Quadrature Mirror Filters
and
Loop Groups

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Abstract: In this paper we want to show, that the finite impulse response quadratic mirror filters (QMF) associated to a tower of grids $\Gamma \subset H = \mathbb{Z}^n$ can be identified with a right coset of the subgroup $\text{Fix}(T_{\Gamma^\perp}, \text{Map}(\mathbb{T}^n \to U(N) : \text{poly})$ of the group of polynomial loops $\text{Map}(\mathbb{T}^n \to U(N) : \text{poly})$ with $N = |H/\Gamma|$. The QMF with some vanishing moments can be identified with cosets of subgroups. The problem to parameterize all finite impulse response QMF in arbitrary space dimensions is now equivalent to factorize all polynomial loops.

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1. Introduction: the case $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Z}$

The kind of theorem we propose to prove in this paper is the easiest illustrated with the one dimensional case. We enclose it here for the sake of clarity although all results are well known. After a short introduction to QMF filters we want to show that the construction of all QMF is equivalent to construct the group of unitary operators that commute with the translations. This in turn will be equivalent to the construction of loops in $U(2)$.

Consider the space of square integrable sequences $L^2(\mathbb{Z})$. At the heart of QMF-filters [1] and discrete wavelet analysis (e.g. [2]) is the idea to split any sequence into an even and odd part; that is we write

$$L^2(\mathbb{Z}) = L^2(2\mathbb{Z}) \oplus L^2(2\mathbb{Z} + 1).$$

The associated orthogonal projectors on $L^2(2\mathbb{Z})$ and on $L^2(2\mathbb{Z} + 1)$ shall be denoted by $\Pi_0$ and $\Pi_1 = T_1\Pi_0 T_1^*$ where $T_n$ is the translation operator. Clearly no information on a sequence $s$ is lost if we split it into its even and odd sub-sequences. In particular the energy $\|s\|^2$ is just the sum of the even and the odd part

$$\|s\|^2 = \|\Pi_0 s\|^2 + \|\Pi_1 s\|^2.$$

We now ask the question whether we can find a pre-treatment of $s$, such that again no information is lost and such that again the energy is conserved. To be more specific we are looking for sequences $\eta_0$ and $\eta_1$ such that the split of the filtered sequences $\eta_0 * s$ and $\eta_1 * s$ is again energy conserving. Thus we want to have

$$\|s\|^2 = \|\Pi_0 (\eta_0 * s)\|^2 + \|\Pi_1 (\eta_1 * s)\|^2,$$

with the convolution product defined as

$$r * s(n) = \sum_{k \in \mathbb{Z}} r(n - k) s(k) = \langle T_n \tilde{r} \mid s \rangle \quad \text{with} \quad \tilde{r}(k) = \overline{r(-k)}$$

But this requirement precisely means that we should have

$$\|s\|^2 = \sum_{k \in \mathbb{Z}} |s(k)|^2$$

$$= \sum_{k \in 2\mathbb{Z}} |\eta_0 * s(k)|^2 + \sum_{k \in 2\mathbb{Z} + 1} |\eta_1 * s(k)|^2$$

$$= \sum_{k \in 2\mathbb{Z}} |\langle T_k \tilde{\eta}_0 \mid s \rangle|^2 + \sum_{k \in 2\mathbb{Z} + 1} |\langle T_k \tilde{\eta}_1 \mid s \rangle|^2.$$

Therefore if we set

$$\phi_0 = \tilde{\eta}_0, \quad \phi_1 = T_1^* \tilde{\eta}_1,$$

then this precisely means that the set of $2\mathbb{Z}$-translated functions satisfies at

$$\{T_k \phi_0 : k \in 2\mathbb{Z}\} \cup \{T_k \phi_1 : k \in 2\mathbb{Z}\} \quad \text{is an o.n.b. of } L^2(\mathbb{Z})$$

(1.2)

This gives rise to a definition
(1.3) Definition. A pair of sequences in \( L^2(\mathbb{Z}) \) satisfying (1.2) is called a QMF-system (=Quadratic Mirror Filter system).

The reason for the name “Quadratic Mirror Filter” becomes clear later on when we rewrite the QMF condition in Fourier space.

By definition a pair of sequences \( \phi_0, \phi_1 \in L^2(\mathbb{Z}) \) is a QMF system if and only if

\begin{enumerate}
  \item the set \( \{T_{2k}\phi_0, T_{2k}\phi_1 : k \in \mathbb{Z}\} \) is complete
  \item \( \langle T_k\phi_i | T_{k'}\phi_{i'} \rangle = \delta(i - i')\delta(k - k') \) for \( i, i' \in \{0, 1\} \) and \( k, k' \in \mathbb{Z} \).
\end{enumerate}

However as we shall see later in theorem (3.7) condition ii is sufficient.

