Exploring the Universe with Dark Light Scalars

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We study the cosmology of a dark sector consisting of (ultra) light scalars. Since the scalar mass is radiatively unstable, a special explanation is required to make it much smaller than the UV scale. There are two well-known mechanisms for the origin of scalar mass. The scalar can be identified as a pseudo-Goldstone boson, whose shift symmetry is explicitly broken by non-perturbative corrections, like the axion. Alternatively, it can be identified as a composite particle like the glueball, whose mass is limited by the confinement scale of the theory. In both cases, the scalar can be naturally light, but interaction behavior is quite different. The lighter the axion (glueball) mass is, the weaker (stronger) the interaction strength is. We consider the dark axion whose shift symmetry is anomalously broken by the hidden non-abelian gauge symmetry. After the confinement of the gauge group, the dark axion and the dark glueball get masses and both form multicomponent dark matter. We carefully consider the effects of energy flow from the dark gluons to the dark axions and derive the full equations of motion for the background and the perturbed variables. The effect of the dark axion-dark gluon coupling on the evolution of the entropy and the isocurvature perturbations is also clarified. Finally, we discuss the gravo-thermal collapse of the glueball subcomponent dark matter after the halos form, in order to explore the potential to contribute to the formation of seeds for the supermassive black holes observed at high redshifts. With simplified assumptions, the glueball subcomponent dark matter with the mass of $\sim 0.01 - 1 \text{MeV}$, and the axion dark matter with the GUT scale decay constant (the mass of $O(10^{-18}) \text{eV}$) can provide the hint on the origin of the supermassive black holes at high redshifts.

I. INTRODUCTION

The discovery of the Higgs boson completed the Standard Model which explains most of the phenomena in the Universe including nuclei and atoms and their interactions. However, various studies based on precise measurements of the cosmic microwave background (CMB), velocity distributions of stars and galaxies, and large-scale structures (LSS) show that only 4% of the Universe is understood by our knowledge of the Standard Models and 96% must be filled with dark matter and dark energy that we do not know their origin.1

Among dark matter candidates, particles classified as ‘weakly interacting massive particle’ (WIMP) have been considered as the best candidate. This is because their freeze-out relic abundance can naturally explain the present amount of dark matter, can be tested by different searches, and their mass scale can be related to new physics that explains why the electroweak scale is stable against various quantum corrections. However, there is no conclusive evidence of WIMP dark matter so far, and the alternative candidates got more attention in recent years. The masses of these candidates are not limited to a narrow range, and various ideas have been proposed particularly focusing on the detection possibility.2

For such a broad range of dark matter mass, no clear guiding principle exists to specify a natural range of it. Especially, it is extremely unnatural to consider a light scalar unless there is a special reason for it. Although theoretical explanations are required, it is quite interesting to note that an ultra-light scalar dark matter is allowed by cosmological/astrophysical observations, because the scalar can act as an oscillating classical field whose averaged equation of state is the same as that of the cold dark matter (CDM). Interesting phenomena also can arise from the field nature of the scalar. The fuzzy dark matter3 and the QCD axion4–7 are good examples. Both candidates, the axion or axion like particle, could be naturally light by the approximate shift symmetry of the axion, which is explicitly broken by the controllable non-perturbative corrections. The consequence of the global symmetry is that the lighter the axion is, the weaker its interaction is. Therefore, the large occupation number of the (ultra) light axion is allowed, and it can be described by the evolution of scalar condensate.

On one hand, there is another way to obtain a light scalar. When the asymptotically free gauge group has confinement at low energy, the gauge fields are confined into the scalar particles, the glueball, whose mass is limited by the confining scale. Unlike in the case of the axion, the lighter the glueballs is, the stronger the scattering cross-section among glueballs is. The number chang-
ing interactions are active, so the occupation number is always limited by its temperature. In this case, the relic density is determined by the freeze-out mechanism. Because of this property, the light dark glueball becomes a good candidate for self-interacting dark matter (SIDM) [8,9], which is one of the ways to make the cored density profile around the center of galaxies [10]. As a subcomponent dark matter, it can also play the role to suppress small scale perturbation [11]. These two mechanisms to obtain a light scalar dark matter provide completely different microscopic nature of dark matter, and yield different predictions for small scale structure.

In this paper we study the cosmology of light scalar dark matter, focusing on the origin of its mass and the consequences associated with it. We cover two mechanisms discussed above at the same time in a minimal set-up: the dark sector consisting of the axion and the confining hidden gauge symmetry without light fermions. In this set-up, the axion scalar potential is dynamically generated by a Chern-Simons type coupling between the axion and the dark gauge field, gluon. As the gauge symmetry confines, the axion and the glueball get masses and become a part of multicomponent dark matter. The idea that the mass of the ultra-light scalar dark matter originates from a confining hidden gauge symmetry was studied in [12,13], but there is no comprehensive study on thermodynamics of the dark gluon fluid and entropy evolution. In Sec. III, we cover two mechanisms to establish basic formalism from the Lagrangian to the dynamical equations of the background and perturbation variables of the coupled axion-gluon fluid. In Sec. III, we set up the model for the evolution of the glueballs and axion as dark matter. The parametric dependence of the relic abundance of the glueballs and the axion is also presented. Sec. IV is devoted to the evolution of the perturbed variables. For the initial conditions, we have three modes: adiabatic perturbation, isocurvature perturbation induced by the initial misalignment of the axion and by the temperature fluctuation of the dark gluon fluid. In Sec. V, the implications of the glueball subcomponent dark matter for the early formation of the supermassive black holes are discussed. Sec. VI is discussion.

II. AXION DARK MATTER AND CONFining DARK SECTOR

A. General description of the model

Our starting Lagrangian of dark sector is composed of the ultra-light axion $\phi$ whose field range is $2\pi f_a$, and the dark gauge symmetry with the confinement scale $\Lambda$ ($\Lambda \ll f_a$). The coupling between the axion and the dark gauge bosons are given as

$$\frac{-L_h}{\sqrt{-g}} = \frac{1}{2} (\partial_{\mu} \phi)^2 + \frac{1}{4} (G_{\mu\nu}^a)^2 + \frac{g^2}{32\pi^2 f_a} G_{\mu\nu}^a \tilde{G}^{\mu\nu} a,$$

where $G_{\mu\nu}$ is the dark gluon field strength and $g$ denotes the dark gauge coupling. For illustration, we take $SU(N)$ as the dark gauge group in our set-up. Below the confinement scale, the dynamics of the gauge fields can be described by the composite bosons, the glueballs. Considering the lightest glueball $\varphi_g$, the effective Lagrangian of $\varphi_g$ can be expanded in $(4\pi/N)(\varphi_g/m_g)$ [20,21,22,23]

$$\frac{-L_{\text{eff}}}{\sqrt{-g}} = \frac{1}{2} (\partial_{\mu} \phi)^2 + V(\phi) + \frac{1}{2} (\partial_{\mu} \varphi_g)^2 + \frac{1}{2} m_g^2 \varphi_g^2 + \frac{a_1}{3!} \left(\frac{4\pi}{N}\right) m_g \varphi_g^3 + \frac{a_2}{4!} \left(\frac{4\pi}{N}\right)^2 \varphi_g^4 + \frac{a_3}{5!} \left(\frac{4\pi}{N}\right)^3 \varphi_g^5 + \cdots,$$

where $m_g$ is the glueball mass of $O(\Lambda)$. The axion also gets a scalar potential by the gluodynamics, which can be written as a power series in $(\phi/Nf_a)^2$ around its minimum ($\phi = 0$) [21,25]

$$V(\phi) = N^2 \Lambda^4 \left( \frac{c_1}{2} \frac{\phi^2}{N^2 f_a^2} + \frac{c_2}{4!} \frac{\phi^4}{N^4 f_a^4} + \cdots \right)$$

$$= \frac{1}{2} m_a^2 \phi^2 + \frac{c_2}{4! c_1} \frac{m_a^2}{N^2 f_a^2} \phi^4 + \cdots.$$

In this expansion, the axion mass is given by

$$m_a^2 = \frac{1}{f_a V_4} \int d^3 x d^4 y \left( \frac{g^2}{32\pi^2} G G(x) \frac{g^2}{32\pi^2} G G(y) \right)_{\phi=0}$$

$$= \left(10^{-12} \sqrt{c_1} \text{eV}^2 \right)^2 \left( \frac{\Lambda}{\text{MeV}} \right)^4 \left( \frac{10^{15} \text{GeV}}{f_a} \right)^2,$$

where $V_4$ is the four-dimensional volume. The $1/N^2$ factor for the quartic term of the axion potential leads to the suppression of the anharmonic effects as long as the initial misalignment of the axion field is $\phi_1 \lesssim f_a$.

The glueball and the axion can be naturally light by lowering the scale of confinement. The difference between the two is that as their masses get smaller, the glueballs...
interact more strongly with each other, whereas the axions interact more feebly due to the additional suppression factor $A/f_a \ll 1$. The glueball is not a lightest particle in our set-up, so it is unstable. In the lattice calculation [26, 27], the glueball mass depends on the axion field value as

$$m_g = m_g(\phi = 0) \left(1 + O\left(\frac{\phi^2}{N^2 f_a^2}\right)\right). \quad (2.5)$$

Through the quadratic and cubic interactions of the glueball, we get the leading interaction for the decay of the glueball to two axions,

$$\frac{m_g^3}{4\pi N^3 f_a^2} \phi^2 \varphi_g. \quad (2.6)$$

From Eq. (2.6), the life-time of the glueball is estimated as $\tau_{\varphi_g} \sim 10^{17}$ Gyr $(f_a/10^{13} \text{ GeV})^6 \text{GeV}/m_g^5$. In the parameter space we will focus on, the glueball is cosmologically stable, so that both axion and glueball are dark matter of the Universe.

For cosmology, we consider the case that the dark sector and the visible sector are thermally decoupled from the beginning. Therefore, dark gluons/glueballs are thermalized by their own interactions at a temperature $T_g$, which is different from the photon temperature $T_p$. Starting from the gluon fluid at high temperatures, as $T_g$ drops and crosses the dark critical temperature $T_{g,c} = \mathcal{O}(\Lambda)$, the confinement phase transition occurs. Below $T_{g,c}$, all gluons are confined into the glueballs, and the evolution is described by the massive glueball fluid. The dark gluon temperature also affects the evolution of the dark axions. The leading term of the axion potential induced by the gluo-thermodynamics is

$$V(T_g, \phi) = \frac{1}{2} m_a^2(T_g) \phi^2, \quad (2.7)$$

where the axion mass $m_a(T_g)$ is well described by the dilute instanton gas approximation in the deconfining phase,

$$m_a(T_g) \approx m_a \left(\frac{T_{g,c}}{T_g}\right)^{\eta_a} \quad \text{for} \quad T_g \gtrless T_{g,c}, \quad (2.8)$$

with $\eta_a = 11N/6 - 2$. For $N = 3(4)$, $\eta_a = 3.5(5.3)$ [28]. After the confinement $(T_g \lesssim T_{g,c})$, the axion mass is saturated as its zero temperature value, $m_a(T_g) \approx m_a$ [29].

