Quadratic Transportation Cost Inequality For Scalar Stochastic Conservation Laws

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Abstract: In this paper, we established a quadratic transportation cost inequality for scalar stochastic conservation laws driven by multiplicative noise. The doubling variables method plays an important role.

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1 Introduction

Fix $T > 0$ and let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in [0, T)}, \{\beta_k(t)\}_{t \in [0, T]} \in \mathbb{N})$ be a stochastic basis. Without loss of generality, here the filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$ is assumed to be complete and $\{\beta_k(t)\}_{t \in [0, T]} (k \in \mathbb{N})$ are one-dimensional i.i.d. real-valued $\{\mathcal{F}_t\}_{t \in [0, T]}$-Wiener processes. The symbol $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$. For fixed $N \in \mathbb{N}$, let $\mathbb{T}^N \subset \mathbb{R}^N$ be the $N$-dimensional torus with the periodic length to be 1. Consider the following Cauchy problem for the scalar conservation laws with stochastic forcing

\[
\begin{aligned}
\frac{du(t, x)}{dt} + \text{div}A(u(t, x))dt &= \sum_{k \geq 1} g_k(x, u(t, x))d\beta_k(t) \quad \text{in } \mathbb{T}^N \times (0, T], \\
u(\cdot, 0) &= \eta(\cdot) \quad \text{on } \mathbb{T}^N.
\end{aligned}
\]

where $u : (\omega, x, t) \in \Omega \times \mathbb{T}^N \times [0, T] \mapsto u(\omega, x, t) := u(x, t) \in \mathbb{R}$ is a random field, the flux function $A : \mathbb{R} \to \mathbb{R}^N$ and the coefficient $g_k(\cdot, \cdot)$ are measurable and fulfill certain conditions (see Section 2 in below). Moreover, the initial value $\eta \in L^\infty(\mathbb{T}^N)$ is a deterministic function.

The purpose of this paper is to establish the quadratic transportation cost inequality for the solution of the stochastic conservation laws. Let us recall the relevant concepts. Let $(X, d)$ be a metric space equipped with the Borel $\sigma$-field $\mathcal{B}$. Let $\mu, \nu$ be two Borel probability measures on the metric space $(X, d)$. The $L^p$-Wasserstein distance between $\mu$ and $\nu$ is defined as

\[
W_p(\nu, \mu) := \left[ \inf \int_{X \times X} d(x, y)^p \pi(dx, dy) \right]^{\frac{1}{p}},
\]
where the infimum is taken over all probability measures $\pi$ on the product space $X \times X$ with marginals $\mu$ and $\nu$. Recall that the Kullback information (or relative entropy) of $\nu$ with respect to $\mu$ is defined by

$$H(\nu|\mu) := \int_X \log \left( \frac{d\nu}{d\mu} \right) d\nu,$$

if $\nu$ is absolutely continuous with respect to $\mu$, and $+\infty$ otherwise.

**Definition 1.1.** We say that a measure $\mu$ satisfies the $L^p$-transportation cost inequality if there exists a constant $C > 0$ such that for all probability measures $\nu$,

$$W_p(\nu, \mu) \leq \sqrt{2CH(\nu|\mu)}.$$  \hfill (1.2)

The case $p = 2$ is referred to as the quadratic transportation cost inequality.

Transportation cost inequalities have close connections with other functional inequalities, e.g. Poincare inequalities, logarithmic Sobolev inequalities, and they also imply the concentration of measure phenomenon.

For a measurable subset $A \subset X$ and $r > 0$, we denote by $A_r$ the $r$-neighborhood of $A$, namely $A_r = \{x : d(x,A) < r\}$. We say that $\mu$ has normal concentration (or Gaussian tail estimates) on $(X,d)$ if there are constants $C, c > 0$ such that for every $r > 0$ and every Borel subset $A$ with $\mu(A) \geq \frac{1}{2}$,

$$1 - \mu(A_r) \leq Ce^{-cr^2}.$$  \hfill (1.3)

The fact that the $L^1$-transportation cost inequality implies normal concentration was obtained in [M1, M2] by Marton and in [T1, T2, T3] by Talagrand. An elegant, simple proof of this fact is also contained in the book [Le]. The connection of the quadratic transportation cost inequality with other functional inequalities was studied in [OV] by Otto and Villani (see also [Le]). For other related interesting works, we refer the reader to [GRS], [LW], [PS], [PS1].

In the past decades, many people established quadratic transportation cost inequalities for various kinds of interesting measures. Let us mention several papers which are relevant to our work. The transportation cost inequalities for stochastic differential equations were obtained by H. Djellout, A. Guillin and L. Wu in [DGW]. The measure concentration for multidimensional diffusion processes with reflecting boundary conditions was considered by S. Pal in [P]. The quadratic transportation cost inequalities for stochastic partial differential equations (SPDEs) driven by Gaussian noise which is white in time and colored in space were obtained by A. S. Ustunel in [U]. In the article [BH], the authors obtained the quadratic transportation cost inequality under the $L^2$-distance for stochastic heat equations. In [KS], the authors established the quadratic transportation cost inequality for more general stochastic partial differential equations (SPDEs) under the $L^2$-distance and under the uniform distance for the case of additive noise. In [SZ], the authors obtained the quadratic transportation cost inequality for stochastic heat equations equations driven by multiplicative space-time white noise under the uniform distance.
On the other hand, both deterministic \((g_t = 0)\) and stochastic conservation laws have been studied extensively by many people. Conservation law is fundamental to our understanding of the space-time evolution laws of interesting physical quantities. For more background on this model, we refer the readers to the monograph [Da], the work of Ammar, Wittbold and Carrillo [AWC] and references therein. As we know, the Cauchy problem for the deterministic first-order PDE (1.1) does not admit any (global) smooth solutions, but there exist infinitely many weak solutions to the deterministic Cauchy problem. To solve the problem of non-uniqueness, an additional entropy condition was added to identify the physical weak solution. Under this condition, the notion of entropy solutions for the deterministic first-order scalar conservation laws was introduced by Kružkov [Kr-1, Kr-2]. The kinetic formulation of weak entropy solution of the Cauchy problem for a general multi-dimensional scalar conservation laws (also called the kinetic system), was derived by Lions, Perthame and Tadmor in [LPT].

In recent years, the stochastic conservation law has been developed rapidly. We refer the reader to the references [K], [VW], [FN], [DWZZ] etc. We particularly mention the paper [DV-1] in which the authors proved the existence and uniqueness of kinetic solution to the Cauchy problem for (1.1) in any dimension. In addition, the long-time behavior of the first-order scalar conservation laws has been studied in the paper [DV-2]. Recently, combining techniques used in the context of kinetic solutions as well as new results on large deviations, Dong et al. [DWZZ] established Freidlin-Wentzell’s type large deviation principles (LDP) for the kinetic solution to the scalar stochastic conservative laws.

The purpose of this paper is to establish the quadratic transportation cost inequality for the kinetic solution of the scalar stochastic conservation laws, which in particular implies the concentration phenomenon of the law of the solution. To our knowledge, the present paper is the first work towards proving the transportation cost inequality directly for the kinetic solutions to the scalar stochastic conservation laws. Due to the lack of viscous term, the kinetic solutions of (1.1) are living in a rather irregular space \(L^1([0, T], L^1(\mathbb{T}^N))\), we will use the doubling variables method as in the work [DV-1]. Differ from [DV-1], we need to deal with the martingale term carefully to derive a proper bound, which can ensure the application of Gronwall inequality to get an appropriate norm estimation (see (3.45)). As an important part of the proof, we also need to make some higher order estimates of the error term than [DV-1], which is nontrivial and completely new.

This paper is organized as follows. In Section 2, we lay out the precise setup for the stochastic conservation law and recall some of the known results. Section 3 is devoted to the proof of the transportation cost inequality.

