Entropic Quantization of Scalar Fields

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Abstract

Entropic Dynamics is an information-based framework that seeks to derive the laws of physics as an application of the methods of entropic inference. The dynamics is derived by maximizing an entropy subject to constraints that represent the physically relevant information that the motion is continuous and non-dissipative. Here we focus on the quantum theory of scalar fields. We provide an entropic derivation of Hamiltonian dynamics and using concepts from information geometry derive the standard quantum field theory in the Schrödinger representation.

1 Introduction

In the entropic dynamics (ED) framework quantum theory is derived as an application of entropic methods of inference. The framework has been applied to non-relativistic particles [1] and to relativistic scalar fields [2] leading to several new insights on foundational issues such as the nature of time and the problem of measurement in quantum mechanics. For example, just as in other forms of dynamics, time is defined so that motion looks simple. In the ED of fields entropic time is defined so that the fields undergo equal fluctuations in equal times—the field fluctuations play the role of a clock. The spatial uniformity of these fluctuations guarantees that time flows at the same rate everywhere. Another appealing insight is the new light entropic methods cast on the interpretation of the infinities typical of quantum field theory. We find that the divergences are not physical but epistemic effects; they are indications that the information that is relevant for the prediction of certain quantities is very incomplete.

In recent work the ED framework has been substantially improved in several respects. The early formulations involved assumptions that seemed ad hoc. Their justification was purely pragmatic — they worked; they led to the right answers. For example, use was made of auxiliary variables the physical interpretation of which remained obscure, and there were further assumptions about the configuration space metric and the form of the quantum potential. In [3] it

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was shown that the auxiliary variables were in fact unnecessary and could be eliminated. More recently, in [4], we have shown that ED can lead to a Hamiltonian dynamics and that the tools of information geometry can be used to provide natural justifications for both the metric of configuration space and for the particular form of the quantum potential. In this paper these improvements — the elimination of auxiliary variables, the derivation of Hamiltonian dynamics, and the introduction of information geometry methods — are extended to the ED of quantum scalar fields.

2 Entropic Dynamics

We wish to study the quantum dynamics of a single scalar field. A field configuration \( \phi(x) \) associates one real degree of freedom to each spatial point \( x \) in three-dimensional Euclidean space. Such a configuration is represented as a point \( \phi \in C \) in an \( \infty \)-dimensional configuration space \( C \) and it is convenient to represent the point \( \phi \) as a vector with infinitely many components denoted \( \phi_x = \phi(x) \). In the ED framework the field \( \phi_x \) has definite values which indicates a major departure from the standard Copenhagen interpretation. However, in general, the values \( \phi_x \) are unknown and the objective is to determine how their probability distribution \( \rho[\phi] \) changes over time. The first goal will be to use the method of maximum entropy to find the probability \( P[\phi' | \phi] \) that the field configuration makes a transition from a configuration \( \phi \) to a neighboring configuration \( \phi' \).

The prior We start from a prior for the transition probability distribution \( Q[\phi' | \phi] \) that expresses extreme ignorance: before any information is taken into account the knowledge of how the field changes at one point \( x \) tells us nothing about how it changes at other points \( x' \). This state of ignorance is represented by a prior that is a product over all space points,

\[
Q[\phi' | \phi] \sim \prod_x Q(\phi'_x | \phi_x). \tag{1}
\]

Furthermore, we assume that for every point \( x \) knowledge about the initial \( \phi_x \) tells us nothing about the final \( \phi'_x \). This is represented by \( Q(\phi'_x | \phi_x) \sim \text{constant} \). Since such constants have no effect on entropy maximization we can set \( Q(\phi'|\phi) = 1 \).

