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A RIGOROUS DERIVATION OF THE EXTENDED KDV EQUATION

MARWA BERJAWI¹, TOUFIC EL ARWADI¹, SAMER ISRAWI²

ABSTRACT. The interesting background and historical development of KdV equations were discussed widely. These equations describe the propagation of water waves in weakly non-linear and weakly dispersive medium. Referring to physical derivation of KdV equations, scientists used to impose shallow water equations, thus the formal or physical derivation of KdV equations. However, these equations have rarely been derived rigorously. The aim of this paper is to giving insight into their rigorous mathematical derivation, instead of only referring to. Thereby, a rigorous derivation of two extended KdV equations: one on the velocity, other on the surface elevation. With this aim in mind, the primary research method for this paper will depend on the definition of consistency. Hence, a rigorous justification of new extended KdV equations will be provided thanks to this definition. This result provides a precise mathematical answer to a question raised by several authors in the last years, that is the verification of the extended KdV equations, derived previously, using formal methods.

1. Introduction

Over the decades, the struggle over energy resources has not stopped. Now, it is universally accepted that fossil fuels are finite, and with the passage of time it will be exhausted. Since it is a matter of time, scientists turned into finding new energy resources. Known as one of the world’s most abundant sources of renewable energy, ocean energy became an innovative solution for new energy challenges. Taking into consideration the huge amount of energy carried by ocean waves across the sea, many different techniques for converting wave energy to electric power have been investigated. However, generated as a result of wind blowing over the ocean surface, ocean energy is the most concentrated form of renewable energy on earth, also it is more predictable and consistent than wind or solar energy. Due to the difficulty and complexity of water waves problems, mathematicians, physicists and oceanographers get to find new asymptotic models with the same accuracy as the original ones, like Green-Naghdi and Boussinessq systems. Thence, the definition of consistency serves mathematicians in the resolution of these models, by deriving consistent equations. One of these non-linear equations is KdV equation, that is the subject of concentrated study to understand the physical phenomena of oceanography. Korteweg and his student de Vries derived their non-linear wave equation to describe the shallow water waves that Russell had observed in 1834. The most renowned KdV equation is (see [10, 4, 5, 6, 7, 8, 9]):

\[ u_t + u_x + \frac{3}{2} \varepsilon uu_x + \frac{\mu}{6} u_{xxx} = 0 \]
that was originally derived for flat bottom. What attracted the focus of scientists in this equation, was its integrability property, and thus its solitons (solitary waves) solutions. Since the derivation of the equation mentioned before, several methods have been used to derive new extended KdV equations, with different ocean conditions and properties. General derivations of this equation were justified with bottom, and with non constant coefficients, with topography (called KdV-top equation) [11, 10]. A formal derivation of KdV equation was provided using Whitham method, in the presence of surface tension [2], all these previous works were done up to \( O(\mu^2) \). Also, using some physical principles, an extended KdV equation was formally derived up to \( O(\mu^3) \) [13]. In the paper at hand, we deal with an irrotational, incompressible, inviscid fluid with a free surface, and constant density, acted on only by gravity. Knowing that \( a \) is the amplitude of the wave, \( \lambda \) is the wave-length of the wave, \( h_0 \) is the reference depth, denote by

\[
\Omega_t = \{(x, z) \in \mathbb{R} \times \mathbb{R}; -h_0 + b < z < \zeta, \}
\]

the domain of the fluid for each time \( t \) where the surface of the fluid is a graph parametrized by \( \zeta \) and its bottom is parametrized by \(-h_0+b\). Consider the following 1D Boussinesq extended system of equations of order \( O(\mu^3) \), with flat bottom:

\[
\begin{aligned}
\partial_t \zeta + [(1 + \varepsilon \zeta) u_x]_x &= 0 \\
u_t + \zeta_x + \varepsilon uu_x &= \frac{\mu}{3} u_{xxx} - \varepsilon \left[ \frac{1}{2} u_x^2 + \frac{1}{3} uu_{xx} \right]_x + \mu^2 \frac{\partial^2}{\partial x^2} \{ uu_{xx} \}
\end{aligned}
\]

