Chern Insulators for Electromagnetic Waves in Electrical Circuit Networks

by

Rafael Haenel

B.Sc., Technical University of Berlin, 2017

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

Master of Science

in

THE FACULTY OF GRADUATE AND POSTDOCTORAL STUDIES
(Physics)

The University of British Columbia
(Vancouver)

August 2019

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The following individuals certify that they have read, and recommend to the Faculty of Graduate and Postdoctoral Studies for acceptance, the thesis entitled:

**Chern Insulators for Electromagnetic Waves in Electrical Circuit Networks**

submitted by **Rafael Haenel** in partial fulfillment of the requirements for the degree of **Master of Science** in **Physics**.

**Examining Committee:**

Marcel Franz, Physics
*Supervisor*

Douglas Bonn, Physics
*Supervisory Committee Member*
Abstract

Periodic networks composed of capacitors and inductors have been demonstrated to possess topological properties with respect to incident electromagnetic waves. In this thesis, we develop an analogy between the mathematical description of waves propagating in such networks and models of Majorana fermions hopping on a lattice. Using this analogy we propose simple electrical network architectures that realize Chern insulating phases for electromagnetic waves. Such Chern insulating networks have a bulk gap for a range of signal frequencies that is easily tunable and exhibit topologically protected chiral edge modes that traverse the gap and are robust to perturbations. The requisite time reversal symmetry breaking is achieved by including a class of weakly dissipative Hall resistor elements whose physical implementation we describe in detail.
Lay Summary

An electrical circuit comprised of an inductor and capacitor exhibits a single resonance frequency. When driven at that frequency, the circuit can generate higher voltages than fed into it. We propose circuits of a periodically repeated two-dimensional pattern. Instead of a single resonance, these circuits possess a number of resonance frequencies that is proportional to the area of the two-dimensional network. We uncover certain topological properties of these resonance spectra. For a range of frequencies, the existence of resonances relies on the presence of edges in the circuit network. Here, the voltage response to a resonant source is large only at the boundaries. Pulses at these frequencies travel in one direction along the boundary only. Distortions of the boundary and component tolerances do not yield qualitatively different results. These findings are in analogy to the physics of so called Chern Insulators in the context of condensed matter physics.
Preface

This thesis is based on the publication

**Chern insulators for electromagnetic waves in electrical circuit networks.**

Rafael Haenel, Timothy Branch, and Marcel Franz, *Physical Review B* **99** 235110 (2019).

The publication resulted from a combined effort of the author and Prof. Marcel Franz. Experimental efforts, that are not part of this thesis, were undertaken by Timothy Branch.

Chapters 1, 3, 4, 5 are taken from above publication and were amended with four additional figures (Figs. 3.4, 3.6, 4.2, 4.4) and slightly more detailed formulations by the author. Chapter 2 has been drafted by the author to supplement the introduction with a broader context.

All figures were prepared by the author and all underlying data was computed by the author.
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Glossary

CAZ  Cartan-Altland-Zirnbauer
EM   electromagnetic
RLC  Resistor-Inductor-Capacitor
SPT  Symmetry Protected Topological
Acknowledgments

I would like to thank Prof. Marcel Franz for his fabulous guidance, advice, and steady support during the two years of my degree.

Further I would like to thank the members of my research group Étienne Lantagne-Hurtubise, Chengshu Li, Oğhuzan Can, Tarun Tumurru, and Stephan Plugge for many inspiring discussions and brilliant help.

Research described in this thesis was supported by NSERC and by CIFAR. Software that facilitated this research was provided by CMC Microsystems.
Chapter 1

Introduction

Topological states of matter in electronic systems exhibit topologically non-trivial bulk band structures accompanied by protected edge or surface modes [3–5]. More generally, the insights gained from the study of electrons in crystalline solids with non-trivial topology can be applied to any physical system whose degrees of freedom are governed by a wave equation. If bulk solutions of the wave equation do not exist in some range of frequencies, the system may be viewed as insulating for these frequencies and may in addition possess topologically protected propagating modes at its boundary. This realization has led to a theoretical study and physical implementation of a wide variety of periodic systems in which topological properties analogous to electronic topological insulators, superconductors, and semimetals are manifest. Most prominent examples of these efforts include photonic [6, 7], acoustic [8, 9], mechanical [10–12], polaritonic [13], and electrical systems [14–17].

In this thesis, we focus on the latter class of topological systems, more specifically, periodic networks comprised of inductors, capacitors, and resistors. These structures, also referred to as topoelectrical circuits [15], have been demonstrated to possess topological properties with respect to the incident electromagnetic (EM) wave signals. In close analogy to electronic tight-binding models, various circuit models realizing classical analogs of quantum spin Hall states [14, 18], Dirac and Weyl semimetals [15, 19], and higher order topological insulators [20–22] have been proposed and some of them have been experimentally characterized.
Conspicuously absent from this list has been the Chern insulator – the analog of the most basic electronic topological phase, the quantum Hall insulator in two dimensions [23]. The reason is simple: networks composed of capacitors and inductors are governed by Maxwell equations which are fundamentally invariant under the time reversal operation $\mathcal{T}$. A Chern insulator, on the other hand, requires broken $\mathcal{T}$ symmetry.

We note that ordinary resistors in a Resistor-Inductor-Capacitor (RLC) network cause dissipation and therefore break $\mathcal{T}$. This, however, hinders the comparison to isolated quantum systems where dynamics are unitary. On the other hand, dissipative networks can provide useful examples of systems studied in the rapidly advancing field of non-Hermitian quantum mechanics [24–27].

In the present work, we circumvent this problem by employing a class of weakly dissipative Hall resistors. These are linear circuit elements whose voltage response to a longitudinal current is predominantly transverse. An ideal Hall resistor introduces strong $\mathcal{T}$ breaking into the circuit without significant dissipation and thus enables construction of the Chern insulator.

The class of EM Chern insulators we introduce here has a bulk gap for EM waves in a range of frequencies but exhibits chiral propagating edge modes that traverse the gap. The edge modes are topologically protected by a non-zero Chern number defined by the bulk band structure of the network and are robust against any imperfections in the network that do not close the gap.

This thesis is organized as follows: In Ch. 2 we give a brief overview of the classification of Symmetry Protected Topological (SPT) matter and discuss the applicability of these concepts outside of the field of condensed matter physics.

In Ch. 3 we introduce three specific circuit network architectures that realize a Chern insulating phase. We do this by taking advantage of a novel mapping that connects the dynamics of a certain class of periodic RLC networks to Hermitian Bloch Hamiltonians describing Majorana fermions in a crystal lattice. Such Hamiltonians are well known to possess Chern-insulating phases. While the possibility of non-trivial topological structure of Kirchhoff’s equations has been previously recognized, it is usually discussed in terms of admittance bands or mapped onto non-Hermitian eigenvalue problems [14, 15]. The description developed in this work offers a more direct analogy to crystalline solids and thus a more transparent
physical interpretation in terms of well understood topological band theory [3–5]. The key physical element required in the realization of our EM Chern insulator architecture is the Hall resistor. We discuss its physical implementation in Ch. 4.
Chapter 2

Symmetry protected topological matter

In this chapter, we review the concept of classification of SPT phases based on homotopy classification of Dirac mass gaps in the presence of global discrete symmetries. I have tried to convey the idea behind this topic in the simplest manner. None of the ideas in this chapter are new and excellent reviews can be found in [1, 28, 29].

2.1 Tight-binding models

A general condensed matter system is modelled by the second-quantized Hamiltonian

\[ H = \sum_{ij} \psi_i^\dagger h_{ij} \psi_j + \sum_{ijkl} V_{ijkl} \psi_i^\dagger \psi_j^\dagger \psi_k \psi_l \]  

(2.1)

and the canonical commutation relations of fermionic creation and annihilation operators

\[ \left\{ \psi_i, \psi_j^\dagger \right\} = \delta_{ij}, \quad \left\{ \psi_i, \psi_j \right\} = 0. \]  

(2.2)
The dynamics of the system is given by the Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle \]  

(2.3)

where states $|\psi\rangle$ are built by application of creation operators $\psi_i^*$ on the ground-state $|0\rangle$ and can be in any superposition due to linearity of (2.3).

The quartic term proportional to $V_{ijkl}$ in Eq. (2.1) describes two-fermion interactions. The solution of interacting many-body systems defines the forefront of present-day condensed matter research. For weak $V_{ijkl}$, solutions are usually approximated analytically using mean-field techniques, (diagrammatic) perturbation theory, or field theoretical methods such as the Renormalization Group [30]. Strong interactions may be solved by Bosonization [31] in one dimension or in arbitrary dimensions using numerical methods, such as Exact Diagonalization, Density Matrix Renormalization Group [32], and Quantum Monte Carlo computations. However, a plethora of interacting phenomena remain elusive to this date.

For many laboratory or real-world materials it is an excellent approximation to neglect the $V_{ijkl}$-terms altogether. We then arrive at the tight-binding model

\[ H = \sum_{ij} \psi_i^* h_{ij} \psi_j. \]  

(2.4)

Given a basis of single-particle eigenstate $\psi_k^* |0\rangle$ corresponding to eigenenergies $\{\epsilon_k\}$, any $N$-particle eigenstate can be constructed as

\[ \psi_{k_1}^* \psi_{k_2}^* \cdots \psi_{k_N}^* |0\rangle, \]  

(2.5)

where the corresponding eigenenergy is simply given by the sum of single-particle energies $E_{k_1 \cdots k_N} = \sum_{i=1}^N \epsilon_{k_i}$.

