QUANTUM COHOMOLOGY OF THE MODULI SPACE OF
STABLE BUNDLES OVER A RIEMANN SURFACE

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ABSTRACT. We determine the quantum cohomology of the moduli space $M_Σ$ of odd
degree rank two stable vector bundles over a Riemann surface $Σ$ of genus $g \geq 1$.
This work together with [10] complete the proof of the existence of an isomorphism
$QH^∗(M_Σ) \cong HF^∗(Σ \times S^1)$.

1. Introduction

Let $Σ$ be a Riemann surface of genus $g \geq 2$ and let $M_Σ$ denote the moduli space of
flat $SO(3)$-connections with nontrivial second Stiefel-Whitney class $w_2$.

This is a smooth symplectic manifold of dimension $6g − 6$. Alternatively, we can
consider $Σ$ as a smooth complex curve of genus $g$. Fix a line bundle $Λ$ on $Σ$ of
degree 1, then $M_Σ$ is the moduli space of rank two stable vector bundles on $Σ$ with
determinant $Λ$, which is a smooth complex variety of complex dimension $3g − 3$. The
symplectic deformation class of $M_Σ$ only depends on $g$ and not on the particular
complex structure on $Σ$.

The manifold $X = M_Σ$ is a positive symplectic manifold with $π_2(X) = \mathbb{Z}$. For such
a manifold $X$, its quantum cohomology, $QH^∗(X)$, is well-defined (see [14] [15] [8] [12]).
As vector spaces, $QH^∗(X) = H^∗(X)$ (rational coefficients are understood), but the
multiplicative structure is different. Let $A$ denote the positive generator of $π_2(X)$,
i.e. the generator such that the symplectic form evaluated on $A$ is positive. Let $N =
c_1(X)[A] ∈ \mathbb{Z}_{> 0}$. Then there is a natural $\mathbb{Z}/2\mathbb{N}\mathbb{Z}$-grading for $QH^∗(X)$, which comes
from reducing the $\mathbb{Z}$-grading of $H^∗(X)$. (For the case $X = M_Σ$, $N = 2$, so $QH^∗(M_Σ)$
is $\mathbb{Z}/4\mathbb{Z}$-graded). The ring structure of $QH^∗(X)$, called quantum multiplication, is a
defformation of the usual cup product for $H^∗(X)$. For $α \in H^p(X)$, $β \in H^q(X)$, we

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define the quantum product of $\alpha$ and $\beta$ as
\[
\alpha \cdot \beta = \sum_{d \geq 0} \Phi_{dA}(\alpha, \beta),
\]
where $\Phi_{dA}(\alpha, \beta) \in H^{p+q-2Nd}(X)$ is given by $< \Phi_{dA}(\alpha, \beta), \gamma >= \Psi^X_{dA}(\alpha, \beta, \gamma)$, the Gromov-Witten invariant, for all $\gamma \in H^{\dim X-p-q+2Nd}(X)$. One has $\Phi_0(\alpha, \beta) = \alpha \cup \beta$. The other terms are the quantum correction terms and they all live in lower degree parts of the cohomology groups. It is a fact [15] that the quantum product gives an associative and graded commutative ring structure.

To define the Gromov-Witten invariant, let $J$ be a generic almost complex structure compatible with the symplectic form. Then for every 2-homology class $dA$, $d \in \mathbb{Z}$, there is a moduli space $\mathcal{M}_{dA}$ of pseudoholomorphic rational curves (with respect to $J$) $f : \mathbb{P}^1 \to X$ with $f_*(\mathbb{P}^1) = dA$. Note that $\mathcal{M}_0 = X$ and that $\mathcal{M}_{dA}$ is empty for $d < 0$. For $d \geq 0$, the dimension of $\mathcal{M}_{dA}$ is $\dim X + 2Nd$. This moduli space $\mathcal{M}_{dA}$ admits a natural compactification, $\overline{\mathcal{M}}_{dA}$, called the Gromov-Uhlenbeck compactification [14] [15, section 3]. Consider now $r \geq 3$ different points $P_1, \ldots, P_r \in \mathbb{P}^1$. Then we have defined an evaluation map $ev : \mathcal{M}_{dA} \to X^r$ by $f \mapsto (f(P_1), \ldots, f(P_r))$.

This map extends to $\overline{\mathcal{M}}_{dA}$, and its image, $ev(\overline{\mathcal{M}}_{dA})$, is a pseudo-cycle [15]. So for $\alpha_i \in H^{p_i}(M_{\Sigma_i})$, $1 \leq i \leq r$, with $p_1 + \cdots + p_r = \dim X + 2Nd$, we choose generic cycles $A_i$, $1 \leq i \leq r$, representatives of their Poincaré duals, and set
\[
(1) \Psi^X_{dA}(\alpha_1, \ldots, \alpha_r) = < A_1 \times \cdots \times A_r, ev(\overline{\mathcal{M}}_{dA}) > = \# ev_{P_1}(A_1) \cap \cdots \cap ev_{P_r}(A_r),
\]
where $\#$ denotes count of points (with signs) and $ev_{P_i} : \mathcal{M}_{dA} \to X$, $f \mapsto f(P_i)$. This is a well-defined number and independent of the particular cycles. Also, as the manifold $X$ is positive, $\text{coker} L_f = H^1(\mathbb{P}^1, f^*c_1(X)) = 0$, for all $f \in \mathcal{M}_{dA}$ (see [14] for definition of $L_f$). By [14] the complex structure of $X$ is generic and we can use it to compute the Gromov-Witten invariants.

Also for $r \geq 2$, let $\alpha_i \in H^{p_i}(M_{\Sigma_i})$, $1 \leq i \leq r$, then
\[
\alpha_1 \cdots \alpha_r = \sum_{d \geq 0} \Phi_{dA}(\alpha_1, \ldots, \alpha_r),
\]
where the correction terms $\Phi_{dA}(\alpha_1, \ldots, \alpha_r) \in H^{p_1+\cdots+p_r-2Nd}(X)$ are determined by $< \Phi_{dA}(\alpha_1, \ldots, \alpha_r), \gamma > = \Psi^X_{dA}(\alpha_1, \ldots, \alpha_r, \gamma)$, for any $\gamma \in H^{\dim X+2Nd-(p_1+\cdots+p_r)}(X)$.

Returning to our manifold $X = M_{\Sigma}$, there is a classical conjecture relating the quantum cohomology $QH^*(M_{\Sigma})$ and the instanton Floer cohomology of the three manifold $\Sigma \times S^1$, $HF^*(\Sigma \times S^1)$ (see [10]). In [1] a presentation of $QH^*(M_{\Sigma})$ was given using physical methods, and in [10] it was proved that such a presentation was a presentation of $HF^*(\Sigma \times S^1)$ indeed. Here we determine a presentation of $QH^*(M_{\Sigma})$ and prove the isomorphism $QH^*(M_{\Sigma}) \cong HF^*(\Sigma \times S^1)$. 

Siebert and Tian have an alternative program [16] to find the presentation of $QH^*(M_\Sigma)$, which goes through proving a recursion formula for the Gromov-Witten invariants of $M_\Sigma$ in terms of the genus $g$.

