Lorentzian spacetimes with constant curvature invariants in four dimensions

Alan Coley\textsuperscript{1}, Sigbjørn Hervik\textsuperscript{1,2} and Nicos Pelavas\textsuperscript{1}

\textsuperscript{1} Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia B3H 3J5, Canada
\textsuperscript{2} Faculty of Science and Technology, University of Stavanger, N-4036 Stavanger, Norway

E-mail: aac@mathstat.dal.ca, pelavas@mathstat.dal.ca and sigbjorn.hervik@uis.no

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Abstract
In this paper we investigate four-dimensional Lorentzian spacetimes with constant curvature invariants (CSI spacetimes). We prove that if a four-dimensional spacetime is CSI, then either the spacetime is locally homogeneous or the spacetime is a Kundt spacetime for which there exists a frame such that the positive boost weight components of all curvature tensors vanish and the boost weight zero components are all constant. We discuss some of the properties of the Kundt–CSI spacetimes and their applications. In particular, we discuss $I$-symmetric spaces and degenerate Kundt–CSI spacetimes.

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1. Introduction

Lorentzian spacetimes for which all polynomial scalar invariants constructed from the Riemann tensor and its covariant derivatives are constant (CSI spacetimes) were studied in [1, 2]. In this paper, we will investigate CSI spacetimes in four dimensions (4D) utilizing a number of the results derived in [3].

The definitions and notation used in this paper will follow those of [1, 2] (and a summary of the algebraic classification of tensors and a summary of curvature operators, necessary for this paper, will be given in the appendices). In particular, we consider a spacetime $\mathcal{M}$ equipped with a metric $g$ and let $I_k$ denote the set of all polynomial scalar invariants constructed from the curvature tensor and its covariant derivatives up to order $k$.

Definition 1.1. (VSI$_k$ spacetimes). $\mathcal{M}$ is called VSI$_k$ if for any invariant $I \in I_k$, $I = 0$ over $\mathcal{M}$.

Definition 1.2. (CSI$_k$ spacetimes). $\mathcal{M}$ is called CSI$_k$ if for any invariant $I \in I_k$, $I$ is constant (i.e. $\partial_\mu I = 0$) over $\mathcal{M}$.
Moreover, if a spacetime is VSI\(_k\) or CSI\(_k\) for all \(k\), we will simply call the spacetime VSI or CSI, respectively. The set of all locally homogeneous spacetimes will be denoted by \(H\). Clearly \(VSI \subset CSI\) and \(H \subset CSI\).

**Definition 1.3. (CSI\(_R\) spacetimes).** Let us denote by CSI\(_R\) all reducible CSI spacetimes that can be built from VSI and \(H\) by (i) warped products, (ii) fibered products and (iii) tensor sums.

**Definition 1.4. (CSI\(_F\) spacetimes).** Let us denote by CSI\(_F\) those spacetimes for which there exists a frame with a null vector \(\ell\) such that all components of the Riemann tensor and its covariants’ derivatives in this frame have the property that (i) all positive boost weight components (with respect to \(\ell\)) are zero and (ii) all zero boost weight components are constant.

Note that CSI\(_R\) \(\subset\) CSI and CSI\(_F\) \(\subset\) CSI. (There are similar definitions for CSI\(_{F,k}\) etc.)

**Definition 1.5. (CSI\(_K\) spacetimes).** Finally, let us denote by CSI\(_K\), those CSI spacetimes that belong to the (higher-dimensional) Kundt class (defined later); the so-called Kundt–CSI spacetimes.

We recall that a spacetime is Kundt on an open neighborhood if it admits a null vector \(\ell\) which is geodesic, non-expanding, shear-free and non-twisting; i.e.

\[
\ell^\mu \ell_{\nu} = 0, \quad \ell_{\mu,\nu} = \ell^{\mu ; \nu} = \ell^{\mu ; \nu}_{[\mu ; \nu]} = 0
\]

(which leads to constraints on the Ricci rotation coefficients in that neighborhood; namely the relevant Ricci rotation coefficients are zero). We note that the Kundt–CSI spacetimes are actually degenerate Kundt spacetimes, in which there exists a common null frame in which the geodesic, expansion-free, shear-free and twist-free null vector \(\ell\) is also the null vector in which all positive boost weight terms of the Riemann tensor and all of its covariant derivatives are zero [3]. We also note that if the Ricci rotation coefficients are all constants then we have a locally homogeneous spacetime.

In addition, we will make extensive use of the algebraic classification of the Riemann tensor (i.e. Riemann type) and its covariant derivatives according to boost weight, as described in detail in [4] (also see appendix A, where the algebraic types of the Riemann and its covariant derivative, \(\nabla\) Riemann, are given). In particular, in 4D the resulting Weyl type is equivalent to the Petrov type and the Ricci type is related to the PP-type (which is related to the Segre type of the Ricci tensor). We will primarily use the terms Petrov type and Segre type\(^3\) here to be consistent with the usual terminology in 4D, and only use the terms Weyl type and Ricci type when referring to results of interest in higher dimensions.

For a Riemannian manifold, there is no distinction between the CSI requirement and local homogeneity: every CSI is locally homogeneous (CSI \(\equiv H\)) [5]. This is not true for Lorentzian manifolds. However, for every CSI spacetime with particular constant invariants there is a homogeneous spacetime (not necessarily unique) with precisely the same constant invariants. From the work in [1] it was conjectured that if a spacetime is CSI then the Riemann tensor (and hence the Weyl tensor) is of type II, III, N or O [4] (part of the CSI\(_F\) conjecture), that if a spacetime is CSI, the spacetime is either locally homogeneous or belongs to the higher-dimensional Kundt–CSI class (the CSI\(_K\) conjecture) and that it can be constructed from locally homogeneous spaces and vanishing scalar invariant (VSI) spacetimes [6] (the CSI\(_R\) conjecture). The various CSI conjectures were proven in three dimensions in [2].

There are a number of important applications of CSI spacetimes. First, there are examples of CSI spacetimes which are exact solutions in supergravity and hence of particular physical

\(^3\) Throughout, Segre type will refer to the Segre type of the Ricci tensor unless otherwise stated.
interest [7], including AdS × S spacetimes [8], generalizations of AdS × S based on different VSI seeds [7] and the AdS gyratons [9]. CSI spacetimes are also related to universal spacetimes [10].

Second, the characterization of CSI spacetimes is useful for investigating the question of when a spacetime can be uniquely characterized by its curvature invariants. This question was addressed in [3], where the class of four-dimensional Lorentzian manifolds that can be completely characterized by the scalar polynomial curvature invariants constructed from the Riemann tensor and its covariant derivatives was determined. In particular, an appropriate set of projection operators was derived from the Riemann tensor and its covariant derivatives which enabled the proof of a number of results, including the main theorem that a metric that is not characterized by its curvature invariants must be of Kundt form.

