Efficient radar detection of weak manoeuvring targets using a coarse-to-fine strategy

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Abstract
Detection of weak manoeuvring targets has always been an important yet challenging task for radar signal processing. One primary reason is that target variable motions within a coherent processing interval generate energy migrations across multiple resolution bins, which severely deteriorate the parameter estimation performance. The problem is compounded by an increasingly complex surveillance environment, as well as more affordable and pervasive than ever small targets such as drones. In this work, a coarse-to-fine strategy for the detection of weak manoeuvring targets is designed. First, a coarse estimation of the chirp parameter in an extended unambiguous range using a novel coprime segmented discrete polynomial-phase transform is coherently integrated and derived. Sparse fractional Fourier transform is then employed to refine the coarse estimation at a significantly reduced computational complexity. The proposed scheme achieves an efficient and reliable weak target detection, while requires no a priori knowledge on the motion parameters, which is verified by the simulation results.

1 | INTRODUCTION

Surveillance radars enable the persistent monitoring of the coverage area and the prompt detection of aerial threats to the public safety. The effectiveness of the countermeasures relies heavily upon how early and accurately the threat warning can be. The radars should be able to successfully detect a wide range of target types, including low radar cross-section and fast manoeuvring flying objects, in the crowded urban area. Today, the detection of weak manoeuvring targets is particularly important, due to the widespread use of highly capable commercial unmanned aerial vehicles (UAVs). Even innocent UAV incidents can cause disruption and harm [1], not to mention those intent on malicious actions.

Radar detection of weak manoeuvring targets inevitably faces a technical dilemma. On the one hand, in order to improve the signal-to-noise ratio (SNR) and facilitate reliable detection and parameter estimation of weak components, observation time should be prolonged. On the other hand, the effect of range and Doppler walks induced by the manoeuvring motion is exacerbated with an extended observation interval. Nevertheless, the vital importance of this problem still encourages many radar researchers to put constant effort into seeking the solutions. Some recent and relevant works are reviewed as follows. Substantial consecutive radar echoes can be integrated incoherently or coherently to improve the SNR. Typical methods in the former category include Radon transform [2], Hough transform [3], track-before-detection algorithm [4,5] etc. Because only echo amplitude is utilised while the phase information is totally ignored, these methods generally suffer from integration gain loss. Alternatively, long-time coherent integration has been shown to be more effective. Taking into account the range and higher-order walks, various algorithms as well as their variant/combinations for coherent detection are proposed. Some important advances in the past 3 years include time-reversing transform [6], location rotation transform [7], discrete polynomial-phase transform (DPT) [8], and Radon Fourier transform (RFT)-based schemes [9–11]. Note that some of these schemes can deal with the problem of arbitrary high-order walks, such as the generalised Radon-Fourier transform (GRFT)
However, the authors argue that the second-order model is sufficient for many detection scenarios and, as will be addressed later, these algorithms usually demand high-dimensional exhaustive searches, which is computationally expensive.

Fractional Fourier transform (FrFT) represents an intuitive and natural domain for the time-frequency analysis of chirp signals, since chirp basis functions are employed to project the original time sequence onto a rotated time axis. We also know that, the scattered electromagnetic energy by multiple manoeuvring targets usually manifests itself as chirps in the radar return signal. Thus, by carefully choosing appropriate FrFT rotation angle, each of these target chirps can be transformed to impulses and exhibits sparsity in the matched fractional Fourier domain (FrFD). In the light of this observation, sparse FrFT (SFrFT) is proposed based on the sampling-type numerical algorithm and efficient detection of uniformly accelerating targets is realised. On this basis, manoeuvring target detection in the presence of strong clutters is considered and a robust SFrFT is proposed. Highly manoeuvring targets are further considered and the high-order phase modulation is handled by sparse fractional ambiguity function (SFrAF) [16]. However, to implement the above algorithms, the prior knowledge on the exact sparsity of ambiguities is considered and a robust SFrFT is proposed based on the sampling-type numerical algorithm and efficient detection of uniformly accelerating targets is realised [14].

On this basis, manoeuvring target detection in the presence of strong clutters is considered and a robust SFrFT is proposed. Highly manoeuvring targets are further considered and the high-order phase modulation is handled by sparse fractional ambiguity function (SFrAF) [16]. However, to implement the above algorithms, the prior knowledge on the exact sparsity of ambiguities is considered and a robust SFrFT is proposed based on the sampling-type numerical algorithm and efficient detection of uniformly accelerating targets is realised [14].

In this work, we periodically extend the difference frequency scope and modify the segmented DPT using coprime sampling. We have presented this idea in a previous initial work [18]. Compared with ref. [18], we consider a more generic parameter setting in this work, where the value of the sampling rate is not an integer multiple of the segment length and the difference frequency can be negative. This adds a high degree of flexibility to the algorithm. In addition, the parameter estimation is further refined using an optimised SFrFT, where the requirement of false-alarm rate in practical radar detection of weak target is considered. The above two stages collectively constitute a novel coarse-to-fine strategy for efficient radar detection of weak manoeuvring targets.

The rest of the article is organised as follows: Section 2 presents the signal model, the proposed algorithm flow, and the detailed description of each module. Simulation results and analyses are given in Section 3. Conclusions are drawn in Section 4.

