Virtual Links with Finite Medial Bikei

Julien Chien∗ Sam Nelson†

Abstract

We consider the question of which virtual knots have finite fundamental medial bikei. We describe and implement an algorithm for completing a presentation matrix of a medial bikei to an operation table, determining both the cardinality and isomorphism class of the fundamental medial bikei, each of which are link invariants. As an example, we compute the fundamental medial bikei for all of the prime virtual knots with up to four classical crossings as listed in the knot atlas.

Keywords: bikei, involutory biquandles, medial bikei, finite presentations

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1 Introduction

In [1] an algebraic structure known as bikei (双圭) was introduced, generalizing the notion of kei (圭) or involutory quandles from [6, 10]. Every unoriented classical or virtual knot or link has an associated fundamental bikei with the property that ambient isotopic knots and links have isomorphic fundamental bikei. Moreover, various quotients of the fundamental bikei obtained by imposing extra algebraic axioms have been defined, each also invariant under ambient isotopy and hence defining invariants of knots and links. In particular, the notion of abelian or medial bikei was defined in [3] for biquandles, which specialize to the case of bikei.

It is natural to ask whether some of these bikei or medial bikei are finite, and thus can be directly used to compare different knots. However, determining which knots have finite bikeis and computing the structure of these bikeis can take some work – given a knot or link diagram, we can obtain a presentation of the fundamental bikei, but comparing isomorphism classes of objects defined by presentations can be nontrivial. In other work such as [5, 9] an algorithm with roots dating back to [11] is used to compute the operation table of a finite algebraic structure given initially by a finite presentation.

In this paper, we describe and implement an algorithm for computing the fundamental medial bikei for virtual knots and links (including the classical case). In particular, we identify some cases when these fundamental medial bikei of a virtual knot or link is finite. The algorithm can be used to prove finiteness and detect the isomorphism classes of fundamental medial bikei for some virtual knots and links. The cardinality of the fundamental medial bikei is an integer-valued invariant of virtual knots and links when finite. For virtual knots and links with the same size medial bikei, the isomorphism class is a generally stronger invariant than the cardinality alone, since isomorphic bikei necessarily have equal cardinality. Moreover, knowing the fundamental medial bikei structure can be helpful in determining which bikei to use for counting invariants and their enhancements, since if two virtual links L, L’ have isomorphic fundamental medial bikei then for any finite medial bikei X the counting invariants Hom(BK(L), X) and Hom(BK(L’), X) will be equal, so we must then use non-medial bikei X if we wish to detect any difference.

The paper is organized as follows. In Section 2 we recall the basics of virtual knots and links. In section 3 we give a brief review of bikei and medial bikei. Section 4 contains a description of our algorithm and collects computational results, and in Section 5 we collect some questions for future research.

∗Email: jchien17@cmc.edu
†Email: Sam.Nelson@cmc.edu. Partially supported by Simons Foundation collaboration grant 316709
2 Virtual Links

We begin with a brief review of virtual knot theory. See [7] for more.

A virtual link is an equivalence class of virtual link diagrams, planar 4-valent graphs with vertices decorated as either classical crossings or virtual crossings as shown

![Classical vs. Virtual Crossings](image)

under the equivalence relation generated by the seven virtual Reidemeister moves:

![Virtual Reidemeister Moves](image)

In these moves, the knots or links in question are identical outside the pictured portion of the diagrams. A virtual link with a single component is a virtual knot.

Virtual links can be interpreted as disjoint unions of simple closed curves in thickened surfaces, i.e. orientable 3-manifolds-with-boundary of the form \( \sigma \times [0,1] \) where \( \sigma \) is a compact orientable surface, up to stabilization moves on \( \sigma \). That is, we can think of drawing the virtual link diagram on a surface with possibly nonzero genus, with each virtual crossing representing a bridge or handle in the surface \( \sigma \) and the classical crossings drawn on \( \sigma \); then, thickening \( \sigma \), the classical crossings represent points where the strands of the virtual link are close together inside the thickened surface, while the virtual crossings result from
compressing $\sigma$ onto genus-zero paper.

A virtual knot is \textit{classical} if it is equivalent to a diagram with no virtual crossings; these correspond to knots in ordinary genus zero three-dimensional space. It is known (see [8] for instance) that if two virtual crossing-free diagrams are equivalent by moves \{I, II, III, vI, vII, vIII, v\} then they are also equivalent via moves \{I, II, III\} only.

