Removing correlations in signals transmitted over a quantum memory channel

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We consider a model of bosonic memory channel, which induces correlations among the transmitted signals. The application of suitable unitary transformations at encoding and decoding stages allows the complete removal of correlations, mapping the memory channel into a memoryless one. However, such transformations, being global over an arbitrary large number of bosonic modes, are not realistically implementable. We then introduce a family of efficiently realizable transformations which can be used to partially remove correlations among errors, and we quantify the reduction of the gap with memoryless channels.

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I. INTRODUCTION

Any physical medium that can store or transfer quantum degrees of freedom can be formally described as a quantum channel, that is, a completely positive and trace-preserving map on the set of quantum states. In information theory, quantum channels are mainly intended as means to convey either classical or quantum information [1]. They model the noisy interaction with the environment, causing losses, dephasing and decoherence. The action of the noise is usually to degrade the information carried by the physical system, hence reducing the performance of quantum communication protocols. A standard strategy to protect the information against noise is the use of quantum error correcting codes [2]. These latter exploit redundancy and dictate the way information should be encoded/decoded in quantum systems. In this scenario, a common assumption is that the noise acts identically and independently at each use of the quantum channel. That is, when a train of signals is sent through the quantum channel, each signal independently experiences the noisy transformation. A quantum channel with this property is said to be memoryless.

Recently, attention has been devoted to quantum memory channels as well (see e.g., [3] for discrete channels and [4] for continuous channels), where the action of the noise at different channel uses is either non indential or non independent. From a physical point of view, that is the case when the typical environmental relaxation times are comparable with the time delay between two signals. From an information theoretical point of view, the presence of correlations in the noise can substantially reduce the efficacy of standard quantum error correcting codes [5].

Then there is a twofold way to deal with information transmission over memory channels: either consider new encoding/decoding procedures according to codes specifically devised for correlated errors [6], or try to remove correlations and then apply standard encoding/decoding procedures. Here, we shall investigate the second line which results unexplored so far. We shall restrict our attention to bosonic memory channels.

In Ref. [7] it has been discussed a technique, called memory unravelling, that allows the removal of correlations in a large class of bosonic memory channels. Actually the channels can be traced back to memoryless ones by global unitary transformations prepended and appended to the channel map. However this is just a mathematical trick useful for evaluating capacities. For practical transmission rates it is unreasonable to realize transformations involving a large number of modes (channel uses). We shall hence introduce unitaries that can be implemented efficiently but only allow a partial reduction of the correlations in the memory channel.

Specifically we shall consider the model of lossy bosonic memory channel introduced and characterized in [8] for which memory unravelling applies. It is known [9] that for the corresponding memoryless attenuating channel a standard encoding/decoding to get close to the classical capacity is given by (single mode) coherent states and dyne measurements [10]. Then, we shall evaluate the performance of such encoding/decoding procedure upon efficient, but partial, removal of correlations. In particular we quantify the reduction of the gap with memoryless channels by means of input-output mutual information.

The article will proceed as follows. In Sec. II we shall introduce the model of bosonic memory channel under scrutiny; in Sec. III we shall consider the unitary pre-processing and post-processing transformations allowing the complete removal of correlations; in Sec. IV we shall introduce an efficient scheme for partially reducing the correlations based on unitaries acting on a finite number of modes. Numerical results about the reduction of the gap with memoryless channels in the case of chaining unitaries acting on two and three modes will be presented in Sec. V. Conclusions will be drawn in Sec. VI.

