Harnack inequality for solutions of the $p(x)$-Laplace equation under the precise non-logarithmic Zhikov’s conditions

Igor Skrypnik\textsuperscript{1,2} · Yevgeniia Yevgenieva\textsuperscript{1,3} \textsuperscript{**}

Received: 8 August 2022 / Accepted: 15 October 2023
© The Author(s) 2023

Abstract
We prove continuity and Harnack’s inequality for bounded solutions to the equation
\[
\text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) = 0, \quad p(x) = \bar{p} + L \frac{\log \log \frac{1}{|x-x_0|}}{\log \frac{1}{|x-x_0|}},
\]
\[
\bar{p} > 1, \quad L > 0,
\]
under the precise non-logarithmic condition on the function $p(x)$.

Mathematics Subject Classification 35B09 · 35B40 · 35B45 · 35B65

1 Introduction and main results

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. In this paper we are concerned with elliptic equations of the type
\[
\text{div} A(x, \nabla u) = 0, \quad x \in \Omega. \tag{1.1}
\]

We suppose that the functions $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ are such that $A(\cdot, \xi)$ are Lebesgue measurable for all $\xi \in \mathbb{R}^n$, and $A(x, \cdot)$ are continuous for almost all $x \in \Omega$. We assume also

Communicated by Andrea Mondino.

\* Igor Skrypnik
ihor.skrypnik@gmail.com

\* Yevgeniia Yevgenieva
yevgeniia.yevgenieva@gmail.com

1 Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine, Batiuk Str. 19, Sloviansk 84116, Ukraine

2 Vasył’ Stus Donetsk National University, 600-richcha Str. 21, Vinnytsia 21021, Ukraine

3 Max Planck Institute for Dynamics of Complex Technical Systems, Sandtorstrasse 1, 39106 Magdeburg, Germany

Published online: 20 November 2023
that the following structure conditions are satisfied

\[
A(x, \xi) \xi \geq K_1 |\xi|^{p(x)}, \quad |A(x, \xi)| \leq K_2 |\xi|^{p(x)-1},
\]

where \(K_1, K_2\) are positive constants,

\[
p(x) = \tilde{p} + \ell(|x - x_0|), \quad \ell(s) = L \frac{\log \log \frac{1}{s}}{\log \frac{1}{s}}
\]

(1.3)

\(\tilde{p} > 1\) and \(L > 0\).

The aim of this paper is to establish basic qualitative properties such as continuity of bounded solutions and Harnack’s inequality for non-negative bounded solutions to equation (1.1).

Before formulating the main results, we say few words concerning the history of the problem. The study of regularity of minima of functionals with non-standard growth has been initiated by Zhikov [51–54, 56], Marcellini [36, 37], and Lieberman [35], and in the last thirty years, the qualitative theory of second order elliptic and parabolic equations with so-called ”logarithmic” condition, i.e. if

\[
\text{osc}_{B_r(x_0)} p(x) \leq \frac{L}{\log \frac{1}{r}}, \quad 0 < r < 1, \quad 0 < L < \infty,
\]

(1.4)

has been actively developed (see e.g. [1–3, 5, 6, 9–16, 20–22, 25–30, 39, 45–47, 49] for references). Equations of this type and systems of such equations arise in various problems of mathematical physics (see e.g. the monographs [8, 25, 40, 50] and references therein).

The case when condition (1.4) is replaced by the condition

\[
\text{osc}_{B_r(x_0)} p(x) \leq \frac{\mu(r)}{\log \frac{1}{r}}, \quad \lim_{r \to 0} \mu(r) = \infty, \quad \lim_{r \to 0} \frac{\mu(r)}{\log \frac{1}{r}} = 0,
\]

(1.5)

differs substantially from the logarithmic case. It turns out that such non-logarithmic condition is a precise condition for the smoothness of finite functions in the corresponding Sobolev space \(W^{1,p(x)}(\Omega)\). Thus this case is extremely interesting to study. But to our knowledge there are only few results in this direction. Zhikov [55] obtained a generalization of the logarithmic condition which guaranteed the density of smooth functions in Sobolev space \(W^{1,p(x)}(\Omega)\). Particularly, this result holds if \(1 < \tilde{p} \leq p(x)\) and

\[
\text{osc}_{B_r(x_0)} p(x) \leq L \frac{\log \log \frac{1}{r}}{\log \frac{1}{r}}, \quad 0 < L \leq \frac{\tilde{p}}{n}.
\]

(1.6)

Later Zhikov and Pastukhova [57] proved higher integrability of the gradient of solutions to the \(p(x)\)-Laplace equation under the same condition.

Interior continuity, continuity up to the boundary and Harnack’s inequality to the \(p(x)\)-Laplace equation were proved in [4, 7] and [48] under condition (1.5) and

\[
\int_0^{\exp(-\gamma \exp(\mu^c(r)))} \frac{dr}{r} = +\infty,
\]

(1.7)

with some numbers \(\gamma, c > 1\). For example, the function \(\mu(r) = L \log \log \log \frac{1}{r}\) satisfies conditions (1.5), (1.7), provided that \(L\) is a sufficiently small positive number.
The results from [4, 7, 48] were generalized in [41, 46] for a wide class of elliptic and parabolic equations with non-logarithmic Orlicz growth. Particularly, it was proved in [46] that under conditions (1.5), (1.7) functions from the correspondent De Giorgi’s $B_1(\Omega)$ classes are continuous and moreover, it was shown that the solutions of the correspondent elliptic and parabolic equations with non-standard growth belong to these classes.

The exponential condition of the type (1.7) was substantially refined in [24]. Particularly, the continuity of solutions to double-phase and degenerate double-phase elliptic equations

$$\text{div} \left( |\nabla u|^{p-2} \nabla u + a(x) |\nabla u|^{q-2} \nabla u \right) = 0, \quad q > p,$$

and

$$\text{div} \left( |\nabla u|^{p-2} \nabla u \left( 1 + \log(1 + |\nabla u|) \right) \right) = 0$$

was proved under the conditions

$$\text{osc}_{B_r(x_0)} a(x) \leq A \mu(r)^q - p r^{q - p}, \quad \text{osc}_{B_r(x_0)} b(x) \leq B \mu(r) r, \quad \int_0^{\mu(r)} \frac{dr}{\mu(r)} = +\infty.$$

Note that the function $\mu(r) = \log \frac{1}{r}$ satisfies the above conditions. In the present paper the continuity and the Harnack’s type inequality have been proved under the conditions similar to (1.6).

Before formulating the main results, let us recall the definition of a bounded weak solution to equation (1.1). We introduce $W(\Omega)$ as a class of functions $u \in W^{1,1}(\Omega)$, such that

$$\int_{\Omega} |\nabla u|^{p(x)} dx < +\infty,$$

and $W_0(\Omega) = W(\Omega) \cap W^{1,1}_0(\Omega)$.

**Definition 1** We say that a function $u \in W(\Omega) \cap L^\infty(\Omega)$ is a bounded weak sub(super)-solution to equation (1.1) if

$$\int_{\Omega} \mathbf{A}(x, \nabla u) \nabla \varphi \, dx \leq (\geq) 0,$$

holds for all non-negative test functions $\varphi \in W_0(\Omega)$.

The following Theorem is the first main result of this paper.

**Theorem 1.1** Let $u$ be a bounded weak solution of Eq. (1.1) and let conditions (1.2), (1.3) be fulfilled, then $u$ is Hölder continuous at point $x_0$.

The next result is a weak Harnack type inequality for non-negative super-solutions.

**Theorem 1.2** Let $u$ be a bounded non-negative weak super-solution to Eq. (1.1), let conditions (1.2), (1.3) be fulfilled. Assume also that

$$\left( \mathbf{A}(x, \xi) - \mathbf{A}(x, \eta) \right) (\xi - \eta) > 0, \quad \xi, \eta \in \mathbb{R}^n, \quad \xi \neq \eta,$$

then there exist numbers $\gamma, \gamma' > 0$ depending only on $n, \bar{p}, K_1, K_2$ and $M = \sup_{\Omega} u$, such that for any $\theta \in (0, \bar{p} - 1)$ there holds

$$\left( \frac{1}{B_{\rho}(x_0)} \int_{B_{\rho}(x_0)} u^\theta \, dx \right)^{\frac{1}{\theta}} \leq \left( \frac{\gamma}{\bar{p} - 1 - \theta} \right)^{\frac{1}{\theta}} \left( \inf_{\frac{1}{\bar{p}}(x_0)} u + \rho \right),$$

(1.10)
provided that \( B_{16\rho}(x_0) \subset \Omega \) and
\[
\frac{1}{\log \log \frac{1}{16\rho}} + \tilde{\gamma} L \frac{\log \log \frac{1}{16\rho}}{\log \frac{1}{16\rho}} \leq 1. \tag{1.11}
\]
The following Theorem is Harnack’s inequality

**Theorem 1.3** Let \( u \) be a bounded non-negative weak sub-solution to Eq. (1.1), let conditions (1.2), (1.3) be fulfilled. Assume also that the monotonicity condition (1.9) holds. Then there exist positive numbers \( \gamma, \tilde{\gamma}_1 \) depending only on \( n, \tilde{p}, K_1, K_2, M \) such that for any \( \theta \in (0, \tilde{p} - 1) \)
\[
\sup_{B_{\frac{\rho}{2}}(x_0)} u \leq \gamma \left( \int_{B_{\rho}(x_0)} u^\theta \, dx \right)^{\frac{1}{\theta}} + \gamma \rho, \tag{1.12}
\]
provided that \( B_{16\rho}(x_0) \subset \Omega \) and
\[
\frac{1}{\log \log \frac{1}{16\rho}} + \tilde{\gamma}_1 L \frac{\log \log \frac{1}{16\rho}}{\log \frac{1}{16\rho}} \leq 1. \tag{1.13}
\]
Particularly, if \( u \) is a bounded non-negative weak solution to Eq. (1.1), then
\[
\sup_{B_{\frac{\rho}{2}}(x_0)} u \leq \gamma \left( \inf_{B_{\rho}(x_0)} u + \rho \right), \tag{1.14}
\]
provided that \( B_{16\rho}(x_0) \subset \Omega \) and
\[
\frac{1}{\log \log \frac{1}{16\rho}} + \max(\tilde{\gamma}, \tilde{\gamma}_1) L \frac{\log \log \frac{1}{16\rho}}{\log \frac{1}{16\rho}} \leq 1, \tag{1.15}
\]
where \( \tilde{\gamma} > 0 \) is the constant defined in Theorem 1.2.

In the present paper, we substantially refine the results of [4, 7, 41, 45, 46, 48]. We would like to mention the approach taken in this paper. To prove the interior continuity we use De Giorgi’s approach. Let us consider the standard De Giorgi’s class \( \text{DG}_{p}(\cdot) (\Omega) \) of functions \( u \) which corresponds to equation (1.1):
\[
\int_{B_{r}(x_0)} |\nabla (u - k)_{\pm}|^{p(x)} \xi^q \, dx \leq \gamma \int_{B_{r}(x_0)} \left( \frac{u - k}{r\sigma} \right)^{p(x)} \, dx, \quad k \in \mathbb{R}, \sigma \in (0, 1), \tag{1.16}
\]
\( B_{16r}(x_0) \subset \Omega \) and \( \xi(x) \) is the correspondent cut-off function for the ball \( B_{r}(x_0) \), namely, \( \xi \in C_0^\infty(B_{r}(x_0)), 0 \leq \xi \leq 1, \xi = 1 \) in \( B_{r(1-\sigma)}(x_0) \), \( |\nabla \xi| \leq (\sigma r)^{-1} \). Using the Young inequality, by conditions (1.5) we have
\[
\int_{B_{r}(x_0)} |\nabla (u - k)_{\pm}|^{p_-} \xi^q \, dx \leq \gamma \sigma^{-\gamma} \mu(r) \int_{B_{r}(x_0)} \left( \frac{u - k}{r} \right)^{p_-} \, dx
\]
\[
+ \gamma |B_{r}(x_0) \cap \{(u - k)_{\pm} > 0\}|, \quad p_- := \min_{B_{r}(x_0)} p(x).
\]
This estimate leads us to condition (1.7) (see, e.g. [45, 46]). It is easy to see that condition (1.7) fails for the function \( \mu(r) = L \log \log \frac{1}{r} \). To avoid this, using the Young inequality and

\( \exists \) Springer
our choice of \( p(x) \) we rewrite inequality (1.16) as

\[
\int_{B_r(x_0)} \left( \frac{M^\pm_r(u, k)}{r} \right)^{\ell(|x-x_0|)} | \nabla (u - k)_\pm |^{\hat{p}} \xi^q \, dx
\]

\[
\leq \gamma \sigma^{-\gamma} \left( \frac{M^\pm_r(u, k)}{r} \right)^{\hat{p}} \int_{B_r(x_0) \cap \{|u-k|_\pm > 0\}} \left( \frac{M^\pm_r(u, k)}{r} \right)^{\ell(|x-x_0|)} \, dx,
\]

\[
M^\pm_r(u, k) := \sup_{B_r(x_0)} (u - k)_\pm.
\]

(1.17)

It appears that the weight \( \left( \frac{M^\pm_r(u, k)}{r} \right)^{\ell(|x-x_0|)} \) satisfies the Muckenhoupt type properties.

