Free field resolution for nonunitary representations of $N=2$ SuperVirasoro

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We study $N=2$ SuperVirasoro SCFT for the generic value of the central charge. The main tool is the nonstandard bosonisation suggested in [1], [2], [3]. The free field resolutions for the irreducible representations are obtained; the characters of these representations are computed. The quantum hamiltonian reduction from the Kac-Moody $\hat{sl}_k(2|1)$ to $N=2 SVir$ is constructed.
1. Introduction

In dealing with N=2 SCFT, we usually restrict ourselves to the study of the unitary theories. One of the reasons is that the representation theory of N=2 SuperVirasoro (from now on called SV\textit{ir}) is much understood for the unitary case only \cite{1,2}. The remarkable feature of these models is the existence of chiral rings \cite{6}.

Recently it was found \cite{1,2,3}, that N=2 superconformal algebra plays an important role in 2d gravity coupled to matter. In fact, any noncritical (as well as critical) string model possesses this symmetry. It is important, that the unitary string string theory considered as an N=2 SCFT is (generically) nonunitary\footnote{Though in some interesting cases it is unitary \cite{7,8}, already in the simplest example of (Vir-)minimal matter coupled to gravity it is not.}. Therefore it is interesting to understand such theories better.

In the second section of this paper we introduce the free field resolutions of all highest weight representations of N=2 SV\textit{ir} using a ”new” bosonisation of \cite{1,2,3}. For the most degenerate representations (parameterized by three integer numbers), the corresponding (Felder type) BRST complex prove to be one-sided infinite. There are also three non-countable families of the less degenerate representations (having only one singular vector in the Verma module).

In \cite{4}, it was suggested, that there exists a hamiltonian reduction from the Kac-Moody $\hat{sl}_k(2|1)$ to N=2 SV\textit{ir}. It was shown that this is the case in the classical limit ($k \to \infty$). Appendix deals with the quantum version of the reduction. The resolution of the vacuum module obtained in the Sec.2 gives us a characterization of the embedding of N=2 SV\textit{ir} into the chiral algebra of the free fields, defined by the bosonisation formulas. Using this characterization we show that the quantum reduction exists indeed.

Finally, it should be noted that we work with N=2 SV\textit{ir} in a basis, naturally appearing in a context of the 2d quantum gravity. In this basis, the algebra is called ”twisted”, although it is the same SV\textit{ir}.

2. The free field resolutions

In this section we consider the bosonisation rules, associated with the used in \cite{1,2} presentation of N=2 SV\textit{ir}. Exactly the same bosonisation was introduced earlier by L. Rozansky \cite{3}. Still we prefer to start with \cite{2}, because it makes the whole construction more...
transparent. So suppose we have a system consisting of the Virasoro algebra ($Vir$) — a matter sector, a free bosonic field $\phi$ with the background charge ($Heis$) — a Liouville sector and a pair of fermions $b, c$ of spins 2,-1 ($Clif$) — the diffeomorphism ghosts (in the brackets are the names of the corresponding chiral algebras). We require the total central charge be equal to zero. Then the currents

$$J(z) =: cb : + 2\alpha_- \partial \phi$$

$$G^+(z) =: c[T_{Vir} + T_\phi + \frac{1}{2} T_{bc}] : -2\alpha_- \partial (c \partial \phi) + \frac{1}{2} (1 - 2\alpha_-^2) \partial^2 c$$

$$G^-(z) = b(z)$$

$$T = T_{Vir} + T_\phi + T_{bc}$$

$$T_\phi = -\frac{1}{4} : (\partial \phi)^2 : + \beta_0 \partial^2 \phi$$

satisfy the OPE of (twisted) N=2 $SVir$ chiral algebra. In fact these formulas give the embedding of N=2 $SVir$ into the tensor product of three other chiral algebras $Vir \otimes Heis \otimes Clif$. It should be stressed that there is such embedding already for the Virasoro matter, with no bosonisation at all. It describes a hidden N=2 superconformal symmetry of any string model. But our aim here is to use the map above to understand better the representation theory of N=2 $SVir$ itself, rather than the structure of the string models.

