THE CHARACTER TABLE OF A SPLIT EXTENSION OF THE HEISENBERG GROUP $H_1(q)$ BY $Sp(2, q)$, $q$ ODD

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ABSTRACT. In this paper we determine the full character table of a certain split extension $H_1(q) \rtimes Sp(2, q)$ of the Heisenberg group $H_1$ by the odd-characteristic symplectic group $Sp(2, q)$.

1. Introduction

In his paper ([Gér]) P. Gérardin constructed the Weil representations of the odd-characteristic symplectic groups using the properties of a certain split extension $H_t(q) \rtimes Sp(2t, q)$ of the Heisenberg group $H_t(q)$ of order $q^{2t+1}$ by the symplectic group $Sp(2t, q)$. In this paper we explicitly determine the character table of this extension, in the case where $t = 1$. A motivation lies in the fact that knowledge of this character table seems to be useful in the study of the restrictions to parabolic subgroups of certain unipotent characters of odd-dimensional orthogonal groups (see [DPW]).

Let $V$ be the column vector space of dimension $2t$ over a finite field $F$ of order $q$, where $q$ is odd, and $V$ is provided with a non-degenerate symplectic form $j$. Given $w \in V$, we denote by $w^*$ the element of the dual space (we think at $w^*$ as a row) such that $w^* w_1 = j(w, w_1)/2$. Let $H_t(q)$ be the group consisting of the matrices

$$h = h_{(w, z)} = \begin{pmatrix} 1 & w^* & z \\ 1 & 1 & w \\ 1 & 1 & 1 \end{pmatrix} \in Mat(2t + 2, F),$$

where $w \in V$ and $z \in F$. We call this group the Heisenberg group of $V$. $H_t(q)$ is obviously a central extension of $(V, +)$ by $(F, +)$. Furthermore, $H_t(q)$ is a two-step nilpotent group of order $q^{2t+1}$ whose center is isomorphic to $F$ (cf. [Gér] Lemma 2.1).

Let $S$ be the symplectic group associated to the form $j$ and, for each $s \in S$, denote by $sw$ the image of $w$ under the natural action of $S$ on $V$. Then, the map $h_{(w, z)} \mapsto h_{(sw, z)}$ defines an automorphism of $H_t(q)$ fixing pointwise $Z(H_t(q))$. Viewed as acting on matrices, this map is the conjugation by the element $s = diag(1, s, 1)$.

Let us denote by $G$ the semidirect product $H_t(q) \rtimes Sp(2t, q)$ defined by the above action of $S$. We want to construct the character table of $G$ in the case where $t = 1$. So, $G = H_1(q) \rtimes Sp(2, q)$. In this case, we can write in a unique way a generic
element \( g \) of \( G \) as
\[
g = g(s,w,z) = s h(w,z) = \begin{pmatrix} 1 & w^* & z \\ s & sw & 1 \end{pmatrix},
\]
where \( s \in S = \text{Sp}(2,q) \) (here we identify \( s \in S \) with \( s \in G \)), \( w \in V \) and \( z \in F \). If \( w = \begin{pmatrix} x \\ y \end{pmatrix} \in V \), then we can take as \( w^* \) the row \( \frac{1}{2}(-y,x) \). Note that \( |G| = q^4(q^2 - 1) \).

2. THE CONJUGACY CLASSES

In the sequel, we denote by \( (g) \) the conjugacy class of \( G \) containing the element \( g \), and by \(|(g)|\) the size of the conjugacy class \( (g) \). The following lemma lists the conjugacy classes of \( G \).

