CLASSICAL SOLUTIONS OF THE BACKWARD PIDE FOR MARKOV MODULATED MARKED POINT PROCESSES AND APPLICATIONS TO CAT BONDS

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ABSTRACT. The objective of this paper is to give conditions ensuring that the backward partial integro differential equation associated with a multidimensional jump-diffusion with a pure jump component has a unique classical solution; that is the solution is continuous, twice differentiable in the diffusion component and differentiable in time. Our proof uses a probabilistic argument and extends the results of Pham [25] to processes with a pure jump component where the jump intensity is modulated by a diffusion process. This result is particularly useful in some applications to pricing and hedging of financial and actuarial instruments, and we provide an example to pricing of CAT bonds.

Keywords: Partial integro differential equations, Classical solution, Markov modulated marked point process, Cauchy problem, CAT Bonds.

1. INTRODUCTION

This paper studies conditions for the existence of smooth solutions for certain partial integro-differential equations (PIDEs) associated with the generator of a Markov jump-diffusion process with at least one pure jump component. In concrete terms, we consider a Markov process of the form $X = (Z, L)$, where $L$ is a pure jump process and $Z$ is a general $d$-dimensional jump diffusion that modulates the local characteristics (jump intensity and jump-size distribution) of $L$. Given functions $c, f$ from $[0, T] \times \mathbb{R}^{d+1}$ to $\mathbb{R}$ and $g$ from $\mathbb{R}^{d+1}$ to $\mathbb{R}$, we define the function $v$ by

$$v(t, x) = \mathbb{E} \left[ \int_t^T e^{-\int_s^t c(u, X_u)du} f(s, X_s)ds + e^{-\int_t^T c(s, X_s)ds} g(X_T) \mid X_t = x \right].$$

The goal of the paper is to give regularity conditions on $c$, $f$ and $g$ and on the generator $\mathcal{L}$ of the process $X$ which ensure that $v$ is a classical solution (i.e. Lipschitz continuous, $C^2$ in $z$, $C^1$ in $t$) of the backward PIDE associated with $\mathcal{L}$.

Jump processes with characteristics modulated by a (jump) diffusion are frequently used in non-life insurance, for instance in the valuation of loss layers or catastrophe bonds (CAT bonds), and

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in specific fields of finance such as credit risk modelling or tick data models for high frequency trading. Functions $v$ of the form (1.1) arise in pricing and hedging problems, where the function $f$ may represent instantaneous dividend payments, function $c$ the discount rate and function $g$ the terminal payoff. The fact that $v$ is a classical solution of the backward PIDE (and not merely a viscosity solution) is essential in this context, as it allows to compute the martingale representation of the price process $(v(t, X_t))_{t \geq 0}$ and, consequently, hedging strategies (see, e.g. Frey [16], or the discussion in Section 3 below). To further illustrate the financial relevance of our results we investigate in detail the pricing and hedging of a CAT bond. CAT bonds are obligations issued by (re)insurance companies to transfer to financial markets the risk of low-frequency high-severity events such as natural catastrophes; if cumulative losses due to such events exceed a (high) threshold, coupons and face value of the bond are reduced. We show that, in a typical actuarial model, the pricing of CAT bonds leads to a PIDE with a pure jump component, and that the hedging requires that the PIDE has a classical solution.

While the PDE characterization of $v$ is well understood in the case of diffusion processes, where $L$ is a second-order differential operator, there are only a few contributions that study the jump-diffusion case, that is when $L$ is an integro differential operator, see for instance Gihman and Skohorod [19], Bensoussan and Lions [2], Pham [25], Davis and Lleo [11]. Moreover, this limited literature does not cover the case where $X$ has a pure jump component, and it makes quite restrictive regularity assumptions on the coefficients appearing in $L$. Most relevant for our analysis are the results of Pham [25, Section 5], where existence and uniqueness of a smooth solution for the backward PIDE is obtained in the case where the process $X$ can be written as the solution of an SDE driven by a Brownian motion and an exogenous Poisson random measure. His analysis relies on two strong assumptions: first, the coefficients in the SDE representation of $X$ satisfy a strong Lipschitz assumption; second, the diffusion part of $L$ is uniformly elliptic. While these two assumptions look quite natural, there are many practically relevant situations where they are not met. For instance, the assumptions that the jumps of the state process $X$ are driven by a Poisson random measure with Lipschitz jump size coefficients excludes models that employ (compound) Cox processes (essentially marked point processes with stochastic jump intensity). Moreover, the ellipticality assumption on the instantaneous covariance matrix of $X$ implies that the diffusion part cannot be degenerate in any direction and hence it excludes processes with a pure jump component. These points are clarified in more details in Remark 2.3.

The goal of this paper is therefore, to extend the results of Pham [25, Section 5] to the more general situation where the multidimensional jump-diffusion process $X$ may have a pure jump component, and to weaken the regularity assumptions on the integral part of the generator $L$ of $X$, so to include the case where the jumps of $X$ are described by a random measure with Markov modulated compensator. Our approach is based on a change of measure argument: loosely speaking we start from a reference probability space where the local characteristics of $L$ are deterministic and we revert to the original model by changing probability. We consider the extended state process given by the pair $(X, \xi)$, where $\xi$ is the martingale density of the measure change. Under the reference probability, the new state process falls under the viscosity modeling framework of
Pham \cite{Pham2000}. Using Bayes formula and the results of Pham \cite{Pham2000} we then obtain that \( v \) is the unique viscosity solution of the backward PIDE associated with the operator \( \mathcal{L} \). Finally, in order to show that \( v \) is also a classical solution, we need to apply a fixed point argument.

The remainder of the paper is organized as follows. In Section 2 we introduce the problem and the main assumptions. In Section 3 we discuss the pricing and hedging of CAT bonds. In Section 4 we construct the process \( X \) via change of measure. Finally, we prove existence and uniqueness for the solution to the backward PIDE in Section 5.

2. Modeling framework and problem formulation

We fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), a time horizon \(T\) and a right continuous and complete filtration \(\mathbb{F}\). Consider measurable functions \(a : [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}\) and \(b : [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{d \times d}\), \(\gamma^Z : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{E} \to \mathbb{R}\) and \(\gamma^L : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{E} \to \mathbb{R}\), where \((\mathbb{E}, \mathcal{E})\) is a separable measurable space. Define the symmetric matrix \(\Sigma(t,z,l) = (\sigma_{i,j}(t,z,l), \ i, j = 1, \ldots, d)\), by

\[
\Sigma(t,z,l) = b(t,z,l)b^\top(t,z,l).
\]

Throughout the paper we use the following notation for partial derivatives: for every function \(h : [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}\) which is \(\mathcal{C}^2\) in \(z\) and continuous in \(l\), we write \(h_{z_i}\) for the first derivatives of \(h\) with respect to \(z_i\) for \(i \in \{1, \ldots d\}\) respectively, \(h_{z_i z_j}\) for second derivatives, for \(i, j \in \{1, \ldots, d\}\), and finally \(h_t\) denotes the first derivative with respect to time.

