INNER GALOIS EQUIDISTRIBUTION
IN S-HECKE ORBITS

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Contents

1. Introduction. 1
2. From Inner equidistribution to Topological closures 10
3. Review of Galois monodromy representations 10
4. The S-Arithmetic lift and Equidistribution 17
5. Focusing criterion and Internality of equidistribution 31
6. Zariski closedness from the S-Mumford-Tate hypothesis. 47
References 47

1. Introduction.

This paper concerns itself with certain special cases of the Zilber-Pink conjecture on unlikely intersections in Shimura varieties and some of its natural generalisations.

Let $(G, X)$ be a Shimura datum, let $K$ be a compact open subgroup of $G(\mathbb{A}_f)$, and let

$$\text{Sh}_K(G, X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$$

be the associated Shimura variety. We refer to [21] and references therein for definitions and facts related to Shimura varieties. Throughout the paper we always assume that $G$ is the generic Mumford-Tate group on $X$. This is a standard convention in the theory of Shimura varieties. We will write a point of $\text{Sh}_K(G, X)$ as $s = (h, t)$ - the double class of the element $(h, t) \in X \times G(\mathbb{A}_f)$.

The Hecke orbit of a point of $\text{Sh}_K(G, X)$ is defined as follows.

**Definition 1.1** (Hecke orbit). Let $s = (h, t)$ be a point of $\text{Sh}_K(G, X)$. We define the Hecke orbit of the point $s$ in $\text{Sh}_K(G, X)$ to be the set

$$\mathcal{H}(s) = \{(h, t \cdot g) \mid g \in G(\mathbb{A}_f)\} \subseteq \text{Sh}_K(G, X).$$

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We refer to [22] for a definition of weakly special subvarieties of $\text{Sh}_K(G, X)$ and related notions. Note that several equivalent definitions (different in flavour) are given in [38], [42] and [41].

The Andrè-Pink-Zannier conjecture is the following statement.

**Conjecture 1.2** (Andrè-Pink-Zannier). Irreducible components of the Zariski closure of any subset $\Sigma$ of $\mathcal{H}(s)$ are weakly special subvarieties.

This conjecture was formulated by Andrè for curves in Shimura varieties in [1], then by Pink for arbitrary subvarieties of mixed Shimura varieties in [25] and independently by Zannier (unpublished). We therefore refer to this conjecture as the Andrè-Pink-Zannier conjecture.

For curves in Shimura varieties of abelian type, conjecture 1.2 was proved by Orr in [23]. Orr also proves in [23] that conjecture 1.2 is a special case of the Zilber-Pink conjecture on unlikely intersections in Shimura varieties.

In this paper we deal with the ‘$S$-Hecke orbit’ version of Conjecture 1.2. We actually prove much stronger statements than the conclusion of 1.2.

1.1. **The $S$-André-Pink-Zannier conjecture.** We consider the following weaker notion of a Hecke orbit.

**Definition 1.3** ($S$-Hecke orbits). Let $S$ be a finite set of prime numbers.

1. Write $(g_\ell)_{\ell}$ an element of $G(\mathbb{A}_f)$ viewed as a restricted product indexed by primes $\ell$. We denote $G_S$ the subgroup of $G(\mathbb{A}_f)$ consisting of elements $(g_\ell)_{\ell}$ such that $g_\ell = 1$ for $\ell \notin S$.

As $S$ is a finite set, we may identify $G_S$ with $\prod_{\ell \in S} G(\mathbb{Q}_\ell)$, or equivalently with $G(\mathbb{Q}_S)$ where $\mathbb{Q}_S = \prod_{\ell \in S} \mathbb{Q}_\ell$.

2. For a point $s = (h, t)$ of $\text{Sh}_K(G, X)$, we define the $S$-Hecke orbit of $s$ to be the subset

$$\mathcal{H}_S(s) = \left\{ (h, t \cdot g) \mid g \in G_S \right\} \subseteq \mathcal{H}(s).$$

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A concise characterisation of weakly special subvarieties, the one studied in [22], is the following: they are the complex algebraic subvarieties of $\text{Sh}_K(G, X)$ which are the images of totally geodesic submanifolds of the symmetric space $X \times \{ t \}$, with some $t \in G(\mathbb{A}_f)$. Each such subvariety is isomorphic to a component of some Shimura variety, that is, an arithmetic quotient of a Hermitian symmetric domain. Weakly special subvarieties containing smooth special points are called special subvarieties. A special subvariety is a component of the image of a morphism of Shimura varieties induced by a morphism of Shimura data. In a certain sense a weakly special subvariety is a translate of a special subvariety.
Let $E$ be a field of definition of $s$. The theory of Shimura varieties (the fact that Hecke correspondences are defined over the reflex field $E(G, X)$) shows that points of $\mathcal{H}(s)$ and $\mathcal{H}_S(s)$ are defined over $\overline{E}$.

We introduce the following definition which will play a central role in what follows.

**Definition 1.4** (S-Shafarevich property). *The point $s$ is said to satisfy the S-Shafarevich property or to be of S-Shafarevich type if for every finite extension $F \subset \overline{E}$ of $E$, $\mathcal{H}_S(s)$ contains only finitely many points defined over $F$.*

We immediately observe that the S-Shafarevich property is invariant by finite morphisms induced by the inclusions of compact open subgroups $K' \subset K$. The conclusions of Conjectures 1.2 and 1.5 as well as those of our main theorems 1.15 and 1.16 are also invariant by replacing $K$ by a subgroup of finite index. Therefore, throughout the paper we always assume the group $K$ to be neat. More precisely, we choose the group $K$ as follows.

1. $K$ is a product $K = \prod K_p$ of compact open subgroups $K_p \subset G(\mathbb{Q}_l)$.
2. Fix $l \geq 3$ and $l \notin S$. We assume that $K_l$ is contained in the group of elements congruent to the identity modulo $l$ with respect to some faithful representation $G \subset \text{GL}_n$.

As noted just above the assumptions cause no loss of any generality but avoid many annoying technicalities. These assumptions will be kept through the paper.

We will examine this property in great detail in section 3. In particular, we will obtain a group theoretic characterisation of this property.

We now formulate the corresponding weaker form of Conjecture 1.2 which will be the main object of study in this paper.

**Conjecture 1.5** (André-Pink-Zannier conjecture for S-Hecke orbits). *Irreducible components of the Zariski closure of any subset of $\mathcal{H}_S(s)$ are weakly special subvarieties.*

In this paper we prove results in the direction of Conjecture 1.5. There are some known cases of this conjecture, all are implied by our results. In particular,

- that Conjecture 1.5 holds whenever $(G, X)$ is of abelian type (proved in [24] by a different method)
- that Conjecture 1.5 holds if $s \in \text{Sh}_K(G, X)$ is a special point (this is a special case of the 'weak André-Oort conjecture' proven in [19]).
The conclusion of our main theorem holds under the assumption that the $S$-Shafarevich property holds. We actually prove equidistribution results that are much stronger than conclusions of Conjecture.

1.2. Galois monodromy properties. Let $E \subset \mathbb{C}$ be a field over which the point $s = (h, t)$ is defined. Let $\mathcal{M} \subset \mathcal{G}$ denote the Mumford-Tate group of $h$. We assume that $E$ contains the reflex field of the Shimura datum $(\mathcal{M}, \mathcal{M}(\mathbb{R}) \cdot h)$.

**Definition 1.6.** (1) There exists a continuous “$S$-adic monodromy representation”

$$\rho_{h,S} : \text{Gal}(\overline{E}/E) \rightarrow M(\mathbb{A}_f) \cap K \cap G_S.$$ which has the property that for any $g \in G_S$ and $\sigma \in \text{Gal}(\overline{E}/E)$,

$$\sigma((h, tg)) = (h, \rho(\sigma) \cdot tg).$$

(2) The $S$-adic monodromy group, which we will denote by $U_S$, is defined as the image of $\rho_{h,S}$:

$$U_S = \rho_{h,S}(\text{Gal}(\overline{E}/E)).$$

This is a compact $S$-adic Lie subgroup of $M(\mathbb{A}_f) \cap G_S \simeq M(\mathbb{Q}_S)$.

(3) The algebraic $S$-adic monodromy group, denoted $H_S$, is defined as the algebraic envelope in $M_{\mathbb{Q}_S}$ of the subgroup $U_S$ of $M(\mathbb{Q}_S)$:

$$H_S = \mathcal{U}^\text{zar}_S.$$ This is an algebraic group over $\mathbb{Q}_S$. We write $H^0_S$ for its neutral component.

**Definition 1.7.** Let $s$ be a point of $\text{Sh}_K(G, X)$ and $E$ a field of definition of $s$ such that the associated $S$-adic monodromy representation

$$\rho : \text{Gal}(\overline{E}/E) \rightarrow M(\mathbb{A}_f) \cap G_S$$

is defined. We say that the point $s$ over $E$ is

(1) of $S$-Mumford-Tate type if $U_S$ is open in $M(\mathbb{A}_f) \cap G_S$;

(2) of $S$-Shafarevich type if for every finite extension $F$ of $E$, there are only finitely many $F$-rational points in $\mathcal{H}_S(s)$ (i.e. property in definition holds);

(3) of $S$-Tate type if $M$ and $H^0_S$ share the same centraliser in $G_S$:

$$Z_{G_S}(H^0_S) = Z_{G_S}(M).$$

We say that the point $s$ over $E$

(4) satisfies $S$-semisimplicity if $H_S$ is a reductive group;

(5) satisfies $S$-algebraicity if the subgroup $U_S$ of $H_S(\mathbb{Q}_S)$ is open.
This latter is equivalent to the Lie algebra of $U_S$ being algebraic (in the sense of Chevalley, cf. \cite[§7]{[5]}), as a $\mathbb{Q}_S$-Lie subalgebra of the Lie algebra of $M_{\mathbb{Q}_S}$.

We will examine the interrelations between these properties in detail in section 3. We will also provide examples where these various assumptions hold and where they do not. In 3 we will make apparent that these properties are very heavily dependent on the choice of field $E$, especially its property of being of finite type over $\mathbb{Q}$ or not.

Note that a point of $S$-Mumford-Tate type is obviously of $S$-Tate type. These properties hold notably for a special point, by definition of a canonical model. We will prove that $S$-Tate type property (3) implies the algebraicity of $U_S$ if $E$ is of finite type. As a consequence, for such $E$, the $S$-Mumford-Tate type property (1) is equivalent to its a priori weaker variant: $H_0^S = M_{\mathbb{Q}_S}$.

Our main results are proved under the $S$-Shafarevich hypothesis (2) which as we will see in section 3 is implied by (and in fact not far from being equivalent to) the $S$-Tate hypothesis (3) together with the $S$-semisimplicity assumption (4). More precisely, the main result of section 3 is that the $S$-Shafarevich property is equivalent to $S$-semisimplicity together with the property that the centraliser of $U_S$ in $G_S$ is compact modulo the centraliser of $M$ in $G_S$.

1.3. Notions related to equidistribution. Let $s \in \text{Sh}_K(G, X)(E)$. Recall that points of $\mathcal{H}(s)$ are defined over $\overline{E}$.

To any $z \in \mathcal{H}(s)$ we attach a probability measure with finite support that we define as follows. In what follows, for a point $z$ in $\text{Sh}_K(G, X)$, we refer to the following set as its Galois orbit:

$$\text{Gal} \left( \overline{E}/E \right) \cdot z = \{ \sigma(z) \mid \sigma \in \text{Gal} \left( \overline{E}/E \right) \}.$$

**Definition 1.8.** Let $z = (x, t)$ be a point of $\mathcal{H}_S(s)$. We define

$$\mu_z = \frac{1}{\# \text{Gal} \left( \overline{E}/E \right) \cdot z} \sum_{\zeta \in \text{Gal} \left( \overline{E}/E \right) \cdot z} \delta_{\zeta}$$

where $\delta_{\zeta}$ is the Dirac mass at $\zeta$.

We introduce a class of groups, arising from the use of Ratner’s theorem.

**Definition 1.9.** A connected $\mathbb{Q}$-subgroup $L$ of $G$ is said to be of $S$-Ratner class if its Levi subgroups are semisimple and for every $\mathbb{Q}$-quasi factor $F$ of this Levi, $F(\mathbb{R} \times \mathbb{Q}_S)$ is not compact\(^2\).

\(^2\)Equivalently such factor $F$ can not be anisotropic simultaneously over $\mathbb{R}$ and every of the $\mathbb{Q}_v$ for $v$ in $S$. 
We denote by $L^\dagger$ the subgroup of $L(\mathbb{R} \times \mathbb{Q}_S)$ generated by the unipotent elements. Then $L$ is of $S$-Ratner class if and only if no proper subgroup of $L$ defined over $\mathbb{Q}$ contains $L^\dagger$.

We denote by $L(\mathbb{R})^+$ the neutral component of $L(\mathbb{R})$ with respect to the Archimedean topology.

This class is slightly more general than [37, Déf. 2.1] and [41, Def. 2.4]. The latter actually corresponds to the case where $S$ is empty, in which case we have $L^\dagger = L(\mathbb{R})^+$.

Let $L$ be subgroup of $G$ of $S$-Ratner class and $(h,t) \in X \times G(\mathbb{A}_f)$, to such data we associate the following subset (actually a real analytic variety):

$$Z_{L,(h,t)} = \{ (l \cdot h,t) \mid l \in L(\mathbb{R})^+ \} \subseteq \text{Sh}_K(G,X).$$

We consider the corresponding generalisation of a notion of weakly special subvariety from [41].

**Definition 1.10.** A weakly $S$-special real submanifold (or subvariety) $Z$ of $\text{Sh}_K(G,X)$ is a subset of the form $Z_{L,(h,t)}$ for some subgroup $L$ of $G$ of $S$-Ratner class and $(h,t)$ in $X \times G(\mathbb{A}_f)$.

Note that the parametrising map $L(\mathbb{R})^+ \xrightarrow{\text{inv image}} Z_{L,(h,t)}$ induces a homeomorphism

$$(\Gamma_{tK} \cap L(\mathbb{R})^+) \setminus L(\mathbb{R})^+/L(\mathbb{R})^+ \cap K_h \to Z_{L,(h,t)}$$

where $\Gamma_{tK}$ is an arithmetic subgroup of $G(\mathbb{Q})$ depending only on $tK$ whose intersection with $L(\mathbb{R})^+$ is an arithmetic subgroup, necessarily a lattice. Indeed, there is a canonical right $L(\mathbb{R})^+$-invariant probability measure on $(\Gamma_{tK} \cap L(\mathbb{R})^+) \setminus L(\mathbb{R})^+$, we denote $\mu_{L,(h,t)}$ its direct image in $Z_{L,(h,t)}$, viewed as a Borel probability measure on $\text{Sh}_K(G,X)$.

**Definition 1.11.** Let $Z$ be a weakly $S$-special real submanifold $Z$ inside of $\text{Sh}_K(G,X)$. The canonical probability $\mu_Z$ with support $Z$, is a measure of the form $\mu_{L,(h,t)}$ with support $Z = Z_{L,(h,t)}$.

The next lemma shows that $\mu_{L,(h,t)}$ is independent of choices.

**Lemma 1.12.** The canonical probability measure $\mu_Z$ is well defined: it depends only on $Z$ and not on the choice of $(L,(h,t))$.

**Proof.** Firstly note that $Z$ is almost everywhere locally isomorphic to its inverse image $\tilde{Z}$ in $X \times G(\mathbb{A}_f)/K$, and $\mu_Z$ is determined by the corresponding locally finite measure $\mu_{\tilde{Z}}$ on $\tilde{Z}$. It will suffice to show that $\mu_{\tilde{Z}}$ actually every Zariski connected subgroup $L$ with semisimple Levi subgroups will be of $S$-Ratner class for some $S$ big enough.
is intrinsic up to a locally constant scaling factor, the latter being characterised by $\mu_Z$ being a probability and $Z$ being connected. Endow $X$ with a $G(\mathbb{R})$-invariant Riemannian structure, which we extend to $X \times G(\mathbb{A}_f)/K$. Then $\mu_Z$ is locally proportional to the volume form of the induced Riemannian structure on $\tilde{Z}$. It suffices to check it for an orbit $L(\mathbb{R})^+ \cdot x$ in $X$. But the $L(\mathbb{R})^+$-invariant measure on $L(\mathbb{R})^+ \cdot x$, the Haar measure, is unique up a factor, and, as $L(\mathbb{R})^+ \leq G(\mathbb{R})$ acts by isometries on $X$, the Riemannian volume form on $L(\mathbb{R})^+ \cdot x$ is a Haar measure. \hfill \Box

1.3.1. **Weakly special subvarieties.** We will be involved with the notion of weakly special subvariety only in the following two statements. The first is a slight generalisation of [41, Prop. 2.6]. This in fact is a direct consequence of the hyperbolic Ax-Lindemann-Weierstrass theorem proven in [18].

**Proposition 1.13** ([18], [39]). The Zariski closure of a weakly $S$-special real submanifold is a weakly special subvariety.

The following is a generalisation of an observation of [41, p. 2].

**Proposition 1.14.** Let $Z = Z_{L, (h, t)}$ be a weakly $S$-special real submanifold of $\text{Sh}_K(G, X)$. If $L$ is normalised by $h$, then $Z$ a weakly special subvariety: it is Zariski closed.

1.4. **Main theorems.** We may finally state our main theorems, which give a stronger form of Conjecture [15] at the cost of $S$-Tate type assumption.

We now state our first main result.

**Theorem 1.15** (Inner Equidistributional S-André-Pink-Zannier). Let $s$ be a point of $\text{Sh}_K(G, X)$ defined over a field $E$ such that the point $s$ is of $S$-Shafarevich type.

Let $(s_n)_{n \geq 0}$ be a sequence of points in the $S$-Hecke orbit $\mathcal{H}_S(s)$ of $s$, and denote $(\mu_n)_{n \geq 0}$ the sequence of measures attached to the $s_n$ as in definition [1.8].

After possibly extracting a subsequence and replacing $E$ by a finite extension, there exists a finite set $\mathcal{F}$ of weakly $S$-special real submanifold $Z$ with canonical probability measure $\mu_Z$ (as defined in Definitions [1.10] and [1.11]) such that

1. the sequence $(\mu_n)_{n \geq 0}$ tightly converges to $\mu_\infty = \frac{1}{|\mathcal{F}|} \sum_{Z \in \mathcal{F}} \mu_Z$, 

\[\text{We quote: “In the case where } h \text{ viewed as a morphism from } S \text{ to } G_\mathbb{R} \text{ factors through } H_\mathbb{R}, \text{ the corresponding real weakly special subvariety has Hermitian structure and in fact is a weakly special subvariety in the usual sense”}.
\]
and for all \( n \geq 0 \), we have \( \text{Supp}(\mu_n) \subseteq \text{Supp}(\mu_\infty) = \bigcup_{Z \in \mathcal{F}} Z \).

(3) If \( s \) is of \( S \)-Mumford-Tate type, then every \( Z \) in \( \mathcal{F} \) is a weakly special subvariety.

We may refer to property (1) as “equidistribution” property – a shorthand for a convergence of measures – and to property (2) by saying this equidistribution is “inner”.

Note that conclusion (3) applies to special points, in which case every \( Z \) in \( \mathcal{F} \) is actually a special subvariety.

We deduce from theorem 1.15 the following theorem, which is more directly related to the André-Pink-Zannier conjecture. Let us stress that, in the deduction process, we need not only (1), but also (2), from Theorem 1.15.

