sign to the canonical angular momentum term generated by the solenoid on transport to its center.

As an example, Burov and Danilov [7] described a symmetrical FTR triplet for an electron energy of 500 MeV and solenoid field strength $B_s = 10.3$ kG; the triplet length is 2.23 m. Any initial triplet thin lens design calculation can use Eq. (2), following a choice of the $\beta_s$ parameter in Eq. (1). Alternatively, it is possible to scale existing designs, as in references [7] or [8], to a desired electron energy and solenoid field. We will adapt as a starting point for the test model to be used in the following scaled version of the Burov-Danilov symmetrical FTR triplet [7], which includes non-zero drift spaces $s_1 = s_4$ (Fig. 1), an electron energy of 5 keV and a solenoid field of 15 G ($\beta_s = 0.319$ m). The resulting overall length of the triplet is 0.213 m. From these initial parameters it is straight forward to obtain a thick lens model that provides the initial conditions for the adjoint optimizations.

The initial FTR optical parameters also provide a computation of the initial beam second moments. If $M_{st}$ is the matrix that transports an initial flat beam through the rotated triplet to the center of the solenoid, then for an initial matrix, $\sum_0$, the sigma matrix at the center of the solenoid is

$$\sum = M_{st} \sum_0 M_{st}^T$$ (3)

where $M_{st}^T$ is the transpose [10, 23]. The generalized second moment equations derived below for the use in the adjoint optimization process relate directly to the elements of $\sum$.

The treatments discussed so far assume no space-charge or self-field effects. In Sec. II we will introduce a general formulation based on a set of (10) differential equations for the evolution with distance of the second moments of the beam particle distribution function. The system allows for the profiles of the magnetic fields of the
focusing elements to be treated; it allows for the orientations, locations and the strength of the quadrupoles to be varied; and it treats self-fields in the approximation that the beam maintains an elliptical transverse cross section of constant density. We will also introduce a system of adjoint equations that allows for the efficient calculation of the change in system performance with changing parameters. In Sec. III we will use the adjoint calculation of the gradient in parameter space to optimize the parameters of a selected model of an FTR transformer under a variety of assumptions and constraints.

II. BASIC MODEL

In this section we first present a system of equations that describes the evolution with distance of the second moments of a charged particle beam distribution in the presence of a combination of transverse forces. These forces include the Lorentz force of a spatially varying solenoidal (axial) magnetic field, the Lorentz force of a superposition of arbitrarily oriented quadrupole magnetic fields, and the electric and magnetic self-force due to the beam’s charge and current densities. These moment equations will be used to simulate the propagation of a beam through a system of magnets, which converts a beam with an elliptical cross section to one with a round cross section. A general figure of merit will be introduced that quantifies how successfully the shape conversion has been made. Subsequently, we will formally perturb this system by making small changes in the parameters defining the focusing forces and assess the changes by evaluating the change in the figure of merit. Such an evaluation would be used in a gradient-based optimization scheme. Then we will introduce an adjoint system of equations that will allow one to calculate compactly the changes in the system due to changes in the focusing parameters.

The moments we consider are averages of products of transverse particle displacements in a Cartesian coordinate system in the lab frame, of motion in which transverse forces are linear in the particles’ displacements from the axis, or are linearly proportional to a particle’s transverse velocity. The final equations will describe the beam’s evolution in the Larmor, or rotating, frame defined by the applied solenoidal magnetic field. We begin the calculation in the lab frame, transform to the Larmor frame, and complete the derivation in the Larmor frame. Beam quantities in the lab frame can always be recovered from the Larmor frame values by application of rotation transformations. These are given in Appendix B.

To start, we write equations for the evolution of the transverse particle displacements in a Cartesian coordinate system in the lab frame,

\[ x_1'' = k_x + k_\Omega y_1' + \frac{1}{2} k_z y_1, \]
\[ y_1'' = k_y - k_\Omega x_1' - \frac{1}{2} k_z x_1, \]

Here \((x_1, y_1)\) are a particle’s transverse displacements in the lab frame, for example, \(x_1\) is the horizontal displacement from the center-line and \(y_1\) is the vertical displacement. A prime denotes differentiation with respect to the axial coordinate \(z\), which measures distance along the center-line. The quantity

\[ k_\Omega(z) = \frac{q B_z(z)}{mc^2 \nu_z}, \]

is the spatial gyration rate due to the axial (solenoidal) magnetic field. The quantities in Eq. (5) are as follows: \(q\) and \(m\) are the particle charge and mass, \(B_z\) is the axial component of the solenoidal magnetic field strength, \(c\) is the speed of light, \(\nu_z\) is the axial velocity and \(\gamma = (1 - \nu_z^2/c^2)^{-1/2}\). (If the beam consists of electrons, \(q = -e\), and if \(B_z > 0\) then \(k_\Omega < 0\).) The quantities \(k_x, k_y\) represent the transverse forces from the quadrupoles and space-charge, and will be defined subsequently. The second terms in Eq. (4a) and (4b) describe the component due to the axial component of the solenoidal field and the transverse velocity, and the third terms describe the Lorentz force due to the radial component of the solenoidal field and the axial velocity. Our next step is to transform the variables to their Cartesian representation in a rotating frame,

\[ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]

where \(\phi(z)\) is an axially dependent rotation phase that we choose to satisfy

\[ \phi' = -\frac{k_\Omega}{2} \]

and which defines the Larmor frame. Substituting Eqs. (6) and (7) into (4a),(4b) and multiplying by the inverse of the rotation matrix appearing in Eq. (6) results in the system

\[ \begin{pmatrix} x'' \\ y'' \end{pmatrix} + \left( \frac{k_\Omega}{2} \right)^2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} k_x \\ k_y \end{pmatrix} \]

of equations that include self-field effects and spatial profiles of focusing fields.

The underlying assumption that will be made in deriving the moment equations is that the beam particles’ trajectories are well described by the paraxial equations
We wish to obtain equations for the average of products of the four variables \((x, x', y, y')\). There are 16 ordered products. However, order does not matter, leaving 10 independent products. We choose combinations of these products that distinguish \(x-y\) symmetric and non-symmetric motion. For spatial moments we choose

\[
\mathbf{Q} = \begin{pmatrix} Q_+ \\ Q_- \\ Q_z \end{pmatrix} = \begin{pmatrix} \langle x^2 + y^2 \rangle/2 \\ \langle x^2 - y^2 \rangle/2 \\ \langle xy \rangle \end{pmatrix}
\]

(9)

Here the angle brackets imply average over the beam distribution function. Accompanying these spatial moments are momentum like moments

\[
\mathbf{P} = \frac{d}{dz} \mathbf{Q} = \begin{pmatrix} P_+ \\ P_- \\ P_z \end{pmatrix} = \begin{pmatrix} \langle xx' + yy' \rangle \\ \langle xx' - yy' \rangle \\ \langle yy' + xx' \rangle \end{pmatrix}
\]

(10)

The angular momentum completes this group of momentum like moments,

\[
L = \langle xy' - yx' \rangle
\]

(11)

Note, that \(L\) is the angular momentum in the rotating frame. The group of 10 moments is completed by three energy-like moments

\[
\mathbf{E} = \begin{pmatrix} E_+ \\ E_- \\ E_z \end{pmatrix} = \begin{pmatrix} \langle x^2 + y^2 \rangle/2 \\ \langle x^2 - y^2 \rangle/2 \\ 2\langle y'x' \rangle \end{pmatrix}
\]

(12)