2 | METHODOLOGY

Key motion parameters of radar manoeuvring targets include range, velocity, and acceleration rate. To estimate the ranges of weak targets, long-time integration is often adopted, which in turn induces range walk (RW) and Doppler walk (DW). Such effects can be non-trivial issues to address. Suppose that the radar transmitter emits a linear frequency modulated (LFM) signal with the form

\[ x(t) = \text{rect}\left(\frac{t}{T_p}\right) \exp\{j\pi\mu t^2\} \cdot \exp\{j2\pi f_c t\}, \]

where \(\text{rect}(\cdot)\) denotes a rectangular function, and \(t\) represents the continuous fast time. Parameters \(T_p\), \(\mu\), and \(f_c\), respectively denote the pulse duration, frequency modulation rate, and carrier frequency. After pulse compression, the received baseline echoes can be expressed as

\[ y(t) = A \cdot \text{sinc}(\pi B_r (t - 2R(t)/c)) \exp\{-j4\pi f_c R(t)/c\}, \]

where \(R(t) = R_0 + vt + at^2\) denotes the instantaneous range between radar and the target. We observe from Equation (2) that, the echo envelope varies with the slow time by following a quadratic function of time after pulse compression, which results in RW and DW. DPT estimates the high-order motion parameter along the curved trajectory. As such, it is able to simultaneously compensate the RW and DW induced by velocity and acceleration without any priori knowledge or exhaustive search, which has been proved in ref. [8]. However, as pointed out by ref. [19], DPT requires phase differentiations to reduce the phase order of the signal by one and the effective signal samples are thus significantly reduced, which can in turn lead to a reduced integration gain and resolution. This fact motivated us to find a way to circumvent the drawback of DPT and make full use of its distinct advantage of highly efficient search-free parameter estimation of weak chirp signals. We can also see from Equation (2) that the manoeuvring signature of the targets is captured by the sparse chirp components in the phase of the echo. Hence, the problem under investigation can be mathematically modelled as sparsity-aware parameter estimation of low-SNR chirp components in the FrFD. Thus, in the sequel, we present our work aiming at this goal.

2.1 | Preliminary

We consider a generic noise-corrupted input signal of \(N\)-point, which is \(K\)-sparse in the FrFD, that is,

\[ s[n] = \sum_{k=1}^{K} A_k \exp\{j2\pi f_k n t_s + j\pi \mu_k (n t_s)^2\} + w[n], \]

where \(n = 0, 1, \ldots, N - 1\) is the discrete time-domain index, and \(t_s\) represents the sampling interval. The term \(w[n]\) denotes the additive complex white noise, which is assumed to be Gaussian, that is, \(w \sim \mathcal{CN}(0, \sigma^2)\). The variables \(A_k, f_k\), and \(\mu_k\), respectively denote the amplitude, the initial frequency, and the chirp rate of the \(k\)-th chirp component, which are the primary parameters of interest in radar measurement.
Recall that the mathematical definition of second-order DPT (2-DPT) is [20].

\[
\text{DPT}_2\{s[n], f, \tau\} = \mathcal{F}\{s[n]s^*[n - \tau]\},
\]

(4)

where \(\mathcal{F}\) and \((\cdot)^*\), respectively denote the discrete Fourier transform (DFT) and conjugate operators. The conjugate multiplication in Equation (4) finds the phase difference between the samples separated by a positive integer spacing \(\tau\).

To better illustrate the principle of 2-DPT, we substitute Equation (3) into Equation (4) and yield Equation (5). From the first term on the right-hand side of Equation (5), we readily observe that, the \(K\) chirp components are converted into \(K\) mono-frequencies, and their locations are determined by the original chirp rates. Hence, a coarse parameter estimation of the chirp signal can be achieved by subsequently performing a DFT to the conjugate product. Note that Equation (5) also implies that the 2-DPT result contains \(K^2 - K\) cross-terms, which are eliminated using a follow-up SFrFT refinement. The remainder of Equation (5) are noise-induced terms.

\[
\text{DPT}_2\{s[n], f, \tau\} = \mathcal{F}\left\{\sum_{k=1}^{K} A_k^2 \exp\{j2\pi f_k t_n - j\pi \mu_k \tau^2 t_n^2 + j2\pi \mu_k \tau t_n (n t_s)\}\right. \\
+ \left. \sum_{p \neq q \in k} A_p A_q \exp\{j2\pi f_q t_n - j\pi \mu_q \tau^2 t_n^2 + j2\pi (f_p - f_q + \mu_q \tau t_n) (n t_s) + j\pi (\mu_p - \mu_q) (n t_s)^2\}\right\}
\]

\[+ w[n] \cdot \sum_{k=1}^{K} A_k \exp\{-j2\pi f_k (n - \tau) t_n - j\pi \mu_k (n - \tau)^2 t_n^2\} \\
+ w^*[n - \tau] \cdot \sum_{k=1}^{K} A_k \exp\{j2\pi f_k n t_s + j\pi \mu_k (n t_s)^2\} + w[n] \cdot w^*[n - \tau]\right\}. \tag{5}
\]

To achieve a reliable parameter estimation of weak chirp signals using 2-DPT, signal segmentation and coherent integration are incorporated. Suppose that we partition the signal into \(P\) non-overlapping segments, with \(L = N/P\) being the length of each segment. Also suppose that we set \(\tau = \frac{L}{2}\) to achieve the optimal estimation precision [20]. As such, the DFT resolutions in the fast/slow-time dimensions, that is, the intra/inter-segment frequency resolutions are respectively

\[
\Delta L = f_s / L \tag{4a}
\]

and

\[
\Delta P = f_s / (L \tau) = 2f_s / N. \tag{4b}
\]

We know from Equation (4a) that, given a fixed sampling rate \(f_s\), a larger segment length \(L\) yields a finer grid for initial frequency estimation of the chirps. Additionally, a larger segment length can obviously produce a higher integration gain. Nevertheless, the maximum observable chirp rate is subject to the unambiguous range of difference frequency estimation given by \([-f_s/(2L), f_s/(2L)]\). This in turn limits the selection of large segment length, because the unambiguous range is narrowed if the segment length is extended. In this context, coprime segmentation is introduced to solve the dilemma.

