Abstract Projective Lines

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Introduction

For $V$ a vector space over a field $k$, one has the Grassmannian manifold $P(V)$ consisting of 1-dimensional linear subspaces of $V$. If $V$ is $n + 1$-dimensional, $P(V)$ is a copy on $n$-dimensional projective space. For $n \geq 2$, $P(V)$ has a rich combinatorial structure, in terms of incidence relations (essentially: the lattice of linear subspaces), in fact, this structure is so rich that one can essentially reconstruct $V$ from the combinatorial structure.

But for $n = 1$, this combinatorial structure (in the form of a lattice), is trivial; as expressed by R. Baer, “A line . . . has no geometrical structure, if considered as an isolated or absolute phenomenon, since then it is nothing but a set of points with the number of points on the line as the only invariant . . . ”, [1] p. 71.

However, it is our contention that a projective line has another kind of structure, making it possible to talk about a projective line as a set equipped with a certain structure, in such a way that isomorphisms (projectivities) between projective lines are bijective maps which preserve this structure.

The structure we describe (Section 2) is that of a groupoid (i.e. a category where all arrows are invertible), and with certain properties. The fact that the coordinate projective line $P(k^2)$, more generally, a projective space of the form $P(V)$ (and also the projective plane in the classical synthetic sense) has such groupoid structure, was observed in [3], and further elaborated on in [2]; we shall recall the relevant notions and constructions from [3] in Section 1 and a crucial observation from [2] in Section 3. The present note may be seen as a completion of some of the efforts of these two papers.
1 Groupoid structure on $P(V)$

Let $k$ be a field and let $V$ a 2-dimensional vector space over $k$. We have a groupoid $L(V)$, whose set of objects is the set of $P(V)$ of 1-dimensional linear subspaces of $V$, and whose arrows are the linear isomorphisms between these. For $A \in P(V)$, the linear isomorphisms $A \to A$ are in canonical bijective correspondence with the invertible scalars,

$$L(V)(A, A) = k^*;$$

on the other hand, if $A$ and $B$ are distinct 1-dimensional subspaces, then the linear isomorphisms $A \to B$ are all of the form “projection from $A$ to $B$ in a certain unique direction $C$”, with $C \in P(V)$ and $C$ distinct from $A$ and $B$. (This also works in higher dimensions, cf. [3] and [2]; one just has to require that $C$ belongs to the 2-dimensional subspace spanned by $A$ and $B$.) This is in fact a bijective correspondence, so $L(V)(A, B)$ is canonically identified with the set $P(V) \setminus \{A, B\}$. Here is a picture from [3]:

![Diagram](image)

The linear isomorphism $A \to B$ thus described, we shall denote $(C : A \to B)$. It is clear that the composite of $(C : A \to B)$ with $(C : B \to A)$ gives the identity map of $A$ (projecting forth and back in the same direction). Also it is clear that $(C : A \to B)$ composes with $(C : B \to D)$ to give $(C : A \to D)$. These equations will appear in the axiomatics for abstract projective lines as the “idempotency laws”, [2] and [3] below.

Also, it is clear that two linear isomorphisms from $A$ to $B$ differ by a scalar $\mu \in k^*$; thus, for $A$ and $B$ distinct, and $(C : A \to B)$ and $(D : A \to B)$, there is a unique scalar $\mu \in k^*$ such that

$$\mu.(C : A \to B) = (D : A \to B) = (C : A \to B).\mu. \quad (1)$$
This scalar \( \mu \) is (for \( A, B, C, D \) mutually distinct) the classical cross-ratio \((A, B; C, D)\), cf. \[3\] (3) and \[2\] Theorem 1.5.3. (For \( A, B, C \) distinct, and \( D = C \), we have \((A, B; C, C) = 1\).) Permuting the four entries (assumed distinct) will change the cross ratio according to well known formulae (see e.g. \[3],\[5\]) which we shall make explicit and take as axioms.

Thus, the groupoid \( L(V) \) which we in this way have associated to a 2-dimensional vector space \( V \) over \( k \) will be an example of an abstract projective line \( L \), in the sense of the next Section.

