Search for $CP$ Violation in Charm at $e^+e^-$ colliders

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In this proceeding, I discuss results from the $BABAR$ and Belle collaborations for searches of direct $CP$ violation in the singly Cabibbo-suppressed decay $D^\pm \to K^+K^\mp\pi^\pm$ from $e^+e^-$ annihilation data collected at a center-of-mass energy at or just below the $\Upsilon(4S)$ resonance. The Belle collaboration measures the $CP$ asymmetry as a function of the production angle of the $D^\pm$ meson in the quasi two-body $D^\pm \to \phi\pi^\pm$ decay. The $BABAR$ experiment studies the entire phase-space with model-independent and model-dependent Dalitz plot analysis techniques to search for $CP$-violating asymmetries in the various intermediate states, in addition to a phase-space integrated measurement as a function of the production angle. No evidence for $CP$ violation is reported from either experiment.

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1 Introduction

Searches for CP violation (CPV) in charm meson decays provide a probe of physics beyond the Standard Model (SM). Singly Cabibbo-suppressed (SCS) decays can exhibit direct CP asymmetries due to interference between tree-level transitions and $|\Delta C| = 1$ penguin-level transitions if there is both a strong and a weak phase difference between the two amplitudes. In the SM, the resulting asymmetries are suppressed by $\mathcal{O}(|V_{ub}/V_{cs}|) \sim 10^{-3}$, where $V_{ij}$ are elements of the Cabibbo-Kobayashi-Maskawa quark-mixing matrix [1]. A larger measured value of the CP asymmetry could be a consequence of the enhancement of penguin amplitudes in $D$ meson decays due to final-state interactions [2, 3], or of new physics [4, 5].

SCS three-body decays of charm mesons are dominated by quasi-two-body decays with resonant intermediate states. Direct CP violation (CPV) can be localized in a particular region of phase-space, and the final-state interactions in these decays may enhance small new physics CP phases. Analysis techniques which utilize the information contained in the Dalitz plot take advantage of the dynamics in these decays to probe for CPV. Both the BABAR and Belle collaborations have analyzed the decay $D^+ \rightarrow K^+K^−\pi^+$ [6] using several complimentary analysis techniques to measure CP violating asymmetries as a function of the Dalitz plot position with sufficient precision to probe for new physics [7, 8].

The Belle collaboration studied CP asymmetries in charged $D^+ \rightarrow \phi\pi^+$ and $D^+_s \rightarrow \phi\pi^+$ decays using 955 fb$^{-1}$ of data recorded with the Belle detector at the KEKB asymmetric energy $e^+e^-$ collider. The study of $D^+ \rightarrow K^+K^−\pi^+$ from BABAR collaboration include a measurement of the integrated CP asymmetry, a comparison of the binned $D^+$ and $D^−$ Dalitz plots, a comparison of the Legendre polynomial moment distributions for the $K^+K^−$ and $K^−\pi^+$ systems, and a comparison of parameterized fits to the Dalitz plots. The analysis is based on a sample of 476 fb$^{-1}$ of data collected with the BaBar detector at the SLAC PEP-II collider.

The production of $D^+$ (and $D^−$) mesons from the $e^+e^− \rightarrow c\bar{c}$ process is not symmetric in $\cos(\theta_{CM})$; this forward-backward (FB) asymmetry, coupled with the asymmetric acceptance of the detector, results in different yields for $D^+$ and $D^−$ events. The FB asymmetry, to first order, arises from the interference of the separate annihilation processes involving a virtual photon and a $Z^0$ boson. The charge asymmetry $A$ in a given interval of $\cos(\theta_{CM})$ by

$$A(\cos(\theta_{CM})) \equiv \frac{N_{D^+}/\epsilon_{D^+} - N_{D^-}/\epsilon_{D^-}}{N_{D^+}/\epsilon_{D^+} + N_{D^-}/\epsilon_{D^-}},$$

where $N_{D^\pm}$ and $\epsilon_{D^\pm}$ are the yield and efficiency, respectively, in the given $\cos(\theta_{CM})$ bin. The FB asymmetry is removed by averaging $A$ over intervals symmetric in
\[
A_{CP} = \frac{A(\cos(\theta_{CM})) + A(-\cos(\theta_{CM}))}{2}
\]  