The evolution of these moments is determined by the following system of equations

\[
\frac{d}{dz} \mathbf{Q} = \mathbf{P},
\]

\[
\frac{d}{dz} \mathbf{P} = \mathbf{E} + \mathbf{O} \cdot \mathbf{Q},
\]

\[
\frac{d}{dz} \mathbf{E} = \mathbf{O} \cdot \mathbf{P} + \mathbf{N} \mathbf{L},
\]

\[
\frac{d}{dz} \mathbf{L} = -\mathbf{N}^\dagger \cdot \mathbf{Q}
\]

(13a-d)

Here the matrix \(\mathbf{O}\) and vector \(\mathbf{N}\) are defined as follows

\[
\mathbf{O} = -\frac{k^2}{2} \frac{\Omega^2}{\Delta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2 \sum_{\text{quads}} K_q \begin{pmatrix} 0 & c_q & -s_q \\ c_q & 0 & 0 \\ -s_q & 0 & 0 \end{pmatrix} + \frac{\Lambda}{Q \Delta} \begin{pmatrix} 1 & c_{qz} & s_{qz} \\ c_{qz} & 1 & 0 \\ -s_{qz} & 0 & 1 \end{pmatrix}
\]

(14)

\[
\mathbf{N} = 2 \sum_{\text{quads}} K_q \begin{pmatrix} 0 \\ s_q \\ c_q \end{pmatrix} - \frac{\Lambda}{Q \Delta} \begin{pmatrix} 0 \\ s_{qz} \\ -c_{qz} \end{pmatrix}
\]

(15)

As shown in Appendix A, this leads to the displayed contributions to the \(\mathbf{O}\) matrix and \(\mathbf{N}\) vector where

\[
K_q(z) = \frac{q B_q'(z)}{mc^2 \gamma^3 z}
\]

(17)

If \(K_q > 0\) the magnet will be defocusing in the lab frame when \(\theta = \psi_q, \psi_q + \pi\), and focusing when \(\theta = \psi_q + \pi/2, \psi_q - \pi/2\). Here, \(\theta\) is the angle a particle’s transverse displacement makes with respect to the \(x_1\) axis. If \(K_q < 0\) these are reversed. In the Larmor frame the orientation of the quadrupole is characterized by the variables

\[
(s_q, c_q) = (\sin (2\phi - 2\psi_q), \cos (2\phi - 2\psi_q))
\]

(18)

where \(\phi(z)\) is the angle resulting from the integration of Eq. (5). We note that with one group of quadrupoles placed separately from the solenoid it is always possible to define the phase such that it vanishes in the region of the quadrupoles.

The self-field term is calculated in the Larmor frame in Appendix B. Here it is assumed that the beam charge density distribution is spatially uniform inside an ellipse.
with differing major and minor radii and tilted at an angle in the Larmor frame. The values of the major and minor radii and the angle are determined by the three spatial moments $(Q_+, Q_-, Q_x)$. The result is the following

$$Q_\Delta = \left[Q_+^2 - (Q_-^2 + Q_x^2)\right]^{1/2},$$  \hspace{1cm} (19a)

$$c_{\alpha} = -\frac{Q_-}{(Q_+ + Q_\Delta)},$$  \hspace{1cm} (19b)

$$s_{\alpha} = -\frac{Q_x}{(Q_+ + Q_\Delta)}.$$  \hspace{1cm} (19c)

The strength of the self-fields is measured by the beam current parameter

$$\Lambda = \frac{eqZ_0I}{4\pi m^2\gamma^3 z_0^3}$$  \hspace{1cm} (20)

with $Z_0 = 377$ Ohms, and $I$ is the beam current.

The equations (13a) - (13d) can be combined to show the following conservation relation

$$\frac{d}{dz} \left[ E \cdot Q + \frac{L^2}{2} - \frac{1}{2} P \cdot P \right] = 0$$  \hspace{1cm} (21)

This does not represent the conservation of a four-dimensional emittance. Rather it is related to the conserved quantity discussed in [24].

We compare the solution of the moment equations with the moments calculated by the PIC code Warp [25] in Fig. 2. For this simulation we consider a symmetric triplet with parameters selected to convert a flat beam to a round one in the absence of space-charge. Parameters for the simulation are given in Table I. The large pipe radius in the list of parameters is chosen to minimize the contributions from image charge forces on the simulations in order to compare the results with the moment equations.

Plotted in Fig. 2 are the computed values of $x_{\text{rms}}^2 = \langle x^2 \rangle = Q_+ + Q_-$ and $y_{\text{rms}}^2 = \langle y^2 \rangle = Q_+ - Q_-$ as functions of $z$ for three values of beam current. The solutions of the moment equations are shown as open circles, and the PIC values are shown as solid and dashed lines. The initial distribution function in the PIC case is taken to be a K-V [26] distribution function. As can be seen, in the zero current case the beam becomes round at the end of the triplet, whereas in the presence of self-fields the roundness is spoiled. In the next section we will show how the optimizations of the triplet parameters can recover the round state of the beam.

We now imagine that Eqs. (13a) - (13d) have been solved for a given set of initial conditions, $(Q, P, E, L) \mid_{z_i}$, and magnetic field parameters and profiles. An assessment of the configuration can be made on the basis of a figure of merit that depends on the final values of the moments,

$$F(Q, P, E, L) \mid_{z_f}$$  \hspace{1cm} (22)

For a flat to round transition, the figure of merit might be a function which when minimized forces the components $Q_x, Q_-$ and $P$ all to zero, signifying the beam is round and its radius will remain constant. This will be discussed in more detail in the next section. We refer to a general solution just described (not necessarily the minimizing one) as a base case.

If a small change is made in the focusing parameters, there will be a commensurately small change in the moments as functions of $z$, and a small change in the figure of merit from the corresponding base case. The small changes in the moments will satisfy a linearization of Eqs. (13) - (15). We denote the small changes with a superscript $(X)$, and we refer to them as true changes.

### Table I. PIC code simulation and FTR lattice parameters.

| Parameter          | Value     |
|--------------------|-----------|
| Beam distribution  | K-V       |
| Particle number    | 40000     |
| Longitudinal step-size | 0.1 (mm)  |
| Grid cell number   | 2048 x 2048 |
| Pipe radius        | 250.0 (mm) |
| Beam energy        | 5 keV     |
| Quadrupole 1 location | 0.0043 (m) |
| Quadrupole 2 location | 0.1066 (m) |
| Quadrupole 3 location | 0.2090 (m) |
| Quadrupole 1,3 strength | -18.236 (T/m) |
| Quadrupole 2 strength | 21.364 (T/m) |
| Quadrupole 1,2,3 length | 0.1 (mm)  |
| Solenoid location  | 0.2133    |
| Solenoid strength  | 15e-4 (T) |
| Initial $\beta_x = \beta_y$ | 0.3187 (m) |
| Initial $\alpha_x = \alpha_y$ | 0         |
| Initial rms $\epsilon_x, \epsilon_y$ | 16.0.16 (mm-mrad) |
These satisfy
\[ \frac{d}{dz} \delta Q^{(X)} = \delta P^{(X)} \]  
(23a)

\[ \frac{d}{dz} \delta P^{(X)} = \delta E^{(X)} + O \cdot \delta Q^{(X)} + \delta O^{(X)} \cdot Q \]  
(23b)

\[ \frac{d}{dz} \delta E^{(X)} = O \cdot \delta P^{(X)} + N \delta L^{(X)} + \delta O^{(X)} \cdot P + \delta N^{(X)} L \]  
(23c)

\[ \frac{d}{dz} \delta L^{(X)} = -N^\dagger \cdot \delta Q^{(X)} - \delta N^\dagger L \cdot Q \]  
(23d)

The resulting change in the figure of merit is
\[ \delta F^{(X)} = \left[ \delta Q^{(X)} \frac{\partial F}{\partial Q} + \delta P^{(X)} \frac{\partial F}{\partial P} + \delta E^{(X)} \frac{\partial F}{\partial E} + \delta L^{(X)} \frac{\partial F}{\partial L} \right]_{\epsilon} \]  
(24)

Again, the superscript \((X)\) signifies that it is a true perturbation, i.e. the result of changing the focusing system: the solenoidal and quadrupole magnetic fields.

