ON THE DECELERATING SHOCK INSTABILITY OF A PLANE-PARALLEL SLAB WITH FINITE THICKNESS

RYOICHI NIH1 AND HIDEYUKI KAMAYA2

nishi@tap.scphys.kyoto-u.ac.jp, kamaya@kusastro.kyoto-u.ac.jp

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ABSTRACT

Dynamical stability of a shock-compressed layer with finite thickness is investigated. It is characterized by self-gravity, structure, and shock conditions at the surfaces of the compressed layer. On one side of the shocked layer, its surface conditions are determined via the ram pressure, while on the other side the thermal pressure supports its structure. When the ram pressure dominates the thermal pressure, we expect deceleration of the shocked layer. In this paper we examine in particular how the stratification of the decelerating layer has an effect on its dynamical stability. Performing the linear perturbation analysis, we derive a dispersion relation that is more general than one previously derived by one of the present authors. It gives us an interesting piece of information about the stability of the decelerating layer: when the deceleration of the layer dominates the self-gravity, the decelerating shock instability (DSI) is efficient, while gravitational instability occurs when the self-gravity works better than the deceleration. The length scales of the two instabilities are the order of the width of the decelerating layer. Their growth timescales are of the order of the free-fall time in the density of the shocked slab. Importantly, they are always incompatible. We also consider the effects of the evolution of the shocked layer. In the early stages of its evolution, only DSI occurs. In the late stages, however, it is possible for the shocked layer to be unstable for the DSI (on a smaller scale) and the gravitational instability (on a larger scale), generally. The onset time of the gravitational instability is of the order of the free-fall time of the external medium. After onset, as shown above, the growth timescale becomes the free-fall time of the dense slab, which is much shorter than the free-fall time of the external medium. Furthermore, we find that there is a stable range of wavenumbers against both the DSI and the gravitational instability between respective unstable wavenumber ranges. These stable modes suggest the ineffectiveness of DSI for the fragmentation of the decelerating slab. Thus, in various cases, the structure of a shocked layer should be determined when its stability is discussed.

Subject headings: hydrodynamics — instabilities — shock waves — stars: formation

1. INTRODUCTION

Hydrodynamic shock waves are expected to play important roles in many astrophysical phenomena, especially in the formation of their structure. Shock waves are induced by supernovae (e.g., Tomisaka 1990), expanding H II regions (e.g., Elmegreen 1989), molecular outflows around protostars (e.g., Nakano, Hasegawa, & Norman 1995), stellar winds from early-type stars (e.g., Stevens, Blondin, & Pollock 1992), and other phenomena. These expanding shock waves sweep up the external matter, and the swept matter will become the materials of the structure at the next stage (e.g., Kimura & Tosa 1988). Previously, it was postulated that such a mechanism acted during the galaxy formation era (e.g., Ikeuchi 1981). In this paper, paying attention to the finite thickness of the shocked layer (cf. Vishniac & Ryu 1989), we reexamine its stability by linear perturbation analysis. In particular, we present a general dispersion relation including the effect of the self-gravity of a shocked slab with finite thickness.

The stability of decelerating shock waves has been investigated, especially in the last two decades. Vishniac (1983) first studied isothermal shock waves expanding spherically in a uniform medium with a thin shell picture. He found that the shocked layer was overstable against linear perturbations and that the smaller wavelength perturbation was more unstable, although plane shock waves in a steady state were believed to be stable against rippling perturbations (Erpenbeck 1962). The instability he found was attendant on decelerating shock waves (e.g., Nishi 1992). Thus, we call it decelerating shock instability (DSI). Importantly, after Vishniac's series of papers (Ryu & Vishniac 1987, 1988, 1991; Vishniac & Ryu 1989) was published, Grun et al. (1991) reported the existence of instability of shock waves in the laboratory. After that, Mac Low & Norman (1993) studied the nonlinear evolution of blast waves and confirmed the results of Grun et al. The self-gravity effect on the nonlinear stage was examined numerically by Yoshida & Habe (1992). Dgani, van Buren, & Noriega-Crespo (1996) also discussed the stability of bow shocks, by means of numerical simulations, and transverse acceleration instability, semianalytically. The evolution of supernova remnants was discussed in Chevalier & Blondin (1995). Vishniac (1994) examined the stability of a slab bounded on both sides by shocks, which is linearly stable but nonlinearly unstable. His analysis was confirmed by numerical simulations (Blondin & Marks 1996). For an extended review, see Vishniac (1995).

