THE MINIMAL RESOLUTION OF A COINTERVAL EDGE IDEAL IS MULTIPLICATIVE

EMIL SKÖLDBERG

Abstract. We show that the minimal resolution of the quotient of the polynomial algebra over a field by a cointerval edge ideal can be given the structure of a DG-algebra.

1. Introduction

To a simple graph $G$ on the vertex set $[n] = \{1, \ldots, n\}$, one can associate an ideal $I_G$ in the polynomial algebra $S = k[x_1, \ldots, x_n]$ over the field $k$, by letting $I_G$ be generated by all monomials $x_i x_j$ such that $ij$ is an edge in $G$; this ideal is known as the edge ideal of $G$. In recent years, the study of edge ideals has enjoyed a great deal of popularity, and several authors have worked on relating the graph-theoretical properties of $G$ to the algebraic properties of $I_G$.

In this paper, we study the minimal resolution of $R_G = S/I_G$ in the case when $G$ is a cointerval graph, which is a graph that is the complement of an interval graph. The resolution can be obtained as a special case either of results by Chen [Che10], or by Dochtermann and Engström [DE12]. Chen constructs the minimal resolution of $R_G$ for all complements of chordal graphs, and since every interval graph is chordal, cointerval ideals are covered. Dochtermann and Engström construct the minimal resolution of $I_G$ for all cointerval $d$-hypergraphs; our cointerval graphs being the case of $d = 2$.

In section 2, we describe the resolution, in section 3 we use algebraic Morse theory to construct a contracting homotopy of the minimal resolution $F_\bullet$, and in section 4 we use this contracting homotopy to construct a map $\mu : F_\bullet \otimes S F_\bullet \to F_\bullet$ which we show gives a commutative and associative multiplication on $F_\bullet$ making it into a DG-algebra.

Not every cyclic module $S/I$ has the property that its minimal resolution is multiplicative, see Avramov [Avr81] for results on homological obstructions to the existence of DG-algebra structures, as well as examples of ideals $I$ such that $S/I$ does not have a multiplicative minimal resolution. For a good survey of much of the early works on the existence and non-existence of multiplicative structures on resolutions, see Miller [Mil92].

Nevertheless, several classes of resolutions of monomial ideals have been found to be multiplicative. Gemeda [Gem76] and Fröberg [Frö79] have independently shown that the Taylor resolution of a monomial ideal is multiplicative; Peeva [Pee96] has shown that for $I$ a stable monomial ideal, the
minimal resolution of $S/I$ is multiplicative, and Sköldberg [Skö11] has shown the corresponding result for matroidal ideals.

2. The resolution and its contracting homotopy

An interval graph is a graph whose vertices correspond to intervals of the real line, and where two vertices are adjacent if the corresponding intervals overlap. A cointerval graph is the complement of an interval graph.

Example 1. Consider the intervals $I_1 = [0, 3]$, $I_2 = [0, 1]$, $I_3 = [2, 3]$ $I_4 = [4, 5]$ as depicted below:

\[ \begin{array}{cccc}
I_1 & & I_4 \\
I_2 & & I_3 \\
\end{array} \]

The corresponding cointerval graph $G$ is thus

We will now describe the minimal resolution $F_\bullet$ of $R_G$ for $G$ a cointerval graph. Dochtermann and Engström have constructed a polyhedral complex that supports the minimal resolution of a cointerval $d$-hypergraph; the resolution we will study is a special case of their construction. It is not hard to see that an interval graph is chordal, so the resolution $F_\bullet$ is also a special case of Chen’s construction of the minimal resolution of $R_G$ for $G$ such that its complement $\bar{G}$ is chordal.

We will in the following assume that the vertex set is $[n]$ and that the vertices are ordered such that if the vertex $i$ corresponds to the interval $[a_i, b_i]$, then $a_i \leq a_j$ whenever $i < j$.

For $i$ a vertex of $G$, its neighbourhood $\text{nbhd}(i)$ is the set of all vertices $j$ such that $ij \in E(G)$. Following Chen, we also define its pre-neighbourhood $\text{pnbhd}(i)$ to be all $j$ in $\text{nbhd}(i)$ with $j < i$. We can then make the following observation.

Lemma 1. Let $i$ and $j$ be vertices in $G$ with $i < j$. Then $\text{pnbhd}(i) \subseteq \text{pnbhd}(j)$.

Proof. If $i < j$ and $k \in \text{pnbhd}(i)$, it means that $[a_k, b_k] \cap [a_i, b_i] = \emptyset$, and thus $b_k < a_i \leq a_j$, so $k \in \text{pnbhd}(j)$.

