A new topological perspective of expanding space-times with applications to cosmology

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Abstract

We discuss the possible role of the Tietze extension theorem in providing a rigorous topological base to the expanding space-time in cosmology. A simple toy model has been introduced to show the analogy between the topological extension from a circle $S$ to the whole space $M$ and the cosmic expansion from a non-zero volume to the whole space-time in non-singular cosmological models. A topological analogy to the cosmic scale factor function has been suggested, the paper refers to the possible applications of the topological extension in mathematical physics.

Keywords: Topological extension; cosmic expansion, singularity.

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1 Introduction and Motivation

Space-time singularities and the accelerating cosmic expansion are two major challenging problems in modern theoretical physics. In order to fully understand and describe what happens near the singularity, a complete quantum theory of gravity is required which is still missing. Due to the absence of such a unified theory, the best available option is to investigate toy models in which quantum effects are considered \cite{1}. Examples of some candidate theories of quantum gravity are Wheeler-DeWitt theory \cite{2}, super-string theory \cite{3}, loop quantum gravity \cite{4}, brane theories \cite{5}, and higher-order gravity \cite{6}. It has been found in \cite{7,8} that loop quantum gravity can be very helpful in eliminating singularities. It has also been shown in \cite{9} that quantum gravity effects can remove singularity. Another attempt to overcome the initial cosmological singularity is represented by the oscillatory cosmological models \cite{10,11,12}. It has also been shown that cosmological models free from singularity can be obtained in string theories \cite{13} and quadratic gravity \cite{14}. Another
singularity-Free Cosmological solution has been obtained in [15]. All such theories and attempts show the need for a deeper understanding of the space-time singularities from both mathematical and physical sides. In a previous publication [16], Ahmed and Rafat explored the problem of space-time singularities from a global topological viewpoint through the topological retraction theory. They have investigated the role that can be played by the retraction theory and suggested a mathematical restriction on the formation of such singularities.

Another big mystery is the reason behind the late-time accelerating cosmic expansion [17, 18, 19] which is still unknown. The existence of ‘Dark Energy’ with negative pressure has been suggested as a possible explanation [20]. This exotic energy can act as a repulsive gravity pushing the universe to expand faster and faster. Modeling the current accelerating expanding universe has been a major subject in both mathematical and physical cosmology. In the present work, we aim to find a solid topological ground for this observational confirmed cosmic expansion. The current failing to understand the transition from decelerating to accelerating cosmic expansion could be due to the incomplete understanding of the mathematical nature of the expanding spaces. In the literature, there is a sharp lack of the global topological study of expanding spaces against the local geometrical modeling. Some applications of topology in general relativity has been studied in [21, 22]. The topological QFT was introduced in [23]. To describe deformations in topology we use the homotopy [24] which has been shown to be very useful in different areas of mathematical physics [25, 26, 27, 28, 29].

The space-time global topology is a major problem in mathematical physics and the question of whether this topology is static or variable together with the expanding space-time is still open [30]. In case it is variable, then the relation between this variable space-time topology and the cosmic space-time expansion is unknown. The possibility of a changing space topology was first introduced in [31], and the topology change in canonical quantum cosmology has been studied in [32]. The quantum changes of topology has been discussed through path integral in [33, 34, 35]. Some approaches to relate topology to cosmology have been also studied in [36, 37, 38, 39, 40] (all of them are metric approaches with a local study to the topology change). In [30], the expansion of a topological space has been defined using ‘fractal topology’ which is a new suggested topological tool through which the detection of continuous deformations of space is allowed.

In the current work, our aim is to provide a solid topological base to the space-time expansion through utilizing the extension theorem in topology. While all previous studies related to the theory of retracts have been pure mathematical studies, Ahmed and Rafat have started a series of publications [41, 42, 43, 44] in which some applications of this topological theory in mathematical physics have been suggested for the first time. In [16], it has been suggested that the retraction theory in algebraic topology can provide a topological description of the gravitational collapse process which leads to the formation of space-time singularities. We have also clarified the similarity between a collapsing physical system (a contracting volume in space-time), and the topological retraction. The retraction and folding of the higher-dimensional Schwarzchid metric have been investigated in [41]. In [42], we have shown the existence of a strong connection between the topological retraction and the holographic principle in quantum gravity where the geometry of the hologram boundary can be explored through this connection. From this point of view, the retraction theory represents a solid topological base to the holographic principle. In [43], we have
used the retraction theory [44, 45] to provide rigorous proof to the existence of deformations and dimensional reduction in black holes/wormholes, and to explain the topological origin of such deformations/dimensional reduction.