Perhaps the main question in virtual knot theory is determining when two virtual knots or links are equivalent. This is generally done via \textit{virtual link invariants}, functions we can compute from virtual link diagrams whose value does not change when the diagram is changed by Reidemeister moves. In the remainder of the paper we will describe our method for computing two invariants, the size and isomorphism class of the fundamental medial bikei of a virtual link.

3 Bikei

We begin this section with a definition (see [1, 4]).

\textbf{Definition 1.} A \textit{bikei} is a set $X$ with two binary operations $\ast, \bar{\ast} : X \times X \rightarrow X$ satisfying for all $x, y, z \in X$

(i) $x \ast x = x \bar{\ast} x$,

(ii) $x \ast (y \bar{\ast} x) = x \bar{\ast} y$ (ii.i)

$x \bar{\ast} (y \ast x) = x \bar{\ast} y$ (ii.ii)

$(x \bar{\ast} y) \bar{\ast} y = x$ (ii.iii)

$(x \ast y) \bar{\ast} y = x$ (ii.iv)

and

(iii) (Exchange Laws)

$(x \bar{\ast} y) \bar{\ast} (z \ast y) = (x \bar{\ast} z) \bar{\ast} (y \bar{\ast} z)$ (iii.i)

$(x \bar{\ast} y) \ast (z \bar{\ast} y) = (x \bar{\ast} z) \bar{\ast} (y \bar{\ast} z)$ (iii.ii)

$(x \ast y) \bar{\ast} (z \ast y) = (x \bar{\ast} z) \ast (y \bar{\ast} z)$ (iii.iii).

\textbf{Example 1.} The integers $\mathbb{Z}$ and integers mod $n \mathbb{Z}_n$ form bikei with operations $x \ast y = 2y - x$ and $x \bar{\ast} y = x$.

\textbf{Example 2.} A module $X$ over the ring $\mathbb{Z}/(t^2 - 1, s^2 - 1, (1 - s)(s - t))$ is a bikei under the operations $x \ast y = tx + (s - t)y$ and $x \bar{\ast} y = sx$, known as an \textit{Alexander bikei}. As a special case, $\mathbb{Z}_n$ is a bikei under $x \ast y = tx + (s - t)y$ and $x \bar{\ast} y = sx$ where we choose $s, t \in \mathbb{Z}_n$ such that $s^2 = t^2 = 1$ and $(1 - s)(s - t) = 0$. 

3
Given a finite bikei $X$, we can represent the bikei structure with a block matrix encoding the operation tables of $\ast, \bar{\ast}$ in the following way: let $X = \{x_1, \ldots, x_n\}$. Then the bikei matrix of $X$ is the $n \times 2n$ matrix whose entry in row $j$ column $k$ is given by $l \in \{1, \ldots, n\}$ where

$$x_l = \begin{cases} x_j \ast x_k & 1 \leq k \leq n \\ x_j \bar{\ast} x_{k+n} & n + 1 \leq k \leq 2n \end{cases}$$

**Example 3.** Let $X = \mathbb{Z}_4$ and set $t = 1$ and $s = 3$; then we have $s^2 = t^2 = 1$ and $(1 - s)(s - t) = (1 - 3)(3 - 1) = 2(2) = 0$, so we have a bikei with operations

$$x \ast y = tx + (s - t)y = x + 2y \quad \text{and} \quad x \bar{\ast} y = sx = 3x.$$

Then using $X = \{1, 2, 3, 4\}$ with the class of $0 \in \mathbb{Z}_4$ represented by 4 so we can start our row and column numbering with 1, $X$ has matrix

\[
\begin{pmatrix}
3 & 1 & 3 & 1 & 3 & 3 & 3 \\
4 & 2 & 4 & 2 & 2 & 2 & 2 \\
1 & 3 & 1 & 3 & 1 & 1 & 1 \\
2 & 4 & 2 & 4 & 4 & 4 & 4
\end{pmatrix}.
\]

**Example 4.** If $X$ is a bikei in which $x \bar{\ast} y = x$ for all $X$, then $X$ is called a kei or involutory quandle. For example, any group $G$ forms a kei called a core kei with $x \ast y = y^{-1}x$ and $x \bar{\ast} y = x$; the group $S_3 = \{x_1 = (1), x_2 = (12), x_3 = (13), x_4 = (23), x_5 = (123), x_6 = (132)\}$ has bikei matrix

