Inverse problems of recovering first-order integro-differential operators from spectra

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\textbf{ABSTRACT}
Inverse spectral problems are studied for first-order integro-differential operators on a finite interval. These problems consist in recovering some components of the kernel from one or multiple spectra. Uniqueness theorems are proved for this class of inverse problems.

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1. Introduction
This paper deals with inverse spectral problems for first-order integro-differential operators in the form

\[ \ell y := iy'(x) + \int_{0}^{x} M(x, t)y(t) \, dt, \]

where $M(x, t)$ is a continuous function called the kernel. Inverse problems of spectral analysis consist in recovering operators from their spectral characteristics. For ordinary differential operators, such problems have been studied fairly completely (see [1–6] and the references therein).

For integro-differential and other classes of nonlocal operators, inverse problems are more difficult for investigation. The main classical methods (transformation operator method and the method of spectral mappings) either are not applicable to them or require essential modifications. Therefore the general inverse problem theory for such operators has not been created yet. At the same time, nonlocal and, in particular, integro-differential operators are being actively studied, because they have many applications in physics, mechanics, biology, etc. (see, e.g. [7]).

The majority of applications are concerned with the second-order integro-differential operators in the form

\[ -y''(x) + \int_{0}^{x} K(x, t)y(t) \, dt \]

or in the more general form with $\int_{0}^{l}$, where $l$ is the interval length. Spectral theory of such operators was developed in connection with investigation of the behavior of particles on a crystalline surface [8, 9] and with the nonlocal theory of elasticity [10]. The first-order operator (1) plays a role of the...
‘square root’ of the second-order operator (2) (see [11]), so the spectral theory of operator (1) also leads to the need to study complex-valued solutions.

The problem of recovering the kernel \( M(x,t) \) or \( K(x,t) \) in the general form depending on two variables is still open. The known studies deal with the kernels of some specific structure. In particular, Buterin [11, 12] developed a method for solving inverse spectral problems for the second-order integro-differential operators with the convolutional kernel \( K(x,t) = P(x-t) \). The same approach was applied to the first-order operator in [13] and to the higher order operators in [14]. Yurko [15] studied an inverse problem for operator (1) with the kernel \( K(x,t) = \alpha v(x)\overline{v(t)} \). Later on, Zolotarev [17] independently investigated direct and inverse spectral problems for the second-order integro-differential operator with the kernel \( K(x,t) = \sum_{k=1}^{n} \alpha_k v_k(x)\overline{v_k(t)} \). Some other fragmentary results of inverse problem theory for integro-differential operators were obtained in [19–21]. Thus, it worth developing methods for dealing with kernels of general structure.

In this paper, we consider operator (1) with an arbitrary continuous kernel \( M(x,t) \). However, it is assumed that some part of this kernel is known a priori. Namely, we consider the kernel in the form

\[
M(x,t) = M_0(x,t) + R(x,t)P(x-t),
\]

where \( M_0(x,t), R(x,t), P(x) \) are continuous functions. We suppose that the functions \( M_0(x,t) \) and \( R(x,t) \) are known and the function \( P(x) \) needs to be constructed. Our goal is to prove the uniqueness of recovering \( P(x) \) from the spectrum. We also prove the uniqueness of reconstruction of \( M(x,t) \) in a more general form by multiple spectra.

Let us proceed with formulations of the main results. Consider the boundary value problem \( Q(M) \) for the integro-differential equation

\[
\ell y := iy'(x) + \int_{0}^{x} M(x,t)y(t)\,dt = \lambda y(x), \quad x \in [0, \pi],
\]

with the condition \( y(\pi) = 0 \). Analogously to a first-order differential equation, it is natural to take one boundary condition for the first-order integro-differential equation (4). For definiteness, we choose the condition \( y(\pi) = 0 \), and we get the boundary value problem having a discrete spectrum. It is also possible to consider other types of conditions, e.g. periodic boundary conditions. Note that, if one takes the condition \( y(0) = 0 \), he gets the initial value problem.