## 2 Framework

In this section, we will lay out the precise setup for the stochastic conservation law and recall some results which will be used later.
2.1 Kinetic solution

We will follow closely the framework of [DV-1]. Let \( \| \cdot \|_{L^p} \) denote the norm of usual Lebesgue space \( L^p(\mathbb{T}^N) \) for \( p \in [1, \infty) \). In particular, set \( H = L^2(\mathbb{T}^N) \) with the corresponding norm \( \| \cdot \|_H \). \( C_b \) represents the space of bounded, continuous functions and \( C^1_b \) stands for the space of bounded, continuously differentiable functions having bounded first order derivative. Define the function \( f(x, t, \xi) := I_{u(x, t) > \xi} \), which is the characteristic function of the subgraph of \( u \). We write \( f := I_{u > \xi} \) for short. Moreover, denote by the brackets \( \langle \cdot, \cdot \rangle \) the duality between \( C^\infty_c(\mathbb{T}^N \times \mathbb{R}) \) and the space of distributions over \( \mathbb{T}^N \times \mathbb{R} \). In what follows, with a slight abuse of the notation \( \langle \cdot, \cdot \rangle \), we denote the following integral by

\[
\langle F, G \rangle := \int_{\mathbb{T}^N} \int_{\mathbb{R}} F(x, \xi)G(x, \xi)dx d\xi, \quad F \in L^p(\mathbb{T}^N \times \mathbb{R}), G \in L^q(\mathbb{T}^N \times \mathbb{R}),
\]

where \( 1 \leq p \leq +\infty, q := \frac{N}{p-1} \) is the conjugate exponent of \( p \). In particular, when \( p = 1 \), we set \( q = \infty \) by convention. For a measure \( m \) on the Borel measurable space \( \mathbb{T}^N \times [0, T] \times \mathbb{R} \), the shorthand \( m(\phi) \) is defined by

\[
m(\phi) := \langle m, \phi \rangle([0, T]) := \int_{\mathbb{T}^N \times [0, T] \times \mathbb{R}} \phi(x, t, \xi)dm(x, t, \xi), \quad \phi \in C_b(\mathbb{T}^N \times [0, T] \times \mathbb{R}).
\]

In the sequel, the notation \( a \leq b \) for \( a, b \in \mathbb{R} \) means that \( a \leq \mathcal{D}b \) for some constant \( \mathcal{D} > 0 \) independent of any parameters.

2.2 Hypotheses

For the flux function \( A \) and the coefficients of (1.1), we assume that

**Hypothesis H** The flux function \( A \) belongs to \( C^2(\mathbb{R}; \mathbb{R}^N) \) and its derivative \( a := A' \) is of polynomial growth with degree \( q_0 > 1 \). That is, there exists a constant \( C(q_0) \geq 0 \) such that

\[
|a(\xi)| \leq C(q_0)(1 + |\xi|^{q_0}), \quad |a(\xi) - a(\zeta)| \leq \Upsilon(\xi, \zeta)|\xi - \zeta|,
\]

where \( \Upsilon(\xi, \zeta) := C(q_0)(1 + |\xi|^{q_0-1} + |\zeta|^{q_0-1}). \)

Moreover, we assume that \( g_k \in C(\mathbb{T}^N \times \mathbb{R}) \) satisfies the following bounds

\[
|g_k(x, u)| \leq C_k^0, \quad \sum_{k \geq 1} |C_k^0|^2 \leq D_0, \quad \sum_{k \geq 1} |C_k^1|^2 \leq D_1,
\]

\[
|g_k(x, u) - g_k(y, v)| \leq C_k^1(|x - y| + |u - v|), \quad \sum_{k \geq 1} |C_k^1|^2 \leq \frac{D_1}{2},
\]

for \( x, y \in \mathbb{T}^N, u, v \in \mathbb{R} \), where \( C_k^0, C_k^1, D_0, D_1 \) are positive constants.

The hypothesis H implies that

\[
G^2(x, u) := \sum_{k \geq 1} |g_k(x, u)|^2 \leq D_0, \quad \sum_{k \geq 1} |g_k(x, u) - g_k(y, v)|^2 \leq D_1(|x - y|^2 + |u - v|^2).
\]
2.3 Kinetic solution

Let us recall the notion of a kinetic solution to equation (1.1) from [DV -1].

**Definition 2.1.** (Kinetic measure) A map \( m \) from \( \Omega \) to the set of non-negative, finite measures over \( \mathbb{T}^N \times [0, T] \times \mathbb{R} \) is said to be a kinetic measure, if

1. \( m \) is measurable, that is, for each \( \phi \in C_b(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \), \( \langle m, \phi \rangle : \Omega \to \mathbb{R} \) is measurable,

2. \( m \) vanishes for large \( \xi \), i.e.,

\[
\lim_{R \to +\infty} \mathbb{B}[m(\mathbb{T}^N \times [0, T] \times B_R^c)] = 0,
\]

where \( B_R^c := \{ \xi \in \mathbb{R}, |\xi| \geq R \} \).

3. for every \( \phi \in C_b(\mathbb{T}^N \times \mathbb{R}) \), the process

\[
(\omega, t) \in \Omega \times [0, T] \mapsto \langle m, \phi \rangle([0, t]) := \int_{\mathbb{T}^N \times [0, t] \times \mathbb{R}} \phi(x, \xi) dm(x, s, \xi) \in \mathbb{R}
\]

is predictable.

**Remark 1.** For any \( \phi \in C_b(\mathbb{T}^N \times \mathbb{R}) \) and kinetic measure \( m \), define \( A_t := \langle m, \phi \rangle([0, t]) \), then a.s., \( t \mapsto A_t \) is a right continuous function of finite variation. Moreover, the function \( A \) has left limits at any point \( t \in (0, T] \). We write \( A_t^+ = \lim_{s \downarrow t} A_s \) and set \( A_0^- = 0 \). As a result, \( A := \langle m, \phi \rangle([0, t]) \), which is càglàd (left continuous with right limits).

**Definition 2.2.** (Kinetic solution) Let \( \eta \in L^\infty(\mathbb{T}^N) \). A measurable function \( u : \mathbb{T}^N \times [0, T] \times \Omega \to \mathbb{R} \) is called a kinetic solution to (1.1) with initial datum \( \eta \), if

1. \( (u(t))_{t \in [0, T]} \) is predictable,

2. for any \( p \geq 1 \), there exists \( C_p \geq 0 \) such that

\[
\mathbb{E}\left( \text{ess sup}_{0 \leq t \leq T} \| u(t) \|_{L^p(\mathbb{T}^N)}^p \right) \leq C_p,
\]

3. there exists a kinetic measure \( m \) such that \( f := I_{u>\xi} \) satisfies: for all \( \varphi \in C^1_c(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \),

\[
\begin{align*}
\int_0^T & \langle f(t), \partial_t \varphi(t) \rangle dt + \langle f_0, \varphi(0) \rangle + \int_0^T \langle f(t), a(\xi) \cdot \nabla \varphi(t) \rangle dt \\
= & - \sum_{k \geq 1} \int_0^T \int_{\mathbb{T}^N} g_k(x, u(t, x)) \varphi(x, t, u(x, t)) dx d\beta_k(t) \\
& - \frac{1}{2} \int_0^T \int_{\mathbb{T}^N} \partial_t \varphi(x, t, u(x, t)) G^2(x, u(t, x)) dx dt + m(\partial_t \varphi), \text{ a.s.,}
\end{align*}
\]

where \( f_0 = I_{u>\xi} \), \( u(t) = u(., t, \cdot) \) and \( G^2 = \sum_{k=1}^\infty |g_k|^2 \).
Let \((X, \mathcal{A})\) be a finite measure space. For some measurable function \(u : X \rightarrow \mathbb{R}\), define \(f : X \times \mathbb{R} \rightarrow [0, 1]\) by \(f(z, \xi) = I_{u(z) > \xi}\) a.e. we use \(\bar{f} := 1 - f\) to denote its conjugate function. Define \(\Lambda_f(z, \xi) := f(z, \xi) - I_{0 > \xi}\), which can be viewed as a correction to \(f\). Note that \(\Lambda_f\) is integrable on \(X \times \mathbb{R}\) if \(u\) is.

It is shown in [DV-1] that almost surely, for each kinetic solution \(u\), the function \(f = I_{u(x,t) > \xi}\) admits left and right weak limits at any point \(t \in [0, T]\), and the weak form (2.8) satisfied by a kinetic solution can be strengthened to be weak only with respect to \(x\) and \(\xi\). More precisely, the following results are obtained.

