For many years, the derivation of
\[ H = -\Delta - Tr \sum_{i<j} [X_i, X_j]^2 \]
(involving finitely many antihermitean traceless \( N \times N \) matrices \( X_i = \sum_{a=1}^{N^2-1} x_{ia} T_a, \ i = 1, \ldots, d \)) as describing a quantized discrete analogue of relativistically invariant surface dynamics in \( \mathbb{R}^{d+1} \) was available only in rather inconvenient forms (handwritten \[1\], too condensed \[2\], typeset \[3\] by a local journal \[3\]). With the supersymmetric analogue of \( H \),
\[ H_{\text{Susy}} = H \cdot \mathbb{1} + if_{abc} x_{jc} \gamma^j \alpha \beta \theta_{\alpha \alpha} \theta_{\alpha \beta} , \]
\( \{ \theta_{\alpha \alpha}, \theta_{\alpha \beta} \} = \delta_{\alpha \beta} \delta_{ab}, \quad (\gamma^j \gamma^k + \gamma^k \gamma^j)_{\alpha \beta} = 2 \delta_{\alpha \beta} \delta_{jk} \), \( \gamma^j \gamma^k \gamma^j = 2 \delta_{jk} \delta_{jk} \), cf \[4, 5\], having become relevant for yet another reason \[6\], - and for the participants of a summer school on Schrödinger–operators, it seemed useful to give a detailed account of \( H \) (Section II) while adding various introductory remarks about surface motions, their relation with hydrodynamics \[7\], as well as diffeomorphism – and relativistic invariance (Section I).

Section III concerns a rather particular, though quite important, question: does \( H_{\text{Susy}} \) (whose spectrum is known to cover the whole positive axis, \( \mathbb{R}^+ \) \[8, 29\]) admit a zero energy bound state, or not? Some work on this is summarized.

Section IV (following \[9\], \[10\], \[11\]) discusses the space of solutions of the differential equations
\[ \dot{X}_a = \epsilon_{abc} X_b X_c - 2X_a \]

\[ i.e. \ H = -\frac{\partial}{\partial \epsilon_{abc}} \frac{\partial}{\partial \epsilon_{abc}} + \frac{i}{2} f^{(N)}_{abc} f^{(N)}_{ae} x_{ib} x_{jc} x_{ib} x_{jc} x_{ic} , \] for each \( N \) and \( d \), being an ordinary Schrödinger–operator on \( \mathbb{R}^{d(N^2-1)} \), with a non–negative quartic potential, given in terms of totally antisymmetric \( su(N) \)–structure–constants
\[ f^{(N)}_{abc} = -TrT_a[T_b, T_c] . \]
for 3 traceless anti–hermitean $N \times N$ matrices $X_a(t), \, t \in (-\infty, +\infty)$, interpolating between different representations of $su(2)$.

Some exercises have been added, and a remark/conjecture concerning 5-commutators.

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Let me start with two relatively simple surface motions,

\[ \overset{\rightarrow}{x} = \overset{\rightarrow}{n}, \quad (1) \]

\[ \dot{\overset{\rightarrow}{x}} = \sqrt{g} \overset{\rightarrow}{n} \quad (= \partial_1 \overset{\rightarrow}{x} \times \partial_2 \overset{\rightarrow}{x}) \quad (2) \]

In both cases the surface is, for each time \( t \), described in a parametric way (by giving \( \overset{\rightarrow}{x}(t, \varphi_1, \varphi_2) \)), i.e. by viewing the surface \( \sum_t \subset \mathbb{R}^3 \) as the image of a (time-dependent) map from some fixed manifold \( \sum \) (specifying the topology; e.g. that of \( S^2 \)) into \( \mathbb{R}^3 \). As long as \( \sum_t \) is non-degenerate, every vector can be decomposed into its components parallel to \( \partial_r \overset{\rightarrow}{x} \) and \( \partial_r \overset{\rightarrow}{x} \) (tangent to \( \sum_t \)) resp.

\[ \overset{\rightarrow}{n} := \frac{\partial_1 \overset{\rightarrow}{x} \times \partial_2 \overset{\rightarrow}{x}}{|\partial_1 \overset{\rightarrow}{x} \times \partial_2 \overset{\rightarrow}{x}|} \quad (3) \]

(normal to \( \sum_t \)); \( \partial_r \overset{\rightarrow}{x} \) stands for \( \frac{\partial \overset{\rightarrow}{x}}{\partial \varphi_r} \), \( r = 1, 2 \), \( g \) denotes the determinant of the \( 2 \times 2 \) matrix \( (g_{rs}) = (\partial_r \overset{\rightarrow}{x} \cdot \partial_s \overset{\rightarrow}{x}) \).

The evolution equation (1), though consisting of coupled non-linear PDE’s for the 3 unknown functions \( x^i(t, \varphi_1, \varphi_2) \) is trivial, as \( \overset{\rightarrow}{n} \) implies \( \overset{\rightarrow}{n} \cdot \dot{\overset{\rightarrow}{n}} = 0 \) and \( \overset{\rightarrow}{n} \cdot \partial_r \overset{\rightarrow}{n} = 0 \), hence

\[ \dot{\overset{\rightarrow}{x}} = \overset{\rightarrow}{n} = 0 \quad (4) \]

\[ \overset{\rightarrow}{x}(t, \varphi_1, \varphi_2) = \overset{\rightarrow}{x}(0, \varphi_1, \varphi_2) + t \overset{\rightarrow}{n}(0, \varphi_1, \varphi_2) \]

as \( \dot{\overset{\rightarrow}{n}} \) has zero component in the direction of \( \overset{\rightarrow}{n} \), as well as parallel to \( \sum_t (\overset{\rightarrow}{n} \partial_r \overset{\rightarrow}{x} = - \overset{\rightarrow}{n} \partial_r \overset{\rightarrow}{x} = - \overset{\rightarrow}{n} \partial_r \overset{\rightarrow}{n}). \)

What about (2)?

Despite the r.h.s. being polynomial in the first derivatives (so, from this point of view, being ‘simpler’ than (1)) the (still non-linear) equation (2) needs quite different techniques to be ‘solved’. One possibility is to note that after interchanging dependent and independent variables, \( t \) (as a function of \( \overset{\rightarrow}{x}! \) turns out to satisfy a linear! (and, in fact, the simplest possible 2nd order) differential equation,

\[ \Delta t(\overset{\rightarrow}{x}) = 0 \quad (5) \]

so the time at which the surface passes a point \( \overset{\rightarrow}{x} \) in space is a harmonic function (resp. \( \sum_t \) level sets of that harmonic function) [12, 13].

What if the normal velocity (= \( \sqrt{g} \) in (2)) is generalized to be an arbitrary (non-linear) function of \( \sqrt{g} \)?

\[ \text{To avoid (hopefully not cause) confusion: strictly speaking, } \sqrt{g} \text{ always has to be divided by some reference-density } \rho \text{ in order to make the evolution equation (2), resp. (6), well defined (see [13] for a full account of this).} \]
As shown in [13], the evolution equations
\[ \dot{x}^i = \alpha(\sqrt{g}) n^i, \] (6)
for arbitrary monotonic function \( \alpha \), and the hypersurface motion taking place in any Riemannian manifold \( N(i = 1 \ldots M + 1) \) if \( x^i = x^i(t, \varphi_1, \ldots, \varphi^M) \), and \( g = \det(\partial_r x^i \partial_s x^j \eta_{ij}(x)) \), \( \eta_{ij} \) = metric on \( N \) can always be converted to a second–order equation for \( t(x) \) (however, only in the case (2) this equation becomes linear).

Furthermore, despite their appearance as first order equations (in the derivatives of the \( x^i \)) they can be viewed as coming from a class of diffeomorphism invariant Hamiltonians [13] [14];
\[ H[x^i, p_j] = \int d^M \varphi \sqrt{g} h(p/\sqrt{g}) \] (7)
\((p = \sqrt{p_i p_j \eta^{ij}(x))})\), restricted to \( C_r := p_i \partial_r x^i \), \( r = 1, \ldots, M \), (the generators of diffeomorphisms, which are constants of the motion) set equal to zero!

Returning now to flat embedding space and \( M = 2 \) (2–surfaces moving in \( \mathbb{R}^3 \)) for simplicity, let me ask the following question: Does there exist a function \( h \) (corresponding to the freedom of choosing \( \alpha \) in (6)) such that the corresponding surface motions are relativistically invariant, i.e. for which \( H \) can be complemented by 9 other functionals of \( \vec{x} \) and \( \vec{p} \), altogether generating the inhomogeneous Poincaré–group?

The answer is ‘Yes’: for \( h(u) = \sqrt{u^2 + 1} \), i.e.
\[ H = \int_{\Sigma(2)} d^2 \varphi \sqrt{\vec{p}^2 + g} \] (8)
one not only has
\[ L_{ij} := \int_{\Sigma(2)} d^2 \varphi (x_i p_j - x_j p_i) \], \( \vec{p} := \int_{\Sigma(2)} d^2 \varphi \vec{p} \) (9)
(generating rotations, and spatial translations), but also the generators of ‘boosts’,
\[ K_i := \int_{\Sigma(2)} d^2 \varphi x_i \sqrt{\vec{p}^2 + g} \]. \(10\)
Restricting to \( C_r := \vec{p} \cdot \partial_r \vec{x} = 0 \) then gives \( \sqrt{\vec{p}^2 + g} = \rho = \text{time–independent density} \), and – using this –
\[ \vec{x} = \pm \sqrt{1 - g/\rho^2} \vec{n} . \] (11)
2–dimensional surfaces moving according to (11) not only correspond to ‘relativistically invariant’ motions, but have the more specific property that they actually sweep out a 3–dimensional manifold \( M \) in Minkowski–space, which has vanishing mean curvature (see e.g. [15], and below).
In any case, comparing (8), (9), (10) with the corresponding expressions for a (finite–dimensional) system of $N$ free (!) relativistic particles

$$ H = \sum_{a=1}^{N} \sqrt{\vec{p}_a^2 + m_a^2} , \quad \vec{P} = \sum_{a} \vec{p}_a $$

$$ \vec{L} = \sum_{a} (\vec{x}_a \times \vec{p}_a) , \quad \vec{K} = \sum_{a} \vec{x}_a \sqrt{\vec{p}_a^2 + m_a^2} , \quad (12) $$

one finds exact correspondence ($a \overset{\hat{\sim}}{=} \varphi^1, \varphi^2$) when replacing the position–independent masses $m_a$, $a = 1, \ldots, N$, by the position–dependent density $\sqrt{g[\vec{x}(\varphi)]}$. While it was proved decades ago [16] that, given certain physical requirements concerning the realization of the 10 generators, any set deviating from (12), but still satisfying the same commutation relations, can, by a sequence of canonical transformations, be brought into the form (12) (so in particular, it is, for finite $N$ not! possible to allow for $x$–dependent masses $m_a$) it should be noted that the way (9)–(10) circumvents the just mentioned No–Interaction theorem for finitely many degrees of freedom, is quite interesting.