Actually, the temperature dependence of the axion potential implies the existence of the energy flow from the gluon fluid to the axions as the temperature decreases. Then, a natural question is whether or not the entropy of the dark gluon also evolves during the energy transfer. In order to make it clearer, let us discuss the gluo-thermodynamics in more detail. The free energy density of the gluon/glueball fluid can be evaluated from the gluon partition function for a given temperature $T_g$. If the topological $\theta$-term $(\theta \equiv \phi/f_a)$ is vanishing, the free energy density $f_g$ is only the function of the temperature as $f_g(T_g) = -p_g$, where $p_g$ is the pressure of the gluon/glueball fluid. The energy $(\rho_g)$ and entropy $(s_g)$ densities are obtained by the thermodynamic relations, $s_g = -df_g/dT_g$ and $\rho_g = T_g s_g - p_g$. On the other hand, the situation is a little bit different for the non-vanishing $\theta$-term. Because the gluon partition function also depends on $\theta$, the free energy density (the negative of the pressure for the gluon fluid with the $\theta$-term) is evaluated as [26]

$$f_{g+\theta}(T_g) = -p_g + V(T_g, \phi), \quad (2.9)$$

where $p_g = -f_g(T_g)$. The second term of the RHS represents the contribution of the vacuum energy density generated by the non-perturbative gluo-thermodynamics. The energy density of the gluon fluid with the $\theta$-term also can be decomposed into the sum of the vacuum energy density and the pure gluonic contribution as $\rho_{g+\theta}(T_g) = \rho_g + V(T_g, \phi)$. Then, the thermodynamics relations provide

$$s_g = -\frac{df_{g+\theta}(T_g)}{dT_g} = -\frac{dp_g}{dT_g} = -\frac{\partial V(T_g, \phi)}{\partial T_g}, \quad (2.10)$$

As a result, the entropy and the energy density of the gluon/glueball fluid depend not only on the temperature but also on the axion field value as Eq. (2.10) and

$$\rho_g = T_g \frac{dp_g}{dT_g} - p_g - T_g \frac{\partial V(T_g, \phi)}{\partial T_g}. \quad (2.11)$$

From the continuity equation of the dark sector, we can explicitly show that the entropy of the dark sector in a comoving volume $s_a \alpha^3$ is conserved for whatever value of $\phi$ and its evolution except the period of the confining phase transition.

Before discussing the explicit evolution of each component, for completeness, we address general equations of motion for axion and gluon as the fluids including their homogenous and perturbation parts.

B. Dynamics of the axion-gluon/glueball fluids

The axion dark matter is described by the evolution of the classical fields, $\phi(x)$. The dark gluon/glueball densities and their perturbations can be parameterized by its temperature evolution $T_g(x)$ and $\phi(x)$ as discussed in the previous section. In this context, $\{\phi(x), T_g(x)\}$ are good variables to derive full equations of motion of dark sector including their perturbations. Considering the fluid description, the evolution of energy densities and pressures are deduced from the evolution of $\phi$ and $T_g$ with the help of the Einstein equations, gluo-thermodynamics and the lattice calculation.
In order to include perturbations, we introduce the gauge, the conformal Newtonian gauge, for the inhomogeneous part of the metric tensor
\[ ds^2 = a(\tau)^2 \left( - (1 + 2\Psi) d\tau^2 + (1 + 2\Phi) d\vec{x}^2 \right). \] (2.12)
where the conformal time \( \tau \) and the conformal Hubble rate \( H \)
\[ \tau = \int \frac{dt}{a}, \quad H \equiv \frac{1}{a} \frac{da}{d\tau} = \frac{a'}{a} = aH \] (2.13)
are introduced. Here, \( H = \dot{a}/a \) is the Hubble expansion rate for the proper time \( t \).

For the evolution of the perturbed variables, the Fourier mode of the axion field \( \delta \phi \) with the wavenumber \( k \) is evolved as
\[ \delta\phi'' + 2H\delta\phi' + \left( k^2 + a^2 \frac{\partial^2 V}{\partial T_g^2} \right) \delta\phi + \left( -\Psi' + 3\Phi' \right) \]
\[ + 2a^2 \frac{\partial V}{\partial \phi} \Psi + a^2 \frac{\partial^2 V}{\partial \phi \partial T_g} \delta T_g = 0. \] (2.22)

The corresponding energy density and pressure are
\[ \delta\rho_a = \frac{\phi'}{a^2} \delta\phi + \delta^2 \Phi \]
\[ \delta p_g = \delta\rho_a - \frac{\partial V}{\partial \phi} \delta\phi - \frac{\partial V}{\partial T_g} \delta T_g, \] (2.23a)
\[ \delta p_a = \delta\rho_a - 2 \frac{\partial V}{\partial \phi} \delta\phi - \frac{\partial V}{\partial T_g} \delta T_g, \] (2.23b)
\[ (\rho_a + p_a)\nu_a = a^{-3} k^3 \phi' \delta\phi, \quad \rho_a \pi_a = 0. \] (2.23c)

Expanding \( \phi \) and \( T_g \) near the homogeneous solutions,
\[ \phi(\tau, \vec{x}) = \phi(\tau) + \delta\phi(\tau, \vec{x}), \] (2.15a)
\[ T_g(\tau, \vec{x}) = T_g(\tau) + \delta T_g(\tau, \vec{x}), \] (2.15b)
equations of motion for the background axion field is given by
\[ \phi'' + 2H\phi' + a^2 \frac{\partial^2 V}{\partial T_g^2} T_g = 0. \] (2.16)

The corresponding energy density and pressure are
\[ \rho_a = \frac{\phi'^2}{2a^2} + V(T_g, \phi), \quad p_a = \frac{\phi'^2}{2a^2} - V(T_g, \phi). \] (2.17)

From Eq. (2.16) and (2.17), the continuity equation for the background axion is obtained
\[ \rho_a' + 3H(1 + \omega_a)\rho_a = \frac{\partial V}{\partial T_g} T_g'. \] (2.18)

where \( \omega_a \equiv p_a/\rho_a \) is the equation of state for the axion. Since the axion-gluon fluid is isolated from the visible sector, the total energy and pressure of dark sector should satisfy the continuity equation without source terms
\[ \rho_{a+g}' + 3H(\rho_{a+g} + p_{a+g}) = 0. \] (2.19)

This leads to the evolution of the gluon/glueball fluid as
\[ \rho_g' + 3H(1 + \omega_g)\rho_g = -\frac{\partial V}{\partial T_g} T_g', \] (2.20)

where \( \omega_g = p_g/\rho_g \). Together with Eq. (2.10), (2.11), we can derive the conservation of the dark entropy as we claimed,
\[ s_g' + 3Hs_g = 0. \] (2.21)
with the isolated axion and gluon/glueball perturbations, the equations contain the terms proportional to $\partial V/\partial T_g$, which originate from the non-perturbative interactions between the gluon and the axion. With the $T_g$ dependent axion potential given by Eq. (2.8), the contribution of these terms becomes larger as $T_g$ approaches to $T_{g,c}$.

The amount of dark gluons can be parameterized by the ratio between the entropies of the dark sector and the visible sector, $s_g/s_{SM}$. Although dealing with the entropy ratio between two sectors would be easier to trace the evolution of the densities, in order to have a more intuitive picture about how cold the gluons are compared to the SM sector, we will use the ratio parameter between the temperatures. Taking a period around the phase transition, the photon temperature when $T_g$ arrives at $T_{g,c}$ is denoted by $T_{\gamma,c}$. Then, we define the ratio parameter $r$ as

$$r = \left(g_{8S}(T_{\gamma,c})s_g/2(N^2 - 1)s_{SM}\right)^{1/3} \approx T_{g,c}/T_{\gamma,c},$$

(2.26)

where $s_g$ ($s_{SM}$) is the entropy density of the gluon fluid (the SM sector), and $g_{8S}$ is the effective number of degrees of freedom in entropy for the SM sector.

Up to now, we have ignored the effect of dissipation induced by the background dark gluon plasma. Although, it is not crucial in our discussion, let us clarify the condition that our assumption is invalid. Including the friction term ($\gamma_{fr}$) for the axion’s motion, the equation of motion of the background axion is written as \[33\]

$$\ddot{\phi} + (3H + \gamma_{fr})\dot{\phi} + m_a^2(T_g)\phi = 0.$$  \hspace{1cm} (2.27)

In the deconfining phase, $\gamma_{fr} = \Gamma_{\text{ sph}}(T_g)/f_a^2T_g$, where the sphaleron rate is estimated as \[34\]

$$\Gamma_{\text{sph}}(T_g) = \int dt \left(\frac{g^2}{32\pi^2}N\tilde{G}(x)\frac{g^2}{32\pi^2}N\tilde{G}(0)\right) \approx \mathcal{O}(0.1 - 1) \left(\frac{g^2N^2}{4\pi}\right)^5 \left(\frac{N^2 - 1}{N}\right)T_g^4,$$  \hspace{1cm} (2.28)

for $g^2/4\pi \lesssim 0.1$. Around the critical temperature, $T_g \sim T_{g,c}$ the gauge coupling can be as large as $g^2N^2/4\pi = \mathcal{O}(1)$. In this regime, the sphaleron rate is expected as $\Gamma_{\text{sph}}(T_{g,c}) \approx T_{g,c}^4$ from the dimensional analysis. After the confining phase transition, no reliable calculation has been done so far. A crude estimation based on the dimensional analysis is that the dissipation rate is at most proportional to the entropy density (or number density) of the glueballs as $\gamma_{fr} \sim s_g/f_a^2$. Because the time dependence of the Hubble rate and the dissipation rate are given as $H \propto a^{-2}(a^{-3/2})$ in radiation dominated era (matter dominated era) and $\gamma_{fr} \propto a^{-3}$, the gluon induced friction term is important only when the temperature of the visible sector becomes larger than

$$T_g > \mathcal{O}(1) \left(\frac{10^9 \text{ GeV}}{N^3}\right) \left(\frac{10^{14} \text{ GeV}}{f_a}\right)^2 \left(\frac{g^2N^2}{4\pi}\right)^{-5}.$$  \hspace{1cm} (2.29)

Comparing this with the temperature when the axion starts to oscillate ($T_\gamma < \text{TeV}$), it is always irrelevant.

