ELECTROMAGNETIC INTERIOR TRANSMISSION EIGENVALUE PROBLEM FOR AN INHOMOGENEOUS MEDIUM WITH A CONDUCTIVE BOUNDARY

YUEBIN HAO

School of Mathematics and Statistics
Beijing Institute of Technology, Beijing 100081, P.R.China

(Communicated by Yuri Latushkin)

Abstract. The interior transmission eigenvalue problem plays a basic role in the study of inverse scattering problems for an inhomogeneous medium. In this paper, we consider the electromagnetic interior transmission eigenvalue problem for an inhomogeneous medium with conductive boundary. Our main focus is to understand the associated eigenvalue problem, more specifically to prove the transmission eigenvalues form a discrete set and show that they exist by employing a variety of variational techniques under various assumptions on the index of refraction.

1. Introduction. In recent years, the interior transmission eigenvalue problem has become an important area of research in inverse scattering theory. Although simply stated, the transmission eigenvalue problem is a non-selfadjoint eigenvalue problem that can not be covered by the standard theory of eigenvalue problems for elliptic equations. Our particular interest is the fact that transmission eigenvalues carry information about the material properties of the scattering object. Besides, the transmission eigenvalues can be determined by the measured scattering data, see [5, 17, 19, 22, 29]. For a connection of the interior transmission problem with the scattering problem we refer the readers to [4, 10, 11, 14, 16, 21].

Up to now, some progress has been made in the study of transmission eigenvalue problem. Regarding the transmission eigenvalue problem for Helmholtz equation, several papers have appeared that address both the question of discreteness and existence of transmission eigenvalues, for details, we refer to the monographs [1, 6, 7, 9, 12, 24, 25, 27, 28] and the references therein. Similarly, some research has been made about the transmission eigenvalues for Maxwell’s equations and plays an important role in application [18, 26]. In [2, 3], discreteness of electromagnetic transmission eigenvalues has been proved under the condition that the magnetic and electric permittivity does not change sign near the boundary. It is known [8] that the transmission eigenvalues for Maxwell’s equations form a discrete set without finite accumulation point and there exists an infinite set of real transmission eigenvalues under some assumptions for the index of refraction. In the current article [23], they extend the result [7] to the much more technical and complicated Maxwell’s system governing the electromagnetic scattering, and prove the discreteness and existence.

2000 Mathematics Subject Classification. Primary: 35Q61, 35P25; Secondary: 35J15.

Key words and phrases. Transmission eigenvalues, conductive boundary condition, existence, discreteness.
of the interior transmission eigenvalues. For more related works, we refer to the monographs [13, 20] and the references therein.

In the current paper, the underlying scattering problem is the scattering of electromagnetic waves by a non-magnetic material situated in homogenous background, which in terms of the electric field reads:

$$\begin{align*}
\text{curl}\text{curl}\ E - k^2 n E &= 0, \quad \text{in} \ R^3 \setminus D, \\
\text{curl}\text{curl}\ E - k^2 n E &= 0, \quad \text{in} \ D, \\
\nu \times E_+ - \nu \times E_- &= 0, \quad \text{on} \ \partial D, \\
\nu \times \text{curl} E_+ + \eta(\nu \times E_+) \times \nu &= \nu \times \text{curl} E_-, \quad \text{on} \ \partial D, \\
\lim_{|x| \to \infty} (\text{curl} E \times x - ik|x|E) &= 0,
\end{align*}$$

where $D \subset R^3$ be a collection of bounded simply-connected domains with $C^\infty$ smooth boundary $\partial D$ and $\nu$ denote the outward unit normal to the boundary $\partial D$. Let $n(x)$ denote the refractive index, $k$ be the wave number and $\eta$ be a boundary parameter. Besides, $E := E' + E^s$ be the total electric field, where $E'$ be the incident electric field and $E^s$ be the scattered electric field. $E_+$ (E_-) denote the limit of $E$ on the surface $\partial D$ from the exterior (interior) of $\partial D$, that is

$$E_\pm(x) := \lim_{h \to 0^+} E(x \pm h \nu(x)).$$

The Silver-Müller radiation condition is satisfied uniformly with respect to $\hat{x} = x/|x|$.

We assume that $D$ is given. The interior transmission eigenvalue problem corresponding to (1) is to determine $k > 0$ such that there exists a nontrivial solution to

$$\begin{align*}
\text{curl}\text{curl}\ E_1 - k^2 n E_1 &= 0, \quad \text{in} \ D, \\
\text{curl}\text{curl}\ E_2 - k^2 E_2 &= 0, \quad \text{in} \ D, \\
\nu \times E_1 - \nu \times E_2 &= 0, \quad \text{on} \ \partial D, \\
\nu \times \text{curl} E_1 - \nu \times \text{curl} E_2 &= \eta(\nu \times E_2) \times \nu, \quad \text{on} \ \partial D.
\end{align*}$$

If the above problem (2) has a nontrivial solution, then $k$ is called the interior transmission eigenvalues. In this paper, we will consider the case where $\eta$ is real-valued and positive. We shall first prove the discreteness and existence of the interior transmission eigenvalues of the system (2). To our best knowledge, those results are new to literature in the study of electromagnetic interior transmission eigenvalue problems.

The rest of the paper is organized as follows. In section 2, we define the interior transmission eigenvalue problem in the appropriate Sobolev spaces and derive its variational form. In section 3, we investigate the spectral properties of the interior transmission eigenvalue problem. We prove the discreteness and the existence of the interior transmission eigenvalues, provided that the real-valued index of refraction $n := n(x)$ in the medium satisfies $0 < n < 1$. Our approach does not work if $n > 1$. In section 4, we obtain monotonicity results for the transmission eigenvalues with respect to the material parameters $n$ and $\eta$.