**Proposition 2.1. (DV-1, Left and right weak limits)** Let \(u\) be a kinetic solution to (1.1) with initial value \(\eta\). Then \(f = I_{u(x,t) > \xi}\) admits, almost surely, left and right limits respectively at every point \(t \in [0, T]\). More precisely, for any \(t \in [0, T]\), there exist kinetic functions \(f^\pm\) on \(\Omega \times \mathbb{T}^N \times \mathbb{R}\) such that \(\mathbb{P}\)-a.s.

\[
\langle f(t - r), \varphi \rangle \rightarrow \langle f^-, \varphi \rangle \tag{2.9}
\]

and

\[
\langle f(t + r), \varphi \rangle \rightarrow \langle f^+, \varphi \rangle \tag{2.10}
\]

as \(r \rightarrow 0\) for all \(\varphi \in C^1_c(\mathbb{T}^N \times \mathbb{R})\). Moreover, almost surely,

\[
\langle f^+ - f^-, \varphi \rangle = -\int_{\mathbb{T}^N \times [0,T] \times \mathbb{R}} \partial_\xi \varphi(x, \xi) I_{\{u\}}(s) dm(x, s, \xi).
\]

In particular, almost surely, the set of \(t \in [0, T]\) fulfilling \(f^+ \neq f^-\) is countable.

For the function \(f = I_{u(x,t) > \xi}\) in Proposition 2.1, define \(f^\pm\) by \(f^\pm(t) = f^{\pm t}\), \(t \in [0, T]\). Since we are dealing with the filtration associated to Brownian motion, both \(f^+\) and \(f^-\) are clearly predictable as well. Also \(f = f^+ = f^-\) almost everywhere in time and we can take any of them in an integral with respect to the Lebesgue measure or in a stochastic integral. However, if the integral is with respect to a measure, typically a kinetic measure in this article, the integral is not well-defined for \(f\) and may differ if one chooses \(f^+\) or \(f^-\).

The following result was proved in [DV-1].

**Lemma 2.1.** The weak form (2.8) satisfied by \(f = I_{u>\xi}\) can be strengthened to be weak only respect to \(x\) and \(\xi\). Concretely, for all \(t \in [0, T]\) and \(\varphi \in C^1_c(\mathbb{T}^N \times \mathbb{R})\), \(f = I_{u>\xi}\) satisfies

\[
\langle f^+(t), \varphi \rangle = \langle f_0, \varphi \rangle + \int_0^t \langle f(s), a(\xi) \cdot \nabla \varphi \rangle ds + \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) G(x, \xi) d\nu_{x,s}(\xi) dx d\beta_k(s) + \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \partial_\xi \varphi(x, \xi) G^2(x, \xi) d\nu_{x,s}(\xi) dx ds - \langle m, \partial_\xi \varphi \rangle([0, t]) \tag{2.11}, \text{ a.s.,}
\]

and we set \(f^+(T) = f(T)\).
Where \( \nu_{x,s}(\xi) = -\partial_\xi f(x, s, \xi) = \delta_{u(x,s)=\xi} \).

**Remark 2.** By making modification of the proof of Lemma 2.1, we have for all \( t \in (0, T] \) and \( \varphi \in C^1_c(\mathbb{T}^N \times \mathbb{R}) \), \( f = I_{u>\xi} \) satisfies

\[
\langle f^-(t), \varphi \rangle = \langle f_0, \varphi \rangle + \int_0^t \langle f(s), a(\xi) \cdot \nabla \varphi \rangle ds + \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,s}(\xi) dx d\beta_k(s) + \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \partial_\xi \varphi(x, \xi) G^2(x, \xi) d\nu_{x,s}(\xi) dx ds - \langle m, \partial_\xi \varphi \rangle([0, t]), \quad \text{a.s.,} \quad (2.12)
\]

and we set \( f^-(0) = f_0 \).

The following well-posedness of (1.1) was established in [DV-1].

**Theorem 2.2.** ([DV-1], Existence and Uniqueness) Let \( \eta \in L^\infty(\mathbb{T}^N) \). Assume Hypothesis H holds, then there is a unique kinetic solution \( u \) to equation (1.1) with initial datum \( \eta \).

### 3 Transportation cost inequality

Let \( \mu \) be the law of the random field solution \( u(\cdot, \cdot) \) of SPDE (1.1), viewed as a probability measure on \( L^1([0, T], L^1(\mathbb{T}^N)) \). First we state a lemma which is essentially proved in [KS] describing the probability measures \( \nu \) that are absolutely continuous with respect to \( \mu \).

Let \( \nu \ll \mu \) on \( L^1([0, T], L^1(\mathbb{T}^N)) \). Define a new probability measure \( Q \) on the filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}) \) by

\[
dQ := \frac{d\nu}{d\mu}(u) \, d\mathbb{P}. \tag{3.13}
\]

Denote the Radon-Nikodym derivative restricted on \( \mathcal{F}_t \) by

\[
M_t := \left. \frac{dQ}{d\mathbb{P}} \right|_{\mathcal{F}_t}, \quad t \in [0, T].
\]

Then \( M_t, t \in [0, T] \) forms a \( \mathbb{P} \)-martingale. A variant of the following result was proved in [KS].

**Lemma 3.1.** There exists an adapted stochastic process \( h = (h(s) = (h_1(s), h_2(s), \cdots) \in \ell^2, s \geq 0) \) such that \( Q - \text{a.s. for all } t \in [0, T] \),

\[
\int_0^T |h_s^2(s)| \, ds < \infty
\]

and \( \overline{\beta}_k : [0, T] \to \mathbb{R} \) defined by

\[
\overline{\beta}_k(t) := \beta_k(t) - \int_0^t h_k(s) \, ds, \tag{3.14}
\]

7
are independent Brownian motions under the measure \( Q \). Moreover,

\[
M_t = \exp \left( \sum_{k=1}^{\infty} \int_0^t h_k(s) \, dB_k(s) - \frac{1}{2} \int_0^t |h_k(s)|^2 \, ds \right), \quad Q \text{-a.s.}, \quad (3.15)
\]

and

\[
H(\nu|\mu) = \frac{1}{2} \mathbb{E}^Q \left[ \int_0^T |\nu|^2(s) \, ds \right], \quad (3.16)
\]

where \( \mathbb{E}^Q \) stands for the expectation under the measure \( Q \).

**Theorem 3.1.** Let \( \eta \in L^\infty(\mathbb{T}^N) \). Assume Hypothesis H holds. Then the law \( \mu \) of the solution of the stochastic conservation law (1.1) satisfies the quadratic transportation cost inequality on the space \( L^1([0, T], L^1(\mathbb{T}^N)) \).

**Proof.** Take \( \nu \ll \mu \) on \( L^1([0, T], L^1(\mathbb{T}^N)) \). Let \( Q \) be the probability measure defined as in (3.13). Let \( h(t) \) be the corresponding stochastic process appeared in Lemma 3.1. Then, by the Girsanov theorem the solution \( u(t) \) of equation (1.1) satisfies the following stochastic partial differential equation (SPDE) under the measure \( Q \),

\[
\begin{cases}
    du^h(t, x) + divA(u^h(t, x))dt = \sum_{k \geq 1} g_k(x, u^h(t, x))dB_k(t) + \sum_{k \geq 1} g_k(x, u^h(t, x))h_k(t)dt & \text{in } \mathbb{T}^N \times (0, T], \\
    u^h(\cdot, 0) = \eta(\cdot) & \text{on } \mathbb{T}^N.
\end{cases} \quad (3.17)
\]

Similar to Lemma 2.1, we can show that the kinetic solution \( u^h \) satisfies that for any \( p \geq 1 \), there exists \( C_p \geq 0 \) such that

\[
\mathbb{E}^Q \left( \text{ess sup}_{0 \leq t \leq T} \|u^h(t)\|_{L^p(\mathbb{T}^N)}^p \right) \leq C_p, \quad (3.18)
\]

and there exists a kinetic measure \( m^h \) such that for all \( t \in [0, T] \) and \( \varphi \in C^1(\mathbb{T}^N \times \mathbb{R}) \), \( f := I_{u^h \gg \xi} \) satisfies

\[
(f^+(t), \varphi) = \langle f_0, \varphi \rangle + \int_0^t \langle f(s), a(\xi) \cdot \nabla \varphi \rangle ds \\
+ \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) v^h_{x,s}(\xi) d\xi d\beta_k(s) \\
+ \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) h_k(s) d\xi d\beta_k(s) \\
+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \partial_\xi \varphi(x, \xi) G^2(x, \xi) d\xi d\beta_k(s) - \langle m^h, \partial_\xi \varphi \rangle([0, t]), \quad \text{a.s.}, \quad (3.19)
\]

where \( v^h_{x,s}(\xi) = -\partial_\xi f(x, s, \xi) = \delta_{u^h(x,s) = \xi} \) and we set \( f^+(T) = f(T) \).

