REALIZATION OF THE Riemann Hypothesis VIA COUPLING CONSTANT SPECTRUM

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Abstract

We present a Non-relativistic Quantum mechanical model, which exhibits the realization of Riemann Conjecture. The technique depends on exposing the S-wave Jost function at zero energy and in identifying it with the Riemann $\xi(s)$ function following a seminal paper of N. N. Khuri.

*I dedicate this note to my teacher, George Sudarshan of the University of Texas at Austin
We begin by recalling the all-too-familiar lore that the Riemann hypothesis has been the Holy Grail of mathematics and physics for more than a century [1]. It asserts that all the zeros of $\xi(s)$ have $\sigma = \frac{1}{2}$, where $s = \sigma \pm it_n$, $n = 1, 2, 3 \ldots \infty$. It is believed all zeros of $\xi(s)$ are simple. The function $\zeta(s)$ is related to the Riemann $\xi(s)$ function via the defining relation [1],

$$\xi(s) = \frac{1}{2}s(s - 1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

so that $\xi(s)$ is an entire function, where

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad s = \sigma + it, \quad \sigma > 1$$

$\zeta(s)$ is holomorphic for $\sigma > 1$ and can have No zeros for $\sigma > 1$. Since $1/\Gamma(z)$ is entire, the function $\Gamma\left(\frac{s}{2}\right)$ is non-vanishing, it is clear that $\xi(s)$ also has no zeros in $\sigma > 1$: the zeros of $\xi(s)$ are confined to the “critical strip” $0 \leq \sigma \leq 1$. Moreover, if $\rho$ is a zero of $\xi(s)$, then so is $1 - \rho$ and since $\xi(s) = \xi(\overline{s})$, one deduces that $\overline{\rho}$ and $1 - \overline{\rho}$ are also zeros. Thus the Riemann zeros are symmetrically arranged about the real axis and also about the “critical line” given by $\sigma = \frac{1}{2}$. The Riemann Hypothesis, then, asserts that ALL zeros of $\xi(s)$ have Re $s = \sigma = \frac{1}{2}$.

We conclude this introductory, well-known remarks with the assertion that every entire function $f(z)$ of order one and "infinite type" (which guarantees the existence of infinitely many Non-zero zeros can be represented by the Hadamard factorization, to wit [1],

$$f(z) = z^m e^{A_0} e^{Bz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n}\right)$$

where ‘m’ is the multiplicity of the zeros (so that $m = 0$, for simple zero).

Finally, $\xi(s) = \xi(1 - s)$ is indeed an entire function of order one and infinite type and it has No zeros either for $\sigma > 1$ or $\sigma < 0$.

We begin our brief note by defining the non-relativistic, quantum mechanical potential model in 3 dimensions.

The zero energy (i.e., $k^2 = 0$, $\frac{2m}{\hbar^2} = 1$) Schrodinger equation reads [for zero angular momentum, $l = 0$ (S-waves)]

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi(r)}{\partial r}\right) + \left(0 - \frac{\lambda}{r^2}\right) \psi(r) = 0$$

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The potential we choose is:

\[ V(r) = \frac{\lambda}{r^2} \quad (1) \]

The canonical change of the independent variable \( \psi(r) \) to \( U(r) \) results in the equation, well-known to every one (!):

\[ U''(r) - \frac{\lambda}{r^2} U(r) = 0 \quad (5) \]

Eq. (5) has the solution:

\[ U(r) = r^s \Rightarrow \psi(r) = r^{s-1} \quad (6) \]

where \( s \) is constrained to satisfy the relation,

\[ \lambda = s(s - 1) \quad (7) \]

For the Regular solution at the origin, i.e., \( \psi(0) = 0 \), we require that Real \( s > 1 \). Eq. (7) will play a crucial role in the upcoming analysis. We note in passing that the Repulsive nature of the real potential, i.e., \( V(r) = \frac{\lambda}{r^2} \), \( \lambda \) real and > 0, requires that there are No bound states. Further due to the scale invariance of the problem, i.e., the potential is scale-free (and we are suppressing the overall factor \( \frac{1}{M} \) in \( V(r) \)). The coupling constant \( \lambda \) is dimensionless.

