On R-duals and the duality principle in Gabor analysis

Diana Stoeva, Ole Christensen

April 14, 2014

Abstract

The concept of R-duals of a frame was introduced by Casazza, Kutyniok and Lammers in 2004, with the motivation to obtain a general version of the duality principle in Gabor analysis. For tight Gabor frames and Gabor Riesz bases the three authors were actually able to show that the duality principle is a special case of general results for R-duals. In this paper we introduce various alternative R-duals, with focus on what we call R-duals of type II and III. We show how they are related and provide characterizations of the R-duals of type II and III. In particular, we prove that for tight frames these classes coincide with the R-duals by Casazza et al., which is desirable in the sense that the motivating case of tight Gabor frames already is well covered by these R-duals. On the other hand, all the introduced types of R-duals generalize the duality principle for larger classes of Gabor frames than just the tight frames and the Riesz bases; in particular, the R-duals of type III cover the duality principle for all Gabor frames.

1 Introduction

The concept of R-duals of a frame was introduced in the paper [2] by Casazza, Kutyniok and Lammers. They showed that the relation between a frame and its R-duals resembles the known results for the connection between a Gabor system \( \{ E_{mbT_{na}g} \}_{m,n \in \mathbb{Z}} \) and \( \{ \sqrt{ab} E_{m/aT_{n/b}g} \}_{m,n \in \mathbb{Z}} \) (see the definition below). This lead the three authors to the natural question whether \( \{ \sqrt{ab} E_{m/aT_{n/b}g} \}_{m,n \in \mathbb{Z}} \) can be realized as an R-dual of \( \{ E_{mbT_{na}g} \}_{m,n \in \mathbb{Z}} \); while
the general question remains open, they were able to confirm this for Gabor frames \( \{ E_{mb}T_{na}g \}_{m,n \in \mathbb{Z}} \) that are either tight or Riesz bases.

In this paper we propose various alternative definitions of R-duals, to be called \textit{R-duals of type II, III, IV}, with focus on type II & III. For this reason we will from now on refer to the R-duals by Casazza & al. as \textit{R-duals of type I}. Among the new types, the R-duals of type II form the smallest set; the R-duals of type III and IV are defined via two steps of relaxation of the conditions. For the case of a tight frame we show that the classes of R-duals of type I, II and III coincide; this is a desirable property because we know that the motivating case of a tight Gabor frames is already well covered by R-duals of type I. Based on a characterization of R-duals of type III we show that for any Gabor frame \( \{ E_{mb}T_{na}g \}_{m,n \in \mathbb{Z}} \) the sequence \( \{ \frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g \}_{m,n \in \mathbb{Z}} \) can be realized as an R-dual of type III of \( \{ E_{mb}T_{na}g \}_{m,n \in \mathbb{Z}} \); for R-duals of type II this is at least possible for all integer-oversampled Gabor frames.

In the rest of this introduction we recall the definition of the R-duals of type I, state the main results from \cite{2}, and prove a new result that characterizes when a tight Riesz sequence is an R-dual of a given tight frame with the same frame bound (Proposition \ref{prop:1.6}). We also provide the necessary background on Gabor frames, most importantly, the duality principle. Finally, for easy reference we state a few general results about frames and Riesz bases. In Section \ref{sec:1} we derive a relationship between \( \{ \frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g \}_{m,n \in \mathbb{Z}} \) and \( \{ E_{mb}T_{na}g \}_{m,n \in \mathbb{Z}} \) that naturally leads us to the definition of the various types of R-duals. Section \ref{sec:2} is devoted to an analysis of the R-duals of type II and the relations between R-duals of type I and II. Section \ref{sec:3} presents the properties of R-duals of type III. We provide a characterization of these R-duals (Theorem \ref{thm:4.4}), and show that for any given Gabor frame \( \{ E_{mb}T_{na}g \}_{m,n \in \mathbb{Z}} \) the sequence \( \{ \frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g \}_{m,n \in \mathbb{Z}} \) can be realized as an R-dual of type III (Corollary \ref{cor:4.5}). In the same section we prove that R-duals of type III enjoy most of the properties that make R-duals of type I attractive.

We note that the duality principle and the work on R-duals in \cite{2} have trigged quite some interest in various directions. The papers \cite{4} by Christensen, Kim, and Kim and \cite{9} by Fan and Shen consider R-duals within the original framework of \cite{2}, in \cite{9} using the concept of adjoint systems. In \cite{8} Dutkay, Han and Larson prove that the duality principle extends to any dual pair of projective unitary representations of countable groups. Finally, X. M. Xiao and Zhu considered an extension of R-duality to Banach spaces in \cite{13}, a work that was followed up by \cite{5} by Christensen, X. C. Xiao and Zhu.
1.1 Basic results on frames and Riesz bases

For easy reference we will collect some of the needed facts about frames and Riesz bases here. Much more information can be found in the standard monographs, see, e.g., [14, 6, 10, 3].

In the entire paper we let $\mathcal{H}$ denote a separable Hilbert space, with the inner product $\langle\cdot,\cdot\rangle$ chosen to be linear in the first entry. We skip the formal definition of a frame and a Riesz basis, which is expected to be well known; we just mention that when we speak about a frame, it is understood that it is a frame for the entire space $\mathcal{H}$. In contrast, a frame sequence is a sequence that is just a frame for the closed span of its elements. We use the same distinction between a Riesz basis (which spans $\mathcal{H}$) and a Riesz sequences (which is a Riesz basis for the closed span of its elements).

It is well known that we can construct a tight frame based on any given frame:

**Lemma 1.1** Let $\{f_i\}_{i \in I}$ be a frame for $\mathcal{H}$, with frame operator $S$. Then $\{S^{-1/2}f_i\}_{i \in I}$ is a tight frame for $\mathcal{H}$ with frame bound 1. If $\{f_i\}_{i \in I}$ is a Riesz basis, then $\{S^{-1/2}f_i\}_{i \in I}$ is an orthonormal basis for $\mathcal{H}$.

For any given frame there is a natural procedure to construct a Riesz basis with the same frame bounds; see, e.g., [3] for a proof of this standard result.

**Lemma 1.2** Let $\{e_i\}_{i \in I}$ be any orthonormal basis for $\mathcal{H}$ and $Q : \mathcal{H} \to \mathcal{H}$ a bounded bijective operator. Then the following holds.

(i) The sequence $\{Qe_i\}_{i \in I}$ is a Riesz basis with frame operator $QQ^*$ and optimal bounds $\frac{1}{||Q||^2}, ||Q||^2$.

(ii) The dual Riesz basis of $\{Qe_i\}_{i \in I}$ is $\{(Q^*)^{-1}e_i\}_{i \in I}$; the frame operator for this sequence is $(QQ^*)^{-1}$ and the optimal bounds are $\frac{1}{||Q||^2}, ||Q^{-1}||^2$.

In particular, if $\{f_i\}_{i \in I}$ is a frame with frame operator $S$ and optimal bounds $A, B$, then $\{S^{1/2}e_i\}_{i \in I}$ is a Riesz basis with frame operator $S$ and optimal bounds $A, B$; the dual Riesz is $\{S^{-1/2}e_i\}_{i \in I}$, with frame operator $S^{-1}$ and optimal bounds $\frac{1}{B}, \frac{1}{A}$.

In case $\{f_i\}_{i \in I}$ is a frame sequence in $\mathcal{H}$ and $V := \overline{\text{span}}\{f_i\}_{i \in I} \neq \mathcal{H}$, the frame operator $S$ can be considered as a bijection on $V$. The following
elementary lemma shows in particular how we can extend $S$ to a bijective and bounded operator on $\mathcal{H}$, while keeping the norm of the operator and its inverse; we will need this result in the analysis of R-duals of type III.

**Lemma 1.3** Let $V$ be a closed subspace of $\mathcal{H}$ and $\Phi : V \to V$ a bounded bijective operator. Define an extension of $\Phi$ to an operator

$$\tilde{\Phi} : \mathcal{H} \to \mathcal{H}, \tilde{\Phi}(x_1 + x_2) := \Phi x_1 + ||\Phi^{-1}||^{-1}x_2, \ x_1 \in V, x_2 \in V^\perp.$$

Then $\tilde{\Phi}$ is bijective and bounded, $||\tilde{\Phi}|| = ||\Phi||, ||\tilde{\Phi}^{-1}|| = ||\Phi^{-1}||$, and

$$\tilde{\Phi}^{-1}(x_1 + x_2) = \Phi^{-1}x_1 + ||\Phi^{-1}||x_2, \ x_1 \in V, x_2 \in V^\perp \quad (1)$$

If $\Phi$ is self-adjoint, then also $\tilde{\Phi}$ is self-adjoint.