2. Main theorems

The main theorem is the following:

Theorem 2.1. A 4D spacetime is CSI if and only if either:

(i) the spacetime is locally homogeneous; or
(ii) the spacetime is a Kundt spacetime for which there exists a frame such that all curvature tensors have the following properties: (a) all positive boost weight components vanish and (b) all boost weight zero components are constants.

Proof. The first part of this theorem follows from the corollary in [3], which says that either a CSI metric is locally homogeneous or it is Kundt.

The second part of this theorem, which essentially says that CSI$_K$ implies CSI$_F$, also follows from the work of [3] by keeping track of the boost weight 0 components of the curvature tensors. The most effective way of seeing that all the boost weight 0 components can be put into a constant form is to utilize the curvature operators. For example, the Petrov type II or D case gives curvature projectors, $\perp_1$ and $\perp_2$, of type $\{1,1\}(11)$ already at the level of the Weyl tensor (see appendix B for the notation and a short review of curvature operators). These projectors are already aligned with the Kundt frame so we immediately obtain that the boost weight 0 components of the Weyl tensor are constants. The idea is that the curvature projector picks out the necessary boost weight 0 components and relates them to invariants (possibly in an algebraic way). The components are thus constants provided the invariants are constants. By doing this for the full Riemann tensor and its derivatives it becomes clear that for a CSI$_K$ spacetime, in the Kundt frame, all boost weight 0 components are determined by the curvature invariants in the sense of [3]. Hence, if the invariants are constants, the boost weight 0 components must also be constants.

Note that all of the CSI conjectures are proven in 4D as a result of this theorem.

Theorem 2.2. For a 4D Lorentzian spacetime,

\[ \text{CSI} \iff \text{CSI}_3. \]

Proof. The result that CSI $\Rightarrow$ CSI$_3$ is trivial. Assume, therefore, that the spacetime is CSI$_3$.

From the calculations in [3] we see that as we reach the third-order invariants, by inspection of all the cases, we either have constructed our timelike projector $\perp_1$ or not. For the cases where we have the projector, the metric is $I$-non-degenerate, and hence we have a spacetime
which is \( \text{CH}_3 \), where \( \text{CH}_2 \) denotes curvature homogeneous of order \( k \).\(^4\)

In \([11]\), the relation between curvature homogeneity and locally homogeneity was studied. In particular, it follows that \( \text{CH}_3 \) implies local homogeneity and consequently CSI.

If we do not have a timelike projector, the spacetime is either Kundt or it is Petrov type \( O \), Segre type \( \{1, 11\} \). Treating the latter case first, the CSI\(_0\) assumption and the Bianchi identity immediately imply that \( \nabla \text{Riem} = 0 \) (and thus the spacetime is symmetric). Hence, this is a locally homogeneous space and thus CSI.

Last, assume the spacetime is Kundt. This part of the proof will be given in section 3. □

3. Kundt–CSI metrics

From the above and using the results of \([1]\), we have now established that a Kundt–CSI spacetime can be written in the form

\[
dx^2 = 2 \, du [dv + H(v, u, x^k) \, du + W_i(v, u, x^k) \, dx^i] + g_{ij}^\perp(x^k) \, dx^i \, dx^j,
\]

where \( dS_\perp^2 = g_{ij}^\perp(x^k) \, dx^i \, dx^j \) is the locally homogeneous metric of the 'transverse' space and the metric functions \( H \) and \( W_i \), requiring CSI\(_0\), are given by

\[
W_i(v, u, x^k) = v W_i^{(1)}(u, x^k) + W_i^{(0)}(u, x^k),
\]

\[
H(v, u, x^k) = v^2 \sigma + v H^{(1)}(u, x^k) + H^{(0)}(u, x^k),
\]

\[
\sigma = \frac{1}{8} (4 \sigma + W^{(1)} \, W_i^{(1)}),
\]

where \( \sigma \) is a constant. The remaining equations for CSI\(_0\) that need to be solved are as follows (hatted indices refer to an orthonormal frame in the transverse space):

\[
W_i^{(1)}_{\hat{[i}, \hat{j]}_j} = a_{ij},
\]

\[
W_i^{(1)}_{\hat{[i}, \hat{j]}} = \frac{1}{2} (W_i^{(1)} W_i^{(1)}) = s_{ij},
\]

and the components \( R_{\hat{i}\hat{j}\hat{k}\hat{l}} \) are all constants (i.e. \( dS_\perp^2 \) is curvature homogeneous). Here the antisymmetric and symmetric constant matrices \( a_{ij} \) and \( s_{ij} \), respectively, are determined from the boost weight 0 components of the Riemann tensor (see equations (9)–(11) later). In four dimensions, \( dS_\perp^2 \) is two dimensional, which immediately implies \( g_{ij}^\perp(x^k) \, dx^i \, dx^j \) is a two-dimensional locally homogeneous space and is, in fact, a maximally symmetric space. Up to scaling, there are (locally) only three such spaces; namely, the sphere, \( S^2 \), the flat plane, \( \mathbb{R}^2 \), and the hyperbolic plane, \( \mathbb{H}^2 \).

Equations (5) and (6) now give a set of differential equations for \( W_i^{(1)} \). These equations uniquely determine \( W_i^{(1)} \) up to initial conditions (which may be free functions in \( u \).

Requiring also CSI\(_1\) gives a set of constraints:

\[
\alpha_j = \sigma W_i^{(1)} - \frac{1}{2} (s_{jj} + a_{jj}) W_i^{(1)} j,
\]

\[
\beta_{ij} = W_i^{(1)} R_{\hat{i}\hat{j}\hat{k}\hat{l}} - W_i^{(1)} a_{ij} + (s_{i[j} + a_{i[j}) W_k^{(1)} j],
\]

where \( \alpha_j \) and \( \beta_{ij} \) are constants determined from the boost weight 0 components of the covariant derivative of the Riemann tensor (see equations (60)–(63) in \([1]\)).

\(^4\) Recall that a \( k \)-curvature homogeneous spacetime is defined as a spacetime for which there exists a frame such that the components of the curvature tensors up to order \( k \) are all constants.
Lemma 3.1. For a 4D Kundt spacetime, CSI implies CSI.

Proof. To show this we therefore need to show that the above equations are sufficient to ensure that the spacetime is CSI. The CSI0 and CSI1 conditions force the boost weight 0 components of Riem and ∇(Riem) to be constants.

