EULER’S CONSTANT, q-LOGARITHMS, AND FORMULAS OF RAMANUJAN AND GOSPER

JONATHAN SONDOW (New York) and WADIM ZUDILIN† (Moscow)

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Abstract. The aim of the paper is to relate computational and arithmetic questions about Euler’s constant $\gamma$ with properties of the values of the $q$-logarithm function, with natural choice of $q$. By these means, we generalize a classical formula for $\gamma$ due to Ramanujan, together with Vacca’s and Gosper’s series for $\gamma$, as well as deduce irrationality criteria and tests and new asymptotic formulas for computing Euler’s constant. The main tools are Euler-type integrals and hypergeometric series.

We recall the definition of Euler’s constant:

$$\gamma = \lim_{n \to \infty} \left( \sum_{\nu=1}^{n} \frac{1}{\nu} - \log n \right)$$

$$= 0.57721566490153286060651209008240243104215933593992\ldots$$

and the fact that the (expected) irrationality of $\gamma$ has not yet been proved.

Recently, the first author gave [9], [10] a construction of $\mathbb{Z}$-linear forms involving $\gamma$ and logarithms; namely, he proved that

$$d_{2n}I_n \in \mathbb{Z} + \mathbb{Z}\gamma + \mathbb{Z}\log(n+1) + \mathbb{Z}\log(n+2) + \cdots + \mathbb{Z}\log(2n),$$

where

$$I_n = \iint_{[0,1]^2} \frac{x^n(1-x)^ny^n(1-y)^n}{(1-xy)|\log xy|} \, dx \, dy$$

$$= \sum_{\nu=n+1}^{\infty} \int_{\nu}^{\infty} \left( \frac{n!}{t(t+1)\cdots(t+n)} \right)^2 \, dt$$

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and \( d_n \) denotes the least common multiple of the numbers 1, 2, \ldots, \( n \). This type of approximation allows deducing several (ir)reducibility criteria for Euler’s constant [9]. In a sense, what follows is inspired by considerations in [9].

A special function that we will require is the \( q \)-logarithm

\[
\ln_q(1 + z) = \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1} z^\nu}{q^\nu - 1}, \quad |z| < q,
\]

for \( q \) real with \( |q| > 1 \) (in fact, our main choices will be \( q = 2 \) in Sections 2–4 and \( q = 3 \) in Section 8, but we let \( q > 1 \) be any integer in Section 9). The function (1) is a \( q \)-extension of the ordinary logarithm function

\[
\log(1 + z) = \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1} z^\nu}{\nu}, \quad |z| \leq 1, \ z \neq -1,
\]

since

\[
\lim_{q \to 1 \atop q > 1} ((q - 1) \ln_q(1 + z)) = \log(1 + z), \quad |z| < 1.
\]

We also mention that the irrationality of the values of the \( q \)-logarithm (1) for \( q \in \mathbb{Z} \setminus \{0, \pm 1\} \) and \( z \in \mathbb{Q}, \ z \leq 1 \), is known [3].

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1. Gosper’s acceleration of Vacca’s series

The following expression for \( \gamma \) is known as Vacca’s series [11]:

\[
\gamma = \sum_{n=1}^{\infty} (-1)^n \frac{\lfloor \log_2 n \rfloor}{n},
\]

where \( \log_q x = (\log x) / (\log q) \) is the ordinary logarithm base \( q \). Gosper [6] has transformed this series into a more rapidly converging series of positive rationals. His method (see Section 6) is to use partial summation to write (2) as the double series

\[
\gamma = \sum_{\nu=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^\nu + k},
\]
apply Euler’s transformation to each inner sum on \( k \), obtaining the double series (7) below, then triangularize it. In this way, he arrives at a version of series (4). Here we give a different proof of (4). To begin, we reverse the order of summation in (3) and write the resulting double series as the integral (5), which is a form of an integral that Bromwich attributes to Catalan ([4], p. 526).

**Theorem 1** (Gosper’s Series). *We have the series for Euler’s constant*

\[
\gamma = \frac{1}{2} + \sum_{\nu=2}^{\infty} \frac{1}{2\nu+1} \sum_{k=1}^{\nu-1} \left( \frac{2^{\nu-k} + k}{k} \right)^{-1}.
\]

**Proof.** After expanding \( 1/(1 + x) \) in a geometric series, termwise integration shows that the formula

\[
\gamma = \int_{0}^{1} \frac{1}{1 + x} \sum_{\nu=1}^{\infty} x^{2\nu-1} \, dx
\]

is equivalent to (3) with the order of summation reversed, which is easily justified by grouping terms in pairs. Integrating by parts \( K+1 \) times gives

\[
\gamma = \sum_{k=0}^{K} \sum_{\nu=1}^{\infty} \frac{1}{2\nu+k+1} \left( \frac{2^{\nu+k}}{k} \right) + \int_{0}^{1} \frac{(K+1)!}{(1+x)^{K+2}} \sum_{\nu=1}^{\infty} 2^{\nu} \left( \frac{2^{\nu} + 1}{2^{\nu} + K} \right) \frac{x^{2\nu+K}}{dx}.
\]

Since the integral is less than

\[
\int_{0}^{1} \frac{1}{(1+x)^{K+2}} \sum_{\nu=1}^{\infty} \frac{1}{2^{\nu}} \, dx = \frac{1}{K+1} \left( 1 - \frac{1}{2^{K+1}} \right),
\]

as \( K \) tends to infinity (6) becomes

\[
\gamma = \sum_{k=0}^{\infty} \sum_{\nu=1}^{\infty} \frac{1}{2^{\nu+k+1}} \left( \frac{2^{\nu+k}}{k} \right).
\]

The inner series with \( k = 0 \) sums to \( 1/2 \), and in the series with \( k > 0 \) we may collect terms (triangularize) as in (4), completing the proof. \( \Box \)

**Remark.** We can regard the series on the right of (2), (4), (7) as base \( q = 2 \) series for Euler’s constant. Gosper indicated [6] the following extensions of (2), (7) to base \( q = 3 \):

\[
\gamma = 2 \sum_{n=1}^{\infty} \frac{\log_{3} n}{n} \cdot \cos \frac{2\pi n}{3} = 2 \sum_{\nu=1}^{\infty} \sum_{k=0}^{\infty} \frac{\cos \frac{\pi(k-1)}{6}}{3^{\nu+(k+1)/2} \left( \frac{3^{\nu+k}}{k} \right)}
\]

(see Sections 5 and 6 for proofs and generalizations), which give faster convergence than formulas (2), (7), respectively.
2. A q-LOGARITHM APPROACH

Lemma 1. Let $q$ be real, $q > 1$. Then for all integers $n \geq 0$ and $k$, with $0 < k < q^{n+1}$, the following identity holds:

\[
\sum_{\nu=n+1}^{\infty} \frac{1}{q^\nu + k} = \frac{1}{k} \ln_q \left(1 + \frac{k}{q^n}\right).
\]

Remark. For $k = 0$, the series in (9) equals

\[
\sum_{\nu=n+1}^{\infty} \frac{1}{q^\nu} = \frac{q^{-(n+1)}}{1 - q^{-1}} = \frac{1}{q^n(q-1)}.
\]

Proof. We have

\[
\sum_{\nu=n+1}^{\infty} \frac{1}{q^\nu + k} = \sum_{\nu=n+1}^{\infty} \frac{1}{q^\nu + k \cdot q^{-\nu}} = \sum_{\nu=n+1}^{\infty} \frac{1}{q^\nu} \sum_{\mu=1}^{\infty} \frac{(-k)^{\mu-1}}{q^{\mu(\mu-1)}} = \sum_{\mu=1}^{\infty} (-k)^{\mu-1} \sum_{\nu=n+1}^{\infty} \frac{q^{-(n+1)\mu}}{1 - q^{-\mu}}
\]

\[
= \frac{1}{k} \sum_{\mu=1}^{\infty} \frac{(-1)^{\mu-1}(k/q^n)^\mu}{q^\mu - 1} = \frac{1}{k} \ln_q \left(1 + \frac{k}{q^n}\right)
\]

as required. □

Remark. P. Sebah [8] has pointed out that in order to compute the q-logarithm, the following formula is much more efficient than (1) or (9):

\[
\ln_q(1 + z) = z \sum_{\nu=1}^{N} \frac{1}{q^\nu + z} + r_N(z), \quad \text{where} \quad r_N(z) = \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1} z^\nu}{q^{N\nu}(q^\nu - 1)},
\]

with $N$ being any positive integer. The proof of the formula is similar to that of Lemma 1.

