Abstract. We consider rational double point singularities (RDPs) that are non-taut, which means that the isomorphism class is not uniquely determined from the dual graph of the minimal resolution. Such RDPs exist in characteristic 2, 3, 5. We compute the actions of Frobenius, and other inseparable morphisms, on $W_n$-valued local cohomology groups of RDPs. Then we consider RDP K3 surfaces admitting non-taut RDPs. We show that the height of the K3 surface, which is also defined in terms of the Frobenius action on $W_n$-valued cohomology groups, is related to the isomorphism class of the RDP.

1. Introduction

1.1. Non-taut rational double points. In this subsection we recall non-taut rational double points.

Rational double point singularities (RDPs for short) are the simplest normal singularities in dimension 2. The fundamental invariant of an RDP is the dual graph of the exceptional divisor of the minimal resolution, which is a Dynkin diagram. In most cases the dual graph determines the isomorphism class of the singularity (in a fixed characteristic $p \geq 0$). Such RDPs are called taut. However in some special cases there are more than one isomorphism classes, in which cases the RDPs are called non-taut.

To describe them we define, for each pair of a characteristic $p \geq 0$ and a Dynkin diagram $S$ (which is $A_N$, $D_N$, or $E_N$), a non-negative integer $r_{\text{max}}(S) = r_{\text{max}}(p, S)$ as follows:

$$r_{\text{max}}(p, S) = \begin{cases} \lfloor N/2 \rfloor - 1 & \text{if } (p, S) = (2, D_N), \\ 1 & \text{if } (p, S) = (2, E_6), \\ 3 & \text{if } (p, S) = (2, E_7), \\ 4 & \text{if } (p, S) = (2, E_8), \\ 1 & \text{if } (p, S) = (3, E_6), (3, E_7), \\ 2 & \text{if } (p, S) = (3, E_8), \\ 1 & \text{if } (p, S) = (5, E_8), \\ 0 & \text{otherwise.} \end{cases}$$
Theorem 1.1 (Artin [Art77]). Let \( p \) and \( S \) as above. Then there exist exactly \( r_{\text{max}} + 1 \) isomorphism classes of RDPs in characteristic \( p \) whose dual graph is a Dynkin diagram of type \( S \). Explicit equations are given and, when \( r_{\text{max}} > 0 \), the isomorphism classes are distinguished by the symbols \( S^r \) \((0 \leq r \leq r_{\text{max}})\), see Table 1 in Section 4.

The isomorphism classes \( S^r \) are ordered in the way that the coindex \( r \) is lower semi-continuous in families of RDPs with the same dual graph \( S \).

1.2. Inseparable maps on \( W_n \)-valued local cohomology groups of non-taut RDPs. In this paper we consider \( W_n \)-valued local cohomology groups \( H^2_{m,A}(W_n(A)) \), and their \( I \)-torsion parts \( H^2_{m,A}(W_n(A))[I] \) for ideals \( I \subset A \), of (mainly non-taut) RDPs \( A \) for some \( n \). We compute the Frobenius actions on certain classes of such cohomology groups. See Section 4.2 for precise results. The behaviors of the Frobenius actions depend heavily on the isomorphism classes of the non-taut RDP, and we can derive Theorem 1.2 from these results.

More thorough studies concerning kernels of powers of Frobenius can distinguish isomorphism classes among the same dual graph, as will be proven in a subsequent work [LMM21] by Liedtke, Martin, and the author. Also we refer to Tanaka’s work [Tan15] concerning relations of tautness of 2-dimensional singularities and other kind of Frobenius-related properties, such as \( F \)-regularity and \( F \)-purity.

We also compute (Section 4.3) the maps induced by \( \mu_p \)- or \( \alpha_p \)-quotient morphisms, which are another kind of inseparable morphisms. This is used to show Theorem 1.3.

1.3. Height of K3 surfaces with non-taut RDPs: main results. The height is an invariant of K3 surfaces in positive characteristic which takes values in \{1, 2, \ldots, 10\} \cup \{\infty\}. Among several characterizations, the most relevant to our purpose is the one by the Frobenius actions on \( W_n \)-valued cohomology groups (see Theorem 5.4). Proposition 6.5 connects this and the computations of Frobenius actions on \( W_n \)-valued local cohomology groups of RDPs, and we obtain the following result on the height of RDP K3 surfaces (by which we mean a proper surface with only RDP singularities whose minimal resolution is a K3 surface in the usual sense).

**Theorem 1.2** (see Theorem 6.6 for a detailed statement). For each Dynkin diagram \( S \) and characteristic \( p \) (with \( r_{\text{max}}(p,S) > 0 \)), we give a subsequence \((r_1, r_2, \ldots, r_l)\) of \((r_{\text{max}}(p,S), \ldots, 2, 1)\) with the following properties. Suppose an RDP K3 surface \( Y \) in characteristic \( p \) admits an RDP of type \( S^r \).

- If \( r > 0 \), then \( \text{ht}(Y) \leq l \) and \( r = r_{\text{ht}(Y)} \).
- If \( r = 0 \), then \( \text{ht}(Y) > l \).

In short, \( \text{ht}(Y) \) determines \( r \), and if \( r > 0 \) then \( r \) determines \( \text{ht}(Y) \).

If \((p, S)\) is not \((2, D_N)\) \((N \geq 8)\) nor \((2, E_8)\), then the subsequence is the entire sequence \((r_{\text{max}}(p,S), \ldots, 2, 1)\).

While the index of an RDP on an RDP K3 surface is bounded by the Picard number of the minimal resolution and hence by 22, this theorem shows the non-existence of certain RDPs (e.g. \( E_8^3 \)) in characteristic 2 with a different reason. In Section 7, we determine which non-taut RDPs are realizable on RDP K3 surfaces (Theorem 7.1).
Now suppose $\pi: X \to Y$ is a $G$-quotient morphism between RDP K3 surfaces, $G \in \{\mu_p, \alpha_p, \mathbb{Z}/p\mathbb{Z}\}$. If $G = \mu_p$ or $G = \alpha_p$, then the “dual” map $\pi': Y^{(1/p)} \to X$ is also a $G'$-quotient with $G' = \mu_p$ or $G' = \alpha_p$, and we can use the local behavior of the pullback maps by $\pi$ and $\pi'$ to relate the singularities of $X$ and $Y$ to the height of $X$ and $Y$. If $G = \mathbb{Z}/p\mathbb{Z}$, then $Y$ always have a non-taut RDP, to which we can apply Theorem 1.2. Thus we obtain the following.

**Theorem 1.3** (see Theorems 6.9 and 6.13 for a detailed statement). Let $\pi: X \to Y$ be a $G$-quotient morphism between RDP K3 surfaces, where $G \in \{\mu_p, \alpha_p, \mathbb{Z}/p\mathbb{Z}\}$. Then we have $\text{ht}(X) = \text{ht}(Y) =: h$, we determine $h$ in terms of $\text{Sing}(Y)$ and $\text{Sing}(X)$, and $h$ is always finite.

As a consequence, we prove (Corollary 6.12) that $G$-quotient of an RDP K3 surface $X$ in characteristic $p$, with $G = \mu_p$ or $G = \alpha_p$, is an RDP K3 surface if and only if $X$ is of finite height.