2. The group of unitary operators with \([U, T_2] = 0\).

We now want to show that the set of QMF-systems has a natural group structure. It turns out to be isomorph to a subgroup of unitary operators acting in the Hilbert space \( L^2(\mathbb{Z}) \). We shall denote the group of all unitary operators acting in \( L^2(\mathbb{Z}) \) by \( \mathcal{U}(L^2(\mathbb{Z})) \). Let \( U \) be a unitary operator acting on sequences in \( L^2(\mathbb{Z}) \). Suppose further that it commutes with the translations by \( 2 \)

\[ [T_2, U] = T_2U - UT_2 = 0, \quad \text{or equivalently} \quad T_2U T_2^* = U \quad (2.1) \]

Because of the second equation this class of operators shall be denoted by \( \text{Fix}(T_2, \mathcal{U}(L^2(\mathbb{Z}))) \). Let now \( \phi_0, \phi_1 \in L^2(\mathbb{Z}) \) be a QMF-system. Then the image of \( \phi_0 \) and \( \phi_1 \) under any unitary \( U \) satisfying (2.1) clearly is again a QMF-system. Indeed as unitary images they are again an orthonormal base and in addition we have for \( i, i' \in \{0, 1\} \) and \( k, k' \in \mathbb{Z} \)

\[ \langle T_{2k}U\phi_i | T_{2k'}U\phi_{i'} \rangle = \langle UT_{2k}\phi_i | UT_{2k'}\phi_{i'} \rangle = \langle T_{2k}\phi_i | T_{2k'}\phi_{i'} \rangle = \delta(i - i')\delta(k - k') \]

Now clearly the sequences \( \phi_0 = \delta, \phi_1 = T_1\delta \) are a QMF-system. We claim that any other QMF-system can be obtained as image of \( \delta \) and \( T_1\delta \) under some unitary operator \( U \) that commutes with the \( 2\mathbb{Z} \)-translates. To see this let \( \phi_0, \phi_1 \) be a QMF-system. Clearly any operator that commutes with the translations with respect to \( 2\mathbb{Z} \) is completely determined by its image on one single fundamental domain \( \mathbb{Z}/2\mathbb{Z} = \{0, 1\} \). Therefore if we set

\[ U\delta = \phi_0, \quad UT_1\delta = \phi_1 \quad (2.2) \]

then there is at most one such unitary \( U \) acting on all of \( L^2(\mathbb{Z}) \) and satisfying \([T_2, U] = 0\). There actually is one and it is given by

\[ U : s \mapsto (I_0s) \ast \phi_0 + (I_1s) \ast (T_1\phi_1). \quad (2.3) \]

It obviously commutes with \( T_2 \) and we only have to check its unitarity. But since it sends the orthonormal base \( \{T_{2k}\delta, T_{2k+1}\delta\} \) into the orthonormal base \( \{T_{2k}\phi_0, T_{2k+1}\phi_1\} \), we are done. Therefore we have shown:
\(2.4\) Theorem. There is a bijection between the group \(\text{Fix}(T_2, \mathcal{U}(L^2(\mathbb{Z})))\) and the QMF-systems \(\phi_0, \phi_1\). This relation is explicitly given by (2.2) and (2.3).

This shows that the QMF have a natural group structure. In particular one can produce new ones from old ones by multiplication of their associated unitary operators.

3. The Fourier space picture

The definition of Fourier transform that we use is the following
\[
\hat{\cdot}: L^2(\mathbb{Z}) \mapsto L^2(\mathbb{T}), \quad \hat{s}(\omega) = \sum_{k \in \mathbb{Z}} s(k) e^{-i\omega k}.
\]

Now the action of \(\Pi_0\) and \(\Pi_1\) in Fourier space is easily computed yielding
\[
\Pi_0: \hat{s}(\omega) \mapsto \frac{1}{2}(\hat{s}(\omega) + \hat{s}(\omega + \pi)), \quad \Pi_1: \hat{s}(\omega) \mapsto \frac{1}{2}(\hat{s}(\omega) - \hat{s}(\omega + \pi)). \tag{3.1}
\]

Now ortho-normality of the \(2\mathbb{Z}\)-translates of a sequence \(\phi_0\) is equivalent to
\[
(\tilde{\phi}_0 \ast \phi_0)(2k) = \delta(k), \quad \text{with} \quad \tilde{\phi}_0(n) = -\phi_0(-n)
\]
which also reads
\[
\Pi_0 (\tilde{\phi}_0 \ast \phi_0) = \delta.
\]

We therefore see from (3.1) that being a QMF-filter system (1.2) implies that
\[
\left|\hat{\phi}_0(\omega)\right|^2 + \left|\hat{\phi}_0(\omega + \pi)\right|^2 = 2, \quad \text{for almost all } \omega.
\]

This explains the name Quadratic Mirror Filter. In the same way we obtain
\[
\left|\hat{\phi}_1(\omega)\right|^2 + \left|\hat{\phi}_1(\omega + \pi)\right|^2 = 2, \quad \text{for almost all } \omega.
\]

Now mutual orthogonality of the respective translates \(T_{2k}\phi_0\) and \(T_{2k'}\phi_1\) is equivalent to
\[
\Pi_0 (\tilde{\phi}_0 \ast \phi_1) = 0
\]
or in Fourier space
\[
\overline{\hat{\phi}_0}(\omega)\hat{\phi}_1(\omega) + \overline{\phi}_0(\omega + \pi)\hat{\phi}_1(\omega + \pi) = 0, \quad \text{for almost all } \omega.
\]

All these relations can be summarized by saying that the matrix
\[
\frac{1}{\sqrt{2}} \begin{bmatrix} \hat{\phi}_0(\omega) & \hat{\phi}_0(\omega + \pi) \\ \hat{\phi}_1(\omega) & \hat{\phi}_1(\omega + \pi) \end{bmatrix}
\]
\[\tag{3.2}\]
is unitary for all \( \omega \). As we shall see in a while (theorem (3.7)) the inverse is also true: any such matrix valued function \( \chi : T \to U(2) \) satisfying

\[
\chi(\omega + \pi) = \chi(\omega) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\] (3.3)
gives rise to a QMF-system. We shall call such a matrix a QMF-matrix. By what we have seen, such a matrix defines a pair of sequences such that their \( 2\mathbb{Z} \) translates are an ortho-normal set. However the completeness still has to be checked.

