Relativistic elliptic matrix tops
and finite Fourier transformations

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Abstract

We consider a family of classical elliptic integrable systems including (relativistic) tops and their matrix extensions of different types. These models can be obtained from the "off-shell" Lax pairs, which do not satisfy the Lax equations in general case but become true Lax pairs under various conditions (reductions). At the level of the off-shell Lax matrix there is a natural symmetry between the spectral parameter $z$ and relativistic parameter $\eta$. It is generated by the finite Fourier transformation, which we describe in detail. The symmetry allows to consider $z$ and $\eta$ on an equal footing. Depending on the type of integrable reduction any of the parameters can be chosen to be the spectral one. Then another one is the relativistic deformation parameter. As a by-product we describe the model of $N^2$ interacting $GL(M)$ matrix tops and/or $M^2$ interacting $GL(N)$ matrix tops depending on a choice of the spectral parameter.

1 Integrable elliptic tops

We consider a special class of the Liouville integrable systems. The simplest example is the (complexified) Euler top in $\mathbb{C}^3$:

$$\dot{\mathbf{S}} = \mathbf{S} \times J(\mathbf{S}),$$

(1.1)

where $\mathbf{S} = (S_1, S_2, S_3)$, $J(\mathbf{S}) = (J_1 S_1, J_2 S_2, J_3 S_3)$ and $\{J_i\}$ (inverse components of inertia tensor in principle axes) are arbitrary constants. Equation (1.1) is simply rewritten in matrix form:

$$\dot{\mathbf{S}} = [\mathbf{S}, J(\mathbf{S})],$$

(1.2)

where

$$S = \frac{1}{2i} \sum_{k=1}^{3} \sigma_k S_k, \quad J(S) = \frac{1}{2i} \sum_{k=1}^{3} \sigma_k S_k J_k,$$

(1.3)

and $\{\sigma_k\}$ are the Pauli matrices. Equation (1.2) admits generalization to Mat($N, \mathbb{C}$)-valued $S$. These model are known as the Euler-Arnold tops [2, 3, 19, 15, 20]. They are not integrable for a generic linear constant operator $J$ but only for very specific choices of $J$. 

1
Elliptic non-relativistic top. In particular, the following model is integrable (and is called elliptic top) \[^{23}^{23}^{23}^{26} \]. Consider the Heisenberg group with generators $Q$ and $\Lambda$: $\zeta Q\Lambda = \Lambda Q$. For $\zeta = \exp(\frac{2\pi i}{N})$, $N \in \mathbb{Z}_+$ it has finite-dimensional representation given by $N \times N$ matrices

$$Q_{kl} = \delta_{kl} \exp\left(\frac{2\pi i}{N}k\right), \quad \Lambda_{kl} = \delta_{k-l+1=0 \mod N}$$

with the property $Q^N = \Lambda^N = 1_{N \times N}$. Define the following basis in $\text{Mat}(N, \mathbb{C})$ \[^{42}^{22} \]:

$$T_\alpha = T_{\alpha_1\alpha_2} = \exp\left(\frac{\pi i}{N} \alpha_1 \alpha_2\right) Q^{\alpha_1} \Lambda^{\alpha_2}, \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_N \times \mathbb{Z}_N.$$  

Then

$$T_\alpha T_\beta = \kappa_{\alpha,\beta} T_{\alpha+\beta}, \quad \kappa_{\alpha,\beta} = \exp\left(\frac{\pi i}{N}(\beta_1 \alpha_2 - \beta_2 \alpha_1)\right),$$

and

$$[T_\alpha, T_\beta] = C_{\alpha,\beta} T_{\alpha+\beta}, \quad C_{\alpha,\beta} = \kappa_{\alpha,\beta} - \kappa_{\beta,\alpha},$$

where we assume $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$. Using the basis \[^{15,5} \] we can define the inverse inertia tensor for the elliptic top. It is given as follows:

$$S = \sum_\alpha \bar{T}_\alpha S_\alpha, \quad J(S) = \sum_\alpha \bar{T}_\alpha S_\alpha J_\alpha, \quad J_\alpha = -\wp(\omega_\alpha), \quad \omega_\alpha = \frac{\alpha_1 + \alpha_2 \tau}{N},$$

where the sums are over $\alpha \in \mathbb{Z}_N \times \mathbb{Z}_N$, the prime means $\alpha \neq 0 \equiv (0,0)$ and $\wp$ is the Weierstrass $\wp$-function with elliptic moduli $\tau$ ($\text{Im}(\tau) > 0$). In $N = 2$ case this model reproduces the Euler top \[^{13} \]. Indeed, in this case we have three components of $S$ ($S_{(1,0)}$, $S_{(0,1)}$ and $S_{(1,1)}$). The values of $J_\alpha$ are not arbitrary since they depend on a single parameter $\tau$. However, one can shift them by the same constant (without changing of equations of motion) and rescale simultaneously by another constant (it is equivalent to rescaling of time variable).

The integrability of \[^{12} \] with $J$ \[^{18} \] follows from existence of the Lax pair with spectral parameter $(z)$ \[^{24}^{23}^{23}^{23}^{8} \]. It is similar to Krichever’s ansatz \[^{7} \] for elliptic Calogero-Moser model:

$$L(z) = \sum_\alpha \bar{T}_\alpha S_\alpha \varphi_\alpha(z, \omega_\alpha), \quad M(z) = \sum_\alpha \bar{T}_\alpha S_\alpha f_\alpha(z, \omega_\alpha),$$

where the sets of function $\varphi_\alpha(z, \omega_\alpha)$ and $f_\alpha(z, \omega_\alpha)$ ($\alpha \neq 0$) are defined by \[^{A.10}, A.12 \].

The Lax equations

$$\dot{L}(z) = [L(z), M(z)]$$

are equivalent to equations of motion of elliptic top \[^{12} \] with $J$ \[^{18} \] identically in $z$. The proof is based on \[^{A.15} \].

Relativistic elliptic top \[^{9}, 10 \] is one-parameter deformation of the elliptic top described above. Its Lax pair

$$L(z) = \sum_\alpha \bar{T}_\alpha S_\alpha \varphi_\alpha(z, \eta + \omega_\alpha), \quad M(z) = -\sum_\alpha \bar{T}_\alpha S_\alpha \varphi_\alpha(z, \omega_\alpha)$$

provides equations of motion \[^{12} \] with

$$J_\alpha^n(S) = \sum_\alpha \bar{T}_\alpha S_\alpha J_\alpha^n, \quad J_\alpha^n = E_1(\eta + \omega_\alpha) - E_1(\omega_\alpha).$$
Indeed, for the Lax pair (1.11) the l.h.s. of the Lax equations (1.10) equals
\[ \dot{L}(z) = \sum_{\alpha} T_{\alpha} \dot{S}_{\alpha} \varphi_{\alpha}(z, \eta + \omega_{\alpha}). \] (1.13)

For the r.h.s. of (1.10) we have:
\[-\sum_{\beta} \sum_{\gamma} \big[ T_{\beta}, T_{\gamma} \big] S_{\beta} S_{\gamma} \varphi_{\beta}(z, \omega_{\beta} + \eta) \varphi_{\gamma}(z, \omega_{\gamma}) = -\sum_{\beta, \gamma} \big[ T_{\beta}, T_{\gamma} \big] S_{\beta} S_{\gamma} \varphi_{\beta}(z, \omega_{\beta} + \eta) \varphi_{\gamma}(z, \omega_{\gamma}) \]
\[\text{since } T_{00} \text{ is identity matrix.} \]
\[\text{(Anti)symmetrization with respect to } \beta \text{ and } \gamma \text{ yields}
\[-\frac{1}{2} \sum_{\beta, \gamma} \big[ T_{\beta}, T_{\gamma} \big] S_{\beta} S_{\gamma} \left( \varphi_{\beta}(z, \omega_{\beta} + \eta) \varphi_{\gamma}(z, \omega_{\gamma}) - \varphi_{\gamma}(z, \omega_{\gamma} + \eta) \varphi_{\beta}(z, \omega_{\beta}) \right) \]
\[-\frac{1}{2} \sum_{\beta, \gamma} C_{\beta, \gamma} T_{\beta+\gamma} S_{\beta} S_{\gamma} \varphi_{\beta+\gamma}(z, \eta + \omega_{\beta+\gamma}) \left( J_{\beta}^{\eta} - J_{\gamma}^{\eta} \right) = \]
\[\sum_{\beta, \gamma} C_{\beta, \gamma} T_{\beta+\gamma} S_{\beta} S_{\gamma} \varphi_{\beta+\gamma}(z, \eta + \omega_{\beta+\gamma}) J_{\gamma}^{\eta} = \sum_{\alpha, \beta} T_{\alpha} \varphi_{\alpha}(z, \eta + \omega_{\alpha}) C_{\beta, \alpha-\beta} S_{\beta} S_{\alpha-\beta} J_{\alpha-\beta}^{\eta}. \] (1.14)

By equating coefficients behind \(T_{\alpha}\) in (1.13) and (1.14) we obtain
\[ \dot{S}_{00} = 0, \quad \dot{S}_{\alpha} = \sum_{\beta} C_{\beta, \alpha-\beta} S_{\beta} S_{\alpha-\beta} J_{\alpha-\beta}^{\eta}, \quad \alpha \neq 0. \] (1.15)

Equations of motion (1.15) are components of the matrix equation (1.2) in the basis (1.5)-(1.7) with \(J_{\eta}^{\eta}\) (1.12).

