Fractional hypergraph isomorphism and fractional invariants

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Abstract

Fractional graph isomorphism is the linear relaxation of an integer programming formulation of graph isomorphism. It preserves some invariants of graphs, like degree sequences and equitable partitions, but it does not preserve others like connectivity, clique and independence numbers, chromatic number, vertex and edge cover numbers, matching number, domination and total domination numbers.

In this work, we extend the concept of fractional graph isomorphism to hypergraphs, and give an alternative characterization, analogous to one of those that are known for graphs. With this new concept we prove that the fractional packing, covering, matching and transversal numbers on hypergraphs are invariant under fractional hypergraph isomorphism. As a consequence, fractional matching, vertex and edge cover, independence, domination and total domination numbers are invariant under fractional graph isomorphism. This is not the case of fractional chromatic, clique, and clique cover numbers. In this way, most of the classical fractional parameters are classified with respect to their invariance under fractional graph isomorphism.

Keywords: fractional isomorphism, fractional graph theory, fractional covering, fractional matching, hypergraphs.
1. Introduction

Graphs $G$ and $H$ are isomorphic, denoted $G \cong H$, when there is a bijection $\phi : V(G) \rightarrow V(H)$ so that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. In other words, graphs $G$ and $H$ are isomorphic if they differ only in the names of their vertices. In terms of matrices, if $A$ and $B$ are the adjacency matrices of $G$ and $H$, then $G \cong H$ if and only if there is a permutation matrix $P$ so that $A = PBP^{-1}$. The relation $A = PBP^{-1}$ can be rewritten as $AP = PB$, and the requirement that $P$ is a permutation matrix can be restated as “$P \cdot \mathbf{1} = 1$, $P^t \cdot \mathbf{1} = 1$, and the entries of $P$ are in $\{0,1\}$”, where $P^t$ denotes the transposed matrix of $P$ and $\mathbf{1}$ stands for a vector of all 1’s. So, the graph isomorphism problem can be viewed as an integer programming feasibility problem, where $A$ and $B$ are given and the unknowns are the coefficients of matrix $P$.

In [30], Ramana, Scheinerman, and Ullman consider a linear relaxation of the integer programming formulation and denote the concept by fractional isomorphism of two graphs. Namely, they drop the requirement that $P$ is a $\{0,1\}$-matrix and simply require the entries in $P$ to be nonnegative. A matrix $S$ whose entries are nonnegative, and whose rows and columns all sum up to 1 (i.e., $S \cdot \mathbf{1} = 1$ and $S^t \cdot \mathbf{1} = 1$) is called a doubly stochastic matrix. Graphs $G$ and $H$ are said to be fractionally isomorphic, $G \cong_f H$, provided there is a doubly stochastic matrix $S$ for which $AS = SB$ where $A$ and $B$ are the adjacency matrices of the graphs $G$ and $H$, respectively.

The concept of fractional isomorphism fits within the more general concept of fractional graph theory, surveyed by Ullman and Scheinerman in [34], in which fractional relaxations of classical (integer) combinatorial optimization problems are studied. Some of the notions involved in the main results about fractional isomorphism are already present in the work of Brualdi [7], Godsil [18], Leighton [22], McKay [26], Mowshowitz [27], and Tinhofer [33], under different names, mainly with the aim of having tools to efficiently reject some instances of the isomorphism problem, whose computational complexity is still open. The current best result is a quasipolynomial time algorithm by Babai [3]. The color refinement procedure, introduced in 1968 by Weisfeiler and Lehman [35], was also related recently to fractional isomorphism in [2].

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It has been proved that the fractional isomorphism is an equivalence relation that generalizes isomorphism, and preserves some invariants of graphs as the number of vertices and edges, and the degree sequence. Indeed, any two \( r \)-regular graphs on \( n \) vertices are fractionally isomorphic. In particular, the disjoint union of two triangles and a cycle of length six are fractionally isomorphic. So, properties like connectivity, clique and independence numbers, chromatic number, vertex, edge, and clique cover numbers, and matching number, are not preserved. There are also similar examples to show that neither are domination and total domination numbers.

In this work, we extend the concept of fractional graph isomorphism to hypergraphs, and give an alternative characterization, analogous to one of those that are known for graphs. With this new concept we prove that the fractional packing, covering, matching and transversal numbers on hypergraphs are invariant under fractional hypergraph isomorphism. As a consequence, fractional matching, vertex and edge cover, independence, domination and total domination numbers are invariant under fractional graph isomorphism. This is not the case of fractional chromatic, clique, and clique cover numbers. In this way, most of the classical fractional parameters are classified with respect to their invariance under fractional graph isomorphism.

2. Definitions and basic results

2.1. Graph theory

All graphs in this work are finite, undirected, and have no loops or multiple edges. For all graph-theoretic notions and notation not defined here, we refer to West [36]. Let \( G \) be a graph. Denote by \( V(G) \) its vertex set, and by \( E(G) \) its edge set. In general, we will assume \( n = |V(G)| \) and \( m = |E(G)| \).

Denote by \( N(v) \) the neighborhood of a vertex \( v \) in \( G \), and by \( N[v] \) the closed neighborhood \( N(v) \cup \{v\} \). If \( X \subseteq V(G) \), denote by \( N(X) \) the set of vertices not in \( X \) having at least one neighbor in \( X \). A vertex \( v \) of \( G \) is universal (resp. isolated) if \( N[v] = V(G) \) (resp. \( N(v) = \emptyset \)).

Denote by \( d(v) \) the degree of a vertex \( v \) of \( G \), i.e., \( d(v) = |N(v)| \), and by \( \Delta(G) \) (resp. \( \delta(G) \)) the maximum (resp. minimum) degree of a vertex in \( G \). Let \( S \) be a subset of the vertex set of a graph \( G \). Let \( d(v, S) \) denote the degree of \( v \) in \( S \), i.e., \( d(v, S) = |N(v) \cap S| \).
The degree sequence of a graph $G$ is the multiset of the degrees of its vertices. A graph is $k$-regular if every vertex has degree $k$, and regular if it is $k$-regular for some $k$.

The adjacency matrix of a graph $G$ with vertices $v_1, \ldots, v_n$ is a matrix $A_G \in \{0, 1\}^{n \times n}$ such that $A_G(i, j) = 1$ if $v_i v_j \in E(G)$ and $A_G(i, j) = 0$, otherwise. If the edges of $G$ are numbered as $e_1, \ldots, e_m$, $m \geq 1$, the vertex-edge incidence matrix of $G$ is a matrix $M_G \in \{0, 1\}^{n \times m}$ such that $M_G(i, j) = 1$ if $v_i$ is one of the endpoints of $e_j$, and $M_G(i, j) = 0$, otherwise.

We will denote by $J_n$ (resp. $J_{n \times m}$) the $(n \times n)$-matrix (resp. $(n \times m)$-matrix) with all its entries equal 1.

Given a graph $G$ and $W \subseteq V(G)$, denote by $G[W]$ the subgraph of $G$ induced by $W$. Denote the size of a set $S$ by $|S|$.

Denote by $C_n$ a chordless cycle on $n$ vertices.

A clique or complete set (resp. stable set or independent set) is a set of pairwise adjacent (resp. non-adjacent) vertices. The size of a maximum size clique in a graph $G$ is called the clique number and denoted by $\omega(G)$. The size of a maximum size independent set in a graph $G$ is called the independence number and denoted by $\alpha(G)$.

A matching of a graph is a set of pairwise disjoint edges (i.e., no two edges share an endpoint). The maximum number of edges of a matching in a graph $G$ is called the matching number and denoted by $\mu(G)$.

A vertex cover is a set $S$ of vertices of a graph $G$ such that each edge of $G$ has at least one endpoint in $S$. Analogously, an edge cover is a set $F$ of edges of a graph $G$ such that each vertex of $G$ belongs to at least one edge of $F$. Denote by $\tau(G)$ (resp. $k(G)$) the size of a minimum vertex (resp. edge) cover of a graph $G$, called the vertex (edge) cover number of $G$.

A clique cover is a set $F$ of cliques of a graph $G$ such that each vertex of $G$ belongs to at least one clique of $F$. Denote by $\theta(G)$ the size of a minimum clique cover of a graph $G$, called the clique cover number of $G$.

A dominating set in a graph $G$ is a set of vertices $S$ such that every vertex in $V(G)$ is either in $S$ or adjacent to a vertex in $S$. The domination number $\gamma(G)$ of a graph $G$ is the size of a smallest dominating set. A total dominating set in a graph $G$ is a set of vertices $S$ such that every vertex in $V(G)$ is adjacent to a vertex in $S$. The total domination number $\Gamma(G)$ of a graph $G$ is the size of a smallest total dominating set.

A coloring of a graph is an assignment of colors to its vertices such that any two adjacent vertices are assigned different colors. The smallest number $t$ such that $G$ admits coloring with $t$ colors (a $t$-coloring) is called the chromatic
number of $G$ and is denoted by $\chi(G)$. A coloring defines a partition of the vertices of the graph into stable sets, called color classes.

A graph is bipartite if it admits a 2-coloring, and a bipartition of it is a partition $(A, B)$ of its vertex set into two stable sets. A bipartite graph $(A \cup B, E)$ is called biregular if the vertices of $A$ have the same degree $a$ and the vertices of $B$ have the same degree $b$ (where not necessarily $a = b$). If we want to make explicit these values, we write $(a, b)$-regular. Given a graph $G$ and two disjoint subsets $A, B$ of $V(G)$, the bipartite graph $G[A, B]$ is defined as the subgraph of $G$ formed by the vertices $A \cup B$ and the edges of $G$ that have one endpoint in $A$ and one in $B$. Notice that $G[A, B]$ is not necessarily an induced subgraph of $G$.

2.2. Matrix theory

We will state here some well known definitions and results from matrix theory which we need in this paper, and can be found, for example, in [20].