Let $m$ be a non-negative integer. Using the rational function

\[
R_m(t) = \frac{m!}{t(t+1) \cdots (t+m)} = \sum_{j=0}^{m} \frac{(-1)^j \binom{m}{j}}{t+j},
\]

for each $\nu \geq 0$ define the convergent series

\[
S_{\nu,m} = \sum_{t=0}^{\infty} (-1)^t R_m(2^\nu + t).
\]
Lemma 2. We have

\[ S_{\nu,m} = 2^m \sum_{k=0}^{\infty} \frac{(-1)^k}{2^\nu + k} - \sum_{j=1}^{m} \binom{m}{j} \sum_{k=0}^{j-1} \frac{(-1)^k}{2^\nu + k}. \]

(We use the standard convention that any sum $\sum_{j=a}^{b}$ is zero if $a > b$.)

Proof. Indeed,

\[ S_{\nu,m} = \sum_{j=0}^{m} \sum_{t=0}^{\infty} \frac{(-1)^{t+j} \binom{m}{j}}{2^\nu + t + j} = \sum_{j=0}^{m} \binom{m}{j} \sum_{t=0}^{\infty} \frac{(-1)^{t+j}}{2^\nu + t + j} = \sum_{j=0}^{m} \binom{m}{j} \sum_{k=j}^{\infty} \frac{(-1)^k}{2^\nu + k} = \sum_{j=0}^{m} \binom{m}{j} \left( \sum_{k=0}^{j-1} - \sum_{k=0}^{\infty} \right) \frac{(-1)^k}{2^\nu + k}, \]

and (12) follows. □

Lemma 3. For the series $S_{\nu,m}$, we have the representations

\[ S_{\nu,m} = (-1)^{2^\nu} \sum_{t=2^\nu}^{\infty} (-1)^t R_m(t) = \int_0^1 \frac{(1-x)^m}{1+x} x^{2^\nu-1} \, dx. \]

Proof. Replace $t$ by $t - 2^\nu$ in (11) to get the first equality in (13). For the second, use the Binomial Theorem to write

\[ R_m(t) = \int_0^1 (1-x)^m x^{t-1} \, dx. \]

Substitute this in the series for $S_{\nu,m}$, interchange summation and integration, and sum the series

\[ \sum_{t=2^\nu}^{\infty} (-1)^{t-2^\nu} x^{t-1} = \frac{x^{2^\nu-1}}{1+x}, \]

arriving at the second equality in (13). To justify termwise integration, replace the geometric series (14) with a finite sum plus remainder $r_N(x) = (-1)^N x^{N+2^\nu-1}/(1+x)$, and note that

\[ \left| \int_0^1 r_N(x) \, dx \right| < \frac{1}{N + 2^\nu} \to 0 \quad \text{as } N \to \infty. \] □

Our final object is the double series

\[ I_{n,m} = \sum_{\nu=n+1}^{\infty} S_{\nu,m}, \]
where \( n \geq 0 \). By Lemma 2 and formula (3), for the quantity (15) we have

\[
I_{n,m} = 2^m \left( \sum_{\nu=1}^{\infty} \sum_{k=0}^{n} \sum_{l=1}^{\nu} \frac{(-1)^k}{2^\nu + k} - \sum_{j=1}^{m} \sum_{\nu=n+1}^{\infty} \sum_{k=0}^{\nu} \frac{(-1)^k}{2^\nu + k} \right)
\]

\[
= 2^m \gamma - 2^m \sum_{\nu=1}^{n} \left( \sum_{l=1}^{\nu-1} \frac{(-1)^{l-1}}{l} - \sum_{j=1}^{m} \sum_{\nu=n+1}^{\infty} \frac{1}{2^\nu} \right)
\]

\[
- \sum_{j=1}^{m} \left( \sum_{\nu=n+1}^{\infty} \sum_{k=1}^{\nu} \frac{(-1)^k}{2^\nu + k} \right).
\]

For simplicity, assume now that \( m \leq 2^{n+1} \). Then using Lemma 1,

\[
(16) \quad I_{n,m} = 2^m \cdot (\gamma + n \log 2) - 2^m \sum_{\nu=1}^{n} \sum_{l=1}^{2^{\nu-1}} \frac{(-1)^{l-1}}{l} - \frac{2^m - 1}{2^n}
\]

\[
- \sum_{j=2}^{m} \left( \sum_{\nu=n+1}^{\infty} \sum_{k=1}^{\nu} \frac{(-1)^k}{k} \ln_2 \left( 1 + \frac{k}{2^n} \right) \right).
\]

We can summarize the result of these computations in the following way.

**Lemma 4.** If \( m \leq 2^{n+1} \), then

\[
(17) \quad I_{n,m} = 2^m \gamma + L_{n,m} - A_{n,m},
\]

where

\[
L_{n,m} = 2^m n \log 2 + \sum_{j=2}^{m} \left( \sum_{\nu=n+1}^{\infty} \sum_{k=1}^{\nu} \frac{(-1)^{k-1}}{k} \ln_2 \left( 1 + \frac{k}{2^n} \right) \right)
\]

and

\[
A_{n,m} = \frac{2^m - 1}{2^n} + \sum_{\nu=1}^{n} \sum_{l=1}^{2^{\nu-1}} \frac{(-1)^{l-1}}{l}
\]

satisfy

\[
d_m L_{n,m} \in \mathbb{Z} \log 2 + \mathbb{Z} \ln_2 \left( 1 + \frac{1}{2^n} \right) + \mathbb{Z} \ln_2 \left( 1 + \frac{2}{2^n} \right) + \cdots + \mathbb{Z} \ln_2 \left( 1 + \frac{m-1}{2^n} \right)
\]

and \( d_{2^n} A_{n,m} \in \mathbb{Z} \).

**Lemma 5.** For the double series \( I_{n,m} \), we have the integral

\[
(18) \quad I_{n,m} = \int_0^1 \frac{(1-x)^m}{1+x} \sum_{\nu=n+1}^{\infty} x^{2^{\nu-1}} \, dx,
\]

which generalizes Catalan’s integral (5) for \( \gamma \).

**Proof.** Integrating termwise (since the series in the integrand is dominated by a geometric series, the justification in the proof of Lemma 3 applies here) and using Lemma 3, the formula follows from the definition of \( I_{n,m} \). The case \( I_{0,0} = \gamma \) (see (16)) is indeed the integral (5). \( \square \)
Lemma 6. For $m$ a positive integer and $r \geq 1$ a real number, we have the bounds

\begin{equation}
\int_0^1 x^{rm-1} (1-x)^m \, dx > \frac{1}{rm} \left( \frac{r^r}{(r+1)^{r+1}} \right)^m
\end{equation}

and, for $0 < x < 1$,

\begin{equation}
x^{rm-1} (1-x)^m < 4 \cdot \left( \frac{r^r}{(r+1)^{r+1}} \right)^m.
\end{equation}

Proof. Euler’s beta integral

\[
\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} \, dx = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}, \quad \text{Re} \, \alpha > 0, \quad \text{Re} \, \beta > 0,
\]

gives

\[
\int_0^1 x^{rm-1} (1-x)^m \, dx = \frac{\Gamma(rm) m!}{\Gamma(rm + m + 1)} = \frac{1}{rm} \prod_{j=1}^m \left( 1 + \frac{rm}{j} \right)^{-1};
\]

denote the product by $\Pi(m, r)$. Using $r \geq 1$ to get

\[
\frac{j + r(m + 1)}{j + rm} < 1 + \frac{1}{m} \leq \left( 1 + \frac{1}{r} \right)^{r/m} \quad \text{for} \quad j = 1, 2, \ldots, m,
\]

we obtain

\[
\frac{\Pi(m + 1, r)}{\Pi(m, r)} = \frac{1}{r + 1} \prod_{j=1}^m \frac{j + rm}{j + r(m + 1)} > \frac{1}{r + 1} \left( 1 + \frac{1}{r} \right)^{-r} = \frac{r^r}{(r+1)^{r+1}},
\]

and by induction on $m$ it follows that

\[
\Pi(m, r) > \left( \frac{r^r}{(r+1)^{r+1}} \right)^m
\]

since the inequality holds for $m = 1$. This proves (19).

