Potentially semi-stable deformations of specified Hodge-Tate type and Galois type

Yong Suk Moon

Abstract

Let $k$ be a perfect field of characteristic $p > 2$, and let $K$ be a finite totally ramified extension of $W(k)[1/p]$. We prove that the locus of potentially semi-stable $\text{Gal}(\overline{K}/K)$-representations of a given Hodge-Tate type and Galois type is a closed subspace of the universal deformation ring, generalizing the result of Kisin (2007) where $k$ is assumed to be finite.

Contents

1 Introduction 1

2 Torsion Representation and Construction of $M_{st}$ 3
  2.1 Potentially Semi-stable Representation and Filtered $(\varphi, N, \Gamma)$-module . . . 3
  2.2 Construction of $M_{st}$ 4
  2.3 Potentially Semi-stable Torsion Representations 7
  2.4 Representation with Coefficient 9

3 Hodge-Tate Type and Galois Type 9
  3.1 Hodge-Tate Type 9
  3.2 Galois Type 17

4 Galois Deformation Ring 20

1 Introduction

Let $k$ be a perfect field of characteristic $p > 2$, and let $W(k)$ be its ring of Witt vectors. Write $K_0 = \text{Frac}(W(k))$, and let $K/K_0$ be a finite totally ramified extension. We fix an algebraic closure $\overline{K}$ of $K$, and let $\mathcal{G}_K := \text{Gal}(\overline{K}/K)$ be the absolute Galois group of $K$.

Let $E/\mathbb{Q}_p$ be a finite extension with residue field $\mathbb{F}$, and let $V_0$ be a finite dimensional $\mathbb{F}$-representation of $\mathcal{G}_K$. Denote by $\mathcal{C}$ the category of local topological $\mathcal{O}_E$-algebras $A$ such
that the natural map $\mathcal{O}_E \to A/\mathfrak{m}_A$ is surjective and the map from $A$ to the projective limit of its discrete artinian quotients is a topological isomorphism. If $V_0$ is absolutely irreducible, then there exists a universal deformation ring $R \in \mathcal{C}$ with a deformation $V_R$ which parametrizes the isomorphism classes of deformations of $V_0$ ([SL97]). Note that $R$ is not necessarily noetherian in general when $k$ is not finite.

In this paper, we study the geometry of the locus of potentially semi-stable representations with a specified Hodge-Tate type $\nu$ and Galois type $\tau$. We show that such a locus cuts out a closed subspace in the following sense:

**Theorem A.** There exists a closed ideal $a_{\nu,\tau} \subset R$ such that the following holds: for any finite flat $\mathcal{O}_E$-algebra $A$ and a continuous $\mathcal{O}_E$-algebra homomorphism $f : R \to A$ (where we equip $A$ with the $(p)$-adic topology), the induced representation $A[\frac{1}{p}] \otimes f_* V_R$ is potentially semi-stable of Hodge-Tate type $\nu$ and Galois type $\tau$ if and only if $f$ factors through the quotient $R/a_{\nu,\tau}$.

When the residue field $k$ is finite, Kisin proved the corresponding result in [Kis07, Theorem 2.7.6]. One of the main steps in [Kis07] is the construction of the projective scheme which parametrizes representations of $E(u)$-height $\leq r$ for a fixed positive integer $r$ (cf. [Kis07 Section 1.2]). It is obtained as a closed subscheme of the affine Grassmannian for the restriction of scalars $\text{Res}_{W(k)/\mathbb{Z}_p} \text{GL}_d$. But this construction does not make sense in general when $k$ is infinite. The main difficulty is that we do not know how to analyze whether the restriction of scalars $\text{Res}_{W(k)/\mathbb{Z}_p}$ for a non-affine scheme over $W(k)$ is representable by an Ind-scheme when $k$ is infinite, even for simple examples such as $\mathbb{P}^1_{W(k)}$.

Another approach to studying the locus cut out by certain $p$-adic Hodge theoretic conditions, motivated by Fontaine’s conjecture in [Fon97], is to analyze torsion representations given as the subquotients of Galois stable lattices satisfying the given conditions. For semi-stable (or crystalline) representations having Hodge-Tate weights in $[0, r]$, this is carried out by Liu in [Liu07]. And in the case $k$ is finite, Liu proved the corresponding result for semi-stable representations of a given Hodge-Tate type in [Liu15].

We use the functor given in [Liu12] from the category of representations semi-stable over a totally ramified Galois extension $K'/K$ to the category of lattices in filtered modules equipped with Frobenius, monodromy, and $\text{Gal}(K'/K)$-action, in order to study the refined structure of a Hodge-Tate type and Galois type of torsion representations. We first study semi-stable representations of a fixed Hodge-Tate type and show that the locus of such representations is $p$-adically closed (cf. Theorem 3.3). This generalizes the corresponding result in [Liu15] to the case $k$ is not necessarily finite. The proof in [Liu15] is based on reducing to the situation when the coefficient field $E$ contains the Galois closure of $K'$, thereby requiring $k$ to be finite. We remove such a restriction using a different argument.

Then, we study potentially semi-stable representations of a given Galois type, and prove that such representations cut out a $p$-adically closed locus (cf. Theorem 3.17).
Acknowledgments

I would like to express my sincere gratitude to Mark Kisin for suggesting me to work on this topic and making many helpful comments. This paper is based on a part of author’s Ph.D. thesis under his supervision. I also wish to thank Brian Conrad and Tong Liu for helpful discussions on this work. I thank the referee for a careful reading of the paper and making helpful suggestions to improve it. I would like to thank the department of Mathematics at Harvard University and Purdue University for their cordial environment. This work was partly supported by Samsung Scholarship Foundation, South Korea.

2 Torsion Representation and Construction of $M_{st}$

We keep the notations as in the introduction. Let $K' \subset \bar{K}$ be a finite totally ramified Galois extension of $K$. In this section, we will first explain the construction of the functor given in [Liu12] from the category of representations semi-stable over $K'$ to the category of lattices in filtered modules equipped with Frobenius, monodromy, and $\text{Gal}(K'/K)$-action. Then, we will explain the result proved in [Liu12] and [Liu15] that one can associate a Hodge-Tate type and Galois type to a torsion representation up to some constant depending only on $K'$.

2.1 Potentially Semi-stable Representation and Filtered $(\varphi, N, \Gamma)$-module

For a $\mathcal{G}_K$-representation $V$ over $\mathbb{Q}_p$, we say $V$ is potentially semi-stable if there exists a finite extension $L \subset \bar{K}$ of $K$ such that $V$ restricted to $\mathcal{G}_L := \text{Gal}(\bar{K}/L)$ is semi-stable. This means precisely that $\dim_{\mathbb{Q}_p} V = \dim_{L_0}(B_{st} \otimes_{\mathbb{Q}_p} V^\vee)^{\mathcal{G}_L}$ where $L_0$ is the maximal unramified subextension of $L/K_0$.

Let $e' = [K' : K_0]$. We fix a uniformizer $\pi$ of $K'$, and let $F(u)$ be the Eisenstein polynomial for $\pi$ over $K_0$. Denote by $\text{Rep}_{\text{pst},K'}$ the category of $\mathcal{G}_K$-representations over $\mathbb{Q}_p$ which become semi-stable over $K'$ (i.e., semi-stable as $\text{Gal}(\bar{K}/K')$-representations). Let $\Gamma = \text{Gal}(K'/K)$ and $\mathcal{G}_{K'} = \text{Gal}(\bar{K}/K')$. Note that $K_0$ is equipped with the natural Frobenius endomorphism $\varphi$.

We consider the category of filtered $(\varphi, N, \Gamma)$-modules whose objects are finite dimensional $K_0$-vector spaces $D$ equipped with:

- a Frobenius semi-linear injection $\varphi : D \to D$,
- $W(k)$-linear map $N : D \to D$ such that $N\varphi = p\varphi N$,
- decreasing filtration $\text{Fil}_i D_{K'}$ on $D_{K'} := K' \otimes_{K_0} D$ by $K'$-sub-vector spaces such that $\text{Fil}_i D_{K'} = D_{K'}$ for $i \ll 0$ and $\text{Fil}_i D_{K'} = 0$ for $i \gg 0$, and
structures. The functor $D$ and the category of weakly admissible lattices in filtered $(\mathcal{O}, V)$ and $(\mathcal{O}, \mathcal{W})$.

Let $\Gamma$ be the category of $(\mathcal{O}, V)$-modules. Theorem 2.2. We first recall the definitions of period rings in $p$-adic Hodge theory.

Definition 2.1. Let $D$ be a filtered $(\varphi, N, \Gamma)$-module. A lattice $M$ in $D$ is a finite free $W(k)$-submodule of $D$ such that $M[p^\ell] := M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong D$, and $\varphi(M) \subset M$, $N(M) \subset M$, and $\gamma(M) \subset M$ for all $\gamma \in \Gamma$. For a lattice $M \subset D$, we equip $M_{K'} := \mathcal{O}_{K'} \otimes_{W(k)} M$ with the natural filtration by $\mathcal{O}_{K'}$-submodules, given by $\text{Fil}^i M_{K'} = M_{K'} \cap \text{Fil}^i D_{K'}$. If $M_1, M_2$ are lattices in filtered $(\varphi, N, \Gamma)$-modules $D_1, D_2$ respectively, then a morphism $f : M_1 \rightarrow M_2$ is a morphism of filtered $(\varphi, N, \Gamma)$-modules.

Note that for a lattice $M$ in a filtered $(\varphi, N, \Gamma)$-module, the associated graded $\mathcal{O}_{K'}$-modules $\text{gr}^i M_{K'} = \text{Fil}^i M_{K'}/\text{Fil}^{i+1} M_{K'}$ is torsion free by the definition of the filtration.

Let $r$ be a positive integer. Denote by $L^r(\varphi, N, \Gamma)$ the category of lattices in filtered $(\varphi, N, \Gamma)$-modules $D$ that satisfy $\text{Fil}^0 D_{K'} = D_{K'}$ and $\text{Fil}^{r-1} D_{K'} = 0$. Let $\text{Rep}_{\mathcal{O}_p}^{p, K', r}$ be the full subcategory of $\text{Rep}_{\mathcal{O}_p}^{p, K'}$ whose objects have Hodge-Tate weights in $[0, r]$, and let $\text{Rep}_{\mathcal{O}_p}^{p, K', r}$ be the category of $\mathcal{G}_K$-stable $\mathbb{Z}_p$-lattices of representations in $\text{Rep}_{\mathcal{O}_p}^{p, K', r}$. The following theorem is proved in [Liu12].

Theorem 2.2. (cf. [Liu12, Theorem 2.3]) There exists a faithful contravariant functor $M_{st}$ from $\text{Rep}_{\mathcal{O}_p}^{p, K', r}$ to $L^r(\varphi, N, \Gamma)$. If we denote by $M_{st} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ the functor $M_{st}$ associated to the isogeny categories, then there exists a natural isomorphism of functors between $M_{st} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $D_{st}^{K'}$.

2.2 Construction of $M_{st}$

We now explain briefly the construction in [Liu12] of the functor $M_{st}$ in Theorem 2.2. We first recall the definitions of period rings in $p$-adic Hodge theory.

Let $S'$ be the $p$-adic completion of the divided power-envelope of $\mathcal{O} = W(k)[u]$ with respect to the ideal $(F(u))$. Denote $S'_{K_0} := S'_{[\frac{1}{p}]}$. Let $C_p$ be the $p$-adic completion of $K$, and let $\mathcal{O}_{C_p}$ be its ring of integers. We define $\mathcal{O}_{C_p}^r := \varprojlim_{x \rightarrow x^p} \mathcal{O}_{C_p}/p$. By the universal property of the ring of Witt vectors $W(\mathcal{O}_{C_p}^p)$, there exists a unique surjection $\theta : W(\mathcal{O}_{C_p}^p) \rightarrow \mathcal{O}_{C_p}$, which lifts the projection $\mathcal{O}_{C_p}^r \rightarrow \mathcal{O}_{C_p}/p$ onto the first factor of the inverse limit. We denote by $B_{dR}^+$ the $\ker(\theta)$-adic completion of $W(\mathcal{O}_{C_p}^p)_{[\frac{1}{p}]}$. Let $A_{cris}$ be the $p$-adic completion of the divided power-envelope of $W(\mathcal{O}_{C_p}^p)$ with respect to $\ker(\theta)$. We fix a compatible system of $p^n$-th roots $\pi_n \in \mathcal{O}_K$ of $\pi$ for non-negative integers $n$, and let $\overline{\pi} := (\pi_n) \in \mathcal{O}_{C_p}^r$. We have an
embedding $\mathcal{G} \subset W(\mathcal{O}_{\mathbb{C}^p})$ mapping $u$ to $[\pi]$, and hence the embeddings $\mathcal{G} \subset S' \subset A_{\text{cris}}$ compatible with Frobenius endomorphisms. Let $B_{\text{cris}}^+ = A_{\text{cris}}[\frac{1}{p}]$. Let $u = \log[\pi]$, and $B_{\text{st}}^+ = B_{\text{cris}}^+[u]$. We also fix a compatible system of primitive $p^n$-th roots of unity $\zeta_{p^n} \in \mathcal{O}_K$ for non-negative integers $n$, and let $\epsilon : = (\zeta_{p^n}) \in \mathcal{O}_{\mathbb{C}^p}$. Let $t = \log[\epsilon] \in B_{\text{dr}}^+$. Note that we also have $t \in A_{\text{cris}}$. Let $B_{\text{dr}} = B_{\text{dr}}[\frac{1}{p}]$, $B_{\text{cris}} = B_{\text{cris}}[\frac{1}{p}]$, and $B_{\text{st}} = B_{\text{st}}[\frac{1}{p}]$.