Consider the solution of the following SPDE:

\[
\begin{cases}
    du(t, x) + divA(u(t, x))dt = \sum_{k \geq 1} g_k(x, u(t, x))dB_k(t) & \text{in } \mathbb{T}^N \times (0, T], \\
    u(\cdot, 0) = \eta(\cdot) & \text{on } \mathbb{T}^N.
\end{cases} \quad (3.20)
\]
By Lemma 3.1, it follows that under the measure $Q$, the law of $(u, u^h)$ forms a coupling of $(\mu, \nu)$. Therefore by the definition of the Wasserstein distance,

$$W_2(\nu, \mu)^2 \leq \mathbb{E}^Q \left[ \left( \int_0^T \int_{\mathbb{T}^N} |u(t, x) - u^h(t, x)| dt dx \right)^2 \right].$$

In view of (3.16), to prove the quadratic transportation cost inequality

$$W_2(\nu, \mu) \leq \sqrt{2CH(\nu|\mu)}, \quad (3.21)$$

it is sufficient to show that

$$\mathbb{E}^Q \left[ \left( \int_0^T \int_{\mathbb{T}^N} |u(t, x) - u^h(t, x)| dt dx \right)^2 \right] \leq C \mathbb{E}^Q \left[ \int_0^T |\mu_{\beta}^2(s)| ds \right] \quad (3.22)$$

when the right hand side of (3.22) is finite.

For simplicity, in the sequel we still denote $\mathbb{E}$ by the symbol $\mathbb{E}$ and denote $\tilde{\beta}_k$ by $\beta_k$. The proof of (3.22) is technical and lengthy. It is divided into the following two propositions.

Following the idea of the proof Proposition 13 in [DV-1] and using the doubling variables method, we have the following result relating the two kinetic solution $u$ and $u^h$. As the proof is very similar to that of Proposition 13 in [DV-1], we omit the proof and refer the reader to [DV-1].

**Proposition 3.2.** Assume Hypothesis H is in place. Let $u$ and $u^h$ be the kinetic solution of (3.20) and (3.17), respectively. Then, for all $0 < t < T$, and non-negative test functions $\rho \in C^\infty(\mathbb{T}^N), \psi \in C^\infty_c(\mathbb{R})$, the corresponding functions $f_1(x, t, \xi) := I_{u^h(x,t)>\xi}$ and $f_2(y, t, \xi) := I_{u(y,t)>\xi}$ satisfy the following

$$\int_{\mathbb{T}^N} \int_{\mathbb{R}^2} \rho(x-y) \psi(\xi - \zeta)(f_1^{\pm}(x, t, \xi) f_2^{\pm}(y, t, \xi) + \tilde{f}_1^{\pm}(x, t, \xi) f_2^{\pm}(y, t, \xi)) d\xi d\zeta dxdy$$

$$\leq \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} \rho(x-y) \psi(\xi - \zeta)(f_1^{+}(x, t, \xi) f_2^{+}(y, t, \xi) + \tilde{f}_1^{+}(x, t, \xi) f_2^{+}(y, t, \xi)) d\xi d\zeta dxdy$$

$$+ I(t) + J(t) + K(t) + H(t), \quad a.s., \quad (3.23)$$

where

$$I(t) = \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} (f_1 \tilde{f}_2 + \tilde{f}_1 f_2)(\alpha(\xi) - a(\zeta)) \cdot \nabla_x a d\xi d\zeta dxdyds,$$

$$J(t) = \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} \alpha \sum_{k \geq 1} |g_k(x, \xi) - g_k(y, \zeta)|^2 dv_{x,s}^{1} \otimes v_{y,s}^{2}(\xi, \zeta) dxdyds,$$

$$K(t) = 2 \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} (g_k(x, \xi) - g_k(y, \zeta)) \rho(x-y) \chi(\xi, \zeta) dv_{x,s}^{1} \otimes v_{y,s}^{2}(\xi, \zeta) dxdyds,$$

$$H(t) = 2 \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} g_k(x, \xi) h_k(s) \rho(x-y) \chi(\xi, \zeta) dv_{x,s}^{1} \otimes v_{y,s}^{2}(\xi, \zeta) dxdyds,$$

with $f_1^{+}(x, t, \xi) = I_{(x,s)>\xi}, f_2^{+}(y, \xi) = I_{(y,s)>\xi}, \alpha = \rho(x-y) \psi(\xi - \zeta), v_{x,s}^{1} = -\delta_{\xi} f_1(s, x, \xi) = \delta^{x}_{(x,s)=\xi}, v_{y,s}^{2} = \delta_{\xi} f_2(s, y, \zeta) = \delta_{u(x,t)=\xi}$ and $\chi(\xi, \zeta) = \int_{-\infty}^{\xi} \psi(\xi' - \zeta) d\xi' = \int_{-\infty}^{\xi} \psi(y) dy$.
The statement \(3.22\) is contained in the next proposition.

**Proposition 3.3.** For \(T > 0\), it holds that

\[
\mathbb{E} \left[ \int_0^T \left\| u^h(t) - u(t) \right\|_{L^1(T^N)} dt \right]^2 \leq C \mathbb{E} \left[ \int_0^T \left| h^2 \right| dt \right],
\]

where \( C = C(T, D_0, D_1) \).

**Proof.** Let \( \rho_\gamma, \psi_\delta \) be approximations to the identity on \( T^N \) and \( \mathbb{R} \), respectively. That is, let \( \rho \in C^\infty(T^N) \), \( \psi \in C^\infty(\mathbb{R}) \) be symmetric non-negative functions such as \( \int_{T^N} \rho = 1 \), \( \int_\mathbb{R} \psi = 1 \) and \( \text{supp} \psi \subset (-1, 1) \). We define

\[
\rho_\gamma(x) = \frac{1}{\gamma^2} \rho\left(\frac{x}{\gamma}\right), \quad \psi_\delta(\xi) = \frac{1}{\delta} \psi\left(\frac{\xi}{\delta}\right).
\]

Letting \( \rho := \rho_\gamma(x-y) \) and \( \psi := \psi_\delta(\xi - \zeta) \) in Proposition 3.2, we get from \(3.23\) that

\[
\int_{(T^N)^2} \int_{\mathbb{R}^2} \rho_\gamma(x-y)\psi_\delta(\xi - \zeta)(f_1^+(x, t, \xi), \tilde{f}_2^+ (y, t, \xi) + \tilde{f}_1^+ (x, t, \xi))d\xi d\zeta dxdy \
\leq \int_{(T^N)^2} \int_{\mathbb{R}^2} \rho_\gamma(x-y)\psi_\delta(\xi - \zeta)(f_1,0 (x, \xi), \tilde{f}_2,0 (y, \xi) + \tilde{f}_1,0 (x, \xi))d\xi d\zeta dxdy \
+ \tilde{I}(t) + \tilde{J}(t) + \tilde{K}(t) + \tilde{H}(t), \quad \text{a.s.},
\]

where \( \tilde{I}, \tilde{J}, \tilde{K}, \tilde{H} \) are the corresponding terms \( I, J, K, H \) in the statement of Proposition 3.2 with \( \rho, \psi \) replaced by \( \rho_\gamma, \psi_\delta \), respectively. For simplicity, we still use the notation:

\[
\chi(\xi, \zeta) = \int_{-\infty}^{\xi - \zeta} \psi_\delta(y)dy.
\]

