Optical chirality exhibited by two axially propagating electromagnetic waves in counter-rotations

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The optical chirality of two axially propagating electromagnetic waves is investigated. These two waves of different nature are in counter-rotations, thereby being non-plane waves. From the resulting spatial distributions of energy and chirality, we find not only enhancements but also cancellations due to the interferences in these hybrid waves. In addition, the roles of the axial wave number and interference phase are illustrated by introducing a proper figure of merit for relevant device performances.

Keywords: optical chirality; counter-rotation; hollow cylinder; axial propagation; interference; device performance

1. Introduction

Perhaps, the simplest pair of scalar parameters characterizing photons is their energy density and optical chirality.\cite{1-4} The energy density per photon characterizes how strong electromagnetic (EM) waves are. On the other hand, optical chirality is one measure of asymmetry of the EM fields. As such, optical chirality provides a way of communicating with material chirality when it comes to light–matter interactions.\cite{5-7} In this study, we examine optical chirality of a particular non-plane-wave field,\cite{2} and its relationships to possible applications in nanotechnology.

Suppose in Figure 1 that two laser beams of a well-defined frequency and of a finite radial extent start respectively from the left and right bottom sides. Rotations are imparted to both waves, for instance, by spiral phase plates (not shown here).\cite{8-10} Afterward, the rotating waves from the lower left side pass through a polarizing lens $P_{TM}$ so that transverse magnetic (TM) waves emerge. These TM waves are then reflected by the mirror $M_{TM}$, thus being illuminated upward. Likewise, the rotating waves from the lower right side pass through another polarizing lens $P_{TE}$ so that transverse electric (TE) waves emerge. These TE waves are then reflected by the mirror $M_{TE}$, thus being illuminated upward as well. The two mixed hybrid waves are assumed to be confined to the hollow cylinder under the assumption of no radial diffusions.\cite{7,11} The interior free space is separated from the exterior dielectric medium by an interface located at $r = R$.\cite{9,12}

\footnotesize

\begin{itemize}
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\end{itemize}

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We remark that the TE and TM waves as defined here are those employed in our previous study. However, the TE and TM waves are defined oppositely in most (but not all) of the existing references. In addition, the TE and TM modes considered here are the radially and azimuthally polarized Bessel beam modes. Our formulations will be scale independent, for instance, roughly over 3 orders of magnitudes. As a typical pair of scales, the operating frequency of EM waves is 3 terahertz, and the diameter of the cylinder shown in Figure 1 is one wavelength or 100 micrometers. Although beams of finite radial extents are under the current investigation, our configuration is different from that of optical fibers. It is because we suppose the hollow cylinder to be a sort of working chamber, say, for single molecules.

This study complements our recent one, where interactions have been investigated between two rotating but axially standing (non-propagating) waves. Such standing waves can be formed, for instance, when a sensing beam propagates in the direction oppositely to an illuminating beam in a configuration of two counter-propagating plane waves. Notice that the transverse direction in Figure 1 refers to the direction normal to the rotations so that the axial direction is considered to be transverse. Hence, the TE and TM waves have nonzero electric and magnetic components in the axial direction, respectively. This definition for wave polarizations is made specific to our study, since axial propagations in this study are added to the rotational but axial standing waves in Lee and Mok. Besides, we remark that the non-reciprocity between the TE and TM waves plays a great role.

Our analytical results are expected to provide fundamental understanding on both manipulations of nanoscale objects and sensing of enantiomers. The asymmetry implied by the optical chirality is related to the angular momentums of photons, which form the operating principles of vortex gears, optical tweezers, stretchers, rotators, and spanners. The Maxwellian stresses by photons exert forces and torques on nano-objects.

Figure 1. A schematic for the TE–TM hybrid waves propagating upwardly through a hollow cylinder. Note: The axial direction is along the $z$-axis, whereas the “in-plane” refers to the $xy$-plane. The superscript “$<$” refers to the interior of a hollow cylinder, whereas the superscript “$>$” refers to the exterior. An interface lies between the interior and exterior.
facilitate manipulations or trapping of nano-objects.[5–7,11] From this perspective, we propose a figure of merit (FOM) for enantiomer-sensing devices by incorporating the strength of axial propagations in addition to the energy and chirality of photons.[3,9] Notwithstanding, the mechanical actions necessary for sorting enantiomers and chiral molecules require additional devices and techniques,[1,3] thus lying out of the main scope of this study. In this aspect, only cursory remarks will be made concerning the enantiomer separation.