### 1.4. Organization of the paper

In Section 2 we recall the definition and basic properties of the rings $W_n(A)$ of truncated Witt vectors. In Section 3 we introduce morphisms between $W_n$-valued local cohomology groups and interpret them in terms of Čech cohomology groups. In Section 4 we carry out explicit computations for inseparable morphisms between RDPs. In Section 5 we recall the definition and basic properties of the height of K3 surfaces. The main results, connecting the height of K3 surfaces and the maps on $W_n$-valued local cohomology groups, will be proved in Section 6. In Sections 7 and 8 we discuss which RDPs are realizable on RDP K3 surfaces, and give examples for all possible non-taut RDPs.

### 2. Rings of truncated ($p$-typical) Witt vectors

We recall the definition and basic properties of the rings $W_n(A)$ of truncated $p$-typical Witt vectors.

Let $p$ be a prime and $A$ an $F_p$-algebra. The ring $W(A)$ of $p$-typical Witt vectors on $A$ is the set $A^n$ equipped with the ring structure satisfying, for each polynomial $P \in \mathbb{Z}[x, y]$, $P((a_0, a_1, \ldots), (b_0, b_1, \ldots)) = (P_0(a_0, b_0), P_1(a_0, b_0, a_1, b_1), \ldots)$, where $P_i \in \mathbb{Z}[x_0, \ldots, x_{i-1}, y_0, \ldots, y_{i-1}]$ is the unique collection of polynomials satisfying, for each $N \in \mathbb{N}$, $w_N(P_0(\ldots), P_1(\ldots), \ldots, P_N(\ldots)) = P(w_N(a_0, a_1, \ldots, a_N), w_N(b_0, b_1, \ldots, b_N))$, where $w_N(t_0, t_1, \ldots, t_N) := \sum_{i=0}^{N} p^i t_i^{p^{N-i}}$ is the so-called $N$-th ghost component. For example, we clearly have $(a_0) + (b_0) = (a_0 + b_0)$ and $(a_0) \cdot (b_0) = (a_0 b_0)$ on $W_1(A) \cong A$, and it follows from the equalities

\[
(a_0^p + pa_1) + (b_0^p + pb_1) = (a_0 + b_0)^p + p(a_1 + b_1) - \frac{(a_0 + b_0)^p - a_0^p - b_0^p}{p},
\]
\[
(a_0^p + pa_1) \cdot (b_0^p + pb_1) = (a_0 b_0)^p + p(a_1 b_0^p + a_0^p b_1 + pa_1 b_1)
\]

that

\[
(a_0, a_1) + (b_0, b_1) = (a_0 + b_0, a_1 + b_1 - \frac{(a_0 + b_0)^p - a_0^p - b_0^p}{p}),
\]
\[
(a_0, a_1) \cdot (b_0, b_1) = (a_0 b_0, a_1 b_0^p + a_0^p b_1 + pa_1 b_1)
\]
on $W_2(A)$.

It follows from [Haz12, 17.1.18] that, for $P(x, y) = x + y$, the $i$-th component $P_i$ of $P((a_0, a_1, \ldots), (b_0, b_1, \ldots))$ is a homogeneous polynomial of $a_0, a_1, \ldots, b_0, b_1, \ldots$ of degree $p^i$ if we declare $a_i$ and $b_i$ to be homogeneous of degree $p^i$.

$W$ is functorial: any homomorphism $f : A \to B$ of $\mathbb{F}_p$-algebras induces a morphism $f : W(A) \to W(B)$ of rings by $f(a_0, a_1, \ldots) = (f(a_0), f(a_1), \ldots)$ that is compatible with $V$ and $R$ defined below. An example is the Frobenius morphism $F : W(A) \to W(A)$ defined as $F(a_0, a_1, \ldots) = (a_0^p, a_1^p, \ldots)$.

The shift morphism, or Verschiebung, $V$ on $W(A)$ is defined as $V(a_0, a_1, \ldots) = (0, a_0, a_1, \ldots)$.

The ring of Witt vectors of length $n$ is the quotient $W_n(A) = W(A)/V^n W(A)$, hence in $W_n(A)$ only the first $n$ components $(a_0, a_1, \ldots, a_{n-1})$ are considered. The Verschiebung induces $V : W_n(A) \to W_{n+1}(A)$. The restriction morphism $R : W_n(A) \to W_{n-1}(A)$ is defined as $R(a_0, a_1, \ldots, a_{n-1}) = (a_0, a_1, \ldots, a_{n-2})$ and is a ring homomorphism. We have an exact sequence

$$0 \to W_n(A) \xrightarrow{V^{n-n'}} W_n(A) \xrightarrow{R} W_{n-n'}(A) \to 0$$

for each $0 < n' < n$.

We use the following equalities in Section 4.

Lemma 2.1. If $x \in W_n(A)$ and $y \in W_{n+m}(A)$, then $V^m(x) \cdot y = V^m(x \cdot F^m(R^m(y)))$.

Lemma 2.2. Let $A$ be an $\mathbb{F}_p$-algebra.

1. In $W_2(A)$ with $p = 2$, write $(t_1 + t_2, 0, 0, \ldots) - (t_2, 0, 0, \ldots) = (S_0(t_1, t_2), S_1(t_1, t_2), \ldots)$ with polynomials $S_i \in \mathbb{F}_2[t_1, t_2]$. Then $S_i$ is homogeneous of degree $2^i$ and we have $S_i \equiv t_1t_2^{2^i-1} \pmod{t_2^i}$.

2. In $W_3(A)$ with $p = 2$, we have

$$\begin{pmatrix} a + b + c \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a + c \\ ab \\ (a + c)^2 + (a + c)b^2 + (a^2 + ab + b^2) \end{pmatrix}.$$

3. In $W_4(A)$ with $p = 2$, we have

$$\begin{pmatrix} a + b \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} b \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ ab \\ ab(a^2 + ab + b^2) \\ ab(a^6 + ab + b^7 + ab^5 + b^6) \end{pmatrix}$$

and

$$(c_0, c_1, c_2 + d_2, c_3) - (0, 0, d_2, 0) = (c_0, c_1, c_2 + c_3 + c_2d_2).$$

4. In $W_2(A)$ with $p = 3$, we have

$$(a + b, 0) - (a, 0) - (b, 0) = (0, ab(a + b)).$$

5. In $W_2(A)$ with $p = 5$, we have

$$(a + b, 0) - (a, 0) - (b, 0) = (0, ab(a + b)(a^2 + ab + b^2)).$$

Proof. Straightforward. \qed

The closed immersion $R^* : \text{Spec } W_{n-1}(A) \to \text{Spec } W_n(A)$ is a homeomorphism if $n \geq 2$. For an $\mathbb{F}_p$-scheme $Z$ and $n \geq 1$, we define $W_n(Z)$ to be the scheme whose underlying topological space is $Z$ and whose structure sheaf is $W_n(\mathcal{O}_Z)$.
Lemma 2.3. If $Z$ is a scheme projective (resp. quasi-projective) over an algebraically closed field $k$, then $W_n(Z)$ is projective (resp. quasi-projective) over $W_n(k)$.