The non-relativistic limit (1.8) is obtained from (1.15) via rescaling of time variable by \(\eta\), taking the limit \(\eta \to 0\) and usage of (A.4) and (A.5).

**Relativization.** There is a simple relation between Lax matrices of the non-relativistic (1.9) and relativistic (1.11) tops. Denote the first one as \(l(z, S)\) and the second one as \(L_{\eta}^{\eta}(z, S)\). Introduce also \(L_{\eta}^{0}(z, S) = 1_{N} S_{0} + l(z, S)\). Then
\[ L_{\eta}^{\eta}(z - \eta, L_{\eta}^{\eta}(\eta, S)) = \phi(z - \eta, \eta) L_{\eta}^{0}(z, S). \] (1.16)

This follows from the definition of the Kronecker function (A.1) because the latter implies
\[ \frac{\varphi_{\alpha}(z, \omega_{\alpha} + \eta)}{\phi(z, \eta)} = \frac{\varphi_{\alpha}(z + \eta, \omega_{\alpha})}{\varphi_{\alpha}(\eta, \omega_{\alpha})}, \quad \alpha \neq 0. \] (1.17)

Relation (1.16) means explicit change of variables
\[ S_{\alpha} \to S_{\alpha} / \varphi_{\alpha}(\eta, \omega_{\alpha}) \] (1.18)

between non-relativistic and relativistic models. At the same time the transformation (1.18) changes eigenvalues of matrix \(S\) which are obviously conserved by dynamics (1.2). In particular, (1.18) can not connect non-relativistic and relativistic models with rank 1 matrices \(S\). The first one is known to be gauge equivalent to the elliptic Calogero-Moser model while the second is gauge equivalent to the Ruijsenaars-Schneider model.

\[\text{1Here and elsewhere we mean } S_{0} = S_{00}, \text{ and } 1_{N} \text{ is the identity matrix of size } N \times N.\]

3
**Z₂ reduction.** The elliptic top (1.8) allows the following reduction:

\[ S_\alpha = S_{-\alpha}, \quad \forall \alpha. \]  

(1.19)

At the level of the Lax matrix (1.9) it is generated by condition

\[ L(-z) = -hL(-z)h^{-1}, \quad h = J\Lambda^{-1}, \quad J_{ij} = \delta_{i,N-j+1}. \]  

(1.20)

Constraints (1.19) follow from (1.20) due to \( h^T \alpha h \) \( = T_{\alpha} \). The general idea of \( Z₂ \) type reduction appeared first in [16]. For details of (1.19)-(1.20) see Appendix in [13]. In relativistic case the set of constraints (1.19) is replaced by

\[ S_\alpha \varphi_{\alpha}(\eta, \omega_{\alpha}) = S_{-\alpha} \varphi_{-\alpha}(\eta, -\omega_{\alpha}), \quad \forall \alpha. \]  

(1.21)

which is in agreement with the transition (1.18). The reduced models are integrable, and their Lax matrices are restrictions of the initial Lax matrices to the submanifolds in the phase spaces defined by (1.19) or (1.21) respectively.

This paper is a kind of a brief review of several previous articles [9, 12, 14, 25, 27]. The aim is have a fresh approach and some new results by focus attention on possible Lax pairs and a symmetry between relativistic and spectral parameters arising from the lattice Fourier transformation.

### 2 Finite Fourier transformation on lattice \( \mathbb{Z}_N \times \mathbb{Z}_N \)

The following set of identities hold true:

\[ \frac{1}{N} \sum_{\alpha} \kappa_{\alpha,\gamma}^2 \varphi_{\alpha}(Nh, \omega_{\alpha} + \frac{z}{N}) = \varphi_{\gamma}(z, \omega_{\gamma} + h) \]  

(2.1)

or, equivalently,

\[ \frac{1}{N} \sum_{\alpha} \kappa_{\alpha,\gamma}^2 \varphi_{\alpha}(z, \omega_{\alpha} + h) = \varphi_{\gamma}(Nh, \omega_{\gamma} + \frac{z}{N}) \]  

(2.2)

with \( \kappa_{\alpha,\gamma} \) (1.6). The proof can be found in [12]. In the limit \( z \to 0 \) the identity (2.2) yields

\[ \frac{1}{N} \sum_{\alpha} \left( E_1(\omega_{\alpha} + h) + 2\pi \partial_i \omega_{\alpha} \right) = E_1(Nh) \]  

(2.3)

for \( \gamma = 0 \) and

\[ \frac{1}{N} \sum_{\alpha} \kappa_{\alpha,\gamma}^2 \left( E_1(\omega_{\alpha} + h) + 2\pi \partial_i \omega_{\alpha} \right) = \varphi_{\gamma}(Nh, \omega_{\gamma}) \]  

(2.4)

for \( \gamma \neq 0 \). The simple pole in (2.2) at \( z = 0 \) is cancelled due to \( 2 \sum_{\alpha} \kappa_{\alpha,\gamma}^2 = N^2 \delta_{\gamma,0} \).

\[ \sum_{\alpha} \kappa_{\alpha,\gamma}^2 = N^2 \delta_{\gamma,0}. \]  

(2.5)

The permutation operator in the basis (1.5) is \( P_{12} = (1/N) \sum_{\alpha} T_\alpha \otimes T_{-\alpha} \). In this respect (2.5) is equivalent to \( P_{12}^2 = 1_N \otimes 1_N \).
Similarly, in the limit $\hbar \to 0$ the identity (2.2) yields
\[
\frac{1}{N} E_1(z) + \frac{1}{N} \sum_{\alpha \neq 0} \kappa_{\alpha,\gamma}^2 \varphi_\alpha(z, \omega_\alpha) = E_1(\omega_\gamma + \frac{z}{N}) + 2\pi i \partial_\tau \omega_\gamma.
\] (2.6)

Taking the limit $\hbar \to 0$ in (2.3) or the limit $z \to 0$ in (2.6) with $\gamma = 0$ we get:
\[
\frac{1}{N} \sum_{\alpha \neq 0} \left( E_1(\omega_\alpha) + 2\pi i \partial_\tau \omega_\alpha \right) = 0,
\] (2.7)

while the limit $\hbar \to 0$ in (2.4) or the limit $z \to 0$ in (2.6) with $\gamma \neq 0$ yields:
\[
\frac{1}{N} \sum_{\alpha \neq 0} \kappa_{\alpha,\gamma}^2 \left( E_1(\omega_\alpha) + 2\pi i \partial_\tau \omega_\alpha \right) = E_1(\omega_\gamma) + 2\pi i \partial_\tau \omega_\gamma.
\] (2.8)

The derivative of (2.3) with respect to $\hbar$ gives
\[
\sum_{\alpha} E_2(\omega_\alpha + \hbar) = N^2 E_2(N\hbar) \quad \text{or} \quad \sum_{\alpha} \varphi(\omega_\alpha + \hbar) = N^2 \varphi(N\hbar).
\] (2.9)

Similarly from (2.4) for $\gamma \neq 0$ we get
\[
\sum_{\alpha} \kappa_{\alpha,\gamma}^2 E_2(\omega_\alpha + \hbar) = -N^2 \varphi(\gamma, \omega_\gamma)(E_1(N\hbar + \omega_\gamma) - E_1(N\hbar) + 2\pi \partial_\tau \omega_\gamma).
\] (2.10)

With the limit $\hbar \to 0$ in (2.9) and the relation (A.5) we obtain
\[
\sum_{\alpha \neq 0} \varphi(\omega_\alpha) = 0 \quad \text{or} \quad \sum_{\alpha \neq 0} E_2(\omega_\alpha) = -\frac{N^2 - 1}{3} \frac{\varphi''(0)}{\varphi'(0)}.
\] (2.11)

The derivative of (2.2) with respect to $\hbar$ at $\hbar = 0$ gives
\[
\frac{1}{2} \left( E_1^2(z) - \varphi(z) \right) + \sum_{\alpha \neq 0} \kappa_{\alpha,\gamma}^2 f_\alpha(z, \omega_\alpha) = \frac{N^2}{2} \left( (E_1(\omega_\gamma + \frac{z}{N}) + 2\pi \partial_\tau \omega_\gamma)^2 - \varphi(\omega_\gamma + \frac{z}{N}) \right).
\] (2.12)

Similarly, from (2.1) we have
\[
\sum_{\alpha} \left( (E_1(\omega_\alpha + \frac{z}{N}) + 2\pi \partial_\tau \omega_\alpha)^2 - \varphi(\omega_\alpha + \frac{z}{N}) \right) = E_1^2(z) - \varphi(z)
\] (2.13)

and
\[
\frac{1}{2} \sum_{\alpha} \kappa_{\alpha,\gamma}^2 \left( (E_1(\omega_\alpha + \frac{z}{N}) + 2\pi \partial_\tau \omega_\alpha)^2 - \varphi(\omega_\alpha + \frac{z}{N}) \right) = f_\gamma(z, \omega_\gamma)
\] (2.14)

for $\gamma \neq 0$. 