The sets $B_i$ which will consist of the basis elements of the resolution are now defined as follows: for the degree 0 part we let $B_0 = \{1\}$ and for the higher degrees we let $B_d$ consist of the symbols $(\sigma|\tau)$ where $\sigma, \tau \subseteq [n]$ such that (1) $\sigma$ and $\tau$ are disjoint and nonempty with $|\sigma \cup \tau| = d + 1$, (2) $\max \sigma < \min \tau$, and (3) $\{i, \min \tau\} \in E(G)$ for all $i \in \sigma$. 


Now we can set $$F_i = \bigoplus_{e \in B_i} S \cdot e$$, and describe the differential in the complex $$F_\bullet$$:

$$F_\bullet : 0 \rightarrow F_r \xrightarrow{d_r} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{e} R_G \rightarrow 0 \quad \text{by}$$

$$d(i|j) = x_i x_j$$

$$d(\sigma|\tau) = \sum_{i \in \sigma} (-1)^{\alpha_1(\sigma, \tau, i)} x_i (\sigma \setminus i|\tau)$$

$$+ \sum_{i \in \tau} (-1)^{\alpha_2(\sigma, \tau, i)} x_i (\sigma|\tau \setminus i)$$

where

$$\alpha_1(\sigma, \tau, i) = |\tau| + |\{j \in \sigma \mid j > i\}| \quad \alpha_2(\sigma, \tau, i) = |\{j \in \tau \mid j > i\}|,$$

and where we interpret non-existent basis elements occurring in the formula as zero. By setting $$\deg_{N^n}(\sigma|\tau) = \deg_{N^n} \left( \prod_{i \in (\sigma \cup \tau)} x_i \right)$$ we get a complex of $$N^n$$-graded modules, since it is clear that the differential respects this grading.

**Theorem 1** (Chen, Dochtermann–Engström). *Given a cointerval graph $$G$$, the complex $$F_\bullet$$ defined above is the minimal free $$N^n$$-graded resolution of $$R_G$*.  

**Proof.** It is easy to see that the complex $$F_\bullet$$ is the chain complex of the polyhedral complex that Dochtermann and Engström describe in [DE12], for the special case of an edge ideal of a cointerval (non-hyper)-graph.

Alternatively, the definition of $$F_\bullet$$ can be seen to agree with Chen’s resolution, [Che10] Construction 3.4 by virtue of the conclusion of Lemma [1] and the last remark in Chen’s construction.  

### 3. A Contracting Homotopy

In this section we will use methods of algebraic Morse theory to define a contracting homotopy on the resolution. The notation we will use is the same as in [Skö11], whither the reader is referred for reference.

In order to construct the contracting homotopy on $$F_\bullet$$, we consider $$F_\bullet$$ to be a based complex of $$k$$-vector spaces with basis elements $$x^\alpha(\sigma|\tau)$$, and we will construct a Morse matching $$M$$ on the directed graph $$\Gamma_{F_\bullet}$$. To help us show that the matching we are about to define is a Morse matching, we partially order the elements of $$B_d$$ by letting $$(\sigma_1|\tau_1) \prec (\sigma_2|\tau_2)$$ if (1) $$\max \tau_1 > \max \tau_2$$, or (2) $$\max \tau_1 = \max \tau_2$$, and $$\min \sigma_1 < \min \sigma_2$$.

We define three sets of edges of $$\Gamma_{F_\bullet}$$: $$M_1$$, $$M_2$$ and $$M_3$$, the union of which will be our partial matching.

First, we let $$M_1$$ consist of the edges

$$x^\alpha(\sigma|\tau \cup j) \rightarrow x^\alpha x_j(\sigma|\tau), \quad j \geq \max(\text{supp } \alpha), j > \max(\tau).$$

There are now two types of unmatched vertices; first we have the vertices $$x^\alpha$$, and then the vertices $$x^\alpha(\sigma|\tau)$$ where $$|\tau| = 1$$ and $$\max(\text{supp } \alpha) \leq \max \tau$$.

Next, we let $$M_2$$ be the edges

$$x^\alpha(i \cup \sigma|j) \rightarrow x^\alpha x_i(\sigma|j)$$
satisfying
\[ i \in \text{nbhd}(j), \quad i < \min \sigma, \quad i \leq \min(\text{supp} \alpha \cap \text{nbhd}(j)) \]
in the induced subgraph on the vertices \( M_0^1 \). The vertices in \((M_1 \cup M_2)^0\) are then all \( x^\alpha \) and the \( x^\alpha(i|j) \) for which \( j \geq \max \text{supp} \alpha \), and \( i \leq \min(\text{supp} \alpha \cap \text{nbhd}(j)) \), so we let \( M_3 \) be the set of edges \( x^\alpha(i|j) \rightarrow x^\alpha x_i x_j, \quad (i|j) \prec \) -minimal such that \( x_i x_j | x^\alpha x_i x_j \).

