Numerical Solution of Magnetohydrodynamic Flow and Heat transfer of Sisko Fluid over an Exponential Stretching Sheet

Yogesh Dadhich 1 and Reema Jain 2
1,2 Department of Mathematics and Statistics, Manipal University Jaipur, Jaipur (Rajasthan), India -303007

Corresponding author: reemajain197@gmail.com

Abstract. The present work explores the numerical solution of magnetohydrodynamic (MHD) flow and heat transfer of Sisko fluid model over an exponential stretching surface. The flow is produced by the stretching of exponential surface in the presence of a uniform magnetic field. The governing equations are converted to nonlinear ordinary differential equations, using some appropriate dimensionless variables. The solution of the reduced nonlinear ordinary differential equations are found by numerical technique using bvp4c solver with MATLAB. The impact of various physical factors like magnetic parameter, material parameter, power law index & Prandtl number on the velocity and temperature profiles are shown graphically and elaborated.

Key words Sisko fluid model, Boundary layer flow, MHD, Exponential stretching sheet, Heat transfer, bvp4c solver.

1. Introduction

The study of non-Newtonian fluids has become progressively significant due to their large number of technological applications including manufacturing of plastic sheets, performance of lubricants and motion of biological fluids. At first, a deep insight was developed by Sarpkaya [1] on magnetohydrodynamic flow of non-Newtonian fluid and then followed by other researchers. Liao [2] made an effort to get an analytical solution of magnetohydrodynamic flows of non-Newtonian fluids over a linearly stretching surface. Academic inquisitiveness and real-world practices have nurtured huge curiosity in finding the solutions of differential equations governing the motion of non-Newtonian fluids therefore various fluid models have been anticipated to show the behavior of non-Newtonian fluids [3, 4, 5]. The prominence of non-Newtonian fluids is notable but the importance of Newtonian fluids can never be overlooked. So in present discussion, the Sisko fluid model is taken into account which can envisage the characteristics of Newtonian as well as non-Newtonian fluid models. Sisko fluid model is primarily an extension of power law model of non-Newtonian fluids and was suggested by Sisko [6] in 1958. It is basically applied in oil industry. Recently, Khan et al. [7] discussed the time independent flow & heat transfer of Sisko fluid model in annular pipe geometry and derived many significant results. They found that the sturdy shear coagulating effects becomes sturdier by enhancing the power law index of fluid. Akyildiz et al. [8] presented their work on the solution of implicit differential equation arising in the time independent flow of Sisko fluid model. Nadeem et al. [9] explored the peristaltic of Sisko fluid in a uniform inclined tube. They studied it for various values of the power law index and concluded that the Newtonian fluid has the finest propelling traits. Munir et al. [10] explored the effects of buoyancy in Sisko fluid with viscous dissipation. Khan et al. [11, 12] also mentioned some interesting
results in the study of Sisko fluid flow over a radially stretching sheet. Mahmood et al. [13] also examined the two dimensional MHD flow and heat transfer of Sisko fluid model over time dependent stretching surface and discussed the effects of various quantities on flow and temperature distributions.

The flow over an exponentially stretching sheet has gained remarkable thought due to its extensive use in manufacturing and technological procedures such as fluid film condensation, aerodynamic extrusion of plastic sheets, refrigeration process of metallic sheets etc. Fazel et al. [14] carried out the research on MHD flow over an exponential radiating stretching sheet and derived some significant results.

After reviewing the above-mentioned literature authors observed that so far no effort is made for the study of Sisko fluid flow over an exponential stretching surface. This investigation is executed to bridge this gap. The present work provides a numerical solution [15] of magnetohydrodynamic (MHD) flow and heat transfer of above mentioned fluid model over an exponential stretching surface. The flow is caused by an exponential stretching sheeting which has not been considered before. For numerical solution, bvp4c solver [13] is used and the impact of various physical measures on fluid velocity and temperature are discussed in length.

