Enhanced quasi-particle current of Bogoliubov phonons in a Bose-Einstein condensate

Shunji Tsuchiya\textsuperscript{1,2} and Yoji Ohashi\textsuperscript{1,2}

\textsuperscript{1}Department of Physics, Keio University, 3-14-1, Hiyoshi, Kohoku-ku, Yokohama, 223-8522, Japan
\textsuperscript{2}CREST(JST), 4-1-8, Honcho, Saitama, 332-0012, Japan

E-mail: tsuchiya@rk.phys.keio.ac.jp, yohashi@rk.phys.keio.ac.jp

Abstract. We discuss tunneling properties of Bogoliubov excitations through a potential barrier. We find that the quasi-particle current carried by Bogoliubov excitations is remarkably enhanced near the potential barrier at low incident energies. This explains the increase of the transmission probability of Bogoliubov phonons in the low energy region. In addition, taking into account of the backreaction of Bogoliubov excitations on Bose condensates, we show that the quasi-particle current twists the phase of the condensate wavefunction, leading to a Josephson supercurrent through the barrier.

1. Introduction

Bogoliubov excitations play important roles for the low energy properties of Bose superfluids \cite{1}. Since the realization of Bose-Einstein condensates (BECs) in ultracold atomic gases, research of Bogoliubov excitations have been one of the main issues in this field \cite{2, 3}.

Recently, tunneling effects of Bogoliubov excitations have attracted much interest \cite{4, 5, 6, 7, 8, 9, 10, 11, 12}. Kovrizhin and co-workers \cite{4, 5, 6} considered tunneling of Bogoliubov excitations between two condensates separated by a potential barrier. They showed that the transmission probability increases at low energies and Bogoliubov excitations exhibit a perfect transmission in the low energy limit. This striking property is called anomalous tunneling \cite{6}. The origin of the anomalous tunneling has been addressed in several papers \cite{6, 7, 8, 9, 10, 11, 12}, and the anomalous tunneling has been extended to various cases \cite{7, 8, 9, 10, 11, 12}.

In this paper, we investigate tunneling properties of Bogoliubov excitations by calculating the quasi-particle current. We show that in the tunneling process the quasi-particle current carried by Bogoliubov excitations is not conserved but enhanced near the potential barrier at low energies due to the supply from the condensate. This can be considered as the origin of the increase of the transmission probability at low energies discussed in \cite{6}. We find that the backreaction effect of Bogoliubov excitations on Bose condensates has to be taken into account for the conservation of total current. As a result, we find that the phase of the condensate wavefunction is twisted by tunneling Bogoliubov excitations, leading to the counter flow of Josephson supercurrent through the barrier.
2. Formalism
We investigate tunneling of Bogoliubov excitations through a potential barrier at $T = 0$ by the Bogoliubov mean-field theory [2]. We consider a BEC in a 3D system separated by a one-dimensional potential barrier which only depends on $x$. We assume excitations are allowed to move only in the $x$-direction. For simplicity, we ignore effects of a harmonic trap.

From the Bogoliubov mean-field theory [2], the field operator for bosons can be written as a sum of the condensate wavefunction $\Psi_0(x)$ and the non-condensate part, as $\hat{\psi}(x) = \Psi_0(x) + \hat{\delta}\hat{\psi}(x)$, where $\hat{\delta}\hat{\psi}(x) = \sum_j[u_j(x)\hat{\alpha}_j - \hat{\alpha}_j^\dagger v_j(x)]$. Here, $\hat{\alpha}_j$ is a creation operator of an excitation in the $j$-th state. ($u_j, v_j$) can be regarded as wavefunctions for excitations.

The condensate wavefunction $\Psi_0(x) = \langle \hat{\psi}(x) \rangle$ satisfies the static Gross-Pitaevskii (GP) equation [2], given by (we set $\hbar = 1$)

$$
\left(-\frac{1}{2m}\frac{d^2}{dx^2} + U(x) + g|\Psi_0|^2\right)\Psi_0 = \mu \Psi_0,
$$

where $m$, $\mu$, and $U(x)$ are the mass of a boson, chemical potential, and a potential barrier, respectively. $g(> 0)$ is a repulsive interaction between bosons. We assume a simple rectangular potential barrier $U(x) = U_0 \theta(d/2 - |x|)$ ($U_0 > 0$). In the ground state, $\Psi_0$ can be taken to be real. The analytic solution of Eq. (1) in the ground state was obtained in Ref. [6]. It is employed in the following calculation.