Every doubly stochastic matrix $S$ can be written as a convex combination of permutation matrices, i.e., $S = \sum_{i \in I} \alpha_i P_i$, where $\sum \alpha_i = 1$, the $\alpha_i$'s are positive and each $P_i$ is a permutation matrix. This convex combination is known as a Birkhoff decomposition of $S$.

Let $A$ and $B$ be square matrices. The direct sum of $A$ and $B$ is the square matrix

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

If $M = A \oplus B$ we say $M$ is decomposable. In general, $M$ is decomposable if there exists $A$, $B$, $P$ and $Q$ such that $P$ and $Q$ are permutation matrices and $M = P(A \oplus B)Q$. If no such decomposition exists, we say that $M$ is indecomposable.

Let $M$ be a $n \times n$ matrix. Let $D(M)$ be a digraph on $n$ vertices $v_1, \ldots, v_n$ with an arc from $v_i$ to $v_j$ if $M_{ij} \neq 0$. We say that $M$ is irreducible when $D(M)$ is strongly connected. Otherwise, we say that $M$ is reducible. Furthermore, we say that a matrix $M$ is strongly irreducible provided $PM$ is irreducible for any permutation matrix $P$.

**Proposition 2.1.** [34] Let $S$ be a doubly stochastic, indecomposable matrix. Then $S$ is also strongly irreducible.
Theorem 2.2. [19] Let $S, R$ be two doubly stochastic matrices of dimensions $n \times n$ with Birkhoff’s decomposition $S = \sum \alpha_i P_i$ and $R = \sum \beta_j Q_j$, respectively. Let $x, y$ be vectors of length $n$

(1) If $y = Sx$ and $x = Ry$, then $y = P_i x$ and $x = Q_j y$ for every $i, j$.

(2) Let $x, y$ as in (1). If, in addition, either $S$ or $R$ is indecomposable, then $x = y = s \cdot 1$ for some scalar $s$.

(3) If $x$ and $y$ are $\{0, 1\}$-vectors and $y = Sx$, then $y = P_i x$ for every $i$.

2.3. Main results on fractional isomorphism

We will survey here the main definitions and results on fractional isomorphism, as they are stated in [30, 34]. Some of them were partially and independently shown in [1, 18, 22, 26, 27, 33], with different notations.

Definition 2.3. Let $G, H$ be graphs and $A_G, A_H$ their adjacency matrices, respectively. We say that $G$ and $H$ are fractionally isomorphic, and we write $G \cong_{f} H$, if there exists a doubly stochastic matrix $S$ such that $A_G S = S A_H$.

Proposition 2.4. The relation $\cong_{f}$ is an equivalence relation that preserves the usual graph isomorphism.

Proposition 2.5. If $G \cong_{f} H$ for two graphs $G$ and $H$ then:

1. $G$ and $H$ have the same number of vertices;
2. $G$ and $H$ have the same number of edges;
3. $G$ and $H$ have the same degree sequence;
4. the adjacency matrices $A_G$ and $A_H$ have the same maximum eigenvalue.

Proposition 2.6. If $G$ and $H$ are two $r$-regular graphs with $n$ vertices then $G \cong_{f} H$.

This implies that the disjoint union of two triangles and a cycle of length six are fractionally isomorphic. However, $2C_3$ is not connected, $\omega(2C_3) = \chi(2C_3) = 3$, $\alpha(2C_3) = \theta(2C_3) = \mu(2C_3) = 2$, and $\tau(2C_3) = k(2C_3) = 4$, while $C_6$ is connected, $\omega(C_6) = \chi(C_6) = 2$, $\alpha(C_6) = \theta(C_6) = \mu(C_6) = 3$, $\tau(C_6) = k(C_6) = 3$. So the properties of connectivity, clique and independence
numbers, chromatic number, edge, vertex, and clique cover numbers, and matching number, are not preserved by fractional isomorphism.

Similarly, \((C_5 \cup C_7) \cong f C_{12}\). However, \(\gamma(C_5 \cup C_7) = 5\), \(\gamma(C_{12}) = 4\), \(\Gamma(C_5 \cup C_7) = 7\), and \(\Gamma(C_{12}) = 6\). So, domination and total domination numbers are not preserved by fractional isomorphism.

The notion of fractional isomorphism is deeply related to the one of equitable partition. We say that a partition \(\{V_1, \ldots, V_s\}\) of \(V(G)\) is equitable provided that for all \(i, j\) and all \(x, y \in V_i\) we have \(d(x, V_j) = d(y, V_j)\). In other words, each of the induced subgraphs \(G[V_i]\) must be regular and each of the bipartite graphs \(G[V_i, V_j]\) must be biregular. It is clear that every graph has an equitable partition: each vertex is a class by itself. If \(G\) is regular, then the singleton \(\{V(G)\}\) is an equitable partition. Equitable partitions of a graph are partially ordered by the usual refinement relation for partitions, i.e., if \(P\) and \(Q\) are partitions of a common set \(S\), we say that \(P\) is a refinement of \(Q\) provided every part of \(P\) is a subset of some part in \(Q\). When \(P\) is a refinement of \(Q\) we also say that \(P\) is finer than \(Q\) and that \(Q\) is coarser than \(P\). The equitable partitions of a graph form a lattice \([26]\). A maximum element of the equitable partition lattice is denoted as a coarsest equitable partition of \(G\).

Theorem 2.7. \([26]\) Let \(G\) be a graph. Then \(G\) has a unique coarsest equitable partition.

Let \(G\) be a graph and let \(P = \{P_1, \ldots, P_p\}\) be an equitable partition of \(V(G)\). The parameters of \(P\) are a pair \((v, D)\) where \(v\) is a \(p\)-vector whose \(i\)-th entry is the size of \(P_i\) and \(D\) is a \((p \times p)\)-matrix whose \(ij\)-entry is \(d(x, P_j)\) for any \(x \in P_i\). We say that equitable partitions \(P\) and \(Q\) of graphs \(G\) and \(H\) have the same parameters if we can index the sets in \(P\) and \(Q\) so that their parameters \((v, D)\) are identical. In such a case we say that \(G\) and \(H\) have a common equitable partition. If, in addition, \(P\) and \(Q\) are coarsest equitable partitions of \(G\) and \(H\), then we say that \(G\) and \(H\) have a common coarsest equitable partition.

Another concept which is central to the understanding of fractional isomorphism is that of the iterated degree sequence of a graph. Let us recall that the degree of a vertex \(v\) in \(G\) is the cardinality of its neighbor set, \(d(v) = |N(v)|\), and the degree sequence of a graph \(G\) is
the multiset of the degrees of its vertices, \( d_1(G) = \{ d(v) : v \in V(G) \} \). The **degree sequence of a vertex** is the multiset of the degrees of its neighbors: \( d_1(v) = \{ d(w) : w \in N(v) \} \). In general, for \( k \geq 1 \), define: \( d_{k+1}(G) = \{ d_k(v) : v \in V(G) \} \), and \( d_{k+1}(v) = \{ d_k(w) : w \in N(v) \} \). The **ultimate degree sequence** of a vertex \( v \) or a graph \( G \) are defined as the infinite lists: \( \mathcal{D}(v) = [d_1(v), d_2(v), \ldots] \), and \( \mathcal{D}(G) = [d_1(G), d_2(G), \ldots] \). The equivalence between having a fractional isomorphism, having a common coarsest equitable partition, and having the same ultimate degree sequence is the main theorem about fractional isomorphism.

**Theorem 2.8.** Let \( G \) and \( H \) be graphs. The following are equivalent.

1. \( G \cong_f H \).

2. \( G \) and \( H \) have a common coarsest equitable partition.

3. \( G \) and \( H \) have some common equitable partition.

4. \( \mathcal{D}(G) = \mathcal{D}(H) \).

3. **Fractional hypergraph isomorphism**

An **hypergraph** \( G \) is a pair \( G = (S, X) \) where \( S = V(G) \) is a finite set and \( X = E(G) \) is a family of subsets of \( S \). The set \( S \) is called the set of **vertices** of the hypergraph. The elements of \( X \) are the **hyperedges** or **edges** of the hypergraph. The **degree** of a vertex is the number of hyperedges which contains it. A hypergraph is **\( r \)-regular** if every vertex has degree \( r \), and it is **\( r \)-uniform** if all the hyperedges are of the same cardinality \( r \). Graphs are the 2-uniform hypergraphs.

For a hypergraph \( G \) with at least one edge, the **vertex-hyperedge incidence matrix** \( M_G \) is the matrix having \( n \) rows (as vertices in the set \( S \)) and \( m \) columns (as hyperedges in the set \( X \)), and such that \( M_{ij} = 1 \) if the vertex \( i \) belongs to the hyperedge \( j \) and 0, otherwise. If the hypergraph \( G \) is a graph (i.e., 2-uniform), \( M_G \) coincides with the usual vertex-edge incidence matrix of the graph, so the notation \( M_G \) is well defined. The hypergraph \( G \) is a graph if and only if every column of \( M_G \) has exactly two 1’s. In other words, \( 1^t \cdot M_G = 2 \cdot 1^t \). On the other hand, it is not difficult to see that \( M_G \cdot 1 = \delta_1 \), where \( \delta_1 \) is the degree of the vertex \( i \) in \( G \). The **dual hypergraph** of a hypergraph \( H \), denoted \( H^* \), is the hypergraph whose vertex-hyperedge
The incidence matrix is \( M_H \). The 2-section of a hypergraph \( H \) is the graph with the same vertex set as \( H \), and such that two vertices are adjacent if and only if there is a hyperedge of \( H \) that contains both of them.