And, finally, we set \( M = M_1 \cup M_2 \cup M_3 \), and we get the unmatched vertices \( M^0 = \{ x^\alpha \mid x^\alpha \notin I_G \} \).

**Lemma 2.** The set \( M \) is a Morse matching on \( \Gamma_{F}^0 \).

**Proof.** It is clear from construction that \( M \) is a partial matching, so we need to show that there are no infinite paths in \( \Gamma_{F}^0 \). We can see that if we have an elementary reduction path from \( x^\alpha(\sigma|\tau) \) to \( x^\beta(\sigma'|\tau') \), then \( (\sigma'|\tau') \prec (\sigma|\tau) \) which shows that the length of the directed path between vertices in the same degree is bounded. \( \square \)

Since \( M \) is a Morse matching with critical vertices \( M^0 \) concentrated in degree 0, we get a contracting homotopy \( \phi \) as in [Skö06, Lemma 2], which can be described in terms of reduction paths, see Jöllenbeck and Welker [JW09] and Sköldberg [Skö11]. We will next define a \( k \)-linear map \( c \) and then show that \( c \) coincides with the contracting homotopy \( \phi \).

We will need to distinguish between three types of basis elements in order to describe \( c \):

- (1) \( x^\alpha \).
- (2) \( x^\alpha(\sigma|\tau) \) where \( |\tau| = 1 \).
- (3) \( x^\alpha(\sigma|\tau) \) where \( |\tau| \geq 2 \).

To the basis element \( x^\alpha(\sigma|\tau) \), we associate sets \( C_1, C_2 \) and \( C_3 \) by

\[
\begin{align*}
C_1 &= \{ i \mid i \in \text{supp} \alpha, i > \max \tau \} \\
C_2 &= \{ i \mid i \in \text{supp} \alpha, i < \min \sigma, \{ i, \min \tau \} \in E(G) \} \\
C_3 &= \{ i \mid i \in \text{supp} \alpha, i < \min \sigma, i < \max \text{supp} \alpha, \{ i, \max \text{supp} \alpha \} \in E(G) \}
\end{align*}
\]

and in case the corresponding set is non-empty, we let \( m_1 = \max C_1, m_2 = \min C_2 \) and \( m_3 = \min C_3 \).

For the basis elements \( x^\alpha \) we now let

\[
c(x^\alpha) = \begin{cases} 
\frac{x^\alpha}{x_i x_j}(i|j), & \text{if } x^\alpha \in I_G, (i|j) \prec \text{-minimal such that } x_i x_j | x^\alpha, \\
0, & \text{otherwise.}
\end{cases}
\]

Turning to the basis elements \( x^\alpha(\sigma|\tau) \) where \( |\tau| = 1, \tau = \{ i \} \) next, we set
the signs, we can see that there are no elementary reduction paths starting in \(v\), which shows that \(v\) is matched with \(\sigma\).)

Let \(\phi\) be the homotopy we get from the Morse matching \(M\); we shall see that \(c = \phi\).

First we look at the basis element \(v = x^\alpha\). We have two cases, if \(x^\alpha \in I_G\), then \(x^\alpha\) is matched with \(v' = x^\alpha/x_i x_j \cdot (ij)\) where \((ij)\) is minimal with respect to \(\prec\). There are no elementary reduction paths originating in \(v'\), so we can conclude that in this case \(c(v) = v' = \phi(v)\). In the case \(x^\alpha \notin I_G\), we have that \(x^\alpha \in M^0\), so \(c(v) = 0 = \phi(v)\).

Next, we turn to elements \(v = x^\alpha(\sigma_0\hat{j})\). If \(C_1 \neq \emptyset\), \(v \in M^-\) and is matched with \(v' = x^\alpha x_i \cdot (\sigma_0 j)\). There is an elementary reduction path from \(v'\) to \(v'' = x^\alpha x_j (m_3 \cup \sigma m_1)\) precisely when \(C_3 \neq \emptyset\). It is easy to see that there are no elementary reduction paths starting in \(v''\), so after verifying the signs, we can see that \(c(v) = \phi(v)\) when \(C_1 \neq \emptyset\). If \(C_1 = \emptyset\), we have that \(x^\alpha(\sigma_0 j) \in M^-\) precisely when \(C_2 \neq \emptyset\), in which case \(v\) is matched with \(v' = x^\alpha x_j (m_2 \cup \sigma j)\) and there are no elementary reduction paths from \(v'\), so \(c(v) = \phi(v)\) in this case as well.

