A HEAWOOD–TYPE RESULT FOR THE ALGEBRAIC
CONNECTIVITY OF GRAPHS ON SURFACES

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ABSTRACT. We prove that the algebraic connectivity $a(G)$ of a graph embedded
on a nonplanar surface satisfies a Heawood–type result. More precisely, it
is shown that the algebraic connectivity of a surface $S$, defined as the supre-
mum of $a(G)$ over all graphs that can be embedded in $S$, is equal to the
chromatic number of $S$. Furthermore, and with the possible exception of the
Klein bottle, we prove that this bound is attained only in the case of the
maximal complete graph that can be embedded in $S$. In the planar case, we
show that, at least for some classes of graphs which include the set of regular
graphs, $a(G)$ is less than or equal to four. As an application of these results
and techniques, we obtain a lower bound for the genus of Ramanujan graphs.

We also present some bounds for the asymptotic behaviour of $a(G)$ for
certain classes of graphs as the number of vertices goes to infinity.

1. INTRODUCTION

In recent years, there has been some work relating the spectral radius of the
adjacency matrix of a graph, $r(G)$, to its genus – see [EZ, Ho1, Ho2]. The idea
behind these results is to combine an appropriate estimate for $r(G)$ in terms of the
number of vertices and edges of the graph, with a direct consequence of Euler’s
formula giving an upper bound for the number of edges of a graph embedded in a
surface $S$. This allows the derivation of a bound for $r(G)$ in terms on the number
of vertices of $G$ and the genus of $S$.

In the continuous case there is also a relation between the eigenvalues of a certain
differential operator and the genus of that surface. More precisely, there exist upper
bounds for the first nontrivial eigenvalue of the Laplace–Beltrami operator on a
surface in terms of its genus. The first of these results was obtained by Hersch
for the case of the sphere [He], and this was later generalized by Yang and Yau to
orientable surfaces of genus $\gamma$:

**Theorem 1.1.** [YY] Let $S$ be an orientable surface of genus $\gamma$ and let $\lambda$ denote
the first nontrivial eigenvalue of the Laplace–Beltrami operator on $S$. Then

$$\lambda \leq \frac{8\pi(1 + \gamma)}{A(S)},$$

where $A(S)$ denotes the area of $S$.

Note that in the case of general manifolds of dimension greater than two, it is
known that no such results are possible [I].

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These results suggest the derivation of similar bounds for the first nontrivial eigenvalue of the Laplacian operator defined on graphs. This quantity is related to the connectivity of a graph \( G \), and was thus christened the algebraic connectivity of \( G \), \( a(G) \), by Fiedler [F].

The main purpose of this paper is to study the maximum possible value of \( a(G) \) for a graph which can be embedded on a surface of genus \( \gamma \). It turns out that this value is equal to the Heawood number which appears in graph colouring problems [GT, RY, WB]. Our main result is that for surfaces of positive genus this maximum possible value is given by the algebraic connectivity of the maximal complete graph that is possible to embed on such a surface and that, with the possible exception of the Klein bottle, this value is attained only for this graph – see Section 3 for details. An immediate consequence is that the algebraic connectivity of a graph of genus \( \gamma \) is bounded from above by the chromatic number of \( S_{\gamma} \). In a sense, this is a surprising result since the algebraic connectivity and the chromatic number are not intimately related in general – see Section 6.

As in the case of map colouring problems, the techniques used to prove the result in the case of positive genus do not apply to the case of the sphere, and so this remains an open problem. We are, however, able to obtain some results under some restrictions which include the case of planar regular graphs. These are presented in Sections 4 and 5.

In Section 7 we show how the results of the paper can be used to obtain estimates for the genus of some graphs. In particular, we apply this to the case of Ramanujan graphs.

These results show that the maximum value of the algebraic connectivity on a given surface is attained for a finite number of vertices. However, it is also of interest to know how \( a(G) \) behaves as the number of vertices becomes large. A first step in this direction is done in Section 8 where we consider the supremum of the algebraic connectivity of certain classes of graphs on a surface of fixed genus as the number of vertices goes to infinity.

