GAUSS QUADRATURE FOR FREUD WEIGHTS, MODULATION SPACES, AND MARCINKIEWICZ-ZYGMUND INEQUALITIES

MARTIN EHLER AND KARLHEINZ GRÖCHENIG

Abstract. We study Gauss quadrature for Freud weights and derive worst case error estimates for functions in a family of associated Sobolev spaces. For the Gaussian weight \( e^{-\pi x^2} \) these spaces coincide with a class of modulation spaces which are well-known in (time-frequency) analysis and also appear under the name of Hermite spaces. Extensions are given to more general sets of nodes that are derived from Marcinkiewicz-Zygmund inequalities. This generalization can be interpreted as a stability result for Gauss quadrature.

1. Introduction

The fame of Gauss quadrature partially comes from the connection to the theory of orthogonal polynomials and the fact that a quadrature rule based on \( n \) points in \( \mathbb{R} \) is exact for polynomials of degree \( \leq 2n - 1 \). Although this is standard material in undergraduate text books on numerical analysis, Gauss quadrature, and quadrature rules in general, remain an active topic of research. In this paper we study the quadrature that is associated to the orthogonal system of Hermite polynomials. This so-called Gauss-Hermite quadrature is a quadrature rule for integrals on the whole real line, whose success and efficiency is not uncontested in the literature. Therefore Gauss-Hermite quadrature is the subject of many recent investigations.

The basic results for Gauss-Hermite quadrature are explicit error bounds in classical style involving a weighted norm of the derivative of the function \[33\]. However, Trefethen \[42\] argues that a significant portion of the nodes and weights is below machine precision and thus numerically irrelevant, and he suggests to ignore large nodes.

In several recent papers, Dick, Irrgeher, Pillichshammer et al. in various combinations \[6, 22, 24\] have introduced function spaces on \( \mathbb{R} \) and \( \mathbb{R}^d \), which they call Hermite spaces, for which they derive strong error estimates. In addition, these authors show that in certain cases digital nets outperform Gauss-Hermite nodes \[6\].

Misspecified settings are considered in \[27\], and adaptive Gauss-Hermite quadrature is discussed in \[26, 31\]. Recently, the trapezoidal rule has shown better performance than Gauss-Hermite nodes, see \[28\]. Further integration problems with Gauss-Hermite nodes are discussed in \[39\], and, for hyperinterpolation on general domains, we refer to \[38\].

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With this background we take another look at Gauss-Hermite quadrature and its
generalizations. Our specific contributions can be summarized as follows:
(i) We study Gauss quadrature and error estimates for Gauss quadrature with
respect to the Freud polynomials which are orthogonal with respect to the weight
function \(e^{-2\pi|x|^{\alpha}}\) for \(\alpha > 1\). This quadrature includes in particular the Gauss-
Hermite quadrature.
(ii) We introduce a scale of function spaces that are adapted to the weight and
lead to natural error estimates for Gauss quadrature. These function spaces can
be viewed as the appropriate versions of Sobolev spaces in the context of orthogo-
nal polynomials. The motivation comes from the study of Marcinkiewiez-Zygmund
inequalities in [13,19,32,34] and the work of Irrgeher et al. [6,22–24]. For the Gauss-
ian weight \(e^{-\pi|x|^2}\) the latter authors have introduced these spaces (Hermite spaces
in their terminology) to study quadrature rules from the point of view of complexity
theory.
As a typical result we offer a (slightly vague) version of the main theorem: Let
\(W(x) = e^{-\pi|x|^{\alpha}}\) for \(\alpha > 1\) and let \(H_n\) be the orthogonal polynomials with
respect to the weight \(W(x)^2\). Then the functions \(h_k = H_kW\) form an orthonormal ba-
sis for \(L^2(\mathbb{R})\). The associated Sobolev space \(H^s\) is defined by the norm
\[\|f\|_{H^s}^2 = \sum_{k=0}^\infty |\langle f, h_k \rangle|^2 (1 + k)^s.\]
Next assume that the set of nodes \(X_n = \{x_1, \ldots, x_N\} \subseteq \mathbb{R}\)
satisfies the Marcinkiewicz-Zygmund inequalities (with weights \(\tau(x_j)\)) for the finite-
dimensional subspace \(\Pi_n = \text{span} \{h_0, \ldots, h_n\}\), i.e.,
\[a\|f\|_2^2 \leq \sum_{j=1}^N |f(x_j)|^2 \tau(x_j) \leq b\|f\|_2^2, \quad \text{for all } f \in \Pi_n.\]
Under a small technical assumption on the spread of \(X_n\), we prove the existence of
weights \(\omega_j \in \mathbb{R}, j = 1, \ldots, N\) such that
\[\sup_{\|f\|_{L^2} \leq 1} \left| \int_{-\infty}^{\infty} f(x)W(x)dx - \sum_{j=1}^N \omega_j f(x_j) \right|^2 \lesssim n^{-s+\frac{4}{3}}.\]
This may also be read as a stability result of Gaussian quadrature. If \(\{x_j\}_{j=1}^n\)
are small distortions of the Gaussian quadrature nodes, then \(\{\omega\}^n_{j=1}\) are only small dis-
tructions (hence positive) of the associated Gaussian quadrature weights. In general
and in contrast to Gauss quadrature the number of nodes \(N\) need not be identical
to the dimension of \(\Pi_n\). Our proof technique is applicable in much more general
settings and the term \(\frac{4}{3}\) in the error estimate is probably an artifact of this generality.
(iii) For \(\alpha = 2\), i.e., for the Gaussian weight \(e^{-\pi x^2}\) we show that the Hermite
spaces of [6,22,24] coincide with a class of well-known function spaces in analysis,
namely the modulation spaces introduced by Feichtinger [9,10] in 1983. Thus the
abstract definition of “Hermite space” or “appropriate Sobolev space” turns out
to be perfectly natural and leads to a class of well-studied and important function
spaces. In fact, the Hermite spaces aka modulation spaces are precisely the Sobolev
spaces with respect to the harmonic oscillator and known as Shubin classes in the
PDE literature [36]. We feel that these identifications are of independent interest,
as they add new tools to the study of numerical quadrature related to Hermite
polynomials. We mention that modulation spaces are the canonical function spaces in time-frequency analysis [16]. They have many applications in the analysis of pseudodifferential operators and even in wireless communications [3,16,17] and are used for the analysis of nonlinear PDE [1], the formulation of uncertainty principles, in the theory of coherent states (Gabor analysis) and many more, see also [11,12]. With some satisfaction, we may now say that modulation spaces are also useful in numerical analysis.

(iv) We extend the error analysis to general sets of nodes beyond Gaussian nodes (which are the zeros of the $n + 1$-st orthogonal polynomial). The advantage is that one can build in redundancy so that a missing node does not spoil the result of the numerical quadrature.

On the technical level, we highlight the role of Marcinkiewicz-Zygmund inequalities in the derivation of quadrature nodes and weights. A fundamental tool are precise and deep estimates from [29,30] for the size of orthogonal polynomials with respect to Freud weights and for the associated Christoffel functions. Specifically, lower bounds for the Christoffel function are used to derive tail estimates for the reproducing kernel of the associated Sobolev spaces.

The outline is as follows: In Section 2, we collect the basic definitions, introduce the class of orthogonal polynomials and the associated Sobolev-type spaces. We derive some preparatory inequalities for the reproducing kernels in these spaces that are used in Sections 3 and 7. We establish error bounds for Gaussian quadrature nodes in Section 3. In Section 4, we prove that the Hermite spaces of [6,22,24] coincide with classical modulation spaces. Some numerical experiments for modulation spaces with exponential weights are presented in Section 5. Sections 6 and 7 are dedicated to error bounds beyond Gaussian nodes. We recast the quadrature problem by means of general Hilbert spaces in Section 6. The actual generalization beyond Gaussian nodes is derived in Section 7.

2. Sparsity classes for Hermite type functions

For $\alpha > 1$, we consider the Freud weight function

$$W(x) = e^{-\pi|x|^\alpha}, \quad x \in \mathbb{R}.$$ 

The orthonormal polynomials with respect to $W^2$ are called Freud polynomials which we denote by $\{H_k\}_{k \in \mathbb{N}}$, so that $\deg(H_k) = k$. Consequently the functions

$$h_k := H_k W, \quad k \in \mathbb{N},$$

form an orthonormal basis for $L^2(\mathbb{R})$. For $f \in L^2(\mathbb{R})$, we denote the coefficient with respect to this basis by

$$\hat{f}_k := \langle f, h_k \rangle_{L^2(\mathbb{R})}, \quad k \in \mathbb{N}.$$  

Fix $s \geq 0$, $p, q > 0$ and consider the inner products

$$\langle f, g \rangle_{H^s} = \sum_{k \in \mathbb{N}} (1 + k^s) \hat{f}_k \overline{\hat{g}_k}, \quad \text{and} \quad \langle f, g \rangle_{E^p_q} = \sum_{k \in \mathbb{N}} e^{qk \alpha} \hat{f}_k \overline{\hat{g}_k}.$$
Then the corresponding norms induce the function spaces
\[
\mathbb{H}^s = \{ f \in L^2(\mathbb{R}) : \| f \|_{\mathbb{H}^s} < \infty \}, \quad s \geq 0,
\]
\[
\mathbb{E}^p_q = \{ f \in L^2(\mathbb{R}) : \| f \|_{\mathbb{E}^p_q} < \infty \}, \quad p, q > 0,
\]
see also [41]. The \( \mathbb{H}^s \) are the appropriate versions of Sobolev spaces for the specific basis \( \{ h_k \}_{k \in \mathbb{N}} \).

For the exponential weights \( e^{qk^p} \), the \( \mathbb{E}^p_q \) are related to Hilbert spaces of analytic functions. We refer to both as Sobolev spaces in the following and to Lemma 4.2 and 4.3 for a more detailed explanation why \( \mathbb{H}^s \) and \( \mathbb{E}^p_q \) are indeed versions of Sobolev spaces.

For the study of the Sobolev spaces \( \mathbb{H}^s \) and \( \mathbb{E}^p_q \) and for the error analysis of quadrature rules we need some detailed estimates on the associated (reproducing) kernels.

**Proposition 2.1.** For \( s > 1 - \frac{1}{\alpha} \) and \( p, q > 0 \), we have
\[
\sum_{k=n}^{\infty} (1 + k)^{-s} |h_k(x)|^2 \lesssim (1 + n)^{-s + 1 - \frac{1}{\alpha}},
\]
\[
\sum_{k=n}^{\infty} e^{-qk^p} |h_k(x)|^2 \lesssim e^{-qn^p} (1 + n)^{\frac{1}{p} - \frac{1}{q} + \max(1-p,0)}.
\]

The constants depend on \( s, \alpha, \) and \( p \), but they are independent of \( x \) and \( n \).

