GRAPHS, $\mathbb{F}_1$-SCHEMES AND VIRTUAL MIXED TATE MOTIVES

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Abstract. In a number of recent works [6, 7], the authors have introduced and studied a functor $F_k$ which associates to each loose graph $\Gamma$ —which is similar to a graph, but where edges with 0 or 1 vertex are allowed — a $k$-scheme, such that $F_k(\Gamma)$ is largely controlled by the combinatorics of $\Gamma$. Here, $k$ is a field, and we allow $k$ to be $\mathbb{F}_1$, the field with one element. For each finite prime field $\mathbb{F}_p$, it is noted in [6] that any $F_k(\Gamma)$ is polynomial-count, and the polynomial is independent of the choice of the field. In this note, we show that for each $k$, the class of $F_k(\Gamma)$ in the Grothendieck ring $K_0(Sch_k)$ is contained in $\mathbb{Z}[L]$, the integral subring generated by the virtual Lefschetz motive.

Résumé. Dans certains ouvrages récents [6, 7], les auteurs ont introduit et étudié un foncteur $F_k$ qui associe à chaque loose graph $\Gamma$ —similaire à un graph mais où les arêtes avec 0 ou 1 sommets sont aussi permis — un $k$-schéma, de telle façon que $F_k(\Gamma)$ est essentiellement contrôlé par la combinatorique de $\Gamma$. Ici, nous considérons $k$ comme un corps, $\mathbb{F}_1$, le corps à un élément, inclus. Pour chaque corps premier $\mathbb{F}_p$, il est remarqué en [6] que tous les $F_k(\Gamma)$ sont polynomial-count et les polynômes sont indépendants du choix du corps. Dans cet article, on prouve que pour chaque $k$, la classe de $F_k(\Gamma)$ dans l’anneau de Grothendieck $K_0(Sch_k)$ est contenu dans $\mathbb{Z}[L]$, l’anneau integral engendré par le motif virtuel de Lefschetz.

1. Introduction

In a series of papers [6, 7, 8], the authors of the present note have studied a functor $F_k$ which associates to each graph a $k$-scheme, where $k$ is any field. In fact, $F_k$ is a functor from the category of “loose graphs” — which are similar to graphs, but allowing edges with 0 or 1 vertex — and $k$ is any field, including the field “with one element,” $\mathbb{F}_1$. The functor $F_{\mathbb{F}_1}$ maps to Deitmar schemes, the core scheme theory over $\mathbb{F}_1$, where a slightly more general definition of Deitmar scheme is used than in [1], in order to give us more flexibility (see the definition of congruence schemes in [2]). The functors obey a number of rules which model the fact that a projective $n$-dimensional $\mathbb{F}_1$-space corresponds naturally to a complete graph on $n+1$ vertices — see Tits [11] and Thas [10] — while an affine $n$-dimensional $\mathbb{F}_1$-space then logically corresponds to a loose graph on one vertex, with $n$ edges through it. The vertex corresponds to the closed point of the space, and the “loose edges” with affine directions — this idea is rooted in the formalism

$$\mathbb{P}^n(k) = \mathbb{A}^n(k) + \mathbb{P}^{n-1}(k).$$

The obtained schemes $F_k(\Gamma)$, with $\Gamma$ a loose graph, are covered by affine $k$-spaces whose intersections are governed by the relations in the loose graph. In [3], we will supply more details concerning the definition of $F_k$.

One of the underlying ideas of [6, 7] is that the schemes $F_k(\Gamma)$ can be studied by using the combinatorial theory of loose graphs, and that some (geometrical, topological) invariants can be easily determined as such. (This idea stems from the note [9], where a related functor was defined, but lacked a for us important property that locally (loose) stars should always correspond to affine spaces.) One example is that for each finite field $k = \mathbb{F}_q$, the number of $\mathbb{F}_q$-rational points of $F_k(\Gamma)$ can be easily determined if $\Gamma$ is a tree, solely from data of $\Gamma$. A second motivation is the fact that

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$\mathcal{F}_k(\Gamma)$, for any loose graph $\Gamma$ and finite field $k$, is defined over $\mathbb{F}_1$ in Kurokawa’s sense [4], and hence comes with a Kurokawa zeta function, as in [4]. This zeta function is independent of the choice of finite field, and in this way, we introduce a new zeta function for all (loose) graphs which is very different than the Ihara zeta function.

1.1. Polynomial countability and virtual mixed Tate motives. In [8] it is shown that for each loose graph $\Gamma$, there exists a polynomial $P(X) \in \mathbb{Z}[X]$ such that for each finite field $k = \mathbb{F}_q \neq \mathbb{F}_1$ we have, with $|\mathcal{F}_q(\Gamma)|_q$ the number of $\mathbb{F}_q$-rational points of $\mathcal{F}_q(\Gamma)$, that

$$|\mathcal{F}_q(\Gamma)|_q = P(q).$$

By definition, this means that the scheme $\mathcal{F}_p(\Gamma)$ is polynomial-count for all primes $p$ [8] Appendix], and moreover, the polynomial $P(X)$ is independent of the chosen finite field. It has been conjectured (and it would follow from one of the Tate conjectures) that this property implies that the class of $\mathcal{F}_q(\Gamma)$ in the Grothendieck ring of schemes $K_0(\text{Sch}_{\mathbb{F}_q})$ is contained in $\mathbb{Z}[L]$, where $L := [A^1(\mathbb{F}_q)]$ is the class of the affine line, that is, that $[\mathcal{F}_q(\Gamma)]$ is a “virtual mixed Tate motive.”