To do so, it is convenient to bosonize the matter by the free field $X$ with the background charge $\alpha_0$. In other words we embed $Vir$ into the Heisenberg algebra generated by $\partial X$ which we denote by $Heis'$ to distinguish it from the Liouville $Heis$. Substituting the bosonized matter stress energy

$$T_{Vir}(z) = \frac{1}{4} : (\partial X)^2 : + \alpha_0 X$$

$$\beta_0^2 - \alpha_0^2 = 1, \quad \alpha_\pm = \alpha_0 \pm \beta_0$$

$$c_{Vir} = 1 - 24\alpha_0^2$$

into (2.1) we finally obtain the bosonisation prescriptions we are going to use. Unlike the standard bosonisation [3], [4], the formulas for $G^+(x)$ and $G^-(z)$ are very asymmetric. It is a whole story at the level of chiral algebras.

For the representations, we take a Fock space $F_{\alpha\beta} = F_\alpha \otimes F_\beta \otimes F_{gh}$. $F_\alpha$ and $F_\beta$ here are the standard Fock modules of $Heis'$ and $Heis$ with vacuums $|\alpha >$ and $|\beta >$ respectively and $F_{gh}$ is a ghosts Fock space (a $Clif$ Verma module) with the vacuum vector $|0 >$ annihilated by

$$c_n |0 >= 0 \quad n > 1, \quad b_n |0 >= 0 \quad n > -2$$

(2.3)
In $F_{gh}$ we take a vector $|0>_{phys}=c_1|0>$ and define the N=2 vacuum as $Ω = |α> ⊗ |β> ⊗ |0>_{phys}$. This procedure is well known in string theory. Here we use it to endow the free field Fock space $F_{αβ}$ with a structure of a highest weight N=2 $SVir$ module:

$$L_nΩ = J_nΩ = G_n^- = 0, \ n ≥ 0$$

$$G_n^+Ω = 0, \ n > 0$$

$$L_0Ω = (−1 + α(α − 2α_0) − β(β − 2β_0))$$

$$J_0Ω = (1 + α − β)$$

(2.4)

There are two screening operators in our bosonisation. One of them is just $E = \oint \partial : e^αX(z) :$. It comes from the bosonisation of the $Vir$ matter. The other one is $F_1 = \oint \partial : e^{−α/2}(X(z)+φ(z)) : [2],[3],[9]$. It is fermionic and local to itself:

$$F_1^2 = 0$$

(2.5)

When taken together, $E$ and $F_1$ form a quantum superalgebra $u_q(n_+(sl(2|1)))$ with $q = e^{πiα_+^2}$. Namely they satisfy the Serre relation:

$$E^2F_1 − (q + q^{-1})EF_1E + F_1E^2 = 0$$

(2.6)

(As usual, the left hand side of (2.6) is to be understood as a part of some formal polynomial in screenings acting on the appropriate state, see below.)

We will use $E,F_1$ to construct a family of maps between the Fock spaces $F_{αβ}$. Any such maps from the Fock space $F_{αβ}$ can be presented by the polynomial in generators $e,f$ of $u_q(n_+(sl(2|1)))$ which is, by the definition, just the element of $u_q(n_+(sl(2|1)))$. The correspondence between the polynomials and the maps is given by the following integral representation. On the monomials $e_1...fe_{i+1}...fe_{j+1}...e_n$ it is defined as the integral

$$\int_{z_1}...\int_{z_n} : e^{α+X(z_1)} ... : b(z_i)e^{-α/2}(X(z_i)+φ(z_i)) : ... : b(z_j)e^{-α/2}(X(z_j)+φ(z_j)) : \nonumber$$

$$... : e^{α+X(z_n)} :: c(0)e^{αX(0)+βφ(0))} :$$

(2.7)

over the nested set of contours on the complex plane $|z_1| > ⋯ |z_n|$ going from $x_0$ to $x_0$ around 0, where $x_0$ is an auxiliary point on the complex plane. Then the definition is