The perturbed matrices \(O^{(X)}, N^{(X)}\) appearing in Eqs. (23b) - (23d) are the result of perturbations to matrices \(O\) and \(N\). These matrices are perturbed due to two effects. One effect is the set of perturbations due to changes in the solenoidal and quadrupole magnetic fields that appear explicitly in the definitions of \(O\) and \(N\). These are being varied in order to optimize the configuration.

A second effect is due to changes in the self-field terms. These changes are expressed through the changes in the spatial moments, \(\delta Q^{(X)}\) in the definitions of Eqs. (19a) - (19c). We write the changes to these matrices as a sum of contributions from explicit changes in the focusing configuration and implicit changes in the self-fields,

\[ \delta O^{(X)} \cdot Q = \delta O^{(X)}_{Q,B} \cdot Q + M_Q \cdot \delta Q^{(X)} \]  
(25a)

\[ \delta O^{(X)} \cdot P = \delta O^{(X)}_{P,B} \cdot P + M_P \cdot \delta Q^{(X)} \]  
(25b)

\[ \delta N^{(X)} = \delta N^{(X)}_{Q,B} + M_N \cdot \delta Q^{(X)} \]  
(25c)

Here, the variables with subscripts \(Q, B\) are the contributions to the changes from the explicit dependence of the matrices on the magnetic focusing parameters. The matrices \(M_Q, M_P, M_N\) are the contributions from the self-field terms in the matrices \(M, N\), which depend on \(Q\) as given by Eqs. (14), (15), and (19). Formulas for these matrices are given in Appendix B.

Changing each parameter will result in a change to the figure of merit. This can be thought of as defining a gradient of the figure of merit in parameter space. Direct evaluation of the change in the figure of merit would require resolving the system of 10 coupled equations \(M_p\) times where \(M_p\) is the number of parameters that could be varied. We can avoid this burden by introducing an adjoint system of equations. Consider a second linear perturbation to the base case, denoted by a superscript \((Y)\). These are the adjoint perturbation equations that we will actually solve.

\[ \frac{d}{dz} \delta Q^{(Y)} = \delta P^{(Y)} \]  
(26a)

\[ \frac{d}{dz} \delta P^{(Y)} = \delta E^{(Y)} + O \cdot \delta Q^{(Y)} \]  
(26b)

\[ \frac{d}{dz} \delta E^{(Y)} = O \cdot \delta P^{(Y)} + N \delta L^{(Y)} + \delta E^{(Y)} \]  
(26c)

\[ \frac{d}{dz} \delta L = -N^\dagger \cdot \delta Q^{(Y)} \]  
(26d)

In Eqs. (26a) - (26d) we use the base case (unperturbed) \(O\) and \(N\) matrices. However, we have added a term in Eq. (26c), \(\delta E^{(Y)}\), which will be chosen, as described subsequently, so as to achieve a desired cancellation of the self-field contributions to the changes in \(O\) and \(N\).

We next form the following combination of true \((X)\) and adjoint \((Y)\) variables,

\[ \epsilon \equiv \delta P^{(Y)} \cdot \delta P^{(X)} - \delta Q^{(X)} \cdot \delta E^{(Y)} - \delta Q^{(X)} \cdot \delta E^{(Y)} - \delta L^{(Y)} \delta L^{(X)} \]  
(27)

differentiate it with respect to \(z\). We evaluate the individual terms in the derivative of Eq. (27) using the product rule and Eqs. (23) and (26). Noting numerous cancellations we arrive at

\[ \frac{d}{dz} \epsilon = \delta P^{(Y)} \cdot \delta Q^{(X)} + \delta L^{(Y)} \delta N^{(X)} \cdot Q + \delta Q^{(Y)} \cdot \delta M \cdot \delta N^{(X)} \cdot \delta P + \delta Q^{(Y)} \cdot \delta N^{(X)} \cdot \delta L \]  
(28)

Let us now consider the quantities \(\delta N^{(X)}\) and \(\delta O^{(X)}\) appearing in Eq. (28). We write them according to the separation defined in Eq. (23). We then have

\[ \frac{d}{dz} \epsilon = \delta P^{(Y)} \cdot M_Q \cdot \delta Q^{(X)} + \delta P^{(Y)} \cdot \delta O^{(X)}_{Q,B} \cdot Q - \delta Q^{(X)} \cdot \delta E^{(Y)} + \delta L^{(Y)} \cdot M_N \cdot \delta Q^{(X)} \]  
\[ + \delta L^{(Y)} \cdot \delta N^{(X)}_{Q,B} - \delta Q^{(Y)} \cdot \delta O^{(X)}_{Q,B} \cdot P - \delta Q^{(Y)} \cdot M_P \cdot \delta Q^{(X)} - \delta Q^{(Y)} \cdot \delta N^{(X)}_{Q,B} \cdot L - \delta Q^{(Y)} \cdot M_N \cdot \delta Q^{(X)} \cdot L \]  
(29)

The next step is to pick \(\delta E^{(Y)}\) to cancel all terms proportional to the unknown \(\delta Q^{(X)}\) in Eq. (29),

\[ \delta E^{(Y)} = \delta P^{(Y)} \cdot M_Q + \delta L^{(Y)} \cdot Q \cdot M_N - \delta Q^{(Y)} \cdot M_P - \delta Q^{(Y)} \cdot M_N L \]  
(30)

This relation is taken to define the quantity \(\delta E^{(Y)}\) that enters the adjoint equations in Eq. (26c). This leaves for
the adjoint relation

\[
\frac{d}{dz} = \delta P^{(Y)} \cdot \delta O_{Q,B}^{(X)} \cdot Q + \delta L^{(Y)} Q \cdot \delta N_{Q,B}^{(X)} \\
- \delta Q^{(Y)} \cdot \delta O_{Q,B}^{(X)} \cdot P - \delta Q^{(Y)} \cdot \delta N_{Q,B}^{(X)} L 
\]

(31)

Integrating over \( z \) from initial to final point

\[
\left. \left( \delta P^{(Y)} \cdot \delta P^{(X)} - \delta Q^{(Y)} \cdot \delta Q^{(X)} - \delta E^{(Y)} - \delta E^{(X)} - \delta L^{(Y)} \delta L^{(X)} \right) \right|_{z=z_i}^{z=z_f} = \int_{z_i}^{z_f} dz \left\{ \delta P^{(Y)} \cdot \delta O_{Q,B}^{(X)} \cdot Q + \delta L^{(Y)} Q \cdot \delta N_{Q,B}^{(X)} \\
- \delta Q^{(Y)} \cdot \delta O_{Q,B}^{(X)} \cdot P - \delta Q^{(Y)} \cdot \delta N_{Q,B}^{(X)} L \right\}
\]

(32)

