The Accelerated Universe and the Moon

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Abstract

Cosmologically motivated theories that explain small acceleration rate of the Universe via modification of gravity at very large, horizon or super-horizon distances, can be tested by precision gravitational measurements at much shorter scales, such as the Earth-Moon distance. Contrary to the naive expectation the predicted corrections to the Einsteinian metric near gravitating sources are so significant that might fall within sensitivity of the proposed Lunar Ranging experiments. The key reason for such corrections is the van Dam-Veltman-Zakharov discontinuity present in linearized versions of all such theories, and its subsequent absence at the non-linear level ala Vainshtein.
1 Generalities

Recent observations suggest that the Universe is accelerating on the scales of the present cosmological horizon \[1\]. This indicates that, either there is a small vacuum energy (or some “effective” vacuum energy), or that conventional laws of Einstein gravity get modified at very large distances, imitating a small cosmological constant \[2, 7, 8\]. The first possibility is unnatural in the view of quantum field theory\[1\], since the required value of vacuum energy \(\sim 10^{-12}\text{GeV}^4\) is unstable under quantum corrections. This unnaturalness goes under the name of the Cosmological Constant Problem.

In this respect the second approach, of modifying gravity in far infrared can be more promising since it is perturbatively stable under quantum corrections. The unnaturally small value of the vacuum energy is replaced by the idea that laws of conventional gravity break down at very large distances, beyond a certain crossover scale \(r_c\). The value of \(r_c\) is perturbatively-stable, i.e. it does not suffer from cut-off sensitive corrections experienced by vacuum energy. This modification at the level of linearized effective four-dimensional equation for the freely propagating metric fluctuations \(h_{\mu\nu}(x)\) about the flat background can be described in the following general way \[2, 7, 8, 19\]:

\[
(\nabla^2 + \frac{1}{r_c^2} f(r_c^2 \nabla^2)) h_{\mu\nu} = 0 \tag{1}
\]

Here \(\nabla^2 = \nabla_\mu \nabla^\mu\) denotes four-dimensional d’Alambertian, and \(f(r_c^2 \nabla^2)/r_c^2\) is an operator that dominates over \(\nabla^2\) only for momenta \(q \ll 1/r_c\). One way to think of this modification is that the effective gravitational coupling (effective Newton’s “constant”)

\[
G_{\text{eff}} = \frac{1}{8\pi M_{\text{Pl}}^2} \left[1 + \frac{f(r_c^2 \nabla^2)}{r_c^2 \nabla^2}\right]^{-1} \tag{2}
\]

becomes dependent on the wave-length. We will assume that the function \(f(r_c^2 \nabla^2)\) is expandable in power series of \(\nabla^2\).

Depending on the precise structure, such models can be divided in the following three categories. The first category is the model \[2\], in which Newtonian gravity turns into the five dimensional \(1/r^2\)-potential at distances \(r \gg r_c\). This effect is due to existence of an infinite-volume flat extra dimension to which gravity “leaks” from the 3-brane, where the conventional particles live. So a brane-observer sees effectively four-dimensional theory of gravity, in which the effective 4D graviton is unstable, but with an arbitrarily large lifetime \(\tau \sim r_c\), over which it decays into five-dimensional continuum of states.

Although extra dimensions play crucial role in formulating the manifestly generally-covariant theory, for our purposes we can equally well use purely 4D language. For the brane observer interested in metric on the brane created by brane sources, there is an effective four-dimensional description in terms of a single four-dimensional

\[^1\]An alternative explanation could be provided by Anthropic approach \[24\]
graviton $h_{\mu\nu}$. Not surprisingly, this effective 4D theory is non-local, but non-local terms only dominate in far infrared. Moreover, if one is interested in estimates of sub-leading corrections to Einstein metric near the gravitating sources, the 4D language is simpler. At the linearised level this 4D theory is given by (1) with

$$f(r_c^2\nabla^2) = r_c\sqrt{-\nabla^2}$$ \hspace{1cm} (3)

In this theory the smallness of cosmological constant is not explained. However, if one postulates that vacuum energy is zero due to some other reason, the model explains the accelerated expansion of the Universe without any need of dark energy [5,6]. The reason, as discovered in [5], is that in [2] the effective Friedmann equation gets modified by additional powers of Hubble parameter $H$, that dominate for very low curvatures (late times). In the leading order the resulting equation can be parameterized as [5, 6]

$$H^2 - \frac{H}{r_c} = \frac{\rho}{3M_{Pl}^2}. \hspace{1cm} (4)$$

At late times, this equation has a self-accelerating cosmological branch with $H = \frac{1}{r_c}$. Observation fix the crossover scale to be $r_c \sim 10^{28}$ cm. Confronted with minimal models where dark energy is a pure vacuum energy, the above model also contains just one parameter ($r_c$ versus the vacuum energy), but the difference is that the value of $r_c$ is insensitive to quantum loops.

