POLYDIFFERENTIALS AND THE DEFORMATION FUNCTOR OF CURVES WITH AUTOMORPHISMS II

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ABSTRACT. We apply the known results on the Galois module structure of the sheaf of polydifferentials in order to study the dimension of the tangent space of the deformation functor of curves with automorphisms. We are able to find the dimension for the case of weakly ramified covers and for the case of the action of a cyclic group of order $p^v$.

1. INTRODUCTION

Let $X$ be a non-singular complete curve of genus $g \geq 2$ defined over an algebraic closed field $k$ of positive characteristic $p > 3$, and let $G$ be a subgroup of the automorphism group of $X$. In [1] J.Bertin, A. Mézard proved that the equivariant cohomology of Grothendieck $H^1(G, T_X)$ is the tangent space of the global deformation functor of smooth curves with automorphisms. The dimension of the $k$-vector space $H^1(G, T_X)$ is a measure of the directions a curve can be deformed together with a subgroup of the automorphism group. The computation of this dimension turns out to be a difficult problem, and in the literature there are results concerning ordinary curves [5], cyclic groups of order $p$ [11]. In [9] the author attempts to compute this space using the low terms of the Lyndon-Hochschild-Serre spectral sequence. This allows us to compute the dimension of $H^1(G, T_X)$ if $G$ is an elementary abelian group i.e. isomorphic to a direct product of cyclic groups of order $p$. The disadvantage of this method is that it involves the computation of the transfer map that depends on the group structure of the decomposition series of the decomposition groups $G(P)$ at the wild ramification points.

The author in [10] proposed an alternative method of approaching the problem of computing the dimension of $H^1(G, T_X)$. Precisely one can prove that for a $p$-group $G$ and for the sheaf $\Omega_X$ of differentials on $X$ the following equation holds:

$$H^1(G, T_X) = H^0(X, \Omega_X^\otimes 2)_G.$$ 

This observation allows us to compute the desired dimension if we know the Galois module structure of the space of 2-holomorphic differentials. The knowledge of the Galois module structure of the space of holomorphic differentials in positive characteristic is still an open problem and there are only partial results in the case of tame covers [7],[15],[12] and when $G$ is a cyclic group of order equal to the characteristic [14]. The computation of the Galois module structure of holomorphic differentials in the tame case does not offer any new information for the computation of $H^1(G, T_X)$ since the classical methods from the theory of Riemann surfaces can be used in this case.

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In [10] the author was able to recover the result of J.Bertin, A. Mégard concerning the dimension of $H^1(G, T_X)$ in the case $G = \mathbb{Z}/p\mathbb{Z}$ by applying the computation of Nakajima [13] concerning the $\mathbb{Z}/p\mathbb{Z}$-module structure of $\Omega_X^2$.

In this article we use the results of B. Köck [8] on weakly ramified covers, i.e. on covers so that $G_2(P) = \{1\}$ for all wild ramified points, in order to determine $\dim_k H^1(G, T_X)$. Since all ordinary curves are weakly ramified [10] th. 2i] we recover the result of Cornelissen and Kato [5] for deformations of ordinary curves.

Many authors [25, 13, 7, 3, 24], in order to study the $k[G]$-module structure of spaces $\Omega_X(-D)$ considered the $k[G]$-module structures of the spaces of semisimple and nilpotent differentials with respect to the Cartier operator. In section 3 we follow this approach. The representation module corresponding to semisimple differentials is studied and the problem of computing the space of covariant differentials is reduced to the problem of computing nilpotent covariant differentials. More precisely we are able to prove that:

\begin{align}
\dim_k H^1(G, T_X) = 2(g_Y - 1) + \gamma_Y - 1 + r + \dim_k \Omega_X(-D)^p_G,
\end{align}

where $g_Y, \gamma_Y$ denote the genus and the $p$-rank of the Jacobian of $Y = X/G$ and $r$ is the number of points of $Y$ ramified in $X \to X/G$.

In his PhD thesis [2] N. Borne proved an equivariant Riemann-Roch theorem and provided an equivariant Euler characteristic $K_0(G, X) \to R_k(G)$, where $K_0(G, X)$ is the Grothendieck group of $G - \mathcal{O}_X$-modules and $R_k(G)$ is the Brauer Character Group. In the case of $p$-groups in positive characteristic this approach does not give us more information than the classical Riemann-Roch theorem: the group $R_k(G) = \mathbb{Z}$ [21, 15.6] and equivariant information is lost.

N. Borne in [3] proposed a refinement of his equivariant $K$-theory by using the ideas of Auslander in modular representation theory. This new theory enables him to compute the Galois module structure of spaces of global sections of linear systems of curves of the form $\mathcal{O}_X(D)$ where $\deg D \geq 2q_X - 2$ and $G$ is a cyclic group of the form $\mathbb{Z}/p^n\mathbb{Z}$. We apply the results of Borne in order to prove:

**Proposition 1.1.** Assume that the group $G = \mathbb{Z}/p^n\mathbb{Z}$ acts on $X$. Let $X_{ram}$ denote the set of ramification points of the cover $\pi : X \to X/G = Y$. For every point $P \in X_{ram}$ let $N_P$ denote the highest jump in the lower ramification filtration and $f(P)$ the highest jump in the upper ramification filtration. Let $\Delta : \mathbb{Z} \to \mathbb{Z}$ denote the map $\Delta : a \mapsto \lfloor a/p \rfloor$, and $\Delta^k$ the composition of $k$ times the map $\Delta$. Let $g_Y$ be the genus of the quotient curve $Y$ and let $r = \#\pi_*(X_{ram})$.