Rotational waves play an important role in enabling optical isolators and circulators.[14] They are also related to the torsion mode, where radiation forces are exerted on wires or rods (either space-fixed or moving) of nano-sized diameters.[5,6] Meanwhile, our analytical results corroborate the importance of interference taking place between two resonators,[15] in terms of electric multipoles.[1,2] The interactions between the axial and azimuthal components are found to play a significant role in the presence of electronic excitations as well.[16]

This paper is organized as follows. Section 2 presents basic formulas. Section 3 offers key results for the energy density. Section 4 discusses optical chirality. Section 5 deals with device performance. Section 6 provides discussions, followed by conclusion. This paper is made self-contained by providing all the intermediate and detailed steps in the Supplemental data. In this respect, consulting [13] is recommended but not required for details.

2. Fundamentals

Figure 1 shows both electric field \( \vec{E} \) and magnetic field \( \vec{H} \), which depend on the cylindrical coordinates \((r, \theta, z)\) and time \(t\). The EM field vectors are decomposable into two orthogonal modes: (i) the TE waves with \( \vec{E} = (0, 0, E_z) \) and \( \vec{H} = (H_r, H_\theta, 0) \), and (ii) the TM waves with \( \vec{E} = (E_r, E_\theta, 0) \) and \( \vec{H} = (0, 0, H_z) \). For each of these separate waves, there will be no axial propagations.[17] Therefore, we could have assumed \( \vec{E}, \vec{H} \propto \exp(\pm im\theta - i\omega t) \), where \( \omega \) is the frequency and the integer \( m \) is the azimuthal mode index (AMI). However, axial propagations are made possible when these two waves are coupled. For such coupled waves, we can hence assume \( \vec{E}, \vec{H} \propto \exp(ik_zz \pm im\theta - i\omega t) \), where \( k_z \) is the axial propagation constant.

By introducing the normalized functions \((f_r, f_\theta, f_z)\) and \((h_r, h_\theta, h_z)\), the total fields are constructed by superposing the TE and TM waves as follows.

\[
\begin{align*}
\vec{E} & = \frac{1}{\sqrt{1+|q|^2}} (f_r e^{im\theta}, f_\theta e^{im\theta}, q f_z e^{-im\theta}) \\
\vec{H} & = \frac{1}{\sqrt{1+|q|^2}} (g h_r e^{-im\theta}, g h_\theta e^{-im\theta}, h_z e^{im\theta})
\end{align*}
\]

As usual, \( \varepsilon \) is the permittivity. Hence, \( \varepsilon \equiv n^2 > 0 \) for dielectric media with \( n \) as the refractive index. Besides, \( \mu = 1 \) is the permeability for non-magnetic materials. In this way, both Gaussian and SI units can be handled without introducing additional normalizing factors. In addition, the complex \( q \) is the coupling coefficient between the TE and TM waves. Hence, the two limits \( q \to 0 \) and \( q = |q| \to \infty \) refer respectively to the pure TM and pure TE waves. The phase of \( q \) is a controllable parameter.[16] Referring to Figure 1, the variables in the interior and exterior are denoted by the superscripts “<” (meaning \( r < R \)) and “>” (meaning \( r > R \)), respectively. For instance, \( \varepsilon^< \) and \( \varepsilon^> \) are singly denoted by \( \varepsilon^{<>}. \)
Hence, the TE and TM waves are assumed to rotate respectively clockwise and counterclockwise.\[14] In a three-dimensional picture, the two respective waves undergo right- and left-handed helical propagations.\[2,3,7,9,11,12] See Part I of Supplemental data and \[13]\ for details.

Additionally, it is expedient to define the following scaled parameters and variables.

\[
k \equiv \frac{\omega}{c}, \quad k_d \equiv \frac{k}{k_d}, \quad \bar{k}_z \equiv \frac{k}{k_d}, \quad \bar{k} \equiv \sqrt{1 - k_d^2},
\]

\[
\rho \equiv kr, \quad \bar{\rho} \equiv r\sqrt{n^2k^2 - k_d^2}, \quad \bar{R} \equiv R\sqrt{n^2k^2 - k_d^2}.
\]

Here, \(c\) is the light speed in vacuum. Notice that the free-space wave number \(k \equiv \omega/c\) does not carry any subscript. Although the full notation \((n^</>, \bar{k}^</>, \bar{\rho}^</>, \bar{R}^</>\)

is more proper depending on the interior and exterior, its short-hand notation \((n, \bar{k}_z, \bar{\rho}, \bar{R})\) will be employed in most of the analytic formulas for simplicity.

