Quantum Brownian Motion of a charged oscillator in a magnetic field coupled to a heat bath through momentum variables

Suraka Bhattacharjee, Urbashi Satpathi, and Supurna Sinha

1 Raman Research Institute, Bangalore-560080, India
2 Department of Chemistry, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel

(Dated: August 30, 2022)

We study the Quantum Brownian motion of a charged particle moving in a harmonic potential in the presence of an uniform external magnetic field and linearly coupled to an Ohmic bath through momentum variables. We analyse the growth of the mean square displacement of the particle in the classical high temperature domain and in the quantum low temperature domain dominated by zero point fluctuations. We also analyse the Position Response Function and the long time tails of various correlation functions. We notice some distinctive features, different from the usual case of a charged quantum Brownian particle in a magnetic field and linearly coupled to an Ohmic bath via position variables.

I. INTRODUCTION

The Quantum Brownian Motion of a particle undergoing diffusion driven by quantum fluctuations has been the focus of study for some time [1–4]. More recently, researchers have investigated the diffusion behaviour of a charged particle in a magnetic field [5–7]. To study this problem, one considers a Quantum Langevin Equation which corresponds to a reduced description of the system in which the coupling with the heat bath is described by two terms: an operator valued random force \( F(t) \) with mean zero and a mean force characterized by a memory function \( \mu(t) \) [8]. Moreover, some of the studies have focused on an anomalous form of these couplings within a similar framework of the Quantum Langevin dynamics [9–11]. In [12, 13], the authors have considered the motion of a harmonically oscillating Brownian particle in the presence of a bath that is affected by a harmonic potential. It has been reported that the fluctuation dissipation theorem still remains valid at equilibrium, however the random forces show dependence on the confining potential. Moreover, the effect of the harmonic potential on the dissipation constant has been determined in cold atom experiments and also in Molecular Dynamics (MD) simulation for a molecular solute particle suspended in a fluid medium [14, 15]. Although, in most of the literature, one considers a conventional form of coupling between the particle and the bath via position variables [1–7], several authors have also considered the case of such a quantum Brownian particle coupled to the bath variables through momentum coordinates [16–23].

In [23] the authors have derived a Quantum Langevin Equation of a charged quantum particle moving in a harmonic potential in the presence of an uniform external magnetic field and linearly coupled to a quantum heat bath through momentum variables. Here we use the Quantum Langevin Equation derived in [23] as a starting point to analyse the growth of the mean square displacement of the particle in the classical high temperature domain and in the quantum low temperature domain dominated by zero point fluctuations. Furthermore, we analyse the long time tails of various correlation functions. In addition to it, we have shown that the random forces in this case are dependent on the confining potential, although we have not considered the effect of the harmonic potential on the bath variables explicitly in the Hamiltonian [12, 13]. The modified force correlations were derived in [23] for the case of a system coupled to a bath via momentum variables and the fluctuation dissipation theorem holds at equilibrium.

The outline of the paper is as follows. In Sec II we discuss the Quantum Langevin Formalism and the position correlation function, the mean square displacement, the Position Response Function, the position-velocity correlation function and the velocity autocorrelation function. In Sec III we discuss the long time behaviour of the various correlation functions mentioned in Sec II. In Sec IV we present our results and discussions. Finally we end the paper with some concluding remarks in Sec V. We present details of some of the calculations in the Appendix.

II. QUANTUM LANGEVIN FORMALISM

The Hamiltonian of a charged Brownian particle in the presence of a magnetic field and a harmonic potential and linearly coupled to a heat bath consisting of \( N \) oscillators, via momentum coordinate, is given by [23]:

\[
H = \frac{1}{2m} \left( \vec{\dot{p}} - q \frac{\vec{A}}{c} \right)^2 + \frac{1}{2m} \omega_0^2 \vec{r}^2 + \sum_{j=1}^{N} \left[ \frac{1}{2m_j} \left( \vec{\dot{p}}_j - g_j \vec{\dot{r}} + \frac{g_j q}{c} \vec{A} \right)^2 + \frac{1}{2} m_j \omega_j^2 \vec{q}_j^2 \right]
\]  

(1)

where, \( m \) is the mass of the Brownian particle, \( \vec{p} \) and \( \vec{r} \) are the momentum and position coordinates of the particle respectively. \( \omega_0 \) is the frequency of the harmonic oscillator potential, \( q \) is the charge of the particle, \( c \) is the speed of light and \( \vec{A} \) is the vector potential pertaining to the magnetic field \( \vec{B} \). \( m_j, \vec{p}_j, \vec{q}_j, \omega_j \) denote the heat bath variables corresponding to the mass, momentum,
position and frequency of the \( j^{th} \) oscillator respectively. \( g_j \) is the coupling strength between the particle and the \( j^{th} \) bath oscillator.

In Eq. (1) the first term pertains to the kinetic energy in the presence of the external magnetic field. The second term pertains to the harmonic oscillator potential and the last term consists of the Hamiltonian of interaction between the charged particle and the bath of oscillators and the bath Hamiltonian. The first step in deriving the Quantum Langevin equation (QLE) of a charged Brownian particle is to write the Heisenberg equations of motion for both the Brownian particle variables and the bath variables. Solving the Heisenberg equations of motion for the bath variables \( q_j, p_j \) and substituting the solutions for the bath variables in the equation of motion of the Brownian particle \( \vec{r}, \vec{p} \), one can arrive at the Quantum Langevin Equation (QLE) of a quantum particle of charge \( q \) moving in a harmonic potential in the presence of an uniform external magnetic field and linearly coupled to a quantum heat bath through momentum variables [23]:

The Quantum Langevin Equation (QLE) of a Brownian particle of charge \( q \) moving in a harmonic potential in the presence of an uniform external magnetic field \( \vec{B} \) and linearly coupled to a quantum heat bath through momentum variables [23]:

\[
m_r \ddot{\vec{r}}(t) + \int (t-t') \vec{r}(t') dt' + m_r \omega_0^2 \vec{r}(t) + \mu_d(t) \vec{r}(0) - \frac{q}{c} (\vec{r}(t) \times \vec{B}) = \vec{F}(t)
\]

where, \( \vec{r}(0) \) is the initial position of the particle at \( t = 0 \), \( m_r \) is the normalized mass of the particle given by:

\[
m_r = m / \left[ 1 + \sum_{j=1}^{N} \frac{g_j^2 m}{m_j} \right]
\]

with \( g_j \) being the coupling strength between the particle and the \( j^{th} \) bath oscillator.

