SOME RESULTS ON THE GENERALIZED BROWNIAN MEANDER

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Abstract. We present some generalizations of the Brownian meander (also with drift) and study also the distribution of its maximum and first passage time. Analogous generalizations are presented for the bridge of the Brownian meander. The last part of the paper is devoted to the sojourn time \( \Gamma_{l,t} \) spent on \([0, \infty)\) during the interval \((0, t)\) under the condition that up to time \( l < t \), \( \min_{0 \leq s \leq l} B(s) > 0 \). We obtain some generalizations of the arcsine law also for the case of the Brownian excursion.

Keywords: Feynman-Kac functional, arcsine law, Bessel process, First passage times

1. Introduction

The Brownian meander is a Brownian motion \( \{B(t), \ t > 0\} \) evolving under the condition that \( \min_{0 \leq s \leq t} B(s) > 0 \). If the additional condition that \( B(t) = v \geq 0 \) is assumed we have the Brownian excursion or the bridge of the Brownian meander. An active research on the Brownian meander was undertaken since the Seventies (Chung \[5\], Kaigh \[9\], Durrett et al. \[6\]) although some important results on this point can be found in the classical book by Itô and McKean \[8\].

We study the \( n \)-fold joint distribution of the Brownian meander also in the case where the Brownian motion has drift \( \mu \). In particular we show that

\[
P\left\{ B^\mu(t) \in dy \mid \min_{0 \leq z \leq t} B^\mu(z) > 0, \ B^\mu(0) = 0 \right\} = \frac{ye^{-(\frac{y-\mu t}{2t})^2}}{\int_0^\infty we^{-(\frac{w-\mu t}{2t})^2}dw} \quad (1.1)
\]

which for \( \mu = 0 \) becomes the well-known Rayleigh distribution

\[
P\left\{ B(t) \in dy \mid \min_{0 \leq z \leq t} B(z) > 0, \ B(0) = 0 \right\} = \frac{y}{t} e^{-\frac{y^2}{2t}} dy \quad (1.2)
\]

Result \((1.2)\) slightly changes under the condition that \( \min_{0 \leq z \leq t} B(z) > v \) and produces a truncated Rayleigh distribution

\[
P\left\{ B(t) \in dy \mid \min_{0 \leq z \leq t} B(z) > v, \ B(0) = 0 \right\} = \frac{y - v}{t} e^{-(\frac{y-v)^2}{2t}} dy \quad y > v \quad (1.3)
\]

Furthermore we analyze the case where the sample paths are positive for the whole time interval \([0, \infty)\). We obtain in this case that

\[
P\left\{ B^\mu(s) \in dy \mid \min_{z \geq 0} B^\mu(z) > 0, \ B^\mu(0) = u \right\} \quad (1.4)
\]
We examine the distribution of the maximum of the Brownian meander and the Brownian excursion, that is we consider

\[ P \left\{ \max_{0 \leq s \leq t} B(s) \leq x \mid \min_{0 \leq z \leq t} B(z) > 0, \ B(0) = u \right\} \] (1.5)

and

\[ P \left\{ \max_{0 \leq s \leq t} B(s) \leq x \mid \min_{0 \leq z \leq t} B(z) > 0, \ B(0) = u, \ B(t) = v \right\} \] (1.6)

In [5] the distribution of (1.6) is analyzed for \( u = v = 0 \) and here we provide details on the distribution of the maximum and its relationship with the first passage time of the Bessel process.

The results on the maximum permit us to study the distribution of the first passage time of the Brownian meander

\[ P \left\{ T_x > s \mid \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0 \right\} \] (1.7)

for \( s < t, s > t \) where

\[ T_x = \inf \{ z : B(z) = x \} \]

A special case of the analysis of the first-passage time is obtained for \( s = t \) and yields

\[ P \left( T_x > t \mid \min_{0 \leq z < \infty} B(z) > 0, B(0) = v \right) = \sum_{k = 0}^{+\infty} (-1)^r e^{-\frac{r^2}{2s}} \] (1.8)

which is related to the distribution of the maximum of the Brownian bridge.

The first-passage time of the Brownian meander conditioned to stay positive for all \( t > 0 \) has the form

\[ P \left\{ T_x > s \mid \min_{0 \leq z < \infty} B(z) > 0, B(0) = 0 \right\} = \] (1.9)

\[ = \frac{2}{\sqrt{2\pi s}} \sum_{k = -\infty}^{+\infty} \int_0^x \left\{ e^{-\frac{(w-2kx)^2}{2s}} - e^{-\frac{x^2(1-2k)^2}{2s}} \right\} \, dw \]

Furthermore we have the following inequalities

\[ \frac{2x}{\sqrt{2\pi s}} \sum_{r=1}^{+\infty} (-1)^r e^{-\frac{r^2}{2s}} \leq P \left\{ T_x > s \mid \min_{0 \leq z < \infty} B(z) > 0, B(0) = 0 \right\} \leq \frac{2x}{\sqrt{2\pi s}} \sum_{r=0}^{+\infty} (-1)^r e^{-\frac{r^2}{2s}} \]

Finally, under the condition that \( \min_{0 \leq z \leq t} B(z) > 0 \) we study the distribution of the sojourn time in an interval \([0, t]\) with \( t > l \). Clearly

\[ \Gamma_{(0, t)} = \Gamma_l + \Gamma_{l, t} \]
and the distribution of
\[ \Gamma_{l,t} = \int_l^t 1_{[0,\infty)}(B(s)) \, ds \]
is obtained explicitly when the starting point is \( B(0) = u \). For \( u \to 0 \) we have the fine result
\[
\lim_{u \to 0} P \left\{ \Gamma_{l,t} \in ds \ \vline \ \min_{0 \leq z \leq t} B(z) > 0, B(0) = u \right\} = \\
= \begin{cases} \\
\frac{1}{\pi} \frac{s}{\sqrt{s(t-l-s)}} \, ds & 0 \leq s < t-l \\
\frac{1}{\sqrt{t}} & s = t-l \\
\end{cases}
\]
The information that in the interval \([0,l]\) the Brownian motion never intersects the zero level implies that the classical arcsine law fails and a positive probability that the Brownian particle continues to wander on the positive half-line up to time \( t \) appears.

The last result of the paper concerns the distribution of the sojourn time \( \Gamma_{l,t} \) for the Brownian excursion, that is
\[
P \left\{ \Gamma_{l,t} \in ds \ \vline \ \min_{0 \leq z \leq t} B(z) > 0, B(0) = u, B(t) = 0 \right\} \quad (1.10)
\]
The limit of \((1.10)\) for \( u \to 0 \) has the form
\[
P \left\{ \Gamma_{l,t} \in ds \ \vline \ \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0, B(t) = 0 \right\} = \\
= \frac{t}{2} \sqrt{\frac{l}{t-l}} \int_0^s \frac{dw}{\sqrt{(t-l-w)^3(l+w)^3}} \, ds \\
= \frac{t}{2} \sqrt{\frac{l}{t-l}} \int_l^{l+s} \frac{dw}{\sqrt{w^3(t-w)^3}} \, ds \quad 0 < s < t-l
\]
In this case no singularity in the distribution appears because of the condition \( B(t) = 0 \) which compels the sample paths to return to zero.

2. Joint Distributions

2.1. Joint distributions (without drift).
We first consider a standard Brownian motion \((B(t) \ , \ t > 0)\).

For the Brownian meander we have the \( n \)-th order joint distribution at times \( 0 < s_1 < s_2 < \ldots < s_n < t \) as
\[
P \left\{ B(s_1) \in dy_1, \ldots, B(s_n) \in dy_n \ \vline \ \min_{0 \leq z \leq t} B(z) > 0, B(0) = u \right\} = \\
\prod_{j=1}^n \left\{ e^{\frac{(y_j-y_{j-1})^2}{2(s_j-s_{j-1})}} - e^{\frac{(y_j+y_{j-1})^2}{2(s_j-s_{j-1})}} \right\} \left( 1 - 2 \int_{y_n}^{\infty} e^{-\frac{w^2}{2(t-s_n)}} \, dw \right) \\
\times \left( 1 - 2 \int_u^{\infty} e^{-\frac{w^2}{2t}} \, dw \right) \\
\]
The one dimensional marginals of (2.1) can be written as

\[ P \left\{ \bigcap_{j=1}^{n} (B(s_j) \in dy_j) \mid \min_{0 \leq z \leq t} B(z) > 0, B(0) = u \right\} = \]

\[ = P \left\{ \bigcap_{j=1}^{n} (B(s_j) \in dy_j, \min_{s_{j-1} \leq z \leq s_j} B(z) > 0) \mid B(0) = u \right\} \times \]

\[ P \left\{ \min_{0 \leq z \leq t} B(z) > 0 \mid B(s_n) = y_n \right\} \times \]

\[ P \left\{ \min_{0 \leq z \leq t} B(z) > 0 \mid B(0) = u \right\} \]

A slight simplification of (2.1) is obtained by letting \( u \to 0 \) and then we can write

\[ P \left\{ B(s_1) \in dy_1, \ldots, B(s_n) \in dy_n \mid \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0 \right\} = \]

\[ = \frac{y_1 \sqrt{t}}{s_1 \sqrt{s_1}} e^{-\frac{y_1^2}{2s_1}} \prod_{j=2}^{n} \left\{ e^{-\frac{(y_j-y_{j-1})^2}{2s_j}} - e^{-\frac{(y_j+y_{j-1})^2}{2s_j}} \right\} \times \]

\[ \times \left( 1 - 2 \int_{y_n}^{\infty} \frac{e^{-\frac{y^2}{2(t-s_n)}}}{\sqrt{2\pi(t-s_n)}} \, dw \right) \]

The one dimensional marginals of (2.1) can be written as

\[ P \left\{ B(s) \in dy \mid \min_{0 \leq z \leq t} B(z) > 0, B(0) = u \right\} = \]

\[ = \frac{e^{-\frac{(y-u)^2}{2s}} - e^{-\frac{(y+u)^2}{2s}}}{\sqrt{2\pi s}} \int_{s}^{t} \left( \frac{e^{-\frac{(y-u)^2}{2s}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(y+u)^2}{2s}}}{\sqrt{2\pi t}} \right) P \left\{ \min_{0 \leq z \leq t} B(z) > 0 \mid B(s) = y \right\} \, dy \]  \hspace{1cm} (2.3)

for \( 0 < s < t \).