From Maxwell’s equations \(-i\varepsilon \hat{E} = \nabla \times \hat{H} \) and \(i\mu \hat{H} = \nabla \times \hat{E} \), both in-plane components \((f_z, f_0)\) and \((h_r, h_0)\) are expressible in terms of the two axial components \((f_z, h_z)\), the scalar Hertz potentials.\[5,7\] With the Helmholtz operator \(\nabla^2_m \equiv d^2/d\rho^2 + \rho^{-1}d/d\rho + \varepsilon \mu - k_0^2 - (m/\rho)^2\), we obtain both \(\nabla^2_m f_z = 0\) and \(\nabla^2_m h_z = 0.\[11\] Let us further introduce the normalized field variables \(F_m^</> (\bar{\rho})\) as follows.

\[
\begin{cases}
    F_m^< (\bar{\rho}) \equiv \frac{J_m(\bar{\rho})}{J_n(\bar{R}^</>)}, & r < \bar{R} \\
    F_m^> (\bar{\rho}) \equiv \frac{H_n^{(1)}(\bar{\rho})}{H_n^{(1)}(\bar{R}^</>)}, & r > \bar{R}.
\end{cases}
\]

Here, \(J_m(\bar{\rho})\) is the Bessel function of first kind suitable for the interior so that \(F_m^< (\bar{\rho})\) takes real values. In comparison, \(H_n^{(1)}(\bar{\rho})\) is the Hankel function of first order suitable for the exterior, thereby indicating the radiation loss to the free space. In addition, we introduce the following logarithmic field gradient, thus representing inhomogeneous fields.\[1,3,14\]

\[
G_m^</> (\bar{\rho}) \equiv \frac{d}{d\rho} \ln \left[ F_m^</> (\bar{\rho}) \right] = \frac{1}{F_m^</> (\bar{\rho})} \frac{d}{d\rho} F_m^</> (\bar{\rho}).
\]

With the two continuity relations \((E_z)_R^< = (E_z)_R^>\) and \((H_z)_R^< = (H_z)_R^>\) across the interface between the interior and exterior, we arrive at a pair of solutions \(f_z(\bar{\rho}) = F_m^</> (\bar{\rho})\) and \(h_z(\bar{\rho}) = NF_m^</> (\bar{\rho})\). Here, the refractive index ratio \(N\) is defined as follows.\[6,14\]

\[
N = \begin{cases}
    1, & r < \bar{R} \\
    n^{-}/n^{+}, & r > \bar{R}.
\end{cases}
\]

In the meantime, the excited waves within the interior entail the following set of jump conditions across the interface.\[12\]

\[
\begin{cases}
    \left( \frac{dE_z}{d\rho} \right)_R^> - \left( \frac{dE_z}{d\rho} \right)_R^< = A \\
    \frac{1}{\varepsilon} \left( \frac{dH_z}{d\rho} \right)_R^+ - \frac{1}{\varepsilon} \left( \frac{dH_z}{d\rho} \right)_R^- = B
\end{cases}
\]

Here, the two parameters \(A\) and \(B\) account for what should take place across the interface in response to the two illuminated EM waves. The detailed expressions for \(A\) and \(B\) could involve, for example, an electrodynamics within the thin layer of finite
shows the scaled thickness.[16] Although their exact forms do not affect the ensuing discussions in this study, they are derived in Part III of Supplemental data.

3. Energy density

Consider the following energy density of EM waves,[1,6,13]

$$w = \frac{1}{(2n)^2} \left( \varepsilon |\vec{E}|^2 + \mu |\vec{H}|^2 \right).$$  \hspace{1cm} (7)

See [2] for the divisor $(2n)^2$ in this equation. The energy density $\hat{w}$ for the co-rotational case has been previously found in Lee and Mok [13] to be

$$\hat{w} = \frac{1}{4} \frac{1}{1 + |q|^2} |F_m^{\subset}/|^2 \left[ \frac{N^2 + |q|^2}{\frac{k_x^2}{k_x^2 + N^2} + |q|^2(N^2 + k_x^2 |q|^2)} \left( \frac{m^2}{n^2} + |G_m^{\subset}|^2 \right) \right].$$  \hspace{1cm} (8)

Hence, $\hat{w}$ depends strongly on the modulus $|q|$ of $q$, but not on its phase. For $k_z \neq 0$, the energy density is highly anisotropic. In comparison, for the counter-rotational case,

$$w = \hat{w} + \frac{m}{\rho} \frac{N}{k_x^4} |F_m^{\subset}/|^2 \text{Im}(G_m^{\subset})^2.$$  \hspace{1cm} (9)

The additional term in Equation (9) results from the interference between the two fields, but it is independent of $q$. See Section S5 of Supplemental data for detailed derivations.