\( \vec{F}(t) \) is the random force given by,

\[
\vec{F}(t) = \sum_{j=1}^{N} g_j m_r \omega_j^2 \vec{q}_j^h(t) \Theta(t)
\]

where, \( \Theta(t) \) is the step function and \( \vec{q}_j^h(t) \) is the solution of the homogeneous equation of motion for the heat bath variables given by:

\[
\vec{q}_j^h(t) = g_j \cos(\omega_j t) \hat{\xi}_j + \frac{p_j(0)}{m_j \omega_j} \sin(\omega_j t)
\]

\( q_j(0) \) and \( p_j(0) \) are the initial position and momentum of the \( j^{th} \) bath particle respectively [23].

\( \vec{F}(t) \) has the following properties:

\[
\langle F_\alpha(t) \rangle = 0
\]

\[
\frac{1}{2} \langle \{ F_\alpha(t), F_\beta(t') \} \rangle = \frac{\hbar \delta_{\alpha\beta}}{2 \pi} \int_{-\infty}^{\infty} d\omega \text{Re}[\mu_d(\omega)] \frac{\omega^2 m_r}{\omega_0^2 m} e^{-i\omega t} \times \coth\left( \frac{\hbar \omega}{2 k_B T} \right) e^{-i\omega t}
\]

\[
\langle [F_\alpha(t), F_\beta(0)] \rangle = \frac{\hbar \delta_{\alpha\beta}}{\pi} \int_{-\infty}^{\infty} d\omega \text{Re}[\mu_d(\omega)] \frac{\omega^3 m_r}{\omega_0^2 m} e^{-i\omega t}
\]

\( \mu(t) = \mu_d(t) + \Gamma \mu_{od}(t) \) is the memory kernel, \( \mu_d \) and \( \mu_{od} \) are the diagonal and off-diagonal parts of the memory kernel \( \mu \).

\[
\mu_d(t) = \sum_{j=1}^{N} \frac{g_j^2 m_r \omega_j^2 \cos(\omega_j t) \Theta(t)}{m_j}
\]

\[
\mu_{od}(t) = \sum_{j=1}^{N} \frac{g_j^2 m_r \omega_j \omega_c \sin(\omega_j t) \Theta(t)}{m_j B}
\]

\[
\Gamma = \begin{pmatrix}
0 & -B_z & -B_y \\
-B_z & 0 & B_x \\
B_y & -B_x & 0
\end{pmatrix}
\]

\( B_x, B_y, B_z \) are the components and \( B \) is the magnitude of the uniform magnetic field \( \vec{B} \). Notice that the diagonal elements of the memory kernel have an explicit dependence on the external harmonic oscillator potential via \( \omega_0 \). The Fluctuation Dissipation Theorem (Eq. (7) and Eq. (8) connects the force correlation to a memory kernel with a diagonal element dependent on the harmonic potential via \( \omega_0 \) [13]. There are several papers in the literature where a similar dependence of the memory kernel on the externally applied potential has been studied [12–15].

Now, one can define an effective random force \( \vec{G}(t) \) as follows: \( \vec{G}(t) = \vec{F}(t) - \mu_d(t) \vec{r}(0) \), where \( \vec{G}(t) \) retains all the statistical properties of \( \vec{F}(t) \) [8, 23].

For the Ohmic model, \( \mu_d(t) = 2 \gamma m \delta(t) \) [6, 7].

Using this, Eq. (2) takes the form:

\[
m_r \ddot{\vec{r}}(t) + m_r \gamma \dot{\vec{r}}(t) + m_r \omega_0^2 \vec{r} - \frac{q}{c} (\vec{r}(t) \times \vec{B}) = \vec{G}(t)
\]

Taking the Fourier transform of Eq. (13) and separating it into \( x \) and \( y \) components we get:

\[
-m_r \omega^2 x + i m \gamma \omega x + m_r \omega_0^2 x - i m \omega \omega y = G_x(\omega)
\]

\[
-m_r \omega^2 y + i m \gamma \omega y + m_r \omega_0^2 y + i m \omega \omega x = G_y(\omega)
\]
where, $\omega_c = \frac{qB}{m}$ and $G_x(\omega)$ and $G_y(\omega)$ are the Fourier transforms of $G_x(t)$ and $G_y(t)$ respectively.