In two other classes of models [7, 8], one can address the cosmological constant problem. Models of [7] are based on higher-dimensional generalization of [2] with $N > 1$ extra dimensions [3]. As a result, the feature of 4D Newtonian gravity turning into the high-dimensional $1/r^{1+N}$-potential for $r \gg r_c$ is shared by $N > 1$ theories. However, having more than one extra dimensions proved to be crucial for cosmological constant. In these theories the large 4D vacuum energy ($E$) curves the four-dimensional space only mildly resulting in an accelerated expansion, with the rate inversely proportional to a power of $E$. In four-dimensional language, gravity is modified in such a way that sources of wavelength $\gg r_c$ gravitate very weakly. Thus although vacuum energy is huge, it actually does not curve the space and can lead to an observable small acceleration.

Finally there is a third class [8,19], which does not refer to any underlying high-dimensional theory, but is rather based on infrared modification of gravity directly in four-dimensions. In this models, $G_{eff}$ is postulated to act as a high-pass filter, so that as in [7] sources of wavelength $\gg r_c$ gravitate extremely weakly.

In the above models, value of $r_c$ may of may not be fixed. For instance, in the models of [7] it is actually fixed to be $r_c \sim 10^{28}$ cm, due to the current lower limits on Newtonian $1/r$ gravity.

The common feature of all the above theories, is that beyond the crossover scale $r_c$ Newtonian $1/r$ gravity gets substantially modified. But, at short distances (near the gravitating sources), all the prediction of Einstein gravity are reproduced up to some small corrections. These corrections are the central interest of the present work.
Given the fact that crossover scale is $r_c$, one could naively conclude that deviations are suppressed by powers of $r / r_c$. If this were the case, then for $r_c \sim 10^{28}$ cm, the corrections would be completely negligible at any distance of potential precision measurements.

Interestingly, the story is very different. Recall that near any gravitating object there is a second (much smaller scale) in the problem, the gravitational radius of the body $r_g$. It was shown in [13] that leading corrections to the gravitational potential $\Psi$ in the model of [2] are strongly enhanced by inverse powers $r_g$ and are given by

$$\epsilon = \frac{\delta\Psi}{\Psi} \sim \frac{r}{r_c} \sqrt{\frac{r}{r_g}}$$

The aim of the present work is to show that in a large class of theories that modify gravity beyond some horizon or super-horizon size distance $r_c$, there are corrections of the type [13] that penetrate at much shorter scales $r_* \ll r_c$, and could be potentially measurable. These corrections can be detected in precision gravitational measurements in systems that are much smaller than $r_c$, e.g., such as the Earth-Moon system. In most interesting cases, such as the expression (5), they are on the border line of existing measurements, and thus can be detected if the existing sensitivities are improved.

The key reason for such an unexpected behavior is that the graviton in the above-discussed theories has extra polarization that also couples to the conserved energy-momentum source, and mediates a scalar-type force. As a result, at the linearized level gravity is of that scalar-tensor type. Therefore whenever linearized approximation is valid the predictions of the above theory differ from those of Einstein gravity by a finite amount, no matter how small is the ratio $r / r_c$.

This effect was originally pointed out in the framework of linearized theory of massive gravity [3], and goes under the name of van Dam - Veltman - Zakharov discontinuity (vDVZ). It lead vDVZ to the conclusion that massive theories of gravity are ruled out. Later the similar effect was observed in the linearized version of the generally-covariant model of [2].

The crucial point [10], however, is that discontinuity is an artifact of the linearized approximation, and is cured by non-linear corrections. In reality, the solution is continuous in the limit $1/r_c \to 0$. This was originally suggested by Vainshtein in [14] and later confirmed by explicit fully-nonlinear analysis in generally covariant model of [2], both in cosmological solutions [14], as well as for localized sources, such as cosmic strings [12] and Schwarzschild [13, 16]. These studies uncover the same persistent pattern. Near the sources the solutions are arbitrarily close to those of Einstein gravity, and continuous for large $r_c$. However, the corrections set in at

2 On AdS space vDVZD can be absent already at the linear level, if graviton mass is smaller than the AdS curvature $2\pi / L$. The roots of this phenomenon can again be attributed to the effect of [10]. Such situation probably is not experimentally interesting, and won’t be discussed here.
distances, (or time scales) much smaller than \( r_c \), which makes them potentially observable even if \( r_c \) is very large (e.g. horizon size) (the relevance of this corrections for the orbit of Jupiter was pointed out in [13]). This makes us think that analogous behavior must take place in any theory with infrared modification of gravity of the form in (1). The key message of our analysis is that in all such theories of interest vDVZ discontinuity indeed disappears at the non-linear level in accordance to [10, 11, 12, 13], and standard predictions of Einstein gravity are recovered near the gravitating sources, where non-linearities are important. But as in [13] for any given localized source, there exists a distance \( r_* \) for which the non-linearities are unimportant and thus the “wrong predictions” take over.

We give a qualitative prescription to estimate the leading order corrections to the Einstein metric using the form of the function \( f(r_c^2 \nabla^2) \) in theories interest. Using this prescription one can derive observational constraints on such theories.