In contrast to the BaBar experiment, Belle utilizes the Cabibbo-favored (CF) \( D_s^+ \to \phi \pi^+ \) decay and measures

\[
\Delta A_{rec} = \frac{N_{D^+} - N_{D^-}}{N_{D^+} + N_{D^-}} - \frac{N_{D_s^+} - N_{D_s^-}}{N_{D_s^+} + N_{D_s^-}}
\]

in order to cancel detector-induced asymmetries and other systematic effects. This CF decay is governed by the CKM matrix elements \( V_{cs}V_{ud}^* \) and is expected to have negligible CP asymmetry \[^9\] \cite{9}, therefore, a measurement of \( \Delta A_{rec} \) probes \( A_{CP}^{D_s^+ \to \phi \pi^+} \).

Both the BaBar and Belle experiment find no evidence for direct CP violation measured as a function of \( \cos(\theta_{CM}) \). The Belle experiment measures

\[
A_{CP}^{D_s^+ \to \phi \pi^+} = (+0.51 \pm 0.28 \pm 0.05)\%
\]

and the BaBar measurement integrated over the entire phase-space is

\[
A_{CP}^{D^+ \to K^+K^-\pi^+} = (+0.37 \pm 0.30 \pm 0.15)\%.
\]

In order to probe for CPV as a function of position of the Dalitz plot, the BaBar experiment measures the asymmetry in intervals of the Dalitz plot either as a function of \((m^2(K^+K^-), m^2(K^-\pi^+))\), \(m(K^+K^-)\) or \(m(K^-\pi^+)\), as well as differences between the magnitudes and phase angles of resonance and non-resonant amplitudes contributing to the decay.

To search for CPV in intervals of \(m^2(K^+K^-)\) versus \(m^2(K^-\pi^+)\), BaBar measures normalized residuals \( \Delta \) for the efficiency-corrected and background-subtracted \( D^+ \) and \( D^- \) Dalitz plots, where \( \Delta \) is defined by

\[
\Delta \equiv \frac{n(D^+) - Rn(D^-)}{\sqrt{\sigma^2(D^+) + R^2\sigma^2(D^-)}},
\]

with \( n(D^+) \) and \( n(D^-) \) the observed number of \( D^+ \) and \( D^- \) mesons in an interval of the Dalitz plot, where \( \sigma(D^+) \) and \( \sigma(D^-) \) are the corresponding statistical uncertainties. \( R \) is the ratio of efficiency-corrected yields of \( D^+ \) and \( D^- \). The results for \( \Delta \) are shown in Fig. 1. Note that the intervals for Fig. 1 are adjusted so that each interval contains approximately the same number of events. BaBar calculates the quantity \( \chi^2/(\nu - 1) = \sum_{i=1}^{\nu} \Delta_i^2/(\nu - 1) \), where \( \nu \) is the number of intervals in the Dalitz plot. BaBar fits the distribution of normalized residuals to a Gaussian function, whose mean and root-mean-squared (RMS) deviation values are consistent with zero and
one, respectively. The $\chi^2 = 90.2$ for 100 intervals with a Gaussian residual mean of 0.08 ± 0.15, RMS deviation of 1.11 ± 0.15, and a consistency at the 72% level that the Dalitz plots do not exhibit CP asymmetry.