Theorem 1.16 (Topological and Zariski \( S \)-André-Pink-Zannier). Let \( s \) be a point of a Shimura variety \( \text{Sh}_K(G, X) \) defined over a field \( E \subseteq \mathbb{C} \). Consider a subset \( \Sigma \subseteq \mathcal{H}_S(s) \) of its \( S \)-Hecke orbit and denote

\[
\Sigma_E = \text{Gal}(\overline{E}/E) \cdot \Sigma = \{ \sigma(x) \mid \sigma \in \text{Gal}(\overline{E}/E), x \in \Sigma \}.
\]

Then

(1) If \( s \) is of \( S \)-Shafarevich type then the topological closure of \( \Sigma_E \) is a finite union of weakly \( S \)-special real submanifolds;

Furthermore, the Zariski closure of \( \Sigma \) is a finite union of weakly special subvarieties.

(2) If \( s \) is of \( S \)-Mumford-Tate type, then the topological closure of \( \Sigma_E \) is a finite union of weakly special subvarieties;

We will prove that (1) holds whenever (1) and (2) from Theorem 1.15 hold for sequences in \( \Sigma \). The second statement will then be the consequence of Proposition 1.13.

When the \( S \)-Mumford-Tate property holds, the conclusion (3) from Theorem 1.15 will imply (2).

1.4.1. A converse statement. Let us end with a statement emphasizing the importance of property (2) of Theorem 1.15. This statement makes precise the idea that property (2) of Theorem 1.15 implies the \( S \)-Shafarevich property.

This shows that the \( S \)-Shafarevich property assumption is essential and optimal.

Proposition 1.17. Let \( s \) be a point in a Shimura variety \( \text{Sh}_K(G, X) \) defined over a field \( E \) and let \( \mathcal{H}_S(s) \) be its \( S \)-Hecke orbit.

Assume that for any sequence \( (s_n)_{n \geq 0} \) in \( \mathcal{H}_S(s) \), for any finite extension \( F \) of \( E \), there is an extracted subsequence for which the associated
measure $\mu_n$ converges weakly to a limit $\mu_\infty$ in such a way that

$$\forall n \geq 0, \text{Supp}(\mu_n) \subseteq \text{Supp}(\mu_\infty).$$

Then $s$ is of $S$-Shafarevich type.

Proof. Assume for contradiction that $s$ is not of $S$-Shafarevich type. Then there is a finite extension $F$ of $E$ such that there is an infinite sequence $(s_n)_{n \geq 0}$ of pairwise distinct $F$-rational points in $\mathcal{H}_S(s)$. After possibly extracting a subsequence, we may assume that this sequence is convergent or is divergent in $\text{Sh}_K(G, X)$.

As these $s_n$ are rational points, the associated measures $\mu_n$ are Dirac masses. We recall that weak convergence of Dirac masses is induced by convergence in the Alexandroff compactification, with the point at infinity corresponding to the zero measure.

If $(s_n)_{n \geq 0}$ is divergent, so is any subsequence, and the measure $\mu_\infty$ will be the 0 measure, in which case (3) may not hold.

If $(s_n)_{n \geq 0}$ converges to $s_\infty$, then $\mu_\infty$ will be the Dirac measure $\delta_{s_\infty}$, and (3) means that $(s_n)_{n \geq 0}$ is a stationary sequence, which it cannot be since the $s_n$ are pairwise distinct. This yields a contradiction. \[\square\]

1.5. Plan of the Article. In Section 2 we explain how to deduce Theorem 1.16 from Theorem 1.15. Section 3 reviews Galois representations, various properties listed before ($S$-Mumford-Tate, $S$-Shafarevich, $S$-Tate, etc) and relations between them. We in particular prove useful and practical group-theoretic characterisation of the $S$-Shafarevich property. We also provide examples and counterexamples of when the properties do and do not hold depending on the field $E$. We believe the contents and results of this section to be of independent interest.

The sections that follow are devoted to the proof of Theorem 1.15:

- Section 4 explain how to reduce to a situation falling under the scope of application of [30]. It ends by invoking [30], which immediately gives us (1) of Theorem 1.15.
- Section 5 then discusses how to get (2) of Theorem 1.15.
- Finally Section 6 treats the stronger conclusion we can reach under the $S$-Mumford-Tate hypothesis.
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2. From Inner equidistribution to Topological closures

In this section we show how to derive Theorem 1.16 from Theorem 1.15. The main result is proposition 2.7 which implies the main theorem of this section:

Theorem 2.1. Conclusions of Theorem 1.15 imply conclusions of Theorem 1.16.

First we need to develop a dimension theory of weakly $S$-special submanifolds.

2.1. Dimension and Measure in chains of weakly $S$-special real submanifolds. We prove here some standard properties about inclusions of weakly $S$-special real submanifolds, involving dimension, that we define, and their canonical measure.

Definition 2.2. Let $Z = Z_{L,(h,t)}$ be a weakly $S$-special real submanifold. Then we define the dimension of $Z$ as the codimension of the stabiliser $K_h \cap L(\mathbb{R})$ of $h$ in $L(\mathbb{R})$. This is also the dimension $L(\mathbb{R})^+ \cdot h$, or equivalently $L(\mathbb{R}) \cdot h$ in $X$, as a semialgebraic set and as a real analytic variety.

Lemma 2.3. (1) The dimension of a weakly $S$-special real submanifold is well defined. If $Z_{L_1,(h_1,t_1)} = Z_{L_2,(h_2,t_2)}$, then

$$\dim (L_1(\mathbb{R})^+ \cdot h_1) = \dim (L_2(\mathbb{R})^+ \cdot h_2)$$

(2) Let $Z_1 \subsetneq Z_2$ be two $S$-special real submanifolds. Then

$$\dim Z_1 < \dim Z_2$$

(3) Let $Z_1$ and $Z_2$ be two weakly $S$-special real submanifolds, such that $\mu_{Z_1}(Z_2) \neq 0$. Then $Z_1 \subseteq Z_2$, and $\mu_{Z_1}(Z_2) = 1$.

One immediately deduces the following.

Corollary 2.4. Let $Z_1 \subsetneq \ldots \subsetneq Z_l$ be a chain of strictly included weakly $S$-special real submanifolds. Then its length $l$ satisfies $l \leq 1 + \dim(G)$.

From this we deduce the following.

Corollary 2.5. Any non empty collection $\mathcal{F}$ of weakly $S$-special real submanifolds, partially ordered by inclusion, has maximal elements, and any element of $\mathcal{F}$ is contained in a maximal element of $\mathcal{F}$. 
Proof of the Corollary (2.5). By induction we may extend every chain in $\mathcal{F}$ to a maximal one. By 2.4, this induction terminates after at most $1 + \dim(G)$ steps.

The last element of a non empty maximal chain is a maximal element. Hence any element $f$, seen as a chain of length one, is part of a non empty maximal chain. The last element of the latter contains $f$ and is maximal in $\mathcal{F}$. If $\mathcal{F}$ has an element, this implies that there is a maximal element. □

Proof. Write $Z_i$ for $Z_{L_i, (h_i, t_i)}$ (for $i = 1$ or $i = 2$).

We assume that the intersection $Z_1 \cap Z_2$ is not empty.

These two subsets $Z_1$ and $Z_2$ of $\text{Sh}_K(G, X)$ are connected, and hence belong to the same connected component of $\text{Sh}_K(G, X)$. This implies, as subsets in $G(A_f)$,

$$G(\mathbb{Q}) \cdot (G(\mathbb{R}) \times t_1K) = G(\mathbb{Q}) \cdot (G(\mathbb{R}) \times t_2K).$$

Left translating $t_1$ with $\gamma \in G(\mathbb{Q})$ and right translating with $k \in K$ we may assume $t_1 = t_2$. We have to substitute accordingly $h_1$ with $\gamma h_1$ and $L_1$ with $\gamma L_1 \gamma^{-1}$. As we have

$$\gamma L_1 \gamma^{-1} \cdot \gamma h_1 = \gamma (L_1 \cdot h_1)$$

this does not change the notion of dimension of $Z_1$.

We now assume that $t_1 = t_2$, which we will denote simply $t$.

Let $\Gamma_{tK}$ be the inverse image in $G(\mathbb{Q})$ of $tKt^{-1}$ with respect to the map $G(\mathbb{Q}) \to G(\mathbb{A}_f)$. This is the arithmetic subgroup such that the previous component of $\text{Sh}_K(G, X)$ belongs to those of $\Gamma_{tK} \setminus X \times \{t\}$. We will identify $X \times \{t\}$ with $X$ for simplicity. The inverse images of $Z_1$ and $Z_2$ in $X$ are $\widetilde{Z}_1 = \Gamma_t \cdot L_1(\mathbb{R})^+ \cdot h_1$ and $\widetilde{Z}_2 = \Gamma_t \cdot L_2(\mathbb{R})^+ \cdot h_2$ respectively.

We may write

$$\widetilde{Z}_1 \cap \widetilde{Z}_2 = \Gamma_{tK} \cdot \left((L_1(\mathbb{R})^+ \cdot h_1) \cap \widetilde{Z}_2\right)$$

and

$$(L_1(\mathbb{R})^+ \cdot h_1) \cap \widetilde{Z}_2 = \bigcup_{\gamma \in \Gamma_t} (L_1(\mathbb{R})^+ \cdot \gamma h_1) \cap (\gamma L_1 \cdot L_2(\mathbb{R})^+ \cdot h_2).$$

Assume first that $\mu_{Z_1}(Z_2) \neq 0$.

By our definition of $\mu_{Z_1}$, the $(\Gamma_{tK} \cap L_1(\mathbb{R})^+)$-saturated set

$$(\Gamma_{tK} \cap L_1(\mathbb{R})^+) \cdot (L_1(\mathbb{R})^+ \cdot h_1) \cap \widetilde{Z}_2$$

is non negligible (Cf. Lemma 2.6 proven below) in $L_1(\mathbb{R})^+ \cdot h_1$ with respect to a Haar measure on the homogeneous $L_1(\mathbb{R})^+$-set $L_1(\mathbb{R})^+ \cdot h_1$. 


But this \((7)\) is again the countable union \((6)\). So there is a \(\gamma \) in \(\Gamma_{tK}\) such that
\[
(L_1(\mathbb{R})^+ \cdot h_1) \cap (\gamma_t \cdot L_2(\mathbb{R})^+ \cdot h_2)
\]
is not negligible. This is a real semi-algebraic subset of the real semi-algebraic set \(L_1(\mathbb{R})^+ \cdot h_1\). We use a cylindrical cellular decomposition of this subset. Subset of codimension 1 are negligible\(^3\). So at least one cell has codimension 0. It must have non-empty interior.

The orbit map \(L_1(\mathbb{R})^+ \to L_1(\mathbb{R})^+ \cdot h_1\) are open maps. So there is an open subset \(U\) in \(L_1(\mathbb{R})^+\) such that \(U \cdot h_1 \subseteq \gamma L_2(\mathbb{R})^+ \cdot h_2\). But \(L_1\) is Zariski connected, and \(U\) is Zariski dense in \(L_1\). Hence
\[
L_1(\mathbb{R})^+ \cdot h_1 \subseteq (L_1 \cdot h_1)(\mathbb{R}) \subseteq (\gamma L_2 \cdot h_2)(\mathbb{R}).
\]

We note that \(L_1(\mathbb{R})^+ \cdot h_1\) is a connected component of \((L_1 \cdot h_1)(\mathbb{R})\). Likewise \(\gamma L_2(\mathbb{R})^+ \cdot h_2\) is a connected component of \((\gamma L_2 \cdot h_2)(\mathbb{R})\). But the connected \(L_1(\mathbb{R})^+ \cdot h_1\) intersects the component \(\gamma L_2(\mathbb{R})^+ \cdot h_2\), hence is contained in it. It follows \(\tilde{Z}_1 \subseteq \tilde{Z}_2\) and finally \(Z_1 \subseteq Z_2\). We have proved the last point of the lemma.

We now turn to the first point.

*We now assume \(Z_1 \subseteq Z_2\) instead of \(\mu_{Z_1}(Z_2) \neq 0\).*

Then certainly \(\mu_{Z_1}(Z_2) \neq 0\). We may, and will, keep the notations above. We have already proved
\[
L_1(\mathbb{R})^+ \cdot h_1 \subseteq \gamma L_2(\mathbb{R})^+ \cdot h_2
\]
for some \(\gamma \) in \(\Gamma_t\). It follows
\[
dim(L_1(\mathbb{R})^+ \cdot h_1) \leq \dim(\gamma L_2(\mathbb{R})^+ \cdot h_2) = \dim(L_2(\mathbb{R})^+ \cdot h_2).
\]
(the equality on the right is easily checked.) If \(Z_1 = Z_2\) we may exchange roles to get a converse comparison, yielding \((4)\): the dimension of \(Z_i\) is well defined. This was the first point of the lemma.

It remains to prove the second point.

*We now assume \(Z_1 \nsubseteq Z_2\).*

Again we keep our notations. We have proved
\[
L_1(\mathbb{R})^+ \cdot h_1 \subseteq \gamma L_2(\mathbb{R})^+ \gamma^{-1} \cdot \gamma h_2.
\]
The reverse inclusion does not hold, as it would, easily, imply \(Z_2 \subseteq Z_1\). We may substitute our base point \(\gamma h_2\) with \(h_1\), as it does belong to the same \(\gamma L_2(\mathbb{R})^+ \gamma^{-1}\) orbit. We deduce
\[
L_1(\mathbb{R})^+ \cdot h_1 \nsubseteq \gamma L_2(\mathbb{R})^+ \gamma^{-1} \cdot h_1.
\]
Assume by contradiction that both sides have same dimension. The orbit \(L_1(\mathbb{R})^+ \cdot h_1\) is closed in \(X\), and *a fortiori* closed in \(L_2(\mathbb{R})^+ \gamma^{-1}\).
h_1. Furthermore \( L_1(\mathbb{R})^+ \cdot h_1 \rightarrow L_2(\mathbb{R})^+ \cdot h_1 \), as a map of differential manifolds, is a submersion at at least a point, by equality of dimensions. It is a submersion everywhere by homogeneity, hence an open map. As a consequence, \( L_1(\mathbb{R})^+ \cdot h_1 \) is not only closed, but open as well in \( L_2(\mathbb{R})^+ \cdot h_1 \). As \( L_2(\mathbb{R})^+ \gamma^{-1} \cdot h_1 \) is connected, we deduce
\[
L_1(\mathbb{R})^+ \cdot h_1 = \gamma L_2(\mathbb{R})^+ \gamma^{-1} \cdot h_1.
\]
That is a contradiction. This ends our proof.

We finish this section by proving a lemma used in the proof above.

**Lemma 2.6.** Endow on \((\Gamma \cap L) \setminus L\) with a Haar measure. Then the inverse of a negligible subset in \((\Gamma \cap L) \setminus L\) is negligible in \(L\), with respect to a Haar measure.

**Proof.** Let \(N\) be a negligible subset in \((\Gamma \cap L) \setminus L\), and \(\tilde{N}\) its inverse image in \(L\). The lemma amounts to proving that
\[
\int_{l \in L} 1_{\tilde{N}} \, dl = 0
\]
where \(1_{\tilde{N}}\) is the characteristic function of \(\tilde{N}\) and \(dl\) is a Haar measure on \(L\). Let \(K\) be a compact subset in \(L\). And consider, as a real function \((\Gamma \cap L) \setminus L \rightarrow \mathbb{R}\),
\[
f : (\Gamma \cap L) \cdot l \mapsto \int_{\gamma \in \Gamma \cap L} 1_{K \cap N}(\gamma l) \, d\gamma,
\]
where \(d\gamma\) is a Haar measure on \((\Gamma \cap L)\). Then we have, see [9, VII §2.1],
\[
\int_{l \in L} 1_{\tilde{N} \cap K} \, dl = \int_{(\Gamma \cap L) \setminus L} f(x) \, dx
\]
where \(dx\) is the quotient Haar measure (cf. loc. cit.) on \((\Gamma \cap L) \setminus L\). But the support of \(f\) is contained in \(N\), hence is negligible. The last integral evaluates as zero. By choosing increasing compact subsets whose union is \(L\), we, by the monotone convergence \(\lim_K 1_{\tilde{N} \cap K} = 1_{\tilde{N}}\), deduce (8).

2.2. **Topological and Zariski closures.** We place ourselves in the situation of Theorem 1.16. In particular we assume that \(s\) is of \(S\)-Shafarevich type.

Let \(\Sigma\) be as in theorem 1.16. It is a countable set, and we write it as \(\Sigma = \{s_n, n \geq 0\}\). Let \((\mu_n)_{n \geq 0}\) be the sequence of probability measures attached to \((s_n)_{n \geq 0}\) as in Definition 1.8. As the \(S\)-Shafarevich hypothesis is assumed, we are permitted to invoke Theorem 1.15 for any infinite subsequence.

Our proof of Theorem 1.16 relies on the following.
Proposition 2.7. We consider the following situation.

— Let $S$ be the set of supports of limits of converging subsequences of $(\mu_n)_{n \geq 0}$.
— Let $Z$ be the collection of $S$-special real submanifolds $Z$ such that the canonical measure $\mu_Z$ occurs in the composition of the limit of a converging subsequence of $(\mu_n)_{n \geq 0}$.
— We endow $Z$ with the partial order induced by inclusion. Let $\mathcal{M}$ be the subset of maximal elements in $Z$.

We have the following.

i) Every support $S$ belonging to $S$ is a finite union of finitely many weakly $S$-special real submanifolds belonging to $Z$. If $s$ is of $S$-Mumford-Tate type, then the $Z$ belonging to $Z$ are actually weakly special subvarieties.

ii) Every element $Z$ of $Z$ is included in a maximal element of $Z$, an element belonging to $\mathcal{M}$.

iii) The subset $\mathcal{M}$ of maximal elements of $Z$ is a finite subset.

iv) Every $Z$ in $Z$, or $S$ in $S$, is contained in the topological closure of $\Sigma_E$.

v) All but finitely many elements of $\Sigma_E$ are in $\bigcup_{Z \in \mathcal{M}} Z$.

vi) The topological closure of $\Sigma_E$ is a finite union of weakly $S$-special real manifolds.

vii) The Zariski closure of $\Sigma_E$ is a finite union of weakly special subvarieties.

To justify the definition of $Z$, we need to show that the $\mu_Z$ that occur in the sum with a nonzero coefficient of a limit measure $\mu$ are defined unambiguously.

This is a consequence of the following:

Lemma 2.8. Any finite set of canonical measures $\mu_Z$ is linearly independent.

Proof. Consider a linear combination $\mu = \lambda_1 \mu_{Z_1} + \ldots + \lambda_n \mu_{Z_n}$. We may compute $\mu(Z)$ by using 2.3. It follows that we recover the coefficient of $\mu_Z$ as the measure $\mu(Z)$ minus the coefficients associated with subvarieties in $Z$. To see it is well defined, we argue by induction on the dimension of $Z$ to check that we thus obtain only finitely many non-zero coefficients, because these agree with the $\lambda_i$. We refer to Corollary 2.1 for justification why this induction is legit.

So the coefficient of $\mu_Z$ in $\mu$ is uniquely defined. \hfill \Box

We proved that 1.15 implies (1) and (2) of Theorem 1.16. We split the proof of [iii] into two lemmas below. Lemma 2.10 is [iii]
of Proposition 2.7. The statement [i] is a direct consequence of Theorem 1.15.