The dimension of $H^1(\mathbb{Z}/p^n\mathbb{Z}, T_X)$ is given by:

\begin{align}
\dim_k H^1(\mathbb{Z}/p^n\mathbb{Z}, T_X) = 3(g_Y - 1) + 2r + \sum_{P \in \pi_*(X_{ram})} (2f(P) + \Delta^k(-2N_P - 2)) .
\end{align}

This result can be obtained also by using Grothendieck’s equivariant theory. One can compute the the group $H^0(Y, \pi_*^\infty(T_X))$ [1, 9] sec 3.] and the local contributions at each wild ramified point given in [11] prop. 4.1.1].

In deformation theory it is an interesting problem to compare the dimension of the tangent space of the deformation functor to the dimension of the versal deformation ring. In the case of ordinary curves and in the $p$-cyclic case the Krull dimension has been computed [5, 11]. In [17] R. Pries studied unobstructed deformations of curves acted on by Abelian groups under the additional assumption that the deformations do not split the branch locus. A comparison of the result of
Pries to the result of proposition [1.1] gives the dimension of the space generated by obstructed deformations and of deformations that split the branch locus.

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2. SPACES OF 2-HOLOMORPHIC DIFFERENTIALS.

We will denote by $\Omega_X$ the sheaf of differentials of $X$. Let $D$ be an effective divisor on the curve $X$ and let $G$ be a finite subgroup of $\text{Aut}(X)$. The set of meromorphic differentials on $X$ is denoted by $\mathcal{M}_X = \{ xdy : x, y \in K(X) \}$, where $K(X)$ denotes the function field of the curve $X$. We will denote by

$$\Omega_X(D) = \{ \omega \in \mathcal{M}_X : \text{div}(\omega) \geq D \} = H^0(X, \Omega_X \otimes \mathcal{O}_X(-D)).$$

and by

$$\mathcal{L}_X(D) = \{ f \in K(X) : \text{div}(f) \geq -D \} = H^0(X, \mathcal{O}_X(D)).$$

We will denote the $k$-dimension of $\mathcal{L}_X(D)$ by $\ell(D)$. The reader should be careful about the notation here. We follow the notation of Serre, used also in the papers of Stichtenoth, Nakajima, Subrao. In the papers of N. Borne [4], E. Köck [8], E. Kani [7], N. Stalder [24] the notation $\Omega_X(D)$ is used for what we write as $\Omega_X(-D)$.

Lemma 2.1. Let $G$ be a $p$-group and consider the cover $X \to X/G$. We assume that the genus of the curve $X$ is $g_X \geq 2$. There is a $G$-invariant differential $\omega$ in $X$, such that $\text{div}(\omega)$ is effective.

Proof. Let $b_1, \ldots, b_r$ be the ramification points of the cover $\pi : X \to Y = X/G$. In order to find $\omega$ we have to select a meromorphic differential $\phi$ of $Y$ and take the pullback $\pi^*(\phi)$. Then $\pi^*(\phi)$ is $G$-invariant and we have to select it so that $\text{div}(\phi) \geq 0$. Fix a meromorphic differential $\phi_1$ on $Y$ and consider the set $f\phi_1$ where $f$ is an arbitrary element in the function field of $Y$. We have that

$$\text{div}(\pi^*(f\phi_1)) = \pi^*\text{div}(f\phi_1) + R \geq 0,$$

where $R$ is the ramification divisor given by

$$R = \sum_{i=1}^r \sum_{\nu=0}^\infty \sum_{P \in b_i} (e_\nu(P) - 1)P.$$

In order to check that the divisor on the left hand side of (2) is positive we push forward again and arrive at

$$\text{div}(f\phi_1) + \sum_{i=1}^r \sum_{\nu=0}^\infty \frac{e_\nu(b_i) - 1}{e_0(b_i)} \geq 0.$$

The later condition is equivalent to $f \in \mathcal{L}_Y(K + A)$, where $A$ is the effective divisor

$$A := \sum_{i=1}^r \sum_{\nu=0}^\infty \left[ \frac{e_\nu(b_i) - 1}{e_0(b_i)} \right] b_i.$$

Using Riemann-Roch on the curve $Y$ we compute

$$\ell(K + A) = \ell(-A) + 2g_Y - 2 + \deg(A) - g_Y + 1 = g_Y - 1 + \deg(A) \geq 1.$$
$e_1(b_i)$ and if $g_Y \geq 1$ we are done. In the case $g_Y = 0$ we use $g_X \geq 2$ and the Riemann-Hurwitz formula to see that there should be either at least two ramified points so $\deg A \geq 2$, or one ramified point with $\sum_{\nu=0}^{\infty} \frac{e_\nu(b_i)-1}{e_\nu(b_i)} > 2$, and in this case also $\deg A \geq 2$.