The Legendre polynomial moments of the cosine of the helicity angle of the $D^\pm$ decay products reflect the spin and mass of the intermediate resonant and nonresonant amplitudes, and the interference effects among them [10]. A comparison of these moments between the $D^+$ and $D^-$ two-body mass distributions provides a model-independent method to search for CP violation in the Dalitz plot, and to study its mass and spin structure. The helicity angle $\theta_H$ for decays $D^+ \rightarrow (r \rightarrow K^+K^-)\pi^+$ via resonance $r$ is defined as the angle between the $K^+$ direction in the $K^+K^-$ rest frame and the prior direction of the $K^+K^-$ system in the $D^+$ rest frame. For decays $D^+ \rightarrow (r \rightarrow K^-\pi^+)K^+$ via resonance $r$, $\theta_H$ is defined as the angle between the $K^-$ direction in the $K^-\pi^+$ system and the prior direction of the $K^-\pi^+$ system in the $D^+$ rest frame.

The Legendre polynomial moment distribution for order $l$ is defined as the efficiency-corrected and background-subtracted invariant two-body mass distribution $m(K^+K^-)$ or $m(K^-\pi^+)$, weighted by the spherical harmonic $Y^0_l[\cos(\theta_H)] = \sqrt{2l + 1}/4\pi P_l[\cos(\theta_H)]$, where $P_l$ is the Legendre polynomial (Fig. 2). The two-body invariant mass interval weight is defined as $W_i^{(l)} \equiv (\sum_j w_{ij}^{(l)S} - \sum_k w_{ik}^{(l)B})/\langle \epsilon_i \rangle$, where $w_{ij}^{(l)} (w_{ik}^{(l)})$ is the value of $Y_l$ for the $j^{th}(k^{th})$ event in the $i^{th}$ interval and $\langle \epsilon_i \rangle$ is the average efficiency for the $i^{th}$ interval. The superscripts $S$ and $B$ refer to the signal and background components, respectively. The uncertainty on $W_i^{(l)}$ is $\sigma_i^{(l)} \equiv \sqrt{\sum_j (w_{ij}^{(l)S})^2 + \sum_k (w_{ik}^{(l)B})^2}/\langle \epsilon_i \rangle^2$. To study differences between the $D^+$ and $D^-$ amplitudes, the quantities $X_i^l$ for $l$ ranging
from zero to seven in a two-body invariant mass interval, where

\[ X_i^l = \frac{(W_i^{(l)}(D^+) - RW_i^{(l)}(D^-))}{\sqrt{\sigma_{i}^{(l)}(D^+)^2 + R^2 \sigma_{i}^{(l)}(D^-)^2}} \]  

(7)

are calculated.

\textit{BABAR} calculates the $\chi^2/\text{ndof}$ over 36 mass intervals in the $K^+K^-$ and $K^-\pi^+$ moments using

\[ \chi^2 = \sum_i \sum_l \sum_{l_1} X_i^{(l_1)} \rho_{i}^{l_1l_2} X_i^{(l_2)}, \]  

(8)

where $\rho_{i}^{l_1l_2}$ is the correlation coefficient between $X_i^{l_1}$ and $X_i^{l_2}$,

\[ \rho_{i}^{l_1l_2} \equiv \frac{\langle X_i^{(l_1)} X_i^{(l_2)} \rangle - \langle X_i^{(l_1)} \rangle \langle X_i^{(l_2)} \rangle}{\sqrt{\langle X_i^{(l_1)} \rangle^2 - \langle X_i^{(l_1)} \rangle^2} \sqrt{\langle X_i^{(l_2)} \rangle^2 - \langle X_i^{(l_2)} \rangle^2}}, \]  

(9)

and where the number of degrees of freedom (ndof) is given by the product of the number of mass intervals and the number of moments, minus one due to the constraint that the overall rates of $D^+$ and $D^-$ mesons be equal. The $\chi^2/\text{ndof}$ is found to be 1.10 and 1.09 for the $K^+K^-$ and $K^-\pi^+$ moments, respectively (for ndof = 287), which corresponds to a probability of 11% and 13%, again respectively, for the null hypothesis (no CPV).
The Dalitz plot amplitude $\mathcal{A}$ can be described by an isobar model, which is parameterized as a coherent sum of amplitudes for a set of two-body intermediate states $r$. Each amplitude has a complex coefficient, i.e.,

$$\mathcal{A}_r[m^2(K^+K^-), m^2(K^-\pi^+)] = \sum_r M_r e^{i\phi_r} F_r[m^2(K^+K^-), m^2(K^-\pi^+)],$$