### 2.2 | Coprime segmented DPT

In the following, a detailed description of the proposed coprime segmented DPT is given. We first briefly introduce the general framework of the proposed scheme. To estimate the parameters of interest, a coarse-to-fine grid-search strategy is employed in this work. The input signal, or the radar return, is first sampled at regular intervals using analogue to digital converters (ADC). In the detection of weak manoeuvring targets, the discretised signal includes both the target of interest and strong noise. To improve the input SNR, we segment and coherently integrate the discretised signal prior to the parameter estimation. Additionally, coprime segment lengths are utilised to enlarge the maximum unambiguous range of chirp rate detection. The coarse estimation is then achieved by using the 2-DPT across segments. Within the significantly narrowed candidate ranges, the fine chirp rates of the input signal are further estimated by searching for the FrFT domains associated with the maximum magnitudes and the false detections induced by the cross-terms are also identified. An optimised SFrFT is used to accelerate the fine search process. The overall proposed algorithm flow is summarised in Figure 1.

#### 2.2.1 | Coprime segmentation

As mentioned above, coprime segmentation is performed before coherent integration to allow parameter estimation of high chirp rate. The mathematical principle is based on Chinese remainder theorem (CRT) [21]. Concretely, we select several positive integers \(L_i\), \(i = 1, \ldots, \rho\) and let them be the segment length, where \(\rho\) is the number of moduli. As such, the total number of non-overlapping segments associated with each segment length is \(P_i = N/L_i\). Thereby given \(n = l + (p - 1)L_i\), for any \(p = 1, 2, \ldots, P_i\) and \(l = 1, 2, \ldots, L_i\),...
FIGURE 1 Flow chart of the proposed weak chirp estimation scheme

Suppose that we perform FFT with respect to \( l \) in each column. Let \( m = 0, 1, \ldots, L_i - 1 \) denote the frequency-domain index. The \([p, l]^{th}\) entry is converted to

\[
\tilde{S}[p, m] = \mathcal{F}\{\tilde{s}[p, l]\} = \sum_{k=1}^{K} A_k \exp\{j2\pi f_k (l + pL_i) t_s + j\pi \mu_k (pL_i t_s)^2\} \cdot H_k[p, m] + \tilde{W}[p, m],
\]

(7)

where \( \tilde{W}[p, m] \) is the DFT of \( \tilde{w}[p, l] \) with respect to \( l \), and

\[
H_k[p, m] = \sum_{l=0}^{L_i - 1} \exp\{j2\pi f_k l t_s + j\pi \mu_k (l^2 + 2pL_i l^2 - 2\pi ml/L_i)\}. \tag{8}
\]

The second exponential function term in Equation (8) captures the Doppler spread inside a single segment, in the amount of \( \mu_kL_i t_s \). If the condition \( L_i \leq 1/(N\xi^2) \max\{|\mu_k|\} \) is satisfied, the effect of the Doppler spread is negligible. As such, Equation (8) can be simplified as

\[
H_k[p, m] \approx \sum_{l=0}^{L_i - 1} \exp\{j2\pi f_k l t_s - j2\pi ml/L_i\} \tag{9}
\]

where \( \xi = \pi(f_{ch} t_s - m/L_i) \). Substituting Equation (8) into Equation (7), we have

\[
\tilde{S}[p, m] = \sum_{k=1}^{K} A_k \frac{\sin(L_i \xi)}{\sin\xi} \cdot \exp\{j(L_i - 1)\xi\}
\]

\[
+ j2\pi f_k pL_i t_s + j\pi \mu_k (pL_i t_s)^2 \} + \tilde{W}[p, m]. \tag{10}
\]

2.2.2 Intra-segment coherent integration

This step is the key to reliable chirp rate estimation in low-SNR cases. The basic idea is inspired by the commonly adopted pulse integrator in radar detection, which achieves integration gain by efficiently combining the returns from multiple pulses, usually accomplished by fast Fourier transforms (FFTs). The coherent integration gain is equal to the number of pulses coherently integrated.

Note that the results of segmented DPT are \( M_i \)-periodic in the difference frequency dimension.

2.2.3 Coarse chirp rate estimation

We learn from the last exponential term in Equation (10) that, the phases of the signals in the same frequency bin of different segments preserve the chirp signature with the chirp rate \( \mu_k \). Combing Equations (5) and (10), we obtain the chirp rate estimator associated with a specific \( L_i \) expressed as

\[
\hat{\mu}_k = \frac{1}{L_i t_s} \arg \max_{f_k} |DPT_2\{\tilde{S}[p, m], f, \tau\}|. \tag{11}
\]

where the phase differencing of the coherently integrated weak signal is accomplished by
\[
\text{DPT}_2\{\tilde{S}[p, m], f, \tau\} \triangleq \mathcal{F}\{\tilde{S}[p, m]S^*[p - \tau, m]\}. \tag{12}
\]

The extended maximum unambiguous range for chirp rate estimation is achieved by leveraging the CRT principle. For ease of exposition, and without loss of generality, we assume that \(K = 1\) and \(\rho = 2\) in the following discussions. As such, we omit the subscript \(k\) in the parameters. Note that, the conclusions and methodology can be easily generalised to more complicated multiple chirp component cases. We readily infer from Equation (11) that, the difference frequency associated with the chirp rate estimator \(\hat{\mu}\) is

\[
f_\mu = \hat{\mu}_s t_s = \hat{\mu}_s t_s/2. \tag{13}
\]

Figure 2 shows the amplitude of the spectrum versus the difference frequency for all the aliased replicas within the unambiguous range of the proposed method. The periods where the spikes corresponding to the true difference frequency locate are highlighted with bright yellow colour. The spikes in the two coordinates only coincide at the true difference frequency.