Every $f \in \mathcal{L}_Y(K+A)$ gives rise to the desired differential $\omega = \pi^*(f\phi_1)$. Moreover we can select $f \in \mathcal{L}_Y(K+A)$ such that is has polar divisor $A$. This imply that the support of $\pi^*(f\phi_1)$ has no intersection with the branch locus.

\[ \text{Proposition 2.2.} \] Assume that the $p$-group $G$ acts on the group $G$. There is an effective and invariant divisor $D^* = \text{div}(p^*(\phi)) = \pi^*(\text{div}(\phi)) + R \geq 0$ so that the module $\Omega_X(-D^*)$ is isomorphic to $H^0(X,\Omega_X^{\otimes 2})$ as a $k[G]$-module.

\[ \text{Proof.} \] According to lemma 2.1 we can select $\phi$ so that $\pi^*(\phi)$ is $G$-invariant and the divisor $D^* := \text{div}(\pi^*(\phi)) = \pi^*(\text{div}(\phi)) + R$ is an effective $G$-invariant divisor. Every differential can be written as $f\phi$ and every 2-differential is an expression of the form $\omega \otimes \phi$, for an other differential $\omega$ on $X$. The space

\[ H^0(X,\Omega_X^{\otimes 2}) = \{ \omega \otimes \phi : \text{div}(\omega \otimes \phi) \geq 0 \} = \{ \omega \in \Omega_X : \text{div}(\omega) \geq -\text{div}(\phi) \} = \Omega_X(-D^*). \]

\[ \square \]

3. THE WEAKLY RAMIFIED CASE

In this section we assume that the cover $X \to X/G$ is weakly ramified, i.e. for every $P \in S$ we have that $G_i(P) = \{1\}$ for all $i \geq 2$. The group $G$ is always assumed to be a $p$-group. The ramification divisor is computed $R = \sum_{P \in S} 2(e_0(P) - 1)P$.

\[ \text{Lemma 3.1.} \] The module $\Omega_X(-D^* - \sum_{P \in S} 3P)$ is a projective $k[G]$-module.

\[ \text{Proof.} \] Let $K$ be a canonical divisor. Let us write

\[ \Omega_X(-D^* - \sum_{P \in S} 3P) = \mathcal{L}_X(K + D + \sum_{P \in S} 3P). \]

Let $D' := K + D^* + \sum_{P \in S} 3P = \sum n_P P$. We compute that

\[ D' = K + D^* + \sum_{P \in S} 3P = 2\pi^*(\text{div}(\phi)) + \sum_{P \in S} 4(e_0(P) - 1) + 3P. \]

Since we have assumed that $G$ is a $p$-group we have that $e_0(P) = e_1(P) = e^{nu}(P)$. Therefore $n_P \equiv -1 \text{ mod } e_0$ for all wild ramified points of $X \to X/G$. On the other hand

\[ H^1(X,\mathcal{O}_X(D')) = \mathcal{L}_X(K - D') = 0, \]

since $\deg(K - D') < 0$. The desired result follows by [9] th. 2.1].

B. Kôck proposed to me the following more abstract approach: By [2] th. 44 there is a $G$-invariant canonical divisor $K_X = \sum_{P} m_P P$ on $X$. The proof of corollary 2.3 in [9] implies that all $m_P \equiv -2 \text{ mod } e_0^{nu}(P)$. Therefore the divisor, $2K_X + \sum_{P \in S} 3P = \sum n_P P$ has $n_P \equiv 2(-2) + 3 = -1 \text{ mod } e_0^{nu}(P)$ as required. \[ \square \]

We can now form the short exact sequence:

\[ (4) \quad 0 \to \Omega_X(-D^*) \to \Omega_X(-D^* - \sum_{P \in S} 3P) \to \frac{\Omega_X(-D^*)}{\Omega_X(-D^*)} \to 0. \]
The short exact sequence of sheaves
\[ 0 \to \Omega_X \otimes \mathcal{O}_X(-D^*) \to \Omega_X \otimes \mathcal{O}_X(-D^* - \sum_{P \in S} 3P) \to \Sigma \to 0 \]
where
\[ \Sigma := \frac{\Omega_X \otimes \mathcal{O}_X(-D^* - \sum_{P \in S} 3P)}{\Omega_X \otimes \mathcal{O}_X(-D^*)} \]
gives rise to a long exact sequence of \( k \)-vector spaces by applying the functor of global sections. This long exact sequence combined with \( H^1(X, \Omega_X \otimes \mathcal{O}_X(-D^*)) = 0 \) allows us to express
\[ H^0(X, \Sigma) = \frac{\Omega_X(-D^* - \sum_{P \in S} 3P)}{\Omega_X(-D^*)} \]
as the direct sum of the stalks \( \Sigma_P \) of the skyscraper sheaf \( \Sigma \) at points \( P \in S \). Let us denote by \( \Sigma' := H^0(X, \Sigma) \).

Equation (\ref{eq:long_exact}) gives the following long exact sequence:
\[ 0 \to H_1(G, \Sigma') \to \Omega_X(-D^*)_G \to \Omega_X(-D^* - \sum_{P \in S} 3P) \to \Sigma'_G \to 0, \]
since \( \Omega_X(-D^* - \sum_{P \in S} 3P) \) is a \( k[G] \)-projective module. Therefore, the desired dimension can be computed:

(5) \[ \dim_k \Omega_X(-D^*)_G = \dim_k H_1(G, \Sigma') + \dim_k \Omega_X(-D^* - \sum_{P \in S} 3P)_G - \dim_k \Sigma'_G. \]

In what follows we will compute every summand on the right hand side of (5).