Lastly, we look at the elements \(v = x^\alpha(\sigma \tau)\) where \(|\tau| \geq 2\). Here we can see that \(v \in M^-\) precisely when \(C_1 \neq \emptyset\), in which case \(v\) is matched with \(v' = x^\alpha x m_1 \cdot (\sigma \tau \cup m_1)\) There are no elementary reduction paths from \(v'\) which shows that \(c(v) = \phi(v)\) for these elements too.

It is clear from the definition that \(c\) respects the multidegree, and since \(c(v) = 0\) for all elements in \(M^+\), we can see that \(c^2 = 0\) and \(c(e) = 0\) for all \(S\)-basis elements in \(B_m\). \(\blacksquare\)

4. The multiplicative structure

Now we are in a position that allows us to define the multiplication making \(F_\bullet\) into a DGA. Just like in [Sk01], we are going to use the following result in the construction.

**Lemma 3.** The map \(c\) is an \(\mathbb{N}^n\)-graded contracting homotopy of \(F_\bullet\) such that \(c^2 = 0\) and \(c(e) = 0\) for all \(e \in \bigcup_i B_i\).

**Proof.** Let \(\phi\) be the homotopy we get from the Morse matching \(M\); we shall see that \(c = \phi\).

First we look at the basis element \(v = x^\alpha\). We have two cases, if \(x^\alpha \in I_G\), then \(x^\alpha\) is matched with \(v' = x^\alpha/x_i x_j \cdot (ij)\) where \((ij)\) is minimal with respect to \(\prec\). There are no elementary reduction paths originating in \(v'\), so we can conclude that in this case \(c(v) = v' = \phi(v)\). In the case \(x^\alpha \notin I_G\), we have that \(x^\alpha \in M^0\), so \(c(v) = 0 = \phi(v)\).

Next, we turn to elements \(v = x^\alpha(\sigma_0\hat{j})\). If \(C_1 \neq \emptyset\), \(v \in M^-\) and is matched with \(v' = x^\alpha x_i \cdot (\sigma_0 j)\). There is an elementary reduction path from \(v'\) to \(v'' = x^\alpha x_j (m_3 \cup \sigma m_1)\) precisely when \(C_3 \neq \emptyset\). It is easy to see that there are no elementary reduction paths starting in \(v''\), so after verifying the signs, we can see that \(c(v) = \phi(v)\) when \(C_1 \neq \emptyset\). If \(C_1 = \emptyset\), we have that \(x^\alpha(\sigma_0 j) \in M^-\) precisely when \(C_2 \neq \emptyset\), in which case \(v\) is matched with \(v' = x^\alpha x_j (m_2 \cup \sigma j)\) and there are no elementary reduction paths from \(v'\), so \(c(v) = \phi(v)\) in this case as well.

Lastly, we look at the elements \(v = x^\alpha(\sigma \tau)\) where \(|\tau| \geq 2\). Here we can see that \(v \in M^-\) precisely when \(C_1 \neq \emptyset\), in which case \(v\) is matched with \(v' = x^\alpha x m_1 \cdot (\sigma \tau \cup m_1)\) There are no elementary reduction paths from \(v'\) which shows that \(c(v) = \phi(v)\) for these elements too.

It is clear from the definition that \(c\) respects the multidegree, and since \(c(v) = 0\) for all elements in \(M^+\), we can see that \(c^2 = 0\) and \(c(e) = 0\) for all \(S\)-basis elements in \(B_m\). \(\blacksquare\)
Lemma 5. Suppose that $X_\bullet$ and $Y_\bullet$ are complexes of $S$-modules, where $X_n = S \otimes_k V_n$ and $Y_n = S \otimes_k W_n$ for $k$-spaces $V_n$ and $W_n$, $n \geq 0$. Furthermore, suppose that $Y_\bullet$ is acyclic, with a contracting homotopy $c$ satisfying $c^2 = 0$. Then, every $S$-linear map $\varphi_0 : X_0 \to Y_0$ has a unique lifting to a chain map $\varphi : X_\bullet \to Y_\bullet$ satisfying $\varphi(V_n) \subseteq \text{Im} c$. This map is defined inductively by

$$\varphi_{n+1}(\bar{x}) = c\varphi_n d(\bar{x}), \quad \bar{x} \in V_{n+1}.$$ 

Proof. This is a special case of [ML63, Theorem IX.6.2].