5
3 Matrix extensions of tops

Let us apply the Fourier transformation (2.1) to the Lax matrix of the relativistic top (1.11), where we replace \( \eta \) by \( \eta/N \):

\[
L(z) = \sum_{\alpha} T_{\alpha} S_{\alpha} \varphi_{\alpha}(z, \omega_{\alpha} + \frac{\eta}{N}) = \sum_{\beta} \left( \frac{1}{N} \sum_{\alpha} \kappa_{\beta\alpha}^{2} T_{\alpha} S_{\alpha} \right) \varphi_{\beta}(\eta, \omega_{\beta} + \frac{z}{N}).
\]  

(3.1)

The arguments \( z \) and \( \eta \) of functions \( \varphi_{\gamma} \) are interchanged but the coefficients behind \( \varphi_{\beta}(\eta, \omega_{\beta} + \frac{z}{N}) \) are not proportional to \( T_{\beta} \), they are full matrices instead. It is then natural to start with a more general matrix

\[
A(z, \eta) = \sum_{\alpha} A_{\alpha} \varphi_{\alpha}(z, \omega_{\alpha} + \frac{\eta}{N}),
\]

(3.2)

where \( A_{\alpha} \in \text{Mat}(K, \mathbb{C}) \). Then from (2.1) we conclude that

\[
A(z, \eta) = \sum_{\beta} \tilde{A}_{\beta} \varphi_{\beta}(\eta, \omega_{\beta} + \frac{z}{N}),
\]

(3.3)

where

\[
\tilde{A}_{\beta} = \frac{1}{N} \sum_{\alpha} \kappa_{\beta\alpha}^{2} A_{\alpha}.
\]

(3.4)

In (3.2)-(3.4) \( N^{2} \) matrices \( A_{\alpha} \) are of arbitrary size \( K \). Notice that if \( K = N \) then the following two reductions to the relativistic \( gl_{N} \) top are allowed:

1. \( A_{\alpha} = T_{\alpha} S_{\alpha} \)
2. \( \tilde{A}_{\alpha} = T_{\alpha} \tilde{S}_{\alpha} \)

(3.5)

In the first case \( z \) is the spectral parameter while \( \eta/N \) is relativistic one. In the second case conversely \( \eta \) is the spectral parameter and \( z/N \) is relativistic one. These two reductions are of course incompatible.

In general case the matrix \( A(z, \eta) \) is not Lax matrix of an integrable system (or, at least, it is unknown) but there are several more examples of integrable reductions besides (3.4).

**Matrix tops.** If \( K = NM \) then the following reductions are available (these are the matrix extensions of the relativistic top [13]):

1. \( A_{\alpha} = T_{\alpha} \otimes S_{\alpha}, S_{\alpha} \in \text{Mat}(M, \mathbb{C}) \),
2. \( \tilde{A}_{\alpha} = T_{\alpha} \otimes \tilde{S}_{\alpha}, \tilde{S}_{\alpha} \in \text{Mat}(M, \mathbb{C}) \),

\[
S_{0} = S_{0}1_{M}, \quad \tilde{S}_{0} = \tilde{S}_{0}1_{M},
\]

(3.6)

\[
\frac{S_{\alpha}}{\varphi_{\alpha}(\eta/N, \omega_{\alpha})} = \frac{S_{\alpha}}{\varphi_{-\alpha}(\eta/N, -\omega_{\alpha})}, \quad \frac{\tilde{S}_{\alpha}}{\varphi_{\alpha}(z/N, \omega_{\alpha})} = \frac{\tilde{S}_{\alpha}}{\varphi_{-\alpha}(z/N, -\omega_{\alpha})}
\]

The left column case is described by the Lax pair

\[
L^{n}(z, S) = \sum_{\alpha} T_{\alpha} \otimes S_{\alpha} \varphi_{\alpha}(z, \omega_{\alpha} + \frac{\eta}{N}), \quad M^{n}(z, S) = -\sum_{\alpha \neq 0} T_{\alpha} \otimes S_{\alpha} \varphi_{\alpha}(z, \omega_{\alpha}),
\]

(3.7)
which provides equations of motion

\[
\dot{S}_\alpha = \sum_{\beta, \gamma; \beta + \gamma = \alpha} \left( \kappa_{\beta, \gamma} S_{\beta} S_{\gamma} - \kappa_{\gamma, \beta} S_{\gamma} S_{\beta} \right) J^{\eta/N}_\gamma, \quad \alpha \neq 0; \quad J^{\eta/N}_\alpha = E_1 \left( \frac{\eta}{N} + \omega_\alpha \right) - E_1 (\omega_\alpha). \tag{3.8}
\]

The constraints (the second and the third lines of the left column of (3.6)) are conserved on equations of motion. They are sufficient conditions for the Lax equations to be fulfilled. When \( M = 1 \) the variables \( S_\alpha \) become scalar \( S_\alpha \) and (3.8) turns into (1.15).

In the Fourier dual case (right column of (3.6)) the roles of \( z \) and \( \eta \) are interchanged. This time \( \eta \) is the spectral parameter, and \( z \) is the relativistic parameter. Equations of motion and the Lax pair are obtained by replacement \((z, \eta, S) \leftrightarrow (\eta, z, \tilde{S})\) in (3.7), (3.8).

**Gaudin like matrix tops on** \( \mathbb{Z}_N \times \mathbb{Z}_N \). Let us return to the general case (3.2) with \( N \) and \( K \) be arbitrary integers. From (3.3) by changing summation index \( \beta \rightarrow -\beta \) we see that \( A(z, \eta) \) as matrix valued function on the elliptic curve with periods \( N, N\tau \) has simple poles at points \( N\omega_\alpha = \alpha_1 + \tau \alpha_2 \) with residues \( \tilde{A}^{-\alpha} \). In this respect it is the model of Gaudin type. Suppose the set of \( N^2 \) matrices \( A^\alpha \in \text{Mat}(K, \mathbb{C}), \alpha \in \mathbb{Z}_N \times \mathbb{Z}_N \) satisfies

\[
1. \quad A^0 = A_0 1_K, \quad \quad 2. \quad \tilde{A}^0 = \tilde{A}_0 1_K,
\]

\[
\frac{A^\alpha}{\varphi_\alpha(z/N, \omega_\alpha)} = \frac{A^{-\alpha}}{\varphi_{-\alpha}(z/N, -\omega_\alpha)} \quad \quad \frac{\tilde{A}^\alpha}{\varphi_\alpha(z/N, \omega_\alpha)} = \frac{\tilde{A}^{-\alpha}}{\varphi_{-\alpha}(z/N, -\omega_\alpha)} \tag{3.9}
\]

Then (similarly to the previous case) the Lax equations with the Lax pair

\[
L^\eta(z, \{ A^\alpha \}) = \sum_\alpha A^\alpha \varphi_\alpha(z, \omega_\alpha + \frac{\eta}{N}), \quad M^\eta(z, \{ A^\alpha \}) = -\sum_{\alpha \neq 0} A^\alpha \varphi_\alpha(z, \omega_\alpha), \tag{3.10}
\]

provides the following equations of motion

\[
\dot{A}^\alpha = \sum_{\beta, \gamma; \beta + \gamma = \alpha} [A^\beta, A^{\gamma}] J^{\eta/N}_\gamma, \quad \alpha \neq 0; \quad J^{\eta/N}_\alpha = E_1 \left( \frac{\eta}{N} + \omega_\alpha \right) - E_1 (\omega_\alpha). \tag{3.11}
\]

Again, the Lax pair and equations of motion for the Fourier dual model are obtained by replacement \((z, \eta, A^0) \leftrightarrow (\eta, z, \tilde{A}^0)\), i.e. the Lax matrices are of the same form due to \( L^\eta(z, \{ A^\alpha \}) = L^\eta(\eta, \{ \tilde{A}^{\alpha} \}) = A(z, \eta) \) but with different reductions constraints, and the \( M \)-matrices are different.

**(In)compatibility of reductions and Euler case.** In general case any reduction of the first type is incompatible with a reduction of the second type. It happens because (3.5) leaves only one non-zero matrix component for each matrix \( A^\alpha \) (or \( \tilde{A}^\alpha \)), while the reductions (3.6) and (3.9) explicitly depend on the corresponding relativistic parameter. The latter plays the role of the spectral parameter in Fourier dual type of reduction but dynamical variables should be independent of the spectral parameter. However, there is a special case when two reductions of different types are compatible. It is \( N = 2 \) case because the second line of (3.9) (or the third line of (3.6)) is fulfilled by itself due to \( \alpha = -\alpha \mod \mathbb{Z}_2 \times \mathbb{Z}_2 \) in this case. Therefore, the conditions given by the second line of (3.9) (or the third line of (3.6) are identities. This case is the matrix extension of the Euler top (1.1)-(1.3). Consider two reductions in (3.9). The matrix
of the Fourier transformation $\kappa_{\alpha,\beta}^2/N$ is of the size $N^2 \times N^2$ because $\alpha, \beta \in \mathbb{Z}_N \times \mathbb{Z}_N$. In $N = 2$ case it is $4 \times 4$:

$$
\begin{pmatrix}
\tilde{A}^{(0,0)} \\
\tilde{A}^{(1,0)} \\
\tilde{A}^{(0,1)} \\
\tilde{A}^{(1,1)}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix} \begin{pmatrix}
A^{(0,0)} \\
A^{(1,0)} \\
A^{(0,1)} \\
A^{(1,1)}
\end{pmatrix}
$$

(3.12)

If all conditions of (3.9) are satisfied then

$$
A^{(0,0)} \sim 1_K \text{ and } \tilde{A}^{(0,0)} = \frac{1}{2}(A^{(0,0)} + A^{(0,1)} + A^{(1,0)} + A^{(1,1)}) \sim 1_K,
$$

(3.13)

and we are left with three matrices $A^{(0,1)}, A^{(1,0)}, A^{(1,1)}$ satisfying $A^{(0,1)} + A^{(1,0)} + A^{(1,1)} \sim 1_K$.