(1)

where \( u \) is the fluid velocity, \( \zeta \) is the surface elevation,

\[
\mu = \frac{h_0^2}{\lambda^2}
\]

and

\[
\varepsilon = \frac{a}{h_0}
\]

are the shallowness and nonlinearity parameters respectively. Recall that the KdV scaling is \( \varepsilon = O(\mu) \), with \( 0 < \varepsilon \leq 1 \) and \( 0 < \mu \ll 1 \). The organization of this paper is as follows: in the second section, a derivation of the extended KdV equation will be done, in the first subsection 2.1, we will derive rigorously an extended KdV equation on the velocity \( u \). In the second subsection 2.2, a rigorous mathematical derivation of extended KdV equation on the surface elevation \( \zeta \) will be provided, and hence a rigorous verification of this imposed equation in [13]. The aim of this paper is to give a rigorous mathematical derivation of the extended KdV equations, up to \( O(\mu^3) \). Concerning the methodology, after the examination of some previous works, we will proceed as in [1], so we will use the definition of consistency to provide these rigorous derivations, which guarantee the relevance of these equations with (1), and serve in the construction of approximate solutions of Boussinesq equations.

2. Derivation of new extended KdV equation

The main goal of this section is to find extended KdV equation on velocity and on surface elevation. First of all we need some notation to this end we set the following remark.

**Remark 1.** For the sake of simplicity, we denote by \( O(\sigma) \) any family of functions \( (f^\sigma) \) such that \( \frac{1}{\sigma} f^\sigma \) remains bounded in \( L^\infty([0, T^*], H^r(\mathbb{R})) \) for all \( \sigma \in [0, 1] \), (and
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for possibly different values of \( r \). The same notation is also used for real numbers, e.g. \( \varepsilon = O(\mu) \), but this should not yield any confusion.

The new derived KdV equation on the velocity will be:

\[
\begin{align*}
\dot{u} + u_x + \frac{3}{2} \varepsilon u u_x + \frac{\mu}{6} u_{xxx} + \mu \varepsilon \left[ \frac{5}{12} uu_{xxx} + \frac{41}{24} u_x u_{xx} \right] + \frac{19}{300} \mu^2 \partial^2_x(u_{xxx}) & = O(\mu^3) \\
\end{align*}
\]

And one on the surface elevation will be:

\[
\begin{align*}
\dot{\zeta} + \zeta_x + \frac{3}{2} \varepsilon \zeta \zeta_x - \frac{3}{8} \varepsilon^2 \zeta^2 \zeta_x + \frac{\mu}{6} \zeta_{xxx} + \mu \varepsilon \left[ \frac{5}{12} \zeta_{xxx} + \frac{23}{24} \zeta_x^2 \right] + \frac{19}{360} \mu^2 \partial^2_x(\zeta_{xxx}) & = O(\mu^3) \\
\end{align*}
\]

2.1. Equation on the velocity. In order to get (2), we will introduce the following equation on \( u \)

\[
\begin{align*}
\dot{u} + u_x + \frac{3}{2} \varepsilon u u_x + \mu \alpha u_{xxx} = \mu \varepsilon \left[ \beta uu_{xxx} + \gamma u_x u_{xx} \right] + \mu^2 \delta \partial^2_x(u_{xxx}) \\
\end{align*}
\]

where \( \alpha, \beta, \gamma, \delta \) are parameters in \( \mathbb{R} \).

Next, we need to find the values of parameters mentioned above, that will be done in three steps.

- **Step 1:**
  From (4) we get:
  \[
  \begin{align*}
  u_x &= -u_t - \frac{3}{2} \varepsilon uu_x - \mu \alpha u_{xxx} + O(\varepsilon, \mu) \\
  u_{xxx} &= -u_{xxx} - \frac{3}{2} \varepsilon \partial^2_x(uu_x) - \mu \alpha \partial^2_x(u_{xxx}) + O(\varepsilon, \mu) \\
  \end{align*}
  \]

Substitute the expression of \( u_{xxx} \) in (4) to get:

\[
\begin{align*}
\dot{u} + u_x + \frac{3}{2} \varepsilon uu_x - \mu \alpha u_{xxx} & = \mu \varepsilon \left[ \beta uu_{xxx} + \gamma u_x u_{xx} \right] + \mu^2 \delta \partial^2_x(u_{xxx}) \\
& = \mu \varepsilon \left[ \beta uu_{xxx} + \gamma u_x u_{xx} \right] + \mu^2 \delta \partial^2_x(u_{xxx}) + O(\mu^3) \\
\end{align*}
\]