Together with \( \ell \) we will consider the operator \( \tilde{\ell} \) of the same form but with a different kernel \( \tilde{M}(x,t) \). We agree that everywhere below if a certain symbol \( \alpha \) denotes an object related to \( \ell \), then \( \tilde{\alpha} \) will denote the analogous object related to \( \tilde{\ell} \).

Let the functions \( M(x,t) \) and \( \tilde{M}(x,t) \) have the form

\[
M(x,t) = M_0(x,t) + R(x,t)P(x-t), \quad \tilde{M}(x,t) = M_0(x,t) + R(x,t)\tilde{P}(x-t),
\]

where \( M_0(x,t), R(x,t), P(x), \tilde{P}(x) \) are continuous functions and

\[
B(x) := \int_{0}^{x} R(\pi-t, x-t)\,dt \neq 0, \quad 0 < x \leq \pi,
\]

Let \( \{\nu_n\} \) and \( \{\tilde{\nu}_n\} \) be the eigenvalues (counted with multiplicities) of the problems \( Q(M) \) and \( Q(\tilde{M}) \), respectively.

Our main result is the following uniqueness theorem.

**Theorem 1.1:** If \( \nu_n = \tilde{\nu}_n \) for all \( n \), then \( P(x) \equiv \tilde{P}(x) \) for \( x \in [0, \pi] \).
Fix $p \in \mathbb{N} \cup \{\infty\}$ and define the set of indices $I_p := \{1, \ldots, p\}$ for $p < \infty$ and $I_p := \mathbb{N}$ for $p = \infty$. Let continuous functions $R_j(x, t)$, $0 \leq t \leq x \leq \pi$, $j \in I_p$ be given, such that

$$\int_0^x R_j(\pi - t, x - t) \, dt \neq 0, \quad 0 < x \leq \pi.$$ 

Let

$$M(x, t) = M_0(x, t) + \sum_{j=1}^p R_j(x, t)P_j(x - t),$$

$$\tilde{M}(x, t) = M_0(x, t) + \sum_{j=1}^p R_j(x, t)\tilde{P}_j(x - t),$$

(7)

where $M_0(x, t), P_j(x), \tilde{P}_j(x)$ are continuous functions for $0 \leq t \leq x \leq \pi$, and for $p = \infty$ the series converge uniformly with respect to $x$ and $t$. Let $\{\nu_{nk}\}$ be the eigenvalues (counted with multiplicities) of the boundary value problem $Q_k := Q(M_k)$ for $k \in I_p$, and let $\{\tilde{\nu}_{nk}\}$ be the eigenvalues of the boundary value problem $\tilde{Q}_k := Q(\tilde{M}_k)$, where

$$M_k(x, t) = M_0(x, t) + \sum_{j=1}^k R_j(x, t)P_j(x - t), \quad \tilde{M}_k(x, t) = M_0(x, t) + \sum_{j=1}^k R_j(x, t)\tilde{P}_j(x - t).$$

By induction, we derive the following corollary from Theorem 1.1.

**Corollary 1.1:** If $\nu_{nk} = \tilde{\nu}_{nk}$ for all $n$ and $k \in I_p$, then $M(x, t) \equiv \tilde{M}(x, t)$.

Note that every continuous function $M(x, t)$ can be represented in form (3) or (7), so Theorem 1.1 and Corollary 1.1 can be applied to any continuous kernel $M(x, t)$.

The paper is organized as follows. In Section 2, we obtain the representation for the solution of Equation (4) in terms of the transformation operator. Note that transformation operators play an important role in spectral analysis of both differential and integro-differential operators (see [1]). Section 3 is devoted to the proof of the uniqueness results.

### 2. Transformation operator

Let $e(x, \lambda)$ be the solution of Equation (4), satisfying the initial condition $e(0, \lambda) = 1$.