We are now in position to describe the utility of the adjoint approach. Suppose we integrate the base case equations, Eqs. (13a) - (13d), forward from \( z_i \) to \( z_f \), and we then integrate the adjoint equations, Eqs. (26a) - (26d), backward in \( z \), starting at \( z_f \). Further, suppose we take the conditions on the adjoint variables at \( z_f \) to be

\[
\delta P^{(Y)}(z_f) = \left. \frac{\partial F}{\partial P} \right|_{z=z_f}, \quad (33a) \\
-\delta E^{(Y)}(z_f) = \left. \frac{\partial F}{\partial Q} \right|_{z=z_f}, \quad (33b) \\
-\delta Q^{(Y)}(z_f) = \left. \frac{\partial F}{\partial E} \right|_{z=z_f}, \quad (33c) \\
\delta L^{(Y)}(z_f) = \left. \frac{\partial F}{\partial E} \right|_{z=z_f} \quad (33d)
\]

where \( F \) is the figure of merit. Then Eqs. (24) and (32) give for the change in the figure of merit due to perturbation \( (X) \)

\[
\delta F^{(X)} = \left. \left( \delta P^{(Y)} \cdot \delta P^{(X)} - \delta Q^{(Y)} \cdot \delta Q^{(X)} - \delta E^{(Y)} \right. \right|_{z=z_i}^{z=z_f} \\
+ \int_{z_i}^{z_f} dz \left\{ \delta P^{(Y)} \cdot \delta O_{Q,B}^{(X)} \cdot Q + \delta L^{(Y)} Q \cdot \delta N_{Q,B}^{(X)} \right\} \\
+ \int_{z_i}^{z_f} dz \left\{ - \delta Q^{(Y)} \cdot \delta O_{Q,B}^{(X)} \cdot P - \delta Q^{(Y)} \cdot \delta N_{Q,B}^{(X)} L \right\}
\]

(34)

Equation (34) can now be used to evaluate the changes in the figure of merit (FoM) \( F \) due to arbitrary, small changes in the initial conditions of the true solution as contained in the first term on the right of Eq. (34). Or it can be used to evaluate changes in the FoM \( F \) due to arbitrary, small changes in the strength or profile of the focusing magnetic fields as contained in the integral term on the right of Eq. (34). These changes can be evaluated by a single integral once the adjoint solution is found without having to resolve the coupled system of 10 equations for each possible change in parameters.

We illustrate the utility of the adjoint method in calculating the gradient of an FoM with respect to variations of parameters in Fig. 3. Here we show the dependence on parameters of a particular FoM that will be introduced.
in the next section. The dependence on parameter values is calculated two ways: first by directly varying the parameter and second by using Eq. (34) to compute the gradient. The top plot in Fig. 3 shows the variation of the FoM with respect to changes in the strengths of the three quadrupoles, while the bottom plot in Fig. 3 shows the dependence on the solenoidal field. The solid straight lines have slopes that are predicted by the adjoint method, while the dashed curved lines show the effect of directly varying the parameter and plotting the change in the FoM. As can be seen, the solid lines are tangent to the dashed lines for vanishing perturbations indicating that the gradient has been calculated correctly. This method of calculating the gradient will be used in the next section to optimize a FTR transformer.

III. OPTIMIZATION

The goal of the optimization is to minimize a given FoM based on the values of the moments at a location \( z = z_f \) inside the solenoid using the adjoint approach. We choose the following function of the beam moments to be our FoM

\[
F = \frac{1}{2} \left[ |P|^2 + k_0^2 (Q_x^2 + Q_y^2) + k_0^{-2} (E_x^2 + E_y^2) \right] k_0^{-2} \left( E_+ - \frac{1}{2} k_0^2 Q_x + \Lambda \right)^2 + (2E_+ Q_x - L^2)^2 \right] (35)
\]

It is a sum of terms quadratic in the moments, each of which should be as small as possible in the optimum FTR transformer. The quantity \( k_0 \) is introduced so that each term has the same units. Here \( k_0 \) is a scaling parameter approximately equal to the inverse of the lattice length. This makes all the terms roughly comparable in magnitude. The choice of each term is made as follows. We would like the beam in the solenoid to be round, \( Q_+ = Q_x = 0 \), and we would like all the nonzero moments to be independent of \( z \). This implies for the spatial moments \( dQ/dz = P \) = 0. If we set the components of \( dP/dz \) = 0, we find \( E_- = E_x = 0 \) is required as well as \( E_+ - k_0^2 Q_x + 2 + \Lambda = 0 \). Finally, the last term is designed to force the trajectories to be as laminar as possible and the rotation to be rigid in the solenoid. It can be motivated as follows. If in the solenoid particles have a radially varying mean rotation rate \( \Omega(r) \) in the Larmor frame then

\[
E_{+0} = \langle \delta x'^2 + \delta y'^2 \rangle + (\Omega_r r^2) (36)
\]

where \( \delta x', \delta y' \) are deviations from the mean rotation. The angular momentum in this case is \( L = \langle \Omega r^2 \rangle \). We thus have, by virtue of the Schwarz inequality

\[
E_+ \geq \langle \Omega r^2 \rangle \geq \langle \Omega r^2 \rangle^2/(r^2) = L^2/(2Q_+) \quad (37)
\]

The second inequality becomes an equality if the rotation is rigid.

The FoM in Eq. (35) is a general function in that it meets all requirements for the FTR transformation. In practice an FoM will also contain extra constraints pertinent to the system being optimized. This can include beam size limits based on a physical aperture size, magnet sizes and locations based on available beamline space, etc. The extra constraints can be incorporated within the FoM or as constraints on the input parameters used for the optimization.

We start with a design based on the symmetric triplet with parameters given by Eqs. (1) and (2). It can be shown from analytic solutions of the moment equations in the thin lens approximations that the entrance and exit conditions in Table II can be achieved for a symmetric triplet oriented at 45 degrees with respect to the long dimension of the incident flat beam. The strengths of the quadrupoles needed to achieve these parameters are given by

\[
d \int_{-\infty}^{\infty} 2K_{1,3}(z)s_{1,3}dz = -\sqrt{1 + \sqrt{2}}
\]

\[
d \int_{-\infty}^{\infty} 2K_{2}(z)s_{2}dz = 2\sqrt{2}/\sqrt{1 + \sqrt{2}} \quad (38)
\]

The values given in Eq. (38) are in agreement with those expressed in Eq. (2). It is interesting to note that for the 45-degree triplet in the absence of space-charge, the 10 moments break into three independent groups, each with a conserved quantity

\[
E_- Q_- + L^2/2 - P_-^2/2 = J_1
\]

\[
(E_+ \pm E_x)(Q_- \pm Q_x) - (P_+ \pm P_x)^2/2 = J_\pm \quad (39)
\]

Also note there are seven quantities which vanish at the exit but only four adjustable parameters, the values of \( E_+ \) and the strengths of the inner and outer quadrupoles displayed in Eq. (38). The ability to satisfy what seems

| \( z = 0 \) | \( z = z_f = 2d \) |
|----------------|----------------|
| \( Q_+ = Q_+(0) \) | \( Q_+ = Q_+(0) \) |
| \( Q_- = Q_-(0) \) | \( Q_- = 0 \) |
| \( Q_x = 0 \) | \( Q_x = 0 \) |
| \( P_+ = 0 \) | \( P_+ = 0 \) |
| \( P_- = 0 \) | \( P_- = 0 \) |
| \( P_x = 0 \) | \( P_x = 0 \) |
| \( E_+ = Q_+(0) \sqrt{d^2(2 + 2\sqrt{2})} \) | \( E_- = Q_-(0) \sqrt{d^2(2 + 2\sqrt{2})} \) |
| \( E_+ = 0 \) | \( E_- = 0 \) |
| \( L = 0 \) | \( L = 2Q_-(0) \sqrt{d\sqrt{1 + \sqrt{2}}} \) |
like three extra conditions then results from the symmetry leading to Eq. (39).