With an eye on the following identity,

\[
\int_{\mathbb{R}} I_{u^h > \xi} \tilde{I}_{u^h > \xi} d\xi = (u^{h^+} - u^+)^+, \quad \int_{\mathbb{R}} \tilde{I}_{u^h > \xi} I_{u^h > \xi} d\xi = (u^{h^+} - u^+)^-,
\]

we will start with the estimate \(3.25\) and eventually let \( \gamma, \delta \) appropriately tend to zero to prove the proposition. For any \( t \in [0, T] \), define the error term

\[
\mathcal{E}_t(\gamma, \delta) := \int_{(T^N)^2} \int_{\mathbb{R}^2} (f_1^+(x, t, \xi), \tilde{f}_2^+ (y, t, \xi) + \tilde{f}_1^+ (x, t, \xi))d\xi dxdy.
\]

\[
\mathcal{E}_t(\gamma, \delta) := \int_{(T^N)^2} \int_{\mathbb{R}^2} \left( f_1^+(x, t, \xi), \tilde{f}_2^+ (y, t, \xi) + \tilde{f}_1^+ (x, t, \xi) \right) d\xi dxdy.
\]
Using \( \int_{\mathbb{R}} \psi_\delta(\xi - \zeta) d\xi = 1 \), \( \int_{\mathbb{R}} \psi_\delta(\xi - \zeta) d\xi = \frac{1}{\delta} \) and \( \int_{V^N \times \mathbb{R}} \rho_\gamma(x - y) dxdy \leq 1 \), we find that

\[
\left| \int_{V^N \times \mathbb{R}} \int_{\mathbb{R}} \rho_\gamma(x - y) f_1^h(y, x, t, \xi) f_2^z(y, x, t, \xi) d\xi dy d\zeta \right|
\geq \int_{V^N \times \mathbb{R}} \int_{\mathbb{R}} \rho_\gamma(x - y) f_1^h(y, x, t, \xi) f_2^z(y, x, t, \xi) d\xi dy d\zeta
\]

Moreover, when \( \gamma \) is small enough, it follows that

\[
\left| \int_{V^N \times \mathbb{R}} \int_{\mathbb{R}} \rho_\gamma(x - y) f_1^h(y, x, t, \xi) f_2^z(y, x, t, \xi) d\xi dy d\zeta \right| \leq 2\delta, \quad a.s. \quad (3.29)
\]

Similarly, we have

\[
\left| \int_{V^N \times \mathbb{R}} \int_{\mathbb{R}} \rho_\gamma(x - y) f_1^h(y, x, t, \xi) f_2^z(y, x, t, \xi) d\xi dy d\zeta \right|
\geq \int_{V^N \times \mathbb{R}} \int_{\mathbb{R}} \rho_\gamma(x - y) f_1^h(y, x, t, \xi) f_2^z(y, x, t, \xi) d\xi dy d\zeta
\]

Moreover, when \( \gamma \) is small enough, it follows that

\[
\left| \int_{V^N \times \mathbb{R}} \int_{\mathbb{R}} \rho_\gamma(x - y) f_1^h(y, x, t, \xi) f_2^z(y, x, t, \xi) d\xi dy d\zeta \right| \leq \sup_{|y| < \gamma} \int_{V^N} f_1^h(y, x, t, \xi) f_2^z(y, x, t, \xi) d\xi dx \]

and

\[
\left| \int_{V^N \times \mathbb{R}} \int_{\mathbb{R}} \rho_\gamma(x - y) f_1^h(y, x, t, \xi) f_2^z(y, x, t, \xi) d\xi dy d\zeta \right| \leq \sup_{|y| < \gamma} \int_{V^N} f_1^h(y, x, t, \xi) f_2^z(y, x, t, \xi) d\xi dx \]

(3.30)
As $\Lambda_F$ is integrable, we have for a countable sequence $\gamma_n \downarrow 0$, (3.30) holds a.s. for all $n$, hence, passing to the limit $n \to \infty$, we get

$$\lim_{n \to \infty} \left| \int_{\mathbb{T}^N} \int_{\mathbb{R}} \rho_{y_n}(x-y)f_1^1(x, t, \xi)f_2^2(y, t, \xi)d\xi dx dy - \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_1^1(x, t, \xi)f_2^2(x, t, \xi)d\xi dx \right| = 0, \ a.s. (3.31)$$

Similarly, it holds that

$$\lim_{n \to \infty} \left| \int_{\mathbb{T}^N} \int_{\mathbb{R}} \rho_{y_n}(x-y)f_1^1(x, t, \xi)f_2^2(y, t, \xi)d\xi dx dy - \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_1^1(x, t, \xi)f_2^2(x, t, \xi)d\xi dx \right| = 0, \ a.s. (3.32)$$

By a similar argument, passing to the limit $t \to 0$, it follows from (3.28)-(3.32) that

$$\lim_{n \to \infty} E_\nu(\gamma_n, \delta_n) = 0, \ a.s.$$ 

Without confusion, from now on, we write

$$\lim_{\gamma, \delta \to 0} E_\nu(\gamma, \delta) = 0, \ a.s. \quad (3.33)$$

In particular, when $t = 0$, it holds that

$$\lim_{\gamma, \delta \to 0} E_0(\gamma, \delta) = 0. \quad (3.34)$$

Now, we will make some estimates for $\tilde{I}(t)$, $\tilde{J}(t)$, $\tilde{K}(t)$ and $\tilde{H}(t)$. We start with $\tilde{I}(t)$. Set

$$\Gamma(\xi, \zeta) = \int_{-\infty}^{\xi} \int_{-\infty}^{\zeta} \gamma(\xi', \zeta')|\xi' - \zeta'|d\xi' d\zeta'.$$

where $\gamma(\xi, \zeta)$ is the function appeared in Hypothesis H. Integration by parts yields that

$$\int_{0}^{t} \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} f_1 \tilde{f}_2(a(\xi) - a(\zeta)) \cdot \nabla_x a d\xi d\zeta dx dy ds$$

$$\leq \int_{0}^{t} \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} f_1 \tilde{f}_2 \gamma(\xi, \zeta) \xi - \zeta \nabla_x a \rho(x-y) \rho'(x-y) d\xi d\zeta dx dy ds$$

$$= - \int_{0}^{t} \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} f_1 \tilde{f}_2 \frac{\partial^2 \gamma(\xi, \zeta)}{\partial \xi \partial \zeta} \nabla_x a \rho(x-y) d\xi d\zeta dx dy ds$$

$$= \int_{0}^{t} \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} \Gamma(\xi, \zeta) d\nu_{x,s} \nabla_x a \rho(x-y) d\xi d\zeta dx dy ds$$

$$\leq C(q_0) t \int_{0}^{t} \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} (1 + |\xi|^q + |\zeta|^q) d\nu_{x,s} \nabla_x a \rho(x-y) d\xi d\zeta dx dy ds$$

where we have used the fact that $a(\cdot)$ is of polynomial growth with degree $q_0$ and (30) in [DV-1]. Namely, we have obtained that

$$\int_{0}^{t} \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} f_1 \tilde{f}_2(a(\xi) - a(\zeta)) \cdot \nabla_x a d\xi d\zeta dx dy ds$$

$$\leq C(q_0) t + C(q_0) t (\operatorname{ess \ sup}_{0 \leq s \leq t} \|a'(s)\|^q_{L^0(\mathbb{T}^N)} + \operatorname{ess \ sup}_{0 \leq s \leq t} \|a(s)\|^q_{L^0(\mathbb{T}^N)}). \ a.s.$$
Similar calculations lead to
\[
\left| \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} \tilde{f}_1 f_2 (a(\xi) - a(\zeta)) \cdot \nabla_x a d\xi d\zeta dxdyds \right|
\leq C(q_0) \delta \gamma^{-1} t + C(q_0) \delta \gamma^{-1} t \left( \text{ess sup}_{0 \leq s \leq t} \|u^h(s)\|_{L^0(\mathbb{T}^N)}^q + \text{ess sup}_{0 \leq s \leq t} \|u(s)\|_{L^0(\mathbb{T}^N)}^q \right). \quad \text{a.s.}
\]