Applying an arbitrary single-particle state $\sum_n \phi_n \psi_n^* |0\rangle$ on Eq. (2.3), we get

\[ i\hbar \sum_n \phi_n \psi_n^* |0\rangle = \sum_{ijn} h_{ij} \psi_i^* \psi_j \phi_n \psi_n^* |0\rangle = \sum_{ij} h_{ij} \phi_j \psi_i^* |0\rangle \]  

(2.6)
from which we extract the condition

\[ i \frac{\partial \phi_i}{\partial t} = \sum_j h_{ij} \phi_j. \]  

(2.7)

This is the Schrödinger equation for a single particle in its first-quantized form. The full solution of the energy spectrum therefore reduces to computation of the eigenvalues \( \epsilon_k \) of the single-particle Hamiltonian \( h_{ij} \), which can then be added in all possible combinations to yield the many-body spectrum.

### 2.1.1 Majorana tight-binding models

For future reference to our circuit models, we will introduce a special type of tight-binding model that describes Majorana Fermions hopping on a lattice. Majorana fermions are particles that are their own anti-particles. In second quantized notation this is expressed by the self-adjointness of the corresponding operators

\[ \gamma^\dagger_j = \gamma_j. \]  

(2.8)

The \( \gamma \) satisfy the commutation relations

\[ \{ \gamma_i, \gamma_j \} = 2 \delta_{ij}. \]  

(2.9)

A general model for non-interacting Majorana fermions on a lattice is defined similarly to Eq. (2.4) by a Hamiltonian of the form

\[ H = \sum_{ij} h_{ij} \gamma_i \gamma_j. \]  

(2.10)

Hermiticity of \( H \) together with relations (2.8) and (2.9) implies that \( h_{ij} \) is purely imaginary and antisymmetric. We henceforth write it as \( h_{ij} = it_{ij} \), where \( t_{ij} \) is a real antisymmetric matrix.

In complete analogy to Eq. (2.7), time evolution of an arbitrary state \( \sum_i \phi_i \gamma_i |0\rangle \) is governed by the corresponding Schrödinger equation

\[ i \frac{\partial \phi_i(t)}{\partial t} = \sum_{ij} h_{ij} \phi_j(t). \]  

(2.11)
Using the property $h_{ij} = it_{ij}$ we see that Eq. (2.11) becomes a purely real-valued wave equation which, therefore, admits purely real solutions $\phi(t)$. Real-valued, dispersing solutions of the Schrödinger equation are referred to as Majorana fermions. They should, however, not be confused with Majorana zero modes, which are localized real-valued solutions, whose corresponding quasi-particles obey non-Abelian exchange statistics.

### 2.2 Classification of symmetry protected topological states of matter

Gapped symmetry protected topological (SPT) states of matter are gapped bulk states with linearly dispersing gapless edge states that are immune to symmetry preserving perturbations as long as they do not close the bulk gap [3, 5]. Remarkably, these states of matter are characterized by a topological invariant that is a bulk property only - a fact that is commonly referred to as the 'bulk-boundary correspondence'.

In this section, we will outline the fundamental concept behind SPT phases and sketch the classification of these phases in the so called 10-fold way [33]. We do this by restricting ourselves from the set of arbitrary Hermitian Hamiltonians to the set of Dirac Hamiltonians, to be defined below. Auspiciously, this restriction does not affect the classification outcomes as gapped SPT phases can always be approximated by a Dirac model at low energies.

#### 2.2.1 Dirac model method

A general Dirac Hamiltonian in $d$ dimensions that describes an insulator can be expressed as

$$H = m\gamma_0 + \sum_{i=1}^{d} k_i \gamma_i,$$  \hspace{1cm} (2.12)

where the $k_i$ are momentum variables. The $\gamma_i$, $i = 1, ..., \tilde{d} \geq d$, are elements of the Clifford algebra $\text{Cl}_{0,\tilde{d}}$ that satisfy the anticommutation relation $\{\gamma_i, \gamma_j\} = 2\delta_{ij}\mathbb{1}$. The spectrum of (2.12) is given by the gapped bands $E = \pm\sqrt{\sum k_i^2 + m^2}$. Each of
these bands is $N/2$-fold degenerate depending on the dimension $N = 2^{(d-1)/2}$ of the matrix representation of the $\gamma_i$. We see that the point $m = 0$ marks a topological phase transition where the gap closes.

We now want to examine the boundary-physics of this minimal model. Without loss of generality we introduce a topological phase boundary perpendicular to the $x_j$-direction by requiring $m = m(x_j)$ to be a monotonic function with

$$m(x_k) = \begin{cases} 
-m_0 & \text{for } x_j \to -\infty \\
0 & \text{for } x_j = 0 \\
m_0 & \text{for } x_j \to \infty.
\end{cases} \quad (2.13)$$

Since this choice of mass-term $m(x_j)$ breaks translational invariance along $x_j$, we replace $k \to -i \partial_j$ and arrive at the field theory

$$H = \gamma_0 (m(x_j) - i \partial_j \gamma_0 y_j) + \sum_{i \neq j} k_i \gamma_i. \quad (2.14)$$

For $k_i = 0$ we find a zero-mode

$$|\psi_0 \rangle = e^{-\int_{x_j}^0 m(x_j') dx_j'} |-\rangle \quad (2.15)$$

that is localized at the phase-boundary. Here, $i \gamma_0 y_j |\rangle = -|\rangle$. Furthermore $i \gamma_0 y_j$ commutes with $H_k = \sum_{i \neq j} k_i \gamma_i$. This means we can choose $|\rangle$ to be an eigenvector of $H_k$ so that

$$H |\psi_0 \rangle = [\gamma_0 (m(x_j) - i \partial_j \gamma_0 y_j) + H_k] |\psi_0 \rangle = H_k |\psi_0 \rangle = \pm \sqrt{\sum_{i \neq j} k_i^2} |\psi_0 \rangle. \quad (2.16)$$

We have found an edge-state that linearly traverses the bulk gap. Note that the existence of this gapless state relies on the absence of a second mass term $m'\gamma'_0$ with $\{\gamma'_0, H\} = 0$. In the presence of such a term, the edge spectrum is gapped out:

$$E = \pm \sqrt{\sum_{i \neq j} k_i^2 + m'^2}. \quad (2.17)$$

In the minimal Dirac-model approach the concept of topological protection there-
fore boils down to the question if such a second mass-term can be found \cite{1,34}. In practice, it is sensible to first restrict oneself to a certain symmetry class, i.e. by requiring that all terms in the Hamiltonian satisfy a specified set of symmetries. The topological classification will naturally depend on the symmetry class.

In the following, we will introduce and motivate the set of symmetries underlying the classification in the so-called tenfold way and discuss the question of second mass terms in the mathematical framework of homotopy classes.

### 2.2.2 Symmetries

We consider tight-binding models introduced above,

\[
H = \sum_{ij} \psi_i^\dagger h_{ij} \psi_j,
\]

(2.18)

where the creation- and annihilations operators satisfy the anti-commutation relation \( \{ \psi_i, \psi_j^\dagger \} = \delta_{ij} \). The symmetries underlying the Cartan-Altland-Zirnbauer \( \text{(CAZ)} \) classification scheme are called time-reversal symmetry \( T \), particle-hole symmetry \( C \), and chiral symmetry \( S = TC \) which is the product of the former two. Chiral symmetry \( S \) is sometimes also referred to as sublattice symmetry in the literature. We will define the corresponding symmetry transformations as a similarity transformation of the second quantized operators \( \psi_i \). The symmetry operations then read as follows:

\[
T : \quad \psi_i \rightarrow T \psi_i T^{-1} = (U_T)^i_j \psi_j \quad (2.19)
\]

\[
C : \quad \psi_i \rightarrow C \psi_i C^{-1} = (U_C)^i_j \psi_j^\dagger \quad (2.20)
\]

\[
S = TC : \quad \psi_i \rightarrow S \psi_i S^{-1} = (U_C U_T)^i_j \psi_j^\dagger. \quad (2.21)
\]

Additionally, we impose \( T \) to be antiunitary, \( TiT = -i \). Particle-hole symmetry \( C \) is a unitary symmetry, hence the product \( S = TC \) is antiunitary.