The rest of the paper is organized as follows. In section 2, we briefly summarize our theoretical results and in section 3 we discuss their relevance for observations, through the anomalous perihelion precession of orbits of planets. The reader interested in purely the phenomenological applications of our results can dismiss the rest of the paper which is devoted to more theoretical considerations. In section 4 we discuss the role of extra graviton polarization in creating the observable effects at intermediate scales in theories with infrared-modified gravity. In section 5 we briefly discuss the applicability range of our results.

\section{Results}

Here we shall briefly summarize our results. We will be interested in a class of generally-covariant theories, containing exclusively spin-2 states, for which the linearized perturbations about the flat space satisfy:

\[
\left( \nabla^2 + \frac{1}{r_c^2} f(r_c^2 \nabla^2) \right) h_{\mu\nu} = -\frac{1}{M_{Pl}^2} \left\{ T_{\mu\nu} - \frac{\beta}{2} \eta_{\mu\nu} T^\alpha_{\alpha} \right\} + \cdots
\]  

(6)

where the ellipsis stands for some derivative-dependent terms that vanish in convolution with conserved sources. The over-all coefficient on r.h.s. was absorbed in redefinition of \( M_{Pl} \). In all theories of interest \( \beta \neq 1 \), as opposed to the case of Einstein theory of massless graviton with two polarizations, in which \( \beta = 1 \). Thus the theory exhibits the analog of vDVZ discontinuity.

Then our analysis suggests the following.

As suggested by Vainshtein, vDVZ is cured at the non-linear level. That is, the solution are continuous in the limit \( r_c \rightarrow \infty \) near the sources. For any gravitating source, with gravitational radius \( r_g \ll r_c \) there is an important intermediate scale in the problem

\[
r_g \ll r_* \ll r_c
\]  

(7)
where \( r_\ast \) is determined from the following equation

\[
\sqrt{\frac{r_g}{r_\ast}} \sim \left( \frac{r_\ast^2}{r_c^2} \right)^{1/2} f \left( \frac{r_c^2}{r_\ast^2} \right) \tag{8}
\]

The various distance scales work as follows. For \( r \ll r_\ast \) the metric produced by the source is nearly Schwarzchild, with the leading correction to the metric that can be estimated as

\[
\delta \Psi \sim \sqrt{\frac{r_g r^3}{r_c^2}} f \left( \frac{r_c^2}{r^2} \right) \tag{9}
\]

This expression, for the particular form of the \( f \)-function given by \( (3) \), correctly reproduces the result of \( [13] \) given in \( (5) \), found by solving the full nonlinear high-dimensional equation of \( [2] \).

As will be discussed below, the above corrections can play the crucial role for testing theories in question in precision gravitational measurements.

In the interval \( r_\ast \ll r \ll r_c \), the linearized approximation is valid. So gravity is still \( 1/r \), but is of scalar-tensor type. Note that the term “scalar-tensor” only refers to the similarity in the tensor structure, rather than to existence of an independent spin-0 state in the theory. Instead the “scalar” admixture comes from the extra polarization of spin-2 graviton.

Finally, for the distances \( r \gg r_c \), gravity becomes unconventional, with the potential given by

\[
\Psi(r) \sim \int d^3q \frac{r_c^2}{f(r_c^2 q^2)} e^{-iqr} \tag{10}
\]

where \( q \) is the three-momentum.

3 Anomalous Perihelion Precession

The considered class of theories predict slight modification of the gravitational potential of a massive body at observed distances according to \( (9) \). This modifications can be observable in the experiments that are sensitive to anomalous perihelion precession of planets. Let \( \epsilon \) be the fractional change of the gravitational potential

\[
\epsilon \equiv \frac{\delta \Psi}{\Psi}, \quad \tag{11}
\]

where \( \Psi = -GM/r \) is the Newtonian potential. The anomalous perihelion precession (the perihelion advance per orbit due to gravity modification) is

\[
\delta \phi = \pi r (r^{-1} \epsilon)' \tag{12}
\]

where \( ' \equiv d/dr \).

Let us apply this to the model of \( [2] \). In this theory, \( [3] \)

\[
\epsilon = -\sqrt{2r_c^{-1}r_g^{-1/2}}r^{3/2}. \tag{13}
\]
The numerical coefficient deserves some clarification. The above coefficient was derived in [13] on Minkowski background. However, non-linearities created by cosmological expansion can further correct the coefficient. One would expect these corrections to scale as powers of \( r_c H \), where \( H \) is the observed value of the Hubble parameter. On the accelerated branch [5], as it’s obvious from (4), \( H \sim 1/r_c \) and thus, one would expect the corrections to be of order one. Recently, a very interesting fact was pointed out by Lue and Starkman [14] that the cosmological background only affects the sign of the coefficient. The sign depends on the particular cosmological branch. It is negative for the standard cosmological branch, and positive for the self-accelerated one. We will restrict ourselves to order of magnitude estimate, but the sign will be very important if the effect is found, since according to [14] it could give information about the cosmological branch.

