Many-body localization (MBL) has been widely investigated for both fermions and bosons, it is however much less explored for anyons. Here we numerically calculate several physical characteristics related to MBL of a one-dimensional disordered anyon-Hubbard model in both localized and delocalized regions. We figure out a logarithmically slow growth of the half-chain entanglement entropy and an area-law rather than volume-law obedience for the highly excited eigenstates in the MBL phase. The adjacent energy level gap-ratio parameter is calculated and it is found to exhibit a Poisson-like probability distribution in the deep MBL phase. By studying a hybridization parameter, we reveal an intriguing effect that the statistics can induce localization-delocalization transition. Several physical quantities, such as the half-chain entanglement, the adjacent energy level gap-ratio parameter, and the critical disorder strength, are shown to be non-monotonically dependent on the anyon statistical angle. Furthermore, a feasible scheme based on the spectroscopy of energy levels is proposed for the experimental observation of these statistically related properties.

I. INTRODUCTION

Many-body localization (MBL) is the interacting analog of single-particle localization and extends the original work of Anderson$^{11}$ with the effects of particle-particle interactions. There are two known classes of closed Many-body systems: ergodic systems and MBL systems. Ergodic systems serve as a heat bath for themselves and thermalize after sufficient long unitary evolution and thus the initial information of the systems is lost$^{12}$. On the contrast, the emergence of local integral of motions caused by disorders such as random potentials or interacting strengths leads to ergodicity breakdown and keeps the system in highly nonthermal states$^{13,14}$. The key ingredient for the many-body localization-delocalization transition is disorder via a mechanism similar to the Anderson localization. Various aspects of MBL systems are theoretically studied in the last few years with great progress, such as a criterion for many-body localization-delocalization phase transition proposed in Ref.$^{15}$ high energy eigenstates with power-law entanglement spectrum in localized region,$^{16}$ and localization-induced real-space renormalization group approach for anyon quantum systems$^{17}$. MBL systems are robust against small perturbations and have potential of storing initial state information for a long time, and hence may be useful for dynamical quantum control and quantum memory devices. Current active experimental searches for MBL have been reported in ultracold atomic$^{18,19}$ and ultracold ion$^{20}$ systems, and superconducting circuits$^{21}$. Fractional statistics that interpolate boson statistics and fermion statistics was first proposed more than forty years ago in two-dimensional systems$^{22}$. The particles obey fractional statistics are anyons, and the many-body wavefunction of the Abelian anyons acquires an additional phase $e^{i\theta}$ when exchanging two anyons on different sites, where $\theta$ denotes the statistical angle. In the limit $\theta \rightarrow 0$, anyons become bosons whose wavefunction remains invariant under particle exchange, and when $\theta \rightarrow \pi$ anyons behave like fermions. Quasiparticles in the two-dimensional fractional quantum Hall effect obey fractional statistics and can be considered as anyons. Anyons play an important role as quasiparticles in topologically ordered states, and may be potentially useful in quantum information processing$^{20,23}$. Fractional statistics was restricted in two-dimensional systems until Haldane introduced arbitrary dimensional fractional statistics$^{24}$. Recently, a one-dimensional Hubbard model of fermions with correlated hopping process has been proposed to realize fractional statistics$^{25}$. Alternative schemes for bosons with occupation-dependent hopping amplitudes by photon-assisted tunneling$^{26}$, Raman-assisted hopping$^{27}$, and lattice-shaking-induced resonant tunneling with potential tilt$^{28}$ have also been proposed to realize anyons in one-dimensional optical lattices$^{29}$. These proposals are based on the fractional Jordan-Wigner transformation by mapping anyons to bosons with a density-dependent tunneling parameter. Some exotic properties of one-dimensional anyon$^{30}$ closely related to the statistical angle have been revealed, such as the statistically induced ground state phase transition$^{20,31,32}$, the asymmetry of two-body correlations in the momentum space$^{32}$, and the spatially asymmetric particle transport of interacting anyons$^{31}$. However, to the best of our knowledge, the MBL properties of anyons in disordered systems have not been studied in literature.

In this paper, we numerically calculate several physical characteristics related to the MBL in a one-dimensional disordered (soft-core) anyon-Hubbard model in both localized and delocalized regions by using the numerical
exact diagonalization (ED) \[23,26\]. Firstly, we present numerical evidences of the existence of the MBL phase in the anyon-Hubbard model. The half-chain entanglement entropy grows quickly in the ergodic phase and logarithmically slow in the localized region, respectively. The area-law growth of steady-state entanglement entropy for highly excited states is also explored. The calculated Poisson-like energy level spacing statistics further indicates that the MBL phase exists in the anyon-Hubbard model with strong disorders, and the mean value of the gap-ratio parameter shows the $\theta$-dependence for various disorder strengths. We also find that the localization length for $\theta = \pi$ is larger than $\theta = 0$. Then by studying a hybridization parameter, we find that a localization-delocalization transition can be induced merely by the anyon statistic angle. Furthermore, several physical quantities, such as the half-chain entanglement, the adjacent energy level gap-ratio parameter, and the critical disorder strength, are found to be non-monotonic functions as the statistical angle. Finally, we propose the scheme based on the spectroscopy of energy levels technique to observe the intrinsic properties of the MBL of anyons in a small system. In our scheme, both the mean value of the gap-ratio parameter and the inverse participation ratio can be extracted from the discrete-time Fourier transform of time-dependent two-point correlation functions.