Finally, in Section 9 we consider some open problems and present some conjectures.

2. Preliminaries

2.1. Notation. We begin by reviewing some concepts from graph theory that will be used in what follows. Let \( G \) be a simple \( n \)-vertex graph \((n \geq 3)\), that is, a graph with \( n \) vertices and no loops nor multiple edges, and denote the sets of vertices and edges by \( V \) and \( E \), respectively. We also denote the number of edges, \(|E|\), by \( e \). The degree \( d_i \) of a vertex is the number of edges having that vertex as one end, and the smallest and the largest of these numbers will be denoted by \( d_{\text{min}} \) and \( d_{\text{max}} \), respectively. The vertex connectivity \( v(G) \) is defined as the minimal number of vertices whose removal (together with their adjacent edges) results in a disconnected graph. The girth of a graph is the length of the shortest cycle in the graph.

The adjacency matrix \( A \) of a graph \( G \) is defined by

\[
A = \{a_{ij}\}_{i,j=1}^n
\]

where \( a_{ij} \) equals one if there is an edge connecting vertices \( i \) and \( j \), and zero otherwise.
The Laplacian of a graph is defined to be the matrix $L = D - A$, where $D$ is the diagonal matrix $\text{diag}(d_1, \ldots, d_n)$. For some basic properties of this and related operators see, for instance, [Ch, CdV, F]. The eigenvalues of the Laplacian will be denoted by $0 = \lambda_1 \leq \lambda_2 \leq \ldots \lambda_n$.

The second eigenvalue $\lambda_2$ is normally denoted by $a(G)$, and is called the algebraic connectivity of the graph.

As far as we are aware, the best bounds for $a(G)$ depending only on the number of vertices and of edges of a graph remain those given by Fiedler in [F]. The bound from that paper which will be of interest here is contained in the following

**Theorem 2.1.** [F] Let $G$ be a simple, connected, $n$–vertex graph with $e$ edges and which is not complete. Then

$$a(G) \leq v(G) \leq d_{\text{min}} \leq \frac{2e}{n}.$$ 

**2.2. Graphs on surfaces.** In what follows a surface $S$ is a compact connected 2–manifold. From the classification of surfaces, we have that $S$ is either homeomorphic to a sphere with $h$ handles in the orientable case, or to the connected sum of $k$ projective planes in the non–orientable case. In this context, it is usual to denote the former by $S_h$ and the latter by $N_k$. The genus $\gamma(S)$ of a surface $S$ is defined to be $1 - \chi(S)/2$, where $\chi(S_h) = 2 - 2h$ and $\chi(N_k) = 2 - k$ is the Euler characteristic of the surface. Whenever a statement applies in both the orientable and non–orientable cases, we shall refer to the surface of genus $\gamma$ by $S_\gamma$, without stating explicitly which case is being considered.

An important aspect of topological graph theory is the study of whether or not it is possible to embed a graph in a given surface. We say that there is an embedding of a graph $G$ in a surface $S$, if there exist a one–to–one mapping of $V$ onto a set of $n$ distinct points in $S$, and a mapping from $E$ to disjoint open arcs in $S$, such that no point in the image of $V$ is contained in the image of an edge, and the image of an edge joining two vertices is an arc joining the corresponding images – see [WB] for this and other related concepts. If a graph $G$ is embedded in a surface $S$, then the set $S \setminus G$ consists of a collection of connected components which are called the faces of $G$. If all the faces of an embedding are homeomorphic to an open disk the graph is said to be cellulary embedded in $S$, and the embedding is called a 2–cell embedding. The orientable (non–orientable) genus of a graph is defined to be the smallest possible genus of an orientable (resp. non–orientable) surface where $G$ is embeddable. We shall refer to the orientable Euler characteristic of a graph as the Euler characteristic of the orientable surface corresponding to the genus of the graph, and similarly for the non–orientable Euler characteristic. Where there is no danger of confusion, or when a result applies in both cases, we shall refer only to the genus or to the Euler characteristic of a graph, without stating explicitly whether it refers to the orientable or non–orientable case.