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2 It is interesting that this operator first has appeared in [3]. Then it was rediscovered in [7] where it was used to furnish maps between ground states of c=1 string
continued to all polynomials by linearity. To be well defined, the integral representation above should not depend on the position of the auxiliary point \( x_0 \). This condition severely restricts the choice of the polynomials, for the fixed \( F_{\alpha\beta} \) excluding all but a finite number of them. Moreover, the condition that there exists a nontrivial map, restricts also the choice for \((\alpha, \beta)\)

Using the Serre relation (2.6) it is easy to see, that monomial in \( e, f_1 \) can have no more than two \( f_1 \) factors. Look first at the case when the polynomial has the order exactly two in \( f_1 \). For the fixed order in \( e \), there is only one such — \( e^n f_1 e f_1 \). Let us think of (2.7) as of its representation. Then if the corresponding map existed, the integral would be well defined and the integrated function be single valued. But now we note that this function is in fact antisymmetric in \( z_i \) and \( z_j \) (because of two \( b \)'s insertions) so the integral equals zero, i.e. the map we started with was in fact trivial. Hence we should deal only with the linear in \( f_1 \) polynomials.

First let us consider three examples of nontrivial maps, which can readily be checked by looking at (2.7). To the monomial \( e^n \) there corresponds a nontrivial map if and only if \( \alpha = \alpha_+ \frac{1-n}{2} + \alpha_- \frac{1-m}{2} \), it is a well known fact from the representation theory of the Virasoro algebra. There is a nontrivial map corresponding to \( f \) if and only \( \alpha - \beta = -\alpha_- k \). Then, if \( \alpha + \beta = -\alpha_- (k+1) + \alpha_+ \) the combination \( x_\alpha e f_1 - f_1 e \) with

\[
x_\alpha = \frac{q^{2\alpha_- \alpha} - q^{-2\alpha_- \alpha}}{q^{2\alpha_- \alpha+1} - q^{-2\alpha_- \alpha-1}} = \frac{[2\alpha_- \alpha]_q}{[2\alpha_- \alpha + 1]_q}
\]

is the map.

We call a map primitive, if it cannot be obtained as a composition of two nontrivial maps. It is clear, that if we know all primitive maps, we know everything. The results of the manipulations with the integral representation (2.7) can be put in the following nice form.

There are only three types of the primitive maps, listed above.

Let us consider the oriented graphs (the diagrams) whose vertices are the Fock spaces \( F_{\alpha\beta} \) and arrows are the primitive maps between them, belonging to our family. It is important that the resulting map between two vertices does not depend on the path we choose to connect them. From now on (unless explicitly stated otherwise) let us restrict

\footnote{This can also be observed from the formulas for the singular vectors in the Verma modules of \( u_q(sl(2|1)) \).}
ourselves to case when $\alpha_+^2$ is not a rational number. The more complicated case of rational $\alpha_+^2$ can be treated by exactly the same technique and will be considered elsewhere \[11\].

Then every $F_{\alpha \beta}$ belongs to the diagram of one of the four types, shown in Fig.1\[12\].

If we knew that the maps which can be expressed in terms of screenings were \textit{all} the maps between the Fock spaces, we could conclude that we know completely the primitive (i.e. singular and cosingular) vectors of the latter. Then a more or less straightforward construction below should provide us with the resolutions we are after. In fact, we cannot \textit{prove} that we have described all the maps, so we have to \textit{assume} that. The general problem of formulating exactly the correspondence between the quantum groups ($u_q(sl(2|1))$ in our case) and the chiral algebras is still open (but cf. \[12\]), so one should think of this assumption as of (very well grounded) hypothesis.

We are ready now to construct the free field resolutions of the irreducible representations of $N=2$ $SVir$. By such resolution we mean the complex of the free field Fock spaces with the cohomology being nontrivial only in the zero degree where it is represented by the irreducible representation. Having a complex we compute its Euler characteristics (character valued, as usual) which turns to be a character of the irreducible $L_{\alpha \beta}$. It is convenient to deal with the normalized characters

$$\tilde{\chi}_{\alpha \beta} = \frac{\chi_{\alpha \beta}}{\chi(F_{\alpha \beta})}, \chi(F_{\alpha \beta}) = Tr_{F_{\alpha \beta}}(q^{L_0}x^{2J_0}) \quad (2.9)$$

\textbf{Case I.} $(\alpha, \beta)$ is generic, the module $F_{\alpha \beta}$ is irreducible. The corresponding complex is therefore trivial, $\tilde{\chi}_{\alpha \beta} = 1$.

