Phase transition of one dimensional bosons with strong disorder

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We study one-dimensional disordered bosons at large commensurate filling. Using a real-space renormalization group approach, we find a new random fixed point which controls a phase transition from a superfluid to an incompressible Mott-glass. The transition can be tuned by changing the disorder distribution even with vanishing interactions. We derive the properties of the transition, which suggest that it is in the Kosterlitz-Thouless universality class, and discuss its physical origin.

Models of interacting bosons subject to quenched disorder are important for understanding many real systems and present a considerable theoretical challenge. Bosons in a disordered potential, for instance, were studied extensively \cite{leggett1993,kivelson1998,leggett2007,affleck1987,affleck1987b} in relation to granular superconductors, Josephson-junction arrays, and $^4$He on porous media. Recently, there has been interest in this problem in the context of experiments with ultra cold atoms \cite{bloch2008,oreg2009}. These may provide a controllable environment for studying quenched disorder and interactions. In this letter we take a new perspective, and utilize the real-space renormalization group (RSRG) method \cite{kadanoff1966,fradkin1991,affleck1987}. Using this method it was shown that random spin chains often exhibit a universal behavior independent of the disorder realization. Using RSRG we describe the properties of a superfluid-insulator transition of the random $O(2)$ quantum rotor model, and find that it exhibits a similar degree of universality.

A standard model of granular superconductors and Josephson arrays in 1-d is the $O(2)$ quantum rotor Hamiltonian:

$$H = \frac{1}{2} \sum_j U_j \left( -i \frac{\partial}{\partial \varphi_j} \right)^2 - \sum_j J_j \cos(\varphi_{j+1} - \varphi_j). \quad (1)$$

where $U_i$ is the grain charging energy, and $J_i$ is the Josephson coupling between grains. For uniform couplings, the zero-temperature quantum partition function of the chain can be mapped to that of the two dimensional classical $x$-$y$ model with unit coupling and effective temperature $T_{\text{eff}} = \sqrt{U/J} \leq 1$. It follows that the pure system displays a Superfluid-Insulator transition of the Kosterlitz-Thouless (KT) universality class at a critical interaction strength $T_{\text{eff}} = 1$. The significance of this difference was illuminated by recent Monte-Carlo simulations of large two dimensional systems, which considered both types of disorder \cite{altman2004}. The Hamiltonian (1) is equivalent to the Bose-Hubbard model at large commensurate filling \cite{affleck1987}, but it differs in some respects from the traditional dirty boson problem \cite{affleck1987,affleck1987b}.\textsuperscript{2} First, the Hamiltonian (1) obeys a local particle hole symmetry. Thus, it describes only the Superfluid-Insulator transition at commensurate filling. In addition, it does not contain the diagonal disorder $iv_i \partial/\partial \varphi_i$. The significance of this difference was illuminated by recent Monte-Carlo simulations of large two dimensional systems, which considered both types of disorder \cite{altman2004}. While the diagonal disorder is expected to ultimately dominate, its effects may be unobservable at practical temperatures or length scales.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1}
\caption{Possible fates of the chain after repeated application of the RG decimation rules. (a) In the superfluid phase sites are joined into ever growing clusters. (b) Insulating phase, where clusters become effectively disconnected at low energy scales.}
\end{figure}
Let us begin our analysis by describing the RSRG method as applied to the random-boson problem. In the spirit of Ref. 12, we define the cutoff energy scale \( \Omega = \max \{ U, J_i \} \). At each RG step, the chain element with largest energy is decimated, renormalizing nearby interaction constants. Two types of steps are possible. If \( \Omega = U_i \) for some site \( i \), this site is assumed frozen in its ground state and the two sites are eliminated or joined into a single one. The charging energy of the new site corresponds to additive recursion relations in terms of capacitances \( C_i \equiv \Omega/U_i \):

\[
\tilde{C}_i = C_i + C_{i+1},
\]

(3)

where \( g_0(\Omega) \equiv g(0, \Gamma) \) and \( f_0(\Gamma) \equiv f(0, \Gamma) \). The first term in each equation describes the flow of the distribution due to a redefinition of the cutoff following an elimination of large couplings. The second term implements the recursion relations, taking account of the renormalized couplings. The last term removes the couplings neighboring the decimated ones and takes care of normalization of the distributions.

