Effective Non-vanishing for Algebraic Surfaces
in Positive Characteristic

Qihong Xie

Dedicated to Professor Musheng Yao on the occasion of his sixtieth birthday

Abstract

We give a theorem on the effective non-vanishing problem for algebraic surfaces in positive characteristic. For the Kawamata-Viehweg vanishing, the logarithmic Kollár vanishing and the logarithmic semipositivity, we give their counterexamples on ruled surfaces in positive characteristic.

1 Introduction

In this paper, we shall consider the following effective non-vanishing problem.

**Problem 1.1.** Let $X$ be a normal proper algebraic variety over an algebraically closed field $k$, and $B = \sum b_i B_i$ an effective $\mathbb{R}$-divisor on $X$ such that $(X, B)$ is Kawamata log terminal. Let $D$ be a nef Cartier divisor on $X$ such that $H = D - (K_X + B)$ is nef and big. Find the smallest positive integer $m$ such that $H^0(X, mD) \neq 0$.

In this problem, we may require the smallest positive integer $m$ is universal in the sense that it depends only on the dimension of $X$. Furthermore, Ambro and Kawamata have conjectured that, if the characteristic of $k$ is zero then $m$ is equal to one, which is called the effective non-vanishing conjecture (cf. [Am99, Ka00]).

**Conjecture 1.2 (Effective Non-vanishing).** With the same assumptions as in Problem 1.1, assume further that char($k$) = 0. Then $H^0(X, D) \neq 0$ holds.

For the convenience of the reader, we give some necessary definitions.

**Definition 1.3.** Let $X$ be a normal proper algebraic variety over an algebraically closed field $k$, and $B = \sum b_i B_i$ an effective $\mathbb{R}$-divisor on $X$. The pair $(X, B)$ is said to be Kawamata log terminal (KLT, for short), or to have Kawamata log terminal singularities, if the following conditions hold:

1. $K_X + B$ is $\mathbb{R}$-Cartier, i.e. $K_X + B$ is an $\mathbb{R}$-linear combination of Cartier divisors;
2. For any birational morphism $f : Y \to X$, we may write $K_Y + B_Y \equiv f^*(K_X + B)$, where $\equiv$ means numerical equivalence, and $B_Y = \sum a_i E_i$ is an $\mathbb{R}$-divisor on $Y$. Then $a_i < 1$ hold for all $i$.

Firstly, it follows from (2) that $[B] = 0$, i.e. $b_i < 1$ for all $i$, where $[B] = \sum [b_i] B_i$ is the round-down of $B$.

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Secondly, this definition is characteristic free, hence it makes sense in positive characteristic as well as Problem 1.1.

Thirdly, provided that \( \text{char}(k) = 0 \) or \( \dim X \leq 2 \), then \( X \) admits a log resolution, i.e. there exists a desingularization \( f : Y \to X \) from a nonsingular variety \( Y \), such that the union of the strict transform \( f^{-1}B \) of \( B \) and the exceptional locus \( \text{Exc}(f) \) of \( f \) has simple normal crossing support. In this time, condition (2) holds for all birational morphisms is equivalent to that it holds for a log resolution of \( X \). Note that the existence of resolution of singularities in positive characteristic is conjectural for higher dimensions.

Let us mention a simple example of KLT pair. Let \( X \) be a nonsingular variety and \( B \) an effective \( \mathbb{Q} \)-divisor on \( X \) such that \( \lceil B \rceil = 0 \) and \( \text{Supp}(B) \) is simple normal crossing. Then \( (X, B) \) is KLT.

Assume that \( B = 0 \) and that \( K_X \) is \( \mathbb{Q} \)-Cartier. We write \( K_Y \equiv f^*K_X + \sum a_i E_i \), where \( E_i \) are all exceptional divisors of \( f \). Then \( X \) is said to be terminal, or have terminal singularities if \( a_i > 0 \) for all \( i \) and for all birational morphisms \( f : Y \to X \). For instance, when \( \dim X = 2 \), \( X \) is terminal equivalent to that \( X \) is nonsingular. Reid and Mori gave the classification of all 3-dimensional terminal singularities.

Similarly, we can give the definitions of other types of singularities, such as canonical, purely log terminal, divisorial log terminal and log canonical. We refer the reader to [KM98] for more details.

Let \( D \) be an \( \mathbb{R} \)-Cartier divisor on \( X \). \( D \) is said to be nef, if \( D.C \geq 0 \) holds for any irreducible proper curve \( C \) in \( X \). \( D \) is said to be nef and big, if \( D \) is nef and if the top self-intersection \( D^n > 0 \) holds for \( n = \dim X \).

The assumptions in Problem 1.1 are standard from the viewpoint of the minimal model theory. We have the following celebrated theorems (cf. [KMM87] Theorems 1-2-5,2-1-1,3-1-1):

**Theorem 1.4 (Kawamata-Viehweg Vanishing).** With the same assumptions as in Conjecture 1.2. Then \( H^i(X, D) = 0 \) holds for any \( i > 0 \).

**Theorem 1.5 (Non-vanishing and Base Point Free).** With the same assumptions as in Conjecture 1.2. Then for any \( m \gg 0 \), \( H^0(X, mD) \neq 0 \) holds and the linear system \( |mD| \) is base point free.

Clearly, Problem 1.1 just to find the minimum of \( m \) in the non-vanishing theorem. From the Kawamata-Viehweg vanishing theorem, it follows that in Conjecture 1.2 \( H^0(X, D) \neq 0 \) equivalent to \( \chi(X, D) \neq 0 \), which shows this conjecture is indeed a topological problem in some sense.

We digress to give the history behind Conjecture 1.2 and to convince the reader that this conjecture is closely related to the minimal model theory.

A nonsingular variety \( X \) of dimension \( n \) is called a Fano \( n \)-fold, if the anticanonical divisor \( -K_X \) is ample. The Fano index of \( X \) is, by definition, the greatest positive integer \( r \) such that \( -K_X = rD \) for some integral divisor \( D \) on \( X \). As is well-known, the classification of Fano \( n \)-folds is one of the most important problems in algebraic geometry, not only because it is interesting in its own right, but also because Fano variety is a kind of outcomes when we run the minimal model program for nonsingular varieties.

We may consider the following problem, whose first part is a very special case of Conjecture 1.2.
Problem 1.6. Let $X$ be a Fano $n$-fold, $r$ the Fano index of $X$, and $D$ an integral divisor such that $-K_X = rD$. Do the following problems have affirmative answers?

(1) $H^0(X, D) \neq 0$;

(2) The general member of $|D|$ is a nonsingular $(n-1)$-fold.

Iskovskikh and Shokurov first studied Fano 3-folds in characteristic zero. Iskovskikh classified Fano 3-folds of the first kind (namely, Fano 3-folds $X$ with the second Betti number $b_2(X) = 1$), under the assumption that Problem 1.6 is true. Shokurov proved that Problem 1.6 is indeed true for all Fano 3-folds of the first kind, and thereby validated Iskovskikh’s classification result of Fano 3-folds. Mori and Mukai classified all Fano 3-folds $X$ with $b_2(X) \geq 2$ by virtue of the extremal ray theory. Later, Fujita (case $r = n-1$) and Mukai (case $r = n-2$) offered an idea to generalize the Iskovskikh and Shokurov’s framework of Fano 3-folds to that of Fano $n$-folds with $n \geq 4$, and Mukai obtained the classification of Fano 4-folds with $r = 2$ provided that Problem 1.6 is true. Thus it is so clear that Problem 1.6 is a basis to the classification of Fano $n$-folds.