(10)

where $M_r$ and $\phi_r$ are real numbers, and the $F_r$ are dynamical functions describing the intermediate resonances [11, 12, 13]. The complex coefficient may also be parameterized in Cartesian form, $x_r = M_r \cos \phi_r$ and $y_r = M_r \sin \phi_r$. The $K^*(892)^0$ is chosen as the reference amplitude in the CP-symmetric and CP-violating fits to the data, such that $M_{K^*(892)^0} = 1$ and $\phi_{K^*(892)^0} = 0$.

The CP-conserving background is modeled using events from the sideband regions of the $D^+$ mass distribution, which is comprised of the $K^*(892)^0$ and $\phi(1020)$ resonance contributions and combinatorial background. The combinatorial background outside the resonant regions has a smooth shape and is modeled with the non-parametric $k$-nearest-neighbor density estimator [14]. The $K^*(892)^0$ and $\phi(1020)$ regions are composed of the resonant structure and a linear combinatorial background, which is parameterized as a function of the two-body mass and the cosine of the helicity angle. The model consists of a Breit-Wigner (BW) PDF to describe the resonant line shape, and a first-order polynomial in mass to describe the combinatorial shape. These are further multiplied by a sum over low-order Legendre polynomials to model the angular dependence.

| Resonance | Mass (MeV/$c^2$) | Width (MeV) |
|-----------|------------------|-------------|
| $K^*(892)^0$ | 895.53 ± 0.17 | 44.90 ± 0.30 |
| $\phi(1020)$ | 1019.48 ± 0.01 | 4.37 ± 0.02 |
| $a_0(1450)$ | 1441.59 ± 3.77 | 268.58 ± 5.28 |
| $K^*(1430)^0$ | 1431.88 ± 5.89 | 293.62 ± 3.83 |
| $K^*(1680)^0$ | 1716.88 ± 21.03 | 319.28 ± 109.07 |
| $f_0(1370)$ | 1221.59 ± 2.46 | 281.48 ± 6.6 |
| $\kappa(800)$ | 798.35 ± 1.79 | 405.25 ± 5.05 |

Assuming no CPV, an unbinned maximum-likelihood fit is performed to determine the relative fractions for the resonances contributing to the decay: $K^*(892)^0$, $K^*(1430)^0$, $\phi(1020)$, $a_0(1450)$, $\phi(1680)$, $K^*_2(1430)^0$, $K^*(1680)^0$, $K^*_1(1410)^0$, $f_2(1270)$,
The mass and width values of several resonances, including the \( \kappa(800) \) and \( \phi(1020) \), are determined in the fit (Table 1). The \( f_0(980) \) resonance is modeled with an effective BW parameterization:

\[
A_{f_0(980)} = \frac{1}{m_0^2 - m^2 - i m_0 \Gamma_0 \rho_{KK}},
\]

(12)
determined in the partial-wave analysis of \( D_s^+ \rightarrow K^+ K^- \pi^+ \) decays [15], where \( \rho_{KK} = 2p/m \) with \( p \) the momentum of the \( K^+ \) in the \( K^+ K^- \) rest frame, \( m_0 = 0.922 \text{ GeV}/c^2 \), and \( \Gamma_0 = 0.24 \text{ GeV} \). The remaining resonances (defined as \( r \rightarrow AB \)) are modeled as relativistic BWs:

\[
\text{RBW}(M_{AB}) = \frac{F_r F_D}{M_r^2 - M_{AB}^2 - i \Gamma_{AB} M_r},
\]

