Variable Besov spaces: continuous version

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Abstract

We introduce Besov spaces with variable smoothness and integrability by using the continuous version of Calderón reproducing formula. We show that our space is well-defined, i.e., independent of the choice of basis functions. We characterize these function spaces by so-called Peetre maximal functions and we obtain the Sobolev embeddings for these function spaces. We use these results to prove the atomic decomposition for these spaces.

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1 Introduction

Function spaces play an important role in harmonic analysis, in the theory of differential equations and in almost every other field of applied mathematics. Some of these function spaces are Besov spaces. The theory of these spaces has been developed in detail in [35] and [36] (and continued and extended in the more recent monographs [37] and [38]), but has a longer history already including many contributors; we do not want to discuss this here. For general literature on function spaces we refer to [1, 5, 19, 29, 30-31, 39] and references therein.

Based on continuous characterizations of Besov spaces, we introduce a new family of function spaces of variable smoothness and integrability. These type of function spaces, initially appeared in the paper of A. Almeida and P. Hästö [4], where several basic properties were shown, such as the Fourier analytical characterization, Sobolev embeddings and the characterization in terms of Nikolskij representations involving sequences of entire analytic functions. Later, the present author characterized these spaces by local means and established the atomic characterization (see [14]). After that, Kempka and Vybíral [26] characterized these spaces by ball means of differences and also by local means, see [28] and [23] for the duality of these function spaces.

This paper is organized as follows. First we give some preliminaries, where we fix some notations and recall some basic facts on function spaces with variable integrability and we give some key technical lemmas needed in the proofs of the main statements. We then define the Besov spaces $\mathcal{B}^{p,q}_{\alpha,\beta}$. We prove a useful characterization of these spaces based on the so called local means. The theorem on local means that is proved for Besov spaces of variable smoothness and integrability is highly technical and its
proof is required new techniques and ideas. Using the results from Sections 3 and 4, we prove in Section 5 the atomic decomposition for $B_{p(\cdot),q(\cdot)}^{\alpha}$. 

2 Preliminaries

As usual, we denote by $\mathbb{N}_0$ the set of all non-negative integers. The notation $f \lesssim g$ means that $f \leq cg$ for some independent constant $c$ (and non-negative functions $f$ and $g$), and $f \approx g$ means that $f \lesssim g \lesssim f$. For $x \in \mathbb{R}$, $[x]$ stands for the largest integer smaller than or equal to $x$.

If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the Lebesgue measure of $E$ and $\chi_E$ denotes its characteristic function. We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of all rapid decreasing functions on $\mathbb{R}^n$, and by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of $\mathcal{S}(\mathbb{R}^n)$. The Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ is denoted by $\mathcal{F}f$. By $c$ we denote generic positive constants, which may have different values at different occurrences. Although the exact values of the constants are usually irrelevant for our purposes, sometimes we emphasize their dependence on certain parameters (e.g., $c(p)$ means that $c$ depends on $p$, etc.). Further notation will be properly introduced whenever needed.

The variable exponents those we consider are always measurable functions $p$ on $\mathbb{R}^n$ with range in $[1, \infty]$. We denote by $\mathcal{P}(\mathbb{R}^n)$ the set of such functions. We use the standard notation:

\[ p^- := \operatorname{ess}\inf_{x \in \mathbb{R}^n} p(x) \quad \text{and} \quad p^+ := \operatorname{ess}\sup_{x \in \mathbb{R}^n} p(x). \]

The variable exponent modular is defined by

\[ \varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx. \]

The variable exponent Lebesgue space $L^{p(\cdot)}$ consists of measurable functions $f$ on $\mathbb{R}^n$ such that $\varrho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$. We define the Luxemburg (quasi)-norm on this space by the formula

\[ \|f\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left( \frac{f}{\lambda} \right) \leq 1 \right\}. \]

A useful property is that $\|f\|_{L^{p(\cdot)}} \leq 1$ if and only if $\varrho_{p(\cdot)}(f) \leq 1$ (see Lemma 3.2.4 from [12]).

A weight function (a weight) is a measurable function $w : \mathbb{R}^n \to (0, \infty)$. Given a weight $w$, we denote by $L^{p(\cdot)}(w)$ the space of all measurable functions $f$ on $\mathbb{R}^n$ such that

\[ \|f\|_{L^{p(\cdot)}(w)} := \left\| f w^{\frac{1}{p'(\cdot)}} \right\|_{L^{p(\cdot)}} < \infty. \]

Let $p,q \in \mathcal{P}(\mathbb{R}^n)$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}_{\geq}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular

\[ \varrho_{\ell^{q(\cdot)}_{\geq}(L^{p(\cdot)})}((f_v)_v) := \sum_{v=1}^{\infty} \inf \left\{ \lambda_v > 0 : \varrho_{q(\cdot)} \left( \frac{f_v}{\lambda_v^{1/q(\cdot)}} \right) \leq 1 \right\}. \]
The (quasi)-norm is defined from this as usual:

\[ \|(f_v)_v\|_{L^p}\left(\cdot\right) := \inf\left\{ \mu > 0 : g_{L^p}\left(\cdot\right) \left(\frac{1}{\mu}(f_v)_v\right) \leq 1 \right\}. \]

If \( q^+ < \infty \), then we can replace (1) by a simpler expression:

\[ g_{L^p}\left(\cdot\right) \left((f_v)_v\right) := \sum_{v=1}^{\infty} \| |f_v|^{q(\cdot)} \|_{L^p}. \]

The case \( p = \infty \) can be included by replacing the last modular by

\[ g_{L^\infty}\left(\cdot\right) \left((f_v)_v\right) := \sum_{v=1}^{\infty} \| |f_v|^{q(\cdot)} \|_{\infty}. \]

We say that a real valued-function \( g \) on \( \mathbb{R}^n \) is \textit{locally log-Hölder continuous} on \( \mathbb{R}^n \), abbreviated \( g \in C_{\text{log}}^\text{loc}(\mathbb{R}^n) \), if for any compact set \( K \) of \( \mathbb{R}^n \), there exists a constant \( c_{\text{log}}(g) > 0 \) such that

\[ |g(x) - g(y)| \leq \frac{c_{\text{log}}(g)}{\log(e + 1/|x - y|)} \]

for all \( x, y \in K \). If

\[ |g(x) - g(0)| \leq \frac{c_{\text{log}}(g)}{\log(e + 1/|x|)} \]

for all \( x \in \mathbb{R}^n \), then we say that \( g \) is \textit{log-Hölder continuous at the origin} (or has a \textit{logarithmic decay at the origin}). We say that \( g \) satisfies the \textit{log-Hölder decay condition}, if there exist two constants \( g_\infty \in \mathbb{R} \) and \( c_{\text{log}} > 0 \) such that

\[ |g(x) - g_\infty| \leq \frac{c_{\text{log}}}{\log(e + |x|)} \]

for all \( x \in \mathbb{R}^n \). We say that \( g \) is \textit{globally log-Hölder continuous} on \( \mathbb{R}^n \), abbreviated \( g \in C_{\text{log}}^\text{loc}(\mathbb{R}^n) \), if it is locally log-Hölder continuous on \( \mathbb{R}^n \) and satisfies the log-Hölder decay condition. The constants \( c_{\text{log}}(g) \) and \( c_{\text{log}} \) are called the \textit{locally log-Hölder constant} and the \textit{log-Hölder decay constant}, respectively. We note that any function \( g \in C_{\text{log}}^\text{loc}(\mathbb{R}^n) \) always belongs to \( L^\infty \).

We define the following class of variable exponents:

\[ \mathcal{P}_{\text{log}}(\mathbb{R}^n) := \left\{ p \in \mathcal{P}(\mathbb{R}^n) : \frac{1}{p} \in C_{\text{log}}(\mathbb{R}^n) \right\}, \]

which is introduced in \cite[Section 2]{13}. We define

\[ \frac{1}{p_{\infty}} := \lim_{|x| \to \infty} \frac{1}{p(x)}, \]

and we use the convention \( \frac{1}{\infty} = 0 \). Note that although \( \frac{1}{p} \) is bounded, the variable exponent \( p \) itself can be unbounded. We put

\[ \Psi(x) := \sup_{|y| \geq |x|} |\varphi(y)| \]
for $\varphi \in L^1$. We suppose that $\Psi \in L^1$. Then it was proved in \cite[Lemma 4.6.3]{11} that if $p \in P_{\log}(\mathbb{R}^n)$, then
\[
\|\varphi \ast f\|_{p(\cdot)} \leq c\|\Psi\|_1\|f\|_{p(\cdot)}
\]
for all $f \in L^{p(\cdot)}$, where
\[
\varphi_{\varepsilon} := \frac{1}{\varepsilon^n} \varphi \left( \frac{\cdot}{\varepsilon} \right), \quad \varepsilon > 0.
\]
We refer to \cite{7} and \cite{9}, where various results on maximal functions on variable Lebesgue spaces are obtained.

We put
\[
\eta_{t,m}(x) := t^{-n}(1 + t^{-1}|x|)^{-m}
\]
for any $x \in \mathbb{R}^n$, $t > 0$ and $m > 0$. Note that $\eta_{t,m} \in L^1$ when $m > n$ and that $\|\eta_{t,m}\|_1 = c(m)$ is independent of $t$. If $t = 2^{-v}$, $v \in \mathbb{N}_0$ then we put
\[
\eta_{2^{-v},m} := \eta_{v,m}.
\]
We refer to the recent monograph \cite{8} for further properties, historical remarks and references on variable exponent spaces.

\section{2.1 Some technical lemmas}

In this subsection we present some useful results. The following lemma is proved in \cite[Lemma 6.1]{11} (see also \cite[Lemma 19]{26}).

\begin{lemma}
Let $\alpha \in C_{\log}^0(\mathbb{R}^n)$, $m \in \mathbb{N}_0$ and let $R \geq c_{\log}(\alpha)$, where $c_{\log}(\alpha)$ is the constant from (2) for $g = \alpha$. Then there exists a constant $c > 0$ such that
\[
t^{-\alpha(x)}\eta_{t,m+R}(x - y) \leq c \, t^{-\alpha(y)}\eta_{t,m}(x - y)
\]
for any $0 < t \leq 1$ and $x, y \in \mathbb{R}^n$. In particular, if $\alpha$ decays logarithmically at the origin, then there exists a constant $c > 0$ such that
\[
t^{-\alpha(x)}\eta_{t,m+R}(x) \leq c \, t^{-\alpha(0)}\eta_{t,m}(x)
\]
for any $0 < t \leq 1$ and $x \in \mathbb{R}^n$.
\end{lemma}

The previous lemma allows us to treat the variable smoothness in many cases as if it were not variable at all. Namely, we can move the factor $t^{-\alpha(x)}$ inside the convolution as follows:
\[
t^{-\alpha(x)}\eta_{t,m+R} * f(x) \leq c \, \eta_{t,m} * (t^{-\alpha(\cdot)} f)(x).
\]

\begin{lemma}
Let $r, N > 0$, $m > n$ and $\theta, \omega \in \mathcal{S}(\mathbb{R}^n)$ with $\text{supp} \, \mathcal{F} \omega \subset B(0, 1)$. Then there exists a constant $c = c(r, m, n) > 0$ such that for all $g \in \mathcal{S}'(\mathbb{R}^n)$, we have
\[
|\theta_N * \omega_N * g(x)| \leq c(\eta_{N,m} * |\omega_N * g|)^{1/r}, \quad x \in \mathbb{R}^n,
\]
where $\theta_N(\cdot) := N^n \theta(N \cdot)$, $\omega_N(\cdot) := N^n \omega(N \cdot)$ and $\eta_{N,m} := N^n (1 + N |\cdot|)^{-m}$.\end{lemma}
The proof of this lemma is given in [16] by using the same arguments of [34] Chapter V, Theorem 5.

The next three lemmas are proved in [11], where the first one tells us that in most circumstances two convolutions are as good as one.

**Lemma 3** For \(v_0, v_1 \in \mathbb{N}_0\) and \(m > n\), we have

\[
\eta_{v_0,m} * \eta_{v_1,m} \approx \eta_{\min(v_0,v_1),m}
\]

with the constant depending only on \(m\) and \(n\).

For \(v \in \mathbb{N}_0\) and \(m = (m_1, ..., m_n) \in \mathbb{Z}^n\), let \(Q_{v,m}\) be the dyadic cube in \(\mathbb{R}^n\):

\[
Q_{v,m} = \{(x_1, ..., x_n) : m_i \leq 2^v x_i < m_i + 1, i = 1, 2, ..., n\}.
\]

As the collection of all such cubes, we define

\[
Q = \{Q_{v,m} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\}.
\]

Then we have:

**Lemma 4** Let \(v \in \mathbb{N}_0\) and \(m > n\). Then for any \(Q \in Q\) with \(l(Q) = 2^{-v}\), \(y \in Q\) and \(x \in \mathbb{R}^n\), we have

\[
\eta_{v,m} \left( \frac{x_Q}{|Q|} \right)(x) \approx \eta_{v,m}(x - y)
\]

with the constant depending only on \(m\) and \(n\), where \(l(Q)\) is the side length of \(Q\).

The next lemma is the Hardy type inequality, which is easy to prove.

**Lemma 5** Let \(0 < a < 1\), \(\sigma \geq 0\) and \(0 < q \leq \infty\). Let \(\{\varepsilon_k\}_k\) be a sequence of positive real numbers, and denote

\[
\delta_k = \sum_{j=-\infty}^{\infty} |k - j|^{\sigma} a^{|k-j|} \varepsilon_j.
\]

Then there exists a constant \(c > 0\) depending only on \(a\) and \(q\) such that

\[
\left( \sum_{k=-\infty}^{\infty} \delta_k^q \right)^{1/q} \leq c \left( \sum_{k=-\infty}^{\infty} \varepsilon_k^q \right)^{1/q}.
\]

Putting

\[
w(Q) := \int_{Q} w(x) \, dx,
\]

we have the following result (see Lemma 3.3 for \(w = 1\) from [13]).
Lemma 6  Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, and $w$ be a weight function on $\mathbb{R}^n$. Then, putting

$$\gamma_m := e^{-4mc(1/p)} \in (0, 1), \quad c(1/p) := \max(c_{\log}(1/p), c_{\log})$$

for every $m > 0$, and

$$p_Q^- := \text{ess-inf}_{z \in Q} p(z)$$

for a cube $Q$ in $\mathbb{R}^n$, we have the inequality:

$$\left( \frac{\gamma_m}{w(Q)} \int_Q |f(y)| w(y) \, dy \right)^{p(x)} \leq c \max\left(1, (w(Q))^{1-p(x) \over r_Q} \right) \frac{1}{w(Q)} \int_Q |f(y)|^{p(y)} w(y) \, dy \leq \frac{c \min(|Q|^m, 1)}{w(Q)} \int_Q \left\{ (e + |x|)^{-m} + (e + |y|)^{-m} \right\} w(y) \, dy$$

for some positive constant $c > 0$ every cube (or ball) $Q$, all $x \in Q$ and all $f \in L^{p(\cdot)}(w)$ with

$$\|f\|_{L^{p(\cdot)}(w)} \leq 1.$$ 

If $Q = (a, b) \subset \mathbb{R}$ with $0 < a < b < \infty$, then, putting

$$\gamma_m := e^{-4mc_{\log}(1/p)}$$

for every $m > 0$, we have the following inequality:

$$\left( \frac{\gamma_m}{w(Q)} \int_Q |f(y)| w(y) \, dy \right)^{p(x)} \leq c \max\left(1, (w(Q))^{1-p(x) \over r_Q} \right) \frac{1}{w(Q)} \int_Q \phi(y) w(y) \, dy \leq \frac{c \omega(m, b)}{w(Q)} \int_Q g(x, y) w(y) \, dy$$

for some positive constant $c > 0$, all $x \in Q$ and all $f \in L^{p(\cdot)}(w)$ with

$$\|f\|_{L^{p(\cdot)}(w)} \leq 1,$$

where we put

$$\omega(m, b) = \min(b^m, 1), \quad \phi(y) = |f(y)|^{p(y)}$$

and

$$g(x, y) = \left( e + \frac{1}{x} \right)^{-m} + \left( e + \frac{1}{y} \right)^{-m},$$

or

$$\omega(m, b) = \min(b^m, 1), \quad \phi(y) = |f(y)|^{p(0)}$$

and

$$g(x, y) = \left( e + \frac{1}{x} \right)^{-m} \chi_{\{z \in Q : p(z) < p(0)\}}(x)$$
with $p \in \mathcal{P}(\mathbb{R})$ being log-Hölder continuous at the origin. In addition, we have the same estimate, when

$$\omega(m, b) = 1, \quad \gamma_m = e^{-mc_{\log}}, \quad \phi(y) = |f(y)|^{p_{\infty}}$$

and

$$g(x, y) = (e + x)^{-m} \chi_{\{z \in Q: p(z) < p_{\infty}\}}(x)$$

with $p \in \mathcal{P}(\mathbb{R})$ satisfying the log-Hölder decay condition.

The proof of Lemma 6 is postponed to appendix.