II. THE MEMORY CHANNEL

As a case study, we consider the Gaussian memory channel introduced and characterized in [8], which models memory effects in linear attenuating media, acting on a set of bosonic modes (e.g., normal modes of the electromagnetic field). In the same spirit of the Refs. [3, 11], the memory channel under scrutiny is defined as a sequence of elementary transformations, which are concatenated through a memory mode. An elementary transformation involves a pair of ingoing bosonic modes, described through the ladder op-
operators \( \{a, a^\dagger\} \) and \( \{c, c^\dagger\} \), and the corresponding outgoing modes \( \{b, b^\dagger\}, \{d, d^\dagger\} \). The modes are coupled via a unitary interaction described by an operator of the form \( U_{\text{BS}} = \exp \left( t_a^a c - h.c. \right) \), modeling an exchange interaction mixing the two modes. In the following we assume the parameter \( \theta \) to be real and positive, yielding the following Heisenberg-picture transformations on the ingoing modes:

\[
\begin{align*}
b &= U_{\text{BS}}^a a U_{\text{BS}} = \cos \theta a - \sin \theta c, \\
d &= U_{\text{BS}}^b c U_{\text{BS}} = \cos \theta c + \sin \theta a,
\end{align*}
\]

(1a, b)

together with their hermitian conjugates. The quantity \( \cos^2 \theta \) is usually referred to as the transmissivity parameter.

To define the action of the quantum memory channel, we consider a sequence of \( n \) consecutive channel uses. Such a sequence is associated with a collection of \( n \) bosonic modes with ladder operators \( \{a_j, a_j^\dagger\} \) representing the channel inputs, and the corresponding channel outputs associated with the ladder operators \( \{b_j, b_j^\dagger\} \). The memory modes, which account for the flux of information from one channel use to the following, are associated to the ladder operators \( \{m_j, m_j^\dagger\} \).

\[
\begin{align*}
b &= \sqrt{\eta} a_k - \sqrt{1 - \eta}(1 - \eta) \sum_{j=1}^{k-1} (\sqrt{\eta})^{k-j-1} a_j \\
&\quad - \sqrt{(1 - \eta)(1 - \eta)} \sum_{j=1}^{k} (\sqrt{\eta})^{k-j} e_j \\
&\quad + \sqrt{\epsilon}(1 - \eta) (\sqrt{\eta})^{-k-1} m_1, \\
&= \sum_j f_{kj} a_j + \sum_j g_{kj} e_j + t_k m_1. 
\end{align*}
\]

(3)

for \( k = 1, \ldots, n \). From Eq. (3) it follows that, for each \( k \), the output mode operator \( b_k \) is a function of the corresponding input mode operator \( a_k \) and of the input mode operators in its past: \( a_j \), with \( k > j \). Hence, a certain amount of information from the \( j \)-th input flows to the \( k \)-th output, for \( k > j \). That is, the input signals from different channel uses interfere at the channel outputs, leading to memory effects in the quantum channel.

Equation (3) can be concisely rewritten in the following form:

\[
b_k = \sum_j f_{kj} a_j + \sum_j g_{kj} e_j + t_k m_1. 
\]

(4)

Finally, to exhaustively define the channel model, we have to fix the initial state of the local environmental modes \( \{e_j, e_j^\dagger\} \). Different choices for the environment states lead to channels with different features. In the following we assume the local environments to be in the vacuum state. With this choice for the environmental states the quantum and classical capacities of the memory channel can be computed exactly [8].

The transmissivity can be related to the ratio between the time delay \( \Delta t \) between to successive channel uses and the typical relaxation time \( \tau_{\text{rel}} \) of the channel environment [12]: for instance we may identify \( \epsilon \approx \exp(-\Delta t/\tau_{\text{rel}}) \). In particular, the model reduces to a memoryless attenuating channel [9] for \( \epsilon = 0 \) (the input \( a_j \) only influences the output \( b_j \)), and to a channel with perfect memory [11] for \( \epsilon = 1 \) (all \( a_j \)’s interacts only with the memory mode). These two limiting settings respectively correspond to the regime \( \Delta t \gg \tau_{\text{rel}} \), and \( \Delta t \ll \tau_{\text{rel}} \). Intermediate configurations are associated with values \( \epsilon \in (0, 1) \) and correspond to “inter-symbol interference” channels, for which the previous input states affect the action of the channel on the current input [13].