There is an important astrophysical phenomenon related to decelerating shock waves. Elmegreen & Lada (1977) proposed a scenario for sequential formation of OB star subgroups. According to their scenario, an H II region formed by a group of OB stars expands in a molecular cloud and a...
shock wave sweeps up the gas. Afterward, the matter swept up by the shock becomes gravitationally unstable and fragmentation occurs. Elmegreen & Elmegreen (1978) studied the stability of an isothermal layer supported by thermal pressure on both sides and showed that the layer is gravitationally unstable and that the growth rate has a maximum for the perturbation wavelength of several times the thickness of the layer. The effect of high external pressure on the self-gravity was examined by Lubow & Pringle (1993). Welter (1982) calculated the growth rate of the perturbation in a layer that is supported by thermal pressure on one side and by balancing ram pressure from a shock on the other side. Voit (1988) investigated an incompressible sheet supported by two unbalanced thermal pressures and showed the coupling of the gravitational instability and the Rayleigh-Taylor instability. The effect of the magnetic field on the stability of a self-gravitating slab was examined precisely by Nagai, Inutsuka, & Miyama (1998).

It is important to note, however, that the shock-compressed layers around H II regions are supported by thermal pressure on one side and ram pressure on the other side and that the two pressures are not always balanced. In this work, we expect that the instability related to a decelerating shock wave plays an important role. Moreover, we must study the instability of the shocked self-gravitating layers since, in the scenario of sequential star formation, the next generation of OB stars is finally formed by self-gravity. Elmegreen (1989) calculated the evolution of an isothermal shocked layer and the growth of the perturbations in the evolving dense layer with self-gravity. The estimated growth timescale is \( \sim 0.25(G \rho_c)^{-1/2} \) and is insensitive to the Mach number. Here \( \rho_c \) is the density of the external (preshock) medium. On the contrary, Vishniac (1983) estimated a relatively short timescale and a small mass scale for gravitational collapse, which are essentially determined by postshock density. The estimate of Nishi (1992) resembles that of Vishniac (1983). The discrepancy originates from the fact that each adopts a different approximation in his analysis. As pointed out by Mac Low & Norman (1993), one of the differences is that Vishniac (1983) assumes transonic turbulence in the shell while Elmegreen (1989) assumes supersonic turbulence with a Mach number close to the shock velocity, and Mac Low & Norman’s numerical work, in which gravity is neglected, may support Vishniac’s assumption. Moreover, very importantly, gravity is not included when the structure of the shocked slab is incorporated in these analyses. Thus, as presented just below, we adopt another assumption about the shocked slab to clarify its dynamical stability.

In the current paper, we investigate the instability of a plane-parallel slab with finite thickness, where the slab is bound by thermal pressure on one side and by ram pressure from a shock on the other side. The pressures on the two sides are dissimilar. We assume incompressibility for the equation of state of the postshock layer (i.e., in the slab), though we assume compressible fluid at the shock surface. This assumption of incompressibility may be reasonable since highly compressed isothermal fluid behaves just like an incompressible fluid, as shown by Elmegreen & Elmegreen (1978). Moreover, as demonstrated by Mac Low & Norman (1993), this approximation may be indicated by the fact that the DSI saturates with transonic transverse flows in the shocked slab. By this assumption we can obtain, in addition, an analytical solution to the perturbation equations, with which we can reveal the precise physics of the instabilities.

In the following sections, we perform linear perturbation analysis in order to study the stability of an incompressible slab without self-gravity of the layer in § 2 and with self-gravity in § 3. We discuss the DSI and gravitational instability of an incompressible slab in § 4. We also discuss the fragmentation process of the slab in § 5 and summarize our results in the final section.

2. THE DSI OF AN INCOMPRESSIBLE SLAB

In order to investigate the stability of the slab of gas swept up by a shock front, we analyze an infinite, plane-parallel shock wave advancing in the negative direction of the z-axis with velocity \( V_s \) in a uniform medium of density \( \rho_c \). We assume incompressibility of the gas except at the shock front, the density in the dense slab being \( \rho_s \), and that the dense layer is confined from the rear by hot and tenuous gas with a finite pressure \( P_r \). As a first step, we ignore the self-gravity of the slab in order to present the dispersion relation of the pure DSI (cf. § 2 of Nishi 1992). We assume that the unperturbed shock front is at \( z = 0 \). At the boundary \( z = 0 \), we impose shock boundary conditions:

\[
[v_\perp] = 0 ,
\]

\[
[P + \rho v_\perp^2] = 0 ,
\]

where \( \rho \) and \( P \) are the density and the pressure and \( v_\perp \) and \( v_\parallel \) are the velocities of the fluid perpendicular and parallel to the shock surface, respectively. The square brackets denote the difference in the enclosed quantity across the shock front.