Another notable aspect is the possibility to abandon the parametric description at this point and describe the surface motion, in Hamiltonian form, purely in terms of diffeomorphism invariant objects. Formally one gets, as a reduced phase–space, the space of shapes $\Sigma$ (as the configuration manifold) together with functions on $\Sigma$ (elements of the cotangentspace at $\Sigma$) as ‘momenta’. In this formulation, the generators of the Poincaré–group read [17]

$$ \mathbb{H}[\Sigma, u] = \int_{\Sigma} \sqrt{u^2 + 1} , \quad \vec{P} = \int_{\Sigma} \vec{n} \ u $$

$$ L_{ij} = \int_{\Sigma} (x_i n_j - x_j n_i) \ u , \quad K_i = \int_{\Sigma} x_i \sqrt{u^2 + 1} . \quad (13) $$

Amazingly, the Poisson structure is the canonical one, meaning that the equations of motion are

$$ \dot{\Sigma} = \frac{\delta \mathbb{H}}{\delta u} = \frac{u}{\sqrt{u^2 + 1}} , \quad \dot{u} = -\frac{\delta \mathbb{H}}{\delta \Sigma} = -\sqrt{u^2 + 1} \ H \quad (14) $$

i.e. (as $\Sigma$, the time–derivative of $\Sigma$, can be identified with the normal velocity, $v$)

$$ \dot{v} = -(1 - v^2) \ H ; \quad (15) $$
here $H$ denotes the mean curvature of $\sum$ (which in a parametric description equals $-g^{rs} \overrightarrow{n} \cdot \partial^2_{rs} \overrightarrow{x}$).

Writing (15) as

$$\frac{\dot{v}}{(1-v^2)^{1/2}} + \frac{H}{(1-v^2)^{1/2}} = 0$$

one finds that the 3–manifold $M_3$ swept out in space–time by a surface moving according to (15) will actually have vanishing mean 3–curvature $H_3$ (the spatial part, proportional to $H$, is cancelled by the curvature of the wordline of the point on $\sum$; the 4–dimensional hypersurface–normal is, up to orientation, $n^\mu = \left( \frac{\dot{v}}{\sqrt{1-v^2}}, \frac{\overrightarrow{n}}{\sqrt{1-v^2}} \right)$).

Conventionally, the problem of finding such 3–manifolds $M_3$ is formulated as considering (in a Diff $M_3$–invariant, parametric, way)

$$S[\overline{x^\mu}] = \text{Vol} (M_3) = \int d^3 \varphi \sqrt{G}$$

$$G = \text{det} \left( \frac{\partial x^\mu}{\partial \varphi^\alpha} \frac{\partial x^\nu}{\partial \varphi^\beta} \eta_{\alpha \beta} \right)_{\alpha, \beta = 0, 1, 2} \tag{16}$$

$$\eta_{\alpha \beta} = \text{diag} (1, -1, -1, -1),$$

a functional of the embedding functions $x^\mu (\varphi^0, \varphi^1, \varphi^2)$ (describing $M_3$), whose first variation ($\dot{=} 0$) gives the equations of motions

$$\frac{1}{\sqrt{G}} \partial_\alpha \sqrt{G} G^{\alpha \beta} \partial_\beta x^\mu = 0. \tag{17}$$

For a direct derivation of (8), resp. (11), from (16), resp. (17), see e.g. [15].

Alternatively, (16) could have been motivated as follows: in order to describe a relativistically invariant motion of an extended object (a surface, in this case), the only chance is to consider the manifold $M$ swept out in space–time, and demand some extremality–property of $M$; considering the Volume–functional is singled out by the fact that, depending only on the first derivatives of $x^\mu$, the equations of motion, (17), are then of second order (other diffeomorphism–invariant functionals like the total curvature of $M$, . . ., would involve second or higher order derivatives of the $x^\mu$, hence leading to higher order equations of motion).

Footnote 5: For the reader familiar with the fact that the first variation of the area–functional gives $H$, the second part of (14) may become intuitively clear, when first dropping the argument $\sqrt{u^2 + 1}$ in $\mathcal{H}$ (and (14)).
Perhaps a third connecting path is worth mentioning: a Lagrangian formulation of (13) would give

\[ S = \int dt \left( \sum vu - \int \sum \sqrt{u^2 + 1} \right) \]

\[ = -\int dt \int \sum \left( \frac{1}{\sqrt{u^2 + 1}} \right) = \sqrt{1 - v^2} = \sqrt{1 - \dot{\Sigma}^2} \]

(18)

\[ = S[\Sigma, \dot{\Sigma}]. \]

Now, describing \( \dot{\Sigma} = \dot{\Sigma}(t) \) as the set of points \((t, \vec{x})\), where some functions \( m(t, \vec{x}) = m(x^\mu) \) vanishes, (18) becomes

\[ S = -\int d^4 x (\delta(m(t, \vec{x})) | \vec{\nabla} m|) \sqrt{1 - \frac{\dot{m}^2}{\nabla m^2}} \]

\[ = -\int d^4 x \delta(m) \sqrt{-\partial_\mu m \partial^\mu m} = S[m], \quad (19) \]

with equations of motion \( \delta(m) \partial_\mu \left( \frac{\partial^\mu m}{\sqrt{-\partial_\mu m \partial^\mu m}} \right) = 0 \), respectively

\[ (\eta^{\mu\nu} \eta^{\rho\lambda} - \eta^{\mu\rho} \eta^{\nu\lambda}) \partial_\mu m \partial_\nu m \partial_\rho m = 0, \quad (20) \]

together with \( m(t, \vec{x}) \equiv 0 \); meaning that first solving (20) and then solving for \( \mathcal{M} := \{(t, \vec{x}) | m(t, \vec{x}) = 0\} \) will yield a 3–manifold with everywhere vanishing mean curvature. To understand why the factors of \( | \nabla m | \) have to enter in \( (\delta(m)| \vec{\nabla} m|) \) and \( (v^2 = \frac{\dot{m}^2}{\nabla m^2}) \) precisely as indicated, making both expressions, hence (20), invariant under \( m \rightarrow F(m) \), one should note that if \( m = 0 \) describes \( \mathcal{M} \), functions of \( m \), like \( F(m) = m^3 \) would yield the same \( \mathcal{M} \). Furthermore note that \( m(t, \vec{x}) = \text{const.} \neq 0 \), with \( m \) solving (20) also yields an extremal 3–manifold. Locally, Minkowski–space is therefore foliated into 3 manifolds, each with vanishing mean curvature, with \( m \) being the parameter orthogonal to the extremal 3–manifolds. In physics, the direct corresponda nce of (20) and (17) was established by Sugamoto [18] and in [19] (see also [20], in particular concerning the factor \( \delta(m) \)).

An Example: Making the Ansatz \( m(t, \vec{x}) = \prod_{\mu=0}^3 m_\mu(x^\mu), \) resp. \( \tilde{m} = \ln m = \ln m_\mu = \sum_{\mu=0}^3 \ln m_\mu = \sum_{\mu=0}^3 \tilde{m}_\mu(x^\mu) \), one can derive [15] that

\[ \hat{\mathcal{M}} := \{(t, \vec{x}) \mid \mathcal{P}(x) \mathcal{P}(y) \mathcal{P}(t) = \mathcal{P}(z)\}, \quad (21) \]

where \( \mathcal{P}(u) \) is an elliptic Weierstrass–function satisfying \( \mathcal{P}^2 = 4 \mathcal{P}(\mathcal{P}^2 - 1) \), defines a periodic \( \mathcal{M} \) with vanishing mean curvature. Viewed as time evolutions of 2–surfaces, (21) corresponds to \( (at = 0, \pm \omega, \pm 2\omega, \ldots) \)
infinite many parallel planes, at distance $\omega$ apart from each other, breaking up into little squares of size $\omega \times \omega$ that grow one scalactite and one scalagmite in the direction perpendicular to the original planes meeting eventually, at times $t = \pm \frac{\omega}{2}, \pm \frac{3\omega}{2}, \ldots$ at the center of each ‘box’ of size $\omega \times \omega \times \omega$.

**Note [19]:**
Just choosing $\varphi^\alpha = x^\alpha$ in (16), leaving $x^3 = z(x^0, x^1, x^2)$ to be determined, yields the same equation than writing $m = m(x^0, x^1, x^2, z(x^0, x^1, x^2))$ in (20); choosing $\varphi^0 = \frac{x^0 + x^3}{2} = \tau, \varphi^r = x^r$ in (16), resp. writing $(\tau + \frac{1}{2} p(\tau, x^1, x^2), x^1, x^2, \tau - \frac{p}{2})$ in (20) both yields a second order equation for $p$, which in first order from (defining $q = (2\dot{p} + (\nabla p)^2)^{-\frac{1}{2}}$) becomes

$$\dot{q} + \vec{\nabla} (q \vec{\nabla} p) = 0 \tag{22}$$

$$\dot{p} = \frac{1}{2} \left( \frac{1}{q^2} - (\vec{\nabla} p)^2 \right),$$

the continuity, and Euler–equation for an inviscid, isentropic irrotational gas, with density $q$, velocity potential $p$, and equation of state (Kármán–Tsien gas) Pressure = $\frac{-1}{\text{Density}}$.

This relation between relativistic membrane motions and 2 + 1–dimensional hydrodynamics, (22), was first derived in [7], and extended to a supersymmetric membrane/liquid in [21].

While it is quite interesting to have all these different formulations (revealing different aspects of one and the same problem, – hopefully leading, one day, to some hidden integrability structure) none of them could, up to now, be quantized. The best route to quantisation, known so far, is the one found 20 years ago (cp. [1], resp. [22]) to be discussed after the following
Excercise 1)  
Choose, in (17), \( \varphi^0 = \frac{x^0 + x^3}{2} =: \tau \), and assume \((\varphi^1, \varphi^2)\) to be chosen such that  
\[ G_{\varphi} = \partial_t \zeta - \frac{\dot{x}}{x} \partial_t x \equiv 0, \]  
where \( \dot{x} = (x^1, x^2) \), \( \cdot = \frac{\partial}{\partial \tau} \) and \( x^0 - x^3 =: \zeta(\tau, \varphi^1, \varphi^2) \).  
Verify that  
\[ \sqrt{G} = \sqrt{g} \sqrt{2 \ddot{x} - \dot{x}} \]  
and show that, as a consequence of (17), i.e.  
\[ \Delta x^\mu = 0, \]  
a) \( \rho := \frac{\sqrt{g}}{\sqrt{2 \ddot{x} - \dot{x}}} \) is independent of \( \tau \) (hint: \( \Delta (x^0 + x^3) = 0 \))  
b) \( \ddot{x} = \frac{1}{\rho} \partial_r \bar{g} \partial_s p \partial_s \bar{x} = \sum_j \{\{\ddot{x}, x_j\}, x_j\} \) (defining \( \{f, g\} := \frac{1}{\rho} \epsilon^{rs} \partial_r f \partial_s g \)).  
Prove that a) and b), together with \( G_{\varphi} = 0 \), automatically imply the “remaining” equation  
\( \Delta \zeta = 0! \) (So that \( \zeta = x^0 - x^3 \), up to a constant, and provided \( \epsilon^{rs} \partial_r \bar{x} \partial_s \bar{x} \equiv 0 \), can simply be thought of as being determined, via \( G_{\varphi} = 0 \), in terms of \( x^1 \) and \( x^2 \), satisfying b).)  