A system \( H \) respects the symmetry \( P \) if \( [H, P] = 0 \). It is often useful to rewrite this as a condition on the first-quantized part \( h_{ij} \) of \( H \) only. We will do this explicitly for the case of time-reversal symmetry \( T \). From
\[ \mathcal{T} \mathcal{H} \mathcal{T}^{-1} = \sum_{ij} \mathcal{T} \psi_i^\dagger h_{ij} \psi_j \mathcal{T}^{-1} = \sum_{ij} \mathcal{T} \psi_i^\dagger (h^*)_{ij} \mathcal{T}^{-1} \psi_j \mathcal{T}^{-1} \]

\[ \mathcal{T} \mathcal{H} \mathcal{T}^{-1} = \sum_{ij} \psi_i^\dagger (U_{ij}^\dagger)^{ij} (h^*)_{ij} \psi_j \mathcal{T}^{-1} \]

\[ \mathcal{T} \mathcal{H} \mathcal{T}^{-1} \mathcal{T} = \sum_{ij} \psi_i^\dagger h_{ij} \psi_j \]

we find the condition

\[ U_{ij}^\dagger h^* U_{ij} = h. \] (2.24)

Similarly, relations for \( \mathcal{C} \) and \( \mathcal{S} \) can be derived. In summary, we have

\[ \mathcal{T} : \quad U_{ij}^\dagger h^* U_{ij} = h \] (2.25)

\[ \mathcal{C} : \quad U_{ij}^\dagger h^* U_{ij} = -h \] (2.26)

\[ \mathcal{S} : \quad U_{ij}^\dagger h U_{ij} = -h. \] (2.27)

If the Hamiltonian \( h \) can be expressed in the momentum basis \( h(k) \), Eqs. (2.25-2.27) must be modified as in

\[ \mathcal{T} : \quad U_{ij}^\dagger h^* (-k) U_{ij} = h(k) \] (2.28)

\[ \mathcal{C} : \quad U_{ij}^\dagger h^* (-k) U_{ij} = -h(k) \] (2.29)

\[ \mathcal{S} : \quad U_{ij}^\dagger h(k) U_{ij} = -h(k). \] (2.30)

where the sign in \( -k \) is a consequence of the anti-unitarity of \( \mathcal{T} \) and \( \mathcal{C} \) in the first-quantized representation.

The CAZ classification scheme now distinguishes between the symmetry classes \( s \) listed in Tab. 2.1. ‘0’ indicates the absence of the corresponding symmetry, whereas ‘1’ indicates its presence. For \( \mathcal{T} \) and \( \mathcal{C} \) one additionally distinguishes between the cases \( U_{ij}^\dagger h_{ij} U_{ij}^* \) = \( \pm 1 \). These are labeled as ‘±’ in Tab. 2.1.

We see that there are 10 symmetry classes, therefore this classification scheme is sometimes referred to as the ‘10-fold’ way. \( \mathcal{T} \) and \( \mathcal{C} \) alone give rise to 9 classes, since either one can be 0, ±1. The tenth class results from the ambiguity when both \( \mathcal{T} \) and \( \mathcal{C} \) are absent. Here, the product \( \mathcal{I} = \mathcal{T} \mathcal{C} \) is no longer well-defined and
\( \mathcal{I} \) can be either absent or present.

The CAZ symmetry classification is only well-defined in the absence of any conserved quantities. Consider, for example, the case where a system has two distinct time-reversal symmetries \( \mathcal{T} \) and \( \mathcal{T}' \). The combined symmetry \( \mathcal{T} \mathcal{T}' \) acts as

\[
(\mathcal{U}^* \mathcal{T} \mathcal{U})^\dagger h (\mathcal{U}^* \mathcal{T} \mathcal{U}) = h
\]

and we notice that the matrix \( \mathcal{U}^* \mathcal{T} \mathcal{U} \) defines a conserved quantity. This means \( h \) can be block-diagonalized, with blocks corresponding to the eigenvalues of \( \mathcal{U}^* \mathcal{T} \mathcal{U} \). The CAZ scheme should then be applied to each block separately.

For the CAZ classification, \( \mathcal{T}, \mathcal{C}, \) and \( \mathcal{S} \) have been selected as the underlying set of symmetries. In principal, an arbitrary set of symmetries can be chosen. In fact, Hamiltonians have been classified with respect to various spatial symmorphic and non-symmorphic symmetries \([35, 36]\). Looking back at Eqs. (2.25-2.27) we can, however, appreciate the motivation behind the particular choice of \( \mathcal{T}, \mathcal{C}, \) and \( \mathcal{S} \). They are the three global symmetries that give rise to (anti-)commutation relations

\[
\{\mathcal{U}_\mathcal{T}, h\} = 0 \\
[\mathcal{U}_\mathcal{T} K, h] = 0 \\
\{\mathcal{U}_\mathcal{C} K, h\} = 0,
\]

where \( K \) denotes the complex conjugation operation. Eqs. (2.32-2.34) are the simplest (anti-)commutation relations that one could write down apart from the usual commutation relation \([U, h] = 0\) which trivially affects the classification scheme as discussed above.
2.2.3 Classification of Dirac mass gaps

The presence of $T$, $C$, and $S$ impose conditions on the $\gamma_i$ of the Hamiltonian 2.12 as can be seen from Eqs. (2.28-2.30). For the mass term they are

\[
T : \quad U_T \gamma_0 U_T^\dagger = \gamma_0 \quad (2.35)
\]

\[
C : \quad U_C \gamma_0 U_C^\dagger = -\gamma_0 \quad (2.36)
\]

\[
S : \quad U_S \gamma_0 U_S^\dagger = -\gamma_0 . \quad (2.37)
\]

For the $\gamma$-matrices of the kinetic terms $k_i \gamma_i$, $i \geq 1$, the signs must be flipped for $T$ and $C$ since these symmetries are anti-unitary in the first-quantized representation. This yields

\[
T : \quad U_T \gamma_i U_T^\dagger = -\gamma_0 \quad (2.38)
\]

\[
C : \quad U_C \gamma_i U_C^\dagger = \gamma_0 \quad (2.39)
\]

\[
S : \quad U_S \gamma_i U_S^\dagger = -\gamma_0, \quad i \geq 1. \quad (2.40)
\]

Equations (2.38-2.40) and the requirement \{ $\gamma_0, \gamma_i$ \} = 0 for $i = 1, \ldots, d$ restrict the mass-term $m \gamma_0$ to some parameter space $\mathcal{R}_{s,d}$. Consider two real-space regions $A, B$ separated by a domain wall. We pick two points $r_A \in A, r_B \in B$ and examine the mass terms $m \gamma_0(r_A), m \gamma_0(r_B) \in \mathcal{R}_{s,d}$ at these points. The two domains $A, B$ are topologically equivalent if we can smoothly deform the two masses into each other, i.e. if there exists a continuous path

\[
m \gamma(t) \in \mathcal{R}_{s,d} \quad \forall t \in [0,1] \quad (2.41)
\]

with $m \gamma(0) = m \gamma_0(r_A)$ and $m \gamma(1) = m \gamma_0(r_B)$. If the masses are not path connected, the two domains are topologically distinct. The number of distinct topological phases is therefore equivalent to the number of path connected regions in $\mathcal{R}_{s,d}$. Mathematically, this is expressed by the 0th homotopy of the parameter space $\pi_0 (\mathcal{R}_{s,d})$ [1].

The results of the computation of homotopy classes for all ten symmetry classes and for dimensions $d = 0$ to 3 are listed in Tab. 2.1. Here, 0 indicates that the parameter space $\mathcal{R}_{s,d}$ is path connected, i.e. only the topologically trivial phase
exists. $\mathbb{Z}_2$-classification implies the existence of a single topological phase next to the trivial one, i.e. $\mathcal{R}_{s,d}$ has two separate path connected regions. Phase boundaries necessarily exhibit gapless modes as discussed in Sec. 2.2.1. $\mathbb{Z}_2$ phases allow for a single pair of such protected edge modes. $\mathbb{Z}$ and $2\mathbb{Z}$ have an infinite number of topologically distinct phases and integer or even-integer number of edge modes may be present, respectively.

### Table 2.1: CAZ classification of symmetry protected topological matter [1].

| Class s | $\mathcal{F}$ | $\mathcal{P}$ | $\mathcal{I}$ | $d = 0$ | $1$ | $2$ | $3$ |
|---------|---------------|---------------|---------------|--------|-----|-----|-----|
| A       | 0             | 0             | 0             | $\mathbb{Z}$ | 0   | $\mathbb{Z}$ | 0   |
| AIII    | 0             | 0             | 1             | 0      | $\mathbb{Z}$ | 0   | $\mathbb{Z}$ | 0   |
| AI      | +             | 0             | 0             | $\mathbb{Z}$ | 0   | 0   | 0   |
| BDI     | +             | +             | 1             | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0   | 0   |
| D       | 0             | +             | 0             | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0   |
| DIII    | -             | +             | 1             | 0      | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| AI      | -             | 0             | 0             | $2\mathbb{Z}$ | 0   | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| CII     | -             | -             | 1             | 0      | $2\mathbb{Z}$ | 0   | $\mathbb{Z}_2$ |
| C       | 0             | -             | 0             | 0      | 0   | $2\mathbb{Z}$ | 0   |
| CI      | +             | -             | 1             | 0      | 0   | 0   | $2\mathbb{Z}$ |

2.2.4 Mass classification for trivial insulators and Chern insulators

The homotopy classification of Dirac mass gaps leading to Tab. 2.1 is a rather abstract concept. To make the topological classification more explicit, we will come back to the end of Sec. 2.2.1 where we concluded that the edge state spectrum of a gapped Dirac Hamiltonian is stable if no second mass terms is allowed by symmetry. This agrees with the homotopy classification in the following way. Let us set the mass at point $\mathbf{r}_A$ in domain $A$ to $m\gamma_0(\mathbf{r}_A) = m\gamma_0$ and at $\mathbf{r}_B$ in domain $B$ to $m\gamma_0(\mathbf{r}_B) = -m\gamma_0$. If no second mass term exist, the only way to deform $m\gamma_0$ into $-m\gamma_0$ is by slowly changing $m$ to $-m$. At some point, however, $m$ must vanish. But since $0 \notin \mathcal{R}_{s,d}$ the condition (2.41) is violated at that point and the two gaps can not be path-connected. If, on the other hand, a second mass term $\gamma'_0$ exists such
that $\gamma_0' \in \mathcal{R}_{s,d}$, the path
\[m\gamma_0 \to (m\gamma_0 + \gamma_0') \to \gamma_0' \to (-m\gamma_0 + \gamma_0') \to -m\gamma_0 \quad (2.42)\]
becomes possible. In this case, domains $A, B$ are topologically equivalent.

