Generating numbers of rings graded by amenable groups

KARL LORENSEN
JOHAN ÖINERT

January 12, 2022

Abstract

A ring $R$ has \textit{unbounded generating number} (UGN) if, for every positive integer $n$, there is no $R$-module epimorphism $R^n \to R^{n+1}$. For a ring $R$ graded by a group $G$ such that $R_1$ has UGN, we investigate conditions under which $R$ must also have UGN. We prove that this occurs in the following two situations: (1) $G$ is amenable, and $R_g$ is a finitely generated free $R_1$-module for every $g \in G$; (2) $G$ is locally finite, and $R_g$ is a finitely generated projective $R_1$-module for every $g \in G$.

We show further that, in each of the cases (1) and (2), neither hypothesis can be omitted. Moreover, (1) leads to several new ring-theoretic characterizations of amenability for a group, including one involving the $K$-theory of its translation rings that partly generalizes a result of G. Elek from 1997.

We also consider rings that do not have UGN; these are said to have \textit{bounded generating number} (BGN). For such a ring $R$, the smallest positive integer $n$ such that there is an $R$-module epimorphism $R^n \to R^{n+1}$ is called the \textit{generating number} of $R$, denoted $\text{gn}(R)$. We describe several classes of examples of a ring $R$ with BGN graded by a group $G$ satisfying (1) or (2) above such that $\text{gn}(R) \neq \text{gn}(R_1)$.

Mathematics Subject Classification (2020): 16S35, 16W50, 16D90, 16P99, 20F65, 43A07

Keywords: graded ring, amenable group, unbounded generating number, UGN, rank condition, generating number, invariant basis number, IBN, crossed product, skew group ring, twisted group ring, translation ring

1 Introduction

In this paper, we explore the link between the property of amenability for groups and a certain condition on rings that generalizes both finiteness and commutativity for nonzero rings. A group is said to be \textit{amenable} if it admits a finitely additive probability measure that is invariant under the action of the group (see [12 §4]). Finite groups and
abelian groups are all amenable; moreover, the class of amenable groups is closed under forming subgroups, quotients, extensions, and direct limits. Noncyclic free groups are the simplest examples of groups that lie outside this class.

Inspired by the Banach-Tarski paradox, the notion of an amenable group was introduced by J. von Neumann [27] in 1929 and has since played a significant role in many areas of mathematics. The widespread interest in amenability has led to a proliferation of characterizations of the property, primarily in analytical or topological terms, but also, notably, in the context of problems involving cellular automata (see [10], [11], [7], [8], and [12]). One purely group-theoretic characterization of amenability was discovered by E. Følner [19] in 1955, and another, using the notion of a paradoxical decomposition, crystallized in earlier work by A. Tarski [31] (see Theorems 3.2 and 3.5, respectively, below).

A description of amenability in terms of group rings emerged in the work of T. Ceccherini-Silberstein, M. Coornaert [10], and L. Bartholdi [8] on cellular automata. The results and arguments in those two papers imply that a group $G$ is amenable if and only if the group ring $KG$ satisfies the strong rank condition for every field $K$. A ring $R$ is said to satisfy the strong rank condition if, for every positive integer $n$, there is no $R$-module monomorphism $R^{n+1} \to R^n$. This characterization of amenability was generalized beyond group rings by P. H. Kropholler and the first author [21].

The present paper investigates the connection between amenability for groups and the property of rings that is dual to the strong rank condition, namely, the property of having unbounded generating number (UGN). In other words, a ring $R$ has UGN if and only if, for every positive integer $n$, there is no $R$-module epimorphism $R^n \to R^{n+1}$. This property is also referred to as the rank condition (see, for instance, [21]). Coined presumably by P. Cohn, the term UGN is particularly apposite since the property is equivalent to the assertion that, for every positive integer $n$, there is a finitely generated $R$-module that cannot be generated by fewer than $n$ elements (see Proposition 2.2((i) $\iff$ (iv)) below).

Every ring $R$ with UGN plainly enjoys the widely studied property of having invariant basis number (IBN), which means that $R^m \cong R^n$ as $R$-modules if and only if $m = n$. However, as shown in [14, §5] and [4, Example 3.19], UGN is a strictly stronger property than IBN. In addition, as opposed to the latter property (see [22], p. 502, Exercise 11), UGN offers the advantage of being Morita invariant (see [16, Exercise 9, §0.1], [4], [2], and Corollary 2.3 below).

It is easy to verify that the strong rank condition implies UGN (see [22, Proposition 1.12]); in particular, every nonzero division ring and every nonzero commutative ring have UGN (see [22, Theorem 1.35] and [22, Corollary 1.38], respectively). Furthermore, any ring that can be mapped to a ring with UGN by a unital ring homomorphism must possess UGN (see Lemma 2.7 below). As a result, every group ring with coefficients in a UGN-ring must have UGN. Hence UGN is a significantly weaker property than the strong rank condition.
We will refer to rings that do not have UGN as having bounded generating number (BGN). If a ring $R$ has BGN, then the smallest integer $n > 0$ for which there is an $R$-module epimorphism $R^n \to R^{n+1}$ is called the generating number of $R$, denoted $\text{gn}(R)$. The generating number of a BGN-ring can also be characterized as the smallest integer $n > 0$ such that every finitely generated $R$-module can be generated by $n$ elements (see Proposition 2.4((i)$\iff$(iv)) below). If $R$ has UGN, then we define $\text{gn}(R) := \aleph_0$.

As far as the authors are aware, the notion of the generating number of a ring has never been studied explicitly before. In the finite case, it is analogous to the concept of the type of a non-IBN-ring (see [23]), and many of the same techniques that have been used to investigate types can be applied to the study of finite generating numbers. This is particularly true of the arguments adduced by G. Abrams in [1]; indeed, our results on finite generating numbers—Theorem B, Corollary D, Theorem F, and Theorem G below—are all proved using methods from his paper.

As in [21], our focus here is on rings that are graded by groups. Recall that, if $G$ is a group, a ring $R$ is graded by $G$, or $G$-graded, if there is a collection $\{R_g : g \in G\}$ of additive subgroups of $R$ such that $R = \bigoplus_{g \in G} R_g$ as additive groups and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. For such a ring, each $R_g$ is referred to as a homogeneous component of $R$. Moreover, $R_1$ is a subring of $R$, called its base ring. If each homogeneous component contains a unit, then $R$ is called a crossed product of $R_1$ by $G$, denoted $R_1 \ast G$. The simplest type of crossed product of $R_1$ by $G$ is the group ring $R_1 G$; more general sorts are skew group rings and twisted group rings of $G$ with coefficients in $R_1$, described in the portion of §2 that is devoted to terminology.

The aim of this paper is to examine the relationship between the generating number of a graded ring and the generating number of its base ring. Our primary objective, in this regard, is to determine conditions that will guarantee that a graded ring must have UGN whenever its base ring does. As mentioned above, this is always true for group rings, but, as we shall see, the situation is very different in general. Our principal result pertains to a ring $R$ that is freely graded by a group $G$, by which we mean that the $R_g$ are all finitely generated free right $R_1$-modules or finitely generated free left $R_1$-modules. Any crossed product is an example of a freely graded ring since, in that case, each $R_g$ is a free (right and left) $R_1$-module of rank one. These are, however, not the only instances of graded rings whose homogeneous components are free of rank one as right and left modules over the base ring; more general examples are crystalline and precrystalline graded rings, discussed in [26] and [29]. There are also graded rings whose homogeneous components are free of finite, unbounded rank over the base ring; some such rings are described in Example 3.4 below.

The main result of this paper is the following theorem.

**Theorem A.** Let $R$ be a ring freely graded by an amenable group $G$. Then $R$ has UGN if and only if $R_1$ has UGN.

In the next three paragraphs, we adumbrate the principal steps in the proof of
Theorem A, highlighting especially the pivotal roles played by crossed products and certain infinite matrix rings, called translation rings. Since unital subrings of UGN-rings must have UGN, we only need to concern ourselves with the “if” part of the theorem. The first step in the proof is to dispose of the case where $R$ is a crossed product; for this, we rely on Følner’s characterization of amenability [19]. The next phase of the proof involves translation rings of amenable groups with coefficients in a UGN-ring. Translation rings were introduced by M. Gromov [20] in 1993 for finitely generated groups and have attracted considerable attention (see, for example, [5], [6], [17], and [30]). For a ring $S$ and a finitely generated group $G$, Gromov employed the right word metric $d : G \times G \to \mathbb{N}$ on $G$ associated to some finite generating set in order to define the translation ring $T(G, S)$ to be the ring of all $G \times G$ matrices $M$ with entries in $S$ for which there exists an integer $r \geq 0$ such that $M(x, y) = 0$ whenever $d(x, y) > r$.

For our purposes, it will be convenient to extend Gromov’s notion to arbitrary groups. For a group $G$ and a ring $S$, we define the translation ring $T(G, S)$ to be the ring of all $G \times G$ matrices $M$ with entries in $S$ for which there is a finite subset $K$ of $G$ such that $M(x, y) = 0$ whenever $y \notin Kx$. It is easy to see that our definition coincides with Gromov’s if $G$ is finitely generated.

As shown in Proposition 2.14 below, $T(G, S)$ is isomorphic to a skew group ring of $G$ with coefficients in the ring $\prod G S$. If $S$ has UGN, then so does $\prod G S$. As a result, the case of Theorem A for crossed products implies that, for $G$ amenable, the ring $T(G, S)$ has UGN whenever $S$ does. From this observation, we deduce the general case of Theorem A by constructing a unital ring embedding of $R$ into $T(G, T(\mathbb{Z}, R_1))$ and invoking the fact that UGN is inherited by unital subrings.

It is important to recognize that the value of Theorem A goes well beyond that derived from its applicability to crossed products. We illustrate this in Example 3.4, where we employ Theorem A to prove that the members of a specific family of freely $\mathbb{Z}$-graded rings have UGN. None of the rings in this family are crossed products with respect to the grading stipulated; in fact, most possess homogeneous components that are free modules of arbitrarily large rank over the base ring.