**Proof.** Note that $||\Phi^{-1}||^{-1} \leq ||\Phi||$. Thus, writing $x \in \mathcal{H}$ as $x = x_1 + x_2$ with $x_1 \in V, x_2 \in V^\perp$,

$$||\tilde{\Phi}x||^2 = ||\Phi x_1||^2 + ||||\Phi^{-1}||^{-1}x_2||^2 \leq ||\Phi||^2 x_1 ||^2 + ||\Phi^{-1}||^{-2} x_2 ||^2 \leq ||\Phi||^2 ||x||^2.$$

From here it is clear that $\Phi$ is bounded and that $||\tilde{\Phi}|| = ||\Phi||$. The other properties are clear by construction. \qed

## 1.2 R-duals of type I

Let $\{e_i\}_{i \in I}$ and $\{h_i\}_{i \in I}$ denote orthonormal bases for $\mathcal{H}$, and let $\{f_i\}_{i \in I}$ be any sequence in $\mathcal{H}$ for which $\sum_{i \in I} |\langle f_i, e_j \rangle|^2 < \infty, \ \forall j \in I$. In [2] the R-dual of $\{f_i\}_{i \in I}$ with respect to the orthonormal bases $\{e_i\}_{i \in I}$ and $\{h_i\}_{i \in I}$ is defined as the sequence $\{\omega_j\}_{j \in I}$ given by

$$\omega_j = \sum_{i \in I} \langle f_i, e_j \rangle h_i, \ j \in I. \quad (2)$$

As mentioned in the introduction we will from now on refer to $\{\omega_j\}_{j \in I}$ in [2] as an R-dual of type I. In [2] the following connections between the properties of $\{f_i\}_{i \in I}$ and $\{\omega_j\}_{j \in I}$ are proved. Note in particular that an R-dual of a frame is a Riesz sequence:

**Theorem 1.4** [2] Let $\{e_i\}_{i \in I}$ and $\{h_i\}_{i \in I}$ denote orthonormal bases for $\mathcal{H}$, and let $\{f_i\}_{i \in I}$ be any sequence in $\mathcal{H}$ for which $\sum_{i \in I} |\langle f_i, e_j \rangle|^2 < \infty$ for all $j \in I$. Define the R-dual $\{\omega_j\}_{j \in I}$ of type I as in [2]. Then the following hold:
(i) For all \( i \in I \), \( f_i = \sum_{j \in I} (\omega_j, h_i) e_j \), i.e., \( \{f_i\}_{i \in I} \) is the R-dual sequence of \( \{\omega_j\}_{j \in I} \) of type I w.r.t. the orthonormal bases \( \{h_i\}_{i \in I} \) and \( \{e_i\}_{i \in I} \).

(ii) \( \{f_i\}_{i \in I} \) is a frame with bounds \( A, B \) if and only if \( \{\omega_j\}_{j \in I} \) is a Riesz sequence in \( H \) with bounds \( A, B \).

(iii) Two Bessel sequences \( \{f_i\}_{i \in I} \) and \( \{g_i\}_{i \in I} \) in \( H \) are dual frames if and only if the associated R-dual sequences \( \{\omega_j\}_{j \in I} \) and \( \{\gamma_j\}_{j \in I} \) of type I, w.r.t. the same choices of orthonormal bases \( \{e_i\}_{i \in I} \) and \( \{h_i\}_{i \in I} \), satisfy

\[
\langle \omega_j, \gamma_k \rangle = \delta_{j,k}, \quad j, k \in I.
\]

(iv) \( \{\omega_j\}_{j \in I} \) is a Riesz basis if and only if \( \{f_i\}_{i \in I} \) is a Riesz basis.

Recall that if \( \{f_i\}_{i \in I} \) is a frame for \( H \), the preframe operator or synthesis operator is defined by

\[
T : \ell^2(I) \to H, \quad T\{c_i\}_{i \in I} = \sum_{i \in I} c_i f_i.
\]

The following result from [2] presents a necessary condition on a sequence \( \{\omega_j\}_{j \in I} \) to be an R-dual of type I of a given frame \( \{f_i\}_{i \in I} \), stated in terms of the dimension of the kernel of \( T \) and the deficit of the sequence \( \{\omega_j\}_{j \in I} \):

**Lemma 1.5** [2] If \( \{f_i\}_{i \in I} \) is a frame with synthesis operator \( T \) and \( \{\omega_j\}_{j \in I} \) is an R-dual of type I of \( \{f_i\}_{i \in I} \), then \( \dim(\ker T) = \dim(\text{span}\{\omega_j\}_{j \in I}^\perp) \).

The dimension condition in Lemma 1.5 will play a crucial role for the various R-duals to be defined in Section 2, see Theorem 4.4. Using Lemma 1.5 we can derive a simple characterization of a Riesz sequence \( \{\omega_j\}_{j \in I} \) being an R-dual of type I of a frame \( \{f_i\}_{i \in I} \) in the tight case:

**Proposition 1.6** Let \( \{f_i\}_{i \in I} \) be a tight frame for \( H \) and let \( \{\omega_j\}_{j \in I} \) be a tight Riesz sequence in \( H \) with the same bound. Denote the synthesis operator for \( \{f_i\}_{i \in I} \) by \( T \). Then \( \{\omega_j\}_{j \in I} \) is an R-dual of \( \{f_i\}_{i \in I} \) of type I if and only if

\[
\dim(\ker T) = \dim(\text{span}\{\omega_j\}_{j \in I}^\perp).
\]

**Proof.** The necessity of the condition in (4) follows from Lemma 1.5. Now let \( \{f_i\}_{i \in I} \) be a tight frame and \( \{\omega_j\}_{j \in I} \) be a tight Riesz sequence with the same bound \( A \), and assume that (4) holds. Take an orthonormal basis \( \{e_i\}_{i \in I} \)
for \( \mathcal{H} \) and observe that \( \{ \sqrt{\lambda} \omega_j \}_{j \in I} \) is an orthonormal sequence. Consider the R-dual \( \{ \nu_j \}_{j \in I} \) of type I of \( \{ f_i \}_{i \in I} \) with respect to the orthonormal bases \( \{ e_i \}_{i \in I} \) and \( \{ h_i \}_{i \in I} \), i.e. \( \nu_j := \sum_{i \in I} \langle f_i, e_j \rangle e_i \), \( j \in I \). By Theorem 1.4 \( \{ \nu_j \}_{j \in I} \) is a tight Riesz sequence with bound \( A \) and hence \( \{ \sqrt{\lambda} \nu_j \}_{j \in I} \) is also an orthonormal sequence. By Lemma 1.5 and (4),

\[
\dim(\text{span}\{ \nu_j \}_{j \in I}) = \dim(\ker T) = \dim(\text{span}\{ \omega_j \}_{j \in I}).
\] (5)

In case \( \text{span}\{ \nu_j \}_{j \in I} = \text{span}\{ \omega_j \}_{j \in I} = \{ 0 \} \), the orthonormality of the sequences \( \{ \sqrt{\lambda} \nu_j \}_{j \in I}, \{ \sqrt{\lambda} \omega_j \}_{j \in I} \), implies that we can define a unitary operator \( U : \mathcal{H} \to \mathcal{H} \) by \( U \nu_j := \omega_j, j \in I \); and in case \( \text{span}\{ \nu_j \}_{j \in I} \neq \{ 0 \} \), letting \( \{ \phi_j \}_{j \in J} \) and \( \{ \psi_j \}_{j \in J} \) be orthonormal bases for \( \text{span}\{ \nu_j \}_{j \in I} \) and \( \text{span}\{ \nu_j \}_{j \in I} \), respectively, Theorem 1.4 has the consequence that we can define a unitary operator \( U : \mathcal{H} \to \mathcal{H} \) by \( U \nu_j := \omega_j, j \in I \), and \( U \phi_j := \psi_j, j \in J \). In both cases,

\[
\omega_j = U \nu_j = U \sum_{i \in I} \langle f_i, e_j \rangle e_i = \sum_{i \in I} \langle f_i, e_j \rangle U e_i, j \in I,
\]

which shows that \( \{ \omega_j \}_{j \in I} \) is an R-dual of \( \{ f_i \}_{i \in I} \) of type I.

Note that in the non-tight case (4) does not imply that \( \{ \omega_j \}_{j \in I} \) is an R-dual of \( \{ f_i \}_{i \in I} \) of type I.