While in the retraction theory space can retract or deformation retract into a subspace, in the current work we need the inverse process with the opposite topological concept given by the Tietze extension theorem. Since such topological extension theorem has not been applied before to physics, this gives another advantage to the current work where it opens the door for interesting applications of this theorem in different areas of mathematical physics. While in the retraction process space $X$ gets continuously shrinked to a subspace $A \subset X$, here we are exploiting the inverse process where we can extend a subspace $A$ to a space $X$. Clearly, this way of thinking implies that as space-time expands, the number of space-time dimensions increases which are quite acceptable from a physical point of view if the big bang theory is correct (It is generally believed that a dimensional reduction occurs near the Planck length [46]).

While the foundations of the theory of retracts were laid by Borsuk [47] who introduced the basic notions, his work had its precedents. The Tietze extension theorem [48] is the most significant among these precedents which proves the extension of continuous functions on a closed subset of normal topological space to the entire space. The Tietze Extension Theorem provides general conditions under which it can be concluded that extensions exist [49]. Specifically, if $A$ is a closed subset of a normal space $X$, and $J \subset R$ is either a closed bounded interval, an open interval, or all of $R$, then this Theorem confirms that every continuous $f : A \rightarrow J$ extends to a continuous function $F : X \rightarrow J$. Tietze theorem was first proved for metric spaces and then Uryson proved his well-known lemma (section 2). For a detailed discussion with an interesting historical review see [50].

The basic motivation behind this work is to find a topological base for the cosmological expansion confirmed by recent observations. Let’s see how the current discussion of the expanding space-time is related to previous works. Firstly, the current work discusses the global topological side while previous studies are interested in the local geometrical research. There is only one topological study on expanding spaces which have been done in [30] through introducing the new concept of 'fractal topology', but in the current paper we use basic algebraic topological notions and theorems with no need to introduce new concepts. On the other hand, the need for a global topological view in describing the expanding space-time has become necessary after the discovery of the challenging problem of the accelerating cosmic expansion. In other words, the incomplete understanding of such accelerating expansion motivates the search for different mathematical models with wider views. Secondly, based on homotopy aspects illustrated in [16], the current work describes an expansion that starts from a non-zero volume and not from a space-time singularity. So, it provides a topological base to a singularity-free cosmic expansion where the absence of the initial singularity is a major advantage in cosmology. Thirdly, the paper represents the first application of the topological extension theorem in cosmology and mathematical physics.

In the current work, we introduce a toy model for the extension from a circle in a metric space (and not from a singularity as we clarified in [16] based on homotopy aspects) to the higher dimensional space. The paper is organized as follows: The introduction and motivation are
described in Sect. 1. The basic definitions and theorems are included in Sect. 2. A quick review of the retraction of the whole space into a lower subspace is given in Sect. 3. A review of the retraction method developed and used in [41, 42, 43] is given Sect. 4 and we refer to the original papers for the details. In Sect. 5, we discuss the extension from $S_i$ to $M$ and cosmological analogy to the continuous function $f$ in Tietze theorem. The last section includes the conclusion.

2 Definitions and theorems

Definition 1. “A subspace $A$ of a topological space $X$ is called a retract of $X$, if there exists a continuous map $r : X \to A$ such that $X$ is open and $r(a) = a$ (identity map), $\forall a \in A$. Because the continuous map $r$ is an identity map from $X$ into $A \subset X$, it preserves the position of all points in $A$” [24].

Definition 2. Deformation retract: “A subset $A$ of a topological space $X$ is said to be a deformation retract if there exists a retraction $r : X \to A$, and a homotopy $f : X \times [0, 1] \to X$ such that [51]: $f(x, 0) = x \ \forall x \in X$, $f(x, 1) = r(x) \ \forall r \in X$, $f(a, t) = a \ \forall a \in A, t \in [0, 1]$.”

2.1 Urysohn’s lemma and Tietze extension theorem [52]

- Let $A$ and $B$ be two disjoint closed subsets of a metric space $X$. Then there exists a continuous function $f : X \to I$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

- Let $F$ be a closed subset of a metric space $X$. Then any continuous function $f : X \to [-1, 1]$ extends over the whole $X$.