We assume that \((\Omega, \mathcal{F}, \mathbb{P})\) supports a RCLL process \(X = (Z, L)\), which is the unique solution of the martingale problem associated with the following (time-inhomogeneous) integro-differential operator \(\mathcal{L}_t\)

\[
\mathcal{L}_t \varphi(x) = \mathcal{L}_t \varphi(z,l) := \sum_{i=1}^{d} a_i(t,z,l) \varphi_{z_i}(z,l) + \frac{1}{2} \sum_{i,j=1}^{d} \sigma_{i,j}(t,z,l) \varphi_{z_i z_j}(z,l) \]

\[
+ \int_{\mathbb{E}} \left( \varphi(z + \gamma^Z(t,z,l,u), l + \gamma^L(t,z,l,u)) - \varphi(z,l) \right) \nu(t, x; du),
\]

for every \((z,l) \in \mathbb{R}^d \times \mathbb{R}\), \(t \in [0, T]\) and every function \(\varphi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}\) which is \(\mathcal{C}^2\) in \(z\) and continuous in \(l\), bounded with bounded derivatives.

It will be shown later that under some regularity conditions (specifically Assumptions 2.2 and 2.3 below) the unique solution to the martingale problem for the generator \(\mathcal{L}_t\) in (2.1) exists, see Corollary 1.2 below. This result is relevant from both a theoretical and also an applied point of view. In fact, problems involving Markov processes \(X = (Z, L)\) with a generator of the form (2.1) are largely used in an actuarial context (see, e.g. Grandell \cite{Grandell1991}, Ceci et al. \cite{Ceci2010}) and in a financial context (see, e.g. Bielecki and Rutkowski \cite{Bielecki2002}, Cartea et al. \cite{Cartea2009}, Colaneri et al. \cite{Colaneri2011}, Frey and Runggaldier \cite{Frey2004}); a specific application is discussed in Section 3 below.

To illustrate our setup we now give the generator for two simple examples.

Example 2.1. First, we consider the case where \(L\) is a time-homogeneous Cox process with intensity \(\lambda(Z_t)\), where \(Z\) is a one dimensional diffusion following the dynamics \(dZ_t = a(Z_t)dt + b(Z_t)dW_t\),
for a Brownian motion $W$. The generator $\mathcal{L}$ of the process $X = (Z, L)$ reads as

$$\mathcal{L}\varphi(x) = \mathcal{L}\varphi(z, l) = a(z)\varphi_z(z, l) + \frac{1}{2}b^2(z)\varphi_{zz}(z, l) + [\varphi(z, l + 1) - \varphi(z, l)]\lambda(z).$$

Second, assume more generally that $L$ is a time-homogenous compound Cox process with jump intensity $\lambda(Z_t)$ and jump size distribution $\mu(du)$ on $\mathbb{R}$. We still assume that $Z$ is a one dimensional diffusion of the same type as before. In this case the generator $\mathcal{L}$ of $X = (Z, L)$ has the form

$$\mathcal{L}\varphi(z, l) = a(z)\varphi_z(z, l) + \frac{1}{2}b^2(z)\varphi_{zz}(z, l) + \int_\mathbb{R} (\varphi(z, l + u) - \varphi(z, l)) \lambda(z)\mu(du);$$

in particular $\nu(z, l; du) = \lambda(z)\mu(du)$. Note that our setup goes beyond compound Cox processes presented in these two examples, since the form of the generator in (2.1) encompasses also models with joint jumps in $L$ and $Z$. This feature can be useful to model self-exciting phenomena.

We continue with the problem formulation. Let $g : \mathbb{R}^{d+1} \to \mathbb{R}$ be a payoff function, $f : [0, T] \times \mathbb{R}^{d+1} \to \mathbb{R}$ a dividend rate function and $c : [0, T] \times \mathbb{R}^{d+1} \to \mathbb{R}$ a discount rate. In the reminder of the paper we work under the following assumptions.

**Assumption 2.2.**

(A0) The functions $a, b, \gamma^Z, \gamma^L, f, g$ and $c$ are continuous.

(A1) The functions $a$ and $b$ are locally Lipschitz in $(t, x)$ and Lipschitz in $x$ for all $t \in [0, T]$.

(A2) There exists a finite measure $\tilde{\nu}(du)$ on $(E, \mathcal{E})$ such that the measure $\nu(t, x; du)$ is equivalent to $\tilde{\nu}(du)$; the Radon Nikodym derivative $\nu(t, x, u) := (d\nu(t, x)/d\tilde{\nu})(u)$ satisfies $\nu(t, x, u) \leq 1$ for all $(t, x, u) \in [0, T] \times \mathbb{R}^{d+1} \times E$.

(A3) The functions $\gamma^Z, \gamma^L$ and $\nu$ satisfy for all $t \in [0, T]$ and $x, y \in \mathbb{R}^{d+1}$.

$$|\gamma^Z(t, x, u) - \gamma^Z(t, y, u)| + |\gamma^L(t, x, u) - \gamma^L(t, y, u)| + |\nu(t, x, u) - \nu(t, y, u)| \leq \rho(u)|x - y|,$$

$$|\gamma^Z(t, x, u)| + |\gamma^L(t, x, u)| \leq \rho(u),$$

where the function $\rho : E \to \mathbb{R}^+$ is such that $\int_E \rho^2(u)\tilde{\nu}(du) < \infty$.

(A4) The function $c$ is bounded and locally Hölder continuous.

(A5) The functions $g$ and $f$ are bounded and satisfy for every $t, s \in [0, T]$ and $x, y \in \mathbb{R}^{d+1}$

$$|f(t, x) - f(s, y)| + |g(x) - g(y)| \leq K(|t - s| + |x - y|).$$

Note that by Assumption (A2), $\sup_{(t,x)\in[0,T] \times \mathbb{R}^{d+1}} \nu(t, x, E) < \tilde{\nu}(E) < \infty$, that is (A2) implies that the jump intensity of $L$ is bounded.

For fixed $l \in \mathbb{R}$ we now introduce the differential operator $\mathcal{L}^*$ by

$$\mathcal{L}^*_{(i,l)} \varphi(z) = \sum_{i=1}^d a_i(t, z, l)\varphi_{z_i}(z) + \frac{1}{2} \sum_{i,j=1}^d \sigma_{i,j}(t, z, l)\varphi_{z_i z_j}(z)$$

(2.2)

for every $z \in \mathbb{R}^d$ and $t \in [0, T]$ and every function $\varphi : \mathbb{R}^d \to \mathbb{R}$ with $\varphi \in \mathcal{C}^2$, bounded with bounded derivatives. Loosely speaking, $\mathcal{L}^*$ is the diffusion part of $\mathcal{L}$. 