Proof. Since $W_n(-)$ preserves open immersions and closed immersions, it suffices to show that $W_n(\mathbb{P}^N)$ is projective. We will show that $W_n(k[x_0, \ldots, x_N])$ is a finitely generated $W_n(k)$-algebra. Indeed, it is generated by the elements $(x_i, 0, \ldots, 0)$ with $0 \leq i \leq N$ and the elements $V^j(x_0^i \ldots x_N^{i_N}, 0, \ldots, 0)$ with $0 < j < n$ and $0 \leq i_k < p^j$.

Lemma 2.4. Suppose $(A, m_A)$ is a Cohen–Macaulay local ring. Then $W_n(A)$ is also Cohen–Macaulay. More precisely, the Teichmüller lift of a regular sequence of $A$ is a regular sequence of $W_n(A)$.

In particular, $\text{Ext}^i_{W_n(A)}(M', W_n(A)) = 0$ if $\text{Supp} M' \subset \{m_A\}$ and $i < \dim A$.

Proof. The former assertion follows from Propositions 16.18, 16.19. (One can also show more generally that $b_1, \ldots, b_N$ in $W_n(A)$ is a regular sequence if $R^{n-1}(b_1), \ldots, R^{n-1}(b_N)$ is a regular sequence in $A$.)

The latter assertion is a consequence of being Cohen–Macaulay.

Also note the equality $W_n(A[x]) = W_n(A)[\frac{1}{x}]$, where $[x]$ is the Teichmüller lift.

3. $W_n$-valued local cohomology groups

Suppose $(A, m_A)$ is a Noetherian Cohen–Macaulay local $k$-algebra of dimension $d$. By Theorems 2.8 and 3.8, we have isomorphisms $\text{lim}_I \text{Ext}^d_A(A/I, A) \cong H^d_{m_A}(A)$, where the limit is taken over $m_A$-primary ideals (i.e. $\text{Supp}(A/I) \subset \{m_A\}$), and we have $H^d_{m_A}(A) = 0$ for $i < d$ and

$$H^d_{m_A}(A) \cong \text{Coker}\left( \bigoplus_{i=1}^d A[[x_1, \ldots, x_i, \ldots, x_d]]^{-1}[x_1 \ldots x_i \ldots x_d]^{-1} \right)$$

for any regular sequence $x_1, \ldots, x_d$ in $m_A$.

Lemma 3.1. Let $(A, m_A)$ be as above. Let $n \geq 1$ be an integer and $I \subset A$ an $m_A$-primary ideal. Let $J := (R^{n-1})^{-1}(I) \subset W_n(A)$, so that $W_n(A)/J \cong A/I$. Then the morphism

$$h = h_I: \text{Ext}^d_{W_n(A)}(A/I, W_n(A)) \to H^d_{m_{W_n(A)}}(W_n(A))$$

is injective, and its image is precisely the submodule $H^d_{m_{W_n(A)}}(W_n(A))[J]$ of the classes annihilated by $J$. Moreover this morphism is compatible with the inclusion of $m_A$-primary ideals $I' \subset I$ and with $V$.

Proof. Compatibility with $V$ is clear.

By replacing $A$ with $W_n(A)$, we may assume $n = 1$ (hence $J = I$).

Note that $\text{Ext}^d_A(-, A)$ is a contravariant left exact functor from the category of $A$-modules of finite length. Hence, for an inclusion $I' \subset I$, $\text{Ext}^d_A(A/I', A) \to \text{Ext}^d_A(A/I, A)$ is injective, and this implies that $h_I$ is injective.

Suppose an element of $H^d_{m_A}(A)$ has annihilator $I$ and is of the form $h(e)$ for $e \in \text{Ext}^d_A(A/I', A)$. Then $I' \subset I$. We want to show that $e$ comes from $\text{Ext}^d_A(A/I, A)$.
Take a sequence $b_1, \ldots, b_N$ generating $I$. Applying $\text{Ext}^d_A(\cdot, A)$ to the exact sequence $\bigoplus_{i=1}^N A/I' \xrightarrow{b_i} A/I' \to A/I \to 0$, we obtain an exact sequence

$$0 \to \text{Ext}^d_A(A/I, A) \to \text{Ext}^d_A(A/I', A) \xrightarrow{b_i} \bigoplus_{i} \text{Ext}^d_A(A/I', A).$$

Since $b_i e = 0$, it follows that $e$ comes from $\text{Ext}^d_A(A/I, A)$. \hfill \Box

**Convention 3.2.** For simplicity, we will write $H^d_{m,A}(W_n(A))[I]$ instead of $H^d_{m,W_n(A)}(W_n(A))[[R_n^{-1}(I)]]$.

We have morphisms

$$V: H^d_{m,A}(W_n(A)) \to H^d_{m,A}(W_{n+1}(A)),$$

$$R: H^d_{m,A}(W_n(A)) \to H^d_{m,A}(W_{n-1}(A)).$$

Suppose $B$ is another Noetherian Cohen–Macaulay local $k$-algebra of dimension $d$ and $f: A \to B$ is a local morphism (an example is the Frobenius morphism $F: A \to A$). Then we have morphisms

$$f: H^d_{m,A}(W_n(A)) \to H^d_{m,B}(W_n(B)).$$

**Lemma 3.3.** The morphisms $f$, $V$, and $R$ induce morphisms

$$f: H^d_{m,A}(W_n(A))[I] \to H^d_{m,B}(W_n(B))[IB],$$

$$V: H^d_{m,A}(W_n(A))[I] \to H^d_{m,A}(W_{n+1}(A))[I],$$

$$R: H^d_{m,A}(W_n(A))[I] \to H^d_{m,A}(W_{n-1}(A))[I].$$

**Proof.** Straightforward. \hfill \Box

For example, the Frobenius morphism induces

$$F: H^d_{m,A}(W_n(A))[I] \to H^d_{m,A}(W_n(A))[I^p].$$

**Lemma 3.4.** Let $A$ be as above, and $x_1, \ldots, x_d \in A$ a regular sequence of $A$. Then the identification

$$H^d_{m,A}(W_n(A)) \cong \text{Coker}\left( \bigoplus_{i=1}^d W_n(A[(x_1 \ldots \hat{x}_i \ldots x_d)^{-1}]) \to W_n(A[(x_1 \ldots x_d)^{-1}]) \right)$$

is compatible with the morphisms $V$ and $R$.

If $f: A \to B$ is as above, and $f(x_1), \ldots, f(x_d)$ is a regular sequence of $B$, then the same assertion holds for $f$. \hfill \Box

4. Computing morphisms on $W_n$-valued local cohomology groups on RDPs

We carry out computations of Frobenius and other inseparable morphisms on $W_n$-valued local cohomology groups on RDPs, which will be used in Section 6 to prove the main theorems. In Section 4.1 we fix notations. In Section 4.2 (Propositions 4.5, 4.6, 4.7) we compute Frobenius morphisms. The simplest is Proposition 4.7, where only $W_1(A) = A$ is needed, while the most complicated is Proposition 4.11, in which $D_N^n$ in characteristic 2 is considered and $W_n$ with unbounded $n$ is needed. In Section 4.3 (Proposition 4.11) we compute $\mu_p$- and $\alpha_p$-quotient morphisms.
4.1. Equations of RDPS.