The statement [ii] is Corollary 2.5 from the previous section.

To prove [iii] we prove two lemmas. The statement [iii] is Lemma 2.10.

**Lemma 2.9.** The set $\mathcal{Z}$ is countable.

**Proof.** Every element of $\mathcal{Z}$ can be associated with the data

- of some group of Ratner class, which is an algebraic subvariety
  over $\mathbb{Q}$ of $G$, hence belong to a countable class;
- of some point in the $S$-Hecke orbit $\Sigma$, which is countable.

As there only finitely many possibilities for these data, we can construct at most countably many elements in $\mathcal{Z}$. \qed

**Lemma 2.10.** The set $\mathcal{M}$ is finite.

**Proof of the last claim.** Assume for contradiction that $\mathcal{M}$ is infinite. It is countable. Hence we can arrange its elements as a sequence $\mathcal{M} = (M_n)_{n \geq 0}$ such that the $M_n$ are the distinct maximal (for inclusion) elements of $\mathcal{Z}$. We arrange $\mathcal{Z}$ likewise in a sequence $(Z_n)_{n \geq 0}$.

Define $S_n = \text{Sh}_K(G, X) \setminus \bigcup_{i < n} M_i$. This is an open subset. By maximality of the $M_i$, we have $\mu_{M_i}(M_j) = 0$ whenever $i \neq j$ by Lemma 2.3 (ii). Hence $S_n$ is of full measure for $\mu_{M_n}$.

We will use a diagonal argument.

By definition, there is a convergent subsequence, say $(\mu_{M_n}^{(n)})_{m \geq 0}$, of the sequence $(\mu_{M_m})_{m \geq 0}$ such that its limit, say $\mu_{M_\infty}$, admits $\mu_{M_n}$ as a component, with some non zero coefficient $\lambda_n$. By convergence, there is some $N_n$ such that for $m \geq N_n$ we have $\mu_{M_n}^{(n)}(S_n) \geq \lambda_n/2$. Write $\mu_{M_n}^{(n)}$. Consequently $\text{Supp} \nu_n$ is not included in $M_1 \cup \ldots \cup M_{n-1}$. We deduce that no finite union of subsets $M$ from $\mathcal{M}$ can support infinitely many of the $\nu_n$. As any $Z$ from $\mathcal{Z}$ is contained in some $M$ from $\mathcal{M}$, by (iii), no finite union of such $Z$ can support infinitely many of the $\nu_n$. A fortiori no $S$ from $\mathcal{S}$ can support infinitely many of the $\nu_n$.

But, by Theorem 1.15 we may extract a subsequence from $(\nu_n)_{n \geq 0}$ which is converging, say with limit $\nu_\infty$, satisfying the conclusions of Theorem 1.15 and

$$\text{Supp}(\nu_n) \subseteq \text{Supp}(\nu_\infty) \in \mathcal{S}.$$  

This yields a contradiction. \qed

The statement [iv] is obvious: the topological closure of $\Sigma_E$ is a closed subset containing the support of the $\mu_n$, and hence contains the support of any limit of a subsequence of $(\mu_n)_{n \geq 0}$. 

Finally, to prove \( v) \), assume for contradiction that there exists an infinite subsequence \((s_n)\) of points of the set \( \Sigma_E \) which are not in \( \bigcup_{Z \in M} Z \). Let \((\mu_n)\) be the associated sequence of measures as defined in 1.8. By theorem 1.15, after possibly extracting a subsequence, we may assume that \((\mu_n)\) converges to a measure \( \mu \) whose support contains \( \text{Supp}(\mu_n) \) for all \( n \). By definition, we have that

\[
s_n \in \text{Supp}(\mu_n) \subseteq \text{Supp}(\mu) \subseteq \bigcup_{S \in S} S = \bigcup_{Z \in Z} Z = \bigcup_{Z \in M} Z.
\]

This contradicts the choice of \( s_n \).

The statement \( \text{vi) \} \) follows directly from \( \text{v) \} \) and statement \( \text{vii) \} \) follows from \( \text{vi) \} \) and the fact that the Zariski closure of a weakly \( S \)-special manifold is a weakly special subvariety (Proposition 1.13).

We have finished proving proposition 2.7 and hence Theorem 2.1. \( \square \)
3. Review of Galois monodromy representations

3.1. Construction of representations. In this section we recall the
construction of Galois representations attached to points in Hecke or-
bits, and then specialise to $S$-Hecke orbits.

The contents of this section are mostly taken from Section 2 of [40].
There it was assumed that $E$ was a number field, however all arguments
carry over verbatim to an arbitrary field of characteristic zero.

Let $Sh(G, X)$ be the Shimura variety of infinite level, it is the profi-
nite cover

$$Sh(G, X)(\mathbb{C}) = \varprojlim_{K} Sh_{K}(G, X)$$

with respect to the finite maps induced by the inclusions of compact
open subgroups. By Appendix to [40], the centre $Z$ of $G$ has the
property that $Z(\mathbb{Q})$ is discrete in $G(\mathbb{A}_{f})$. It follows (Theorem 5.28 of
[21]), we have

$$Sh(G, X)(\mathbb{C}) = G(\mathbb{Q}) \setminus (X \times G(\mathbb{A}_{f})) .$$

The scheme $Sh(G, X)$ is endowed with the right action of $G(\mathbb{A}_{f})$ which
is defined over the reflex field $E(G, X)$.

We let

$$\pi : Sh(G, X) \longrightarrow Sh_{K}(G, X)$$

be the natural projection. The Shimura varieties $Sh(G, X)$ and $Sh_{K}(G, X)$
are defined over the reflex field $E(G, X)$ and so is the map $\pi$.

Let $s = (h, t)$ be a point of $Sh_{K}(G, X)$ defined over a field $E$. Lemma
2.1 of [40] shows that the fibre $\pi^{-1}(s)$ has a transitive fixed point free
right action of $K$. Explicitly

$$\pi^{-1}(s) = (h, tK)$$

and the action of $K$ is the obvious one. Furthermore, $Gal(\overline{E}/E)$ acts on
$\pi^{-1}(s)$ and this action commutes with that of $K$. This is a consequence
of the theory of canonical models of Shimura varieties (see [21] and
[12]). By elementary group theoretic Lemma 2.4 of [40], we obtain a
morphism

$$\rho_{s} : Gal(\overline{E}/E) \longrightarrow K$$

such that the Galois action on $\pi^{-1}(s)$ is described as follows:

$$\sigma((h, tk)) = (h, k\rho_{s}(\sigma)).$$

This representation is continuous since an open subgroup $K'$ of $K$ is of
finite index and hence $\rho_{s}^{-1}(K')$ contains $Gal(\overline{E}/F)$ for a finite extension
$F/E$.

The representation $\rho_{s}$ has the following fundamental property. Let
$M$ be the Mumford-Tate group of $h$. To $M$ one associates the Shimura
datum \((M, X_M)\) with \(X_M = M(\mathbb{R}) \cdot h\). Let \(E_M\) be the reflex field of \((M, X_M)\) and \(E'\) the subfield of \(\overline{E}\) generated by \(E\) and \(E_M\).

**Proposition 3.1.** We have

\[
\rho_s(\text{Gal}(\overline{E}/E')) \subset M(\mathbb{A}_f) \cap K.
\]

**Proof.** This is Proposition 2.9 of [40]. This proposition is stated with \(E\) a number field, however the proof goes through without any changes for an arbitrary field of characteristic zero. \(\square\)

The representation \(\rho_s\) describes the action of \(\text{Gal}(\overline{E}/E)\) on the Hecke orbit \(\mathcal{H}(s)\). Let \(s' = (h, t)\) be a point of \(\mathcal{H}(s)\). Consider the point \(\tilde{s}' = (h, t)\) of \(\pi^{-1}(s)\). Let \(\sigma \in \text{Gal}(\overline{E}/E)\). Since the action of \(G(\mathbb{A}_f)\) is defined over \(E(G, X)\), we have

\[
\sigma((h, t)) = (\sigma(h, t)) \cdot g = (h, \rho_s(\sigma)t)g = (h, \rho_s(\sigma)t)g
\]

By applying \(\pi\) to this relation and using the fact that \(\pi\) is defined over \(E(G, X)\), we obtain:

\[
\sigma((h, t)) = (h, \rho_s(\sigma)t).
\]

We now describe the representation \(\rho_{s, S}\) and Galois action the \(S\)-Hecke orbit \(\mathcal{H}_S(s)\). Recall that \(K\) is product of compact open subgroups \(K_p\) of \(G(\mathbb{Q}_p)\). Let \(K_S := \prod_{p \in S} K_p\) and \(K^S = \prod_{p \not\in S} K_p\). We denote by \(p_S\) the projection map

\[
p_S: G(\mathbb{A}_f) \longrightarrow G_S.
\]

Clearly \(p_S(K) = K_S\).

**Definition 3.2.** We define

\[
\rho_{s, S} := p_S \circ \rho_s: \text{Gal}(\overline{E}/E) \longrightarrow K_S.
\]

Let

\[
\text{Sh}_{K^S}(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K^S.
\]

This is a scheme defined over \(E(G, X)\) endowed with a continuous right \(G_S\) action and a morphism \(\pi^S: \text{Sh}(G, X) \longrightarrow \text{Sh}_{K^S}(G, X)\) defined over \(E(G, X)\).

The maps \(\pi^S: \text{Sh}(G, X) \longrightarrow \text{Sh}_{K^S}(G, X)\) and \(\pi_S: \text{Sh}_{K^S}(G, X) \longrightarrow \text{Sh}_K(G, X)\) are defined over \(E(G, X)\). Furthermore,

\[
\pi = \pi_S \circ \pi^S.
\]

Contemplation of these properties and the properties of \(\rho_{s, S}\) show the following:
Theorem 3.3. The morphism
\[ \rho_{s,S} : \text{Gal}(\overline{E}/E) \longrightarrow K_S \]
we have constructed above has the following properties:

1. The representation \( \rho_{s,S} \) is continuous.
2. Let \( s' = (h, tg) \) (with \( g \in G_S \)) be a point in \( \mathcal{H}_S(s) \). Then for any \( \text{Gal}(\overline{E}/E) \), we have
\[ \sigma(s') = (h, \rho_{s,S}(\sigma)tg). \]
3. After replacing \( E \) by \( EE_M \), we have
\[ \rho_{s,S}(\text{Gal}(\overline{E}/E)) \subset M(\mathbb{A}_f) \cap K_S = M(\mathbb{Q}_S) \cap K \]
(intersection taken inside \( G(\mathbb{A}_f) \)).

3.2. Properties of \( S \)-Galois representations. In this section we examine in detail the properties.

3.2.1. General assumptions. For the sake of the ease of reading, we recall the general situation. Let \( S \) be a finite set of places of \( \mathbb{Q} \). Let \( (G, X) \) be a Shimura datum, normalised so that \( G \) is the generic Mumford-Tate group on \( X \) and \( K \) a compact open subgroup of \( G(\mathbb{A}_f) \) satisfying the following conditions:

1. \( K \) is a product \( K = \prod K_p \) of compact open subgroups \( K_p \subset G(\mathbb{Q}_l) \).
2. There exists an \( l \geq 3 \) and \( l \notin S \) such that \( K_l \) is contained in the group of elements congruent to the identity modulo \( l \) with respect to some faithful representation \( G \subset \text{GL}_n \). In particular, this implies that \( K \) is neat.

We let \( s = (h, t) \) a point of \( \text{Sh}_K(G, X) \) defined over a field \( E \). We let \( M \) be the Mumford-Tate group of \( h \). In what follows, for ease of notation, we will write \( M \) for \( M(\mathbb{Q}_S) \) or \( M(\mathbb{Q}) \) when it is clear from the context what is meant.

As we have seen earlier in this section, there exists a continuous “\( S \)-adic monodromy representation” (we have if necessary replaced \( E \) with \( EE_M \)):
\[ \rho_{h,S} : \text{Gal}(\overline{E}/E) \longrightarrow M(\mathbb{A}_f) \cap K_S = M(\mathbb{Q}_S) \cap K_S \]
which has the property that for any \( t \in G_S \) and \( \sigma \in \text{Gal}(\overline{E}/E) \),
\[ \sigma((h, tg)) = (h, \rho(\sigma) \cdot tg). \]

We let \( U_S \subset M(\mathbb{Q}_S) \) be the image of \( \rho_{(h,S)} \) and \( H_S \) the algebraic monodromy group i.e. the Zariski closure of \( U_S \). It is immediate that properties are invariant by replacing \( K \) by an open subgroup and \( E \).
by a finite extension (or equivalently $U_S$ by an open subgroup). After replacing $E$ by a finite extension, we assume that $U_S$ (and hence $H_S$) are connected.

3.2.2. The algebraicity property. In this section we consider the $S$-algebraicity property. All results of this section are under the assumption that $E$ is of finite type over $\mathbb{Q}$.

The centre of the $S$-adic monodromy. The first result is that when $E$ is of finite type, the $S$-Mumford-Tate property (and hence the $S$-algebraicity) hold for abelian $S$-adic representations.

**Proposition 3.4.** Let $M^{\text{der}}$ be the derived group of $M$ and $M^{\text{ab}} = M/M^{\text{der}}$ its maximal abelian quotient. We consider the quotient map

$$\pi : M(\mathbb{Q}_S) \to M^{\text{ab}}(\mathbb{Q}_S).$$

(1) If $s$ is of $S$-Tate type, then the centre $Z(U_S)$ of $U_S$ is contained in the centre of $M(\mathbb{Q}_S)$.

(2) If $s$ is of $S$-Tate type and $S$-simplicity holds, then we have $\pi(Z(U_S))$ is open in $\pi(U_S)$.

(3) If $E$ is of finite type over $\mathbb{Q}$, then $\pi(U_S)$ is open in $M^{\text{ab}}(\mathbb{Q}_S)$.

In particular, when $M$ is abelian (i.e. the point $s$ is special), the $S$-Mumford-Tate property holds.

The last remark is a straightforward consequence of the reciprocity law for canonical models of Shimura varieties. We will rely on this law for the proof of (3).

**Proof.** Let us prove (1). The center of $U_S$ is the intersection of $U_S$ with its centraliser $Z_{G_S}(U_S)$. By the $S$-Tate property we have $Z_{G_S}(U_S) = Z_{G_S}(M)$. We also have $U_S \subseteq M(\mathbb{Q}_S)$. It follows

$$U_S \subseteq M(\mathbb{Q}_S) \subseteq M(\mathbb{Q}_S) \cap Z_{G_S}(M)$$

but the latter is just the centre $Z(M)$ of $M$.

We now prove (2). Let $x \in \pi(U_S)$. Write $x = \pi(y)$ for $y \in U_S$. Since semisimplicity holds, there exists an integer $n$ (depending only on $U_S$ but not $y$) such that

$$y^n = z \cdot t$$

with $z \in Z_{G_S}(U_S)$ (by (1)) and $t \in U_S^{\text{der}} \subseteq M^{\text{der}}(U_S)$. Thus $x^n = \pi(y^n) = \pi(z) \in \pi(Z(U_S))$. This proves (2).

We prove (3). As $M$ is reductive, the restriction of $\pi$ to $Z(M)$ is an isogeny. Since $S$ is a finite set of primes, it follows that the induced map $Z(M)(\mathbb{Q}_S) \to M^{\text{ab}}(\mathbb{Q}_S)$ has finite kernel and an open
image. It will hence suffice to prove that the image of $U_S \cap Z(M)$ is open in $M^{ab}(\mathbb{Q}_S)$.

In the Shimura variety, associated with $M^{ab}$ every point is a special point. So is the image of $s$. The associated Galois representation is 

$$\pi \circ \rho_{s,S} : \text{Gal}(\overline{E}/E) \to M(\mathbb{Q}_S) \to M^{ab}(\mathbb{Q}_S).$$

By the reciprocity law for the canonical model of the Shimura variety associated with $M^{ab}$, this representation factors through 

$$\text{Gal}(\overline{E}/E(M, X_M)) \to M^{ab}(\mathbb{Q}_S).$$

and the image of the latter is open (of index at most $[F : E_M]$). As $E$ is of finite type, the algebraic closure $F$ of the number field $E_M E(M, X_M)$ in $E$ is finite. Then the image of $\text{Gal}(\overline{E}/E)$ in $\text{Gal}(\overline{Q}/E_M)$ is the open subgroup $\text{Gal}(\overline{Q}/F)$. It follows that the image in $M^{ab}(\mathbb{Q}_S)$ of $\text{Gal}(\overline{Q}/E_M)$ is open (of index at most $[F : E_M]$) in that of $\text{Gal}(\overline{Q}/E_M)$. This concludes the proof of the proposition. 

\[\square\]

\textit{S-Tate property and S-algebraicity.} We prove here that when $E$ is of finite type over $\mathbb{Q}$, the S-Tate property implies the S-algebraicity. Let $\mathfrak{m}$ be the Lie algebra of $M_{\mathbb{Q}_S}$, let $\mathfrak{u}$ be the Lie algebra of $U_S$, a $\mathbb{Q}_S$-Lie subalgebra of $\mathfrak{m}$. Let $H$ be the $\mathbb{Q}_S$-algebraic envelope of $U_S$, and $\mathfrak{h}$ its Lie algebra.

We refer to [?, BorelLAG] and [31] for the following.

**Proposition 3.5.** The following are equivalent:

- the subgroup $U_S$ of $H(\mathbb{Q}_S)$ is open (for the $S$-adic topology);
- the Lie algebra $\mathfrak{u}$ of $U_S$ is an algebraic Lie subalgebra of the Lie algebra $\mathfrak{m}$ of $M$, in the sense of Chevalley as in [5] II.§7; 
- the Lie algebras $\mathfrak{u}$ of $U_S$ and $\mathfrak{h}$ of $H$ are the same: $\mathfrak{u} = \mathfrak{h}$.

If these properties hold we say that $U_S$ is an algebraic Lie subgroup.

**Proposition 3.6.** Assume that Lie algebras $\mathfrak{u}$ is of $S$-Tate type and satisfies $S$-semisimplicity. If $E$ is an extension of finite type of $\mathbb{Q}$, then the Lie subgroup $U_S$ is algebraic in the sense of the previous proposition, that is $s$ satisfies the $S$-algebraicity in the sense of definition [1, 7][5].

**Proof.** We first use the $S$-semisimplicity to note that the adjoint action of $\mathfrak{u}$ on itself is semisimple, that is $\mathfrak{u}$ is a reductive Lie algebra by the definition used in [1] I §6.4 Déf. 4. It follows that we may decompose $\mathfrak{u}$ as a direct sum $\mathfrak{u} = [\mathfrak{u}, \mathfrak{u}] + \mathfrak{z}$ of its derived Lie algebra $[\mathfrak{u}, \mathfrak{u}]$ and its centre $\mathfrak{z}$, by [1] Cor. (b) to Prop. 5. It is enough to show that

\[Of \text{ dimension 0, related to the space of connected components of } \text{Sh}(G, X).\]
both \([u, u]\) and \(z\) are algebraic by \([5\text{, Cor. 7.7 (1) and (3)}]\). The derived Lie algebra \([u, u]\) is algebraic by \([5\text{, Cor. 7.9}]\).