Theorem A says that, for a ring $R$ freely graded by an amenable group $G$, we have

$$\text{gn}(R_1) = \aleph_0 \iff \text{gn}(R) = \aleph_0.$$ 

One might, therefore, ask whether these two generating numbers always coincide if $G$ is amenable. The answer, however, is resoundingly negative, as shown by our next theorem.

**Theorem B.** Let $m, n$ be positive integers with $m \leq n$. Then there exist a finite group $G$, a ring $R$, and a crossed product $R \ast G$ such that $\text{gn}(R) = n$ and $\text{gn}(R \ast G) = m$.

The hypothesis that $G$ is amenable in Theorem A is necessary, for translation rings of nonamenable groups always have BGN. Indeed, the following stronger statement is
true; it is essentially due to G. Elek [17], although our formulation differs somewhat from his.

**Theorem C.** (Elek) Let $G$ be a nonamenable group, $R$ a ring, and $T := T(G, R)$. Then $T^2 \cong T$ as $T$-modules, so that $\text{gn}(T) = 1$.

Theorems A and C allow us to articulate the four ring-theoretic characterizations of amenability listed below.

**Corollary A.** The following five assertions are equivalent for a group $G$.

(i) $G$ is amenable.
(ii) Every ring freely graded by $G$ has UGN if its base ring does.
(iii) Every crossed product of a UGN-ring by $G$ has UGN.
(iv) Every skew group ring of $G$ with coefficients in a UGN-ring has UGN.
(v) For every UGN-ring $R$, the translation ring $T(G, R)$ has UGN.

It is worth mentioning that there is no counterpart to the characterizations (iii) and (iv) pertaining to twisted group rings. We show this in Corollary B below by proving that any twisted group ring of a free-by-amenable group with coefficients in a UGN-ring must have UGN. It is an open question whether the corresponding statement is true for the twisted group ring of an arbitrary group.

**Corollary B.** Let $G$ be a free-by-amenable group, $R$ a ring, and $R \ast G$ a twisted group ring. Then $R \ast G$ has UGN if and only if $R$ has UGN.

**Open Question 1.1.** Let $G$ be an arbitrary group and $R$ a ring. If $R$ has UGN, must every twisted group ring $R \ast G$ also have UGN?

Theorems A and C also give rise to the following characterizations of amenability for a group in terms of the $K$-theory of its translation rings. In the statement of the theorem, and throughout this paper, we define the map $\kappa_R : \mathbb{Z} \to K_0(R)$ for a ring $R$ to be the group homomorphism that maps 1 to the image of the isomorphism class of the free $R$-module of rank one in $K_0(R)$.

**Corollary C.** Let $G$ be a group and $R$ a ring with UGN. If $T := T(G, R)$, then the following three statements are equivalent:

(i) $G$ is amenable;
(ii) $\kappa_T$ is injective;
(iii) $\kappa_T \neq 0$. 

5
The implication (i) \(\implies\) (iii) in Corollary C generalizes the corresponding part of the main result of [17]. Specifically, that paper establishes that, if \(G\) is a finitely generated amenable group and \(S\) is the subring of \(T(G, \mathbb{R})\) consisting of all the matrices with bounded entries, then \(\kappa_S\) is nonzero.

The above results are proved in \(\S 3\). In \(\S 4\), we turn to rings \(R\) that are projectively graded by a group \(G\), meaning that the \(R_g\) are all finitely generated projective right \(R_1\)-modules or finitely generated projective left \(R_1\)-modules. An important instance where this occurs is when \(R\) is strongly graded by \(G\), that is, when \(R_gR_h = R_{gh}\) for all \(g, h \in G\) (see [25, Theorem 3.1.1]). For other examples of projectively graded rings, we refer the reader to [28]. One usually expects projectively graded rings to behave like freely graded rings; however, with regard to UGN, they diverge profoundly. We illustrate this with the following theorem that describes an example of a ring projectively graded by \(\mathbb{Z}\) (an amenable group) that does not have UGN even though its base ring does.

**Theorem D.** Let \(R\) be a UGN-ring and \(L\) the \(R\)-algebra generated by the sets \(E := \{e_1, \ldots, e_n\}\) and \(E^* := \{e_1^*, \ldots, e_n^*\}\), where \(n > 1\), subject only to the following relations.

(i) The elements of \(R\) commute with the generators in \(E \cup E^*\).
(ii) \(e_i^*e_j = \delta_{ij}\) for \(i, j = 1, \ldots, n\).
(iii) \(\sum_{i=1}^n e_i e_i^* = 1\).

Then the following two statements hold.

- \(L^n \cong L\) as \(L\)-modules, so that \(\text{gn}(L) = 1\).
- There is a strong (and therefore projective) \(\mathbb{Z}\)-grading on \(L\) such that \(L_0\) has UGN.

The \(R\)-algebra \(L\) in Theorem D is a special case of what is known as a Leavitt path algebra (see [3]). Moreover, the \(\mathbb{Z}\)-grading referred to in the theorem corresponds to the conventional \(\mathbb{Z}\)-grading on such algebras (see [3, §2.1]).

We can use Theorem D to show that it is possible to find a strongly \(\mathbb{Z}\)-graded ring with any possible generating number such that the base ring has UGN.

**Corollary D.** For any positive integer \(m\), there exists a ring \(T\) strongly (and therefore projectively) graded by \(\mathbb{Z}\) such that \(T_0\) has UGN and \(\text{gn}(T) = m\).

Theorem D demonstrates that the statement of Theorem A becomes false if the hypothesis that the grading is free is replaced by the one that the grading is projective. However, if, in addition to making the grading projective, we insist that \(G\) be locally finite, then we will arrive at a true statement.

**Theorem E.** Let \(R\) be a ring projectively graded by a locally finite group \(G\). Then \(R\) has UGN if and only if \(R_1\) has UGN.
As we shall see, Theorem E is quite easy to deduce from the Morita invariance of UGN (see Corollary 2.3).

Theorem B shows that the equality between generating numbers described in Theorem E fails to hold when these are finite. In our next result, we establish the existence of rings exhibiting this phenomenon that are graded by an arbitrary finite group.

**Theorem F.** Let $G$ be a finite group. For any two positive integers $m \leq n$, there is a ring $R$ strongly (and therefore projectively) graded by $G$ such that $gn(R_1) = n$ and $gn(R) = m$.

We conclude the paper by showing that the hypothesis that the grading is projective in Theorem E cannot be dropped.

**Theorem G.** Let $G$ be a finite group. For any integer $n > 0$, there exists a ring $R$ graded by $G$ such that $R_1$ has UGN and $gn(R) = n$.

**Acknowledgement.** The authors are grateful to Peter Kropholler for posing the question that became the subject of this paper, as well as for pointing out the possible relevance of Morita theory to this matter.

2 Notation and preliminary results

In this section, we discuss some elementary facts about UGN, generating numbers, and translation rings that are required for our main results. First we describe the notation and terminology that we employ throughout the paper.

**Notation and Terminology**

*General notation.*
$\mathbb{Z}^+$ represents the set of positive integers.

For any objects $a, b$,

$$
\delta(a, b) := \begin{cases} 
1 & \text{if } a = b \\
0 & \text{if } a \neq b.
\end{cases}
$$

For $i, j \in \mathbb{Z}^+$, we write $\delta_{ij} := \delta(i, j)$.

*Rings and modules.*
All rings, subrings, and ring homomorphisms are assumed to be unital.

Let $R$ be a ring. Then $R^*$ denotes the group of units (invertible elements) in $R$, and $Z(R)$ is the center of $R$. Moreover, $R^{opp}$ represents the ring opposite to $R$, that is, the
ring with the same set of elements as \( R \) and same addition, but whose multiplication \( \circ \)
is defined by \( r \circ s := sr \) for all \( r, s \in R^{\text{op}} \).

The term module without the modifier left will always mean right module.

If \( R \) is a ring, then \( \mathcal{M}_R \) is the category of \( R \)-modules.

Let \( R \) be a ring graded by a group \( G \) and \( r \in R \). If \( r = \sum_{g \in G} r_g \) with \( r_g \in R_g \) for all \( g \in G \), then the support of \( r \), denoted \( \text{Supp}(r) \), is defined by
\[
\text{Supp}(r) := \{ g \in G : r_g \neq 0 \}.
\]

Crossed products, skew group rings, and twisted group rings.

Let \( G \) be a group and \( R \) a ring. Let \( \sigma : G \to \text{Aut}(R) \) and \( \omega : G \times G \to R^* \) be functions. Write \( g \cdot r := (\sigma(g))(r) \) for \( g \in G, r \in R \). If \( \sigma \) is a group homomorphism, then we call \( \sigma \) an action of \( G \) on \( R \). If \( \sigma \) is the trivial group homomorphism, it is referred to as the trivial action.

The quadruple \((G, R, \sigma, \omega)\) is called a crossed system if the maps \( \sigma \) and \( \omega \) satisfy the following three conditions for any \( g,h,k \in G \) and \( r \in R \):

(i) \( g \cdot (h \cdot r) = \omega(g, h)((gh) \cdot r)(\omega(g, h))^{-1} \);
(ii) \( \omega(g, h) \omega(gh, k) = (g \cdot \omega(h, k)) \omega(g, hk) \);
(iii) \( \omega(g, 1_G) = \omega(1_G, g) = 1_R \).

If \((G, R, \sigma, \omega)\) is a crossed system, then the set of formal sums \( \sum_{g \in G} r_g g \), in which \( r_g \in R \) for all \( g \in G \) and \( r_g = 0 \) for all but finitely many \( g \in G \), can be made into a ring by defining addition componentwise and multiplication according to the rule
\[
(r_g g)(r_h h) := r_g (g \cdot r_h) \omega(g, h) (gh)
\]
for all \( g,h \in G \) and \( r_g, r_h \in R \) (see [25, Proposition 1.4.1]). This ring is denoted \( R \star^\omega G \), or simply \( R \star G \); it is a crossed product with respect to the obvious \( G \)-grading. Moreover, every crossed product of \( R \) by \( G \) is isomorphic to a ring of this form (see [25, Proposition 1.4.2]).