**Proof.** We define the Christoffel function associated to the orthonormal basis \( h_k \) by
\[
\Lambda_n := \frac{1}{\sum_{k=0}^{n} |h_k|^2}.
\]
Note that \( \Lambda_n \) differs from the usual Christoffel function \( (\sum_{k=0}^{n} H_k^2)^{-1} \) by a factor \( W^{-2} \). The precise growth of \( \Lambda_n \) is investigated in [29]. In particular, [29, Eq. (1.13)] leads to the estimate
\[
\sum_{k=0}^{n} |h_k|^2 \lesssim n^{1 - \frac{1}{\alpha}}, \quad n \geq 1.
\]

(i) Polynomial weights: Since \( H_0 \) is constant and \( h_0 \) is bounded, without loss of generality, we may assume \( n \geq 1 \). Choose \( M \in \mathbb{N} \), such that
\[
2^M \leq n < 2^{M+1}.
\]

\(^1\)We write \( \lesssim \) if the left-hand-side is bounded by a constant times the right-hand-side. If \( \lesssim \) and \( \gtrsim \) both hold, then we write \( \asymp \).
Then dyadic summation and the bound for the Christoffel function \cite{2} yield
\begin{align*}
\sum_{k=n}^{\infty} (1 + k)^{-s} |h_k(x)|^2 &\leq \sum_{m=M}^{\infty} \sum_{2^m \leq k < 2^{m+1}} 2^{-sm} |h_k(x)|^2 \\
&\leq \sum_{m=M}^{\infty} 2^{-sm} 2(m+1)(1-\frac{1}{n}) \\
&\leq 2^{-sM} 2(M+1)(1-\frac{1}{n}) \sum_{m=M}^{\infty} 2^{-s(m-M)} 2(m-M)(1-\frac{1}{n}) \\
&\leq n^{-s+1-\frac{1}{n}} \sum_{m=0}^{\infty} 2^{-m(s-1+\frac{1}{n})}.
\end{align*}

Since \(s > 1 - \frac{1}{n}\), the series \(\sum_{m=0}^{\infty} 2^{-m(s-1+\frac{1}{n})}\) converges.

(ii) Exponential weights: Here we use the following estimate for the supremum norm of the orthogonal basis from \cite{29}, Eq. (1.7)],
\begin{equation}
(3) \sup_{x \in \mathbb{R}} |h_k(x)|^2 \lesssim k^\frac{1}{3} - \frac{1}{\alpha}, \quad k \geq 1.
\end{equation}

This bound leads to
\begin{align*}
\sum_{k=n}^{\infty} e^{-qk^p} |h_k(x)|^2 &\lesssim \sum_{k=n}^{\infty} e^{-qk^p} \frac{k^{\frac{1}{3} - \frac{1}{\alpha}}}{k^{\frac{1}{p}}} \\
&\lesssim \sum_{m=M}^{\infty} e^{-qm} (m + 1) \frac{\frac{1}{3} - \frac{1}{\alpha}}{m^{\frac{1}{p} - 1}} \\
&\lesssim e^{-qM} (M + 1)^{\frac{\frac{1}{3} - \frac{1}{\alpha}}{\frac{1}{p} - 1}} \sum_{m=M}^{\infty} e^{-q(m-M)} \left( \frac{m + 1}{M + 1} \right)^{\frac{\frac{1}{3} - \frac{1}{\alpha}}{\frac{1}{p} - 1}} \\
&\lesssim e^{-qM} (M + 1)^{\frac{\frac{1}{3} - \frac{1}{\alpha}}{\frac{1}{p} - 1}} \sum_{m=0}^{\infty} e^{-qm} \left( 1 + \frac{m}{1 + M} \right)^{\frac{\frac{1}{3} - \frac{1}{\alpha}}{\frac{1}{p} - 1}}.
\end{align*}

Since \(1 + \frac{m}{1 + M} \leq 1 + m\), the final sum is bounded independently of \(M\) and hence independently of \(n\). According to \((M + 1) \asymp n^p\), we obtain
\begin{align*}
\sum_{k=n}^{\infty} e^{-qk^p} |h_k(x)|^2 &\lesssim e^{-qM} (M + 1)^{\frac{\frac{1}{3} - \frac{1}{\alpha}}{\frac{1}{p} - 1}} \lesssim e^{-qM} n^{\frac{1}{3} - \frac{1}{\alpha} + 1 - \frac{1}{p}}.
\end{align*}
For $p \geq 1$, the block size of the partitioning (4) satisfies $(m + 1)^{\frac{1}{p}} - m^\frac{1}{p} \approx 1 = m^0$. Then the above calculations lead to
\[
\sum_{k=n}^{\infty} e^{-qk^p} |h_k(x)|^2 \lesssim e^{-qn^p} n^{\frac{1}{3} - \frac{1}{p}},
\]
which concludes the proof.

In the case of $p < \frac{1}{3}$, it is beneficial to apply (2) instead of (3) in the above proof. Then the exponent $\frac{1}{3} - \frac{1}{\alpha} + 1 - p$ in Proposition 2.1 can even be replaced with $1 - \frac{1}{\alpha}$.

Using Proposition 2.1, we see easily that the Sobolev spaces $\mathbb{H}^s$ and $\mathbb{E}_q^p$ are reproducing kernel Hilbert spaces.

Lemma 2.2. (i) For $s > 1 - \frac{1}{\alpha}$, $\mathbb{H}^s$ is a reproducing kernel Hilbert space with reproducing kernel
\[
K_{\mathbb{H}^s}(x, y) = \sum_{k \in \mathbb{N}} (1 + k)^{-s} h_k(x) h_k(y).
\]
(ii) For $p, q > 0$, $\mathbb{E}_q^p$ is a reproducing kernel Hilbert space with reproducing kernel
\[
K_{\mathbb{E}_q^p}(x, y) = \sum_{k \in \mathbb{N}} e^{-qk^p} h_k(x) h_k(y).
\]

Proof. This follows immediately from the fact that the series defining $K_{\mathbb{H}^s}$ and $K_{\mathbb{E}_q^p}$ converge uniformly on $\mathbb{R} \times \mathbb{R}$ and that $(1 + k)^{-s/2} h_k, k \in \mathbb{N}$, is an orthonormal basis for $\mathbb{H}^s$ with respect to its inner product. Likewise $e^{-qk^p/2} h_k, k \in \mathbb{N}$, is an orthonormal basis for $\mathbb{E}_q^p$.

3. Error bounds for Gauss-Hermite quadrature

In this section, we derive error bounds for Gauss-Hermite quadrature in $\mathbb{H}^s$ and $\mathbb{E}_q^p$. The quadrature nodes are the $n$ zeros of $h_n$, which we denote by $X_n$, so that $\# X_n = n$. The quadrature weights are the Christoffel numbers
\[
\omega(x) = \Lambda_n(x) W(x) = \left( \sum_{k=0}^{n} |h_k(x)|^2 \right)^{-1} W(x), \quad x \in X_n,
\]
 cf. [35]. These are positive and satisfy the exact quadrature relations
\[
\int_{-\infty}^{\infty} h_k(x) W(x) dx = \sum_{x \in X_n} \omega(x) h_k(x), \quad k \leq 2n - 1.
\]
For a continuous function $f : \mathbb{R} \to \mathbb{R}$, the Gauss-Hermite quadrature formula is then
\[
Q_n(f) = \sum_{x \in X_n} \omega(x) f(x),
\]
and $Q_n(f)$ provides an approximation of the actual integral $\int_{-\infty}^{\infty} f(x) W(x) dx$.

Note that we formulate the Gauss quadrature with respect to the orthonormal basis $\{h_k\}$ and the measure $W(x) dx$ instead of the orthogonal polynomials $H_k$ and the measure $W(x)^2 dx$. This requires some tiny adaptation of the classical formulas.

In the following we bound the error for functions in $\mathbb{H}^s$ and $\mathbb{E}_q^p$.
Theorem 3.1. For $1 < \alpha \in 2\mathbb{N}$ with $s > 1 - \frac{1}{\alpha}$ and $p, q > 0$, we have

\[
\sup_{f \in \mathbb{H}^s} \left\| \int_{-\infty}^{\infty} f(x)W(x)dx - \sum_{x \in X_n} \omega(x)f(x) \right\|^2 \lesssim n^{-s+1-\frac{1}{\alpha}} \quad (8)
\]

\[
\sup_{f \in \mathbb{E}^p\mathbb{E}^q} \left\| \int_{-\infty}^{\infty} f(x)W(x)dx - \sum_{x \in X_n} \omega(x)f(x) \right\|^2 \lesssim e^{-q(2n)^p}n^{-\frac{1}{\alpha}+\max(1-p,0)}
\]

For the proof of Theorem 3.1, we require some bounds on the sum of the Christoffel numbers. In the proof of [22, Proposition 1], a suitable bound is derived for the case $\alpha = 2$, but the idea still works for all even integers $\alpha = 2k, k \in \mathbb{N}$.

Lemma 3.2. If $1 < \alpha \in 2\mathbb{N}$, then we have

\[
\sum_{x \in X_n} \omega(x) \leq 2. \quad (9)
\]

Proof. According to [21, Eq. (8.4.6)] (with weight $W^2$ and $f = W^{-1}$), there are $\xi_n \in \mathbb{R}$ and $c_n \geq 0$, such that

\[
\int_{-\infty}^{\infty} W(x)dx = \sum_{x \in X_n} \omega(x) + c_n \partial^{2n}W^{-1}(\xi_n).
\]

For even integers $\alpha > 1$, one may check by induction that $\partial^{2n}W^{-1} \geq 0$.