Solving Eqs.(14) and (15), we get $x(\omega)$ and $y(\omega)$ as:

\[
x(\omega) = \frac{-(m_r^2 - m_r^2 + im\omega)(-\mathcal{G}_x) - i\omega\omega_m(-\mathcal{G}_y)}{m^2\omega^2(\omega^2 - \omega_r^2)^2} \quad (16)
\]

\[
y(\omega) = \frac{-(m_r^2 - m_r^2 + im\omega)(-\mathcal{G}_y) + i\omega\omega_m(-\mathcal{G}_x)}{m^2\omega^2(\omega^2 - \omega_r^2)^2} \quad (17)
\]

The position autocorrelation function as defined in Refs.[7, 24] is given by:

\[
C_x(\omega) = \frac{1}{2}\langle \{x(\omega), x^*(\omega)\} \rangle \quad (18)
\]

\[
C_y(\omega) = \frac{1}{2}\langle \{y(\omega), y^*(\omega)\} \rangle \quad (19)
\]

We have made use of the force correlation functions mentioned in the Appendix A (see Eq.(7)), Eq.(16-17) and Eq. (18-19) to get $C_x(\omega) = C_y(\omega) = C(\omega)$ as:

\[
C(\omega) = \left(\frac{\hbar m_r}{\omega_0^2}\right) \omega^3 \left[\frac{1}{m^2}(\omega_0^2 - \omega^2)^2 + \frac{1}{m^2}(\omega_0^2 + \omega^2)\frac{\omega_0}{\Omega_{th}}\right] \coth\left(\frac{\omega}{\Omega_{th}}\right)
\]

\[
= \left[\frac{m}{m^2}(\omega_0^2 - \omega^2)^2 + \frac{m}{m^2}(\omega_0^2 + \omega^2)^2\right] - 4\omega^2\omega_c^2 (\omega_0^2 - \omega^2)^2
\]

\[
\Omega_{th} = \frac{2k_BT}{\hbar}
\]

According to the Fluctuation Dissipation Theorem (FDT) in the Fourier domain [7, 24–26]:

\[
\frac{1}{\hbar} C(\omega) = \coth\left(\frac{\omega}{\Omega_{th}}\right) R''(\omega) \quad (21)
\]

where $R''(\omega)$ is the imaginary part of the Response Function, which is defined as the measure of the response of a system to an external perturbing force. In the time domain, the displacement of a particle in response to a perturbing force can be expressed as the convolution of the response function and the applied force.

In this paper, we study the position autocorrelation function, mean square displacement, response function, position-velocity correlation function and velocity autocorrelation function in the context of a quantum Brownian motion of a charged oscillator in a magnetic field coupled to a bath via momentum coupling.

(i) The Mean Square Displacement (M.S.D) is defined by [6]:

\[
M.S.D = 2(C(0) - C(t)) \quad (22)
\]
\[ R''(\omega) = \frac{2m_c}{m_r \omega_p^2} \omega^3 \left[ \frac{1}{m_r} (\omega_0^2 - \omega^2)^2 + \frac{1}{m_r^2} \omega^2 (\omega_c^2 + \gamma^2) \right] \]

\[ \left\{ \frac{m_r}{m_c} (\omega_0^2 - \omega^2)^2 + \frac{m_r^2}{m_r} \omega^2 (\omega_c^2 + \gamma^2) \right\}^2 - 4\omega^2 \omega_c^2 (\omega_0^2 - \omega^2)^2 \]

(27)

The real part \( R'(\omega) \) of the Position Response Function can be calculated from the imaginary part \( R''(\omega) \) using the Kramers Kronig relation given by [7, 28]:

\[ R'(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\omega' R''(\omega')}{(\omega'^2 - \omega^2)} d\omega' \]

(28)

\[ R'(\omega') \] has poles at \( \omega' = p_j \) (for \( j=1,...,4 \)) and at \( \omega' = \pm \omega \). The residues at \( \omega \) and \( -\omega \) cancel out. Thus \( R'(\omega) \) is derived using the Cauchy’s residue theorem:

\[ R'(\omega) = (2\pi i) \sum_{j=1}^{4} \left[ \text{Res} \left\{ \frac{\omega' R''(\omega')}{\pi(\omega'^2 - \omega^2)}, p_j \right\} \right] \]

(29)

The total Position Response Function is then calculated as:

\[ R(\omega) = R'(\omega) + iR''(\omega) \]

(30)

Then the Position Response Function \( R(t) \) in the time domain is easily derived via the Fourier transform of \( R(\omega) \) obtained from the above formalism:

\[ R(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\omega)e^{-i\omega t} d\omega \]

(31)

This calculation shows that the response function exhibits a decay from a finite value at \( t=0 \), which is distinct from the position coupling case which shows a rise in the response function starting from \( t=0 \) and settling to a constant as \( t \) increases.

For the simplest case, one may consider the particle in the absence of a magnetic field, i.e, \( \omega_c = 0 \) and the response function works out to be:

\[ R(t) = \left\{ \frac{m_r}{m_r \gamma p} \cos \left( \frac{\omega_p t}{2} \right) + \left( \omega_p^2 - 2\omega_0^2 \right) \sin \left( \frac{\omega_p t}{2} \right) \right\} \]

\[ \times e^{\frac{-m_r \gamma t}{2m_r}} \]

(32)

where, \( \omega_p = \sqrt{4\omega_0^2 - \frac{m_r^2}{m_r}} \).

It can be seen from Eq. (32) that, at \( t=0 \), \( R(t) \) reduces to \( \frac{m_r}{m_r \gamma p} \) and we notice a subsequent exponential decay at long times, in the over-damped regime. However, it can be seen that in the presence of a magnetic field \( (\omega_c \neq 0) \), the decay is followed by an oscillatory behaviour when \( \omega_c > \gamma \).

Interestingly, this form of the Position Response Function is similar to that of the Velocity Response Function that one gets in the case of a coupling to the bath via position variables. It could perhaps be due to the fact that the momentum and the position are related via a time derivative and this leads to the difference that we observe between the two cases (the case of a bath coupled via position variables and the one in which the bath is coupled via momentum variables).