The chromatic number of a surface $S$, $\kappa(S)$, is the maximum chromatic number of all graphs which can be embedded in $S$. In an analogous way, we define the algebraic connectivity of a surface as

$$A(S) = \sup_{G \in G(S)} a(G),$$

where $G(S)$ is the set of all graphs that can be embedded in $S$. 


Another question of interest is the study of the asymptotic behaviour of \( a(G) \) for families of graphs embedded in a surface as the number of vertices goes to infinity. Given an infinite family \( \mathcal{F} \) of graphs embeddable on a surface \( S \) we define the asymptotic algebraic connectivity of \( S \) by
\[
\mathcal{A}_\infty^\mathcal{F}(S) := \sup \left[ \limsup_{n \to \infty} a(G_n) \right],
\]
where \( G_n \in \mathcal{F}, |G_n| = n, \) and the supremum is taken over all possible such sequences – when \( \mathcal{F} = \mathcal{G}(S), \) we omit the subscript and write \( \mathcal{A}_\infty(S). \)

2.3. Auxiliary results. The basic result that allows us to relate the algebraic connectivity of a graph embedded in a surface to its genus is the following consequence of Euler’s formula:

**Theorem 2.2.** Let \( G \) be a simple, connected, \( n \)-vertex graph with \( e \) edges and girth \( g(\geq 3) \). If \( G \) is embeddable in a surface \( S \) with characteristic \( \chi \), then
\[
e \leq \frac{g}{g-2} \left[ n - \chi \right].
\]

For a proof see, for instance, \([WB]\).

Combining this with Theorem 2.1 yields the following result, which will be used in the sequel.

**Corollary 2.3.** Let \( G \) be a non–complete graph of Euler characteristic \( \chi \). Then
\[
a(G) \leq \frac{2g}{g-2} \frac{n - \chi}{n}.
\]

We will also need a relation between the vertex connectivity of a graph and its genus. This is given by the following theorem, due to Cook.

**Theorem 2.4.** Let \( G \) be a graph of nonpositive Euler characteristic \( \chi \). Then
\[
v(G) \leq C(S_\gamma) := \left\lfloor \frac{5 + \sqrt{49 - 24\chi}}{2} \right\rfloor.
\]

As mentioned in \([PZ]\), this still holds in the case of the projective plane (non–orientable genus one). Cook has improved this result when \( G \) does not contain triangles. Here we shall only make use of this in the planar case.

**Theorem 2.5.** Let \( G \) be a planar graph of girth \( g \). Then
\[
v(G) \leq \begin{cases} 5 & \text{if } g = 3, \\ 3 & \text{if } g = 4, 5, \\ 2 & \text{if } g \geq 6. \end{cases}
\]

Finally, let \( S \) be a surface of genus \( \gamma \). Then the maximal complete graph of genus \( \gamma, \) which we denote by \( K^\gamma, \) is the complete graph on the largest possible number of vertices that can be embedded in \( S. \)

A theorem of Ringel and Youngs gives the orientable genus of complete graphs.

**Theorem 2.6.** The orientable genus of the complete graph \( K_p \) \((p \geq 3)\) is given by
\[
\gamma(K_p) = \left\lfloor \frac{(p-3)(p-4)}{12} \right\rfloor.
\]
In the case of non–orientable genus, the corresponding result is due to Ringel.

**Theorem 2.7.** The non–orientable genus of the complete graph \( K_p \) \((p \geq 3)\) is given by

\[
\gamma(K_p) = \left\lceil \frac{(p-3)(p-4)}{6} \right\rceil,
\]

with the exception of \( K_7 \) for which we have \( \gamma(K_7) = 3 \).