In Sect. 2 we define the correspondent weighted De Giorgi’s classes by inequalities (1.17) and prove the Hölder continuity at point \( x_0 \) for the functions which belong to these classes.

The main difficulty arising in the proof of the Harnack type inequalities is related to the so-called theorem on the expansion of positivity. Roughly speaking, having information on the measure of the “positivity set” of a so-called theorem on the expansion of positivity. Roughly speaking, having information on the measure of the “positivity set” of a

\[
\{ x \in B_r(\bar{x}) : u(x) \geq s \} \geq \alpha(r) | B_r(\bar{x}) |, \quad \alpha(r) = \gamma^{-1} \exp(-\mu^p(r)),
\]

with some \( r > 0, s > 0 \) and \( \gamma > 1 \), and using the standard De Giorgi’s or Moser’s arguments, we inevitably arrive at the estimate

\[
u(x) \geq \gamma^{-1} s \exp\left( - \gamma \exp(\mu^c(r)) \right), \quad x \in B_{2r}(\bar{x}),
\]

with some \( \gamma, c > 1 \). This estimate leads us to condition (1.7) (see, e.g. [41, 46]). Note that we can not use the classical approach of Krylov and Safonov [31], DiBenedetto and Trudinger [19], as it was done in [9] under the logarithmic conditions. We also can not use the local clustering lemma of DiBenedetto, Gianazza and Vespri [17] (see also [18, 49]). Difficulties arise not only due to the constant \( \alpha(r) \) which depends on \( r \), but also when an additional term, that couldn’t be estimated, occurs during the process of iteration from \( B_r(\bar{x}) \) to \( B_{\rho}(x_0) \). To overcome it, we use a workaround that goes back to Mazya [38] and Landis [33, 34] papers.

We will demonstrate our approach on the \( p \)-Laplacian. Fix \( x_0 \in \Omega \) and let \( 0 < r < \rho, E \subset B_r(x_0) \subset B_{\rho}(x_0), B_{16\rho}(x_0) \subset \Omega \) and consider solution \( v := v(x, s) \) of the following problem:

\[
\text{div}( | \nabla v |^{p-2} \nabla v ) = 0, \quad x \in D := B_{16\rho}(x_0) \setminus E,
\]

(1.18)

\[
v - s \psi \in W^{1,p}_0(D),
\]

(1.19)

where \( s > 0 \) is some fixed number, and \( \psi \in W^{1,p}_0(B_{16\rho}(x_0)), \psi = 1 \) on \( E \).

By the well-known estimate (see e.g. [23]) we have

\[
\inf_{B_{4\rho}(x_0) \setminus B_{2\rho}(x_0)} v \geq \gamma^{-1} s \left( \frac{C_p(E)}{\rho^{n-p}} \right)^{\frac{1}{p-1}},
\]

where \( C_p(E) \) is a capacity of the set \( E \). By the Poincare inequality from the previous we obtain

\[
\inf_{B_{4\rho}(x_0) \setminus B_{2\rho}(x_0)} v \geq \gamma^{-1} s \left( \frac{|E|}{\rho^n} \right)^{\frac{1}{p-1}},
\]

(1.20)
Let $u$ be a non-negative bounded super-solution to the $p$-Laplace equation in $\Omega$ and construct the set

$$E(\rho, s) := B_\rho(x_0) \cap \{ u > s \}, \quad 0 < s < \sup_{\Omega} u.$$ 

Consider also a solution $v$ of the problem (1.18), (1.19) with $E$ replaced by $E(\rho, s)$. Then since $u \geq v$ on $\partial D$, by the maximum principle and by (1.20) we obtain

$$m(2\rho) := \inf_{B_{2\rho}(x_0)} u \geq \gamma^{-1} s \left( \frac{|E(\rho, s)|}{\rho^p} \right)^{\frac{1}{p-1}},$$

which by standard arguments yields for any $\theta \in (0, p-1)$

$$\int_{B_{\rho}(x_0)} u^\theta \, dx = |B_{\rho}(x_0)|^{-1} \theta \int_0^\infty E(\rho, s) s^{\theta-1} \, ds \leq m^\theta(2\rho)$$

$$+ \gamma m^{p-1}(2\rho) \int_{m(2\rho)}^\infty s^{\theta-p} \, ds \leq \frac{\gamma}{p-1-\theta} m^\theta(2\rho),$$

from which the weak Harnack type inequality follows.

In Sects. 3, 4 we adapt this simple idea to the case of $p(x)$-Laplacian with non-logarithmic growth. The weight

$$\left( \frac{\widehat{M}_{\rho_1}(v)}{\rho_1} \right)^{\ell(|x-x_0|)}, \quad \widehat{M}_{\rho_1}(v) := \sup_{B_{16\rho}(x_0) \setminus B_{\rho_1}(x_0)} v, \quad \rho < \rho_1 < 16\rho$$

which naturally arises in the proof of Theorem 1.2 also satisfies a Muckenhoupt-type conditions.

**Remark 1** It was unexpected for authors that the modulus of continuity and the constants in the Harnack type inequalities do not depend on the additional term $\log \log \frac{1}{r}$ (usually, there is a dependency, see e.g. [7, 24, 46, 48]).

The rest of the paper contains the proof of the above theorems.

## 2 Elliptic DG classes, proof of Theorem 1.1

In this Section we define the following De Giorgi’s classes.

**Definition 2** We say that a measurable function $u : B_R(x_0) \to \mathbb{R}$ belongs to the elliptic class $DG(B_R(x_0))$ if $u \in W^{1,\tilde{p}}(B_R(x_0)) \cap L^\infty(B_R(x_0))$, ess sup $|u| \leq M$ and there exists numbers $1 < \tilde{p} < q, c_1 > 0$ such that for any ball $B_{kR}(x_0) \subset B_R(x_0)$, any $k \in \mathbb{R}, |k| < M$, any $\sigma \in (0, 1)$, for any $\xi \in C_0^\infty(B_k(x_0)), 0 \leq \xi \leq 1, \xi = 1$ in $B_{R(1-\sigma)}(x_0), |\nabla \xi| \leq (\sigma r)^{-1}$, $|\nabla \xi| \leq (\sigma r)^{-1}$. Springer
the following inequalities hold:

$$\int_{A_{k,r}^\pm \cap \mathcal{S}} \left( \frac{M_r^\pm(u,k)}{r} \right)^{\ell(|x-x_0|)} |\nabla u| \zeta dx \leq c_1 \sigma^{-q} \left( \frac{M_r^\pm(u,k)}{r} \right)^{\bar{p}} \zeta q dx,$$

(2.1)

here \((u - k)_{\pm} := \max\{\pm(u - k), 0\}\), \(A_{k,r}^\pm := B_r(x_0) \cap \{(u - k)_{\pm} > 0\}\),
\(M_r^\pm(u,k) := \sup_{B_r(x_0)} (u - k)_{\pm}\) and \(\ell(|x - x_0|) := L \log \log \frac{1}{|x - x_0|}, \quad L > 0\).

We refer to the parameters \(c_1, n, \bar{p}, q\) and \(M\) as our structural data, and we write \(\gamma\) if it can be quantitatively determined a priori in terms of the above quantities. The generic constant \(\gamma\) may change from line to line.

Our main result of this Section reads as follows:

**Theorem 2.1** Let \(u \in DG(B_R(x_0))\), then \(u\) is Hölder continuous at \(x_0\).

We note that the solutions of Eq. (1.1) belong to the corresponding \(DG(B_R(x_0))\) classes, provided that \(B_{2R}(x_0) \subset \Omega\). We test identity (1.8) by \(\phi = (u - k)_{\pm} \zeta^q(x)\), by the Young inequality we obtain

$$\int_{A_{k,r}^\pm} |\nabla u|^{p(x)} \zeta^q dx \leq \gamma \int_{A_{k,r}^\pm} \left( \frac{u - k}{\sigma r} \right)^{p(x)} dx$$

$$\leq \gamma \sigma^{-\gamma} \left( \frac{M_r^\pm(u,k)}{r} \right)^{\bar{p}} \zeta q dx,$$

From this, using again the Young inequality

$$\int_{A_{k,r}^\pm} \left( \frac{M_r^\pm(u,k)}{r} \right)^{\ell(|x-x_0|)} |\nabla u| \zeta dx \leq \int_{A_{k,r}^\pm} |\nabla u|^{p(x)} dx + \int_{A_{k,r}^\pm} \left( \frac{M_r^\pm(u,k)}{r} \right)^{p(x)} dx,$$

from which the required (2.1) follows.

**2.1 Auxiliary propositions**

For \(k \in \mathbb{R}\) and \(0 < r < R\) set \(w_r^\pm(x,u,k) := \left( \frac{M_r^\pm(u,k)}{r} \right)^{\ell(|x-x_0|)}\), further we need the following lemmas
Lemma 2.2 There exists $C > 0$ depending only on the data, such that for any $u \in \text{DG}(B_R(x_0))$ and for any $t > 0$ the following inequalities hold

$$r^{-n} \int_{B_r(x_0)} w_r^\pm(x, u, k) \, dx \left( r^{-n} \int_{B_r(x_0)} \left[w_r^\pm(x, u, k)\right]^{-t} \, dx \right)^{\frac{1}{t}} \leq \gamma^{1 + \frac{1}{t}}, \quad (2.2)$$

$$r^{-n} \int_{B_r(x_0)} \left[w_r^\pm(x, u, k)\right]^{1 + t} \, dx \leq \gamma^{\frac{1}{1 + t} + 1} r^{-n} \int_{B_r(x_0)} w_r^\pm(x, u, k) \, dx, \quad (2.3)$$

provided that

$$\frac{1}{\log \log \frac{1}{r}} + t C L \frac{\log \log \frac{1}{r}}{\log \frac{1}{r}} \leq 1, \quad \text{and} \quad r \leq M_r^\pm(u, k) \leq 1. \quad (2.4)$$

Proof To prove inequalities (2.2), (2.3) we just need to check

$$\gamma^{-1} \left( \frac{M_r^\pm(u, k)}{r} \right)^{-t\ell(r)} \leq r^{-n} \int_{B_r(x_0)} \left[w_r^\pm(x, u, k)\right]^{-t} \, dx \leq \gamma \left( \frac{M_r^\pm(u, k)}{r} \right)^{-t\ell(r)}, \quad (2.5)$$

$$\gamma^{-1} \left( \frac{M_r^\pm(u, k)}{r} \right)^{t\ell(r)} \leq r^{-n} \int_{B_r(x_0)} \left[w_r^\pm(x, u, k)\right]^{t} \, dx \leq \gamma \left( \frac{M_r^\pm(u, k)}{r} \right)^{t\ell(r)}, \quad (2.6)$$

for $t > 0$. The left inequality in (2.5) and the right inequality in (2.6) are obvious due to the fact that $\ell(|x - x_0|)$ is increasing if $x \in B_r(x_0)$ and $r$ is sufficiently small. Let us check the right inequality in (2.5). Integrating by parts, using the fact that $\frac{\log \log \frac{1}{r}}{\log \frac{1}{r}}$ is increasing on the interval $(0, r)$, we obtain

$$\int_{B_r(x_0)} \left[w_r^\pm(x, u, k)\right]^{-t} \, dx = \gamma \int_{0}^{r} \left( \frac{M_r^\pm(u, k)}{r} \right)^{-t\ell(s)} s^{n-1} \, ds$$

$$\leq \gamma r^n \left( \frac{M_r^\pm(u, k)}{r} \right)^{-t\ell(r)}$$

$$+ \gamma t L \log \frac{M_r^\pm(u, k)}{r} \int_{0}^{r} \left( \frac{M_r^\pm(u, k)}{r} \right)^{-t\ell(s)} \frac{\log \log \frac{1}{r}}{\log^2 \frac{1}{r}} s^{n-1} \, ds$$

$$\leq \gamma r^n \left( \frac{M_r^\pm(u, k)}{r} \right)^{-t\ell(r)}$$

$$+ \gamma t L \log \frac{M_r^\pm(u, k)}{r} \log \frac{1}{r} \int_{0}^{r} \left( \frac{M_r^\pm(u, k)}{r} \right)^{-t\ell(s)} s^{n-1} \, ds$$

$$\leq \gamma r^n \left( \frac{M_r^\pm(u, k)}{r} \right)^{-t\ell(r)} + \frac{1}{2} \int_{B_r(x_0)} \left[w_r^\pm(x, u, k)\right]^{-t} \, dx,$$
provided that \( \gamma t L \frac{\log \log r}{\log r} \leq 1/2 \), \( r \leq M_r^\pm (u, k) \leq 1 \), from which the required inequality follows.