We now optimize the parameters describing the magnetic fields by using the FoM in Eq. (35). We consider the FoM to be a function of the elements of a list of parameters $a$. The elements of the list include the strengths, locations, and orientations of the quadrupoles as well as the strength and location of the solenoid. We use a simple steepest descent algorithm in which we calculate the gradient of $F$ in the space of parameters $\mathbf{a}$, $\nabla_a F(\mathbf{a})$ and then adjust the values of the parameters by moving from the current set of parameters along the line of steepest descent.

To calculate the gradient we set the conditions at $z = z_f$ for the adjoint variables in Eqs. (33a) - (33d) according to the current values of the parameters, $a_n$. This then allows the solution for the adjoint variables and the integration in Eq. (34) to be carried out in order to find the variation in the FoM due to a perturbation in any of the focusing magnets in the system (contained within the $\mathbf{O}$ and $\mathbf{N}$ matrices). We then update the parameters according to

$$a_{n+1} = a_n - \gamma \nabla_a F(\mathbf{a}) \bigg|_{a_n}$$

The step size $\gamma$ is adjusted iteratively according to a simple algorithm based on successive values of $F(a_{n+1})$. Once the values of the parameters $a_{n+1}$ are determined the gradient is recomputed and Eq. (40) is reapplied. The procedure is repeated until the FoM stops decreasing.

A termination criterion is used in cases where the FoM does not reach a minimum fast enough. Meaning if the improvement in the FoM drops below a certain threshold, then the optimization terminates. A relative tolerance level of around $10^{-7}$ is the cut-off point in the optimizations. At this level the trade-off in computation time for further reduction of FoM is too great. In our case the threshold is good enough for the lattice configurations being investigated. Depending on the lattice accuracy requirements for various experimental setups, the relative tolerance thresholds would need to be adjusted accordingly.

As a first example of the technique, three optimizations are run to optimize a FTR lattice in the presence of self-fields. A 5 keV beam is used with 0 mA (no self-fields), 1 mA, and 5 mA beam currents. For these currents the self-field parameter takes on values $\Lambda = 0, 2.13e^{-5},$ and $1.06e^{-4}$. The importance of self-fields is estimated from Eq. (20) to be quantified by the value of $d^2\mathcal{A}/Q_z(0) = 0, 0.104,$ and $0.518$. The moment values in Table III are used to create a horizontally flat beam with an initial emittance ratio of 100/1.

The lattice consists of a quadrupole triplet with a solenoid at the end. A set of 11 input parameters are tuned for the optimization: the positions and magnet strengths of the three quadrupoles and solenoid along with the rotation angle of each quadrupole. An initial guess at parameters for the 0 mA case is used as the starting values for the optimization. The final optimized results for 0 mA run are then used as initial values for

FIG. 4. RMS beam size, $\langle x^2 \rangle$, $\langle y^2 \rangle$, plotted as a function of the longitudinal coordinate through a FTR transformer. Top row shows the initial results with no optimization. Bottom row shows the optimized results. Scaled quadrupole ($K_q$) and solenoid ($k_\Omega$) field profiles are also displayed for reference. Results are for a 5 keV beam.
FIG. 5. Input parameters being optimized vs the number of iterations. This is for the optimized results in Fig. 4. The percent change of each input parameter from beginning to end is show in Table IV. The FoM vs iterations is also plotted as a dashed black line. The left vertical axis is for the input parameters while the right side vertical axis is for the FoM.

TABLE III. Initial moment values when solving the equations from Eq. 13

| Initial Moments \((z = 0)\) | Value  |
|---------------------------|--------|
| \(Q^+\)                   | 2.58e-6 |
| \(Q^-\)                   | 2.52e-6 |
| \(E^+\)                   | 5.07e-5 |
| \(E^-\)                   | 4.97e-5 |

the optimizations with added in self-fields. We can think of the added self-fields as perturbations on the 0 mA solution. Thus, the best starting values for the self-field optimizations are the results obtained from the 0 mA, no self-fields, solution. Results are shown in Fig. 4.

Fig. 4 shows that by using the adjoint technique, we are able to find and optimize solutions even in the presence of large self-field forces. The solutions are found relatively fast as the input parameters quickly converge to optimized values. In comparison, a brute force method of running PIC simulations over the full parameter space and calculating gradients would take orders of magnitude more computation time. Figure 5 shows the change in input parameters vs iterations for the three optimizations shown in Fig. 4. Table IV shows the percent change in each input parameter over the full optimization runs shown in Fig 4. In general, the second quadrupole has the largest change over the course of the different optimization runs. It is interesting to note that in the case of the solenoid, the optimization preferred to push the starting location of the magnet upstream when tuning for large self-field forces. This would at times create optimized solutions where the solenoid was too close to the third quadrupole resulting in overlapping fields. To fix this a constraint was placed on how far the optimization could shift the solenoid starting location; this prevented moving the solenoid fields too close to other magnets and still resulted in successful optimized solutions. This can be seen in the 5 mA results in Fig. 5 with one of the input parameter lines constant across the iterations; this is the solenoid starting location being prevented from changing.

While the optimizations in Fig. 4 use hardedge magnet models, it is not limited to such profiles. The magnet strengths, \(k_Ω\) and \(K_q\), are continuous variables in the moment equations, and as such, any profile can be used. This may be useful for unique and nonlinear magnet profiles that can not necessarily be represented in matrix form for single particle tracking or would take too long to simulate in PIC tracking codes. Figure 6 shows an optimization result using Gaussian like magnet profiles similar to the quadrupole fields available in UMER’s magnets [27]. The solenoid field also has a fringe field edge included in its profile. With these continuous profiles the optimization is still able to find acceptable solutions.

Certain lattice configurations require the quadrupole triplet to maintain a fixed 45-degree transverse rotation angle. While such an FTR solution is possible for a beam with no self-fields, it is not necessarily true when self-

TABLE IV. Percent change of the 11 magnet parameters as a result of running the optimization. The values are for the optimized results in Fig. 4.

| Parameters            | 0mA % change | 1mA % change | 5mA % change |
|-----------------------|--------------|--------------|--------------|
| Solenoid Location     | 3.0          | 6.0          | 6.0          |
| Solenoid Strength     | 1.8          | 4.2          | 12.0         |
| Quad 1 location       | 0.1          | 0.3          | 1.0          |
| Quad 2 location       | 1.6          | 6.8          | 23.3         |
| Quad 3 location       | 1.0          | 2.6          | 7.5          |
| Quad 1 strength       | 1.0          | 4.0          | 13.5         |
| Quad 2 strength       | 3.7          | 7.3          | 16.0         |
| Quad 3 strength       | 3.7          | 4.6          | 8.3          |
| Quad 1 rotation       | 2.5          | 5.3          | 11.9         |
| Quad 2 rotation       | 2.7          | 3.5          | 4.9          |
| Quad 3 rotation       | 1.5          | 3.0          | 6.7          |
FIG. 6. RMS beam size vs longitudinal position for an optimized result that used non hardedge magnet profiles. A beam with 5 keV energy and zero current was used.

fields are added. Figure 7 shows the FoM vs iterations for three different optimization runs. Each run is repeated using fixed and varied quadrupole rotation angles. In the case with fixed angles and self-field forces, the optimization of the FoM converges early in the iterations limiting the reduction of the FoM. When angles are varied and free to rotate, the FoM can be further minimized, showing the necessity of having to adjust quadrupole rotations in the presence of self-fields in order to better meet the FTR requirements. Figure 7 also shows that in the case of no self-field forces (0 mA beam) the FoM reduction does not plateau when optimizing with fixed rotation angles. This conveys that the FTR transformation can be done with fixed 45-degree quadrupole rotation angles if there are no self-fields, but once self-fields are added the quadrupole rotations angles need to be adjusted to compensate.