We get

\[ \delta \phi = (3\pi/4)\epsilon. \]  

(14)

Numerically, the gravitational radius of the Earth is \( r_g = 0.886 \text{cm} \), the Earth-Moon distance is \( r = 3.84 \times 10^{10} \text{cm} \), the gravity modification parameter that gives the observed acceleration without dark energy \( r_c = 6 \text{ Gpc} \). We get the theoretical precession

\[ \delta \phi = 1.4 \times 10^{-12}. \]  

(15)

This is to be compared to the accuracy of the precession measurement by the lunar laser ranging. Today the accuracy is \( \sigma_\phi = 2.4 \times 10^{-11} \) and no anomalous precession is detected at this accuracy [17]. In the future a tenfold improvement of the accuracy is expected [18].

As noted in [14], anomalous Martian precession is also of interest for testing with cosmologically interesting values of \( r_c \). For \( r_c = 6 \text{ Gpc} \) we get \( \delta \phi_{Mars} = 4 \times 10^{-11} \). This is to be compared to the accuracy of \( \sigma_{\phi_{Mars}} = 9 \times 10^{-11} \) which might become possible as a result of the Pathfinder mission [15].

Let us now consider some generalizations. In [2] \( f(r_c^2 \nabla^2) \) has the form (3). Consider now the minimal modification of the form

\[ f(r_c^2 \nabla^2) = (r_c^2 \nabla^2)^{1-\gamma}. \]  

(16)

According to (9) this modification of the graviton propagator gives the fractional change of the gravitational potential analogous to (3):

\[ \epsilon \sim r_c^{-1-\gamma} r_g^{-1/2} r^{3/2+\gamma}. \]  

(17)

The corresponding anomalous lunar perihelion precession is

\[ \delta \phi \sim r_c^{-1-\gamma} r_g^{-1/2} r^{3/2+\gamma}. \]  

(18)

\[ ^3\text{We thank these authors for sharing their preliminary results with us.} \]
Observations of accuracy $\sigma_\phi$ can therefore test gravity theories with

$$r_c < r \left( \frac{r}{\sigma_\phi^2 r_g} \right)^{\frac{1}{2(1+\gamma)}}. \quad (19)$$

One can speculate that in the absence of additional scales the gravity theories that produce self-acceleration, without vacuum energy should have $r_c \sim \text{few Gpc}$. Then the lunar precession accuracy of $\sigma_\phi \sim 10^{-12}$ will tests the $\gamma = 0$ theory [2]. The dependence of the right-hand side of (3) on $\gamma$ is very strong, and cosmologically interesting theories with $\gamma < 0$ are ruled out by current observations, while theories with $\gamma > 0$ are not testable by the solar system observations.

4 vDVZ Discontinuity and its Absence

Some time ago van Dam and Veltman, and Zakharov [9] suggested that the solar system observations rule out the possibility of non-zero graviton mass, no matter how small. Their conclusion was based on a linearized theory of massive graviton with the following action

$$S = S_{EL} + \int d^4 x \left( \frac{M_{Pl}^2 m_g^2}{2} \left( h_{\mu\nu}^2 - (h_{\mu}^\nu)^2 \right) + \frac{1}{2} h_{\mu\nu} T^{\mu\nu} \right) \quad (20)$$

The first term on the r.h.s. is the standard Einstein action expanded to a quadratic order in the metric fluctuations about the flat space $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The mass term in (20), has the Pauli-Fierz form [23], which is the only possible ghost-free combination quadratic in $h_{\mu\nu}$.

Discontinuity is due to the fact that massive graviton contains five degrees of freedom (five polarizations) as opposed to two polarizations in the massless case. Three out of the five polarizations couple to the conserved energy momentum source, leading to an additional scalar attraction at distances $r \ll \frac{1}{m_g}$, as compared to the massless case. Since the additional degree of freedom couples differently to relativistic and non-relativistic sources, the effect is not merely reducible to the rescaling of the Newton’s constant $G_N$, and is observable at the level of one-graviton exchange. Thus, at this level the theory is discontinuous in the limit $m_g \to 0$.

The amplitude of the lowest tree-level exchange by a single massless graviton between two sources with energy-momentum tensors $T_{\mu\nu}$ and $T'_{\alpha\beta}$ is (the tilde sign denotes the quantities which are Fourier transformed to momentum space):

$$A_{\text{massless}} = -\frac{8\pi G_N}{q^2} \left( \tilde{T}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \tilde{T}_\beta^\beta \right) \tilde{T}'^{\mu\nu}. \quad (21)$$

In the massive case this amplitude takes the form:

$$A_{\text{massive}} = -\frac{8\pi G_N}{q^2 + m_g^2} \left( T_{\mu\nu} - \frac{1}{3} \eta_{\mu\nu} T_\beta^\beta \right) \tilde{T}'^{\mu\nu}. \quad (22)$$
The additional scalar attraction for massive case, is reflected in the difference in the tensor structure. This difference can not be eliminated by simple redefinition of parameters and is finite, for arbitrarily small graviton mass.