**Lemma 7** Let $\{f_v\}_{v \in \mathbb{N}}$ be a sequence of measurable functions on $\mathbb{R}^n$ and $p \in \mathcal{P}(\mathbb{R}^n)$. Let $q \in \mathcal{P}(\mathbb{R})$ be log-Hölder continuous at the origin. Then

$$\left\| \left( t^{-\frac{1}{q(t)}} \left\| f_v \right\|_{q(t)} \chi_{[2^{-v}, 2^{1-v})} \right) \right\|_{L^q(\mathbb{R}^n)} \approx \left( \sum_{v=1}^{\infty} \left\| f_v \right\|_{q(0)}^{q(t)} \right)^{\frac{1}{q(0)}},$$

where the implicit positive constants are independent of $f_v, v \in \mathbb{N}$.

**Proof.** We divide the proof into two steps.

**Step 1.** Let $\{f_v\}_{v \in \mathbb{N}}$ be a sequence of measurable functions such that the right-hand side is less than or equal to 1. We will prove that

$$\sum_{v=1}^{\infty} \int_{2^{-v}}^{2^{1-v}} \left\| f_v \right\|_{q(t)}^{q(t)} \frac{dt}{t} \lesssim 1.$$  

Our estimate clearly follows from the inequality

$$\int_{2^{-v}}^{2^{1-v}} \left\| f_v \right\|_{q(t)}^{q(t)} \frac{dt}{t} \lesssim \left\| f_v \right\|_{q(0)}^{q(t)} + 2^{-v} =: \delta_v$$

for any $v \in \mathbb{N}$. This claim can be reformulated as showing that

$$\int_{2^{-v}}^{2^{1-v}} \left( \delta_v^{-\frac{1}{q(t)}} \right) \left\| f_v \right\|_{q(t)}^{q(t)} \frac{dt}{t} \lesssim 1.$$

Let us prove that

$$\delta_v^{-\frac{1}{q(t)}} \left\| f_v \right\|_{q(t)} \lesssim 1$$

for any $t \in [2^{-v}, 2^{1-v}]$ and any $v \in \mathbb{N}$. We use the log-Hölder continuity of $q$ at the origin to show that

$$\delta_v^{-\frac{1}{q(t)}} \approx \delta^{-1}, \quad t \in [2^{-v}, 2^{1-v}], v \in \mathbb{N}.$$  

Therefore, from the definition of $\delta_v$, we find that

$$\delta_v^{-\frac{1}{q(t)}} \left\| f_v \right\|_{q(0)} \lesssim 1.$$  

Hence we obtain the desired estimate.
Step 2. Let \( \{ f_v \}_{v \in \mathbb{N}} \) be a sequence of measurable functions such that the left-hand side is less than or equal to 1. We will prove that
\[
\sum_{v=1}^{\infty} \| f_v \|_{p(\cdot)}^{q(0)} \lesssim 1.
\]

Clearly, the estimate follows from the inequality:
\[
\| f_v \|_{p(\cdot)}^{q(0)} \lesssim \int_{2^{-v}}^{2^{1-v}} \| f_v \|_{p(\cdot)}^{q(\tau)} \frac{d\tau}{\tau} + 2^{-v} =: \delta_v
\]
for any \( v \in \mathbb{N} \). This claim can be reformulated as showing that
\[
(\delta_v^{-1/q(0)} \| f_v \|_{p(\cdot)})^{q(0)} = \left( \frac{1}{\log 2} \int_{2^{-v}}^{2^{1-v}} \delta_v^{-1/q(0)} \| f_v \|_{p(\cdot)} \frac{d\tau}{\tau} \right)^{q(0)} \lesssim 1.
\]

By Lemma 6, we have
\[
\left( \frac{\gamma_m}{\log 2} \int_{2^{-v}}^{2^{1-v}} \delta_v^{-1/q(0)} \| f_v \|_{p(\cdot)} \frac{d\tau}{\tau} \right)^{q(t)} \lesssim \int_{2^{-v}}^{2^{1-v}} \delta_v^{-1/q(0)} \| f_v \|_{p(\cdot)} \frac{d\tau}{\tau} + 1,
\]
where \( \gamma_m = e^{-2mc_{\log(1/p)}} \) and \( m > 0 \). We use the log-Hölder continuity of \( q \) at the origin to show that
\[
\delta_v^{-1/q(0)} \approx \delta_v^{-1}, \quad \tau \in [2^{-v}, 2^{1-v}], v \in \mathbb{N}.
\]
Therefore, from the definition of \( \delta \), we find that
\[
\int_{2^{-v}}^{2^{1-v}} \delta_v^{-1} \| f_v \|_{p(\cdot)} \frac{d\tau}{\tau} \lesssim 1
\]
for any \( v \in \mathbb{N} \), which implies that
\[
(\delta_v^{-1/q(0)} \| f_v \|_{p(\cdot)})^{q(0)} \lesssim 1
\]
for any \( v \in \mathbb{N} \). The proof of Lemma 7 is complete.

By a similar argument as in Lemma 7, we have:

**Lemma 8** Let \( \{ f_v \}_{v \in \mathbb{N}} \) be a sequence of measurable functions on \( \mathbb{R}^n \), \( p \in \mathcal{P}(\mathbb{R}^n) \) and \( \alpha \in C_{\log}^{\infty}(\mathbb{R}^n) \). Let \( q \in \mathcal{P}(\mathbb{R}) \) be log-Hölder continuous at the origin. Then
\[
\left( \left\| \left( t^{-\alpha(\cdot)-\frac{1}{q(\tau)}} f_v \chi_{Q_v} \right)_{\ell^1(\cdot)} \chi_{[2^{-v}, 2^{1-v}]} \right\|_{\ell^1_{\log}(\cdot)} \right)_{v} \lesssim \left( \sum_{v=1}^{\infty} \left\| 2^{v\alpha(\cdot)} f_v \chi_{Q_v} \right\|_{p(\cdot)}^{q(0)} \right)^{\frac{1}{q(0)}},
\]
where \( Q_v, v \in \mathbb{N} \), are dyadic cubes on \( \mathbb{R}^n \) with \( \ell(Q_v) = 2^{-v} \), and the implicit positive constants are independent of \( f_v, v \in \mathbb{N} \).

The next two lemmas are the continuous version of the Hardy type inequality, where the second lemma for constant exponents is from [27].
Lemma 9 Let $p \in \mathcal{P} (\mathbb{R}^n)$. Let $q \in \mathcal{P} (\mathbb{R})$ be log-Hölder continuous at the origin with $1 \leq q^- \leq q^+ < \infty$. Let $\{f_v\}_{v \in \mathbb{N}}$ be a sequence of measurable functions on $\mathbb{R}^n$. For all $x \in \mathbb{R}^n$, $v \in \mathbb{N}$ and all $\delta > 0$, let

$$g_v(x) = \sum_{k=1}^{\infty} 2^{-|k-v|\delta} f_k(x).$$

Then there exists a positive constant $c$, independent of $\{f_v\}_{v \in \mathbb{N}}$ such that

$$\left\| \left( t^{-\frac{1}{q(v)}} \|g_v\|_{p(v)} \chi_{[2^{-v},2^{1-v}]} \right) v \|_{L^q_v(L^p_v)} \right\| \leq c \left\| \left( t^{-\frac{1}{q(v)}} \|f_v\|_{p(v)} \chi_{[2^{-v},2^{1-v}]} \right) v \|_{L^q_v(L^p_v)} \right\|.$$

Proof. The required estimate follows from Lemmas 5 and 7.

Lemma 10 Let $s > 0$, and let $q \in \mathcal{P} (\mathbb{R})$ be log-Hölder continuous at the origin with $1 \leq q^- \leq q^+ < \infty$. Let $\{\varepsilon_t\}_t$ be a sequence of positive measurable functions. Let

$$\eta_t = t^s \int_0^1 \tau^{-s} \varepsilon \frac{d\tau}{\tau} \quad \text{and} \quad \delta_t = t^{-s} \int_t^\infty \tau^s \varepsilon \frac{d\tau}{\tau}.$$

Then there exists a constant $c > 0$ depending only on $s$, $q^-$, $c_{log}(q)$ and $q^+$ such that

$$\|\eta_t\|_{L^v_q((0,1],q_v)} + \|\delta_t\|_{L^v_q((0,1],q_v)} \leq c \|\varepsilon_t\|_{L^v_q((0,1],q_v)}.$$

Proof. This lemma is proved in [10], and here, we give an alternative proof. We suppose that $\|\varepsilon_t\|_{L^v_q((0,1],q_v)} \leq 1$. Notice that

$$\|\eta_t\|_{L^v_q((0,1],q_v)} \approx \left\| \left( t^{-\frac{1}{q(v)}} \eta_l \chi_{[2^{-v},2^{1-v}]} \right) v \|_{L^q_v(L^p_v)} \right\|.$$

We see that

$$\int_{2^{-v}}^1 \tau^{-s} \varepsilon \frac{d\tau}{\tau} = \sum_{i=0}^{v-1} \int_{2^{-v}}^{2^{i+1}-v} \tau^{-s} \varepsilon \frac{d\tau}{\tau} = \sum_{i=0}^{v-1} \int_{2^{-v}}^{2^{i+1}-v} \tau^{-s} \varepsilon \frac{d\tau}{\tau}.$$

Let $\sigma > 0$ be such that $q^+ < \sigma$. We have

$$\left( \sum_{j=1}^v \int_{2^{-j}}^{2^{1-j}} \tau \varepsilon \frac{d\tau}{\tau} \right)^{q(t)/\sigma} \leq \sum_{j=1}^v \left( \int_{2^{-j}}^{2^{1-j}} \tau \varepsilon \frac{d\tau}{\tau} \right)^{q(t)/\sigma} \leq \sum_{j=1}^v 2^{j \cdot q(t)/\sigma} \left( \int_{2^{-j}}^{2^{1-j}} \varepsilon \frac{d\tau}{\tau} \right)^{q(t)/\sigma} = 2^{v \cdot q(t)/\sigma} \sum_{j=1}^v 2^{(j-v)q(t)/\sigma} \left( \int_{2^{-j}}^{2^{1-j}} \varepsilon \frac{d\tau}{\tau} \right)^{q(t)/\sigma}.$$

By Hölder’s inequality, we estimate this expression by

$$2^{v \cdot q(t)/\sigma} \left( \sum_{j=1}^v 2^{(j-v)q(t)/\sigma} \left( \int_{2^{-j}}^{2^{1-j}} \varepsilon \frac{d\tau}{\tau} \right)^{q(t)} \right)^{1/\sigma}.$$
By Lemma 5, we find an $m > 0$ such that

$$
\left(\frac{1}{(v - j + 1) \log 2} \int_{2^{-v}}^{2^{1-j}} \varepsilon_\tau \chi_{[2^{-j}, 2^{1-j}]}(\tau) \frac{d\tau}{\tau}\right)^{q(t)} \lesssim \frac{1}{v - j + 1} \int_{2^{-j}}^{2^{1-j}} \varepsilon_\tau^{q(\tau)} \frac{d\tau}{\tau} + 2^{-jm}
$$

for any $v \geq j$ and any $t \in [2^{-v}, 2^{1-v}] \subset [2^{-v}, 2^{1-j}]$. Therefore, we get

$$
\eta_t^{q(t)} \lesssim \sum_{j=1}^{v} 2^{(j-v)q^-/\sigma} (v - j + 1)^{q^+ - 1} \int_{2^{-j}}^{2^{1-j}} \varepsilon_\tau^{q(\tau)} \frac{d\tau}{\tau} + h_v
$$

for any $t \in [2^{-v}, 2^{1-v}]$, where

$$
h_v = \sum_{j=1}^{v} 2^{(j-v)q^-/\sigma} (v - j + 1)^{q^+} 2^{-jm}, \quad v \in \mathbb{N}.
$$

Observe that

$$
\int_{2^{-v}}^{2^{1-v}} \frac{dt}{t} = \log 2.
$$

Then

$$
\int_{2^{-v}}^{2^{1-v}} \eta_t^{q(t)} \frac{dt}{t} \lesssim \sum_{j=1}^{v} 2^{(j-v)q^-/\sigma} (v - j + 1)^{q^+ - 1} \int_{2^{-j}}^{2^{1-j}} \varepsilon_\tau^{q(\tau)} \frac{d\tau}{\tau} + h_v.
$$

Applying Lemma 5, we get

$$
\sum_{v=1}^\infty \int_{2^{-v}}^{2^{1-v}} \eta_t^{q(t)} \frac{dt}{t} \lesssim \sum_{j=1}^\infty \int_{2^{-j}}^{2^{1-j}} \varepsilon_\tau^{q(\tau)} \frac{d\tau}{\tau} + c \lesssim 1
$$

by taking $m$ large enough such that $m > 0$. Hence, we get

$$
\|\eta_t\|_{L^q((0,1], \frac{dt}{t})} \lesssim 1.
$$

Now we prove that

$$
\|\delta_t\|_{L^q((0,1], \frac{dt}{t})} \lesssim 1.
$$

Notice that

$$
\|\delta_t\|_{L^q((0,1], \frac{dt}{t})} \approx \left\| \left( t^{-\frac{1}{q+}} \delta_t \chi_{[2^{-v}, 2^{1-v}]} \right) v \right\|_{L^{q^+}((0,1])}.
$$

We have

$$
\int_0^{2^{-v}} \tau^s \varepsilon_\tau \frac{d\tau}{\tau} = \sum_{i=-\infty}^{-v} \int_{2^{i}}^{2^{i+1}} \tau^s \varepsilon_\tau \frac{d\tau}{\tau} = \sum_{j=v}^\infty \int_{2^{-j}}^{2^{1-j}} \tau^s \varepsilon_\tau \frac{d\tau}{\tau}.
$$

Let $\sigma > 0$ be such that $q^+ < \sigma$. We have

$$
\left( \sum_{j=v}^\infty \int_{2^{-j}}^{2^{1-j}} \tau^s \varepsilon_\tau \frac{d\tau}{\tau} \right)^{q(t)/\sigma} \leq \sum_{j=v}^\infty \left( \int_{2^{-j}}^{2^{1-j}} \tau^s \varepsilon_\tau \frac{d\tau}{\tau} \right)^{q(t)/\sigma} \leq \sum_{j=v}^\infty 2^{-jsq(t)/\sigma} \left( \int_{2^{-j}}^{2^{1-j}} \varepsilon_\tau \frac{d\tau}{\tau} \right)^{q(t)/\sigma} = 2^{-vsq(t)/\sigma} \sum_{j=v}^\infty 2^{(v-j)sq(t)/\sigma} \left( \int_{2^{-j}}^{2^{1-j}} \varepsilon_\tau \frac{d\tau}{\tau} \right)^{q(t)/\sigma}.
$$
Again, by Hölder’s inequality, we estimate this expression by
\[
2^{-vq(t)/\sigma} \left( \sum_{j=v}^{\infty} 2^{(v-j)q(t)/\sigma} \left( \int_{2^{-j}}^{2^{-j+1}} \varepsilon_{\tau} \frac{d\tau}{\tau} \right)^{q(t)} \right)^{1/\sigma}.
\]

Applying again Lemmas 1 and 6, we get
\[
\left( \frac{1}{(j - v + 1) \log 2} \int_{2^{-j}}^{2^{-j+1}} \varepsilon_{\tau} \chi_{[2^{-j}, 2^{-j+1}]}(\tau) \frac{d\tau}{\tau} \right)^{q(t)} \lesssim \frac{1}{j - v + 1} \int_{2^{-j}}^{2^{-j+1}} \varepsilon_{\tau}^{q(t)} \frac{d\tau}{\tau} + 2^{-vm}
\]
for any \( j \geq v \) and any \( t \in [2^{-v}, 2^{-1-v}] \subset [2^{-j}, 2^{-v}] \). Therefore, we deduce that
\[
\delta_{t}^{q(t)} \lesssim \sum_{j=v}^{\infty} 2^{(v-j)q(t)/\sigma} (j - v + 1)^{q+1} \int_{2^{-j}}^{2^{-j+1}} \varepsilon_{\tau}^{q(t)} \frac{d\tau}{\tau} + f_v
\]
for any \( t \in [2^{-v}, 2^{-1-v}] \), where
\[
f_v = 2^{-vm}, \quad v \in \mathbb{N}.
\]

Again, observe that
\[
\int_{2^{-v}}^{2^{-v+1}} \frac{dt}{t} = \log 2.
\]

Then
\[
\int_{2^{-v}}^{2^{-v+1}} \delta_{t}^{q(t)} \frac{dt}{t} \lesssim \sum_{j=v}^{\infty} 2^{(v-j)q(t)/\sigma} (j - v + 1)^{q+1} \int_{2^{-j}}^{2^{-j+1}} \varepsilon_{\tau}^{q(t)} \frac{d\tau}{\tau} + f_v.
\]

By taking \( m \) large enough such that \( m > 0 \) and again by Lemma 5 we get
\[
\sum_{v=1}^{\infty} \int_{2^{-v}}^{2^{-v+1}} \delta_{t}^{q(t)} \frac{dt}{t} \lesssim \sum_{j=1}^{\infty} \int_{2^{-j}}^{2^{-j+1}} \varepsilon_{\tau}^{q(t)} \frac{d\tau}{\tau} + c \lesssim 1.
\]

The proof of Lemma 10 is completed by the scaling argument. □

The following lemma is from [33, Lemma 1].