Combining the above inequalities, we get
\[
|\bar{I}(t)| \leq C(q_0) \delta \gamma^{-1} t + C(q_0) \delta \gamma^{-1} t \left( \text{ess sup}_{0 \leq s \leq t} \|u^h(s)\|_{L^0(\mathbb{T}^N)}^q + \text{ess sup}_{0 \leq s \leq t} \|u(s)\|_{L^0(\mathbb{T}^N)}^q \right). \quad \text{a.s.}
\]

By (2.5) in Hypothesis H, we see that
\[
\bar{J}(t) = \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} a \sum_{k \geq 1} |g_k(x, \xi) - g_k(y, \zeta)|^2 d\nu_{\xi,s}^1 \otimes \nu_{\zeta,s}^2 (\xi, \zeta) dxdyds
\leq D_1 \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} \rho_\gamma(x - y)|x - y|^2 \int_{\mathbb{R}^2} \psi_\delta(\xi - \zeta) d\nu_{\xi,s}^1 \otimes \nu_{\zeta,s}^2 (\xi, \zeta) dxdyds
+ D_1 \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} \rho_\gamma(x - y) \psi_\delta(\xi - \zeta)|\xi - \zeta|^2 d\nu_{\xi,s}^1 \otimes \nu_{\zeta,s}^2 (\xi, \zeta) dxdyds
=: \bar{J}_1(t) + \bar{J}_2(t).
\]

Noting that
\[
\int_{\mathbb{R}^2} \psi_\delta(\xi, \zeta) d\nu_{\xi,s}^1 \otimes \nu_{\zeta,s}^2 (\xi, \zeta) \leq \delta^{-1}, \quad \text{a.s.,}
\int_{\mathbb{T}^N} \rho_\gamma(x - y)|x - y|^2 dxdy \leq \gamma^2,
\]
we have
\[
\bar{J}_1(t) \leq D_1 \delta^{-1} \gamma^2 t. \quad \text{a.s.} \quad (3.35)
\]

For the term \( \bar{J}_2 \), we have
\[
\bar{J}_2 \leq \delta D_1 \int_0^t \int_{\mathbb{T}^N} \int_{|\xi - \zeta| \leq \delta} \rho_\gamma(x - y) \psi_\delta(\xi - \zeta)|\xi - \zeta| d\nu_{\xi,s}^1 \otimes \nu_{\zeta,s}^2 (\xi, \zeta) dxdyds
\leq \delta D_1 C_\psi t, \quad \text{a.s.}, \quad (3.36)
\]
where \( C_\psi := \sup_{\xi \in \mathbb{R}} \|\psi(\xi)\| \). (3.35) and (3.36) together yield
\[
\bar{J}(t) \leq D_1 \delta^{-1} \gamma^2 t + D_1 C_\psi t, \quad \text{a.s.}
\]

By Hölder inequality and (2.4), we get
\[
\bar{H}(t) \leq 2 \int_0^t |h(s)|_{L^q} \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} \left( \sum_{k \geq 1} |g_k(x, \xi)|^2 \right)^{\frac{1}{2}} \rho_\gamma(x - y)\chi(\xi, \zeta) d\nu_{\xi,s}^1 \otimes \nu_{\zeta,s}^2 (\xi, \zeta) dxdyds
\leq 2D_0^\frac{1}{q} \int_0^t |h(s)|_{L^q} \int_{\mathbb{T}^N} \rho_\gamma(x - y) dxdyds
\leq 2D_0^\frac{1}{q} \int_0^t |h(s)|_{L^q} ds, \quad \text{a.s.}
\]

13
where we have used the fact that $\chi(\xi, \zeta) \leq 1$.

Combining all the above estimates, we deduce that

$$
\int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho_y(x-y)\psi_0(\xi - \zeta)(f_1^x(x, t, \xi)\bar{f}_2^y(y, t, \zeta) + \bar{f}_1^x(x, t, \xi)f_2^y(y, t, \zeta))d\xi d\zeta dy
$$

$$
\leq \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho_y(x-y)\psi_0(\xi - \zeta)(f_{1,0}(x, \xi)\bar{f}_{2,0}(y, \zeta) + \bar{f}_{1,0}(x, \xi)f_{2,0}(y, \zeta))d\xi d\zeta dy
$$

$$
+ D_1 \delta^{-1} \gamma^2 t + D_1 C_\psi \delta t + C(q_0)\delta \gamma^{-1} t + 2D_0^2 \int_0^t |h(s)|_{L^2} ds
$$

$$
+ C(q_0)\delta \gamma^{-1} t(\text{ess sup } ||u^h(s)||_{L^0(\mathbb{T}^N)} + ||u(s)||_{L^0(\mathbb{T}^N)}) + \bar{K}(t), \quad a.s.. \quad (3.37)
$$

For $s \in (0, T)$, set

$$
R(s) := \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho_y(x-y)\psi_0(\xi - \zeta)(f_1^x(x, s, \xi)\bar{f}_2^y(y, s, \zeta) + \bar{f}_1^x(x, s, \xi)f_2^y(y, s, \zeta))d\xi d\zeta dy.
$$

Then, we deduce from (3.37) that

$$
\text{ess sup } R(s) \leq \int_{\mathbb{T}^N} \int_{\mathbb{R}} (f_{1,0}\bar{f}_{2,0} + \bar{f}_{1,0}f_{2,0})d\xi dx + \mathcal{E}_0(\gamma, \delta)
$$

$$
+ D_1 \delta^{-1} \gamma^2 t + D_1 C_\psi \delta t + C(q_0)\delta \gamma^{-1} t + 2D_0^2 \int_0^t |h(s)|_{L^2} ds
$$

$$
+ C(q_0)\delta \gamma^{-1} t(\text{ess sup } ||u^h(s)||_{L^0(\mathbb{T}^N)} + ||u(s)||_{L^0(\mathbb{T}^N)}) + \sup_{0 \leq s \leq T} |\bar{K}(s)|, \quad a.s.,
$$

where $\lim_{\gamma, \delta \to 0} \mathcal{E}_0(\gamma, \delta) = 0$.

Taking the $L^2(\Omega)$-norm on both sides and using Hölder inequality, we get that

$$
\left(\mathbb{E}\text{ess sup } ||R(s)||_{L^2(\mathbb{T}^N)} \right)^{\frac{1}{2}} \leq \int_{\mathbb{T}^N} \int_{\mathbb{R}} (f_{1,0}\bar{f}_{2,0} + \bar{f}_{1,0}f_{2,0})d\xi dx + \mathcal{E}_0(\gamma, \delta)
$$

$$
+ D_1 \delta^{-1} \gamma^2 t + D_1 C_\psi \delta t + C(q_0)\delta \gamma^{-1} t + 2t^2 D_0^2 \left(\mathbb{E} \int_0^t |h(s)|_{L^2}^2 ds\right)^{\frac{1}{2}}
$$

$$
+ C(q_0)\delta \gamma^{-1} tR + \left(\mathbb{E} \sup_{s \in [0, t]} |\bar{K}(s)|^2\right)^{\frac{1}{4}}, \quad (3.38)
$$

where

$$
\mathcal{R} := \left\{ \left(\mathbb{E}\text{ess sup } ||u^h(s)||_{L^0(\mathbb{T}^N)}^{2q_0}\right)^{\frac{1}{2}} + \left(\mathbb{E}\text{ess sup } ||u(s)||_{L^0(\mathbb{T}^N)}^{2q_0}\right)^{\frac{1}{2}} \right\}.
$$

In view of (2.7) and (3.18), we have

$$
\mathcal{R} < +\infty. \quad (3.39)
$$
With the help of (3.26), (3.28) and (3.29), we deduce that

\[ \int_{0}^{\infty} \int_{\mathbb{R}^{2}} |\chi(\xi, \zeta)\rho_{\gamma}(x-y)(g_{k}(x, \xi) - g_{k}(y, \zeta)) d\nu_{x,r}^{1} \otimes \nu_{y,r}^{2}(\xi, \zeta) d\sigma_{r}(r)|^{2} dr \]