Excercise 2)  
Consider now the case \( d = 2 \) (4-dimensional Minkowski-space)  
a) Show that the equations of motion 1b can be written as  
\[ \begin{align*}  
\dot{x}_1 &= p_1, & \ddot{p}_1 &= \{x_1, x_2\} \times_{x_2} \\
\dot{x}_2 &= p_2, & \ddot{p}_2 &= \{x_2, x_1\} \times x_1,  
\end{align*} \]  
where  
\[ \{f, g\} := \frac{1}{\rho} \epsilon^{rs} \partial_r f \partial_s g \]  
b) Changing independent variables from \( t(= \varphi^0), \varphi^1, \varphi^2 \) to \( x^0 := t, x^1, x^2 \) show that \( \{f, x_r\} = -J \epsilon_{rs} \frac{\partial f}{\partial x_s} \), where \( J(x) \) and \( f(x) \) are \( \{x_1, x_2\} \) resp. \( f(\varphi) \) when expressed in the new variables. Verify that the constraint \( \epsilon_{rs} \partial_r \ddot{x} \partial_s \ddot{x} \) therefore becomes the condition  
\[ \partial_{x_1} p_2 - \partial_{x_2} p_1 = 0, \]  
which can easily be solved:  
\[ \ddot{p} = \nabla p(x) \]  
c) Calculate \( \frac{\partial f}{\partial \varphi_0} \{x_1, x_2\} \) in 2 different ways (via \( \{p_1, x_2\} + \{x_1, p_2\} \), as well as using  
\[ \partial_{\varphi_0} = \partial_{x_0} + \ddot{\bar{p}} \cdot \nabla \]  
to obtain the continuity equation  
\[ \ddot{q} + \nabla (q \nabla \ddot{p}) = 0 \]  
for the ‘velocity-potential’ \( \ddot{p} \) and the ‘density’  
\[ q := \frac{1}{\ddot{p}} \]  
d) Convert the equations involving \( \ddot{p} \) (cp. 2a) into ‘Euler’s equation’,  
\[ \ddot{p} = \frac{1}{2} \left( \frac{1}{q^2} - (\nabla p)^2 + c \right) \]
for an irrotational, inviscid, isentropic gas whose pressure is $\frac{-1}{\text{density}}$.

e) Choosing the constant $c$ to be zero, verify that the equations obtained (cp. 2c, d) are the ones that follow from

$$H = \frac{1}{2} \int d^2 x (q(\nabla p)^2 + \frac{1}{q})$$

when $q$ and $p$ are treated as (hence ARE) canonically conjugate.
Consider the theory of a time-dependent \( M \)-dimensional extended object \( \Sigma_M(t) \) in \( D = d + 2 \) dimensional Minkowski-space, defined by requiring the Volume-functional

\[
S[x^\mu] = - \int d\varphi^0 d^M \varphi \sqrt{G} = - \text{Vol}(\mathcal{M})
\]

\[
G = (-)^M \det \left( \frac{\partial x^\mu}{\partial \varphi^\alpha} \frac{\partial x^\nu}{\partial \varphi^\beta} \eta_{\mu\nu} \right)_{\alpha,\beta = 0,1,\ldots,M}
\]

to be stationary under small variations of the embedding functions \( x^\mu(\varphi^0, \varphi^1, \ldots, \varphi^M), \mu = 0,1,\ldots,d+1 \) (describing the \( M+1 \) dimensional world volume \( \mathcal{M} \)). (23) is invariant under diffeomorphisms \( \varphi^\alpha \rightarrow \tilde{\varphi}^\alpha(\varphi^\beta) \), and inhomogeneous Lorentztransformations

\[
x^\mu \rightarrow \tilde{x}^\mu = \Lambda^\mu_\nu x^\nu + d^\mu, \quad \Lambda^\mu_\nu \Lambda^\nu_\rho \eta_{\mu\rho} = \eta_{\mu'\nu'} = \text{diag}(1,-1,\ldots,-1).
\]

Choosing

\[
\varphi^0 = \frac{x^0 + x^{d+1}}{2} =: \tau
\]

while denoting \( x^0 - x^{d+1} \) by \( \zeta \), differentiation with respect to \( \tau \) by \( \cdot \), \( (x^1,\ldots,x^d) \) by \( \vec{x} \), and \( G_{0r} = \partial_r \zeta - \vec{x} \partial_r \vec{x} \) by \( u_r \), the metric (induced from \( \mathbb{R}^{1,d+1} \)) on \( \mathcal{M} \) reads

\[
(G_{\alpha\beta}) = \begin{pmatrix}
2\dot{\zeta} - \frac{\dot{\vec{x}}^2}{\vec{x}} & u_1 & \cdots & u_M \\
u_1 & \\
\vdots & \\
\vdots & \\
u_M & -g_{rs}
\end{pmatrix}
\]

\[
g_{rs} = \partial_r \vec{x} \cdot \partial_s \vec{x},
\]

hence

\[
G = g(2\dot{\zeta} - \frac{\dot{\vec{x}}^2}{\vec{x}} + u_r u_s g^{rs}) =: g \cdot \Gamma
\]

with

\[
g = \det(g_{rs})_{r,s=1\ldots M}, \quad g^{rs} g_{su} = \delta^r_u.
\]
So

\[ S = S[\vec{x}, \zeta] = \int d\tau d^d \varphi \mathcal{L} \]

\[ \mathcal{L} = -\sqrt{g/\Gamma} \]  \hspace{1cm} (27)

can be thought of as the action, resp. Lagrange density, of an ordinary field theory involving the \( d + 1 \) fields \( x^1, \ldots, x^d, \zeta \). Defining canonical momenta,

\[ \vec{p}: = \frac{\delta \mathcal{L}}{\delta \dot{x}} = \sqrt{g/\Gamma} (\dot{x} + \partial_{\tau} \vec{x} u_{s} \rho^s) \]

\[ \pi: = \frac{\delta \mathcal{L}}{\delta \zeta} = -\sqrt{g/\Gamma} \]  \hspace{1cm} (28)

the Hamiltonian density reads

\[ \mathcal{H}: = \Pi \dot{\zeta} + \vec{p} \cdot \dot{\vec{x}} + \sqrt{g/\Gamma} \]

\[ = \sqrt{g/\Gamma} \{ \Gamma + \vec{x}^2 + \vec{x} \partial_{\tau} \vec{x} u_{s} \rho^s - \dot{\zeta} \} \]

\[ = \sqrt{g/\Gamma} \{ \dot{\zeta} + \partial_{\tau} \zeta u_{s} \rho^s \}. \]  \hspace{1cm} (29)

Instead of now trying to eliminate the velocities in favour of the momenta, simply note that (29) is the same as

\[ \mathcal{H} = \frac{-\vec{p}^2 + g}{-2\pi} \]  \hspace{1cm} (30)

\[ (= 1/2 \sqrt[4]{g/\Gamma} \{ (\vec{x} + \partial_{\tau} \vec{x} u_{s} \rho^s)^2 + \Gamma \} = \ldots = (29)). \]  

As \( \mathcal{H} \) does not contain \( \zeta \) (!), the classical equation of motion \( \ddot{\pi} = -\frac{\delta \mathcal{H}}{\delta \zeta} = 0 \) imply that \( \pi \) is a constant (density),

\[ \pi = -\eta \rho(\varphi), \quad \int d^M \varphi \rho(\varphi) = 1, \]  \hspace{1cm} (31)

so that the Hamiltonian

\[ H(\cdot) = \int \mathcal{H} = \frac{1}{2\eta} \int \Sigma(M) d^M \varphi \frac{-\vec{p}^2 + g}{\rho(\varphi)} \]  \hspace{1cm} (32)

becomes polynomial in the dynamical fields \( (\vec{x} \text{ and } \vec{p}) \)- which is the crucial difference to all the Hamiltonian formulations mentioned in section I (they all contain square-roots). But what has happened to \( \zeta \)!!
(After all, as it is part of the description of the manifold \( M \), it can’t just disappear altogether). The point is that actually the transition to (32) is not quite as straightforward as pretended, due to the presence of the constraints \( \Phi_r \) (reflecting the remaining time-dependent spatial reparametrisation invariance after the partial gauge-fixing (24))

\[
\Phi_r := \pi \partial_r \zeta + \ddot{p} \partial_r \dddot{x} \equiv 0 \\
r = 1, \ldots, M;
\]

(so strictly speaking (32) is only defined up to terms \( \int dM \varphi u^r \Phi_r \); putting them to zero, as done in (32), amounts to the further gauge choice \( G_{0r} = u_r = 0 \) (cp. the excercise at the end of the section I) - which explains the further reduction of the invariance group to volume-preserving Diffeomorphisms of \( \Sigma_M \) in (32). On the other hand, (33) precisely resolves the puzzle of the missing \( \zeta \) as, given (31), it allows to determine \( \zeta \) (up to a constant which is canonically conjugate to \( \eta \)) in terms of the dynamical fields \( \dddot{x} \), as long as the integrability conditions

\[
\Phi_{rs} := \partial_r(\dddot{p} / \rho)\partial_s \dddot{x} - \partial_s(\dddot{p} / \rho)\partial_r \dddot{x} \equiv 0
\]

hold (-which are consistent with (32), as the \( \Phi_{rs} \) are the generators of the above mentioned symmetry group \( SDiff \sum_M \) of (32)). Finally noting that \( H(-) \) is generating translations in \( \tau \) (just as \( \dot{P} := \int \ddot{p} \) generates translations in \( \dddot{x} \), and \( P_+ := -\int \pi dM \varphi = \eta \) in \( x^0 - x^3 \)), one sees that the relativistically invariant

\[
\mathcal{M}^2 := P^\mu P_\mu \equiv 2P_+P_- - \dddot{P}^2
\]

takes the simple form

\[
\mathcal{M}^2 = \int \frac{d^M \varphi}{\rho(\varphi)}(\dddot{P}^2 + g) - \dddot{P}^2
\]

which may be rewritten in the following ways:

first of all, \( g = \sum_{i_1 < i_2 < \ldots < i_M} \{x_{i_1}, x_{i_2}, \ldots, x_{i_M}\}^2 \),

where the multilinear bracket is defined, for any set of \( M \) differentiable functions on \( \sum_M \), as

\[
\{x_{i_1}, \ldots, x_{i_M}\} := \frac{1}{\rho} \sum_{i_1 \ldots i_M \rho} \frac{\partial x_{i_1}}{\partial \varphi^{i_1}} \cdots \frac{\partial x_{i_M}}{\partial \varphi^{i_M}}.
\]