Thus, we have

\[
\sum_{x \in X_n} \omega(x) \leq \int_{-\infty}^{\infty} W(x)dx = \int_{-\infty}^{\infty} e^{-\pi|x|^{1/\alpha}}dx = 2\pi^{-\frac{1}{\alpha}} \Gamma(1 + \frac{1}{\alpha}) \leq 2,
\]

which concludes the proof. \(\square\)

We now derive Theorem 3.1

Proof of Theorem 3.1. Since $\mathbb{H}^s$ is a reproducing kernel Hilbert space, point evaluation is continuous, and therefore

\[
T : f \mapsto \int_{-\infty}^{\infty} f(x)W(x)dx - \sum_{x \in X_n} \omega(x)f(x)
\]

is a continuous linear functional on $\mathbb{H}^s$. The left-hand-side of the first inequality (8) in Theorem 3.1 is the square $\|T\|^2$ of the norm of $T$. By the Riesz representation theorem $T$ can be identified with a function in $\mathbb{H}^s$. Since $\{(1+k)^{-\frac{1}{2\alpha}}h_k\}_{k \in \mathbb{N}}$ is an orthonormal basis for $\mathbb{H}^s$, the norm of $T$ is given by

\[
\|T\|^2 = \sum_{k=0}^{\infty} |T((1+k)^{-\frac{1}{2\alpha}}h_k)|^2
\]

\[
= \sum_{k=0}^{\infty} (1+k)^{-s} \left| \int_{-\infty}^{\infty} h_k(x)W(x)dx - \sum_{x \in X_n} \omega(x)h_k(x) \right|^2.
\]
Since the quadrature rule is exact for $k \leq 2n - 1$ by (7), the latter sum starts at $k = 2n$. Furthermore, $h_0 = c_0 W$ is orthogonal to $h_k$ for $k \geq 1$, and we obtain a closed form of the error:

$$\sum_{k=0}^{\infty} |T((1 + k)^{-\frac{\alpha}{2}} h_k)|^2 = \sum_{k=2n}^{\infty} |(1 + k)^{-s} \sum_{x \in X_n} \omega(x) h_k(x)|^2$$

$$= \sum_{x,y \in X_n} \omega(x) \omega(y) \sum_{k=2n}^{\infty} (1 + k)^{-s} h_k(x) h_k(y) .$$

An application of Cauchy-Schwartz and the kernel estimate of Proposition 2.1 lead to

$$\sum_{k=2n}^{\infty} (1 + k)^{-s} h_k(x) h_k(y) \leq \left( \sum_{k=2n}^{\infty} (1 + k)^{-s} |h_k(x)|^2 \sum_{k=2n}^{\infty} (1 + k)^{-s} |h_k(y)|^2 \right)^{1/2}$$

$$\leq (1 + 2n)^{-s+\frac{1}{2}} .$$

We are left with

$$\sum_{x,y \in X_n} \omega(x) \omega(y) \leq 4$$

by Lemma 3.2. Consequently, $\|T\|^2 \lesssim n^{-s+\frac{1}{2}}$, which concludes the proof of (8) for $\mathbb{H}^s$.

The proof for $\mathbb{E}_p^q$ is derived analogously. We note that $e^{-qk^p/2} h_k, k \in \mathbb{N}$, is an orthonormal basis for $\mathbb{E}_p^q$ and find as above that $T$ on $\mathbb{E}_p^q$ satisfies

$$\|T\|^2 = \sup_{f \in \mathbb{E}_p^q, \|f\|_{\mathbb{E}_p^q} \leq 1} \left| \int_{-\infty}^{\infty} f(x) W(x) dx - \sum_{x \in X_n} \omega(x) f(x) \right|^2$$

$$= \sum_{x,y \in X_n} \omega(x) \omega(y) \sum_{k=2n}^{\infty} e^{qk^p} h_k(x) h_k(y) .$$

Again, (10) with Proposition 2.1 concludes the proof. □

**Remark 3.1.** Let us now consider $\alpha = 2$ only. We expect that the optimal exponent for Gaussian quadrature in $\mathbb{H}^s$ is $-s$ and the additional $\frac{1}{2}$ is an artifact of our proof technique, cf. [6,28]. The case $\mathbb{E}_p^q$ with $p \geq 1$ is also covered by results in [22], whereas we have the additional factor $n^{-\frac{1}{2}}$. While [22] addresses multivariate functions in a rather general setting, the case $0 < p < 1$ is excluded there.

### 4. MODULATION SPACES

In this section we restrict ourselves to the exponent $\alpha = 2$ and investigate the Sobolev-type spaces associated to the Gaussian weight $W(x) = e^{-\pi x^2}$. Our main insight is the identification of the spaces $\mathbb{H}^s$ with a class of well-known function spaces from analysis that have been used in time-frequency analysis, for the analysis of pseudodifferential operators, for the description of uncertainty principles, etc.
See [1, 5, 11, 12, 16] for an exposition of their theory and their relevance. Certain modulation spaces have been introduced in [6, 22–24] under the name Hermite spaces.

For $\alpha = 2$, $h_0$ is the normalized Gaussian
\[ \varphi(x) = 2^{\frac{1}{4}}e^{-\pi x^2}, \]
and the associated orthogonal polynomials are the Hermite polynomials, the associated orthonormal basis for $L^2(\mathbb{R})$ is the Hermite basis $\{h_k\}$.

We first define a special family of modulation spaces. These are defined by imposing a norm on the short-time Fourier transform. Recall that the short-time Fourier transform of $f \in L^2(\mathbb{R})$ (with respect to the Gaussian $\varphi$) is given by
\[ V_\varphi f(x, \xi) = \int_{-\infty}^{\infty} f(t) \varphi(t-x)e^{-2\pi i\xi t} dt, \quad x, \xi \in \mathbb{R}. \]

We apply the standard identification $(x, \xi) \simeq z = x + i\xi \in \mathbb{C}$. Then a weighted $L^2$-norm with respect to a polynomial or an exponential weight function leads to the norms
\[
\begin{align*}
\|f\|_{M^s}^2 &= \int_{\mathbb{R}^2} (1 + |z|^2)^s |V_\varphi f(z)|^2 dz, \\
\|f\|_{M^s_{e}}^2 &= \int_{\mathbb{R}^2} e^{s|z|^2} |V_\varphi f(z)|^2 dz, \\
\|f\|_{M^s_{e2}}^2 &= \int_{\mathbb{R}^2} e^{s|z|^2}|V_\varphi f(z)|^2 dz,
\end{align*}
\]
and the modulation spaces
\[
\begin{align*}
M^s &= \{ f \in L^2(\mathbb{R}) : \|f\|_{M^s} < \infty \}, \quad s \geq 0, \\
M^s_{e} &= \{ f \in L^2(\mathbb{R}) : \|f\|_{M^s_{e}} < \infty \}, \quad s \geq 0, \\
M^s_{e2} &= \{ f \in L^2(\mathbb{R}) : \|f\|_{M^s_{e2}} < \infty \}, \quad \pi > s \geq 0.
\end{align*}
\]

The following theorem identifies the Sobolev spaces $H^s$ and $E^p_q$ as modulation spaces with respect to a polynomial or exponential weight function. We feel that the existing results for modulation spaces could be useful for numerical analysts.

**Theorem 4.1.** For $\alpha = 2$, the following identities hold with equivalent norms:

(i) $H^s = M^s$, for $s \geq 0$,

(ii) $E^{\frac{1}{q}} = M^s_{e}$, for $q = \frac{s}{\sqrt{\pi}}$ and $s \geq 0$,

(iii) $E^{1}_{q} = M^s_{e2}$, for $q = \ln(\frac{\pi}{\pi - s})$ and $\pi > s \geq 0$.

**Proof.** We use the Bargmann transform to translate the norms to an equivalent norm for a subspace of the Bargmann-Fock space $\mathcal{F}$ [14]. This space consists of all entire functions $F$ such that
\[
\|F\|_{2}^2 = \int_{\mathbb{C}} |F(z)|^2 e^{-\pi |z|^2} dz
\]

\[\text{In the standard notation } M^s \text{ is the modulation space } M^2_{\nu_s} \text{ with } \nu_s(z) = (1 + |z|^2)^s \text{ and } M^s_{e} \text{ is } M^2_{w_{\nu_s}} \text{ with } w_{\nu_s}(z) = e^{s|z|}. \text{ Since we use only modulation spaces that are Hilbert spaces, we have simplified the notation.} \]
is finite. The normalized monomials \( \{ \sqrt{\frac{\pi}{k!}} z^k \}_{k \in \mathbb{N}} \) form an orthonormal basis for \( \mathcal{F} \) with respect to the inner product

\[
\langle F, G \rangle_{\mathcal{F}} = \int_{\mathbb{C}} F(z) \overline{G(z)} e^{-\pi|z|^2} dz.
\]

The Bargmann transform defined by

\[
Bf(z) = 2^{1/4} e^{-\pi z^2/2} \int_{-\infty}^{\infty} f(x) e^{2\pi zx} e^{-\pi x^2} dx
\]

is a unitary operator from \( L^2(\mathbb{R}) \) onto \( \mathcal{F} \) and maps the Hermite functions to the normalized monomials, i.e.,

\[
(13) \quad B : L^2(\mathbb{R}) \rightarrow \mathcal{F}, \quad h_k \mapsto \sqrt{\frac{\pi}{k!}} z^k.
\]

With the identification \((x, \xi) \simeq z = x + i\xi \in \mathbb{C}\), the connection of \( B \) to the short-time Fourier transform is given by the formula

\[
(14) \quad V_{\varphi} f(z) = e^{-ix\xi} (Bf)(\bar{z}) e^{-\frac{\varphi}{2} |z|^2},
\]

cf. \[14, 18\]. For the proof we use the Hermite expansion

\[
f = \sum_{k=0}^{\infty} \hat{f}_k h_k
\]

of \( f \in L^2(\mathbb{R}) \). The main point of the proof is the fact that the monomials \( z^k, k \in \mathbb{N} \), are still orthogonal with respect to the weighted inner product \( \int \overline{F(z)} G(z) w(z) e^{-\pi|z|^2} dz \) for an arbitrary rotation-invariant weight function \( w \).