This question is the objective of the present note.

1.2. The present note. In this note, we obtain the following result.

**Theorem 1.1.** Let $\Gamma$ be any loose graph, and let $k \neq \mathbb{F}_1$ be any finite field. Then the class $[\mathcal{F}_k(\Gamma)] \in K_0(\text{Sch}_k)$ is a virtual mixed Tate motive.

Our proof uses the process of “surgery,” which is a stepwise procedure devised in [8] and performed on a loose graph, in which each step consists of replacing an edge with 2 vertices from a prescribed set of edges, by two edges with only one vertex. The local dimension of the corresponding $(k)$-scheme rises, and at the end of the process one winds up with a tree. By doing a local analysis, which refines a part of the “Affection Principle” obtained in [8], we will show that the classes of $k$-schemes that arise in consecutive steps, always differ by elements in $\mathbb{Z}[L]$. Finally, using precise results for trees, which were already obtained in [8], we then finish the proof.

2. Deitmar schemes

We consider an “$\mathbb{F}_1$-ring” $A$ to be a multiplicative commutative monoid with an extra absorbing element 0. Let $\text{Spec}(A)$ be the set of all prime ideals of $A$ together with a Zariski topology. This topological space endowed with a structure sheaf of $\mathbb{F}_1$-rings is called an affine Deitmar scheme, and is also denoted by $\text{Spec}(A)$. We define a monoidal space to be a pair $(X, \mathcal{O}_X)$ where $X$ is a topological space and $\mathcal{O}_X$ is a sheaf of $\mathbb{F}_1$-rings defined over $X$. A Deitmar scheme is then a monoidal space such that for every point $x \in X$ there exists an open subset $U \subseteq X$ such that $(U, \mathcal{O}_X|_U)$ is isomorphic to an affine Deitmar scheme. For a more detailed definition of Deitmar schemes and the structure sheaf of $\mathbb{F}_1$-rings, we refer to [1].

2.1. Affine space. Let $A := \mathbb{F}_1[X_1, \ldots, X_n]$ be the monoidal ring in $n$ variables; then the $n$-dimensional affine space over $\mathbb{F}_1$ is defined as the monoidal space $\text{Spec}(A)$ and denoted by $\mathbb{A}_p^n$ or $\mathbb{A}_p^n(\mathbb{F}_1)$. There is one closed point, and to each unknown $X_i$ corresponds a coordinate axis (“direction”).

2.2. Projective space. The $n$-dimensional projective space is defined as the projective scheme $\text{Proj}(\mathbb{F}_1[x_0, \ldots, x_n]) := \mathbb{P}_p^n$, or $\mathbb{P}_p^n(\mathbb{F}_1)$. There are $n+1$ closed points, and a linear subspace of dimension $r$ contains $r+1$ of these points; in particular, a projective subline has 2 closed points. Combinatorially, one depicts $\mathbb{P}_p^n(\mathbb{F}_1)$ as a complete graph on $n+1$ vertices. The combinatorial affine $n$-space over $\mathbb{F}_1$ then arises by deleting a complete subgraph on $n$ vertices. So one obtains a loose graph with one vertex and $n$ loose edges.

2.3. Embedding theorem. Let $\Gamma$ be a loose graph. The embedding theorem of [8] observes that $\Gamma$ can be seen as a subgeometry of the combinatorial projective $\mathbb{F}_1$-space $\mathbb{P}(\Gamma)$, where $\Gamma$ and $\mathbb{P}(\Gamma)$ are constructed as in section [8]. Applying $\mathcal{F}_k$ (for any field $k$ including $\mathbb{F}_1$), cf. the next section, one obtains that $\mathcal{F}_k(\Gamma)$ is embedded in $\mathbb{P}(\Gamma)$ (now seen as a scheme).
3. The functors \( \mathcal{F}_k \)

We briefly describe how one can associate a Deitmar scheme to a loose graph \( \Gamma \) through the functor \( \mathcal{F}_k \), with \( k \) any field, including \( \mathbb{F}_1 \). The main thing is that the functor must obey a set of rules, namely:

- **COV** If \( \Gamma \subset \tilde{\Gamma} \) is a strict inclusion of loose graphs, \( \mathcal{F}_k(\Gamma) \) also is a proper subscheme of \( \mathcal{F}_k(\tilde{\Gamma}) \).
- **LOC-DIM** If \( x \) is a vertex of degree \( m \in \mathbb{N}^\times \) in \( \Gamma \), then there is a neighborhood \( \Omega \) of \( x \) in \( \mathcal{F}_k(\Gamma) \) such that \( \mathcal{F}_k(\Gamma)|_{\Omega} \) is an affine space of dimension \( m \).
- **CO** If \( K_m \) is a sub complete graph on \( m \) vertices in \( \Gamma \), then \( \mathcal{F}_k(K_m) \) is a closed sub projective space of dimension \( m - 1 \) in \( \mathcal{F}_k(\Gamma) \).
- **MG** An edge without vertices should correspond to a multiplicative group.