Recalling (2.3) in Hypothesis H

\[ \int_{0}^{\infty} \sum_{k \geq 1} \left[ \int_{\mathbb{R}^{2}} |g_{k}(x, \xi) - g_{k}(y, \zeta)| \rho_{y}(x-y) \chi(\xi, \zeta) d\nu_{x,r}^{1} \otimes \nu_{y,r}^{2}(\xi, \zeta) d\sigma_{r}(r) \right]^{2} dr \]

it follows from (3.40) that

\[ \mathbb{E} \left[ \sup_{s \in [0,t]} |\tilde{K}(s)| \right]^{2} \leq D_{1} \left[ \gamma + \int_{\mathbb{R}^{2}} |u^{h,\pm} - u^{\pm}| \rho_{y}(x-y) d\sigma_{r}(r) \right]^{2} dr \]

Since

\[ \int_{\mathbb{R}^{2}} |x-y| \rho_{y}(x-y) \chi(\xi, \zeta) d\nu_{x,r}^{1} \otimes \nu_{y,r}^{2}(\xi, \zeta) d\sigma_{r}(r) \leq \gamma, \quad a.s. \]

it follows that

\[ \mathbb{E} \left[ \sup_{s \in [0,t]} |\tilde{K}(s)| \right]^{2} \leq D_{1} \left[ \gamma + \int_{\mathbb{R}^{2}} |u^{h,\pm} - u^{\pm}| \rho_{y}(x-y) d\sigma_{r}(r) \right]^{2} dr \]

With the help of (3.26), (3.28) and (3.29), we deduce that

\[ \int_{\mathbb{R}^{2}} |u^{h,\pm}(x, r) - u^{\pm}(y, r)| \rho_{y}(x-y) d\sigma_{r}(r) \]

\[ = \int_{\mathbb{R}^{2}} \left( (u^{h,\pm}(x, r) - u^{\pm}(y, r))^{+} + (u^{h,\pm}(x, r) - u^{\pm}(y, r))^{-} \right) \rho_{y}(x-y) d\sigma_{r}(r) \]

\[ = \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} (f_{1}^{\pm}(x, r, \xi) f_{2}^{\pm}(y, r, \xi) + f_{1}^{\pm}(x, r, \xi) f_{2}^{\pm}(y, r, \xi)) \rho_{y}(x-y) d\xi d\sigma_{r}(r) \]

\[ \leq 4 \delta + \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{2}} (g_{1}^{\pm}(x, r, \xi) g_{2}^{\pm}(y, r, \xi) + g_{1}^{\pm}(x, r, \xi) g_{2}^{\pm}(y, r, \xi)) \rho_{y}(x-y) \psi_{\delta}(x - \zeta) d\xi d\delta \]

\[ = 4 \delta + R(r), \quad a.s. \]

Combining (3.41) and (3.42), we obtain that

\[ \mathbb{E} \left[ \sup_{s \in [0,t]} |\tilde{K}(s)| \right]^{2} \leq D_{1} \left[ \gamma + 4 \delta + R(r) \right]^{2} dr \]

\[ \leq 2D_{1} \left[ \gamma + 4 \delta \right]^{2} dr + \int_{0}^{t} \left[ R(r) \right]^{2} dr \]

\[ \leq 2D_{1} t \gamma + 4 \delta \right]^{2} + 2D_{1} \left( \int_{0}^{t} R^{2}(r) dr \right). \]
Applying Gronwall inequality to (3.44), we get

\[ \left( \mathbb{E} \left[ \left| \text{ess sup } R(s) \right|^2 \right]^{\frac{1}{4}} \right)^\frac{1}{2} \]

\[ \leq \int_{\mathbb{T}^N} \int_{\mathbb{R}} (f_{1,0} \tilde{f}_{2,0} + \tilde{f}_{1,0} f_{2,0}) d\xi dx + \mathcal{E}_0(\gamma, \delta) + D_1 \delta^{-1} \gamma^2 T + D_1 C_\phi \delta T + C(q_0) \delta \gamma^{-1} T \]

\[ + 2t^\frac{1}{2} D_0^\frac{1}{2} \left( \mathbb{E} \int_0^T |h(s)|^2 ds \right)^{\frac{1}{2}} + C(q_0) \delta \gamma^{-1} RT + 2t^\frac{1}{2} D_1^\frac{1}{2} (\mathbb{E} \left( \int_0^T R^2(s) ds \right))^\frac{1}{2}. \]

Squaring the above inequality, we get

\[ \mathbb{E} \left[ \text{ess sup } R(s) \right]^2 \]

\[ \leq \left[ \int_{\mathbb{T}^N} \int_{\mathbb{R}} (f_{1,0} \tilde{f}_{2,0} + \tilde{f}_{1,0} f_{2,0}) d\xi dx + \mathcal{E}_0(\gamma, \delta) + D_1 \delta^{-1} \gamma^2 T + D_1 C_\phi \delta T + C(q_0) \delta \gamma^{-1} T \]

\[ + 2t^\frac{1}{2} D_0^\frac{1}{2} \left( \mathbb{E} \int_0^T |h(s)|^2 ds \right)^{\frac{1}{2}} + C(q_0) \delta \gamma^{-1} RT + 2t^\frac{1}{2} D_1^\frac{1}{2} (\mathbb{E} \left( \int_0^T R^2(s) ds \right))^\frac{1}{2} \]

\[ + 2D_1 \int_0^T \left( \mathbb{E} \left[ \text{ess sup } R(s) \right]^2 \right) dr. \]  

(3.44)

Applying Gronwall inequality to (3.44), we get

\[ \left( \mathbb{E} \left[ \left| \text{ess sup } R(s) \right|^2 \right]^{\frac{1}{4}} \right)^\frac{1}{2} \]

\[ \leq e^{D_1 T} \left[ \int_{\mathbb{T}^N} \int_{\mathbb{R}} (f_{1,0} \tilde{f}_{2,0} + \tilde{f}_{1,0} f_{2,0}) d\xi dx + \mathcal{E}_0(\gamma, \delta) + D_1 \delta^{-1} \gamma^2 T + D_1 C_\phi \delta T + C(q_0) \delta \gamma^{-1} T \]

\[ + 2t^\frac{1}{2} D_0^\frac{1}{2} \left( \mathbb{E} \int_0^T |h(s)|^2 ds \right)^{\frac{1}{2}} + C(q_0) \delta \gamma^{-1} RT + 2t^\frac{1}{2} D_1^\frac{1}{2} (\mathbb{E} \left( \int_0^T R^2(s) ds \right))^\frac{1}{2} \]

\[ + C(q_0) \delta \gamma^{-1} RT + 2t^\frac{1}{2} D_1^\frac{1}{2} (\mathbb{E} \left( \int_0^T R^2(s) ds \right))^\frac{1}{2} \leq \mathbb{E} \left[ \text{ess sup } R(s) \right]. \]  

(3.45)

Let

\[ Q(s) := \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} \rho_\gamma(x-y) \psi_\gamma(\xi - \zeta) (f_{2}^\pm(s, x, \xi)) f_{2}^\pm(s, y, \xi) + \tilde{f}_{2}^\pm(s, x, \xi) f_{2}^\pm(s, y, \xi) d\zeta d\xi dx dy. \]

Applying the same arguments to \( f_{2}^\pm \) and \( \tilde{f}_{2}^\pm \) (in this case, \( h = 0 \)), we can show that

\[ \left( \mathbb{E} \left[ \text{ess sup } Q(s) \right]^2 \right)^{\frac{1}{2}} \leq e^{D_1 T} \left[ \mathcal{E}_0(\gamma, \delta) + D_1 \delta^{-1} \gamma^2 T + D_1 C_\phi \delta T + C(q_0) \delta \gamma^{-1} T \right. \]

\[ \left. + C(q_0) \delta \gamma^{-1} RT + 2t^\frac{1}{2} D_1^\frac{1}{2} (\mathbb{E} \left( \int_0^T R^2(s) ds \right))^\frac{1}{2} \right]. \]  

(3.46)

