Markov’s inequality on Koornwinder’s domain in $L^p$ norms

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Abstract. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : |x| < y + 1, x^2 > 4y\}$. We prove that the optimal exponent in Markov’s inequality on $\Omega$ in $L^p$ norms is 4.

Keywords: Markov inequality; $L^p$ norms; Markov exponent

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1 Introduction

Throughout this paper $\mathcal{P}(\mathbb{R}^N)$ ($\mathcal{P}_n(\mathbb{R}^N)$, respectively) denotes the set of algebraic polynomials of $N$ variables with real coefficients (with total degree at most $n$). We begin with the definition of multivariate Markov’s inequality.

Definition 1.1 Let $E \subset \mathbb{R}^N$ be a compact set. We say that $E$ admit Markov’s inequality if there exist constants $M, r > 0$ such that for every polynomial $P \in \mathcal{P}(\mathbb{R}^N)$ and $j \in \{1, 2, \ldots, N\}$

$$\left\| \frac{\partial P}{\partial x_j} \right\|_E \leq M(\deg P)^r \|P\|_E$$

where $\| \cdot \|_E$ is the supremum norm on $E$.

A compact set $E$ with this property is called a Markov set. The inequality (1) is a generalization of the classical inequality proved by A. A. Markov in 1889, which gives such estimate on $[-1, 1]$. The theory of Markov inequality and its generalizations is still the active and fruitful area of approximation theory (see, for instance, [5, 7, 17]). For a given compact set $E$, an important problem is to determine the minimal constant $r$ in (1). This can be used to minimize the loss of regularity in problems concerning the linear extension of classes of $C^\infty$ functions with restricted growth of derivatives (see [24, 25]). Such an $r$ is so-called Markov’s exponent of $E$ (see [4] for more detail on this matter). In the case of supremum norm various information about Markov’s exponent is known (see, e.g., [8, 11, 21, 26, 27]). The Markov type inequalities were also studied in $L^p$ norms (see [11, 6, 12, 13, 14, 15]). In this case the question of Markov’s exponent problem is much more complex. In particular, to the best of our knowledge, there is no example of a compact set in $\mathbb{R}^N$ with cusps for which Markov’s exponent (with respect to the Lebesgue measure) is known. The attempts to solve this problem led, among others, to a so-called Milówka–Ozorka identity (see [3, 22, 23] for discussion). The aim of this note is to give such an example. More precisely,
we show that, in the notation above, Markov’s exponent of $\Omega$ in $L^p$ norms is 4. Here $\Omega = \{(x, y) \in \mathbb{R}^2 : |x| < y + 1, x^2 > 4y\}$ which is depicted in the Figure 1. Since $\Omega$ is the region for Koornwinder orthogonal polynomials (first type), see [18, 19], we call this set Koornwinder’s domain.

2 Some weighted polynomial inequalities on simplex

The following lemma will be particularly useful in the proof of our main result.

Lemma 2.1 Let $S = \{(x_1, x_2) \in \mathbb{R}^2 : -1 < x_1 < x_2 < 1\}$, $w(x_1, x_2) = x_2 - x_1$ and $1 \leq p \leq \infty$. Then there exists a positive constant $C(S, w)$ such that, for every $P \in \mathcal{P}_n(\mathbb{R}^2)$, we have

$$\left\| \frac{\partial P}{\partial x_i} \right\|_{L^p(S, w)} \leq Cn^2\|P\|_{L^p(S, w)} \quad (i = 1, 2). \quad (2)$$

Proof. We start with $p = \infty$. Since $\overline{S}$ is a convex body in $\mathbb{R}^2$, the result of Wilhelmsen [30] gives

$$\max \left\{ \left\| w \frac{\partial P}{\partial x_1} - P \right\|_{L^\infty(S)}, \left\| w \frac{\partial P}{\partial x_2} + P \right\|_{L^\infty(S)} \right\} \leq \frac{2(n + 1)^2}{\delta_S} \|wP\|_{L^\infty(S)},$$

where $\delta_S$ is the width of the convex body (the minimal distance between parallel supporting hyperplanes). Therefore by Lemma 3 from [13], there is a constant $\kappa > 0$ such that, for all $P \in \mathcal{P}_n(\mathbb{R}^2)$,

$$\left\| w \frac{\partial P}{\partial x_i} \right\|_{L^\infty(S)} \leq \frac{2(\kappa \delta_S + 1)(n + 1)^2}{\delta_S} \|P\|_{L^\infty(S, w)} \quad (i = 1, 2). \quad (3)$$