\(^6\)just insert (28), to verify (33)
Secondly, expanding the fields \( x_i(t, \varphi^1, \ldots, \varphi^M) \) and their conjugate momenta \( p_i = \rho p_i Y_\alpha \) in terms of basis functions \( Y_\alpha, \alpha = 0, 1, \ldots, \) on \( \sum_M, \int d^M \varphi Y_\alpha Y_\beta = \delta_{\alpha\beta} \), (33) becomes

\[
\mathbb{M}^2 = p_{i\alpha} p_{i\alpha} + \frac{1}{M!} g_{\alpha\beta_1 \ldots \alpha M} x_i \{ x_j, x_i \} \ldots x_i \beta_1 \ldots x_i \beta_M ,
\]

where the ‘structure-constants’

\[
g_{\alpha\beta_1 \ldots \alpha M} := \int d^M \varphi Y_\alpha \{ Y_{\beta_1}, \ldots, Y_{\alpha M} \}
\]

encode all the information about the fact that one is dealing with a certain \( M \)-dimensional object (a sphere \( S^M \) or a torus \( T^M \), or \ldots); the sum over \( \alpha, \beta \) is defined to run only from 1 to \( \infty \) (rather than 0 to \( \infty \); due to the \( \overleftarrow{P}^2 \) subtraction, and \( \{ \ldots, \} \) containing only derivatives, all zeromodes drop out). The potential \( \mathbb{M}^2 - p_{i\alpha} p_{i\alpha} \) is a homogeneous polynomial of degree \( 2M \), with the notable property that in each term, each coordinate appears at most quadratically. While for \( M > 2 \) the question of how to quantize (38) is mostly open (see [23]), let us come back to the original case of interest, to membranes; we then have (dropping the factor \( \frac{1}{2 \eta} \) in (32), resp. not making any notational distinction between \( H(-) \) and \( \mathbb{M}^2 \),

\[
H = p_{i\alpha} p_{i\alpha} + \frac{1}{2} g_{\alpha\beta\gamma} g_{\alpha', \beta', \gamma'} x_i \beta x_j \gamma x_i \beta' x_j \gamma'
\]

\[
= \int_{\Sigma^2} d^2 \varphi \rho (\overleftarrow{P}^2 + \sum_{i<j} \{ x_i, x_j \}^2) - \overrightarrow{P}^2
\]

with

\[
g_{\alpha\beta\gamma} := \int_{\Sigma^2} d^2 \varphi Y_\alpha \left( \frac{\partial Y_\beta}{\partial \varphi^1} \frac{\partial Y_\gamma}{\partial \varphi^2} - \frac{\partial Y_\gamma}{\partial \varphi^1} \frac{\partial Y_\beta}{\partial \varphi^2} \right)
\]

being structure constants (in the basis corresponding to the \( Y_\alpha \)) of the infinite-dimensional Lie algebra \( \text{sdiff} \sum_{(2)} \) of divergence-free vectorfields (resp. functions, modulo constants) on the space \( \Sigma^2 \) (parametrizing the surface). (40) has to be supplemented by the constraints (\( \alpha = 1, 2, \ldots \))

\[
g_{\alpha\beta\gamma} x_i \beta x_j \gamma p_{j\gamma} = 0 .
\]

\( SO(d + 1, 1) \) - invariance of this (classical) theory was proven by Goldstone [24].

Formally, one could now try to define a ‘quantum theory of relativistic surfaces’ by putting ‘hats’ on all \( x \)'s and \( p \)'s, and demanding
canonical commutation relations,
\[ [\hat{x}_j, \hat{p}_\alpha] = i\hbar\delta_{jk}\delta_{\alpha\beta}; \]
representing \( \hat{p}_\beta \) by \(-i\hbar\frac{\partial}{\partial x^\beta}\), one would thus arrive at an (a priori ill-defined) infinite-dimensional Schrödinger operator
\[ -\Delta_\infty + V_\infty. \]
Fortunately, there exists a (symmetry maximally preserving) regulation procedure\(^7\) amounting to the following

Theorem:
For each \( \sum_2 \) there exists a basis \( \{Y_\alpha\}_{\alpha=1}^\infty \) of \( \text{sdiff} \sum_2 \), and a basis \( \{T_a(N)\}_{a=1}^{N^2-1} \) of \( \text{su}(N) \) such that
\[ \lim_{N \to \infty} \text{Tr}(-[T_a(N), T_b(N)]T_c(N)) = g_{abc} \quad \forall a,b,c. \] (43)
One could therefore consider the class of \( \text{SU}(N) \) invariant matrix models \( (X_i := x^a_i T_a(N)) \)
\[ H_N := \sum_{i=1}^{d} \sum_{a=1}^{N^2-1} p_{ia} p_{ia} + \frac{1}{2} f^{(N)}_{abc} f^{(N)}_{ab'c'} x_{ib} x_{jc} x_{ib'} x_{jc'} \] (44)
as approximating (40), with
\[ f^{(N)}_{abc} x_{jb} p_{jc} = 0 \] (45)
 replacing (42). These finite-dimensional models can be quantized without any problem, thus arriving at (1)
\[ \hat{H}_N = -\Delta(N) - Tr \sum_{i<j} [X_i, X_j]^2, \] (46)
\( N \to \infty \), as a quantized discrete analogue of a relativistic membrane theory in \( \mathbb{R}^{1,d+1} \).
\( \hat{H}_N \) commutes with the operators
\[ \hat{K}^{(N)}_a := -f^{(N)}_{abc} x_{jb} \frac{\partial}{\partial x_{jc}} \] (47)
a = 1 \ldots N^2 - 1
while \( [\hat{K}_a^{(N)}, \hat{K}_b^{(N)}] = i f^{(N)}_{abc} \hat{K}_c^{(N)} \); it is therefore consistent to restrict to square-integrable wavefunctions \( \psi(x^a_i) \) that are annihilated by (47),\(^8\)
\(^7\)originally discovered in the case of a 2-sphere [1], 10 years later realized for \( T^2 \) [25], conjectured to hold for higher genus surfaces [26], and finally proven, for general Kähler manifolds, in [27].
and because of (43), this is the physical Hilbert-space to be considered.

Let me now prove (43) for $S^2$ and $T^2$: In the case of the 2-sphere the basis of $su(N)$ is obtained by considering the usual spherical harmonics $\{Y_{lm}(\theta, \varphi)\}_{m=|l|,\ldots, l}$, writing them as harmonic homogeneous polynomials in $x_1 = r \sin \theta \cos \varphi, x_2 = r \sin \theta \sin \varphi, x_3 = r \cos \varphi$ (restricted to $r^2 = \vec{x} = 1$):

$$Y_{lm}(\theta, \varphi) = \sum_{a_1 \ldots a_l} c_{a_1 \ldots a_l}^{(m)} x_{a_1} \ldots x_{a_l} |_{\vec{x} = 1} = 1$$ (48)

(where the tensor $c_{a_1 \ldots a_l}$ is by definition traceless and totally symmetric) and then replacing the commuting variables $x_a$ by generators $X_a$ of the $N-$dimensional irreducible (spin $s = \frac{N-1}{2}$) representation of $su(2)$, to obtain $N^2 - 1$ $N \times N$ matrices defined by [1]

$$T_{lm}^{(N)} = \gamma_{N\ell} \sum_{a_1 \ldots a_l} c_{a_1 \ldots a_l}^{(m)} X_{a_1} \ldots X_{a_l}; \quad (49)$$

$$l = 1 \ldots N^2 - 1$$

$$m = -l, \ldots, +l$$

instead of having $X_1^2 + X_2^2 + X_3^2 = \frac{N^2 - 1}{4} \mathbb{1}$, which makes the explicit proof of (43) quite involved, it is easier to choose $\vec{x} = 1$, i.e.

$$[X_a, X_b] = \frac{2i}{\sqrt{N^2 - 1}} \epsilon_{abc} X_c$$ (50)

and

$$\gamma_{N\ell} = -i \sqrt{\frac{N^2 - 1}{4}}.$$

As the Poisson-bracket $\{,\}$ on $S^2$ can be thought of as coming from

$$\{f(\vec{x}), g(\vec{x})\}_{\mathbb{R}^3} = \epsilon_{abc} \partial_a f \partial_b g \ x_c ,$$

$\{Y_{lm}, Y_{l'm'}\}$ can be computed from $\{r^l Y_{lm}, r^{l'} Y_{l'm'}\}_{\mathbb{R}^3}$

$$= \sum_{a_1 \ldots a_l, b_1 \ldots b_{l'}} c_{a_1 \ldots a_l}^{(m)} c_{b_1 \ldots b_{l'}}^{(m')} \{x_{a_1} \ldots x_{a_l}, x_{b_1} \ldots x_{b_{l'}}\}_{\mathbb{R}^3}$$

by using the derivation property of $\{,\}$, and

$$\{x_a, x_b\} = \epsilon_{abc} x_c ,$$ (52)

as well as then decomposing the resulting polynomial of degree $l+l'-1$ into harmonic homogeneous ones. Calculating

$$[T_{lm}^{(N)}, T_{l'm'}^{(N)}] = -\frac{N^2 - 1}{4} \sum_{a_1 \ldots a_l, b_1 \ldots b_{l'}} c_{a_1 \ldots a_l}^{(m)} c_{b_1 \ldots b_{l'}}^{(m')} [X_{a_1} \ldots X_{a_l}, X_{b_1} \ldots X_{b_{l'}}] ,$$ (53)
MEMBRANES AND MATRIX MODELS

the first step is identical to the above, while any further use of the commutation-relations \((50)\), necessary to obtain the desired traceless totally symmetric tensors, introduces factors of \(\frac{1}{\sqrt{N^2-1}}\); hence the agreement of \(f^{(N)}_{\alpha\beta\gamma}\) and \(g_{\alpha\beta\gamma}\) to leading order in \(N\). Concrete expressions for the matrix elements of the \(T_{lm}^{(N)}\) in terms of hypergeometric functions, and related discrete orthogonal polynomials, are given in \([28]\).

For the 2–torus, things are even easier. Taking \(Y_m := -e^{i(m_1\phi_1+m_2\phi_2)}, \quad (m_1, m_2) = \hat{m} \in \mathbb{Z}^2\), the relevant basis of \(su(N), N\) odd, is given by (cp. \([25]\))

\[
T_m^{(N)} := \frac{iN}{4\pi M} \omega^{\hat{m}m_2}g^{m_1}h^{m_2}
\]

where

\[
\omega = e^{\frac{4\pi i M}{N}},
\]

\[
g = \begin{pmatrix}
1 & \omega & 0 \\
\omega & \omega^2 & 0 \\
0 & \cdots & \omega^{N-1}
\end{pmatrix}, \quad h = \begin{pmatrix}
0 & 1 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
1 & 0 & \cdots & \cdots
\end{pmatrix},
\]

and \(M \in \mathbb{N}\) having no common divisor with \(N\). Using the basic relation \(h \cdot g = \omega g \cdot h\) it is very easy to verify that

\[
[T_m^{(N)}, T_n^{(N)}] = \frac{N}{2\pi M} \sin \left(\frac{2\pi M}{N} (\hat{m} \times \hat{n}) \right) T_{\hat{m}+\hat{n}}^{(N)}.
\]

The trigonometric structure constants (of \(g\ell(n, \mathbb{C})\) in the basis \(\{T_m^{(N)}\}_{m_1, m_2 = -\frac{N-1}{2}, \ldots, \frac{N-1}{2}}\) indeed converge (for \(N \to \infty, M\) fixed) to

\(\hat{m} \times \hat{n} = m_1 n_2 - m_2 n_1\), i.e. those of \(s\, \text{diff} T^2\) in the basis \(\{-e^{i\phi}\}_{m \in \mathbb{Z}^2}\). To demonstrate the subtleness of this (non–inductive) limit note that if \(M = M(N)\) is chosen such that \(\lim_{N \to \infty} \frac{M(N)}{N} = \Lambda \in [0, \frac{1}{4})\), one obtains infinite dimensional Lie–Algebras \(L_\Lambda\) with commutation–relations

\[
[T_{\hat{m}}^{\hat{n}}, T_{\hat{n}}^{\hat{m}}] = \frac{1}{2\pi \Lambda} \sin \left(2\pi \Lambda (\hat{m} \times \hat{n}) \right) T_{\hat{m}+\hat{n}}^{\hat{m}}
\]

which for irrational \(\Lambda\) are all non–isomorphic (see e.g. \([39]\)) and (up to \(T_0^{-}\) simple (for rational \(\Lambda\), \(L_\Lambda = p/q\) contains a large ideal \(I\) of finite codimension, with \(L_\Lambda/I \cong g\ell(N, \mathbb{C})\).
Excercise 3: (cp. [22], [28])
Assuming that the $T^{(N)}_{\ell m}$, like the $Y_{\ell m}$, transform as spherical tensors, leading to orthogonality relations

$$\text{Tr} T^{\dagger}_{\ell m} T_{\ell' m'} = \delta_{\ell \ell'} \delta_{mm'} r^2_N(\ell) ,$$

calculate $r_N(\ell)$ (from (49)/(50)) by using that

$$r^\ell Y_{\ell \ell} = (-)^\ell \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{(2\ell)!}{2^{2\ell}(\ell)!^2}} (x_1 + ix_2)^\ell ,$$

i.e.

$$r^2_N(\ell) = \frac{2\ell+1}{4\pi} \frac{2\ell!}{2^{2\ell}(\ell)!^2} \frac{N^2 - 1}{4} \text{Tr}((X_1 + iX_2)^\ell (X_1 - iX_2)^\ell)$$

$$= \cdots = \frac{(N+\ell)!}{16\pi (N-\ell-1)! (N^2-1)^{\ell-1}}$$

(in quite a few “fuzzy-sphere-articles” this non-trivial $\ell$-dependence has been forgotten).

