Research Article
Padé-Sumudu-Adomian Decomposition Method for Nonlinear Schrödinger Equation

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The main purpose of this paper is to solve the nonlinear Schrödinger equation using some suitable analytical and numerical methods such as Sumudu transform, Adomian Decomposition Method (ADM), and Padé approximation technique. In many literatures, we can see the Sumudu Adomian decomposition method (SADM) and the Laplace Adomian decomposition method (LADM); the SADM and LADM provide similar results. The SADM and LADM methods have been applied to solve nonlinear PDE, but the solution has small convergence radius for some PDE. We perform the SADM solution by using the function \( P_{LM} \) called double Padé approximation. We will provide the graphical numerical simulations in 3D surface solutions of each application and the absolute error to illustrate the efficiency of the method. In our methods, the nonlinear terms are computed using Adomian polynomials, and the Padé approximation will be used to control the convergence of the series solutions. The suggested technique is successfully applied to nonlinear Schrödinger equations and proved to be highly accurate compared to the Sumudu Adomian decomposition method.

1. Introduction

In science and engineering, most problems can be represented by linear or nonlinear ordinary or partial differential equations. To provide more important information about the scientific questions, it is necessary to solve these problems accurately. To achieve this goal, various researches have been done to find more efficient solutions to these problems. Many analytical and numerical methods have been established. Different powerful mathematical techniques such as Adomian Decomposition Method (ADM) \([1–3]\), Sumudu transform \([4]\), Sumudu Adomian Decomposition method (SADM) \([5–7]\), homotopy perturbation method (HPM), Laplace decomposition method (LDM) \([8]\), Padé approximant \([9–13]\), Laplace transform combined with Padé approximation \([14]\), homotopy perturbation Sumudu transform method (HPSTM) \([15]\), Homotopy Padé Approximate \([16]\), Modified Adomian Decomposition Method \([17–19]\), Homotopy Laplace transform \([20]\), Akbari-Ganji’s Method (AGM) \([21–23]\), and Hopf bifurcation \([24]\) have been proposed to obtain exact and approximate analytic solutions. Inspired by many researchers in this field, we offer a new method called the Padé Sumudu Adomian decomposition method (PSADM) for solving the nonlinear Schrödinger equations in the present paper. We are mentioning that the proposed method is a combination of the Sumudu Adomian decomposition method and Padé approximation. Padé approximation has been applied for rational series solutions in many areas; the technique was developed around 1890 by Henri Padé \([11]\). The existence and the convergence of subsequences were proved in \([12]\) by Baker. We have known that Padé approximants show high performance over series approximations, and the diagonal Padé approximation is bounded. The Padé approximation can be used to control the convergence of the series. It can be seen in many papers; the Padé approximants give better numerical results than approximation by the polynomial.

The Adomian Decomposition Method (ADM) was introduced in 1980 by George Adomian, a new robust method for solving a nonlinear functional equation. The technique has been applied to broad class of stochastic and deterministic problems in biology, physics, and chemical.
Wazwaz and Salah introduced the modified Adomian decomposition method; the series solution converges rapidly. The algorithm proposed by Wazwaz for calculating Adomian polynomials for all forms of nonlinearity is not easy to implement due to its massive size of algebraic calculations, complicated trigonometric terms.

Choi and Shin proposed a new symbolic implementation code [25]; this technique was a developed method proposed by Wazwaz. The decomposition method has been shown to solve easily, effectively, and accurately a large class of linear and nonlinear ordinary differential equation and has been applied to obtain formal solutions to a broad category of the partial differential equations of fractional orders, when initial and boundary conditions have been given. Abbaoui and Cherruault [26] proved the convergence of the Adomian method for differential and operator equations.

In the year 1990, the Sumudu transform was introduced by Gamog K. Watugala to solve differential equations and control engineering problems. The Sumudu transform is an integral transform similar to the Laplace-Carson transforms [27, 28] with the substitution $p = 1/u$. In Sinhala language, the word Sumudu means “smooth.” The Sumudu transform method can only solve linear PDE. In many papers, we can see the modified Sumudu decomposition method [29], the Sumudu Adomian Decomposition Method (SADM) which is a combination of Sumudu transform and Adomian Decomposition Method. The method is applied to solve nonlinear PDE. For some PDE, the SADM solution is not accurate in large domain.

In this paper, we will apply our proposed method (PSADM) for solving the most common form of the nonlinear Schrödinger equation to approximate the solution numerically. These applications demonstrate the efficiency and show the advantage of PSADM to be more powerful than SADM.

2. Preliminaries

2.1. The Sumudu Transform. Sumudu transform was shown to have unit preserving properties and may be used to solve a frequency domain problem in engineering.

