THE $L^2$ RESTRICTION OF THE EISENSTEIN SERIES TO A GEODESIC SEGMENT

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Abstract. We study the $L^2$ norm of the Eisenstein series $E(z, 1/2 + iT)$ restricted to a segment of a geodesic connecting infinity and an arbitrary real. We conjecture that on slightly thickened geodesics of this form, the Eisenstein series satisfies restricted QUE. We prove a lower bound that matches this predicted asymptotic. We also prove an upper bound that nearly matches the lower bound assuming the Riemann Hypothesis (unconditionally, the sharp upper bound holds for almost all $T$). Finally, we show the restricted QUE conjecture for geodesics with rational endpoints.

1. Introduction

1.1. Background and statements of results. The behavior of eigenfunctions of the Laplacian on an arithmetic surface $\Gamma \backslash \mathbb{H}$ is a rich subject with connections to analytic number theory and spectral geometry. One of the foundational questions in the area is the Quantum Unique Ergodicity (QUE) conjecture of Rudnick and Sarnak [RS] which is now a theorem of Lindenstrauss [L] (complete in the compact setting), with input by Soundararajan [So] to cover congruence subgroups.

It is desirable to understand the (non)-localization of Laplace eigenfunctions on a negatively curved finite volume manifold, measured via $L^p$ restriction norms. For instance, one may fix a curve on $\Gamma \backslash \mathbb{H}$, and study the analytic behavior of the eigenfunction restricted to the curve. Such questions were initially raised by Reznikov [R] who used representation theory techniques for their study. Later, Ghosh, Reznikov, and Sarnak [GRS] obtained estimates for the $L^2$ norm of a Maass form along special curves such as horocycles or geodesic segments, with the application of counting sign changes of the Maass form along the curve (see also work of J. Jung [J] for further progress in this direction). Marshall [M] has used the amplification method to estimate the $L^2$ norm of a Maass form restricted to an arbitrary geodesic segment, with a power saving over the local bound. Toth and Zelditch [TZ] and Dyatlov and Zworski [DZ] have shown a quantum ergodic restriction theorem that in particular holds on a generic geodesic in a compact hyperbolic surface.

The author [Y] recently showed that the Eisenstein series $E_T(z) = E(z, 1/2 + iT)$ for $\Gamma = SL_2(Z)$ equidistributes along the geodesic connecting 0 and $i\infty$ (this is an analog of QUE but along the curve). The Eisenstein series is a good test case for more advanced questions since it behaves in many respects like the cusp forms, yet is more explicit and easier to handle. In this paper, we study the behavior of the Eisenstein series restricted to a geodesic segment of the form $\{x + iy : \beta > y > \alpha > 0\}$. This is not a simple modification of our previous work and the tradeoff in generality is that the results here are not as complete as in [Y].

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There are two overriding differences when one generalizes $x$ to be arbitrary instead of $x = 0$. To aid in this discussion, we quickly gather some notation from [Y]. Let $E^*_T(z) = \frac{\theta(1/2 + iT)}{[\theta(1/2 + iT)]^2} E_T(z)$, where $\theta(s) = \pi^{-s} \Gamma(s) \zeta(2s)$. This makes $E^*_T$ real-valued for $z \in \mathbb{H}$. The Fourier expansion of $E^*_T$ is

$$E^*_T(x + iy) = \mu y^{1/2 + iT} + \pi y^{1/2-iT} + \rho^*(1) \sum_{n \neq 0} \frac{\tau_{iT}(n)e(nx)}{|n|^{1/2}} V_T(2\pi |n| y).$$

Here $\rho^*(1) = (2/\pi)^{1/2} |\theta(1/2 + iT)|^{-1}$, $\mu = \frac{\theta(1/2 + iT)}{[\theta(1/2 + iT)]^2}$, $V_T(y) = \sqrt{y} K_{iT}(y)$, and $\tau_{iT}(n) = \sum_{ab=|n|} (a/b)^iT$. The most important difference is that when $x = 0$, the coefficients $e(nx) \tau_{iT}(n)$ of $E^*_T$ are multiplicative, while for general $x \in \mathbb{R}$ they are not. This causes some crucial aspects of [Y] to break down, and we are only able to partially replace these with methods that work for general $x$.

The other significant difference comes from an asymmetry between $n > 0$ and $n < 0$ in (1.1). Note that the $K$-Bessel function in the Fourier expansion only depends on $|n|$, which has the following practical effect. When one considers a $y$-integral of $|E^*_T(x + iy)|^2$, analyzed by squaring out and integrating the Fourier expansion term-by-term, there will be diagonal terms and off-diagonal terms. The diagonal parts will include a sum with say $n_1 = n_2$ as well as terms with $n_1 = -n_2$, the latter of which will then lead to sums of the form $\sum_n |\tau_{iT}(n)|^2 e(2nx)$. It is also necessary to treat the opposite sign off-diagonal terms of the form $\sum_{h \neq 0} \sum_n \tau_{iT}(n)\tau_{iT}(n+h)e((2n + h)x)$. It is quite difficult to estimate such sums because the weight is highly oscillatory (for general $x$, that is–the case $x = 0$ for example does not have this problematic feature). We are able to sidestep this particular problem by “thickening” the geodesic segment with an extremely short $x$-integral. This $x$-integral is required to be barely longer than the Planck scale of length $1/T$; on smaller scales, the Eisenstein series is roughly constant. This thickening procedure gives just enough extra savings to bound the terms with opposite signs, but has no practical effect on the terms with the same sign.

Now we state our main results. Suppose that $\psi$ is a smooth function with support on $[\alpha, \beta]$. Define

$$I_\psi(x, T) = \int_0^\infty \psi^2(y)|E_T(x + iy)|^2 \frac{dy}{y}.$$

To get our bearings, we quote two results from the literature. Suppose that $\phi : SL_2(\mathbb{Z}) \backslash \mathbb{H} \to \mathbb{R}$ is smooth and compactly-supported. Luo and Sarnak [LS] showed

$$\int_{SL_2(\mathbb{Z}) \backslash \mathbb{H}} \phi(z)|E_T(z)|^2 \frac{dxdy}{y^2} = \frac{3}{\pi} \log(1/4 + T^2) \int_{SL_2(\mathbb{Z}) \backslash \mathbb{H}} \phi(z) \frac{dxdy}{y^2} + o(\log T),$$

as $T \to \infty$. The interested reader can find a full main term with a power saving error term in [Y] (5.35). The author [Y] showed

$$I_\psi(0, T) = 2\frac{3}{\pi} \log(1/4 + T^2) \int_0^\infty \psi^2(y) \frac{dy}{y} + o(\log T).$$

Again, one has a fully developed main term with a power saving error term. The Eisenstein series (squared) is twice as large on the distinguished geodesic $x = 0$ accounting for the opposite sign diagonal terms which are smaller for typical $x$. 
For $\gamma > 0$, define
\[
I_{\psi,\gamma}(x_0, T) = \int_0^\infty \int_{|x-x_0| \leq \frac{2}{T}} \psi^2(y)|E_T(x + iy)|^2 dx \frac{dy}{y},
\]
which we wish to compare to the (weighted) measure of the set, that is,
\[
m_{\psi,\gamma} = \int_0^\infty \int_{|x-x_0| \leq \frac{2}{T}} \psi^2(y) dy.
\]
Here $I^*$ represents the thickened geodesic integral. Note $m_{\psi,\gamma} \approx \frac{\gamma}{T}$, for $\psi$ fixed.

Our first result is a lower bound of the correct order of magnitude.

**Theorem 1.1.** Suppose $\gamma = \gamma(T) \to \infty$ as $T \to \infty$, arbitrarily slowly. Then uniformly in $x_0 \in \mathbb{R}$, we have as $T \to \infty$,
\[
I_{\psi,\gamma}(x_0, T) \geq \frac{3}{\pi} \log(1/4 + T^2)m_{\psi,\gamma}(1 + o(1)).
\]

To partially complement Theorem 1.1, we have the following upper bound.

**Theorem 1.2.** Uniformly in $x \in \mathbb{R}$, we have
\[
I_{\psi}(x, T) \ll (\log T)^2.
\]
Alternatively, we have
\[
I_{\psi}(x, T) \ll \log T \left(1 + \frac{(\log \log T)^2}{|\zeta(1+2iT)|^2}\right).
\]
Assuming the Riemann Hypothesis, we have $|\zeta(1+2iT)|^{-1} \ll \log \log T$ (see [14.8.3]), and so
\[
I_{\psi}(x, T) \ll (\log T)(\log \log T)^4.
\]
It is also known that $|\zeta(1+2iT)|^{-1} \ll \log \log T$ for “almost all” $T$ (see [GS]), so that (1.9) is almost always stronger than (1.8).

One may expect that the Eisenstein series equidistributes along any fixed geodesic.

**Conjecture 1.3.** Under the same conditions as Theorem 1.1, we have
\[
I_{\psi,\gamma}^*(x_0, T) = \frac{3}{\pi} \log(1/4 + T^2)m_{\psi,\gamma}(1 + o(1)).
\]

The averaging over $x$ in Conjecture 1.3 is important because some individual geodesics have more mass than others (e.g. $x = 0$ as we mentioned earlier). This averaging is enough to wash out the exceptional behavior on $x = 0$.

The extremely thin $x$-integral appearing in Conjecture 1.3 means that the methods used to prove QUE for Eisenstein series are not suitable for this set. Harmonic analysis on $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ is not appropriate on sets of this shape.

We are able to report some partial progress on Conjecture 1.3. To do so, we need to describe the earliest stages of analysis of $I_{\psi}(x, T)$. By a use of Parseval’s formula (in the Mellin transform setting), $I_{\psi}(x, T)$ decomposes as
\[
I_{\psi}(x, T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F_{x,T}(it)|^2 dt,
\]
where
\[
F_{x,T}(it) = \int_{-\infty}^{\infty} \psi(y)|E_T(x + iy)|^2 dy.
\]
where
\begin{equation}
F_{x,T}(s) = \int_{0}^{\infty} \psi(y)y^{s} E^{*}_{T}(x + iy) \frac{dy}{y}.
\end{equation}

We will then insert the Fourier expansion (1.1) and calculate the \(y\)-integral in terms of gamma functions (see (2.1) below). One arrives at an integral of the form
\begin{equation}
|\zeta(1 + 2iT)|^{-2} \int_{|t| \leq T} (1 + |t - T|)^{-1/2} T^{-1/2} \left| \sum_{n \neq 0} \frac{\tau_{T}(n)e(nx)}{|n|^{1/2+it}} k(n, t, T) \right|^{2} dt,
\end{equation}
as well as some other terms of lesser importance. Here \(k\) is an innocuous weight function, which restricts the support of \(n\) to \(|n| \approx T^{1/2}(1 + |t - T|)^{1/2}\). The integral naturally decomposes into pieces of the form \(|t - T| \approx \Delta\) where \(1 \ll \Delta = o(T)\), as well as the "bulk" range with \(|t| \leq T(1 - o(1))\). The diagonal term in the bulk range is responsible for the main term. This part of the \(t\)-integral is also where the difficulties lie with the opposite sign off-diagonal terms. The device of incorporating the extra \(x\)-integral allows us to bound the opposite sign terms trivially. For \(\Delta = o(T)\), we do not expect to obtain a main term, and so we can use Cauchy’s inequality to separate the positive and negative values of \(n\), thereby avoiding this tricky problem in this range of \(\Delta\’s\).

Define
\begin{equation}
I(\Delta, T, x, N) = \int_{-\infty}^{\infty} w_{1}\left(\frac{T - t}{\Delta}\right) \left| \sum_{n \geq 1} \frac{\tau_{T}(n)e(nx)}{n^{1/2+it}} w_{2}\left(\frac{n}{N}\right) \right|^{2} dt,
\end{equation}
where \(w_{1}\) and \(w_{2}\) are some fixed smooth compactly-supported weight functions, and \(N = \sqrt{\Delta T}\). Roughly speaking, to turn Theorem 1.1 into an asymptotic formula, we need to show that
\begin{equation}
\sum_{1 \ll \Delta = o(T)} (\Delta T)^{-1/2} I(\Delta, T, x, N) = o(|\zeta(1 + 2iT)|^{2} \log T).
\end{equation}
This is the same as showing
\begin{equation}
I(\Delta, T, x, N) = o(|\zeta(1 + 2iT)|^{2} N),
\end{equation}
on average over \(\Delta\) ranging over the \(O(\log T)\) dyadic segments. The mean value theorem for Dirichlet polynomials shows that \(I(\Delta, T, x, N) \ll |\zeta(1 + 2iT)|^{2}(\Delta + N) \log T\), which gives rise to (1.8). This shows that to prove Conjecture 1.3 we need to save slightly more than \(\log T\) in our analysis of \(I(\Delta, T, x, N)\).

Estimating (1.15) is the main subject of interest in this paper. The qualitative behavior of the integral changes drastically as \(\Delta\) ranges from 1 to \(T\). We are able to estimate it satisfactorily for \(\Delta\) sufficiently large, or sufficiently small, with uniformity in \(x\).

**Theorem 1.4.** Suppose \(\gamma = \gamma(T) \rightarrow \infty\). Then
\begin{equation}
\frac{1}{2\pi} \int_{|t| \leq T-C(\varepsilon)T^{25/27+\varepsilon}} \int_{|x-x_{0}| \leq \frac{1}{7}} |F_{x,T}(it)|^{2} dt = \frac{3}{\pi} \log(1/4 + T^{2}) m_{\psi,\gamma}(1 + O(\gamma^{-1/5} + \log \log T/\log T)).
\end{equation}
Theorem 1.4 implies Theorem 1.1 by positivity. Actually, to prove Theorem 1.1 we only need to evaluate the integral over \(|t| \leq T(1 - o(1))\) which is somewhat easier.

For the smaller values of \(|t| \neq T|\), we have

**Theorem 1.5.** Let \(0 < \varepsilon < 1/4\). Then

\[
(1.19) \quad \int_{T^{1/2-\varepsilon} \leq |t| \leq T^{1/2+\varepsilon}} |F_{x,T}(it)|^2 dt \ll \varepsilon.
\]

The very smallest values of \(|t| \neq T|\) require a slightly different proof, and we claim this estimate with

**Proposition 1.6.** We have

\[
(1.20) \quad \int_{|t-T| \leq T^{1/100}} |F_{x,T}(it)|^2 dt \ll (\log T)^\varepsilon.
\]

In light of these results, we see that in order to prove Conjecture 1.3 we need to show

\[
\int_{T^{1/2-\varepsilon} \leq |t-T| \leq T^{25/27+\varepsilon}} |F_{x,T}(it)|^2 dt = o(\log T).
\]

It would therefore be of great interest to close the gap to treat these medium-range values of \(\Delta\).

In [Y], Conjecture 1.3 was proved in the special case \(x_0 = 0\) (without requiring the \(t\)-integral). The key difference for this case is that \(e(nx_0) = 1\) is multiplicative. More generally, if \(x_0\) is rational, then \(e(nx_0)\) can be expressed as a short linear combination of multiplicative functions (changing basis to Dirichlet characters), which leads to the following

**Proposition 1.7.** Suppose that \(x_0\) is rational. Then if \(\gamma \rightarrow \infty\), but constrained by \(\gamma \ll T^\delta\) for some small \(\delta > 0\), then

\[
(1.21) \quad I_{\psi,\gamma}^*(x_0,T) = \frac{3}{\pi} \log(1/4 + T^2)m_{\psi,\gamma}(1 + o(1)).
\]

1.2. **Outline and overview.** In this section, we give a simplified and informal presentation of the method of proof of the primary results of this paper. The comments here should not be taken literally, but instead should serve as a guide for reading the rest of the paper.

As mentioned earlier in the introduction, we have

\[
(1.22) \quad I_{\psi}(x,T) \approx |\zeta(1 + 2iT)|^{-2} \int_0^T (1 + |T - t|)^{-1/2}T^{-1/2} \left| \sum_n \frac{\tau_{iT}(n)e(nx)}{|n|^{1/2+it}} \right|^2 dt.
\]

In reality, the sum over \(n\) contains a weight function depending on \(t,T,n\), and we have only displayed the part of the integral with \(t \geq 0\), the negative values of \(t\) being similar by symmetry. Cutting the \(t\)-integral into regions \(|t - T| > \Delta\) with \(1 \ll \Delta \ll T\), we have

\[
(1.23) \quad I_{\psi}(x,T) \approx |\zeta(1 + 2iT)|^{-2} \sum_{1 \ll \Delta \ll T} \left( \Delta T \right)^{-1/2} \left( \sum_{|n| \approx N} \left| \frac{\tau_{iT}(n)e(nx)}{|n|^{1/2+it}} \right|^2 \right),
\]

where

\[
(1.24) \quad N = (\Delta T)^{1/2}.
\]

The mean value theorem for Dirichlet polynomials, along with a bound on the average size of \(|\tau_{iT}(n)|^2\) (see Lemma 2.3 below) essentially shows that

\[
(1.25) \quad I_{\psi}(x,T) \ll \sum_{1 \ll \Delta \ll T} (\Delta T)^{-1/2}(\Delta + (\Delta T)^{1/2}) \log T \ll \log^2 T.
\]
This (roughly) explains how to derive (1.8), and in light of Conjecture 1.3 indicates that it is desirable to save slightly more than a factor \( \log T \) over this trivial bound. More precisely, the diagonal terms correspond to the term \( \Delta \) inside the parentheses, and the term \( (\Delta T)^{1/2} \gg N \) corresponds to the off-diagonal terms. One observes that the diagonal terms do indeed give a final bound of \( \log T \) to \( I_\psi(x, T) \), and a more careful analysis (which is easy to perform for the diagonal terms) leads to the asymptotic predicted in Conjecture 1.3.