Excercise 4: (cp. [43])
In terms of hermitian $N \times N$ matrices $X_j$, the classical equations of motion corresponding to (44)/(45) are

$$\ddot{X}_i = - \sum_{j=1}^d [[X_i, X_j], X_j]$$

$$\sum_{j=1}^d [X_j, \dot{X}_j] = 0 .$$

Verify the following types of solutions:

A)

$$X_i(t) = x(t) r_{ij}(t) M_j$$

$$= x(t) (\cos \varphi(t) \cdot \vec{M}, \sin \varphi(t) \cdot \vec{M}, 0 \ldots 0)$$

with

$$\frac{1}{2} x^2 + \frac{\lambda}{4} x^4 + \frac{L^2}{2} \frac{1}{x^2} = \text{const}$$

$$x^2 \dot{\varphi} = L ,$$

$$\sum_{a=1}^{\tilde{d}} [[M_a, M_b], M_b] = \lambda M_a \quad (\ast)$$

and

I)

$$[M_a, M_b] = f_{abc} M_c, \quad f_{abc} f_{cb a'} = -\lambda f_{aa'}$$
II)
\[
\vec{M} = \frac{1}{\sqrt{2}} \left( \frac{g + g^{-1}}{2}, \frac{g - g^{-1}}{2i}, \frac{h + h^{-1}}{2}, \frac{h - h^{-1}}{2i}, 0 \ldots 0 \right)
\]
(g and h as in (55)).

III)
\[
M_1 = \frac{\pi M}{iN} (T_{k\ell} + T_{-k-\ell} + T_{-k\ell} + T_{k-\ell})
\]
\[
M_2 = -\frac{\pi M}{iN} (T_{k\ell} - T_{-k-\ell} - T_{-k\ell} + T_{k-\ell})
\]
\[
M_3 = -\frac{\pi M}{N} (T_{k\ell} - T_{-k-\ell} + T_{-k\ell} - T_{k-\ell})
\]
\[
M_4 = -\frac{\pi M}{N} (T_{-k\ell} - T_{-k-\ell} - T_{-k\ell} - T_{k-\ell})
\]
(T\_k\_1 as in (54)).

B)
\[
X_j(t) = \sum_{\alpha} r_\alpha(t) (\vec{E}_\alpha)_j
\]
with
\[
\vec{E}_1 = \frac{2\pi M}{iN} (T_{m_1} + T_{-m_1}, -i(T_{m_1} - T_{-m_1}), 0 \ldots 0)
\]
\[
\vec{E}_2 = \frac{2\pi M}{iN} (0, 0, T_{m_2} + T_{-m_2}, -i(T_{m_2} - T_{-m_2}), 0 \ldots 0)
\]
\[
\vdots
\]
\[
n_\alpha(t) = -4 r_\alpha \sum_{\beta} \sin^2 \left( \frac{2\pi M}{N} (\vec{m}_\alpha \times \vec{m}_\beta) \right) r_\beta^2.
\]
Let me now turn to the supersymmetric extension \([4, 5]\) of the bosonic
matrix model (44) which, following \([6]\), has been intensively studied
over the past few years, as a candidate for ‘\(M–\)theory’\

\[
H_{\text{Susy}} = p_{tA}p_{tA} + \frac{1}{2} f_{ABC} f_{AB'C'} q_s B q_{sB'} q_{tC'} + i q_t c f_{ABC} \gamma^t_{\alpha\beta} \Theta_{\alpha A} \Theta_{\beta B},
\]

(1)

\(\alpha, \beta = 1, \ldots, s_d (= 2, 4, 8 \text{ or } 16)\); \(f_{ABC}\) = structure constants of \(SU(N)\)
(real, antisymmetric),
\[\gamma^s \gamma^t + \gamma^t \gamma^s = 2 \delta^{st} \] \(\mathbb{I}_{s_d \times s_d}\), \(\gamma^s\) real, symmetric;

the dynamical degrees of freedom satisfy canonical (anti)commutation
relations,
\[
[q_{tA}, p_{sB}] = i \delta_{AB} \delta_{ts}, \quad \{\Theta_{\alpha A}, \Theta_{\beta B}\} = \delta_{\alpha \beta} \delta_{AB}.
\]

(2)

\(H_{\text{Susy}}\) commutes with the generators of \(SU(N), Spin(d)\) and super-symmetry,
\[
J_A = f_{ABC} (q_s B p_{sC} - \frac{1}{2} i \Theta_{\alpha B} \Theta_{\alpha C})
\]
\[
J_{st} = q_s A p_{tA} - q_{tA} p_{sA} - \frac{1}{4} i \Theta_{\alpha A} \gamma^t_{\alpha \beta} \Theta_{\beta A}
\]
\[
Q_\beta = (p_{tA} \gamma^t_{\beta A} + \frac{1}{2} f_{ABC} q_s B q_{tC} \gamma^s_{\beta A}) \Theta_{\alpha A}
\]

while
\[
\{Q_\beta, Q_{\beta'}\} = \delta_{\beta \beta'} H + 2 \gamma^t_{\beta \beta'} q_{tA} J_A,
\]

(3)

(4)

and all states \(\Psi\) are required to be \(SU(N)\)–invariant, \(J_A \Psi = 0\) (so
that, on the physical Hilbert space \(\mathcal{H}, H_{\text{Susy}}\) is twice the square of each
\(Q_\beta\)).

In verifying (4), \(d = 2, 3, 5\) and 9 are singled out when calculating the
mixed \(pqq\) terms, which yield (only) the fermionic (last term) part of
(1) and the ‘weakly zero’–part (proportional to the \(J_A\)) in (4) provided
\[
\gamma^s_{\alpha \beta} \gamma^s_{\alpha' \beta'} + \gamma^s_{\alpha' \beta} \gamma^s_{\alpha \beta'} + \gamma^s_{\alpha \beta} \gamma^s_{\alpha' \beta'} + \gamma^s_{\alpha' \beta} \gamma^s_{\alpha \beta'} = 2 (\delta_{\alpha \alpha'} \gamma^t_{\beta \beta'} - \delta_{\beta \beta'} \gamma^t_{\alpha \alpha'})
\]

(5)

– which (setting e.g. \(\beta = \beta'\) and summing) implies
\[
s_d = 2(d - 1); \quad (6)
\]

together with the reality condition on the \(\gamma'\)s (to have \(H_{\text{Susy}}\) and \(Q_\beta\)
hermitean) these hold only when \(d = 2, 3, 5\) or 9.

The spectrum of \(H_{\text{Susy}} \geq 0\) is known to cover the whole positive
axis \(\mathbb{R}^+ [8,29]\) and it is conjectured that for \(d = 9\) there exists a unique
normalizable state for each \(N\), while not for all other cases \((d = 2, 3, 5)\).

\*Sorry for the change of notation; \(su(N)\) indices are now denoted by: \(A, B, C = 1, \ldots, N^2 - 1,\) transverse space–time indices by \(s, t = 1, \ldots, d (= 2, 3, 5 \text{ or } 9)\).
For $N=2$, the latter was proven in [30] and for $d=9$ the precise form of a (unique) asymptotic solution was derived in [31,32]. Writing [32]

$$q_{tA} = r e_A E_t + y_{tA}$$

for configurations with almost vanishing potential energy ($V = 0$ for $y_{tA} = 0; E_t E_t = 1 = e_A e_A, r \to \infty$) the asymptotic wavefunction takes the form

$$\Psi \sim r^{-\kappa} e^{-r y^2/2} |F^{\perp}\rangle |F^{\parallel}\rangle$$

with $\kappa = 0, -1, 6$ for $d=3, 5, 9$ (respectively), $y^2 = y_{tA} y_{tA}$, $|F^{\parallel}\rangle$ only involving $\Theta_{tA} e_A$ and $E_s$, and $|F^{\perp}\rangle$ involving all other fermionic degrees of freedom (and $E_s$).

For $d=9$, $r^4 e^{-r y^2/2} |F^{\perp}\rangle$ corresponds to the ground state of a system of 16 supersymmetric harmonic oscillators, and

$$|F^{\parallel}\rangle = (E_s E_t - \frac{1}{9} \delta_{st})|44; st\rangle,$$

where $|44; st\rangle$ is the only spin(9) representation in the 256–dimensional fermionic Hilbert–space $H^{||} (= 44 \oplus 84 \oplus 128$, arising from the 16–dimensional spinor $\Theta^||_A := e_A \Theta_{tA}$) that can be contracted with a bosonic spin(9) representation (made out of the $E_s$’s) to form a spin(9)–singlet.

For $N > 2$, the asymptotic wavefunction, in accordance with (8), is speculated to factorize into the ground state of a system of supersymmetric harmonic oscillators, times a supersymmetric spin 9 (and Weyl!–) invariant wavefunction, $\Psi_c$, involving only the Cartan–subalgebra degrees of freedom, $\Theta_{tA}$ and $q_{tk}$ ($k = 1, 2, \ldots, N-1$), and annihilated by effective (free) supercharges

$$Q_{\beta} := -i \frac{\partial}{\partial q_{tk}} \gamma^{\alpha}_{\beta\alpha} \Theta_{tA}.$$  

Unlike the $N=2$ case, where $r^{-9} |F^{||}\rangle \sim \partial_s \partial_t (\frac{1}{r^7}) |44; st\rangle$ was the only spin(9) invariant wavefunction\footnote{In order to explicitly check that it is supersymmetric one may use that $\Theta^||_A |44; st\rangle = \gamma^s_{\alpha\beta} |t\beta\rangle + \gamma^t_{\alpha\beta} |s\beta\rangle$, which leads to a fermionic $[us]$ resp. $[ut]$ anti-symmetry, giving indeed zero, when contracted with $\partial_s \partial_t \partial_u (\frac{1}{r^7})$ [35].} there are abundantly many harmonic wavefunctions,

$$\Psi_c = \sum_{\ell,S,R,m} r^{-2\ell-(N-1)d+2-S \times R} \Psi_{\ell m}(\vec{q}_1, \ldots, \vec{q}_{N-1})|S \times R; m\rangle$$

for $N \geq 3$ [36,37]; $S$ and $R$ label the irreducible representations of the permutation group $S_N$ (= the Weyl group of $SU(N)$) and spin(9), $\Psi_{\ell m}^{S \times R}$ is a harmonic (homogeneous of degree $\ell$) polynomial in the $d(N-1)$ bosonic variables $q_{tk}, |S \times R; m\rangle, m = 1 \ldots \dim (S \times R)$, is a corresponding $S \times R$ representation made out of the fermions, and $r^{-2\ell-(N-1)d+2}$ is the (unique) power making $\Psi_c$ a decaying harmonic
function. The (non–trivial) question then is: which of the wavefunctions (11) is annihilated by (10)?