Rule (MG) implies that we have to work with a more general version of Deitmar schemes since \( (\mathbb{A}^1, \mathbb{G}_m) \) is not defined in Deitmar scheme theory.) The reader can find a more detailed explanation of this association in [7].

A very simple way to explain \( \mathcal{F}_k \) is as follows: first, for any loose star \( S \) (one vertex plus a number \( n \) of loose edges through it), \( \mathcal{F}_k(S) \) is the affine \( k \)-space of dimension \( n \). Here, a *loose edge* is an edge with 0 or 1 vertex. Now let \( \Gamma \) be any connected loose graph, and let \( \overline{\Gamma} \) be the graph-theoretical completion of \( \Gamma \), that is, \( \overline{\Gamma} \) is the graph one obtains by adding a new vertex on any loose edge of \( \Gamma \). Say that \( \overline{\Gamma} \) has \( m + 1 \) vertices. Let \( \mathbb{P}(\overline{\Gamma}) \) be the projective \( k \)-space of dimension \( m \) defined by these vertices (see [2]); then \( \mathcal{F}_k(\Gamma) \) is the union in \( \mathbb{P}(\overline{\Gamma}) \otimes_{\text{Spec}(k)} \text{Spec}(k) \) of the \( k \)-affine spaces defined by the stars which are defined by each vertex, without the closed points which correspond to the vertices which were added to obtain \( \overline{\Gamma} \). If we choose coordinates (in \( \mathbb{P}(\overline{\Gamma}) \)) such that each such vertex has as coordinates a vector in \( \{0,1\}^{m+1} \) with precisely one nonzero entry, then for each \( k \), \( \mathcal{F}_k(\Gamma) \) can be described explicitly analytically. The disconnected case is easily derived from the connected case.

**Theorem 3.1 ([7]).** The map \( \mathcal{F}_{\mathbb{F}_1} \) is a functor from the category of loose graphs to the category of Deitmar congruence schemes. Moreover, for any finite field \( k \) (or \( \mathbb{Z} \)), the lifting map \( \mathcal{F}_k(\cdot) = \mathcal{F}_{\mathbb{F}_1}(\cdot) \otimes_{\text{Spec}(\mathbb{F}_1)} \text{Spec}(k) \) is also a functor.

Let \( \Gamma \) be a loose graph and \( \mathcal{F}_{\mathbb{F}_1}(\Gamma) \) be the Deitmar scheme associated to it. Let us call \( v_1, \ldots, v_k \) the vertices of \( \Gamma \) and \( \mathbb{A}_{v_i} \) the affine space associated to \( v_i \), \( 1 \leq i \leq k \).

**Lemma 3.2 ([5]).** For all \( 1 \leq r, s, \leq k \), \( \mathbb{A}_{v_r} \cap \mathbb{A}_{v_s} \neq \emptyset \) if and only if \( v_r \) and \( v_s \) are adjacent vertices in \( \Gamma \).

4. Grothendieck ring of schemes of finite type over \( \mathbb{F}_1 \)

The \( \text{Spec} \)-construction on monoids allows us to have a scheme theory over \( \mathbb{F}_1 \) defined in an analogous way to the classical scheme theory over \( \mathbb{Z} \). This also allows us to define the *Grothendieck ring of schemes over \( \mathbb{F}_1 \).*

**Definition 4.1.** The Grothendieck ring of schemes of finite type over \( \mathbb{F}_1 \), denoted as \( K_0(\text{Sch}_{\mathbb{F}_1}) \), is generated by the isomorphism classes of schemes \( X \) of finite type over \( \mathbb{F}_1 \), \( [X]_{\mathbb{F}_1} \), with the relation

\[ [X]_{\mathbb{F}_1} = [X \setminus Y]_{\mathbb{F}_1} + [Y]_{\mathbb{F}_1} \]

for any closed subscheme \( Y \) of \( X \) and with the product structure given by

\[ [X]_{\mathbb{F}_1} \cdot [Y]_{\mathbb{F}_1} = [X \times_{\mathbb{F}_1} Y]_{\mathbb{F}_1}. \]

We denote by \( \mathbb{L} = [\mathbb{A}_{\mathbb{F}_1}^1]_{\mathbb{F}_1} \) the class of the affine line over \( \mathbb{F}_1 \).
5. Counting Polynomials

Let $\Gamma$ be a loose tree and $\mathcal{F}_\Gamma(\Gamma)$ its corresponding Deitmar scheme. The next result gives us information about the class of $\mathcal{F}_\Gamma(\Gamma)$ in the Grothendieck ring of Deitmar schemes of finite type, $K_0(\text{Sch}_\mathbb{F}_1)$. We will sometimes use the notation $[\Gamma]_{\mathcal{F}_\Gamma}$ for the class of $\mathcal{F}(\Gamma)$ in $K_0(\text{Sch}_{\mathbb{F}_1})$ (also when $\Gamma$ is a general loose graph). We adapt the same notation over fields $k$.