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 6 & 5 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 4 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 4 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 3 & 2 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 6 & 6 & 5 & 1 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
6 & 5 & 5 & 5 & 1 & 6 & 6 & 6 & 6 & 6 & 6 & 6
\end{pmatrix}.
\]

**Definition 2.** A bikei $X$ is medial or abelian if it satisfies

\begin{align*}
(x \ast y) \ast (z \ast w) &= (x \ast z) \ast (y \ast w) \quad (m.i) \\
(x \ast y) \bar{\ast} (z \ast w) &= (x \bar{\ast} z) \ast (y \ast w) \quad (m.ii) \\
(x \bar{\ast} y) \bar{\ast} (z \ast w) &= (x \bar{\ast} z) \bar{\ast} (y \bar{\ast} w) \quad (m.iii)
\end{align*}

for all $x, y, z, w \in X$.

**Example 5.** Alexander bikei are always medial, since we have

\begin{align*}
(x \ast y) \ast (z \ast w) &= t(tx + (s - t)y + (s - y)(tz + (s - t)w) \\
&= t^2x + t(s - t)(y + z) + (s - t)^2w \\
&= t(tx + (s - t)z) + (s - y)(ty + (s - t)w) \\
&= (x \ast z) \ast (y \ast w)
\end{align*}

so (m.i) is satisfied,

\begin{align*}
(x \ast y) \bar{\ast} (z \ast w) &= s(tx + (s - t)y) \\
&= t(sx) + (s - t)(sy) \\
&= (x \bar{\ast} z) \ast (y \bar{\ast} w)
\end{align*}

so (m.ii) is satisfied, and

\begin{align*}
(x \bar{\ast} y) \bar{\ast} (z \ast w) &= s^2x = (x \bar{\ast} z) \bar{\ast} (y \bar{\ast} w)
\end{align*}

so (m.iii) is satisfied.
Example 6. The core bikei of a group $G$ need not be medial if $G$ is non-abelian, since (m.i) requires
\[(x \star y) \star (z \star w) = wz^{-1}wy^{-1}xy^{-1}wz^{-1}w = wy^{-1}wz^{-1}wy^{-1}w = (x \star z) \star (y \star w).\]
Indeed, for $G = S_3$ consider $x = (12)$, $y = (13)$, $z = (23)$ and $w = 1$; then
\[(x \star y) \star (z \star w) = wz^{-1}wy^{-1}xy^{-1}w = (23)(13)(12)(13)(23) = (23)\]
while
\[(x \star z) \star (y \star w) = wy^{-1}wz^{-1}xz^{-1}w = (13)(23)(12)(13)(23) = (23) \neq (13).\]

Let $D$ be an unoriented virtual link diagram representing a virtual link $L$. The fundamental bikei of $L$ is the set of equivalence classes of bikei words in a set $X$ of generators corresponding one to one with the semiarcs of $D$, i.e., the portions of $D$ between crossing points, modulo the equivalence relation generated by the bikei axioms and the crossing relations of $D$. More precisely, let $X = \{x_1, \ldots, x_n\}$ be a set of generators, one for each semiarcs in $D$. Then define a set $W(X)$ recursively by the rules

(i) $x \in X$ implies $x \in W(X)$ and
(ii) $x, y \in W(X)$ implies $x \star y, x \star y \in W(X)$.

Note that since the operations $\star, \circ$ are not associative, we need parentheses, e.g., $x_1 \in W(X)$ and $x_2 \star x_3 \in W(X)$ implies $x_1 \star (x_2 \star x_3) \in W(X)$ etc. The free bikei on $X$ is the set of equivalence classes of elements of $W(X)$ under the equivalence relation generated by the bikei axioms; that is, we have $x \star x \sim x \star x$, $(x \star y) \star y \sim x$, etc.

Then the fundamental bikei $BK(L)$ is the set of equivalence classes of bikei words under the stronger equivalence relation generated by both the bikei axioms and the crossing relations, namely

\[
\begin{array}{c}
\text{We can specify such a bikei with a presentation } \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle \text{ listing generators } x_1, \ldots, x_n \text{ and relations } r_1, \ldots, r_m. \text{ These relations are the equivalences generating the equivalence relation other than the bikei axioms, which we don’t list explicitly. Two such presentations describe isomorphic bikei if and only if they are related by Tietze moves, which come in two types:}
\end{array}
\]

(i) adding or removing a generator $x$ and relation of the form $x \sim W$ where $W$ is a word in the other generators not involving $x$, and
(ii) adding or removing a consequence of the other relations.