**Convention 4.1.** For each non-taut RDP $A$, we often work under an isomorphism $A \cong k[[x, y, z]]/(f)$, where $f$ is the polynomial as in Table 1. In the case of $D^r_N$, we use two different equations. All equations are taken from [Art77], except for the equation with the term $zx^m - r$ in the case of $D^r_N$.

**Setting 4.2.** Suppose a non-taut RDP $A$ and an integer $j \geq 1$ satisfy one of the following, and define an ideal $I_j \subset A$ accordingly.

- $j = 1$, and $I_1 = m_A \subset A$ is the maximal ideal.
- $1 \leq j \leq \lfloor N/2 \rfloor - 1$, $A$ is an RDP of type $D^r_N$ in characteristic 2, and $I_j \subset A$ consists of the elements whose vanishing order at the $2j$-th component of the exceptional divisor of the minimal resolution of $A$ is at least $2j$.
- $j = 2$, $A$ is an RDP of type $E^r_8$ in characteristic 2, and $I_j \subset A$ consists of the elements whose vanishing order at the $4$-th component of the exceptional divisor of the minimal resolution of $A$ is at least $8$.

Here, in the case of $D^r_N$ or $E^r_8$, the components are ordered in a way that the 1-st component is the end of the longest branch of the Dynkin diagram, and the $i$-th component ($i \leq N - 2$ or $i \leq 5$ respectively) is the unique component of distance $i - 1$ from the 1-st component. (In the case of $D^r_4$ the longest branch is not unique, but still the 2-j-th component, $j = 1$, is well-defined.)

**Lemma 4.3.** Suppose $A$, $j$, $I_j$ are as above. Fix an isomorphism $A \cong k[[x, y, z]]/(f)$ with $f$ is as in Table 1. Then,

1. We have $I_j = (x, y^j, z)$.
2. The class $[\varepsilon]$ of $\varepsilon := x^{-1}y^{-j}z$ is a generator of the $A$-module $H^2_{m_A}(A)[I_j]$ with $\text{Ann}([\varepsilon]) = I_j$.

**Proof.** 1. Straightforward.
2. Since $A$ is Gorenstein, we have $\dim_k H^2_{m_A}(A)[I_j] = \dim_k \text{Ext}^2(A/I_j, A) = \dim_k A/I_j = j$. Hence it suffices to check that, in $\text{Coker}(A[x^{-1}] \oplus A[y^{-1}] \to A((xy)^{-1}))$, the classes $[x\varepsilon], [y^j\varepsilon], [z\varepsilon]$ are trivial and $[y^{-1}\varepsilon]$ is nontrivial. Straightforward. \hfill \square

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**Table 1. Equations of non-taut RDPS**

| $p$ | equation |
|-----|----------|
| 2   | $D^r_{2m}$ $0 \leq r \leq m - 1$ $z^2 + x^2y + xy^m + zxy^{m-r}$ |
| 2   | $D^r_{2m}$ $z^2 + x^2y + xy^m$ |
| 2   | $D^r_{2m+1}$ $0 \leq r \leq m - 1$ $z^2 + x^2y + zxy^m + zxy^{m-r}$ |
| 2   | $D^r_{2m+1}$ $z^2 + x^2y + zxy^m$ |
| 2   | $E_r^0$ $r = 1, 0$ $z^2 + x^2 + y^2z + bxyz$ $b = 1, 0$ |
| 2   | $E_r^7$ $r = 3, 2, 1, 0$ $z^2 + x^2 + y^5 + \beta$ $\beta = zxy, zy^3, zx^2y, 0$ |
| 2   | $E_r^8$ $r = 4, 3, 2, 1, 0$ $z^2 + x^2 + y^5 + \beta$ $\beta = zxy, zy^3, zy^2, zxy^3, 0$ |
| 3   | $E_r^0$ $r = 1, 0$ $z^2 + x^2 + y^4 + bx^2y^2$ $b = 1, 0$ |
| 3   | $E_r^7$ $r = 1, 0$ $z^2 + x^2 + y^3 + bx^2y^2$ $b = 1, 0$ |
| 3   | $E_r^8$ $r = 2, 1, 0$ $z^2 + x^3 + y^5 + \lambda x^2y^2$ $\lambda = 1, y, 0$ |
| 5   | $E_r^8$ $r = 1, 0$ $z^2 + x^3 + y^5 + (b/2)xy^4$ $b = 2, 0$ |
Lemma 4.4. Let $A = k[[x, y, z]]/(f)$ be a local ring such that $x, y$ is a regular sequence, and let $j \geq 1$ an integer. Let $e = x^{-1}y^{-j}z \in A[[xy]]$, $I = (x, y^j, z) \subset A$, and $e = \langle e, 0, \ldots, 0 \rangle \in H^2_{m_A}(W_n(A))$. To show that $e$ belongs to $H^2_{m_A}(W_n(A))[I]$, it suffices to show that $F(R(e)) = 0$ (in $H^2_{m_A}(W_{n-1}(A))$) and $ze = 0$.

Proof. It suffices to check $(a_0, 0, \ldots, 0) \in \text{Ann}(e)$ for $a_0 \in \{x, y^j, z\}$ and $V(b) \in \text{Ann}(e)$ for $b \in W_{n-1}(A)$. The former assertion is obvious for $a_0 = x, y^j$ and is assumed for $a_0 = z$. For the latter assertion we have $V(b) \cdot e = V(b \cdot F(R(e))) = 0$ since $F(R(e)) = 0$. \qed

4.2. Frobenius morphisms. Let $A$ be a non-taut RDP. In this section we compute the Frobenius images of certain elements of the local cohomology groups $H^2_{m_A}(W_n(A))[I_j]$, where $I_j \subset A$ are the $m_A$-primary ideals introduced in Setting 4.2.

Proposition 4.5. Let $A$ be an RDP of type $D_N^r$ in characteristic $p = 2$. Let $j \geq 1$ and $n \geq 1$ be integers and assume

- $\lfloor N/2 \rfloor \geq C_1(n, j) := 2j + (2^{n-1} - 1)(2j - 1) = 2^{n-1}(2j - 1) + 1$, and
- either $n = 1$ or $\lfloor N/2 \rfloor - r \geq C_1(n - 1, j)$.

Let $I_j$ be the ideal defined as in Setting 4.4. Then there is an element $e \in H^2_{m_A}(W_n(A))[I_j]$ satisfying the following two conditions:

- its restriction $R^{N-1}(e) \in H^2_{m_A}(A)[I_j]$ is a generator, and
- its image $F(e)$ by the Frobenius map
  $$F: H^2_{m_A}(W_n(A))[I_j] \to H^2_{m_A}(W_n(A))[I_j^{(p)}]$$

  satisfies, letting $a := \lfloor N/2 \rfloor - r - C_1(n, j)$,
  $$F(e) = \begin{cases} 
  0 & (if \ a \geq 0), \\
  V^{n-1}(e') & (if \ a < 0)
  \end{cases}$$

for some generator $e' \in H^2_{m_A}(A)[I_{-a}]$.