2. Preliminaries and variational formulation.

2.1. Preliminaries. In order to formulate our transmission eigenvalue problem more precisely, we need to introduce some related spaces.

Let $D \subset R^3$ be defined as above. We denote by $(\cdot, \cdot)_D$ the $L^2(D)^3$ scalar product and consider the Hilbert spaces

$$H(\text{curl}, D) := \{ u \in L^2(D)^3 : \text{curl} u \in L^2(D)^3 \},$$
$H_0(\text{curl}, D) := \{ u \in H(\text{curl}, D) : \nu \times u = 0 \text{ on } \partial D \}$
equipped with the scalar product 
$$(u, v)_{H(\text{curl}, D)} = (u, v)_D + (\text{curl} u, \text{curl} v)_D$$
and the corresponding norm $\| \cdot \|_{H(\text{curl}, D)}$. Next we define 
$$H^2(\text{curl}, D) := \{ u \in H(\text{curl}, D) : \text{curl} u \in H(\text{curl}, D) \},$$
$$H^2_0(\text{curl}, D) := \{ u \in H_0(\text{curl}, D) : \text{curl} u \in H(\text{curl}, D) \}$$
equipped with the scalar product 
$$(u, v)_{H^2(\text{curl}, D)} = (u, v)_{H(\text{curl}, D)} + (\text{curl} u, \text{curl} v)_{H(\text{curl}, D)}$$
and the corresponding norm $\| \cdot \|_{H^2(\text{curl}, D)}$.

The electromagnetic interior transmission eigenvalue problem reads as follows: for given functions $n \in C^1(D)$ and $\eta \in L^\infty(\partial D)$, find $k > 0$ and $\mathbf{E}_1, \mathbf{E}_2 \in L^2(D)^3$ such that $\mathbf{E}_1 - \mathbf{E}_2 \in H^2_0(\text{curl}, D)$ and $\mathbf{E}_1, \mathbf{E}_2$ satisfies (2). For analytical considerations we put the following hypotheses on $n, \eta$. 

(H1) $n(x) \in C^1(D)$ is real-valued and $0 < n(x) < 1$. 

(H2) $\eta(x) \in L^\infty(\partial D)$ is real-valued such that $\eta > 0$ a.e. on $\partial D$.

2.2. Variational formulation. In this subsection we are going to derive the variational form of the interior transmission problem (2).

We now let $\tilde{\mathbf{E}} \in H^2_0(\text{curl}, D)$ denote the difference $\mathbf{E}_1, \mathbf{E}_2$. By the first and second identities in (2) we have that $\tilde{\mathbf{E}}$ satisfies 

$$(\text{curl} \text{curl} \tilde{\mathbf{E}} - k^2 n \tilde{\mathbf{E}} = k^2 (n-1) \mathbf{E}_2, \text{ in } D) \tag{3}$$
or 

$$(\text{curl} \text{curl} - k^2)(n-1)^{-1}(\text{curl} \text{curl} \tilde{\mathbf{E}} - k^2 n \tilde{\mathbf{E}}) = 0, \text{ in } D. \tag{4}$$

We also get the boundary conditions

$$\nu \times \tilde{\mathbf{E}} = 0, \text{ on } \partial D \tag{5}$$
and

$$\eta^{-1}(\nu \times \text{curl} \tilde{\mathbf{E}}) = (\nu \times \mathbf{E}_2) \times \nu, \text{ on } \partial D. \tag{6}$$

By the boundary conditions (5) and (6), together with Green’s second vector theorem, we have

$$\int_{\partial D} \eta^{-1}(\nu \times \text{curl} \tilde{\mathbf{E}}) \cdot (\nu \times \text{curl} \mathbf{F}) \, ds$$
$$= \int_{\partial D} [(\nu \times \mathbf{E}_2) \times \nu] \cdot (\nu \times \text{curl} \mathbf{F}) \, ds$$
$$= \int_{\partial D} \mathbf{E}_2 \cdot (\nu \times \text{curl} \mathbf{F}) \, ds = - \int_{\partial D} (\nu \times \mathbf{E}_2) \cdot \text{curl} \mathbf{F} \, ds$$
$$= \int_{D} \mathbf{E}_2 \cdot \text{curl} \text{curl} \mathbf{F} - \mathbf{F} \cdot \text{curl} \text{curl} \mathbf{E}_2 \, dx$$
$$= \int_{D} \mathbf{E}_2 \cdot \text{curl} \text{curl} \mathbf{F} - k^2 \mathbf{F} \cdot \mathbf{E}_2 \, dx.$$
According to (3) and (4), we obtain
\[
\int_{\partial D} \eta^{-1}(\nu \times \text{curl} \tilde{E}) \cdot (\nu \times \text{curl} F) ds \\
= \int_D \left( \frac{1}{k^2} (n-1)^{-1}(\text{curl} \text{curl} \tilde{E} - k^2 n \tilde{E}) \right) \cdot \text{curl} \text{curl} F \\
- ((n-1)^{-1}(\text{curl} \text{curl} \tilde{E} - k^2 n \tilde{E})) \cdot F dx \\
= \frac{1}{k^2} \int_D (n-1)^{-1}(\text{curl} \text{curl} \tilde{E} - k^2 n \tilde{E}) \cdot (\text{curl} \text{curl} F - k^2 F) dx.
\]

