ON THE COMMUTATOR MAP
FOR
REAL SEMISIMPLE LIE ALGEBRAS

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Abstract. We find new sufficient conditions for the commutator map of a real semisimple Lie algebra to be surjective. As an application, we prove the surjectivity of the commutator map for all simple algebras except $su_{p,q}$ ($p$ or $q > 1$), $so_{p,p+2}$ ($p$ odd or $p = 2$), $u_{2m+1}(\mathbb{H})$ ($m \geq 1$) and $E_{III}$.

1. Introduction and statement of results

Let $\mathfrak{g}$ be a semisimple Lie algebra. The commutator map
$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \ (X, Y) \mapsto [X, Y],$$
is known to be surjective if $\mathfrak{g}$ is split. This result, due to Gordon Brown [1], is valid for all infinite fields and for finite fields of sufficiently big cardinality. If $\mathfrak{g}$ is any real semisimple Lie algebra then its complexification $\mathfrak{g}^\mathbb{C}$ is split over $\mathbb{C}$. Therefore, for any $Z \in \mathfrak{g}$ there exist $X_1, X_2, Y_1, Y_2 \in \mathfrak{g}$, such that $Z = [X_1 + iX_2, Y_1 + iY_2]$. Hence $Z$ is the sum of two commutators, namely, $Z = [X_1, Y_1] - [X_2, Y_2]$. To the best of author’s knowledge, the surjectivity of the commutator map is in general not established. On the other hand, there is no example where two commutators in the above formula are indeed essential, so that a presentation of $Z$ as one commutator does not exist.

Our goal is to obtain new sufficient conditions for the surjectivity of the commutator map.

Theorem 1.1. Let $\mathfrak{g}$ be a semisimple real Lie algebra, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition, $\mathfrak{a}$ a maximal abelian subspace in $\mathfrak{p}$, and $\mathfrak{m} = \mathfrak{z}_\mathfrak{k}(\mathfrak{a})$ the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. If $\mathfrak{m}$ is semisimple then the commutator map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \ (X, Y) \mapsto [X, Y]$, is surjective.

If $\mathfrak{g}$ is split then $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{m} = \{0\}$. If $\mathfrak{g}$ is compact then $\mathfrak{a} = \{0\}$ and $\mathfrak{m} = \mathfrak{g}$. Therefore we have the following corollary.

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Corollary 1.2. If $g$ is split or compact then the commutator map is surjective.

As we noted above, the result in the split case is known, see [1]. In the compact case Corollary 1.2 is easily deduced from Theorem 3.4 in [2]. Our proof in both cases is new. It is not even necessary to invoke Theorem 1.1 in full generality, see Corollaries 3.2 and 3.3.

Clearly, the question of surjectivity of the commutator map reduces to simple Lie algebras. Apart from real split, compact and complex simple algebras, we have the following list, see e.g. [4] for the notations.

Theorem 1.3. The commutator map is surjective for classical algebras $\mathfrak{sl}_m(\mathbb{H})(m \geq 2), \mathfrak{so}_{p,q}(1 \leq p \leq q, q \neq p + 2), \mathfrak{u}_{2m}^*(\mathbb{H})(m \geq 2), \mathfrak{sp}_{p,q}(1 \leq p \leq q)$ and for exceptional algebras EIV, EVI, EVII, EIX, FII.

Recall that $g$ is said to be of inner type if the Cartan involution of $g$ is an inner automorphism or, equivalently, if $\mathfrak{k}$ has full rank. The list of simple algebras of inner type comprises all Hermitian algebras as well as $\mathfrak{so}_{p,q}$ with $p$ or $q$ even, $p, q \neq 2$, $\mathfrak{sp}_{p,q}$, and exceptional algebras EII, EV, EVI, EVIII, EIX, FII, FIII, G.

Theorem 1.4. If $g$ is a simple non-Hermitian algebra of inner type then the commutator map of $g$ is surjective. In particular, the commutator map is surjective for $\mathfrak{so}_{p,p+2}$ ($p > 2$, $p$ even) and EII.