Opening the square and integrating, we have

\[
I_\psi(x, T) \approx |\zeta(1 + 2iT)|^{-2} \sum_\Delta \frac{\Delta}{(\Delta T)^{1/2}} \sum_{|m|, |n| \gg N} \tau_{\Delta T}(n) \tau_{\Delta T}(m) e((n - m)x) / |mn|^{1/2}|m/n|^{iT}.
\]

It is difficult to deal with the terms with \( m \) and \( n \) of opposite signs, because one encounters a shifted convolution sum with a highly oscillatory weight function. Even the opposite diagonal case with \( m = -n \) is problematic. One may partially circumvent this problem by introducing an additional \( x \)-integral that is very short (of length \( \gamma/T \) where \( \gamma = \gamma(T) \to \infty \) arbitrarily slowly with \( T \)). The point is that the \( x \)-integral gives a small savings from the oscillation of these opposite sign terms. Indeed, we have

\[
\int_{|x - x_0| \leq \frac{\gamma}{T}} e((n - m)x) dx \ll \frac{1}{N},
\]

provided \( m, n \gg N \) and have opposite signs. Meanwhile, the trivial bound on the integral is \( \frac{\gamma}{T} \), so this technique gives some savings for \( N \) somewhat larger than \( T/\gamma \) (so this breaks down for values of \( \Delta \) somewhat smaller than \( T \)). For this reason, we rearrange (1.23) in the following way. For \( \Delta \gg \eta T \), where \( \eta = \eta(T) \to 0 \) slowly with \( T \) (\( \eta \) will depend on \( \gamma \)), we use this method to bound the opposite sign terms in a satisfactory way. In this same range of \( \Delta \), we can solve the shifted divisor sum

\[
\sum_{n > N} \tau_{\Delta T}(n) \tau_{\Delta T}(n + h),
\]

with a power-saving error term (indeed, the result of [Y] allows for \( \Delta \gg T^{25/27 + \varepsilon} \)). In all, this gives an asymptotic for the part of \( I_\psi(x, T) \), with \( \Delta \gg \eta T \), with the additional \( x \)-integral. By the positivity of the right hand side of (1.23), for proving a lower bound, we can drop all the other values of \( \Delta \). This is the reasoning that leads to Theorem 1.1.

In light of Conjecture 1.3, it should then suffice to give an upper bound only for the right hand side of (1.23) with \( \Delta \ll \eta T \) (since the unconditional lower bound of Theorem 1.1 comes from \( \Delta \gg \eta T \), and matches the asymptotic of Conjecture 1.3). This pleasant fact allows one to simplify the analysis by using the inequality \( |\sum_{n \neq 0} a_n|^2 \leq 2|\sum_{n > 0} a_n|^2 + 2|\sum_{n < 0} a_n|^2 \). Using this allows us to essentially replace the right hand side of (1.26) with \( m, n > 0 \), when \( \Delta \ll \eta T \). Thus, for all values of \( \Delta \ll \eta T \), one can completely avoid the opposite sign problem (indeed, (1.27) is trivial for smaller values of \( \Delta \), so one cannot rely on this idea in this range).

The solution of the shifted divisor sum allows for all \( \Delta \gg T^{25/27 + \varepsilon} \) to be acceptable. This explains the range of applicability of Theorem 1.4. In estimating this shifted divisor sum, there is also an unexpectedly delicate matter with an off-diagonal main term, given by (4.23) below. A trivial bound shows only that this main term contributes to \( I(\Delta, T, x, N) \) an amount that is \( O((\Delta T)^{1/2}|\zeta(1 + 2iT)|^2) \), which just barely fails to satisfy the desired bound.
However, a finer analysis of the sum over the shift $h$ gains a slight saving which allows us to save a factor of $\log T$ on average over the dyadic sum over $\Delta$.

When $\Delta$ is small, the shifted divisor sum approach breaks down. Another approach to the problem is to open up the divisor function $\tau_T(n) = \sum_{ab=n}(a/b)T$, and use the sums over $a$ and $b$ in some way. Arguments in this vein appear in Section 4. For example, we have that for a given value of $N$, that

$$
(1.29) \quad \int_{|t-T|\simeq \Delta} \left| \sum_{n \leq N} \frac{\tau_T(n)e(nx)}{n^{1/2+it}} \right|^2 \ll \log T \sum_{B \text{ dyadic}} \int_{|t-T|\simeq \Delta} \left| \sum_{a \geq A} \sum_{b \leq B} \frac{e(abx)}{a^{1/2+it-bb^{1/2+it}}b^{1/2+it+iT}} \right|^2,
$$

where $AB \asymp N$ (observe that once $b$ is localized by $b \asymp B$, then automatically $a \asymp N/B$). This comes from breaking up the sum over $a$ and $b$ into dyadic segments, and applying Cauchy’s inequality. It turns out that if $A \gg \Delta^{1/2+\varepsilon}$, then one can get a savings from Poisson summation on $a$ inside the square (and prior to any $t$-integration), which can then be combined with savings from the $t$-integral. Similarly, if $B \gg T^{1/2+\varepsilon}$, then there is savings from the $b$-sum. Since $AB = N = (\Delta T)^{1/2}$, this means $A = \Delta^{1/2+o(1)}$ and $B = T^{1/2+o(1)}$ is the main region of interest. Actually, a careful analysis shows that one may focus on $A = \Delta^{1/2}(\log T)^O(1)$ and $B = T^{1/2}(\log T)^O(1)$.

On the other hand, if $A$ is not too large, then one can use Cauchy’s inequality to move the sum over $a$ to the outside, obtaining

$$
(1.30) \quad \int_{|t-T|\simeq \Delta} \left| \sum_{a \geq A} \sum_{b \leq B} \frac{e(abx)}{a^{1/2+it-bb^{1/2+it}}b^{1/2+it+iT}} \right|^2 \ll \int_{|t-T|\simeq \Delta} \sum_{a \geq A} \left| \sum_{b \leq B} \frac{e(abx)}{b^{1/2+it+iT}} \right|^2.
$$

The diagonal alone is of size $AB$ which is then roughly $\Delta^{3/2}$ (for $A \asymp \Delta^{1/2}$) which is better than $N = (\Delta T)^{1/2}$ if $\Delta \ll T^{1/2-\varepsilon}$. The off-diagonal terms may be satisfactorily bounded because the $t$-integral detects close values, say $b+h, b$ with now $h \neq 0$, and the summand is still an oscillatory function of $b$, so exponential sum estimates will cover the off-diagonal terms. This explains the range of applicability in Theorem 1.5.

The arguments of Section 5 are also used in the alternative upper bound of (1.10). Once we know that the important ranges are $A = \Delta^{1/2}(\log T)^O(1)$ and $B = T^{1/2}(\log T)^O(1)$, we can use Cauchy’s inequality in a more efficient way than in (1.29), to avoid losing the factor $\log T$. The idea is to first apply Cauchy’s inequality to separate the values of $b$ with $B = T^{1/2}(\log T)^O(1)$ from the rest. These other values of $B$ still retain the factor $\log T$ (which comes from counting the number of dyadic values of $B$) in (1.29), but we are able to save a large power of $\log T$ to more than compensate for this. Since there are only $O(\log \log T)$ values of $B$ with $B = T^{1/2}(\log T)^O(1)$, we can essentially replace (1.29) with

$$
(1.31) \quad \int_{|t-T|\simeq \Delta} \left| \sum_{n \geq N} \frac{\tau_T(n)e(nx)}{n^{1/2+it}} \right|^2 \ll \log T \sum_{B=T^{1/2}(\log T)^O(1)} \sum_{B \text{ dyadic}} \int_{|t-T|\simeq \Delta} \left| \sum_{a \geq A} \sum_{b \leq B} \frac{e(abx)}{a^{1/2+it-bb^{1/2+it}}b^{1/2+it+iT}} \right|^2 + O(N(\log T)^{-10}).
$$

In addition, the Poisson summation arguments alluded to following (1.29) show

$$
(1.32) \quad \int_{|t-T|\simeq \Delta} \frac{1}{N} \left| \sum_{a \geq A} \sum_{b \leq B} \frac{e(abx)}{b^{1/2+it+iT}} \right|^2 \ll (\log \log T) \min \left( \frac{B^2}{T}, \frac{T}{B^2} \right),
$$
which is $O(\log \log T)$ even after summing over dyadic values of $B$. This method does not recover the factor $|\zeta(1 + 2iT)|^2$, and leads to (1.10).

One may also attempt to estimate (1.28) with absolute values, as in Holowinsky’s work [Ho]. Although this is an intriguing possibility, I have been unable to improve on Theorem 1.2 (there is no hope to prove Conjecture 1.3 this way) using this method. See Section 7.2 for further discussion on this matter.

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2. Notation and initial developments

2.1. Notation. Recall $V_T(y) = \sqrt{y}K_{iT}(y)$. By a well-known formula for the Mellin transform of the $K$-Bessel function, we have

$$\gamma_{V_T}(1/2 + s) := \int_0^\infty V_T(2\pi y) y^s dy = 2^{-3/2} \pi^{-s} \Gamma\left(\frac{1/2 + s + iT}{2}\right) \Gamma\left(\frac{1/2 + s - iT}{2}\right).$$

It is also helpful to recall

$$\gamma_{V_T^2}(1 + s) := \int_0^\infty V_T(2\pi y)^2 y^s dy = 2^{-2} \pi^{-s} \frac{\Gamma(\frac{1+s+2iT}{2})\Gamma(\frac{1+s}{2})^2\Gamma(\frac{1+s-2iT}{2})}{\Gamma(1+s)},$$

and

$$Z(s, E_T) := \sum_{n=1}^\infty \frac{\tau_{iT}(n)^2}{n^s} = \frac{\zeta(s - 2iT)\zeta(s + 2iT)\zeta(s)^2}{\zeta(2s)}.$$

2.2. Miscellaneous lemmas. Here we collect some basic tools used throughout this paper.

**Lemma 2.1** (Vinogradov-Korobov). For some $c > 0$ and for any $|t| \gg 1$, $1 - \frac{c}{(\log |t|)^{2/3}} \leq \sigma \leq 1$, we have

$$\zeta(\sigma + it) \ll (\log |t|)^{2/3},$$

and

$$\frac{\zeta'}{\zeta}(1 + it) \ll (\log |t|)^{2/3+\varepsilon},$$

For a reference, see [IK, Corollary 8.28, Theorem 8.29].

We also need the mean value theorem for Dirichlet polynomials of Montgomery and Vaughan [MV].

**Lemma 2.2** (Montgomery-Vaughan). Let $a_n$ be an arbitrary complex sequence. We have

$$\int_0^U \left| \sum_{n \leq N} a_n n^{-iu} \right|^2 du = \sum_{n \leq N} |a_n|^2 (U + O(n)).$$
2.3. Initial developments. We will calculate $F_{x,T}(s) = F(s)$ here, and then perform some easy approximations. Inserting (1.1) into (1.13), using (2.1) and the Mellin inversion formula for $\tilde{\psi}(y) = \frac{1}{2\pi i} \int_{(1)} \tilde{\psi}(-u)y^u du$, we obtain

\begin{equation}
F(s) = \mu \tilde{\psi}(1/2 + s + iT) + P\tilde{\psi}(1/2 + s - iT) \\
+ \frac{\rho^*(1)}{(2\pi)^s} \sum_{n \neq 0} \frac{\tau_T(n)e(nx)}{|n|^{1/2+s}} \frac{1}{2\pi i} \int_{(1)} \tilde{\psi}(-u)|n|^{-u}\gamma_{\sigma_T}(1/2 + s + u) du.
\end{equation}

Next we develop some simple approximations. First, note that

\begin{equation}
|\rho^*(1)|^2 = \frac{2/\pi}{\cosh(\pi T)}.
\end{equation}

By (2.1), combined with (2.8), we have that the integral appearing in (2.7) is exponentially small if $|\text{Im}(s + u) - T| \gg (\log^2 T)$. Since $\tilde{\psi}$ has rapid decay in the imaginary direction, we have that $F(it) \ll T^{-100}$, say, for $|t| \geq T + T^\varepsilon$.

The size of $\text{Im}(s)$ is the most important basic parameter in the analysis of $I_{\psi}(x,T)$.

We can simplify $F(s)$ in the range $|\text{Im}(s)| \leq T - T^\varepsilon$. Let

\begin{equation}
F_0(s) = \rho^*(1)\gamma_{\sigma_T}(1/2 + s) \sum_{n \neq 0} \frac{\tau_T(n)e(nx)}{|n|^{1/2+s}} \psi\left(\frac{\sqrt{|s + iT||s - iT|}}{2\pi |n|}\right).
\end{equation}

Lemma 2.3. For each $\varepsilon > 0$ there exists $\delta > 0$ so that

\begin{equation}
\int_{|t| \leq T - T^\varepsilon} |F(it)|^2 dt = \int_{|t| \leq T - T^\varepsilon} |F_0(it)|^2 dt + O(T^{-\delta}).
\end{equation}

Proof. In the range $|t| \leq T - T^\varepsilon$, Stirling gives

\begin{equation}
\Gamma\left(\frac{1/2 + s + u \pm iT}{2}\right) \sim \Gamma\left(\frac{1/2 + s \pm iT}{2}\right) \left(\frac{1/2 + s \pm iT}{2}\right)^{u/2} \left(1 + \sum_{j=1}^{\infty} \frac{P_{j,\pm}(u)}{(s \pm iT)^j}\right),
\end{equation}

where $P_{j,\pm}$ is a polynomial, and where the meaning of $\sim$ here is as an asymptotic expansion. Therefore, since $|s \pm iT| \gg T^\varepsilon$, we can truncate the expansion at $j \asymp \varepsilon^{-2}$ with a very small error term. By (2.11), we have

\begin{equation}
\gamma_{\sigma_T}(1/2 + s) \sim (2\pi)^{-u}|s + iT|^{u} |s - iT|^{u} \gamma_{\sigma_T}(1/2 + s) \prod_{j=1}^{\infty} \left(1 + \sum_{j=1}^{\infty} \frac{P_{j,\pm}(u)}{(s \pm iT)^j}\right).
\end{equation}

Therefore, for $s = it$, $|t| \leq T - T^\varepsilon$,

\begin{equation}
F(s) = \left[ \rho^*(1)\gamma_{\sigma_T}(1/2 + s) \sum_{n \neq 0} \frac{\tau_T(n)e(nx)}{|n|^{1/2+s}} \right]
\frac{1}{2\pi i} \int_{(1)} \tilde{\psi}(-u)\left(\frac{\sqrt{|s + iT||s - iT|}}{2\pi |n|}\right)^u \prod_{j=1}^{\infty} \left(1 + \sum_{1 \leq j \leq \varepsilon^{-2}} \frac{P_{j,\pm}(u)}{(s \pm iT)^j}\right) du + O(T^{-100}).
\end{equation}
The leading term in the $u$-integral is calculated exactly via (2.9), so that
\begin{equation}
F(s) = \sum_{0 \leq i \leq e^{-2}} F_i(s) + O(T^{-100}),
\end{equation}
say. The lower-order terms in the asymptotic expansion with $l \geq 1$ involve linear combinations of derivatives of $\psi$ in place of $\psi$, and multiplied by factors of the form $(s + iT)^{-\frac{3}{2}l} (s - iT)^{-\frac{3}{2}j}$, and are hence all of the same essential shape yet smaller by these additional factors. In fact, one can show that $\int_{|t| \leq T^{-T^\varepsilon}} |F_0(it)|^2 dt \ll T^{\varepsilon'}$, using only the mean value theorem for Dirichlet polynomials (Lemma 2.2 below), and by similar reasoning, for $j \geq 1$, we have $\int_{|t| \leq T^{-T^\varepsilon}} |F_j(it)|^2 dt \ll T^{-\delta}$, for some $\delta > 0$. □

Next we smoothly decompose the $t$-integral into pieces of the form $|t \mp T| \asymp \Delta$ where $T^\varepsilon \ll \Delta \ll T$. It will also be necessary to treat the integral of $|F(it)|^2$ with $|t \mp T| \ll T^\varepsilon$ for which see Section 6 below. As a first-order approximation, we mention that for such $t$, we have
\begin{equation}
t^k \frac{d^k}{dt^k} \cosh(\pi T)|\gamma_{\nu_T}(1/2 + it)|^2 \ll (\Delta T)^{-1/2}, \quad k = 0, 1, 2, \ldots.
\end{equation}

For the purposes of calculating diagonal terms, it is convenient to record the following.

**Lemma 2.4.** Let $w$ be a fixed smooth, compactly-supported function. Suppose $\log N \gg (\log T)^{2/3+\delta}$, for some fixed $\delta > 0$. Then
\begin{equation}
\sum_{n=1}^{\infty} \tau_{iT}(n)^2 w\left(\frac{n}{N}\right) = \frac{\zeta(1 + 2iT)^2}{\zeta(2)} [\tilde{w}(1)N \log N + O(N(\log T)^{2/3+\varepsilon})].
\end{equation}

**Proof.** By Mellin inversion and (2.3), the sum on the left hand side of (2.16) is
\begin{equation}
\frac{1}{2\pi i} \int_{(2)} N^s \tilde{w}(s) \frac{\zeta(s - 2iT)\zeta(s + 2iT)\zeta(s)^2}{\zeta(2s)} ds.
\end{equation}

We evaluate the integral by the standard method of moving the contour, in this case to $\text{Re}(s) = \sigma = 1 - \frac{c}{(\log T)^{2/3}}$, for some constant $c > 0$. We first calculate the residue at $s = 1$. Using
\begin{equation}
N^s = 1 + (s - 1) \log N + O((s - 1)^2),
\end{equation}
and the Vinogradov-Korobov bound (Lemma 2.1), one calculates that the residue at $s = 1$ is consistent with the right hand side of (2.16). The residues at $s = 1 \pm 2iT$ are negligible because $\tilde{w}$ has rapid decay in the imaginary direction.