**Lemma 3.2.** \[ \dim_k \Omega_X(-D^* - \sum_{P \in S} 3P)_G = 3(g_Y - 1) + 3r. \]

**Proof.** Using the theorem of Riemann-Roch we compute:

(6) \[ \dim_k \Omega_X(-D^* - \sum_{P \in S} 3P) = 2g_X - 2 + |G|(2g_Y - 2) + \sum_{P \in S} (2(e_0(P) - 1) + 3) - g_X + 1 + \dim_k L_X(-D^* - \sum_{P \in S} 3P). \]

But \( \deg(-D^* - \sum_{P \in S} 3P) < 0 \) therefore \( \dim_k L_X(-D^* - \sum_{P \in S} 3P) = 0. \) Riemann-Hurwitz implies that

(7) \[ 2g_X - 2 = |G|(2g_Y - 2) + 2 \sum_{P \in S} (e_0(P) - 1). \]

By combining (6), (7) we obtain:

\[ \dim_k \Omega_X(-D^* - \sum_{P \in S} 3P) = 3|G|(g_Y - 1) + 3r|G|. \]

Since \( \Omega_X(-D^* - \sum_{P \in S} 3P) \) is projective it is of the form \( k[G]^a \), where \( a = 3(g_Y - 1) + 3r \), and each \( k[G] \) direct summand contributes 1 to the dimension of \( \Omega_X(-D^* - \sum_{P \in S} 3P)_G \). The desired result follows. \( \Box \)

Let us now study the space \( \Sigma' \) as a \( G \)-module: The differential \( \pi^*(\phi) \) defined in lemma \( \ref{lemma:inverse_image} \) is an invariant differential and we have that \( \div(\pi^*(\phi)) = D^* \). Every differential \( \omega \) can be written as \( \omega = f \pi^*(\phi) \), Therefore,

\[ \Omega_X(-D^*) = \{ \div f \pi^*(\phi) > -D^* \} \cong \mathcal{O}_X(2D^*), \]
and
\[ \Omega_X(-D^* - \sum_{P \in S} 3P) \cong O_X(2D^* + \sum_{P \in S} 3P), \]
where the last two isomorphisms are isomorphisms of \( k[G] \)-modules.

Let \( O(P) = \{ gP : g \in G \} \) denote the orbit of \( P \) under the action of the group \( G \). The set \( S \) of ramification points can be written as a disjoint union of orbits of points of \( X \):
\[ S = \bigcup_{j=1}^r O(P_j), \]
for a selection \( P_1, \ldots, P_r \) of points of \( X \). We can write \( H^0(X, \Sigma) \) as
\[ (8) \quad \Sigma' = \bigoplus_{j=1}^r \bigoplus_{P \in O(P_j)} \Sigma_P, \]
\( \Sigma_P \) is the stalk of \( \Sigma \) at \( P \). Thus, (8) can be written as
\[ (9) \quad \Sigma' = \bigoplus_{j=1}^r \text{Ind}^G_{G(P_j)} \Sigma_{P_j}. \]

Shapiro lemma [26 6.3.2, p.171] implies that
\[ (9) \quad H_\ast(G, \Sigma') = \bigoplus_{j=1}^r H_\ast(G, \text{Ind}^G_{G(P_j)} \Sigma_{P_j}) = \bigoplus_{j=1}^r H_\ast(G(P_j), \Sigma_{P_j}). \]

Let \( P \) be a ramified point. We have assumed that the cover is weakly ramified we have that \( G_2(P) = \{1\} \). This implies that for a local uniformizer \( t_P \) at \( P \) we have
\[ (10) \quad G(P) \ni g : t_P \mapsto t_P \left( 1 + \alpha_1(g)t_P + \sum_{\nu=2}^\infty \alpha_\nu(g)t_P^\nu \right), \]
where \( \alpha_1(g) \neq 0 \). The map \( g \mapsto \alpha_1(g) \) is a homomorphism and allows us to see \( G(P) \) as a finite dimensional \( \mathbb{F}_p \)-vector subspace of \( k \).

The quotient \( \Sigma_P \) is then generated as a \( k \)-vector space by elements
\[ \Sigma_P = \left\{ \omega_1 := \frac{1}{t_P^{4e(P)-1}}, \omega_2 := \frac{1}{t_P^{4e(P)-2}}, \omega_3 := \frac{1}{t_P^{4e(P)-3}} \right\} \]
and the action is given by:
\[ (11) \quad \sigma : \frac{1}{t_P^{4e(P)-1}} \mapsto \frac{1 + \alpha_1(g)t_P + \alpha_2(g)t_P^2}{t_P^{4e(P)-1}}, \quad \frac{1}{t_P^{4e(P)-2}} \mapsto \frac{1 + \alpha_1(g)t_P}{t_P^{4e(P)-2}} + \frac{\alpha_2(g)t_P}{t_P^{4e(P)-2}} + \frac{1}{t_P^{4e(P)-3}}, \]
since \((-4e(P)+1) = 1 \) and \((-4e(P)+1) = 0 \).
\[ (12) \quad \sigma : \frac{1}{t_P^{4e(P)-2}} \mapsto \frac{1 + 2\alpha_1(g)t_P}{t_P^{4e(P)-2}} = \frac{1}{t_P^{4e(P)-2}} + 2\alpha_1(g) \frac{1}{t_P^{4e(P)-3}} \]
and
\[ (13) \quad \sigma : \frac{1}{t_P^{4e(P)-3}} \mapsto \frac{1}{t_P^{4e(P)-3}}. \]

**Lemma 3.3.** We have that \( \dim_k \Sigma'_G = H_0(G, \Sigma') = r \).
Proof. Let \( P \) be one of the \( \{P_1, \ldots, P_r\} \). Let \( \omega \in \Sigma_P \). Using equations \([11],[12],[13]\) we observe that there is only one invariant element in \( \Sigma_P \), therefore the images of the linear maps \( (g - 1) \omega \) are all isomorphic and two dimensional generated by the elements \( \omega_2, \omega_3 \) and so we arrive at:

\[
\dim_k H_0(G(P), \Sigma_P) = \dim_k \frac{\Sigma_P}{g\omega - \omega} = 1.
\]

For the global computation we use \([9]\) in order to obtain:

\[
\dim_k \Sigma'_G = H_0(G, \Sigma') = r.
\]

\[\square\]