When running the minimal model program started from a nonsingular variety $X$ of dimension $n \geq 3$, we have to consider the singularities. It turns out that the category of terminal varieties is suitable for running the minimal model program. Namely, for any nonsingular variety $X$, by virtue of extremal divisorial contractions or flips, finally we can obtain a minimal model or a Mori fiber space which belongs to this category and is birational to $X$. This statement is called the minimal model conjecture, and was already proved for dimension $n = 3, 4$ and char($k$) = 0. When considering the pair $(X, B)$ with $B$ a suitable $\mathbb{Q}$-divisor on a nonsingular variety $X$, the corresponding statement is called the log minimal model conjecture, and a suitable category is the category of varieties with KLT singularities.

From the viewpoint of the (log) minimal model program, we should consider a Fano variety with suitable singularities, since it appears as the general fiber of some (log) Mori fiber space.

Definition 1.7. Let $X$ be a normal proper variety of dimension $n$. $X$ is called a terminal $\mathbb{Q}$-Fano $n$-fold, if $X$ is terminal and $-K_X$ is ample.

Let $B$ be an effective $\mathbb{Q}$-divisor on $X$. $(X, B)$ is called a KLT $\mathbb{Q}$-Fano pair, if $(X, B)$ is KLT and $-(K_X + B)$ is ample.

The Fano index of a terminal $\mathbb{Q}$-Fano $n$-fold $X$ is the greatest rational number $r$ such that $-K_X \sim_\mathbb{Q} rD$ for some Cartier divisor $D$ on $X$. The Fano index of a KLT $\mathbb{Q}$-Fano pair $(X, B)$ is the greatest rational number $r$ such that $-(K_X + B) \sim_\mathbb{Q} rD$ for some Cartier divisor $D$ on $X$.

The classification of terminal $\mathbb{Q}$-Fano $n$-folds is more difficult than that of nonsingular Fano $n$-folds for $n \geq 3$. It seems impossible to classify all KLT $\mathbb{Q}$-Fano pairs. However, we can consider the similar one to Problem 1.6 whose first part is also a special case of Conjecture 1.2 (we omit the terminal $\mathbb{Q}$-Fano case below).

Problem 1.8. Let $(X, B)$ be a KLT $\mathbb{Q}$-Fano pair of dimension $n$, $r$ the Fano index of $(X, B)$, and $D$ a Cartier divisor such that $-(K_X + B) \sim_\mathbb{Q} rD$. Do the following problems have affirmative answers?

(1) $H^0(X, D) \neq 0$;

(2) Let $X' \in |D|$ be the general member. Then $(X', B|_{X'})$ has KLT singularities.
Theorem 1.9. With the same assumptions as in Problem 1.8 assume further that \( r > n - 3 \) and \( \text{char}(k) = 0 \). Then Problem 1.8 is true.

Note that if \( n \leq 3 \) then the assumption \( r > n - 3 \) is trivial. Thus Problem 1.8 is true for all KLT \( \mathbb{Q} \)-Fano pair of dimension \( n \leq 3 \). When \( n = 4 \), \( B = 0 \) and \( X \) has only Gorenstein canonical singularities, Kawamata dealt with the case \( r = 1 \) by showing that \( H^0(X, D) \neq 0 \) holds and that the general member \( X' \in |D| \) has also only Gorenstein canonical singularities (cf. [Ka00, Theorem 5.2]).

Let us return to the argument of Conjecture 1.2. It is easily verified in the curve case by using the Riemann-Roch theorem. The surface case, which is absolutely nontrivial, was proved by Kawamata (cf. [Ka00, Theorem 3.1]), by means of the following so-called logarithmic semipositivity theorem (we omit its general statement and only give a special case where the base space is 1-dimensional, cf. [Ka00, Theorem 1.2]).

Theorem 1.10 (Logarithmic Semipositivity). Let \( X \) be a normal proper variety over an algebraically closed field \( k \) with \( \text{char}(k) = 0 \), and \( B \) an effective \( \mathbb{Q} \)-divisor on \( X \) such that \( (X, B) \) is KLT. Let \( f : X \to C \) be a surjective morphism to a smooth curve \( C \). Let \( D \) be a Cartier divisor on \( X \) such that \( D \sim_{\mathbb{Q}} K_{X/C} + B \). Then \( f_* \mathcal{O}_X(D) \) is a semipositive locally free sheaf on \( C \).

A locally free sheaf \( \mathcal{E} \) on \( C \) is said to semipositive, if for any morphism \( g : C' \to C \) from a smooth curve \( C' \) to \( C \), and for any quotient line bundle \( \mathcal{L} \) of \( g^* \mathcal{E} \) on \( C' \), we have \( \deg \mathcal{L} \geq 0 \) holds.

For the higher dimensional cases, the effective non-vanishing conjecture is still open, and only a few results are known. We list them in the following remark.

Remark 1.11. With the same assumptions as in Conjecture 1.2

(1) If \( (X, B) \) is assumed to be log canonical, but not KLT, then we can reduce this case to the KLT case of lower dimension, by means of Kawamata’s subadjunction theorem and the Nadel vanishing theorem (cf. [Am99, Appendix]). So we only need to treat the KLT case from the beginning.

(2) If the irregularity \( q(X) := h^1(X, \mathcal{O}_X) > 0 \), then we can reduce this case to the lower dimensional case, by virtue of the Fourier-Mukai transform. Thus for 3-folds \( X \), it remains to prove Conjecture 1.2 when \( q(X) = 0 \). On the other hand, by the same technique, we can show that Conjecture 1.2 holds for such varieties which are birational to an abelian variety (cf. [Xie]).

(3) Assume further that \( B = 0 \) and that \( X \) is a terminal 3-fold. Then Conjecture 1.2 holds provided that the second Chern class \( c_2(X) \) is pseudo-effective (cf. [Xie05, Proposition 4.3]).

In this paper, we shall consider Problem 1.1 for algebraic surfaces in positive characteristic. There are some motivations to deal with this case. Firstly, both the Kodaira type vanishing theorems and the semipositivity theorem do not hold in general. Secondly, as for index 1 cover, the same as what is true in \( \text{char}(k) = 0 \) can be false in \( \text{char}(k) > 0 \). For instance, locally, Kawamata gave counterexamples which show that the index 1 cover of a log terminal surface is not necessarily of canonical singularities when \( \text{char}(k) = 2 \) or 3 (cf. [Ka99]). Globally, for the Kawamata-Viehweg vanishing, the logarithmic Kollár vanishing (see below), and the logarithmic semipositivity, there
are counterexamples on ruled surfaces (cf. Examples 3.6, 3.7, 3.9, 3.10). Thirdly, there are several kinds of pathological surfaces appearing in the classification theory.

We recall the logarithmic Kollár vanishing theorem for the convenience of the reader (cf. [Ko95, Theorem 10.19]).

**Theorem 1.12 (Logarithmic Kollár Vanishing).** Let $f : X \rightarrow Y$ be a surjective morphism between normal proper varieties over an algebraically closed field $k$ with char($k$) = 0. Let $B$ be an effective $\mathbb{Q}$-divisor on $X$ such that $(X, B)$ is KLT. Let $D$ be a Cartier divisor on $X$, and $M$ a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $Y$, such that $D \equiv K_X + B + f^*M$. Then $H^i(Y, R^j f_* O_X(D)) = 0$ for any $i > 0$ and any $j \geq 0$.