4. GL$_N \times$ GL$_M$ models

In this section we describe GL$_N \times$ GL$_M$ models. They are generalizations of the relativistic top (1.11)-(1.12), where dependence on the parameter $\eta$ is similar to dependence on the spectral parameter $z$. The study of such models is inspired by $R$-matrix description of integrable tops, and existence of the symmetric $R$-matrix with (almost) symmetric dependence on the spectral parameter and the Planck constant. We review briefly these constructions, and then describe GL$_N \times$ GL$_M$ models in much the same way as in the previous section. As a by-product of the $R$-matrix construction we have simple formulation (4.5) of the Fourier transformations (2.1).

**R-matrix description.** The Lax pairs, equations of motion and Hamiltonian description of integrable tops under consideration can be formulated in terms of $GL(N)$ quantum $R$-matrix $R^h_{12}(z) \in \text{Mat}(N, \mathbb{C})^{\otimes 2}$ satisfying the following properties:

unitarity: $R^h_{12}(z)R^h_{21}(-z) \sim 1_N \otimes 1_N$,

associative Yang-Baxter equation [6, 21]:

$$
R^h_{12}(z_1 - z_2)R^h_{23}(z_2 - z_3) = R^h_{13}(z_1 - z_3)R^h_{12}(z_1 - z_2) + R^h_{23}(z_2 - z_3)R^h_{13}(z_1 - z_3),
$$

(4.1)

classical limit (expansion near $\hbar = 0$):

$$
R^h_{12}(z) = \frac{1}{\hbar} 1 \otimes 1 + r_{12}(z) + \hbar m_{12}(z) + O(\hbar^2).
$$

Namely, it can be shown that with the properties (4.1) the Lax pair

$$
L^n(z, S) = \text{tr}_2(R^n_{12}(z)S_2), \quad M^n(z, S) = -\text{tr}_2(r_{12}(z)S_2)
$$

(4.2)

provides equations of motion (1.2)

$$
\dot{L}^n(z, S) = [L^n(z, S), M^n(z, S)], \quad \forall z \quad \iff \quad \dot{S} = [S, J^n(S)].
$$

(4.3)

In the elliptic case the $R$-matrix is the Belavin’s one [4]. Being written in the form

$$
R^h_{12}(z) = \sum_{\alpha} T_\alpha \otimes T_{-\alpha} \varphi_\alpha(z, \omega_\alpha + \hbar)
$$

(4.4)
it reproduces a Lax pair \( (1.11) \) via \( (4.2) \). Details of this construction can be found in \([13]\). It is based on the fact that \( R \)-matrix \( (4.4) \) with properties \( (1.1) \) can be viewed as matrix generalization of the Kronecker function \( (A.1) \) \([6, 11, 12, 27]\). In particular, the trivial property \( \phi(z, h) = \phi(h, z) \) of the Kronecker function is generalized to

\[
R_{12}^h(z)P_{12} = R_{12}^{2/N}(Nh), \quad P_{12} = \frac{1}{N} \sum_\alpha T_\alpha \otimes T_{-\alpha}, \tag{4.5}
\]

where \( P_{12} \) is the permutation operator. The latter relation \( (4.5) \) is equivalent to the set of finite Fourier transformations \( (2.1) \).

**Symmetric \( R \)-matrix.** Functions \( \varphi_\alpha(z, \omega_\alpha + \eta) \) entering \( R \)-matrix \( (4.4) \) as well as the Lax matrix of the relativistic top \( (1.11) \) and its matrix extension \( (3.2) \) have a single simple pole at \( z = 0 \) on the elliptic curve \( \Sigma \) with coordinate \( z \) and moduli \( \tau \) (and periods \( 1, \tau \)). On the other hand, as functions of \( \eta \) they have simple poles at \( \eta = -\omega_\alpha \). In \( GL(N) \times GL(M) \) case there is a possibility to make dependence on \( z \) and \( \eta \) in much the same way.

Let \( N \) and \( M \) be positive integers. For simplicity we also assume that \( N \) and \( M \) are coprime \( \text{g.c.d.}(N, M) = 1 \). Consider the following set of \( N^2M^2 \) functions

\[
\Phi_{\alpha, \tilde{\alpha}}(z, \eta) = \exp \left( 2\pi i (z + N\tilde{\omega}_{\tilde{\alpha}}) \frac{\alpha_2}{N} + 2\pi i N \frac{\tilde{\omega}_{\tilde{\alpha}}}{M} \right) \phi(z + N\tilde{\omega}_{\tilde{\alpha}}, \eta + \omega_\alpha) =
\]

\[
= \exp \left( 2\pi i N \frac{\tilde{\omega}_{\tilde{\alpha}}}{M} \right) \varphi_\alpha(z + N\tilde{\omega}_{\tilde{\alpha}}, \eta + \omega_\alpha), \tag{4.6}
\]

where

\[
\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_N \times \mathbb{Z}_N, \quad \omega_\alpha = \frac{\alpha_1 + \alpha_2 \tau}{N}, \quad \tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2) \in \mathbb{Z}_M \times \mathbb{Z}_M, \quad \tilde{\omega}_{\tilde{\alpha}} = \frac{\tilde{\alpha}_1 + \tilde{\alpha}_2 \tau}{M}.
\]

We will keep tildes for the variables related to \( \text{Mat}(M, \mathbb{C}) \). The functions \( \Phi_{\alpha, \tilde{\alpha}}(z, \eta) \) are almost symmetrical (but not completely) with respect to simultaneous interchanging of \( z \leftrightarrow \eta \) and \( N \leftrightarrow M \). The reason for non-symmetric dependence comes from our additional requirement on \( \Phi_{\alpha, \tilde{\alpha}}(z, \eta) \) to be periodic with respect to shifts of discrete variables:

\[
\Phi_{(\alpha_1 + N, \alpha_2), \tilde{\alpha}}(z, \eta) = \Phi_{(\alpha_1, \alpha_2 + N), \tilde{\alpha}}(z, \eta) = \Phi_{\alpha, \tilde{\alpha}}(z, \eta), \tag{4.7}
\]

\[
\Phi_{\alpha, (\tilde{\alpha}_1 + M, \tilde{\alpha}_2)}(z, \eta) = \Phi_{\alpha, (\tilde{\alpha}_1, \tilde{\alpha}_2 + M)}(z, \eta) = \Phi_{\alpha, \tilde{\alpha}}(z, \eta).
\]

The properties \( (4.7) \) can be verified using \( (A.6) \). With these properties the functions \( \Phi_{\alpha, \tilde{\alpha}}(z, \eta) \) are well defined on \( \alpha \in \mathbb{Z}^2, \tilde{\alpha} \in \mathbb{Z}^2 \) — they are periodic with periods \( N \) and \( M \). The latter is equivalent to the definition of the functions on \( \mathbb{Z}_N^2 \times \mathbb{Z}_M^2 \) and convenient for summation (subtraction) of indices.

Analogues of identities \( (A.13) \) and \( (A.14) \) are easily obtained. Due to \( (4.6) \) \( \Phi_{\alpha, \tilde{\alpha}}(z, 0) = \varphi_\alpha(z + N\tilde{\omega}_{\tilde{\alpha}}, \omega_\alpha) \), and we get the following identity from \( (A.13) \):

\[
\Phi_{\beta, \tilde{\beta}}(z, \eta) \Phi_{\gamma, \tilde{\gamma}}(z, 0) = \Phi_{\beta, \tilde{\beta} - \gamma}(0, \eta) \Phi_{\beta + \gamma, \tilde{\gamma}}(z, \eta) + \Phi_{\gamma, \tilde{\gamma} - \beta}(0, 0) \Phi_{\beta + \gamma, \tilde{\beta}}(z, \eta). \tag{4.8}
\]