When either typical \( \zeta \gg 1 \) or \( \beta \gg 1 \) one can use \( \delta(\zeta_1 + \zeta_2 + 2 - \zeta) \approx \delta(\zeta_1 + \zeta_2 - \zeta) \) in (4). Then the master equations are solved by the ansatz:

\[
f(\zeta, \Gamma) = f_0(\Gamma) e^{-\zeta f_0(\Gamma)}, \quad g(\beta, \Gamma) = g_0(\Gamma) e^{-\beta g_0(\Gamma)},
\]

(5)

where \( f_0 \) and \( g_0 \) obey:

\[
\frac{df_0(\Gamma)}{d\Gamma} = f_0(\Gamma) - f_0(\Gamma)g_0(\Gamma), \quad \frac{dg_0(\Gamma)}{d\Gamma} = -g_0(\Gamma)f_0(\Gamma).
\]

(6)

Thus,

\[
f_0(\Gamma) = g_0(\Gamma) - \ln g_0(\Gamma) + A.
\]

(7)

It is interesting to note that Eqs. (6) acquire precisely the form of the KT flow equations \[14\] when written in terms of \( y = \sqrt{f_0} \) and \( x = g_0 \). The integration constant \( A \) controls the flow as depicted in Fig. 2. Distributions while the hopping connecting the new site with its neighbors is unchanged.

Repeated decimations gradually reduce the energy scale from its initial value \( \Omega_0 \) to a lower energy scale \( \Omega \). Depending on the initial distributions of the couplings, two scenarios emerge. In the first, sites are joined together into ever growing superfluid clusters (see Fig. 1(a)) leading to a superfluid phase. In the second, a growing number of sites are eliminated and form disconnected clusters leading to an insulating phase (Fig. 1(b)).

It is convenient to describe the progression of the RG transformations and the cutoff energy scale with the parameter \( \Gamma \equiv \log(\Omega_0/\Omega) \). In addition, we define the dimensionless coupling \( \zeta_i = C_i - 1 \) and \( \beta_i = \log(\Omega/J_i) \), characterized by probability distributions \( f(\zeta_i, \Gamma) \) and \( g(\beta_i, \Gamma) \[15\]. Their flow with the decreasing energy scale is given by the master equations:

\[
\frac{\partial f(\zeta, \Gamma)}{\partial \Gamma} = (1 + \zeta) \frac{\partial f(\zeta, \Gamma)}{\partial \zeta} + g_0(\Gamma) \int d\zeta_1 d\zeta_2 f(\zeta_1, \Gamma) f(\zeta_2, \Gamma) \delta(\zeta_1 + \zeta_2 + 1 - \zeta) + f(\zeta, \Gamma) (f_0(\Gamma) + 1 - g_0(\Gamma)),
\]

\[
\frac{\partial g(\beta, \Gamma)}{\partial \Gamma} = \frac{\partial g(\beta, \Gamma)}{\partial \beta} + f_0(\Gamma) \int d\beta_1 d\beta_2 g(\beta_1, \Gamma) g(\beta_2, \Gamma) \delta(\beta_1 + \beta_2 - \beta) + g(\beta, \Gamma) (g_0(\Gamma) - f_0(\Gamma)),
\]

(4)

where \( A > -1 \) flow to \( f_0 \rightarrow \infty \) and \( g_0 \rightarrow 0 \). This corresponds to vanishing Josephson coupling \( U \rightarrow \Omega \), namely an insulating phase. For \( A < -1 \) and \( g_0 > 1 \) the flow approaches a line of fixed points with a non universal \( g_0(\infty) = 1 + \alpha, \alpha > 0 \) and \( f_0(\infty) = 0 \). This corresponds to an array of Josephson junctions with no charging energy and a power-law distribution of couplings \( p(J) \propto (J/\Omega)^\alpha \). The critical flow occurs when \( A = -1 \). It terminates at the unstable fixed point \( g^* = g_0(\infty) = 1 \) \( (\alpha = 0) \) and \( f^* = f_0(\infty) = 0 \). Note that even a system with vanishingly small charging energy and random tunneling can be tuned through the critical point by changing the disorder distribution. When \( \alpha < 0 \) the system flows to the insulating phase.

