Vortex lattice stability and phase coherence in three-dimensional rapidly rotating Bose condensates

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We establish the general equations of motion for the modes of a vortex lattice in a rapidly rotating Bose-Einstein condensate in three dimensions, taking into account the elastic energy of the lattice and the vortex line bending energy. As in two dimensions, the vortex lattice supports Tkachenko and gapped sound modes. In contrast, in three dimensions the Tkachenko mode frequency at long wavelengths becomes linear in the wavevector for any propagation direction out of the transverse plane. We compute the correlation functions of the vortex displacements and the superfluid order parameter for a homogeneous Bose gas of bounded extent in the axial direction. At zero temperature the vortex displacement correlations are convergent at large separation, but at finite temperatures, they grow with separation. The growth of the vortex displacements should lead to observable melting of vortex lattices at higher temperatures and somewhat lower particle number and faster rotation than in current experiments. At zero temperature a system of large extent in the axial direction maintains long range order-parameter correlations for large separation, but at finite temperatures the correlations decay with separation.

I. INTRODUCTION

Rapidly rotating Bose-Einstein condensates reveal a triangular lattice of quantized vortices [1, 2, 3, 4]. As shown analytically in Refs. [5, 6, 7, 8, 9] and numerically in Refs. [10, 11, 12], and observed in Refs. [3, 4] in trapped atomic gases, the lattice supports Tkachenko modes, in which the vortices undergo primarily transverse, but slightly elliptically polarized, oscillations about their home positions. Superfluid liquid helium II is always in a regime where the Tkachenko frequencies, \( \omega_T \), are linear in the wavevector \( k \) at long wavelengths. However, by rotating the atomic condensates sufficiently rapidly one enters a regime where the excitations in the plane transverse to the rotation axis obey \( \omega_T \propto k^2 \). As shown in Refs. [7, 9] such soft modes in a two-dimensional system lead to decrease of phase coherence at large separations as the lattice rotational frequency, \( \Omega \), approaches the transverse harmonic trap frequency, \( \omega_\perp \), even at zero temperature, \( T \). At \( T = 0 \), the vortex lattice is stable and ordered until quantum fluctuations become sufficiently large to melt the lattice; however, at finite temperature, the lattice lacks long-range order in two dimensions [3].

In this paper we examine the stability of the vortex lattice and the long range phase coherence in a fully three dimensional rotating BEC, taking into account contributions of vortex line bending to the lattice modes. This analysis is an extension of that given in Ref. [9] for excitations in two dimensions. We first formulate the basic equations governing the modes for a general density profile, and then show that a uniform system has two elasto-hydrodynamic modes: a gapped sound mode (referred to as an inertial mode in [8, 9]), whose frequency is relatively isotropic; and a Tkachenko mode, whose frequency, \( \omega_T \), depends strongly on the direction of propagation. As we find, \( \omega_T \) is quadratic at long wavelengths only in the transverse plane. For given \( k \), \( \omega_T \) becomes linear as the wavevector \( \vec{k} \) moves out of the transverse plane by an angle \( \delta \theta(k) \), which vanishes as \( k \to 0 \).

In analyzing the implications of the modes for lattice stability and phase coherence we assume for simplicity that the system is uniform in the axial direction, and of thickness \( Z \); we do not attempt to model realistic condensate geometry here (as in [10, 11, 12, 13]). As a consequence of the stiffening of the Tkachenko mode as the wavevector moves out of the transverse plane, the correlations of the vortex displacements at zero temperature remain convergent at large distances; at finite temperatures, however, the displacement correlations, as in He-II [14], grow logarithmically with increasing separation, as long as the system is bounded in the axial direction. The order-parameter correlations in three dimensions are finite at large transverse separations at \( T = 0 \) for infinite thickness, but fall algebraically for \( Z \to 0 \). At finite temperature, the order-parameter correlations decay algebraically with transverse separation for infinite \( Z \), and exponentially for small \( Z \), as in He-II. Analogous issues are also encountered in the vortex lattices of superconductors [15].