(i) We first consider \( M^s \) and obtain

\[
\|f\|_{M^s}^2 = \int_{\mathbb{R}^2} \left| \sum_{k=0}^{\infty} \hat{f}_k \sqrt{\frac{\pi}{k!}} z^k \right|^2 (1 + |z|^2)^s dz = \int_{\mathbb{R}^2} \left| \sum_{k=0}^{\infty} \hat{f}_k e^{-ix\xi} (Bh_k)(\bar{z}) \right|^2 (1 + |z|^2)^s e^{-\pi|z|^2} dz
\]

\[
= \int_{\mathbb{R}^2} \left| \sum_{k=0}^{\infty} \hat{f}_k \sqrt{\frac{\pi}{k!}} z^k \right|^2 (1 + |z|^2)^s e^{-\pi|z|^2} dz = \sum_{k,l} \hat{f}_k \hat{f}_l \sqrt{\frac{\pi}{k!}} \sqrt{\frac{\pi}{l!}} \int_{\mathbb{R}^2} z^k \overline{z^l} (1 + |z|^2)^s e^{-\pi|z|^2} dz.
\]

Since \( (1 + |z|^2)^s \) is rotationally invariant, the monomials are still orthogonal with respect to \( (1 + |z|^2)^s e^{-\pi|z|^2} dz \) and we obtain

\[
\|f\|_{M^s}^2 = \sum_{k=0}^{\infty} |\hat{f}_k|^2 \sqrt{\frac{\pi}{k!}} \int_{\mathbb{R}^2} |z|^{2k} (1 + |z|^2)^s e^{-\pi|z|^2} dz.
\]
The use of polar coordinates and the substitution $t = \pi r^2$ lead to

$$\frac{\pi^k}{k!} \int_{\mathbb{R}^2} |z|^{2k} (1 + |z|^2)^s e^{-\pi|z|^2} \, dz = \frac{2\pi^{k+1}}{k!} \int_0^\infty r^{2k+1} \left(1 + r^2\right)^s e^{-\pi r^2} \, dr$$

$$= \frac{1}{k!} \int_0^\infty t^k (1 + \frac{t}{\pi})^s e^{-t} \, dt$$

$$\propto \frac{1}{k!} \int_0^\infty t^k (1 + \frac{t^s}{\pi^s}) e^{-t} \, dt$$

$$= \frac{1}{\Gamma(k+1)} \left(\Gamma(k+1) + \frac{\Gamma(k+s+1)}{\pi^s}\right).$$

The standard asymptotics

$$\frac{\Gamma(k+s+1)}{\Gamma(k+1)} \asymp k^s,$$

cf. [8, 5.11.12], lead to

$$\frac{\pi^{k+1}}{k!} \int_0^\infty t^k (1+t^s) e^{-\pi t} \, dt \asymp (1+k^s) \asymp (1+k)^s.$$

Consequently,

$$\|f\|^2_{M^s} \asymp \sum_{k=0}^\infty (1+k)^s |\hat{f}_k|^2 = \|f\|_{\mathbb{H}^s},$$

which concludes the proof of the identification of $M^s$ with $\mathbb{H}^s$.

(iii) Because it is easier, we prove (iii) next. As before

$$\|f\|^2_{M^s_{e^2}} = \int_{\mathbb{R}^2} |V^s f(z)|^2 |e^s| |z|^2 \, dz$$

$$= \sum_{k=0}^\infty |\hat{f}_k|^2 \frac{\pi^k}{k!} \int_{\mathbb{R}^2} |z|^{2k} e^{-(\pi-s)|z|^2} \, dz.$$

Polar coordinates and the substitution $u = (\pi-s)r^2$ lead to

$$\frac{\pi^k}{k!} \int_{\mathbb{R}^2} |z|^{2k} e^{-(\pi-s)|z|^2} \, dz = \frac{\pi^k}{k!} \int_0^\infty r^{2k} e^{-(\pi-s)r^2} 2\pi r \, dr$$

$$= \frac{2\pi^{k+1}}{k!(\pi-s)^{k+1}} \int_0^\infty u^k e^{-u} \, du$$

$$= \left(\frac{\pi}{\pi-s}\right)^{k+1}.$$

Consequently,

$$\|f\|_{M^s_{e^2}} = \sum_{k=0}^\infty |\hat{f}_k|^2 \left(\frac{\pi}{\pi-s}\right)^{k+1} = \frac{\pi}{\pi-s} \sum_{k=0}^\infty |\hat{f}_k|^2 \exp \left(\ln\left(\frac{\pi}{\pi-s}\right) k\right) = \frac{\pi}{\pi-s} \|f\|_{\mathbb{E}^1_q}.$$

This concludes the proof of the identity $M^s_{e^2} = \mathbb{E}^1_q$ with $q = \ln(\frac{\pi}{\pi-s}).$
(ii) The proof for $M_s^e$ is more involved and requires detailed formulas and estimates for hypergeometric functions. We cite freely the digital library of mathematical functions DLMF [8]. As above, we obtain

$$\|f\|_{M_s^e}^2 = \int_{\mathbb{R}^2} |V_\varphi f(z)|^2 e^{s|z|} \, dz = \sum_{k=0}^{\infty} |\hat{f}_k|^2 \frac{\pi^k}{k!} \int_{\mathbb{R}^2} |z|^{2k} e^{s|z|} e^{-|z|^2} \, dz.$$ 

The use of polar coordinates and the identity $\cosh(x) + \sinh(x) = e^x$ lead to

$$\frac{\pi^k}{k!} \int_{\mathbb{R}^2} |z|^{2k} e^{s|z|} e^{-|z|^2} \, dz = \frac{2\pi^{k+1}}{k!} \int_0^\infty r^{2k+1} e^{-r^2} \, dr + \frac{2\pi^{k+1}}{k!} \int_0^\infty r^{2k+1} e^{-r^2} \sinh(sr) \, dr.$$ 

For $p, q \in \mathbb{N}$, denote $_pF_q$ the hypergeometric function and observe

$$\cosh(sr) = \, _0F_1\left(\frac{1}{2}, \frac{sr^2}{2}\right),$$

$$\sinh(sr) = \, _0F_1\left(\frac{3}{2}, \frac{sr^2}{2}\right) sr.$$ 

According to DLMF [8, 16.5.3], for $a \in \mathbb{C}$ with positive real part, we have

$$\, _1F_1(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} e^{-t} {}^0F_1(b, zt) \, dt,$$

$$= \frac{2\pi^a}{\Gamma(a)} \int_0^\infty r^{2a-1} e^{-r^2} {}^0F_1(b, zr^2) \, dr,$$

where the second equality is due to the substitution $r = \sqrt{\frac{t}{z}}$. This leads to

$$\frac{2\pi^{k+1}}{k!} \int_0^\infty r^{2k+1} e^{-r^2} \cosh(sr) \, dr = \, _1F_1(k + 1; \frac{1}{2}, \frac{s^2}{4\pi})$$

and

$$\frac{2\pi^{k+1}}{k!} \int_0^\infty r^{2k+1} e^{-r^2} \sinh(sr) \, dr = \frac{2s\pi^{k+1}}{k!} \int_0^\infty r^{2k+2} e^{-r^2} \, dr \, _1F_1(3; \frac{3}{2}, \frac{s^2}{4\pi}).$$

For large $a$,

$$\, _1F_1(a, b, z^2) \approx (4a - 2b) \frac{\Gamma(k + 3)}{\Gamma(k + 1)} I_{b-1} \left( z\sqrt{4a - 2b} \right),$$

cf. [37, Eq. (4.6.42) in Section 4.6.1], where $I_{b-1}$ denotes the modified Bessel function. (Note that there is a square missing in [40, Eq. (2.2)]). According to DLMF [8, 10.40.1] we have, for large $x$,

$$I_\nu(x) \approx \frac{e^x}{\sqrt{2\pi x}}.$$
Here, we only consider \( b = \frac{1}{2} \) or \( b = \frac{3}{2} \), so that we need these asymptotics for \( \nu = \pm \frac{1}{2} \), in which case we even have
\[
I_{\pm \frac{1}{2}}(x) = \frac{e^x \mp e^{-x}}{\sqrt{2\pi x}}.
\]
Therefore, we obtain
\[
1F_1(k+1, \frac{1}{2}; \frac{s^2}{4\pi}) \asymp (4k+3)^\frac{1}{2} \frac{e^{\frac{s}{2\sqrt{\pi}} \sqrt{4k+3}}}{\sqrt{2\pi \frac{s}{2\sqrt{\pi}} \sqrt{4k+3}}} \asymp e^{\frac{s}{2\sqrt{\pi}} \sqrt{k}}.
\]
Since \( \frac{\Gamma(k+\frac{3}{2})}{\Gamma(k+1)} \asymp \sqrt{k} \), we have
\[
\frac{\Gamma(k+\frac{3}{2})}{\Gamma(k+1)} 1F_1(k, 3; 2; \frac{s^2}{4\pi}) \asymp k^{\frac{1}{2}}(4k+3)^{-\frac{1}{2}} \frac{e^{\frac{s}{2\sqrt{\pi}} \sqrt{4k+3}}}{\sqrt{2\pi \frac{s}{2\sqrt{\pi}} \sqrt{4k+3}}} \asymp e^{\frac{s}{2\sqrt{\pi}} \sqrt{k}},
\]
which concludes the proof for \( M_\alpha^s \).

The identification \( M^s = H^s \) in the case \( \alpha = 2 \) is also mentioned in [25, Eq. (1.5)] without proof. We also point out that our quadrature bounds for \( M_\alpha^s \) in Theorem 3.1 are not covered by [22], see also Remark 3.1.

To complete the discussion of modulation spaces (or Hermite spaces), we add two relevant characterizations that explain why we consider them as Sobolev spaces where Sobolev space is understood as the domain of a differential operator.

Consider the differential operator \( Lf(x) = -\frac{1}{2}\pi f''(x) + 2\pi x^2 f(x) \). In quantum mechanics \( L \) is the Schrödinger operator of the harmonic oscillator. Its eigenfunctions are precisely the Hermite functions, and we have
\[
Lh_n = (2n+1)h_n
\]
in our normalization of the Hermite functions, see [14]. Then we have the following identification.

**Lemma 4.2.** \( H^s = \text{dom } L^{s/2} = L^{s/2}(L^2(\mathbb{R})) \) with equivalence of norms.

**Proof.** Since \( L \) is diagonalized by the Hermite basis, it follows that \( L^{s/2}f \in L^2(\mathbb{R}) \) if and only if \( \sum_{k=0}^{\infty} |\langle f, h_k \rangle|^2 (1+2k)^s < \infty \), and thus \( \|L^{s/2}f\|_2 \asymp \|f\|_{H^s} \). \( \Box \)

In this context \( H^s \) is referred to as the Shubin class, and as always \( H^s \) coincides with the domain of much more general pseudodifferential operators. See Shubin’s book [36, Ch. 25.3] for a detailed exposition.

Next, for \( s \geq 0 \) let \( L^2_s \) be the weighted \( L^2 \)-space defined by the norm \( \|f\|_{L^2_s}^2 = \int_{\mathbb{R}} |f(x)|^2 (1+|x|)^{2s} \, dx \) and let \( \mathcal{F}L^2_s = \{ f \in L^2 : \hat{f} \in L^2_s \} \) be its image under the Fourier transform. Note that \( \mathcal{F}L^2_s \) is the standard Sobolev space on \( \mathbb{R} \). The following identification is mentioned in [24,20].