## 2 Abstract projective lines: axiomatics

Let \( k \) be a field. By a \( k\)-groupoid, we understand a groupoid \( L \) which is transitive (i.e. the hom set \( L(A, B) \) is non-empty, for any pair of objects \( A, B \) in \( L \)), and such that all vertex groups \( L(A, A) \) are identified with the (commutative, multiplicative) group \( k^* \) of non-zero elements of the field \( k \). We assume that \( k^* \) is central in \( L \) in the sense that for all \( f : A \rightarrow B \) and \( \lambda \in k^* = L(A, A) = L(B, B) \), \( \lambda.f = f \cdot \lambda \). (We compose from the left to the right.)

A \( k \)-functor between \( k \)-groupoids is a functor which preserves \( k^* \) in the evident sense.

We now define the notion of abstract projective line over \( k \); it is to be a \( k \)-groupoid \( L \), equipped with the following kind of structure (\( L \) denotes the set of objects of \( L \)):

for any two different objects \( A, B \in L \), there is given a bijection between the set \( L(A, B) \) and the set \( L\backslash\{A, B\} \) satisfying some equational axioms: the idempotence laws \([2]\) and \([3]\), and the permutation laws \([4], \ldots, [7]\). To state these laws, we use, as in Section 1, the notation:

if \( C \in L\backslash\{A, B\} \), then the arrow \( A \rightarrow B \) corresponding to it (under the assumed bijection) by \( (C : A \rightarrow B) \), or just by \( C \), if \( A \) and \( B \) are clear from the context (say, from a diagram).

Here are the first set of equations that we assume (the “idempotence equations”) (we compose from left to right): Let \( A, B, F \) be mutually distinct, then

\[
(F : A \rightarrow B).(F : B \rightarrow A) = 1 \in k^*
\]  

(2)
and for $A, B, C, F$ mutually distinct

$$(F : A \to B)(F : B \to C) = (F : A \to C). \tag{3}$$

The permutation laws which we state next are concerned with the crucial notion of cross ratio: If $A, B, C, D$ are four distinct elements of $L$, we let $(A, B; C, D)$ be the unique scalar (element of $k^*$) such that

\[
\begin{array}{c}
A \quad C \\
\downarrow \quad \downarrow \\
1 \quad (A, B; C, D) \\
\downarrow \quad \downarrow \\
A \quad D \quad B
\end{array}
\]

commutes; also, $(A, B; C, D)$ makes sense if $C = D$, and in this case equals $1 \in k^*$. This scalar is called the cross ratio of the 4-tuple $A, B, C, D$.

Since the elements of $L$ both appear as objects of $L$ and as labels of arrows of $L$, the four entries (assumed distinct) in a cross ratio expression can be permuted freely by the 24 possible permutations of four letters. We assume the standard equation formulas for these permutation instances of a given cross ratio $\mu = (A, B; C, D)$; they give six values,

$$\mu, \frac{1}{\mu}, 1 - \mu, \frac{1}{1 - \mu}, \frac{\mu}{\mu - 1}, \frac{\mu - 1}{\mu},$$

see e.g. [6] p. 8 or [5] 0.2. The equations are

$$(A, B; C, D) = (B, A; D, C) = (C, D; A, B) = (D, C; B, A), \tag{4}$$

and the following equations, where $\mu$ denotes $(A, B; C, D)$,

$$(A, B; D, C) = \mu^{-1}; \tag{5}$$

$$(A, C; B, D) = 1 - \mu; \quad (A, C; D, B) = \frac{1}{1 - \mu}. \tag{6}$$

\[\text{1\textsuperscript{ Convenience, as well as continuity, prompts us to define } (A, B; C, C) = 1 \text{ and } (A, B; C, B) = 0: this is consistent with determinant formulas for cross ratios in } P(k^2) \text{ to be given later. In fact, one may consistently define } (A, B; C, D) \text{ whenever } A \neq D \text{ and } B \neq C; (A, A; C, D) = (A, B; C, C) = 1, \text{ and } (A, B; A, D) = (A, B; C, B) = 0. \]
\[(A, D; B, C) = \frac{\mu - 1}{\mu}; \quad (A, D; C, B) = \frac{\mu}{\mu - 1}.\]  \tag{7}

(This set of equations is not independent.) We had not needed to be so specific about these “permutation equations”, since we shall only need the following consequence: if a map \(\Phi : L \rightarrow L'\) preserves cross ratios of the form \((A, B; C, D)\) for some distinct \(A, B, C, D\), then it also preserves any other cross ratio in which the entries are \(A, B, C, D\) in some other order.

