Anisotropic bispectrum of curvature perturbations from primordial non-Abelian vector fields

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Abstract. We consider a primordial $SU(2)$ vector multiplet during inflation in models where quantum fluctuations of vector fields are involved in producing the curvature perturbation. Recently, a lot of attention has been paid to models populated by vector fields, given the interesting possibility of generating some level of statistical anisotropy in the cosmological perturbations. The scenario we propose is strongly motivated by the fact that, for non-Abelian gauge fields, self-interactions are responsible for generating extra terms in the cosmological correlation functions, which are naturally absent in the Abelian case. We compute these extra contributions to the bispectrum of the curvature perturbation, using the $\delta N$ formula and the Schwinger-Keldysh formalism. The primordial violation of rotational invariance (due to the introduction of the $SU(2)$ gauge multiplet) leaves its imprint on the correlation functions introducing, as expected, some degree of statistical anisotropy in our results.

We calculate the non-Gaussianity parameter $f_{NL}$, proving that the new contributions derived from gauge bosons self-interactions can be important, and in some cases the dominant ones. We study the shape of the bispectrum and we find that it turns out to peak in the local configuration, with an amplitude that is modulated by the preferred directions that break statistical isotropy.

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1. Introduction

The standard cosmological model is based on the idea of inflation as driven by a single slowly-rolling scalar field whose primordial quantum fluctuations are responsible for both CMB perturbations and the large scale structures of the universe [1]. Current observations indicate that the universe on large scales is homogeneous and isotropic. The degree of deviation from this smooth background is provided by the temperature fluctuations of the CMB which are of the order of $10^{-5}$ and are almost scale-independent [2, 3, 4, 5, 6]. The standard inflationary scenario predicts the observed power spectrum of cosmic fluctuations. On the other hand, there is still a lot of room for alternative scenarios without ruining the current agreement with observations. Alternatives include, for example, multifield models [7, 8, 9, 10, 11, 12, 13], the curvaton scenario [14, 15, 16, 17, 18, 19] and theories with non-canonical Lagrangians as k-inflation [20, 21], DBI inflation [22, 23] or ghost inflation [24]. So far, observations have not provided us with the precision measurements which would allow to discriminate among all these different models, but new projects, such as the Planck [25] satellite which was just launched, can potentially reach the levels of precision necessary for this purpose. One important feature of the CMB anisotropies to be decoded is the degree of non-Gaussianity [26, 27]. A random field is defined as Gaussian if all the information is contained in its two-point correlation function. In single-field, slow-roll inflation the level of non-Gaussianity is very small since the connected parts of higher order correlation functions are proportional to powers of the slow-roll parameters [28, 29, 30]. If a larger non-Gaussianity will be detected in the future, this could then rule out the minimal model. Recently another alternative to the standard inflationary scenario has attracted some attention where primordial vector fields play a non-negligible role during inflation [31, 32, 33, 34, 35, 36, 37, 38, 39, 40]. An interesting possibility is that these extra fields might be responsible for generating an observable level of non-Gaussianity [36, 38, 35]. This is not the only motivation to consider such models. First of all vector fields are present in particle physics models and that suggests that one has to include them in a theory of the early universe. This can be done without necessarily affecting current observations, for example keeping their contribution to the energy density of the universe way below the one from the inflaton. However, a peculiarity of the vector fields, which the scalar fields do not possess, is to violate rotational invariance as some observations seem to suggest. Violation of rotational invariance would affect the correlation functions introducing some degree of statistical anisotropy. The power spectrum is constrained in this sense if a single preferred direction of space is involved [41]. This is something to keep in mind for example when constructing models of vector inflation. The isotropic background expansion can be preserved to a good approximation in different ways, such as by keeping the energy density of the vector subdominant w.r.t. the one of the inflaton, or by considering the existence of many randomly oriented vectors [33]. Nevertheless, the presence of the vector fields introduces statistical anisotropy in the power spectrum and in the bispectrum as well [36]. On the other hand, no observational constraints are currently available for...
the two-point function if more than one preferred direction is involved. Similarly, for higher order correlators, no obervational constraints of this kind are available.

The effects of a primordial vector field on the power spectrum and bispectrum of the curvature fluctuations $\zeta$ were recently investigated in [36, 38] using the $\delta N$ formalism [42, 43, 44, 45] both in vector inflation and in the vector curvaton models, and also in [35] for a model of hybrid inflation.

Up to now only primordial Abelian vector fields were considered. In this paper we wish to explore a more realistic model of gauge interactions of an $SU(2)$ vector multiplet. The reason for considering a non-Abelian theory relies on the fact that, differently from what happens in the Abelian case, the vector field components are involved in self-interactions as described by cubic and quartic terms in the Lagrangian. Gauge self-interactions naturally produce additional contributions to the bispectrum w.r.t. the Abelian case. The computation of these new contributions represents the main purpose of our work.

The paper is organized as follows: in Section 2 we introduce the $\delta N$ formula for the curvature perturbation and the Lagrangian of our system; in Section 3 we carry out the calculations of all the terms that do not require the intervention of the Schwinger-Keldysh formula; in Section 4 we derive the contribution to the bispectrum that arises from gauge bosons self-interactions; in Section 5 we study the shape of the bispectrum; in Section 6 we estimate the non-Gaussianity parameter $f_{NL}$; in Section 7 we draw our conclusions. We include four Appendices: A, on background and first order perturbation equations for the gauge multiplet; B, where we derive the expression of the number of e-foldings of inflation and its derivatives in the presence of vector fields; C, which collects lengthy expressions for functions appearing in the final results for the bispectrum; D, where we give some details about the dependence of the bispectrum from the angles between wave and gauge vectors in a sample spatial configuration.

2. Bispectrum of the curvature perturbation in the $\delta N$ formalism

In the the $\delta N$ formalism the curvature perturbation $\zeta(\vec{x})$ at a given time $t$ can be interpreted as a geometrical quantity indicating the fluctuations in the local expansion of the universe; in fact, if $N(\vec{x}, t^*, t)$ is the number of e-foldings of expansion evaluated between times $t^*$ and $t$, where the initial hypersurface is chosen to be flat and the final one is uniform density, we have

$$\zeta(\vec{x}, t) = N(\vec{x}, t^*, t) - N(t^*, t) \equiv \delta N(\vec{x}, t).$$ (1)

The number of e-foldings $N(\vec{x}, t^*, t)$ depends on all the fields and their perturbations on the initial slice. In principle, since the fields are governed by second order differential
equations, it should also depend on their first time derivative, but if we assume that slow- 
roll conditions apply, then the time derivatives will not count as independent quantities. 
In a theory which includes vector fields, spatial isotropy is not expected to be preserved 
and therefore the power spectrum and the bispectrum can in principle depend both on 
the length adn direction of the wavevectors \( \vec{k}_i \), rather than just on their moduli 

\[
\langle \zeta_{\vec{k}_1}(t) \zeta_{\vec{k}_2}(t) \rangle = (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2) P_\zeta(\vec{k}), \\
\langle \zeta_{\vec{k}_1}(t) \zeta_{\vec{k}_2}(t) \zeta_{\vec{k}_3}(t) \rangle = (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3).
\]

(2) \hspace{1cm} (3)

In the next section, we are going to spell out the \( \delta N \) expansion in terms of the fields of 
our theory; later we will replace it into Eq.(3) and calculate the bispectrum to tree-level.

2.1. Specializing to our theory

Our theory includes a scalar field \( \phi \) playing the role of the inflaton and an \( SU(2) \) gauge 
multiplet \( A^a_\mu \) (\( a=1,2,3 \)) non-minimally coupled to gravity. The metric of an FRW flat 
spacetime \( ds^2 = dt^2 - a^2 \delta_{ij} dx^i dx^j \) is employed and held unperturbed. The action is 

\[
S = \int d^4x \sqrt{-g} \left[ \frac{m^2 R}{2} - \frac{1}{4} g^{\mu\alpha} g^{\nu\beta} \sum_{a=1,2,3} F^a_{\mu\nu} F^a_{\alpha\beta} - \frac{M^2}{2} g^{\mu\nu} \sum_{a=1,2,3} B^a_\mu B^a_\nu + L_\phi \right], 
\]

(4)

where \( L_\phi \) is the scalar field Lagrangian and 

\[
F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + g_e \varepsilon^{abc} B^b_\mu B^c_\nu 
\]

(5)

(we name \( g_e \) the gauge coupling). Notice that \( B^a_\mu \) are the comoving fields, the physical 
fields being given by \( A^a_\mu = \left( B^a_\mu, (1/a) B^a_i \right) \). The coupling between the gauge bosons and 
gravity is “hidden” in the effective mass \( M \): by definition \( M^2 = m_0^2 + \xi R \), where \( m_0 \) is the 
gauge bosons mass and \( \xi \) is a numerical factor; using the slow-roll approximation, 
the Ricci scalar is \( R = -6 \left[ \frac{\dot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right] \simeq -12 H^2 \).

The quantum fluctuations for the vector fields can be expanded in terms of creation and 
anihilation operators (this is always correct if we are dealing with free gauge boson, 
their background equations of motion being of Klein-Gordon type) 

\[
\delta A^a_\lambda(\vec{x}, \eta) = \int \frac{d^3 q}{(2\pi)^3} e^{i \vec{q} \cdot \vec{x}} \sum_{\lambda=L,R,I} \left[ e^\lambda_\eta(q) a^a_\lambda(q, \eta) + e^\lambda_{\eta}(q) \left( a^a_\lambda(q) \right)^\dagger \right] \delta A^a_\lambda(q, \eta), 
\]

(6)

where 

\[
\left[ a^a_\lambda(q), \left( a^a_\lambda(q) \right)^\dagger \right] = (2\pi)^3 \delta_{\lambda,\lambda'} \delta_{a,a'} \delta^{(3)}(\vec{q} - \vec{q}') 
\] 

(7)

\( d\eta = dt/a(t) \) is the conformal time and \( \lambda \) labels left, right and longitudinal polarization 
states.

The perturbation of the number of e-folds on large scales is determined by the perturbations of the fields at the initial time and the complete \( \delta N \) formula then reads
\[ \zeta(\vec{x}, t) = N_\phi \delta \phi + N_a^\mu \delta A^\mu_a + \frac{1}{2} N_{\phi \phi} (\delta \phi)^2 + \frac{1}{2} N_{ab}^{\mu \nu} \delta A^\mu_a \delta A^\nu_b + N_{\phi a}^\mu \delta \phi \delta A^\mu_a + \ldots \] (8)

where

\[ N_\phi \equiv (\frac{\partial N}{\partial \phi})_{\tau^*}, \quad N_a^\mu \equiv (\frac{\partial N}{\partial A^\mu_a})_{\tau^*}, \quad N_{\phi a}^\mu \equiv (\frac{\partial^2 N}{\partial \phi \partial A^\mu_a})_{\tau^*} \] (9)

and so on. \( N \) is the number of e-foldings between the time \( \tau^* \) on the initial hypersurface and the final time \( \eta \) at which the bispectrum of the \( \zeta \) is being evaluated. A convenient choice, that we will follow for the rest of the paper, is to take \( \eta^* \) to be close to the time when a given perturbation mode of wavenumber \( k \) crosses out the horizon. The coefficients in Eq. (9) will be calculated in Appendix B. Notice that the e-folding numbers and their derivatives are defined w.r.t. the physical fields, whereas the action (4) is written in terms of the comoving fields \( B^a_\mu \).

Plugging the \( \delta N \) expansion into Eq. (3), we get an infinite series of terms. Retaining the tree-level ones (i.e. the terms that are formally equivalent to the product of two power spectra), we are left with

\[ \langle \zeta_{k_1}(t) \zeta_{k_2}(t) \zeta_{k_3}(t) \rangle_{\text{tree}} = \frac{3}{2} N^2 N_{\phi \phi} \int d^3x \langle \delta \phi_{k_1} \delta \phi_{k_2} \rangle \int d^3q_1 d^3q_2 e^{-i \vec{q}_1 \cdot \vec{x}} \delta \phi_{\vec{q}_1} \delta \phi_{\vec{q}_2} \rangle_{\tau^*} \]

\[ + \frac{3}{2} N_a^i N_b^j N_c^{kl} \langle \delta A^a_{i,k_1} \delta A^b_{j,k_2} \rangle \int d^3q_1 d^3q_2 e^{-i \vec{q}_1 \cdot \vec{x}} \delta \phi_{\vec{q}_1} \delta \phi_{\vec{q}_2} \rangle_{\tau^*} \]

\[ + 6 N_{\phi} N^i_a N^j_b \langle \delta \phi_{k_1} \delta \phi_{k_2} \rangle \int d^3q_1 d^3q_2 e^{-i \vec{q}_1 \cdot \vec{x}} \delta \phi_{\vec{q}_1} \delta \phi_{\vec{q}_2} \rangle_{\tau^*} \]

\[ + N_{\phi}^2 \langle \delta \phi_{k_1} \delta \phi_{k_2} \rangle_{\tau^*} \]

\[ + N_a^i N_b^j N_c^k \langle \delta A^a_{i,k_1} \delta A^b_{j,k_2} \delta A^c_{k,k_3} \rangle_{\tau^*}. \] (10)

In the previous expansion we have not included terms that would require some kind of coupling between scalar and vector fields, i.e. terms that are proportional to \( N_\phi N_a^i N_b^j, N^2 N_{\phi \phi a}, N_\phi N_a^i N_{\phi \phi b}, N^2 N_a^i, N_{\phi \phi a} N_{\phi \phi b} \) and \( N_\phi N_{\phi \phi a} N_{\phi \phi b} \). Also, the terms that are proportional to \( N_0^a \equiv (\partial N)/(\partial A^0_a) \) do not appear in the previous equation since the \( A^0_a \) fields are set equal to zero. The reason for this will be explained in details in Appendix A (see discussion after Eq (A.7)); here we anticipate that, with a zero expectation value for the temporal components of the background gauge fields and with some upper bounds on the SU(2) coupling constant of the theory, it is possible to obtain a slow-roll evolution for the \( A^a_0 \) fields during inflation.