It remains to prove that \(z\) is an algebraic Lie subalgebra. We use the \(S\)-Tate type property, to note that the centraliser of \(U\) is included in the centraliser of \(M\). We infer \(z \subseteq z_M(\mathbb{Q}_S)\). As \(E\) is of finite type, we may apply Prop. 3.4 we have seen that \(U_S\) contain an open subgroup of the centre of \(M(\mathbb{Q}_S)\). We have conversely \(z \supseteq z_M(\mathbb{Q}_S)\). Finally, \(z\) is the algebraic lie subalgebra \(z_M\). □

\(N.B.:\) This algebraicity statement is similar to Bogomolov’s algebraicity result \([4\text{, Th. 1}]\) for abelian varieties. The reduction to the case of abelian Lie algebras is very similar. We rely on \(S\)-Tate property and the theory of canonical models to treat the abelian case.

For similar algebraicity results see \([35\text{, 133. Th. p. 4}]\), and notably its subsequent Corollary.

### 3.2.3. Characterisation of the \(S\)-Shafarevich property.

We prove that \(S\)-Shafarevich property \(\mathbf{1.7} (2)\) is equivalent to the conjonction of the \(S\)-semi-simplicity property \(\mathbf{1.7} (4)\) and a weakening of the \(S\)-Tate property (as defined in \(\mathbf{1.7} (3)\)). This is amounts to a group theoretic characterisation of the \(S\)-Shafarevich property, which is essential for proving the main theorems of this paper and which, we believe, is also of independent interest.

**Proposition 3.7.** Let \(s\) be a point of \(\text{Sh}_K(G, X)\) and \(E\) a field of definition of \(s\) such that the associated \(S\)-adic monodromy representation \(\rho : \text{Gal}(\overline{\mathbb{F}}/E) \to M(\mathbb{A}_f) \cap G_S\)

is defined, and the image \(U_S\) of \(\rho\) is Zariski-connected. Let \(Z_{G_S}(M)\) and \(Z_{G_S}(U_S)\) denote the centraliser in \(G_S\) of the Mumford-Tate group \(M\) of \(s\) and of the image \(U_S\) of \(\rho\) respectively.

The point \(s\) is of \(S\)-Shafarevich type if and only if it satisfies \(S\)-semisimplicity and, furthermore, \(Z_{G_S}(M) \setminus Z_{G_S}(U_S)\) is compact.

In particular the \(S\)-Shafarevich property is implied by the conjunction of \(S\)-semisimplicity and \(S\)-Tate properties.

We will split the proof into several steps. Let us consider a sequence \(s_n = (h, g_n)\) with \(g_n \in G_S\) of points in the \(S\)-Hecke orbit \(\mathcal{H}_S(s)\) of \(s\).

We start with an easy Lemma about the homogeneous structure of the \(S\)-Hecke orbit \(\mathcal{H}_S(s)\).
Lemma 3.8. We embed $Z_G(M)(\mathbb{Q})$ into $G(\mathbb{A}_f)$ (at the finite places only) and $G(\mathbb{Q})$ into $G(\mathbb{A})$. Then the application

$$Z_G(M)(\mathbb{Q}) \backslash Z_G(M)(\mathbb{Q}) \cdot G_S \cdot K/K \rightarrow \mathcal{H}_S(s) = G(\mathbb{Q}) \backslash G(\mathbb{Q}) \cdot (\{h\} \times G_S) \cdot K/K,$$

which maps the double class of $g$ to $(\overline{h}, g)$, is a bijection.

Proof. The surjectivity is immediate, by the very definition of $\mathcal{H}_S(s)$. We prove the injectivity. We start with the identity $(\overline{h}, g) = (\overline{h}, g')$. Equivalently there is $q$ in $G(\mathbb{Q})$ and $k$ in $K$ such that

$$q \cdot (h, g) \cdot k = (h, g').$$

From $qh = h$, we infer that $q$ is in the stabiliser of $h$. As $q$ is in $G(\mathbb{Q})$, it belongs to the biggest $\mathbb{Q}$-subgroup in the stabiliser of $h$, which is $Z_G(M)(\mathbb{Q})$. We conclude by observing

$$q \cdot g \cdot k = g'.$$

□

In the following Lemma, we show how the $S$-simplicity hypothesis can be used to work out a group theoretic condition on $g_n$ for $s_n$ to be defined over $E$.

Lemma 3.9. Assume that the $S$-simplicity holds.

There exists a compact subset $C \subset G_S$ such that if a point $(\overline{h}, g)$ of $\mathcal{H}_S(s)$ is defined over $E$, then

$$g \in Z_G(U_S) \cdot C.$$

Proof. The point $(\overline{h}, g)$ is defined over $E$ if and only if, for every element $u$ of $U_S$, we have

$$(h, ug) = (h, g).$$

This means that there exists a $q$ in $G(\mathbb{Q})$ and $k$ in $K$, depending on $u$, such that

$$qh = h, \quad \text{i.e.} \quad q \in Z_G(M)(\mathbb{Q}),$$

and $qug = gk$, \quad \text{i.e.} \quad qu = gkg^{-1}$.

Thus $qu$ belongs to the group $gKg^{-1}$. Any power $(qu)^n$ belongs to the same group. As $q$ centralises $U_S$, we have $(qu)^n = u^n q^n$. It follows

$$q^n = u^{-n} \cdot gk^n g^{-1}.$$

In particular all powers of $q$ belongs to the compact set $U_S \cdot gKg^{-1}$. They also belong to the discrete set $Z_G(M)(\mathbb{Q})$ in $G(\mathbb{A}_f)$. It must be that $q$ is a torsion element of $Z_G(M)(\mathbb{Q})$. 

Actually all torsion elements of $Z_G(M)(\mathbb{Q})$ satisfy $q^N = 1$ for a uniform order $N > 0$: we may embed $Z_G(M)$ in a linear group $GL(D)$ and apply Lemma 3.10. We hence have

(10) \[ u^N = q^N u^N = g k^N g^{-1}, \] whence $g^{-1} u^N g \in K$.

Let $V = \{ u^N | u \in U \}$. This is a neighbourhood of the neutral element in $U$, as the $N$-th power map has a non-zero differential at the origin: it is the multiplication by $N$ map on the Lie algebra. It follows that $V$ is Zariski dense in $U$ (recall that $U$ is Zariski connected).

We deduce from (10) above that $g$ belongs to the transporteur, for the conjugation right-action, of $V$ to $K$

\[ T = \{ t \in G_S | t^{-1} V t \subseteq K \}. \]

The Zariski closed subgroup generated by $V$ is the same as the one generated by $U$, and is a reductive group by hypothesis. This is the essential hypothesis we need to invoke \cite{29} Lemma D.2, according to which there exists a compact subset $C$ of $G_S$ such that

\[ g \in T \subseteq Z_G(U_S) \cdot C. \]

We are done, but from the fact that actually loc. cit. works only for one ultrametric place at a time. But arguing with the projections $G_p$, $U_p$, $V_p$, $K_p$ and $T_p$ of $G_S$, $U_S$, $V$, $K$ and $T$ in $G_p$ at a some place $p$ in $S$, we can prove as above that

\[ T_p \subseteq \{ t \in G_p | t^{-1} V_p t \subseteq K_p \} \subseteq Z_G(U_p) \cdot C_p \]

for a compact subset $C_p$ of $G_p$, and conclude, with $C$ the product compact set

\[ T \subseteq \prod_{p \in S} T_p \subseteq \prod_{p \in S} Z_G(U_p) \cdot C_p = Z_G(U_S) \cdot \prod_{p \in S} C_p = Z_G(U_S) \cdot C. \]

The following standard fact was used in the preceding proof.

**Lemma 3.10.** Consider a general linear group $GL(D, \mathbb{Q})$ over the field $\mathbb{Q}$ of rational numbers. Then there is an integer $N(D)$ such that for any $g$ in $GL(D, \mathbb{Q})$ of finite order its power $g^{N(D)}$ is the neutral element.

**Proof.** Let $g$ be a torsion element. Every complex eigenvalue of $g$ is some root of unity $\zeta$. Let $d$ be the order of $\zeta$. As the cyclotomic polynomials are irreducible over $\mathbb{Q}$, it must be that $g$ has at least $\phi(d)$ eigenvalues, the algebraic conjugates of $\zeta$. We hence have $\phi(d) \leq D$.

It is known that $\phi(n)$ diverges to infinity as $n$ diverges to infinity (one has $n^{1-\epsilon} = o(\phi(n))$ for instance). There is a largest integer $N$ such that $\phi(N) \leq D$.
We have necessarily \( d \leq N \). It follows \( d \mid N! \), and hence \( \zeta^{N!} = 1 \). The only eigenvalue of \( g^{N!} \) is 1. The power \( g^{N!} \) is unipotent. It is also of finite order, and we must have that \( g^{N!} \) is the neutral element. We can take \( N(D) = N! \). \( \square \)

We now prove one implication in the second part of Proposition 3.7.

Lemma 3.11. Assume now that \( S \)-simplicity holds and that \( Z_{G_S}(M) \) is cocompact in \( Z_{G_S}(U_S) \). Then \( s \) is of \( S \)-Shafarevich type.

Proof. Let \((h, g_n)\) be a sequence of points, in the \( S \)-Hecke orbit \( \mathcal{H}_S(s) \), defined over a finite extension \( F \) of \( E \). Our aim is to show this sequence can take at most finitely many distinct values. After replacing \( U_S \) by an open subgroup of finite index, we may assume that \( F = E \), which translates into the property

\[
\forall u \in U_S, \quad (h, u \cdot g_n) = (h, g_n)
\]

for all \( n \).

Let us write

\[
Z = Z_G(M), \quad Z(\mathbb{Q}_S) = Z_{G_S}(M),
\]

\[
Z(\mathbb{Q}) = Z_G(M)(\mathbb{Q}) \subseteq Z(\mathbb{A}_f) = Z_G(M)(\mathbb{A}_f)
\]

Lemma 3.9 shows that elements \( g_n \) are contained in \( T = Z_{G_S}(U_S) \cdot C \) for some compact subset \( C \) of \( G_S \). By hypothesis, \( Z_{G_S}(M) \backslash Z_{G_S}(U_S) \) is compact. By the adelic version of Godements’s compactness criterion it is also true that \( Z(\mathbb{Q}) \backslash Z(\mathbb{A}_f) \) is compact as well, as \( Z \) is \( \mathbb{R} \)-anisotropic up to the centre of \( G \).

We remark that \( Z(\mathbb{A}_f) \) normalises \( Z_{G_S}(U_S) \). As the latter is a place by place product it can be checked place by place: at places in \( S \) the projection of \( Z(\mathbb{A}_f) \) is contained in \( Z_{G_S}(M) \) which is itself contained in \( Z_{G_S}(U_S) \); at other places \( Z_{G_S}(U_S) \) has only trivial factors. We hence have a homomorphism

\[
Z_{G_S}(M) \backslash Z_{G_S}(U_S) \longrightarrow Z(\mathbb{A}_f) \backslash Z_{G_S}(U_S) \cdot Z(\mathbb{A}_f),
\]

which is surjective, with compact source, hence has compact image.

Incorporating with the compactness of \( Z(\mathbb{Q}) \backslash Z(\mathbb{A}_f) \) we infer the compactness of

\[
Z(\mathbb{Q}) \backslash Z_{G_S}(U_S) \cdot Z(\mathbb{A}_f)
\]

and follows the compactness of the subset

\[
Z(\mathbb{Q}) \backslash Z(\mathbb{A}_f) \cdot Z_{G_S}(U_S) \cdot C = Z(\mathbb{Q}) \backslash Z(\mathbb{A}_f) \cdot T
\]

of \( Z(\mathbb{Q}) \backslash G(\mathbb{A}_f) \) (we note that this quotient is separated, as \( Z(\mathbb{Q}) \) is discrete in \( G(\mathbb{A}_f) \)).
As $K$ is open, we deduce that

$$Z(Q) \backslash Z(A_f) \cdot T \cdot K/K$$

is finite. To sum it up we have

$$Z(Q) \cdot g_n \cdot K \in Z(Q) \backslash Z(A_f) \cdot T \cdot K/K \subseteq Z(Q) \backslash Z(Q) T \cdot K/K.$$ 

Now, the double coset on the right characterises $(h, g_n)$, by Lemma 3.8. Finally $(h, g_n)$ can take at most $\#Z(Q) \backslash Z(Q) T \cdot K/K$ distinct values. □

To prove the other inclusion, let us first prove that the $S$-Shafarevich property implies $S$-semisimplicity. This is done in the following lemma.

**Lemma 3.12.** The $S$-Shafarevich property implies the $S$-semisimplicity property, namely that the group $H_S$ is reductive.

**Proof.** We can argue place by place: indeed the place by place Shafarevich hypothesis is weaker than $S$-Shafarevich property, and the reductivity of $H_S$ can be checked place by place. We may assume for simplicity that $S$ consists of only one finite place.

We will prove the contrapositive statement, namely that for non reductive $H_S$ the $S$-Shafarevich property cannot hold. Let us denote the unipotent radical of $H$ by $N_H$. That $H_S$ is reductive means that $N_H$ is trivial. We assume it is not the case.

We apply [6, Proposition 3.1] to the unipotent subgroup $N_H \subset G_S$. There exists a parabolic subgroup $P$ of $G_S$ such that $N_H$ is contained in the unipotent radical $N_P$ of $P$ and the normaliser of $N_H$ in $G_S$ is contained in $P$. In particular, $H_S$ is contained in $P$.

By [36, Prop 8.4.5] there exists a cocharacter $y : G_m \rightarrow G_S$ of $G_S$ over $Q_S$ such that

$$P = \{ g \in G_S : Ad_{y(t)}(g) \text{ converges as } t \rightarrow \infty \}$$

and (cf. [36 Th. 13.4.2(i)], [20 §2.2 Def. 2.3/Prop. 2.5])

$$N_P = \{ g \in G_S : Ad_{y(t)}(g) \text{ converges to } e \text{ as } t \rightarrow \infty \}.$$ 

Moreover, the centraliser of $y$ in $P$ is a Levi factor $L$ of $P$. It follows that for all $p = \lambda \cdot n$ in $P$, with $l$ in $L$ and $n$ in $\text{Rad}^u(P)$, the limit $\lim_{t \rightarrow \infty} Ad_{y(t)}(p)$ is the factor $l$.

We will contradict the $S$-Shafarevich property by showing that the family of the $(h, y(t))$, as $t$ diverges to infinity,

- describes infinitely many points in the $S$-Hecke orbit of $s$,
– and that these points are all defined on a common finite extension of $E$.

We address the first statement.

By the non triviality of $N_H$, we may pick an element $u$ in $N_H$ distinct from $e$. It follows that the conjugacy class $C(u)$ of $u$ in $G_S$ does not contain $e$. As $Z_G(M)$ centralises $H_S$ and its subgroup $N_H$, the orbit map at $u$ for the conjugation action factors through $G_S/Z_G(M)$. We have a map

$$c : G_S/Z_G(M) \xrightarrow{gZ_G(M)\mapsto\text{Ad}_g(u)} C(u).$$

We can deduce by contradiction that $y(t)$ is not bounded modulo $Z_G(M)$ as $t$ diverges to $\infty$. Assume not. Then $y(t)Z_G(M)$ would have some accumulation point $gZ_G(M)$. Hence $c(y(t)Z_G(M))$ would have the accumulation point $c(gZ_G)$, which belongs to $C(u)$ and hence is distinct from $e$. This contradicts the fact that $c(y(t)Z_G(M))$ converges to $e$.

The Hecke orbit of $s$ can be identified with $Z_G(M)(\mathbb{Q})\backslash G_S/K_S$ through the quotient of the map $g \mapsto (s, g)$ on $G_S$. Let us claim that $Z_G(M)(\mathbb{Q})y(t)K_S$ describes infinitely many cosets. It is sufficient that $Z_G(M)y(t)K_S$ does so. If not, then $Z_G(M)y(t)$ would be contained in finitely many right $K_S$ orbits, that is in a bounded set of $Z_G(M)$-cosets, which cannot be, as we already proved. This proves the statement.

We address the second statement, investigating the field of definition of these $S$-Hecke conjugates of $s$. We use that the extension of definition of a point $(s, g)$ is associated with the finite quotient $gU_Sg^{-1}K_S/K_S$ of $\text{Gal}(\overline{E}/E)$.

As $U_S$ is topologically of finite type, it will be sufficient to show that $\# y(t)U_S y(t)^{-1}K_S/K_S$ is bounded as $t$ diverges to $\infty$. It will even be sufficient that $y(t)U_S y(t)^{-1}$ remains in a bounded subset $C$ of $G_S$, as then we have the bound $\# y(t)U_S y(t)^{-1}K_S/K_S \leq \# C K_S/K_S$.

On $P$ the family of functions $\text{Ad}_{y(t)}$ converges simply to the projection onto the Levi factor $L$ of $P$. Let us prove the claim that this convergence is uniform on compacts subsets of $P$, including $U_S$. As $\text{Ad}_{y(t)}$ acts on factor of the Levi decomposition $P = LM$ of $P$ separately, it will suffice to argue for $L$ and $\text{Rad}^a(P)$ separately.

– It is immediate for $L$, on which $\text{Ad}_{y(t)}$ is the identity, independently from $t$.
– We turn to the unipotent group $N_P$. We may argue at the level of Lie algebras, on which the corresponding action $\text{ad}_{y(t)}$ is linear. Yet again we may argue separately, this time with
respect to the decomposition into eigenspaces. On a given eigenspace, $ad_{y(t)}$ acts by a negative power of $t$, which converges uniformly to 0 as $t$ diverges to $\infty$ on any bounded subset.

This proves the claim.

We conclude that $Ad_{y(t)}$ is a uniformly bounded family on $U_S$ as $t$ diverges to $\infty$: there is a bounded set $C$ that contains $Ad_{y(t)}(U_S)$ for big enough $t$. We obtained bounded subset that we sought. This proves the second statement.

We have proven that the $S$-Shafarevich property cannot hold. □

We can now conclude the proof of the implication.

**Lemma 3.13.** The $S$-Shafarevich property implies that $Z_{G_S}(M)$ is cocompact in $Z_{G_S}(H_S)$.

**Proof.** Assume that $Z_G(M) \backslash Z_{G_S}(U_S)$ is not compact, and let us disprove the $S$-Shafarevich property. Possibly substituting $E$ with a finite extension thereof, we may assume $U_S \subseteq K$. We will prove that

\[ \mathcal{H} = \{(h, z) | z \in Z_{G_S}(U_S)\} \]

is an infinite set of points defined over $E$.

Let $(h, z)$ be such a point. For $\sigma$ in $\text{Gal}(\overline{E}/E)$ with image $u$ in $U_S$ we have

\[ \sigma((h, z)) = (h, uz) = (h, zu) = (h, z), \]

by definition of $U_S$, by the fact that $z$ commutes with $U_S$, and that $U_S$ is contained in $K$ respectively. It follows that these points are defined over $E$.

By Lemma 3.8 we have a bijection

\[ \mathcal{H} \cong Z_G(M)(\mathbb{Q}) \backslash Z_G(M)(\mathbb{Q}) \cdot Z_{G_S}(U_S) \cdot K/K. \]

We need to to prove this is an infinite set. As the following map of double quotients

\[ Z(\mathbb{Q}) \backslash Z(\mathbb{Q}) \cdot Z_{G_S}(U_S) \cdot K/K \rightarrow Z(\mathbb{A}_f) \backslash Z(\mathbb{A}_f) \cdot Z_{G_S}(U_S) \cdot K/K \]

is a surjection, it is sufficient to prove its image is infinite. As $K$ is compact, it is enough that

\begin{equation}
Z(\mathbb{A}_f) \backslash Z(\mathbb{A}_f) \cdot Z_{G_S}(U_S)
\end{equation}

be unbounded in $Z(\mathbb{A}_f) \backslash G(\mathbb{A}_f)$.