In order to define a fractional hypergraph isomorphism notion, we seek for a matrix equation characterizing the (hyper)graph isomorphism in terms of incidence matrices instead of adjacency matrices. Indeed, for two graphs \( G \) and \( H \), \( G \cong H \) if and only if there exists permutation matrices \( P_1 \) and \( P_2 \) such that \( P_1M_G = M_H P_2^t \) and \( M_GP_2 = P_1^tM_H \), and these equations hold also for hypergraphs. We will linearly relax these conditions in the same way as in the fractional graph isomorphism definition.

**Definition 3.1.** Let \( G \) and \( H \) be hypergraphs. We write \( G \equiv H \) if either \( G \) and \( H \) have the same number of vertices and no hyperedges, or their vertex-hyperedge incidence matrices \( M_G \) are such that there exist two doubly stochastic matrices \( S_1 \) and \( S_2 \) so that \( S_1M_G = M_H S_2^t \) and \( M_G S_2 = S_1^tM_H \).

It is clear that if two matrices \( M_G \) and \( M_H \) follow the conditions in the last definition, then they have the same dimensions (both have \( n \) rows and \( m \) columns, for some \( n, m \)).

**Proposition 3.2.** The relation \( \equiv \) is an equivalence relation.

**Proof.** It is straightforward for the case with no hyperedges. So we assume there is at least one hyperedge. **Reflexivity:** \( I_nM_G = M_G I_m \) and \( M_G I_m = I_n^tM_G \), where \( I_n = I_n^t \) and \( I_m = I_m^t \) are the identity matrices of orders \( n \) and \( m \), respectively (and \( n \) and \( m \) are the number of vertices and hyperedges of \( G \), respectively).

**Symmetry:** evident because renaming \( S_1^t = U_1 \) and \( S_2^t = U_2 \) we obtain
\[
U_1M_H = M_GU_2 \quad \text{and} \quad M_HU_2 = U_1^tM_G.
\]

**Transitivity:** if \( S_1M_G = M_H S_2^t \) and \( M_G S_2 = S_1^tM_H \), and \( S_3M_G = M_K S_4^t \) and \( M_G S_4 = S_3^tM_K \), we have \( S_1M_G S_4 = M_H S_2^t S_4 \). Also \( S_1M_G S_4 = S_1 S_4^t M_K \), so we obtain \( M_H S_2^t S_4 = S_1 S_4^t M_K \). In the same way, we have \( S_3 M_G S_2 = S_3 S_1^t M_H \) and also \( S_3 M_G S_2 = M_K S_4^t S_2 \), so we obtain \( S_3 S_1^t M_H = M_K S_4^t S_2 \). Renaming \( U_1 = S_2 S_4 \) and \( U_2 = S_3 S_1 \), we have \( M_H U_1 = U_2^t M_K \) and \( U_2 M_H = M_K U_1^t \). (The product of two doubly stochastic matrices is a doubly stochastic matrix). \( \square \)
3.1. The relation \( \equiv \) for 2-uniform hypergraphs

We will show that the \( \equiv \) relation for 2-uniform hypergraphs is equivalent to the fractional isomorphism relation for graphs. It is straightforward for the case of graphs with no edges, so, from now on, we will assume that the graphs have at least one edge. First we will demonstrate it for regular graphs, then for bipartite biregular graphs, and finally for general graphs.

**Proposition 3.3.** If \( G \) and \( H \) are \( n \)-vertex \( r \)-regular graphs. Then \( G \equiv H \).

**Proof.** As we mentioned above, we may assume \( r > 0 \). We will compute explicitly the matrices \( S_1 \) and \( S_2 \). Let us recall that \( J_s \) (resp. \( J_{s \times t} \)) is the \((s \times s)\)-matrix (resp. \((s \times t)\)-matrix) with all its coefficients equal to 1. Let \( M_G \) and \( M_H \) be the vertex-edge incidence matrices of the graphs \( G \) and \( H \). The dimensions of these matrices are \( n \times a \), where \( a = \frac{nr}{2} \) is the number of edges of \( G \) and also of \( H \). We compute \( S_1 M_G \) and \( M_H S_2^t \), where \( S_1 = S_1^t = \frac{1}{n} J_n \) and \( S_2 = S_2^t = \frac{1}{a} J_a \).

On the one hand, \( S_1 M_G = \frac{1}{n} J_n M_G = \frac{1}{n} 2 J_{n \times a} \), where the number 2 appears because every column of \( M_G \) adds up to 2 (it is a graph).

On the other hand, \( M_H S_2^t = M_H \frac{1}{a} J_a = \frac{1}{a} r J_{n \times a} \), where \( r \) appears because every row of \( M_H \) adds up to \( r \) (there are exactly \( r \) 1’s in every row of an \( r \)-regular graph).

Since \( a = \frac{nr}{2} \), we obtain \( \frac{2}{n} = \frac{r}{a} \). So, \( S_1 M_G = M_H S_2^t \). And, in the same way, \( M_G S_2 = M_G \frac{1}{a} J_a = \frac{1}{a} r J_{n \times a} \) and \( S_1^t M_H = \frac{1}{n} J_n M_H = \frac{1}{n} 2 J_{n \times a} \), which coincide. \( \Box \)

**Proposition 3.4.** Let \( G \) and \( H \) be \((b, c)\)-regular bipartite graphs, each of them having \( n \) vertices of degree \( b \) and \( m \) of degree \( c \) (so, \( G \cong_f H \)). Then \( G \equiv H \).

**Proof.** Let \( G \) and \( H \) be \((b, c)\)-regular bipartite graphs, each of them having \( n \) vertices of degree \( b \) and \( m \) of degree \( c \). Then the rows of \( M_G \) and \( M_H \) can be reordered so that, for each of them, the first \( n \) rows correspond to the vertices of degree \( b \). Let \( a = nb = mc \) be the number of edges of \( G \) and \( H \), which we assume greater than zero. After the reordering, \( M_G \) can be divided into two matrices of \( n \times a \) and \( m \times a \) (one below the other) where in each of these matrices there is only one 1 per column, the first matrix has \( b \) 1’s per row, and the second matrix has \( c \) 1’s per row. Namely,
\[ M_G = \begin{bmatrix} M_{G,1} \\ M_{G,2} \end{bmatrix} \quad \text{and} \quad M_H = \begin{bmatrix} M_{H,1} \\ M_{H,2} \end{bmatrix}. \]

We will compute explicitly the matrices \( S_1 \) and \( S_2 \). We define

\[ S_1 = \frac{1}{n} J_n \oplus \frac{1}{m} J_m = \begin{bmatrix} \frac{1}{n} J_n & 0 \\ 0 & \frac{1}{m} J_m \end{bmatrix} = S_1^t \quad \text{and} \quad S_2 = \frac{1}{a} J_a = S_2^t. \]

Then,

\[ S_1 M_G = \begin{bmatrix} \frac{1}{n} J_n & 0 \\ 0 & \frac{1}{m} J_m \end{bmatrix} \begin{bmatrix} M_{G,1} \\ M_{G,2} \end{bmatrix} = \begin{bmatrix} \frac{1}{n} J_{n \times a} \\ \frac{1}{m} J_{m \times a} \end{bmatrix} \quad \text{and} \quad M_H S_2^t = \begin{bmatrix} M_{H,1} \\ M_{H,2} \end{bmatrix} \frac{1}{a} J_a = \begin{bmatrix} b J_{n \times a} \\ c J_{m \times a} \end{bmatrix}. \]

Since \( \frac{b}{a} = \frac{1}{n} \) and \( \frac{c}{a} = \frac{1}{m} \), the right hand side expressions coincide, so \( S_1 M_G = M_H S_2^t \). In the same way,

\[ M_G S_2 = \begin{bmatrix} M_{G,1} \\ M_{G,2} \end{bmatrix} \frac{1}{a} J_a = \begin{bmatrix} b J_{n \times a} \\ c J_{m \times a} \end{bmatrix} \quad \text{and} \quad S_1^t M_H = \begin{bmatrix} \frac{1}{n} J_n & 0 \\ 0 & \frac{1}{m} J_m \end{bmatrix} \begin{bmatrix} M_{H,1} \\ M_{H,2} \end{bmatrix} = \begin{bmatrix} \frac{1}{n} J_{n \times a} \\ \frac{1}{m} J_{m \times a} \end{bmatrix}. \]

Again, the right hand side expressions coincide, so \( M_G S_2 = S_1^t M_H \).

\[ \square \]

**Proposition 3.5.** Let \( G \) be a graph with the equitable coarsest partition consisting of two parts. Suppose \( G \) can be described with the following parameters: \( v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \) and \( D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \), with \( v_1 D_{12} = v_2 D_{21} \). Let \( H \) be another graph with the same parameters \( (G \cong_f H) \). Then \( G \equiv H \).