Thus we conclude that (2) holds when $p = \infty$. Now, for each $1 \leq p < \infty$, it is clear that

$$\left\| \frac{\partial P}{\partial x_i} \right\|_{L^p(S, w)} \leq \sum_{j=0}^2 \left( \int_{D_j} \left| \frac{\partial P}{\partial x_i}(x_1, x_2) \right|^p (x_2 - x_1) \, dx_1 \, dx_2 \right)^{1/p}$$
where
\[ D_0 = \{(x_1, x_2) \in \mathbb{R}^2: -1 < x_1 < 0, x_1 + 1 < x_2 < 1\}, \]
\[ D_1 = \{(x_1, x_2) \in \mathbb{R}^2: -1 < x_1 < 0, x_1 < x_2 < x_1 + 1\}, \]
\[ D_2 = \{(x_1, x_2) \in \mathbb{R}^2: 0 < x_2 < 1, x_2 - 1 < x_1 < x_2\}. \]

We shall show that there is a constant \( \tilde{C} > 0 \) such that, for all \( P \in \mathcal{P}(\mathbb{R}^2) \),
\[
\left\| \frac{\partial P}{\partial x_i} \right\|_{L^p(D_j, w)} \leq \tilde{C}(\deg P)^2 \|P\|_{L^p(S, w)}, \quad j = 0, 1, 2. \tag{4}
\]

Since \( D_0 \) is a bounded convex set and \( w \) is bounded away from zero on \( D_0 \), we have (see \[9, 12, 13, 20\])
\[
\left( \int_{D_0} \left| \frac{\partial P}{\partial x_i}(x_1, x_2) \right|^p (x_2 - x_1) \, dx_1 dx_2 \right)^{1/p} \leq C_0(\deg P)^2 \|\, P\|_{L^p(D_0, w)} \leq C_0(\deg P)^2 \|\, P\|_{L^p(S, w)}.
\]

Now consider the case \( j = 1 \). The integral is then
\[
\left( \int_{D_1} \left| \frac{\partial P}{\partial x_i}(x_1, x_2) \right|^p (x_2 - x_1) \, dx_1 dx_2 \right)^{1/p}.
\]

We perform the change of variables \( t = x_1, s = x_2 - x_1 \). The integral becomes
\[
\left( \int_{-1}^0 \int_{0}^1 \left| \frac{\partial P}{\partial x_i}(t, s + t) \right|^p s \, ds \, dt \right)^{1/p}.
\]

Define \( Q(t, s) = P(t, s + t) \). Then
\[
\frac{\partial Q}{\partial t}(t, s) - \frac{\partial Q}{\partial s}(t, s) = \frac{\partial P}{\partial x_1}(t, s + t), \quad \frac{\partial Q}{\partial s}(t, s) = \frac{\partial P}{\partial x_2}(t, s + t).
\]

Hence, (using Goetgheluck’s result—see \[10\])
\[
\int_0^1 \left| \frac{\partial P}{\partial x_2}(t, s + t) \right|^p s \, ds \leq C_1^p(\deg Q)^{2p} \int_0^1 |Q(t, s)|^p s \, ds.
\]

Therefore
\[
\left\| \frac{\partial P}{\partial x_2} \right\|_{L^p(D_1, w)} \leq \left( \int_{-1}^0 \left[ C_1^p(\deg P)^{2p} \int_0^1 |Q(t, s)|^p s \, ds \right] \, dt \right)^{1/p}
\]
\[
= C_1(\deg P)^2 \left( \int_{-1}^0 \int_0^1 |P(t, s + t)|^p s \, ds \, dt \right)^{1/p}
\]
\[
\leq C_1(\deg P)^2 \|\, P\|_{L^p(S, w)}.
\]

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On the other hand,
\[
\left\| \frac{\partial P}{\partial x_1} \right\|_{L^p(D_1,w)} \leq \left( \int_{-1}^{0} \int_{0}^{1} \left| \frac{\partial Q}{\partial t}(t,s) \right|^p s \, ds \, dt \right)^{1/p} + \left( \int_{-1}^{0} \int_{0}^{1} \left| \frac{\partial Q}{\partial s}(t,s) \right|^p s \, ds \, dt \right)^{1/p}.
\]

