Relativistic polytropic spheres with electric charge: Compact stars, compactness and mass bounds, and quasiblack hole configurations

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We study the static stellar equilibrium configurations of uncharged and charged spheres composed by a relativistic polytropic fluid, and compare with those of spheres composed by a non-relativistic polytropic fluid, the later case already being studied in a previous work [J. D. Arbañil, P. S. Lemos, V. T. Zanchin, Phys. Rev. D 88, 084023 (2013)]. For the two fluids under study, it is assumed an equation of state connecting the pressure \( p \) and the energy density \( \rho \). In the non-relativistic fluid case, the connection is through a non-relativistic polytropic equation of state, \( p = \omega \rho ^\gamma \), with \( \omega \) and \( \gamma \) being respectively the polytropic constant and the polytropic exponent. In the relativistic fluid case, the connection is through a relativistic polytropic equation of state, \( p = \omega \rho ^\gamma \), with \( \delta = \rho - p/2(\gamma - 1) \), and \( \delta \) being the rest mass density of the fluid. For the electric charge distribution, in both cases, we assume that the charge density \( \alpha \) is proportional to the energy density \( \rho \). \( \rho = \alpha \rho \) with \( \alpha \) being a constant such that \( 0 \leq |\alpha| \leq 1 \). The study is developed by integrating numerically the hydrostatic equilibrium equation, i.e., the modified Tolman-Oppenheimer-Volkoff equation for the charged case. Some properties of the charged spheres such as mass, total electric charge, radius, redshift, and the speed of sound are analyzed. The dependence of such properties with the polytropic exponent is also investigated. In addition, some limits that arise in general relativity, such as the Chandrasekhar limit, the Oppenheimer-Volkoff limit, the Buchdahl bound and the Buchdahl-Andréasson bound, i.e., the Buchdahl bound for the electric case, are studied. As in a charged non-relativistic polytropic sphere, the charged relativistic polytropic sphere with \( \gamma \rightarrow \infty \) and \( \alpha \rightarrow 1 \) saturates the Buchdahl-Andréasson bound, thus indicating that it reaches the quasiblack hole configuration. We show by means of numerical analysis that, as expected, the major differences between the two cases appear in the high energy density region.

1. INTRODUCTION

1.1. Uncharged spheres: Equations of state and mass bounds

In the study of stars, both in Newtonian gravitation and in General Relativity, it is usual to model the matter inside the star by a perfect fluid. Such a fluid is fully characterized by its energy density \( \rho \) and pressure \( p \), besides the speed of sound in it. In general, to close the system of equations, and additional relation is needed. Usually, an equation of state relating the pressure to the energy density of a fluid in a very simple way is specified. Since Eddington [1], a polytropic equation of state has been assumed to build analytically simple star models. Such an equation relates the pressure and energy density by a power law of the form

\[ p = \omega \rho ^\gamma, \tag{1} \]

where \( \omega \) and \( \gamma \) are respectively the polytropic constant and the polytropic exponent. Such a relation, which we call EoS 1, is derived in Newtonian fluid mechanics, in which case \( \rho \) is the mass density, but it is a good approximation for relativistic fluids as long as the energy density is sufficiently small (see, e.g., [2]).

The equation of state (1) has been used in several contexts. A fact of interest here is that the first bound for the mass of a compact object was established, when studying white dwarfs, by using such a polytropic equation of state [3, 4]. In order to study the configurations of white dwarfs composed by a relativistically degenerate electron gas in a very simple manner, Chandrasekhar [3, 4] used EoS 1 [see Eq. (1)] with \( \gamma = 4/3 \). Applying the laws of Newtonian gravitation, he found that the radius of the configuration decreases with growing of the energy density, and it shrinks to zero for a mass of approximately \( 1.44 M_\odot \). This is the Chandrasekhar limit.

As in Newtonian gravitation, in the context of general relativity there are also mass bounds for compact objects. Studies in this direction were performed by Tolman [5] and Oppenheimer and Volkoff [6]. In their works, they showed that a mass limit can be also achieved in neutron stars. This mass limit, known as Oppenheimer-Volkoff limit, appears when the neutron star pressure is sufficiently large, since it contributes to the mass-energy of the system turning the gravitational field large enough that it cannot be counteracted by the pressure itself. It is worth mentioning that this limit, as well as that the Chandrasekhar limit, have also been determined by Landau using heuristic arguments, see [7]. In their works, Tolman [5] and Oppenheimer and Volkoff [6] developed a consistent method to describe a star in equilibrium configuration. This method is prone to numerical integration. Once defined the matter distribution, they wrote...
the gradient pressure in a very convenient form. This equation is known as hydrostatic equilibrium equation or Tolman-Oppenheimer-Volkoff (TOV) equation. To allow a complete description of the stars, all these equilibrium configurations can be connected smoothly with the Schwarzschild vacuum exterior solution.

The polytropic equation of state (1) and the TOV equation of hydrostatic equilibrium were used together for the first time by Tooper [8]. He discussed the structure of polytropic stars (polytropes) through the numerical integration of TOV equation. Despite that this equation of state describes spherical objects in a very simple manner, its use has some drawbacks. At very high pressures, it leads to obtain values of the sound speed higher than the speed of light, violating the principle of causality. Thus, it is understood that a generalization of the EoS 1 is required. The most reasonable generalization of the polytropic equation of state was determined by Tooper in [9]. He showed that the pressure and the energy density of the generalized polytropic equation of state (EoS 2) obey the relations

\[ p = \omega \delta^\gamma, \]
\[ \rho = \delta + p/(\gamma - 1), \]

respectively, where \( \delta \) represents the rest mass density and \( \gamma \) is the polytropic exponent. Equations of state of this form (2) have been used to study neutron stars, in which the neutrons are non-relativistic, and in white dwarfs, in which the electron gas is extremely relativistic (see, e.g., [9]). For white dwarf models, where the fluid pressure is small in comparison to the energy density, EoS 1 is equivalent to EoS 1, because in that situation we may neglect the pressure in the second term on the right-hand-side of Eq. (2), and take \( \delta \simeq \rho \). The equilibrium configurations determined with the EoS 2 are named by Thorne as the relativistic polytropic models or relativistic polytropes (for short) [2], henceforth, these names will be used throughout this work. It is important to mention that a brief comparison between non-relativistic polytropes and relativistic polytropes without and with cosmological constant have been considered respectively in [10] (see also [11, 12]) and [13], and in the presence of anisotropy in [14].

1.2. Charged spheres and the TOV method

The first analyses on charged objects by means of the TOV method were developed by Bekenstein in [15]. He generalized the hydrostatic equilibrium equation, i.e., the TOV equation, to include the effects electric charges and electrostatic fields. From then on, different works addressing the influence of electric charge in the structure of compact objects were reported. Among them, we find the studies of the influence of the electric charge in the equilibrium configurations of compact stars where the fluid follows the EoS 1, e.g., see [16–19]. In Refs. [17, 18] the authors focused on studying the effects of the electric charge on the structure of compact cold stars. In these works, the modified TOV equation was solved considering the EoS 1 with \( \gamma = 5/3 \) and a charge density proportional to the energy density, \( \rho_e = \alpha \rho \) (\( \alpha \) being a constant that obeys the constraint \( 0 \leq \alpha \leq 1 \)). Arbañil, Lemos and Zanchin (ALZ) in [19] also studied the structure of electrically charged objects considering the EoS 1, for different \( \gamma \), and with the charge distribution \( \rho_e = \alpha \rho \). The authors found that extremely charged polytropic stars with \( \gamma \to \infty \) are structures with the total charge \( Q \) close to the total mass \( M \), \( Q \simeq M \), and the total radius \( R \) close to the gravitational radius \( R_+ \), \( R \simeq R_+ \simeq M \). This indicates that the solutions are close to the quasiblack hole configurations, i.e., structures with \( Q = M \) and \( R = M \) and quasi-horizons (see, e.g., [20]). All the aforementioned charged static equilibrium configurations are matched smoothly with the Reissner-Nordström vacuum exterior solution.

1.3. Compactness bounds and quasiblack hole configurations

The solutions of compact objects found in general relativity are connected with the Buchdahl bound [21]. This bound states that the radius \( R \) and the gravitational mass \( M \) of a sphere of perfect fluid in hydrostatic equilibrium, in which the energy density is non-increasing outward, satisfies the inequality \( R/M \geq 9/4 \). If a star shrinks to a size that violates this bound, it eventually turns into a black hole. This bound is saturated by the interior Schwarzschild solution in the limit of infinite central pressure [22] (see also [23]). This is the Schwarzschild interior limit, which saturates the Buchdahl bound in the sense that an incompressible fluid with an infinite central pressure gives the upper limit of the bound, \( R/M = 9/4 \). The Buchdahl bound is a general result, i.e., it is independent of the equation of state used.