### III. EVOLUTION HISTORY

#### A. Evolution of the background gluon and glueballs

In Eqs. (2.18–2.25), we establish the continuity equations for the background and perturbative variables based on the equations of motion of the axion field, and gluo-thermodynamics. On one hand, the dependence on the temperature of $\rho_g$ and $\rho_p$ should be clarified. It is particularly easy for $T_g \gg T_{g,c}$ and $T_g \ll T_{g,c}$. In the limit of $T_g \gg T_{g,c}$, gluons are just relativistic fluid. Therefore,

$$p_g = \frac{\pi^2}{45}(N^2 - 1)T_g^4,$$

$$\rho_g = \frac{\pi^2}{15}(N^2 - 1)T_g^4 = 3\rho_g.$$  \hspace{1cm} (3.1)

and the axion-gluon interaction is negligible because

$$\frac{\rho_a}{\rho_g} \partial \ln \rho_g = \frac{2\eta_a\rho_a}{\rho_g} \approx \frac{0.7c_1\rho_a}{\rho_g} \left(T_{g,c}/T_g\right)^{2\eta_a + 4}.$$  \hspace{1cm} (3.2)

However, as the gluon temperature approaches $T_{g,c}$, Eq. (3.1) does not hold. The true dependence on the gluon temperature can only be figured out by the lattice calculation \[35\ [37]. We adopt the lattice data with $\theta = 0$, and deduce the densities for nonzero $\theta$ case based on the data of gluon/glueball pressure $\rho_g(T_g)$.

At $T_{g,c}$, the dark gluons are combined into dark glueballs with masses of multiples of the confining scale $\Lambda$. \[38\ [41]. Here, $\Lambda$ is defined with a certain regularization scheme. Taking the $\overline{MS}$ scheme, the lattice results with the functional method \[42\ [43\ show that

$$\frac{T_{g,c}}{\Lambda} \approx 1.2 + \frac{0.27}{N^2}. $$  \hspace{1cm} (3.3)

The phase transition is first-order if $N > 2$. In order to understand how long the phase transition happens, we compare the energy density in the deconfined phase with that of the confined phase at $T_{g,c}$. The former is the energy density of the gluon fluid of $O(0.1N^2T_{g,c}^4)$, and the latter is the sum of the tower of the glueballs of $O(0.1T_{g,c}^4)$. Thus, as $N$ increases, larger latent heat is released and the transition period becomes longer. Let us shortly discuss how we evaluate the energy density of the glueballs.

Since the lightest glueball mass is calculated as \[39\ [41\]

$$\frac{m_g}{T_{g,c}} = (5 - 6)\left(1 + \mathcal{O}(1/N^2)\right),$$  \hspace{1cm} (3.4)

all glueballs are non-relativistic at $T_{g,c}$. With the help of the spectral density $\hat{\rho}(m)$, the energy density of glueballs can be written as

$$\rho_g(T_{g,c}) = \int_0^\infty \frac{dm}{\hat{\rho}(m)} = \frac{(mT_{g,c})^{3/2}}{2\pi} e^{-m/T}. $$  \hspace{1cm} (3.5)

$\hat{\rho}(m)$ is approximated by the sum of the discrete low-lying resonances with masses $m_g(J PC)$, where $J$ (angular momentum), $P$ (parity), and $C$ (charge conjugation) are the
eigenvalues of the states, and the continuum spectrum of the Hagedorn tower

\[ \hat\rho(m) \simeq \sum_{m \leq m_{\text{th}}} (2J + 1) \delta (m - m_g(J^PC)) + \frac{nN}{m} \left( \frac{2\pi T_H}{3m} \right)^3 e^{m/T_H} \Theta (m - m_{\text{th}}), \]  

(3.6)

where \( n = 2, nN \geq 2 \), and the threshold mass \( m_{\text{th}} \) is usually chosen as \( 2m_g \). The Hagedorn temperature \( T_H \) is related with \( T_{g,c} \) as \( 43 \)

\[ \frac{T_H}{T_{g,c}} \simeq 1.16 - \frac{0.8}{N^2}. \]  

(3.7)

Note that even if \( N \) increases, the number of degrees of freedom of the glueballs does not increase. Therefore, the contribution from the Hagedorn glueballs is insensitive to \( N \) and becomes \( \mathcal{O}(0.01 - 0.1)T_{g,c}^4 \). The low-lying glueball contributions are \( \mathcal{O}(10 - 50)\% \) of it. We adopt the spectrum of the glueballs in Ref. 39.

The lower limit of the actual transition period is given by the period assuming the quasi-equilibrium transition, i.e., the pressures of deconfining/confining phases are equal and the latent heat is released adiabatically as the Universe expands. In this case, the temperatures of the confining and deconfining phases are the same and maintained at around \( T_{g,c} \). The duration of the phase transition is estimated by the conservation of the dark entropy:

\[ a_{cf} = a_{ci} \left( \frac{s_{\text{gluon}}(T_{g,c})}{s_{\text{glueball}}(T_{g,c})} \right) \simeq a_{ci} N^{2/3}, \]  

(3.8)

where \( a_{cf} \) (\( a_{ci} \)) is the scale factor when the phase transition ends (starts), \( s_{\text{gluon}}(T_{g,c}) \) (\( s_{\text{glueball}}(T_{g,c}) \)) denotes the entropy density of the dark gluon (glueball) at \( T_g = T_{g,c} \).

The phase transition becomes stronger first-order as \( N \) increases. Thus, the additional entropy is generated during the transition and makes the glueballs hotter than the previous estimation. However, this effect is negligible unless \( N \gg 4T \), because the nucleation temperature is just around \( T_{g,c} \) and the strong interactions of the gluon and glueball fluids provide a large friction coefficient for the bubble wall propagation. For \( N = 3 \), \( a_{cf} \simeq 2a_{ci} \) is obtained numerically, which is well matched with our parametric estimation \( N^{2/3} \). The assumption of dark entropy conservation will be kept in the following discussion.

After the transition, the Hagedorn contribution becomes negligible as the scale factor increases by an order of magnitude compared to \( a_{cf} \). The contribution of the low-lying glueballs maintains chemical equilibrium by two-to-two and three-to-two scatterings whose rates are estimated as

\[ \sigma_{2 \to 2V} \simeq \frac{v_f(4\pi/N)^4}{32\pi m_g^2} \]  

\[ \sigma_{3 \to 2g} \simeq \frac{(4\pi/N)^6}{(4\pi)^3 m_g^2}, \]  

(3.9)

where \( v_f \) is the relative velocity of the final particles from the scattering. The detailed description of the evolution of low-lying stable glueballs is provided in [16]. If the \( P \) and \( T \) violating topological \( \theta \)-term is absent, these glueballs can be classified as the eigenstates of \( J^PC \). In our set-up, because of the axion’s motion, the time dependent mixings can arise between different \( P \) eigenstates, e.g., between \( 0^{++} \) and \( 0^{-+} \). As a result, the terms like

\[ \Delta \mathcal{L} \sim m_g \frac{\phi^2}{f_a} \varphi_+^2 \varphi_-^2 \]  

(3.10)

may be generated. It turns out that theses mixings do not lead to the decay of heavier state to the light states by the kinematic reason. We expect that the stability of low lying glueballs is not altered by the axion’s evolution.

As the universe expands, the \( 3 \to 2 \) process freezes out when the most of the \( 2 \to 2 \) processes still active. This is because the interaction rate of the \( 3 \to 2 \) process is proportional to the square of the number density of the glueballs, while that of the \( 2 \to 2 \) processes is linearly proportional to the number density of the glueballs. Therefore, the relative chemical equilibrium holds between different glueball states. Since the \( 3 \to 2 \) process of the lightest glueball keeps the chemical potential of the lightest glueball to be zero, the chemical potential of all glueball states is zero until the \( 3 \to 2 \) process of the lightest glueball freezes out. Before the freeze-out of \( 3 \to 2 \) interactions, a single glueball state with a mass \( m_g \) in thermal equilibrium is given by

\[ \rho_g(T_g) = m_g \left( \frac{m_g T_g}{2\pi} \right)^{3/2} e^{-m_g/T_g} \left( 1 + \frac{27}{8} T_g m_g \Theta \left( \frac{T_g^2}{m_g^2} \right) \right), \]  

\[ p_g(T_g) = \frac{T_g}{2\pi} \left( \frac{m_g T_g}{2\pi} \right)^{3/2} e^{-m_g/T_g} \left( 1 + \frac{15}{8} T_g m_g \Theta \left( \frac{T_g^2}{m_g^2} \right) \right). \]  

(3.11)

The total energy density and pressure of the glueballs can be obtained by summing Eq. (3.11) over all low-lying stable glueballs. It is found that the contributions other than the lightest glueball become negligible for \( T_g \lesssim 0.5T_{g,c} \).

A distinguishing properties of the dark matter with the three-to-two self-interaction is the scaling behavior as the Universe expands. The temperature of the glueballs drops much slower than that of the photon during its chemical equilibrium maintaining \( s_g \propto 1/a^3 \), so \( T_g \sim 1/\ln a \). The energy density

\[ \rho_g \simeq T_g s_g \propto \frac{1}{a^3 \ln a}, \]  

(3.12)

drops faster that of a cold dark matter, since the three-to-two self-interaction converts the mass energy to the kinetic energy. This behavior ends when the process freezes out at \( T_g = T_{g,f-o} \) with

\[ \rho_g^2(T_{g,f-o}) \simeq \frac{(3m_g T_{g,f-o}) (H(T_g = T_{g,f-o}))}{(s_{3\to2}v_f^2)}. \]  

(3.13)

After that, the glueballs act as free-streaming particles. Using Eq. (3.13) and the dark entropy conservation, the
FIG. 1. An example of the evolution of the energy density for the case of the axion-dominated (left) and the glueball-dominated (right). After the confinement, the number-changing self-interaction of the glueballs reduces its total number only logarithmically. In the axion-dominated scenario, the energy density of the axion is $O(\rho_a/N^2)$ at the time of confinement and dominates that of the glueballs afterward. In the glueball-dominated scenario, the energy density of the gluon and the glueballs is much larger than that of the axion for all epoch.

FIG. 2. Parametric dependence of the relic abundance of the glueball and the axion for $\Omega_{\text{DM}}h^2 = 0.11$. $m_a$ is the zero-temperature axion mass and $T_{\text{g,fo}} \simeq T_{g,c}/r$ is the photon temperature when the confining phase transition of the dark sector starts. For the region above the line $R(r, f_a) = 1$, the oscillation of the axion starts earlier than the confinement phase transition and the glueball is the dominant dark matter component with the mass $m_g \simeq 6 r T_{g,c}$, while below $R(r, f_a) = 1$, the axion starts to oscillate after the transition. Here, we did not impose the constraints from the current bound, which are discussed in text.