Finally we bound the integral along the line $\sigma$. We use the Vinogradov-Korobov bound again, which leads to an error of size $N(\log T)^{4/3} \exp(-c\frac{\log N}{(\log T)^{2/3}})$, which is bounded by the error term appearing in (2.16). □

For ease of use, we also directly quote the following,
Proposition 2.5 ([Y], Theorem 8.1). Suppose that \( w(x) \) is a smooth function on the positive reals supported on \( Y \leq x \leq 2Y \) and satisfying \( w^{(j)}(x) \leq_j (P/Y)^j \) for some parameters \( 1 \leq P \leq Y \). Let \( \theta = 7/64 \), and set \( R = P + \frac{\tau_{Tm}}{Y} \). Then for \( m \neq 0 \), \( R \leq T/(TY)^\delta \), we have

\[
(2.19) \quad \sum_{n \in \mathbb{Z}} \tau_{iT}(n)\tau_{iT}(n+m)w(n) = M.T + E.T.,
\]

where

\[
(2.20) \quad M.T. = \sum_{\pm} \frac{|\zeta(1+2iT)|^2}{\zeta(2)} \sigma_{-1}(m) \int_{\max(0,-m)}^{\infty} (x+m)^{\mp iT}x^{\pm iT}w(x)dx,
\]

and

\[
(2.21) \quad E.T. \ll (|m|^\theta T^{64YR^2} + T^{64YR^2})TY^\varepsilon.
\]

Furthermore, with \( R = P + \frac{Tm}{YT} \), we have

\[
(2.22) \quad \sum_{1 \leq |m| \leq M} |E.T.| \ll (M^T Y^R R^2 + MT^2 Y^R R^2)TY^\varepsilon.
\]

Remark. The bound (2.22) roughly means that the Ramanujan-Petersson conjecture, i.e., \( \theta = 0 \), holds on average over \( m \).

3. The largest \( \Delta \) regime: extracting the main term

The purpose of this section is to show the following

Proposition 3.1. Let \( \delta > \eta(T) \gg (\log T)^{-\delta} \) for some fixed small \( \delta > 0 \). Then

\[
(3.1) \quad \int_{|t| \leq T^{-\eta(T)T}} |F_0(it)|^2 \frac{dt}{2\pi} = \frac{3}{\pi} \log(1/4 + T^2) \int_{0}^{\infty} \psi^2(y) \frac{dy}{y} + R(x,T)
\]

\[
+ O(\eta(T)^{1/2} \log T + |\log \eta(T)|),
\]

where \( R(x,T) \) is a term satisfying, for any \( U \geq 1 \),

\[
(3.2) \quad \left| \int_{|x-x_0| \leq \frac{1}{T}} R(x,T)dx \right| \ll \frac{\eta^{-2}(T) \log T}{T}.
\]

Furthermore, under the same conditions,

\[
(3.3) \quad \int_{|t| \leq T^{-\eta(T)T}} |F(it)|^2 dt \ll \log T.
\]

This accounts for the main bulk range of \( t \)'s which provide a main term as well as an additional term (this is \( R(x,T) \) which we can show is small with a tiny extra \( x \)-averaging).

From Proposition 3.1, we may deduce Theorem 5.1 as follows. We have

\[
I_{\psi,\gamma}(x_0, T) = \int_{|x-x_0| \leq \frac{1}{T}} \int_{\infty} \left| F_{x,T}(it) \right|^2 \frac{dt}{2\pi} \geq \int_{|x-x_0| \leq \frac{1}{T}} \int_{|t| \leq T^{-\eta(T)T}} \left| F_{x,T}(it) \right|^2 \frac{dt}{2\pi}
\]

\[
= \frac{3}{\pi} \log(1/4 + T^2)m_{\psi,\gamma} \left[ 1 + O(\eta(T)^{1/2} + \frac{\log \log T}{\log T}) \right] + O(\left| \int_{|x-x_0| \leq \frac{1}{T}} R(x,T)dx \right|).
\]

The bound (3.2) implies \( \int_{|x-x_0| \leq \frac{1}{T}} R(x,T)dx \ll m_{\psi,\gamma}(\log T)(\gamma T)^{-1} \), recalling that \( m_{\psi,\gamma} \asymp \gamma/T \). Letting \( \gamma \to \infty \) very slowly (in particular, we may assume \( \gamma \ll (\log T)^{\delta} \), some \( \delta > 0 \),

and then setting $\eta = \gamma^{-2/5}$, we conclude the proof. In fact, this argument exhibits the error term in Theorem 1.4 but to fully prove Theorem 1.4 we need to study $T^{25/27+\varepsilon} \ll |t+T| \leq \eta T$ also.

**Proof of Proposition 1.4** For shorthand, set $\eta = \eta(T)$. Let $W(t)$ be a function that is 1 on $|t| \leq T - c_1 \eta T$, and 0 on $|t| \geq T - c_2 \eta T$, where here $c_1 > c_2 > 0$ are positive constants to be chosen later, and satisfying $W^{(j)}(t) \ll (\eta T)^{-j}$.

We have

$$
(3.5) \quad \int_{-\infty}^{\infty} W(t) |F_0(it)|^2 dt = \frac{|\rho^*(1)|^2}{\cosh(\pi T)} \sum_{m,n} \frac{\tau_{\text{T}}(m) \tau_{\text{R}}(n) e((n-m)x)}{|mn|^{1/2}} J(m, n),
$$

where

$$
(3.6) \quad J(m, n) = \int_{-\infty}^{\infty} W(t) |\gamma_{\text{T}}(1/2 + it)|^2 \cosh(\pi T) \psi \left( \frac{\sqrt{T^2 - t^2}}{2\pi |n|} \right) \psi \left( \frac{\sqrt{T^2 - t^2}}{2\pi |m|} \right) \frac{1}{|n|} dt.
$$

In the right hand side of (3.5), write

$$
(3.7) \quad \int_{-\infty}^{\infty} W(t) |F_0(it)|^2 = M.T. + E(x, T) + R(x, T),
$$

where $M.T.$ corresponds to the terms with $m = n$ (both positive and negative), $R(x, T)$ corresponds to the terms with $m$ and $n$ of opposite signs, and $E(x, T)$ are the terms with $m$ and $n$ of the same sign, and $m \neq n$. We will show that

$$
(3.8) \quad M.T. = \frac{3}{\pi} \log(1/4 + T^2) \left( \int_{0}^{\infty} \psi^2(y) \frac{dy}{y} + O(\eta^{1/2}) \right),
$$

that $R(x, T)$ satisfies (3.2), and that

$$
(3.9) \quad E(x, T) \ll |\log \eta|.
$$

At this point we can also explain how to remove the smoothing factor $W$. We have that

$$
(3.10) \quad \int_{|t| \leq T - \eta T} |F_0(it)|^2 dt \leq \int_{-\infty}^{\infty} W(t) |F_0(it)|^2 + \int_{-\infty}^{\infty} (W_+(t) - W(t)) |F_0(it)|^2 dt, \quad \int_{|t| \leq T - \eta T} |F_0(it)|^2 dt \geq \int_{-\infty}^{\infty} W(t) |F_0(it)|^2 + \int_{-\infty}^{\infty} (W_-(t) - W(t)) |F_0(it)|^2 dt,
$$

where $W_+$ and $W_-$ satisfy the same properties as $W$, but with different choices of constants $c_1$, and such that $W_+ \geq W \geq W_-$. We use the decomposition (3.7) to treat the term $\int_{-\infty}^{\infty} W(t) |F_0(it)|^2 dt$ appearing on each line above. The second integral appearing on each line we shall treat slightly differently, as follows. Instead of using the formula (3.5) (but with $W$ replaced by $|W - W_{\pm}|$), we first apply Cauchy’s inequality in the form $|\sum_{n \neq 0} a_n|^2 \leq 2|\sum_{n \geq 1} a_n|^2 + 2|\sum_{n \leq -1} a_n|^2$. After this, we then open the square and perform the integral. The same estimates on $E(x, T)$ and $M.T.$ apply (in fact the “main term” here is $O(\eta^{1/2} \log T)$ since the right hand side of (3.8) does not depend on $W$), and the gain is that the opposite sign terms do not appear. That is, (3.8) and (3.9) imply

$$
(3.11) \quad \int_{-\infty}^{\infty} |W_\pm(t) - W(t)| |F_0(it)|^2 dt \ll \eta^{1/2} \log T + |\log \eta|.
$$
The same idea here of using Cauchy’s inequality shows \( \int_{-\infty}^{\infty} W(t)|F_0(it)|^2dt \ll \log T + E(x, T) \), which by (3.9) implies (3.3).

Now we analyze \( J(m, n) \). Note that \( m \) and \( n \) are supported on

\[
\eta^{1/2} T \ll |m|, |n| \ll T.
\]

By Stirling’s approximation (see (2.15)), we have that \( J(m, n) = \int_{-\infty}^{\infty} j(m, n, t, T)(|m|/|n|)^i dt \), where \( j \) is a function satisfying \( \frac{\partial}{\partial \tau} j(m, n, t, T) \ll \eta^{-1/2} T^{-1} (\eta T)^{-k} \). Thus by standard integration by parts, if \( |m| \neq |n| \),

\[
J(m, n) \ll_k \frac{\eta^{-1/2}}{\eta T \log(|m|/|n|)^k}, \quad k = 0, 1, 2, \ldots
\]

To get our bearings, we shall check the size of the trivial bound on the off-diagonal terms with \( |m| \neq |n| \) (it will turn out to be just barely insufficient). For this we use the simple bound \( |xy| \leq |x|^2 + |y|^2 \), (3.13), and symmetry, giving

\[
\sum_{m \neq n} \left| \frac{\tau_{iT}(m)\tau_{iT}(n)}{mn^{1/2}} \right| |J(m, n)| \leq \sum_{m \neq n} \frac{\left| \tau_{iT}(m) \right|^2}{|m|} \frac{\eta^{-1/2}}{\eta T \log(|m|/|n|)^k} \left( 1 + \frac{\eta T|h|}{|m|} \right)^2,
\]

where \( h = |m| - |n| \). Summing over \( h \) trivially and using Lemma 2.4 we obtain that (3.14) is

\[
\ll \eta^{-3/2} T^{-1} \sum_{|n| \ll T} \left| \tau_{iT}(n) \right|^2 \ll \eta^{-3/2} |\zeta(1 + 2iT)|^2 \log T.
\]

Recalling (2.8), in all we have shown

\[
\frac{|\rho^*(1)|^2}{\cosh(\pi T)} \sum_{m \neq n} \frac{\left| \tau_{iT}(m)\tau_{iT}(n) \right|}{mn^{1/2}} |J(m, n)| \ll \eta^{-3/2} \log T.
\]

We will provide an improved estimate on the off-diagonal terms which will give (3.9), but prior to that we bound \( R(x, T) \) and evaluate the main term.

Recall that \( R(x, T) \) denotes the terms on the right hand side of (3.5) such that \( m \) and \( n \) have opposite signs. Further write \( R(x, T) = R'(x, T) + R_0(x, T) \) corresponding to the terms with \( |m| \neq |n| \) and \( |m| = |n| \), respectively. Then changing variables and using symmetry, we have

\[
R'(x, T) = 2 \frac{|\rho^*(1)|^2}{\cosh(\pi T)} \Re \sum_{m \neq n > 0} \frac{\tau_{iT}(m)\tau_{iT}(n)e((m + n)x)}{|mn|^{1/2}} J(m, n).
\]

The bound (3.16) obviously shows that \( R'(x, T) \ll \eta^{-3/2} \log T \), and so we need a small savings in the \( x \)-integral. For this, one may directly calculate that for any \( U > 0 \) that

\[
\left| \int_{|x-x_0| \leq U^{-1}} e(nx) dx \right| \leq \frac{1}{\pi |n|},
\]

so, recalling that \( |n| \gg \eta^{1/2} T \), we obtain (3.2) for \( R' \), precisely,

\[
\left| \int_{|x-x_0| \leq \frac{1}{T}} R'(x, T) dx \right| \ll \frac{\log T}{\eta^2 T}.
\]
Next we evaluate $M.T.$ defined by
\begin{equation}
M.T. = 2\left| \frac{\rho^*(1)}{2\pi} \right|^2 \int_{-\infty}^{\infty} W(t) |\gamma_{V_T}(1/2 + it)|^2 \sum_{n=1}^{\infty} \psi^2 \left( \frac{\sqrt{T^2 - t^2}}{2\pi |n|} \right) dt.
\end{equation}
We need a slight generalization of Lemma 2.4, so we proceed directly. Letting $\Psi = \psi^2$, we have that $M.T.$ equals
\begin{equation}
2\left| \frac{\rho^*(1)}{2\pi} \right|^2 \int_{-\infty}^{\infty} W(t) |\gamma_{V_T}(1/2 + it)|^2 \frac{1}{2\pi i} \int_{c+i\infty}^{c-i\infty} \tilde{\Psi}(-u)(T^2 - t^2)^{\frac{u}{2}} (2\pi)^{-u} Z(1 + u, E_T) du dt.
\end{equation}
We shift the contour to the left, to $\text{Re}(u) = -\frac{1}{2}$. On the new contour we get a bound of the form $T^{-\frac{1}{2} + \frac{1}{3} + \varepsilon}$, by Weyl’s subconvexity bound. By a similar computation as in Lemma 2.4 (in the proof it is helpful to remember $|t - T| \gg \eta T$ from the support of $W$, and $\eta \gg (\log T)^{-\delta}$), the residue at $u = 0$ gives
\begin{equation}
2\left| \frac{\rho^*(1)}{2\pi} \right|^2 \frac{\zeta(1 + 2iT)}{\zeta(2)} \left[ (\log T) \tilde{\Psi}(0) + O((\log T)^{2/3 + \varepsilon}) \right] \int_{-\infty}^{\infty} W(t) |\gamma_{V_T}(1/2 + it)|^2 dt.
\end{equation}
Note $\tilde{\Psi}(0) = \int_{0}^{\infty} \psi^2(y) \frac{dy}{y}$. We also claim
\begin{equation}
\frac{1}{2\pi} \int_{-\infty}^{\infty} W(t) |\gamma_{V_T}(1/2 + it)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\gamma_{V_T}(1/2 + it)|^2 dt + O(\eta^{1/2} \exp(-\pi T)).
\end{equation}
This follows from the estimates
\begin{equation}
\int_{T-\eta T}^{T} T^{-1/2}(1 + |t - T|)^{-1/2} dt \ll \eta^{1/2},
\end{equation}
and
\begin{equation}
\int_{T}^{\infty} \cosh(\pi T) |\gamma_{V_T}(1/2 + it)|^2 dt \ll T^{-1/2},
\end{equation}
using (2.15). To calculate the constant on the right hand side of (3.22), we first recall the Mellin formula
\begin{equation}
\frac{1}{2\pi} \int_{-\infty}^{\infty} |\gamma_{V_T}(1/2 + it)|^2 dt = \gamma_{V_T}^2(1).
\end{equation}
One then easily checks, using (2.2) and the formulas following (1.1), that
\begin{equation}
\frac{\zeta(1 + 2iT)}{\zeta(2)} |\rho^*(1)|^2 \gamma_{V_T}^2(1) = \frac{3}{\pi}.
\end{equation}
Thus we have shown (3.8).

Finally, we consider the “pseudo”-main term that arises from $|m| = |n|$ but with $m = -n$, which we denoted $R_0(x, T)$. A similar computation to that for $M.T.$ shows that $R_0(x, T) \ll \log T$, and (3.2) holds for $R_0$. Thus, (3.2) holds for $R = R_0 + R'$.

To complete the proof of Proposition 3.1, we need an improved bound (compared to (3.16)) on the off-diagonal terms, $E(x, T)$. It suffices to bound the terms with both $m, n > 0$, the case where both $m, n < 0$ following from this by symmetry. We shall use Proposition 2.5 to treat the terms with $m = n + h, h \neq 0$. Technically to meet the hypotheses of the proposition we need a weight function supported on a dyadic interval, so we apply a dyadic partition of unity to the off-diagonal positive values of $m, n$ on the right hand side of (3.5). Taking one
such part of the partition with say \( n \asymp N \), we get that \( E(x, T) \) is a sum (over \( N \)) of terms of the form

\[
S(x, N, T) := |\zeta(1 + 2iT)|^2 \sum_{0 < |h| < T^\varepsilon} N^{-1} e(hx) \sum_n \tau_{iT}(n) \tau_{iT}(n + h) g(h, n),
\]

where \( g \) is a function satisfying

\[
\frac{\partial^k}{\partial u^k} g(h, u) \ll \frac{N}{T} \left(1 + \frac{N|h|}{T}\right)^{-100} N^{-k}.
\]

Here this estimate arises by a slight modification of (3.13), using now that if \( |n| \asymp N \), then \( |t - T| \asymp N^{2} \). By Proposition 2.5 and its following remark, \( S(x, N, T) \) equals a main term plus an error term of size \( O(T^{-1/12 + \varepsilon}) \) (here we use that \( N = T^{1+o(1)} \)). The main term takes the form

\[
\sum_{\pm} |\zeta(1 + 2iT)|^2 \sum_{0 < |h| < T^\varepsilon} N^{-1} \int_{\max(0, -h)}^{\infty} (r + h)^{\pm iT} r^{\mp iT} g(h, r) dr,
\]

which by trivial estimations using (3.29) is \( \ll |\zeta(1 + 2iT)|^2 \). Since the number of dyadic segments is \( O(|\log \eta|) \), we have thus shown (3.9). We will give a more elaborate analysis of this off-diagonal term following (4.23) below. \( \square \)

4. LARGE \( \Delta \) REGIME VIA SHIFTED DIVISOR SUM

Complementing Proposition 3.1, we have

**Proposition 4.1.** Suppose that \( \delta > \eta(T) \gg (\log T)^{-\delta} \) for some fixed small \( \delta > 0 \), and

\[
T^{3/2 + \varepsilon} \ll \Delta \leq \eta(T) T.
\]

Then for some \( \delta' > 0 \), we have

\[
\int_{\frac{1}{2} \Delta \leq |t + iT| \leq \Delta} |F_0(it)|^2 dt \ll \frac{\Delta^{1/2}}{T^{1/2}} \log T + T^{-\delta'} + S_\Delta,
\]

where \( S_\Delta \) is a nonnegative quantity satisfying

\[
\sum_{1 \leq \Delta \ll T \text{ \textup{dyadic}}} S_\Delta \ll 1.
\]

Remark. There is a possibility to allow somewhat smaller values of \( \Delta \) by exploiting additional savings in the sum over \( h \). However, the improvement is modest and the work is technical, so we have omitted this line of discussion.