The rest of this paper is organized as follows. In Sec. II, we introduce the anyon-Hubbard model and its mapping to the Bose-Hubbard model with an occupation-dependent gauge field through the Jordan-Wigner transformation. Section III is devoted to investigating the difference of ergodic and localized phases, studying the statistically induced localization-delocalization transition, and revealing the non-monotonic dependence of critical disorder strength on the statistical angle. In Sec. IV, we propose the methods to experimentally observe the MBL in the system. A brief summary is presented in Sec. V.

II. MODEL AND METHODS

Let us first briefly introduce the anyon-Hubbard model and the fractional Jordan-Wigner transformation which exactly maps the anyon model to the boson model. The interacting anyon-Hubbard model in one-dimensional lattice reads

$$
\hat{H}^a = -J \sum_{j=1}^{L-1} (\hat{a}_j^\dagger \hat{a}_{j+1} + h.c.) + U \sum_{j=1}^{L} \hat{n}_j (\hat{n}_j - 1),
$$

(1)

where $J$ is the tunneling amplitude, $L$ is the lattice size, $U$ is the on-site interaction strength, and $\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$ is the anyon number operator on site $j$ with $\hat{a}_j^\dagger \hat{a}_j$ being anyon creation (annihilation) operator on site $j$. This model in the clean case has been studied in Refs. \[23,26\]. Here we consider disorders by adding random on-site potential $\sum_j h_j \hat{n}_j$ in $\hat{H}^a$, where $h_j \in [-W, W]$ and $W$ is the disorder strength. Anyons obey the generalized commutation relations

$$
\hat{a}_j \hat{a}_l^\dagger = e^{-i\theta \text{sgn}(j-l)}\hat{a}_l^\dagger \hat{a}_j = \delta_{jl},
$$

(2)

$$
\hat{a}_j \hat{a}_l = e^{i\theta \text{sgn}(j-l)}\hat{a}_l \hat{a}_j,
$$

(3)

where $\theta$ is the particle statistical angle and sgn is sign operator with sgn(0) = 0. Thus exchange particles between different sites will rise an additional phase factor $e^{i\theta}$ in the many-body wavefunction and particles on the same site behave the same as bosons. Anyons in the one-dimensional system can be mapped to bosons by the fractional Jordan-Wigner transformation\[26\]

$$
\hat{a}_j = b_j e^{i\theta \sum_{l=1}^{j-1} \hat{n}_l}, \quad \hat{a}_j^\dagger = e^{-i\theta \sum_{l=1}^{j-1} \hat{n}_l} b_j^\dagger,
$$

(4)

where $\hat{b}_j$ ($\hat{b}_j^\dagger$) is boson annihilation (creation) operator. By making use of this anyon-boson mapping, the anyon-Hubbard model with on-site potential disorders can be rewritten under the boson operators

$$
\hat{H}^b = -J \sum_{j=1}^{L-1} (\hat{b}_j^\dagger \hat{b}_{j+1} e^{i\theta \hat{n}_j} + h.c.)
$$

$$
+ U \sum_{j=1}^{L} \hat{n}_j (\hat{n}_j - 1) + \sum_{j} h_j \hat{n}_j.
$$

(5)

Several schemes have been proposed to realize the Hamiltonian \[5\] in the absence of disorders with ultracold atoms in optical lattices \[23,24,26\]. Remarkably, the occupation-dependent synthetic gauge fields \[32,44\] as the key ingredient and additional disordered potential \[31,12\] have been experimentally achieved.

Below we implement an occupation-dependent tunneling scheme in the ED method to numerically handle the conditional-hopping Boson-Hubbard model. In the ED calculation, we use QuSpin\[16\] with a modified Hamiltonian builder which inserts additional $e^{\pm i\hat{n}_j \theta}$ for all matrix elements of tunneling terms based on the occupation number $n_j$. The particle number can be directly read out from the constructed Fock state basis in the particle-conserving manifold. In the following numerical simulations, we set $J = 1$ as the energy scale and use $U = 1$ or 2 in order to investigate the soft-core anyon cases. Open boundary condition is assumed in all of our numerical calculations.