Let us now explicitly verify some results of the classification table [2.1] that we will refer back to when we discuss circuit models in Ch. 3. We start with class BDI in $d = 2$. Without loss of generality we choose the time-reversal operator to be represented by $U_T = \sigma_x$ and particle-hole symmetry by $U_C = 1$. Note that both symmetries square to $+1$. Chiral symmetry is necessarily present with $U_S = \sigma_z$.

The simplest 2-dimensional Dirac Hamiltonian requires a $4 \times 4$ representation of the $\gamma$-matrices. We choose them as $\{\sigma_x, \sigma_y, \sigma_z \tau_x, \sigma_z \tau_y, \sigma_z \tau_z\}$. The Dirac model satisfying $T, C,$ and $S$ can hence only be
\[h = k_1 \sigma_z \tau_x + k_2 \sigma_z \tau_z. \quad (2.43)\]
Of the remaining three $\gamma$-matrices $\sigma_y$ and $\sigma_z \tau_y$ satisfy the mass term conditions Eqs. (2.38-2.40). Therefore, two distinct mass terms exist and we conclude that class BDI is topologically trivial in $d = 2$.

We now relax the symmetry constraints by breaking time-reversal symmetry $T$ as well as chiral symmetry $S$. This brings us into class D. A minimal Dirac Hamiltonian satisfying $C$ with $U_C = 1$ is
\[h = k_1 \sigma_z + k_2 \sigma_z. \quad (2.44)\]
The only mass-term available is $m\sigma_y$. Therefore, class D possesses a topological phase in two dimensions. To differentiate between $\mathbb{Z}$ and $\mathbb{Z}_2$ classification, we have to examine if more than one pair of stable edge modes can exist. We do this by including multiple copies of the Hamiltonian,
\[h' = (k_1 \sigma_x + k_2 \sigma_z) \otimes \mathbb{1}_n, \quad (2.45)\]
and then checking if additional, higher-dimensional mass terms can be found next to $m\sigma_y \otimes \mathbb{1}_n$. It is clear that Eq. (2.45) does not allow for any second symmetry-
preserving mass term, independent of the dimension of the identity $\mathbb{1}_n$. Consequently, any number of edge modes is topologically protected and we assign the classification $\mathbb{Z}$ to class D. Note that above arguments also hold in the absence of the symmetry $\mathcal{C}$, so that class A is expected to have the same topological classification. Comparing with Tab. 2.1 we confirm that our arguments indeed lead to the correct result.

Topologically non-trivial Hamiltonians in class A or D are referred to as Chern insulators. Their main characteristic is the unidirectional propagation of their edge modes. The topological phase of a bulk model can be identified by computation of the Chern number, a topological invariant, which is given by the integral

$$Q^{(n)} = -\frac{1}{2\pi} \int_{BZ} d\mathbf{k} \left( \frac{\partial A^{(n)}_y}{\partial k_x} - \frac{\partial A^{(n)}_x}{\partial k_y} \right)$$

of the Berry connection

$$A^{(n)}_j(\mathbf{k}) = i \langle \Psi_n(\mathbf{k}) | \partial_{k_j} | \Psi_n(\mathbf{k}) \rangle, \quad \text{for } j = x, y$$

over the Brillouin zone.

### 2.3 Do we need quantum mechanics?

We have started this chapter by introducing a fermionic tight-binding model (2.4) using second-quantized fermionic operators. These operators $\psi_i, \psi_i^\dagger$ obey anti-commutation relations, Eq. (2.2), that make the problem inherently quantum-mechanical.

However, for the homotopy classification we simply made use of the first quantized single particle Hamiltonian $h_{ij}$. The only two ingredients were an extensive set of conserved quantities $\mathbf{k}$ originating from translational invariance of the underlying system and a spectral gap in the eigenvalues $\varepsilon_\mathbf{k}$ of the matrix $h_{ij}$. In fact, the first condition can even be relaxed as topological protection also holds in the case of weakly translational-invariance-breaking perturbations.

We can therefore conclude that quantum mechanics, i.e. the non-commutative behavior of fermionic operators, is not a necessary ingredient for the scheme of
topological classification. For that reason SPT classification extends to a wide variety of problems that possess a dispersion relation and can be formulated as an eigenvalue problem.
Chapter 3

Chern insulators from RLC networks

We have motivated in Ch. 2 that SPT phases occur outside the realm of condensed-matter physics. We now enter the main part of this thesis and study three models of periodic circuit networks that realize Chern insulating phases.

3.1 General setup and a toy model

![Figure 3.1](image)

**Figure 3.1:** (a) Square RLC lattice toy model realizing the Chern insulator for EM waves. A unit cell is marked by gray background. (b) The Hall resistor element with four side terminals and one central terminal.

The simplest RLC network capable of exhibiting non-trivial topology is depicted in Fig. 3.1(a). It consists of an array of five-terminal Hall elements, denoted
by gray diamonds, arranged in a square lattice. The central terminal of each Hall element is connected to ground via a capacitor $C$ while the side terminals connect to neighboring Hall resistors through inductors $L$.

The five-terminal Hall element is characterized by its resistance tensor $\hat{R}$, defined by the relation

$$
\begin{pmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4 \\
\end{pmatrix} =
\begin{pmatrix}
R_1 & R_4 & R_3 & R_2 \\
R_2 & R_1 & R_4 & R_3 \\
R_3 & R_2 & R_1 & R_4 \\
R_4 & R_3 & R_2 & R_1 \\
\end{pmatrix}
\begin{pmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4 \\
\end{pmatrix}.
$$
(3.1)

Here, the voltages $V_i$ are measured with respect to the central terminal and the directionality of currents $I_i$ is indicated in Fig. 3.1(b). We note that Eq. (3.1) is the most general parametrization of $\hat{R}$ under fourfold rotational symmetry.

The description of the EM signal propagating through the circuit requires the definition of three dynamical variables: voltage $V_{rrr}(t)$ across each capacitor, and two currents $I_{x}^{rrr}(t)$ and $I_{y}^{rrr}(t)$ flowing through the inductors in each unit cell labeled by vector $rrr$. They are denoted by red and green labels in Fig. 3.1(a), respectively. Then, Kirchhoff’s laws yield the following coupled system of linear differential equations:

$$
C \frac{\partial V_r}{\partial t} = I_{r-\hat{x}}^{x} + I_{r-\hat{y}}^{y} - I_{r}^{x} - I_{r}^{y},$
$$
$$
V_{r} - V_{r+\hat{x}} = L \frac{\partial I_{r}^{x}}{\partial t} + [\hat{R} \cdot I_{r}]_3 - [\hat{R} \cdot I_{r+\hat{x}}]_1,$n$$
$$
V_{r} - V_{r+\hat{y}} = L \frac{\partial I_{r}^{y}}{\partial t} + [\hat{R} \cdot I_{r}]_4 - [\hat{R} \cdot I_{r+\hat{y}}]_2.$
(3.2)

The first of these equations expresses current conservation for each Hall element and the remaining two relate the voltage differences between neighboring unit cells to the corresponding currents, through the usual constitutive relations for inductors and resistors. $I_r = (I_{r-\hat{x}}^{x}, I_{r-\hat{y}}^{y}, -I_{r}^{x}, -I_{r}^{y})^T$ is a vector of currents flowing into the Hall resistor at position $r$.

We begin by considering the case of a non-resistive network, i.e. $\hat{R} = 0$. Then Eqs. (3.2) exhibit invariance under $\mathcal{T}$ which sends $t \rightarrow -t$ and reverses all currents,
Figure 3.2: (a) Effective Majorana tight-binding model corresponding to the RLC network toy model with ideal Hall elements. Tunneling matrix elements between sublattices $a, b, c$ are labeled by straight lines, arrows indicate directionality. Here, $\gamma = R_H \sqrt{C/L}$. (b) Sketch of boundary conditions used for calculations in the strip geometry.

$(I_r^x, I_r^y) \rightarrow (-I_r^x, -I_r^y)$. In addition, because voltages and currents are by definition real-valued, Eqs. (3.2) are trivially invariant under complex conjugation.