The following are the main theorems in this paper, which give a partial answer to Problem 1.1 for algebraic surfaces in positive characteristic.

**Theorem 1.13.** With the same assumptions as in Problem 1.1, assume further that dim $X = 2$ and char($k$) > 0. Then we have

1. $H^0(X, D) \neq 0$ holds except possibly in the following cases:
   - (C) $X$ is a ruled surface with $h^1(O_X) \geq 2$;
   - (D-I) $X$ is a quasi-elliptic surface with $\chi(O_X) < 0$;
   - (D-II) $X$ is a surface of general type with $\chi(O_X) < 0$.

2. In Case (C), $H^0(X, 2D) \neq 0$ always holds. Furthermore, if either $X$ is relatively minimal or $D$ is not big, then $H^0(X, D) \neq 0$ holds.

**Theorem 1.14.** For the Kawamata-Viehweg vanishing, the logarithmic Kollár vanishing and the logarithmic semipositivity, there are counterexamples on ruled surfaces in any positive characteristic.

We always work over an algebraically closed field $k$ of characteristic $p > 0$ unless otherwise stated. For the classification theory of surfaces in positive characteristic, we refer the reader to [M69, BM] or [Ba01]. For the definitions and results related to the minimal model theory, we refer the reader to [KMM87, KM98]. We use $\equiv$ to denote numerical equivalence, $\sim_\mathbb{Q}$ to denote $\mathbb{Q}$-linear equivalence, and $[B] = \sum [b_i]B_i$ to denote the round-down of a $\mathbb{Q}$-divisor $B = \sum b_iB_i$.

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## 2 Reduction to Cases

First of all, we give an easy reduction to Problem 1.1 in the surface case.

**Proposition 2.1.** With the same assumptions as in Problem 1.1, assume that dim $X = 2$. Then we may assume that $X$ is smooth projective, $B$ is a $\mathbb{Q}$-divisor and $H$ is ample.

**Proof.** Let $f : Y \rightarrow X$ be the minimal resolution of $X$. We may write $K_Y = f^*K_X + \sum a_iE_i$, where $E_i$ are exceptional curves of $f$ and $-1 < a_i \leq 0$ for all $i$. Let $B' = f^*B - \sum a_iE_i \geq 0$. Then $K_Y + B' = f^*(K_X + B)$. It is easy to see that $(Y, B')$ is also
KLT. Note that $H' = f^*D - (K_Y + B')$ is nef and big, and $H^0(X, D) \neq 0$ is equivalent to $H^0(Y, f^*D) \neq 0$. On the other hand, by Kodaira’s Lemma, we may assume that $B'$ is a $\mathbb{Q}$-divisor and $H'$ is ample by adding a sufficiently small $\mathbb{R}$-divisor to $B'$. 

Therefore we consider the following problem in what follows.

**Problem 2.2.** Let $X$ be a smooth projective surface over an algebraically closed field $k$ of characteristic $p > 0$, $B = \sum_{i=1}^m b_i B_i$ an effective $\mathbb{Q}$-divisor on $X$ such that $(X, B)$ is KLT. Let $D$ be a nef divisor on $X$ such that $H = D - (K_X + B)$ is ample. Does $H^0(X, D) \neq 0$ hold?

Secondly, we have the following easy criterion for non-vanishing.

**Lemma 2.3.** If $\chi(X, D) > 0$, then $h^0(X, D) > 0$.

**Proof.** We have that $h^2(X, D) = h^0(X, K_X - D) = h^0(X, -H - B) = 0$ by Serre duality, hence the conclusion is obvious. 

**Case (A).** $D \equiv 0$, hence $-(K_X + B)$ is ample.

It follows from Serre duality that $h^2(X, \mathcal{O}_X) = h^0(X, K_X) = 0$. We shall show that $h^1(X, \mathcal{O}_X) = 0$ by the following two lemmas.

**Lemma 2.4.** Assume that we are in Case (A). Then $\overline{\mathcal{NE}}(X) = \sum \mathbb{R}_+[l_i]$, where $l_i$ are rational curves on $X$ (not necessarily extremal).

**Proof.** For any $C \not\subset \operatorname{Supp} B$, we have $-K_X.C > B.C \geq 0$. On the other hand, we have $-K_X.B_i > (\sum b_j B_j).B_i \geq b_i B_i^2$. If $B_i^2 \geq 0$, then $-K_X.B_i > 0$. If $B_i^2 < 0$, then $-(K_X + b_i B_i).B_i > (\sum_{j \neq i} b_j B_j).B_i \geq 0$ and $2 - 2p_a(B_i) = -(K_X + B_i).B_i > -(K_X + b_i B_i).B_i > 0$. Hence $p_a(B_i) = 0$ and $B_i \cong \mathbb{P}^1$.

By permutation of the indices, we may assume that $B_i^2 < 0$ for $1 \leq i \leq s$, where $0 \leq s \leq m$. By the cone theorem (cf. [Mori 1982, Theorem 1.4]), we have

$$\overline{\mathcal{NE}}(X) = \sum \mathbb{R}_+[l_i] + \overline{\mathcal{NE}}_{K_X + \varepsilon L \geq 0}(X),$$

where $l_1, \ldots, l_r$ are extremal rational rays and $L = -(K_X + B)$.

We claim that $\overline{\mathcal{NE}}(X) = \sum_{i=1}^r \mathbb{R}_+[l_i] + \sum_{j=1}^s \mathbb{R}_+[B_j]$. Indeed, for any curve $C$, we may write $C = \lim(\sum a_i l_i + \sum c_j z_j)$, where $a_i \geq 0, c_i \geq 0, z_k \in \mathcal{NE}_{K_X + \varepsilon L \geq 0}(X)$ are irreducible curves on $X$, and $\lim$ means the limit of vectors under the usual topology of $\overline{\mathcal{NE}}(X)$. By definition, for each $k$ we have

$$(K_X - \varepsilon(K_X + B)).z_k = (1 - \varepsilon)K_X.z_k - \varepsilon B.z_k \geq 0,$$

$$K_X.z_k \geq \frac{\varepsilon}{1 - \varepsilon} B.z_k.$$

If $z_k \not\subset \operatorname{Supp} B$, then $K_X.z_k \geq 0$, a contradiction. Hence $z_k = B_j$ for some $1 \leq j \leq n$. If $B_j^2 \geq 0$, then $K_X.z_k \geq 0$, a contradiction. Hence $z_k = B_j \cong \mathbb{P}^1$ for some $1 \leq j \leq s$. Therefore $C = \lim(\sum_{i=1}^r a_i l_i + \sum_{j=1}^s c_j B_j) = \sum_{i=1}^r a_i l_i + \sum_{j=1}^s c_j B_j$. 

**Lemma 2.5.** Assume that we are in Case (A). Let $\alpha : X \to A$ be the Albanese map of $X$. Then $q(X) := \dim A = 0$ and $h^1(\mathcal{O}_X) = 0$. 


Proof. Let $M$ be an ample divisor on $A$. By Lemma 2.4, for any curve $C$ on $X$, we may write $C \equiv \sum a_i l_i$, where $a_i \geq 0$ and $l_i$ are rational curves on $X$. Since $A$ contains no rational curves, $\alpha(l_i)$ is a point for each $i$. Then

$$\alpha^*M.C = \alpha^*M.(\sum a_i l_i) = \sum a_i \alpha^*M.l_i = 0,$$

hence $\alpha(C)$ is also a point. Thus $\alpha$ is constant and $q(X) = 0$.