The following numerical results are produced with the pair of refractive indices $n^x = 1$ and $n^z = 2$, thus modeling a hollow cylinder carved out from a solid dielectric medium. In this way, nano-objects placed within the hollow cylinder are free to move. The radius of the cylindrical interface is chosen to be $kR = 5$ so that EM waves of just enough spatial features can be sculpted.[1,3] In addition, we focus on the counter-rotational case in this study.[13] Moreover, we are mostly concerned with $m = 1$, which is the non-trivial and lowest AMI. Most of the time, $q = 2^{-1/2}(1 + i)$ is assigned so that $|q| = 1$. Notice that the range $1 < k_{z0}$ referring to evanescent waves is not considered here. Since $n^x = 1$ and $n^z = 2$, the condition $0 < k_{z0} < 1$ for the guided waves leads to the following different ranges of $k = \sqrt{1 - k_z^2}$ in the interior and exterior, respectively.

$$\begin{cases} 
0 < \sqrt{1 - k_{z0}^2} \leq 1, & r < R \\
\sqrt{0.75} < \sqrt{1 - \frac{(k_{x0})^2}{2}} < 1, & r > R
\end{cases}$$  \hspace{1cm} (10)

Figure 2 shows the scaled $w^{1/4}$ rather than $w$ for a better visual recognition. Both panels are plotted against $\rho = kr = \omega r / c$, since they are independent of $\theta$. It is prescribed that $q = 2^{-1/2}(1 + i)$. The five curves in Figure 2(a) are obtained for $m = 0, 1, 2, 3,$ and 4 with a fixed $k_{z0} = 0.75$. In comparison, the five curves in Figure 2(b) are obtained for $k_{z0} = 0.1, 0.25, 0.5, 0.75,$ and $0.9$ with $m = 1$. As a result, the red dotted curve marked with $m = 1$ in Figure 2(a) is identical to the similar curve marked with $k_{z0} = 0.75$ in Figure 2(b).

We find in Figure 2(a) a nonzero on-axis energy at $\rho = 0$ in the interior not only for $m = 0$ but also for $m = 1$.[10] Therefore, the axis is singular only for $|m| > 1$. Equally important is the existence of energy extrema in the interior. For instance, the vertical downward arrow points to a single local minimum for the curve with $m = 1$. On the
other hand, $w_m^e$ in the exterior exhibits little variations with differing $m$, while it decreases monotonically with $\rho \to \infty$. Figure 2(b) exhibits that $w_m^e$ in the interior undergoes a relatively large variation with $k_{z0}$, particularly, as $k_{z0} \to 1$. This issue will be further discussed later with Figure 5 and Movie 1 (Movie 1 given in supplemental data). In comparison, $w_m^e$ in the exterior decreases rather uniformly as $\rho \to \infty$ with a negligible dependence on $k_{z0}$ as in panel (a).

Figure 3 displays three-dimensional relief plots of $w^{1/4}$. They are drawn on the $(kx, ky)$-plane over $−7 \leq kx, ky \leq 7$. The prescribed data are $q = 2^{-1/2}(1 + i)$ and $k_{z0} = 0.75$. Here $w^{1/4}$ is independent of the azimuthal direction so that these contours are azimuthally symmetric. The AMI is prescribed such that $m = 1$ in Figure 3(a) and $m = 3$ in Figure 3(b). As mentioned for Figure 2(a), the on-axis energy does not vanish in the hat-like Figure 3(a), whereas it takes on the zero value in the funnel-like Figure 3(b). Suppose inside the hollow cylinder on each panel that there are two

Figure 3. Three-dimensional relief plots of the energy density $w^{1/4}$ drawn on the $(kx, ky)$-plane. The prescribed data are $q = 2^{-1/2}(1 + i)$, $k_{z0} = 0.75$, and (a) $m = 1$ and (b) $m = 3$. 

Figure 2. The energy density $w^{1/4}$ plotted against $\rho = kr = \omega r/c$ with $q = 2^{-1/2}(1 + i)$. (a) For $k_{z0} = 0.75$, curves are drawn for $m = 0, 1, 2, 3, 4$; and (b) For $m = 1$, curves are drawn for $k_{z0} = 0.1, 0.25, 0.5, 0.75, 0.9$. 