But

\[
\begin{align*}
\beta uu_{xxx} + \gamma u_x u_{xx} &= \beta uu_{xxx} + \beta u_x u_{xx} - \beta u_x u_{xx} + \gamma u_x u_{xx} \\
& = \beta uu_{xxx} - \frac{3}{2} \varepsilon uu_x - \mu \alpha uu_{xxx} \\
& = \beta uu_{xxx} - \frac{3}{2} \varepsilon uu_x - \mu \alpha uu_{xxx} \\
& = \left[ \beta uu_{xxx} + \gamma - \beta \right] u_x^2 \\
\end{align*}
\]

Then

\[
\begin{align*}
\dot{u} + u_x + \frac{3}{2} \varepsilon uu_x - \mu \alpha uu_{xxx} & = \mu \varepsilon \left[ \beta uu_{xxx} + \gamma uu_x u_{xx} \right] + \mu^2 \delta \partial^2_x(u_{xxx}) + O(\mu^3) \\
\end{align*}
\]

One can get

\[
\begin{align*}
\dot{u} + u_x + \frac{3}{2} \varepsilon uu_x - \mu \alpha uu_{xxx} & = \mu \varepsilon \left[ \beta uu_{xxx} + \gamma uu_x u_{xx} \right] + \mu^2 \delta \partial^2_x(u_{xxx}) + O(\mu^3) \\
\end{align*}
\]
where
\[ a = \frac{3\alpha}{2} + \beta; \quad b = \frac{3\alpha + \gamma - \beta}{2}; \quad c = \delta + \alpha^2. \]

To find another equation on \(a, b, c\) and \(\alpha\) we need to use the equations of (1).

**Step 2:**
Let \(v\) be a smooth enough function such that \(\zeta = u + \varepsilon v\). Then (1) becomes

\[ u_t + u_x + (\varepsilon v)_x + \varepsilon u_x = \frac{\mu}{3} u_{xxt} - \mu \varepsilon \left[ \frac{1}{2} u_x^2 + \frac{1}{3} u_{xx} \right]_x + \frac{\mu^2}{45} \partial_x^2(u_{xxt}) \]

We know from (5) that

\[ u_t + u_x + \frac{3}{2} uu_x - \mu \alpha u_{xxt} - \mu \varepsilon [auu_{xx} + bu^2_x]_x - \mu^2 c \partial_x^2(u_{xxx}) = O(\mu^3) \]

Then

\[ (\varepsilon v)_x + u_t + u_x + \frac{3}{2} \varepsilon uu_x - \mu \alpha u_{xxt} - \mu \varepsilon [auu_{xx} + bu^2_x]_x - \mu^2 c \partial_x^2(u_{xxx}) \]

\[ = u_t + u_x + (\varepsilon v)_x + \varepsilon uu_x + \frac{\varepsilon}{2} uu_x - \mu \alpha u_{xxt} - \mu \varepsilon [auu_{xx} + bu^2_x]_x - \mu^2 c \partial_x^2(u_{xxx}) \]

So we can deduce up to \(O(\mu^3)\)

\[ (\varepsilon v)_x = \frac{\varepsilon}{2} uu_x + \mu \left( \frac{1}{3} - \alpha \right) u_{xxt} - \mu \varepsilon \left[ a + \frac{1}{3} \right] uu_{xx} + \mu \varepsilon \left[ b + \frac{1}{2} \right] u^2_x \]

\[ = \mu \varepsilon \left( \frac{1}{45} + \varepsilon \right) \partial_x^2(u_{xxx}) + O(\mu^3) \]

Hence

\[ \varepsilon v = \frac{\varepsilon}{4} u^2 + \mu \left( \frac{1}{3} - \alpha \right) u_x - \mu \varepsilon \left[ a + \frac{1}{3} \right] uu_{xx} + \mu \varepsilon \left[ b + \frac{1}{2} \right] u^2_x \]

\[ - \mu^2 \left( \frac{1}{45} + c \right) \partial_x(u_{xxx}) \]

(6)

Now, we will use the first equation of (1).

**Step 3:**
Put \(\zeta = u + \varepsilon v\) in (1) that is

\[ u_t + u_x + 2\varepsilon uu_x + (\varepsilon v)_t + \varepsilon^2 (uv)_x = 0 \]

From (5) we have

\[ u_t = -u_x + O(\varepsilon, \mu) \]

\[ u_{xt} = -u_{xx} + O(\varepsilon, \mu) \]