The charged static equilibrium solutions found in an Einstein-Maxwell system are related with the Buchdahl bound for the electric case [24], i.e., with the Buchdahl-Andrèasson bound, in which the hydrostatic equilibrium configuration satisfies the condition \( R/M \geq 9/\left(1 + \sqrt{1 + 3Q^2/R^2}\right)^2 \). When \( Q = R \), the Buchdahl-Andrèasson bound is saturated, i.e., we obtain \( R = M \) and also \( Q = M \). In other words, this bound is saturated by a quasiblack hole configuration. As shown in Ref. [25], the Buchdahl-Andrèasson bound is saturated by the Gul'offe solutions [26] for charged spheres in the limit where the central pressure attains arbitrarily large values, in full analogy to the Schwarzschild interior limit. As far as we know, this is the only solutions that saturates such a bound. As verified in Refs. [19, 23], charged fluids satisfying the non-relativistic polytropic equation of state and a charged incompressible fluid do not saturate the Buchdahl-Andrèasson bound.

It is important to stress that the quasiblack hole limit is found using different equations of state and different distributions of electric charge. Such limiting solutions have been found, e.g, for an incompressible fluid, i.e.,
\( \rho \) = constant, with a distribution of electric charge which follows a particular function of the radial coordinate \([27–29]\), and when the charge density is proportional to the energy density \([19, 23]\). They are also obtained in works that use an equation of state for electrically charged dust, i.e., \( p = 0 \) \([30, 31]\). These objects are also obtained in \([25, 32, 33]\) where it is considered the Cooperstock-De la Cruz-Florides equation of state \([26, 34, 35]\). The general properties of quasiblack holes are defined in \([20, 30]\).

1.4. This work

We are interested in comparing equilibrium configurations of charged fluid spheres in the presence or absence of electric charge, obtained from the two equations of state cited above, namely the non-relativistic polytropic equation (1) and the relativistic polytropic equation (2). For short, we refer to the respective configurations as non-relativistic polytropic stars (or non-relativistic polytropes), and relativistic polytropic stars (or relativistic polytropes). Very compressed objects and the compactness bounds and quasiblack hole limits are the main objects of interest here. Let us mention once again that the major part of the analysis in the case of the non-relativistic polytropic equation of state was performed in Ref. \([19]\). The main aim now is the relativistic polytropic stars (or non-relativistic polytropes), and the comparison to the non-relativistic polytropic equation is also done here. For the sake of comparison with previous works, the distribution of electric charge in the structure of the star is assumed to follow the equation \( \rho_e = \alpha \rho \). Some features mentioned previously are investigated in this paper. For these objects we study the Chandrasekhar limit, the Oppenheimer-Volkoff limit, the Buchdahl bound, the Buchdahl-Andréasson bound, and the quasiblack hole limit. The speed of sound throughout a given sphere and the redshift at the surface of the sphere are also investigated.

The article is structured as follows. In Sec. 2 we write the TOV equation with the inclusion of the electric charge. To complete the set of equations, we also present the equations of state to be used, as well as the charge density profile and the boundary conditions. Sec. 3 is dedicated to compare the structure of charged non-relativistic polytropes with the charged relativistic polytropes for different values of polytropic exponent \( \gamma \). We analyze the Chandrasekhar limit, the Oppenheimer-Volkoff limit, the Buchdahl bound and the Buchdahl-Andréasson bound. We present the dependence of the mass, the radius, and the charge of the charged spheres as a function of the polytropic exponent. We also present the dependence of the mass, radius and charge against the charge fraction. Some physical properties of the fluid for an arbitrarily large polytropic exponent \( \gamma \) are given in Sec. 4. The dependence of the speed of sound as a function the polytropic exponent is accomplished in Sec. 5. Section 6 is devoted to study the quasiblack hole limit and the redshift on the surface of a quasiblack hole. In every section we present the new results for the relativistic polytropic spheres and, for a better comparison, the results of the non-relativistic polytropic spheres are also reviewed. In Sec. 7 we conclude.

Finally, it is worth mentioning that, unless otherwise stated, geometric units shall be used throughout the text, so that \( c = 1 = G \).

2. General relativistic charged perfect fluid

2.1. Equations of structure

With the purpose of analyzing the properties of static charged perfect fluid distributions we take the line element, in Schwarzschild coordinates, as

\[
ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,
\]

with the metric potentials \( B(r) \) and \( A(r) \) depending on the radial coordinate \( r \) only. The Einstein-Maxwell equations furnish the following non-identically zero equations

\[
\frac{dq(r)}{dr} = 4\pi\rho_e(r) r^2 \sqrt{A(r)},
\]

and

\[
\frac{1}{A(r)} \left[ 1 - \frac{r}{A(r)} \frac{dA(r)}{dr} \right] = 1 - 8\pi r^2 \left[ \rho(r) + \frac{q^2(r)}{8\pi r^4} \right],
\]

\[
\frac{1}{A(r)} \left[ 1 + \frac{r}{B(r)} \frac{dB(r)}{dr} \right] = 1 + 8\pi r^2 \left[ \rho(r) - \frac{q^2(r)}{8\pi r^4} \right].
\]

Function \( q(r) \) represents the electric charge within a sphere of radius \( r \), \( \rho_e(r) \) is the electric charge density, and \( \rho(r) \) and \( p(r) \) stand respectively for the energy density and the pressure of the fluid.

We now introduce the mass function \( m(r) \) through the relation

\[
A^{-1}(r) = 1 - \frac{2m(r)}{r} + \frac{q^2(r)}{r^2}.
\]

Considering this mass function (7), we have that Eq. (5) can be written in the form

\[
\frac{dm(r)}{dr} = 4\pi \rho(r) r^2 + \frac{q(r)}{r} \left[ \frac{dq(r)}{dr} \right].
\]

This differential equation represents the continuity equation, i.e., the mass-energy conservation.

An additional relation may be obtained from the Bianchi identity \((\nabla_{\mu} T^{\mu\nu} = 0)\) which, with metric (3), yields

\[
\frac{dB(r)}{dr} = \frac{2 B(r)}{p(r) + \rho(r)} \left[ \frac{q(r)}{4\pi r^4} \frac{dq(r)}{dr} - \frac{dp(r)}{dr} \right].
\]

Replacing Eqs. (4) and (6) into Eq. (9) we obtain the modified TOV equation with the inclusion of electric charge \([15]\),

\[
\frac{dp}{dr} = -(p + \rho)A \left( 4\pi pr + \frac{m}{r^2} - \frac{q^2}{r^5} \right) + \rho_e \sqrt{A} \frac{q}{r^2}.
\]
where, with the purpose of simplifying the equation, the explicit dependence of the variables on the radial coordinate was removed, i.e., we have written \( p(r) = p, \rho(r) = \rho_c, m(r) = m, q(r) = q, \) and \( A(r) = A. \) Taking \( q = 0 \) in this equation, the original TOV \([5, 6]\) equation is recovered.

### 2.2. The equation of state and charge density profile

In order to look for equilibrium solutions, it is necessary to solve simultaneously equations \((4), (7), (8), \) and \((10)\). These four equations contain six variables \( q(r), A(r), m(r), \rho(r), p(r), \) and \( \rho_c(r), \) forming an incomplete set of equations. To complete the system, as usual, an equation of state relating the pressure with the energy density and, for the charged fluid, a relation defining the charge density profile are supplemented. Once closed the system of equations, together with the boundary conditions imposed in the system, these are solved numerically.

As stated earlier, we employ two different polytropic equations of state to connect the fluid pressure and the fluid energy density. In the following, the equation of state \((1)\) (EoS 1) and its respective results are called case 1, while the equation \((2)\) (EoS 2) and its respective results are called case 2.

As in previous works, e.g., \([16–19, 37, 38]\), the electric charge distribution is considered proportional to the energy density, as follows

\[
\rho_c = \alpha \rho, \tag{11}
\]

where \( \alpha \) is a dimensionless constant that we call the charge fraction, which is constrained to the interval \( \alpha \in [0, 1) \).