Freeze-out temperature is evaluated as

$$T_{g,fo} \simeq \frac{m_g}{T_{g,fo}} + \frac{5}{4} \ln \frac{m_g}{T_{g,fo}} + \frac{3}{4} \ln \frac{m_g}{\text{MeV}}$$

$$\simeq 28.2 + \frac{3}{2} \ln \frac{r}{0.01} - \frac{7}{2} \ln \frac{N}{3},$$

(3.14)

If it happens during radiation dominated era. As a specific example, for $N = 3$, $r = 0.01$, and $m_g = 1 \text{ MeV}$, we get

$$T_{g,fo} \simeq 0.04 m_g \simeq 0.2 T_{g,c}.$$  

(3.15)

The relation $T_{g,fo} = O(0.2)T_{g,c}$ is not much sensitive to the values of $r$ and $m_g$ that we are interested in.

During the evolution of the glueballs, the photon temperature also evolves. When the dark glueballs freeze-out, the photon temperature becomes

$$T_{\gamma,fo} \simeq 3 \text{ keV} \left( \frac{N}{3} \right)^{1/2} \left( \frac{0.01}{r} \right)^{3/2} \left( \frac{m_g}{\text{MeV}} \right)^{5/4}. $$

(3.16)

We can also easily evaluate the case that the freeze-out of the dark glueball happens after the matter-radiation equality for $m_g < \text{keV}$.

So far, we have specified all history of the gluons and glueballs in order to identify the time dependence of the glueball temperature $T_g(\tau)$ for Eq. (2.25). For the final relic density of the glueballs, it can be evaluated in a much simpler way from the conservation of the entropy.
of dark sector as
\[ \Omega_g h^2 \simeq 0.014 \left( \frac{N^2 - 1}{10} \right) \left( \frac{r}{0.01} \right)^3 \left( \frac{T_{g,fo}}{10\text{keV}} \right) \left( \frac{3.94}{g_* S(T_{\gamma,c})} \right) \]
\[ \simeq 0.014 \left( \frac{N^2 - 1}{10} \right) \left( \frac{r}{0.01} \right)^4 \left( \frac{T_{\gamma,c}}{5\text{MeV}} \right) \left( \frac{3.94}{g_* S(T_{\gamma,c})} \right) \]
\[ \simeq 0.12 \left( \frac{N^2 - 1}{10} \right) \left( \frac{r}{0.003} \right)^3 \left( \frac{m_g}{100\text{MeV}} \right) \left( \frac{3.94}{g_* S(T_{\gamma,c})} \right). \]

(3.17)

B. Evolution of the background axion

The lattice studies provide the coefficient of each term in perturbation expansion of the axion scalar potential Eq. (2.3) as [17].

\[ c_1 \simeq 0.3 + \frac{1}{N^2}, \quad c_2 \simeq -2.7 c_1. \]  
(3.18)

One interesting feature of the axion potential is that it is not a single branch, but multiple \((N)\) branches, where for each branch the period of the scalar potential is \(2\pi N f_a\) [18]. The general expression of the scalar potential for a \(k\)th branch is

\[ V_k = N^2 \Lambda^4 h \left( \frac{\phi}{N f_a} + \frac{2\pi k}{N} \right) k = 1, \cdots, N. \]  
(3.19)

where \(h(\psi)\) is the \(2\pi\)-periodic function. At present, calculation of the full shape of the potential is not available. Besides the lattice study, the scalar potential for the part of axion field range was studied in the large \(N\) limit using the holographic description of the pure \(SU(N)\) gauge theory [49–50]. The 't Hooft coupling \(\lambda = g^2 N\) at the KK scale in the dual gravity theory can be matched with that deduced from the axion potentials of the lattice calculation Eq. (3.18). We find that \(\lambda = 10 - 20\) gives a reasonable matching.

At high temperatures of the gluons, the instanton approximation for the axion potential is valid, and there is a single branch. During the phase transition, branches will emerge, and the axion can be located in a different branch in a different patch of the Universe. If each branch provides a stable axion trajectory, we have to consider the effect of them seriously.

Following the approach of the holographic description as in [49], we estimate the tunneling rate between \(k\)th to \(k - 1\)th branches.

\[ \Gamma_{\text{tunneling}} \propto e^{-S(k \to k-1)}, \]  
(3.20)

where the Euclidean action is

\[ S(k \to k-1) = O(10^{-11}) N \frac{(N/k)^3}{1 + (c_1 k)^2 / N^2}. \]  
(3.21)

This can be significantly large only when \(N \gtrsim 10^4\). Therefore, in our consideration, all branches with higher energy densities are quite unstable, and the transition to the lowest energy state will occur almost immediately. As a result, the effective potential of the axion is well described by

\[ V(\phi) = \min_k V_k(\phi), \]  
(3.22)

and one can think the evolution of the axion within the range \(2\pi f_a\). Without worrying about the effect of other branches, Eq. (2.16) gives

\[ \phi'' + 2H \phi' + a^2 m_a^2(T_g) \phi = 0, \]  
(3.23)

for \(\phi \gtrsim f_a\). If the second term of the LHS is much larger than the third term, the axion field is approximately constant because of the large Hubble friction. This is the slow-roll limit. In the opposite case, the axion field oscillates with the oscillation frequency \(m_a(T_g)\). Such evolution can be well approximated by the simple transition at \(a = a_{osc}\) to satisfy \(3H = a m_a(T_g)\) \((3H = m_a(T_g))\).

For each epoch,

\[ \phi(\tau) \simeq \phi_i \equiv f_a \theta_i \quad (a < a_{osc}) \]  
(3.24)

\[ \simeq A(\tau) \cos \left( \int d\tau a(\tau) m_a(T_g(\tau)) \right), \quad (a > a_{osc}) \]

where \(\theta_i\) is the initial misalignment angle of the axion field, \(A(\tau)\) is slowly varying function with \(A' / A \ll 1\). The axion acts like dark energy during \(a < a_{osc}\), while for \(a > a_{osc}\), the axion plays the role of cold dark matter because \((w_a) \simeq 0\) by averaging out the fast oscillation. The initial axion value \(\phi_i\) is not deterministic. Since both \(\theta_i \ll 1\), and \(|\theta_i - \pi| \ll 1\) need some tuning or special model building, here we take

\[ \theta_i = O(1). \]  
(3.25)

Since the axion’s mass depends on the history of the dark gluons (Eq. (2.8)), there are two characteristic scales which determine the evolution history of the axion: \(a_{osc}\) (onset of the axion oscillation) and \(a_{c1}\) (onset of the confining phase transition). As the scale factor approaches \(a_{c1}\), the contribution of the gluons to the axion’s potential becomes substantial, and the axion mass is saturated. The evolution of the axion mass is smooth unless \(N\) is very large. There is no significant distinction between the deconfined and confined phases.

With the help of the lattice data, we find that the following quantity

\[ R(r, f_a) \equiv \left( \frac{r}{0.01} \right)^2 \left( \frac{6 \times 10^{13}\text{GeV}}{f_a} \right) \]  
(3.26)

determines whether or not the axion starts to oscillate before the confining transition. If \(R(r, f_a) > 1\), the axion starts to oscillate before the phase transition. The corresponding photon temperature becomes

\[ T_{\gamma,osc} \simeq R(r, f_a)^{3/4} T_{\gamma,c}. \]  
(3.27)

Otherwise \((R(r, f_a) < 1)\), the axion oscillation happens after the phase transition when \(T_{\gamma,c}\) becomes

\[ T_{\gamma,osc} \simeq R(r, f_a)^{1/2} T_{\gamma,c}. \]  
(3.28)
For the evolution of the axion density, the initial density of the axion at $T_s = T_{\gamma,osc}$ is approximated as
\[ \rho_a \simeq \frac{1}{2} m_a^2 (T_g, osc) f_a^2 \theta^2. \] (3.29)

After that, the axion field will oscillate with the time dependent frequency. For the combination
\[ N_a = \frac{a^2 \rho_a}{m_a (T_g)}. \] (3.30)

Eq. (2.18) gives
\[ N_a' + \left( 3H + \frac{m_a (T_g (\tau))}{m_a (T_g (\tau))} \right) w_a N_a = 0. \] (3.31)

Since the fast oscillation of the axion field means $\langle w_a \rangle = 0$, $N_a' \simeq 0$ and $N_a$ is nearly conserved, i.e. $\rho_a/m_a (T_g) \propto 1/a^3$. Therefore, if $R(r, f_a) > 1$, the axion starts to oscillate before the confining phase transition ($T_{\gamma, osc} > T_{\gamma,c}$), and the present relic density of the axion dark matter becomes
\[ \Omega_a h^2 \simeq 0.8 \times 10^{-3} \theta^2 \left( \frac{r}{0.01} \right)^4 \left( \frac{T_{\gamma,c}}{5 \text{ MeV}} \right) R(r, f_a)^{-\frac{3+2\theta}{4-\theta}}. \] (3.32)

If $R(r, f_a) < 1$, the axion oscillates after the confining phase transition ($T_{\gamma, osc} < T_{\gamma,c}$). The corresponding axion dark matter density is given by
\[ \Omega_a h^2 \simeq 0.8 \times 10^{-3} \theta^2 \left( \frac{r}{0.01} \right)^4 \left( \frac{T_{\gamma,c}}{5 \text{ MeV}} \right) R(r, f_a)^{-\frac{2}{4}} \]
\[ \simeq 0.05 \theta^2 \left( \frac{r}{0.01} \right) \left( \frac{T_{\gamma,c}}{5 \text{ MeV}} \right) \left( \frac{f_a}{10^{15} \text{ GeV}} \right)^{3/2} \]
\[ \simeq 0.15 \left( \frac{m_a}{10^{-22} \text{ eV}} \right)^{1/2} \left( \frac{f_a}{10^{17} \text{ GeV}} \right)^2. \] (3.33)

As shown in Eq. (3.32), the glueballs dominate the total energy density of dark matter if the axion oscillation happens earlier than the confining phase transition. The reason is simply that initial axion energy density before the oscillation is bounded by the confining scale $\Lambda^4$. On the hand, if $r^2/f_a$ becomes small enough, so that the axion starts to oscillate after the phase transition, the axion becomes a dominant component of dark matter.