**Lemma 2.1:** There exists a function $G(x, t)$ continuous for $0 \leq t \leq x$ such that

$$e(x, \lambda) = \exp(-i\lambda x) + \int_0^x G(x, t) \exp(-i\lambda t) \, dt$$

(8)

and

$$G(x, 0) = 0, \quad G(x, x) = i \int_0^x M(t, t) \, dt.$$
**Proof:** One can easily show that the function \( e(x, \lambda) \) satisfies the integral equation

\[
 e(x, \lambda) = \exp(-i\lambda x) + i \int_0^x \exp(-i\lambda(x - t)) \, dt \int_0^t M(t, \tau) e(\tau, \lambda) \, d\tau. \tag{9}
\]

Let us solve Equation (9) by the method of successive approximations:

\[
 e(x, \lambda) = \sum_{n=0}^\infty e_n(x, \lambda), \tag{10}
\]

where

\[
 e_0(x, \lambda) = \exp(-i\lambda x), \tag{11}
\]

\[
 e_{n+1}(x, \lambda) = i \int_0^x \exp(-i\lambda(x - t)) \, dt \int_0^t M(t, \tau) e_n(\tau, \lambda) \, d\tau, \quad k \geq 0. \tag{12}
\]

We will show by induction that

\[
 e_n(x, \lambda) = \int_0^x G_n(x, t) \exp(-i\lambda t) \, dt, \quad n \geq 1, \tag{13}
\]

where \( G_n(x, t) \) are continuous functions, and \( G_n(x, 0) = 0 \).

Indeed, for \( n = 1 \) we have

\[
 e_1(x, \lambda) = i \int_0^x dt \int_0^t M(t, \tau) \exp(-i\lambda(x - t + \tau)) \, d\tau
 = i \int_0^x dt \int_x^t M(t, s + t - x) \exp(-i\lambda s) \, ds.
\]

Interchanging the order of integration, we obtain

\[
 e_1(x, \lambda) = i \int_0^x \exp(-i\lambda s) \, ds \int_x^t M(t, s + t - x) \, dt = \int_0^x G_1(x, s) \exp(-i\lambda s) \, ds,
\]

and, consequently, (13) is proved for \( n = 1 \), where

\[
 G_1(x, t) = i \int_{x-t}^x M(s, t + s - x) \, ds. \tag{14}
\]

Suppose now that (13) is valid for a certain \( n \geq 1 \). Then, substituting (13) into (12), we get

\[
 e_{n+1}(x, \lambda) = i \int_0^x dt \int_0^t M(t, \tau) \, d\tau \int_0^{\tau} G_n(\tau, \xi) \exp(-i\lambda \xi) \, d\xi
 = i \int_0^x \exp(-i\lambda(x - t)) \, dt \int_0^t M(t, \tau) \, d\tau \int_{x-t}^{x-t+\tau} G_n(\tau, s + t - x) \exp(-i\lambda s) \, ds.
\]
Interchanging the order of integration, we obtain
\[ e_{n+1}(x, \lambda) = \int_0^x G_{n+1}(x, t) \exp(-i\lambda t) \, dt, \]
where
\[ G_{n+1}(x, t) = i \int_{x-t}^x ds \int_{t+s-x}^s M(s, \tau)G_n(\tau, t + s - x) \, d\tau. \]  \hspace{1cm} (15)

Relations (14) and (15) imply that
\[ G_n(x, 0) = 0 \text{ for } n \geq 1. \]
Substituting (13) into (10) and using (11), we arrive at (8), where
\[ G(x, t) = \sum_{n=1}^{\infty} G_n(x, t). \]  \hspace{1cm} (16)

Let us prove the convergence of the series (16). Recall that the function \( M(x, t) \) is continuous and denote \( a := \max_{0 \leq t \leq x \leq \pi} M(x, t) \).