For sufficiently large thermally excited vortex displacements, the vortex lattice is expected to melt. In current experiments on rapidly rotating Bose-Einstein condensates [2, 4] the vortex arrays are ordered. However, as we argue on the basis of the Lindemann criterion, with reasonable decreases in particle numbers and increases in rotation rates and temperature, thermal melting of the vortex lattice could be observable. Unlike in the case of quantum melting of the lattice at zero temperature [2, 9] where the system makes a transition eventually to a highly correlated state,
thermally induced melting is classical in nature. Since the system is quasi-two dimensional, we expect the melting to be defect-mediated \cite{25}, accompanied by observable dislocations leading to a disordered array of vortices.

II. ELASTOHYDRODYNAMIC MODES

The basic equations describing two-dimensional motions in the system, were derived in Ref. \cite{9} using an elasto-hydrodynamic approach. In constructing a full three dimensional description here we also include the bending tension of the vortex lines and fluid velocities in the axial direction. As in \cite{9}, we describe the deviations in the transverse plane of the vortices from their equilibrium positions by the continuum displacement field, \( \epsilon(r,t) \). We assume that the vortex lattice is rotating at an angular frequency \( \Omega \), and work in the frame co-rotating with the lattice. To take into account the bending of the vortices, we allow the displacement field to vary in the \( z \) direction (parallel to the rotation axis). We ignore contributions of the normal fluid at finite temperature here.

As required by conservation of vorticity, the velocity in the rotating frame, \( \mathbf{v} \), is related to the phase, \( \Phi \), of the order parameter and the vortex displacement field, by

\[
\mathbf{v} + 2\Omega \times \epsilon = \frac{\hbar}{m} \nabla \Phi.
\]

The time derivative of this equation is the superfluid acceleration equation,

\[
m \left( \frac{\partial \mathbf{v}}{\partial t} + 2\Omega \times \dot{\epsilon} \right) = \hbar \nabla \dot{\Phi} = -\nabla \mu_s,
\]

where \( \mu_s \) is the chemical potential of the system.

The equation of particle conservation is, as usual,

\[
\frac{\partial n}{\partial t} + \nabla \cdot j = 0,
\]

where \( n(r) \) and \( j = n \mathbf{v} \) are the local particle density and particle current. Conservation of momentum in three dimensions has the same form as in \cite{9},

\[
m \left( \frac{\partial j}{\partial t} + 2\Omega \times j \right) + \nabla P = -\sigma - \zeta,
\]

where \( P \) is the pressure of the superfluid. The elastic stress, \( \sigma \), is given by the derivative \( \delta E / \delta \epsilon \) of the elastic energy with respect to the displacement. We include an external force, \( \zeta \), for later convenience in calculating correlation functions.

The total energy density of the system is,

\[
E_{\text{total}}(r) = \frac{1}{2} mn|\mathbf{v}|^2 + E_{\text{internal}} + E_{\text{el}}(r).
\]

where the pressure, \( E_{\text{internal}} \), includes the effects of the centrifugal force and external potential along with the interaction energy, and the elastic energy, with line bending, is given by

\[
E_{\text{el}}(r) = 2C_1 (\nabla \cdot \mathbf{e})^2 + C_2 \left[ \left( \frac{\partial \epsilon_x}{\partial x} - \frac{\partial \epsilon_y}{\partial y} \right)^2 + \left( \frac{\partial \epsilon_x}{\partial y} + \frac{\partial \epsilon_y}{\partial x} \right)^2 \right] + \frac{\tau}{2} \left| \frac{\partial \epsilon_z}{\partial z} \right|^2 + \gamma n^0 \mathbf{e} \cdot \nabla_{\perp} n.
\]

Here \( C_1 \) and \( C_2 \) are respectively the compressional and shear moduli of the vortex lattice, and \( \tau \) is the vortex line bending tension. The final term describes the coupling of the particle and vortex densities \cite{16}; the coupling constant \( \gamma \) is a function of rotation rate. In an incompressible fluid, \( C_2 = n\hbar \Omega / 8 = -C_1 \) and \( \gamma = (\pi \hbar^2 / m) \ln(\ell / \xi) \), where \( \ell \) and \( \xi \) are the radii of the Wigner-Seitz cell and core of any given vortex, respectively; in the limit where only the lowest Landau level (LLL) in the Coriolis force is occupied (\( \Omega \rightarrow \omega_L \)), \( C_1 = 0, C_2 \approx 0.1191 m \hbar s^2 n \) and \( \gamma = \pi \hbar^2 / m \) \cite{16, 17}. For vortex core size, \( \xi \), small compared with the spacing between vortices,