**Proof.** Like for the previous propositions, we define explicitly doubly stochastic matrices \( S_1 \) and \( S_2 \). Let

\[ S_1 = \frac{1}{v_1} J_{v_1} \oplus \frac{1}{v_2} J_{v_2} = \begin{bmatrix} \frac{1}{v_1} J_{v_1} & 0 \\ 0 & \frac{1}{v_2} J_{v_2} \end{bmatrix} = S_1^t \]

and

\[ S_2 = \frac{1}{a_1} J_{a_1} \oplus \frac{1}{a_2} J_{a_2} \oplus \frac{1}{a_{12}} J_{a_{12}} = \begin{bmatrix} \frac{1}{a_1} J_{a_1} & 0 & 0 \\ 0 & \frac{1}{a_2} J_{a_2} & 0 \\ 0 & 0 & \frac{1}{a_{12}} J_{a_{12}} \end{bmatrix} = S_2^t \]
where \( a_1 = \frac{v_1 D_{11}}{2} \), \( a_2 = \frac{v_2 D_{12}}{2} \), and \( a_{12} = v_1 D_{12} = v_2 D_{21} \) are the number of edges of \( G \) and \( H \) within the first part, the second part, and between both parts, respectively. If some of them is zero, we do not include the corresponding term in the direct sum. So, \( M_G \) and \( M_H \) have the same dimensions \( n \times a \), where \( n = v_1 + v_2 \) and \( a = a_1 + a_2 + a_{12} \). And, reordering them, we can write in blocks:

\[
M_G = \begin{bmatrix} M_{G,11} & 0 & M_{G,12} \\ 0 & M_{G,22} & M_{G,21} \end{bmatrix}
\]

where \( M_{G,11} \) (of dimension \( v_1 \times a_1 \)) corresponds to the \( D_{11} \)-regular induced subgraph of \( v_1 \) vertices, \( M_{G,22} \) (of dimension \( v_2 \times a_2 \)) corresponds to the \( D_{22} \)-regular induced subgraph of \( v_2 \) vertices, and \( M_{G,12} \) and \( M_{G,21} \) (of dimension \( v_1 \times a_{12} \) and \( v_2 \times a_{12} \), respectively) correspond to the bipartite \((D_{12}, D_{21})\)-regular subgraph joining them. Notice that \( M_{G,11} \cdot 1 = D_{11} \cdot 1 \), \( 1^t \cdot M_{G,11} = 2 \cdot 1^t \), \( M_{G,22} \cdot 1 = D_{22} \cdot 1 \) and \( 1^t \cdot M_{G,22} = 2 \cdot 1^t \). Also, \( M_{G,12} \cdot 1 = D_{12} \cdot 1 \), \( 1^t \cdot M_{G,12} = 1^t \), \( M_{G,21} \cdot 1 = D_{21} \cdot 1 \) and \( 1^t \cdot M_{G,21} = 1^t \). Then,

\[
S_1 M_G = \begin{bmatrix} \frac{1}{v_1} J_{v_1} & 0 \\ 0 & \frac{1}{v_2} J_{v_2} \end{bmatrix} \begin{bmatrix} M_{G,11} & 0 & M_{G,12} \\ 0 & M_{G,22} & M_{G,21} \end{bmatrix}
= \begin{bmatrix} \frac{1}{v_1} J_{v_1} & 0 & \frac{1}{v_1} J_{v_1 \times a_{12}} \\ 0 & \frac{1}{v_2} J_{v_2} & \frac{1}{v_2} J_{v_2 \times a_{12}} \end{bmatrix}
\]

and

\[
M_H S_2^t = \begin{bmatrix} M_{H,11} & 0 & M_{H,12} \\ 0 & M_{H,22} & M_{H,21} \end{bmatrix} \begin{bmatrix} \frac{1}{a_1} J_{a_1} & 0 & 0 \\ 0 & \frac{1}{a_2} J_{a_2} & 0 \\ 0 & 0 & \frac{1}{a_{12}} J_{a_{12}} \end{bmatrix}
= \begin{bmatrix} \frac{D_{11}}{a_1} J_{v_1 \times a_1} & 0 & \frac{D_{12}}{a_{12}} J_{v_1 \times a_{12}} \\ 0 & \frac{D_{22}}{a_2} J_{v_2 \times a_2} & \frac{D_{21}}{a_{12}} J_{v_2 \times a_{12}} \end{bmatrix}
\]

Since \( \frac{2}{v_1} = \frac{D_{11}}{a_1} \), \( \frac{2}{v_2} = \frac{D_{22}}{a_2} \), \( \frac{D_{12}}{a_{12}} = \frac{1}{v_1} \), and \( \frac{D_{21}}{a_{12}} = \frac{1}{v_2} \), the expressions above coincide. Also,

\[
S_1^t M_H = \begin{bmatrix} \frac{1}{v_1} J_{v_1} & 0 \\ 0 & \frac{1}{v_2} J_{v_2} \end{bmatrix} \begin{bmatrix} M_{H,11} & 0 & M_{H,12} \\ 0 & M_{H,22} & M_{H,21} \end{bmatrix}
= \begin{bmatrix} \frac{1}{v_1} J_{v_1} & 0 & \frac{1}{v_1} J_{v_1 \times a_{12}} \\ 0 & \frac{1}{v_2} J_{v_2} & \frac{1}{v_2} J_{v_2 \times a_{12}} \end{bmatrix}
\]

and

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are equal. It is not hard to check that the equations hold still when some of $a_1$, $a_2$, $a_{12}$ are zero and the corresponding terms are missing in the direct sum. □

3.1.1. General Case

We show in the following two theorems that two graphs $G$ and $H$ are fractionally isomorphic if and only if $G \equiv H$.

**Theorem 3.6.** Let $G$ and $H$ be two fractionally isomorphic graphs ($G \cong_f H$). Then $G \equiv H$.

**Proof.** If $G \cong_f H$, both graphs have the same number of vertices (say $n$) and edges (say $a$), and the same coarsest equitable partition. Thus, they can be described with the same parameters $v$ and $D$ where $v$ is a vector of $k$ numbers corresponding to the sizes of the $k$ parts of the graph and $D$ is a $(k \times k)$-matrix whose coefficients correspond to the degrees of connection within each part and between parts. Let $t = k + \binom{k}{2} = \frac{k(k+1)}{2}$. We can classify the edges of each graph into at most $t$ types, according to the blocks in which they have their endpoints. Let $a_i = \frac{\nu_i D_{ii}}{2}$ for $1 \leq i \leq k$ be the number of internal edges of the $i$-th part, and $a_{ij} = \nu_i D_{ij}$ for $1 \leq i < j \leq k$ be the number of edges joining the $i$-th and the $j$-th parts. Reordering rows and columns, we can divide $M_G$ (whose dimension is $n \times a$) into at most $kt$ blocks. For the first (at most) $k$ groups of columns, there is one non-zero block, namely $M_{G,ii}$ for $1 \leq i \leq k$ such that $D_{ii} > 0$. The block $M_{G,ii}$ has dimension $v_i \times a_i$ and corresponds to the regular component $G[V_i]$ with at least one edge. It holds $M_{G,ii} \cdot 1 = D_{ii} \cdot 1$ and $1^t \cdot M_{G,ii} = 2 \cdot 1^t$. In the remaining groups of columns, there are two non-zero blocks per group, namely $M_{G,ij}$ and $M_{G,ji}$, for $1 \leq i < j \leq k$ such that $D_{ij} > 0$, which correspond to the bipartite subgraph $G[V_i, V_j]$ with at least one edge. The blocks $M_{G,ij}$ and $M_{G,ji}$ are of dimension $v_i \times a_{ij}$ and $v_j \times a_{ij}$, respectively. In this case, $M_{G,ij} \cdot 1 = D_{ij} \cdot 1$,
$1^t \cdot M_{G,ij} = 1^t$, $M_{G,ji} \cdot 1 = D_{ji} \cdot 1$, and $1^t \cdot M_{G,ji} = 1^t$. We define $S_1 = \frac{1}{v_1} J_{v_1} \oplus \cdots \oplus \frac{1}{v_k} J_{v_k}$ and $S_2 = \bigoplus_{1 \leq i \leq k : a_i > 0} \frac{1}{a_i} J_{a_i} \oplus \bigoplus_{1 \leq i < j \leq k : a_{ij} > 0} \frac{1}{a_{ij}} J_{a_{ij}}$ (this last part ordered lexicographically by $(i,j)$). Then $S_1 = S_1^t$, $S_2 = S_2^t$, and

\[
S_1 M_G = \left[ \begin{array}{cccccccc} \frac{1}{v_1} J_{v_1} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{v_2} J_{v_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{v_{k-1}} J_{v_{k-1}} & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{v_k} J_{v_k} \end{array} \right] \left[ \begin{array}{cccccccc} M_{G,11} & 0 & \cdots & 0 & M_{G,12} & M_{G,13} & \cdots & 0 \\ 0 & M_{G,22} & \cdots & 0 & M_{G,21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & M_{G,kk} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & M_{G,(k-1)k} \end{array} \right] = \left[ \begin{array}{cccccccc} \frac{1}{v_1} J_{v_1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{v_2} J_{v_2} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \frac{1}{v_k} J_{v_k} \end{array} \right] \left[ \begin{array}{cccccccc} M_{H,11} & 0 & \cdots & 0 & M_{H,12} & M_{H,13} & \cdots & 0 \\ 0 & M_{H,22} & \cdots & 0 & M_{H,21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & M_{H,kk} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & M_{H,(k-1)k} \end{array} \right]
\]

\[
M_H S_2^t = \left[ \begin{array}{cccccccc} M_{H,11} & 0 & \cdots & 0 & M_{H,12} & M_{H,13} & \cdots & 0 \\ 0 & M_{H,22} & \cdots & 0 & M_{H,21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & M_{H,kk} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & M_{H,(k-1)k} \end{array} \right] \left[ \begin{array}{cccccccc} \frac{1}{a_1} J_{a_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_2} J_{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \frac{1}{a_{k-1}} J_{a_{k-1}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \frac{1}{a_k} J_{a_k} \end{array} \right] = \left[ \begin{array}{cccccccc} \frac{1}{a_1} J_{a_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_2} J_{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \frac{1}{a_{k-1}} J_{a_{k-1}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \frac{1}{a_k} J_{a_k} \end{array} \right]
\]

So, $S_1 M_G = M_H S_2^t$ because $\frac{1}{v_i} = \frac{D_{ii}}{a_i}$ for $1 \leq i \leq k$, $a_i > 0$ and $\frac{1}{v_i} = \frac{D_{ii}}{a_i} \frac{1}{a_{ij}}$ for $1 \leq i < j \leq k$, $a_{ij} > 0$. In the same way, we have $S_1^t M_H = M_G S_2$. 

We will now show that if two hypergraphs $G$ and $H$ are graphs (2-uniform hypergraphs) and $G \equiv H$, then $G \cong_f H$. To do this, we use the same ideas as in the main theorem of fractional isomorphism.
Theorem 3.7. If $G \equiv H$ and $G$ and $H$ are graphs then $G \cong_I H$.