Note that the following inequalities hold (cf. [BM]):

$$0 \leq h^1(O_X) - q(X) \leq pg(X) = h^2(O_X) = 0.$$

Hence $h^1(O_X) = q(X) = 0$.

In total, in Case (A), we have $\chi(X, D) = \chi(O_X) = 1 > 0$. As a corollary, we know that any smooth projective surface with a log Fano structure is rational.

**Case (B).** $D \not\equiv 0$ and either

(I) $\kappa(X) \geq 0$ and $\chi(O_X) \geq 0$, or

(II) $X$ is a ruled surface and $q(X) \leq 1$, hence $\chi(O_X) \geq 0$.

In Case (B), by the Riemann-Roch theorem, we have

$$\chi(X, D) = \frac{1}{2}D(D - K_X) + \chi(O_X) = \frac{1}{2}D(H + B) + \chi(O_X) > 0.$$

Let us consider the remaining cases. Assume that $X$ is not contained in Cases (A) or (B). Let $Y$ be a relatively minimal model of $X$. If $\kappa(Y) = -\infty$, then $Y$ must be a $\mathbb{P}^1$-bundle with $c_2(Y) < 0$, which is Case (C).

**Case (C).** $D \not\equiv 0$. There exist a smooth curve $C$ with $g(C) \geq 2$ and a surjective morphism $f : X \to C$ such that $X$ is a ruled surface over $C$.

In characteristic zero, it is well-known that if $c_2(X) < 0$, then $X$ is ruled. A similar result holds in positive characteristic due to Raynaud and Shepherd-Barron (cf. [SB91, Theorem 7]).

**Theorem 2.6.** Let $X$ be a smooth surface over an algebraically closed field $k$ of positive characteristic. If $c_2(X) < 0$, then $X$ is uniruled. In fact, there exist a smooth curve $C$ and a surjective morphism $f : X \to C$ such that the geometric generic fiber of $f$ is a rational curve.

If $\kappa(Y) = 0$, then $c_2(Y) \geq 0$ by the explicit classification (cf. [BM]), hence $\chi(O_X) = \chi(O_Y) \geq 0$, such $X$ are contained in Case (B-I). If $\kappa(Y) = 1$ and $c_2(Y) = 12\chi(O_Y) < 0$, then $Y$ must be a quasi-elliptic surface by the classification theory and Theorem 2.6. The last one is the case that $X$ is of general type with $\chi(O_X) < 0$. Therefore we have the following Case (D).

**Case (D).** $D \not\equiv 0$. There exist a smooth curve $C$ and a surjective morphism $f : X \to C$ such that $\chi(O_X) < 0$ and either

(I) the geometric generic fiber of $f$ is a rational curve with an ordinary cusp, or

(II) the geometric generic fiber of $f$ is a rational curve, and $X$ is of general type.
In char($k) = 0$, Case (D) cannot occur, and Case (C) is settled by Kawamata by using the logarithmic semipositivity theorem (cf. [Ka00 Theorem 3.1]). Note that Case (D-I) can occur only if char($k) = 2$ or $3$ (cf. [BM]), and the explicit examples have been given by Raynaud and Lang (cf. [Ra78, La79]). For Case (D-II), we can restrict our attention to a small class by [SB91, Theorem 8], however no example is known so far.

We shall discuss Case (C) in §3 and §4.

3 Some Counterexamples

When char($k) = p > 0$, it is well-known that the Kodaira vanishing does not hold on surfaces in general. However, the Kodaira vanishing does hold on ruled surfaces, which was first proved by Tango (cf. [Ta72]). In fact, we have the following theorem given by Mukai (cf. [Mu79]).

**Theorem 3.1.** Let $X$ be a smooth projective surface over an algebraically closed field $k$ of characteristic $p > 0$. If the Kodaira vanishing does not hold on $X$, then $X$ must be a quasi-elliptic surface or a surface of general type.

Furthermore, we may ask whether the Kawamata-Viehweg vanishing holds on ruled surfaces. This problem is important because the Kawamata-Viehweg vanishing gives a sufficient condition for the effective non-vanishing in Case (C). Roughly speaking, the vanishing of $H^1(X, D)$ implies the non-vanishing of $H^0(X, D)$ by virtue of the Fourier-Mukai transform. This idea was first used in [CH02, CCZ05]. We recall the following theorem due to Mukai (cf. [Mu81 Theorem 2.2]).

**Theorem 3.2.** Let $A$ be an abelian variety, $\hat{A}$ its dual abelian variety, $\mathcal{P}$ the Poincaré line bundle on $A \times \hat{A}$. Then the Fourier-Mukai transform $\Phi_{A \to \hat{A}}: D(A) \to D(\hat{A})$,

$\mathcal{F}^\bullet \mapsto R\pi_1^*(\pi_\hat{A}^*\mathcal{F}^\bullet \otimes \mathcal{P})$ is an equivalence of derived categories.

Let $\mathcal{F}$ be a coherent sheaf on $A$. Assume that $H^i(A, \mathcal{F} \otimes P) = 0$ for all $P \in \text{Pic}^0(A)$ and all $i \neq 0$. Then the dual sheaf $\hat{\mathcal{F}} = \Phi_{A \to \hat{A}}(\mathcal{F})$ is a locally free sheaf on $\hat{A}$ of rank $h^0(A, \mathcal{F})$.

**Proposition 3.3.** Assume that we are in Case (C), and that $H^1(X, D + f^*P) = 0$ for any $P \in \text{Pic}^0(C)$. Then $H^0(X, D) \neq 0$ holds.

**Proof.** Let $\alpha : X \to A = \text{Alb}(X)$ be the Albanese map of $X$. Then $\alpha(X) = C \subset A$. Let $\mathcal{F} = \alpha_*\mathcal{O}_X(D)$ be the coherent sheaf on $A$. Then we have that $H^i(X, D + \alpha^*P) = R^i\alpha_*(D + \alpha^*P) = 0$ for any $P \in \text{Pic}^0(A)$ and any $i > 0$ by the assumption and easy computations. It follows from the Leray spectral sequence that $H^i(A, \mathcal{F} \otimes P) = 0$ for any $P \in \text{Pic}^0(A)$ and any $i > 0$, hence by Theorem 3.2 its dual $\hat{\mathcal{F}}$ is a locally free sheaf of rank $h^0(A, \mathcal{F}) = h^0(X, D)$. If $H^0(X, D) = 0$, then $\hat{\mathcal{F}} = 0$, hence $\mathcal{F} = 0$. Next we prove that $\mathcal{F} \neq 0$. Consider the general fiber $F$ of $f : X \to C$, then the stalk of $\mathcal{F}$ at the general point of $C$ is isomorphic to $H^0(F, D|_F) \neq 0$ since $D$ is nef and $F \cong \mathbb{P}^1$.

**Remark 3.4.** Proposition 3.3 gives a new proof of the surface case of Conjecture 1.2 since the Kawamata-Viehweg vanishing theorem holds in characteristic zero.
Even if the Kodaira vanishing holds on ruled surfaces, we cannot expect that the Kawamata-Viehweg vanishing holds on ruled surfaces in general. Next we shall give some counterexamples for the Kawamata-Viehweg vanishing on ruled surfaces. The constructions are similar to, however generalize those to some extent, which were given by Raynaud to yield the counterexamples for the Kodaira vanishing on quasi-elliptic surfaces and general type surfaces (cf. [Ra78]).