The boost weight 0 components of Riemann are

\[
R_{0011} = -\sigma, \quad R_{00ij} = a_{ij},
\]

(9)

\[
R_{01ij} = \frac{1}{2}(s_{ij} + a_{ij}),
\]

(10)

\[
R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}).
\]

(11)

These components split into boost weight 0 components of the Weyl and Ricci tensors. From these we can construct curvature operators in the way described in [3] and appendix B. Combining both the Weyl operator and the Ricci operator we get projectors of the following types, which we consider in turn. In the following, the Segre-like notation refers to the set of curvature projectors [3].

- \{(1, 111)\}: this is the general case and it implies that \((s_{ij} + a_{ij})\) has two distinct eigenvalues. Further, from the CSI1 criterion, we see that \(W_i^{(1)}\) is an eigenvector of this matrix, unless it is a constant. If it is a constant, we immediately get CSI. Therefore, assume it is an eigenvector. Again we get that either it is a constant, in which case this implies CSI, or it is not constant and \(\alpha_i = 0\). We also note that we also must have \(\beta_{ijk} = 0\) and, in fact, \((\nabla Riem)_0 = 0\). Therefore, since we cannot acquire higher boost weight components by taking the covariant derivatives (due to the fact that it is Kundt), this also implies CSI.

- \{(1, 111)\}: by considering \((\nabla Riem)_0\), we get two possibilities. If \((\nabla Riem)_0 \neq 0\), we can always construct a second spacelike projector so that we have a set \{(1, 111)\}. This would again imply that \(W_i^{(1)}\) is a constant; hence, this spacetime is CSI. If \((\nabla Riem)_0 = 0\), also get a CSI spacetime (for the same reason as above).

- \{(1, 111)\}: this case must have \((C)_0 = 0\). Therefore, the Ricci tensor is the only tensor with boost weight 0 components. By calculating \(\nabla(Riem)\), we get \((\nabla Riem)_0 = 0\) and hence this is a CSI spacetime.

- \{(1, 111)\}: this again gives \((\nabla Riem)_0 = 0\) and hence this is a CSI spacetime.

\[\square\]

Since we have restricted ourselves to four dimensions, \(g_{ij}^{\perp}(x^k)dx^i dx^j\) is a two-dimensional locally homogeneous space. Up to scaling, there are (locally) only the sphere, \(S^2\), the flat plane, \(E^2\), and the hyperbolic plane, \(H^2\). To delineate all the cases we need to integrate the above-mentioned equations in terms of the remaining functions \(W_i^{(1)}(u, x^i)\). To solve for these we need to consider these cases in turn.

Below we have listed the various cases in terms of the transverse metric \(dx_\perp^2 = g_{ij}^{\perp}(x^k)dx^i dx^j\), the Segre type of the Ricci tensor and the one-form \(W^{(1)} \equiv W_i^{(1)} dx^i\). The constant \(\sigma\) is a freely specifiable constant, unless stated otherwise.

3.1. The sphere, \(S^2\)

Segre type \([2(11)]\) or \([1, 1)(11)\]. Here,

\[
dx_\perp^2 = dx^2 + \frac{1}{\lambda^2} \sin^2(\lambda x) dy^2, \quad W^{(1)} = 0.
\]

(12)

The possible Petrov types are D or II (possibly III, N or O in degenerate cases).
Segre type \{(211)\}, \{(31)\} or \{(1, 111)\}. Here, we have several cases. They all have \(\sigma > 0\) and are as follows:

\[
dx^2 = dx^2 + \frac{1}{\sigma} \sin^2(\sqrt{\sigma} x) dy^2,
\]

where

1. \(W^{(1)} = 2\sqrt{\sigma} \tan(\sqrt{\sigma} x) dx\).
2. \(W^{(1)} = 2\sqrt{\sigma} \cot(\sqrt{\sigma} x) dx + \tan(\sqrt{\sigma} y) dy\).
3. \(W^{(1)} = 2\sqrt{\sigma} \cot(\sqrt{\sigma} x) dx - \cot(\sqrt{\sigma} y) dy\).

The possible Petrov types are III, N or O.

### 3.2. The Euclidean plane, \(\mathbb{E}^2\)

Here we can set

\[
dx^2 = dx^2 + dy^2.
\]

**Segre type \{(211)\} or \{(1, 111)\}\**

1. \(W^{(1)} = 2\sigma dx, \quad 0 \neq \sigma \neq -r^2 \neq 0\).

   The possible Petrov types are D or II.

**Segre type \{(211)\}, \{(31)\} or \{(1, 111)\}\**

1. \(\sigma > 0\): \(W^{(1)} = 2\alpha dx\).
2. \(\sigma < 0\): \(W^{(1)} = 2\beta e^{-qx} dy\).
3. \(\sigma < 0\): \(W^{(1)} = 2\sqrt{\sigma} \coth(\sqrt{\sigma} |x|) dx\).
4. \(\sigma < 0\): \(W^{(1)} = 2\sqrt{\sigma} |x| dx\).

   Here, the possible Petrov types are III, N and O.

**Segre type \{(211)\} or \{(1, 111)\}\**

1. \(W^{(1)} = 0\).

   Possible Petrov types are D and II.

**Segre type \{(211)\}, \{(31)\} or \{(1, 111)\}\**

1. \(\sigma = 0\): \(W^{(1)} = 2\epsilon dx, \quad \epsilon = 0, 1\). This is the VSI case so the possible Petrov types are III, N and O.

### 3.3. The hyperbolic plane, \(\mathbb{H}^2\)

**Segre type \{(211)\} or \{(1, 111)\}\**

1. \(dx^2_1 = dx^2 + e^{-2q^t} dy^2, \quad W^{(1)} = \alpha dx + \beta e^{-q^t} dy\).

   The possible Petrov types are D and II (or III, N in degenerate cases).

**Segre type \{(211)\}, \{(31)\} or \{(1, 111)\}\**

1. \(dx^2_1 = dx^2 + e^{-2q^t} dy^2, \quad W^{(1)} = \alpha dx + \beta e^{-q^t} dy, \quad \sigma = -q^2 \pm \frac{1}{2} \sqrt{(\alpha^2 + \beta^2)(\beta^2 + (\alpha - 2q)^2)}\).

   The possible Petrov types are D and II (or III, N in degenerate cases).
Segre type \([2(11)]\) or \([(1, 1)(11)]\)

1. \(dx_1^2 = dx^2 + e^{-2q_1} dy^2,\quad W^{(1)} = 2q dx,\quad \sigma \neq -q^2.\)
2. \(dx_1^2 = dx^2 + e^{-2q_1} dy^2,\quad W^{(1)} = 0,\quad \sigma \neq -q^2.\)

The possible Petrov types are D and II (or III, N, O in degenerate cases).