3. A Heawood–type result for the algebraic connectivity

The main theorem of the paper is the following

**Theorem 3.1.** Let \( G \) be a graph of genus \( \gamma \) and nonpositive Euler characteristic \( \chi \). Then \( a(G) \leq a(K_\gamma) \). More precisely, and with the exception of the Klein bottle (non–orientable genus two), we have that

\[
a(G) \leq a(K_\gamma) = H(S) := \left\lfloor \frac{7 + \sqrt{49 - 24\chi}}{2} \right\rfloor,
\]

with equality if and only if \( G = K_\gamma \).

In the case of the Klein bottle, we have that \( a(G) \leq a(K_6) = 6 < H(S) = 7 \).

For planar graphs, \( a(G) \leq 5 \).

The number \( H(S) \) is Heawood’s number for the surface \( S \), and is related to the chromatic number of a surface – see [GT], for instance. As was pointed out in the Introduction, there is a great similarity between this result and the corresponding result for the colouring of graphs known as the Heawood map–colouring problem. In fact, a straightforward corollary to this theorem is that the algebraic connectivity of a graph of genus \( \gamma \) is bounded from above by the chromatic number \( \kappa(S_\gamma) \).

**Corollary 3.2.** Let \( G \) be a nonplanar graph of genus \( \gamma \). Then \( a(G) \leq \kappa(S_\gamma) \). Furthermore, if \( G \neq K_\gamma \) then \( a(G) \leq a(K_\gamma) - 1 \), except possibly in the case of the Klein bottle.

In terms of the algebraic connectivity of a surface, this may be stated as follows.

**Corollary 3.3.** For any surface \( S \) of positive genus \( \gamma \) we have that

\[
\mathcal{A}(S) = \kappa(S).
\]

Furthermore, if \( G \neq K_\gamma \) is a graph embedded on a surface \( S \), then \( a(G) \leq \mathcal{A}(S) - 1 \), except possibly in the case of the Klein bottle.

**Proof of Theorem 3.1.** Combining Theorem 2.4 with Fiedler’s bound we have that for a nonplanar orientable graph \( G \)

\[
a(G) \leq \nu(G) \leq C(S) = H(S) - 1.
\]

On the other hand, we have from Theorem 2.6 that for each value of \( \gamma \) the complete graph \( K_{H(S)} \), can be embedded in \( S_\gamma \). Since \( K_{H(S)} = H(S) \), we have that for noncomplete graphs

\[
a(G) \leq C(S) = H(S) - 1 = a(K_{H(S)}) - 1,
\]

which proves the result, as well as the second part of Corollary 3.3.
In the planar case the maximum possible vertex connectivity is five, and thus \( a(G) \leq 5 \).

In the non–orientable case, and for \( \gamma \) larger than two, we proceed in a similar way to obtain the result for noncomplete graphs, now using Theorem 2.7.

For the projective plane (\( \gamma = 1 \)), the result follows in the same way by using the fact that Theorem 2.4 extends to this case, and that the maximal complete graph is now \( K_6 \).

In the case of the Klein bottle (\( \gamma = 2 \)), we have that the maximal complete graph that can be embedded there is \( K_6 \). From Theorem 2.4 we obtain that for noncomplete graphs \( a(G) \leq v(G) \leq C(2) = 6 = a(K_6) \), proving the result in this case.

### 4. Planar graphs

We will begin by obtaining some bounds on the algebraic connectivity which apply to general graphs. The main argument which will be used is a bound based on a test function similar to that used in the proof of Cheeger’s inequality – see [CdV], for instance. The idea is that it should be possible to improve on bounds based on \( v(G) \) in cases where there are clusters of vertices for which a sufficient large number of the edges join vertices within the cluster. In order to do this, we need the following definition.

**Definition 4.1.** Let \( G \) be a graph and \( H \) be a proper nonempty subset of \( V(G) \), formed by the vertices \( y_i, i = 1, \ldots, m \). The degree of the subset \( H \), is defined as

\[
d(H) = \sum_{i} \left( d_i - \tilde{d}_i \right),
\]

where \( \tilde{d}_i \) is the number of edges joining two vertices \( y_i \) and \( y_j, 1 \leq i, j \leq m \).