**Lemma 1.** Let \( F = GF(q) \), \( q \) odd, and let \( F^* = \langle \nu \rangle \) be the multiplicative group of \( F \). Set
\[
\mathcal{A}(z) = \begin{pmatrix} 1 & 1 & z \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},
\]
\[
\mathcal{C}(z) = \begin{pmatrix} 1 & -1 & z \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix}, \quad \mathcal{D}(z) = \begin{pmatrix} 1 & -1 & z \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix},
\]
\[
\mathcal{G}_m(z) = \begin{pmatrix} 1 & b^m & z \\ b^m & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathcal{H}(z) = \begin{pmatrix} 1 & 1 & z \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},
\]
\[
\mathcal{I}(z) = \begin{pmatrix} 1 & 1 & z \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathcal{L}_m = \begin{pmatrix} 1 & 0 & \frac{1}{q} \mu^m \\ 1 & 0 & \mu^m \\ 1 & 0 & \mu^{m+1} \end{pmatrix}, \quad \mathcal{M}_m = \begin{pmatrix} 1 & 0 & \frac{1}{q} \mu^m \\ 1 & 0 & \mu^m \\ 1 & 0 & \mu^{m+1} \end{pmatrix},
\]
where \( z \in F, \ 1 \leq k \leq \frac{q^2 - 3}{2}, \ 1 \leq m \leq \frac{q - 1}{2} \) and \( b \) is an element of order \( q + 1 \) (a ‘Singer cycle’) of \( \text{Sp}(2,q) \). These are elements of \( G \), and \( G \) admits exactly \( q^2 + 5q \) conjugacy classes \( (g) \) with representative \( g \), as listed in the Table below.
Proof. Let \( g_1 = g(s_1, w_1, z_1) \) and \( g_2 = g(s_2, w_2, z_2) \) be two generic elements of \( G \). Then
\[
g_1 g_2 g_1^{-1} = g(s_1 s_2^{-1}, s_1 (w_2 - w_1 + s_1^{-1} w_1), z_2 - (w_2 + s_1^{-1} w_1 + s_2 w_2)^* w_1).
\]
It easily follows that if \( g_1 \) is conjugate to \( g_2 \) in \( G \), then \( s_1 \) is conjugate to \( s_2 \) in \( S \). Moreover, if \( z_1 \neq z_2 \), then the elements \( g(s_1, 0, z_1) \) and \( g(s_2, 0, z_2) \) cannot be conjugate in \( G \). Observe that \( g_1 \in C_G(g_2) \) if and only if
\[
\begin{align*}
s_1 &\in C_S(s_2) \\
w_2 + s_2^{-1} w_1 &= w_1 + s_1^{-1} w_2 \\
w_1^2(s_2 w_2) &= w_2^2(s_1 w_1)
\end{align*}
\]
Let us consider the elements \( \mathcal{A}(z) = g(1, 0, z) \), \( z \in F \). It is straightforward to see that \( Z(G) = Z(H_1(q)) = \{ \mathcal{A}(z) : z \in F \} \cong (F, +) \). Therefore, each of these \( q \) elements of \( G \) forms a central class of order \( 1 \). In particular, \( \mathcal{A}(0) \) is the identity of \( G \).

Now, let us consider the element \( g(1, w, 0) = \mathcal{B} \in H_1(q) \setminus Z(H_1(q)) \). Then, \( |C_G(\mathcal{B})| = q^2 \), i.e. \( |(\mathcal{B})| = q(q^2 - 1) \). Since
\[
g(s_1, w_1, z_1) \mathcal{B} g(s_1, w_1, z_1)^{-1} = g(s_1 w_1 - 2 w^* w_1),
\]
it turns out that the elements of \( H_1(q) \setminus Z(H_1(q)) \) form a single conjugacy class \((\mathcal{B})\) of \( G \).

Set \( g = g(s, 0, z) \in \{ \mathcal{C}(z), \mathcal{D}(z), \mathcal{E}(z), \mathcal{F}(z), \mathcal{G}(z) \} \). Recall (e.g., see [Dor §38]) that \( S \) admits elements \( b \) of order \( q + 1 \), the so-called ‘Singer cycles’. As observed before, for different values of \( z \) and \( s \) the elements \( g(s, 0, z) \) belong to \( q^2 + q \) distinct conjugacy classes of \( G \). Now, an element \( g(s, w_1, z_1) \) belongs to \( C_G(g) \) if and only if
\[
\begin{align*}
s_1 &\in C_S(s) \\
s w_1 &= w_1
\end{align*}
\]
Since \( s \) does not have eigenvalue \( 1 \), the condition \( s w_1 = w_1 \) implies \( w_1 = 0 \). It follows that \( |C_G(g)| = q|C_S(s)| \), and using the information about the centralizers of elements of \( S \) contained in [Dor §38], we obtain the results listed in the statement of the lemma.