We now use Eqs. (6) together with the asymptotic forms of \( g_0(\Gamma) \) and \( f_0(\Gamma) \) to show that the transition has KT-like universal properties and dynamical exponent \( z = 1 \). First, we establish a connection between energy and length scales. As the RG flow proceeds, more sites are either eliminated or joined into larger superfluid clusters. One can see that the total number of sites at scale \( \Gamma \) decreases as

\[
N(\Gamma) = N_0 e^{-\int_{f_0 + g_0}^{f_0} df^*},
\]

(8)

The length scale associated with \( \Gamma \) is the size of superfluid clusters (or bonds between them). In units of the
original lattice constant it is given by \( l(\Gamma) = N_0/N(\Gamma) = \exp(f_0^T(f_0 + g_0)d\Gamma') \). Solving the flow Eqs. (6) near the fixed point, \( g_0^* = 1 \), \( f_0^* = 0 \) we identify typical time and length scales, which characterize the critical flow:

\[
\tau_c \propto e^{-1/\sqrt{\epsilon}}, \quad \xi_c \propto e^{1/\sqrt{\epsilon}},
\]

where \( \epsilon = 1 + A > 0 \) is the tuning parameter. The exponential divergence toward the critical point suggests a transition of the KT universality class with a dynamical exponent \( z = 1 \).

Using the universal distributions we found above we can calculate several observables. For example, to find the compressibility, we apply the RG to a finite chain down to the scale \( \Gamma_1 \), where only a single site is left \((N(\Gamma_1) = 1)\). The compressibility extracted from the renormalized Hamiltonian of the remaining site \( H_1 = U n^2 \) then reads:

\[
\kappa = \left( \frac{1}{N_0 U} \right) \approx \frac{1}{N_0 \Omega(\Gamma_1)} \int_0^\infty (\zeta + 1)f_0(\Gamma_1)e^{-f_0(\Gamma_1)\kappa_d}\zeta, \tag{10}
\]

Using Eqs. (8), and (9) to derive the asymptotic forms of \( f_0(\Gamma) \) and \( g_0(\Gamma) \), we find in the insulating regime, \( \kappa_{in} \propto (\ln N_0)/N_0 \), which vanishes in the thermodynamic limit. Similarly, in the SF side \( \kappa_{sf} = k/\Omega_0 \), where \( k \) is a positive non-universal constant, that depends on initial distributions. It follows that \( \kappa \) is discontinuous at the critical point. The gap in the SF phase obeys \( \Delta = \Omega(\Gamma_1)f_0(\Gamma_1)(-\ln f_0(\Gamma_1)) \sim \sqrt{2\epsilon/\ln N_0} \). In the insulating phase, the gap is \( \Delta \sim \Omega(\Gamma_1) \sim 1/\ln N_0 \). The gap vanishes in this case because of rare large clusters that have arbitrarily small charging energy. Thus, we identify the insulating phase with a Mott glass.

We have implemented the RG scheme numerically to verify that the solution (5) is an attractor of the flow for generic initial disorder. We started from chains with \( 2.5 \times 10^6 \) sites, and different forms of initial disorder distributions. The general features of the flow diagram seem insensitive to the type of distributions used. A specific example calculated from various box distributions of \( J \) and \( U \) is given in Fig. 3. The result clearly shows that the distributions converge to the solutions (5), demonstrating that they are stable attractors of the RG flow.

The positions of the minima marked by the dashed line in Fig. 3 agree with those predicted by the first correction to (6) from expansion of the \( \delta \)-function in Eq. (1) to lowest order in \( 1/\zeta \).

The validity of the recursion relations (2) and (3) relies on the distribution of \( J \)'s being wide in the sense that \( J \ll \Omega \). Clearly this does not hold at the critical point where \( J = \Omega/2 \). While the relation (2) can be controlled by the smallness of \( \langle U \rangle \) near the critical point, corrections to (3) produce next nearest neighbor Josephson couplings of order \( (J)^2/\Omega \). To justify the RG scheme used here, we need to show that the longer range terms are irrelevant. The corresponding analysis is rather complicated and we defer it to a future work. Instead, we rederive the phase transition and explain its nature using an alternative physical argument.