III. MANY-BODY LOCALIZATION

In this section, we study the localization properties of the one-dimensional disordered anyon-Hubbard model and reveal their dependence of the anyon statistical angle $\theta$. The distinctions of entanglement growth, the dependence of half-chain entanglement on the system size, and the many-body energy level statistics indicate that both ergodic and localized phases exist in the anyon system. We show that the statistical angle $\theta$ has non-monotonic
influence on the entanglement entropy and the mean value of the adjacent energy level gap-ratio parameter, which is consisting with the non-monotonic dependence of $\theta$ for the critical disorder strength.

### A. Half-chain entanglement

We first study the half-chain entanglement of many-body states in the anyon-Hubbard model. It was revealed that very weak interactions can significantly change the growth of entanglement in nonequilibrium many-body states driven by disordered Hamiltonians. The entanglement entropy $S_{\text{ent}}$ can be defined as the von Neumann entropy

$$S_{\text{ent}} = -\text{Tr} \rho_A \log \rho_A - \text{Tr} \rho_B \log \rho_B$$

of the reduced density matrix of either side labeled by $A$ and $B$. $S_{\text{ent}}$ shows a characteristic logarithmically slow growth in the MBL phase and the saturate value is unbounded in the thermodynamic limit. Here we consider a bipartition of equal half chain $L_A = L_B = L/2$, and observe the logarithmic growth of $S_{\text{ent}}$ in our disordered anyon-Hubbard model. In our simulations, we implement the Chebyshev polynomials to approximate the action of matrix exponential $|\psi(t + \Delta t)| \approx e^{-iH\Delta t} |\psi(t)|$ at the time $t$, which can efficiently access the dynamical properties of soft-core anyons (at half filling). The half-chain entanglement entropy is also calculated under this invariant subspace in order to avoid the diagonalization of large reduced density matrix.

In Fig. 1 (a), we present the growth of half-chain entanglement entropy $S_{\text{ent}}(t)$ for $L = 10$ anyon-Hubbard model in both ergodic and deep in the localized region for several statistical angles. At half filling, we consider the initial state $|\psi(0)\rangle$ prepared in a product state where every even site is filled by an anyon. The time evolution of $S_{\text{ent}}(t)$ is obtained by averaging over 2000 disorder realizations, with the results shown in Fig. 1 (a). We can see that the entanglement entropies quickly increase from the initial time for both weak disorders (solid lines) and strong disorders (dashed lines), which correspond to the expansion of the wave package. The growth of half-chain entanglement entropy cross from dephasing-dominated to transport-dominated dynamics and then increases logarithmically slow in time for all three simulated statistical angles for strong disorders, while it grows quickly and approaches the saturate value for weak disorders.

As the steady-state entanglement entropy $S_{\text{ent}}$ scales differently in the MBL and ergodic phases, we study its dependence on the system size for highly excited eigenstates in the two phases. By using the shift-invert spectral transformation $(H^b - E_{\text{shift}})^{-1}$ along with Krylov subspace methods with an energy shift $E_{\text{shift}}$, we obtain those excited eigenstates nearest to $E_{\text{shift}} = 0$ up to $L = 12$. We plot the averaged entanglement entropy $S_{\text{ent}}$ as a function of $L$ for two different disorder strengths $W = 2$, 12 and three different statistical angles $\theta = 0, 0.5\pi, \pi$ (Fig. 1(b)). Here $S_{\text{ent}}$ is averaged over 20000, 10000, 2000, 200 disorder realizations for $L = 6, 8, 10, 12$, respectively. In the ergodic phase for weak disorder ($W = 2$, solid lines), the steady-state entanglement entropies of highly excited eigenstates increase significantly with system size for all three statistical angles. The reason for the non-linear dependence here lies in the fact that those eigenstates nearest to $E_{\text{shift}} = 0$ are not locating at the same position in the spectrum for different system size and disorder realizations. For strong disorder ($W = 12$, dashed lines), the entanglement entropies show very weak dependence on system size. This phenomenon reveals the area-law entanglement in the deep MBL phase, which is different from the volume-law in the ergodic phase. The typical eigenstates of an ergodic system exhibit thermal volume-law entanglement according to the eigenstate thermalization hypothesis, and this volume law will be broken down by strong enough disorders and the entanglement entropy scales with the area between two bipartite sub-
systems $A$ and $B$, which means $S_{\text{ent}}$ is approximately independent of the system size for one-dimensional systems. This MBL eigenstates are short-range entangled and locally correlated near the boundary of two subsystems.

We further calculate $S_{\text{ent}}$ of highly excited states as a function of the statistical angle $\theta$, with the results shown in Fig. 2 (c). Here $S_{\text{ent}}$ is calculated from states nearest to $E_{\text{shift}} = 0$ of the $L = 10$ anyon-Hubbard systems and is averaged over 2000 disorder realizations with $W = 8$ [other parameters are the same with those in Fig. 2 (b)]. The half-chain entanglement entropy shows non-monotonic relation with $\theta$, which first grows then decreases when increasing $\theta$ with $S_{\text{ent}}(\theta = \pi)$ larger than $S_{\text{ent}}(\theta = 0)$. Note that $S_{\text{ent}}$ changes from volume law to area law when the eigenstate is localized and can reflect the localization property in some aspects.