It is useful to notice that the terms in the first four lines of (10) are also present in the Abelian case, while the one in the last line is strictly non-Abelian: it provides a non-zero contribution only thanks to gauge bosons self-interactions.
3. Power spectrum and the ‘Abelian’ terms of the bispectrum

It is useful to write the expression for the power spectrum of the curvature perturbation in the presence of a vector multiplet. Let us define

\[ P_{ab}^\pm \equiv (1/2)(P_{R}^{ab} \pm P_{L}^{ab}), \quad (11) \]

where

\[ P_{R}^{ab} \equiv \delta_{ab} \delta A_{R}^{a}(k, t^*) \delta A_{R}^{b}(k, t^*), \quad (12) \]
\[ P_{L}^{ab} \equiv \delta_{ab} \delta A_{L}^{a}(k, t^*) \delta A_{L}^{b}(k, t^*), \quad (13) \]
\[ P_{long}^{ab} \equiv \delta_{ab} \delta A_{long}^{a}(k, t^*) \delta A_{long}^{b}(k, t^*) \quad (14) \]

(\( \delta A \) indicating the eigenfunctions for the gauge fields). The power spectrum of \( \zeta \) can then be written as

\[ P_{\zeta}^{\pm}(\vec{k}) = N_{\phi}^2 P_{\phi}(k) + N_{a}^{i} N_{b}^{j} \left[ \delta_{ij} P_{+}^{ab} + \hat{k}_{i} \hat{k}_{j} \left( P_{long}^{ab} - P_{+}^{ab} \right) + i \epsilon_{ijk} \hat{k}_{k} P_{+}^{ab} \right] \]

\[ = P_{iso}^{\pm}(k) \left[ 1 + g^{ab} \left( \hat{k} \cdot \vec{N}_{a} \right) \left( \hat{k} \cdot \vec{N}_{b} \right) + i s^{ab} \hat{k} \cdot \left( \vec{N}_{a} \times \vec{N}_{b} \right) \right], \quad (15) \]

where

\[ g^{ab} \equiv \frac{P_{long}^{ab} - P_{+}^{ab}}{N_{\phi}^2 P_{\phi} + \left( \vec{N}_{c} \cdot \vec{N}_{d} \right) P_{+}^{cd}}, \quad (16) \]
\[ s^{ab} \equiv \frac{P_{+}^{ab}}{N_{\phi}^2 P_{\phi} + \left( \vec{N}_{c} \cdot \vec{N}_{d} \right) P_{+}^{cd}} \quad (17) \]

and we have factorized the isotropic part of the power spectrum

\[ P_{iso}^{\pm}(k) \equiv N_{\phi}^2 P_{\phi}(k) + \left( \vec{N}_{c} \cdot \vec{N}_{d} \right) P_{+}^{cd}. \quad (18) \]

In Eq. 15 \( g^{ab} \) and \( s^{ab} \) represent the amplitude of statistical anisotropy, along the preferred directions specified by the vectors \( \vec{N}_{a} \equiv N_{a}^{i} \) and \( \vec{N}_{b} \). Eq.(18) can also be written in the form

\[ P_{iso}^{\pm}(k) = N_{\phi}^2 P_{\phi}(k) \left[ 1 + \beta_{cd} \frac{P_{+}^{cd}}{P_{\phi}} \right], \quad (19) \]

introducing the parameter

\[ \beta_{cd} \equiv \frac{\vec{N}_{c} \cdot \vec{N}_{d}}{N_{\phi}^2}. \quad (20) \]

Let us now move to the bispectrum. The first three lines of Eq.(10) can be easily evaluated
It is convenient to rewrite the expression in the second line of Eq.\((21)\) as follows (see \([36]\)). Notice that if gauge indices are suppressed in Eq.\((21)\), the Abelian expression is recovered.

\[
\begin{align*}
    B_\zeta(k_1, k_2, k_3)_{tree} & \equiv \frac{1}{2} N^i_\phi N_{i\phi} \left[ P_{\phi}(k_1) P_\phi(k_2) + \text{perms.} \right] \\
    & + \frac{1}{2} N^i_a N^j_b N^{kl}_{cd} \left[ \Pi^{ac}_{ik}(k_1) \Pi^{bd}_{jl}(k_2) + \text{perms.} \right] \\
    & + \frac{1}{2} N_\phi N^i_\alpha N^j_{\phi\beta} \left[ P_{\phi}(k_1) \Pi^{ab}_{ij}(k_2) + \text{perms.} \right],
\end{align*}
\]  

(21)

where we have defined

\[
\Pi^{ab}_{ij}(k) \equiv T^{even}_{ij}(k) P^{ab}_+ + i T^{odd}_{ij}(k) P^{ab}_- + T^{long}_{ij}(k) P^{ab}_L.
\]

(22)

In the previous equations, \(T^{even}_{ij}, \ T^{odd}_{ij}\) and \(T^{long}_{ij}\) are defined as in \([36]\)

\[
\begin{align*}
    T^{even}_{ij}(k) & \equiv e^L_i(k) e^L_j(k) + e^R_i(k) e^R_j(k), \\
    T^{odd}_{ij}(k) & \equiv i \left[ e^L_i(k) e^L_j(k) - e^R_i(k) e^R_j(k) \right], \\
    T^{long}_{ij}(k) & \equiv e^I_i(k) e^I_j(k),
\end{align*}
\]

(23)\-(25)

with \(e^L_i(k) \equiv \frac{1}{\sqrt{2}} (\cos \theta \cos \phi - i \sin \phi, \cos \theta \sin \phi + i \cos \phi, -\sin \theta)\), \(e^R_i(k) = e^L_i(k)\) and \(e^I_i(k) = \hat{k} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\), so

\[
\begin{align*}
    T^{even}_{ij}(k) & = \delta_{ij} - \hat{k}_i \hat{k}_j, \\
    T^{odd}_{ij}(k) & = \epsilon_{ijk} \hat{k}_k, \\
    T^{long}_{ij}(k) & = \hat{k}_i \hat{k}_j.
\end{align*}
\]

(26)\-(28)

Notice that if gauge indices are suppressed in Eq.\((21)\), the Abelian expression is recovered (see \([36]\)).

It is convenient to rewrite the expression in the second line of Eq.\((21)\) as follows

\[
N^i_a N^j_b N^{kl}_{cd} \Pi^{ac}_{ik}(k_1) \Pi^{bd}_{jl}(k_2) = \left( N^i_a N^{ac}_{ik} \right) N^{kl}_{cd} \left( N^j_b N^{bd}_{jl} \right) = M^c_k N^{kl}_{cd} M^d_l,
\]

(29)

where

\[
\tilde{M}^c(k) \equiv P^{ac}_+ (k) \left[ \tilde{N}_a + p^{ac}(k) \hat{k} \left( \hat{k} \cdot \tilde{N}_a \right) + i q^{ac}(k) \hat{k} \times \tilde{N}_a \right]
\]

(30)

and we define

\[
p^{ac}(k) \equiv \frac{P^{ac}_{long} - P^{ac}_+}{P^{ac}_+},
\]

(31)

\[
q^{ac}(k) \equiv \frac{P^{ac}_-}{P^{ac}_+}.
\]

(32)

Similarly, the third line in Eq.\((21)\) becomes

\[
N_\phi N^i_a N^j_{\phi\beta} P_\phi(k_1) \Pi^{ab}_{ij}(k_2) = N_\phi N^i_a N^j_{\phi\beta} P_\phi(k_1) M^a_i(k_2).
\]

(33)
The contribution from the fourth line in Eq.(10) can be calculated using the Schwinger-Keldysh formalism [46, 47, 48]. It is a well-known result that the sum of this last term and the one from the first line of Eq.(10) is proportional to the slow-roll parameters [28, 29].

It is important to notice that, in models where \( P_{ab}^{long} = P_{ab}^+ \) and \( P_{ab}^- = 0 \), the coefficients \( g_{ab}^c, s_{ab}^c, p_{ab}^c \) and \( g_{ab}^d \) are all equal to zero and therefore both the power spectrum and the Abelian part of the bispectrum of \( \zeta \) become isotropic. In general \( P_{ab}^- = 0 \), unless there is some mechanism producing parity violation for \( A_i \). The condition \( P_{ab}^{long} = P_{ab}^+ \) is more subtle. It could be realized if the longitudinal and the transverse parts of the vector fields evolved in the same way [36].

4. Calculation of the ‘non-Abelian’ terms of the bispectrum

The last line of Eq.(10) also requires the Schwinger-Keldysh formula

\[
\langle \Omega | \Theta(t) | \Omega \rangle = \left\langle 0 \left| T \left( e^{i \int_0^t H_I(t')dt'} \right) \right| \Theta_I(t \right) \left| T \left( e^{-i \int_0^t H_I(t')dt'} \right) \right| 0 \right\rangle, \tag{34}
\]

where \( \Theta(t) \) is a field operator, \( | \Omega \rangle \) represents the vacuum of the interaction theory, \( T \) and \( \bar{T} \) are time-ordering and anti-time-ordering operators. All the fields are in the interaction picture, as the subscript \( I \) indicates.

Our next step will then be to write explicitly the action for the gauge bosons and look at the detailed expression of their interaction Hamiltonian. Using the definition of \( F_{\mu \nu}^a \), the action for the gauge fields becomes

\[
S_{A_\mu} = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} g^{\mu \alpha} g^{\nu \beta} \left( \partial_\mu B_\alpha^a \partial_\beta B_\beta^a - \partial_\mu B_\nu \partial_\beta B_\alpha^a \right) \right.
\]

\[
- g_c \delta_{abc} g^{\mu \alpha} g^{\nu \beta} \left( \partial_\mu B_\alpha^a \right) B_\beta^b B_\beta^c - \frac{1}{4} g_c \delta_{eab} \partial_\mu g^{\mu \gamma} \partial_\nu g^{\nu \beta} B_\gamma^b B_\alpha^c B_\beta^a B_\alpha^c B_\beta^a - \frac{M^2}{2} g^{\mu \nu} B_\mu B_\nu - \left. \right], \tag{35}
\]

The interaction Hamiltonian to third order in the gauge field perturbations is thus made up of two contributions coming respectively from the third and fourth order interactions in the lagrangian

\[
H_{int} = g_c \delta_{abc} \left( \partial_\mu B_\nu^a \right) \delta B_\mu^b \delta B_\nu^c + g_c \delta_{eab} \partial_\mu \delta B_\mu^a \delta B_\nu^b \delta B_\nu^c + g_c \delta_{eab} \delta B_\mu^a \delta B_\mu^b \delta B_\nu^c. \tag{36}
\]

The correction to the bispectrum (10) due to these interactions has the form

\[
B_3(\vec{k}_1, \vec{k}_2, \vec{k}_3) \supset N_{A_i}^a N_{A_j}^b N_{A_k}^c \int d\eta \left[ \epsilon_{abc} \Pi_{ab}^m (\vec{k}_1) \Pi_{bc}^l (\vec{k}_2) \Pi_{bc}^{m\prime} (\vec{k}_3) \left[ g_c \delta_{abc} k_{1I} + g_c \delta_{eab} \epsilon_{abc} B_1^d \right] \right. \]

\[
+ \text{perms.} + \text{c.c.} \tag{37}
\]

where

\[
\Pi_{ij}^{ab} (\vec{k}) \equiv T_{ij}^{\text{even}} (\vec{k}) \tilde{P}_{ij}^{ab} + i T_{ij}^{\text{odd}} (\vec{k}) \tilde{P}_{ij}^{ab} + T_{ij}^{\text{long}} (\vec{k}) \tilde{P}_{ij}^{ab} \tag{38}
\]
and $\tilde{P}_\pm \equiv (1/2)(\tilde{P}_R \pm \tilde{P}_L)$, $\tilde{P}_R$ being defined as the product of the eigenfunctions $\delta B_R(k, \eta^*)$ and $\delta B_R^*(k, \eta)$, and so on for $\tilde{P}_L$ and $\tilde{P}_{long}$. Notice that $\eta$ is the time at which the three point function for $\zeta$ is being evaluated. Also, we dropped the gauge indices in $\tilde{P}$, since the perturbations have the same time and momentum dependence for the different gauge fields. This last point can be discussed looking at Eq. (A.15) for the perturbations of $\delta B$: the equations for the free fields (which are the ones we need to compute the power spectrum and the bispectrum) are obtained from (A.15) setting $g_c = 0$, i.e. suppressing the interaction terms, and thus we get the same equation for all the components of the gauge multiplet.