(7)
Multiply (6) by $uv$, derive with respect to $x$ and use (7) to get
\[
\varepsilon^2 uv = \frac{\varepsilon^2}{4} u^3 + \mu \varepsilon \left( \frac{1}{3} - \alpha \right) uu_{xt} + O(\mu^3)
\]
\[
\varepsilon^2 (uv)_x = \frac{3}{4} \varepsilon^2 u^2 u_x + \mu \varepsilon \left( \frac{1}{3} - \alpha \right) (uu_{xt})_x + O(\mu^3)
\]
\[
= \frac{3}{4} \varepsilon^2 u^2 u_x - \mu \varepsilon \left( \frac{1}{3} - \alpha \right) (uu_{xx})_x + O(\mu^3)
\]

Next, deriving (6) with respect to $t$ one can get
\[
(\varepsilon v)_t = \frac{\varepsilon}{2} uu_t + \mu \left( \frac{1}{3} - \alpha \right) uu_{tt} - \mu \left[ \left( a + \frac{1}{3} \right) uu_{xx} + \left( b + \frac{1}{2} \right) u_x^2 \right]_t
\]
\[\quad - \mu^2 \left( \frac{1}{45} + c \right) \partial^2_{xt}(u_{xxx}) + O(\mu^3)
\]

From (5) we get
\[
u_t = -u_x - \frac{3}{2} \varepsilon uu_x + \mu \alpha uu_{xt} + O(\varepsilon, \mu)
\]
\[
u_{xtt} = -uu_{tt} - \frac{3}{2} \varepsilon \partial_t uu_x + \mu \alpha \partial_{xt} uu_{xt} + O(\varepsilon, \mu)
\]
\[
u_{xxt} = -uu_{xx} - \frac{3}{2} \varepsilon \partial_x uu_x + \mu \alpha \partial^2_x (uu_{xt}) + O(\varepsilon, \mu)
\]

Remark that
\[
\partial_t (uu_{xt}) = \partial^2_t (u_{xxx}) + O(\varepsilon, \mu)
\]
\[
\partial_{xt} (u_{xxx}) = -\partial^2_t (u_{xxx}) + O(\varepsilon, \mu)
\]

Also we have
\[
\partial^2_{xt}(uu_x) = -\partial_x (uu_t + uu_{xt}) = -\partial_x (u_x^2 + uu_x) = -\partial^2_t (uu_x) + O(\varepsilon, \mu)
\]

Then
\[
u_{xtt} = -uu_{xx} + \frac{3}{2} \varepsilon \left[ u^2_x + uu_{xx} \right]_x + \mu \alpha \partial^2_t (u_{xxx}) + O(\varepsilon, \mu)
\]

Gathering all the computations done above one can get
\[
(\varepsilon v)_t = -\frac{\varepsilon}{2} uu_x - \frac{3}{4} \varepsilon^2 u^2 u_x + \mu \varepsilon \left( \frac{1}{3} - \alpha \right) uu_{xx} - \mu \left( \frac{1}{3} - \alpha \right) uu_{xt}
\]
\[\quad + \mu \varepsilon \left( a + \frac{1}{3} \right) uu_{xx} + \mu \varepsilon \left( b + \frac{1}{2} \right) u_x^2 + \mu \varepsilon \left( \frac{1}{45} + c \right) \partial^2_{xt}(u_{xxx}) + O(\mu^3)
\]

But
\[
\mu \varepsilon \frac{\alpha}{2} uu_{xx} = -\mu \varepsilon \frac{\alpha}{2} uu_{xxx} + O(\mu^3) = -\mu \varepsilon \frac{\alpha}{2} \left[ uu_{xx} - \frac{1}{2} u_x^2 \right] + O(\mu^3)
\]
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So
\[
(\varepsilon v)_t = -\frac{\varepsilon}{2} uu_x - \frac{3}{4} \varepsilon^2 u^2 u_x - \mu \left( \frac{1}{3} - \alpha \right) u_{xxx}
+ \mu \varepsilon \left[ \left( a - 2\alpha + \frac{5}{6} \right) uu_{xx} + \left( 1 + b - \frac{5\alpha}{4} \right) u_x^2 \right]_x
+ \mu^2 \left( -\alpha \left( \frac{1}{3} - \alpha \right) + c - \frac{1}{45} \right) \partial_x^2 (u_{xxx}) + O(\mu^3)
\]

Substitute \( \varepsilon^2 (uv)_x \) and \((\varepsilon v)_t \) in (1) to get

\[
u_t + u_x + 3 \varepsilon u u_x + \mu \left( \alpha - \frac{1}{3} \right) u_{xxx}
= \mu \varepsilon \left[ \left( \alpha - a - \frac{1}{2} \right) uu_{xx} + \left( -1 - b + \frac{5\alpha}{4} \right) u_x^2 \right]_x
+ \mu^2 \left( -\alpha \left( \frac{1}{3} - \alpha \right) - c - \frac{1}{45} \right) \partial_x^2 (u_{xxx}) + O(\mu^3)
\]