### B. Energy level statistics

The adjacent energy levels of a many-body Hamiltonian show different spectral statistics in the localized and ergodic phases. In the ergodic phase, the energy levels of large amounts of disorder realizations are described by random matrix theory, particularly by Gaussian orthogonal ensemble (GOE) for real symmetric matrices and Gaussian unitary ensemble (GUE) for complex hermitian matrices. In the MBL phase, nearby eigenstates that localized in the Fock space without level repulsion do not interact with each other and the nearest energy levels show Poisson statistics. For those ED solvable finite size systems, energy levels usually vary smoothly between GOE/GUE and Poisson statistics when increasing the disorder strength $W$. In order to avoid energy unfolding, a dimensionless gap-ratio parameter can be used to characterize statistics between adjacent energy level gaps. The gap-ratio parameter is defined as

$$r_n = \min\{\delta_n, \delta_{n-1}\}/\max\{\delta_n, \delta_{n-1}\}.$$  

where $\delta_n = E_{n+1} - E_n$ is the adjacent energy level gap. The Poisson distribution of $r$ is $p(r) = 2/(1+r)^2$ and has the mean value $\langle r \rangle_p = 2 \ln 2 - 1$.

We numerically calculate the gap-ratio parameter of anyon-Hubbard model for weak and strong disorder strengths and three different statistical angles, with the results shown in Fig. 2. In Fig. 2 (a), the probability distribution of gap-ratio parameter $p(r)$ in the MBL phase ($W = 12$, dashed lines) shows Poisson like behavior while $p$ in the ergodic phase ($W = 2$, solid lines) has probability distribution of GOE ($\theta = 0, \pi$) or GUE ($\theta = 0.5\pi$). The green dashed curve is exactly the Poisson distribution plotted as a guide to the eye. Here the system size is limited to $L = 10$ whose Hilbert space is 2002 in the half-filling manifold and 2000 disorder realizations are averaged. For different statistical angles, the probability distribution $p(r)$ behaves similarly for strong disorders but distinguishable in the ergodic phase. For the weak disorder strength, anyons with statistical angle $\theta = 0$ or $\pi$ are more likely to be localized than $\theta = 0.5\pi$.

To further investigate the energy level statistics, we analyze the relationship among the mean value of gap-ratio parameter $\langle r \rangle$, system size $L$, and disorder strength $W$. It is revealed in other many-body systems that the mean value $\langle r \rangle$ is a universal function of $(W - W_c)L^{1/\nu}$, where $W_c$ is the critical value of ergodic-MBL transition and $\nu$ is a critical exponent. We fit this universal function in Fig. 2 (b) for three different system sizes, $L = 8, 10, 12$, averaged from 10000, 2000, 200 disorder realizations respectively with the statistical angle $\theta = 0$. By choosing $W_c \approx 5.5$ and $\nu \approx 1.1$ we can see that these three curves approximately collapse to the same curve which stands for a universal function $\langle r(W; L) \rangle = f((W - W_c)L^{1/\nu})$. 

**Fig. 2.** (Color online) (a) The probability distribution of gap-ratio parameter $p(r)$ for $L = 10$ anyon-Hubbard model with two different disorder strengths $W = 2, 12$ and three statistical angles $\theta = 0, 0.5\pi, \pi$. Green dashed line is the Poisson distribution and is plotted as a guide to the eye. (b) The finite size scaling of mean value $\langle r \rangle$ with $\theta = 0$, $\langle r \rangle$ collapse to a universal function $\langle r(W, L) \rangle = f((W - W_c)L^{1/\nu})$ for different system sizes $L = 8, 10, 12$. (c) The mean value $\langle r \rangle$ as a function of statistical angle $\theta$ for different disorder strength $W$s in $L = 10$ systems. Other parameters are chosen as $J = 1$, $U = 2$, and all data is obtained by averaging over 20000, 10000, 2000, and 200 disorder realizations for $L = 6, L = 8, L = 10$, and $L = 12$ systems respectively in the half-filling manifold.
The dotted line indicates the Poisson limit $\langle r \rangle_p \approx 0.386$, and it is clear that the mean value $\langle r \rangle$ tends to this limit when increasing disorder strength $W$. We also find similar behaviors for other statistical angles, but the corresponding critical disorder strength $W_c$ is quantitatively different. Due to the small system size and limited disorder realizations available in the ED method, we are unable to figure out the difference of critical exponent $\nu$ for different statistical angle $\theta$.