4.1. Bispectrum from third-order interactions

Let us begin with the bispectrum contribution from Eq.(37) at the lowest order in the coupling $g_c$

$$
\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle_{\eta^*} \supseteq N_{a_1} N_{b_1} N_{c_1} (\delta A^a_{i_1} (k_1) \delta A^b_{j_1} (k_2) \delta A^c_{k_1} (k_3)) =
N_{a_1} N_{b_1} N_{c_1} \frac{\delta(3)(k_1 + k_2 + k_3)}{a^3(\eta^*)} \varepsilon^{abc} \delta_{a a_1} \delta_{b b_1} \delta_{c c_1}
\times \int_{-\infty}^{\eta^*} d\eta' a^4(\eta') \tilde{\Pi}_{i il} \tilde{\Pi}_{j jm} \tilde{\Pi}_{k kn} g_{sm} g_{ln} i k_s^{(1)} g_c + \text{perms. + c.c } \tag{39}
$$

where $g^{ij}$ is the unperturbed background metric. The factor $ik_s^{(1)}$ originates from the spatial derivative in the interaction term considered, whereas $a^4$ come from $\sqrt{-g}$ together with having moved from cosmic to conformal time $dt = ad\eta$. In order to write Eq.(39), we used the third order interaction Hamiltonian

$$
H^{(3)}_{int} = g_c \varepsilon^{abc} \left[ g^{ik} g^{jl} \left( \partial_i \delta B^a_j \right) \delta B^b_k \delta B^c_l + g^{ij} \left( \partial_i \delta B^a_0 \right) \delta B^b_j \delta B^c_0 + g^{ij} \left( \partial_0 \delta B^a_i \right) \delta B^b_0 \delta B^c_j \right]. \tag{40}
$$

The calculation of the tree-level contributions due to the last two vertices in Eq.(40) will not be carried out since they contain the time components $\delta B^a_0$, which turn out to be affected by instabilities. In fact, $\delta B_0$ is a non-dynamical mode that, for the free-field case, can be expressed in terms of the longitudinal component as follows [31]

$$
\delta B_0 = -i \frac{\partial_i (\vec{k} \cdot \delta \vec{B})}{k^2 + (aM)^2}. \tag{41}
$$

The longitudinal modes of the theory have negative kinetic energy and are therefore considered unstable. In addition to this, their equations of motion suffer from singularities around the horizon crossing time. However, a regular solution was provided in [36], which we are going to employ when handling these modes (Eq. (58)). The factor $1/a^3(\eta^*)$ outside the integral accounts for the fact that the action is written in terms of the comoving fields $B^a_i$ whereas the bispectrum of $\zeta$ is written in terms of
the bispectrum of the physical fields $A_i^a$.
In our model, $\hat{P}_- = 0$ (i.e. $\hat{P}_R = \hat{P}_l$), therefore the integrand of Eq. (39) can be written as the sum of products of $\hat{P}_+ = \hat{P}_R$ and $\hat{P}_\text{long}$ weighted by products of coefficients $T^{\text{even}}$ and $T^{\text{long}}$

$$T^{\text{EEE}}_{ijk} = k_1 \left[ \delta_{ik}k_{j1} - k_{i1}k_{j2}k_{k1} - k_{i3}k_{j2}k_{k3} - \delta_{ik}(k_1 \cdot k_2)(k_1 \cdot k_3)(k_1 \cdot k_3)(k_1 \cdot k_2) \right]$$

$$T^{\text{EEl}}_{ijk} = k_1 \left[ k_{i3}k_{j1}k_{k3} - k_{i3}k_{j2}k_{k3}(k_1 \cdot k_2) - k_{i1}k_{j2}k_{k3}(k_1 \cdot k_2) \right]$$

$$T^{\text{EEl}}_{ijk} = k_1 \left[ k_{i1}k_{j2}k_{k3}(k_1 \cdot k_3) - k_{i1}k_{j2}k_{k3}(k_1 \cdot k_2) \right]$$

$$T^{\text{EIl}}_{ijk} = k_1 \left[ k_{i3}k_{j2}k_{k3}(k_1 \cdot k_3) - k_{i3}k_{j2}k_{k3}(k_1 \cdot k_2) \right]$$

$$T^{\text{ElE}}_{ijk} = k_1 \left[ k_{i1}k_{j3}k_{k3}(k_1 \cdot k_2) - k_{i1}k_{j3}k_{k3}(k_1 \cdot k_3) \right]$$

$$T^{\text{ElL}}_{ijk} = k_1 \left[ k_{i1}k_{j3}k_{k3}(k_1 \cdot k_2) - k_{i1}k_{j3}k_{k3}(k_1 \cdot k_3) \right]$$

where where we used the notation $T^{\text{EEE}}_{ijk} = T^{\text{EEl}}_{il}T^{\text{EEl}}_{jm}T^{\text{EEl}}_{kn}\delta^{sm}\delta^{tn}k_s$ and so on.

The next step before carrying out the final integration over time in Eq. (39) is to sum over all the permutations separately for each one of the eight $T^{\alpha\beta\gamma}_{ijk}$ coefficients. Because of the antisymmetric properties of $\varepsilon^{abc}$, the sums provide results of the form $\varepsilon^{abc}S_{ab}$, where $S_{ab}$ is a symmetric tensor. We thus find out that, independently of the results of time integrals (i.e. independently of the particular form of the wavefunctions for transverse and longitudinal modes), the contribution to the bispectrum from third-order interactions is zero.

Let us then move to considering the quartic interaction.

### 4.2. Bispectrum from fourth-order interactions

The contribution from the quartic interaction is

$$\langle \zeta_{i1} \zeta_{i2} \zeta_{i3} \rangle \sim (2\pi)^3 \left( \sum_{i=1,2,3} \vec{k}_i g_c^2 \frac{1}{H} \left( \frac{H x^s}{k} \right)^3 N^i_a N^i_b N^i_c A_i^a A_i^b A_i^c \varepsilon^{abc} \varepsilon^{abc} \right)$$

$$\times \sum_{\alpha,\beta,\gamma} \left( \int dx \right) T_{\alpha\beta}^{\omega} T_{\gamma}^{\rho} T_{\delta}^{\tau} + \text{perms.} + \text{c.c.}$$

$$= (2\pi)^3 \delta^3 \left( \sum_{i=1,2,3} \vec{k}_i g_c^2 \frac{1}{H} \left( \frac{H x^s}{k} \right)^3 \sum_{\alpha,\beta,\gamma} K_{\alpha\beta\gamma} + \text{perms.} \right)$$

(50)
with \( x = -k\eta \). \((Hx^*)^3\) comes from the \( 1/a^3(\eta^*) \) factor outside the time integral and the \( 1/H \) factor comes from having a factor of \( a \) inside the time integral, i.e. \( \int d\eta a(\eta) = -\int dx/(Hx) \). This factor originates from the background field appearing inside the integral (the interaction term is quartic and can be developed as a product of three perturbations times a background field), i.e. \( B_i^a(\eta) = a(\eta)A_i^a(\eta) \), where \( A \) is the slowly-rolling physical field (and hence taken out of the integral). Notice that, once again, we are setting \( A_0^a = 0 \), so terms proportional to \( N_0^a \) do not appear in the formulas.

The indices \( \alpha, \beta, \gamma \) as usual run over the longitudinal and the transverse components. \( \int dx^{\alpha\beta\gamma} \) is a shorthand for the integral over the wavefunctions associated with the internal legs times the wavefunctions associated with the external ones. The definition of the symbol \( K_{\alpha\beta\gamma} \) can be read from the first two lines in Eq.(50). It is important to sum over all of the permutations, i.e. over all the possible field contractions, which we are going to list below

\[
\begin{align*}
c_1^{\alpha\beta\gamma} &= T_{ipjl}^{\alpha\beta\gamma} \delta_{a'd'} \delta_{c'd'} \\
c_2^{\alpha\beta\gamma} &= T_{ipjl}^{\alpha\beta\gamma} \delta_{a'd'} \delta_{b'd'} \\
c_3^{\alpha\beta\gamma} &= T_{iljl}^{\alpha\beta\gamma} \delta_{a'd'} \delta_{c'd'} \\
c_4^{\alpha\beta\gamma} &= T_{iljl}^{\alpha\beta\gamma} \delta_{a'd'} \delta_{b'd'} \\
c_5^{\alpha\beta\gamma} &= T_{iljl}^{\alpha\beta\gamma} \delta_{b'd'} \delta_{c'd'} \\
c_6^{\alpha\beta\gamma} &= T_{iljl}^{\alpha\beta\gamma} \delta_{c'd'} \delta_{b'd'} \end{align*}
\]

(51) \( \ldots \) (56)

Let us write the expression of the wavefunctions of the gauge fields. We are going to work in the regime where \( M^2 = -2H^2 \), i.e. we take \( m_0 \ll H \) and \( \xi = 1/6 \). This is the situation where the transverse mode of the \( \delta \vec{B} \) field is governed by an equation of motion similar to the one of a light scalar field in unperturbed FRW background (see Eq.(A.16)). The solution is very well known and, after matching with the vacuum solution at early times, is given by

\[
\delta B^\parallel = -\frac{\sqrt{\pi x}}{2\sqrt{k}} \left[ J_{3/2}(x) + iJ_{-3/2}(x) \right].
\]

(57)

where \( x \equiv -k\eta \). We have dropped the gauge index since the equation of motion is unique for all of the gauge bosons (see Eq.(A.16) and (A.17) and discussion below Eq. (38)). As to the longitudinal mode, there is an on-going discussion about its instability [36, 37, 39, 40]; according to some authors [37, 39], there are two sources of instability: the first one is due to the fact that the equation of motion (A.17) is singular for a given mode \( k \) equal to the effective mass of the vector field; the second one arises from the field behaving like a 'ghost', i.e. having a negative energy. In spite of the divergence affecting the linearized equation of motion, the authors of [36] provided two independent exact solutions to (A.17). The initial conditions were formulated after rescaling the vector field with the introduction of \( \delta \vec{B} \equiv (a|M|/k)\delta B^\parallel \), which allows for the well-known initial conditions \( \delta \vec{B} = e^{-ik\eta}/\sqrt{2k} \) at very early times. We are going to use the solution
and the initial condition prescription of [36], so the longitudinal mode function acquires the following form

\[
\delta B^i = \frac{1}{2\sqrt{k}} \left( x - \frac{2}{x} + 2i \right) e^{ix}.
\]  

(58)

The terms \( K_{\alpha\beta\gamma} \) below (permutations are included) then read

\[
K_{EEE} = -\frac{I_{EEE}}{24k^6 k_1^2 k_2^2 k_3^2 x^5} \left[ A_{EEE} + (B_{EEE} \cos x^* + C_{EEE} \sin x^*) E_i x^* \right]
\]  

(59)

\[
K_{III} = -\frac{I_{III}}{192k^9 k_1^3 k_2^3 k_3^3 x^5} \left[ A_{III} + (B_{III} \cos x^* + C_{III} \sin x^*) E_i x^* \right]
\]  

(60)

\[
K_{IIE} = \frac{I_{IIE}}{96k^8 k_1^3 k_2^3 k_3^3 x^5} \left[ A_{IIE} + (B_{IIE} \cos x^* + C_{IIE} \sin x^*) E_i x^* \right]
\]  

(61)

\[
K_{EEI} = -\frac{I_{EEI}}{48k^7 k_1^3 k_2^3 k_3^3 x^3} \left[ A_{EEI} + (B_{EEI} \cos x^* + C_{EEI} \sin x^*) E_i x^* \right]
\]  

(62)

where \( A, B, C \) and \( D \) are functions (to be provided in Appendix C) of \( x^* \) and of the momenta \( k_i \equiv |\vec{k}_i|, \ i = 1, 2, 3 \) and \( E_i \) is the exponential-integral function.

