Forecasting the resource of mechanical system by the Theory of Catastrophes

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Abstract. The article considers the formation of a mathematical model for changing the technical state of a mechanical system using the Theory of Catastrophes. During the operation of the mechanical system, failures gradually accumulate, causing a decrease in working capacity and machine failure. The Theory of Catastrophes allows developing models for changing the technical condition of a mechanical system for various operating modes and operating conditions. In the field of the space of technical states of a mechanical system containing critical features, the coordinates of the point of the limiting state and possible failure of the part are determined.

1. Introduction

One of the most important tasks in the design and manufacture of engineering products is to evaluate and predict the durability of the main structural elements and the machine as a whole. The solution to this problem is usually based on the results of engineering calculations or on statistical data on the reliability of a mechanical engineering product obtained under operating conditions of a machine or its analogue [1].

Any structural element of the machine can be represented in the form of a mechanical system, the main properties of which are organization, controllability and relativity (hierarchy) [1].

In situations of complete uncertainty and lack of statistical information about the technical condition of a mechanical system, predicting the moment of reaching the limit state and the occurrence of failure is possible using the theory of catastrophes [2]. The state space of any mechanical system in general can be described by the following equation:

$$F_i(\varphi; t; X_a) = 0, \quad 1 \leq i \leq n; \quad 1 \leq a \leq k;$$

where $\varphi$ - variable parameters of the system state; $X_a$ - coordinates of the system in space $R^n$; $t$ - time; $C_a$ - control parameters.

The task is to determine the dependence of the technical condition of the mechanical system on the control parameters. This is necessary to justify the values of the control parameters at which the state of the system will change.

The technical state of a mechanical system can be described using a function $S(X, C)$. We can assume that this function depends on a number of control parameters $C_1, C_2, \ldots, C_k$. The stability matrix $\mathbf{S}_{ij}$ and its eigenvalues will obviously depend on these parameters. It must be borne in mind that with a
certain combination of control parameter values, a zero value of the stability matrix can be obtained: 
\[ S_{ij} = 0. \]

The zero value of the stability matrix corresponds to the moment of transition of the system from an operational state to an inoperative one, which leads to a failure of the mechanical system. In this case, 
\[ \text{Det } S_{ij} = 0. \]

Moreover, the conditions necessary for the applicability of the Morse lemma \( \Delta S = 0, \text{Det } S_{ij} \neq 0 \) are not satisfied, and the potential function at the equilibrium point cannot be represented in canonical form:

\[ S = \sum_{i=1}^{n} \lambda_i y_i^2 \]  

(2)

To represent the potential function in this case, the Thom’s splitting lemma is used [2,3,4], which allows representing the function in the form of two components:

\[ S_{(x; c)} = f(y_i(x; c), \ldots, y_l(x; c); c) + \sum_{i=l+1}^{n} \lambda_i(c)[y_i(x)]^2 \]  

(3)

The first component describes in canonical form a potential function at a non-Morse critical point, which determines the behavior during the transition of the system to a qualitatively new (limiting, inoperative) state. The coordinates of the critical point \( y_1(x; c), \ldots, y_l(x; c) \) correspond to \( l \) vanishing eigenvalues. Functions \( \lambda_1(c), \ldots, \lambda_l(c) y_1(x; c), \ldots, y_l(x; c) \) are smooth functions of \( n \) variables of state \( x \) and \( c \) control parameters of \( y \).

The second component includes coordinates \( y_{l+1}(x), \ldots, y_n(x) \) that correspond to non-zero eigenvalues of functions \( \lambda_{l+1}(c), \ldots, \lambda_n(c) \). These functions are also smooth functions of the initial variables of state \( X \).

In the works of R. Thom, a canonical decomposition of a potential function is presented, for which an important property is characteristic:

If \( X_o \) is a non-Morse critical point of the potential function of the family \( S(X, C) \) for \( C = C_{np} \), then from an open neighborhood of a point \( (X_o, C_{np}) \) in space \( R^n \otimes R^k \):

\[ S = \text{Cat}(l, k) + \sum_{i=l+1}^{n} \lambda_i(c)y_i^2 \]  

(4)

The function \( \text{Cat}(l, k) \) is called the catastrophe function. In this case, a catastrophe means a sudden change in the technical state of a mechanical system, which manifests itself in its sudden loss of operability (occurrence of failure) with a smooth change in external conditions [4,5].

In expression (4), \( l \) - the dimension of zero space \( S_{ij} \) at a non-Morse critical point; \( K \) - the number of control parameters. Thus, the catastrophe function \( \text{Cat}(l, k) \) is a function of \( l \) system state variables and \( K \) control parameters. At the moment when in space \( R^k \) the parameters of the technical state of the mechanical system reach the limit values: \( C = C_{np} \), and the mathematical control parameters \( a \) take zero values, the catastrophe function will be reduced to catastrophe growth \( \text{C\sigma}(l) \).