An important special case of a crossed product arises when \( \omega \) is the trivial map, meaning \( \omega(g, h) = 1_R \) for all \( g,h \in G \). In this case, \((G, R, \sigma, \omega)\) is a crossed system if and only if \( \sigma \) is an action of \( G \) on \( R \). When \( \omega \) is trivial and \( \sigma \) an action, the crossed product \( R \star^\omega G \) is referred to as the skew group ring of \( G \) with coefficients in \( R \) that is associated to the action \( \sigma \).

A second important case is where \( \sigma \) is the trivial action. In this case, a map \( \omega : G \times G \to R^* \) satisfies (i) if and only if \( \text{Im } \omega \subseteq Z(R)^* \). Moreover, in the presence of this containment, the map \( \omega \) fulfills (ii) if and only if \( \omega \) is a 2-cocycle \( G \times G \to Z(R)^* \). If, in
addition, (iii) holds, then we call \( \omega \) a normalized 2-cocycle. If \( \sigma \) is the trivial action and \( \omega \) a normalized 2-cocycle \( G \times G \to Z(R)^* \), then \( R \ast_\omega^\sigma G \) is called the twisted group ring of \( G \) with coefficients in \( R \) that is associated to the normalized 2-cocycle \( \omega \). Furthermore, the twisted group ring \( R \ast_\omega^\sigma G \) is isomorphic to the group ring \( RG \) if and only if the cohomology class of \( \omega \) in \( H^2(G, Z(R)^*) \) is zero (see [25, §1.5, Exercise 10]).

**Monoids.**

For any ring \( R \), \( \mathcal{P}(R) \) is the abelian monoid consisting of the isomorphism classes of finitely generated projective \( R \)-modules under the operation of forming direct sums.

For an abelian monoid \( M \), we write \( x \leq y \) for \( x, y \in M \) if there exists \( z \in M \) such that \( x + z = y \).

For any two positive integers \( n \) and \( k \), \( C(n,k) \) denotes the monoid with presentation

\[
C(n,k) := \langle a : (n+k)a = na \rangle.
\]

**Matrices.**

If \( S \) and \( T \) are sets, an \( S \times S \) matrix \( M \) with entries in \( T \) is a function \( M : S \times S \to T \).

For an \( m \times n \) matrix \( M \) with \( m, n \in \mathbb{Z}^+ \), we write \( M_{ij} := M(i,j) \) for \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \).

The ring of \( n \times n \) matrices with entries in a ring \( R \) will be denoted \( M_n(R) \).

The transpose of a (finite or infinite) matrix \( A \) is denoted \( A^t \).

**Partitions of sets.**

We use the symbol \( \coprod \) for disjoint unions of sets; in other words,

\[
C = A \coprod B
\]

will mean \( C = A \cup B \) and \( A \cap B = \emptyset \). Similarly, if \( \{ A_k : k \in K \} \) is a family of sets indexed by a set \( K \), then

\[
B = \coprod_{k \in K} A_k
\]

signifies \( B = \bigcup_{k \in K} A_k \) and \( A_k \cap A_{k'} = \emptyset \) for any pair \( k, k' \in K \) with \( k \neq k' \).

We begin our discussion of preliminary results with an elementary lemma that is crucial for understanding rings with UGN.

**Lemma 2.1.** Let \( R \) be a ring, \( A \) an \( R \)-module, and \( n \in \mathbb{Z}^+ \). If there is an \( R \)-module epimorphism \( A^n \to A^{n+1} \), then there is an \( R \)-module epimorphism \( A^n \to A^m \) for every \( m \in \mathbb{Z}^+ \).
Proof. Let \( \phi : A^n \to A^{n+1} \) be an \( R \)-module epimorphism. We will show by induction on \( k \) that there is an \( R \)-module epimorphism \( A^n \to A^{n+k} \) for every \( k \in \mathbb{Z}^+ \). Since there is clearly an \( R \)-module epimorphism \( A^n \to A^m \) when \( m \leq n \), this will prove the lemma.

Suppose that there is an \( R \)-module epimorphism \( \psi : A^n \to A^{n+k-1} \) for \( k > 1 \). Then we have the chain of \( R \)-module epimorphisms

\[
A^n \xrightarrow{\psi} A^{n+k-1} \cong A^n \oplus A^{k-1} \xrightarrow{\xi} A^{n+1} \oplus A^{k-1} \cong A^{n+k},
\]

where \( \xi(a,b) := (\phi(a),b) \) for all \( a \in A^n \) and \( b \in A^{k-1} \). The composition of the maps in the chain is an \( R \)-module epimorphism \( A^n \to A^{n+k} \).

Lemma 2.1 allows us to establish several alternative characterizations of rings with BGN. These are all well known and can be found, stated less formally, in the works of Cohn (see, for instance, [15], §1.4).

**Proposition 2.2.** For a ring \( R \), the following five statements are equivalent.

(i) \( R \) has BGN.

(ii) There is an \( R \)-module epimorphism \( R^n \to R^m \) for some \( m,n \in \mathbb{Z}^+ \) with \( n < m \).

(iii) There is an integer \( n > 0 \) such that there is an \( R \)-module epimorphism \( R^n \to R^m \) for every \( m \in \mathbb{Z}^+ \).

(iv) There is an integer \( n > 0 \) such that every finitely generated \( R \)-module is a homomorphic image of \( R^n \).

(v) There is a finitely generated \( R \)-module \( A \) such that every finitely generated \( R \)-module is a homomorphic image of \( A \).

Proof. The equivalence of (iv) and (v) is trivial, as are the implications (i) \( \implies \) (ii), (iii) \( \implies \) (iv), and (iv) \( \implies \) (i). Moreover, the implication (ii) \( \implies \) (iii) follows from Lemma 2.1.

Statement (v) in Proposition 2.2 is significant in that it is expressed in purely categorical terms. As a consequence, its equivalence to (i) implies that BGN, and perforce UGN, is Morita invariant, thus furnishing a succinct, positive answer to a question in an exercise by Cohn [16], Exercise 9, §0.1. For an alternative proof that focuses instead on the structure of the monoid \( P(R) \) for a ring \( R \), see P. Ara’s argument in [4], §2.

**Corollary 2.3.** Let \( R \) and \( S \) be rings that are Morita equivalent. Then \( R \) has UGN if and only if \( S \) has UGN.

Proof. Let \( F : \mathcal{M}_R \to \mathcal{M}_S \) be a category equivalence. It is well known that \( F \) must preserve the property of finite generation for modules, as well as the surjectivity of morphisms (see, for example, [22], §18A)). As a result, it follows from Proposition 2.2((i) \( \iff \) (v)) that \( R \) has UGN if and only if \( S \) does.
Reasoning similar to that employed in proving Proposition 2.2 can be invoked to establish the following set of equivalences concerning the generating number of a ring.

**Proposition 2.4.** Let $R$ be a ring and $n \in \mathbb{Z}^+$. The following four statements are equivalent.

(i) $\text{gn}(R) = n$.

(ii) The integer $n$ is the smallest positive integer such that there is an $R$-module epimorphism $R^n \to R^m$ for some integer $m > n$.

(iii) The integer $n$ is the smallest positive integer such that there is an $R$-module epimorphism $R^n \to R^m$ for every $m \in \mathbb{Z}^+$.

(iv) The integer $n$ is the smallest positive integer such that every finitely generated $R$-module is a homomorphic image of $R^n$.

**Remark.** We point out that the generating number of a ring fails to be a Morita invariant: see Lemma 2.9 and Proposition 2.12 below.

The next lemma is useful in that it allows us to formulate the conditions in Proposition 2.4, as well as (i)-(iii) in Proposition 2.2, in terms of matrices. Because the property is well known and the proof is very straightforward, we leave it to the reader.

**Lemma 2.5.** Let $R$ be a ring and $m,n \in \mathbb{Z}^+$. There is an $R$-module epimorphism $R^n \to R^m$ if and only if there are an $m \times n$ matrix $A$ with entries in $R$ and an $n \times m$ matrix $B$ with entries in $R$ such that $AB = I_m$.

Lemma 2.5 enables us to establish the following four fundamental lemmas about generating numbers (Lemmas 2.6, 2.7, 2.9, and 2.10). The special cases of Lemmas 2.6, 2.7, and 2.10 for infinite generating numbers are all well known (see [14], [16] §0.1, and [22] §1C).

**Lemma 2.6.** Let $R$ be a ring. Then $\text{gn}(R) = \text{gn}(R^{\text{op}})$. In particular, $R$ has UGN if and only if $R^{\text{op}}$ has UGN.

**Proof.** If $A$ is an $m \times n$ matrix with entries in $R$ and $B$ an $n \times p$ matrix with entries in $R$, we employ $A \circ B$ to denote the product of $A$ and $B$ where the entries are multiplied in $R^{\text{op}}$. This means

$$A \circ B = (B^t A^t)^t. \quad (2.1)$$

Since $(R^{\text{op}})^{\text{op}} = R$, we only need to show that $\text{gn}(R^{\text{op}}) \leq \text{gn}(R)$. This is plainly true if $\text{gn}(R) = \aleph_0$, so we assume that $k := \text{gn}(R)$ is finite. By Proposition 2.4((i) $\implies$ (ii)) and Lemma 2.5, there are an integer $l > k$, an $l \times k$ matrix $P$ with entries in $R$, and a $k \times l$ matrix $Q$ with entries in $R$ such that $PQ = I_l$. From (2.1) we obtain $Q^t \circ P^t = (PQ)^t = I_l$. Therefore, appealing to Lemma 2.5 and Proposition 2.4((ii) $\implies$ (i)), we deduce $\text{gn}(R^{\text{op}}) \leq k$. \qed
Remark. Lemma 2.6 means that the UGN and generating number notions are left-right symmetric; in other words, any of their characterizations in terms of right $R$-modules can also be expressed using left $R$-modules. Note that this is not the case for the strong rank condition (see [22, Remark 1.32]).