The factors \( I_{\alpha\beta\gamma} \) are given by

\[
I_{EEE} \equiv \varepsilon^{aa'b'} \varepsilon^{ac'e} \left[ 6 \left( \vec{N}^{a'} \cdot \vec{N}^{c'} \right) \left( \vec{N}^{b'} \cdot \vec{A} \right) + \left( \vec{N}^{b'} \cdot \vec{A} \right) \left( -2 \left( \hat{k}_3 \cdot \vec{N}^{a'} \right) \left( \hat{k}_3 \cdot \vec{N}^{c'} \right) - 2 \left( \hat{k}_1 \cdot \vec{N}^{a'} \right) \left( \hat{k}_1 \cdot \vec{N}^{c'} \right) \right) \right] \left( \hat{k}_1 \cdot \hat{k}_3 \right) + (1 \leftrightarrow 2) + (3 \leftrightarrow 2)
\]  

(63)

\[
I_{III} \equiv \varepsilon^{aa'b'} \varepsilon^{ac'e} \left[ \left( \hat{k}_1 \cdot \vec{N}^{a'} \right) \left( \hat{k}_3 \cdot \vec{N}^{b'} \right) \left( \hat{k}_2 \cdot \vec{N}^{c'} \right) \left( \hat{k}_1 \cdot \hat{k}_3 \right) \left( \hat{k}_3 \cdot \vec{A} \right) - \left( \hat{k}_3 \cdot \vec{N}^{a'} \right) \left( \hat{k}_2 \cdot \vec{N}^{b'} \right) \left( \hat{k}_1 \cdot \vec{N}^{c'} \right) \left( \hat{k}_3 \cdot \hat{k}_2 \right) \left( \hat{k}_3 \cdot \vec{A} \right) \right] + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)
\]  

(64)

\[
I_{IIE} \equiv \varepsilon^{aa'b'} \varepsilon^{ac'e} \left[ \left( \vec{N}^{b'} \cdot \vec{A} \right) \left( \hat{k}_1 \cdot \vec{N}^{a'} \right) \left( \hat{k}_2 \cdot \vec{N}^{c'} \right) \left( \hat{k}_1 \cdot \hat{k}_2 \right) \left( \hat{k}_3 \cdot \vec{N}^{c'} \right) \right] \hat{k}_1 \cdot \hat{k}_2
\]  

+ \left[ \left( \vec{N}^{b'} \cdot \vec{A} \right) \left( \hat{k}_1 \cdot \vec{N}^{a'} \right) \left( \hat{k}_2 \cdot \vec{N}^{c'} \right) \left( \hat{k}_1 \cdot \hat{k}_2 \right) \left( \hat{k}_3 \cdot \vec{N}^{c'} \right) \right] + (1 \leftrightarrow 2)
\]  

- \left[ \left( \vec{N}^{b'} \cdot \vec{A} \right) \left( \hat{k}_2 \cdot \vec{N}^{a'} \right) \left( \hat{k}_1 \cdot \vec{N}^{c'} \right) \right] \hat{k}_2 \cdot \hat{k}_1
\]  

+ (1 \leftrightarrow 3) + (2 \leftrightarrow 3)
\]  

(65)

\[
I_{EEI} \equiv \varepsilon^{aa'b'} \varepsilon^{ac'e} \left[ 4 \left( \vec{N}^{b'} \cdot \vec{A} \right) \left( \vec{N}^{c'} \cdot \vec{A} \right) \left( \vec{N}^{a'} \cdot \vec{N}^{c'} \right) \left( \vec{N}^{b'} \cdot \vec{N}^{a'} \right) \left( \hat{k}_2 \cdot \vec{N}^{c'} \right) \right]
\]  

+ \left[ \left( \vec{N}^{b'} \cdot \vec{A} \right) \left( \hat{k}_2 \cdot \vec{N}^{a'} \right) \left( \vec{N}^{c'} \cdot \vec{A} \right) \left( \vec{N}^{a'} \cdot \vec{N}^{c'} \right) \right] \hat{k}_1 \cdot \hat{k}_3
\]  

+ (2 \leftrightarrow 1) + (2 \leftrightarrow 3)
\]  

- \left[ \left( \vec{N}^{b'} \cdot \vec{A} \right) \left( \vec{N}^{a'} \cdot \vec{N}^{c'} \right) \left( \vec{N}^{b'} \cdot \vec{N}^{a'} \right) \left( \vec{N}^{c'} \cdot \vec{N}^{a'} \right) \right] + (1 \leftrightarrow 2) + (2 \leftrightarrow 3) + (1 \leftrightarrow 3)
\]
\[ \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \geq (2\pi)^3 \frac{1}{H} \sum_{i=1,2,3} k_i^2 g_i^4 \frac{1}{k} \left( \frac{H}{k} \right)^3 \epsilon^{aa'b'} \epsilon^{ac'e} \left( \vec{N}^a \cdot \vec{N}^c \right) \left( \vec{N}^b' \cdot \vec{A}^e \right) \times \frac{K_{EEE}}{I_{EEE}} + \text{perms.} \] \] 

The way this result is achieved can be easily seen from the expression of the \( T^a T^b T^c \)
products, from which $K_{\alpha\beta\gamma}$ were evaluated. We list these products for one of the permutations

$$T^E_{il}T^{E*}_{jp}T^E_{kp} = \delta_{il}\delta_{jp} + \delta_{il}(-k_{j3}k_{k3} - k_{j2}k_{k2} + k_{j2}k_{k3}k_{23}) - k_{il}(\delta_{jk}k_{i1} - k_{133} - k_{122} + k_{123}k_{23})$$

$$T^E_{il}T^{E*}_{jp}T^l_{kp} = \delta_{il}(k_{j3}k_{k3} - k_{j2}k_{k2}k_{23}) - k_{il}(k_{133} - k_{123}k_{23})$$

$$T^l_{il}T^{E*}_{jp}T^E_{kp} = \delta_{il}(k_{j2}k_{k2} - k_{j2}k_{k3}k_{23}) - k_{il}(k_{122} - k_{123}k_{23})$$

$$T^E_{il}T^{l*}_{jp}T^E_{kp} = k_{il}(\delta_{jk}k_{i1} - k_{133} - k_{122} + k_{123}k_{23})$$

$$T^l_{il}T^{l*}_{jp}T^E_{kp} = k_{il}(k_{122} - k_{123}k_{23})$$

$$T^l_{il}T^{E*}_{jp}T^l_{kp} = k_{il}(k_{133} - k_{123}k_{23})$$

$$T^l_{il}T^{l*}_{jp}T^l_{kp} = k_{il}k_{123}k_{23}$$

where $k_{abc} \equiv k_{ia}k_{jb}k_{kc}$ and $k_{ab} \equiv \vec{k}_a \cdot \vec{k}_b$ ($a, b, c = 1, 2, 3$ running over the external momenta). Notice that summing over the eight terms in the equations above only leaves $\delta_{il}\delta_{jp}$.

5. Shape of the bispectrum

As we saw in Eq.(50), the bispectrum in the presence of an SU(2) gauge multiplet can be written in terms of isotropic parts, i.e. functions of $x^*$ and of the moduli of the external wave vectors $k_i$, weighted by anisotropic coefficients $I_{\alpha\beta\gamma}$ which depend on the angles between the (wave and gauge) vectors. One way of studying the profile of the bispectrum is to analyse the isotropic parts separately in order to understand what is the preferred configuration, for example if it resembles the local or the equilateral shapes [52]. Once the answer to that is found, we calculate the anisotropic coefficients in that specific configuration. Let us employ the variables $x_2 \equiv k_2/k_1$ and $x_3 \equiv k_3/k_1$ as in [52]. Plotting the isotropic part, one can notice that for most of the functions $K_{\alpha\beta\gamma}/I_{\alpha\beta\gamma}$, which sum up to provide the bispectrum in (50), the regions with the highest values are around $x_2 = 1$, $x_3 = 0$ (see plots in Fig.1). The graphs that don’t have their maxima in this region, i.e. $r_{llE}$ and $r_{lll}$, which peak around $x_2 = x_3 = 1$, and $r_{llE}$, which peaks around $x_2 = x_3 = 0.5$, show negligible amplitudes w.r.t. the previous graphs. Therefore we can safely state that the bispectrum peaks in the ‘local’ configuration $k_2 \sim k_1$, $k_3 \ll k_1$. In fact this result can be understood recalling that the contribution to the bispectrum comes from the quartic interaction terms in Eq. (36), which are local in space, and do not involve gradient terms of the fields.

We can now calculate the anisotropic coefficients in the local configuration. We have the freedom to pick up a coordinate system $(\hat{x}, \hat{y}, \hat{z})$ with $\hat{k}_1$ and $\hat{k}_2$ pointing along $\hat{z}$ and $\hat{k}_3$ along $\hat{x}$. In the general case of three differently oriented vectors $\vec{N}_a$, the coefficients $I_{\alpha\beta\gamma}$ depend on a number of angular variables that is bigger than two and, therefore, they are not easy to plot. What we can do is looking at some particular cases with
Figure 1. Plot of $r_{\alpha\beta\gamma} \equiv \Theta(x_2 - x_3)\Theta(x_3 - 1 + x_2)x_2^2x_3^2R_{\alpha\beta\gamma}(x_2, x_3)$, where we define $R_{\alpha\beta\gamma} = k_{\gamma}^0(K_{\alpha\beta\gamma}/I_{\alpha\beta\gamma})$. The Heaviside step functions $\Theta$ help restricting the plot domain to the region $(x_2, x_3)$ that is allowed for the triangle $\vec{k_1} + \vec{k_2} + \vec{k_3} = 0$ (in particular, we set $x_3 < x_2$). We also set $x^* = 1$. 
specific orientations of the $\vec{N}_a$ w.r.t one another and to the coordinate frame we built from the wave-vectors.

Let us consider the simplest situation with the three vectors $\vec{N}_a$ aligned along a given direction of space with spherical coordinates $(\theta, \phi)$. It is easy to show that in this situation all of the coefficients $I_{\alpha\beta\gamma}$ are equal to zero, so the bispectrum contribution from (50) becomes zero as well. Notice that this does not apply either to the power spectrum or to the Abelian part of the bispectrum, which retain a non-zero anisotropic part (see Eqs.(15) and (30)) when the vectors $\vec{N}_a$ are parallel to each other.

There are several other situations that can be taken into account, for example one of the $\vec{N}_a$ being aligned along or perpendicular to one of the $\hat{k}_i$ and the other two generically oriented, two of the $\vec{N}_a$ aligned along $\hat{k}_1$ and $\hat{k}_3$ respectively and the third one generically oriented and so on. Our next step is to provide a plot of the bispectrum complete of its isotropic and anisotropic parts in the favoured configuration (in our case the local one). For convenience while plotting, let us normalize the vector $\vec{N}_a$ so that they all have the same length and let us restrict our attention to situations where the total number of angles the $I_{\alpha\beta\gamma}$ coefficients depend on does not exceed two. One possibility is for example given by the following case

\begin{align*}
\vec{N}_3 &= N_A(0, 0, 1) \\
\vec{N}_1 &= \vec{N}_2 = N_A(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta),
\end{align*}

where, as mentioned above, we are working in the coordinate frame with $\hat{k}_3 = \hat{x}$ and $\hat{k}_1 = \hat{k}_2 = \hat{z}$ and we took $N_A \equiv |\vec{N}_a|$ for all $a = 1, 2, 3$. Let us introduce the angle $\delta$ between $\vec{N}_{1,2}$ and $\hat{k}_3$. Then from Eq.(50) we have

\begin{equation}
B_\zeta \supset g_c^2 H^2 \sum_{\alpha\beta\gamma} I_{\alpha\beta\gamma}(\theta, \delta) R_{\alpha\beta\gamma}(x^*, x_2, x_3).
\end{equation}

The functions $R_{\alpha\beta\gamma}(x_2, x_3)$ are plotted in Fig.1 for $x^* = 1$ and their analytic expressions are provided in Appendix C. See Appendix D for the expressions of the coefficients $I_{\alpha\beta\gamma}(\theta, \delta)$.

A plot of the ‘non-Abelian’ bispectrum normalized to the ratio $(g_c^2 H^2 m^2 N_A^4)/(k_1^0 k_2^0 k_3^0)$ is given in Fig.2 for fixed values of $x^*, x_2$ and $x_3$ (see Appendix D for its analytic expression).

6. Calculation of the non-Gaussianity parameter $f_{NL}$

The non-Gaussianity of a given theory of inflation and cosmological perturbations can be studied by looking at the expression of the non-linearity $f_{NL}$. This parameter is defined by the ratio of the bispectrum to the square of the isotropic part (as for example given in Eq.(18)) of the power spectrum.
Figure 2. Plot of \( f(\theta, \delta) \equiv [(B_\zeta(\theta, \delta, x^*, x_2, x_3)x_2^2x_3^2k_1^6)/(g_c^2H^2m^2N_A^4)](x^* = 1, x_2 = 0.9, x_3 = 0.1) \). As we can see, the graph dependence from \( \theta \) is extremely weak; in fact, the dominant contribution from Eq.(D.9) can be written as \( f(\theta, \delta) = -140 \cos^2 \delta \).

\[
\frac{6}{5} f_{NL} = \frac{B_\zeta(k_1, k_2, k_3)}{P_{\zeta}^{iso}(k_1)P_{\zeta}^{iso}(k_2) + \text{perms.}}.
\]

As we saw in the previous sections, the complete expression for the bispectrum is made up of an ‘Abelian’ and a ‘non-Abelian’ parts, which can be read respectively from Eq.(21) and (50). We are going to rewrite their sum below in a compact expression

\[
B_\zeta(k_1, k_2, k_3) = B_\zeta^A(k_1, k_2, k_3) + B_\zeta^{NA}(k_1, k_2, k_3).
\]

Our next step will be to calculate the contributions from the two terms on the right hand side of the previous equation to the \( f_{NL} \) parameter and establish a comparison between them. The 'Abelian' part of the bispectrum was given in Eq.(21) as the contribution of three different pieces

\[
B_\zeta^A(k_1, k_2, k_3) = B_\zeta^{A1} + B_\zeta^{A2} + B_\zeta^{A3},
\]

where

\[
B_\zeta^{A1} \approx (N_{\phi})^2 N_{\phi\phi} P^2,
\]

\[
B_\zeta^{A2} \approx (N_A)^2 N_{AA} P^2,
\]

\[
B_\zeta^{A3} \approx N_{\phi} N_A N_{\phi A} P^2,
\]

where the symbol \( \approx \) stresses the fact that we are interested in the overall order of magnitude of the quantities above in terms of the parameters of the theory, without worrying about the details of numerical factors of order one. Also, we now indicate the power spectrum of the scalar field perturbation by \( P \). At late times this is proportional to the power spectrum of the gauge fields. This can be seen by looking at the transverse mode function for \( \delta B \) in Eq.(57) and at the longitudinal mode in Eq.(58), which, after
taking the limit $x \to 0$, give $P_+ = P$ and $P_{long} = 2P$. We shortened $N_a^i$ by $N_A$, $N_{ab}^{ij}$ by $N_{AA}$ and $N_{φa}^i$ by $N_{φA}$ in order to make the expressions simpler, suppressing the gauge and the vector indices. Notice that in the theory we have chosen, no interactions occur between the scalar and the gauge fields, therefore we can set $N_{φA} \simeq 0$.