If $0 < x < 1$, then

\[
x^{rm-1} (1-x)^m < (x^r (1-x))^{m-1/r} \leq \left( \frac{r^r}{(r+1)^{r+1}} \right)^{m-1/r}
\]

\[
= \frac{(r+1)^{1+1/r}}{r} \cdot \left( \frac{r^r}{(r+1)^{r+1}} \right)^m
\]

and (20) follows, using $r \geq 1$. \qed
Lemma 7. If \( n \geq 0 \) is an integer and \( r \) is a real number satisfying \( 1 \leq r \leq 2^{n+1} \), then we have the bounds

\[
\frac{1}{2r(m+1)} \left( \frac{r^r}{(r+1)^{r+1}} \right)^{m+1} < I_{n,m} < 6 \left( \frac{r^r}{(r+1)^{r+1}} \right)^m \quad \text{for} \quad m = \left\lfloor \frac{2^{n+1}}{r} \right\rfloor.
\]

In particular, for \( r \geq 1 \) fixed and \( m = \lfloor 2^{n+1}/r \rfloor \geq 1 \), the quantity \( I_{n,m} \) is positive and decreases exponentially at the rate

\[
\lim_{n \to \infty} \log \frac{I_{n,\lfloor 2^{n+1}/r \rfloor}}{\lfloor 2^{n+1}/r \rfloor} = r \log r - (r+1) \log(r+1).
\]

Proof. Put \( m = \lfloor 2^{n+1}/r \rfloor \) in Lemma 5. Then \( m+1 > 2^{n+1}/r \), hence

\[
I_{n,m} > \int_0^1 \frac{(1-x)^m}{1+x} x^{2^{n+1}-1} \, dx > \frac{1}{2} \int_0^1 x^{r(m+1)-1}(1-x)^{m+1} \, dx,
\]

and the lower bound in (21) follows from (19) (with \( m+1 \) in place of \( m \)).

Using \( m \leq 2^{n+1}/r \) and relations (20) and (5), we have

\[
I_{n,m} \leq \int_0^1 \frac{(1-x)^m}{1+x} x^{rm-1} \sum_{\nu=n+1}^{\infty} x^{2^{\nu} - 2^{n+1}} \, dx
\leq \int_0^1 x^{rm-1}(1-x)^m \frac{1}{1+x} \left( 1 + \sum_{\nu=1}^{\infty} x^{2^{\nu}-1} \right) \, dx
\leq 4 \left( \frac{r^r}{(r+1)^{r+1}} \right)^m (\log 2 + \gamma),
\]

and the upper bound in (21) follows from the inequality \( 4(\log 2 + \gamma) < 6 \). Since (21) implies (22), we are done. \( \square \)

Remark 1. If \( 2^{n+1}/r \) is an integer, then we may replace \( m+1 \) by \( m \) in the lower bound in (21).

Remark 2. As \( r \to \infty \), the right-hand side of (22) equals \(-\log r + O(1)\), hence its absolute value tends to infinity with \( r \).

3. Irrationality tests for Euler’s constant

Theorem 2 (Rationality Criterion for \( \gamma \)). The fractional part of \( d_{2^n} L_{n,m} \) equals \( d_{2^n} I_{n,m} \) for all \( n \) and \( m \) with

\[
n \text{sufficiently large and} \quad 2^{n-1} \leq m \leq 2^{n+1},
\]

if and only if Euler’s constant is a rational number.
Proof. With $2^{n-1} \leq m \leq 2^{n+1}$ set $r = 2^{n+1}/m$, so that $1 \leq r \leq 4$. By Lemma 7, we have $I_{n,m} < 6\rho(r)^m = 6\rho(r)^{2^{n+1}/r}$, where $\rho(r) = r^r/(r+1)^{r+1}$. Straightforward verification shows that $\rho(r)^{2/r}$ is increasing for $r > 0$, and satisfies

$$0 < \rho(r)^{2/r} < \frac{1}{e}$$

for $0 < r < r_0 = 5.6213305349\ldots$, where $\rho(r_0)^{2/r_0} = 1/e$. Therefore $I_{n,m} < 6(\rho(4)^{1/2})^{2^n}$ and $\rho(4)^{1/2} < 1/e$. Since the Prime Number Theorem implies that for any $\varepsilon > 0$ we have $d_N = O((e^{1+\varepsilon})^N)$ as $N \to \infty$, we conclude that $0 < d_{2^n}I_{n,m} < 1$ for $n$ and $m$ satisfying (23).

Now multiply (17) by $d_{2^n}$ and write the result as

$$d_{2^n}L_{n,m} = d_{2^n}I_{n,m} + d_{2^n}(A_{n,m} - 2^m\gamma).$$

Since $d_{2^n}A_{n,m} \in \mathbb{Z}$ by Lemma 4, the theorem follows. \qed

Remark. The argument shows that the Criterion holds with (23) replaced by the more general conditions $n \geq n(r_1)$ and $[2^{n+1}/r_1] \leq m \leq 2^{n+1}$, where $r_1$ is any number between 1 and $r_0$.

**Theorem 3** (Irrationality Test for $\gamma$). If the fractional part of $d_{2^n}L_{n,2^n}$ satisfies

$$\{d_{2^n}L_{n,2^n}\} > 6 \cdot \left(\frac{128}{729}\right)^{2^{n-1}},$$

infinitely often, then Euler’s constant is irrational. In fact, the inequality for a given $n > 0$ implies that if $\gamma$ is a rational number, then its denominator does not divide $2^{2^n}d_{2^n}$.

Proof. It follows from [7], Theorem 13, that $d_{2N} < 8^N$ for all integers $N \geq 1$ (see [9], Lemma 3). Combined with the upper bound in Lemma 7 for $r = 2$, this gives

$$0 < d_{2^n}I_{n,2^n} < 8^{2^n-1} \cdot 6 \cdot \left(\frac{4}{27}\right)^{2^{n-1}} = 6 \cdot \left(\frac{128}{729}\right)^{2^{n-1}}$$

for $n \geq 1$. The first part of the theorem follows by the Rationality Criterion for $\gamma$, and the second part by formula (24) with $m = 2^n$. \qed

Remark 1. Condition (25) in Theorem 3 can be replaced by the simpler condition

$$\{d_{2^n}L_{n,2^n}\} > 5^{-2^{n-1}}$$

if $n > 1$. For $n > 4$, this follows from the inequalities $128/729 = 0.17558299\ldots < 1/5$ and $6 \cdot (128/729)^{14} < 5^{-14}$. For $n = 2, 3, 4$, it follows from the proof of Theorem 3 by
replacing the bound $d_{2m} < 8^m$ with the exact values $d_4 = 12, d_8 = 840, d_{16} = 720720,$ for which $d_{2n} \cdot 6 \cdot (4/27)^{2n} < 5^{2^n-1}$.

Remark 2. Using the Rationality Criterion for $\gamma$, we may generalize the Irrationality Test as follows: $\gamma$ is irrational if there exist infinitely many integers $n$ and $m$ such that $2^{n-1} \leq m \leq 2^{n+1}$ and

$$\{d_{2n}L_{n,m}\} > 8^{2^n-1} \cdot 6 \cdot \left(\frac{r}{r+1}\right)^m,$$

where $r = \frac{2^{n+1}}{m}$.