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nano-particles as indicated by two small circles with red boundaries. Let us further assume that such nano-particles are transported by the actions of the optical energy.\cite{9}

The two nano-particles are likely to follow the respective trajectories of broken curves, thereby ending up at the locations indicated by the arrows. Therefore, nano-particles are likely to be settled at the radially finite positions in Figure 3(a), whereas they would be attracted onto the axial location.\cite{9}

According to the forthcoming Figures 4–6, each particle should possess a distinct optical chirality depending its location. In this way, enantiomers exhibiting different chiralities can be sensed and maybe sorted out with additional provisions.

4. Optical chirality

The optical chirality is given by

$$C \equiv -(2n)^{-1} \text{Im}\left( \frac{\mathbf{E}^* \cdot \mathbf{H}}{\mathbf{H}^*} \right).$$

\cite{2–4}

The dividing factor $(2n)$ conforms to the similar factor in defining $w$ in Equation (7).\cite{2} In fact, by reverting to $\mathbf{B} \equiv \mu \mathbf{H} \equiv \mathbf{H} + \mathbf{M}$ involving magnetic field and magnetization, we have

$$C \equiv -(2n)^{-1} \text{Im}\left( \frac{\mathbf{E}^*}{\mathbf{H}^*} \mathbf{H}^{-1} \mathbf{B} \right).$$

In words, the optical chirality is one manifestation of the magnetic property of a given medium in response to an applied electric field. With this more general formula, diverse phenomena associated with, say, anisotropic or inhomogeneous magnetic permeability $\mu^{-1}(r, \theta)$ can be handled. In this study, we are however dealing only with isotropic and constant permeability as a first approximation.

Section S6 of Supplemental data shows for the counter-rotational cases that

$$C = \frac{1}{2} \frac{1}{1 + |q|^2} \left| \frac{F_m^<}{1 - k_z^2} \right|^2 \times \left[ N \left( 1 - k_z^2 \frac{|G_m^<|^2}{|G^<|^2} \right) + \frac{2k_z}{1 - k_z^2} (N^2 - 1) \right] \frac{\text{Im}(qe^{-2im\theta})}{\text{Re}(qe^{-2im\theta})} \right]. \tag{11}$$

As a result, $q$ is essential for a nonzero optical chirality. via $q \equiv |q| \exp(i\phi)$, $\text{Re}(qe^{-2im\theta}) = |q| \cos(\phi - 2m\theta)$ and $\text{Im}(qe^{-2im\theta}) = |q| \sin(\phi - 2m\theta)$ in Equation (11), respectively. These even and odd modes imply standing waves in the respective azimuthal directions.\cite{3,8} As expressed by the factor $qe^{-2im\theta}$ in Equation (11), $C$ depends explicitly on the azimuthal angle $\theta$.\cite{7}

![Figure 4](image-url)  

Figure 4. Contour plots of the scaled optical chirality $\text{sgn}(C)|C|^{1/5}$ on the $(k_x, k_y)$-plane for the counter-rotational case. The data are $m = 1$, $q = 2^{-1/2}(1 + i)$, and $k_0 = 0.75$. The signs of $C$ are indicated in several typical locations. Panel (a) spans over a smaller square window $-5 \leq k_x, k_y \leq 5$, whereas panel (b) is made over a larger one $-15 \leq k_x, k_y \leq 15$. 


Figure 4 shows a contour plot of $\text{sgn}(C)|C|^{1/5}$ on the $(kx, ky)$-plane over a smaller square window $-5 \leq kx, ky \leq 5$ in Figure 4(a) and a larger one $-15 \leq kx, ky \leq 15$ in Figure 4(b). It is prescribed that $m = 1$, $q = 2^{-1/2}(1 + i)$, and $k_{z0} = 0.75$. Therefore, Figure 4 corresponds to $C$ discussed for the two dotted curves in Figure 3(a). The color bar on the right of each panel indicates the levels of $\text{sgn}(C)|C|^{1/5}$. On each panel, there are hence both zones of positive (in yellowish colors) and negative (in bluish colors) values. We added both upward and downward arrows on the left of each vertical color bar to clarify the signs of $C$. Notice that $C$ does change its sign across the interior–exterior interface at $kR = 5$.

Instead, there are two kinds of curves in Figure 4, across which the sign of $C$ is altered. Firstly, consider the four straight radial lines, across which $C$ undergoes its sign changes in the azimuthal direction. Hence, there are four pie-shaped zones, each spanning 90° degrees in the azimuthal direction. It is because the pattern repeats itself twice (namely, $2|m|$ times) with $m = 1$ in the azimuthal direction. As seen from Equation (11), none of these four radial lines coincides with the line $\theta = 0$, since $\text{Re}(q) = \text{Im}(q) = 2^{-1/2}$ from the input value of $q = 2^{-1/2}(1 + i)$. In fact, they are shifted by $\phi = 22.5^\circ$ from the respective azimuthal locations obtainable for $q = 1$.