(13)

where \( \Gamma_{AB} \) is a function of the mass \( M_{AB} \), the momentum \( p_{AB} \) of either daughter in the \( AB \) rest frame, the spin of the resonance, and the resonance width \( \Gamma_R \). The form factors \( F_r \) and \( F_D \) model the underlying quark structure of the parent particle of the intermediate resonances. Our model for the \( K^- \pi^+ \) \( S \)-wave term consists of the \( \kappa(800) \), the \( \overline{K}_0(1430)^0 \), and a nonresonant amplitude. Different parameterizations for this term [16, 17] do not provide a better description of data. The resulting fit fractions are summarized in Table 2. A \( \chi^2 \) value is defined as

\[
\chi^2 = \sum_i \frac{(N_i - N_{MC_i})^2}{N_{MC_i}}
\]

(14)

where \( N_{\text{bins}} \) denotes 2209 intervals of variable size. The \( i^{th} \) interval contains \( N_i \) events (around 100), and \( N_{MC_i} \) denotes the integral of the Dalitz-plot model within the
Figure 3: $D^{\pm} \rightarrow K^+K^-\pi^\pm$ Dalitz plot and fit projections assuming no CPV, with the regions used for model-independent comparisons indicated as boxes. The A/B boundary is at $m_{K\pi} = 0.6$ GeV$^2$/c$^4$, the B/C boundary at $m_{K\pi} = 1.0$ GeV$^2$/c$^4$, and the C/D boundary at $m_{KK} = 1.3$ GeV$^2$/c$^4$. In the fit projections, the data are represented by points with error bars and the fit results by the histograms. The normalized residuals below each projection, defined as $(N_{Data} - N_{MC})/\sqrt{N_{MC}}$, lie between $\pm 5\sigma$. The horizontal lines correspond to $\pm 3\sigma$. The goodness-of-fit $\chi^2$/ndof = 1.21 for ndof = 2165. The distribution of the data in the Dalitz plot, the projections of the data and the model of the Dalitz plot variables, and the one-dimensional residuals of the data and the model, are shown in Fig. 3.
Table 2: Fit fractions of the resonant and nonresonant amplitudes in the isobar model fit to the data. The uncertainties are statistical.

| Resonance         | Fraction (%) |
|-------------------|--------------|
| $K^*(892)^0$      | 21.15 ± 0.20 |
| $\phi(1020)$     | 28.42 ± 0.13 |
| $K_0^*(1430)^0$  | 25.32 ± 2.24 |
| NR                | 6.38 ± 1.82  |
| $\kappa(800)$    | 7.08 ± 0.63  |
| $a_0(1450)^0$    | 3.84 ± 0.69  |
| $f_0(980)$       | 2.47 ± 0.30  |
| $f_0(1370)$      | 1.17 ± 0.21  |
| $\phi(1680)$     | 0.82 ± 0.12  |
| $K^*_1(1410)$    | 0.47 ± 0.37  |
| $f_0(1500)$      | 0.36 ± 0.08  |
| $a_2(1320)$      | 0.16 ± 0.03  |
| $f_2(1270)$      | 0.13 ± 0.03  |
| $K_2^*(1430)$    | 0.06 ± 0.02  |
| $K^*(1680)$      | 0.05 ± 0.16  |
| $f_0(1710)$      | 0.04 ± 0.03  |
| $f'_2(1525)$     | 0.02 ± 0.01  |
| Sum              | 97.92 ± 3.09 |