The fluid equations for a general first-order perturbation in the slab are

\[
\frac{dv_i}{dt} = -\nabla \left( \frac{P_\parallel}{\rho_s} \right) ,
\]

and

\[
-\nabla \cdot (\rho_s v_i) = 0 ,
\]

where \( P \) is the pressure of the fluid and the subscripts 0 and 1 refer to unperturbed and perturbed quantities, respectively. All perturbed quantities are proportional to \( e^{ikz+\omega t} \), so we can rewrite equations (3) and (4) as

\[
[i\omega + (u\partial_z)]v_{ix} + ik \left( \frac{P_\parallel}{\rho_s} \right) = 0 ,
\]

\[
[i\omega + (u\partial_z)]v_{iz} + \partial_z \left( \frac{P_\parallel}{\rho_s} \right) = 0 ,
\]

and

\[
[i\omega + (u\partial_z)]v_{ix} + \partial_z v_{iz} = 0 ,
\]

where \( u = (\rho_s/\rho_r) V_s \) is \( v_0 \) in the slab and \( v_{ix} \) and \( v_{iz} \) are the \( x \)- and \( z \)-components, respectively, of \( v_\perp \).

According to Vishniac & Ryu (1989), we get the following three (approximate) solutions with two different assumptions. For the first set of assumptions, we assume that \( u\partial_z \ll i\omega \). Then we can ignore the terms in parentheses and find a combination of the two independent solutions given by

\[
P_\perp = P_+ e^{kz} + P_- e^{-kz} ,
\]
The growth rate is normalized by \( c/H \), i.e., \( \Omega = \omega(c/H) \). In the figure, \( \Omega_r \) is a real part of \( \Omega \), while \( \Omega_i \) is an imaginary part of \( \Omega \). The wavenumber is expressed as \( K = kH \). (a) Positive real parts are shown; the dotted line corresponds to the positive solutions of (20) and the solid line to its negative solutions. (b) The two imaginary parts of both modes coincide.

To find the third solution, we consider the case where the perturbed quantities have a large gradient and \( u_0 \) terms are important. By inspection the third solution is

\[
\frac{P_1}{\rho_s} = 0 ,
\]

and

\[
v_{1x} = A \exp \left( -\frac{i\omega}{u} z \right) ,
\]

and

\[
v_{1z} = Au \frac{k}{\omega} \exp \left( -\frac{i\omega}{u} z \right) .
\]

If the perturbation mode is unstable, then this solution will fall rapidly to zero away from \( z = 0 \). Since we are interested only in the case where perturbation grows, we also assume that \(- i\omega \gg (H/u)^{-1}\), where \( H \) is the thickness of the shocked slab and \(- i\omega \) means the growth rate of the perturbation.

We shall determine the boundary conditions. Assuming very high compressibility at the shock surface, we can suppose \( P_s = \rho_s V_s^2 \). Thus, we integrate the equation of hydrostatic equilibrium (Goldreich & Lynden-Bell 1965) and find the pressure boundary conditions to first order:

\[
P_1(0) = (1 - \beta) \frac{P_s}{\sigma_0} \rho_s \eta - 2\rho_c V_S v_{1z}(0) \quad (14)
\]

and

\[
P_1(H) = (1 - \beta) \frac{P_s}{\sigma_0} \rho_s \zeta ,
\]

where \( \beta (\equiv P_s/P_s) \) is the pressure ratio at the surface of the slab, \( \sigma_0 \) is the surface density of the slab, and \( \eta (\equiv z_{1s} - 0) \) and \( \zeta (\equiv z_{1s} - H) \) are the displacements of the shock and trailing surfaces, respectively. The displacements are \( \eta \approx -iv_{1s}(0)/\omega \) and \( \zeta \approx -iv_{1s}(H)/\omega \). To derive our dispersion relation, we neglect the second term of the right-hand side of equation (14) (see, e.g., Nishi 1992). With the shock boundary conditions, we set the following boundary condition:

\[
v_{1x}(0) = k \frac{v_{1s}}{\omega} ,
\]

Applying the boundary conditions, we have

\[
P_+ + P_- e^{-2kH} = (1 - \beta) \frac{c^2}{H^2 \omega^2} kH(P_+ - P_- e^{-2kH}) ,
\]

and

\[
\frac{P_+}{\rho_s} + \frac{P_-}{\rho_s} = (1 - \beta) \frac{c^2}{H^2 \omega^2} kH \left( \frac{P_+}{\rho_s} - \frac{P_-}{\rho_s} - iAu \right) ,
\]
and
\[
\left(1 - \frac{c^2 k^2}{\omega^2}\right)A = i \frac{V_s k^2}{\omega^2} \left(\frac{P_+}{\rho_s} - \frac{P}{\rho_s}\right),
\]
where \(c^2 \equiv u V_s = \rho_s V_s^2 / \rho_s^2\).