We will use this proposition (in the proof of Theorem 0.6) only in the following cases.

- $\lfloor N/2 \rfloor - r > 2^{n-1}$ and $j = 1$. In this case $a \geq 0$.
- $\lfloor N/2 \rfloor - r = 2^{n-1}(2j - 1)$ and $r > 0$. In this case $a = -1$.

Proof. Let $e = x^{-1}y^{-j}z$ (with respect to the equation specified below), and consider the class $e = \langle (e, 0, \ldots, 0) \rangle \in H^2_{m_A}(W_n(A))$. To show $e \in H^2_{m_A}(W_n(A))[I_j]$, it suffices (by Lemma 4.4) to check that $ze = 0$ and $F(R(e)) = 0$, and both follow from the computation of $F(e)$ below (using $ze = xyF(e)$).

We will discuss the two cases $N = 2m$ and $N = 2m + 1$ ($D_{2m}^r$ and $D_{2m+1}^r$) in a parallel way. We may assume $A \cong k[[x, y, z]]/(f)$, where

$$f = \begin{cases} 
  z^2 + x^2y + xy^m + zxy^{m-r} & (D_{2m}^r), \\
  z^2 + x^2y + zy^m + zxy^{m-r} & (D_{2m+1}^r),
  \end{cases}$$

and then we have $I_j = (x, y^j, z)$. Let

$$\lambda := y^{m-r-j}, \ \varepsilon := \frac{z}{xy^j}, \ \eta := \frac{1}{y^{2j-1}}, \ \xi := \begin{cases} 
  \frac{y^{m-2j}}{y^{2j-1}} & (D_{2m}^r), \\
  \frac{y^{m-2j}z}{x^2} & (D_{2m+1}^r),
  \end{cases}$$

then we have $I_j = (x, y^j, z)$. Let
in $A[(xy)^{-1}]$. Then we have $\varepsilon^2 + \eta + \xi + \lambda \varepsilon = 0$.

As in Lemma 2.2.1, define polynomials $S_i \in k[t_1, t_2]$ by

$$(t_1 + t_2, 0, \ldots) - (t_2, 0, \ldots) = (S_0(t_1, t_2), S_1(t_1, t_2), \ldots),$$

and let $Q_i := S_i(\xi + \lambda \varepsilon, \eta) (0 \leq i \leq n - 1)$. We claim that

$$Q_i \equiv \begin{cases} 0 & (\text{if } i < n - 1), \\ \eta^{2^{n-1}-1} \lambda \varepsilon & (\text{if } i = n - 1) \end{cases} \pmod{A[x^{-1}]}.$$ 

By Lemma 2.2.1, $Q_i$ is a linear combination of monomials $\xi^{i_1}(\lambda \varepsilon)^{i_2} \eta^{i_3}$ with $i_1 + i_2 + i_3 = 2^i$ and $(i_1, i_2, i_3) \neq (0, 0, 2^i)$. Let $c(i_1, i_2, i_3) := (m - 2j)i_1 + (m - r - 2j)i_2 + (-2j - 1)i_3$, so that $\xi^{i_1}(\lambda \varepsilon)^{i_2} \eta^{i_3} \in y^{c(i_1, i_2, i_3)}A[x^{-1}]$. We shall show that $c(i_1, i_2, i_3) \geq 0$ for all such $(i_1, i_2, i_3)$ except $(0, 1, 2^{n-1} - 1)$.

- If $i_1 \geq 1$, then
  $$c(i_1, i_2, i_3) \geq c(0, 1, 2^i - 1)$$
  $$= m - 2j - (2^i - 1)(2j - 1)$$
  $$= m - C_1(i + 1, j) \geq m - C_1(n, j),$$

  which is $\geq 0$ by assumption.

- If $i_2 \geq 1$ and $i < n - 1$ (hence $n \geq 2$), then
  $$c(i_1, i_2, i_3) \geq c(0, 1, 2^i - 1)$$
  $$= m - r - j - (2^i - 1)(2j - 1) - j$$
  $$= m - r - C_1(i + 1, j) \geq m - r - C_1(n - 1, j),$$

  which is $\geq 0$ by assumption.

- If $i_2 \geq 2$ and $i = n - 1 \geq 1$ (hence $n \geq 2$), then
  $$c(i_1, i_2, i_3) \geq c(0, 2, 2^{n-1} - 2) = 2c(0, 1, 2^{n-2} - 1),$$

  which is $\geq 0$ by the previous case.

For the remaining term $\eta^{2^{n-1}-1} \lambda \varepsilon$, which appears in $Q_{n-1}$ with coefficient 1 by Lemma 2.2.1, we have

$$\eta^{2^{n-1}-1} \lambda \varepsilon = y^{m-r-j-(2^{n-1}-1)(2j-1)} = y^{m-r-C_1(n,j)+j} = y^a+j,$$

where $a$ as in the statement. Hence $\eta^{2^{n-1}-1} \lambda \varepsilon = x^{-1} y^a z$. Therefore we have

$$F(\varepsilon, 0, \ldots, 0) = (\varepsilon^2, 0, \ldots, 0)$$

$$= (\xi + \lambda \varepsilon + \eta, 0, \ldots, 0)$$

$$\equiv (\xi + \lambda \varepsilon + \eta, 0, \ldots, 0) - (\eta, 0, \ldots, 0) \pmod{W_n(A[y^{-1}])}$$

$$= (Q_0, \ldots, Q_{n-1})$$

$$\equiv (0, \ldots, 0, x^{-1} y^a z) \pmod{W_n(A[x^{-1}])}.$$ 

If $a \geq 0$ then $x^{-1} y^a z \in A[x^{-1}]$, and if $a < 0$ then $x^{-1} y^a z$ is a generator of $H^2_{m_A}(A)[I_{-a}]$. 

**Proposition 4.6.** Let $A$ be an RDP of type $E_8^1$ (resp. $E_8^2$) in characteristic $p = 2$. Then there exists an element $e \in H^2_{m_A}(A)[I_2]$, $e \notin H^2_{m_A}(A)[m_A]$, where $I_2$ is defined as in Setting 4.2 such that $F(e)$ is a generator of $H^2_{m_A}(A)[m_A]$ (resp. $F(e) = 0$).
Proof. We may assume $A = k[[x, y, z]]/(z^2 + x^3 + y^3 + bxz^3)$, where $b = 1$ (resp. $b = 0$). Then $I_9 = (x, y^2, z)$. Let $e = [e] \in A$ with $e := x^{-1}y^{-1}z$. Since $e^2 = y^{-4}x + x^{-2}y + byz \equiv byz \pmod{A[x^{-1}] + A[y^{-1}]}$, we have $F(e) = b[ye]$, which is a generator of $H^2_{m,A}(A)[m_A]$ (resp. 0).

Proposition 4.7. Suppose a prime $p$, a Dynkin diagram $S$, and a positive integer $n$ satisfy one of the following.

1. $p = 2, S = E_7, E_8, n \leq 3$.
2. $p = 3, S = E_8, n \leq 2$.
3. $p = 2, S = E_6, n = 1$.
4. $p = 3, S = E_6, E_7, n = 1$.
5. $p = 5, S = E_8, n = 1$.