To summarize, Corollary 1.2, Theorem 1.3 and Theorem 1.4 show that the only real simple algebras, for which the surjectivity of the commutator map is an open question, are $\mathfrak{su}_{p,q}$ ($p$ or $q > 1$), $\mathfrak{so}_{p,p+2}$ ($p$ odd or $p = 2$), $\mathfrak{u}_{2m+1}^*(\mathbb{H})$ ($m \geq 1$) and EIII, with well-known overlaps between series.

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2. Preliminaries

We gather here several known facts which will be used later on. Let $\mathfrak{z}(A)$ denote the centralizer of an element $A$ in a Lie algebra $g$.

Lemma 2.1. Assume $g$ has a non-degenerate bilinear form $\beta$. Then

$\text{Im ad}(A) = \mathfrak{z}(A)^\perp$,

where the orthogonal complement is taken with respect to any such form $\beta$. In particular, this complement is independent of $\beta$.

Proof. For any $B \in \mathfrak{z}(A)$ and any $X \in g$ one has

$\beta(B, [A, X]) = \beta([B, A], X) = 0$,

showing that $\mathfrak{z}(A) \subset \text{Im ad}(A)^\perp$. Since $\beta$ is non-degenerate, this inclusion is in fact an equality by the dimension argument. \qed
Lemma 2.2. Let $V$ be a real vector space and $\Gamma \subset \text{GL}(V)$ a finite linear group acting without fixed vectors. Then the convex hull of any $\Gamma$-orbit contains $0 \in V$.

Proof. Assigning mass 1 to every point of an orbit $\Gamma \cdot v$, we see that the mass center is a fixed vector in the convex hull $\text{Conv}(\Gamma \cdot v)$. Hence $0 \in \text{Conv}(\Gamma \cdot v)$. □

Let $g = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of a real semisimple Lie algebra $g$, let $a \subset \mathfrak{p}$ be a Cartan subspace, and let $K$ be the adjoint group $\exp(\text{ad} \mathfrak{k}) \subset \text{GL}(g)$. The Weyl group acting on $a$ is denoted by $W$. The following result is due to B.Kostant [3].

Lemma 2.3. Consider the orthogonal decomposition $\mathfrak{p} = a + a^\perp$ with respect to the Killing form and let $\pi : \mathfrak{p} \to a$ be the corresponding projection map. Then for any $X \in \mathfrak{p}$ the intersection $K \cdot X \cap a$ is a single $W$-orbit and one has

$$\pi(K \cdot X) = \text{conv}(K \cdot X \cap a).$$

Proof. The proof is found in [3], see Prop.2.4 for the first statement and Thm.8.2 for the second one. □

Corollary 2.4. For any $X \in \mathfrak{p}$ there exists an element $k \in K$, such that $k \cdot X \in a^\perp$.

Proof. The convex hull of a Weyl group orbit contains 0 by Lemma 2.2. Therefore, by Lemma 2.3 there exists $k \in K$, such that $\pi(k \cdot X) = 0$. □

3. Commutator map $\mathfrak{k} \times \mathfrak{p} \to \mathfrak{p}$

We keep the above notations. Recall that an element $A \in \mathfrak{p}$ is called regular if $\mathfrak{z}(A) \cap \mathfrak{p}$ is a Cartan subspace.

Theorem 3.1. For any $X \in \mathfrak{p}$ there exist $Y \in \mathfrak{k}$ and a regular $A \in \mathfrak{p}$, such that $X = [Y, A]$.