We have, arguing as before, that there exists constants \( \hat{C}_1, C_1 \) such that for every polynomial \( Q \in P(\mathbb{R}^2) \)
\[
\int_{-1}^{0} \left| \frac{\partial Q}{\partial t}(t,s) \right|^p dt \leq \hat{C}_1^p (\deg Q)^{2p} \int_{-1}^{0} \left| Q(t,s) \right|^p dt,
\]
\[
\int_{0}^{1} \left| \frac{\partial Q}{\partial s}(t,s) \right|^p s \, ds \leq C_1^p (\deg Q)^{2p} \int_{0}^{1} \left| Q(t,s) \right|^p s \, ds.
\]

Therefore we see immediately that
\[
\left\| \frac{\partial P}{\partial x_1} \right\|_{L^p(D_1,w)} \leq \left( \int_{0}^{1} \left[ \hat{C}_1^p (\deg P)^{2ps} \int_{-1}^{0} \left| Q(t,s) \right|^p dt \right] ds \right)^{1/p} + \left( \int_{-1}^{0} \left[ C_1^p (\deg P)^{2ps} \int_{0}^{1} \left| Q(t,s) \right|^p s \, ds \right] dt \right)^{1/p}.
\]

Thus we finally have
\[
\left\| \frac{\partial P}{\partial x_1} \right\|_{L^p(D_1,w)} \leq \hat{C}_1 (\deg P)^2 \| P \|_{L^p(D_1,w)} + C_1 (\deg P)^2 \| P \|_{L^p(D_1,w)}
\]
\[
\leq (\hat{C}_1 + C_1)(\deg P)^2 \| P \|_{L^p(S,w)}.
\]

A similar result for \( D_2 \) obtains if one considers the substitution \( t = x_2, s = x_2 - x_1 \) and polynomial \( \hat{Q}(t,s) = P(t-s,t) \). We omit the details. Thus we have shown that, if \( \hat{C} = 2 \max \{ C_0, \hat{C}_1, C_1, \hat{C}_2, C_2 \} \), then (4) holds. That completes the proof. Now we shall prove the following weighted Schur-type inequality.

**Lemma 2.2 (with previous notation).** Let \( d \) be a natural number. Then, for every \( A \subset \bar{S} \) and \( R \in \mathcal{P}_k(\mathbb{R}^2) \), satisfying the condition
\[
\{ \text{there exists } \alpha \in \mathbb{N}^2 \text{ such that } \alpha_1 + \alpha_2 \leq d \text{ and } |R^{(\alpha)}(x)| \geq m > 0 \ (x \in A) \}
\]
one can find a constant \( C_d \) such that, for any \( \epsilon > 0 \) and every \( P \in \mathcal{P}_n(\mathbb{R}^2) \), we have
\[
\| P \|_{L^p(A,w)} \leq C_d m^{-1} \epsilon^{-1} (n + k)^{2d} \| PR \|_{L^p(S,w)} + \epsilon \| P \|_{L^p(S,w)}. \tag{5}
\]

**Proof.** The idea of the proof comes from [13]. Thus we proceed by induction on the length of \( \alpha \). If \( \alpha_1 = \alpha_2 = 0 \), then
\[
|P(x)| \leq m^{-1} |P(x)R(x)| \text{ for } x \in A.
\]
Thus we can share

On the other hand, if

Therefore by the preceding lemma,

if

\[ x \\in \mathbb{R}^n \]

\[ \alpha \]

denotes the length of \( \alpha \). Let

\[ I = \{ (\beta_1, \beta_2) \in \mathbb{N}^2 : 0 < |\beta|, 0 \leq \beta_1 \leq \alpha_1, 0 \leq \beta_2 \leq \alpha_2 \} \.

Notice that the set \( I \) contains at most \( \frac{(d_0+1)(d_0+2)}{2} - 1 \) elements. By Leibniz’s rule, if \( x \in A \), then

\[
|P(x)| \leq m^{-1} \left[ |(PR)^{(\alpha)}(x)| + \sum_{\beta \in I} \left( \frac{\alpha}{\beta} \right) |R^{(\alpha-\beta)}(x)| |P^{(\beta)}(x)| \right].
\]