We denote by $O_{\mathcal{E}}$ the $p$-adic completion of $\mathcal{G}[\frac{1}{p}]$, and let $E = \text{Frac}(O_{\mathcal{E}})$. Let $\hat{E}_{ur}$ be the $p$-adic completion of the maximal unramified subextension of $E$ in $W(\text{Frac}(O_{\mathbb{C}^p})[\frac{1}{p}])$, and $O_{\mathcal{E}}$ its ring of integers. We let $\mathcal{G}_{ur} = O_{\mathcal{E}} \cap W(\mathcal{O}_{\mathbb{C}^p})$.

We let $K'_{\infty} = \bigcup_{n=1}^{\infty} K'_{\infty}^n$ and $K'_{\infty} = \bigcup_{n=1}^{\infty} K'_{n}(\zeta_{p^n})$. Let $K'_{c} = K'_{\infty} \cap K'_{0}$, which is the Galois closure of $K'_{\infty}$ over $K'$. Let $\mathcal{G} = \text{Gal}(K'_{c}/K')$, $\mathcal{G}_{\infty} = \text{Gal}(K/K'_{\infty})$, and $\mathcal{H}_{K'} = \text{Gal}(K'_{c}/K'_{\infty})$. Write 

$$t(i) = \frac{q(i)}{i!}$$

where $q(i)$ is defined by $i = q(i)(p - 1) + r(i)$ with $0 \leq r(i) < p - 1$. We define

$$\mathcal{R}_{K_0} := \{ \sum_{i=0}^{\infty} a_i t(i) \mid a_i \in S'_{K_0}, \ a_i \to 0 \ p\text{-adically as } i \to \infty \}.$$ 

We have a natural map $\nu : W(\mathcal{O}_{\mathbb{C}^p}) \to W(\hat{k})$ induced by the projection $\mathcal{O}_{\mathbb{C}^p} \to \hat{k}$, which can be seen to extend uniquely to $\nu : B_{\text{cris}}^+ \to W(\hat{k})[\frac{1}{p}]$. For any subring $A \subset B_{\text{cris}}^+$, write $I_+A := A \cap \ker(\nu)$. We have $I_+ \mathcal{G} = u \mathcal{G}$ and

$$I_+S' = \{ \sum_{i=1}^{\infty} \frac{b_i}{[\frac{i}{p}]} u^i \mid b_i \in W(k), \ b_i \to 0 \ p\text{-adically as } i \to \infty \}.$$ 

Define $\hat{\mathcal{R}} = W(\mathcal{O}_{\mathbb{C}^p}) \cap \mathcal{R}_{K_0}$ and $I_+ = I_+ \hat{\mathcal{R}}$. The following lemma is proved in [Lin10].

**Lemma 2.3.** ([Lin10, Lemma 2.2.1])

1. $\hat{\mathcal{R}}$ (resp. $\mathcal{R}_{K_0}$) is a $\varphi$-stable $\mathcal{G}$-algebra as a subring in $W(\mathcal{O}_{\mathbb{C}^p})$ (resp. $B_{\text{cris}}^+$).
2. $\hat{\mathcal{R}}$ and $I_+$ (resp. $\mathcal{R}_{K_0}$ and $I_+ \mathcal{R}_{K_0}$) are $\mathcal{G}_{K'}$-stable. The $\mathcal{G}_{K'}$-actions on $\hat{\mathcal{R}}$ and $I_+$ (resp. $\mathcal{R}_{K_0}$ and $I_+ \mathcal{R}_{K_0}$) factor through $\mathcal{G}_{\infty}$.
3. $\mathcal{R}_{K_0}/I_+ \mathcal{R}_{K_0} \cong K_0$ and $\hat{\mathcal{R}}/I_+ \cong S'/I_+S' \cong \mathcal{G}/u \mathcal{G} \cong W(k)$.

Let $r$ be a positive integer. A Kisin module of height $r$ is a pair $(\mathcal{M}, \varphi_{\mathcal{M}})$ where $\mathcal{M}$ is a finite free $\mathcal{G}$-module, and $\varphi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is a $\varphi$-semi-linear map such that the cokernel of the induced map $1 \otimes \varphi_{\mathcal{M}} : \varphi^{*}(\mathcal{M}) \to \mathcal{M}$ is killed by $F(u)^r$. A morphism between two
Kisin modules $\mathcal{M}_1, \mathcal{M}_2$ is a morphism as $\mathcal{S}$-modules compatible with $\varphi_{\mathcal{M}_1}$. Let $\text{Mod}_\mathbb{A}(\varphi)$ denote the category of Kisin modules of height $r$. For $(\mathcal{M}, \varphi_{\mathcal{M}}) \in \text{Mod}_\mathbb{A}(\varphi)$, we write $\hat{\mathcal{M}} = \hat{\mathcal{R}} \otimes_{\mathcal{S}, \varphi, \mathcal{M}} \hat{\mathcal{M}}$. The Frobenius $\varphi_{\hat{\mathcal{M}}}$ on $\hat{\mathcal{M}}$ naturally extends to $\hat{\mathcal{M}}$ by $\varphi_{\hat{\mathcal{M}}}(a \otimes m) = \varphi_{\hat{\mathcal{M}}}(a) \otimes \varphi_{\mathcal{M}}(m)$.

**Definition 2.4.** A $(\varphi, \hat{\mathcal{G}})$-module of height $r$ is a triple $(\mathcal{M}, \varphi_{\hat{\mathcal{M}}}, S_{\mathcal{G}K})$ satisfying the following:

- $(\mathcal{M}, \varphi_{\mathcal{M}})$ is Kisin module of height $r$.
- $\hat{\mathcal{G}}_{\mathcal{M}}$ denotes a $\hat{\mathcal{R}}$-semi-linear $\hat{\mathcal{G}}$-action on $\hat{\mathcal{M}}$ which commutes with $\varphi_{\mathcal{M}}$ and induces a trivial action on $\mathcal{M}/I_{+}\mathcal{M}$.
- Considering $\mathcal{M}$ as a $\varphi(\mathcal{S})$-submodule of $\hat{\mathcal{M}}$, we have $\mathcal{M} \subset \hat{\mathcal{M}}^{H_{\mathcal{K}^\text{av}}}$.

A morphism between two $(\varphi, \hat{\mathcal{G}})$-modules $\mathcal{M}_1, \mathcal{M}_2$ of height $r$ is a morphism in $\text{Mod}_\mathbb{A}(\varphi)$ which commutes with the $\hat{\mathcal{G}}$-actions. We denote by $\text{Mod}_\mathbb{A}(\varphi, \hat{\mathcal{G}})$ the category of $(\varphi, \hat{\mathcal{G}})$-modules of height $r$. For $\hat{\mathcal{M}} \in \text{Mod}_\mathbb{A}(\varphi, \hat{\mathcal{G}})$, we associate a $\mathbb{Z}_p[\mathcal{G}_{\mathcal{K}^\text{av}}]$-module $\hat{T}^\vee(\mathcal{M}) := \text{Hom}_{\hat{\mathcal{S}}, \hat{\varphi}}(\hat{\mathcal{M}}, W(\mathcal{O}^b_{\mathcal{C}_p}))$ with $\mathcal{G}_{\mathcal{K}^\text{av}}$-action given by $g(f)(x) = g(f(g^{-1}(x)))$ for $g \in \mathcal{G}_{\mathcal{K}^\text{av}}$, $f \in \hat{T}^\vee(\mathcal{M})$. Here, $\mathcal{G}_{\mathcal{K}^\text{av}}$-action on $\hat{\mathcal{M}}$ is given by $\hat{\mathcal{G}}$-action on $\hat{\mathcal{M}}$. Moreover, for $\mathcal{M} \in \text{Mod}_\mathbb{A}(\varphi)$, we associate a $\mathbb{Z}_p[\mathbb{G}_{\mathcal{S}}]$-module $T^\vee(\mathcal{M}) := \text{Hom}_{\mathbb{S}, \mathbb{S}}(\mathcal{M}, \mathcal{S}^w)$ similarly. The main result proved in [Liu10] is the following.

**Theorem 2.5.** (cf. [Liu10], Theorem 2.3.1, Proposition 3.1.3)]

1. $\hat{T}^\vee$ induces an anti-equivalence between $\text{Mod}_\mathbb{A}(\varphi, \hat{\mathcal{G}})$ and the category of $\mathcal{G}_{\mathcal{K}^\text{av}}$-stable $\mathbb{Z}_p$-lattices in semi-stable representations of $\mathcal{G}_{\mathcal{K}^\text{av}}$ having Hodge-Tate weights in $[0, r]$.

2. $\hat{T}^\vee$ induces a natural $W(\mathcal{O}^b_{\mathcal{C}_p})$-linear injection

$$i : W(\mathcal{O}^b_{\mathcal{C}_p}) \otimes \hat{\mathcal{M}} \to W(\mathcal{O}^b_{\mathcal{C}_p}) \otimes \mathbb{Z}_p \hat{T}(\mathcal{M})$$

such that $i$ is compatible with Frobenius maps and $\mathcal{G}_{\mathcal{K}^\text{av}}$-actions on both sides. Here, $\hat{T}(\mathcal{M}) := \text{Hom}_{\mathbb{Z}_p}(\hat{T}^\vee(\mathcal{M}), \mathbb{Z}_p)$.

3. There exists a natural isomorphism $T^\vee(\mathcal{M}) \cong \hat{T}^\vee(\hat{\mathcal{M}})$ of $\mathbb{Z}_p[\mathbb{G}_{\mathcal{S}}]$-modules.

To construct the functor $M_{st}$, we establish a connection between $(\varphi, \hat{\mathcal{G}})$-modules and filtered $(\varphi, N, \Gamma)$-modules. Let $V \in \text{Rep}_{\mathbb{Q}_p}^{\text{st}, \mathcal{K}^\text{av}}$, and let $T \subset V$ be a $\mathcal{G}_{\mathcal{K}^\text{av}}$-stable $\mathbb{Z}_p$-lattice. By Theorem 2.5, there exists a unique $\mathcal{M} \in \text{Mod}_\mathbb{A}(\varphi, \hat{\mathcal{G}})$ such that $\hat{T}^\vee(\mathcal{M}) = T$ as $\mathbb{Z}_p[\mathcal{G}_{\mathcal{K}^\text{av}}]$-modules. Let $\mathcal{D} := S_{\mathcal{K}_0} \otimes_{\mathcal{S}, \varphi, \mathcal{M}} \mathcal{M}$ equipped with the Frobenius endomorphism given by $\varphi_{\mathcal{D}} = \varphi_{S_{\mathcal{K}_0}} \otimes \varphi_{\mathcal{M}}$. Let $D = \mathcal{D}/(I_{+}S_{\mathcal{K}_0})\mathcal{D}$, which is a finite $\mathcal{K}_0$-vector space equipped with the Frobenius induced from $\varphi_{\mathcal{D}}$. By [Bre97, Proposition 6.2.1.1], there exists a unique section $s : D \to \mathcal{D}$ compatible with the Frobenius morphisms on both sides. Thus, $\mathcal{D} = S_{\mathcal{K}_0} \otimes_{\mathcal{K}_0} D$. 


if we identify $D$ with $s(D)$. So $B^+_{\text{cris}} \otimes_{\mathcal{M}} \hat{\mathcal{M}} \cong B^+_{\text{cris}} \otimes_{K_0} D$, and the map $i$ given in Theorem 2.5 (2) induces a natural injection $D \hookrightarrow B^+_{\text{cris}} \otimes_{Z_p} T^\vee$ where $T^\vee := \text{Hom}_{Z_p}(T, Z_p)$.

On the other hand, the functor $D^K \rightarrow$ induces an injection $i : B^+_{\text{st}} \otimes_{K_0} D^K (V) \rightarrow B^+_{\text{st}} \otimes_{Z_p} V^\vee$

such that $i$ is compatible with Frobenius, monodromy, filtration, and $G_{K'}$-action on both sides. The following is proved in [Liu12].