Proof. By Corollary 2.4 we may assume that $X \in a^\perp$. Take any regular element $A \in a$. Its centralizer is of the form $\mathfrak{z}(A) = \mathfrak{z}(A) \cap \mathfrak{k} + \mathfrak{a}$. Since $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to the Killing form, we have $X \in \mathfrak{z}(A)^\perp$. Therefore $X = [Z, A]$ for some $Z \in \mathfrak{g}$ by Lemma 2.1. Now write $Z$ as the sum $Z = Y + Y'$, where $Y \in \mathfrak{k}, Y' \in \mathfrak{p}$. Then $[Y', A] \in \mathfrak{k}$ and, in fact, $[Y', A] = 0$, hence $X = [Y, A]$. □

Corollary 3.2. If $g$ is split then for any $X \in g$ there exist $Y \in g$ and a regular $A \in \mathfrak{p}$, such that $X = [Y, A]$. 

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Proof. Let \( X = X_1 + X_2 \), where \( X_1 \in \mathfrak{k}, X_2 \in \mathfrak{p} \). As in the proof of Theorem 3.1 we may assume that \( X_2 \in \mathfrak{a}^\perp \). But \( X_1 \) is also in \( \mathfrak{a}^\perp \), hence \( X \in \mathfrak{a}^\perp \). Since \( \mathfrak{g} \) is split, \( \mathfrak{a} \) is a Cartan subalgebra in \( \mathfrak{g} \). Thus \( \mathfrak{a} = \mathfrak{z}(\mathfrak{A}) \) for any regular \( \mathfrak{A} \in \mathfrak{a} \), and our assertion follows from Lemma 2.1. \( \square \)

**Corollary 3.3.** Any element of a compact semisimple Lie algebra is a commutator of two elements and one of these two can be chosen regular.

*Proof.* Let \( \mathfrak{k} \) be a compact semisimple subalgebra. We can apply Theorem 3.1 to the algebra \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \). Here \( \mathfrak{p} = i\mathfrak{k} \), and so for any \( \mathfrak{k} \in \mathfrak{k} \) we get a presentation \( \mathfrak{i} \mathfrak{k} = [\mathfrak{Y}, \mathfrak{i} \mathfrak{A}] \) or, equivalently, \( X = [\mathfrak{Y}, \mathfrak{A}] \), where \( \mathfrak{Y}, \mathfrak{A} \in \mathfrak{k} \) and \( \mathfrak{A} \) is regular. \( \square \)

## 4. Proofs of main results

We shall prove a theorem which is slightly more precise than Theorem 1.1. Recall that \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \) is the Cartan decomposition of a semisimple Lie algebra \( \mathfrak{g} \) and \( \mathfrak{m} \) is the centralizer of a Cartan subspace \( \mathfrak{a} \subset \mathfrak{p} \) in \( \mathfrak{k} \). Choose a Cartan subalgebra \( \mathfrak{t} \) in the reductive algebra \( \mathfrak{m} \). The orthogonal complement to a subspace \( \mathfrak{V} \subset \mathfrak{g} \) with respect to the Killing form is denoted by \( \mathfrak{V}^\perp \). The Killing form is positive definite on \( \mathfrak{p} \), negative definite on \( \mathfrak{k} \), and \( \mathfrak{p} \) and \( \mathfrak{k} \) are mutually orthogonal. For \( \mathfrak{V} \subset \mathfrak{k} \), resp. \( \mathfrak{V} \subset \mathfrak{p} \), we write \( \mathfrak{V}^\perp \) instead of \( \mathfrak{V}^\perp \cap \mathfrak{k} \), resp \( \mathfrak{V}^\perp \) instead of \( \mathfrak{V}^\perp \cap \mathfrak{p} \).

**Theorem 4.1.** If \( \mathfrak{m} \) is semisimple then any element of \( \mathfrak{g} \) is contained in the image of \( \text{ad}(\mathfrak{A}) \) for some regular element \( \mathfrak{A} \in \mathfrak{g} \).