Let \( C \) be a constant so that (2) holds. We set

\[ B_0 = \{ x \in A : |R^{(\alpha-\beta)}(x)| \leq \frac{m\epsilon}{\eta^2} (Cn^2)^{-|\beta|}, \beta \in I \}, \]

where \( \eta = \frac{(d_0+1)(d_0+2)}{2} \). Then, for each \( x \in B_0 \), we have

\[
|P(x)| \leq m^{-1}|(PR)^{(\alpha)}(x)| + \frac{\epsilon}{\eta^2} \sum_{\beta \in I} (Cn^2)^{-|\beta|} |P^{(\beta)}(x)|.
\]

This yields

\[
\|P\|_{L^p(B_0, w)} \leq m^{-1}\|(PR)^{(\alpha)}\|_{L^p(B_0, w)} + \frac{\epsilon}{\eta^2} \sum_{\beta \in I} (Cn^2)^{-|\beta|} \|P^{(\beta)}\|_{L^p(B_0, w)}
\]

\[
\leq m^{-1}\|(PR)^{(\alpha)}\|_{L^p(S, w)} + \frac{\epsilon}{\eta^2} \sum_{\beta \in I} (Cn^2)^{-|\beta|} \|P^{(\beta)}\|_{L^p(S, w)}.
\]

Therefore by the preceding lemma,

\[
\|P\|_{L^p(B_0, w)} \leq m^{-1}C^{\alpha}(n+k)^{2\alpha}\|PR\|_{L^p(S, w)} + \frac{\epsilon}{\eta}\|P\|_{L^p(S, w)}.
\]

On the other hand, if \( x \in A \setminus B_0 \), then there exists \( \beta \in I \) such that

\[
|P^{(\alpha-\beta)}(x)| > \left( \frac{\alpha}{\beta} \right)^{-1} \frac{m\epsilon(Cn^2)^{-|\beta|}}{\eta^2}.
\]

Thus we can share \( x \in A \setminus B_0 \) into at most \( \eta - 1 \) disjoint subsets \( B_j \) such that, for every \( x \in B_j \), there exists an index \( \beta \) for which (6) holds. Therefore, since \( |\beta| > 0 \), on each \( B_j \), replacing \( \epsilon \) by \( \frac{\epsilon}{\eta} \), we conclude by induction that

\[
\|P\|_{L^p(B_j, w)} \leq (Cn^2)^{\delta}|\eta^2\frac{m\epsilon}{\eta^2} (Cn^2)^{-|\beta|} \sum_{\beta \in I} \left( \frac{\eta}{\epsilon} \right)^{d_0-1}
\]

\[
\times (n+k)^{2(d_0-|\beta|)}\|PR\|_{L^p(S, w)} + \frac{\epsilon}{\eta}\|P\|_{L^p(S, w)}.
\]
Since \( A = \bigcup_j B_j \) we see that
\[
\| P \|_{L^p(A,w)} \leq C_d m^{-d_0} (n + k)^{2d_0} \| PR \|_{L^p(S,w)} + \epsilon \| P \|_{L^p(S,w)}
\]
with
\[
C_d = C^{2d_0} + \left( \frac{(d_0 + 1)(d_0 + 2)}{2} \right)^{d_0 + 1} C_{d_0 - 1} \sum_{\beta \in I} (\alpha \beta)^{|\beta|},
\]
which completes the induction and the proof.

3 Main result

Our main result reads as follows:

**Theorem 3.1** Let \( p \geq 1 \). Then there exists constant \( C = C(\Omega, p) \) such that for every polynomial \( P \in \mathcal{P}_n(\mathbb{R}^2) \) we have
\[
\max \left\{ \left\| \frac{\partial P}{\partial x} \right\|_{L^p(\Omega)}, \left\| \frac{\partial P}{\partial y} \right\|_{L^p(\Omega)} \right\} \leq C n^4 \| P \|_{L^p(\Omega)}. \tag{7}
\]

**Proof.** Let us first prove the inequality (7) with respect to the second variable. Let \( P \in \mathcal{P}_n(\mathbb{R}^2) \). Then the integrals
\[
\int_{\Omega} \left| \frac{\partial P}{\partial y}(x,y) \right|^p \, dx \, dy, \quad \int_{\Omega} |P(x,y)|^p \, dx \, dy
\]
become, under a change of variables \( x = u + v, \ y = uv \),
\[
\int_S \left| \frac{\partial P}{\partial y}(u+v,uv) \right|^p (v-u) \, du \, dv, \quad \int_S |P(u+v,uv)|^p (v-u) \, du \, dv
\]
where \( S = \{(u,v) \in \mathbb{R}^2: -1 < u < v < 1\} \). Let us define polynomial \( Q(u,v) = P(u+v,uv) \). Then
\[
(v-u) \frac{\partial P}{\partial y}(u+v,uv) = \frac{\partial Q}{\partial u}(u,v) - \frac{\partial Q}{\partial v}(u,v).
\]
We now see, using Lemma 2.1 that
\[
\left\| (v-u) \frac{\partial P}{\partial y}(u+v,uv) \right\|_{L^p(S,w)} = \left\| \frac{\partial Q}{\partial u} - \frac{\partial Q}{\partial v} \right\|_{L^p(S,w)} \leq C(p,S)(2n)^2 \| Q \|_{L^p(S,w)}.
\]