Similarly, we rewrite the 'non-Abelian' part of the bispectrum in a simplified fashion that leaves out numerical factors and complicated functions of the external momenta

$$B_ζ^{NA}(\vec{k}_1, \vec{k}_2, \vec{k}_3) \simeq g_ζ^2 H^2 A (N_A)^3,$$

(79)

where we now call $A$ the background value of the spatial part of the gauge fields (suppressing all indices).

Let us now rewrite the isotropic part of the power spectrum

$$P_ζ^{iso} \simeq (N_φ)^2 P (1 + β),$$

(80)

where $β \equiv (N_A/N_φ)^2$.

Before proceeding to the evaluation of $f_{NL}$, we need to make more assumptions about the scenario we are dealing with, since the derivatives of $N$ are model-dependent. Two possible choices are the vector curvaton model (introduced for the first time in [31], see also [34] for one of the possible realizations of this paradigm) and vector inflation [33]. The former is the analogue of the (scalar) curvaton mechanism: a vector field plays the role of the curvaton, i.e. a field whose energy density is subdominant during inflation and, after the inflaton has decayed, it is is mainly responsible for the production of the curvature perturbation. In vector inflation, the universe exponential expansion is driven by vector fields, which need to be spatially oriented in such a way as to avoid large scale anisotropies; one possible field configuration is a triplet of orthogonal vector fields, another one consists in a large number of randomly oriented vector fields. In our model we are dealing with a triplet of vector fields, so if we assumed the fields were orthogonally oriented w.r.t one another, vector inflation would be a possible scenario to work in. A third possibility can be to consider a scalar field driven inflation in the presence of the $SU(2)$ gauge multiplet and to assume that the latter is undergoing slow-roll and its energy density is subdominant w.r.t. the energy density of the scalar inflaton.

This way we keep the vector fields in the game, without imposing any restriction on their orientation in space and, at the same time, we avoid unwanted anisotropies in the power spectrum. We are going to consider both this last possibility, which we dare refer to as 'vector inflation' for simplicity, and the curvaton scenario. In 'vector inflation' $N_a^i$ is given by Eq.(B.12), from which we get

$$N_a^i = \frac{A_a^i}{2m^2_P},$$

(81)

$$N_{ab}^{ij} = \frac{δ_{ab}δ^{ij}}{2m^2_P}.$$

(82)

In the curvaton scenario [17]
\[ N^i_a = \frac{2}{3} r \frac{A^i_a}{\sum_b |A|^2}, \quad \text{(83)} \]
\[ N^{ij}_{ab} = \frac{1}{3} r \frac{\delta_{ab} \delta^{ij}}{\sum_c |A|^2}, \quad \text{(84)} \]

where \( r \equiv (3 \rho_A)/(3 \rho_A + 4 \rho_\phi) \), \( \rho_A \) and \( \rho_\phi \) being respectively the energy density of the vector fields and of the inflaton at the time where the vector field(s) decay.

We are now ready to evaluate \( f_{NL} \) as the sum of three contributions corresponding to \( B^{\text{tree}}_{\zeta, A_1} \), \( B^{\text{tree}}_{\zeta, A_2} \) and \( B^{\text{tree}}_{\zeta, NA} \)

\[ f_{NL} \approx f^{A_1}_{NL} + f^{A_2}_{NL} + f^{NA}_{NL}, \quad \text{(85)} \]

where

\[ f^{A_1}_{NL} = \frac{1}{(1 + \beta)^2} \frac{N_{\phi\phi}}{N_{\phi}^2}, \quad \text{(86)} \]
\[ f^{A_2}_{NL} = \frac{\beta}{(1 + \beta)^2} \frac{N_{AA}}{(N_{\phi})^2}, \quad \text{(87)} \]
\[ f^{NA}_{NL} = \frac{\beta^2}{(1 + \beta)^2} g^2_c (m_H)^2. \quad \text{(88)} \]

Numerical coefficients of order one were not reported in the previous equations. \( m \) is a quantity with the dimensions of a mass and proportional to the Planck mass in the case of vector inflation and to \( A_{tot}/\sqrt{r} \) (where we define \( A_{tot} = \sqrt{\sum_b |A|^2} \) from Eq.(84)) in the vector curvaton model.

We summarize our results for the two models in Table 1, where we used the expressions \( N_{\phi} \approx \left( m_P \sqrt{\epsilon_\phi}\right)^{-1} \) and \( N_{\phi\phi} \approx m_P^{-2} \), with \( \epsilon_\phi \equiv \left( \dot{\phi}^2 \right)/(2m_P^2H^2) \). The background values \( \phi \) and \( H \) are all meant at time \( x = x^* \).

| Scenario       | \( f^{A_1}_{NL} \)                                      | \( f^{A_2}_{NL} \)                                      | \( f^{NA}_{NL} \)                                      |
|----------------|--------------------------------------------------------|--------------------------------------------------------|--------------------------------------------------------|
| v.inflation    | \( \frac{\epsilon_\phi}{(1+\left(\frac{A}{m_P}\sqrt{\epsilon_\phi}\right)^2)^2} \) | \( \frac{\epsilon^2_\phi}{(1+\left(\frac{A}{m_P}\sqrt{\epsilon_\phi}\right)^2)^2} \left(\frac{A}{m_P}\right)^2 \) | \( \frac{\epsilon^2_\phi g^2_c}{(1+\left(\frac{A}{m_P}\sqrt{\epsilon_\phi}\right)^2)^2} \left(\frac{A^2}{m_P H}\right)^2 \) |
| v.curvaton     | \( \frac{\epsilon_\phi}{(1+\left(\frac{A m_P}{A_{tot}}\right)^2 \epsilon_\phi r^2)^2} \) | \( \frac{\epsilon^2_\phi}{(1+\left(\frac{A m_P}{A_{tot}}\right)^2 \epsilon_\phi r^2)^2} \left(\frac{A m_P}{A_{tot}}\right)^2 \) | \( \frac{\epsilon^2_\phi r^2 g^2_c}{(1+\left(\frac{A m_P}{A_{tot}}\right)^2 \epsilon_\phi r^2)^2} \left(\frac{A^2 m_P^2}{A_{tot} H}\right)^2 \) |

The ratios between the non-Abelian and the Abelian parts of the \( f_{NL} \) parameter are

\[ \frac{f^{NA}_{NL}}{f^{A_1}_{NL}} \approx \epsilon_\phi g^2_c \left(\frac{A}{H}\right)^2 \left(\frac{A}{m_P}\right)^2, \quad \frac{f^{NA}_{NL}}{f^{A_2}_{NL}} \approx \epsilon_\phi r^2 \left(\frac{A}{H}\right)^2 g^2_c \left(\frac{A}{A_{tot}}\right)^2, \quad \text{(89)} \]
in vector inflation and

$$f_{NL}^{NA} \simeq \frac{F_1^2}{e_\phi} \left( \frac{A}{H} \right)^2, \quad f_{NL}^{NA} \simeq \frac{F_2^2}{e^2_{\phi}} \left( \frac{A}{m_P} \right)^2, \quad (90)$$

for the vector curvaton.

When estimating the order of magnitude of $f_{NL}$ as calculated above, it is important to remember that our results in Eqs. (86) through (90) and in Table 1 have been derived neglecting all vector and gauge indices. Also, these expressions involve the background value of the slowly-rolling fields $A_i^a$ in a multfield space and so $f_{NL}$ depends on the whole specific background configuration. In this respect, different possibilities can be considered and it turns out that, depending on the chosen configuration, the non-Abelian contributions to $f_{NL}$ can subdominant or dominant w.r.t. the Abelian ones. Notice that the ratios $g_\phi A/H = (g_\phi A/m_0)(m_0/H)$ for the different gauge fields are bounded from above, as shown in Eqs. (A.9) through (A.11). There is a very large region of the parameter space of the background gauge field configurations for which this upper bound is much larger than one and, therefore, it does not prevent $f_{NL}^{NA}$ from dominating over $f_{NL}^{A_1}$.

Here is an example: suppose that two of the gauge fields, for instance $\vec{B}^1$ and $\vec{B}^2$ are aligned, about equal in magnitude ($|\vec{B}^1| \simeq |\vec{B}^2|$) and that $|\vec{B}^{1,2}| \gg |\vec{B}^3|$. In this case, by looking at Eqs. (A.9) and (A.10), it is easy to realize that the upper bounds can be much larger than one. Therefore, the ratios in Eqs. (89) and (90) can be much bigger than one depending, for example, on how $A/H$ compares to the other quantities appearing in these expressions (notice: the theory puts no constraints on $A/H$ which can in principle be a very large number). Obviously, inspecting Eqs. (A.9) through (A.11), if the $A^a$’s are not approximately aligned and their magnitude $A$ is roughly the same, then the upper bound $g_\phi \ll H/A$ holds.

As to the absolute value of $f_{NL}^{NA}$, expanding around $\beta \ll 1$, from Eq.(88) we have

$$f_{NL}^{NA} \simeq \beta^2 g_\phi^2 \left( \frac{m_P}{H} \right)^2, \quad f_{NL}^{NA} \simeq \frac{\beta g_\phi^2}{r^2} \left( \frac{A}{m_P} \right)^2, \quad (91)$$

respectively for vector inflation and the vector curvaton model. The ratio $m_P/H$ is of order $10^9$ and $A/H$ could be much bigger than one as well, so in principle $f_{NL}^{NA}$ can be much larger than one in both models. Equivalently, from Table 1, $f_{NL}^{NA}$ can be put in the following form (as a function of the slow-roll parameter $\epsilon_\phi$)

$$f_{NL}^{NA} \simeq \begin{cases} \frac{g_\phi^2}{r^2} \left( \frac{A}{m_P} \right)^4 \left( \frac{H}{m_P} \right)^2 & \text{vector inflation} \\ \frac{g_\phi^2}{r^2} \left( \frac{m_P}{A} \right)^2 \left( \frac{m_P}{H} \right)^2 & \text{vector curvaton} \end{cases}$$

7. Overview and conclusions

We have considered a triplet of $SU(2)$ vector bosons non-minimally coupled to gravity, Eq. (4), in order to have $M^2 \simeq -2H^2$. Using the $\delta N$ and the Schwinger-Keldysh formalisms we have computed the contributions to the curvature perturbation three-point
correlation function arising from the gauge fields self-interactions. These interactions are of two kinds, third-order interactions (proportional to one power of the $SU(2)$ coupling $g_c$) and fourth-order ones (proportional to $g_c^2$). The former provide a vanishing contribution to the bispectrum because of the antisymmetric properties of the Levi-Civita tensor appearing in the Lagrangian, the result being independent of the particular form of the wave functions appearing in the gauge bosons operator expansions. The quartic interactions produce instead a non-zero contribution, Eq. (50). This result is anisotropic if the wavefunctions of the transverse and longitudinal modes of the vector fields are different from each other, which seems to be the case by looking at their equations of motion, Eqs. (A.16) and (A.17).

The ongoing debate about the instability of the longitudinal mode sheds some doubts about the true physical meaning of the mode. Our analysis, in this respect, is pretty general, being mainly focused on studying the non-Gaussianity effects in non-Abelian theories. In addition to that, we tried to also use a more general approach when dealing with longitudinal modes, with a parametrization of the wavefunctions in terms of the transverse mode and of an unknown function $n(x)$, Eq. (68), which, to a first approximation, can be set equal to a constant near horizon crossing, $n(x) = n(x^*)$. A particular case of this parametrization occurs when the longitudinal and transverse mode functions coincide, which offers an interesting result, i.e. the isotropization of the full bispectrum, both the ‘non-Abelian’ and the ‘Abelian’ parts, Eq. (69) and (21), respectively. We are aware that the particular Lagrangian we used represents only one possible model of primordial vector fields. Other theories have been proposed, which do not present any kind of instability (see for example, [55, 36, 35, 56]).