Compare the equations (5) and (8) to deduce

\[
\alpha = \frac{1}{6}; a = \frac{1}{6}; b = -\frac{19}{48}; c = \frac{1}{40}.
\]

Doing some calculations one can get

\[
\beta = -\frac{5}{12}; \gamma = -\frac{41}{24}; \delta = -\frac{19}{360}.
\]

2.2. Equation on the surface elevation. In this subsection, we are going to derive the extended KdV equation on the surface elevation \( \zeta \) as a rigorous verification of a previous work [13], where it was imposed formally. For this purpose, let us introduce the following equation

\[
\zeta_t + \zeta_x + \frac{3}{2} \varepsilon \zeta \zeta_x - \frac{3}{8} \varepsilon^2 \zeta^2 \zeta_x + \mu \alpha \zeta_{xxx}
= \mu \varepsilon \left[ \beta \zeta_{xxx} + \gamma \zeta_x \zeta_{xx} \right] + \mu^2 \delta \zeta_2 \zeta_x + O(\mu^3)
\]

where \( \alpha, \beta, \gamma, \delta \) are parameters in \( \mathbb{R} \), and will be determined next.

- **Step1:**

  From (9) one can get

  \[
  \zeta_x = -\zeta_t - \frac{3}{2} \varepsilon \zeta \zeta_x + \frac{3}{8} \varepsilon^2 \zeta^2 \zeta_x - \mu \alpha \zeta_{xxx} + O(\varepsilon, \mu)
  \]

  \[
  \zeta_{xxx} = -\zeta_{xxx} - \frac{3}{2} \varepsilon \partial_x^2 (\zeta_x) + \frac{3}{8} \varepsilon^2 \partial_x^2 (\zeta^2 \zeta_x) - \mu \alpha \partial_x^2 (\zeta_{xxx}) + O(\varepsilon, \mu)
  \]

  Substitute \( \zeta_{xxx} \) in (9) to get

  \[
  \zeta_t + \zeta_x + \frac{3}{2} \varepsilon \zeta \zeta_x - \frac{3}{8} \varepsilon^2 \zeta^2 \zeta_x - \mu \alpha \zeta_{xxx} - \frac{3}{2} \alpha \varepsilon \zeta \zeta_x - \mu^2 \alpha^2 \partial_x^2 (\zeta_{xxx})
  = \mu \varepsilon \left[ \beta \zeta_{xxx} + \gamma \zeta_x \zeta_{xx} \right] + \mu^2 \delta \zeta_2 \zeta_x + O(\mu^3)
  \]

  Use the fact that

  \[
  \beta \zeta_{xxx} + \gamma \zeta_x \zeta_{xx} = \left[ \beta \zeta_{xxx} + \gamma \zeta_{xxx} \right]_x
  \]
To get the following equation

\begin{equation}
\frac{\partial \zeta_t + \frac{3}{2} \zeta \zeta_x - \frac{3}{8} \varepsilon^2 \zeta^2 \zeta_x - \mu \alpha \zeta_{xxt}}{\partial t} = \mu \varepsilon [a \zeta_{x xx} + b \zeta_x^2]_x + \mu^2 \partial_x^2 (\zeta_{xxx}) + O(\mu^3)
\end{equation}

where

\[
a = \frac{3\alpha}{2} + \beta; \quad b = \frac{3\alpha + \gamma - \beta}{2}; \quad c = \delta + \alpha^2.
\]

**Step 2:**
Let \(w\) be a smooth enough function such that \(u = \zeta + \varepsilon w\).
Compute

\[(1 + \varepsilon \zeta) u = (1 + \varepsilon \zeta)(\zeta + \varepsilon w) = \zeta + \varepsilon \zeta^2 + (1 + \varepsilon \zeta)(\varepsilon w)\]

Then the first equation of (1) gives

\[
\zeta_t + \zeta_x + 2\varepsilon \zeta \zeta_x + [(1 + \varepsilon \zeta)(\varepsilon w)_x] = 0
\]