Define \(r_i\) as the remainders of the division of \(f_\mu\) by \(M_i\), that is,

\[
f_\mu \equiv r_i \mod M_i, \tag{14}
\]

where \(\mod\) denotes the modulo operation, which wraps the numbers (e.g., \(r_i\) in Equation (14)) when they reach a modulus value (\(M_i\) in Equation (14)). In the context of difference frequency estimation illustrated in Figure 2, we readily see that \(r_i\) represents the estimated difference frequency in the principle region, that is, the range \([-M_i, M_i]\), of the \(M_i\)-periodic segmented DPT result \(\left\{ \text{DPT}_2\{\tilde{S}[p, m], f, \tau\} \right\}_{i=1}^\rho\). Thus, one straightforward way to estimate the true difference frequency among the aliased replicas is to write down all the possible candidates in the unambiguous range, which are \(M_i\) apart, and choose the closest match in the \(\rho\) candidate sets. Alternatively, we can recur to the closed-form CRT to analytically derive the true difference frequency from the segmented DPT results, which is described as follows.

Let \(M\) be the greatest common divisor (gcd) of \(\{M_i\}\) and define

\[
\Gamma_i = M_i/M. \tag{15}
\]

In addition, we define another parameter \(\Gamma\) as the least common multiple (lcm) of \(\Gamma_i\) and we have

\[
\gamma_i = \Gamma/\Gamma_i. \tag{16}
\]

Parameters \(M\) and \(\Gamma\) directly control the maximum unambiguous range and resolution of the difference frequency for chirp rate estimation, which are respectively given as \(M_\Gamma\) and \(M\) (both in Hz). The parameters \(\gamma_i\) and \(\Gamma_i\) are apparently coprime. As such, the modular multiplicative inverse of \(\gamma_i\) modulo \(\Gamma_i\) exists, which is represented by \(\gamma_i^{-1}\), that is,

\[
\gamma_i\gamma_i^{-1} \equiv 1 \mod \Gamma_i. \tag{17}
\]

We further define the quotient

\[
q_i = [r_i/M], \tag{18}
\]

where the symbol \([\cdot]\) represents the floor operation. As such, we have the following relationship for noiseless conditions:

\[
r_i = q_iM + r', \tag{19}
\]

where \(r'\) denotes the common remainder of \(r_i\) modulo \(M\). For corrupted remainders \(\tilde{r}_i\) in noisy conditions, their common remainders are obtained by

\[
\tilde{r}_i = \tilde{r}_i - M[\tilde{r}_i/M]. \tag{20}
\]

For arbitrary real number \(r \in (0, M)\), the average value of \(\tilde{r}_i\) is computed as

\[
\bar{r} = \arg \min_r \sum_{i=1}^\rho \left( \tilde{r}_i - r - \text{round}\left(\frac{\tilde{r}_i - r}{M}\right) \right)^2, \tag{21}
\]

where the term \(\text{round}\left(\frac{\tilde{r}_i - r}{M}\right)\) only takes the values \(-1, 0, 1\) for reasonable inputs, which ensures that the values of \(r\) and \(\tilde{r}_i\) are bounded by the value \(M\), such that the averaged \(\bar{r}\) and the corrupted \(\tilde{r}_i\)’s are close. Equation (21) performs average by finding the best least-square fit via minimising the sum of the distances between the raw \(\tilde{r}_i\)’s and the averaged \(\bar{r}\).

On the other hand, the folding factor \(N_0\) can be determined as per the CRT formula, as

\[
N_0 = \left( \sum_{i=1}^\rho \gamma_i^{-1}, q_i \right) \mod \Gamma. \tag{22}
\]

Hence, \(f_\mu\) can be uniquely reconstructed from

\[
\hat{f}_\mu = N_0 M + \bar{r}. \tag{23}
\]

2.2.4 CRT for general parameter settings

Although derivation of difference frequency using closed-form CRT is more robust and accurate, the limitation is obvious. As
will be shown in Section 3, the slow-time sampling rate \( M_i \) as a quotient of the sampling rate divided by the segment length can be a non-integer, and the difference frequency to be estimated can be a negative value. Nevertheless, closed-form CRT is only applicable to positive integers. Hence, we must consider CRT for more general inputs to cope with a broader range of applications. The idea is quite straightforward: (1) If the detected difference frequency is negative, we simply add \( M_i \) to fold it back to a positive region; (2) If \( M_i \) is not an integer, we use a scaling factor to convert it to an integer.

Concretely, the scaling factor is defined as:

\[
\beta \triangleq f_\mu \left/ \prod_{i=1}^{p} L_i. \right.
\]  

(24)

The scaled slow-time sampling rate becomes

\[
M_i \leftarrow M_i \left/ \beta \right. = \prod_{j \neq i} L_j.
\]  

(25)

It is not hard to see that the scaled \( M_i \) is guaranteed to be an integer as long as all \( L_i \)'s are integers. The detailed generalised algorithm flow for CRT-based coarse chirp rate estimation is summarised in Algorithm 1.