For example, the case $m = 2^n$ is Theorem 3, and the case $m = 2^{n+1}$ can be stated: a sufficient condition for irrationality of $\gamma$ is that

$$\{d_{2n}L_{n,2^{n+1}}\} > 6 \cdot 2^{-5 \cdot 2^n - 1} \text{ infinitely often.}$$

For any $\varepsilon > 0$, we may further refine the Test by replacing $8^{2^n-1} \cdot 6$ with $e^{(1+\varepsilon)2^n}$ in (26).

Example. P. Sebah [8] has calculated that

$$\{d_{2\cdot 12}L_{12,2\cdot 12}\} = 0.178346164 \ldots$$

Since the last number exceeds $5^{2^{11}}$, if $\gamma$ is a rational number $a/b$, then $b$ is not a divisor of $2^{4096} \cdot d_{4096}$; in particular, $|b| \geq 4099$ (since the numbers $4097 = 17 \cdot 241$ and $4098 = 2 \cdot 3 \cdot 683$ divide $d_{4096}$).

Remark. Similar irrationality tests for $\gamma$, but using ordinary logarithms instead of 2-logarithms, are proved in [9], where an example shows that no divisor of $\left(\frac{2^{20000}}{10000}\right) \cdot d_{20000}$ can be a denominator of $\gamma$. The present example yields additional information because of the high power of 2.

4. Computing Euler’s constant

Theorem 4. The following asymptotic formulas as $n \to \infty$ are valid:

$$\gamma = \frac{A_{n,2^{n+1}} - L_{n,2^{n+1}}}{2^{2^n+1}} + O(2^{-6 \cdot 2^n}),$$

$$\gamma = \frac{A_{n,2^n} - L_{n,2^n}}{2^{2^n}} + O\left(\frac{1}{27}\right)^{2^n},$$

$$\gamma = \frac{A_{n,2^{n-1}} - L_{n,2^{n-1}}}{2^{2^n-1}} + O\left(\frac{128}{3125}\right)^{2^{n-1}},$$

$$\gamma = \frac{1}{2}A_{n,1} - n \log 2 + O(2^{-n}).$$
Proof. For (27), put \( m = 2^{n+1} \) in (17) and use the upper bound in (21) with \( r = 1 \). For (28), (29), take \( m = 2^n, 2^{n-1} \) and \( r = 2, 4 \), respectively. Finally, let \( m = 1 \) in Lemma 4 to get \( I_{n,1} = 2\gamma + 2n \log 2 - A_{n,1} \). From (21) with \( r = 2^{n+1} \), we obtain

\[
0 < I_{n,1} < 6 \cdot \frac{(2^{n+1})^{2^{n+1}}}{(2^{n+1} + 1)^{2^{n+1}+1}} = \frac{6}{2^{n+1} + 1} \left( 1 + \frac{1}{2^{n+1}} \right)^{-2^{n+1}} < 2^{-n+1}
\]

for \( n \geq 0 \), proving more than (30). \( \square \)

Remark. Using (28) and his calculation of \( L_{n,2^n} \) for \( n = 12 \), P. Sebah [8] has computed \( \gamma \) correct to 4631 decimal places, which is the accuracy \((2/27)^{2^{12}}\) predicted. Formula (27) gives almost 60% more digits of \( \gamma \), but requires computing about twice as many 2-logarithms. Similarly, (29) yields nearly 39% fewer digits than (28), but involves only half as many 2-logarithms. Letting \( m \) continue down to 1, we reach formula (30), which doesn’t involve any 2-logarithms.

5. Ramanujan’s base \( q \) integral for Euler’s constant

For integer \( q \geq 2 \), Ramanujan (see [2] for proofs and references) gave the formula for Euler’s constant

\[
\gamma = \int_0^1 \left( \frac{q}{1 - x^q} - \frac{1}{1 - x} \right) \sum_{\nu=1}^{\infty} x^{q^\nu-1} \, dx,
\]

which reduces to Catalan’s integral (5) when \( q = 2 \). Equation (31) implies the following base \( q \) generalization of Vacca’s series (2) (see [2], Theorem 2.6):

\[
\gamma = \sum_{n=1}^{\infty} \frac{\sigma_n [\log_q n]}{n},
\]

where

\[
\sigma_n = \sigma_{n,q} = \begin{cases} q - 1 & \text{if } q \mid n, \\ -1 & \text{if } q \nmid n. \end{cases}
\]

The case \( q = 3 \) gives the first equality in (8).

Let \( \epsilon = \epsilon_q \) be a fixed primitive \( q \)-th root of unity (for instance, \( \epsilon = e^{2\pi i/q} \)). Using

\[
\sigma_n = -1 + \sum_{l=0}^{q-1} \epsilon^{nl} = \sum_{l=1}^{q-1} \epsilon^{nl},
\]

we can represent formula (32) as

\[
\gamma = \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} \frac{[\log_q n] \epsilon^{nl}}{n},
\]

which can be regarded as another base \( q \) generalization of Vacca’s series (2).
6. A base $q$ generalization of Gosper’s Series

Following Gosper’s proof [6] of (7), we write (34) as

$$\gamma = \sum_{l=1}^{q-1} \sum_{\nu=1}^{\infty} \sum_{k=0}^{\infty} \frac{\epsilon^{kl}}{q^\nu + k} = \sum_{l=1}^{q-1} \sum_{\nu=1}^{\infty} \frac{1}{q^\nu} \cdot \, _2F_1 \left( 1, q^\nu + 1 \mid \epsilon^l \right)$$

and apply Euler’s transform (see, e.g., [1], Section 2.4, formula (1))

$$2F_1 \left( \alpha, \beta \mid z \right) = \frac{1}{(1-z)^{\alpha}} \cdot 2F_1 \left( \alpha, \gamma - \beta \mid \frac{-z}{1-z} \right),$$

yielding

$$\gamma = \sum_{l=1}^{q-1} \sum_{\nu=1}^{\infty} \frac{1}{q^\nu} \cdot \frac{1}{1-\epsilon^l} \cdot \, _2F_1 \left( 1, 1 \mid q^\nu + 1 \mid \frac{-\epsilon^l}{1-\epsilon^l} \right)$$

$$= \sum_{l=1}^{q-1} \sum_{\nu=1}^{\infty} \frac{1}{q^\nu} \sum_{k=0}^{\infty} \left( q^\nu + k \right)^{-1} \frac{(-1)^k \epsilon^{kl}}{(1-\epsilon^l)^{k+1}}.$$

Substituting $x = 1$ into the expansion

$$\prod_{j=1}^{q-1} (1 - \epsilon^j x) = \frac{1-x^q}{1-x} = 1 + x + x^2 + \cdots + x^{q-1}$$

leads to

$$\frac{1}{1-\epsilon^l} = \frac{1}{q} \cdot \prod_{j=1 \atop j \neq l}^{q-1} (1 - \epsilon^j)$$

(where the empty product in the case $q = 2$ equals 1). Therefore, we arrive at the following result.

Theorem 5. We have the base $q \geq 2$ accelerated series

$$\gamma = \sum_{\nu=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \chi_q(k)}{q^\nu + k + 1 \left( q^\nu + k \right)},$$

where

$$\chi_q(k) = \sum_{l=1}^{q-1} \epsilon^{kl} \prod_{j=1 \atop j \neq l}^{q-1} (1 - \epsilon^j)^{k+1},$$

a symmetric polynomial in the roots of the polynomial $1 + x + x^2 + \cdots + x^q$, is an integer-valued function independent of the choice of primitive $q$-th root of unity $\epsilon$. 
Remark. From (35) we have
\[ e^{kl} \prod_{\substack{j=1 \atop j \neq l}}^{q-1} (1 - e^j)^{k+1} = e^{kl} \frac{q^{k+1}}{(1 - e^l)^{k+1}}, \]
and it follows that
\[ |\chi_q(k)| \leq (q - 1) \cdot \frac{q^{k+1}}{1 - e^{2\pi i/q}} = (q - 1) \left( \frac{q}{2 \sin \frac{\pi}{q}} \right)^{k+1}. \]
Thus we have the relation
\[ \limsup_{k \to \infty} |\chi_q(k)|^{1/k} \leq \frac{q}{2 \sin \frac{\pi}{q}}, \]
which shows how the convergence of series (34) has been accelerated in (36).