### III. REMOVING CORRELATIONS

As it has been shown in [8], a unitary transformation exists which completely remove the correlations in the memory channel. Such optimal transformation can be constructed by considering the coefficients \( f_{kj} \) appearing in Eq. (4), which express the linear relations between the input and output field operators. For any \( n \), these coefficients define a \( n \times n \) matrix.
\[ f, \text{ which admits a singular value decomposition:} \]
\[ f = \mathcal{V}^\dagger \sqrt{\Delta} \mathcal{U}, \tag{5} \]

where \( \mathcal{U}, \mathcal{V} \) are \( n \times n \) unitary matrices and \( \Delta = \text{diag}(\eta_1, \eta_2, \ldots, \eta_n) \) is diagonal with non-negative entries. The unitary matrices play the role of unraveling the correlations. In the Heisenberg picture, it maps the input and output operators in

\[ A_k := \sum_j U_{kj} a_j, \tag{6} \]
\[ B_k := \sum_j V_{kj} b_j. \tag{7} \]

It follows that these collective input and output field operators satisfy the identities

\[ B_k = \sqrt{\eta_k} A_k + \sum_{l,j} V_{kl} g_{lj} e_j + \sum_l V_{kl} l m_1, \tag{8} \]

from which it is apparent that the \( k \)-th collective output signal is only influenced by the \( k \)-th input signal, i.e., the correlations due to the inter-symbol interference have been completely removed. Another source of correlated noise can be represented by the noisy terms associated to the environmental modes with operators \( e_j, m_1 \). However, if these modes are not populated (or, if they are in a thermal state) their presence does not induce additional correlation terms. In fact, Eq. (8) can be written in the following form [3]:

\[ B_k = \sqrt{\eta_k} A_k + \sqrt{1 - \eta_k} E_k, \tag{9} \]

where \( \{ E_k, E_k^\dagger \} \) are effective canonical field operators associated to the environment and memory. Equation (9) describes the bosonic memory channel as the direct product of uncorrelated attenuated channels, characterized by the transmissivities \( \{ \eta_k \} \), which are in turn obtained as the singular value of the matrix \( f \). In this way, we have shown that the memory channel is unitary equivalent to a noisy channel with independent (although non-identical) noise. This mapping has been used in [3] (see also [2]) to compute the capacities of the memory channel.

From an operational point of view, if the optimal transformations are physically implemented, standard error correcting codes, designed for uncorrelated noise, can be applied with high effectiveness. From this point of view, the above unitaries are not only mathematical tools for computing the channel capacities, but they are actual pre-processing and post-processing of information that can be used to improve the performance of standard error correcting codes. Once the unitaries have to be physically implemented, one can pose the question of quantifying the resources needed for their implementation. First of all, one can notice that the above unitaries can be constructed by combining elementary gates coupling at most two bosonic modes (such as beam-splitters and phase-shifters). According to [14], a unitary over \( n \) channel uses require at most \( n(n+1)/2 \) elementary transformations. However, the decomposition provided by [14] requires the action of beam-splitter transformations coupling arbitrary pairs of channel outputs. It is hence clear that to apply these transformations, one first has to wait for \( n \) channel uses and then start the process. For example, for the perfect memory channel, \( \epsilon = 1 \), the memory channel is unitary itself, and it can be perfectly inverted by waiting the \( n \)-th channel output and then reverse its action.

In the following we consider a more efficient though not perfect procedure. Namely, we consider alternative unitary operators constructed by composition of elementary gates in a number scaling linearly with the number of channel uses. For any \( n \), these pre-processing and post-processing unitaries only couple a fixed number \( \ell \) of consecutive input and output modes.