Therefore,
\[
0 = - \int_D (n-1)^{-1}(\text{curl} \text{curl} \tilde{E} - k^2 n \tilde{E}) \cdot (\text{curl} \text{curl} F - k^2 F) dx \\
+ k^2 \int_{\partial D} \eta^{-1}(\nu \times \text{curl} \tilde{E}) \cdot (\nu \times \text{curl} F) ds.
\]

Furthermore, by Green’s first vector theorem, we obtain
\[
0 = - \int_D (n-1)^{-1}(\text{curl} \text{curl} \tilde{E} - k^2 n \tilde{E}) \cdot (\text{curl} \text{curl} F - k^2 F) dx \\
+ k^2 \int_D \text{curl} \tilde{E} \cdot \text{curl} F dx - k^4 \int_D \tilde{E} \cdot F dx \\
+ k^2 \int_{\partial D} \eta^{-1}(\nu \times \text{curl} \tilde{E}) \cdot (\nu \times \text{curl} F) ds.
\]

Therefore, the variational formulation of the interior transmission problem (3)-(6) becomes: find \( \tilde{E} \in H_0^2(\text{curl}, D) \) satisfies (8) for all \( F \in H_0^2(\text{curl}, D) \).

The functions \( E_2 \) and \( E_1 \) are related to \( \tilde{E} \) through
\[
E_2 = \frac{1}{k^2} (n-1)^{-1}(\text{curl} \text{curl} \tilde{E} - k^2 n \tilde{E}), \\
E_1 = \tilde{E} + E_2 = \frac{1}{k^2} (n-1)^{-1}(\text{curl} \text{curl} \tilde{E} - k^2 \tilde{E}).
\]

**Definition 2.1.** The values of \( k > 0 \) are said to be a transmission eigenvalue if the electromagnetic interior transmission problem (2) has a nontrivial solution \( E_1 \in L^2(D)^3 \) and \( E_2 \in L^2(D)^3 \) such that \( E_1 - E_2 \in H_0^2(\text{curl}, D) \). If \( k > 0 \) is a transmission eigenvalue, we call the solution \( \tilde{E} \in H_0^2(\text{curl}, D) \) of (8) the corresponding eigenfunction.

3. **Spectral property of the electromagnetic interior transmission eigenvalue problem.** In this section, we investigate the spectral properties of the interior transmission eigenvalue problem (2). We will prove the discreteness and the existence of interior transmission eigenvalue by considering \( 0 < n < 1 \) in \( D \). Next, we denote by \( n_* = \inf_{x \in D} n(x) \) and \( n^* = \sup_{x \in D} n(x) \).
3.1. Discreteness of the transmission eigenvalues. Let us define the following sesquilinear forms

\[
\mathcal{A}_k(\mathbf{E}, \mathbf{F}) = -\int_D (n - 1)^{-1}(\text{curl}\text{curl} \mathbf{E} - k^2 \mathbf{E}) \cdot (\text{curl}\text{curl} \mathbf{F} - k^2 \mathbf{F}) \, dx \\
+ k^2 \int_D \text{curl} \mathbf{E} \cdot \text{curl} \mathbf{F} \, dx + k^4 \int_D \mathbf{E} \cdot \mathbf{F} \, dx \\
+ k^2 \int_{\partial D} \eta^{-1}(\nu \times \text{curl} \mathbf{E}) \cdot (\nu \times \text{curl} \mathbf{F}) \, ds
\]

and

\[
\mathcal{B}(\mathbf{E}, \mathbf{F}) = 2 \int_D \mathbf{E} \cdot \mathbf{F} \, dx.
\]

Then the interior transmission problem in the variational form now consists of finding \( \mathbf{E} \in H^2_0(\text{curl}, D) \) such that

\[
\mathcal{A}_k(\mathbf{E}, \mathbf{F}) - k^4 \mathcal{B}(\mathbf{E}, \mathbf{F}) = 0, \quad \text{for all } \mathbf{F} \in H^2_0(\text{curl}, D).
\]

Using the Riesz representation theorem we define two bounded linear operators

\[
A_k : H^2_0(\text{curl}, D) \\
\rightarrow H^2_0(\text{curl}, D) \quad \text{and} \quad B : H^2_0(\text{curl}, D) \\
\rightarrow H^2_0(\text{curl}, D) \quad \text{by}
\]

\[
(A_k \mathbf{E}, \mathbf{F})_{H^2_0(\text{curl}, D)} := \mathcal{A}_k(\mathbf{E}, \mathbf{F}) \quad \text{and} \quad (B \mathbf{E}, \mathbf{F})_{H^2_0(\text{curl}, D)} := \mathcal{B}(\mathbf{E}, \mathbf{F}).
\]

The main result of this section can be stated as follows.

**Theorem 3.1.** Assume that \( n, \eta \) satisfies \((H_1), (H_2)\), respectively. And moreover, \( 0 < n(x) < 1 \) satisfies \( \|\nabla n(x)/n(x)\|_{L^\infty(D)} \ll 1 \) a.e. in \( D \). Then the set of transmission eigenvalues is at most discrete. Moreover, the only accumulation point for the set of transmission eigenvalues is \( +\infty \).

We will give two lemmas before proving the main theorem and assume that \( 0 < n_+ < n < n^* < 1 \).