\textbf{Case II.}

$$\alpha_n = \alpha_+ \frac{1-n}{2} + \alpha_- \frac{1-m}{2} \quad (2.10)$$

$\beta$ is generic. This case essentially reduces to the well known theory for the Virasoro algebra. The map $E^n$ is surjective, its kernel is a submodule in $F_{\alpha \beta}$ generated by the highest weight vector. It gives $L_{m,n}(Vir) \otimes F_{\beta} \otimes F_{gh}$. By our assumption this is the only map \textit{from} $F_{\alpha \beta}$ and the only map \textit{to} $F_{\alpha+n \alpha} \beta$; there is no maps \textit{to} $F_{\alpha \beta}$ or \textit{from} $F_{\alpha+n \alpha} \beta$. Therefore the

\[4\] We mark the Fock space $F_{\alpha \beta}$ by the type of the $u_q(sl(2|1))$ Verma module it corresponds to and the irreducible representation $L_{\alpha \beta}$ by the type of the Fock space having the same highest weight.

\[5\] In fact, the construction goes parallel to that of the BGG resolution for $u_q(sl(2|1))$. 

5
submodule in $F_{\alpha\beta}$ generated by the highest weight vector is irreducible. The quotient of $F_{\alpha\beta}$ by this submodule is also irreducible and coincides with $F_{\alpha+n\alpha_+^,\beta}$. The character is
\[ \tilde{\chi}_{\alpha\beta} = 1 - q^{nm} \] (2.11)

This was the typical example of the argument using our basic assumption. It is always implied below.

**Case III**.
\[ \alpha - \beta = -\alpha_- k \] (2.12)

When $k \geq -1$ the map $F_1$ sends the highest weight vector of $F_{\alpha,\beta}$ to a nonzero element, which generate in $F_{\alpha-\alpha_-,\beta-\alpha_+^,\beta}$ a proper submodule $SF_{\alpha-\alpha_-,\beta-\alpha_+^,\beta}$. There is no other maps into $F_{\alpha-\alpha_-,\beta-\alpha_+^,\beta}$, so $SF_{\alpha-\alpha_-,\beta-\alpha_+^,\beta}$ is the only proper submodule. Therefore it must coincide with the kernel of the map $F_1$ from $F_{\alpha-\alpha_-,\beta-\alpha_+^,\beta}$ to $F_{\alpha-\alpha_+,\beta-\alpha_+^,\beta}$. This means that the diagram III is exact — the image of the ingoing arrow coincide with the kernel of the outgoing arrow. In particular it imply that the composition of two arrows is zero. But this is certainly true as it is just $F_1^2 = 0$ (2.3). Althogh trivial, this fact should be considered as a consistency check for our assumption.

The diagram III has already a natural structure of the complex. The graded components are just the Fock spaces at the vertices and the differentials are given by the arrows. This complex is infinite in both directions. We now it is exact, so its cohomology is trivial. Therefore another (less trivial) consistency check is to compute its Euler character to make sure it equals zero. Easy to see, it is zero indeed. To obtain the resolution of $L_{\alpha,\beta}$ one cuts the diagram by the arrow going from $F_{\alpha,\beta}$ to obtain two complexes with the equal cohomology (so there are two resolutions in fact), one can use either of them. The character is
\[ \tilde{\chi}_{\alpha\beta} = \frac{1}{1 + x^{-1}q^{k+1}} \] (2.13)