Secondly, consider the white circle at $\rho \equiv kr \approx 1.8$ in Figure 4(a). On each one-quarter pie, $C$ changes from negative to positive values in the radial direction or vice versa. We added a circle in bright blue at $\rho \equiv kr \approx 3.37$ in Figure 4(a), where the local minimum of energy density is located as seen from the downward arrows in Figure 2 on the curves for $m = 1$. This local minimum is related to the function $G_m^{\omega'}(\hat{\rho})$ in Equation (11). Consequently, the nano-objects settled on that circle of $\rho \equiv kr \approx 3.37$ should possess a certain $C$ if their azimuthal angle is additionally specified. Regarding Figure 4(b), $C$ varies mildly as $\rho \equiv kr$ increases. It has been found from more extensive numerical experiments that the sign changes in the radial direction get more frequent with increasing $kR$.

Figure 5 displays three panels with $k_{z0} = 0.5, 0.75$, and 0.99 as written on top of each panel. We arbitrarily set $C = 0$ for $\rho \equiv kr > kR = 5$ in the exterior, since the hollow cylinder is of our major interest. Because $m = 1$ and $q = 2^{-1/2}(1 + i)$, Figure 5(b) is

![Figure 5](image-url)
identical to the interior of Figure 4(a). The differences among the three panels of Figure 5 are as follows. The optical chirality remains same-signed in each one-quarter pie in both Figure 5(a) and (c), whereas it undergoes sign changes even inside each pie as mentioned for Figure 4(a). When comparing Figure 5(a) with Figure 5(c), it is found however that the sign on each pie is opposite to the corresponding one. Throughout the three panels, we notice that the four radial boundaries do not move because of the same $q = 2^{-1/2}(1 + i)$. The situation in Figure 5(b) would be less useful for enantiomer sensing due to its mixed optical chirality in the radial direction. As a reference, we remark that the phase changes between Figures 5(a) and (c) are caused by the difference in the $k_z$.

Movie 1 in supplemental data shows the effects of $k_z$ over $0 \leq k_z < 0.999$ by a sequence of frames, three of which are presented in Figure 5. The increments $\Delta k_z$ are properly adjusted for better visual effects, in order to exhibit three salient features discussed for Figure 4(a). As roughly seen from Movie 1 (Movie 1 given in supplemental data), frames similar to Figure 5(a) prevail over $0 \leq k_z < 0.707$. In comparison, frames similar to Figure 5(c) can be seen over $0.905 < k_z < 1$. In between, they are similar to Figure 5(b). Consequently, the wave-number bandgap $0.707 < k_z < 0.905$ is at least undesirable for enantiomer sensing and possibly separation. It is also called an isolation bandgap.

Movie 2 in supplemental data shows the effects of the coupling coefficient $q = |q| \exp(i\phi)$ with $m = 1$ and $k_{z0} = 0.75$. Here, the magnitude is set at $|q| = 1$, and the phase angle $\phi$ is varied in steps of $5^\circ$ over $0^\circ \leq \phi \leq 360^\circ$. Hence, the frame with $\phi = 45^\circ$ (or $\phi = \frac{\pi}{4}$) in Movie 2 (Movie 2 given in supplemental data) is identical to Figure 5(b), because $q = \exp(\frac{\pi}{4}i) = 2^{-1/2}(1 + i)$. The first frame with $\phi = 0^\circ$ exhibits its distinct distributions of $C$ in each one-quarter pie as explained for Figure 4(a). The whole pattern undergoes just azimuthal rotations as $\phi$ is increased. By the way, the pattern completes one full turn with the last frame with $\phi = 360^\circ$.

In addition, we made a variation in $|q|$ for a fixed phase, say, $\phi = 45^\circ$. In this respect, notice that the term involving $N^2 - 1$ in Equation (11) disappears in the interior where $N = 1$. As a consequence, $C \propto \left(1 + |q|^2\right)^{-1}|q|$ in the interior from Equation (11), thus leading to $|C|$ being maximal at $|q| = 1$. However, $C$ does not show any phase change as $|q|$ is varied.