We have now stated what we mean by an abstract projective line \(L\). For (iso-)morphisms (“projectivities”) between such: Let \(L\) and \(L'\) be abstract projective lines with object sets (underlying sets) \(L\) and \(L'\), respectively. By an isomorphism \(L \rightarrow L'\) of projective lines, we understand a bijective map \(\phi : L \rightarrow L'\) with the property that if we put
\[
\bar{\phi}(F : A \rightarrow B) := (\phi(F) : \phi(A) \rightarrow \phi(B)),
\]
(and \(\bar{\phi}(\lambda) = \lambda\) for any scalar \(\lambda \in k^*\)), then \(\bar{\phi}\) commutes with composition, i.e. it defines a functor \(L \rightarrow L'\) (preserving scalars, i.e. a \(k\)-functor). The noticeable aspect of the category \(L\) of abstract projective lines, with (iso)morphisms as just defined, is that the “underlying” functor \(L \rightarrow L\) (from \(L\) to the category of sets) is a faithful functor, so that it makes sense to say whether a given function \(L \rightarrow L'\) is a morphism (projectivity) or not.

As always in such situations, it is convenient to use the same notation for the object itself, and its underlying set; so we henceforth do not have to distinguish notationally between \(L\) and \(L\).

Cross ratio was defined as a special case of composition; projectivities, in the sense defined here, commute with composition, since they are functors. Hence it is clear that projectivities preserve cross ratios.

In an (abstract) projective line \(L\), one may draw some diagrams that are meaningless in more general categories, like the following square (whose commutativity actually can be proved on basis of the axiomatics):

\[
\begin{array}{ccc}
A & C & B \\
B & & A \\
C & & C \\
\end{array}
\]

\[\begin{array}{c}
-1
\end{array}\]  \tag{8}
(where \( A, B, C \) are three distinct points in \( L \)). The commutativity of this diagram, for \( L = P(V) \), expresses an evident geometric fact that one sees by contemplating the figure (from [3], p. 3):

\[ \text{The existence of this diagram shows that “cross ratios do not immediately encode all the geometry” of projective lines; for, no cross ratio (except 1) can be concocted out of just three distinct points; four are needed.} \]

### 3 Three-transitivity

The “Fundamental Theorem” for projective lines derived from 2-dimensional vector spaces is: for any two lists of three distinct points, there is a unique projectivity taking the points of the first list to the points of the second. This theorem, we shall prove holds for abstract projective lines.

Let \( L \) and \( L' \) be abstract projective lines over the field \( k \).

**Theorem 1 (Fundamental Thorem)** Given three distinct points \( A, B, C \) in \( L \), and given similarly \( A', B', C' \) three distinct distinct points in \( L' \). Then there is a unique projectivity \( \phi : L \to L' \) taking \( A \) to \( A' \), \( B \) to \( B' \) and \( C \) to \( C' \).

**Proof.** For \( D \) distinct from \( A, B, C \), we put \( \phi(D) := D' \), where \( D' \) is the unique element in \( L' \) with \( (A', B'; C'; D') = (A, B; C, D) \); equivalently \( D' \) is determined by the equation \( (C' : A' \to B')(A, B; C, D) = (D' : A' \to B') \). By construction and the permutation equations, \( \phi \) preserves cross ratios.
of any distinct 4-tuple, three of whose entries are $A, B, C$. Next, by the idempotence equations (2) and (3),
\[(A, B; D; E) = (A, B; D, C)(A, B; C, E),\]
and similarly for the $A', \ldots, E'$. Each of the cross ratios on the right have three entries from the original set $A, B, C$, and so are preserved, hence so is the cross ratio on the left hand side, $(A, B; D, E)$. So we conclude that any cross ratio, two of whose entries are $A$ and $B$, is preserved. Next,
\[(A, D; E, F) = (A, D; E, B)(A, D; B, F),\]
and similarly for the $A', \ldots, F'$, so we conclude that any cross ratio with $A$ as one of its entries is preserved. Finally,
\[(D, E; F, G) = (D, E; F, A)(D, E; A, G),\]
and similarly for the $A', \ldots, G'$, so we conclude that all cross ratios are preserved.