Proof. We know that $S_1M_G = M_HS_2'$ and $M_GS_2 = S_1'M_H$ for some double stochastic matrices $S_1$ and $S_2$. Firstly, we will show that we can assume that $S_1$ and $S_2$ have a structure of blocks: $S_1 = A_1 \oplus \ldots \oplus A_k$ and $S_2 = B_1 \oplus \ldots \oplus B_a$ where every $A_i$, $B_i$ is indecomposable (so strongly irreducible, by Proposition 2.1). The reason is that if $PS_1Q = A_1 \oplus \ldots \oplus A_k$ with $P$ and $Q$ permutation matrices and $RS_2T = B_1 \oplus \ldots \oplus B_a$ with $R$ and $T$ permutation matrices, we can write:

$$PS_1M_GR^t = PM_HS_2'R^t$$

then

$$(PS_1Q)(Q^tM_GR^t) = (PM_HT)(T^tS_2'R^t)$$

but we have

$$Q^tM_RS_2T = Q^tS_1'M_HT$$

and then

$$(Q^tM_GR^t)(RS_2T) = (Q^tS_1'P^t)(PM_HT).$$

So, when we interchange rows and columns of $S_1$ and $S_2$ to obtain a structure of blocks, $M_G$ and $M_H$ also interchange rows and columns following these rules: $M_G = Q^tM_GR^t$ and $M_H = PM_HT$. We denote $M_G$ and $M_H$ to these new matrices $M'_G$ and $M'_H$.

Using the conditions $S_1M_G = M_HS_2'$ and $M_GS_2 = S_1'M_H$, we obtain:

$$\begin{bmatrix}
A_1 & 0 & \ldots & 0 \\
0 & A_2 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & A_k
\end{bmatrix} \begin{bmatrix}
G_{11} & G_{12} & \ldots & G_{1a} \\
G_{21} & G_{22} & \ldots & G_{2a} \\
\ldots & \ldots & \ldots & \ldots \\
G_{k_1} & G_{k_2} & \ldots & G_{ka}
\end{bmatrix} = \begin{bmatrix}
H_{11} & H_{12} & \ldots & H_{1a} \\
H_{21} & H_{22} & \ldots & H_{2a} \\
\ldots & \ldots & \ldots & \ldots \\
H_{k_1} & H_{k_2} & \ldots & H_{ka}
\end{bmatrix} \begin{bmatrix}
B'_1 & 0 & \ldots & 0 \\
0 & B'_2 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & B'_a
\end{bmatrix}
$$

$$\begin{bmatrix}
G_{11} & G_{12} & \ldots & G_{1a} \\
G_{21} & G_{22} & \ldots & G_{2a} \\
\ldots & \ldots & \ldots & \ldots \\
G_{k_1} & G_{k_2} & \ldots & G_{ka}
\end{bmatrix} \begin{bmatrix}
B_1 & 0 & \ldots & 0 \\
0 & B_2 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & B_a
\end{bmatrix} = \begin{bmatrix}
A'_1 & 0 & \ldots & 0 \\
0 & A'_2 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & A'_k
\end{bmatrix} \begin{bmatrix}
H_{11} & H_{12} & \ldots & H_{1a} \\
H_{21} & H_{22} & \ldots & H_{2a} \\
\ldots & \ldots & \ldots & \ldots \\
H_{k_1} & H_{k_2} & \ldots & H_{ka}
\end{bmatrix}
$$

Then $A_iG_{ij} = H_{ij}B'_j$ and $G_{ij}B_j = A'_iH_{ij}$. Let us call $d_{ij}(G) = G_{ij} \cdot 1$ and $d_{ij}(H) = H_{ij} \cdot 1$. We compute $A_iG_{ij} \cdot 1 = H_{ij}B'_j \cdot 1 = H_{ij} \cdot 1$.

Then $A_i d_{ij}(G) = d_{ij}(H)$ and as $G_{ij} \cdot 1 = G_{ij}B_j \cdot 1 = A'_iH_{ij} \cdot 1$ we have $d_{ij}(G) = A'_i d_{ij}(H)$. But using Theorem 2.2 we have that
\[ d_{ij}(G) = d_{ij}(H) = c \cdot 1 \] for some scalar \( c \). On the other hand, renaming \( u_{ij}(G) = 1^t \cdot G_{ij} \) and \( u_{ij}(H) = 1^t \cdot H_{ij} \), we obtain that \( u_{ij}(G) = 1^t \cdot G_{ij} = 1^t \cdot A_{ij} G_{ij} = 1^t \cdot A_{ij} B_j = u_{ij}(H) B_j \) and \( u_{ij}(G) B_j = 1^t \cdot G_{ij} B_j = 1^t \cdot A_{ij} H_{ij} = 1^t \cdot H_{ij} = u_{ij}(H) \). And again using Theorem 2.2 for \( u_{ij}(G)^t \) and \( u_{ij}(H)^t \), it holds that \( u_{ij}(G) = u_{ij}(H) = e \cdot 1^t \). For 2-uniform hypergraphs, this last value \( e \) could only be 0, 1 or 2. The matrices \( M_G \) and \( M_H \) are divided into blocks \( G_{ij} \) and \( H_{ij} \) of the same dimension where every row adds up the same and every column adds up the same (that only could be 2, 1 or 0 in the case of 2-uniform hypergraphs). Since the columns of \( M_G \) and \( M_H \) add up to 2, we have the following cases: if the sum of the columns of \( G_{ij} \) (and also of \( H_{ij} \)) adds up to 2, we have the vertices of the regular classes, if the sum of the columns of \( G_{ij} \) adds up to 0 then \( G_{ij} \) is the null matrix (so also is \( H_{ij} \)) and if the sum of the columns of \( G_{ij} \) adds up to 1, we have another \( G_{kj} \) where the sum is also 1, we have the connections between class \( i \) and class \( k \) (and the same happens to \( H_{ij} \)). In conclusion, \( S_1 \) and \( S_2 \) induce the same equitable partition of the vertices and edges of \( G \) and \( H \), respectively, so by Theorem 2.8, \( G \cong_f H \).

As we show that, for graphs, \( G \cong_f H \) if and only if \( G \equiv H \), we will use the notation \( G \cong_f H \) from now on, instead of \( G \equiv H \). The concept of fractional isomorphism can be extended to hypergraphs, and we denote \( G \cong_f H \) if \( G \equiv H \).

Likewise, in the case of graphs, the conditions that define the fractional hypergraph isomorphism can be verified using a linear programming model of polynomial size in the size of the hypergraphs: \( G \cong_f H \) if and only if there exist \( S_1 \) and \( S_2 \) doubly stochastic matrices such that \( S_1 M_G = M_H S_2^t \) and \( M_G S_2 = S_1^t M_H \). By the polynomiality of linear programming [21], the recognition of the fractional isomorphism of hypergraphs is polynomial.

3.1.2. Basic properties of fractional isomorphism of hypergraphs

We present here the basic properties of fractional isomorphism of hypergraphs, which are similar to those of graphs but also involve hyperedge sizes, that are implicit for graphs since graphs are 2-uniform.

**Proposition 3.8.** If \( G \cong_f H \) for two hypergraphs \( G \) and \( H \) then:

1. \( G \) and \( H \) have the same number of vertices;
2. \( G \) and \( H \) have the same number of hyperedges;
3. $G$ and $H$ have the same degree sequence;
4. $G$ and $H$ have the same multiset of hyperedge sizes;
5. $G^* \cong_f H^*$ (their dual hypergraphs are fractionally isomorphic).

Proof. If $G \cong_f H$ then there exist $S_1$ and $S_2$ doubly stochastic matrices such that $S_1M_G = M_HS_2^t$ and $M_GS_2 = S_1^tM_H$. In particular, to make the matrix products well defined, $M_G$ and $M_H$ have to have the same dimensions, so $G$ and $H$ have the same number of vertices and hyperedges.

The degree sequence $d_G$ of $G$ can be obtained by multiplying $M_G \cdot 1$, while the multiset of hyperedge sizes $u_G$ can be obtained by multiplying $1^t \cdot M_G$.

\[
M_GS_2 \cdot 1 = S_1^tM_H \cdot 1
\]
\[
M_G \cdot 1 = S_1^t \cdot d_H
\]
\[
d_G = S_1^t \cdot d_H
\]

Similarly, $S_1 \cdot d_G = d_H$. By Theorem 2.2, $d_G$ is a permutation of $d_H$.

Analogously,

\[
1^t \cdot S_1M_G = 1^t \cdot M_HS_2^t
\]
\[
1^t \cdot M_G = u_H^t \cdot S_2^t
\]
\[
u_G^t = u_H^t \cdot S_2^t
\]
\[
u_G = S_2 \cdot u_H
\]

Similarly, $S_2^t \cdot u_G = u_H$. By Theorem 2.2, $u_G$ is a permutation of $u_H$.

Finally, since $M_{G^*} = M_{G'}^t$, $M_{H^*} = M_{H'}^t$, and transposing the equations we have $M_G^tS_1^t = S_2M_H^t$ and $S_2^tM_G^t = M_H^tS_1^t$, it follows that $G^* \cong_f H^*$.

As a corollary, we have the following.

**Corollary 3.9.** Let $G$ and $H$ be hypergraphs. If $G \cong_f H$ and $G$ is a graph, then $H$ is also a graph.
So, we can strengthen Theorem 3.7, requiring only that $G$ be a graph, because $H$ will necessarily be a graph.

We can also extend the concept of equitable partition to hypergraphs.

Let $H$ be a hypergraph. Let $P = \{V_1, \ldots, V_s, X_1, \ldots, X_r\}$ be a partition of $V(H)$ and $E(H)$ (each $V_i$ is a subset of $V(H)$ and each $X_j$ of $E(H)$, $r$ can be zero if $H$ has no hyperedges). The partition $P$ is *equitable* if, for every $1 \leq i \leq s$ and every $1 \leq j \leq r$, every vertex of $V_i$ belongs to the same number of hyperedges of $X_j$, and every hyperedge of $X_j$ contains the same number of vertices of $V_i$.

