PERTURBATION THEORY
OF OBSERVABLE LINEAR SYSTEMS

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Abstract. The present work is motivated by asymptotic control theory for a system of linear oscillators: the problem is to design common bounded scalar control for damping all oscillators in asymptotically minimal time. Motion of the system is described in terms of a canonical system similar to that of the Pontryagin maximum principle. We consider evolution equation for adjoint variables as a perturbed observable linear system. Due to the perturbation, the unobservable part of the state trajectory cannot be recovered exactly. We estimate the recovering error via the $L_1$-norm of perturbation. This allows us to prove that the control makes the system approach the equilibrium with a strictly positive speed.

Keywords: linear system, controllability, observability
MSC 2010: 93B03, 93B07, 93B52.

1. Introduction

1.1. Exact and approximate minimum time problem. The subject of the present paper has grown out of study of linear controllable dynamical system

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{V} = \mathbb{R}^N, \quad u \in \mathbb{U} = \mathbb{R}, \quad |u| \leq 1,$$

where $A$ is a diagonalizable matrix with purely imaginary spectrum. The system is a standard model for control of oscillations.

Our basic problem is the time-optimal damping of a given initial state of system. Optimal trajectory is to be found as the steepest descent in the direction of the gradient of the cost function, aka momentum.

Definition 1. The reachable set $\mathcal{D}(T)$ is the set of ends at time instant $T$ of all admissible trajectories of the system starting at the given manifold at zero time.

The level sets of the cost functions are boundaries of the reachable set of the system with respect to the backward time. The direction of the gradient (momentum) is normal to the boundary of the reachable set.

Thus, the optimal control has the form

$$u(x) = -\text{sign}(B, p(x)), \quad p = \frac{\partial T}{\partial x}(x)$$

where $p$ is normal to the reachable set $\mathcal{D}(T(x))$ at point $x$.

One of the basic results of pertaining to system says that the reachable set $\mathcal{D}(T)$ equals asymptotically as $T \to \infty$ to the set $T\Omega$, where $\Omega$ is a fixed convex body.

The basic idea of our control design is to substitute the set $T\Omega$ for $\mathcal{D}(T)$, and the normal to this set for momentum. If the phase vector $x$ lies in the boundary of $T\Omega$, then

$$x = T \frac{\partial H_\Omega}{\partial p}(p)$$
for a momentum \( p = p(x) \) and time \( T \). Here \( H_\Omega \) is the support function of \( \Omega \) \([1]\). In
the case at hand the support function \( H_\Omega \) is differentiable, and equation (3) defines
the direction of the vector \( p \) and the factor \( T \) uniquely, because of the smoothness
of the boundary of \( \Omega \) \([1], [5]\). Thus, control is given by
\[
(4) \quad u(x) = -\text{sign} \langle B, p(x) \rangle,
\]
where angular brackets stand for the standard scalar product in \( \mathbb{R}^N \).

1.2. Polar-like coordinate system. Here we define a polar-like coordinate sys-
tem, well suited for representation of motion under the control \( u \). Write the phase
vector \( x \) in the form \( x = \rho \phi \), where \( \rho > 0 \) and \( \phi \in \omega = \partial \Omega \). In these coordinates
equations of the motion have the form
\[
(5) \quad \dot{\rho} = -\left| \left( \frac{\partial \rho}{\partial x} B \right) \right|, \quad \dot{\phi} = A\phi + \frac{1}{\rho} \left( Bu + \phi \left| \left( \frac{\partial \rho}{\partial x} B \right) \right| \right).
\]

It should be noted, that \( \rho = T \) and \( \phi = \partial H_\Omega(p)/\partial p \) in terms of equation (3),
\( \rho(x) \) is a norm of vector \( x \), i.e., is a homogenous degree one convex function of \( x \),
which is strictly positive for \( x \neq 0 \).

The function \( \rho \) is invariant under free (uncontrolled) motion of our system, i.e.
\( \langle \partial \rho/\partial x, A x \rangle = 0 \). Moreover, we can show that the hessian \( \partial^2 \rho/\partial x^2 \) is bounded on
sphere \( |x| = 1 \).

