Numerical Methods for Modeling Binary Neutron Star Systems

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Alan C Calder†, F Douglas Swesty†‡ and Edward Y M Wang†§

† National Center for Supercomputing Applications, University of Illinois, Urbana, IL 61801, USA
‡ Department of Astronomy, University of Illinois, Urbana, IL 61801, USA
§ Department of Physics, Washington University in St. Louis, St. Louis, MO 63130, USA and Department of Physics and Astronomy, State University of New York at Stony Brook, Stony Brook, NY 11794, USA

Abstract. We present initial results of our study of numerical methods for modeling neutron star mergers (NSMs) with simulations that perform the full hydrodynamic evolution required to capture tidal effects, particularly in the last several orbits. Our simulations evolve the Euler equations using a modification of the ZEUS-2D algorithm (Stone and Norman 1992). We describe some of the difficulties of modeling NSMs and our approaches to these difficulties, and we discuss the motivation for the choice of performing simulations in a co-rotating reference frame. Our results establish what effects the choices of gravity coupling and reference frame have on the numerical accuracy of the simulation.

1. Introduction

1.1. Scientific motivation

Neutron star mergers (NSMs) are excellent astrophysical laboratories for the study of relativistic astrophysics, gravitational wave astronomy, and nuclear astrophysics, and are thought to be a possible source of observed gamma ray bursts. General relativity predicts that binary systems of compact objects, such as NS-NS systems, will emit energy in the form of gravitational radiation. This loss of energy will lead to the in-spiral and coalescence of the binary system, and gravitational waves produced in these events are expected to be observed by gravitational wave detectors such as LIGO (Abramovici et al. 1991).

1 E-mail: calder@ncsa.uiuc.edu
The limited range of frequencies of such gravitational wave detectors makes in-spiraling compact binaries the only sources of gravitational waves expected to be observed with high accuracy (Thorne 1995). Post-Newtonian methods are adequate for the prediction of waveforms for the early stage of the in-spiral, but the prediction of waveforms in the later stages of the merger, when tidal effects and neutron star structure become important, requires a full three-dimensional numerical simulation.

Information about the neutron star equation of state as well as upper bounds on neutron star masses may be obtained from observed gravitational waveforms. Additionally, the spiral arms of the coalescing neutron stars may be a site of $r$-process nucleosynthesis since bombardment of nuclei by free neutrons is expected to occur in this environment (Lattimer et al 1977). NSMs are a suggested source for the mysterious gamma-ray bursts observed in recent years. These events are thought to release energy on the order of their gravitational binding energy, $\approx 10^{53}$ erg, which is comparable to the estimated gamma-ray burst energies, $\approx 10^{51}$ (Quashnock 1996 and Rees 1997) to $10^{53}$ erg (Woods and Loeb 1994). A popular model for bursts at cosmological distances is the relativistic fire-ball. NSMs are likely candidates for the source of relativistic fire-balls, but the mechanism by which the fire-ball develops has yet to be determined. Simulations of NSMs can test the consistency of the merger energetics and time scales with the estimated burst energies and observed time scales.

1.2. Newtonian hydrodynamics and gravitational waveform calculation

The simulations presented in this work are purely Newtonian, though we have performed additional simulations that include a radiation reaction (Wang et al 1998). We assume that the neutron stars can be described as a compressible Newtonian fluid. The equations of Newtonian hydrodynamics in a reference frame rotating at $\omega$ are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial (\rho e)}{\partial t} + \nabla \cdot (\rho e \mathbf{v}) = -(P + Q) \nabla \cdot \mathbf{v}$$

$$\frac{\partial v_i}{\partial t} + \nabla \cdot (v_i \rho \mathbf{v}) = -\nabla (P + Q) - \rho \nabla \Phi - 2\rho \omega \times \mathbf{v} - \rho \omega \times (\omega \times r)$$

where $\mathbf{r}$ is the position, $\rho$ is the density, $\mathbf{v}$ is the velocity of the fluid, $e$ is the specific internal energy, and $P$ is the pressure. $Q$ is an artificial viscous stress to model the sub-resolution micro-physics occurring across shock fronts. The potential $\Phi$ is the Newtonian gravitational potential. The equations are discretized onto a staggered Eulerian mesh by a method similar to that employed in the ZEUS codes (Stone and Norman 1992). The set of equations is closed by an equation of state, which for this work is that of an ideal gas,