**Assumption 2.3.** For every fixed \( l \in \mathbb{R} \) and every bounded and Lipschitz continuous function \( F : [0, T] \times \mathbb{R}^{d+1} \to \mathbb{R} \) the Cauchy problem

\[
\psi_t(t, z, l) + \mathcal{L}^*_t(t, l) \psi(t, z, l) + F(t, z, l) = c(t, z, l) \psi(t, z, l) \quad (t, z) \in [0, T] \times \mathbb{R}^d,
\]

\[
\psi(T, z, l) = g(z, l), \quad z \in \mathbb{R}^d,
\]

has a unique bounded classical solution.

Sufficient conditions for Assumption 2.3 to hold are given, for instance, in Friedman [18, Chapter 1]. They amount to assuming further to (A0)–(A5) in Assumption 2.2 that the functions \( a(t, z, l) \) and \( b(t, z, l) \) are bounded and that the matrix \( \Sigma(t, z, l) \) is uniformly elliptic in \( z \) on \( \mathbb{R}^d \), that is, there exists \( \tilde{C} > 0 \) such that for any \( \zeta \in \mathbb{R}^d \), \( \zeta^T \Sigma(t, z, l) \zeta \geq \tilde{C} \| \zeta \|^2 \) for every \( (t, z, l) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \). In the case where \( \mathcal{L}^* \) is the generator of an affine diffusion (not necessarily strictly elliptic), existence and uniqueness of the solution of the Cauchy problem in Assumption 2.3 is discussed, for instance in Cordoni and Di Persio [9].

The following theorem is the main result of the paper.

**Theorem 2.4.** Let (A0)–(A5) in Assumption 2.2 and Assumption 2.3 hold. Then the function \( v \) given by

\[
v(t, z, l) = \mathbb{E} \left[ \int_t^T e^{-\int_t^s f(u, Z_u)du} f(s, Z_s, L_s)ds + e^{-\int_t^T c(s, Z_s)ds} g(Z_T, L_T) \mid (Z_t, L_t) = (z, l) \right],
\]

is bounded, continuous on \( [0, T] \times \mathbb{R}^d \times \mathbb{R} \), Lipschitz in \( x = (z, l) \) uniformly in \( t \) and, for fixed \( l \), \( C^1 \) in \( t \) and \( C^2 \) in \( z \). Moreover \( v \) is a classical solution of the Cauchy problem

\[
\begin{align*}
\psi_t(t, z, l) + \mathcal{L}_t \psi(t, z, l) + f(t, z, l) = c(t, z, l) \psi(t, z, l), \quad (t, z, l) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}, \\
v(T, z, l) = g(z, l), \quad (z, l) \in \mathbb{R}^d \times \mathbb{R}.
\end{align*}
\]

Note that in Theorem 2.4, we obtain different degrees of regularity in \( z \) and \( l \): \( v \) is only Lipschitz in \( l \) but \( C^2 \) in \( z \). This reflects that there is a diffusion component in \( Z \) so that the transition kernel of \( X \) has a smoothing effect in the \( z \) direction, whereas no such smoothing can be expected in the \( l \) direction. The proof of the theorem is given in Section 5 and uses the results on viscosity solutions of Pham [25]. However, these results cannot be applied directly, as our setting does not fall under the assumptions of Pham [25], particularly due to the fact that the jump part of the process \( X \) is not driven by a Poisson random measure with Lipschitz coefficients but instead we have a random measure with Markov modulated compensator. Hence an intermediate step is

\[\text{1}\] Although the reader may convey that the process \( X \) is essentially a multidimensional jump-diffusion, with a possibly degenerate diffusion part, it is fundamental in our analysis to disentangle its (non-degenerate) jump-diffusion part \( Z \) and its pure jump part \( L \). Indeed, the process \( Z \) allows to define the operator \( \mathcal{L}^* \), and hence the problem in Assumption 2.3 for which existence and uniqueness of the solution is retrieved by classical literature, and for instance, implied by the property that the diffusion coefficient is uniformly elliptic. The main consequence of this fact is that the value function in the \( z \) component inherits a higher degree of regularity due to the smoothing effect of the diffusion. A pure jump process \( L \), instead, cannot bring more regularity than continuity to the value function in the component \( l \).
needed. Precisely, we will construct the model via a change of measure, in the same spirit as the
reference probability approach in nonlinear filtering. This is discussed in Section 4.

**Remark 2.5 (Contribution of the paper).** In order to clarify the contribution of our paper we
compare our setting with that of [25]. Section 5 of that paper is dedicated to the analysis of
smooth solutions for the Cauchy problem associated to a linear parabolic partial-integro differential
operator. There are two reasons why we cannot directly apply those results. First, [25, Section
5] assumes that the diffusion part of the process $X$ is uniformly elliptic (Assumption (H0) of
that paper). Here we relax this hypothesis and consider the case where the diffusion part can
be degenerate in some direction: indeed, ellipticity is needed only in the component $Z$ (see the
discussion after Assumption 2.3), and hence the process $X$ may have a pure jump component
as well, which is represented by the process $L$. Second, in our paper we relax the regularity
assumptions on the jump coefficients in both $Z$ and $L$. As in Pham [25], we consider jump size
functions $\gamma^Z$ and $\gamma^L$ that are Lipschitz. However, we do not restrict to the case where the jump
times are generated by a Poisson random measure with deterministic compensator but allow also
for stochastic jump intensities. In fact, although it might be possible to write our general jump
measure in terms of a Poisson random measure, this induces a transformation on the jump size
coefficients that would no longer satisfy the Lipschitz conditions. We elaborate on this point in
Example 2.6 below. We would like to underline once more that our extensions are relevant for
applications in insurance and finance, since problems involving jump-diffusion state processes with
degenerate diffusions in some directions and pure jump components driven by random measures
with Markov modulated compensators are frequently used in these fields.

**Example 2.6 (Example 2.1 continued).** We now give conditions ensuring that Theorem 2.4 applies
to the case of a compound Cox process. Assume that the functions $a(\cdot)$, $b(\cdot)$ and $\lambda(\cdot)$ are Lipschitz
and that $\lambda(z) \leq \overline{\lambda} < \infty$ for all $z \in \mathbb{R}$. In that case the generator $L$ of $X = (Z, L)$ satisfies the
regularity conditions from Assumption 2.2. In fact, we may choose $E = \mathbb{R}$, $\gamma^Z = 0$, $\gamma^L(t, z, u) = u$
and the reference measure $\tilde{\nu}(du) = \overline{\lambda}\mu(du)$ so that the density $d\nu(l, z; du)/d\tilde{\nu}(du)$ is given by the
Lipschitz function $\nu(z, l, u) = \lambda(z)/\overline{\lambda} \leq 1$. A sufficient condition for Assumption 2.3 to hold is
that the functions $a(\cdot)$ and $b(\cdot)$ are bounded and that $b^2(z) \geq \underline{b}$ for some $\underline{b} > 0$. Next, we explain
why a compound Cox process is not covered by the original results of Pham [25, Section 5]. Pham
[25] considers an integro differential operator with integral term of the form