The paper is organised as follows. In section 2 we review the ordinary cohomology ring of $M_\Sigma$. In section 3 the moduli space of lines (rational curves representing $A$) in $M_\Sigma$ is described. This makes possible to compute the Gromov-Witten invariants $\Psi_{M_\Sigma A}$, which determine the first quantum correction terms of the quantum products in $QH^*(M_\Sigma)$. Section 4 is devoted to this task. In [3] Donaldson uses this information alone to determine $QH^*(M_\Sigma)$ in the case of genus $g=2$. It is somehow natural to expect that this idea can be developed in the general case $g \geq 3$. In section 5 we give an explicit presentation of $QH^*(M_\Sigma)$ for $g \geq 3$ (theorem 20), concluding the proof of $QH^*(M_\Sigma) \cong HF^*(\Sigma \times S^1)$ (corollary 21). The two main ingredients that we make use of are the $\text{Sp}(2g,\mathbb{Z})$-decomposition of $H^*(M_\Sigma)$ under the action of the mapping class group (not ignoring the non-invariant part as it was customary) and a recursion similar to that in [16] (lemma 17). The difference with [16] lies in the fact that we fix the genus, so that we do not need to compare the Gromov-Witten invariants for moduli spaces of Riemann surfaces of different genus. Finally in section 6 we discuss the cases $g=1$ and $g=2$, which are slightly different.

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2. Classical cohomology ring of $M_\Sigma$

Let us recall the known description of the homology of $M_\Sigma$ [6] [17] [10]. Let $U \rightarrow \Sigma \times M_\Sigma$ be the universal bundle and consider the Künneth decomposition of

$$c_2(\text{End}_0 U) = 2[\Sigma] \otimes \alpha + 4\psi - \beta,$$

with $\psi = \sum \gamma_i \otimes \psi_i$, where $\{\gamma_1, \ldots, \gamma_{2g}\}$ is a symplectic basis of $H^1(\Sigma; \mathbb{Z})$ with $\gamma_i\gamma_{i+g} = [\Sigma]$ for $1 \leq i \leq g$ (also $\{\gamma_i^\#\}$ will denote the dual basis for $H_1(\Sigma; \mathbb{Z})$). Here we can suppose without loss of generality that $c_1(U) = \Lambda + \alpha$ (see [17]). In terms of the map $\mu : H_\ast(\Sigma) \rightarrow H^{4-\ast}(M_\Sigma)$, given by $\mu(a) = -\frac{1}{4} p_1(\mathfrak{g}_U)/a$ (here $\mathfrak{g}_U \rightarrow \Sigma \times M_\Sigma$ is the associated universal $SO(3)$-bundle, and $p_1(\mathfrak{g}_U) \in H^4(\Sigma \times M_\Sigma)$ its first Pontrjagin class), we have

$$\begin{align*}
\alpha &= 2\mu(\Sigma) \in H^2 \\
\psi_i &= \mu(\gamma_i^\#) \in H^3, \quad 1 \leq i \leq 2g \\
\beta &= -4\mu(x) \in H^4
\end{align*}$$

where $x \in H_0(\Sigma)$ is the class of the point, and $H^i = H^i(M_\Sigma)$. These elements generate $H^*(M_\Sigma)$ as a ring [6] [19], and $\alpha$ is the positive generator of $H^2(M_\Sigma; \mathbb{Z})$. We
can rephrase this as saying that there exists an epimorphism

\[ A(\Sigma) = \mathbb{Q}[\alpha, \beta] \otimes \Lambda(\psi_1, \ldots, \psi_{2g}) \to H^*(M_{\Sigma}) \]

(the notation \( A(\Sigma) \) follows that of Kronheimer and Mrowka [7], although it is slightly different). Recall that \( \deg(\alpha) = 2, \deg(\beta) = 4 \) and \( \deg(\psi_i) = 3 \).

The mapping class group Diff(\( \Sigma \)) acts on \( H^*(M_{\Sigma}) \), with the action factoring through the action of \( \text{Sp}(2g, \mathbb{Z}) \) on \( \{\psi_1\} \). The invariant part, \( H^*_I(M_{\Sigma}) \), is generated by \( \alpha, \beta \) and \( \gamma = -2 \sum_{i=0}^g \psi_i \psi_{i+g} \). Then there is an epimorphism

\[ \mathbb{Q}[\alpha, \beta, \gamma] \to H^*_I(M_{\Sigma}) \]

which allows us to write

\[ H^*_I(M_{\Sigma}) = \mathbb{Q}[\alpha, \beta, \gamma]/I_g, \]

where \( I_g \) is the ideal of relations satisfied by \( \alpha, \beta \) and \( \gamma \). From [17], a basis for \( H^*_I(M_{\Sigma}) \) is given by the monomials \( \alpha^a \beta^b \gamma^c \), with \( a + b + c < g \). For \( 0 \leq k \leq g \), the primitive component of \( \Lambda^k H^3 \) is

\[ \Lambda^k H^3 = \ker(\gamma^{g-k+1} : \Lambda^k H^3 \to \Lambda^{2g-k+2} H^3). \]

The spaces \( \Lambda^k H^3 \) are irreducible \( \text{Sp}(2g, \mathbb{Z}) \)-modules, i.e. the transforms of any nonzero element of \( \Lambda^k H^3 \) under \( \text{Sp}(2g, \mathbb{Z}) \) generate the whole of it. The description of the ideals \( I_g \) and the cohomology ring \( H^*(M_{\Sigma}) \) is given in the following

**Proposition 1 ([17] [6]).** Define \( q_0^i = 1, q_0^2 = 0, q_0^3 = 0 \) and then recursively, for all \( r \geq 1 \),

\[
\begin{align*}
q_{r+1}^1 &= \alpha q_r^1 + r^2 q_r^2 \\
q_{r+1}^2 &= (\beta + (-1)^{r+1}) q_r^1 + \frac{2r}{r+1} q_r^3 \\
q_{r+1}^3 &= \gamma q_r^1
\end{align*}
\]

Then \( I_g = (q_g^1, q_g^2, q_g^3) \subset \mathbb{Q}[\alpha, \beta, \gamma], \) for all \( g \geq 1 \). Note that \( \deg(q_1^1) = 2g, \deg(q_1^2) = 2g + 2 \) and \( \deg(q_1^3) = 2g + 4 \). Moreover the \( \text{Sp}(2g, \mathbb{Z}) \)-decomposition of \( H^*(M_{\Sigma}) \) is

\[ H^*(M_{\Sigma}) = \bigoplus_{k=0}^{g-1} \Lambda^k H^3 \otimes \mathbb{Q}[\alpha, \beta, \gamma]/I_{g-k}. \]

This proposition allows us to find a basis for \( H^*(M_{\Sigma}) \) as follows. Let \( \{x_i^{(k)}\}_{i \in B_k} \) be a basis of \( \Lambda^k H^3 \), \( 0 \leq k \leq g - 1 \). Then

\[ \{x_i^{(k)} \alpha^a \beta^b \gamma^c / k = 0, 1, \ldots, g - 1, a + b + c < g - k, i \in B_k\} \]

is a basis for \( H^*(M_{\Sigma}) \). If we set

\[ x_0^{(k)} = \psi_1 \psi_2 \cdots \psi_k \in \Lambda^k H^3, \]

then proposition 1 says that a complete set of relations satisfied in \( H^*(M_{\Sigma}) \) are \( x_0^{(k)} q_{g-k}^i, i = 1, 2, 3, 0 \leq k \leq g, \) and the \( \text{Sp}(2g, \mathbb{Z}) \) transforms of these.
3. Holomorphic lines in $M_\Sigma$

In order to compute the Gromov-Witten invariants $\Psi^A_M$, we need to describe the space of lines, i.e. rational curves in $M_\Sigma$ representing the generator $A \in H_2(M_\Sigma; \mathbb{Z})$,

$$\mathcal{M}_A = \{ f : \mathbb{P}^1 \to M_\Sigma/f \text{ holomorphic, } f_*(\mathbb{P}^1) = A \}.$$