(iii) The Position-Velocity Correlation Function \( (C_{v}(t)) \) is derived from the position autocorrelation function [7]:

\[ C_{p-v}(t) = \frac{d}{dt} C(t) \]

(33)

In the Fourier space:

\[ C_{p-v}(\omega) = -i\omega C(\omega) \]

(34)

Using Eqs.(20), (33) and (34):

\[ C_{p-v}(t) = C_{p-v_1}(t) + C_{p-v_2}(t) \]

(35)

where,

\[ C_{p-v_1}(t) = (-i) \sum_{j=1}^{4} (\text{Res}[C_{p-v}(\omega)e^{(-i\omega t)}, p_j]) \]

(36)

\[ C_{p-v_2}(t) = (ih\pi \Omega_{th}^2) \sum_{n=1}^{\infty} n R''(-in\pi \Omega_{th}) e^{(-n\pi \Omega_{th} t)} \]

(37)

(iv) Similar to the case of the position-velocity correlation function, the Velocity Autocorrelation Function \( (C_v) \) is calculated as follows [7]:

\[ C_v(t) = -\frac{d^2}{dt^2} C(t) \]

(38)

In the Fourier space:

\[ C_v(\omega) = \omega^2 C(\omega) \]

(39)

Using Eqs.(20), (38) and (39):

\[ C_v(t) = C_{v_1}(t) + C_{v_2}(t) \]

(40)

where,

\[ C_{v_1}(t) = (-i) \sum_{j=1}^{4} (\text{Res}[C_v(\omega)e^{(-i\omega t)}, p_j]) \]

(41)

\[ C_{v_2}(t) = (ih\pi \Omega_{th}^3) \sum_{n=1}^{\infty} n^2 R''(-in\pi \Omega_{th}) e^{(-n\pi \Omega_{th} t)} \]

(42)
III. LONG TIME BEHAVIOUR OF THE CORRELATION FUNCTIONS

In this section, we have analyzed the long time behaviour of the correlation functions at finite low temperatures and extended the results to get the behaviour in the zero temperature Quantum regime.

A. Position Autocorrelation Function

As discussed in the previous section, the position autocorrelation function at any temperature is given by Eq.(23). At low temperatures, exp(−γt) falls off faster than exp(−nπΩth) and the long time behaviour of the correlation function is determined by C2(t) (see Appendix for details). Therefore, at finite low temperatures at long times we get:

\[ C(t) = C_2(t) = (-i\hbar\Omega_{th}) \sum_{n=1}^{\infty} R''(-in\pi\Omega_{th}) \text{exp}(-n\pi\Omega_{th}t) \]  
\[ \text{(43)} \]

Now, expanding the imaginary part of the response function R''(ω) in a power series using Eq.(20) and Eq.(21), we get:

\[ R''(\omega) = \frac{\gamma}{m_r\omega_0^6} \omega^3 + \frac{(2m_r^2\omega_0^2 - m^2(\gamma^2 - 3\omega_0^2))\gamma}{m_r^4\omega_0^{10}} \omega^5 + \frac{(3m_r^4\omega_0^4 - 4m^2m_r^2\omega_0^2(\gamma^2 - 3\omega_0^2) + m^4(5\omega_0^4 - 10\omega_0^2\gamma^2 + \gamma^4))}{m_r^6\omega_0^{12}} \times \left( \frac{\gamma m_r}{\omega_0^2} \right) \omega^7 + \ldots \ldots \]  
\[ \text{(44)} \]

Using the above power series expansion, we obtain from Eq.(43) [2, 24]:

\[ C(t) = -i(\hbar\Omega_{th}) \left[ \frac{\gamma}{m_r\omega_0^6} (-i\pi\Omega_{th})^3 \sum_{n=1}^{\infty} n^3 \text{exp}(-n\pi\Omega_{th}t) + \frac{\gamma(2m_r^2\omega_0^2 - m^2(\gamma^2 - 3\omega_0^2))}{m_r^4\omega_0^{10}} (-i\pi\Omega_{th})^5 \sum_{n=1}^{\infty} n^5 \text{exp}(-n\pi\Omega_{th}t) \right] + \ldots \ldots \]  
\[ \text{(45)} \]

\[ C(t) = \frac{\hbar^3\Omega_{th}^4}{m_r\omega_0^6} e^{\pi\Omega_{th}t} (1 + 4e^{\pi\Omega_{th}t} + e^{2\pi\Omega_{th}t}) - \frac{\hbar^5\Omega_{th}^6}{m_r^3\omega_0^{10}} \frac{\gamma(2m_r^2\omega_0^2 - m^2(\gamma^2 - 3\omega_0^2))}{m_r^4\omega_0^{10}} \times \frac{e^{\pi\Omega_{th}t}(1 + 26e^{\pi\Omega_{th}t} + 66e^{2\pi\Omega_{th}t} + 26e^{3\pi\Omega_{th}t} + e^{4\pi\Omega_{th}t})}{(e^{\pi\Omega_{th}t} - 1)^6} + \ldots \ldots \]  
\[ \text{(46)} \]

At finite temperatures and long times, \( \pi\Omega_{th}t \gg 1 \) and thus we get an exponential decay of the position autocorrelation function:

\[ C(t) = A_c \text{exp}[-\pi\Omega_{th}t] \]  
\[ \text{(47)} \]

where,

\[ A_c = \frac{\hbar^3\Omega_{th}^4}{m_r\omega_0^6} - \frac{\hbar^5\Omega_{th}^6}{m_r^3\omega_0^{10}} \frac{\gamma(2m_r^2\omega_0^2 - m^2(\gamma^2 - 3\omega_0^2))}{m_r^4\omega_0^{10}} + \ldots \ldots \]  
\[ \text{(48)} \]