Next, let us consider elements \( g = g(s, 0, z) \in \{ \mathcal{H}(z), \mathcal{I}(z) \} \). We argue as above, but note that this time \( s \) does admit the eigenvalue \( 1 \). This implies that in (1)
Finally, let us consider elements \( g = g(s, w, 0) \in \{ L_m, M_m \} \), where
\[
  s = \begin{pmatrix} 1 & \epsilon & 0 \\ \epsilon & 1 & 0 \end{pmatrix}, \quad w = \begin{pmatrix} \nu_n & 0 \\ 0 & 0 \end{pmatrix},
\]
and \( \epsilon \in \{1, \nu\} \). Easy calculations show that if \( 1 \leq m \leq \frac{q^2+1}{2} \) the elements \( g \) belong to distinct conjugacy classes of \( G \). An element \( g(s_1, w_1, z_1) \) belongs to \( C_G(g) \) if and only if
\[
  \left\{ \begin{array}{l}
    s_1 \in C_S(s) = \left\{ \begin{pmatrix} a & 0 \\ c & a \end{pmatrix} : a = \pm 1, c \in F \right\} \\
    w + s^{-1}w_1 = w_1 + s_1^{-1}w \\
    w_1^*(sw) = w^*(s_1w_1)
  \end{array} \right..
\]
Since the condition \( w + s^{-1}w_1 = w_1 + s_1^{-1}w \) implies \( a = 1 \), it follows that \( g(s_1, w_1, z_1) \) can be chosen in \( q^2 \) different ways. Thus, \( |C_G(g)| = q^2 \), i.e. \( |(L_m)| = |(M_m)| = q^2(q^2 - 1) \).

So far, we have found \( q^2 + 5q \) distinct conjugacy classes, adding up to \( |G| \) elements. Thus, we are done. \( \square \)

3. The character table

First of all, we observe that the character table of \( SL(2, q) \cong Sp(2, q) \cong G/H_1(q) \) is well-known, e.g., see [Dor, §38], to which we refer for notation and all the information needed in the sequel.

Next, note that, as \( Z(G) = \{ z : z \in F \} \), for any irreducible character \( \chi \) of \( G \)
\[
  \chi(\mathcal{E}(z)) = \frac{\chi(\mathcal{A}(z))}{\chi(1)} \chi(\mathcal{E}(0))
\]
for all \( z \in F \). The same holds for the classes \( (\mathcal{D}_k(z)), (\mathcal{E}(z)), (\mathcal{F}(z)), (\mathcal{M}_n(z)), (\mathcal{H}(z)) \) and \( (\mathcal{F}(z)) \). So, in the character table we only report the values of a character on \( \mathcal{E}(0), \mathcal{D}_k(0) \) and so on.

Since \( G/H_1(q) \cong SL(2, q) \), knowledge of the character table of \( SL(2, q) \) gives us by inflation \( q + 4 \) characters: namely \( 1_G, \eta_1, \eta_2, \xi_1, \xi_2, \theta_j \) \((1 \leq j \leq \frac{q-1}{2})\), \( \psi \) and \( \chi_i \) \((1 \leq i \leq \frac{q^2-1}{2})\).

Next, we construct \( q + 1 \) distinct irreducible characters of \( G \) having degree \( q \). Denote by \( \lambda \) a fixed non-trivial character of \( Z(G) \cong (F, +) \). Clearly, each of the \( q \) linear characters of \( Z(G) \) can be parametrised as \( \lambda_u (u \in F) \), where \( \lambda_u(z) = \lambda(uz) \) for all \( z \in F \). In particular, \( \lambda_0 = 1_{Z(G)} \). We know by [GZ, Lemma 1.2] that \( H_1(q) \) has exactly \( q - 1 \) non-linear irreducible characters \( \hat{\lambda}_u \), defined as
\[
  \hat{\lambda}_u(h) = \begin{cases} 
    q\lambda_u(h) & \text{if} \ h \in Z(H_1(q)) \\
    0 & \text{if} \ h \notin Z(H_1(q))
  \end{cases} \quad (u \in F^\times).
\]

Furthermore, by [GZ, Theorem 2.4] the characters \( \hat{\lambda}_u \) can be extended to \( G \). We denote such extensions by \( \omega_u \) \((u \in F^\times)\). The values taken by the characters \( \omega_u \) on the elements of \( S \) can be found in [Sze, Proposition 2]. Namely:
As follows.