Segre type \([(211)], [3(1)]\) or \([(1, 111)]\). For all of these \(\sigma < 0\) and we set \(\sigma = -q^2.\)

1. \(dx_1^2 = dx^2 + e^{-2q_1} dy^2,\quad W^{(1)} = 2q dx + \frac{2\epsilon}{y} dy,\quad \text{where } \epsilon = 0, 1.\)
2. \(dx_1^2 = dx^2 + e^{-2q_1} dy^2,\quad W^{(1)} = 0.\)
3. \(dx_2^1 = dx^2 + \frac{1}{q^2} \sinh^2(qx) dy^2,\quad W^{(1)} = -2q \tanh(qx) dx.\)
4. \(dx_1^2 = dx^2 + \frac{1}{q^2} \sinh^2(qx) dy^2,\quad W^{(1)} = 2q [\coth(qx) dx - \tanh(qy) dy].\)
5. \(dx_1^2 = dx^2 + \frac{1}{q^2} \cosh^2(qx) dy^2,\quad W^{(1)} = 2q \coth(qx) dx.\)
6. \(dx_1^2 = dx^2 + \frac{1}{q^2} \cosh^2(qx) dy^2,\quad W^{(1)} = 2q [-\tanh(qx) dx + \coth(qy) dy].\)

For case (2) the Petrov types can be II or D, while all other cases have Petrov types III, N or O.

### 4. 4D locally homogeneous Lorentzian spaces

From theorem 2.1 and the discussion in section 1, it follows that every 4D CSI spacetime is either locally homogeneous or Kundt. In the last section we presented all of the CSI–Kundt spacetime metrics. Let us next briefly discuss the possible 4D locally homogeneous spacetimes.

A homogeneous space can be considered as a coset manifold \(M = G/H\), where \(G\) is a Lie group and \(H\) is a closed Lie subgroup of \(G\), equipped with an invariant metric. For a homogeneous space the geometry is determined locally at a point, and consequently to a homogeneous space, \(M = G/H\), we can associate a homogeneous triple \((g, h, \langle - , - \rangle)\), where \(g\) is the Lie algebra of \(G\), \(h \subset g\), and \(\langle - , - \rangle\) is a scalar product (the restriction of the invariant metric \(g\) at a point) on a vector space complement \(m\) such that \(g = h \oplus m\). In the Lorentzian case, the metric \(g\) is a Lorentzian metric and this requires that \(h \subset \mathfrak{so}(n - 1, 1)\), where \(n = \dim M\). The classification of locally homogeneous Lorentzian spaces can thus be reduced to finding such homogeneous triples up to isometry.

In four dimensions, the classification of such triples up to isometry is not complete. However, there are some partial classification results. In [13], Komrakov classifies all homogeneous pairs \((g, h)\) with \(\dim h \geq 1\), and in [14] Komrakov provides all invariant Lorentzian metrics to these homogeneous pairs. These lists are quite extensive and, if correct, in principle gives all locally homogeneous spaces with non-trivial isotropy group. Due to the shear number of these triples the possible geometries they describe have not been fully explored. However, the possible solutions to the Einstein–Maxwell equations were explored in [14] and the Ricci-flat geometries were considered in detail in [15].

The four-dimensional homogeneous triples where \(\dim h = 0\) can be considered to correspond to homogeneous spaces being a four-dimensional Lie group equipped with a left-invariant Lorentzian metric. The Lie algebras of dimension 4 are well known (see, e.g., [16]); these can be equipped with a left-invariant metric in a standard manner [17].

An example of a Lie group equipped with a left-invariant metric is the following vacuum Petrov type I metric:
\[ ds^2 = -e^{\alpha z} \left( \cos \left[ \frac{\sqrt{3}}{2} qz \right] dx - \sin \left[ \frac{\sqrt{3}}{2} qz \right] dt \right)^2 + e^{\beta z} \left( \sin \left[ \frac{\sqrt{3}}{2} qz \right] dx + \cos \left[ \frac{\sqrt{3}}{2} qz \right] dt \right)^2 + e^{-2qz} dy^2 + dz^2. \]

5. \( I \)-symmetric spaces

Let us introduce the concept of an \( I \)-symmetric space, which is defined as a spacetime having the same scalar polynomial curvature invariants as that of a symmetric space. This implies that, by considering the scalar invariants only, we cannot distinguish an \( I \)-symmetric space from that of a symmetric space (similarly, we can think of CSI spacetimes as \( I \)-homogeneous spaces).

Recall that a symmetric space is defined as the vanishing of the first covariant derivative of the Riemann tensor; i.e., for a symmetric space \( R_{\alpha \beta \gamma \delta} = 0 \). This implies that the only possible non-vanishing invariants of an \( I \)-symmetric space are the zeroth-order invariants. In addition, all zeroth-order invariants are constant (since the covariant derivative of the Riemann tensor is zero, for any invariant \( I \) constructed from the Riemann tensor it follows that \( \nabla I = 0 \) and hence \( I \) is constant). Therefore, these spaces have a very simple set of invariants, which may be useful in applications in other theories of gravity.

The symmetric spaces in 4D are all classified and are one of the following:

1. maximally symmetric space (Segre type \( (1, 111) \), Petrov type O);
2. product of a three-dimensional maximally symmetric space and \( \mathbb{R} \) (Segre type \( (1, 111) \) or \( (1, (111)) \), Petrov type O);
3. product of two 2-spaces of constant curvature (Segre type \( (1, 1)(11) \), Petrov type D or O);
4. a Cahen–Wallach (CW) plane wave spacetime (Segre type \( (211) \), Petrov type N).

An \( I \)-symmetric space must therefore have the same scalar invariants as these symmetric spaces; in particular, it must be CSI. Indeed, the \( I \)-symmetric spaces are classified as follows:

**Proposition 5.1.** A spacetime is \( I \)-symmetric if and only if it belongs to one of the following classes:

1. Symmetric spaces.
2. The following Kundt–CSI spacetimes described in section 3:
   
   (i) section 3.1: All.
   
   (ii) section 3.2: Segre types \( \{(211)\}, \{(31)\}, \{(1, 111)\}, \{(1, 1)(11)\}, \{(211)\}, \{(31)\}, \{(211)\}, \{(31)\}\) and \( \{(1, 111)\} \).
   