**Lemma 11** Let \( \rho, \mu \in \mathcal{S}(\mathbb{R}^n) \), and \( M \geq -1 \) an integer such that
\[
\int_{\mathbb{R}^n} x^{\alpha} \mu(x) dx = 0
\]
for all \( |\alpha| \leq M \). Then for any \( N > 0 \), there is a constant \( c(N) > 0 \) such that
\[
\sup_{z \in \mathbb{R}^n} \left| t^{-n} \mu(t^{-1} \cdot) * \rho(z) \right| (1 + |z|)^N \leq c(N) t^{M+1}.
\]
3 Variable Besov spaces

In this section we present the definition of Besov spaces of variable smoothness and integrability, and prove the basic properties in analogy to the Besov spaces with fixed exponents. Select a pair of Schwartz functions \( \Phi \) and \( \varphi \) satisfying

\[
\text{supp } \mathcal{F}\Phi \subset \{ x \in \mathbb{R}^n : |x| < 2 \}, \quad \text{supp } \mathcal{F}\varphi \subset \left\{ x \in \mathbb{R}^n : \frac{1}{2} < |x| < 2 \right\}
\]

and

\[
\mathcal{F}\Phi(\xi) + \int_0^1 \mathcal{F}\varphi(t\xi) \frac{dt}{t} = 1, \quad \xi \in \mathbb{R}^n.
\]

Such a resolution (5) and (6) of unity can be constructed as follows. Let \( \mu \in \mathcal{S}(\mathbb{R}^n) \) be such that \( |\mathcal{F}\mu(\xi)| > 0 \) for \( \frac{1}{2} < |\xi| < 2 \). There exists \( \eta \in \mathcal{S}(\mathbb{R}^n) \) with

\[
\text{supp } \mathcal{F}\eta \subset \{ x \in \mathbb{R}^n : \frac{1}{2} < |x| < 2 \}
\]

such that

\[
\int_0^\infty \mathcal{F}\mu(t\xi) \mathcal{F}\eta(t\xi) \frac{dt}{t} = 1, \quad \xi \neq 0
\]

(see [6], [22] and [24]). We set \( \mathcal{F}\varphi = \mathcal{F}\mu \mathcal{F}\eta \) and

\[
\mathcal{F}\Phi(\xi) = \left\{ \begin{array}{ll}
\int_1^\infty \mathcal{F}\varphi(t\xi) \frac{dt}{t} & \text{if } \xi \neq 0, \\
1 & \text{if } \xi = 0.
\end{array} \right.
\]

Then \( \mathcal{F}\Phi \in \mathcal{S}(\mathbb{R}^n) \), and as \( \mathcal{F}\eta \) is supported in \( \{ x \in \mathbb{R}^n : \frac{1}{2} < |x| < 2 \} \), we see that \( \text{supp } \mathcal{F}\Phi \subset \{ x \in \mathbb{R}^n : |x| < 2 \} \).

Now we define the spaces under consideration.

**Definition 1** Let \( \alpha : \mathbb{R}^n \to \mathbb{R} \), \( p \in \mathcal{P}(\mathbb{R}^n) \) and \( q \in \mathcal{P}(\mathbb{R}) \). Let \( \{ \mathcal{F}\Phi, \mathcal{F}\varphi \} \) be a resolution of unity and we put \( \varphi_t = t^{-n}\varphi(t) \). The Besov space \( B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \) is the collection of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that

\[
\|f\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|\mathcal{F}\Phi * f\|_{p(\cdot)} + \left\| t^{-\alpha(\cdot)}(\varphi_t * f) \right\|_{p(\cdot)} < \infty.
\]

When \( q = \infty \), the Besov space \( B^{\alpha(\cdot)}_{p(\cdot),\infty} \) consist of all distributions \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that

\[
\|\mathcal{F}\Phi * f\|_{p(\cdot)} + \sup_{t \in (0,1]} \| t^{-\alpha(\cdot)}(\varphi_t * f) \|_{p(\cdot)} < \infty.
\]

One recognizes immediately that \( B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \) is a normed space, and if \( \alpha, p \) and \( q \) are constants, then \( B^{\alpha}_{p,q} \) is the usual Besov spaces. For general literature on function spaces of variable smoothness and integrability we refer to [2-4, 10-11, 13-15, 18, 23, 25-26, ...]
Now, we are ready to show that the definition of these function spaces is independent of the chosen resolution \( \{ F_\Phi, F_\varphi \} \) of unity. This justifies our omission of the subscript \( \Phi \) and \( \varphi \) in the sequel.

**Theorem 1** Let \( \{ F_\Phi, F_\varphi \} \) and \( \{ F_\Psi, F_\psi \} \) be two resolutions of unity, and \( p \in \mathcal{P}_{\text{log}}(\mathbb{R}^n) \) and \( \alpha \in C_{\text{log}}(\mathbb{R}^n) \). Let \( q \in \mathcal{P}(\mathbb{R}) \) be log-Hölder continuous at the origin. Then

\[
\| f \|_{B^{\alpha}_{p(\cdot),q(\cdot)}} \approx \| f \|_{B^{\alpha}_{p(\cdot),q(\cdot)}}.
\]

**Proof.** It is sufficient to show that there exists a constant \( c > 0 \) such that for all \( f \in B^{\alpha}_{p(\cdot),q(\cdot)} \) we have

\[
\| f \|_{B^{\alpha}_{p(\cdot),q(\cdot)}} \leq c \| f \|_{B^{\alpha}_{p(\cdot),q(\cdot)}}.
\]

Interchanging the roles of \( (\Psi, \psi) \) and \( (\Phi, \varphi) \) we obtain the desired result. We have

\[
F_\Phi(\xi) = F_\Phi(\xi)F_\Psi(\xi) + \int_{1/4}^{1} F_\Phi(\xi)F_\psi(\tau\xi)\frac{d\tau}{\tau}
\]

and

\[
F_\varphi(t\xi) = \int_{t/4}^{\min(1,4t)} F_\varphi(t\xi)F_\psi(\tau\xi)\frac{d\tau}{\tau} + \left\{ \begin{array}{ll}
0, & \text{if } 0 < t < \frac{1}{4}; \\
F_\varphi(t\xi)F_\Psi(\xi), & \text{if } \frac{1}{4} \leq t \leq 1.
\end{array} \right.
\]

for any \( \xi \in \mathbb{R}^n \). Then we see that

\[
\Phi * f = \Phi * \Psi * f + \int_{1/4}^{1} \Phi * \psi_\tau * f d\frac{d\tau}{\tau}
\]

and

\[
\varphi_t * f = \int_{t/4}^{\min(1,4t)} \varphi_t * \psi_\tau * f d\frac{d\tau}{\tau} + \left\{ \begin{array}{ll}
0, & \text{if } 0 < t < \frac{1}{4}; \\
\varphi_t * \Psi * f, & \text{if } \frac{1}{4} \leq t \leq 1.
\end{array} \right.
\]

Since \( p \in \mathcal{P}_{\text{log}}(\mathbb{R}^n) \), the convolution with a radially decreasing \( L^1 \)-function is bounded on \( L^p(\cdot) \):

\[
\| \Phi * f \|_{p(\cdot)} \approx \| \Psi * f \|_{p(\cdot)} + \| \psi_\tau * f \|_{p(\cdot)} d\frac{d\tau}{\tau} \approx \| f \|_{B^{\alpha}_{p(\cdot),q(\cdot)}},
\]

and

\[
\| t^{-\alpha(\cdot)-1/q(t)} (\varphi_t * f) \|_{p(\cdot)} \approx \int_{t/4}^{\min(1,4t)} \| t^{-\alpha(\cdot)-1/q(\tau)} (\psi_\tau * f) \|_{p(\cdot)} d\frac{d\tau}{\tau} + \left\{ \begin{array}{ll}
0, & \text{if } 0 < t < \frac{1}{4}; \\
\| \varphi_t * \Psi * f \|_{p(\cdot)}, & \text{if } \frac{1}{4} \leq t \leq 1,
\end{array} \right.
\]

where we used

\[
t^{-\alpha(x)-1/q(t)} \approx 1, \quad x \in \mathbb{R}^n.
\]

If \( \frac{1}{4} \leq t \leq 1 \), then

\[
t^{-1/q(t)} = \left( \frac{t}{\tau} \right)^{-1/q(t)} \tau^{-1/q(t)} \leq \tau^{-1/q(t)} \leq \tau^{-1/q(\tau)}, \quad \frac{t}{4} \leq \tau \leq \min(1, 4t),
\]

\[
t^{-1/q(t)} = \left( \frac{t}{\tau} \right)^{-1/q(t)} \tau^{-1/q(t)} \leq \tau^{-1/q(t)} \leq \tau^{-1/q(\tau)}, \quad \frac{t}{4} \leq \tau \leq \min(1, 4t),
\]

\[
t^{-1/q(t)} = \left( \frac{t}{\tau} \right)^{-1/q(t)} \tau^{-1/q(t)} \leq \tau^{-1/q(t)} \leq \tau^{-1/q(\tau)}
\]
and
\[ t^{-\alpha(x)} \lesssim (1 + t^{-1}|x - y|)^{c_{\log(\alpha)}t^{-\alpha(y)}}, \quad x, y \in \mathbb{R}^n \]
by Lemma 1 and again the fact that the convolution with a radially decreasing \( L^1 \)-function is bounded on \( L^p(\cdot) \). Hölder’s inequality gives that for any \( \frac{1}{4} \leq t \leq 1 \)
\[
\left\| t^{-\alpha(-\frac{1}{4t})}(\varphi_t \ast f) \right\|_{p(\cdot)} \lesssim \left\| f \right\|_{B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \left\| \chi_{[t/4, \min(1, 4t)]} \right\|_{q'(\cdot)} + \left\| \Psi * f \right\|_{p(\cdot)} \lesssim \left\| f \right\|_{B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}.
\]
Taking the \( L^q(\cdot)((1/4, 1]) \)-norm we obtain the desired estimate. In case \( 0 < t < \frac{1}{4} \) we obtain
\[
\left\| t^{-\alpha(\cdot)}(\varphi_t \ast f) \right\|_{p(\cdot)} \lesssim \int_{t/4}^{4t} \min\left(\frac{\tau}{t}, (\frac{t}{\tau})^s\right) \left\| \tau^{-\alpha(\cdot)}(\varphi_t \ast f) \right\|_{p(\cdot)} \frac{d\tau}{t}, \quad s > 0.
\]
Taking again the \( L^q(\cdot)((0, \frac{1}{4})] \)-norm and using Lemma 10 we conclude that
\[
\left\| \left\| t^{-\alpha(\cdot)}(\varphi_t \ast f) \right\|_{p(\cdot)} \right\|_{L^q(\cdot)((0, \frac{1}{4})], \frac{4}{t})} \lesssim \left\| \left\| \tau^{-\alpha(\cdot)}(\varphi_t \ast f) \right\|_{p(\cdot)} \right\|_{L^q(\cdot)((0, \frac{4}{t})], \frac{4}{t})} \lesssim \left\| f \right\|_{B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}.
\]
Notice that the case \( q := \infty \) can be easily solved. The proof of Theorem 1 is now finished.

**Remark 1** Let \( \alpha : \mathbb{R}^n \to \mathbb{R} \), \( p \in \mathcal{P}(\mathbb{R}^n) \) and \( q \in \mathcal{P}(\mathbb{R}) \), with \( 1 \leq q^- \leq q^+ < \infty \). Let \( \{ \mathcal{F} \Phi, \mathcal{F} \varphi \} \) be a resolution of unity. We set
\[
\left\| f \right\|_{B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} := \left\| \Phi * f \right\|_{p(\cdot)} + \left( \int_0^1 \left\| t^{-\alpha(\cdot)}(\varphi_t \ast f) \right\|_{p(\cdot)} \chi_{[2^{-v}, 2^{1-v}]} dt \right)^{\frac{q(0)}{q(0)}} < \infty.
\]
Then
\[
\left\| f \right\|_{B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \approx \left\| f \right\|_{B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}.
\]

Now we present the main result in this section.

**Theorem 2** Let \( \{ \mathcal{F} \Phi, \mathcal{F} \varphi \} \) be a resolution of unity, \( p \in \mathcal{P}(\mathbb{R}^n) \), \( q \in \mathcal{P}(\mathbb{R}) \) and \( \alpha \in \mathcal{C}^{\log}_{\text{loc}}(\mathbb{R}^n) \). The Besov space \( B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)} \) is the collection of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that
\[
\left\| f \right\|_{B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} := \left\| \Phi * f \right\|_{p(\cdot)} + \left( \int_0^1 \left\| t^{-\alpha(\cdot)}(\varphi_t \ast f) \right\|_{p(\cdot)} \frac{dt}{t} \right)^{\frac{q(0)}{q(0)}} < \infty.
\]
Let \( q \) be log-\( \text{Hölder} \) continuous at the origin. Then \( B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)} = B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)} \) with equivalent norms.
Proof. We divide the proof into two steps.

Step 1. We prove that $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \hookrightarrow B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$, which is equivalent to

$$\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \lesssim \|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}$$

for any $f \in B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$. By the scaling argument, it suffices to consider the case $\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} = 1$ and show that the modular of $f$ on the left-hand side is bounded. In particular, we show that

$$S := \int_0^1 \|t^{-\alpha(\cdot)}(\varphi_t \ast f)\|_{p(\cdot)}^{q(\cdot)} \frac{dt}{t} \lesssim 1.$$

We write

$$S = \int_0^{1/4} \|t^{-\alpha(\cdot)}(\varphi_t \ast f)\|_{p(\cdot)}^{q(\cdot)} \frac{dt}{t} + \int_{1/4}^1 \|t^{-\alpha(\cdot)}(\varphi_t \ast f)\|_{p(\cdot)}^{q(\cdot)} \frac{dt}{t} =: K + M.$$

To prove that $M \lesssim 1$, let $\{\mathcal{F}_\varphi, \mathcal{F}_\psi\}$ be a resolution of unity. We find that for any $\frac{1}{4} \leq t \leq 1$

$$\varphi_t \ast f = \int_{t/4}^1 \varphi_t \ast \psi_{\tau} \ast f \frac{d\tau}{\tau} + \varphi_t \ast \psi \ast f.$$

Using Lemma 1, we obtain

$$t^{-\alpha(x)} \lesssim (1 + t^{-1} |x - y|)^{c_{\cos(\alpha)} t^{-\alpha(y)} \lesssim (1 + t^{-1} |x - y|)^{c_{\cos(\alpha)} (1-y)}$$

for any $\tau \in \left[\frac{1}{4}, \min(1, 4t)\right]$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and hence, we get

$$\|t^{-\alpha(\cdot)}(\varphi_t \ast \psi_t \ast f)\|_{p(\cdot)} \lesssim \|t^{-\alpha(\cdot)}(\eta_{t,N} \ast \psi_t \ast f)\|_{p(\cdot)} \lesssim \|\tau^{-\alpha(\cdot)}(\psi_t \ast f)\|_{p(\cdot)}$$

for any $\frac{1}{4} \leq t \leq 1$, where we used the fact that the convolution with a radially decreasing $L^1$-function is bounded on $L^{p(\cdot)}$ (by taking $N > 0$ large enough). Similarly, we obtain

$$\|t^{-\alpha(\cdot)}(\varphi_t \ast \psi \ast f)\|_{p(\cdot)} \lesssim \|\psi \ast f\|_{p(\cdot)}, \quad \frac{1}{4} \leq t \leq 1.$$

By using (11) and Hölder’s inequality, we see that

$$\int_{t/4}^1 \|t^{-\alpha(\cdot)}(\varphi_t \ast \psi_t \ast f)\|_{p(\cdot)} \frac{d\tau}{\tau} \lesssim \int_{t/4}^1 \|\tau^{-\alpha(\cdot)}(\psi_t \ast f)\|_{p(\cdot)} \frac{d\tau}{\tau} \lesssim \left(\int_{1/16}^1 \|\tau^{-\alpha(\cdot)}(\psi_t \ast f)\|_{p(\cdot)}^{q(0)} \frac{d\tau}{\tau}\right)^{\frac{1}{q(0)}} \lesssim 1$$

for any $\frac{1}{4} \leq t \leq 1$. Therefore, we get

$$M \lesssim \sup_{t \in \left[\frac{1}{4}, 1\right]} \|t^{-\alpha(\cdot)}(\varphi_t \ast f)\|_{p(\cdot)}^{q(\cdot)} \lesssim 1.$$
Now we estimate $K$. We write

$$K = \sum_{v=3}^{\infty} \int_{2^{-v}}^{2^{1-v}} \|t^{-\alpha(\cdot)}(\varphi_t * f)\|_{p(\cdot)}^{q(t)} \frac{dt}{t}.$$ 

Let us prove that

$$\int_{2^{-v}}^{2^{1-v}} \|t^{-\alpha(\cdot)}(\varphi_t * f)\|_{p(\cdot)}^{q(t)} \frac{dt}{t} \lesssim \int_{2^{-v-2}}^{2^{1-v}} \|\tau^{-\alpha(\cdot)}(\psi_{\tau} * f)\|_{p(\cdot)}^{q(0)} \frac{d\tau}{\tau} + 2^{-v} = \delta$$

for any $v \geq 3$. This claim can be reformulated as showing that

$$\int_{2^{-v}}^{2^{1-v}} \left(\delta^{-\frac{1}{q'(0)}} \|t^{-\alpha(\cdot)}(\varphi_t * f)\|_{p(\cdot)}^{q(t)}\right) \frac{dt}{t} \lesssim 1.$$