Example 7. The pictured virtual trefoil has fundamental bikei with four generators $x_1, x_2, x_3, x_4$ and relations $x_2 = x_1 \star x_3$, $x_4 = x_3 \circ x_1$, $x_3 = x_1 \circ x_2$ and $x_4 = x_2 \star x_1$. 

\[
\begin{array}{c}
\text{Example 6. The core bikei of a group } G \text{ need not be medial if } G \text{ is non-abelian, since (m.i) requires}
\end{array}
\]
\[ B(K) = \langle x_1, x_2, x_3, x_4 \mid x_2 = x_1 \pm x_3, x_4 = x_3 \mp x_1, x_3 = x_1 \mp x_2, x_4 = x_2 \mp x_1 \rangle \]

Via Tietze moves, every bikei presentation can be put into short form, in which every relation has the form \( x_j = x_k \pm x_n \) or \( x_j = x_k \mp x_n \) where \( x_j, x_k \) and \( x_n \) are generators as opposed to longer words. Given a presentation in short form, we can write the presentation in matrix format with zeroes in positions without corresponding relations.

**Example 8.** Consider the bikei presented by \( \langle x_1, x_2 \mid (x_1 \mp x_2) \pm x_1 = x_2 \mp x_1 \rangle \). We can put this in short form by introducing new generators and corresponding relations. Via a Tietze I move,

\[ \langle x_1, x_2, x_3 \mid x_3 = x_1 \mp x_2, \quad x_3 \pm x_1 = x_2 \mp x_2 \rangle \]

which then becomes

\[ \langle x_1, x_2, x_3, x_4 \mid x_3 = x_1 \mp x_2, \quad x_3 \pm x_1 = x_4, \quad x_4 = x_2 \mp x_2 \rangle. \]

Then in matrix form, we have presentation

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

4 **Computation of the Fundamental Medial Bikei**

If a bikei \( X \) is finite, then its operation matrix is a short form presentation matrix. Hence, we can play the following game: start with a medial bikei presentation, put it in short form and write its matrix, then use the medial bikei axioms to fill in entries by adding relations which are consequences of the other relations, i.e. via Tietze II moves, in the hopes of completing the table. If the table cannot be completed, we can select a zero entry in the table, assign to it a new generator and relation (a Tietze I move), a new row and column in each matrix corresponding to the new generator, and repeat the process of trying to fill in entries.

During this process of filling in the matrices, we may encounter situations where we need to assign a number to an entry in the matrix which is already nonzero; if the two values for the position are equal, then there is nothing to do, but if they are distinct, then we have found that the two generators assigned to that position are equivalent. We can then systematically replace all instances of one generator with the other. Thus, we can loop over sets of generators, filling in entries using the medial bikei axioms, keeping a working list of which generators are equal, with the matrix growing or shrinking as new generators are consolidated or added. Since there is no guarantee that the presented object is finite, this process may not terminate, but if it does, we have a proof that the object is finite in the form of its complete operation tables. To address the issue of non-terminating searches, we include a parameter defining the maximal size of matrix and terminate the computation when the matrices reach this size.

In terms of pseudocode:

```plaintext
//inputting the medial bikei presentation MB and maximal size L into the function: Bikei(MB, L)
M = shortFormMatrix(MB) // convert the medial bikei to matrix form
I = 0
Do while (M still has entries that equal 0) and (size(M) < L)
  Do while (ChangedM != M)
    M = ChangedM
    sameGenerators = {} // keep track of equivalent generators
```
Do while (I <= size(M))
    ChangedM = firstBikeAxiom(M, I, sameGenerators)
    J = 0
    Do while (J <= size(M))
        ChangedM = secondBikeAxiom(ChangedM, I, J, sameGenerators)
        Z = 0
        Do while (Z <= size(M))
            ChangedM = thirdBikeAxiom(ChangedM, I, J, Z, sameGenerators)
            W = 0
            Do while (W <= size(M))
                ChangedM = medialBikeAxiom(ChangedM, I, J, Z, W, sameGenerators)
            End-Do
        End-Do
    End-Do
End-Do
If (size(sameGenerators) > 0)
    ChangedM = reduce(ChangedM, sameGenerators)  // merge equivalent entries
End-If
End-Do
M = AssignNewGenerators(M)  // assigns new generator to zero entries and expand M
End-Do
return M

Our implementation in Python is available at www.esotericka.org.