**Lemma 3.2.** The operator \( A_k \) is coercive.

**Proof.** Taking specifically \( \mathbf{F} = \mathbf{E} \in H^2_0(\text{curl}, D) \), we have

\[
(A_k \mathbf{E}, \mathbf{E})_{H^2_0(\text{curl}, D)}
\]

\[
= \int_D (1 - n)^{-1}|\text{curl}\text{curl} \mathbf{E} - k^2 \mathbf{E}|^2 \, dx + k^2 \|\text{curl} \mathbf{E}\|_{L^2(D)}^2 \\
+ k^4 \|\mathbf{E}\|_{L^2(D)}^2 + k^2 \|\eta^{-\frac{1}{2}}(\nu \times \text{curl} \mathbf{E})\|_{L^2(\partial D)}^2
\]

(9)

\[
= \int_D (1 - n)^{-1}|\text{curl}\text{curl} \mathbf{E}|^2 + 2k^2 \text{Re}\{\mathbf{E} \cdot \text{curl}\text{curl} \mathbf{E}\} + k^4 |\mathbf{E}|^2 \, dx \\
+ k^2 \|\text{curl} \mathbf{E}\|_{L^2(D)}^2 + k^4 \|\mathbf{E}\|_{L^2(D)}^2 + k^2 \|\eta^{-\frac{1}{2}}(\nu \times \text{curl} \mathbf{E})\|_{L^2(\partial D)}^2.
\]

Setting \( \gamma = \frac{1}{1-n} \) and using the equality

\[
\gamma X^2 - 2\gamma XY + (1 + \gamma)Y^2 = \mu \left( Y - \frac{\nu}{\mu} X \right)^2 + \left( \gamma - \frac{\nu^2}{\mu} \right) X^2 + (1 + \gamma - \mu)Y^2,
\]

\[
\mu.
\]

\[
\text{Re}\{\mathbf{E} \cdot \text{curl}\text{curl} \mathbf{E}\}
\]

\[
and
\]

\[
\|\text{curl}\text{curl} \mathbf{E}\|_{L^2(D)}^2
\]

\[
\|\text{curl} \mathbf{E}\|_{L^2(D)}^2
\]

\[
\|\eta^{-\frac{1}{2}}(\nu \times \text{curl} \mathbf{E})\|_{L^2(\partial D)}^2.
\]

\[
\gamma X^2 - 2\gamma XY + (1 + \gamma)Y^2
\]

\[
\mu.
\]
for $X = \|\text{curl}\nabla\bar{\mathbf{E}}\|_{L^2(D)}$, $Y = k^2\|\bar{\mathbf{E}}\|_{L^2(D)}$ and arbitrary $\mu > 0$, then (9) becomes

$$(A_k\bar{\mathbf{E}}, \bar{\mathbf{E}})_{H_0^2(\text{curl}, D)} \geq \gamma\|\text{curl}\nabla\bar{\mathbf{E}}\|_{L^2(D)}^2 - 2k^2\gamma\|\bar{\mathbf{E}}\|_{L^2(D)}^2 \|\text{curl}\bar{\mathbf{E}}\|_{L^2(D)}^2$$

$$+ k^4(1 + \gamma)\|\bar{\mathbf{E}}\|_{L^2(D)}^2 + k^2\|\text{curl}\bar{\mathbf{E}}\|_{L^2(D)}^2$$

$$+ k^2\|\nabla\phi - \frac{1}{2}(\nu \times \text{curl}\bar{\mathbf{E}})\|_{L^2(\partial D)}^2$$

$$\geq (\gamma - \frac{\gamma^2}{\mu})\|\text{curl}\nabla\bar{\mathbf{E}}\|_{L^2(D)}^2 + k^4(1 + \gamma - \mu)\|\bar{\mathbf{E}}\|_{L^2(D)}^2$$

$$+ k^2\|\text{curl}\bar{\mathbf{E}}\|_{L^2(D)}^2 + k^2\|\nabla\phi - \frac{1}{2}(\nu \times \text{curl}\bar{\mathbf{E}})\|_{L^2(\partial D)}^2,$$

where $\gamma < \mu < \gamma + 1$. For such an $\mu$, we conclude that there exists a constant $C > 0$ such that

$$(A_k\bar{\mathbf{E}}, \bar{\mathbf{E}})_{H_0^2(\text{curl}, D)} \geq C\|\bar{\mathbf{E}}\|_{H_0^2(\text{curl}, D)}^2,$$

for all $\bar{\mathbf{E}} \in H_0^2(\text{curl}, D)$, which proves that $A_k$ is coercive. \qed

**Lemma 3.3.** The operator $B$ is compact on $H_0^2(\text{curl}, D)$.

Proof. Let $\bar{\mathbf{E}}_n \in H_0^2(\text{curl}, D)$ be a bounded sequence. We can extract a subsequence, that we abusively denote by $\bar{\mathbf{E}}_n$, that converges weakly to some $\bar{\mathbf{E}}_0 \in H_0^2(\text{curl}, D)$.

For $\bar{\mathbf{E}} \in H_0^2(\text{curl}, D)$ and by the identity $\text{div}(n\bar{\mathbf{E}}) = n\text{div}\bar{\mathbf{E}} + \nabla n \cdot \bar{\mathbf{E}}$, we have

$$\|\text{div}(n\bar{\mathbf{E}})\|_{L^2(D)} \leq \|n\text{div}\bar{\mathbf{E}}\|_{L^2(D)}^2 + \|\nabla n \cdot \bar{\mathbf{E}}\|_{L^2(D)}^2$$

$$\leq C\|\text{div}\bar{\mathbf{E}}\|_{L^2(D)}^2 + C\|\bar{\mathbf{E}}\|_{L^2(D)}^2$$

under the hypotheses (H1) on $n$. From (2) we can see that $\text{div}(n\mathbf{E}_1) = 0$ and $\text{div}\mathbf{E}_2 = 0$ in $D$. Due to the identity $\bar{\mathbf{E}} = \mathbf{E}_1 - \mathbf{E}_2$ and $\text{div}(n\mathbf{E}_1) = n\text{div}\mathbf{E}_1 + \nabla n \cdot \mathbf{E}_1$ in $D$. Then, we have

$$\|\text{div}\bar{\mathbf{E}}\|_{L^2(D)}^2 = \|\text{div}\mathbf{E}_1\|_{L^2(D)}^2 \leq \|\nabla n(x)/n(x)\|_{L^\infty(D)}^2\|\mathbf{E}_1\|_{L^2(D)}^2.$$  \hfill (11)