$$\int_E [\varphi(z + \gamma^Z(z, l, u), l + \gamma^L(z, l, u)) - \varphi(z, l)]\nu(du) \quad (2.4)$$

for a finite measure $\nu(du)$ that is independent of the state $x$ of the process, and assumes that
the functions $\gamma^Z$ and $\gamma^L$ satisfy the Lipschitz condition (A3) from Assumption 2.2. It is indeed possible to write the integral part of the generator of a compound Cox process in the form (2.4) if we
construct $L$ via thinning. For doing this, we start from a compound Poisson process $\mathcal{T}$ with constant
jump intensity $\overline{\lambda}$ and jump size distribution $\mu$. At each jump time $\mathcal{T}_n$ of $\mathcal{T}$ we independently sample
a standard uniform random variable $V$ and we retain the jump if $V \leq \frac{\lambda(Z_{\mathcal{T}_n})}{\overline{\lambda}}$. This corresponds to
a representation (2.4) for the integral part of the generator of $X$: take $E = [0, 1] \times \mathbb{R}$ with elements
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$u = (v, w)$, let $\nu(du) = \nu(dvdw) = \sum \nu(du) \mu(dw)$, and put

$$\gamma^L(z, u) = \gamma^L(z, v, w) = w \mathbb{1}_{\{u \leq \lambda(z)\}}.$$

Note however, that the function $\gamma^L$ is not continuous and, in particular, it does not satisfy the Lipschitz condition (A3). Hence, the results of Pham [25] do not apply to a compound Cox process, not even his results on viscosity solutions, where the fact that $L$ is a pure jump process is not an issue.

3. Pricing and hedging of a CAT bond

In order to show that our results are relevant in insurance and finance we now discuss the problem of pricing and hedging of a catastrophe bond (CAT bond). CAT bonds are obligations with short maturities, (usually one to three years) issued by insurance and reinsurance companies to transfer to financial markets the risk of extreme losses in non-life insurance. The payoff of a CAT bond depends on some underlying loss index $L$ that measures the losses in a given pool of insurance contracts with specified loss type, geographical loss area and reporting period. If the loss index stays below a given threshold, investors (the buyers of the CAT bond) receive coupons and the face value at maturity; if the loss index is higher than the threshold, coupon payments are reduced and the face value is repaid only partially. Buyers accept the additional risk for a generous rate of return.

There exist many variations of CAT bonds, see for instance Cox and Pedersen [10], Lee and Yu [23], Jarrow [22], Jaimungal and Chong [21]. In this paper we model the loss index as a pure jump process $L$. We ignore coupon payments for simplicity, and we assume that at the payoff of the bond at the maturity $T$ is given by the face value $F$, reduced by the payoff of a loss layer on $L_T$ with attachment points $K_1$ and $K_2 = K_1 + \delta F$ for some $\delta \in [0, 1]$. Formally, the payoff at maturity is thus given by $g(L_T)$ for the bounded and Lipschitz continuous function

$$g(l) = F - \left( (l - K_1)^+ - (l - K_2)^+ \right); \quad (3.1)$$

in particular, for $L_T \leq K_1$ the bond pays the face value in full and for $L_T > K_2$ the bond has payoff $(1 - \delta)F$. The percentage $\delta$ plays the same role as the loss given default in credit risk. The goal is to determine the price of the CAT bond (in Section 3.1) and a self-financing hedging strategy that allows to cover for the interest rate risk (in Section 3.2). Due to market incompleteness we address the hedging problem via a quadratic approach. Under standard assumptions on the model dynamics the price of the bond is given by the solution of a PIDE with a pure jump component as in (2.3). We show that a classical solution of this PIDE is needed to determine the hedging strategy, as the latter involves the use of derivatives of the pricing function.

Comments. Of course other payoff functions than the one in (3.1) could be considered as well, provided that they are consistent with Assumption A5. For instance, Jaimungal and Chong [21] considers call and put options on the loss index. More generally, our analysis could also be

\[\text{The payoff of a call option is unbounded, but call options can usually be replaced by put options via put-call parity.}\]
extended to models as in Jarrow [22] where the “default time” $\tau$ of a CAT bond (the time when coupons and the repayment of the face value are reduced) is modeled as a doubly stochastic random time whose intensity might depend on the loss index.

Jaimungal and Chong [21] characterize the price of the CAT bond in terms of a PIDE that is similar to our pricing PIDE (Equation (3.4) below). However they simply assume the existence of a smooth solution and apply Fourier transform to compute the solution numerically. We mention a few other hedging problems in insurance and finance where smooth solutions of a PIDE of the type (2.3) play a crucial role: Ceci et al. [6] consider classical solution of a PIDE to determine the hedge ratio of unit-linked life insurance contracts under partial information; Ceci et al. [7] study the hedging of reinsurance counterparty credit risk; the hedging of derivatives in a high frequency data setting is considered in Frey and Runggaldier [17]. In all these papers the existence of a classical solution of the PIDE is assumed and not established. By giving sufficient conditions for the existence of a smooth solution to the pricing PIDE our paper puts the computation of hedge ratios in these papers on a sound mathematical footing.

3.1. The pricing problem. We introduce the dynamics of the loss index and of the short rate of interest. For this we fix a probability space $(\Omega, \mathcal{F}, Q)$ with a complete and right continuous filtration $\mathcal{F}$, and we interpret $Q$ as the pricing measure. Markets for CAT bonds are incomplete, and therefore, the choice of the pricing measure $Q$ is a delicate modelling issue involving also the real-world measure $P$; see for instance the discussion in Ceci et al. [7]. However, this question is not central to the present paper so that we specify the dynamics of all model quantities directly under $Q$. We also fix a time horizon $T$ and assume that $\mathcal{F} = \mathcal{F}_T$.

We begin with the loss index. We assume that the loss index is modeled as $L_t = \sum_{n=1}^{N_t} U_n$ and that the loss amounts $(U_n)_{n \in \mathbb{N}}$ are given by a sequence of independent identically distributed nonnegative random variables with distribution $\mu(du)$ on $[0, \infty)$. Following Jaimungal and Chong [21] we assume that the $N = (N_t)_{t \in [0,T]}$ is a point process with intensity $\lambda(t, Z_1^t)$ for a positive, bounded and Lipschitz function $\lambda : [0, T] \times \mathbb{R}^+ \times \mathbb{R} \to (0, \bar{\lambda}]$, with $\bar{\lambda} > 0$. The factor $Z_1^t = (Z_1^t)_{t \in [0,T]}$ that affects the intensity has dynamics $dZ_1^t = a_1(b_1 - Z_1^t)dt + \sqrt{1 - \rho^2}\sigma_1 dW_1^t + \rho\sigma_1 dW_2^t + \gamma Z_1^t dL_t \tag{3.2}$ for independent $Q$ Brownian motions $(W_1^t)_{t \in [0,T]}$ and $(W_2^t)_{t \in [0,T]}$ and constants $a_1, b_1, \gamma \geq 0$, $\sigma_1 > 0$, $\rho \in (-1, 1)$. Note that (3.2) allows for exogenous fluctuations in the loss intensity (due to the diffusion term) and for self-excitation: a loss event (a jump of $L$) causes an upward jump in the loss intensity which in turn raises the likelihood of future losses. This effect is dampened over time by the mean reverting drift. The intensity dynamics illustrates the rich modelling possibilities under Theorem [2.4]