**Theorem 5.1** ([6]). Let $\Gamma$ be a loose tree. Let $D$ be the set of degrees $\{d_1, \ldots, d_k\}$ of $V(\Gamma)$ such that $1 < d_1 < d_2 < \ldots < d_k$ and let $n_i$ be the number of vertices of $\Gamma$ with degree $d_i$, $1 \leq i \leq k$. We call $E$ the number of vertices of $\Gamma$ with degree 1 and $I = \sum_{i=1}^{k} n_i - 1$. Then

$$[\Gamma]_{\mathcal{F}_\Gamma} = \sum_{i=1}^{k} n_i \mathbb{L}^{d_i} - I \cdot \mathbb{L} + I + E. \quad (5)$$

6. Surgery

In order to inductively calculate the counting polynomial of a $\mathbb{Z}$-scheme coming from a general loose graph, we introduced a procedure called surgery, in [6]. In each step of the procedure we “resolve” an edge, so as to eventually end up with a tree in much higher dimension. One has to keep track of how the counting polynomial changes in each step.

**6.1. Resolution of edges.** Let $\Gamma$ be a loose graph, and let $e$ be an edge with two distinct vertices $v_1, v_2$. The resolution of $\Gamma$ along $e$, denoted $\Gamma_e$, is the loose graph obtained from $\Gamma$ by deleting $e$, and adding two new loose edges (each with one vertex) $e_1$ and $e_2$, where $v_i \in e_i$, $i = 1, 2$.

The following theorem reduces the computation of the alteration of the number of $k$-rational points after resolving an edge, to a local problem. In the statement, $d(\cdot, \cdot)$ stands for the distance function in a graph.

**Theorem 6.1** (Affection Principle, [6]). Let $\Gamma$ be a finite connected loose graph, let $xy$ be an edge on the vertices $x$ and $y$, and let $S$ be a subset of the vertex set. Let $k$ be any finite field, and consider the $k$-scheme $\mathcal{F}_k(\Gamma)$. Then $\cap_{s \in S} \mathcal{A}_s$, where $\mathcal{A}_s$ is the local affine space corresponding to the vertex $s \in S$, changes when one resolves the edge $xy$ only if $\cap_{s \in S} \mathcal{A}_s$ is contained in $\mathbb{P}_{x,y}$, the projective subspace of $\mathbb{P}(\Gamma) \otimes_{\mathbb{F}_1} k$ generated by $\mathbb{B}(x, 1) \cup \mathbb{B}(y, 1)$, where $\mathbb{B}(x, 1) = \{ v \in V(\Gamma) \mid d(v, x) \leq 1 \}$.

In terms of counting polynomials, we have the following theorem, in which $\big| \cdot \big|_k$ denotes the number of $k$-rational points.

**Corollary 6.2** (Polynomial Affection Principle, [6]). Let $\Gamma$ be a finite connected loose graph, let $\Gamma_{xy}$ be the loose graph after resolving the edge $xy$ and let $k$ be any finite field. Then in $K_0(\text{Sch}_k)$ we have

$$\big| \Gamma \big|_k - \big| \Gamma_{xy} \big|_k = \big| \Gamma_{P_{x,y}} \big|_k - \big| \Gamma_{xy \mid P_{x,y}} \big|_k. \quad (6)$$

6.2. Counting Polynomial for General Loose Graphs. To compute the counting polynomial of a scheme coming from a loose graph $\Gamma$ we choose a spanning loose tree $T$ of $\Gamma$ and resolve in $\Gamma$ all edges not belonging to $T$. This yields a loose tree $\overline{T}$ in which we apply the map defined in Theorem 5.1 so as to obtain a counting polynomial for it. Take an edge $e$ now that was resolved and consider the loose graph $\overline{T}$ in which all other edges except $e$ are resolved, i.e., $\overline{T}$ is the next-to-last step in the procedure of obtaining $\overline{T}$. Thanks to Corollary 6.2, we can compute the counting polynomial for $\overline{T}$ by restricting to $P_e$ (for the concrete formulas of the Affection Principle we refer to [6, section 11]). By repeating this process as many times as edges were resolved, we inductively obtain the counting polynomial for $\mathcal{F}_\Gamma(\Gamma)$.

**Proposition 6.3** ([6]). Let $\Gamma$ be a loose graph and let $T$ and $\overline{T}$ be defined as above. Then the counting polynomial in $K_0(\text{Sch}_{\mathbb{F}_1})$ of $\mathcal{F}(\overline{T})$ is independent of the choice of the spanning loose tree $T$ of $\Gamma$. 