To understand the origin of the fixed point distributions we employ two simple toy models (see also Ref. 7). First, consider the insulating fixed point. We assume that the Josephson couplings take the values \( J_1 = 1 \) with probability \( q \), and \( J_1 = 0 \) with probability \( 1 - q \). We also assume that the charging \( U_i = u \ll 1 \) is uniform. Such a chain consists of disconnected clusters and can describe only the insulating phase. The cluster sizes are distributed according to \( P(N) = q^N \), and the effective interaction is given by \( U(N) = u/N \). From this
we immediately see that $f(\zeta)$ is indeed given by (5) with $f_0 = \ln |q|$. If we now start eliminating sites with large $U$ from the chain, we will exactly reproduce the flow equation in (6). Clearly, the insulating phase is stable towards weak Josephson couplings between the clusters. Therefore the disconnected chain used in this toy model is fairly generic once we deal with an insulator.

We can describe the superfluid fixed line by a chain with uniform Josephson couplings $J_i = \epsilon << 1$ and charging energies with a binary distribution: $U_i = 1$ with probability $q$, and $U_i = 0$ with probability $1 - q$. We use perturbation theory to eliminate the sites with $U_i = 1$. Two remaining sites, separated by a block of $N$ eliminated sites are effectively connected by $J = \epsilon^N$ (see Fig. 4). In addition, the size distribution of blocks of a cluster of size $l$ is given by $\Delta(l) \sim \Delta_0/l$. Because clusters are independent from each other, the distribution of their sizes is simply $P(l) = L^{-1} \exp(-l/L)$, the average charging energy on the coarse grained lattice is $\Delta \sim \Delta_0 \ln L/L$. Putting everything together we find:

$$\frac{\Delta}{\Omega} \sim \left( \frac{\Delta_0}{\Omega_0} \right)^\alpha \ln \left( \frac{\Omega_0}{\Omega} \right) \left( \frac{\Delta_0}{\Omega_0} \right)^{\Gamma} e^{-\alpha \Gamma}. \quad (12)$$

Charging energy is thus relevant for $\alpha \leq 0$, and irrelevant for $\alpha > 0$, in agreement with the RG flows in Fig. 2.

The flow towards an insulator for $\alpha < 0$ implies that the superfluid stiffness is suppressed exponentially with the system size. But when $\alpha = 0^+$, $\rho(L) \sim 1/\ln L$ where $L$ is the system size. This is because the smallest $J_i$ on a chain is of order $1/L$ and serves as a lower cutoff for the average $(1/J_i)$. Thus in finite but large chains the stiffness appears to jump at the point $\alpha = 0$, in analogy with KT transition in a pure system. Disorder suppresses this jump logarithmically with increasing size. Note that in the superfluid regime, $\alpha > 0$, the collective modes can be described by a harmonic chain with off diagonal disorder (see e.g. Ref. [17]). In particular, all finite energy phonons in the disordered superfluid are localized [18].

In this letter we studied a disordered 1-d O(2) quantum rotor model. Using RSRG analysis, we found a strong-randomness fixed point that controls a transition between an incompressible Mott glass and a superfluid phase. The new phase transition is consistent with KT universality class and has a dynamical exponent $z = 1$. Surprisingly, this fixed point lies on the classical axis corresponding to vanishing charging energy and uniform distribution of Josephson couplings. Unlike other fixed points found using RSRG, this one does not belong to the infinite randomness universality class. Nevertheless, we manage to find the physical properties of the model in the critical region and in the insulating and superfluid phases. One can also use the (universal) solutions of the RSRG flow we found to calculate additional thermodynamic quantities, as well as finite temperature properties of the system. This we leave for future work.

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[19] We do not imply that the distributions of \( \zeta \) and \( \beta \) are independent of each other. Rather there is a joint distribution function \( F(\zeta, \beta, \Gamma) \), and
\[
\int d\beta F(\zeta, \beta, \Gamma),
\]
g(\( \beta, \Gamma \)) = \( \int d\zeta F(\zeta, \beta, \Gamma) \).