A Numerical Example about the Geometric Approach to the Output Regulation Problem with Stability for Linear Switching Systems

Elena Zattoni* Anna Maria Perdon○ Giuseppe Conte○

Abstract

This note presents a numerical example worked out in order to illustrate the solution to the output regulation problem with quadratic stability for linear switching systems derived in [1].

Index Terms

Switching systems, output regulation, quadratic stability, geometric approach, linear matrix inequalities.

I. INTRODUCTION

In [1], the classical output regulation problem for linear time-invariant systems (see, e.g., [2], [3]) has been given an extended formulation in the case of linear switching systems. The solution of the problem with stability has been achieved through the geometric approach [4], [5], along the lines first proposed in [6] for linear time-invariant systems. The main feature of the geometric solution developed in [1] is allowing the structural issue (i.e., the requirement that the regulation error goes to zero uniformly under arbitrary switching) to be dealt with separately from the stability issue (i.e., the requirement that the regulation loop satisfies specific stability conditions under arbitrary switching). In this way, different stability specifications, such as asymptotic stability, exponential stability, or quadratic stability, only impact on few, precisely-defined, aspects of the solution. In particular, the synthesis procedure presented in [1] refers to quadratic stability and makes an extensive use of linear matrix inequalities [7]–[10]. The scope of this technical note is to help the reader in the implementation of the proposed synthesis procedure by illustrating the single steps with a numerical example.

II. NOTATION

The symbol $\mathbb{R}$ stands for the set of real numbers. Matrices and linear maps are denoted by upper-case letters, like $A$. The image of $A$ is denoted by $\text{Im } A$. The transpose of $A$ is denoted

*E. Zattoni is with the Department of Electrical, Electronic and Information Engineering “Guglielmo Marconi”, Alma Mater Studiorum · University of Bologna, Viale Risorgimento 2, 40136 Bologna, Italy. E-mail: elena.zattoni@unibo.it
○A. M. Perdon and G. Conte are with the Department of Information Engineering, Polytechnic University of Marche, Via Brecce Bianche, 60131 Ancona, Italy. E-mail: perdon@univpm.it, gconte@univpm.it
by $A^T$. Vector spaces and subspaces are denoted by calligraphic letters, like $\mathcal{V}$. The symbols $I_n$, $O_{m \times n}$, and $0_n$ are respectively used for the identity matrix of dimension $n$, the $m \times n$ zero matrix, and the $n$-dimensional zero vector (subscripts are omitted when the dimensions can be inferred from the context). The symbol $M > 0$, where $M \in \mathbb{R}^{n \times n}$ is symmetric, means that $M$ is positive-definite: i.e., $x^T M x > 0$ for all nonzero $x \in \mathbb{R}^n$. Similarly, $M < 0$ means that $M$ is negative-definite. The symbol $\lambda_{\text{max}}(M)$ denotes the maximal eigenvalue of the matrix $M = M^T$. Similarly, $\lambda_{\text{min}}(M)$ denotes the minimal eigenvalue of $M$.

III. A Numerical Example

The aim of this section is to illustrate a computational framework for the synthesis procedure developed in [1], with the help of a numerical example. The basic tools that will be used are the subspace computation algorithms of the Geometric Approach Toolbox, first appeared in [5] and now available on-line in an upgraded version, and the LMI solvers of the Robust Control Toolbox [11]. The variables will be displayed in scaled fixed point format with five digits, although the computation will be made in floating point precision.

Let $\Sigma_{\sigma(t)}$, defined by (1) in [1], be a discrete-time sample-data linear switching system, with sampling time $T_s = 0.1$ s. Let $\mathcal{I} = \{1, 2\}$ and

$$A_1 = \begin{bmatrix} 0.6 & 0 & 0 & 0 & 0.1 & 0 \\ 1 & 1 & -0.3 & 0 & -0.2 & 0 \\ 0 & 0.4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.9 & 0.25 & -0.4 \\ 0 & 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & -0.2 & 0.7 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & 0 \\ 0.4 & 1 & 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 0 & 2.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.6 & 0 & 0 & 0 & -0.3 & 0 \\ 1 & 1 & -0.3 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & -0.4 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 0 & 2.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$
Let the exogenous system \( \Sigma_{g, \sigma(t)} \), defined by (3) in [1], have the following matrices

\[
A_{g,1} = A_{g,2} = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
E_{g,1} = E_{g,2} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}.
\]

Namely, the internal model of the ramp signal is replicated in the exogenous system dynamics a number of times equal to the number of the outputs of the to-be-controlled system, so that independent reference signals can be obtained for each output, based on the exosystem initial state.