### 2.3. The boundary conditions and the exterior line element

The complete set of differential equations are provided with a set of additional constraints, allowing to find the equilibrium solutions. For the non-relativistic polytropic fluids, case 1, the conditions at the center of the spheres are \( q(r = 0) = 0, m(r = 0) = 0, \) and \( p(r = 0) = \rho_c. \) For the relativistic polytropic fluids, case 2, the conditions at the center of the spheres are \( q(r = 0) = 0, m(r = 0) = 0, \) and \( \delta(r = 0) = \delta_c. \) In both cases, the surface of the objects is found when \( p(r = R) = 0. \) The input data for the numerical calculation are the central energy density \( \rho_c \) and the central rest mass density \( \delta_c, \) respectively, for case 1 and case 2, the polytropic constant \( \omega, \) the polytropic exponent \( \gamma, \) and the charge fraction \( \alpha. \)

In both cases, the interior solution connects smoothly with the exterior solution given by the Reissner-Nordström metric

\[
ds^2 = -F(r)dT^2 + \frac{dr^2}{F(r)} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right), \tag{12}
\]

with \( F(r) = 1 - 2M/r + Q^2/r^2. \) The total mass and the total charge of the sphere are represented by \( M \) and \( Q, \) respectively. The time \( T \) is proportional to the inner time \( t, \) and the radial coordinate \( r \) is identical to the interior region. The full set boundary conditions at the surface of the sphere are \( p(R) = 0 \) (this condition is used to determine the radius of the star), \( m(R) = M, q(R) = Q, \) and the continuity of the metric functions \( B(R) = 1/A(R) = F(R). \)

### 3. The structure of relativistic charged polytropic spheres

#### 3.1. General remarks

Here the structure of charged spheres is analyzed for different values of the exponent \( \gamma, \) and for different values of \( \rho_c \) and \( \delta_c, \) respectively, in the case 1 and in the case 2.

In order to make a proper comparison of our results with those found in the literature, some similar considerations have to be made, both for the choice of the polytropic constant as for the normalization used in the numerical integration. Therefore, for the two equations of state used, following Ref. \([19]\), we take the polytropic constant as \( \omega = 1.47518 \times 10^{-3} \times 78266 \times 10^{15}[\text{kg/m}^3]^{1-\gamma}. \) For such a choice, the normalization factor adopted during the numerical integration of the TOV equation is \( \rho_0 = 1.78626 \times 10^{15}[\text{kg/m}^3] \) for the case 1, and \( \delta_0^* = 1.78626 \times 10^{15}[\text{kg/m}^3] \) for the case 2. Then, during the numerical calculations, the polytropic constant is written as

\[
\omega = \bar{\omega} \rho_0^{1-\gamma}, \quad \omega = \bar{\omega} \delta_0^{*1-\gamma}, \tag{13}
\]

respectively, for the EoS 1 and EoS 2. The dimensionless polytropic constant is now \( \bar{\omega} = 1.47518 \times 10^{-3}. \)

For the numerical solutions, the equations of structure \((4), (7), (8), \) and \((10)\), the equation of state, the charge density profile, and the boundary conditions are written in a dimensionless form. For the non-relativistic polytropic case, Eq. \((1)\), this was done in Ref. \([19]\), while for the relativistic polytropic case, Eq. \((2)\), the normalized equations are shown in Appendix A. Once the values of \( \alpha, \gamma, \) and \( \rho_c \) (or \( \delta_c, \) depending on the case) has been fixed, the system of equations are solved numerically by using the fourth-order Runge-Kutta method.

For convenience of the numerical analysis we restore the gravitational constant, \( G = 7.42611 \times 10^{-28}[\text{m/kg}], \) but keep the speed of light \( c = 1. \)

Along this section, we compare the equilibrium configurations of charged relativistic polytropes to the charged non-relativistic polytropes studied in Ref. \([19]\). Additionally, some particular limits, such as the Chandrasekhar limit, the Oppenheimer-Volkoff limit, the Buchdahl bound, the Buchdahl-Andréasson bound, and the quasiblack hole limit are tested for the new equation of state (case 2). To study the Chandrasekhar and Oppenheimer-Volkoff limits, the polytropic exponent is fixed and the interval of both the central energy density and the central rest mass density is varied.
from $10^{13}$ [kg/m$^3$] to $10^{20}$ [kg/m$^3$]. In turn, to study the
the Buchdahl and the Buchdahl-Andrëasson bounds and
quasiblack hole limit, we fixed the central energy density
in $10 \rho_0$ in case 1 and the central rest mass density $10 \delta_0$
in the case 2. For those densities, the largest value of the
polytropic exponent that produces good numerical
results is 17.0667 for the case 1, and 17.1109 for the case
2. Thus, in order to realize a comparison between the
results found for the two equations of state, case 1 (1) and
case 2 (2), we take the values of the polytropic exponent
in the same range $4/3 \leq \gamma \leq 17.0667$.

3.2. The radius against the mass for fixed
polytropic exponent: The Chandrasekhar and the
Oppenheimer-Volkoff limits

![Graph showing radius vs. mass for different values of the polytropic exponent.]

FIG. 1: The radius of the sphere against the mass for $\gamma =
4/3$ and three values of charge fractions, as indicated. The
top (bottom) panel is for the EoS 1 (EoS 2). The central
energy density and the central rest mass density, respectively,
are in the interval $[10^{13} (\text{kg/m}^3), 10^{20} (\text{kg/m}^3)]$. In both cases
the Chandrasekhar and Oppenheimer-Volkoff mass limits are
found. Notice that these limits do not depend on the equation
of state.

Let us start investigating the behavior of the radius
and the mass of the charged fluid spheres for different
central densities. Figs. 1 and 2 contain the curves for the
radius as a function of the mass of the spheres (normal-
ized to the Sun’s mass $M_\odot$) for $\gamma = 4/3$ and $\gamma = 5/3$,
respectively, and three charge fraction values $\alpha = 0.0,
0.5$ and $0.9$. The upper panel in each figure shows the
results for the non-relativistic polytropes (case 1), while
the lower panel contains the results determined for rela-
tivistic polytropes (case 2). The central energy density
and the central rest mass density are both varied from
$10^{13}$ [kg/m$^3$] to $10^{20}$ [kg/m$^3$].

In Fig. 1, for $\gamma = 4/3$, the curves in top panel, case
1, indicate that the mass and the radius of the spheres
decrease with increasing of the central energy density.
Similar behavior is shown by the curves in the bottom
panel, case 2, showing that the mass and the radius of
the relativistic polytropes decrease with the increase of
the central rest mass density. Moreover, the radius is an
increasing function of the mass, the smaller values of the
energy (rest mass) density correspond to the higher val-
ues of $R(M)$, on the right end of each curve. In addition,
we note that the Chandrasekhar limit is found at zero ra-
adius, and the Oppenheimer-Volkoff limit appears at the
point where the vertical lines turn to the left. It is clear
the influence of the electric charge. For $\alpha = 0.9$ the mass
of the stars for the same central density are about three
times larger than for $\alpha = 0.0$. The radius, of course, also
grows with the charge fraction approximately at the same
rate as the mass, almost independently of the equation
of state.

In Fig 2, for $\gamma = 5/3$, the curves in the top panel, case
1, show that the radius of the spheres decreases with the
mass, while the central energy density grows. Similar
behavior is shown by the curves in the bottom panel,
case 2, the radius of the relativistic polytropic spheres
decreases with the mass, while the central rest mass den-
sity grows. For high values of $\rho_c$ and $\delta_c$ we observe that
the curves in both panels present a spiraling behavior.
When $\gamma = 5/3$ only the Oppenheimer-Volkoff limit ap-
ppears in the point where the inclined lines are folded to
the left. For $\gamma \neq 4/3$, the Chandrasekhar limit does not
appear. The radius and mass grow with the charge factor
as for other polytropic exponents, the growth rate being
approximately independent of the equation of state cho-
sen.

It is worth mentioning that, irrespective the $\alpha$ used in
Figs. 1 and 2, we find that the masses of (non-relativistic)
polytropes and relativistic polytropes are very close to
each other for low and equal values of $\rho_c$ and $\delta_c$. For
instance, in the case of $\gamma = 4/3$, $\alpha = 0.9$, and $\rho_c =$
\[ \delta_c = 10^{13}[\text{kg/m}^3] \] we obtain a mass of 1.8173 \( M_\odot \) in the case 1, and a mass of 1.8154 \( M_\odot \) in the case 2. Independently of the \( \alpha \) employed, the difference of these masses becomes more apparent when the value of \( \rho_c = \delta_c \) is incremented. For example, still in the case for \( \gamma = 4/3, \alpha = 0.9 \), but now with \( \rho_c = \delta_c = 10^{20} \text{[kg/m}^3] \), the mass of the sphere in the case 1 is 0.57874 \( M_\odot \), and in the case 2 it is 0.53214 \( M_\odot \), a difference of about 19\%. From this result we understand that the EoS 1 and EoS 2 are not equivalent for large values of energy (rest mass) density, as expected.