The constraints The actual information about evolution is introduced through constraints. The first piece of information is that the evolution of the fields is continuous. This means that at first we need only consider a small change; later

\( \infty \)-infinite dimensional spaces are complicated objects. We make no claim of mathematical rigor and follow the standard assumptions, notation, and practices of the subject. To be definite we can, for example, assume that the fields are initially defined on a discrete lattice (which makes the dimension of \( C \) infinite but countable) and that the continuum is eventually reached in the limit of vanishing lattice constant.
we will consider how a large change is achieved as a result of many small changes. For each \( x \) the field will change by a small amount from \( \phi_x \) to \( \phi'_x = \phi_x + \Delta \phi_x \) and we impose that the expected squared change is

\[
\langle \Delta \phi_x^2 \rangle = \int D\phi' P[\phi'|\phi] (\Delta \phi_x)^2 = \kappa_x ,
\]

where \( \int D\phi \) denotes a functional integration over \( \mathcal{C} \). This is an infinite number of constraints; one for each point \( x \). The constant \( \kappa_x \) is some small number and a continuous motion will be eventually achieved by letting \( \kappa_x \rightarrow 0 \). To reflect the translational invariance of three-dimensional Euclidean space we will set \( \kappa_x = \kappa \) independent of \( x \).

The constraints (2) lead to an evolution that is completely isotropic in \( \mathcal{C} \). Directionality is introduced assuming the existence of a “potential” \( \Lambda = \Lambda[\phi] \) and imposing a constraint on the expected displacement \( \langle \Delta \phi \rangle \) along the functional gradient of \( \Lambda \),

\[
\langle \Delta \phi \rangle \cdot \nabla \Lambda[\phi] = \int D\phi' P[\phi'|\phi] \int d^3x \Delta \phi_x \frac{\delta \Lambda}{\delta \phi_x} = \kappa' ,
\]

where \( \delta/\delta \phi_x \) denotes the functional derivative and \( \kappa' \) is a constant independent of \( \phi \).

**Entropy maximization**  We seek the transition probability distribution \( P[\phi'|\phi] \) that maximizes the relative entropy

\[
S[P, Q] = -\int D\phi' P[\phi'|\phi] \log \frac{P[\phi'|\phi]}{Q[\phi'|\phi]} \tag{4}
\]

subject to the constraints (2), (3), and normalization. For \( Q[\phi'|\phi] = 1 \) the resulting distribution is Gaussian,

\[
P[\phi'|\phi] = \frac{1}{\zeta} \exp \left[ -\int d^3x \left( \frac{\alpha_x}{2} (\Delta \phi_x)^2 - \alpha' \frac{\delta \Lambda}{\delta \phi_x} \Delta \phi_x \right) \right] ,
\]

where \( \alpha_x \) and \( \alpha' \) are Lagrange multipliers, and \( \zeta \) is a normalization constraint. Since by translation invariance we had \( \kappa_x = \kappa \), the corresponding multipliers \( \alpha_x \) must also be independent of \( x \) so that \( \alpha_x = \alpha \). Furthermore, since both the potential \( \Lambda \) and the constant \( \kappa' \) are so far unspecified we can, without loss of generality, absorb \( \alpha' \) into \( \Lambda \) which amounts to setting \( \alpha' = 1 \). The resulting transition probability is

\[
P[\phi'|\phi] = \frac{1}{Z} \exp \left[ -\frac{\alpha}{2} \int d^3x \left( \Delta \phi_x - \frac{1}{\alpha} \frac{\delta \Lambda}{\delta \phi_x} \right)^2 \right] \tag{5}
\]

where \( Z \) is a new normalization constant. In eq.(5) we see that \( \kappa \rightarrow 0 \) is recovered as \( \alpha \rightarrow \infty \).
Drift and fluctuations  The transition probability (5) shows that a small change \( \Delta \phi_x \) can be written as an expected drift plus a fluctuation, \( \Delta \phi_x = \langle \Delta \phi_x \rangle + \Delta w_x \). The expected drift is given by

\[
\langle \Delta \phi_x \rangle = \int D\phi' \Delta \phi_x P[\phi'|\phi] = \frac{1}{\alpha} \frac{\delta \Lambda}{\delta \phi_x} .
\]