**Lemma 2.7.** Let $R$ and $S$ be rings such that there is a ring homomorphism $\phi : R \to S$. Then $\text{gn}(R) \geq \text{gn}(S)$. In particular, if $S$ has UGN, then so does $R$.

**Proof.** For any $m \times n$ matrix $M$ with entries in $R$, we let $\phi M$ denote the $m \times n$ matrix with entries in $S$ such that $(\phi M)_{ij} := \phi(M_{ij})$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. The desired inequality obviously holds if $\text{gn}(R) = \aleph_0$; hence we suppose that $\text{gn}(R)$ is finite and write $k := \text{gn}(R)$. By Proposition 2.4((i) $\implies$ (ii)) and Lemma 2.5, there are an integer $l > k$, an $l \times k$ matrix $A$ with entries in $R$, and a $k \times l$ matrix $B$ with entries in $R$ such that $AB = I_l$. This implies $(\phi A)(\phi B) = I_l$. It follows, then, from Lemma 2.5 and Proposition 2.4((ii) $\implies$ (i)) that $\text{gn}(S) \leq k$. 

Lemma 2.7 furnishes the following property of group rings.

**Corollary 2.8.** Let $R$ be a ring and $G$ a group. Then $\text{gn}(RG) = \text{gn}(R)$. In particular, $RG$ has UGN if and only if $R$ has UGN.

**Proof.** There is a ring embedding $R \to RG$, and the augmentation map is a ring homomorphism $RG \to R$. Hence the conclusion follows from Lemma 2.7.

**Lemma 2.9.** Let $R$ be a ring with BGN, and let $n := \text{gn}(R)$. If $m$ is a positive integer divisor of $n$, then $\text{gn}(M_m(R)) = \frac{n}{m}$.

**Proof.** Because $M_m(R)$ possesses a subring that is isomorphic to $R$, Lemma 2.7 implies that $M_m(R)$ has BGN. Put $p := \text{gn}(M_m(R))$. Appealing to Proposition 2.4((i) $\implies$ (ii)) and Lemma 2.5, we obtain an integer $q > p$, a $q \times p$ matrix $A$ with entries in $M_m(R)$, and a $p \times q$ matrix $B$ with entries in $M_m(R)$ such that $AB = I_q$. This gives rise to an $mq \times mp$ matrix $A'$ with entries in $R$ and an $mp \times mq$ matrix $B'$ with entries in $R$ such that $A'B' = I_{mq}$. Consequently, by Lemma 2.5 and Proposition 2.4((ii) $\implies$ (i)), we have $n \leq mp$.

Set $r := \frac{n}{m}$. Referring to Proposition 2.4((i) $\implies$ (iii)) and Lemma 2.5, we can acquire an integer $k > r$, a $km \times n$ matrix $P$ with entries in $R$, and an $n \times km$ matrix $Q$ with entries in $R$ such that $PQ = I_{km}$. This set-up can then be converted into a $k \times r$ matrix $P'$ with entries in $M_m(R)$ and an $r \times k$ matrix $Q'$ with entries in $M_m(R)$ such that $P'Q' = I_k$. It follows, then, from Lemma 2.5 and Proposition 2.4((ii) $\implies$ (i)) that $p \leq r$. Combining this with the result from the previous paragraph yields $p = r$.

**Lemma 2.10.** Let $\{R_\alpha : \alpha \in I\}$ be a directed system of rings. If $R := \varinjlim R_\alpha$, then $\text{gn}(R) = \min\{\text{gn}(R_\alpha) : \alpha \in I\}$.

In particular, $R$ has UGN if and only if $R_\alpha$ has UGN for every $\alpha \in I$.
Proof. Lemma 2.7 implies \( \text{gn}(R) \leq \text{gn}(R_\alpha) \) for all \( \alpha \in I \). The desired conclusion will therefore follow if we show that \( \text{gn}(R_\alpha) \leq \text{gn}(R) \) for some \( \alpha \in I \). Since this statement holds trivially if \( \text{gn}(R) = \aleph_0 \), suppose that \( \text{gn}(R) \) is finite and set \( p := \text{gn}(R) \). Invoking Proposition 2.4((i) \implies (ii)) and Lemma 2.5, we obtain an integer \( q > p \), a \( q \times p \) matrix \( A \) with entries in \( R \), and a \( p \times q \) matrix \( B \) with entries in \( R \) such that \( AB = I_q \). This means that, for some \( \alpha \in I \), there are a \( q \times p \) matrix \( A' \) with entries in \( R_\alpha \) and a \( p \times q \) matrix \( B' \) with entries in \( R_\alpha \) such that \( A'B' = I_q \). Therefore, appealing to Lemma 2.5 and Proposition 2.4((ii) \implies (i)), we conclude that \( \text{gn}(R_\alpha) \leq p \).

Our next result describes the relationship between the gene rating number of a direct product of rings and the generating numbers of the individual factors. The proof is very straightforward; nonetheless, as with the previous four lemmas, we provide the details for the sake of completeness.

**Lemma 2.11.** Let \( \{ R_\alpha : \alpha \in I \} \) be a family of rings indexed by a set \( I \), and write \( R := \prod_{\alpha \in I} R_\alpha \). Then

\[
\text{gn}(R) = \sup \{ \text{gn}(R_\alpha) : \alpha \in I \}.
\]

**Proof.** That \( \text{gn}(R) \) is an upper bound for the set \( S := \{ \text{gn}(R_\alpha) : \alpha \in I \} \) follows by applying Lemma 2.7 to the projection maps \( R \to R_\alpha \). Suppose that \( b \in \mathbb{Z}^+ \cup \{ \aleph_0 \} \) is an arbitrary upper bound for \( S \). We wish to establish that \( \text{gn}(R) \leq b \). This is plainly true if \( b = \aleph_0 \), so assume that \( b \) is finite. For each \( \alpha \in I \), there is an \( R_\alpha \)-module epimorphism \( \phi_\alpha : R_\alpha^b \to R_\alpha^{b+1} \). The maps \( \phi_\alpha \), then, induce an \( R \)-module epimorphism \( \phi : \prod_{\alpha \in I} R_\alpha^b \to \prod_{\alpha \in I} R_\alpha^{b+1} \). Moreover, since \( \prod_{\alpha \in I} R_\alpha^k \cong R^k \) as \( R \)-modules for every \( k \in \mathbb{Z}^+ \), the map \( \phi \) induces an \( R \)-module epimorphism \( R^b \to R^{b+1} \). Thus \( \text{gn}(R) \leq b \), and so \( \text{gn}(R) = \sup S \).

In our final preliminary result about generating numbers, we observe that every positive integer can be realized as the generating number of some ring.

**Proposition 2.12.** For any integer \( n > 0 \), there is a ring \( R \) such that \( \text{gn}(R) = n \).

We will deduce Proposition 2.12 from the following important result of G. Bergman [9, Theorem 6.2], which will also serve as the foundation for our proof of Theorem G in §4.

**Theorem 2.13.** (Bergman) Let \( M \) be a finitely generated abelian monoid with distinguished element \( I \neq 0 \) such that the following two properties are satisfied.

(i) For all \( x, y \in M \), if \( x + y = 0 \), then \( x = y = 0 \).

(ii) For each \( x \in M \), there exists \( \lambda \in \mathbb{Z}^+ \) such that \( x \leq \lambda I \).

Then a ring \( R \) exists such that there is a monoid isomorphism \( \theta : M \to \mathcal{P}(R) \) with \( \theta(I) = [R] \).
Proof of Proposition 2.12. Let $k \in \mathbb{Z}^+$. The abelian monoid $C(n,k)$ satisfies the hypotheses of Theorem 2.13 where the distinguished element is the generator $a$. Thus there are a ring $R$ and a monoid isomorphism $\theta : C(n,k) \to \mathcal{P}(R)$ with $\theta(a) = [R]$. Observe that $n$ is the smallest positive integer such that $(n+1)a \leq na$ in the monoid $C(n,k)$. As a result, $n$ is also the smallest positive integer such that $R^{n+1}$ is an $R$-module direct summand of $R^n$. In other words, $\text{gu}(R) = n$. \hfill $\Box$

We conclude this section with a result of a different flavor, which will play a key role in the proof of Theorem A. This proposition describes how every translation ring can be viewed as a skew group ring. It is similar to \cite[Theorem 4.28]{30} and is well known in the case of a finite group (see, for instance, \cite[Example 1.3.6]{25}).

**Proposition 2.14.** For any ring $R$ and group $G$, the ring $T(G,R)$ is isomorphic to the skew group ring $\prod_G R \rtimes G$, where the action of $G$ on $\prod_G R$ is defined by

$$(g \cdot f)(x) := f(g^{-1}x) \quad \text{for} \quad g,x \in G.$$ 

**Proof.** For an arbitrary element $g$ of $G$, define $A_g$ to be the matrix in $T(G,R)$ such that $A_g(x,y) := 1$ if $y = g^{-1}x$ and $A_g(x,y) := 0$ if $y \neq g^{-1}x$. Moreover, for each function $f : G \to R$, let $D_f$ be the $G \times G$ diagonal matrix with $D_f(x,x) := f(x)$ for all $x \in G$.

An easy calculation establishes

$$(A_g D_f A_g^{-1})(x,y) = \begin{cases} f(g^{-1}x) & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} = D_{g \cdot f}(x,y)$$

for every $f : G \to R$ and $g \in G$. This means that there is a ring homomorphism $\phi : \prod_G R \rtimes G \to T(G,R)$ such that $\phi(f) = D_f$ for every function $f : G \to R$ and $\phi(g) = A_g$ for every $g \in G$.