**Example 1.7** Let \( \{ z_i \}_{i=1}^\infty \) be an orthonormal basis for \( \mathcal{H} \). Consider the frame (actually a Riesz basis) \( \{ f_i \}_{i=1}^\infty = \{ \sqrt{2} z_1, z_2, \sqrt{2} z_3, \sqrt{2} z_4, \sqrt{2} z_5, \sqrt{2} z_6, \ldots \} \) and the Riesz basis \( \{ g_j \}_{j=1}^\infty = \{ \sqrt{2} z_1, z_2, z_3, z_4, z_5, \ldots \} \). The sequences \( \{ f_i \}_{i=1}^\infty \) and \( \{ g_j \}_{j=1}^\infty \) both have the optimal bounds \( A = 1, B = 2 \), but \( \{ g_j \}_{j=1}^\infty \) is not an R-dual of \( \{ f_i \}_{i=1}^\infty \) of type I. In fact, assume that there exist orthonormal bases \( \{ e_i \}_{i=1}^\infty \) and \( \{ h_i \}_{i=1}^\infty \) for \( \mathcal{H} \) so that \( g_j = \sum_{i=1}^\infty \langle f_i, e_j \rangle h_i, \forall j \in \mathbb{N} \). Then \( z_j = g_j = \sum_{i=1}^\infty \langle f_i, e_j \rangle h_i \), for every \( j \geq 2, j \in \mathbb{N} \), which implies that \( \langle z_j, h_i \rangle = \langle f_i, e_j \rangle \) for all \( i, j \in \mathbb{N}, j \geq 2 \). Thus

\[
1 = \| z_j \|^2 = \sum_{i=1}^\infty | \langle z_j, h_i \rangle |^2 = \sum_{i=1}^\infty | \langle f_i, e_j \rangle |^2 = \\
= \sum_{i=1}^\infty | \langle z_i, e_j \rangle |^2 + \sum_{i \in \{1,3,4,5,\ldots\}} | \langle z_i, e_j \rangle |^2 = 1 + \sum_{i \in \{1,3,4,5,\ldots\}} | \langle z_i, e_j \rangle |^2.
\]

It follows that \( z_1 \perp e_j \) and \( z_3 \perp e_j \) for all \( j \in \mathbb{N}, j \geq 2 \), which is a contradiction. Thus \( \{ g_j \}_{j=1}^\infty \) is not an R-dual of \( \{ f_i \}_{i=1}^\infty \) of type I. \( \square \)
1.3 Gabor analysis

For parameters $a, b \in \mathbb{R}$, define the operators $T_a$ and $E_b$ on $L^2(\mathbb{R})$ by $T_a f(x) = f(x - a)$ and $E_b f(x) = e^{2\pi ibx} f(x)$, respectively. The Gabor system generated by a fixed function $g \in L^2(\mathbb{R})$ and some $a, b > 0$ is the collection of functions $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$. Remember that when we speak about $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ being a frame, it is understood that we mean a frame for the space $L^2(\mathbb{R})$. For a detailed discussion of the role of systems $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ in time-frequency analysis we refer to the monograph [10]; their frame properties are in focus in the monograph [3].

One of the deepest results in Gabor analysis is the duality principle, which was discovered at the same time by Janssen [11], Daubechies, Landau, and Landau [7], and Ron and Shen [12]:

**Theorem 1.8** [7, 11, 12] Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Then the Gabor system $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is a frame with bounds $A, B$ if and only if $\{\frac{1}{\sqrt{ab}} E_{m/a} T_{n/b} g\}_{m,n \in \mathbb{Z}}$ is a Riesz sequence with bounds $A, B$.

The similarity between Theorem 1.4(ii) and Theorem 1.8 leads to the obvious question whether there is a connection between the duality principle and the R-duals of type I. In [2] it was shown that at least for two important special cases of Gabor frames $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$, the sequence $\{\frac{1}{\sqrt{ab}} E_{m/a} T_{n/b} g\}_{m,n \in \mathbb{Z}}$ is actually an R-dual of $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$:

**Theorem 1.9** [2] Assume that the frame $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is either tight or a Riesz basis. Then $\{\frac{1}{\sqrt{ab}} E_{m/a} T_{n/b} g\}_{m,n \in \mathbb{Z}}$ can be realized as an R-dual of type I of $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$.

It is still an open question whether Theorem 1.9 holds for other classes of Gabor frames. We will not provide any new results about this; instead, our purpose is to introduce other types of R-duals that cover the duality principle for larger classes of Gabor frames than the two classes in Theorem 1.9. The R-duals of type III (to be defined in Section 2 and analyzed in Section 4) turns out to cover all Gabor frames, and also keep essential parts of the properties R-duals of type I, as stated in Theorem 1.4.

Let us end this section with a few results from Gabor analysis that will be used repeatedly. The first one is due to Balan, Casazza, and Heil. Recall that a frame $\{f_i\}_{i \in I}$ has infinite excess if infinitely many elements can be removed while the remaining sequence is still a frame. Also, the deficit of a sequence $\{f_i\}_{i \in I}$ in $\mathcal{H}$ is the number $\dim(\text{span}\{f_i\}_{i \in I})$. 

7
Lemma 1.10 [1] Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Then the following hold.

(i) If $ab < 1$ and $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame, then $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ has infinite excess.

(ii) If $ab > 1$, then $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ has infinite deficit.

The following well-known result will be used at several places. Recall that for any frame, the frame operator $S$ is a positive bounded operator, and thus has a unique positive square-root $S^{1/2}$.

Lemma 1.11 Let $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ be a Bessel sequence in $L^2(\mathbb{R})$, with frame operator $S$. Then $S$ commutes with the operators $E_{mb}T_{na}$, $m,n \in \mathbb{Z}$. If $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame, then also the operators $S^{1/2}$ and $S^{-1/2} = (S^{-1})^{1/2}$ commute with the operators $E_{mb}T_{na}$, $m,n \in \mathbb{Z}$.

Recall that a Gabor frame $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is said to be integer-oversampled if $ab = 1/K$ for some $K \in \mathbb{N}$. For our purpose the importance of integer-oversampled Gabor frames lies in the fact that for such frames the operators $E_{mb}T_{na}$, $m,n \in \mathbb{Z}$, form a subclass of $E_{mb}T_{na}$, $m,n \in \mathbb{Z}$ and therefore commute with the frame operator of $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ and its various powers.

2 Towards generalized R-duals

We have already noticed that for Gabor frames $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ that are either tight or Riesz bases, the sequence $\{\frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g\}_{m,n \in \mathbb{Z}}$ can be realized as an R-dual of $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ of type I. We will now show that we can cover a larger class of Gabor frames by replacing the orthonormal bases in the definition of the R-dual by other types of sequences.

Proposition 2.1 Assume that $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is either a tight Gabor frame or an integer-oversampled Gabor frame, with frame operator $S$. Then there exist Riesz bases $\{x_{m,n}\}_{m,n \in \mathbb{Z}}, \{y_{m,n}\}_{m,n \in \mathbb{Z}}$ for $L^2(\mathbb{R})$ such that

$$\frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g = \sum_{m',n' \in \mathbb{Z}} \langle E_{m'b}T_{n'a}g, x_{m,n} \rangle y_{m',n'}.$$  \hspace{1cm} (6)

More explicitly, there exist orthonormal bases $\{e_{m,n}\}_{m,n \in \mathbb{Z}}$ and $\{h_{m,n}\}_{m,n \in \mathbb{Z}}$ such that (6) holds with $x_{m,n} = S^{-1/2}e_{m,n}, y_{m,n} = S^{1/2}h_{m,n}$.  

8
Proof. Let us first consider the case where $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is an integer-oversampled Gabor frame. Then $\{E_{mb} T_{na} S^{-1/2} g\}_{m,n \in \mathbb{Z}}$ is a tight frame for $L^2(\mathbb{R})$ with frame bound 1. Let $\{e_{m,n}\}_{m,n \in \mathbb{Z}}$ be an orthonormal basis for $L^2(\mathbb{R})$ consisting of bounded and compactly supported functions. Then, according to the proof of Theorem 1.9(ii) in [2] there exists a unitary operator $U : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ such that

$$\frac{1}{\sqrt{ab}} E_{m/a} T_{n/b} S^{-1/2} g = \sum_{m',n' \in \mathbb{Z}} \langle E_{m'b} T_{n'a} S^{-1/2} g, e_{m,n} \rangle U e_{m',n'}.$$  

(7)

Since $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is a frame, the operator $S^{-1/2}$ commutes with the operators $E_{mb} T_{na}$ for all $m,n \in \mathbb{Z}$. Since $ab = 1/K$ for some $K \in \mathbb{N}$, the operators $E_{m/a} T_{n/b}$ form a subclass of the operators $E_{mb} T_{na}$, $m,n \in \mathbb{Z}$, and therefore (7) implies that

$$\frac{1}{\sqrt{ab}} S^{-1/2} E_{m/a} T_{n/b} g = \sum_{m',n' \in \mathbb{Z}} \langle S^{-1/2} E_{m'b} T_{n'a} g, e_{m,n} \rangle U e_{m',n'}.$$  

Thus $\frac{1}{\sqrt{ab}} E_{m/a} T_{n/b} g = \sum_{m',n' \in \mathbb{Z}} \langle E_{m'b} T_{n'a} g, S^{-1/2} e_{m,n} \rangle S^{1/2} U e_{m',n'}$, which proves (7), with sequences $x_{m,n}$ and $y_{m,n}$ as stated in the proposition.

Let us now assume that $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ is a tight frame, with frame bound $A$. Then $S = AI$. By Theorem 1.9 there exist orthonormal bases $\{e_{m,n}\}_{m,n \in \mathbb{Z}}$ and $\{h_{m,n}\}_{m,n \in \mathbb{Z}}$ such that

$$\frac{1}{\sqrt{ab}} E_{m/a} T_{n/b} g = \sum_{m',n' \in \mathbb{Z}} \langle E_{m'b} T_{n'a} g, e_{m,n} \rangle h_{m',n'}.$$  

(8)

Using that $S^{1/2} = A^{1/2} I$ and $S^{-1/2} = A^{-1/2} I$, (8) can clearly be written on the form (6), as claimed. □

Proposition 2.1 leads to several natural ways of alternative definitions of R-duals. We will call the new types for R-duals of type II, III, IV, respectively. Note that at the moment we have not motivated the definition of R-duals of type III; the reason for the definition will become clear in Section 4 where we prove a statement of similar spirit as Proposition 2.1 but without the assumption on $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ being integer-oversampled. The resulting expansion places the general Gabor case within the framework of R-duals of type III.
Definition 2.2 Let \( \{f_i\}_{i \in I} \) be a frame for \( \mathcal{H} \) with frame operator \( S \).