Then we find the dispersion relation to be
\[
\omega^2 = \frac{c^2}{H^2} \left\{ \frac{k^2 H^2}{2} \pm k H \left[ \frac{k^2 H^2}{4} - \left(1 - \beta \right) \coth (kH) (1 - \beta)^2 \right]^{1/2} \right\}. \quad (20)
\]

The numerical estimate of (20) is presented in Figure 1, where the growth rate is normalized by \(c/H\), i.e., \(\Omega = \omega/(c/H)\). The wavenumber is expressed as \(K = kH\), and \(\beta = 0.5\) is adopted in the same figure. As clearly shown in this figure, there is a set of DSI modes (the other set is damping mode).

We evaluate the limiting case of a long wavelength \((KH \to 0)\). In the case of \(\beta(1 - \beta) \neq 0\), we have the growth rate as
\[
\frac{\Gamma}{c/H} \approx 2^{-1/2} \beta^{1/4} (1 - \beta)^{1/4} (kH)^{1/2}. \quad (21)
\]

This growth rate is equal to the long-wavelength limit of equation (2.35) of Nishi (1992) with the assumption \(H = \sigma_0 / \rho_s\).

### 3. THE DSI OF AN INCOMPRESSIBLE SLAB WITH SELF-GRAVITY

With the effect of self-gravity, the equation of fluid motion for a general first-order perturbation is
\[
\frac{d\psi_i}{dt} = \nabla \left( \psi_1 - \frac{P_1}{\rho_s} \right), \quad (22)
\]
where \(\psi_1\) is a perturbed component of the gravitational potential. All perturbed quantities are also set to be proportional to \(e^{ik(z + \omega t)}\) in this section. Then we can rewrite equations with \(\psi_1\) as
\[
\begin{align*}
[i\omega + (u \partial_z)] v_{1x} - i k \left( \psi_1 - \frac{P_1}{\rho_s} \right) & = 0, \quad (23) \\
[i\omega + (u \partial_z)] v_{1z} - \partial_z \left( \psi_1 - \frac{P_1}{\rho_s} \right) & = 0, \quad (24)
\end{align*}
\]
and
\[
iv_{1x} + \partial_z v_{1z} = 0. \quad (25)
\]

As in the previous section, we first assume that \(u \partial_z \ll i\omega\). Namely, we ignore the advection terms and then find a combination of the two independent solutions given by
\[
\psi_1 = \psi_+ e^{kz} + \psi_- e^{-kz}, \quad (26)
\]
\[
P_1 = P_+ + k \frac{P_+ - \psi_+}{\rho_s} e^{kz} - k \frac{P_+ - \psi_-}{\rho_s} e^{-kz}, \quad (27)
\]
\[
v_{1x} = -k \frac{P_+ - \psi_+}{\rho_s} e^{kz} - k \frac{P_+ - \psi_-}{\rho_s} e^{-kz}, \quad (28)
\]
and
\[
v_{1z} = ik \frac{P_+ - \psi_+}{\rho_s} e^{kz} - ik \frac{P_+ - \psi_-}{\rho_s} e^{-kz}. \quad (29)
\]

Next, since we are concerned with the large gradient of the perturbed quantities, we must treat the \(u \partial_z \) terms. By inspection, this third solution is
\[
\frac{P_1}{\rho_s} - \psi_1 = 0, \quad (30)
\]
\[
v_{1x} = A \exp \left( - \frac{i\omega}{\omega} z \right), \quad (31)
\]
and
\[
v_{1z} = A k \exp \left( - \frac{i\omega}{\omega} z \right). \quad (32)
\]

Since we are interested only in the case where perturbation grows, we assume that \(-i\omega / \omega \gg H / u\) again.

We must determine the boundary conditions. We have the perturbed column densities induced by the displacement of the surface as
\[
\sigma_s = -\rho_s \eta e^{-kz} e^{ik(x + \omega t)}, \quad (33)
\]
and
\[
\sigma_t = \rho_s \zeta e^{-kz} e^{ik(x + \omega t)}. \quad (34)
\]

Since the fluid is incompressible, we can determine potential perturbation, \(\psi_1\), from the surface density distributions \(\sigma_s\) and \(\sigma_t\) alone. With the constraints \(\eta, \zeta \ll 2\pi / k\), we have the potentials due to \(\sigma_s\) and \(\sigma_t\) as
\[
\psi_s = -\frac{2\pi G \rho_s}{k} \eta e^{-kz} e^{ik(x + \omega t)}, \quad (35)
\]
and
\[
\psi_t = -\frac{2\pi G \rho_s}{k} \zeta e^{-k(z - \omega t)} e^{ik(x + \omega t)}. \quad (36)
\]