**Definition 1** [4]. (Sumudu transform). The Sumudu transform is defined over the function set:

$$E = \{ f(t) | \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\text{d} |t|/\tau_2}, t \in (-1)^t \times [0, \infty) \},$$

by the formula

$$F(v) = S[f](v) = S[f(t)](v) = \int_{0}^{\infty} f(vt)e^{-vt}dt, v \in (-\tau_1, \tau_2).$$

Theorem 1. (Sumudu theorems for multiple differentiations). Let $f \in E$ and let $F_n(v)$ denote the Sumudu transform of the $n$th-order derivative $f^{(n)}(t)$ of $f(t)$. Then, for $n \geq 1$,

$$F_n(v) = \frac{F(v)}{v^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{n-k}}.$$  

Then, the inverse of the Sumudu transform function $F(v)$ of the function $f(t)$ is given by:

$$f(t) = S^{-1}[F(v)](t) = \frac{1}{2\pi i} \int_{C}^{\text{Reg}} e^{st}F \left( \frac{1}{s} \right) \frac{ds}{s}.$$  

Theorem 2. (Sumudu theorems for multiple differentiations). Let $F_n(v)$ denote the Sumudu transform of the $n$th-order derivative $f^{(n)}(t)$ of $f(t)$. Then, for $n \geq 1$,

$$F_n(v) = \frac{F(v)}{v^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{n-k}}.$$  

Then, the inverse of the Sumudu transform function $F(v)$ of the function $f(t)$ is given by:

$$f(t) = S^{-1}[F(v)](t) = \frac{1}{2\pi i} \int_{C}^{\text{Reg}} e^{st}F \left( \frac{1}{s} \right) \frac{ds}{s}.$$  

2.2. The Adomian Decomposition Method ([1, 2]). George Adomian introduced a new powerful method known as the Adomian decomposition method (ADM) for solving nonlinear functional equations. The Adomian decomposition method has been applied to a various class of initial value or boundary value problems and even stochastic systems, such as ordinary-differential equations, partial-differential equations, integro-differential equations, fractional differential equations, and stochastic differential equations. The method has shown good results in supplying analytical approximation.

Let recall the basic principles of the Adomian decomposition method by solving the general nonlinear equation in the form

$$L_1u(x,t) + L_2u(x,t) + R(u(x,t)) + G(u(x,t)) = F(x,t),$$

where $c_1$ and $c_2$ are constants. About the Sumudu transform $S[\cdot]$, the following theorems hold.

**Theorem 3** [4]. Let us $F(v)$ be the Sumudu transform of $f(t)$ such that

(i) $(F(1/s))/s$ is a meromorphic function, with singularities having $\text{Re } (s) < \gamma$, and

(ii) There exists a circular region $\Gamma$ with radius $R$ and positive constants, $M$ and $K$, with

$$|F(1/s)/s| < MR^{-K}. \quad (5)$$

Then, the inverse of the Sumudu transform function $F(v)$ of the function $f(t)$ is given by:

$$f(t) = S^{-1}[F(v)](t) = \frac{1}{2\pi i} \int_{C}^{\text{Reg}} e^{st}F \left( \frac{1}{s} \right) \frac{ds}{s}.$$  

Let us denote by $S_t[f](v)$ the Sumudu transform with respect to $t$ and $S_v^{-1}[-](t)$ the inverse of the Sumudu transform with respect to $v$. By using integral by part, we have the following theorem.

**Theorem 4.** Assume $F(x, v)$ is the Sumudu transform $S_v^{-1}[f(x, t)](v)$. Then, we have the relation

$$S_t \left[ \frac{\partial f(x, t)}{\partial x} \right] (v) = \frac{\partial}{\partial x} \left[ F(x, v) \right].$$  

(7)
where $L_t$ is the highest order differential in $t$, $L_x$ is the highest order differential in $x$, $R$ contains the remaining linear terms of lower order derivative terms, $G(u(x,y))$ is the nonlinear term, and $F(x,t)$ is the inhomogeneous or forcing term.

The decision as to choose which operator $L_t$ or $L_x$ should be used to solve the problem depends on two conditions:

(i) The operator of lowest order should be selected to minimize the size of computational work

(ii) The selected operator of lowest order should be the best known conditions to accelerate the evaluation of the components of the solution

Assuming that the operator $L_t$ satisfies the two bases of selection.

Take $L_t^{-1}$ in both sides of (8) gives

$$u(x,t) = \Phi_0 + L_t^{-1}F(x,t) - L_t^{-1}L_xu(x,t) - L_t^{-1}G(u(x,t)) - L_t^{-1}Ru(x,t),$$

with

$$\Phi_0 = \begin{cases} u(x,0) & \text{for } L_t = \frac{\partial}{\partial t}, \\ u(x,0) + tu_t(x,0) & \text{for } L_t = \frac{\partial^2}{\partial t^2}. \end{cases}$$

(9)

The Adomian decomposition method consists in looking for the solution in the series form, $u = \sum_{n=0}^{\infty} A_n$. The nonlinear operator is decomposed as

$$G(u(x,t)) = \sum_{n=0}^{\infty} A_n,$$

(10)

where $A_n$ are Adomian polynomials of $u_0, u_1, u_2, \cdots, u_n$ which calculated by:

$$A_n = \frac{1}{n!} \frac{d^n}{dt^n} \left[ g \left( \sum_{m=0}^{\infty} \lambda^m u_m \right) \right] , \text{ for } n = 0, 1, 2, \cdots$$