We have now described the bijection $\phi : L \rightarrow L'$, and proved that it preserves cross ratio of any four distinct points. To prove that it is a projectivity, in the sense defined, we need to argue that the corresponding $\bar{\phi}$ preserves composition of arrows. This is essentially an argument from [2] 2-4-4, which we make explicit:

**Proposition 1** If a bijection $\phi : L \rightarrow L'$ preserves cross ratio formation, then $\bar{\phi}$ preserves composition.

It suffices to prove that commutative triangles go to commutative triangles. If the three vertices of the triangle agree, these arrows are scalars $\in k^*$, and $\bar{\phi}$ preserves scalars. If two, but not all three, vertices agree, one arrow is a scalar, and commutativity of the triangle expresses that this scalar is the cross ratio (or its inverse) of the four points that appear as the two vertices and those two labels (likewise points in $L$) that appear on the non-scalar arrows in the triangle; this is a an immediate consequence of the definition (11), possibly combined with the idempotence law (2). We conclude that composites of this form are likewise preserved by $\bar{\phi}$. Finally, we consider the case where the three vertices of the triangle are distinct, so the three arrows in the triangle are of the form $(E : A \rightarrow B), (F : B \rightarrow C), \text{ and } (G : A \rightarrow C)$
with $A, B, C$ distinct. Consider $(E : A \to B).(F : B \to C).(G : C \to A)$, displayed as the top composite in the diagram

All squares commute; the lower right hand rectangle commutes because of the idempotence law $3$ (the two $F$’s combine into one). The lower composite is 1, because of an idempotence law (refidem1). So we conclude by $(2)$:

$$(C, A; G, F).(B, A; F, E) = 1 \iff (E : A \to B).(F : B \to C).(G : C \to A) = 1.$$ 

Multiplying on the right by $G : A \to C$ (which is inverse to $G : C \to A$), we conclude

$$(C, A; G, F).(B, A; F, E) = 1 \iff (E : A \to B).(F : B \to C) = (G : A \to C).$$

Thus commutativity of diagrams can be expressed in terms of cross ratio. Hence since cross ratio are preserved, the composite of $(E : A \to B)$ and $(F : B \to C)$ is preserved by $\phi$. This proves the existence assertion of the Theorem. The uniqueness is clear, since a projectivity preserves cross ratios, so that we are forced to define $\phi(D)$ as the $D' \in L'$ with $(A', B'; C', D') = (A, B; C, D)$.

### 4 L = P($k^2$) as an abstract projective line

The content of the present Section is mostly classical, but it emphasizes the category aspects of $P(k^2)$. Non-zero vectors in $k^2$ are denoted $a = (a_1, a_2)$, $b = (b_1, b_2)$ etc. and the 1-dimensional linear subspaces of $k^2$ spanned by $a$
is denoted $A$; similarly, $b$ spans $B$, etc; $A, B, \ldots$ are the points of the set $L = P(k^2)$. We now have available the precious tool of determinants of $2 \times 2$ matrices. We denote the determinant whose rows (or columns) are $a, b$ by the symbol $|a, b|$.

Given distinct $A, B, C$, spanned by $a, b, c$, respectively. We describe the linear map “projection from $A$ to $B$ in the direction of $C$” by describing its value on $a \in A$; this value, being in $B$, is of the form $\lambda \cdot b$ for some unique scalar $\lambda \in k^*$, and an elementary calculation with linear equation systems (say, using Cramer’s rule) gives that $\lambda = \frac{|c, a|}{|c, b|}$. Thus

$$ (C : A \to B)(a) = \frac{|c, a|}{|c, b|} \cdot b \quad (9) $$

is the basic formula. We can calculate the value of the composite $(C : A \to B). (D : B \to E)$ on $a \in A$; it takes $a \in A$ into