5. Device performance

Notice in Figure 5 that the maximum in $|C|$ over the whole interior varies from panel to panel as seen by the changing color bars. This uneven order of magnitude gets worse in Movie 1 (Movie 1 given in supplemental data) when we vary $k_{z0}$ over its whole range of $0 < k_{z0} < 1$ in consultation of Equation (10). Therefore, $C$ appears inadequate as a FOM for the performance of an enantiomer separator. Another FOM is the ratio of output to input, namely, $C/w$, being the optical chirality per energy density of illuminating beams. The ratio $C/w$ has been introduced by Ref. [2], as relevant to the enantioselectivity.

A more proper FOM is thus the following average over the cross-sectional area.

$$\left\langle \frac{|C|}{w} \right\rangle_{\text{avg}} = \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} \frac{|C|}{w} \, d\theta \, dr.$$  (12)

Note in particular that $|C|$ is employed within the integrand instead of $C$ to guard against the self-cancellations due to azimuthal periodicity as seen from Figures 4 and 5.
along with Movies 1 and 2 (Movies 1 and 2 given in supplemental data). In this respect, \( C_{TC} \equiv (\omega w)^{-1}C \) has been suggested by Tang and Cohen (TC) as another FOM for enantiomer-sensing devices.[3] Literally, \( C_{TC} \) refers to the optical chirality per energy density modified by the operating frequency. In addition, we find that our \( w \) encompasses both electric and magnetic parts of the energy density on equal footing as in Equation (7), whereas twice the electric part of the energy density is employed in Tang and Cohen [3].

In contrast to the standing waves for sensing devices,[3] we are faced with axially propagating waves for sensing operations. We propose hence the following FOM simply by multiplying \( C/\omega \) with \( k_0z \), which is representative of axial propagations.

\[
\eta = k_0 \frac{C}{\omega} = k_0^C C_{TC}. \tag{13}
\]

Figure 6 displays the three-dimensional relief plots of \( \eta \). They are drawn on the \((kx, ky)\)-plane over \(-7 \leq kx, ky \leq 7\). The prescribed data is \( m = 1, q = 2^{-1/2}(1 + i) \), and \( k_{z0} = 0.75 \). For better visual effects, \( \eta \) in the exterior for \( kr > kR = 5 \) is intentionally set to zero. Now, \( \eta \) exhibits a strong dependence on the azimuthal direction as with Figures 4 and 5. Here, the reddish and bluish colors denote respectively positive and negative values, whereas the stronger colors refer to larger magnitudes. Refer to Figure 3(a) for the corresponding \( w \). The two small circles with attached trajectories denote the nano-objects being settled at the locations pointed to respectively by the arrows at the ends of such trajectories. It is seen from comparing Figures 3(a) and 6 that a particular nano-object ending up at a particular location is associated with a certain optical chirality.

Movie 3 (Movie 3 given in supplemental data) in supplemental data displays a sequence of frames as Figure 6 with increasing values of \( k_{z0} \) at intervals employed for both Movies 1 and 2 (Movies 1 and 2 given in supplemental data). Notice further that \( \max(\eta) = -\min(\eta) \) and the levels are found from numerical computations to lie within the range \(|\eta| < 0.4\) for all the frames in Movie 3 (Movie 3 given in supplemental data).

Meanwhile, the cross-sectional average \( \eta_{avg} \) is defined from \( \eta \) through integrations as for Equation (12). Moreover, it is theoretically and numerically confirmed that \(|C/\omega < 1\), thus giving rise to \( 0 \leq (|C/\omega|)_{avg} < 1 \). As a result, \( 0 \leq \eta < 1 \) and \( 0 \leq \eta_{avg} < 1 \). Consequently, \( \eta_{avg} \) can serve as an efficiency for a device,[9] with \( \eta \) as a local efficiency.
For the counter-rotational case, Figure 7 shows both $C_{jj} = w(\theta)$ and $\eta_{avg}$ in red and blue colors, respectively. Once $C_{jj} = w(\theta)$ is computed, $g_{avg}/C_{17}$ is computed easily by multiplying it with $k_{z0}$. The prescribed data is $kR = 5$, $m = 1$, and $q = 2^{-1/2}(1 + i)$ for the counter-rotational case.

Figure 7. The optical chirality per energy density ($|C|/w)_{avg}$ and the efficiency $\eta_{avg}$ averaged over the cross-sectional disk. Both are plotted against $k_{z0}$ (the lower horizontal scale) and its inverse $(ck_z)^{-1}\omega$ (the upper horizontal scale). The data is $kR = 5$, $m = 1$, and $q = 2^{-1/2}(1 + i)$ for the counter-rotational case.