Let $A$ be an RDP in characteristic $p$ of type $S^r$, $0 \leq r \leq r_{\max}(p, S) + 1 - n$. Then there is an element $e \in H^2_{m,A}(W_n(A))[m_A]$ whose restriction $R^{n-1}(e)$ is a generator of $H^2_{m,A}(A)[m_A]$ and satisfying

$$F(e) = \begin{cases} 0 & (\text{if } r < r_{\max}(p, S) + 1 - n), \\ V^{n-1}(e') & (\text{if } r = r_{\max}(p, S) + 1 - n) \\ \end{cases}$$

for some generator $e' \in H^2_{m,A}(A)[m_A]$.

Proof. In each case we consider the class $e = [(\varepsilon, 0, \ldots, 0)], \varepsilon = x^{-1}y^{-1}z$. To show $e \in H^2_{m,A}(W_n(A))[m_A]$, it suffices (by Lemma 4.3) to check that $ze = 0$ and $F(R(e)) = 0$. The former is clear if $n = 1$ (since $z^2 \in (x, y)$), easy if $p = 2$ using the computation of $F(e)$ (since $ze = xyF(e)$), and straightforward in the remaining case ($p = 3, S = E_8, n = 2$). The latter follows from the computation of $F(e)$.

Case 11: $E_7^r$ (resp. $E_8^r$) in characteristic 2. We may assume $A = k[[x, y, z]]/(f)$, $f = z^2 + x^3 + y^3 + \beta$ (resp. $f = z^2 + x^3 + y^5 + \beta$), with $\beta$ as in Convention 4.1 (Table 1). Let

$$\varepsilon := \frac{z}{xy}, \xi := \frac{y}{x} (\text{resp. } \xi := \frac{y^3}{x^2}, \eta := \frac{x}{y^2}, \omega := \frac{\beta}{x^2y^2},)$$

so we have $\varepsilon^2 + \eta + \xi + \omega = 0$. We have

$$\omega = \varepsilon, \frac{yz}{x^2}, \frac{z}{y}, 0 \quad (\text{if } A \text{ is } E_7^r, r \geq 3, 2, 1, 0)$$

(resp. $\omega = \varepsilon, \frac{yz}{x^2}, \frac{y^2}{x}, 0 \quad (\text{if } A \text{ is } E_8^r, r = 4, 3, 2, 1, 0)$).

Suppose $n = 1$. Then $F(\varepsilon) = \varepsilon^2 = \varepsilon + \xi + \eta + \omega$, all monomials of which belong to $A[x^{-1}] \cup A[y^{-1}]$ except precisely for $\varepsilon$ in the case $r = r_{\max}$, which is equal to $\varepsilon$.

Suppose $n = 2$ and $r \leq r_{\max} - 1$. We compute

$$F(\varepsilon, 0) = (\varepsilon^2, 0) = (\xi + \eta + \omega, 0)$$

$$\equiv (\varepsilon + \eta + \omega) - (\xi, 0) - (\eta, 0) \pmod{W_2(A[x^{-1}]) + W_2(A[y^{-1}])}$$

$$= (\omega, \xi + \eta + \xi \omega + \eta \omega).$$

Then the 0-th component belongs to $A[x^{-1}]$ or $A[y^{-1}]$, and all monomials in the 1-st component belong to $A[x^{-1}] \cup A[y^{-1}]$ except precisely for $\eta \omega$ in the case $r = r_{\max} - 1$, which is equal to $\varepsilon$.\]
Suppose \( n = 3 \) and \( r \leq r_{\text{max}} - 2 \). We compute, by using Lemma 2.2 with \((a, b, c) = (\eta, \xi, \omega)\) (resp. \((a, b, c) = (\xi, \eta, \omega)\)),

\[
F(\varepsilon, 0, 0) = (\varepsilon^2, 0, 0) = (\eta + \xi + \omega, 0, 0) \\
= (\eta + \xi + \omega, \eta\omega) - (\xi, \xi\omega, 0) \quad \text{(mod } W_3(A[x^{-1}])) \\
= (\eta + \omega, \eta\xi, (\eta + \xi)\xi + (\eta + \omega)\xi^3 + (\eta^2 + \eta\omega + \omega^2)\xi^2) \\
\]

(resp. \(F(\varepsilon, 0, 0) = (\varepsilon^2, 0, 0) = (\xi + \eta + \omega, 0, 0)\))

\[
= (\xi + \eta + \omega, \xi\eta) - (\eta, \eta\omega, 0) \quad \text{(mod } W_3(A[y^{-1}])) \\
= (\xi + \omega, \xi\eta, (\xi + \omega)^3\eta + (\xi + \omega)\eta^3 + (\xi^2 + \xi\omega + \omega^2)\eta^2)) \\
\]

Then the 0-th and 1-st component belong to \(A[y^{-1}]\) (resp. \(A[x^{-1}]\)), and all monomials in the expansion of the 2-nd component belong to \(A[x^{-1}] \cup A[y^{-1}]\) except precisely for \(\eta\omega \varepsilon^2\) (resp. \(\xi\omega\eta^2\)) in the case \(r = r_{\text{max}} - 2\), which is equal to \(\varepsilon\).

**Case (2):** \(E_0^8\) in characteristic 3. We may assume \(A = k[[x, y, z]]/(f)\), \(f = -z^2 + x^3 + y^5 + \lambda x^2 y^2\), where \(\lambda = 1, y, 0\) for \(r = 2, 1, 0\) respectively. Let \(\varepsilon := \frac{x}{xy}, \eta := \frac{y}{x}, \xi := \frac{y}{x^2}\), so we have \(\varepsilon^2 = \xi + \eta + \lambda\).

Suppose \(n = 1\). Then \(F(\varepsilon) = \varepsilon^3 = \xi\varepsilon + \eta\varepsilon + \lambda\varepsilon\), all monomials of which belong to \(A[x^{-1}] \cup A[y^{-1}]\) except precisely for \(\lambda\varepsilon\) in the case \(r = r_{\text{max}}\), which is equal to \(\varepsilon\).

Suppose \(n = 2\) and \(r \leq r_{\text{max}} - 1 = 1\). We compute, by using Lemma 2.2 with \(f = 1\),

\[
F(\varepsilon, 0) = (\varepsilon^3, 0) = (\varepsilon^3, \varepsilon + \eta, \varepsilon, 0) \\
\equiv ((\xi + \lambda)\varepsilon + \eta, \varepsilon, 0) - ((\xi + \lambda)\varepsilon, 0, 0) \quad \text{(mod } W_2(A[x^{-1}]) + W_2(A[y^{-1}])) \\
= (0, \eta\xi, (\xi + \eta + \lambda)\varepsilon) \\
= (0, \eta\varepsilon(\xi + \lambda)(\xi + \eta + \lambda)^2). \\
\]

Write \(\lambda = by\), where \(b = 1, 0\) for \(r = 1, 0\). For the 1-st component, we have

\[
\varepsilon\eta(\xi + \lambda)(\eta + \xi + \lambda)^2 = \frac{z}{x^3 y^3}(x \cdot (y^3 + bx^2 y) \cdot (x^3 + y^5 + bx^2 y^3)^2) \\
= \frac{z}{x^5 y^5}(y^2 + bx^2)(2x^3 y^5 + 2bx^5 y^3 + 4x^6 + y^{10} + b^2 x^4 y^6 + 2bx^2 y^8) \\
= \frac{z}{x^9 y^6}(4bx^5 y^5 + 2x^3 y^7 + 2bx^7 y^3) + (y^2 + bx^2)(x^6 + y^{10} + b^2 x^4 y^6 + 2bx^2 y^8) \\
\equiv be \quad \text{(mod } A[x^{-1}] + A[y^{-1}]). \\
\]

where \{\ldots\} indicates that the terms inside belong to \((x^6, y^6) \subset k[[x, y]]\) and are thus negligible.