**Lemma 3.4.** We have that \( \dim_k H_1(G, \Sigma') = \sum_{j=1}^r \log_p |G(P_j)| - r \)

Proof. Let \( P \) be one of the \( \{P_1, \ldots, P_r\} \). We will use the normalised bar resolution defined in \([26, 6.5, \text{p. } 177]\) in order to compute \( H_1(G, \Sigma_P) \). Recall that \( k[G(P)] \cong B_0 \) and \( B_0 \) is generated by the symbol \([\_]\), \( B_1 \) is the free \( k[G(P)]\)-module on the set of symbols \( \{[g] : g \in G(P)\setminus\{1\}\} \), and \( B_2 \) is the free \( k[G(P)]\)-module on the set of symbols \( \{[g,h] : g, h \in G(P)\setminus\{1\}\} \). For the differential maps we have \( \partial_2 : B_1 \to B_0 \) and \( \partial_2[g] = (g - 1)[\_] \), and \( \partial_2 : B_2 \to B_1, \partial_2[g,h] = [gh] + [g] \). Higher \( B_n \) can be similarly defined but we don’t need them here. The group \( H_1(G, \Sigma_P) \) is defined by the homology at position 1 of the chain complex \( \Sigma_P \otimes B_+ \). We have:

\[
H_1(G, \Sigma_P) = \frac{\ker(\partial_1 : \Sigma_P \otimes B_1 \to \Sigma_P \otimes B_0)}{\text{im}(\partial_2 : \Sigma_P \otimes B_2 \to \Sigma_P \otimes B_1)}.
\]

We will focus first on the study of the local components \( \Sigma_P \). Let \( \omega_3 = \frac{1}{t_p^{p(r-1)}} \), \( \omega_2 = \frac{1}{t_p^{r-1}} \), \( e_1 = \frac{1}{t_p^{r-1-2}} \). Observe that the space generated by \( \omega_3 \otimes [g] \) is in \( \ker \partial_1 \), since

\[
\partial_1(\omega_3 \otimes [g]) = \omega_3(g - 1) \otimes [\_] = 0.
\]

On the other hand

\[
\partial_2(\omega_2 \otimes [g,h]) = \omega_2 \otimes (g[h] - [gh] + [g]) = \omega_2 g \otimes [h] - \omega_2 \otimes [gh] + \omega_2 \otimes [g] = \\
= \alpha_1(g)\omega_3 \otimes [h] + \omega_2 \otimes (g[h] - [gh] + [g]).
\]

Since \( \omega_2, \omega_3 \) are linear independent and \( \alpha_1(g) \neq 0 \), we obtain that all elements of the form \( \omega_3 \otimes [h], \ h \in G(P) \) are 0 in the homology group \( H_1(G(P), \Sigma_P) \).

Observe that an element \( \sum_{g \in G(P)} \lambda_g \omega_2 \otimes [g] \) is in the kernel of \( \partial_1 \) if and only if \( \sum_{g \in G(P)} \lambda_g \alpha_1(g) = 0 \), since we have assumed that \( p \neq 2 \). On the other hand

\[
\partial_2(\omega_1 \otimes [g,h]) = \omega_1 \otimes (g[h] - [gh] + [g]) = \omega_2 \alpha_1(g) \otimes [h] + \omega_1([h] - [gh] + [g]),
\]

therefore \( \omega_2 \otimes [h] \) is zero in the homology group \( H_1(G, \Sigma_P) \).

Consider an element

\[
\omega = \sum_{g \in G(P)} \lambda_g \omega_1 \otimes [g] + \sum_{g \in G(P)} \mu_g \omega_2 \otimes [g] + \sum_{g \in G(P)} \nu_g \omega_3 \otimes [g],
\]

and the image \( \partial_1 \omega \) given by

\[
\partial_1(\omega) = \sum_{g \in G(P)} \alpha_1(g)\lambda_g \omega_2 \otimes [\_] + \sum_{g \in G(P)} (2\alpha_2(g)\lambda_g + \mu_g \alpha_1(g)) \omega_3 \otimes [\_].
\]

\[7\]
Equation (16) gives two necessary conditions for \( \omega \in \ker \partial_1 \), namely:

\[
\sum_{g \in G(P)} (2\alpha_2(g)\lambda_g + \mu_g\alpha_1(g)) = 0
\]

and

\[
\sum_{g \in G(P)} \alpha_1(g)\lambda_g = 0.
\]

We have seen that \( H_1(G(P), \Sigma_P) \) is generated by the images of elements of the form \( \omega \otimes \{g\} \). On the other hand, equation (15) implies that:

\[
(17) \quad \omega \otimes \{gh\} = \omega \otimes \{g\} \oplus \omega \otimes \{h\} \text{ in } H_1(G(P), \Sigma_P).
\]

The groups \( G(P) \) are elementary Abelian and can be written as \( G(P) = \bigoplus_{t=1}^{t} g_i \mathbb{F}_p \).

Equation (17) implies that \( H_1(G, \Sigma_P) \subset \omega \otimes \langle \{g_i\} : i = 1, \ldots, t \rangle_k \).

For a linear combination \( \sum_{i=1}^{t} \lambda_i \omega \otimes \{g_i\} \) we have that it is in \( \ker \partial_1 \) if and only if \( \sum_{i=1}^{t} \lambda_i = 0 \). Therefore:

\[
\dim_k H_1(G, \Sigma_P) = \log_p |G(P)| - 1.
\]

The global contribution is computed using (15) and equals to

\[
H_1(G, \Sigma') = \sum_{j=1}^{r} \log_p |G(P_j)| - r,
\]

i.e., the desired result.