Lemma 2.2 tells us that
\[
\left\| \frac{\partial P}{\partial y}(u+v,uv) \right\|_{L^p(S,w)} \leq C_1(p,S)(2n)^2 \left\| (v-u) \frac{\partial P}{\partial y}(u+v,uv) \right\|_{L^p(S,w)}.
\]
Hence
\[ \left\| \frac{\partial P}{\partial y} (u + v, uv) \right\|_{L^p(S, w)} \leq CC_1 (2n)^4 \| Q \|_{L^p(S, w)}. \]
That completes the proof of (7) for the derivative of \( P \) with respect to \( y \). To prove the remaining part we need to consider the polynomials \( uQ \) and \( vQ \). Then
\[ (v - u) \frac{\partial P}{\partial x} (u + v, uv) = \frac{\partial vQ}{\partial v} (u, v) - \frac{\partial uQ}{\partial u} (u, v). \]
Hence
\[ \left\| (v - u) \frac{\partial P}{\partial x} (u + v, uv) \right\|_{L^p(S, w)} \leq C (2n + 1)^2 \left( \| vQ \|_{L^p(S, w)} + \| uQ \|_{L^p(S, w)} \right) \]
\[ \leq C' (2n + 1)^2 \| Q \|_{L^p(S, w)}. \]
Thus using an argument similar to the one that we carry out in detail in the previous case, one can obtain the desired estimate.

**Remark 3.1**  In the same fashion, we may prove that there exists a positive constant \( C_1 \) such that for every \( P \in P_n(\mathbb{R}^2) \) we have
\[ \max \left\{ \left\| \frac{\partial P}{\partial x} \right\|_{L^p(\Delta_l)}, \left\| \frac{\partial P}{\partial y} \right\|_{L^p(\Delta_l)} \right\} \leq C_1 n^{2l} \| P \|_{L^p(\Delta_l)} \]  \( \quad \) (8)
where \( \Delta_l = \{(x, y) \in \mathbb{R}^2 : |x|^{1/l} + |y|^{1/l} \leq 1 \} \) and \( l \) is a positive odd number.

4 Sharpness of the exponents

In fact, according to [2], it is enough to prove sharpness in the supremum norm. The discussion here is based on unpublished work of M. Baran. Let us consider following sequence of polynomials
\[ P_k(x, y) = \left[ \frac{1}{k} T_k \left( \frac{2 - x}{4} \right) \right]^5 \left( \frac{1 + x + y}{4} \right) \]
where \( T_k \) is the \( k \)th Chebyshev polynomial of the first kind. Note that the degree of a polynomial \( P_k \) is equal \( 5k - 4 \). Since
\[ \left| \frac{1}{k} T'_k (1 - x) \right| \leq \frac{1}{\sqrt{x}} \text{ for every } x \in (0, 1] \text{ and } \]
\[ \frac{1 + x + y}{4} \leq \left( \frac{1}{2} + \frac{x}{4} \right)^2 \text{ for } (x, y) \in \Omega, \]
we may conclude that
\[ |P_k(x, y)| \leq \left| \frac{1}{k} T'_k \left( \frac{2 - x}{4} \right) \right| \sqrt{\frac{1}{2} + \frac{x}{4}} \left| \frac{1}{k} T_k \left( \frac{2 - x}{4} \right) \right| \leq \frac{1}{k} \| T_k \|_{[-1, 1]} = k \]
for any \((x, y) \in \Omega\). On the other hand,
\[
\left| \frac{\partial P_k}{\partial y} (-2, 1) \right| = \frac{1}{4} \left| \frac{1}{k} T'_k(1) \right|^5 = \frac{k^5}{4} \geq \frac{k^4}{4} \| P_k \|_\Omega.
\]