We need only to show that

$$\delta^{-\frac{1}{q'(0)}} \|t^{-\alpha(\cdot)}(\varphi_t * f)\|_{p(\cdot)} \lesssim 1.$$

As before, we find that for any $v \geq 3$ and $t \in [2^{-v}, 2^{1-v}]$

$$\varphi_t * f = \int_{t/4}^{4t} \varphi_{\tau} * \psi_{\tau} * f d\tau.$$

Using (10) and the fact that the convolution with a radially decreasing $L^1$-function is bounded on $L^p(\cdot)$, we obtain

$$\|t^{-\alpha(\cdot)}(\varphi_t * \psi_\tau * f)\|_{p(\cdot)} \lesssim \|t^{-\alpha(\cdot)}(\varphi_t, \psi_\tau * f)\|_{p(\cdot)} \lesssim \|\tau^{-\alpha(\cdot)}(\psi_\tau * f)\|_{p(\cdot)}.$$

Since $q$ is log-Hölder continuous at the origin, we find that

$$\delta^{-\frac{1}{q'(0)}} \approx \delta^{-\frac{1}{q'(0)}} \quad v \geq 3 \quad t \in [2^{-v}, 2^{1-v}]$$

with constants independent of $t$ and $v$. Therefore, by taking $N$ large enough and Hölder’s inequality we get

$$\delta^{-\frac{1}{q'(0)}} \|t^{-\alpha(\cdot)}(\varphi_t * f)\|_{p(\cdot)} \lesssim \int_{t/4}^{4t} \delta^{-\frac{1}{q'(0)}} \|\tau^{-\alpha(\cdot)}(\psi_\tau * f)\|_{p(\cdot)} \frac{d\tau}{\tau} \lesssim \left( \int_{t/4}^{4t} \delta^{-\frac{1}{q'(0)}} \|\tau^{-\alpha(\cdot)}(\psi_\tau * f)\|_{p(\cdot)}^{q(0)} \frac{d\tau}{\tau} \right)^{\frac{1}{q'(0)}}$$

$$\lesssim \int_{2^{-v-2}}^{2^{1-v}} \left(\delta^{-\frac{1}{q'(0)}} \|\tau^{-\alpha(\cdot)}(\psi_\tau * f)\|_{p(\cdot)}^{q(0)} \right) \frac{d\tau}{\tau} \lesssim 1,$$

which follows immediately from the definition of $\delta$. Thus we conclude from (12) that $K \lesssim 1$.

**Step. 2.** We will prove that $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \hookrightarrow B_{p(\cdot),q(0)}^{\alpha(\cdot)}$; which is equivalent to

$$\|f\|_{B_{p(\cdot),q(0)}^{\alpha(\cdot)}} \lesssim \|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}.$$
for any $f \in B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$. By the scaling argument, we see that it suffices to consider the case $\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} = 1$ and show that the modular of $f$ on the left-hand side is bounded. In particular, we show that

$$H := \int_0^1 \|t^{-\alpha(\cdot)}(\varphi_t * f)\|_{p(\cdot)}^{q(0)} \frac{dt}{t} \lesssim 1.$$

We write

$$H = \int_0^{1/4} \|t^{-\alpha(\cdot)}(\varphi_t * f)\|_{p(\cdot)}^{q(0)} \frac{dt}{t} + \int_{1/4}^1 \|t^{-\alpha(\cdot)}(\varphi_t * f)\|_{p(\cdot)}^{q(0)} \frac{dt}{t} =: T + D.$$

We use the same notations as in Step 1. We find that

$$\int_{1/4}^1 \|t^{-\alpha(\cdot)}(\varphi_t * f)\|_{p(\cdot)}^{q(0)} \frac{dt}{t} \lesssim \int_{1/16}^1 \|t^{-\alpha(\cdot)}(\psi_T * f)\|_{p(\cdot)}^{q(0)} \frac{dt}{t} \lesssim \|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \|I_{[1/16,1]}\|_{q'(\cdot)} \leq 1$$

for any $\frac{1}{4} \leq t \leq 1$. Therefore, we get $D \lesssim 1$. To estimate $T$ we need only to prove

$$\int_{2^{-v}}^{2^{1-v}} \|t^{-\alpha(\cdot)}(\varphi_t * f)\|_{p(\cdot)}^{q(0)} \frac{dt}{t} \lesssim \int_{2^{-v-2}}^{2^{1-v}} \|t^{-\alpha(\cdot)}(\psi_T * f)\|_{p(\cdot)}^{q(0)} \frac{dt}{t} + 2^{-v} = \delta$$

for any $v \geq 3$. This estimate can be obtained by repeating the above arguments. The proof of Theorem 2 is complete. 

Let $a > 0$, $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $f \in S'(\mathbb{R}^n)$. Then we define the Peetre maximal function as follows:

$$\varphi_t^{*,a} t^{-\alpha(\cdot)} f(x) := \sup_{y \in \mathbb{R}^n} \frac{|\varphi_t * f(y)|}{(1 + t^{-1} |x-y|)^a}, \quad t > 0$$

and

$$\Phi^{*,a}(x) := \sup_{y \in \mathbb{R}^n} \frac{|\Phi * f(y)|}{(1 + |x-y|)^a}.$$

We now present a fundamental characterization of spaces under consideration.

**Theorem 3** Let $\alpha \in C_{\text{log}}^0(\mathbb{R}^n)$, $p \in \mathcal{P}_{\text{log}}(\mathbb{R}^n)$ and $a > \frac{n}{p}$. Let $q \in \mathcal{P}([0,\infty))$ be a resolution of unity. Then

$$\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} := \|\Phi^{*,a} f\|_{p(\cdot)} + \|\varphi_t^{*,a} t^{-\alpha(\cdot)} f\|_{p(\cdot)} \|_{L^q([0,\infty),\frac{dt}{t})}$$

is the equivalent norm in $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$.

**Proof.** It is easy to see that for any $f \in \mathcal{S}'(\mathbb{R}^n)$ with $\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} < \infty$ and any $x \in \mathbb{R}^n$ we have

$$t^{-\alpha(x)} |\varphi_t * f(x)| \leq \varphi_t^{*,a} t^{-\alpha(\cdot)} f(x).$$

This shows that the right-hand side in (7) is less than or equal to (13).
We will prove that there is a constant $C > 0$ such that for every $f \in \mathcal{B}^{\alpha(\cdot)}_{p(\cdot), q(\cdot)}$

\begin{align}(14) \quad \|f\|_{\mathcal{B}^{\alpha(\cdot)}_{p(\cdot), q(\cdot)}} \leq C \|f\|_{\mathcal{B}^{\alpha(\cdot)}_{p(\cdot), q(\cdot)}}.
\end{align}

By Lemmas 1 and 2 the estimate

\begin{align}(15) \quad t^{-\alpha(y)}|\varphi_t \ast f(y)| & \leq C_1 t^{-\alpha(y)} \left(\eta_{t, \sigma t^{-\alpha}} \ast |\varphi_t \ast f|^p(y)\right)^{1/p} \\
& \leq C_2 \left(\eta_{t, \sigma \log(\alpha)} \ast (t^{-\alpha(y)}|\varphi_t \ast f|^p(y))^{1/p}\right).
\end{align}

is true for any $y \in \mathbb{R}^n$, $\sigma > n/p^-$ and $t > 0$. Now dividing both sides of (15) by $(1 + t^{-1}|x - y|)^a$, in the right-hand side we use the inequality

\begin{align}(1 + t^{-1}|x - y|)^{-a} \leq (1 + t^{-1}|x - z|)^{-a} (1 + t^{-1}|y - z|)^a, \quad x, y, z \in \mathbb{R}^n,
\end{align}

while in the left-hand side we take the supremum over $y \in \mathbb{R}^n$, we find that for all $f \in \mathcal{B}^{\alpha(\cdot)}_{p(\cdot), q(\cdot)}$ any $t > 0$ and any $\sigma > \max(n/p^-, a + c\log(\alpha))$

\begin{align}(16) \quad |\varphi_{t, a} t^{-\alpha(\cdot)} f(x)| \leq C_2 \left(\eta_{t, \sigma \log(\alpha)} \ast (t^{-\alpha(y)}|\varphi_t \ast f|^p(y))^1/p\right),
\end{align}

where $C_2 > 0$ is independent of $x, t$ and $f$. Applying the $L^{p(\cdot)}$-norm and using the fact the convolution with a radially decreasing $L^1$-function is bounded on $L^{p(\cdot)}$, we get

\begin{align}(17) \quad \|\varphi_{t, a} t^{-\alpha(\cdot) - 1/q(t)} f\|_{L^p(\cdot)} \leq \|t^{-\alpha(\cdot) - 1/q(t)} \varphi_t \ast f\|_{L^p(\cdot)}, \quad t \in (0, 1].
\end{align}

Taking $L^{q(\cdot)}((0, 1])$-norm, we conclude the desired estimate. The proof of Theorem 3 is complete. ■

In order to formulate the main result of this section, let us consider $k_0, k \in S(\mathbb{R}^n)$ and $S \geq -1$ an integer such that for an $\varepsilon > 0$

\begin{align}(16) \quad |\mathcal{F}k_0(\xi)| > 0 \quad \text{for} \quad |\xi| < 2\varepsilon, \\
(17) \quad |\mathcal{F}k(\xi)| > 0 \quad \text{for} \quad \frac{\varepsilon}{2} < |\xi| < 2\varepsilon
\end{align}

and

\begin{align}(18) \quad \int_{\mathbb{R}^n} x^{\alpha} k(x) dx = 0 \quad \text{for any} \quad |\alpha| \leq S.
\end{align}

Here (16) and (17) are Tauberian conditions, while (18) states that moment conditions on $k$. We recall the notation

\begin{align}(19) \quad k_t(x) := t^{-n} k(t^{-1} x) \quad \text{for} \quad t > 0.
\end{align}

For any $a > 0$, $f \in S'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we denote

\begin{align}(19) \quad k_{t, a} \ast t^{-\alpha(\cdot)} f(x) := \sup_{y \in \mathbb{R}^n} t^{-\alpha(y)} \left|k_t * f(y)\right|, \quad j \in \mathbb{N}_0.
\end{align}

Usually $k_t \ast f$ is called local mean.

We are now able to state the so called local mean characterization of $\mathcal{B}^{\alpha(\cdot)}_{p(\cdot), q(\cdot)}$ spaces.
Theorem 4 Let $\alpha \in C^{\log}_{\text{loc}}(\mathbb{R}^n)$ and $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$. Let $q \in \mathcal{P}(\mathbb{R})$ be log-Hölder continuous at the origin, $a > \frac{n}{p}$ and $\alpha^+ < S + 1$. Then

\begin{equation}
\|f\|_{B^\alpha_p} := \|k_0^{\ast,a} f\|_p + \|\|k_t^{\ast,a} t^{-\alpha} f\|_p\|_{L^p((0,1], \frac{dt}{t})}
\end{equation}

and

\begin{equation}
\|f\|''_{B^\alpha_p} := \|k_0 * f\|_p + \|\|t^{-\alpha}(k_t * f)\|_p\|_{L^p((0,1], \frac{dt}{t})},
\end{equation}

are equivalent norms on $B^\alpha_p$.

**Proof.** The idea of the proof is from V. S. Rychkov [33]. The proof is divided into three steps.

**Step 1.** Let $\varepsilon > 0$. Take any pair of functions $\varphi_0$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that

\[ |\mathcal{F}\varphi_0(\xi)| > 0 \quad \text{for} \quad |\xi| < 2\varepsilon, \]
\[ |\mathcal{F}\varphi(\xi)| > 0 \quad \text{for} \quad \frac{\varepsilon}{2} < |\xi| < 2\varepsilon. \]

We prove that there is a constant $c > 0$ such that for any $f \in B^\alpha_p$

\begin{equation}
\|f\|_{B^\alpha_p} \leq c \|\varphi_0^{\ast,a} f\|_p + \|\|\varphi_t^{\ast,a} t^{-\alpha} f\|_p\|_{L^p((0,1], \frac{dt}{t})}. \tag{22}
\end{equation}

Let $\Lambda, \lambda \in \mathcal{S}(\mathbb{R}^n)$ such that

\begin{equation}
\text{supp } \mathcal{F}\Lambda \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\varepsilon\}, \quad \text{supp } \mathcal{F}\lambda \subset \{\xi \in \mathbb{R}^n : \varepsilon/2 < |\xi| < 2\varepsilon\}, \tag{23}
\end{equation}

\[ \mathcal{F}\Lambda(\xi)\mathcal{F}\varphi_0(\xi) + \int_0^1 \mathcal{F}\lambda(\tau\xi)\mathcal{F}\varphi(\tau\xi) \frac{d\tau}{\tau} = 1, \quad \xi \in \mathbb{R}^n. \]

In particular, for any $f \in B^\alpha_p$ the following identity is true:

\begin{equation}
f = \Lambda * \varphi_0 * f + \int_0^1 \lambda * \varphi_t * f \frac{d\tau}{\tau}. \tag{24}
\end{equation}

Hence we can write

\[ k_t * f = k_t * \Lambda * \varphi_0 * f + \int_0^1 k_t * \lambda * \varphi_t * f \frac{d\tau}{\tau}, \quad t \in (0, 1]. \]

We have

\begin{equation}
t^{-\alpha(y)} |k_t * \lambda * \varphi_t * f(y)| \leq t^{-\alpha(y)} \int_{\mathbb{R}^n} |k_t * \lambda (z)| |\varphi_t * f(y - z)| \, dz. \tag{25}
\end{equation}

First, let $t \leq \tau$. Writing for any $z \in \mathbb{R}^n$

\[ k_t * \lambda (z) = t^{-n} k_{\frac{z}{t}} * \lambda \left( \frac{z}{\tau} \right), \]

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we deduce from Lemma 11 that for any integer $S \geq -1$ and any $N > 0$ there is a constant $c > 0$ independent of $t$ and $\tau$ such that
\[
|k_t * \lambda_{\tau}(z)| \leq c \frac{\tau^{-n} \left( \frac{t}{\tau} \right)^{S+1}}{(1+\tau^{-1}|z|)^N}, \quad z \in \mathbb{R}^n.
\]

Hence, the right-hand side of (25) is estimated from above by
\[
ct^{-\alpha(y)\tau^{-n}} \left( \frac{t}{\tau} \right)^{S+1} \int_{\mathbb{R}^n} \left( 1 + \tau^{-1}|z| \right)^{-N} |\varphi_{\tau} * f(y - z)| \, dz
\]
\[
= ct^{-\alpha(y)\tau^{-n}} \left( \frac{t}{\tau} \right)^{S+1} t^{-\alpha(y)} \eta_{\tau,N} * |\varphi_{\tau} * f|(y).
\]

By using Lemma 1, we estimate
\[
t^{-\alpha(y)} \eta_{\tau,N} * |\varphi_{\tau} * f|(y) \leq \left( \frac{\tau}{t} \right)^{\alpha^+} \eta_{\tau,N-c\log(\alpha)} * (\tau^{-\alpha(y)} |\varphi_{\tau} * f|)(y)
\]
\[
\leq \left( \frac{\tau}{t} \right)^{\alpha^+} \varphi_{\tau,\alpha \tau^{-\alpha(\cdot)}} f(y) \eta_{\tau,N-a-c\log(\alpha)}
\]
\[
\leq c \left( \frac{\tau}{t} \right)^{\alpha^+} \varphi_{\tau,\alpha \tau^{-\alpha(\cdot)}} f(y),
\]
for any $N > n + a + c\log(\alpha)$ and any $t \leq \tau$.