An eikonal-type equation holds for the function \( \rho = \rho(x) \)
\[
(6) \quad H_\Omega(p) = 1, \quad p = \frac{\partial \rho}{\partial x}.
\]
It is “dual” to the equation \( \rho(\partial H_\Omega/\partial p) = 1 \) of the surface \( \omega \).

1.3. Problem statement. The following problem was addressed in \([1]\): does this
control makes us approach zero with a positive speed? It turns us eful to measure
distance to the target by means of the norm \( \rho \). Precisely speaking, suppose that
\( x(t) \) is a trajectory and \( \rho_t = \rho(x(t)) \), then, the question is: Is it true that
\[
(7) \quad \rho_0 - \rho_T \geq cT,
\]
where \( c > 0 \) By means of theory presented below (Section 2) we can give a positive
answer provided that \( \rho_t > M \) for \( t \in [0, T] \), where \( M \) is a sufficiently large positive
constant.

In view of \([5], [9]\), we have to show that \( L_1 \)-norm of \( \langle p, B \rangle \) in the interval \([0, T]\)
is greater than \( cT \). Time evolution of vector \( p = \partial \rho/\partial x \) is described by
\[
(8) \quad \dot{p} = -A^*p + \frac{\partial^2 \rho}{\partial x^2} Bu = -A^*p + \tilde{B}u.
\]

We note that the remainder \( \tilde{B}u \) is small provided the the state vector \( x \) moves at
large distance from the origin, because \( \frac{\partial^2 \rho}{\partial x^2} \) is a homogeneous function of degree \(-1 \)
so that \( \frac{\partial^2 \rho}{\partial x^2} = O(1/|x|) \).

Our idea is to interpret (8) as the homogenous linear system
\[
(9) \quad \dot{p} = -A^*p
\]
perturbed by \( \tilde{B}u \), and consider \( \langle p, B \rangle \) as a partial observation of the state vector
\( p \). It allows us to embed our problem in the framework of theory of observable
linear systems. The theory predicts that the \( L_1([0, T]) \)-norms of functions \( p(t) \)
and \( \langle p(t), B \rangle \) are of the same order of magnitude, and this leads to the required in-
equality \( L_1([0, T]) \)-norm \( \langle p, B \rangle \) is \( \geq cT \).
2. Perturbation theory

The subject of the Kalman linear observation theory [6],[7] is a linear time-invariant system
\[ \dot{x} = Ax, \quad y = Cx, \]
which is observed, so that the vector \( y \) is the observation result. Here, \( A \) and \( C \) are constant matrices.

**Definition 2.** The system is said to be completely observable, if the knowledge of the curve \( y(t) \) in an open time interval allows to recover \( x(t) \) uniquely.

We consider a perturbed situation where the observed vector has the same structure, but the vector \( x \) satisfies the perturbed equation
\[ \dot{x} = Ax + f, \quad y = Cx. \]
Then, it is impossible to recover \( x \) from knowledge of \( y \) precisely, but, if the perturbation \( f \) is small, we can do this with a small error.

More quantitatively, the error size is described by the following:

**Theorem 1.** Suppose \( \dot{x} = Ax, y = Cx \) is a completely observable time-invariant linear system. Then, for a solution \( z \) of \( \dot{z} = Az + f \) in the interval \( I \) of integer length the a priori estimate
\[ \int_I |z| dt \leq c \left( \int_I |Cz| dt + \int_I |f| dt \right) \]
holds, where the constant \( c \) does not depend on the interval \( I \).

It is clear by summation over adjacent intervals of length 1 that it suffices to consider the case \( I = [0, 1] \). We present two proofs of the theorem. Both are based on the following lemma:

**Lemma 1.** Under assumptions of Theorem 1 consider the map
\[ \Phi : z \mapsto [y, f] = [Cz, \dot{z} - Az] \]
from \( \mathbb{W} = W^{1,1} \otimes \mathbb{R}^n \) to \( L = L_1 \otimes \mathbb{R}^m \oplus L_1 \otimes \mathbb{R}^n \) and its image \( L \). Then, there exists a pair of linear ordinary differential operators \( P = P \left( \partial_t \right) \) and \( Q = Q \left( \partial_t \right) \) with constant coefficients such that \( (y, f) \in L \) iff \( Py = Qf \). Moreover, the degrees of polynomials \( P, Q \) are \( \leq n \).