Observe that the dimension of the tangent space of the deformation functor can now be computed by combining (16) and lemmata 3.2, 3.3, 3.4:

\[
\dim_k H^1(G, \mathcal{T}_X) = 3g_Y - 3 + r + \sum_{j=1}^{r} \log_p |G(P_j)|.
\]

Notice, that if the curve \( X \) is ordinary and \( G = \text{Aut}(X) \) then the cover \( X \to X/G \) is weakly ramified [16, th. 2i] and the above result coincides with the result of G. Cornelissen and F. Kato [17] on deformations of ordinary curves.

4. The \( p \)-rank representation.

Let \( D \) be an effective divisor on a curve \( X \). On the spaces \( \Omega_X(-D) \) one can define the action of the Cartier operator. For an introduction to all necessary material the interested reader may consult [20], [25], [13], [24]. There is the following decomposition of the above space

\[
\Omega_X(-D) = \Omega_X(-D)^s \bigoplus \Omega_X(-D)^n
\]

where \( \Omega_X(-D)^s, \Omega_X(-D)^n \) are the spaces of semisimple and nilpotent differentials with respect to the Cartier operator. The above decomposition is compatible with the \( G \)-action. For the \( k[G] \)-module \( \Omega_X(-D)^n \) of nilpotent elements little seems to be known. On the other hand the \( k[G] \)-module \( V_D := \Omega_X(-D)^s \) was studied by many authors ([13, 7, 3, 24]). The \( k[G] \)-module \( V_D \) is called in the literature the \( p \)-rank representation. In general, we have the following decomposition:

\[
V_D = \text{core}(V_D) \bigoplus \bigoplus_{S \in \text{irr } G} P_G(S)^t(G,D,S).
\]
where $S$ runs over the set of equivalent classes of irreducible representations, $P_G(S)$ denotes the projective cover of $S$, and $b(G, D, S) \in \mathbb{N}$ are called the Borne invariants corresponding to $G, D, S$ [24].

Since $G$ is a $p$-group and $k$ is assumed to be of characteristic $p > 0$ the only irreducible representation is the trivial one and has projective cover $k[G]$ [21, 15.6]. Moreover $\text{core}(V_{D^*}) = 0$ since $D^*$ is non empty and contains all ramification points [13, 24, 4.5]. Therefore, $V_{D^*} = k[G]_{B(G, D^*, k)}$, where $B(G, D^*, k)$ is an integer.

**Proposition 4.1.** With the above notation $B(G, D^*, k) = 2(g_Y - 1) + \gamma_Y - 1 + r$, where $g_Y, \gamma_Y$ are the genus and the $p$-rank of the Jacobian of the curve $Y = X/G$, $r$ is the number of points that are ramified in the cover $X \rightarrow Y = X/G$.

**Proof.** Let us denote by $D^*_{\text{red}}$ the divisor that has the same support with $D^*$, so that for all prime divisors $P$ we have $v_P(D^*_{\text{red}}) > 0 \Rightarrow v_P(D^*_{\text{red}}) = 1$. According to [25, p. 175]

$$\dim_k \Omega_X(-D)^s = \dim_k \Omega_X(-D^*_{\text{red}})^s = \dim_k \Omega_X(0)^s + \deg D^*_{\text{red}} - 1. \tag{18}$$

But the space $\Omega_X(0)^s$ of semisimple regular differentials is of dimension $\gamma_X$, and the degree of $D^*_{\text{red}}$ is equal to $2(g_Y - 1)|G| + r$.

The Deuring-Shafarevich formula ([11], [25], [6], [23], [13]) relates the $p$-ranks $\gamma_X, \gamma_Y$:

$$\gamma_X - 1 = |G| \left( \gamma_Y - 1 + \sum_{i=1}^{r} \left(1 - \frac{1}{e_0(b_i)} \right) \right).$$

This combined with (18) gives us that

$$\dim_k V_{D^*} = \dim_k \Omega_X(-D^*)^s = |G|(\gamma_Y - 1 + 2(g_Y - 2) + r),$$

and since $V_{D^*}$ is projective we have that $V_{D^*} = k[G]^{\gamma_Y - 1 + 2(g_Y - 2) + r}$, and the desired result follows. $\square$

The above computation allows us to compute the dimension of the space of nilpotent elements. Indeed, the dimension of the space $\Omega_X(-D^*)$ is computed to be equal to

$$\Omega_X(-D^*) = (3g_Y - 3)|G| + |G| \sum_{i=1}^{r} \sum_{\nu=0}^{\infty} \frac{e_\nu(b_i) - 1}{e_0(b_i)}.$$

Therefore, we have that

$$\dim_k \Omega(-D^*)^n = |G|(g_Y - \gamma_Y) + |G| \sum_{i=1}^{r} \sum_{\nu=0}^{\infty} \frac{e_\nu(b_i) - 1}{e_0(b_i)} - r.$$  

If the curve $X$ is ordinary i.e. $g_Y = \gamma_Y$ then $e_\nu(b_i) = 0$ for all $\nu \geq 2$ and the above formula gives us

$$\dim_k \Omega_X(-D^*)^n = |G| \left( \sum_{i=1}^{r} (1 - \frac{2}{e_0(b_i)}) \right).$$

Since this dimension is not divisible by $|G|$ the module $\Omega_X(-D^*)^n$ can not be projective.