(i) Let \( \{e_i\}_{i \in I} \) and \( \{h_i\}_{i \in I} \) denote orthonormal bases for \( \mathcal{H} \). The R-dual of type II of \( \{f_i\}_{i \in I} \) with respect to \( \{e_i\}_{i \in I} \) and \( \{h_i\}_{i \in I} \) is the sequence \( \{\omega_j\}_{j \in I} \) given by

\[
\omega_j = \sum_{i \in I} \langle f_i, S^{-1/2}e_j \rangle S^{1/2}h_i, \quad j \in I.
\]

(ii) Let \( \{e_i\}_{i \in I} \) and \( \{h_i\}_{i \in I} \) denote orthonormal bases for \( \mathcal{H} \) and \( Q : \mathcal{H} \to \mathcal{H} \) be a bounded bijective operator with \( \|Q\| \leq \sqrt{\|S\|} \) and \( \|Q^{-1}\| \leq \sqrt{\|S^{-1}\|} \). The R-dual of type III of \( \{f_i\}_{i \in I} \) with respect to the triplet \( (\{e_i\}_{i \in I}, \{h_i\}_{i \in I}, Q) \), is the sequence \( \{\omega_j\}_{j \in I} \) defined by

\[
\omega_j := \sum_{i=1}^{\infty} \langle S^{-1/2}f_i, e_j \rangle Qh_i.
\]

(iii) Let \( \{e_i\}_{i \in I} \) and \( \{h_i\}_{i \in I} \) denote Riesz bases for \( \mathcal{H} \). The R-dual of type IV of \( \{f_i\}_{i \in I} \) with respect to \( \{e_i\}_{i \in I} \) and \( \{h_i\}_{i \in I} \) is the sequence \( \{\omega_j\}_{j \in I} \) given by

\[
\omega_j = \sum_{i \in I} \langle f_i, e_j \rangle h_i, \quad j \in I.
\]

Note that in the definition of R-duals of type III the action of the operator \( S^{-1/2} \) is written on \( f_i \) instead of \( e_j \). This implies that R-duals of type III also are defined if we only assume that \( \{f_i\}_{i \in I} \) is a frame sequence and interpret the frame operator as a bijection on \( \text{span}\{f_i\}_{i \in I} \). In contrast, R-duals of type II only exist when \( \{f_i\}_{i \in I} \) is a frame for \( \mathcal{H} \).

The conditions on the operator \( Q \) in Definition 2.2(ii) means that \( \{Qh_i\}_{i \in I} \) is a Riesz basis for \( \mathcal{H} \) with bounds \( \|S^{-1/2}\|^{-1}, \|S\| \), which according to Lemma 1.2 are the optimal bounds for the sequence \( \{S^{1/2}h_i\}_{i \in I} \). Thus, the R-duals of type II are contained in the class of R-duals of type III. It is obvious that the R-duals of type III are contained in the class of R-duals of type IV. It is also clear that R-duals of type I are contained in the class of R-duals of type IV.

Example 3.2, Example 4.6 and Theorem 4.4 provide further information about the relationship between the various classes; in order for the reader to get a quick overview of what to come, we summarize these results in Figure 1. We also note already now that Proposition 3.1 and Proposition 4.2 will show that for a tight frame, the classes of R-duals of type I, II and III coincide.
A general analogue of R-duals of type IV have been considered in Banach spaces in the papers [13] and [5]. In particular, the results by Xiao and Zhu in [13] imply that in the framework of Definition 2.2 (iii), 
\[ f_i = \sum_{j \in I} \langle \omega_j, \tilde{h}_i \rangle \tilde{e}_j, \]
where \( \{ \tilde{e}_i \}_{i \in I} \) and \( \{ \tilde{h}_i \}_{i \in I} \) are the dual Riesz bases of \( \{ e_i \}_{i \in I} \) and \( \{ h_i \}_{i \in I} \), respectively; furthermore, if \( \{ f_i \}_{i \in I} \) is a frame then \( \{ \omega_j \}_{j \in I} \) is a Riesz sequence, with a lower bound that is the multiple of the lower bound of \( \{ f_i \}_{i \in I} \) and the lower bounds for the two Riesz sequences \( \{ e_i \}_{i \in I}, \{ h_i \}_{i \in I} \) (and a similar result for the upper bound). These properties are getting too far away from the properties we know for R-duals of type I, so we will not discuss type IV in more detail in this paper.

3 R-duals of type II

The purpose of this section is to state some of the most important properties of R-duals of type II and to compare the properties of the R-duals of type I and II. Other results are on Section 4, which cover the more general case of R-duals of type III. Also note that Theorem 4.4 contains a characterization of the R-duals of type II.

First we notice that for tight frames, the classes of R-duals of type I and II coincide:

**Proposition 3.1** Assume that \( \{ f_i \}_{i \in I} \) is a tight frame for \( \mathcal{H} \). Then the classes of R-duals of type I and II coincide.

**Proof.** Since the frame operator \( S = A I \) for some \( A > 0 \), \( S^{1/2} = A^{1/2} I \) and \( S^{-1/2} = A^{-1/2} I \). Thus, for any orthonormal bases \( \{ e_i \}_{i \in I} \) and \( \{ h_i \}_{i \in I} \), we have 
\[ \sum_{i \in I} \langle f_i, e_j \rangle h_i = \sum_{i \in I} \langle f_i, S^{-1/2} e_j \rangle S^{1/2} h_i, \]
which proves the result. \( \square \)
In general the R-duals of type I and II constitute two different classes. The following example exhibits an R-dual of type II of a specific frame, which is not an R-dual of type I; in Example 4.6 we will use the same construction to find an R-dual of type I which is not an R-dual of type II.

**Example 3.2** Let \( \{e_i\}_{i=1}^{\infty} \) be an orthonormal basis for \( \mathcal{H} \). Consider the frame \( \{f_i\}_{i=1}^{\infty} = \{e_1, e_1, e_2, e_3, e_4, \ldots\} \), which has optimal bounds 1, 2. Note that \( Se_1 = 2e_1, Se_i = e_i, e \geq 2 \). A simple calculation of the R-dual of type II of \( \{f_i\}_{i=1}^{\infty} \) with respect to \( \{e_i\}_{i=1}^{\infty} \) and \( \{h_i\}_{i=1}^{\infty} = \{e_i\}_{i=1}^{\infty} \) shows that \( \{\omega_j\}_{j=1}^{\infty} = \{e_1 + \sqrt{2} e_2, e_3, e_4, \ldots\} \). In particular, \( \{\omega_j\}_{j=1}^{\infty} \) has the optimal bounds 1, 3/2. By Theorem 1.3 the R-duals of type I have the same optimal bounds as the given frame, which shows that \( \{\omega_j\}_{j=1}^{\infty} \) cannot be an R-dual of type I of \( \{f_i\}_{i=1}^{\infty} \).

It follows immediately from Proposition 2.1 that for any Gabor frame \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) that is either tight or integer-oversampled, the sequence \( \{\sqrt{ab} E_{m/a}T_{n/b}g\}_{m,n \in \mathbb{Z}} \) can be realized as an R-dual of type II. In the general setting of frames in Hilbert spaces we now show that R-duals of type II enjoy many of the properties that make R-duals of type I attractive, see Theorem 1.4. This is the content of the following proposition, as well as Proposition 4.3 and Theorem 4.4 in the next section.

**Proposition 3.3** Let \( \{f_i\}_{i \in I} \) be a frame for \( \mathcal{H} \) with frame operator \( S \) and let \( \{\omega_j\}_{j \in I} \) be the R-dual of type II of \( \{f_i\}_{i \in I} \) with respect to some orthonormal bases \( \{e_i\}_{i \in I} \) and \( \{h_i\}_{i \in I} \). The following statements hold.

(i) \( f_i = \sum_{j \in I} \langle \omega_j, S^{-1/2} h_i \rangle S^{1/2} e_j, \forall i \in I \).

(ii) Let \( \{\gamma_j\}_{j \in I} \) denote the R-dual of type II of the canonical dual frame \( \{S^{-1/2} f_i\}_{i \in I} \) with respect to \( \{e_i\}_{i \in I} \) and \( \{h_i\}_{i \in I} \). Then \( \{\omega_j\}_{j \in I} \) and \( \{\gamma_j\}_{j \in I} \) are biorthogonal.