**Lemma 4.2.** Let \( G \) be a graph on \( n \) vertices and let \( H \) be a proper nonempty subset of \( V(G) \) with \( m \) vertices and degree \( d(H) \). Then

\[
a(G) \leq \frac{d(H)n}{m(n - m)}.
\]

**Proof.** By the variational formulation for the eigenvalues of \( G \), we have that

\[
a(G) \leq \frac{\sum_{x \sim y} [f(x) - f(y)]^2}{\sum_{x \in G} f^2(x)},
\]

for all \( f \) such that

\[
\sum_{i=1}^{n} f(x_i) = 0,
\]

and where \( x \sim y \) denotes that the vertices \( x \) and \( y \) are adjacent. Let \( f : V \to \mathbb{R} \) be defined by

\[
f(x) = \begin{cases} 
\frac{n}{m}, & x \in H \\
-1, & x \in G \setminus H.
\end{cases}
\]
We then have that $f$ is orthogonal to the vector with all entries equal to 1, and thus, by (4.1), it follows that

$$a(G) \leq \frac{d(H) \left( \frac{n}{m} \right)^2}{m \left( \frac{n}{m} - 1 \right)^2 + n - m}$$

$$= \frac{d(H)n^2}{m^2 \left( \frac{n^2}{m} - n \right)}$$

$$= \frac{d(H)n}{m(n - m)}.$$

\[\square\]

**Theorem 4.3.** Let $G$ be a planar graph with $d_{\text{max}}$ smaller than or equal to five. Then $a(G) \leq 4$.

**Proof.** If $G$ does not have a triangle, then by Theorem 2.5 $a(G) \leq 3$. Assume thus that $G$ has a triangle $T$. Then $d(T) \leq 3 \times 5 - 6 = 9$ and hence, from Lemma 4.2, it follows that

$$a(G) \leq \frac{3n}{n - 3}.$$

We thus have that $a(G) > 4$ implies $n < 12$. On the other hand, for a planar graph to have $a(G)$ greater than four, it must have $d_{\text{min}}$ equal to five. Since $d_{\text{max}}$ is less than or equal to five, from Theorem 2.2

$$\sum_{i=1}^{n} d_i = 5n = 2e \leq 6(n - 2),$$

and we obtain that for this to happen the graph must have at least twelve vertices. \[\square\]

5. Regular graphs

Since for planar graphs $d_{\text{min}}$ is less than or equal to five, Theorem 4.3 implies that for planar regular graphs $a(G)$ is less than or equal to four. We shall now consider the restriction to some particular cases of regular graphs.

We begin by obtaining a general bound for regular graphs (not necessarily planar) of a given girth.

**Theorem 5.1.** Let $G$ be a regular graph of girth $g$ smaller than $n$ and Euler characteristic $\chi$. Then

$$a(G) \leq \frac{2n}{(n - g)(g - 2)} \left( 2 - \frac{g\chi}{n} \right).$$
Proof. Take $H$ in Lemma 4.2 to correspond to the cycle $C_g$. Then $d(H) = g(d-2)$ and

$$a(G) \leq \frac{(d-2)n}{n-g}$$

$$= 2 \left( \frac{\frac{g}{g-2}}{n} - \chi \right) \frac{n}{n-g}$$

$$= \frac{2}{g-2} \left( 2 - \frac{g\chi}{n} \right) \frac{n}{n-g}.$$

Note that for a fixed girth and sufficiently large $n$ this is better than Corollary 2.3 which is obtained from Fiedler’s bound.

In the case of planar graphs $\chi$ equals two and this bound takes the following simple form.

**Corollary 5.2.** Let $G$ be a regular planar graph of girth $g$ smaller than $n$. Then

$$a(G) \leq \frac{4}{g-2}.$$

When the girth is equal to $n$, we have the case of 2−regular graphs (the cycle $C_g$) for which it is easy to see that, with the exception of $K_3$, the algebraic connectivity is always less than or equal to two, this value being attained for the cycle $C_4$. A similar result holds for planar cubic graphs.