Let us fix some notation. Let $J$ denote the Jacobian variety of $\Sigma$ parametrising line bundles of degree 0 and let $L \to \Sigma$ be a fixed line bundle of degree 1 on $\Sigma$. Fix the line bundle $F$ with the notations of [9], being a good wall as described in [9]. First, note that the results in [9] use the hypothesis of $F$ having geometric genus $P = \sum \frac{g}{2}$ and canonical bundle $K \equiv -2\Sigma + (2g - 2)\mathbb{P}^1$. Recall that $A$ is a fixed line bundle of degree 1 on $\Sigma$. Fix the line bundle $L = \Lambda \otimes O_{\mathbb{P}^1}(1)$ on $S$ (we omit all pull-backs) with $c_1 = c_1(L) \equiv \mathbb{P}^1 + \Sigma$, and put $c_2 = 1$. The ample cone of $S$ is $\{ a\mathbb{P}^1 + b\Sigma / a, b > 0 \}$. Let $H_0$ be a polarisation close to $\mathbb{P}^1$ in the ample cone and $H$ be a polarisation close to $\Sigma$, i.e. $H = \Sigma + t\mathbb{P}^1$ with $t$ small. We wish to study the moduli space $\mathcal{M} = \mathcal{M}_H(c_1, c_2)$ of $H$-stable bundles over $S$ with Chern classes $c_1$ and $c_2$.

**Proposition 2.** $\mathcal{M}$ can be described as a bundle $\mathbb{P}^{2g-1} \to \mathcal{M} = \mathbb{P}(\mathcal{E}_\zeta^\vee) \to J$, where $\mathcal{E}_\zeta$ is a bundle on $J$ with $\text{ch} \mathcal{E}_\zeta = 2g + 8\omega$. So $\mathcal{M}$ is compact, smooth and of the expected dimension $6g - 2$. The universal bundle $\mathcal{V} \to S \times \mathcal{M}$ is given by

$$0 \to O_{\mathbb{P}^1}(1) \otimes L \otimes \lambda \to \mathcal{V} \to \Lambda \otimes L^{-1} \to 0,$$

where $\lambda$ is the tautological line bundle for $\mathcal{M}$.

**Proof.** For the polarisation $H_0$, the moduli space of $H_0$-stable bundles with Chern classes $c_1, c_2$ is empty by [13]. Now for $p_1 = -4c_2 + c_1^2 = -2$ there is only one wall, determined by $\zeta \equiv -\mathbb{P}^1 + \Sigma$ (here we fix $\zeta = 2\Sigma - L = \Sigma - c_1(\Lambda)$ as a divisor), so the moduli space of $H$-stable bundles with Chern classes $c_1, c_2$ is obtained by crossing the wall as described in [9]. First, note that the results in [9] use the hypothesis of $-K$ being effective, but the arguments work equally well with the weaker assumption of $\zeta$ being a good wall [9, remark 1] (see also [5] for the case of $q = 0$). In our case, $\zeta \equiv -\mathbb{P}^1 + \Sigma$ is a good wall (i.e. $\pm \zeta + K$ are both not effective) with $l_\zeta = 0$. Now with the notations of [9], $F$ is a divisor such that $2F - L \equiv \zeta$, e.g. $F = \Sigma$. Also $\mathcal{F} \to S \times J$ is the universal bundle parametrising divisors homologically equivalent to $F$, i.e. $\mathcal{F} = L \otimes O_{\mathbb{P}^1}(1)$. Let $\pi : S \times J \to J$ be the projection. Then $\mathcal{M} = E_\zeta = \mathbb{P}(\mathcal{E}_\zeta^\vee)$, where

$$\mathcal{E}_\zeta = \mathcal{E}xt^1_\pi(\mathcal{O}(L - \mathcal{F}), \mathcal{O}(\mathcal{F})) = R^1\pi_* (\mathcal{O}(\zeta) \otimes \mathcal{L}^2).$$
Actually \( \mathcal{M} \) is exactly the set of bundles \( E \) that can be written as extensions
\[
0 \to \mathcal{O}_\mathbb{P}^1(1) \otimes L \to E \to \Lambda \otimes L^{-1} \to 0
\]
for a line bundle \( L \) of degree 0. The Chern character is computed in \([9, \text{section} \ 3]\) to be \( \text{ch} \mathcal{E}_\mathcal{Z} = 2g + e_{K-2\mathcal{Z}}, \) where \( e_\alpha = -2(\mathbb{P}^1 \cdot \alpha)\omega \) (the class \( \Sigma \) defined in \([9, \text{lemma} \ 11]\) is \( \mathbb{P}^1 \) in our case). Finally, the description of the universal bundle follows from \([9, \text{theorem} \ 10]\). \( \square \)

**Proposition 3.** There is a well defined map \( \mathcal{M}_A \to \mathcal{M} \).

*Proof.* Every line \( f : \mathbb{P}^1 \to M_\Sigma \) gives a bundle \( E = (\text{id}_\Sigma \times f)^* \mathcal{U} \) over \( \Sigma \times \mathbb{P}^1 \) by pulling-back the universal bundle \( \mathcal{U} \to \Sigma \times M_\Sigma \). Then for any \( t \in \mathbb{P}^1 \), the bundle \( E|_{\Sigma \times t} \) is defined by \( f(t) \). Now, by equation (2), \( p_1(E) = p_1(\mathcal{U}|_{\Sigma \times A}) = -2a(A) = -2. \) Since \( c_1(E) = (\text{id}_\Sigma \times f)^*c_1(\mathcal{U}) = \Lambda + \Sigma \), it must be \( c_2 = 1 \). To see that \( E \) is \( H \)-stable, consider any sub-line bundle \( L \hookrightarrow E \) with \( c_1(L) \equiv a[\mathbb{P}^1] + b\Sigma \). Restricting to any \( \Sigma \times t \subset \Sigma \times \mathbb{P}^1 \) and using the stability of \( E|_{\Sigma \times t} \), one gets \( a \leq 0 \). Then \( c_1(L) \cdot \Sigma < \frac{c_1(E) \cdot \Sigma}{2} \), which yields the \( H \)-stability of \( E \) (recall that \( H \) is close to \( \Sigma \)). So \( E \in \mathcal{M} \). \( \square \)

Now define \( N \) as the set of extensions on \( \Sigma \) of the form
\[
0 \to L \to E \to \Lambda \otimes L^{-1} \to 0,
\]
for \( L \) a line bundle of degree 0. Then the groups \( \text{Ext}^1(\Lambda \otimes L^{-1}, L) = H^1(L^2 \otimes \Lambda^{-1}) = H^0(L^{-2} \otimes \Lambda \otimes K) \) are of constant dimension \( g \). Moreover \( H^0(L^2 \otimes \Lambda^{-1}) = 0 \), so the moduli space \( N \) which parametrises extensions like (8) is given as \( N = \mathbb{P}(\mathcal{E}^\vee) \), where \( \mathcal{E} = \mathcal{E}_{\mathcal{X}}|_p(\Lambda \otimes \mathcal{L}^{-1}, \mathcal{L}) = R^1p_*(\mathcal{L}^2 \otimes \Lambda^{-1}), \) \( p : \Sigma \times J \to J \) the projection. Then we have a fibration \( \mathbb{P}^{g-1} \to N = \mathbb{P}(\mathcal{E}^\vee) \to J \). The Chern character of \( \mathcal{E} \) is
\[
\text{ch} (\mathcal{E}) = \text{ch} (R^1p_*(\mathcal{L}^2 \otimes \Lambda^{-1})) = -\text{ch} (p_*(\mathcal{L}^2 \otimes \Lambda^{-1})) =
- p_*(((\text{ch} \mathcal{L})^2 (\text{ch} \Lambda)^{-1} \text{Todd} T_{\Sigma})) =
- p_*((1 + c_1(\mathcal{L}) + \frac{1}{2}c_1(\mathcal{L})^2)^2(1 - \Lambda)(1 - \frac{1}{2}K)) =
- p_*(1 - \frac{1}{2}K + 2c_1(\mathcal{L}) - 4\omega \otimes [\Sigma] - \Lambda) = g + 4\omega.
\]
It is easy to check that all the bundles in \( N \) are stable, so there is a well-defined map
\[
i : N \to M_\Sigma.
\]