- (Urysohn’s lemma) Let $A$ and $B$ be two nonempty disjoint closed subsets of a normal space $X$. Then there exists a continuous function $f : X \to I$ such that $f(A) = 0$ and $f(B) = 1$. This lemma is a key ingredient in the proof of the Tietze extension theorem.

- (Tietze extension theorem) While Urysohn’s lemma proves that on a normal topological space disjoint closed subsets may be separated by continuous functions, Tietze extension theorem proves that such continuous functions extend from closed subsets of normal topological space to the whole space. The formal statement is given as follows: Let $A$ be a closed subset of a normal space $X$. Let $f : A \to [-1, 1]$ be a continuous function. Then, $f$ has a continuous function $F : X \to [-1, 1]$ such that $F|_A = f$. (The statement of the Tietze theorem remains true if we replace the segment $[-1, 1]$ by a circle $S^1$ [52], this note is essential for the current work as we will see).

- Tietze extension theorem for metric spaces Let $A$ be a closed subset of a metric space $X$; then every continuous $f : A \to [-1, 1]$ extends to a continuous function $F : X \to [-1, 1]$.

- The Tietze extension theorem is used to prove the general existence theorem about retractions, details of the proof can be found in [49].
It is also helpful, before leaving this section, to summarize the work we have done in [16]. First, we performed a retraction to a 5D metric $M$ onto 4D circles $S_i \subset M$. Such 4D circles can still be retracted to a point. However, by defining the appropriate homotopy, the existence of a deformation retract on $M$ has been proved which means that the circles can not continue shrinking into a point or a ‘singularity’. From a physical viewpoint, such shrinking of the space into a point is analogous to a universe collapsing into a singularity. Consequently, the deformation retracts we have proved on $M$ stops the formation of the singularity. In the current work, and considering the same cosmological space-time, we are interested in extending the space from a lower-dimensional circle $S$ to the whole space. From a physical viewpoint, such extension is analogous to a continuously expanding universe from a non-zero volume.

3 A quick review to the retraction of the whole space into a lower subspace.

In this section, we briefly summarize the basic idea and main result obtained in [16] without rewriting the details of calculations again in the current work. We have performed a retraction of the following five-dimensional (5D) cosmological space-time $M$ into subspaces considering only the flat case supported by recent observations [55, 56, 57]:

$$ds^2 = B^2 dt^2 - A^2 \left( \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right) - dy^2,$$

where $\kappa$ is the 3D curvature index ($k = \pm 1, 0$). The reasons behind our choice of this Ricci-flat space-time are: 1- it represents an extension to the FRW solutions [53]. 2- the special importance of Ricci-flat manifolds in gravity and geometry [54]. We then obtained the following relations for the coordinates

$x_o = \pm \left( \frac{2n^3(K - y)^2 + n^2(5K^2 - 2K y - 3y^2) + 4n(K + \frac{1}{2}y)^2 + K(K + 2y)}{2t^{2n/(4n^3 - 4n^2 - n + 1)}} \right)^{\frac{1}{2}} + C_o$  (2)

$x_1 = \pm \sqrt{Ar^2 + C_1}$, $x_2 = \pm \sqrt{A^2r^2\theta^2 + C_2}$, $x_3 = \pm \sqrt{A^2r^2 \sin^2 \theta \phi^2 + C_3}$, $x_4 = \pm \sqrt{\frac{1}{2}y^2 + C_4}$,

where $C_i$ are the integration constants. Then, After using Euler-Lagrange equations to explore the geodesics, we obtained the following set of equations directly from (2) for $\phi = 0$ and $\theta = 0$ respectively

$x_o^{\phi=0} = \pm \left( \frac{2n^3(K - y)^2 + n^2(5K^2 - 2K y - 3y^2) + 4n(K + \frac{1}{2}y)^2 + K(K + 2y)}{2t^{2n/(4n^3 - 4n^2 - n + 1)}} \right)^{\frac{1}{2}} + C_o$  (3)