2. Mathematical Modelling
Here we discuss a time independent, 2D laminar flow of Sisko fluid caused by an exponential stretching surface. The fluid is assumed incompressible and electrically conducting. The x-axis is taken in horizontal direction and y-axis in vertical direction i.e. normal to the sheet. A magnetic field of uniform strength $B_0$, is worked normal to the sheet and is small enough so induction effect can be overlooked. The sheet is spread along x-axis and y axis is taken normal to it. The horizontal and vertical components of velocity are taken as $u$ & $v$ respectively. The flow is produced by the stretching of the sheet in horizontal direction. The physical model of the current problem is shown in the Figure 1.

![Flow configuration of the current problem](image)

The governing boundary layer equations i.e. equation of continuity, equation of momentum and equation of energy for two dimensional of Sisko fluid are presented as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$  \hspace{1cm} (1)

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\sigma_0 a^2 u}{\rho} \frac{\partial^2 u}{\partial y^2} - b \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right)$$  \hspace{1cm} (2)

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho C_p} \frac{\partial^2 T}{\partial y^2} \quad \text{where} \quad \alpha = \frac{k}{\rho C_p}$$  \hspace{1cm} (3)

The corresponding boundary conditions are:

at $y = 0$, $u = u_w = u_0 e^{x/L}$; $T = T_w = T_\infty + T_0 e^{x/2L}$; $v = 0$  \hspace{1cm} (4)

at $y \to \infty$, $u = 0$, $T \to T_\infty$  \hspace{1cm} (5)

Here, $a$ & $b$ are material constants; $u_w = u_0 e^{x/L}$ is the stretching velocity of the sheet and $T_w = T_\infty + T_0 e^{x/2L}$ sheet temperature where $u_0$ & $T_0$ are constants $T_\infty$ is the free stream and $L$ is the characteristic length. $\alpha = \frac{k}{\rho C_p}$ is the thermal diffusivity; $C_p$ is the specific heat of the fluid and $k$ is thermal conductivity.
The equation of continuity is satisfied by introducing a stream function $\Psi(x, y)$ such that
\[ u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \]

To get the non-dimensional form of above mentioned equations and boundary conditions, the following transformations are used by Sajid et al. [16]:
\[ \eta = y_0 \left[ \frac{u_0}{2lp} \right] e^{x/2L} ; \]
\[ u = u_0 e^{x/L} f'(\eta) ; \]
\[ v = -\frac{u_0 \theta}{2L} e^{x/2L} [f(\eta) + \eta f'(\eta)] ; \]
\[ \theta(\eta) = \frac{\tau - \tau_{\infty}}{\tau_{\infty} - \tau_{\infty}} ; \]
where $\eta$ is the similarity parameter; $f(\eta)$ is dimensionless stream function; $\theta(\eta)$ is dimensionless temperature and $f'(\eta)$ shows the derivative of $f(\eta)$.

Applying similarity variables equations (2)-(5) are transformed into the following nonlinear ordinary differential equations:
\[ Af''' + n(-f'')^{n-1}f''' - M^2 f' - 2f f'' + ff''' = 0 \] \hspace{1cm} (7)
\[ \theta'' = P_r (f' - f \theta') \] \hspace{1cm} (8)

and the reduced boundary conditions are
at $\eta = 0$ : $f'(\eta) = 1$; $f(\eta) = 0$; $\theta(\eta) = 1$
for $\eta \to \infty$ : $f'(\eta) = 0$; $\theta(\eta) = 0$ \hspace{1cm} (9)

here $A = \frac{\rho e^{2n+1}}{Re_a} = \frac{a}{2 \omega \rho}$ is the material parameter; $M = \frac{2 \sigma B_0^2 L}{\rho u_w}$ is the magnetic field parameter; $P_r = \frac{\sigma}{\alpha}$ is the Prandtl number; $Re_a = \frac{\rho x u_w}{a}$ & $Re_b = \frac{\rho x u_w}{b}$ are local Reynolds numbers.