Formula (2.3) further simplifies for \( s \to t \) and takes the form

\[ P \left\{ B(t) \in dy \mid \min_{0 \leq z \leq t} B(z) > 0, B(0) = u \right\} = \]

\[ = \frac{e^{-\frac{(y-u)^2}{2t}} - e^{-\frac{(y+u)^2}{2t}}}{\sqrt{2\pi t}} \int_{t}^{t} \left( \frac{e^{-\frac{(y-u)^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(y+u)^2}{2t}}}{\sqrt{2\pi t}} \right) P \left\{ \min_{0 \leq z \leq t} B(z) > 0 \mid B(0) = u \right\} \, dy \]  \hspace{1cm} (2.4)

Instead for \( u \to 0 \) formula (2.3) yields

\[ P \left\{ B(s) \in dy \mid \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0 \right\} = \]

\[ = \sqrt{\frac{t}{s}} \int_{s}^{t} \left( 1 - 2 \int_{y}^{\infty} e^{-\frac{y^2}{2(t-s)}} \sqrt{2\pi(t-s)} \, dy \right) e^{-\frac{y^2}{2t}} \, dy \]

\[ = e^{-\frac{t}{2s}} \int_{s}^{t} \left( 1 - 2 \int_{y}^{\infty} e^{-\frac{y^2}{2(t-s)}} \sqrt{2\pi(t-s)} \, dy \right) \, dy \]  \hspace{1cm} (2.5)
From (2.4) or (2.5) we obtain finally the well-known fact that
\[
P\left\{ B(t) \in dy \left| \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0 \right. \right\} = \frac{y}{t} e^{-\frac{y^2}{2t^2}} \, dy \quad y > 0
\]  \tag{2.6}

As a particular case of (2.1) we can write the bivariate distribution of Brownian meander which reads
\[
P\left\{ B(s_1) \in dy_1, B(s_2) \in dy_2 \left| \min_{0 \leq z \leq t} B(z) > 0, B(0) = u \right. \right\} =
\begin{align*}
&\left[ e^{-\frac{(y_1 - y_1)^2}{2s_1}} - e^{-\frac{(y_1 + y_1)^2}{2s_1}} \right] \left[ e^{-\frac{(y_2 - y_1)^2}{2(t-s_1)}} - e^{-\frac{(y_2 + y_1)^2}{2(t-s_1)}} \right] \\
&\int_{-y_2}^{y_2} e^{-\frac{y^2}{2(t-s_2)}} \, dw \\
&1 - 2 \int_{u}^{\infty} e^{-\frac{w^2}{2t^2}} \, dw
\end{align*}
\]  \tag{2.7}

From (2.7) or (2.2) we obtain, for \( u \to 0 \)
\[
P\left\{ B(s_1) \in dy_1, B(s_2) \in dy_2 \left| \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0 \right. \right\} =
\begin{align*}
&\frac{y_1}{s_1} e^{-\frac{y_1^2}{2s_1}} \sqrt{t} \left[ e^{-\frac{(y_1 - y_1)^2}{2(t-s_1)}} - e^{-\frac{(y_1 + y_1)^2}{2(t-s_1)}} \right] \\
&\int_{-y_2}^{y_2} e^{-\frac{y^2}{2(t-s_2)}} \, dw \, dy_1 \, dy_2 \quad y_1, y_2 > 0
\end{align*}
\]  \tag{2.8}

By letting \( s_2 \to t \) we obtain
\[
P\left\{ B(s_1) \in dy_1, B(t) \in dy \left| \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0 \right. \right\} =
\begin{align*}
&\frac{y_1}{s_1} \sqrt{t} e^{-\frac{y_1^2}{2s_1}} \left[ e^{-\frac{(y_1 - y_1)^2}{2(t-s_1)}} - e^{-\frac{(y_1 + y_1)^2}{2(t-s_1)}} \right] \\
&\, dy_1 \quad y_1, y > 0
\end{align*}
\]  \tag{2.9}

By integrating out \( y_1 \) from (2.9) we retrieve the Rayleigh distribution
\[
P\left\{ B(t) \in dy \left| \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0 \right. \right\} =
\begin{align*}
&\int_{0}^{\infty} \frac{y_1}{s_1} \sqrt{t} e^{-\frac{y_1^2}{2s_1}} \left[ e^{-\frac{(y_1 - y_1)^2}{2(t-s_1)}} - e^{-\frac{(y_1 + y_1)^2}{2(t-s_1)}} \right] \\
&\, dy_1 \\
&= \frac{y}{t} e^{-\frac{y^2}{2t^2}} \, dy \quad y > 0
\end{align*}
\]  \tag{2.10}

Formula (2.8) permits us to check that the Brownian meander is a Markov process since
\[
P\left\{ B(s_1) \in dy_1, B(s_2) \in dy_2 \left| \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0 \right. \right\} =
\begin{align*}
&P\left\{ B(s_1) \in dy_1 \left| \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0 \right. \right\} \\
&\left[ e^{-\frac{(y_2 - y_1)^2}{2(t-s_1)}} - e^{-\frac{(y_2 + y_1)^2}{2(t-s_1)}} \right] \\
&\int_{-y_2}^{y_2} e^{-\frac{y^2}{2(t-s_2)}} \, dw \\
&\int_{-y_1}^{y_1} e^{-\frac{y^2}{2(t-s_1)}} \, dy
\end{align*}
\]
distribution of an absorbing Brownian motion on

\[ \text{It is possible to extend the results of section 2.1 to the case where a Brownian Brownian meander with drift.} \]

2.2. Brownian meander with drift.
It is possible to extend the results of section 2.1 to the case where a Brownian motion \((B^\mu(t), t \geq 0)\) with constant drift \(\mu\) is considered. In this case we need the distribution of an absorbing Brownian motion on \((0, +\infty)\) starting at point \(u\) and an absorbing barrier at the origin, that is

\[
P \left\{ B^\mu(s) \in dy \mid \min_{0 \leq z \leq s} B^\mu(z) > 0, B^\mu(0) = u \right\} = \left( e^{-\frac{(y-u-\mu)^2}{2s}} - e^{-2\mu u \frac{(y+\mu)^2}{2s}} \right) dy
\]

In view of (2.10) we have that

\[
P \left\{ \bigcap_{j=1}^{n} (B^\mu(s_j) \in dy_j) \mid \min_{0 \leq z \leq t} B^\mu(z) > 0, B^\mu(0) = u \right\} = \prod_{j=1}^{n} \left[ e^{-\frac{(y_j-y_{j-1}-\mu(s_j-s_{j-1}))^2}{2(s_j-s_{j-1})}} \right. \\
\left. - e^{-2\mu y_j \frac{(y_j+y_{j-1}-\mu(s_j-s_{j-1}))^2}{2(s_j-s_{j-1})}} \right] dy_j \times
\]

\[
P \left\{ \min_{0 \leq s_n \leq t} B^\mu(z) > 0 \mid B^\mu(s_n) = y_n \right\} \times
\]

\[
P \left\{ \min_{0 \leq z \leq t} B(z) > 0 \mid B(0) = u \right\}
\]

Clearly (2.11) coincides with (2.1) for \(\mu = 0\). For the one-dimensional distribution in view of (2.10) we have that

\[
P \left\{ B^\mu(s) \in dy \mid \min_{0 \leq z \leq t} B^\mu(z) > 0, B^\mu(0) = u \right\}
\]

\[
= \left[ e^{-\frac{(y-u-\mu)^2}{2s}} - e^{-2\mu u \frac{(y+\mu)^2}{2s}} \right] dy \times
\]

\[
\int_{0}^{\infty} e^{-\frac{(y-u-\mu t)^2}{2(t-s)}} \frac{1}{2\pi(t-s)} dw - e^{-2\mu y} \int_{0}^{\infty} e^{-\frac{(y+u-\mu t)^2}{2(t-s)}} \frac{1}{2\pi(t-s)} dw
\]

\[
\int_{0}^{\infty} e^{-\frac{(u-u-\mu t)^2}{2t}} \frac{1}{2\pi t} dw - e^{-2\mu u} \int_{0}^{\infty} e^{-\frac{(u+u-\mu t)^2}{2t}} \frac{1}{2\pi t} dw
\]

For \(u \to 0\) (2.12) converges to

\[
\lim_{u \to 0} P \left\{ B^\mu(s) \in dy \mid \min_{0 \leq z \leq t} B^\mu(z) > 0, B^\mu(0) = u \right\}
\]
By taking the limit for

While if we take the limit for

Theorem 2.1.

From (2.12) we are able to extract the following result:

for \(0 < y < \infty\), \(t > 0\). For the check it suffices to make a change of variable inside the square brackets and let \(s \to t\). We also note that the Brownian meander is a Markovian process by the same arguments as in the previous section. From (2.12), we are able to extract the following result:

Theorem 2.1.

Proof. By taking the limit for \(t \to \infty\) in (2.12), for \(\mu \leq 0\), we have that

While if we take the limit for \(t \to \infty\) in (2.12), for \(\mu \leq 0\) we obtain

\[
\lim_{t \to \infty} P \left\{ B^\mu(s) \in dy \left| \min_{0 \leq z \leq t} B^\mu(z) > 0, B^\mu(0) = u \right. \right\} / dy =
\]

\[
\lim_{t \to \infty} P \left\{ B^\mu(s) \in dy \left| \min_{0 \leq z \leq t} B^\mu(z) > 0, B^\mu(0) = u \right. \right\} / dy =
\]
\[ = e^{(y-u) - \frac{\mu^2 s}{2}} \left[ \frac{e^{-\frac{(y-u)^2}{2s}} - e^{-\frac{(y+u)^2}{2s}}}{\sqrt{2\pi s}} \right] \cdot \frac{1 - e^{-2\mu y}}{1 - e^{-2\mu u}} \]

\[ = e^{-\frac{\mu^2 s}{2}} e^{\mu y} - e^{-\mu y} \left[ \frac{e^{-\frac{(y-u)^2}{2s}} - e^{-\frac{(y+u)^2}{2s}}}{\sqrt{2\pi s}} \right] \]

\[ \square \]

The surprising fact is that distribution (2.15) is affected by the drift only when \( \mu > 0 \). In the other case (\( \mu \leq 0 \)) the downward drive of the drift is canceled by the condition that \( \min_{z \geq 0} B(z) > 0 \).

**Remark 2.1.** In order to check that in the first case of (2.15) the distribution integrates to one it suffices to show that

\[ \frac{1}{\sqrt{2\pi s}} \int_{0}^{\infty} e^{\mu y} \left( e^{-\frac{(y-u)^2}{2s}} - e^{-\frac{(y+u)^2}{2s}} \right) dy + \]

\[ - \frac{1}{\sqrt{2\pi s}} \int_{0}^{\infty} e^{-\mu y} \left( e^{-\frac{(y-u)^2}{2s}} - e^{-\frac{(y+u)^2}{2s}} \right) dy = e^{-\frac{\mu^2 s}{2}} \left( e^{\mu y} - e^{-\mu y} \right) \]

**Remark 2.2.** We have that

\[ \lim_{\mu \to 0} e^{-\frac{\mu^2 s}{2}} e^{\mu y} - e^{-\mu y} \left[ \frac{e^{-\frac{(y-u)^2}{2s}} - e^{-\frac{(y+u)^2}{2s}}}{\sqrt{2\pi s}} \right] = \frac{y e^{-\frac{(y-u)^2}{2s}} - e^{-\frac{(y+u)^2}{2s}}}{u} \frac{1}{\sqrt{2\pi s}} \]

i.e. the densities in (2.15) are continuous w.r.t. the \( \mu \) parameter for every \( y > 0, t > 0 \).