(6)

The expected fluctuations are such that

\[
\langle \Delta w_x \rangle = 0 \quad \text{and} \quad \langle \Delta w_x \Delta w_{x'} \rangle = \frac{1}{\alpha} \delta_{xx'} ,
\]

(7)

where \( \delta_{xx'} = \delta(x - x') \). Since \( \Delta w_x \sim \alpha^{-\frac{1}{2}} \) while \( \langle \Delta \phi_x \rangle \sim \alpha^{-1} \) we see that for large \( \alpha \) the fluctuations dominate the dynamics.

Entropic Time  In ED time is introduced as a book-keeping device to keep track of the accumulation of small changes. This involves introducing a notion of instants that are distinct and ordered, and defining the interval or duration between them. For details see [1][5]. The result is that if \( \rho_t[\phi] \) refers to a probability distribution at a given instant, which we label \( t \), then entropic time is constructed by defining the next instant, labelled \( t' \), in terms of a distribution \( \rho_{t'}[\phi'] \) given by

\[
\rho_{t'}[\phi'] = \int D\phi P[\phi'|\phi] \rho_t[\phi]
\]

(8)

where \( P[\phi'|\phi] \) is given by (5). This definition readily lends itself to an iterative process in which time is constructed instant by instant: \( \rho_{t'} \) is constructed from \( \rho_t \), \( \rho_{t''} \) is constructed from \( \rho_{t'} \), and so on. This process defines the dynamics.

It remains to specify the interval \( \Delta t \) between two successive instants \( t \) and \( t' \) and the idea is captured by Wheeler’s slogan: time is defined so that motion (or, in our case, the evolution of the fields) looks simple. For small changes the dynamics is dominated by the fluctuations, eq.(7). It is therefore convenient to define duration so that the fluctuations are simple. Let

\[
\alpha = \frac{1}{\eta \Delta t} \quad \text{so that} \quad \langle \Delta w_x \Delta w_{x'} \rangle = \eta \Delta t \delta_{xx'} ,
\]

(9)

where \( \eta \) is a constant (which will eventually be regraduated into \( \hbar \)) that fixes the units of time relative to those of \( \phi \).

Thus, just as in Newtonian mechanics time is defined so that a free particle travels equal distances in equal times, in the ED of fields time is defined so that the fields undergo equal fluctuations in equal times. The translation invariance (\( \alpha_x = \alpha \)) guarantees that time flows at the same rate everywhere.

The information geometry of configuration space  To each point \( \phi \in \mathcal{C} \) we can associate a probability distribution \( P[\phi'|\phi] \). Therefore \( \mathcal{C} \) is a statistical
manifold and up to an arbitrary global scale factor its geometry is uniquely
determined by the information metric,
\[ \gamma_{xx'} = C \int D\phi' P[\phi'|\phi] \frac{\delta \log P[\phi'|\phi]}{\delta \phi_x} \frac{\delta \log P[\phi'|\phi]}{\delta \phi_{x'}} , \]  
(10)
where \( C \) is an arbitrary positive constant. (See e.g., [2].) For short steps \( (\alpha \to \infty) \) a straightforward substitution of (5) using (9) yields
\[ \gamma_{xx'} = \frac{C}{\eta \Delta t} \delta_{xx'} . \]  
(11)
We see that as \( \Delta t \to 0 \) we have \( \gamma_{xx'} \to \infty \). The reason is that as the distributions \( P[\phi'|\phi] \) and \( P[\phi'|\phi + \Delta \phi] \) become more sharply peaked it becomes increasingly easier to distinguish one from the other which means the information distance between them diverges. To define a distance that remains meaningful for arbitrarily small \( \Delta t \) it is convenient to choose \( C = \eta \Delta t \). Thus the metric \( \gamma_{xx'} \) of the configuration space \( C \) is a straightforward generalization of the metric \( \delta_{ij} \) of Euclidean space and the distance \( \Delta \ell \) between two slightly different configurations \( \phi \) and \( \phi + \Delta \phi \) is
\[ \Delta \ell^2 = \int d^3x d^3x' \delta_{xx'} \Delta \phi_x \Delta \phi_{x'} = \int d^3x (\Delta \phi_x)^2 . \]  
(12)
In [2] this choice of distance was merely postulated; here it is justified from information geometry, the assumptions implicit in (1), (2), and translation invariance.