### 3.3. The structure dependence of the relativistic charged spheres on the polytropic exponent

#### 3.3.1. Mass of the spheres against the polytropic exponent

The numerical results obtained for the mass of the charged fluid spheres as a function of polytropic exponent \( \gamma \) produce the graphs of Fig. 3. The behavior of the ratio \( M/M_\odot \) for some values of the charge fraction \( \alpha \) and for the two different equations of state under consideration is seen in that figure. As in the case of Figs. 1 and 2, the top panel is for the EoS 1 (case 1), Eq. (1), and with the central energy density \( \rho_c = 1.78266 \times 10^{16} \text{[kg/m}^3] \), and the bottom panel is for the EoS 2 (case 2), Eq. (2), and with the central rest mass density \( \delta_c = 1.78266 \times 10^{16} \text{[kg/m}^3] \). The polytropic exponent considered in both cases is in the interval \( 4/3 \leq \gamma \leq 17.0667 \). For low values of \( \gamma \), the masses found in both cases are very close to each other. This result is expected since, in this regime, i.e., taking into account that the central energy density is not very high, the relativistic effects on the equation of state for the fluid are small and EoS 1 and EoS 2 are equivalent. In both cases, we observe that the mass increases very fast with the polytropic exponent. For instance, analyzing the mass in the points \( \gamma = 4/3 \) and 17.0667 for \( \alpha = 0.5 \), we obtain that it grows approximately 36, 431\% in case 1, and around 34, 242\% in case 2. In turn, for \( \alpha = 0.99 \) we obtain that the mass grows at about 488\% for EoS 1, and almost 456\% for EoS 2. The growth of the mass with the polytropic exponent \( \gamma \) is explained in the same way for both equations of state, since a larger central pressure \( \rho_c \) is obtained with a higher \( \gamma \). In both cases, EoS 1 and EoS 2, the mass also grows with the increase of charge fraction (see, also, Fig. 7). Again we note only a small difference between the results of the two equations of state. The masses of the non-relativistic polytropes (case 1) are of the order of 10\% larger than the masses of the relativistic stars (case 2).

#### 3.3.2. Radius of the relativistic spheres against the polytropic exponent

The radius to mass ratio of the sphere against the polytropic exponent is shown in Fig. 4, where we plot the ratio \( R/M \) versus \( \gamma \) for two values of charge fraction, \( \alpha = 0.5 \) and 0.99, and for the two equations of state. The results for the EoS 1 with the central energy density \( \rho_c = 1.78266 \times 10^{16} \text{[kg/m}^3] \) are shown in the top panel, while the results for EoS 2 with the central rest mass density \( \delta_c = 1.78266 \times 10^{16} \text{[kg/m}^3] \) are shown in the bottom panel. As in the other figures of the present section, the polytropic exponent values are between 4/3 and 17.0667.

It can be observed in Fig. 4 that the ratio \( R/M \) decreases with the increment of \( \gamma \), reaching its minimum value at the maximum value of the polytropic exponent \( \gamma = 17.0667 \). In the uncharged case, \( \alpha = 0.0 \), we see...
that the minimum value of the radius to mass ratio is approximately $R/M = 2.279$ in the EoS 1 case, and $R/M = 2.352$ in the EoS 2 case. From these results we understand that if we extrapolate the polytropic exponent $\gamma$ to infinity, so to reach the incompressible fluid configuration, the Buchdahl bound [21] is saturated in the case 1. However in the EoS 2 case the upper limit of the Buchdahl bound is not attained (see also Refs. [19, 23]). The different result found in the EoS 2 case may be explained by observing that the effects of a large (infinite) central pressure is counterbalanced by the effects of a large (infinite) energy density and, as consequence, by a large attractive gravitational force, preventing the object to reach the maximum compactness set by the Buchdahl bound. The main point that may explain this different degree of compactness is that the central energy density is finite in case 1, while it diverges in case 2. Notwithstanding, in the extremely charged case, $\alpha = 0.99$, we have $R/M \approx 1.027$ for the case EoS 1, and $R/M \approx 1.025$ for the case EoS 2. These values of $R/M$ are close to the maximum compactness of a charged object, $R/M = 1.0$. From this we understand that for large (infinite) values of $\gamma$ the Buchdahl-Andréasson bound [24] is saturated in the limit $\alpha \to 1$.

For a better visualization of the results shown in Fig. 4, the relation $R/M$ for the highest values of $\gamma$ we have obtained are shown in the Fig. 5. From this figure it can be seen in detail the extreme limits for $\alpha = 0.0$ as well as for $\alpha = 0.99$. For the case without charge, in the top panel, unlike what is shown in the bottom panel, we see that the Buchdahl limit is close to be attained. In turn, for $\alpha = 0.99$, on the top as well as in the bottom panel, we see that the Buchdahl-Andréasson bound is close to be saturated (see Sect. 4 for more details).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{The ratio $R/M$ against the polytropic exponent. This is an amplification of the region of Fig 4 corresponding to high values of $\gamma$. The data are the same as in that figure.}
\end{figure}

3.3.3. Charge of the spheres against the polytropic exponent

The charge to mass ratio $(Q/M)$ as a function of the polytropic exponent for two values of charge fraction, $\alpha = 0.5$ and $\alpha = 0.99$, is plotted in Fig. 6. As in the previous figures, the curves in the top panel are obtained for the EoS 1 with $\rho_c = 1.78266 \times 10^{16}[\text{kg/m}^3]$, while the curves in the bottom panel are for the EoS 2 with $\delta_e = 1.78266 \times 10^{16}[\text{kg/m}^3]$. The behavior of the curves indicate that the relation $Q/M$ grows with $\gamma$ and $\alpha$, and it is essentially the same for the two equations of state. Note that in the extreme case, with $\alpha = 0.99$ and $\gamma = 17.0667$, the values of the ratio $Q/M$ are very close to unity. The largest values of $Q/M$ for case 1 and case 2 are respectively 0.999793 and 0.999814. This fact suggests that the quasiblack hole regime is about to be reached in both cases, and we investigate this point in detail later on.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.png}
\caption{The ratio $Q/M$ as a function of $\gamma$ for two values of the charge fraction, 0.5 and 0.99. The top panel is for the polytropic equation of state (EoS 1) with $\rho_c = 1.78266 \times 10^{16}[\text{kg/m}^3]$, while the bottom panel is for the relativistic polytropic equation of state (EoS 2) with $\delta_e = 1.78266 \times 10^{16}[\text{kg/m}^3]$.}
\end{figure}

3.4. The structure dependence of the relativistic charged spheres on the charge fraction

3.4.1. The mass of the spheres as a function of the charge fraction

The ratio $M/M_\odot$ as a function of the charge fraction $\alpha$ is presented in Fig. 7, for the case 1 with $\rho_c = 1.78266 \times 10^{16}[\text{kg/m}^3]$ exhibited in the top panel, and for the case 2 with $\delta_e = 1.78266 \times 10^{16}[\text{kg/m}^3]$ displayed in the bottom panel. Two values of the polytropic exponent are considered, 4/3 and 17.0667. In both fluid types, we see the smooth growth of the mass with the charge fraction and also with the polytropic exponent.
The behavior of the curves indicate that the relation $M/M_{\odot}$ is essentially the same for the two equations of state.

FIG. 7: Mass of the spheres versus the charge fraction $\alpha$ for two values of the polytropic exponent, $\gamma = 4/3$ and $\gamma = 17.0667$. The top panel is for the polytropic equation of state (EoS 1), with the central energy density $\rho_c = 1.78266 \times 10^{16}$[kg/m$^3$]. The bottom panel is for the relativistic polytropic equation of state (EoS 2), with the central rest mass density $\delta_c = 1.78266 \times 10^{16}$[kg/m$^3$]. It is seen from the figure that, for both equations of state, the ratio $R/M$ decreases with the increment of the charge fraction $\alpha$. Note also that in the extreme case, with $\alpha = 0.99$ and $\gamma = 17.0667$, the ratio $R/M$ is close to unity for both equations of state. In fact, the values of $R/M$ are 1.02676 and 1.02514 in the EoS 1 case and in the EoS 2 case, respectively.

3.4.2. The radius of the spheres as a function of the charge fraction

The radius to mass ratio $(R/M)$ as a function of the charge fraction $\alpha$ for two values of the polytropic exponent, $\gamma = 4/3$ and $\gamma = 17.0667$, is presented in Fig. 8. The top panel is for the EoS 1 with the central density $\rho_c = 1.78266 \times 10^{16}$[kg/m$^3$], while the bottom panel is for the EoS 2 with the central rest mass density $\delta_c = 1.78266 \times 10^{16}$[kg/m$^3$]. Note also that in the extreme case, with $\alpha = 0.99$ and $\gamma = 17.0667$, the ratio $R/M$ is close to unity for both equations of state.