Combining Proposition 3.1 and its following discussion with Proposition 4.1, we derive

\[
\int_{|t| \leq T - C(\varepsilon) T^{25/27 + \varepsilon}} \int_{|x - x_0| \leq \frac{T}{2}} |F_0(it)|^2 dt = \frac{3}{\pi} \log(1/4 + T^2) m_{\psi, \gamma}
\]

\[+ O\left( \left\| \int_{|x - x_0| \leq \frac{T}{2}} R(x, T) \right\| \right) + O\left( \frac{\gamma \log T}{T} \left( \eta^{1/2} + \frac{|\log \eta|}{\log T} \right) \right).
\]

Recall \( m_{\psi, \gamma} \asymp \frac{T}{\pi} \). Using (3.2), and choosing \( \eta = \gamma^{-2/5} \), we derive Theorem 1.4. There is one technical point that was glossed over in the above proof, which is the restriction that \( \eta \gg (\log T)^{-\delta} \) for some fixed \( \delta > 0 \). This will hold under the additional assumption that
\( \gamma \ll (\log T)^{\delta'} \) for some fixed small \( \delta' > 0 \). Once Theorem 1.4 is proved with this restriction on \( \gamma \), it then applies without the restriction by additivity of the integral and the fact that the estimate is uniform in \( x \) so that a larger interval can be decomposed as a union of smaller ones (each of length \( \ll (\log T)^{\delta'} \)).

We begin with some simplifications of a general nature.

**Lemma 4.2.** Suppose that \( T^\varepsilon \ll \Delta \leq \eta(T)T \), where \( \eta(T) > 0 \) is sufficiently small. Then

\[
(4.5) \quad \int_{\frac{\Delta}{2} \leq |t - T| \leq \Delta} |F_0(it)|^2 \, dt \ll \sum_{\pm} \left| \frac{\zeta(1 + 2iT)}{\sqrt{\Delta T}} \right|^2 I(\Delta, T, \pm x, N) + T^{-100},
\]

where

\[
(4.6) \quad I(\Delta, T, x, N) = \int_{-\infty}^\infty w_1 \left( \frac{T - t}{\Delta} \right) \left| \sum_{n \geq 1} \frac{\tau_{iT}(n)e(nx)}{n^{1/2 + it}} w_2 \left( \frac{n}{N} \right) \right|^2 \, dt, \quad N = \sqrt{\Delta T},
\]

and \( w_1, w_2 \) are certain fixed smooth weight functions with compact support on the positive reals.

**Proof of Lemma 4.2.** The essential part of the proof is a separation of variables argument. We will need variations on this argument in other parts of the paper, so we provide full details here.

By symmetry, we may consider the portion of the integral with \( t > 0 \). Using Stirling’s approximation (2.15), and Cauchy’s inequality in the form \( |\sum_{n \neq 0} a_n| \leq 2 |\sum_{n > 0} a_n|^2 + 2 |\sum_{n > 0} a_{-n}|^2 \), we obtain

\[
(4.7) \quad \int_{\frac{\Delta}{2} \leq |t - T| \leq \Delta} |F_0(it)|^2 \, dt \ll \frac{|\zeta(1 + 2iT)|^2}{\sqrt{\Delta T}} \max_{\pm} \int_{\frac{\Delta}{2} \leq |t - T| \leq \Delta} \left| \sum_{n \geq 1} \frac{\tau_{iT}(n)e(nx)}{n^{1/2 + it}} \psi \left( \frac{\sqrt{T^2 - t^2}}{2\pi n} \right) \right|^2 \, dt.
\]

The support on \( \psi \) means that the \( n \)-sum is supported on \( n \sim N \), where

\[
(4.8) \quad N = \sqrt{\Delta T}.
\]

This integral is not yet quite in the form \( I(\Delta, T, x, N) \). For this, we multiply \( \psi \) by a redundant factor \( w_2(n/N) \), such that \( w_2(n/N) = 1 \) for all \( n \) in the support of \( \psi \), yet such that \( w_2 \) is smooth of compact support. Then we use \( \psi(y) = \frac{1}{2\pi} \int_0^\infty \tilde{\psi}(-u) y^u du \), and apply Cauchy-Schwarz to the \( u \)-integral, obtaining

\[
(4.9) \quad \left| \sum_n a_n \psi \left( \frac{\sqrt{T^2 - t^2}}{2\pi n} \right) \right|^2 \ll \int_{-\infty}^\infty |\tilde{\psi}(iu)| \left| \sum_n a_n n^{-iu} \right|^2 \, du,
\]

using the fact that \( \int_{-\infty}^\infty |\tilde{\psi}(iv)| \, dv \ll 1 \) (the reader may recall that \( \psi \) was fixed at the beginning of the paper and is independent of all relevant parameters).
In our application, we may truncate the $u$-integral at $|u| \leq \frac{\Delta}{100}$, leading to an error that is at most $O(T^{-100})$, using the rapid decay of $\tilde{\psi}$, and the fact that $\Delta \gg T^\varepsilon$. That is,

\begin{equation}
\int_{\frac{1}{2}\Delta \leq |t-T| \leq \Delta} \left| \sum_{n \geq 1} \frac{\tau_T(n)e(nx)}{n^{1/2+it}} \psi\left(\frac{\sqrt{T^2-t^2}}{2\pi n}\right) \right|^2 dt \lesssim \int_{\frac{1}{2}\Delta \leq |t-T| \leq \Delta} \left| \int_{|u| \leq \frac{\Delta}{100}} |\tilde{\psi}(-iu)| \left| \sum_{n \geq 1} \frac{\tau_T(n)e(nx)}{n^{1/2+it+iu}} w_2\left(\frac{n}{N}\right) \right|^2 dt du + O(T^{-100}).
\end{equation}

Next we change variables $t \to t-u$ and over-extend the $t$-range of integration to $\frac{9}{100}\Delta \leq |t-T| \leq \frac{10}{100}\Delta$. In this way the variables $u$ and $t$ are separated, and the $u$-integral can be bounded again using $\int_{-\infty}^{\infty} |\tilde{\psi}(iv)|dv \ll 1$.

Finally, by attaching a smooth weight function $w_1\left(\frac{T-t}{\Delta}\right)$ we obtain the desired result. □

**Proof of Proposition 4.1.** The basic approach is similar to that of Proposition 3.1, in that we shall solve a shifted divisor problem.

In light of Lemma 4.2 we need to show

\begin{equation}
I(\Delta, T, x, N) \ll |\zeta(1+2iT)|^2(\Delta \log T + NT^{\varepsilon} + NS\Delta),
\end{equation}

for $\Delta \gg T^{25/27+\varepsilon}$, and some $\varepsilon > 0$. Opening the square and integrating in $t$, we obtain

\begin{equation}
I(\Delta, T, x, N) = \sum_{m,n \geq 1} \tau_T(m) \tau_T(n)e((m-n)x) \frac{w_2\left(\frac{m}{N}\right)}{\sqrt{mn}} \int_{-\infty}^{\infty} w_1\left(\frac{T-t}{\Delta}\right) \left(\frac{n}{m}\right)^{it} dt.
\end{equation}

By Lemma 2.4 we have that the terms with $m = n$ contribute to $I(\Delta, T, x, N)$ an amount

\begin{equation}
\ll \Delta |\zeta(1+2iT)|^2 \log T,
\end{equation}

which leads to the first bound in (4.11).

Our main job is to bound the terms with $m \neq n$, in which case we set $m = n+h$. By simple estimations, we then have that these terms contribute to $I(\Delta, T, x, N)$ an amount at most

\begin{equation}
\frac{\Delta}{N} \sum_{h \neq 0} e(hx) \sum_n \tau_T(n) \tau_T(n+h) K(n, h),
\end{equation}

where

\begin{equation}
K(n, h) = \frac{w_2\left(\frac{n}{N}\right)}{\left(\frac{n}{N}\right)^{1/2}} \frac{w_2\left(\frac{n+h}{N}\right)}{\left(\frac{n+h}{N}\right)^{1/2}} \left(\frac{n+h}{n}\right)^{-iT} \hat{w}_1\left(-\frac{\Delta}{2\pi} \log \left(\frac{n+h}{n}\right)\right).
\end{equation}

In particular, $K(n, h)$ is supported on $n \asymp N$ and satisfies

\begin{equation}
\frac{\partial^j}{\partial u^j} K(u, v) \ll R^j N^{-j} \left(1 + \frac{\Delta|v|}{N}\right)^{-A}, \quad j = 0, 1, 2, \ldots,
\end{equation}

with $A > 0$ arbitrary, and where

\begin{equation}
R = \Delta^{-1} T.
\end{equation}
Proposition \[2.5\] gives

\[
(4.18) \quad \frac{\Delta}{N} \sum_{h \neq 0} e(hx) \left( \sum_{n} \tau_{iT}(n) \tau_{iT}(n + h) K(n, h) - M.T. \right) \ll T^{\frac{1}{2} + \varepsilon} N^{\frac{3}{4}} R^{2} + T^{\frac{1}{2} + \varepsilon} N^{\frac{1}{4}} R^{\frac{3}{2}},
\]

where \( M.T. \) is a main term (off-diagonal) to be discussed presently. A short calculation shows that the two error terms in \((4.18)\) are \( O((\Delta T)^{1/2} T^{-\delta}) \) for \( \Delta \gg T^{\frac{1}{2} + \varepsilon} \). It turns out that the main terms are surprisingly difficult to estimate with uniformity in \( x \), and we next turn to this problem.

The off-diagonal main term is defined by

\[
(4.19) \quad M.T. = \sum_{\pm} \frac{|\zeta(1 + 2iT)|^2}{\zeta(2)} \Delta \sum_{h \neq 0} e(hx) \sigma_{1}(h) \int_{\max(0, -h)}^{\infty} (y + h) e^{\pm iT} y^{\pm iT} K(y, h) dy.
\]

These two terms then give to \( I(\Delta, T, x, N) \) an amount

\[
(4.20) \quad \int_{0}^{\infty} \int_{0}^{\frac{\Delta}{2 \pi}} w_{2}(\frac{y}{N}) w_{2}(\frac{y + h}{N}) \left( \frac{y + h}{y} \right)^{-iT} e^{\pm iT} \hat{w}_{1}( - \frac{\Delta}{2 \pi} \log \left( \frac{y + h}{y} \right) ) dy.
\]

When the choice of sign makes the exponent of \( (y + h)/y \) be \(-2iT\), then a simple integration by parts argument shows that

\[
(4.21) \quad \int_{0}^{\infty} \int_{0}^{\frac{\Delta}{2 \pi}} w_{2}(\frac{y}{N}) w_{2}(\frac{y + h}{N}) \left( \frac{y + h}{y} \right)^{-2iT} e^{\pm iT} \hat{w}_{1}( - \frac{\Delta}{2 \pi} \log \left( \frac{y + h}{y} \right) ) dy \ll N \left( 1 + \frac{|h|T}{N} \right)^{-10}.
\]

Hence this off-diagonal term contributes to \( I(\Delta, T, x, N) \) an amount that is

\[
(4.22) \quad \ll |\zeta(1 + 2iT)|^2 \frac{\Delta}{N} \sum_{h \neq 0} N \left( 1 + \frac{|h|T}{N} \right)^{-10} \ll |\zeta(1 + 2iT)|^2 \Delta \frac{N}{T} \ll |\zeta(1 + 2iT)|^2 \Delta,
\]

since \( N \ll T \), which is more than satisfactory for \((4.11)\). The other main term is much more subtle, and gives

\[
(4.23) \quad MT_{OD} := \frac{|\zeta(1 + 2iT)|^2}{\zeta(2)} \frac{\Delta}{N} \sum_{h \neq 0} e(hx) \sigma_{1}(h) \int_{0}^{\infty} \int_{0}^{\frac{\Delta}{2 \pi}} w_{2}(\frac{y}{N}) w_{2}(\frac{y + h}{N}) \left( \frac{y + h}{y} \right)^{-1} e^{\pm iT} \hat{w}_{1}( - \frac{\Delta}{2 \pi} \log \left( \frac{y + h}{y} \right) ) dy.
\]

We will show here

\[
(4.24) \quad MT_{OD} \ll |\zeta(1 + 2iT)|^2 (\Delta \log T + NS_{\Delta}),
\]

where \( S_{\Delta} \) is a function satisfying \((4.3)\). It is quite easy to show with a trivial bound that

\[
MT_{OD} \ll |\zeta(1 + 2iT)|^2 N, \quad \text{so } (4.24) \text{ amounts to a logarithmic savings with the sum over } \Delta \text{ in dyadic segments (there are of course } O(\log T) \text{ such values of } \Delta \text{ over which to sum).}
\]
first simple approximation step, by taking Taylor expansions we have that
\[
\begin{align*}
(4.25) \quad & \int_0^{\infty} w_2\left(\frac{y}{N}\right) \frac{w_2\left(\frac{y+h}{N}\right)}{\left(\frac{y}{N}\right)^{1/2} \left(\frac{y+h}{N}\right)^{1/2}} \overline{w_1}\left( -\frac{\Delta}{2\pi} \log \left( \frac{y+h}{y} \right) \right) dy \\
&= \int_0^{\infty} \frac{w_2\left(\frac{y}{N}\right)}{\left(\frac{y}{N}\right)^{1/2}} \overline{w_1}\left( -\frac{\Delta}{2\pi} y \right) dy + O\left( |h| \left( 1 + \frac{|h| \Delta}{N} \right)^{-100} \right),
\end{align*}
\]
and hence by a trivial estimate on the \( h \)-sum, we have
\[
(4.26) \quad MT_{OD} = \frac{|\zeta(1+2iT)|^2}{\zeta(2)} \sum_{h \neq 0} e(hx) \sigma_{-1}(h) \int_0^{\infty} \frac{w_2(y)^2}{y} \overline{w_1}\left( -\frac{\Delta}{2\pi N y} \right) dy + O\left( \frac{N}{\Delta} |\zeta(1+2iT)|^2 \log T \right).
\]
This error term is certainly \( O(NT^{-\delta}) \), since \( \Delta \gg T^{25/27+\varepsilon} \). By changing variables, we have
\[
(4.27) \quad MT_{OD} \ll |\zeta(1+2iT)|^2N \int_0^{\infty} \frac{w_2(y)^2}{y} \left| \frac{\Delta}{N} \sum_{h \neq 0} \sigma_{-1}(h)e(hx)\overline{w_1}\left( -\frac{\Delta}{N y} \right) \right| dy + O(NT^{-\delta}).
\]
We need to control the sum over \( h \), on average over \( \Delta \) running over dyadic segments. We pause our estimations of \( MT_{OD} \) to state the following.

**Lemma 4.3.** Let \( w_1 \) be a smooth function supported on the positive reals, and for \( x \in \mathbb{R} \) and \( H \gg 1 \), define
\[
(4.28) \quad Q(x, H) = H^{-1} \sum_{h \neq 0} \sigma_{-1}(h)e(hx)\overline{w_1}(h/H).
\]
Then
\[
(4.29) \quad \sum_{1 \leq H < \infty \atop H \text{ dyadic}} |Q(x, H)| \ll 1,
\]
where the implied constant depends only on \( w_1 \) and is uniform in \( x \).

Lemma 4.3 suffices to show (4.24), as \( H = y_N^2 = y_{\Delta}^{1/2} \), and \( \Delta \) runs over dyadic segments. This means that \( H \) can be partitioned into two classes of dyadic segments (taking even powers and odd powers separately, for example).

Having accounted for all the terms, the proof of Proposition 4.1 is complete. \( \square \)

**Proof of Lemma 4.3.** We open \( \sigma_{-1}(h) \) as a convolution, obtaining
\[
(4.30) \quad Q(x, H) = H^{-1} \sum_{a \geq 1} a^{-1} \sum_{b \neq 0} e(abx)\overline{w_1}\left( \frac{ab}{H} \right).
\]
For larger values of \( a \), say \( a > K \geq 1 \), we estimate trivially, obtaining
\[
(4.31) \quad H^{-1} \sum_{a > K} a^{-1} \sum_{b \neq 0} e(abx)\overline{w_1}\left( \frac{ab}{H} \right) \ll \frac{1}{K}.
\]
For \( a \leq K \), we apply Poisson summation, getting
\[
(4.32) \quad H^{-1} \sum_{a \leq K} a^{-1} \sum_{b \neq 0} e(abx)\overline{w_1}\left( \frac{ab}{H} \right) = \sum_{a \leq K} a^{-1} \left( -\overline{w_1}(0) \frac{1}{H} + \frac{1}{a} \sum_{\nu \in \mathbb{Z}} w_1(H(x - \frac{\nu}{a})) \right).
\]
Thus
\begin{equation}
Q(x, H) = \sum_{a \leq K} a^{-2} \sum_{\nu \in \mathbb{Z}} w_1 \left( H \left( x - \frac{\nu}{a} \right) \right) + O \left( K^{-1} + \frac{\log(eK)}{H} \right).
\end{equation}

Suppose that \( w_1 \) is supported on \([\alpha, \beta]\) where \( 0 < \alpha < \beta \). If we assume that \( K \leq \varepsilon H \) for \( \varepsilon^{-1} > 2\beta \), then the sum over \( \nu \) will capture at most one value of \( \nu \), which must be the integer nearest \( ax \). More precisely, we have that the sum over \( \nu \) vanishes unless \( \alpha \leq \|ax\| H^{-1} \leq \beta \). For each \( a \), there are at most \( 1 + \log_2(\beta/\alpha) \) dyadic values of \( H \) such that the sum over \( \nu \) does not vanish. Thus
\begin{equation}
\sum_{H \gtrsim \varepsilon^{-1}} \sum_{\nu \in \mathbb{Z}} \left| w_1 \left( H \left( x - \frac{\nu}{a} \right) \right) \right| \ll 1.
\end{equation}

We set \( K \approx \varepsilon H \), and by reversal of order of summation combined with (4.34), we derive
\begin{equation}
\sum_{H \gtrsim \varepsilon^{-1}} \sum_{a \leq \varepsilon H} a^{-2} \sum_{\nu \in \mathbb{Z}} \left| w_1 \left( H \left( x - \frac{\nu}{a} \right) \right) \right| \ll \sum_{a=1}^{\infty} a^{-2} \ll 1.
\end{equation}

Similarly, for \( H \ll \varepsilon^{-1} \) we use only the trivial bound \( Q(x, H) \ll 1 \), which needs to be used \( O(\varepsilon) \) times. Thus we obtain that (4.29) holds. \( \square \)

5. The Small \( \Delta \) Regime

5.1. Statement of results. One goal of this section is to collect some results which together prove Theorem 1.5. Much of our work here is also crucial in the proof of Theorem 1.2. Recall \( I(\Delta, T, x, N) \) is defined by (1.6). One of the main results of this section is

**Proposition 5.1.** Suppose \( T^\varepsilon \ll \Delta \ll T^{1/2-\varepsilon} \), for some \( \varepsilon > 0 \). Then
\begin{equation}
I(\Delta, T, x, N) \ll N|\zeta(1+2iT)|^2((\log T)^{-10} + S_\Delta),
\end{equation}
where \( S_\Delta \) has the properties described in Proposition 4.1.