$$P = (\Gamma - 1) E,$$

where $E = \rho e$ is the energy density and $\Gamma$ is the adiabatic index. Future simulations will make use of the realistic equation of state of Lattimer and Swesty (Lattimer et al 1985 and Lattimer and Swesty 1990). The Euler equations, equations (1)-(3), are solved in two steps, an advection update and a source update. The advection step makes use of second-order van Leer monotonic advection with Norman’s consistent advection method.
(Norman 1980) to evolve the left-hand sides of equations (1)–(3). The source update calculates the sources and sinks of the right-hand side of equations (2) and (3). The solution of Poisson’s equation
\[ \nabla^2 \Phi(r) = 4\pi G \rho(r) \] (5)
gives the gravitational potential \( \Phi \).

The gravitational waveforms are calculated from the quadrupole approximation. In this approximation, the strain amplitude of the gravitational wave radiation in the transverse traceless gauge is
\[ h_{TT}^{TT} = \frac{2}{r} \frac{G}{c^4} \ddot{I}_{lm}^{TT}, \] (6)
where \( r \) is the distance to the source, \( c \) is the speed of light, \( G \) is the gravitational constant. \( \ddot{I}_{lm}^{TT} \) is second time derivative of the transverse, traceless part of the reduced quadrupole moment, \( I_{lm} \),
\[ I_{lm} = \int \rho (x_l x_m - \frac{1}{3} \delta_{lm} r^2) \, d^3r, \] (7)
in Cartesian coordinates. The second time derivatives of \( I_{lm} \) are calculated without numerical time differentiation (Finn and Evans 1990 and Rasio and Shapiro 1992). Along the z-axis, the two polarizations of \( h^{TT} \) are given by
\[ h_+ = \frac{G}{c^4} \frac{1}{r} (\ddot{I}_{xx} - \ddot{I}_{yy}), \] (8)
\[ h_\times = \frac{G}{c^4} \frac{2}{r} \ddot{I}_{xy}. \] (9)

1.3. Coupling gravity to the hydrodynamics

Our code, V3D, performs the advection step before the source step that updates the Lagrangian terms (the terms on the right hand side of the hydrodynamics equations). This ordering, advection before the source update, allows the choice of computing the right-hand side of the Poisson equation, \( 4\pi G \rho \), with the density at the old time step (time lagged), the new time step (time advanced), or the average of the two densities (time centered). The finite difference expressions for these choices are
\[ (\nabla^2 \Phi)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} = 4\pi G \rho^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} \] t. l.
\[ (\nabla^2 \Phi)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} = 4\pi G \rho^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} \] t. a.
\[ (\nabla^2 \Phi)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} = 4\pi G \left( \frac{\rho^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} + \rho^{n+1}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}}{2} \right) \] t. c.

1.4. Additional method details

Our numerical method requires that material be on all of the grid; therefore, we include a hot, low density (\( \approx 1 \text{ g/cm}^3 \)) atmosphere as a background. The high temperature of the atmosphere gives the material enough energy to avoid falling onto the surface of the star and forming a shock, which would require unacceptably small time steps to evolve. We find thin atmospheres (1 - 10\(^3\)g/cm\(^3\)) have no effect on the simulation
dynamics. Early simulations made use of a full multi-grid W-cycle algorithm to solve (3), but recent simulations (some of which are presented below) made use of a Fast Fourier Transform (FFT) method to solve (3). The solution of the Poisson equation requires the calculation of the potential at the grid boundaries. A direct summation over density to get the potential on the boundary is computationally expensive so the calculation is performed over a coarser grid with the contributions of zones containing less than $10^{-5} M_\odot$ neglected. This method works well for “clumpy” mass distributions (e.g. two stars), and was tested against an entire grid summation.

We should note that many simulations were performed to test the different simulation methods studied during the development of our method and to determine the optimal choices of parameters such as Courant factor and grid resolution. Full details of our method and complete results will appear in Swesty et al (1998).

2. Results

The goal of this study was to determine the effects of the choices of gravity coupling and reference frame on the dynamics of the simulations. The simulations that are described as co-rotating were performed in a reference frame initially co-rotating with the stars, and the simulations described as fixed frame were performed in a fixed inertial reference frame. We use as a measure of these effects the binary separation, the conservation of energy and angular momentum, and the gravitational waveform. All simulations began with the same initial conditions describing two $1.4M_\odot$, $\gamma = 2$ polytropes with an initial separation of $4.0R_\ast$ with $R_\ast = 9.56$ km. This initial separation is large enough that the simulations should maintain stable orbits. These simulations made use of a Courant factor of 0.4 and had 129 grid points along each axis, which with a simulation cube size of 63.5 km gave a resolution of 0.5 km per zone.

