Spectral convergence bounds for classical and quantum Markov processes

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Introduction
  Motivation
  Definitions

Spectral bounds from a function space based approach
  Bounding functions of an operator
  Main result: spectrum and convergence

Conclusions and References
Classical and quantum Markov chains

Markov chain: Description of time-homogenous probabilistic evolution.

\[ \mathcal{X} \xrightarrow{T} \mathcal{X} \xrightarrow{T} \mathcal{X} \xrightarrow{T} \mathcal{X} \cdots \xrightarrow{T} \mathcal{X} \]

\[
\rho \mapsto T(\rho) \mapsto T^2(\rho) \mapsto T^3(\rho) \cdots \mapsto T_\infty(\rho)
\]

\( \mathcal{X} \): state space, \( \rho \): state of system, 
\( T \): transition map, \( T_\infty \): asymptotic evolution
Classical and quantum Markov chains

Markov chain: Description of time-homogenous probabilistic evolution.

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\[ \rho \rightarrow \mathcal{T}(\rho) \rightarrow \mathcal{T}^2(\rho) \rightarrow \mathcal{T}^3(\rho) \rightarrow \cdots \rightarrow \mathcal{T}\infty(\rho) \]

\[ \mathcal{X} : \text{state space,} \quad \rho : \text{state of system,} \quad \mathcal{T} : \text{transition map,} \quad \mathcal{T}\infty : \text{asymptotic evolution} \]

**Classical:**

- \( \mathcal{X} = \mathbb{R}^d \)
- \( \rho \): vector with non-negative components, sum to 1
- \( \mathcal{T} \): stochastic matrix

**Quantum:**

- \( \mathcal{X} = \{ X \in \mathbb{C}^{d\times d} | X = X^\dagger \} \)
- \( \rho \): positive semi-definite trace-one matrix
- \( \mathcal{T} \): trace-preserving and completely positive map
Approaching Asymptotic behavior

In many cases one is interested, when asymptotic behavior sets in:

**Classical:**
- Algorithms close to correct?
- Shuffling random?
  - Stability of fixed point of evolution
  - Cut-off phenomena

**Quantum:**
- Dissipative state preparation and computation
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In this talk we consider convergence properties of classical and quantum Markov chains. How is the *spectrum* of $T$ related to $\|T^n - T_\infty\|$?
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In this talk we consider convergence properties of classical and quantum Markov chains.
How is the *spectrum* of $\mathcal{T}$ related to $\|\mathcal{T}^n - \mathcal{T}_\infty\|$?
Mathematical primer

Linear maps $\mathcal{M}$:

- $\sigma(\mathcal{M}) = \{\lambda_1, \ldots, \lambda_d\}$ spectrum of $\mathcal{M}$ with spectral radius $\mu_\mathcal{M}$,
- $m_\mathcal{M}(z) = \prod_i (z - \lambda_i)^{k_i}$ minimal polynomial of $\mathcal{M}$: smallest degree non-zero poly. with $m_\mathcal{M}(\mathcal{M}) = 0$
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- Spectral radius $\mu = 1$
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$$\mathcal{T}_\infty := \sum_{|\lambda_i| = 1} \lambda_i \mathcal{P}_i$$

via Jordan decomposition: $\mathcal{T} = \sum_i (\lambda_i \mathcal{P}_i + \mathcal{N}_i)$, $\mathcal{P}_i$ spectral projector, $\mathcal{N}_i$ nilpotent.
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- $\mathcal{T}^n - \mathcal{T}_\infty^n = (\mathcal{T} - \mathcal{T}_\infty)^n$
Linear algebraic bounds

Use $\|T^n - T^n_\infty\| = \|(T - T_\infty)^n\|$ and Jordan/ Schur decompositions of $T - T_\infty$. 
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**Jordan:**