\[
\tau = -\frac{1}{2} n \hbar \Omega \ln(\Omega / \Omega_C),
\]
where $\Omega_c \sim \hbar/m\xi^2 > \Omega$. More generally, we can estimate $\tau$ from the local energy of the vortex within a unit cell, $\sim |\hbar \nabla f(r_\perp)|^2/2m$, where $f(r_\perp)$ is the order parameter within the cell, normalized so that its average over a unit cell is unity. From Ref. [18], we have,

$$\tau = n\hbar\Omega a(\Omega),$$

(8)

where

$$a = \int \left\{ \left( \frac{\partial f}{\partial \rho} \right)^2 + \frac{f^2}{\rho^2} \right\} \frac{d^2 \rho}{2\pi},$$

(9)

where $\rho$ is the transverse radius measured from the center of the cell, and the integration is over the cell. For small rotation, Eq. (8) agrees with Eq. (7), while in the LLL limit, $a \to 1.7$ [18].

From Eq. (9), we find,

$$\sigma = -4\nabla_\perp(C_1 \nabla_\perp \cdot \epsilon) - 2(\nabla \cdot C_2 \nabla_\perp) \epsilon - \frac{\partial}{\partial z} \left( \frac{\tau \epsilon}{\partial z} \right) + \gamma n_0^1 \nabla_\perp n,$$

(10)

If the density of the system is non-uniform the gradients act on the elastic constants as well. The elasto-hydrodynamic modes of the system follow from the superfluid acceleration equation and conservation laws of momentum and particle number given above. These equations are valid for general density dependence in equilibrium.

We are interested in the small variations of wavevector $k$ and frequency $\omega$, in the velocity, displacement field, and density. We obtain two coupled equations. Taking the Fourier transform of Eq. (46), we have,

$$\omega^2 - s^2 k^2 - \frac{\hbar^2 k^4}{4m^2} \delta n + \left(4n^2 + \frac{\gamma n_0 nk^2}{m} \right) ik \cdot \epsilon = \frac{i}{m} k \cdot (\sigma + \zeta),$$

(11)

where $ik \cdot \sigma = ((4C_1 + 2C_2)k^2 + \tau k^2)ik \cdot \epsilon - \gamma n_0^1 k^2 \delta n$. Fourier transforming Eq. (47), and using the curl of Eq. (11) for $v_\perp$ together with the equation of continuity, we have

$$\omega^2 - \frac{2C_2}{nm} k^2 - \frac{\tau}{nm} k^2 - 4\Omega^2 k^2 \frac{\delta n}{nk^2} \frac{ik^2}{mnk^2} k \cdot (\sigma + \zeta).$$

(12)

From Eq. (11), the longitudinal and transverse components, $\hat{k} \cdot \epsilon \equiv \epsilon_L$ and $(\hat{k} \times \epsilon)_z \equiv \epsilon_T$, respectively, are related by,

$$i\omega \left(1 - \frac{\gamma n_0 k^2}{\pi m(\omega^2 - s^2 k^2)} \right) (\hat{k} \times \epsilon)_z = 2\Omega \left(1 + \frac{2(C_2 + 2C_1)k^2 + \tau k^2}{4\Omega^2 mn} + \frac{\gamma n_0^1 k^4}{4\Omega^2 mn^2(\omega^2 - s^2 k^2)} \right) \hat{k} \cdot \epsilon,$$

(13)

where we have the modified speed of sound, $s^2 = s^2 + \hbar^2 k^2/4m^2$. For $k \to 0$, we have $i\omega(\hat{k} \times \epsilon)_z = 2\Omega \hat{k} \cdot \epsilon$.

Equations (11) and (12) give the frequencies and mode functions for the rapidly rotating BEC; eliminating $\delta n$ and $\hat{k} \cdot \epsilon$ we find the secular equation for the mode frequencies,

$$D(k, \omega) \equiv \omega^4 - \alpha \omega^2 + \beta = (\omega^2 - \omega_I^2)(\omega^2 - \omega_T^2) = 0,$$

(14)

with

$$\alpha = 4\Omega^2 + s^2 k^2 + \frac{(4C_1 + 4C_2)k^2}{mn} \frac{2\tau}{mn} k^2 + \frac{2\gamma n_0^1}{m} k^2,$$