Similarly,

\[
\int_{B_r(x_0)} \left[ w_r^\pm (x, u, k) \right] dx \geq \gamma r^n \left( \frac{M_r^\pm (u, k)}{r} \right)^{\ell(r)} \\
- \gamma t L \log \frac{M_r^\pm (u, k)}{r} \int_0^r \left( \frac{M_r^\pm (u, k)}{r} \right)^{\ell(s)} \frac{\log \log \frac{1}{s}}{\log \frac{1}{r}} s^{n-1} ds \\
\geq \gamma r^n \left( \frac{M_r^\pm (u, k)}{r} \right)^{\ell(r)} \left( 1 - \gamma t L \frac{\log \log r}{\log r} \right),
\]

from which the left inequality in (2.6) follows, provided that \( \gamma t L \frac{\log \log r}{\log r} \leq 1/2 \), \( r \leq M_r^\pm (u, k) \leq 1 \), which completes the proof of the lemma. \( \square \)

In the sequel we also need the following lemma

**Lemma 2.3** There exist \( C_1 > 0 \), \( \kappa_1 > 1 \) such that for any \( u \in DG(B_R(x_0)) \) and any \( \varphi \in W_0(B_R(x_0)) \) the following inequality holds

\[
\frac{1}{S_{u, k, r}^\pm (B_r(x_0))} \int_{B_r(x_0)} w_r^\pm (x, u, k) \left| \varphi \right|^{\kappa_1 \hat{p}} \, dx \\
\leq \gamma \left( r^{\hat{p}} \frac{1}{S_{u, k, r}^\pm (B_r(x_0))} \int_{B_r(x_0)} w_r^\pm (x, u, k) \left| \nabla \varphi \right|^{\hat{p}} \, dx \right)^{\kappa_1}
\]

provided that

\[
\frac{1}{\log \log \frac{1}{r}} + C_1 L \frac{\log \log \frac{1}{r}}{\log \frac{1}{r}} \leq 1, \quad \text{and} \quad r \leq M_r^\pm (u, k) \leq 1.
\]

Here

\[
S_{u, k, r}^\pm (F) := \int_F w_r^\pm (x, u, k) \, dx, \quad F \subset \mathbb{R}^n.
\]

**Proof** Inequality (2.7) is a consequence of (2.5), (2.6) and Sobolev embedding theorem. Indeed, using the Hölder inequality, if \( 0 < \delta \leq \frac{\hat{p}}{p+n} \) and \( 1 < \kappa_1 < \frac{n(1-\delta)}{p-n(1-\delta)} \), we obtain with

\[
t = \frac{n(1-\delta)}{n(1-\delta)-\kappa_1(n-\hat{p}(1-\delta))} > 1
\]
\[
\int_{B_r(x_0)} w_r^\pm(x, u, k) |\varphi|^{\kappa_1\bar{\rho}}\, dx \\
\leq \left( \int_{B_r(x_0)} \left[ w_r^\pm(x, u, k) \right]^t\, dx \right)^{\frac{1}{t}} \left( \int_{B_r(x_0)} |\varphi|^{\frac{\bar{\rho}(1-\delta)}{n(1-\delta)}}\, dx \right)^{\frac{\kappa_1 n \bar{\rho}(1-\delta)}{n(1-\delta)}} \\
\leq \gamma \left( \int_{B_r(x_0)} \left[ w_r^\pm(x, u, k) \right]^t\, dx \right)^{\frac{1}{t}} \left( \int_{B_r(x_0)} |\nabla \varphi|\bar{\rho}\, dx \right)^{\frac{\kappa_1}{1-\delta}} \\
\times \left( \int_{B_r(x_0)} w_r^\pm(x, u, k) |\nabla \varphi|\bar{\rho}\, dx \right)^{\kappa_1} \\
\leq \gamma r^{\kappa_1} \left( \int_{B_r(x_0)} w_r^\pm(x, u, k)\, dx \right)^{1-\kappa_1} \left( \int_{B_r(x_0)} w_r^\pm(x, u, k) |\nabla \varphi|\bar{\rho}\, dx \right)^{\kappa_1}.
\]

Choosing \( C_1 = t \), we arrive at the required (2.7), which completes the proof of the lemma. \( \square \)

### 2.2 De Giorgi Type Lemma

Let \( B_{8r}(x_0) \subset B_R(x_0) \) and let \( \mu_r^+ \geq \text{ess sup}_{B_r(x_0)} u, \ \mu_r^- \leq \text{ess inf}_{B_r(x_0)} u, \ \omega_r := \mu_r^+ - \mu_r^- \).

**Lemma 2.4** Let \( u \in DG(B_R(x_0)) \) and fix \( \xi \in \left( 0, \frac{1}{2M} \right) \). Then there exists \( \nu \in (0, 1) \) depending only on \( n, \bar{\rho}, q, c_1 \) and \( M \), such that if

\[
S_{u, \mu_r^- - \xi \omega_r, r}^+(B_r(x_0)) : u \geq \mu_r^- - \xi \omega_r \leq \nu S_{u, \mu_r^- - \xi \omega_r, r}^+(B_r(x_0)), \quad (2.9)
\]

then either

\[
\xi \omega_r \leq 4r, \quad (2.10)
\]

or

\[
u S_{u, \mu_r^- + \xi \omega_r, r}^-(B_r(x_0)) : u \leq \mu_r^- + \xi \omega_r \leq \nu S_{u, \mu_r^- + \xi \omega_r, r}^-(B_r(x_0)), \quad (2.13)
\]

provided that

\[
\frac{1}{\log \log \frac{1}{r}} + C_1 L \frac{\log \log \frac{1}{r}}{\log \frac{1}{r}} \leq 1, \quad (2.12)
\]

where \( C_1 \) is the constant defined in Lemma 2.3.

Likewise, if

\[
\nu S_{u, \mu_r^+ - \xi \omega_r, r}^-(B_r(x_0)) : u \leq \mu_r^+ - \xi \omega_r \leq \nu S_{u, \mu_r^+ - \xi \omega_r, r}^-(B_r(x_0)), \quad (2.13)
\]

\( \square \) Springer
then either (2.10) holds, or

\[ u(x) \geq \mu_r^+ - \frac{\xi}{4} \omega_r \quad \text{for a.a. } x \in B_2 r(x_0), \]

provided that (2.12) is valid.

**Proof** We provide the proof of (2.11), while the proof of (2.14) is completely similar. For \( j = 0, 1, 2, \ldots \) we set \( r_j := \frac{r}{2^{k_j}} \) and \( k_j := \mu_r^+ - \frac{\xi}{2} \omega_r - \xi \omega_r 2^{-j-1} \), let \( \gamma \) be chosen to satisfy (2.11). We note that \( r_j \) is chosen to satisfy (2.12) provided that (2.12) is valid.

To prove our next result we need the following lemma.

2.3 Expansion of the Positivity

If (2.10) is violated, then condition (2.8) holds due to (2.12) and the choice of \( \gamma \) and \( r_j \). Further we will assume that \( \sup_{B_{2 r_j}(x_0)} (u - k_j)_+ \geq \frac{\xi}{4} \omega_r \), because otherwise inequality (2.11) is evident. We note that

\[ \gamma^{-1} \omega_r(x, u, k_j) \leq \omega_r(x, u, \mu_r^+ - \xi \omega_r) \leq \gamma \omega_r(x, u, k_j), \quad x \in B_{r_j}(x_0). \]

If (2.10) is violated, then condition (2.8) holds due to (2.12) and the choice of \( \gamma, r_j \). So, by Lemma 2.3 and inequality (2.1) we have

\[
(k_j - k_{j+1}) \gamma \omega_r(x, u, k_j) \leq \gamma \omega_r(x, u, \mu_r^+ - \xi \omega_r) \leq \gamma \omega_r(x, u, k_j), \quad x \in B_{r_j}(x_0).
\]

which implies

\[
y_{j+1} := \frac{S_{u, \mu_r^+ - \xi \omega_r, r}(A_{j+1})}{S_{u, \mu_r^+ - \xi \omega_r, r}(B_r(x_0))} \leq \gamma^{2 j y_j} y_j^{2 - \frac{1}{\gamma_1}},
\]

from which by standard arguments (see e.g. [32]) the required (2.11) follows, provided that \( \gamma \) is chosen to satisfy \( \gamma \leq \gamma^{-1} \). This completes the proof of the lemma. \( \square \)

2.3 Expansion of the Positivity

To prove our next result we need the following lemma.
Lemma 2.5 Let \( k < l, 0 < \delta < 1 - \frac{1}{p}, u \in DG(B_R(x_0)), \varphi \in W^{1, \tilde{p}(1-\delta)}(B_r(x_0)), \) then
\[
(l - k) \frac{S^+_{u,k,r}(A^+_{k,r})}{S^+_{u,l,r}(B_r(x_0))} \frac{|A^-_{k,r}|}{|B_r(x_0)|} \leq \gamma r^{1 - \frac{n}{p(1-\delta)}} \left( \int_{A^+_{k,r}\setminus A^-_{k,r}} |\nabla \varphi| \tilde{p}(1-\delta) \, dx \right)^{\frac{1}{p(1-\delta)}},
\] (2.15)
\[
(l - k) \frac{S^+_{u,k,r}(A^-_{k,r})}{S^+_{u,l,r}(B_r(x_0))} \frac{|A^+_{k,r}|}{|B_r(x_0)|} \leq \gamma r^{1 - \frac{n}{p(1-\delta)}} \left( \int_{A^-_{k,r}\setminus A^+_{k,r}} |\nabla \varphi| \tilde{p}(1-\delta) \, dx \right)^{\frac{1}{p(1-\delta)}},
\] (2.16)
provided that
\[
\frac{1}{\log \log \frac{1}{r}} + \frac{L C}{\tilde{p}(1-\delta) - 1} \leq 1, \quad \text{and} \quad r \leq M^+_R(u, l), \ M^-_R(u, k) \leq 1,
\] (2.17)
here \( C > 1 \) is the constant, defined in Lemma 2.2.

Proof Let \( \{v\}_r = \frac{1}{B_r(x_0)} \int_{B_r(x_0)} v \, dx \). Using the Poincaré inequality and inequality (2.3) with \( t = \frac{1}{p(1-\delta) - 1} \) we get
\[
\int_{B_r(x_0)} w^+_r(x, u, l) |v - \{v\}_r| \, dx
\]
\[
\leq \gamma \left( \int_{B_r(x_0)} [w^+_r(x, u, l)]^{\tilde{p}(1-\delta)} \, dx \right)^{1 - \frac{1}{\tilde{p}(1-\delta)}} \left( \int_{B_r(x_0)} |v - \{v\}_r|^{\tilde{p}(1-\delta)} \, dx \right)^{\frac{1}{\tilde{p}(1-\delta)}}
\]
\[
\leq \gamma r^{1 - \frac{n}{p(1-\delta)}} S^+_{u,l,r}(B_r(x_0)) \left( \int_{B_r(x_0)} |\nabla v|^{\tilde{p}(1-\delta)} \, dx \right)^{\frac{1}{p(1-\delta)}}.
\]
Take \( v = 0 \), if \( \varphi < k, v = \varphi - k \), if \( k < \varphi < l, v = l - k \), if \( \varphi > l \). We evidently have \( \{v\}_r \leq (l - k) \frac{|A^+_{k,r}|}{|B_r(x_0)|} \), hence
\[
\int_{B_r(x_0)} w^+_r(x, u, l) |v - \{v\}_r| \, dx \geq (l - k) \left( 1 - \frac{|A^+_{k,r}|}{|B_r(x_0)|} \right) \int_{A^+_{k,r}} w^+_r(x, u, l) \, dx,
\]
from which the required inequality (2.15) follows. The proof of (2.16) is completely similar. \( \square \)

Lemma 2.6 (Expansion of the Positivity) Let \( u \in DG(B_R(x_0)), \) fix \( \xi \in (0, \frac{1}{2M}) \) and assume that with some \( \alpha \in (0, 1) \) there holds
\[
|\{x \in B_r(x_0) : u(x) \geq \mu_+ - \xi \omega_r\}| \leq \frac{1}{1 - \alpha} |B_r(x_0)|.
\] (2.18)
Then there exists number \( s_* \) depending only on \( n, \tilde{p}, q, c_1, M, \alpha \) and \( \xi \) such that either
\[
\omega_r \leq 2^{s_*+1} r,
\] (2.19)
or
\[
u(x) \leq \mu_+ - 2^{-s_*-1} \omega_r \quad \text{for a.a.} \ x \in B_{\frac{r}{2}}(x_0), \] (2.20)
provided that
\[
\frac{1}{\log \log \frac{1}{r}} + L (s_n + 1) \frac{\log \log \frac{1}{r}}{\log \frac{1}{r}} \leq 1.
\] (2.21)

Likewise, if
\[
| \{ x \in B_r(x_0) : u(x) \leq \mu_- + \xi \omega_r \} | \leq (1 - \alpha) | B_r(x_0) |,
\] (2.22)
then there exists number \( s_n \) depending only on \( n, \bar{p}, q, c_1, M, \alpha \) and \( \xi \), such that either (2.19) holds or
\[
u(x) \geq \mu_- + 2^{-s_n} \omega_r \text{ for a.a. } x \in B_{2r}(x_0),
\] (2.23)
provided that (2.21) holds.