Now we wish to construct the space of lines in \( N \). Note that \( \pi_2(N) = \pi_2(\mathbb{P}^{g-1}) = \mathbb{Z} \), as there are no rational curves in \( J \). Let \( L \in \pi_2(N) \) be the positive generator. We want to describe
\[
\mathcal{N}_L = \{ f : \mathbb{P}^1 \to N/f \text{ holomorphic}, f_*[\mathbb{P}^1] = L \}.
\]
For the projective space $\mathbb{P}^n$, the space $H_1$ of lines in $\mathbb{P}^n$ is the set of algebraic maps $f : \mathbb{P}^1 \to \mathbb{P}^n$ of degree 1. Such an $f$ has the form $f[x_0, x_1] = [x_0u_0 + x_1u_1], [x_0, x_1] \in \mathbb{P}^1$, where $u_0$, $u_1$ are linearly independent vectors in $\mathbb{C}^{n+1}$. So

$$H_1 = \mathbb{P} \{ (u_0, u_1)/u_0, u_1 \text{ are linearly independent} \} \subset \mathbb{P}((\mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1})^\vee) = \mathbb{P}^{2n+1}.$$ 

The complement of $H_1$ is the image of $\mathbb{P}^n \times \mathbb{P}^1 \cong \mathbb{P}^{2n+1}$, $([u], [x_0, x_1]) \mapsto [x_0u, x_1u]$, which is a smooth $n$-codimensional algebraic subvariety. So $\mathcal{N}_L$ can be described as the fibration

$$
\begin{array}{ccc}
H_1 & \to & \mathcal{N}_L \\
\cap & \cap & \cap \\
\mathbb{P}^{2g-1} & \to & \mathbb{P}((\mathcal{E} \oplus \mathcal{E})^\vee) & \to & J \\
\end{array}
$$

Remark 4. Note that $\mathcal{E}_\zeta = R^1\pi_*(\mathcal{O}(\zeta) \otimes \mathcal{L}^2) = R^1\pi_*(\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{L}^2 \otimes \Lambda^{-1}) = H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes R^1\pi_*(\mathcal{L}^2 \otimes \Lambda^{-1}) = H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{E} \cong \mathcal{E} \oplus \mathcal{E}$. So $\mathfrak{M} = \mathbb{P}((\mathcal{E} \oplus \mathcal{E})^\vee)$, canonically.

**Proposition 5.** The map $i : N \to M_{\Sigma}$ induces a map $i_* : \mathcal{N}_L \to \mathcal{M}_A$. The composition $\mathcal{N}_L \to \mathcal{M}_A \to \mathfrak{M}$ is the natural inclusion of (10).

**Proof.** The first assertion is clear as $i$ is a holomorphic map. For the second, consider the universal sheaf on $\Sigma \times N$,

$$
0 \to \mathcal{L} \otimes U \to \mathcal{E} \to \Lambda \otimes \mathcal{L}^{-1} \to 0,
$$

where $U = \mathcal{O}_N(1)$ is the tautological bundle of the fibre bundle $\mathbb{P}^{g-1} \to N \to J$. Any element in $\mathcal{N}_L$ is a line $\mathbb{P}^1 \hookrightarrow N$, which must lie inside a single fibre $\mathbb{P}^{g-1}$. Restricting (11) to this line, we have an extension

$$0 \to L \otimes \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{E} \to \Lambda \otimes \mathcal{L}^{-1} \to 0$$

on $S = \Sigma \times \mathbb{P}^1$, which is the image of the given element in $\mathfrak{M}$ (here $L$ is the line bundle corresponding to the fibre in which $\mathbb{P}^1$ sits). Now it is easy to check that the map $\mathcal{N}_L \to \mathfrak{M}$ is the inclusion of (10).

**Corollary 6.** $i_*$ is an isomorphism.

**Proof.** By proposition 5, $i_*$ has to be an open immersion. The group $PGL(2, \mathbb{C})$ acts on both spaces $\mathcal{N}_L$ and $\mathcal{M}_A$, and $i_*$ is equivariant. The quotient $\mathcal{N}_L/PGL(2, \mathbb{C})$ is compact, being a fibration over the Jacobian with all the fibres the Grassmannian $Gr(\mathbb{C}^2, \mathbb{C}^{g-1})$, hence irreducible. As a consequence $i_*$ is an isomorphism.

**Remark 7.** Notice that the lines in $M_{\Sigma}$ are all contained in the image of $N$, which is of dimension $4g-2$ against $6g-6 = \dim M_{\Sigma}$. They do not fill all of $M_{\Sigma}$ as one would naively expect.
4. Computation of $\Psi_A^{Ms}$

The manifold $N$ is positive with $\pi_2(N) = \mathbb{Z}$ and $L \in \pi_2(N)$ is the positive generator. Under the map $\iota : N \to M_S$, we have $\iota_* L = A$. Now $\dim N = 4g - 2$ and $c_1(N)[L] = c_1(\mathbb{P}^{g-1})[L] = g$. So quantum cohomology of $N$, $QH^*(N)$, is well-defined and $\mathbb{Z}/2g\mathbb{Z}$-graded. From corollary 6, it is straightforward to prove

**Lemma 8.** For any $\alpha_i \in H^{p_i}(M_S)$, $1 \leq i \leq r$, such that $p_1 + \cdots + p_r = 6g - 2$, it is $\Psi_A^{Ms}(\alpha_1, \ldots, \alpha_r) = \Psi_L^{N}(i^* \alpha_1, \ldots, i^* \alpha_r)$. □

It is therefore important to know the Gromov-Witten invariants of $N$, i.e. its quantum cohomology. From the universal bundle (11), we can read the first Pontrjagin class $p_1(g) = -8[\Sigma] \otimes \omega + h^2 - 2[\Sigma] \otimes h + 4h \cdot c_1(L) \in H^4(\Sigma \times N)$, where $h = c_1(U)$ is the hyperplane class. So on $N$ we have

$$
\begin{align*}
\alpha &= 2\mu(\Sigma) = 4\omega + h \\
\psi_i &= \mu(\gamma_i) = -h \cdot \phi_i \\
\beta &= -4\mu(x) = h^2
\end{align*}
$$

Let us remark here that $h^2$ denotes ordinary cup product in $H^*(N)$, a fact which will prove useful later. Now let us compute the quantum cohomology ring of $N$. The cohomology of $J$ is $H^*(J) = \Lambda H_1$, where $H_1 = H_1(\Sigma)$. Now the fibre bundle description $\mathbb{P}^{2g-1} \to N = \mathbb{P}(\mathcal{E}^\vee) \to J$ implies that the usual cohomology of $N$ is $H^*(N) = \Lambda H_1[h]/ < h^g + c_1 h^{g-1} + \cdots + c_g = 0 >$, where $c_i = c_i(\mathcal{E}) = \frac{i}{2g} \omega^i$, from (9).