Furthermore, we numerically obtain the mean value $\langle r \rangle$ as a non-monotonic function of $\theta$, which is depicted in Fig. 3(c) for four different disorder strengths, with the parameters $J = 1$, $U = 2$, and $L = 10$ in the half-filling manifold. For moderate and strong disorder strength ($W = 4$, red upward triangle; $W = 6$, green downward triangle; $W = 8$, blue square), the mean values $\langle r \rangle$ show a non-monotonic relation with $\theta$, similar to those observed from Fig. 1(c) and Fig. 4(c). It is also obvious that the values of $\langle r \rangle$ are larger for $\theta = \pi$ than those for $\theta = 0$, which implies that the fermions (anyons with $\theta = \pi$ at the anyon-Hubbard model) is more difficult than bosons (anyons with $\theta = 0$) to be localized, which is consistent with the results shown in Fig. 1(c). For weak disorder $W = 2$ (black circle), the mean value $\langle r \rangle$ forms plateau with $\langle r \rangle \approx 0.53$ at $\theta = 0$ and $\theta = \pi$. For $\theta$s in between 0 and $\pi$, the Hamiltonian becomes complex hermitian, and $\langle r \rangle \approx 0.6$.

We now present a heuristic argument to understand the $\theta$-dependence localization features. Consider the simplest case with $2 \times 2$ hermitian matrix, to make a level crossing, three parameters are necessary to be controllable for the unitary transformation (when off-diagonal elements are complex), while only two are sufficient in the orthogonal case (when the off-diagonal element is real). The level crossing resistance for complex hermitian Hamiltonians are greater than real symmetric ones, and the mean value $\langle r \rangle$ of GUE is larger than GOE. Similarly, the parameter space of eigenstates for complex hermitian Hamiltonians is larger than real symmetric ones. So anyons with statistical angle $\theta$ in between $\theta = 0$ and $\theta = \pi$ are more robust to disorders.

C. Localization length

The localization length is one of the standard measures of localization in single-particle systems. When considering interactions, it is difficult to derive the localization length exactly. The interacting localization length can be extracted from two-particle correlations, which is given by

$$C^{(2)}(i, j) = \langle \psi_n | a_i^\dagger a_j^\dagger a_j a_i | \psi_n \rangle,$$

where $| \psi_n \rangle$ is the $n$-th eigenstate of the many-body Hamiltonian. Then the distance dependent

$$C^{(2)}(d) = \sum_i C^{(2)}(i, i + d)/(L - d)$$

is the average of two-particle correlation with the same distance $j - i = d$. Near the localized phase, this two-particle correlation falls off exponentially with the distance

$$C^{(2)}_n(d) \sim e^{-|d|/\xi}$$

(10)

where $\xi$ is defined as the localization length in the interacting system. We calculate $C^{(2)}_n(d)$ in the two-particle manifold with the parameters $J = 1$, $U = 2$, $L = 14$, and $| \psi_n \rangle$ chosen to be the eigenstate at the middle of the energy spectrum. It is clear from Fig. 3 that $C^{(2)}_n(d)$ approximately falls off exponentially when increasing the distance $d$. For the same disorder strength (solid lines for $W = 8$ and dashed lines for $W = 4$), the decay rate of $C^{(2)}_n(d)$ is the largest and smallest for $\theta = 0$ and $\theta = \pi$, respectively. This means the localization length $\xi(\theta = \pi) > \xi(\theta = 0.4\pi) > \xi(\theta = 0.2\pi) > \xi(\theta = 0)$, and the anyon-Hubbard model with $\theta = \pi$ is harder to localize than $\theta = 0$.

D. Statistically induced localization-delocalization transition

Up to now, we have shown the existence of MBL phase in the one-dimensional anyon-Hubbard model, and find that the physical quantities non-monotonically depends on the statistical angle $\theta$. In this subsection, we uncover an intriguing phenomenon that the anyonic statistics may induce localization-delocalization transition at a fixing disorder strength $W$.

In order to detect the localization-delocalization transition in many-body interacting systems, we adopt the hybridization parameter $G(\epsilon, L)$ introduced in Ref. The
hybridization parameter is given by

$$G(\epsilon, L) = \log \frac{|\langle \psi_{n+1} | V | \psi_n \rangle|}{E_{n+1} - E_n},$$ \hspace{1cm} (11)

where $\epsilon = (E_n - E_{\text{min}})/(E_{\text{max}} - E_{\text{min}})$ is the energy density with $E_n$ in ascending order, $E_n = E_n + |\langle \psi_n | V | \psi_n \rangle|$ is the modified energy, $|\psi_n\rangle$ is the eigenstate corresponding to energy $E_n$, and $E_{\text{max}}$ ($E_{\text{min}}$) is the highest excited (ground) energy, and $V$ is a perturbation operator. This parameter characterizes the hybridization of nearest eigenstates induced by the perturbation. Typically, $G(\epsilon, L) \propto -\kappa L$, and $\kappa = 0$ separates localized states ($\kappa > 0$) from delocalized states ($\kappa < 0$). Thus $dG(\epsilon, L)/d\theta$ can be used to detect the localization-delocalization transition. Here we choose $V = a_L^{-1} a_{L+1}^+$ as the perturbation operator and calculate $G(\epsilon, L)$ as a function of statistical angle $\theta$ for different system size $L$s, choosing other perturbation operators to lead similar results. $\epsilon$ is fixed to where the delocalized phase is most robust in the whole energy spectrum.