---

\( ^3 \)The \( M \)-matrix from \( (4.2) \) differs from the one \( (1.11) \) by scalar component \( 1_N E_1(z)S_{(0,0)} \) which is cancelled out from the Lax equations.
It is valid for $\gamma \neq 0$ and $\tilde{\beta} \neq \tilde{\gamma}$. If $\tilde{\beta} = \tilde{\gamma}$ then from (A.14) we have
\[
\Phi_{\beta,\tilde{\beta}}(z, \eta) \Phi_{\gamma,\tilde{\gamma}}(z, 0) = \\
= \Phi_{\beta+\gamma,\tilde{\beta}}(z, \eta)(E_1(z + N\tilde{\omega}_{\tilde{\beta}}) + E_1(\eta + \omega_{\beta}) + E_1(\omega_{\gamma}) - E_1(z + \eta + \omega_{\beta+\gamma} + N\tilde{\omega}_{\tilde{\beta}})).
\] (4.9)
Alternatively,
\[
\Phi_{\beta,\tilde{\beta}}(z, \eta) \Phi_{\gamma,\tilde{\gamma}}(0, \eta) = \Phi_{\beta-\gamma,\tilde{\beta}}(z, 0) \Phi_{\gamma-\beta,\tilde{\gamma}}(z, \eta) + \Phi_{\gamma-\beta,\tilde{\gamma}}(0, 0) \Phi_{\beta,\tilde{\beta}+\gamma}(z, \eta),
\] (4.10)
which is valid for $\tilde{\gamma} \neq 0$ and $\beta \neq \gamma$. In other cases
\[
\Phi_{\beta,\tilde{\beta}}(z, \eta) \Phi_{\beta,\tilde{\gamma}}(0, \eta) = \\
= \Phi_{\beta,\tilde{\beta}+\gamma}(z, \eta)(E_1(z + N\tilde{\omega}_{\tilde{\beta}}) + E_1(N\tilde{\omega}_{\tilde{\gamma}}) + E_1(\eta + \omega_{\beta}) - E_1(z + \eta + N\tilde{\omega}_{\tilde{\beta}+\gamma} + \omega_{\beta})).
\] (4.11)
The set of functions (4.6) allows to define the so-called symmetric $R$-matrix [14]:
\[
\mathcal{R}_{12,\tilde{1}2}(z, h) = \sum_{\alpha \in \mathbb{Z}_N \times \mathbb{Z}_N} \sum_{\delta \in \mathbb{Z}_M \times \mathbb{Z}_M} \Phi_{\alpha,\delta}(z, h) T_{\alpha} \otimes T_{-\alpha} \otimes \tilde{T}_{\alpha} \otimes \tilde{T}_{-\alpha},
\] (4.12)
where $\{\tilde{T}_\alpha\}$ is the basis in $\text{Mat}(M, \mathbb{C})$ [15]-[17] with $N$ replaced by $M$. It satisfies the unitarity condition $\mathcal{R}_{12,\tilde{1}2} \mathcal{R}_{21,\tilde{1}2} = 1_N \otimes 1_N \otimes 1_M \otimes 1_M$ and the associative Yang-Baxter equation
\[
\mathcal{R}_{12,\tilde{1}2} \mathcal{R}_{23,\tilde{3}2} = \mathcal{R}_{13,\tilde{3}2} \mathcal{R}_{12,\tilde{1}3} + \mathcal{R}_{23,\tilde{3}1} \mathcal{R}_{13,\tilde{1}2},
\] (4.13)
where $\mathcal{R}_{ab,\tilde{a}b}(z_a - z_b, h_a - h_b)$. These are the first and the second conditions from (4.11). The third condition (classical limit) is not fulfilled because the simplest rational analogue of (4.12) is
\[
\mathcal{R}_{12,\tilde{1}2}(z, h) = M 1_N \otimes 1_N \otimes \tilde{P}_{\tilde{1}2} + N P_{12} \otimes \tilde{1}_M \otimes 1_M.
\] (4.14)
So, the expansion of $\mathcal{R}_{12,\tilde{1}2}(z, h)$ (4.12) near $h = 0$ does not starts with $1_{NM} \otimes 1_{NM}$ as well as similar expansion near $z = 0$. At the same time in $M = 1$ case (4.12) reproduces $R_{12}^h(z)$ (4.4), and for $N = 1$ it yields $R_{12}^z(h)$:
\[
R_{12}^h(z) \quad \text{and} \quad R_{12}^z(h)
\] (4.15)
In this respect $\mathcal{R}_{12,\tilde{1}2}(z, h)$ is an intermediate case between $GL(NM)$-valued Belavin’s $R$-matrices $R_{12}^h(h)$ and $R_{12}^z(z)$. More precisely, since $\mathbb{Z}_{NM} \cong \mathbb{Z}_N \times \mathbb{Z}_M$ ($N$ and $M$ are coprime) we can perform the Fourier transformation on sublattice $\mathbb{Z}_N \times \mathbb{Z}_N$ or $\mathbb{Z}_M \times \mathbb{Z}_M$:
\[
\mathcal{R}_{12,\tilde{1}2}(z, h) P_{12} \otimes \tilde{1}_M \otimes 1_M = R_{12}^{z/N}(Nh),
\] (4.16)
\[
\mathcal{R}_{12,\tilde{1}2}(z, h) 1_N \otimes 1_N \otimes \tilde{P}_{\tilde{1}2} = R_{12}^{h/M}(Mz),
\] where in r.h.s. we mean $R$-matrix [4.4] with $N := NM$. 10
GL_N × GL_M models. The fact that the classical limit of (4.12) is not of the form required in (4.1) (in both z and \(\hbar\)) means that the ansatz (4.2) does not work. However, it is not of great importance for matrix extensions because of additional constraints. For example, the Lax pair of the matrix top (3.7) is related through (4.2) with R-matrix \(R_{12,\bar{1}2}(z) = R_{12}^\eta(z) \otimes \bar{T}_{12}\), which has the wrong first term in the classical limit. This problem is solved by imposing constraints (3.6). The same phenomenon takes place for \(\mathcal{R}_{12,\bar{1}2}(z, \hbar)\).

Consider the following matrix-valued function on \(\Sigma_{z,\tau} \times \Sigma_{N,M}\):

\[
\mathcal{A}(z, \eta) = \sum_{\alpha, \bar{\alpha}} A^{\alpha, \bar{\alpha}} \Phi_{\alpha, \bar{\alpha}}(z, \eta) \in \text{Mat}(K, \mathbb{C}), \quad \alpha \in \mathbb{Z}_N^x, \bar{\alpha} \in \mathbb{Z}_M^x.
\] (4.17)

Suppose first that the spectral parameter is \(z\). Using the definitions (4.6) and (A.10) rewrite \(\Phi_{\alpha, \bar{\alpha}}(z, \eta)\) as follows:

\[
\Phi_{\alpha, \bar{\alpha}}(z, \eta) = \tilde{k}_{\alpha, \gamma, \bar{\alpha}}^2 \exp(2\pi iz \frac{\omega_\alpha}{N}) \varphi_{\alpha, N\gamma}(\eta + \omega_\alpha, z + N\tilde{\omega}_\alpha), \quad \tilde{k}_{\alpha, \gamma, \bar{\alpha}} = \exp(\pi i \frac{\tilde{\alpha}_1 \omega_2 - \omega_1 \alpha_2}{M}).
\] (4.18)

By applying the Fourier transform on \(\mathbb{Z}_M \times \mathbb{Z}_M\) (2.2) we get

\[
\Phi_{\alpha, \bar{\alpha}}(z, \eta) = \frac{1}{M} \sum_{\bar{\alpha}} \tilde{k}_{\alpha, N\gamma, \bar{\alpha}}^2 \varphi_{\alpha, N\gamma}(Mz, \omega_\alpha + \eta + \frac{\eta}{M}),
\] (4.19)

where \(\tilde{\omega}_\gamma + \frac{\omega_\alpha + \eta}{M} = \frac{1}{M} \omega_{\alpha + N\gamma} \in \mathbb{Z}_M \otimes \tau \mathbb{Z}_M\). Therefore, denoting \(\alpha + N\bar{\gamma} = a \in \mathbb{Z}_M^x\) we come to

\[
\mathcal{A}(z, \eta) = \sum_{\alpha \in \mathbb{Z}_N^x} \tilde{A}^a \varphi_a(Mz, \omega_\alpha + \frac{\eta}{M}), \quad \tilde{A}^a = \tilde{A}^{\alpha + N\bar{\gamma}} = \frac{1}{M} \sum_{\bar{\alpha} \in \mathbb{Z}_M^x} \tilde{k}_{\alpha, N\gamma, \bar{\alpha}}^2 \mathcal{A}^{\alpha, \bar{\alpha}},
\] (4.20)

i.e. to the matrix function of form (3.12) with \(N := NM\), which become Lax matrix in cases (3.5), (3.6) and (3.9).

Suppose now that the spectral parameter in (4.17) is \(\eta\). Similarly to the previous case, apply the Fourier transform (2.1) to \(\Phi_{\alpha, \bar{\alpha}}(z, \eta)\) on the lattice \(\mathbb{Z}_N \times \mathbb{Z}_N\):

\[
\Phi_{\alpha, \bar{\alpha}}(z, \eta) = \frac{1}{N} \exp(2\pi i N\eta \tilde{\omega}_\alpha) \sum_{\bar{\alpha}} \tilde{k}_{\alpha, \gamma, \bar{\alpha}}^2 \varphi_{\gamma, N\gamma}(N\eta + \omega_\alpha, \omega_\gamma + \frac{\omega_\gamma}{N}).
\] (4.21)

Again \(\tilde{\omega}_\alpha + \omega_\gamma = \frac{1}{N} \tilde{\omega}_{N\alpha + M\gamma} \in \mathbb{Z}_M \otimes \tau \mathbb{Z}_M\). Therefore, denoting \(M\gamma + N\bar{\alpha} = a \in \mathbb{Z}_M^x\) we have

\[
\mathcal{A}(z, \eta) = \sum_{\alpha \in \mathbb{Z}_N^x} \tilde{A}^a \varphi_a(N\eta, \omega_\alpha + \frac{\omega_\gamma}{N}), \quad \tilde{A}^a = \tilde{A}^{N\bar{\alpha} + M\gamma} = \frac{1}{N} \sum_{\bar{\alpha} \in \mathbb{Z}_M^x} \tilde{k}_{\gamma, \alpha, \bar{\alpha}}^2 A^{\alpha, \bar{\alpha}},
\] (4.22)

i.e. the matrix function of the form (3.12) with \(N := NM\).