**Lemma 4.3.** \( M^s = H^s = L^2_s(\mathbb{R}) \cap \mathcal{F}L^2_s(\mathbb{R}) \).

Finally we mention that the modulation spaces \( \mathbb{H}^s = \mathcal{M}^s \) and \( E_{\alpha}^{\frac{1}{2}} = M_\alpha^s \) are invariant with respect to time-frequency shifts defined by \( f_{y,\eta}(x) = e^{2\pi i \eta x} f(x-y) \) (shift by \( y \) and modulation by \( \eta \)): if \( f \in \mathbb{H}^s \), then \( f_{y,\eta} \in \mathbb{H}^s \) for all \( y, \eta \in \mathbb{R} \) and similarly for \( M_\alpha^s \). This is easy to see from the definition of \( M^s \), but is not at all
obvious when using the $H^s$-norm. This invariance property is just one example of how time-frequency methods facilitate the investigation of general Hermite spaces.

5. Numerical experiments

In this section we make the error estimates of Theorem 3.1 more explicit for the modulation spaces and compare them with numerical simulations. We provide some numerical experiments for Gaussian quadrature nodes $X_n$ with associated Christoffel weights $\{\omega(x)\}_{x \in X_n}$ in the modulation spaces $M^s_{e^2}$ and $M^s_{\| \cdot \|_\infty}$. We omit the case $M^s_{\| \cdot \|_\infty}$ with polynomial weights, as it was already presented in [6]. Note that the results in [6] suggest a decay rate of $O(n^{-s})$ instead of the theoretical bound $O(n^{-s+\frac{1}{2}})$ proved in Theorem 3.1.

5.1. The modulation space $M^s_{e^2}$. We choose $t > 1$ and consider the (slightly modified) Mehler kernel

$$K_t(x, y) = \sum_{k=0}^\infty t^{-(k+1)} h_k(x) h_k(y) = \sqrt{\frac{2}{t^2 - 1}} e^{\frac{t^2}{2}(4txy - (t^2+1)(x^2+y^2))}. \tag{15}$$

In view of the identification $M^s_{e^2} = E^1_q$, (5) in Lemma 2.2 with $t = \frac{\pi}{\pi - s}$ shows that $K_t$ is the reproducing kernel of $M^s_{e^2}$.

We now use a standard identity for the worst case integration error in terms of the reproducing kernel, see, e.g., [15] or [7, Prop. 2.11]. We have

$$\sup_{f \in M^s_{e^2}, \| f \|_{e^2} \leq 1} \left| \int_{-\infty}^\infty f(x) W(x) dx - \sum_{x \in X_n} \omega(x) f(x) \right|^2 = \int_{-\infty}^\infty \int_{-\infty}^\infty K_t(x, y) W(x) dx W(y) dy + \sum_{x, y \in X_n} \omega(x) \omega(y) K_t(x, y)$$

$$- 2 \sum_{x \in X_n} \omega(x) \int_{-\infty}^\infty K_t(x, y) W(y) dy. \tag{16}$$

For the Mehler kernel (15) we can make the worst case error more explicit. Since $W = 2^{-1/4} h_0$ and

$$\int_{-\infty}^\infty W^2(x) dx = \int_{-\infty}^\infty e^{-2\pi x^2} dx = \frac{1}{\sqrt{2}},$$

the orthogonality of the Hermite functions implies

$$\int_{-\infty}^\infty K_t(x, y) W(x) dx = \sum_{k=0}^\infty t^{-k-1} \int_{-\infty}^\infty h_k(x) W(x) dx h_k(y)$$

$$= \sum_{k=0}^\infty t^{-k-1} 2^{-1/4} \langle h_k, h_0 \rangle h_k(y) = t^{-1} W(y).$$
A further integration with respect to \( y \) yields
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_t(x, y) W(x) dx \ W(y) dy = t^{-1} \int_{-\infty}^{\infty} W(y)^2 dy = \frac{1}{\sqrt{2}t}.
\]

For the third term in (16) we use the exact quadrature formula (7) for \( k = 0 \) and obtain that
\[
-2 \sum_{x \in X_n} \omega(x) \int_{-\infty}^{\infty} K_t(x, y) W(y) dy = -2t^{-1} \sum_{x \in X_n} \omega(x) W(x)
\]
\[
= -2t^{-1} \int_{-\infty}^{\infty} W^2(x) dx = -\frac{2}{\sqrt{2}t}.
\]

Consequently, we now have
\[
WCE(n, M_{q,2}^2) := \sup_{f \in M_{q,2}^2: \|f\|_{M_{q,2}^2} \leq 1} \left| \int_{-\infty}^{\infty} f(x) W(x) dx - \sum_{x \in X_n} \omega(x) f(x) \right|^2
\]
\[
= \sum_{x, y \in X_n} \omega(x) \omega(y) K_t(x, y) - \frac{1}{\sqrt{2}t}.
\]

By Theorem 4.1(iii) the modulation space \( M_{q,2}^2 \) coincides with the Sobolev-type space \( E^1_q \) with parameter \( q = \ln \frac{\pi}{\pi - s} = \ln t \). For the case \( p = 1 \) the theoretical prediction of Theorem 3.1 ensures that the decay rate of the worst case error (17) is of the order
\[
WCE(n, M_{q,2}^2) = O \left( e^{-q(2n)^r} \right) = O \left( (e^{-q})^{2n} \right) = O \left( t^{-2n} \right),
\]
so that \( \log_{10} WCE(n, M_{q,2}^2) \leq C - 2n \log_{10}(t) \). This means that the logarithm of the worst case quadrature error from (17) must stay below a line of slope \(-2 \log_{10}(t)\).

In the numerical experiments we have evaluated the explicit formula (17) for various values of \( t \), namely \( t_1 = \frac{\pi}{\pi - s} = \frac{5}{4} \) and \( t_2 = \frac{\pi}{\pi - s} = \frac{50}{59} \). Note that the closed form (15) of the Mehler kernel \( K_t \) permits a direct evaluation of \( WCE(n, M_{q,2}^2) \).

We then let \( n \) run through the odd integers from 3 to 41 and plot the resulting line, see Figure II. We observe that the actual slope is slightly steeper than the theoretical prediction.

5.2. The modulation space \( M_e^* \). We use the equivalence of norms in \( M_e^* = E^1_q \) with \( q = \frac{\pi}{\pi - s} \) proved in Theorem 4.1(ii). By the proof of Theorem 3.1 in particular (12), the worst case quadrature error in \( M_e^* \) is given by
\[
WCE(n, M_e^2) := \sup_{f \in M_e^2: \|f\|_{M_e^2} \leq 1} \left| \int_{-\infty}^{\infty} f(x) W(x) dx - \sum_{x \in X_n} \omega(x) f(x) \right|^2
\]
\[
\propto \sum_{x, y \in X_n} \omega(x) \omega(y) \sum_{k=2n}^{\infty} e^{-q \sqrt{x}} h_k(x) h_k(y).
\]
Figure 1. Logarithmic plot of the Gaussian quadrature error in $M_{s_i}^2$ against $n$ for $s_i = (1 - \frac{1}{t_i})\pi$ with $t_1 = \frac{5}{4}$ and $t_2 = \frac{59}{49}$ for $i = 1, 2$. The blue line is a linear least squares fit of the actual evaluation of the worst case quadrature error (17) at odd $n$ marked in red. The slope of the least squares fit is slightly steeper than the theoretical slope $-2\log_{10}(t_1) \approx -0.20$ and $-2\log_{10}(t_2) \approx -0.02$.

Theorem 3.1 ensures that (18) decays at least as $e^{-q\sqrt{2n}}$. Thus

$$\log_{10} WCE(n, M_{s_i}^2) \leq C + \log_{10}(e^{-q\sqrt{2n}}) = C - \sqrt{2}q\sqrt{n}\log_{10}(e).$$

This means that the logarithm of the worst case error (18) must stay below a line of slope $-\sqrt{2}q\frac{1}{\sqrt{\pi}}\log_{10}(e)$ when plotted against $\sqrt{n}$.

An additional difficulty arises in numerical experiments, because the term $\sum_{k=2}^{\infty} e^{-q\sqrt{2}h_k(x)h_k(y)}$ in (18) cannot be evaluated exactly. Nonetheless, the series converges so fast that a simple truncation yields sufficient numerical accuracy. Indeed, we evaluate a truncation of (18) for several small values of $n$ in Figure 2. Our plots show a slightly steeper slope than $-\sqrt{2}q\frac{1}{\sqrt{\pi}}\log_{10}(e)$ when plotted against $\sqrt{n}$.

We must point out though that the calculations are less stable than our previous experiments in Section 5.1, where no truncation was needed.

6. Interlude: Abstract Quadrature in General Hilbert Spaces

In this section, we recast the quadrature problem by means of abstract Hilbert space concepts. This reformulation will help us to generalize Theorem 3.1 beyond Gaussian quadrature nodes and beyond even parameters $\alpha$ in the Freud weight $W(x) = e^{-\pi|x|^\alpha}$.

The general set-up is as follows: We assume that $\{u_k : k \in \mathbb{N}\}$ is an orthonormal basis for a separable Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$. For $f \in \mathcal{H}$, we define

$$\hat{f}_k := \langle f, u_k \rangle, \quad k \in \mathbb{N}.$$
Figure 2. Logarithmic plot of the Gaussian quadrature error in $M_{\infty}$ with $s_1 = 1$ and $s_2 = \frac{1}{2}$ against $\sqrt{n}$. The blue line is a linear least squares fit of the actual evaluation of the truncation of the worst case quadrature error (18) at $\sqrt{n}$ marked in red. The slope of the least squares fit is slightly steeper than $-\sqrt{2^{a_n}} \frac{\log_{10}(e)}{\sqrt{\pi}} \approx -0.35$ and $-\sqrt{2^{s_2}} \frac{\log_{10}(e)}{\sqrt{\pi}} \approx -0.17$, respectively.