For \( \mu > 0 \) if we let \( u \to 0 \) we get

\[ \lim_{u \to 0} P \left\{ B^\mu(s) \in dy \mid \min_{z \geq 0} B^\mu(z) > 0, B^\mu(0) = u \right\} = \]

\[ = \frac{y e^{-\frac{(y-u)^2}{2s}} - e^{-\frac{(y+u)^2}{2s}}}{\mu \sqrt{2\pi s}} dy \quad y > 0, \ s > 0 \]

Furthermore for \( \mu \to 0 \) and \( u \to 0 \) we obtain from (2.15) as well as from (2.16)

\[ \lim_{u \to 0} P \left\{ B^\mu(s) \in dy \mid \min_{z \geq 0} B^\mu(z) > 0, B^\mu(0) = u \right\} = \]

\[ = \sqrt{\frac{2}{\pi s}} y^2 e^{-\frac{y^2}{2s}} dy \quad y > 0, \ s > 0 \]

We note that the density (2.17) satisfies the parabolic equation

\[ \frac{\partial u}{\partial s} = \frac{1}{2} \frac{\partial^2 u}{\partial y^2} - \frac{1}{y} \frac{\partial u}{\partial y} + \frac{1}{y^2} u \]
Since the law of the first passage time of a Brownian motion $u(y, s) = \frac{ye^{-y^2/2}}{\sqrt{2\pi s}}$, with Laplace transform $\int_0^\infty e^{-\gamma s}u(y, s)\, ds = e^{-y\sqrt{2\gamma}}$, is a solution to the fractional equation

$$\frac{\partial u}{\partial y} = -\frac{1}{t} \frac{\partial^{1/2} u}{\partial s^{1/2}},$$

we can conclude that $v(y, s) = 2yu(y, s)$, defined in (2.17), also solves the following fractional equation

$$\frac{\partial v}{\partial y} = \frac{v}{2y} - \frac{1}{t} \frac{\partial^{1/2} v}{\partial s^{1/2}},$$

where the fractional derivative must be understood in the Riemann-Liouville sense.

3. Some generalizations of the Brownian meander

Assuming that $(B(s), 0 < s < t)$ develops under the condition that

$$\min_{0 \leq z \leq t} B(z) > v > 0 \quad (3.1)$$

produces some slight extensions of the distributions of Section 2

$$P\{B(s) \in dy \mid \min_{0 \leq z \leq t} B(z) > v, B(0) = u\} = \frac{e^{-(y-u)^2/2s} - e^{-(2v-y-u)^2/2s}}{\sqrt{2\pi s}} \left[ 1 - 2P(B(t-s) > y - v) \right] \int_{y-v}^{\infty} \frac{e^{w^2/2(t-s)}}{\sqrt{2\pi(t-s)}} \, dw \, dy \quad (3.2)$$

For $u \to v$ we have a generalization of (2.4) in the following form

$$\lim_{u \to v} P\{B(s) \in dy \mid \min_{0 \leq z \leq t} B(z) > v, B(0) = u\} = \frac{t}{s} \left( y - v \right) e^{-(y-v)^2/2t} \left[ 1 - 2 \int_{y-v}^{\infty} \frac{e^{w^2/2(t-s)}}{\sqrt{2\pi(t-s)}} \, dw \right] \, dy \quad y > v \quad (3.3)$$

For $s \to t$ we extract from (3.3) a truncated Rayleigh probability density

$$\lim_{u \to v} \lim_{s \to t} P\{B(s) \in dy \mid \min_{0 \leq z \leq t} B(z) > v, B(0) = u\} = \frac{y-v}{t} e^{-(y-v)^2/2t} \, dy \quad y > v \quad (3.4)$$

4. Bridge of the Brownian meander

A Brownian motion developing in the time interval $[0, t]$ under the conditions that $B(0) = u, B(t) = v$ and $\min_{0 \leq z \leq t} B(z) > 0$ is a bridge of the Brownian meander usually called Brownian excursion. The one-dimensional, generalized distribution of the bridge of the Brownian meander is, for $y, u, v > 0$:

$$P\{B(s) \in dy \mid \min_{0 \leq z \leq t} B(z) > 0, B(0) = u, B(t) = v\} = \lim_{u \to v} P\{B(s) \in dy \mid \min_{0 \leq z \leq s} B(z) > 0|B(0) = u\} \times$$

$$= \int_{y-v}^{\infty} \frac{e^{w^2/2(t-s)}}{\sqrt{2\pi(t-s)}} \, dw \quad y > v \quad (4.1)$$
Another representation of the bridge of the Brownian meander involves the three-

\[
\frac{e^{-\frac{(y-u)^2}{2u}} - e^{-\frac{(y+u)^2}{2u}}}{\sqrt{2\pi u}} \cdot \frac{e^{-\frac{(w-y)^2}{2(t-s)}} - e^{-\frac{(w+y)^2}{2(t-s)}}}{\sqrt{2\pi (t-s)}} \, dy
\]

This means that between the Brownian excursion and the Brownian meander there

\[
\mathbb{P}\left\{ \min_{0 \leq z \leq t} B(z) > v, B(t) \in dv \mid B(0) = u \right\} = \frac{e^{-\frac{(y-u)^2}{2u}} - e^{-\frac{(y+u)^2}{2u}}}{\sqrt{2\pi u}} \cdot \frac{e^{-\frac{(w-y)^2}{2(t-s)}} - e^{-\frac{(w+y)^2}{2(t-s)}}}{\sqrt{2\pi (t-s)}} \, dy
\]

A substantial simplification of (4.1) is obtained by letting \( u \to 0 \), as follows

\[
\lim_{u \to 0} \mathbb{P}\left\{ \min_{0 \leq z \leq t} B(z) > 0, B(t) = v \right\} = \mathbb{P}\left\{ \min_{0 \leq z \leq t} B(z) > 0, B(t) = v \right\} = \int_0^t \mathbb{P}\left\{ B(s) \in dy \mid \min_{0 \leq z \leq t} B(z) > 0, B(0) = u, B(t) = v \right\} \, dy
\]

By letting \( v \to 0 \) we get the distribution

\[
\lim_{v \to 0} \mathbb{P}\left\{ B(s) \in dy \mid \min_{0 \leq z \leq t} B(z) > 0, B(t) = v \right\} = \frac{2}{\sqrt{2\pi}} \left( \frac{t}{s(t-s)} \right)^{3/2} y^2 e^{-\frac{y^2}{2s(t-s)}} \, dy
\]

The distribution (4.3) coincides with that of a Bessel process in \( \mathbb{R}^3 \) with components

\[ B_j \left( \frac{t}{s(t-s)} \right), \quad j = 1, 2, 3. \]

We note that (4.3) coincides with the density (2.17) at time \( \frac{s(t-s)}{t} \), i.e.

\[
\mathbb{P}\left\{ B(s) \in dy \mid \min_{0 \leq z \leq t} B(z) > 0, B(t) = 0 \right\} = \mathbb{P}\left\{ B^\mu \left( \frac{s(t-s)}{t} \right) \in dy \mid \min_{z \geq 0} B^\mu(z) > 0, B(0) = 0 \right\}
\]

This means that between the Brownian excursion and the Brownian meander there

\[
\mathbb{P}\left\{ \min_{0 \leq z \leq t} B(z) > 0, B(t) \in dv\left| B(0) = u \right. \right\} = \frac{e^{-\frac{(y-u)^2}{2u}} - e^{-\frac{(y+u)^2}{2u}}}{\sqrt{2\pi u}} \cdot \frac{e^{-\frac{(w-y)^2}{2(t-s)}} - e^{-\frac{(w+y)^2}{2(t-s)}}}{\sqrt{2\pi (t-s)}} \, dy
\]

For the Brownian meander \( B^{mc}(s) \), \( s > 0 \) the following representation holds (see [12])

\[
P\left\{ \sqrt{\frac{1}{t} \left[ P^2_1(s) + P^2_2(s) + P^2_3(s) \right]} < y \right\} = 4\pi \int_0^{y\sqrt{2}} \rho^2 e^{-\frac{\rho^2}{2(t-s)}} \frac{1}{\sqrt{2\pi s(t-s)}} \, d\rho
\]
We can prove (4.5) by direct calculations as follows. Clearly for \( 0 < y, 0 < s < 1 \) we havra that
\[
P \left\{ \frac{|B(T_0 + s(t - T_0))|}{\sqrt{t - T_0}} < y \right\} =
\int_0 \int_{y\sqrt{1-z}} \int_{-y\sqrt{1-z}} P \{ B(z + s(t - z)) \in dw | T_0 = z \}
\]
where
\[
P(T_0 \in dz) = \frac{dz}{\pi \sqrt{z(t - z)}}, \quad 0 < z < t
\]
From (4.6), we can write the density of \( B^{me}(s) \), \( 0 < s < 1 \) as
\[
P \left\{ \frac{|B(T_0 + s(t - T_0))|}{\sqrt{t - T_0}} \in dy \right\} =
2 \int_0^t \sqrt{t - z} P \{ T_0 \in dz \} \left\{ P \{ B(z + s(t - z)) \in d(y\sqrt{1-z}) | T_0 = z \} \right\} dy
\]
We observe that
\[
P \{ B(z + s(t - z)) \in dy | T_0 = z \} =
\frac{1}{2} \left\{ \frac{1}{2} P \{ B(z + s(t - z)) \in dy | B(z) = 0, \min_{z \leq w \leq t} B(w) > 0 \} + \right.
\frac{1}{2} \left\{ P \{ B(z + s(t - z)) \in dy | B(z) = 0, \max_{z \leq w \leq t} B(w) < 0 \} \right\}
\]
\[
= \frac{1}{2} \left\{ \sqrt{t - z} y \int_{s(t - z)}^{y\sqrt{1-z}} \int_{-y\sqrt{1-z}} e^{\frac{w^2}{2(1-s)t}} \frac{e^{\frac{w^2}{2(1-s)t}}}{2\pi(1-s)} dw \cdot e^{\frac{y^2}{2(1-s)t}} dy \right\}
\]
where in the last step we applied (2.5) with suitably adapted parameters that is \( t \) must be replaced by \( t - z \) and \( s \) by \( s(t - z) \). Clearly for \( y > 0 \) we must consider that \( \max_{z \leq w \leq t} B(w) < 0 \} \cap \{ B(z + s(t - z)) \in dy \} = \emptyset \) for \( y > 0 \).