**Proof.** (i) By Lemma 1.2 and (9), \( \langle \omega_j, S^{-1/2} h_i \rangle = \langle f_i, S^{-1/2} e_j \rangle \) for every \( i, j \in I \); thus, expanding \( f_i \) with respect to the dual pair of Riesz bases \( \{S^{-1/2} e_j\}_{j \in I}, \{S^{1/2} e_j\}_{j \in I} \) yields that

\[
 f_i = \sum_{j \in I} \langle f_i, S^{-1/2} e_j \rangle S^{1/2} e_j = \sum_{j \in I} \langle \omega_j, S^{-1/2} h_i \rangle S^{1/2} e_j, \forall i \in I.
\]
(ii) The frame operator of \( \{S^{-1}f_i\}_{i \in I} \) is \( S^{-1} \), so by definition
\[
\gamma_n = \sum_{k \in I} \langle S^{-1}f_k, S^{1/2}e_n \rangle S^{-1/2}h_k, \quad n \in I.
\]
Therefore, for \( j, n \in I \),
\[
\langle \omega_j, \gamma_n \rangle = \sum_{i \in I} \sum_{k \in I} \langle f_i, S^{-1/2}e_j \rangle \langle S^{1/2}e_n, S^{-1}f_k \rangle \langle S^{1/2}h_i, S^{-1/2}h_k \rangle.
\]
By Lemma 1.2 \( \{S^{1/2}h_i\}_{i \in I} \) and \( \{S^{-1/2}h_i\}_{i \in I} \) are biorthogonal; thus,
\[
\langle \omega_j, \gamma_n \rangle = \sum_{i \in I} \langle f_i, S^{-1/2}e_j \rangle \langle S^{1/2}e_n, S^{-1}f_i \rangle = \langle S^{1/2}e_n, S^{-1/2}e_j \rangle = \delta_{n,j},
\]
which completes the proof. \( \square \)

4 R-duals of type III

The main result in this section is a characterization of the R-duals of type II and III (Theorem 4.4). As a consequence we show that for any Gabor frame \( \{E_m T_n g\}_{m,n \in \mathbb{Z}} \), the sequence \( \{a^{1/2} E_{m/a} T_{n/b} g\}_{m,n \in \mathbb{Z}} \) can be realized as an R-dual of type III (Corollary 4.5). We will also derive some key properties of R-duals of type III and relate them to the properties of R-duals of type I.

We first note that R-duals of type III have an obvious characterization in terms of R-duals of type I. We leave the proof to the reader:

**Lemma 4.1** Let \( \{f_i\}_{i \in I} \) be a frame sequence with frame operator \( S \), \( \{\omega_j\}_{j \in I} \) a Riesz sequence, and \( Q : \mathcal{H} \to \mathcal{H} \) a bounded bijective operator with \( \|Q\| \leq \sqrt{\|S\|} \) and \( \|Q^{-1}\| \leq \sqrt{\|S^{-1}\|} \). Then the following are equivalent:

(i) \( \{\omega_j\}_{j \in I} \) is an R-dual of type III of \( \{f_i\}_{i \in I} \) w.r.t. the operator \( Q \);

(ii) \( \{Q^{-1}\omega_j\}_{j \in I} \) is an R-dual of type I of \( \{S^{-1/2}f_i\}_{i \in I} \).

In particular, if \( \{\omega_j\}_{j \in I} \) is an R-dual of \( \{f_i\}_{i \in I} \) of type III w.r.t. \( Q \), then \( \{Q^{-1}\omega_j\}_{j \in I} \) is an orthonormal sequence.

We have already seen that for a tight frame, the classes of R-duals of type I and II coincide. In this special case the type III duals give the same class:
Proposition 4.2 Assume that \( \{f_i\}_{i \in I} \) is a tight frame for \( \mathcal{H} \). Then the classes of R-duals of type I and III coincide.

Proof. Denote the frame operator for \( \{f_i\}_{i \in I} \) by \( S \). Then \( S = AI \) for some \( A > 0 \). Let \( \{e_i\}_{i \in I} \) and \( \{h_i\}_{i \in I} \) be any orthonormal bases for \( \mathcal{H} \).

Take a bounded bijective operator \( Q \) which has the property \( \|Q\| \leq \sqrt{\|S\|} = A^{1/2} \) and \( \|Q^{-1}\| \leq \sqrt{\|S^{-1}\|} = A^{-1/2} \). By Lemma 1.2 \( \{Qh_i\}_{i \in I} \) is a tight Riesz basis with bound \( A \), which implies that it has the form \( \{A^{1/2}u_i\}_{i \in I} \) for some orthonormal basis \( \{u_i\}_{i \in I} \). Thus the R-dual of \( \{f_i\}_{i \in I} \) of type III with respect to the triplet \( \{e_i\}_{i \in I}, \{h_i\}_{i \in I}, Q \) is

\[
\omega_j = \sum_{i \in I} \langle S^{-1/2}f_i, e_j \rangle Qh_i = \sum_{i \in I} \langle A^{-1/2}f_i, e_j \rangle A^{1/2}u_i = \sum_{i \in I} \langle f_i, e_j \rangle u_i, \ j \in I,
\]

which is an R-dual of type I of \( \{f_i\}_{i \in I} \).

Now consider the R-dual of \( \{f_i\}_{i \in I} \) of type I w.r.t. \( \{e_i\}_{i \in I} \) and \( \{h_i\}_{i \in I} \),

\[
\nu_j = \sum_{i \in I} \langle f_i, e_j \rangle h_i = \sum_{i \in I} \langle A^{-1/2}f_i, e_j \rangle A^{1/2}h_i = \sum_{i \in I} \langle S^{-1/2}f_i, e_j \rangle S^{1/2}h_i, \ j \in I;
\]

this is clearly an R-dual of type III of \( \{f_i\}_{i \in I} \).

For R-duals of type III we now state the analogue of the properties in Theorem 1.4 (ii) and (iv).

Proposition 4.3 Let \( \{f_i\}_{i \in I} \) be a frame sequence and \( \{\omega_i\}_{i \in I} \) an R-dual of \( \{f_i\}_{i \in I} \) of type III. Then the following hold.

(i) \( \{f_i\}_{i \in I} \) is a frame if and only if \( \{\omega_j\}_{j \in I} \) is a Riesz sequence; in the affirmative case the bounds for \( \{f_i\}_{i \in I} \) are also bounds for \( \{\omega_j\}_{j \in I} \).

(ii) \( \{f_i\}_{i \in I} \) is a Riesz sequence if and only if \( \{\omega_j\}_{j \in I} \) is a frame; in the affirmative case the bounds for \( \{f_i\}_{i \in I} \) are also bounds for \( \{\omega_j\}_{j \in I} \).

(iii) \( \{\omega_j\}_{j \in I} \) is a Riesz basis if and only if \( \{f_i\}_{i \in I} \) is a Riesz basis.

Proof. (i) Assume first that \( \{f_i\}_{i \in I} \) is a frame. Lemma 1.2 and \( (10) \) yield that for any finite scalar sequence \( \{c_j\} \),

\[
\| \sum_j c_j \omega_j \| \leq \| \sum_j c_j \sum_{i \in I} \langle f_i, S^{-1/2}e_j \rangle Qh_i \| = \| \sum_{i \in I} \langle S^{-1/2}f_i, \sum_j c_j e_j \rangle Qh_i \|
\leq \|Q\| \| \{\langle S^{-1/2}f_i, \sum_j c_j e_j \rangle\}_{i \in I} \|_\ell^2 = \sqrt{\|S\|} \| \sum_j c_j e_j \|_\mathcal{H}
= \sqrt{\|S\|} \| \{c_j\} \|_\ell^2.
\]
The lower bound is proved in the same way.

On the other hand assume that \( \{\omega_j\}_{j \in I} \) is a Riesz sequence. Then the sequence \( \{\nu_j\}_{j \in I} \) given by \( \nu_j = Q^{-1}\omega_j = \sum_{i \in I} \langle S^{-1/2} f_i, e_j \rangle h_i \) is also a Riesz sequence, which, by Theorem 4.1, implies that \( \{S^{-1/2} f_i\}_{i \in I} \) is a frame, in particular, that \( \{S^{-1/2} f_i\}_{i \in I} \) is total in \( \mathcal{H} \). Since the frame operator and its powers are bijections on \( \text{span}\{f_i\}_{i \in I} \), this implies that \( \text{span}\{f_i\}_{i \in I} = \mathcal{H} \), i.e., \( \{f_i\}_{i \in I} \) is a frame for \( \mathcal{H} \).

For the proof of (ii), assume that \( \{f_i\}_{i \in I} \) is a Riesz sequence. By Lemma 4.1, \( \{Q^{-1}\omega_j\}_{j \in I} \) is an R-dual of type I of the 1-tight Riesz sequence \( \{S^{-1/2} f_i\}_{i \in I} \), which by Theorem 4.1 implies that \( \{Q^{-1}\omega_j\}_{j \in I} \) is a tight frame for \( \mathcal{H} \), with bound 1. It follows that \( \{\omega_j\}_{j \in I} \) is a frame for \( \mathcal{H} \) with optimal bounds \( 1/\|Q^{-1}\| \geq 1/\|S^{-1}\| \), \( \|Q\| \leq \|S\| \).