For $N = 3$, the answer was guessed in [37] and proven in [38]:

$$\Psi_c = \left( \prod_{\beta=1}^{16} Q_\beta \right) \left( r^{-16} |1 \times 1\rangle \right).$$

(12)

Assuming that also for higher odd $N$ the fermionic Hilbertspace will contain a unique Weyl– and $spin(9)$ invariant state, one would guess that

$$\Psi_c = \left( \prod_{\beta=1}^{16} Q_\beta \right) \left( r^{-9(N-1)+2} |1 \times 1\rangle_N \right).$$

(13)

for odd $N \geq 3$ (but note that even for $N = 3$ the proof that (12) is non–vanishing does, a priori, not exclude the existence of other non–zero, invariant, wavefunctions, annihilated by all $Q_\alpha$).

What about the exact form of the full (non–asymptotic) wavefunction? If one wants to represent the fermionic degrees of freedom by creation and annihilation operators, e.g. by defining

$$\lambda_{\alpha A} := \frac{1}{\sqrt{2}} (\Theta_{\alpha A} + i \Theta_{\alpha+\frac{1}{2}} s_{d,A})$$

$$\frac{\partial}{\partial \lambda_{\alpha A}} = \frac{1}{\sqrt{2}} (\Theta_{\alpha A} + i \Theta_{\alpha+\frac{1}{2}} s_{d,A}) = \lambda_{\alpha A}^\dagger$$

(14)

(where now $\alpha$ takes only half a many values as before!) explicit $Spin(d)$ invariance is lost, respectively nonlinearly realized [5]; some of the generators $J_{st}$ will not respect the degree of homogeneity (mix the various components) of

$$\Psi = \psi + \psi_{\alpha A} \lambda_{\alpha A} + \frac{1}{2} \psi_{\alpha A \beta B} \lambda_{\alpha A} \lambda_{\beta B} + \ldots + \frac{1}{\Lambda!} \psi_{\alpha_1 A_1, \ldots, \alpha_\Lambda A_\Lambda} \lambda_{\alpha_1 A_1} \ldots \lambda_{\alpha_\Lambda A_\Lambda}.$$  

(15)

On the other hand, taking linear combinations of the Herimitan supercharges to form half as many nilpotent ones (on $H$), and their Hermitian conjugates, the resulting supersymmetry algebra [5]

$$\{Q_\alpha, Q_{\beta}^\dagger\} \approx \delta_{\alpha \beta} H$$

$$\{Q_\alpha, Q_\beta\} \approx 0 \approx \{Q_\alpha^\dagger, Q_\beta^\dagger\}$$

(16)

will imply very simple properties,

$$(M \cdot \lambda)^2 = 0 , \quad (D \cdot \lambda)^2 = 0 , \quad \{M \cdot \lambda, D \cdot \partial_\lambda\} \approx 0$$

(17)
when writing \([33]\)

\[
Q_\beta = M^{(\beta)}_{\alpha\beta} \lambda_{\alpha A} + D^{(\beta)}_{\alpha A} \frac{\partial}{\partial \lambda_{\alpha A}} = M_{\alpha A} \lambda_{\alpha} + D_{\alpha A} \partial_{\lambda_{\alpha}} = M \cdot \lambda + D \cdot \partial_{\lambda},
\]

\[
Q^{\dagger}_{\beta} = M^{(\beta)\dagger}_{\alpha\beta} \partial_{\lambda_{\alpha A}} + D^{(\beta)\dagger}_{\alpha A} \lambda_{\alpha A} = M^{\dagger} \partial_{\lambda} + D^{\dagger} \lambda .
\]  

The differential equations obtained componentwise from \(Q_\beta \Psi = 0,\) \(Q^{\dagger}_{\beta} \Psi = 0,\)

\[
D_{a_{2k}} \psi_{a_1,\ldots,a_{2k}} = (2k - 1) M_{[a_1 \psi_{a_2,\ldots,a_{2k-1}]}}
\]

\[
M^{\dagger}_{a_{2k}} \psi_{a_1,\ldots,a_{2k}} = (2k - 1) D_{[a_1 \psi^{\dagger}_{a_2,\ldots,a_{2k-1}]}},
\]  

were first (recursively) solved \([33]\) in the form

\[
\psi_{2k} = \chi_{2k}^{[h]} + \chi_{2k}^{[in]}
\]

\[
\psi_{2k} = \psi_{2k}^{(h)} + \psi_{2k}^{(in)}
\]

where ‘\(h\)’ indicates the general solution of the corresponding homogeneous equation, and ‘\(in\)’ a particular solution of the inhomogeneous equation (given in \([33]\)).

This procedure of solving \(Q \Psi = 0\) or \(Q^{\dagger} \Psi = 0\) can be written more elegantly by defining \([34]\)

\[
A := I^{\dagger} \cdot \lambda D^{\dagger} \cdot \lambda, \quad B := I \cdot \partial_{\lambda} D \cdot \partial_{\lambda}
\]

where \(I_{\alpha A}\) is chosen such that

\[
\{M \cdot \lambda, I \cdot \partial_{\lambda}\} = -I .
\]  

Formally, all solutions of \(Q^{\dagger} \Psi = 0,\) respectively \(Q \chi = 0,\) are of the form

\[
\Psi = (I - A)^{-1} \psi^{(h)}, \quad \text{respectively} \quad \chi = (I - B)^{-1} \chi^{[h]}
\]  

with

\[
(M^{\dagger} \partial_{\lambda}) \psi^{(h)} = 0 , \quad (M \cdot \lambda) \chi^{[h]} = 0
\]  

(as if \(Q^{\dagger} \Psi = 0,\) let \(\Psi^{(h)} := \Psi - A \Psi\) and verify that \(M^{\dagger} \partial_{\lambda} \Psi + D^{\dagger} \lambda \Psi + I^{\dagger} \lambda M^{\dagger} \partial_{\lambda} D^{\dagger} \lambda \Psi = 0),\) while it is also not difficult to show that each \(\Psi\) of the form \(27\) does solve \(Q^{\dagger} \Psi = 0,\) respectively \(Q \chi = 0:\)

\[
Q(I - B)^{-1} \chi^{[h]} = (I - B)^{-1} Q \chi^{[h]}
\]

\[
+ (I - B)^{-1} [M \cdot \lambda + D \cdot \partial_{\lambda}, I \cdot \partial_{\lambda} D \cdot \partial_{\lambda} (I - B)^{-1} Q \chi^{(h)}
\]

\[
= (I - B)^{-1} (D \partial_{\lambda} (I - B)
\]

\[
+ \{ M \cdot \lambda, I \cdot \partial_{\lambda}\} D \cdot \partial_{\lambda} I \cdot \partial_{\lambda} [M \cdot \lambda, D \cdot \partial_{\lambda}]
\]

\[
+ D \cdot \partial_{\lambda} B) (I - B)^{-1} \chi^{[h]} \approx 0 .
\]
Excercise 5: (cp. [40], [41])

Calculate \( \{ Q_\beta, Q_{\beta'} \} \) when adding

\[
m_3 \sum_{i=1}^{3} q_{iA} (\gamma^{123} \gamma^i)_{\beta\alpha} \Theta_{\alpha A} + m_6 \sum_{\mu=4}^{9} q_{\mu A} (\gamma^{123} \gamma^\mu)_{\beta\alpha} \Theta_{\alpha A}
\]

to the r.h.s. of \( Q_\beta \) in (3).

In particular, verify that the anticommutator of the above terms with the derivative part in the supercharges yields

\[
m_3 (-i(q_{iA} \nabla_{jA'} - q_{jA} \nabla_{iA}) (\gamma^{123} \gamma^i)_{\beta\beta'}) + m_6 (-i(q_{\mu A} \nabla_{\nu A} - q_{\nu A} \nabla_{\mu A}) (\gamma^{123} \gamma^\mu)_{\beta\beta'}) + 2i \frac{\Theta_{\alpha A}}{32} \Theta_{\alpha' A} \left\{ (-6m_3 + 12m_6) \delta_{\beta\beta'} \gamma_{\alpha\alpha'}^{123} + (-m_3 + 6m_6) (\gamma^{123} \gamma^i)_{\beta\beta'} \gamma_{\alpha\alpha'}^i + (3m_3 + 2m_6) (\gamma^{123} \gamma^\mu)_{\beta\beta'} \gamma_{\alpha\alpha'}^\mu \right\}
\]

(use that the \((\beta \leftrightarrow \beta')\) symmetric, \([\alpha \leftrightarrow \alpha']\) antisymmetric, part of \(\gamma_{\beta\alpha'}^{123} (\gamma^i)_{\beta\alpha}, \text{ resp. } \gamma_{\beta\alpha'}^{123} (\gamma^\mu)_{\beta\alpha}, \text{ can only contain terms proportional to the ones written above inside the curly bracket); recalling that the coefficient of \(\Theta \gamma \Theta\) in the generators of Spin (3), resp. Spin (6), should be \(\frac{1}{4}\), relative to \(q \nabla - q \nabla\) (cp. (3)), deduce that if (and only if) \(m_6\) is equal to \(-\frac{1}{2} m_3\), the above expression, apart from contributing to the Hamiltonian (a term \(-\frac{3m_3}{4} i \Theta \gamma^{123} \Theta\)), can be written solely in terms of symmetry-generators

\[
\left( m_3 J_{ij} (\gamma^{123} \gamma^i)_{\beta\beta'} - \frac{m_3}{2} J_{\mu\nu} (\gamma^{123} \gamma^\mu)_{\beta\beta'} \right).
\]

The derivative-free extra contributions to (4) are all proportional to \(\delta_{\beta\beta'}\), adding to the potential in (1)

\[
m_3^2 q_{iA} q_{iA} + m_3^2 q_{\mu A} q_{\mu A} + m_3 f_{ABC} \epsilon_{ijk} q_{iA} q_{jB} q_{kC}
\]

\[
= Tr \left\{ \frac{m_3^2}{4} X^\mu \ X^\mu + (m_3 X_j - i \epsilon_{jkl} X_k X_l)^2 \right\}
\]

\[
+ \frac{1}{2} Tr [X_j, X_k] [X_j, X_k];
\]

here, \(X = i q_{A} T_{A}\).