(11)

The first few Adomian polynomials are:

$$A_0 = G(u_0),$$

$$A_1 = u_1 G'(u_0),$$

$$A_2 = u_2 G'(u_0) + \frac{1}{2!} u_t^2 G''(u_0),$$

$$A_3 = u_3 G'(u_0) + u_1 u_2 G''(u_0) + \frac{1}{3!} u_t^3 G'''(u_0).$$

(12)

The decomposition method consists to identifying:

$$u_0 = \Phi_0 + L_t^{-1}F(x,t),$$

$$u_n = -L_t^{-1}L_xu_{n-1}(x,t) - L_t^{-1}Ru_{n-1}(x,t) - A_{n-1},$$

(13)

for $n = 1, 2, \cdots$. After calculating $u_0, u_1, u_2, \cdots, u_n$, we obtain the series solution $u$ as

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots.$$  

(14)

2.3. The Padé Approximation. Padé approximation has been applied for rational series solutions in more papers. It is known that Padé approximants show high performance over series approximations. Padé approximants give better numerical results than approximation by the polynomial. The Padé approximate provides some advantage for controlling the convergences of approximation series.

A rational approximation of $f(x)$ on $[a, b]$ is the quotient of two polynomials $R_L(x)$ and $Q_M(x)$ of degrees $L$ and $M$, respectively. We use the notation $[L/M]_f(x)$ or $P^{[L/M]}_x[f]$ to denote the quotient:

$$\frac{R_L(x)}{Q_M(x)}, x \in [a, b].$$

(15)

The idea of choosing the $R_L(x)$ and $Q_M(x)$ is to make the maximum error $\max_x |f(x) - P^{[L/M]}_x[f]|$ as small as possible. For a given amount of computational effort, one can usually construct a rational approximation that has a smaller overall error on $[a, b]$ than a polynomial approximation. For sufficiently smooth function $f$, we can properly choose the polynomials $R_L$ and $Q_M$ such that the algebraic accuracy degree of the Padé approximation is $L + M$.

Definition 5. Let $f$ be function of two variables $x$ and $t$. We defined two-dimensional Padé approximation $P^{[L/M]}_{x}[f](x,t)$ of the function $f$ as

$$P^{[L/M]}_{x}[f](x,t) = P^{[L/M]}_{x}[f^\lambda(t)][x](x,t),$$

(16)

where $P^{[L/M]}_{x}[f](x,t)$ denote the $[L,M]$-order Padé approximation of $f(x,t)$ with respect to the variable $t$, and $P^{[L/M]}_{x}[f^\lambda(t)][x](x,t)$ denote the $[L,M]$-order Padé approximation of $f(x,t)$ with respect to the variable $x$.

If $M = L$, we will denote the diagonal Padé approximation of order $M$ by $P^{[M,M]}_{x}[f](x,t)$ and called $[M,M]$-order Padé approximation or $M$-Padé approximation of $f(x,t)$.

3. SADM and PSADM for Solving Nonlinear PDEs

In this section, we take the following PDE problem as an example to show the Sumudu Adomian decomposition method (SADM) and the Padé Sumudu Adomian decomposition method (PSADM) for solving nonlinear PDE problems.

$$L_tu(x,t) + L_xu(x,t) + R(u(x,t)) + G(u(x,t)) = F(x,t),$$

(17)
with the initial condition $u(x, 0) = h(x)$, where $L_1$ is the first order differential in $t$, $L_2$ is the highest order differential in $x$, $R$ contains the remaining linear terms of lower order derivative terms, $G(u(x, y))$ is the nonlinear term, and $F(x, t)$ is the inhomogeneous or forcing term.

We write the procedures of SADM and PSADM for solving (17) as follows.

**Step 1.** Take the Sumudu transform to equation (17) to get

$$S_t[u(x, t)](v) = h(x) + vS_t[-L_2 u(x, t) - R(u(x, t)) - G(u(x, t)) + F(x, t)](v).$$

**Step 2.** Apply the inverse of the Sumudu transform to the above equation to obtain

$$u(x, t) = S_v^{-1}[h(x)](t) + S_v^{-1}[vS_t[-L_2 u(x, t) - R(u(x, t)) - G(u(x, t)) + F(x, t)](v)](t) \cdots$$

**Step 3.** Use Adomian decomposition method to decompose the nonlinear function $G(u)$ and the $u$, respectively, as

$$G(u) = \sum_{n=0}^{\infty} A_n,$$

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t).$$

**Step 4.** Write the equation in the form

$$\sum_{n=0}^{\infty} u_n(x, t) = S_v^{-1}[h(x)](t) + \sum_{n=0}^{\infty} S_v^{-1}[v][-L_2 u_n(x, t) - R(u_n(x, t)) - G(u_n(x, t)) + F(x, t) - A_n](v)](t),$$

which lead to the SADM recursive relations:

$$u_0(x, t) = S_v^{-1}[h(x)](t),$$

$$u_k(x, t) = S_v^{-1}[v][-L_2 u_{k-1}(x, t) - R(u_{k-1}(x, t)) + F(x, t) - A_{k-1}](v)](t).$$