Hence
\[
P \left\{ B(z + s(t - z)) \in d(y\sqrt{1-z}) | T_0 = z \right\} =
\frac{\sqrt{t - z} y \sqrt{t - z}}{s(t - z)} \left\{ y \sqrt{t - z} \int_{-y\sqrt{1-z}}^{y\sqrt{1-z}} \frac{e^{\frac{w^2}{2(1-s)t}}}{2\pi(1-s)} dw \cdot e^{\frac{y^2}{2(1-s)t}} dy \right\}
\]
By inserting (4.10) into (4.8) we have that
\[
P \left\{ \frac{|B(T_0 + s(t - T_0))|}{\sqrt{t - T_0}} \in dy \right\} =
\frac{1}{2} \left\{ \frac{1}{2} P \{ B(z + s(t - z)) \in dy | B(z) = 0, \min_{0 \leq w \leq t} B(w) > 0 \} + \right.
\frac{1}{2} \left\{ P \{ B(z + s(t - z)) \in dy | B(z) = 0, \max_{0 \leq w \leq t} B(w) < 0 \} \right\}
\]
\[ = 2 \int_0^t P \{ T_0 \in dz \} \frac{y}{s^2} e^{\frac{y^2}{2s}} \int \frac{w}{\sqrt{2\pi}} e^{\frac{w^2}{2}} \, dw \]
\[ = 2y \sqrt{s^3} e^{\frac{y^2}{2s}} \int -\frac{y}{\sqrt{2\pi}} \, dw \, dy \]

Clearly (4.11) coincides with (2.5) for \( t = 1 \). The representation (4.5) of the meander in \([0,1]\) is independent from \( t \). In a sense the condition \( \min_{0 \leq z \leq I} B(z) > 0 \) is here replaced by the last passage time \( T_0 \) through zero.

If we want to represent the Brownian meander in an arbitrary time interval \([0,t]\) we perform the transformation

\[ \left\{ \begin{array}{l}
    s = \frac{s'}{t} \\
y = \frac{y'}{\sqrt{t}}
\end{array} \right. \]

and thus (4.11) is converted into (2.5) with \( 0 < s' < t \). Therefore a representation of the meander in \([0,t]\) is given by

\[ \sqrt{t} \left| B \left( T_0 + \frac{s}{t} (T - T_0) \right) \right| \text{i.d.} = B^{me}(s) \quad 0 < s < t \]

where \( T \) is an arbitrary (possibly random) time and \( T_0 = \sup \{ z < T : B(z) = 0 \} \).

A representation for the Brownian meander in \([0,1]\) similar to (4.5) is presented in [12] in the form

\[ B^{ex}(s) = \frac{|B(T_0 + s(T^+ - T_0))|}{\sqrt{T^+ - T_0}} \quad 0 < s < 1 \]

where \( T^+ = \inf \{ s > 1 : B(s) = 0 \} \) and can be extended to an arbitrary time interval \([0,t]\). Distributions (4.2), (4.3) can be traced back to [8], pages 75, 76 for the case of an excursion in intervals \([s,t]\) with \( 0 < s < t < 1 \).

5. Bridge of the meander with drift

Formulas of section [12] can be easily generalized to the case where a Brownian motion with a constant drift \( \mu \) is considered. The main tool is the distribution of an absorbing Brownian motion on \([0,\infty)\), with the absorbing barrier placed at \( x = 0 \). With this at hand we can write

\[ P \left\{ B^{\mu}(s) \in dy \mid \min_{0 \leq z \leq t} B^{\mu}(z) > 0, B^{\mu}(0) = u, B^{\mu}(t) = v \right\} = (5.1) \]

\[ = \sqrt{\frac{t}{2\pi s(t-s)} \left[ e^{-(y-u)^2/2s} - e^{-2\mu(u-y)/\sqrt{2\pi}} \right] \times \int e^{-2\mu y e^{-(y-u)^2/2s}} \, dy} \]
\[
\times \left[ e^{-\frac{(v-\mu(t-s)-y)^2}{2(t-s)}} - e^{-2\mu y} e^{-\frac{(v-\mu(t-s)+y)^2}{2(t-s)}} \right] \, dy
\]

From (5.1) for \( u \to 0 \) we extract the following result

\[
\lim_{u \to 0} P \left\{ B^\mu(s) \in dy \mid \min_{0 \leq z \leq t} B^\mu(z) > 0, B^\mu(0) = u, B^\mu(t) = v \right\} = (5.2)
\]

Finally we have that

\[
P \left\{ B^\mu(s) \in dy \mid \min_{0 \leq z \leq t} B^\mu(z) > 0, B^\mu(0) = 0, B^\mu(t) = 0 \right\} = (5.3)
\]

Therefore

\[
P \left\{ B^\mu(s) \in dy \mid \min_{0 \leq z \leq t} B^\mu(z) > 0, B^\mu(0) = 0, B^\mu(t) = 0 \right\} = P \left\{ B(s) \in dy \mid \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0, B(t) = 0 \right\}
\]

This means that the effect of the drift is canceled by the conditioning in the Brownian meander when \( B^\mu(0) = B^\mu(t) = 0 \). Formula (5.2) shows instead that if \( B^\mu(t) = v > 0 \) and \( B^\mu(0) = 0 \) (or vice versa) the distribution of the meander is affected by the drift \( \mu \). We observe that, instead

\[
P \left\{ \max_{0 \leq s \leq t} B^\mu(s) > \beta \mid B^\mu(t) = \eta \right\} = e^{-\frac{2\beta(\beta-\eta)}{t}}
\]

is independent of \( \mu \) also when \( \eta \neq 0 \) (see Beghin and Orsingher [1] formula (2.1) of Remark 2.1).

Moreover we note that for \( t \to \infty \) (5.3) tends to (2.13), that is the condition that \( (B(t) = 0) \) as \( t \to \infty \) does not affect the distribution of the Brownian excursion.

6. THE DISTRIBUTION OF THE MAXIMUM OF THE BROWNIAN MEANDER

We start this section by considering the distribution of the maximum of the generalized Brownian meander, that is

\[
P \left\{ \max_{0 \leq s \leq t} B(s) < y \mid \min_{0 \leq z \leq t} B(z) > x, B(0) = u \right\} = (6.1)
\]

\[
P \left\{ x < \min_{0 \leq z \leq t} B(z) < \max_{0 \leq z \leq t} B(z) < y \mid B(0) = u \right\}
\]

\[
P \left\{ \min_{0 \leq z \leq t} B(z) > x \mid B(0) = u \right\}
\]

\[
P \left\{ \max_{0 \leq s \leq t} B(s) > \beta \mid B(0) = \eta \right\} = e^{-\frac{2\beta(\beta-\eta)}{t}}
\]
For obtaining the explicit expression of (6.1) we need the following trivariate distribution of \( \{B(t), \max_{0 \leq s \leq t} B(s) < y \} \) which reads (when \( B(0) = 0 \))

\[
P(B(t) \in dw, x < \min_{0 \leq s \leq t} B(s) < \max_{0 \leq s \leq t} B(s) < y | B(0) = 0) =
\]

\[
= \frac{dw}{\sqrt{2\pi t}} \left\{ \sum_{k = -\infty}^{+\infty} e^{-\frac{(w-2k(y-x))^2}{2t}} - \sum_{k = -\infty}^{+\infty} e^{-\frac{(2x-w+2k(y-x))^2}{2t}} \right\}
\]

(6.2)

From (6.2) we can calculate (6.1) and write

\[
P(x < \min_{0 \leq s \leq t} B(s) < \max_{0 \leq s \leq t} B(s) < y | B(0) = u)
\]

\[
= \int_x^y P(B(t) \in dw, x < \min_{0 \leq s \leq t} B(s) < \max_{0 \leq s \leq t} B(s) < y | B(0) = u)
\]

\[
= \int_x^y P(B(t) \in dw, x < \min_{0 \leq s \leq t} B(s) < \max_{0 \leq s \leq t} B(s) < y - u | B(0) = 0)
\]

\[
= \sum_{k = -\infty}^{+\infty} \int_x^y \frac{dw}{\sqrt{2\pi t}} \left\{ e^{-\frac{(w-u-2k(y-x))^2}{2t}} - e^{-\frac{(2x-w+2k(y-x))^2}{2t}} \right\}
\]

Hence

\[
P\left\{ \max_{0 \leq s \leq t} B(s) < y \ \mid \ \min_{0 \leq s \leq t} B(s) > x, B(0) = u \right\} =
\]

\[
= \sum_{k = -\infty}^{+\infty} \int_x^y \frac{dw}{\sqrt{2\pi t}} \left\{ e^{-\frac{(w-u-2k(y-x))^2}{2t}} - e^{-\frac{(2x-w+2k(y-x))^2}{2t}} \right\}
\]

\[
1 - 2P(B(t) > u - x)
\]

(6.3)

Some interesting results emerge when \( u \to x \). In effect the l’Hôpital’s rule applied to (6.4) yields

\[
\lim_{u \to x} P\left\{ \max_{0 \leq s \leq t} B(s) < y \ \mid \ \min_{0 \leq s \leq t} B(s) > x, B(0) = u \right\} =
\]

\[
= \sum_{k = -\infty}^{+\infty} \int_x^y \frac{dw}{\sqrt{2\pi t}} e^{-\frac{(w-x-2k(y-x))^2}{2t}} e^{-\frac{(w-x+2k(y-x))^2}{2t}}
\]

(6.5)
Bridge of the Brownian meander as Formula (6.3) permits us also to obtain the distribution of the maximum of the Brownian meander (for $x \to 0$). In this case we have that

$$P_k = \sum_{r \text{ even}} e^{-\frac{r^2(y-x)^2}{2t}} - \sum_{r \text{ odd}} e^{-\frac{r^2(y-x)^2}{2t}}$$

From (6.5) we can extract the following relationship between the maximum of the Brownian meander (for $x \to 0$) and the maximum of the Brownian bridge

$$P \left\{ \frac{1}{2} \max_{0 \leq z \leq t} B(z) < y \left| \min_{0 \leq s \leq t} B(s) > 0, B(0) = 0 \right. \right\} = P \left\{ \max_{0 \leq z \leq t} |B(z)| < y \left| B(0) = 0, B(t) = 0 \right. \right\}$$

$$= \sum_{r = -\infty}^{+\infty} (-1)^r e^{-\frac{2r^2y^2}{t}} \quad y > 0$$

Formula (6.3) permits us also to obtain the distribution of the maximum of the bridge of the Brownian meander as

$$P \left\{ \max_{0 \leq z \leq t} B(z) < y \left| \min_{0 \leq s \leq t} B(s) > 0, B(0) = 0, B(t) = v \right. \right\} = \frac{P \left\{ B(t) \in dv, x < \min_{0 \leq s \leq t} B(s) < \max_{0 \leq s \leq t} B(z) < y \left| B(0) = u \right. \right\}}{P \left\{ \min_{0 \leq z \leq t} B(z) > x, B(t) \in dv \left| B(0) = u \right. \right\}}$$

$$= \frac{\sum_{k=-\infty}^{+\infty} e^{-\frac{(v-u-2k(y-x))^2}{2t}} - \sum_{k=-\infty}^{+\infty} e^{-\frac{2(v-u)+(v-u)+2k(y-x))^2}{2t}}}{e^{-\frac{(v-u)^2}{2t}} - e^{-\frac{(v+u-2x)^2}{2t}}}$$