Equations (3.2) can be recast in the form of a Schrödinger equation $i \partial_t \phi_i = \sum_j h_{ij} \phi_j$ with the wavefunction $\phi_i$ containing voltages and currents and $h_{ij}$ the Hermitian Hamiltonian matrix. We can further exploit translational invariance of the network by expanding currents and voltages in terms of plane waves

$$
V_r(t) = \sum_k e^{j(\omega t - k \cdot r)} \gamma_k / \sqrt{C},
$$

$$
I_r^\alpha(t) = \sum_k e^{j(\omega t - k \cdot r)} \mathcal{I}_k^\alpha / \sqrt{L},
$$

(3.3)

where $\alpha = x, y$ and the rescaling is made for convenience. Equations (3.2) reduce to a $3 \times 3$ Hermitian eigenvalue problem $\sum_j (h_k)_{j} \phi_{k_j} = \omega_k \phi_{k_i}$, where

$$
\phi_k = \begin{pmatrix} \gamma_k^x \\ \gamma_k^y \\ \gamma_k^z \end{pmatrix}, \quad h_k = \frac{1}{\sqrt{LC}} \begin{pmatrix} 0 & \Gamma_x & \Gamma_y \\ \Gamma_x & 0 & 0 \\ \Gamma_y & 0 & 0 \end{pmatrix},
$$

(3.4)

and $\Gamma_\alpha = i(1 - e^{ik_\alpha})$.

$h_k$ is formally identical to a tight-binding model of Majorana fermions (cf. Sec. 2.1.1) on the Lieb lattice. The underlying Bravais lattice of the Lieb lattice is the square lattice with lattice constant $a$. It has three sublattices. One sublattice is
Figure 3.3: (a) Bulk band structure of the circuit network. The dashed line corresponds to $\gamma = R_H \sqrt{C/L} = 0$ while the solid line corresponds to $\gamma = 0.25$ which gives a gap $\Delta = \omega_0$, where $\omega_0 = 1/\sqrt{LC}$. (b) Spectrum of a strip of width $W = 10$ with open boundary conditions along $y$ for $\gamma = 0.25$. Boundary conditions are chosen as indicated in Fig. 3.2(b). The color scale indicates the average distance $\langle y \rangle$ measured from the center of the strip of the eigenstate belonging to the eigenvalue $\omega_k$. The states inside the bulk gap $\Delta$ are localized near the opposite edges of the system.

placed on the Bravais lattice points, the remaining two sublattices are located on links of neighboring Bravais lattice sites, i.e. they are shifted by $(0, a/2), (a/2, 0)$, respectively. The effective electronic unit cell with imaginary hopping parameters is sketched in Fig. 3.2(a). The correspondence with Majorana as opposed to complex fermions follows from the fact that the original wave equation (3.2) is purely real-valued as is the time-domain Schrödinger equation for Majorana fermions of Eq. (2.11). We would like to emphasize that non-trivial Majorana physics in the condensed matter context relies on the existence of an exponentially large many-body Hilbert space. Here, braiding of Majoranas is represented by non-Abelian unitary operations acting on states in that Hilbert space. However, in the case of a classical circuit, no such many-body Hilbert space exists.

The spectrum of $h_k$ consists of one zero mode $\omega_{k,0} = 0$, and two non-zero
eigenvalues of the form

\[ \omega_{k,\pm} = \pm \frac{1}{\sqrt{LC}} \sqrt{|\Gamma_x|^2 + |\Gamma_y|^2} \]

\[ = \pm \frac{2}{\sqrt{LC}} \sqrt{\sin^2(\frac{k_x}{2}) + \sin^2(\frac{k_y}{2})}. \]  (3.5)

It can be checked that the states belonging to the \( \omega_{k,0} = 0 \) eigenvalue correspond to static patterns of currents in the network consistent with current conservation and zero voltages. These will be damped in the presence of arbitrary resistance and are of no interest to us. The two branches in Eq. (3.5) define the propagating modes of the system. They are gapless and linearly dispersing near \( \mathbf{k} = 0 \), as illustrated in Fig. 3.3(a). Only the positive-frequency branch is physical; the negative branch appears because the ansatz in Eq. (3.3) permits complex-valued solutions while voltages and currents are strictly real.

In the Bloch Hamiltonian formulation time reversal symmetry \( T \) and charge conjugation symmetry \( C \) may be expressed as

\[ T : \quad U \mathcal{T} h^\dagger_{-\mathbf{k}} U^\dagger_{-\mathcal{T}} = h_{\mathbf{k}}, \]

\[ C : \quad h^*_{-\mathbf{k}} = -h_{\mathbf{k}}. \]  (3.6)

with \( U \mathcal{T} = \text{diag}(1, -1, -1) \). Both \( \mathcal{T} \) and \( C \) square to +1 and thus define the BDI class in the CAZ classification. In two spatial dimensions class BDI supports only topologically trivial gapped phases, as listed in Tab. 2.1 Therefore, we must break time reversal symmetry to enable a topological phase in this system. (The \( C \) symmetry derives from real-valuedness of Eq. (3.2) and therefore, like the analogous symmetry present in a generic superconductor, cannot be broken by a physical perturbation.) When \( \mathcal{T} \) is broken, the system belongs to class D which has an integer topological classification in \( d = 2 \). The corresponding topological invariant is the Chern number \( c \) and its non-zero values label distinct Chern insulating phases.

To proceed, we now include a non-zero resistance tensor defined by Eq. (3.1).
The Bloch Hamiltonian describing the network becomes

\[ h_k = \frac{1}{\sqrt{LC}} \begin{pmatrix} 0 & \Gamma_x & \Gamma_y \\ \Gamma_x^* & L_k^x & M_k + N_k \\ \Gamma_y^* & M_k^* - N_k^* & L_k^y \end{pmatrix}, \quad (3.7) \]

with

\[ L_k^x = 2i \sqrt{\frac{C}{L}} (R_3 \cos k_x - R_1), \]
\[ M_k = -\frac{i}{2} \sqrt{\frac{C}{L}} (R_4 - R_2)(1 + e^{ik_y})(1 + e^{-ik_y}), \]
\[ N_k = -\frac{i}{2} \sqrt{\frac{C}{L}} (R_4 + R_2)(1 - e^{ik_y})(1 - e^{-ik_y}). \]

Time-reversal is explicitly broken whenever \( \hat{R} \) is non-zero. We observe that the Hamiltonian (3.7) remains Hermitian only when \( L_k^x \) and \( N_k \) both vanish for all \( k \). This requires \( R_1 = R_3 = 0 \) and \( R_4 = -R_2 \). Under these conditions the resistance tensor (3.1) becomes purely off-diagonal and antisymmetric. This form signifies a purely transverse, non-dissipative response – an “ideal Hall resistor”. It is important to note that the resistance tensor, Eq. (3.1), is not invertible in this limit. As a consequence, the current response to applied voltages is ill-defined. However, we can still achieve sensible results by keeping a small non-zero dissipative component \( R_1 = R_3 = R \). This causes the network Hamiltonian to become non-Hermitian and results in weak damping of the ac signal. Topological properties of the system should not be affected as we explicitly illustrate below. Large non-Hermitian components could lead to new interesting topological phases.

We now focus on the approximately Hermitian limit and define the Hall parameter \( R_H = R_4 = -R_2 \). The bulk spectrum corresponding to the Hamiltonian (3.7) is illustrated by blue lines in Fig. 3.3(a). It develops a gap \( \Delta = 4R_H \sqrt{C/L} \omega_0 \) at \( k = 0 \), where \( \omega_0 = 1/\sqrt{LC} \). Since the term \( M_k \), responsible for the gap formation, is odd under time reversal, we expect the gapped phase to be topologically non-trivial. An explicit calculation indeed indicates a non-zero Chern number \( c = \text{sgn} R_H \) for the negative frequency band. Numerical calculation of the spectrum in a strip geometry confirms the existence of a single chiral edge mode traversing the gap, as
Figure 3.4: Spectral function (3.9) for a circuit with parameters $\gamma = 0.25$ in the absence (left) and presence (right) of 30 percent box disorder for $R, L, C$ parameters. The lifetime broadening parameter is chosen as $\eta = 0.03 \omega_0$.

To examine the finite-size behaviour of the system and to test the stability of the edge modes against disorder, we numerically compute the spectral function

$$A(k, \omega) = -2Im \sum_j G^R_j(k, \omega), \quad j = a, b, c$$

(3.9)

for a finite circuit of $10 \times 10$ unit cells and the same parameters as in the finite Hall-resistor computation of Fig. 3.3(a). Here, $G^R_j$ is the retarded Green’s function

$$G^R_j(k, \omega) = -i \int_0^\infty dt e^{i\omega t} \phi_{kj}(t) \phi^\dagger_{kj}(0)$$

(3.10)

and $j$ indexes current- and voltage degrees of freedoms of the wavefunction. The result, plotted in the left panel of Fig. 3.4, reveals a bulk-bandstructure in agreement with the diagonalization of the translationally invariant model. Additionally, we notice the quantization of energy levels due to the finite size of the circuit, and

shown in Fig. 3.3(b).
observe linearly dispersing edge modes close to the Γ-point. The spectral function for the same parameters with an additional assumption of 30 percent randomness in \( R, L, C \) parameters is plotted in the right panel of Fig. 3.4. Evidently, the disorder washes out the bulk bands but has little effect on the edge states. This is a consequence of topological protection.