(15)

$$\beta = s^2 \left(4\Omega^2 k^2 + \frac{2k^2}{mn} (2C_1 k^2 + 2C_2 (k^2 + 2k^2)) + \frac{\tau k^2}{mn} (k^2 + 2k^2)^2 \right) \left(\frac{\gamma n_0}{m} \right)^2 k^2.$$

The second term in $\beta$ is only significant for $\hbar\Omega >> ms^2$. The mode with frequency $\omega_I$ is the gapped sound mode, and with frequency $\omega_T$, the Tkachenko mode. In the soft limit $\hbar\Omega \gg sk$, $\alpha^2 \gg 4\beta$ and we find, $\omega_I^2 \approx \alpha$ and $\omega_T^2 \approx \beta/\alpha$. Explicitly, in the incompressible limit,

$$\omega_I^2 \approx \frac{s^2}{4\Omega^2 k^2 + h\Omega (k^4 + 2k^2 (k^2 + 2k^2) \ln(\Omega/\Omega_c)) /4m + (h\Omega \ln(\Omega/\Omega_c)/(2m)) k^4 + 2k^2}{4\Omega^2 + s^2 k^2},$$

(16)

and in the LLL limit,

$$\omega_T^2 \approx \frac{(h\Omega/m)^2 k^2}{k^2 + 2k^2 + 1.7(h\Omega k^2/m)(k^2 + 2k^2) + 0.24s^2 k^2 (k^2 + 2k^2)}$$

(17)
where we have used the values of $C_1$, $C_2$, $\tau$, and $\gamma$ in the LLL limit.

Figures 1 and 2 display the gapped sound mode frequency and the Tkachenko mode frequency, respectively, as a function of the wavenumber, $k$. In the plots, we use parameters consistent with the experiments of Refs. [3, 4] in $^{87}$Rb: $a_s = 4.76$ nm and $\omega_\perp = 8.3 \times 2\pi$ Hz.

From Figure 1, frequencies of the gapped sound mode do not vary significantly with $\theta$ in the incompressible limit shown in Figure 1a, since the elastic contributions to this mode are small relative to the hydrodynamic terms. However, in the lowest Landau limit, the mode frequency does change with direction, Figure 1b, with the different values of the azimuthal angle, $\theta$, in terms of which $k_\perp = k \sin \theta$ and $k_z = k \cos \theta$. Since $\hbar \Omega >> ms^2$ in this limit, the line tension is
FIG. 2: Dispersion relations for the Tkachenko mode in three dimensions. The curves are labelled by the azimuthal angle of the wavevector, measured from the rotation axis. Panel (a) shows the characteristic structure in the regime where $\bar{h}\Omega$ is small compared with $ms^2$. The parameters are taken for a cloud of $^{87}$Rb with density $10^{14}\text{cm}^{-3}$ and $\Omega = 0.9(8.3 \times 2\pi) \text{Hz}$, for which $\bar{h}\Omega/ms^2 \approx 0.01$. The line bending tension is described by Eq. (7). Panel (b) corresponds to the regime where $\bar{h}\Omega$ is large compared with $ms^2$. The graph is calculated for a density of $10^{11}\text{cm}^{-3}$ and $\Omega = 0.99(8.3 \times 2\pi) \text{Hz}$ so that $\bar{h}\Omega/ms^2 \approx 12$, with $\tau = 1.7n\bar{h}\Omega$. The lattice Debye wavevector corresponds to $sk/2\Omega = 1/\sqrt{2}$; thus the elasto-hydrodynamic description is valid only for $(sk/2\Omega)\sin \theta$ well to the left of this point.
the coupling of the particle and vortex densities with the line tension terms with the product of the compression and plane, there is another crossover from linear to quadratic for plane, the modes is purely quadratic, and it is determined by the sound velocity. For modes out of the transverse plane, the system sizes along the azimuthal axis and in plane (∥k∥) is defined in Eq. (14). The coupling of the particle and vortex densities only comes into play in the

\[
\langle \epsilon_L \epsilon_L \rangle = \frac{\omega^2 - s^2 k^2}{nm D(k, \omega)} \left( \frac{\omega^2 - s^2 k^2 + 2C_2 k^2 + \tau k^2}{\omega^2 - s^2 k^2} \right),
\]