We see that for $k \geq -1$ the formula (2.13) can naturally be interpreted as a character of the representation with the highest weight $(\Delta_{\alpha\beta}, q_{\alpha\beta})$. But for $k < -1$ the identical transformation (2.13) $\rightarrow \frac{xq^{-k-1}}{1+xq^{-k-1}}$ shows that the character we compute now correspond to the weight $(\Delta_{\alpha+^{\alpha_+},\beta+^{\alpha_+}}, q_{\alpha+^{\alpha_+},\beta+^{\alpha_+}})$. The reason for this phenomena is simple. For $k < -1$ the map $F_1$ kills the highest weight vector of $F_{\alpha,\beta}$ and sends some vector $w \in F_{\alpha,\beta}$ to the highest vector of $F_{\alpha-\alpha_-,\beta-\alpha_+^,\beta}$. Hence each Fock space has one cosingular vector $w_{\alpha\beta}$ and the irreducible representation is a submodule of $F_{\alpha,\beta}$ generated by the highest weight.
vector of the Fock space. Now, to obtain a resolution of $L_{\alpha\beta}$ we should cut the diagram $\text{III}_-$ by the arrow coming into $F_{\alpha\beta}$. The character is

$$\tilde{\chi}_{\alpha\beta} = \frac{1}{1 + x q^{-k-1}}$$  \hspace{1cm} (2.14)

**Case III$_+$**

$$\alpha + \beta = \alpha_- (k+1) + \alpha_+$$  \hspace{1cm} (2.15)

One can repeat everything that have been said about $\text{III}_-$. The only subtlety here is to check that the composition of two conseuctive maps in the diagram is zero. The reader should convince oneself it is true using (2.6),(2.8) and simple $q$-polynomial identities. The formulas for the characters are given by the same formulas (2.13),(2.14).

**Case IV$_-$** — the conditions II and $\text{III}_-$ (IV$_-$) are met simultaneously.

$$\alpha_{nm} = \alpha_+ \frac{1-n}{2} + \alpha_- \frac{1-m}{2}$$
$$\beta_{nmk} = \alpha_+ \frac{1-n}{2} + \alpha_- \frac{1-m + 2k}{2}$$  \hspace{1cm} (2.16)

We denote $F_{\alpha_{nm}\beta_{nmk}}$ by $F_{nmk}$. We know everything already about the maps in the diagram. First consider the resolution of the representation with $n \geq 0$, i.e. belonging to the left column in Fig.1. To obtain a resolution we should again cut the diagram by the horizontal line crossing the arrow above $(\alpha\beta)$ for $k \geq -1$ or below $(\alpha\beta)$ for $k < -1$. Holding the upper half end up with a ”ladder” shown in the Fig.2. The structure of the complex $\{C^r, d(r)\}_{r \geq 0}$ is given by

$$C^0 = F_{n+1} \ m \ k, \ C^r = F_{n+1+r} \ m \ k \oplus F_{-(n+r)} \ m \ k$$
$$d(0) = E^{n+1} \oplus F_1, \ d(r) = \begin{pmatrix} F_1 & 0 \\ E^{n+1+r} & x_{n+r} E F_1 - F_1 E \end{pmatrix}$$  \hspace{1cm} (2.17)

$$x_{l+1} = (q + q^{-1}) - \frac{1}{x_l}, \ x_0 = q + q^{-1}$$  \hspace{1cm} (2.18)

The character is

$$k \geq -1 \hspace{0.5cm} \tilde{\chi}_{\alpha\beta} = \frac{1 - q^{m(n+1)} + q^{m-k-1}(1 - q^{mn})}{(1 + x^{-1} q^{k+1})(1 + x q^{m-k-1})}$$
$$k < -1 \hspace{0.5cm} \tilde{\chi}_{\alpha\beta} = \frac{1 - q^{mn} + q^{m-k-1}(1 - q^{m(n-1)})}{(1 + x q^{-k-1})(1 + x q^{m-k-1})}$$  \hspace{1cm} (2.19)

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6 Compare to $k \geq -1$ when the irr. rep. was a *quotient* of the Fock space.

7 It is a good exercise to check, using the Serre relation, that $d(r + 1) d(r) = 0$. 

7
Now about the representations with \( n < 0 \) (in the right column in Fig.1). The image of the map \( E^n \) in such Fock space is generated by the highest weight vector. Thus we infer that all these modules are of type \( III_+ \) with \( k' = m - k - 1 \).