The inverse-square potential acts like a centrifugal term in the free Schroedinger equation. It is a straightforward exercise in undergraduate physics to determine the corresponding \( S \)-wave phase shift for scattering solution [2]:

\[ \delta = \frac{\pi}{4} - \frac{\pi \nu}{2} \quad (8) \]

where

\[ \nu = \sqrt{\frac{1}{4} + \lambda} > \frac{1}{2} \quad (9) \]

We note in passing that for the free case, \( \lambda = 0 \), \( \nu = \frac{1}{2} \) and the phase shift does indeed vanish and the \( S \) matrix is unity:

\[ S = \exp(2i\delta) \quad (10) \]
It is important to emphasize that the phase shift $\delta$ in Eq. (8) and hence also the $l = 0$ partial wave scattering matrix in Eq. (10) are **independent of energy** and the $S$ matrix is thus manifestly scale invariant. (There are no bound states).

We go on to summarize the seminal idea of Khuri (and Chadan) [3]. It is well known that the $S$ wave Jost function is identified via the defining relation,

$$S(k) = \frac{F_-(k)}{F_+(k)}$$

(11)

where $F_+(k)$ is holomorphic in the upper half complex $k$ plane and $F_-(k)$ is holomorphic in the lower half of $k$ plane, where $k$ is the momentum (in the center-of-mass frame).

Thus,

$$F_+(k) = S^{-1}(k)F_-(k), \quad Imk = 0,$$

$$-\infty < k < \infty$$

(12)

and

$$\text{det} \ S(k) \neq 0, \quad Imk = 0$$

(13)

Khuri’s idea is the following: the $S$-wave Jost function for a potential $\lambda V^* = \frac{\lambda}{r^2}$

(14)

is an entire function of $\lambda$ with an infinite number of zeros extending to infinity. For a repulsive potential $V$ and at **zero energy**, these zeros of the coupling constant $\lambda$, will all be **real** and **negative** (see the elaboration of this fact later on), i.e., $\lambda_n(k^2 = 0) < 0$.

By rescaling $\lambda$ such that $\lambda_n < -\frac{1}{4}$ and changing variables to $s$, with $\lambda = s(s - 1)$, it follows that as a function of $s$, the $S$-wave Jost function has only (!) zeros on the line $s_n = \frac{1}{2} + it_n$. Thus, if we can “find” a **repulsive** potential $V^*$ whose coupling constant spectrum coincides with the Riemann zeros of $\xi(s)$, this will unambiguously establish the holy grail of physics and mathematics! Khuri commented that **“This will be a very difficult and unguided search.”** That said, we wish to make the crucial observation: For
the potential at hand, i.e., \( V(r) = \lambda V^*(r) \), \( V^*(r) = \frac{1}{r} \), \( \lambda \) real & positive, the coupling constant \( \lambda \) must \textbf{necessarily} satisfy the equality,

\[
\lambda = s(s-1)
\]  

(15)

In other words, there is “no need to change variables!” We proceed by summarizing the “solution” to the boundary value problem defined by Eq. (12) for the S wave Jost function.

In the case of uncoupled partial waves the solution to the boundary value problem formulated via Eq. (12) has been given by Krutov, Muravyev and Troitsky [4] in 1996. The “Solution” reads:

\[
F_\pm(k) = \Pi_\pm(k) \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(S^{-1}(k)\Pi^2(k))}{k - k \mp io} dk'\right)
\]  

(16)

where

\[
\Pi_\pm(k) = \prod_{j=1}^{m'} \frac{k \mp ik_j}{k \pm ik_j}, \quad k_j > 0, \quad j = 1, 2, \ldots, m' \quad (m' < \infty)
\]  

(17)

\[
\Pi_\pm(k) \equiv 1, \quad m' = 0
\]  

(18)

where \( m' \) is the number of bound states.

It is well-known, of course that

\[
\frac{1}{k' - k \mp io} = P \frac{1}{k' - k} \pm i\pi\delta(k' - k)
\]  

(19)

Proceeding further, we set the Jost function (S wave), at zero energy

\[
F_+(s(s-1); k^2 = 0) \equiv \chi(s)
\]  

(20)

The simplification of Eq. (16) is immediate. One finds easily that

\[
F_+ = \chi(s) = \exp(-2i\delta), \quad S(s) = \exp(2i\delta) \quad \text{and} \quad F_- = 1
\]  

(21)