Here \( W^{1,1} \) is a Sobolev space. This immediately implies the following

**Corollary 1.** The image \( L \) of the map \( \Phi \) is closed in \( L \).

Indeed, the condition \( Py = Qf \) defines a closed subspace in the space of (pairs of vector valued) distributions.

Now it is easy to derive Theorem 1 from Corollary 1. The map \( \Phi : \mathbb{W} \to L \) is a continuous linear map. By Corollary 1 the image \( L \) is closed in \( L \). The observability condition means that the kernel of the map \( \Phi \) is zero. Hence, one can apply the Banach inverse operator theorem and conclude that
\[ |z|_1 \leq c(|Cz|_0 + |\dot{z} - Az|_0). \]
Here, \( c \) is the norm of the inverse operator \( \Phi^{-1} \), and
\[ |z|_n = \sum_{k=0}^n \int_0^1 \left| \frac{\partial^k z}{\partial t^k} \right| dt \]
is the standard Sobolev norm in \( W^{n,1}(\{0, 1\}) \). The conclusion of Theorem 1 is an obvious weakening of inequality (12).
Another proof of Theorem 1 is more constructive and lengthy. We present it only for the case of scalar observation \((m = 1)\), because this simplifies our arguments. It starts with the following weakening of inequality \((11)\).

**Lemma 2.** Suppose \(\dot{z} = Ax, y = Cx\), where \(A : \mathbb{R}^n \to \mathbb{R}^n, C : \mathbb{R}^n \to \mathbb{R}\), is a completely observable time-invariant linear system. Then, for any solution \(z\) of equation \(\dot{z} = Ax + f\) a priori estimate

\[
|z|_0 \leq c (|Cz|_0 + |f|_n)
\]

holds, where \(C\) is a positive constant.

In this case \(m = 1\) the polynomial \(P(A)\) is scalar and monomial of order \(n\). We rewrite equation \(Py = Qf\) of Lemma 1 in the form

\[
P(A) y = g,
\]

where \(|g|_0 \leq C|f|_n\), and this reduces the proof of Lemma 2 to a priori estimate

\[
|y|_n \leq c (|y|_0 + |g|_0),
\]

resembling basic estimates in the \(L^p\)-theory of elliptic equations \([10]\), for any solution of \((13)\). We prove the estimate by induction with respect to the order \(n\) of the operator \(P\). For \(n = 1\) we have a scalar equation

\[
ay = g,
\]

with constant \(a\), which implies

\[
|\frac{\partial y}{\partial t}|_0 \ll |y|_0 + |g|_0.
\]

The latter inequality is equivalent to \((14)\) for \(n = 1\). To perform the induction step we write \(P\) in the form \(\prod_{k=1}^n (\frac{\partial}{\partial t} + \alpha_k)\), and define \(v = (\frac{\partial}{\partial t} + \alpha_1)y\). By induction we obtain that \(|v|_{n-1} \ll |v|_0 + |g|_0\). This immediately implies that

\[
|y|_n \ll |y|_1 + |g|_0.
\]

In order to arrive at \((14)\) we invoke the Kolmogorov-type inequality

\[
\int_0^1 |\frac{\partial y}{\partial t}| \, dt \ll \left( \int_0^1 \left| \frac{\partial^2 y}{\partial t^2} \right| \, dt \right)^{1/n} \left( \int_0^1 |y| \, dt \right)^{(n-1)/n}
\]

proved in \([8]\), which implies inequality of the form

\[
\int_0^1 |\frac{\partial y}{\partial t}| \, dt \leq c \int_0^1 \left| \frac{\partial^2 y}{\partial t^2} \right| \, dt + C \varepsilon \left( \int_0^1 |y| \, dt \right),
\]

where \(C\) is a fixed constant, and \(\varepsilon > 0\) is arbitrary. This last inequality follows also from compactness of the Sobolev spaces imbedding \(W^{n,1} \hookrightarrow W^{1,1}\) for \(n > 1\). Thus, \(|y|_1 \leq c |y|_0 + \frac{C}{\varepsilon} |g|_0\), and we obtain \((14)\) from \((15)\) provided that the chosen \(\varepsilon\) is sufficiently small. This proves the induction step and Lemma 2.