Every hypergraph has a trivial equitable partition: each vertex and each hyperedge is a class by itself. If $H$ is uniform and regular, then $\{V(H), E(H)\}$ is an equitable partition.

Notice that, if we have an equitable partition $P = \{V_1, \ldots, V_s, X_1, \ldots, X_r\}$ and, for some $1 \leq j, j' \leq r$ and for every $1 \leq i \leq s$, the hyperedges of $X_j$ and $X_{j'}$ have the same number $n_i$ of vertices of the class $V_i$, then we can define a coarser equitable partition replacing $X_j$ and $X_{j'}$ by $X_j \cup X_{j'}$.

In particular, if $G$ is a graph, for every equitable partition $P = \{V_1, \ldots, V_s, X_1, \ldots, X_r\}$, we have a coarser one with the same vertex partition and so that each edge set is either the set of all edges having both endpoints in some vertex set $V_i$, or the set of all edges having one endpoint in $V_i$ and the other in $V_j$, for some $i, j$. So, the concept of equitable partition for hypergraphs coincides in graphs with the traditional definition.

For hypergraphs, the *parameters* of an equitable partition $P = \{V_1, \ldots, V_s, X_1, \ldots, X_r\}$ are a triple $(v, D, U)$ where $v$ is a $s$-vector whose $i$-th entry is the size of $V_i$, $D$ and $U$ are $(s \times r)$-matrices such that $D_{ij}$ is the number of hyperedges of $X_j$ to which a vertex of $V_i$ belongs, and $U_{ij}$ is the number of vertices of $V_i$ that an hyperedge of $X_j$ contains. When $r > 0$, the number of edges $a_{ij}$ in each $X_j$, which is always greater than zero, is then $\frac{v_i D_{ij}}{U_{ij}}$, for any $i$ such that $U_{ij} > 0$.

We say that equitable partitions $P$ and $Q$ of hypergraphs $G$ and $H$ have the same parameters if we can index the sets in $P$ and $Q$ so that their parameters $(n, D, U)$ are identical. In such a case we say that $G$ and $H$ have a *common equitable partition*.

Following the main ideas in the proofs of Theorems 3.6 and 3.7, we can prove the following result.
Theorem 3.10. Let $G$ and $H$ be hypergraphs. Then $G \cong_f H$ if and only if $G$ and $H$ have a common equitable partition.

Proof. Suppose $G$ and $H$ have a common equitable partition with parameters $(v, D, U)$. In particular, they have the same number of vertices and hyperedges. Let $a_j$ be the number of edges of $X_j$, for $1 \leq j \leq r$. Recall that for every $1 \leq i \leq s$, $U_{ij}a_j = D_{ij}v_i$. Reordering rows and columns according to the partition on each hypergraph, we can divide $M_G$ into $sr$ blocks. For each block $M_{G,ij}$ of dimension $v_i \times a_j$, with $1 \leq i \leq s$, $1 \leq j \leq r$, $M_{G,ij} \cdot 1 = D_{ij} \cdot 1$ and $1^t \cdot M_{G,ij} = U_{ij} \cdot 1^t$.

We define $S_1 = \frac{1}{v_1}J_{v_1} \oplus \cdots \oplus \frac{1}{v_s}J_{v_s} \ (S_1 = S_1^t)$ and $S_2 = \frac{1}{a_1}J_{a_1} \oplus \cdots \oplus \frac{1}{a_r}J_{a_r} \ (S_2 = S_2^t)$. Then

$$S_1M_G = \begin{bmatrix}
\frac{1}{v_1}J_{v_1} & 0 & \cdots & 0 \\
0 & \frac{1}{v_2}J_{v_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{v_s}J_{v_s}
\end{bmatrix}
\begin{bmatrix}
M_{G,11} & M_{G,12} & \cdots & M_{G,1r} \\
M_{G,21} & M_{G,22} & \cdots & M_{G,2r} \\
\vdots & \vdots & \ddots & \vdots \\
M_{G,s1} & M_{G,s2} & \cdots & M_{G,sr}
\end{bmatrix}
$$

$$= \begin{bmatrix}
U_{11}J_{v_1 \times a_1} & U_{12}J_{v_1 \times a_2} & \cdots & U_{1r}J_{v_1 \times a_r} \\
U_{21}J_{v_2 \times a_1} & U_{22}J_{v_2 \times a_2} & \cdots & U_{2r}J_{v_2 \times a_r} \\
\vdots & \vdots & \ddots & \vdots \\
U_{s1}J_{v_s \times a_1} & U_{s2}J_{v_s \times a_2} & \cdots & U_{sr}J_{v_s \times a_r}
\end{bmatrix}
$$

$$M_H S_2^t = \begin{bmatrix}
M_{H,11} & M_{H,12} & \cdots & M_{H,1r} \\
M_{H,21} & M_{H,22} & \cdots & M_{H,2r} \\
\vdots & \vdots & \ddots & \vdots \\
M_{H,s1} & M_{H,s2} & \cdots & M_{H,sr}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{a_1}J_{a_1} & 0 & \cdots & 0 \\
0 & \frac{1}{a_2}J_{a_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{a_r}J_{a_r}
\end{bmatrix}
$$

$$= \begin{bmatrix}
D_{a_1}J_{a_1 \times a_1} & D_{a_1}J_{a_1 \times a_2} & \cdots & D_{a_1}J_{a_1 \times a_r} \\
D_{a_2}J_{a_2 \times a_1} & D_{a_2}J_{a_2 \times a_2} & \cdots & D_{a_2}J_{a_2 \times a_r} \\
\vdots & \vdots & \ddots & \vdots \\
D_{a_r}J_{a_r \times a_1} & D_{a_r}J_{a_r \times a_2} & \cdots & D_{a_r}J_{a_r \times a_r}
\end{bmatrix}
$$

So, $S_1M_G = M_HS_2^t$ because $\frac{U_{ij}}{v_i} = \frac{D_{ij}}{a_j}$ for $1 \leq i \leq s$ and $1 \leq j \leq r$. In the same way, we have $S_1^tM_H = M_GS_2$.

Now, suppose $G \cong_f H$. We know that $S_1M_G = M_HS_2^t$ and $M_GS_2 = S_1^tM_H$ for some double stochastic matrices $S_1$ and $S_2$. Repeating
the reasoning in the proof of Theorem 3.7, we can assume that $S_1$ and $S_2$ have a structure of blocks: $S_1 = A_1 \oplus \ldots \oplus A_s$ and $S_2 = B_1 \oplus \ldots \oplus B_r$ where every $A_i$, $B_i$ is strongly irreducible. Let $v_i$ be the number of rows and columns of $A_i$, for $1 \leq i \leq s$, and $a_i$ be the number of rows and columns of $B_i$, for $1 \leq i \leq r$. We can partition $M_G$ and $M_H$ into $(v_i \times a_j)$-blocks $G_{ij}$ and $H_{ij}$, respectively. Using the conditions $S_1 M_G = M_H S_2^t$ and $M_G S_2 = S_1^t M_H$, we obtain:

$$
\begin{bmatrix}
A_1 & 0 & \ldots & 0 \\
0 & A_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & A_s
\end{bmatrix}
\begin{bmatrix}
G_{11} & G_{12} & \ldots & G_{1r} \\
G_{21} & G_{22} & \ldots & G_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
G_{s1} & G_{s2} & \ldots & G_{sr}
\end{bmatrix}
= 
\begin{bmatrix}
H_{11} & H_{12} & \ldots & H_{1r} \\
H_{21} & H_{22} & \ldots & H_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
H_{s1} & H_{s2} & \ldots & H_{sr}
\end{bmatrix}
\begin{bmatrix}
B_1^t & 0 & \ldots & 0 \\
0 & B_2^t & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & B_r^t
\end{bmatrix}
$$

Then $A_i G_{ij} = H_{ij} B_i^t$ and $G_{ij} B_j = A_i^t H_{ij}$. Let us call $d_{ij}(G) = G_{ij} \cdot 1$ and $d_{ij}(H) = H_{ij} \cdot 1$. We compute $A_i G_{ij} \cdot 1 = H_{ij} B_i^t \cdot 1 = H_{ij} \cdot 1$.

Then $A_i d_{ij}(G) = d_{ij}(H)$ and as $G_{ij} \cdot 1 = G_{ij} B_j \cdot 1 = A_i^t H_{ij} \cdot 1$ we have $d_{ij}(G) = A_i^t d_{ij}(H)$. But using Theorem 2.2, we have that $d_{ij}(G) = d_{ij}(H) = D_{ij} \cdot 1$ for some scalar $D_{ij}$. On the other hand, renaming $u_{ij}(G) = 1^t \cdot G_{ij}$ and $u_{ij}(H) = 1^t \cdot H_{ij}$, we obtain that $u_{ij}(G) = 1^t G_{ij} = 1^t A_i G_{ij} = 1^t H_{ij} B_i^t = u_{ij}(H) B_i^t$ and $u_{ij}(G) B_j = 1^t G_{ij} B_j = 1^t A_i H_{ij} = 1^t H_{ij} = u_{ij}(H)$. And again using Theorem 2.2 for $u_{ij}(G)^t$ and $u_{ij}(H)^t$, it holds that $u_{ij}(G) = u_{ij}(H) = U_{ij} \cdot 1^t$ for some scalar $U_{ij}$. Thus, the partition of the vertices and of the hyperedges of $G$ and $H$ induced by the blocks $G_{ij}$ and $H_{ij}$ of the respective incidence matrices is a common equitable partition of $G$ and $H$. 

\[ \square \]

**Corollary 3.11.** If $G$ and $H$ are two $k$-uniform $r$-regular hypergraphs with $n$ vertices, then $G \cong_f H$.

Notice that the fact that two hypergraphs are fractionally isomorphic, does not imply that their 2-sections are fractionally isomorphic graphs. Consider the following 4-uniform 2-regular hypergraphs on 8 vertices $v_1, \ldots, v_8$ (thus, fractionally isomorphic). The hypergraph $H$ has...
hyperedges \{v_1, v_2, v_3, v_4\}, \{v_3, v_4, v_5, v_6\}, \{v_5, v_6, v_7, v_8\}; the hypergraph \(G\) has hyperedges \{v_1, v_2, v_3, v_4\}, \{v_1, v_2, v_3, v_8\}, \{v_4, v_5, v_6, v_7\}, \{v_5, v_6, v_7, v_8\}. The 2-section of \(H\) is 5-regular, while the 2-section of \(G\) has 6 vertices of degree 4 and 2 vertices of degree 6, so they are not fractionally isomorphic.