**Definition 3.5.** Let $C$ be a smooth projective curve over an algebraically closed field $k$ of characteristic $p > 0$. Let $f \in K(C)$ be a rational function on $C$.

$$n(f) := \deg \left( \frac{(df)}{p} \right),$$

where $(df) = \sum_{x \in C} v_x(df)x$ is the divisor associated to the rational differential 1-form $df$. We denote $K^p(C) = \{ f^p \mid f \in K(C) \}$.

$$n(C) := \max \{ n(f) \mid f \in K(C), f \notin K^p(C) \}.$$

If $f \notin K^p(C)$, then $(df)$ is a canonical divisor on $C$ with degree $2(g - 1)$. It is easy to see that $n(C) \leq [2(g - 1)/p]$.

**Example 3.6.** There do exist smooth projective curves $C$ such that $n(C) > 0$ for each characteristic $p > 0$.

1. Let $h \geq 3$ be an odd integer, $p \geq 3$. Let $C$ be the projective completion at infinity of the affine curve defined by $y^2 = x^{ph} + x^{p+1} + 1$. It is easy to verify that $C$ is a smooth hyperelliptic curve and that $(d(y/x^p)) = (ph - 3)z_{\infty}$, where $z_{\infty}$ is the infinity point of $C$ (cf. [Sh94, Ch. III, §6.5]). Hence $n(C) = n(y/x^p) = h - 1 > 0$.

2. (cf. [Ra78]). Let $h > 2$ be an integer. Let $C$ be the projective completion at infinity of the Artin-Schreier cover of the affine line defined by $y^{hp-1} = x^p - x$. It is easy to verify that $C$ is a smooth curve of genus $g$ with $2(g - 1) = p(h(p - 1) - 2)$, and that $(dy) = p(h(p - 1) - 2)z_{\infty}$, where $z_{\infty}$ is the infinity point of $C$. Hence $n(C) = n(y) = h(p - 1) - 2 > 0$.

3. (cf. [Ta72a]). Let $C \subset \mathbb{P}^2$ be the curve defined by $x_0^{p+1} = x_1x_2(x_0^{p-1} + x_1^{p-1} - x_2^{p-1})$, where $p \geq 3$. We can show that $C$ is smooth and that $n(C) = n(x_0/x_1) = p - 2 > 0$.

**Example 3.7.** Let $C$ be a smooth projective curve over an algebraically closed field $k$ of characteristic $p > 0$. If $n(C) > 0$, then there are a $\mathbb{P}^1$-bundle $f : X \rightarrow C$, an effective $\mathbb{Q}$-divisor $B$ and an integral divisor $D$ on $X$ such that $(X, B)$ is KLT and $H = D - (K_X + B)$ is ample. However $H^1(X, D) \neq 0$.

Let $F : C \rightarrow C$ be the Frobenius map. We have the following exact sequences of $\mathcal{O}_C$-modules:

1. $0 \rightarrow \mathcal{O}_C \rightarrow F_\ast \mathcal{O}_C \rightarrow \mathcal{B}^1 \rightarrow 0$
2. $0 \rightarrow \mathcal{B}^1 \rightarrow F_\ast \Omega^1_C \rightarrow \Omega^1_C \rightarrow 0$

where $\mathcal{B}^1$ is the image of the map $F_\ast(d) : F_\ast \mathcal{O}_C \rightarrow F_\ast \Omega^1_C$, and $c$ is the Cartier operator (cf. [Ta72a]).

Let $\mathcal{L} = \mathcal{O}_C(L)$ be a line bundle on $C$. Tensor \cite{2} by $\mathcal{O}_C(-L)$, we have:

$$0 \rightarrow \mathcal{B}^1(-L) \rightarrow F_\ast(\Omega^1_C(-F^\ast L)) \xrightarrow{c(-L)} \Omega^1_C(-L) \rightarrow 0.$$
Thus $H^0(C, B^1(-L)) = \{ df | f \in K(C), (df) \ge pL \}$. Since $n(C) > 0$, there exists an $f_0 \in K(C)$ such that $n(f_0) = \deg([df_0]/p) = n(C) > 0$. Let $L = ([df_0]/p)$. Then $\deg L = n(C) > 0$ and $(df_0) \ge pL$, hence $0 \neq df_0 \in H^0(C, B^1(-L))$, and we can regrad the line bundle $L = \mathcal{O}_C(L) \subset B^1$.

Tensor $[1]$ by $L^{-1}$ and take cohomology, we have:

$$0 \to H^0(C, B^1(-L)) \overset{\eta}{\to} H^1(C, L^{-1}) \overset{E}{\to} H^1(C, L^{-p}).$$

Since $\eta$ is injective, we may take the element $0 \neq \eta(df_0) \in H^1(C, L^{-1})$, which determines the following extension sequence:

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{L} \to 0.$$  \hfill (3)

Pull back the exact sequence $[3]$ by the Frobenius map $F$, we have the following split exact sequence:

$$0 \to \mathcal{O}_C \to F^*\mathcal{E} \to \mathcal{L}^p \to 0,$$  \hfill (4)

since the obstruction of extension of $[1]$ is just $F^*\eta(df_0) = 0$.

Let $X = \mathbb{P}(\mathcal{E})$ be the $\mathbb{P}^1$-bundle over $C$, $f : X \to C$ the projection, $\mathcal{O}_X(1)$ the tautological line bundle. The sequence $[3]$ determines a section $E$ of $f$ such that $\mathcal{O}_X(E) \cong \mathcal{O}_X(1)$, and $E$ corresponds to a section $s \in H^0(X, \mathcal{O}_X(1)) = H^0(C, \mathcal{E})$ which is the image of 1 under the map $H^0(C, \mathcal{O}_C) \to H^0(C, \mathcal{E})$. The sequence $[1]$ induces an exact sequence:

$$0 \to \mathcal{O}_C \to F^*\mathcal{E} \otimes \mathcal{L}^{-p} \to \mathcal{L}^{-p} \to 0,$$

which determines a section $t \in H^0(X, \mathcal{O}_X(p) \otimes f^*\mathcal{L}^{-p})$ through the maps $H^0(C, \mathcal{O}_C) \to H^0(C, F^*\mathcal{E} \otimes \mathcal{L}^{-p}) \to H^0(C, \mathcal{O}_X(p) \otimes f^*\mathcal{L}^{-p}) = H^0(X, \mathcal{O}_X(p) \otimes f^*\mathcal{L}^{-p})$. The section $t$ determines an irreducible curve $C'$ on $X$ such that $\mathcal{O}_X(C') \cong \mathcal{O}_X(p) \otimes f^*\mathcal{L}^{-p}$. It is easy to verify that both $E$ and $C'$ are smooth over $k$, and $E \cap C' = \emptyset$.

(†) Assume that $p \ge 3$.

Let $B = \frac{1}{p}C'$, $D = K_X + \frac{1}{p^2}E + \frac{1-p}{2}f^*L = \frac{p-3}{2}E + f^*(K_C + \frac{1-p}{2}L)$. Then $H = D - (K_X + B) = \frac{1}{p}(E + f^*L)$. It is easy to see that $(X, B)$ is KLT. Since $E^2 = \deg \mathcal{E} = \deg \mathcal{L} > 0$, $E$ is a nef divisor on $X$. On the other hand, $E$ is $f$-ample, hence $H$ is an ample $\mathbb{Q}$-divisor on $X$. Next we show that $H^1(X, D) \neq 0$.