$$ \frac{|c, a|}{|c, b|} \cdot \frac{|d, b|}{|d, e|} e \in E. \quad (10) $$

In particular, if $E = A$, $a \in A$ goes into $(A, B; C, D). a$, where

$$ (A, B; C, D) := \frac{|c, a|}{|c, b|} \cdot \frac{|d, b|}{|d, a|} $$

(\text{using } |c, a| = -|a, c|, \text{ and similarly for the other factors}). This is the standard cross ratio $(A, B; C, D)$, and the standard permutation rules follow by known determinant calculations, as do the idempotency laws. So $P(k^2)$ is indeed an abstract projective line, in our sense.

In $L = P(k^2)$, we describe the points $A \in L$ by homogeneous coordinates $[a_1 : a_2]$, where $a$ is any vector spanning $A$. It is convenient to select three particular points $V, H, \text{ and } D$ (for “vertical”, “horizontal”, and “diagonal”, respectively):

$$ V = [0 : 1], H = [1 : 0], D = [1, 1]. $$

For any point $X$ distinct from $V$, there exists a unique $x \in k$ so that $X = [1 : x]$. Thus, the $x \in k$ corresponding to $H$ and $D$ are 0 and 1, respectively.
For $X$ distinct from $V$, the corresponding $x \in k$ may be calculated in terms of a cross ratio,

$$x = (V, H; D, X),$$

again by an easy calculation with determinants. Thus $\mathbb{L} \setminus V$ has, by the chosen conventions, been put in 1-1 correspondence with the affine line $k$, so

$$\mathbb{L} = \{V\} + k;$$

$V$ is the “point at infinity” of the (“vertical”) copy $\{(1, x) \mid x \in k\}$ of the affine line $k$ inside $k^2$.

The Fundamental Theorem then has the following

**Corollary 1** For every abstract projective line $\mathbb{L}$ over $k$, there exists an isomorphism (“projective equivalence”) with the projective line $P(k^2)$.

(The isomorphism claimed is not unique, unless $k$ is the 2-element field.) To prove the Corollary, pick three distinct points $A, B, C$ in $\mathbb{L}$, and let $\phi$ be the unique projectivity (as asserted by the Theorem) to $P(k^2)$ sending $A$ to $[1 : 0]$, $B$ to $[0 : 1]$, and $[C]$ to $[1 : 1]$.

The isomorphism/projectivity $\phi$ described in this Corollary depends on the choice of $A, B, C$, and so is not canonical. However, it allows us to perform calculations in $\mathbb{L}$ using coordinates, in the form of such projective equivalence $\mathbb{L} \cong P(k^2)$.

Let us for instance prove commutativity of (8). It suffices to prove that it holds in $\mathbb{L} = P(k^2)$. For, then it follows from the Fundamental Theorem that it also holds for three distinct points in an abstract projective line $\mathbb{L}$.

So consider points $A, B, C$ in $P(k^2)$. Using (10), we see that the composite $(C : A \rightarrow B). (A : B \rightarrow C)$ takes $a \in A$ into

$$\frac{|c, a|}{|c, b|} \frac{|a, b|}{|a, c|} c,$$

and since $|c, a| = -|a, c|$, two factors cancel except for the sign, and we are left with

$$-\frac{|a, b|}{|c, b|} c = -\frac{|b, a|}{|b, c|} c$$

which is the value of $-(B : A \rightarrow C)$ on $a$. (See [2] 1-4-2 for a more coordinate free proof.)
To complete the comparison with the classical “coordinate-” projective line $P(k^2)$, we need to compare projectivities in our sense (functors) with classical projectivities, meaning maps $P(k^2) \to P(k^2)$ that are “tracked” by linear automorphisms $k^2 \to k^2$.

Let $f : k^2 \to k^2$ be such linear automorphism. Then it defines a map $P(f) : P(k^2) \to P(k^2)$ by $[a_1 : a_2] \mapsto [f(a_1) : f(a_2)]$. We shall see that this map preserves composition of arrows, hence is a functor; for, by (9), $f(C : A \to B)$ takes $f(a) \in P(f)(A)$ to

$$f(c) = f(a) \cdot f(b)$$

(using the product rule for determinants and then cancelling the four occurrences of the determinant of $f$ that appear). The fact that composition is preserved is then a consequence of the formula (10).