Assumption 1 in [1] is satisfied, since the switching system \( \Sigma_{\sigma(t)} \) is quadratically stable under arbitrary switching. In fact, the LMIs \( A_i^T Q A_i - Q < 0 \) hold for all \( i \in \mathcal{I} \) with \( Q \) given, e.g., by

\[
Q = \begin{bmatrix}
6.5135 & 0.2046 & -0.2647 & -0.0873 & -0.7930 & -0.0446 \\
0.2046 & 1.0830 & -1.0809 & 0.0752 & 0.0876 & -0.1392 \\
-0.2647 & -1.0809 & 2.8705 & -0.1276 & -0.2097 & 0.2743 \\
-0.0873 & 0.0752 & -0.1276 & 2.0635 & 0.4823 & -0.6029 \\
-0.7930 & 0.0876 & -0.2097 & 0.4823 & 6.5706 & -0.7474 \\
-0.0446 & -0.1392 & 0.2743 & -0.6029 & -0.7474 & 6.1649
\end{bmatrix},
\]

which is symmetric and positive-definite. Moreover, Assumption 2 in [1] is satisfied by the switching system \( \Sigma_{e,\sigma(t)} \), defined according to (5)–(7) in [1]. In fact, \( \Sigma_{e,\sigma(t)} \) is quadratically stabilizable under arbitrary switching by the linear output injection matrices

\[
G_{e,1} = \begin{bmatrix}
0.0116 & -0.0035 \\
0.0490 & -0.0183 \\
0.0469 & 0.0082 \\
-0.0057 & 0.0525 \\
0.0043 & 0.0036 \\
-0.0043 & -0.0248 \\
-0.6393 & 0.0644 \\
-0.1291 & 0.0184 \\
-0.0118 & -1.3235 \\
-0.0008 & -0.3628
\end{bmatrix}, \quad G_{e,2} = \begin{bmatrix}
0.0074 & -0.0078 \\
0.0500 & -0.0157 \\
0.0346 & -0.0016 \\
-0.0084 & 0.0500 \\
0.0042 & 0.0059 \\
-0.0016 & -0.0229 \\
-0.6393 & 0.0649 \\
-0.1293 & 0.0185 \\
-0.0113 & -1.3235 \\
-0.0007 & -0.3627
\end{bmatrix},
\]

which have been derived through the solution of the LMIs (A.8), as specified in Appendix A of [1]. Hence, in order to solve Problem 1 in [1] (i.e., the autonomous regulator problem with quadratic stability or, briefly, ARPQS), Conditions (i), (ii) of Theorem 2 in [1] are checked. The
maximal robust controlled invariant subspace $V_R^*$ is given by

$$V_R^* = \text{Im} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.3363 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.7071 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -0.9417 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0.7071 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$ 

The subspace $\mathcal{P}$ is defined as

$$\mathcal{P} = \text{Im} \begin{bmatrix} I_6 \\ O_{4 \times 6} \end{bmatrix},$$

according to (19) in [1]. Therefore, Condition (i) of Theorem 2 in [1] is satisfied. Moreover, taking a suitable set of state feedbacks satisfying (39) of Lemma 2 in [1], one gets that also Condition (ii) of Theorem 2 in [1] is met. In particular, a subspace $\mathcal{V}$, computed along the lines sketched in the proof of Theorem 2 in [1], and, therefore, satisfying Conditions (i), (ii) of Theorem 1 in [1] is

$$\mathcal{V} = \text{Im} \begin{bmatrix}
0.1009 & 0 & -0.9196 & -0.1875 \\
0.8661 & -0.1326 & -0.9219 & -2.5469 \\
0.3363 & 0 & 0 & 0 \\
0 & -0.7071 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0.1009 & 0.1768 & -1.1875 & 2.3125 \\
0.9417 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & -0.7071 & 0 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}.$$ 

Consequently, a set $\{F_{e,i}, i \in I\}$ of state feedbacks, defined according to Lemma 1 in [1], consists of the matrices

$$F_{e,1} = \begin{bmatrix}
-0.0998 & 0.0551 & -0.1138 & -0.0053 & 0.3356 & 0.0230 & -0.0018 & 0.0137 & 0.0007 & -0.0685 \\
-0.2954 & 0.2055 & -0.4266 & -0.0302 & 0.9029 & 0.1199 & 0.0251 & 0.4149 & 0.0216 & -0.1157 \\
0.1318 & -0.1232 & 0.2606 & -0.0053 & -0.5643 & 0.0546 & -0.0104 & -0.0841 & -0.0330 & -0.6033
\end{bmatrix}.$$
\[ F_{e,2} = \begin{bmatrix}
0.1502 & -0.1593 & 0.3113 & 0.0374 & 0.3058 & -0.1429 & 0.0346 & 0.1784 & -0.0433 & 0.0470 \\
-0.0537 & 0.1395 & -0.2776 & -0.0426 & -0.0045 & 0.1620 & 0.0021 & 0.2035 & 0.0569 & 0.1044 \\
0.3953 & -0.2748 & 0.5710 & 0.0130 & 0.0096 & -0.0227 & -0.0019 & -0.0949 & -0.0421 & -0.4456 
\end{bmatrix} \]