**Proposition 2.6.** (cf. [Liu12] Proposition 2.6, Corollary 2.7, 2.8) There exists a unique $K_0$-linear isomorphism $i : D^K (V) \rightarrow D$ such that $i$ is compatible with the Frobenius morphisms on both sides and makes the following diagram commutative:

$$
\begin{array}{ccc}
D^K (V) & \longrightarrow & B^+_{\text{st}} \otimes_{Z_p} T^\vee \\
\downarrow i & & \downarrow \text{mod } u \\
D & \longrightarrow & B^+_{\text{cris}} \otimes_{Z_p} T^\vee
\end{array}
$$

Furthermore, such $i$ is functorial.

Note that $\mathcal{M}/u\mathcal{M} \cong \phi^*(\mathcal{M})/u\phi^*(\mathcal{M}) \subset \mathcal{D}/I_i S_{K_i} \mathcal{D} = D$.

We set $M_{\text{st}} (T) \subset D^K_{\text{st}} (V)$ to be the inverse image of $\phi^*(\mathcal{M})/u\phi^*(\mathcal{M})$ under the isomorphism $i : D^K_{\text{st}} (V) \rightarrow D$ given in Proposition 2.6. $M_{\text{st}} (T)$ is a finite free $W(k)$-lattice in $D^K_{\text{st}} (V)$ stable under Frobenius. Furthermore, it is proved in [Liu12], Corollary 2.15 that $M_{\text{st}} (T)$ is stable under $G_{K'}$-action and $N$ on $D^K_{\text{st}} (V)$. Thus, $M_{\text{st}} (T)$ is a lattice of the filtered $(\phi, N, \Gamma)$-module $D^K_{\text{st}} (V)$. And the association $M_{\text{st}} (\cdot)$ is a contravariant functor from $\text{Rep}_{Z_p, K'}$ to $L^* (\phi, N, \Gamma)$ since the isomorphism $i$ in Proposition 2.6 is functorial.

### 2.3 Potentially Semi-stable Torsion Representations

We now associate torsion filtered $(\phi, N, \Gamma)$-modules to potentially semi-stable torsion representations. Denote by $\text{Rep}_{\text{tor}, K', r}$ the category of torsion representations $L$ semi-stable over $K'$ and of height $r$, in a sense that there exist lattices $\mathcal{L}_1, \mathcal{L}_2 \in \text{Rep}_{Z_p, K', r}$ with a $G_{K'}$-equivariant injection $j : \mathcal{L}_1 \hookrightarrow \mathcal{L}_2$ such that $L \cong \mathcal{L}_2/j(\mathcal{L}_1)$ as $Z_p[G_{K'}]$-modules, and $L$ is killed by some power of $p$. Morphisms between two torsion representations in $\text{Rep}_{\text{tor}, K', r}$ are morphisms of $Z_p[G_{K'}]$-modules. We call such $(\mathcal{L}_1, \mathcal{L}_2, j)$ a lift of $L$. We will sometimes denote simply by $j$ a lift of $L$. Note that a lift of $L \in \text{Rep}_{\text{tor}, K', r}$ is not unique. Let $L, L' \in \text{Rep}_{\text{tor}, K', r}$ with lifts $(\mathcal{L}_1, \mathcal{L}_2, j), (\mathcal{L}_1', \mathcal{L}_2', j')$ respectively. If $f : L \rightarrow L'$ is a
We define the filtration on $\mathcal{M}_\phi$ and $\mathcal{M}_\psi$. We set the following exact sequence of the associated graded modules:

\[ L_{st}morphism in \text{Rep}^{\text{pst},K',r}_{\text{tor}}, \text{we say a morphism } \tilde{f} : \mathcal{L}_2 \to \mathcal{L}'_2 \text{ in } \text{Rep}^{\text{pst},K',r}_{\text{tor}} \text{ is a lift of } f \text{ if } \tilde{f}(j(\mathcal{L}_1)) \subset j'(\mathcal{L}'_1) \text{ and } \tilde{f} \text{ induces } f.\]

We denote by $M^{\text{fil},r}_{\text{tor}}(\phi,N,\Gamma)$ the category whose objects are finite $W(k)$-modules $M$ killed by some power of $p$ and endowed with the following structures:

- a Frobenius semilinear morphism $\phi : M \to M$,
- $W(k)$-linear map $N : M \to M$ satisfying $N\phi = p\phi N$,
- $W(k)$-linear $\Gamma$-action on $M$ which commutes with $\phi$ and $N$, and
- $M_{K'} := \mathcal{O}_{K'}\otimes_{W(k)}M$ has decreasing filtration by $\mathcal{O}_{K'}$-submodules such that $\text{Fil}^0 M_{K'} = M_{K'}$ and $\text{Fil}^r M_{K'} = 0$. Also, $\gamma(\text{Fil}^r M_{K'}) \subset \text{Fil}^r M_{K'}$ for any $\gamma \in \Gamma$.

Morphisms in $M^{\text{fil},r}_{\text{tor}}(\phi,N,\Gamma)$ are $W(k)$-linear maps compatible with above structures. For $L \in \text{Rep}^{\text{pst},K',r}_{\text{tor}}$ with a lift $j : \mathcal{L}_1 \hookrightarrow \mathcal{L}_2$, we can associate an object $M_{st,j}(L) \in M^{\text{fil},r}_{\text{tor}}(\phi,N,\Gamma)$ as follows. By Theorem 2.2, we have the morphism $M_{st,j}(L) : M_{st}(\mathcal{L}_2) \to M_{st}(\mathcal{L}_1)$ in $L'(\phi,N,\Gamma)$ corresponding to $j$, and $M_{st,j}(L)$ is injective by [Liu12, Corollary 3.8]. We set $M_{st,j}(L) = M_{st}(\mathcal{L}_1)/M_{st}(j)(M_{st}(\mathcal{L}_2))$. Then $M_{st,j}(L)$ has natural endomorphisms $\psi$ and $N$, and $\Gamma$-action induced from $M_{st}(\mathcal{L}_1)$. Furthermore, tensoring by $\mathcal{O}_{K'}$ on $M_{st,j}(L)$ gives the following exact sequence:

\[ 0 \to \mathcal{O}_{K'} \otimes_{W(k)} M_{st}(\mathcal{L}_2) \to \mathcal{O}_{K'} \otimes_{W(k)} M_{st}(\mathcal{L}_1) \to \mathcal{O}_{K'} \otimes_{W(k)} M_{st,j}(L) \to 0.\]

We define the filtration on $M_{st,j}(L)_{K'}$ by $\text{Fil}^r M_{st,j}(L)_{K'} := q(\text{Fil}^r M_{st}(\mathcal{L}_1)_{K'})$. This gives $M_{st,j}(L)$ a structure as an object in $M^{\text{fil},r}_{\text{tor}}(\phi,N,\Gamma)$. By the snake lemma, we further have the following exact sequence of the associated graded modules:

\[ 0 \to \text{gr}^i(M_{st}(\mathcal{L}_2)_{K'}) \to \text{gr}^i(M_{st}(\mathcal{L}_1)_{K'}) \to \text{gr}^i(M_{st,j}(L)_{K'}) \to 0.\]

If $f : L \to L'$ is a morphism in $\text{Rep}^{\text{pst},K',r}_{\text{tor}}$ with a lift $\tilde{f} : (\mathcal{L}_1,\mathcal{L}_2,j) \to (\mathcal{L}'_1,\mathcal{L}'_2,j')$, then it induces a morphism $M_{st,j}(f) : M_{st,j}(L) \to M_{st,j}(L)$ in $M^{\text{fil},r}_{\text{tor}}(\phi,N,\Gamma)$.

Note that the above construction depends on the choice of the lift of $L$. However, the following theorem, which can be deduced directly from [Liu15] and [Liu12], shows that the construction depends on lifts only up to a constant.

**Theorem 2.7.** There exists a constant $c$ depending only on $F(u)$ and $r$ such that the following statement holds: for any morphism $f : L \to L'$ in $\text{Rep}^{\text{pst},K',r}_{\text{tor}}$ with lifts $j,j'$ of $L,L'$ respectively, there exists a morphism $\tilde{h} : M_{st,j'}(L') \to M_{st,j}(L)$ in $M^{\text{fil},r}_{\text{tor}}(\phi,N,\Gamma)$ such that

- if there exists a morphism of lifts $\tilde{f} : j \to j'$ which lifts $f$, then $\tilde{h} = p^r M_{st,j}(f)$,
• let $f' : L \to L''$ be a morphism in $\text{Rep}_{\text{tor}}^{\text{pst}}(K', r)$, $j''$ a lift of $L''$, and $\tilde{h}' : M_{\text{st}, j''}(L'') \to M_{\text{st}, j'}(L')$ the morphism in $M_{\text{tor}}^\varphi, N, \Gamma$ associated to $f', j'$, and $j''$. If there exists a morphism of lifts $\tilde{g} : j \to j''$ which lifts $f' \circ f$, then $\tilde{h} \circ \tilde{h}' = p^{2e} M_{\text{st}, \tilde{g}}(f' \circ f)$.

Proof. It follows directly from [Liu12, Theorem 3.1] and [Liu15, Theorem 2.13, Remark 2.1.5].

The following corollary is immediate.

Corollary 2.8. (cf. [Liu12, Corollary 3.2], [Liu15, Corollary 2.1.4]) With notations as in Theorem 2.7, assume that $f : L \to L'$ is an isomorphism with the inverse $f^{-1} : L' \to L$. Let $\tilde{h}_1 : M_{\text{st}, j}(L) \to M_{\text{st}, j'}(L')$ be the morphism as in Theorem 2.7 associated to $f^{-1}, j$, and $j'$. Then $\tilde{h} \circ \tilde{h}_1 = p^{2e} \text{Id}$ on $M_{\text{st}, j}(L)$ and $\tilde{h}_1 \circ \tilde{h} = p^{2e} \text{Id}$ on $M_{\text{st}, j'}(L')$. Furthermore, the similar statement holds for the induced morphisms on $\text{gr}^i(M_{\text{st}, j}(L)_K)$ and $\text{gr}^i(M_{\text{st}, j'}(L')_K)$.

2.4 Representation with Coefficient

Let $A$ be a $\mathbb{Z}_p$-algebra, and denote by $\text{Rep}_A^{\text{pst}, K', r}$ the subcategory of $\text{Rep}_{\mathbb{Z}_p}^{\text{pst}, K', r}$ whose objects are $A$-modules such that $G_K$-actions are $A$-linear. Morphisms in $\text{Rep}_A^{\text{pst}, K', r}$ are morphisms of $A[\mathcal{G}_K]$-modules. Let $\text{Rep}_{\text{tor}, A}^{\text{pst}, K', r}$ be the subcategory of $\text{Rep}_{\text{tor}}^{\text{pst}, K', r}$ whose objects have lifts in $\text{Rep}_{\mathbb{Z}_p}^{\text{pst}, K', r}$, and the morphisms are $A[\mathcal{G}_K]$-module morphisms. For $L \in \text{Rep}_{\text{tor}, A}^{\text{pst}, K', r}$ having a lift $j : \mathcal{L}_1 \hookrightarrow \mathcal{L}_2$ in $\text{Rep}_A^{\text{pst}, K', r}$, note that $M_{\text{st}}(\mathcal{L}_1)$ and $M_{\text{st}}(\mathcal{L}_2)$ are naturally $A \otimes \mathbb{Z}_p \mathcal{W}(k)$-modules, and thus so is $M_{\text{st}, j}(L)$.

Proposition 2.9. Let $f : L \to L'$ be a morphism in $\text{Rep}_{\text{tor}, A}^{\text{pst}, K', r}$, and let $j$ and $j'$ be lifts in $\text{Rep}_A^{\text{pst}, K', r}$ of $L$ and $L'$ respectively. Then, the associated morphism $\tilde{h} : M_{\text{st}, j'}(L') \to M_{\text{st}, j}(L)$ in $M_{\text{tor}}^\varphi, N, \Gamma$ as in Theorem 2.7 is a morphism of $A \otimes \mathbb{Z}_p \mathcal{W}(k)$-modules.

Proof. It follows immediately from [Liu12, Proposition 3.13] and [Liu15, Lemma 4.2.4].

3 Hodge-Tate Type and Galois Type

3.1 Hodge-Tate Type

Let $E$ be a finite extension of $\mathbb{Q}_p$, and let $B$ be a finite $E$-algebra. Let $V_B$ be a free $B$-module of rank $d$ equipped with a continuous $G_K$-action. Suppose that as a representation of $G_K$, $V_B$ is semi-stable over $K'$, i.e., $V_B \in \text{Rep}_{\text{pst}}^{K', r}$. Then $V_B$ is de Rham over $K$, and we set $D_{\text{dR}}^K(V_B) = (B_{\text{dR}} \otimes_{\mathbb{Q}_p} V_B')^\varphi$. For any $E$-algebra $A$, we write $A_K := A \otimes \mathbb{Q}_p K$.

Lemma 3.1. (cf. [Liu15, Lemma 4.1.2])
1. Let $B'$ be a finite $B$-algebra, and write $V_{B'} = B' \otimes_B V_B$. Then $D^K_{\text{dR}}(V_{B'}) \cong B' \otimes_B D^K_{\text{dR}}(V_B)$, and $\text{gr}^i(D^K_{\text{dR}}(V_{B'})) \cong B' \otimes_B \text{gr}^i(D^K_{\text{dR}}(V_B))$.