On the other hand, every projectivity $\phi : P(k^2) \to P(k^2)$ (in our sense) is of the form $P(f)$ for some linear automorphism $f : k^2 \to k^2$ (which is in fact unique modulo $k^*$). Let $\phi(H) = A$, $\phi(V) = B$ and $\phi(D) = C$. Pick non-zero vectors $a \in A$, $b \in B$ and $c \in C$. The linear automorphism $f : k^2 \to k^2$ with matrix

$$f = \begin{bmatrix} a_1 & \lambda b_1 \\ a_2 & \lambda b_2 \end{bmatrix},$$

where

$$\lambda := -\frac{|c, a|}{|c, b|}$$

has the property that it takes $(1, 0)$ to $a$, hence $P(f)$ takes $H$ to $A$; it takes $(0, 1)$ to $\lambda b$, hence $P(f)$ takes $V$ to $B$; and finally, some calculation with Cramer’s rule, say, shows that $f$ takes $(1, 1)$ into a multiple of $c$, so $P(f)$ takes $D$ to $C$. Since $\phi$ and $P(f)$ both are projectivities, and they agree on $H, V$, and $D$, they agree everywhere, by the Fundamental Theorem. This proves that every projectivity $\phi : P(k^2) \to P(k^2)$ (functor) is indeed tracked by a linear automorphism $k^2 \to k^2$.

**Remark.** The projectivity $\phi : P(k^2) \to P(k^2)$ tracked by a linear automorphism $f : k^2 \to k^2$ with matrix $[\alpha_{ij}]$ is also classically described as the **fractional linear transformation**

$$x \mapsto \frac{\alpha_{21} + \alpha_{22}x}{\alpha_{11} + \alpha_{12}x}.$$

This refers to the identification of $x \in k$ with $[1 : x] \in P(k^2)$. 

11
5 Structures on punctured projective lines

Proposition 2 Given a projective line $L$, and given $A \in L$. Then $L \setminus \{A\}$ carries a canonical structure of an affine line.

**Proof.** We first consider the case where $L = P(k^2)$, and where $A = V = [0 : 1])$. Now $P(k^2) \setminus V$ is identified with $k$ via $x \mapsto [1 : x]$, and structure of affine line on $k$ gives by this identification a structure of affine line on $P(k^2) \setminus V$. Using the Fundamental Theorem, the general result now follows if we can to prove that a projectivity $P(k^2) \to P(k^2)$ which fixes the point $V = [0 : 1]$ preserves affine combinations of the remaining points. A projectivity which fixes $V = [0 : 1]$ is tracked by a $2 \times 2$ lower triangular matrix, or in terms of fractional linear transformations on $k$, by a function of the form $x \mapsto (\alpha_{21} + \alpha_{22}x)/\alpha_{12}$, and this is an affine map $k \to k$, hence preserves affine structure (i.e. preserves linear combinations whose coefficient sum is 1).

Proposition 3 Given a projective line $L$, and given $A, B \in L$ with $A, B$ distinct. Then $L \setminus \{A\}$ carries a canonical structure of abstract vector line, with $B$ as 0.

**Proof.** This is analogous to the proof of the previous Proposition. The requirement that not only $V$, but also $H$ is preserved implies that the fractional linear transformation $(\alpha_{21} + \alpha_{22}x)/\alpha_{12}$ considered in the previous proof must have $\alpha_{21} = 0$, and so is of the form $x \mapsto \alpha_{22}x/\alpha_{12}$, and hence is linear in $x$.

Finally, Proposition 4 Given a projective line $L$, and given $A, B, C \in L$, mutually distinct. Then $L \setminus \{A\}$ carries canonical structure of vector line with chosen basis vector, with $B$ as 0 and $C$ as the chosen basis vector.

This last proposition is a reformulation of the Fundamental Theorem.