The conclusion concerning the dimension is that

$$\dim_k H^1(G, T_X) = 2(g_Y - 1) + \gamma_Y - 1 + r + \dim_k \Omega_X(-D^*)^n. \tag{19}$$
5. Borne Theory

Let $V_j$ denote the indecomposable $k[G]$-module of dimension $j$. Denote by $V$ the $k[G]$-module with $k$-basis $\{e_1, \ldots, e_p\}$ and action given by $\sigma e_\ell = e_\ell + e_{\ell-1}$, $e_0 = 0$. Then, $V_j$ is the subspace of $V$ generated by $\{e_1, \ldots, e_j\}$.

Following Borne we define:

**Definition 5.1.** Let $\pi : X \to Y$ be a Galois cover of curves defined over $k$ with Galois group $G \cong \mathbb{Z}/p^e\mathbb{Z}$. For a ramified point $P$ of $X$ we define $N_P$ so that $\sigma(t_P) - t_P$ has valuation $N_P + 1$, where $t_P$ is a local uniformizer at $P$. Let $X_{ram}$ denote the set of ramification locus of the above cover. For every $0 \leq \alpha \leq p - 1$ we define a map $\pi_*^\alpha : \text{Div}_X \to \text{Div}_Y$ by

$$
\pi_*^\alpha D = \left[ \frac{1}{p^\alpha} D - \alpha \sum_{P \in X_{ram}} N_P P \right],
$$

where $[\cdot]$ denotes the integral part of a divisor, taken coefficient by coefficient.

We will use the following

**Theorem 5.2.** Suppose that $X$ is acted on faithfully by the cyclic $p$-group $G \cong \mathbb{Z}/p^e\mathbb{Z}$. We break the cyclic extension $X \to X/G$ to a sequence of cyclic p-extensions by defining for every $1 \leq n \leq v$ the cover $X_n = X/H_n$ where $H_n$ is the unique subgroup of $G$ of order $n$. We set $X_0 = X$. Let $\pi_n : X_{n-1} \to X_n$ denote the canonical morphism. Let $D$ be a $G$-invariant divisor on $X$, then $H^0(X, \mathcal{O}_X(D)) \cong \bigoplus_{j=1}^v V_j^{\otimes m_j}$. Suppose that deg$(D) \geq 2g_X - 2$. Then, the integers $m_j$ are given by

$$
m_j = \deg(\pi_*^\alpha_1 \cdots \pi_*^{\alpha_{j-1}} D) - \deg(\pi_*^{\alpha_1(j+1)} \cdots \pi_*^{\alpha_{j-1}(j+1)} D) \text{ if } 1 \leq j \leq p^\nu - 1,
$$

$$m_{p^\nu} = 1 - g_X + \deg(\pi_*^{p^\nu - 1} \cdots \pi_*^{p - 1} D),
$$

where for $1 \leq j \leq p^\nu$ the integers $\alpha_0(j), \ldots, \alpha_{j-1}(j)$ are the digits of the $p$-adic expansion of $j - 1$ i.e.,

$$
j - 1 = \sum_{h=0}^{v-1} \alpha_h(j)p^h, \text{ with } 0 \leq \alpha_h(j) \leq p - 1.
$$

**Proof.** [4] th. 7.25

We would like to consider the space $\Omega_X(-D^*) = \mathcal{O}_X(K + D^*) = \mathcal{O}_X(2D^*)$, where $D^* = \pi^*(\text{div}(\phi)) + \sum_{P \in X_{ram}} \sum_{j=0}^{\infty} e_i(P) - 1)P$. Notice that deg$(2D^*) = 4g_X - 4 \geq 2g_X - 2$ so theorem [4] is applicable.

Define

$$
D(j) := \pi_*^{\alpha_0(j)} \cdots \pi_*^{\alpha_{j-1}(j)} 2D^*.
$$

We have $m_j = \deg(D(j)) - \deg(D(j+1))$ for $1 \leq j \leq p^\nu - 1$. Since $\dim_k H_0(G, V_j) = \dim_k(V_j)_G = 1$ the desired dimension is given by:

$$
(20)
\dim_k \mathcal{O}_X(2D^*)_G = \sum_{j=1}^{p^\nu} m_j = \deg(D(1)) - \deg(D(p^\nu)) + 1 - g_X + \deg(\pi_*^{p^\nu - 1} \cdots \pi_*^{p - 1} 2D^*).
$$

Observe that the $p$-adic expansion of $p^\nu - 1$ is

$$
p^\nu - 1 = p - 1 + (p - 1)p + \cdots (p - 1)p^{p-1},
$$
The divisor $2D$ can be written as

$$2\pi^*\text{div}(\phi) + \sum_{P \in X_{\text{ram}}} \sum_{i=0}^{\infty} (2\epsilon_i(P) - 2)P.$$ 

Since the divisor $\pi^*\text{div}(\phi)$ has empty intersection with the ramification locus, we have

$$\pi^*_* (2D^*) = (\pi_\nu \circ \pi_{\nu-1} \circ \cdots \circ \pi_2)^* 2\text{div}(\phi) + \sum_{P \in \pi_1(X_{\text{ram}})} \left[ \frac{1}{P} \sum_{i=0}^{N_P} (2\epsilon_i(P) - 2) \right] P = (\pi_{\nu} \circ \pi_{\nu-1} \circ \cdots \circ \pi_2)^* 2\text{div}(\phi) + \sum_{P \in \pi_1(X_{\text{ram}})} \Delta \left( \sum_{i=0}^{N_P} (2\epsilon_i(P) - 2) \right) P.$$ 