   (iii) section 3.3: Segre types \( \{2(11)\} \) with \( W^{(1)} = 0 \), \( \{(1, 1)(11)\} \) with \( W^{(1)} = 0 \), \( \{(211)\} \), \( \{(31)\} \) and \( \{(1, 111)\} \).

The proof of this is straightforward by considering all the cases in section 3 separately. We note that many examples from the literature are \( I \)-symmetric (for example, the Siklos spacetime [22], the AdS gyron [24, 25] and many more [1, 7]).

Similarly, we can define a \( k \)-th order \( I \)-symmetric space as a spacetime having the same invariants as a \( k \)-symmetric spacetime. Recall that a \( k \)-symmetric spacetime is defined by the requirement that \( R_{\alpha \beta \gamma \delta ; \mu \nu \rho \sigma} = 0 \). Defining the set of \( k \)-order \( I \)-symmetric spacetimes as \( \mathcal{I} \mathcal{S} \mathcal{Y} \mathcal{m}^k \), we get the sequence of inclusions:

\[ \mathcal{V} \mathcal{S} \mathcal{I} \subset \mathcal{I} \mathcal{S} \mathcal{Y} \mathcal{m}^1 \subset \cdots \subset \mathcal{I} \mathcal{S} \mathcal{Y} \mathcal{m}^k. \]
In [26] the class of two-symmetric spacetimes was investigated. It was found that a two-symmetric spacetime is either a symmetric spacetime or admits a covariantly constant null vector (CCNV).\(^5\) In the first instance, we refer to the list of symmetric spacetimes given above; therefore, this subclass of two-symmetric spacetimes are necessarily CSI (locally homogeneous). In particular, they have curvature invariants that are constant at zeroth order and vanishing at first and higher order. Evidently, in this case the following classes are equivalent: \(T_{\text{Sym}}^2 = T_{\text{Sym}}^1\). In the second instance, the two-symmetric spacetimes admitting a CCNV define a subclass of the Brinkmann metrics (which are themselves a subclass of the Kundt metrics)

\[
ds^2 = 2\, du[du + H(u, x^k)\, du + W_i(u, x^k)\, dx^i] + g_{ij}(u, x^k)\, dx^i\, dx^j,
\]

(13)

where the CCNV is \(\ell = \partial_v\). A study of spacetimes admitting a CCNV shows that, in general, the Riemann tensor (of metric (13)) and all of its covariant derivatives are of type II or less [19]. Moreover, all boost weight 0 components of \(\nabla^{(k)}(\text{Riem})\), \(k \geq 0\), arise solely from the Riemannian metric of the transverse space, and more importantly \(\nabla^{(k)}(\text{Riem}) \cdot \ell = 0\) for all \(k \geq 0\). This last property describes the form of the non-vanishing components of Riemann and its derivatives in a null tetrad with \(\ell\) being one of the null frame vectors. Therefore, a two-symmetric CCNV spacetime implies that the transverse space of (13) is two-symmetric; thus from [26] it follows that \(g_{ij}\) is, in fact, a family of Riemannian symmetric spaces parameterized by \(u\). That is, \(V_{ij}\) (Riem) has vanishing boost weight 0 components, \(R_{ijkl;m} = 0\), resulting in \(V(\text{Riem})\) of type III, N or O (this last algebraic type corresponds to a symmetric CCNV spacetime). In a two-symmetric CCNV spacetime, curvature invariants at first and higher order vanish. In addition, since the zeroth-order invariants are determined only by the transverse space metric, which are Riemannian symmetric spaces, it follows that at any point in a two-symmetric CCNV spacetime there exists a symmetric CCNV spacetime having the same set of constant zeroth-order invariants. However, a two-symmetric CCNV spacetime need not have constant invariants on a neighborhood and, in general, it is therefore not symmetric. In this sense we have \(T_{\text{Sym}}^1 \subseteq T_{\text{Sym}}^2\). Equality only occurs when \(u\) is a non-essential coordinate in the transverse metric; that is, \(u\) does not parameterize inequivalent Riemannian symmetric spaces \(g_{ij}\) thus a coordinate transformation exists such that \(g_{ij}(u, x^k) \rightarrow g_{ij}(x^k)\).

6. Degenerate Kundt–CSI spacetimes

Non-locally homogeneous CSI spacetimes are contained in the degenerate Kundt class. A degenerate Kundt spacetime is a spacetime that admits an aligned null vector which is geodesic, shear-free, twist-free and expansion-free such that the Riemann tensor and all of its covariant derivatives are of type II (or more special) in the same aligned (kinematic) frame [27]. If the Riemann tensor is of type N, III or O in a degenerate Kundt spacetime, they must be aligned (with the kinematic frame) and from the theorems of [6] the spacetime is VSI. This implies that a non-VSI degenerate Kundt spacetime must be of proper Riemann type II or type D. In particular, a proper (i.e. non-VSI) CSI spacetime is of Riemann type II or D. A degenerate Kundt spacetime is not \(T\)-non-degenerate and hence it is not locally characterized by its scalar curvature invariants (which are all constant in the CSI case) [3].

A CSI spacetime is either degenerate Kundt or locally homogeneous. Consider a CSI spacetime in which the Riemann tensor and all of its covariant derivatives are aligned and of

\(^5\) Therefore, a two-symmetric spacetime is either locally homogeneous (and hence, CSI–Kundt) or there exists a covariantly constant null vector (CCNV–Kundt) (a similar result has been conjectured for the set of \(k\)-symmetric spacetimes). Note that a two-symmetric spacetime need not be CSI (which is locally homogeneous or Kundt), in which case a CCNV exists.
algebraic type D (i.e. of type D to ‘all orders’ or, in short, type $D^k$, see appendix A). Recall that a type D tensor has only boost weight 0 components and these components consequently comprise the curvature invariants. Indeed, although such a spacetime is degenerate Kundt and hence not $I$-non-degenerate, it is exceptional in the sense that it is locally homogeneous.

**Theorem 6.1.** A 4D type $D^k$ CSI spacetime, in which the Riemann tensor $\text{Riem}$ and $\nabla^k(\text{Riem})$ are all simultaneously of type D, is locally homogeneous.