Figure 1 shows the locations of the star centers and the location of the center of mass for six NSM simulations that began from the same initial data. The locations of the stars initially on the left-hand side of each frame are shown as a dashed line, and the locations of the stars initially on the right-hand side of each frame are shown as a dotted line. The three left-hand side panels are of simulations performed in the co-rotating reference frame, and the right hand side panels are of simulations performed in the fixed reference frame. The locations of the co-rotating simulations have been mapped back into a fixed reference frame for comparison. In the figure, the instability of some of the simulations appears as inward spiral deviations from perfectly circular orbits. The three simulations performed in the fixed frame (right-hand side panels) all demonstrate significant orbital instabilities. The three simulations performed in the co-rotating frame were all more stable than their fixed frame counter parts. Simulations with time lagged gravity were the most unstable for both fixed and co-rotating simulations. The most stable simulation (the upper left panel) was performed in the co-rotating frame with time advanced gravity. It shows very little deviation after 5 orbits (12.5 ms). The least stable (the lower right panel) was performed in the fixed frame with time lagged gravity. In it, the stars merged in little more than one orbit (2.0 ms).

Figure 2 shows a comparison between the first 6.0 ms of simulations with time advanced gravity in a co-rotating frame (solid line) and in a fixed frame (dashed line). The plots are from the same simulations as the time advanced simulations in Figure 1.
Figure 1. Star locations and centers of mass from six NSM simulations from the same initial conditions. From top to bottom are results from simulations with time advanced, time centered, and time lagged gravity. The images in the left column are from co-rotating simulations, and those in the right column are from fixed frame simulations.
Figure 2. Results of two time advanced gravity simulations for 6.0 ms from the same initial conditions. Plotted in the left panel is the angular momentum at five density thresholds as functions of time for a simulation in a fixed frame (dashed lines) and a simulation in an initially co-rotating frame (solid lines). The right panel shows the $h_x$ waveform for the fixed frame simulation (dashed line) and the initially co-rotating frame simulation (solid line).

The panel on the left of Figure 2 shows the angular momentum at five density thresholds. Each line represents the angular momentum of material with density greater than a particular threshold density. Conserved angular momentum would appear as a set of straight horizontal lines. The plot shows that the simulation in the fixed frame does not conserve angular momentum as well as the simulation in the co-rotating frame. Comparison of energy conservation between the two simulations shows a similar result. The panel on the right shows the $h_x$ waveform of the two simulations. While the waveforms have the same amplitude, it is readily apparent that the two become out of phase by approximately 1/4 orbit within less than three orbits. This significant difference shows that the choice of reference frame can have a strong impact on the accuracy of a NSM simulation.

3. Conclusions

Our results show that the simulation dynamics of NSMs are very sensitive to the finite difference scheme with which gravity is coupled to the hydrodynamics. We find that simulations with a time advanced matter-gravity coupling are most stable in both co-rotating and fixed frame simulations. The co-rotating frame simulations are not as sensitive, however, to the coupling of gravity to the hydrodynamics. Conservation of energy and angular momentum are important tests, and all future numerical calculations of NSMs should reveal how well these quantities are conserved. We find that simulations in a reference frame initially co-rotating with the stars conserve energy and angular momentum well.
Accurate waveform calculations are imperative for gravitational wave astronomy, and consequently the waveforms must be free from numerical artifacts. Our results show that the waveforms of simulations in a fixed frame can be very different from simulations with the same initial conditions but performed in a co-rotating frame. We find that waveforms become almost \(1/4\) orbit out of phase within 6 ms in the most stable case of time advanced gravity. The principle difference between fixed and co-rotating frame simulations is that fixed frame simulations dissipate angular momentum much more quickly than co-rotating frame simulations. It is the requisite numerical advection and the need to resolve a rapidly varying gravitational field in a fixed frame that dissipates angular momentum leading to the very different waveforms.

Based on these results, we conclude that NSM simulations should be performed in a co-rotating reference frame. Numerical effects seem to dominate the evolution of simulations in a fixed reference frame. Further work is in progress to compare these calculations to those done with other hydrodynamic algorithms such as the piecewise parabolic method (Colella and Woodward 1984).

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