Let $\mu = \mu_{T - T_\infty}$ and $d_\mu$ largest Jordan block for $\mu$. There are $n$-independent $C_1, C_2 > 0$ such that

$$C_1 \mu^{n-d_\mu+1} n^{d_\mu-1} \leq \| T^n - T^n_\infty \| \leq C_2 \mu^{n-d_\mu+1} n^{d_\mu-1},$$
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**Schur:** (for quantum channels)

$$\|T^n - T^n_\infty\|_\diamond \leq 2d^{3/2}(\mu + 2d^{1/2})^{d^2-1} n^{d^2-1} \mu^{n-d^2+1}. $$
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Both bounds are not satisfactory: Jordan only qualitative, Schur too bad.
Certain spaces of analytic functions:

- $\text{Hol}(\mathbb{D})$: space of analytic functions on complex unit disc.
- $H^p \subset \text{Hol}(\mathbb{D})$ with $p > 0$: Hardy spaces

\[
H^p = \{ f \in \text{Hol}(\mathbb{D}) | \|f\|_{H^p}^p := \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\phi})|^p \, d\phi < \infty \}\]

- $W \subset \text{Hol}(\mathbb{D})$: Wiener algebra of absolutely convergent Taylor series

\[
W = \{ f = \sum_{k \geq 0} \hat{f}(k)z^k | \sum_{k \geq 0} |\hat{f}(k)| < \infty \}. \]
Power-bounded operators obey Wiener functional calculus

\[ \mathcal{M} \text{ power-bounded iff } \| \mathcal{M}^n \| \leq C \ \forall n \in \mathbb{N}. \text{ Examples:} \]

- \( \mathcal{T} \) quantum channel: \( \| \mathcal{T}^n \|_\diamond = 1 \)
- \( \mathcal{T} \) classical stochastic matrix: \( \| \mathcal{T}^n \|_{1 \rightarrow 1} = 1 \)
- \( \mathcal{T} - \mathcal{T}_\infty: \| (\mathcal{T} - \mathcal{T}_\infty)^n \|_\diamond = \| \mathcal{T}^n - \mathcal{T}_\infty^n \|_\diamond \leq \| \mathcal{T}^n \|_\diamond + \| \mathcal{T}_\infty^n \|_\diamond = 2 \)
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Suppose want to bound \( \| f(\mathcal{M}) \| \),
\( f \in \mathcal{W} = \{ f = \sum_{k \geq 0} \hat{f}(k) z^k | \sum_{k \geq 0} |\hat{f}(k)| < \infty \} \):
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**Observation I:**

\[
\| f(\mathcal{M}) \| = \| \sum_{k \geq 0} \hat{f}(k)\mathcal{M}^k \| \leq \sum_{k \geq 0} |\hat{f}(k)| \| \mathcal{M}^k \| \leq C \sum_{k \geq 0} |\hat{f}(k)| = C \| f \|_{\mathcal{W}}
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\]

**Observation II:**

\[
\| f(\mathcal{M}) \| = \| (f + mMg)(\mathcal{M}) \| \leq C \| f + m\mathcal{M}g \|_\mathcal{W} \ \forall g \in \mathcal{W}
\]
Bounding functions of operators

Thus, \( \|f(M)\| \leq C \inf_{g \in W} \|f + mMg\|_W \)

\( \leftrightarrow \) framework for spectral bounds on norm of function of operator:
Bounding functions of operators

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\( \rightarrow \) framework for spectral bounds on norm of function of operator:

- Find “good” function space for given class of operators
- Use above to shift problem to function space
- Find bound in function space e.g. choose “good” \( h \) with

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- Find “good’’ function space for given class of operators
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Examples:

- \( M \) Hilbert space contraction, then
  \( \| f(M) \| \leq \inf_{g \in H^\infty} \| f + m_M g \|_{H^\infty} \quad \forall f \in H^\infty \)
- \( T \) quantum channel, then [Nik06] \( \| T^{-1} \|_\diamond \leq \sqrt{2ed} / (\prod_i |\lambda_i|) \)
- \( T \) quantum channel, then

\[
\| T^n - T^n \|_\diamond = \| (T - T_\infty)^n \|_\diamond \leq 2 \inf_{g \in W} \| z^n + g \cdot m(T - T_\infty) \|_W
\]
Main result: Spectrum and convergence