For the counter-rotational case, Figure 7 shows both $|C|/w)_{avg}$ and $\eta_{avg}$ in red and blue colors, respectively. Once $|C|/w)_{avg}$ is computed, $\eta_{avg} = k_{z0}(|C|/w)_{avg}$ is computed easily by multiplying it with $k_{z0}$. The prescribed data is $kR = 5$, $m = 1$, and $q = 2^{-1/2}(1 + i)$. Both are plotted against $k_{z0}$ on the linear scale drawn below the display box. Because $k_{z0} = k_{z0} = c k_{z0}/\omega$ defined in Equation (2), the inverse-linear scale for the frequency $(ck_z)^{-1}\omega$ is shown above the display box as well. The maximum of $\eta_{avg}$ occurs at $k_{z0} \approx 0.48$ as indicated by the blue circle. For larger $k_{z0}$, both $|C|/w)_{avg}$ and $\eta_{avg}$ are relatively small so that this $k_{z0}$-range is not suitable for good performances of an enantiomer sensing device. The horizontal thick arrow in Figure 7 marks the forbidden $k_{z0}$-bandgap as mentioned for Figure 5, because of the mixed chirality of a given one-quarter pie. It corresponds hence to the frequency bandgap as well.

The trends shown in Figure 7 are appreciably altered (if not all) for co-rotational cases. In addition, we obtain another set of trends as we make changes in the coupling coefficient $q$ for both rotational cases. For space reasons, in-depth discussions on these parametric studies are provided in Section 7 of Supplemental data.

6. Discussions

Our EM fields presented in Figure 1 display a special feature that a nonzero on-axis intensity is obtained for the dipoles with the AMI one. In contrast, the on-axis intensities vanish for higher-order multipoles with larger AMIs, thus meaning the central nulls.[10] These characteristics of our EM fields should have a relevance to the passive nano-particles immersed in our EM fields. It is because the particle’s dipolar resonance is associated with the continuum of frequency dispersions. In comparison, the particle’s multipolar resonances are associated with the peaked frequency dispersions.[15] In this regard, the resonance exhibited by nano-particles with a mixture of dipoles and multipoles should respond differently to our EM fields depending on the latter’s AMIs.[1,2] From the four one-quarter pies displayed in Figures 4–6, we expect quadruple helices as trajectories of the Poynting vector flows.[13] In this respect, the optical chirality is
closely associated with the polarization state.\[4,11,16\] which is in turn related to the photon spins. Both polarizations and spins will be treated in detail by future publications when we deal with the angular momentum of photons with the current wave configuration.\[13\]

As regards device realizations, a number of items should be taken into account.\[3\] Firstly, the finite axial size of the cylinder would give rise to an edge effect. Secondly, sharply confining the two waves within the specified hollow cylinder will be a challenging task.\[16\] Thirdly, all kinds of losses including material dissipation and beam divergence should be accounted for.\[11\] Fourthly, for a specified $q$, the two laser beams as proposed in Figure 1 maintain a certain coherent phase relationship between the two and this relationship (viz. phase purity) be not altered during their respective passages through various optical components.\[14\] A spatial coherence between the two laser beams is assumed as well.

We have examined energy density and optical chirality of EM waves by finding an exact solution to the Maxwell’s equations in the cylindrical geometry. Two waves were assumed to propagate in the same axial direction, while undergoing counter-rotations. We considered coherent and non-dissipative interactions between the transverse electric and magnetic fields. The resulting complicated yet interesting features arise essentially from the spatially inhomogeneous field variables.\[3\] The co-rotational case with axially propagating waves has not been presented here, whereas non-propagating waves in co-rotations have been studied in Lee and Mok \[13\]. In this respect, notice that Supplemental data provides all the formulas for both rotational cases for the sake of completeness. For both rotational cases, the whole set of necessary devices remain the same, except the need of operating the angular momentum-generating parts just in reverse.

Through our study, we hope thus to have given at least a partial answer to the question posed in Yang et al. \[1\]: “Could there be CW (continuous-wave) solutions to Maxwell’s equations that show greater enantioselectivity than do circularly polarized plane waves?” In this respect, the key characteristics of the EM fields under current study are spatial inhomogeneity and non-plane waves.

7. Conclusions
Firstly, the effects of axial propagations are quite different from those of the azimuthal rotations. Secondly, the interference between the transverse electric and magnetic waves exhibits subtle effects concerning the phase angle between them, thereby revealing mutual reinforcements and cancellations. Thirdly, the interior and exterior exhibit quite different characters in their sensitivities to parametric variations. All these finding would contribute to better understanding the nano-objects immersed in EM waves and relevant micro-devices and enantiomer sensors.

Disclosure statement
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