Some simplification of (6.7) is obtainable by letting $x \to 0$ so that

$$P \left\{ \max_{0 \leq z \leq t} B(z) < y \left| \min_{0 \leq s \leq t} B(s) > 0, B(0) = u, B(t) = v \right. \right\}$$

$$= \sum_{k=-\infty}^{+\infty} e^{-\frac{(v-u-2k(y))^2}{2t}} - e^{-\frac{2(v-u-2k(y)^2)}{2t}}$$

$$u, v, y > 0$$

A further and substantial reduction of the formulas above is obtainable by also letting $u \to 0$. In this case we have that
\[ P \left\{ \max_{0 \leq z \leq t} B(z) < y \bigg| \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0, B(t) = v \right\} \] (6.9)

\[ = \sum_{k=-\infty}^{+\infty} \frac{v - 2ky}{v} e^{-\frac{(v-2ky)^2}{2t}} \]

\[ = \sum_{k=-\infty}^{+\infty} \frac{v - 2ky}{v} e^{-\frac{2k^2y^2 + 2kvy}{t}} \quad y > v \]

For (6.9) it is easy to show that for \( y \to \infty \) the distribution converges to one and for \( y \to 0 \) tends to zero as an inspection of the series

\[ \sum_{k=-\infty}^{\infty} (1 - 2k)e^{-\frac{2ky^2}{t}}(k-1) = 1 + \sum_{k\neq 0} (1 - 2k)e^{-\frac{2ky^2}{t}}(k-1) \]

shows. The problem arises when also \( v \to 0 \) which cannot be obtained as a term-by-term derivation of (6.9). The result of the distribution of the maximum obtained in different ways writes

\[ P \left\{ \max_{0 \leq z \leq t} B(z) < y \bigg| \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0, B(t) = 0 \right\} \]

\[ = 1 + 2 \sum_{k=1}^{+\infty} \left[ 1 - \frac{(2ky)^2}{t} \right] e^{-\frac{2k^2y^2}{t}} \] (6.10)

If we write (6.9) as

\[ P \left\{ \max_{0 \leq z \leq t} B(z) < y \bigg| \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0, B(t) = v \right\} \]

\[ = 1 + \sum_{k=-\infty}^{+\infty} \frac{v - 2ky}{v} e^{-\frac{2k^2y^2 + 2kvy}{t}} \quad y > v \]

and then let \( v \to 0 \) and apply De l'Hôpital's rule term by term we obtain (6.10) which was derived by a different reasoning by [9], for example.

The proof that (6.10) is a distribution function can be carried out from the product expansion of the sine function.

\[ \frac{\sin z}{z} = \prod_{k=1}^{\infty} \left( 1 - \frac{z^2}{k^2\pi^2} \right) \quad z \in \mathbb{C} \] (6.12)

For \( z = ix \) we derive from (6.12)

\[ \frac{\sinh x}{x} = \prod_{k=1}^{\infty} \left( 1 + \frac{x^2}{k^2\pi^2} \right) \]

and then

\[ \ln \sinh x = \ln x + \sum_{k=1}^{\infty} \ln \left( 1 + \frac{x^2}{k^2\pi^2} \right) \] (6.13)
From (6.13) we obtain that
\[
\cosh x = \frac{1}{x} + 2 \sum_{k=1}^{\infty} \frac{1}{x^2 + k^2 \pi^2} \tag{6.14}
\]
and after some manipulations we get
\[
\frac{1}{e^x - 1} = -\frac{1}{2} + \frac{1}{x} + 2 \sum_{k=1}^{\infty} \frac{1}{x^2 + (2k\pi)^2} \tag{6.15}
\]
By deriving (6.15) we arrive at the following formula
\[
\frac{e^x}{(e^x - 1)^2} = \frac{1}{x^2} + 2 \sum_{k=1}^{\infty} \frac{x^2 - (2k\pi)^2}{x^2 + (2k\pi)^2} \tag{6.16}
\]
It is convenient to work with
\[
P \left\{ \left( \max_{0 \leq z \leq t} B(z) \right)^2 < y \left| \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0, B(t) = 0 \right. \right\} \tag{6.17}
\]
\[
= 1 + 2 \sum_{k=1}^{\infty} \left( 1 - \frac{(2k)^2 y}{2t} \right) e^{-\frac{4k^2 y}{x^2}}
\]
The Laplace-Stjeltjes transform of (6.17) is
\[
\lambda \int_0^\infty e^{-\lambda y} \left[ 1 + 2 \sum_{k=1}^{\infty} \left( 1 - \frac{(2k)^2 y}{2t} \right) e^{-\frac{4k^2 y}{x^2}} \right] dy \tag{6.18}
\]
\[
= 1 + \sum_{k=1}^{\infty} \frac{2\lambda}{\lambda + 2k^2 \frac{2t}{x^2}} - 2\lambda \sum_{k=1}^{\infty} \frac{(2k)^2 y}{2t} \frac{1}{(\lambda + 2k^2 \frac{2t}{x^2})^2}
\]
\[
= 1 + 2\lambda \sum_{k=1}^{\infty} \frac{\lambda - 2k^2 \frac{2t}{x^2}}{(\lambda + 2k^2 \frac{2t}{x^2})^2}
\]
If we put \( \lambda = x^2, \frac{2t}{x^2} = (2\pi)^2 \), the last term in (6.17) writes
\[
1 + 2x^2 \sum_{k=1}^{\infty} \frac{x^2 - (2k\pi)^2}{x^2 + (2k\pi)^2}
\]
If we multiply both members of (6.16) by \( x^2 \) we get that
\[
\frac{x^2 e^x}{(e^x - 1)^2} = 1 + 2x^2 \sum_{k=1}^{\infty} \frac{x^2 - (2k\pi)^2}{x^2 + (2k\pi)^2} \tag{6.19}
\]
Instead if we put \( x = 2\pi \sqrt{\frac{\lambda}{2t}} \), formula (6.19) becomes
\[
1 + 2\lambda \sum_{k=1}^{\infty} \frac{\lambda - 2k^2 \frac{2t}{x^2}}{(\lambda + 2k^2 \frac{2t}{x^2})^2} = 2\pi^2 \lambda e^{-\pi \sqrt{2\lambda t}} \left( 1 - e^{-\pi \sqrt{2\lambda t}} \right)^2 \tag{6.20}
\]
where \( \tau \) is the first passage-time of a Bessel process in \( \mathbb{R}^3 \).
We recall that the Laplace transform of the first-passage time of a Bessel process of order $\nu$ (see for example [7]) is

$$\mathbb{E} e^{-\lambda \tau_{0,b}} = \frac{(b\sqrt{2\lambda})^\nu}{2^{\nu} \Gamma(\nu + 1)} I_{\nu}(b\sqrt{2\lambda}) \quad (6.21)$$

where, for us, $\nu = 1/2$ and thus $I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x$. The Laplace transform of (6.21) for $\nu = 1/2$ becomes

$$\mathbb{E} e^{-\lambda \tau_{0,b}} = 2b\sqrt{2\lambda} e^{-b\sqrt{2\lambda}} 1 - e^{-2b\sqrt{2\lambda}}$$

and coincides with (6.19) for $x = \pi \sqrt{2\lambda t}$ and thus for the level $b = \frac{\pi \sqrt{\lambda}}{2}$.

7. On the first passage time of the Brownian meander

In this section we give an explicit distribution of the first passage time $T_x$ of a Brownian meander through a point $x > 0$. In order to study $P\{T_x > s | \min_{0 \leq z \leq t} B(z) > 0, B(0) = u\}$ we must distinguish two cases, that is $s < t$ or $s > t$ since we assume that $\{\min_{0 \leq z \leq t} B(z) > 0\}$.

In the first case, for $s < t$ we have that

$$P\left\{ T_x > s \left| \min_{0 \leq z \leq t} B(z) > 0, B(0) = u \right. \right\} =$$

$$= \int_0^x \left[ P\left\{ \max_{0 \leq z \leq s} B(z) < x, \min_{0 \leq z \leq s} B(z) > 0, B(s) \in dy | B(0) = u \right. \right] \times$$

$$P\left\{ \min_{s \leq z \leq t} B(z) > 0 | B(0) = u \right\} \right\}$$

$$= \int_0^x \sum_{k=\infty}^{+\infty} \frac{1}{\sqrt{2\pi} s} \left( e^{-\frac{(y-u-2k\pi)^2}{2s}} - e^{-\frac{(y+u+2k\pi)^2}{2s}} \right) \frac{1 - 2P(B(t-s) > y)}{1 - 2P(B(t) > u)} dy$$

For $u \to 0$ we have that

$$P(T_x > s \left| \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0 \right.) =$$

$$= 2 \sqrt{\frac{t}{s}} \sum_{k=\infty}^{+\infty} \int_0^x \frac{(y-2k\pi)}{s} e^{-\frac{(y-2k\pi)^2}{2s}} \int_0^y e^{-\frac{w^2}{2\pi(t-s)}} \, dw \, dy$$

$$= 2 \sqrt{\frac{t}{s}} \sum_{k=\infty}^{+\infty} \int_0^x \frac{e^{-\frac{w^2}{2\pi(t-s)}}}{\sqrt{2\pi(t-s)}} \int_w^x \frac{(y-2k\pi)}{s} e^{-\frac{(y-2k\pi)^2}{2s}} \, dy \, dw$$

$$= 2 \sqrt{\frac{t}{s}} \sum_{k=\infty}^{+\infty} \int_0^x \frac{e^{-\frac{w^2}{2\pi(t-s)}}}{\sqrt{2\pi(t-s)}} \left\{ e^{-\frac{(w-2k\pi)^2}{2s}} - e^{-\frac{w^2}{2s}} \right\} \, dw \quad (7.2)$$