Proposition 5.1 then implies Theorem 1.5 via Lemma 4.2.

It is of great interest to extend the range of \( \Delta \) in Proposition 5.1 especially as large as \( T^{25/27+\varepsilon} \), since that would prove Conjecture 1.3. The methods in this section rely on opening the divisor function \( \tau_T(n) = \sum_{ab=n} (a/b)^T \) and do not rely on the spectral theory of automorphic forms but rather abelian harmonic analysis. The relative sizes of \( a \) and \( b \) play a major role in our estimations, and so we wish to localize \( a \) and \( b \) at this early stage. To this end, define
\begin{equation}
I_{A,B} = \int_{-\infty}^{\infty} w \left( \frac{t}{\Delta} \right) \left| \sum_{a,b} \frac{e(abx)}{a^{1/2-it}b^{1/2+2iT-it}} f \left( \frac{a}{A} \right) \left( \frac{b}{B} \right) g \right|^2 \, dt,
\end{equation}
where \( A, B \gg 1 \), and \( w, f, g \) are fixed smooth compactly-supported weight functions. For small values of \( b \), it is better to not use a dyadic partition. Instead, let \( B_0 \approx T^\alpha \) for some small \( 0 < \alpha < 1/4 \), and let \( g_0 \) be a smooth function such that \( g_0(y) = 1 \) for \( y < 1 \), and \( g_0(y) = 0 \) for \( y > 2 \). Then set
\begin{equation}
I_{B_0} = \int_{-\infty}^{\infty} w \left( \frac{t}{\Delta} \right) \left| \sum_{a,b} \frac{e(abx)}{a^{1/2-it}b^{1/2+2iT-it}} g_0 \left( \frac{b}{B_0} \right) w_2 \left( \frac{ab}{N} \right) \right|^2 \, dt.
\end{equation}
The values of $b$ near $T^{1/2}$ are also somewhat exceptional, and it is helpful to employ a configuration taking this into account. To this end, define

$$B_1^- = \frac{T^{1/2}}{(\log T)^{100}}, \quad B_1^+ = T^{1/2}(\log T)^{100}.\tag{5.4}$$

We elucidate a connection between $I(\Delta, T, x, N)$, $I_{A,B}$, and $I_{B_0}$, with the following

**Lemma 5.2.** With certain $w, f, g, g_0, B_0$ as above, and with $AB \asymp N = \sqrt{\Delta T}$, we have

$$I(\Delta, T, x, N) \ll I_{B_0} + \log \log T \sum_{B_1^- \ll B \ll B_1^+} I_{A,B} + \log T \left( \sum_{B_0 \ll B \ll B_1^-} + \sum_{B_1^+ \ll B \ll N} \right) I_{A,B} + \frac{N \log T}{\Delta^{100}}.\tag{5.5}$$

**Proof of Lemma 5.2.** In the definition of $I(\Delta, T, x, N)$ (that is, (4.6)), write $\tau_{it}(n) = \sum_{ab=n} (a/b)^{it}$, and then apply a partition of unity to the $b$-sum. In this way, we have

$$I(\Delta, T, x, N) = \int_{-\infty}^\infty w_1\left(\frac{t}{\Delta}\right) \left| \sum_j \sum_{a,b} \frac{\epsilon(abx)g_j(b)}{a^{1/2-it}b^{1/2+2it^2-it}w_2\left(\frac{ab}{N}\right)} \right|^2,\tag{5.6}$$

where $g_j$ are elements of the partition. We do not want to immediately apply Cauchy’s inequality to take the sum over $j$ to the outside, because this loses a factor of $\log T$ which is fatal. However, in some ranges of the parameters, we save a large power of $\log T$ that is able to compensate for this, so we are free to lose $\log T$ to simplify the analysis. This discussion helps motivate the following decomposition.

First we choose $g_0(b/B_0)$ to be one element of the partition of unity in the $b$-sum. Next we simply apply Cauchy’s inequality to separate this part of the partition from all the other terms. This explains the presence of $I_{B_0}$ in (5.5); the main point here is we did not lose the factor $\log T$.

We may assume that $g_0$ is such that $1 - g_0(g/B_0)$ equals a dyadic partition of unity (meaning, each $g_j(b) = g(b/B)$, say with $b \asymp B$, where $AB = N$, and $B$ runs over a dyadic sequence), since we could start with a dyadic partition of unity of this type, and then write $g_0$ as a finite sum of elements of this dyadic partition. Having located $b \asymp B$ with $g(b/B)$, $a$ is automatically localized by $a \asymp A$ by the support of $w_2$. We are then free to multiply by $f(a/A)$ where $f$ is smooth of compact support, and such that $f(a/A) = 1$ for all $a$ in the support of $g(b/B)w_2(ab/N)$. In all, this shows that we may write

$$I(\Delta, T, x, N) \ll I_{B_0} + \int_{-\infty}^\infty w_1\left(\frac{t}{\Delta}\right) \left| \sum_{B_0 \ll B \ll N} \sum_{a,b} \frac{\epsilon(abx)f(a/A)g(b/B)}{a^{1/2-it}b^{1/2+2it^2-it}w_2\left(\frac{ab}{N}\right)} \right|^2.\tag{5.7}$$

Next we separate the sum over $B$ into two more blocks, those with $B_1^- \ll B \ll B_1^+$, and the complement. There are $\ll \log \log T$ dyadic values of $B$ in the first block, and $\ll \log T$ in the second block. Applying Cauchy’s inequality first to separate the two blocks, and then again within each block, shows an estimate almost of the form (5.5), the only difference being that instead of $I_{A,B}$, we have an expression of the form

$$\int_{-\infty}^\infty w_1\left(\frac{t}{\Delta}\right) \left| \sum_{a,b} \frac{\epsilon(abx)f(a/A)g(b/B)}{a^{1/2-it}b^{1/2+2it^2-it}w_2\left(\frac{ab}{N}\right)} \right|^2.\tag{5.8}$$
That is, we have a weight function of the form \( f(a/A)g(b/B)w_2(ab/N) \), and we would like to omit \( w_2 \) in order to separate the variables further. Using a method very similar to that in the proof of Lemma 4.2, we can do this, at the cost of an error term of size \( \Delta^{-100}N \log T \).

In the application to \( I(\Delta, T, x, N) \) with \( \Delta \gg T^\varepsilon \), we have \( AB = N = (\Delta T)^{1/2} \). However, for some technical reasons, when \( \Delta \ll T^\varepsilon \), we need a minor generalization of the range of \( A, B \). For this reason, in the rest of this section we let

\[
AB = M, \quad \text{where} \quad T^{1/2-\varepsilon} \ll M \ll T^{1-\varepsilon}.
\]

Before developing more intricate bounds on \( I_{A,B} \), we record a “trivial” bound:

**Lemma 5.3.** We have

\[
I_{A,B} \ll (\Delta + M) \log T.
\]

**Proof.** By the mean value theorem for Dirichlet polynomials (Lemma 2.2), we have

\[
I_{A,B} \ll \left( \frac{\Delta}{M} + 1 \right) \sum_{n \leq M} |\tau_{iT}(n; A, B)|^2,
\]

where we define

\[
\tau_{iT}(n; A, B) = \sum_{ab = n} \left( \frac{a}{b} \right)^iT f(a/A)g(b/B).
\]

If we had not localized the variables \( a \asymp A, b \asymp B \), \( \tau_{iT}(n; A, B) \) would simply be \( \tau_{iT}(n; A, B)w_2(n/N) \), and Lemma 2.4 would give a bound of the desired shape (multiplied by \( |\zeta(1+2iT)|^2 \)). Instead, we may use the trivial bound \( |\tau_{iT}(n; A, B)| \leq \tau_0(n; A, B) \), and by elementary considerations,

\[
\sum_{n \leq N} |\tau_0(n; A, B)|^2 \ll \sum_{b_1, b_2 \asymp B} \sum_{n \equiv 0 \pmod{[b_1, b_2]}} 1.
\]

To complete the proof, we calculate easily that

\[
\sum_{b_1, b_2 \asymp B} \frac{1}{[b_1, b_2]} \ll \sum_{d \leq B} d^{-1} \sum_{b_1, b_2 \asymp B/d} \frac{1}{b_1 b_2} \ll \log B. \quad \Box
\]

The rest of this section contains lemmas that show various improvements over Lemma 5.3. For ease of reference, we collect a few lemmas together, deferring the proofs until later in this section.

**Lemma 5.4.** Suppose that for some \( \varepsilon > 0 \), we have

\[
\Delta T^\varepsilon \ll B \ll T^{1-\varepsilon},
\]

and \( 1 \ll \Delta \ll T^{1-\varepsilon} \). Then there exists \( \delta > 0 \) (depending on \( \varepsilon \)) so that

\[
I_{A,B} \ll MT^{-\delta}.
\]

The dependence of \( \delta \) on \( \varepsilon \) is not explicit because it relies on iterated uses of Van der Corput’s A-Process.

**Lemma 5.5.** Suppose \( 1 \ll \Delta \ll T^{1-\varepsilon} \), and \( B \ll T^{1-\varepsilon} \). Then

\[
I_{A,B} \ll \left( \Delta + \frac{MT}{B^2} \right) \log \log T.
\]

This bound is \( O(MT^{-\delta}) \) for \( B \gg T^{1/2+\varepsilon} \) and \( \Delta \ll MT^{-\delta} \).
Lemma 5.6. Suppose $(\log T)^{100} \ll \Delta \ll T^{1-\varepsilon}$ and $B \gg T^\varepsilon$. Then for some $\delta > 0$ (depending on $\varepsilon$), we have

\begin{equation}
I_{A,B} \ll M|\zeta(1+2iT)|^2(\log T)^{-50} + \Delta \frac{B^2}{M} \log \log T + \Delta \log \log T.
\end{equation}

Finally we require a result valid for $B \ll T^\varepsilon$.

Lemma 5.7. Suppose that $B_0 = T^\alpha$, for some fixed $0 < \alpha < 1/4$, and $(\log T)^{100} \ll \Delta \ll T^{1-\varepsilon}$. Then with $S_\Delta$ satisfying the same properties as in Proposition 4.4, we have

\begin{equation}
I_{B_0} \ll M|\zeta(1+2iT)|^2(\log T)^{-50} + S_\Delta + \Delta \log T.
\end{equation}

Remark. In truth, the range of applicability of the method of proof of Lemma 5.7 is for values of $b$ at most slightly larger than $a$.

Before proceeding to the proofs of these lemmas, we describe how they combine to prove Proposition 5.1. We need to bound all the terms on the right hand side of (5.5). Lemma 5.7 treats $I_{B_0}$. We need to bound $I_{A,B}$ with an acceptable bound. Next we analyze $I_{A,B}$. In case $T^\alpha \ll B \ll T^{1/2-\varepsilon}$, we apply Lemma 5.6. In case $B \gg T^{1/2-\varepsilon}$, we may apply Lemma 5.4, since we are assuming $\Delta \ll T^{1/2-\varepsilon'}$ for some $\varepsilon' > 0$. We have therefore covered all the ranges of $B$, showing Proposition 5.1.

The astute reader may notice that we did not require Lemma 5.5. Nevertheless, it is conceptually satisfying, since when combined with Lemmas 5.6 and 5.7, it shows that we can handle all values of $B$ satisfactorily except for $B = T^{1/2}(\log T)^{O(1)}$, and hence $A = \Delta^{1/2}(\log T)^{O(1)}$. The key point that allows us to treat all values of $B$, for $T^\varepsilon \ll \Delta \ll T^{1/2-\varepsilon}$, is that Lemma 5.4 is applicable in the range $B \gg \Delta T^\varepsilon$ which overlaps the otherwise problematic region $B = T^{1/2+o(1)}$, at least for $\Delta \ll T^{1/2-\varepsilon}$. Anyway, we shall use Lemma 5.5 in the proof of (5.19).

Using these bounds, we may also conclude

Corollary 5.8. Suppose $T^\varepsilon \ll \Delta \ll T^{1-\varepsilon}$. Then

\begin{equation}
I(\Delta, T, x, N) \ll N \left( \frac{|\zeta(1+2iT)|^2}{(\log T)^{10}} + |\zeta(1+2iT)|^2 S_\Delta + (\log \log T)^2 \right).
\end{equation}

Remark. The first term in (5.20) may be dropped by comparison to the third term with $(\log \log T)^2$.

For this, we use (5.5). Lemma 5.7 treats $I_{B_0}$, as in the proof of Proposition 5.1, leading to the first two terms in (5.20). For the terms with $I_{A,B}$ we take the minimum of the bounds stated in Lemmas 5.5 and 5.6. Thus we obtain

\begin{equation}
\sum_{B_0 \ll B \ll B_1^+} I_{A,B} \ll N \frac{|\zeta(1+2iT)|^2}{(\log T)^{50}} + N \log \log T \sum_{B_0 \ll B \ll B_1^+} \min \left( \frac{\Delta B^2}{N^2}, \frac{T}{B^2} \right).
\end{equation}

Recall that $\frac{\Delta B^2}{N} = \frac{B^2}{N}$, and so the dyadic sums satisfy $\sum_{B \ll T^{1/2}} \frac{B^2}{T} \ll 1$, and $\sum_{B \gg T^{1/2}} \frac{T}{B^2} \ll 1$. Therefore, the bound on this range of $B$’s is acceptable for Corollary 5.8. For the other values of $B$ with $B \gg B_1^+$ or $B \ll B_1^-$, we lose a factor $\log T$ (the one directly appearing in (5.5)), but we gain a large power of $\log T$ back since $B \ll B_1^-$ or $B \gg B_1^+$, so the bound on these terms is already included on the right hand side of (5.20).

From Corollary 5.8 by summing over the $O(\log T)$ values of $\Delta$, we may conclude
Proof of Lemma 5.4.

This follows by combining Lemmas 5.5 and 5.6 again, and noting that \( N |\zeta(1+2iT)|^2 (\log T)^{-50} \) is negligible compared to the displayed bound. \[
5.2. \text{Proof of Lemma 5.4.} \]

By Cauchy’s inequality applied to the inner \( a \)-sum, we derive

\[
I_{A,B} \ll \int_{-\infty}^{\infty} e(t) \sum_a f(a/A) \left( \frac{b}{b_{1/2} + 2iT - iu} \right)^2 \left( \frac{1 + \log(b/b_1)}{2\pi} \right) \ dt.
\]

Then we square out and execute the integral, getting

\[
I_{A,B} \ll \Delta \sum_{b_1, b_2} \sum_a f(a/A) \frac{(b_2/b_1)^{2iT}}{(b_1 b_2)^{1/2}} \ g \left( \frac{b_1}{B} \right) \ g \left( \frac{b_2}{B} \right) \ \hat{w} \left( \frac{\Delta}{2\pi} \log(b_2/b_1) \right) \ e(ax(b_1 - b_2)).
\]

For the diagonal \( b_1 = b_2 \), there is no possible cancellation, and we get a bound of size \( O(\Delta A) \), which is \( \ll ABT^{-\epsilon} \), by \( 5.13 \).

We need to bound the off-diagonal terms also. For \( b_1 \neq b_2 \), we change variables \( b_1 = b, b_2 = b + h \) to obtain

\[
I_{A,B} \ll \Delta \sum_{b_1, b_2} \sum_a f(a/A) e(-axh) \sum_b \left( \frac{b + h}{b} \right)^{-2iT} \ W(h, b),
\]

with

\[
W(h, b) = \frac{g \left( \frac{b}{B} \right)}{(b/B)^{1/2}} \ g \left( \frac{b + h}{B} \right)^{1/2} \ \hat{w} \left( \frac{\Delta}{2\pi} \log \left( \frac{b + h}{b} \right) \right).
\]

Note that the trivial bound applied to the off-diagonal terms gives \( I_{A,B} \ll \Delta A + AB \), so we need to show there is some cancellation. It is easy to see that

\[
\frac{\partial^j}{\partial y^j} W(h, y) \ll B^{-j} \left( 1 + \frac{|h|}{B} \right)^{-C}.
\]

for arbitrary \( C > 0 \). More importantly, the phase function \( \phi(y) = -2T \log((y + h)/y) \) satisfies

\[
\left| \frac{d^j}{dy^j} \phi(y) \right| \leqslant B^{-j} F, \quad \text{where} \quad F = \frac{T|h|}{B}.
\]

Thus by [GK, Theorem 2.9] (alternatively, see [IK, Theorem 8.4] for a bound with more stringent hypotheses on \( F \)) we have that

\[
\sum_b \left( \frac{b + h}{b} \right)^{-2iT} \ W(h, b) \ll_q B \left( \frac{F}{B^{q+2}} \right)^{\frac{1}{q-2}} + \frac{B}{F},
\]

where \( q \in \{0, 1, 2, \ldots \} \) is at our disposal, and \( Q = 2^q \). Therefore, as long as \( T^\eta \ll F \ll B^k \) for some \( \eta > 0 \) and \( k \in \mathbb{N} \), we may claim a power saving in the \( b \)-sum (this is initially
given as a power saving in \(B\), but since \(B \gg T^\varepsilon\), it is a power saving in \(T\) also. Note that \(F \geq \frac{T}{B} \gg T^\varepsilon\), so the lower bound on \(F\) is satisfied, and also we may assume \(|h| \ll B T^\varepsilon\), by (5.28), in which case \(F \ll \frac{T}{2} T^\varepsilon \ll T^{1+\varepsilon}\). In particular, we have \(F \ll B^k\) for some \(k\) depending on \(\varepsilon > 0\). Therefore, we may claim the bound (5.16).