Examples. Series (36) is a generalization of Gosper's series (7), which is the case \( q = 2 \). When \( q = 3 \), taking \( \epsilon = e^{2\pi i/3} \) we obtain
\[
\chi_3(k) = e^{2\pi ki/3} (1 - e^{-2\pi i/3})^{k+1} + e^{-2\pi ki/3} (1 - e^{2\pi i/3})^{k+1} \\
= e^{2\pi ki/3} \cdot 3^{k+1}/2 e^{\pi(k+1)i/6} + e^{-2\pi ki/3} \cdot 3^{k+1}/2 e^{-\pi(k+1)i/6} \\
= 3^{(k+1)/2} \cdot 2 \cos \frac{\pi(k-1)}{6},
\]
which proves the second equality in (8). When \( q = 4 \), we take \( \epsilon = e^{\pi i/2} = i \) and obtain
\[
\chi_4(k) = i^k 2^{k+1} (1 + i)^{k+1} + (-1)^{k} (1 - i)^{k+1} (1 + i)^{k+1} + (-i)^{k} 2^{k+1} (1 - i)^{k+1} \\
= 2^{3(k+1)/2+1} \cdot \cos \frac{\pi(3k+1)}{4} + (-1)^k 2^{k+1},
\]
which is an integer-valued function of \( k \).

7. Base \( q \) integrals

There are several ways to generalize integrals (18) for \( I_{n,m} \) and (31) for \( \gamma \) simultaneously. One way is to define
\[ I_{n,m,q} = \int_0^1 \left( \frac{q}{1 - xq} - \frac{1}{1 - x} \right) (1 - x)^m \sum_{\nu=n+1}^{\infty} x^{\nu-1} \, dx. \]
Then \( I_{n,m,2} = I_{n,m} \) and \( I_{0,0,q} = \gamma \), and the analog of Lemma 7 with \( r = q \) and \( m = q^n \),
\[ \lim_{n \to \infty} \frac{\log I_{n,q^n,q}}{q^n} = q \log q - (q + 1) \log(q + 1), \]
can be easily derived by the methods of Section 2 (cf. Lemma 9 below). We may expand integral (37) as in the proof of Lemma 4. Namely, with \( R_m(t) \) defined in (10), and \( \sigma_t \) in (33), we have
\[
\frac{q}{1-x^q} - \frac{1}{1-x} = \frac{(q-1) - x - x^2 - \cdots - x^{q-1}}{1-x^q} = \sum_{t=0}^{\infty} \sigma_t x^t,
\]
hence
\[
I_{n,m,q} = \sum_{t=0}^{\infty} \sum_{\nu=n+1}^{\infty} \int_0^1 x^{q^\nu+t-1} (1-x)^m \, dx = \sum_{\nu=n+1}^{\infty} \sum_{t=0}^{\infty} \sigma_t R_t(q^\nu + t) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{\nu=n+1}^{\infty} \sum_{k=j}^{\infty} \frac{\sigma_{k-j}}{q^\nu + k}.
\]
Unfortunately, the last representation shows that \( I_{n,m,q} \) involves not only logarithms and \( q \)-logarithms but also “generalized Euler constants”
\[
\gamma_{j,q} = \sum_{\nu=1}^{\infty} \sum_{k=j}^{\infty} \frac{\sigma_{k-j}}{q^\nu + k}, \quad j = 0, 1, \ldots, q-1;
\]
the equality \( \gamma_{0,q} = \gamma \) follows from (32) just as (3) follows from Vacca’s series (2) (the case \( q = 2 \)). We may avoid the appearance of generalized Euler constants \( \gamma_{j,q} \neq \gamma \) in the particular case \( q = 3 \) (details are in Section 8), but not for \( q > 3 \) (see Section 10).

However, there is a way to generalize integral (18) for \( I_{n,m} \) that solves this problem, namely, by defining
\[
I'_{n,m,q} = \int_0^1 \left( \frac{q}{1-x^q} - \frac{1}{1-x} \right) \left( q - \frac{1-x^q}{1-x} \right)^m \sum_{\nu=n+1}^{\infty} x^{q^\nu-1} \, dx
\]
with the properties \( I'_{n,m,2} = I_{n,m} \) and \( I'_{0,0,q} = \gamma \), as for (37). For integral \( I'_{n,m,q} \) in (40), we may generalize the results of Sections 2 and 4, but the poor asymptotics of the integral make these generalizations useless for irrationality testing. For example, the choice \( m = (q^n - 1)/(q - 1) \) gives the inclusions
\[
d_{q^n} I'_{n,m,q} \in \mathbb{Z} + \mathbb{Z} \gamma + \mathbb{Z} \log q + \mathbb{Z} \ln_q \left( 1 + \frac{1}{q^n} \right) + \mathbb{Z} \ln_q \left( 1 + \frac{2}{q^n} \right) + \cdots
\]
(see Lemma 10 below) and the asymptotics
\[
\lim_{n \to \infty} \frac{\log I'_{n,(q^n-1)/(q-1),q}}{q^n} > -1
\]
(see Lemma 13); this makes it impossible to use \( I'_{n,m,q} \) to give irrationality tests for \( q > 2 \) similar to those of Theorems 2 and 3 for \( q = 2 \). Instead we use \( I'_{n,m,q} \) to extend Theorem 4 to a base \( q > 2 \) asymptotic formula for \( \gamma \); details are in Section 9.
8. Base 3 irrationality testing

Taking \( q = 3 \) in (31) gives one of Ramanujan’s formulas for Euler’s constant

\[
\gamma = \int_0^1 \frac{2 + x}{1 + x + x^2} \sum_{\nu=1}^{\infty} x^{3\nu-1} \, dx
\]

(see [2], Corollary 2.3 and the equation following (2.7)), and taking \( q = 3 \) in (37) gives the base 3 integral

\[
I_{n,m,3} = \int_0^1 \frac{2 + x}{1 + x + x^2} (1 - x)^m \sum_{\nu=n+1}^{\infty} x^{3\nu-1} \, dx.
\]

(42)

It turns out that, for \( m \) a multiple of 6, we can extend the results of Sections 2–4 to base \( q = 3 \), as follows.

Since

\[
\frac{(1 - x)^6}{1 + x + x^2} = 28 - 34x + 21x^2 - 7x^3 + x^4 - \frac{27}{1 + x + x^2},
\]

by induction we have

\[
\frac{(1 - x)^{6k}}{1 + x + x^2} = Q_k(x) + \frac{(-3)^{3k}}{1 + x + x^2},
\]

where \( Q_k(x) \) is the polynomial defined by the recursion

\[
Q_0(x) = 0,
\]

(44)

\[
Q_{k+1}(x) = (1 - x)^6 Q_k(x) + (-3)^{3k} (28 - 34x + 21x^2 - 7x^3 + x^4) \quad \text{for } k \geq 0.
\]

From (42), (43), (41), for \( m = 6k \) we obtain

\[
I_{n,m,3} = \int_0^1 (2 + x) \left( Q_k(x) + \frac{(-3)^{3k}}{1 + x + x^2} \right) \left( \sum_{\nu=1}^{\infty} - \sum_{\nu=1}^{n} \right) x^{3\nu-1} \, dx
\]

\[
= (-3)^{3k} \left( \gamma - \int_0^1 \frac{2 + x}{1 + x + x^2} \sum_{\nu=1}^{n} x^{3\nu-1} \, dx \right)
\]

\[
+ \int_0^1 (2 + x) Q_k(x) \sum_{\nu=n+1}^{\infty} x^{3\nu-1} \, dx.
\]

By division, we find that

\[
\frac{(2 + x)x^{3\nu-1}}{1 + x + x^2} = 1 + x - \sum_{\mu=1}^{3\nu-1-1} (2x^{3\mu-1} - x^{3\mu} - x^{3\mu+1}) - \frac{1 + 2x}{1 + x + x^2}
\]

for \( \nu \geq 1 \), where the sum is zero if \( \nu = 1 \). Therefore,

\[
\int_0^1 \frac{2 + x}{1 + x + x^2} \sum_{\nu=1}^{n} x^{3\nu-1} = n \left( \frac{3}{2} - \log 3 \right) - \sum_{\nu=2}^{n} \sum_{\mu=1}^{3\nu-1} \left( \frac{2}{3\mu} - \frac{1}{3\mu+1} - \frac{1}{3\mu+2} \right)
\]

(46)
for $n \geq 1$, where the double sum vanishes if $n = 1$.