(18)

\[
\langle \epsilon_T \epsilon_T \rangle = \frac{\omega^2 - s^2 k^2}{nm D(k, \omega)} \left( 1 + \frac{2(2C_1 + C_2) k^2 + \tau k^2}{4\Omega^2 n m} - \frac{\gamma n k^2}{4m^2 \Omega^2 (\omega^2 - s^2 k^2)} \right),
\]

(19)

\[
\langle \epsilon_L \epsilon_T \rangle = \langle \epsilon_T \epsilon_L \rangle^* = \frac{i\omega(\omega^2 - s^2 k^2)}{2\Omega nm D(k, \omega)} \left( 1 - \frac{\gamma n k^2}{m(\omega^2 - s^2 k^2)} \right).
\]

(20)

where \(D(k, \omega)\) is defined in Eq. (10). The coupling of the particle and vortex densities only comes into play in the transverse-longitudinal correlation functions. These off-diagonal terms are not important for calculating the real space elastic displacement correlations.

We consider a system bounded between two planes perpendicular to the rotational axis with periodic boundary conditions in the \(z\) direction; \(Z\) is the distance of separation of the two planes. In the limit, \(\Omega \gg s/R_\perp\), where \(R_\perp\) is the transverse size of the system, the real space correlation of the vortex displacements is

\[
\langle |\epsilon(r) - \epsilon(r')|^2 \rangle \approx \frac{h}{Z} \sum_{k_z} \int \frac{1 - \cos k \cdot R}{(\omega_T^2 - \omega_T^2)mn} \left[ \frac{s^2 k^2}{\omega_T} (1 + 2f(\omega_T)) \right] \frac{d^2 k_\perp}{(2\pi)^2} + \text{regular terms.}
\]

(21)

where \(R = r - r' = (R_\perp, R_z)\), and \(f(\omega)\) is the Bose distribution function, \(f(\omega) = (e^{\beta \hbar \omega} - 1)^{-1}\). The \(k_z = 0\) term is the two dimensional result; the displacement correlation function in three dimensions is augmented by the \(k_z \neq 0\)
terms in the sum. At $T = 0$, the right side of Eq. (21) is convergent. However, depending on the geometry, the softness of the Tkachenko mode, i.e., $\omega_T \propto k^2 \ell$, can lead to an infrared divergence as $|R_\perp| \to \infty$ of the finite temperature part of the first term on the right side of Eq. (21). The regular terms include contributions from the gapped sound mode which, because of the gap, do not diverge.

Extracting the most divergent parts of the correlation function, taking the leading terms in the Tkachenko mode frequencies, we find $|R_\perp| \gg \ell$,

$$\langle |e^z(r) - e^z(r')|^2 \rangle = \frac{Tm\Omega}{4\pi\hbar C^2_2} \ln \frac{\ln(\hbar C^2_2 Z)}{\ln\left(\frac{|R_\perp|^2}{4} \right)} ,$$

where $\ell = (\pi n_r)^{-1/2}$ is the radius of the Wigner-Seitz cell of each vortex. If $Z$ is finite, then the displacement correlations diverge when $|R_\perp| \to \infty$. On the other hand, for $Z \ll R_\perp^2 / \ell$, the right side of Eq. (22) is proportional to $\ln N_v(R_\perp)$, where $N_v(|R_\perp|) = (|R_\perp|/\ell)^2$ is the number of vortices within radius $R_\perp$ — the result for two dimensions [9]. In this limit, no motion is excited in the $z$ direction. For $Z \sim |R_\perp| \to \infty$, the displacement correlations become independent of $|R_\perp|$ and $Z$, and the vortex lattice maintains long range order, with the vortices oscillating a finite degree about their equilibrium positions, due to quantum and thermal fluctuations.