Since we have already established the equivalence between QMF-systems and unitary operators \( U \) that commute with \( T_2 \) let us now exploit this relation by writing the action of the unitary operator \( U \) in Fourier space. From (2.3) it follows that we have

\[
U : \hat{s}(\omega) \mapsto \frac{1}{2} (\hat{s}(\omega) + \hat{s}(\omega + \pi)) \hat{\phi}_0(\omega) + \frac{1}{2} (\hat{s}(\omega) - \hat{s}(\omega + \pi)) \hat{\phi}_1(\omega) e^{i\omega}
\]

But now the action of \( U \) on \( \hat{s}(\omega + \pi) \) looks similar and therefore if we introduce the vector

\[
\begin{bmatrix} \hat{s}(\omega) \\ \hat{s}(\omega + \pi) \end{bmatrix}
\]

we may write

\[
U : \begin{bmatrix} \hat{s}(\omega) \\ \hat{s}(\omega + \pi) \end{bmatrix} \mapsto \frac{1}{2} \begin{bmatrix} \hat{\phi}_0(\omega) + \hat{\phi}_1(\omega) e^{i\omega} & \hat{\phi}_0(\omega) - \hat{\phi}_1(\omega) e^{i\omega} \\ \hat{\phi}_0(\omega + \pi) - \hat{\phi}_1(\omega + \pi) e^{i\omega} & \hat{\phi}_0(\omega + \pi) + \hat{\phi}_1(\omega + \pi) e^{i\omega} \end{bmatrix} \begin{bmatrix} \hat{s}(\omega) \\ \hat{s}(\omega + \pi) \end{bmatrix}
\]

or equivalently

\[
U : \begin{bmatrix} \hat{s}(\omega) \\ \hat{s}(\omega + \pi) \end{bmatrix} \mapsto \frac{1}{2} \begin{bmatrix} \hat{\phi}_0(\omega) & \hat{\phi}_1(\omega) \\ \hat{\phi}_0(\omega + \pi) & \hat{\phi}_1(\omega + \pi) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ e^{i\omega} & -e^{i\omega} \end{bmatrix} \begin{bmatrix} \hat{s}(\omega) \\ \hat{s}(\omega + \pi) \end{bmatrix}
\]

This is a product of two unitary matrices and hence unitary itself. It is a matrix valued function \( \eta : T \to U(2) \) that satisfies at

\[
\eta(\omega + \pi) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \eta(\omega) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

The space of such mappings forms a group under pointwise multiplication. Vice versa suppose we are given such a matrix-valued function. Upon setting

\[
U : \begin{bmatrix} \hat{s}(\omega) \\ \hat{s}(\omega + \pi) \end{bmatrix} \mapsto \eta(\omega) \begin{bmatrix} \hat{s}(\omega) \\ \hat{s}(\omega + \pi) \end{bmatrix}
\] (3.4)

this defines an operator acting on \( \hat{s}(\omega) \) and the above condition precisely assures that this operator is well defined as can be seen upon replacing \( \omega \) with \( \omega + \pi \). It clearly is unitary. And the translation operator \( T_2 \) becomes diagonal

\[
T_2 : \begin{bmatrix} \hat{s}(\omega) \\ \hat{s}(\omega + \pi) \end{bmatrix} \mapsto \begin{bmatrix} e^{-2i\pi \omega} & 0 \\ 0 & e^{-2i\pi \omega} \end{bmatrix} \begin{bmatrix} \hat{s}(\omega) \\ \hat{s}(\omega + \pi) \end{bmatrix}
\]

and therefore \( U \) commutes with \( T_2 \). We therefore have shown
(3.5) Theorem. There is a bijection of the subgroup of all unitary operators acting in $L^2(\mathbb{Z})$ with $[T_2, U] = 0$ and the group of measurable mappings $\eta : \mathbb{T} \rightarrow U(2)$ satisfying (almost everywhere)

$$\eta(\omega + \pi) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \eta(\omega) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (3.6)$$

The correspondence is given by (3.4).

Suppose now that $\chi : \mathbb{T} \rightarrow U(2)$ satisfies (3.3). Since

$$\rho(\omega) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{i\omega} \\ 1 & -e^{i\omega} \end{bmatrix}$$

satisfies at

$$\rho(\omega + \pi) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rho(\omega),$$

it follows that the combination $\xi(\omega) = \rho(\omega) \chi(\omega)$ satisfies (3.6). It therefore gives rise to a unitary operator which implies that $\chi(\omega)$ is coming from a QMF-system. To summarize we have shown

(3.7) Theorem. There is a bijection between QMF-systems and measurable functions $\chi : \mathbb{T} \rightarrow U(2)$ satisfying (almost everywhere)

$$\chi(\omega + \pi) = \chi(\omega) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (3.8)$$

it reads explicitly

$$\begin{bmatrix} \hat{\phi}_0(\omega) & \hat{\phi}_0(\omega + \pi) \\ \hat{\phi}_1(\omega) & \hat{\phi}_1(\omega + \pi) \end{bmatrix} = \chi(\omega) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \chi(\omega + \pi) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

4. QMF and loop groups.

A map from the torus $\mathbb{T}$ to the space of unitary matrices $U(2)$ is called a closed loop in $U(2)$. The space of all loops, $\text{Map}(\mathbb{T} \rightarrow U(2))$, is a group under pointwise multiplication (e.g. [5]).