Now assume that \( \{\omega_j\}_{j \in I} \) is a frame for \( \mathcal{H} \). Then the frame \( \{Q^{-1}\omega_j\}_{j \in I} \) is an R-dual of type I of \( \{S^{-1/2} f_i\}_{i \in I} \), which by Theorem 4.1 implies that \( \{S^{-1/2} f_i\}_{i \in I} \) is a Riesz sequence. Therefore, \( \{f_i\}_{i \in I} \) is a Riesz sequence.

In order to prove (iii), assume that \( \{f_i\}_{i \in I} \) is a Riesz basis for \( \mathcal{H} \). By (ii), \( \{\omega_j\}_{j \in I} \) is a Riesz sequence. It remains to prove the completeness of \( \{\omega_j\}_{j \in I} \). Let \( x \in \mathcal{H} \) and \( \langle x, \omega_j \rangle = 0 \) for every \( j \in I \). Then for every \( j \in I \),

\[
0 = \sum_{i \in I} \langle S^{-1/2} e_j, f_i \rangle \langle x, Q h_i \rangle = \langle e_j, \sum_{i \in I} \langle Q h_i, x \rangle S^{-1/2} f_i \rangle,
\]

which implies that \( \sum_{i \in I} \langle Q h_i, x \rangle S^{-1/2} f_i = 0 \). Therefore, \( \langle Q h_i, x \rangle = 0 \) for every \( i \in I \), which implies that \( x = 0 \). Thus \( \{\omega_j\}_{j \in I} \) is a Riesz basis.

Now assume that \( \{\omega_j\}_{j \in I} \) is a Riesz basis for \( \mathcal{H} \). Consider the sequence \( \{\nu_i\}_{i \in I} \) given by \( \nu_i = \sum_{j \in I} \langle \tilde{\omega}_j, Q h_i \rangle S^{-1/2} e_j \), \( i \in I \), where \( \{\tilde{\omega}_j\}_{j \in I} \) is the canonical dual of \( \{\omega_j\}_{j \in I} \). We will now show that \( \{\nu_j\}_{j \in I} \) is biorthogonal to \( \{f_i\}_{i \in I} \). Note that by (i) we know that \( \{f_i\}_{i \in I} \) is a frame. We will now use a representation of \( f_i \), to be proved in Theorem 4.7 (i). Using that \( f_i = \sum_{j \in I} \langle \omega_j, (Q^*)^{-1} h_i \rangle S^{1/2} e_j \), for every \( i, k \in I \) we have

\[
\langle \nu_i, f_k \rangle = \sum_{j \in I} \langle \tilde{\omega}_j, Q h_i \rangle \langle S^{-1/2} e_j, \sum_{\ell \in I} \langle \omega_{\ell}, (Q^*)^{-1} h_k \rangle S^{1/2} e_{\ell} \rangle = \sum_{j \in I} \langle \tilde{\omega}_j, Q h_i \rangle \sum_{\ell \in I} \langle (Q^*)^{-1} h_k, \omega_{\ell} \rangle \langle S^{-1/2} e_j, S^{1/2} e_{\ell} \rangle.
\]
Using Lemma 1.2, it now follows that
\[ \langle \nu_i, f_k \rangle = \sum_{j \in I} \langle \tilde{\omega}_j, Qh_i \rangle \langle (Q^*)^{-1}h_k, \omega_j \rangle = \langle \sum_{j \in I} \langle (Q^*)^{-1}h_k, \omega_j \rangle \tilde{\omega}_j, Qh_i \rangle = \langle \sum_{j \in I} \langle (Q^*)^{-1}h_k, \omega_j \rangle \tilde{\omega}_j, Qh_i \rangle = \langle (Q^*)^{-1}h_k, Qh_i \rangle = \delta_{k,i}. \]

The fact that \( \{ f_i \}_{i \in I} \) is a frame and has a biorthogonal sequence now implies that \( \{ f_i \}_{i \in I} \) is a Riesz basis. □

Note that in contrast with the statement for R-duals of type I (Theorem 1.4), Proposition 4.3 (i) only states that frame bounds for \( \{ f_i \}_{i \in I} \) are also bounds for \( \{ \omega_j \}_{j \in I} \). This statement can not be strengthened, as demonstrated by Example 3.2, even if \( \{ \omega_j \}_{j \in I} \) is an R-dual of type III with respect to \( Q = S^{1/2} \) which satisfies \( \| Q \| = \| S \|^{1/2} \) and \( \| Q^{-1} \| = \sqrt{\| S^{-1} \|} \). However, if \( \{ \omega_j \}_{j \in I} \) is assumed to be an R-dual of type III of \( \{ f_i \}_{i \in I} \) with respect to \( Q \), then in Proposition 4.3 (ii) and (iii), \( \{ \omega_j \}_{j \in I} \) keeps the optimal frame bounds of \( \{ f_i \}_{i \in I} \).

We now prove a characterization of R-duals of type II and III:

**Theorem 4.4** Let \( \{ f_i \}_{i \in I} \) be a frame for \( \mathcal{H} \), let \( \{ \omega_j \}_{j \in I} \) be a Riesz sequence in \( \mathcal{H} \) and assume that the bounds of \( \{ f_i \}_{i \in I} \) are also bounds for \( \{ \omega_j \}_{j \in I} \). Denote the synthesis operator for \( \{ f_i \}_{i \in I} \) by \( T \). Then the following hold.

(i) \( \{ \omega_j \}_{j \in I} \) is an R-dual of type II of \( \{ f_i \}_{i \in I} \) if and only if \( \{ S^{-1/2} \omega_j \}_{j \in I} \) is an orthonormal system and

\[ \dim(\ker T) = \dim(\text{span}\{ \omega_j \}_{j \in I}^\perp) \quad (12) \]

(ii) \( \{ \omega_j \}_{j \in I} \) is an R-dual of type III of \( \{ f_i \}_{i \in I} \) if and only if (12) holds.

(iii) The class of type I duals of \( \{ f_i \}_{i \in I} \) is contained in the class of type III duals.

**Proof.** We first prove (ii). Assume that \( \{ \omega_j \}_{j \in I} \) is an R-dual of type III of \( \{ f_i \}_{i \in I} \) with respect to some orthonormal bases and some bounded bijective operator \( Q \). By Lemma 1.1 this implies that \( \{ Q^{-1} \omega_j \}_{j \in I} \) is an R-dual of type I of \( \{ S^{-1/2} f_i \}_{i \in I} \). Since the synthesis operator for \( \{ S^{-1/2} f_i \}_{i \in I} \) equals \( S^{-1/2} T \), its kernel equals the kernel of \( T \); thus, Lemma 1.5 implies that

\[ \dim(\ker T) = \dim(\text{span}\{ Q^{-1} \omega_j \}_{j \in I}^\perp) = \dim(\text{span}\{ \omega_j \}_{j \in I}^\perp), \]
i.e., (12) holds. On the other hand, assume that (12) holds, and denote the frame operator for \( \{\omega_j\}_{j \in I} \) by \( S_\Omega \). Then \( \{S^{-1/2}_\Omega \omega_j\}_{j \in I} \) is a tight Riesz sequence, and \( \{S^{-1/2}f_i\}_{i \in I} \) is a tight frame, both of them with bound 1. Considering again the synthesis operator \( S^{-1/2}T \) for \( \{S^{-1/2}f_i\}_{i \in I} \), the assumption (12) implies that

\[
\dim(\ker (S^{-1/2}T)) = \dim(\ker T) = \dim(\text{span}\{\omega_j\}_{j \in I}) = \dim(\text{span}\{S^{-1/2}_\Omega \omega_j\}_{j \in I}).
\]

Thus, by Proposition 1.6 \( \{S^{-1/2}_\Omega \omega_j\}_{j \in I} \) is an R-dual of type I of \( \{S^{-1/2}f_i\}_{i \in I} \), i.e., there exist orthonormal bases \( \{e_i\}_{i \in I} \) and \( \{h_i\}_{i \in I} \) such that

\[
S^{-1/2}_\Omega \omega_j = \sum_{i \in I} \langle f_i, S^{-1/2}e_j \rangle h_i, \quad j \in I.
\] (13)

Now consider the extension \( \tilde{S}^{1/2}_\Omega \) of \( S^{1/2}_\Omega \) to an operator on \( \mathcal{H} \), as in Lemma 1.3. Then \( \|\tilde{S}^{1/2}_\Omega\| = \|S^{1/2}_\Omega\| \leq \|S\|^{1/2} \) and \( \|(S^{1/2}_\Omega)^{-1}\| = \|S^{-1/2}_\Omega\| \leq \|S^{-1}\|^{1/2} \).

Applying the operator \( S^{1/2}_\Omega \) to the representation (13) we get

\[
\omega_j = \sum_{i \in I} \langle f_i, S^{-1/2}e_j \rangle \tilde{S}^{1/2}_\Omega h_i,
\]

which shows that \( \{\omega_j\}_{j \in I} \) is an R-dual of type III of \( \{f_i\}_{i \in I} \).