Use (10) to compute

\[
[(1 + \varepsilon \zeta)(\varepsilon w)]_x + \zeta_t + \zeta_x + \frac{3}{2} \varepsilon \zeta \zeta_x - \frac{3}{8} \varepsilon^2 \zeta^2 \zeta_x - \mu \alpha \zeta_{xxt} - \mu \varepsilon [a \zeta_{x xx} + b \zeta_x^2]_x - \mu^2 \partial_x^2 (\zeta_{xxx})
\]

\[
= \zeta_t + \zeta_x + 2\varepsilon \zeta \zeta_x + [(1 + \varepsilon \zeta)(\varepsilon w)]_x - \frac{\varepsilon}{2} \zeta \zeta_x - \frac{3}{8} \varepsilon^2 \zeta^2 \zeta_x - \mu \alpha \zeta_{xxt} - \mu \varepsilon [a \zeta_{x xx} + b \zeta_x^2]_x
\]

\[- \mu^2 \partial_x^2 (\zeta_{xxx}) + O(\mu^3)
\]

Then

\[
[(1 + \varepsilon \zeta)(\varepsilon w)]_x = \frac{\varepsilon}{2} \zeta \zeta_x - \frac{3}{8} \varepsilon^2 \zeta^2 \zeta_x - \mu \alpha \zeta_{xxt} - \mu \varepsilon [a \zeta_{x xx} + b \zeta_x^2]_x - \mu^2 \partial_x^2 (\zeta_{xxx}) + O(\mu^3)
\]

Hence

\begin{equation}
(1 + \varepsilon \zeta)(\varepsilon w) = -\frac{\varepsilon}{4} \zeta^2 - \frac{1}{8} \varepsilon^2 \zeta^3 - \mu \alpha \zeta_{xxt} - \mu \varepsilon [a \zeta_{x xx} + b \zeta_x^2]_x - \mu^2 \partial_x^2 (\zeta_{xxx}) + O(\mu^3)
\end{equation}

**Step 3:**
Here we will use the second equation of (1).
Since we need to keep the terms \(\zeta_t\) and \(\zeta_x\), and since the latter derived term is \((1 + \varepsilon \zeta)(\varepsilon w)\) and not \(\varepsilon w\), we will multiply the second equation of (1) by \((1 + \varepsilon \zeta)\). Hence, one can get

\begin{equation}
(1 + \varepsilon \zeta) u_t + \zeta_x + \varepsilon \zeta \zeta_x + (1 + \varepsilon \zeta) \varepsilon u u_x
\end{equation}

\[
= \frac{\mu}{3} \zeta_{xxt} + \frac{\mu}{3} (\varepsilon w)_{xxt} + \frac{\mu \varepsilon}{3} \zeta_{xxt} - \mu \varepsilon \left[ \frac{1}{2} u_x^2 + \frac{1}{3} u u_x \right]_x
\]

\[
+ \frac{\mu^2}{45} \partial_x^2 (\zeta_{xxx}) + O(\mu^3)
\]

For the right hand side of (12):

\[
\frac{\mu}{3} (\varepsilon w) = -\frac{\mu \varepsilon}{12} \zeta - \mu \frac{\alpha}{3} \zeta_{xxt} + O(\mu^3)
\]

\[
\frac{\mu}{3} (\varepsilon w)_{xxt} = -\frac{\mu \varepsilon}{12} \partial_{xxt} (\zeta^2) - \mu \frac{\alpha}{3} \partial_{xxt} (\zeta_{xxt}) + O(\mu^3)
\]
Recall that (10) gives:
\[ \zeta_t = -\zeta_x + O(\varepsilon, \mu) \]
\[ \zeta_{xxt} = -\zeta_{xxx} + O(\varepsilon, \mu) \]

Also
\[ \partial_{xxt}(\zeta_{xt}) = \partial_x^2(\zeta_{xxx}) + O(\varepsilon, \mu) \]

Therefore
\[ \frac{\mu}{3}(\varepsilon w)_{xxt} = \frac{\mu \varepsilon}{6} [\zeta_x^2 + \zeta_{xx}]_x - \frac{\mu^2 \alpha}{3} \partial_x^2(\zeta_{xxx}) + O(\mu^3) \]

Now, use
\[ \zeta_{xxt} = -\zeta_{xxx} + O(\varepsilon, \mu) \]
to get
\[ \frac{\mu \varepsilon}{3} \zeta_{xxt} = -\frac{\mu \varepsilon}{3} \zeta_{xxx} + O(\mu^3) = -\frac{\mu \varepsilon}{3} \left[ \zeta_{xx} - \frac{1}{2} \zeta_x^2 \right]_x + O(\mu^3) \]