When the vortex displacements become sufficiently large, due to quantum and thermal fluctuations, the lattice is expected to melt. To determine the mean square displacements of the vortices in a realistic trapping geometry requires solving Eqs. (11)-(13) and (10), taking the spatially varying density into account. As a first estimate, we employ the dispersion relation for the modes in uniform geometry, evaluated at the center of the trap [22], and find

$$\langle \epsilon^2 \rangle / \ell^2 \sim \Gamma \left[ N \left( 1 - \frac{\Omega^2}{\omega_T^2} \right) \right]^{-4/15} \left( \frac{T}{T_c} \right) \ln[N_v(R_\perp)] ,$$

(23)

where

$$\Gamma = \frac{8}{15^{3/5} \zeta(3)} \left[ \frac{a_s b}{d_\perp} \right]^{2/5} \left( \frac{\omega_x}{\omega_\perp} \right)^{11/15} ;$$

(24)

here $d_\perp$ is the transverse oscillator length, $b \gg 1$ is the Abrikosov parameter, describing the renormalization of $g$ arising from density variations within the vortex cells [7, 8, 18], we have taken $C_2 = n\hbar\Omega/8$; and $T_c \approx \hbar / (N\omega_\perp^2 - \Omega_2^2) / \zeta(3))^{1/3}$ is the Bose-Einstein condensation temperature [24].

For the JILA experiments [3, 4], $\Gamma \approx 0.07$. Thus the Lindemann criterion for melting of the vortex lattice [23], $\langle \epsilon^2 \rangle / \ell^2 \approx 0.07$, would indicate that the currently observed lattices should be ordered, as observed, but at higher temperatures, and somewhat lower particle number and faster rotation rates, one could observe thermal melting of the vortex lattice. For example, with $N \approx 10^4$ and $\omega_\perp / \omega_T \approx 0.999$, Eq. (23) gives $\langle \epsilon^2 \rangle / \ell^2 \approx 0.2T/T_c$, so that one is in the range of melting. The drop in $C_2$ in the LLL region, which is not included in this estimate, would further increase the vortex displacements. In the LLL limit, quantum fluctuations contribute to the vortex displacements an amount $\langle \epsilon^2 \rangle / \ell^2 \approx 1.22/\nu$, where $\nu = N/N_v$ is the filling factor [5, 9]. The ratio of the mean square displacements due to thermal fluctuations to those due to quantum fluctuations is

$$\frac{\langle \epsilon^2 \rangle_{\text{thermal}}}{\langle \epsilon^2 \rangle_{\text{quantum}}} \sim 10^{3/2} \nu^{-1/3} (T/h\Omega) \ln N_v .$$

As $\nu$ decreases, this ratio grows, and eventually thermal fluctuations, at a given temperature, will dominate over quantum fluctuations.

The Lindemann criterion gives only a rough approximation for determining the point of vortex lattice melting. Correlation functions of the type in (22) with logarithmic terms are characteristic of two dimensional systems that exhibit a Kosterlitz-Thouless transition, suggesting that the thermal melting of the vortex lattice is defect-mediated [25]. Experimental determination of the breakdown of the vortex lattice order, and the detailed conditions under which the lattice melts, would provide valuable insight. We discuss this theoretical avenue in a future article [26].

We now derive the phase-phase correlation function, which we write in terms of displacement and velocity correlations using the Fourier transform of Eq. (11) as [9]:

$$\langle \Phi \Phi^* \rangle = \frac{m^2}{\hbar^2 k^2} \left[ \langle v_L v_L^* \rangle - 2\Omega (\langle \epsilon_T v_L^* \rangle + \langle v_L \epsilon_T^* \rangle) + 4\Omega^2 (\epsilon_T \epsilon_T^*) \right] .$$

(26)

The density-density correlation function,

$$\langle \delta n \delta n^* \rangle = \frac{nk^2}{mD(k, \omega)} \left[ \omega^2 - 4\Omega \frac{k^2}{k^2} - 2C_2 \frac{k^2}{nm} \left( 1 + \frac{k^2}{k^2} \right) - 4C_1 \frac{k^2}{nm} \left( \frac{k^2}{k^2} \right) - \tau \frac{k^2}{nm} \left( 1 + \frac{k^2}{k^2} \right) \right] ,$$

(27)
together with the equation of continuity, gives
\[ \langle v_L v_L^* \rangle = \frac{\omega^2}{n k^2 D(k, \omega)} \left[ \frac{1}{nm} \right]^{\frac{1}{nm}} - \frac{1}{nm} \]
\[ = \frac{\omega^2}{nm D(k, \omega)} \left[ \omega^2 - 4\Omega^2 \frac{k^2}{k^2} - \frac{2C_2}{nm} \left( 1 + \frac{k^2}{k^2} \right) - \frac{4C_1 k^2}{nm} \left( \frac{k^2}{k^2} \right) - \tau k^2 \right] - \frac{1}{nm}. \]  
(28)