If we identify $\mathbb{T}$ with $\mathbb{R}/2\pi\mathbb{Z}$ then the loop condition reads

$$\xi(\omega) \in U(2), \quad \xi(\omega + 2\pi) = \xi(\omega)$$

The set of functions $\xi : \mathbb{T} \rightarrow U(2)$ such that $T_\pi \xi(\omega) = \xi(\omega + \pi) = \xi(\omega)$ form a (non normal) subgroup that we shall denote by $\text{Fix}(T_\pi, \text{Map}(\mathbb{T} \rightarrow U(2)))$. This group may be identified with $\text{Map}(\mathbb{T} \rightarrow U(2))$ itself upon replacing $\xi(\omega)$.
with $\xi(\omega/2)$. Next consider the matrices of the special form (3.2), or what is the same a loop $\chi(\omega)$ satisfying (3.3). The set of such loops does not form a group. However this set of loops is a coset; to be more precise let $\chi_1(\omega)$ and $\chi_2(\omega)$ be two such loops. Then $\xi(\omega) = \chi_1(\omega) \chi_2^*(\omega)$ satisfies at

$$\xi(\omega + \pi) = \chi_1(\omega + \pi) \chi_2^*(\omega + \pi) = \chi_1(\omega) \chi_2^*(\omega) = \xi(\omega),$$

and is therefore an element of $\text{Fix}(T_{\pi}, \text{Map}(T \rightarrow U(2)))$. In addition all loops in this subgroup can be obtained that way since for $\xi \in \text{Fix}(T_{\pi}, \text{Map}(T \rightarrow U(2)))$ we may set

$$\chi_2(\omega) = \xi^*(\omega) \chi_1(\omega),$$

and this loop satisfies at (3.8) and it gives back $\xi$ by setting $\xi(\omega) = \chi_1(\omega) \chi_2^*(\omega)$.

To summarize we have shown

(4.1) Theorem. The set of QMF-loops (3.2) is a right coset of $\text{Fix}(T_{\pi}, \text{Map}(T \rightarrow U(2)))$. Therefore let $\chi(\omega)$ be any QMF-matrix (3.2). Then all others can be obtained in a unique way as

$$\eta(\omega) = \chi(\omega) \rho(\omega) \quad \text{where } \rho \text{ runns through } \text{Fix}(T_{\pi}, \text{Map}(T \rightarrow U(2))).$$

Let us now look at the other combination; that is consider the set of loops $\xi(\omega) = \chi_1^*(\omega) \chi_2(\omega)$. Note that this set is not just the conjugate set of the previous one since $\chi^*(\omega)$ does not satisfy (3.3) any more. We rather have now

$$\xi(\omega + \pi) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \xi(\omega) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This is again a (non normal) subgroup of all loops. Its elements are called twisted loops. If we introduce the operator $\sigma$ acting on the loop group via conjugation

$$\sigma : \rho(\omega) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rho(\omega) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

then we can write for the group of twisted loops $\text{Fix}(\sigma T_{\pi}, \text{Map}(T \rightarrow U(2)))$. Again we have that the set of QMF is a coset which this time is a left-coset.

(4.2) Theorem. The set of QMF-loops (3.2) is a left coset of $\text{Fix}(\sigma T_{\pi}, \text{Map}(T \rightarrow U(2)))$. Therefore let $\chi(\omega)$ be any QMF-matrix (3.2). Then all other QMF-matrices can be obtained in a unique way as

$$\eta(\omega) = \rho(\omega) \chi(\omega) \quad \text{where } \rho \text{ runns through } \text{Fix}(\sigma T_{\pi}, \text{Map}(T \rightarrow U(2))).$$
5. Some subclasses of QMF.

We now shall consider various subgroups of the whole loop-group and their associated QMF-systems. The correspondence is made in the spirit of the preceding theorems; that is the set of QMF matrices will be identified with a right coset of a given subgroup of all loops.

a) **The smooth loops.** Here smoothness means that every matrix element of $\xi(\omega)$ is a function in $C^\infty(\mathbb{T})$. The space of all smooth loops, $\text{Map}(\mathbb{T} \to U(2) : C^\infty)$, is a Lie group under pointwise multiplication (e.g. [5]). As we know already these loops correspond to QMF-systems $\phi_0, \phi_1$, whose Fourier transforms are in $C^\infty(\mathbb{T})$, or what is the same, whose sequences are arbitrary well localized in the sense that

$$
\sup_{n \in \mathbb{Z}} (1 + |n|)^\alpha |\phi_i(n)| < \infty, \quad \text{for all } \alpha > 0, \quad i \in \{0, 1\}.
$$

Since

$$
\omega \to \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{i\omega} \\ 1 & -e^{i\omega} \end{bmatrix}
$$

is a smooth loop that is at the same time a QMF loop, we see that all highly localized QMF-systems are obtained as a right coset

$$
\omega \to \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{i\omega} \\ 1 & -e^{i\omega} \end{bmatrix} \eta(\omega)
$$

where $\eta$ runs through $\text{Fix}(T_\pi, \text{Map}(\mathbb{T} \to U(2) : C^\infty))$. This last group is isomorphic to the group of smooth loops $\text{Map}(\mathbb{T} \to U(2) : C^\infty)$.

b.) **The polynomial loops.** This is the class of loops for which the matrix elements of $\xi(\omega)$ are trigonometric polynomials (= Laurent polynomials in $z = e^{i\omega}$). We shall denote this set by $\text{Map}(\mathbb{T} \to U(2) : \text{poly})$. The set of QMF-systems with finite impulse response (= their convolution with a delta sequence has only a finite number of non-vanishing terms) is a right-coset of $\text{Fix}(T_\pi, \text{Map}(\mathbb{T} \to U(2) : \text{poly}))$. This last group is isomorphic to the group of all polynomial loops $\text{Map}(\mathbb{T} \to U(2) : \text{poly})$.