The amplitudes $(u_j, v_j)$ are determined by the Bogoliubov equations [2]:

$$
\begin{pmatrix}
\hat{\hbar} & -g\Psi_0^2 \\
-g^2\Psi_0 & \hbar
\end{pmatrix}
\begin{pmatrix}
u_j \\
v_j
\end{pmatrix}
= E_j \tau_3
\begin{pmatrix}
u_j \\
v_j
\end{pmatrix},
$$

where $\hat{\hbar} = -\frac{1}{2m}\frac{d^2}{dx^2} + U(x) + 2g|\Psi_0|^2 - \mu$. $\tau_3$ is the Pauli matrix, and $E_j$ is the excitation energy. In a uniform system, we obtain the chemical potential $\mu = gn_0$ from Eq. (1) and the well-known Bogoliubov excitation spectrum $E = \sqrt{\xi_p(\xi_p + 2gn_0)}$ from Eq. (2), where $n_0$ is the condensate density and $\xi_p = p^2/2m$.

We assume that an incident Bogoliubov excitation with energy $E$ is coming from the left side of the barrier. The asymptotic solution of Eq. (2) is give by

$$
\begin{cases}
\begin{pmatrix}
u \\
v
\end{pmatrix} = \begin{pmatrix}
a \\
b
\end{pmatrix} e^{ikx} + r \begin{pmatrix}
a \\
b
\end{pmatrix} e^{-ikx} + A \begin{pmatrix}
-b \\
a
\end{pmatrix} e^{\kappa x}, & (x \to -\infty), \\
\begin{pmatrix}
u \\
v
\end{pmatrix} = \begin{pmatrix}
a \\
b
\end{pmatrix} e^{ikx} + B \begin{pmatrix}
-b \\
a
\end{pmatrix} e^{-\kappa x}, & (x \to \infty),
\end{cases}
$$

where $k = \sqrt{2m\sqrt{E^2 + (gn_0)^2} - gn_0}$, $\kappa = \sqrt{2m\sqrt{E^2 + (gn_0)^2} + gn_0}$, $a = \sqrt{(\sqrt{E^2 + (gn_0)^2}/E + 1)/2V}$, and $b = \sqrt{(\sqrt{E^2 + (gn_0)^2}/E - 1)/2V}$ ($V$ is the volume of the system). Here, $r$ and $t$ are the reflection and transmission amplitudes, respectively. They satisfy the condition $|r|^2 + |t|^2 = 1$ [6]. The last terms in the right hand sides of Eq. (3) describe the localized components. We note that they appear only in an inhomogeneous system.

Assuming the asymptotic solution in Eq. (3), we numerically solve the Bogoliubov equation Eq. (2) by the finite element method.

3. Anomalous tunneling and quasi-particle current
The calculated transmission probability $W \equiv |t|^2$ and phase shift $\delta \equiv \arg(t)$ are shown in Fig. 1 as functions of the incident energy $E$ for various sets of $(d, U_0)$. One can clearly see that, at low
incident energies, $W(\delta)$ increases (decreases) as $E$ decreases. In the low energy limit ($E \to 0$), a perfect transmission is achieved as $W \to 1$ and $\delta \to 0$. Thus, the potential barrier turns out to be transparent for low-energy Bogoliubov phonons irrespective of the detail of the barrier. This was first shown by Kovrizhin et al. [4] and referred to as anomalous tunneling [6].

![Figure 1](image1.png) ![Figure 2](image2.png)

**Figure 1.** Transmission probability $W$ (upper panel) and phase shift $\delta$ (lower panel) of Bogoliubov excitations as functions of the incident energy $E$.

**Figure 2.** Excess quasi-particle current $\Delta J_q(x)$ (upper panel) and source term $S(x)$ (lower panel) when $(d, U_0) = (\xi, 10\mu)$.

We investigate the tunneling properties of low-energy Bogoliubov phonons by calculating the quasi-particle current $[11] J_q = J_u - J_v$, where $J_u = (1/m)\text{Im}(u^* \partial_x u)$ and $J_v = (1/m)\text{Im}(v^* \partial_x v)$. From Eq. (2), the continuity equation for quasi-particles is given by $\partial_x J_q = S$, where $S = -4g \text{Im}(\Psi^0_0 u^* v)$ describes the source for quasi-particles. This continuity equation means that the number of quasi-particles (Bogoliubov excitations) is not conserved if $S$ is finite in an inhomogeneous system. (Note that $S = 0$ and $J_q$ is conserved in a uniform system.) Indeed, we will see that this is the case for our tunneling problem.