**Theorem 5.3.** Every planar cubic graph other than $K_4$ has

$$a(G) \leq 2.$$

Proof. If the girth $g$ is larger than or equal to four, the result follows from the bound in Corollary 2.2. Assume thus that $g$ equals three. Then $G$ has a triangle and proceeding as in the proof of Theorem 1.2 but now using the fact that the degree is three, we obtain that

$$a(G) \leq \frac{n}{n-3}.$$

This will be greater than two provided that $n$ is less than 6. Since we are interested in cubic graphs, the case of $n$ equal to five is excluded and the only cubic graph when $n$ equals four is $K_4$.

The result is sharp, in the sense that the bound is attained by both the 3−prism and the cube graphs.

6. **Algebraic connectivity and chromatic numbers**

In general, and without any further assumptions, we cannot expect a deep relation between $a(G)$ and $\kappa(G)$, in the sense that there exist graphs for which $a(G) < \kappa(G)$ (any connected graph with $a(G)$ less than two), graphs for which $a(G) = \kappa(G)$ (complete graphs), and graphs for which $a(G) > \kappa(G)$ (any complete bipartite graph $K_{p,q}$ with $p$ and $q$ both greater than two). However, it is possible to prove the following
Theorem 6.1. Let $G$ be a noncomplete graph on $n$ vertices and with chromatic number $\kappa(G)$. Then

$$a(G) \leq n - \left\lfloor \frac{n}{\kappa(G)} \right\rfloor.$$ 

Proof. The graph $G$ can be divided into subsets $X_i, i = 1, \ldots, \kappa(G)$, such that there are no edges connecting vertices within each set $X_i$. This means that the complementary graph of $G$, $G^c$, contains the complete graphs $K_{n_i}$, where $n_i = |X_i|, i = 1, \ldots, \kappa(G)$. We also have that at least one of the numbers $n_i$ is greater than $\left\lceil \frac{n}{\kappa(G)} \right\rceil$ and, since $G$ is not complete, greater than one. Hence

$$\lambda_n(G^c) \geq \left\lceil \frac{n}{\kappa(G)} \right\rceil,$$

from which the result follows, since $a(G) \leq n - \lambda_n(G^c)$. \hfill \Box

In the special case of bichromatic graphs of a given Euler characteristic it is possible to improve on this result.

Theorem 6.2. If $G$ is a graph on $n$ vertices with $\kappa(G) = 2$ and Euler characteristic $\chi$, then

$$a(G) \leq 4 \frac{n - \chi}{n}.$$ 

Proof. Since $G$ is bichromatic, its girth must be greater than three. The result now follows from Corollary 2.3. \hfill \Box

As a consequence, we obtain that bichromatic planar graphs satisfy $a(G) < 4$. However, in this case it is possible to obtain a better bound using Theorem 2.5.

Theorem 6.3. If $G$ is a bichromatic planar graph, then $a(G) \leq 3$.

Proof. It follows as before but now using Theorem 2.5 instead of Corollary 2.3. \hfill \Box

It is, of course, also possible to use the results in [2, 4] for the case of general orientable genus and where the graph’s girth is greater than or equal to four to obtain further bounds for bichromatic graphs. However, these will not be as good as those in Theorem 6.3, at least for $\gamma$ larger than one and sufficiently large $n$.

In general, bichromatic graphs can have an arbitrarily high algebraic connectivity, as can be seen from the case of the complete bipartite graph $K_{p,q}$, for which $\kappa(K_{p,q})$ equals two, while $a(K_{p,q}) = \min\{p,q\}$. Note that when $p$ equals $q$ this gives equality in the case of Theorem 6.1.