Consider the Leray spectral sequence $E_2^{ij} = H^i(C, R^jf_*?) \Rightarrow H^{i+j}(X, ?)$. Since $E_2^{ij} = 0$ for $i \ge 2$, by the five term exact sequence we have

$$H^1(X, D) \cong H^1(X, -H - B)^\vee \cong H^0(C, R^1f_*\mathcal{O}_X(-\frac{p+1}{2}) \otimes \mathcal{L}^{p-1})^\vee.$$

By the relative Serre duality,

$$R^1f_*\mathcal{O}_X(-\frac{p+1}{2})^\vee \cong f_*\mathcal{O}_X(\frac{p+1}{2}) \otimes \mathcal{O}_X/C) = f_*\mathcal{O}_X(\frac{p-3}{2} \otimes f^*\mathcal{L}) = S^{(p-3)/2}(\mathcal{E}) \otimes \mathcal{L}.

Since $S^{(p-3)/2}(\mathcal{E})$ has a quotient sheaf $\mathcal{L}^{(p-3)/2}$, $R^1f_*\mathcal{O}_X(-\frac{p+1}{2})$ has a subsheaf $\mathcal{L}^{-(p-1)/2}$. Thus $H^1(X, D) \cong H^0(C, \mathcal{O}_C)^\vee = k$, which is desired.

(†) Assume that $p = 2$.  

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Let $B = \frac{2}{3} C', D = K_X + 2E - f^*L = f^*K_C$, $H = D - (K_X + B) = \frac{1}{3}(2E + f^*L)$. It is easy to verify that $(X, B)$ is KLT and $H$ is $\mathbb{Q}$-ample. By the same argument, we have $H^1(X, D) = H^1(X, f^*\omega_C) \cong H^1(X, \omega_{X/C})^\vee \cong H^0(C, R^1 f_*\omega_{X/C})^\vee \cong H^0(C, \mathcal{O}_C)^\vee = k$, which is desired.

By Examples 3.6-7, there do exist counterexamples for the Kawamata-Viehweg vanishing on ruled surfaces. Furthermore, it is easy to verify that $D$ is nef and $|D| \neq \emptyset$ in both cases. Hence it follows that the Kawamata-Viehweg vanishing is a sufficient but not a necessary condition for the effective non-vanishing in Case (C).

Examples 3.6-7 also give the counterexamples for the $\mathbb{Q}$-divisor version of the Kawamata-Viehweg vanishing (cf. [Ko86, Theorem 1-2-3]). Indeed, we can take $D - (K_X + B)$ as the required $\mathbb{Q}$-divisor. However, it is unknown whether there exist counterexamples for its nef and big version mentioned below. So it is interesting to take the following problem into account, which is compared with Theorem 3.3.

**Problem 3.8.** Let $X$ be a smooth projective surface over an algebraically closed field $k$ of characteristic $p > 0$. Let $D$ be an integral divisor on $X$ such that $D - K_X$ is nef and big. Assume that $X$ is neither quasi-elliptic nor of general type. Does $H^1(X, D) = 0$ hold?

In arbitrary characteristic, the Kollár vanishing of $f_*\omega_X$ for ruled surfaces is trivial (cf. [Ko86, Theorem 2.1]). However, we shall give counterexamples for the logarithmic Kollár vanishing in positive characteristic (cf. Theorem 1.12).

**Example 3.9.** Let $C$ be a smooth projective curve over an algebraically closed field $k$ of characteristic $p > 0$. If $n(C) > 0$, then there are a $\mathbb{P}^1$-bundle $f : X \to C$, an effective $\mathbb{Q}$-divisor $B'$ and an integral divisor $D$ on $X$ such that $(X, B')$ is KLT and $D \sim Q K_X + B' + f^*M$, where $M$ is an ample $\mathbb{Q}$-divisor on $C$. However $H^1(C, f_*\mathcal{O}_X(D)) \neq 0$.

It is just Example 3.7. We use the same notation and assumptions as in Example 3.7. When $p \geq 3$, let $B' = \frac{1}{2}(E + C')$ and $M = \frac{1}{2}L$. When $p = 2$, let $B' = \frac{2}{3}(E + C')$ and $M = \frac{1}{3}L$. Then $D \sim Q K_X + B' + f^*M$. It follows from $R^1 f_*\mathcal{O}_X(D) = 0$ and the Leray spectral sequence that $H^1(C, f_*\mathcal{O}_X(D)) \cong H^1(X, D) \neq 0$.

In characteristic zero, Kawamata settled Case (C) by means of the logarithmic semipositivity theorem (cf. Theorem 1.10). In arbitrary characteristic, the semipositivity of $f_*\omega_{X/C}$ is trivial for ruled surfaces, however we shall give counterexamples for the logarithmic semipositivity in positive characteristic.

**Example 3.10.** Let $C$ be a smooth projective curve over an algebraically closed field $k$ of characteristic $p > 0$. Assume that $n(C) > 0$ and that the following condition holds:

(*) If $p = 2$, then $\frac{1}{2}L$ is integral, where $L = [(df_0)/p]$ is the divisor on $C$ for some rational function $f_0 \in K(C)$ such that $n(f_0) = n(C)$.

Then there are a $\mathbb{P}^1$-bundle $f : X \to C$, an effective $\mathbb{Q}$-divisor $B'$ and an integral divisor $D'$ on $X$ such that $(X, B')$ is KLT and $D' \sim Q K_{X/C} + B'$. However $f_*\mathcal{O}_X(D')$ is not semipositive.

When $p \geq 5$, the counterexample is just Example 3.7. We use the same notation and assumptions as in Example 3.7. Since $H$ is ample, we can take a general member $M \in |nH|$ for $n$ sufficiently large and divisible such that $M$ is irreducible and smooth, and $B' = B + \frac{1}{n}M$ has simple normal crossing support, hence $(X, B')$ is KLT. Let $D' = D - f^*K_C$. Then $D' \sim Q K_{X/C} + B'$. Since $D'|_F = D|_F$ is nef hence basepoint free on $F \cong \mathbb{P}^1$, the canonical homomorphism $f^* f_* \mathcal{O}_X(D') \to \mathcal{O}_X(D')$ is surjective.
If \( f_*O_X(D') \) were semipositive, then we would have \( D' = D - f^*K_C \) is nef on \( X \). However, \((D - f^*K_C)C' = \frac{3-p}{2} f^*L.C' < 0\), this is absurd. Hence \( f_*O_X(D') \) never be semipositive.

For \( p < 5 \), we need to modify Example 3.7 slightly. When \( p = 3 \), let \( B = \frac{5}{6}C' \), \( D = E + f^*(K_C - L) \). Then \((X, B)\) is KLT and \( H = D - (K_X + B) = \frac{1}{2}(E + f^*L) \) is ample. However \( D' = D - f^*K_C \) satisfies \( D'.C' = (E - f^*L)C' = -f^*L.C' < 0 \), hence \( f_*O_X(D') \) never be semipositive.

When \( p = 2 \), we need the additional assumption \((*)\) mentioned above. Let \( B = \frac{5}{6}C' \), \( D = E + f^*(K_C - \frac{1}{2}L) \). Then \((X, B)\) is KLT and \( H = D - (K_X + B) = \frac{1}{9}(4E + f^*L) \) is ample. However \( D' = D - f^*K_C \) satisfies \( D'.C' = (E - \frac{1}{3}f^*L)C' = -\frac{1}{3}f^*L.C' < 0 \), hence \( f_*O_X(D') \) never be semipositive. \( \square \)

Note that the assumption \((*)\) can be realized by Example 3.6(2). Indeed, let \( h - 2 \) be a positive integer divisible by 3, then we are done. Hence there do exist counterexamples for the logarithmic semipositivity on ruled surfaces.