**Proof.** In [3] it was shown that if there exists a frame in which all of the positive boost weight terms of the Riemann tensor and its covariant derivatives $\nabla^{(k)}(\text{Riem})$ are zero (in this frame) in 4D, then it is Kundt$^6$. This is obviously true in 4D for type $D^k$ (as it is a special case of above). From [3] we can deduce that CSI$_0$ implies CH$_0$. From the proof of lemma 3.1, it can also be seen that assuming CSI$_1$ and that $\nabla(\text{Riem})$ is of type D, implies CH$_1$ (since $\nabla(\text{Riem})$ has only boost weight 0 components) . In fact, at this stage, we already have curvature operators of type $\{((1, 1)(11))$ or $((1, 1)11)$. By Singer’s theorem [28] the first case implies local homogeneity, while for the latter case we have to consider $\nabla^2(\text{Riem})$. However, in this case we already have all the curvature operators necessary to determine all the independent components of $\nabla^2(\text{Riem})$ (since this is also of type $D$).$^7$ Hence, CSI$_2$ implies CH$_2$ and consequently the spacetime is also locally homogeneous. □

These spacetimes, even though they are not $I$-non-degenerate, are in some sense ‘characterized’ by their constant curvature invariants, at least within the class of type $D^k$ CSI spacetimes. In general, there are many degenerate Kundt–CSI metrics (that are not type $D^k$) with the same set of constant invariants, $I$. In this case there is at least one $\nabla^k(\text{Riem})$ which is proper type II and thus has negative boost weight terms; this Kundt–CSI metric will have precisely the same scalar curvature invariants as the corresponding type $D^k$ CSI metric (which has no negative boost weight terms). Therefore, there is a distinguished or a ‘preferred’ metric with the same set of constant invariants, $I$; namely, the corresponding type $D^k$ locally homogeneous CSI metric, which is distinguished within the class of algebraic type $D^k$ CSI spacetimes.

It is plausible that the following is true: if $\nabla^{(k)}(\text{Riem})$ is type $D^k$ then every boost weight 0 component of $\nabla^{(k)}(\text{Riem})$ is, in principle, expressible in terms of $k$th-order curvature invariants. As a corollary, if $\nabla^{(k)}(\text{Riem})$ is type $D^k$ and CSI then the spacetime is locally homogeneous.

In proving theorem 6.1, we note that every Riemann type D, CH$_2$ spacetime is locally homogeneous [11]. Thus, if a spacetime is type $D^2$ and CSI$_2$, then by showing it is CH$_2$ gives the desired result.

It is not hard to show that if the Riemann tensor is of type D, then all boost weight 0 components, $\Psi_2$, $\Phi_{11}$, $\Phi_{02}$ and $\Lambda$ (using the Newman–Penrose (NP) notation) can be expressed in terms of 0th-order curvature invariants; hence CSI$_0$ implies CH$_0$. Calculating $\nabla(\text{Riem})$ is now somewhat simpler since all of these components are just products of constant boost weight 0 curvature scalars and certain spin coefficients. When $\nabla(\text{Riem})$ is of type D a refinement of the above statement, that CH$_2$ implies local homogeneity, can be made by noting the following facts. Let $G_k$ denote the isotropy group of $\nabla^{(k)}(\text{Riem})$, then for Riemann type D, $G_0$ is either two-dimensional consisting of boost and spins if $\Phi_{02} = 0$, or one-dimensional consisting of only boosts if $\Phi_{02} \neq 0$. It was shown in [12] that for the Riemann type D case with $\Phi_{02} = 0$, if $\nabla(\text{Riem})$ is CH$_1$ then $G_1$ cannot be one dimensional. Therefore, if $G_1$ is two dimensional,

6 This can be seen from [3] by inspection of all the cases.

7 Note that since this is also of type D the curvature operators obtained from $\nabla(\text{Riem})$ can be used to determine all boost weight 0 components of $\nabla^2(\text{Riem})$ in terms of scalar polynomial invariants up to second order.
by Singer’s theorem the spacetime is locally homogeneous. Hence, in this case a proper CH$_1$

must have (if it exists) a frame that is completely fixed by $\nabla$(Riem). If we suppose that $\nabla$(Riem) is type D (in addition to Riemann type D), then since the only non-vanishing components are boost weight 0 we always have, at least, a boost isotropy. Thus $G_1$ cannot be trivial, implying there is no proper CH$_1$. Therefore, if a spacetime has Riemann and $\nabla$(Riem)

type D and is CH$_1$ then it is locally homogeneous.

Note that the order in the above statements can be reduced and since the proof of

theorem 6.1 shows that CSI$_1$ implies CH$_1$, we have as a result:

Corollary 6.2. Every 4D CSI$_1$ spacetime in which Riem and $\nabla$(Riem) are simultaneously of type D is locally homogeneous.

In particular, in 4D if the spacetime is Petrov type D, then if $\nabla$(Riem) is type D the spacetime is Kundt. For Petrov type O, there are several Segre types: for Segre types $\{(1, 111)\}$ and $\{(1, 1)(11)\}$ (PP-type D), if $\nabla$(Riem) is type D the spacetime is Kundt, and for Segre types $\{(1, 111)\}$ or $\{(1, 111)\}$ the spacetime is symmetric and so $\nabla$(Riem) = 0. Therefore, if the spacetime is CSI and Riem and $\nabla$(Riem) are of type D, it follows that the spacetime is Kundt.

This establishes that spacetimes satisfying the conditions of corollary 6.2 are members of the Kundt class, and hence this result is already covered by lemma 3.1 as a special case when Riem and $\nabla$(Riem) are of type D.

7. Discussion

In this paper we have proven that if a 4D spacetime is CSI, then either the spacetime is locally homogeneous or the spacetime is a Kundt spacetime. A number of partial results can be deduced from previous work, some of which are presented in appendix C. We have also discussed the properties of the Kundt-CSI spacetimes. The CSI spacetimes are of particular interest since they are solutions of supergravity or superstring theory, when supported by appropriate bosonic fields [7]. It is plausible that a wide class of CSI solutions are exact solutions to string theory non-perturbatively [18].

In the context of string theory, it is of considerable interest to study higher-dimensional Lorentzian CSI spacetimes. In particular, a number of $N$-dimensional CSI spacetimes are known to be solutions of supergravity theory when supported by appropriate bosonic fields [7]. It is known that AdS$_d \times$ S$^{N-d}$ (in short AdS $\times$ S) is an exact solution of supergravity (and preserves the maximal number of supersymmetries) for certain values of $(d, N)$ and for particular ratios of the radii of curvature of the two space forms [8]. There are a number of other CSI spacetimes known to be solutions of supergravity and admit supersymmetries; namely, generalizations of AdS $\times$ S, generalizations of the chiral null models [23] and generalizations of the AdS gyraton [9].