In contrast, at \( T \to 0 \), we expand \( e^{\pi\Omega_{th}t} \) as a Taylor series and we get a power law behaviour, similar to the case of a particle coupled to a bath via a position coordinate coupling [2, 24], albeit, with higher power exponents.

\[ C(t) = A_{1c}t^{-4} + A_{2c}t^{-6} + \ldots \]  
\[ \text{(49)} \]

where,

\[ A_{1c} = \frac{6\hbar\gamma}{2\pi m_r\omega_0^2}, \]
\[ A_{2c} = \frac{-120\hbar\gamma(2m_r^2\omega_0^2 - m^2(\gamma^2 - 3\omega_0^2))}{2\pi m_r^2\omega_0^2} \]  

It is evident from Eq.(49) that at long times the position autocorrelation function falls off as \( t^{-4} \), which is faster than the \( t^{-2} \) fall off of the position autocorrelation function of a particle coupled to the heat bath via a position coordinate coupling [24].

B. Position-Velocity Correlation Function

The position-velocity correlation function is calculated using Eq.(33) and the long time behaviour is determined at low temperatures.

At finite temperatures, the position velocity correlation function is derived from Eq.(47):

\[ C_{p-v}(t) = -\pi\Omega_{th} A_c(t) \text{exp}[\pi\Omega_{th}t] \]
\[ = A_{c_{p-v}}(t) \text{exp}[\pi\Omega_{th}t] \]  
\[ \text{(50)} \]

where \( A_{c_{p-v}}(t) = -\pi\Omega_{th} A_c(t) \).

At \( T \to 0 \),

\[ C_{p-v}(t) = A_{1c_{p-v}}t^{-5} + A_{2c_{p-v}}t^{-7} + \ldots \]  
\[ \text{(51)} \]

where,

\[ A_{1c_{p-v}} = -4A_{1c}, \]
\[ A_{2c_{p-v}} = -6A_{2c} \]

C. Velocity Autocorrelation Function

The velocity autocorrelation function is calculated using Eq.(38) and the long time behaviour is determined at low temperatures.

At finite temperatures, the velocity autocorrelation function is derived from Eq.(47):

\[ C_v(t) = -\pi^2\Omega_{th}^2 A_c(t) \text{exp}[\pi\Omega_{th}t] \]
\[ = A_{v_{c}}(t) \text{exp}[\pi\Omega_{th}t] \]  
\[ \text{(52)} \]
where, \( A_{c_v}(t) = -\pi^2 \Omega_{th}^2 A_c(t) \).

At \( T \to 0 \),

\[
C_v(t) = A_{1c_v} t^{-6} + A_{2c_v} t^{-8} + \ldots
\]

(53)

where,

\[
A_{1c_v} = 5A_{1c_p} = -20A_{1c},
\]

\[
A_{2c_v} = 7A_{2c_p} = -42A_{2c}
\]

IV. RESULTS AND DISCUSSIONS

In this section, we have plotted and analyzed our results based on the graphical representations of the derived quantities.

We have used dimensionless variables in our graphical representations. In Fig.(1) we have plotted the scaled total response function \( R(t)/R(0) \) as a function of scaled time \( \omega_0 t \) for under-damped and over-damped cases. Figs.(2) and (3) display the mean square displacement (MSD), scaled by the magnetic length squared \( (r_B^2 = \frac{\hbar c}{eB}) \) and the scaled correlation functions at a finite temperature and very low temperature \( (T \to 0) \) respectively for various damping regimes.

---

FIG. 1: The scaled response function \( \left( \frac{R(t)}{R(0)} \right) \) versus time for \( \omega_0 = 1, m=1, m_r=0.5 \): (a) \( \omega_c=10, \gamma = 2 \); (b) \( \omega_c=1.2, \gamma = 10 \)
FIG. 2: The scaled mean square displacement \( \left( \frac{MSD}{r_B^2} \right) \) and scaled correlation function plots as a function of time at a finite temperature for \( \omega_0 = 1 \), \( m = 1 \), \( m_r = 0.5 \), \( \Omega_{th} = 0.03 \): (a) \( \frac{MSD}{r_B^2} \) for \( \omega_c = 10 \), \( \gamma = 2 \); (b) \( \frac{MSD}{r_B^2} \) for \( \omega_c = 1.2 \), \( \gamma = 10 \); (c) \( \frac{C(t)}{C(t_0)} \) for \( \omega_c = 10 \), \( \gamma = 2 \); (d) \( \frac{C(t)}{C(t_0)} \) for \( \omega_c = 1.2 \), \( \gamma = 10 \) in log-log scale; (e) \( \frac{C_{p-v}(t)}{C_{p-v}(t_0)} \) for \( \omega_c = 10 \), \( \gamma = 2 \); (f) \( \frac{C_{p-v}(t)}{C_{p-v}(t_0)} \) for \( \omega_c = 1.2 \), \( \gamma = 10 \) in log-log scale; (g) \( \frac{C_{v}(t)}{C_{v}(t_0)} \) for \( \omega_c = 10 \), \( \gamma = 2 \); (h) \( \frac{C_{v}(t)}{C_{v}(t_0)} \) for \( \omega_c = 1.2 \), \( \gamma = 10 \) in log-log scale. (The orange dashed lines in the plots represent the long time exponential behaviours of the correlation functions).
FIG. 3: The scaled mean square displacement \(\frac{MSD}{r_B^2}\) and scaled correlation function plots as a function of time at very low temperature for \(\omega_0 = 1, m=1, m_r=0.5, \Omega_{th}=0.0005\): (a) \(\frac{MSD}{r_B^2}\) for \(\omega_c=10, \gamma = 2\); (b) \(\frac{MSD}{r_B^2}\) for \(\omega_c=1.2, \gamma = 100\); (c) \(\frac{C(t)}{C(t_0)}\) for \(\omega_c=10, \gamma = 2\); (d) \(\frac{C(t)}{C(t_0)}\) for \(\omega_c=1.2, \gamma = 100\) in log-log scale; (e) \(\frac{C_{p-v}(t)}{C_{p-v}(t_0)}\) for \(\omega_c=10, \gamma = 2\); (f) \(\frac{C_{p-v}(t)}{C_{p-v}(t_0)}\) for \(\omega_c=1.2, \gamma = 100\) in log-log scale; (g) \(\frac{C_v(t)}{C_v(t_0)}\) for \(\omega_c=10, \gamma = 2\); (h) \(\frac{C_v(t)}{C_v(t_0)}\) for \(\omega_c=1.2, \gamma = 100\) in log-log scale (The scaling time \(t_0 = \frac{1}{\Omega_{th}}\)). [The orange dashed lines in the plots represent the long time exponential behaviours of the correlation functions.]
In Fig.(1a), we notice that the Position Response Function in the under-damped regime initially shows an oscillatory behaviour starting from a finite value and then settles to zero with increasing time. On the other hand, the Response Function in the over-damped regime exhibits an exponential decay from \( \frac{2}{m \omega_0^2} \) at \( t=0 \). As discussed in the previous section, this kind of behaviour resembles the Velocity Response Function, obtained as the derivative of the Position Response Function for coupling through position coordinate. The sharp contrast stems from the \( \omega^3 \) dependence of force correlations (see Eq.(8)) in the case of a bath coupled via momentum variables, which is in contrast to the case of a bath coupled via position variables, where the force correlation varies linearly with \( \omega \).