**The Fokker-Planck equation** The dynamics expressed by the integral equation (8) can be rewritten in differential form. The result is a functional Fokker-Planck equation (see e.g., [5]) that takes the form of a continuity equation,
\[ \frac{\partial_t \rho_t [\phi]}{\partial t} = -\int d^3x \frac{\delta}{\delta \phi_x} (\rho_t [\phi] v_x [\phi]) . \]  
(13)
(The combination \( \int d^3x \frac{\delta}{\delta \phi_x} \) is the functional version of the divergence.) The velocity \( v_x [\phi] \) with which probabilities propagate in configuration space is called the current velocity. It is given by
\[ v_x [\phi] = b_x [\phi] + u_x [\phi] , \]  
(14)
where
\[ b_x [\phi] = \frac{\langle \Delta \phi_x \rangle}{\Delta t} = \eta \frac{\delta \Lambda}{\delta \phi_x} \quad \text{and} \quad u_x [\phi] = -\eta \frac{\delta \log \rho^{1/2}}{\delta \phi_x} , \]  
(15)
are the drift and the osmotic velocities. The current velocity \( v_x [\phi] \) can be written as the functional gradient of a scalar functional \( \Phi \),
\[ v_x [\phi] = \frac{\delta \Phi}{\delta \phi_x} \quad \text{where} \quad \frac{\Phi[\phi]}{\eta} = \Lambda [\phi] - \log \rho^{1/2} [\phi] . \]  
(16)
Incidentally, it is convenient to introduce a functional $H[\rho, \Phi]$ on $C$ in order to write the Fokker-Planck equation as a functional derivative in $C$,

$$\partial_t \rho[\phi] = \frac{\Delta H[\rho, \Phi]}{\Delta \Phi[\phi]}.$$  

(17)

(For a useful and brief description of functional calculus in configuration space see [6].) Using (13), equation (17) is easily integrated. The result is

$$H[\rho, \Phi] = \int D\phi \int d^3x \frac{1}{2} \rho \left( \frac{\delta \Phi}{\delta \phi} \right)^2 + F[\rho],$$  

(18)

where $F[\rho]$ is an integration constant. In what follows we will assume that $F[\rho]$ is independent of time. We emphasize that eq. (17) does not reflect a new assumption or a new dynamical principle; it is merely a rewriting of (13).

3 Non-dissipative Diffusion

The Fokker-Planck equation (13) describes a standard diffusion process, it does not describe quantum systems. As discussed in [1][4] the solution to this problem is to modify the constraints: instead of $\Lambda[\phi]$ being an externally prescribed potential we allow it to represent a dynamical field on $C$. The appropriate constraint consists in demanding that at each instant of time the potential $\Lambda$, or equivalently the related quantity $\Phi$ in (16), is updated in such a way that a certain functional – that we will call “energy” – remains constant. It turns out that the appropriate “energy” is the functional $H[\rho, \Phi]$ given in eq. (18). Thus, the dynamics consists of the coupled non-dissipative evolution of $\rho[\phi]$ and $\Phi[\phi]$.