Using the optimized parameters from the final FoM values within Fig 7, a plot of the RMS beam size is made through the FTR to see the effects of having the quadrupole rotation angles fixed and varied. Results are shown in Fig 8. The most notable observation is how much larger the round beam becomes at the end of the FTR when using varied quadrupole rotation angles vs fixed angles. This difference in beam size also grows as a function of the strength of the self-fields.

All optimizations up to this point have used the same initial beam conditions displayed in Table II. These initial conditions are needed in order to setup a successful FTR transformation. However, some level of deviation from these initial conditions is expected when trying to experimentally setup such a transformation; it might be difficult to exactly match the beam to the required initial conditions. Is it then possible to use the optimization routine and adjust the magnet parameters in order to correct for inaccurate initial beam conditions? Figure 9 shows these results and proves that such corrections are possible.

Figure 9 has optimized solutions for three beam currents: 0 mA, 1 mA, and 5 mA. The analysis starts by taking optimized results and randomly perturbing the initial emittance of the beam by changing moment values: $E_{\pm}(0), E_{\mp}(0)$. The perturbations are about 50% from the ideal value listed in Table II. The beam size through the FTR transformation with the perturbed initial conditions is then plotted in Fig. 9. An optimization routine is run, and magnet parameters are tuned to correct for such a perturbation; these are also displayed in the same plot. Figure 9 shows that bad initial conditions can be corrected with and without self-field forces. The size of the beam at the end of the FTR in the corrected
FIG. 9. RMS beam size is plotted for optimized results with ideal initial conditions (blue circles). Initial conditions are purposely perturbed to break the optimized results (orange line). Magnet parameters are re-optimized to correct for the perturbed initial conditions (green line).

case matches close to the beam size before perturbations were applied.

IV. CONCLUSION

In conclusion, this paper demonstrates a novel method of using the adjoint approach to successfully model and optimize a Flat-to-Round or Round-to-Flat lattice transformer in the presence of space-charge forces. The developed system of continuous moment equations shows excellent agreement with PIC code simulation results. The gradient-descent algorithm developed with these moment equations and the adjoint approach is then able to successfully optimize various transformer systems. The optimization can correct for space-charge forces introduced into the beam as well as for inaccurate initial beam conditions. The correction is achieved by tuning a set of quadrupole and solenoid magnet parameters within the lattice. Because of the adjoint approach to the problem, such optimizations are performed at significantly reduced computational costs. Traditional methods of brute force simulations to solve the same problem would require considerably more computational resources and time.

For future work, a set of fully kinetic moment equations will be developed. The adjoint approach will continue to be studied in its usefulness at addressing new and existing problems within the accelerator physics field in a meaningful way.

V. ACKNOWLEDGEMENTS

This work was supported by DOE-HEP award No. DE-SC0010301.

Appendix A: COUPLED MOMENT EQUATIONS

In this appendix we outline the steps leading to Eqs. (13)-(15). We start with the definitions of the moments in Eqs. (9) - (12). Equation (13a), the derivative of $Q$ with respect to distance $z$, follows directly from the definition of the components of $P$ in Eq. (10). Differentiating the vector $P$ and angular momentum $L$ with respect to $z$ gives,

$$P'_x = E_x + \langle xx'' + yy'' \rangle,$$  
(A1a)

$$P'_y = E_y - \langle xx'' + yy'' \rangle,$$  
(A1b)

$$P'_z = E_z + \langle xy'' + yx'' \rangle,$$  
(A1c)

$$L' = \langle xy'' - yx'' \rangle$$  
(A1d)

where the moments $E$ are defined in Eq. (12). Differentiating the moments $E$ with respect to axial distance gives,

$$E'_x = 2 \langle x'^{'}x'' + y'y'' \rangle,$$  
(A2a)

$$E'_y = 2 \langle x'^{'}x'' - y'y'' \rangle,$$  
(A2b)

$$E'_z = 2 \langle x'y'' + y'x'' \rangle,$$  
(A2c)

Evaluation of each of the averages on the right sides of Eqs. (A1) and (A2) requires inserting expressions for the second derivatives of $x$ and $y$ in the Larmor Frame. For this we use Eq. (9).

The first contribution to the averages in Eqs. (A1) and (A2) comes from the solenoidal field contribution to Eq. (9),

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = -\left( \frac{k_q}{2} \right)^2 \begin{pmatrix} x \\ y \end{pmatrix}$$  
(A3)
This when inserted in Eqs. (A1) and (A2) gives rise to the first term in the expression for the matrix \( O \) in Eq. (14).

The second contribution to the averages in Eqs. (A1) and (A2) comes from the quadrupole fields. To evaluate those terms we write the second derivatives in the Larmor frame in the following way

\[
\begin{pmatrix}
  x''_Q \\
  y''_Q
\end{pmatrix} = R^{-1}(\phi) \sum_{\text{Quads}} K_Q \begin{pmatrix}
  \cos 2\psi & \sin 2\psi \\
  \sin 2\psi & -\cos 2\psi
\end{pmatrix} R(\phi) \begin{pmatrix}
  x \\
  y
\end{pmatrix} \tag{A4}
\]

Here the Larmor variables, \( x \) and \( y \), are first transformed back to the lab frame, where Eq. (17) describes the quadrupole fields. The transverse Lorentz force is computed and the lab frame, and the resulting acceleration is back to the lab frame, where Eq. (17) describes the quadrupole fields. To evaluate those terms we write the second derivatives in the Larmor frame in the following way

\[
\begin{pmatrix}
  x''_Q \\
  y''_Q
\end{pmatrix} = R^{-1}(\phi) \sum_{\text{Quads}} K_Q \begin{pmatrix}
  \cos 2\psi & \sin 2\psi \\
  \sin 2\psi & -\cos 2\psi
\end{pmatrix} R(\phi) \begin{pmatrix}
  x \\
  y
\end{pmatrix} \tag{A4}
\]

The transformation from beam frame to Larmor frame coordinates is thus given by

\[
\begin{pmatrix}
  x \\
  y
\end{pmatrix} = R(\phi) \begin{pmatrix}
  x_2 \\
  y_2
\end{pmatrix} \tag{A6}
\]

The beam is assumed to have uniform density within an elliptical cross section in the beam frame. Thus, the beam frame moments satisfy,

\[
\begin{align*}
\langle x^2 \rangle &= a^2/4, \quad \langle y^2 \rangle = b^2/4, \quad \langle xy \rangle = 0 \tag{A7}
\end{align*}
\]

This implies through application of Eq. (A6) for the Larmor frame moments,

\[
\begin{align*}
\langle x^2 \rangle &= \frac{1}{8} \left[ (a^2 + b^2) + \cos(2\alpha)(a^2 - b^2) \right] \\
\langle y^2 \rangle &= \frac{1}{8} \left[ (a^2 + b^2) - \cos(2\alpha)(a^2 - b^2) \right] \\
\langle xy \rangle &= \frac{1}{8} (a^2 - b^2) \sin(2\alpha) \tag{A8}
\end{align*}
\]