**Step 5.** Deduct the SADM approximation solution $u_{SADM} = u(x, t, j)$:

$$u(x, t, j) = u_0 + u_1 + \cdots + u_j.$$

**Step 6.** The $[L, M]$-order PSADM solution $u_{PSADM} = u(x, t, j, [L, M])$ is given by

$$u(x, t, j, [L, M]) = P_{[L, M]}[u_{SADM}](x, t),$$

if $L = M$, we denote $M$-PSADM solution by

$$u(x, t, j, M) = P_{[M/M]}[u_{SADM}](x, t).$$

### 4. SADM and PSADM for Solving General Nonlinear Schrödinger Equation

Like Newton’s law in classical physics, the Schrödinger equation is the fundamental equation of quantum physics. It is used to describe the various problems in quantum optics, chemistry, atomic physics, biology, plasma physics, and recently in finance.

To illustrate the basic idea of this method, we consider a general nonlinear Schrödinger equation with the initial condition of the form:

$$i \frac{\partial u}{\partial t} + \Delta u + g(u) = 0,$$

$$u(x, 0) = h(x) \in L^2,$$

where $g$ represents the general nonlinear term and $h$ is a given analytical function.

Using the differentiation property of the Sumudu transform and above initial condition, we have

$$S_t\left[i \frac{\partial u}{\partial t}(v) + S_t[\Delta u + g(u)](v)\right] = 0,$$

$$S_t\left[i \frac{\partial u}{\partial t}(v) = S_t[\Delta u + g(u)](v),$$

$$S_t[u](v) - u(x, 0) = iS_t[\Delta u + g(u)](v),$$

$$S_t[u](v) = h(x) + ivS_t[\Delta u + g(u)](v),$$

$$u = S_v^{-1}[h(x)](t) + S_v^{-1}[ivS_t[\Delta u + g(u)](v)](t).$$

Let us represent the solution $u$ as the infinite series

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t),$$

and the nonlinear term $g$ as

$$g[u(x, t)] = \sum_{n=0}^{\infty} A_n,$$

where $A_n$ are Adomian polynomials of $u_0, u_1, u_2, \cdots, u_n$ which is calculated by:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ g \left( \sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0}, n = 0, 1, 2, \cdots.$$
The general formula can be simplified. By using (30), for \( n = 0, 1, 2, 3, 4 \cdots \) Adomian polynomials \( A_n \) are given by

\[
A_0 = g(u_0), \\
A_1 = u_1 g'(u_0), \\
A_2 = u_2 g'(u_0) + \frac{1}{2!} u_1^2 g''(u_0), \\
A_3 = u_3 g'(u_0) + u_1 u_2 g''(u_0) + \frac{1}{3!} u_1^3 g'''(u_0), \\
A_4 = u_4 g'(u_0) + \left( \frac{1}{2!} u_1^2 + u_1 u_3 \right) g''(u_0) + \frac{1}{4!} u_1^4 g^{(4)}(u_0) + \frac{1}{4!} u_2^2 u_3 g''(u_0) + \frac{1}{4!} u_1^4 g^{(4)}(u_0). \\
\]

(32)

Other polynomials can be generated in a similar manner. Then, we have the following algorithm for solving (26) and (27).

\[
\sum_{n=0}^{\infty} u_n(x, t) = S_v^{-1}[h(x)](t) + S_v^{-1} \left\{ \sum_{n=0}^{\infty} \Delta u_n(x, t) \right\} \left( v \right) + \sum_{n=0}^{\infty} A_n \left( v \right) \right\}(t). \\
\]

(33)

On comparing both sides, we let

\[
u_0(x, t) = S_v^{-1}[h(x)](t), \\
u_1(x, t) = S_v^{-1} [ivS_v \Delta u_0(x, t) + A_0](v)(t), \\
u_2(x, t) = S_v^{-1} [ivS_v \Delta u_1(x, t) + A_1](v)(t), \\
\vdots \\
u_n(x, t) = S_v^{-1} [ivS_v \Delta u_{n-1}(x, t) + A_n](v)(t), \quad n \geq 0.
\]

(34)

After calculating the value of \( u_0, u_1, u_2, \cdots, u_n, \cdots \), we obtain the series solution \( u \) as

\[
u(x, t) = u_0 + u_1 + u_2 + u_3 + \cdots.
\]

(35)

Denote by \( u_{SADM} = u(x, t, j) \), the Sumudu Adomian Decomposition Method solution is obtain by using \( u_0, u_1, u_2, u_3, u_4, \cdots, u_j \)

\[
u(x, t, j) = u_0 + u_1 + u_2 + u_3 + u_4 + \cdots + u_j.
\]

(36)

Then, the \([M, M]\)-order Padé Sumudu Adomian Decomposition Method solution is given by

\[
u(x, t, j, M) = P[M, M]u_{SADM}(x, t).
\]

(37)

4.1. Application 1. Consider the Schrödinger equation of the form:

\[
\frac{\partial u}{\partial t} + i\Delta u + (k|u|^\alpha + w(x))u = 0,
\]

(38)

\[
u(x, 0) = h(x),
\]

(39)

\( k \) is real numbers, \( \alpha = 2 \), \( w(x) \) is function of \( x \), and \( h(x) \) is analytical function.