To this purpose, we compute

\[
\begin{align*}
\omega & = \sum_{t \in F} \lambda(-t^2/2), \\
\omega_1 & = \sum_{t \in F} \lambda_u(-t^2/2) = \left(\frac{u}{F}\right)Q(\lambda)
\end{align*}
\]

(it turns out that \(|Q(\lambda)|^2 = q\).

We are left to compute the values of the \(\omega_u\)'s on the classes \((\mathcal{L}_m)\) and \((\mathcal{M}_m)\). To this purpose, we compute

\[
1 = (\omega_u, \omega_u)_G = q^4(q^2 - 1) + q^2(q^2 - 1) \sum_{m=1}^{2^s-1} (|\omega_u(\mathcal{L}_m)|^2 + |\omega_u(\mathcal{M}_m)|^2).
\]

This implies that \(\omega_u(\mathcal{L}_m) = \omega_u(\mathcal{M}_m) = 0\), for all \(1 \leq m \leq \frac{q-1}{2}\).

It is easy to verify that the characters \(\omega_u\eta_1, \omega_u\eta_2, \omega_u\xi_1, \omega_u\xi_2, \omega_u\theta_j, \omega_u\psi\) and \(\omega_u\chi\) \((u \in 2^\times)\) are pairwise distinct irreducible characters of \(G\).

At this stage, \(q \) irreducible characters of \(G\) are still missing. We construct them as follows.

Let us consider the Sylow \(p\)-subgroup \(K\) of \(G\) consisting of the matrices of shape

\[
k_{(a,x,y,z)} = \begin{pmatrix}
1 & -y/2 & x/2 & z \\
1 & a & x + ay & y \\
1 & 1 & & \\
1 & & & \\
\end{pmatrix},
\]

where \(a, x, y \in F\). Define the linear characters \(\mu_{u_1,u_2}(k_{(a,x,y,z)}) = \lambda_{u_1}(a)\lambda_{u_2}(y) = \lambda(u_1a + u_2y)\), where, as above, \(\lambda_u\) denotes the non-trivial linear character of \(\mathbb{Z}(G)\) associated to \(u \in F^x\) (in particular, \(\mu_{0,0} = 1_K\)).

We consider the induced characters \(\mu_{u_1,u_2}^G\).

First of all, note that \((\mathcal{E}(z)) \cap K = \emptyset\) and that the same holds also for \((\mathcal{R}(z)), (\mathcal{E}(z)), (\mathcal{F}(z))\) and \((\mathcal{M}(z))\). So, the value of \(\mu_{u_1,u_2}^G\) on these classes is 0, whereas the value on \(\mathcal{A}(z)\) is

\[
\mu_{u_1,u_2}^G(\mathcal{A}(z)) = \frac{q^4(q^2 - 1)}{q^4} = q^2 - 1.
\]

To compute \(\mu_{u_1,u_2}^G(\mathcal{R})\), we observe that if \(s = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \in S\) and \(g = g_{(s,w,z)}\), then \(\mu_{u_1,u_2}(gRg^{-1}) = \lambda_u(0)\lambda_{u_2}(s_{21}) = \lambda_{u_2}(s_{21})\). So, if \(s_{21} = 0\) the matrix \(s\) can be chosen in \(q(q-1)\) ways, whereas if we fix \(s_{21} \neq 0\), \(s\) can be chosen in \(q^2\) different ways. For \(u_2 \neq 0\) we obtain

\[
\begin{align*}
\mu_{u_1,u_2}^G(\mathcal{R}) & = \frac{q^3[q(q-1) + q^2 \sum_{s_{21} \neq 0} \lambda_{u_2}(s_{21})]}{q^4} \\
& = \frac{q^3[q(q-1) - q^2]}{q^4} = -1
\end{align*}
\]
Next, we look at the classes \((\mathcal{H}(z))\). The matrices \(s\) such that \(g\mathcal{H}(z)g^{-1} \in K\) are of shape \(\begin{pmatrix} s_{11} & s_{12} \\ -1/s_{12} & 0 \end{pmatrix}\). It follows that

\[
\mu_{u_1,u_2}(g\mathcal{H}(z)g^{-1}) = \lambda_{u_1}(-s_{12}^2)\lambda_{u_2}(0)
\]

and therefore

\[
\mu_{u_1,u_2}^G(\mathcal{H}(z)) = \frac{q^3q}{q^4} \sum_{t \neq 0} \lambda_{u_1}(-t^2) \lambda_{u_2}(-\nu t^2) = -1 + Q(\lambda_{2u_1}) = -1 + \left(\frac{2}{F}\right)Q(\lambda_u)
\]