**Algorithm 1** CRT-based coarse chirp rate estimation for general parameter settings

1: **Inputs:**  
   \( f_\mu, \{L_i\}_{i=1}^{p}, \left\{ \left( \text{DPT}_2(\hat{S}[p,m], f_\mu, \tau) \right) \right\}_{i=1}^{p} \)  
2: Observe and obtain the large frequency entries \( \{r_i\}_{i=1}^{p} \) from \( \left\{ \left( \text{DPT}_2(\hat{S}[p,m], f_\mu, \tau) \right) \right\}_{i=1}^{p} \)  
3: if \( f_\mu \mod L_i \neq 0 \) then  
4: Compute the scaled \( M_i \) using Equation (25)  
5: Scale \( \{r_i\}_{i=1}^{p} \leftarrow \{r_i\}_{i=1}^{p} / \beta \)  
6: scale_flag \leftarrow 1  
7: else  
8: Compute \( M_i \) using (6)  
9: scale_flag \leftarrow 0  
10: end if
11: Obtain \( M = \gcd \{\{M_i\}_{i=1}^{p}\} \)  
12: Compute \( \Gamma \) using Equation (15)  
13: Obtain \( \Gamma \) = \( \text{lcm} \{\{\Gamma_i\}_{i=1}^{p}\} \)  
14: Compute \( \gamma_\mu \) using Equation (16)  
15: Compute \( \gamma_\mu^{-1} \) using Equation (17)  
16: if \( \exists i = 1, 2, \ldots, p, \text{ such that } r_i \leq 0 \) then  
17: \( r_i \leftarrow r_i + M_i \)  
18: end if
19: Compute \( q_i \) using Equation (18)  
20: Compute \( r^\prime \) using Equations (19)–(21)  
21: Compute \( N_0 \) using Equation (22)  
22: Reconstruct \( \hat{f}_\mu \) using Equation (23)  
23: if \( \hat{f}_\mu > M\Gamma/2 \) then  
24: \( \hat{f}_\mu \leftarrow \hat{f}_\mu - M\Gamma \)  
25: end if
26: if scale_flag == 1  
27: \( \hat{f}_\mu \leftarrow \beta \hat{f}_\mu \)  
28: end if  
29: \( \hat{\mu} = 2\hat{f}_\mu f_c / N \)

It is important to note that, for initial frequency estimation, if we let the frequency spacing between the two closest spaced components be \( \Delta f_{\text{min}} \), the coprime segment length set \( \{L_i\} \) should satisfy \( \max \{L_i\} \geq f_c / \Delta f_{\text{min}} \) to separate these two components. On the other hand, the maximum unambiguous range of initial frequency estimation is \( f_c / 2 \), which is independent of \( \{L_i\} \). For difference frequency estimation, the frequency resolution and the maximum unambiguous range are \( M \) and \( M\Gamma \), respectively. As \( M \) and \( \Gamma \) are derived from \( \gcd \) and \( \text{lcm} \) of the values that are related to \( L \), no explicit closed-form expression can describe the direct quantitative relationship between \( L \) and \( M \) as well as \( \Gamma \). As such, the selection criterion for \( L \) which compromises the estimated range and resolution of the chirp rate cannot be yielded.
Sparse FrFT

Algorithm 2 Optimised sparse FrFT

1: Inputs: 
   \( s[n], \alpha_k = \arccot(-2\pi\mu_k) \), \( P_{ta} \) 
   \( \{\zeta\} \) (A set of random positive odd integers) 
   \( g[n] \) (A \( W \)-point flat window) 
   \( B \) (A positive proper divisor of \( N \))

2: Initialise:
   \( b_z[m] \leftarrow [\frac{g}{\zeta} m] \) (A hash function)
   \( a_{\delta}[m] \leftarrow \zeta m - \frac{1}{2} b_z[m] \) (A offset function)
   \( \zeta = \alpha\sqrt{2W/N} \cdot \ln P_{ta} \) (Detection threshold)

3: for \( k = 1: K \) do
4:   \( a = \tilde{a}_k \pm j \cdot \Delta\alpha \), \( j = 1, 2, \ldots \)
5:   \( x[n] \leftarrow s[n] \cdot \phi[n] \)
6:   for loop_cnt \( \leq \) num_loop do
7:     \( f[n] \leftarrow x[n] \mod N \)
8:   \( y[n] \leftarrow f[n] \cdot g[n] \)
9:   \( Y[m] \leftarrow \mathcal{F}\left\{ \sum_{i=1}^{W/B} y[n+i-B]\right\} \)
10: \( J \leftarrow \{ m \in [1,B] : |Y(m)| > \zeta \} \)
11: \( I \leftarrow \{ m \in [1,N] : b_z[m] \in J \} \)
12: for loop_cnt \( > \) num_loc_loop do
13:   \( \tilde{X}[m] \leftarrow Y(b_z[m]) \cdot \exp(-j\pi a_z[m] \frac{W}{N}) \)
14: end for
15: end for
16: \( \tilde{X}[m] \leftarrow \text{median of } \tilde{X}[m] \)
17: \( \mathcal{F}_a^s[n] = \tilde{X}[m] \cdot \Phi[m] \)
18: Find and record the optimal \( \alpha \) such that the large coefficient reaches its maximum
19: end for

Equation (5) indicates that the result of 2-DPT can be corrupted by undesirable cross-terms. Additionally, the estimated chirp rate can be considerably inaccurate due to the high noise level and the coarse resolution grid. In the light of these limitations, we further refine the parameter estimation in a significantly narrowed range in the FrFD. The step size of the fine search can be chosen as per the chirp rate resolution of 2-DPT and the definition of the rotation angle \( \alpha \) in the FrFT, which is [14].

\[
\Delta\alpha = \frac{8\pi\sin^2\alpha}{(Nt_s)^2} \tag{26}
\]

The concept of SFrFT stem from an efficient sampling-type numerical algorithm, frequently referred to as the Pei’s algorithm. An order-\( \alpha \) discrete FrFT (DFrFT) \( \mathcal{F}^a \) of an input signal \( s[n] \) is defined as [22].