Example 9.

The unknot at first may seem to have infinite fundamental medial bikei since it has presentation $\langle x_1 | \rangle$ with a single generator and an empty list of relations. However, writing this in matrix form, we have

$$ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, $$

and applying our procedure, we add a new generator $x_2 = x_1 \ast x_1$:

$$ \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. $$

Then axiom (i) says $x \ast x = x \ast x$ and gives us

$$ \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. $$

and axioms (ii.iii) and (ii.iv), i.e., $(x \ast y) \ast y = x$ and $(x \ast y) \ast y = x$ give us

$$ \begin{bmatrix} 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}. $$
Then axiom (ii.i), $x \prescript{*}{y}{\bar{x}} = x \bar{y}$, says

$$1 \bar{2} = 1 \bar{(2 \bar{1})} = 1 \bar{1} = 2$$

and we have

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix};$$

similarly axiom (ii.ii), $x \bar{y} (y \bar{x}) = x \bar{y}$

$$1 \bar{2} = 1 \bar{(2 \bar{1})} = 1 \bar{1} = 2$$

and our matrix is

$$\begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Lastly, another application of axioms (ii.iii) and (ii.iv), i.e., $(x \bar{y}) \bar{y} = x$ and $(x \bar{y}) \bar{y} = x$, yields

$$\begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

which says the fundamental medial bikei (in fact, plain fundamental bikei since we’ve not used the medial condition) of the unknot is the two element bikei given by $\mathbb{Z}_2$ with $x \bar{y} = x \bar{y} = x + 1$. We note that this presentation can be read directly from the unknot diagram below:

![Unknot Diagram](attachment://unknot_diagram.png)

**Example 10.** Consider the diagram below of the unknot:

![Unknot Diagram](attachment://unknot_diagram.png)

We obtain the presentation matrix

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The axiom (i) now says that $x_3 = x_2$ since we have $x_1 \bar{x_1} = x_3$ and $x_1 \bar{x_1} = x_2$; then we must replace every instance of $3$ with $2$ and replace each instance of $j > 3$ (as row number, column number, and entry)
with \( j - 1 \). Note that in general this entails merging columns 2 and 3 and rows 2 and 3 in both matrices, which can trigger further merging. In this case we have
\[
\begin{bmatrix}
2 & 0 & 0 & 2 & 0 & 0 \\
0 & 3 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Then axioms (ii.iii) and (ii.iv) give us
\[
\begin{bmatrix}
2 & 0 & 0 & 2 & 0 & 0 \\
1 & 3 & 0 & 1 & 3 & 0 \\
0 & 2 & 0 & 0 & 2 & 0
\end{bmatrix}.
\]

Then as before, (ii.i) says
\[
1 \ast 2 = 1 \ast (2 \ast 1) = 1 \ast 1 = 2
\]
so we have
\[
\begin{bmatrix}
2 & 2 & 0 & 2 & 0 & 0 \\
1 & 3 & 0 & 1 & 3 & 0 \\
0 & 2 & 0 & 0 & 2 & 0
\end{bmatrix}
\]
but then (ii.iii) says we must have \( x_2 = x_3 \); reducing, we have
\[
\begin{bmatrix}
2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]
as before.

**Example 11.** The virtual trefoil knot in example 7 has presentation matrix
\[
\begin{bmatrix}
0 & 0 & 2 & 0 & 0 & 3 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

We can then fill in entries using the medial bikei axioms. For instance, bikei axiom (ii.iv) says \((x \ast y) \ast y = x\), so since \(x_2 \ast x_1 = x_4\) this says \(x_4 \ast x_1 = x_2\) and our matrix becomes
\[
\begin{bmatrix}
0 & 0 & 2 & 0 & 0 & 3 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Repeating this with the other bikei axioms, we obtain the matrix
\[
\begin{bmatrix}
2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 4 & 1 & 4 & 1 \\
2 & 2 & 2 & 2 & 3 & 3 & 3 & 3
\end{bmatrix}
\]
together with the requirement that \(x_1 = x_4\) and \(x_2 = x_3\), which collapses to
\[
\begin{bmatrix}
2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1
\end{bmatrix}.
\]

Hence, the fundamental medial bikei of the virtual trefoil is the same as that of the unknot, and the invariant in this case does not detect the nontriviality of the virtual trefoil.
Example 12. The virtual knot \textit{4.71} below