In a similar way, one also obtains that

\[
\mu_{u_1,u_2}^G(\mathcal{I}(z)) = \sum_{t \neq 0} \lambda_{u_1}(-\nu t^2).
\]

In particular, for \(u_1 = 0\) we get \(\mu_{0,u_2}^G(\mathcal{H}(z)) = \mu_{0,u_2}^G(\mathcal{I}(z)) = q - 1\), whereas for \(u_1 \neq 0\), we get \(\mu_{u_1,u_2}^G(\mathcal{I}(z)) = -1 - Q(\lambda_{2u_1})\).

The value of \(\mu_{u_1,u_2}^G\) on \(\mathcal{L}_m\) is obtained in the same way as above: \(s\) has the same shape as in the case \(\mathcal{H}(z)\), but \(\mu_{u_1,u_2}(g\mathcal{L}_m g^{-1}) = \lambda_{u_1}(-s_{12}^2)\lambda_{u_2}\left(\frac{\nu m}{s_{12}}\right)\). Thus,

\[
\mu_{u_1,u_2}^G(\mathcal{L}_m) = \frac{q^3q}{q^4} \sum_{t \neq 0} \lambda_{u_1}(-t^2) \lambda_{u_2}(-\nu t^2) \frac{-\nu^m t}{t} = \sum_{t \neq 0} \lambda \left( -\frac{u_1t^3 + u_2\nu^m}{t} \right).
\]

Similarly, in the case of \(\mathcal{M}_m\), we obtain

\[
\mu_{u_1,u_2}^G(\mathcal{M}_m) = \frac{q^3q}{q^4} \sum_{t \neq 0} \lambda_{u_1}(-\nu t^2) \lambda_{u_2}(-\nu^m t) \frac{-\nu^m t}{t} = \sum_{t \neq 0} \lambda \left( -\frac{u_1t^3 + u_2\nu^m}{t} \right).
\]

In particular, for \(u_1 = 0\) we get \(\mu_{0,u_2}^G(\mathcal{L}_m) = \mu_{0,u_2}^G(\mathcal{M}_m) = -1\).

Set \(\kappa_0 = \mu_{0,1}^G\). Computing \((\kappa_0, \kappa_0)_G\), one sees that \(\kappa_0\) is irreducible. Furthermore, for all \(u_1, u_2 \in F^\times\), \(\kappa_0\) is different from any of the \(\mu_{u_1,u_2}^G\)’s because

\[
(\kappa_0)(\mathcal{H}(0)) + (\kappa_0)(\mathcal{I}(0)) = 2q - 2 \neq \mu_{u_1,u_2}^G(\mathcal{M}(0)) + \mu_{u_1,u_2}^G(\mathcal{I}(0)) = -2.
\]

Next, we show that we can always pick \(q - 1\) pairwise distinct irreducible characters among the \(\mu_{u_1,u_2}^G\)’s. For instance, we can take as \((u_1, u_2)\) the pairs \((1, \nu^m)\) and \((\nu, \nu^m)\), where \(1 \leq \nu \leq \nu^{-1} / 2\). Set \(\kappa_{1,n} = \mu_{1,\nu^m}^G, \kappa_{\nu,n} = \mu_{\nu,\nu^m}^G\). We start showing that these characters are irreducible.

Use of Mackey’s formula implies that

\[
(\kappa_{1,n}, \kappa_{1,n})_G = \sum_{r \in R} (\mu_{1,\nu^m}, r \mu_{1,\nu^m})_K \kappa_{1,n} K,
\]
where $\mathcal{R}$ is a complete set of representatives for the double cosets of $K$ in $G$. As $\mathcal{R}$ we can choose the set $\{s(\alpha), \overline{s}(\beta) \mid \alpha, \beta \in F^\times\}$, where
\[
s = s(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix}, \quad \overline{s} = \overline{s}(\beta) = \begin{pmatrix} 0 & -1/\beta \\ \beta & 0 \end{pmatrix} \in S.
\]

Note that $|KsK| = q^4$ and $|K\overline{s}K| = q^5$. Since the $\mu_{1,\nu}^n$'s are linear characters, it suffices to show that for $r \neq s(1)$, the restrictions of $\mu_{1,\nu}^n$ and $\overline{r}\mu_{1,\nu}^n$ to $K \cap rK$ are distinct.