The Larmor frame values for the elements of \( Q \) are then related to the radii \( a \) and \( b \) and the angle \( \alpha \) by

\[
\begin{align*}
Q_+ &= \frac{1}{2} \langle x^2 + y^2 \rangle = \frac{1}{8}(a^2 + b^2) \\
Q_- &= \frac{1}{2} \langle x^2 - y^2 \rangle = \frac{\cos(2\alpha)}{8}(a^2 - b^2) \\
Q_x &= \langle xy \rangle = \frac{\sin(2\alpha)}{8}(a^2 - b^2) \tag{A9}
\end{align*}
\]

**Appendix B: Self-Fields and Transformations**

The effective electric potential due to a beam with elliptical cross section far from conducting boundaries is given by

\[
\Phi = -\frac{2\Lambda}{(a+b)} \left[ \frac{x_2^2}{a} + \frac{y_2^2}{b} \right] = -\frac{2\Lambda}{ab} \left[ \frac{1}{2} (x^2 + y^2) + \frac{b-a}{b+a} \left( \frac{\cos(2\alpha)}{2} (x^2 - y^2) + \sin(2\alpha)xy \right) \right] \tag{B1}
\]

where the first expression is in the beam frame and the second expression has been transformed to the Larmor frame. The current strength parameter \( \Lambda \) is defined in Eq. (20). The relation between the effective potential and the space-charge potential \( \phi_{sc} \) is \( \Phi = q\phi_{sc}/(mv^2\gamma^5) \). One of the three powers of \( \gamma \) in the denominator of the effective potential accounts for the effective relativistic mass increase of the beam particles, and the other two powers accounts for the self magnetic field. The accelerations due to self-fields then follow from

\[
\begin{pmatrix}
  x''_A \\
  y''_A
\end{pmatrix} = - \left( \frac{\partial}{\partial x} \right) \Phi(x, y) \tag{B2}
\]

When Eq. (B2) is inserted in Eqs. (A1) and (A2), and the parameters \( a, b, \) and \( \alpha \) are expressed in terms of the
moments \( Q \) using Eq. (A9), the result is the self-field contributions to the matrix \( O \) and vector \( N \) appearing in Eqs. (14) and (15) along with definitions in Eq. (19).

Equations (13a)-(13d) describe the evolution of the moments in the Larmor frame. If the values of the moments in the lab frame are desired, these can be recovered by the following transformations,

\[
\begin{align*}
Q_{\text{lab}} &= R_1 \cdot Q \\
P_{\text{lab}} &= R_1 \cdot P + 2\phi' R_2 \cdot Q \\
E_{\text{lab}} &= R_1 \cdot E + 2\phi' R_2 \cdot P + 2\phi'^2 R_3 \cdot Q + 2\phi' 1_+ L \\
L_{\text{lab}} &= L + 2\phi' Q_+
\end{align*}
\]

(B3a)--(B3d)

Here the following matrices are defined

\[
R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(2\phi) & -\sin(2\phi) \\ 0 & \sin(2\phi) & \cos(2\phi) \end{bmatrix}
\]

(B4a)

\[
R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin(2\phi) & -\cos(2\phi) \\ 0 & \cos(2\phi) & -\sin(2\phi) \end{bmatrix}
\]

(B4b)

\[
R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos(2\phi) & \sin(2\phi) \\ 0 & \sin(2\phi) & \cos(2\phi) \end{bmatrix}
\]

(B4c)

\[
1_+ = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

(B4d)

The adjoint equation (26c) has an added term \( \delta \dot{E}^{(Y)} \), which is added to cancel out the terms in Eq. (28) that come from the dependence of the matrix \( O \) and vector \( N \) on the changes in the self-fields. These changes are proportional to changes in the moments \( \delta Q^{(X)} \). To represent these changes in the \( O \) matrix we first separate the changes due to the self-fields (\( \Lambda \)) from the changes in the magnetic focusing parameters \( (Q, B) \),

\[
\delta O^{(X)} = \delta O_{Q,B}^{(X)} + \delta O_{\Lambda}^{(X)}.
\]

We then construct a matrices \( M_P \) and \( M_Q \) that satisfy

\[
\begin{align*}
\delta O_{\Lambda}^{(X)} \cdot P &= M_P \cdot \delta Q^{(X)} \\
\delta O_{\Lambda}^{(X)} \cdot Q &= M_Q \cdot \delta Q^{(X)}
\end{align*}
\]

(B5a)--(B5b)

The vector \( N \) is treated similarly.

We focus now on \( M_P \). We start by forming the vector

\[
R_P = O_{\Lambda} \cdot P = \Lambda \frac{Q}{Q_{\Delta}} \begin{pmatrix} P_+ - \frac{Q}{Q_{\Delta} + Q_+} P_+ - \frac{Q}{Q_{\Delta} + Q_+} P_x \\ \frac{Q}{Q_{\Delta} + Q_+} P_+ + P_+ - \frac{Q}{Q_{\Delta} + Q_+} P_x + P_x \end{pmatrix}
\]

(B6)

The elements of the matrix \( M_P \) are generated by differentiating each element of vector \( R_P \) with respect to each element of vector \( Q \),

\[
M_P = \left[ \frac{\partial R_P}{\partial Q} \right]^T
\]

(B7)

To keep track of terms we decompose \( M_P \) according to

\[
M_P = V_1 U_1^T + V_2 U_2^T + V_3 U_3^T + V_4 U_4^T
\]

(B8)

where

\[
V_1 = -\Lambda \frac{Q}{Q_{\Delta}} \begin{pmatrix} P_+ - \frac{Q}{Q_{\Delta} + Q_+} P_+ - \frac{Q}{Q_{\Delta} + Q_+} P_x \\ \frac{Q}{Q_{\Delta} + Q_+} P_+ + P_+ - \frac{Q}{Q_{\Delta} + Q_+} P_x + P_x \end{pmatrix}
\]

(B9a)

\[
U_1^T = \begin{pmatrix} \frac{\partial Q}{\partial Q_{\Delta}} \frac{\partial Q}{\partial Q_{+}} \frac{\partial Q}{\partial Q_{x}} \end{pmatrix} = \frac{1}{Q_{\Delta}} (Q_+ Q_- Q_x)
\]

(B9b)

\[
V_2 = \Lambda \frac{Q}{Q_{\Delta}(Q_{\Delta} + Q_+)^2} \begin{pmatrix} Q_+ P_+ \\ Q_- P_+ \\ Q_x P_+ \end{pmatrix}
\]

(B9c)

\[
U_2^T = \begin{pmatrix} \frac{\partial Q}{\partial Q_{\Delta}} \frac{\partial Q}{\partial Q_{+}} \frac{\partial Q}{\partial Q_{x}} \end{pmatrix} = U_1^T + (1 0 0)
\]

(B9d)

\[
V_3 = -\Lambda \frac{Q}{Q_{\Delta}(Q_{\Delta} + Q_+)} \begin{pmatrix} P_+ \\ 0 \\ 0 \end{pmatrix}
\]

(B9e)

\[
V_4 = -\Lambda \frac{Q}{Q_{\Delta}(Q_{\Delta} + Q_+)} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

(B9f)

The matrices \( M_Q \) and \( M_N \) are constructed in the same way.
Appendix C: SCALING PROPERTIES

Finally, we discuss the scaling properties of our system of moment equations. Let us suppose that we have found a solution to the moment equations in the form of functions

\[ Q_0(z_0), P_0(z_0), E_0(z_0), O_0(z_0), N_0(z_0), L_0(z_0) \]  

(C1)

It can then be verified that a scaled solution that is also satisfying the moment equations is

\[
\begin{align*}
Q(z) &= \lambda Q_0(\epsilon z) \\
P(z) &= \lambda P_0(\epsilon z) \\
E(z) &= \lambda^2 E_0(\epsilon z) \\
O(z) &= \epsilon^2 O_0(\epsilon z) \\
N(z) &= \epsilon^2 N_0(\epsilon z) \\
L(z) &= \lambda L_0(\epsilon z)
\end{align*}
\]  

(C2)

Here \( \epsilon, \lambda \) are arbitrary constants allowing for separate scaling of the axial dependence and size of the beam. If \( \epsilon > 1 \) the scaled solution is shorter in spatial length than the original solution, and if \( \epsilon < 1 \) the scaled solution is longer than the original. If \( \lambda > 1 \) the scaled beam is larger. Of course this is subject to the requirement that the transverse forces remain linear in displacements.