The ensemble Hamiltonian and its conservation  
To impose a non-dissipative diffusion we demand the conservation of the functional $H[\rho, \Phi]$,  

$$\frac{dH[\rho, \Phi]}{dt} = \int D\phi \left[ \frac{\Delta H}{\Delta \Phi} \partial_t \Phi + \frac{\Delta H}{\Delta \rho} \partial_t \rho \right] = 0.$$  

(19)

Using (17)

$$\frac{dH[\rho, \Phi]}{dt} = \int D\phi \left[ \partial_t \Phi + \frac{\Delta H}{\Delta \rho} \right] \partial_t \rho = 0.$$  

(20)

This condition must be satisfied at all times $t$ and for arbitrary choices of the initial values of $\rho$ and $\Phi$. From (13) this means that (20) must hold for arbitrary choices of $\partial_t \rho$ which implies that the integrand of (20) must vanish. Therefore,

$$\partial_t \Phi = -\frac{\Delta H}{\Delta \rho} \quad \text{and} \quad \partial_t \rho = \frac{\Delta H}{\Delta \Phi},$$  

(21)

which we recognize as a functional form of Hamilton’s equations with the conserved functional $H[\rho, \Phi]$ playing the role of Hamiltonian.
The Schrödinger functional equation

The Fokker-Planck equation together with the conservation of \( H[\rho, \Phi] \) leads to a Hamiltonian structure regardless of the choice of \( F[\rho] \). However, as discussed in [4], quantum theory is reproduced only for a special choice of \( F[\rho] \),

\[
F[\rho] = \int D\phi \int d^3x \left[ \rho V(\phi_x, \nabla \phi_x) + \frac{\xi}{\rho} \left( \frac{\delta \rho}{\delta \phi_x} \right)^2 \right]. \tag{22}
\]

In the first term \( V(\phi_x, \nabla \phi_x) \) is a potential energy density to be discussed further below. The second term is usually called the “quantum” potential. It is the functional trace of the Fisher information and its origin in information geometry is discussed in [4]. \( \xi \) is a positive constant that controls the effect of the quantum potential.

As a matter of convenience we can combine the two variables \( \rho[\phi] \) and \( \Phi[\phi] \) into a single complex variable, \( \Psi_k[\phi] = \rho_{1/2} e^{i k \Phi/\eta} \), where \( k \) is an arbitrary positive constant. The pair of Hamilton’s equations (21) can then be combined into a single non-linear equation for the wave functional \( \Psi_k[\phi] \),

\[
i \frac{\eta}{k} \partial_t \Psi_k[\phi] = \int d^3x \left[ -\frac{\eta^2}{2k^2} \frac{\delta^2}{\delta \phi_x^2} + \left( \frac{\eta^2}{2k^2} - 4\xi \right) \frac{1}{|\Psi_k|} \frac{\delta^2 |\Psi_k|}{\delta \phi_x^2} + V \right] \Psi_k[\phi].
\]

Different choices of the arbitrary \( k \) lead to different but equivalent descriptions of the same theory. Let us therefore take advantage of the arbitrariness of \( k \) and choose the simplest and most convenient description. This is achieved for the value \( \tilde{k} = (\eta^2/8\xi)^{1/2} \) that leads to the linear Schrödinger equation,

\[
i \hbar \partial_t \Psi[\phi] = \int d^3x \left[ -\frac{\hbar^2}{2} \frac{\delta^2}{\delta \phi_x^2} + V \right] \Psi[\phi], \tag{23}
\]

where we have identified \( \eta/\tilde{k} = \hbar \) and dropped the index \( k \) so that \( \Psi = \rho^{1/2} e^{i \Phi/\hbar} \). This is quantum field theory in the Schrödinger representation and one can now proceed in the usual way to introduce a Hilbert space, operators, and all the standard machinery of quantum mechanics. For example, the commutator of the field \( \phi_x \) and its conjugate momentum is

\[
[\phi_x, \frac{\hbar}{k} \frac{\delta}{\delta \phi_{x'}}] = i \hbar \delta_{xx'}.
\]