First, we look at the double cosets $K\overline{s}(\beta)K$. For all $\overline{s}k \in K \cap \overline{s}K$, we have $\mu_{1,\nu}^n(\overline{s}k) = \lambda_{\nu^n}(\beta x)$ and $\overline{s}\mu_{1,\nu}^n(\overline{s}k) = \mu_{1,\nu}^n(k) = \lambda_{\nu^n}(y)$. It follows that, if $\mu_{1,\nu}^n = \overline{s}\mu_{1,\nu}^n$, then $\lambda_{\nu^n}(\beta x) = \lambda_{\nu^n}(y)$, for all $x, y \in F$. In particular, for $x = 0$, we have $\lambda_{\nu^n}(y) = 1$ for all $y \in F$, i.e. $\text{Ker}(\lambda_{\nu^n}) = \mathcal{Z}(H_1(q))$, forcing $\nu^n = 0$, a contradiction.

Next, we look at the double cosets $Ks(\alpha)K$. For all $s^*k \in K \cap s^*K$, we have $\mu_{1,\nu}^n(s^*k) = \lambda_1(\alpha\alpha^2)\lambda_{\nu^n}(\frac{y}{\alpha})$ and $s^*\mu_{1,\nu}^n(s^*k) = \mu_{1,\nu}^n(k) = \lambda_1(\alpha)\lambda_{\nu^n}(y)$. It follows that, if $\mu_{1,\nu}^n = s^*\mu_{1,\nu}^n$, then $\lambda_1(\alpha\alpha^2)\lambda_{\nu^n}(\frac{y}{\alpha}) = \lambda_1(\alpha)\lambda_{\nu^n}(y)$, for all $a, y \in F$. In particular, for $y = 0$, we get $\lambda_{\alpha^2} = \lambda_1$, and so $\alpha = 1$. Clearly, for $\alpha = 1$, the two restrictions are the same character. This proves that the characters $\kappa_{1,n}$ are irreducible.

In the same way, we can prove that the characters $\kappa_{\nu,n}$ are also irreducible. To conclude, we are left to show that the characters $\kappa_{1,n}$ and $\kappa_{\nu,n}$ are pairwise distinct. This can be obtained proving that $(\kappa_{d,n}, \kappa_{d_1,n_1}) = 0$, for $d, d_1 \in \{1, \nu\}$, $1 \leq n \leq \frac{q-1}{2}$ and $(d, n) \neq (d_1, n_1)$. As above, we exploit Mackey's formula. The double cosets $K\overline{s}(\beta)K$ are dealt with in the same way as before. In the case of the double cosets $Ks(\alpha)K$, for $d = d_1$ we can argue as before. In the case $(d, d_1) = (1, \nu)$, if the restrictions of $\mu_{1,\nu}^n$ and $\mu_{\nu,\nu_1}$ are the same, then
\[
\lambda_1(\alpha\alpha^2)\lambda_{\nu^n}(\frac{y}{\alpha}) = \lambda_{\nu}(a)\lambda_{\nu_1}(y)
\]
for all $a, y \in F$. In particular, for $y = 0$, we get $\lambda_{\alpha^2} = \lambda_{\nu}$, a contradiction, since $\nu$ is not a square in $F$.