Since

$$
\omega \to \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{i\omega} \\ 1 & -e^{i\omega} \end{bmatrix}
$$

is a polynomial loop that defines a QMF, only the last statement has to be shown yet. To see this note that the subgroup of polynomial loops in $\text{Fix}(T_\pi, \text{Map}(\mathbb{T} \to U(2) : \text{poly})$ is isomorphically mapped onto the group of all polynomial loops via the identification $\eta(\omega) \to \eta(\omega/2)$. Indeed from $\eta(\omega + \pi) = \eta(\omega)$ it follows easily that $\eta(\omega)$ must be a polynomial in $e^{2i\pi \omega}$.

c.) **The vanishing moment loops.** Consider the subset of smooth loops that satisfy

$$
\xi(\omega) = 1 + o(\omega^n) \quad (\omega \to 0).
$$

(5.1)
This is again a normal subgroup of $\text{Map}(\mathfrak{g} \to U(2) : C^\infty)$. On the other hand consider the class of highly localized QMF-systems such that

$$
\begin{bmatrix}
\frac{1}{\sqrt{2}} \hat{\phi}_0(\omega) & \hat{\phi}_0(\omega + \pi) \\
\hat{\phi}_1(\omega) & \hat{\phi}_1(\omega + \pi)
\end{bmatrix} = \mathbb{1} + o(\omega^n) \quad (\omega \to 0),
$$

or what is the same that

$$
\hat{\phi}_0(\omega) = \sqrt{2} + o(\omega^n), \quad \hat{\phi}_1(\omega) = o(\omega^n) \quad (\omega \to 0).
$$

The second condition is equivalent to the fact that the first $n$ moments of $\phi_1$ vanish

$$
\sum_{m \in \mathbb{N}} m^p \phi_1(m) = 0, \quad p = 0, 1, \ldots, n.
$$

Again the subclass of such QMF-matrices is a right coset of the subgroup of smooth loops in $\text{Fix}(T_\pi, \text{Map}(\mathfrak{g} \to U(2) : C^\infty))$ satisfying (5.1). The analogue statement holds for the polynomial loops and the finite impulse response QMF systems.

6. The factorization problem.

Because of its particular importance in signal processing we shall have a closer look at the polynomial loops. In particular we shall give a technique how to generate all finite impulse response QMF-systems.

Given that the polynomial loops are an infinite dimensional Lie group, the question arises whether or not there is a small family of loops that generate all polynomial loops. Equivalently the problem is how to generate in an economic way all finite impulse response QMF. Such a factorization exists as can be seen by general theorems about loop groups (e.g. in [5]). Instead of reproducing the demonstration given there we present a different, more geometric construction.

As we have seen we may equally well consider the group of unitary operators that commute with the translates with respect to the sub-lattice $2\mathbb{Z}$.

Consider the fundamental domain $\Gamma = \{0, 1\}$ of $2\mathbb{Z}$ in $\mathbb{Z}$. Recall that a fundamental domain is a set $\Gamma \subset \mathbb{Z}$ whose translates with respect to $2\mathbb{Z}$ are a mutually disjoint cover of $\mathbb{Z}$. Suppose that $U$ is a unitary operator acting in $L^2(\mathbb{Z})$ that commutes with $T_2$. Clearly by translation invariance such an operator is completely determined by its action on the fundamental domain $\Gamma$. We now pick a unitary 2 by 2 matrix $u$ in $U(2)$. It acts in a natural way in the two-dimensional vector space $L^2(\mathbb{Z})$ that we consider in a natural way as a subspace of $L^2(\mathbb{Z})$. Now consider the well defined operator $U_u$, with $[U_u, T_2] = 0$, whose restriction to the fundamental domain $\Gamma$ is given by $u$. Since this operator preserves the $L^2$-norm in each of the translated fundamental domains we find that $U_u$ is unitary. We therefore have found a family of unitary operators that commute with the translations by 2. This family is indexed by points in the 4-dimensional manifold $U(2)$.

This family together with the translation operators $T_1$ actually generates all finite impulse response QMF.
(6.1) Theorem. Let \( U : L^2(\mathbb{Z}) \rightarrow L^2(\mathbb{Z}) \) be a unitary operator with \([U, T_2] = 0\). Suppose in addition that the associated QMF \( \phi_0 = U \delta_0 \) and \( \phi_1 = U \delta_1 \) is of finite impulse response with length not larger than \( 2N \) (=each sequence has at most \( 2N \) non-zero coefficients). Then \( U \) can be factorized as follows

\[ U = T_p U_{u_{2N}} T_1 U_{u_{2N-1}} \ldots T_1 U_{u_1} \text{ or} \]

with some \( p \in \mathbb{Z} \). If all coefficients are real valued then the \( u_n \) can be chosen in \( O(2) \).