Furthermore, it is easy to verify that \( D \) is nef and \(|D| \neq \emptyset\) in both cases, hence it follows that the logarithmic semipositivity is a sufficient but not a necessary condition for the effective non-vanishing in Case (C).

Let us compare the two approaches for proving the effective non-vanishing conjecture for surfaces in characteristic zero. Of course, we only need to treat Case (C). Since the semipositivity theorem can be deduced from the Kollár vanishing theorem (cf. [Ko86, Corollary 3.7]), the approach provided by Kawamata gives the diagram (1), and Proposition 3.3 gives the diagram (2) as follows:

\[
\begin{array}{ccc}
\text{Kollár vanishing} & \text{effective non-vanishing} & \text{Kodaira vanishing} \\
\text{semipositivity} & \xrightarrow{\text{cyclic cover}} & \xleftarrow{\text{cyclic cover}} \text{Kawamata-Viehweg vanishing} \\
\end{array}
\]

In characteristic zero, the vanishing theorem is the start point of both approaches, and the cyclic cover trick plays a more important role in both proofs. However, Examples 3.7 and 3.10 show that, to some extent, the cyclic cover trick does not behave well in positive characteristic. It will turn out in the next section that without the cyclic cover trick, we could not deal with the case \( B \neq 0 \) effectively.

## 4 Ruled Surface Case

Firstly, there is a partial answer to the effective non-vanishing in Case (C), whose proof is numerical, hence valid in positive characteristic (cf. [Am99, Proposition 4.1(2a)]).

**Proposition 4.1.** Let \( F \) be the general fiber of \( f : X \to C \). If \( H.F > 1 \), then \( H^0(X, D) \neq 0 \) (This is true even if \( H = D - (K_X + B) \) is nef and big).

Proposition 4.1 guarantees the non-vanishing for the absolute case, i.e. \( B = 0 \) and \( D - K_X \) is nef and big, since \( H.F \geq -K_X.F = 2 > 1 \). Hence we have to consider the case \( B \neq 0 \).

Secondly, we shall prove the following theorem as a first step.

**Theorem 4.2.** In Case (C), assume furthermore that \( X \) is relatively minimal. Then \( H^0(X, D) \neq 0 \).
Let us fix some notation. Assume that \( f : X = \mathbb{P}(E) \to C \) is a \( \mathbb{P}^1 \)-bundle over \( C \) associated to a normalized rank 2 locally free sheaf \( E \) on \( C \). Let \( e = -\text{deg} \, E \), \( E \) the canonical section of \( f \) with \( E^2 = -e \), \( F \) the fiber of \( f \). Note that the proof of Theorem 4.2 is also numerical, and that we only need the condition \([B] = 0\), so the KLT assumption of \((X, B)\) is unnecessary.

Assume that \( e \geq 0 \). It is easy to see that if \( L \equiv aE + bF \) is an irreducible curve on \( X \), then either \( L = E, F \) or \( a > 0, b \geq ae \geq 0 \). Hence \( L^2 = a(2b - ae) \geq 0 \) in the latter case. In other words, if \( L^2 < 0 \) then \( L = E \) and \( e > 0 \). We may write \( B = aE + B' \) with \( E \not\subseteq \text{Supp} \, B' \). Then \( B' \) is nef, \( H + B' = D - (K_X + aE) \) is ample and \((H + B').F = (D - K_X - aE).F \geq 2 - a > 1 \). By Proposition 4.1, we have \( H^0(X, D) \neq 0 \).

It remains to deal with the case \( e < 0 \). Let \( B = \sum_{i \in I} b_iB_i \). If \( B^2_i \geq 0 \), then \( B_i \) is a nef divisor on \( X \), and we can move \( b_iB_i \) from \( B \), add \( b_iB_i \) to \( H \) and keep \( D \) unchanged to consider the non-vanishing problem. Hence we may assume that \( B^2_i < 0 \) for all \( i \in I \). Since \( B_i \) are numerically independent and \( \rho(X) = 2 \), we have \(|I| \leq 1 \). Indeed, if \( B_1, B_2 \) are distinct components of \( B \), then we may write \( F \equiv c_1B_1 + c_2B_2 \), where \( c_i \) are rational numbers and at least one of \( c_i \) is positive. If both \( c_i > 0 \), then both \( B_i \) is ample, hence \( B_i = F \), a contradiction. If \( c_1 > 0, c_2 \leq 0 \), then \( F.B_1 = c_1B_1^2 + c_2B_2B_1 < 0 \), a contradiction.

Therefore we have only to consider the following case:

**Case (C-M).** Let \( f : X \to C \) is a \( \mathbb{P}^1 \)-bundle over a smooth curve \( C \) of genus \( g \geq 2 \) with invariant \( e < 0 \). Let \( D \neq 0 \) be a nef divisor on \( X \), \( B = cG \), where \( 0 < c < 1 \) and \( G \) is an irreducible curve on \( X \) with \( G^2 < 0 \), such that \( H = D - (K_X + B) \) is ample.

We need an easy lemma (cf. [Ha77], Ch. V, Ex. 2.14):

**Lemma 4.3.** With the same assumptions as in Case (C-M).

(i) If \( G \equiv xE + yF \) is an irreducible curve \( \neq E, F \), then either \( x = 1, y \geq 0 \), or \( 2 \leq x \leq p - 1, y \geq xe/2 \), or \( x \geq p, y \geq xe/2 + 1 - g \).

(ii) If \( D \equiv aE + bF \) is ample, then \( a > 0, b > ae/2 \).

**Lemma 4.4.** Assume that we are in Case (C-M). Then \( H^0(X, D) \neq 0 \).

**Proof.** Let \( D \equiv aE + bF \), \( G \equiv xE + yF \). Then \( a \geq 0, b \geq ae/2 \) and \( x, y \) satisfy the condition mentioned in Lemma 4.3(i). We have \( H \equiv aE + bF + 2E + (2 - 2g + e)F - cxE - cyF = (a + 2 - cx)E + (b + 2 - 2g + e - cy)F \). Since \( H \) is ample, the following conditions hold by Lemma 4.3(ii):

\[
a + 2 - cx > 0, \quad b + 2 - 2g + e - cy > \frac{1}{2}(a + 2 - cx)e.
\]

By Lemma 4.3(i) and the later inequality, we have

\[
b - \frac{1}{2}ae > 2g - 2 + c(y - \frac{1}{2}xe) > (2 - c)(g - 1) > g - 1.
\]

It follows from the Riemann-Roch theorem that

\[
\chi(X, D) = \frac{1}{2}D(D - K_X) + \chi(O_X) = (a + 1)(b - \frac{1}{2}ae + 1 - g) > 0,
\]

which also completes the proof of Theorem 4.2. \( \square \)
Let $X$ be a smooth projective surface, $B$ an effective $\mathbb{Q}$-divisor such that $(X, B)$ is KLT. Let $D$ be a nef divisor on $X$ such that $H = D - (K_X + B)$ is ample. Next, we consider the reduction of the effective non-vanishing problem for the triple $(X, B; D)$ under the $(-1)$-curve contractions.