On the other hand, it is also possible to keep $a(G)$ bounded while making the chromatic number as large as desired. To see this, consider the graph $G$ obtained from $K_n$ by adding a vertex which is connected to one single vertex in $K_n$, that is, $G = (K_{n-1} \cup \{x\}) + K_1$. The spectrum of this graph is given by $(0, 1, n, \ldots, n, n+1)$ (and thus $a(G) = 1$), while its chromatic number grows with $n$.

These two examples show that unless we impose an extra restriction such as fixing the genus of the graph, we should not expect a close relation between the algebraic connectivity and the chromatic number of a graph.
7. A LOWER BOUND FOR THE GENUS OF RAMANUJAN GRAPHS

It is possible to use the bounds in Corollary 3.2 to obtain estimates for the genus of a given graph, provided one has a lower bound for $a(G)$ – this is actually just a consequence from the fact that $a(G)$ is less than or equal to $C(S)$. Here we apply this to the case of Ramanujan graphs – see [Ch, LPS].

Theorem 7.1. Let $G$ be a Ramanujan graph on $n$ vertices and of degree $d$, with $9 \leq d \neq n - 1$. Then its orientable genus satisfies

$$\gamma \geq \left\lceil \frac{(2d - 4\sqrt{d - 1} - 5)^2 - 1}{48} \right\rceil.$$ 

Proof. The algebraic connectivity of a Ramanujan graph satisfies (see [Ch], page 97, for instance)

$$a(G) \geq d - 2\sqrt{d - 1}.$$ 

Since by Corollary 3.2 the algebraic connectivity of a noncomplete graph which can be embedded on a surface $S$ satisfies

$$a(G) \leq a(K^2) - 1 = H(S) - 1 = C(S),$$

it follows that

$$\sqrt{49 - 24\chi} \geq 2d - 4\sqrt{d - 1} - 5.$$ 

From this we see that if $d$ is smaller than 9 then nothing can be concluded, while for $d$ greater than or equal to 9 we obtain the desired result.

A similar result can be obtained for the case of the nonorientable genus in the same way, except that then one has to consider the exceptional case of genus two separately.

8. ASYMPTOTIC BEHAVIOUR OF $a(G)$

From Corollary 3.2 we have that the algebraic connectivity of a graph embedded in a surface $S$ is bounded from above independently of the number of vertices of the graph and, in fact, if $S$ is neither the sphere nor the Klein bottle, then $\mathcal{A}^\infty(S) \leq A(S) - 1$. A better bound for $\mathcal{A}^\infty(S)$ is given by Corollary 2.3 which yields that $\mathcal{A}^\infty(S)$ is less than or equal to 6, independently of $S$.

On the other hand, by considering the sequence of (planar) double wheel graphs, that is, $G_n = C_n + 2K_1$, for which

$$a(G_n) = \min\{ a(C_n) + 2, a(2K_1) + n \} = \min\{ 4 - 2\cos(2\pi/n), n \},$$

we obtain that $\mathcal{A}^\infty(S)$ is greater than or equal to two and so we have that

$$2 \leq \mathcal{A}^\infty(S) \leq 6.$$ 

We shall now restrict our attention to certain classes of graphs. An immediate consequence of Theorem 5.1 is a bound for the asymptotic connectivity of regular graphs with a given fixed girth $g$.

Theorem 8.1. Let $\mathcal{G}_{r,g}(S)$ be the set of all regular graphs with girth equal to a fixed number $g$ that can be embedded in a given surface $S$. Then

$$\mathcal{A}^\infty_{\mathcal{G}_{r,g}}(S) \leq \frac{4}{g - 2}.$$
Note that the dependence of the bound in Theorem 5.1 is not monotonic in \( g \) and so we cannot obtain directly a uniform bound which would allow us to conclude that the asymptotic connectivity of regular graphs is less than or equal to four. However, it is possible to prove this combining Theorem 5.1 with Corollary 2.3.