Excercise 6: (cp. [34], [42])

Let \(\lambda_a\) and \(\lambda_a^\dagger = \frac{\partial}{\partial \lambda_a}\) \((a = 1, \ldots , \Lambda)\) be fermionic creation, resp. annihilation-operators; i.e. \(\{\lambda_a, \lambda_b\} = 0 = \left\{ \frac{\partial}{\partial \lambda_a}, \frac{\partial}{\partial \lambda_b} \right\}, \left\{ \lambda_a, \frac{\partial}{\partial \lambda_b} \right\} = \delta_{ab}, \left. \frac{\partial}{\partial \lambda_a} \right|_0 = 0 \text{ (for all } a, \text{ and } b)\).
Define \([34]\)

\[ * := \sum_{m=0}^{\Lambda} \frac{(-)^m}{m!(\Lambda - m)!} \varepsilon_{a_1...a_\Lambda} \lambda_{a_1} \cdot \ldots \cdot \lambda_{a_m} \partial_{\lambda_{a_\Lambda}} \ldots \partial_{\lambda_{a_{m+1}}} \cdot \]

Show that on the \(p\)-fermion-sector, i.e. on states \(|\Psi_p\rangle = \Psi_{a_1...a_p} \lambda_{a_1} \cdot \ldots \cdot \lambda_{a_p} |0\rangle\), one has

\[ \partial_{\lambda_a} * |\Psi_p\rangle = (-)^{\Lambda-p} * \lambda_a |\Psi_p\rangle \]
\[ \lambda_a * |\Psi_p\rangle = (-)^{\Lambda-p+1} * \partial_{\lambda_a} |\Psi_p\rangle , \]

as well as \(*^2 |\Psi_p\rangle = (-)^{p(\Lambda-p)+\Lambda} |\Psi_p\rangle\); thus, (note the sign-errors in equations (17)-(19) of \([34]\))

\[ H_{ab} \lambda_a \frac{\partial}{\partial \lambda_b} * = * H_{ab} \lambda_a \frac{\partial}{\partial \lambda_b} \]
\[ (F_{ab} \lambda_a \lambda_b + G_{ab} \partial_{\lambda_a} \partial_{\lambda_b}) * = - * (F_{ab} \lambda_a \lambda_b + G_{ab} \partial_{\lambda_a} \partial_{\lambda_b}) . \]

For the \(SU(N)\)-invariant matrix-models (1)

\[ F_{\alpha A, \beta B} = \delta_{\alpha \beta} f_{ABE}(q_{d-1,E} + i q_{d,E}) \]
\[ G_{\alpha A, \beta B} = - F^*_{\alpha A, \beta B} \]
\[ H_{\alpha A, \beta B} = H_{\alpha A, \beta B}(q_{j,E})_{j=1,...,d-2} . \]

Therefore, if one defines \(\tilde{q}_{dE} := -q_{dE}, \tilde{q}_{s\neq d,E} = q_{s,E}\), \(H\) will commute with the joint action of \(*\) and \(\tilde{\phantom{q}}\), hence \(H(*\tilde{\Psi}(q)) = 0\) if \(H \Psi(q) = 0\). For \(d = 2\) and even \(N\) this means (cp. \([42]\); there, \(*\) is taken to be the Hodge-operator – which corresponds to taking \((-)^m(\Lambda-m)\), instead of \((-)^m\), in the above definition of \(*\) ) that the corresponding index (of \(H\)) trivially vanishes, as for odd \(\Lambda\) \(*\tilde{\Psi}(\tilde{q})\) will be fermionic if \(\Psi(q)\) is bosonic (and vice versa).
IV

Consider $3$ traceless, antihermitean $N \times N$ matrices $X_a(t)$, $t \in (-\infty, +\infty)$, developing in time according to the equations

$$\dot{X}_a = \epsilon_{abc} X_b X_c - m X_a.$$  \hfill (1)

The stationary points of this flow are representations of $su(2)$, i.e. $X_a = m J_a$,

$$[J_a, J_b] = \epsilon_{abc} J_c.$$  \hfill (2)

The question is: given $2$ such representations, $\rho_+$ and $\rho_-$, under which circumstances do there exist solutions $X_a(t)$ of (1) approaching the representation $\rho_+$ as $t \to +\infty$ and (being conjugate to) $\rho_-$ as $t \to -\infty$?

Denoting the space of such solutions by $\mathcal{M}(\rho_-, \rho_+)$, Kronheimer [10], in parts building on work of Slodowy [9], proved that

$$\mathcal{M}(\rho_-, \rho_+) = \mathcal{N}(\rho_-) \cap S(\rho_+),$$  \hfill (3)

where the r.h.s. is well known from singularity theory related to Lie algebras [9]. In the main part of this lecture, based on joint work with C. Bachas and B. Pioline (see [11]; in particular concerning the physical relevance of (1), (3)) I will discuss (3):

Take

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$  \hfill (4)

as generators of $sl(2, \mathbb{C})$, the complexification of $su(2)$; denote by $H_\pm := \rho_\pm(h)$, $X_\pm := \rho_\pm(x)$, $Y_\pm := \rho_\pm(y)$, the corresponding $N \times N$ matrices in the representation $\rho_\pm$, i.e. satisfying the same commutation relations as those following from (4),

$$[x, y] = h, \quad [h, x] = 2x, \quad [h, y] = -2y.$$  \hfill (5)

$\mathcal{N}(\rho_\pm)$ is then defined as the orbit of $Y_\pm$ under the complexified gauge group, $SU(N)_{\mathbb{C}} = SL(N, \mathbb{C})$:

$$\mathcal{N}(\rho_\pm) := \{ g Y_\pm g \mid g \in SL(N, \mathbb{C}) \}$$  \hfill (6)

while

$$S(X_\mp) = Y_\mp + Z(X_\mp)$$  \hfill (7)

where

$$Z(X_\mp) := \{ A \in sl(N, \mathbb{C}) \mid [A, X_\mp] = 0 \}$$  \hfill (8)

is the centralizer of $X_\mp$.

Example ($N = 3$):
Let $\rho_-$ be the irreducible 3-dimensional representation of $su(2)$, and $\rho_+ = 2 \oplus 1$ the direct sum of the irreducible 2-dimensional one, and the trivial 1-dimensional (putting all $J_a = 0$). Then one has

\[
Y_- = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y_+ = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
X_- = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
H_- = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad H_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}.
\]

In this example,

\[
S(\rho_+) = \left\{ s = \begin{pmatrix} a & b & c \\ 1 & a & 0 \\ 0 & e & -2a \end{pmatrix} \left| a, b, c, e \in \mathbb{C} \right. \right\},
\]

as can be found either by a simple explicit computation, or using the general fact that $sl(N, \mathbb{C})$ decomposes, under the adjoint action of $\rho_+ \ominus \rho_-$, into irreducible representation spaces of $sl(2, \mathbb{C})$, each of which contains exactly one 1-parameter family of elements of $Z(X_+)$; in the above case, $sl(3, \mathbb{C})$ decomposes, under the action of $(Y_+ = E_{21}, X_+ = E_{12}, H_+ = E_{11} - E_{22})$ into one 3-dimensional representation space ($\rho_+$ itself, contributing $\mathbb{C} \cdot X_+$ to $Z(X_+)$), two 2-dimensional ones: spanned by $E_{23}$ and $[E_{12}, E_{23}] = E_{13} \in Z(E_{12})$, resp. $E_{31}$ and $[E_{12}, E_{31}] = -E_{32} \in Z(E_{12})$, and one 1-dimensional one ($\mathbb{C} \cdot (E_{11} + E_{22} - 2E_{33}) \in Z(E_{12})$). Instead of computing $N(\rho_-)$ explicitly, $N(\rho_-) \cap S(\rho_+)$ can, in the above example, be determined by simply demanding $s^3 = 0$, $s^2 \neq 0$ for the elements in (10); this gives $b = -3a^2$, $ec = 8a^3$, i.e.

\[
N(\rho_-) \cap S(\rho_+) = \left\{ \begin{pmatrix} a & -3a^2 & c \\ 1 & a & 0 \\ 0 & e & -2a \end{pmatrix} \left| a, c, e \in \mathbb{C} \right. \right\}.
\]

According to (3), $M(\rho_-, \rho_+)$ is therefore the 4-dimensional (singular) space (11). Let me now sketch (part of) the proof of (3) (cp [10]): One first ‘gauges’ (1) by introducing a 4-th traceless, antihermitean, $N \times N$ matrix, $X_0$, and going over to the equations

\[
\dot{X}_a + [X_0, X_a] = \frac{1}{2} \epsilon_{abc} [X_b, X_c] - mX_a.
\]
Due to their invariance under
\[ X_a \to \tilde{X}_a = U(t)X_a U^{-1}(t), \]
\[ X_0 \to \tilde{X}_0 = UX_0 U^{-1} - \dot{U} U^{-1}, \]
(13)
a solution \( \tilde{X}_a \) of (1) may be obtained from a solution \( X_a \) of (12) by choosing \( U \) in (13) such that \( \tilde{X}_0 = 0 \). (12) is then split into one complex equation (from now on, \( m = 2 \))
\[ \dot{\hat{\beta}} + 2\beta + 2[\alpha, \beta] = 0, \]
(14)
and one real equation,
\[ \frac{d}{dt} (\alpha + \alpha^\dagger) + 2(\alpha + \alpha^\dagger) + 2[\alpha, \alpha^\dagger] + 2[\beta, \beta^\dagger] = 0. \]
(15)
Due to \( \alpha := \frac{1}{2}(X_0 - iX_3) \) and \( \beta := -\frac{1}{2}(X_1 + iX_2) \) no longer having to obey any (anti)hermiticity conditions, the gauge-invariance of (14) is enhanced to complex (!) gauge transformations
\[ \alpha \to g\alpha g^{-1} - \frac{1}{2} \dot{g}g^{-1} \quad g \in SL(N, \mathbb{C}) \]
(16)
\[ \beta \to g\beta g^{-1}. \]
Kronheimer [1] then proved that any solution of (14) (with the required boundary conditions) is gauge equivalent to
\[ \alpha_-(t) = \frac{1}{2}H_-, \quad \beta_-(t) = Y_- \quad \text{for } t \in (-\infty, 0] \]
\[ \alpha_+(t) = \frac{1}{2}H_+, \quad \beta_+(t) = Y_+ + e^{-2t}e^{-t \text{ad} H_+}Z_+ \]
for \( t \in [0, +\infty), \]
(17)
with \( Z_+ \in Z(X_+). \) Stated the other way round (actually 0 may be replaced by any finite time, in (17)): for any given solution \((\alpha, \beta)\) of (14) there exist \( g_+ \) and \( g_- \) (approaching the identity, resp. a constant group element, at \( t = +\infty \), resp. \( t = -\infty \)) such that, for any finite \( t \),
\[ \beta = g_+^{-1}(Y_+ + e^{-(2 + \text{ad} H_+)}t Z_+)g_+ \]
\[ 2\alpha = g_+^{-1}H_+ g_+ + g_+^{-1}\dot{g}_+ \]
(18)
AND
\[ \beta = g_-^{-1}Y_- g_- \]
\[ 2\alpha = g_-^{-1}H_- g_- + g_-^{-1}\dot{g}_-. \]
This means that for any finite \( t \),
\[ Y_+ + e^{-(2 + \text{ad} H_+)}t Z_+, \]
(19)
which is \( \in S(\rho_+) \), must be gauge-equivalent to \( Y_- \), i.e. must be \( \in \mathcal{N}(\rho^-) \). Putting \( t = 0 \), and assuming (proved in [10]) that the gauge-invariance can always be used to satisfy (15), this gives the desired correspondence between solutions of (1), interpolating between \( \rho_- \) and \( \rho_+ \), and \((Y_+ + Z(X_+)) \cap \mathcal{N}(Y_-)\). Letting \( t \to +\infty \), while noting that (2+
ad $H_+$ is strictly positive on $Z(X_+)$, one finds that $Y_+$ (hence $\mathcal{N}(\rho_+)$) must actually be contained in the closure of $\mathcal{N}(\rho_-)$ (for $\mathcal{M}(\rho_-, \rho_+)$ to be non-empty). If this condition is fulfilled, the dimension of $\mathcal{M}$, due to $S_+$ and $\mathcal{N}_-$ meeting transversely, can be computed as follows:

$$\dim(S_+ \cap \mathcal{N}_-) = \dim S_+ + \dim \mathcal{N}_- - \dim(S_+ \cap \mathcal{N}_-)$$

$$= \dim S_+ - \dim \mathcal{N}_- .$$

As a general $N \times N$ $SU(2)$ representation is a direct sum of irreducible ones,

$$\rho = \sum_{j \in \mathbb{N}/2} n_j[j]$$

with $\dim[j] = 2j + 1 \in \mathbb{N}, \rho_\pm$ correspond to partitions $P_\pm$ of $N$, hence are associated to Young–tableaux(s) $T_\pm$, the number of boxes in row $k$ corresponding to the dimension of the $k$’s representation in (21), — having ordered them with the largest $[j]$ first. The condition that $\mathcal{N}(g_+) \subset \mathcal{N}(g_-)$ then translates (see e.g. [45, 11]) into the condition that, for each $p = 1, 2, \ldots$, the number of boxes contained in the first $p$ columns must not decrease (when going from $T_-$ to $T_+$). Another way to state the same condition is to compare the null–spaces of powers of $Y_-$ to those of $Y_+$ (which, for each given power, must not decrease).

*In the previous example one would have

$$[H_+ = E_{11} - E_{22}, \quad X_+ = \begin{array}{c|c}
E_{13} & 2X_+ \\
E_{32} & 1 \cdot E_{13} \\
E_{11} + E_{22} - 2E_{33} & 1 \cdot E_{32} \\
\end{array}, \quad E_{33} = 0$$

*
Excercise 7: In [44] it has been shown that if \( X_j = \partial_1 f_j \partial_2 - \partial_2 f_j \partial_1 \) (\( j = 1, \ldots, 5 \)) are 5 divergence-free vectorfields, their totally antisymmetrized product \( s_5(X_1, \ldots, X_5) \) will, in contrast with all other \( M \)-commutators (\( M > 2 \)), again be a (generally non-zero) divergence-free vector-field, and
\[
ad s_5(X_1, \ldots, X_5) = s_5(ad X_1, \ldots, ad X_5) ,
\]
as well as
\[
[X_6, s_5(X_1, \ldots, X_5)] = \sum_{j=1}^5 s_5(X_1, \ldots, X_{j-1}, [X, X_j], X_{j+1}, \ldots, X_5) ;
\]
correspondingly one may define
\[
\{f_1, \ldots, f_5\}_5 := \begin{vmatrix}
\partial_1 f_1 & \partial_1 f_2 & \partial_1 f_3 & \partial_1 f_4 & \partial_1 f_5 \\
\partial_2 f_1 & \cdots & \cdots & \cdots & \partial_2 f_5 \\
\partial_2^2 f_1 & \cdots & \cdots & \cdots & \partial_2^2 f_5 \\
\partial_2^2 f_1 & \cdots & \cdots & \cdots & \partial_2^2 f_5 \\
\partial_2^2 f_1 & \cdots & \cdots & \cdots & \partial_2^2 f_5 \\
\end{vmatrix}
\]
for functions on \( T^2 \), with \( \{\ldots\}_5 \) and the ordinary Poisson-bracket \( \{f, g\} := \partial_1 f \partial_2 g - \partial_2 f \partial_1 g \) satisfying various (mixed) identities, – which will be quite relevant for the understanding of 2- and 5-branes in \( M \)-theory.

Conjecture (Approximation of 5-commutator of functions by 5-commutator of matrices):

Let \( \gamma(\tilde{m}_1, \ldots, \tilde{m}_5) \) be defined via
\[
\{f_{\tilde{m}_1}, \ldots, f_{\tilde{m}_5}\}_5 = \gamma(\tilde{m}_1, \ldots, \tilde{m}_5) e^{i \sum_{j=1}^5 \tilde{m}_j \tilde{\varphi}} ,
\]
\( f_{\tilde{m}_j} = e^{i \tilde{m}_j \tilde{\varphi}} \), and \( T_{\tilde{m}} := w^{\frac{1}{2}} m_1 m_2 g^{m_1} h^{m_2} \) (with \( w, g, h \) as in II (55); \( M = 1 \) for simplicity); then, as \( N \to \infty \),
\[
s_5(T_{\tilde{m}_1}, \ldots, T_{\tilde{m}_5}) = C_N \gamma(\tilde{m}_1, \ldots, \tilde{m}_5) T_{\sum_{j=1}^5 \tilde{m}_j} \left( 1 + 0 \left( \frac{1}{N} \right) \right) .
\]
References.

[1] J. Hoppe; “Quantum Theory of a Massless Relativistic Surface . . . ” (1982): MIT Ph.D. Thesis.
    J. Goldstone; unpublished.
[2] J. Hoppe; “Quantum Theory of a Relativistic Surface” in “Constraint’s Theory and Relativistic Dynamics”, World Scientific 1987 (eds. G. Longhi, L. Lusanna).
[3] J. Hoppe; “Elementary Particle Research Journal” (Kyoto) 80, 3 (1989).
[4] R. Flume; *Ann. Phys.* 164 (1985) 189.
    M. Claudson, M. Halpern; *Nucl. Phys.* 250 (1985) 689.
    M. Baake, P. Reinicke, V. Rittenberg; *J. Math. Phys.* 26 (1985) 1070.
[5] B. de Wit, J. Hoppe, H. Nicolai; *Nucl. Phys.* B 305 (1988) 545.
[6] E. Witten; *Nucl. Phys.* B 460 (1996) 335.
    T. Banks, W. Fischler, S. Shenker, L. Susskind; *Phys. Rev. D* 55 (1997) 5112.
[7] M. Bordemann, J. Hoppe; *Phys. Lett.* B 317 (1993) 315.
[8] B. de Wit, M. Lüscher, H. Nicolai; B 320 (1989) 135.
[9] P. Slodowy; *Lecture Notes in Mathematics* 815, Springer 1980.
[10] P. Kronheimer; *J. Diff. Geom.* 32 (1990) 473.
[11] C. Bachas, J. Hoppe, B. Pioline; hep–th/007067.
[12] J. Hoppe; *Phys. Lett.* B 335 (1994) 41.
[13] M. Bordemann, J. Hoppe; *J. Math. Phys.* 39 (1998) 683.
[14] J. Hoppe; hep–th/9407103.
[15] J. Hoppe; hep–th/9503069.
[16] D.G. Currie, T.F. Jordan, E.C.G. Sudarshan; *Rev. Mod. Phys.* 35 (1965) 350; and H. Leutwyler; *Nuovo Cim.* 37 (1965) 556.
[17] J. Hoppe, T. Ratiu; *Class and Quant. Gravity* 14 (1997) L45.
[18] A. Sugamoto; *Nucl. Phys.* B 215 (1983) 381.
[19] M. Bordemann, J. Hoppe; *Phys. Lett.* 325 (1994) 359.
[20] T.R. Morris; From first to second quantized string theory; *Phys. Lett. B* 202:222 (1988); and D.L. Gee, T.R. Morris; From first to second quantized string theory. 3. Gauge fixing and quantization; *Nucl. Phys. B* 331:694 (1990).
[21] J. Hoppe; hep–th/9311059.
[22] J. Hoppe; MIT Ph.D. Thesis (see [1]). Since 1999 available under http://www.aei-potsdam.mpg.de/~hoppe
[23] J. Hoppe; *Helvetia Physica Acta* Vol. 70 (1997) hep/th
[24] J. Goldstone; unpublished notes (1985).
[25] D. Fairlie, P. Fletcher, C. Zachos; *Phys. Lett.* B 218 (1989) 203.
    J. Hoppe; *Phys. Lett.* B 215 (1988) 706.
[26] M. Bordemann, J. Hoppe, P. Schaller, M. Schlichenmaier; *Comm. Math. Phys.* 138 (1991) 209.
[27] M. Bordemann, E. Meinrenken, M. Schlichenmaier; CMP 165 (1994) 281.
[28] J. Hoppe, S.–T. Yau; Matrix Harmonics on $S^2$, Comm.Math.Phys. 195 (1998) 67.
[29] A. Smilga; Nucl. Phys. B 266 (1986) 45.
A. Smilga; Super–Yang–Mills quantum mechanics and supermembrane spectrum (1990) Proc. 1989 Trieste Conf. ed M. Duff, C. Pope and E. Sezgin (Singapore: World Scientific).
[30] J. Fröhlich and J. Hoppe; Commun. Math. Phys. 191 (1998) 613.
J. Hoppe and S.–T. Yau; Absence of zero energy states in the simplest $d = 3$ ($d = 5$?) matrix models, hep–th/9806152.
J. Fröhlich, G.M. Graf, D. Hasler, J. Hoppe and S.–T. Yau; Asymptotic form of zero energy wave functions in supersymmetric matrix models, hep–th/9904182.
[31] M. Halpern, C. Schwartz; Int.J.Mod.Phys. A 13 (1998) 4367.
[32] G.M. Graf and J. Hoppe, Asymptotic ground state for 10–dimensional reduced supersymmetric $SU(2)$ Yang–Mills theory, hep–th/980580.
[33] J. Hoppe; On the construction of zero energy states in supersymmetric matrix models, I + II, hep–th/9709132.
[34] J. Hoppe; On the construction of zero energy states in supersymmetric matrix models, III, hep–th/9711033.
[35] V. Kac, A. Smilga; Nucl.Phys. B 571 (2000) 515.
[36] M. Bordemann, J. Hoppe, R. Suter; Zero Energy States for $SU(N)$; . . . , hep–th/9909191.
[37] J. Hoppe; Asymptotic Zero Energy States for $SU(N \geq 3)$, hep–th/9912163.
[38] J. Hoppe, J. Plefka; The Asymptotic Groundstate of $SU(3)$ Matrix Theory, hep–th/0002107.
[39] J. Hoppe, P. Schaller; Phys.Lett. B 237 (1990) 407.
[40] D. Berenstein, J. Maldacena, H. Nastase; Strings in flat space and $pp$ waves from $\mathcal{N} = 4$ Super Yang Mills; hep–th/0202011.
[41] K. Dasgupta, M.M. Sheikh-Jabbari, M. van Raamsdonk; Matrix Perturbation Theory For M-Theory On a PP-Wave; hep-th/0205185.
[42] G.M. Graf, D. Hasler, J. Hoppe; Vanishing index for supersymmetric 2-matrix model with odd dimensional gauge group; hep–th/0205283.
[43] J. Hoppe; Some Classical Solutions of Membrane Matrix Model Equations; Proceedings of the Cargèse Nato Advanced Study Institute, May 1997, Kluwer 1999.
[44] A.S. Dzhumadil’daev; N-commutators of vector fields; math. RA/0203036.
[45] H. Kraft, C. Procesi; Comm.Math.Helv. 57 (1982) 539.