To evaluate the last integral in (45), assume that $m = 6k < 3^{n+1}$ and let $a_{0,k}, \ldots, a_{6k-1,k} \in \mathbb{Z}$ denote the coefficients in the polynomial

$$
\sum_{j=0}^{6k-1} a_{j,k}x^j = (2 + x)Q_k(x).
$$

(47)

Noting that $a_{0,k} = (-3)^{3k}56k$ by (44), and using Lemma 1, we have

$$
\int_0^1 (2 + x)Q_k(x) \sum_{\nu=n+1}^\infty x^{3^\nu-1} \, dx = \sum_{j=0}^{6k-1} a_{j,k} \sum_{\nu=n+1}^\infty \frac{1}{3^\nu + j}
$$

$$
= (-1)^k \frac{28k}{3^{n-3k}} + \sum_{j=1}^{6k-1} \frac{a_{j,k}}{j} \ln_3 \left(1 + \frac{j}{3^n}\right).
$$

(48)

Formulas (45), (46), (48) imply the following base 3 version of Lemma 4.

**Lemma 8.** If $m < 3^{n+1}$ is a multiple of 6, say $m = 6k$, then

$$
I_{n,m,3} = (-3)^{3k} \gamma + L_{n,m,3} - A_{n,m,3},
$$

(49)

where

$$
L_{n,m,3} = (-3)^{3k} n \log 3 + \sum_{j=1}^{m-1} \frac{a_{j,k}}{j} \ln_3 \left(1 + \frac{j}{3^n}\right)
$$

and

$$
A_{n,m,3} = (-3)^{3k} \left(\frac{3n}{2} - \frac{28k}{3^n} - \sum_{\nu=2}^n \sum_{\mu=1}^{3^{\nu-1}-1} \left(\frac{2}{3\mu} - \frac{1}{3\mu + 1} - \frac{1}{3\mu + 2}\right)\right)
$$

satisfy

$$
d_m L_{n,m,3} \in \mathbb{Z} \log 3 + \mathbb{Z} \ln_3 \left(1 + \frac{1}{3^n}\right) + \mathbb{Z} \ln_3 \left(1 + \frac{2}{3^n}\right) + \cdots + \mathbb{Z} \ln_3 \left(1 + \frac{m-1}{3^n}\right)
$$

and $d_m A_{n,m,3} \in \mathbb{Z}$, and the $a_{j,k}$ are integers determined by (47) and (44).

**Lemma 9.** We have the bounds

$$
\frac{1}{40m + 90 \left(\frac{3^3}{4^4}\right)^m} < I_{n,m,3} < 3 \cdot \left(\frac{3^3}{4^4}\right)^m \quad \text{for} \quad m = 3^n - 3 \geq 0
$$

(50)

and the limit

$$
\lim_{n \to \infty} \frac{\log I_{n,3^n-3,3}}{3^n} = 3 \log 3 - 4 \log 4 = -2.24934057 \ldots
$$

(51)
Proof. Following the proof of Lemma 7, with \( m = 3^n - 3 \) we have

\[
I_{n,m,3} > \int_0^1 \frac{2 + x}{1 + x + x^2} (1 - x)^m x^{3^{n+1} - 1} \, dx > \int_0^1 x^{3m+8} (1 - x)^m \, dx
\]

\[
= \frac{(3m+8)! m!}{(4m+9)!} > \frac{(3/4)^8}{4m+9} \cdot \left( \frac{4m}{m} \right)^{-1} > \frac{1}{40m+90} \left( \frac{3^3}{4^4} \right)^m
\]

and

\[
I_{n,m,3} = \int_0^1 \frac{2 + x}{1 + x + x^2} (1 - x)^m x^{3m+8} \sum_{\nu=n+1}^{\infty} x^{3\nu-3^{n+1}} \, dx
\]

\[
< \int_0^1 x^{3m+8} (1 - x)^m \frac{2 + x}{1 + x + x^2} \left( 1 + \sum_{\nu=1}^{\infty} x^{3\nu-1} \right) \, dx
\]

\[
< \left( \frac{3^3}{4^4} \right)^m (2 + \gamma) < 3 \cdot \left( \frac{3^3}{4^4} \right)^m.
\]

This proves the lemma. \( \square \)

**Theorem 6.** We have the asymptotic formula for Euler’s constant

\[
\gamma = (-1)^{n-1} \frac{A_{n,3^n-3} - L_{n,3^n-3}}{3(3^n-3)/2} + O \left( \left( \frac{5/2}{64} \right)^{3^n} \right) \quad \text{as} \ n \to \infty.
\]

Proof. Let \( m = 6k = 3^n - 3 \) for \( n > 0 \). Then \( (-1)^{3k} = (-1)^{n-1} \) and the formula follows from (49) and (51). \( \square \)

**Theorem 7.** Euler’s constant is rational if and only if the fractional part of \( d_{3^n} L_{n,3^n-3} \) equals \( d_{3^n} I_{n,3^n-3} \) for all large \( n \).

Proof. Multiply (49) by \( d_{3^n} \) and write the result as

\[
d_{3^n} L_{n,m,3} = d_{3^n} I_{n,m,3} + d_{3^n} (A_{n,m,3} - (-3)^k \gamma).
\]

By the Prime Number Theorem and Lemma 9, for \( m = 3^n - 3 \) and any \( \varepsilon > 0 \) we have

\[
d_{3^n} I_{n,m,3} = O \left( (3^3 e^{1+\varepsilon}/4^4)^m \right) \quad \text{as} \ n \to \infty.
\]

Thus \( 0 < d_{3^n} I_{n,3^n-3} < 1 \) for all large \( n \). Since \( d_{3^n} A_{n,3^n-3} \in \mathbb{Z} \), the theorem follows. \( \square \)

**Theorem 8.** If the inequality

\[
\left\{ d_{3^n} L_{n,3^n-3,3} \right\} > 3 \cdot \left( \frac{3}{4} \right)^{3^n+1}
\]

holds infinitely often, then \( \gamma \) is irrational. In fact, the inequality for a given \( n > 0 \) implies that no divisor of \( 3^{3^n} d_{3^n} \) can be a denominator of \( \gamma \).

Proof. The inequalities \( d_{3N} \leq d_{4N} < 8^{2N} \) and the upper bound in (50) imply that

\[
d_{3^n} I_{n,3^n-3,3} < 8^{2 \cdot 3^{n-1}} \cdot 3 \cdot \left( \frac{3^3}{4^4} \right)^{3^n} = 3 \cdot \left( \frac{3}{4} \right)^{3^{n+1}}
\]

for \( n \geq 1 \). Using Lemma 8, this proves the second part of the theorem, which implies the first. \( \square \)
9. Base $q$ asymptotic formulas for Euler’s constant

Denoting by $F_q(x)$, $G_q(x)$ the polynomials

$$F_q(x) = 1 + x + x^2 + \cdots + x^{q-1} = \frac{1 - x^q}{1 - x},$$

$$G_q(x) = (q - 1) + (q - 2)x + \cdots + x^{q-2} = \frac{q - F_q(x)}{1 - x},$$

we may write the integral (40) as

$$I'_{n,m,q} = \int_0^1 \frac{G_q(x)}{F_q(x)} (q - F_q(x)) \left( \sum_{\nu=1}^n - \sum_{\nu=1}^n \right) x^{\nu-1} \, dx.$$  

Expanding the binomial and using Ramanujan’s formula $I_{0,0,q} = \gamma$, we obtain

$$I'_{n,m,q} = q^m \left( \gamma - \int_0^1 \frac{G_q(x)}{F_q(x)} \sum_{\nu=1}^n x^{\nu-1} \, dx \right)$$

$$+ \int_0^1 G_q(x) \sum_{j=1}^m (-1)^j \binom{m}{j} q^{m-j} F_q(x)^{j-1} \sum_{\nu=n+1}^\infty x^{\nu-1} \, dx.$$ 

In the first integrand, we find by dividing that

$$\frac{G_q(x)x^{\nu-1}}{F_q(x)} = F_{q-1}(x) + \sum_{\mu=1}^{q^{\nu-1}-1} (F_q(x) - q)x^{\mu-1} - \frac{F_q'(x)}{F_q(x)}$$

for $\nu \geq 1$, the summation being zero if $\nu = 1$. Therefore,

$$\int_0^1 \frac{G_q(x)}{F_q(x)} \sum_{\nu=1}^n x^{\nu-1} \, dx = n \left( \sum_{\nu=2}^{q-1} \frac{1}{\nu} - \log q \right) - \sum_{\nu=2}^n \sum_{\mu=1}^{q^{\nu-1}-1} \left( \frac{q - 1}{q\mu} - \sum_{\lambda=1}^{q-1} \frac{1}{q\mu + \lambda} \right)$$

for $n \geq 1$, where the double summation vanishes if $n = 1$.