### 5.3. Proof of Lemma 5.5

We may certainly assume \(B \gg T^{1/2} (\log T)^{-50}\), since otherwise the bound is trivial. The basic idea is that Poisson summation in \(b\) inside the absolute square gives a savings by reducing the length of summation (provided \(B\) is slightly larger than \(T^{1/2}\)). Furthermore, this process is compatible with the cancellation from the \(t\)-integration.

We have, with \(G(u) = u^{-1/2} g(u)\), that

\[
\sum_b \frac{e(abx)}{b^{1/2+2iT-it}} g\left(\frac{b}{B}\right) = B^{-1/2} \sum_{l \in \mathbb{Z}} \int_{-\infty}^{\infty} e(axu + lu - \frac{(2T-t)}{2\pi} \log u) G(u/B) du.
\]

Write the phase here as \(\phi(u) = (ax+l)u - \frac{2T-t}{2\pi} \log u\), which satisfies

\[
\phi'(u) = (ax+l) - \frac{2T-t}{2\pi u}.
\]

We first claim that the integral is small unless there is a stationary point. The second term in the derivative of the phase is \(\asymp \frac{T}{B}\), and repeated integration by parts (see [BKY, Lemma 8.1]) shows that the integral is small (e.g., \(O(T^{-100})\)) unless \((ax+l) \asymp \frac{T}{B}\). By periodicity, we may as well assume \(|x| \leq 1/2\), in which case \(a|x| \leq a \ll \frac{M}{B} = o(T/B)\) (recall (5.3)), and therefore this means \(l \asymp \frac{T}{B}\). We think of \(l+ax\) as a perturbation of \(l\). The stationary point occurs at \(2\pi u_0 = \frac{2T-t}{l+ax}\). By [BKY, Proposition 8.2], we get an asymptotic expansion for the \(b\)-sum in the form

\[
\sum_b \frac{e(abx)}{b^{1/2+2iT-it}} g\left(\frac{b}{B}\right) = \sum_l \frac{e^{i\varphi(t,T)}}{(l+ax)^{1/2-2iT-it}} P\left(\frac{2T-t}{B(l+ax)}\right) + O(T^{-100}),
\]

where \(P\) is a function supported on \(v \asymp 1\), satisfying \(P^{(j)}(v) \ll_j 1\). The phase here takes the form

\[
e^{i\varphi(t,T)}
\]

for some function \(\varphi\) independent of \(a\) and \(l\). Altogether, this shows

\[
I_{A,B} = \int_{-\infty}^{\infty} w\left(\frac{t}{\Delta}\right) \left| \sum_{a,l} P\left(\frac{2T-t}{B(l+ax)}\right) f\left(\frac{a}{A}\right) \right|^2 dt + O(T^{-50}).
\]

Since \(\frac{|w|}{T} \ll \frac{A}{T} \ll T^{-\varepsilon}\), and \(\frac{d|w|}{T} \ll \frac{A}{T^2} = \frac{M}{T} \ll T^{-\varepsilon}\), we can take Taylor expansions to obtain

\[
P\left(\frac{2T-t}{B(l+ax)}\right) = P\left(\frac{2T}{Bl}\right) + \sum_{j,k} P_{j,k}\left(\frac{2T}{Bl}\right) \left(\frac{t}{T}\right)^j \left(\frac{ax}{l}\right)^k + O(T^{-100}),
\]

where \(P_{j,k}\) is a finite linear combination of derivatives of \(P\), with \(j+k \geq 1\), and the total number of terms in the sum bounded by a function of \(\varepsilon\). By a use of Cauchy’s inequality,
we then have

\[ I_{A,B} \ll \int_{-\infty}^{\infty} w\left(\frac{t}{\Delta}\right) \left| \sum_{a,l} P\left(\frac{2T}{B}\right) f\left(\frac{a}{\Delta}\right) a^{1/2-it} (l + ax)^{1/2-2iT+it} \right|^2 dt + O(T^{-50}), \]

where by abuse of notation, \( P \) is meant to represent \( P \) or one of the \( P_{j,k} \) (of course the case of \( P \) is the hardest for us to estimate, as the terms with \( j + k \geq 1 \) will be smaller than this by a factor \( \ll T^{-\epsilon} \), so the notation is not that abusive after all).

Change variables \( a \to da \) and \( l \to dl \) where now \( (a,l) = 1 \). Define the coefficients

\[ \gamma_{a,l} = \sum_{d=1}^{\infty} P\left(\frac{2T}{B}\right) f\left(\frac{ad}{\Delta}\right) d^{1-2iT}, \]

so that

\[ I_{A,B} \ll \int_{-\infty}^{\infty} w\left(\frac{t}{\Delta}\right) \left| \sum_{(a,l)=1} \gamma_{a,l} a^{1/2-it} (l + ax)^{1/2-2iT+it} \right|^2 dt + O(T^{-50}). \]

Note that \( \gamma_{a,l} = 0 \) unless \( a \asymp \frac{MT}{T} \). Of course we have the trivial bound \( |\gamma_{a,l}| \ll 1 \), but in many ranges this can be improved:

\[ |\gamma_{a,l}| \ll (\log T)^{-50}, \quad \text{if} \quad l \ll \frac{T}{B} (\log T)^{-2/3-\epsilon}. \]

This follows from the Vinogradov-Korobov bound (Lemma 2.1). Consequently, we have

\[ \sum_{a,l} \frac{|\gamma_{a,l}|^2}{al} \ll \log \log T. \]

Similarly, if we define \( \gamma_X = \max_{t \geq X} |\gamma_{a,l}| \) (with no restriction on \( a \)), then \( \sum_X \gamma_X \ll \log \log T \).

Now we continue with the estimation of \( I_{A,B} \). By opening the square and performing the \( t \)-integral, we obtain with a trivial bound that

\[ I_{A,B} \ll \Delta \sum_{(a_1,l_1)=1} \frac{|\gamma_{a_1,l_1} \gamma_{a_2,l_2}|}{(a_1a_2l_1l_2)^{1/2}} \left| \hat{w}\left(\frac{\Delta}{2\pi} \sum_{a,l} \frac{a_1(l_2 + ax)}{a_2(l_1 + ax)} \right) \right| + O(T^{-50}). \]

Here we used that \( l + ax \asymp l \). Note that \( a_1(l_2 + ax) - a_2(l_1 + ax) = a_1l_2 - a_2l_1 \). Thus we obtain

\[ I_{A,B} \ll \Delta \sum_{(a_1,l_1)=1} \frac{|\gamma_{a_1,l_1} \gamma_{a_2,l_2}|}{(a_1a_2l_1l_2)^{1/2}} \left( 1 + \frac{\Delta |a_1l_2 - a_2l_1|}{a_2l_1} \right)^{-100} + O(T^{-50}). \]

The diagonal contribution, that is, the solutions with \( a_1l_2 = a_2l_1 \), in fact have \( a_1 = a_2 \) and \( l_1 = l_2 \), and so contribute at most

\[ \ll \Delta \sum_{(a,l)=1} \frac{|\gamma_{a,l}|^2}{al} \ll \Delta \log \log T, \]

by (5.41). Next we estimate the off-diagonal contributions. For simplicity, let us localize the variables dyadically, say \( l_1 \asymp X_1 \) and \( l_2 \asymp X_2 \), where now \( X_1 \) and \( X_2 \) range over a dyadic set of numbers \( \ll T/B \). To ease the exposition, let us also only count the solutions with \( |a_1l_2 - a_2l_1| \ll \frac{2B}{\Delta} \); the larger values do not lead to a larger bound, because of the
rapid decay of the weight function. Observe that \( \frac{a_1 - a_2}{l_1} \geq \frac{1}{l_2} \simeq (X_1X_2)^{-1} \). Therefore, given coprime \( a_1, l_1 \) with \( l_1 \asymp X_1 \), the number of coprime \( a_2, l_2 \) with \( l_2 \asymp X_2 \), and satisfying \( \frac{|a_1 - a_2|}{l_1 l_2} \ll \frac{M}{l_3} \), is at most \( X_1X_2 \frac{M}{l_3} \) (since their spacing is at least \( (X_1X_2)^{-1} \)). Therefore, the contribution of the off-diagonal terms is

\[
(5.45) \quad \ll \sum_{X_1X_2 \text{ dyadic}} \frac{\Delta \gamma_{X_1} \gamma_{X_2}}{T} \frac{T X_1^2}{M X_2} X_1 X_2 = \sum_{X_1X_2 \text{ dyadic}} \gamma_{X_1} \gamma_{X_2} \frac{T X_1^2}{M}.
\]

We have \( \sum_{X_1 \text{ dyadic}} \gamma_{X_1} X_1^2 \ll (T/B)^2 \), and \( \sum_{X_2 \text{ dyadic}} \gamma_{X_2} \ll \log \log T \). In all, the off-diagonal terms contribute to \( I_{A,B} \) at most

\[
(5.46) \quad \ll \frac{TM}{B^2} \log \log T.
\]

Combining (5.44) and (5.46) completes the proof.

5.4. Proof of Lemma 5.6. The initial steps of the proof are quite similar to that of Lemma 5.5 except that we apply Poisson summation in \( a \) instead of \( b \). The later steps are different because we encounter a much more difficult counting argument in this problem. We may assume \( B \ll \Delta^{-1/2} M (\log T)^{100} \), equivalently, \( A \gg \Delta^{1/2} (\log T)^{-100} \), since otherwise the bound is trivial. We have

\[
(5.47) \quad \sum_{a=1}^{\infty} \frac{e(abx)}{a^{1/2-it}} f(a/A) = \sum_{q \in \mathbb{Z}} \int_{-\infty}^{\infty} e((bx-q)u + \frac{t}{2\pi} \log u) u^{-1/2} f(u/A) du.
\]

By [BKY] Lemma 8.1, the integral is \( O(A \Delta^{-100}) \) unless \( |bx-q| \simeq \frac{\Delta}{A} \), that is,

\[
(5.48) \quad \left| x - \frac{q}{b} \right| \simeq \frac{\Delta}{M}.
\]

Assuming (5.48) holds, the integral can be asymptotically evaluated using stationary phase. The conditions of [BKY] Proposition 8.2 are not quite met if \( \Delta \) is small compared to \( A \). Instead, we may quote [Hu] Lemma 5.5.6 (the reader may also try to derive a weighted version of [IK] Corollary 8.15). In this way, we have

\[
(5.49) \quad \sum_{a=1}^{\infty} \frac{e(abx)}{a^{1/2-it}} f(a/A) = \frac{(t/e)^i}{ \Delta^{1/2}} \sum_{q \in \mathbb{Z}} P\left(\frac{t}{A(q-bx)}\right) + O\left(\frac{A^{1/2}}{\Delta^{1/2}}\right),
\]

where as in the proof of Lemma 5.5, \( P(v) \) is a smooth function supported on \( v \simeq 1 \), satisfying \( P^{(j)}(v) \ll 1 \) (in fact, \( P \) here is a constant multiple of \( f \); obtaining a stronger error term requires altering the weight function slightly, but the quoted error term is strong enough for us here). By a use of Cauchy-Schwarz to separate the main term from the error term, and trivially integrating over \( t \) in the error term, we derive

\[
(5.50) \quad I_{A,B} \ll \int_{-\infty}^{\infty} w\left(\frac{t}{\Delta}\right) \left| \sum_{b,q} \frac{g(b/B) P\left(\frac{t}{A(q-bx)}\right)}{b^{1/2+2it} - it(q-bx)^{1/2+it}} \right|^2 dt + O(AB\Delta^{-2}).
\]

The error term is satisfactory.

By analogy with the mean value theorem for Dirichlet polynomials, one might expect that \( I_{A,B} \ll (\Delta + \frac{B\Delta}{T})^T \), and note \( \frac{B\Delta}{T} = \Delta^2 = M^2 \Delta^2 \) (and if \( M = N = 0 \), this is \( N^2 B^2 \)). However, there are some differences so this guess is not entirely reliable, yet partially hints at the truth.
Next we perform some simplifications, towards separation of variables. Write \((b, q) = d\), and set \(b = db_0\), \(q = dq_0\) where now \((q_0, b_0) = 1\). In the summation, we enforce the condition \((5.48)\) (of course \(q/b = q_0/b_0\)), which is redundant to the support of \(P\). We then obtain \((5.51)\)

\[
I_{A,B} \ll \int_{-\infty}^{\infty} w\left(\frac{t}{\Delta}\right) \left| \sum_d \frac{1}{d^{1+2iT}} \sum_{\substack{(b_0, q_0) = 1 \\ |x - \frac{q_0}{b_0}| \leq M^{-1}\Delta}} \frac{g(db_0/B)P\left(\frac{t}{Ad(q_0 - b_0 x)}\right)}{b_0^{1/2 + 2iT - it}(q_0 - b_0 x)^{1/2 + it}} \right|^2 dt + O(AB\Delta^{-2}).
\]

By the same separation of variables method as in the proof of Lemma \(4.2\) we can remove the weight function \(P\) by a Mellin transform. Let us say that the integration variable attached to \(P\) is \(v\). Changing variables \(t \rightarrow t - v\), and letting \(2T' = 2T + v\), we obtain, for some smooth compactly-supported function \(w_3\), \((5.52)\)

\[
I_{A,B} \ll \max_{|T' - T| \ll \Delta^\varepsilon} \int_{-\infty}^{\infty} w_3\left(\frac{t}{\Delta}\right) \left| \sum_d \frac{1}{d^{1+2iT'}} \sum_{\substack{(b_0, q_0) = 1 \\ |x - \frac{q_0}{b_0}| \leq M^{-1}\Delta}} \frac{g(db_0/B)}{b_0^{1/2 + 2iT' - it}(q_0 - b_0 x)^{1/2 + it}} \right|^2 dt + \frac{M}{\Delta^2}.
\]

Define the following coefficients \(\alpha_b\) by

\[
(5.53) \quad \alpha_b = b^{-2iT'} \sum_d \frac{g(db/B)}{d^{1+2iT'}},
\]

so with this notation

\[
(5.54) \quad I_{A,B} \ll \max_{|T' - T| \ll \Delta^\varepsilon} \int_{-\infty}^{\infty} w_3\left(\frac{t}{\Delta}\right) \left| \sum_{\substack{(b_0, q_0) = 1 \\ |x - \frac{q_0}{b_0}| \leq M^{-1}\Delta}} \frac{\alpha_{b_0}}{b_0^{1/2 - it}(q_0 - b_0 x)^{1/2 + it}} \right|^2 dt + \frac{M}{\Delta^2}.
\]

We claim the following bounds on \(\alpha_b\):

\[
(5.55) \quad \alpha_b \ll \begin{cases} (\log T)^{-100} & \text{ for } b \ll \frac{B}{(\log T)^{2/3 + \varepsilon}} \\ T^{-\delta} & \text{ for } b \ll B^{1/2}. \end{cases}
\]

Here \(\delta > 0\) depends on the size of \(\varepsilon > 0\), where \(B \gg T^\varepsilon\). The first bound follows by using a Mellin transform, and the Vinogradov-Korobov bound (Lemma \(2.1\)). The latter bound comes from \([\text{GK}, \text{Theorem 2.9}]\). Of course, we also have the trivial bound \(\alpha_b \ll 1\) which is the best we may claim for \(b \gg B(\log T)^{-2/3}\). One easy deduction from \((5.55)\) is that

\[
(5.56) \quad \sum_{b \in B} \frac{|\alpha_b|}{b} \ll \log \log X,
\]

coming from the range where we only may claim the trivial bound on \(\alpha_b\).
Next we open the square and perform the integration. For notational simplicity, we drop
the max over $T'$ by assuming that $T'$ is chosen to maximize the expression. Then we obtain
\begin{equation}
I_{A,B} \ll \Delta \sum_{(b_1,q_1)=1}^{\alpha_{b_1}} \frac{\alpha_{b_1}}{(b_1(q_1 - b_1)x))^{1/2}} \sum_{(b_2,q_2)=1}^{\alpha_{b_2}} \frac{\alpha_{b_2}}{(b_2(q_2 - b_2)x)^{1/2}} \tilde{w}_3 \left( \frac{\Delta}{2\pi} \log \left( \frac{b_2(q_1 - b_1)x}{b_1(q_2 - b_2)x} \right) \right),
\end{equation}
plus $O(M\Delta^{-2})$. We give up any cancellation, and bound everything with absolute values.
It is easy to see that under the assumption \((5.48)\), we have
\begin{equation}
\tilde{w}_3 \left( \frac{\Delta}{2\pi} \log \left( \frac{b_2(q_1 - b_1)x}{b_1(q_2 - b_2)x} \right) \right) \ll \left( 1 + M \left| \frac{q_1}{b_1} - \frac{q_2}{b_2} \right| \right)^{-10}.
\end{equation}
Therefore, we have
\begin{equation}
I_{A,B} \ll M \sum_{(b_1,q_1)=1}^{\alpha_{b_1}} \sum_{(b_2,q_2)=1}^{\alpha_{b_2}} \frac{|\alpha_{b_1}\alpha_{b_2}|}{b_1b_2} \left( 1 + M \left| \frac{q_1}{b_1} - \frac{q_2}{b_2} \right| \right)^{-10} + M\Delta^{-2}.
\end{equation}
Consider the diagonal contribution to \((5.59)\), that is, the terms with $q_1 = q_2$ and $b_1 = b_2$.
These terms contribute to $I_{A,B}$ an amount that is
\begin{equation}
\ll M \sum_{b \ll B} \frac{|\alpha_b|^2}{b^2} \left( 1 + \frac{\Delta b}{M} \right).
\end{equation}
Using \((5.56)\), the part with $\frac{\Delta b}{M}$ contributes $O(\Delta \log \log T)$. On the other hand, with the
trivial bound $\alpha_b \ll 1$ for $b \gg B^{1/2}$, and with $\alpha_b \ll T^{-\delta}$ for $b \ll B^{1/2}$, we obtain that
$M \sum_{b \ll B} b^{-2} |\alpha_b|^2 = O(MT^{-\delta})$, for some $\delta > 0$. Thus the diagonal terms are bounded in
accordance with Lemma \((5.6)\).