In the case of an anisotropic bispectrum we showed that it can be written in terms of isotropic parts, i.e. functions of the moduli of the external wave vectors $k_i$, modulated by anisotropic coefficients $I_{\alpha\beta\gamma}$ which depend on the angles between the (wave and gauge) vectors. We studied the shape of the ‘isotropic parts’ of the bispectrum, Eq. (50) and Eq. (59) through (62), which turned out to peak in the local momenta configuration (see plots in Fig. 1). Using this finding, we analysed the ‘anisotropic’ part, Eq. (63) through (66) for different spacial configurations of the gauge vectors. A limiting case is represented by the three components of the $SU(2)$ gauge group being all aligned with one another: because of the presence of the Levi-Civita tensors and with simple symmetry consideration, the contribution from (50) to the bispectrum in this case is proven to be zero. This result could be interpreted as the Abelian limit of our theory, since the three components of the $SU(2)$ multiplet are all identifiable with a unique spatial direction.

Another example of vector fields configuration was provided in Eqs. (70)-(71). For this particular case, a complete plot of the bispectrum was given in Fig. 2 for the local configuration. It contains the information about the anisotropy of the bispectrum, showing how its amplitude is modulated according to the angles between the wavevectors $\vec{k}_i$ and
the preferred directions induced by the vector fields.

Finally we have calculated the parameter $f_{NL}$, estimating the level of non-Gaussianity. We analyzed the different contributions to $f_{NL}$, i.e. the ‘Abelian’, Eqs. (86) and (87), and the ‘non-Abelian’ ones, Eq.(88). We compared them both for vector inflation and for the vector curvaton models and for different background configurations. It turns out that one contribution can be dominant w.r.t. the other or viceversa, depending on the given background configuration, Eqs. (89) and (90). Focusing on the order of magnitude of the ‘non-Abelian’ contribution, we noticed that it can be much bigger than one, for a large region of parameter space, Eq.(91).

In a forthcoming paper [54], we will study the trispectrum of the curvature perturbation in the same vector fields populated model, calculate its magnitude and shapes and and investigate the relationship between the $f_{NL}$ and $\tau_{NL}$ parameters of the theory.

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Appendix A. Background and first order perturbation equations for the
gauge fields

The equations of motion for the gauge fields have been completely derived for the $U(1)$
case in [31]. We are going to carry out a similar calculation for the $SU(2)$ case

$$\frac{1}{\sqrt{-g}} \partial_\mu \left[ \sqrt{-g} g^{\mu \nu} \partial_\nu \left( F_{\alpha \beta}^{(AB)a} + g_c \varepsilon^{abc} B_0^a B_0^b \right) \right] + M^2 g^{\mu \nu} B_0^a$$

$$+ g_c \varepsilon^{abc} g^{\gamma \nu} \partial_\nu \left( F_{\gamma \delta}^{(AB)b} B_0^c + g_c \varepsilon^{abc} \varepsilon^{b' c'} g^{\nu \alpha} \partial_\alpha B_0^b B_0^b \right) B_0^c = 0$$

(A.1)

where $F_{\mu \nu}^{(AB)a} \equiv \partial_\mu B_0^a - \partial_\nu B_0^a$.

The $\nu = 0$ component of the equations of motion is

$$\partial_j \dot{B}_j^a - \partial_j \partial_0 B_0^a + a^2 M^2 B_0^a + g_c \varepsilon^{abc} \left[ - \left( \partial_j B_j^b \right) B_0^c - 2 B_j^b \partial_j B_0^c - \dot{B}_j^b B_j^c 
\right.

\left. + g_c \varepsilon^{b' c'} B_j^b B_0^b B_0^{c'} \right] = 0$$

(A.2)

where $B_0^a = B_0^a(x,t)$.

Let us now move to the spatial ($\nu = i$) part of (A.1)

$$\ddot{B}_i^a + H \dot{B}_i^a - \frac{1}{a^2} \partial_j \partial_j B_i^a + M^2 B_i^a - \partial_i \dot{B}_0^a - H \partial_i B_0^a + \frac{1}{a^2} \partial_j \partial_j B_j^a$$

$$+ g_c \varepsilon^{abc} \left[ H B_0^a B_i^c + \dot{B}_0^a B_i^c + B_0^b \dot{B}_i^b \right] - g_c \frac{\varepsilon^{abc}}{a^2} \left[ \left( \partial_i B_j^b \right) B_0^c + B_j^b \partial_i B_i^c \right]$$

$$+ g_c \varepsilon^{abc} \left[ \left( \partial_i B_j^b \right) B_0^c - \dot{B}_j^b B_i^c \right] - g_c \frac{\varepsilon^{abc}}{a^2} \left[ \left( \partial_i B_j^b \right) B_j^c - \left( \partial_i B_j^b \right) B_j^c \right]$$

$$+ g_c \varepsilon^{abc} \varepsilon^{b' c'} \left[ B_0^a B_i^b B_i^c - \frac{g_c^2}{a^2} \varepsilon^{abc} \varepsilon^{b' c'} \left[ B_j^b B_0^b B_i^c \right] \right] = 0$$

(A.3)

If we contract Eq.(A.1) with $\partial_\nu$, we get the integrability condition

$$(aM)^2 \dot{B}_0^a - M^2 \partial_i B_i^a + 3H \left( \partial_0 \partial_0 B_0^a - \partial_i \dot{B}_i^a \right) + g_c \varepsilon^{abc} \left[ 2H \left( \partial_i B_i^b \partial_i B_i^c + B_i^b \partial_i B_i^c + \dot{B}_0^b \dot{B}_0^c \right) \right.$$  

$$- \partial_0 \partial_i B_i^c B_0^c + \partial_0 \partial_0 B_0^a + \partial_0 \dot{B}_0^a + \frac{1}{a^2} \left( B_i^b \partial_i B_i^c + \partial_i B_i^b \partial_j B_i^c + B_j^b \partial_i B_i^c + \partial_0 B_0^a \right)$$

$$- \partial_0 \partial_i B_i^b B_0^c + \partial_i \dot{B}_0^a + \partial_0 \partial_0 B_0^a + \partial_0 \dot{B}_0^a + \frac{1}{a^2} \left( B_i^b \partial_i B_i^c + \partial_i B_i^b \partial_j B_i^c + B_j^b \partial_i B_i^c + \partial_0 B_0^a \right)$$

$$- \partial_0 \partial_i B_i^b B_0^c + \partial_i \dot{B}_0^a + \partial_0 \partial_0 B_0^a + \partial_0 \dot{B}_0^a + \frac{1}{a^2} \left( B_i^b \partial_i B_i^c + \partial_i B_i^b \partial_j B_i^c + B_j^b \partial_i B_i^c + \partial_0 B_0^a \right)$$

(A.4)

which reduces to Eq.(7) of [31] in the Abelian case.

Combining Eq.(A.4) with Eq.(A.2) we get

$$(aM)^2 \dot{B}_0^a - M^2 \partial_i B_i^a + 3H \left( a^2 M^2 B_0^a + g_c \varepsilon^{abc} \left[ - \left( \partial_j B_j^b \right) B_0^c - 2 B_j^b \partial_j B_0^c - \dot{B}_j^b B_j^c + g_c \varepsilon^{b' c'} \dot{B}_j^b B_0^b B_0^{c'} \right] \right)$$

$$+ g_c \varepsilon^{abc} \left[ 2H \left( \partial_i B_i^b \partial_i B_i^c + B_i^b \partial_i B_i^c + \dot{B}_0^b \dot{B}_0^c \right) \right.$$  

$$- \partial_0 \partial_i B_i^c B_0^c + \partial_0 \dot{B}_0^a + \partial_0 \partial_0 B_0^a + \partial_0 \dot{B}_0^a + \frac{1}{a^2} \left( B_i^b \partial_i B_i^c + \partial_i B_i^b \partial_j B_i^c + B_j^b \partial_i B_i^c + \partial_0 B_0^a \right)$$

$$- \partial_0 \partial_i B_i^b B_0^c + \partial_i \dot{B}_0^a + \partial_0 \partial_0 B_0^a + \partial_0 \dot{B}_0^a + \frac{1}{a^2} \left( B_i^b \partial_i B_i^c + \partial_i B_i^b \partial_j B_i^c + B_j^b \partial_i B_i^c + \partial_0 B_0^a \right)$$

$$- \partial_0 \partial_i B_i^b B_0^c + \partial_i \dot{B}_0^a + \partial_0 \partial_0 B_0^a + \partial_0 \dot{B}_0^a + \frac{1}{a^2} \left( B_i^b \partial_i B_i^c + \partial_i B_i^b \partial_j B_i^c + B_j^b \partial_i B_i^c + \partial_0 B_0^a \right)$$

$$+ \frac{1}{a^2} \left( B_i^b \partial_i B_i^c + \partial_i B_i^b \partial_j B_i^c + B_j^b \partial_i B_i^c + \partial_0 B_0^a \right) \right] = 0$$
Plugging this into Eq.(A.3) we get

\[
\dot{B}_n + H \dot{B}_n - \frac{1}{a^2} \partial_j \partial_j B^a_n + M^2 B_n^a + 2H \partial_n B^a_0 \\
- \frac{1}{(aM)^2} \partial_n \left[ - 3H \left( g_{c}^{\epsilon abc} \left[ - \left( \partial_j B^b_j \right) B^c_0 - 2B^b_j \partial_j B^c_0 - \dot{B}^b_j B^c_0 + g_{c}^{\epsilon c b b'} B^b_j B^b_j \right] \right) \\
+ g_{c}^{\epsilon abc} \left[ 2H \left( \partial_j B^b_j B^c_0 + B^b_j \partial_j B^c_0 + \dot{B}^b_j B^c_j \right) - \partial_j B^b_j \dot{B}^c_j \\
- \dot{B}^b_j B^c_j + \partial_j \dot{B}^b_j B^c_0 - \partial_j B^b_j \dot{B}^c_0 - \partial_j \dot{B}^b_j B^c_j + \frac{1}{a^2} \left( B^b_j \partial_j B^c_0 + \partial_j B^b_j \partial_j B^c_0 + B^b_j \partial_j \partial_j B^c_0 + \partial^2 B^b_j B^c_j \\
- \partial_j \partial_j B^b_j B^c_j \right) \right] + g_{c}^{\epsilon abc} \left[ a^2 \left( \dot{B}^b_j B^b_j \dot{B}^c_0 + B^b_j \dot{B}^b_j \dot{B}^c_0 + B^b_0 \dot{B}^b_0 \dot{B}^c_0 \right) + 2H B^b_0 \dot{B}^b_0 \dot{B}^c_0 \\
- B^b_j \dot{B}^b_j B^c_j - B^b_j \dot{B}^b_0 B^c_0 - B^b_j \dot{B}^b_0 B^c_0 - \partial_j B^b_0 B^b_j B^c_0 - B^b_0 \partial_j B^b_0 B^c_0 - B^b_0 \partial_j B^b_0 B^c_0 \\
+ \frac{1}{a^2} \left( \partial_j B^b_j B^b_j B^c_0 + B^b_j \partial_j B^c_0 + B^b_0 \partial_j B^c_0 \dot{B}^c_0 \right) \right] \right] \\
+ g_{c}^{\epsilon abc} \left[ H B^b_0 B^c_n + \dot{B}^b_0 B^c_n + B^b_0 \dot{B}^c_n - \frac{g_{c}^{\epsilon abc}}{a^2} \left( \left( \partial_j B^b_j \right) B^c_0 + B^b_j \partial_j B^c_0 \right) \right] \\
+ g_{c}^{\epsilon abc} \left[ \left( \partial_j B^b_j \right) B^c_0 - \dot{B}^b_j B^c_0 \right] - g_{c}^{\epsilon abc} \left( \partial_j B^b_j \right) B^c_n - \frac{g_{c}^{\epsilon abc}}{a^2} \left( \partial_j B^b_j \right) B^c_n \\
+ g_{c}^{\epsilon abc} \epsilon^{b b' c'} \left[ B^b_0 B^b_0 \dot{B}^c_0 \right] - \frac{g_{c}^{\epsilon abc}}{a^2} \epsilon^{b b' c'} \left[ B^b_0 B^b_0 \dot{B}^c_0 \right] = 0. \tag{A.6}
\]

Let us consider the background part of the vector fields, i.e. \( \partial_i B^a_i = 0 \). Then from Eq.(A.2)

\[
a^2 M^2 B^a_0 + g_{c}^{\epsilon abc} \left[ - \dot{B}^b_j B^c_0 + g_{c}^{\epsilon c b b'} B^b_j B^b_j \dot{B}^c_0 \right] = 0. \tag{A.7}
\]