We model the short rate of interest by $r_t = r(Z_2^t)$ for an increasing bounded and Lipschitz continuous function $r : \mathbb{R} \to \mathbb{R}$. For concreteness we assume that $Z_2^t = (Z_2^t)_{t \in [0,T]}$ follows a Vasicek
model
\[ dZ_t^2 = a_2(b_2 - Z_t^2)dt + \sigma_2 dW_t^2, \quad Z_0^2 \in \mathbb{R}^+, \]
with parameters \(a_2, b_2, \sigma_2 > 0\). Moreover we fix two constants \(\underline{r} < \overline{r}\) and we let \(r(z) = (z \vee \underline{r}) \wedge \overline{r}\). Assuming that the factor processes \(Z^1\) and \(Z^2\) are correlated depicts a certain dependence between losses and the market. Such effects are empirically observed: for instance natural catastrophes, or unexpected events such as the recent Covid-19 pandemic, affect both the loss index (typically by increasing the intensity of loss events) and the performance of financial markets.

We assume that a riskless zero coupon bond with maturity \(T\) and a CAT bond with maturity \(T\) and payoff \(g(L_T)\) are traded. The price of the zero coupon bond is given by
\[ P_t = \mathbb{E}^Q[e^{-\int_0^T r(s)ds} \mid \mathcal{F}_t]. \]
Using Theorem 2.4, we get that \(P_t = p(t, Z_t^2)\) for a function \(p(t, z_2) \in C^{1,2}([0, T] \times \mathbb{R})\) which solves
\[
p_t(t, z_2) + \mathcal{L}_t^{Z_2} p(t, z_2) = r(z_2)p(t, z_2), \quad (t, z) \in [0, T) \times \mathbb{R}, \quad \text{and} \quad p(T, z_2) = 1, \quad (3.3)
\]
where \(\mathcal{L}_t^{Z_2} \varphi(t, z_2) = a_2(b_2 - z_2)\varphi_{z_2}(t, z_2) + \frac{1}{2}\sigma_2^2\varphi_{z_2,z_2}(t, z_2)\), is the generator of the process \(Z_2\).\(^3\)
Moreover, we have the bond price dynamics
\[ dP_t = P_t r(Z_t^2)dt + P_t \beta(t, Z_t^2)dW_t^2 \]
where \(\beta(t, Z_t^2) = \sigma_2 p_{z_2}(t, Z_t^2)/p(t, Z_t^2)\). We underline that, for \(\underline{r}\) small and \(\overline{r}\) large we may identify \(Z_t^2\) and \(r_t\): the function \(p(t, r)\) can be approximated by the explicit bond price formula in the Vasicek model and \(\beta(\cdot)\) depends only on time; see Filipovic [13, Ch. 5 and Ch. 10].

Let \(X = (Z, L)\), where \(Z = (Z^1, Z^2)\) is the two dimensional factor process. By risk-neutral pricing the price of the CAT bond is given by
\[ P_t^{\text{CAT}} = p^{\text{CAT}}(t, X_t) =: \mathbb{E}^Q\left[e^{-\int_0^T r(s)ds} g(L_T) \mid \mathcal{F}_t \right]. \]
To identify the PIDE that characterizes the function \(p^{\text{CAT}}\) via Theorem 2.4, we first determine the generator of \(X\). Let \(z = (z_1, z_2)\) and define the measure \(\nu(t, z)\) on \(\mathbb{R}\) by \(\nu(t, z; du) = \lambda(t, z_1)\mu(du)\). Then the process \(X\) is Markov with generator
\[
\mathcal{L}_t \varphi(x) = a_1(b_1 - z_1)\varphi_{z_1}(z, l) + \frac{\sigma_1^2}{2}\varphi_{z_1,z_1}(z, l) + a_2(b_2 - z_2)\varphi_{z_2}(z, l) + \frac{\sigma_2^2}{2}\varphi_{z_2,z_2}(z, l) + \rho\sigma_1\sigma_2\varphi_{z_1,z_2}(z, l) + \int_{[0,\infty)} (\varphi(z_1 + \gamma^Z u, z_2, l + u) - \varphi(z_1, z_2, l)) \nu(t, z, du).
\]
By Theorem 2.4, the function \(p^{\text{CAT}}\) is the unique classical solution of the PIDE
\[
p_t^{\text{CAT}}(t, z, l) + \mathcal{L}_t p_t^{\text{CAT}}(t, z, l) = r(z_2)p_t^{\text{CAT}}(t, z, l), \quad (t, z, l) \in [0, T) \times \mathbb{R}^2 \times [0, +\infty), \quad (3.4)
\]
\[
p^{\text{CAT}}(T, z, l) = g(l), \quad (z, l) \in \mathbb{R} \times [0, +\infty). \quad (3.5)
\]
\(^3\)Notice that this result is covered by the classical Feymann-Kac formula, as the process \(Z^2\) follows a one dimensional diffusion with regular coefficients.
3.2. The hedging strategy. In this section we address the problem of finding a hedging strategy for the CAT bond that allows the bond holder to eliminate the interest rate risk of the bond. We recall that the market is incomplete since there are no financial instruments that allow to hedge the risk due to insurance losses. In this setting we apply a quadratic hedging criterion, namely, mean variance hedging, that allows to identify the unique self-financing strategy that covers the interest rate risk and minimizes the difference between the portfolio value and the value of the CAT bond at maturity in $L^2$-sense. We now formalize the hedging problem. Let $(h_t)_{t \geq 0} = (h^0_t, h^1_t)_{t \geq 0}$ be a self financing strategy, where $h^0$ represents the investment in the money market account with the price $(P^0_t)_{t \geq 0}$ and $h^1$ is the number of shares invested in the bond with the price $(P_t)_{t \geq 0}$. The discounted value of the strategy $h$ is given by

$$\tilde{V}(h) = v_0 + \int_0^t h^1_s d\tilde{P}_s$$

with $\tilde{P} = P/P^0$ being the discounted value of the bond. A self financing strategy $h$ is admissible if it is $\mathbb{F}$-predictable and satisfies $\mathbb{E} \left[ \int_0^T (h^1_t)^2 P^2_t dt \right] < \infty$.