### 3.1.3. Bipartite representation and fractional isomorphism

It is known that every hypergraph can be described by a bipartite graph where the two parts of the bipartition correspond to the vertices and hyperedges of the hypergraph, respectively.

A possible question arises with this correspondence: Do two fractionally isomorphic hypergraphs correspond to fractionally isomorphic bipartite graphs? And if we have two bipartite graphs that arise from hypergraphs, if they are fractionally isomorphic, are the hypergraphs fractionally isomorphic?

Given a hypergraph \(G\), we may construct a bipartite graph \(B_G\) where the first part of \(B_G\) is in correspondence with the vertices of \(G\) and the second part of \(B_G\) is in correspondence with the hyperedges of \(G\). We have an edge in \(B_G\) that links a vertex in the first part to a vertex in the second part if and only if the corresponding vertex of \(G\) belongs to the corresponding hyperedge in \(G\). If \(M_G\) is the vertex-hyperedge incidence matrix of \(G\) then it is straightforward that \(A = \begin{bmatrix} 0 & M_G \\ M_G^t & 0 \end{bmatrix}\) is the adjacency matrix of \(B_G\).

**Proposition 3.12.** Let \(G\) and \(H\) be two fractionally isomorphic hypergraphs \(G \cong f H\) and let \(B_G\) and \(B_H\) be the two bipartite graphs that correspond to \(G\) and \(H\), respectively. Then \(B_G \cong f B_H\).

**Proof.** Since \(G\) and \(H\) are fractionally isomorphic, then there exist doubly stochastic matrices \(S_1\) and \(S_2\) such that \(S_1 M_G = M_H S_2\) and \(M_G S_2 = S_1 M_H\).

We define \(A = \begin{bmatrix} 0 & M_G \\ M_G^t & 0 \end{bmatrix}\), \(B = \begin{bmatrix} 0 & M_H \\ M_H^t & 0 \end{bmatrix}\) and \(S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}\). The matrix \(S\) is doubly stochastic, \(S A = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} 0 & M_G \\ M_G^t & 0 \end{bmatrix} = \begin{bmatrix} 0 & S_1 M_G \\ S_1 M_G^t & 0 \end{bmatrix}\), and \(B S = \begin{bmatrix} 0 & M_H \\ M_H^t & 0 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} = \begin{bmatrix} 0 & M_H S_1 \\ M_H S_1^t & 0 \end{bmatrix}\). Then \(S A = B S\) and \(B_G \cong f B_H\). \(\Box\)
On the other hand, the converse is not true: $B_G \cong_f B_H$, does not imply that $G$ and $H$ are fractionally isomorphic. Indeed, for every hypergraph $H$, $B_H = B_{H^*}$ (as graphs, even if the sets have different meaning in the representation) and, in general, $H$ and $H^*$ have different number of vertices and edges, so they are not fractionally isomorphic.

We can find also counterexamples with the same number of vertices and hyperedges. Let $H_1$ be the complete graph on four vertices and $H_2$ be the 5-vertex graph consisting on an induced path of four vertices plus a universal vertex (known as gem). Both the disjoint unions $H_1 \cup H_2^*$ and $H_1^* \cup H_2$ have 11 vertices and 11 hyperedges. It is clear that $B_{H_1 \cup H_2^*} = B_{H_1^* \cup H_2}$ (as graphs). However, their degree sequences differ: $d_1(H_1 \cup H_2^*) = \{2, 2, 2, 2, 2, 2, 3, 3, 3\}$ while $d_1(H_1^* \cup H_2) = \{2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 4\}$, so they are not fractionally isomorphic.

4. Fractional invariants

In this section we will show that several fractional invariants of graphs and hypergraphs are preserved by fractional (hyper)graph isomorphism. We will deal with fractional versions of packing (independent set), edge covering, matching, and transversal (vertex covering), which have been widely studied in the literature (see, for example, [1, 4, 6, 8, 9, 12, 13, 14, 15, 16, 17, 23, 24, 25, 28, 29, 31, 32, 34]).

Given a hypergraph $H = (S, X)$, a covering of $H$ is a collection of hyperedges $X_1, X_2, \ldots, X_j$ so that $S = X_1 \cup \cdots \cup X_j$. The least $j$ for which this is possible, the smallest size of a covering, is called the covering number of $H$ and is denoted $k(H)$. An element $s \in S$ is called exposed if it is in no hyperedge. If $H$ has an exposed vertex, then no covering of $H$ exists and $k(H) = \infty$. The covering problem can be formulated as an integer program, 

\[
\text{minimize } 1^t \cdot x \text{ subject to } M_H \cdot x \geq 1, \ x \in \{0, 1\}^{|X|}. \]

Furthermore, the variables in this and subsequent linear programs are tacitly assumed to be nonnegative. This integer problem can be relaxed to calculate $k_f(H)$ (the fractional covering number of $H$), as the linear program “minimize $1^t \cdot x$ subject to $M_H \cdot x \geq 1, \ x \geq 0$” (each $x_i$ can take any real nonnegative value). Any feasible solution of the integer program is also a feasible solution of the linear program, so $k(H) \geq k_f(H)$.

A packing of an hypergraph $H = (S, X)$ is a subset of vertices $Y \subseteq S$ with the property that no two elements of $Y$ are in the same hyperedge of $H$. The packing number $p(H)$ is the maximum number of elements that a
packing can have, an can be formulated by the integer program “maximize $1^t y$ subject to $M^t_H y \leq 1$, $y \in \{0, 1\}^{|S|}$”. Again, relaxing this problem, we can compute $p_f(H)$ (the fractional packing number of $H$) as the following linear program: “maximize $1^t y$ subject to $M^t_H y \leq 1$, $y \geq 0$”. In the same way, we have that $p_f(H) \geq p(H)$. Notice also that if $H$ has an exposed vertex $i$, then $p_f(H) = \infty$, since the value of $x_i$ is not upper bounded.

The linear programs above are dual, so $p_f(G) = k_f(G)$ (see, for example, [11]). Then, we have $p(G) \leq p_f(G) = k_f(G) \leq k(G)$.

In a graph $G$ without isolated vertices (seeing it as a 2-uniform hypergraph), the packing number $p(G)$ corresponds to the independence number $\alpha(G)$. So, we can define $\alpha_f(G) = p_f(G)$. In the same way, $k(G)$ is the minimum number of edges we can choose to cover every vertex in $G$, the edge cover number of $G$, and $k_f(G)$ is a fractional version of it.

Given a graph $G$, we can construct a hypergraph $H_G$ where the hyperedges of $H_G$ are the independent sets of $G$. Given $A_G$, the adjacency matrix of $G$, we can compute $M_{H_G}$. In this way, the chromatic number of $G$ corresponds to the covering number of $H_G$ ($\chi(G) = k(H_G)$). In the same way, we can define $\chi_f(G) = k_f(H_G)$ and it is possible to show that this definition coincides with other ways to define the fractional chromatic number of $G$ [34]. The clique number of $G$ corresponds to the packing number of $H_G$ ($\omega(G) = p(H_G)$), and the fractional clique number $\omega_f(G)$ is defined as $p_f(H_G)$. In particular, $\omega(G) \leq \omega_f(G) = \chi_f(G) \leq \chi(G)$.

We can also consider the dual hypergraph $H^*$ of the hypergraph $H$. A matching of $H$ is a set of pairwise disjoint hyperedges, and the matching number of $H$, denoted $\mu(H)$, is the maximum pairwise disjoint number of hyperedges of $H$. So, $\mu(H)$ is the packing number of $H^*$. Also, we can define $\mu_f(H) = p_f(H^*)$, and this definition coincides with other ways to define the fractional matching number of graphs and hypergraphs [5, 34]. A graph $G$ admits a perfect fractional matching when $\mu_f(G) = \frac{1}{2}|V(G)|$.

A transversal of $H$ is a set of vertices such that each hyperedge contains at least one of them, and the transversal number of $H$, denoted $\tau(H)$, is the minimum cardinality of a transversal of $H$. So, $\tau(H) = k(H^*)$. Also, we can define $\tau_f(H) = k_f(H^*)$. For graphs, the transversal (integer or fractional) is better known as vertex cover of the graph.

The main result in this section is the following.

**Theorem 4.1.** Let $G$ and $H$ be hypergraphs. If $G \cong_f H$, then $k_f(G) = k_f(H)$.
Proof. Since \( G \cong_f H \), there exist doubly stochastic matrices \( S_1 \) and \( S_2 \) such that \( S_1 M_G = M_H S_2^t \) and \( M_G S_2 = S_1^t M_H \). We write the following linear programs:

P1: “Minimize \( a = 1^t \cdot z \) subject to \( M_H z \geq 1 \) with \( z \geq 0 \).”

P2: “Minimize \( b = 1^t \cdot w \) subject to \( S_1^t M_H w \geq 1 \) with \( w \geq 0 \).”

P3: “Minimize \( c = 1^t \cdot u \) subject to \( M_G S_2 u \geq 1 \) with \( u \geq 0 \).”