Let us accept for now that the map

\begin{equation}
Z(\mathbb{Q}_S) \backslash G(\mathbb{Q}_S) \rightarrow Z(\mathbb{A}_f) \backslash G(\mathbb{A}_f),
\end{equation}

induced by the closed immersion $\mathbb{Q}_S \rightarrow \mathbb{A}_f$, is itself a closed immersion. Moreover $Z_{G_S}(U_S)$ is closed in $G(\mathbb{Q}_S)$ and contains $Z(\mathbb{Q}_S)$. Hence the
group $Z(Q_S) \setminus Z_{G,S}(U_S)$ embeds as a closed subset of $Z(Q_S) \setminus G(Q_S)$. It is also non compact by hypothesis. Its image in $Z(A_f) \setminus G(A_f)$ is closed and non compact. It is hence unbounded. But this image is (13). This concludes

We prove now that (14) is a closed immersion. It is certainly the case of $Z \setminus G \rightarrow Z \setminus A_f$ as it induced by the closed immersion $Q_S \rightarrow A_f$ (we may embed $Z/G$ as a closed subvariety in an affine space). Moreover the map $Z(Q_S) \setminus G(Q_S) \rightarrow (Z \setminus G)(Q_S)$ is the kernel of the continuous map $(Z \setminus G)(Q_S) \rightarrow H^1(Q_S; Z)$, hence a closed immersion. The image $F$ of a closed subset of $Z(Q_S) \setminus G(Q_S)$ in $(Z \setminus G)(A_f)$ is hence closed. The inverse image of $F$ by the continuous map $Z(A_f) \setminus G(A_f) \rightarrow (Z \setminus G)(A_f)$ is a fortiori closed. The map (14) is then a closed map. Clearly, it is furthermore injective. It is finally a closed immersion. This concludes. □

This finishes the proof of Proposition 3.7.

3.3. Shimura varieties of abelian type. In this section we show that when $E$ is of finite type over $Q$ and $(G, X)$ is a Shimura datum of abelian type, all the properties except possibly $S$-Mumford-Tate hold.

Proposition 3.14. Assume that $Sh_K(G, X)$ is a Shimura variety of abelian type and that $E$ is a field of finite type.

Then the $S$-semisimplicity, $S$-Tate (and hence $S$-Shafarevich) and $S$-algebraicity hold for all points of $Sh_K(G, X)(E)$.

Proof. By definition of a Shimura variety of abelian type, there exists a Shimura subdatum $(G', X) \subset (GSp_{2g}, H_g)$ with a central isogeny $\theta: G' \rightarrow G$.

There exists a compact open subgroup $K' \subset G'(A_f)$ such that $Sh_{K'}(G', X)$ is a subvariety of $A_{g,3}$ (the fine moduli scheme of abelian varieties with level 3 structure) and there is a finite morphism $Sh_{K'}(G', X) \rightarrow Sh_K(G, X)$.

For the purposes of proving the $S$-Tate property we may assume that all our Shimura varieties are defined over $E$.

Let $x = (s, t) \in Sh_K(G, X)(E)$ and $x'$ a point of $Sh_{K'}(G', X)$ over a finite extension of $E$. Without loss of generality in the proof of the $S$-Tate property we replace $E$ by this finite extension and hence assume that $s'$ is defined over $E$.

The point $s'$ corresponds to an abelian variety $A$ defined over $E$. Let $V_S(A) = \prod_{l \in S} V_l(A)$ be the product of $l$-adic Tate modules attached to
A for \( l \in S \). The module \( V_l(A) \) is endowed with an action of \( \text{Gal}(\overline{E}/E) \) and with a symplectic action of \( \text{GSp}_{2g}(\mathbb{Q}_S) \). Following the arguments of Remark 2.8 of [40], we see that the action of \( \text{Gal}(\overline{E}/E) \) is given by the representation

\[
\rho'_S : \text{Gal}(\overline{E}/E) \to G'_S \subset \text{GSp}_{2g}(\mathbb{Q}_S).
\]

Note that in [40] there the authors suppose the field \( E \) to be a number field and \( S \) to consist of one prime. However, all arguments adapt verbatim in our situation.

We denote by \( M' \) the Mumford-Tate of \( x' \) and by \( H'_S \) the image of \( \rho'_S \).

By Falting’s theorem (Tate conjecture for abelian varieties), the group \( H'_S \) is reductive. Note that Falting’s theorem holds for abelian varieties over finitely generated fields over \( \mathbb{Q} \) (see Chapter VI of [15]).

Again, by the Tate conjecture, we have

\[
Z_{\text{GSp}_{2g,S}}(M') = Z_{\text{GSp}_{2g,S}}(H'_S)
\]

and therefore

\[
Z_{G'_S}(M') = Z_{G'_S}(H'_S).
\]

Let \( M \) be the Mumford-Tate group of \( x \) and let \( H_S \) be Zariski closure of the image of \( \rho_S \), the representation of \( \text{Gal}(\overline{E}/E) \) attached to \( x \).

Recall that we have a central isogeny \( \theta : G' \to G \). We naturally have

\[
M = \theta(M'), \quad H_S = \theta(H'_S).
\]

In particular, \( H_S \) is semisimple.

Since \( \theta \) is a central isogeny and hence commutes with conjugation, the equality \( Z_{G'_S}(M') = Z_{G'_S}(H'_S) \) commutes with conjugation.

3.4. Dependence on the field \( E \). We have seen that for a Shimura variety of abelian type and when \( E \) is of finite type all properties 1.7 (except \( S \)-Mumford-Tate) hold.

In this section we will show that these properties fail even when \( (G, X) \) is of abelian type when the field \( E \) is not of finite type.

Let \( A \) be an elliptic curve over \( \mathbb{Q} \) without complex multiplication. By a celebrated theorem of Serre, the image of the adelic representation of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) attached to \( A \) is open in \( \text{GL}_2(\mathbb{Z}) \). For any \( S \), all our properties hold for \( A \) over \( \mathbb{Q} \) (or any finite type extension of \( \mathbb{Q} \)).

Choose a prime \( l \) such that \( \rho_l \) surjects onto \( \text{GL}_2(\mathbb{Z}_l) \). Take \( S = \{ l \} \).

Let \( B \) be the standard Borel in \( \text{GL}_2(\mathbb{Z}_l) \) (upper triangular matrices) and let \( E \) be an extension corresponding to the subgroup \( \rho^{-1}_l(B) \) of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). The extension \( E \) is not of finite type over \( \mathbb{Q} \).
We immediately see that $S$-Mumford-Tate property does not hold for $A/E$ (the image of Galois is $B$). The $S$-semisimplicity also fails since the Zariski closure of $B$ is not reductive. The $S$-Tate property holds however. Indeed the centraliser of $B$ is the centre of $\text{GL}_2$ which is of course also the centraliser of $\text{GL}_2$ in itself.

Finally, let us see directly that the $S$-Shafarevich property fails. Write $(s, 1)$ the point of $\text{Sh}_{\text{GL}_n}(\mathbb{Z})((\text{GL}_2, \mathbb{H}^+))$ corresponding to $A$.

Consider elements $g_n = \begin{pmatrix} l^n & 0 \\ 0 & 1 \end{pmatrix}$ for $n > 0$.

Note that

$$g_n^{-1}B g_n \subset B$$

and therefore the sequence of points $(s, g_n)$ is defined over $E$. This sequence of points is obviously infinite - it corresponds to all elliptic curves isogeneous (over $\mathbb{C}$) to $A$ by a cyclic isogeny of degree $l^n$.

We now give an example where $S$-algebraicity fails. In the previous situation, consider the subgroup of $\text{GL}_2(\mathbb{Z})$ defined by $(a \ b \\ 0 \ a)$ where $x \in \mathbb{Z}_l$. Note that this group is Zariski dense in the group $(a \ b \\ 0 \ a)$ where $a, b \in \mathbb{Z}_l$. However it is not open in this group and therefore $S$-algebraicity fails.

Note that in the previous example semi-simplicity did not hold. However, we can construct an example where $S$-semi-simplicity does hold, but the $S$-algebraicity fails.

Let $\alpha$ be the Liouville number, $\alpha = \sum_{n=0}^{\infty} l^n$. This is an element of $\mathbb{Z}_l$ which is transcendental over $\mathbb{Q}$. We refer to [2] and references therein for more details.

Consider the subgroup $(e^x \ 0 \\ 0 \ e^x)$ where $x \in \mathbb{Z}_l$ and as before $E$ the extension corresponding to this subgroup. The Zariski closure is the diagonal torus therefore the $S$-semisimplicity holds. However the $S$-algebraicity fails. This example is analogous to the one given after Theorem 2.3 in [8].

4. The $S$-Arithmetic Lift and Equidistribution

In this section we start working towards proving Theorem 1.15. Our aim here is to translate our problem (i.e. Theorem 1.15) about the probabilities $\mu_n$ on $\text{Sh}_K(G, X)$ into a problem of equidistribution on an $S$-arithmetic homogeneous space of a semisimple algebraic group over $\mathbb{Q}$, of the kind studied [30], in order to apply the main theorem thereof. More precisely, we will use the maps $\pi_G$ of (16) and $\tau$ of (35). We warn here that the map $\pi$ we construct in this section is not the same as the one considered in the previous section.
We work with a point \( s = (h, t) \) of \( \text{Sh}_K(G, X) \) and its \( S \)-Hecke orbit \( \mathcal{H}_S(s) \) as in the statement of Theorem 1.15.

We also have a sequence of points \( s_n = (h, t \cdot g_n) \) of \( \mathcal{H}_S(s) \) with some \( g_n \in G_S \). The elements \( g_n \) are not necessarily uniquely defined by the points \( s_n \) and will actually be subject to modification, without changing \( s_n \), in the course of the proof.

For any \( g \in G_S \), we may change the representative \( (h, t) \) of \( s \) into the representative \( (h, t \cdot g) \), provided that we change accordingly each \( g_n \) into \( g^{-1} g_n \). Neither the \( S \)-Hecke orbit \( \mathcal{H}_S(s) \), nor the set of points \( \{s_n\} \) are changed by such substitutions. The group \( U_S \) is left unchanged too. Hence the \( S \)-Shafarevich property of \( s \) remains valid under this substitution of \( s \) with \( (h, tg) \).

Lastly, since the conclusion of Theorem 1.15 we may “extract a subsequence”, we may, whenever necessary, replace \( (g_n)_{n \geq 0} \) by a subsequence.

For technical reasons we will need to assume that \( H_S \cap G^{\text{der}} \) to be \( \mathbb{Q}_S \)-Zariski connected. It is sufficient that \( U_S \cap G^{\text{der}}(\mathbb{Q}_S) \) be contained in \( (H_S \cap G^{\text{der}}(\mathbb{Q}_S))^0 \). This can be achieved by passing to a subgroup of finite index in \( U_S \), that is to a finite extension of \( E \). We note that this assumption will still be fulfilled after passing again to a finite extensions of \( E \). We recall that the statement of Theorem 1.15 allows passing to finite extensions of \( E \).

4.1. The reductive \( S \)-arithmetic lift. Since the equidistribution theorems of \([30]\) apply to \( S \)-arithmetic homogeneous spaces, we need to “lift” our situation, from the base Shimura variety of level \( K \) to such a space.

In this section we construct a map \( \pi_G \) below (see (16)), from an \( S \)-arithmetic homogeneous space of the reductive group \( G \) to \( \text{Sh}_K(G, X) \).

We then introduce probabilities \( \tilde{\mu}'_n \) which are “lifts” of the probabilities \( \mu_n \), that is such that we have the compatibility (18) by direct image.

4.1.1. The \( S \)-arithmetic map. For convenience we identify the subgroup \( G_S \subseteq G(\mathbb{A}_f) \) with its image \( G(\mathbb{Q}_S) \). Let us consider the “orbit maps”

\[
\omega_{tK} : G_S \xrightarrow{g \mapsto g \cdot tK} G(\mathbb{A}_f)/K \quad \text{at the coset } tK,
\]

and \( \omega_h : G(\mathbb{R}) \xrightarrow{g \mapsto g \cdot h} X \) at the point \( h \).

Together these induce the following map

\[
\overline{\omega(h, tK)} : G(\mathbb{R} \times \mathbb{Q}_S) \xrightarrow{\omega_h \times \omega_{tK}} X \times G(\mathbb{A}_f) \xrightarrow{(h, t) \mapsto (h, t)} \text{Sh}_K(G, X).
\]
Equivalently, for any element \((g_R, g_S) \in G(\mathbb{R}) \times G(\mathbb{Q}_S) \simeq G(\mathbb{R} \times \mathbb{Q}_S),\)
\[\omega(h, tK)(g_R, g_S) = (g_R \cdot h, g_S \cdot t)\].

Remark. The Shimura variety \(\text{Sh}_K(G, X)\) has finitely many geometrically connected components. Each component is a quotient of a the hermitian symmetric domain \(X^+\) by an arithmetic subgroup of \(G(\mathbb{Q})\). The image of \(\omega(h, tK)\) consists of the union the components this image intersects.

The product \(G_S \cdot tKt^{-1}\) is open in \(G(\mathbb{A}_f)\). As \(G_S\) is a normal subgroup in \(G(\mathbb{A}_f)\), this product is a subgroup. We define then following \(S\)-arithmetic subgroup of \(G(\mathbb{Q})\):
\[(15) \quad \Gamma_G := G(\mathbb{Q}) \cap (G_S \cdot tKt^{-1})\]
Equivalently we may define, as the intersection, inside of \(G(\mathbb{A})\),
\[\Gamma_G = (G(\mathbb{R}) \cdot G_S \cdot tKt^{-1}) \cap G(\mathbb{Q}).\]
This \(\Gamma_G\) depends on \(tK\) and \(S\) though we don’t specify it to simplify notation.

The map \(\omega(h, tK)\) is left invariant under the action of \(\Gamma_G\): this map factors through a map
\[(16) \quad \pi_G = \pi_{(h, tK)} : \Gamma_G \backslash G(\mathbb{R} \times \mathbb{Q}_S) \rightarrow \text{Sh}_K(G, X).\]

Let \(K_h\) be the stabiliser of \(h\) in \(G(\mathbb{R})\) and \(K_S = K \cap G_S\), so that \(tK_S t^{-1}\) is the stabiliser of the coset \(tK\) in \(G_S\). We may further factor \(\pi_{(h, tK)}\) as the composition

- of the quotient map
\[\Gamma_G \backslash G(\mathbb{R} \times \mathbb{Q}_S) \rightarrow \Gamma_G \backslash (G(\mathbb{R}) \times G(\mathbb{Q}_S)) / (K_h \times tK_S t^{-1})\]

by the right action of the compact group \(K_h \times tK_S t^{-1}\);

- followed by a closed open immersion into \(\text{Sh}_K(G, X)\), that is the inclusion of a union of components.

\(^7\)This is an instance of a “\(S\)-adic packet” of components extracted out of the “adelic packet” of all components. For instance, if the Shimura variety is reduced to a class group (and hence finite), viewed as a Galois group, we obtain cosets of the subgroup generated by the Frobenius from the places in \(S\).

\(^8\)We understand “\(S\)-arithmetic” in the sense that there is a faithful \(\mathbb{Q}\)-linear representation \(\rho : G \rightarrow \text{GL}(n)\), such that the groups \(\rho(G)(\mathbb{Q}) \cap \text{GL}(n, \mathbb{Z}[(1/\ell)_{\ell \in S}])\) and \(\rho(\Gamma)\) are commensurable. For \(S = \emptyset\) we recover the usual notion of arithmetic subgroup.

We will define \(\Gamma\) in (32), as an \(S\)-arithmetic subgroup of \(G^\text{der}\).
4.1.2. Lift of probabilities. Recall from (2) of Definition 1.6 that we denote $U_S$ denote the $S$-adic monodromy group associated with $(h, t)$. Since $U_S$ is a compact group, it supports a Haar probability. Let $\mu_{U_S}$ be the direct image of this Haar probability in the $S$-arithmetic quotient

$$U_S \hookrightarrow G(\mathbb{R} \times \mathbb{Q}_S) \to \Gamma_G \backslash G(\mathbb{R} \times \mathbb{Q}_S).$$

The right translate of $\mu_{U_S}$ by $g_n$ is denoted by

$$\tilde{\mu}_{n} = \mu_{U_S} \cdot g_n.$$  \hfill (17)

We now show that the $\tilde{\mu}_{n}$ are lifts of the $\mu_{n}$.

**Lemma 4.1.** Pushing forward the measures $\tilde{\mu}_{n}$ from (17) along the map $\pi_G$ from (16) we get

$$\pi_G(\tilde{\mu}_{n}) = \mu_{n}.$$  \hfill (18)

**Proof.** The map $\alpha$ defined by the commutativity of the diagram

$$\text{(19) } \text{Gal}(\overline{E}/E) \xrightarrow{\rho} U_S \xrightarrow{\sigma \mapsto \Gamma_G \sigma} G(\mathbb{R} \times \mathbb{Q}_S) \xrightarrow{x \mapsto x \cdot g_n} \Gamma_G \backslash G(\mathbb{R} \times \mathbb{Q}_S) \xrightarrow{\pi_G} \text{Sh}_K(G, X)$$

sends $\sigma$ to $(h, \rho(\sigma)t g_n)$. By the defining properties (9) of $\rho$ and of $g_n$, we may rewrite $\alpha(\sigma) = \sigma \cdot (h, t g_n) = \sigma \cdot s_n$. In other words $\alpha$ is the orbit map at $s_n$ for the action of $\text{Gal}(\overline{E}/E)$ on $\text{Sh}_K(G, X)$. In particular its image is the Galois orbit of $s_n$, which is $\text{Supp}(\mu_{s_n})$ in the notation of (2). The map $\alpha$ is clearly left $\text{Gal}(\overline{E}/E)$-equivariant. Consequently the direct image of the Haar probability, say $\mu_E$, on $\text{Gal}(\overline{E}/E)$ is an invariant probability on $\text{Supp}(\mu_{s_n})$, necessarily the Haar probability of the transitive $\text{Gal}(\overline{E}/E)$-space $\text{Supp}(\mu_{s_n})$. The counting probability $\mu_n = \mu_{s_n}$ is invariant by permutation, hence is $\text{Gal}(\overline{E}/E)$-invariant. It must equal $\mu_n$ by the uniqueness of the Haar probability:

$$\alpha_*(\mu_E) = \mu_n.$$  \hfill (19)

As (19) is commutative and pushforwards are functorial, we may factor

$$\beta_*(\rho_*) = (\beta \circ \rho)_* = \alpha_*.$$  \hfill (20)

The representation $\rho$ is a continuous map of compact groups, so the direct image $\rho_*(\mu_E)$ is the Haar measure on the image $\rho(\text{Gal}(\overline{E}/E))$. This image is $U_S$ by definition. It follows that $\mu_n$ is the direct image of the Haar probability measure $\rho_*(\mu_E)$ on $U_S$ through this $\beta$. 
We defined $\mu_{U_S}$ as the image of the Haar measure of $U_S$ in the first occurrence in (19) of $\Gamma_G \backslash G(\mathbb{R} \times \mathbb{Q}_S)$, and $\tilde{\mu}_{n'}$ as the direct image in the second occurrence. By the same functoriality argument as above, the compatibilities
\[ \mu_n = \gamma_*(\mu_{U_S}) = \pi_{G*}(\tilde{\mu}_{n'}) = \beta_*(\rho_*(\mu_E)) = \alpha_*(\mu_E), \]
including the identity (18), follow. □

4.2. Passing to the derived subgroup. Theorems from [30] require that the group $G$ be semisimple. We will reduce to this case by passing from $G$ to $G^{\text{der}}$. In this section, we modify our lifting probabilities $\tilde{\mu}_{n'} = \mu_{U_S} \cdot g_n$ in two ways in order to be able to make this assumption and thus recover the setting of [30].