In the second integrand in (54), let $b_{0,m,q}, \ldots, b_{m(q-1)-1,m,q} \in \mathbb{Z}$ denote the coefficients of the polynomial

$$\sum_{l=0}^{m(q-1)-1} b_{l,m,q} x^l = G_q(x) \sum_{j=1}^m (-1)^j \binom{m}{j} q^{m-j} F_q(x)^{j-1}.$$ 

Note that $b_{0,m,q} = (q - 1)((q - 1)^m - q^m)$ by (52). If $m \leq q^{n+1}/(q - 1)$, then by Lemma 1 the last integral in (54) equals

$$\int_0^1 \sum_{l=0}^{m(q-1)-1} b_{l,m,q} x^l \sum_{\nu=n+1}^\infty x^{\nu-1} \, dx = \sum_{l=0}^{m(q-1)-1} b_{l,m,q} \sum_{\nu=n+1}^\infty \frac{1}{q^{\nu+l}}$$

$$= \frac{(q - 1)^m - q^m}{q^n} + \sum_{l=1}^{m(q-1)-1} \frac{b_{l,m,q}}{l} \ln_q \left( 1 + \frac{l}{q^n} \right).$$

Equations (54)–(57) imply the following generalization of Lemma 4 to base $q$. 
Lemma 10. If \( m \leq q^{n+1}/(q-1) \), then

\[
I_{n,m,q}' = q^m \gamma + L_{n,m,q}' - A_{n,m,q}',
\]

where

\[
L_{n,m,q}' = q^m n \log q + \sum_{l=1}^{(m(q-1))^{-1}} b_{l,m,q} \ln \left(1 + \frac{l}{q^n}\right)
\]

and

\[
A_{n,m,q}' = \frac{q^m - (q-1)^m}{q^n} + q^m \left( \sum_{\mu=1}^{q-1} n - \sum_{\nu=2}^{n} \sum_{\mu=1}^{q^n-1-1} \left( \frac{q-1}{q\mu} - \sum_{\lambda=1}^{q-1} \frac{1}{q\mu + \lambda} \right) \right)
\]

satisfy

\[
d_{m(q-1)} L_{n,m,q}' \in \mathbb{Z} \log q + \mathbb{Z} \ln \left(1 + \frac{1}{q^n}\right) + \mathbb{Z} \ln \left(1 + \frac{2}{q^n}\right) + \cdots
\]

\[
+ \mathbb{Z} \ln \left(1 + \frac{m(q-1)-1}{q^n}\right)
\]

and \( d_q A_{n,m,q}' \in \mathbb{Z} \), and the \( b_{l,m,q} \) are integers determined by (56) and (52).

To optimize the asymptotics of \( I_{n,m,q}'/q^m \) in the range of Lemma 10, and thus obtain the smallest error term in our formula for \( \gamma \), we choose \( m = (q^n - 1)/(q - 1) \) in the remainder of this section. Let \( f_q(x) \) denote the polynomial

\[
f_q(x) = (q - F_q(x)) x^{q(q-1)} = (q - 1) x^{q(q-1)} - \sum_{\nu=1}^{q-1} x^{q(q-1)+\nu}.
\]

Lemma 11. If \( q \geq 2 \), there exists \( x_q \) between 0 and 1 such that

\[
f_q(x_q) = \max_{0 \leq x \leq 1} f_q(x).
\]

Moreover, we have the asymptotic formula

\[
\int_0^1 f_q(x)^m x^{q-1} \, dx = f_q(x_q)^m (1 + o(1)) \quad \text{as } m \to \infty
\]

and the bounds

\[
\frac{1}{(q+1)e} < f_q(x_q) < \frac{1}{2e}.
\]

Proof. Since

\[
\frac{f_q'(x)}{x^{q(q-1)-1}} = q(q-1)^2 - \sum_{\nu=1}^{q-1} (q(q-1) + \nu)x^\nu
\]
is decreasing for \( x \geq 0 \), positive at \( x = 0 \) and negative at \( x = 1 \), it follows that \( f_q'(x) \) vanishes at exactly one point \( x = x_q \) in the open interval \((0, 1)\). Then (59) holds with this choice of \( x_q \), since \( f_q(0) = f_q(1) = 0 \) and \( f_q(x) > 0 \) for \( 0 < x < 1 \). Differentiating (62), we substitute \( f_q'(x_q) = 0 \) and find that \( f_q''(x_q) < 0 \); by Laplace’s method (see, e.g., [5], Section 4.2) we obtain (60).

Since 

\[
q - F_q(x) = (q - 1) \left(1 - \frac{x + x^2 + \cdots + x^{q-1}}{q - 1}\right),
\]

the inequalities

\[
(x \cdot x^2 \cdots x^{q-1})^{1/(q-1)} \leq \frac{x + x^2 + \cdots + x^{q-1}}{q - 1} \leq x \quad \text{for } 0 \leq x \leq 1
\]
yield

\[
(q - 1)(1 - x)x^{q(q-1)} \leq f_q(x) \leq (q - 1)(1 - x^{q/2})x^{q(q-1)}.
\]

Using the relations

\[
\frac{1}{(r + 1)e} < \max_{0 \leq y \leq 1} ((1 - y)y^r) = \frac{r^r}{(r + 1)^{r+1}} < \frac{1}{re} \quad \text{for } r > 0,
\]

we set \( y = x, r = q(q-1) \) and deduce the lower bound in (61), then set \( y = x^{q/2}, r = 2(q-1) \) and obtain the upper bound. \( \Box \)

The next result with \( q = 2 \) is comparable to Lemma 7 with \( m = 2^n - 1 \).

**Lemma 12.** For \( q \geq 2 \), we have the bounds

\[
0 < I'_{n,m,q} < \frac{q}{(2e)^m} \quad \text{with} \quad m = \frac{q^n - 1}{q - 1}
\]

and

\[
-1 - \log(q + 1) < \lim_{n \to \infty} \frac{\log I'_{n,(q^n-1)/(q-1),q}}{(q^n - 1)/(q - 1)} < -1 - \log 2.
\]

**Proof.** Let \( m = (q^n - 1)/(q - 1) \), so that \( x^{q^n+1-1} = x^{(mq+1)(q-1)} \). Using (53), (59) and Ramanujan’s formula \( I'_{0,0,q} = \gamma \), we have

\[
0 < I'_{n,m,q} = \int_0^1 \frac{G_q(x)}{F_q(x)} (q - F_q(x))^m x^{(mq+1)(q-1)} \sum_{\nu=n+1}^{\infty} x^{q^\nu - q^n+1} \, dx
\]

\[
< \int_0^1 f_q(x)^m \frac{G_q(x)}{F_q(x)} \left(1 + \sum_{\nu=1}^{\infty} x^{q^\nu - 1}\right) \, dx
\]

\[
\leq f_q(x_q)^m(c_q + \gamma), \quad \text{where} \quad c_q = \int_0^1 \frac{G_q(x)}{F_q(x)} \, dx,
\]
and (63) follows using $c_q + \gamma < (q - 1) + \gamma < q$ and the upper bound in (61).