Proof. Let \( \phi_0 = U \delta_0 \) and \( \phi_1 = U \delta_1 \) be the associated QMF system. By an overall translation we may suppose that the support of \( \phi_0 \) and \( \phi_1 \) is contained in \([0, 2N - 1]\). By the QMF-property we have

\[ \langle \phi_0 \mid T_{2(N-1)} \phi_0 \rangle = \langle \phi_1 \mid T_{2(N-1)} \phi_1 \rangle = \langle \phi_0 \mid T_{2(N-1)} \phi_1 \rangle = \langle \phi_1 \mid T_{2(N-1)} \phi_0 \rangle = 0 \]

unless \( N = 1 \). This shows that the four vectors

\[ e_0^t = [\phi_0(0), \phi_0(1)], \quad e_0^r = [\phi_0(2N - 2), \phi_0(2N - 1)] \]
\[ e_1^t = [\phi_1(0), \phi_1(1)], \quad e_1^r = [\phi_1(2N-2), \phi_1(2N-1)] \]

are pairwise orthogonal. We thus can find a two by two matrix \( u_N \in SU(2) \) such that

\[ \langle u_N e_0^t \mid [1, 0] \rangle = \langle u_N e_1^t \mid [1, 0] \rangle = \langle u_N e_0^r \mid [0, 1] \rangle = \langle u_N e_1^r \mid [0, 1] \rangle = 0. \]

Therefore \( U_{u_N} \phi_0 \) and \( U_{u_N} \phi_1 \) are supported by a subset of \([1, 2N - 2]\). We therefore may repeat the procedure with \( T_1^* U_{u_N} \phi_0 \) and \( T_1^* U_{u_N} \phi_1 \) and so on until \( N = 1 \). Here we are left with a two dimensional vector space with two orthogonal vectors. We may again find a \( u_1 \in U(2) \) that maps these two vectors to \([1, 0]\) and \([0, 1]\). Upon reverting all operations we have found the desired factorization. The case where all sequences are real valued follows the same way.

We note that in the one dimensional case a different factorization has been obtained before in [4].

7. The general case

We now want to formulate the results of the preceding section for general lattices; by a lattice we mean a subgroup of \( \mathbb{Z}^n \) that is isomorph to \( \mathbb{Z}^n \).

Let \( \Gamma \) be a lattice in \( H = \mathbb{Z}^n \). Let \( N = \lvert H/\Gamma \rvert \) be the number of elements in the quotient group. As before we define

(7.1) Definition. A system of \( N \) sequences \( \phi_k \in L^2(H) \), \( k = 0, 1, \ldots, N-1 \) is called a QMF-system if and only if the set of all \( \Gamma \)-translates of these sequences is an orthonormal basis of the whole space \( L^2(H) \).
More explicitly we want to have

\[ \{ T_m \phi_k : k = 0, 1, \ldots, N - 1, m \in \Gamma \} \]

is an ortho-normal basis of \( L^2(H) \).

which is again equivalent to

1) \( T_m \phi_k : k = 0, 1, \ldots, N - 1, m \in \Gamma \) is complete

2) \( \langle T_m \phi_k | T_{m'} \phi_{k'} \rangle = \delta(m - m') \delta(k - k') \).

As we shall see again condition \( ii \) alone is sufficient.

Consider now the group of unitary operators acting in the Hilbert space \( L^2(H) \) that commute with the translates with respect to the sub-grid \( \Gamma \subset H \)

\[ [K, T_m] = K T_m - T_m K \quad \text{for all } k \in \Gamma \]

with the translation operator defined as

\[ T_m : L^2(H) \rightarrow L^2(H), \quad s(\cdot) \mapsto s(\cdot - m). \]

Let \( \{ \phi_k \} \) be a QMF-system. We claim that the image of a QMF-system under a unitary operator \( K \) that commutes with the \( \Gamma \)-translates is again a QMF-system. Indeed as image of an orthonormal basis it is again an ortho-normal basis and in addition we have as before

\[ \langle T_{k'} K \phi_i | T_k \phi_i' \rangle = \langle K T_k \phi_i | K T_k' \phi_i' \rangle = \delta(i - i') \delta(k - k'). \]

Now consider the quotient \( H/\Gamma \) that we may identify with a fundamental domain for the action of \( \Gamma \) on \( H \). It contains \( N = |H/\Gamma| \) points. Clearly the family of delta sequences \( \{ T_k \delta : k \in H/\Gamma \} \) is a QMF-system. But its image is by what we have said again a QMF-system. Therefore upon setting

\[ \phi_k = K T_k \delta, \quad k \in H/\Gamma \quad (7.2) \]

we get a whole family of QMF-systems. Actually we get all of them that way. Indeed every unitary operator that commutes with the \( \Gamma \)-translations is uniquely determined by the its image of the delta sequences \( T_k \delta \) with \( k \in H/\Gamma \). Let us call these functions again \( \phi_k \). We then can recover \( K \) explicitly by means of the following: let \( \Pi_\Gamma \) be the orthogonal projector on \( L^2(\Gamma) \) taken in the obvious way as a subspace of \( L^2(H) \). We now set

\[ K : s \mapsto \sum_{k \in \Gamma} \langle T_k \Pi_\Gamma T_k^* s | T_k^* \phi_k \rangle = \sum_{k \in \Gamma} \langle \Pi_\Gamma T_k s | T_k^* \phi_k \rangle \quad (7.3) \]

This operator clearly commutes with the \( \Gamma \)-translations and its image of the \( \delta_k \) with \( k \in H/\Gamma \) are just the \( \phi_k \). But now the \( \{ \phi_k \} \) are an QMF-system and the operator as defined by (7.3) is therefore unitary. We therefore have shown
(7.4) **Theorem.** There is a bijection between QMF-systems and the group of unitary operators $U$ acting in $L^2(H)$ such that

$$[U, T_m] = 0, \quad \text{for all } m \in \Gamma.$$  

This bijection is explicitly given by (7.2) and (7.3).