**Proof** We provide the proof of (2.20), while the proof of (2.23) is completely similar. We set \( k_s := \mu_r^+ - \frac{\omega_r}{2s}, s = \left\lfloor \log \frac{1}{\xi} \right\rfloor + 1, \ldots, s_n - 1 \), where \( s_n \) is large enough to be chosen later, \([a] \) denotes the integer part of a number \( a \). We will assume that \( \sup_{B_{2r}(x_0)} (u - k_{s_n})_+ \geq \frac{\omega_r}{2s_n + 1} \), since otherwise inequality (2.20) is evident. If inequality (2.19) is violated, then Lemma 2.6 with \( l = k_{s+1} \) and \( k = k_s \) yields
\[
\frac{\omega_r}{2s+1} \frac{S_{\alpha,k_s+1,r}(A_{k_s+1,r}^+)}{S_{\alpha,k_s+1,r}^+(B_r(x_0))} \leq \gamma(\alpha) r^{1 - \frac{n}{p(1 - \delta)}} \left( \int_{A_{k_s,r}^+ \setminus A_{k_s+1,r}^+} | \nabla u | \bar{\rho}^{(1 - \delta)} \, dx \right)^{\frac{1}{p(1 - \delta)}}
\]
\[
\leq \gamma(\alpha) r^{1 - \frac{n}{p(1 - \delta)}} \left( \int_{A_{k_s,r}^+ \setminus A_{k_s+1,r}^+} \left[ w_{\alpha}^+(x, u, k_s) \right]^{\frac{1 - \delta}{\delta}} \, dx \right)^{\frac{\delta}{p(1 - \delta)}} \times \left( \int_{A_{k_s,r}^+} w_{\alpha}^+(x, u, k_s) | \nabla u | \bar{\rho} \, dx \right)^{\frac{1}{p}}.
\]

From this, by inequality (2.1) we obtain
\[
\frac{S_{\alpha,k_s+1,r}(A_{k_s+1,r}^+)}{S_{\alpha,k_s+1,r}^+(B_r(x_0))} \leq \gamma(\alpha) r^{-\frac{n}{p(1 - \delta)}} \left( \int_{A_{k_s,r}^+ \setminus A_{k_s+1,r}^+} \left[ w_{\alpha}^+(x, u, k_s) \right]^{\frac{1 - \delta}{\delta}} \, dx \right)^{\frac{\delta}{p(1 - \delta)}} \left[ S_{\alpha,k_s,r}^+(B_{2r}(x_0)) \right]^\frac{1}{p}.
\]

By our choice and (2.21) we have
\[
\left( \frac{M_{\alpha}^+(u, k_{s_n})}{M_{\alpha}^+(u, k_{s_n})} \right)^{\ell(x - x_0)} \leq 2^{(s_n + 1 - s) \ell(x - x_0)} \leq 2^{L(s_n + 1) \frac{\log \log \frac{1}{r}}{\log \frac{1}{r}} \leq 2, \ x \in B_r(x_0),
\]
therefore \( S^+_{u,k_{s+1},r}(A^+_{k_{s+1},r}) \geq \gamma^{-1} S^+_{u,k_{s},r}(A^+_{k_{s},r}) \) and \( S^+_{u,k_{s+1},r}(B_r(x_0)) + S^+_{u,k_{s},r}(B_{2r}(x_0)) \leq \gamma S^+_{u,k_{s},r}(B_r(x_0)), s = \left[ \log \frac{1}{\xi} \right] + 1, \ldots, s_0 - 1, \) so from the previous relation we have

\[
\left[ S^+_{u,k_{s},r}(A^+_{k_{s},r}) \right]^{\frac{p(1-\delta)}{s}} \leq \gamma(\alpha) r^{-\frac{\delta}{2}} \left[ S^+_{u,k_{s},r}(B_r(x_0)) \right]^{\frac{p(1-\delta)}{s}} \int_{A^+_{k_{s+1},r}} \left[ u^+_r(x,u,k_{s}) \right]^{-\frac{1}{r}} dx.
\]

Summing up these inequalities over \( s = \left[ \log \frac{1}{\xi} \right] + 1, \ldots, s_0 - 1, \) we conclude that

\[
\left( s_0 - \left[ \log \frac{1}{\xi} \right] - 1 \right) \left[ S^+_{u,k_{s},r}(A^+_{k_{s},r}) \right]^{\frac{p(1-\delta)}{s}} \leq \gamma(\alpha) r^{-\frac{\delta}{2}} \left[ S^+_{u,k_{s},r}(B_r(x_0)) \right]^{\frac{p(1-\delta)}{s}} \int_{B_r(x_0)} \left[ u^+_r(x,u,k_{s}) \right]^{-\frac{1}{r}} dx.
\]

Using inequality (2.2) from the last inequality we arrive at

\[
S^+_{u,k_{s},r}(A^+_{k_{s},r}) \leq \gamma(\alpha) \left( s_0 - \left[ \log \frac{1}{\xi} \right] - 1 \right)^{-\frac{\delta}{p(1-\delta)}} S^+_{u,k_{s},r}(B_r(x_0)).
\]

Choosing \( s_0 \) by the condition \( \gamma(\alpha) \left( s_0 - \left[ \log \frac{1}{\xi} \right] - 1 \right)^{-\frac{\delta}{p(1-\delta)}} = \nu \) and using Lemma 2.4 we obtain (2.20), which proves Lemma 2.6.

\[\square\]

### 2.4 Proof of Theorems 1.1, 2.1

To complete the proof of Theorems 1.1 and 2.1 we fix \( R \) by the condition

\[
\frac{1}{\log \log \frac{1}{R}} + L(s_n + 1) \frac{\log \log \frac{1}{R}}{\log \frac{1}{R}} \leq 1,
\]

where \( s_n \) is the number defined in Lemma 2.6, and assume that the following two alternative cases are possible:

\[
\left| \left\{ x \in B_r(x_0) : u(x) \geq \mu_r^+ - \frac{\omega_r}{2s_0} \right\} \right| \leq \frac{1}{2} \left| B_r(x_0) \right|, \quad s_0 \geq 2 + \left[ \log M \right],
\]
or

\[
\left| \left\{ x \in B_r(x_0) : u(x) \leq \mu_r^- + \frac{\omega_r}{2s_0} \right\} \right| \leq \frac{1}{2} \left| B_r(x_0) \right|
\]

for any \( 0 < r < \rho < R \). Assume, for example, the first one holds. Then by Lemma 2.6 we obtain

\[
\omega_r \leq \left( 1 - 2^{-s_0-1} \right) \omega_r + 2^{s_0+1} r.
\]

Iterating this inequality, we have

\[
\omega_r \leq \gamma M \left( \frac{r}{\rho} \right)^{\beta} + \gamma \rho, \quad \beta = \beta(s_n) \in (0, 1).
\]

This completes the proof of Theorems 1.1 and 2.1.
3 Upper and lower estimates of auxiliary solutions

In this Section we prove upper and lower bounds for auxiliary solutions $v := v(x, m)$ to the problem

$$\text{div}A(x, \nabla v) = 0, \quad x \in \mathcal{D} := B_{16\rho}(x_0) \setminus E, \quad E \subset B_\rho(x_0),$$

$$v - m\psi \in W_0(\mathcal{D}),$$

where $0 < m \leq M$ is some fixed number, and $\psi \in W_0(B_{16\rho}(x_0))$, $\psi = 1$ on $E$. The existence of the solutions $v$ follows from the general theory of monotone operators. We will assume that the following integral identity holds:

$$\int_{\mathcal{D}} A(x, \nabla v) \nabla \varphi \, dx = 0 \quad \text{for any} \quad \varphi \in W_0(\mathcal{D}). \quad (3.1)$$

Testing (3.1) by $\varphi = (v - m)_+$ and by $\varphi = v_-$ and using condition (1.9), we obtain that $0 \leq v \leq m \leq M$.

For $\rho < \rho_1 < \rho_2 \leq 16\rho$ we set:

$$K(\rho_1, \rho_2) := B_{\rho_2}(x_0) \setminus B_{\rho_1}(x_0), \quad M_{\rho_1}(v) := \sup_{K(\rho_1, 16\rho)} v,$$

$$w_{\rho_1}(x, v) := \left(1 + \frac{M_{\rho_1}(v)}{\rho_1}\right)^{1/16}, \quad \hat{w}_{\rho_1}(x, v) := \int_{K(\rho_1, \rho_2)} w_{\rho_1}(x, v) \, dx.$$

Note that similarly to (2.5), (2.6), for all $\rho_1 \in (\rho, 16\rho)$ there hold

$$\gamma^{-1} \left(1 + \frac{M_{\rho_1}(v)}{\rho_1}\right)^{-t\ell(16\rho)} \leq \int_{K(\rho_1, 16\rho)} \left[\hat{w}_{\rho_1}(x, v)\right]^{-t} \, dx$$

$$\leq \gamma \left(1 + \frac{M_{\rho_1}(v)}{\rho_1}\right)^{-t\ell(16\rho)}, \quad t > 0, \quad (3.2)$$

$$\gamma^{-1} \left(1 + \frac{M_{\rho_1}(v)}{\rho_1}\right)^{t\ell(16\rho)} \leq \int_{K(\rho_1, 16\rho)} \left[\hat{w}_{\rho_1}(x, v)\right]^t \, dx$$

$$\leq \gamma \left(1 + \frac{M_{\rho_1}(v)}{\rho_1}\right)^{t\ell(16\rho)}, \quad t > 0, \quad (3.3)$$

provided that

$$\frac{1}{\log \log \frac{1}{16\rho}} + t \tilde{C}L \log \left(1 + \frac{M}{\rho}\right) \log \log \frac{1}{16\rho} \leq 1, \quad (3.4)$$

where $\tilde{C} = \max(C, C_1)$ and $C, C_1$ are the constants defined in Lemmas 2.2, 2.3. Therefore Lemmas 2.2, 2.3 continue to hold in $K(\rho_1, 16\rho)$ with $u^+(x, u, k)$ replaced by $\hat{w}_{\rho_1}(x, v)$.

To formulate our results, we need the notion of the capacity. Let $E \subset B_r(x_0) \subset B_\rho(x_0)$ and for any $m > 0$ set

$$C_{\rho(\cdot)}(E, B_{16\rho}(x_0); m) := \frac{1}{m} \inf_{\varphi \in \mathfrak{M}(E)} \int_{B_{16\rho}(x_0)} \left|m \nabla \varphi\right|^{p(x)} \, dx,$$
where the infimum is taken over the set $\mathfrak{M}(E)$ of all functions $\varphi \in W_0(B_{16\rho}(x_0))$ with $\varphi \geq 1$ on $E$. If $m = 1$, this definition leads to the standard definition of $C_{p(\cdot)}(E, B_{16\rho}(x_0))$ capacity (see, e.g. [3]).

### 3.1 Upper bound for the function $v$

We note that in the standard case (i.e. if $L = 0$) the upper bound for the function $v$ was proved in [42] (see also [43, Chap. 8, Sec. 3], [44]).