P4: “Minimize \( d = 1^t \cdot v \) subject to \( M_G v \geq 1 \) with \( v \geq 0 \).”

Let \( a^*, b^*, c^* \), and \( d^* \) be the optimal values of each of the problems. It is straightforward that P2 and P3 are the same problem using that \( S_1^t M_H = M_G S_2 \), thus \( b^* = c^* \). On the other hand, \( a^* \geq b^* \) because every feasible solution of P1 is a feasible solution of P2: if \( z \) satisfies \( M_H z \geq 1 \), then \( S_1^t(M_H z) \geq S_1^t \cdot 1 = 1 \) (\( S_1 \) is a doubly stochastic matrix, and in particular its entries are nonnegative). In the same way, \( c^* \geq d^* \), because for every solution \( u \) of P3 we have a feasible solution \( v \) of P4 with the same objective function: if \( u \) satisfies \( M_G S_2 u \geq 1 \), then defining \( v = S_2 u \), we obtain \( M_G v \geq 1 \) and \( 1^t \cdot v = 1^t \cdot S_2 u = 1^t \cdot u \). So, we have \( a^* \geq b^* = c^* \geq d^* \) and \( a^* = k_f(H) \) and \( d^* = k_f(G) \), then \( k_f(H) \geq k_f(G) \).

Using the same idea, we will write two additional linear programs:

P5: “Minimize \( e = 1^t \cdot x \) subject to \( S_1 M_G x \geq 1 \) with \( x \geq 0 \).”

P6: “Minimize \( f = 1^t \cdot y \) subject to \( M_H S_2^t y \geq 1 \) with \( y \geq 0 \).”

Let \( e^* \) and \( f^* \) be their optimal values. It holds that \( d^* \geq e^* = f^* \geq a^* \), so \( k_f(G) \geq k_f(H) \). Therefore, \( k_f(H) = k_f(G) \). □

As a consequence, we have the following.

Corollary 4.2. Let \( G \) and \( H \) be hypergraphs. If \( G \cong_f H \), then \( p_f(G) = p_f(H) \), \( \mu_f(G) = \mu_f(H) \), and \( \tau_f(G) = \tau_f(H) \).
By definition, $\mu_f(G) = \tau_f(G) = \tau_f(H) = \mu_f(H)$. □

In terms of graphs, we have the following.

**Corollary 4.3.** Let $G$ and $H$ be graphs. If $G \cong_f H$, then they have the same fractional independence number, fractional edge and vertex covering numbers, and fractional matching number. In particular, $G$ has a perfect fractional matching if and only if $H$ has a perfect fractional matching.

A sufficient condition for a graph to admit a perfect fractional matching is the following. If the coarsest equitable partition $P = V_1, \ldots, V_k$ of a graph $G$ is such that for every $1 \leq i \leq k$, $G[V_i]$ is $r_i$-regular with $r_i > 0$, then $G$ has a perfect fractional matching. Namely, we assign a value $\frac{1}{r_i}$ to every internal edge of $V_i$, and 0 to the edges that join different parts of $P$.

Concerning the fractional independence number, there are other ways of defining a fractional version of the independence number of a graph. For example, relaxing the clique formulation of it [10]. That is, the fractional packing number of the hypergraph whose vertices are the vertices of the graph and whose hyperedges are the cliques of the graph. We will denote this fractional relaxation of the independence number of a graph $G$ by $\alpha_c^e(G)$. The dual problem for this formulation is the (fractional) clique cover of a graph (denoted by $\theta(G)$, resp. $\theta_f(G)$), instead of the (fractional) edge cover.

Notice that for the disjoint union of two triangles, $2 = \alpha(2C_3) \leq \alpha_f^e(2C_3) = \tau_f(2C_3) \leq \theta(2C_3) = 2$, so $\alpha_f^e(2C_3) = \tau_f(2C_3) = 2$, and for the cycle of length six, $3 = \alpha(C_6) \leq \alpha_f^e(C_6) = \theta_f(C_6) \leq \theta(C_6) = 3$, so $\alpha_f^e(C_6) = \theta_f(C_6) = 3$. Therefore, the fractional clique cover and the fractional relaxation of the clique formulation of the independence number are not invariant under fractional graph isomorphism.

Also, for the disjoint union of two triangles, $3 = \omega(2C_3) \leq \omega_f(2C_3) = \chi_f(2C_3) \leq \chi(2C_3) = 3$, so $\omega_f(2C_3) = \chi_f(2C_3) = 3$, and for the cycle of length six, $2 = \omega(C_6) \leq \omega_f(C_6) = \chi_f(C_6) \leq \chi(C_6) = 2$, so $\omega_f(C_6) = \chi_f(C_6) = 2$. Therefore, the fractional chromatic and clique numbers are not invariant under fractional graph isomorphism.

The domination number and the total domination number of a graph can be viewed also as the covering numbers of associated hypergraphs. Given a graph $G$, let $H_{N(G)}$ be the hypergraph whose vertex set is $V(G)$ and whose hyperedges are the open neighborhoods of the vertices of $G$. Then it is not difficult to see that $\Gamma(G) = k(H_{N(G)})$. Alternatively, if we consider the
hypergraph $H_{N[G]}$, whose hyperedges are the closed neighborhoods of the vertices of $G$, then $\gamma(G) = k(H_{N[G]})$. The fractional domination number and fractional total domination number are defined, as in other cases, as $\gamma_f(G) = k(H_{N[G]})$ and $\Gamma_f(G) = k_f(H_{N[G]})$, and this coincides with other ways of defining fractional domination \[34\].

Domke proved the following result.

**Theorem 4.4.** If $G$ has $n$ vertices and is $k$-regular, then $\gamma_f(G) = \frac{n}{k+1}$ and $\Gamma_f(G) = \frac{n}{k}$.

In particular, two $n$-vertex $k$-regular graphs have the same fractional domination and total domination numbers. We generalize this to fractionally isomorphic graphs.

**Theorem 4.5.** Let $G$ and $H$ be fractionally isomorphic graphs. Then $\gamma_f(G) = \gamma_f(H)$ and $\Gamma_f(G) = \Gamma_f(H)$.

**Proof.** Let $A$ and $B$ be square symmetric matrices such that there exists a double stochastic matrix $S$ with $AS = SB$. Then, defining $S_1 = S^t$, $S_2 = S$, it holds $S_1 A = S^t A = (A^t S)^t = (AS)^t = (SB)^t = B^t S^t = BS^t = BS_2^t$, and $AS_2 = AS = SB = S^t B$. Notice that if $G$ (and thus $H$) has an isolated vertex, then $\Gamma_f(G) = \Gamma_f(H) = \infty$. Otherwise, the vertex-hyperedge incidence matrix of the hypergraph $H_{N[G]}$ of $G$ is exactly its adjacency matrix $A_G$, and the same holds for $H$. In particular, these are symmetric matrices. Since $G \cong_f H$, then there exists a double stochastic matrix $S$ with $A_G S = S A_H$. By the observation above, there exist double stochastic matrices $S_1$ and $S_2$ such that $S_1 A_G = A_H S_2^t$, and $A_G S_2 = S_1^t A_H$. Thus, $H_{N[G]} \cong_f H_{N[H]}$ and by Theorem 4.4, $\Gamma_f(G) = k_f(H_{N[G]}) = k_f(H_{N[H]}) = \Gamma_f(H)$.

Similarly, independently of the existence of isolated vertices, the vertex-hyperedge incidence matrix of the hypergraph $H_{N[G]}$ of $G$ is $A_G + I$, where $I$ is the identity matrix of the appropriate dimension (and the same for $H$). These are also symmetric matrices, and we have

\[
A_G S = S A_H \\
A_G S + S = S A_H + S \\
(A_G + I) S = S (A_H + I)
\]

As for the previous case, this implies $H_{N[G]} \cong_f H_{N[H]}$ and by Theorem 4.4, $\gamma_f(G) = k_f(H_{N[G]}) = k_f(H_{N[H]}) = \gamma_f(H)$. \[26\]
5. Conclusions

The computational complexity of the recognition problem of graph isomorphism is an open question. It leads to define relaxations of the usual isomorphism that can be computed efficiently, in order to discard negative instances. One of these relaxations is the fractional isomorphism. Two graphs $G$ and $H$ are fractionally isomorphic if there exists a double stochastic matrix $S$ such that $AS = SB$, where $A$ and $B$ are the adjacency matrices of $G$ and $H$, respectively. The recognition problem of fractional isomorphism has polynomial complexity, as it can be modeled by linear programming, which is polynomial [21].

A question this work tried to answer is whether there exists a relationship between the fractional graph isomorphism and some fractional parameters, such as fractional packing and covering.

To compute these parameters, it is necessary to work with the vertex-edge incidence matrix instead of the adjacency matrix. So, it is natural to ask how to describe the fractional graph isomorphism in terms of these matrices. We obtain that $G$ and $H$ are fractionally isomorphic if and only if there exist doubly stochastic matrices $S_1$ and $S_2$ such that $S_1M_G = M_HS_2^t$ and $M_GS_2 = S_1^tM_H$, with $M_G$ and $M_H$ the vertex-edge incidence matrices of $G$ and $H$, respectively. This later definition can be extended to hypergraphs, and with this definition, the recognition of the fractional hypergraph isomorphism has polynomial complexity as well. We showed that the fractional covering and packing number, and the fractional matching and transversal number are invariants of the fractional hypergraph isomorphism. In particular, for graphs, the fractional edge cover and independence number, and the fractional matching and vertex cover number are invariants of the fractional isomorphism. Moreover, the fractional domination and total domination are invariants of the fractional graph isomorphism. This is not the case of fractional chromatic, clique, and clique cover numbers. In this way, most of the classical fractional parameters are classified with respect to their invariance under fractional graph isomorphism.

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