The decomposition group $G_0(P)$ is a cyclic subgroup of the whole group $G$, therefore $G_0(P) = \mathbb{Z}/p^{k(P)}\mathbb{Z}$. Observe that a point $P \in X_{\text{ram}}$ is fully ramified in all covers $X_i \to X_{i+1}$ for $i \leq k(P)$. We can see that

$$D(1) = 2\text{div}(\phi) + \sum_{P \in \pi_1(X_{\text{ram}})} \sum_{i=1}^{k(P)} \Delta^{k(P)} \left( \sum_{i=0}^{N_P} (2\epsilon_i(P) - 2) \right) P.$$ 

Let

$$1 \leq i_1 \leq i_2 \leq \cdots \leq i_{k(P)} = N_P,$$

be the sequence of the jumps in the ramification filtration at the point $P$, i.e.

$$G_{i_k} \supseteq G_{i_{k+1}} \text{ and } G_{i_{k(P)+1}} = \{1\}.$$ 

**Lemma 5.4.** There are strictly positive integers $a_0, a_1, \ldots, a_{k-1}$, so that the sequence of jumps for the ramification filtration for the cyclic group $\mathbb{Z}/p^k\mathbb{Z}$ is of the form:

$$i_\nu = \sum_{\mu=0}^{\nu-1} a_\mu p^\mu.$$
In particular, for all \( \nu \geq 2 \)

\[
i_{\nu} - i_{\nu-1} = a_{\nu-1} p^{\nu-1}
\]

**Proof.** This is a direct consequence of the Hasse-Arf theorem for Abelian groups and it is explained in the example that appears on page 76 in [22]. \( \square \)

Notice that we have \( k(P) \) jumps in the ramification filtration since \( G_{i_k}/G_{i_k+1} \) is elementary Abelian, therefore isomorphic to \( \mathbb{Z}/p\mathbb{Z} \). We set \( d_P = \sum_{i=0}^{N_p} (2e_i(P) - 2) \), and we compute:

\[
d_p = \sum_{i=0}^{i_1} (2e_i(P) - 2) + \sum_{i=i_1+1}^{i_2} (2e_i(P) - 2) + \cdots + \sum_{i=k(p)}^{i_{k(p)-1}+1} (2e_i(P)(P) - 2).
\]

Since \( e_i = p^{k(P)-i+1} \) for all \( 1 \leq \ell \leq k(P) \) we have that

\[
d_P = 2(p^{k(P)} - 1)(i_1 + 1) + 2(p^{k(P)} - 1)(i_2 - i_1) + \cdots + 2(p - 1)(i_{k(P)} - i_{k(P)-1}) = \sum_{\nu=1}^{k(P)} 2(p^{k(P)} - \nu + 1)(i_\nu - i_{\nu-1}), \text{ where } i_0 = -1.
\]

Using lemma 5.3 we obtain that there are integers \( a_0(P), a_1(P), \ldots, a_{k(P)-1}(P) \) so that

\[
d_p = 2p^{k(P)} - 2 + \sum_{\nu=1}^{k(P)} 2p^{k(P)-\nu+1}p^{\nu-1}a_{\nu-1}(P) + \sum_{\nu=1}^{k(P)} 2(-i_\nu + i_{\nu-1}) = 2p^{k(P)} + p^{k(P)} 2 \sum_{\nu=1}^{k(P)} a_{\nu-1} - 2N_p - 2
\]

One can compute [22] exam. p.72] that the sum \( f(P) := \sum_{\nu=1}^{k(P)} a_{\nu-1}(P) \) is the highest jump in the upper ramification filtration. Using \( (22) \) we compute that

\[
\Delta^{k(P)}(d_p) = 2f(P) + \Delta^k(-2N_p - 2).
\]

Combining all the above we arrive at

\[
deg D(1) = 4g_Y - 4 + 2\#\pi_* (\text{X}_{\text{ram}}) + \sum_{P \in \pi_* (\text{X}_{\text{ram}})} (2f(P) + \Delta^k(-2N_p - 2))
\]

The desired result follows:

\[
dim_k \mathcal{O}(2D^*) = 3(g_Y - 1) + 2\#\pi_* (\text{X}_{\text{ram}}) + \sum_{P \in \pi_* (\text{X}_{\text{ram}})} (2f(P) + \Delta^k(-2N_p - 2)).
\]

Observe that if \( G = \mathbb{Z}/p\mathbb{Z} \) the above result coincides with the computation of [11, 10].

**Example:** Let \( \sigma_i = \sum_{\nu=0}^{i-1} a_{\nu} i = 0, \ldots, k(P) \) be the jumps in the upper ramification filtration at the wild ramified point \( P \). It is known that \( \sigma_{i+1} = p\sigma_i \) or \( \sigma_{i+1}/p \geq \sigma_i \) and \( p \nmid \sigma_{i+1} \). Assume that all jumps in the ramification filtration are of the form \( \sigma_{i+1} = p\sigma_i \). In this case one can prove by induction that \( a_i = a_0 p^{i-1}(p-1) \), and we can compute

\[
f(P) + \Delta^k(-2N_p - 2) = \sum_{\nu=0}^{k(P)-1} a_{\nu} + \left[ -2 \sum_{\nu=0}^{k(P)-1} a_{\nu} p^{\nu-1} - 2 \right] =
\]
\[ k(p) - 1 \sum_{\nu=0}^{k(p)-1} a_0(p-1)p^{\nu-1} + \left\lfloor \frac{-2 \sum_{\nu=0}^{k(p)-1} (p-1)p^{2\nu-1} - 2}{p^{k(p)}} \right\rfloor \]

\[ = a_0(p^{k(p)} - 1) + \left\lfloor -a_02p^{2k(p)} - 1 - 2p^{k(p)} \right\rfloor. \]

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