7. The main result

Let $\Gamma$ be any loose graph, and let $\mathcal{F}_k(\Gamma)$ be the corresponding $k$-scheme, with $k$ any field (including $\mathbb{F}_1$). Our proof goes by induction on the number $N$ which is the sum of the number of edges and the number of vertices. For small values of $N$, the main theorem, which is stated below for the sake of convenience, is easy to obtain.

**Theorem 7.1.** Let $\Gamma$ be any loose graph, and let $k \neq \mathbb{F}_1$ be any finite field. Then the class $[\mathcal{F}_k(\Gamma)] \in K_0(\text{Sch}_k)$ is a virtual mixed Tate motive.

We now proceed with the proof. Note that we may suppose w.l.o.g. that $\Gamma$ is connected (and note that resolving an edge on a connected loose graph not necessarily yields again a connected loose graph). Note also that resolution of edges on trees is not defined.

7.1. Local lemmas. The following lemma is easy to prove:

**Lemma 7.2.** In $K_0(\text{Sch}_k)$, we have that $[\mathcal{F}_k(\Gamma)] \in \mathbb{Z}[L]$ if and only if $[\mathcal{F}_k(\Gamma)] \in \mathbb{Z}[L]$, with $\Gamma$ the underlying graph of $\Gamma$.  

The underlying graph of $\Gamma$ is the subgraph of $\Gamma$ induced on the vertices. By Lemma 7.2 we may thus suppose that $\Gamma$ is a graph. Now suppose $e = xy$ is an edge, with $x$ and $y$ its vertices. Resolve the edge $xy$ to obtain $\Gamma_{xy}$ (this is a loose graph).

**Remark 7.3.** Notice that intersecting with a projective space commutes with the functor $\mathcal{F}_k(\cdot)$. We will prove this remark in the following lemma.

**Lemma 7.4.** Let us denote by $P = K_V$ the complete graph defined on a subset $V$ of vertices of $\Gamma$ and let us call $P_k$ the $k$-projective space defined by $P$. Then $\mathcal{F}_k(\Gamma) \cap P_k = \mathcal{F}_k(\Gamma \cap P)$.

Proof. Before starting the proof let us denote by $S_w$ the loose star of a vertex $w$ of $\Gamma$, that is, the loose subgraph of $\Gamma$ formed by the vertex $w$ and all its incident edges.

It is easy to check that $\mathcal{F}_k(\Gamma \cap P)$ is a subscheme of $\mathcal{F}_k(\Gamma) \cap P_k$, since $\Gamma \cap P$ is a subgraph of both $\Gamma$ and $P$. Consider now a point $x \in \mathcal{F}_k(\Gamma) \cap P_k$. Then, from the definition of $\mathcal{F}_k$ (see §7), $x$ belongs to $\text{Spec}(A_w) \cap P_k$, for a vertex $v \in \Gamma$. The latter scheme is defined by the part of the loose star $S_w \subseteq v$ inside $P$, i.e., by $S_v \cap P$. This concludes the proof since $S_v \cap P$ is a subgraph of $\Gamma \cap P$ and so $x \in \mathcal{F}_k(S_v \cap P) \subseteq \mathcal{F}_k(\Gamma \cap P)$.  

The following lemma, in the spirit of Corollary 6.2, shows that we can restrict ourselves to local considerations.

**Lemma 7.5.** In $K_0(\text{Sch}_k)$, we have that $[\mathcal{F}_k(\Gamma)] - [\mathcal{F}_k(\Gamma_{xy})] \in \mathbb{Z}[L]$ if and only if $[\mathcal{F}_k(\Gamma \cap P_{x,y})] - [\mathcal{F}_k(\Gamma_{xy} \cap P_{x,y})] \in \mathbb{Z}[L]$.

Proof. In order to prove the statement, we will compute the difference of classes $[\mathcal{F}_k(\Gamma)] - [\mathcal{F}_k(\Gamma_{xy})]$. Thanks to the remark above and the relative topology on $\mathcal{F}_k(\Gamma)$ and $\mathcal{F}_k(\Gamma_{xy})$, we can deduce that both $\mathcal{F}_k(\Gamma \cap P_{x,y})$ and $\mathcal{F}_k(\Gamma_{xy} \cap P_{x,y})$ are closed in $\mathcal{F}_k(\Gamma)$ and $\mathcal{F}_k(\Gamma_{xy})$, respectively. Then, by the relations in the appropriate Grothendieck ring of schemes, we have that:

\begin{equation}
\begin{cases}
[\mathcal{F}_k(\Gamma)] = [\mathcal{F}_k(\Gamma \cap P_{x,y})] + [\mathcal{F}_k(\Gamma \setminus (\Gamma \cap P_{x,y}))], \\
[\mathcal{F}_k(\Gamma_{xy})] = [\mathcal{F}_k(\Gamma_{xy} \cap P_{x,y})] + [\mathcal{F}_k(\Gamma_{xy} \setminus (\Gamma_{xy} \cap P_{x,y}))].
\end{cases}
\end{equation}