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

which after application of our algorithm becomes

\[
\begin{bmatrix}
5 & 5 & 5 & 5 & 5 & 5 & 5 & 4 & 2 & 2 & 5 & 4 \\
6 & 6 & 6 & 6 & 6 & 6 & 3 & 6 & 1 & 1 & 3 & 6 \\
4 & 4 & 4 & 4 & 4 & 4 & 2 & 5 & 4 & 4 & 2 & 5 \\
3 & 3 & 3 & 3 & 3 & 3 & 6 & 1 & 3 & 3 & 6 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 6 & 6 & 1 & 3 \\
2 & 2 & 2 & 2 & 2 & 2 & 4 & 2 & 5 & 5 & 4 & 2 \\
\end{bmatrix}
\]

This bikei is isomorphic to the Cartesian product of the unknot’s bikei with the bikei obtained from the three-element Takasaki kei \((\mathbb{Z}_3 \text{ with } x \overline{y} = 2x + 2y \text{ and } x \overline{y} = x)\) with

\[(x_1, x_2) \overline{y} (y_1, y_2) = (x_1 + 1, x_2) \quad \text{and} \quad (x_1, x_2) \overline{y} (y_1, y_2) = (x_1 + 1, 2x_2 + 2y_2)\]

after applying a vertical mirror image, i.e., switching \(\overline{y}\) and \(\overline{y} \). Then since this is not the same bikei we obtained for the unknot, this example shows the fundamental medial bikei distinguishing this virtual knot from the unknot.

Example 13. We computed the fundamental medial bikei for all virtual knots on the virtual knot table in [2]. Of the 116 virtual knots on the list, most have the same medial fundamental bikei as the unknot computed above, but twenty-five do not; these break down into three isomorphism classes. According to our python computations.

- The virtual knots numbered 4.61 through 4.77 at [2], up to vertical mirror image, all have fundamental medial bikei isomorphic to the six-element bikei listed in Example 12.

- The virtual knots 3.6, 3.7, 4.98 and 4.99 all have fundamental medial bikei with 18 elements isomorphic to the fundamental medial bikei of the trefoil knot 3.6, and

- The virtual knots 4.105, 4.106, 4.107 and 4.108 all have fundamental medial bikei with 50 elements isomorphic to the fundamental medial bikei of the figure eight knot 4.108.
Example 14. For examples with multicomponent virtual links, we do not have a convenient table analogous to the virtual knot table in [2], but we note that our algorithm gives the eight-element medial bikei
\[\begin{array}{cccccccc}
2 & 2 & 1 & 1 & 2 & 1 & 2 & 5 \\
1 & 1 & 2 & 2 & 1 & 2 & 1 & 7 \\
6 & 6 & 4 & 4 & 6 & 4 & 3 & 3 \\
8 & 8 & 3 & 8 & 3 & 8 & 4 & 3 \\
7 & 7 & 5 & 5 & 7 & 5 & 7 & 1 \\
3 & 3 & 8 & 8 & 3 & 8 & 6 & 8 \\
5 & 5 & 7 & 7 & 5 & 5 & 7 & 5 \\
4 & 4 & 6 & 6 & 4 & 6 & 8 & 6 \\
\end{array}\]
for the virtual Hopf link, while the unlink of two components has fundamental medial bikei given by the free medial bikei on two generators; this is infinite since it has the free Alexander kei on two generators \(\Lambda[x, y]\) where \(\Lambda = \mathbb{Z}[t]/(t^2 - 1)\) as a quotient.

5 Questions

We conclude with a few questions for future research.

On the computational side, we have noticed that when choosing a zero entry to fill in with a new generator, the choice of which zero to fill in matters greatly for the speed of completion of the procedure. We implemented a system in which each zero receives a score based on the number of other entries which would be filled in as a result of filling in the zero in question; this seems to provide better results than simply selecting the zero based on position in the dictionary ordering, despite the associated computational overhead. However, in cases where the bikei is infinite, this may slow down the procedure from reaching the exit size for the matrix. What is the optimal strategy for zero selection?

On the mathematical side, the main question is what happens when we remove the medial condition or replace it with a different condition; which virtual links have finite fundamental bikei?

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DEPARTMENT OF MATHEMATICAL SCIENCES
CLAREMONT MCKENNA COLLEGE
850 COLUMBIA AVE.
CLAREMONT, CA 91711