For the proof of (i), assume that \( \{\omega_j\}_{j \in I} \) is an R-dual of \( \{f_i\}_{i \in I} \) of type II. Since R-duals of type II are special cases of the R-duals of type III, the above argument shows that (12) holds. Also, by definition \( \{S^{-1/2}_\Omega \omega_j\}_{j \in I} \) is an R-dual of type I of \( \{S^{-1/2}f_i\}_{i \in I} \), which is a tight frame with frame bound 1; thus, Theorem 1.5 implies that \( \{S^{-1/2}_\Omega \omega_j\}_{j \in I} \) is a tight Riesz sequence with bound 1, i.e., an orthonormal system.

For the proof of the other implication, if \( \{S^{-1/2}_\Omega \omega_j\}_{j \in I} \) is an orthonormal system and (12) holds, we can repeat the proof for (ii) with the operator \( S_\Omega \) replaced by \( S \), and arrive at

\[
S^{-1/2}_\Omega \omega_j = \sum_{i \in I} \langle f_i, S^{-1/2}e_j \rangle h_i, \quad j \in I.
\]

Applying the operator \( S^{1/2}_\Omega \) to this proves (ii). Finally, (iii) is a direct consequence of Lemma 1.3 and (ii). \( \square \)

Theorem 4.4 has several immediate consequences. First, we can now prove the claimed result for Gabor frames:
Corollary 4.5 Let \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) be a Gabor frame for \( L^2(\mathbb{R}) \). Then \( \{\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g\}_{m,n \in \mathbb{Z}} \) can be realized as an R-dual of type III of \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \).

Proof. Let \( T \) denote the synthesis operator associated with the frame \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \). Since \( \{\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g\}_{m,n \in \mathbb{Z}} \) is a Riesz sequence with the same frame bounds as \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \), Theorem 4.4 says that it is enough to show that
\[
\dim(\ker T) = \dim(\text{span}\{\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g\}_{m,n \in \mathbb{Z}}).
\] (14)
Since \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) is a frame for \( L^2(\mathbb{R}) \), we have \( ab \leq 1 \). If \( ab < 1 \), then \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) is an overcomplete frame, which according to Lemma 4.4 has infinite excess, i.e., \( \dim(\ker T) = \infty \). Also, \( \{\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g\}_{m,n \in \mathbb{Z}} \) is a Riesz sequence with infinite deficit, which shows that (14) holds. On the other hand, if \( ab = 1 \), then \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) is a Riesz basis, so \( \dim(\ker T) = 0 \). Also, for \( ab = 1 \), \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} = \{\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g\}_{m,n \in \mathbb{Z}} \), so clearly \( \dim(\text{span}\{\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g\}_{m,n \in \mathbb{Z}}) = 0 \). Thus (14) holds for any Gabor frame \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) and the proof is completed.

With the insight gained by Theorem 4.4 we can now provide further relations between the R-duals of type I, II, and III. First we give an easy example of a frame \( \{f_i\}_{i=1}^\infty \) with an R-dual of type I which is not an R-dual of type II, as well as a frame with an R-dual of type III which is neither an R-dual of type I nor an R-dual of type II.

Example 4.6 (i) Consider the frame \( \{f_i\}_{i=1}^\infty \) in Example 3.2 and let \( \{\nu_j\}_{j=1}^\infty \) be the R-dual of type I of \( \{f_i\}_{i=1}^\infty \) with respect to \( \{e_i\}_{i=1}^\infty \) and \( \{h_i\}_{i=1}^\infty = \{e_i\}_{i=1}^\infty \). Then \( \{\nu_j\}_{j=1}^\infty = \{e_1 + e_2, e_3, e_4, e_5, \ldots\} \). A calculation shows that \( \{S^{-1/2}\nu_j\}_{j=1}^\infty = \{\frac{1}{\sqrt{2}}e_1 + e_2, e_3, e_4, e_5, \ldots\} \), which is not orthonormal. Thus, Theorem 4.4(i) implies that \( \{\nu_j\}_{j=1}^\infty \) is not an R-dual of type II of \( \{f_i\}_{i=1}^\infty \).

(ii) Consider the frame \( \{f_i\}_{i=1}^\infty \) and the Riesz basis \( \{g_j\}_{j=1}^\infty \) in Example 1.7. Denote the frame operator of \( \{f_i\}_{i=1}^\infty \) by \( S \). Since \( S^{-1/2}g_3 = 1/\sqrt{2}g_3 \), the sequence \( \{S^{-1/2}\nu_j\}_{j=1}^\infty \) is not orthonormal, which by Theorem 4.4(i) implies that \( \{g_j\}_{j=1}^\infty \) is not an R-dual of type II of \( \{f_i\}_{i=1}^\infty \). By Theorem 4.4(ii), \( \{g_j\}_{j=1}^\infty \) is an R-dual of type III of \( \{f_i\}_{i=1}^\infty \).

If \( \{\omega_j\}_{j \in I} \) is an R-dual of type III of a frame \( \{f_i\}_{i \in I} \) with respect to orthonormal bases \( \{e_i\}_{i \in I}, \{h_i\}_{i \in I} \) and a bounded bijective operator \( Q \), there
is a natural way to define an R-dual of the canonical dual frame \( \{ S^{-1} f_i \}_{i \in I} \). In fact, the frame operator associated with \( \{ S^{-1} f_i \}_{i \in I} \) is \( S^{-1} \), and the bijective operator \( \tilde{Q} := (Q^*)^{-1} \) satisfies that

\[
||\tilde{Q}|| = ||Q^{-1}|| \leq \sqrt{||S^{-1}||}, \quad \text{and} \quad ||\tilde{Q}^{-1}|| = ||Q|| \leq \sqrt{||S^{-1}|^{-1}||}.
\]

We have already noticed that \( \{(Q^*)^{-1} h_i\}_{i \in I} \) is the canonical dual Riesz basis of \( \{ h_i \}_{i \in I} \). We call the R-dual of type III of \( \{ S^{-1} f_i \}_{i \in I} \) with respect to the orthonormal bases \( \{ e_i \}_{i \in I}, \{ h_i \}_{i \in I} \) and the operator \( (Q^*)^{-1} \) for the canonical R-dual of type III of \( \{ S^{-1} f_i \}_{i \in I} \). Specifically, it is the sequence \( \{ \gamma_j \}_{j \in I} \), where

\[
\gamma_j = \sum_{i \in I} \langle S^{-1} f_i, S^{1/2} e_j \rangle (Q^*)^{-1} h_i = \sum_{i \in I} \langle S^{-1/2} f_i, e_j \rangle (Q^*)^{-1} h_i. \tag{15}
\]

For R-duals of type III we will now prove an analogue of Proposition \ref{prop:canonical_dual}. Let \( \{ f_i \}_{i \in I} \) be a frame and \( \{ \omega_i \}_{i \in I} \) an R-dual of \( \{ f_i \}_{i \in I} \) of type III, w.r.t. orthonormal bases \( \{ e_i \}_{i \in I}, \{ h_i \}_{i \in I} \) and a bounded bijective operator \( Q \). Denote the frame operator of \( \{ f_i \}_{i \in I} \) by \( S \). Then the following hold:

(i) \( f_i = \sum_{j \in I} \langle \omega_j, (Q^*)^{-1} h_i \rangle S^{1/2} e_j, \forall i \in I \).

(ii) The R-dual of type III of \( \{ f_i \}_{i \in I} \) with respect to some orthonormal bases \( \{ e_i \}_{i \in I}, \{ h_i \}_{i \in I} \), and an operator \( Q \), is biorthogonal to the canonical R-dual of type III of \( \{ S^{-1} f_i \}_{i \in I} \).

Proof. (i) Using that \( \langle f_i, S^{-1/2} e_j \rangle = \langle S^{-1/2} f_i, e_j \rangle = \langle \omega_j, (Q^*)^{-1} e_j \rangle \), we have

\[
f_i = \sum_{j \in I} \langle f_i, S^{-1/2} e_j \rangle S^{1/2} e_j = \sum_{j \in I} \langle \omega_j, (Q^*)^{-1} e_j \rangle S^{1/2} e_j,
\]

as claimed.