*Proof.* Let \( M = \exp(\text{ad}(\mathfrak{m})) \subset K \). We can apply Corollary 2.4 to the semisimple algebra \( \mathfrak{m} + i\mathfrak{m} \) and the the Cartan subspace \( i\mathfrak{k} \subset i\mathfrak{m} \). Thus, given \( \mathfrak{Y} \in \mathfrak{m} \) we can find \( \mathfrak{m} \in M \) such that \( \mathfrak{m} \cdot \mathfrak{Y} \) is orthogonal to \( \mathfrak{k} \) with respect to the Killing form of \( \mathfrak{m} \). However, the orthogonal complement of \( \mathfrak{k} \) in \( \mathfrak{m} \) is the same for all invariant non-degenerate bilinear forms on \( \mathfrak{m} \), because it coincides with the image of \( \text{ad}(\mathfrak{A}) \) for any regular \( \mathfrak{A} \in \mathfrak{k} \).

In particular, \( \mathfrak{m} \cdot \mathfrak{Y} \in \mathfrak{t}^\perp \cap \mathfrak{m} \).

Starting with an arbitrary \( \mathfrak{X} \in \mathfrak{g} \), write \( \mathfrak{X} = \mathfrak{Y} + \mathfrak{Y}' + \mathfrak{Z} \), where \( \mathfrak{Y} \in \mathfrak{m}, \mathfrak{Y}' \in \mathfrak{m}^\perp \) and \( \mathfrak{Z} \in \mathfrak{p} \). Our goal is to present \( \mathfrak{X} \) as a commutator. In doing so, we can replace \( \mathfrak{X} \) by a conjugate element. By Corollary 2.4 we may assume that \( \mathfrak{Z} \in \mathfrak{a}^\perp \). Now choose \( \mathfrak{m} \in M \) for the given \( \mathfrak{Y} \in \mathfrak{m} \) as above. Then \( \mathfrak{m} \cdot \mathfrak{Y} \in \mathfrak{t}^\perp \) and \( \mathfrak{m} \cdot \mathfrak{Y}' \in \mathfrak{m}^\perp \). Since \( \mathfrak{m} \) preserves \( \mathfrak{a} \) and therefore also \( \mathfrak{a}^\perp \), we have \( \mathfrak{m} \cdot \mathfrak{Z} \in \mathfrak{a}^\perp \). In the decomposition \( \mathfrak{m} \cdot \mathfrak{X} = \mathfrak{m} \cdot \mathfrak{Y} + \mathfrak{m} \cdot \mathfrak{Y}' + \mathfrak{m} \cdot \mathfrak{Z} \) the first and the second summand are in \( \mathfrak{k} \) and in \( \mathfrak{t}^\perp \), whereas the last one is in \( \mathfrak{p} \) and in \( \mathfrak{a}^\perp \). Since \( \mathfrak{k} \) and \( \mathfrak{p} \) are orthogonal with respect to the Killing form of \( \mathfrak{g} \), it follows that \( \mathfrak{m} \cdot \mathfrak{X} \in \mathfrak{t}^\perp \cap \mathfrak{a}^\perp = (\mathfrak{t} + \mathfrak{a})^\perp \).
Finally, $\mathfrak{t} + \mathfrak{a}$ is a Cartan subalgebra in $\mathfrak{g}$. Therefore $m \cdot X \in \text{Im}\text{ad}(A)$ for any regular $A \in \mathfrak{t} + \mathfrak{a}$ by Lemma 2.1.

Consider the root system $\Delta$ of $\mathfrak{g}^C$ with respect to the Cartan subalgebra $\mathfrak{h} = (\mathfrak{t} + \mathfrak{a})^C$. The roots vanishing on $\mathfrak{a}$ are called compact, the remaining roots are called noncompact. The set of compact, resp. noncompact, roots is denoted by $\Delta_c$, resp. $\Delta_{nc}$. Let $\theta : \mathfrak{g}^C \to \mathfrak{g}^C$ be the involutive automorphism acting as $\text{Id}$ on $\mathfrak{t}^C$ and as $-\text{Id}$ on $\mathfrak{p}^C$. One can choose an ordering of $\Delta$ in such a way that the root $\theta^T(\alpha)$ is negative if $\alpha \in \Delta_{nc}$ is positive. I. Satake [5] showed then that for every simple root $\alpha \in \Delta_{nc}$ there is another simple root $\alpha' \in \Delta_{nc}$, such that