2. $D^K_{\text{dR}}(V_B)$ is a free $B_K$-module of rank $d$.

Proof. (1) is proved in [Liu15, Lemma 4.1.2]. For (2), since $D^K_{\text{dR}}(V_B) = K \otimes_{K_0} D^K_{\text{st}}(V_B)$, it suffices to prove that $D^K_{\text{st}}(V_B)$ is a free $B \otimes_{Q_p} K_0$-module of rank $d$. For any finite $B$-algebra $B'$, we can show similarly as in (1) that $D^K_{\text{st}}(V_{B'}) \cong B' \otimes_B D^K_{\text{st}}(V_B)$. Let $B_{\text{red}} = B/\mathcal{N}(B)$ where $\mathcal{N}(B)$ denotes the nilradical of $B$. $B_{\text{red}}$ is a reduced Artinian ring, so there exists a ring isomorphism $B_{\text{red}} \cong \prod_{j=1}^m E_j$ for some field $E_j$ finite over $E$. $E_j \otimes_{Q_p} K_0$ is isomorphic to a finite direct product of fields, so $D^K_{\text{st}}(V_{E_j}) \cong E_j \otimes_{B_{\text{red}}} D^K_{\text{st}}(V_{B_{\text{red}}})$ is finite projective as an $E_j \otimes_{Q_p} K_0$-module. Note that the Frobenius morphism on $K_0$ extends $E_j$-linearly to $E_j \otimes_{Q_p} K_0$, and the extended Frobenius permutes the maximal ideals of $E_j \otimes_{Q_p} K_0$ transitively. Therefore, $D^K_{\text{st}}(V_{E_j})$ is a free $E_j \otimes_{Q_p} K_0$-module of rank $d$, and $D^K_{\text{st}}(V_{B_{\text{red}}}) = B_{\text{red}} \otimes_B D^K_{\text{st}}(V_B)$ is a free $B_{\text{red}} \otimes_{Q_p} K_0$-module of rank $d$.

Let $\{e_1, \ldots, e_d\}$ be a $B_{\text{red}} \otimes_{Q_p} K_0$-basis of $D^K_{\text{st}}(V_{B_{\text{red}}})$, and choose a lift $\hat{e}_i \in D^K_{\text{st}}(V_B)$ of $e_i$. By Nakayama's lemma, $\{\hat{e}_1, \ldots, \hat{e}_d\}$ generate $D^K_{\text{st}}(V_B)$ as a $B \otimes_{Q_p} K_0$-module. Thus, we have a surjection of $B \otimes_{Q_p} K_0$-modules

$$f : \bigoplus_{i=1}^d (B \otimes_{Q_p} K_0) \cdot \hat{e}_i \twoheadrightarrow D^K_{\text{st}}(V_B).$$

As a $K_0$-vector space, $\dim_{K_0} D^K_{\text{st}}(V_B) = d \cdot \dim_{Q_p} B$. Thus, $f$ is an isomorphism, and $D^K_{\text{st}}(V_B)$ is a free $B \otimes_{Q_p} K_0$-module of rank $d$. \qed

Let $D_E$ be a finite $E$-vector space such that $D_{E,K} := D_E \otimes_{Q_p} K$ is equipped with a decreasing filtration $\text{Fil}^i D_{E,K}$ of $E \otimes_{Q_p} K$-modules and $\{i \mid \text{gr}^i D_{E,K} \neq 0\} \subset \{0, \ldots, r\}$. We denote $\nu = (D_{E,K}, \{\text{Fil}^i D_{E,K}\}_{i=0, \ldots, r})$. We say that $V_B$ has Hodge-Tate type $\nu$ if $\text{gr}^i D^K_{\text{dR}}(V_B) \cong B \otimes_E \text{gr}^i D_{E,K}$ as $B_K$-modules for all $i$.

Remark. We can consider a Hodge-Tate type as a conjugacy class of Hodge-Tate cocharacter in the following way. The Hodge-Tate period ring is given by $B_{\text{HT}} = C_p[X, X^{-1}]$ where $G_K$ acts on $X$ via the cyclotomic character $\chi$. When $V_B$ is de Rham and thus Hodge-Tate, we have the isomorphism $\alpha_{\text{HT}} : D^K_{\text{HT}}(V_B) \otimes_K B_{\text{HT}} \xrightarrow{\cong} V_B \otimes_{Q_p} B_{\text{HT}}$, and $D^K_{\text{HT}}(V_B) \cong \bigoplus_i \text{gr}^i D^K_{\text{dR}}(V_B)$ as graded $B \otimes_{Q_p} K$-modules. By base change of $\alpha_{\text{HT}}$ via the map $B_{\text{HT}} \to C_p$ given by $X \mapsto 1$, we have the isomorphism $D^K_{\text{HT}}(V_B) \otimes_K C_p \cong V_B \otimes_{Q_p} C_p$. This gives the grading on $V_B \otimes_{Q_p} C_p$ whose graded pieces are $B \otimes_{Q_p} K$-modules. Therefore, we get the induced map $G_m \to \text{Res}_{B/Q_p}(GL_B(V_B))$ over $C_p$, and we can think of a Hodge-Tate type as a conjugacy class of Hodge-Tate cocharacter.

When $K/Q_p$ is finite and $E$ contains the Galois closure of $K$, then we can also consider a Hodge-Tate type as induced from embeddings of $K$ into $E$, but this point of view does
not work in our case where $K/Q_p$ is allowed to be infinite. Note also that Hodge-Tate type does not make sense if we allow both $B$ and $K$ to be infinite over $Q_p$, so we always assume $B/Q_p$ is finite.

**Lemma 3.2.** For a finite $B$-algebra $B'$, $V_{B'}$ has Hodge-Tate type $\mathfrak{v}$ if $V_B$ has Hodge-Tate type $\mathfrak{v}$.

*Proof.* It follows immediately from Lemma 3.1. \qed

The goal of this subsection is to prove the following theorem:

**Theorem 3.3.** (cf. [Liu15, Theorem 4.3.4]) There exists a constant $c_1$ depending only on $K', r$, and $d$ such that the following statement holds:

Let $A$ and $A'$ be finite flat $\mathcal{O}_E$-algebras and let $\rho : G_K \to GL_d(A)$ and $\rho' : G_K \to GL_d(A')$ be representations such that $\rho \in \text{Rep}_{\text{pst}, K', r}$ and $\rho' \in \text{Rep}_{\text{ pst}, K', r}$. Suppose that there exist an ideal $I \subset A$ such that $A/I$ is killed by a power of $p$ and an $O_E$-linear surjection $\beta : A' \to A/I$ such that $A/I \otimes_A \rho \cong A/I \otimes_{\beta, A'} \rho'$ as $A[G_K]$-modules. Let $V$ be the free $A[\frac{1}{p}]$-module of rank $d$ equipped with the $G_K$-action corresponding to $\rho \otimes_{Z_p} Q_p$, and similarly let $V'$ be corresponding to $\rho' \otimes_{Z_p} Q_p$. If $I \subset p s A$ and $V'$ has Hodge-Tate type $\mathfrak{v}$, then $V$ also has Hodge-Tate type $\mathfrak{v}$.

When $k$ is further assumed to be finite, Theorem 3.3 is proved in [Liu15, Theorem 4.3.4]. The proof in [Liu15] is based on reducing to the case when $E$ contains the Galois closure of $K'$, and thus require $k$ to be finite. We remove such a restriction in the following.

Since $E$ is a finite extension of $Q_p$, we have a ring isomorphism $E_K = E \otimes_{Q_p} K \cong \prod_{j=1}^{s} H_j$ for some fields $H_j$ finite over $K$. Note that each $H_j$ is an $E_K$-algebra via $E_K \cong \prod_{j=1}^{s} H_i \otimes_{K} H_j$ where $q_i$ is the natural projection onto the $j$-th factor. For any $E_K$-module $M$, we write $M_j := M \otimes_{E_K} H_j$. Then $M \cong \oplus_{j=1}^{s} M_j$. For a filtered $E_K$-module $D_K$, we denote $(\text{Fil}' D_K)_j$ and $(\text{gr}_j D_K)_j$ by $\text{Fil}'_j D_K$ and $\text{gr}_j D_K$ respectively. Since any finite $E_K$-module is projective, we have $\text{gr}_j D_K \cong \text{Fil}'_j D_K/\text{Fil}'_{j+1} D_K$. Write $B_{H_j} := B \otimes_{E} H_j$.

**Lemma 3.4.** (cf. [Liu15, Lemma 4.1.4]) With notations as above, $V_B$ has Hodge-Tate type $\mathfrak{v}$ if and only if $\text{gr}_j D_K^{\text{gr}_j D_K^{\text{gr}_j D_K}}(V_B)$ is $B_{H_j}$-free and rank$B_{H_j} \text{gr}_j D_K^{\text{gr}_j D_K^{\text{gr}_j D_K}}(V_B) = \dim_{H_j} \text{gr}_j D_{E, K}$ for all $j = 1, \ldots, s$ and $i \in \mathbb{Z}$.

*Proof.* This follows from the same argument as in the proof of [Liu15, Lemma 4.1.4]. \qed

$B_{\text{red}} = B/\mathcal{N}(B)$ is a reduced Artinian $E$-algebra, so $B_{\text{red}} \cong \prod_{l=1}^{m} E_l$ for some field $E_l$ finite over $E$. We set $V_{E_l} = E_l \otimes_{B} V_B$.

**Lemma 3.5.** (cf. [Liu15, Proposition 4.1.5]) $V_B$ has Hodge-Tate type $\mathfrak{v}$ if and only if $V_{E_l}$ has Hodge-Tate type $\mathfrak{v}$ for each $l = 1, \ldots, m$. 

11
Proof. This follows from the same argument as in the proof of [Liu15, Proposition 4.1.5].

The following lemma is useful when we consider an extension of the coefficient field $E$.

**Lemma 3.6.** Let $H$ be a field, and let $C$ be a field (possibly of an infinite degree) over $H$. Let $H'$ be a finite extension of $H$, and let $R$ and $T$ be finite extensions of $H'$. If $M$ is a $C \otimes_H R$-module such that $M \otimes_{H'} T$ is a finite free $C \otimes_H R \otimes_{H'} T$-module, then $M$ is finite free over $C \otimes_H R$.

**Proof.** $M$ is a finite projective $C \otimes_H R$-module, and there exists a surjection $f : M \otimes_{H'} T \to M$ of $C \otimes_H R$-modules having a section. Let $\{e_1, \ldots, e_n\}$ be a basis of $M \otimes_{H'} T$ over $C \otimes_H R \otimes_{H'} T$. Let $N := \oplus_{i=1}^n (C \otimes_H R) \cdot f(e_i)$. Then the natural map $N \to M$ of $C \otimes_H R$-modules is an injection since $\{e_1, \ldots, e_n\}$ is a basis of $M \otimes_{H'} T$ over $C \otimes_H R \otimes_{H'} T$. Furthermore, $\dim_C N = \dim_C M$, so it is bijective.

Let $L$ be a finite extension of $E$, and write $B_L := L \otimes_E B$. Given $v$ as above, let $v' = (D_L := L \otimes_E D, \{\Fil i D_{L,K} = L \otimes_E \Fil i D_{E,K}\}_{i=0,\ldots,r})$.

**Lemma 3.7.** (cf. [Liu15 Lemma 4.1.6]) With notations as above, $V_B$ has Hodge-Tate type $v$ if and only if $V_{B_L} := B_L \otimes_B V_B$ has Hodge-Tate type $v'$.

**Proof.** Given Lemma 3.6, it follows from the same argument as in the proof of [Liu15 Lemma 4.1.6].

**Lemma 3.8.** Suppose we have an injection $B \hookrightarrow B'$ of finite $E$-algebras. If $V_{B'} = B' \otimes_B V_B$ has Hodge-Tate type $v$, then also $V_B$ has Hodge-Tate type $v$.

**Proof.** We have an induced injection of finite $E$-algebras $B_{\text{red}} \hookrightarrow B'_{\text{red}}$. By Lemma 3.3, we can reduce to the case when $B$ and $B'$ are fields. Then it follows from Lemma 3.4 and Lemma 3.6.

As we will apply the functor $M_{\text{st}}$ to $G_K$-representations semi-stable over $K'$, we need to consider $D_{\text{dr}}^K(V_B) := (B_{\text{dr}} \otimes_{\mathbb{Q}_p} V_B')^\text{\text{dR}}$. Note that $D_{\text{cr}}^K(V_B) = D_{\text{cr}}^K(V_B) \otimes_K K'$. Thus, by essentially the same argument as in the proof of Lemma 3.7, we see that $V_B$ has Hodge-Tate type $v$ if and only if $\gr i D_{\text{dr}}^K(V_B) \cong B \otimes_E \gr i D_{E,K'}$ as $B_{K'}$-modules for all $i$. Here, $D_{E,K'} := D_E \otimes_{\mathbb{Q}_p} K' = D_{E,K} \otimes_K K'$ which has the induced filtration from $D_{E,K}$.