In Fig. 4 (a), we plot $G(\epsilon, L)$ for half-filling anyon-Hubbard model with $J = 1$, $U = 2$, $W = 7$ and four different $L$s ($L = 6$, black circle; $L = 8$, upward triangle; $L = 10$, blue downward triangle; $L = 12$, red square). 20000, 10000, 2000, 200 disorder realizations are averaged for $L = 6, 8, 10, 12$, respectively. The hybridization parameter $G(\epsilon, L)$ decreases (increases) when enlarge system size $L$ for small (large) $\theta$ and the crossover indicates the localization-delocalization transition where $\kappa = 0$. It is clear from Fig. 4 (a) that there is a transition from localized to delocalized phase transition at $\theta \approx 0.47\pi$ when disorder strength is fixed at $W = 7$. To further clarify the different critical disorder strength $W_c$ for different statistical angle $\theta$, we calculate $G(\epsilon, L)$ as a function of disorder strength $W$ with the parameters $J = 1$, $U = 2$ and the results are plotted in Fig. 4 (b). The crossover for different system size happens at $W \approx 6.5$ and $W \approx 7.6$ for $\theta = 0.2\pi$ and $\theta = 0.5\pi$, respectively. From this aspect, the critical disorder strength $W_c$ also shows $\theta$ dependence.

It is found that the critical disorder strength $W_c$ also has a non-monotonic relationship with the statistical angle $\theta$. Firstly, we examine the critical disorder strength $W_c$ as a function of statistical angle $\theta$ by using the hybridization parameter $G(\epsilon, L)$. As shown in Fig. 4 (c) with the same parameters as Fig. 4 (b), the critical disorder $W_c$ first increases with anyonic statistics $\theta$ and then decreases when $\theta \gtrapprox 0.7\pi$. The critical value for $\theta = 0$ is $W_c \approx 5.4$, while the value for $\theta = \pi$ is much larger ($W_c \approx 7.3$). Such non-monotonic relation for the critical localization-delocalization transition value $W_c(\theta)$ is similar as from other quantities related to the ergodic-MBL transition [see Fig. 1 (c) and Fig. 2 (c)].

IV. PROPOSAL OF EXPERIMENTAL OBSERVATIONS

A recently developed many-body spectroscopy technique is able to resolve the energy levels of the interacting system and makes it possible to observe the properties of the MBL in a realistic experiment setup. We simulate the spectroscopy of energy levels in $L = 9$ sites systems with a maximum of two anyons which has 45 energy levels in the two-particle manifold, and then we derive the mean value ($r$) and inverse participation ratio (IPR) as functions of statistical angle $\theta$. The key idea is recording the response of the system after a local perturbation as a function of time and using spectrum analysis to reveal the characteristic modes of the system. We here consider only the two-particle energy manifold; although it is the simplest case for interacting systems, some typical features of the many-body localization emerge. The initial state is prepared in a product state

$$|\psi_0\rangle_{m,n} = \cdots \left( |0\rangle + |1\rangle \right)_{m} \cdots \left( |0\rangle + |1\rangle \right)_{n} \cdots,$$ \hspace{1cm} (12)

where sites $m$ and $n$ are in the superposition of $|0\rangle$ and $|1\rangle$, and all other sites are in $|0\rangle$ states. The state evolved
at time $t$ reads
\[
|\psi(t)\rangle_{m,n} = \frac{1}{2} |\text{Vac}\rangle + \frac{1}{2} \sum_{\beta} C_{m,n}^{\beta} e^{-\frac{E_{\beta}^{(2)} - E_{\beta}^{(1)}}{\hbar} t} |\phi_{\beta}^{(2)}\rangle + \frac{1}{2} \sum_{\alpha} (C_{m}^{\alpha} + C_{n}^{\alpha}) e^{-\frac{E_{\alpha}^{(1)} - E_{\alpha}^{(2)}}{\hbar} t} |\phi_{\alpha}^{(1)}\rangle,
\]
where $|\phi_{\beta}^{(2)}\rangle$ is the $\beta$-th eigenstate in the two-particle manifold with the corresponding energy $E_{\beta}^{(2)}$ and $C_{m,n}^{\beta} = \langle \phi_{\beta}^{(2)} | 1_{m}, 1_{n} \rangle$. $E_{\alpha}^{(1)}$, $|\phi_{\alpha}^{(1)}\rangle$, and $C_{m}^{\alpha}$ are counterparts in the single-particle manifold and are irrelevant in this simulation. The two-point correlation of a two-particle lowering operator can be expressed as
\[
\chi_2(m, n) = \frac{1}{4} \sum_{\beta} |C_{m,n}^{\beta}|^2 e^{-\frac{E_{\beta}^{(2)} - E_{\beta}^{(1)}}{\hbar} t}.
\]
(15)