This kind of a behaviour is expected in the case of a bath coupled via momentum variables since the position and momentum are conjugate variables connected by a time derivative.

One noticeable distinctive feature of the plots of the MSD versus \( t \) is the fact that the MSD settles to a constant value (2C(0)) at long times over the entire range of temperatures. This feature survives even when one removes the harmonic confinement. This is strikingly different from the case of a bath with a position coordinate coupling, where the MSD exhibits a linear behaviour with time in the absence of a harmonic potential (i.e. for \( \omega_0 = 0 \)) and settles to a constant value in the presence of a harmonic confinement (i.e. for a nonzero \( \omega_0 \)) [6]. Similar to the Position Response Function, this behaviour of the MSD is attributed to the decay of position autocorrelation function (\( C(t) \)) to zero, with increasing time, due to the fact that the force correlation (see Eq.(8)) goes as \( \omega^3 \).

We have also plotted the correlation functions at finite and \( T \to 0 \) temperatures and analyzed the long time behaviours of the correlation functions, similar to our previously reported work on the quantum Brownian motion of a particle coupled to a bath via a position coordinate coupling [24].

At finite temperatures, the correlation functions display an exponential decay at long times as discussed in the previous section. However, at \( T \to 0 \), we notice a transition from a power law to an exponential behaviour around a time scale set by \( \frac{\hbar}{2\pi k_B T} \) [2, 24]. This can be clearly seen from the above plots, where, the orange dashed lines in (Figs.(1c)-(1h)) represent the long time exponential behaviour at finite temperatures, whilst the orange dashed curves in (Figs.(2c)-(2h)) represent the power law tails at long times for \( T \to 0 \). It can be noticed that the \( t^{-4}, t^{-5}, t^{-6} \) power law decays of the position autocorrelation function (\( C(t) \)), position-velocity correlation function (\( C_{p-v}(t) \)) and velocity autocorrelation function (\( C_v(t) \)) respectively are sharper compared to the case of a position coordinate coupling (\( t^{-2} \) for \( C(t) \), \( t^{-3} \) for \( C_{p-v}(t) \), \( t^{-4} \) for \( C_v(t) \)) [24]. Hence, it can be inferred from the behaviours of the MSD and the correlation functions, that the effect of harmonic confinement is more pronounced for a particle coupled to the heat bath via momentum coordinate coupling. It may also be noted that, the \( t^{-4} \) decay of the position autocorrelation function in the case of a coupling to the bath via a momentum coordinate is exactly same as the \( t^{-4} \) power law decay of the velocity autocorrelation function in the case of a coupling to the bath via a position coordinate coupling.

V. CONCLUSION

We have derived the mean square displacement, position response function and the position autocorrelation, position-velocity correlation and velocity autocorrelation functions of a charged harmonically confined Quantum Brownian particle in the presence of a magnetic field and attached to a heat bath via a momentum coordinate coupling. There are quite a few studies on the case of a momentum coordinate coupling to a bath of a Quantum Brownian particle [16–23]. Nevertheless, the study of a system coupled to a heat bath via a momentum coordinate coupling still remains a relatively unexplored area in the context of Quantum Brownian motion, in particular in the context of a charged harmonically confined Quantum Brownian particle coupled to a heat bath. In [23] the Quantum Langevin Equation for a charged harmonic oscillator coupled to a heat bath via a momentum coordinate coupling and in the presence of a magnetic field has been derived. We have, however, gone far beyond that and derived the MSD, the position response function and the correlation functions and analyzed their long time behaviours at finite and near zero temperatures.

Here, we notice two distinctive features in the behaviours of the MSD and position response function (\( R(t) \)), which can be seen in Figs.(1a,b), (2a,b) and (3a,b). The MSD settles down to a constant value at long times, which persists even in the absence of a confining potential, whereas, \( R(t) \) shows an exponential decay, starting from a finite value at \( t=0 \). These behaviours of the MSD and \( R(t) \) are in sharp contrast to the case of a particle coupled to a bath via a position coupling [7] and the origin of these anomalous behaviours can be traced to the \( \omega^3 \) dependence of the force correlation functions as given in Eq.(8). Moreover, it can be surmised that the position response function in the case of a coupling via a momentum coordinate behaves as the velocity response function for the conventional case of a particle coupled to a bath via a position coordinate coupling, as position and momentum are conjugate variables connected by a derivative.