A similar calculation shows that, for \(Q_k = \left[ \frac{1}{k} T'_k \left( \frac{1+y}{2} \right) \right]^5 \left( \frac{x^2}{4} - y \right)\),
\[
\| Q_k \|_\Omega \leq k \quad \text{and} \quad \left| \frac{\partial Q_k}{\partial x} (2, 1) \right| = k^5.
\]

Let \(P_n^{(\alpha, \beta)}\) denote the Jacobi polynomials. In order to prove sharpness of (8), we consider the sequence of polynomials \(W_n(x, y) = y P_n^{(\alpha, \alpha)}(x)\). Thus
\[
\int_{\Delta} \left| \frac{\partial W_n}{\partial y} (x, y) \right|^p \, dx dy = 2 \int_{-1}^1 \left| P_n^{(\alpha, \alpha)}(x) \right|^p \left( 1 - |x|^{1/l} \right)^l \, dx,
\]
\[
\int_{\Delta} |W_n(x, y)|^p \, dx dy = \frac{2}{p+1} \int_{-1}^1 \left| P_n^{(\alpha, \alpha)}(x) \right|^p \left( 1 - |x|^{1/l} \right)^{(p+1)l} \, dx.
\]

By the well known symmetry relation (see [29], Chap. IV)
\[
P_n^{(\alpha, \beta)}(x) = (-1)^n P_n^{(\beta, \alpha)}(-x).
\]

we find that
\[
\int_{\Delta} \left| \frac{\partial W_n}{\partial y} (x, y) \right|^p \, dx dy = 4 \int_0^1 \left| P_n^{(\alpha, \alpha)}(x) \right|^p \left( 1 - x^{1/l} \right)^l \, dx,
\]
\[
\int_{\Delta} |W_n(x, y)|^p \, dx dy = \frac{4}{p+1} \int_0^1 \left| P_n^{(\alpha, \alpha)}(x) \right|^p \left( 1 - x^{1/l} \right)^{(p+1)l} \, dx.
\]

Now Bernoulli’s inequality, for each positive integer \(l\) and \(x \in [0, 1]\), implies that
\[
\left( 1 - \frac{x}{l} \right)^l \leq (1 - x^{1/l})^l \leq (1 - x)^l.
\]

Hence, if \(n \to \infty\), then
\[
\frac{\int_{\Delta} \left| \frac{\partial W_n}{\partial y} (x, y) \right|^p \, dx dy}{\int_{\Delta} |W_n(x, y)|^p \, dx dy} \sim \frac{\int_0^1 \left| P_n^{(\alpha, \alpha)}(x) \right|^p \left( 1 - x^{1/l} \right)^l \, dx}{\int_0^1 \left| P_n^{(\alpha, \alpha)}(x) \right|^p \left( 1 - x \right)^{(p+1)l} \, dx}.
\]

We may now apply the result of Szegő (see [29], Chap. VII) to get
\[
\int_0^1 \left| P_n^{(\alpha, \alpha)}(x) \right|^p \left( 1 - x \right)^l \, dx \sim n^{\alpha p - 2l - 2} \quad \text{whenever} \quad 2l < \mu_{\alpha, p}, \quad (9)
\]
\[
\int_0^1 \left| P_n^{(\alpha, \alpha)}(x) \right|^p \left( 1 - x \right)^{(p+1)l} \, dx \sim n^{\alpha p - 2(p+1)l - 2}, \quad 2(p+1)l < \mu_{\alpha, p}, \quad (10)
\]

where \(\mu_{\alpha, p} = \alpha p - 2 + p/2\). If \(2(p+1)l < \mu_{\alpha, p}\), we can combine (9) and (10) to see that
\[
\frac{\left\| \frac{\partial W_n}{\partial y} \right\|_{L^p(\Delta)}}{\| W_n \|_{L^p(\Delta)}} \sim n^{2l}.
\]

That is what we wished to prove.
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References

[1] M. Baran, New approach to Markov inequality in \( L^p \) norms, in: Approximation theory. In memory of A.K. Varma, I. Govil et al. – editors, M. Dekker, Inc., New York–Basel–Hong Kong (1998) 75–85.

[2] M. Baran, A. Kowalska, Generalized Nikolskii’s property and asymptotic exponent in Markov’s inequality, arXiv:1706.07175 (2017).

[3] M. Baran, A. Kowalska, B. Milówka and P. Ozorka, Identities for a derivation operator and their applications, Dolomites Res. Notes Approx. 8 (2015) 102–110.