The remaining cases: We may assume \(A = k[[x, y, z]]/(f)\) with

\[
f = \begin{cases} 
  z^2 + x^3 + y^2 z + bxyz & (p, S) = (2, E_0) \\
  -z^2 + x^3 + y^4 + bx^2 y^2 & (p, S) = (3, E_0) \\
  -z^2 + x^3 + xy^3 + bx^2 y^2 & (p, S) = (3, E_7) \\
  z^2 + x^3 + y^5 + (b/2)xy^4 & (p, S) = (5, E_8) 
\end{cases}
\]

\]
for some \( b \in k \) with \( b = 0 \) if \( r = 0 \) and \( b \neq 0 \) if \( r = 1 \). Let \( \varepsilon := x^{-1}y^{-1}z \). It suffices to show that \( \varepsilon^p - \varepsilon = \eta + \xi \) for some \( \eta \in A[y^{-1}] \) and \( \xi \in A[x^{-1}] \). We take

\[
\eta := \begin{cases} 
  y^{-2}x, \\
  y^{-3}z, \\
  y^{-3}z, \\
  y^{-5}xz,
\end{cases} \quad \xi := \begin{cases} 
  x^{-2}z, \\
  x^{-3}yz, \\
  x^{-2}z, \\
  x^{-5}(y^5 + bxy^4 + (b^2/4)x^2y^3 + 2x^3)z.
\end{cases}
\]

\[ \square \]

**Remark 4.8.** A consequence of Proposition 4.5 (\( n = j = 1 \)) and Proposition 4.7 (\( n = 1 \)) is that a non-taut RDP is \( F \)-injective (i.e. the Frobenius action on \( H^2_{\text{max}}(A) \) is injective) if and only if \( r = r_{\text{max}} \). Tanaka [Tan15, Section 5.2], using Fedder’s criterion for \( F \)-purity, observed that a non-taut RDP is \( F \)-pure if and only if \( r = r_{\text{max}} \). Note that, for Gorenstein singularities, \( F \)-injectivity is equivalent to \( F \)-purity [TW18].

Finally we note the following relation between RDPs connected by partial resolutions (although we do not need it in this paper). If \( Z \) is an RDP surface with an RDP \( z \) of type \( S \) and \( S' \subset S \) is a subdiagram, then the minimal resolution \( \rho: \tilde{Z} \to Z \) of \( Z \) at \( z \) factors through the contraction \( \rho': \tilde{Z} \to Z_{S'} \) of \( S' \subset S = \text{Exc}(\rho), \) and \( Z_{S'} \) is an RDP surface. We say that \( Z_{S'} \to Z \) is the **partial resolution** corresponding to \( S' \subset S \). If \( S' \) is connected and non-empty, then \( Z_{S'} \) has a single RDP above \( z \), which is of type \( S' \).

**Convention 4.9.** We abuse the notation and say that an RDP of type \( S \) is of type \( S^0 \) if \( r_{\text{max}}(p, S) = 0 \), so that the coindex \( r \) of an RDP is always defined.

**Lemma 4.10.** Let \( S' \subset S \) be a non-empty connected subdiagram of a Dynkin diagram \( S \). Let \( Z \) be an RDP of type \( S' \) and \( Z_{S'} \to Z \) be the partial resolution corresponding to \( S' \subset S \). Suppose \( z' \) is of type \( S'' \). Then we have \( r' = \max\{0, r - (r_{\text{max}}(S) - r_{\text{max}}(S'))\} \).

In other words, we have \( r_{\text{max}}(S') - r' = r_{\text{max}}(S) - r \) if this equality is achieved by a non-negative integer \( r' \), and \( r' = 0 \) otherwise.

**Proof.** We may assume that the number of components of \( S' \) is one less than that of \( S \). If \( r_{\text{max}}(S') = 0 \) then the assertion is trivial. So we may assume \( r_{\text{max}}(S') \geq 0 \).

If \( (S, S') \) is \( (E_8, E_7) \) or \( (E_7, D_6) \), then the partial resolution is the blow-up at the closed point. In the other cases, the partial resolution is the blow-up at the ideals \((x, y^2, z), (y, z), \) or \((x, z),\) as displayed in Table 2 with respect to the equations given in Table 1. One can check that this blow-up is dominated by the thrice blow-up \( Z_3, \) where \( Z_0 := Z \) and \( Z_{i+1} := \text{Bl}_{\text{Sing}(Z_i)} Z_i, \) hence it is indeed a partial resolution. A straightforward computation proves the assertion in each case. \[ \square \]

### 4.3. \( \mu_p^* \) and \( \alpha_p \)-quotient morphisms.

**Proposition 4.11.** Suppose a prime \( p \), a group scheme \( G \), a Dynkin diagram \( S \), and a positive integer \( n \) satisfy one of the following.

1. \( p \) is arbitrary, \( G = \mu_p, S = A_{p-1}, n = 1. \)
2. \( p = 2, G = \alpha_p, S = D_{2n}, n \geq 2. \)
3. \( p = 2, G = \alpha_p, S = E_8, n = 4. \)
4. \( p = 3, G = \alpha_p, S = E_6, n = 2. \)
**Table 2. Partial resolutions of RDPs**

| p  | S    | S'   | r_{max}(S) | r_{max}(S') | equation of S' | add: |
|-----|------|------|------------|-------------|----------------|------|
| 2   | D_{2m} | D_{2m-1} | m - 1      | m - 2       | z^2 + x^2y + xy^m + \ldots | z/y |
| 2   | D_{2m+1} | D_{2m} | m - 1      | m - 1       | z^2 + x^2y + y^m + \ldots | z/y |
| 2   | E_8   | E_7   | 4          | 3           | z^2 + x^3 + y^2 + \ldots | x/y, z/y |
| 2   | E_8   | D_7   | 4          | 2           | z^2 + x^3 + y^3 + \ldots | z/x, y^2/x |
| 2   | E_7   | D_6   | 3          | 2           | z^2 + x^3 + y^3 + \ldots | x/y, z/y |
| 2   | E_7   | E_6   | 3          | 1           | z^2 + x^3 + y^2z + \ldots | z/x |
| 3   | E_8   | E_7   | 2          | 1           | -z^2 + x^3 + y^5 + \ldots | x/y, z/y |
| 3   | E_7   | E_6   | 1          | 1           | -z^2 + x^3 + xy^3 + \ldots | z/x |

(5) \( p = 5, \ G = \alpha_p, \ S = E_8, \ n = 2 \).