**Case \( IV_+ \) —** the conditions II and \( III_+ \) are met simultaneously.

\[
\alpha_{nm} = \alpha_+ \frac{1 - n}{2} + \frac{1 - m}{2} \\
\beta_{nmk} = \alpha_+ \frac{1 + n}{2} + \frac{-1 + m + 2(k + 1)}{2}
\]

(2.20)

One just repeats what was said about \( IV_+ \) (probably it is better to take a bottom half of the cut diagram to construct a resolution of the representations with \( n > 0 \) in this case). The representations with \( n < 0 \) are of type \( III_- \). The formulas for the characters (2.19) are applicable.

Thus, using the basic assumption we come up with the consistent results for the resolutions and characters.\(^8\) Still it is desirable to have additional evidence these results are true. Fortunately, there is another way to prove that we have a resolution, but it works only for the generic values of \( \alpha_2^2 \) and only for some representations. Following [13], [14], we consider a classical limit \((\alpha_2^2 \to 0)\). Rescaling the generators of the Heisenberg algebras and the screenings we end up with the action of the (classical) \( n_+(sl(2|1)) \) just on \((F_{\alpha\beta})_{\text{class}}\), if the latter limits exist. (The problem is that as \( \alpha_2^+ \to 0, \alpha_2^- \to \infty \), so for example for the \( IV \) one should restrict \( m \) to 1 to and \( k \) to 0 to have the meaningful limits). The BRST complexes above become just the BGG complexes of \( sl(2|1) \), to be more exact, they have the structure of \( \text{Hom}(BGG(\alpha, \beta), (F_{\alpha\beta})_{\text{class}}) \). Hence the cohomology of these complexes is just that of the superalgebra \( n_+(sl(2|1)) \) with the coefficients in \((F_{\alpha\beta})_{\text{class}}\). But one can check (it was known for \( E_{\text{class}} \) and is fairly obvious for \((F_1)_{\text{class}}\) that the action of \( E_{\text{class}}, (F_1)_{\text{class}} \) on \((F_{\alpha\beta})_{\text{class}}\) is co-free. It implies that \( H_{d_{\text{class}}}^0(C) \) — the invariants of \( n_+(sl(2|1)) = (L_{\alpha\beta})_{\text{class}} \) and the higher cohomologies are trivial. Now, using the semicontinuity of the cohomology of the algebraic family of complexes we conclude that for the generic value of \( \alpha_2^+ \) the higher cohomologies of the BRST complexes above are still trivial and \( H_{d_{\text{class}}}^0(C) \) is still \( L_{\alpha\beta} \).

An important representation which has a classical limit and therefore surely has a free field resolution (of type IV) is a vacuum representation. We have seen, that it coincides

\(^8\) An expert reader must have noticed already that \( I – IV \) are equivalent to saying that there is always a resolution of \( N=2 \) \( S\text{vir} \) ir. rep. by the modules \((\text{Vir}-\text{ir. rep.}) \otimes (\text{Fock}_\phi) \otimes (\text{Fock}_{gh})\). It is in this form that \( I – IV \) are generalized to the rational \( c \).
with the intersection of the kernels of two screenings $E, F_1$ acting on the vacuum Fock space $F_{00}$. On the other hand, the vacuum irreducible representation is a vertex operator algebra (VOA) \cite{15} of $N=2 SVir$. Therefore, we have proved that

As a chiral algebra $N=2 SVir$ in the product $Heis' \otimes Heis \otimes Clif$ is characterized as a centralizer of two screening operators $E, F_1$.

3. Conclusion

We have considered the quantum group approach to the representation theory of $N=2 SVir$. From the point of view of the pure mathematics, our main assumption is but a hypothesis. However, we have seen that it gives consistent results and can be vindicated for the generic central charges $c$. It is natural to conjecture that in fact it holds for all values of $c$. Then for the rational we can construct the (Felder type) resolutions simply looking at the representations of $u_q(sl(2|1))$ in the roots of unity. Another obvious generalization is for the $N=2 W$ algebras.

Then, it is interesting to study the fusion rules of the $N=2$ SCFT, corresponding to the representation theory we have described. It appears that it can be done using essentially the same technique \cite{11}.

Appendix Hamiltonian reduction.