Before proceeding with the derivation of the equations of motion for the background and the field perturbations, it is necessary to make some comments on Eqs. (A.6) and (A.7). One approximation that we have been using in this paper is allowing the fields to undergo slow-roll during inflation. One possible way to achieve this is by restricting the parameter space of the background gauge fields through the request that their temporal components should be much smaller than the spatial ones, \( B^b_0 \ll |B^b_0|/a(t) \), \( b = 1, 2, 3 \), and, in addition to that, assuming \( B^b_0 \simeq B^b_0 \), \( b, c = 1, 2, 3 \). With these assumptions, the temporal component can be factored out in Eq. (A.7), using the approximation \( \dot{B}^b_0 \simeq H B^b_0 \) (valid in a slow-roll regime). A solution to (A.7) is then given by \( B_0 = 0 \). Adopting this solution and plugging it in Eq (A.6), it is easy to show that a slow-roll equation of motion for the physical fields

\[
\ddot{A}^a_i + 3H \dot{A}^a_i + m^2 A^a_i = 0 \tag{A.8}
\]

follows from (A.6) if \( \dot{H} \ll m^2 \) and
Finally from Eq.(A.6) we get
\[
\begin{align*}
\left(\frac{g_c A^1}{m_0}\right)^2 & \ll \left|\frac{(A^1)^2}{(A^2)^2 + (A^3)^2 - (A^3)^2 \cos^2 \theta_{13} - (A^2)^2 \cos^2 \theta_{12}}\right|, \\
\left(\frac{g_c A^2}{m_0}\right)^2 & \ll \left|\frac{(A^2)^2}{(A^1)^2 + (A^3)^2 - (A^3)^2 \cos^2 \theta_{23} - (A^1)^2 \cos^2 \theta_{12}}\right|, \\
\left(\frac{g_c A^3}{m_0}\right)^2 & \ll \left|\frac{(A^3)^2}{(A^1)^2 + (A^2)^2 - (A^2)^2 \cos^2 \theta_{23} - (A^1)^2 \cos^2 \theta_{13}}\right|,
\end{align*}
\]  

are satisfied. In the equations above, we defined \( A^a \equiv |\vec{A}^a| \) and \( \cos \theta_{ab} \equiv \hat{A}^a \cdot \hat{A}^b, a \) and \( b \) running over the gauge indices. The quantities appearing on the right-hand sides of Eqs.(A.9) through (A.11) can be either large or small w.r.t. one, depending on the specific background configuration, i.e. on the moduli of the gauge fields and the angles \( \theta_{ab} \).

Suppose now the conditions described above are all met, then from Eq (A.6), in terms of the comoving fields, we have
\[
\ddot{B}_i^a + H \dot{B}_i^a + M^2 B_i^a = 0. \tag{A.12}
\]

Let us now derive the equations for the perturbations. Eq.(A.2) becomes
\[
\begin{align*}
\partial_j \dot{B}_j^a - \partial^2 \delta B_0^a + a^2 M^2 \delta B_0^a + g_c \varepsilon^{abc} \left[ - \partial_j \delta B_j^b B_0^c - 2 B_j^b \partial_j \delta B_0^c - \delta \dot{B}_j^b B_j^c - \dot{B}_j^b \delta B_j^c \\
+ g_c \varepsilon^{b'c'} \left( \delta B_j^b B_0^{b'} B_j^{c'} + B_j^b \delta B_0^{b'} B_j^{c'} + B_j^b B_0^{b'} \delta B_j^{c'} \right) \right] = 0 \tag{A.13}
\end{align*}
\]

Eq.(A.1) for the field perturbations gives
\[
\begin{align*}
\delta \ddot{B}_i^a + H \dot{\delta B}_i^a - \frac{1}{a^2} \partial_j \partial_j \delta B_i^a + M^2 \delta B_i^a + \frac{1}{a^2} \partial_i \partial_j \delta B_j^a - H \partial_i \delta B_0^a - H \partial_0 \delta B_i^a \\
- \frac{g_c}{a^2} \varepsilon^{abc} \left[ (\partial_j \delta B_j^b) B_i^c + B_j^b \partial_j \delta B_i^c \right] - \frac{g_c}{a^2} \varepsilon^{abc} \left[ (\partial_j \delta B_j^b) B_i^c - (\partial_j \delta B_j^c) B_i^b \right] \\
- \frac{g_c}{a^2} \varepsilon^{abc} \varepsilon^{b'c'} \left[ \delta B_j^b B_j^{b'} B_i^{c'} + B_j^b \delta B_j^{b'} B_i^{c'} + B_j^b B_j^{b'} \delta B_j^{c'} \right] \\
+ g_c \varepsilon^{abc} \left[ H \left( B_0^b \delta B_i^c + \delta B_0^b B_i^c \right) + \dot{B}_j^b \delta B_j^c + \dot{B}_j^b B_j^c + \delta B_0^b \dot{B}_j^c + B_0^b \dot{B}_j^c \right] \\
+ g_c \varepsilon^{abc} \left[ \partial_0 \delta B_0^b \delta B_i^c - \dot{B}_0^b \delta B_i^c - \dot{B}_0^b \delta B_i^c \right] \\
+ g_c^2 \varepsilon^{abc} \varepsilon^{b'c'} \left[ \delta B_j^b B_j^{b'} B_i^{c'} + B_j^b \delta B_j^{b'} B_i^{c'} + B_j^b B_j^{b'} \delta B_j^{c'} \right] = 0 \tag{A.14}
\end{align*}
\]

Finally from Eq.(A.6) we get
\[
\begin{align*}
\delta \ddot{B}_i^a + H \dot{\delta B}_i^a - \frac{1}{a^2} \partial^2 \delta B_i^a + M^2 \delta B_i^a + 2H \partial_i \delta B_0^a + (\sim g_c \text{ terms}) = 0. \tag{A.15}
\end{align*}
\]

When calculating \( n \) – point functions for the gauge bosons, the eigenfunctions we need are provided by free-field solutions, i.e. by solutions of Eq.(A.15) with \( g_c \) being set to zero. This is exactly the Abelian limit, in fact in this case Eq.(A.15) corresponds to (18) of [31] and can be decomposed into a transverse and a longitudinal part.
\[
\begin{align*}
\partial_0^2 + H \partial_0 + M^2 + \left(\frac{k}{a}\right)^2 \delta \vec{B}^T = 0 \\
\partial_0^2 + \left(1 + 2 \frac{k^2}{k^2 + (aM)^2}\right) H \partial_0 + M^2 + \left(\frac{k}{a}\right)^2 \delta \vec{B}^|| = 0
\end{align*}
\tag{A.16, A.17}
\]

where the time derivatives are intended w.r.t. cosmic time.

**Appendix B. Calculation of the number of e-foldings of single-(scalar)field driven inflation in the presence of a vector multiplet**

Let us consider the complete Lagrangian of our theory as in Eq.(4). Let us assume that the $SU(2)$ gauge multiplet undergoes slow-roll as well as the scalar field but the latter provides the dominant part of the energy density of the universe. This last hypothesis is necessary in order to produce isotropic inflation (i.e. in order for the anisotropy in the expansion that the vector fields introduce to be negligible w.r.t. the isotropic contribution from the scalar field). The expression of the number of e-foldings is

\[
N = N_{\text{scalar}} + N_{\text{vector}} = N_{\text{scalar}} + \frac{1}{4m_P^2} \sum_{a=1,2,3} \vec{A}^a \cdot \vec{A}^a.
\tag{B.1}
\]

The previous expression can be easily derived from the equations of motion of the system neglecting terms that are proportional to the $SU(2)$ coupling constant $g_c$ and assuming slow-roll conditions for both the scalar the gauge fields.

The starting point is represented by Einstein equations

\[
H^2 = \frac{8\pi G}{3} \left( \rho_{\text{scalar}} + \rho_{\text{vector}} \right).
\tag{B.2}
\]

where we split the energy density into a scalar and a vector contribution. In slow-roll approximation, $\rho_{\text{scalar}} \sim V(\phi)$. Let us calculate $\rho_{\text{vector}}$. The energy momentum tensor for the gauge bosons

\[
T_{\mu\nu} = \frac{\delta L}{\delta g^{\mu\nu}} - g_{\mu\nu} L
\tag{B.3}
\]

where, as a remainder, $L = -(1/4)g^{\mu\alpha}g^{\nu\beta}F^a_{\mu\nu}F^a_{\alpha\beta} + (M^2/2)g^{\mu\nu}B^a_\mu B^a_\nu$. So we get

\[
T_{00} = \frac{\dot{B}_i^a \dot{B}_i^a}{2a^2} + \frac{m_0^2}{2a^2}B_i^a B_i^a + \frac{m_0^2}{2}B_0^a B_0^a - \frac{H}{a^2} \dot{B}_i^a B_i^a + \frac{H^2}{2a^2} B_i^a B_i^a
\]

\[
+ \frac{g_c^2}{2a^2} \varepsilon^{abc} \dot{B}_i^a B_0^b B_i^c + \frac{g_c^2}{4a^4} \varepsilon^{abc} \varepsilon^{ab'c'} B_0^a B_i^b B_0^b B_i^c
\]

\[
+ \frac{g_c^2}{4a^4} \varepsilon^{abc} \varepsilon^{ab'c'} B_i^a B_j^b B_i^b B_j^c
\tag{B.4}
where sums are taken over all repeated indices. Let us write this in terms of the physical fields

$$T_{00}^\text{vector} = \frac{\dot{A}_a^a A_i^a}{2} + \frac{m_0^2}{2} (A_a^a A_i^a + A_i^a A_0^a) + g_c \varepsilon^{abc} \left( H A_a^c + \dot{A}_a^c \right) A_0^b A_i^c + \frac{g_c^2}{2} \varepsilon^{abc} \varepsilon^{ab'} c' A^b_0 A_i^c A_i^{a'} + \frac{g_c^2}{4} \varepsilon^{abc} \varepsilon^{ab'} c' A_i^b A_j^c A_i^{a'} A_j^{a'} \tag{B.5}$$

If we neglect the non-Abelian contribution and we set $A_a^0 = 0$, we are left with the Abelian result [33]

$$T_{00}^\text{vector} = \frac{\dot{A}_a^a A_i^a}{2} + \frac{m_0^2}{2} A_i^a A_i^a \tag{B.6}$$

The equation of motion for the background vector multiplet $\vec{A}^a$ can be derived from Eq.(A.12)

$$\ddot{A}_i^a + 3 H \dot{A}_i^a + m_0^2 A_i^a = 0. \tag{B.7}$$

which is equal to the equation of a light scalar field of mass $m_0$, if $m_0 \ll H$. If the conditions for accelerated expansions are met, Eq.(B.7) reduces to

$$3 H \dot{A}_i^a + m_0^2 A_i^a \sim 0. \tag{B.8}$$

We are now ready to derive Eq.(B.1). Let us start from the definition of $N$ and keep in mind Eq.(B.2), where we are assuming the existence of a scalar fields $\phi$ in de-Sitter with a separable potential governed by the usual (background) equation

$$\ddot{\phi} + 3 H \dot{\phi} + V' = 0 \tag{B.9}$$

and slowly rolling down their potential. Then we have

$$N = \int_t^t H dt' = \int_t^t H^2 dt' = 8\pi G \int_t^t \frac{V(\phi)}{3H} dt' + 8\pi G \int_t^t \frac{V(A)}{3H} dt'$$

$$= 8\pi G \int_t^t \frac{V(\phi)}{3H} d\phi + 8\pi G \sum_a \int_t^{\phi(t)} \left( \frac{m_0^2}{2} \right) \frac{A_i^a A_i^a}{3H} dA^a dt' \tag{B.10}$$

where $A^a \equiv \vec{A}^a \cdot \vec{A}^a$. So

$$N = 8\pi G \int_{\phi(t')} \frac{V(\phi)}{3H} d\phi + 8\pi G \sum_a \int_{A^a(t')} \left( \frac{m_0^2}{4} \right) \frac{A_i^a A_i^a}{3H} A_j^a A_j^a dA^a$$

$$= -\frac{1}{m_P^2} \int_{\phi(t')} V d\phi + \frac{1}{m_P^2} \sum_a \int_{A^a(t')} \left( \frac{m_0^2}{4} \right) \frac{A_i^a A_i^a}{(-m_0^2)} A_j^a A_j^a dA^a$$

$$= -\frac{1}{m_P^2} \int_{\phi(t')} V d\phi - \left( \frac{1}{4m_P^2} \right) \sum_a \int_{A^a(t')} dA^a \tag{B.11}$$
after using the slow-roll conditions. Eq.(B.1) is thus recovered.
In the final expression for the bispectrum then we can substitute

\[
N_i^i \equiv \frac{dN}{dA_i^i} = \left( \frac{1}{2m_P^3} \right) A_i^a
\]

where the derivatives are as usual calculated at the initial time \( \eta^* \). The upper limits in integrals such as the ones in Eq.(B.11) depend on the chosen path in field space and so they also depend on the initial field configuration. It is important to notice though, as also stated in [53], that if the final time is chosen to be approaching (or later than) the end of inflation, the fields are supposed to have reached their equilibrium values and so \( N \) becomes independent of the field values at the final time \( t \). Eq.(B.12) is thus recovered.