Apply differentiation property of the Sumudu transform, we obtain:

\[
S_v \left[ \frac{\partial u}{\partial t} \right] (v) = S_v [(k |u|^\alpha + w(x))u](v),
\]

\[
v \frac{\partial u}{\partial t} (v) = -S_v [(k |u|^\alpha + w(x))u](v),
\]

\[
S_v |u|(v) - u(x, 0) = S_v [(k |u|^\alpha + w(x))u](v),
\]

\[
v = S_v^{-1}[h(x)](t) + S_v^{-1} \left\{ ivS_v \Delta u + (k |u|^\alpha + w(x))u \right\}(v),
\]

\[
u = S_v^{-1}[h(x)](t) + S_v^{-1} \left\{ ivS_v \Delta u + uw(x) + k |u|^\alpha u \right\}(v).
\]

(40)

Let us represent the solution as an infinite series given below

\[
u(x, t) = \sum_{n=0}^{\infty} u_n(x, t).
\]

(41)

And the nonlinear term can be decomposed as:

\[
g|u(x, t)| = |u|^\alpha u = \sum_{n=0}^{\infty} A_n,
\]

(42)

For given \( \alpha = 2 \)

\[
g|u(x, t)| = |u|^2 u = \sum_{n=0}^{\infty} A_n,
\]

(43)

where \( A_n, n \geq 0 \), is given by:

\[
A_0 = u_0^2 \bar{u}_0, \\
A_1 = 2u_1 u_0 \bar{u}_0 + \bar{u}_1 u_0^2, \\
A_2 = 2u_0 u_2 \bar{u}_0 + \bar{u}_1 u_0 + 2u_0 u_1 \bar{u}_1 + u_0^2 \bar{u}_2, \\
A_3 = 2u_0 u_3 \bar{u}_0 + 2u_0 u_2 \bar{u}_1 + 2u_0 u_1 \bar{u}_2 + 2u_0 \bar{u}_1 u_2 + \bar{u}_1^2 \bar{u}_3.
\]

(44)

Case 1. \( w(x) = 0, k = 2 \), and \( h(x) = e^{ix} \).
Equation (38) becomes:

\[ \frac{\partial u}{\partial t} + \Delta u + 2|u|^2 u = 0, \quad u(x, 0) = e^{ix}. \]  

(45)

From (42) and (43), we obtain:

\[ \sum_{n=0}^{\infty} u_n(x, t) = e^{ix} + S_A^{-1} \left\{ \sum_{n=0}^{\infty} S_i [\Delta u_n + 2A_n](v) \right\}(t). \]  

(46)

Then, we deduce the recursive relations:

\[ u_0(x, t) = e^{ix}, \quad u_n(x, t) = S_A^{-1} [ivS_i [\Delta u_{n-1} + 2A_{n-1}](v)](t), \quad n \geq 1. \]  

(47)

Then, we obtain:

\[ A_0 = e^{ix}, \quad A_1 = ite^{ix}, \quad A_2 = -\frac{t^2}{2} e^{ix}, \quad A_3 = -\frac{t^3}{3!} e^{ix}. \]  

(48)

Accordingly, the series solution is given by

\[ u(x, t) = e^{ix} \left( 1 + it + \frac{1}{2!} (it)^2 + \frac{1}{3!} (it)^3 + \ldots \right). \]  

(49)

The algorithm is coded by the symbolic computation software Mathematica.

We know \( u(x, t) = e^{ix(t+1)} \) is the exact solution for the problem.

Figures 1(a), 1(c), and 1(e) show, respectively, the real part of Sumudu-Adomian-Decomposition Method solution \( u_{SADM} = u(x, t, 15) \), real part of Padé Sumudu-Adomian-Decomposition Method solution \( u_{PSADM} = u(x, t, 15, 7) \), and real part of exact solution \( u(x, t) \) in domain \( D = [0, 2] \times [0, 2] \).

Figures 1(b), 1(d), and 1(f) show, respectively, the imaginary part of Sumudu-Adomian-Decomposition Method solution \( u_{SADM} = u(x, t, 15) \), imaginary part of Padé Sumudu-Adomian-Decomposition Method solution \( u_{PSADM} = u(x, t, 15, 7) \), and imaginary part of exact solution \( u(x, t) \) in domain \( D = [0, 2] \times [0, 2] \).