Next, let $t \geq \tau$. Then, again by Lemma 11, we have for any $z \in \mathbb{R}^n$ and any $L > 0$
\[
|k_t * \lambda_{\tau}(z)| = t^{-n} \left| k * \lambda_{\tau} \left( \frac{z}{t} \right) \right| \leq c \frac{t^{-n} \left( \frac{z}{t} \right)^{M+1}}{(1+\tau^{-1}|z|)^L},
\]
where an integer $M \geq -1$ is taken arbitrarily large, since $D^\alpha \mathcal{F} \lambda(0) = 0$ for all $\alpha$. Therefore, for $t \geq \tau$, the right-hand side of (25) can be estimated from above by
\[
ct^{-\alpha(y)} t^{-n} \left( \frac{\tau}{t} \right)^{M+1} \int_{\mathbb{R}^n} \left( 1 + t^{-1}|z| \right)^{-L} |\varphi_{\tau} * f(y - z)| \, dz
\]
\[
= ct^{-\alpha(y)} t^{-n} \left( \frac{\tau}{t} \right)^{M+1} \eta_{\tau,L} * |\varphi_{\tau} * f|(y).
\]

We have for any $t \geq \tau$
\[
\left( 1 + t^{-1}|z| \right)^{-L} \leq \left( \frac{t}{\tau} \right)^L \left( 1 + \tau^{-1}|z| \right)^{-L}.
\]

Then, again, the right-hand side of (25) is dominated by
\[
ct^{-\alpha(y)} \left( \frac{\tau}{t} \right)^{M-L+1+n} \eta_{\tau,L} * |\varphi_{\tau} * f|(y)
\]
\[
\leq c \left( \frac{\tau}{t} \right)^{M-L+1+n+\alpha^-} \eta_{\tau,L-c\log(\alpha)} * (\tau^{-\alpha(\cdot)} |\varphi_{\tau} * f|)(y)
\]
\[
\leq c \left( \frac{\tau}{t} \right)^{M-L+1+n+\alpha^-} \varphi_{\tau,\alpha \tau^{-\alpha(\cdot)}} f(y) \eta_{\tau,L-a-c\log(\alpha)}
\]
\[
\leq c \left( \frac{\tau}{t} \right)^{M-L+1+n+\alpha^-} \varphi_{\tau,\alpha \tau^{-\alpha(\cdot)}} f(y),
\]
where in the first inequality we have used Lemma 1 (by taking \( L > n + a + c\log(\alpha) \)). Let us take \( M > L - \alpha^- + a - n \) to estimate the last expression by
\[
c \left( \frac{\tau}{\alpha} \right)^{a+1} \varphi^* \tau^{-\alpha} f(y),
\]
where \( c > 0 \) is independent of \( t, \tau \) and \( f \). Further, note that for all \( x, y \in \mathbb{R}^n \) and all \( t, \tau \in (0, 1] \)
\[
\varphi^* \tau^{-\alpha} f(y) \leq \varphi^* \tau^{-\alpha} f(x)(1 + \tau^{-1} |x - y|)^a
\]
\[
\leq \varphi^* \tau^{-\alpha} f(x) \max \left( 1, \left( \frac{t}{\tau} \right)^a \right)(1 + t^{-1} |x - y|)^a.
\]
Hence
\[
\sup_{y \in \mathbb{R}^n} \frac{t^{-\alpha(y)} |k_t * \lambda_t * \varphi \tau * f(y)|}{(1 + t^{-1} |x - y|)^a} \leq C \varphi^* \tau^{-\alpha} f(x) \times \begin{cases} \left( \frac{t}{\tau} \right)^{\alpha} & \text{if} \quad t \leq \tau, \\ \left( \frac{\tau}{t} \right)^{\alpha} & \text{if} \quad t \geq \tau. \end{cases}
\]
Using the fact that for any \( z \in \mathbb{R}^n \), any \( N > 0 \) and any integer \( S \geq -1 \)
\[
|k_t * \Lambda(z)| \leq c \frac{t^{S+1}}{(1 + |z|)^N},
\]
we obtain by the similar arguments that for any \( t \in (0, 1] \)
\[
\sup_{y \in \mathbb{R}^n} \frac{t^{-\alpha(y)} |k_t * \Lambda * \varphi_0 * f(y)|}{(1 + t^{-1} |x - y|)^a} \leq C t^{S+1-\alpha} \varphi_0^* \alpha f(x).
\]
Hence for all \( f \in B^a_{\alpha} \), any \( x \in \mathbb{R}^n \) and any \( t \in (0, 1] \), we get
\[
(26) \quad k_t^* \tau^{-\alpha} f(x) \leq C t^{S+1-\alpha} \varphi_0^* \alpha f(x) + C \int_0^1 \min \left( \left( \frac{t}{\tau} \right)^{S+1-\alpha}, \frac{\tau}{t} \right) \varphi^* \tau^{-\alpha} f(x) \frac{d\tau}{\tau}.
\]
Also we have for any \( z \in \mathbb{R}^n \), any \( N > 0 \) and any integer \( M \geq -1 \)
\[
|k_0 * \lambda_t(z)| \leq c \frac{\tau^{M+1}}{(1 + |z|)^N}
\]
and
\[
|k_0 * \Lambda(z)| \leq c \frac{1}{(1 + |z|)^N}.
\]
As before, we get for any \( x \in \mathbb{R}^n \)
\[
(27) \quad k_t^* \alpha f(x) \leq C \varphi_0^* \alpha f(x) + C \int_0^1 \tau \varphi^* \tau^{-\alpha} f(x) \frac{d\tau}{\tau}.
\]
In (26) and (27) taking the \( L^p \)-norm and then using Lemma 10 we get (22).

**Step 2.** We prove in this step that there is a constant \( c > 0 \) such that for any \( f \in B^a_{\alpha} \)
\[
(28) \quad \|f\|_{B^a_{\alpha}} \leq c \|f\|_{B^a_{\alpha}}.
\]
Analogously to (23), (24) we find two functions $\Lambda, \psi \in S(\mathbb{R}^n)$ such that

$$\mathcal{F}\Lambda(t\xi)\mathcal{F}k_0(t\xi) + \int_0^1 \mathcal{F}\psi(t\tau\xi)\mathcal{F}k(t\tau\xi) \frac{d\tau}{\tau} = 1, \quad \xi \in \mathbb{R}^n,$$

and for all $f \in B^{(\alpha)}_{p,q}$ and $t \in (0, 1]$

$$f = \Lambda_t * (k_0)_t * f + \int_0^t \psi_h * k_h * f \frac{dh}{h}.$$  

Hence

$$k_t * f = \Lambda_t * (k_0)_t * k_t * f + \int_0^t k_t * \psi_h * k_h * f \frac{dh}{h}.$$  

Writing for any $z \in \mathbb{R}^n$

$$k_t * \psi_h(z) = t^{-n}(k * \psi_{\frac{z}{t}})(\frac{z}{t}),$$

we deduce from Lemma 11 that for any integer $K \geq -1$ and any $M > 0$ there is a constant $c > 0$ independent of $t$ and $h$ such that

$$|k_t * \psi_h(z)| \leq c \frac{t^{-n}(\frac{h}{t})^{K+1}}{(1 + t^{-1}|z|)^M}, \quad z \in \mathbb{R}^n.$$

Analogous estimate

$$|\Lambda_t * (k_0)_t(z)| \leq c \frac{t^{-n}}{(1 + t^{-1}|z|)^M}, \quad z \in \mathbb{R}^n,$$

is obvious. From this it follows that

$$t^{-\alpha(y)} |k_t * f(y)| \leq c t^{-\alpha(y)} \eta_{t,M} * |k_t * f|(y)$$

$$+ \int_0^t \left( \frac{h}{t} \right)^{K+1+\alpha^-} h^{-\alpha(y)} \eta_{h,M} * |k_h * f|(y) \frac{dh}{h}.$$  

Since

$$(1 + t^{-1}|y - z|)^{-M} \leq \left( \frac{t}{h} \right)^M (1 + h^{-1}|y - z|)^{-M}, \quad y, z \in \mathbb{R}^n,$$

by Lemma 11 and Hölder’s inequality, we obtain

$$t^{-\alpha(y)} |k_t * f(y)|$$

$$\leq c t^{-\alpha(y)} \eta_{t,M} * |k_t * f|(y) + \int_0^t \left( \frac{h}{t} \right)^{K+1+\alpha^-} h^{-\alpha(y)} \eta_{h,M} * |k_h * f|(y) \frac{dh}{h}$$

$$\leq c \left( \eta_{t,ap^-} * t^{-\alpha(y)p^-} |k_t * f|^{p^-}(y) \right)^{1/p^-}$$

$$+ \int_0^t \left( \frac{h}{t} \right)^{K+1+\alpha^-} \left( \eta_{h,ap^-} * h^{-\alpha(y)p^-} |k_h * f|^{p^-}(y) \right)^{1/p^-} \frac{dh}{h}.$$  

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by taking $M > a + n + c_{\log}(\alpha)$. Using the elementary inequality:

$$
(1 + t^{-1}|x - y|)^{-a} \leq (1 + t^{-1}|x - z|)^{-a} (1 + t^{-1}|y - z|)^a
$$

$$
\leq \left( \frac{t}{h} \right)^a (1 + h^{-1}|x - z|)^{-a} (1 + h^{-1}|y - z|)^a,
$$
and again Hölder’s inequality, we get

$$
(k_t^{*,a} t^{-\alpha} f(x))^{p^-} \leq c \eta_t * t^{-\alpha} |k_t * f|^{p^-}(x)
$$

$$
+ c \int_0^t \left( \frac{h^N}{t} \right) \eta_h * h^{-\alpha} |k_h * f|^{p^-}(x) dh,
$$
where $N > 0$ can be still be taken arbitrarily large. Similarly, we obtain

$$
(k_0^{*,a} f(x))^{p^-} \leq c \eta_1 * |k_0 * f|(x) +
$$

$$
\int_0^1 h^N \eta_{h,a} * h^{-\alpha} |k_h * f|^{p^-}(x) dh.
$$

Observe that

$$
\left\| \eta_{h,a} * h^{-\alpha} |k_h * f|^{p^-} \right\|_{p(\cdot)/p^-} \lesssim \left\| h^{-\alpha} k_h * f \right\|_{p(\cdot)},
$$

$$
\left\| \eta_{h,a} * |k_0 * f|^{p^-} \right\|_{p(\cdot)/p^-} \lesssim \left\| k_0 * f \right\|_{p(\cdot)},
$$
and

$$
\left\| \eta_{t,a} * t^{-\alpha} |k_t * f|^{p^-} \right\|_{p(\cdot)/p^-} \lesssim \left\| t^{-\alpha} k_t * f \right\|_{p(\cdot)},
$$
for any $h \in (0, t]$ and any $0 < t \leq 1$. Then (30), with power $1/p^-$, in $L^{p(\cdot)}$-norm is bounded by

$$
\left\| k_0 * f \right\|_{p(\cdot)} + \int_0^1 h^N \left\| h^{-\alpha} k_h * f \right\|_{p(\cdot)} dh \lesssim \|f\|_{B^{p(\cdot)}(0, 1)}^{p^-},
$$

since $N$ can be still be taken arbitrarily large. In (29), taking the $L^{p(\cdot)/p^-}$-norm, using the above estimates, taking the $1/p^-$ power and then the $L^{p(\cdot)}((0, 1], \frac{dt}{t})$-norm, we find from Lemma 10 that

$$
\|f\|_{B^{p(\cdot)}(0, 1)}^{p^-} \leq c \|f\|_{B^{p(\cdot), q(\cdot)}(0, 1)}^{p^-}.
$$

**Step 3.** We will prove in this step that for all $f \in B^{p(\cdot), q(\cdot)}_{p(\cdot)}$, the following estimate is true:

$$
\|f\|_{B^{p(\cdot), q(\cdot)}_{p(\cdot)}} \leq c \|f\|_{B^{p(\cdot), q(\cdot)}_{p(\cdot)}} \leq c \|f\|_{B^{p(\cdot), q(\cdot)}_{p(\cdot)}}.
$$

Let $\{\mathcal{F}\Phi, \mathcal{F}\varphi\}$ be a resolution of unity. The first inequality is proved by the chain of the estimates

$$
\|f\|_{B^{p(\cdot), q(\cdot)}_{p(\cdot)}} \leq c \|\Phi * f\|_{p(\cdot)} + \left\| \|\varphi_t^{*,a} t^{-\alpha} f\|_{p(\cdot)} \right\|_{L^{q(\cdot)}((0, 1], \frac{dt}{t})}
$$

$$
\leq c \|\Phi * f\|_{p(\cdot)} + \left\| t^{-\alpha} \varphi_t * f\|_{p(\cdot)} \right\|_{L^{q(\cdot)}((0, 1], \frac{dt}{t})} \leq c \|f\|_{B^{p(\cdot), q(\cdot)}_{p(\cdot)}}.
$$
where the first inequality is (22), see Step 1, the second inequality is (28) (with \( \varphi \) and \( \varphi_0 \) instead of \( k \) and \( k_0 \)), see Step 2, and finally the third inequality is obvious. Now the second inequality can be obtained by the following chain

\[
\|f\|_{B^\alpha_{p,(\cdot, q)(\cdot)}} \leq c \|\Phi^{*, \alpha} f\|_{p(\cdot)} + \|\|\varphi^{*, \alpha} L^{-\alpha} f\|_{p(\cdot)}\|_{L^q((0,1], \frac{dt}{t})} \\
\leq c \|f\|_{B^\alpha_{p,(\cdot, q)(\cdot)}} \leq c \|f\|_{B^\alpha_{p,(\cdot, q)(\cdot)}},
\]

where the first inequality is obvious, the second inequality is (22) in Step 1, with the roles of \( k_0 \) and \( k \) respectively \( \varphi_0 \) and \( \varphi \) interchanged, and finally the last inequality is (28) in Step 2. Thus, Theorem 4 is proved. ■

From Theorems 2 and 4 we have:

**Theorem 5** Let \( \alpha \in C^{\log}_{\text{loc}}(\mathbb{R}^n) \) and \( p \in \mathcal{P}^{\log}(\mathbb{R}^n) \). Let \( q \in \mathcal{P}(\mathbb{R}) \) be log-Hölder continuous at the origin, \( a > \frac{n}{p} \) and \( \alpha^+ < S + 1 \). Let \( f \in B^{\alpha(\cdot)}_{p(\cdot), q(\cdot)} \),

\[
\|f\|_{B^{\alpha(\cdot)}_{p(\cdot), q(\cdot)}} := \|k_0^{*, \alpha} f\|_{p(\cdot)} + \|\|k_0^{*, \alpha} L^{-\alpha} f\|_{p(\cdot)}\|_{L^q((0,1], \frac{dt}{t})} \\
\text{and}
\]

\[
\|f\|_{B^{\alpha(\cdot)}_{p(\cdot), q(\cdot)}} := \|k_0 f\|_{p(\cdot)} + \|t^{-\alpha}(k_0 f)\|_{p(\cdot)}\|_{L^q((0,1], \frac{dt}{t})}. 
\]

Then

\[
\|f\|_{B^{\alpha(\cdot)}_{p(\cdot), q(\cdot)}} \approx \|f\|_{B^{\alpha(\cdot)}_{p(\cdot), q(\cdot)}} \approx \|f\|_{B^{\alpha(\cdot)}_{p(\cdot), q(\cdot)}} \approx \|f\|_{B^{\alpha(\cdot)}_{p(\cdot), q(\cdot)}} \approx \|f\|_{B^{\alpha(\cdot)}_{p(\cdot), q(\cdot)}}.
\]

4 Embeddings

For the spaces \( B^{\alpha(\cdot)}_{p(\cdot), q(\cdot)} \) introduced above we want to show some embedding theorems. We say a quasi-Banach space \( A_1 \) is continuously embedded in another quasi-Banach space \( A_2 \), if \( A_1 \subset A_2 \) and there exists a constant \( c > 0 \) such that

\[
\|f\|_{A_2} \leq c \|f\|_{A_1}
\]

for all \( f \in A_1 \). Then we write

\( A_1 \hookrightarrow A_2 \).

We begin with the following elementary embeddings.