In order to get from the estimate of Lemma 2 to \((11)\) we denote by \(z_g\), where \(g \in L_1\), the solution \(\dot{z}_g = \alpha z_g + g\) with the initial condition \(z_g(0) = z(0)\). It is clear, e.g., from the Cauchy formula, that \(|z - z_g|_0 \ll |f - g|_0\), and, therefore, \(|\beta z - \beta z_g|_0 \ll |f - g|_0\). Thus, the following strengthening of the Lemma 2 holds:

**Lemma 3.** Suppose that \(\dot{z} = Ax, y = Cx\) is a completely observable time-invariant linear system. Then, for a solution \(z\) of \(\dot{z} = Ax + f\) in the interval \([0, 1]\) the a priori estimate \(|z|_0 \ll |\beta z|_0 + \inf\{|f - g|_0 + |g|_n\}\) take place, where \(\inf\) is taken over \(g \in W^{n,1}\), and \(n\) is the dimension of the phase space.
It is obvious, that \(\inf\{ |f - g|_0 + |g|_n \} \leq |f|_0\) and we arrive at estimate (11) from Lemma [3].

Now it remains to prove Lemma 1. Note that the change of parameters
\[
A \mapsto A + \gamma C, \quad C \mapsto C, \quad A \mapsto \delta A \delta^{-1}, \quad C \mapsto C \delta,
\]
where \(\gamma\) is an arbitrary matrix, and \(\delta\) is an arbitrary invertible matrix does not affect the validity of Lemma 1 because it corresponds to substitutions \(f \mapsto f + \gamma y,\) and \(z \mapsto \delta z.\) In view of the Brunovsky normal form [9], and the Kalman duality between controllability and observability we may assume that the observable system takes the form of direct sum of systems with a scalar observation of the form
\[
\begin{align*}
\dot{z}_1 &= z_2 + f_1 \\
\vdots & \\
\dot{z}_{n-1} &= z_n + f_{n-1} \\
\dot{z}_n &= f_n
\end{align*}
\]
\[y = z_1.\]

This allows to prove by induction that the defining relation of Lemma 1 holds in the form of direct sum of equations of the form
\[
\frac{\partial^n}{\partial t^n}y = \sum_{k=0}^{n} \frac{\partial^k}{\partial t^k}f_{n-k}.
\]

3. Application

We regard equation (8) as a perturbed completely observable linear system (10), where the phase vector \(x = p,\) matrices \(A = -A^*, \ C = B^*,\) observation \(y = B^*p = \langle p, B \rangle,\) and perturbation \(f = \tilde{B}u.\) Assume, that in the entire time interval \(I\) of integer length \(T\) the motion of the state vector \(x\) takes place within the domain \(\rho(x) \geq C.\) Then \(|f| = O(1/C)\) in the entire interval. Moreover, eikonal equation (6) holds for \(p,\) and, therefore, \(1 \ll |p|\) and \(T \ll \int_I |p|dt\) (here \(\ll\) is the Vinogradov symbol, meaning \(O(\text{RHS}).\) The estimate of the main Theorem 1 gives that
\[
T \ll \int_I |p|dt \ll \int_I |\langle p, B \rangle|dt + \frac{1}{C} T.
\]
By taking a sufficiently large constant \(C = C(A,B)\) we obtain, that
\[
T \ll \int_I |\langle p, B \rangle|dt.
\]

This is the inequality (7) in another notation. To be clear, we restate the result:

**Theorem 2.** Suppose that system (11) moves from the level set \(\rho = M\) to the level set \(\rho = N\) under control (4), and \(M, N \geq C(A,B),\) in the time interval of integer length \(T,\) where \(C(A,B)\) is a (sufficiently large) constant, depending only on parameters of system (11). Then, \(T \leq c(M - N),\) where \(c = c(A,B)\) is a strictly positive constant.

**Acknowledgements**

This work was supported by Russian Foundation for Basic Research (14-08-00606).
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