More general supergravity CSI solutions have been constructed by taking a homogeneous (Einstein) spacetime, $(M_\text{Hom}, \tilde{g})$, of Kundt form and generalizing to an inhomogeneous spacetime, $(M, g)$, by including arbitrary Kundt metric functions [7]. In addition, product manifolds of the form $M \times K$, where $M$ is an Einstein space with negative constant curvature and $K$ is a (compact) Einstein–Sasaki spacetime, can give rise to supergravity CSI spacetimes. The supersymmetric properties of CSI spacetimes have also been studied [19]. It is known
that, in general, if a spacetime admits a Killing spinor it necessarily admits a null or timelike Killing vector (KV). Therefore, a necessary (but not sufficient) condition for a particular supergravity solution to preserve some supersymmetry is that the spacetime possesses such a KV.

In future work, motivated by the physical interest of higher-dimensional CSI spacetimes, we shall discuss possible higher-dimensional generalizations to the results presented in this paper. In particular, we hope to prove a higher-dimensional version of theorem 2.1 and to generalize theorem 6.1 (and thus show that a Kundt–CSI is CSI\textsubscript{F} in arbitrary dimensions) \cite{29}. The first step is to investigate the curvature operators in higher dimensions and to classify these for the various algebraic types. With the aid of these operators it is then hoped that the results obtained in 4D here can also be shown to be true in higher dimensions.

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**Appendix A. Algebraic classification**

Given a covariant tensor \( T \) with respect to a Newman–Penrose tetrad (or null frame) \( \{ \ell, n, m, \bar{m} \} \), the effect of a boost \( \ell \mapsto e^{\lambda} \ell, n \mapsto e^{-\lambda} n \) allows \( T \) to be decomposed according to its boost weight,

\[
T = \sum_b (T)_b, \quad (A.1)
\]

where \((T)_b\) denotes the boost weight \( b \) components of \( T \). Recall that the boost weight \( b \) components are defined as those components, \( T_{ab\ldots d} \), of \( T \) that transform according to

\[
T_{ab\ldots d} \mapsto e^{b\lambda} T_{ab\ldots d},
\]

under the aforementioned boost.

An algebraic classification of tensors \( T \) has been developed \cite{4} which is based on the existence of certain normal forms of (A.1) through successive application of null rotations and spin boosts. In the special case where \( T \) is the Weyl tensor in four dimensions, this classification reduces to the well-known Petrov classification. However, the boost weight decomposition can be used in the classification of any tensor \( T \) in arbitrary dimensions. As an application, a Riemann tensor of type G has the following decomposition,

\[
R = (R)_{+2} + (R)_{+1} + (R)_0 + (R)_{-1} + (R)_{-2}, \quad (A.2)
\]

in every null frame. A Riemann tensor is algebraically special if there exists a frame in which certain boost weight components can be transformed to zero; these are summarized in tables A1 and A2.

In particular, we see that a type D Riemann tensor has \( R = (R)_0 \). Similarly, a type D \( \nabla \) Riemann tensor has \( \nabla R = (\nabla R)_0 \). If this holds for all covariant derivatives of the Riemann tensor (i.e. \( \nabla^{(k)} R = (\nabla^{(k)} R)_0 \), for all \( k \)), then we call the spacetime type D\textsuperscript{k}.

**Appendix B. Curvature operators and curvature projectors**

A curvature operator, \( T \), is a tensor considered as a (pointwise) linear operator

\[
T : V \mapsto V,
\]
for some vector space, $V$, constructed from the Riemann tensor, its covariant derivatives and the curvature invariants.

The archetypical example of a curvature operator is obtained by raising one index of the Ricci tensor. The Ricci operator is consequently a mapping of the tangent space $T_p\mathcal{M}$ into itself:

$$R \equiv (R^\mu_\nu) : T_p\mathcal{M} \mapsto T_p\mathcal{M}.$$  

Another example of a curvature operator is the Weyl tensor, considered as an operator, $C \equiv (C^\rho_{\mu\nu})$, mapping bivectors onto bivectors.

For a curvature operator, $T$, consider an eigenvector $v$ with eigenvalue $\lambda$; i.e., $Tv = \lambda v$. If $d = \text{dim}(V)$ and $n$ is the dimension of the spacetime, then the eigenvalues of $T$ are $GL(d)$ invariant. Since the Lorentz transformations, $O(1, n-1)$, will act via a representation $\Gamma \subset GL(d)$ on $T$, the eigenvalue of a curvature operator is an $O(1, n-1)$-invariant curvature scalar. Therefore, curvature operators naturally provide us with a set of curvature invariants (not necessarily polynomial invariants) corresponding to the set of distinct eigenvalues: $\{\lambda_A\}$. Furthermore, the set of eigenvalues are uniquely determined by the polynomial invariants of $T$ via its characteristic equation. The characteristic equation, when solved, gives us the set of eigenvalues and hence these are consequently determined by the invariants.

We can now define a number of associated curvature operators. For example, for an eigenvector $v_A$ so that $Tv_A = \lambda_A v_A$, we can construct the annihilator operator:

$$P_A \equiv (T - \lambda_A 1).$$
Considering the Jordan block form of $T$, the eigenvalue $\lambda_A$ corresponds to a set of Jordan blocks. These blocks are of the form:

$$B_A = \begin{bmatrix}
\lambda_A & 0 & 0 & \cdots & 0 \\
1 & \lambda_A & 0 & \cdots & \\
0 & 1 & \lambda_A & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & \lambda_A
\end{bmatrix}.$$  

There might be several such blocks corresponding to an eigenvalue $\lambda_A$; however, they are all such that $(B_A - \lambda_A 1)$ is nilpotent and hence there exists an $n_A \in \mathbb{N}$ such that $P^{n_A}_A$ annihilates the whole vector space associated with the eigenvalue $\lambda_A$.

This implies that we can define a set of operators $\perp_A$ with eigenvalues 0 or 1 by considering the products

$$\prod_{B \neq A} P^{n_B}_B = \Lambda_A \perp_A,$$

where $\Lambda_A = \prod_{B \neq A} (\lambda_B - \lambda_A)^{n_B} \neq 0$ (as long as $\lambda_B \neq \lambda_A$ for all $B$). Furthermore, we can now define

$$\perp_A \equiv 1 - (1 - \perp_A)^{n_A}$$

where $\perp_A$ is a curvature projector. The set of all such curvature projectors obeys

$$1 = \perp_1 + \perp_2 + \cdots + \perp_A + \cdots, \quad \perp_A \perp_B = \delta_{AB} \perp_A. \tag{B.1}$$

We can use these curvature projectors to decompose the operator $T$:

$$T = N + \sum_A \lambda_A \perp_A. \tag{B.2}$$

The operator $N$ therefore contains all the information not encapsulated in the eigenvalues $\lambda_A$. From the Jordan form we can see that $N$ is nilpotent; i.e., there exists an $n \in \mathbb{N}$ such that $N^n = 0$. In particular, if $N \neq 0$, then $N$ is a negative/positive boost weight operator which can be used to lower/raise the boost weight of a tensor.