6 The canonical bundles

For each $A \in L$, we have a canonical structure of affine line on $L \setminus \{A\}$. So over $L$, we have a bundle $A \to L$ of affine lines, whose fibre over $A \in L$ is the affine line $L \setminus \{A\}$. This bundle trivializes canonically over the covering $L^{(3)} \to L$, where $L^{(3)}$ denotes the set of triples $A, B, C$ of mutually distinct points, and where the exhibited map is given by $(A, B, C) \mapsto A$. 


The cocycle associated to this trivialization takes values in the group of affine automorphisms of $k$, which is a semidirect product of $(k, +)$ with $(k^*, \cdot)$. If $h_{A,B,C} : L\{A\} \rightarrow k$ is the restriction of the unique projectivity which takes $A$ to $[0 : 1] = V$, $B$ to $[1 : 0] = H$ and $C$ to $D = [1 : 1]$, then by construction of the affine structure on $L\{A\}$, $h_{A,B,C}$ is an affine isomorphism. If $(A, B, C)$ and $(A, B', C')$ are two triples of distinct points (same $A$!), then the value of the desired cocycle on this pair is the affine isomorphism $h_{A,B,C}^{-1} \circ h_{A,B',C'} : k \rightarrow k$. Its value on $0$ is $h_{A,B',C'}(0)$, which is the cross ratio $(A', B'; C', B)$, and similarly its value on $1$ is the cross ratio $(A', B'; C', C)$, so that the value of the cocycle on the pair $(A, B, C), (A, B', C')$ is
\[( (A, B'; C', B), (A, B'; C', C)) \in k \ltimes k^* \]
(with $(A, B'; C', B')$ by definition $= 0$).

For each $A, B \in L$, mutually distinct, we have a canonical vector line structure on $L\{A\}$, with underlying affine line the one just described, and with $B$ as $0$. So over $L^{(2)}$ (= set of pairs of distinct points $A, B$ in $L$), we have canonically a vector line bundle, whose fibre over $(A, B)$ is this vector bundle just described. It trivializes canonically over the covering $L^{(3)} \rightarrow L^{(2)}$ given by $(A, B, C) \mapsto (A, B)$ (here $L^{(3)}$ is the set of triples of distinct points in $L$).

The $k^*$-valued cocycle describing the associated principal bundle associates to $(A, B, C), (A, B', C')$ the cross ratio $(A', B; C, C')$. The fact that this is a cocycle is the idempotence laws for cross ratio.

Finally, over $L^{(3)}$, we have a bundle of vector-lines-with-chosen basis-vector. This bundle is already itself trivial, since a vector line with chosen basis vector is uniquely isomorphic to $k$ with $1 \in k$ as the chosen element.

### 7 Stacks of projective lines

The notion of projective line, and of morphism (= isomorphism = projectivity) between such, as described here, is a (1-sorted) first order theory. This immediately implies that the notion of a bundle of projective lines over a space $M$ makes sense, and in fact, such bundles pull back along maps, and descend along surjections, so projective line bundles form canonically a stack over the base category of sets, or, with suitable modifications, over the base category of spaces, say. Continuity, or other forms of cohesion, will usually
follow by the the fact that the constructions employed are canonical, as in [4], Section A.5). The study of bundles of of projective lines in the category of schemes, from [5], was the input challenge for the present work, and I hope to push further into loc. cit. using the abstract-projective-line concepts.)

**Example.** Let $k$ denote the field of three elements $\mathbb{Z}_3$. Every 4-element $L$ set carries a unique structure of projective line over this $k$. We invite the reader to construct this structure (a groupoid with 4 objects, and each hom-set a 2-element set); the composition laws follow from the idempotence equations; the cross ratio of the the four distinct points (in any order) is $-1$.

(Another argument: the group $PGL(2; \mathbb{Z})$ has 24 elements, which is also the number of permutations of a 4-element set, hence every permutation is a projectivity.)

It follows that for any space $M$, and for any 4-fold covering $E \to M$, the bundle $E \to M$ is uniquely a bundle of projective lines over $k$. Clearly, such $E \to M$ need not have a section $M \to E$, so does not come about from a bundle of affine lines over $M$, by completing the fibres by points at infinity (the fibrewise infinity points would provide a cross section).

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