From equations (26), (35), and (36) we find
\[
\psi_+ = -\frac{2\pi G \rho_s}{k} \zeta e^{-kH} \quad (37)
\]
and
\[
\psi_- = -\frac{2\pi G \rho_s}{k} \eta. \quad (38)
\]

For the pressure boundary conditions, we find, to first order,
\[
P_1(0) = -2\pi G \rho_s^2 H \eta + (1 - \beta) \frac{P_s}{\sigma_0} \rho_s \eta - 2\rho_e V_s v_{1z}(0) \quad (39)
\]
and
\[
P_1(H) = 2\pi G \rho_s^2 H \zeta + (1 - \beta) \frac{P_s}{\sigma_0} \rho_s \zeta. \quad (40)
\]

We also neglect the third term of the right-hand side of equation (39), just as we derive equation (20). Applying the shock boundary conditions gives the next boundary condition:
\[
v_{1z}(0) = \frac{k}{\omega} V_s v_{1z}(0). \quad (41)
\]
Applying the boundary conditions, we get
\[ \psi_+ = \frac{2\pi G \rho_s}{\omega^2} \left[ \left( \frac{P_+}{\rho_s} - \psi_+ \right) - \left( \frac{P_-}{\rho_s} - \psi_- \right) e^{-2kH} \right], \tag{42} \]
\[ \psi_- = -\frac{2\pi G \rho_s}{\omega^2} \left[ \left( \frac{P_+}{\rho_s} - \psi_+ \right) - \left( \frac{P_-}{\rho_s} - \psi_- \right) - iAu \right], \tag{43} \]
\[ \frac{P_+}{\rho_s} + \frac{P_-}{\rho_s} e^{-2kH} = (\tilde{\alpha} + 1) \frac{2\pi G \rho_s \kappa H}{\omega^2} \]
\[ \times \left[ \left( \frac{P_+}{\rho_s} - \psi_+ \right) - \left( \frac{P_-}{\rho_s} - \psi_- \right) e^{-2kH} \right], \tag{44} \]
\[ \frac{P_+}{\rho_s} + \frac{P_-}{\rho_s} = (\tilde{\alpha} - 1) \frac{2\pi G \rho_s \kappa H}{\omega^2} \]
\[ \times \left[ \left( \frac{P_+}{\rho_s} - \psi_+ \right) - \left( \frac{P_-}{\rho_s} - \psi_- \right) - iAu \right], \tag{45} \]
and
\[ \left( 1 - \frac{c^2 k^2}{\omega^2} \right) = \frac{i V_s \kappa e}{\omega^2} \left[ \left( \frac{P_+}{\rho_s} - \psi_+ \right) - \left( \frac{P_-}{\rho_s} - \psi_- \right) \right], \tag{46} \]

where \( \tilde{\alpha} \equiv (1 - \beta) P_0/(2\pi G \rho_s \kappa) \) is the ratio of the deceleration of the whole slab to the acceleration of the slab's self-gravity at the surface.

From equations (42)-(45), we have
\[ B_+ + B_- = (\tilde{\alpha} - 1) \frac{2\pi G \rho_s \kappa H}{\omega^2} (B_+ - B_- - iAu) \]
\[ -\frac{2\pi G \rho_s}{\omega^2} (B_- - B_+ e^{-2kH} + iAu e^{-2kH}) \tag{47} \]
and
\[ B_+ + B_- e^{-2kH} = (\tilde{\alpha} + 1) \frac{2\pi G \rho_s \kappa H}{\omega^2} (B_+ - B_- e^{-2kH}) \]
\[ -\frac{2\pi G \rho_s}{\omega^2} (B_+ - B_- e^{-2kH} + iAu), \tag{48} \]

where \( B_+ \equiv P_+ / \rho_s - \psi_+ \) and \( B_- \equiv P_- / \rho_s - \psi_- \), respectively. Then we can rewrite equations (47) and (48) as
\[ \left[ \frac{\omega^2}{2\pi G \rho_s} - (\tilde{\alpha} - 1)kH \right] B_+ \]
\[ + \left[ \frac{\omega^2}{2\pi G \rho_s} + (\tilde{\alpha} - 1)kH + (1 - e^{-2kH}) \right] B_- \]
\[ + [1 + (\tilde{\alpha} - 1)kH] iAu = 0 \tag{49} \]
and
\[ \left[ \frac{\omega^2}{2\pi G \rho_s} - (\tilde{\alpha} + 1)kH + (1 - e^{-2kH}) \right] B_+ \]
\[ + e^{-2kH} \left[ \frac{\omega^2}{2\pi G \rho_s} + (\tilde{\alpha} + 1)kH \right] B_- + e^{-2kH} iAu = 0 \tag{50} \]

and equation (46) as
\[ c^2 k^2 B_+ - c^2 k^2 B_- + (\omega^2 - c^2 k^2) iAu = 0. \tag{51} \]