We now let $\mu$ be the map $\mu : F_\bullet \otimes_S F_\bullet \to F_\bullet$ that is the lifting of the canonical isomorphism $\mu_0 : F_0 \otimes_S F_0 = S \otimes_S S \to S = F_0$ using the contracting homotopy $c$ from the previous section. This will be our proposed product on $F_\bullet$, so we will henceforth write $x \cdot y$ for $\mu(x \otimes y)$.

Lemma 5. For all basis elements $x, y$ of $F_\bullet$ we have

\begin{enumerate}
  \item $d(x \cdot y) = d(x) \cdot y + (-1)^{|x|} x \cdot d(y)$.
  \item $x \cdot y = (-1)^{|x||y|} y \cdot x$.
  \item $1 \cdot x = x \cdot 1 = x$.
\end{enumerate}

Proof. Claim (1) just expresses that $\mu$ is a chain map. For (2), let $\tau : F_\bullet \otimes_S F_\bullet \to F_\bullet \otimes_S F_\bullet$ be defined on basis elements $x, y$ by $\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x$. Now $\mu$ and $\mu \circ \tau$ are chain maps lifting the same map in degree 0 and both mapping basis elements to $\text{Im} c$, so by Lemma 4 they must be equal. Claim (3) is proven by induction on the degree of $x$. \qed

Let us now define a map $\partial : F_n \to F_{n-1}$, $n \geq 1$, by

$$\partial(ij) = x_i x_j$$

$$\partial(\sigma|\tau) = x_{\max(|\tau|)}(\sigma|\tau \wedge \max(\tau)) - (-1)^{|\tau|+|\sigma|} x_{\min(|\tau|)}(\sigma \wedge \min(|\sigma|)|\tau),$$

again treating any non-existent basis elements occurring as zero.

Its usefulness comes from that we can replace the real differential $d$ by $\partial$ when reasoning about the multiplication, as the following lemma shows.

Lemma 6. For basis elements $(\sigma_1|\tau_1)$, $(\sigma_2|\tau_2)$ we have

$$c(d((\sigma_1|\tau_1)) \cdot (\sigma_2|\tau_2)) = c(\partial((\sigma_1|\tau_1)) \cdot (\sigma_2|\tau_2))$$

Proof. Consider the difference

$$c(d((\sigma_1|\tau_1)) \cdot (\sigma_2|\tau_2)) - c(\partial((\sigma_1|\tau_1)) \cdot (\sigma_2|\tau_2)) = c((d((\sigma_1|\tau_1)) - \partial((\sigma_1|\tau_1))) \cdot (\sigma_2|\tau_2))$$

A term occurring in $d((\sigma_1|\tau_1)) - \partial((\sigma_1|\tau_1))$ is either of the form $x_i(\sigma_1 \wedge i|\tau_1)$ where $i > \min(|\sigma_1|)$ or $x_i(\sigma_1|\tau_1 \wedge i)$ where $i < \max(|\tau_1|)$.

Now assume that $x_k(\sigma_3|\tau_3)$ occurs in a product $(\sigma_1 \wedge i|\tau_1) \cdot (\sigma_2|\tau_2)$ or $(\sigma_1|\tau_1 \wedge i) \cdot (\sigma_2|\tau_2)$, and assume further that $c(x_k(\sigma_3|\tau_3)) \neq 0$. This means that either (i), $i \geq k$ and $i > \max(|\tau_3|)$, which implies that $i = \max(|\tau_3|)$, or (ii), $|\tau_3| = 1$, and $i < \min(|\sigma_3|)$. Now, in case (ii), if $k < i$, we would have that $\tau_3 = \{t\}$ where $t = \max(\tau_1 \cup \tau_2)$, so by Lemma 4 it would be the case that $k \in \text{pubhd}(t)$, but then $x_k(\sigma_3|\tau_3) \in M^+$ which contradicts that $x_k(\sigma_3|\tau_3) \in M^-$. Thus $i \leq k$, so $i = \min(|\sigma_1|)$. \qed
We will now give an explicit description of the multiplication in the simplest non-trivial case.