Considering the Ricci operator, or the Weyl operator, we can show that (where the type refers to either Ricci type or Weyl type):

- **Type I**: $N = 0, \lambda_A \neq 0$.
- **Type D**: $N = 0, \lambda_A \neq 0$.
- **Type II**: $N^2 = 0, \lambda_A \neq 0$.
- **Type III**: $N^3 = 0, \lambda_A = 0$.
- **Type N**: $N^2 = 0, \lambda_A = 0$.
- **Type O**: $N = 0, \lambda_A = 0$.

Consider a curvature projector $\perp : T_p \mathcal{M} \mapsto T_p \mathcal{M}$. Then, for a Lorentzian spacetime there are four categories:

1. Timelike: for all $v^\mu \in T_p \mathcal{M}$, $v_\nu (\perp)_{\nu \mu} v^\mu \leq 0$.
2. Null: for all $v^\mu \in T_p \mathcal{M}$, $v_\nu (\perp)_{\nu \mu} v^\mu = 0$.
3. Spacelike: for all $v^\mu \in T_p \mathcal{M}$, $v_\nu (\perp)_{\nu \mu} v^\mu \geq 0$.
4. None of the above.
We can consider a complete set of curvature projectors, $\perp_A : T_p M \mapsto T_p M$, which can be of any of the aforementioned categories, and we use a Segre-like notation to characterize the set with a comma separating time and space. For example, $\{1, 111\}$ means that we have four projectors: one timelike and three spacelike. A bracket indicates that the image of the projectors are of dimension 2 or higher; e.g., $\{((1, 1)11)\}$ means that we have two spacelike operators, and one with a two-dimensional image. If there is a null projector, we automatically have a second null projector. Given an NP frame $\{\ell_\mu, n_\mu, m_\mu, \bar{m}_\mu\}$, then a null projector can typically be

$$\perp_1 = -\ell^\mu n_\mu.$$ 

Note that $\perp_1 = \perp_1$, but it is not symmetric. Therefore, acting from the left and right gives two different operators. Indeed, defining

$$\perp_2 = g^{\alpha\beta} g_\alpha^\mu g_\beta^\nu (\perp_1)^\mu_\nu,$$

we get a second null projector being orthogonal to $\perp_1$. The existence of null projectors enables us to pick out certain null directions; however, note that the null operators, with respect to the aforementioned NP frame, are of boost weight 0 and so cannot be used to lower/raise the boost weights. In particular, considering the combination $\perp_1 + \perp_2$ we see that the existence of null projectors implies the existence of projectors of type $\{(1, 1)11\}$.

### Appendix C. Previous work

A number of partial results on 4D CSI spacetimes can be deduced from previous work. For example, it can be shown that in 4D Petrov (Weyl) type I and PP-(Ricci) type I Lorentzian CSI spacetimes are locally homogeneous, which is consistent with the above analysis.

Consider first the PP-type I case. In this case, there is a frame in which the Ricci tensor has components that are all constant and can be put into a canonical form [2], and the result easily follows.

Let us next consider the Petrov type I case. Immediately we know that local (zeroth-order) curvature homogeneity implies local homogeneity in this case [11]. From [20] it follows that if a 4D spacetime is of Petrov type I it can be classified according to its rank and it is either:

(i) general curvature class $A$ or
(ii) curvature class $C$ with restricted Segre type (see [20] or [3]).

Now, suppose the components of the Riemann tensor $R^a_{bcd}$ are given in a coordinate domain $U$ with metric $g$. In case (i), where the curvature class is of type A, for any other metric $g'$ with the same components $R^a_{bcd}$ it follows that $g'_{ab} = \alpha g_{ab}$ (where $\alpha$ is a constant). We can then pass to the frame formalism and determine the frame components of the Riemann tensor. The Petrov type I case is completely backsolvable [21] and hence the frame components are completely determined by the zeroth-order scalar invariants, and it follows that the spacetime is locally homogeneous in this case.

Let us now consider case (ii), where the curvature class is $C$. Again, let us suppose that the $R^a_{bcd}$ are given in $U$ with metric $g$. If $g'$ is any other metric with the same $R^a_{bcd}$, it follows that $g'_{ab} = \alpha g_{ab} + \beta k_a k_b$ (where $\alpha$ and $\beta$ are constants). The equation

$$R^a_{bcd} k_d = 0 \quad (C.1)$$

has a unique (up to scaling) non-trivial solution for $k \in T_p M$. If $R^a_{bcd} c k_a \neq 0$, then $\beta = 0$ and the metric is determined up to a constant conformal factor (and the holonomy type is $R_{15}$).\footnote{As defined, for example, in [20].}
This is similar to the first case discussed above (but now some information on the covariant derivative of the Riemann tensor is necessary; e.g., \( I_2 \equiv [R^{abcd.;e} R_{abcd.;e} - 4 R^{ab.;e} R_{ab.;e} + R^{a} R_{a}] \neq 0 \)). Hence, the spacetime is locally homogeneous.

If \( R^{abcd.;e} k_a = 0 \), then \( R^{abcd.;e} k_a = 0 \), and since equation (C.1) has a unique solution, \( k_a \) is recurrent. If \( k_a \) is null, the spacetime is algebraically special, and since we assume that the Petrov type is I, this is not possible. Hence, \( k_a \) is (a) timelike (TL) or (b) spacelike (SL) and is, in fact, covariantly constant (CC). In case (iia), the spacetime admits a TL CC vector field \( k_a \). The holonomy is \( R_{13} \), with a TL holonomy invariant subspace which is non-degenerately reducible, and \( M \) is consequently locally \((1 + 3)\) decomposable (and static). All of the non-trivial components of the Riemann tensor and its covariant derivatives are constructed from the 3D positive definite metric, and if the spacetime is CSI it is consequently locally homogeneous.

In case (iib), the spacetime admits a SL CC vector field \( k_a \). The holonomy is \( R_{10} \), there exists a holonomy invariant SL vector \( k_a \) which is non-degenerately reducible and \( M \) is thus locally \((3 + 1)\) decomposable. Classification now reduces to the classification of a subclass of 3d Lorentzian spacetimes, and it follows from [1] that the spacetime is locally homogeneous.

The analysis could proceed in a similar fashion on a case-by-case basis (according to Petrov and/or Segre type); however, it is not clear that a complete proof could be obtained in this way. In addition, the analysis presented above in the main text of the paper is easier to apply and is more readily applicable to higher-dimensional generalizations.

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