**IV. PARTIAL UNRAVELING OF CORRELATIONS**

We consider a scheme in which, upon \( n \) uses of the channel, pre-processing and post-processing unitaries are prepended and appended to the memory channel. We consider the case of canonical unitaries, hence the input modes are transformed as follows

\[ A_k := \sum_j \mathcal{U}^{(\ell)}_{kj} a_j, \tag{10} \]
\[ B_k := \sum_j \mathcal{V}^{(\ell)}_{kj} b_j. \tag{11} \]

In order to consider unitaries which can be applied in an efficient way, we assume that the pre-processing and post-processing unitaries are obtained as the concatenation of elementary unitary transformations \( U_\ell, V_\ell \), which only couple \( \ell \) consecutive modes, in such a way that \( \ell - 1 \) of the output modes of each elementary transformation enter the next one. This is depicted in Figs. [3][4] for the case of unitaries of \( \ell = 2 \) and \( \ell = 3 \). We refer to the integer \( \ell \) as the depth of the unitary transformations. Denoting \( I_m \) the identity transformation acting on \( m \) consecutive modes, we then have

\[ \mathcal{U}^{(\ell)} = (I_{n-\ell} \otimes U_\ell) \cdots (I_1 \otimes U_\ell \otimes I_{n-\ell}) (U_\ell \otimes I_{n-\ell}) \]
\[ \mathcal{V}^{(\ell)} = (I_{n-\ell} \otimes V_\ell) \cdots (V_\ell \otimes I_{n-\ell}) \]

![FIG. 2: Pre-processing and post-processing of depth \( \ell = 2 \), involving unitary transformations which couple two consecutive bosonic modes at the input of the memory channel (pre-processing), and two consecutive output modes (post-processing).](image-url)
unitaries lead to the input-output relations

$$B_k = \sum_j \hat{f}_{kj} A_j + \sum_j \hat{g}_{kj} e_j + \hat{t}_k m_1 ,$$  
(12)

where $\hat{f}_{kj} = \sum_{i,m} V_{ki} V_{mi} U^{(t)\dagger}_{mj}$, $\hat{g}_{kj} = \sum_{i,j} V_{ki} g_{ij}$, and $\hat{t}_k = \sum_l V_{kl} t_l$.

Then, we can optimize the choice of the unitaries of a given depth, in terms of a suitable performance quantifier. Here we would like to quantify the amount of correlations between the $k$-th output mode and the $k$-th input mode, and between the $k$-th output mode and all the other input modes. In the case in which $U^{(t)} = U$, $V^{(t)} = U$, we have the unraveling of correlations, that is, the correlations between the $k$-th output and $k$-th input are maximal, and the $k$-th output is uncorrelated with the other inputs. Clearly, by increasing the depth of the unitaries, one gets higher and higher performances, since the case of unitaries with depth equal to $\ell$ is obtained as a special case of the unitaries with depth equal to $\ell' > \ell$.

We are going to use the mutual information as a quantifier of correlations. To define the mutual information, we need to consider a specific instance of encoding/decoding procedure for classical communication via the memory channel. From Ref. [9] we know that for the corresponding memoryless at-

For extracting classical information from the quantum states, we consider the heterodyne measurement [10] described by the POVM elements

$$E_{\gamma_k} = \frac{|\gamma_k\rangle\langle\gamma_k|}{\pi} .$$  
(15)

Since, in the chosen encoding scheme, the $k$-th output is $|\beta_k\rangle$, the probability of measuring the amplitude $\gamma_k$ is

$$P(\gamma_k|\beta_k) = \text{Tr}(E_{\gamma_k}|\beta_k\rangle\langle\beta_k|)$$

$$= \frac{|\langle\gamma_k|\beta_k\rangle|^2}{\pi} \simeq \exp \left( -|\gamma_k - \beta_k|^2 \right) .$$  
(16)

We can now compute the probability density that, given the coherent state $|\alpha_k\rangle$ is sent at the $k$-th channel use, the amplitude $\gamma_k$ is measured at the output. That conditional probability density is obtained by integrating the Eq. (16) over all the $\alpha_l$ with $l \neq k$, i.e.,

$$P(\gamma_k|\alpha_k) \simeq \int \exp \left( -|\gamma_k - \beta_k|^2 \right) \prod_{l \neq k} P(|\alpha_l\rangle) d^2\alpha_l .$$  
(17)