As the quantum cohomology has the same generators as the usual cohomology and the relations are a deformation of the usual relations [18], it must be $h^g + c_1 h^{g-1} + \cdots + c_g = r$ in $QH^*(N)$, with $r \in \mathbb{Q}$. As in [15, example 8.5], $r$ can be computed to be 1. So

$$
\tag{13} QH^*(N) = \Lambda H_1[h]/ < h^g + c_1 h^{g-1} + \cdots + c_g = 1 > .
$$

**Lemma 9.** For any $s \in H^{2g-2i}(J)$, $0 \leq i \leq g$, denote by $s \in H^{2g-2i}(N)$ its pullback to $N$ under the natural projection. Then the quantum product $h^{2g-1+i} s$ in $QH^*(N)$ has component in $H^{4g-2}(N)$ equal to $\frac{(-8)^i}{g^i} \omega^i \wedge s$ (the natural isomorphism $H^{4g-2}(N) \cong H^{2g}(J)$ is understood).

**Proof.** First note that for $s_1, s_2 \in H^*(J)$ such that their cup product in $J$ is $s_1 s_2 = 0$, then the quantum product $s_1 s_2 \in QH^*(N)$ vanishes. This is so since every rational line in $N$ is contained in a fibre of $\mathbb{P}^{2g-1} \to J \to N$.

Next recall that $h^{g-1+i} s$ has component in $H^{4g-2}(N)$ equal to $s_i(\mathcal{E}) \wedge s = \frac{(-4)^i}{g^i} \omega^i \wedge s$. Then multiply the standard relation (13) by $h^{g-1+i} s$ and work by induction on $i$. For $i = 0$ we get $h^{2g-1} s = h^{g-1} s$ and the assertion is obvious. For $i > 0$,

$$
h^{2g-1+i} s + h^{2g-2+i} c_1 s + \cdots + h^{2g-1} c_i s = h^{g-1+i} s.
$$
So the component of $h^{2g-1+i}s$ in $H^{4g-2}(N)$ is

$$-\sum_{j=1}^{i} \frac{(-8)^{i-j}}{(i-j)!} \omega^{i-j} c_{j}s + \frac{(-4)^{i}}{i!} \omega^{i}s = \frac{(-8)^{i}}{i!} \omega^{i}s - \sum_{j=0}^{i} \frac{(-8)^{i-j} 4^{j}}{(i-j)!} \omega^{j}s + \frac{(-4)^{i}}{i!} \omega^{i}s = \frac{(-8)^{i}}{i!} \omega^{i}s.$$  

\[\Box\]

**Lemma 10.** Suppose $g > 2$. Let $\alpha^{a_1} \beta^{b_1} \psi_{\psi_1} \cdots \psi_{\psi_r} \in \mathbb{A}(\Sigma)$ have degree $6g-2$. Then

$$\Psi_{\mathbb{L}}(\alpha^{a_1}, \beta^{b_1}, \psi_1, \ldots, \psi_r) = (4 \omega + X)^{a} (X^{2})^{b} \psi_{\psi_1} \cdots \psi_{\psi_r} X^{r}, [J],$$

evaluated on $J$, where $X^{2g-1+i} = \frac{(-8)^{i}}{i!} \omega^{i} \in H^{*}(J)$.

**Proof.** By definition the left hand side is the component in $H^{4g-2}(N)$ of the quantum product $\alpha^{a_1} \beta^{b_1} \psi_{\psi_1} \cdots \psi_{\psi_r} \in \mathcal{QH}^{*}(N)$. From (12), this quantum product is $(4 \omega + h^{a})(h^{b})(-h \phi_{\psi_1}) \cdots (-h \phi_{\psi_r})$, upon noting that when $g > 2$, $\beta = h^{2}$ as a quantum product as there are no quantum corrections because of the degree. Note that $r$ is even, so the statement of the lemma follows from lemma 9. \[\Box\]

Now we are in the position of relating the Gromov-Witten invariants $\Psi_{\mathbb{A}}^{\mathbb{M}}$ with the Donaldson invariants for $S = \Sigma \times \mathbb{P}^{1}$ (for definition of Donaldson invariants see [4] [7]).

**Theorem 11.** Suppose $g > 2$. Let $\alpha^{a_1} \beta^{b_1} \psi_{\psi_1} \cdots \psi_{\psi_r} \in \mathbb{A}(\Sigma)$ have degree $6g-2$. Then

$$\Psi_{\mathbb{A}}^{\mathbb{M}}(\alpha^{a_1}, \beta^{b_1}, \psi_1, \ldots, \psi_r) = (-1)^{g-1} D_{S,H}^{a}((2 \Sigma)^{a}(-4pt)^{b} \gamma_{\psi_1}^{#} \cdots \gamma_{\psi_r}^{#}),$$

where $D_{S,H}^{a}$ stands for the Donaldson invariant of $S = \Sigma \times \mathbb{P}^{1}$ with $w = c_{1}$ and polarisation $H$.

**Proof.** By definition, the right hand side is $\epsilon_{S}(c_{1}) < \alpha^{a_1} \beta^{b_1} \psi_{\psi_1} \cdots \psi_{\psi_r}, [\mathfrak{M}] >$, where $\alpha = 2 \mu(\Sigma) \in H^{2}(\mathfrak{M}), \beta = -4 \mu(x) \in H^{1}(\mathfrak{M}), \psi_{i} = \mu(\gamma_{i}^{#}) \in H^{3}(\mathfrak{M})$. Here the factor $\epsilon_{S}(c_{1}) = (-1)^{g-1} \frac{\kappa \epsilon_{S}^{2}}{2} = (-1)^{g-1}$ compares the complex orientation of $\mathfrak{M}$ and its natural orientation as a moduli space of anti-self-dual connections [4]. By [9, theorem 10], this is worked out to be $(-1)^{g-1} < (4 \omega) (X^{2})^{b} \psi_{\psi_1} \cdots \psi_{\psi_r} X^{r}, [J], >$, where $X^{2g-1+i} = s_{i}((\mathcal{E}_{\psi})^{a} \omega^{i}$. Thus the theorem follows from lemmas 8 and 10. \[\Box\]

**Remark 12.** The formula in theorem 11 is not right for $g = 2$, as in such case, the quantum product $h^{2} \in \mathcal{QH}^{*}(N)$ differs from $\beta$ by a quantum correction.

**Remark 13.** Suppose $g \geq 2$ and let $\alpha^{a_1} \beta^{b_1} \psi_{\psi_1} \cdots \psi_{\psi_r} \in \mathbb{A}(\Sigma)$ have degree $6g-6$. Then

$$\Psi_{0}^{\mathbb{M}}(\alpha^{a_1}, \beta^{b_1}, \psi_1, \ldots, \psi_r) = \epsilon_{S}(\mathbb{P}^{1}) < \alpha^{a_1} \beta^{b_1} \psi_{\psi_1} \cdots \psi_{\psi_r}, [\mathfrak{M}] > =$$

$$-D_{S,H}^{a}((2 \Sigma)^{a}(-4pt)^{b} \gamma_{\psi_1}^{#} \cdots \gamma_{\psi_r}^{#}),$$

as the moduli space of anti-self-dual connections on $S$ of dimension $6g-6$ is $\mathfrak{M}$. 

5. Quantum cohomology of $M_{\Sigma}$

It is natural to ask to what extent the first quantum correction determines the full structure of the quantum cohomology of $M_{\Sigma}$. In [3], Donaldson finds the first quantum correction for $M_{\Sigma}$ when the genus of $\Sigma$ is $g = 2$ and proves that this is enough to find the quantum product. Now it is our intention to show how the Gromov-Witten invariants $\Psi^{M_{\Sigma}}$ determine completely $QH^*(M_{\Sigma})$. First we check an interesting fact.