On the other hand, from (5.27), it follows that

\[ \left( \mathbb{E} \left[ \text{ess sup } \int_{\mathbb{T}^N} \int_{\mathbb{R}} (f_{1}^\pm(s, x, \xi)) f_{1}^\pm(s, x, \xi) d\xi dx \right]^2 \right)^{\frac{1}{2}} \]

\[ \leq \left( \mathbb{E} \left[ \text{ess sup } |\mathcal{E}_4(\gamma, \delta)| \right] \right)^{\frac{1}{2}} + \left( \mathbb{E} \left[ \text{ess sup } R(s) \right]^2 \right)^{\frac{1}{2}}. \]  

(3.47)
Now we will provide estimates for $\left( E \left[ \text{ess sup}_{0 \leq s \leq T} |E_s(\gamma, \delta)| \right] \right)^{\frac{1}{2}}$. For any $s \in (0, T)$, we write

$$E_s(\gamma, \delta) = \int_{(T \times \Omega)^2} \int_{\mathbb{R}^2} \left( f_1^s(x, s, \xi) \bar{f}_2^s(y, s, \xi) + \bar{f}_1^s(x, s, \xi) f_2^s(y, s, \xi) \right) \rho_y(x-y) \psi_\delta(\xi-\zeta) \, dxdy \, d\xi d\zeta$$

$$- \int_{(T \times \Omega) \times \mathbb{R}} \left( f_1^s(x, s, \xi) \bar{f}_2^s(x, s, \xi) + \bar{f}_1^s(x, s, \xi) f_2^s(x, s, \xi) \right) \, d\xi dx$$

$$= \left[ \int_{(T \times \Omega)^2} \int_{\mathbb{R}^2} \rho_y(x-y)(f_1^s(x, s, \xi) \bar{f}_2^s(y, s, \xi) + \bar{f}_1^s(x, s, \xi) f_2^s(y, s, \xi)) \, dxdy \right]$$

$$- \int_{(T \times \Omega) \times \mathbb{R}} \left( f_1^s(x, s, \xi) \bar{f}_2^s(x, s, \xi) + \bar{f}_1^s(x, s, \xi) f_2^s(x, s, \xi) \right) \, d\xi dx$$

$$+ \left[ \int_{(T \times \Omega)^2} \int_{\mathbb{R}^2} \rho_y(x-y)(f_1^s(x, s, \xi) \bar{f}_2^s(y, s, \xi) + \bar{f}_1^s(x, s, \xi) f_2^s(y, s, \xi)) \rho_y(x-y) \psi_\delta(\xi-\zeta) \, dxdy \, d\xi d\zeta \right]$$

$$- \int_{(T \times \Omega)^2} \int_{\mathbb{R}^2} \rho_y(x-y)(f_1^s(x, s, \xi) \bar{f}_2^s(y, s, \xi) + \bar{f}_1^s(x, s, \xi) f_2^s(y, s, \xi)) \, dxdy \right]$$

$$=: H_1 + H_2.$$

By (3.28) and (3.29), we have

$$|H_2| \leq 4\delta, \quad \text{a.s.} \quad (3.48)$$

On the other hand,

$$|H_1| \leq \left| \int_{(T \times \Omega)^2} \int_{\mathbb{R}^2} \rho_y(x-y) \int_{\mathbb{R}^2} I_{\psi_\delta(x,s) > \xi} (I_{\psi_\delta(x,s) \leq \zeta} - I_{\psi_\delta(x,s) \leq \xi}) \, d\xi \, dxdy \right|$$

$$+ \left| \int_{(T \times \Omega)^2} \int_{\mathbb{R}^2} \rho_y(x-y) \int_{\mathbb{R}^2} I_{\psi_\delta(x,s) \leq \xi} (I_{\psi_\delta(x,s) > \zeta} - I_{\psi_\delta(x,s) \leq \xi}) \, d\xi \, dxdy \right|$$

$$\leq 2 \int_{(T \times \Omega)^2} \int_{\mathbb{R}^2} \rho_y(x-y) u^+(x, s) - u^+(y, s) \, dxdy, \quad \text{a.s.}$$

Using (3.28) and (3.29) again, it follows that

$$\int_{(T \times \Omega)^2} \int_{\mathbb{R}^2} \rho_y(x-y) u^+(x, s) - u^+(y, s) \, dxdy$$

$$= \int_{(T \times \Omega)^2} \int_{\mathbb{R}^2} \rho_y(x-y)(f_1^s(x, s, \xi) \bar{f}_2^s(y, s, \xi) + \bar{f}_1^s(x, s, \xi) f_2^s(y, s, \xi)) \, d\xi \, dxdy$$

$$\leq \int_{(T \times \Omega)^2} \int_{\mathbb{R}^2} \rho_y(x-y) \psi_\delta(\xi-\zeta) (f_1^s(x, s, \xi) \bar{f}_2^s(y, s, \xi) + \bar{f}_1^s(x, s, \xi) f_2^s(y, s, \xi)) \, d\xi \, dxdy + 4\delta$$

$$= Q(s) + 4\delta, \quad \text{a.s.}$$

Hence,

$$|H_1| \leq 2Q(s) + 8\delta, \quad \text{a.s.} \quad (3.49)$$

Collecting (3.48) and (3.49) yields

$$|E_s(\gamma, \delta)| \leq 2Q(s) + 12\delta, \quad \text{a.s.},$$

17
By (3.46), we deduce that

\[
(B \|E \|_{L^1(T)}^2)^{1/2} \\
\leq (B \|E \|_{L^1(T)}^2)^{1/2} + \delta \\
\leq e^{D T} \left[ E_{0}(\gamma, \delta) + D_{1} D T + C(q_{0}) \gamma^{-1} T \\
+ C(q_{0}) \gamma^{-1} T + 2 D_{1} D T |\gamma + 4\delta| \right] + \delta.
\]

Combining (3.45) and [3.50], we deduce from (3.47) that

\[
(B \|E \|_{L^1(T)}^2)^{1/2} \\
\leq e^{D T} \left[ E_{0}(\gamma, \delta) + D_{1} D T + C(q_{0}) \gamma^{-1} T \\
+ 2 D_{1} D T |\gamma + 4\delta| \right] + \delta.
\]

Note that we have \( f_{1}(x, s, \xi) = I_{\eta} \) and \( f_{2}(x, s, \xi) = I_{\eta} \) with initial data \( f_{1,0} = I_{\eta} \) and \( f_{2,0} = I_{\eta} \), respectively. In view of (3.26), we can rewrite the above inequality as

\[
(B \|E \|_{L^1(T)}^2)^{1/2} \leq r(\gamma, \delta),
\]

where

\[
\begin{align*}
\delta &= \gamma^{1/2} \\
\Rightarrow \delta &= \gamma^{1/2},
\end{align*}
\]

we have

\[
\begin{align*}
r(\gamma, \delta) &= e^{D T} \left[ E_{0}(\gamma, \delta) + 2 D_{1} D T + C(q_{0}) \gamma^{-1} T \\
&+ 2 D_{1} D T |\gamma + 4\delta| \right] + \delta.
\end{align*}
\]

Let \( \gamma \to 0 \) to get

\[
\lim_{\gamma \to 0} r(\gamma, \delta) \leq 2 D_{1} D_{0} e^{D T} \left( E \int_{I} |h(s)|_{L^{2}(T)}^2 \right)^{1/2}.
\]
Therefore, we deduce from (3.51) that
\[
\left( \mathbb{E} \left[ \text{ess sup}_{0 \leq s \leq T} ||u^{h,s}(s) - u^*(s)||_{L^1(\mathbb{R}^N)}^2 \right] \right)^{\frac{1}{2}} \leq 2 DT^2 D_0 e^{D_1 T} \left( \mathbb{E} \int_0^T |h(s)|^2 \mu(ds) \right)^{\frac{1}{2}},
\]
which implies
\[
\mathbb{E} \left[ \int_0^T |u^h(t) - u(t)||_{L^1(\mathbb{R}^N)}^2 dt \right]^2 \leq 4 D^2 D_0 T^3 e^{2 D_1 T} \mathbb{E} \int_0^T |h(s)|^2 \mu(ds).
\]
(3.53)
We complete the proof.

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