3.4.3. The charge of the spheres as a function of the charge fraction

The charge to mass ratio $Q/M$ versus the charge fraction $\alpha$ is presented in Fig. 9, for two values of the polytropic exponent, $\gamma = 4/3$ and $\gamma = 17.0667$. The top panel shows the results for the EoS 1 with central energy density $1.78266 \times 10^{16}$[kg/m$^3$]. The bottom panel shows the results for EoS 2 with the central rest-mass density $1.78266 \times 10^{16}$[kg/m$^3$]. The curves indicate that $Q/M \to 1$ for $\alpha \to 1$. As discussed below, this signals the facts that the Buchdahl-Andrèasson bound and the quasiblack hole limit is about to be reached for large charge fraction.
4. PROPERTIES OF RELATIVISTIC POLYTROPIC FLUID SPHERES WITH INFINITELY LARGE POLYTROPIC EXPONENT

4.1. Large polytropic exponent and incompressible fluids

Analyzing equations EoS 1 (1) and EoS 2 (2), it can be seen that in both cases a small $\gamma$ provides a low pressure and a large $\gamma$ leads to a high pressure.

As shown above, at low fluid pressures (low exponents $\gamma$) the EoS 1 is equivalent to the EoS 2. On the other hand, at high fluid pressures (large exponents $\gamma$), EoS 1 and EoS 2 yield completely different results. In order to highlight the differences between these two equations of state for large values of $\gamma$, a comparison between them in that region must be performed. Even though the analysis of the polytropic equation of state (EoS 1) is such a limit was developed in [19], for convenience we rewrite the relevant parts of that analysis here. Let us call $p_0$ a particularly chosen value of the central pressure and consider it as a normalization factor for the pressure. Hence, using Eq. (1) it follows,

$$\lim_{\gamma \to \infty} \frac{p}{p_0} = \lim_{\gamma \to \infty} \left( \frac{\rho}{\rho_0} \right)^{\gamma} = \left\{ \begin{array}{ll} \infty, & \text{if } \rho > \rho_0; \\ 0, & \text{if } \rho < \rho_0; \end{array} \right. \tag{14}$$

and

$$\lim_{\gamma \to \infty} \frac{p}{\rho_0} = \lim_{\gamma \to \infty} \left( \frac{p}{\rho_0} \right)^{1/\gamma} = 1. \tag{15}$$

where $p_0 = \omega \rho_0^\gamma$. This limit conducts to an incompressible (constant energy-density $\rho$) fluid, as in the Schwarzschild interior solution [22], besides the addition of a constant electric charge density, since we have also

$$\lim_{\gamma \to \infty} \frac{\rho_e}{\rho_{e0}} = \lim_{\gamma \to \infty} \frac{\rho}{\rho_0} = \lim_{\gamma \to \infty} \left( \frac{p}{p_0} \right)^{1/\gamma} = 1. \tag{16}$$

Such an electrified Schwarzschild interior solution was investigated in [23].

Now the fluid quantities in the case of EoS 2 are normalized as $p/p_0^\gamma$, $\rho/\rho_0^\gamma$ and $\delta/\delta_0^\gamma$, where $\delta_0^\gamma$, $\rho_0^\gamma$ and $p_0^\gamma$ are normalization factors. These factors are related by $p_0^\gamma = \omega \delta_0^{\gamma\gamma}$ and $\rho_0 = \delta_0^\gamma + p_0^\gamma/(\gamma - 1)$. Hence, we get

$$\lim_{\gamma \to \infty} \frac{p}{p_0^\gamma} = \lim_{\gamma \to \infty} \left( \frac{\delta}{\delta_0^\gamma} \right)^{\gamma} = \left\{ \begin{array}{ll} 0, & \text{if } \delta < \delta_0^\gamma; \\ \infty, & \text{if } \delta > \delta_0^\gamma; \end{array} \right. \tag{17}$$

and we have

$$\lim_{\gamma \to \infty} \frac{\rho}{\rho_0^\gamma} = \lim_{\gamma \to \infty} \frac{\rho_e}{\rho_{e0}^\gamma} = \left\{ \begin{array}{ll} \delta \delta_0^\gamma, & \text{if } \delta < \delta_0^\gamma; \\ \infty, & \text{if } \delta > \delta_0^\gamma, \end{array} \right. \tag{18}$$

with

$$\frac{\rho}{\rho_0^\gamma} = \frac{\delta}{\delta_0^\gamma} \left( \frac{\delta}{\delta_0^\gamma} \right)^\gamma + \frac{\rho_0^\gamma}{1 + \omega^{-1} - \omega}, \tag{19}$$

where $\omega = 1.47518 \times 10^{-3}$ and we used Eq. (13).

For the normalized rest mass density we get

$$\lim_{\gamma \to \infty} \frac{\delta}{\delta_0^\gamma} = \lim_{\gamma \to \infty} \left( \frac{p}{p_0} \right)^{1/\gamma} = 1. \tag{20}$$

Therefore, the limit of high polytropic exponents of the relativistic equation of state (EoS 2) does not yield an incompressible fluid. It gives a constant rest-mass density, and in the instance when the pressure may assume arbitrarily large values, it gives an infinitely large energy density too, and it gives a constant energy density in a second instance when the pressure vanishes. This second situation is not interesting for the present analysis.

It is also worth mentioning that, since we assume the relation $\rho_c = \alpha \rho$, the conditions given by Eq. (18) is also fulfilled by the charge density in its normalized form, $\rho_c/\rho_0^\gamma$. Notice also that the polytropic constant $\omega$ plays an important role in the normalization of the relativistic polytropic equation of state. Since it depends upon $\gamma$, the normalization adopted according to Eq. (13) implies the results presented in Eqs. (18) and (19).

On the basis of results previously reported in this work, we know that when $\rho_c < \rho_0$ and $\delta_0 < \delta_0^\gamma$ in the cases 1 and 2, respectively, there are no equilibrium solutions for the polytropic spheres (see also [19]) and neither for the relativistic polytropic spheres with infinitely large polytropic exponents. On the other hand, as shown in [19] and confirmed in the present study, in the limit $\gamma \to \infty$ for case 1, when $\rho_c > \rho_0$, we have that the polytropic stars have constant energy densities and infinitely large central pressures. For these (non-relativistic) polytropic star configurations it was found that the Buchdahl bound is saturated, thus, in the limit of zero electric charge, reaching the limit $R/M = 9/4$. In turn, from the results presented here for the EoS 2 (case 2), in the limit $\gamma \to \infty$, and with $\delta_0^\gamma > \delta_0^\gamma$, the relativistic polytropic star configurations have both the central pressures and the central energy densities becoming infinitely large. For these relativistic polytropic configurations, we have that the Buchdahl bound is far from being saturated. For the extremely charged case, however, we have seen that the use of the equations of state EoS 1 and EoS 2, with $\gamma \to \infty$, allows the stars to saturate the Buchdahl-Andréasson bound with $R/M \simeq 1$. These solutions correspond to quasi-black holes. These important results are investigated in more detail in the following section.

The above analytical results regarding the equations of state of the non-relativistic polytropic, as well as of the relativistic polytropic fluid, at very large polytropic exponents can be confirmed resorting to the numerical calculation. For instance, the behavior of the energy density $\rho(r)/\rho_0$ and of the fluid pressure $p(r)/\rho_0$ with the radial coordinate $r$ in case 1 is shown in the top panel of Fig. 10. As before, the central energy density for EoS 1 is $\rho_c = 10 \rho_0 = 1.78266 \times 10^{16}[\text{kg}/\text{m}^3]$. The bottom panel of the figure shows the comportment of the energy density $\rho(r)/\rho_0^\gamma$, pressure $p(r)/\rho_0^\gamma$, and rest-mass density $\delta(r)/\delta_0^\gamma$ of the relativistic polytropic fluid (case 2) against the radial coordinate, with the central rest-mass density...
given by $\delta_c = 10\delta_0^* = 1.78266 \times 10^{16}$[kg/m$^3$]. For every plots in Fig. 10, the same charge fraction $\alpha = 0.99$ and the same polytropic exponent $\gamma = 17.0667$ were used.