Theorem (Szehr, Reeb, Wolf [SRW13])

Suppose $\|T^n\| \leq C \forall n \in \mathbb{N}$. Let $m = m_{T-T_\infty}$ be minimal polynomial and $\mu$ spectral radius of $T - T_\infty$. Then, for $n > \frac{\mu}{1-\mu}$ we have

$$\|T^n - T_\infty\| \leq \mu^n R(\mu, m, n) \prod_{m/(z-\lambda_D)} \frac{1 - (1 + \frac{1}{n})\mu|\lambda_i|}{\mu - |\lambda_i| + \frac{\mu}{n}},$$

where $R(\mu, m, n) = \frac{4Ce^2 \sqrt{|m|}(|m|+1)}{(1-(1+\frac{1}{n})\mu)^{3/2}}.$
Comparison to Schur and Jordan

To compare, note that

\[
\frac{1 - (1 + \frac{1}{n})\mu|\lambda_i|}{\mu - |\lambda_i| + \frac{\mu}{n}} \leq \frac{n}{\mu}(1 - \mu^2).
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i) Jordan:
- If $|\lambda_i| = \mu$ then catch factor $\frac{n}{\mu}$. Hence, Jordan bound is direct corollary.
- Advantage: Found quantitative bound since specified constants
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ii) Schur:
- In case of worst spectrum find Schur bound as corollary.
- Advantage: *Exponential* improvement in dimension prefactor even for worst spectrum
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Conclude: New bound outperforms Jordan and Schur
Some words about proof

Sufficient to bound \( \inf_{g \in \mathcal{W}} \left\| z^n + g \ m(T - T_\infty) \right\|_{\mathcal{W}}. \)

1. Interpolation problem [Nik09]:
\[ \inf_{g \in \mathcal{W}} \left\| z^n + g \ m(T - T_\infty) \right\|_{\mathcal{W}} = \inf_{h \in \mathcal{W}} \left\{ \| h \|_{\mathcal{W}} \mid h(\lambda_i) = \lambda_i^n \right\} \]
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   \]

2. Choose good representative: \( r \in (0, 1) \) and
   \[
   h_r(z) = \sum_k \lambda_k^n \tilde{B}(rz) \left(1 - r^2|\lambda_k|^2\right) \prod_{j \neq k} \frac{1 - r^2 \bar{\lambda}_j \lambda_k}{r \lambda_k - r \lambda_j}
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3. Bound in terms of Hardy norm:
   
   $\| h_r \|_{\mathcal{W}} \leq \sqrt{\sum_{k \geq 0} |\hat{h}(k)|^2 \sqrt{\frac{1}{1 - r^2}}} = \| h \|_{H^2} \sqrt{\frac{1}{1 - r^2}} \leq \| h \|_{H^\infty} \sqrt{\frac{1}{1 - r^2}}$
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   \]

4. Express \( h \) as contour integral and \( s \in (\mu, 1) \)
   \[
   \| h \|_{H^\infty} \leq \frac{s^{n+1}}{2\pi (n+1)} \sup_{|z|=1} \int_\gamma \left| \frac{1}{\tilde{B}_r(\lambda)(z-r \lambda)} \right|' \| d\lambda \|
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\]

5. Use Spijker Inequality. Let \( |\lambda| = (1 + 1/n)\mu \)
\[
\| T^n - T_\infty^n \| \leq \sqrt{\frac{1}{1-r^2}} \frac{\mu^{n+1} |m| + 1}{nr|m|(1-r(1+1/n)\mu)} \sup_{\lambda} \left| \prod_i \frac{1 - \bar{\lambda}_i r^2 \lambda}{\lambda - \lambda_i} \right|
\]
Conclude:

- New framework for spectral bounds
- New convergence estimate even for classical Markov chains
- Outperform classical convergence estimates

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