where the constant term comes from the commutation relation between the current density and the particle density. Furthermore, using Eq. (45) and the equation of continuity,
\[ \langle v_L \epsilon_T \rangle = \frac{2\Omega^2}{\omega^2 - s^2 k^2} (\epsilon_T \epsilon_T) - i \frac{\gamma n_0^0 \omega k^2}{m (\omega^2 - s^2 k^2)} \langle \epsilon_L \epsilon_T \rangle \]  
(30)
\[ \langle \epsilon_T v_L^* \rangle = \frac{2\Omega^2}{\omega^2 - s^2 k^2} (\epsilon_T \epsilon_T) + i \frac{\gamma n_0^0 \omega k^2}{m (\omega^2 - s^2 k^2)} \langle \epsilon_T \epsilon_L \rangle \]  
(31)

We thus find the correlation function of the phase:
\[ \langle \Phi(\vec{r}) - \Phi(\vec{r}^\prime) \rangle^2 \approx \frac{1}{Z} \sum_{k_z} \int \frac{1}{\omega^2 - \omega_T^2} \left( \omega^2 + \omega_T^2 \right) \left( 1 + \frac{k^2}{k^2} \right) \frac{d^2 k}{(2\pi)^2} + \text{regular terms} \]  
(33)

At \( T = 0 \), this correlation function is convergent as \( Z \to \infty \), but for \( Z \to 0 \) it grows logarithmically with increasing separation [9]. At finite temperatures, for \( |\vec{R}| \gg \ell \),
\[ \langle \Phi(\vec{r}) - \Phi(\vec{r}^\prime) \rangle^2 \approx \frac{T}{\pi \hbar^2} \left( \frac{2\Omega^2 \sqrt{C_2}}{\Omega^2 nm} \right) \frac{d^2 k}{k} + C(\frac{k^2}{2\Omega^2 nm})^{-\frac{1}{2}}, \]  
(34)

where \( k^2_\perp = 4\pi n_e \) is the Debye wavelength, cutting off modes with wavelengths less than the lattice spacing, and
\[ \Lambda^2 = |R_\perp|^2 + R_z \sqrt{C_2/2\Omega^2 nm} \]  
(35)

is the square of the effective infrared cutoff. For \( Z \gg R_\perp^2 / \ell \), Eq. (34) increases logarithmically as \( |\vec{R}| \to \infty \):
\[ \langle \Phi(\vec{r}) - \Phi(\vec{r}^\prime) \rangle^2 \approx \frac{T}{\pi \hbar^2} \sqrt{\frac{2\Omega^2 \sqrt{C_2}}{\Omega^2 nm}} \left( k^2_\perp + k^2_\parallel \right)^{-\frac{1}{2}}. \]  
(36)

where \( \epsilon = (T/\pi \hbar^2) \sqrt{\Omega^2 \sqrt{C_2}} \). If we assume that the fluctuations of the order parameter, \( \Psi \), are Gaussianly distributed, then
\[ \langle \delta \Psi(\vec{r}) \delta \Psi(\vec{r}^\prime) \rangle \sim e^{-\frac{1}{2} \langle (\Phi(\vec{r}) - \Phi^*(\vec{r}^\prime) )^2 \rangle} \sim \langle k_D^2 \rangle^{-\frac{1}{2}}. \]  
(37)

On the other hand, for \( Z \ll \hbar / ms \), Eq. (34) gives
\[ \langle \Phi(\vec{r}) - \Phi(\vec{r}^\prime) \rangle^2 \approx \frac{2\Omega^2 m^2 \tau}{3\pi \hbar^2 C_2 Z} \Lambda^2, \]  
(38)

which implies an exponential loss of long range order as \( |\vec{R}| \to \infty \). Thus, thermal fluctuations can have a stronger effect on destroying phase coherence than quantum fluctuations, which depend only logarithmically on \( \Lambda \). For finite temperatures in the LLL limit, the ratio of the contribution to \( \langle (\Phi(\vec{r}) - \Phi^*(\vec{r}^\prime) )^2 \rangle \) from thermal fluctuations to the contribution from quantum fluctuations is \( \sim 10^3 \nu^3 (\hbar/\Omega) N_e(f(|R_\perp|)/ln N_e(|R_\perp|), which increases with |R_\perp|. \)