It is obvious that the single-particle component is projected out, and we can reveal $|C_{m,n}^{\beta}|^2$ and $E_{\beta}^{(2)}$ by the discrete-time Fourier transform
\[
\chi_2^{f}(m, n)[k] = \frac{1}{N} \sum_{l=1}^{N} e^{-2\pi ikl/N} \chi_2(m, n)[l],
\]
(16)
where $N = T/\tau$ is the number of data sampled from evolution time $T$ with sampling interval $\tau$. By varying $m$ and $n$ in the initial state, the confidence of detecting each energy level is enhanced.

In Fig. 5(a), we plot $\sum_{m,n} |\chi_2(m, n)|$ as a function of energy (frequency) for a single disorder realization of anyon-Hubbard model with the parameters $J = 1$, $U = 1$, $W = 3$, and $\theta = 0.5\pi$. The discrete-time Fourier transform is preformed from data obtained by 2000 time step samplings, and 44 eigenenergies can be resolved from the peaks labeled by red crosses. In Fig. 5(b), we present the mean value $\langle r \rangle$ as a function of $\theta$ calculated from the peak positions of the discrete-time Fourier transform of $N = 2000$ samplings in evolution time $T$ (red upward triangle labeled), $N = 2000$ samplings in evolution time $10T$ (blue downward triangle labeled), and ED results (black circle labeled), all data is averaged over 100 disorder realizations. The non-monotonic relation $\langle r(\theta) \rangle$ can be clearly seen from the Fourier transform of $N = 2000$ samplings. Here the values of $\langle r \rangle$ are larger than ED results due to the energy level missing. One can measure after a longer evolution time and resolve the spectrum more accurately, but it requires much longer decoherence time of the experimental system.

**V. CONCLUSION**

In summary, we have explored the localization properties of the ergodic and localized phases in the one-dimensional disordered anyon-Hubbard model. Several physical characteristics, such as the half-chain entanglement, the adjacent energy level gap-ratio parameter, and
the critical disorder strength, have been numerically calculated. It is found that these localization characters are non-monotonically dependent on the anyon statistical angle. Furthermore, we have demonstrated that the statistics can induce localization-delocalization transition by studying the hybridization parameter. Finally, the possibility of observing these statistically related properties in the experiments is explored based on the numerical simulation of spectroscopy of energy levels.

ACKNOWLEDGMENTS

This work was supported by the NKDRP of China (Grant No. 2016YFA0301800), the NAF (Grants No. U1830111 and No. U1801661), the NSFC (Grant No. 11704132), the Key-Area Research and Development Program of Guangdong Province (Grant No. 2019B030330001), and the Key Program of Science and Technology of Guangzhou (Grant No. 20180420055).
1 W. Zhang, S. Greschner, E. Fan, T. C. Scott, and Y. Zhang, Ground-state properties of the one-dimensional unconstrained pseudo-anyon hubbard model, Phys. Rev. A 95, 053614 (2017)

2 C. Stirte, S. C. Srivastava, and A. Eckardt, Floquet realization and signatures of one-dimensional anyons in an optical lattice, Phys. Rev. Lett. 117, 205303 (2016)

3 L. Wang, L. Wang, and Y. Zhang, Quantum walks of two interacting anyons in one-dimensional optical lattices, Phys. Rev. A 90, 063618 (2014)

4 Y. Hao, Y. Zhang, and S. Chen, Ground-state properties of one-dimensional anyon gases, Phys. Rev. A 78, 023631 (2008)

5 J. Arcila-Forero, R. Franco, and J. Silva-Valencia, Critical points of the anyon-hubbard model, Phys. Rev. A 94, 013611 (2016)

6 Z.-W. Zuo, G.-L. Li, and L. Li, Statistically induced topological phase transitions in a one-dimensional superlattice anyon-hubbard model, Phys. Rev. B 97, 115126 (2018)

7 L. Wang, L. Wang, and Y. Zhang, Quantum walks of two interacting anyons in one-dimensional optical lattices, Phys. Rev. A 90, 063618 (2014)

8 F. Liu, J. R. Garrison, D.-L. Deng, Z.-X. Gong, and A. V. Gorshkov, Asymmetric particle transport and light-cone dynamics induced by anyonic statistics, Phys. Rev. Lett. 121, 250404 (2018)