$x_1^{\phi=0} = \pm \sqrt{Ar^2 + C_1}$, $x_2^{\phi=0} = \pm \sqrt{A^2r^2\theta^2 + C_2}$, $x_3^{\phi=0} = \pm \sqrt{C_3}$, $x_4^{\phi=0} = \pm \sqrt{\frac{1}{2}y^2 + C_4}$.
Since
\[ ds^2 = x_1^2 + x_2^2 + x_3^2 - x_o^2 > 0. \] (4)
is satisfied, this retraction leads to a circle \( S_1 \subset M \) \cite{41, 42, 43, 16}. For \( \theta = 0 \), we get
\[
\begin{align*}
x_o^{\theta=0} &= \pm \left( \frac{2n^3(K - y)^2 + n^2(5K^2 - 2Ky - 3y^2) + 4n(K + \frac{1}{2}y)^2 + K(K + 2y)}{2t^{2n}(4n^3 - 4n^2 - n + 1)} + C_o \right)^{\frac{1}{2}}, \\
x_1^{\theta=0} &= \pm \sqrt{Ar^2 + C_1}, \quad x_2^{\theta=0} = \pm \sqrt{C_2}, \quad x_3^{\theta=0} = \pm \sqrt{C_3}, \quad x_4^{\theta=0} = \pm \sqrt{\frac{1}{2}y^2 + C_4}.
\end{align*}
\] (5)

Which also leads to a circle \( S_2 \subset M \). So, we have a retraction of \( M \) defined as \( R : M \Rightarrow S_i, \ i = 1, 2 \) which proves the following theorem

**Theorem 3.1.** Some types of the geodesic retractions of the 5D cosmological space-time \( M \) are circles \( S_i \subset M \).

### 4 Extension from \( S_i \) to \( M \) and the cosmological analogy to the continuous function \( f \) in Tietze theorem

Having obtained all the required tools (Theorems, definitions, and proof for the existence of a retraction from \( M \) to \( S_i \)), extending the subspace (expanding it) back from \( S_i \) to \( M \) should now be straightforward and can be considered as a topological base to a cosmological expansion from a non-zero volume (the lower-dimensional subspace at the beginning of time) to the whole higher-dimensional space-time (the volume of the current universe).

We recall that The statement of the Tietze extension theorem remains true if we replace the segment \([-1, 1]\) by a circle \( S \) \cite{52} (section 2.1). Starting from a circle \( S \) in a metric space \( M \), and choosing the spherical coordinates as a frame of work along with the time \( t \), the following inequality is valid in \( M \)
\[ ds^2 = x_1^2 + x_2^2 + x_3^2 - x_o^2 > 0. \] (6)

From the existence of the retraction (section 3), there exists a set of constants \( C_i \) such that the coordinates \( x_1, x_2, x_3, x_4 \) and \( x_o \) can be expressed as in (5), i.e. :
\[
\begin{align*}
x_o^{\theta=0} &= \pm \left( \frac{2n^3(K - y)^2 + n^2(5K^2 - 2Ky - 3y^2) + 4n(K + \frac{1}{2}y)^2 + K(K + 2y)}{2t^{2n}(4n^3 - 4n^2 - n + 1)} + C_o \right)^{\frac{1}{2}}, \\
x_1^{\theta=0} &= \pm \sqrt{Ar^2 + C_1}, \quad x_2^{\theta=0} = \pm \sqrt{C_2}, \quad x_3^{\theta=0} = \pm \sqrt{C_3}, \quad x_4^{\theta=0} = \pm \sqrt{\frac{1}{2}y^2 + C_4}.
\end{align*}
\] (7)

Which still can be extended, by appropriate choices for the constants, and generalized to the case
when $\theta \neq 0$ to get the form (2)

$$x_o = \pm \left( \frac{2n^3(K - y)^2 + n^2(5K^2 - 2K y - 3y^2) + 4n(K + \frac{1}{2}y)^2 + K(K + 2y) + C_o}{2t^{2n}(4n^3 - 4n^2 - n + 1)} \right)^{\frac{1}{n}}$$ (8)

$$x_1 = \pm \sqrt{Ar^2 + C_1}, \quad x_2 = \pm \sqrt{A^2 r^2 \theta^2 + C_2}, \quad x_3 = \pm \sqrt{A^2 r^2 \sin^2 \theta \phi^2 + C_3}, \quad x_4 = \pm \sqrt{\frac{1}{2}y^2 + C_4}.$$ 

So, we are back to the $5D$ metric of the expanding space-time (1). An important point is that what makes the appropriate choices of constants always guaranteed is the existence of a deformation retract from $M$ to $S_i$ described in section 3. The general existence theorem about retraction can always be proved using the Tietze extension theorem as we have mentioned in section 2 [49].