Furthermore, the formalism discussed here can be applied to other bath models and for generalized forms of couplings, for instance for a coupling strength varying as \( \omega^3 \) at low frequency for various values of \( \delta \) and at all temperatures [4]. The change in the low frequency behaviours of the coupling strength is supposed to change the low frequency correlation function and the long time behaviours as we notice the anomalous behaviours of the
MSD and R(t), due to the $\omega^3$ dependence of the force correlation in the Fourier domain. This type of anomalous coupling can be realized practically, for example, when one considers the effect of black body electromagnetic field in Josephson junction [18, 29].

We expect the results of our study to induce the experimentalists to test them in a state of the art cold atom laboratory, along the lines of the experiments carried out for anomalous diffusion of cold atoms [30–33].

**APPENDIX**

The position autocorrelation function is given by:

$$C(t) = C_1(t) + C_2(t)$$

where,

$$C_1(t) = (-i) \sum_{j=1}^{4} \left( \text{Res}[C(\omega)\exp(-i\omega t), p_j] \right)$$

$$C_2(t) = (-i\hbar\omega_{th}) \sum_{n=1}^{\infty} R''(-in\pi\omega_{th})\exp(-n\pi\omega_{th}t)$$

$\text{Res}[C(\omega)\exp(-i\omega t), p_j]$ is the residue of $C(\omega)\exp(-i\omega t)$ corresponding to pole at $p_j$.

\[
I_1 = \text{Res}[C(\omega)\exp(-i\omega t), p_1] = \left[ \frac{e^{it\left|m\omega_c+i\gamma\right|^2+\sqrt{4m_0^2\omega_0^2+m_0^2(\omega_c+i\gamma)^2}}}{2m_r} + \frac{2im_r^2\omega_0^2 + m \left( \sqrt{4m_0^2\omega_0^2+m_0^2(\omega_c+i\gamma)^2} - m(\omega_c+i\gamma) \right) \left( \frac{\gamma m}{\omega_0^2} \right)}{8mm^2\gamma \sqrt{4m_0^2\omega_0^2 + m_0^2(\omega_c+i\gamma)^2}} \right] \times \coth \left( \frac{\sqrt{4m_0^2\omega_0^2 + m_0^2(\omega_c+i\gamma)^2} - m(\omega_c+i\gamma)}{2m_r\omega} \right)
\]

\[
I_2 = \text{Res}[C(\omega)\exp(-i\omega t), p_2] = \left[ \frac{ie^{it\left|m\omega_c+i\gamma\right|^2+\sqrt{4m_0^2\omega_0^2+m_0^2(\omega_c+i\gamma)^2}}}{2m_r} + \frac{2m_r^2\omega_0^2 + m \left( \sqrt{4m_0^2\omega_0^2+m_0^2(\omega_c+i\gamma)^2} + m(\omega_c+i\gamma) \right) \left( \frac{\gamma m}{\omega_0^2} \right)}{8mm^2\gamma \sqrt{4m_0^2\omega_0^2 + m_0^2(\omega_c+i\gamma)^2}} \right] \times \coth \left( \frac{\sqrt{4m_0^2\omega_0^2 + m_0^2(\omega_c+i\gamma)^2} + m(\omega_c+i\gamma)}{2m_r\omega} \right)
\]

\[
I_3 = \text{Res}[C(\omega)\exp(-i\omega t), p_3] = \left[ \frac{e^{it\left|m\omega_c-i\gamma\right|^2+\sqrt{4m_0^2\omega_0^2+m_0^2(\omega_c-i\gamma)^2}}}{2m_r} + \frac{2im_r^2\omega_0^2 + m \left( \sqrt{4m_0^2\omega_0^2+m_0^2(\omega_c-i\gamma)^2} + m(\omega_c-i\gamma) \right) \left( \frac{\gamma m}{\omega_0^2} \right)}{8mm^2\gamma \sqrt{4m_0^2\omega_0^2 + m_0^2(\omega_c-i\gamma)^2}} \right] \times \coth \left( \frac{\sqrt{4m_0^2\omega_0^2 + m_0^2(\omega_c-i\gamma)^2} + m(\omega_c-i\gamma)}{2m_r\omega} \right)
\]
Therefore, the position autocorrelation function is given by:

\[
C(t) = (-i) \left( I_1 + I_2 + I_3 + I_4 \right) (-i\hbar \omega_{ih}) \sum_{n=1}^{\infty} R''(-in\pi\omega_{ih}) \exp(-n\pi\omega_{ih}t) \\
= (-i) \left( I_1 + I_2 + I_3 + I_4 \right) + \frac{\hbar \pi^2 \omega^4_{ih}}{m \nu_\theta} \left( e^{n\pi\omega_{ih}t} + e^{i n\pi\omega_{ih}t} \right) - \frac{\hbar \pi^4 \omega^6_{ih} \left( 2m^2 \omega^2_0 - m^2 (\gamma^2 - 3\omega^2) \right)}{m \nu_\theta} e^{n\pi\omega_{ih}t} + \ldots
\]

(61)