**Lemma 7.** Let \((s_1|t_1)\) and \((s_2|t_2)\) be basis elements of degree 1 in \(F_{\bullet}\). Then

\[
(s_1|t_1) \ast (s_2|t_2) = \begin{cases}
  x(s_2|t_2 t_1) + x_{t_1}(s_1 s_2|t_1) & t_1 > t_2, s_1 < s_2, \\
  x(s_2|t_2 t_1) & t_1 > t_2, s_1 = s_2, \\
  x(s_2|t_2 t_1) - x_{t_1}(s_2 s_1|t_1) & t_1 > t_2, s_1 > s_2, \\
  x_{t_1}(s_1 s_2|t_2) & t_1 = t_2, s_1 < s_2, \\
  0 & t_1 = t_2, s_1 = s_2, \\
  -x_{t_1}(s_2 s_1|t_1) & t_1 = t_2, s_1 > s_2, \\
  x_{t_1}(s_1 s_2|t_2) - x_{t_2}(s_1|t_1 t_2) & t_1 < t_2, s_1 < s_2, \\
  -x_{s_1}(s_1|t_1 t_2) & t_1 < t_2, s_1 = s_2, \\
  -x_{s_2}(s_1|t_1 t_2) - x_{t_1}(s_2 s_1|t_2) & t_1 < t_2, s_1 > s_2.
\end{cases}
\]

**Proof.** By the definition of the product map

\[
(s_1|t_1) \ast (s_2|t_2) = c(x_{s_1}x_{t_1}(s_2|t_2)) - c(x_{s_2}x_{t_2}(s_1|t_1))
\]

from which the statement follows by using the definition of \(c\). \(\square\)

**Lemma 8.** Let \((\sigma_1|\tau_1)\) and \((\sigma_2|\tau_2)\) be basis elements of \(F_{\bullet}\). If \(x_k(\sigma_3|\tau_3)\) occurs in \((\sigma_1|\tau_1) \ast (\sigma_2|\tau_2)\), then \(\sigma_3 \subseteq \sigma_1 \cup \sigma_2\) and \(\tau_3 \subseteq \tau_1 \cup \tau_2\).

**Proof.** We use induction over \(d = \text{deg}(\sigma_1|\tau_1) + \text{deg}(\sigma_2|\tau_2)\). If \(d = 2\), the claim follows from Lemma 7. If \(d \geq 3\), we look at \(c(\partial((\sigma_1|\tau_1)) \ast (\sigma_2|\tau_2))\). In the case of \(\text{deg}(\sigma_1|\tau_1) = 1\), so \((\sigma_1|\tau_1) = (i|j)\), it is equal to \(c(x_{i,j}(\sigma_2|\tau_2))\). The only terms that could occur in \(c(x_{i,j}(\sigma_2|\tau_2))\) are \(x_{i}(\sigma_1 \cup \sigma_2|\tau_2), x_{i}(\sigma_2|\tau_2 \cup j)\) and \(x_{i}(i \cup \sigma_2|\tau_2 \cup l \cup j)\), all of which satisfy the statement of the lemma.

Next we turn to the case of \(\text{deg}(\sigma_1|\tau_1) \geq 2\), and, letting \(s = \min \sigma_1, t = \max \tau_1\), we consider

\[
c(\partial((\sigma_1|\tau_1)) \ast (\sigma_2|\tau_2)) = c(x_{s}(\sigma_1|\tau_1 \setminus t) \ast (\sigma_2|\tau_2)) \pm c(x_{s}(\sigma_1 \setminus s|\tau_1) \ast (\sigma_2|\tau_2)).
\]

First, suppose that \(x_{i}(\sigma_4|\tau_4)\) occurs in \((\sigma_1|\tau_1 \setminus t) \ast (\sigma_2|\tau_2)\), then the only terms that can occur in \(c(x_{i}x_{j}(\sigma_4|\tau_4))\) are \(v_1 = x_{i}(\sigma_4|\tau_1 \cup t)\) and \(v_2 = x_{m}(l \cup \sigma_4|l)\). Note that if \(v_2\) occurs, we must have \(l = \min(\sigma_1 \cup \sigma_2 \cup \tau_1 \cup \tau_2)\), so \(l \in \sigma_1 \cup \sigma_2\). Next suppose that \(x_{i}(\sigma_4|\tau_4)\) occurs in \((\sigma_1 \setminus s|\tau_1) \ast (\sigma_2|\tau_2)\), the only term that can occur in \(c(x_{i}x_{j}(\sigma_4|\tau_4))\) is then \(v_3 = x_{i}(s \cup \sigma_4|\tau_4)\). By induction, we have in both cases that \(\sigma_4 \subseteq \sigma_1 \cup \sigma_2\) and \(\tau_4 \subseteq \tau_1 \cup \tau_2\), so all of \(v_1, v_2\) and \(v_3\) satisfy the conclusions of the lemma, and since the multiplication is graded commutative, the above argument also shows that all terms occurring in \(c((\sigma_1|\tau_1) \ast \partial((\sigma_2|\tau_2)))\) also satisfy the conclusion of the lemma, and thus, by invoking Lemma 7 we have shown that all terms occurring in \((\sigma_1|\tau_1) \ast (\sigma_2|\tau_2)\) satisfy the conclusion of the lemma, and we are done. \(\square\)