To any sequence $\lambda = (\lambda_k)_{k \in \mathbb{N}}$ such that $\lambda_k > 0$ and $\lim \lambda_k = \infty$, we introduce the Hilbert space

$$H_\lambda := \{ f \in H : \sum_{k \in \mathbb{N}} \lambda_k |\hat{f}_k|^2 < \infty \}$$

with the inner product

$$\langle f, g \rangle_{H_\lambda} := \sum_{k \in \mathbb{N}} \hat{f}_k \overline{\hat{g}_k} \lambda_k, \quad f, g \in H_\lambda.$$ 

In analogy to polynomials of degree $n$, we define the finite-dimensional subspaces

$$\Pi_n := \text{span}\{ u_k : 1 \leq k \leq n \}.$$ 

Our main assumption — and this assumption is non-trivial — is the existence of (a sequence of) positive semi-definite sesquilinear forms $\langle \cdot, \cdot \rangle_n$ on $H_\lambda$ that are continuous (with respect to $\| \cdot \|$) and satisfy the following property: we assume that there is $a_n > 0$ such that

$$a_n \| f \|^2 \leq \| f \|^2_n, \quad f \in \Pi_n,$$

where $\| f \|_n = \langle f, f \rangle_n^{1/2}$. Consequently $\langle \cdot, \cdot \rangle_n$ is an inner product on $\Pi_n$ and we refer to it as a semi-inner product on $H_\lambda$. We may think of $\langle \cdot, \cdot \rangle_n$ as the discretization of some inner product via a quadrature rule of the form $\langle f, g \rangle_n = \sum_{j=1}^n f(x_j)\overline{g(x_j)}w_j$. The assumption (21) implies that the operator $S_n : \Pi_n \to \Pi_n$ defined by

$$S_n f = \sum_{k=0}^n \langle f, u_k \rangle_n u_k$$
is invertible on \( \Pi_n \). In the language of frame theory \( S_n \) is the frame operator associated to the basis \( \{ u_1, \ldots, u_n \} \) of \( \Pi_n \).

The goal is now to approximate the linear functional

\[
\langle f, u_0 \rangle
\]

for given \( f = \sum_{k=1}^{\infty} f_k u_k \in \mathcal{H}_\lambda \) by an expression containing the semi-inner product \( \langle \cdot, \cdot \rangle_n \). The underlying idea is that \( \langle \cdot, \cdot \rangle_n \) is simpler than the original inner product on \( H \).

We start with some simple consequences of these definitions that are guided by analogous observations in the context of frame theory [3], where the semi-inner products are supposed to be inner products.

**Lemma 6.1.** Let \( Q_n : \mathcal{H}_\lambda \to \Pi_n \) be defined by

\[
Q_n f = \sum_{k=0}^{n} \langle f, u_k \rangle_n S_n^{-1} u_k .
\]

(i) Then \( Q_n \) is the orthogonal projection from \( \mathcal{H}_\lambda \) onto \( \Pi_n \) with respect to the semi-definite inner product \( \langle \cdot, \cdot \rangle_n \) and also satisfies \( Q_n f = \sum_{k=0}^{n} \langle f, S_n^{-1} u_k \rangle_n u_k \).

(ii) \( Q_n f \) is the unique solution to the optimization problem

\[
\arg \min_{p \in \Pi_n} \| f - p \|_n , \quad \text{for } f \in \mathcal{H}_\lambda .
\]

(iii) If \( f \in \mathcal{H}_\lambda \) and \( g \in \Pi_n \), then

\[
\langle Q_n f, g \rangle = \langle f, S_n^{-1} g \rangle_n .
\]

(\textit{Note the use of different inner products on both sides!})

**Proof.**

(i) Define the extension of \( S_n \) to \( \mathcal{H}_\lambda \) by \( \tilde{S}_n : \mathcal{H}_\lambda \to \Pi_n \) by

\[
\tilde{S}_n f = \sum_{k=0}^{n} \langle f, u_k \rangle_n u_k .
\]

Then \( Q_n = S_n^{-1} \tilde{S}_n \) is defined on all of \( \mathcal{H}_\lambda \). Its restriction to \( \Pi_n \) is \( Q_n|_{\Pi_n} = S_n^{-1} \tilde{S}_n|_{\Pi_n} = S_n^{-1} S_n = \text{Id}_{\Pi_n} \). Thus \( Q_n \) is the identity on \( \Pi_n \), and consequently, \( Q_n^2 = Q_n \) and \( Q_n \) is a projection. The kernel of \( Q_n \) equals the orthogonal complement of \( \Pi_n \) with respect to \( \langle \cdot, \cdot \rangle_n \), so \( Q_n \) is the orthogonal projection with respect to \( \langle \cdot, \cdot \rangle_n \).

Next we observe that \( S_n \) and \( \tilde{S}_n \) are self-adjoint with respect to \( \langle \cdot, \cdot \rangle_n \).

The alternative representation of \( Q_n f \) follows from \( Q_n f = 0 \) for \( f \in \Pi_n \) and \( Q_n f = S_n S_n^{-1} f = \sum_{k=1}^{n} \langle S_n^{-1} f, u_k \rangle_n u_k \) and the self-adjointness of \( S_n^{-1} \) (of course, this is just the usual formalism of frame theory [3]).

(ii) follows because \( Q_n \) is the orthogonal projection from \( \mathcal{H}_\lambda \) onto \( \Pi_n \) with respect to \( \langle \cdot, \cdot \rangle_n \).

(iii) By definition of \( Q_n \) and (i) we obtain

\[
\langle Q_n f, g \rangle = \sum_{k=1}^{n} \langle f, S_n^{-1} u_k \rangle_n \langle u_k, g \rangle = \langle f, S_n^{-1} g \rangle_n ,
\]

since \( g = \sum_{k=1}^{n} \langle g, u_k \rangle u_k \in \Pi_n . \)

\( \square \)
Coming back to abstract quadrature, we may take \( \langle Q_n f, u_0 \rangle \) as an approximation for the functional \( \langle f, u_0 \rangle \). In view of the identity

\[ \langle Q_n f, u_0 \rangle = \langle f, S_n^{-1} u_0 \rangle_n, \]

from (24), this means that \( \langle f, S_n^{-1} u_0 \rangle_n \) is an approximation of \( \langle f, u_0 \rangle \) that requires only knowledge of \( u_0, \ldots, u_n \) and the “simpler” inner product \( \langle \cdot, \cdot \rangle_n \), but we do not need to evaluate \( \langle \cdot, \cdot \rangle \) directly. In this sense \( \langle f, S_n^{-1} u_0 \rangle_n \) is an abstract quadrature rule.

To quantify the approximation error, we introduce the error function

\[ \phi_\lambda(n) := \sum_{k=n+1}^{\infty} \lambda_k^{-1} \| u_k \|_n^2 \in [0, \infty]. \]

The worst case error for the abstract quadrature rule is then given by the following estimate.

**Theorem 6.2.** We have

\[ \sup_{f \in \mathcal{H}_\lambda \atop \|f\|_{\mathcal{H}_\lambda} \leq 1} | \langle f, u_0 \rangle - \langle f, S_n^{-1} u_0 \rangle_n |^2 \leq \frac{\phi_\lambda(n)}{a_n}. \]

**Proof.** We observe that \( T f = \langle f, u_0 \rangle - \langle f, S_n^{-1} u_0 \rangle_n \) is a continuous linear functional on \( \mathcal{H}_\lambda \). As in the proof of Theorem 5.1, \( T \) can be identified with a vector in \( \mathcal{H}_\lambda \) by the Riesz representation theorem. Since \( \{ \lambda_k^{-\frac{1}{2}} u_k \}_{k \in \mathbb{N}} \) is an orthonormal basis for \( \mathcal{H}_\lambda \), the norm of the functional \( T \) is given by

\[ \| T \|_{\mathcal{H}_\lambda} = \sum_{k \in \mathbb{N}} \lambda_k^{-1} \| T(u_k) \|^2. \]

This leads to

\[ \| T \|_{\mathcal{H}_\lambda}^2 = \sup_{f \in \mathcal{H}_\lambda \atop \|f\|_{\mathcal{H}_\lambda} \leq 1} | \langle f, u_0 \rangle - \langle f, S_n^{-1} u_0 \rangle_n |^2 \]

\[ = \sum_{k=0}^{\infty} \lambda_k^{-1} | \langle u_k, u_0 \rangle - \langle u_k, S_n^{-1} u_0 \rangle_n |^2 \]

\[ = \sum_{k=n+1}^{\infty} \lambda_k^{-1} | \langle u_k, S_n^{-1} u_0 \rangle_n |^2 \]

\[ \leq \sum_{k=n+1}^{\infty} \lambda_k^{-1} \| u_k \|_n^2 \| S_n^{-1} u_0 \|^2_n \]

\[ = \| S_n^{-1} u_0 \|^2_n \phi_\lambda(n). \]

In this chain of identities we have used (24) in the form \( \langle u_k, S_n^{-1} u_0 \rangle_n = \langle Q_n u_k, u_0 \rangle = \langle u_k, u_0 \rangle = \delta_{k,0} \) for \( k \leq n \).

To derive the bound \( \| S_n^{-1} u_0 \|^2_n \leq a_n^{-1} \), we recall that \( S_n \) is self-adjoint with respect to \( \langle \cdot, \cdot \rangle_n \) and positive semi-definite. Hence, the same holds for \( S_n^{-1} \) and its square root. Using (24) again for \( g \in \Pi_n \), we obtain

\[ \| S_n^{-1/2} g \|^2_n = \langle S_n^{-1/2} g, S_n^{-1/2} g \rangle_n = \langle g, S_n^{-1/2} g \rangle_n = \langle g, g \rangle = \| g \|^2 \leq a_n^{-1} \| g \|^2_n. \]

This implies

\[ \| S_n^{-1} u_0 \|^2_n \leq a_n^{-1} \| S_n^{-1/2} u_0 \|^2_n = a_n^{-1} \| u_0 \|^2 = a_n^{-1}, \]

which completes the proof.
Remark 6.1. We may replace $u_0$ with any element $g \in \Pi_n$ and obtain
\[
\sup_{\|f\|_{H_\lambda} \leq 1} |\langle f, g \rangle - \langle f, S_n^{-1}g \rangle_n| \leq \frac{\phi_\lambda(n)}{a_n}\|g\|^2.
\]
If $a_n = 1$ and (21) is replaced by the identity $\|f\|^2 = \|f\|^2_n$, then $S_n$ is the identity operator, and
\[
\sup_{\|f\|_{H_\lambda} \leq 1} |\langle f, g \rangle - \langle f, g \rangle_n| \leq \phi_\lambda(n)\|g\|^2.
\]

7. The use of MZ-inequalities in quadrature

In this section we extend the error estimates of Theorem 3.1 considerably so that they hold (i) for more general point sets beyond Gaussian quadrature nodes and (ii) for the full range of Freud weights $W(x) = e^{-\pi|x|^{\alpha}}$ for all $\alpha > 1$. For this we will apply the results about abstract quadrature rules from Section 6.