Now we wish to use the last result of the section 2 to prove that $N=2 SVir$ (as a chiral algebra) can be obtained from the Kac-Moody $\hat{sl}_k(2|1)$ by the Hamiltonian reduction, as it was suggested in \cite{2}. To do this, we consider a BRST reduction complex. We take the chiral algebra $\hat{sl}_k(2|1)$

\[
B^+(z)B^-(0) = \frac{1}{z}H(0) + \frac{k}{z^2}, \quad H(z)H(0) = \frac{2k}{z^2}
\]

\[
S(z)B_\pm(0) = \mp \frac{1}{z}B_\pm(0), \quad F_1^+(z)F_1^-(0) = \frac{1}{z}S(0) + \frac{k}{z^2}
\]

\[
S(z)S(0) = 0, \quad F_2^+(z)F_2^-(0) = \frac{1}{z}(S(0) + H(0)) + \frac{k}{z^2}
\]

\[
H(z)S(0) = -\frac{k}{z^2}
\]
and add to it four ghost-antighost pairs. Two bosonic — \((\beta, \gamma)\) and \((\beta', \gamma')\) and two fermionic — \((b, c)\) and \((b', c')\). Then we put the constraints encoded in the BRST reduction operator

\[ d_{\text{BRST}} = \oint \{ c' (B^+ - 1) + \beta (F_1^+ - b) + \beta' F_2^+ + c' \beta \gamma' \} \]  

(Hamiltonian reduction.2)

To compute the cohomology (in a fashion similar to that used in \([13], [14]\)) it is convenient to decompose BRST operator \(d_{\text{BRST}} = d_0 + d_1\)

\[ d_0 = \oint (c' B^+ + \beta F_1^+ + \beta' F_2^+ + c' \beta \gamma') \]  

(Hamiltonian reduction.3)

\[ d_1 = - \oint (c' + \beta b) \]

into two pieces and then use the spectral sequence technique.

The cohomology of \(d_0\) can easily be computed. It is generated (as a chiral algebra) by the currents \(c'(z)\), \(\beta(z)\), \((b(z), c(z))\) and

\[ \tilde{H} = H + 2 : b' c' : + : \beta' \gamma' : - : \beta \gamma : \]  

\[ \tilde{S} = S : b' c' : - : \beta' \gamma' : \]  

(Hamiltonian reduction.4)

They are not algebraically independent — in \(H^*_d\) the identities \([d_0, B^-(z)] = 0\), \(\{d_0, F_1^-(z)\} = 0\) give the relations

\[ : \tilde{H} c' : + (k + 1) \partial c' = 0 \]  

(Hamiltonian reduction.5)

\[ : \tilde{S} \beta : + (k + 1) \partial \beta = 0 \]

Introducing the new variables \((\alpha_+ = \frac{1}{\sqrt{2(k+1)}})\)

\[ X(z) = \alpha_+ \tilde{H}(z) \]  

\[ \phi(z) = - \alpha_+ (\tilde{H}(z) + 2 \tilde{S}(z)) \]  

(Hamiltonian reduction.6)

we see that on \(E_1^0\) (by definition this subspace contains no ghosts \(c'(z), \beta(z)\), so it is just \(\text{Heis}^! \otimes \text{Heis} \otimes \text{Clif}\) of \(X(z), \phi(z)\) and \((b, c)\)) the currents \(c(z), \beta(z)\) act as the vertex operators \(c'(z) = : e^{\alpha_+ X(z_1)} :, \beta(z) = : e^{-\frac{\alpha_+}{2} (X(z_1) + \phi(z_1))} :\). Thus the cohomology of \(d_1\) restricted to \(E_1^0\) is the the centralizer of these two screenings and therefore coincides with \(\text{N=2 SVir}\). On the other hand, because of the relations

\[ c'(z) = [d_1, S(z)] \]  

\[ \beta(z) = \{d_1, c(z)\} \]  

(Hamiltonian reduction.7)

the cohomology of \(d_1\) on the compliment to \(E_1^0\) is zero.

Finally, the cohomology of the BRST reduction complex we described is given by \(\text{N=2 SVir}\).
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