– Firstly we substitute the translating element $g_n$ in (17) with another one which comes from the derived group $G^{\text{der}}(\mathbb{Q}_S)$, thus constructing a probability $\tilde{\mu}_{n''}$. This first step might require passing to a subsequence and altering $t$.

– Secondly, we replace the compact subgroup $U_S$ of $G(\mathbb{Q}_S)$ by a compact subgroup $\Omega$ of $G^{\text{der}}(\mathbb{Q}_S)$, thus producing $\tilde{\mu}_{n''}$. This step may require passing to a finite extension of $E$.

We recall that we beforehand ensured that $U_S \cap G^{\text{der}}(\mathbb{Q}_S)$ is $\mathbb{Q}_S$-Zariski connected.

As, for simplicity reasons, the reference [30] deals semisimple groups $G$, instead of general reductive, we need to carry out the reduction to the semisimple case. A reader uninterested in subtle technical details may skip directly to the next section 4.3.

4.2.1. Some finite index open subgroups. We first note that
\[ G^{\text{der}}(\mathbb{Q}_S) \cdot Z(\mathbb{Q}_S) \text{ is a finite index open subgroup of } G_S. \]

Proof. As $S$ is finite, we may work place by place, in which case it is sufficient to refer to [26, §3.2, Cor. 3 p. 122, §6.4 Cor. 3 p. 320]. □

In this section we will use the notation:
\[ \Gamma_Z = (Z(\mathbb{R}) \cdot \Gamma_G) \cap Z(\mathbb{Q}_S). \]
The finiteness of the class group of the torus $Z$ tells us that
\[ (Z(\mathbb{Q}_S) \cap K) \cdot \Gamma_Z \]
is an open subgroup of finite index in $Z(\mathbb{Q}_S)$. We deduce, combining with (20), that
\[ G^{\text{der}}(\mathbb{Q}_S) \cdot (Z(\mathbb{Q}_S) \cap K) \cdot \Gamma_Z \]
is a finite index open subgroup of $G_S$. 

4.2.2. Reducing to \( g_n \in G^\text{der.}(\mathbb{Q}_S) \). The sequence of right cosets relative to \((g_n)_{n \geq 0}\), induced by \((g_n)_{n \geq 0}\), that is
\[
X \left( g_n \cdot \left( G^\text{der.}(\mathbb{Q}_S) \cdot (Z(\mathbb{Q}_S) \cap K) \cdot \Gamma_Z \right) \right)_{n \geq 0},
\]
can, by finiteness of the index below, be decomposed into at most
\[
N(G, \Gamma_G) = \left[ G^\text{der.}(\mathbb{Q}_S) \cdot (Z(\mathbb{Q}_S) \cap K) \cdot \Gamma_Z : G_S \right]
\]
constant subsequences. (This index only depends on \( G \) and \( \Gamma_G \) as hinted.)

After possibly passing to a subsequence, we may assume the sequence \((23)\) has a constant value. This value is the right coset of \( g_0 \). Replacing \( t \) by \( tg_0 \), we may assume that \( g_0 = 1 \), and that the right coset of \( g_n \) is the neutral coset. Equivalently, for every \( n \geq 0 \),
\[
\exists (d_n, z_n, \gamma_n) \in G^\text{der.}(\mathbb{Q}_S) \times (Z(\mathbb{Q}_S) \cap K) \times \Gamma_Z, \ g_n = d_n \cdot z_n \cdot \gamma_n.
\]
We will see that we may assume \( z_n = \gamma_n = 1 \), and replace \( g_n \) by \( d_n \), without interfering with the compatibility \((18)\). To achieve this, similarly to \((17)\), we set:
\[
\tilde{\mu}_n'' = \mu_{U_S} \cdot d_n.
\]

**Lemma 4.2.** The direct image along \( \pi_G \) of the above \( \tilde{\mu}_n'' \) is given by
\[
\pi_G \cdot (\tilde{\mu}_n'') = \mu_n.
\]

Lemma 4.2 is deduced from \((18)\) using following invariance properties.

**Lemma 4.3.** Let \( \mu \) be any probability measure on \( \Gamma_G \backslash G(\mathbb{R}) \times G_S \).

1. For any \( \gamma \) in \( \Gamma_G \cap Z(\mathbb{R} \times \mathbb{Q}_S) \) we have
\[
\mu \cdot \gamma = \mu.
\]
2. For any \( k \in K_h \times K \), we have
\[
\pi_G \cdot (\mu \cdot k) = \pi_G \cdot (\mu).
\]
3. For every \( g \in G(\mathbb{Q}_S) \), and every \( z \in K \cap Z(\mathbb{Q}_S) \) or \( z \in Z(\mathbb{R}) \),
\[
\pi_G \cdot (\mu \cdot z \cdot g) = \pi_G \cdot (\mu \cdot g).
\]

**Proof of 4.1.** Note that \( \Gamma_G \cap Z(\mathbb{R} \times \mathbb{Q}_S) \) acts trivially on \( \Gamma_G \backslash G(\mathbb{R}) \times G_S \), as can be checked pointwise, for some \( \gamma \in Z(\mathbb{R} \times \mathbb{Q}_S) \) and \( \gamma \in \Gamma_S \), with
\[
\Gamma_G \cdot (g_\mathbb{R}, g_S) \cdot \gamma = \Gamma_G \cdot \gamma \cdot (g_\mathbb{R}, g_S) = \Gamma_G \cdot (g_\mathbb{R}, g_S).
\]
By “transport of structure” it acts trivially its measure space.  \( \Box \)
Proof of (2). As $\pi_G$ factors through the right action of $K_h \times K$, we have
\begin{equation}
\forall k \in K \times K_h, \pi_G(x \cdot k) = \pi_G(x).
\end{equation}
Equivalently $\pi_{G*}(\delta_x \cdot k) = \pi_{G*}(\delta_x)$. Concerning $\mu$ such as in the statement, we may compute
\begin{equation}
\pi_{G*}(\mu \cdot k) = \pi_{G*} \left( \int \delta_x \cdot k \mu(x) \right) = \int \pi_{G*}(\delta_x \cdot k) \mu(x) = \pi_{G*}(\mu).
\end{equation}
We used linearity and continuity of $\pi_{G*}$ on bounded measures, seen for instance as continuous linear form on continuous functions. □

Proof of (3). As $z \in Z(\mathbb{R} \times \mathbb{Q}_S)$, we may substitute $z \cdot g = g \cdot z$. Replacing $\mu$ by $\mu \cdot g$ we may omit $g$. We have $z \in Z(\mathbb{R}) \leq K_h$ or $z \in Z(\mathbb{Q}_S) \cap K \leq K$. In either case we may apply (28b). □

Proof of Lemma 4.2. We note from definition (21) that
\begin{equation}
\Gamma_Z \subseteq Z(\mathbb{R}) \cdot (\Gamma_G \cap Z(\mathbb{R} \times \mathbb{Q}_S)).
\end{equation}
We may decompose accordingly
\begin{equation}
\gamma_n = \zeta_n \cdot c_n
\end{equation}
with $\zeta_n \in Z(\mathbb{R})$ and $c_n \in \Gamma_G \cap Z(\mathbb{R} \times \mathbb{Q}_S)$. We have
\begin{equation}
\mu_{U_S} \cdot g_n = \mu_{U_S} \cdot d_n \cdot z_n \cdot \zeta_n \cdot c_n = \mu_{U_S} \cdot d_n \cdot z_n \cdot \zeta_n
\end{equation}
where we may omit $c_n$ in the right-hand side thanks to (28a). Applying (28c) to $z = \zeta_n$ and to $z = z_n$, we deduce
\begin{equation}
\pi_{G*}(\mu_{U_S} \cdot d_n \cdot z_n \cdot \zeta_n) = \pi_{G*}(\mu_{U_S} \cdot d_n \cdot z_n) = \pi_{G*}(\mu_{U_S} \cdot d_n).
\end{equation}
The Lemma then follows from (18). □

We conclude by noting that
\begin{equation}
(h, tg_n) = s_n = (h, td_n).
\end{equation}
We may follow the same proof as that of Lemma 4.2, but with $\delta(h, tg_n)$ instead of $\mu_{U_S}$. We are now reduced to the case $g_n \in G^{\text{der}}(\mathbb{Q}_S)$.

\footnote{We recall that our spaces are ‘polish’ (separable and metrizable) and hence they are Radon spaces: Borel probability measures are inner regular (cf [10] INT Ch. IX, §3 n°3 Prop. 3).}
4.2.3. **Passing from** $U_S \leq G_S$ **to** $\Omega \leq G^{\text{der}}(Q_S)$. We now turn to the matter of replacing $U_S$ by a subgroup $\Omega$ of $G^{\text{der}}(Q_S)$.

Let
\begin{equation}
\tilde{U}_S := U_S \cdot (K \cap Z(Q_S)).
\end{equation}

Note that this is a compact group. We define
\begin{equation}
\Omega = \tilde{U}_S \cap G^{\text{der}}(Q_S).
\end{equation}

This $\Omega$ is a compact group, and therefore carries a Haar probability. Let $\mu_\Omega$ be the direct image of this Haar probability in the $S$-arithmetic quotient
\begin{equation}
\Gamma_G \backslash G(\mathbb{R} \times Q_S).
\end{equation}

Let
\begin{equation}
\Gamma = \Gamma_G \cap G^{\text{der}}(\mathbb{R} \times Q_S).
\end{equation}

This is an $S$-arithmetic group in $G^{\text{der}}(\mathbb{R} \times Q_S)$, and a lattice by the $S$-arithmetic form of Borel and Harish-Chandra theorem (e.g. [16] by Godement-Weil, cf [20, §5.4]). We dropped the index in $\Gamma$, as this will be the $S$-arithmetic lattice involved when applying (30), which is denoted by $\Gamma$ in loc. cit.

We identify the semisimple arithmetic quotient space
\begin{equation}
\Gamma \backslash G^{\text{der}}(\mathbb{R} \times Q_S)
\end{equation}

with its image via the natural embedding $\Gamma \cdot g \mapsto \Gamma_G \cdot g$ into the reductive arithmetic quotient space (31). The support of the probability measure $\mu_\Omega$ is contained in $\Gamma \backslash G^{\text{der}}(\mathbb{R} \times Q_S)$.

We now define
\begin{equation}
\tilde{\mu}_n''' = \mu_\Omega \cdot g_n,
\end{equation}
as a probability measure on $\Gamma \backslash G^{\text{der}}(\mathbb{R} \times Q_S)$.

Let now
\begin{equation}
\pi : \Gamma \backslash G^{\text{der}}(\mathbb{R} \times Q_S) \to \text{Sh}_K(G, X)
\end{equation}

be the restriction of $\pi_G$.

We have thus reduced ourselves to the case $G = G^{\text{der}}$.

4.2.4. **Passing to a splitting extension and the lifting property.** Ensuring that these $\tilde{\mu}_n'''$ are still lifts of the $\mu_n$, as in (27), will require a bit of extra work.

Recall from (20) that $G^{\text{der}}(Q_S) \cdot Z(Q_S)$ is an open subgroup of $G_S$. By its construction in (29), the group $\tilde{U}_S$ contains an open subgroup of $Z(Q_S)$. The Lie algebra of $\tilde{U}_S$ hence contains that of $Z(Q_S)$, and is
then the sum of the latter with the Lie algebra of \( \Omega \). Equivalently, the product

\[
\left( \tilde{U}_S \cap Z(\mathbb{Q}_S) \cap K' \right) \cdot \Omega
\]

is an open subgroup of \( \tilde{U}_S \); which has finite index, as \( U_S \) is compact. After replacing \( E \) by the finite extension corresponding to this subgroup, we may assume that

\[
\tilde{U}_S = \left( \tilde{U}_S \cap Z(\mathbb{Q}_S) \cap K' \right) \cdot \Omega,
\]

which implies

\[
\tilde{U}_S \cap Z(\mathbb{Q}_S) \subseteq Z(\mathbb{Q}_S) \cap K' \subseteq Z(\mathbb{Q}_S) \cap K.
\]

Thus replacing \( E \), we did not change \( \Omega \), nor the associated \( \tilde{\mu}_n''' \). We can now prove:

**Lemma 4.4.** The direct images of the measures \( \tilde{\mu}_n''' \) from \( \text{(34)} \) above, along the map \( \pi \) from \( \text{(35)} \), are given by

\[
\forall n \geq 0, \quad \pi_* (\tilde{\mu}_n''') = \mu_n.
\]

This follows from \( \text{(26)} \) and the following equality, proven below,

\[
\pi_* (\tilde{\mu}_n''') = \pi_{G_*} (\tilde{\mu}_n''').
\]

**Proof of \( \text{(40)} \).** Let \( U = \tilde{U}_S \cap Z(\mathbb{Q}_S) \). Note that the map

\[
\Omega \times U \to \Omega \cdot U = \tilde{U}_S,
\]

is a continuous map of compact groups. It is surjective in view of \( \text{(37)} \). The image of the Haar probability measure is a probability which is invariant under the image of the map. This is hence the Haar probability measure on \( \tilde{U}_S \). The direct image measure is actually a convolution of measure. Pushing into \( \Gamma_G \backslash G(\mathbb{R} \times \mathbb{Q}_S) \), this convolution, in integral form, is defined as

\[
\mu_{\tilde{U}_S} = \int_{u \in U} \mu_{\Omega} \cdot u \, du \quad \text{on} \quad \Gamma_G \backslash G(\mathbb{R} \times \mathbb{Q}_S)
\]

where the differential notation \( du \) denotes Haar probability measure on \( U \), and \( \mu_{\tilde{U}_S} \) is the direct image of the Haar probability on \( \tilde{U}_S \) in the \( S \)-arithmetic space of \( G^{\text{der}} \). Similarly we can prove

\[
\mu_{\tilde{U}_S} = \int_{k \in K \cap Z(\mathbb{Q}_S)} \mu_{U_S} \cdot k \, dk
\]
From (3) of Lemma 4.3, we get, with $dz$ the Haar probability on $K \cap Z(\mathbb{Q}_S)$,

$$
\pi_{G^*}(\mu_{\tilde{U}_S} \cdot g_n) = \pi_{G^*}\left( \left( \int_{z \in K' \cap Z(\mathbb{Q}_S)} \mu_{U_S} \cdot z \; dz \right) \cdot g_n \right) 
$$

$$
= \pi_{G^*}\left( \int_{z \in K' \cap Z(\mathbb{Q}_S)} \mu_{U_S} \cdot z \cdot g_n \; dz \right) 
$$

$$
= \int_{z \in K' \cap Z(\mathbb{Q}_S)} \pi_{G^*}(\mu_{U_S} \cdot z) \cdot g_n \; dz 
$$

$$
= \int_{z \in K' \cap Z(\mathbb{Q}_S)} \pi_{G^*}(\mu_{U_S} \cdot g_n) \; dz 
$$

In the same manner, we prove

$$(41) \quad \pi_{G^*}(\mu_{\tilde{U}_S} \cdot g_n) = \pi_{G^*}(\mu_{\Omega} \cdot g_n).$$

Finally, using Definitions (34), (35) we prove (40)

$$(42) \quad \pi^*(\tilde{\mu}_{m}^n) = \pi_{G^*}(\mu_{\Omega} \cdot g_n) = \pi_{G^*}(\mu_{\tilde{U}_S} \cdot g_n) = \pi_{G^*}(\mu_{U_S} \cdot g_n). \quad \square$$

4.2.5. Conclusion. We now have a sequence $(\tilde{\mu}_{m}^n)_{n \geq 0}$ which lifts the sequence $(\mu_n)_{n \geq 0}$, and is the translate $\mu_{\Omega} \cdot g_n$ of a probability $\mu_{\Omega}$ coming from $\Omega \leq G^\text{der}(\mathbb{Q}_S)$ by elements $g_n$ from $G^\text{der}(\mathbb{Q}_S)$, with respect to the semisimple group $G^\text{der}$. This lifted setting is the setting of [30]. We will now need to verify the hypothesis of loc. cit.

4.3. The analytic stability hypothesis. In this section we will further alter the sequence $(\tilde{\mu}_{m}^n)_{n \geq 0}$ in order to be able to apply the results of [30] to it.

To apply results of [30], the sequence $(g_n)_{n \geq 0}$ must satisfy a technical “analytic stability” hypothesis of [30], which very loosely speaking means that the “direction” in which the $g_n$ diverge is “not too close to that of the centraliser of $\Omega$”.

Recall that $M$ denotes the Mumford-Tate group of $s$, and that $H_S$ is the $\mathbb{Q}_S$-algebraic envelope of the $S$-adic monodromy group $U_S$.

Since $s$ satisfies the $S$-Shafarevich assumption, by Proposition 3.7, we know that

$$Z_G(M)(\mathbb{Q}_S) \Z_G(H_S).$$

is compact. Then the closed subspace

$$Z_G^{\text{der}}(M)(\mathbb{Q}_S) \Z_G^{\text{der}}(H_S)$$

is compact.
INNER GALOIS EQUIDISTRIBUTION IN $S$-HECKE ORBITS 41

is also compact. We may find a compact subset $C$ of $Z_{G^\der}^S(H_S)$ such that
\begin{equation}
Z_{G^\der}^S(H_S) = Z_{G^\der}(M)(\mathbb{Q}_S) \cdot C.
\end{equation}

We will denote $Z_S = Z_{G^\der}^S(H_S)$. Hence
\begin{equation}
Z_{G^\der}(\mathbb{R} \times \mathbb{Q}_S)(\Omega) = G^\der(\mathbb{R}) \times Z_S.
\end{equation}

Recall that the centraliser $Z_{G^\der}(M)$ is $\mathbb{R}$-anisotropic, as it is contained in the centraliser of the point $h \in X$, which is compact subgroup. It is a fortiori a $\mathbb{Q}$-anisotropic group.

We are in a trivial instance of Godement compactness criterion. There is a compact subset $F \subset Z_{G^\der}^S(M)(\mathbb{R} \times \mathbb{Q}_S)$ (a “fundamental set”) such that
\begin{equation}
Z_{G^\der}^S(M)(\mathbb{R} \times \mathbb{Q}_S) = (Z_{G^\der}^S(M) \cap \Gamma_S) \cdot F.
\end{equation}
Combining with (43) we get
\begin{equation}
Z_S = (Z_{G^\der}^S(M) \cap \Gamma_S) \cdot F \text{ with } F = F \cdot C
\end{equation}
a compact subset of $Z_S$.