In the second case, for $s > t$, we have that

$$P\left\{ T_x > s \left| \min_{0 \leq z \leq t} B(z) > 0, B(0) = u \right. \right\} =$$

$$= 2 \sqrt{\frac{t}{s}} \sum_{k=\infty}^{+\infty} \int_0^x \frac{e^{-\frac{w^2}{2\pi(t-s)}}}{\sqrt{2\pi(t-s)}} \left\{ e^{-\frac{(w-2k\pi)^2}{2s}} - e^{-\frac{w^2}{2s}} \right\} \, dw \quad (7.3)$$
\[ P\left\{ 0 < \min_{0 \leq z \leq t} B(z) < \max_{0 \leq z \leq s} B(z) < x \mid B(0) = u \right\} = \]
\[ \int_0^x \left[ P\left\{ \max_{0 \leq z \leq t} B(z) < x, \min_{0 \leq z \leq t} B(z) > 0, B(t) \in dy \mid B(0) = u \right\} \right] \times \]
\[ P\left\{ \min_{0 \leq z \leq t} B(z) > 0 \mid B(0) = u \right\} \]
\[ = \int_0^x \sum_{k=-\infty}^{+\infty} \frac{1}{\sqrt{2 \pi t}} \left( e^{-\frac{(y-u-2kx)^2}{2t}} - e^{-\frac{(y-u+2kx)^2}{2t}} \right) \frac{1 - 2P(B(s-t) > x-y)}{1 - 2P(B(t) > u)} \ dy \]

By letting \( u \to 0 \) we extract from (7.3) the following probability

\[ P\left\{ T_x > s \mid \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0 \right\} = \]

\[ = 2 \sum_{k=-\infty}^{+\infty} \frac{1}{\sqrt{2 \pi t}} \int_0^x \frac{e^{-\frac{(y-2kx)^2}{2t}}}{t} \ dy \int_0^{x-y} \frac{e^{-\frac{w^2}{2(s-t)}}}{\sqrt{2\pi(s-t)}} \ dw \]

\[ = 2 \sum_{k=-\infty}^{+\infty} \left[ \frac{1}{\sqrt{2 \pi t}} \int_0^x \frac{e^{-\frac{w^2}{2(s-t)}}}{\sqrt{2\pi(s-t)}} \ dw \right] \left( e^{-\frac{(2kx)^2}{2t}} - e^{-\frac{(u+x(2k-1))^2}{2t}} \right) \]

For \( s \uparrow t \) (7.2) converges to the same limit of (7.4) for \( s \downarrow t \) that is

\[ P\left\{ T_x > s \mid \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0 \right\} = \sum_{r=-\infty}^{+\infty} (-1)^r e^{-r^2t^2} \]

which corresponds to (6.4) with \( x = 0 \).

**Remark 7.1.** For \( t \to \infty \) we obtain from (7.4) that

\[ P\left\{ T_x > s \mid \min_{0 \leq z < \infty} B(z) > 0, B(0) = 0 \right\} = \]

\[ = \frac{2}{\sqrt{2\pi s}} \sum_{k=-\infty}^{+\infty} \int_0^x \left\{ e^{-\frac{(w-2kx)^2}{2s}} - e^{-\frac{x^2(1-2k)^2}{2s}} \right\} \ dw \]

Clearly for \( x \to \infty \) we obtain from (7.5) that

\[ \lim_{x \to \infty} P\left\{ T_x > s \mid \min_{0 \leq z < \infty} B(z) > 0, B(0) = 0 \right\} = \frac{2}{\sqrt{2\pi s}} \int_0^{+\infty} e^{-\frac{w^2}{2s}} \ dw = 1 \]

because all the terms in the series converge to zero for \( x \to \infty \) except when \( k = 0 \). Since the the Gaussians \( e^{-\frac{(w-2kx)^2}{2s}} \) have means on the negative half-line for \( k \leq 0 \)
and maximum on the positive half-line for \( k > 0 \), we can write, after some simple calculations that

\[
P \left\{ T_x > s \ \bigg| \ \min_{0 \leq z < \infty} B(z) > 0, B(0) = 0 \right\} =
\frac{2}{\sqrt{2\pi}s} \left\{ \sum_{k=-\infty}^{0} \int_{0}^{\infty} \left( e^{-\frac{(w-2k\pi)^2}{2s}} - e^{-\frac{w^2(1-2k)^2}{2s}} \right) dw \right\} + \sum_{k=1}^{\infty} \int_{0}^{\infty} \left( e^{-\frac{(w-2k\pi)^2}{2s}} - e^{-\frac{w^2(1-2k)^2}{2s}} \right) dw
\]

\[
\leq \frac{2x}{\sqrt{2\pi}s} \sum_{k=0}^{\infty} \left\{ e^{-\frac{(2k\pi)^2}{2s}} - e^{-\frac{2(1-2k)^2}{2s}} \right\}
\]

Analogously we provide a lower bound for (7.5). Thus we have the following inequalities

\[
\frac{2x}{\sqrt{2\pi}s} \sum_{r=1}^{\infty} (-1)^r e^{-\frac{r^2}{2s}} \leq P \left\{ T_x > s \ \bigg| \ \min_{0 \leq z < \infty} B(z) > 0, B(0) = 0 \right\} 
\]

(7.6)

Thanks to (7.6) we are able to ascertain that (7.5) respects the constraint that

\[
\lim_{s \to \infty} P \left\{ T_x > s \ \bigg| \ \min_{0 \leq z < \infty} B(z) > 0, B(0) = 0 \right\} = 0
\]

8. Sojourn time of the Brownian meander

We here study the time spent by the Brownian particle on \([0, \infty)\) up to time \( t \) under the condition that \( \min_{0 \leq z \leq l} B(z) > 0 \), for \( l < t \). In symbols we want to study the distribution of

\[
\Gamma_{(0,t)} = l + \Gamma_{l,t} = l + \int_{l}^{t} 1_{[0,\infty)}(B(s)) ds
\]

(8.1)

Clearly the condition that \( \min_{0 \leq z \leq l} B(z) > 0 \), with the assumption that \( B(0) = u \), \( u > 0 \), exerts its effect on the distribution of the sojourn time \( \Gamma_{l,t} \).

It is well-known that

\[
P \left\{ \Gamma_t \in ds \bigg| B(0) = B(t) = 0 \right\} = \frac{ds}{t} \quad 0 < s < t
\]

(8.2)

where

\[
\Gamma_t = \int_{0}^{t} 1_{[0,\infty)}(B(s)) ds
\]

In the analysis of (8.1) we need to extend result (8.2), in particular we must generalize it as follows

\[
P \left\{ \Gamma_t \in ds \bigg| B(0) = u, B(t) = 0 \right\} =
\]

(8.3)
Hence we have that

\[ \sqrt{\frac{t}{2\pi}} e^{\frac{x^2}{2t}} \int_0^s \frac{ue^{-\frac{w^2}{2}}}{w^3(t-w)^3} \, dw \] 

\[ 0 \leq s \leq t \,, \quad u \in \mathbb{R}^+ \]

We give a brief sketch of the derivation of (8.3), based on the conditional Feynman-Kac functional. In Beghin et al. [2], Theorem 3.1, the case of Brownian motion with drift is examined. We consider that

\[ \psi(x, t) = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} \mathbb{E} \left( e^{-\beta \Gamma_t} \mid B(0) = u, B(t) = x \right) \]  

(8.4)

satisfies the Cauchy problem

\[
\begin{aligned}
\frac{\partial \psi}{\partial t} &= \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} - \beta \mathbb{1}_{[0,\infty)}(x) \psi \quad x > 0, \quad u > 0, \quad t > 0 \\
\psi(x, 0) &= \delta(x-u)
\end{aligned}
\]  

(8.5)

The time-Laplace transform of \( \psi \) has the form

\[
\Psi(x, \lambda) = \int_0^\infty e^{-\lambda t} \psi(x, t) \, dt
\]  

(8.6)

\[
\Psi(x, \lambda) = \begin{cases} \sqrt{2} \frac{e^{-u\sqrt{2(\lambda+\beta)}-x\sqrt{2\lambda}}}{\sqrt{(\lambda+\beta)+\sqrt{\lambda}}} & x \leq 0 \\ \frac{e^{-(u-x)\sqrt{2(\lambda+\beta)}}}{\sqrt{2(\lambda+\beta)}} + \frac{\sqrt{\lambda+\beta} - \sqrt{\lambda}}{\sqrt{\lambda+\beta} + \sqrt{\lambda}} \frac{e^{-(x+u)\sqrt{2(\lambda+\beta)}}}{\sqrt{2(\lambda+\beta)}} & 0 < x < u \\ \frac{e^{-(x-u)\sqrt{2(\lambda+\beta)}}}{\sqrt{2(\lambda+\beta)}} + \frac{\sqrt{\lambda+\beta} - \sqrt{\lambda}}{\sqrt{\lambda+\beta} + \sqrt{\lambda}} \frac{e^{-(x+u)\sqrt{2(\lambda+\beta)}}}{\sqrt{2(\lambda+\beta)}} & x \geq u \end{cases}
\]

For \( x = 0 \) and by considering that

\[
\int_0^\infty e^{-(\lambda+\beta)w} \frac{ue^{-\frac{w^2}{2\beta}}}{\sqrt{2\pi w^3}} \, dw = e^{-u\sqrt{2(\lambda+\beta)}}
\]

and

\[
\int_0^\infty e^{-\lambda w} \frac{1 - \beta}{\beta \sqrt{2\pi w^3}} \, dw = \frac{2}{\sqrt{2(\lambda+\beta)} + \sqrt{2\lambda}}
\]

the inverse Laplace transform of (8.6) (first branch) becomes

\[
\psi(0, t) = \frac{1}{2\pi \beta} \int_0^t \frac{ue^{-\frac{w^2}{2\beta}}}{\sqrt{w^3(t-w)^3}} \left( e^{-\beta w} - e^{-\beta t} \right) \, dw.
\]  

(8.7)

Hence we have that

\[
\mathbb{E} \left( e^{-\beta \Gamma_t} \mid B(0) = u, B(t) = 0 \right) =
\]  

(8.8)

\[
= \sqrt{\frac{2\pi t e^{\frac{x^2}{2t}}}{2\pi \beta}} \int_0^t \frac{ue^{-\frac{w^2}{2\beta}}}{\sqrt{w^3(t-w)^3}} \left( e^{-\beta w} - e^{-\beta t} \right) \, dw
\]

\[
= \sqrt{\frac{\pi e^{\frac{x^2}{2t}}}{2\pi}} \int_0^t \frac{ue^{-\frac{w^2}{2\beta}}}{\sqrt{w^3(t-w)^3}} \int_w^t e^{-\beta z} \, dz \, dw
\]

\[
= \sqrt{\frac{\pi e^{\frac{x^2}{2t}}}{2\pi}} \int_0^\infty e^{-\beta s} \left( \int_0^s \frac{ue^{-\frac{w^2}{2\beta}}}{\sqrt{w^3(t-w)^3}} \, dw \right) \, ds
\]
We have that
\[
P\left\{ \Gamma_t \in ds \middle| B(0) = u, B(t) = 0 \right\} =
\begin{align*}
&= ds \frac{\sqrt{te^{2\pi}}}{\sqrt{2\pi}} \int_0^s \frac{ue^{-\frac{y^2}{2\pi}}}{\sqrt{w^3(t-w)^3}} \, dw \\
&= ds \sqrt{te^{2\pi}} \int_0^s \frac{P\{T_u \in dw\}}{\sqrt{(t-w)^3}}
\end{align*}
\]  
where \( T_u = \inf \{ t \colon B(t) = u \} \).