**Theorem 6** Let \( \alpha_0, \alpha_1 \in C^{\log}_{\text{loc}}(\mathbb{R}^n) \) and \( p \in \mathcal{P}^{\log}(\mathbb{R}^n) \). Let \( q_0, q_1 \in \mathcal{P}(\mathbb{R}) \) satisfy the log-Hölder decay condition at the origin. If \( (\alpha_0 - \alpha_1)^- > 0 \), then

\[
B^{\alpha_0(\cdot)}_{p(\cdot), q_0(\cdot)} \hookrightarrow B^{\alpha_1(\cdot)}_{p(\cdot), q_1(\cdot)}. 
\]

**Proof.** Let \( \{\mathcal{F} \Phi, \mathcal{F} \varphi\} \) and \( \{\mathcal{F} \Psi, \mathcal{F} \psi\} \) be two resolutions of unity. We have

\[
\varphi_t * f = \int_{t/4}^{\min(1, 4t)} \varphi_t * \psi_t * f \frac{d\tau}{\tau} + h_t, \quad t \in (0, 1] \\
geq h_t, 
\]

where the first inequality is (22), see Step 1, the second inequality is (28) (with \( \varphi \) and \( \varphi_0 \) instead of \( k \) and \( k_0 \)), see Step 2, and finally the third inequality is obvious. Now the second inequality can be obtained by the following chain
where
\[ h_t = \begin{cases} 
0, & \text{if } 0 < t < \frac{1}{4}, \\
\varphi_t \ast \Psi \ast f, & \text{if } \frac{1}{4} \leq t \leq 1.
\end{cases} \]

From the fact that the convolution with a radially decreasing \( L^1 \)-function is bounded on \( L^p(\cdot) \), we deduce that
\[
\left\| t^{-\alpha_1(\cdot)} g_t \right\|_{p(\cdot)} \lesssim t^{(\alpha_0 - \alpha_1)(\cdot)} \int_{t/4}^{\min(1,4t)} \left\| \tau^{-\alpha_0(\cdot)} (\psi_\tau \ast f) \right\|_{p(\cdot)} \frac{d\tau}{\tau}
\]
\[
\lesssim t^{(\alpha_0 - \alpha_1)(\cdot)} \left\| \left\| \tau^{-\alpha_0(\cdot)} (\psi_\tau \ast f) \right\|_{p(\cdot)} \right\|_{L^{q_0}(\cdot)}(0,1) \frac{1}{4} \left\| L^{q_0}(\cdot)(t/4,\min(1,4t),\frac{d\tau}{\tau}) \right\|_{L^{q_0}(\cdot)((0,1),\frac{d\tau}{\tau})} \lesssim t^{(\alpha_0 - \alpha_1)(\cdot)} \left\| f \right\|_{B^{\alpha_0(\cdot)}_{p(\cdot),q_0(\cdot)}},
\]
where we have used H"older's inequality. Similarly, we find that
\[
\left\| t^{-\alpha_1(\cdot)} h_t \right\|_{p(\cdot)} \lesssim \begin{cases} 
0, & \text{if } 0 < t < \frac{1}{4}; \\
\|\Psi \ast f\|_{p(\cdot)}, & \text{if } \frac{1}{4} \leq t \leq 1.
\end{cases}
\]

Taking the \( L^{q_1(\cdot)}((0,1),\frac{d\tau}{\tau}) \)-norm we obtain the desired estimate. The estimation of \( \|\Phi \ast f\|_{p(\cdot)} \) can be obtained by the decomposition (8). The proof of Theorem 6 is complete. □

**Theorem 7** Let \( \alpha \in C^l_{\text{loc}}(\mathbb{R}^n) \) and \( p \in \mathcal{P}^l_{\text{loc}}(\mathbb{R}^n) \). Let \( q_0, q_1 \in \mathcal{P}(\mathbb{R}) \) satisfy the log-H"older decay condition at the origin. If \( q_0(0) \leq q_1(0) \), then
\[
B^{\alpha(\cdot)}_{p(\cdot),q_0(\cdot)} \hookrightarrow B^{\alpha(\cdot)}_{p(\cdot),q_1(\cdot)}.
\]

**Proof.** Let \( f \in B^{\alpha(\cdot)}_{p(\cdot),q_0(\cdot)} \). By the scaling argument, it suffices to consider the case \( \|f\|_{B^{\alpha(\cdot)}_{p(\cdot),q_0(\cdot)}} = 1 \) and show that the modular of \( f \) on the left-hand side is bounded. In particular, by Theorem 2 we show that
\[
\int_0^1 \left\| t^{-\alpha(\cdot)} (\varphi_t \ast f) \right\|_{q_1(\cdot)}^{q_0(0)} \frac{dt}{t} \lesssim 1.
\]

We use the notation in the previous theorem. We need only to prove that
\[
\left\| t^{-\alpha(\cdot)} (\varphi_t \ast f) \right\|_{p(\cdot)} \lesssim 1
\]
for any \( t \in (0,1] \). We have
\[
\varphi_t \ast f = \int_{t/4}^{\min(1,4t)} \varphi_t \ast \psi_\tau \ast f \frac{d\tau}{\tau} + h_t, \quad t \in (0,1].
\]

From Lemma 11 and the fact the convolution with a radially decreasing \( L^1 \)-function is bounded on \( L^p(\cdot) \), we deduce that
\[
\left\| t^{-\alpha(\cdot)} (\varphi_t \ast f) \right\|_{p(\cdot)} \lesssim \int_{t/4}^{\min(1,4t)} \left\| \tau^{-\alpha(\cdot)} (\psi_\tau \ast f) \right\|_{p(\cdot)} \frac{d\tau}{\tau} + \left\| t^{-\alpha(\cdot)} h_t \right\|_{p(\cdot)},
\]

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Proof. We prove that
\[ \| t^{-\alpha(t)} h_t \|_{p(t)} \lesssim \left\{ \begin{array}{ll} 0, & \text{if } 0 < t < \frac{1}{4}; \\
\| \Psi * f \|_{p(t)}, & \text{if } \frac{1}{4} \leq t \leq 1, \end{array} \right. \]
which implies that
\[ \| t^{-\alpha(t)} h_t \|_{p(t)} \lesssim 1 \]
for any \( t \in (0, 1] \). Finally, by using Hölder’s inequality we find that
\[ \int_{t/4}^{\min(1,4t)} \| \tau^{-\alpha(\tau)} (\psi_\tau * f) \|_{p(\tau)} \frac{d\tau}{\tau} \lesssim \left( \int_{t/4}^{\min(1,4t)} \| \tau^{-\alpha(\tau)} (\psi_\tau * f) \|_{p(\tau)}^{q_0(0)} \frac{d\tau}{\tau} \right)^{\frac{1}{q_0(0)}} \lesssim 1, \]
and hence, Theorem 7 is proved. ■

We next consider embeddings of Sobolev type. It is well known that
\[ B^{\alpha_0}_{p_0/q} \hookrightarrow B^{\alpha_1}_{p_1/q}, \]
if \( \alpha_0 - n/p_0 = \alpha_1 - n/p_1 \), where \( 0 < p_0 < p_1 \leq \infty \) and \( 0 < q \leq \infty \) (see e.g. [35, Theorem 2.7.1]). In the following theorem we generalize these embeddings to variable exponent case.

**Theorem 8** Let \( \alpha_0, \alpha_1 \in C^1_{\text{loc}}(\mathbb{R}^n) \) and \( p_0, p_1 \in \mathcal{P}^1(\mathbb{R}^n) \). Let \( q \in \mathcal{P}(\mathbb{R}) \) be log-Hölder continuous at the origin. If \( \alpha_0 \geq \alpha_1 \) and \( \alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)} \), then
\[ B^{\alpha_0}_{p_0(q)} \hookrightarrow B^{\alpha_1}_{p_1(q)}. \]

**Proof.** We prove that
\[ \| f \|_{B^{\alpha_1}_{p_1(q)}} \lesssim \| f \|_{B^{\alpha_0}_{p_0(q)}} \]
for any \( f \in B^{\alpha_0}_{p_0(q)} \). Let us prove that
\[ \| t^{-\alpha_1(t)} (\varphi_t * f) \|_{p_1(t)} \lesssim \| t^{-\alpha_0(t)} (\varphi_t * f) \|_{p_0(t)} = \delta. \]
This is equivalent to
\[ \| \delta^{-1} t^{-\alpha_1(t)} (\varphi_t * f) \|_{p_1(t)} \lesssim 1. \]
By Lemma 2 we have for any \( m > n, d > 0 \)
\[ |\varphi_t * f(x)| \leq c(\eta_{t,m} * |\varphi_t * f|^d(x))^{1/d}. \]
Hence
\[
\delta^{-1} t^{\frac{n}{p_1(x)} - \alpha_1(x)} |\varphi_t * f(x)| \\
\leq c(\eta_{t,m - c\log(\alpha_1 - c\log(1/p_1))} * \delta^{-d} t^{-\alpha_1(x)} d_{p_1(t)} (\varphi_t * f)^d(x))^{1/d} \\
\leq c \left| t^{\frac{n}{p_1(t)} - \alpha_1(x) \eta_{t,m - c\log(\alpha_1 - c\log(1/p_1))} / d(x - \cdot)} \right|_{h_1(\cdot)} \| \delta^{-d} t^{-\alpha_0(t)} (\varphi_t * f) \|_{p_0(t)},
\]
where
\[ \frac{1}{d} = \frac{1}{p_0(\cdot)} + \frac{1}{h(\cdot)}. \]
The second norm on the right hand side is bounded by 1 due to the choice of δ. To show that the first norm is also bounded, we investigate the corresponding modular:

\[ \eta_{t_n} (m - c_{\log}(\alpha_1 - c_{\log}(1/p_1))/d(x - \cdot)) \]

\[ = \int_{\mathbb{R}^n} \left( 1 + t_1^{-1} |x - z| \right)^{(m - c_{\log}(\alpha_1 - c_{\log}(1/p_1)))h(z)/d} dz < \infty \]

for \( m > 0 \) large enough. Now we have

\[ \left| \delta^{-1} t^{-\alpha_1(x)} \varphi_t \ast f(x) \right|_{p_1(x)} \]

\[ = \left| \delta^{-1} t^{-\alpha_1(x)} \varphi_t \ast f(x) \right|_{p_1(x) - p_0(x)} \left| \delta^{-1} t^{-\alpha_1(x)} \varphi_t \ast f(x) \right|_{p_0(x)} \]

\[ \lesssim \left| \delta^{-1} t^{-\alpha_1(x)} \varphi_t \ast f(x) \right|_{p_0(x)}. \]

Integrating this inequality over \( \mathbb{R}^n \) and taking into account the definition of \( \delta \), we conclude that the claim is true. The proof of Theorem 8 is complete by taking in (32) the \( L^q((0, 1], dt) \)-norm.

Let \( \alpha \in C^{log}_{\text{loc}}(\mathbb{R}^n) \) and \( p \in \mathcal{P}^{\log}(\mathbb{R}^n) \). Let \( q \in \mathcal{P}(\mathbb{R}) \) be log-Hölder continuous at the origin. From (31), we obtain

\[ B^{\alpha^{(\cdot)}}_{p^{(\cdot)},q^{(\cdot)}} \hookrightarrow B^{\alpha^{(\cdot)} + n/p^+ - n/p}_{p^{(\cdot)} + q^{(\cdot)}} \hookrightarrow B^{\alpha^{(\cdot)} + n/p^+ - n/p}_{p^{(\cdot)} + q^{(\cdot)}} \rightarrow S'((\mathbb{R}^n)), \]

where \( 0 < \varepsilon < (\alpha + n/p^+ - n/p) \). Let \( \alpha_0 \in \mathbb{R} \) be such that \( \alpha_0 > (\alpha + n/p^+ - n/p)^+ \). We have

\[ S(\mathbb{R}^n) \hookrightarrow B^{\alpha_0}_{p^{(\cdot)} + q^{(\cdot)}} \hookrightarrow B^{\alpha_0 - n/p^+ + n/p}_{p^{(\cdot)} + q^{(\cdot)}} \hookrightarrow B^{\alpha^{(\cdot)}}_{p^{(\cdot)},q^{(\cdot)}}. \]

Thus we have:

**Theorem 9** Let \( \alpha \in C^{log}_{\text{loc}}(\mathbb{R}^n) \) and \( p \in \mathcal{P}^{\log}(\mathbb{R}^n) \). Let \( q \in \mathcal{P}(\mathbb{R}) \) be log-Hölder continuous at the origin. Then

\[ S(\mathbb{R}^n) \hookrightarrow B^{\alpha^{(\cdot)}}_{p^{(\cdot)},q^{(\cdot)}} \hookrightarrow S'((\mathbb{R}^n)). \]

## 5 Atomic decomposition

The idea of atomic decompositions goes back to M. Frazier and B. Jawerth in a series of papers [17], [18] (see also [37]). The main goal of this section is to prove an atomic decomposition result for \( B^{\alpha^{(\cdot)}}_{p^{(\cdot)},q^{(\cdot)}} \). Atoms are the building blocks for the atomic decomposition.

**Definition 2** Let \( K, L + 1 \in \mathbb{N}_0 \) and \( \gamma > 1 \). A \( K \)-times continuous differentiable function \( a \in C^K(\mathbb{R}^n) \) is called \([K,L]\)-atom centered at \( Q_{v,m} \), \( v \in \mathbb{N}_0 \) and \( m \in \mathbb{Z}^n \), if

\[ (33) \quad \text{supp} \ a \subseteq \gamma Q_{v,m}, \]

\[ (34) \quad |D^\beta a(x)| \leq 2^v(|\beta| + 1/2), \quad \text{for} \quad 0 \leq |\beta| \leq K, x \in \mathbb{R}^n, \]

and

\[ (35) \quad \int_{\mathbb{R}^n} x^\beta a(x) dx = 0 \quad \text{for} \quad 0 \leq |\beta| \leq L \text{ and } v \geq 1. \]
If the atom $a$ is located at $Q_{v,m}$, i.e., if it fulfills (33), then we denote it by $a_{v,m}$. For $v = 0$ or $L = -1$ the moment conditions (35) is not required.

For proving the decomposition by atoms we need the following lemma, see Frazier and Jawerth [17, Lemma 3.3].

**Lemma 12** Let $\{\mathcal{F} \Phi, \mathcal{F} \varphi\}$ be a resolution of unity and let $\varrho_{v,m}$ be an $[K,L]$-atom. If $\lambda \in \mathbb{N}_0$ and $2^{-j} \leq t \leq 2^{1-j}$, then

$$|\varphi_t * \varrho_{v,m}(x)| \leq c 2^{(v-j)K+vn/2} (1 + 2^v |x - x_{Q_{v,m}}|)^{-M}$$

if $v \leq j$ and

$$|\varphi_t * \varrho_{v,m}(x)| \leq c 2^{(j-v)(L+n+1)+vn/2} (1 + 2^j |x - x_{Q_{v,m}}|)^{-M}$$

if $v \geq j$, where $M$ is sufficiently large and $\varphi_t = t^{-n} \varphi(\frac{t}{t})$. Moreover

$$|\Phi * \varrho_{v,m}(x)| \leq c 2^{-v(L+n+1)+vn/2} (1 + |x - x_{Q_{v,m}}|)^{-M}.$$

Let $p \in \mathcal{P}(\mathbb{R}^n)$, $q \in \mathcal{P}(\mathbb{R})$ and $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$. Then for any complex valued sequences $\lambda = \{\lambda_{v,m} \in \mathbb{C}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ we define

$$b_{p(\cdot), q(\cdot)}^{\alpha(\cdot)} := \left\{ \lambda : \|\lambda\|_{b_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} < \infty \right\},$$

where

$$\|\lambda\|_{b_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} := \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{0,m} \chi_{0,m} \right\|_{p(\cdot)} + \left\| \left\{ \|t^{-\alpha(\cdot)+n/2} - \frac{1}{|t|^\alpha} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \|_{p(\cdot)} \chi_{[2^{-v},2^{v-1}]} \right\}_v \right\|_{L^q(\mathbb{R}^n)}.$$

Here $\chi_{v,m}$ is the characteristic function of the cube $Q_{v,m}$. Let $p \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha \in C^\log_{\text{loc}}(\mathbb{R}^n)$. Let $q \in \mathcal{P}(\mathbb{R})$ be log-Hölder continuous at the origin. From Lemma 8 we have

$$\|\lambda\|_{b_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \approx \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{0,m} \chi_{0,m} \right\|_{p(\cdot)} + \left( \sum_{v=1}^\infty \|2^{v(\alpha(\cdot)-n/2)} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \|_{p(\cdot)} \right)^{q(0)} \left\| \chi_{(0)} \right\|_{L^q(\mathbb{R}^n)}.$$

Now we are now in a position to state the atomic decomposition theorem.

**Theorem 10** Let $\alpha \in C^\log_{\text{loc}}(\mathbb{R}^n)$ and $p \in \mathcal{P}^\log(\mathbb{R}^n)$. Let $q \in \mathcal{P}(\mathbb{R})$ be log-Hölder continuous at the origin with $1 \leq q^- \leq q^+ < \infty$. Let $K, L + 1 \in \mathbb{N}_0$ such that

$$(36) \quad K \geq [\alpha^+] + 1,$$

and

$$(37) \quad L \geq \max(-1, [-\alpha^-]).$$
Then \( f \in S'(\mathbb{R}^n) \) belongs to \( b_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \), if and only if it is represented as

\[
(38) \quad f = \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \varrho_{v,m}, \quad \text{converging in } S'(\mathbb{R}^n),
\]

where \( \varrho_{v,m} \) are \([K,L]-\)atoms and \( \lambda = \{ \lambda_{v,m} \in \mathbb{C} | v \in \mathbb{N}_0, m \in \mathbb{Z}^n \} \in b_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}. \) Furthermore, inf \( \| \lambda \|_{b_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} \) is an equivalent norm in \( b_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \), where the infimum is taken over admissible representations \((38)\).

**Proof.** The idea of proof comes from [17, Theorem 6] and [16, Theorem 4.3]. The proof of convergence of \((38)\) in \( S'(\mathbb{R}^n) \) is postponed to appendix.

We divide the proof into three steps.