Obviously, one gets
\[ \mu \varepsilon \left[ \frac{1}{2} u_x^2 + \frac{1}{3} uu_{xx} \right]_x = \mu \varepsilon \left[ \frac{1}{2} \zeta_x^2 + \frac{1}{3} \zeta_{xx} \right]_x + O(\mu^3) \]

Using the identity
\[ \partial_x^2(\zeta_{xt}) = -\partial_x^2(\zeta_{xxx}) + O(\varepsilon, \mu) \]
one can deduces
\[ \frac{\mu^2}{45} \partial_x^2(\zeta_{xxt}) = -\frac{\mu^2}{45} \partial_x^2(\zeta_{xxx}) + O(\mu^3) \]

Gathering all the informations found above in the right hand side of (12) we get
\[ \frac{\mu}{3} \zeta_{xxt} + \frac{\mu}{3}(\varepsilon w)_{xxt} + \frac{\mu \varepsilon}{3} \zeta_{xxt} - \mu \varepsilon \left[ \frac{1}{2} u_x^2 + \frac{1}{3} uu_{xx} \right]_x + \frac{\mu^2}{45} \partial_x^2(\zeta_{xxt}) = \frac{\mu}{3} \zeta_{xxt} - \mu \varepsilon \left[ \frac{1}{2} \zeta_{xx} + \frac{1}{6} \zeta_x^2 \right]_x - \mu^2 \left( \frac{\alpha}{3} + \frac{1}{45} \right) \partial_x^2(\zeta_{xxx}) + O(\mu^3) \]

For the left hand side of (12):
\[ \varepsilon uu_x = \varepsilon \zeta_x + [(\varepsilon w)(\varepsilon \zeta)]_x + O(\mu^3) \]
\[ (1 + \varepsilon \zeta)(\varepsilon uu_x) = \varepsilon \zeta_x + [(\varepsilon w)(\varepsilon \zeta)]_x + \varepsilon^2 \zeta^2 \zeta_x + O(\mu^3) \]

We have
\[ (\varepsilon w)(\varepsilon \zeta) = -\frac{\varepsilon^2}{4} \zeta^3 - \mu \varepsilon \zeta_x x + O(\mu^3) \]
\[ [(\varepsilon w)(\varepsilon \zeta)]_x = -\frac{3 \varepsilon^2}{4} \zeta^2 \zeta_x - \mu \varepsilon \zeta_x x + O(\mu^3) \]
\[ = -\frac{3 \varepsilon^2}{4} \zeta^2 \zeta_x + \mu \varepsilon \zeta_x x + O(\mu^3) \]
\[(1 + \varepsilon \zeta)(\varepsilon u_{xx}) = \varepsilon \zeta_x + \frac{\varepsilon^2}{4} \zeta^2_x + \mu \varepsilon \alpha (\zeta_{xxx})_x + O(\mu^3)\]

Next,
\[(1 + \varepsilon \zeta)u_t = (1 + \varepsilon \zeta)(\varepsilon w)_t\]
\[= \zeta_t + (\varepsilon w)_t + \varepsilon \zeta w_t + e^2 \zeta w_t\]
\[= \zeta_t + \varepsilon \zeta w_t + [(1 + \varepsilon \zeta)(\varepsilon w)]_t - \varepsilon^2 \zeta w\]
\[= \zeta_t + \varepsilon \zeta w + [(1 + \varepsilon \zeta)(\varepsilon w)]_t + \varepsilon^2 \zeta w + O(\mu^3)\]

Multiply (11) by \(\varepsilon \zeta_x\) to get
\[\varepsilon^2 \zeta_x w = -\frac{\varepsilon^2}{4} \zeta^2_x - \mu \varepsilon \alpha (\zeta_x \zeta_{xt}) + O(\mu^3) = -\frac{\varepsilon^2}{4} \zeta^2_x + \mu \varepsilon \alpha \left[\zeta^2_x\right]_x + O(\mu^3)\]

Also
\[\varepsilon \zeta_t + [(1 + \varepsilon \zeta)(\varepsilon w)]_t\]
\[= \frac{\varepsilon}{2} \zeta - \frac{3}{8} \varepsilon^2 \zeta^2_t - \mu \alpha \zeta_{xxt} - \mu \varepsilon \left[a \zeta_{xx} + b \zeta^2_x\right]_t - \mu^2 \partial_x^2 (\zeta_{xxx}) + O(\mu^3)\]