It follows from (14) that \(|G_1(x, t)| \leq at\). By induction, we obtain the estimate
\[ |G_n(x, t)| \leq \frac{a^n t^n \pi^{n-1}}{n!}, \quad n \geq 1, \quad 0 \leq t \leq x \leq \pi. \] \hspace{1cm} (17)

Indeed, using (15) and (17), we derive
\[ |G_{n+1}(x, t)| \leq \int_{x-t}^x ds \int_{t+s-x}^s a \cdot \frac{a^n (t+s-x)^n \pi^{n-1}}{n!} d\tau \]
\[ = \int_{x-t}^x a^{n+1} (t+s-x)^n \pi^{n-1} \frac{d}{(n+1)!} (x-t) \, ds \leq \frac{a^{n+1} t^{n+1} \pi^n}{(n+1)!}. \]

In view of (17), the series (16) converges absolutely and uniformly for \( 0 \leq t \leq x \leq \pi \), and the function \( G(x, t) \) is continuous. Moreover, \( G(x, 0) = 0 \) and
\[ G(x, x) = i \int_0^x M(t, t) \, dt. \]

Lemma 2.1 is proved. \( \blacksquare \)

3. Uniqueness

In this section, we prove the uniqueness Theorem 1.1.

The eigenvalues \( \{v_n\} \) of the boundary value problem \( Q(M) \) coincide with the zeros of the entire function \( \Delta(\lambda) := e(\pi, \lambda) \). Therefore, the set of the eigenvalues is at most countable, but it can be finite and even empty, which is shown by the following examples. Theorem 1.1 is valid for all these cases.

Example 3.1: Let \( M(x, t) \equiv 0 \). Then \( e(x, \lambda) = \exp(-i\lambda x) \) and the characteristic function \( \exp(-i\lambda \pi) \) has no zeros. Hence, \( \{v_n\} = \emptyset \). Therefore, Theorem 1.1, in particular, implies that, if \( M_0(x, t) \equiv 0 \), \( R(x, t) \equiv 1 \), and the spectrum is the empty set, then \( P(x) \equiv 0 \). So the empty spectrum uniquely specifies the function \( P(x) \).
Example 3.2: Let $M(x, t) \equiv i$. Put $z(x) := \int_0^x y(t) \, dt$. Then Equation (4) takes the form

$$z''(x) + z(x) = -i\lambda z'(x), \quad x \in [0, \pi].$$

(18)

The eigenvalues of the problem $Q(M)$ coincide with the eigenvalues of the boundary value problem for Equation (18) with boundary conditions $z(0) = z(\pi) = 0$. Calculations imply that the spectrum has the form $\{\pm \sqrt{n^2 - 1}\} \cap \mathbb{N}$. This example shows that the spectrum can be countable.

Using (8) and Hadamard’s factorization theorem, we obtain that the specification of the zeros $\{\nu_n\}$ uniquely determines the function $\Delta_1(\lambda)$. Under the assumptions of the theorem this yields

$$e(\pi, \lambda) \equiv \tilde{e}(\pi, \lambda).$$

Let the function $\psi(x, \lambda)$ be the solution of the equation

$$\ell^* \psi := -i\psi'(x, \lambda) + \int_x^\pi M(t, x)\psi(t, \lambda) \, dt = \lambda \psi(x, \lambda)$$

(19)

under the condition $\psi(\pi, \lambda) = 1$. We multiply the relation $\tilde{e}(\pi, \lambda) = \lambda \tilde{e}(x, \lambda)$ by $\psi(x, \lambda)$, then subtract relation (19) multiplied by $\tilde{e}(x, \lambda)$, and integrate with respect to $x$:

$$\int_0^\pi \psi(x, \lambda) \, dx \int_0^x (M(x, t) - \tilde{M}(x, t))\tilde{e}(t, \lambda) \, dt = i(\tilde{e}(\pi, \lambda) - \psi(0, \lambda)).$$

(20)

In particular, we get $\varphi(\pi, \lambda) \equiv \psi(0, \lambda)$. We note that the function $w(\pi, \lambda) := \psi(\pi - x, \lambda)$ satisfies the relations

$$iw'(x, \lambda) + \int_0^x M(\pi - t, \pi - x)w(t, \lambda) \, dt = \lambda w(x, \lambda), \quad w(0, \lambda) = 1.$$  

Using (20), we derive

$$\int_0^\pi \psi(x, \lambda) \, dx \int_0^x (M(x, t) - \tilde{M}(x, t))\tilde{e}(t, \lambda) \, dt \equiv 0.$$  

(21)