**Lemma 3.1** There exists positive number $\gamma_1$ depending only on the data such that if

$$\rho \frac{C_{p(\cdot)}(E, B_{16\rho}(x_0), m)}{\hat{S}_v(K(\frac{3}{2}\rho, 16\rho))} \geq 1,$$

then

$$\gamma_{-1}\left(\rho^{\hat{p}}\frac{C_{p(\cdot)}(E, B_{16\rho}(x_0), m)}{\hat{S}_v(K(\frac{3}{2}\rho, 16\rho))}\right)^{\frac{1}{p+1}} \leq \hat{\mathcal{M}}_{\frac{3}{2}\rho}(v),$$

$$\leq \gamma\left(\rho^{\hat{p}}\frac{C_{p(\cdot)}(E, B_{16\rho}(x_0), m)}{\hat{S}_v(K(\frac{3}{2}\rho, 16\rho))}\right)^{\frac{1}{p+1}},$$

provided that

$$\frac{1}{\log \log \frac{1}{16\rho}} + \gamma_1 L \frac{\log \log \frac{1}{16\rho}}{\log \frac{1}{16\rho}} \leq 1.$$  \hspace{1cm} \text{(3.7)}

**Proof** First, we prove inequality on the right-hand side of (3.6). Fix $\sigma \in (0, 1/4)$ and for any $s \in (5/4\rho, 2\rho(1 - \sigma))$, $j = 0, 1, 2, \ldots$, set $\rho_j^{(1)} := s(1 + \sigma - \sigma 2^{-j})$, $K_j := K(\rho_j^{(1)}, 16\rho)$, $k_j := k - k 2^{-j}$, $k > 0$, $A_j := K_j \cap \{v > k_j\}$, $M_j := \sup_{K_j} v$ and let $\xi_j \in C^\infty(B_{16\rho}(x_0))$, $0 \leq \xi_j \leq 1$, $\xi_j = 0$ in $B_{\rho_j^{(1)}}(x_0)$, $\xi_j = 1$ in $K_{j+1}$, $|\nabla \xi_j \lfloor \leq \gamma 2^j (\sigma \rho)^{-1}$. Further, we will assume that

$$\hat{\mathcal{M}}_{\frac{3}{2}\rho}(v) \geq \frac{3}{2}\rho,$$

since otherwise, by (3.5) inequality (3.6) is evident, moreover this inequality yields

$$\left(\frac{\hat{\mathcal{M}}_{\rho_1}(v)}{\rho_1}\right)^{\ell(|x-x_0|)} \leq \hat{\omega}_{\rho_1}(x, v) \leq \gamma\left(\frac{\hat{\mathcal{M}}_{\rho_1}(v)}{\rho_1}\right)^{\ell(|x-x_0|)}, \text{ } x \in \mathcal{D}.$$

Testing (3.1) by $\varphi = (v - k_{j+1})_+ \xi_j^q$ and using the Young inequality we obtain

$$\int_{K_j \cap \{v > k_{j+1}\}} |\nabla v|^{p(x)} \xi_j^q \, dx$$

$$\leq \gamma \sigma^{-\gamma} 2^{ij} \rho^{-\hat{p}} \int_{A_j} \left(\frac{M_j}{\rho_j^{(1)}}\right)^{\ell(|x-x_0|)} (v - k_j)_+ \, dx.$$
\[
\begin{align*}
&\leq \gamma \sigma^{-\gamma} 2^{j\gamma} \rho^{-\tilde{p}} \int_{A_j} \left( \frac{M_0}{\rho_0^{(1)}} \right)^{\frac{\ell(x-x_0)}{\bar{p}}} (v - k_j)^{\bar{p}} dx \\
&= \gamma \sigma^{-\gamma} 2^{j\gamma} \rho^{-\tilde{p}} \int_{A_j} \hat{w}_{\rho_0^{(1)}}(x, v) (v - k_j)^{\bar{p}} dx.
\end{align*}
\]

Using again the Young inequality, assuming that \( k > \varepsilon_0 M_0 \), where \( \varepsilon_0 \in (0, 1) \) is small enough, from the previous we have

\[
\int_{K_j \cap \{ v > k_j + 1 \}} \hat{w}_{\rho_0^{(1)}}(x, v) |\nabla v|^\tilde{p} \zeta_j^q dx
\leq \gamma \sigma^{-\gamma} 2^{j\gamma} \rho^{-\tilde{p}} \int_{A_j} \hat{w}_{\rho_0^{(1)}}(x, v) (v - k_j)^{\bar{p}} dx
\]

Choose \( \gamma_1 > 0 \) large enough, by our assumption Lemma 2.3 is applicable, therefore from this we obtain

\[
y_{j+1} := \int_{A_{j+1}} \hat{w}_{\rho_0^{(1)}}(x, v) (v - k_{j+1})^{\bar{p}} dx
\leq \gamma \varepsilon_0^{-\gamma} \sigma^{-\gamma} 2^{j\gamma} \left[ \hat{S}_v \left( K \left( \rho_0^{(1)}, 16\rho \right) \right) \right]^{-\left(1 - \frac{1}{\varepsilon_1} \right)} \left( \int_{A_{j+1}} \hat{w}_{\rho_0^{(1)}}(x, v) dx \right)^{1 - \frac{1}{\varepsilon_1}}
\times \int_{A_j} \hat{w}_{\rho_0^{(1)}}(x, v) (v - k_j)^{\bar{p}} dx
\leq \gamma \varepsilon_0^{-\gamma} \sigma^{-\gamma} 2^{j\gamma} \left[ \hat{S}_v \left( K \left( \rho_0^{(1)}, 16\rho \right) \right) \right]^{-\left(1 - \frac{1}{\varepsilon_1} \right)}
\times \kappa^{-\tilde{p} \left(1 - \frac{1}{\varepsilon_1} \right)} y_j^{-\frac{1}{\varepsilon_1}}, \quad j = 0, 1, 2, ...
\]

Hence, setting \( M_{\sigma} := M_\infty \), by standard arguments (see, e.g. [32]) and by our choice, we arrive at

\[
M_{\sigma} \bar{p} \leq \varepsilon_0 \bar{p} M_0 \bar{p} + \gamma \varepsilon_0^{-\gamma} \sigma^{-\gamma} \left[ \hat{S}_v \left( K \left( \rho_0^{(1)}, 16\rho \right) \right) \right]^{-1} \int_{K \left( \rho_0^{(1)}, 16\rho \right)} \hat{w}_{\rho_0^{(1)}}(x, v) v^{\bar{p}} dx. \quad (3.8)
\]
Let us estimate the second term on the right-hand side of (3.8). For this we set $v_{M_0} := \min\{v, M_0\}$, by Lemma 2.3 we have for any $\varepsilon \in (0, 1)$

$$\int_{K(\rho_0^{(1)}, 16\rho)} \hat{w}_{(1)}(x, v) v_{M_0} \, dx = \int_{K(\rho_0^{(1)}, 16\rho)} \hat{w}_{(1)}(x, v) v_{M_0} \, dx$$

$$\leq \gamma \rho^\hat{p} \int_{K(\rho_0^{(1)}, 16\rho)} \hat{w}_{(1)}(x, v) |\nabla v_{M_0}|^\hat{p} \, dx$$

$$\leq \frac{\varepsilon \sigma^\gamma \rho^\hat{p}}{\gamma} \int_{K(\rho_0^{(1)}, 16\rho)} \left[ \hat{w}_{(1)}(x, v) \right]^\rho(x) \, dx$$

$$+ \frac{\gamma \rho^\hat{p}}{\varepsilon^\gamma \sigma^\gamma} \int_{\mathcal{D}} |\nabla v_{M_0}|^p(x) \, dx$$

$$= \frac{\varepsilon \sigma^\gamma}{\gamma} M_0^\hat{p} \hat{S}_v \left( K\left(\rho_0^{(1)}, 16\rho\right) \right) + \frac{\gamma \rho^\hat{p}}{\varepsilon^\gamma \sigma^\gamma} \int_{\mathcal{D}} |\nabla v_{M_0}|^p(x) \, dx. \quad (3.9)$$

Collecting the last two inequalities we obtain

$$M_0^\hat{p} \leq (\varepsilon_0^\hat{p} + \frac{\varepsilon}{\varepsilon_0^\hat{p}}) M_0^\hat{p} + \frac{\gamma \rho^\hat{p}}{\varepsilon^\gamma \sigma^\gamma} \left[ \hat{S}_v \left( K\left(\rho_0^{(1)}, 16\rho\right) \right) \right]^{-1} \int_{\mathcal{D}} |\nabla v_{M_0}|^p(x) \, dx. \quad (3.9)$$

Let us estimate the second term on the right-hand side of (3.9). Let $\psi \in \mathcal{M}(E)$ be such that

$$\int_{\mathcal{D}} |m \nabla \psi|_E^p \, dx \leq C_{p(\cdot)}(E, B_{16\rho}(x_0); m) + \rho^n \leq C_{p(\cdot)}(E, B_{16\rho}(x_0); m) + \gamma \hat{S}_v \left( K\left(\rho_0^{(1)}, 16\rho\right) \right).$$

Testing identity (3.1) by $\varphi = v - m\psi$, by the Young inequality we obtain

$$\int_{\mathcal{D}} |\nabla v|_E^p \, dx \leq \gamma \int_{B_{16\rho}(x_0)} |m \nabla \psi|_E^p \, dx$$

$$\leq \gamma m \left( C_{p(\cdot)}(E, B_{16\rho}(x_0); m) + \hat{S}_v \left( K\left(\rho_0^{(1)}, 16\rho\right) \right) \right).$$

Testing (3.1) by $\varphi = v_{M_0} - \frac{M_0}{m} v$, using the Young inequality and the previous inequality, we have

$$\int_{\mathcal{D}} |\nabla v_{M_0}|_E^p \, dx \leq \gamma \frac{M_0}{m} \int_{\mathcal{D}} |\nabla v|_E^p \, dx$$

$$\leq \gamma M_0 \left( C_{p(\cdot)}(E, B_{16\rho}(x_0); m) + \hat{S}_v \left( K\left(\rho_0^{(1)}, 16\rho\right) \right) \right).$$
This inequality, the Young inequality and (3.9) imply that
\[
M_{\sigma}^\rho \leq (\epsilon_0^\rho + \frac{\epsilon}{\epsilon_0^\rho}) M_0^\rho + \gamma \varepsilon_0^{-\gamma} \gamma^{-\gamma} \sigma^{-\gamma} M_0 \left( \rho^\rho \frac{C_{p(\cdot)}(E, B_{16\rho}(x_0); m)}{\mathcal{S}_v(K(\rho_1^{(1)}, 16\rho))} + \rho^\rho \right)
\]
\[
\leq (2\epsilon_0^\rho + \frac{\epsilon}{\epsilon_0^\rho}) M_0^\rho + \gamma \varepsilon_0^{-\gamma} \gamma^{-\gamma} \sigma^{-\gamma} \left\{ \left( \rho^\rho \frac{C_{p(\cdot)}(E, B_{16\rho}(x_0); m)}{\mathcal{S}_v(K(\hat{\rho}, 16\rho))} \right)^{\frac{\rho}{\rho_1}} + \rho^\rho \right\},
\]
Iterating the last inequality, choosing \( \epsilon_0 \) and then \( \epsilon = \varepsilon(\epsilon_0) \) small enough, by (3.5) we arrive at
\[
\hat{M}_{\frac{3}{2}\rho}(v) \leq \gamma \left( \rho^\rho \frac{C_{p(\cdot)}(E, B_{16\rho}(x_0); m)}{\mathcal{S}_v(K(\frac{3}{2}\rho, 16\rho))} \right)^{\frac{1}{\rho_1}} + \gamma \rho
\]
\[
\leq \gamma \left( \rho^\rho \frac{C_{p(\cdot)}(E, B_{16\rho}(x_0); m)}{\mathcal{S}_v(K(\frac{3}{2}\rho, 16\rho))} \right)^{\frac{1}{\rho_1}},
\]
which completes the proof of the lemma.

Now we prove inequality on the left-hand side of (3.6). Let \( \zeta_1 \in C_0^\infty(B_{4\rho}(x_0)), 0 \leq \zeta_1 \leq 1, \zeta_1 = 1 \) in \( B_{2\rho}(x_0), |\nabla \zeta_1| \leq \frac{\gamma}{\rho} \). Testing (3.1) by \( \varphi = v - m \zeta_1^{\rho} \), using the Young inequality, we obtain for any \( \varepsilon_1 > 0 \)
\[
\int_D |\nabla v|^{p(x)} \, dx \leq \gamma \frac{m}{\rho} \int_{K(2\rho, 4\rho)} |\nabla v|^{p(x)-1} \zeta_1^{\rho-1} \, dx
\]
\[
\leq \gamma \frac{m}{\varepsilon_1 \rho} \int_{K(2\rho, 4\rho)} |\nabla v|^{p(x)} \, dx + \gamma \frac{m}{\rho} \int_{K(2\rho, 4\rho)} \zeta_1^{\rho-1} \, dx.
\]
Let \( \zeta_2 \in C_0^\infty(K(\frac{3}{2}\rho, 6\rho)), 0 \leq \zeta_2 \leq 1, \zeta_2 = 1 \) in \( K(2\rho, 4\rho), |\nabla \zeta_2| \leq \frac{\gamma}{\rho} \). Testing (3.1) by \( \varphi = v \zeta_2^\rho \) and using the Young inequality, we estimate the first term on the right-hand side of the previous inequality as follows:
\[
\int_{K(2\rho, 4\rho)} |\nabla v|^{p(x)} \, dx \leq \gamma \int_{K(\frac{3}{2}\rho, 6\rho)} \left( \frac{v}{\rho} \right)^{p(x)} \, dx \leq \frac{\gamma}{\rho^p} \int_{K(\frac{3}{2}\rho, 16\rho)} \hat{w}_{3\rho}(x, v) \, dx.
\]
Combining the last two inequalities and using the definition of capacity, we obtain
\[
C_{p(\cdot)}(E, B_{16\rho}(x_0); m) \leq \frac{1}{m} \int_D |\nabla v|^{p(x)} \, dx
\]
\[
\leq \frac{\gamma}{\varepsilon_1 \rho^{p+1}} \int_{K(\frac{3}{2}\rho, 16\rho)} \hat{w}_{3\rho}(x, v) \, dx + \frac{\gamma}{\rho} \int_{K(2\rho, 4\rho)} \zeta_1^{\rho-1} \, dx.
\]
\begin{equation}
(3.10)
\end{equation}
Choose \( \varepsilon_1 \) from the condition \( \varepsilon_1 = \frac{\hat{M}_{\frac{3}{2}\rho}(v)}{\rho} \), then inequality (3.10) yields
\[
C_{p(\cdot)}(E, B_{16\rho}(x_0); m) \leq \gamma \hat{S}_v(K(\frac{3}{2}\rho, 16\rho)) \frac{\hat{M}_{\frac{3}{2}\rho}(v)^{\rho-1}}{\rho^p}.
\]
from which the required inequality follows, this completes the proof of the lemma. □

3.2 Lower bound for the function $v$

Further we need the following lemma.