$$\theta^T(\alpha) = -\alpha' - \sum c_{\alpha\beta}\beta,$$

where $\beta$ ranges over simple roots in $\Delta_c$ and $c_{\alpha\beta}$ are non-negative integers. The mapping $\alpha \mapsto \alpha'$ is involutive. The Satake diagram of $\mathfrak{g}$ is defined as the Dynkin diagram of $\mathfrak{g}^C$, on which the compact roots are denoted by black circles, the non-compact roots are denoted by white circles, and the white circles corresponding to $\alpha$ and $\alpha' \neq \alpha$ are joined by two-pointed arrows.

**Theorem 4.2.** Let $\mathfrak{z}(m)$ be the center of $\mathfrak{m}$. Then $\dim \mathfrak{z}(m)$ equals the number of two-pointed arrows on the Satake diagram.

**Proof.** Let $r_c$ and $r_{nc}$ be the numbers of compact and non-compact simple roots, respectively. Write $l = r_c + r_{nc}$ for the rank of $\mathfrak{g}^C$. Clearly,

$$\mathfrak{m}^C = \mathfrak{t}^C + \sum_{\alpha \in \Delta_c} (\mathfrak{g}^C)_\alpha.$$ 

Let $\alpha_1, \ldots, \alpha_c$ be all compact simple roots. Then

$$\{ H \in \mathfrak{h} \mid \alpha_1(H) = \ldots = \alpha_c(H) = 0 \} = \mathfrak{z}(m)^C + \mathfrak{a}^C.$$ 

Therefore

$$l - r_c = \dim \mathfrak{z}(m) + \dim \mathfrak{a}.$$ 

On the other hand, $\dim \mathfrak{a}$ is the number of simple restricted roots, i.e., $r_{nc}$ minus the number of two-pointed arrows.

**Proof of Theorem 1.3.** All Satake diagrams of real simple Lie algebras are listed, see e.g. [4], Table 5. The diagrams without two-pointed arrows are easily found.

Finally, Theorem 1.4 follows from a more general result, which we now prove.

**Theorem 4.3.** Let $\mathfrak{g}$ be a real semisimple Lie algebra of inner type. Then any element of $[\mathfrak{k}, \mathfrak{k}] + \mathfrak{p}$ is the commutator of two elements of $\mathfrak{g}$.
Proof. Let $c$ be the center of $\mathfrak{k}$, so that $\mathfrak{k} = c + [\mathfrak{k}, \mathfrak{k}]$. Let $\mathfrak{t}$ be a Cartan subalgebra in $[\mathfrak{k}, \mathfrak{k}]$. Take $X \in [\mathfrak{k}, \mathfrak{k}]$ and $Y \in \mathfrak{p}$. Applying Corollary 2.4 as in the proof of Theorem 4.1 we can find $k \in K$, so that $k \cdot X$ is orthogonal to $\mathfrak{t}$ with respect to the Killing form of the semisimple algebra $[\mathfrak{k}, \mathfrak{k}]$. But the orthogonal complement to $\mathfrak{t}$ in $[\mathfrak{k}, \mathfrak{k}]$ is the same for all invariant non-degenerate bilinear forms on $[\mathfrak{k}, \mathfrak{k}]$ and, in particular, for the restriction of the Killing form of $\mathfrak{g}$. It follows that $k \cdot X \in \mathfrak{t}^\perp$. On the other hand, $k \cdot X \in c^\perp$, hence $k \cdot X \in (c + \mathfrak{t})^\perp$. Since we also have $k \cdot Y \in \mathfrak{p} = \mathfrak{t}^\perp \subset (c + \mathfrak{t})^\perp$, it follows from Lemma 2.1 that $k \cdot (X + Y) \in \text{Im} \text{ad}(A)$ for any regular $A$ in the Cartan subalgebra $c + \mathfrak{t} \subset \mathfrak{g}$.

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