We also have

\[ I'_{n,m,q} > \int_0^1 \frac{G_q(x)}{F_q(x)} (q - F_q(x))^m x^{(m+1)(q-1)} \, dx > \frac{q-1}{2} \int_0^1 f(x)^m x^{q-1} \, dx. \]

From (65), (66), (60) we deduce that

\[ \lim_{n \to \infty} \log I'_{n,(q^n-1)/(q-1),q}^{(q^n-1)/(q-1)} = \log f(x_q), \]

and (61) gives (64). \(\Box\)

**Remark.** The exact value of the constant $c_q$ is $-\psi(1/q) - \log q - \gamma$, as another formula of Ramanujan shows (see [2], Proposition 2.1).

**Theorem 9** (Base $q$ Asymptotic Formula for $\gamma$). If $q \geq 2$ and $m = (q^n - 1)/(q-1)$, then

\[ \gamma = \frac{A'_{n,m,q} - L'_{n,m,q}}{q^m} + \delta_{m,q} \quad \text{with} \quad 0 < \delta_{m,q} < \frac{q}{(2eq)^m}. \]

**Proof.** Set $\delta_{m,q} = I'_{n,m,q}/q^m$ and use (58) and (63). \(\Box\)

**Remark.** For $q = 2$, one could use $I'_{m,n,q}$ with $m = (q^n - 1)/(q-1)$ to derive a rationality criterion and an irrationality test for $\gamma$ (but they are already contained in Theorems 2 and 3 as the case $m = 2^n - 1$, since $I'_{n,2^n-1,2} = I_{n,2^n-1}$ and $L'_{n,2^n-1,2} = L_{n,2^n-1}$). For $q > 2$, this is not possible, the required inequality $d_q^n I'_{n,m,q} < 1$ not being valid, since

\[ \lim_{n \to \infty} \frac{\log d_q^n}{q^n} < 1 \]

(from the Prime Number Theorem) and the following observation imply that $d_q^n I'_{n,m,q}$ tends to infinity with $n$.

**Lemma 13.** If $q \geq 3$, then

\[ \lim_{n \to \infty} \frac{\log I'_{n,(q^n-1)/(q-1),q}}{q^n} > -1. \]

**Proof.** By (64), we have

\[ \lim_{n \to \infty} \frac{\log I'_{n,(q^n-1)/(q-1),q}}{q^n} = \frac{1}{q-1} \lim_{n \to \infty} \frac{\log I'_{n,(q^n-1)/(q-1),q}}{(q^n-1)/(q-1)} \]

\[ > 1 + \log(q+1) \quad \text{for} \quad q \geq 2. \]
Since the lower bound is increasing, and exceeds $-1$ when $q = 4$, this proves (68) for $q > 3$. For $q = 3$, we use (67) to replace the bound by the exact value

$$\frac{1}{2} \log f_3(x_3) = -0.90997390 \ldots > -1,$$

where $x_3 = 0.86304075 \ldots$ is the positive root of $12 - 7x - 8x^2$, which is the polynomial (62) with $q = 3$. □

**Remark.** Alternatively, for $q$ sufficiently large (68) follows from (67) and the formula

$$\lim_{q \to \infty} f_q(x_q) = \frac{1}{2e},$$

which in turn follows from the upper bound in (61) and the fact that the limit, as $q \to \infty$, of the lower bound in

$$\max_{0 \leq x \leq 1} f_q(x) \geq f_q(x)|_{x = (1 - 1/q)^{1/q}} = \left(q - \frac{1/q}{1 - (1 - 1/q)^{1/q}}\right) \cdot \left(1 - \frac{1}{q}\right)^{q-1}$$

is $1/(2e)$.

10. Concluding remarks

Any polynomial $P_m(x) \in \mathbb{Z}[x]$ can be used to generalize integral (18) to base $q$ as follows:

$$\int_0^1 \left(\frac{q}{1 - x^q} - \frac{1}{1 - x}\right) P_m(x) \sum_{\nu = n+1}^{\infty} x^{q\nu-1} \, dx. \tag{69}$$

Integrals (37) and (40) are particular cases of construction (69). The choice of $P_m(x)$, the “damping factor,” is based on the requirements:

(i) to ensure the appearance of $\gamma$, and no other $\gamma_{j, q}$ from (39), in the corresponding linear forms;

(ii) to produce sufficiently good asymptotics (upper estimates).

The choice $P_m(x) = (1 - x)^m$ gives the best possible answer (38) to feature (ii); that is why we use it in Sections 2–4 for $q = 2$, and Section 8 for $q = 3$. The following lemma and the arguments in Section 7 show that $(1 - x)^m$ cannot be used for $q > 3$, because of requirement (i).

**Lemma 14.** If $q > 3$, the equality

$$\left(1 - x\right)^m = C + Q(x) \cdot \frac{1 - x^q}{1 - x} \tag{70}$$

is impossible for any $m > 0$, constant $C$ and polynomial $Q(x)$. 

Proof. For $q > 1$, equality (70) implies the system of equalities $(1 - \epsilon^j)^m = C$ for $j = 1, 2, \ldots, q - 1$, where $\epsilon$ is a primitive $q$-th root of unity. In particular, the value of $|1 - \epsilon^j|$ is the same for all $j = 1, 2, \ldots, q - 1$, which is possible only when $q = 2$ or $q = 3$. □

One can avoid this difficulty by introducing linear combinations of powers of $1 - x$. More precisely, there exist (integer) constants $c_{m,0}, c_{m,1}, \ldots, c_{m,q-1}$ such that

$$c_{m,0}(1 - x)^m + c_{m,1}(1 - x)^{m-1} + \cdots + c_{m,q-1}(1 - x)^{m-q+1} = C + Q(x) \cdot \frac{1 - x^q}{1 - x}.$$ 

However, we cannot choose the left-hand side as the polynomial $P_m(x)$, because the constants grow so much faster than $m$ that requirement (ii) fails.

References

1. W. N. Bailey, Generalized Hypergeometric Series, Cambridge Math. Tracts, vol. 32, Cambridge Univ. Press, Cambridge, 1935; 2nd reprinted edition, Stechert-Hafner, New York–London, 1964.
2. B. C. Berndt and D. C. Bowman, Ramanujan’s short unpublished manuscript on integrals and series related to Euler’s constant, Constructive, Experimental, and Nonlinear Analysis (Limoges, 1999) (M. Thera, ed.), CMS Conf. Proc., vol. 27, Amer. Math. Soc., Providence, RI, 2000, pp. 19–27.
3. P. Borwein, On the irrationality of $\sum \frac{1}{q^{n+r}}$, J. Number Theory 37 (1991), 253–259.
4. T. Bromwich, An Introduction to the Theory of Infinite Series, 2nd edition, Macmillan, London, 1926.
5. N. G. de Bruijn, Asymptotic Methods in Analysis, Dover Publications, New York, 1981.
6. R. W. Gosper, Jr., Personal communication (7 May 2002).
7. J. B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, Illinois J. Math. 6 (1962), 64–94.
8. P. Sebah, Personal communication (17 December 2002).
9. J. Sondow, Criteria for irrationality of Euler’s constant, Proc. Amer. Math. Soc. (to appear).
10. J. Sondow, A hypergeometric approach, via linear forms involving logarithms, to irrationality criteria for Euler’s constant, CRM Conference Proceedings of CNTA 7 (May 2002) (submitted).
11. G. Vacca, A new series for the Eulerian constant $\gamma = -\sum \frac{1}{n}$, Quart. J. Pure Appl. Math. 41 (1909–10), 363–364.

209 West 97th Street
New York, NY 10025 USA
URL: http://home.earthlink.net/~jsondow/
E-mail address: jsondow@alumni.princeton.edu

Moscow Lomonosov State University
Department of Mechanics and Mathematics
Vorobiovy Gory, GSP-2, Moscow 119992 RUSSIA
URL: http://wain.mi.ras.ru/index.html
E-mail address: wadim@ips.ras.ru