**Lemma 3.2** Let condition (3.5) holds, then there exists $\varepsilon \in (0, 1)$ depending only on the data such that

$$
\int_{K(\frac{3}{2} \rho, 16 \rho)} \hat{w}_{\frac{2}{\bar{p}}} \chi \left[ v \geq \varepsilon \left( \rho^\bar{p} \frac{C_{p(\cdot)}(E, B_{16 \rho}(x_0); m)}{\hat{S}_v(K(\frac{3}{2} \rho, 16 \rho))} \right)^{\frac{1}{\bar{p}-1}} \right] dx 
$$

$$
\geq \gamma^{-1} \hat{S}_v \left( K \left( \frac{3}{2} \rho, 16 \rho \right) \right), \tag{3.11}
$$

provided that inequality (3.7) holds, here $\chi[F]$ is the characteristic function of the set $F$.

**Proof** To prove (3.11) we use inequality (3.10). Choose $\varepsilon_1$ from the condition $\varepsilon_1 = \frac{\varepsilon_1}{\bar{p}}$, $\bar{\varepsilon}_1 \in (0, 1)$, then by Lemma 3.1 the terms on the right-hand side of (3.10) are estimated as follows

$$
\frac{\gamma}{\rho} \int_{K(2 \rho, 4 \rho)} \varepsilon_1^{p(x)-1} dx \leq \frac{\gamma}{\rho} \bar{\varepsilon}_1 \hat{M}_{\frac{3}{2} \rho}(v)^{-1} \hat{S}_v \left( K \left( \frac{3}{2} \rho, 16 \rho \right) \right)
$$

$$
\leq \gamma \bar{\varepsilon}_1 C_{p(\cdot)}(E, B_{16 \rho}(x_0); m). \tag{3.12}
$$

Similarly, by Lemma 3.1

$$
\frac{\gamma}{\varepsilon_1 \rho} \int_{K(\frac{3}{2} \rho, 16 \rho)} \hat{w}_{\frac{2}{\bar{p}}} (x, v) v^{\bar{p}} dx
$$

$$
\leq \gamma \frac{\varepsilon_1}{\bar{\varepsilon}_1} C_{p(\cdot)}(E, B_{16 \rho}(x_0); m)
$$

$$
+ \gamma \frac{C_{p(\cdot)}(E, B_{16 \rho}(x_0); m)}{\hat{S}_v(K(\frac{3}{2} \rho, 16 \rho))}
$$

$$
\times \int_{K(\frac{3}{2} \rho, 16 \rho)} \hat{w}_{\frac{2}{\bar{p}}} (x, v) \chi \left[ v \geq \varepsilon \left( \rho^\bar{p} \frac{C_{p(\cdot)}(E, B_{16 \rho}(x_0); m)}{\hat{S}_v(K(\frac{3}{2} \rho, 16 \rho))} \right)^{\frac{1}{\bar{p}-1}} \right] dx. \tag{3.13}
$$

Collecting estimates (3.10), (3.12), (3.13) we obtain

$$
C_{p(\cdot)}(E, B_{16 \rho}(x_0); m) \leq \left( \gamma \bar{\varepsilon}_1 + \gamma \frac{\varepsilon_1}{\bar{\varepsilon}_1} \right) C_{p(\cdot)}(E, B_{16 \rho}(x_0); m)
$$

$$
+ \gamma \frac{C_{p(\cdot)}(E, B_{16 \rho}(x_0); m)}{\hat{S}_v(K(\frac{3}{2} \rho, 16 \rho))}
$$

$$
\times \int_{K(\frac{3}{2} \rho, 16 \rho)} \hat{w}_{\frac{2}{\bar{p}}} (x, v) \chi \left[ v \geq \varepsilon \left( \rho^\bar{p} \frac{C_{p(\cdot)}(E, B_{16 \rho}(x_0); m)}{\hat{S}_v(K(\frac{3}{2} \rho, 16 \rho))} \right)^{\frac{1}{\bar{p}-1}} \right] dx.
$$
Choosing $\tilde{\varepsilon}_1$ by the condition $\gamma \tilde{\varepsilon}_1 = \frac{1}{4}$ and then choosing $\varepsilon$ by the condition $\gamma \frac{\varepsilon}{\tilde{\varepsilon}_1} = \frac{1}{4}$, from the previous we arrive at

$$
\int_{K(\frac{3}{2} \rho, 16 \rho)} \tilde{w}_2^\rho(x, v) \chi \left[ v \geq \varepsilon \left( \rho \tilde{\rho} \frac{C_{\rho(\cdot)}}{S_v} (E, B_{16 \rho}(x_0); m) \right) \right] \frac{1}{\rho^{1\gamma_1}} \textrm{d}x \geq \gamma^{-1} S_v \left( K(\frac{3}{2} \rho, 16 \rho) \right),
$$

which completes the proof of the lemma. \hfill \square

The following lemma is the main result of this Section

**Lemma 3.3** There exists $\bar{\varepsilon} \in (0, 1)$ depending only on the data such that either

$$
\bar{\varepsilon} m \left( \frac{\rho}{|B_{\rho}(x_0)|} \right)^{\frac{1}{\gamma_1}} \leq \rho, \tag{3.14}
$$

or

$$
| \{ K(\frac{3}{2} \rho, 16 \rho) : v \geq \bar{\varepsilon} m \left( \frac{\rho}{|B_{\rho}(x_0)|} \right)^{\frac{1}{\gamma_1}} \} | \geq \gamma^{-1} | K(\frac{3}{2} \rho, 16 \rho) |, \tag{3.15}
$$

provided that inequality (3.7) holds.

**Proof** Lemma 3.3 is a consequence of Lemma 3.2, for this we first estimate the capacity of the set $E$ from below. Let $\varphi \in W_0(B_{16 \rho}(x_0)), \varphi = 1$ on $E$, then by Lemmas 2.2, 2.3, 3.1 and using the evident inequalities $\gamma^{-1} S_v \left( K(\frac{3}{2} \rho, 16 \rho) \right) \leq \int_{B_{16 \rho}(x_0)} \tilde{w}_2^\rho(x, v) \textrm{d}x \leq \gamma S_v \left( K(\frac{3}{2} \rho, 16 \rho) \right)$ we have

$$
m^\rho |E| \leq m^\rho \int_{B_{16 \rho}(x_0)} \varphi^\rho \textrm{d}x
$$

$$
\leq \left( \int_{B_{16 \rho}(x_0)} \tilde{w}_2^\rho(x, v) \left( m \varphi \right)^{\rho \infty} \textrm{d}x \right)^{\frac{1}{\gamma_1}} \left( \int_{B_{16 \rho}(x_0)} \left( \tilde{w}_2^\rho(x, v) \right)^{-\frac{1}{\gamma_1}} \textrm{d}x \right)^{1-\frac{1}{\gamma_1}}
$$

$$
\leq \gamma \rho \tilde{\rho} \left[ S_v \left( K(\frac{3}{2} \rho, 16 \rho) \right) \right]^{\frac{1}{\gamma_1}-1} \left( \int_{B_{16 \rho}(x_0)} \left( \tilde{w}_2^\rho(x, v) \right)^{-\frac{1}{\gamma_1}} \textrm{d}x \right)^{1-\frac{1}{\gamma_1}}
$$

$$
\times \int_{B_{16 \rho}(x_0)} \tilde{w}_2^\rho(x, v) \left| \nabla (m \varphi) \right|^{\tilde{\rho}} \textrm{d}x
$$

$$
\leq \gamma \rho \tilde{\rho} \left[ S_v \left( K(\frac{3}{2} \rho, 16 \rho) \right) \right]^{\frac{1}{\gamma_1}-1} \left( \int_{B_{16 \rho}(x_0)} \left( \tilde{w}_2^\rho(x, v) \right)^{-\frac{1}{\gamma_1}} \textrm{d}x \right)^{1-\frac{1}{\gamma_1}}
$$

$$
\times \left( \int_{B_{16 \rho}(x_0)} \tilde{w}_2^\rho(x, v) \left( 1 + \frac{M_2^\rho(v)}{\rho} \right)^{\tilde{\rho}} \textrm{d}x + \int_{B_{16 \rho}(x_0)} \left| \nabla (m \varphi) \right|^{p(x)} \textrm{d}x \right)
$$

$$
\leq \gamma \rho \tilde{\rho}^{n+1} + \gamma m \left( \frac{\rho \tilde{\rho}^{n+1}}{S_v \left( K(\frac{3}{2} \rho, 16 \rho) \right)} \right) C_{\rho(\cdot)}(E, B_{16 \rho}(x_0), m)
$$

Springer
\[ + \gamma \frac{\rho^{\hat{p}+n}}{S_v \left( K \left( \frac{3}{2} \rho, 16 \rho \right) \right)} \int_{B_{16\rho}(x_0)} |\nabla (m \varphi) |^{p(x)} \, dx. \tag{3.16} \]

Since \( \varphi \) is arbitrary, estimate (3.16) yields
\[
m^{\hat{p}} | E | \leq \gamma \rho^{\hat{p}+n} + \gamma m \frac{\rho^{\hat{p}+n}}{S_v \left( K \left( \frac{3}{2} \rho, 16 \rho \right) \right)} C_{p(\cdot)}(E, B_{16\rho}(x_0), m).
\]

If inequality (3.14) is violated then
\[
m^{\hat{p}} \frac{| E |}{| B_\rho(x_0) |} \geq m^{\hat{p}} \left( \frac{| E |}{| B_\rho(x_0) |} \right)^{\frac{\hat{p}}{p-1}} \geq \left( \frac{\rho}{\bar{\varepsilon}} \right)^{\hat{p}},
\]
so, if \( \bar{\varepsilon} \) is sufficiently small, from the previous we arrive at
\[
\frac{\rho^{\hat{p}}}{S_v \left( K \left( \frac{3}{2} \rho, 16 \rho \right) \right)} C_{p(\cdot)}(E, B_{16\rho}(x_0), m) \geq \gamma^{-1} m^{\hat{p}-1} \frac{| E |}{| B_\rho(x_0) |}.
\]

And hence
\[
\rho \frac{C_{p(\cdot)}(E, B_{16\rho}(x_0); m)}{S_v \left( K \left( \frac{3}{2} \rho, 16 \rho \right) \right)} \geq \gamma^{-\hat{p}} \rho^{1-\hat{p}} m^{\hat{p}-1} \frac{| E |}{| B_\rho(x_0) |} \geq \gamma^{-\hat{p}} \bar{\varepsilon}^{-\hat{p}} \geq \gamma_0,
\]
provided that (3.14) is violated and \( \bar{\varepsilon} \) is sufficiently small. Now we use Lemma 3.2 for this

we set \( F := \{ K \left( \frac{3}{2} \rho, 16 \rho \right) : v \geq \bar{\varepsilon} m \left( \frac{| E |}{| B_\rho(x_0) |} \right)^{\frac{1}{p-1}} \} \). We have by Lemmas 2.2 and 3.2
\[
\gamma^{-1} \leq \frac{\rho^{\hat{p}}}{S_v \left( K \left( \frac{3}{2} \rho, 16 \rho \right) \right)} \int_{K \left( \frac{3}{2} \rho, 16 \rho \right)} \tilde{w}^{\frac{1}{2}}(x, v) \chi(F) \, dx
\]
\[
\leq \frac{\frac{1}{2}}{S_v \left( K \left( \frac{3}{2} \rho, 16 \rho \right) \right)} \left( \int_{K \left( \frac{3}{2} \rho, 16 \rho \right)} [\tilde{w}^{\frac{1}{2}}(x, v)]^2 \, dx \right)^{\frac{1}{2}} \leq \gamma \left( \frac{| F |}{| K \left( \frac{3}{2} \rho, 16 \rho \right) |} \right)^{\frac{1}{2}},
\]
which completes the proof of the lemma.