According to (10) and (11), we obtain

$$\|\text{div}(n\bar{\mathbf{E}})\|_{L^2(D)}^2 < +\infty$$

under the assumption that $\|\nabla n(x)/n(x)\|_{L^\infty(D)} < 1$. Then, it is easy to check that $H_0^2(\text{curl}, D)$ is a subspace of $H(\text{curl}, \text{div}, D)$ (see Appendix B in [15]), under the boundary condition $\nu \times \bar{\mathbf{E}} = 0$ and the embedding of $H(\text{curl}, \text{div}, D)$ into $L^2(D)$ is compact, (see Proposition B.2. in [15]).

From the definition of $B$ and the Cauchy-Schwarz inequality, we have

$$\|B(\bar{\mathbf{E}}_n - \mathbf{E}_0)\|_{H_0^2(\text{curl}, D)} \leq 2\|\bar{\mathbf{E}}_n - \mathbf{E}_0\|_{L^2(D)}.$$  

One deduces that $B\bar{\mathbf{E}}_n$ converges strongly to $B\mathbf{E}_0$, we have the corresponding conclusion. \qed

Now, we complete the proof of Theorem 3.1.

**Proof of Theorem 3.1.** To prove the discreteness of transmission eigenvalues we use the analytic Fredholm theory ([14]). By Lemma 3.2 and 3.3, we know that the operator $A_k - k^4B$ is Fredholm with index zero. The transmission eigenvalues are the values of $k > 0$ for which $I - k^4A_k^{-1}B$ has a nontrivial kernel. To apply the analytic Fredholm theorem, it remains to show that $I - k^4A_k^{-1}B$ or $A_k - k^4B$ is injective for at least one $k$. To this end, we recall the Poincaré inequality

$$\|\mathbf{E}\|_{L^2(D)} \leq C_0(\|\text{curl}\mathbf{E}\|_{L^2(D)} + \|\text{div}\mathbf{E}\|_{L^2(D)}),$$  \hfill (12)
for all $\tilde{E}$ satisfies $\nu \times \tilde{E} = 0$ on $\partial D$, where constant $C_0$ is independent of $\tilde{E}$ (see [26]). According to (11) and (12), we have

$$A_k(\tilde{E}, \tilde{E}) - k^4 B(\tilde{E}, \tilde{E})$$

$$= \int_D (1 - n)^{-1} |\text{curl}(\nu \times \tilde{E}) - k^2 \tilde{E}|^2 \text{d}x + k^2 \| \text{curl} \tilde{E} \|_{L^2(D)}^2$$

$$- k^4 \| \tilde{E} \|_{L^2(D)}^2 + k^2 \| \nu \times \text{curl} \tilde{E} \|_{L^2(\partial D)}^2$$

$$\geq k^2 (1 - k^2 C_0) \| \text{curl} \tilde{E} \|_{L^2(D)}^2 - k^4 C_0 \| \text{div} \tilde{E} \|_{L^2(D)}^2$$

$$\geq k^2 (1 - k^2 C_0) \| \text{curl} \tilde{E} \|_{L^2(D)}^2 - k^4 C_0 \| \nabla n(x)/n(x) \|_{L^\infty(D)}^2 \| \text{curl} \tilde{E} \|_{L^2(D)}^2.$$  

If $\| \nabla n(x)/n(x) \|_{L^\infty(D)} \ll 1$, we deduce that $A_k(\tilde{E}, \tilde{E}) - k^4 B(\tilde{E}, \tilde{E}) > 0$ for all $k > 0$ such that $k^2 < \frac{1}{C_0}$. Hence, $A_k - k^4 B$ is injective for such $k$ and the analytical Fredholm theory implies that the set of transmission eigenvalues is discrete and from the analyticity with $+\infty$ and the only possible accumulation point.  

3.2. Existence of the transmission eigenvalues. In this subsection, we want to prove the existence of the electromagnetic interior transmission eigenvalues. If we consider the generalized eigenvalue problem

$$A_k \tilde{E} - \lambda(k) B \tilde{E} = 0, \quad \tilde{E} \in H_0^1(\text{curl}, D),$$

(13)

which is known to have an infinite sequence of eigenvalues $\lambda_j(k)$, $j \in \mathbb{N}$, then the transmission eigenvalues are the solutions $\lambda_j(k) = k^4$ of (13), $j \in \mathbb{N}$, under the assumption $n = 1 < 0$. We prove the existence of infinitely many transmission eigenvalues using [9, Theorem 2.3]. We recall this key result in the following lemma.

**Lemma 3.4** ([9], Theorem 2.3). Let $\tau \mapsto A_\tau$ be a continuous mapping from $(0, +\infty)$ to the set of self-adjoint positive definite bounded linear operators on the Hilbert space $U$ and assume that $B$ is a self-adjoint non-negative compact linear operator on $U$. We assume that there exists two positive constants $\tau_0$ and $\tau_1$ such that

1. $A_{\tau_0} - \tau_0 B$ is positive on $U$.

2. $A_{\tau_1} - \tau_1 B$ is non-positive on a $m$ dimensional subspace of $U$.