Equation (38) becomes:

\[ \frac{\partial u}{\partial t} + \Delta u + 6|u|^2 u = 0, \quad u(x, 0) = e^{ix}, \]  

(50)

then

\[ |u|^2 u = \sum_{n=0}^{\infty} A_n, \]

\[ u = S_i^{-1} \left\{ e^{ix} \right\}(t) + S_i^{-1} \left\{ ivS_i [\Delta u + 6|u|^2 u](v) \right\}(t), \]

\[ |u|^2 u = \sum_{n=0}^{\infty} A_n, \]

\[ u = e^{ix} + \sum_{n=0}^{\infty} S_i^{-1} \left\{ ivS_i [\Delta u + 6|u|^2 u](v) \right\}(t), \]

\[ u_0(x, t) = e^{ix}, \quad u_1(x, t) = S_i^{-1} \left\{ ivS_i [\Delta u_0 + 6A_0](v) \right\}(t), \]

\[ u_2(x, t) = S_i^{-1} \left\{ ivS_i [\Delta u_1 + 6A_1](v) \right\}(t), \]

\[ u_n(x, t) = S_i^{-1} \left\{ ivS_i [\Delta u_{n-1} + 6A_{n-1}](v) \right\}(t), \quad n \geq 0, \]

\[ A_0(u) = e^{ix}, \quad A_1 = -3it e^{ix}, \quad A_2 = -\frac{9t^2}{2} e^{ix}, \quad A_3 = \frac{27t^3}{3!} e^{ix}. \]  

(51)
Figure 1: (a–d) SADM and PSADM solutions using 15 terms. (e, f) Exact solutions.
Figure 2: (a–d) SADM and PSADM solutions using 15 terms. (e, f) Exact solutions.
Accordingly, the series solution is given by

\[ u(x, t) = e^{3x} \left( 1 - 3it + \left(\frac{3it}{2!}\right)^2 - \left(\frac{3it}{3!}\right)^3 + \cdots \right). \]  

(52)

The algorithm is coded by the symbolic computation software Mathematica.

We know \( u(x, t) = e^{3i(x-t)} \) is the exact solution of the problem.

Figures 4(a), 4(c), and 4(e) show, respectively, the real part of Sumudu-Adomian-Decomposition Method solution \( u_{SADM} = u(x, t, 20) \), real part of Padé Sumudu-Adomian-Decomposition Method solution \( u_{PSADM} = u(x, t, 20, 7) \), and real part of exact solution \( u(x, t) \) in domain \( D = [0, 2] \times [0, 2] \).

Figures 4(b), 4(d), and 4(f) show, respectively, the imaginary part of Sumudu-Adomian-Decomposition Method solution \( u_{SADM} = u(x, t, 20) \), imaginary part of Padé Sumudu-Adomian-Decomposition Method solution \( u_{PSADM} = u(x, t, 20, 7) \), and imaginary part of exact solution \( u(x, t) \) in domain \( D = [0, 2] \times [0, 2] \).

Figures 6(a) and 6(b) show, respectively, the absolute error for real part and imaginary part of the Padé Sumudu Adomian Decomposition solution \( u(x, t, 20, 8) \).

Figures 6(c) and 6(d) show, respectively, the absolute error for real part and imaginary part of the Padé Sumudu Adomian Decomposition solution \( u(x, t, 20, 7) \).

Figures 6(e) and 6(f) show, respectively, the absolute error for real part and imaginary part of the Sumudu Adomian Decomposition solution \( u(x, t, 20) \).

We can see in this case that the \([7, 7]\)-order PSADM solution is better than SADM solution in domain \( D = [0, 5] \).
Figure 4: (a–d) SADM and PSADM solutions using 20 terms. (e, f) Exact solutions.
Figure 5: (a–d) SADM and PSADM solutions using 20 terms. (e, f) Exact solutions.
Figure 6: Absolute errors.
Figure 7: (a–d) SADM and PSADM solutions using 15 terms. (e, f) Exact solutions.
Figure 8: (a–d) SADM and PSADM solutions using 20 terms. (e, f) Exact solutions.
× [0, 5], and both methods give almost the same result in domain \( D = [0, 2] \times [0, 2] \). The graph of absolute error shows the PSADM solution can be performed by choosing different value of \( M \).

4.2. Application 2: One-Dimensional Nonlinear Schrödinger Equation with Harmonic Oscillator. The nonlinear Schrödinger equation with harmonic oscillator described by \( u \) with identical initial condition can be expressed as

\[
\frac{\partial u}{\partial t} - \frac{i}{2m} \Delta u + \frac{i}{2} k x^2 u + i |u|^2 u = 0,
\]

(53)

\[
u(x, 0) = e^{ix},
\]

(54)

where \( u \) is the wave function, \( m \) is the mass of the particle, \( i \) is the imaginary unit to describe motion, and \( k \) is the spring constant.

In this section, we solve the One-Dimensional Nonlinear Schrödinger Equation with Harmonic Oscillator using Padé Sumudu decomposition.