Given the weight $W(x) = e^{-\pi|x|^{\alpha}}$ and the associated orthogonal polynomials $H_k$, $k = 0, \ldots$, we choose the orthonormal basis
\[
u_k = h_k = H_k W, \quad k \in \mathbb{N}
\]
as the orthonormal basis for $L^2(\mathbb{R})$. Let us emphasize again that $h_k$ depends on $\alpha$.

The sequence of weights for the definition of the associated Sobolev spaces $H_\lambda$ are either of polynomial type $\lambda_k = (1+k)^s$ or of exponential type $\lambda_k = e^{qk^p}$. At least for $\alpha = 2$, these abstractly defined spaces coincide with a class of well-known function spaces in analysis, see Section 4.

The associated sequence of finite-dimensional subspaces are the spaces $\Pi_n = \text{span}\{1, x, \ldots, x^n\} W$ consisting of weighted polynomials of degree $\leq n$.

Next we consider a sequence of nodes $X_n \subseteq \mathbb{R}$ and of nonnegative weights $\{\tau(x)\}_{x \in X_n}$ and for every $n$ define a semi-inner product by
\[
\langle f, g \rangle_n = \sum_{x \in X_n} \tau(x)f(x)g(x), \quad \|f\|^2_n = \langle f, f \rangle_n.
\]
Here is the decisive condition: $X_n$ must be a Marcinkiewicz-Zygmund family and every $X_n$ must satisfy a Marcinkiewicz-Zygmund inequality (in different terminology: a sampling inequality), i.e., there exist $0 < a_n \leq b_n < \infty$ such that
\[
\|f\|^2_{L^2(\mathbb{R})} \leq \sum_{x \in X_n} |f(x)|^2\tau(x) = \|f\|^2_n \leq b_n\|f\|^2_{L^2(\mathbb{R})}, \quad \text{for all } f \in \Pi_n.
\]
Assumption (27) is non-trivial. In view of the vast literature on Marcinkiewicz-Zygmund inequalities we take the existence of Marcinkiewicz-Zygmund families for granted. For general constructions, we refer to [13,32,34], for a simple direct derivation of quadrature rules from Marcinkiewicz-Zygmund inequalities see [10].

To apply Theorem 6.2 we need one more property of the orthogonal polynomials associated to the Freud weights. Let $m_n$ be the Mhaskar-Rahmanov-Saff numbers defined by
\[
m_n = m_{n, \alpha} = \frac{2}{\sqrt{\pi}} \left( \frac{\Gamma(\frac{\alpha}{2})^2}{4\Gamma(\alpha)} \right)^{\frac{1}{2}} n^\frac{1}{\alpha},
\]
Then, according to [29], there is $L > 0$ such that the zeros of $h_n$, which are the nodes for Gauss quadrature chosen in Section 3, are contained in an interval bounded by $m_n(1 + \ln^{-\frac{3}{2}})$. It is therefore natural to assume that a Marcinkiewicz-Zygmund family used for a quadrature rule should satisfy a similar restriction. We may, for instance, think of $X_n$ to be distorted Gaussian nodes, so that the following result can be read as a stability result for Gaussian quadrature.

We now state the generalization of Theorem 3.1 beyond Gaussian nodes and beyond even $\alpha$.

**Theorem 7.1.** Let $W(x) = e^{-\pi |x|^\alpha}$ for $\alpha > 1$ arbitrary, and let $H^s$ and $E_q^p$ be the associated Sobolev spaces. Suppose that the Marcinkiewicz-Zygmund inequalities (27) hold for $X_n \subseteq \mathbb{R}$ and assume that, for some $L > 0$,

$$\max_{x \in X_n} |x| \leq m_n(1 + L \ln^{-\frac{3}{2}}).$$

Define the quadrature weights

$$\omega(x) = \tau(x)(S_n^{-1}W)(x), \quad x \in X_n.$$ 

Then we have, for $s > 1 - \frac{1}{\alpha}$ and $p, q > 0$,

$$\sup_{\|f\|_{H^s} \leq 1} \left| \int_{-\infty}^{\infty} f(x)W(x)dx - \sum_{x \in X_n} \omega(x)f(x) \right|^2 \leq \frac{b_n}{a_n} n^{-s+\frac{3}{2}},$$

$$\sup_{\|f\|_{E_q^p} \leq 1} \left| \int_{-\infty}^{\infty} f(x)W(x)dx - \sum_{x \in X_n} \omega(x)f(x) \right|^2 \leq \frac{b_n}{a_n} e^{-qn^p} \cdot n^{\frac{3}{2}+\max(1-p,0)}$$

with a constant independent of $n$.

**Proof.** According to Theorem 6.2, we need to derive suitable bounds for the error function $\phi$ defined in (25) for the weight sequences $\lambda_k = (1+k)^s$ and $\lambda_k = e^{qk^p}$. For $H^s$ and $E_q^p$ these are

$$\phi^s(n) = \sum_{k=n+1}^{\infty} (1+k)^{-s} \sum_{x \in X_n} \tau(x)|h_k(x)|^2$$

$$\phi^p_q(n) = \sum_{k=n+1}^{\infty} e^{-qk^p} \sum_{x \in X_n} \tau(x)|h_k(x)|^2.$$ 

By interchanging the summation and applying Proposition 2.1 we obtain

$$\phi^s(n) = \sum_{x \in X_n} \tau(x) \sum_{k=n+1}^{\infty} (1+k)^{-s}|h_k(x)|^2 \lesssim \left( \sum_{x \in X_n} \tau(x) \right) n^{-s+1-\frac{1}{s}},$$

$$\phi^p_q(n) = \sum_{x \in X_n} \tau(x) \sum_{k=n+1}^{\infty} e^{-qk^p}|h_k(x)|^2 \lesssim \left( \sum_{x \in X_n} \tau(x) \right) e^{-qn^p} \cdot n^{\frac{1}{2}+\max(1-p,0)}.$$ 

We still need a bound for $\sum_{x \in X_n} \tau(x)$. We apply the Marcinkiewicz-Zygmund inequality (27) to each $h_k, k \leq n$, and recall that $\sum_{k=0}^{n} h_k^2 = \Lambda_n^{-1}$ is the reciprocal of
the Christoffel function. We obtain
\[ \sum_{x \in X_n} \tau(x) h_k(x)^2 \leq b_n \int_{-\infty}^{\infty} h_k(x)^2 \, dx = b_n, \]
so that
\[ \sum_{x \in X_n} \tau(x) \Lambda_n^{-1}(x) \leq b_n \int_{-\infty}^{\infty} \Lambda_n^{-1}(x) \, dx = b_n(n + 1). \]  
Consequently
\[ \sum_{x \in X_n} \tau(x) = \sum_{x \in X_n} \tau(x) \Lambda_n^{-1}(x) \Lambda_n(x) \leq b_n(n + 1) \sup_{x \in X_n} \Lambda_n(x). \]

With the hypothesis \(|x| \leq m_n(1 + Ln^{-\frac{2}{3}})\) in (28), the estimates for \(\Lambda_n\) from [29] imply that
\[ \Lambda_n(x) \lesssim n^{\frac{1}{12} + \frac{2}{3}}, \]
where the constant may depend on \(\alpha\) and \(L\). Hence, we obtain
\[ \sum_{x \in X_n} \tau(x) \lesssim b_n n^{\frac{1}{12} + \frac{2}{3}}, \]
which leads to
\[ \phi^\sigma(n) \lesssim b_n n^{-s + \frac{4}{3}}, \]
\[ \phi^\pi(n) \lesssim b_n e^{-q^p \cdot n^{\frac{2}{3} + \max(1-p,0)}}. \]
This concludes the proof since all assumptions of Theorem 6.2 are satisfied. \(\square\)

A few remarks are in order.

**Remark 7.1.** (i) In the literature it is usually assumed that the Marcinkiewicz-Zygmund inequalities hold with uniform constants. In this case there are \(a\) and \(b\) such that
\[ 0 < a \leq a_n \leq b_n \leq b < \infty, \]
and the fraction \(\frac{b_n}{a_n} \leq \frac{b}{a}\) dissolves into the hidden constant of \(\lesssim\).

(ii) The weights in the Marcinkiewicz-Zygmund inequalities \(\tau(x)\) for \(x \in X_n\) are nonnegative by assumption. In general, however, the quadrature weights \(\omega(x)\) defined by (29) could possibly be negative. However, due to continuity arguments, if \(X_n\) are distorted Gaussian nodes, then \(\{\omega(x)\}_{x \in X_n}\) are still positive for sufficiently small distortions.

(iii) The case of Gauss quadrature can be viewed as a special case of Theorem 7.1. In this case the nodes for \(X_{n+1}\) are the zeros of \(h_{n+1}\). Endowed with the weights
\[ \tau(x) = \Lambda_n(x), \quad x \in X_{n+1}, \]
they satisfy the Marcinkiewicz-Zygmund inequalities with equality \(a_n = b_n = 1\). Indeed, if \(f = pW \in \Pi_n\) with a polynomial of degree \(n\), then \(|p|^2 W \in \Pi_{2n}\) and the exactness of \(X_{n+1}\) on \(\Pi_{2n}\) with \(\omega(x) = \Lambda_n(x)W(x)\) means that
\[ \sum_{x \in X_{n+1}} |p(x)|^2 W(x) \omega(x) = \int_{\mathbb{R}} |p(x)|^2 W(x)^2 \, dx. \]
Written in terms of \( f \in \Pi_n \), this is the Marcinkiewicz-Zygmund inequality
\[
\sum_{x \in X_{n+1}} |f(x)|^2 \Lambda_n(x) W(x)^{-2} = \int_{\mathbb{R}} |f(x)|^2 \, dx ,
\]
with constants \( a_n = b_n = 1 \). Therefore, \( S_n \) is the identity operator, and the quadrature weights \( \{\omega(x)\}_{x \in X_{n+1}} \) defined by (29) coincide with the Christoffel weights in (6). Thus, Theorem 7.1 contains Gauss quadrature as a special case, but it yields slightly weaker bounds than Theorem 3.1.

(iv) The proof of Theorem 7.1 reveals two key ingredients for deriving quadrature bounds. We need
- upper and lower Marcinkiewicz-Zygmund inequalities, i.e., sampling theorems for the finite-dimensional subspaces \( \Pi_n \), and
- upper and lower bounds on Christoffel functions.

Lower Marcinkiewicz-Zygmund inequalities are used in the abstract Theorem 6.2 to reduce the problem of finding suitable bounds on \( \phi^s \) and \( \phi^s_p \). The lower Christoffel bounds are used in the proof of Proposition 2.1 to ensure (30) and (31) hold. Then upper Marcinkiewicz-Zygmund inequalities in (32) combined with upper Christoffel bounds in (33) are used to bound the sum of quadrature weights in (33).