Then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1, \ldots, m$, has at least one solution in $[\tau_0, \tau_1]$ where $\lambda_j(\tau) = \tau$ is the $j^{th}$ eigenvalue (counting multiplicity) of $A_\tau$ with respect to $B$, i.e., $\ker(A_\tau - \lambda_j(\tau) B) \neq 0$.

The existence of the interior eigenvalues is given in the following theorem.

**Theorem 3.5.** Assume that $n, \eta$ satisfies $(H_1)$, $(H_2)$, respectively. And moreover, $0 < n(x) < 1$ a.e. in $D$. Then there exists an infinite discrete set of transmission eigenvalues.

**Proof.** In the proof of Theorem 3.1 we have shown that for $k_0^2 < \frac{1}{C_0}$, where $C_0$ is given in (12), $A_{k_0} - k_0^4 B$ is positive in $H_0^1(\text{curl}, D)$. Hence, the first assumption in Lemma 3.4 is satisfied. Now we try to find $k_1$ such that $A_{k_1} - k_1^4 B$ is non-positive in a subspace of $H_0^1(\text{curl}, D)$.

Let $B^j_r = B(x_j, r) := \{ x \in \mathbb{R}^3 : |x - x_j| < r \}, j = 1, \ldots, M(r)$ and $r > 0$. Define $M(r)$ as the number of disjoint balls $B^j_r$, i.e., $\overline{B^j_r} \cap \overline{B^k_r} \neq 0$, with $r$ small enough such that $\overline{B^j_r} \subset D$. We denote by $k_1$ the first transmission eigenvalue corresponding to the interior transmission problem for $B^j_r$ for all $j = 1, \ldots, M(r)$ with index of
refraction \( n^* \) which is known to exist [14]. Let \( \tilde{E}_j \in H_0^2(\text{curl}, B_j^\ell) \), \( j = 1, \ldots, M(r) \) be the corresponding eigenvector which satisfies

\[
\int_{B_j^\ell} (1 - n^*)^{-1} (\text{curl}\,\text{curl}\,\tilde{E}_j - k_1^2 n^* \tilde{E}_j) \cdot (\text{curl}\,\text{curl}\,\mathbf{F} - k_1^2 \mathbf{F}) \, dx = 0, \tag{14}
\]

for all \( \mathbf{F} \in H_0^2(\text{curl}, B_j^\ell) \). We denote by \( \tilde{E}_j^+ \in H_0^2(\text{curl}, D) \) the extension of \( \tilde{E}_j \) by zero to the whole of \( D \) and we define a \( M(r) \)-dimensional subspace of \( H_0^2(\text{curl}, D) \) by \( \mathbf{V} := \text{span}\{ \tilde{E}_j^+, 1 < j < M(r) \} \). Since for \( j \neq m \), \( \tilde{E}_j^+ \) and \( \tilde{E}_m^+ \) have disjoint support, for \( \tilde{E} = \sum_{j=1}^{M(r)} \xi_j \tilde{E}_j \in \mathbf{V} \), we have

\[
A_{k_1}(\tilde{E}_j^+, \tilde{E}_j^*) - k_1^4 \mathcal{B}(\tilde{E}_j^+, \tilde{E}_j^*)
\]

\[
= \sum_{j=1}^{M(r)} |\xi_j|^2 \left( \int_D (1 - n)^{-1} |\text{curl}\,\text{curl}\,\tilde{E}_j - k_1^2 \tilde{E}_j|^2 \, dx + k_1^2 \int_D |\text{curl}\,\tilde{E}_j|^2 \, dx \right)
- k_1^4 \int_D |\tilde{E}_j^+|^2 \, dx + k_1^2 \int_{\partial D} |\nu \times \text{curl}\,\tilde{E}_j|^2 \, ds \tag{15}
\]

\[
\leq \sum_{j=1}^{M(r)} |\xi_j|^2 \left( \int_{B_j^\ell} (1 - n^*)^{-1} |\text{curl}\,\text{curl}\,\tilde{E}_j - k_1^2 \tilde{E}_j|^2 \, dx + k_1^2 \int_{B_j^\ell} |\text{curl}\,\tilde{E}_j|^2 \, dx \right)
- k_1^4 \int_{B_j^\ell} |\tilde{E}_j^+|^2 \, dx \right.
- k_1^4 \int_{B_j^\ell} |\text{curl}\,\tilde{E}_j|^2 \, dx \right).
\]

By Green’s first vector theorem and \( \nu \times \tilde{E}_j = 0 \), we have

\[
\int_{B_j^\ell} |\text{curl}\,\tilde{E}_j|^2 \, dx = \text{Re} \left\{ \int_{B_j^\ell} \text{curl}\,\tilde{E}_j \cdot \tilde{E}_j \, dx \right\}.
\]

Substituting (16) into (15), together with (14), we have

\[
A_{k_1}(\tilde{E}_j^+, \tilde{E}_j^*) - k_1^4 \mathcal{B}(\tilde{E}_j^+, \tilde{E}_j^*)
\]

\[
\leq \sum_{j=1}^{M(r)} |\xi_j|^2 \left( \int_{B_j^\ell} (1 - n^*)^{-1} (\text{curl}\,\text{curl}\,\tilde{E}_j - k_1^2 \tilde{E}_j) \cdot (\text{curl}\,\text{curl}\,\tilde{E}_j - k_1^2 \tilde{E}_j) \, dx \right) = 0.
\]

Thus, the second assumption in Lemma 3.4 is satisfied. We conclude that there exists \( M(r) \) transmission eigenvalues in \( \left[ \sqrt{\frac{1}{C_0}}, k_1 \right] \). Letting \( r \to 0 \), we have that \( M(r) \to \infty \) and thus we can now deduce that there exists an infinite set of transmission eigenvalues.