Fig. 1 shows the evolution of dark matter densities for both glueball and axion domination scenarios. Fig. 2 shows the parametric dependence of the current relic abundance of the dark matter. The axes are represented by $T_{\gamma,c}$, the photon temperature when the confining phase transition of the dark gauge sector starts, and $m_a$, the zero temperature axion mass, defined in Eq. (2.18) with Eq. (4.11), respectively. As we discussed, the dark matter today is dominated by the axion in the region $T_{\gamma, osc} < T_{\gamma,c}$. The glueballs are dominant dark matter today in the region $T_{\gamma,c} < T_{\gamma, osc}$.

There are various astrophysical observations to constraint the mass of the glueball and axion dark matter. We shortly summarize the bounds on it. When the glueball dominates dark matter, its self-interaction gives observable effects if the scattering rate is large enough to reach the isothermal density profile inside the halo. It can be also detectable from the merger of dark matter halos, because the glueballs will be slowed down during the collision, which leads to the offset between the dark matter and the collisionless components like stars. The cross-section of the glueball like self-interacting dark matter is bounded as $\sigma_{\text{SI}} < \mathcal{O}(0.5 - 5) \text{ cm}^2/\text{g}$. (3.34)

In terms of the glueball mass, the glueball should have the mass greater than $\mathcal{O}(50)\text{MeV}$ if it is the dominant component of dark matter. The phenomenology of heavier glueball dark matter was studied in [52].

If the axion is the dominant dark matter component, there is also the lower bound on the axion mass due to its fuzziness. The de Broglie wavelength becomes astrophysical scale if $m_a$ is around $10^{-22} \text{ eV}$, and suppresses the structure formation. The ultra-light axion dark matter can act like waves that are bound to or interact with each other by gravity inside the halo, which leads to the formation of solitonic cores and macroscopic quasiparticles moving around the center. These structures can have a great influence on the motion of stars. All these considerations give the strong constraint on the axion mass in the range $m_a \lesssim 10^{-22} - 10^{-20} \text{eV}$ (53, 54 and references therein). There is another constraint due to the observation of highly spinning black holes, even if the axion is not dark matter. That is because if the axion mass is close to the inverse of the size of the spinning black hole, a superradiance phenomenon occurs and parts of black hole’s mass and spin are removed by the produced axion cloud. Current observations of the supermassive black hole with the mass of $10^6 - 10^7 M_\odot$ provide interesting constraints within the axion mass range $10^{-20} - 10^{-16} \text{eV}$ (55 and references therein). However, it is sensitive to the axion self-interaction, the constraint is model dependent.

When the dark matter is mostly composed of the axions, $\Omega_a h^2 \approx 0.11$, and the freeze-out of the glueball happens after the BBN, the fraction of the dark glueball subcomponent dark matter becomes
\[ f_g \equiv \frac{\Omega_g}{\Omega_{\text{DM}}} \approx 0.28 \left( \frac{N^2 - 1}{10^{0.01}} \right) \left( \frac{r}{0.01} \right)^3 \left( \frac{10^{15} \text{ GeV}}{f_a} \right)^{3/2} \]
\[ \approx 0.02 \left( \frac{N^2 - 1}{10} \right) \left( \frac{r}{0.01} \right)^3 \left( \frac{m_a}{0.05 \text{ MeV}} \right). \] (3.35)

One can think that there is no strong constraint on the subcomponent glueball dark matter for $f_g \lesssim 0.1$. However, the evolution of the glueball dark matter after structures form may alter the cosmological history of the Universe around $z = 7 - 15$ as discussed in Sec. IV.
IV. PERTURBATIONS

We now study the evolution of the cosmological perturbations for the axion and glueball dark matter. Both have non-trivial features compared to the CDM. For example, the transition of the nature of the axion from dark energy to dark matter modifies the early ISW effect of the cosmic microwave background [56]. The perturbation at scales smaller than the effective de Broglie wavelength is suppressed by its wave nature [3, 57, 58]. For glueballs, the number-changing self-interaction also disturbs the growth of the density perturbation of the scales which enter the horizon well before the freeze-out [11, 59].

On one hand, in our set-up, the dark sector is decoupled from the visible sector and the origin of their abundance can be totally different from that of the Standard Model particles. Let us provide a simple example for it. As the coupling from the visible sector and the origin of their number-changing self-interaction also disturbs, the initial amount of dark sector particles can be generated by a completely different mechanism than that of the Standard Model particles. For example, if the inflation Hubble parameter $H_I$ is given as $m_s \lesssim H_I \ll M_Q$, the $U(1)_{PQ}$ symmetry is not restored, and $s$ and $a$ are all light particles during inflation. The stochastic fluctuations of the scalar field during inflation $\delta \sim H_I/2\pi$ can yield its initial abundance after inflation. From Eq. (4.4), these scalars will eventually decay mostly to axions and partially to dark gluons as $\Gamma_a (s \to gg) \sim (N_g^2/8\pi^2)^2$ with a total decay rate $\Gamma_a \sim m_a^3/8\pi f_a^2$. Gluons will be quickly thermalized and form thermal bath with a temperature $T_g$, while the axions are just redshifted. Although this is just one of the production mechanisms for dark sector, it gives a good motivation to study the isocurvature perturbation of dark matter from the initial dark gluon temperature fluctuation $\delta T_g,i$. In this case, in addition to the adiabatic perturbation, there are two sources of the isocurvature perturbation. One is the fluctuation of the axion field during the inflation $\delta \phi_s$, and the other is the fluctuation of the gluon temperature $\delta T_g,i$, which is induced by initial perturbation of the decaying scalar as Eq. (4.4). Because the dark axions and the dark gluons are coupled with each other by $\partial V/\partial T_a$, in Eq. (2.25), both perturbations could be important for the final isocurvature perturbation of dark matter.

Based on the evolution of the background dark axion and dark gluon/gluaball, we solve the equations for the density perturbations focusing on the effect of isocurvature perturbation transfer and obtain the approximated solutions for the super-horizon modes ($k = 0$), in order to understand the parametric dependence more clearly.

A. Adiabatic perturbation

For the evolution of the multicomponent fields or fluids, the perturbations can be decomposed into the curvature (adiabatic) perturbation and the isocurvature (entropy) perturbations. The adiabatic perturbation is the modes perturbed along the direction of the background evolution, so that

$$S_{XY} = \mathcal{H} \left( \frac{\delta \rho_X}{\rho_X} - \frac{\delta \rho_Y}{\rho_Y} \right) = 0.$$  (4.5)

for any different species $X$ and $Y$ [60, 62]. $S_{XY}$ is the relative entropy perturbation, whose name can be easily understood from thermodynamics. For an isolated species which satisfies the continuity equation $\rho' X + 3H(\rho_X + p_X) = 0$, the perturbation of the entropy density $s_X$ is given as $\delta s_X/s_X = \mathcal{H} \delta \rho_X/\rho_X$, hence $S_{XY} = 3 \mathcal{H} (s_X/s_Y)$. The adiabatic mode can be described by the evolution of the comoving curvature perturbation $63,$

$$\mathcal{R} = \Phi - \frac{\mathcal{H}(\Phi' - \mathcal{H}\Psi)}{\mathcal{H}' - \mathcal{H}^2}.$$  (4.6)

The corresponding initial condition for the adiabatic mode is derived as

$$\Psi_i = -\Phi_i = -\frac{2}{3} \mathcal{R}_i, \quad \delta_{\gamma,i} = \frac{4}{3} \mathcal{R}_i,$$  (4.7a)

$$\delta_{a,i} = -\frac{2\mathcal{R}_i}{3} \mathcal{R}_i, \quad \delta_{g,i} = \frac{4}{3} \mathcal{R}_i,$$  (4.7b)

where $\delta_{\gamma}$ is for the photon fluid, the index $i$ indicates the time at which the initial perturbation is defined. Here we use the axion potential and mass in Eq. (2.3).
From Eq. [2.18] and [2.20], we derive the solutions for the super-horizon modes in radiation-dominated era.

\[
\Psi = -\Phi = -\frac{2}{3}R_i, \quad \delta_\gamma = \frac{4}{3}R_i, \quad (4.8a)
\]

\[
\delta_a = \left[ (1 + w_a) - (1 - w_a) \frac{m'_a(T_g(\tau))}{3Hm_a(T_g(\tau))} \right] R_i, \quad (4.8b)
\]

\[
\delta_g = \left[ (1 + w_g) + (1 - w_a) \rho_a \frac{m'_a(T_g(\tau))}{3Hm_a(T_g(\tau))} \right] R_i \quad (4.8c)
\]

where \(m'_a(T_g(\tau)) \equiv dm_a(T_g(\tau))/d\tau\). As the scale factor becomes larger than \(a_{ci}\), the terms proportional to \(m'_a\) is rapidly vanishing. One can show that from the continuity equation for the coupled gluon-axion fluid,

\[
\Delta \left( \rho_a \delta_a + \rho_g \delta_g \right)_{a = a_{ci}} = 0, \quad (4.9)
\]

the detailed evolution of \(m_a(T_g)\) around \(a = a_{ci}\) does not lead to the different final result.

Eq. (4.8) states that the adiabatic perturbation shares same form as \(\delta_X = (1 + w_X)R_c\) after the confining phase transition of the dark sector, and no history dependence happens, because \(S_{XY} = 0\) holds under the time evolution for the super-horizon modes. This is the characteristic feature of the adiabatic perturbation.

**B. Isocurvature perturbation**

The isocurvature perturbation is a mode perturbed along a direction orthogonal to the direction of background evolution. Taking \(S_X\) as

\[
S_X = \mathcal{H} \left( \frac{\delta \rho_X}{\rho_X} - \frac{\delta \rho_{tot}}{\rho_{tot}} \right), \quad (4.10)
\]

we can trace the evolution of the individual component of the isocurvature perturbation. At the linear perturbation level, the curvature perturbation cannot generate the isocurvature perturbations, and it is also conserved on the super-horizon scales \([61]\). Thus, for the evolution of isocurvature perturbations, we can safely take \(\mathcal{R}_i = 0\), so that \(\Phi_i = 0, \Psi_i = 0, \) and \(\delta \rho_{tot,i} = 0\) as the initial condition at high temperatures, and solve the perturbation equations of the density contrast of \(X\) with the initial nonzero \(\delta_{X,i}\).

The actual evolution of the density perturbation can be numerically calculated based on Eq. [2.25] and compared with the CMB and matter power spectrum. The form of the perturbation becomes particularly simple, if both the axion and glueballs becomes CDM-like well before the matter-radiation equality. In this case, the constraints on the isocurvature perturbation can be easily provided by comparing the analytic formula in super-horizon limit with the criteria of the Planck 2018 [1]. Therefore in this section, let us focus on this case.