We note in passing that since the partial wave S matrix is independent of \( k \) (energy) due to scale invariance, one can set \( k = 0 \) in simplifying Eq. (16). Recall that there are no bound states either, \( m' = 0 \) and \( \Pi_\pm(k) \equiv 1. \)
Since \( \chi(s) \) defined in Eq. (20) is entire \([5]\) and has only zeros on the real line \( \text{Re} \ s_n = \frac{1}{2} \) (see clarification of this later) and the number of non-zero zeros are infinite on the “critical” line \( \text{Re} \ s_n = \frac{1}{2} \), the Hadamard factorization Eq. (3) is valid, i.e.,

\[
\chi(s) = e^{A}e^{Bs} \prod_{n=1}^{\infty} \left( 1 - \frac{s}{s_n} \right) \exp \left( \frac{s}{s_n} \right)
\]

(22)

It will turn out that both the constants \( A \) and \( B \) can be identified (see below) and also that \( \chi(s) \) has no zeros either for \( s > 1 \) or \( s < 0 \).

From Eq. (8), Eq. (9) and Eq. (10), we obtain

\[
\nu = \frac{1}{2} - \frac{2}{\pi} \delta = \frac{1}{2} + \frac{1}{\pi} |\ln \chi(s)| > \frac{1}{2}
\]

(23)

where we have set (Eq. (21))

\[
\ln \chi(s) = +i|\ln \chi(s)|
\]

(24)

in order to satisfy that \( \delta < 0 \)

\[
\delta = -\frac{1}{2}|\ln \chi(s)|
\]

(25)

[Recall that for a repulsive potential, \( \lambda > 0 \) the phase shift has to be negative!]

\[
\therefore \lambda = s(s - 1) = \nu^2 - \frac{1}{4}
\]

(26)

giving [From Eq. (23), Eq. (25)]

\[
\lambda = \frac{1}{\pi}|\ln \chi(s)| + \frac{1}{\pi^2}|\ln \chi(s)|^2 > 0
\]

(27)

where \( \lambda \) is real positive and \( s \) real and > 1.

From Eq. (26) and Eq. (27), we obtain \( \chi(s) \) for real \( s > 1 \):

\[
|\ln \chi(s)| = \pi(s - 1) > 0 \quad s, \text{ real} & > 1
\]

(28)

\[
\therefore \chi(s) = e^{\pi(s-1)}, \quad s > 1, \text{ real} \quad \chi(s) = e^{-\pi s}, \quad s < 0, \text{ real}
\]

(29) (30)
\[ A = -\pi, \quad B = \pi \quad \text{for} \quad s > 1, \quad \text{real (from Eq. (22))} \]

Notice, as promised that \( \chi(s) \) is entire (and order one) and has No zeros for \( s > 1 \) and \( s < 0 \). It is worth noticing that since \( \lambda = s(s - 1) \) is invariant under \( s \rightarrow 1 - s \), the forms of \( \chi(s) \) in Eq. (29) and Eq. (30) are consistent.

We now comment on the **all-important** restriction that for a real, repulsive potential the \( S \)-wave Jost function \( \lambda V \) is an entire function of \( \lambda \) with an infinite number of zeros extending to infinity and these zeros will all be real and negative. [This can be verified following Khuri See his Eq. (1.1) and Eq. (1.2)] i.e.,

\[
[\text{Im}\lambda_n(i\tau)] \int_0^\infty |f(\lambda_n(i\tau)); i\tau; r|^2 dr = 0 \quad (31)
\]

where \( k = i\tau \).

We observe that Eq. (31) does not have the potential \( V(r) \) under the integration sign. This comes about by examining the Schrödinger equation for \( f(k, r) \) [which is identical to the case of one dimension] and its complex conjugate \( f^*(k, r) \), and by performing the “canonical” operation, i.e., multiplying the equation for \( f(k, r) \) by \( f^*(k, r) \), subtracting and then importantly multiplying the resulting equation by \( r^2 \). This “gets rid of” the potential term on R.H.S. Subsequently, one integrates by parts the equation thus obtained, making use of the fact that the Wronskian of \( f(k, r) \) and \( f^*(k, r) \) is independent of \( r \) (because they satisfy the same Schrödinger equation: See Chaden and Sabatier, [3]). The end result is Eq. (31) above, the potential \( V(r) \) not present (in contrast to Khuri’s Eq. (1.2)). The rationale behind this is to ensure that the integral in Eq. (31) is now finite, under the stated conditions (See Eq. (36 ’) and Eq. (36 ’’)). If the potential \( V(r) = \lambda r^2 \) were present, then the corresponding integral in Eq. (31) would diverge!