Instead, a set \( \{ G_{e,i}, i \in I \} \) of output injections, defined as in the if-part of the proof of Theorem 1 in [1], consists of the matrices \( G_{e,1} \) and \( G_{e,2} \) previously computed. Finally, the matrices of the switching regulator \( \Sigma_{r,\sigma(t)} \), defined by (8) in [1], are determined according to (21)–(22) in [1].

A simulation on the given switching system \( \Sigma_{\sigma(t)} \) and the given exogenous system \( \Sigma_{g,\sigma(t)} \) is run with the following data. The time goes from 0 s to 10 s, so that, the total number of samples is 100. The switching signal \( \sigma(t) \) is defined by

\[
\sigma(t) = \begin{cases} 
1, & \text{with } t = 0, \ldots, 29, \\
2, & \text{with } t = 30, \ldots, 69, \\
1, & \text{with } t = 70, \ldots, 99.
\end{cases}
\]

Namely, the active mode of \( \Sigma_{\sigma(t)} \) is \( \Sigma_1 \), between 0 s and 2.9 s as well as between 7 s and 9.9 s, while the active mode is \( \Sigma_2 \) between 3 s and 6.9 s. Figure 1 shows the graphs of the error for the two outputs. The graphs clearly reveal the different dynamics before and after the time 3 s, when the first switch occurs, while the effect of the second switch, at the time 7 s, is less evident, mainly because of the smaller values of the error, which goes to zero as the time increases.

### IV. Conclusion

The implementation of the procedure for synthesizing a switching compensator achieving asymptotic tracking of the reference signal and quadratic stability of the closed-loop presented in [1] has been illustrated by means of a numerical example. References have been provided for the computational tools that have been used.

### References

[1] E. Zattoni, A. M. Perdon, and G. Conte, “The output regulation problem with stability for linear switching systems: A geometric approach,” *Automatica*, vol. 49, no. 10, pp. 2953–2962, October 2013, http://dx.doi.org/10.1016/j.automatica.2013.07.005.

[2] B. A. Francis and W. M. Wonham, “The internal model principle of control theory,” *Automatica*, vol. 12, pp. 457–463, 1976.

[3] B. A. Francis, “The linear multivariable regulator problem,” *SIAM Journal on Control and Optimization*, vol. 15, no. 3, pp. 486–505, May 1977.

[4] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 3rd ed. New York: Springer-Verlag, 1985.

[5] G. Basile and G. Marro, *Controlled and Conditioned Invariants in Linear System Theory*. Englewood Cliffs, New Jersey: Prentice Hall, 1992.

[6] G. Marro, “Multivariable regulation in geometric terms: Old and new results,” in *Colloquium on Automatic Control*, ser. Lecture Notes in Control and Information Sciences, C. Bonivento, G. Marro, and R. Zanasi, Eds. Berlin / Heidelberg: Springer, 1996, vol. 215, pp. 77–138.
Fig. 1. Graphs of the error – amplitude vs. time (s)

[7] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, ser. SIAM Studies in Applied and Numerical Mathematics. Philadelphia, PA: Society for Industrial and Applied Mathematics, 1994.

[8] E. Feron, “Quadratic stabilizability of switched linear systems via state and output feedback,” Center for Intelligent Control Systems, Providence, RI, Tech. Rep. CICS-P-468, 1996.

[9] G. Garcia, J. Bernussou, and D. Arzelier, “Stabilization of an uncertain linear dynamic system by state and output feedback: A quadratic stabilizability approach,” *International Journal of Control*, vol. 64, no. 5, pp. 839–858, 1996.

[10] L. El Ghaoui and S.-I. Niculescu, Eds., *Advances in Linear Matrix Inequality Methods in Control*, ser. Advances in Design and Control. Philadelphia, PA: Society for Industrial and Applied Mathematics, 2000.

[11] G. Balas, R. Chiang, A. Packard, and M. Safonov, “LMI Solvers,” in Robust Control Toolbox™– Getting Started Guide R2012b. Natick, MA: The MathWorks Inc., 2012, pp. 1–22.