In Fig.2, the excess quasi-particle current $\Delta J_q(x) = J_q(x) - J_q(x = -\infty)$ and source term $S(x)$ are plotted for several incident energies for $(d, U_0) = (\xi, 10\mu)$. We find that $S$ is positive (negative) on the left (right) side of the potential barrier. From the relation $\Delta J_q = \int_{-\infty}^{x} S(y) \, dy$, $J_q$ is supplied and enhanced near the potential barrier at low energies. This enhancement of $J_q$ becomes remarkable as the incident energy becomes smaller, which leads to the increase of the transmission probability of Bogoliubov excitations in Fig.1. In fact, comparing Fig.2 with Fig.1, one finds that the energy at which $W$ starts increasing with decreasing $E$ in Fig.1 coincides with the energy at which $\Delta J_q$ starts being pronounced in Fig.2 ($E \simeq 0.1\mu$).

### 4. Josephson counterflow

The excess quasiparticle current $J_q$ is supplied from the condensate [11]. It was pointed out in [11] that the Bogoliubov mean-field theory based on Eqs. (1) and (2) do not satisfy the conservation of total current given by $J = J_s + \sum_j J_{q_j}(\hat{\alpha}_j^\dagger \hat{\alpha}_j) - \sum_j J_{v_j}$, where $J_s = (1/m)\text{Im}(\Psi^0_0 \partial_x \Psi^0_0)$ is the supercurrent carried by a condensate. This inconsistency was eliminated by considering the backreaction of Bogoliubov excitations on Bose condensates. The GP equation is modified by adding the quasi-particle contribution in the anomalous average term.
The conservation of the total current requires the counter-flow of supercurrent 
\[ \Delta J_s = -\sum_j J_{ij} \langle \hat{a}_j \hat{\alpha} \rangle \] induced at the barrier region.

The spatial variation of the normalized phase shift \( N_0 \theta(x) \) (\( N_0 \) is the number of condensate atoms) is plotted in the inset of Fig.3, under the assumption that the condensate wavefunction is changed by a phase \( \theta(x) \) by the backreaction of quasi-particles. One can see that \( \theta(x) \) sharply changes at the barrier region, and a phase difference \( \phi \) exists between the condensates across the barrier. This clearly shows that the tunneling Bogoliubov phonons twist the relative phase of the condensates on the left and right sides. The induced supercurrent \( \Delta J_s \) can be regarded as a Josephson current due to the phase difference \( \phi \). In fact, the supercurrent under the barrier region \( \Delta J_s(x = 0) \) is proportional to the phase difference \( \phi \) at low energies.

![Figure 3. Phase difference across the barrier as a function of the energy \( E \). \( N_0 \) is the number of condensate atoms. The spatial variation of the condensate phase \( N_0 \theta(x) \) for \((d, U_0) = (\xi, 10\mu)\) and \( E = 0.01\mu \) is shown in the inset.](image)

In Fig. 3, the normalized phase difference \( N_0 \phi \) is plotted as a function of \( E \). It shows that \( \phi \) is enhanced for low energy Bogoliubov phonons. This implies that at low energies the coupling effect of Bogoliubov phonons and Bose condensates becomes large, because of the collective nature of phonon excitations. Thus, low energy Bogoliubov phonons create a large phase difference. We also note that \( \phi \) increases as the potential barrier gets higher in Fig.3. Since the phase of the condensates across the barrier becomes weaker for a high potential barrier, the phase of the condensate wavefunction is easily twisted by tunneling Bogoliubov excitations.

5. Conclusion

We have studied tunneling effects of Bogoliubov excitations at \( T = 0 \). We found the anomalous enhancement of the quasi-particle current supplied from the condensate near a potential barrier at low incident energies. This enhancement explains the increase of the transmission probability of Bogoliubov excitations at low energies in [6]. We also found that the tunneling Bogoliubov excitations twist the condensate phase and induce the Josephson supercurrent across the barrier.

S.T. thanks I. Danshita, K. Kamide, S. Inouye, and F. Dalfovo for useful discussions.

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