Now consider the off-diagonal terms where $q_1/b_1 \neq q_2/b_2$. Suppose that $|\frac{q_1}{b_1} - \frac{q_2}{b_2}| \ll M^{-1}$
(the more general case where the difference is $\propto CM^{-1}$ for $C \gg 1$ is similar, and the
decay of $\tilde{w}_3$ more than compensates for the extra terms, so we omit the proof for this case).
Further restrict to say $b_1 \approx X_1$, and $b_2 \approx X_2$, where the $X_i$ range over dyadic segments,
with $1 \ll X_i \ll B$. We claim the number of solutions in $(q_1,b_1)$ to \((5.48)\) with $b_1 \sim X_1$
is $\ll 1 + \frac{\Delta X_1^2}{N}$. To see this, note that the term “1” accounts for a potential solution (this
certainly cannot be improved since given $q,b$ there exist $x$ satisfying \((5.48)\)). Once such a
solution is found, we can estimate the number of others by noting that the spacing between
two reduced fractions of denominator of size $X_1$ is $\gg X_1^{-2}$, so the total number of such
fractions that can be packed into an interval of length $L$ is bounded from above by $1 + LX_1^2$.
Similarly, for each choice of $(q_1,b_1)$, the number of $(q_2,b_2)$ with $0 < |\frac{q_2}{b_2} - \frac{q_1}{b_1}| \ll M^{-1}$
is $\ll \frac{X_1X_2}{M}$. Let $\alpha_{X_i} = \max_{b \ll X_i} |\alpha_b|$. Similarly to \((5.56)\), the bounds \((5.55)\)
imply that $\sum_{1 \ll X = 2^i \ll B} \alpha_X \ll \log \log T$. Thus the bound on the off-diagonal terms is
\begin{equation}
\ll \sum_{X_1,X_2} M \left( 1 + \frac{\Delta X_1^2}{M} \right) \frac{\alpha_{X_1}\alpha_{X_2}X_1X_2}{X_1X_2} \ll \left( \log \log T + \frac{\Delta B^2}{M} \right) \log \log T,
\end{equation}
which completes the proof.
5.5. **Proof of Lemma 5.7**. When \( b \) is very small (and so \( a \) is very large) then we treat the problem in a similar way to the early steps of [DFI2]. We go back to the original definition (5.3), open the square, and execute the integral, obtaining

\[
I_{b_0} = \Delta \sum_{a_1,a_2,b_1,b_2} w_2\left(\frac{a_1b_1}{N}\right) w_2\left(\frac{a_2b_2}{N}\right) g_0\left(\frac{b_1}{B_0}\right) g_0\left(\frac{b_2}{B_0}\right) e\left((a_1b_1 - a_2b_2)x\right) \frac{\Delta}{2\pi} \log \frac{a_1b_1}{a_2b_2}.
\]

Our plan is to solve the equation \( a_1b_1 - a_2b_2 = h \neq 0 \) via Poisson summation in \( a_2 \) (mod \( b_1 \)). The contribution from \( h = 0 \) is \( O(\Delta \log T) \), by a trivial bound. First we extract the greatest common divisor \( d = (b_1, b_2) \) which necessarily divides \( h \). In this way we obtain

\[
\Delta \sum_{h \neq 0} e(hx) \sum_{d|h} \sum_{(b_1, b_2) = 1} g_0\left(\frac{db_1}{B_0}\right) g_0\left(\frac{db_2}{B_0}\right) R(b_1, b_2, h),
\]

where

\[
R(b_1, b_2, h) = \sum_{a_1b_1 - a_2b_2 = \frac{h}{d}} w_2\left(\frac{a_1b_1}{M}\right) w_2\left(\frac{a_2b_2}{M}\right) \frac{\Delta}{2\pi} \log \left(1 + \frac{h/d}{a_2b_2}\right).
\]

We next re-interpret the equation \( a_1b_1 - a_2b_2 = \frac{h}{d} \) as a congruence \( a_2 \equiv -\frac{h}{d} \) (mod \( b_1 \)), and replace every occurrence of \( a_1 \) with \( \frac{h+a_2b_2}{b_1} \). Therefore by Poisson summation in \( a_2 \) (mod \( b_1 \)), we have

\[
R(b_1, b_2, h) = \frac{1}{b_1} \sum_{l \in \mathbb{Z}} e\left(\frac{lb_2(h/d)}{b_1}\right)
\]

\[
\int_{0}^{\infty} w_2\left(\frac{h+lrb_2}{M}\right) w_2\left(\frac{drb_2}{M}\right) \frac{\Delta}{2\pi} \log \left(1 + \frac{h/d}{rb_2}\right) e\left(-\frac{l}{b_1}\right) dr.
\]

Here the integral is of the form \( \tilde{q}(y) = \int q(r)e(-ry)dr \) where \( q \) is a function supported on \( r \approx \frac{M}{db_2} \ll \frac{M}{b_1} \), satisfying \( q^{(j)}(r) \ll \left(\frac{M}{db_2}\right)^{-j} \), and with \( y = \frac{l}{b_1} \). Thus \( \tilde{q}(y) \ll_j \left(\frac{M}{db_2}\right)^{-j} \frac{M}{db_2} \). Now we have, with \( y = l/b_1 \), that

\[
\frac{|y|M}{db_2} = \frac{|l|M}{db_1b_2} \gg |l|\frac{Md}{B_0^2} \gg |l|MT^{-2\alpha},
\]

and since we assume \( M \gg T^{1/2-\varepsilon} \), and \( \alpha < 1/4 \), if \( l \neq 0 \), then \( \tilde{q}(l/b_1) \) is very small. For this reason, we only need to consider \( l = 0 \).

Remark. Although there is a possibility for further analysis of the terms with \( l \neq 0 \), I have so far been unable to improve on Proposition 4.1 using this approach. The reason is that Lemmas 5.5 and 5.6 already indicate that the important ranges are \( A = \Delta^{1/2+o(1)} \), and \( B = T^{1/2+o(1)} \). The results in the literature on this problem ([DFI1] and [BC]) are nontrivial if \( A \) is not much smaller than \( B \). This means that \( \Delta \) needs to be \( \gg T^{1-\delta} \) for some small \( \delta > 0 \) (the recent work of Bettin and Chandee [BC] perhaps allows any \( \delta < 1/14 \)). Meanwhile, in Section 4 we have given strong estimates in the range \( \Delta \gg T^{25/27+\varepsilon} \) using spectral methods.
Now we return to the proof. The terms with \( h \neq 0 \) contribute to \( I_{B_0} \) an amount

\[
\Delta \sum_{h \neq 0} e(hx) \sum_{d|h} \sum_{(b_1, b_2) = 1} \frac{g_0 \left( \frac{db_1}{B_0} \right) g_0 \left( \frac{db_2}{B_0} \right)}{(db_1)^{1/2 + 2iT} (db_2)^{1/2 - 2iT}} w_2 \left( \frac{h + db_2 x}{M} \right) w_2 \left( \frac{dr b_2}{M} \right) \tilde{w}_1 \left( -\frac{\Delta}{2\pi} \log \left( 1 + \frac{h/d}{rb_2} \right) \right) dr,
\]

plus a small error of size \( O(T^{-100}) \). By changing variables \( r \to \frac{r}{db_2} \), we obtain that (5.67) equals

\[
\Delta \sum_{h \neq 0} e(hx) \sum_{d|h} \sum_{(b_1, b_2) = 1} \frac{g_0 \left( \frac{db_1}{B_0} \right) g_0 \left( \frac{db_2}{B_0} \right)}{(b_1)^{1/2 + 2iT} (b_2)^{1/2 - 2iT}} \int_0^\infty \frac{w_2 \left( \frac{r}{M} \right) w_2 \left( \frac{r + h}{M} \right)}{(r(r + h))^{1/2}} \tilde{w}_1 \left( -\frac{\Delta}{2\pi} \log \left( 1 + \frac{h}{r} \right) \right) dr.
\]

At this point, the integral is independent of \( b_1 \) and \( b_2 \). Note that if we restrict attention to \( d \geq D > 1 \), then by a trivial bound, we obtain

\[
\Delta \sum_{d \geq D} \frac{1}{d \log^2 B_0} \ll \frac{M}{D} \log^2 T.
\]

We set \( D = B_0^{1/2} \), so that this error term is satisfactory.

Next we claim that if \( d \ll D = B_0^{1/2} \), then

\[
\sum_{(b_1, b_2) = 1} \frac{g_0 \left( \frac{db_1}{B_0} \right) g_0 \left( \frac{db_2}{B_0} \right)}{(b_1)^{1/2 + 2iT} (b_2)^{1/2 - 2iT}} = \frac{\zeta(1 + 2iT)^2}{\zeta(2)} + O_c((\log T)^{-c}),
\]

where \( c > 0 \) is arbitrarily large. For this, we simply apply a double Mellin transform, obtaining that the left hand side of (5.70) equals

\[
\left( \frac{1}{2\pi i} \right)^2 \int_{(1)} \int_{(1)} \frac{B_0}{d} \zeta(s_1 + s_2) \tilde{g}_0(s_1) \tilde{g}_0(s_2) \frac{(1 + 2iT + s_1)\zeta(1 - 2iT + s_2)}{\zeta(2 + s_1 + s_2)} ds_1 ds_2.
\]

Integrating by parts once, we have that for \( \text{Re}(s) > 0 \),

\[
\tilde{g}_0(s) = -\frac{1}{s} \int_0^{\infty} g'(x)x^s dx.
\]

Since \( g_0 \) is identically 1 for \( 0 \leq x < 1 \), \( \tilde{g}_0 \) has a pole at \( s = 0 \) (and nowhere else). Furthermore, \( \text{Res}_{s=0} \tilde{g}_0(s) = -\int_0^{\infty} g'(x) dx = 1 \). We move the \( s_1 \) integral to \( \sigma_1 = -\frac{\alpha}{(\log T)^{2/3}} \), some \( \alpha > 0 \), followed by the same process for the \( s_1 \) integral. By the rapid decay of \( \tilde{g}_0 \), the residues at \( s_1 = -2iT \) and \( s_2 = 2iT \) are very small. Using Lemma 2.1 (Vinogradov-Korobov), we obtain that the left hand side of (5.70) equals

\[
\frac{|\zeta(1 + 2iT)|^2}{\zeta(2)} + O((\log T)^{4/3}) \exp \left( -\beta \frac{\log B_0}{(\log T)^{2/3}} \right),
\]

for some constant \( \beta > 0 \). Since we assume \( B_0 \gg T^\epsilon \), this error term is \( O((\log T)^{-c}) \) for \( c > 0 \) arbitrary.
Applying (5.70) to (5.68) and extending the $d$-sum back to all the divisors of $h$ (without making a new error term), we obtain for arbitrary $c > 0$

\[(5.74)\]

\[I_B = \Delta \frac{|\zeta(1+2iT)|^2}{\zeta(2)} \int_0^\infty \sum_{h \neq 0} \sigma_1(h)e(hx) \frac{w_2\left(\frac{r}{M}\right) w_2\left(\frac{x+h}{M}\right)}{(r(r+h))^{1/2}} \tilde{w_1}\left(-\frac{\Delta}{2\pi} \log \left(1 + \frac{h}{r}\right)\right) dr + O_c\left(\frac{N}{(\log T)^c}\right).\]

At this point we can use Taylor expansions, (essentially we think of $\frac{h}{M} \ll \frac{1}{\Delta}$, even though this is not literally true), giving now

\[(5.75)\]

\[I_B = \Delta \frac{|\zeta(1+2iT)|^2}{\zeta(2)} \int_0^\infty \frac{w_2(r)^2}{r} \sum_{h \neq 0} \sigma_1(h)e(hx) \tilde{w_1}\left(-\frac{\Delta h}{2\pi Mr}\right) dr + O\left(\frac{M|\zeta(1+2iT)|^2}{\min(\Delta, (\log T)^{50})}\right).\]

By comparison to the expression for $MT_{OD}$ given by (4.26), we have shown that these terms equal $M \cdot MT_{OD}$ plus a satisfactory error term. Since we already showed (4.24), the proof here is now complete.

6. THE SMALLEST $\Delta$ REGIME

In this section we show Proposition 1.6, that is,

\[(6.1)\]

\[\int_{|t-T| \leq T^{1/100}} |F(it)|^2 dt \ll (\log T)^\varepsilon.\]

The previous methods do not cover this range because Stirling’s formula is no longer applicable. Instead, we go back to the original definition (2.7). Obviously,

\[(6.2)\]

\[\int_{-\infty}^\infty |\tilde{\psi}(1/2 + it \pm iT)|^2 dt \ll 1,\]

so that by a use of Cauchy-Schwarz, we have

\[(6.3)\]

\[\int_{|t-T| \leq T^{1/100}} |F(it)|^2 dt \ll 1 + \frac{1}{|\zeta(1+2iT)|^2} \sum_{\Delta \ll T^{1/100}} K(\Delta, T, x),\]

where

\[(6.4)\]

\[K(\Delta, T, x) = \int_{|t-T| \geq \Delta} \left| \sum_{n \geq 1} \frac{\tau_i(n)e(nx)}{n^{1/2+it}} k(n, t, T) \right|^2 dt,\]

and where

\[(6.5)\]

\[k(n, t, T) = \frac{1}{2\pi i} \int_{(\sigma)} \tilde{\psi}(-u)n^{-u} \cosh(\pi T/2) \gamma_{V_2}(1/2 + it + u) du, \quad \sigma > -\frac{1}{2}.\]

When $\Delta \approx 1$, we need to re-interpret $|t-T| \approx \Delta$ as $|t-T| \ll 1$ (we do not make new notation for this range since we will shortly extend this range to $|t-T| \ll (\log T)^{100}$ in (6.8)).

Next we split the $n$-sum into three ranges: $n \ll N_0^-$, $N_0^- \ll n \ll N_0^+$, and $n \gg N_0^+$, where

\[(6.6)\]

\[N_0^- = \frac{(\Delta T)^{1/2}}{(\log T)^2}, \quad N_0^+ = (\Delta T)^{1/2}(\log T)^\varepsilon.\]
Let us call these three intervals $\mathcal{N}_i$, $i = 1, 2, 3$, respectively. To simplify the analysis, we suppose that these three ranges are detected by a smooth partition of unity, say $\gamma_1 + \gamma_2 + \gamma_3 = 1$, with $\gamma_i$ respective to $\mathcal{N}_i$.

Our goal is to relate $K(\Delta, T, x)$ to expressions to which we may apply the results of Section 6.

To this end, we have

**Lemma 6.1.** Suppose that $\Delta \gg (\log T)^{100}$. Then

$$K(\Delta, T, x) \ll \int_{|t-T| < \Delta} (\Delta T)^{-\frac{1}{2}} \left| \sum_{N_0^- < n < N_0^+} \frac{\tau_{i_*,}(n)e(nx)\gamma_2(n)}{n^{1/2+it}} \right|^2 dt + \frac{|\zeta(1+2iT)|^2}{(\log T)^2}. \quad (6.7)$$

If $\Delta \ll (\log T)^{100}$, then

$$K(\Delta, T, x) \ll \epsilon \int_{|t-T| < (\log T)^{100}} T^{-\frac{1}{2}} \left| \sum_{N_0^- < n < N_0^+} \frac{\tau_{i_*,}(n)e(nx)\gamma_2(n)}{n^{1/2+it}} \right|^2 dt + \frac{|\zeta(1+2iT)|^2}{(\log T)^2}. \quad (6.8)$$

The error terms here contribute $\ll (\log T)^{-1}$ to (6.1), after summing over the $O(\log T)$ values of $\Delta$ in (6.3), which is satisfactory.