Thus, in equations (49), (50), and (51), we have the dispersion relation
\[ \omega^6 + 2\pi G \rho_s \left[ 2 - \frac{\tilde{\alpha}}{1 - \beta} k^2 H^2 - 2 \coth (k\kappa H) \right] \omega^4 \]
\[ + (2\pi G \rho_s)^2 \left[ (\tilde{\alpha} + 1) - \frac{\tilde{\alpha}}{1 - \beta} \coth [k\kappa H]^3 - \frac{\tilde{\alpha}}{1 - \beta} k^2 H^2 \right. \]
\[ - (\tilde{\alpha}^2 - 1)k^2 H^2 - 2kH + (1 - e^{-2kH}) \right] \omega^2 = 0. \tag{52} \]

We neglect the mode where \( \omega = 0 \) because of the assumption of \( -i\omega \gg H/u \). Finally, then, we obtain the dispersion relation
\[ \frac{\omega^2}{2\pi G \rho_s} = \frac{\tilde{\alpha}}{1 - \beta} - \frac{k^2 H^2}{2} + \coth (k\kappa H) - 1 \]
\[ \pm \left\{ \left[ \frac{\tilde{\alpha}}{1 - \beta} - \frac{k^2 H^2}{2} + \coth (k\kappa H) - 1 \right]^2 \right. \]
\[ -(1 + \tilde{\alpha}) \frac{\tilde{\alpha}}{1 - \beta} \coth (k\kappa H)^3 + \frac{\tilde{\alpha}}{1 - \beta} k^2 H^2 \]
\[ -(1 - \tilde{\alpha}^2)k^2 H^2 + 2kH + (e^{-2kH} - 1)^{1/2}. \tag{53} \]

4. DISCUSSION

First, we investigate the limiting cases of the dispersion relation of equation (53). To the limit of the long wavelength, i.e., \( kH \rightarrow 0 \), we have
\[ \frac{\omega^2}{2\pi G \rho_s} \approx \pm kH(1 - \alpha^2)^{1/2}, \quad \alpha \equiv \left( \frac{\beta}{1 - \beta} \right)^{1/2} \tilde{\alpha}. \tag{54} \]

From equation (54), we can classify the characteristics of the instability by value of \( \alpha \). In the case of \( \alpha > 1 \), the layer is overstable for long-wavelength perturbation. When the self-gravity of the slab does not play important roles, i.e., in the limits of \( \tilde{\alpha} \) and \( \alpha \rightarrow \infty \), we have the dispersion relation of equation (20). On the other hand, in the case of \( \alpha < 1 \), the layer is unstable for gravity modes.

The numerical presentation of equation (53) is given in Figures 2, 3, and 4, where the growth rate is normalized by (2\pi G \rho_s)^{0.5}, i.e., \( \Omega = \omega/(2\pi G \rho_s)^{0.5} \). The wavenumber is expressed as \( K = kH \) in all three figures. In Figure 2, we display the cases of \( \alpha = 1.2 \) and \( \beta = 0.2, 0.4, 0.6, \) and 0.8. As in Figure 1, we thus find the DSI dispersion relations in Figure 2. For the same value of \( \alpha \), \( \beta \) is larger, the unstable region becomes narrower, and the growth rate becomes smaller. This is because large \( \beta \) means inefficiency of the deceleration at the surface of the slab. In the case of the adopted parameters, none of the instability is due to self-gravity.

In Figure 3, the dispersion relations of \( \alpha = 0.7 \) and \( \beta = 0.2, 0.4, 0.6, \) and 0.8 are presented. Since \( \alpha \) is smaller than unity, we expect gravitational instability in a range of small wavenumbers. Indeed, we find the gravitational instability for all \( \beta \) in a range of small \( K \). As indicated in this figure, the growth rate of the gravitational instability depends almost entirely on \( \alpha \). On the other hand, we also
obtain DSI in larger $K$ except in the case of $\beta = 0.8$. We find again that the growth rate of the DSI does not depend only on $\alpha$ but also on $\beta$. Incidentally, we would like to comment here that all three cases ($\beta = 0.2$, 0.4, and 0.6) are stable for intermediate values of $K$ in Figure 3; that is, both DSI and gravitational instability do not occur for a special range of wavenumbers. For the case of $\beta = 0.8$, moreover, the DSI mode is always stable because of small $\tilde{x}$. These points are

Fig. 2.—Numerical expression of eq. (53). The growth rate is normalized by $(2\pi G\rho)^{0.5}$. We draw $\Omega_r$ as a real part of $\Omega$, while $\Omega_i$ as an imaginary part of $\Omega$. The wavenumber is expressed as $K = kH$. We adopt $\alpha = 1.2$ and $\beta = 0.2$ (solid line), 0.4 (dashed line), 0.6 (dotted line), and 0.8 (dot-dashed line). Only DSI modes are found for the four examples.