\[
\square
\]

4. Monotonicity of the transmission eigenvalues. For this section we turn our attention to proving that the first transmission eigenvalue can be used to determine information about the material parameters \( n \) and \( \eta \). To this end, we will show that the first transmission eigenvalue is a monotonic function with respect to the functions \( n \) and \( \eta \). From the monotonicity we will obtain a uniqueness result for
a homogeneous refractive index and homogeneous conductive boundary parameter. Recall that the transmission eigenvalues satisfy
\[ \lambda_j(k; n, \eta) - k^4(n, \eta) = 0, \quad \text{for} \quad 0 < n < 1. \] (17)
and the first transmission eigenvalue is the smallest root of (17) for \( \lambda_1(k; n, \eta) \).
Notice that \( \lambda_1(k; n, \eta) \) satisfies for \( \tilde{E} \neq 0 \)
\[ \lambda_1(k; n, \eta) = \min_{\tilde{E} \in H^2_0(\text{curl}, D)} \frac{A_k(\tilde{E}, \tilde{E})}{B(\tilde{E}, \tilde{E})}, \quad \text{for} \quad 0 < n < 1. \] (18)
It is clear that \( \lambda_1(k; n, \eta) \) is a continuous function of \( k \in (0, \infty) \). Notice that the minimizers of (18) are the eigenfunctions corresponding to \( \lambda_1(k; n, \eta) \). We will denote the first transmission eigenvalue by \( k_1(n, \eta) \).

**Theorem 4.1.** Assume that \( 0 < n_2 \leq n_1 < 1 \) and \( 0 < \eta_1 \leq \eta_2 \), then we obtain that \( k_1(n_2, \eta_2) \leq k_1(n_1, \eta_1) \). Moreover, if the inequalities for the parameters \( n \) and \( \eta \) are strict, then the first transmission eigenvalue is strictly monotone with respect to \( n \) and \( \eta \).

**Proof.** Assume that \( k_1 = k_1(n_1, \eta_1) \) and \( k_2 = k_1(n_2, \eta_2) \). Therefore, for all \( \tilde{E} \in H^2_0(\text{curl}, D) \) such that \( \|\tilde{E}\|_{L^2(D)} = 1 \). Recall that the definition of \( A_k(\tilde{E}, \tilde{E}), B(\tilde{E}, \tilde{E}) \) and (18), we have
\[
2\lambda_1(k_1; n_2, \eta_2) \leq \int_D (1 - n_2)^{-1}|\text{curl}\tilde{E} - k_1^2\tilde{E}|^2dx + k_1^2\|\text{curl}\tilde{E}\|^2_{L^2(D)} \\
+ k_1^4\|\tilde{E}\|^2_{L^2(D)} + k_1^2\int_{\partial D} \eta_2^{-1}|\nu \times \text{curl}\tilde{E}|^2ds \\
\leq \int_D (1 - n_1)^{-1}|\text{curl}\tilde{E} - k_1^2\tilde{E}|^2dx + k_1^2\|\text{curl}\tilde{E}\|^2_{L^2(D)} \\
+ k_1^4\|\tilde{E}\|^2_{L^2(D)} + k_1^2\int_{\partial D} \eta_1^{-1}|\nu \times \text{curl}\tilde{E}|^2ds.
\]
Now, let \( \tilde{E}_1 \) be the normalized transmission eigenfunction such that \( \|\tilde{E}_1\|_{L^2(D)} = 1 \) corresponding with the eigenvalue \( k_1 \). Notice that (18) gives
\[
2\lambda_1(k_1; n_1, \eta_1) = \int_D (1 - n_1)^{-1}|\text{curl}\tilde{E}_1 - k_1^2\tilde{E}_1|^2dx + k_1^2\|\text{curl}\tilde{E}_1\|^2_{L^2(D)} \\
+ k_1^4\|\tilde{E}_1\|^2_{L^2(D)} + k_1^2\int_{\partial D} \eta_1^{-1}|\nu \times \text{curl}\tilde{E}_1|^2ds,
\]
since \( \tilde{E}_1 \) is the minimizer of (18) for \( n = n_1 \) and \( \eta = \eta_1 \). This yields \( \lambda_1(k_1; n_2, \eta_2) \leq \lambda_1(k_1; n_1, \eta_1) \) and (17) gives
\[ \lambda_1(k_1; n_1, \eta_1) - k_1^4 = 0. \]
Therefore \( \lambda_1(k_1; n_2, \eta_2) - k_1^4 \leq 0 \). By the proof of Theorem 3.1, for all \( k^2 \) sufficiently small we have that \( A_k(\tilde{E}, \tilde{E}) - k^4B(\tilde{E}, \tilde{E}) > 0 \). This implies that there exists \( \delta > 0 \) such that for any \( k^4 < \delta \) that \( \lambda_1(k_1; n_2, \eta_2) - k^4 \geq 0 \) holds. By the continuity we have that \( \lambda_1(k_1; n_2, \eta_2) - k^4 \) has at least one root in the interval \( [\sqrt{\delta}, k_2] \). Since \( k_2 \) is the biggest root of \( \lambda_1(k_1; n_2, \eta_2) - k^4 \) we conclude that \( k_2 \leq k_1 \). Then the proof of Theorem 4.1 is completed. \( \square \)
By the proof of the previous result we have the following uniqueness result for a homogeneous media and homogeneous boundary parameter $\eta$ from the strict monotonicity of the first transmission eigenvalue.