**Lemma 14.** Let $g \geq 3$. Then $\gamma = -2 \sum \psi_i \psi_{i+g}$ as elements in $QH^*(M_{\Sigma})$ (i.e. using the quantum product in the right hand side).

**Proof.** Let $\gamma = -2 \sum \psi_i \psi_{i+g} \in QH^*(M_{\Sigma})$. In principle, it is $\gamma = \gamma + s \alpha$, for some $s \in \mathbb{Q}$. Let us show that $s = 0$. Multiplying by $\alpha^{3g-4}$, we have $\gamma \alpha^{3g-4} = \gamma \alpha^{3g-4} + sa^{3g-3}$. Considering the component in $H^{0g-6}(M_{\Sigma})$ and using lemma 8, we have

$$-2 \sum \Psi^N_L(\alpha, (3g-4), \alpha, \psi_i, \psi_{i+g}) = \Psi^N_L(\alpha, (3g-4), \alpha, \gamma) + s < \alpha^{3g-3}, [M_{\Sigma}] > .$$

Now, in $N$, $\gamma$ is the cup product $-2 \sum \phi_i \psi_{i+g} h^2$. It is easy to check that this coincides with the quantum product $-2 \sum \phi_i \psi_{i+g} h^2$. For $g > 3$ it is evident because of the degree. For $g = 3$ there might be a quantum correction in $H^0(N)$, but this is $-2 \sum \Psi^N_L(\phi_i, \psi_{i+g}, h, h, pt) = 0$ (since lines are contained in the fibres). Now lemma 10 and its proof imply that $-2 \sum \Psi^N_L(\alpha, \ldots, \alpha, \psi_i, \psi_{i+g}) = \Psi^N_L(\alpha, \ldots, \alpha, \gamma)$, so $s = 0$. $\square$

We are pursuing to prove an isomorphism between $QH^*(M_{\Sigma})$ and $HF^*(\Sigma \times S^1)$, the instanton Floer homology of the three manifold $\Sigma \times S^1$. First recall the main result contained in [10].

**Theorem 15 ([10]).** Define $R^1_0 = 1$, $R^2_0 = 0$, $R^3_0 = 0$ and then recursively, for all $r \geq 1$,

$$
\begin{cases}
R^1_{r+1} = \alpha R^1_r + r^2 R^2_r \\
R^2_{r+1} = (\beta + (-1)^{r+1}8) R^1_r + \frac{2r}{r+1} R^3_r \\
R^3_{r+1} = \gamma R^1_r
\end{cases}
$$

Put $I_r' = (R^1_r, R^2_r, R^3_r) \subset \mathbb{Q}[\alpha, \beta, \gamma]$, $r \geq 0$. Then the $Sp(2g, \mathbb{Z})$-decomposition of $HF^*(\Sigma \times S^1)$ is

$$HF^*(\Sigma \times S^1) = \bigoplus_{k=0}^{g-1} \Lambda^k H^3 \otimes \mathbb{Q}[\alpha, \beta, \gamma]/I_{g-k}'.$$

The elements $R^1_r$, $R^2_r$ and $R^3_r$ are deformations graded mod 4 of $q^1_r$, $q^2_r$ and $q^3_r$, respectively. This means that we can write

$$(14) \quad R^i_r = \sum_{j \geq 0} R^i_{r,j},$$
where \( \deg(R^i_{r,j}) = \deg(q^i_r) - 4j, j \geq 0, \) and \( R^i_{r,0} = q^i_r. \) In the case of \( QH^*(M_\Sigma) \) we shall have

**Proposition 16.** The \( \text{Sp}(2g, \mathbb{Z}) \)-decomposition of \( QH^*(M_\Sigma) \) is

\[
QH^*(M_\Sigma) = \bigoplus_{k=0}^{g-1} \Lambda^k_0 H^3 \otimes \mathbb{Q}[\alpha, \beta, \gamma]/J_{g-k},
\]

where \( J_r \) is generated by three elements \( Q^1_r, Q^2_r \) and \( Q^3_r, \) which are deformations graded mod 4 of \( q^1_r, q^2_r \) and \( q^3_r, \) respectively.

**Proof.** The action of \( \text{Sp}(2g, \mathbb{Z}) \) on \( M_\Sigma \) being symplectic (see [16, section 3.1]), we have an epimorphism of rings (with \( \text{Sp}(2g, \mathbb{Z}) \)-actions) like in (3)

\[
A_\Sigma \to QH^*(M_\Sigma).
\]

This induces an epimorphism on the invariant parts

\[
\mathbb{Q}[\alpha, \beta, \gamma] \to QH^*_I(M_\Sigma),
\]

where \( \gamma = -2 \sum_{i=0}^{g} \psi_i \psi_{i+g} \) (see lemma 14). Therefore we have maps

\[
(15) \quad \Lambda^k_0(\psi_1, \ldots, \psi_{2g}) \otimes \mathbb{Q}[\alpha, \beta, \gamma] \to QH^*(M_\Sigma).
\]

Let \( V_k \) be the image of the map (15). As \( \Lambda^k_0 H^3, 0 \leq k \leq g-1, \) are inequivalent irreducible \( \text{Sp}(2g, \mathbb{Z}) \)-modules, the subspaces \( V_k \) are pairwise orthogonal. On the other hand, the existence of the basis (6) of \( H^*(M_\Sigma) \) and the results in [18] imply that \( \{x_i^{(k)} a^b b^{e} / k = 0, 1, \ldots, g-1, a + b + c < g-k, i \in B_k \} \) (where quantum products are now understood) is a basis of \( QH^*(M_\Sigma). \) So the subspaces \( V_k \) generate \( QH^*(M_\Sigma), \) i.e.

\[
(16) \quad QH^*(M_\Sigma) = \bigoplus_{k=0}^{g-1} V_k.
\]

Actually this decomposition coincides with the decomposition (5). This is proved by giving a definition of \( V_k \) independent of the ring structure (cup product or quantum product). For instance, say that \( V_k \) is the space generated by elements which are orthogonal to \( V_0, \ldots, V_{k-1} \) and such that the \( \text{Sp}(2g, \mathbb{Z}) \)-module generated by them have dimension equal to \( \dim \Lambda^k_0 H^3. \)

Our second purpose is to describe the kernel of (15), i.e. the relations satisfied by the elements of \( \Lambda^k_0 H^3, \alpha, \beta \) and \( \gamma. \) The results in [18] imply that we only need to write the relations of \( V_k \subset H^*(M_\Sigma) \) in terms of the quantum product. Fix \( k, \) and recall \( x_i^{(k)} = \psi_1 \psi_2 \cdots \psi_k \in \Lambda^k_0 H^3 \) from (7). By section 2 the relations in \( V_k \) are given
by \( x_0^{(k)} q_g^{(k)} \), \( i = 1, 2, 3 \), and its \( \text{Sp}(2g, \mathbb{Z}) \)-transforms. We rewrite these relations in terms of the quantum product, using the basis (6), as
\[
(17) \quad x_0^{(k)} q_g^{(k)} = \sum_{a+b+c<g-k} x_{abc} \alpha^a \beta^b \gamma^c \in QH^*(M_\Sigma)
\]
where \( x_{abc} \in \Lambda_0^k H^3 \), and the monomials in the right hand side have degree strictly less than the degree of the left hand side. Now we want to prove that \( x_{abc} \) are all multiples of \( x_0^{(k)} \). Suppose not. Then it is easy to see that there exists \( \phi \in \text{Sp}(2g, \mathbb{Z}) \) satisfying \( \phi(x_0^{(k)}) = x_0^{(k)} \) and \( \phi(x_{abc}) \neq x_{abc} \). Consider (17) minus its transform under \( \phi \). This is a relation between the elements of the basis of \( V_k \), which is impossible.