The crucial point to verify is that the integral in Eq. (31) is **non-vanishing** and finite. Only then, can one conclude that

\[
\text{Im}\lambda_n(i\tau) = 0. \quad (32)
\]

We proceed to demonstrate this as follows.

The \( S \) wave Jost solution \( f(k, r) \) for the potential \( V(r) = \frac{\lambda}{r^2} \) \( (\lambda > 0) \) is well-known [2]:

\[
f(k, r) = \sqrt{\frac{\pi kr}{2}} e^{i(\frac{\pi}{4} \nu + \frac{\pi}{4})} H_\nu^{(1)}(kr) \quad (33)
\]
where $H^{(1)}_\nu(kr)$ is Hankel function of $I$ kind.

One **cannot** obtain the Jost function by taking the limit $r \to 0$ in Eq. (33) because this limit does **not** exist. This only works for **REGULAR** POTENTIALS! (See Chadan and Sabatier, Page 10, Eq. (I.3.6).) We can bypass this conundrum, because we already “know” the $S$ wave Jost function $F_+(k) = F_+(0)$ from Eqs. (8), (9) and (21).

We observe in passing that the Jost solution Eq. (33) has the required asymptotic behavior, i.e.,

$$f(k, r) \xrightarrow{r \to \infty} e^{ikr}$$

Since

$$H^{(1)}_\nu(z) \sim \sqrt{2/\pi z} e^{iz - \frac{\pi \nu}{2} - \frac{\pi}{4}},$$

it is now straightforward to check that

$$\int_0^\infty |f(\lambda_n(i\tau); i\tau; r)|^2 dr < \infty$$

by plugging in Eq. (33) for $f(k, r)$.

$$V(r) = \frac{\lambda}{r^2}$$

Eq. (31), (33) give

$$[Im\lambda_n(i\tau)\tau] \frac{2}{\pi} \int_0^\infty r K^2_\nu(\tau r) dr = 0$$

where

$$\int_0^\infty r K^2_\nu(\tau r) dr = \frac{1}{8 \tau^2} \frac{\pi \nu}{\sin \pi \nu}$$

**: Eq. (32) follows from Eq. (36'), for \(\tau \neq 0 \ (\tau > 0) \ [9]$$

$$Im\lambda_n(i\tau) = 0, \quad \nu > \frac{1}{2}, \quad \nu \neq 1, 2, 3 \ldots \infty.$$  (32)

We now follow Khuri [3]:

All zeros of $\lambda_n(i\tau)$ must be **real** $$(\tau > 0) \ “But \ \lambda_n(i\tau) \ must \ be \ negative \ since \ the \ potential \ [\lambda_n(i\tau)V^*] \ will \ have \ a \ bound \ state \ at \ E = -\tau^2 \ and \ that \ could \ not \ happen \ if \ V^* \geq 0 \ and \ \lambda_n(i\tau) > 0. \ Hence \ by \ continuity, \ \lambda_n(0), \ for$$
all \( n \), is real and negative. The zero energy coupling spectrum lies on the negative real line for \( V^* \geq 0 \).” In our case, in view of Eq. (7), i.e.,

\[
\lambda = s(s - 1) \text{ is mandatory!}
\]

Thus, we conclude unequivocally that \( \chi(s) \) has all its infinite number of nonzero zeros on the critical line. And can be set equal to Riemann’s \( \xi \) function!

This establishes the Holy grail of physics and mathematics!!

We conclude by commenting on the precise relation between the \( S \) wave Jost function, \( \chi(s) \) and Riemann’s \( \xi(s) \) function. Both are entire, order one and infinite type and possess identical infinite number of non-zero zeros only on the critical line \( \text{Re} \ s = \frac{1}{2} \) and both have no zeros either for \( \sigma > 1 \) or \( \sigma < 0 \). Their ratio, by Hadamard’s theorem is given by

\[
\frac{\chi(s)}{\xi(s)} = e^{\alpha + \beta s}, \quad \text{all } s
\]

Since [7]

\[
\xi(s) = \frac{1}{2} e^{b_0 s} \prod_n \left( 1 - \frac{s}{s_n} \right) e^{s_n}, \quad \text{all } s
\]

where

\[
b_0 = -\frac{1}{2} \gamma - 1 - \frac{1}{2} \ln 4\pi
\]

(\( \gamma \) is Euler’s constant)

We obtain the identification,

\[
\alpha = \ln 2 - \pi
\]

and \( \beta = \pi + \frac{1}{2} \gamma + 1 - \frac{1}{2} \ln 4\pi \)

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