Conversely, we can start with \(A(x, y)\) in the form (3.2):

\[
A(x, y) = \sum_{\alpha \in \mathbb{Z}_N^x} A^\alpha \varphi_\alpha(x, \omega_\alpha + y) =
\] (4.23)
\[ \sum_{\alpha \in \mathbb{Z}_N^2} \sum_{\tilde{\alpha} \in \mathbb{Z}_M^2} A^{\alpha, \tilde{\alpha}} \exp(2\pi i x (M\alpha_2 + N\tilde{\alpha}_2) / NM) \phi(x, \omega_\alpha + \tilde{\omega}_\tilde{\alpha} + y), \]

and apply the Fourier transformations on \( \mathbb{Z}_N^2 \) and \( \mathbb{Z}_M^2 \):

\[ \exp(2\pi i x (M\alpha_2 + N\tilde{\alpha}_2) / NM) \phi(x, \omega_\alpha + \tilde{\omega}_\tilde{\alpha} + y) = \frac{1}{N} \sum_{\gamma \in \mathbb{Z}_N^2} \kappa_{\gamma, \alpha}^2 \Phi_{\gamma, \alpha}(Ny, x N), \] (4.24)

\[ \exp(2\pi i x (M\alpha_2 + N\tilde{\alpha}_2) / NM) \phi(x, \omega_\alpha + \tilde{\omega}_\tilde{\alpha} + y) = \frac{1}{M} \sum_{\gamma \in \mathbb{Z}_M^2} \kappa_{\gamma, \alpha}^2 \tilde{\Phi}_{\gamma, \alpha}(My, x M), \] (4.25)

where \( \tilde{\Phi} \) is obtained from \( \Phi \) (4.6) by interchanging \( N \) and \( M \).

First, perform the Fourier transform on \( \mathbb{Z}_N^2 \) (4.24). Then

\[ A(x, y) = \sum_{\gamma \in \mathbb{Z}_N^2} \sum_{\tilde{\alpha} \in \mathbb{Z}_M^2} \tilde{A}^{\gamma, \tilde{\alpha}} \Phi_{\gamma, \tilde{\alpha}}(Ny, x N), \quad \tilde{A}^{\gamma, \tilde{\alpha}} = \frac{1}{N} \sum_{\alpha \in \mathbb{Z}_N^2} \kappa_{\gamma, \alpha}^2 A^{\alpha, \tilde{\alpha}}. \] (4.26)

Second, perform the Fourier transform on \( \mathbb{Z}_M^2 \) (4.25). Then

\[ A(x, y) = \sum_{\alpha \in \mathbb{Z}_N^2} \sum_{\gamma \in \mathbb{Z}_M^2} \tilde{A}^{\alpha, \gamma} \tilde{\Phi}_{\alpha, \gamma}(My, x M), \quad \tilde{A}^{\alpha, \gamma} = \frac{1}{M} \sum_{\tilde{\alpha} \in \mathbb{Z}_M^2} \kappa_{\gamma, \tilde{\alpha}}^2 A^{\alpha, \tilde{\alpha}}. \] (4.27)

In both cases (4.26) and (4.27) we can substitute \( x = z, y = \eta \) or \( x = \eta, y = z \). These two substitutions are related by the Fourier transform (3.3), (3.4) on \( \mathbb{Z}^{\times 2}_{NM} \). Thus, there are four natural possible forms of the matrix function (3.2), and (4.17) is one of these.

Matrix Gaudin models. The matrix (4.17) can be viewed as matrix generalization of the special Gaudin models with marked points at \( \{ \omega_\alpha \} \) or \( \{ \tilde{\omega}_\tilde{\alpha} \} \). Indeed, the Gaudin model based on the relativistic top (1.11) is defined by the Lax matrix\(^6\)

\[ L(z) = \sum_{k=1}^n \sum_{\alpha} T_\alpha S^k_\alpha \varphi_\alpha(z - z_k, \eta + \omega_\alpha). \] (4.28)

The set of matrices \( S^k = \sum_{\alpha} T_\alpha S^k_\alpha \) are residues at \( z = z_k \). In our case the marked points \( z_k \) are given by the set \( \{ \omega_\alpha, \alpha \in \mathbb{Z}_N^2 \} \) or \( \{ \tilde{\omega}_\tilde{\alpha}, \alpha \in \mathbb{Z}_M^2 \} \). To be precise, consider (4.26) with \( y = z/N \) and \( x = N \eta \) (\( z \) is supposed to be spectral parameter). Impose reduction

\[ \tilde{A}^{\gamma, \tilde{\alpha}} = \exp(-2\pi i \eta \tilde{\alpha}_2) S^{-N\tilde{\alpha}_2}_\gamma T_\gamma. \] (4.29)

Then the matrix (4.26) acquires the form (4.28) with marked points at \( z = -N \tilde{\omega}_\tilde{\alpha} \). Similarly, consider (4.27) with \( y = z/M \) and \( x = M \eta \) (\( z \) is again the spectral parameter) and impose reduction

\[ \tilde{A}^{\alpha, \gamma} = \exp(-2\pi i \eta \alpha_2) S^{-M\alpha}_\gamma T_\gamma. \] (4.30)

\(^6\)This model is the quasi-classical limit of generalized spin chain with inhomogeneous parameters \( z_k \) and \( L(z) \) being the monodromy matrix. In quantum case \( L(z) \) satisfies exchange relations which provides quadratic algebras of Sklyanin’s type [24, 25].
Then the matrix (4.26) acquires the form (4.28) with marked points at $z = -M\omega_\alpha$. The same reductions can be made for the case when $\eta$ is treated as spectral parameter. Thus, the off-shell (unreduced) matrix (4.23) contains both - matrix extension of $GL(M)$ Gaudin model with $N^2$ marked points at $\omega_\alpha$ and $GL(N)$ Gaudin model with $M^2$ marked points at $\tilde{\omega}_\tilde{\alpha}$. Moreover, in both cases any one of the parameters ($z$ or $\eta$) can be considered as the spectral.

5 Conclusion and discussion

In this paper we consider a wide class of integrable relativistic elliptic tops and their different matrix extensions. The simplest model

$$\dot{S}_{00} = 0, \quad \dot{S}_\alpha = \sum'_\beta C_{\beta,\alpha-\beta} S_\beta S_{\alpha-\beta} J^n_\alpha - \beta, \quad \alpha \neq 0, \quad J^n_\alpha = E_1(\eta + \omega_\alpha) - E_1(\omega_\alpha)$$

is a generalization of the Euler-Arnold top for multi-dimensional (complexified) space. For rank one matrix $S$ it is gauge equivalent to the elliptic Ruijsenaars-Schneider model with relativistic parameter $\eta$. Contrary to the many-body systems the set of functions defining the underlying bundle (the Lax matrix is its section) is independent of dynamical variables. This allows to treat the spectral and relativistic parameters on equal terms.

First we review results of [9] and [13]. The main point of that constructions is that underlying $R$-matrices should satisfy the associative Yang-Baxter equation (4.1) together with some additional requirements such as skew-symmetry (or unitarity) and the classical limit should begin with the identity matrix. The latter condition can be violated. For example, equations (3.8) or (3.11)

$$\dot{A}_\alpha = \sum_{\beta, \gamma : \beta + \gamma = \alpha} [A^\beta, A^\gamma] J^{\eta/N}_\gamma, \quad \alpha \neq 0; \quad J^{\eta/N}_\alpha = E_1(\frac{\eta}{N} + \omega_\alpha) - E_1(\omega_\alpha).$$

are related to $R$-matrices multiplied by permutation operator in "matrix" space $(R_{12}^h(z) \otimes \tilde{P}_{1,2})$. Such object has certainly no classical expansion beginning with identity matrix. However additional reductions (3.9) allow to keep the Lax equations.

The most symmetric form (with respect to $z$ and $\eta$) comes from another example of the $R$-matrix (4.12) - the so-called symmetric $R$-matrix [14]. It turns into $R^h_{12}(z)$ when $M = 1$ and to $R^h_{12}(\hbar)$ when $N = 1$. Moreover, such $R$-matrices satisfy the associative Yang-Baxter equation in the form (4.13), but again have wrong local expansion (4.14) near $z = 0$ or $\hbar = 0$. Our aim is to describe an integrable Lax pairs based on the symmetric $R$-matrix, and keep possible symmetry between a pair of parameters $z$, $\eta$ implying that any of them can be chosen to be the spectral. For this purpose we are going to use well-known [22] relation (4.5) $R^h_{12}(z)P_{12} = R^{z/N}_{12}(N\hbar)$. For the Belavin’s $R$-matrix written as (4.4) it provides a set of discrete Fourier transformations (2.1) and its degenerations given in Section 2.

Consider the Lax matrix related to symmetric $R$-matrix. It is given by the matrix-valued function on $\Sigma_{z,\tau} \times \Sigma_{\eta,\tau}$:

$$L(z, \eta) = \sum_{\alpha \in Z^2_N} \sum_{\tilde{\alpha} \in Z^2_M} A^{\alpha, \tilde{\alpha}} \Phi_{\alpha, \tilde{\alpha}}(z, \eta) \in \text{Mat}(K, \mathbb{C}),$$

In transition between these two cases one should rescale $z \leftrightarrow Nz/M$ and $\eta \leftrightarrow M\eta/N$. 