The isocurvature perturbation of dark matter is expressed as

\[
(\hat{\delta}_{DM})_{iso} = \frac{\Omega_a}{\Omega_{DM}} (\hat{\delta}_a)_{iso} + \frac{\Omega_g}{\Omega_{DM}} (\hat{\delta}_g)_{iso}, \quad (4.11)
\]

where ‘hat’ denotes the Gaussian random variables satisfying \((\delta_{a,i}, \delta_{g,i}) = 0\), and

\[
\begin{bmatrix}
(\hat{\delta}_a)_{iso} \\
(\hat{\delta}_g)_{iso}
\end{bmatrix} = \begin{bmatrix}
T_{aa} & T_{ag} \\
T_{ga} & T_{gg}
\end{bmatrix} \begin{bmatrix}
\hat{\delta}_{a,i} \\
\hat{\delta}_{g,i}
\end{bmatrix}, \quad (4.12)
\]

with \(T_{aa} \simeq T_{gg} \simeq 1\). The off-diagonal elements of the transfer matrix, \(T_{ag}\) and \(T_{ga}\) are determined by Eq. [2.25] as follows.

1. **Induced by the initial displacement of the axion field**

   The evolution of the isocurvature perturbation induced by an initial density perturbation of the axion \(\delta_{a,i} = 2 \delta \phi_i/\phi_i\) can be described by the input value of \(\delta_{a,i}\) with the condition \(\mathcal{R}_i = 0\) and the associated solutions from Eq. (2.25). For \(T_g \gg T_{g,c}, \partial V/\partial T_g \approx 0\) and

   \[
   \Psi_i = -\Phi_i = 0, \quad \delta_{\gamma,i} = 0, \quad \delta_{g,i} = 0. \quad (4.13)
   \]

   In the axion dominated dark matter scenario \((\Omega_a \gg \Omega_g)\), the dominant contribution is trivial: \((\hat{\delta}_{DM})_{iso} \simeq \hat{\delta}_{a,i}\). In the opposite case \((\Omega_a \ll \Omega_g)\), the relevant equation for the glueball isocurvature perturbation induced by that of the axion is

   \[
   \delta_{g} + 3\mathcal{H}(c_g^2 - w_g)\delta_{g} \simeq \frac{(w_a - 1)\rho_a m'_a(T_g(\tau))}{\rho_g} \frac{\partial}{\partial T_g} \left( 1 + 3\mathcal{H}c_g^2 T_g \right) \delta_{a,i}, \quad (4.14)
   \]

   where \(c_g^2 \equiv (1 + 3\mathcal{H}c_g^2 T_g)\). Note that the combination of \(1 + 3\mathcal{H}c_g^2 T_g\) vanishing in the limit of \(\rho_a/\rho_g \gg 0\) as

   \[
   T_g \simeq \left( \frac{\partial \rho_g}{\partial T_g} \right)^{-1} \frac{d\rho_g}{dT_g} \simeq 3\mathcal{H}c_g^2. \quad (4.15)
   \]

   This implies that the transfer matrix element \(T_{ga}\), which represents the effect of the initial axion isocurvature perturbation to that of the glueball dark matter, is of \(O(\Omega_a^2/\Omega_g^2)\). It is the consequence of the conservation of the gluon/glueball entropy during the evolution. Therefore, the dominant contribution to the isocurvature perturbation of dark matter is just that from the axion sub-component dark matter. In summary,

   \[
   \begin{cases}
   (\hat{\delta}_{DM})_{iso} \simeq \hat{\delta}_{a,i} & \text{for } \Omega_g \gg \Omega_a, \\
   (\hat{\delta}_{DM})_{iso} \simeq \frac{\Omega_a}{\Omega_{DM}} \hat{\delta}_{a,i} & \text{for } \Omega_a \ll \Omega_g.
   \end{cases} \quad (4.16)
   \]

2. **Induced by the initial fluctuation of the gluon temperature**

   For the initial fluctuation of the gluon temperature \(\delta_{g,i} = 4\delta T_{g,i}/T_{g,i}\), the condition \(\mathcal{R}_i = 0\) and \(\delta \phi_i = 0\)
for the perturbative variables give the following initial values

\[
\begin{align*}
\Psi_i &= -\Phi_i = 0, \\
\delta_{v,i} &= -\frac{1}{2} T_{g,i}, \\
\delta_{a,i} &= \frac{\eta_a}{2} \delta_{g,i}.
\end{align*}
\]  

(4.17a)

In the glueball-dominated dark matter case, the contribution of the axion is suppressed by its energy density, so \( \delta_{\text{DM}} \approx \delta_{g,i} \). In the opposite case (\( \Omega_g \ll \Omega_a \)), the effect of the gluon temperature fluctuation to the axion perturbation can be captured by Eq. (4.22). There are three stages of the axion evolution: (I) slow-rolling period \( (H \gg m_a(T_g)) \), (II) confining phase transition with the saturation of the axion mass \( m_a(T_g) = m_a \), (III) axion oscillating period \( (H \ll m_a) \). For (I), the axion mass term is negligible and

\[
\delta \phi'' + 2H \delta \phi' + a^2 \phi \left( \frac{3 \eta_g^2 \, dm_a^2(T_g)}{4 \ln T_g} \right) \delta_{g,i} \approx 0.
\]  

(4.18)

The solution becomes

\[
\frac{\delta \phi}{\phi} \approx \frac{\eta_a}{2(2\eta_a + 4)} \left( \frac{m_a^2(T_g)}{H^2} \right) \frac{\delta_{g,i}}{a^2}.
\]  

(4.19)

After the confining phase transition, the perturbation of the axion in the periods (II), (III) obeys the equation of motion without \( \delta m_a \),

\[
\delta \phi'' + 2H \delta \phi' + a^2 m_a^2 \delta \phi = 0.
\]  

(4.20)

The general solution to Eq. (4.20) can be written as the sum of the Bessel functions

\[
\delta \phi = \frac{4H}{m_a} \sum_{\lambda = \pm} \delta \phi_{\lambda} J_{\lambda/4} \left( \frac{m_a}{2 H} \right),
\]  

(4.21)

with the constant coefficients \( \delta \phi_{\pm} \). Matching Eq. (4.19) and Eq. (4.21) at \( a = a_c \) determines \( \delta \phi_{\pm} \) and gives the solution for (II) and (III). In the period (III), \( \delta_a \) is given by

\[
\delta_a \approx \frac{\eta_a}{2\eta_a + 4} \left( \frac{m_a}{H_{c1}} \right)^2 \delta_{g,i}.
\]  

(4.22)

where \( H_{c1} \) is the Hubble rate at \( a = a_c \). Note that the transfer matrix element \( T_{g} \) is suppressed by the factor of \( m_a^2 / H_{c1}^2 \) whenever the axion dominates the dark matter, but no further suppression happens. Therefore,

\[
\begin{align*}
(\hat{\delta}_{\text{DM}})_{\text{iso}} &\approx \hat{\delta}_{g,i} \quad \text{for} \quad \Omega_g \ll \Omega_a, \\
(\hat{\delta}_{\text{DM}})_{\text{iso}} &\approx \left( \frac{\eta_a}{2\eta_a + 4} \frac{m_a}{H_{c1}} \right)^2 \frac{\Omega_g}{\Omega_{\text{DM}}} \hat{\delta}_{g,i} \quad \text{for} \quad \Omega_g \ll \Omega_a.
\end{align*}
\]  

(4.23)

C. Bound on the isocurvature perturbation

Since \( \delta \phi_i \) and \( \delta T_{g,i} \) are independent random fluctuations, the power spectrum can be decomposed as

\[
P(k, z) = P_{RR}(k, z) + \sum_{X = a, g} P_{II,X}(k, z),
\]  

(4.24)

where \( X = a, g \) stand for the isocurvature perturbations induced by \( \delta a_i \) and \( \delta g_i \), respectively. For the decomposition of the power spectrum as \( P(k, z) = (2\pi^2/k^3)P(k)T(k, z) \), we can match the primordial spectrum \( P(k) \) with the values we obtained for the super-horizon modes in the previous section. From the observations, the adiabatic mode is nearly scale-independent. For the isocurvature perturbations produced during the inflation, we can also naturally assume they are nearly scale-invariant,

\[
P_{RR} = A_s \left( \frac{k}{k_*} \right)^{n_s - 1}, \quad P_{II,X} = A_X \left( \frac{k}{k_*} \right)^{n_X - 1}
\]  

(4.25)

where \( k_* \) is the pivot scale of the wave number, and parameters \( \{A_X, n_X\} \) are constants.

Observation of the CMB presents the upper bound on the isocurvature perturbation \[1, 64–66\]. The constraint is expressed by the bound on the isocurvature fraction \( \beta_{\text{iso}} \), which is defined by

\[
\beta_{\text{iso}}(k) = \frac{P_{II}(k)}{P_{RR}(k) + P_{II}(k)}
\]  

(4.26)

where \( P_{RR} \) and \( P_{II} \) are the power spectra defined in Eq. (4.25). For density perturbations, \( P_{RR} = k^2 \langle \mathcal{R}_i^2 \rangle / 2\pi^2 \) and \( P_{II} = \sum_X k^2 \langle \delta_{\text{DM}} \rangle^2 / 2\pi^2 \). We focus on the large scales to constrain the primordial perturbation from the CMB data. The constraint on \( \beta_{\text{iso}} \) for the pivot scale is given by \[1\]

\[
\beta_{\text{iso}}(k_* = 0.002 \text{ Mpc}^{-1}) < 0.035.
\]  

(4.27)

We can compare this constraint with the values of \( \beta_{\text{iso}} \) in our scenario.