In both models the pressure inside the spheres starts with the same value at the center of the sphere and decreases monotonically with the radial coordinate. The pressure starts with a very high value at the origin, $r = 0$, and reaches its minimum value on the surface of the sphere, at $r = R$. On the other hand, note that the energy density for case 2 has a completely different behavior when compared to case 1. In case 1, the energy density is nearly constant, starting with $\rho(r)/\rho_0 = 10$ at $r = 0$ and decreasing very slowly with $r$ until the surface of the sphere at $r = R$, where it reaches its minimum value. For case 2, the energy density starts with the high value $\rho(r)/\rho_0^* = 10^{18.3}$ at the center of the object, and varies rapidly with the radial coordinate to reach a value close to zero at the surface the object $r = R$. Finally, in reference to the rest-mass density function, shown in the bottom panel of Fig. 10, we see that it is approximately a constant, starting with the value $\delta(r)/\delta_0^* = 10$ at $r = 0$ and decaying very slowly toward the surface of the sphere $r = R$.

4.2. Large polytropic exponents: The Buchdahl bound, the Buchdahl-Andréasson bound, and the quasiblack hole limit

The existence of upper bounds for compact objects is one of the remarkable predictions of general relativity. The upper limit established by Buchdahl [21] in the case of uncharged fluid spheres was extended to include electric charged fluid spheres by Andréasson [24]. Our main concern here is testing these bounds for the relativistic polytropic spheres (case 2). So, we search in the parameter space, namely varying the central mass-density $\delta_c$, the charge fraction $\alpha$, and the polytropic exponent $\gamma$, for the most compressed objects. The outcome of such a search is that the extremely compressed spheres are found for large polytropic exponents. The central mass-density is not important, while the charge fraction is relevant but not essential since the Buchdahl bound is found for zero charge. The extremely compressed objects are found for large $\gamma$, but the compactness ratio $R/M$ depends also on $\alpha$, varying from $R/M = 9/4$ at $\alpha = 0$ to $R/M \approx 1$ for $\alpha = 0.99$.

Figure 11 shows the behavior of the ratio $R/M$ as a function of $Q/M$ for the most compressed stellar static objects that follow from the EoS 1 (solid line) with $\rho_c = 1.78266 \times 10^{16}$[kg/m$^3$], and for EoS 2 with $\delta_c = 1.78266 \times 10^{16}$[kg/m$^3$], as indicated. The Buchdahl-Andréasson bound is also shown for comparison (dashed line). This bound is saturated only in the limit of large charge fraction, $\alpha \to 1$, for which the quasiblack hole limit is reached.

$$\frac{R}{M} \geq \frac{9}{\left(1 + \sqrt{1 + 3Q^2/R^2}\right)^2},$$

(21)

is also depicted in Fig. 11. This equation reproduces the Buchdahl bound for $Q = 0$, viz, $R/M \geq 9/4$, and deliver the extremal compactness for $Q = M$, i.e., $2.25 \geq R/M \geq 1$. Notice that the two curves for the polytropic spheres (the solid and the dotted curves) appear above the Buchdahl-Andréasson bound for all $Q/R$. 

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**FIG. 10:** Top panel: The radial dependency of the normalized functions $p(r)/p_0$ and $\rho(r)/\rho_0$ with $\rho_c = 1.78266 \times 10^{16}$[kg/m$^3$] in case 1. Bottom panel: The radial dependency of the normalized $p(r)/p_0^*$, $\rho(r)/\rho_0^*$ and $\delta(r)/\delta_0^*$ with $\delta_c = 1.78266 \times 10^{16}$[kg/m$^3$] in case 2. In both cases it was used $\alpha = 0.99$ and $\gamma = 17.0667$. The normalization factors are $p_0 = \rho_0 = 1.78266 \times 10^{15}$[kg/m$^3$], $p_0^* = \rho_0^* = 2.62974 \times 10^{12}$[kg/m$^3$], and $\delta_0^* = 1.78282 \times 10^{15}$[kg/m$^3$].

**FIG. 11:** The most compressed objects found numerically for EoS 1 with $\rho_c = 1.78266 \times 10^{16}$[kg/m$^3$], and for EoS 2 with $\delta_c = 1.78266 \times 10^{16}$[kg/m$^3$], as indicated. The Buchdahl-Andréasson bound is also shown for comparison (dashed line). This bound is saturated only in the limit of large charge fraction, $\alpha \to 1$, for which the quasiblack hole limit is reached.
In the limit of zero charge, $Q/R \to 0$, the ratio $R/M$ for the non-relativistic polytropic spheres (case 1) approaches the upper limit of the Buchdahl bound $R/M \to 9/4 = 2.25$ better than for the relativistic polytropic spheres (case 2). Noticing that the limit of infinitely high polytropic exponents yields an incompressible fluid in case 1, the previous results of Ref. [23] (see also [19]) assure that the Buchdahl bound is saturated by the uncharged fluid spheres in such a case. The numerical calculation does not reach the ceiling value $M/R = 9/4$ since the method employed here does not allow to go beyond $\gamma = 17.0667$. In the same limit of zero electric charge, the curve for the relativistic polytropic spheres (case 2) fails to converge to $R/M = 9/4$. In fact, the values shown in Fig. 11 at $Q = 0$ are $R/M \simeq 2.28$ for the EoS 1, and $R/M \simeq 2.35$ for the EoS 2. Thus, the Buchdahl bound is not saturated by the uncharged fluid spheres with the relativistic polytropic equation of state (case 2).

On the other side of the parameter space, for large charge fractions, $\alpha \to 1$, the two curves for charged spheres converge to the Buchdahl-Andrèasson line. This means that the two equations of state model very compressed objects that saturate the Buchdahl-Andrèasson bound in such a limit. The three lines converge to the quasiblack hole limit $R = M = Q$. As a matter of fact, in the cases analyzed here, the maximum value of the charge fraction is $\alpha = 0.99$ rather than $\alpha = 1.0$, since we have not found static equilibrium solutions (the numerical method does not converge) for $\alpha$ larger than 0.99. For this value of $\alpha$ we have found $R/M \simeq Q/M \simeq 1.02676$ in case 1, and $R/M \simeq Q/M \simeq 1.02514$ in case 2.

Let us stress that in Fig. 11 the three lines showed are very close to each other in the region $Q/R \simeq 1.0$. However, these lines do not coincide, thus, indicating that the values of $R/M$ shown by the dotted line and the solid line are near but always larger than those shown by the dashed line. The numerical results indicate that the three lines shall coincide just in the limit $\alpha \to 1$ with $\gamma \to \infty$.

5. THE SPEED OF SOUND IN RELATIVISTIC POLYTROPIC CHARGED SPHERES

The aim here is to verify the limit, if there is one, where causality may be violated, as done in Ref. [19] for the non-relativistic polytropes.

The speed of sound in a compressible fluid is determined through the relation $c_s^2 = dp/d\rho$. For the non-relativistic polytropic equation of state (EoS 1), this gives

$$c_s^2 = \gamma \omega \rho^\gamma = \frac{\gamma p}{\rho}. \tag{22}$$

For the relativistic polytropic equation of state (EoS 2), we get

$$c_s^2 = \frac{dp}{d\rho} = \frac{\gamma p}{\rho + p}. \tag{23}$$

First we comment on the dependence of the speed of sound in terms of the energy density. It is well known that, for any $\gamma > 1$, EoS 1 violates the causality condition ($c_s^2 \leq 1$) for large energy densities. Namely, if $\gamma \omega \rho^{\gamma-1} > 1$ then, since $\omega$ is a positive constant parameter, for sufficiently large $\rho$, it gives $c_s^2 > 1$ for any given $\gamma > 1$ and $\omega > 0$. On the other hand, it is also known that EoS 2 does not violate the constraint $c_s^2 \leq 1$ for large $\rho$. In fact, by taking the limit of large energy densities of the ratio $\gamma p/(\rho + p)$ it yields $c_s^2 = \gamma (\gamma - 1)/2$. One then sees that $c_s^2$ equals unity for $\gamma = 2$. Therefore, as also known, the relativistic equation of state does not violate causality for $\gamma$ in the interval $1 \leq \gamma \leq 2$.

Now we comment on the dependence of the speed of sound in terms of the polytropic exponent. Since the speed of sound decreases toward the surface of the sphere, as happens to the pressure, to see if the velocity of sound exceeds the speed of light it is only necessary to analyze the speed of sound in the center of the objects. The dependence of the central (at $r \to 0$) speed of sound upon the polytropic exponent is shown in Fig. 12, where we plot the results for the non-relativistic polytropic equation of state (EoS 1)) with $\rho_c = 1.78266 \times 10^{16}[kg/m^3]$ (top panel), and for the relativistic polytropic equation of state (EoS 2) with $\delta_c = 1.78266 \times 10^{16}[kg/m^3]$ (bottom panel). We determine that the speed of sound $c_s$ in the center of the spheres reaches the speed of light at $\gamma \simeq 3.31120$ for the EoS 1, and at $\gamma \simeq 3.52364$ for the EoS 2.