By the results of [28] and [27, Partie 2], there exists a subset
\begin{equation}
Y = \{1_{\mathbb{R}}\} \times Y_S \subseteq G_S,
\end{equation}
where $1_{\mathbb{R}}$ denotes the neutral element of $G(\mathbb{R})$, such that
- we have a “$S$-adic Mostow decomposition”
\begin{equation}
G^\der(\mathbb{R} \times \mathbb{Q}_S) = Z_{G^\der}(\Omega)(\mathbb{R} \times \mathbb{Q}_S) \cdot (\{1_{\mathbb{R}}\} \times Y_S),
\end{equation}
which at finite places becomes
\begin{equation}
G^\der(\mathbb{Q}_S) = Z_S \cdot Y_S,
\end{equation}
- any sequence $(g_n)_{n \geq 0}$ in $\{1_{\mathbb{R}}\} \times Y_S$ satisfies the “analytic stability” property\(^{10}\) of [30] with respect to $\Omega$ (for $y^{-1}$ in $Y_S$ as we deal with left arithmetic quotient $\Gamma \backslash G$). Here we used that the $\mathbb{Q}_S$-algebraic envelope $H_S \cap G^\der_S$ of $\Omega$ is reductive.

\(^{10}\)Namely the $y^{-1}$ in $Y_S^{-1}$ are such that for any representation $\rho : G \to GL(N, \mathbb{Q}_S)$, there is a constant $c = c(\rho, \Omega)$ such that for any vector $v \in \mathbb{Q}_S^N$, the action of $y^{-1}$ one cannot shrink $\Omega \cdot V$ uniformly by a factor bigger than $c$: one has
\begin{equation}
\sup_{\omega \in \Omega} ||y^{-1} \cdot \omega \cdot v|| \geq c \cdot ||v||
\end{equation}
with respect, say, to some standard product norm on $\mathbb{Q}_S^N$. See footnote\(^{11}\).
Moreover we may, in $G_{\det}(\mathbb{R} \times \mathbb{Q}_S)$, multiply $\{1_R\} \times Y_S$ on the left by a compact subset of $G_{\det}(\mathbb{Q}_S) \times Z_S$ (and on the right by any compact subset of $G$) and the product will still satisfy previous properties.

This is the case of the subset

\[ Y' = F \cdot (\{1_R\} \times Y_S) \cdot Z_G(M)(\mathbb{R}) \subset G_{\det}(\mathbb{R} \times \mathbb{Q}_S). \]

Let us now decompose $g_n$ according to (47)

\[ \forall n \geq 0, \exists z_n \in Z_G(M)(\mathbb{Q}_S), \exists y_n \in Y_S, \ 0 \leq z_n \cdot y_n, \]

and then decompose $z_n$ inside $Z_G(M)(\mathbb{R} \times \mathbb{Q}_S)$ with respect to the decomposition (47),

\[ z_n = \gamma_n \cdot f_n, \text{ with } \gamma_n \in \Gamma_S \cap Z_G(M)(\mathbb{R} \times \mathbb{Q}_S) \text{ and } f_n \in F. \]

We recall from (28a) that

\[ \mu_\Omega \cdot \gamma_n = \mu_\Omega. \]

Substituting $g_n = \gamma_n \cdot f_n \cdot y_n$ with $f_n \cdot y_n$ we may, and will, assume that $\gamma_n = 1$. But doing so we lose the property that $g_n \in G_S$, as $\gamma_n$ may be non trivial at the real place. Instead we define

\[ g'_n = \gamma_n^{-1} \cdot g_n \cdot \gamma_n, \]

where $\gamma_n$ is the real place factor of $\gamma_n$. We then have $g'_n \in G_S$ as we ensured it be trivial at the real place. Define, in our final attempt to lift the $\mu_n$,

\[ \widetilde{\mu}_n = \mu_\Omega \cdot g'_n. \]

We observe that definition (51) agrees with (49): we have

\[ \forall n \geq 0, g'_n \in Y'. \]

**Proposition 4.5.** The sequence $(g'_n)_{n \geq 0}$ satisfies the analytic stability hypothesis required by [30] Theorem 3. Furthermore, we have

\[ \pi_* (\mu_\Omega \cdot g'_n) = \mu_n. \]

**Proof of 54.** The analytic stability hypothesis is given by (53). It remains to prove the lifting compatibility (54).

We have, using (30), and recalling definition (34),

\[ \widetilde{\mu}_n = \mu_\Omega \cdot \gamma_n^{-1} \cdot g_n \cdot \gamma_n = \mu_\Omega \cdot g_n \cdot \gamma_n = \mu_n'' \cdot \gamma_n. \]

---

We refer to footnote 10. Let $F$ resp. $C$ be a compact subset of $Z_S$ resp. $G$. For $(\gamma, y, f, v) \in C \times Y_S \times F \times \mathbb{Q}_S^N$ we have $\gamma^{-1} \cdot y^{-1} \cdot f^{-1} \cdot \omega \cdot v = \gamma^{-1} \cdot y^{-1} \cdot \omega \cdot f^{-1} \cdot v$. Let $c_1$ and $c_2$ be the maximum of the operator norm of $\rho(\gamma)$ resp. $\rho(f)$ for $\gamma$ in $C$ resp. $f$ in $F$. For instance one has $||v|| = ||f \cdot (f^{-1} \cdot v)|| \leq c_2 \cdot ||f^{-1} \cdot v||$. We have $||\gamma^{-1} \cdot y^{-1} \cdot f^{-1} \cdot \omega \cdot v|| = ||\gamma^{-1} \cdot y^{-1} \cdot \omega \cdot f^{-1} \cdot v|| \geq c_1 \cdot ||y^{-1} \cdot \omega \cdot f^{-1} \cdot v|| \geq c_1 \cdot ||f^{-1} \cdot v|| \geq c_1 \cdot c_2 \cdot ||v||$. This proves (48) of footnote 10 for $F \cdot Y_S \cdot C$ with the constant $c_1 \cdot c \cdot c_2$. 

---
But recall that $\gamma_n, R \in \mathbb{Z}^G_{\text{der}}(M)(\mathbb{R})$ a compact group which is included in the stabiliser $K_h$ of $h$ in $X$. Similarly to (28c), we deduce that the measures $\tilde{\mu}_n$ and $\tilde{\mu}''_n$, though they maybe different as measures on $\Gamma_S \setminus G^\text{der.}(\mathbb{R} \times \mathbb{Q}_S)$, will satisfy

$$\pi_{G^*}(\tilde{\mu}_n) = \pi_{G^*}(\tilde{\mu}''_n).$$

We are done: by recalling Lemma 4.4, and substituting $\pi$ for $\pi_G$, as we are dealing with measures supported on $\Gamma \setminus G^\text{der.}(\mathbb{R} \times \mathbb{Q}_S)$. □

4.4. Invoking the Theorem of Richard-Zamojksi. We now are in a position to apply [30, Theorem 3]. We summarize here how to invoke [30, Theorem 3].

4.4.1. Reviewing the hypotheses.

4.4.1.1. About the amiant group $G$ and its $S$-arithmetic quotient.

The semisimple algebraic group $G$ of [30] is the derived subgroup $G^\text{der.}$ of our initial $G$. The finite set of places considered in [30] is our set $S$ together with the archimedean place. The $S$-arithmetic lattice is our $\Gamma$ defined by (32) and (15). This defines the ambient $S$-arithmetic space $\Gamma \setminus G^\text{der.}(\mathbb{R} \times \mathbb{Q}_S)$ in our notations.

4.4.1.2. About the piece of orbit $\Omega$. The $\Omega$ of [30] is $\{1, \mathbb{R}\} \times \Omega$ here. The image $U_S$ of the representation of the Galois group is a compact $S$-adic Lie subgroup of $M(\mathbb{Q}_S)$. It follows that $\Omega$, as defined in (29) and (30), is a bounded $S$-adic Lie subgroup. It is Zariski connected is indeed a bounded $S$-adic Lie subgroup. We postpone the proof that its $\mathbb{Q}_S$-algebraic envelope $H$ is a Zariski connected subgroup of $G_S$ to 4.4.3 below.

4.4.1.3. About the sequence of translates. The measure $\mu_\Omega$ to be translated fits into the scope of study of [30].

The translating elements $g_n$ of [30] are our $g'_n$. The “analytic stability” hypothesis has been taken care of in the preceding section.

4.4.2. Applying Theorem of [27]. As a consequence of this analytic stability hypothesis, we may apply [27, Théorème 1.3, Exp. VI, p. 121] with $H = H_S$ and $Y_S = \{y_n\}$ and $f$ the characteristic function of $\Omega$ to the sequence of translated probabilities

$$\tilde{\mu}_n = \mu_\Omega \cdot g'_n$$

on the $S$-arithmetic homogeneous space

$$\Gamma \setminus G^\text{der.}(\mathbb{R} \times \mathbb{Q}_S).$$

Thus the sequence $(\tilde{\mu}'_n)_{n \geq 0}$ is tight: any subsequence contains a subsequence converging to a probability measure.
In particular, after possibly extracting a subsequence, we may assume the sequence \((\tilde{\mu}_n')_{n \geq 0}\) is tightly convergent.

We may now invoke \cite[Theorem 3]{30}.  

4.4.3. The envelope of \(\Omega\). We recall that the \(\mathbb{Q}_S\)-algebraic envelope of \(\bar{U}_S\), the algebraic monodromy group, is denoted \(H_S\), and that \(H_S\) is a reductive group. Here we will determine the \(\mathbb{Q}_S\)-algebraic envelope of \(\Omega\), in terms of that of \(H_S\). We discuss why it is a reductive group, and how to ensure it is Zariski connected.

Firstly the \(\mathbb{Q}_S\)-algebraic envelope \(\tilde{H}_S\) of \(\tilde{U}_S\) is the algebraic group generated by \(U_S\) and \(K \cap Z(\mathbb{Q}_S)\). The \(\mathbb{Q}_S\)-algebraic envelope of \(Z(\mathbb{Q}_S) \cap K\) is a Zariski open subgroup \(\tilde{Z}\) of \(Z(\mathbb{Q}_S)\): one has \(Z^0 \leq \tilde{Z} \leq Z\). Then

\[
\tilde{H}_S = H_S \cdot \tilde{Z}.
\]

Let \(H\) be the \(\mathbb{Q}_S\)-algebraic envelope \(\Omega\). One has \(H \leq G^\text{der}_S\) and \(H \leq \tilde{H}_S\) as \(\Omega\) is contained in both groups.

Let \(u, \tilde{u}, \omega, 3, h, \tilde{h}\) be the Lie algebras of \(U, \bar{U}_S, \Omega, Z, H_S\) and \(H\). All sums and direct sums will be sum of linear spaces, and the resulting sums will be Lie algebras. We have \(\tilde{u} = u + 3\). The decomposition \(g = g^\text{der} \oplus 3\) induces \(\tilde{u} = \omega \oplus 3\). Taking algebraic envelopes of Lie subalgebras is compatible with sums. We get

\[
\tilde{h}_S = h_S + 3 = h \oplus 3,
\]

and deduce \(h = \tilde{h}_S \cap g^\text{der}\). This establishes that \(H\) is a Zariski open subgroup of \(\tilde{H}_S \cap G^\text{der}_S\).

The product map \((h, z) \mapsto h \cdot z\) is an homomorphism, as \(H\) and \(Z\) commute. At the level of the groups, we deduce an isogeny

\[
H^0 \times Z^0 \to H^0_S \cdot Z^0.
\]

Recall that \(H_S\) and \(Z\) are reductive. Hence \(H^0_S \times Z^0\) is reductive too, and so is its quotient \(H^0_S \cdot Z^0\). The finite cover \(H^0 \times Z^0\) must then be reductive, and so is its direct factor \(H^0\). This establishes that \(H\) is a reductive group.

The Zariski neutral component \(H^0\) of \(H\) is that \((\tilde{H}_S \cap G^\text{der}_S)^0\), and we have \(H = H^0\) if and only if

\[
\Omega \subseteq H^0(\mathbb{Q}_S) = (\tilde{H}_S \cap G^\text{der}_S)^0(\mathbb{Q}_S).
\]

As \(H^0(\mathbb{Q}_S)\) is open in \(H(\mathbb{Q}_S)\), there is a neighbourhood \(V\) of \(H^0(\mathbb{Q}_S)\) in \(G_S\) that does not meet \(H(\mathbb{Q}_S) \setminus H^0(\mathbb{Q}_S)\). Note that \(H^0\) does only depend on \(H_S\), not on \(\Omega\) or \(K\). Provided \(K\) and \(U_S\) are sufficiently small, we may ensure that \(\Omega \subseteq U_S \cdot K \subseteq V\). In such a case, the group \(H\) is Zariski connected.
4.5. **The equidistribution property.** We apply [30, Theorem 3] in the setting recalled in §4.4. Then, possibly passing to a subsequence of \((\tilde{\mu}_n)_{n \geq 0}\), there exist

- a \(\mathbb{Q}\)-subgroup \(L \leq G^{\text{der}}\) of \(S\)-Ratner type (as in Definition 1.9),
- and elements \(n_\infty, g_\infty \in G^{\text{der}}(\mathbb{R} \times \mathbb{Q}_S)\)

such that the limit

\[
\lim_{n \to \infty} \tilde{\mu}_n = \mu_{L^\vee} n_\infty \ast \nu \cdot g_\infty
\]

is the translate by \(g_\infty\) of the convolution of the translated probability measure \(\mu_{L^\vee} \cdot n_\infty\) by the Haar probability \(\nu\) on \(\Omega\). In terms of Radon measures on \(\Gamma_S \backslash G^{\text{der}}(\mathbb{R} \times \mathbb{Q}_S)\), applied to an arbitrary bounded continuous test function \(f\) in \(C^b(\Gamma_S \backslash G^{\text{der}}(\mathbb{R} \times \mathbb{Q}_S))\), this means

\[
\lim_{n \to \infty} \int f \tilde{\mu}_n = \int f \mu_{L^\vee} n_\infty \ast \nu \cdot g_\infty = \int_{\omega \in \Omega} \int_{l \in \Gamma_S^L} f(l \cdot n_\infty \cdot \omega \cdot g_\infty) \mu_{L^\vee}(l) \nu(\omega).
\]

This proves the equidistribution conclusion (1) of Theorem 1.15.

4.5.1. **Getting rid of \(n_\infty\).** After replacing by a subsequence, all the conclusions of [30, Theorem 3] hold. In particular we also know that \(n_\infty\) belongs to the closure of \((\Gamma \cap N) \cdot (Z_{G^\text{der}}(\Omega) \cap N)\). We claim that

\[
(\Gamma \cap N) \cdot (Z_{G^\text{der}}(\Omega) \cap N)
\]

is already a closed subset.

**Proof.** By the S-Shafarevich hypothesis, the quotient

\[
Z_{G_S}(\Omega)/Z_{G_S}(M)
\]

is compact. It follows that for any subgroup \(Z\) of \(Z_{G_S}(\Omega)\) the quotient

\[
Z/(Z \cap Z_{G_S}(M)) \subseteq Z_{G_S}(\Omega)/Z_{G_S}(M)
\]

is compact. In particular, for \(Z = Z_N(\Omega) = N \cap Z_{G^\text{der}}(\Omega)\), we may write

\[
(\Gamma \cap N) \cdot (Z_{G^\text{der}}(\Omega) \cap N) = (\Gamma \cap N) \cdot (Z_{G^\text{der}}(M) \cap N) \cdot C
\]

for a compact subset \(C\). The closedness of (57), that is of

\[
(\Gamma \cap N) \cdot (Z_{G^\text{der}}(\Omega) \cap N) = (\Gamma \cap N) \cdot (Z_{G^\text{der}}(M) \cap N) \cdot C
\]

will follow from that of

\[
(\Gamma \cap N) \cdot (Z_{G^\text{der}}(M) \cap N).
\]

Let \(F\) be a finite generating subset of \(M\), as a topological group for the Zariski topology, defined over \(\mathbb{Q}\). Then the orbit map for the adjoint action

\[
G^\text{der}_S \xrightarrow{g \mapsto (gfg^{-1})_{f \in F}} (G^\text{der}_S)^F
\]
embeds $G_S/Z_{G_S}^\text{der}(\Omega)$ as a subvariety of the affine variety $G_S^\text{der}$. (In particular $G_S/Z_{G_S}^\text{der}(\Omega)$ is quasi-affine). We can linearise this action by choosing a faithful representation $G_S^\text{der} \to GL(N)$ and embedding $GL(N)^F$ into $V = M^F_N$ a Cartesian power of the corresponding matrix space.

Let $p$ be a generator of $\det l \subseteq \bigwedge^{\dim L} g$, the maximal exterior power of the Lie algebra of $L$. Then $N$ is defined as the stabiliser of $p$ in $G_S^\text{der}$. The orbit map

$$G_S^\text{der} \xrightarrow{g \mapsto g \cdot p = \bigwedge^{\dim L(ad_g)}(p)} \bigwedge^{\dim L} g$$

at $p$ embeds $G/N$ into the affine variety $W = \bigwedge^{\dim L} g$.

We deduce, with the product map, an embedding

$$G/(Z_{G_S}^\text{der}(\Omega) \cap N) \to V \times W,$$

the orbit map at $((f)_{f \in F}, p)$.

By definition $\Gamma$ stabilises an lattice $\Lambda$ in the free $\mathbb{Q}_S$ module $V \times W$ defined over $\mathbb{Q}$, which is arithmetic, that made of $\mathbb{Q}$ rational elements. Up to scaling we may assume $((f)_{f \in F}, p) \in \Lambda$. It follows that

$$(\Gamma \cap N) \cdot ((f)_{f \in F}, p)$$

which is a subset of $\Lambda$, is discrete, and in particular closed. It follows that its inverse image in $G_S^\text{der}$, which is (59) is closed as well. □

We may write $n_\infty = \gamma \cdot z$ with $\gamma \in \Gamma \cap N$ and $z \in Z_{G_S}^\text{der}(\Omega)$. We claim that

$$\mu_{L^\dagger} \cdot \gamma = \mu_{L^\dagger},$$

which is proven below. We view $\nu$ as a measure on $G_S^\text{der}$ supported on $\overline{\Omega}$. (Actually $\overline{\Omega} = \Omega$). We also have

$$z \cdot \nu = \nu \cdot z.$$

So we may rewrite the limit measure

$$\mu_{L^\dagger} \cdot n_\infty \ast \nu \cdot g_\infty = \mu_{L^\dagger} \ast \nu \cdot z \cdot g_\infty.$$

Subsituting $g_\infty = z \cdot g_\infty$ we may assume $n_\infty = 1$.