\[
\begin{align}
\mathcal{F}^a\{s[n]\} & \triangleq \\
S[m], \alpha = 2D\pi, \\
S[-m], \alpha = (2D+1)\pi, \\
\Phi[m], \alpha = 2D\pi + (0, \pi), \\
\Phi[m], \alpha = 2D\pi + (-\pi, 0), \\
\end{align} \tag{27a}
\]

\[
\Phi[n] = \exp\left\{ \frac{1}{2}(\cot\alpha)n^2\ell_s \right\}, \tag{27b}
\]

\[
\Phi[m] = \exp\left\{ \frac{m^2\nu_c^2}{2\tan\alpha} \right\} \sqrt{\text{sgn}(\sin\alpha - j\cos\alpha) M}, \tag{27c}
\]

where \( D \) and the \( \text{sgn}(\cdot) \), respectively denote an arbitrary integer and a signum function. The output are discrete points in the FrFD with a sampling interval \( u_c \). Exhaustive parameter search of chirp rate using the above DFrFT formula is computationally prohibitive and, thus, demands substantial time/hardware resources. A sparsity-aware randomised algorithm is proposed to reduce the computational load by incorporating permutation mechanism and subsampled FFT operations. In the context of weak manoeuvring target detection, to guarantee a false-alarm rate of \( P_{fa} \) in the fractional frequency estimation, Neyman–Pearson (NP) detection is applied and detection threshold is set accordingly [23]. The optimised SFrFT is summarised in Algorithm 2. Note that \( J \) and \( I \), respectively denote the sets that contain the coordinates of the large entries in the sub-sampled and original spectra.

3 | NUMERICAL SIMULATIONS

In this section, numerical simulations are provided to showcase the advanced performance of the proposed coarse-to-fine detection scheme for weak manoeuvring targets. In both simulations, we assume that the radar operates at a centre frequency of \( f_c = 3 \) GHz, and the pulse repetition interval is \( 0.125 \) ms which corresponds to a sampling rate of \( f_s = 8000 \) Hz. The overall signal length is \( N = 32,768 \) and the input SNR is 0 dB. We take the first 20,160 points and use the proposed coprime segmented DPT to obtain a coarse estimation of the chirp rate. The segment lengths in the two coprime segmentation processes are \( L_1 = 105 \) and \( L_2 = 120 \). Thus, the corresponding slow-time sampling rates are \( M_1 = 76.2 \) Hz and \( M_2 = 66.7 \) Hz, respectively. The maximum unambiguous range of the detectable chirp rate is \([-266.7, 266.7] \) Hz.

3.1 | Mono-chirp case

In the first simulation example, we consider a single target with an initial velocity of \( v = 50 \) m/s and an acceleration rate of \( a = -10 \) m/s\(^2\). We know from Equation (2) that the Doppler frequency induced by target accelerating motion is derived as

\[
f_d = \frac{2v_t}{\lambda} = \frac{2(v + at)}{c/f_c} = f_{ina} + \mu t, \tag{28}
\]

where \( v_t \) represents the instantaneous velocity of a target at time instant \( t \) and \( c \) denotes the speed of light. As such, the corresponding initial frequency and chirp rate are \( f_{ina} = 2f_a/v \) and \( f_a = 1000 \) Hz and \( \mu = 2f_a/c = -200 \) Hz/s, respectively. We
learn from Equation (13) that the true difference frequency is $f_\mu = -252$ Hz, which lies within the observable range. The signal in each segment is zero-padded to 2048 points.

The results of the segmented DPT with coprime segment lengths $L_1$ and $L_2$ are given in Figures 3 and 4, respectively. Although the regions where the detected difference frequencies locate are distinct, it is somewhat difficult to confidently pick a specific point to represent the peak estimation in Figures 3(a) and 4(a). Hence, we further apply a Gaussian window function to smooth the DPT maps and we have Figures 3(b) and 4(b). To better illustrate the necessity of Gaussian smoothing, we slice the DPT maps at the initial frequency of 1000 Hz and show their side views, which are given in Figures 3(c) and 4(c). The side views of the smoothed DPT maps in Figures 3(b) and 4(b) and the raw segmented DPT maps in Figures 3(a) and 4(a) are respectively represented by the red thick lines and the blue thin lines in Figures 3(c) and 4(c). We can see that a coarse estimation of the chirp rate can be obtained more easily from the smoothed DPT maps. The maximum amplitudes in Figures 3(b) and 4(b) are detected, and we record the associated difference frequencies $f_1 = -23.29$ Hz and $f_2 = 14.71$ Hz. We further derive the corresponding true difference frequency candidates in the unambiguous range respectively as $\{-251.86, -175.67, -99.48, -23.29, 52.90, 129.09, 205.28\}$ Hz and $\{-251.95, -185.29, -118.62, -51.95, 14.71, 81.38, 148.04, 214.71\}$ Hz. By matching the numbers in the two candidate sets, we find the nearest values $-251.86$ and $-251.95$ Hz. As such, we infer from Equation (13) that the coarse estimation of the chirp rate is around $-199.93$ Hz/s.