We now come to the Fourier space picture. The Fourier transform of a sequence $s$ over $H = \mathbb{Z}^n$ is defined as

$$\hat{s}(\omega) = \sum_{m \in H} e^{-i\langle \omega | m \rangle}, \quad \omega \in \mathbb{T}^n,$$

Now consider a sub-lattice $\Gamma \subset H$. Recall that the dual lattice $\Gamma^\perp$ of $\Gamma$ is defined as

$$k \in \Gamma^\perp \iff e^{i\langle k | m \rangle} = 1 \quad \text{for all } m \in \Gamma,$$

where the scalar-product is the one naturally inherited from $\mathbb{R}^n$. The quotient group $\Gamma^\perp / H^\perp$ may be identified with a subgroup of $\mathbb{T}^n$.

We now want to rephrase the QMF-condition in Fourier space. Recall that the Poisson summation formula states that the projection operator $\Pi_{\Gamma}$ reads in Fourier space

$$\Pi_{\Gamma} : \hat{s}(\omega) \mapsto \frac{1}{|H/\Gamma|} \sum_{\xi \in \Gamma^\perp / H^\perp} \hat{s}(\omega + \xi). \quad (7.5)$$

As before, the orthogonality relations can also be written

$$\Pi_{\Gamma}(\hat{\phi}_k * \hat{\phi}_{k'}) = \delta(k - k') \delta, \quad \text{with } \overline{\hat{\phi}_k}(p) = \overline{\hat{\phi}_k}(-p)$$

Therefore from (7.5) we have

$$\sum_{\xi \in \Gamma^\perp / H^\perp} \overline{\hat{\phi}_k}(\omega + \xi) \hat{\phi}_k(\omega + \xi) = |H/\Gamma| \delta(k - k').$$

To put it still differently, the ortho-normality condition reads

$$M_{k,\xi}(\omega) = \frac{1}{\sqrt{|H/\Gamma|}} \hat{\phi}_k(\omega + \xi), \quad k \in \Gamma / H, \ \xi \in \Gamma^\perp / H^\perp \quad (7.6)$$

is unitary for every $\omega \in \mathbb{T}^n$. Again the converse is true also. To see this consider again the unitary operator $K$ as given by (7.3). In Fourier space it acts as follows

$$K : \hat{s}(\omega) \mapsto \frac{1}{|H/\Gamma|} \sum_{k \in \Gamma / H} \sum_{\xi \in \Gamma^\perp / H^\perp} \hat{s}(\omega + \xi) e^{i\langle \omega + \xi | k \rangle} \hat{\phi}_k(\omega)$$

But the functions $\hat{s}(\omega + \rho), \rho \in \Gamma^\perp / H^\perp$ have a similar transformation behavior, namely
If we introduce now the vector $N$-dimensional vector with components $\widehat{s}_\xi(\omega) = \widehat{s}(\omega + \xi)$, $\xi \in \Gamma^\perp/H^\perp$ then the action of $K$ becomes just matrix multiplication:

$$K : s_\rho(\omega) \mapsto \sum_{\xi \in \Gamma^\perp/H^\perp} K_{\rho,\xi}(\omega) s_\xi(\omega)$$

with

$$K_{\rho,\xi}(\omega) = \frac{1}{|H/\Gamma|} \sum_{k \in \Gamma} e^{i(\omega + \xi | k)} \widehat{\varphi}_k(\omega + \rho).$$

The matrix can still be simplified by writing

$$K_{\rho,\xi}(\omega) = \sum_{k \in \Gamma} F_{k,\xi}(\omega) M_{k,\rho}(\omega)$$

with $M_{k,\rho}(\omega)$ the unitary matrix given by (7.6) and

$$F_{k,\xi}(\omega) = |H/\Gamma|^{-1/2} e^{i(\omega + \xi | k)}$$

is the complex conjugated of the QMF-matrix associated to the QMF-system $\delta_k, k \in H/\Gamma$. Therefore in particular this matrix is unitary and therefore the matrix $K_{\rho,\xi}(\omega)$ is unitary.

To better understand the structure of this kind of matrix valued function as in (7.6) we introduce the permutation representation of $\Gamma^\perp/H^\perp$; that is

$$R(\xi) = R_{\omega,\rho}(\xi) = \delta_{\omega,\xi + \rho}$$

Now the loops of the kind (7.6) are precisely the loops $M(\omega) = M_{k,\xi}(\omega)$ that satisfy at

$$M(\omega + \rho) = M(\omega) R^*(\rho), \quad \rho \in \Gamma^\perp/H^\perp. \quad (7.8)$$

The matrix valued function $K$ defining $U$ instead satisfies at

$$K(\omega + \rho) = R(\rho) K(\omega) R^*(\rho). \quad (7.9)$$

Vice versa let $\chi$ be a function from $\mathbb{T}^n$ with values in $U(N)$ that satisfies the above condition. Consider the associated operator defined through its action on the vector $\widehat{s}_\xi(\omega) = \widehat{s}(\omega + \xi)$ for each $\omega$ via

$$U : \widehat{s}_\xi(\omega) \mapsto \sum_{\rho \in H^\perp} \chi_{\xi,\rho}(\omega) \widehat{s}_\rho(\omega)$$

Since by hypothesis we have $\chi(\omega + \rho) = R(\rho) \chi(\omega) R^*(\rho)$, and hence the operation is well defined as operator on a function. In addition the operator induced
by $\chi(\omega)$ via its action on the $\hat{s}(\omega + \xi)$ is clearly unitary since it is invertible and it preserves the energy. Again the translation operator is diagonal

$$T_k : \hat{s}_\xi(\omega) \rightarrow e^{-i\langle \omega | k \rangle} \hat{s}_\xi(\omega), \quad k \in \Gamma$$

and thus the operator $U$ commutes with the $\Gamma$-translates.