**Appendix C. Complete expressions for the functions appearing in the bispectrum from quartic interactions**

We list below the complete expressions for the functions \( A_{\alpha\beta\gamma} \), \( B_{\alpha\beta\gamma} \) and \( C_{\alpha\beta\gamma} \) (\( \alpha, \beta, \gamma = R, l \)) appearing in Eq.(59) through (62)

\[
A_{EEE} \equiv k x^{*2} \left( - k^2 (k_1^3 + k_2^3 + k_3^3 - 4k_1k_2k_3) - k^3 (k_2k_3 + k_1k_2 + k_1k_3) \right) \\
+ k_1 k_2 k_3 (k_1^2 + k_2^2 + k_3^2 - 2k_2k_3 - k_1k_2 - k_1k_3) x^{*2} \tag{C.1}
\]

\[
B_{EEE} \equiv \left( k_1^3 + k_2^3 + k_3^3 \right) x^{*3} \left( - k^3 + k_1 k_2 k_3 x^{*2} \right) \tag{C.2}
\]

\[
C_{EEE} \equiv - k \left( k_1^3 + k_2^3 + k_3^3 \right) x^{*2} \left( - k^2 + (k_2k_3 + k_1k_2 + k_1k_3) x^{*2} \right) \tag{C.3}
\]

\[
A_{III} \equiv 16k^6 \left[ 2k^3 (k_1 + k_2)(k_1 + k_3)(k_2 + k_3) - 3 \left( k_1^4 (k_2 + k_3)^2 + k_2^2 k_3^2 (k_2 + k_3)^2 \right) \right. \\
+ k_1^2 \left( 2k_2^3 + 9k_2^2 k_3 + 9k_2k_3^2 + 2k_3^3 \right) + k_1k_2k_3 (2k_2^3 + 9k_2^2 k_3 + 9k_2k_3^2 + 2k_3^3) \\
+ k_2^2 \left( k_2^4 + 9k_2^2 k_3^2 + 19k_2^2 k_3^2 + 9k_2k_3^3 + k_3^4 \right) \left] x^{*2} - 8k^5 \left[ -3k_1^2 k_2^2 k_3^2 \right. \\
+ 2k^4 (k_1 + k_2)(k_1 + k_3)(k_2 + k_3) + 2k^2 k_1 k_2 k_3 (k_2k_3 + k_1k_2 + k_1k_3) \\
- 2k^3 (k_2k_3 + k_1k_2 + k_1k_3)^2 + 6k_1k_2k_3 (k_1^3 (k_2 + k_3) + k_2k_3 (k_2 + k_3)^2 \\
+ 2k_1^2 (k_2^2 + 3k_2k_3 + k_3^2) + k_1 (k_2^3 + 6k_2^2 k_3 + 6k_2k_3^2 + k_3^3)) - 3k \left( k_2^4 (k_2 + k_3)^2 \right) \\
+ k_2 k_3 (k_2 + k_3)^2 + k_1^2 \left( 2k_2^2 + 9k_2^2 k_3 + 9k_2k_3^2 + 2k_3^3 \right) + k_1 k_2 k_3 \\
x \left( 2k_2^3 + 9k_2^2 k_3 + 9k_2k_3^2 + 2k_3^3 \right) + k_1^2 \left( k_2^4 + 9k_2^2 k_3^2 + 19k_2^2 k_3^2 + 9k_2k_3^3 + k_3^4 \right) \left] \right) x^{*4} \\
+ 4k^2 \left[ -3k_2^4 k_3^2 (k_2 + k_3)^2 - 3k_1^6 (k_2 + k_3)^4 - 3k_1^5 (k_2 + k_3)^2 \right. \\
x \left( 2k_2^2 + 11k_2 k_3 + 2k_3^3 + k_2 k_3 (-2k + 11k_3) \right) \\
- 3k_1^4 (k_2^2 + 15k_2^2 k_3 + 8k_2k_3^2 (-4k + 15k_3) + 2k_3^2 k_3^2 (-8k + 47k_3) \\
+ k_2^2 k_3 (-4k + 61k_3) + k_2^2 k_3 (k_2^2 - 16k_3^3 + 61k_3^3)) - 3k_1^2 k_2 k_3 (k_2 + k_3) \\
x \left( 4k_2^4 + 2k^2 k_2 k_3 + 31k_2^2 k_3 + 63k - 2k_2^2 + 31k_2 k_3^2 + 4k_3^4 \right) \\
\right] \]

\]

}\]
\[-2k(k_2^3 + 7k_2^2k_3 + 7k_2k_3^2 + k_3^3) - k_1k_2^2k_3^2\left(-2k^5 - 6k_2k_3(k_2 + k_3)^2\right) + 3k_2k_3(k_2^3 + 15k_2^2k_3 + 15k_2k_3^2 + 4k_3^3) - k_2^2k_2k_3\left(2k^4k_2k_3 - 2k^5(k_2 + k_3)\right) + 3k_2^2k_2k_3(k_2 + k_3)^2 - 12k_2k_3(k_2^3 + 4k_2^2k_3 + 4k_2k_3^2 + k_3^3) + 3k_2^3k_3(6k_2^4 + 35k_2^3k_3 + 61k_2^2k_3^2 + 35k_2k_3^3 + 6k_3^4)\bigg]\right]x^6 + 6k_2^2k_3^2k_4^2\left[k_1^4(k_2 + k_3)^2 + k_2^2(k_2^2 + k_3)^2 + k_3^4\right]x^8

\[B_{lll} \equiv 8k^4\left(k_1^3 + k_2^3 + k_3^3\right)x^3\left(4k^4(k_1 + k_2 + k_3) - 2k^2(k_1 + k_2)(k_1 + k_3)\right)\times\left(k_2 + k_3\right)x^2 + k_1k_2k_3(k_2k_3 + k_1k_2 + k_1k_3)x^4\right)\] (C.4)

\[C_{lll} \equiv -4k^3\left(k_1^3 + k_2^3 + k_3^3\right)x^2\left(8k^6 - 4k^4(k_1 + k_2 + k_3)^2 x^2 + 2k^2(k_2k_3 + k_1k_2)
+ k_1k_3\right)^2 x^4 - k_1k_2k_3x^6\right)\] (C.5)

\[A_{lll} \equiv 4k^5\left[2k^3(k_1 + k_2)(k_1 + k_2 + k_3) - 2k^2\left(k_1^3 + k_2^3 + 4k_2^2k_3\right)
- 2k_3^3 + 4k_1^2(k_2 + k_3) + 4k_1k_2(k_2 + 2k_3)\right] + 3k_1k_2\left(2k_2k_3(k_2 + k_3)
+ k_2^3(k_2 + 2k_3) + k_1(k_2^2 + 6k_2k_3 + 2k_3^2)\right)\bigg]\right]x^2

\[B_{lll} \equiv 2k^2\left(k_1^3 + k_2^3 - 2k_3^3\right)\left(4k^4k_1 + 4k^4k_2 + 4k^4k_3 + \left[-2k^2(k_1 + k_2)\right)\times\left(k_2k_3 + k_1k_2 + k_1k_3\right)\bigg\right]x^5

\[C_{lll} \equiv 2k^3\left(k_1^3 + k_2^3 - 2k_3^3\right)\left[4k^4x^2 - 2k^2(k_1 + k_2)(k_1 + k_2 + k_3)x^4\right.
+ k_1k_2\left(2k_2k_3 + k_1(k_2 + 2k_3)\right)\bigg\right]x^6\] (C.7)

\[A_{lll} \equiv 2k^4\left[-2k^4 - 2k^2 - 2k_3^2k_3 - k^2k_3^2 + 2k^2k_3^2 + k^2 - 2k_3^2(k_2 + k_3)
+ 3k_2k_3(2k_2 + k_3) + k_2k_3(-2k^2 + kk_3 + 4k^2)\right]
+ 2k_1k_2k_3(k_2^3 + 3k_2^2k_3 + 3k_2k_3^2 + k_3^3) + k^2(2k_2^4 + 9k_2^3k_3 + 9k_2^2k_3^2 + 2k_3^3) + k_1k_2k_3
\times\left(2k_2^3 + 9k_2^2k_3 + 9k_2k_3^2 + 2k_3^3\right) + k_2^2(2k_2^4 + 9k_2^3k_3 + 9k_2^2k_3^2 + 9k_2k_3^3 + k_3^4)
+ k^2(k_2^3 + k_2^2k_3 + k_3^2) - 2k\left(k_2^3k_3^2(k_2 + k_3) + k_1^3(k_2 + k_3)^2\right)
+ 2k_1k_2k_3(k_2^3 + 3k_2^2k_3 + 3k_2k_3^2 + k_3^3) + k_1^2(k_2^3 + 6k_2^2k_3 + 6k_2k_3^2 + k_3^3)\times^2\] (C.8)

\[B_{lll} \equiv 2k^3\left(k_1^3 + k_2^3 - 2k_3^3\right)\left[4k^4x^2 - 2k^2(k_1 + k_2)(k_1 + k_2 + k_3)x^4\right.
+ k_1k_2\left(2k_2k_3 + k_1(k_2 + 2k_3)\right)\bigg\right]x^6\] (C.9)
\[ + k_1 \left( -2k_2^2 + 6k_2^2k_3 + 9k_2k_3^2 + 4k_3^3 - 2k^2(k_2 + k_3) \\
+ k_2k_3(2k_2 + k_3) \right) + k^2k_3 \left[ 2k_2^2k_2k_3 + 2k_4^2(2k_2 + k_3) \\
- 3k_2^2k_3(2k_2 + 2k_2k_3 + k_3^2) + 2k_2k_3^2(2k_2^2 - 3k_2k_3^2 - k_3^3) \\
+ k_1(4k_2^2 + 2k_2k_3 + 2k_2^3k_3 - 6k_2^2k_3^2 - 5k_2k_3^3 - k_3^4) \right] x^2 \\
- 3k_2^2k_3^2x^4 \] (C.10)

\[ B_{EEI} \equiv -k^2(2k_1^2 + 2k_2^2 - k_3^2) \left[ 2k^3x^* + \left[ -2k_1k_2k_3 - k_1k_3^2 - k_2k_3^2 \right] x^3 \right] \] (C.11)

\[ C_{EEI} \equiv k(2k_1^3 + 2k_2^3 - k_3^3) \left[ 2k^4 + \left[ -2k_1^2k_2 - 2k_2^2k_1k_3 - 2k_2^3k_3 - k_2^2k_3^2 \right] x^2 \\
+ k_1k_2k_3^2x^4 \right] \] (C.12)

where \( k \equiv k_1 + k_2 + k_3 \).

Appendix D. Profile of the bispectrum: detailed expressions of the plotted functions

We are going to study the profile of the 'non-Abelian' contribution to the bispectrum in terms of the angles between gauge and wave vectors in the local configuration and for the set up considered in Eq.(70) and (71). The coefficients \( I_{\alpha\beta\gamma} \) then become

\[ I_{EEE} = m^2N_A^4 \left[ -20 - 24 \cos \delta + 2 \cos \theta - 12 \cos^2 \delta + 12 \cos^2 \theta - 2 \cos^2 \theta \cos \delta + 2 \cos \delta \cos^3 \theta \right], \] (D.1)

\[ I_{III} = m^2N_A^4 [4 \cos^2 \delta], \] (D.2)

\[ I_{IEE} = m^2N_A^4 [4 - 2 \cos \theta - 6 \cos^2 \theta - 4 \cos^2 \delta], \] (D.3)

\[ I_{IEI} = m^2N_A^4 [ -4 \cos^2 \delta], \] (D.4)

\[ I_{IEI} = m^2N_A^4 [ -4 \cos^2 \delta], \] (D.5)

\[ I_{EEI} = m^2N_A^4 [ -2 \cos^2 \delta], \] (D.6)

\[ I_{IEI} = m^2N_A^4 [4 - 4 \cos^2 \theta - 8 \cos^2 \delta], \] (D.7)

\[ I_{IEE} = m^2N_A^4 [2 + \cos \theta - 3 \cos^2 \theta - 4 \cos^2 \delta]. \] (D.8)

where \( m^2 \equiv \langle A \rangle /\langle N_A \rangle \), \( A \) being the background value of the \( \tilde{A}_a \)’s evaluated at horizon crossing.

Fig.2 shows a plot of the 'non-Abelian' bispectrum normalized to the ratio \( (g_c^2 H^2 m^2 N_A^4)/(k_1^6 x_2^2 x_3^2) \), as a function of the angles \( \theta \) and \( \delta \) and for fixed values of \( x^* \), \( x_2 \) and \( x_3 \)

\[ B(\theta, \delta) \simeq g_c^2 H^2 m^2 N_A^4 \left[ \cos^2 \delta(8 \cos \theta - 1.4 \times 10^3) + 3 \cos \delta(\cos^3 \theta - \cos^2 \theta - 11) \\
- 11 \cos 2\delta - 40 - 6 \cos 2\theta - \cos \theta(3 \cos^2 \theta - 30 \cos \theta - 10) \right] \] (D.9)
where we set $x^* = 1$, while $x_2$ and $x_3$ were chosen in the 'squeezed' region, $x_2 = 0.9$ and $x_3 = 0.1$.

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