To allow for the possibility of CPV in the decay, resonances with a fit fraction of at least 1% (see Table 2) are permitted to have different $D^+$ and $D^-$ magnitudes and phase angles in the decay amplitudes ($A$ or $\bar{A}$). A simultaneous fit is performed to the $D^+$ and $D^-$ data, where each resonance has four parameters: $M_r$, $\phi_r$, $r_{CP}$, and $\Delta \phi_{CP}$. The CP-violating parameters are $r_{CP} = \frac{|M_r|^2 - |\bar{M}_r|^2}{|M_r|^2 + |\bar{M}_r|^2}$ and $\Delta \phi_{CP} = \phi_r - \bar{\phi}_r$. In the case of $S$-wave resonances in the $K^+K^-$ system, which make only small contributions to the model, used instead are the Cartesian-form of the CP parameters, $\Delta x$ and $\Delta y$, to parameterize the amplitudes and asymmetries. This choice of parameterization removes or eliminates technical problems with the fit. For these resonances, the parameters $x_r(D^\pm) = x_r \pm \Delta x_r/2$ and $y_r(D^\pm) = y_r \pm \Delta y_r/2$ are introduced. The masses and widths determined in the initial fit (shown in Table 1) are fixed, while the remaining parameters are determined in the fit. Table 3 summarizes the CP asymmetries, i.e., either the polar-form pair ($r_{CP}, \Delta \phi_{CP}$) or the Cartesian pair ($\Delta x_r, \Delta y_r$). Figure 4 shows the difference between the Dalitz-plot projections of the
Table 3: $CP$-violating parameters from the simultaneous Dalitz plot fit. The first uncertainties are statistical and the second are systematic.

| Resonance      | $r_{CP}$ (%) | $\Delta \phi$ (°) |
|----------------|--------------|-------------------|
| $K^-(892)^0$   | 0. (FIXED)   | 0. (FIXED)        |
| $\phi(1020)$   | 0.35$^{+0.82}_{-0.82}$ ± 0.60 | 7.43$^{+3.55}_{-3.56}$ ± 2.35 |
| $K^0(1430)^0$  | $-9.40^{+5.65}_{-5.36}$ ± 4.42 | $-6.11^{+3.29}_{-3.24}$ ± 1.39 |
| NR             | $-14.30^{+11.67}_{-12.57}$ ± 5.98 | $-2.56^{+7.01}_{-6.17}$ ± 8.91 |
| $\kappa(800)$  | $2.00^{+5.09}_{-4.96}$ ± 1.85 | $2.10^{+2.42}_{-2.45}$ ± 1.01 |
| $a_0(1450)^0$  | $5.07^{+6.86}_{-6.54}$ ± 9.39 | $4.00^{+4.04}_{-3.96}$ ± 3.83 |

| $\Delta x$     | $\Delta y$   |
|----------------|--------------|
| $f_0(980)$      | $-0.199^{+0.106}_{-0.110}$ ± 0.084 | $-0.231^{+0.100}_{-0.105}$ ± 0.079 |
| $f_0(1370)$     | $0.019^{+0.049}_{-0.048}$ ± 0.022 | $-0.0045^{+0.037}_{-0.036}$ ± 0.016 |

Figure 4: The difference between the $D^+$ and $D^-$ Dalitz plot projections of data (points) and of the fit (cyan band). The width of the band represents the ±1 standard deviation statistical uncertainty expected for the size of our data sample.

$D^+$ and $D^-$ decays, for both the data and the fit. It is evident from the figure that both the charge asymmetry of the data and fit are consistent with zero and with each other.

In summary, the BaBar and Belle collaborations have studied the SCS $D^+ \rightarrow K^+K^-\pi^+$ decay using complimentary analysis techniques to measure $CP$ violating asymmetries and search for new physics beyond the SM. The Belle measurement probed for $CPV$ in the dominant quasi two-body SCS decay mode $D^+ \rightarrow \phi\pi^+$ and CF mode $D_s^+ \rightarrow \phi\pi^+$, resulting in a precise measurement with small systematic uncertainties. The BaBar measurement took advantage of the Dalitz plot decay to probe for $CPV$ in all regions of the phase-space, making use of both model-dependent and model-independent techniques. No $CPV$ is observed in either measurement.
Further studies to improve the description of the Dalitz plot may provide a deeper understanding of the dynamics in three-body decays and \textit{CPV} in charm decays.

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