We will prove that the last terms on the right-hand side of the equations are the same. Let $\Gamma' = \Gamma \cap P_{x,y}$ and $\Gamma_{xy}' = \Gamma_{xy} \cap P_{x,y}$. Note that thanks to the Affection Principle (see Theorems 6.1 and 6.2 Lemma 11.5), in order to compare the classes of $\mathcal{F}_k(\Gamma) \setminus \mathcal{F}_k(\Gamma')$ and $\mathcal{F}_k(\Gamma_{xy}) \setminus \mathcal{F}_k(\Gamma_{xy}')$ in $K_0(\text{Sch}_k)$, we (only) need to take into account the local affine spaces in $\mathcal{F}_k(\Gamma) \setminus \mathcal{F}_k(\Gamma_{xy})$ associated to vertices of $\Gamma \setminus \Gamma_{xy}$ which are at distance at most one from the loose graph $\Gamma \setminus \Gamma_{xy}$ (since vertices at distance strictly more than one, give rise to affine spaces that remain unchanged through resolution of $xy$).
From the definition of \( \Gamma' \), it is easy to see that both vertices \( x \) and \( y \) are at least at distance two from any vertex of \( \Gamma \setminus \Gamma' \), which implies that

\[
\begin{align*}
\left( \mathcal{F}_k(\Gamma) \setminus \mathcal{F}_k(\Gamma') \right) \cap A_x &= \emptyset, \\
\left( \mathcal{F}_k(\Gamma) \setminus \mathcal{F}_k(\Gamma') \right) \cap A_y &= \emptyset.
\end{align*}
\]

As resolving the edge \( xy \) only changes locally the affine spaces \( A_x \) and \( A_y \) in \( \mathcal{F}(\Gamma) \) (more precisely in \( \mathcal{F}(\Gamma') \)), and as the distance between \( x \) (or \( y \)) and \( \Gamma \setminus \Gamma' \) is preserved through resolution, this process does not affect the local affine spaces in \( \mathcal{F}_k(\Gamma) \setminus \mathcal{F}_k(\Gamma') \), nor the intersection of any two of them. Notice that in the case of vertices \( v \in \Gamma' \) at distance one from \( \Gamma \setminus \Gamma' \), possible changes of the affine space \( A_v \) in \( \mathcal{F}(\Gamma) \) by resolution of \( xy \) do not affect the scheme \( \mathcal{F}_k(\Gamma) \setminus \mathcal{F}_k(\Gamma') \); changes only occur in the completion \( \overline{A_v} \cap \mathcal{F}(\Gamma') \).

It is now easy to observe that there is a natural isomorphism between \( \mathcal{F}_k(\Gamma) \setminus \mathcal{F}_k(\Gamma') \) and \( \mathcal{F}_k(\Gamma_{xy}) \setminus \mathcal{F}_k(\Gamma_{xy}') \) induced by the graph morphism

\[
\gamma : \Gamma'' \rightarrow \Gamma_{xy},
\]
where \( \Gamma'' \) (respectively \( \Gamma_{xy}' \)) is the subgraph of \( \Gamma \) (respectively \( \Gamma_{xy} \)) defined on the vertex set \( V(\Gamma) \setminus V(\Gamma') \cup \{ v \in V(\Gamma) \mid d(v, \Gamma \setminus \Gamma') = 1 \} \) (respectively \( V(\Gamma_{xy}) \setminus V(\Gamma_{xy}') \cup \{ v \in V(\Gamma_{xy}) \mid d(v, \Gamma_{xy} \setminus \Gamma_{xy}') = 1 \} \)), and where \( \gamma \) acts as the identity on vertices. This implies that both classes in \( K_0(\mathrm{Sch}_k) \) are equal. We can conclude now that

\[
[\mathcal{F}_k(\Gamma)] - [\mathcal{F}_k(\Gamma_{xy})] = [\mathcal{F}_k(\Gamma \cap P_{xy})] - [\mathcal{F}_k(\Gamma_{xy} \cap P_{xy})].
\]

By Lemma 7.5, we may suppose that \( \Gamma = \Gamma \cap P_{xy} \).

We now slightly refine the Affection Principle from [6].

**Lemma 7.6.** Let \( \Gamma \) be a graph, \( xy \) an edge with vertices \( x \) and \( y \) and \( \Gamma_{xy} \) the graph after resolving the edge \( xy \). Let \( u, v \) be two vertices of \( \Gamma \) and consider \( A_u \) and \( A_v \), the local affine spaces at \( u \) and \( v \) in \( \mathcal{F}_k(\Gamma) \). The intersection \( A_u \cap A_v \) and the union \( A_u \cup A_v \) change after resolution only if \( u, v \in \{ x, y \} \cup (x^+ \cap y^+) \).