(ii) Using the expression for the canonical R-dual of type III in (15),

\[
\langle \omega_j, \gamma_n \rangle = \sum_{i \in I} \langle f_i, S^{-1/2} e_j \rangle Q h_i, \sum_{k \in I} \langle S^{-1} f_k, S^{1/2} e_n \rangle (Q^*)^{-1} h_k
\]

\[
= \sum_{i \in I} \sum_{k \in I} \langle f_i, S^{-1/2} e_j \rangle \langle S^{1/2} e_n, S^{-1} f_k \rangle Q h_i, (Q^*)^{-1} h_k
\]

\[
= \sum_{i \in I} \langle f_i, S^{-1/2} e_j \rangle \langle S^{1/2} e_n, S^{-1} f_i \rangle
\]

\[
= \sum_{i \in I} \langle S^{1/2} e_n, S^{-1} f_i \rangle f_i, S^{-1/2} e_j \rangle = \langle S^{1/2} e_n, S^{-1/2} e_j \rangle = \delta_{n,j},
\]

as desired. \qed
Observe that Theorem 4.4(ii) only claims biorthogonality between a given R-dual of type III of \( \{ f_i \}_{i \in I} \) and the canonical R-dual of type III of \( \{ S^{-1} f_i \}_{i = 1}^{\infty} \). That is, in contrast with the situation for R-duals of type I, see Theorem 1.4, we do not deal with arbitrary dual frames. The following simple example shows that the biorthogonality actually might fail in that case:

**Example 4.8** We return to Example 3.2 where we considered an orthonormal basis \( \{ e_i \}_{i = 1}^{\infty} \) and the frame \( \{ f_i \}_{i = 1}^{\infty} = \{ e_1, e_1, e_2, e_3, e_4, \ldots \} \). The dual frames are exactly the frames on the form \( \{ g_i \}_{i = 1}^{\infty} = \{ \alpha e_1, (1-\alpha)e_1, e_2, e_3, e_4, \ldots \} \), for some \( \alpha \in \mathbb{C} \); the R-dual of type III of \( \{ g_i \}_{i = 1}^{\infty} \) with respect to \( \{ e_i \}_{i = 1}^{\infty} \) and \( \{ h_i \}_{i = 1}^{\infty} = \{ e_i \}_{i = 1}^{\infty} \) is \( \{ \gamma_j \}_{j = 1}^{\infty} = \{ \alpha e_1 + \frac{1-\alpha}{\sqrt{|\alpha|^2 + |1-\alpha|^2}} e_2, e_3, e_4, \ldots \} \). In Example 3.2 we found an R-dual of type III, \( \{ \omega_j \}_{j = 1}^{\infty} \), of \( \{ f_i \}_{i = 1}^{\infty} \). It is easy to see that \( \{ \omega_j \}_{j = 1}^{\infty} \) and \( \{ \gamma_j \}_{j = 1}^{\infty} \) are biorthogonal if and only if \( \alpha + \frac{1-\alpha}{\sqrt{2|\alpha|^2 + |1-\alpha|^2}} = 1 \); for \( \alpha \in \mathbb{R} \), this holds if and only if \( \alpha = 1 \) or \( \alpha = 1/2 \). Thus, in general \( \{ \omega_j \}_{j = 1}^{\infty} \) and \( \{ \gamma_j \}_{j = 1}^{\infty} \) are not biorthogonal.

For frames \( \{ f_i \}_{i \in I} \) and Riesz sequences \( \{ \omega_j \}_{j \in I} \) with exactly the same bounds, we can now show that \( \{ \omega_j \}_{j \in I} \) is an R-dual of type III of \( \{ f_i \}_{i \in I} \) if and only if \( \{ f_i \}_{i \in I} \) is an R-dual of type III of \( \{ \omega_j \}_{j \in I} \). Again, this is a property that resembles what we know for R-duals of type I from Theorem 1.4(i).

**Proposition 4.9** Let \( \{ f_i \}_{i \in I} \) be a frame and let \( \{ \omega_j \}_{j \in I} \) be a Riesz sequence with the same optimal bounds. Then \( \{ \omega_j \}_{j \in I} \) is an R-dual of type III of \( \{ f_i \}_{i \in I} \) if and only if \( \{ f_i \}_{i \in I} \) is an R-dual of type III of \( \{ \omega_j \}_{j \in I} \).

**Proof.** Denote the frame operators of \( \{ f_i \}_{i \in I} \) and \( \{ \omega_j \}_{j \in I} \) by \( S \) and \( S_{\Omega} \), respectively. First assume that \( \{ \omega_j \}_{j \in I} \) is an R-dual of type III of \( \{ f_i \}_{i \in I} \) with respect to some triplet \( \{ e_i \}_{i \in I}, \{ h_i \}_{i \in I}, Q \). It follows from the proof of Theorem 4.4(ii) that \( \{ \omega_j \}_{j \in I} \) is an R-dual of type III of \( \{ f_i \}_{i \in I} \) with respect to the triplet \( \{ e_i \}_{i \in I}, \{ h_i \}_{i \in I}, S_{\Omega}^{1/2} \), where \( S_{\Omega}^{1/2} \) is the operator defined by Lemma 1.3. Now Theorem 4.7(i) and Lemma 1.3 imply that

\[
f_i = \sum_{j \in I} \langle \omega_j, (S_{\Omega}^{1/2})^{-1} h_i \rangle S_{\Omega}^{1/2} e_j = \sum_{j \in I} \langle S_{\Omega}^{-1/2} \omega_j, h_i \rangle S_{\Omega}^{1/2} e_j, \forall i \in I.
\]

Furthermore, \( \| S_{\Omega}^{1/2} \| = \sqrt{\| S \|} = \sqrt{\| S_{\Omega} \|} \) and \( \| S_{\Omega}^{-1/2} \| = \sqrt{\| S^{-1} \|} = \sqrt{\| S_{\Omega}^{-1} \|} \), which implies that \( \{ f_i \}_{i \in I} \) is an R-dual of type III of \( \{ \omega_j \}_{j \in I} \) with respect to the triplet \( \{ h_i \}_{i \in I}, \{ e_i \}_{i \in I}, S_{\Omega}^{1/2} \).
Now assume that $\{f_i\}_{i \in I}$ is an R-dual of type III of $\{\omega_j\}_{j \in I}$. Using techniques as in the proof of Theorem 4.4(ii), one can prove that (12) holds. Now Theorem 4.4(ii) implies that $\{\omega_j\}_{j \in I}$ is an R-dual of type III of $\{f_i\}_{i \in I}$. □

Notice that when $\{f_i\}_{i \in I}$ is a frame for $\mathcal{H}$ and $\{\omega_j\}_{j \in I}$ is a Riesz sequence in $\mathcal{H}$ with the same optimal bounds, one has a "symmetry" representation: if (12) holds, then there exist orthonormal bases $\{e_i\}_{i \in I}$ and $\{h_i\}_{i \in I}$ for $\mathcal{H}$ so that $\{\omega_j\}_{j \in I}$ is the $R$-dual of type III of $\{f_i\}_{i \in I}$ with respect to the triplet $(\{e_i\}_{i \in I}, \{h_i\}_{i \in I}, \overline{S^{1/2}_\Omega})$, and $\{f_i\}_{i \in I}$ is the R-dual of type III of $\{\omega_j\}_{j \in I}$ with respect to the triplet $(\{h_i\}_{i \in I}, \{e_i\}_{i \in I}, S^{1/2})$, where $S$ and $S^{1/2}_{\Omega}$ are as in Proposition 4.9.

Acknowledgment: Diana Stoeva was supported by the Austrian Science Fund (FWF) START-project FLAME (‘Frames and Linear Operators for Acoustical Modeling and Parameter Estimation’; Y 551-N13). She is grateful for the hospitality of the Technical University of Denmark, where most of the work on the paper was done.

References

[1] Balan, R., Casazza, P., and Heil, C.: Deficits and excesses of frames. Adv. Comp. Math. 18 (2003), 93–116.

[2] Casazza, P., Kutyniok, G., and Lammers, M.: Duality principles in abstract frame theory. J. Fourier Anal. Appl. 10 no. 4, (2004), 383–408.

[3] Christensen, O.: Frames and bases. An introductory course. Birkhäuser 2008.

[4] Christensen, O., Kim, H.O., and Kim, R.Y.: On the duality principle by Casazza, Kutyniok, and Lammers. J. Fourier Anal. Appl. 17 (2011), 640–655.

[5] Christensen, O., Xiao, X. C., and Zhu, Y. C.: Characterizing R-duality in Banach spaces. Acta Mathematica Sinica, Eng. Series 29 no.1 (2013), 75–84.

[6] Daubechies, I.: Ten lectures on wavelets. SIAM, Philadelphia, 1992.
[7] Daubechies, I., Landau, H. J., and Landau, Z.: *Gabor time-frequency lattices and the Wexler-Raz identity*. J. Fourier Anal. Appl. 1 (1995), 437–478.

[8] Dutkay, D., Han, D., and Larson, D.: *A duality principle for groups*. J. Funct. Anal. **257** (2009), 1133–1143.

[9] Fan, Z., and Shen, Z.: *Dual Gramian analysis: duality principle and unitary extension principle*. Preprint, 2013.

[10] Gröchenig, K.: *Foundations of time-frequency analysis*. Birkhäuser, Boston, 2000.

[11] Janssen, A. J. E. M.: *Duality and biorthogonality for Weyl-Heisenberg frames*. J. Fourier Anal. Appl. 1 no. 4 (1995), 403–436.

[12] Ron, A. and Shen, Z.: *Weyl-Heisenberg systems and Riesz bases in \( L^2(\mathbb{R}^d) \)*. Duke Math. J. **89** (1997), 237–282.

[13] Xiao, X. M. and Zhu, Y. C.: *Duality principles of frames in Banach spaces*. Acta. Math. Sci. Ser. A. Chin. **29** (2009), 94–102.

[14] Young, R.: *An introduction to nonharmonic Fourier series*. Academic Press, New York, 1980 (revised first edition 2001).

Diana T. Stoeva  
Acoustics Research Institute  
Wohllebengasse 12-14  
Vienna A-1040, Austria  
Email: dstoeva@kfs.oeaw.ac.at

Ole Christensen  
Technical University of Denmark  
DTU Compute  
2800 Lyngby, Denmark  
Email: ochr@dtu.dk