Let $K_1 \subset E$ be the maximal unramified subextension over $\mathbb{Q}_p$. Then $K_1 = W(k_1)[\frac{1}{p}]$ for some finite field $k_1$, and $E/K_1$ is totally ramified. Choose a uniformizer $\varpi_E \in E$, and let $\hat{F}(u)$ be its Eisenstein polynomial over $K_1$. Let $G(u)$ be a monic irreducible polynomial in $\mathbb{Q}_p[u]$ such that $K_1 \cong \mathbb{Q}_p[u]/G(u)\mathbb{Q}_p[u]$, and let $G(u) = \prod_{j=1}^m G_j(u)$ be the decomposition into monic irreducible polynomials in $K_0[u]$. Note that $G_j(u) \in W(k)[u]$. Denote by $\bar{G}_j(u) \in k[u]$ the reduction of $G_j(u)$ mod $p$. Then $\bar{G}_j(u)$ is irreducible in $k[u]$ and $k[u]/\bar{G}_j(u)k[u] \cong l_j$ for a finite extension $l_j/k$. By the Chinese remainder theorem,
\[ W(k_1) \otimes \mathbb{Z}_p W(k) \cong \prod_{j=1}^{m} W(l_j). \]
Since \( \tilde{F}(u) \) is irreducible over \( W(l_j)[\frac{1}{p}] \) for each \( j \), we have \( E \otimes_{\mathbb{Q}_p} K_0 \cong \prod_{j=1}^{m} L_j \) and \( O_E \otimes \mathbb{Z}_p W(k) \cong \prod_{j=1}^{m} O_{L_j} \) where \( L_j := (W(l_j)[\frac{1}{p}])(\varpi_E) \).

For each \( j = 1, \ldots, m \), let \( F(u) = \prod_{s=1}^{t_j} F_{j,s}(u) \) be the decomposition of \( F(u) \) into monic irreducible polynomials in \( L_j[u] \), and choose a root \( \varpi_{j,s} \) of \( F_{j,s}(u) \) for each \( s \). Then

\[
L_j \otimes_{K_0} K' \cong \prod_{s=1}^{t_j} L_j[u]/F_{j,s}(u)L_j[u] \cong \prod_{s=1}^{t_j} T_{j,s}
\]
where \( T_{j,s} := L_j(\varpi_{j,s}) \). Thus, we have ring isomorphisms

\[
E \otimes_{\mathbb{Q}_p} K' \cong (E \otimes_{\mathbb{Q}_p} K_0) \otimes_{K_0} K' \cong \prod_{j=1}^{m} \prod_{s=1}^{t_j} T_{j,s}.
\]

Let \( t = \sum_{j=1}^{m} t_j \). After re-indexing the fields \( T_{j,s} \), we have \( E \otimes_{\mathbb{Q}_p} K' \cong \prod_{s=1}^{t} T_j \), and the statement analogous to Lemma 3.4 holds for \( \text{D}^{\text{dir}}(V_B) \).

Let \( O_{E,K'} := O_E \otimes_{\mathbb{Z}_p} O_{K'} \). The projection \( q_s : E_{K'} \to T_s \) induces the map \( O_{E,K'} \to O_{T_s} \), and we have the natural map \( q : O_{E,K'} \to \prod_{s=1}^{t} O_{T_s} \). Denote by \( v_p \) the \( p \)-adic valuation normalized by \( v_p(p) = 1 \).

**Lemma 3.9.** There exists a positive integer \( c' \) depending only on \( K_0 \) and \( F(u) \) such that \( p^c(\prod_{s=1}^{t} O_{T_s}) \subset q(O_{E,K'}) \).

**Proof.** For a field \( L \) finite over \( K_0 \), let \( F(u) = \prod_{s=1}^{w} F_s(u) \) be the decomposition of \( F(u) \) into monic irreducible polynomials in \( L[u] \), and choose a root \( \varpi_s \) of \( F_s(u) \) for each \( s \). We have

\[
L \otimes_{K_0} K' \cong \prod_{s=1}^{w} L[u]/F_s(u)L[u] \cong \prod_{s=1}^{w} L'_s
\]
where \( L'_s := L(\varpi_s) \). Let \( q'_s : L \otimes_{K_0} K' \to L'_s \) be the composition of the above isomorphism followed by the projection onto the \( s \)-th factor. Then \( q'_s \) induces a surjection \( O_L \otimes_{\mathbb{Z}_p} O_{K'} \to O_{L'_s} \) and we have the natural map \( O_L \otimes_{\mathbb{Z}_p} O_{K'} \to \prod_{s=1}^{w} O_{L'_s} \). Under this map, \( \prod_{h \neq s} F_h(\pi) \) maps to \( (0, \ldots, 0, \prod_{h \neq s} F_h(\varpi_s), 0, \ldots, 0) \) whose components are 0 except the \( s \)-th component. Write \( v_p(\prod_{h \neq s} F_h(\varpi_s)) = \frac{a}{b} \) for some relatively prime integers \( a, b \).

Then \( (\prod_{h \neq s} F_h(\varpi_s))^b = p^a x \) for some \( x \in O_{L'_s} \) with \( v_p(x) = 0 \). Thus, \( (0, \ldots, 0, p^a, 0, \ldots, 0) \), whose components are 0 except the \( s \)-th component, lies in the image of \( O_L \otimes_{\mathbb{Z}_p} O_{K'} \) under the above map.

Repeating this argument for all \( s = 1, \ldots, w \) and considering all possible decompositions of \( F(u) \) into irreducible factors over some finite field over \( K_0 \), we see that there exists a positive integer \( c' \) depending only on \( K_0 \) and \( F(u) \) such that for any \( L \) finite over \( K_0 \), if we write \( L \otimes_{K_0} K \cong \prod_{s=1}^{w} L(\varpi_s) \) as above, then for each \( s, (0, \ldots, 0, p^c, 0, \ldots, 0) \) whose components are 0 except the \( s \)-th component lies in the image of \( O_L \otimes_{\mathbb{Z}_p} O_{K'} \). Applying this for each \( L_j \), we get the result. \( \square \)
Corollary 3.10. Let $M$ be a torsion free $\mathcal{O}_{E,K'}$-module. Then the torsion part of $M_s := M \otimes_{\mathcal{O}_{E,K',q_s}} \mathcal{O}_{T_s}$ is killed by $p^c$, where $c'$ is the constant given in Lemma 3.9.

Proof. Let $M' = \bigoplus_{s=1}^t M_s$. By Lemma 3.9 there exist morphisms of $\mathcal{O}_{E,K'}$-modules $q_M : M \to M'$ and $s_M : M' \to M$ such that $q_M \circ s_M = p^c \text{Id}_{M'}$. Let $x$ be a torsion element in $M'$. Then $s_M(x) = 0$, so $p^c x = q_M(s_M(x)) = 0$. \hfill \Box

Let $C$ be a finite flat $\mathcal{O}_E$-algebra, and let $\Lambda \in \text{Rep}_{C}^{\text{st},K',r}$ such that $\Lambda$ is a finite free $C$-module of rank $d$. Thus, $\mathcal{O}_E$ is finite free over $C$. By Nakayama’s lemma, we have a surjection

$$\bigoplus_{i=1}^d \mathfrak{S}_C \cdot e_i \twoheadrightarrow \mathfrak{M}$$

of $\mathfrak{S}_C$-modules. $\Lambda$ is a free $\mathbb{Z}_p$-module of rank $[C : \mathbb{Z}_p]d$, so $\mathfrak{M}$ is free over $\mathfrak{S}$ of rank $[C : \mathbb{Z}_p]d$. Thus, $f$ is an isomorphism. \hfill \Box

Suppose that $C$ is good and that there exists an ideal $J \subset C$ such that $C/J \cong \mathcal{O}_E/p^b \mathcal{O}_E$ for some positive integer $b$. For $s = 1, \ldots, t$, we set $C^{[1]}_{p^s} := (C^{[1]}_{p^s} \otimes_{\mathbb{Q}_p} \mathcal{K}^t) \otimes_{\mathcal{O}_{E,K',q_s}} \mathcal{T}_s$, and define $d_s := \text{rank}_{C^{[1]}_{p^s}}(\mathfrak{g}^0_s(D^R_{\mathfrak{tr}}(A^{[1]}_{p^s}))))$. Denote $\text{Fil}^i_0L_{K'} := \text{Fil}^iL_{K'} \otimes_{\mathcal{O}_{E,K',q_s}} \mathcal{T}_s$, and similarly for the graded modules. By Lemma 3.11, $\text{Fil}^0_0L_{K'}$ is free over $C_s := C_{K'} \otimes_{\mathcal{O}_{E,K',q_s}} \mathcal{T}_s$, of rank $d$.

Lemma 3.12. (cf. [Liu15, Lemma 4.2.7]) Suppose that $d_s \neq 0$. Let $l$ be a positive integer satisfying $b \geq ld + 1$. Then there exists $x \in \text{gr}^0_sL_{K'}/J \text{gr}^0_sL_{K'}$ such that $p^l x \neq 0$.  

14
Proof. This follows from essentially the same argument as in the proof of [Liu15, Lemma 4.2.7]. For any $C$-module $M$, denote $M/JM$ by $M/J$. We have the following right exact sequence:

$$\Fil^1_sL_{K'} \to \Fil^0_sL_{K'} \to \gr^0_sL_{K'} \to 0.$$  

Let $\Fil^1_sL_{K'}$ be the image of $\Fil^1_sL_{K'}$ in $\Fil^0_sL_{K'}$ under the first map in the above sequence. We then obtain the following right exact sequence

$$\Fil^1_sL_{K'}/J \to \Fil^0_sL_{K'}/J \to \gr^0_sL_{K'}/J \to 0.$$

Denote $\tilde{M} := \Fil^0_sL_{K'}/J$ and let $\tilde{N} \subset \tilde{M}$ be the submodule given by the image of $\Fil^1_sL_{K'}/J$. Then $\tilde{M}/\tilde{N} = \gr^0_sL_{K'}/J$.

Suppose that $p'$ annihilates $\tilde{M}/\tilde{N}$. By Lemma 3.11, $\tilde{M}$ is a finite free $\mathcal{O}_{T_\alpha}/p'b\mathcal{O}_{T_\alpha}$-module of rank $d$. Let $\tilde{\pi}_s$ be a uniformizer of $\mathcal{O}_{T_\alpha}$. Then there exists an $\mathcal{O}_{T_\alpha}/p'b\mathcal{O}_{T_\alpha}$-basis $\tilde{e}_1, \ldots, \tilde{e}_d$ of $\tilde{M}$ such that

$$\tilde{N} \cong \bigoplus_{i=1}^d ((\mathcal{O}_{T_\alpha}/p'b\mathcal{O}_{T_\alpha}) \cdot (\tilde{\pi}_s^{a_i} \tilde{e}_i))$$

for some nonnegative integers $a_i$. We have $\tilde{\pi}_s^{a_i} \mid p'$ for all $i = 1, \ldots, d$. Let $e_1, \ldots, e_d$ be a $C_s$-basis of $\Fil^0_sL_{K'}$ which lifts $\tilde{e}_1, \ldots, \tilde{e}_d$. For $i = 1, \ldots, d$, let $y_i \in \Fil^1_sL_{K'}$ which lifts $\tilde{\pi}_s^{a_i} \tilde{e}_i$. If $X$ denotes the $d \times d$-matrix such that $(y_1, \ldots, y_d) = (e_1, \ldots, e_d)X$, then $\det(X) = \tilde{\pi}_s^{a_0 + j}$ with $a = \sum_{i=1}^d a_i$ and $j \in J$. Since $b \geq ld + 1$, we have $\tilde{\pi}_s \not\equiv 0$ in $C_s/J$, and thus $\det(X) \not\equiv 0$ in $C_s$. On the other hand, let $\tilde{z}_1, \ldots, \tilde{z}_d$ be a $C[\frac{1}{p_i}]_s$-basis of $\gr^0_s(D^K_{dR}(\Lambda[\frac{1}{p_i}]_s))$. We have $\det(X)(e_1, \ldots, e_d) \subset \Fil^1_s(D^K_{dR}(\Lambda[\frac{1}{p_i}]_s))$, and therefore $\det(X)\tilde{z}_i = 0$. This gives a contradiction.

Proof of Theorem 3.3. Given above results, Theorem 3.3 follows from essentially the same argument as in the proof of [Liu15, Theorem 4.3.4], except that we do not reduce to the case where $E$ contains the Galois closure of $K'$. We also remark that the proof of [Liu15] reduces to the case $A'$ is local. But $A'$ is not necessarily finite over $\mathcal{O}_{E}$ after such reduction, which has been overlooked in [Liu15]. This is a very minor gap, and we remedy it by only reducing to the case $A'$ is good.