We can also compute the joint probability distribution

$$P(\gamma_k, \alpha_k) = P(\gamma_k|\alpha_k) P(|\alpha_k\rangle)$$

$$= \int \exp \left( -|\gamma_k - \beta_k|^2 \right) \prod_l P(|\alpha_l\rangle) d^2\alpha_l .$$  
(18)

Using the explicit expression of the amplitude, $\beta_k = \sum_l \hat{f}_{kl} \alpha_l$, it is straightforward to compute the Gaussian integrals in Eqs. (17), (18).

From the probability distribution $P(\gamma_k, \alpha_k)$, it is easy to calculate the mutual information of variables $\gamma_k$ and $\alpha_k$. The mutual information is (with a bit of abuse of notation)

$$I_k = H[P(\gamma_k)] + H[P(\alpha_k)] - H[P(\gamma_k, \alpha_k)] ,$$  
(19)

where $H$ denotes the Shannon entropy, and

$$P(\gamma_k) = \int P(\gamma_k, \alpha_k) P(|\alpha_k\rangle) d^2\alpha_k .$$  
(20)

Analogously, one can compute the mutual information between the measured amplitude at $k$-th output and the amplitude of the coherent states $\alpha_h$ with $h \neq k$. In order to do that, let us denote $\tilde{\alpha}_k$ the complex vector whose entries are the amplitudes $\{\alpha_l\}_{l \neq k}$. Hence, the conditional probability that the output measured amplitude has value $\alpha$ is

$$P(\gamma_k|\tilde{\alpha}_k) \simeq \int \exp \left( -|\gamma_k - \beta_k|^2 \right) P(|\alpha_k\rangle) d^2\alpha_k ,$$  
(21)

from which

$$P(\gamma_k, \tilde{\alpha}_k) = P(\gamma_k|\tilde{\alpha}_k) \prod_{l \neq k} P(|\alpha_l\rangle) .$$  
(22)

These Gaussian probability density distributions can explicitly written in terms of the matrix coefficients $f_{kl}$ in Eq. (12). The mutual information between the measured amplitude $\gamma_k$
at the $k$-th channel output and the all the input amplitudes $\alpha_i$, with $l \neq k$, is
\[
I'_k = H[P(\gamma_k)] + H[P(\tilde{\alpha}_k)] - H[P(\gamma_k, \tilde{\alpha}_k)].
\] (23)

Explicitly, we have
\[
I_k = \frac{1}{2} \left[ \log_2 M_{11} + \log_2 M_{22} - \log_2 \det \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \right],
\] (24)
\[
I'_k = \frac{1}{2} \left[ \log_2 M_{11} + \log_2 M_{33} - \log_2 \det \begin{pmatrix} M_{11} & M_{13} \\ M_{31} & M_{33} \end{pmatrix} \right],
\] (25)
where $M_{ij}$ are the elements of the matrix
\[
M = \begin{pmatrix} 1 + N(f_{kk}^2 + \mu^2) & Nf_{kk} & N \mu \\ Nf_{kk} & N & 0 \\ N\mu & 0 & N \end{pmatrix},
\] (26)
with
\[
\mu = \sqrt{\sum_{j \neq k} |f_{kj}|^2}.
\] (27)

By increasing the value of $k$, the mutual informations quickly converge to the limiting functions
\[
I := \lim_{k \to \infty} I_k, \quad I' := \lim_{k \to \infty} I'_k.
\] (28) (29)

Our aim is to find optimal pre-processing and post-processing unitaries, for any given $\ell$, that can maximize $I$ and minimize $I'$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Left: a depth-2 canonical unitary realized as beam-splitter transformation. Right: a depth-3 canonical unitary realized as a network of three beam-splitters, according to Euler decomposition.}
\end{figure}

V. RESULTS

As examples, we present results for the optimization in the case of unitaries of depth $\ell = 2$ and $\ell = 3$. For the case of depth-2 unitaries, we assume $\mathcal{U}^{(2)}$ and $\mathcal{V}^{(2)}$ of the form
\[
\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}
\] (30)
with $\theta_{U}$ and $\theta_{V}$ respectively.