3. Numerical Solution
The numerical solutions of equations (7)-(8) under the given boundary conditions (9)-(10) are obtained by applying bvp4c MATLAB solver. It is a numerical technique which can help us to solve a given set of ordinary differential equations. In order to implement the bvp4c coding, the system of equations (7)-(10) are converted to first order differential equations (11)-(17) as appended below:

\[ f = f(1) \] \hspace{1cm} (11)
\[ f' = f'(1) = f(2) \] \hspace{1cm} (12)
\[ f'' = f''(2) = f(3) \] \hspace{1cm} (13)
\[ \theta = f(4) \] \hspace{1cm} (14)
\[ \theta' = f'(4) = f(5) \] \hspace{1cm} (15)
\[ f''' = f'(3) = \frac{M^2 f(2)+2[f(2)]^{2}-f(1)f(3)}{[4+n(-f(3))^{n-1}]} \] \hspace{1cm} (16)
\[ \theta'' = f'(4) = P_r [f(4)f(2) - f(1)f(5)] \] \hspace{1cm} (17)

with boundary conditions
at $\eta = 0$ : $f_0(2) = 1$, $f_0(1) = 0$, $f_0(4) = 1$
for $\eta \to \infty$ : $f_\infty(2) = 0$, $f_\infty(4) = 0$. \hspace{1cm} (18)

Since, we have five ODEs therefore five initial conditions are required to get the solution. But in equations (18)-(19), we have three initial conditions only. So, appropriate initial guesses are supposed for $f_0(3)$ and $f_0(5)$. Also, we deal with the boundary condition at infinity by imposing it at a finite point, namely $\eta_\infty = 10$ i.e. the domain of the problem is restricted to $[0, 10]$. The computed solution will converge if the error in the calculated values of $f_\infty(2)$ and $f_\infty(4)$ is less than $10^{-6}$. If it is more than error tolerance then initial assumptions are changed and the process is repeated till the solution criterion is achieved.
4. Result & discussion

In order to get the definite perception of the stated problem, velocity and temperature profiles are shown graphically for significant physical quantities. All the results are calculated for Pseudoplastic i.e. shear thinning fluid (n < 1), Newtonian fluid (n = 1) and Dilatant i.e. shear thickening fluid (n > 1) fluid. These numerical solutions are obtained using bvp4c solver with MATLAB.

The influence of magnetic field $M$ on fluid velocity is discussed in figures 2(a-c) for shear thinning, Newtonian and shear thickening fluids. It is clear from these plots that an increase in the parameter $M$ causes a sharp decline in velocity of the fluid. The reason is, when transverse magnetic field is applied to an electrically conducting fluid then it produces a resistive type of force which is identified as Lorentz force. So, the resistive force becomes more powerful when $M$ increases. Thus fluid motion is resisted and hence the velocity.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Variation of velocity for changed values of $M$}
\end{figure}

Figures 3(a-c) depict the influence of magnetic field on temperature distribution for various types of fluids. It is evident that fluid temperature boosts with the increase of magnetic field parameter $M$. The reason is, when magnetic field strength is increased, it increases resistive force i.e. Lorentz force. Consequently, more heat is generated in the fluid which gives increase in temperature.
Influence of Prandtl number $P_r$ is discussed in figures 4(a-c) for different values of $n$ i.e. shear thinning, Newtonian and shear thickening fluids. It is evident from these graphs that an increment in Prandtl number $P_r$, causes decline in the fluid temperature. Since, the large Prandtl number shows less thermal diffusivity and high momentum transportation. So, the fluid with large $P_r$ causes low heat diffusion and high momentum transportation which results in the increment of convective heat transfer and reduction of conductive heat transfer. As a consequence, the fluid temperature declines.
Figures 4(a-c) demonstrate that the fluid velocity increases with the increase of the value of Sisko fluid parameter $A$ (material parameter). Since, the material parameter is inversely proportional to the consistency index $b$ i.e. viscosity of fluid. So increase in $A$, reduces the viscosity of fluid, so smaller resistive force is offered to the fluid motion. Hence, the fluid velocity increases.