At this point the potential \( V(\phi_x, \nabla \phi_x) \) is essentially arbitrary. A useful form is obtained by doing a Taylor expansion about weak fields and gradients and then imposing the rotational and Lorentz symmetries required by the experimental evidence,

\[
V(\phi_x, \nabla \phi_x) = \frac{1}{2}(\nabla \phi_x)^2 + \frac{1}{2} m^2 \phi_x^2 + \lambda' \phi_x^3 + \lambda'' \phi_x^4 + \ldots \tag{24}
\]

The various coefficients represent mass and other coupling constants. We conclude that the ED framework reproduces the Schrödinger representation of the standard relativistic quantum theory of scalar fields. [7]
4 Discussion

Setting \( \lambda' = \lambda'' = \ldots = 0 \) the Schrödinger equation,

\[
i \hbar \partial_t \Psi = \frac{1}{2} \int d^3x \left[ -\hbar^2 \frac{\delta^2}{\delta \phi_x^2} + (\partial \phi_x)^2 + m^2 \phi_x^2 \right] \Psi ,
\]

(25)

reproduces the quantum theory of free real scalar fields [7] and all the standard results can now be obtained using conventional methods (see e.g., [8]). For example, choosing units such that \( \hbar = c = 1 \), a standard calculation of the ground state gives a Gaussian functional,

\[
\Psi_0[\phi] = \frac{1}{Z_0^{1/2}} e^{-iE_0 t} \exp \left[ -\frac{1}{2} \int d^3x \int d^3y \, \phi(\bar{x}) G(\bar{x}, \bar{y}) \phi(\bar{y}) \right] ,
\]

(26)

where

\[
G(\bar{x}, \bar{y}) = \int \frac{d^3k}{(2\pi)^3} \omega_k e^{i\bar{k} \cdot (\bar{x} - \bar{y})} , \quad \text{with} \quad \omega_k = (\vec{k}^2 + m^2)^{1/2} .
\]

(27)

The energy of the ground state is

\[
E_0 = \langle H \rangle_0 = \frac{1}{2} \int d^3x \, G(\bar{x}, \bar{x}) = \int d^3x \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \omega_k
\]

(28)

is both infrared and ultraviolet divergent. The vacuum expectation value of the field at any point \( \bar{x} \) vanishes while its variance diverges,

\[
\langle \phi(\bar{x}) \rangle = 0 \quad \text{and} \quad \text{Var} \left[ \phi(\bar{x}) \right] = \langle \phi^2(\bar{x}) \rangle_0 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} .
\]

(29)

Note, however, that what diverges here are not the physical fields but the uncertainty in our predictions. ED recognizes the role of incomplete information: the theory is completely unable to predict the field value at a sharply localized point. The theory does, however, offer meaningful predictions for other quantities. For example, the equal time correlations between two field variables \( \phi(\bar{x}) \) and \( \phi(\bar{y}) \) are [8],

\[
\langle \phi(\bar{x}) \phi(\bar{y}) \rangle_0 = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\bar{k} \cdot (\bar{x} - \bar{y})}}{2\omega_k} = \frac{m}{4\pi^2 |\bar{x} - \bar{y}|} K_1 \left( m |\bar{x} - \bar{y}| \right)
\]

(30)

where \( K_1 \) is a modified Bessel function.

**Conclusion**  Entropic dynamics provides an alternative method of quantization — entropic quantization. In the ED framework a quantum theory is a non-dissipative diffusion in the configuration space.

The entropic quantization of scalar fields yields the standard predictions of quantum field theory. At this early point in the development the advantages
of the entropic approach do not lie in any new predictions (at least not yet) but rather in the suitability of the formalism to be extended beyond the domain in which ED is equivalent to the current quantum field theory and in the new insights it offers on matters of interpretation. More specifically, concerning entropic time: In the ED of fields, the field fluctuations provide the clock and entropic time is defined so that field fluctuations are uniform in space and time. Concerning the nature of particles: fields are real, particles are just some peculiar spatial correlations in the field. Concerning the divergences: they are the expected consequence of handling incomplete information. Some predictions will be certain, some will be uncertain, and some may even be infinitely uncertain.

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