Some calculations show that
\[
\lim_{u \to 0} P\{T_u \in dw\} = \delta(w)
\]
we have that
\[
\lim_{u \to 0} P\left\{ \Gamma_t \in ds \middle| B(0) = u, B(t) = 0 \right\} = \frac{ds}{t} 
\]
0 < s < t

Some calculations show that
\[
\int_0^t P\left\{ \Gamma_t \in ds \middle| B(0) = u, B(t) = 0 \right\} = \frac{\sqrt{t}}{2\pi} \int_0^t ds \int_0^s \frac{ue^{-\frac{y^2}{2\pi}}}{\sqrt{w^3(t-w)^3}} \, dw =
\]
\[
\sqrt{\frac{t}{2\pi}} e^{2\pi} \int_0^t \frac{ue^{-\frac{y^2}{2\pi}}}{\sqrt{w^3(t-w)^3}} \, dw \int_w^t ds = \sqrt{\frac{t}{2\pi}} e^{2\pi} \int_0^t \frac{ue^{-\frac{y^2}{2\pi}}}{\sqrt{w^3(t-w)^3}} \, dw =
\]
\[
\sqrt{\frac{t}{2\pi}} e^{2\pi} \int_0^t \frac{ue^{-\frac{y^2}{2\pi}}}{\sqrt{w^3(t-w)^3}} \, dw = \sqrt{\frac{t}{2\pi}} e^{2\pi} \int_0^t \frac{ue^{-\frac{y^2}{2\pi}}}{\sqrt{w^3(t-w)^3}} \, dw =
\]
\[
\sqrt{\frac{t}{2\pi}} e^{2\pi} \int_0^t \frac{ue^{-\frac{y^2}{2\pi}}}{\sqrt{w^3(t-w)^3}} \, dw = \sqrt{\frac{t}{2\pi}} e^{2\pi} \int_0^t \frac{ue^{-\frac{y^2}{2\pi}}}{\sqrt{w^3(t-w)^3}} \, dw =
\]
\[
\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{ue^{-\frac{y^2}{2\pi}}}{\sqrt{w^3(t-w)^3}} \, dw = 1
\]

This is important because it shows that the sojourn time of the Brownian motion with \( B(t) = 0 \) is an absolutely continuous r.v. while for the free Brownian motion starting at \( u \neq 0 \) there is a positive probability of staying on the half line \([0, \infty)\) for the entire time interval.

If we consider the Brownian motion \( B \) in the interval \([0, t]\) subject to the condition that \( \min_{0 \leq z \leq t} B(z) > 0 \), for \( l < t \), the total time spent by \( B \) on the positive half-line in \([0, \infty)\) is \( l + \Gamma_{l,t} \).

In the next theorem we are able to give the explicit distribution of \( \Gamma_{l,t} \).

**Theorem 8.1.** For a standard Brownian motion \( (B(t), \ t > 0) \) we have that
\[
\lim_{u \to 0} P\left\{ \Gamma_{l,t} \in ds \middle| \min_{0 \leq z \leq t} B(z) > 0, B(0) = u \right\} =
\begin{cases}
\frac{1}{\pi \sqrt{s(t-l-s)}} \frac{s}{s+l} ds & 0 \leq s < t-l \\
\sqrt{\frac{l}{t}} & s = t-l
\end{cases}
\]
Proof. We start by considering
\[
P\left\{ \Gamma_{t,t} \in ds \mid \min_{0 \leq s \leq t} B(z) > 0, B(0) = u \right\} =
\]
\[
\int_0^\infty P\left\{ \Gamma_{t,t} \in ds, B(l) \in dw \mid \min_{0 \leq s \leq l} B(z) > 0, B(0) = u \right\} =
\int_0^\infty P\left\{ B(l) \in dw \mid \min_{0 \leq s \leq l} B(z) > 0, B(0) = u \right\} P\left\{ \Gamma_{t,t} \in ds \mid B(l) = w \right\}
\]
In order to evaluate (8.11) we need the following result valid for examination two different cases.

Result (8.12) shows that for a positive starting point \( x > 0 \) there is a positive probability of never crossing the zero level. Thus in order to evaluate (8.11) we must examine two different cases.

If we consider the absolutely continuous component of (8.12) we have that, for \( s < t \),
\[
P\left\{ \Gamma_{t,t} \in ds \mid \min_{0 \leq s \leq l} B(z) > 0, B(0) = u \right\} =
\int_0^\infty \frac{e^{-\frac{w^2}{2t}}}{\pi \sqrt{s(t-s)}} \left[ e^{-\frac{(w-u)^2}{2t}} - e^{-\frac{(w+u)^2}{2t}} \right] \frac{w}{\sqrt{2\pi t}} dw
ds
1 - 2 \int_u^\infty \frac{e^{-\frac{w^2}{2t}}}{\sqrt{2\pi t}} dw
\]
By taking the limit for \( u \to 0 \) of (8.13) we arrive at
\[
P\left\{ \Gamma_{t,t} \in ds \mid \min_{0 \leq s \leq l} B(z) > 0, B(0) = u \right\} =
\int_0^\infty \frac{e^{-\frac{w^2}{2t}}}{\pi \sqrt{s(t-s)}} \frac{w}{\sqrt{2\pi t}} e^{-\frac{w^2}{2}} dw
= \frac{ds}{\pi} \sqrt{\frac{s}{t-l}} \frac{1}{s+l} 0 \leq s < t - l
\]
In the second case we have that, in view of (2.4) and the second formula of (8.12),
\[
P\left\{ \Gamma_{t,t} = t - l \mid \min_{0 \leq s \leq l} B(z) > 0, B(0) = u \right\} =
\int_0^\infty P\left\{ B(l) \in dy \mid \min_{0 \leq s \leq l} B(z) > 0, B(0) = u \right\} P\left\{ \Gamma_{t,t} = t - l \mid B(l) = y \right\}
= \frac{1}{1 - 2 \int_u^\infty \frac{e^{-\frac{w^2}{2t}}}{\sqrt{2\pi t}} dw} \int_0^\infty \left[ e^{-\frac{(w-u)^2}{2t}} - e^{-\frac{(w+u)^2}{2t}} \right] \frac{2dy}{\sqrt{2\pi(t-l)}} \int_0^y \frac{e^{-\frac{w^2}{2(t-l)}}}{\sqrt{2\pi(t-l)}} dw
\]
For $u \to 0$ the limit of (8.15) becomes

\[ P \left\{ \Gamma_{l,t} \in [t-l, t] \mid \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0 \right\} = \int_0^\infty \frac{2dy}{\sqrt{2\pi(t-l)}} \left( \int_y^\infty e^{-\frac{y^2}{2(t-l)}} dw \right) \frac{y}{t} e^{-\frac{y^2}{2t}} \]

\[ = \frac{2}{\sqrt{2\pi(t-l)}} \int_0^\infty e^{-\frac{y^2}{2(t-l)}} \frac{y}{t} e^{-\frac{y^2}{2t}} dy \]

\[ = \frac{2}{\sqrt{2\pi(t-l)}} \sqrt{\frac{l(t-l)}{t}} \int_0^\infty e^{-\frac{w^2}{2t}} dw = \sqrt{\frac{l}{t}} \]

\[ \square \]

**Remark 8.1.** We check that the distribution (8.10) integrates to one. The first term of (8.10) has weight equal to

\[ \frac{1}{\pi} \int_0^{t-l} \sqrt{\frac{s}{t-l-s}} \frac{ds}{t+l+s} = \frac{1}{\pi} \int_0^{t-l} \sqrt{\frac{y}{t-l+y(t-l)y}} dy \]

\[ = \frac{2}{\pi} \int_0^{\frac{t-l}{\pi}} \sin^2 \varphi \frac{t \sin^2 \varphi + l \cos^2 \varphi}{t} d\varphi \]

\[ = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{t \tan^2 \varphi + l}{t \tan^2 \varphi + l} d\varphi \]

\[ = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{t \tan^2 \varphi + l}{t \tan^2 \varphi + l} \frac{dy}{1+y^2} \]

\[ = \frac{2}{\pi} \left[ \int_0^\infty \frac{dy}{1+y^2} - (t-l) \int_0^\frac{l}{t} \frac{dy}{1+y^2} \right] = 1 - \frac{\sqrt{l}}{t} \]

For growing values of $l$ the probability of the discrete component of (8.10) tends to one because in a short time interval the particle is unable to cross the zero level. In other words, for $l \to 0$, from (8.14) we recover the arcsine law.

**Remark 8.2.** The distribution (8.10) can be regarded as the distribution of the sojourn time on $[0, \infty)$ of a Brownian motion starting from a random initial point possessing Rayleigh distribution. In other terms we have that

\[ P \left\{ \Gamma_{l,t} \in ds \mid \min_{0 \leq z \leq t} B(z) > 0, B(0) = 0 \right\} = \int_0^\infty P \left\{ \Gamma_{0,t} \in ds \mid \min_{0 \leq z \leq \frac{u}{l}} B(z) > 0, B(0) = u \right\} \frac{u}{l} e^{-\frac{u^2}{2t}} du \]