**Lemma 9.** For basis elements \((\sigma_1|\tau_1), (\sigma_2|\tau_2)\) in \(F_{\bullet}\), we have that if \(x_k(\sigma_3|\tau_3)\) occurs in the product \((\sigma_1|\tau_1) \ast (\sigma_2|\tau_2)\), then \(\max \tau_3 = \max(\tau_1 \cup \tau_2)\) and \(|\tau_3| \geq |\tau_1| + |\tau_2| - 1\).

**Proof.** For the first claim we observe that if \(\max \tau_3 \neq \max(\tau_1 \cup \tau_2)\), then \(k = \max(\tau_1 \cup \tau_2)\), which would imply that \(x_k(\sigma_3|\tau_3) \in M^\circ\), but since \(x_k(\sigma_3|\tau_3) \in \text{Im} \, c\), we know that \(x_k(\sigma_3|\tau_3) \in M^+\).
Lemma 10. Let $(\sigma_1|\tau_1), (\sigma_2|\tau_2)$ and $(\sigma_3|\tau_3)$ be basis elements of $F_\bullet$, then $(\sigma_1|\tau_1) \star ((\sigma_2|\tau_2) \star (\sigma_3|\tau_3)) \in \text{Im } c$.

Proof. Suppose $x_j(\sigma_4|\tau_4)$ occurs in $(\sigma_2|\tau_2) \star (\sigma_3|\tau_3)$ and furthermore that $x_k(\sigma_5|\tau_5)$ occurs in $(\sigma_1|\tau_1) \star (\sigma_4|\tau_4)$.

Suppose that $x_jx_k(\sigma_5|\tau_5) \notin \text{Im } c$. Then we must have that $c(x_jx_k(\sigma_5|\tau_5)) \neq 0$, which can only happen if $c(x_j(\sigma_5|\tau_5)) \neq 0$. Since $\max \tau_5 = \max(\tau_1 \cup \tau_4) = \max(\tau_1 \cup \tau_2 \cup \tau_3)$, we have that $j < \min \sigma_5$ and that $|\tau_5| = 1$, so by lemma $9$ this means that $|\tau_i| = 1$ for $1 \leq i \leq 4$ and we can define $m_i$, $1 \leq i \leq 5$ by \{m_i\} = \tau_i.

It cannot be the case that $k < \min \sigma_5$, since that would imply that one of $km_1$, $km_2$, or $km_3$ is in $E(G)$, and thus, by Lemma $1$ and Lemma $8$ that $km_5 \in E(G)$ which would mean that $x_k(\sigma_5|\tau_5) \in M^-$. This means that $\min \sigma_5 \leq \min \sigma_4$.

Therefore we can conclude that $j < \min \sigma_5 \leq \min \sigma_4$, so we have that one of $jm_2$ and $jm_3$ is in $E(G)$, so $jm_4 \in E(G)$, and $x_j(\sigma_4|\tau_4) \in M^-$ which contradicts that $x_j(\sigma_4|\tau_4) \in \text{Im } c$ and thus is in $M^+$. \hfill \Box

Theorem 2. For a cointerval graph $G$, the minimal resolution $F_\bullet$ of $I_G$ is a DGA over $S$.

Proof. Lemma $5$ gives that the proposed multiplication has a unit, satisfies the Leibniz rule and is graded commutative. It thus remains to see associativity. To this end we look at the two chain maps

$$\mu \circ (\mu \otimes 1), \mu \circ (1 \otimes \mu) : F_\bullet \otimes_S F_\bullet \otimes_S F_\bullet \longrightarrow F_\bullet.$$ 

Since they agree in degree $0$, Lemma $4$ tells us that it is enough to show that the images of basis elements under both maps lie in $\text{Im } c$.