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where $N$ and $M$ are coprime and $K$ is arbitrary. It has $M^2$ simple poles in variable $z$ at $-N\tilde{\omega}_\alpha$, $\tilde{\alpha} \in \mathbb{Z}_M^2$ and $N^2$ simple poles in variable $\eta$ at $-\omega_\alpha$, $\alpha \in \mathbb{Z}_N^2$. The answer is as follows. Let $z$ be the spectral parameter. Introduce $M$-matrix

$$M(z) = -\sum_{\alpha \neq 0} \sum_{\tilde{\alpha}} A^{\alpha, \tilde{\alpha}} \Phi_{\alpha, \tilde{\alpha}}(z, 0) - \sum_{\tilde{\alpha}} A^{0, \tilde{\alpha}} E_1(z + N\tilde{\omega}_\tilde{\alpha})$$

(5.4)

The l.h.s. of the Lax equations $\dot{L}(z, \eta) = [L(z, \eta), M(z)]$ is the quasi-periodic function in $z$ but its r.h.s. is not quasi-periodic due to the property of $E_1$ function. This is why we require the following condition:

$$\sum_{\tilde{\alpha}} A^{0, \tilde{\alpha}} \sim 1_K.$$  

(5.5)

Using calculations from Appendix B we get the following equations of motion:

$$\dot{A}^{\alpha, \tilde{\alpha}} = \sum_{\beta \neq \alpha, \gamma \neq \tilde{\alpha}} \left( [A^{\gamma, \alpha}, A^{\alpha - \gamma, \beta}] \Phi_{\alpha - \gamma, \beta - \tilde{\alpha}}(0, \eta) + [A^{\gamma, \alpha}, A^{\alpha - \gamma, \beta}] \Phi_{\gamma, \tilde{\alpha} - \tilde{\alpha}}(0, \eta) \right)$$

$$+ \sum_{\tilde{\gamma} \neq \tilde{\alpha}} [A^{0, \tilde{\gamma}}, A^{\alpha, \tilde{\alpha}}] \left( \exp(2\pi i N\tilde{\alpha}_2 M) E_1(N\tilde{\omega}_{\tilde{\gamma} - \tilde{\alpha}}) + \Phi_{\alpha - \gamma, \tilde{\beta} - \tilde{\alpha}}(0, \eta) \right)$$

$$+ \sum_{\gamma \neq 0} [A^{\gamma, \alpha}, A^{\alpha - \gamma, \tilde{\alpha}}] (E_1(\eta) + E_1(\omega_\gamma) - E_1(\eta + \omega_\gamma + N\tilde{\omega}_{\tilde{\alpha}}))$$

$$+ \sum_{\gamma \neq 0} [A^{\gamma, \tilde{\alpha}}, A^{\alpha - \gamma, \tilde{\alpha}}] (E_1(\omega_\gamma) - E_1(\omega_\gamma + \eta)).$$

(5.6)

When $M = 1$ all the terms are absent except the last line, which reproduces (5.2). The whole system of equations (5.6) can be considered as generalization of (5.2) describing $M^2$ interacting $GL(N)$ matrix tops.

Recall that previously we had some special reduction conditions of types (3.9)-(3.11). In fact we have them here as well. Indeed, we may perform the Fourier transformation over sublattice $\tilde{\alpha} \in \mathbb{Z}_M^2 \subset \mathbb{Z}_N^2$. Then it easy to see that our requirement $A^0 \sim 1_K$ from (5.5) is just the one $A^0 \sim 1_K$ from (3.9) written in the case $N := NM$ after Fourier transformation over $\mathbb{Z}_M^2$. The rest of conditions are hardly visible from the above calculations, but we know that the do exist. Using notations of (4.19), (4.20) we may write them as

$$A^{\alpha + N\tilde{\gamma}}_{\varphi^{\alpha + N\tilde{\gamma}}} = \frac{1}{M} \sum_{\tilde{\alpha}} \hat{\kappa}_{\alpha + N\tilde{\gamma}, \tilde{\alpha}} A^{\alpha, \tilde{\alpha}}.$$  

(5.7)

In a similar way one can use the matrix (5.3) as the Lax matrix with spectral parameter $\eta$. The corresponding model describes $N^2$ interacting $GL(M)$ matrix tops, and acquires other set of reduction conditions. Viewed in this way we deal with a spectral surface in $(\lambda, z, \eta) \in \mathbb{C}^3$:

$$\det(\lambda - L(z, \eta)) = 0$$

reductions of type 1,  \quad \checkmark 

$$\det(\lambda - L^\eta(z)) = 0, \quad M = M(z)$$

reductions of type 2.  \quad \checkmark 

$$\det(\lambda - \hat{L}^z(\eta)) = 0, \quad M = \hat{M}(\eta)$$

(5.8)

Integrable cases appear after we choose a parameter (say $z$) be the spectral one, and fix the parameter $\eta = \text{const}$. Then, with additional reduction constraints such slice gives a spectral curve of integrable system (5.6).
In the end let us remark again that matrix (5.3) has $M^2$ simple poles in variable $z$ at $-N\tilde{\omega}_\alpha$, $\tilde{\alpha} \in \mathbb{Z}^2$ and $N^2$ simple poles in variable $\eta$ at $-\omega_\alpha$, $\alpha \in \mathbb{Z}^2$. The described model in both case of (5.8) is a matrix extension of the Gaudin-like model. When $z$ is spectral parameter indices $\mathbb{Z}^2$ numerates the marked points while $\mathbb{Z}^2$ are the matrix indices. Contrariwise, when $\eta$ is the spectral parameter, then $\mathbb{Z}^2$ numerates the marked points while $\mathbb{Z}^2$ are the matrix indices. A link between Gaudin type models (and/or integrable spin chains) which interchanges the indices counting marked points and the matrix indices is known as the spectral duality [1] (rational Gaudin – rational Gaudin), [17] (trigonometric Gaudin – XXX spin chain), [18] (XXZ spin chain – XXZ spin chain). In our case there is no direct relation between these two models since the reductions in (5.8) are different. Such relation exists at the level of the off-shell matrix (5.3) only, where it is given by the discrete Fourier transformation.

6 Appendix A: elliptic functions

The Kronecker function on elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z})$ with moduli $\tau$ ($\text{Im} \tau > 0$):

$$\phi(\eta, z) = \frac{\vartheta'(0)\vartheta(\eta + z)}{\vartheta(\eta)\vartheta(z)} = \sum_{k \in \mathbb{Z}} \exp \left( \pi i \tau (k + \frac{1}{2})^2 + 2\pi i (z + \frac{1}{2})(k + \frac{1}{2}) \right).$$

(A.1)

Next function is its derivative:

$$f(z, u) \equiv \partial_u \phi(z, u) = \phi(z, u) (E_1(z + u) - E_1(u)),$$

(A.2)

where the first Eisenstein function is used:

$$E_1(z) = \vartheta'(z)/\vartheta(z), \quad E_1(-z) = -E_1(z).$$

(A.3)

The second Eisenstein function:

$$E_2(z) = -\partial_z E_1(z), \quad E_2(-z) = E_2(z)$$

(A.4)

The functions (A.3), (A.4) are simply related to the Weierstrass $\zeta$- and $\wp$-functions:

$$E_1(z) = \zeta(z) + \frac{z}{3} \frac{\vartheta'''(0)}{\vartheta'(0)}, \quad E_2(z) = \wp(z) - \frac{1}{3} \frac{\vartheta'''(0)}{\vartheta'(0)}.$$  

(A.5)

Quasiperiodic properties:

$$\phi(x + 1, y) = \phi(x, y), \quad \phi(x + \tau, y) = \exp(-2\pi i y)\phi(x, y)$$

$$E_1(z + 1) = E_1(z), \quad E_1(z + \tau) = E_1(z) - 2\pi i, \quad E_2(z + 1) = E_2(z + \tau) = E_2(z).$$

(A.6)

The Fay identity:

$$\phi(z, q)\phi(w, u) = \phi(z - w, q)\phi(w, q + u) + \phi(w - z, u)\phi(z, q + u)$$

(A.7)

Its degenerations:

$$\phi(z, q)\phi(w, q) = \phi(z + w, q)(E_1(z) + E_1(w) + E_1(q) - E_1(z + w + q)),$$

(A.8)

8The type of relation we are discussing here was first observed for the Toda chain [5].
\[ \phi(z, x)f(z, y) - \phi(z, y)f(z, x) = \phi(z, x + y)(\varphi(x) - \varphi(y)). \]  

(A.9)

The following set of \( N^2 \) functions is widely used:

\[ \varphi_\alpha(z, \eta + \omega_\alpha) = \exp(2\pi i z \partial_\tau \omega_\alpha) \phi(z, \eta + \omega_\alpha), \]

where

\[ \omega_\alpha = \frac{\alpha_1 + \alpha_2 \tau}{N}, \quad \partial_\tau \omega_\alpha = \frac{\alpha_2}{N}, \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_N \times \mathbb{Z}_N. \]  

(A.11)

The index \( \alpha \) in \( \varphi_\alpha \) stands for the exponential factor depending on \( \alpha \) and the first argument. Similarly,

\[ f_\alpha(z, \omega_\alpha) = \exp(2\pi i z \partial_\tau \omega_\alpha) f(z, \omega_\alpha), \quad (\alpha_1, \alpha_2) \neq (0, 0). \]  

(A.12)