**Step 1.** Assume that \( f \in b_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \) and let \( \Phi \) and \( \varphi \) satisfy

\[
(39) \quad \text{supp } F \Phi \subset \overline{B(0,2)} \text{ and } |F \Phi(\xi)| \geq c \text{ if } |\xi| \leq \frac{5}{3}
\]

and

\[
(40) \quad \text{supp } F \varphi \subset \overline{B(0,2)} \setminus B(0,1/2) \text{ and } |F \varphi(\xi)| \geq c \text{ if } \frac{3}{5} \leq |\xi| \leq \frac{5}{3}.
\]

There exist functions \( \Psi \in S(\mathbb{R}^n) \) satisfying \((39)\) and \( \psi \in S(\mathbb{R}^n) \) satisfying \((40)\) such that

\[
f = \Psi * \Phi * f + \int_0^1 \psi_t * \varphi_t * f \frac{dt}{t}
\]

(see Section 3). Using the definition of the cubes \( Q_{v,m} \), we obtain

\[
f(x) = \sum_{m \in \mathbb{Z}^n} \int_{Q_{0,m}} \Phi(x-y) \Psi * f(y) dy + \sum_{v=1}^{\infty} \sum_{m \in \mathbb{Z}^n} \int_{2^{-v}}^{2^{1-v}} \int_{Q_{v,m}} \varphi_t(x-y) \psi_t * f(y) dy \frac{dt}{t}.
\]

We define for every \( v \geq 0, t \in [2^{-v}, 2^{1-v}] \) and all \( m \in \mathbb{Z}^n \)

\[
(41) \quad \lambda_{v,m} = C_{\varphi} \left( \int_{2^{-v}}^{2^{1-v}} \int_{Q_{v,m}} |\psi_t * f(y)|^2 dy \frac{dt}{t} \right)^{1/2},
\]

where

\[
C_{\varphi} = \max \{ \sup_{|y| \leq c} |D^\alpha \varphi(y)| : |\alpha| \leq K \}, \quad c > 0.
\]

Define also

\[
(42) \quad \varrho_{v,m}(x) = \begin{cases} 
\frac{1}{\lambda_{v,m}} \int_{2^{-v}}^{2^{1-v}} \int_{Q_{v,m}} \varphi_t(x-y) \psi_t * f(y) dy \frac{dt}{t} & \text{if } \lambda_{v,m} \neq 0, \\
0 & \text{if } \lambda_{v,m} = 0.
\end{cases}
\]

Similarly we define for every \( m \in \mathbb{Z}^n \) the numbers \( \lambda_{0,m} \) and the functions \( \varrho_{0,m} \) taking in \((41)\) and \((42)\) \( v = 0 \) and replacing \( \psi_t \) and \( \varphi \) by \( \Psi \) and \( \Phi \), respectively. Let us now
check that such \( \varrho_{v,m} \) are atoms in the sense of Definition 2. Note that the support and moment conditions are clear by (39) and (40), respectively. It thus remains to check (34) in Definition 2. We have

\[
|D^3_x \varrho_{v,m}(x)| \\
\leq \frac{1}{\lambda_{v,m}} \int_{2^{-v}}^{2^1-v} \left( \int_{Q_{v,m}} |D^3_x \varphi_t(x-y)|^2 dy \right)^{1/2} \left( \int_{Q_{v,m}} |\psi_t * f(y)|^2 dy \right)^{1/2} \frac{dt}{t} \\
\leq \frac{1}{\lambda_{v,m}} \left( \int_{2^{-v}}^{2^1-v} \int_{Q_{v,m}} |D^3_x \varphi_t(x-y)|^2 dy \right)^{1/2} \left( \int_{2^{-v}}^{2^1-v} \int_{Q_{v,m}} |\psi_t * f(y)|^2 dy \right)^{1/2} \frac{dt}{t} \\
\leq \frac{1}{C_\varphi} \left( \int_{2^{-v}} t^{-2(n+|\beta|)} \int_{Q_{v,m}} \left| D^3_x \varphi_t \left( \frac{x-y}{t} \right) \right|^2 \frac{dy dt}{t} \right)^{1/2} \\
\leq 2^{v(|\beta|+n/2)}.
\]

The modifications for the terms with \( v = 0 \) are obvious.

\textit{Step 2.} Next we show that there is a constant \( c > 0 \) such that

\[
\| \lambda \|_{B^\alpha(q)} \leq c \| f \|_{B^\alpha(q)}.
\]

For that reason we exploit the equivalent norms given in Theorem 4 involving Peetre’s maximal function. Let \( v \geq 0 \). Taking into account that \( |x-y| \leq c \, 2^{-v} \) for \( x, y \in Q_{v,m} \) we obtain

\[
2^{v(\alpha - \alpha(x))} \leq \frac{c_{\log}(\alpha)^v}{\log(e + 1/|x-y|)} \leq \frac{c_{\log}(\alpha)^v}{\log(e + 2^{-v}/c)} \leq c, \quad t \in [2^{-v}, 2^{1-v}],
\]

if \( v \geq \lceil \log_2 c \rceil + 2 \). If \( 0 < v < \lceil \log_2 c \rceil + 2 \), then \( 2^{v(\alpha - \alpha(y))} \leq 2^{v(\alpha - \alpha^+)} \leq c \). Hence, we see that

\[
t^{-\alpha(x)} |\psi_t * f(y)| \leq c \, t^{-\alpha(y)} |\psi_t * f(y)|
\]

for any \( x, y \in Q_{v,m} \) any \( v \in N \) and any \( t \in [2^{-v}, 2^{1-v}] \). Hence, we deduce that

\[
\sum_{m \in \mathbb{Z}^n} \lambda_{v,m} t^{-(\alpha(x)+n/2)} \chi_{v,m}(x) \\
\leq C_\varphi \sum_{m \in \mathbb{Z}^n} t^{-\alpha(x)} \sup_{t \in [2^{-v}, 2^{1-v}]} \sup_{y \in Q_{v,m}} |\psi_t * f(y)| \chi_{v,m}(x) \\
\leq c \sum_{m \in \mathbb{Z}^n} t^{-\alpha(y)} \sup_{|y-x| \leq c \, 2^{-v}} |\psi_t * f(y)| (1 + t^{-1} |y-x|)^a \chi_{v,m}(x) \\
\leq c \psi_t^{*,a} t^{-\alpha} f(x) \sum_{m \in \mathbb{Z}^n} \chi_{v,m}(x) = c \psi_t^{*,a} t^{-\alpha} f(x),
\]

where we have used \( \sum_{m \in \mathbb{Z}^n} \chi_{v,m}(x) = 1 \). Similarly, we obtain

\[
\sum_{m \in \mathbb{Z}^n} \lambda_{0,m} \chi_{0,m}(x) \leq c \, \Psi^{*,a} f(x),
\]
which implies that
\[ \| \lambda \|_{B^p_{\alpha} (L^q (\cdot))} \leq c \| f \|_{B^p_{\alpha} (L^q (\cdot))}, \]
by Theorem 4 (with the equivalent norm (9)).

Step 3. Assume that \( f \) can be represented by (38), with \( K \) and \( L \) satisfying (36) and (37), respectively. We show that \( f \in B^p_{\alpha} (L^q (\cdot)) \), and that for some \( c > 0 \),
\[ \| f \|_{B^p_{\alpha} (L^q (\cdot))} \leq c \| \lambda \|_{B^p_{\alpha} (L^q (\cdot))}. \]

We use the equivalent norm given in (9). The arguments are similar to those in [16]. We write
\[ f = \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_v m \varphi_v m, \]
\[ = \sum_{m \in \mathbb{Z}^n} \lambda_0 m \varphi_0 m + \sum_{v=1}^{j} \sum_{m \in \mathbb{Z}^n} \lambda_v m \varphi_v m + \sum_{v=j+1}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_v m \varphi_v m. \]

From Lemmas 4 and 12, we have for any \( M \) sufficiently large, \( t \in [2^{-j}, 2^{1-j}] \) and any \( 0 \leq v \leq j \)
\[ \sum_{m \in \mathbb{Z}^n} t^{-\alpha (x)} \| \lambda_v m \| \| \varphi_v m (x) \| \leq 2^{(v-j)(K - \alpha^+)} \sum_{m \in \mathbb{Z}^n} 2^{v (\alpha (x) - n / 2)} \| \lambda_v m \| \eta_v m (x - x Q_v m) \]
\[ \leq 2^{(v-j)(K - \alpha^+)} \sum_{m \in \mathbb{Z}^n} 2^{v (\alpha (x) + n / 2)} \| \lambda_v m \| \eta_v m \chi_v m (x). \]

Put
\[ I_{t,j} := t^{-\frac{\alpha (x)}{n}} \sum_{v=1}^{j} 2^{(v-j)(K - \alpha^+)} \eta_v m * \left( 2^{v (\alpha (\cdot) + n / 2)} \sum_{m \in \mathbb{Z}^n} \| \lambda_v m \| \chi_v m \right) \| \|_{p (\cdot)} \chi_{[2^{-j}, 2^{1-j}] (\cdot)} \]}
Since
\[ 2^{v \alpha (x)} \eta_v m \chi_v m \lesssim \eta_v m * 2^{v \alpha (x)} \chi_v m \]
by Lemma 4 for \( t = M - c \log (\alpha) \), and since \( K > \alpha^+ \), we apply Lemma 9 to obtain
\[ \left\| I_{t,j} \right\|_{L^p (\cdot)} \lesssim \left\| \left( t^{-\frac{\alpha (x)}{n}} \right) \left( \sum_{v=0}^{j} 2^{(v-j)(K - \alpha^+)} \sum_{m \in \mathbb{Z}^n} \lambda_v m \chi_v m \right) \| \|_{p (\cdot)} \chi_{[2^{-j}, 2^{1-j}] (\cdot)} \right\|_{L^p (\cdot)} \]
\[ \lesssim \left\| \left( t^{-\frac{\alpha (x)}{n}} \right) \left( 2^{j (\alpha (\cdot) + n / 2)} \sum_{m \in \mathbb{Z}^n} \lambda_j m \chi_j m \right) \| \|_{p (\cdot)} \chi_{[2^{-j}, 2^{1-j}] (\cdot)} \right\|_{L^p (\cdot)} \]
\[ \lesssim \| \lambda \|_{B^p_{\alpha} (L^q (\cdot))}. \]
For \( v = 0 \), we have
\[
\left\| \left( \left\| t^{-\frac{1}{\varphi(t)}} 2^{-j(K-\alpha^+)} \eta_{0,M} \ast \left( \sum_{m \in \mathbb{Z}^n} |\lambda_{0,m}| \chi_{0,m} \right) \right\|_{p(t)} \chi_{[2^{-j},2^{1-j}]} \right\|_{L^\infty} \leq \sum_{m \in \mathbb{Z}^n} |\lambda_{0,m}| \chi_{0,m} \right\|_{p(t)} \chi_{[2^{-j},2^{1-j}]} \right\|_{L^\infty} \]
\[
\leq \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{0,m} \chi_{0,m} \right\|_{p(t)} \left\| \left( \left\| t^{-\frac{1}{\varphi(t)}} 2^{-j(K-\alpha^+)} \chi_{[2^{-j},2^{1-j}]} \right\|_{L^\infty} \right) \right\|_{L^\infty} \]
\[
\leq \left\| |\lambda| b_{p(t)} \cdot \right\|_{L^\infty}.
\]
Now from Lemma 12 we have for any \( M \) sufficiently large, \( t \in [2^{-j}, 2^{1-j}] \) and \( v \geq j \)
\[
\sum_{m \in \mathbb{Z}^n} t^{-\alpha(x)} |\lambda_{v,m}| \left| \varphi_t \ast \varphi_{v,m} (x) \right| \leq 2^{j-v(L+1+n/2)} \left( \sum_{m \in \mathbb{Z}^n} 2^j |\lambda_{v,m}| \eta_{j,M}(x - x \chi_{v,m}) \right) \leq 2^{j-v(L+1+n/2)} \left( \sum_{m \in \mathbb{Z}^n} 2^j |\lambda_{v,m}| \eta_{j,M} \ast \eta_{v,M}(x - x \chi_{v,m}) \right)
\]
where we used Lemma 3 in the last step, since \( \eta_{j,M} = \eta_{\text{min}(v,j),M} \). Again by Lemma 4 we have
\[
\eta_{j,M} \ast \eta_{v,M}(x - x \chi_{v,m}) \leq 2^v \eta_{j,M} \ast \eta_{v,M} \chi_{v,m}(x).
\]
Therefore, we estimate as
\[
\sum_{m \in \mathbb{Z}^n} t^{-\alpha(x)} |\lambda_{v,m}| \left| \varphi_t \ast \varphi_{v,m} (x) \right| \leq c 2^{j-v(L+1-n/2)} \left( \sum_{m \in \mathbb{Z}^n} 2^j |\lambda_{v,m}| \eta_{j,M} \ast \eta_{v,M} \chi_{v,m}(x) \right)
\]
by Lemmas 11 and 3 with \( t = M - c_{\log}(\alpha) \). Since the convolution with a radially decreasing \( L^1 \)-function is bounded on \( L^p(\cdot) \), it follows that
\[
\left\| \left( \left\| t^{-\frac{1}{q(t)}} \sum_{v=j}^{\infty} 2^{j-v} H \right\| \eta_{j,T} \ast \eta_{v,T} \ast \left( \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m} \right) \chi_{[2^{-j},2^{1-j}]} \right\|_{L^p(\cdot)} \right\|_{L^\infty(\cdot)}
\]
is bounded by
\[
\left\| \left( \left\| t^{-\frac{1}{q(t)}} \sum_{v=j}^{\infty} 2^{j-v} H \right\| \left( \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m} \right) \chi_{[2^{-j},2^{1-j}]} \right\|_{L^p(\cdot)} \right\|_{L^\infty(\cdot)}
\]
where we put \( H := L + 1 + \alpha^- \). Observing that \( H > 0 \), an application of Lemma 9 yields that the last expression is bounded by
\[
c \left\| \left( \left\| t^{-\frac{1}{q(t)}} \right\| \left( \sum_{m \in \mathbb{Z}^n} |\lambda_{j,m}| \chi_{j,m} \right) \chi_{[2^{-j},2^{1-j}]} \right\|_{L^p(\cdot)} \right\|_{L^\infty(\cdot)} \leq \left\| \lambda \right\| b_{p(t),\varphi(t)} \cdot
\]

Clearly, \( \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| |\Phi \ast \varrho_{v,m}(x)| \) is bounded by
\[
c \sum_{v=0}^{\infty} 2^{-v(L+1+\alpha^-)} \eta_{0,T} \ast \eta_{v,T} \ast \left[ 2^{v(\alpha^-)+n/2} \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m} \right](x).
\]

Taking the \( L^p(\cdot) \)-norm and the fact the convolution with a radially decreasing \( L^1 \)-function is bounded on \( L^p(\cdot) \), we get
\[
\|
\Phi \ast f
\|_{p(\cdot)} \lesssim \sum_{v=0}^{\infty} 2^{-v(L+1+\alpha^-)} \left[ 2^{v(\alpha^-)+n/2} \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m} \right] \|_{p(\cdot)} \lesssim \| \lambda \|_{b^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} ^{\alpha(\cdot)} ,
\]
where
\[
\| \lambda \|_{b^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \sup_{v \geq 0} \left\| t^{-(\alpha^-)+n/2} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \right\|_{p(\cdot)} \| L^q(\cdot)[2^{-v},2^{1-v}].\frac{q}{p} \|.
\]

Indeed, we see from Hölder’s inequality that
\[
\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \|_{p(\cdot)} \lesssim 1 \sum_{v=0}^{\infty} 2^{-v(L+1+\alpha^-)} \left[ 2^{v(\alpha^-)+n/2} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \right] \|_{p(\cdot)} \lesssim \| \lambda \|_{b^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} ^{\alpha(\cdot)} ,
\]
where \( q'(\cdot) \) is the conjugate exponent of \( q(\cdot) \). Our estimate follows from the embedding
\[
b^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow b^{\alpha(\cdot)}_{p(\cdot),q(\cdot)},
\]
and hence, the proof of Theorem 10 is complete. \( \blacksquare \)

6 Appendix

Here we present the proof of Lemma 6 and the convergence of (38). 

Proof of Lemma 6 Let \( p \in P^{\log}(\mathbb{R}^n) \) with \( 1 \leq p^- \leq p^+ < \infty \), and recall that
\[
\tilde{p}_Q = \text{ess-inf}_{z \in Q} p(z).
\]

Define \( q \in P^{\log}(\mathbb{R}^n \times \mathbb{R}^n) \) by
\[
q(x,y) := \max \left( \frac{1}{p(x)}, \frac{1}{p(y)}, 0 \right).
\]

We split \( f(y) \) into three parts:
\[
f_1(y) = f(y) \chi_{\{z \in Q: |f(z)| > 1\}}(y), \quad f_2(y) = f(y) \chi_{\{z \in Q: |f(z)| \leq 1, p(z) \leq p(x)\}}(y),
\]
\[
f_3(y) = f(y) \chi_{\{z \in Q: |f(z)| \leq p(x)\}}(y).
\]
\[ f_3(y) = f(y)\chi_{\{z \in Q : |f(z)| \leq 1, p(z) > p(x)\}}(y). \]

Then we can write
\[
\left( \frac{\gamma_m}{w(Q)} \int_Q |f(y)| w(y) \, dy \right)^{p(x)} \leq \max(3^{p^{-1}}, 1) \sum_{i=1}^{3} \left( \frac{\gamma_m}{w(Q)} \int_Q |f_i(y)| w(y) \, dy \right)^{p(x)} = \max(3^{p^{-1}}, 1)(I_1 + I_2 + I_3).
\]

**Estimation of I_1.** By Hölder’s inequality, we see that
\[
I_1 \leq \gamma_m^{p(x)} \left( \frac{1}{w(Q)} \int_Q |f_1(y)|^{p_{\tilde{Q}}} w(y) \, dy \right)^{\frac{p(x)}{p_{\tilde{Q}}}}.
\]

Since \(|f_1(y)| > 1\), we have
\[
|f_1(y)|^{p_{\tilde{Q}}} \leq |f_1(y)|^{p(y)} \leq |f(y)|^{p(y)}
\]
and thus we conclude that
\[
I_1 \leq \gamma_m^{p(x)} \left( \frac{1}{w(Q)} \int_Q |f(y)|^{p(y)} w(y) \, dy \right)^{\frac{p(x)}{p_{\tilde{Q}}}} \left( \frac{1}{w(Q)} \int_Q |f(y)|^{p(y)} w(y) \, dy \right) \leq \gamma_m^{p(x)} \left( \frac{1}{w(Q)} \int_Q |f(y)|^{p(y)} w(y) \, dy \right),
\]
where we used the fact that
\[
\int_Q |f(y)|^{p(y)} w(y) \, dy \leq 1.
\]

Obviously, if \( Q = (a, b) \subset \mathbb{R}, \) with \( 0 < a < b < \infty, \) we have the same estimate for \( p_{\tilde{Q}} \) replaced by \( p^- \).