This result goes under the name of vDVZ discontinuity, and if true, would be an extremely powerful result, as it would rule out not only the possibility of massive gravity, but much wider classes of theories that modify gravity in far-infrared.

However, the story is not so straightforward as we shall now discuss. Shortly after vDVZ observation, Vainshtein [10] suggested that vDVZ discontinuity was an artifact of the linearized approximation, and would be absent in fully non-linear theory. That is, vDVZ result was obtained at one-graviton exchange level, that is in the first order in $G_N$ expansion. This corresponds to solving the linearized equation

$$\frac{\delta S_{El}}{\delta h_{\mu\nu}} + m_g^2 (h_{\mu\nu} - \eta_{\mu\nu} h_{\alpha}^{\alpha}) = \frac{T^{\mu\nu}}{M_{Pl}^2}$$

(23)

However, Vainshtein noted that perturbative expansion in $G_N$ breaks down in the zero $m_g$ limit. However, he also showed that if one takes into account non-linearities, then for small $m_g$ the perturbative expansion in powers of $m_g$ can be organized, in which case discontinuity can disappear. Since no fully non-linear generally-covariant theory for massive graviton was known, Vainshtein used a theory obtained by non-linear completion of only the first, Einstein, term in r.h.s. of (23):

$$G_{\mu\nu} + m_g^2(h_{\mu\nu} - \eta_{\mu\nu} h_{\alpha}^{\alpha}) = 8\pi G_N T_{\mu\nu}$$

(24)

where $G_{\mu\nu}$ is the Einstein tensor. In which case solving for $g_{00} = e^{\nu(r)}$ and $g_{rr} = e^{\lambda(r)}$ components of the spherically-symmetric metric outside massive body, he found:

$$\nu(r) = -\frac{r_g}{r} + O \left( m_g^2 \sqrt{r_g r^2} \right) , \quad \lambda(r) = \frac{r_g}{r} + O \left( m_g^2 \sqrt{r_g r^2} \right) ,$$

(25)

Note that in the same parametrization the standard Schwarzschild solution of the massless theory takes the following form:

$$\nu^{Schw}(r) = -\lambda^{Schw}(r) = \ln \left( 1 - \frac{r_g}{r} \right) = -\frac{r_g}{r} - \frac{1}{2} \left( \frac{r_g}{r} \right)^2 + \ldots ,$$

(26)

Here $r_g \equiv 2G_N M$ is the gravitational radius of the source of mass $M$.

This expression is fully continuous in $m_g$ and reproduces Einsteinian results in the zero-mass limit. Thus vDVZ has indeed disappeared at the non-linear level. Notice the two important facts. First the sub-leading corrections to the metric are
non-analytic in \( r_g \). Secondly, the deviations from the standard Einstein gravity become important at distance

\[
 r_\ast = \frac{(m_g r_g)^{1/5}}{m_g},
\]

which is parametrically shorter than the Compton wave-length of graviton. This is not surprising, if we recall that vDVZ was cured by non-linearities that are most important in the neighborhood of the heavy sources (large \( r_g \)). As a result the heavier is the source, the larger is the critical distance at which linear approximation takes over.

The above results however, may not sound completely satisfactory, since the mass term in (24) was still kept at the linearized level, and as a result theory was not fully generally-covariant. As we shall see, the above results nevertheless persists in theories, which are fully non-linear and generally-covariant. There too the discontinuity is absent and corrections can penetrate at distances much shorter than the scale at which linearized gravity gets modified. The latter fact gives possibility of experimentally testing such theories through precision measurements at the relatively short distances.

An example of generally-covariant theory that modifies gravity in far infrared, and which exhibits vDVZ at the linearized level is given by the following action \[2,3\]

\[
 S = \frac{M_{Pl}^2}{4r_c} \int d^4x \, dy \sqrt{|g^{(5)}|} \, R + \frac{M_{Pl}^2}{2} \int d^4x \, \sqrt{|g|} \, R(x) .
\]

(28)

Where \( g^{(5)}_{AB} \) is 5D metric tensor, \( A, B \) are five-dimensional indexes, and \( R \) is the five-dimensional Ricci scalar, \( g_{\mu\nu} \) denotes the induced metric on the brane which we take as

\[
 g_{\mu\nu}(x) \equiv g^{(5)}_{\mu\nu}(x, y = 0) , \quad \mu, \nu = 0, 1, 2, 3 ,
\]

(29)

neglecting the brane fluctuations.

We assume that our observable 4D world is confined to a brane which is located at the point \( y = 0 \) in extra fifth dimension. That is the energy-momentum tensor of 4D matter has the form \( T_{\mu\nu}(x) \delta(y) \).

Although, the underlying theory is high-dimensional, from the point of view of a 4D observer localized on the brane, Newtonian gravitational potential is just the usual \( 1/r \) gravity, which gets modified to \( 1/r^2 \) only at very large distances \( r \gg r_c \).

The gravitational potential between the two bodies located on the brane can be read-off from the form of the Greens function (it is convenient to work in momentum space in the four world-volume directions and in position space with respect to the transverse coordinate \( y \)).