In conclusion, the desired character table of $G$ can be described as follows:
\[
\begin{array}{|c|c|c|c|c|c|}
\hline
& 1 & \mathcal{A}(z) & \mathcal{B} & \mathcal{C}(0) & \mathcal{D}_k(0) \\
\hline
\eta_1 & \frac{q-1}{2} & \frac{q-1}{2} & \frac{q-1}{2} & \frac{q-1}{2} & \frac{q-1}{2} \\
\eta_2 & \frac{q+1}{2} & \frac{q+1}{2} & \frac{q+1}{2} & \frac{q+1}{2} & \frac{q+1}{2} \\
\xi_1 & \frac{q+1}{2} & \frac{q+1}{2} & \frac{q+1}{2} & \frac{q+1}{2} & \frac{q+1}{2} \\
\xi_2 & \frac{q+1}{2} & \frac{q+1}{2} & \frac{q+1}{2} & \frac{q+1}{2} & \frac{q+1}{2} \\
\theta_j & q & q & q & q & q \\
\psi & q & q & q & q & q \\
\chi_i & q+1 & q+1 & q+1 & q+1 & q+1 \\
\kappa_0 & q^2 - 1 & q^2 - 1 & q^2 - 1 & q^2 - 1 & q^2 - 1 \\
\kappa_{1,n} & q^2 - 1 & q^2 - 1 & q^2 - 1 & q^2 - 1 & q^2 - 1 \\
\kappa_{n,n} & q^2 - 1 & q^2 - 1 & q^2 - 1 & q^2 - 1 & q^2 - 1 \\
\omega_{u} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} \\
\omega_{u,\eta_1} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} \\
\omega_{u,\eta_2} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} \\
\omega_{u,\xi_1} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} \\
\omega_{u,\xi_2} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} & \frac{q}{q(q-1)} \\
\omega_{u,\theta_j} & q(q-1) & q(q-1) & q(q-1) & q(q-1) & q(q-1) \\
\omega_{u,\psi} & q^2 & q^2 & q^2 & q^2 & q^2 \\
\omega_{u,\chi_i} & q(q+1) & q(q+1) & q(q+1) & q(q+1) & q(q+1) \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
& 1 & \mathcal{A}(z) & \mathcal{B} & \mathcal{C}(0) & \mathcal{D}_k(0) \\
\hline
\eta_1 & \frac{-1 - \sqrt{77}}{2} & \frac{-1 - \sqrt{77}}{2} & \frac{-1 - \sqrt{77}}{2} & \frac{-1 - \sqrt{77}}{2} & \frac{-1 - \sqrt{77}}{2} \\
\eta_2 & \frac{-1 + \sqrt{77}}{2} & \frac{-1 + \sqrt{77}}{2} & \frac{-1 + \sqrt{77}}{2} & \frac{-1 + \sqrt{77}}{2} & \frac{-1 + \sqrt{77}}{2} \\
\xi_1 & \frac{1 + \sqrt{77}}{2} & \frac{1 + \sqrt{77}}{2} & \frac{1 + \sqrt{77}}{2} & \frac{1 + \sqrt{77}}{2} & \frac{1 + \sqrt{77}}{2} \\
\xi_2 & \frac{1 - \sqrt{77}}{2} & \frac{1 - \sqrt{77}}{2} & \frac{1 - \sqrt{77}}{2} & \frac{1 - \sqrt{77}}{2} & \frac{1 - \sqrt{77}}{2} \\
\theta_j & (-1)^{j+1} & (-1)^{j+1} & (-1)^{j+1} & (-1)^{j+1} & (-1)^{j+1} \\
\psi & 0 & 0 & 0 & 0 & 0 \\
\chi_i & (-1)^{i} & (-1)^{i} & (-1)^{i} & (-1)^{i} & (-1)^{i} \\
\kappa_0 & 0 & 0 & 0 & 0 & 0 \\
\kappa_{1,n} & 0 & 0 & 0 & 0 & 0 \\
\kappa_{n,n} & 0 & 0 & 0 & 0 & 0 \\
\omega_{u} & \delta & \delta & \delta & \delta & \delta \\
\omega_{u,\eta_1} & \frac{1 - \sqrt{77}}{2} & \frac{1 - \sqrt{77}}{2} & \frac{1 - \sqrt{77}}{2} & \frac{1 - \sqrt{77}}{2} & \frac{1 - \sqrt{77}}{2} \\
\omega_{u,\eta_2} & \frac{1 + \sqrt{77}}{2} & \frac{1 + \sqrt{77}}{2} & \frac{1 + \sqrt{77}}{2} & \frac{1 + \sqrt{77}}{2} & \frac{1 + \sqrt{77}}{2} \\
\omega_{u,\xi_1} & \frac{1 + \sqrt{77}}{2} & \frac{1 + \sqrt{77}}{2} & \frac{1 + \sqrt{77}}{2} & \frac{1 + \sqrt{77}}{2} & \frac{1 + \sqrt{77}}{2} \\
\omega_{u,\xi_2} & \frac{1 - \sqrt{77}}{2} & \frac{1 - \sqrt{77}}{2} & \frac{1 - \sqrt{77}}{2} & \frac{1 - \sqrt{77}}{2} & \frac{1 - \sqrt{77}}{2} \\
\omega_{u,\theta_j} & (-1)^{j+1} & (-1)^{j+1} & (-1)^{j+1} & (-1)^{j+1} & (-1)^{j+1} \\
\omega_{u,\psi} & 0 & 0 & 0 & 0 & 0 \\
\omega_{u,\chi_i} & (-1)^{i} & (-1)^{i} & (-1)^{i} & (-1)^{i} & (-1)^{i} \\
\hline
\end{array}
\]
|       | $\mathcal{I}(0)$ | $\mathcal{Z}_m$ | $\mathcal{M}_m$ |
|-------|-----------------|-----------------|-----------------|
| $1_G$ | 1               | 1               | 1               |
| $\eta_1$ | $(-1+i\sqrt{3})$ | $(-1+i\sqrt{3})$ | $(-1+i\sqrt{3})$ |
| $\eta_2$ | $(-1-i\sqrt{3})$ | $(-1-i\sqrt{3})$ | $(-1-i\sqrt{3})$ |
| $\xi_1$ | $(1-i\sqrt{3})/2$ | $(1+i\sqrt{3})/2$ | $(1+i\sqrt{3})/2$ |
| $\xi_2$ | $(1+i\sqrt{3})/2$ | $(1-i\sqrt{3})/2$ | $(1-i\sqrt{3})/2$ |
| $\theta_j$ | -1       | -1       | -1       |
| $\psi$ | 0               | 0               | 0               |
| $\chi_i$ | 1               | 1               | 1               |