If the axion dominates dark matter, \( \Omega_g \ll \Omega_a \approx \Omega_{\text{DM}} (R(r, f_a) < 1 \text{ for Eq. (3.26)}) \), the fraction \( \beta_{\text{iso}} \) can be expressed as

\[
\beta_{\text{iso}} \approx \frac{\delta_{g,i}^2}{\mathcal{R}_i^2} + \left( \frac{\eta_a}{2\eta_a + 4} \frac{m_a}{H_{c1}} \right)^2 \frac{\Omega_g}{\Omega_{\text{DM}}} \left( \frac{\delta_{g,i}^2}{\mathcal{R}_i^2} \right)
\]  

(4.28)

where

\[
\frac{m_a}{H_{c1}} \approx 3R(r, f_a), \quad \frac{\Omega_g}{\Omega_{\text{DM}}} \approx (N^2 - 1)R(r, f_a)^{3/2}.
\]  

(4.29)

Because \( R(r, f_a) < 1 \), the term proportional to \( \Omega_g / \Omega_{\text{DM}} \) is always the dominant contribution in the second term of the RHS. In the opposite case, if the glueball is the main dark matter component, \( \Omega_a \ll \Omega_g \approx \Omega_{\text{DM}} (R(r, f_a) > 1) \), we have

\[
\beta_{\text{iso}} \approx \frac{\delta_{g,i}^2}{\mathcal{R}_i^2} + \left( \frac{\Omega_a}{\Omega_{\text{DM}}} \right)^2 \left( \frac{\delta_{g,i}^2}{\mathcal{R}_i^2} \right)
\]  

(4.30)
where
\[ \frac{\Omega_a}{\Omega_{DM}} \simeq \frac{1}{(N^2 - 1)R(r, f_a)^{3+\epsilon_a} + \rho_{\text{sol}}}. \]

Although the coupling between the axion and the gluon is key to the amount of axion dark matter, the contributions of the same coupling to the isocurvature perturbation is always subdominant. The perturbation is close to the sum of the independent elements, so it allows a large isocurvature perturbation of the subcomponent dark matter. The effect of such a large isocurvature perturbation is not clear yet. Since the glueballs are strongly self-interacting particles, it may provide non-trivial effects if the glueball is the subcomponent dark matter with a large isocurvature perturbation.

In the following section, we study the somewhat different aspect of the subcomponent glueball dark matter in the late time Universe.

V. SUBCOMPONENT GLUEBALL DM: FORMATION OF SUPERMASSIVE BLACK HOLE

The subcomponent self-interacting dark matter can play a certain role in formation of supermassive black holes (SMBH). If the self-interaction is strong enough, the gravo-thermal collapse of the subcomponent dark matter can occur at the center of the dark matter halo, leading to the black hole formation at high redshifts \( z \gtrsim 7 \) [14]. This can provide a possible explanation of the SMBH observations with masses around \( 10^9M_\odot \) at \( z \sim 7 \). From the quasar observations, we have the list of SMBHs (\( z_{\text{obs}}, M_{\text{BH}} \)) as J1342+0928 (7.54, 7.8 \times 10^8M_\odot), J1120+0641 (7.09, 2.0 \times 10^9M_\odot), J2348-3054 (6.89, 2.1 \times 10^9M_\odot) and also J0100+2802 (6.3, 1.2 \times 10^{10}M_\odot) [16–19].

In the standard mechanism on the formation and growth of black holes, SMBHs efficiently increase their mass by the accretion of baryonic material. However, the rate is limited, because the radiation pressure slows down the absorption. The maximal growth rate is captured by the Salpeter time based on the Eddington limit [67–69],
\[ t_{\text{Sal}} = \frac{\epsilon \sigma_t}{4\pi G m_p} = \left( \frac{\epsilon}{0.1} \right) 45 \text{ Myr}, \]
where \( \epsilon \) is the efficient factor, \( m_p \) is the proton mass. If the seed black hole is generated at \( t_i \) with a mass \( M_{\text{seed}} \),
\[ M_{\text{BH}}(t) \lesssim M_{\text{seed}} e^{t/t_{\text{rel}}}. \]
If the seed black hole is formed at \( z = 15 \), the maximal black hole mass becomes \( (2 - 6) \times 10^9M_{\text{seed}} \) at \( z = 7 \). If the seed is formed at \( z = 30 \), its mass becomes \( (6 - 10) \times 10^9M_{\text{seed}}. \) Therefore in order to explain the SMBHs with masses of \( \mathcal{O}(10^9M_\odot) \) at \( z \sim 7 \), a seed mass should be greater than \( (10^4 - 10^8)M_\odot. \) This is quite challenging in the standard theory of black hole formation.

On the other hand, by solving the gravo-thermal fluid equations [14] and performing \( N \)-body simulation [15] with the assumption that the host halo is isolated, it is shown that such a heavy seed black hole could be generated from the gravo-thermal collapse of the subcomponent dark matter. Given the NFW density profile
\[ \rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2}, \]
for the dominant DM component, the seed black hole is formed with the mass
\[ M_{\text{seed}} \simeq \beta f_g M_h, \]
when the age of the Universe becomes
\[ t(z_{\text{col}}) = t(z_i) + \Delta t_{\text{col}}. \]
Here, \( M_h \) is the mass of the host halo, \( z \) corresponds to the redshift. \( t(z_i) \) is the time when the virialized dark matter halo is isolated as we assume. \( \Delta t_{\text{col}} \) is the duration of the gravo-thermal collapse of the subcomponent dark matter for given initial conditions. \( \beta_1 \) and \( \Delta t_{\text{col}} \) are both calculated numerically. The fraction factor \( \beta_1 \simeq 0.025/(\ln(1 + c) - c/(1 + c)) \) in [14], where \( c \) is the concentration of the NFW profile \( (M_h = 4\pi \rho_s r_s^3(\ln(1 + c) - c/(1 + c)) \), and \( \beta_1 \simeq 0.006 \) in [15]. By comparing the dark matter halo density profiles of two papers, we find that both results are well matched. The formation period \( \Delta t_{\text{col}} \) is estimated as the form
\[ \Delta t_{\text{col}} \simeq \beta_1 f_g^{-2} t_{\text{rel}} \]
where \( \beta_2 \simeq 456(480), p = 0 \) (2) in [14] [15], and the apparent relaxation time of the subcomponent dark matter at \( t = t(z_i) \) is defined as
\[ t_{\text{rel}} \equiv \frac{m_g}{f_g \sigma_g \rho_s v_s} \]
\[ = 0.28 \text{ Myr} \left( \frac{10 \text{ cm}^2/\text{g}}{f_g \sigma_g/m_g} \right) \left( \frac{10^9M_\odot/\text{km}^3}{\rho_s} \right)^{3/2} \left( \frac{3 \text{ kpc}}{r_s} \right). \]
\( \sigma_g \) is the elastic scattering cross-section between two subcomponent dark matters (dark glueballs in our case), \( v_s \) is the virialized velocity at \( r = r_s \). Then the seed black hole can form after the period
\[ \Delta t_{\text{col}} \simeq 130 \text{ Myr} \left( \frac{10 \text{ cm}^2/\text{g}}{f_g \sigma_g/m_g} \right) \]
\[ \times \left( \frac{10^9M_\odot/\text{km}^3}{\rho_s} \right)^{3/2} \left( \frac{3 \text{ kpc}}{r_s} \right). \]
Note that \( \Delta t_{\text{col}} \) can be shorter than the age of the Universe for a given \( z, t(z) \simeq 550 \text{ Myr}(\frac{10^9M_\odot}{1+z^2})^{3/2}. \) Therefore, for the isolated halo with a mass \( M_h = 10^{12}M_\odot, \) \( f_g \sigma_g/m_g \gtrsim (1 - 10) \text{ cm}^2/\text{g}, \) and \( f_g \leq 0.001 \) – 0.01 can explain the SMBH around \( z \sim 7 \). We illustrate the formation of the seed black hole and its growth history in Fig. [3] for the halo mass \( M_h = 10^{12}M_\odot. \)

In our scenario, the dark glueball dark matter provides a such strongly interacting subcomponent dark matter.
the axion, its decay constant is the parameters of the dominant component of dark matter (curve or on the shaded area in which the observations are explained by slower growth of the seed black hole. The time of collapse ($z_{coll}$) and the mass of the seed black hole $M_{seed}$ are determined by model parameters $\{f_g, \sigma_g/m_g\}$ or $\{m_g, r\}$.

Since $\beta_2$ and $p$ are directly estimated in N-body simulation, we take the result of $15$ ($\beta_2 = 480, p = 2$) as the benchmark value. Then, the relevant combination of the model parameters is $f_g^3\sigma_g/m_g$, which is estimated as

$$f_g^3\sigma_g/m_g = \left(\frac{3}{N}\right)^4 \left(\frac{f_g}{m_g}\right)^3 \approx 40 \text{ cm}^2/\text{g} \left(\frac{3}{N}\right)^4 \left(\frac{N^2 - 1}{10}\right)^3 \left(\frac{r}{0.005}\right)^9. \quad (5.9)$$

For the final expression, Eq. (3.35) is used. Because it is very sensitive to $r$, the ratio parameter is nearly predicted from the explanation of the SMBH at high $z$. The corresponding allowed range of the glueball mass is also provided as $m_g = O(0.05)\text{MeV}$ for $f_g = O(0.001)$. As to the parameters of the dominant component of dark matter, the axion, its decay constant is $f_a = O(10^{16})\text{GeV}$ and the axion mass becomes $m_a = O(10^{-18})\text{eV}$. This is safe from the current fuzzy dark matter constraints. Interestingly, this axion mass is also related with the supermassive black hole with the mass of $M_{BH} \sim 10^7M_\odot$ through superradiance as we discussed before. The axions can be efficiently generated from the spinning black hole by superradiant amplification. During the amplification, the axion also takes away the sizable amount of the black hole’s angular momentum, which gives the contradiction to the observation [89]. However, if the self-interaction among the axions is sizable, they will collapse before the axion cloud is saturated [70], and the loss of the angular momentum is limited. For $m_a \sim 10^{-18}\text{eV}$, the GUT scale decay constant provides a sizable axion self-interaction to trigger bosonovae. Therefore, the constraint may not be applied directly.

Several simplifications are used in the previous discussion. Let us discuss possible caveats and alternative history of the seed black hole formation. The host halo mass is taken as $10^{12}M_\odot$. This is because the halo mass is expected to be greater than $O(10^3)$ times the mass of its SMBH [71, 72]. In N-body simulations [73, 74], the comoving number density of the dark cold matter halos with $M_h \geq 10_{12}M_\odot$ is evaluated as $(10^{-5} - 10^{-6})(\text{Mpc})^{-3}$ at $z = 7$. Thus, the halo is also heavy enough to coincide with the fact that observations of SMBH around $z = 7$ are rare.

However, since we consider the formation of the seed black hole at higher redshifts ($z > 7$), the existence of such (isolated) heavy halo is questionable. If we extrapolate the halo mass function obtained by the N-body simulation [75], the comoving number density of the halos with $M_h \geq 10_{12}M_\odot$ becomes $(10^{-8} - 10^{-9})(\text{Mpc})^{-3}$ at $z = 10$, and $10^{-15}(\text{Mpc})^{-3}$ at $z = 15$. In this context, the issue of formation of heavy seed black holes is just transferred to the problem of supermassive halo formation at high redshifts.