Neglecting the tensorial structure of the propagator the scalar part of the Green function has the following form \[4\]:

\[
 \tilde{G}(q, y = 0) = \frac{1}{M_{Pl}^2} \frac{1}{q^2 + \sqrt{q^2 + r_c^2}} ,
\]

(30)
For the static gravitational potential $\Psi(r)$, one gets at short distances, i.e., when $r \ll r_c$

$$\Psi(r) = -\frac{1}{8\pi^2 M_{Pl}^2} \frac{1}{r} \left\{ \frac{\pi}{2} + \left[ -1 + \gamma - \ln \left( \frac{r_c}{r} \right) \right] \left( \frac{r}{r_c} \right) + O(r^2) \right\}.$$  \hspace{1cm} (31)

Here $\gamma \simeq 0.577$ is the Euler constant. The leading term in this expression has the familiar $1/r$ scaling of the four-dimensional Newton law. For $r \gg r_c$ one finds:

$$\Psi(r) = -\frac{1}{16\pi^2 M_{Pl}^2} \frac{1}{r^2} + O\left( \frac{1}{r^3} \right).$$  \hspace{1cm} (32)

The long distance potential scales as $1/r^2$ in accordance with the 5D Newton law.

The above model (28) exhibits the vDVZ discontinuity in the one-graviton tree-level exchange. This can be seen directly from the $\mu\nu$ components of linearized Einstein equation for bulk metric fluctuations about the flat 5D metric, which after gauge fixing can be brought into the following form [2]

$$\left( \frac{1}{r_c} \left( \nabla^2 - \partial_y^2 \right) + \delta(y) \nabla^2 \right) h_{\mu\nu} = -\frac{1}{M_{Pl}^2} \left\{ T_{\mu\nu} - \frac{1}{3} \eta_{\mu\nu} T^\alpha_\alpha \right\} \delta(y) + \delta(y) \partial_\mu \partial_\nu h^\alpha_\alpha.$$  \hspace{1cm} (33)

The first term on r.h.s of this equation has a structure which is identical to that of a massive 4D graviton. The second term, $\partial_\mu \partial_\nu$, vanishes whenever it is contracted with the conserved energy-momentum tensor, and thus plays no role at one graviton exchange level. As a result, the amplitude of interaction of two test sources takes the form:

$$A(q) \propto \frac{\tilde{T}^\mu_\nu T^\nu_\mu - \frac{1}{3} \tilde{T}^\mu_\mu \tilde{T}_\nu^\nu}{q^2 + \frac{q^2}{r_c}},$$  \hspace{1cm} (34)

where $q \equiv \sqrt{q^2}$. We see that the tensor structure is the same as in the case of the massive 4D theory, see Eq. (22), which signals vDVZ discontinuity.

The discontinuity however is cured at the nonlinear level. This was demonstrated explicitly both for cosmological solution [11], as well as for cosmic strings [12] and Schwarzschild [13], [16]. We shall concentrate on the latter case. The solution of [13] has the form:

$$\nu(r) = -\frac{r_g}{r} + O \left( \frac{1}{r_c \sqrt{r_g r^3}} \right), \hspace{1cm} \lambda(r) = \frac{r_g}{r} + O \left( \frac{1}{r_c \sqrt{r_g r^3}} \right).$$  \hspace{1cm} (35)

This expression exhibits some interesting behavior. First, we see that the two features observed by Vainshtein persist here, but there are some peculiarities. The correction is non-analytic in $r_g$, and moreover it becomes important at distances much closer than $r_c$. This is in accordance with Vainshtein case. However, there are important differences which are crucial for observations. Notice that, one would
naively expect that if Vainshtein’s analysis is correct, then the above expression should be obtained from Vainshtein case by substitution $m_g \to 1/r_c$. The reason is that, although we are not dealing with massive gravity in the strict sense, nevertheless, $r_c$ is the scale at which $1/r$-law of linearized approximation breaks down. It is true that in our case it is replaced by $1/r^2$ rather than by exponentially suppressed Youkawa potential, but this difference naively seems inessential. So naively one would conclude that $r_c$ should play the role of $m_g$ in controlling the strength of correction. However, this is not the case. Instead, the expression of ref [13] is obtained from Vainshtein by substitution:

$$m_g^2 \to \frac{1}{r r_c}$$  \hspace*{1cm} (36)

We shall now try to understand why this is the case, but let us first explain non-analyticity of the correction in $r_g$. The above theory contains two parameters $M_{Pl}$ and $r_c$. vDVZ happens at the linear level due to the term that is proportional to $1/r_c$, and is cured by the non-linear corrections proportional to $M_{Pl}^2$. These non-linear corrections make sure that metric gets standard Einstein form near the sources. The deviation from the standard metric is due to the fact that linearized approximation becomes good again and $1/r_c$-terms take over. Thus, corrections to the metric are due to the fact that one-particle exchange dominates at large scales. Then the leading corrections should be proportional to $1/r_c$. Close to the source the metric can be expanded in powers of $1/r_c$, e.g.