**Proof.** By Cauchy's inequality, it suffices to bound the three ranges $\mathcal{N}_i$ separately. For $n \ll N_0^-$ (that is, $i = 1$), we take $\sigma = -\frac{1}{2}$, and for $n \gg N_0^+$ (i.e., $i = 3$), we take $\sigma = 1/\epsilon^2$ (large). In each $n$-range, we then move the $u$-integral outside and apply Cauchy-Schwarz in the following form (similarly to (4.9) above):

$$\int_t \left| \left( \sum_{n \in \mathcal{N}_i} \frac{\tau_{i_*,}(n)e(nx)\gamma_i(n)}{n^{1/2+it}} \right) \frac{1}{2\pi i} \int_{(\sigma)} \tilde{\psi}(-u)n^{-u} \cosh(\pi T/2)\gamma_{i_*,}(1/2+it+u) du \right|^2 dt$$

$$\ll \int_t \left[ \left( \int_v |\tilde{\psi}(-v)| \cosh(\pi T)\gamma_{i_*,}(1/2+it+v) \right|^2 dv \right] \left( \int_u |\tilde{\psi}(-u)| \left| \sum_{n \in \mathcal{N}_i} \frac{\tau_{i_*,}(n)e(nx)\gamma_i(n)}{n^{1/2+it+u}} \right|^2 du \right] dt. \quad (6.9)$$

We next argue that

$$\int_v |\tilde{\psi}(-v)| \cosh(\pi T)\gamma_{i_*,}(1/2+it+v) \right|^2 dv \ll (1 + |t-T|)^{\sigma - \frac{1}{4}}(1 + |t+T|)^{\sigma - \frac{1}{4}}. \quad (6.10)$$

By Stirling, the left hand side above is bounded by

$$\int_{-\infty}^{\infty} |\tilde{\psi}(-\sigma - iy)|(1 + |t+T+y|)^{\sigma - \frac{1}{4}}(1 + |t-T+y|)^{\sigma - \frac{1}{4}} dy. \quad (6.11)$$

The main range of the integral is $|y| \leq 1 + \frac{1}{2}|t-T|$, which by a trivial bound leads to the claimed bound. If $|y| \geq 1 + \frac{1}{2}|t-T|$, then the rapid decay of $\tilde{\psi}$ can be used to show that this part of the integral is negligible by comparison. Hence

$$K(\Delta, T, x) \ll \sum_{i=1}^{3} \int_{|t-T| < \Delta} (\Delta T)^{\sigma - \frac{1}{4}} \int_{-\infty}^{\infty} |\tilde{\psi}(-\sigma - iy)| \left| \sum_{n \in \mathcal{N}_i} \frac{\tau_{i_*,}(n)e(nx)\gamma_i(n)}{n^{1/2+\sigma_i+it+iy}} \right|^2 dy dt. \quad (6.12)$$

To focus on the important parts here, we shall apply the mean value theorem for Dirichlet polynomials (combined with Lemma 2.4) in some ranges where this is acceptable. In (6.12),
with \( i = 1 \) (so \( \sigma_1 = -1/4 \)), we have the bound

\[
(\Delta T)^{-3/4} \sum_{n \in \mathbb{N}_0} \frac{\tau_T(n)^2}{n^{1/2}} (\Delta + n) \ll \frac{\zeta(1 + 2iT)^2 \log T}{(\Delta T)^{3/4}} (\Delta N_0^-)^{1/2} + (N_0^-)^{3/2} \ll \frac{\zeta(1 + 2iT)^2}{(\log T)^2},
\]

consistent with Lemma 6.1. A similar (even stronger) bound holds for \( \mathcal{N}_3 \), and we omit the details as it is even easier than the above case (it saves an arbitrary power of \( \log T \)). Therefore,

\[
K(\Delta, T, x) \ll \int_{|t - T| = \Delta} (\Delta T)^{-\frac{1}{2}} \int_{-\infty}^{\infty} |\tilde{\psi}(-iy)| \sum_{N_0^- \ll n \ll N_0^+} \frac{\tau_T(n)\epsilon(nx)\gamma_2(n)}{n^{1/2 + it + iy}}^2 dy \ dt
+ \frac{\zeta(1 + 2iT)^2}{(\log T)^2}.
\]

If \( \Delta \gg (\log T)^{100} \), then we can use an argument as in the proof of Lemma 6.2 to truncate the \( y \)-integral at \( \varepsilon \Delta \), and then over-extend the \( t \)-integral slightly. The error term obtained in this way saves a large power of \( \Delta \) which is acceptable for these values of \( \Delta \). This proves (6.7).

Next suppose \( \Delta \ll (\log T)^{100} \). In this situation, we truncate the \( y \)-integral at \( (\log T)^\varepsilon \), which by the rapid decay of \( \tilde{\psi} \) introduces an error term that saves an arbitrarily large power of \( \log T \) over the trivial bound, which is satisfactory. We then use a simple over-estimate in the following shape:

\[
(\Delta T)^{-1/2} \int_{|t - T| = \Delta} |f(t)|^2 dt \leq T^{-1/2} \int_{|t - T| \leq (\log T)^{100}} |f(t)|^2 dt.
\]

At this point, the \( t \)-integral is longer than the \( y \)-integral, so we can again change variables \( t \to t - y \) and double the length of \( t \)-integration to separate the variables \( t \) and \( y \). This proves (6.8).

Now we are almost in a position to apply the results from Section 5 to prove (6.1). First suppose \( \Delta \gg (\log T)^{100} \). We need to apply a dyadic partition of unity inside the \( n \)-sum, and apply Cauchy’s inequality to focus on one dyadic segment at a time. Since \( \log(N_0^+/N_0^-) \ll \log \log T \), there are only \( O(\log \log T) \) such dyadic segments to consider. Then Lemmas 5.4 and 5.7 show

\[
K(\Delta, T, x) \ll |\zeta(1 + 2iT)|^2 (\log T)^\varepsilon ((\log T)^{-50} + S_\Delta),
\]

which is satisfactory (recall that \( \sum_\Delta S_\Delta \ll 1 \)). The factor \( (\log T)^\varepsilon \) comes about because \( \frac{N_0^+}{(\Delta T)^{1/2}} \ll (\log T)^\varepsilon \).

Next suppose \( \Delta \ll (\log T)^{100} \). Again we can apply Lemmas 5.4 and 5.7 to obtain the same bound as (6.16), however, there is a technical difficulty in showing \( \sum_\Delta S_\Delta \ll 1 \) here, because in Lemma 4.3 we assumed that \( w_1 \) has support on the positive reals. Here the weight function attached to the \( t \)-integral has compact support on \( \mathbb{R} \), but not \( (0, \infty) \). Nevertheless, it is easy to show that \( Q(x, H) \ll 1 \) for an individual \( H \gg 1 \) and \( w_1 \) a fixed Schwartz-class function, which in this context means \( S_\Delta \ll 1 \). Since there are only \( O(\log \log T) \) dyadic values of \( \Delta \) with \( \Delta \ll (\log T)^{100} \), this bound is satisfactory for (6.1).
7. Upper bounds

7.1. Proofs. In this section, we prove Theorem 1.2. First we recall two “endpoint” bounds. One, where \(|t - T| \ll T^{1/100}\) is given by Proposition 1.6. For the other end, combining Propositions 3.1 and 4.1 gives

\[
\int_{|t| \leq CT^{26/27}} |F(it)|^2 dt \ll \log T.
\]

Next we consider the intermediate ranges \(T^{1/100} \ll \Delta \ll T^{26/27}\). By Lemma 4.2, we need to bound

\[
\sum_{T^\epsilon \ll \Delta \ll T^{1-\epsilon}} |\zeta(1 + 2iT)|^{-2}(\Delta T)^{-1/2}I(\Delta, T, x, N).
\]

The mean value theorem for Dirichlet polynomials (Lemma 2.2), combined with Lemma 2.4 gives that (7.2) is

\[
\ll \sum_{T^\epsilon \ll \Delta \ll T^{1-\epsilon}} |\zeta(1 + 2iT)|^{-2}(\Delta T)^{-1/2} \sum_{n \asymp \Delta^{1/2}} |\tau_{iT}(n)|^2 \left( \frac{\Delta}{n} + 1 \right) \ll \sum_{T^\epsilon \ll \Delta \ll T^{1-\epsilon}} \log T \ll \log^2 T.
\]

This proves (1.8). To show (1.9), we instead apply Corollary 5.9.

7.2. Remarks on the sieve method for shifted convolution sums. Here we give an informal discussion of estimates for \(I(\Delta, T, x, N)\) using Holowinsky’s sieve method [Ho]. Consider the range \(T^\epsilon \ll \Delta \ll T^{1-\epsilon}\) (we could restrict \(\Delta\) further to \(T^{1/2-\epsilon} \ll \Delta \ll T^{25/27+\epsilon}\), but this is not helpful for this discussion). By (4.12), we have

\[
\ll \sum_{T^\epsilon \ll \Delta \ll T^{1-\epsilon}} |\zeta(1 + 2iT)|^{-2} \sum_{n \asymp (\Delta T)^{1/2}} |\tau_{iT}(n)|^2 \left( \frac{\Delta}{n} + 1 \right) \ll \sum_{T^\epsilon \ll \Delta \ll T^{1-\epsilon}} \log T \ll \log^2 T.
\]

Recall we desire to estimate (7.2). The term \(h = 0\) gives a term of size \(\Delta |\zeta(1 + 2iT)|^2 \log T\), which is more than acceptable, so we focus on \(h \geq 1\). By [FI, Theorem 15.6] which gives a slightly different version of a result of Holowinsky [Ho], we have

\[
I(\Delta, T, x, N) \ll \frac{\Delta}{N} \sum_{h} \left( 1 + \frac{|h| \Delta}{N} \right)^{-100} \sum_{n \asymp N} |\tau_{iT}(n)\tau_{iT}(n+h)|.
\]

Recall we desire to estimate (7.2). The term \(h = 0\) gives a term of size \(\Delta |\zeta(1 + 2iT)|^2 \log T\), which is more than acceptable, so we focus on \(h \geq 1\). By [FI, Theorem 15.6] which gives a slightly different version of a result of Holowinsky [Ho], we have

\[
\ll \sum_{n \asymp N} |\tau_{iT}(n)| \tau_{iT}(n+h) \ll \frac{h}{\varphi(h)} NE_{iT}(N)^2 L_{iT}(N)^6,
\]

where

\[
E_{iT}(x) = \frac{1}{\log x} \prod_{p \leq x} \left( 1 + \frac{|\tau_{iT}(p)|}{p} \right),
\]

and \(L_{iT}(N) \ll \log \log N\). One can show

\[
E_{iT}(N) \ll (\log T)^{1/3} |\zeta(1 + 2iT)|^{7/9} |\zeta(1 + 4iT)|^{-1/9}.
\]

To see this, we begin with the inequality

\[
|\tau_{iT}(p)| \leq \frac{1}{18} (8 + 11 |\tau_{iT}(p)|^2 - |\tau_{iT}(p)|^4).
\]

Thus

\[
|\tau_{iT}(p)| \leq \frac{1}{18} (24 + 7 (p^{2iT} + p^{-2iT}) - (p^{4iT} + p^{-4iT}),
\]
which implies
\[ (7.10) \quad E_{iT}(x) \ll (\log x)^{-2} \prod_{p \leq x} \left( 1 + \frac{1}{p} \right)^{4/3} \prod_{p \leq x} \left| 1 + \frac{1}{p^{1+2iT}} \right|^{7/9} \prod_{p \leq x} \left| 1 + \frac{1}{p^{1+4iT}} \right|^{-1/9}. \]

Of course, \( \prod_{p \leq x} (1 + p^{-1}) \ll \log x \). It follows from standard methods from Davenport’s book and \((2.5)\) that
\[ (7.11) \quad \sum_{p \leq x} p^{-1-2iT} = \log (1 + 2iT) + O(1), \]

provided \( \log x \gg (\log T)^{2/3+\varepsilon} \). This shows \((7.1)\).

Therefore, we have
\[ (7.12) \quad \frac{I(\Delta, T, x, N)}{|\zeta(1 + 2iT)|^2 (\Delta T)^{1/2}} \ll \left( \frac{(\log T)^{1/3+\varepsilon}}{|\zeta(1 + 2iT)|^{11/9} |\zeta(1 + 4iT)|^{1/9}} \right). \]

The problem with using this is that the best bound for the \( \zeta \)'s that we may claim is \( |\zeta(1 + it)|^{-1} \ll (\log |t|)^{2/3+\varepsilon} \) which gives a weaker bound than that in \((7.3)\). Alternatively, one may say that \( |\zeta(1 + it)|^{-1} \ll (\log \log |t|) \) for almost all \( t \), but then one arrives at a bound worse than \((1.9)\).

This line of thought indicates that one should choose a different polynomial inequality than \((7.8)\), in order to optimize the total power of \( \log T \) that occurs on the right hand side of \((7.12)\). We performed a computer search for the optimal choice available from the bounds \( |\zeta(1 + it)|^{\pm 1} \ll (\log |t|)^{\eta+\varepsilon} \), with \( \eta = 2/3 \) and with polynomials of degree \( \leq 6 \), and found no improvement on \((7.3)\). Numerical experiments indicated that an improvement on \((7.3)\) is possible with \( \eta < 1/2 \), but in this case \((1.9)\) also improves on \((1.8)\).

8. Rational \( x \): the proof of Proposition 1.7

As mentioned in the remark following Proposition 1.6 to show Proposition 1.7, all that is required is to show
\[ (8.1) \quad \int_{T^{1/2-\varepsilon} \ll |t-T| \ll T^{5/27+\varepsilon}} |F_{x,T}(it)|^2 dt = o(\log T), \]

for \( |x - x_0| \leq \frac{\gamma}{T} \), and \( x_0 \in \mathbb{Q} \). By Lemma 1.2 it suffices to show
\[ (8.2) \quad \sum_{T^{49} \ll \Delta \ll T^{94}} \frac{I(\Delta, T, x, N)}{|\zeta(1 + 2iT)|^2 (\Delta T)^{1/2}} \ll 1. \]

Using the facts that \( n \asymp N = (\Delta T)^{1/2} \ll T^{.97} \), and \( \gamma \) is bounded by a small power of \( T \), we have \( e(nx) = e(nx_0) + O(T^{-0.1}) \), for all \( n, x \) appearing in \((8.2)\). Therefore, it suffices to show \((8.2)\) for \( x \) replaced by \( x_0 \).

In the course of proof, we shall need the following

**Proposition 8.1** (Jutila-Motohashi [JM]). Let \( 1 \ll U \ll V \), and let \( q \) be a positive integer. Then
\[ (8.3) \quad \int_{V \leq |u| \leq V + U} \sum_{\chi \pmod{q}} |L(1/2 + it, \chi)|^4 du \ll q^{1+\varepsilon}(U + V^{2/3})^{1+\varepsilon}, \]

This is a generalization of a result of Iwaniec [Iw]. We also require the following change of basis formula.
Lemma 8.2. Suppose that \( c_n \) is a finite sequence of complex numbers, \( q \geq 1 \) is an integer, and \( (a, q) = 1 \). Then
\[
\sum_{n} c_n e\left(\frac{an}{q}\right) = \sum_{d \mid q} \frac{1}{\phi(q/d)} \sum_{\chi \pmod{q/d}} \tau(\chi) \chi(a) \sum_{n} c_{dn} \chi(n).
\]

If \( q \geq 1 \) is an integer and \( b \in \mathbb{Z} \), then writing \( b = (b, q)b', q = (b, q)q' \), we obtain
\[
e\left(\frac{b}{q}\right) = e\left(\frac{b'}{q'}\right) = \frac{1}{\phi(q')} \sum_{\chi \pmod{q'}} \tau(\chi) \chi(b').
\]

This quickly gives the Lemma by writing \((n, q) = d\) and changing the order of summation.

Proof of (8.1). Suppose \( x_0 = a/q \), with \((a, q) = 1\). We will show
\[
I(\Delta, T, \frac{a}{q}, N) \ll_q N|\zeta(1 + 2iT)|^2 S_{\Delta} + NT^{-\delta},
\]
where \( \delta > 0 \) and \( S_{\Delta} \) has the properties stated in Proposition 4.1. This is sufficient for (8.1).

Using Lemma 8.2, we have
\[
I(\Delta, T, \frac{a}{q}, N) = \int_{-\infty}^{\infty} w_1\left(\frac{t}{\Delta}\right) \left| \sum_{d \mid q} \sum_{\chi \pmod{q/d}} \frac{\tau(\chi) \chi(a)}{\phi(q/d)} \sum_{n \geq 1} \frac{\tau_{iT}(dn) \chi(n)}{(dn)^{1/2-\sigma+iT}} w_2\left(\frac{dn}{N}\right) \right|^2 \, dt.
\]

The inner sum over \( n \) has a generating function
\[
\sum_{n=1}^{\infty} \frac{\tau_{iT}(dn) \chi(n)}{(dn)^s} = L(s + iT, \chi)L(s - iT, \chi)Z_{\alpha}(s),
\]
where \( Z_{\alpha}(s) \) is a finite Euler product over prime powers dividing \( d \), satisfying \( Z_{\alpha}(s) \ll d^{-\sigma+\epsilon} \), uniformly for \( \text{Re}(s) = \sigma \geq .01 \). By the Mellin transform method, we have
\[
I(\Delta, T, \frac{a}{q}, N) = \int_{-\infty}^{\infty} w_1\left(\frac{t}{\Delta}\right) \left| \sum_{d \mid q} \sum_{\chi \pmod{q/d}} \frac{\tau(\chi) \chi(a)}{\phi(q/d)} \right|^2 \, dt.
\]

We move the contour to the left, to the line \( \text{Re}(s) = 0 \). When \( \chi = \chi_0 \) is the principal character, then we cross poles at \( s = 1/2 + it - 2iT \), and at \( s = 1/2 + it \). The former pole is very small since \( \tilde{w}_2 \) is \( O(T^{-100}) \) at this point. The latter pole requires closer scrutiny, which we defer for a moment. Applying Cauchy-Schwarz, we see that the new line of integration gives
\[
\ll q \sum_{d \mid q} \sum_{\chi \pmod{q/d}} \int_{-\infty}^{\infty} w_1\left(\frac{t}{\Delta}\right) \left| \int_{(0)} \tilde{w}_2(s) \left| L(1/2 + s - it + 2iT, \chi)L(1/2 + s - it, \chi)Z_{\alpha}(1/2 - it + iT + s) \right|^2 \right| ds \, dt.
\]

Using Cauchy-Schwarz again, and Proposition 8.1, we bound this part by
\[
q^{1+\epsilon} T^c (\Delta^{1/2} + T^{1/3}) \Delta^{1/2}.
\]

Since \( q \) is fixed, and \( \Delta \ll T^{.94} \), this provides a power saving over \( (\Delta T)^{1/2} \), so this is more than sufficient towards the proof of (8.1).
Now we turn to the pole from $s = 1/2 + it$ that occurs for $\chi = \chi_0$. Conceptually, it should be clear that the main term could have been calculated using the known residues of the Estermann function and should therefore agree with other main terms that we have encountered in a variety of ways throughout this work. Nevertheless, it is easiest to carry forward directly. We have that this pole contributes to $I(\Delta, T, \frac{2}{q}, N)$ an amount

$$\ll N|\zeta(1 + 2iT)|^2 \int_{-\infty}^{\infty} w_1\left(\frac{t}{\Delta}\right)|\tilde{w}_2(1/2 - it)|^2 dt.$$  

Here we need to use that $\sum_\Delta w_1(t/\Delta) \ll 1$, uniformly in $t \in \mathbb{R}$, where the sum is over dyadic values of $\Delta$. Hence we may interpret this as an expression of the form $N|\zeta(1 + 2iT)|^2 S_\Delta$. $\square$

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