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APPENDIX

In this Appendix, we derive Eqs. (11)-(13). Dividing Eq. (4) by \( n \), using the Gibbs-Duhem relation at \( T = 0 \), \( \nabla P = n \nabla \mu_s \), and subtracting it from Eq. (2), we find

\[
2m\Omega \times (\dot{e} - v) = \frac{1}{n}(\sigma + \zeta),
\]

expressing the balance of the Coriolis force on the left with the elastic stress and external perturbation. The curl and divergence of this equation yield,

\[
\hat{z} \nabla \cdot (\dot{e} - v) - \frac{\partial}{\partial z} (\dot{e} - v) = \frac{1}{2nm} \nabla \times (\sigma + \zeta),
\]

and

\[
(\nabla \times \dot{e})_z + 2\Omega \nabla \cdot \epsilon = -\frac{1}{2nm} \nabla \cdot (\sigma + \zeta).
\]

For the rapidly rotating BEC, the energy to add a particle includes the interaction energy, \( \frac{1}{2}gn^2 \), the external potential, the quantum pressure, and the elastic energy on account of the coupling of the particle and vortex densities. Thus,

\[
\mu_s = V_{\text{eff}} + gn - \frac{\hbar^2}{2m\sqrt{n}} \nabla^2 \sqrt{n} + \frac{\delta \xi_{\perp}}{\delta n}
\]

\[
= V_{\text{eff}} + gn - \frac{\hbar^2}{2m\sqrt{n}} \nabla^2 \sqrt{n} - \gamma n^0_\nu \nabla \cdot \epsilon,
\]

where

\[
V_{\text{eff}}(r) = V(r) - \frac{m}{2} \Omega^2 r^2_\perp
\]

is the trapping potential, \( V \), plus the centrifugal potential. For the following calculations, we deal with a uniform particle density, which is the case if the system is in a harmonic trap with a radial trap frequency equal to the rotation rate, or if the superfluid is incompressible and in a cylindrical container.

We add perturbations of the density and velocity to linear order. From the Thomas-Fermi approximation, \( \mu_s^{(0)} = V_{\text{eff}} + gn_0 \) in equilibrium, we have \( \delta \mu_s = ms^2 \delta n - \gamma n^0_\nu \nabla \cdot \epsilon \), where \( s = \sqrt{gn/m} \) is the speed of sound. Taking the divergence of Eq. (2) and using the equation of continuity, we obtain

\[
\left( -\frac{\partial^2}{\partial t^2} + s^2 \nabla^2 - \frac{\hbar^2}{4m^2} \nabla^4 \right) \delta n = 2n\Omega (\nabla \times \dot{e})_z + \frac{\gamma n^0_\nu}{m} \nabla^2 (\nabla \cdot \epsilon)
\]

\[
= \left( \frac{\gamma n^0_\nu}{m} \nabla^2 - 4n\Omega^2 \right) (\nabla \cdot \epsilon) - \frac{1}{m} \nabla \cdot (\sigma + \zeta),
\]

where we note that \( \nabla \cdot \epsilon = \nabla \cdot \epsilon \). The density variations are coupled to the transverse motion of the displacement field via the Coriolis force and to the longitudinal displacement via the coupling of the particle and vortex densities.

Taking the time derivative of Eq. (10), using the curl of Eq. (10) and Eq. (41), and neglecting terms of the elastic moduli squared, we obtain

\[
\left( \frac{\partial^2}{\partial t^2} + \frac{2C_2}{nm} \nabla^2_\perp + \frac{\tau}{nm} \frac{\partial^2}{\partial z^2} \right) \nabla \cdot \epsilon = \frac{1}{n} \frac{\partial^2}{\partial t^2} \delta n + \frac{\gamma n^0_\nu}{(2\Omega nm)^2} \left( 2C_2 \nabla^2_\perp + \tau \frac{\partial^2}{\partial z^2} \right) \nabla^2_\perp \delta n + \frac{\partial^2}{\partial z \partial t} \delta v_z.
\]

From the curl of the curl of Eq. (1), we have an expression for \( v_z \):

\[
\nabla^2 v_z = -\frac{1}{n} \frac{\partial^2}{\partial z \partial t} \delta n - 2\Omega \frac{\partial^2}{\partial z^2} (\nabla \times \epsilon)_z
\]

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