In view of (1.2) the initial point has distribution coinciding with the random position of the Brownian meander at time $l$. 
if we let where we applied (8.3) in the last step. The probability (8.17) considerably simplifies

\[ P\left\{ \Gamma_{l,t} \in ds \mid \min_{0 \leq z \leq l} B(z) > 0, B(0) = u, B(t) = 0 \right\} = \]

\[ = \int_0^\infty P\left\{ \min_{0 \leq z \leq l} B(z) > 0, B(l) \in dy \mid B(0) = u \right\} \times \]

\[ \times P\left\{ \Gamma_{l,t} \in ds \mid B(l) \in dy, B(t) = 0 \right\} P\left\{ B(t) \in d0 \mid B(l) = y \right\} \times \]

\[ \times \left( \int_0^\infty P\left\{ \min_{0 \leq z \leq l} B(z) > 0, B(l) \in dy \mid B(0) = u \right\} P\left\{ B(t) \in d0 \mid B(l) = y \right\} \right)^{-1} \]

\[ = \int_0^\infty \left[ \frac{e^{-\frac{(y-u)^2}{2t}} - e^{-\frac{(y+u)^2}{2t}}}{\sqrt{2\pi t}} \right]^2 \left[ \sqrt{\frac{t-1}{2\pi}} e^{\frac{y^2}{2(t-1)}} \int_0^u \frac{ye^{-\frac{y^2}{2(t-1)}}}{\sqrt{u^3(t-1-w)^3}} dw \right] e^{-\frac{y^2}{2\pi(t-1)}} dy \]

\[ \int_0^\infty \left[ \frac{e^{-\frac{(y-u)^2}{2t}} - e^{-\frac{(y+u)^2}{2t}}}{\sqrt{2\pi t}} \right]^2 \frac{e^{-\frac{y^2}{2\pi(t-1)}}}{\sqrt{2\pi(t-1)}} dy \]

where we applied (8.3) in the last step. The probability (8.17) considerably simplifies if we let \( u \to 0 \). In this case we have that

\[ P\left\{ \Gamma_{l,t} \in ds \mid \min_{0 \leq z \leq l} B(z) > 0, B(0) = 0, B(t) = 0 \right\} / ds = \]

\[ = \int_0^\infty ye^{-\frac{y^2}{2t}} \sqrt{\frac{t-1}{2\pi}} \int_0^u \frac{ye^{-\frac{y^2}{2(t-1)}}}{\sqrt{u^3(t-1-w)^3}} dw dy \]

\[ = \frac{t}{l(t-1)} \int_0^u \frac{dw}{\sqrt{w^3(t-l-w)^3}} \int_0^\infty \frac{y^2 e^{-\frac{y^2}{2\pi(t-1)}}}{\sqrt{2\pi}} dy \]

\[ = \frac{t}{2l\sqrt{t-1}} \int_0^u \frac{dw}{\sqrt{w^3(t-l-w)^3}} \left( \sqrt{\frac{l}{l+w}} \right)^3 \]

\[ = \frac{t\sqrt{l}}{2\sqrt{l-1}} \int_0^u \frac{dw}{\sqrt{(t-l-w)^3(l+w)^3}} \]
\begin{equation*}
= \frac{t}{2} \sqrt{\frac{t}{t-l}} \int_l^{l+s} dw \quad 0 < s < t-l
\end{equation*}

We now check that

\[ \int_0^{t-l} P\left\{ \Gamma_{l,t} \in ds \left| \min_{0 \leq z \leq l} B(z) > 0, B(0) = 0, B(t) = 0 \right. \right\} = 1 \]

In view of (8.18) we have that

\begin{align*}
&\frac{t\sqrt{t}}{2\sqrt{t-l}} \int_0^{t-l} ds \int_0^s \frac{dw}{\sqrt{(t-l-w)^3(l+w)^3}} = \\
&= \frac{t\sqrt{t}}{2\sqrt{t-l}} \int_0^{t-l} dw \int_w^l \frac{ds}{\sqrt{(t-l-w)^3(l+w)^3}} \\
&= \frac{t\sqrt{t}}{2\sqrt{t-l}} \int_0^{t-l} \frac{dw}{\sqrt{(l+w)^3(t-l-w)}} \\
&= \frac{t}{2} \sqrt{\frac{t}{t-l}} \int_l^t \frac{dw}{\sqrt{w^3(t-w)}}
\end{align*}

With the change of variable \( w = t \sin^2 \varphi \) the last integral becomes

\begin{align*}
&\int_0^{t-l} P\left\{ \Gamma_{l,t} \in ds \left| \min_{0 \leq z \leq l} B(z) > 0, B(0) = 0, B(t) = 0 \right. \right\} = \\
&= \frac{t}{2} \sqrt{\frac{t}{t-l}} \int_{\arcsin \sqrt{\frac{2t}{t}} \cos \varphi}^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{t \cos \varphi \sin^3 \varphi \sqrt{t}}} \\
&= \sqrt{\frac{l}{t-l}} \int_{\arcsin \sqrt{\frac{2t}{t}}}^{\frac{\pi}{2}} \frac{d\varphi}{\sin^2 \varphi} \\
&= \sqrt{\frac{l}{t-l}} \left[ -\cos \varphi \right]_{\arcsin \sqrt{\frac{2t}{t}}}^{\frac{\pi}{2}} \\
&= \sqrt{\frac{l}{t-l}} \left( \sqrt{\frac{1-t}{l}} \right) = 1
\end{align*}

Remark 8.3. The probability distribution of \( \Gamma_{l,t} \) [8.18] can be worked out explicitly as follows

\begin{align*}
P\left\{ \Gamma_{l,t} \in ds \left| \min_{0 \leq z \leq l} B(z) > 0, B(0) = 0, B(t) = 0 \right. \right\} / ds = \\
&= \frac{t}{2} \sqrt{\frac{t}{t-l}} \int_l^{l+s} dw \quad \sqrt{w^3(t-w)^3} \\
&= \frac{t}{2} \sqrt{\frac{t}{t-l}} \int_{\arcsin \sqrt{\frac{1+t}{t}}}^{\frac{\pi}{2}} \frac{d\varphi}{\sin^2 \varphi \cos^2 \varphi} \\
&= \frac{1}{l} \sqrt{\frac{l}{t-l}} \left\{ \sqrt{\frac{1+s}{t-l}} - \sqrt{\frac{l}{t-l}} - \sqrt{\frac{t-l-s}{t+l}} + \sqrt{\frac{t-l}{l}} \right\}
\end{align*}
where

\[ h \]

We can evaluate the mean value of

\[ \text{We observe that for} \]

In view of (8.19) we can write (8.21) as

\[ f_{\Gamma_{l,t}} = H \cdot \left\{ h(s) - \frac{1}{h(s)} + K \right\} \]

Thus

\[ \frac{df_{\Gamma_{l,t}}}{ds} = H \left\{ h'(s) + \frac{h'(s)}{h^2(s)} \right\} = H \cdot h'(s) \left\{ 1 + \frac{1}{h^2(s)} \right\} \]

where \( h(s) = \sqrt{\frac{t+s}{t-t}} \). Furthermore we have that

\[ \lim_{s \to 0^+} f_{\Gamma_{l,t}} = 0 \quad \lim_{s \to t^-} f_{\Gamma_{l,t}} = +\infty \]

This shows that the assumption that \( \min_{0 \leq z \leq t} B(s) > 0 \) implies that in \([l,t]\) the Brownian particle spends on the half-line an increasingly long time.

For \( l = \frac{t}{2} \) the density (8.19) simplifies as

\[ P \left\{ \Gamma_{\frac{t}{2},t} \in ds \left| \min_{0 \leq z \leq \frac{t}{2}} B(z) > 0, B(0) = 0, B(t) = 0 \right. \right\} = \]

\[ = \frac{4s}{t\sqrt{t^2 - 4s^2}} ds \quad 0 < s < t \]

We observe that for \( l = 0 \) we retrieve in (8.19) the uniform distribution.

We can evaluate the mean value of \( \Gamma_{l,t} \) which becomes

\[ \mathbb{E} \left( \Gamma_{l,t} \left| \min_{0 \leq z \leq l} B(z) > 0, B(0) = B(t) = 0 \right. \right) = \]

\[ = \frac{t}{2} \sqrt{\frac{l}{t-l}} \left( \frac{\pi}{2} - \arcsin \sqrt{\frac{l}{t}} \right) + \frac{t-2l}{2} \]

In view of (8.19) we can write (8.21) as

\[ \mathbb{E} \left( \Gamma_{l,t} \left| \min_{0 \leq z \leq l} B(z) > 0, B(0) = B(t) = 0 \right. \right) = \]

\[ = \frac{t}{2} \sqrt{\frac{l}{t-l}} \int_0^{t-l} \int_0^s \frac{dw}{\sqrt{(t-l-w)^3(l+w)^3}} ds = \]

\[ = \frac{t}{2} \sqrt{\frac{l}{t-l}} \int_0^{t-l} \frac{dw}{\sqrt{(t-l-w)^3(l+w)^3}} \int_w^{t-l} s ds = \]

\[ = \frac{t}{4} \sqrt{\frac{l}{t-l}} \int_0^{t-l} \frac{t-l+w}{\sqrt{(t-l-w)(l+w)^3}} dw \quad (w + l = t \sin^2 \varphi) \]

\[ = \frac{1}{2} \sqrt{\frac{l}{t-l}} \int_{\arcsin \sqrt{\frac{l}{t}}}^{\frac{\pi}{2}} \left\{ \frac{t-2l}{\sin^2 \varphi} + t \right\} d\varphi \]

\[ = \frac{t}{2} \sqrt{\frac{l}{t-l}} \left( \frac{\pi}{2} - \arcsin \sqrt{\frac{l}{t}} \right) + \frac{t-2l}{2} \]
For \( l = 0 \) \( \mathbb{E} \Gamma_{0,t} = \frac{t}{2} \) while for \( l = \frac{t}{2} \) we have \( \mathbb{E} \Gamma_{\frac{t}{2},t} = \frac{t^2}{8} \). For this case the mean total amount of time spent on \((0, \infty)\) by the Brownian particle is therefore \( \frac{t}{2} (1 + \frac{\pi}{4}) \). The distribution function of \( \Gamma_{1,t} \) writes

\[
P \left\{ \Gamma_{1,t} < s \mid \min_{0 \leq z \leq l} B(z) > 0, B(0) = B(t) = 0 \right\} = \frac{1}{t} \sqrt{\frac{l}{t-l}} \int_0^s \left[ \frac{t-2l}{\sqrt{l(t-l)}} + \frac{\sqrt{l+s}}{\sqrt{t-l-s}} - \frac{\sqrt{l-l-s}}{\sqrt{t+s}} \right] \, ds
\]

From (8.22) it emerges that

\[
P \left\{ \Gamma_{1,t} < \bar{s} \mid \min_{0 \leq z \leq l} B(z) > 0, B(0) = B(t) = 0 \right\} = \frac{\bar{s}(t-2l)}{t(t-l)} - 2\sqrt{\frac{l}{t-l}} \int_{\arcsin \sqrt{\frac{t}{t-l}}}^{\arcsin \sqrt{\frac{l}{t-l}}} \cos^2 \varphi \, d\varphi = \frac{\bar{s}(t-2l)}{t(t-l)} - 2\sqrt{\frac{l}{t-l}} \left\{ \frac{\arcsin \sqrt{\frac{t}{t-l}}}{\arcsin \sqrt{\frac{l}{t-l}}} \right\}
\]

that is the random ratio \( \Gamma_{1,t} \) is asymptotically uniform on \([0, 1]\).

From (8.22) it emerges that

\[
\lim_{t \to \infty} P \left\{ \Gamma_{1,t} < \bar{s} \mid \min_{0 \leq z \leq l} B(z) > 0, B(0) = B(t) = 0 \right\} = \bar{s} \quad 0 < \bar{s} < 1
\]

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