$$g_{00} (r, r_c, r_g) = 1 + \frac{r_g}{r} + \frac{1}{r_c} r_g^{\alpha} r^{1-\alpha} + \text{higher order terms}$$  \hspace*{1cm} (37)

where $\alpha$ is some positive power. Now, we know that for the fixed $r$, for weak sources, the linearized approximation is valid, and thus the effect of extra graviton polarization must be reintroduced. Thus, for fixed $r$, the second term on r.h.s. of (37) must dominate in the limit $r_g \to 0$. This can only happen if $\alpha < 1$, which explains non-analyticity of the correction. To summarize shortly, given the fact that difference in the tensor structure of the metric is introduced by dominance of one-particle exchanges, which are proportional to $1/r_c$, and dominate for small $r_g$, the corrections to Schwarzchild cannot be analytic in $r_g$.

To understand which quantity plays the role of $m_g$, the following observation is useful. If we are interested in making 4D metric on the brane continuously approach Schwarzschild in the limit $1/r_c \to 0$, it is enough to have non-linear action on the brane only, and keep high-dimensional part of the action linear. In other words for curing vDVZ the non-linear interactions in the bulk are unimportant. That is, for obtaining Schwarzchild solution in the leading order on the brane the following equation is enough

$$M_{Pl}^2 \left( \frac{1}{r_c} G_{\mu \nu}^L + \delta(y) G_{\mu \nu} \right) = \delta(y) T_{\mu \nu}$$  \hspace*{1cm} (38)
Where, $G_{\mu\nu}^L$ is the five-dimensional Einstein tensor, linearized on a flat background. This system describes the five dimensional graviton $h_{\mu\nu}(x, y)$ that freely propagates in the bulk, and has non-linear self-couplings only on the brane. These brane-localized self couplings are most important for determining the metric on the brane near the source localized on the brane. Diagrammatically, this fact can be understood as follows. The nonlinear corrections that cure vDVZ discontinuity correspond to all the tree-level diagrams with virtual graviton lines that end on the sources localized on the brane. The self-interaction vertexes of virtual gravitons can be located both on the brane or in the bulk. The bulk non-linearities come from the $1/r_c$-suppressed bulk curvature term and are subleading. So the diagrams for which graviton self-interaction vertices are not located on the brane are sub-leading in $1/r_c$-expansion. Theory with linearized bulk action (38) gives rise only to the diagrams in which all graviton vertices are located on the brane. Since these diagrams are dominant, the eq (38) is suffices to cure vDVZ discontinuity. Now let us compare the five-dimensional theory defined in by eq. (38) to an effective four-dimensional theory of four-dimensional graviton $h_{\mu\nu}(x)$ defined by the following equation

$$M^2_{Pl} \left( G_{\mu\nu} + \frac{1}{r_c} \sqrt{\gamma^2} (h_{\mu\nu} - \eta_{\mu\nu}h^0_\alpha) \right) = T_{\mu\nu}$$

(39)

These can be treated as two independent theories, but some sub-class of amplitudes in the two cases are equal. In fact the interaction amplitudes between brane-localized sources with no graviton emission in 5D theory are equal to the ones of 4D one. Notice that the two theories are designed in such a way, that the Greens function of five-dimensional theory (38) is equal to the Greens function of the five-dimensional theory defined in by eq. (38) to an effective four-dimensional theory of four-dimensional graviton $h_{\mu\nu}(x)$ defined by the following equation

$$\langle h_{\mu\nu}(x) h_{\alpha\beta}(x') \rangle = \langle h_{\mu\nu}(x, y = 0) h_{\alpha\beta}(x', y = 0) \rangle$$

(40)

Also all the nonlinear interactions in two theories are the same. Given these facts, it is obvious that any interaction amplitude between the brane-localized sources $T_{\mu\nu}(x_1), T_{\mu\nu}(x_2), ..., T_{\mu\nu}(x_n)$, (in which no gravitons are emitted in the final state) in five dimensional theory (38) will be equal to a similar amplitude in four-dimensional theory (38). For instance consider the lowest three-level interaction among the three sources with a single intermediate three-graviton vertex $V(x)\delta(y)$:

$$A \sim \int d^4 x_1 d^4 y_1 d^4 x_2 d^4 y_2 d^4 x_3 d^4 y_3 T^{\mu\nu}(x_1)\delta(y_1) T^{\alpha\beta}(x_2)\delta(y_2) T^{\gamma\rho}(x_3)\delta(y_3)$$

$$V_{\mu'\nu',\alpha'|\beta'|\gamma'|\rho'}(x)\delta(y) \langle h_{\mu\nu}(x_1 y_1) h_{\mu'\nu'}(x, y) \rangle \langle h_{\alpha\beta}(x_2, y_2) h_{\alpha'\beta'}(x, y) \rangle \langle h_{\gamma\rho}(x_3, y_3) h_{\gamma'\rho'}(x, y) \rangle =$$

$$\int d^4 x_1 d^4 x_2 d^4 x_3 \rightarrow \mu'\nu',\alpha'|\beta'|\gamma'|\rho' T^{\mu\nu}(x_1) T^{\alpha\beta}(x_2) T^{\gamma\rho}(x_3) V_{\mu'\nu',\alpha'|\beta'|\gamma'|\rho'}(x) \langle h_{\mu\nu}(x_1) h_{\mu'\nu'}(x) \rangle \langle h_{\alpha\beta}(x_2) h_{\alpha'\beta'}(x) \rangle \langle h_{\gamma\rho}(x_3) h_{\gamma'\rho'}(x) \rangle$$