Let $g : X \to Y$ be a contraction of a $(-1)$-curve $l \subset X$. Assume that there exists a divisor $D_Y$ on $Y$ such that $D = g^*D_Y$ (this condition is equivalent to $D.l = 0$). It is easy to verify that $D_Y$ is nef. Let $B_Y := g_*B$ be the strict transform of $B$. Then $B_Y$ is also an effective divisor with $[B_Y] = 0$. We may write

$$K_X + B = g^*(K_Y + B_Y) + dl,$$

where $d > 0$ since $D - (K_X + B)$ is ample. It follows from $d > 0$ that $(Y, B_Y)$ is again KLT. Let $C$ be an irreducible curve on $Y$, it is easy to verify that $(D_Y - (K_Y + B_Y)).C = (D - (K_X + B)).g^*C > 0$ and that $(D_Y - (K_Y + B_Y))^2 = (D - (K_X + B))^2 + d^2 > 0$, hence $H_Y = D_Y - (K_Y + B_Y)$ is also ample by the Nakai-Moishezon criterion.

**Definition 4.5.** Given a triple $(X, B; D)$. Let $g : X \to Y$ be a birational morphism to a smooth projective surface $Y$. Assume that $D$ is $g$-trivial, i.e. there exists a divisor $D_Y$ on $Y$ such that $D = g^*D_Y$. Then the induced triple $(Y, B_Y; D_Y)$ is called the reduction model of $(X, B; D)$.

Note that $H^0(X, D) = H^0(Y, D_Y)$, hence the reduction model does give a reduction to the effective non-vanishing problem. As an application of Theorem 4.2, we know that in Case (C) if $(X, B; D)$ admits a relatively minimal reduction model, then the effective non-vanishing holds.

**Remark 4.6.** In general, given a birational morphism $g : X \to Y$, e.g. $Y$ is a relatively minimal model of $X$, even if $D$ is not $g$-trivial, we also can define $D_Y = g_*D$ as the push-out of algebraic cycles. It is easy to verify that $D_Y$ is nef and $D_Y - (K_Y + B_Y)$ is ample. However this model is not good, since the pair $(Y, B_Y)$ is not necessarily KLT, and in general, $H^0(X, D) = H^0(Y, D_Y)$ does not hold by observing the following two examples.

**Example 4.7.** Let $X$ be a smooth projective surface, $B_1$ a $(-1)$-curve and $B_2, B_3$ smooth curves on $X$ such that $B_1, B_2, B_3$ intersect transversally at one point $p \in X$. Let $B = \frac{2}{5}B_1 + \frac{4}{5}B_2 + \frac{2}{5}B_3$. Then we can verify that the pair $(X, B)$ is KLT by blowing up at $p$. Let $g : X \to Y$ be the contraction of $B_1$. Then the pair $(Y, B_Y)$ is not KLT since the discrepancy of the exceptional divisor with center $p$ is $-\frac{11}{10} \leq -1$.

Let $X = \mathbb{F}_1$ be the Hirzebruch surface, $g : X \to Y = \mathbb{P}^2$ the contraction of the $(-1)$-section, $D$ the fiber of $X$ over $\mathbb{P}^1$. Then $D_Y = g_*D$ is a line in $\mathbb{P}^2$. It is easy to see that $H^0(X, D) < H^0(Y, D_Y)$.

Let us return to the argument of the effective non-vanishing problem on ruled surfaces.

**Lemma 4.8.** Assume that we are in Case (C). Let $F$ be the general fiber of $f : X \to C$. If $D.F \leq 1$, then $H^0(X, D) \neq 0$ holds.

**Proof.** By Theorem 4.2 we may assume that $X$ is not relatively minimal.

1. $D.F = 0$

By assumption, $X$ contains a $(-1)$-curve $l$ which is contained in some fiber $F_0$ of $f$. The inequality $0 \leq D.l \leq D.F_0 = 0$ implies that $D.l = 0$. We may consider the
contraction $g : X \to Y$ of $l$ and the reduction model $(Y, B_Y; D_Y)$. Since $D_Y.F = D.F = 0$, finally we can obtain a relatively minimal reduction model by induction.

(2) $D.F = 1$

By a similar way, we may contract all $(−1)$-curves $l$ with $D.l = 0$, at last, to obtain a reduction model $(Y, B_Y; D_Y)$ such that $D_Y$ is positive on any $(−1)$-curve on $Y$. We claim that $Y$ is relatively minimal. Otherwise, there would exist a $(−1)$-curve $l_0$ contained in some fiber $F_0 = \sum_{i=0}^r l_i$ such that all of $l_i$ are smooth rational curves with negative self-intersections. The inequality $0 < D_Y.l_0 \leq D_Y.F_0 = D.F = 1$ implies that $D_Y.l_0 = 1$ and $D_Y.l_i = 0$ for all $i > 0$, hence $l_i$ are not $(−1)$-curves and $K_Y.l_i \geq 0$ for all $i > 0$. Thus we have $−2 = K_Y.F_0 = K_Y.l_0 + \sum_{i=1}^r K_Y.l_i \geq −1$, a contradiction. □

Remark 4.9. The proof of the case $D.F = 1$ in Lemma 4.8 has already appeared in that of Proposition 4.1(2b) of [Am99]. However, there is a mistake in the remaining argument of the relatively minimal case. So we give a complete proof here for the convenience of the reader.

Due to an idea of Ambro, we can give the following partial results.

**Proposition 4.10.** In Case (C), $H^0(X, 2D) \neq 0$ always holds. Furthermore, assume that the Iitaka dimension $\kappa(X, −K_X) \geq 0$ or the numerical dimension $\nu(D) = 1$. Then $H^0(X, D) \neq 0$ holds.

**Proof.** By Lemma 4.8 we may assume that $a = D.F \geq 2$. Apply [Am99, Lemma 4.2] to $D/a$, then we have $\chi(\mathcal{O}_X) \geq −D(D + aK_X)/2a^2$, hence

$$
\chi(X, 2D) \geq D(2D − K_X) − \frac{1}{2a^2} D(D + aK_X)
$$

$$
= \frac{2a + 1}{2a} D((2 − \frac{1}{a})D − K_X)
$$

$$
= \frac{2a + 1}{2a} D((1 − \frac{1}{a})D + H + B) > 0.
$$

If $\kappa(X, −K_X) \geq 0$, then we have $D.K_X \leq 0$, hence

$$
\chi(X, D) \geq \frac{1}{2} D(D − K_X) − \frac{1}{2a^2} D(D + aK_X)
$$

$$
= \frac{a + 1}{2a} D((1 − \frac{1}{a})D − K_X)
$$

$$
= \frac{a^2 − 1}{2a^2} D(H + B) − \frac{a + 1}{2a^2} D.K_X > 0.
$$

If $\nu(D) = 1$, then $D$ is nef but not big, i.e. $D^2 = 0$. Hence $D.(−K_X) = D(H + B) > 0$, which implies $H^0(X, D) \neq 0$. □

If we denote (CR) the subcase of (C) where $D$ is nef and big and $\kappa(X, −K_X) = −\infty$, then it remains to deal with Problem [12] in Case (CR) and Case (D). It is expected that $H^0(X, D) \neq 0$ should hold in Case (CR). Until now, we cannot say anything for quasi-elliptic surfaces and general type surfaces whose Euler characteristics are negative. It is expected that in Case (D), the universal integer $m$ should be greater than 1. We shall treat these in a subsequent paper.
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DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, 2-12-1 OKAYAMA, MEGURO, TOKYO 152-8551, JAPAN

E-mail address: xie@math.titech.ac.jp