**Theorem 8.2.** Let \( G_r(S) \) be the set of all regular graphs that can be embedded in \( S \). Then

\[
\lambda_\infty^\infty(G_r(S)) \leq 4.
\]

**Proof.** If \( g \) is equal to three, Theorem 5.1 gives that

\[
a(G) \leq \frac{2n}{n-3} \left( 2 - \frac{3\chi}{n} \right) = \frac{4n - 6\chi}{n - 3}.
\]

On the other hand, if the girth is larger than three, then by Corollary 2.3 it follows that

\[
a(G) \leq \frac{4(n - \chi)}{n}.
\]

Since both bounds converge to four as \( n \) goes to infinity, the result follows.

Combining this with a bound of Alon and Boppana, we obtain the following bound for the specific case of \( d \)--regular graphs for a fixed integer \( d \).

**Corollary 8.3.** For a given integer \( d \) greater than or equal to two, let \( G^d(S) \) denote the set of \( d \)--regular graphs that can be embedded in \( S \). We have that

\[
\lambda_\infty^\infty(G^d(S)) \leq \begin{cases} 
    d - 2\sqrt{d-1}, & d \leq 10 \\
    4, & d > 10.
\end{cases}
\]

**Proof.** From a result by Alon and Boppana quoted in [A] – see also [Ch] –, we have that for a \( d \)--regular graph

\[
a(G) \leq d - 2\sqrt{d-1} + O(\log_d n)^{-1},
\]

from which it follows that \( \lambda_\infty^\infty(G^d(S)) \leq d - 2\sqrt{d-1} \). This is smaller than four for \( d \) smaller then or equal to 10. For \( d \) larger than 10, we use the bound from Theorem 8.2.

9. Discussion

As in the proof of Theorem 4.3, we see that for \( a(G) \) to be greater than four then a planar graph must have at least twelve vertices, since this is a condition for the minimum degree of a planar graph to be equal to five. We thus have that \( a(G) \leq 4 \) both when \( n \) is smaller than twelve or when \( d_{max} \) is smaller than or equal to five. Both this and the similarity with colouring problems suggest that part of Theorem 7.2 extends to planar graphs, that is, that the maximal algebraic connectivity of a planar graph is four, the algebraic connectivity of \( K_4 \), but we haven’t been able to prove it. However, if that is the case, then the maximal algebraic connectivity will not be uniquely attained in this case, since the octahedron (the join of \( 2K_1 \) with the cycle on four vertices, \( 2K_1 \mathbin{+} C_4 \)), has as its spectrum \((0, 4, 4, 4, 6, 6)\). We have the following conjecture

**Conjecture 1.** If \( G \) is a planar graph, then \( a(G) \leq 4 \), with equality if and only if \( G = K_4 \) or \( G = 2K_1 \mathbin{+} C_4 \). Furthermore, if \( G \) is neither of these graphs, then \( a(G) \) is less than or equal to three.
Regarding chromatic numbers, note that $\kappa(2K_1+4)$ is three, and so it is possible to have a planar graph whose chromatic number is three while its algebraic connectivity equals four, although if the conjecture above holds, then this will be the only graph for which this will happen.

As we have seen, for bichromatic planar graphs we must have that $a(G)$ can be at most three. In fact, we conjecture that this value can be improved.

**Conjecture 2.** Every planar bichromatic graph has $a(G) \leq 2$.

In the case of the Klein bottle, and since it is possible to have equality between $C(N_2)$ and $a(K_6)$, we have not been able to prove that equality holds only for the complete graph $K_6$. However, we believe this to be the case.

**Conjecture 3.** For the Klein bottle (the non-orientable case of genus two), $a(G)$ is 6 if and only if $G = K_6$.

Finally, regarding the upper bound for the asymptotic algebraic connectivity in the general case, we remark that this is most likely far from being optimal, since we have used methods which are based mainly on local properties of graphs. We conjecture that the value of the asymptotic algebraic connectivity in the general case is independent of the (fixed) surface $S$, and that it will in fact be equal to two, the algebraic connectivity of the double wheel graph.

**Conjecture 4.** $A_\infty(S) \equiv 2$.

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