References:
[1] H. Grabert, U. Weiss, and P. Talkner, Quantum theory of the damped harmonic oscillator. Z. Physik B - Condensed Matter 55, 87 (1984).
[2] R. Jung, G. Ingold, and H. Grabert. Long-time tails in quantum brownian motion. Phys. Rev. A 32, 2510 (1985).
[3] S. Sinha, Decoherence at absolute zero. Physics Letters A 228, 1 (1997).
[4] C. Aslandul, N. Pottier, and D. Saint-James, Quantum dynamics of a damped free particle. J. Phys. France 48, 1871 (1987).
[5] G. W. Ford, J. T. Lewis, and R. F. O’Connell. Quantum langevin equation. Phys. Rev. A 37, 4419 (1988).
[6] U. Satpathi and S. Sinha, Quantum Brownian motion in a magnetic field. Transition from monotonic to oscillatory behaviour. Physica A: Statistical Mechanics and its Applications 506, 692 (2018).
[7] S. Bhattacharjee, U. Satpathi, and S. Sinha. Quantum langevin dynamics of a charged particle in a magnetic field. Response function, position–velocity and velocity autocorrelation functions. Pramana 96, 53 (2022).
[8] X. L. Li, G. W. Ford, and R. F. O’Connell. Magnetic-field effects on the motion of a charged particle in a heat bath. Phys. Rev. A 41, 5287 (1990).
[9] K. Fa, Anomalous diffusion in a generalized langevin equation. Journal of Mathematical Physics 50, 083301 (2009).
[10] S. Mckinley and H. Nguyen, Anomalous diffusion and the generalized langevin equation. Siam. J Math. Anal 50, 5109–5160 (2018).
[11] G. Didier and H. Nguyen, The generalized langevin equation in harmonic potentials: Anomalous diffusion and equipartition of energy. Communications in Mathematical Physics (2022).
[12] V. Lisý and J. Tóthová, Generalized langevin equation and the fluctuation-dissipation theorem for particle-bath systems in a harmonic field. Results in Physics 12, 1212 (2019).
[13] J. Tóthová and V. Lisý, Brownian motion in a bath affected by an external harmonic potential. Physics Letters A 395, 127220 (2021).
[14] J. Berner, B. Müller, J. Gomez-Solano, M. Krüger, and C. Bechinger, Oscillating modes of driven colloids in overdamped systems. Nature Communications 9, 999 (2018).
[15] J. O. Daldrop, B. G. Kowalk, and R. R. Netz. External potential modifies friction of molecular solutes in water. Phys. Rev. E 7, 041065 (2017).
[16] A. Caldeira and A. Leggett, Path integral approach to quantum brownian motion. Physica A: Statistical Mechanics and its Applications 121, 587 (1983).
[17] A. J. Leggett, Quantum tunneling in the presence of an arbitrary linear dissipation mechanism. Phys. Rev. B 30, 1208 (1984).
[18] A. Cuccoli, A. Fabini, V. Tognetti, and R. Vaia. Quantum thermodynamics of systems with anomalous dissipative coupling. Phys. Rev. E 64, 066124 (2001).
[19] Z.-W. Bai, J.-D. Bao, and Y.-L. Song. Classical and quantum diffusion in the presence of velocity-dependent coupling. Phys. Rev. E 72, 061105 (2005).
[20] J.-D. Bao and Y.-Z. Zhuo, Anomalous dissipation: Strong non-markovian effect and its dynamical origin. Phys. Rev. E 71, 010102 (2005).
[21] J.-D. Bao, Y.-Z. Zhuo, F. A. Oliveira, and P. Hänggi, Intermediate dynamics between newton and langevin. Phys. Rev. E 74, 061111 (2006).
[22] J. Ankerhold and E. Pollak, Dissipation can enhance
quantum effects, Phys. Rev. E 75, 041103 (2007).

[23] S. Gupta and M. Bandyopadhyay, Quantum langevin equation of a charged oscillator in a magnetic field and coupled to a heat bath through momentum variables, Phys. Rev. E 84, 041133 (2011).

[24] S. Bhattacharjee, U. Satpathi, and S. Sinha, Long time tails in quantum brownian motion of a charged particle in a magnetic field, arXiv: 2201.09712 (2022).

[25] B. U. Felderhof, On the derivation of the fluctuation-dissipation theorem, Journal of Physics A: Mathematical and General 11, 921 (1978).

[26] J. Weber, Fluctuation dissipation theorem, Phys. Rev. 101, 1620 (1956).

[27] U. Satpathi and S. Sinha, Non-equilibrium quantum langevin dynamics of orbital diamagnetic moment, Journal of Statistical Mechanics: Theory and Experiment 2019, 063106 (2019).

[28] C. F. Bohren, What did Kramers and Kronig do and how did they do it?, European Journal of Physics 31, 573 (2010).

[29] F. Sols and I. Zapata, Effect of qed fluctuations on the dynamics of the macroscopic phase, in New Developments on Fundamental Problems in Quantum Physics, edited by M. Ferrero and A. van der Merwe (Springer Netherlands, Dordrecht, 1997) pp. 403–413.

[30] G. Afek, J. Coslovsky, A. Courvoisier, O. Livneh, and N. Davidson, Observing power-law dynamics of position-velocity correlation in anomalous diffusion, Phys. Rev. Lett. 119, 060602 (2017).

[31] B. G. Klappauf, W. H. Oskay, D. A. Steck, and M. G. Raizen, Experimental study of quantum dynamics in a regime of classical anomalous diffusion, Phys. Rev. Lett. 81, 4044 (1998).

[32] Y. Sagi, M. Brook, I. Almog, and N. Davidson, Observation of anomalous diffusion and fractional self-similarity in one dimension, Phys. Rev. Lett. 108, 093002 (2012).

[33] T. H. Solomon, E. R. Weeks, and H. L. Swinney, Observation of anomalous diffusion and lévy flights in a two-dimensional rotating flow, Phys. Rev. Lett. 71, 3975 (1993).