Let $H = g(L_T)/P^0_T$ be the discounted payoff of the CAT bond. The hedging problem consists on finding a self-financing strategy $h^* = (h^0*, h^1*)$ with initial value $v_0$ which minimizes the quadratic hedging error

$$\mathbb{E}^{\mathbb{Q}} \left[ \left( H - \tilde{V}_T(h) \right)^2 \right],$$

over the set of all admissible strategies. The minimizer $h^*$ is called the mean variance hedging strategy. Since the process $\tilde{P}$ is a square integrable martingale, it is well known (see for instance Föllmer and Sondermann [15] or Schweizer [26]) that the optimal strategy can be determined using the Galtchouk–Kunita–Watanabe decomposition of the discounted CAT bond price process $\tilde{P}^{\text{CAT}} = P^{\text{CAT}}/P^0$ with respect to the discounted zero coupon bond price $\tilde{P}$. This decomposition is given by

$$\tilde{P}_t^{\text{CAT}} = P_0^{\text{CAT}} + \int_0^t \theta_s d\tilde{P}_s + O_t$$

where $(O_t)_{t \in [0,T]}$ is a martingale null at $t = 0$ and orthogonal to $\tilde{P}^{\mathbb{F}}$. The process $h^1* = \theta_t$ for all $t \in [0, T]$, provides the mean variance hedging strategy. Moreover, at time $T$ it holds that $\tilde{P}^{\text{CAT}} - \tilde{V}_T(h) = O_T$, so that $O_T$ represents the hedging error.

To characterize the hedging strategy $(\theta_t)_{t \in [0,T]}$, we derive the martingale representation of the (discounted) CAT bond price process. Recall that $P_t^{\text{CAT}} = p^{\text{CAT}}(t, Z_t, L_t)$, where $p^{\text{CAT}}(t, z, l)$

---

4 This condition guarantees that the discounted value process $V(h)$ is a square integrable martingale.

5 Two martingales $M^1$ and $M^2$ are said to be orthogonal if the product $M^1 M^2$ is a martingale, or, equivalently, if $M^1$ and $M^2$ have zero predictable quadratic covariation, i.e. $\langle M^1, M^2 \rangle_t = 0$ for all $t \geq 0$. 
satisfies the PIDE (3.4), with the final condition (3.5). Hence,
\[
\frac{d\tilde{P}_{t}^{\text{CAT}}}{\tilde{P}_{t}^{\text{CAT}}} = \sqrt{1 - \rho^2\beta_1^{\text{CAT}}(t, Z_t, L_t)}dW_t^1 + \left(\rho\beta_1^{\text{CAT}}(t, Z_t, L_t) + \beta_2^{\text{CAT}}(t, Z_t, L_t)\right) dW_t^2
\]
\[
+ \int_{[0, +\infty)} \left(\alpha^{\text{CAT}}(t, Z_t^1 + \gamma^Z u, Z_t^2, L_t - u) - 1\right) \left(m(dt, dz) - \nu(t, Z_t^1, Z_t^2, du)\right)
\]
where \(\beta_i^{\text{CAT}}(t, z, l) = \sigma_i p_{z_1}^{\text{CAT}}(t, z, l)/p^{\text{CAT}}(t, z, l)\) for \(i = 1, 2\), \(\alpha^{\text{CAT}}(t, z_1, \gamma, z_2, l + u) = p^{\text{CAT}}(t, z_1 + \gamma, z_2, l + u)/p^{\text{CAT}}(t, z_1, z_2, l)\), and the measure \(m(dt, du)\) denotes the jump random measure of the process \(L\). Since \(d\tilde{P}_t = \tilde{P}_t\beta(t, Z_t^2)dW_t^2\), we get
\[
\frac{d\tilde{P}_t^{\text{CAT}}}{\tilde{P}_t^{\text{CAT}}} = \left(\rho\beta_1^{\text{CAT}}(t, Z_t, L_t) + \beta_2^{\text{CAT}}(t, Z_t, L_t)\right) d\tilde{P}_t^\rho + dM_t
\]  
(3.7)
where \((M_t)_{t \in [0, T]}\) is a martingale null at \(t = 0\) and orthogonal to the process \(\tilde{P}\) \(^6\). Comparing equation (3.6) and (3.7) we obtain the optimal strategy
\[
h_t^* = \theta(t, z_1, z_2, l) = \frac{p_{z_2}^{\text{CAT}}(t, z_1, z_2, l)}{p_{z_2}(t, z_2)} + \frac{\rho_1}{\rho_2} \frac{p_{z_2}^{\text{CAT}}(t, z_1, z_2, l)}{p_{z_2}(t, z_2)},
\]
and the hedging error \(O_t = M_t\) for all \(t \in [0, T]\).

Notice that \(\theta\) is written in terms of the derivative of the functions \(p\) and \(p^{\text{CAT}}\), which means that the computation of the hedging strategy requires that these functions are regular and hence a classical solution of the PDE (3.3) and the PIDE (3.4), respectively.

4. Construction via change of measure

We start from a probability space \((\Omega, \mathcal{F}, \tilde{P})\) with a filtration \(\mathbb{F}\), that supports a \(d\)-dimensional-Brownian motion \(W\) and a Poisson random measure \(N(dt, du)\) on \([0, T] \times E\) with \((\mathbb{F}, \tilde{P})\)-compensator \(\tilde{v}(du)dt\), where \(\tilde{v}\) and \(E\) are as in (A2) of Assumption 2.2. Let \(X = (Z, L)\) be the unique strong solution to the following system of SDEs
\[
dZ_t = a(t, X_t)dt + b(t, X_t)dW_t + \int_E \gamma^Z(t, X_t^-, u)N(dt, du), \quad Z_0 = z \in \mathbb{R}^d,
\]  
(4.1)
\[
dL_t = \int_E \gamma^L(t, X_t^-, u)N(dt, du), \quad L_0 = l \in \mathbb{R},
\]  
(4.2)

\(^6\)The process \((M_t)_{t \in [0, T]}\) satisfies
\[
dM_t = \tilde{P}_t^{\text{CAT}} \sqrt{1 - \rho^2\beta_1^{\text{CAT}}(t, Z_t, L_t)}dW_t^1
\]
\[
+ \tilde{P}_t^{\text{CAT}} \int_{[0, +\infty)} \left(\alpha^{\text{CAT}}(t, Z_t^1 + \gamma^Z u, Z_t^2, L_t - u) - 1\right) \left(m(dt, dz) - \nu(t, Z_t^1, Z_t^2, du)\right).
\]
where the functions $a$, $b$, $\gamma^z$ and $\gamma^l$ satisfy (A0), (A1) and (A3) in Assumption 2.2. The process $X$ is Markov under $\tilde{P}$ with the generator

$$\tilde{L}_t \varphi(z,l) = \sum_{i=1}^{d} a_i(t,z,l) \varphi_{z_i}(z,l) + \frac{1}{2} \sum_{i,j=1}^{d} \sigma_{i,j}(t,z,l) \varphi_{z_i,z_j}(z,l)$$

$$+ \int_{\mathcal{E}} [\varphi(z + \gamma^z(t,z,l,u), l + \gamma^l(t,z,l,u)) - \varphi(z,l)] \tilde{\nu}(du) ,$$

for every $(z,l) \in \mathbb{R}^d \times \mathbb{R}$ and every $t \in [0,T]$ and every function $(z,l) \to \varphi(z,l)$, $C^2$ in $z$ and continuous in $l$, bounded with bounded derivatives.