The coarse chirp rate estimation can also be analytically derived using Algorithm 1 as follows. Since we observe from Figures 3 and 4 that $f_1 = -23.29$ Hz and $f_2 = 14.71$ Hz, we have $r_1 = -23.29$ Hz and $r_2 = 14.71$ Hz. As we know that $f_i \mod L_i \neq 0$, that is, $M_i$’s are not integers, we scale $M_i$ and $r_i$ with $\beta = 8000/120/105$. We thus obtain $M_1 = 120, M_2 = 105, r_1 = -36.7, r_2 = 23.1$, and $\beta = \gcd(M_i) = 15$. As per Equation (15), we further have $\Gamma_1 = 120/15 = 8, \Gamma_2 = 105/15 = 7$, and $\Gamma = 1 \text{ cm} \Gamma_i = 56$. At this point, we know that the maximum unambiguous range of difference frequency estimation is $M_1 \Gamma = 840$. Next, we derive from Equations (16) and (17) that $\gamma_1 = 7, \gamma_2 = 8, \gamma_1^{-1} = 7, \gamma_2^{-1} = 1$. Since $r_1 = -36.7 < 0$, we add a corresponding $M_i$ to it, which yields $r_1 = -36.7 + 120 = 83.3$. From Equation (18), we have $q_1 = [83.3/15] = 5$ and $q_2 = [23.1/15] = 1$. Additionally, from Equations (19)–(21), we have $\hat{r} = \arg\min\{(8.3 - r)^2 + (8.1 - r)^2\} = 8.2, N_0 = (7 * 7 * 5 + 1) \mod 56 = 29$. We finally reconstruct $\hat{f}_\mu = 29 * 15 + 8.2 = 443.2$ Hz, which is apparently larger than $M_1 \Gamma / 2 = 420$ Hz. Hence, the true difference frequency is $\hat{f}_\mu = (443.2 - 840) \cdot \beta = 251.94$ Hz and the coarse estimation of the chirp rate is approximately $-199.95$ Hz/s. This estimate tallies with that of the previous approach.
The results of DFrFT and SFrFT using the rotation angle matched with chirp rate $-199.95$ Hz/s are provided in Figure 5(a). Fine search using the step size (26) is conducted and we find the maximum amplitude at the chirp rate $-200$ Hz/s, as shown in Figure 5(b). We also readily see from Figure 5 that, the results of SFrFT can achieve a good approximation of DFrFT. Meanwhile, the computational complexity is significantly reduced. By contrast, if we instead adopt the original DPT-SFrFT method [17] in the first example, the unambiguous range degrades to $[-38.1, 38.1]$ Hz or $[-33.3, 33.3]$ Hz when choosing $L = 105$ or 120. The true difference frequency $f_\mu = -252$ Hz becomes unobservable in both cases.

### 3.2 Multi-chirp case

In the following, the multi-component resolution performance of the proposed method is further investigated. We consider three targets that perform uniformly variable motion during the observation period. The initial velocities and acceleration rates of the targets are respectively $v = [50, 49.5, -50]$ (m/s) and $a = [-5, -6, -7]$ (m/s$^2$). That is, the corresponding initial frequencies and chirp rates are respectively $f_{0i} = [1000, 990, -1000]$ (Hz) and $\mu = [-100, -120, -140]$ (Hz/s). Bearing in mind that, in implementing the coprime segmented DPT, the true difference frequencies are $f_\mu = [-126, -151.2, -176.4]$ (Hz). As such, the remainders of the three true difference frequencies corresponding to $M_1 = 76.2$ Hz and $M_2 = 66.7$ Hz are $[26.38, 1.18, -24]$ and $[7.33, -17.87, 23.60]$ (Hz), respectively. As shown in Figures 6(a) and (c), we observe several regions of high amplitudes in the segmented DPT maps. Again, we apply the Gaussian window to the results in Figures 6(a) and (c) and obtain the smoothed results in Figures 6(b) and (d). Note that the cross terms arise due to the interaction of two closely valued components in the initial frequency dimension. It is reasonable to expect that if the input SNR is lower, the cross terms can be difficult to be identified. We record the reminder estimation $[26.45, 1.08, -24.18]$ (Hz) and $[7.39, -17.71, 23.57]$ (Hz) associated with the coprime segment lengths $L_1$ and $L_2$ from Figures 6(b) and (d). In the worst cases, the record can contain some cross terms. Then, if we derive the candidate sets in the entire unambiguous range and pair the candidates, the process will be very difficult since too many spectrum lines present. In contrast, Algorithm 2 offers a much clearer way to estimate the true difference frequencies. In Table 1, we list all the key intermediate results in the process of coarse chirp rate estimation. It can be seen that the estimated difference frequencies are accurate.

Next, we use SFrFT to exclude possible false detections and obtain a finer estimation. The concentrated peak and the maximum amplitude for each chirp component are recorded, and the associated accurate estimation of the chirp rates is in turn deduced. Figure 7 presents matched-order DFrFTs/SFrFTs corresponding to the chirp rates of $[-99.95, -119.98, -140.08]$ (Hz/s). For the mismatched orders including those corresponds to the cross terms, no concentrated peak can be found in the fractional Fourier spectrum. For instance, if we mistakenly match 26.45 Hz with $-17.71$ Hz or 1.08 Hz with $7.39$ Hz, we obtain the chirp rate estimation $143.30$ and $60.05$ Hz/s. If we further compute the DFrFTs, we have the results shown in Figure 8, where the energy of the signal components disperses over several fractional Fourier frequency bins and the amplitudes fall sharply from 150 $+$ in Figure 7 to around 8 in Figure 8. This clearly indicates that the estimated chirp rates are completely false.

### 3.3 Performance analyses

#### 3.3.1 Computational complexity

The number of complex multiplications involved in the computation of segmented DPT is $\frac{\rho L_2}{2} \log_2 L_1$. As generally $\rho = 2$.
**Figure 6** Multiple target detection scenario. (a) Segmented DPT with $L_1$. (b) Smoothed results of (a). (c) Segmented DPT with $L_2$. (d) Smoothed results of (c)

**Table 1** Intermediate results of coarse difference frequency estimation in the multi-chirp case

| Raw $r_1$ | Legal $r_1$ | $q_i$ | $r^*$ | $N_0$   | Raw $f_μ$ | $f_μ$   |
|------------|-------------|-------|-------|---------|----------|---------|
| [26.45, 1.08, −24.18] | [41.66, 1.70, 81.92] | [2, 0, 5] |       |         |         |         |
| [7.39, −17.71, 23.57] | [11.64, 77.11, 37.12] | [0, 5, 2] | [11.65, 1.90, 7.02] | [42, 40, 37] | [641.65, 601.90, 562.02] | [−125.94, −151.17, −176.50] |