Experimental characterization of a finite size network can be given through two-point impedance measurements which are conveniently described by the circuit Green’s function formalism. To this end one writes the frequency-domain Kirchhoff law for current conservation in the matrix form

\[
0 = \sum_{\mathbf{r}'} Y_{\mathbf{rr}'}(\omega) V_{\mathbf{r}'}(\omega),
\]

which defines the admittance tensor \( Y_{\mathbf{rr}'}(\omega) \) and the voltage distribution \( V_{\mathbf{r}} \) corresponding to an eigenmode. Solutions to this equation exist only for \( \det Y(\omega) = 0 \), which yields the resulting eigenspectrum \( \omega_k \) equivalent to Eq. (3.5). The circuit Green’s function \( G_{\mathbf{rr}'}(\omega) = [Y(\omega)^{-1}]_{\mathbf{rr}'} \) describes the voltage response of the network at point \( \mathbf{r} \) to a driving current profile \( I_{\mathbf{rr}'}^{\text{drive}}(\omega) \) at frequency \( \omega \) according to

\[
V_{\mathbf{r}}^{\text{resp}}(\omega) = \sum_{\mathbf{r}'} G_{\mathbf{rr}'}(\omega) I_{\mathbf{rr}'}^{\text{drive}}(\omega).
\]

In analogy to condensed matter systems, where the complete characterization of a non-interacting system is contained in the time-ordered two-point correlation function \( \langle \mathcal{T} \psi(\mathbf{r}, t) \psi^\dagger(\mathbf{r}', t') \rangle \), full experimental knowledge of \( G_{\mathbf{rr}'}(\omega) \) provides a complete characterization of the electrical circuit. We can therefore expect topologically non-trivial behavior to be evident in a circuit’s two-point impedance.

In Fig. 3.5 we demonstrate this explicitly by plotting the voltage profile \( V_{\mathbf{r}}^{\text{resp}} \) induced by a current with frequency \( \omega \) injected at the boundary of a 10 × 10 network. As an example of possible dissipative dynamics we include a non-zero \( R \) component of the resistance tensor \( \hat{R} \) and quantify the strength of dissipation by a dimensionless parameter \( \epsilon = R/R_H \). For the frequency inside the bulk bandgap the signal is seen to propagate along the boundary of the system and in one direction only, consistent with the chiral nature of the gapless edge mode. Parameter \( \epsilon \) clearly controls the lengthscale over which the signal is damped.
Figure 3.5: Voltage response $V_{\text{resp}}^r(\omega)$ induced by a current with in-gap frequency $\omega = \Delta/2$ injected at a node marked by green cross of the $10 \times 10$ network with $\gamma = 0.25$ for various values of the dissipative resistance $R$ characterized by parameter $\epsilon = R/R_H$.

Figure 3.6: (left) Spectrum of circuit for $\gamma = 0.25$ and $\epsilon = 0.1$. (right) Voltage as measured in the bulk (black) or at the edge (blue) after current has been injected at an edge-site.

A more quantitative analysis of this behavior is shown in Fig. 3.6. Here, we plot the voltage measured at a bulk site (black) and an edge site (blue) after a current of frequency $\omega$ has been injected at an edge side. As expected, the edge-to-bulk signal is close to zero for frequencies inside the bulk gap, as such signals can only propagate around the edge and not enter the bulk. The edge-to-edge signal is prevalent for the whole frequency range since propagating modes are available throughout the whole plotted frequency range. Dips and spikes of the two-point signal are the result of finite size energy level quantization.

Finally, we investigate the propagation of such signals in the time domain. To
Figure 3.7: Time evolution of a localized Gaussian wave packet of frequency width \((\Delta \omega)/\omega_0 = 0.35\) excited at the boundary. The simulation models disorder by assuming a capacitor and inductor device tolerance of 30\%. Color scale corresponds to the weight of the wavefunction on the circuit node. The signal travels along the boundary and circumvents the boundary defect indicated in white.

Figure 3.8: Plot of voltage profile along boundary sites as function of time that shows the constant group velocity of the wave packet.

this end we excite a Gaussian wave packet with the frequency width \((\Delta \omega)/\omega_0 = 0.35\), spatially localized around an edge site, and unitarily evolve it in time with the propagator \(U = \exp(-iht)\). The corresponding simulation for a non-dissipative network with \(\gamma = 1\) and assuming \(\pm 30\%\) randomness in \(L\) and \(C\) values is shown in Fig. 3.7. The edge signal propagates unidirectionally along the circuit boundary, even in the presence of boundary defects. A plot of voltage profile along the network boundary as a function of time in Fig. 3.8 reveals approximately constant group velocity of the wave packet.
The circuit described above illustrates the mathematical correspondence between periodic RLC networks and tight-binding Hamiltonians with non-trivial topology. Our approach allows for the mapping of the differential equations governing the RLC network onto a simple Bloch equation with similar models analyzed in the condensed-matter literature [37]. The non-trivial ingredient required to break time reversal symmetry is the five-terminal Hall element described by the resistance tensor, Eq. (3.1). However, as we will discuss in Ch. 4 its experimental realization is not straightforward. For this reason, we may regard the above network as an instructive but unphysical toy model. Next, we will describe two different network architectures which have well-defined experimental implementations and are only slightly more complex.

### 3.2 Chern insulator on the square lattice

![Square RLC lattice network with four-terminal Hall elements](image)

**Figure 3.9:** Square RLC lattice network with four-terminal Hall elements. Voltage nodes (red) and currents (green) are labeled for the unit cell (gray background) at position $r$.

Consider the network depicted in Fig. 3.9(a). It has a square lattice symmetry and contains four inductors, three capacitors, and one Hall resistor per unit cell. The Hall resistor is now in a four-terminal configuration. We characterize it by the
Hall admittance tensor \( \hat{Y} \) that relates input currents to terminal voltages via \( I = \hat{Y}V \). In its idealized version it is

\[
\begin{pmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4
\end{pmatrix} = \frac{1}{R_H} \begin{pmatrix}
0 & -1 & 0 & 1 \\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4
\end{pmatrix}.
\] (3.13)

Currents and voltages are labeled as shown previously in Fig. 3.1(b) with the difference that no central terminal exists. We note that \( \hat{Y} \) has rank 2 and is therefore not invertible. We can reduce (3.13) to a set of two linearly independent equations by realizing that it conserves current for pairs of opposing terminals, that is, for any voltage input the currents satisfy \( I_1 = -I_3 \) and \( I_2 = -I_4 \). In electrical circuit theory this is known as the port condition. Two opposing terminals define a port. A full description of the Hall element is then achieved in terms of two currents through the ports, \( I_1 \) and \( I_2 \), and two voltages across the ports, \( V_1 - V_3 \) and \( V_2 - V_4 \). The corresponding resistance tensor is

\[
\hat{R} = \hat{Y}^{-1} = \begin{pmatrix}
0 & R_H \\
-R_H & 0
\end{pmatrix}.
\] (3.14)

We note that the circuit element corresponding to the above resistance matrix is in fact well known in electrical engineering literature as the gyrator [38]. This device, together with the resistor and the capacitor, defines a basis of linear circuit elements. All other network elements can be composed from the aforementioned three.

The degrees of freedom describing the network in Fig. 3.9(a) can be chosen as three voltages on the capacitors and four currents flowing through the inductors, forming a seven-component vector \( \Psi_r = (V_A^r, V_B^r, V_C^r, I_1^r, I_2^r, I_3^r, I_4^r)^T \). To preserve the fourfold rotational symmetry of the network, we take capacitances on B and C sublattices to be equal, \( C_B = C_C = C \), and further set \( C_A = C/g^2 \) with \( g \) a dimensionless parameter. All inductors have inductance \( L \).

The corresponding Bloch Hamiltonian follows from current conservation for all nodes and Kirchhoff’s second law for the potential difference between two
nodes connected through an inductor. It can be represented as a $7 \times 7$ matrix of the form
\[
 h_k = \frac{1}{\sqrt{LC}} \begin{pmatrix}
 M_k & P_k \\
 P_k^\dagger & \hat{0}
\end{pmatrix},
\]
(3.15)
where $\hat{0}$ is a $4 \times 4$ matrix with all elements zero and $P_k$ denotes the $4 \times 3$ matrix
\[
 P_k = i \begin{pmatrix}
 -g & ge^{-ik_x} & -g & ge^{-ik_y} \\
 1 & -1 & 0 & 0 \\
 0 & 0 & 1 & -1
\end{pmatrix}.
\]
(3.16)
The $3 \times 3$ matrix $M_k$ contains time reversal breaking terms due to the presence of the Hall element,
\[
 M_k = \begin{pmatrix}
 0 & 0 & 0 \\
 0 & 0 & m_k \\
 0 & m_k^* & 0
\end{pmatrix},
\]
(3.17)
with $m_k = \frac{i}{R_H} \sqrt{\frac{L}{C}} (1 - e^{ik_x})(1 - e^{-ik_y})$.

![Figure 3.10](image)

**Figure 3.10:** Eigenmode spectrum of the network for $g = 1/\sqrt{2}$, with (solid lines) and without (dashed lines) the Hall element. The gap parameter is $\gamma = \sqrt{C/LR_H} = 5\sqrt{2}$ and we have defined an overall frequency scale $\omega_0 = \sqrt{2/LC}$. Strip-diagonalization of the network with $g = 1/\sqrt{2}$ and $\gamma = 5\sqrt{2}$. Colorscale indicates the average distance $\langle y \rangle$ measured from the center of the strip of the eigenstate belonging to the eigenvalue $\omega_k$.