(v) Optimality: How sharp are the estimates of Theorems 3.1 and 7.1?

For the weight function \( W(x) = e^{-\pi x^2} \) (\( \alpha = 2 \)) and the Sobolov spaces \( \mathbb{H}^s \) it was shown in [6] that no choice of points and weights can do better than \( O(n^{-2s}) \). This bound is achieved by points derived from digital nets up to some logarithmic factor (at least when \( n \) is a power of a prime).

For quadrature in \( \mathbb{E}^q_p \) with \( \alpha = 2 \) and \( p \geq 1 \), according to [22], no choice of points and weights can do better than \( e^{-q(2n)^p} 4^{-n} n^{-2} \).

In view of the generality of our techniques, our results come fairly close to the results that are optimal in special cases.

(vi) Theorem 7.1 holds for a more general class of Borel measures instead of just point measures. We assume that \( \nu_n \) is a sequence of Borel measures that define the semi-inner products
\[
\langle f, g \rangle_n = \int_{\mathbb{R}} f(x) g(x) d\nu_n(x) ,
\]
such that (a) the Marcinkiewicz-Zygmund inequalities (27) are satisfied and (b) the support condition
\[
\max_{x \in \text{supp}(\nu_n)} |x| \leq m_n \big( 1 + L n^{-\frac{1}{2}} \big)
\]
holds for some \( L > 0 \). Then the conclusion of Theorem 7.1 holds for the quadrature rule \( \int_{-\infty}^{\infty} f(x)(S_n^{-1} W)(x) d\nu_n(x) \).

**Numerics.** Fix \( \alpha = 2 \) and \( W(x) = e^{-\pi x^2} \) and let \( X_{n+1} \) be the Gaussian nodes with weights \( \{\tau(x)\}_{x \in X_{n+1}} \) in (36). We now choose perturbed nodes \( \tilde{X}_{n+1} \) by
\[
\tilde{x} := x + \epsilon(x), \quad x \in X_{n+1} .
\]
If the perturbations $\epsilon(x)$ are sufficiently small, then the $\tilde{X}_{n+1}$ still satisfy the Marcinkiewicz-Zygmund inequalities (27) with the same weights $\{\tau(x)\}_{x \in X_{n+1}}$, e.g., by [3]. According to (29), we define

$$\omega(\tilde{x}) = \tau(x)(S_n^{-1}W(\tilde{x})), \quad x \in X_{n+1}.$$ 

The quadrature error in $M^e$ is determined analogously to Section 5.2, so that the proof of Theorem 3.1 leads to

$$\sup_{f \in M^e, \|f\|_{M^e} \leq 1} \left| \int_{-\infty}^{\infty} f(x)W(x)dx - \sum_{\tilde{x} \in \tilde{X}_{n+1}} \omega(\tilde{x})f(\tilde{x}) \right|^2 \asymp \sum_{\tilde{x}, \tilde{y} \in \tilde{X}_{n+1}} \omega(\tilde{x})\omega(\tilde{y}) \sum_{k=n+1}^{\infty} e^{-\frac{n}{\sqrt{k}}}h_k(\tilde{x})h_k(\tilde{y}).$$

For the quadrature error in $M^s$, we obtain

$$\sup_{f \in M^s, \|f\|_{M^s} \leq 1} \left| \int_{-\infty}^{\infty} f(x)W(x)dx - \sum_{\tilde{x} \in \tilde{X}_{n+1}} \omega(\tilde{x})f(\tilde{x}) \right|^2 \asymp \sum_{\tilde{x}, \tilde{y} \in \tilde{X}_{n+1}} \omega(\tilde{x})\omega(\tilde{y}) \sum_{k=n+1}^{\infty} (1 + k)^{-s}h_k(\tilde{x})h_k(\tilde{y}).$$

For numerical experiments in Figure 3 we have truncated the infinite series in (38) and (39) and then plotted the worst case error as a function of $n$ and $\sqrt{n}$ respectively.

For $M^e$ in particular, we observed some stability issues. For $M^s$ nonetheless, the results suggest that the actual decay rate is $n^{-s}$ and the additional $\frac{4}{3}$ in Theorem 7.1 is simply an artifact of our proof.

**Appendix A. Multivariate setting**

Using tensor products the results of the previous sections can be extended to several variables. Fix $\alpha > 1$, write $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, and define the tensor weight $W_d$ on $\mathbb{R}^d$ as

$$W_d(x) = W(x_1) \cdots W(x_d), \quad x \in \mathbb{R}^d,$$

where $W(x_i) = e^{-\pi|x_i|^\alpha}$. The tensor product of the generalized Hermite functions is denoted by

$$h_k(x) = h_{k_1}(x_1) \cdots h_{k_d}(x_d), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{N}^d,$$

and we still keep the notation $\hat{f}_k = \langle f, h_k \rangle_{L^2(\mathbb{R}^d)}$. Then $\{h_k : k \in \mathbb{N}^d\}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$. 
For $s \geq 0$ and $p, q > 0$, the multivariate modulation spaces $\mathbb{H}^s(\mathbb{R}^d)$ and $\mathbb{E}^p_q(\mathbb{R}^d)$ are defined by the norms

$$
\|f\|_{H^s(\mathbb{R}^d)}^2 = \sum_{k \in \mathbb{N}^d} (1 + k_1)^s \cdots (1 + k_d)^s |\hat{f}_k|^2,
$$

$$
\|f\|_{E^p_q(\mathbb{R}^d)}^2 = \sum_{k \in \mathbb{N}^d} e^{-q(k_1^p + \cdots + k_d^p)} |\hat{f}_k|^2.
$$

We choose the Cartesian product of the Gaussian quadrature nodes $X_n$, i.e., $X_{n,d} = X_n \times \cdots \times X_n = \{ x \in \mathbb{R}^d : x_j \in X_n \}$ and the quadrature weights $\omega_d(x) = \prod_{j=1}^d \omega(x_j), x_j \in X_n$, where $\omega$ is taken from (6). Then for $s > 1 - \frac{1}{\alpha}$ and $p, q > 0$, 

![Logarithmic plot of the quadrature errors for distorted Gaussian nodes against $\sqrt{n}$ and $n$, respectively.](image)

**Figure 3.** Logarithmic plot of the quadrature errors (38) and (39) for distorted Gaussian nodes against $\sqrt{n}$ and $n$, respectively. (blue) $|\epsilon(x)| = \frac{1}{10}$, (red) $|\epsilon(x)| = \frac{1}{5}$ in (37).
one derives

\begin{align}
\text{(40)} & \quad \sup_{f \in L^2(\mathbb{R}^d)} \left| \int_{\mathbb{R}^d} f(x) W(x) dx - \sum_{x \in X_n} \omega(x) f(x) \right|^2 \lesssim n^{-s+1-\frac{1}{q}}, \\
\text{(41)} & \quad \sup_{f \in E^a_q(\mathbb{R}^d)} \left| \int_{\mathbb{R}^d} f(x) W_\alpha(x) dx - \sum_{x \in X_n} \omega(x) f(x) \right|^2 \lesssim e^{-q(2n)^p} \cdot n^{\frac{1}{a} - \frac{1}{a}}.
\end{align}

Note that \( X_{n,d} \) consists of \( N = (n+1)^d \) points, so that the asymptotic error is actually \( O(N^{-\left(\frac{1}{q}+\frac{1}{a}-\frac{1}{a}\right)} \) in dimension \( d \) for \( \mathbb{H}^s(\mathbb{R}^d) \). We may also consider different \( \alpha_1, \ldots, \alpha_d \) for each dimension in the tensor product. Then (40) and (41) hold for \( \alpha \) replaced by \( \alpha_{\text{max}} = \max(\alpha_1, \ldots, \alpha_d) \) for all \( s > 1 - \frac{1}{\alpha_{\text{max}}} \).

For the normalized multivariate Gaussian window function

\[ \varphi_d(x) = 2^\frac{d}{2} e^{-\pi \|x\|^2}, \quad x \in \mathbb{R}^d, \]

the short-time Fourier transform of \( f \in L^2(\mathbb{R}^d) \) is

\[ V_{\varphi_d} f(x, \xi) = \int_{\mathbb{R}^d} f(t) \varphi_d(t-x) e^{-2\pi i (\xi \cdot t)} dt, \quad x, \xi \in \mathbb{R}^d. \]

For \( f \in L^2(\mathbb{R}^d) \), the modulation norms

\begin{align*}
\|f\|_{M^s(\mathbb{R}^d)}^2 & = \int_{\mathbb{R}^{2d}} |V_{\varphi_d} f(z)|^2 (1 + |z_1|^2)^s \cdots (1 + |z_d|^2)^s \, dz, \\
\|f\|_{M_\alpha^s(\mathbb{R}^d)}^2 & = \int_{\mathbb{R}^{2d}} |V_{\varphi_d} f(z)|^2 e^{\|z\|^2} \, dz, \\
\|f\|_{M_\alpha^\infty(\mathbb{R}^d)}^2 & = \int_{\mathbb{R}^{2d}} |V_{\varphi_d} f(z)|^2 e^{\|z\|^2} \, dz,
\end{align*}

where \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) denote the standard 1 and 2-norm on \( \mathbb{C}^d \), lead to the respective multivariate modulation spaces \( M^s(\mathbb{R}^d) \), \( M_\alpha^s(\mathbb{R}^d) \), and \( M_\alpha^\infty(\mathbb{R}^d) \). Since

\[ V_{\varphi_d} h_k = V_{\varphi_1} h_{k_1} \otimes \cdots \otimes V_{\varphi_d} h_{k_d}, \quad k \in \mathbb{N}^d, \]

they carry the tensor structure, and one may deduce, for \( \alpha = 2, \)

\begin{align*}
M^s(\mathbb{R}^d) & = \mathbb{H}^s(\mathbb{R}^d), \quad s \geq 0, \\
M_\alpha^s(\mathbb{R}^d) & = \mathbb{E}_q^\frac{s}{\sqrt{\pi}}(\mathbb{R}^d), \quad q = \frac{s}{\sqrt{\pi}}, \quad s \geq 0, \\
M_\alpha^\infty(\mathbb{R}^d) & = \mathbb{E}_q^1(\mathbb{R}^d), \quad q = \ln(\frac{\pi}{s}), \quad \pi > s \geq 0.
\end{align*}

with equivalent norms.

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