We first reduce to the case where $A = \mathcal{O}_{E}$ and $A'$ is good. For this, let $B := A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $B' := A' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. We have $B_{\text{red}} = B/N(B) \cong \prod_j E_j$ and $B'_{\text{red}} \cong \prod_j E'_j$ for some $E_j, E'_j$ finite over $E$. Let $L$ be a finite Galois extension of $E$ containing all Galois closures of $E_j, E'_j$. Denote $O_L \otimes_{\mathbb{Q}_p} (*)$ by $(*)_{O_L}$ for $(*)$ being $A, A', \rho, \rho', I$, and $\beta$. Note that $(A_{\mathcal{O}_L}[\frac{1}{p_i}]_{\text{red}})$ is the natural map $\psi_l : A_{\mathcal{O}_L} \to (A_{\mathcal{O}_L}[\frac{1}{p_i}]) \to L$ to the $l$-th factor of $\prod_i L$. By Lemma 3.5 and 3.7, it suffices to show (assuming $I \subset p^c A$ for a suitable constant $c_1$) that $L \otimes_{\psi_l A_{\mathcal{O}_L}} (\rho)_{\mathcal{O}_L}$ has Hodge-Tate type $\varphi$. Let $A_l = \psi_l(A_{\mathcal{O}_L})$.
and \( I_t = \psi_t(I_{O_L}) \). \( \psi_t : A_{O_L} \to A_t \subseteq L \) is a morphism of \( O_L \)-algebras, so \( A_t = O_L \) (and analogously for \( A'_t \)), and we have a natural projection \( \gamma_t : A_{O_L}/I_{O_L} \to A_t/I_t \). Thus, by replacing \( E \) by \( L \) and \( A' \) by \( A'_{O_L} \), we can assume that \( A = O_E \) and that \( A' \) is good.

Let \( T \) denote the torsion representation \( A/I \otimes_A \rho \cong A'/I' \otimes_{A'} \rho' \in \text{Rep}^\text{pss}_K \) where \( I' = \ker(\beta) \). We denote by \( j \) and \( j' \) the two lifts \( \rho \) and \( \rho' \) of \( T \) respectively. Write \( L_K := M_{st}(\rho)K \), \( L_{K'} := M_{st}(\rho')K' \), \( M_K := M_{st,j}(T)K \), and \( M_{K'} := M_{st,j'}(T)K' \). We have \( \text{gr}^s_i M_K \cong \text{gr}^s_i L_K/1\text{gr}^s_i L_K \) and \( \text{gr}^s_i M_{K'} \cong \text{gr}^s_i L_{K'}/1\text{gr}^s_i L_{K'} \) for \( s = 1, \ldots, t \). By Corollary 2.8 and Proposition 2.9, there exist morphisms of \( \text{Tor}_s \)-modules \( g^i_s : \text{gr}^s_i M_K \to \text{gr}^s_i M_{K'} \) and \( h^i_s : \text{gr}^s_i M_{K'} \to \text{gr}^s_i M_K \) such that \( g^i_s \circ h^i_s = p^{2\epsilon} \text{Id}|_{\text{gr}^s_i M_{K'}} \), and \( h^i_s \circ g^i_s = p^{2\epsilon} \text{Id}|_{\text{gr}^s_i M_K} \).

Now, we set \( \bar{c} = \bar{c}(K', r, d) := (2c + c')d + 1 \) where \( c \) and \( c' \) are given as in Theorem 2.7 and Lemma 3.9 respectively. Assume \( I \subset p^{\bar{c}} A = p^{\bar{c}} O_E \). We claim that if \( \text{gr}^0_s(D_{\text{dr}}^i(V')) \neq 0 \), then \( \text{gr}^0_s(D_{\text{dr}}^i(V)) \neq 0 \). Suppose \( \text{gr}^0_s(D_{\text{dr}}^i(V)) = 0 \). By Corollary 3.10, \( \text{gr}^0_s M_{K'} \) is killed by \( p^{\bar{c}} \). But by Lemma 3.12, there exists \( x \in \text{gr}^0_s M_{K'} \) such that \( p^{\bar{c}} x \neq 0 \). We have a contradiction since \( p^{\bar{c}} + \bar{c} x \neq 0 \).

On the other hand, let \( B' = A'[1] \), and denote \( d_0 = \dim_{\text{Tor}} \text{gr}^0_s(D_{\text{dr}}^i(V)) \). We claim (assuming \( I \subset p^{c} O_E \)) that \( d_0 \leq \dim_{\text{Tor}} \text{gr}^0_s(D_{\text{dr}}^i(V')) \). For this, note that as an \( \text{Tor}_s \)-module, \( \text{gr}^0_s L_{K'} = N_{\text{tor}} \oplus N \) where \( N_{\text{tor}} \) is the torsion submodule of \( \text{gr}^0_s L_{K'} \) and \( N \) is a finite free \( \text{Tor}_s \)-module of rank \( d_0 \). By Corollary 3.10,

\[
\text{gr}^0_s M_{K'} \cong N_{\text{tor}} \oplus \bigoplus_{i=1}^{d_0} \text{Tor}_s/I \text{Tor}_s.
\]

Let \( \bar{N} := p^{\bar{c}} \bigoplus_{i=1}^{d_0} \text{Tor}_s/I \text{Tor}_s \). Then \( p^{\bar{c}} \text{gr}^0_s M_{K'} = \bar{N} \), again by Corollary 3.10, and therefore \( h^0_i(\text{gr}^0_s(\bar{N})) \cong \bigoplus_{i=1}^{d_0} p^{2\epsilon+c'} \text{Tor}_s/I \text{Tor}_s \). Since \( p^{\bar{c}} \text{gr}^0_s M_{K'} \) surjects onto \( h^0_i(p^{\bar{c}} \text{gr}^0_s M_{K'}) \) and \( g^0_i(\bar{N}) \subset p^{\bar{c}} \text{gr}^0_s M_{K'} \), we have by Corollary 3.10 that the \( \text{Tor}_s \)-rank of \( p^{\bar{c}} \text{gr}^0_s L_{K} \) is at least \( d_0 \). Thus, the \( \text{Tor}_s \)-rank of \( \text{gr}^0_s L_{K'} \) is at least \( d_0 \), and \( \dim_{\text{Tor}} \text{gr}^0_s(D_{\text{dr}}^i(V')) \geq d_0 \).

Hence, assuming \( I \subset p^{\bar{c}} O_E \), we have \( \text{gr}^0_s(D_{\text{dr}}^i(V)) \neq 0 \) if and only if \( \text{gr}^0_s(D_{\text{dr}}^i(V')) \neq 0 \).

For the last step, we set \( c_1 = \bar{c}(K', dr, d) \) and assume \( I \subset p^{c_1} O_E \). It suffices to show that for each \( i \),

\[
\dim_{\text{Tor}} \text{gr}^i_s(D_{\text{dr}}^i(V)) = \text{rank}_{B^i_s} \text{gr}^i_s(D_{\text{dr}}^i(V')).
\]

Suppose that the above equality fails for some \( i \), and let \( i_* \) be the smallest such number. Write \( d_i = \dim_{\text{Tor}} \text{gr}^i_s(D_{\text{dr}}^i(V)) \) and \( d'_i = \text{rank}_{B^i_s} \text{gr}^i_s(D_{\text{dr}}^i(V')) \). Suppose first \( d_i > d'_i \). We set \( t_1 = \sum_{i \leq i_*} d_i \) and \( t_2 = \sum_{i > i_*} i d_i \). Let \( \bar{i} = \max\{i \mid \sum_{j \leq i} d'_j \leq t_1\} \) and \( t' = \sum_{i \leq \bar{i}} d'_i \). Then \( i_* \leq \bar{i} \) and \( t' \leq t_1 \). Let

\[
t'' = (\sum_{i \leq \bar{i}} i d'_i) + (t_1 - t') + (\bar{i} + 1).
\]

We have \( t_2 < t'' \). Moreover, \( t_2 \) (resp. \( t'' \)) is the smallest \( i \) such that \( \text{gr}^i_s(D_{\text{dr}}^i(\bigwedge^{t_1} V)) \) (resp. \( \text{gr}^i_s(D_{\text{dr}}^i(\bigwedge^{t_1} V')) \)) is nontrivial. Let \( \chi \) be a crystalline character such that \( \text{gr}^i_s(D_{\text{dr}}^i(\chi)) \neq 0 \)
It follows from essentially the same proof as for Lemma 3.1.

\[ \text{Proof.} \]

Let \( \chi \) be a representation of the inertia group \( I_K \) with an open kernel. Note that there exists an \( I_K \)-representation \( D_{\text{pst}}(V_B) \) of finite unramified extensions of \( K \) for all \( \sigma \in \text{ker}(\chi) \). Let \( \chi \) denote the inertia subgroup of \( G \) equipped with a potentially semi-stable continuous \( G \)-module of rank \( d \). The Frobenius action commutes with the \( I_K \)-action, so \( \text{tr}(\sigma|D_{\text{pst}}(V_B)) \in B \) for all \( \sigma \in I_K \).

Let \( D_E \) be an \( E \)-vector space of dimension \( d \), and let \( D_{E,K} = D_E \otimes_{\mathbb{Q}_p} K \) equipped with a filtration giving a Hodge-Tate type \( \mathbf{v} \). Fix a representation

\[ \tau : I_K \to \text{End}_E(D_E) \]

with an open kernel. Note that there exists an \( I_K \)-stable \( \mathcal{O}_E \)-lattice in \( D_E \), so \( \text{tr}(\tau(\sigma)) \in \mathcal{O}_E \) for all \( \sigma \in I_K \). We say \( V_B \) has Galois type \( \tau \) if the \( I_K \)-representation \( D_{\text{pst}}(V_B) \) is equivalent to \( \tau \), i.e., \( \text{tr}(\sigma|D_{\text{pst}}(V_B)) = \text{tr}(\tau(\sigma)) \) for all \( \sigma \in I_K \).

Let \( L/K \) be a finite Galois extension contained in \( \bar{K} \) such that \( I_L \subset \text{ker}(\tau) \). Here, \( I_L \) denotes the inertia subgroup of \( \mathcal{G}_L \). \( D^{L}_{\text{st}}(V_B) = (B_{\text{st}} \otimes_{\mathbb{Q}_p} V_B^\vee)^{\mathcal{G}_L} \) is an \( L_0 \)-vector space where \( L_0 \) is the maximal unramified subextension of \( K_0 \) contained in \( L \). If \( V_B \) is semi-stable over \( L_0 \), then \( D_{\text{pst}}(V_B) \cong K_0^{\text{ur}} \otimes_{L_0} D^{L}_{\text{st}}(V_B) \). Therefore, \( V_B \) has Galois type \( \tau \) if and only if \( V_B \) is semi-stable over \( L_0 \) and \( \text{tr}(\sigma|D^{L}_{\text{st}}(V_B)) = \text{tr}(\tau(\sigma)) \) for all \( \sigma \in I_{L/K} \), where \( I_{L/K} \) is the inertia subgroup of \( \text{Gal}(L/K) \).

3.2 Galois Type

We now study the Galois types of potentially semi-stable representations. As in Section 3.1, let \( E \) be a finite field over \( \mathbb{Q}_p \), and let \( B \) be a finite \( E \)-algebra. Let \( V_B \) be a free \( B \)-module of rank \( d \) equipped with a potentially semi-stable continuous \( G \)-action. Let

\[ D_{\text{pst}}(V_B) = \lim_{\mathcal{K} \subset \bar{\mathcal{K}}} (B_{\text{st}} \otimes_{\mathbb{Q}_p} V_B^\vee)^{\mathcal{G}_{\mathcal{K}}}, \]

where the limit goes over finite extensions \( \mathcal{K} \) of \( K \) contained in \( \bar{K} \). Denote by \( K_0^{\text{ur}} \) the union of finite unramified extensions of \( K_0 \) contained in \( \bar{K} \). We have \( \dim_{K_0^{\text{ur}}} D_{\text{pst}}(V_B) = \dim_{\mathbb{Q}_p} V_B \).

**Lemma 3.13.** Let \( B' \) be a finite \( B \)-algebra, and write \( V_{B'} = B' \otimes_B V_B \). Then \( V_{B'} \) is potentially semi-stable as a \( \mathcal{G}_K \)-representation, and \( D_{\text{pst}}(V_{B'}) \cong B' \otimes_B D_{\text{pst}}(V_B) \). If \( V_B \) becomes semi-stable over \( L \supset K \), then so does \( V_{B'} \). Furthermore, \( D_{\text{pst}}(V_B) \) is a free \( B \otimes_{\mathbb{Q}_p} K_0^{\text{ur}} \)-module of rank \( d \).

**Proof.** It follows from essentially the same proof as for Lemma 3.1.\( \square \)

\( D_{\text{pst}}(V_B) \) is equipped with a \( K_0^{\text{ur}} \)-semilinear action of \( \mathcal{G}_K \), and thus a \( K_0^{\text{ur}} \)-linear action of the inertia group \( I_K \). The Frobenius action commutes with the \( I_K \)-action, so \( \text{tr}(\sigma|D_{\text{pst}}(V_B)) \in B \) for all \( \sigma \in I_K \).