Plugging (A.10) into (A.7), (A.8) and (A.9) we get:

\[ \varphi_\beta(x, \eta + \omega_\beta) \varphi_\gamma(y, \omega_\gamma) = \]

\[ = \varphi_\beta(x - y, \eta + \omega_\beta) \varphi_{\beta + \gamma}(y, \eta + \omega_{\beta + \gamma}) + \varphi_\gamma(y - x, \omega_\gamma) \varphi_{\beta + \gamma}(x, \eta + \omega_{\beta + \gamma}), \]

\[ \varphi_\beta(z, \eta + \omega_\beta) \varphi_\gamma(z, \omega_\gamma) = \]

\[ = \varphi_{\beta + \gamma}(z, \eta + \omega_{\beta + \gamma})(E_1(z) + E_1(\eta + \omega_\beta) + E_1(\omega_\gamma) - E_1(z + \eta + \omega_{\beta + \gamma})) \]

and

\[ \varphi_\beta(z, \omega_\beta) f_\gamma(z, \omega_\gamma) - \varphi_\gamma(z, \omega_\gamma) f_\beta(z, \omega_\beta) = \varphi_{\beta + \gamma}(z, \omega_{\beta + \gamma})(\varphi(\omega_\beta) - \varphi(\omega_\gamma)). \]  

(A.15)

### 7 Appendix B

Consider the r.h.s. of the Lax equation \( \dot{L}(z, \eta) = [L(z, \eta), M(z)] \) for the Lax pair (5.3), (5.4):

\[ [L(z, \eta), M(z)] = \]

\[ \sum_{\beta, \tilde{\beta}, \gamma} \sum_{\gamma \neq 0} [A^{\gamma, \tilde{\gamma}}, A^{\beta, \tilde{\beta}}] \Phi_{\beta, \tilde{\beta}}(z, \eta) \Phi_{\gamma, \tilde{\gamma}}(z, 0) + \sum_{\beta, \tilde{\beta}, \gamma} [A^{0, \tilde{\gamma}}, A^{\beta, \tilde{\beta}}] \Phi_{\beta, \tilde{\beta}}(z, \eta) E_1(z + N\tilde{\omega}_\gamma) \]  

(B.1)

Both sums are subdivided into two parts: with \( \tilde{\beta} = \tilde{\gamma} \) and \( \tilde{\beta} \neq \tilde{\gamma} \):

\[ [L(z, \eta), M(z)] = \]

\[ \sum_{\beta, \tilde{\beta} \neq \gamma} \left( \sum_{\gamma \neq 0} [A^{\gamma, \tilde{\gamma}}, A^{\beta, \tilde{\beta}}] \Phi_{\beta, \tilde{\beta}}(z, \eta) \Phi_{\gamma, \tilde{\gamma}}(z, 0) + [A^{0, \tilde{\gamma}}, A^{\beta, \tilde{\beta}}] \Phi_{\beta, \tilde{\beta}}(z, \eta) E_1(z + N\tilde{\omega}_\gamma) \right) \]

(B.2)

\[ + \sum_{\beta, \tilde{\beta}} \left( \sum_{\gamma \neq 0} [A^{\gamma, \tilde{\beta}}, A^{\beta, \tilde{\beta}}] \Phi_{\beta, \tilde{\beta}}(z, \eta) \Phi_{\gamma, \tilde{\beta}}(z, 0) + [A^{0, \tilde{\beta}}, A^{\beta, \tilde{\beta}}] \Phi_{\beta, \tilde{\beta}}(z, \eta) E_1(z + N\tilde{\omega}_\beta) \right) \]

In the second sum in the last line we rename \( \beta \) into \( \gamma \) (mention that the terms with \( \beta = 0 \) are cancelled out). The first sum in the last line is subdivided into two parts: with \( \beta = 0 \) and \( \beta \neq 0 \):

\[ [L(z, \eta), M(z)] = \]

(B.3)
Finally, we have
\[
\sum_{\beta, \tilde{\beta} \neq \gamma} \left( \sum_{\gamma \neq 0} [A^{\gamma, \tilde{\gamma}}, A^{\beta, \tilde{\beta}}] \Phi_{\beta, \tilde{\beta}}(z, \eta) \Phi_{\gamma, \tilde{\gamma}}(z, 0) + [A^{0, \tilde{\gamma}}, A^{\beta, \tilde{\beta}}] \Phi_{\beta, \tilde{\beta}}(z, \eta) E_1(z + N\tilde{\omega}) \right)
\]
\[\quad + \sum_{\beta} \sum_{\gamma \neq 0} \left( [A^{\gamma, \tilde{\beta}}, A^{0, \tilde{\beta}}] \Phi_{0, \tilde{\beta}}(z, \eta) \Phi_{\gamma, \tilde{\beta}}(z, 0) + [A^{0, \tilde{\gamma}}, A^{\gamma, \tilde{\beta}}] \Phi_{\gamma, \tilde{\beta}}(z, \eta) E_1(z + N\tilde{\omega}_\beta) \right)
\]
\[\quad + \sum_{\beta \neq 0, \gamma \neq 0} [A^{\gamma, \tilde{\beta}}, A^{\beta, \tilde{\beta}}] \Phi_{\beta, \tilde{\beta}}(z, \eta) \Phi_{\gamma, \tilde{\beta}}(z, 0) .
\]

Consider the r.h.s. of (B.3) in more detail. The first sum in its first line is simplified by (4.8), and we leave the second term in the first line as is. The second line is simplified via (4.9). The last line is also modified through (4.9) likewise it was made in (1.14):

\[
\sum_{\beta \neq 0, \gamma \neq 0} [A^{\gamma, \tilde{\beta}}, A^{\beta, \tilde{\beta}}] \Phi_{\beta, \tilde{\beta}}(z, \eta) \Phi_{\gamma, \tilde{\beta}}(z, 0) = \frac{1}{2} \sum_{\beta \neq 0, \gamma \neq 0} [A^{\gamma, \tilde{\beta}}, A^{\beta, \tilde{\beta}}] \left( \Phi_{\beta, \tilde{\beta}}(z, \eta) \Phi_{\gamma, \tilde{\beta}}(z, 0) - \Phi_{\gamma, \tilde{\beta}}(z, \eta) \Phi_{\beta, \tilde{\beta}}(z, 0) \right) = (B.4)
\]

\[
\frac{1}{2} \sum_{\beta \neq 0, \gamma \neq 0} [A^{\gamma, \tilde{\beta}}, A^{\beta, \tilde{\beta}}] \Phi_{\beta, \tilde{\beta}}(z, \eta)(E_1(\eta + \omega_\beta) + E_1(\omega_\gamma) - E_1(\eta + \omega_\gamma) - E_1(\omega_\beta)) = \sum_{\beta \neq 0, \gamma \neq 0} [A^{\gamma, \tilde{\beta}}, A^{\beta, \tilde{\beta}}] \Phi_{\beta, \tilde{\beta}}(z, \eta)(E_1(\eta + \omega_\beta) - E_1(\omega_\beta)) .
\]

Finally, we have

\[
[L(z, \eta), M(z)] = \sum_{\beta, \tilde{\beta} \neq \gamma} \sum_{\gamma \neq 0} [A^{\gamma, \tilde{\gamma}}, A^{\beta, \tilde{\beta}}] \left( \Phi_{\beta, \tilde{\beta}}(0, \eta) \Phi_{\gamma, \tilde{\gamma}}(z, \eta) + \Phi_{\gamma, \tilde{\gamma}}(0, 0) \Phi_{\beta, \tilde{\beta}}(z, \eta) \right)
\]

\[\quad + \sum_{\beta, \tilde{\beta} \neq \gamma} [A^{0, \tilde{\gamma}}, A^{\beta, \tilde{\beta}}] \Phi_{\beta, \tilde{\beta}}(z, \eta) E_1(z + N\tilde{\omega}_\gamma)
\]

\[\quad + \sum_{\beta} \sum_{\gamma \neq 0} [A^{\gamma, \tilde{\beta}}, A^{0, \tilde{\beta}}] \Phi_{\gamma, \tilde{\beta}}(z, \eta) \left( E_1(\eta) + E_1(\omega_\gamma) - E_1(z + \eta + \omega_\gamma + N\tilde{\omega}_\beta) \right)
\]

\[\quad + \sum_{\beta \neq 0, \gamma \neq 0} [A^{\gamma, \tilde{\beta}}, A^{\beta, \tilde{\beta}}] \Phi_{\beta, \tilde{\beta}}(z, \eta)(E_1(\eta + \omega_\beta) - E_1(\omega_\beta)) .
\]

The r.h.s. has simple poles at \(z = -N\tilde{\omega}_\alpha, \tilde{\alpha} \in \mathbb{Z}^2 \) (apparent pole at \(z = -\eta - \omega_\gamma - N\tilde{\omega}_\beta \) in the third line is cancelled by the zero of \(\Phi_{\gamma, \beta}(z, \eta)\)). In order to get equations of motion we are going to compare its residues with those of functions \(\Phi_{\gamma, \beta}(z, \eta)\) entering \(\dot{L}(z, \eta)\). Recall that

\[
\operatorname{Res}_{z = -N\tilde{\omega}_\alpha} \Phi_{\alpha, \tilde{\alpha}}(z, \eta) = \exp(2\pi i \eta N\tilde{\alpha}_2/M) .
\]

This gives the equations of motion (5.6).

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