**Proof.** First let us note that if \( A_u \cap A_v = \emptyset \), then \( A_u \cup A_v \) is stable under resolution. Consider now a vertex \( u \in x^+ \setminus ((x^+ \cap y^+) \cup \{ y \}) \). Then, it is clear that neither \( A_u \) nor \( A_v \) in \( \mathcal{F}_k(\Gamma) \) changes after resolving the edge \( xy \). The latter is not affected by the resolution since the edge \( xy \) is not in the graph \( \Gamma \cap \overline{B}(u,1) \). The same holds for vertices \( u \in y^+ \setminus ((x^+ \cap y^+) \cup \{ x \}) \). To simplify notation we will write from now on only \( \overline{A_u} \) instead of \( \overline{A_u} \cap \mathcal{F}_k(\Gamma) \) and we consider it embedded in the ambient space of \( \mathcal{F}_k(\Gamma) \).

Now suppose that \( u \in x^+ \cap y^+ \); then the graph defined by \( xy \) is a subgraph of the "part at infinity" of the graph completion of \( S_u \), the star associated to \( u \). This implies that locally at \( u \) the changes that occur by resolving \( xy \) are contained in \( \overline{A_u} \), so the local affine space at \( u \) also remains invariant under resolution of \( xy \).

Observe that \( A_u \cap A_v \) and \( A_u \cup A_v \) are controlled by \( A_u \), \( A_v \), \( \overline{A_u} \), \( \overline{A_v} \) and \( \overline{A_u} \cap \overline{A_v} \). So, if \( u, v \notin x^+ \cap y^+ \), then indeed \( A_u \cap A_v \) and \( A_u \cup A_v \) are stable under resolution. In the case where one of \( u, v \in x^+ \cap y^+ \) and \( u, v \notin x, y \), changes under resolution will be controlled by \( \overline{A_u} \cap \overline{A_v} \) and \( \overline{A_u} \setminus \overline{A_v} \). This implies that changes in \( \overline{A_u} \cap \overline{A_v} \) are contained in \( \overline{A_u} \setminus \overline{A_v} \), so \( \overline{A_u} \cap \overline{A_v} \) and \( \overline{A_u} \cup \overline{A_v} \) are also stable after resolving \( xy \).

Suppose now that \( v = x \) and \( u \in x^+ \setminus ((x^+ \cap y^+) \cup \{ y \}) \). Then \( A_u \) and \( \overline{A_u} \) are stable after resolution. From the graph theoretical point of view, it is easy to see that \( S_u \cap \overline{A_u} \) (as a subgraph of \( \Gamma \)) remains invariant after resolving the edge \( xy \) (considering the same intersection inside \( \Gamma_{xy} \)). Since \( A_u \cap A_x \subset \overline{A_u} \cap \overline{A_x} \), we deduce that indeed \( A_u \cap A_x \) and \( A_u \cup A_x \) are also stable under resolution. The same reasoning holds when \( v = y \) and \( u \in y^+ \setminus ((x^+ \cap y^+) \cup \{ x \}) \). This concludes the proof. ■
Remark 7.7. Notice that the reason why the previous spaces are not affected by the resolution of the edge $xy$ comes as a direct consequence of the fact that the (loose) subgraphs of $\Gamma$ defining them do not contain the edge $xy$.

Remark 7.8. Note that the equality of the last terms in the right-hand sides of (7) in Lemma 7.5 can also be obtained by applying Lemma 7.6.

7.2. “Full cones”. We first handle a useful specific case of graphs.

Lemma 7.9. Suppose that either $y^\perp = (x^\perp \cap y^\perp) \cup \{x\}$, or $x^\perp = (x^\perp \cap y^\perp) \cup \{y\}$. Then $[\mathcal{F}_k(\Gamma)] \in \mathbb{Z}[L]$.

Proof. Suppose w.l.o.g. that $y^\perp = (x^\perp \cap y^\perp) \cup \{x\}$; then for all vertices $v$ in $\Gamma$, we have that either $v = x$ or $v \sim x$. It follows immediately that

$$[\mathcal{F}_k(\Gamma)] = [A_x] + [\mathcal{F}_k(x^\perp \cap \Gamma)].$$

By induction, applied on the second term in the right-hand side, the lemma follows. \qed

7.3. Without external edges. We assume that there are no edges $uv$ with $u \in x^\perp \setminus ((x^\perp \cap y^\perp) \cup \{y\})$ and $v \in y^\perp \setminus ((x^\perp \cap y^\perp) \cup \{x\})$ (call such edges “external”) — the case where such edges exist will be handled separately below.

We also suppose that $y^\perp \neq (x^\perp \cap y^\perp) \cup \{x\}$ and $x^\perp \neq (x^\perp \cap y^\perp) \cup \{y\}$, since otherwise the statement is already true by the previous subsection.