**Corollary 1.** 1. If it is known that $0 < n < 1$ is a constant refractive index with $\eta$ known and fixed, then $n$ is uniquely determined by the first transmission eigenvalue. 2. If $0 < n < 1$ is known and fixed with $\eta$ a constant, then the first transmission eigenvalue uniquely determines $\eta$.

**Acknowledgments.** The author would like to express his gratitude to the reviewers for the valuable comments and suggestions which served to improve the manuscript.

**REFERENCES**

[1] O. Bondarenko, I. Harris and A. Kleefeld, The interior transmission eigenvalue problem for an inhomogeneous media with a conductive boundary, *Appl. Anal.*, 96 (2017), 2–22.

[2] F. Cakoni, H. Haddar and S. Meng, Boundary integral equations for the transmission eigenvalue problem for Maxwell’s equations, *J. Integral Equations Appl.*, 27 (2015), 375–406.

[3] L. Chesnel, Interior transmission eigenvalue problem for Maxwell’s equations: the T-coercivity as an alternative approach, *Inverse Probl.*, 28 (2012), 065005, 14.

[4] F. Cakoni and D. Colton, *Qualitative Methods in Inverse Scattering Theory*, Interaction of Mechanics and Mathematics. Springer-Verlag, Berlin, 2006.

[5] F. Cakoni, D. Colton and H. Haddar, On the determination of Dirichlet or transmission eigenvalues from far field data, *C. R. Math. Acad. Sci. Paris*, 348 (2010), 379–383.

[6] F. Cakoni, D. Colton and H. Haddar, The interior transmission problem for regions with cavities, *SIAM J. Math. Anal.*, 42 (2017), 145–162.

[7] F. Cakoni, A. Cossonnière and H. Haddar, Transmission eigenvalues for inhomogeneous media containing obstacles, *Inverse Probl. Imaging*, 6 (2012), 373–398.

[8] F. Cakoni, D. Gintides and H. Haddar, The existence of an infinite discrete set of transmission eigenvalues, *SIAM J. Math. Anal.*, 42 (2010), 237–255.

[9] F. Cakoni and H. Haddar, On the existence of transmission eigenvalues in an inhomogeneous medium, *Appl. Anal.*, 88 (2009), 475–493.

[10] F. Cakoni and H. Haddar, Transmission eigenvalues [Editorial], *Inverse Probl.*, 29 (2013), 100201, 3.

[11] F. Cakoni and H. Haddar, Transmission eigenvalues in inverse scattering theory, Inverse problems and applications: inside out. II, *Sci. Res. Inst. Publ.*, 60 (2013), 529–580, Cambridge Univ. Press, Cambridge.

[12] A. Cossonnière and H. Haddar, Surface integral formulation of the interior transmission problem, *J. Integral Equations Appl.*, 25 (2013), 341–376.

[13] A. Cossonnière and H. Haddar, The electromagnetic interior transmission problem for regions with cavities, *SIAM J. Math. Anal.*, 43 (2011), 1698–1715.

[14] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, volume 93 of Applied Mathematical Sciences, third edition. Springer, New York, 2013.

[15] C. Hazard and M. Lenoir, On the solution of time-harmonic scattering problems for Maxwell’s equations, *SIAM J. Math. Anal.*, 27 (1996), 1597–1630.

[16] D. Colton, L. Päivärinta and J. Sylvester, The interior transmission problem, *Inverse Probl. Imaging*, 1 (2007), 13–28.

[17] G. Giorgi and H. Haddar, Computing estimates of material properties from transmission eigenvalues, *Inverse Probl.*, 28 (2012), 055009, 23.

[18] H. Haddar, The interior transmission problem for anisotropic Maxwell’s equations and its applications to the inverse problem, *Math. Methods Appl. Sci.*, 27 (2004), 2111–2129.

[19] I. Harris, F. Cakoni and J. Sun, Transmission eigenvalues and non-destructive testing of anisotropic magnetic materials with voids, *Inverse Probl.*, 30 (2014), 035016, 21.

[20] H. Haddar and S. Meng, The spectral analysis of the interior transmission eigenvalue problem for maxwells equations, *arXiv:1707.04815v2*. 
A. Kirsch and N. Grinberg, *The Factorization Method for Inverse Problems*, volume 36 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2008.

A. Kirsch and A. Lechleiter, The inside-outside duality for scattering problems by inhomogeneous media, *Inverse Probl.*, 29 (2013), 104011, 21.

J. Li, X. Li, H. Liu and Y. Wang, Electromagnetic interior transmission eigenvalue problem for inhomogeneous media containing obstacles and its applications to near cloaking, 

[arXiv:1701.05301v1](https://arxiv.org/abs/1701.05301v1).

E. Lakshtanov and B. Vainberg, Ellipticity in the interior transmission problem in anisotropic media, *SIAM J. Math. Anal.*, 44 (2012), 1165–1174.

E. Lakshtanov and B. Vainberg, Applications of elliptic operator theory to the isotropic interior transmission eigenvalue problem, *Inverse Probl.*, 29 (2013), 104003, 19.

P. Monk, *Finite Element Methods for Maxwell’s Equations*, Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2003.

L. Robbiano, Spectral analysis of the interior transmission eigenvalue problem, *Inverse Probl.*, 29 (2013), 104001, 28.

J. Sylvester, Discreteness of transmission eigenvalues via upper triangular compact operators, *SIAM J. Math. Anal.*, 44 (2012), 341–354.

F. Yang and P. Monk, The interior transmission problem for regions on a conducting surface, *Inverse Probl.*, 30 (2014), 015007, 34.

Received March 2019; revised September 2019.

E-mail address: hybgxj980223427@163.com