| $\kappa_0$ | $q-1$ | $-1$ | $-1$ |
| $\kappa_{1,n}$ | $-1 - (\frac{1}{F})Q(\lambda)$ | $\sum_{t \in F} \lambda(-q^{t_1 + \mu n + m})$ | $\sum_{t \in F} \lambda(-q^{t_1 + \mu n + m})$ |
| $\kappa_{2,n}$ | $-1 + (\frac{1}{F})Q(\lambda)$ | $\sum_{t \in F} \lambda(-q^{t_1 + \mu n + m})$ | $\sum_{t \in F} \lambda(-q^{t_1 + \mu n + m})$ |

| $\omega_u$ | $-Q(\lambda_u)$ | 0 | 0 |
| $\omega_u \eta_1$ | $(1+i\sqrt{3})Q(\lambda_u)$ | 0 | 0 |
| $\omega_u \eta_2$ | $(1-i\sqrt{3})Q(\lambda_u)$ | 0 | 0 |
| $\omega_u \xi_1$ | $(1+i\sqrt{3})Q(\lambda_u)$ | 0 | 0 |
| $\omega_u \xi_2$ | $(1-i\sqrt{3})Q(\lambda_u)$ | 0 | 0 |
| $\omega_u \theta_j$ | $Q(\lambda_u)$ | 0 | 0 |
| $\omega_u \psi$ | 0 | 0 | 0 |
| $\omega_u \chi_i$ | $-Q(\lambda_u)$ | 0 | 0 |

**Notations.** 1 ≤ $i, k ≤ \frac{2m-1}{2}$, 1 ≤ $j, m, n ≤ \frac{2m-1}{2}$. $\delta = (-1)^{\frac{2m-1}{2}}$, $\rho = e^{\frac{2\pi i}{F}}$, $\sigma = e^{\frac{2\pi i}{F}}$. $F = GF(q)$, $F^\times = \langle \nu \rangle$, $u \in F^\times$. $\lambda$ is a (fixed) non-trivial character of $\mathbf{Z}(G)$. $\lambda_u$ is the linear character of $\mathbf{Z}(G)$ defined by $\lambda_u(z) = \lambda(uz)$ for all $z \in \mathbf{Z}(G)$.

$$Q(\lambda) = \sum_{t \in F} \lambda(-t^2/2), \quad Q(\lambda_u) = \left(\frac{1}{F}\right)Q(\lambda).$$

For all $\chi \in Irr(G)$, we have (but this is omitted from the Table)

$$\chi^\ast(\mathcal{E}(z)) = \frac{\chi(\mathcal{E}(z))}{\chi(\mathcal{E}(0))}\chi(\mathcal{E}(0)),$$

and likewise for the other conjugacy classes.

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