First applying Sumudu transform to both sides of equation (42) as follows:

\[
S_t \left[ \frac{\partial u}{\partial t} \right] (v) - i S_t \left[ \frac{1}{2m} \Delta u \right] (v) + i S_t \left[ \frac{1}{2} k x^2 u \right] (v) + i S_v \left[ |u|^2 u \right] (v) = 0.
\]

(55)

From the properties of Sumudu transform of the first derivative and substituting the initial conditions (54) and (55) becomes

\[
S_t [u](v) - u(x, 0) - i S_t \left[ \frac{1}{2m} \Delta u \right] (v) + i S_t \left[ \frac{1}{2} k x^2 u \right] (v) + i S_v \left[ |u|^2 u \right] (v) = 0.
\]

(56)
By applying inverse Sumudu transform and substituting initial condition, we obtain

\[ u = S_v^{-1} \left[ e^{ix} \right] (t) + S_v^{-1} \left[ \frac{1}{2m} \Delta u \right] (v) \left( t \right) \]
\[ - S_v^{-1} \left\{ ivS_t \left[ \frac{1}{2} kx^2 u - |u|^2 u \right] \right\} (v) \] \hspace{1cm} (57)

Applying Adomian decomposition, we get

\[ \sum_{n=0}^{\infty} u_n(x, t) = S_v^{-1} \left[ e^{ix} \right] (t) + S_v^{-1} \left[ \frac{1}{2m} \Delta \sum_{n=0}^{\infty} u_n \right] (v) \left( t \right) \]
\[ - S_v^{-1} \left\{ ivS_t \left[ \frac{1}{2} kx^2 \sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} A_n \right] \right\} (v) \] \hspace{1cm} (58)

where \( A_n \) are the Adomian Polynomials of \((u_0, u_1, u_2, \cdots, u_n)\) to replace the nonlinear term \( |u|^2 u = u^2 \bar{u} \) by \( \sum_{n=0}^{\infty} A_n \) and \( \bar{u} \) is the conjugate of \( u \). Then, we obtain

\[ u_0(x, t) = S_v^{-1} \left[ e^{ix} \right] (t), \]
\[ u_1(x, t) = S_v^{-1} \left[ ivS_t \left[ \frac{1}{2m} \Delta u_0 \right] \right] (v) \left( t \right) \]
\[ - S_v^{-1} \left\{ ivS_t \left[ \frac{1}{2} kx^2 u_0 - A_0 \right] \right\} (v) \] \hspace{1cm} (59)
\[ \vdots \]
\[ u_n(x, t) = S_v^{-1} \left[ ivS_t \left[ \frac{1}{2m} \Delta u_{n-1} \right] \right] \]
\[ - S_v^{-1} \left\{ ivS_t \left[ \frac{1}{2} kx^2 u_{n-1} - A_{n-1} \right] \right\} (v) \] \hspace{1cm} (59)

Figure 10: SADM and PSADM solutions using 3 terms.
Now, we obtain

\[
\begin{align*}
\mathbf{u}_0(x,t) &= e^{it}, \\
\mathbf{u}_1(x,t) &= -ite^{it} - \frac{ikx^2te^{it}}{2}, \\
\mathbf{u}_2(x,t) &= -\frac{t^2}{2} \left( \frac{e^{it}}{2m} - \frac{kx^2e^{it}}{2m} - \frac{e^{it}}{2m} - \frac{kx^2e^{it}}{2m} \right) \\
\ &- \frac{k^2x^4e^{it}}{4m}, \\
\mathbf{u}_3(x,t) &= \frac{t^3}{3!} \left( \frac{3ie^{it}}{4m^2} + \frac{3ikx^2e^{it}}{8m^2} + \frac{idte^{it}}{8m} + \frac{3kxe^{it}}{2m} - \frac{7ike^{it}}{4m^2} - \frac{3ike^{it}}{2m} + \frac{3ikxe^{it}}{2m} + \frac{ik^2x^2e^{it}}{8} \\
&+ \frac{3ik^2x^4e^{it}}{8m} + \frac{idte^{it}}{8m} + \frac{3kxe^{it}}{2m} + \frac{3ikx^2e^{it}}{2m} + \frac{ik^2x^2e^{it}}{8m} \right). \\
\end{align*}
\]  

(60)

The solution is given by

\[
\begin{align*}
\mathbf{u}(x,t) &= e^{it} \left( 1 - \frac{it}{2m} - \frac{ikx^2t}{2} - \frac{e^{it}}{2m} + \frac{ke^{it}}{2m} - \frac{t^2}{2m} - \frac{1}{4m^2} + \frac{k}{2m} + \frac{ikx^2}{2m} - \frac{k^2x^4}{4} - kx^2 - 1 \right) \\
&+ \frac{t^3}{3!} \left( \frac{3i}{4m^2} + \frac{3ik^2x^4}{8m^2} + \frac{i}{8m} + \frac{3kx^2}{2m} - \frac{7ik}{4m^2} - \frac{3ik}{2m} + \frac{3ik^2x^4}{2m} + \frac{ik^3x^4}{4} + i \\
&+ \frac{3kx^2}{m} - \frac{7ik^2x^2}{4m} - \frac{3ik^2x^4}{4m} \right). \\
\end{align*}
\]  

(61)

For more simplicity, we will find the numerical approximation given \(k = 0\) and \(k = 1\).
Case 1. $k = 0$.

\begin{align*}
  u_0 &= e^{ix}, \\
  u_1 &= \left( -\frac{it}{2m} - it \right) e^{ix}, \\
  u_2 &= \frac{1}{2!} \left( -\frac{it}{2m} - it \right)^2 e^{ix}, \\
  u_3 &= \frac{1}{3!} \left( -\frac{it}{2m} - it \right)^3 e^{ix}, \\
  u_4 &= \frac{1}{4!} \left( -\frac{it}{2m} - it \right)^4 e^{ix}, \\
  u_5 &= \frac{1}{5!} \left( -\frac{it}{2m} - it \right)^5 e^{ix}.
\end{align*}

For a given $m = 1$, we know the exact solution for the problem is $u(x, t) = e^{ix-\lambda^2/2t}$.