(41)

The last expression represents a similar amplitude evaluated in theory of eq. (38). Generalization of the above relation to more complicated diagrams with arbitrary
number of internal vertexes and external sources is trivial. As a result any such amplitude in high-dimensional theory eq. (38) has an equal counterpart in four-dimensional one. Thus, if for example, we are interested in the metric created by the brane-localized source on the brane in theory defined by eq. (38), we can instead solve eq. (39) and get the correct answer. But for solving perturbatively for Schwarzschild metric, the equation (39) in the leading order is equivalent to that of Vainshtein [24] in which $m_g^2$ is substituted by $1/r_c$.

Now we can generalize this result and give a simple qualitative prescription that gives a possibility to estimate the sub-leading correction to Schwarzschild in other theories of interest, in which Einstein gravity is modified in far infrared by adding some operators that at the linearized level behave as

$$G_{\mu\nu} + \frac{1}{r_c^2} f(r_c^2 \nabla^2)(h_{\mu\nu} - \eta_{\mu\nu} h_\alpha^\alpha) = 0$$

(42)

Then subleading correction to the metric can be estimated from (25) by substitution

$$m_g^2 \to \frac{1}{r_c^2} f\left(\frac{r_c^2}{r^2}\right)$$

(43)

where action of the operator should be understood in terms of the eigenvalues of $\nabla$. For instance:

$$f(r_c^2 \nabla^2) \frac{1}{r} = \int d^3p \frac{f(r_c^2 p^2)}{p^2} e^{-irp}$$

(44)

It should be noted that we have neglected corrections of the same order in $1/r_c$ coming from the nonlinear completion of the mass term. Thus our results is just an order of magnitude estimate. In principle there could be theories in which the additional terms exactly cancel the corrections to Schwarzschild we computed although this is not the case in the theories of [2, 3].

5 Applicability

We shall now briefly formulate the applicability range of our analysis. Consider a theory which, in some gauge, at the linearized level reduces to eq. (6). The criteria for the presence of discontinuity ($\beta \neq 1$) is the existence of the following spectral representation:

$$\frac{1}{q^2 + f(r_c^2 q^2)} = \int_0^\infty ds \frac{\rho(s)}{s + q^2}$$

(45)

with a semi-positive definite spectral function $\rho(s)$, which is requirement for the absence of unphysical negative norm states. In such a case the equation (6) is derivable from the action

$$S = \int_0^\infty ds \left[ S_{El}^{(s)} + \int d^4x \left( \frac{s}{2} ((h_{\mu\nu}^{(s)})^2 - (h_{\mu\nu})^2) + \sqrt{\rho(s)} h_{\mu\nu}^{(s)} T^{\mu\nu} \right) \right]$$

(46)
where $h_{\mu\nu} = \int ds \sqrt{\rho(s)} h_{\mu\nu}^{(s)}$. This action describes the linearized theory of continuum of free massive gravitons with masses $m^2 = s$. They all couple to the same conserved source $T^\nu_\mu$ but with mass-dependent coupling $\sqrt{\rho(s)}/2M_{Pl}$. Therefore the propagator of this theory is simply an integral over the continuum of the massive propagators taken with the weights $\rho(s)$

$$\mathcal{A}(q) = -8\pi G_N \int_0^\infty ds \rho(s) \frac{\tilde{T}^{\mu\nu} \tilde{T}_{\mu\nu} - \frac{c(s)}{3} \tilde{T}_\mu \tilde{T}^{\mu\nu}}{s + q^2} \quad \text{(47)}$$

where in our parametrization $c(0) = 3/2$ and $c(s \neq 0) = 1$.

A useful illustrative example, in which vDVZ can be seen from the above reasoning is the model of [2]. There the representation (46) has a simple physical meaning [3]. It is just an expansion into the continuum of massive Kaluza-Klein states. In that case the spectral function is given by $\rho(s) = \frac{1}{\sqrt{s(4 + r^2 s)}}$. In this language the presence of vDVZ discontinuity in [2] is trivially understood. Since each Kaluza-Klein mode is a massive spin-2 particle, each of them exhibits vDVZ discontinuity by default. Thus exchange by tower of Kaluza-Klein states exhibits the same discontinuity as the individual exchanges do.

## 6 Conclusion

There are strong motivations coming from cosmology for modifying the standard Einstein gravity at large distances. A wide class of such gravity theories can be tested by astronomical observations of the solar system.

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