Figures 5(a-c) depict the impact of Sisko fluid parameter $A$ (material parameter) on fluid temperature. The falloff in temperature of fluid is observed with an increase in the material parameter $A$. 

**Figure 4(a-c).** Variation of temperature for changed values of $P_f$.

**Figure 5(a-c).** Variation of velocity for changed values of material parameter $A$.
Figure 6(a-c). Variation of temperature for changed values of material parameter $A$

It is also clear from figures 7(a-b) that the rate of fluid flow diminishes as the value of flow behavior index $n$ shoots up for $n \geq 1$. As an increment in $n$ makes the fluid becomes thicker, hence resist the flow. Whereas for $n < 1$, we can see two unlike phenomena i.e. in the close vicinity of the sheet, fluid velocity rises while away from the sheet it drops with increase of $n$.

Figure 7(a-b). Variation of velocity for changed values of power law index $n$

Figures 8(a-b) are plotted in order to show the effect of flow behavior index $n$ on fluid temperature. There is drop in temperature of fluid with respect to rise in the power law index $n$. 

7
Figure 8(a-b). Variation of temperature for changed values of power law index $n$

5. Conclusion
Here, magnetohydrodynamic (MHD) flow and heat transfer of incompressible Sisko fluid over an exponential stretching surface is analysed. The flow is considered steady and in the presence of a uniform transverse magnetic field. The solution of the reduced equations are obtained using numerical technique. Few major findings are mentioned here:

- Effects of material parameter $A$ and magnetic field parameter $M$ on velocity profile are totally different. An increase of magnetic field parameter $M$ decreases velocity while increase of $A$ causes increase in velocity.
- As power law index $n$ increases, the velocity decreases for $n \geq 1$ and temperature decreases.
- An increase in magnetic field parameter $M$, gives rise in temperature field.
- The impact of $Pr$ and flow behavior index $n$ is to decrease the temperature field.
- Progression in material parameter $A$ is to reduce the temperature profile.

References
[1] Sarpkaya T 1961 AIChE Journal 7 324-328
[2] Liao S J 2003 J Fluid Mech. 488 189-212
[3] Ahmed J, Begum A and Shahzad A 2016 Results in Physics 6 973–981
[4] Adamu Gizachew and Bandari Shankar 2018 Advances in Applied Sciences 3(3) 34-42
[5] Ahmad K and Ishak A 2016 Malaysian Journal of Mathematical Sciences 10(S) 311–23
[6] Sisko A W 1958 Industrial and. Engineering Chemistry 50 1789-92
[7] Khan M, Munawar S and Abbaspandy S 2010 International Journal of Heat and Mass Transfer 53 1290-97
[8] Akyildiz F T, Vajravelu K, Mohapatra R N, Sweet E, and Van Gorder R A 2009 Applied Mathematics and Computation 210 189-96
[9] Nadeem S and Akbar N S 2010 Acta Mechanica Sinica 26 (5) 675-83
[10] Munir A, Shahzad A and Khan M 2015 Chemical Engineering Research and Design 97 120-27
[11] Khan M and Shahzad A 2012 International Journal of Nonlinear Mechanics 47 999–07
[12] Khan M and Shahzad 2013 Quaestiones Mathematicae 36 137–51
[13] Mahmood T, Shahzad A and Iqbal Z 2017 Results in Physics 7 832-42
[14] Fazel Mabood and Khan W A 2017 Journal of King Saud University-Engineering Phy. 29 68-74
[15] Gireeshaa B J, Mahanthesh B, Manjunatha P T and Gorla R S R 2015 Journal of Nigerian Math Society 34 267–85
[16] Sajid M and Hayat T 2008 Int. Communications in Heat and Mass Transfer 35(3) 347-56