Figures 7(a), 7(c), and 7(e) show, respectively, the real part of Sumudu-Adomian-Decomposition Method solution $u_{SADM} = u(x, t, 15)$, real part of Padé Sumudu-Adomian-Decomposition Method solution $u_{PSADM} = u(x, t, 15, 7)$, and real part of exact solution $u(x, t)$ in domain $D = [0, 2] \times [0, 2]$. Figures 7(b), 7(d), and 7(f) show, respectively, the imaginary part of Sumudu-Adomian-Decomposition Method solution $u_{SADM} = u(x, t, 15)$, imaginary part of Padé Sumudu-Adomian-Decomposition Method solution $u_{PSADM} = u(x, t, 15, 7)$, and imaginary part of exact solution $u(x, t)$ in domain $D = [0, 2] \times [0, 2]$.

Figures 8(a), 8(c), and 8(e) show, respectively, the real part of Sumudu-Adomian-Decomposition Method solution $u_{SADM} = u(x, t, 15)$, real part of Padé Sumudu-Adomian-Decomposition Method solution $u_{PSADM} = u(x, t, 15, 7)$, and real part of exact solution $u(x, t)$ in domain $D = [0, 10] \times [0, 10]$.

Figures 8(b), 8(d), and 8(f) show, respectively, the imaginary part of Sumudu-Adomian-Decomposition Method solution $u_{SADM} = u(x, t, 15)$, imaginary part of Padé Sumudu-Adomian-Decomposition Method solution $u_{PSADM} = u(x, t, 15, 7)$, and imaginary part of exact solution $u(x, t)$ in domain $D = [0, 10] \times [0, 10]$.

Figures 9(a) and 9(b) show, respectively, the absolute error for real part and imaginary part of the Sumudu Adomian Decomposition solution $u(x, t, 15)$. Figures 9(c) and 9(d) show, respectively, the absolute error for real part and imaginary part of the Padé Sumudu Adomian Decomposition solution $u(x, t, 15, 7)$.

We can see in this case that the $[7, 7]$-order PSADM solution is better than SADM solution in domain $D = [0, 10] \times [0, 1]$; both methods give almost the same result in $D = [0, 2] \times [0, 2]$. The graph of absolute error shows the high accuracy of PSADM compared to SADM.

Case 2. $k = 1$. In computation, we use 6-order Padé approximation and compute only 3 terms. Figures 10(a) and 10(c) show, respectively, the real part of the SADM solution and the real part of the PSADM in domain $D = [-2, 2] \times [0, 1]$.

Figures 10(b) and 10(d) show the imaginary part of the solution, respectively, using Sumudu Adomian decomposition method and Padé Sumudu Adomian decomposition method in domain $D = [-2, 2] \times [0, 1]$.

Figures 11(a) and 11(c) show the real part of the solution using, respectively, Sumudu Adomian decomposition method and Padé Sumudu Adomian decomposition method in domain $D = [-10, 10] \times [0, 1]$.

Figures 11(b) and 11(d) show where the imaginary part of the solution using, respectively, Sumudu Adomian decomposition method and Padé Sumudu Adomian decomposition method in domain $D = [-10, 10] \times [0, 1]$

5. Conclusion

This work presented the application of Padé Sumudu Adomian Decomposition method as a technique with high potential to solve nonlinear Schrödinger equations. The Padé Sumudu Adomian decomposition provides an ingenious avenue for controlling the convergence of approximation series. The Padé Sumudu Adomian decomposition method (PSADM) for solving nonlinear Schrödinger equation shown is computationally efficient in our applications. It is worth mentioning that the result is important to researchers in the field of quantum optics, atomic physics, plasma physics, chemistry, biology, finance, and mechanics. This approach can also be generalized to investigate more complicated nonlinear partial differential equations that can only be solved numerically. It should be noted that:

(i) For different values of $M$, we obtain different solutions; some of them are not better than the SADM solutions

(ii) For some PDE the PSADMs, $u_{PSADM} = P_{[M/M]}[u_{SADM}]$ obtained by using the diagonal Padé approximation are not the best; the PSADMs, $u_{PSADM} = P_{[L/M]}[u_{SADM}]$, for $L=M$ give better approximation than the diagonal Padé approximation

(iii) For $L=0$ and $M=0$, we have polynomial approximation (series expansion)

Data Availability

There are no available data. The Mathematica software is used to plot all the graphs.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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