Another straightforward calculation shows that, for any function $f : G \to R$ and $g \in G$,

$$(D_f A_g)(x,y) = \begin{cases} f(x) & \text{if } y = g^{-1}x \\ 0 & \text{if } y \neq g^{-1}x \end{cases}.$$ 

Hence, if $f_1, \ldots, f_r : G \to R$ are functions and $g_1, \ldots, g_r \in G$, then

$$\left( \phi \left( \sum_{i=1}^r f_i g_i \right) \right)(x,y) = \left( \sum_{i=1}^r D_{f_i} A_{g_i} \right)(x,y) = \begin{cases} f_i(x) & \text{if } y = g_i^{-1}x \\ 0 & \text{if } y \neq g_i^{-1}x \end{cases}.$$ 

From this, we ascertain that $\phi$ is bijective. \hfill $\Box$

**Remark.** It is a routine exercise to show that the skew group ring described in Proposition 2.14 is isomorphic to the one obtained using the action $(g \cdot f)(x) := f(xg)$. Moreover, this can be used to show that $T(G,R)$ is isomorphic to the ring of all $G \times G$ matrices $M$ with entries in $R$ for which there is a finite subset $K$ of $G$ such that $M(x,y) = 0$ whenever $y \notin xK$. 

14
3 Free gradings

In this section, we prove our main results about freely graded rings. As related in §1, we first establish Theorem A for the case of a crossed product.

**Proposition 3.1.** Let $G$ be an amenable group and $R$ a ring. Any crossed product $R \rtimes G$ has UGN if and only if $R$ has UGN.

The proof of Proposition 3.1 is based on the following version of Følner’s [19] characterization of amenability (see [12] Exercise 4.26).

**Theorem 3.2 (Følner).** A group $G$ is amenable if and only if, for every finite subset $K$ of $G$ and every $\epsilon > 0$, there is a finite subset $F$ of $G$ such that $|KF| < (1 + \epsilon)|F|.$

**Proof of Proposition 3.1.** The “only if” part follows from Lemma 2.7. We will establish the “if” statement by proving its contrapositive. Suppose that $S := R \rtimes G$ has BGN. In other words, there is a right $S$-module epimorphism $\phi : S^n \to S^m$ for some positive integers $m > n$. Let $K \subseteq G$ be the union of the supports of all the entries in the matrix representation of $\phi$ with respect to the standard bases. Write $U := K \cdot F$. Applying Theorem 3.2, we obtain a finite subset $F$ of $G$ such that $n|U| < m|F|.$

For any positive integer $r$ and subset $V$ of $G$, let $S^n_{rV}$ be the $R$-submodule of $S^n$ consisting of all the elements with support contained in $V$. In addition, let $\pi_{rV} : S^n \to S^n_{rV}$ be the projection map, an $R$-module homomorphism. We claim that $S^m_{rF} \subseteq \pi_{mF} \phi(S^n_{rU}).$ To show this, let $y \in S^m_{rF}$. Then $y = \phi(x + x'),$ where $x \in S^n_{rU}$ and $x' \in S^n$ with $\text{Supp}(x') \cap U = \emptyset$. But $\text{Supp}([\phi(x')]) \subseteq K \cdot \text{Supp}(x')$. Thus $\text{Supp}([\phi(x')]) \cap F = \emptyset$; that is, $\pi_{mF} \phi(x') = 0$. Therefore $y = \pi_{mF} \phi(x),$ proving the claim.

Let $\phi_U : S^n_U \to S^m$ be the restriction of $\phi$ to $S^n_U$, which is an $R$-module homomorphism. Hence, in view of the claim made in the preceding paragraph, the composition $\pi_{mF} \phi_U : S^n_U \to S^m_{rF}$ is an $R$-module epimorphism. Moreover, $S^n_U \cong R^n|U|$ and $S^m_{rF} \cong R^m|F|$ as $R$-modules. Consequently, we have an $R$-module epimorphism $R^n|U| \to R^m|F|$. It follows, then, from Proposition 2.2((ii) $\implies$ (i)) that $R$ has BGN.

**Corollary 3.3.** Let $G$ be an amenable group and $R$ a ring. Then $T(G, R)$ has UGN if and only if $R$ has UGN.

**Proof.** As a consequence of Lemma 2.11, we have that $R$ has UGN if and only if $\prod G R$ does. However, in view of Propositions 2.14 and 3.1, the latter statement is equivalent to $T(G, R)$ possessing UGN.

From Corollary 3.3 we deduce the general case of Theorem A.
Proof of Theorem A. Again, the “only if” part follows from Lemma 2.7. To prove the converse, we assume that \( R_1 \) has UGN. In view of Lemma 2.6, we only need to consider the case where the homogeneous components of \( R \) are free right \( R_1 \)-modules of finite rank. Our argument will make use of the set \( \Lambda \) of all \( G \times G \) matrices \( M \) with the following two properties.

1. For each pair \( g, h \in G \), the entry \( M(g, h) \) lies in \( \text{Hom}_{R_1}(R_h, R_g) \).
2. For each \( h \in G \), \( M(g, h) = 0 \) for all but finitely many \( g \in G \).

Notice that \( \Lambda \) has the structure of a ring and \( \Lambda \cong \text{End}_{R_1}(R) \).

Let \( \Lambda' \) be the subring of \( \Lambda \) consisting of all those matrices \( M \in \Lambda \) for which there is a finite subset \( K \) of \( G \) such that \( M(g, h) = 0 \) for \( h \notin K g \). The canonical ring embedding \( R \to \text{End}_{R_1}(R) \) induces a ring embedding \( \theta : R \to \Lambda \). We claim that \( \text{Im} \theta \subseteq \Lambda' \). To see this, let \( g \in G \) and \( r \in R_g \). Then \( \theta(r_g)(x, y) = 0 \) for \( y \neq g^{-1} x \). Therefore \( \theta(r_g) \in \Lambda' \), proving the claim.

For any integer \( k \) and positive integer \( n \), we employ \( \rho_n(k) \) to denote the unique integer in the interval \( [1, n] \) that is congruent to \( k \) modulo \( n \). Moreover, for each \( g \in G \), let \( n_g \) be the rank of \( R_g \) as a free \( R_1 \)-module and fix a basis for \( R_g \). Next, for every pair \( g, h \in G \) and map \( \phi \in \text{Hom}_{R_1}(R_h, R_g) \), let \( M^{(g,h)}_\phi \) be the \( n_g \times n_h \) matrix with entries in \( R_1 \) that represents \( \phi \) with respect to the bases selected for \( R_g \) and \( R_h \). Furthermore, for each pair \( g, h \in G \), define the map \( \Psi_{g,h} : \text{Hom}_{R_1}(R_h, R_g) \to T(\mathbb{Z}, R_1) \) by the formula

\[
(\Psi_{g,h}(\phi))(i,j) := \begin{cases} \frac{M^{(g,h)}_\phi(\rho_{n_g}(i),\rho_{n_h}(j))}{(k_h+1)n_g+(k+1)n_h} & \exists k \in \mathbb{Z} : kn_g + 1 \leq i \leq (k+1)n_g \text{ and } kn_h + 1 \leq j \leq (k+1)n_h \\ 0, & \text{otherwise} \end{cases} 
\]

for all \( \phi \in \text{Hom}_{R_1}(R_h, R_g) \) and \( i, j \in \mathbb{Z} \).

Observe that the maps \( \Psi_{g,h} \) enjoy the three properties listed below.

1. \( \Psi_{g,h} \) is an injective additive map for all \( g, h \in G \).
2. \( \Psi_{g,g'}(\phi_1)\Psi_{g',g''}(\phi_2) = \Psi_{g,g''}(\phi_1\phi_2) \) for all \( g, g', g'' \in G \), \( \phi_1 \in \text{Hom}_{R_1}(R_{g'}, R_g) \), \( \phi_2 \in \text{Hom}_{R_1}(R_{g''}, R_{g'}) \).
3. \( \Psi_{g,1}(1_{R_g}) = 1_{T(\mathbb{Z}, R_1)} \) for every \( g \in G \).

Finally, define the map \( \Psi : \Lambda' \to T(G, T(\mathbb{Z}, R_1)) \) by

\[
(\Psi(A))(g, h) := \Psi_{g,h}(A(g, h))
\]

for all \( A \in \Lambda' \) and \( g, h \in G \). Then properties (i)-(iii) ensure that \( \Psi \) is a ring monomorphism. But Corollary 3.3 implies that the ring \( T(G, T(\mathbb{Z}, R_1)) \) has UGN. Therefore the same is true for \( \Lambda' \) and hence also for \( R \) (see Lemma 2.7).

We illustrate the extent of Theorem A’s utility by applying the result to a family of \( \mathbb{Z} \)-graded rings most of which are far from crossed products, their homogeneous components being free of finite, unbounded rank as modules over the base ring.
**Example 3.4.** Let $S$ be a ring with UGN. Take $a_1, \ldots, a_n, b_1, \ldots, b_n \in S$ such that each $a_i$ is a unit. We define $R$ to be the $S$-algebra generated by $x_1, \ldots, x_n, y$ subject only to the following relations:

1. $sx_i = x_is$ and $sy = ys$ for all $s \in S$ and $i = 1, \ldots, n$;
2. $yx_i = a_ix_iy + b_i$ for $i = 1, \ldots, n$.

The ring $R$ is a generalization of the well-known Weyl algebra as well as the quantum Weyl algebra (see, for instance, [29, Examples 3, 4]).

Set up a $\mathbb{Z}$-grading on $R$ by assigning $\deg(x_i) := 1$ for $i = 1, \ldots, n$ and $\deg(y) := -1$.

Standard results in universal algebra (see, for example, [13, p. 159, Theorem 9.3]) allow us to conclude that $R$ is freely generated as an $S$-module by each of the two sets $\mathcal{A}$ and $\mathcal{A}'$ defined below.

\[
\mathcal{A} := \left\{ x_{i_1} \ldots x_{i_k} y^l : l \geq 0, \ 1 \leq i_1, \ldots, i_k \leq n \right\} \cup \left\{ y^l : l \geq 0 \right\};
\]

\[
\mathcal{A}' := \left\{ y^l x_{i_1} \ldots x_{i_k} : l \geq 0, \ 1 \leq i_1, \ldots, i_k \leq n \right\} \cup \left\{ y^l : l \geq 0 \right\}.
\]

As a result, $R_0$ is freely generated as an $S$-module by each of the sets

\[
\mathcal{B} := \left\{ x_{i_1} \ldots x_{i_k} y^k : k > 0, \ 1 \leq i_1, \ldots, i_k \leq n \right\} \cup \left\{ 1 \right\},
\]

\[
\mathcal{B}' := \left\{ y^k x_{i_1} \ldots x_{i_k} : k > 0, \ 1 \leq i_1, \ldots, i_k \leq n \right\} \cup \left\{ 1 \right\}.
\]

From the properties above, it is apparent that, for $m \in \mathbb{Z}^+$, $R_m$ is freely generated as an $R_0$-module by the set

\[
\left\{ x_{i_1} \ldots x_{i_m} : 1 \leq i_1, \ldots, i_m \leq n \right\},
\]

and that $R_{-m}$ is freely generated as an $R_0$-module by the solitary element $y^m$. In other words, the grading that we have defined on $R$ is free.