Despite the fact that, in both kinds of fluids, the sound speed surpasses the speed of light for sufficiently high values of the Polytropic exponent $\gamma$, these solutions are interesting because, in such a limit, the fluids become incompressible and the quasiblack hole limit is reached.
6. THE QUASIBLACK HOLE LIMIT OF A RELATIVISTIC POLYTROPIC CHARGED SPHERE

6.1. Basic properties and the quasiblack hole limit

From the results reported in Ref. [19] and reproduced here in the top panels of Figs. 4 and 6, it is verified that the non-relativistic polytropes (EoS 1) with charge fraction \( \alpha = 0.99 \) and polytropic exponent \( \gamma = 17.0667 \) are very close to the quasiblack hole configuration. Here, we have verified that a similar situation happens for the relativistic charged polytropes (EoS 2), as seen in the bottom panels of the cited figures. For \( \alpha = 0.99 \) and \( \gamma = 17.0667 \) one has \( R \simeq M \simeq Q \) (with \( R, M \) and \( Q \) expressed in geometric units), indicating that these objects are also quasiblack holes. In fact, it was argued in [19] that charged polytropic spheres in the limit of infinitely large polytropic exponent and charge fraction reaching unity are quasiblack holes. Now we check if the relativistic polytropic equation with \( \alpha \rightarrow 1 \) and \( \gamma \rightarrow \infty \) yields quasiblack holes too. For this purpose, following the defining properties of a static quasiblack hole put forward in Ref. [20], the potential metrics \( A(r) \) and \( B(r) \) are analyzed. In addition, we make a comparison of the behavior of the metric potentials for both cases (the EoS 1 and the EoS 2), at the largest values \( \alpha = 0.99 \) and \( \gamma = 17.0667 \), as a function of the radial coordinate. The central energy density and central rest-mass density are \( \rho_c = 10\rho_0 = 1.78266 \times 10^9[\text{kg/m}^3] \) and \( \delta_c = \delta_0^* = 1.78266 \times 10^9[\text{kg/m}^3] \) for Eos 1 and EoS 2, respectively.

The inverse of the metric function \( A(r) \) versus the radial coordinate \( r \) is plotted in Fig. 13 for the EoS 1 (top panel) and the EoS 2 (bottom panel). Near the origin \( (r \sim 0) \) there is a sharp difference between the two shown curves. In the case 2, function \( 1/A(r) \) presents a jump from 1 to about 0.823 at the central region. This is due to the fact that, close to the center, the electric charge \( q(r) \) and mass \( m(r) \) grow rapidly with \( r \) due to the very large values of the central energy and charge densities. Nevertheless, in case 1 their growth is smooth since the respective densities are not very high. Function \( A^{-1}(r) \) decreases with the increasing of the radial coordinate, reaching its minimum value, namely, \( A^{-1}(R) \sim 0 \), at the surface of the object. Such a small value signals that the object is close to a quasiblack hole configuration.

The interior metric is matched to the exterior Reissner-Nordström metric, i.e., \( A^{-1}(r) = 1 - 2M/R + Q^2/R^2 \), from what follows that the quasihorizon is present.

The metric function \( B(r) \) is shown in Fig. 14, for the cases 1 (top panel) and 2 (bottom panel). Note in the figure that function \( B(r) \) assumes values close to zero in the interior of the sphere, i.e., \( B(r) \rightarrow 0 \) in the whole interval \( 0 \leq r \leq R \). This feature also reveals that we are close to the quasiblack hole configuration. Since the interior solution is matched to the exterior vacuum Reissner-Nordström solution, it follows we have \( B(R) = A^{-1}(R) = 1 - 2M/R + Q^2/R^2 \sim 0 \), confirming once again the presence of a quasi-horizon.

Besides the defining properties of the metric potentials, as just checked, in the case of static charged spacetimes, another important property of quasiblack holes is the existence of an extremal limit for the ratio \( Q/M \). As already mentioned, for large Polytropic exponent and charge fraction close to unity, we have found that the radius, the total mass, and the total charge of the relativistic polytropic spheres are close to each other \( (R \simeq M \simeq Q) \). Then, considering the two solutions of equation \( F(r) = 1 - 2M/r + Q^2/r^2 = 0 \), which are \( r_{\pm} = M \pm \sqrt{M^2 - Q^2} \), we get \( r_+ \simeq r_- \simeq M \simeq Q \simeq R \). Moreover, the numerical analysis shows that the extremal
bound $R = Q = M$ is continuously approached with the increasing of the polytropic exponent and, in particular, of the charge fraction $\alpha$. This means that the radius of the charged matter distribution is reaching the gravitational radius from above, $R \gtrsim r_+$, assuring the solution is regular, static, and very close to extremality, i.e., the quasiblack hole with pressure limit is being attained.

Table I presents the mass $M$, the charge $Q$, the radius $R$, and their relations for the relativistic polytropes (EoS 2) with $\alpha = 0.99$, and for two values of the polytropic exponent, $\gamma = 17.0667$ and $\gamma = 17.1109$. These are the highest values of $\gamma$ the numerical procedure yielded results without convergence problems. For comparison, the same quantities for the EoS 1 case are also listed in the table (row A). By analyzing rows B and C it is seen that $R/M$ and $Q/M$ are closer to unity in the EoS 2 case than in the EoS 1 case. Based on these results we note that the relativistic polytropic spheres with an infinitely large polytropic exponent and charge fraction approaching unity attain the quasiblack hole limit, i.e., approach $R/M = Q/M = 1.0$, faster than the non-relativistic polytropic spheres do.

6.2. The redshift at the surface of a quasiblack hole

![Diagram](image)

FIG. 15: The redshift function $B(R)^{-1/2} - 1$ at the surface of the spheres for two values of the polytropic exponent, $\gamma = 4/3$ and 17.0667, for the EoS 1 with $\rho_c = 1.78266 \times 10^{16} [\text{kg/m}^3]$ (top panel), and for the EoS 2 with $\delta_c = 1.78266 \times 10^{16} [\text{kg/m}^3]$ (bottom panel). The charge fraction varies in the range $0 < \alpha < 1$.

To be complete, we calculate the quantity $B(R)^{-1/2} - 1$ for both the non-relativistic and the relativistic charged polytropes, and taking two values of the polytropic exponent, $\gamma = 4/3$ and 17.0667. The expression $B(R)^{-1/2} - 1$ gives the redshift at the surface of the star, which is defined in the usual way by the fractional difference between the light wave frequency at the surface of the star (at $r = R$) with respect to infinity (at $r \to \infty$). The dependence of the redshift as a function of the charge fraction is plotted in Fig. 15 for the two equations of stated investigated in the present work. As expected from previous works on the non-relativistic polytropes [20, 23], the redshift at the surface of the quasiblack hole limit is infinitely large. Numerically we determine values of the redshift of about 100 in the cases with $\alpha = 0.99$ and $\gamma = 17.0667$. Again the results for the EoS 2 are very close to those for the EoS 1 (see [23]), but the redshift is a little higher for the relativistic polytropes (EoS 2).

7. CONCLUSIONS

We compared the stellar structure configurations of charged objects made of a non-relativistic polytropic fluid [case 1, see Eq. (1)] with those composed by a relativistic polytropic fluid [case 2, see Eq. (2)] in the Maxwell-Einstein theory. For the two cases analyzed, i.e., for the non-relativistic polytropic and relativistic polytropic cases, we used respectively the equation of state $p = \omega \rho^\gamma$ (EoS 1, case 1), and $p = \omega \delta^\gamma$, with $\delta = \rho - p/(\gamma - 1)$ (EoS 2, case 2). The parameters $\omega$ and $\gamma$ represent respectively the polytropic constant and polytropic exponent. The chosen value of $\omega$ is such that for $\gamma = 5/3$ the results found in [17–19] for polytropic stars are reproduced. The configurations studied are assumed to be composed by a spherically symmetric distribution of a charged perfect fluid, and by an exterior vacuum region described by Reissner-Nordström metric. We assumed a charge density profile directly proportional to the energy density, of the form $\rho_c = \alpha \rho$ (with $\alpha$ being the charge fraction). By varying the fundamental parameters of each model, we analyzed some limits found in general relativity, such as the Chandrasekhar limit, the Oppenheimer-Volkoff limit, the Buchdahl bound, and the Buchdahl-Andréasson bound and the quasiblack hole limit.