**Proof of the claim.** Recall that $L$ is a normal subgroup of $N$. Consequently $\Gamma \cap L(\mathbb{R} \times \mathbb{Q}_S)$ is normal in $\Gamma \cap N(\mathbb{R} \times \mathbb{Q}_S)$. The subgroup $L(\mathbb{R} \times \mathbb{Q}_S)^+$ is normalised by $N(\mathbb{R} \times \mathbb{Q}_S)$, hence by $\Gamma \cap N(\mathbb{R} \times \mathbb{Q}_S)$. We have seen both factors of

$$L(\mathbb{R} \times \mathbb{Q}_S)^+ \cdot (\Gamma \cap L(\mathbb{R} \times \mathbb{Q}_S))$$

are normalised by $\Gamma \cap N(\mathbb{R} \times \mathbb{Q}_S)$. As $\Gamma \cap N(\mathbb{R} \times \mathbb{Q}_S)$ acts continuously, it normalises the closure of this product, namely $L^\dagger$. 
By definition $\mu_L^\dagger$ is the right $L^\dagger$-invariant probability on $\Gamma \backslash \Gamma L^\dagger$. It follows that $\mu_L^\dagger \gamma$ is the right $\gamma^{-1}L^\dagger \gamma$-invariant probability on $\Gamma \backslash \Gamma L^\dagger \gamma$. We just have seen $\gamma^{-1}L^\dagger \gamma = L^\dagger$. But we also have

$$\Gamma \backslash \Gamma L^\dagger \gamma = \Gamma \backslash \Gamma \gamma^{-1}L^\dagger \gamma = \Gamma \backslash \Gamma L^\dagger.$$ 

It follows that $\mu_L^\dagger \gamma = \mu_{\gamma^{-1}L^\dagger \gamma} = \mu_L^\dagger$. \hfill \Box

5. **Focusing criterion and Internality of equidistribution**

In this section we prove conclusion (2) of Theorem 1.15, the inclusion of supports of the measures $\mu_n$ in the support of the limiting measure. This inclusion of supports is required in the reasoning of section 2.

**Lemma 5.1.** For all $n$ large enough, we have the inclusion

$$\text{Supp}(\mu_n) \subset \pi(\text{Supp}(\mu_\infty))$$

of closed subsets in $\text{Sh}_K(G, X)$.

5.1. **A special case.** Let us first treat the case where $g'_n$ belongs to $\bigcap_{\omega \in \Omega} \omega L^\dagger \omega^{-1}$. This is the typical situation featuring the dynamics explicited by [30, Theorem 3]. We will see later in this section how to reduce ourselves to this case.

We first note that $\Omega$ is a compact subgroup. Therefore $\Omega = \overline{\Omega}$ and

$$\text{Supp}(\nu) = \Gamma \backslash \Gamma \cdot \overline{\Omega} = \Gamma \backslash \Gamma \cdot \overline{\Omega}$$

and

$$\bigcap_{\omega \in \Omega} \omega L^\dagger \omega^{-1} = \bigcap_{\overline{\Omega}} \omega L^\dagger \omega^{-1}.$$

For any $\omega$ in $\Omega$ we rewrite

$$\Gamma \cdot \omega \cdot g'_n = \Gamma \cdot (\omega \cdot g'_n \cdot \omega^{-1}) \omega \in \Gamma L^\dagger \omega.$$

It follows that

$$\text{Supp}(\tilde{\mu}_n) = \Gamma \backslash \Gamma \Omega \cdot g'_n \subseteq \Gamma \backslash \Gamma L^\dagger \Omega = \text{Supp}(\mu_{L^\dagger} \star \nu).$$

It is now enough to observe that

$$\mu_\infty = \pi_*(\mu_{L^\dagger} \star \nu),$$

which are canonical measures supported on a union of real weakly $S$-special subvarieties associated with $L$.

5.2. **The focusing criterion’s factorisation.** We recall that we have a limiting distribution

$$\lim_{n \to \infty} \tilde{\mu}_n = \mu_L^\dagger \star \nu \cdot g_\infty$$

where $\tilde{\mu}_n = \mu_\Omega \cdot g'_n$ satisfies

$$\mu_n = \pi_*(\tilde{\mu}_n).$$
We ensured that $g'_n$ belongs to $G^\text{der}(\mathbb{Q}_S)$ (viewed as a subgroup of $G(\mathbb{R} \times \mathbb{Q}_S)$).

We want to prove that, after possibly extracting a subsequence, $\text{Supp}(\mu_n) = \text{Supp}(\pi_*(\tilde{\mu}_n))$ is contained in $\text{Supp}(\mu_\infty) = \text{Supp}(\pi_*(\tilde{\mu}_\infty))$.

As we are allowed to extract subsequences we may use the more stringent conclusions of [30, Theorem 3], the focusing criterion, according to which we may factor

\[ g'_n = l_n \cdot f_n \cdot b_n \]

where $l_n \in \bigcap_{\omega \in \Omega} \omega L(\mathbb{R} \times \mathbb{Q}_S) \omega^{-1}$, where $(b_n)_{n \geq 0}$ is a bounded sequence and where $f_n \in N \cap Z_{G^\text{der}}(\Omega)$.

We note that such a decomposition occurs if and only if it occurs place by place. In particular, as the real component of $g'_n$ is the neutral element, we may, and we will, substitute the real components of $l_n$, of $f_n$ and of $b_n$ by the neutral element and still have a decomposition a above. To summarize: without loss of generality, we may assume

\[ l_n, f_n, b_n \in G^\text{der}_S. \]

Moreover, the $S$-Shafarevich hypothesis implies that $Z_{G^\text{der}_S}(M)$ is cocompact in $Z_{G^\text{der}_S}(\Omega)$.

**Lemma 5.2.** The subgroup $Z_{G^\text{der}_S}(M) \cap N$ of $Z_{G^\text{der}_S}(\Omega) \cap N$ is cocompact.

**Proof.** Note that $Z_{G^\text{der}_S}(\Omega)$, as it is a centraliser, is an algebraic subgroup of $G_S$. Let $Z^+$ be the maximal isotropic connected $\mathbb{Q}_S$-subgroup of $Z_{G^\text{der}_S}(\Omega)$ (the product of the non compact factors of the factor groups $Z_{G^\text{der}_S}(\Omega) \cap G(\mathbb{Q}_p)$). This is a normal subgroup of $Z_{G^\text{der}_S}(\Omega)$. The isotropic factors are generated by unipotent subgroup and split tori. Hence every regular function on $Z^+$ which is bounded is actually constant. It also contained in every cocompact algebraic reductive subgroup of $Z_{G^\text{der}_S}(\Omega)$ (the isotropic factors are generated by unipotent subgroup and split tori, on there which have no nontrivial bounded map). \hfill \Box

The quotient $(Z_{G^\text{der}_S}(\Omega) \cap N)/(Z_{G^\text{der}_S}(M) \cap N)$ can be identified, as a group, to $NZ_{G^\text{der}_S}(M)/Z_{G^\text{der}_S}(M)$.

### 5.3. Getting rid of the bounded factor $b_n$.

The bounded factors $b_n$ are just translating, at the “infinite level” $\Gamma \backslash G^\text{der}(\mathbb{R} \times \mathbb{Q}_S)$, the situation. They involve no asymptotic dynamical feature, and as they belong to $G_S$, they will essentially be killed at finite level, modulo $K$. Here are the details.
Hence $\mu$ for every bounded measure say, $b$. We may assume possibly extracting further, we may assume that this is a constant sequence.

We consider the algebraic subgroups of $G$, $\Gamma \backslash G(\mathbb{R} \times \mathbb{Q})$ we have $\pi(x \cdot b_n) = \pi(x)$. Hence

$$\pi_*(\mu \cdot b_n) = \pi_*(\mu)$$

for every bounded measure $\mu$ on $\Gamma \backslash G(\mathbb{R} \times \mathbb{Q})$. In particular

$$\mu_n = \pi_* (\hat{\mu}_n) = \pi_*(\mu_\Omega \cdot l_n \cdot f_n \cdot b_n) = \pi_*(\mu_\Omega \cdot l_n \cdot f_n).$$

In order to prove, in $\Gamma \backslash G(\mathbb{R} \times \mathbb{Q}) K/K \subseteq \text{Sh}_K(G, X)$

$$\text{Supp}(\pi_*(\hat{\mu}_n)) = \text{Supp}(\mu_n) \subseteq \text{Supp}(\mu_\infty)$$

we may substitute $\hat{\mu}_n = \mu_\Omega \cdot l_n \cdot f_n \cdot b_n$ with $\mu_\Omega \cdot l_n \cdot f_n$. In other terms we may assume $b_n = 1$.

5.4. Getting rid of the centralising factor $f_n$. Actually the element $n_\infty$, that we got rid of, is related to the $f_n$. Part of the argumentation here will parallel the one that dealt with $n_\infty$.

As $f_n$ belongs to both $Z_{G^{\text{der}}(\mathbb{Q})}(\Omega)$ and the normaliser of $L^\dagger$, we have

$$f_n M f_n^{-1} = \bigcap_{\omega \in \Omega} f_n \omega f_n^{-1} \cdot f_n L^\dagger f_n^{-1} \cdot f_n \omega^{-1} f_n^{-1} = \bigcap_{\omega \in \Omega} \omega(f_n L^\dagger f_n^{-1}) \omega^{-1}.$$

the element $l_n^\prime = f_n^{-1} \cdot l_n \cdot f_n$ belongs to $\bigcap_{\omega \in \Omega} \omega L^\dagger \omega^{-1}$ as much as $l_n$ does. And so does $l_n^\prime = \omega \cdot l_n^\prime \cdot \omega^{-1}$. Let us now rewrite

$$\Gamma \backslash \omega \cdot l_n \cdot f_n = \Gamma \backslash \omega \cdot l_n \cdot f_n = \Gamma \backslash \gamma f_n \cdot \omega \cdot l_n^\prime = \Gamma \backslash \Gamma f_n \cdot l_n^\prime \cdot \omega.$$

We consider the algebraic subgroups of $G^{\text{der}}(\mathbb{R} \times \mathbb{Q})$ given by

\begin{align*}
(61a) & \quad N' = N \cap Z_G(M) \\
(61b) & \quad M = \bigcap_{\omega \in \Omega} \omega L^\dagger \omega^{-1}.
\end{align*}

We note that the coset $\Gamma \backslash \gamma f_n \cdot l_n^\prime$ in $\Gamma \backslash G^{\text{der}}(\mathbb{R} \times \mathbb{Q})$ belongs to the subspace $\Gamma \backslash N' \simeq (\Gamma \cap N') \backslash N'$. As $M$ is normalised by $N'$, we may consider the quotient space $(\Gamma \cap N') \backslash N' \cdot M/M$
We may hence consider the double coset 

$$(\Gamma \cap N') \cdot f_n \cdot M.$$ 

We claim that this sequence is bounded in $(\Gamma \cap N') \setminus N'M/M$.

**Proof.** By contradiction, assume not. Then this sequence of double cosets diverges to infinity in $(\Gamma \cap N') \setminus N'L^\perp/L^\perp$. A fortiori $(\Gamma \cap N') \cdot f_n \cdot l$ diverges to infinity, uniformly for every $l$ in $L^\perp$. As $(\Gamma \cap N') \setminus N' \cong \Gamma \setminus \Gamma \cdot N' \subseteq \Gamma \setminus G^\text{der}(\mathbb{R} \times \mathbb{Q}_S)$ is a closed immersion, hence proper, the $\Gamma \cdot f_n \cdot l''$ diverges uniformly to infinity in $\Gamma \setminus G^\text{der}_S$. As $\Omega$ is bounded, so does the $\Gamma \cdot f_n \cdot l'' \cdot \omega$. It follows that

$$\lim \tilde{\mu}_n = 0,$$

(strongly on every compact) which is not the case. \qed

Possibly extracting a subsequence we may assume that $(\Gamma \cap N') \cdot f_n \cdot M$ is convergent, with some limit, say $(\Gamma \cap N') n_{\infty} M$ with $n_{\infty}$ in $N'$. We may write, inside $N'$,

$$\gamma_n^{-1} \cdot f_n \cdot \lambda_n = n_{\infty} \cdot \beta_n$$

with $\gamma_n$ in $\Gamma \cap N'$, with $\lambda_n$ in $M$ and $(\beta_n)_{n \geq 0}$ a bounded sequence in $N'$. We rearrange

$$\Gamma \setminus \Gamma \cdot \omega \cdot l_n \cdot f_n = \Gamma \setminus \Gamma \gamma_n \cdot \omega \cdot l''_n \cdot \lambda'_n \cdot n_{\infty} \cdot \beta_n.$$

where $\omega \gamma_n = \gamma_n \omega$, where $l''_n = \gamma_n^{-1} l_n \gamma_n$ and $\lambda'_n = (n_{\infty} \cdot \beta_n) \lambda_n \cdot (n_{\infty} \cdot \beta_n)^{-1}$. We may factor $\Gamma \gamma_n = \Gamma$.

6. **Zariski closedness from the S-Mumford-Tate hypothesis.**

We put ourselves in the situation of Section 5. We will use the S-Mumford-Tate property to reach the stronger conclusion (3) of Theorem 1.15, namely that we obtain actual weakly special subvarieties.

6.1. **Normalisation by rational monodromy.**

**Proposition 6.1.** Assume that the algebraic monodromy subgroup $H_S$ is definable over $\mathbb{Q}$, that is of form $H_S = H_{\mathbb{Q}_S}$ for some $\mathbb{Q}$-subgroup $H$ of $G$.

Then the subgroup of Ratner class $L$ in $G$ is normalised by $H$. 
Proof. We assume that the sequence \((\mu_n)\) converges to a measure \(\mu_\infty\) associated to a \(\mathbb{Q}\)-group \(L\). We know that
\[ g_n = L^H \cdot (N \cap Z_G(H)) \cdot O(1) \]
with
\[ L^H = \cap hLh^{-1} \]
and
\[ N \subset N_G(L). \]
We want to show that \(L = L^H\).
Without loss of generality, we may assume that \(O(1) = 1\).
We have that \(N \cap Z_G(H)\) is defined over \(\mathbb{Q}\) and
\[ N \cap Z_G(H) = N \cap Z_G(H) \cap \Gamma \times \mathcal{F}. \]
As \(Z_G(H)\) is \(\mathbb{Q}\)-anisotropic, \(N \cap Z_G(H)\) is \(\mathbb{Q}\)-anisotropic and therefore \(\mathcal{F}\) is compact.
We write
\[ g_n = l_nf_n \]
with \(f_n \in (N \cap Z_G(H))(\mathbb{R} \times \mathbb{Q}_S)\) and we write
\[ f_n = \gamma_n \cdot \phi_n \]
with \(\gamma_n \in Z_G(H) \cap \Gamma\) and \(\phi_n \in \mathcal{F}\).
As \(\mathcal{F}\) is compact, after extraction, we may assume that \(\phi_n\) is convergent and we may assume that the limit is one.
We can therefore assume that
\[ g_n = l_n \cdot \gamma_n \]
and \(g_n\) stabilises the closed set \(\Gamma \backslash \Gamma L^H \cdot U\) (it is closed because \(L^H\) is defined over \(\mathbb{Q}\))
and
\[ \Gamma \backslash \Gamma L^H U \text{ contains the support of } \mu_n. \]
Therefore \(\Gamma \backslash \Gamma L^H U\) contains \(Supp(\mu_\infty)\), hence \(\dim L^H \geq \dim(Supp(\mu_\infty)) = \dim(L)\).
With the assumption of the Mumford-Tate, we show that \(L\) is normalised by \(M = H\). \(\square\)

Under the \(S\)-Mumford-Tate type hypothesis we immediately deduce the following.

**Corollary 6.2.** Assume moreover that \(s\) is of \(S\)-Mumford-Tate type, that is \(M = H\).
Then \(L(\mathbb{R})\) is normalised by \(h\).
6.2. A criterion for a weakly $S$-special real submanifold to be a weakly special subvariety. In this section we show that stronger conclusions 3 and 2 of the main theorems 1.15 and 1.16 respectively hold under the assumption that the $S$-Mumford-Tate hypothesis holds. These conclusions follow from the following proposition.

**Proposition 6.3.** Let $(G, X)$ be a Shimura datum and $h$ a point in a connected component $X^+$ of $X$. Let $L \subset G$ be a subgroup such that $L_R$ is normalised by $h(S)$.

Then the image of $L(\mathbb{R})^+ \cdot x \subseteq X^+$ in $\Gamma \backslash X$ is a weakly special subvariety, where $\Gamma$ is any congruence arithmetic subgroup of $G$.

**Proof.** First observe that $L_R$ is normalised by $h(\sqrt{-1})$ which induces a Cartan involution on $G_{\text{ad}}$. Therefore $L_R$ is reductive (see [32, Th. 4.2]).

Let $N = N(L)^0$ be the neutral component of the normaliser of $L$ in $G$. Because $L$ is reductive, so are $N$ and its centraliser $Z_G(L)$, and we have

$$N = Z_G(L) \cdot L.$$  

Note that $h$ factors through $N_R$.

We have a almost-product decomposition of semisimple groups

$$N^\text{der} = Z_G(L)^\text{der} \cdot L^\text{der}.$$  

Let $Z^c$ (resp. $Z^{nc}$) denote the almost product of the almost $\mathbb{Q}$-factors of $Z_G(L)$ which are $\mathbb{R}$-compact (resp. which are not $\mathbb{R}$-compact). Using the analogous notation for $L$, we have

$$Z_G(L)^\text{der} = Z^{nc} \cdot Z^c \quad \text{and} \quad L^\text{der} = L^{nc} \cdot L^c.$$  

By [37, Lemme 3.7], $x$ factors through $H = Z(N)Z^{nc}L^{nc}$.

Let $X_H = H(\mathbb{R}) \cdot h$. By [37, Lemme 3.3], $(H, X_H)$ is a Shimura subdatum of $(G, X)$.

Note that

$$H^\text{ad} = Z^{nc,ad} \times L^{nc,ad}$$  

and write $h^\text{ad} = (h_1, h_2)$ in this decomposition.

We have

$$X_H^{ad} = X_1 \times X_2$$  

where $X_1 = Z^{nc,ad}(\mathbb{R}) \cdot h_1$ and $X_2 = L^{nc,ad}(\mathbb{R}) \cdot h_2$.

By Lemma 3.3 of [37], both $(Z^{nc,ad}, X_1)$ and $(L^{nc,ad}, X_2)$ are Shimura data.

The image of $L(\mathbb{R})^+ \cdot h$ in $X_{H^{ad}}$ is $\{h_1\} \times X_2$, as in [32, Section 2]. This finishes the proof. □

For the sake of completeness, we give an alternative proof of the statement [?].
Alternative proof. The symmetry $s_h$ of $X$ at $x$ is induced by the Cartan involution given by the conjugation action of $h(i)$. But $h(i)$ normalises $L(\mathbb{R})$, hence normalises its neutral component $L(\mathbb{R})^+$, from which we get that $L(\mathbb{R})^+ \cdot h$ under $s_h$: it is symmetric at $h$.

Moreover $L(\mathbb{R})^+$ is normalised by $h(U(1))$, whose conjugation induces the complex structure on the tangent $T_hX$ space of $X$ at $h$. We deduce that the tangent space of $L(\mathbb{R})^+ \cdot h$ at $h$ is complex subspace of $T_hX$.

Notice that $L$ is normalised by $lhl^{-1}$ for every $l \in L(\mathbb{R})^+$. By the argument above, $L(\mathbb{R})^+ \cdot h$ is symmetric at every point and has a complex tangent space at every point: it is a symmetric quasi-complex subspace.

By [17], this quasi-complex structure is a complex structure: $L(\mathbb{R})^+ \cdot h$ is a symmetric holomorphic subvariety.

By a theorem of Baily-Borel [3], the arithmetic quotients

$$(\Gamma \cap L(\mathbb{R})^+) \setminus L(\mathbb{R})^+ \cdot h \text{ and } \Gamma \setminus X$$

are quasi-projective varieties. And by a theorem of Borel [7], the embedding into $\Gamma \setminus X$ is algebraic.

It implies that it is a totally geodesic subvariety in the sense of [22], that is a weakly special subvariety in our terminology. □

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