Define the function $\phi : R_0 \to S$ by, for any $r \in R_0$, letting $\phi(r)$ be the coefficient of 1 in the expression of $r$ as an $S$-linear combination of the elements of $\mathcal{B}$. Plainly, $\phi$ is additive, and $\phi(1_{R_0}) = 1_S$. Notice further that, whenever any product of two elements of $\mathcal{B} - \{1\}$ is written as a linear combination of elements of $\mathcal{B}$ with coefficients in $S$ by repeatedly using the relations (2), the coefficient of 1 in that linear combination will be 0. From this observation we can see that $\phi$ is multiplicative and therefore a ring homomorphism. Thus, according to Lemma 2.7, $R_0$ must have UGN, and so, by Theorem A, $R$ also possesses UGN. Moreover, we are unaware of any way to deduce this property of $R$ without appealing to Theorem A.

Next we prove Theorem B about finite generating numbers.
Proof of Theorem B. By Proposition 2.12, there exist rings $S$ and $T$ with generating numbers $m$ and $n$, respectively. Put $U := M_n(T)$. According to Lemma 2.9, we have $\text{gn}(U) = 1$. Let $G$ be a finite group of order $n$ and notice that $T(G,T) \cong U$ as rings. Proposition 2.14 implies, then, that $U$ is isomorphic to a skew group ring of $G$ with coefficients in $\prod G T$. Write $V := U \times S$ and $R := (\prod G T) \times S$. Define a $G$-grading on $V$ by setting $V_g := U_g \times S$ for all $g \in G$. Then $V$ is a crossed product with respect to this grading, and $V_1 = R$. Furthermore, applying Lemma 2.11, we obtain $\text{gn}(R) = n$ and $\text{gn}(V) = m$. \hfill $\Box$

We now prove Theorem C. As stated in the introduction, this result is essentially contained in [17, §3] (see also the proof of [18, Proposition 2.2]). Though framed differently, our argument is fundamentally the same as the one there. The proof relies on the following characterization of amenability.

Theorem 3.5 ([12, Theorem 4.9.1]). A group $G$ is not amenable if and only if there is a finite subset $K$ of $G$ as well as two families $\{A_k : k \in K\}$ and $\{B_k : k \in K\}$ of subsets of $G$ such that

$$G = \coprod_{k \in K} A_k = \coprod_{k \in K} B_k = \left( \prod_{k \in K} kA_k \right) \coprod \left( \prod_{k \in K} kB_k \right).$$

Such a decomposition of a group is called a paradoxical decomposition.

Proof of Theorem C. Assume that we have a paradoxical decomposition of $G$ as described in Theorem 3.5. For every $g \in G$, let $\alpha(g)$ be the unique element of $K$ such that $g \in A_{\alpha(g)}$, and let $\beta(g)$ be the unique element of $K$ such that $g \in B_{\beta(g)}$. Observe that the functions $g \mapsto \alpha(g)g$ and $g \mapsto \beta(g)g$ from $G$ to $G$ are both injective.

Now define the elements $A$ and $B$ of $T := T(G,R)$ as follows:

$$A(g,h) := \begin{cases} 1 & \text{if } h = \alpha(g)g \\ 0 & \text{if } h \neq \alpha(g)g \end{cases}, \quad B(g,h) := \begin{cases} 1 & \text{if } h = \beta(g)g \\ 0 & \text{if } h \neq \beta(g)g \end{cases}$$

for all $g,h \in G$. Our aim is to establish the equations (3.1) and (3.2) below; these will imply $T^2 \cong T$ as $T$-modules.

$$\begin{pmatrix} A & B^t \end{pmatrix} = \begin{pmatrix} 1_T & 0 \\ 0 & 1_T \end{pmatrix}. \quad (3.1)$$

$$\begin{pmatrix} A^t & B^t \end{pmatrix} = \begin{pmatrix} 1_T \end{pmatrix}. \quad (3.2)$$

We verify (3.1) by first looking at the $(g,g')$ entry of $AA^t$ for two arbitrary elements $g,g'$ of $G$.

$$(AA^t)(g,g') = \sum_{h \in G} A(g,h)A(g',h).$$
If $g = g'$, then this sum is plainly equal to 1. If $g \neq g'$, then $\alpha(g)g \neq \alpha(g')g'$, which means $(AA^t)(g, g') = 0$. Hence $AA^t = 1_T$. Similarly, $BB^t = 1_T$.

Next we examine

$$(AB^t)(g, g') = \sum_{h \in G} A(h, g)B(g', h),$$

where $g, g'$ are arbitrary elements of $G$. Since $\alpha(g)g \neq \beta(g')g'$ for every pair $g, g'$, we have $AB^t = 0$. Also, by the same token, $BA^t = 0$. We have shown, then, that (3.1) holds.

Now we calculate the products $A^tA$ and $B^tB$.

$$(A^tA)(g, g') = \sum_{h \in G} A(h, g)A(h, g')$$

for all $g, g' \in G$. This sum is 1 if there is an $h \in G$ such that $g = g' = \alpha(h)h$; otherwise it is 0. Hence $A^tA$ is diagonal with

$$(A^tA)(g, g) = \begin{cases} 1 & \text{if } g \in \bigcup_{k \in K} kA_k \\ 0 & \text{if } g \in \bigcup_{k \in K} kB_k \end{cases}$$

for every $g \in G$. Similarly, $B^tB$ is diagonal, and

$$(B^tB)(g, g) = \begin{cases} 0 & \text{if } g \in \bigcup_{k \in K} kA_k \\ 1 & \text{if } g \in \bigcup_{k \in K} kB_k \end{cases}$$

for all $g \in G$. Equation (3.2), then, follows.

Next we prove Corollaries A and B.

**Proof of Corollary A.** The implication (i) $\implies$ (ii) is Theorem A. That (ii) $\implies$ (iii) $\implies$ (iv) is obvious, and that (iv) $\implies$ (v) follows from Proposition 2.14 and Lemma 2.11. Finally, the implication (v) $\implies$ (i) is a consequence of Theorem C.

**Proof of Corollary B.** Suppose that $R$ has UGN. The group $G$ has a free normal subgroup $F$ such that $Q := G/F$ is amenable. This means that the twisted group ring $R \ast G$ is isomorphic to a crossed product of a twisted group ring $R \ast F$ by the group $Q$. Since $F$ is free, we have $H^2(F, Z(R)*) = 0$, which implies $R \ast F \cong RF$ as rings. Hence, by Corollary 2.8, $R \ast F$ possesses UGN. It follows, then, from Theorem A that $R \ast G$ has UGN.

We conclude this section by establishing Corollary C. For this, we require the following two lemmas relating the IBN property to the functor $K_0$.

**Lemma 3.6 ([16 Proposition 0.3.4(i)])**, Let $R$ be a ring. Then $R$ has IBN if and only if $\kappa_R$ is injective.
Lemma 3.7. Let $R$ be a ring. There exists a positive integer $n$ such that $R^n \cong R^{n+1}$ as $R$-modules if and only if $\kappa_R = 0$.

To prove Lemma 3.7, we will make use of the following elementary fact about Grothendieck completions of abelian monoids.

Lemma 3.8. Let $M$ be an abelian monoid and $C(M)$ its Grothendieck completion. If $m \in M$, then $[m] = [0]$ in $C(M)$ if and only if there exists $n \in M$ such that $m + n = n$.

Proof of Lemma 3.7. Assume that $R^n \cong R^{n+1}$ for some $n \in \mathbb{Z}^+$. From the relation $R \oplus R^n \cong R^n$, it follows, by Lemma 3.8, that $\kappa_R = 0$. This proves the “only if” part.

To establish the converse, assume $\kappa_R = 0$. Then Lemma 3.8 implies that there exists a finitely generated projective $R$-module $P$ such that $R \oplus P \cong P$. For some $n \in \mathbb{Z}^+$, $P$ is a direct summand in $R^n$. This means $R \oplus R^n \cong R^n$; that is, $R^{n+1} \cong R^n$.

Proof of Corollary C. We remind the reader that UGN implies IBN. In view of this, the implication (i) $\implies$ (ii) follows from Corollary A((i) $\implies$ (v)) and Lemma 3.6. Furthermore, (ii) $\implies$ (iii) is trivial, and (iii) $\implies$ (i) is a consequence of Lemma 3.7 and Theorem C.

4 Projective gradings

In this section, we investigate projectively graded rings. We begin by establishing Theorem D and Corollary D, which show that the statement of Theorem A is false if the grading is assumed to be projective rather than free. The proof of Theorem D makes use of the following well-known, elementary property.

Proposition 4.1. Let $R$ be a ring, and let $S$ be an $R$-algebra that is generated by elements $\epsilon_{ij}$ for $i, j = 1, \ldots, n$ satisfying the following four conditions:

(i) $r \epsilon_{ij} = \epsilon_{ij} r$ for all $r \in R$ and $i, j = 1, \ldots, n$;
(ii) $\text{Ann}_R(\epsilon_{ij}) = 0$ for all $i, j = 1, \ldots, n$;
(iii) $\epsilon_{ij} \epsilon_{km} = \delta_{jk} \epsilon_{im}$ for all $i, j, k, m = 1, \ldots, n$;
(iv) $\sum_{i=1}^n \epsilon_{ii} = 1$.