**Estimation of I_2.** Again by Hölder’s inequality, we see that
\[
I_2 \leq \gamma_m^{p(x)} \frac{1}{w(Q)} \int_Q |f_2(y)|^{p(x)} w(y) \, dy =: J.
\]

Since \(|f_2(y)| \leq 1\), it follows that
\[
|f_2(y)|^{p(x)} \leq |f_2(y)|^{p(y)}
\]
and hence,
\[
J \leq \frac{1}{w(Q)} \int_Q |f(y)|^{p(y)} w(y) \, dy,
\]
which is true if \( Q = (a, b) \subset \mathbb{R}, \) with \( 0 < a < b < \infty. \) Thus we conclude that
\[
I_2 \leq \frac{1}{w(Q)} \int_Q |f(y)|^{p(y)} w(y) \, dy.
\]

**Estimation of I_3.** Again by Hölder’s inequality, we see that
\[
(44) \quad \left( \frac{\gamma_m}{w(Q)} \int_Q |f_3(y)| w(y) \, dy \right)^{p(x)} \leq \frac{1}{w(Q)} \int_Q (\gamma_m |f_3(y)|)^{p(x)} w(y) \, dy.
\]

Observe that
\[
\frac{1}{p(x)} = \frac{1}{p(y)} + \frac{1}{p(x)} - \frac{1}{p(y)} \leq \frac{1}{p(y)} + \frac{1}{q(x, y)}, \quad x, y \in Q,
\]
where \( q(x, y) \) is defined in (43). Therefore,

\[
(\gamma_m |f_3(y)|)^p \leq |f_3(y)|^p + \gamma_m q^{(x,y)}, \quad x, y \in Q,
\]

by using Young’s inequality. Hence the right member of (44) is dominated by

\[
\frac{1}{w(Q)} \int_Q \left( |f(y)|^p + \gamma_m q^{(x,y)} \right) \chi_{\{z \in Q: |f(z)| \leq 1, p(z) > p(x)\}}(y)w(y)\,dy
\]

\[
\leq \frac{1}{w(Q)} \int_Q \left( |f(y)|^p + \gamma_m q^{(x,y)} \right) w(y)\,dy.
\]

Now observe that

\[
\frac{1}{q(x, y)} \leq \frac{1}{s(x)} + \frac{1}{s(y)},
\]

where \( \frac{1}{s(z)} = \left| \frac{1}{p(z)} - \frac{1}{p_\infty} \right| \). We have

\[
\gamma_m q^{(x,y)} = \gamma_m q^{(x,y)/2} q^{(x,y)/2}
\]

and

\[
\gamma_m q^{(x,y)/2} \leq \gamma_m s^{(x)/4} + \gamma_m s^{(y)/4},
\]

again by Young’s inequality. We suppose that \(|Q| < 1\). Since \( p \in \mathcal{P}^{\log}(\mathbb{R}^n) \), we have

\[
\frac{1}{q(x, y)} \leq \left| \frac{1}{p(x)} - \frac{1}{p(y)} \right| \leq c \frac{1/p}{-\log |Q|}.
\]

Hence,

\[
\gamma_m q^{(x,y)/2} = e^{-2mc(1/p)q(x,y)} \leq |Q|^m.
\]

If \(|Q| \geq 1\), then we use \( \gamma_m q^{(x,y)/2} \leq 1 \) which follows from \( \gamma_m < 1 \). We have

\[
\frac{1}{s(x)} = \left| \frac{1}{p(x)} - \frac{1}{p_\infty} \right| \leq c \frac{1/p}{\log(e + |x|)}.
\]

Hence,

\[
\gamma_m s^{(x)/4} = e^{-mc(1/p)s(x)} \leq (e + |x|)^{-m}, \quad x \in Q.
\]

Similarly, we obtain

\[
\gamma_m s^{(y)/4} = e^{-mc(1/p)s(y)} \leq (e + |y|)^{-m}, \quad y \in Q.
\]

Summarizing the estimates obtained now, we conclude the required inequality (4).

We now turn to prove (4) with \( Q = (a, b) \subset \mathbb{R} \),

\[
\omega(m, b) = \min(b^m, 1), \quad \phi(y) = |f(y)|^p(y)
\]

and

\[
g(x, y) = \left( e + \frac{1}{x} \right)^{-m} + \left( e + \frac{1}{y} \right)^{-m}
\]

with \( p \in \mathcal{P}(\mathbb{R}) \) being log-Hölder continuous at the origin. Recall that

\[
I_3(x) \leq \frac{1}{w(Q)} \int_Q \left( |f(y)|^p + \gamma_m q^{(x,y)} \right) w(y)\,dy.
\]
Observe that
\[
\frac{1}{q(x, y)} = \max \left( \frac{1}{p(x)} - \frac{1}{p(y)}, 0 \right) \leq \frac{1}{h(x)} + \frac{1}{h(y)},
\]
where \( \frac{1}{h(\cdot)} = \left| \frac{1}{p(\cdot)} - \frac{1}{p(0)} \right| \) and \( x, y \in Q = (a, b) \), with \( 0 < a < b < \infty \). We have
\[
\gamma^q_m(x, y) = \gamma^{q(x, y)/2}_m \gamma^{q(x, y)/2}_m \leq \gamma^{h(x)/4}_m \gamma^{h(y)/4}_m,
\]
again by Young’s inequality. We suppose that \( b < 1 \). Then for any \( x, y \in Q = (a, b) \), with \( 0 < a < b < \infty \), we have
\[
\frac{1}{q(x, y)} \leq \frac{1}{h(x)} + \frac{1}{h(y)} \leq \frac{c_{\log}(1/p)}{\log(e + \frac{1}{x})} + \frac{c_{\log}(1/p)}{\log(e + \frac{1}{y})} \leq \frac{2c_{\log}(1/p)}{\log(e + \frac{1}{b})},
\]
since \( p \in P(\mathbb{R}) \) is log-Hölder continuous at the origin. Hence,
\[
\gamma^q_m(x, y)/2 = e^{-2mc_{\log}(1/p)q(x, y)} \leq b^m.
\]
If \( b \geq 1 \), then \( \gamma^q_m(x, y)/2 \leq 1 \), which follows again from \( \gamma_m < 1 \). We have
\[
\gamma^{h(x)/4}_m = e^{-mc_{\log}(1/p)} \leq e^{-m\log(e + \frac{1}{x})} = \left( e + \frac{1}{x} \right)^{-m} , \quad x \in Q.
\]
Similarly, we obtain
\[
\gamma^{h(y)/4}_m \leq \left( e + \frac{1}{y} \right)^{-m} , \quad y \in Q.
\]
We now turn to prove (4) with \( Q = (a, b) \subset \mathbb{R} \),
\[
\phi(y) = |f(y)|^{p(0)} , \quad g(x, y) = \left( e + \frac{1}{x} \right)^{-m} \chi_{\{z \in Q : p(z) < p(0)\}}(x),
\]
\( \omega(m, b) = \min(b^m, 1) \) and \( p \in P(\mathbb{R}) \) being log-Hölder continuous at the origin. We split \( f(y) \) into two parts
\[
\begin{align*}
 f_1(y) &= f(y)\chi_{\{z \in Q : |f(z)| > 1\}}(y) \\
 f_2(y) &= f(y)\chi_{\{z \in Q : |f(z)| \leq 1\}}(y)
\end{align*}
\]
for any \( y \in Q \). Then we can write
\[
\left( \frac{\gamma_m}{w(Q)} \int_Q |f(y)| w(y) dy \right)^{p(x)} \leq \max(2^{p^+ - 1}) \sum_{i=1}^2 \left( \frac{\gamma_m}{w(Q)} \int_Q |f_i(y)| w(y) dy \right)^{p(x)} = \max(2^{p^+ - 1}) (J_1(x) + J_2(x))
\]
for any \( x \in Q = (a, b) \) with \( 0 < a < b < \infty \). As in the estimation of \( I_1 \) we obtain
\[
J_1(x) \leq (w(Q))^{1 - \frac{p(x)}{p}} \left( \frac{1}{w(Q)} \int_Q |f(y)|^{p(0)} w(y) dy \right).
\]
We write
\[
\begin{align*}
 J_2(x) &= J_2(x)\chi_{\{z \in Q : p(z) \geq p(0)\}}(x) + J_2(x)\chi_{\{z \in Q : p(z) < p(0)\}}(x) \\
 &= J_3(x) + J_4(x).
\end{align*}
\]
By Hölder’s inequality, we have

$$J_3(x) \leq \gamma_m^{p(x)} \frac{1}{w(Q)} \int_Q |f_2(y)|^{p(0)} w(y) dy,$$

since $|f_2(y)| \leq 1$ and $p(x) \geq p(0)$. Again by Hölder’s inequality, we have

$$J_4(x) \leq \frac{1}{w(Q)} \int_Q \left( \gamma_m \left| f(y) \right|^{p(x)} w(y) \right) dy \chi_{\{z \in Q : p(z) < p(0)\}}(x).$$

Now, thanks to Young’s inequality, the last term is dominated by

$$J_4(x) \leq \frac{1}{w(Q)} \int_Q \left( \left| f(y) \right|^{p(0)} + \gamma_m^{\sigma(x,0)} \right) w(y) dy \chi_{\{z \in Q : p(z) < p(0)\}}(x),$$

where

$$\frac{1}{\sigma(x,0)} = \frac{1}{p(x)} - \frac{1}{p(0)}, \quad x \in \{z \in Q : p(z) < p(0)\}.$$

We have

$$\frac{1}{\sigma(x,0)} \leq \frac{c_{\log} (1/p)}{\log(e + \frac{1}{x})} \leq \frac{c_{\log} (1/p)}{\log(e + \frac{1}{b})}, \quad x \in \{z \in Q : p(z) < p(0)\},$$

since $p \in \mathcal{P}(\mathbb{R})$ is log-Hölder continuous at the origin. Then

$$\gamma_m^{\sigma(x,0)/2} = e^{-2\sigma(x,0)c_{\log}(1/p)} \leq (e + \frac{1}{x})^{-m}, \quad x \in \{z \in Q : p(z) < p(0)\},$$

and

$$\gamma_m^{\sigma(x,0)/2} \leq \begin{cases} b^m, & \text{if } b < 1; \\ 1, & \text{if } b \geq 1. \end{cases}$$

We now turn to prove (4) with $Q = (a, b) \subset \mathbb{R},$

$$\omega(m, b) = 1 \quad \gamma_m = e^{-mc_{\log}}, \quad \phi(y) = |f(y)|^{p_{\infty}}$$

and

$$g(x, y) = (e + x)^{-m} \chi_{\{z \in Q : p(z) < p_{\infty}\}}(x)$$

with $p \in \mathcal{P}(\mathbb{R})$ satisfying the log-Hölder decay condition. We employ the same notation as in the proof of (4). We need only to estimate $J_2$. We write

$$J_2(x) = J_2(x) \chi_{\{z \in Q : p(z) \geq p_{\infty}\}}(x) + J_2(x) \chi_{\{z \in Q : p(z) < p_{\infty}\}}(x) = J_5(x) + J_6(x).$$

By Hölder’s inequality, we have

$$J_5(x) \leq \gamma_m^{p(x)} \frac{1}{w(Q)} \int_Q |f_2(y)|^{p_{\infty}} w(y) dy,$$

since $|f_2(y)| \leq 1$ and $p(x) \geq p_{\infty}$. Again by Hölder’s inequality, we have

$$J_6(x) \leq \frac{1}{w(Q)} \int_Q \left( \gamma_m \left| f(y) \right|^{p(x)} w(y) \right) dy \chi_{\{z \in Q : p(z) < p_{\infty}\}}(x).$$
Again by Young’s inequality, the last term is dominated by
\[
J_6(x) \leq \frac{1}{w(Q)} \int_Q \left( |f(y)|^{p_\infty} + \gamma_m^{\sigma(x,p_\infty)} \right) w(y) dy \chi_{\{z \in Q : p(z) < p_\infty\}}(x),
\]
where
\[
\frac{1}{\sigma(x,p_\infty)} = \frac{1}{p(x)} - \frac{1}{p_\infty}, \quad x \in \{z \in Q : p(z) < p_\infty\}.
\]
We have
\[
\frac{1}{\sigma(x,p_\infty)} \leq \frac{c_{\log}}{\log(e + x)}, \quad x \in \{z \in Q : p(z) < p_\infty\},
\]
since \( p \in \mathcal{P}(\mathbb{R}) \) satisfies the log-Hölder decay condition. Then
\[
\gamma_m^{\sigma(x,p_\infty)} = e^{-\sigma(x,p_\infty)m_{\log}} 
\leq (e + x)^{-m}, \quad x \in \{z \in Q : p(z) < p_\infty\}.
\]
Hence the lemma is proved.

Proof of the convergence of (38). As a by-product of the previous proof, we can prove the convergence of (38) by employing the same method as in [16, Theorem 4.3].

Let \( \varphi \in \mathcal{S}^c(\mathbb{R}^n) \). By (33), (34) and (35), and the Taylor expansion of \( \varphi \) up to order \( L \) with respect to the off-points \( x_{Q,v,m} \), we obtain for fixed \( v \)
\[
\int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \varphi_v(y) \varphi(y) dy
= \int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \varphi_v(y) \left( \varphi(y) - \sum_{|\beta| \leq L} (y - x_{Q,v,m})^\beta \frac{D^\beta \varphi(x_{Q,v,m})}{\beta!} \right) dy.
\]
The last factor in the integral can be uniformly estimated from the above by
\[
c 2^{-v(L+1)}(1 + |y|^2)^{-M/2} \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^{M/2} \sum_{|\beta| \leq L+1} |D^\beta \varphi(x)|,
\]
where \( M > 0 \) is at our disposal. Let \( 0 < t < 1 \) and \( s(x) = \alpha(x) + \frac{m}{p(x)}(t - 1) \) be such that \( L + 1 > -s(\cdot) > -\alpha(\cdot) \). Since \( \varphi_v \) are \([K,L]-\)atoms, then for every \( S > 0 \), we have
\[
|\varphi_v(y)| \leq c 2^{m/2} \left( 1 + 2^v |y - x_{Q,v,m}| \right)^{-S}.
\]
Therefore, we find that
\[
\left| \int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \varphi_v(y) \varphi(y) dy \right|
\leq c 2^{-v(L+1)} \int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} 2^{m/2} |\lambda_{v,m}| \frac{1 + |y|^2}{{(1 + 2^v |y - x_{Q,v,m}|)^S}} dy.
\]
Applying Lemma 4, we obtain
\[
\sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \left( 1 + 2^v |y - x_{Q,v,m}| \right)^{-S} \lesssim \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \eta_{v,S} \chi_{v,m}(y).
\]
We split $M$ into $R + S$. Since we have in addition the factor $(1 + |y|^2)^{-S/2}$, Hölder’s inequality and $(1 + |y|^2)^{-R/2} \lesssim (1 + |h|^2)^{-R/2}$ give that the term $\left| \int_{\mathbb{R}^n} \cdots dy \right|$ is dominated by

$$c 2^{-v(L+1)} \sum_{h \in \mathbb{Z}^n} (1 + |h|^2)^{-R/2} \left\| \eta_{v,S} \ast \left( \sum_{m \in \mathbb{Z}^n} 2^{m/2} |\lambda_{v,m}| x_{v,m} \right) \right\|_{p(x)/t}$$

$$\lesssim c 2^{-v(L+1+s(x))} \sup_{j \geq 0} \left\| 2^{(s(x)+n/2)j} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} X_{j,m} \right\|_{p(x)/t}$$

$$= c 2^{-v(L+1+s(x))} \sup_{j \geq 0} g_j$$

for some $x \in \mathbb{R}^n$ and by taking $R$ large enough. We see from Hölder’s inequality that

$$g_j = \frac{1}{\log 2} \int_{2^{1-j}}^{2^{1-j}} \frac{g_j d\tau}{\tau} \leq \frac{1}{\log 2} \int_{2^{-j}}^{2^{-j}} \left\| \tau^{-(s(x)+n/2)} \right\|_{L^q([2^{-j}, 2^{1-j}])} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} X_{j,m} \right\|_{L^q([2^{-j}, 2^{1-j}])} d\tau$$

$$\lesssim \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} X_{j,m} \right\|_{L^q([2^{-j}, 2^{1-j}])} \left\| 1 \right\|_{L^{q'}([2^{-j}, 2^{1-j}])}$$

$$\lesssim \| \lambda \|_{B^s_{p(x)/t, q(x)}},$$

The convergence of (38) is now clear from the fact that $L + 1 + s(x) > 0$ and the embedding

$$B^\alpha_{p(x), q(x)} \hookrightarrow B^s_{p(x)/t, q(x)} \hookrightarrow \tilde{B}^s_{p(x)/t, q(x)}.$$

The proof is complete.

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