**Figure 7** FrFT spectra and local zoomed results using conventional DFrFT and SFrFT with different matched orders. (a) Matched order of chirp rate estimation $-99.95$ Hz/s. (b) Matched order of $-119.98$ Hz/s. (c) Matched order of $-140.08$ Hz/s
is sufficient for coprime segmentation, the overall computational complexity of coprime-segmented DPT-based coarse chirp rate estimation is $O(N \log_2 L)$. On the other hand, the computational complexity of SFrFT is $O((\log_2 N) \cdot \sqrt{K \log_2 N})$. Therefore, the total computational load adds up to $O(N \log_2 L + \log_2 N \cdot \sqrt{K \log_2 N})$ in the proposed two-stage scheme. By comparison, chirp parameter estimation based on traditional discrete FrFT algorithm requires a computational complexity of $O(N^2 N_s \log_2 N)$, where $N_s$ denotes the number of exhaustive searches of the fractional order. RFT requires an exhaustive search of the blind velocity, and its computational cost is associated with the number of targets. For $K$ targets, the computational complexity of RFT is $O(K N_1 N_2 N)$, where $N_2$ is the number of exhaustive searches of the velocity. Radon FrFT (RFrFT) [10] uses three-dimensional exhaustive searches on the basis of discrete FrFT. Thus, the computational complexity of RFrFT is $O(M N_1 N_2 N_3 \log_2 N)$, where $N_3$ is the number of acceleration searches. Radon Iv’s distribution (RLVD) [24] also employs three-dimensional exhaustive searches to remove RW and DW, and the computational cost is $O(M N_1 N_2 \log_2 N)$. We readily conclude that the proposed method is with the lowest complexity among all the competitive approaches.

3.3.2 MSE of Chirp Rate Estimation

It has been proved in ref. [25] that the mean-square error (MSE) of the original-DPT-based estimate of $\mu$ can be written as

$$\text{MSE}_{\text{DPT}}(\mu) \approx \frac{48}{N^3 (N^2 - 4)} \left[ \left( 1 + \frac{1}{\text{SNR}_m} \right)^2 - 1 \right]. \quad (29)$$

By comparison, the proposed Algorithm 1 estimates the chirp rate by applying CRT to the result of the segmented DPT. While segmented DPT achieves an $L$-fold SNR gain by coherent integration, CRT is essentially a coordinate transformation, which has no impact on the estimation MSE. As such, the MSE of the Algorithm 1-based estimate of $\mu$ can be written as

$$\text{MSE}_{\text{A1}}(\mu) \approx \frac{48}{N^3 (N^2 - 4)} \left[ \left( 1 + \frac{1}{L \cdot \text{SNR}_m} \right)^2 - 1 \right]. \quad (30)$$

The Cramér–Rao lower bound (CRLB) for unbiased estimation of $\mu$ is given by

$$\text{CRLB}_\mu = \frac{15}{N^4 \text{SNR}_n}. \quad (31)$$

In Figure 9, we draw the curves of the MSE of the chirp rate estimation versus different input SNRs ranging from $-20$ to $10$ dB. Note that the LVD curve is also included for reference, which is bounded by [26].

$$\text{MSE}_{\text{LVD}}(\mu) \approx \frac{294}{\pi^2 N^2 \text{SNR}_n}. \quad (32)$$

It can be concluded from Figure 9 that, both DPT-based schemes outperform the LVD method in terms of MSE, and their performance gap grows narrower with the improvement of input SNR. It is noteworthy that, the proposed Algorithm 1 can achieve a significantly enhanced estimation accuracy in the low-SNR cases compared to the original DPT.

3.3.3 Detection probability

We further examine the detection performance of the proposed method and a comparison with the existing approaches is drawn
by $10^4$ Monte Carlo trials of constant false alarm (CFAR) detection. In the simulation, we consider a single point target and use the same radar and motion parameter settings as in Section 3.2. Several competitive methods such as the classic moving target detection (MTD), RFT [9], RFrFT [10], and RLVD [24] are included in the detection probability evaluation. The simulated radar returns are contaminated by the zero-mean white Gaussian noise and the input SNRs after the range compression are set as $-30$ to $15$ dB. The false alarm ratio is set as $P_{fa} = 10^{-4}$. The probabilities of detection of the five detectors versus different SNRs are presented in Figure 10. It can be concluded from Figure 10 that, if we pick the same probability of detection $P_d = 70\%$, the SNR threshold for the proposed method is respectively about 4, 15, and 24 dB lower than RFrFT, RFT, and MTD. The detection performance of MTD is the worst since it cannot compensate either RW or DW. RFT can handle RW and improve the integration gain to some extent, it still suffers from integration loss due to DW. Though RFrFT can mitigate the DW induced by the target’s acceleration like RLVD and the proposed method do, it obtains a lower noise margin than the latter two methods. The detection performance of RLVD is slightly better than the proposal (less than 1 dB), but its computational complexity is significantly higher.

4 CONCLUSION

In the presented work, a coarse-to-fine strategy is designed to help radar operators to observe weak and highly manoeuvring targets that are otherwise difficult to detect. The target-scattered energy in the radar return is coherently integrated in the FrFD using an efficient sparsity-aware numerical algorithm termed optimised SFrFT. The optimised SFrFT allows custom configuration of false-alarm rate. But prior knowledge on exact sparsity and matched rotation angle is also required. Hence, we perform a segmented DPT and extend the maximum unambiguous range for chip rate estimation using co-prime slow-time sampling beforehand. The potential of the proposed two-stage scheme is showcased in the numerical simulations.

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