8. Loop groups and QMF

A map $\varphi$ from the torus $T^n$ into the group $U(N)$ is called a loop. The set of loops is again a group under pointwise multiplication. All matrix valued functions we have encountered so far are elements of such a loop group. The set of loops satisfying at (7.9) is again a group. It is called the group of twisted loops. Upon conjugating a twisted loop with $F$ defined in (7.7) we see that the twisted loops are isomorph to the set of $\Gamma^\perp$-periodic loops

$$\eta(\omega + \rho) = \eta(\omega), \quad \rho \in \Gamma^\perp.$$  

This group is denoted by $\text{Fix}(T_{\Gamma^\perp}, \text{Map}(T^n \rightarrow U(N)))$. It is again isomorphic to the whole loop-group. Indeed by hypothesis $\Gamma$ is isomorphic to $H$. Thus $\Gamma^\perp$ is isomorphic to $H^\perp$. Let $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear, invertible map who sends $\Gamma^\perp$ onto $H^\perp$. Then any loop in $\text{Fix}(T_{\Gamma^\perp}, \text{Map}(T^n \rightarrow U(N)))$ is mapped via $\eta(\omega) \rightarrow \eta(\Phi(\omega))$ into a loop in $\text{Map}(T^n \rightarrow U(N)))$, and this defines the isomorphism we where looking for. As in the simplest case that we have discussed in great detail, the QMF-matrices are right cosets of $\text{Fix}(T_{\Gamma^\perp}, \text{Map}(T^n \rightarrow U(N)))$ and of its analogue sub-groups respectively. The argumentation follows exactly the same line as before and the details are left to the reader. To summarize we have shown

(8.1) Theorem. There is an explicit bijection between

i) the QMF-filters

ii) the QMF-loops: $M(\omega \rho) = M(\omega) R^*(\rho), \quad \rho \in \Gamma^\perp$

iii) the group of unitary operators with $[K, S^H_x] = 0, \quad x \in \Gamma$

iv) the twisted loops: $B(\omega \rho) = R(\rho) B(\omega) R^*(\rho), \quad \rho \in \Gamma^\perp$

iiv) the untwisted loops: $A(\omega \rho) = A(\omega), \quad \rho \in \Gamma^\perp$

iiiv) the whole loop group

More precisely the groups iii, iv, iiv and iiiv are isomorphic. The set of QMF-loops is a left coset with respect to iv and a right coset with respect to iiv.

We only have a closer look at the polynomial loops or what is the same, the space of finite impulse response QMF-systems or equivalently the subgroup of unitary operators in $L^2(H)$ that commute with the $\Gamma$-translates and that leave invariant the space of compactly supported sequences.
**Theorem.** The group \( \text{Fix}(T_{\Gamma^\perp}, \text{Map}(\mathbb{T}^n \to U(N) : \text{poly})) \) is isomorphic to the group \( \text{Map}(\mathbb{T}^n \to U(N) : \text{poly}) \).

**Proof.** Let \( \eta \in \text{Fix}(T_{\Gamma^\perp}, \text{Map}(\mathbb{T}^n \to U(N) : \text{poly})) \). Then

\[
\eta(\omega) = \sum_{k \in \mathbb{Z}} U_k e^{i(k|\omega)}
\]

where the coefficients are matrices in \( U(N) \) only a finite number of which is different from 0 so that the sum is finite. In addition we have \( \eta(\omega + \xi) = \eta(\omega) \) for all \( \xi \in \Gamma^\perp \). This implies

\[
\left\{ \sum_{k \in \Gamma} + \sum_{k \not\in \Gamma} \right\} U_k e^{i(k|\omega+\xi)} = \left\{ \sum_{k \in \Gamma} + \sum_{k \not\in \Gamma} \right\} U_k e^{i(k|\omega)}.
\]

Since \( e^{i(k|\xi)} = 1 \) in the first sum, it follows that

\[
\sum_{k \not\in \Gamma} U_k (e^{i(k|\xi)} - 1) e^{i(k|\omega)} = 0,
\]

which implies, since \( e^{i(k|\xi)} \neq 1 \) in this sum, that \( U_k = 0 \) for \( k \not\in \Gamma \). Therefore the group isomorphism defined above gives the desired isomorphism for polynomial loops. \(\square\)

**The factorization problem.** In more than one dimension it really is a problem. Only in the case \( n = 1 \) and arbitrary \( N \) factorization theorems for the polynomial loops are known. In all other cases the problem is open. However there is a general construction to obtain a large quantity of polynomial loops. It uses once more the equivalence between the unitary operators acting in \( L^2(H) \) and the loop groups. As in the easiest case we may define a unitary operator \( U^F_u \) whose restriction to some fundamental domain \( F = \Gamma/H \)—consisting of \( N \) non-congruent points modulo \( \Gamma \)—coincides with the unitary map \( u \in U(N) \). This defines a whole manifold of finite impulse response QMF-systems and hence of polynomial loops. By changing the fundamental domain we obtain another family of such unitary operators. In general the operators corresponding to different fundamental domains do not commute and thus one can compose them to obtain nontrivial new ones. The kind of factorization we propose therefore is the same as already proposed in [3].

\[
U = \prod_{p=0}^{m} U_{u_p}^{F_p}, \quad \text{with} \quad u_p \in U(N), \ F_p \text{ some fundamental domain}
\]

whether or not this family is exhaustive is not known yet. In any case it is a subgroup of all polynomial loops.
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