Recalling that the metric (1) represents an extension to FLRW solutions, this topological extension from simple circles to the whole cosmological space-time provides a rigorous topological base for any non-singular FLRW cosmological model in which the universe starts the expansion from a non-zero volume. A non-singular cyclic cosmological model has been introduced in [12] where the universe expands from a non-zero volume with a scale factor given by

$$a(t) = A \exp \left[ \frac{2}{\sqrt{c^2 - m^2}} \arctan \left( \frac{c \tan \left( \frac{kt}{2} \right) + m}{\sqrt{c^2 - m^2}} \right) \right]$$ (9)

where $A$ and $c$ are integration constants. The scale factor (9), as a function of cosmic time, corresponds topologically to the continuous extendable function $f$ in Tietze extension theorem. This cosmic function grows from a non-zero volume, where its initial radius can be easily calculated from the model to the whole universe before shrinking back again to the initial radius. The evolution of $a(t)$ is shown in figure [1(a)] and we can see that it has a non-zero value at the beginning of time $t = 0$. The cosmological model in [58] is another FLRW type model free from the initial singularity developed in the framework of General Relativity with a perfect fluid. The singularity-free solution of the model is represented by the scale factor function as

$$a(\tau) = a_o \left[ 1 + \left( \frac{\tau}{T_o} \right)^2 \right]^{\frac{1}{3(1-\omega)}}$$ (10)

Where $a_o$ and $T_o$ are constants, and $\omega$ is the equation of state parameter. The evolution of $a(\tau)$ is shown in figure [1(b)].
Cosmological models free from initial singularity may provide solutions to the problems of standard cosmology such as the flatness problem, the homogeneity problem, and the generation of primordial perturbations (see for example [59]). The initial singularity is a major undesirable feature in the standard FLRW cosmology where no geometrical or physical description of space-time is available at the beginning of time. For this reason, there are many modified theories of gravity in the literature that tried to eliminate the initial Big-Bang singularity problem. Examples of such generalized gravity theories are: higher-order gravity [6], loop quantum gravity [60], superstring [61] the brane-world gravity [62], and others. Some theories succeeded to introduce cosmological scenarios in which the cosmic evolution can be extended through the initial singularity such as the cyclic scenario [63] and the pre-big-bang scenario [64]. In [65], a simple modification of General Relativity theory has been considered where there are no space-time singularities at the classical level in FRW and Kasner universes. This opens the door to having a gravitational theory free from singularities in general.

Although the analysis introduced in this work has been done based on the Ricci-flat metric (1) which represents an extension to the FRW solutions, This analogy is general and can be still valid for other non-singular cosmological models. For example, a more realistic singularity-free theory for early and late-time accelerated expansion which covers whole universe history has been introduced in [66].

5 Conclusion

Following previous studies where some applications of the deformation retract have been suggested, in the current work, we have studied the analogy between the topological extension and cosmic expansion. A toy model has been introduced using a cosmological metric where the expansion starts from a non-zero volume. The paper represents the first application of the Tietze extension theorem in mathematical physics and opens the door for more applications in other physical systems.
Our main point of view behind developing such analogy is to look for a deeper understanding to the mathematical nature of expanding space-times. It is known that implementing new mathematical definitions and theorems into physics increases our ability to see the problem from different sides which motivates exploring new ideas. Because topology covers the global view with rigorous definitions, there have been so many successful attempts to utilize topological notions in Modern physics (we have mentioned some examples in the introduction). The need for a global topological description of expanding space-times has become important after the discovery of the challenging problem of the accelerating cosmic expansion. The incomplete understanding of such accelerating expansion could be due to an incomplete understanding to the mathematical nature of expanding spaces (In the literature, there is a sharp lack of the global topological study of expanding spaces against the local geometrical modeling).

Although we have clarified the connection between the topological extension and the study of expanding spaces by introducing a simple toy model, this opens the door for more applications of this connection. Suggesting such a new connection can help to understand the expanding space-times from a different mathematical side and, consequently, attacking some related open questions from a different view. many pure topological concepts, such as homotopy, have been used in modern physics and found to be so useful in providing proofs and simplifying calculations.

One of the main achievements of the current work is that it represents the first application of the topological extension theorem in Physics. The analysis we have provided is general and not restricted to cosmology, we have suggested a topological base for all expanding space-times which can be used in other fields of physics.

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