\[ \square \]

4 Harnack’s inequality, proof of Theorems 1.2 and 1.3

4.1 Weak Harnack inequality, proof of Theorem 1.2

For \( 0 < s < M \) set \( E(\rho, s) := \{ B_\rho(x_0) : u \geq s \} \). As it was mentioned in Sect. 1 Theorem 1.2 is a simple consequence of the following lemma

**Lemma 4.1.** Let \( u \) be a non-negative bounded super-solution to equation (1.1) in \( \Omega \) and let condition (1.9) be fulfilled, then there exist positive numbers \( C_2, C_3 \) depending only on the data such that
\[
| E(\rho, s) | \leq C_2 | B_\rho(x_0) | s^{1-\hat{p}} (\rho + \inf_{B_\rho(x_0)} u)^{\hat{p}-1}, \tag{4.1}
\]
provided that $B_{16\rho}(x_0) \subset \Omega$ and
\[
\frac{1}{\log \log \frac{1}{16\rho}} + C_3 \, L \, \frac{\log \log \frac{1}{16\rho}}{\log \frac{1}{16\rho}} \leq 1. \tag{4.2}
\]

**Proof** We construct the solution $v$ of the problem (3.1) in $\mathcal{D} = B_{16\rho}(x_0) \setminus E(\rho, s)$, since $u \geq v$ on $\partial \mathcal{D}$, by (1.9) $u \geq v$ in $\mathcal{D}$. First we use Lemma 3.3, if inequality (3.14) is violated, i.e. if
\[
\bar{\varepsilon} \, s \left( \frac{|E(\rho, s)|}{|B_{\rho}(x_0)|} \right)^{\frac{1}{p-1}} \geq \rho, \tag{4.3}
\]
by Lemma 3.3 there holds
\[
| \left\{ B_{16\rho}(x_0) : u \geq \bar{\varepsilon} \, s \left( \frac{|E(\rho, s)|}{|B_{\rho}(x_0)|} \right)^{\frac{1}{p-1}} \right\} | \\
\leq | \left\{ K(\frac{3}{2}\rho, 16\rho) : u \geq \bar{\varepsilon} \, m \left( \frac{|E(\rho, s)|}{|B_{\rho}(x_0)|} \right)^{\frac{1}{p-1}} \right\} | \\
\geq | \left\{ K(\frac{3}{2}\rho, 16\rho) : v \geq \bar{\varepsilon} \, s \left( \frac{|E(\rho, s)|}{|B_{\rho}(x_0)|} \right)^{\frac{1}{p-1}} \right\} | \geq \gamma^{-1} | B_{16\rho}(x_0) |,
\]
provided that
\[
\frac{1}{\log \log \frac{1}{16\rho}} + \gamma_1 \, L \, \frac{\log \log \frac{1}{16\rho}}{\log \frac{1}{16\rho}} \leq 1. \tag{4.4}
\]
Now we use Lemma 2.6 with $r = 16\rho$, $\mu_- = 0$ and $\xi \omega_r = \bar{\varepsilon} \, s \left( \frac{|E(\rho, s)|}{|B_{\rho}(x_0)|} \right)^{\frac{1}{p-1}}$, we obtain that
\[
u(x) \geq 2^{-s_\ast^{-1}} \, s \left( \frac{|E(\rho, s)|}{|B_{\rho}(x_0)|} \right)^{\frac{1}{p-1}}, \quad x \in B_{8\rho}(x_0), \tag{4.5}
\]
provided that
\[
\frac{1}{\log \log \frac{1}{16\rho}} + s_\ast \, L \, \frac{\log \log \frac{1}{16\rho}}{\log \frac{1}{16\rho}} \leq 1. \tag{4.6}
\]
Choosing $C_2$, $C_3$ sufficiently large, collecting (4.3)–(4.6) we arrive at (4.1), which completes the proof of the lemma. \qed
To complete the proof of Theorem 1.2 set \( m(\rho/2) := \rho/2 + \inf_{B_{\rho/2}(x_0)} u \), then by Lemma 4.1

for \( \theta \in (0, \tilde{p} - 1) \) we have

\[
\int_{B_{\rho}(x_0)} u^\theta \, dx = \theta \mid B_{\rho}(x_0) \mid^{-1} \int_0^\infty \mid E(\rho, s) \mid s^{\theta-1} \, ds \\
\leq m^\theta(\rho/2) + \theta \mid B_{\rho}(x_0) \mid^{-1} \int_{m(\rho/2)}^\infty \mid E(\rho, s) \mid s^{\theta-1} \, ds \\
\leq m^\theta(\rho/2) + \gamma m^{\tilde{p}-1}(\rho/2) \int_{m(\rho/2)}^\infty s^{\theta-\tilde{p}} \, ds \leq \gamma \frac{\rho}{\tilde{p} - 1 - \theta} m^\theta(\rho/2),
\]

provided that

\[
\frac{1}{\log \log \frac{1}{16\rho}} + C_3 L \frac{\log \log \frac{1}{16\rho}}{\log \frac{1}{16\rho}} \leq 1,
\]

which completes the proof of Theorem 1.2.

4.2 Proof of Theorem 1.3

The proof of Theorem 1.3 is almost standard.

For fixed \( \sigma \in (0, 1/8), s \in (3/4 \rho, 7/8 \rho), k > 0 \) and \( j = 0, 1, 2, \ldots \) set

\[
k_j := k - 2^{-j}, \quad \rho_j := s(1 - \sigma + \sigma 2^{-j}), \quad \tilde{\rho}_j := \frac{1}{2}(\rho_j + \rho_{j+1}), \quad B_j := B_{\rho_j}(x_0),
\]

\( \tilde{B}_j := B_{\tilde{\rho}_j}(x_0) \) and let \( M_0 := \sup_{B_\infty} u, \ M_\sigma := \sup_{B_0} u. \) Denote by \( \zeta_j \) a non-negative piece-wise smooth cutoff function in \( \tilde{B}_j \) that equals one on \( B_j + 1, \) such that \( |\nabla \zeta_j| \leq \gamma \frac{2^j}{\sigma \rho}. \)

Set also \( w_0(x) := \left( 1 + \frac{M_0}{\rho_0} \right)^{\ell(|x-x_0|)} \) and \( w_0(F) := \int_F w_0(x) \, dx. \) Evidently, we have

\[
\left( \frac{M_0}{\rho_0} \right)^{\ell(|x-x_0|)} \leq w_0(x) \leq \gamma \left( \frac{M_0}{\rho_0} \right)^{\ell(|x-x_0|)}, \text{ if } M_0 \geq \rho_0.
\]

Note that similarly to (2.5), (2.6) there hold

\[
\gamma^{-1}\left( 1 + \frac{M_0}{\rho_0} \right)^{-t\ell(\rho_0)} \leq \int_{B_0} w_0^{-t}(x) \, dx \leq \gamma\left( 1 + \frac{M_0}{\rho_0} \right)^{-t\ell(\rho_0)}, \quad t > 0, \quad (4.7)
\]

\[
\gamma^{-1}\left( 1 + \frac{M_0}{\rho_0} \right)^{t\ell(\rho_0)} \leq \int_{B_0} w_0^t(x) \, dx \leq \gamma\left( 1 + \frac{M_0}{\rho_0} \right)^{t\ell(\rho_0)}, \quad t > 0, \quad (4.8)
\]

provided that

\[
\frac{1}{\log \log \frac{1}{\rho_0}} + t \tilde{C} L \log \left( 1 + \frac{M_0}{\rho_0} \right) \frac{\log \log \frac{1}{\rho_0}}{\log^2 \frac{1}{\rho_0}} \leq 1,
\]

\( \tilde{C} \) Springer
where \( \bar{C} = \max(C, C_1) \) and \( C, C_1 \) are the constants defined in Lemmas 2.2, 2.3. Therefore Lemmas 2.2, 2.3 continue to hold in \( B_0 \) with \( w_0^\tilde{p}(x, u, k) \) replaced by \( w_0(x) \).

Further we will assume that \( M_0 \geq \rho_0 \).

Test identity (1.8) by \( \varphi = (u - k_{j+1})^+ \), then

\[
\int_{B_j} |\nabla (u - k_{j+1})^+|^{p(x)} \xi_j^q \, dx \leq \gamma 2^{j\nu} \int_{B_j} \left( \frac{u - k_{j+1}}{\sigma \rho} \right)^{p(x)} \, dx
\]

\[
\leq \gamma \sigma^{-\gamma} 2^{j\nu} \frac{1}{\rho^p} \int_{B_j} w_0(x) (u - k_j)^{\tilde{p}} \, dx.
\]

From this by the Young inequality, assuming that \( k > \varepsilon_0 M_0, \varepsilon_0 \in (0, 1) \) is small enough, we obtain

\[
\int_{B_j} w_0(x) |\nabla (u - k_{j+1})^+|^{\tilde{p}} \xi_j^q \, dx \leq \gamma \left( \frac{M_0}{\rho} \right)^{\tilde{p}} \int_{B_j \cap \{ u > k_{j+1} \}} w_0(x) \, dx
\]

\[
+ \gamma \sigma^{-\gamma} 2^{j\nu} \frac{1}{\rho^p} \int_{B_j} w_0(x) (u - k_{j+1})^{\tilde{p}} \, dx
\]

\[
\leq \gamma \sigma^{-\gamma} 2^{j\nu} \left( \frac{M_0}{\rho} \right)^{\tilde{p}} \int_{B_j \cap \{ u > k_j \}} w_0(x) \, dx
\]

\[
+ \gamma \sigma^{-\gamma} 2^{j\nu} \frac{1}{\rho^p} \int_{B_j} w_0(x) (u - k_{j+1})^{\tilde{p}} \, dx
\]

\[
\leq \gamma \varepsilon_0^{\gamma - \gamma} 2^{j\nu} \frac{1}{\rho^p} \int_{B_j} w_0(x) (u - k_{j+1})^{\tilde{p}} \, dx,
\]

provided that (4.9) holds. From this similarly to (3.8) we obtain

\[
M_\sigma \leq \varepsilon_0^{\frac{1}{\tilde{p}}} M_0 + \gamma \varepsilon_0^{\gamma} (\sigma^{-\gamma} \left( [w_0(B_0)]^{-1} \int_{B_0} w_0(x) u^{\tilde{p}} \, dx \right)^{\frac{1}{\tilde{p}}}). \tag{4.10}
\]

Let us estimate the second term on the right-hand side of (4.10), using Lemma 2.2 we obtain for any \( 0 < \theta < \tilde{p} \) and any \( \varepsilon \in (0, 1) \)

\[
\left( [w_0(B_0)]^{-1} \int_{B_0} w_0(x) u^{\tilde{p}} \, dx \right)^{\frac{1}{\tilde{p}}}
\]

\[
\leq M_0^{\frac{1 - \frac{\theta}{\tilde{p}}}{2^{\theta}}} \left( [w_0(B_0)]^{-1} \int_{B_0} w_0(x) u^{\theta} \, dx \right)^{\frac{1}{\tilde{p}}}
\]

\[
\leq \varepsilon M_0 + \gamma \varepsilon^{-\gamma} \left( [w_0(B_0)]^{-1} \int_{B_0} w_0(x) u^{\theta} \, dx \right)^{\frac{2}{\tilde{p}}}.\]
\[ \leq \varepsilon M_0 + \gamma \varepsilon^{-\gamma} [w_0(B_0)]^{-\frac{2}{\beta}} \left( \int_{B_0} w_0^2(x) \, dx \right)^{\frac{1}{\beta}} \left( \int_{B_0} u^\theta \, dx \right)^{\frac{1}{\beta}} \]

\[ \leq \varepsilon M_0 + \gamma \varepsilon^{-\gamma} \left( \rho^{-n} \int_{B_0} u^\theta \, dx \right)^{\frac{1}{\beta}} , \]

which together with (4.10) yield

\[ M_\sigma \leq \left( \frac{1}{\beta} \varepsilon_0 + \varepsilon \right) M_0 + \gamma \varepsilon_0^{-\gamma} \varepsilon^{-\gamma} \sigma^{-\gamma} \left( \rho^{-n} \int_{B_0} u^\theta \, dx \right)^{\frac{1}{\beta}} . \]

Choosing \( \varepsilon_0, \varepsilon \) small enough, iterating this inequality and taking into account our choices we arrive at

\[ \sup_{B_{\rho/2(x_0)}} u \leq \gamma \left( \rho^{-n} \int_{B_0} u^\theta \, dx \right)^{\frac{1}{\beta}} + \gamma \rho , \]

provided that (4.9) is valid. This proves inequality (1.12).