Let $u \neq y$ be any vertex in $x^\perp \setminus (x^\perp \cap y^\perp)$; let $e := uy$. Let $\Gamma^e$ be the graph $\Gamma$ without the edge $e$ (while not deleting $u$ and $x$); similarly, we define $\Gamma^e_x$. As $\Gamma^e$ is a subgraph of $\Gamma$, induction implies that $[\mathcal{F}_k(\Gamma^e)] \in \mathbb{Z}[L]$. Also, by Lemma 7.2, $[\mathcal{F}_k(\Gamma_{xy})] \in \mathbb{Z}[L]$ if and only if $[\mathcal{F}_k(\Gamma_{xy})] \in \mathbb{Z}[L]$, and by induction, the latter expression is true since $\Gamma_{xy}$ is a subgraph of $\Gamma$. In the same way, $[\mathcal{F}_k(\Gamma_{xy})] \in \mathbb{Z}[L]$. Now consider $[\mathcal{F}_k(\Gamma)] - [\mathcal{F}_k(\Gamma^e)]$. Then obviously $[\mathcal{F}_k(\Gamma)] - [\mathcal{F}_k(\Gamma^e)] \in \mathbb{Z}[L]$ if and only if $[\mathcal{F}_k(\Gamma_{xy})] \in \mathbb{Z}[L]$; by Lemma 7.3, this holds if and only if $[\mathcal{F}_k(\Gamma \cap P_{x,u})] - [\mathcal{F}_k(\Gamma_{xu} \cap P_{x,u})] \in \mathbb{Z}[L]$. Now by our assumption, we have

$$\begin{align*}
\Gamma \cap P_{x,u} &\neq \Gamma; \\
\Gamma_{xu} \cap P_{x,u} &\neq \Gamma_{xu},
\end{align*}$$

so that induction yields that $[\mathcal{F}_k(\Gamma)] - [\mathcal{F}_k(\Gamma^e)] \in \mathbb{Z}[L]$. Since $[\mathcal{F}_k(\Gamma^e)] \in \mathbb{Z}[L]$, it follows that $[\mathcal{F}_k(\Gamma)] \in \mathbb{Z}[L]$. \qed

7.4. With external edges. Now suppose $\Gamma$ has external edges. Suppose $\Gamma'$ is the subgraph of $\Gamma$ which one obtains by deleting one chosen external edge $e = uv$. By induction we know that $[\mathcal{F}_k(\Gamma')] \in \mathbb{Z}[L]$. Then

$$[\mathcal{F}_k(\Gamma)] = \mathcal{F}_k(\Gamma') \prod \left((A_u \cup A_v) \setminus (\cup_{s \in \Gamma'} A'_s)\right),$$

where $A_w$ is the local affine space at $w$ in $\mathcal{F}_k(\Gamma)$, and $A'_t$ is the local affine space at $t$ in $\mathcal{F}_k(\Gamma')$ (note that $\cup_{s \in \Gamma'} A'_s = \mathcal{F}_k(\Gamma')$). Then

$$[\mathcal{F}_k(\Gamma)] = [\mathcal{F}_k(\Gamma')] + \left((A_u \cup A_v) \setminus (\cup_{s \in \Gamma'} A'_s)\right).$$

Doing the same for $\Gamma_{xy}$, we obtain that

$$[\mathcal{F}_k(\Gamma_{xy})] = [\mathcal{F}_k(\Gamma_{xy}')] + \left((A_u \cup A_v) \setminus (\cup_{s \in \Gamma_{xy}'} A'_s)\right),$$

where all the local affine spaces are now considered in $\Gamma_{xy}$ or $\Gamma_{xy}'$.\]
By Lemma 7.6, we have that (A) = (B). For, 
\((A_u \cup A_v) \cap A'_x = A_u \cap A'_x\) and 
\((A_u \cup A_v) \cap A'_y = A_v \cap A'_y\) do not change when resolving \(xy\), and if \(w \in x^+ \cap y^+\), then \((A_u \cup A_v) \cap A'_w\) also does not change through resolution. All the other cases are covered by Lemma 7.6.

After applying induction, we now get that \([\mathcal{F}_k(\Gamma)] - [\mathcal{F}_k(\Gamma^\prime)] \in \mathbb{Z}[L]\. \square

7.5. End of the proof of Theorem 7.1. Starting from a connected loose graph \(\Gamma\), we first note that if \(\Gamma\) is a loose tree, the result follows from Theorem 5.1. So suppose that \(\Gamma\) is not a loose tree. Choose any edge \(xy\) that is contained in a loose spanning tree \(T\), and resolve \(xy\). In the above we have shown that

\[ [\mathcal{F}_k(\Gamma)] - [\mathcal{F}_k(\Gamma_{xy})] \in \mathbb{Z}[L]. \tag{14} \]

Now there are two ways to proceed.

(A) Carry out surgery on the scheme \(\mathcal{F}_k(\Gamma)\) (using the loose tree \(T\)), and eventually wind up with a scheme \(\mathcal{F}_k(T)\), where \(T\) is a loose tree containing \(T\), cf. \S 6.2. We have seen that \([\mathcal{F}_k(T)] \in \mathbb{Z}[L]\) in Theorem 5.1. Since by (14) each difference between Grothendieck classes of consecutive steps is an element of \(\mathbb{Z}[L]\), we conclude that the same is true for the initial class \([\mathcal{F}_k(\Gamma)]\) as well.

(B) Use the induction hypothesis to conclude that \([\mathcal{F}_k(\Gamma_{xy})] \in \mathbb{Z}[L]\), so that \(\mathcal{F}_k(\Gamma) \in \mathbb{Z}[L]\).

This concludes the proof of Theorem 1.1. \square

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