The mode spectrum of the circuit consists of seven bands. Charge-conjugation
symmetry $C$ constraints the bands to come in pairs of opposite frequency and the unpaired band to be confined to $\omega_k = 0$. In the absence of the Hall resistor, time reversal symmetry enforces degeneracies at $k = (0,0)$ and $(\pi, \pi)$ as follows

$$
\begin{align*}
\omega_{(0,0)} &= \omega_0(0, \pm 0, \pm 1, \pm \sqrt{1 + 2g^2}), \\
\omega_{(\pi,\pi)} &= \omega_0(0, \pm 1, \pm 1, \pm \sqrt{2}g).
\end{align*}
$$

Here, we have defined $\omega_0 = \sqrt{2/LC}$. The Hall resistor breaks $\mathcal{T}$ and splits the degeneracy at $(\pi, \pi)$. The quadratic band crossing thus acquires a gap and the two bands become topologically non-trivial with the Chern number $c = \pm \text{sgn}(R_H)$. Since $M_{(0,0)} = 0$ the degeneracy at the $\Gamma$ point remains intact.

For an arbitrary $g$ and $R_H$ one thus expects the network to realize a Chern insulator. A situation of special interest occurs for $g = 1/\sqrt{2}$. In the absence of the Hall resistor, three bands then touch at $(\pi, \pi)$ and the middle band is completely flat; see Fig. 3.10(a). The Hall resistor separates the three bands and makes the top and bottom bands topological with Chern number $c = \pm \text{sgn}(R_H)$. The flat band remains trivial with $c = 0$. This is confirmed by numerical diagonalization of Hamiltonian Eq. (3.15) on a strip geometry with translational invariance along $\hat{x}$, shown in Fig. 3.10(b). We clearly observe chiral edge modes. We further analyze the admittance
properties of the network by calculating the circuit Green’s function in a finite system and plotting the voltage response to a current injected at a single node. For these calculations, we assume that the inductors are weakly resistive and characterize their resistance $R_L$ by a parameter $\varepsilon = R_L/R_H$. The resulting Hamiltonian becomes weakly non-Hermitian and the propagating waves are damped.

Fig. 3.11(a) shows the voltage response to a current of in-gap-frequency $\omega$ that is injected at a bulk site. The voltage profile is localized around the node of injection. If we tune the frequency out of the bulk gap, the voltage signal propagates through the whole circuit, independent of the point of injection, as shown in panel (b). To demonstrate the topological nature of the edge transport, we include a defect on the circuit’s left boundary and excite the edge mode of the circuit at an in-gap frequency, cf. panel (c). As expected the signal propagates around the defect by following the distorted edge. This does not change qualitatively when we introduce bulk disorder (panel d), which we model by including a 17% randomness in $L$, $C$, $\varepsilon$, and $R_H$ values, larger than typical tolerances of commercially available electronic components.

### 3.3 Chern insulator on the honeycomb lattice

The graph structure of RLC networks in principle allows for engineering of arbitrary lattice models. Here, we briefly discuss a circuit whose tight-binding analog is similar to the Haldane model on the honeycomb lattice [23], which was historically the first model realizing the Chern insulator in electronic systems. A unit cell is schematically shown in Fig. 3.12(a). Each of the two sublattices of the honeycomb lattice contains a node that is connected to ground through a capacitor $C$. Nearest-neighbor nodes are connected by inductors $L$. Second neighbors within a hexagonal plaquette each connect to a three-terminal Hall resistor.

The three-terminal Hall resistor is described by a three-fold rotationally symmetric resistance tensor whose idealized, non-dissipative form is defined by the relation

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 & R_H \\ -R_H & 0 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}. \quad (3.19)$$
Figure 3.12: (a) Unit cell of a network realizing the honeycomb lattice model. Three next-nearest neighbors within each plaquette connect to a three-terminal Hall element. (b) Band structure of a strip with zig-zag termination in \( \hat{y} \)-direction for \( \gamma = R_H \sqrt{C/L} = 10 \sqrt{2} \), where \( \omega_0 = \sqrt{1/LC} \). Colorscale shows mean value of the distance of the corresponding eigenfunction from the center of the strip.

While the above resistance tensor is formally equivalent to the matrix in Eq. (3.14), the port condition does not apply for a triangular Hall element. Instead \( \hat{R} \) relates input currents at two of the three terminals to the corresponding terminal voltages. The potential at the third terminal is gauged to zero and the corresponding current is determined by current conservation.

The network in Fig. 3.12(a) has a tight-binding representation and topological structure closely related to Haldane’s celebrated lattice model of the Chern insulator [23]. The Bloch Hamiltonian is a 5 \( \times \) 5 matrix which takes the same form as Eq. (3.15), where now

\[
M_k = \begin{pmatrix} m_k & 0 \\ 0 & -m_k \end{pmatrix}, \quad P_k = i \begin{pmatrix} -1 & -1 & -1 \\ 1 & e^{i k \cdot a_1} & e^{-i k \cdot a_2} \end{pmatrix}.
\]

(3.20)

Here, \( m_k = 2 R_H^{-1} \sqrt{L} \sum_{\alpha=1}^{3} \sin(k \cdot a_\alpha) \) and \( a_\alpha \) are Bravais lattice vectors as shown in the inset of Fig. 3.9(b). Similar to graphene the band structure has a pair of Dirac points located at the corners of the hexagonal Brillouin zone but they now occur at non-zero frequency. The band crossings are protected by a combination of \( \mathcal{T} \) and the lattice inversion symmetry. Inclusion of the Hall resistors breaks \( \mathcal{T} \).
and creates a Chern insulator. Numerical diagonalization of the Hamiltonian in the
strip geometry confirms the existence of the chiral edge modes traversing the gap;
see Fig. 3.12(b).
Chapter 4

Hall resistor implementation

Non-trivial physics in the models studied in Ch. 3 above relies on Hall resistors characterized by a transverse voltage response to a longitudinal current that is odd under time reversal. We now discuss concrete physical implementations of these elements using the classical Hall effect in metals and simulated Hall effects achieved through active circuit elements.

4.1 Hall-resistor implementation by classical Hall effect

4.1.1 Hall effect, galvanic coupling

We consider a metal or semiconductor film in a perpendicular magnetic field $B_\perp$. In such a setting the microscopic current response is accurately described by Ohm’s law, $j = \sigma E$, where the material’s conductivity takes the form \[ \sigma = \frac{\sigma_0}{1 + \frac{\sigma_0}{R_H^2} B^2} \left( \begin{array}{cc} 1 & -\sigma_0 R_H B_\perp \\ \sigma_0 R_H B_\perp & 1 \end{array} \right) \] (4.1)

Here, $\sigma_0$ is the zero-field conductivity and $R_H$ is the Hall coefficient. For $\sigma_0 R_H B \gg 1$, the microscopic current response to a potential gradient is predominantly transverse. One might be tempted to assume that a device depicted in Fig. 4.1 could therefore serve as a near-ideal Hall resistor. As our simulations below illustrate, this unfortunately is not the case because of the phenomenon of geometric magne-
Figure 4.1: Finite element simulation of the current and voltage distributions in a two-dimensional Hall plate in a perpendicular magnetic field $B_\perp$ with current density $j$ driven by potential difference $V = 1$ V from left to right terminal. White lines follow the electric field $E$, black arrows denote the direction of the current flow $j$. At zero field (left panel) $j \parallel E$ and there is no voltage drop $V_H$ between the top and the bottom terminal. For weak fields (middle) $|E||j| > j \cdot E > 0$ and a small Hall voltage $V_H < V$ is observed. At high fields (right) $j \perp E$ and $V_H \approx V$. In the infinite $B_\perp$-field limit the electric field diverges at the two contact points marked by red arrows.

We use Comsol Multiphysics finite element software to numerically solve the current conservation equation $\nabla \cdot \sigma \nabla V = 0$ for the four-terminal geometry depicted in Fig. 4.1. We choose insulating boundary conditions for the edges as well as top and bottom terminals and drive a longitudinal current $I$ by a voltage difference $V$ from the left to the right terminal. The resulting potential distribution is plotted as a color scale, electric field lines are white, and the current flow is denoted by black arrows. For $B_\perp = 0$, the current flow is parallel to $E$ and the potential difference between top and bottom terminals is $V_H = 0$. As one increases the magnetic field, $j$ and $E$ span the Hall angle $\theta_H$ and one measures a finite Hall voltage $V_H$. For constant current flow, $V_H$ increases linearly with $B_\perp$ as shown in Fig. 4.2(a). Naively, one could expect that $V_H \gg V$ for large enough field $B_\perp$. This is the necessary condition for the realization of an ideal Hall element. However, as one can see in the right panel of Fig. 4.1, the Hall voltage saturates at $V_H = V$.

This effect is commonly known as two-terminal resistance and may be interpreted as a geometrical magnetoresistance. For the diamond geometry it estab-
Figure 4.2: (left) Linear dependence of Hall-voltage as a function of magnetic field. (right) The longitudinal voltage shows a dependence on the magnetic field that is linear in the strong-field limit.

lishes magnetic field dependence of the longitudinal resistance that is linear for large fields, see Fig. 4.2(b). For constant current, $V_H$ and $V$ then show the same linear behavior at high fields, precluding the desired limit $V_H \gg V$. In fact, it has been shown that, on general grounds, $V_H \leq V$ for arbitrarily shaped three-, four-, and six-terminal geometry and arbitrary magnetic field [40].