If $\{E_{ij} : i, j = 1, \ldots, n\}$ is the set of standard matrix units in $M_n(R)$, then there is an $R$-algebra isomorphism $\phi : M_n(R) \to S$ such that $\phi(E_{ij}) = \epsilon_{ij}$ for $i, j = 1, \ldots, n$.

Proof of Theorem D. Let $\mathcal{M}(E)$ be the multiplicative submonoid of $L$ generated by $E$. Then $\mathcal{M}(E)$ is free on $E$. For any $\alpha = \epsilon_{i_1} \cdots \epsilon_{i_l}$, we call $l$ the length of $\alpha$, denoted $l(\alpha)$, and write $\alpha^* := e_{i_1}^* e_{i_{l-1}}^* \cdots e_{i_1}^*$. In addition, we define $l(1) := 0$ and $1^* := 1$. Finally, we write $\mathcal{M}_l(E)$ for the subset of $\mathcal{M}(E)$ consisting of all the elements of length $l$. With this notation, we can generalize the relations (ii) and (iii) among the hypotheses of Theorem D as follows:
(1) $\alpha^*\beta = \delta(\alpha, \beta)$ for all $\alpha, \beta \in \mathcal{M}(E)$ with $l(\alpha) = l(\beta)$;

(2) $\sum_{\alpha \in \mathcal{M}_l(E)} \alpha \alpha^* = 1$ for all $l \geq 0$.

We set up the $\mathbb{Z}$-grading on $L$ by, for any $k \in \mathbb{Z}$, defining $L_k$ to be the $R$-linear span of the set of elements of the form $\alpha\beta^*$ such that $\alpha, \beta \in \mathcal{M}(E)$ and $l(\alpha) - l(\beta) = k$. Moreover, relations (1) and (2) ensure that this grading is strong (see [25, Proposition 1.1.1(3)]).

That $L^0 \cong L$ as $L$-modules follows from the defining relations of $L$. To establish that $L_0$ has UGN, we let $L^1_{0}$ be the subalgebra of $L_0$ consisting of all $R$-linear combinations of elements of the form $\alpha\beta^*$ where $\alpha, \beta \in \mathcal{M}_l(E)$. Then $L_0$ is the union of the chain

$$L^0_0 \subset L^1_0 \subset L^2_0 \subset \cdots,$$

with the containments following easily from relation (iii). Our intent is to apply Proposition 4.1 to show that, for every $l \geq 0$, $L^1_0 \cong M_{n^l}(R)$. This will mean that each subalgebra $L^k_0$ has UGN, implying, by Lemma 2.10, that $L_0$ does too.

To construct the desired isomorphism, we first choose a bijection $\sigma : \{1, \ldots, n^l\} \to \mathcal{M}_l(E)$ and write $\epsilon_{ij} := \sigma(i)(\sigma(j))^*$ for $i, j = 1, \ldots, n^l$. Then (1) and (2), respectively, imply the following two equations:

- $\epsilon_{ij}\epsilon_{km} = \delta_{jk}\epsilon_{im}$ for $i, j, k, m = 1, \ldots, n^l$;
- $\sum_{i=1}^{n^l} \epsilon_{ii} = 1$.

Now let $\{E_{ij} : i, j = 1, \ldots, n^l\}$ be the set of standard matrix units in $M_{n^l}(R)$. Referring to Proposition 4.1, we see that there is an $R$-algebra isomorphism $\phi : M_{n^l}(R) \to L^1_0$ such that $\phi(E_{ij}) = \epsilon_{ij}$ for $i, j = 1, \ldots, n^l$. \hfill $\square$

**Remark.** The interpretation of $L_0$ as a direct limit of matrix rings that plays a central role in our proof of Theorem D is originally due to G. Abrams and P. N. Ánh [2] (in the case where $R$ is a field).

**Remark.** In light of the hypothesis about the grading in Theorem A, we point out that, in Theorem D, $L_k$ is a finitely generated free right $L_0$-module if $k > 0$ and a finitely generated free left $L_0$-module if $k < 0$.

**Proof of Corollary D.** Let $L$ be the $\mathbb{Z}$-graded ring defined in Theorem D. By Proposition 2.12, there is a ring $S$ with $\operatorname{gn}(S) = m$. Put $T := L \times S$ and grade $T$ by $\mathbb{Z}$ by defining $T_k := L_k \times S$ for every $k \in \mathbb{Z}$. Since the grading on $L$ is strong, the same is true for the grading on $T$. Moreover, Lemma 2.11 implies that $T_0$ has UGN and $\operatorname{gn}(T) = m$. \hfill $\square$
Remark. We point out that it is not possible to find a Leavitt path algebra over a field that enjoys the properties of the ring $T$ in Corollary D. The reason is that any Leavitt path algebra with BGN that is over a field must have generating number one (see [4, Remark 3.17]).

Next we establish Theorem E, dealing with projective gradings by locally finite groups. For the concepts from Morita theory invoked in the proof, we refer the reader to [22, §18].

Proof of Theorem E. Since the “only if” part follows immediately from Lemma 2.7, we proceed directly to the “if” statement. In view of Lemma 2.6, we only need to consider the case where the homogeneous components of $R$ are finitely generated projective right $R_1$-modules. Also, because UGN is inherited by direct limits (see Lemma 2.10), it suffices to treat the case where $G$ is finite. This means that $R$ is finitely generated and projective as an $R_1$-module. Since $R_1$ is an $R_1$-module direct summand in $R$, we have further that $R$ is a generator for $M_{R_1}$. These properties make $R$ a progenerator in $M_{R_1}$, rendering $\text{End}_{R_1}(R)$ Morita equivalent to $R_1$. It follows, then, from Corollary 2.3 that $\text{End}_{R_1}(R)$ has UGN. Therefore, being isomorphic to a subring of this endomorphism ring, $R$ must also have UGN.

Below we prove Theorem F; our argument tracks very closely the proof of [1, Theorem B], although there the group employed for the grading is $Z/2$, rather than an arbitrary finite group.

Proof of Theorem F. Invoking Proposition 2.12, we let $S$ be a ring with $\text{gn}(S) = n$. Then $\text{gn}(M_n(S)) = 1$ by Lemma 2.9. Put $k := |G|$ and let $l \in \mathbb{Z}^+$ such that $nl > k - 1$. Set $p := nl - k + 1$, and define a family $\{A_g : g \in G\}$ of $S$-modules as follows: $A_1 := S^p$; $A_g := S$ for $g \neq 1$. Write $A := \bigoplus_{g \in G} A_g$ and $T := \text{End}_S(A)$. This means $T \cong M_{nl}(S)$, implying that there is a ring embedding $M_n(S) \to T$. Therefore, by Lemma 2.7, we have $\text{gn}(T) = 1$.

The ring $T$ may be viewed as the ring of all $G \times G$ matrices $P$ such that $P(g, h) \in \text{Hom}_S(A_h, A_g)$ for every pair $g, h \in G$. This allows us to equip $T$ with a $G$-grading by, for each $g \in G$, defining $T_g$ to be the set of all matrices $P \in T$ such that $P(x, y) = 0$ if $y \neq g^{-1}x$. Moreover, it is easy to check that this grading is strong.

We observe that the following isomorphisms between rings hold:

$$T_1 \cong \text{End}_S(S^p) \times \text{End}_S(S) \times \cdots \times \text{End}_S(S) \cong M_p(S) \times S \times \cdots \times S.$$  

Since there is a ring embedding $S \to M_p(S)$, it follows from Lemma 2.7 that $\text{gn}(M_p(S)) \leq n$. Hence we obtain $\text{gn}(T_1) = n$ by applying Lemma 2.11. Now let $U$ be a ring with $\text{gn}(U) = m$. Put $R := T \times U$ and endow $R$ with a strong $G$-grading by defining $R_g := T_g \times U$ for $g \in G$. Then Lemma 2.11 implies $\text{gn}(R_1) = n$ and $\text{gn}(R) = m$.  

22
We now use the methods employed for [1, Theorem A] to prove Theorem G, thus demonstrating that the hypothesis that the ring is projectively graded in Theorem E cannot be omitted. Like the approach in [1], ours is based upon Theorem 2.13. In addition, we rely on Lemmas 4.2 and 4.3 below. The first, inspired by [24, Proposition 3.3], describes an elementary, and undoubtedly well-known, property of projective modules that can be used to determine the generating numbers of their endomorphism rings. The second lemma introduces a certain abelian monoid to which we will apply Theorem 2.13 when we prove Theorem G.

**Lemma 4.2.** Let $R$ be a ring and $P$ a projective $R$-module. Write $S := \text{End}_R(P)$. Then the following two statements are equivalent for any $m,n \in \mathbb{Z}^+$.

(i) There is an $S$-module epimorphism $S^n \to S^m$.

(ii) There is an $R$-module epimorphism $P^n \to P^m$.

**Proof.** Assume that there is an $S$-module epimorphism $S^n \to S^m$. This induces an $R$-module epimorphism $S^n \otimes_S P \to S^m \otimes_S P$. However, $S \otimes_S P \cong P$ as $R$-modules, so that we have an $R$-module epimorphism $P^n \to P^m$. This proves (i) $\implies$ (ii).

Assume that there is an $R$-module epimorphism $\phi : P^n \to P^m$. Since $P$ is projective, the map $\phi$ induces an $S$-module epimorphism $\text{Hom}_R(P, P^n) \to \text{Hom}_R(P, P^m)$. But, for every $k \in \mathbb{Z}^+$, $\text{Hom}_R(P, P^k) \cong S^k$ as $S$-modules. Therefore we have an $S$-module epimorphism $S^n \to S^m$. This establishes (ii) $\implies$ (i). \qed

**Lemma 4.3.** For any triple $n,k,l$ of positive integers, let $M(n,k,l)$ be the abelian monoid generated by $u, x_1, \ldots, x_l, y_1, \ldots, y_l$ subject only to the relations

$$(n+k)(u+x_1+\cdots+x_l) = n(u+x_1+\cdots+x_l), \quad x_i + y_i = u \text{ for } i = 1, \ldots, l.$$ \hspace{1cm} (4.1)

Then, for any $j = 1, \ldots, l$ and any two positive integers $\lambda, \mu$, $\lambda x_j \leq \mu x_j \implies \lambda \leq \mu$.