9 S.-L. Zhu, Z.-D. Wang, Y.-H. Chan, and L.-M. Duan, Topological boson-mott insulators in a one-dimensional optical superlattice, Phys. Rev. Lett. 110, 075303 (2013)

10 J. M. Zhang and R. X. Dong, Exact diagonalization: the boson–hubbard model as an example, European Journal of Physics 31, 591 (2010)

11 P. Weinberg and M. Bukov, QuSpin: a Python Package for Dynamics and Exact Diagonalisation of Quantum Many Body Systems part I: spin chains, SciPost Phys. 2, 003 (2017)

12 Y.-L. Chen, G.-Q. Zhang, D.-W. Zhang, and S.-L. Zhu, Simulating bosonic chern insulators in one-dimensional optical superlattices, Phys. Rev. A 101, 013627 (2020)

13 V. Lienhard, P. Scholl, S. Weber, D. Barredo, S. de Lésséulc, R. Bai, N. Lang, M. Fleischhauer, H. P. Bichler, T. Lahaye, and A. Browaeys, Realization of a density-dependent peierls phase in a synthetic, spin-orbit coupled rydberg system, Phys. Rev. X 10, 021031 (2020)

14 F. Grg, K. Sandholzer, J. Minguzzi, R. Desbuquois, M. Messer, and T. Esslinger, Realization of density-dependent peierls phases to engineer quantized gauge fields coupled to ultraslow matter, Nat. Phys. 15, 1161 (2019)

15 L. W. Clark, B. M. Anderson, L. Feng, A. Gaj, K. Levin, and C. Chin, Observation of density-dependent gauge fields in a bose-einstein condensate based on micromotion control in a shaken two-dimensional lattice, Phys. Rev. Lett. 121, 030402 (2018)

16 F. Meinert, M. Mark, K. Lauber, A. Daley, and H.-C. Ngele, Floquet engineering of correlated tunneling in the bose-hubbard model with ultracold atoms, Phys. Rev. Lett. 116, 205301 (2016)

17 S. Paekel, T. Kihler, A. Swoboda, S. R. Mannan, U. Schollwöck, and C. Hubig, Time-evolution methods for matrix-product states, Ann. Phys. 411, 167998 (2019)

18 J. H. Bardarson, F. Pollmann, and J. E. Moore, Unbounded growth of entanglement in models of many-body localization, Phys. Rev. Lett. 109, 017202 (2012)

19 M. Serbyn, Z. Papić, and D. A. Abanin, Universal slow growth of entanglement in interacting strongly disordered systems, Phys. Rev. Lett. 110, 260601 (2013)

20 J. C. Mason and D. C. Handscomb, Chebyshev polynomials (Chapman and Hall/CRC, 2002)

21 V. Hernandez, J. E. Roman, and V. Vidal, Slepce: A scalable and flexible toolkit for the solution of eigenvalue problems, ACM Trans. Math. Softw. 31, 351 (2005)

22 M. Serbyn, Z. Papić, and D. A. Abanin, Local conservation laws and the structure of the many-body localized states, Phys. Rev. Lett. 111, 127201 (2013)

23 T. Devakul and R. R. P. Singh, Early breakdown of area-law entanglement at the many-body delocalization transition, Phys. Rev. Lett. 115, 187201 (2015)

24 H. Wu, M. Vallières, D. H. Feng, and D. W. L. Sprung, Gaussian-orthogonal-ensemble level statistics in a one-dimensional system, Phys. Rev. A 42, 1027 (1990)

25 G. Livan, M. Novaes, and P. Vivo, Introduction to random matrices: theory and practice, Vol. 26 (Springer, 2018)

26 F. Haake, Quantum signatures of chaos, in Quantum Coherence in Mesoscopic Systems (Springer, 1991) pp. 583–595.

27 V. Oganesyan and D. A. Huse, Localization of interacting fermions at high temperature, Phys. Rev. B 75, 155111 (2007)

28 A. Edelman and N. R. Rao, Random matrix theory, Acta Numerica 14, 233 (2005).

29 M. Schulz, C. A. Hooley, R. Moessner, and F. Pollmann, Stark many-body localization, Phys. Rev. Lett. 122, 040606 (2019)

30 E. van Nieuwenburg, Y. Baum, and G. Refael, From bloch oscillations to many-body localization in clean interacting systems, Proceedings of the National Academy of Sciences 116, 9269 (2019)

31 H.-Z. Xu, F.-H. Liu, S.-Y. Zhang, G.-C. Guo, and M. Gong, Stochastic and long-distance level spacing statistics in many-body localization (2019), arXiv:1901.09570 [cond-mat.stat-mech]

32 Y. Lahini, Y. Bromberg, D. N. Christouloudes, and Y. Silberberg, Quantum correlations in two-particle anderson localization, Phys. Rev. Lett. 105, 163905 (2010)