Looking at the expressions for the matrices \( O, N \) in Eq. \((\text{C2})\), there are a number of conditions that must be satisfied to give the required \( \epsilon^2 \) scaling. The solenoidal field contribution requires,

\[
k_{\Omega}(z) = \epsilon k_{\Omega_0}(\epsilon z)
\]

(C3)

This means a shorter solution requires a stronger solenoidal field. The scaled solution gives for the phase

\[
\phi(z) = \phi_0(\epsilon z)
\]

(C4)

Thus, the values of the phase in the locations of the quadrupoles are preserved under the scaling. For the quadrupoles, strict application of the scaling gives

\[
K_q(z) = \epsilon^2 K_{q_0}(\epsilon z)
\]

(C5)

However, in the thin lens approximation only the integrated value of the quadrupole field matters

\[
K = \int dz K_q(z) = \epsilon \int \epsilon dz K_{q_0}(\epsilon z) = \epsilon K_0
\]

(C6)

In this approximation the strength of the quadrupole field also scales inversely with length.

The self-field contribution to the \( O \) and \( N \) matrices can also be preserved. We note that these contributions scale as

\[
O, N \propto \Lambda/Q
\]

(C7)

So to preserve the solution we require

\[
O \propto \Lambda/Q = \Lambda/(\epsilon \lambda Q_0) = \epsilon^2 O_0 \propto \epsilon^2 \Lambda_0/Q_0
\]

(C8)

As a result, the current parameter scales as

\[
\Lambda = \lambda \epsilon^3 \Lambda_0
\]

(C9)

Thus, scaling length or amplitude requires changing beam current to maintain space-charge influence.

---

[1] S. Director and R. Rohrer, “The generalized adjoint network and network sensitivities,” IEEE Transactions on Circuit Theory 16, 318–323 (1969).
[2] N. K. Nikolova, M. H. Bakr, and J. W. Bandler, “Sensitivity analysis of network parameters with electromagnetic frequency-domain simulators,” IEEE Transactions on Microwave Theory and Techniques 54, 670–681 (2006).
[3] A. Jameson, “Optimum aerodynamic design using cfd and control theory,” CFD Review 3 (1995), 10.2514/6.1995-1729.
[4] T.M. Antonsen, D. Chernin, and J.J. Petillo, “Adjoint approach to beam optics sensitivity based on hamiltonian particle dynamics,” Physics of Plasmas 26, 013109 (2019).
[5] T.M. Antonsen, B.L. Beaudoin, L. Dovlatyan, and I. Haber, “Adjoint Approach to Accelerator Lattice Design,” in Proc. NAPAC’19, North American Particle Accelerator Conference No. 4 (JACoW Publishing, Geneva, Switzerland, 2019) pp. 376-378.
[6] Ya. Derbenev, Adapting optics for high-energy electron cooling, Tech. Rep. UM-HE-98-04-A (1998).
[7] A.V. Burov and V.V. Danilov, An Insertion to eliminate horizontal temperature of high-energy electron beam, Tech. Rep. FERMILAB-TM-2043 (1998).
[8] A.V. Burov and S. Nagaitsev, Covariant-Snyder parameters of beam adapters, Tech. Rep. FERMILAB-TM-2114 (2000).
[9] A.V. Burov, S. Nagaitsev, and Ya. Derbenev, “Circular modes, beam adapters, and their applications in beam optics,” Phys. Rev. E 66, 016503 (2002).
[10] A.W. Chao and M. Tigner, Handbook of Accelerator Physics and Engineering (World Scientific, Singapore, 1999).
[11] Ya. Derbenev, Advanced optical concepts for electron cooling,” Nucl. Instrum. Meth. A 441, 223–233 (2000).
[12] R. Brinkmann, Ya. Derbenev, and K. Flottmann, A flat beam electron source for linear colliders, Tech. Rep. DESY-TESLA-99-09 (1999).
[13] R. Brinkmann, Y. Derbenev, and K. Flöttmann, “A low emittance, flat-beam electron source for linear colliders,” Phys. Rev. ST Accel. Beams 4 (2001), 10.1103/PhysRevSTAB.4.053501.

[14] D. Edwards et al., “The Flat beam experiment at the FNAL photoinjector,” eConf C000821, MOB13 (2000), arXiv:physics/0008042.

[15] D. Edwards et al., “Status of flat electron beam production,” in PAC 2001. Proceedings of the 2001 Particle Accelerator Conference (Cat. No.01CH37268), Vol. 1 (2001) pp. 73–75.

[16] S. Wang, I. Corlett, S. Lidia, J. Staples, and A. Zholents, “Flat beam production in low energy injectors,” in Proceedings of the 2003 Particle Accelerator Conference, Vol. 5 (2003) pp. 3198–3200.

[17] S. Lidia, “Emittance compensation studies of photoinjector beams with angular momentum,” in Proceedings of the 2003 Particle Accelerator Conference, Vol. 3 (2003) pp. 2089–2091.

[18] A. Ody, P. Musumeci, J. Maxson, D. Cesar, R. England, and K. Wootton, “Flat electron beam sources for DLA accelerators,” Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment 865, 75–83 (2017).

[19] F.W. Cropp V et al., “Maximizing 2-D Beam Brightness Using the Round to Flat Beam Transformation in the Ultralow Charge Regime,” in Proc. NAPAC’19, North American Particle Accelerator Conference No. 4 (JACoW Publishing, Geneva, Switzerland, 2019) pp. 986–989.

[20] A. Wolski, “Alternative approach to general coupled linear optics,” Phys. Rev. ST Accel. Beams 9, 024001 (2006).

[21] E. Thrane et al., “Photoinjctor production of a flat electron beam,” in 21st International Linear Accelerator Conference (2002) pp. 308–310.

[22] Y. Sun, Angular-momentum-dominated electron beams and flat-beam generation, Ph.D. thesis, Chicago U. (2005).

[23] K. Kim, “Round-to-flat transformation of angular-momentum-dominated beams,” Phys. Rev. ST Accel. Beams 6, 104002 (2003).

[24] G. Rangarajan, F. Neri, and A. Dragt, “Generalized emittance invariants,” in Proceedings of the 1989 IEEE Particle Accelerator Conference, .’ Accelerator Science and Technology, Vol. 2 (1989) pp. 1280–1282.

[25] A. Friedman et al., “Computational Methods in the Warp Code Framework for Kinetic Simulations of Particle Beams and Plasmas,” IEEE Trans. Plasma Sci. 42, 1321–1334 (2014).

[26] I. M. Kapchinsky and V. V. Vladimirsky, in Proceedings, International Conference on High Energy Accelerators (CERN, Geneva, Switzerland, 1959) p. 274.

[27] W. W. Zhang, S. Bernal, H. Li, T. Godlove, R. A. Kishek, P. G. O’Shea, M. Reiser, V. Yun, and M. Venturini, “Design and field measurements of printed-circuit quadrupoles and dipoles,” Phys. Rev. ST Accel. Beams 3, 122401 (2000).