Therefore (17) can be rewritten as
\[
x_0^{(k)} (q_g^{(k)} + Q_g^{(k)+1} + Q_g^{(k)+2} + \cdots) = 0,
\]
where \( \deg Q_g^{(k)+j} = \deg q_g^{(k)} - 4j, j \geq 1 \). This finishes the proof. \( \square \)

**Lemma 17.** \( \gamma J_k \subset J_{k+1} \subset J_k \), for \( k = 0, 1, \ldots, g - 1 \).

**Proof.** Let \( f \in J_k \subset \mathbb{Q}[\alpha, \beta, \gamma] \). By definition (proposition 16) this means that the quantum product \( \psi_1 \cdots \psi_g f = 0 \). Using the action of \( \text{Sp}(2g, \mathbb{Z}) \) we have \( \psi_1 \cdots \psi_g^{(k)} f = 0 \), for \( g-k \leq i \leq g \). Thus \( \psi_1 \cdots \psi_{g-k+1} \gamma f = 0 \), i.e. \( \gamma f \in J_{k+1} \). For the second inclusion, let \( f \in J_{k+1} \). Then \( \psi_1 \cdots \psi_{g-k+1} f = 0 \) and hence \( \psi_1 \cdots \psi_g f = 0 \), i.e. \( f \in J_k \). \( \square \)

**Proposition 18.** There are numbers \( c_r, d_r \in \mathbb{Q} \), \( 1 \leq r \leq g - 1 \), such that for \( 0 \leq r \leq g - 1 \) it is
\[
\begin{align*}
Q_{r+1}^1 &= \alpha Q_r^1 + r^2 Q_r^2 \\
Q_{r+1}^2 &= (\beta + c_{r+1})Q_r^1 + \frac{2r}{r+1} Q_r^3 \\
Q_{r+1}^3 &= \gamma Q_r^1 + d_{r+1} Q_r^2
\end{align*}
\]

**Proof.** Completely analogous to the proof of [10, theorem 10]. \( \square \)

**Proposition 19.** For all \( 1 \leq r \leq g \), \( c_r = (-1)^{g-r+1} 8 \) and \( d_r = 0 \).

**Proof.** We write \( R_g^{(i)} = \sum_{j \geq 0} R_g^{i-j} \) and \( Q_g^{(i)} = \sum_{j \geq 0} Q_g^{i-j} \), as in (14), for \( 0 \leq k \leq g-1 \). Then \( R_g^{(i)-k,0} = Q_g^{i-k,0} = \delta_g^{(k-i,0)} \). The coefficients \( c_r \) and \( d_r \) are determined by the first correction term \( Q_{r+1}^1 \) of \( Q_r^1 \). By the definition of \( R_g^{(i)} \) in theorem 15, we only need to check that \( R_g^{(i)-k,1} = (-1)^g Q_g^{(i-k-1,1)} \), for \( i = 1, 2, 3, 0 \leq k \leq g-1 \).

Fix \( i \) and \( k \). Recall \( x_0^{(k)} = \psi_1 \cdots \psi_k \in \Lambda_0^k H^3 \). By theorem 15, \( x_0^{(k)} R_g^{i-k} = 0 \in HF^*(\Sigma \times S^1) \). Pick an arbitrary \( f = \alpha^a \beta^b \psi_1 \cdots \psi_r \in \Lambda(\Sigma) \) of degree \( 6g-2 - \deg(x_0^{(k)} q_g^{(k)}) \). In \( HF^*(\Sigma \times S^1) \) the pairing \( \langle x_0^{(k)} R_g^{i-k}, f \rangle = 0 \), i.e. \( D_S^{(w,\Sigma)}(x_0^{(k)} R_g^{i-k} f) = 0 \).
where $R_{g-k}^i = R_{g-k}^i(2\Sigma, -4x, -2\sum \gamma_i^# \gamma_{g+i}^#)$, and analogously for $\tilde{f}$ and $\bar{x}_0^{(k)}$ (for the notation $D^{(w, \Sigma)}$ see [10]). This means that

$$D^{\tilde{w}}_{S,H}(\tilde{x}_0^{(k)} \tilde{R}_g^{i-k,1,1} \tilde{f}) + D^{\tilde{w}}_{S,H}(\bar{x}_0^{(k)} \bar{R}_g^{i-k,0,0} \bar{f}) = 0.$$ 

From theorem 11 and remark 13 we have that the component in $H^{6g-6}(M_\Sigma)$ of the quantum product $-x_0^{(k)} R_{g-k,1}^i f + (-1)^{g-1} x_0^{(k)} R_{g-k,0}^i \bar{f} \in QH^*(M_\Sigma)$ vanishes, i.e.

$$- < x_0^{(k)} R_{g-k,1}^i, f > + (-1)^{g-1} < x_0^{(k)} R_{g-k,0}^i, \bar{f} > = 0$$

in $QH^*(M_\Sigma)$.

On the other hand, proposition 16 says that $x_0^{(k)} Q_{g-k}^i = 0 \in QH^*(M_\Sigma)$. Multiplying by $f$, $x_0^{(k)} Q_{g-k}^i f = 0$, so the component in $H^{6g-6}(M_\Sigma)$ of the quantum product $x_0^{(k)} Q_{g-k,1}^i f + x_0^{(k)} Q_{g-k,0}^i \bar{f} \in QH^*(M_\Sigma)$ is zero. Thus

$$< x_0^{(k)} Q_{g-k,1}^i, f > + < x_0^{(k)} Q_{g-k,0}^i, \bar{f} > = 0$$

in $QH^*(M_\Sigma)$.

Equations (18) and (19) imply together that

$$< x_0^{(k)} Q_{g-k,1}^i, f > = (-1)^g < x_0^{(k)} R_{g-k,1}^i, f >,$$

for any $f \in A(\Sigma)$ of degree $6g - 2 - \deg(x_0^{(k)} Q_{g-k}^i) = 6g - 6 - \deg(x_0^{(k)} Q_{g-k,1}^i)$. As we are considering the pairing on classes of complementary degree, equation (20) holds in $H^*(M_\Sigma)$ as well. By section 2, $Q_{g-k,1}^i \equiv (-1)^g R_{g-k,1}^i (\mod I_{g-k})$. Considering the degrees, it must be $Q_{g-k,1}^1 \equiv (-1)^g R_{g-k,1}^1$ and $Q_{g-k,1}^2 \equiv (-1)^g R_{g-k,1}^2$. For $i = 3$, the difference $Q_{g-k,1}^3 - (-1)^g R_{g-k,1}^3$ is a multiple of $q_{g-k}$. The vanishing of the coefficient of $\alpha^{g-k}$ for both $Q_{g-k}^3$ and $R_{g-k}^3$ (see theorem 15 and equation (17)) implies $Q_{g-k,1}^3 = (-1)^g R_{g-k,1}^3$. 