On one hand, based on N-body simulations, we can define $M_h(z)$ at a given $z$ such that the comoving number density of the halos with their masses greater than $M_h(z)$ is given by $10^{-6}(\text{Mpc})^{-3}$. Then, $M_h(z)$ is evaluated as $10_{12}M_\odot$ at $z = 7$, $10_{11}M_\odot$ at $z = 10$, and $10_{10}M_\odot$ at $z = 15$. It is more natural to think the possibility that when the seed black hole is formed, the mass of the host halo is smaller than $10_{12}M_\odot$, although it is still one of the heaviest halos at $z_i$. These heaviest halos get bigger and bigger by mergers with nearby smaller halos or by accretion of the gases. The actual merger history is quite complex, but the heaviest halo is likely to remain the heaviest. In this sense, we consider $M_h(z)$ as the evolution of the host halo mass, and estimate the growth rate $\Gamma_h(z)$ as

$$\Gamma_h(z) = \frac{1}{M_h(z)} \frac{dM_h(z)}{dt} \approx \frac{4}{\tilde{t}(z)}. \quad (5.10)$$

The last equality holds numerically for $7 \lesssim z \lesssim 15$. The black hole growth rate by the accretion of baryons is much greater than the halo growth rate. However, the halo mass is still hierarchically larger than the black hole mass during the evolution.

The another important feature is that in terms of the halo mass, the relaxation time defined by Eq. (5.7) depends on $z$, $c$ and $M_h$ as

$$t_{rel} \propto \left(\frac{\ln(1 + c) - \frac{c}{1 + c}}{(1 + z)^2 c^2 M_h^2}\right). \quad (5.11)$$

The concentration parameter $c$ also depends on the halo mass and the redshift. The recent N-body simulation [76] calculates the concentration parameter $c(M_h, z)$ as the function of $M_h$ and $z$ in a wide range of $M_h$.
and z. With the reasonable extrapolation, we find $c(10^{10} M_\odot, 13.8) \approx c(10^{11} M_\odot, 9.5) \approx c(10^{12} M_\odot, 7) = 4 - 5$. Thus, $c(M_h(z), z)$ does not have significant dependence. Including all these considerations, Fig. 4 shows the apparent gravo-thermal collapse period $\Delta t_{\text{col}}$ as the function of $M_h$ and $z$ in the unit of $(f_g^3 \sigma_g/m_g)^{-1}$. The formation of the seed black hole is more efficient for heavier halos at a given $z$. In order to see whether or not the early formation of the seed is preferred ($z$-dependence), we have to compare $\Delta t_{\text{col}}$ with the Hubble time. Numerically, we find that the $z$ dependence of $\Delta t_{\text{col}}$ for $M_h = M_h(z)$ approximately scales as $1/(1 + z)^{1.5}$. In the range $z = 7 - 15$ like the Hubble time. Therefore, if the seed black hole can form, the formation happens at earlier time with a smaller mass.

Even if $\Delta t_{\text{col}}$ is shorter than the age of the Universe at $z_1$, the isolated halo assumption may not be valid if the period of the gravo-thermal collapse is longer than the halo growth time scale $1/\Gamma_h$. The general expectation is that the merger process will hinder the gravo-thermal collapse. We consider the conservative criterion for the formation of the seed black hole as

$$\Gamma_h(z) \Delta t_{\text{col}}(z) \lesssim 4 \Delta t_{\text{col}}(z)/t(z) \lesssim 1. \quad (5.12)$$

This condition means that the seed black hole can only form when the collapse process is faster than the growth rate of the halo mass. We take $z_1 = 15$ as the initial redshift for the virialized heaviest host halo. Then, Eq. (5.12) is satisfied if

$$\frac{f_g^3 \sigma_g}{m_g} \gtrsim 40 \text{cm}^2/\text{g}. \quad (5.13)$$

After the seed black hole is formed around $z = 15$, its mass is exponentially growing, and it becomes $M_{\text{BH}} = 10^9 M_\odot$ at $z = 7$ if the fraction of the glueball dark matter is given as $f_g = 2 \times 10^{-4}$. This is the case of the fastest growth, so the lower bound of $f_g$ to explain current observations of the SMBHs is given by

$$f_g \gtrsim 2 \times 10^{-4}. \quad (5.14)$$

So far, we have ignored the effect of the number-changing interactions of the dark glueballs during the gravo-thermal collapse. If the number-changing process becomes efficient as the density increases, the sizable pressure of the glueballs may disturb the collapse. To simplify our discussion, in terms of the temperature of the glueball dark matter ($T_g$) inside the dark matter halo ($r < r_s$), there are two totally different sources to increase $T_g$. One is the gravo-thermal collapse. Because the gravitationally bound system has a negative specific heat, as heat flows outward, the glueballs become more and more concentrated in a smaller volume with a larger virial velocity. This results in temperature increasing, and leads to the collapse as the heat outflow accelerates. On the other hand, the $3 \to 2$ scatterings directly produce the large kinetic energies of the daughter glueballs as $E_{\text{kin}} \approx m_g/2$, respectively. These energies will be redistributed among glueballs within the relaxation time, so that the overall glueball temperature will increase compared to the virial temperature, and inhibit to collapse.

In order to figure out the condition that the gravo-thermal collapse can start, we require a criterion that the rate of glueball temperature increase is small enough to satisfy

$$\frac{\Delta t_{\text{col}}}{T_g} \left( \frac{dT_g}{dt} \right)_{3\to2} \ll 1. \quad (5.15)$$

The temperature increase rate by the three-to-two scatterings is estimated for the given glueball density $\rho_g$ and the velocity $v_g$

$$\frac{1}{T_g} \left( \frac{dT_g}{dt} \right)_{3\to2} = \xi_{\text{eff}} (\sigma_{3\to2} v_g^2) n_g^2 m_g \rho_g \gtrsim \frac{v_{3\to2}^2}{m_g^2 (v_g^2)}, \quad (5.16)$$

where $T_g = m_g (v_g^2)$, $\xi_{\text{eff}}$ is the efficiency factor of the energy redistribution. $\xi_{\text{eff}}$ could be suppressed if the mean-free path of the glueball is much larger than the size of the core. In our case, most of the glueballs are trapped by the elastic scattering, so $\xi_{\text{eff}} \approx 1$. Before the gravo-thermal collapse accelerates, the glueball density and the velocity are not much changed. For $\rho_g = f_g \rho_s$, $v_g = v_s$,

$$\frac{\Delta t_{\text{col}}}{T_g} \left( \frac{dT_g}{dt} \right)_{3\to2} \approx 0.06 \left( \frac{10^{-3}}{f_g} \right) \left( \frac{3}{N} \right)^2 \left( \frac{10^{-3}}{v_s} \right)^2 \times \left( \frac{keV}{m_g} \right)^4 \left( \frac{\rho_s}{10^{12} M_\odot / \text{kpc}^2} \right). \quad (5.17)$$
Therefore, we expect that the gravo-thermal collapse for the SMBH would not be triggered if \( m_g < 10^3 \) keV.

If \( m_g \) is much larger than \( \mathcal{O}(\text{keV}) \), the three-to-two interaction is not effective before the gravo-thermal collapse happens. The gravo-thermal collapse begins to accelerate after \( \Delta t_{\text{col}} \). During the collapse, the diffusion of the dark matter mass is inefficient, and the glueballs concentrate their mass of \( \mathcal{O}(M_{\text{seed}}) \) around the center by increasing the core density and its temperature \( T \). Then, the number changing interaction becomes gradually important. It is not clear how it affects the last stage of the gravo-thermal collapse (the formation of the seed black hole). This is because the temperature increase rate caused by gravo-thermal collapse is not known yet for such a high mass density of the core. We leave it for future work.

There is also the lower bound on the glueball mass from the cosmological evolution. If the glueball is light enough, it becomes a warm or hot dark matter, so that its speed around \( z = 7 - 15 \) is greater than the escape velocity of the halo. This means that the subcomponent dark matter is not clustered, and cannot provide a good initial condition. After the dark glueball freeze-out, its velocity scales as \( 1/\alpha \). The corresponding redshifted glueball velocity at a given \( z \) is

\[
v_g(z) \approx 10^{-3} \left(\frac{1 + z}{16}\right) \left(\frac{r}{0.001}\right) \left(\frac{100 \text{ eV}}{m_g}\right) \frac{3}{2} \tag{5.18}
\]

if the freeze-out happens before the epoch of matter-radiation equality, and

\[
v_g(z) \approx 10^{-3} \left(\frac{1 + z}{16}\right) \left(\frac{r}{0.001}\right) \left(\frac{100 \text{ eV}}{m_g}\right) \frac{4}{3} \tag{5.19}
\]

if the freeze-out happens in the dark matter dominated era. In order to explain the SMBH formation, this value should be hierarchically smaller than the virial velocity \( v_* \sim 10^{-3} \) during the period \( z = 7 - 15 \). In our scenario, \( r \) is nearly fixed as 0.005. See Eq. (5.9). This implies the lower bound on \( m_g \) as 100 eV.

**VI. DISCUSSION**

We have studied the cosmological evolution of dark light scalars, whose masses and interactions originate from the approximate global symmetry and the non-perturbative dynamics of the hidden gauge symmetry. One is the feebly interacting dark axion, and the other is the strongly interacting dark glueball. Both can be dark matter if they are light enough. The equations of motion are derived and evaluated to identify the dark matter abundance and the perturbation evolution induced by the coupling between the axion and the dark gluon. We also explore the possibility that the subcomponent glueball dark matter contributes to the formation of the supermassive black hole at redshift \( z \sim 7 \).

Although we have dealt with the problems as closely as possible, there are still many questions that have not been covered by this paper. What would be the observable consequences of the first-order confining phase transition? In our discussion, we ignore gravitational wave productions during the confining phase transition, because it is weakly first-order unless \( N \) is very large. However, if the phase transition happens around the recombination era, it may leave a footprint on the CMB. What is the exact form of the axion scalar potential and the effect of self-interactions? The scalar potential of the axion is not a simple cosine form, and a multi-branch structure may provide the nontrivial effects if the axion is produced around the spinning supermassive black hole by superradiant amplification. What is the correct period of the gravo-thermal collapse when the fraction of the subcomponent dark matter is small enough? So far, there is no intensive study on the gravo-thermal collapse of the subcomponent dark matter for such a small fraction. The empirical form of the collapse time scale \( \Delta t_{\text{col}} \) should be confirmed for \( f_g \ll 10\% \) and higher scattering cross-sections. What is the effect of the number changing interactions of the glueball dark matter for the final stage of the black hole formation? During the gravo-thermal collapse, one may think of the possibility that the defining phase transition occurs, because of the large density of the glueball dark matter inside the core. It would be very interesting to study the implication of such a microscopic nature of the dark matter for the final formation of the black hole.

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