Let \( L/K \) be a finite Galois extension contained in \( \bar{K} \) such that \( I_L \subset \text{ker}(\tau) \). Here, \( I_L \) denotes the inertia subgroup of \( \mathcal{G}_L \). \( D^{L}_{\text{st}}(V_B) = (B_{\text{st}} \otimes_{\mathbb{Q}_p} V_B^\vee)^{\mathcal{G}_L} \) is an \( L_0 \)-vector space where \( L_0 \) is the maximal unramified subextension of \( K_0 \) contained in \( L \). If \( V_B \) is semi-stable over \( L_0 \), then \( D_{\text{pst}}(V_B) \cong K_0^{\text{ur}} \otimes_{L_0} D^{L}_{\text{st}}(V_B) \). Therefore, \( V_B \) has Galois type \( \tau \) if and only if \( V_B \) is semi-stable over \( L_0 \) and \( \text{tr}(\sigma|D^{L}_{\text{st}}(V_B)) = \text{tr}(\tau(\sigma)) \) for all \( \sigma \in I_{L/K} \), where \( I_{L/K} \) is the inertia subgroup of \( \text{Gal}(L/K) \).
Lemma 3.14. Let $\alpha : B \to B'$ be an $E$-algebra morphism between finite $E$-algebras. Suppose $V$ is semi-stable over $L$. Then for all $\sigma \in I_{L/K}$, we have $\text{tr}(\sigma|D^L_{st}(V_{B'})) = \alpha(\text{tr}(\sigma|D^L_{st}(V_B)))$. In particular, if $V_B$ has Galois type $\tau$, then so does $V_{B'}$. If $\alpha$ is injective, then the converse is also true, i.e., $V_B$ has Galois type $\tau$ if and only if $V_{B'}$ has Galois type $\tau$.

Proof. $D_{pst}(V_{B'}) \cong B' \otimes_B D_{pst}(V_B)$ by Lemma 3.13 so

$$\text{tr}(\sigma|D^L_{st}(V_{B'})) = \alpha(\text{tr}(\sigma|D^L_{st}(V_B)))$$

for all $\sigma \in I_{L/K}$. The remaining statements follow immediately. \hfill $\Box$

Consider the case when $B$ is local. If $E'$ is its residue field, then $E'$ is finite over $E$ and $B$ is naturally an $E'$-algebra. Note that the $I_K$-action on $D_{pst}(V_B)$ has an open kernel. Since the cohomology of a finite group with coefficients in $E' \otimes \mathbb{Q}_p K_0^{ur}$ is trivial in all positive degrees, it follows from the deformation theory that the representation $D_{pst}(V_B)$ arises from a representation over $E' \otimes \mathbb{Q}_p K_0^{ur}$. Thus, $V_B$ has Galois type $\tau$ if and only if $V_{E'} = E' \otimes_B V_B$ has Galois type $\tau$. For a general finite $E$-algebra $B$, we have isomorphisms $B \cong \prod_{i=1}^n B_{m_i}$ and $B_{\text{red}} \cong \prod_{i=1}^m E_i$, where $m_1, \ldots, m_n$ are the maximal ideals of $B$ and $E_i = B_{m_i}/m_i B_{m_i}$. Let $V_{E_i} = E_i \otimes_B V_B$. The following lemmas are analogous to Lemma 3.5 and 3.7.

Lemma 3.15. $V_B$ has Galois type $\tau$ if and only if $V_{E_i}$ has Galois type $\tau$ for each $i = 1, \ldots, n$.

Proof. It follows directly from Lemma 3.14. \hfill $\Box$

Lemma 3.16. Let $E'$ be a finite extension of $E$, and let $B_{E'} = E' \otimes_B B$ and $V_{B_{E'}} = B_{E'} \otimes_B V_B$. Then $V_B$ has Galois type $\tau$ if and only if $V_{B_{E'}}$ has Galois type $\tau$.

Proof. Since the natural map of $E$-algebras $B \to B_{E'}$ is injective, it follows from Lemma 3.14. \hfill $\Box$

The following theorem is essential in studying the locus of representations with a given Galois type.

Theorem 3.17. Let $\tau$ be a Galois type, and let $L/K$ be a finite Galois extension in $\bar{K}$ over which $\tau$ becomes trivial. Let $A$ be a finite flat $O_E$-algebra and $\rho : G_K \to \text{GL}_d(A)$ be a Galois representation such that $\rho \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable over $L$ having Hodge-Tate weights in $[0, r]$.

Suppose that for each positive integer $n$, there exist a finite flat $O_E$-algebra $A_n$, a Galois representation $\rho_n : G_K \to \text{GL}_d(A_n)$, and an $O_E$-linear surjection $\beta_n : A_n \to A/p^n A$ such that $A/p^n A \otimes_A \rho \cong A/p^n A \otimes_{\beta_n, A_n} \rho_n$ as $A[G_K]$-modules, and that $\rho_n \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable over $L$ having Hodge-Tate weights in $[0, r]$ and Galois type $\tau$.

Then $\rho \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ also has Galois type $\tau$.
Proof. Let $B = A[\frac{1}{p}]$. We have $B_{\text{red}} \cong \prod_i E_i$ for some finite extensions $E_i/E$. Let $H/E$ be a finite Galois extension containing the Galois closures of $E_i$ for all $i$. We write $A_{O_H} = O_H \otimes_{O_E} A$. Then $(A_{O_H}[\frac{1}{p}])_{\text{red}} \cong H \otimes_E B_{\text{red}} \cong H \otimes_E \prod_i E_i$. Since $H$ contains the Galois closures of $E_i$ for all $i$, $H \otimes_E E_i \cong \prod_j H$ with $E_i$ embedding to $H$ differently. This induces the natural map $\psi_A : A_{O_H} \to A_{O_H}[\frac{1}{p}] \to H$ to the $l$-th factor of $\prod_j H$. Let $A_l = \psi_l(A_{O_H})$. Since $\psi_l : A_{O_H} \to A_l \subset H$ is a morphism of $O_H$-algebras, $A_l = O_H$. By Lemma 3.15 and 3.16 it suffices to show that $H \otimes_{\psi_A, A_{O_H}} (A_{O_H} \otimes_A \rho)$ has Galois type $\tau$. Therefore, we may and will replace $A$ by $O_H$, $\rho$ by $O_H \otimes_{\psi_A, A_{O_H}} (A_{O_H} \otimes_A \rho)$, $A_n$ by $O_H \otimes_{O_E} A_n$, and replace $\beta_n$ and $\rho_n$ accordingly.

Denote by $L_0$ the maximal unramified extension of $K_0$ contained in $L$. Note that $I_{L/K} \cong I_{L/K_0}$. Applying the results of Section 2 with $(L_0, K_0, L)$ in place of $(K_0, K, K')$, we get the associated lattice $M_{st}(\rho) \in L^r(\varphi, N, \text{Gal}(L/KL_0))$ in $D_{st}^L(\rho \otimes_{Z_p} Q_p)$. By the proof of Lemma 3.11 $M_{st}(\rho)$ is a free $O_H \otimes \mathbb{Z}_p O_{L_0}$-module of rank $d$. Thus, for all $\sigma \in I_{L/K}$, $\text{tr}(\sigma D_{st}^L(\rho \otimes_{Z_p} Q_p)) = \text{tr}(\sigma M_{st}(\rho)) \in O_H$.

Now, fix a positive integer $n$, and let $B_n = A_n[\frac{1}{p}]$. $B_{n,\text{red}} \cong \prod_i E_i$ for some finite extensions $F_i/E$. Let $H'/H$ be a finite Galois extension which contains the Galois closures of $F_i$ for all $i$. Similarly as in the proof of Theorem 3.3 we see that $O_{H'} \otimes_{O_H} A_n$ is good (as defined in Section 3.1). Thus, $M_{st}(O_{H'} \otimes_{O_H} \rho_n) \in L^r(\varphi, N, \text{Gal}(L/KL_0))$ is a free $(O_{H'} \otimes_{O_H} A_n) \otimes_{\mathbb{Z}_p} O_{L_0}$-module of rank $d$, for which we fix a choice of bases. Let $\sigma \in I_{L/K}$, and for $i = 1, 2$, let $C_i$ be the $d \times d$ matrix with coefficients in $O_{H'}/p^n O_{H'} \otimes_{\mathbb{Z}_p} O_{L_0}$ which represents the $\sigma$-action on $M_{st,j}(T)$ with respect to the chosen bases. By Corollary 2.8 and Proposition 2.9 there exist $I_{L/K}$-equivariant $O_{H'} \otimes_{\mathbb{Z}_p} O_{L_0}$-module morphisms $g_1 : M_{st,j_1}(T) \to M_{st,j_2}(T)$ and $g_2 : M_{st,j_2}(T) \to M_{st,j_1}(T)$ such that $g_1 \circ g_2 = p^n \sigma |_{M_{st,j_1}(T)}$ and $g_2 \circ g_1 = p^n \sigma |_{M_{st,j_2}(T)}$ where $\sigma$ is a constant depending on the Eisenstein polynomial for $L/L_0$ and $r$. For $i = 1, 2$, let $D_i$ be the $d \times d$ matrix with coefficients in $O_{H'}/p^n O_{H'} \otimes_{\mathbb{Z}_p} O_{L_0}$ representing $g_i$. Then $D_1 D_2 = D_2 D_1 = p^n \sigma |_{D_1 C_1}$. Thus,

$$\text{tr}(C_2 D_1 D_2) = \text{tr}(D_1 C_1 D_2) = \text{tr}(D_2 D_1 C_1),$$

i.e., $p^n \text{tr}(C_1) = p^n \text{tr}(C_2)$ in $O_{H'}/p^n O_{H'}$. Since $\text{tr}(\sigma |_{M_{st}(O_{H'} \otimes_{O_H} \rho_n)}) = \text{tr}(\tau(\sigma)) \in O_E$ and $\text{tr}(\sigma |_{M_{st}(O_{H'} \otimes_{O_H} \rho)}) = \text{tr}(\sigma |_{M_{st}(\tilde{\rho})}) \in O_H$, we have

$$\text{tr}(\sigma |_{M_{st}(\tilde{\rho})}) - \text{tr}(\tau(\sigma)) \in p^n - c^\sigma O_H.$$
Since this holds for all positive integers $n$, we have $\text{tr}(\sigma|_{M_{\text{st}}(\rho)}) = \text{tr}(\tau(\sigma))$. 

\section{Galois Deformation Ring}

We now construct the quotient of the universal deformation ring which corresponds to the locus of potentially semi-stable representations of a given Hodge-Tate type and Galois type. Let $E/\mathbb{Q}_p$ be a finite extension with residue field $\mathbb{F}$. Denote by $\mathcal{C}$ the category of topological local $\mathcal{O}_E$-algebras $A$ satisfying the following conditions:

- The natural map $\mathcal{O}_E \to A/m_A$ is surjective.
- The map from $A$ to the projective limit of its discrete artinian quotients is a topological isomorphism. 

Note that the first condition implies $\mathbb{F}$ is also the residue field of $A$. The second condition is equivalent to the condition that $A$ is complete and its topology can be given by a collection of open ideals $\mathfrak{a}$ for which $A/\mathfrak{a}$ is artinian. Morphisms in $\mathcal{C}$ are continuous $\mathcal{O}_E$-algebra homomorphism.

**Proposition 4.1.** ([SL97, Proposition 2.4]) Suppose $A$ is a noetherian ring in $\mathcal{C}$. Then the topology on $A$ is equal to the $m_A$-adic topology, and $A$ is $m_A$-adically complete. Furthermore, every $\mathcal{O}_E$-algebra homomorphism $A \to A'$ with $A'$ in $\mathcal{C}$ is continuous.

Let $V_0$ be a continuous $\mathbb{F}$-representation of $G_K$ having rank $d$. For $A \in \mathcal{C}$, a deformation of $V_0$ in $A$ is an isomorphism class of continuous $A$-representations $V$ of $G_K$ satisfying $\mathbb{F} \otimes_A V \cong V_0$ as $\mathbb{F}[G_K]$-modules. We denote by Def$(V_0, A)$ the set of such deformations. A morphism $A \to A'$ in $\mathcal{C}$ induces a map $f_* : \text{Def}(V_0, A) \to \text{Def}(V_0, A')$ sending the class of a representation $V$ over $A$ to the class of $A' \otimes_{f,A} V$. Assume $V_0$ is absolutely irreducible. Then, the following is proved in [SL97].

**Proposition 4.2.** (cf. [SL97, Theorem 2.3]) There exists a universal deformation ring $R \in \mathcal{C}$ and a deformation $V_R \in \text{Def}(V_0, R)$ such that for all rings $A \in \mathcal{C}$, we have a bijection

$$\text{Hom}_\mathcal{C}(R, A) \xrightarrow{\sim} \text{Def}(V_0, A)$$

(4.1)

given by $f \mapsto f_*(V_R)$. The ring $R$ is noetherian if and only if $\dim_{\mathbb{F}} \mathcal{H}^1(G_K, \text{End}_{\mathbb{F}}(V_0))$ is finite.