Collecting estimates (1.10), (1.12) with \( \theta = \frac{1}{2}(\bar{\rho} - 1) \), we arrive at

\[ \sup_{B_{\rho/2(x_0)}} u \leq \gamma \left( \inf_{B_{\rho/2(x_0)}} u + \rho \right) , \]

which completes the proof of Theorem 1.3.

**Author Contributions** All authors contributed equally to this work.

**Funding** Open Access funding enabled and organized by Projekt DEAL. The research of the authors was supported by Project 0120U100178 from the National Academy of Sciences of Ukraine. The research of the first author was also supported by Grant EFDS-FL2-08 of the European Federation of Academies of Sciences and Humanities (ALLEA). The research of the second author was also supported by Project 0119U1020890 from the National Academy of Sciences of Ukraine.

**Declarations**

**Conflict of interest** Not applicable.

**Consent for publication** All authors consented for publication.

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.
References

1. Alkhutov, Yu.A.: The Harnack inequality and the Hölder property of solutions of nonlinear elliptic equations with a nonstandard growth condition (Russian), Differ. Uravn. 33(12), 1651–1660 (1997); translation in Differ. Equ. 33(12), 1653–1663 (1997) (1998)
2. Alkhutov, Yu.A.: On the Hölder continuity of \( p(x) \)-harmonic functions. Sb. Math. 196(1–2), 147–171 (2005)
3. Alkhutov, Yu.A., Krasheninnikova, O.V.: Continuity at boundary points of solutions of quasilinear elliptic equations with a nonstandard growth condition. Izv. Ross. Akad. Nauk Ser. Mat. 68(6), 3–60 (2004) (in Russian)
4. Alkhutov, Yu.A., Krasheninnikova, O. V.: On the continuity of solutions of elliptic equations with a variable order of nonlinearity (Russian), Tr. Mat. Inst. Steklova 261 (2008), Differ. Uravn. i Din. Sist. 7–15; translation in Proc. Steklov Inst. Math. 261(1–10) (2008)
5. Alkhutov, Yu.A., Surnachev, M.D.: A Harnack inequality for a transmission problem with \( p(x) \)-Laplacian. Appl. Anal. 98(1–2), 332–344 (2019)
6. Alkhutov, Yu.A., Surnachev, M.D.: Harnack’s inequality for the \( p(x) \)-Laplacian with a two-phase exponent \( p(x) \). J. Math. Sci. (N.Y.) 244(2), 116–147 (2020)
7. Alkhutov, Yu.A., Surnachev, M.D.: Behavior at a boundary point of solutions of the Dirichlet problem for the \( p(x) \)-Laplacian (Russian). Algebra i Analiz 31(2), 88–117 (2019); translation in St. Petersburg Math. J. 31(2), 251–271 (2020)
8. Antontsev, S.N., Díaz, J.I., Shmarev, S.: Energy methods for free boundary problems. Applications to nonlinear PDEs and fluid mechanics. In: Progress in Nonlinear Differential Equations and their Applications, vol. 48. Birkhauser Boston, Inc., Boston (2002)
9. Baroni, P., Colombo, M., Mingione, G.: Harnack inequalities for double phase functionals. Nonlinear Anal. 121, 206–222 (2015)
10. Baroni, P., Colombo, M., Mingione, G.: Non-autonomous functionals, borderline cases and related function classes. St. Petersbg. Math. J. 27, 347–379 (2016)
11. Baroni, P., Colombo, M., Mingione, G.: Regularity for general functionals with double phase. Calc. Var. Partial Differ. Eq. 57, 62 (2018)
12. Benyaiche, A., Harjulehto, P., Hästö, P., Karpinnen, A.: The weak Harnack inequality for unbounded supersolutions of equations with generalized Orlicz growth. J. Differ. Equ. 275, 790–814 (2021)
13. Buryachenko, K.O., Skrypnik, I.I.: Local continuity and Harnack’s inequality for double-phase parabolic equations. Potential Anal. 56(1), 137–164 (2022)
14. Colombo, M., Mingione, G.: Bounded minimisers of double phase variational integrals. Arch. Rational Mech. Anal. 218(1), 219–273 (2015)
15. Colombo, M., Mingione, G.: Regularity for double phase variational problems. Arch. Rational Mech. Anal. 215(2), 443–496 (2015)
16. Colombo, M., Mingione, G.: Calderon–Zygmund estimates and non-uniformly elliptic operators. J. Funct. Anal. 270, 1416–1478 (2016)
17. DiBenedetto, E., Gianazza, U., Vespri, V.: Local clustering of the non-zero set of functions in \( W^{1,1}(E) \). Rend. Lincei Mat. Appl. 17, 223–225 (2006)
18. DiBenedetto, E., Gianazza, U., Vespri, V.: Harnack’s Inequality for Degenerate and Singular Parabolic Equations, x+278 pp. Springer, New York (2012)
19. DiBenedetto, E., Trudinger, N.S.: Harnack inequalities for quasi-minima of variational integrals. Ann. Inst. Henri Poincare 1(4), 295–308 (1984)
20. Diening, L., Harjulehto, P., Hästö, P., Růžička, M.: Lebesgue and Sobolev spaces with variable exponents. In: Lecture Notes in Mathematics, 2017, x+509 pp. Springer, Heidelberg (2011)
21. Fan, X.: A Class of De Giorgi Type and Hölder Continuity of Minimizers of Variational with \( m(x) \)-Growth Condition. Lanzhou University, Lanzhou (1995)
22. Fan, X., Zhao, D.: A class of De Giorgi type and Hölder continuity. Nonlinear Anal. 35, 295–318 (1999)
23. Gariepy, R., Ziemer, W.P.: A regularity condition at the boundary point for solutions of quasilinear elliptic equations. Arch. Ration. Mech. Anal. 67, 25–39 (1977)
24. Hadzhi, O.V., Skrypnik, I.I., Voitovych, M.V.: Interior continuity,continuity up to the boundary and Harnack’s inequality for double-phase elliptic equations with non-logarithmic growth. Math. Nachrichten (in press)
25. Harjulehto, P., Hästö, P.: Orlicz Spaces and Generalized Orlicz Spaces. In: Lecture Notes in Mathematics, vol. 2236, p. X+169. Springer, Cham (2019). https://doi.org/10.1007/978-3-030-15100-3
26. Harjulehto, P., Hästö, P.: Boundary regularity under generalized growth conditions. Z. Anal. Anwend. 38(1), 73–96 (2019)
27. Harjulehto, P., Hästö, P., Lee, M.: Hölder continuity of quasiminimizers and $\omega$-minimizers of functionals with generalized Orlicz growth. Ann. Sc. Norm. Super Pisa Cl. Sci 5 XXII(2), 549–582 (2021)
28. Harjulehto, P., Hästö, P., Toivanen, O.: Hölder regularity of quasiminimizers under generalized growth conditions. Calc. Var. Partial Differ. Equ. 56(2), Art. 22, 26 pp (2017)
29. Harjulehto, P., Kuusi, T., Lukkari, T., Marola, N., Parviainen, M.: Harnack’s inequality for quasiminimizers with non-standard growth conditions. J. Math. Anal. Appl. 344(1), 504–520 (2008)
30. Krasheninnikova, O.V.: On the continuity at a point of solutions of elliptic equations with nonstandard growth condition. In: Proceedings of the Steklov Institute of Mathematics no. 1(236), pp. 193–200 (2002)
31. Krylov, N.V., Safonov, M.V.: A property of the solutions of parabolic equations with measurable coefficients. Izv. Akad. Nauk SSSR Ser. Mat. 44(1), 161–175 (1980) (in Russian)
32. Ladyzhenskaya, O.A., Ural’tseva, N.N.: Linear and Quasilinear Elliptic Equations. Nauka, Moscow (1973)
33. Landis, E.M.: Some questions in the qualitative theory of second-order elliptic equations (case of several independent variables). Uspehi Mat. Nauk 18, no. 1 (109), 3–62 (1963) (in Russian)
34. Landis, E.M.: Second Order Equations of Elliptic and Parabolic Type. In: Translations of Mathematical Monographs, vol. 171. American Mathematical Society, Providence (1998)
35. Lieberman, G.M.: The natural generalization of the natural conditions of Ladyzhenskaya and Ural’tseva for elliptic equations. Commun. Partial Differ. Equ. 16(2–3), 311–361 (1991)
36. Marcellini, P.: Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions. Arch. Rational Mech. Anal. 105(3), 267–284 (1989)
37. Marcellini, P.: Regularity and existence of solutions of elliptic equations with $p, q$-growth conditions. J. Differ. Equ. 90(1), 1–30 (1991)
38. Maz’ya, V.G.: Behavior, near the boundary, of solutions of the Dirichlet problem for a second-order elliptic equation in divergent form. Math. Notes Acad. Sci. USSR 2, 610–617 (1967)
39. Ok, J.: Harnack inequality for a class of functionals with non-standard growth via De Giorgi’s method. Adv. Nonlinear Anal. 7(2), 167–182 (2018)
40. Růžička, M.: Electrorheological fluids: Modeling and Mathematical Theory. Lecture Notes in Mathematics, vol. 1748. Springer, Berlin (2000)
41. Surnachev, M.D.: On Harnack’s inequality for the non-logarithmic Zhikov’s condition. J. Evolut. Equ. 22(2), 45 (2022)
42. Surnachev, M.D.: On Harnack’s inequality for $p(x)$-Laplacian (Russian), 69, 1–32 (2018). Keldysh Institute Preprints https://doi.org/10.20948/prepr-2018-69
43. Surnachev, M.D.: On the weak Harnack inequality for the parabolic $p(x)$- Laplacian. Asymptot. Anal. (2021). https://doi.org/10.3233/ASY-211746
44. Skrypnik, I.V.: Selected works, In: Problems and Methods. Mathematics. Mechanics. Cybernetics, vol. 1, Naukova Dumka, Kiev (2008) (in Russian)
45. Skrypnik, I.I., Voitovych, M.V.: Classes of De Giorgi, Ladyzhenskaya and Ural’tseva and their applications to elliptic and parabolic equations with nonstandard growth. Ukr. Mat. Visn. 16(3), 403–447 (2019)
46. Skrypnik, I.I., Voitovych, M.V.: $B_1$ classes of De Giorgi-Ladyzhenskaya-Ural’tseva and their applications to elliptic and parabolic equations with generalized Orlicz growth conditions. Nonlinear Anal. 202, 112135 (2021)
47. Skrypnik, I.I.: Harnack’s inequality for singular parabolic equations with generalized Orlicz growth under the non-logarithmic Zhikov’s condition. J. Evolut. Equ. 22(2), 45 (2022)
48. Surnachev, M.D.: On Harnack’s inequality for $p(x)$-Laplacian (Russian), 69, 1–32 (2018). Keldysh Institute Preprints https://doi.org/10.20948/prepr-2018-69
49. Surnachev, M.D.: On the weak Harnack inequality for the parabolic $p(x)$- Laplacian. Asymptot. Anal. (2021). https://doi.org/10.3233/ASY-211746
50. Weickert, J.: Anisotropic diffusion in image processing. In: European Consortium for Mathematics in Industry, B.G. Teubner, Stuttgart (1998)
51. Zhikov, V.V.: Questions of convergence, duality and averaging for functionals of the calculus of variations (Russian). Izv. Akad. Nauk SSSR Ser. Mat. 47(5), 961–998 (1983)
52. Zhikov, V.V.: Averaging of functionals of the calculus of variations and elasticity theory (Russian). Izv. Akad. Nauk SSSR Ser. Mat. 50(4), 675–710, 877 (1986)
53. Zhikov, V.V.: On Lavrentiev’s phenomenon. Russ. J. Math. Phys. 3(2), 249–269 (1995)
54. Zhikov, V.V.: On some variational problems. Russ. J. Math. Phys. 5(1), 105–116 (1997) (1998)
55. Zhikov, V.V.: On the density of smooth functions in Sobolev-Orlicz spaces (Russian). Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 310 (2004), Kraev. zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 35 [34], 67–81, 226; translation in J. Math. Sci. (N.Y.) 132(3), 285–294 (2006)
56. Zhikov, V.V., Kozlov, S.M., Oleinik, O.A.: Homogenization of Differential Operators and Integral Functionals. Springer, Berlin (1994)
57. Zhikov, V.V., Pastukhova, S.E.: On the improved integrability of the gradient of solutions of elliptic equations with a variable nonlinearity exponent (Russian). Mat. Sb. 199(12), 19–52 (2008); translation in Sb. Math. 199(11–12), 1751–1782 (2008)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.