Using the Radon Nikodym density $\nu(t,x,u) = (d\nu(t,x)/d\tilde{\nu})(u)$ introduced in (A2) of Assumption 2.2 we define the process $\xi = (\xi_{t})_{t \in [0,T]}$ as the stochastic exponential

$$\xi_{t} = 1 + \int_{0}^{t} \xi_{s-} \int_{\mathcal{E}} (\nu(s,X_{s-},u) - 1) (N(ds,du) - \tilde{\nu}(du)ds), \quad t \in [0,T]. \quad (4.3)$$

Applying the Doléans-Dade exponential formula we get that

$$\xi_{t} = \prod_{T_n \leq t} \nu(T_n, X_{T_n-}, U_n) \exp \left( \int_{0}^{t} \int_{\mathcal{E}} (1 - \nu(s,X_{s-},u)) \tilde{\nu}(du)ds \right), \quad t \in [0,T],$$

where here $(T_n, U_n)_{n \geq 1}$ is the sequence of jump times and corresponding jump sizes of the measure $N(dt,du)$.

In the sequel we will need the following lemma.

**Lemma 4.1.** The process $\xi = (\xi_{t})_{t \in [0,T]}$ is bounded and satisfies $\xi_{t} \leq e^{\tilde{\nu}(E)t} \tilde{P}$-a.s., for every $t \in [0,T]$. Let $P$ be the probability measure equivalent to $\tilde{P}$ defined by $dP \big|_{\mathcal{F}_t} = \xi_{t}$. Then under $P$, $N(dt,du)$ is a random measure with compensator $\nu(t,X_{t-},du)dt$ and $W$ is an $(\mathbb{F},P)$-Brownian motion.

**Proof.** Since $\nu(t,x,u) \leq 1$ by (A2) in Assumption 2.2 using the exponential form of $\xi$, we get that

$$\xi_{t} \leq \exp \left( \int_{0}^{t} \tilde{\nu}(E)ds \right) = e^{\tilde{\nu}(E)t}, \quad P\text{-a.s., } t \in [0,T].$$

Hence, the process $\xi$ is a true martingale as it is a bounded local martingale with $\mathbb{E}[\xi_{T}] = 1$, where $\mathbb{E}$ denotes the expectation under the probability measure $\tilde{P}$. All the other claims follow directly from the Girsanov Theorem for marked point processes, see, e.g. Brémaud [1, Theorem VIII.2].

**Corollary 4.2.** Under Assumptions 2.2 and 2.3 there exists a unique solution of the martingale problem for the operator $\mathcal{L}_t$ given in (2.1).

**Proof. Existence.** By Lemma 4.1 we know that $W$ is an $(\mathbb{F},P)$-Brownian motion and that the random measure $N(du,dt)$ has the compensator $\nu(t,X_{t-},du)dt$. Hence, for any function $\varphi$:
\[ [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \text{ which is } C^1 \text{ in } t, C^2 \text{ in } z, \text{ continuous in } l, \text{ bounded with bounded derivatives, by applying Itô’s lemma we get that} \]
\[
\varphi(t, Z_t, L_t) = \varphi(0, Z_0, L_0) + \int_0^t \mathcal{L}_u \varphi(u, Z_u, L_u) + M_t
\]
where \( M \) is the martingale given by
\[
M_t = \int_0^t b(s, X_s) \varphi_{z_1}(s, X_s) dW_s
\]
\[
+ \int_0^t \int_E (\varphi(s, Z_{s-} + \gamma(s, X_{s-}, u), L_{s-} + \gamma^L(s, X_{s-}, u))) (N(du, ds) - \nu(s, X_{s-}, du)ds),
\]
so that under \( P \), the process \( X = (Z, L) \) solves the martingale problem associated with \( \mathcal{L}_t \).

**Uniqueness.** Here we rely on the well known result that the martingale problem for \( \mathcal{L} \) has a unique solution if the marginal distributions of any solution process \( X \) are uniquely determined, see for instance Ethier and Kurtz [14, Proposition 4.7 and Remark 4.8, Chapter 4]. Now by Theorem 2.4, we get existence of a smooth solution to the backward PIDE with \( c(t, x) = 0 \) and \( f(t, x) = 0 \) for all bounded and Lipschitz continuous terminal conditions \( g \), which immediately implies uniqueness of the marginal distributions of \( X \) and hence the result. □

5. **Proof of Theorem 2.4**

This section is devoted to the proof of the main result of this paper. We denote \( \tilde{\nu}(E) = \tilde{\lambda} \) and define the set \( \bar{D} \subseteq [0, T] \times \mathbb{R}^{d+1} \times \mathbb{R}^+ \) as
\[
\bar{D} = \left\{ (t, x, \xi) \in [0, T] \times \mathbb{R}^{d+1} \times \mathbb{R}^+ : \xi \leq e^{\tilde{\lambda} t} \right\}.
\]
For \((t, x, \xi) \in \bar{D}\), let the function \( \tilde{v} \) be defined as
\[
\tilde{v}(t, x, \xi) := \tilde{E} \left[ \xi_T \left( e^{-\int_0^T c(s, X_s)ds} g(X_T) + \int_0^T e^{\int_s^T c(u, X_u)du} f(s, X_s)ds \right) | X_t = x, \xi_t = \xi \right].
\]
where we recall that \( \tilde{E} \) indicates the expectation under the probability measure \( \tilde{P} \). By applying Bayes formula we get that for every \((t, x, \xi) \in \bar{D}\),
\[
v(t, x) = \frac{\tilde{v}(t, x, \xi)}{\xi}.
\]
The remainder of the proof is organised as follows: in Step 1 we consider the triple \( \tilde{X} = (Z, L, \xi) \) with the generator \( \tilde{\mathcal{L}}^{\tilde{X}} \), and we use the results on viscosity solutions of Pham [25] to show that \( \tilde{v} \) is a Lipschitz continuous viscosity solution of a backward equation involving the operator \( \tilde{\mathcal{L}}^{\tilde{X}} \). From this we can conclude in Step 2 that \( v \) is a Lipschitz continuous viscosity solution of the original backward PIDE (2.3). Finally, in Step 3 we use a fixed point argument to establish that \( v \) is also a classical solution of that equation.
Step 1. First we show that for \((t, x, \xi) \in \bar{D}\),

\[
\tilde{v}(t, x, \xi) = \mathbb{E} \left[ \xi_T e^{- \int_t^T c(s, X_s) \, ds} g(X_T) + \int_t^T \xi_s e^{\int_s^t c(u, X_u) \, du} f(s, X_s) \, ds \mid X_t = x, \xi_t = \xi \right]. \tag{5.2}
\]