In the general case, the mathematical \( a_1, \ldots, a_k \) and physical \( C, \ldots, C_k \) control parameters of the state of the system are connected through a smooth transformation with a non-degenerate Jacobian:

\[ \text{Cat } \left. \text{Det } \frac{\partial a_\alpha(c)}{\partial a_\beta} \right| \neq 0, \quad 1 \leq \alpha, \beta \leq K \]  

(5)
A mathematical description of the process of changing the state of a mechanical system at the moment the last transitions from an operational state to an inoperative one reveals the function of the catastrophe germ: \( C_{\sigma}(I) \). To determine the moment of failure of a mechanical system, suppose that its technical condition is adequately described by a complex indicator \( D \). The density distribution functions of the indicator values for the operational \( (D_1) \) and inoperative \( (D_2) \) states of the mechanical system are shown in figure 1.

The value of the complex indicator at which the transition of a mechanical system from an operational state to an inoperative state with subsequent failure is carried out will be called the limiting one. In the space of technical states of a mechanical system, the value \( D_{pr} \) of the complex indicator corresponds to a certain point at which the system is in a boundary state, i.e. can be attributed both to the operational and inoperative states. This point is critical. Thus, the limiting state of a mechanical system arises at the corresponding mathematical critical point in the state space of the system [6].

We introduce a mathematical parameter \( \lambda \), which is a variable of the technical state of a mechanical system and corresponding to a complex physical parameter \( D \). The value \( \lambda \) depends on the control mathematical parameter \( \alpha \). Then the dependence \( \psi(\alpha) \) in the vicinity of the critical point can be represented by the curve shown in figure 2.
The upper branch of the state curve shown in figure 2 characterizes the change in system performance under the influence of a control parameter \( a \). Quantitative changes in the parameters \( a \) and \( X \) cause a qualitative change in the state of the system, which is expressed in the transition of the parameter \( X \) to a new level — the lower branch of the state curve. The values of the technical state parameter \( X \) related to the lower branch of the state curve indicate that the system is in an inoperative state. The plot of the curve \( A - B \) illustrates the process of transition of the system from an operational state to an inoperative one. A critical point corresponding to a physical critical point \( D_{mp} \) is located on this site.

To analyze the technical condition of a complex mechanical system, we represent the state space in the form of a certain surface \( S \) in coordinates \( a, b, X \), where \( a \) and \( b \) are the control parameters, \( X \) is the system operability indicator, see figure 3.

![Figure 3](image_url)

**Figure 3.** The state space of a mechanical system with two control parameters.

The space of possible states of the system \( S \) in figure 3 is divided into two parts by a fold. The moment of transition of the system from an operational state to an inoperative state is represented in the form of a fold.

For mechanical systems, the technical state of which is characterized by two independent determining parameters, the state space \( S \) in its simplest form can be represented by a cubic surface in a three-dimensional coordinate system \( R^3 \):

\[
X^3 = a + bX
\]  

(6)

The horizontal projection of the fold \( \Phi \) on the plane \( \Gamma \) gives two curves that limit the area of the projection of the fold. When the system reaches its limit state, the curves intersect at a critical point called the catastrophe point. Differentiating expression (6) by \( X \), we obtain the equation of the curve of the projection of the fold \( \Phi \) on the horizontal plane \( \Gamma \):

\[
3X^2 = b
\]  

(7)

In the theory of catastrophes, the projection of a fold \( \Phi \) onto a plane \( \Gamma \) is called a bifurcation. The term “bifurcation” means a splitting, and is used to denote the process of a qualitative change in the technical condition of a mechanical system as a result of a change in determining factors [3,4]. The bifurcation equation is obtained by a joint solution of equations (6) and (7):
Thus, the parts $S_1$ and $S_2$ of the state spaces of the system located outside the fold $\Phi$ are represented by the relation $3X^2 \geq b$. The part of the system state space enclosed within the fold is described by the expression $3X^2 < b$. The space enclosed between the lower and upper branches of the surface $S$ characterizes the process of transition of the system from an operational to an inoperative state.

Equation (8) allows determining the limiting values of the determining parameters, upon reaching which there is a qualitative change in the state of the system associated with a change in its internal structure [7]. In other words, this equation shows at what combinations of values of the determining parameters the mechanical system passes from an operational to an inoperative state with subsequent failure.

2. Conclusions
The mathematical apparatus of the theory of catastrophes allows one to obtain bifurcation equations for mechanical systems of any complexity, the state of which is determined by one or more control parameters. Examples of such systems are the structural elements of transport and technological machines, in predicting the durability of which the theory of catastrophes can be useful [8].

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