It may seem puzzling that a Hall element is dissipative in the limit $\mathbf{j} \perp \mathbf{E}$. After all, the dissipated power is $P = \int \mathbf{E} \cdot d\mathbf{j}$ and should vanish when $\mathbf{j} \perp \mathbf{E}$. But $P$ can still be non-zero if the electric field strength diverges at some point in the sample. In fact, it is known that the two-terminal resistance arises at two points near the terminals where the boundary conditions change from galvanic to electrically insulating. At these points, the electric field diverges. In our setup the points with divergent field strength are marked by red arrows in the rightmost panel of Fig. 4.1.

4.1.2 Hall effect, capacitive coupling

Viola and DiVincenzo proposed an elegant way [41] to circumvent the problem of diverging electric fields outlined above. They showed that a near ideal Hall resistor can be achieved by replacing galvanic contacts by capacitive coupling to the terminals. The resulting setup, illustrated in Fig. 4.3 yields solutions of the EM
field equations that are well behaved on the whole resistor geometry. Explicitly, their resulting impedance tensor for a two-port geometry Fig. 4.3(a) in the limit $\mathbf{j} \perp \mathbf{E}$ has the form

$$\hat{R}(\omega) = R_H \begin{pmatrix} 1 & -i \cot \left( \frac{1}{2} \omega C_L R_H \right) \\ -1 & -i \cot \left( \frac{1}{2} \omega C_L R_H \right) \end{pmatrix}.$$  \hspace{1cm} (4.2)

Here, $C_L$ is the capacitance of a single contact. All contacts are assumed to have the same capacity for simplicity. The anti-symmetric structure of the tensor implies that no energy is dissipated. For a discrete set of perfect “gyration” frequencies

$$\omega_n = \frac{\pi}{C_L R_H} (2n + 1), \quad n = 0, 1, \ldots,$$  \hspace{1cm} (4.3)

diagonal elements vanish and the above tensor describes an ideal Hall resistor.

The impedance tensor of the capacitively contacted Hall element is intrinsically dependent on the drive frequency $\omega$ and this dependence is fundamentally non-linear. This prevents a description in terms of the Bloch equation with a simple frequency-independent Hamiltonian but one can still use the circuit Green’s function method to describe a periodic LC network with these elements. The cor-
Figure 4.4: Plot of $\log |\det Y(\omega, k)|$ for ideal Hall-resistor (top row) and capacitively coupled Hall-resistor (bottom row) for infinite geometry (left column) and strip-geometry (right column), respectively. Dark colors show divergences of the log where the eigenmode equation $\det Y = 0$ is satisfied. For the top row the same parameters as in Fig. 3.10 are assumed, for the bottom row calculation we have $g = 1/\sqrt{2}$, $C_L = 1/5$, $R_H = 5/4$ in units where $L = C = 1$.

The corresponding admittance tensor is given by

$$
\hat{Y} = \begin{pmatrix}
-i\omega C & \frac{4}{i\omega L} \\
\frac{1+e^{-i\delta_x}}{i\omega L} & -i\omega C - \frac{2}{i\omega L} + i\frac{\tan(\omega C R_H)}{2R_H} \xi + \eta^* \\
\frac{1+e^{-i\delta_y}}{i\omega L} & 0 \\
\frac{1+e^{-i\delta_y}}{i\omega L} & -i\omega C - \frac{2}{i\omega L} + i\frac{\tan(\omega C R_H)}{2R_H} \xi + \eta^*
\end{pmatrix}
$$

(4.4)
where $\xi_i = 2(1 - \cos k_i)$ and $\eta = e^{-ik_x} + e^{ik_y} - e^{-ik_x + ik_y} - 1$. In the bottom left panel of Fig. 4.4, we show a plot of $\log \det \hat{Y}$. Values of $\omega, k$, for which the eigenmode equation $\det \hat{Y} = 0$ is satisfied, are plotted by dark lines. We recognize three bulk bands that are distorted from the ideal Hall resistor calculation, shown for comparison in the top left panel. In the right column, we repeat the calculation for an infinite strip that is 10 unit cells wide. In both, the case of ideal Hall resistor and capacitively coupled Hall element, we observe edge modes between the second and third band. This suggests the realization of Chern phase.

A simple physical description of the Viola-DiVincenzo setup that gives Eq. (4.2) relies on the dynamics of the magnetoplasmon edge mode in the Hall effect device [41]. It will work equally well in the four- and three-terminal configuration, but the five-terminal configuration that is required for our toy model analyzed in Sec. 3.1 cannot be realized in this manner.

### 4.2 Ideal Hall-resistor from operational amplifiers

From the discussion above we infer that the realization of Hall resistors by the classical (or quantum) Hall effect can be achieved by capacitively coupling the Hall resistors in a strong magnetic field to the circuit. Nevertheless, this realization is not entirely practical if one wishes to operate the network at room temperature and use moderate magnetic fields.

On the other hand, four-terminal Hall resistors that satisfy the port condition are well known among electrical engineers as 'gyrators' and various other implementations have been conceived [42–46]. A notable example is the realization using operational amplifiers. Such gyrating circuits are discussed in standard textbooks [47].

Here we describe a specific realization of the simulated ideal Hall resistor inspired by the recent work of Hofmann et al. [2]. It can be used in either four- or three-terminal configuration required for the Chern insulating networks discussed in Secs. 3.2 and 3.3, but not in the five-terminal configuration. Construction of the Hall element is based on the building block depicted in Fig. 4.5(a). It consists of an operational amplifier and three resistors.

Assuming the operational amplifier behaves as an ideal op-amp, i.e. it has infi-
Figure 4.5: (a) Circuit element called "negative impedance converter", composed of three resistors and one operational amplifier, introduced in Ref. [2]. It can be used to construct an ideal Hall element in two-port configuration (b) as implemented in the square lattice Chern insulating network discussed Sec. 3.2, or in a three-terminal configuration (c) required in the honeycomb network of Sec. 3.2.

finite input impedance and infinite open-loop gain, we can easily derive the behavior of the circuit element in Fig. 4.5(a). Since the op-amp is operated in feedback configuration, it will hold the potentials at its inputs equal, \( V_{in} = V_1 \). Moreover, due to its infinite input impedance, it will only inject a current at its output and no current will flow into its inputs. We immediately see that \( I_{in} = -I_{out} \) as a consequence. On the other hand \( I_{out} = \frac{V_{in} - V_{out}}{R_H} \), so that the full description is

\[
\begin{pmatrix}
I_{in} \\
I_{out}
\end{pmatrix} = \frac{1}{R_H} \begin{pmatrix}
-1 & 1 \\
1 & -1
\end{pmatrix} \begin{pmatrix}
V_{in} \\
V_{out}
\end{pmatrix}.
\]

Remarkably, arranged in a two-port configuration as depicted in Fig. 4.5(b) or three-terminal configuration in Fig. 4.5(c), these elements precisely realize the respective ideal Hall resistors required for our proposed Chern insulator networks.

Operational amplifiers are commercially available at low cost and can operate in a wide range of frequencies, voltages and power settings. Experimental realiza-
tion of the Chern-insulating networks using the simulated Hall elements depicted in Fig. 4.5 should therefore be easily achievable.
Chapter 5

Discussion and summary

In this thesis, we proposed periodic RLC networks that function as Chern insulators for electromagnetic signals in a broad range of frequencies tunable by adjusting the values of inductance $L$, capacitance $C$, and Hall resistance $R_H$ of the circuit elements. The design is guided by exploiting an analogy between equations governing the EM fields in periodic RLC networks and tight-binding models for Majorana fermions which are known to possess topologically non-trivial phases. Our approach maps Kirchhoff’s laws describing the network onto a Hermitian eigenvalue problem in crystal momentum space where the eigenvalues correspond to frequency modes of the network. Topological properties of the network are then inferred transparently in direct analogy to condensed matter Hamiltonians.

Explicitly, we have proposed three different network architectures realizing Chern insulating phases for EM signals. The required time reversal symmetry breaking is achieved by including Hall resistors which are non-reciprocal circuit elements also known in engineering literature as gyrators. These may be implemented as capacitively contacted metallic or semiconductor films in an external magnetic field or as simple circuits with resistors and off-the-shelf operational amplifiers. In the latter implementation, the time reversal symmetry is broken by the external source of power required to operate the amplifiers. Nevertheless, due to the feedback structure, the operational amplifiers are operated in the linear response regime and the simulated Hall devices can be regarded as linear circuit elements.

Topological properties of the networks proposed in this work are manifest in
the chiral edge modes traversing the gap in the bulk spectrum. These edge modes give rise to unidirectionally propagating voltage and current signals along the network boundary. They are topologically protected by the bulk topological invariant (the integer Chern number) and cannot be removed by any deformation of the boundary. In addition, the edge modes are robust against a moderate amount of bulk disorder, as realized, e.g., by a random spread in the parameters characterizing the individual network elements.

Chern insulating EM networks provide a highly tunable experimental environment. Scale invariance of Maxwell’s equations allows for engineering of band gaps and edge modes in a wide frequency range. Moreover, the flexible graph nature of such networks removes any restriction on dimensionality or locality. Consequently, exotic synthetic materials of arbitrary dimension and connectivity may be designed. In addition to possible engineering applications, Chern insulating EM networks may be established as a teaching resource in university laboratory courses and demonstrations.
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