Note that if $K/\mathbb{Q}_p$ is not finite, then $R$ is not necessarily noetherian in general.

We fix a Hodge-Tate type $\mathfrak{v}$ and Galois type $\tau$, and let $L/K$ be a finite Galois extension over which $\tau$ becomes trivial. Let $\mathcal{C}^0$ be the full subcategory of $\mathcal{C}$ consisting of artinian rings. Abusing the notation, we write $V \in \text{Def}(V_0, A)$ for a continuous $A$-representation $V$ to mean that $\mathbb{F} \otimes_A V \cong V_0$. For $A \in \mathcal{C}^0$ and a $G_K$-representation $V_A \in \text{Def}(V_0, A)$, we
say $V_A$ is potentially semi-stable of type $(\nu, \tau)$ if there exist a finite flat $O_E$-algebra $B$, a surjection $g : B \to A$ of $O_E$-algebras, and a continuous $B$-representation $V_B$ of $G_K$ such that $V_B \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is potentially semi-stable having Hodge-Tate type $\nu$ and Galois type $\tau$, and $A \otimes_{g,B} V_B \cong V_A$ as $A[G_K]$-modules. For $A \in \mathcal{C}$, denote by $S_{\nu,\tau}(A)$ the subset of $\text{Def}(V_0, A)$ consisting of the isomorphism classes of representations $V_A$ such that $A/a \otimes_A V_A$ is potentially semi-stable of type $(\nu, \tau)$ for all open ideals $a \subset A$.

**Proposition 4.3.** For any $\mathcal{C}$-morphism $f : A \to A'$, we have $f_*(S_{\nu,\tau}(A)) \subset S_{\nu,\tau}(A')$. There exists a closed ideal $a_{\nu,\tau}$ of the universal deformation ring $R$ such that the map (4.1) induces a bijection $\text{Hom}_\mathcal{C}(R/a_{\nu,\tau}, A) \cong S_{\nu,\tau}(A)$.

**Proof.** We check the conditions in [SL97, Section 6]. Let $f : A \hookrightarrow A'$ be an inclusion of artinian rings in $\mathcal{C}$, and let $V_A \in \text{Def}(V_0, A)$ be a representation. We first claim that $V_A \in S_{\nu,\tau}(A)$ if and only if $V_{A'} := A' \otimes_{f,A} V_A \in S_{\nu,\tau}(A')$. Suppose that $V_A \in S_{\nu,\tau}(A)$. Then there exist a finite flat $O_E$-algebra $B$, a surjection $g : B \to A$, and a $B$-representation $V_B$ such that $V_B \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is potentially semi-stable having Hodge-Tate type $\nu$ and Galois type $\tau$, and $A \otimes_{g,B} V_B \cong V_A$. There exists a surjection $f' : A[x_1, \ldots, x_n] \to A'$ of $O_E$-algebras extending $f$ such that $f'(x_i) \in \mathfrak{m}_{A'}$ for each $i$. Let $I_{m,A} \subset A[x_1, \ldots, x_n]$ denote the ideal generated by the $m$-th degree homogeneous polynomials with coefficients in $A$. Since $A'$ is artinian, $f'(I_{m,A}) = 0$ for a sufficiently large $m$, and $f'$ induces a surjection $A[x_1, \ldots, x_n]/I_{m,A} \to A'$ for such $m$. Thus, we have surjective homomorphisms of $O_E$-algebras

$$g' : B' := B[x_1, \ldots, x_n]/I_{m,B} \to A[x_1, \ldots, x_n]/I_{m,A} \to A'.$$

Note that $B'$ is a finite flat $O_E$-algebra. Let $V_{B'} = B' \otimes_B V_B$. Then $V_{B'} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable over $L$. By Lemma 3.2 and 3.14, it has Hodge-Tate type $\nu$ and Galois type $\tau$. $A' \otimes_{g',B'} V_{B'} \cong V_{A'}$, so $V_{A'} \in S_{\nu,\tau}(A')$.

Conversely, suppose $V_{A'} \in S_{\nu,\tau}(A')$. Then there exist a finite flat $O_E$-algebra $B'$, a surjection $g' : B' \to A'$, and a $B'$-representation $V_{B'}$ such that $V_{B'} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is potentially semi-stable having Hodge-Tate type $\nu$ and Galois type $\tau$, and we have an isomorphism $h : A' \otimes_{g',B'} V_{B'} \cong V_{A'}$ of $A'$-representations. Since $O_E$ is henselian, $B'$ is a finite product of local rings flat over $O_E$. By Lemma 3.2 and Lemma 3.14 we can take $B'$ to be a local ring, since $A'$ is local. Thus, we can lift the isomorphism $h$ to an isomorphism of $B'$-modules $h_1 : V_{B'} \cong V_{B'}$ such that the composite

$$A' \otimes_{g',B'} V_{B'} \xrightarrow{id \otimes h^{-1}} A' \otimes_{g',B'} V_{B'} \xrightarrow{h} V_{A'}$$

is the identity map of $A'$-modules.

Let $B$ be the kernel of the composition of morphisms $B' \xrightarrow{g'} A' \to A'/f(A)$. Then $B$ is a finite flat $O_E$-algebra, and we have the surjection $g : B \to A$ of $O_E$-algebras induced from $g'$. Let $V_B$ be the kernel of the following composite of morphisms

$$V_{B'} \xrightarrow{h_1^{-1}} V_{B'} \to A' \otimes_{g',B'} V_{B'} \xrightarrow{h} V_{A'} \to A'/f(A) \otimes_{A'} V_{A'}.$$
Then $V_B$ is a continuous $B$-representation of $\mathcal{G}_K$ such that $B' \otimes_B V_B \cong V_B$ and $A \otimes_{g,B} V_B \cong V_A$, since $V_{A'} = A' \otimes_{f,A} V_A$. By the main theorem for semi-stable representations in [Liu07], $V_B \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable over $L$. It has Hodge-Tate type $v$ and Galois type $\tau$ by Lemma 3.8 and 3.14, and therefore $V_A \in S_{v,\tau}(A)$.

Now, for $A \in \mathcal{C}$ and a representation $V_A \in \text{Def}(V_0, A)$, suppose $a_1, a_2 \subseteq A$ are open ideals such that $A/a_i \otimes_A V_A \in S_{v,\tau}(A/a_i)$ for $i = 1, 2$. We claim that $A/(a_1 \cap a_2) \otimes_A V_A \in S_{v,\tau}(A/(a_1 \cap a_2))$. There exist a finite flat $O_E$-algebra $B_i$, a surjection $g_i : B_i \to A/a_i$, and a $B_i$-representation $V_{B_i}$ such that $V_{B_i} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is potentially semi-stable having Hodge-Tate type $v$ and Galois type $\tau$, and that $A/a_i \otimes g_i, B_i, V_{B_i} \cong A/a_i \otimes_A V_A$. Let $V_{B_1 \times B_2}$ be the $(B_1 \times B_2)$-representation corresponding to $V_{B_1} \oplus V_{B_2}$. Note that $V_{B_1 \times B_2} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is potentially semi-stable having Hodge-Tate type $v$ and Galois type $\tau$. Consider the natural inclusion $A/(a_1 \cap a_2) \subseteq A/a_1 \times A/a_2$. Let $B$ be the kernel of the composite of morphisms

$$B_1 \times B_2 \xrightarrow{g_1 \times g_2} A/a_1 \times A/a_2 \to (A/a_1 \times A/a_2)/(A/(a_1 \cap a_2)).$$

Then $B$ is a finite flat $O_E$-algebra, and we have the surjection $g : B \to A/(a_1 \cap a_2)$ induced from $g_1 \times g_2$. Let $V_B$ be the kernel of the composite of morphisms

$$V_{B_1 \times B_2} \to (A/a_1 \times A/a_2) \otimes_{g_1 \times g_2, B_1 \times B_2} V_{B_1 \times B_2} \cong (A/a_1 \times A/a_2) \otimes_A V_A$$

and

$$(A/a_1 \times A/a_2) \otimes_A V_A \to (A/a_1 \times A/a_2)/(A/(a_1 \cap a_2)) \otimes_A V_A.$$

Then $V_B$ is a continuous $B$-representation of $\mathcal{G}_R$ such that $(B_1 \times B_2) \otimes_B V_B \cong V_{B_1 \times B_2}$ and $A/(a_1 \cap a_2) \otimes_{g,B} V_B \cong A/(a_1 \cap a_2) \otimes_A V_A$. By the main theorem for semi-stable representations in [Liu07], $V_B \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable over $L$. It has Hodge-Tate type $v$ and Galois type $\tau$ by Lemma 3.8 and 3.14, Thus, $A/(a_1 \cap a_2) \otimes_A V_A \in S_{v,\tau}(A/(a_1 \cap a_2))$. Therefore, $V_A \in S_{v,\tau}(A)$.

Finally, we prove the main theorem.

**Theorem 4.4.** Let $A$ be a finite flat $O_E$-algebra, and let $f : R \to A$ be a continuous $O_E$-algebra homomorphism (where we equip $A$ with the (p)-adic topology). Then the induced representation $A[1/p] \otimes_{f,R} V_R$ is potentially semi-stable of Hodge-Tate type $v$ and Galois type $\tau$ if and only if $f$ factors through the quotient $R/a_{v,\tau}$.

**Proof.** Let $A_1 = f(R) \subseteq A$. Then $A_1$ is a finite flat $O_E$-algebra and local. We equip $A_1$ with the (p)-adic topology. Then $A_1 \in \mathcal{C}$, and the map $f : A \to A_1$ is a morphism in $\mathcal{C}$. Let $V_{A_1} = A_1 \otimes_{f,R} V_R$ and $V_A = A \otimes_{f,R} V_R \cong A \otimes_{A_1} V_{A_1}$.

Suppose that $V_A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is potentially semi-stable of Hodge-Tate type $v$ and Galois type $\tau$. By the main theorem for semi-stable representations in [Liu07], $V_{A_1} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable over $L$. By Lemma 3.8 and 3.14, $V_{A_1} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ has Hodge-Tate type $v$ and Galois type $\tau$. Thus, $V_{A_1} \in S_{v,\tau}(A_1)$, and $f$ factors through $R/a_{v,\tau}$ by Proposition 4.3.
Conversely, suppose $f$ factors through $R/a_{v,\tau}$. Then $V_{A_1} \in S_{v,\tau}(A_1)$ by Proposition 4.3, so $A_1/p^n \otimes_{A_1} V_{A_1}$ is potentially semi-stable of type $(v, \tau)$ for each $n \geq 1$. By the main theorem for semi-stable representations in [Liu07], $V_{A_1} \otimes_{\mathbb{Z}_p} Q_p$ is semi-stable over $L$. And by Theorem 3.3 and 3.17, $V_{A_1} \otimes_{\mathbb{Z}_p} Q_p$ has Hodge-Tate type $v$ and Galois type $\tau$. Thus, $V_A \otimes_{\mathbb{Z}_p} Q_p$ is potentially semi-stable of Hodge-Tate type $v$ and Galois type $\tau$. 

References

[Bre97] Christophe Breuil, *Représentations $p$-adiques semi-stables et transversalité de Griffiths*, Math. Ann. 307 (1997), no. 2, 191–224.

[CF00] Pierre Colmez and Jean-Marc Fontaine, *Construction des représentations $p$-adiques semi-stables*, Invent. Math. 140 (2000), no. 1, 1–43.

[Fon94] Jean-Marc Fontaine, *Représentations $l$-adiques potentiellement semi-stables*, Astérisque 223 (1994), 321–347.

[Fon97] ———, *Deforming semistable Galois representations*, Proc. Nat. Acad. Sci. U.S.A. 94 (1997), 11138–11141.

[Kis07] Mark Kisin, *Potentially semi-stable deformation rings*, J. Amer. Math. Soc. 21 (2007), no. 2, 513–547.

[Liu07] Tong Liu, *Torsion $p$-adic Galois representations and a conjecture of Fontaine*, Ann. Sci. École Norm. Sup. 40 (2007), no. 4, 633–674.

[Liu10] ———, *A note on lattices in semi-stable representations*, Math. Ann. 346 (2010), no. 1, 117–138.

[Liu12] ———, *Lattices in filtered $(\phi, N)$-modules*, J. Inst. of Math. of Jussieu 11 (2012), no. 3, 659–693.

[Liu15] ———, *Filtration associated to torsion semi-stable representations*, Ann. Inst. Fourier 65 (2015), no. 5, 1999–2035.

[SL97] Bart De Smit and Hendrick W. Lenstra, *Explicit construction of universal deformation rings*, Modular Forms and Fermat’s Last Theorem (New York), Springer-Verlag, 1997, pp. 313–326.