Fig. 3.—Same as Fig. 2 but for $\alpha = 0.7$. For small $K$ we find gravitational instability, while DSI occurs for large $K$. For intermediate values of $K$, we find just the stable modes. The case of $\beta = 0.8$ does not have DSI even in large ranges of $K$. Of course, $\Omega_i = 0$ for the gravitationally unstable modes.
In Figure 4, we present the dispersion relation for $a = 0$. As is clearly shown by the solid lines, there is only gravitational instability. In other words, we get no DSI mode. Thus, for the case of $a = 0$, the self-gravity mainly controls the instability. Indeed, if $a \rightarrow 0$, we obtain

$$\omega^2 = 2\pi G \rho \left[ -(1 \pm e^{-kh}) + \frac{(e^{kh/2} \pm e^{-kh/2})^2}{e^{kh} - e^{-kh}} kH \right].$$

This is the same dispersion relation given by Goldreich & Lynden-Bell (1965). Thus, so long as $a$ is somewhat different from unity, the order of the growth rate of the gravitational instability becomes and the scale length of the most unstable wavelength becomes $H$ (see also Fig. 3), which is almost in agreement with the results of the high-pressure case of Elmegreen & Elmegreen (1978). To obtain this mode, it is important to incorporate the structure of the slab.

5. FRAGMENTATION OF THE SHOCKED SLAB

In the previous considerations, we assumed that $a$ is always constant in each case. However, in reality, $a$ changes with the evolution of the shocked slab (e.g., Elmegreen 1989). In this section, then, we shall discuss the evolution effects (e.g., Nishi 1992). From the definition, we find $a \propto P_S/\sigma_0^2 \propto (V_S/\sigma_0)^2$. Since the shocked layer sweeps up the matter, the column density increases and shock velocity decreases with time; hence $a$ decreases. Thus, the character of the instability in the slab changes at time $t = t_{cr}$, when $a$ becomes unity. In the early stages of the evolution of the shocked slab, the result of the analysis incorporating the structure of the shocked slab; therefore, they do not appear in Nishi (1992).

Previously, Yoshida & Habe (1992) discussed the nonlinear stage of DSI and gravitational instability under similar conditions to those adopted in this paper. According to them, the gravitational instability has a longer wavelength than DSI. This property is understood through our linear analysis since both modes are incompatible in the dispersion relation derived in equation (53). Thus, our result is in accord with the calculations given by Yoshida & Habe. One might think that DSI does not have a bending shocked layer since we cannot find it clearly in Figure 9 of Yoshida & Habe (1992). However, although the thickness of our slab is nearly constant in our linear analysis, in the numerical results of Yoshida & Habe, the shocked slab has a few times larger thickness than the initial value. If the bounding effect were to be tighter in Yoshida & Habe (1992), they would obtain our bending property of DSI in their simulations (cf. Chevalier & Theys 1975).
shocked layer, $t < t_{cr}$, only the DSI can occur in that layer. In the late stages, $t > t_{cr}$, however, it is possible for the shocked layer to be unstable for the DSI (on a smaller scale) and the gravitational instability (on a larger scale), generally. Finally, at $t \gg t_{cr}$, $x$ approaches zero; then the layer becomes stable against the DSI and unstable only for the gravitational instability, as in equation (55). In the following paragraphs, we shall estimate and discuss the evolution of the shocked slab.

If the shock velocity decreases as $V_s \propto t^p (-1 < p < 0)$, we can estimate $t_{cr}$ as follows. The column density at time $t$ is estimated as

$$\sigma_0 = q \rho_e R, \quad \text{where} \quad R = \frac{V_s t}{p + 1}.$$  

(56)

Here, $q$ is the factor depending on the shape of the shock wave and, of course, $q = 1$ is the case for the plane shock wave. The distance of the shock surface from the origin is denoted as $R$. From the definition of $t_{cr}$, $\sigma(t = t_{cr}) = 1$, and $P_s = \rho_e V_s^2$, we obtain the following relation for $t_{cr}$:

$$t_{cr} = \frac{(p + 1)\beta^{1/4}(1 - \beta)^{1/4}}{q(2\pi G \rho_e)^{1/2}}.$$  

(57)

Thus, $t_{cr}$, which is the time for the onset of the gravitational instability, does not depend on the initial shock velocity. Although $\beta$ is a function of time, the product of $\beta^{1/4}$ and $(1 - \beta)^{1/4}$ depends only very weakly on $\beta$. Indeed, for the range $\beta = 0.1 - 0.9$, which is a large enough range for a realistic situation, $\beta^{1/4}(1 - \beta)^{1/4}$ has values only in the range 0.55–0.71. Thus, even for a realistic situation, the estimation of $t_{cr}$ with the assumption of $\beta$ being constant is fairly valid. That is to say, we estimate $t_{cr}$ as being on the order of the free-fall time of the external matter.