Relations (5) and (21) imply that

$$\int_0^\pi \psi(x, \lambda) \, dx \int_0^x R(x, t)(P(x - t) - \tilde{P}(x - t))\tilde{e}(t, \lambda) \, dt \equiv 0.$$  

(22)

The left-hand side of (22) can be transformed as follows:

$$\int_0^\pi \psi(x, \lambda) \, dx \int_0^x R(x, t)(P(x - t) - \tilde{P}(x - t))\tilde{e}(t, \lambda) \, dt$$

$$= \int_0^\pi \psi(x, \lambda) \, dx \int_0^x R(x, x - t)(P(t) - \tilde{P}(t))\tilde{e}(x - t, \lambda) \, dt$$

$$= \int_0^\pi (P(t) - \tilde{P}(t)) \, dt \int_0^\pi R(x, x - t)\psi(x, \lambda)\tilde{e}(x - t, \lambda) \, dx$$

$$= \int_0^\pi (P(\pi - x) - \tilde{P}(\pi - x)) \, dx \int_0^\pi R(\pi - t, x - t)w(t, \lambda)\tilde{e}(x - t, \lambda) \, dt.$$
Consequently, relation (22) implies

$$\int_{0}^{\pi} (P(\pi - x) - \tilde{P}(\pi - x)) z(x, \lambda) \, dx \equiv 0, \quad (23)$$

where

$$z(x, \lambda) := \int_{0}^{x} R(\pi - t, x - t) w(t, \lambda) \tilde{e}(x - t, \lambda) \, dt. \quad (24)$$

In view of Lemma 2.1, we have

$$w(x, \lambda) = \exp(-i\lambda x) + \int_{0}^{x} K_{1}(x, t) \exp(-i\lambda t) \, dt,$$

$$\tilde{e}(x, \lambda) = \exp(-i\lambda x) + \int_{0}^{x} K_{2}(x, t) \exp(-i\lambda t) \, dt,$$

where $K_{j}(x, t)$ are continuous functions. Substituting the latter relations into (24), we get

$$z(x, \lambda) = B(x) \exp(-i\lambda x) + \int_{0}^{x} R(\pi - t, x - t) \, dt \int_{0}^{t} K_{1}(t, \tau) \exp(-i\lambda(x - t + \tau)) \, d\tau + \int_{0}^{x} R(\pi - t, x - t) \, dt \int_{0}^{x-t} K_{2}(x - t, \xi) \exp(-i\lambda(t + \xi)) \, d\xi + \int_{0}^{x} R(\pi - t, x - t) \, dt \int_{0}^{t} K_{1}(t, \tau) \exp(-i\lambda \tau) \, d\tau \int_{0}^{x-t} K_{2}(x - t, \xi) \exp(-i\lambda \xi) \, d\xi,$$

and consequently,

$$z(x, \lambda) = B(x) \exp(-i\lambda x) + \int_{0}^{x} K(x, t) \exp(-i\lambda t) \, dt,$$

where $B(x)$ is defined in (6) and $K(x, t)$ is a continuous function. Substituting the above expression into (23), we obtain

$$\int_{0}^{\pi} (P(\pi - x) - \tilde{P}(\pi - x)) \left( B(x) \exp(-i\lambda x) + \int_{0}^{x} K(x, t) \exp(-i\lambda t) \, dt \right) \, dx \equiv 0.$$

We rewrite the latter relation in the form

$$\int_{0}^{\pi} \exp(-i\lambda x) \left( B(x)(P(\pi - x) - \tilde{P}(\pi - x)) + \int_{x}^{\pi} K(t, x)(P(\pi - t) - \tilde{P}(\pi - t)) \, dt \right) \, dx \equiv 0.$$

Consequently, we get

$$B(x)(P(\pi - x) - \tilde{P}(\pi - x)) + \int_{x}^{\pi} K(t, x)(P(\pi - t) - \tilde{P}(\pi - t)) \, dt \equiv 0.$$

Hence, in view of (6), we have $P(x) \equiv \tilde{P}(x)$. Theorem 1.1 is proved.

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