**Proof.** We will make use of the monoid homomorphism $\phi : M(n,k,l) \to C(n,k)$ such that $\phi(u) := \phi(y_i) := a$ and $\phi(x_i) := 0$ for $i = 1, \ldots, l$. In addition, we will employ the monoid homomorphisms $\psi_j : M(n,k,l) \to \mathbb{Z}$, for $j = 1, \ldots, l$, defined by

$\psi_j(u) := -1, \quad \psi_j(x_i) := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{and} \quad \psi_j(y_i) := \begin{cases} -2 & \text{if } i = j \\ -1 & \text{if } i \neq j \end{cases}.$

The hypothesis $\lambda x_j \leq \mu x_j$ implies that there are nonnegative integers $\lambda_1, \ldots, \lambda_l, \alpha, \beta$ such that $\lambda_j \geq \lambda$ and

$$\lambda_1 x_1 + \cdots + \lambda_l x_l + \alpha u + \beta y = \mu x_j.$$ \hspace{1cm} (4.1)

Applying $\phi$ to equation (4.1) yields $(\alpha + \beta)a = 0$, which means $\alpha = \beta = 0$. We now apply $\psi_j$ to (4.1), obtaining $\lambda_j = \mu$. Thus $\lambda \leq \mu$. \qed

23
Proof of Theorem G. We are given an arbitrary finite group $G$ and an arbitrary positive integer $n$. Set $l := |G| - 1$. Taking $k$ to be another arbitrary positive integer, let $M(n,k,l)$ be the abelian monoid defined in Lemma 4.3. The monoid homomorphisms $\phi : M(n,k,l) \to C(n,k)$ and $\psi_j : M(n,k,l) \to \mathbb{Z}$, for $j = 1, \ldots, l$, used in the proof of that lemma can also be employed to show that, if $\alpha_1, \ldots, \alpha_l, \beta, \gamma$ are nonnegative integers, then

$$\sum_{i=1}^{l} \alpha_i x_i + \beta u + \gamma y = 0 \iff \alpha_i = \beta = \gamma = 0 \quad \forall i = 1, \ldots, l.$$ 

This observation, combined with the fact that, for every $t \in M(n,k,l)$, $t \leq \lambda u$ for some $\lambda \in \mathbb{Z}^+$, allows us to invoke Theorem 2.13. We thereby acquire a ring $S$ such that there is a monoid isomorphism $\theta : M(n,k,l) \to \mathcal{P}(S)$ with $\theta(u) = \{S\}$.

For each $i = 1, \ldots, l$, let $X_i$ be a finitely generated projective $S$-module such that $[X_i] = \theta(x_i)$. Also, let $\sigma : G - \{1\} \to \{1, \ldots, l\}$ be a bijection. For each $g \in G$, write $A_g := X_{\sigma(g)}$ for $g \neq 1$ and $A_1 := S$. Put $A := \bigoplus_{g \in G} A_g$ and $R := \text{End}_S(A) = \text{End}_S(S \oplus \bigoplus_{i=1}^{l} X_i)$. The ring $R$ may be viewed as the ring of all $G \times G$ matrices $P$ such that $P(g,h) \in \text{Hom}_S(A_h,A_g)$ for every pair $g,h \in G$. We can therefore endow $R$ with a $G$-grading by, for each $g \in G$, defining $R_g$ to be the set of all matrices $P \in R$ such that $P(x,y) = 0$ if $y \neq g^{-1}x$. Notice that we then have

$$R_1 \cong S \times \text{End}_S(X_1) \times \cdots \times \text{End}_S(X_l)$$

as rings.

Let $i = 1, \ldots, l$. Write $T_i := \text{End}_S(X_i)$ and let $p \in \mathbb{Z}^+$. Lemma 4.3 implies $(p+1)x_i \not\leq px_i$. As a result, $X_i^{p+1}$ is not an $S$-module direct summand in $X_i^p$. Hence, by Lemma 4.2, there is no $T_i$-module epimorphism $T_i^p \to T_i^{p+1}$. In other words, $T_i$ has UGN for $i = 1, \ldots, l$. Thus, by Lemma 2.11 (or Lemma 2.7), $R_1$ possesses UGN.

It remains to show that $\text{gn}(R) = n$. To accomplish this, we observe that $n$ is the smallest positive integer such that, in the cyclic monoid $C(n,k)$, $(n+1)a \leq na$. Hence, referring to the homomorphism $\phi : M(n,k,l) \to C(n,k)$, we conclude that $n$ is also the smallest positive integer such that $(n+1)(u + \sum_{i=1}^{l} x_i) \leq n(u + \sum_{i=1}^{l} x_i)$ in $M(n,k,l)$. Therefore the integer $n$ may be further characterized as the smallest positive integer such that $\left(S \oplus \bigoplus_{i=1}^{l} X_i \right)^{n+1}$ is an $S$-module direct summand in $\left(S \oplus \bigoplus_{i=1}^{l} X_i \right)^n$. An appeal to Lemma 4.2, then, permits us to conclude that $n$ is also the smallest positive integer such that there is an $R$-module epimorphism $R^n \to R^{n+1}$. In other words, we have $\text{gn}(R) = n$. \hfill $\square$

References

[1] G. Abrams. Invariant basis number and types for strongly graded rings. J. Algebra 237 (2001), 32-37.
2. G. Abrams and P. N. Ánh. Some ultramatricial algebras which arise as intersections of Leavitt algebras. *J. Algebra Appl.* 1 (2002), 357-363.

3. G. Abrams, P. Ara, and M. Siles Molina. *Leavitt Path Algebras.* Lecture Notes in Mathematics 2191. Springer, 2017.

4. G. Abrams, T. G. Nam, and N. T. Phuc. Leavitt path algebras having unbounded generating number. *J. Pure Appl. Algebra* 221 (2017), 1322-1343.

5. P. Ara, K. C. O’Meara, and F. Perera. Stable finiteness of group rings in arbitrary characteristic. *Adv. Math.* 170 (2002), 224-238.

6. P. Ara, K. C. O’Meara, and F. Perera. Gromov translation algebras over discrete trees are exchange rings. *Trans. Amer. Math. Soc.* 356 (2003), 2067-2079.

7. L. Bartholdi. Gardens of Eden and amenability on cellular automata. *J. Eur. Math. Soc.* 12 (2010), 241-248.

8. L. Bartholdi. Amenability of groups is characterized by Myhill’s theorem (with an appendix by D. Kielak). *J. Eur. Math. Soc.* 21 (2019), 3191-3197.

9. G. M. Bergman. Coproducts and some universal ring constructions. *Trans. Amer. Math. Soc.* 200 (1974), 33-88.

10. T. Ceccherini-Silberstein and M. Coornaert. The Garden of Eden theorem for linear cellular automata. *Ergodic Theory Dynam. Systems* 26 (2006), 53-68.

11. T. Ceccherini-Silberstein and M. Coornaert. Induction and restriction of cellular automata. *Ergodic Theory Dynam. Systems* 29 (2009), 371-380.

12. T. Ceccherini-Silberstein and M. Coornaert. *Cellular Automata and Groups.* Springer, 2010.

13. P. Cohn. *Universal Algebra.* Harper & Row, 1965.

14. P. Cohn. Some remarks on the invariant basis property. *Topology* 5 (1966), 215-228.

15. P. Cohn. *Skew Fields: Theory of General Division Rings.* Encyclopedia of Mathematics and its Applications 57. Cambridge, 1995.

16. P. Cohn. *Free Ideal Rings and Localization in General Rings.* Cambridge, 2006.

17. G. Elek. The $K$-theory of Gromov’s translation algebras and the amenability of discrete groups. *Proc. Amer. Math. Soc.* 125 (1997), no. 9, 2551-2553.
[18] G. Elek. On algebras that have almost finite dimensional representations. *J. Algebra Appl.* **4** (2005), no. 2, 179-186.

[19] E. Følner. On groups with full Banach mean value. *Math. Scand.* **3** (1955), 243-254.

[20] M. Gromov. *Asymptotic Invariants of Infinite Groups*, London Mathematical Society Lecture Note Series **182**. Cambridge, 1993.

[21] P. H. Kropholler and K. Lorensen. Group-graded rings satisfying the strong rank condition. *J. Algebra* **539** (2019), 326-338.

[22] T. Y. Lam. *Lectures on Modules and Rings*. Springer, 1999.

[23] W. G. Leavitt. The module type of a ring. *Trans. Amer. Math. Soc.* **103** (1962), 113-130.

[24] C. Năstăsescu, B. Torrecillas, and F. Van Oystaeyen. IBN for graded rings. *Comm. Algebra* **28** (2000), 1351-1360.

[25] C. Năstăsescu and F. Van Oystaeyen. *Methods of Graded Rings*, Lecture Notes in Math. **1836**. Springer, 2004.

[26] E. Nauwelaerts, and F. Van Oystaeyen. Introducing crystalline graded algebras. *Algebr. Represent. Theory* **11** (2008), 133-148.

[27] J. von Neumann. Zur allgemeinen Theorie des Masses. *Fund. Math.* **13** (1929), 73-116.

[28] P. Nystedt, J. Öinert, and H. Pinedo. Epsilon-strongly graded rings, separability and semisimplicity. *J. Algebra* **514** (2018), 1-24.

[29] J. Öinert and S. Silvestrov. Commutativity and ideals in pre-crystalline graded rings. *Acta Appl. Math.* **108** (2009), 603-615.

[30] J. Roe. *Lectures on Course Geometry*. Amer. Math. Soc, 2003.

[31] A. Tarski. Algebraische Fassung des Massproblems. *Fund. Math.* **31** (1938), 47-66.

**Karl Lorensen**
Department of Mathematics and Statistics
Pennsylvania State University, Altoona College
Altoona, PA 16601, USA
E-mail: kql3@psu.edu

26
Johan Öinert
Department of Mathematics and Natural Sciences
Blekinge Institute of Technology
SE-37179 Karlskrona, Sweden
E-mail: johan.oinert@bth.se