Indeed this follows from the sequence of equalities

\[
\begin{align*}
\mathbb{E} \left[ \xi_T \int_t^T e^{\int_s^t c(u, X_u) \, du} f(s, X_s) \, ds \mid X_t = x, \xi_t = \xi \right] &= \int_t^T \mathbb{E} \left[ \xi_T e^{\int_s^t c(u, X_u) \, du} f(s, X_s) \mid X_t = x, \xi_t = \xi \right] \, ds \\
&= \int_t^T \mathbb{E} \left[ \xi_s e^{\int_s^t c(u, X_u) \, du} f(s, X_s) \mid X_t = x, \xi_t = \xi \right] \, ds \\
&= \mathbb{E} \left[ \int_t^T \xi_s e^{\int_s^t c(u, X_u) \, du} f(s, X_s) \, ds \mid X_t = x, \xi_t = \xi \right]
\end{align*}
\]

where we get the first and third equalities by applying the Fubini Theorem, since \(\xi_T, c\) and \(f\) are bounded, and the second equality follows from the tower rule when conditioning on \(\mathcal{F}_s\).

We now consider the triple \(\tilde{X} = (Z, L, \xi)\). Under Assumptions 2.2 the process \(\tilde{X}\) is a strong solution of the system of SDEs \((4.1) - (4.2) - (4.3)\), driven by an exogenous Poisson random measure. By Lemma 4.1, the process \(\xi\) is bounded and therefore we may consider the system on the state space \(\bar{D}\). Denote by \(\hat{\phi}^\tilde{X}\) the \(\hat{P}\)-Markov generator of the process \(\tilde{X}\). It holds that

\[
\hat{\phi}^\tilde{X}_i \varphi(z, l, \xi) = \sum_{i=1}^d a_i(t, z, l) \varphi_{z_i}(z, l, \xi) + \frac{1}{2} \sum_{i,j=1}^d \sigma_{i,j}(t, z, l) \varphi_{z_i, z_j}(z, l, \xi) \\
- \left( \xi \int_E (\nu(t, z, l, u) - 1) \tilde{v}(du) \right) \varphi_{\xi}(z, l, \xi) \\
+ \int_E \left[ \varphi \left( z + \gamma^Z(t, z, l, u), l + \gamma^L(t, z, l, u), \xi \nu(t, z, l, u) \right) - \varphi(z, l, \xi) \right] \tilde{v}(du)
\]

for every \((t, z, l, \xi) \in \bar{D}\) and for every function \((z, l, \xi) \to \varphi(z, l, \xi)\) which is bounded, \(C^2\) in \(z\), continuous in \(l\) and \(C^1\) in \(\xi\). Here \(\varphi_{\xi}\) indicates the first derivative of \(\varphi\) with respect to \(\xi\).

Note that the system \((4.1) - (4.2) - (4.3)\) satisfies Conditions (2.1)–(2.6) in Pham [25]. Indeed, Conditions (2.1), (2.2), (2.5) and (2.6) follow directly from our assumptions (A.1), (A.2), (A.4) and (A.5); Conditions (2.3) and (2.4) follow from (A.2), (A.3) and from the fact that on \(\bar{D}\) the mapping

\[
(t, x, \xi) \to \xi \left( \nu(t, x, u) - 1 \right)
\]

is bounded and Lipschitz, as it is a product of two bounded Lipschitz functions. Therefore we can now apply Pham [25, Theorem 3.1 and Proposition 3.3] and get that the function \(\tilde{v}\) in equation \((5.1)\) is continuous in \(\bar{D}\) and Lipschitz in \((x, \xi)\), uniformly in \(t\) (i.e. \(\tilde{v} \in W^1(\bar{D})\)). Moreover \(\tilde{v}\) is a
Step 2. Let \( \phi : [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \) be a smooth function and define the function \( \tilde{\phi} : \bar{D} \to R \) by \( \tilde{\phi}(t, z, l, \xi) = \phi(t, z, l)\xi \). Then for every \((t, z, l, \xi) \in \bar{D}\) we have that

\[
\mathcal{L}^X_{i} \tilde{\phi}(t, z, l, \xi) = \xi \left\{ \sum_{i=1}^{d} a_i(t, z, l)\phi_{z_i}(t, z, l) + \frac{1}{2} \sum_{i,j=1}^{d} \sigma_{i,j}(t, z, l)\phi_{z_i,z_j}(t, z, l) \right. \\
+ \left. \int_{\mathcal{E}} \left[ \phi(t, z + \gamma^{Z}(t, z, l, u), l + \gamma^{L}(t, z, l, u)) - \phi(t, z, l) \right] \nu(t, z, l; du) \right\},
\]

and this is of course equal to \( \xi L_{i} \phi(t, z, l) \). Consequently we see that \( \tilde{v} \) is a viscosity solution of the original backward PIDE (2.3).

Step 3. We finally want to show that function \( v \) is a classical solution of the backward PIDE (2.3) and hence, in particular, that \( v \) is \( C^1 \) in \( t \) and \( C^2 \) in \( z \). For this we modify the fixed point argument used in the proof of Pham [25, Proposition 5.3]. Define for a bounded function \( \varphi : [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \) that is Lipschitz in \((z, l)\), uniformly in \( t \), the function \( F[\varphi] \) as

\[
F[\varphi](t, z, l) := \int_{\mathcal{E}} \left[ \varphi(t, z + \gamma^{Z}(t, z, l, u), l + \gamma^{L}(t, z, l, u)) - \varphi(t, z, l) \right] \nu(t, z, l; du).
\]

Using (A2) and (A3) in Assumption 2.2 it is easily seen that \( F[\varphi] \) is Lipschitz in \((z, l)\), uniformly in \( t \).

Recall the definition of the differential operator \( \mathcal{L}^*_{(t, l)} \) from equation (2.2). It follows from Pham [25, Lemma 2.1] that \( v \) is also a viscosity solution of the backward equation

\[
v_t(t, z, l) + \mathcal{L}^*_{(t, l)} v(t, z, l) + F[\varphi](t, z, l) = c(t, z, l)\varphi(t, z, l) + f(t, z, l), \quad (t, z) \in [0, T] \times \mathbb{R}^d \tag{5.3}
\]

(see Pham [25, page 22] for details). Note that equation (5.3) is a linear parabolic partial differential equation (and not a PIDE), and a bounded classical solution \( u(t, z, l) \) to (5.3) exists by Assumption 2.3. Now \( u \) is clearly also a viscosity solution of (5.3). Uniqueness results for viscosity solutions of linear parabolic PDEs imply that \( u = v \) and the regularity of \( v \) follows.

Finally, uniqueness of classical solutions of (2.3) follows, for instance, from Pham [25, Proposition 5.2].

\footnote{Notice that Pham [25, Proposition 5.3] cannot be applied directly to the function \( \tilde{v} \), since that result requires uniform ellipticity of the diffusion coefficient of \( X \) (which is not satisfied in our setup due to the presence of the pure jump component \( L \)).}
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