The optimization is then performed over the two angles $\theta_{U}$, $\theta_{V}$, which are the parameters of the beam-splitter transformations underlying the canonical unitaries, see Fig. 4.

For the case of depth-3 unitaries, we consider canonical unitaries $\mathcal{U}^{(3)}$ and $\mathcal{V}^{(3)}$ acting on three modes, parameterized according to Euler decomposition,
\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \gamma & \sin \gamma \\ 0 & -\sin \gamma & \cos \gamma \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{pmatrix}
\] (31)
with $(\gamma, \theta, \phi)_U$ and $(\gamma, \theta, \phi)_V$ respectively.

These transformations represent a small interferometer composed by a network of three beam-splitters, as depicted in Fig. 5. The mutual informations are then optimized over the six angles $(\gamma, \theta, \phi)_U$ and $(\gamma, \theta, \phi)_V$.

For a given value of $N$, the maximum of $I$ and the minimum of $I'$ over the beam-splitter parameters are plotted in Figs. 5, 6, as function of $\ell$ and for different values of $\eta$. It is worth remarking that the optimal choice of the parameters jointly maximizes $I$ and minimizes $I'$. The figures show, as it has been remarked above, that better performances are obtained for higher value of $\ell$.

Several comments are in order. First of all, the mutual information enhancement and reduction are less pronounced for $\eta$ close to 1: as the channel gets closer to the ideal one, memory effects tends to disappear. Secondly, if $\eta$ is close to 0, the major increase and decrease in the mutual informations are for $\ell = 2$, while the differences in the mutual informations between the case with $\ell = 2$ and $\ell = 3$ are relatively small. Finally, for intermediate value of $\eta$, higher values of $\ell$ are more effective for higher values of $\epsilon$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{The mutual information $I$ vs $\epsilon$, for $N = 1$. From bottom to top, without pre-processing and post-processing, with pre-processing and post-processing of depth $\ell = 2$, and with pre-processing and post-processing of depth $\ell = 3$. From left to right, the three plots refer to $\eta = 0.1$, 0.5, 0.9.}
\end{figure}
FIG. 6: The mutual information $I'$ vs $\epsilon$, for $N = 1$. From top to bottom, without pre-processing and post-processing, with pre-processing and post-processing of depth $\ell = 2$, and with pre-processing and post-processing of depth $\ell = 3$. From left to right, the three plots refer to $\eta = 0.1, 0.5, 0.9$. 

VI. CONCLUSION

We have considered a model of quantum communication channel with memory, i.e., characterized by non i.i.d. noise. We have studied the possibility of using standard encoding/decoding strategies devised for i.i.d. errors by removing correlations introduced by the quantum memory channel. For the considered model, it is possible to identify pre-processing and post-processing unitary transformations which allow the complete removal of correlations, mapping the correlated noises into independent (although non identical) ones. However, since these unitaries act globally on a train of transmitted signals, their implementation cannot be efficient, since one has to wait the transmission of long strings of signals before the decoding process can start. We have hence considered some examples of unitary transformations with finite depth, involving two or three consecutive signals, permitting the efficient although partial reduction of the correlations introduced by the memory channel. According to suitable a quantifier such as the mutual information, the correlation reduction increases with increasing depth of the unitary transformations, but already at small depths it appears significant.

By increasing the depth of the pre-processing and post-processing unitaries the reduction of correlations gets improved, as these unitaries become closer and closer to those allowing the complete removal of correlations. Hence, the depth could be used to bound the error probability of standard codes.

Finally it is worth noticing that the introduced unitaries provide an online pre-processing and post-processing procedure that combined with standard codes resembles the convolutional codes [15]. A deeper investigation of this parallelism is left for future work.

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