Let $e_1, e_2$ and $e_3$ be basis elements of $F_\bullet$. If any of them is of degree zero, and thus equal to $1$, it is by Lemma $5$ obvious that $e_1 \star (e_2 \star e_3)$ and $(e_1 \star e_2) \star e_3$, lie in $\text{Im } c$, so let us assume that this is not the case. Then, by Lemma $10$ we know that $e_1 \star (e_2 \star e_3) \in \text{Im } c$

and that

$$(e_1 \star e_2) \star e_3 = (-1)^{|e_3|(|e_1|+|e_2|)}e_3 \star (e_1 \star e_2) \in \text{Im } c.$$ \hfill \Box

We conclude by calculating the full DGA-structure on the resolution of the graph from Example $1$.

Example 2. Continuing with our example, we have the following $S$-basis elements in the resolution:


| Degree | Basis elements |
|--------|----------------|
| 1      | (1|4), (2|3), (2|4), (3|4) |
| 2      | (12|4), (13|4), (23|4), (2|34) |
| 3      | (123|4) |

We can now get the products of elements of degree 1 from Lemma 7. Since the product is graded commutative we have zeros on the diagonal, and elements below the diagonal are the negative of their transposes, so we do not include them in the table.

\[
\begin{array}{cccc}
* & (1|4) & (2|3) & (2|4) & (3|4) \\
(1|4) & x_1(2|34) + x_3(12|4) & x_4(12|4) & x_4(13|4) & x_4(23|4) \\
(2|3) & -x_2(2|34) & x_3(23|4) - x_3(2|34) & x_4(23|4) & x_4(34|4) \\
(2|4) & & & & x_4(34|4) \\
(3|4) & & & & -x_4(123|4) \\
\end{array}
\]

Next we can compute the products of an element of degree 1 with an element of degree 2. From the \(N^n\)-homogeneity of \(\ast\) it follows that \((\sigma_1|\tau_1)\ast(\sigma_2|\tau_2) = 0\) if \(|(\sigma_1 \cup \tau_1) \cap (\sigma_2 \cup \tau_2)| \geq 2\), so we leave those entries blank in the table, and only include entries which need to be calculated.

\[
\begin{array}{cccc}
* & (12|4) & (13|4) & (23|4) & (2|34) \\
(1|4) & & -x_4(123|4) & 0 & x_4(34|4) \\
(2|3) & 0 & x_3(123|4) & & x_4(123|4) \\
(2|4) & & x_4(123|4) & & x_4(123|4) \\
(3|4) & & & & -x_4(123|4) \\
\end{array}
\]

References

[Avr81] Luchezar L. Avramov, Obstructions to the existence of multiplicative structures on minimal free resolutions, Amer. J. Math. 103 (1981), no. 1, 1–31. MR 601460 (82m:13011)

[Che10] Ri-Xiang Chen, Minimal free resolutions of linear edge ideals, J. Algebra 324 (2010), no. 12, 3591–3613. MR 2735401 (2011m:13023)

[DE12] Anton Dochtermann and Alexander Engström, Cellular resolutions of cointerval ideals, Math. Z. 270 (2012), no. 1-2, 145–163. MR 2875826 (2012m:13039)

[Frö79] Ralf Fröberg, Some complex constructions with applications to Poincaré series, Séminaire d’Algèbre Paul Dubreil 31ème année (Paris, 1977–1978), Lecture Notes in Math., vol. 740, Springer, Berlin, 1979, pp. 272–284. MR 563509 (81f:13008)

[Gem76] Demissu Gemeda, Multiplicative structure of finite free resolutions of ideals generated by monomials in an \(R\)-sequence, Ph.D. thesis, Brandeis University, 1976, p. 69. MR 2626146

[JW09] Michael Jöllenbeck and Volkmar Welker, Minimal resolutions via algebraic discrete Morse theory, Mem. Amer. Math. Soc. 197 (2009), no. 923, vi+74. MR 2488864 (2009m:13017)

[Mill92] Matthew Miller, Multiplicative structures on finite free resolutions, Free resolutions in commutative algebra and algebraic geometry (Sundance, UT, 1990),
References

[ML63] Saunders Mac Lane, Homology, Academic Press Inc., Publishers, New York, 1963. MR 28 #122

[Pee96] Irena Peeva, 0-Borel fixed ideals, J. Algebra 184 (1996), no. 3, 945–984. MR 97g:13018

[Skö06] Emil Sköldberg, Morse theory from an algebraic viewpoint, Trans. Amer. Math. Soc. 358 (2006), no. 1, 115–129 (electronic). MR 2171225 (2006e:16013)

[Skö11] ______, Resolutions of modules with initially linear syzygies, ArXiv e-prints (2011).

School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland

E-mail address: emil.skoldberg@nuigalway.ie
URL: http://www.maths.nuigalway.ie/~emil/