Putting all together we have proved the following

**Theorem 20.** The quantum cohomology of $M_\Sigma$, for $\Sigma$ a Riemann surface of genus $g \geq 3$, has a presentation

$$QH^*(M_\Sigma) = \bigoplus_{k=0}^{g-1} \Lambda^k H^3 \otimes \mathbb{Q}[\alpha, \beta, \gamma]/J_{g-k}.$$ 

where $J_r = (Q_r^1, Q_r^2, Q_r^3)$ and $Q_r^i$ are defined recursively by setting $Q_0^1 = 1, Q_0^2 = 0, Q_0^3 = 0$ and putting for all $r \geq 0$

$$\begin{cases}
Q_{r+1}^1 = \alpha Q_r^1 + r^2 Q_r^2 \\
Q_{r+1}^2 = (\beta + (-1)^{r+g+1} 8) Q_r^1 + \frac{2r}{r+1} Q_r^3 \\
Q_{r+1}^3 = \gamma Q_r^1
\end{cases}$$
Corollary 21. Let $\Sigma$ be a Riemann surface of genus $g \geq 3$. Then there is an isomorphism

$$QH^*(M_\Sigma) \xrightarrow{\sim} HF^*(\Sigma \times S^1).$$

For $g$ even, the isomorphism sends $(\alpha, \beta, \gamma) \mapsto (\alpha, \beta, \gamma)$. For $g$ odd, the isomorphism sends $(\alpha, \beta, \gamma) \mapsto (\sqrt{-1} \alpha, -\beta, -\sqrt{-1} \gamma)$.

Proof. This is a consequence of the descriptions of $QH^*(M_\Sigma)$ and $HF^*(\Sigma \times S^1)$ in theorem 20 and theorem 15, respectively. □

Remark 22. Alternatively, we can say that for any $g \geq 1$ there is an isomorphism $QH^*(M_\Sigma) \xrightarrow{\sim} HF^*(\Sigma \times S^1)$, taking $(\alpha, \beta, \gamma) \mapsto (\sqrt{-1}^g \alpha, \sqrt{-1}^{2g} \beta, \sqrt{-1}^{3g} \gamma)$.

6. The cases $g = 1$ and $g = 2$

Let us review the cases of genus $g = 1$ and $g = 2$ in the view of theorem 20. These cases are somehow atypical, as the generators precise the introduction of quantum corrections, a fact already noted in [1].

Example 23. Let $\Sigma$ be a Riemann surface of genus $g = 1$. Then $M_\Sigma$ is a point and we can write

$$QH^*(M_\Sigma) = \mathbb{Q}[\alpha, \hat{\beta}, \gamma]/(\alpha, \hat{\beta} + 8, \gamma),$$

where we have defined $\hat{\beta} = \beta - 8$. This agrees with theorem 20 but with corrected generators. Again $QH^*(M_\Sigma) \xrightarrow{\sim} HF^*(\Sigma \times S^1)$, where $(\alpha, \hat{\beta}, \gamma) \mapsto (\sqrt{-1} \alpha, -\beta, -\sqrt{-1} \gamma)$.

Example 24. Let $\Sigma$ be a Riemann surface of genus $g = 2$. The quantum cohomology ring $QH^*(M_\Sigma)$ has been computed by Donaldson [3], using an explicit description of $M_\Sigma$ as the intersection of two quadrics in $\mathbb{P}^5$. Let $h_2$, $h_4$ and $h_6$ be the integral generators of $QH^2(M_\Sigma)$, $QH^4(M_\Sigma)$ and $QH^6(M_\Sigma)$, respectively. Then, with our notations, $\alpha = h_2$, $\beta = -4h_4$ and $\gamma = 4h_6$ (see [1]). Define $\hat{\gamma} = -2 \sum \psi_i \psi_{i+g} \in QH^*(M_\Sigma)$. The computations in [3] yield $\hat{\gamma} = \gamma - 4\alpha$ (compare with lemma 14). Put $\hat{\beta} = \beta + 4$. It is now easy to check that the relations found in [3] can be translated to

$$QH^*(M_\Sigma) = \left( H^3 \otimes \mathbb{Q}[\alpha, \hat{\beta}, \hat{\gamma}]/(\alpha, \hat{\beta} - 8, \hat{\gamma}) \right) \oplus \mathbb{Q}[\alpha, \hat{\beta}, \hat{\gamma}]/(Q_1^2, Q_2^3, Q_2^3),$$

where $Q_1^2 = \alpha^2 + \hat{\beta} - 8$, $Q_2^2 = (\hat{\beta} + 8)\alpha + \hat{\gamma}$ and $Q_2^3 = \alpha \hat{\gamma}$ (defined exactly as in theorem 20, but with corrected generators). Now $QH^*(M_\Sigma) \xrightarrow{\sim} HF^*(\Sigma \times S^1)$, where $(\alpha, \hat{\beta}, \hat{\gamma}) \mapsto (\sqrt{-1} \alpha, -\beta, -\sqrt{-1} \gamma)$.

The artificially introduced definition of $\hat{\beta}$ is due to the same phenomenon which causes the failure of lemma 10 for $g = 2$, i.e. the quantum product $h^2$ differs from $\beta$ in (12) (defined with the cup product) because of a quantum correction in $QH^*(N)$ which appears when $g = 2$. 
References

1. M. Bershadsky, A. Johansen, V. Sadov and C. Vafa, Topological reduction of 4D SYM to 2D $\sigma$-models, Preprint, 1995.
2. U. Desale and S. Ramanan, Classification of vector bundles of rank 2 on hyperelliptic curves, *Inventiones Math.* **38** 1976, 161-185.
3. S. K. Donaldson, Floer homology and algebraic geometry, *Vector bundles in algebraic geometry*, London Math. Soc. Lecture Notes Series, **208** Cambridge University Press, Cambridge, 1995, 119-138.
4. S. K. Donaldson and P. B. Kronheimer, *The geometry of 4-manifolds*, Oxford University Press, 1990.
5. G. Ellingsrud and L. Göttsche, Variation of moduli spaces and Donaldson invariants under change of polarisation, *Journal reine angew. Math.* **467** 1995, 1-49.
6. A. D. King and P. E. Newstead, On the cohomology ring of the moduli space of rank 2 vector bundles on a curve, Liverpool Preprint, 1994.
7. P. B. Kronheimer and T. S. Mrowka, Embedded surfaces and the structure of Donaldson’s polynomial invariants, *Jour. Differential Geometry*, **41** 1995, 573-734.
8. D. McDuff and D. A. Salamon, J-holomorphic curves and quantum cohomology, Preprint.
9. V. Muñoz, Wall-crossing formulae for algebraic surfaces with $q > 0$, alg-geom/9709002.
10. V. Muñoz, Ring structure of the Floer cohomology of $\Sigma \times S^1$, dg-ga/9710029.
11. S. Piunikhin, Quantum and Floer cohomology have the same ring structure, MIT Preprint, 1994.
12. S. Piunikhin, D. Salamon and M. Schwarz, Symplectic Floer-Donaldson theory and quantum cohomology, Warwick Preprint, 1995.
13. Z. Qin, Moduli of stable sheaves on ruled surfaces and their Picard groups, *Jour. Reine ange Math.* **433** 1992, 201-219.
14. Y. Ruan, Topological sigma model and Donaldson type invariants in Gromov theory, *Duke Math. Jour.* **83** 1996, 461-500.
15. Y. Ruan and G. Tian, A mathematical theory of quantum cohomology, *Jour. Diff. Geom.* **42** 1995, 259-367.
16. B. Siebert, An update on (small) quantum cohomology, Preprint, 1997.
17. B. Siebert and G. Tian, Recursive relations for the cohomology ring of moduli spaces of stable bundles, *Proceedings of 3rd Gökova Geometry-Topology Conference 1994*.
18. B. Siebert and G. Tian, On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator, submitted to Duke Math. Journal.
19. M. Thaddeus, Conformal field theory and the cohomology of the moduli space of stable bundles, *Jour. Differential Geometry*, **35** 1992, 131-150.