In our analysis, the effect of shell expansion is not included. For the case of spherical expansion, the shell expansion effect may be important. The onset time of the gravitational instability is estimated by comparing the divergence rate of the expanding shells to the gravitational attraction rate for the expanding shells by Ostriker & Cowie (1981; in the cosmological case) and by McCray & Kafatos (1987; in the supershell case). Since we assume a large compression ratio at the shock surface, the estimate for the snowplow phase (eq. [22] of McCray & Kafatos) should be compared with ours. The onset time for this case is somewhat shorter than the free-fall time of the external medium for the typical case. Thus, it may be correct that $t_{cr}$ is somewhat shorter than the free-fall time of the external medium even for the expanding shell case, although more detailed investigation is necessary to determine this definitely.

Since the DSI saturates with transonic transverse flows in the shocked slab (Mac Low & Norman 1993), only with the effect of the DSI, the shocked slab hardly fragments. Thus,
we should consider the coupling of the gravitational instability with the DSI to investigate the fragmentation process of the evolving shocked layer. First, we consider the most unstable perturbation of the DSI in an early stage ($t < t_{self}$). Considering the constant wavelength, we find that the perturbation grows because of the DSI, but it may be stable in the later stages as long as the perturbation is linear (see Figs. 2, 3, and 4). Next, we consider the case in which perturbation is unstable for the gravitational instability in the later stages. That perturbation can grow because of the DSI in the early stages. Thus, the DSI might be related to the fragmentation process via the gravity, which occurs in later stages. It is important to note that, in the intermediate stages, the perturbation becomes stable against both instabilities, as shown in Figure 3. Then we find that it is difficult for the DSI to connect directly with the gravitational instability during the evolution of the slab. The calculations by Yoshida & Habe (1992) may suggest the stabilization of the instabilities, i.e., the perturbation is damped when the wavelength is in this stable range probably because of the evolution effect (e.g., Vishniac 1983; Elmegreen 1989). Thus, as far as the stable range is significant, the effect of the DSI on structure formation may not be important. However, Yoshida & Habe do not find the growing bending mode for all parameters, although Mac Low & Norman clearly present the evolution of the bending mode. Thus, there is a possibility that the numerical resolution of Yoshida & Habe is not high enough. Further investigation including the non-linear stage is necessary.

6. SUMMARY

In this paper, we present the dispersion relation of a shocked slab with self-gravity. According to this relation, when the deceleration dominates the self-gravity, the so-called DSI is efficient. Their growth rates are a function of the scale length, which is determined from the efficiency of the deceleration. Thus, we confirm previous research qualitatively. The slab becomes gravitationally unstable when the gravitational energy is larger than the thermal energy of the layer, which is the result of the ram pressure at the shocked surface. In other words, when deceleration is inefficient, self-gravity works well. Interestingly, even if the slab is gravitationally unstable in the long wavelength, it is not gravitationally unstable in the short wavelength. Instead of gravitational instability, DSI occurs. This suggests that a smaller scale structure than the Jeans length may be a result of the DSI. The growth rate of the gravitational instability depends almost entirely on $\alpha$, while that of the DSI depends not only on $\alpha$ but also on $\beta$. The growth timescale of both instabilities is of the order of the free-fall time for the slab density, while the onset time of the gravitational instability is of the order of the free-fall time of the external medium. Thus, within our treatment, fragmentation by gravitational instability becomes possible only after about the free-fall time of the external medium. The new analytical point in this paper is that we find stable modes in the middle wave-numbers. This result is obtained since we are concerned with the structure of a shocked slab with finite thickness (cf. Nishi 1992). This stable range may suggest the ineffectiveness of DSI for the fragmentation of a shocked slab via evolutionary effects (Vishniac 1983; Elmegreen 1989; Yoshida & Habe 1992). Thus, we should determine the structure of the shocked layer when we study the properties of the instabilities precisely.

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Note that $\alpha$ decreases and $H$ increases (i.e., $K$ increases for a constant $k$) with time evolution.