[4] M. Baran, W. Pleśniak, Markov’s exponent of compact sets in \( C^n \), Proc. Amer. Math. Soc. 123 (1995) 2785–2791.

[5] L. Białas-Cież, J.P. Calvi, A. Kowalska, Polynomial inequalities on certain algebraic hypersurfaces, J. Math. Anal. Appl. 459 (2) (2018) 822–838.

[6] P. Borwien, T. Erdélyi, Polynomials and Polynomial Inequalities, Springer, New York 1995.

[7] A. Brudnyi, Bernstein Type Inequalities for Restrictions of Polynomials to Complex Submanifolds of \( \mathbb{C}^N \), J. Approx. Theory, 225 (2018) 106–147.

[8] R.A. DeVore, G.G. Lorentz, Constructive Approximation, Springer-Verlag, 1993.

[9] Z. Ditzian, Multivariate Bernstein and Markov inequalities, J. Approx. Theory 70(3) (1992) 273–283.

[10] P. Goetgheluck, Polynomial inequalities and Markov’s inequality in weighted \( L^p \) spaces, Acta Math. Acad. Sci. Hungar. 33 (1979) 325–331.

[11] P. Goetgheluck, Une inégalité polynômiale en plusieurs variables, J. Approx. Theory 40 (1984) 161–172.

[12] P. Goetgheluck, Markov’s inequality on Locally Lipschitzian compact subsets of \( \mathbb{R}^n \) in \( L^p \) spaces, J. Approx. Theory 49 (1987) 303–310.

[13] P. Goetgheluck, Polynomial inequalities on general subsets of \( \mathbb{R}^N \), Coll. Mat. 57 (1989) 127–136.

[14] P. Goetgheluck, On the problem of sharp exponents in multivariate Nikolskii-type inequalities, J. Approx. Theory 77 (1994) 167–178.

[15] E. Hille, G. Szegő, J. Tamarkin, On some generalisation of a theorem of A. Markoff, Duke Math. J 3 (1937) 729–739.
[16] J. Karamata, Sur une inégalité relative aux fonctions convexes, Publ. Math. Univ. Belgrade 1 (1932) 145–148.

[17] S. Kalmykov, B. Nagy, Higher Markov and Bernstein inequalities and fast decreasing polynomials with prescribed zeros, J. Approx. Theory, 226 (2018) 34–59.

[18] T.H. Koornwinder, Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent partial differential operators, I, II, Proc. Kon. Akad. v. Wet., Amsterdam 36 (1974) 48–66.

[19] T.H. Koornwinder, Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent partial differential operators, III, IV, Proc. Kon. Akad. v. Wet., Amsterdam 36 (1974) 357–381.

[20] A. Kroo, On Bernstein–Markov-type inequalities for multivariate polynomials in $L_q$-norm, J. Approx. Theory 159 (2009) 85–96.

[21] G.V Milovanović, D.S. Mitrinović, T.M. Rassias, Topics in polynomials: extremal problems, inequalities, zeros, World Scientific Publishing, River Edge, NJ, 1994.

[22] B. Milówka, Markov’s inequality and a generalized Pleśniak condition, East J. Approx. 11 (2005) 291–300.

[23] B. Milówka, Markov’s property for derivatives of order k, PhD thesis, 2006, 45 pp. (in Polish).

[24] W. Pawłucki, W. Pleśniak, Markov’s inequality and $C^\infty$ functions on sets with polynomial cusps, Math. Ann. 275 (1986) 467–480.

[25] W. Pleśniak, Markov’s inequality and the existence of an extension operator for $C^\infty$ functions, J. Approx. Theory 61 (1990) 106–117.

[26] W. Pleśniak, Recent progress in multivariate Markov inequality, Approximation theory, Monogr. Textbooks Pure Appl. Math., Dekker, New York, (1998) 449–464.

[27] Q.I. Rachman, G. Schmeisser, Analytic Theory of Polynomials, Oxford University Press, Oxford 2002.

[28] E.M. Stein, Interpolation in polynomial classes and Markoff’s inequality, Duke Math. J. 24 (1957) 467–476.

[29] G. Szegö, Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ. 23, Amer. Math. Soc., Providence, 1975.

[30] D.R. Wilhelmsen, A Markov inequality in several dimensions, J. Approx. Theory 11 (1974) 216–220.