Recall that
\[\zeta_t = -\zeta_x - \frac{3}{2} \varepsilon \zeta_x + \mu \alpha \zeta_{xxt} + O(\varepsilon, \mu)\]

To compute
\[\partial_x^2 (\zeta_{xxx}) = -\partial_x^2 (\zeta_{xxx}) + O(\varepsilon, \mu)\]
\[\partial_x^2 (\zeta_{xxt}) = \partial_x^2 (\zeta_{xxx}) + O(\varepsilon, \mu)\]
\[\partial_x^2 (\zeta_{xx}) = \partial_x^2 (\zeta_{xxx}) = -\left[\zeta^2_x + \zeta_{xx}\right]_x + O(\varepsilon, \mu)\]

And
\[\zeta_{xxt} = -\zeta_{xxt} - \frac{3}{2} \varepsilon \partial_x (\zeta_x) + \mu \alpha \partial_x (\zeta_{xxt}) + O(\varepsilon, \mu)\]
\[= -\zeta_{xxt} + \frac{3}{2} \varepsilon \left[\zeta^2_x + \zeta_{xx}\right]_x + \mu \alpha \partial_x^2 (\zeta_{xxx}) + O(\varepsilon, \mu)\]

Therefore
\[\varepsilon \zeta_t + [(1 + \varepsilon \zeta)(\varepsilon w)]_t\]
\[= -\frac{\varepsilon}{2} \zeta_x - \frac{3}{8} \varepsilon^2 \zeta^2_x + \mu \alpha \zeta_{xxt} + \mu \varepsilon \alpha \left[\zeta^2_x\right]_x + \mu \varepsilon \left[a \zeta_{xx} + b \zeta^2_x\right]_x + \mu^2 (c - \alpha^2) \partial_x^2 (\zeta_{xxx}) + O(\mu^3)\]

Hence
\[(1 + \varepsilon \zeta)u_t + \zeta_x + e \zeta \zeta_x + (1 + \varepsilon \zeta)\varepsilon u_{xx}\]
\[= \zeta_t + \zeta_x + 3 \varepsilon \zeta \zeta_x - \frac{3}{8} \varepsilon^2 \zeta^2_x + \mu \alpha \zeta_{xxt}\]
\[+ \mu \varepsilon \left[a \zeta_{xx} + \left(b - \frac{3 \alpha}{4}\right) \zeta^2_x\right] + \mu^2 (c - \alpha^2) \partial_x^2 (\zeta_{xxx}) + O(\mu^3)\]
Eventually, gathering (14) and (13) we get the equation
\[
\zeta_t + \zeta_x + \frac{3}{2} \varepsilon \zeta_x - \frac{3}{8} \varepsilon^2 \zeta_x + \mu \left( \alpha - \frac{1}{3} \right) \zeta_{xx} t
\]
\[
= \mu \varepsilon \left[ \left( \alpha - a - \frac{1}{2} \right) \zeta_{xx} + \left( \frac{3\alpha}{4} - b - \frac{1}{6} \right) \zeta_x \right]
\]
\[
+ \mu^2 \left( \alpha^2 - c - \frac{\alpha}{3} - \frac{1}{45} \right) \partial_x^2 (\zeta_{xxx}) + O(\mu^3)
\]
(15)

Comparing (10) and (15) one gets
\[
\alpha = \frac{1}{6}; a = -\frac{1}{6}; b = -\frac{1}{48}; c = -\frac{1}{40}.
\]

Do some calculations to deduce
\[
\beta = -\frac{5}{12}; \gamma = -\frac{23}{24}; \delta = \frac{19}{360}.
\]

3. Conclusion

The ocean energy sector is all about innovation and has been evidence of some notable progress. Many studies in this domain have been done, and many questions have given insight into new studies. After providing a rigorous derivation of KdV equation on \( u \), and a rigorous verification of one on \( \zeta \), for flat bottom, new lights have been casted on some future researches, the extended KdV equation on the velocity could be derived with the presence of surface tension effect, this study could be done using the pseudo-differential operator theory. Also, we can solve this equation explicitly using sine-cosine method on a bounded space domain, or a numerical framework could be done using the finite element method. Finally, we would study its well-posedness employing the modified-energy method.

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1 Department of mathematics and Computer science, Faculty of science, Beirut Arab University, Beirut, Lebanon
E-mail address: berjawi.marwa77@gmail.com

2 Department of mathematics, Faculty of science 1, Lebanese University, Beirut, Lebanon