First the analysis was done by varying the central energy density $\rho_c$ (for case 1), the central rest mass density $\delta_c$ (for case 2), and comparing the physical parameters (radius, mass, and charge) of respective equilibrium solutions. A few different values of the polytropic exponent and of the charge fraction were considered in such an analysis. The study confirmed that the two equations of state yield significantly different results just in the limit of high energy densities.

The configurations of the objects were also analyzed by varying the polytropic exponent $\gamma$, from 4/3 to a considerably high value. In this situation, the central energy density (for case 1) and the central rest mass density (for case 2) were kept fixed to ten times the normalization values, i.e., $\rho_c = 10 p_0$ and $\delta_c = 10 \delta_0$, with $p_0 = \delta_0 = 1.78266 \times 10^{16} [\text{kg/m}^3]$. Using such values, we varied $\gamma$ from 4/3 to 17.0667 and 17.1109, respectively, in case 1 and case 2. Higher values of $\gamma$ introduced numerical convergence problems. For the sake of comparison between the results in case 1 and 2, we considered $\gamma = 17.0667$ as a maximum value for the polytropic ex-
TABLE I: The values of the mass $M$, charge $Q$, and radius $R$ of the charged polytropic spheres, in geometric units, with the corresponding values of $R/M$ and $Q/M$, for $\alpha = 0.99$ and $\gamma = 17.0667$ are shown in rows A and B, respectively, for case 1 and case 2. The values of $M$, $Q$, $R$, $R/M$, and $Q/M$, for the EoS 2 (case 2) and for the polytropic exponent $\gamma = 17.1109$ are shown in row C.

| EoS | $\gamma$ | $M \times 10^5$ [m] | $Q \times 10^5$ [m] | $R \times 10^5$ [m] | $R/M$ | $Q/M$ |
|-----|----------|----------------------|----------------------|----------------------|--------|--------|
| A   | 1        | 17.0667              | 2.27478              | 2.27431              | 1.02676| 0.999793|
| B   | 2        | 17.0667              | 2.09502              | 2.09463              | 1.02514| 0.999813|
| C   | 2        | 17.1109              | 2.09662              | 2.09623              | 1.02512| 0.999814|

The charge faction parameter $\alpha$ was varied from zero to very close to unity, $\alpha = 0.99$. A value higher than this also implied in numerical convergence problems. Again the structure of the resulting equilibrium solutions are almost the same for both models of fluids.

In the regime of high polytropic exponents, we tested the various bounds for extremely compact objects. In fact, for the unchanged case ($\alpha = 0.0$), in the case 1, we have that for $\gamma = 17.0667$ the Buchdahl bound is saturated, i.e., the Schwarzschild interior limit is attained. However, in case 2 the Buchdahl bound is far from being saturated. In the extremely charged case ($\alpha = 0.99$), and yet with $\gamma = 17.0667$, we have that the radius $R$, the mass $M$, and the charge $Q$ of the objects are approximately the same, $R \simeq M \simeq Q$. This result is obtained for both equations of state. This result together with the specific characteristics of the potential metrics, i.e., $A^{-1}(R) \rightarrow 0$ and $B(r) \rightarrow 0$ with $r \leq R$, points the presence of a quasiblack hole.

The surface redshift of the extremely compact solutions, including the quasiblack hole limit, were analyzed. The results show higher redshifts for relativistic polytropes (case 2) than for non-relativistic polytropes (case 1).

The dependence of the sound speed $c_s$ on the polytropic exponent at the center of the compact objects was also studied. In both cases $c_s$ reaches values higher than the speed of light for sufficiently high polytropic indexes.

Finally, we emphasize that the aim of this work was to analyze the structure of relativistic polytropes by comparing to non-relativistic polytropes, with particular interest in the upper bounds of compactness established within the theory of general relativity. This is in complement of previous works by us whose results were reported in Refs. [19, 23]. The conclusion of this investigation is that the Buchdahl-Andéasson bound is not saturated in full neither by polytropic stars nor by incompressible stars. On the other hand, as shown in Ref. [25], that bound is saturated by the Guilfoyle [26] solutions, which assumes different conditions on the fluid quantities. This result suggests that a different equation of state for the charged fluid, associated to an alternative charge density profile, may lead to solutions that saturate that important bound, besides reaching the quasiblack hole limit. The analysis of such situations is left for future investigations.

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Appendix A: Equations of structure in dimensionless form

For the numerical calculations, the equations of structure are written in a dimensionless form. In Ref. [19] this was done for non-relativistic polytropes. Here we present the normalized equations of structure for relativistic polytropes only.

The normalized radial coordinate $\xi(r)$ is defined by

$$\xi = r\sqrt{\frac{4\pi\delta_c}{\mu(\xi)}} ,$$

where we have put $c = 1$ and also $G = 1$. Similarly, the electric charge function $q(r)$, the mass function $m(r)$, and the rest-mass density function $\rho(r)$ are replaced by the new normalized variables, $\mu(\xi)$, $v(\xi)$ and $\vartheta(\xi)$, respectively, defined by

$$\mu(\xi) = q(r)\sqrt{\frac{4\pi\delta_c}{\xi^2}} ,$$

$$v(\xi) = m(r)\sqrt{\frac{4\pi\delta_c}{\xi}} ,$$

$$\vartheta(\xi) = \left(\frac{\delta(r)}{\delta_c}\right)^{\gamma} ,$$

where $\delta_c$ represents the central rest mass density. In terms of $\vartheta$ and $\delta_c$, the pressure and the energy density are now given by relations $p(r) = \omega\delta^{-\gamma}_c\vartheta(\xi)$ and $\rho(r) = \delta_c\left[\vartheta^{1/\gamma} + \omega\delta^{-1}_c\vartheta/(\gamma - 1)\right]$, respectively.
With these new variables, the equations of structure (4), (8), and (10) provide,

\[
\frac{d\mu}{d\xi} = -\frac{2\mu}{\xi} + \frac{\alpha \vartheta^{1/\gamma} + \alpha (\gamma - 1)^{1-\vartheta} \vartheta}{\sqrt{1 - \frac{2v}{\xi} + \xi^2 \mu^2}}, \tag{A5}
\]

\[
\frac{dv}{d\xi} = \left[ \frac{\vartheta^{1/\gamma} + \omega \delta^1 \vartheta}{(\gamma - 1)} \right] \left[ \xi^2 + \frac{\alpha \xi^3 \mu}{\sqrt{1 - \frac{2v}{\xi} + \xi^2 \mu^2}} \right], \tag{A6}
\]

\[
\frac{d\vartheta}{d\xi} = -\xi \left[ \alpha \vartheta + \omega^{-1} \delta^{-1} \vartheta^{1/\gamma} \right] \left[ \frac{\omega \delta^{-1} \vartheta - \mu^2 + \frac{v}{\xi^3}}{1 - \frac{2v}{\xi} + \xi^2 \mu^2} \right] + \frac{\alpha \mu \omega^{-1} \delta^{-1} \vartheta^{1/\gamma} + \alpha (\gamma - 1)^{-1} \omega \vartheta}{\sqrt{1 - \frac{2v}{\xi} + \xi^2 \mu^2}}, \tag{A7}
\]

where Eq. (11) was used to eliminate the charge density.

In order to get an equilibrium solution, the coupled equations (A5)–(A7) are solved simultaneously, through numerical integration. After determining \(\mu(\xi), v(\xi),\) and \(\vartheta(\xi),\) the other function \(B(\xi)\) and \(A(\xi)\) are found from the equations

\[
\frac{dB}{d\xi} = 2\xi B \left[ \frac{\omega \delta^{-1} \vartheta - \mu^2 + \frac{v}{\xi^3}}{1 - \frac{2v}{\xi} + \xi^2 \mu^2} \right], \tag{A8}
\]

\[
A^{-1} = 1 - \frac{2v}{\xi} + \xi^2 \mu^2. \tag{A9}
\]

The boundary conditions assumed at the center of the sphere (\(\xi = 0\)) are: \(\mu(0) = 0, v(0) = 0,\) and \(\vartheta(0) = 1.\) The value of \(\xi\) at the surface of the object is determined by the condition \(\vartheta(\xi_s) = 0,\) where \(\xi_s\) is identified as the normalized radius at the surface of the sphere. The integration of Eqs. (A5), (A6), and (A7) is stopped when the value of \(\vartheta\) changes sign, from positive to negative. Once obtained the corresponding values of \(\xi_s,\) \(\mu(\xi_s),\) and \(v(\xi_s),\) the values of the radius \(R,\) the mass \(M,\) and the charge \(Q\) of the object are determined using relations (A1), (A2), and (A3), respectively.

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