Let \( \pi: \text{Spec} \ B \to \text{Spec} \ A \) be a \( G \)-quotient map from a smooth point \( B \) to an RDP \( A \) of type \( S^0 \) in characteristic \( p \), with \( \text{Fix}(G) = \{m_B\} \). Then there is an element \( e \in H^2_m(A) \mathbb{A} \) whose restriction \( R^{n-1}(e) \) is a generator of \( H^2_m(A) \mathbb{A} \) and satisfying \( \pi^*(e) = V^{n-1}(e') \) for a generator \( e' \in H^2_m(B) \mathbb{A} \).

**Proof.** In each case the assumptions determine \( \text{Spec} \ B \to \text{Spec} \ A \) up to isomorphism by [Mat22, Theorem 3.3(1)]. In each case we consider the class of the form \( e = [(e, 0, \ldots, 0)], \ e = x^{-1}y^{-1}z^2 \). We can check \( e \in H^2_m(A) \mathbb{A} \) as in the beginning of the proof of Proposition 4.57.

Case (1): \( A_{p-1} \) in characteristic \( p \). We may assume \( B = k[[X, Y]] \) and \( A = k[[x, y, z]]/(z^p - xy) \) with \( x = X^p, \ y = Y, \ z = XY \). Then it is clear that \( [x^{-1}y^{-1}z^{-1}] \) and \( \pi^*([x^{-1}y^{-1}z^{-1}]) = [X^{-1}Y^{-1}] \) are generators of \( H^2_m(A) \mathbb{A} \) and \( H^2_m(B) \mathbb{A} \) respectively.

Case (2): \( D_{p}^0 \) in characteristic 2. We may assume \( B = k[[X, Y]] \) and \( A = k[[x, y, z]]/(z^2 + x^2y + xy^{2n-1}) \) with \( x = X^2, \ y = Y^2, \ z = X^2Y + XY^{2n-1} \). Let \( \varepsilon = x^{-1}y^{-1}z_2 \). We compute, by using Lemma 2.21 as in the proof of Proposition 4.55

\[
\pi^*(\varepsilon, 0, \ldots, 0) = \left( \frac{X^2Y + XY^{2n-1}}{X^2y^2}, 0, \ldots, 0 \right) = \left( \frac{1}{Y} + \frac{Y^{2n-1}}{X}, 0, \ldots, 0 \right) \\
= \left( \frac{1}{Y} + \frac{Y^{2n-1}}{X}, 0, \ldots, 0 \right) - \left( \frac{1}{Y}, 0, \ldots, 0 \right) \\
= \left( \xi_0, \ldots, \xi_{n-2}, \xi_n \right) + \frac{Y^{2n-1} - 1}{X} \left( \frac{1}{Y} \right)^{2n-1-1} \\
\equiv V^{n-1} \left( \frac{1}{XY} \right) \pmod{W_n(B[x^{-1}]) + W_n(B[y^{-1}] this is a generator of H^2_m(B)[m_B].

Case (3): \( E_8^p \) in characteristic 2. We may assume \( B = k[[X, Y]] \) and \( A = k[[x, y, z]]/(z^2 + x^2 + y^2) \) with \( x = X^2, \ y = Y^2, \ z = X^3 + Y^2 \). Let \( \varepsilon = x^{-1}y^{-1}z \).
We compute, by using Lemma 2.2(3).

\[
\pi^*(\varepsilon, 0, 0, 0) = \left( \frac{X^3 + Y^5}{X^2 Y^2}, 0, 0, 0 \right) = \left( \frac{X}{Y^2} + \frac{Y^3}{X^2}, 0, 0, 0 \right)
\equiv \left( \frac{X}{Y^2} + \frac{Y^3}{X^2}, 0, 0, 0 \right) - \left( \frac{Y^3}{X^2}, 0, 0, 0 \right)
\equiv \left( 0, \frac{Y^7 + Y^2 X^3}{X^3} - \frac{Y^3}{X^2} \right) + \frac{X^2 Y^5 + X^5}{Y^{11}}
\equiv \left( 0, \xi_1, \frac{Y^7 + Y^2 X^3}{X^3} + \frac{X}{Y^3}, \xi_3 + \eta_3 \right) - \left( 0, 0, \frac{X}{Y^3}, 0 \right)
\equiv \left( 0, \xi_1, \xi_2, \xi_3 + \eta_3 + \frac{Y^7 + Y^2 X^3}{X^3} \right)
\equiv Y^3 \left( \frac{1}{XY} \right) \pmod{W_4(B[x^{-1}]) + W_4(B[y^{-1}])},
\]

where \( \xi_i \in B[x^{-1}] \) and \( \eta_i \in B[y^{-1}] \).

Case (4): \( E_b^0 \) in characteristic 3. We may assume \( B = k[[Y, Z]] \) and \( A = k[[x, y, z]]/(x^2 + x^3 + y^2) \) with \( x = Z^2 - Y^4, \ y = Y^3, \ z = Z^3 \). We interpret the local cohomology groups using the regular sequence \( x, y \). Let \( \varepsilon = x^{-1} y^{-1} z \). We compute, by using Lemma 2.2(4) and the equality \( Z^2 = x + Y^4 \),

\[
\pi^*(\varepsilon, 0) = \left( \frac{Z^3}{x Y^3}, 0 \right) = \left( \frac{Z}{Y^3} + \frac{Z Y}{x}, 0 \right)
\equiv \left( \frac{Z}{Y^3} + \frac{Z Y}{x}, 0 \right) - \left( \frac{Z}{Y^3}, 0 \right) - \left( \frac{Z Y}{x}, 0 \right)
\equiv \left( 0, \frac{Z Y Z^3}{x Y^3} \right) - \left( 0, \frac{Z(x + Y^4)^2}{x^2 Y^5} \right) = \left( 0, \frac{Z \cdot (-x Y^4 + \{x^2 Z + Y^8 Z\})}{x^2 Y^5} \right)
\equiv \left( 0, -\frac{Z}{x Y} \right) \pmod{W_2(B[x^{-1}]) + W_2(B[y^{-1}])}
\equiv V(\beta),
\]

where \( \{\ldots\} \in (x^2, Y^8) \) are negligible and \( \beta := -x^{-1} Y^{-1} Z \). Then \( e' := [\beta] \) is a generator of \( H^2_{m_B}(B)[m_B] \) since \( \text{Ann}(\beta) = (Y, Z) = m_B \).

Case (5): \( E_b^0 \) in characteristic 5. We may assume \( B = k[[X, Z]] \) and \( A = k[[x, y, z]]/(z^2 + x^3 - y^2) \) with \( x = X^5, \ y = Y^2 + X^3, \ z = Z^5 \). We interpret the local cohomology groups using the regular sequence \( x, y \). Let \( \varepsilon = x^{-1} y^{-1} z \). We
compute, by using Lemma \[22\] and the equality \(Z^2 = y - X^3\),
\[
\pi^*(\varepsilon,0) = \left(\frac{Z^5}{X^5y}, 0\right) = \left(\frac{Z(y - 2X^3)}{X^5} + \frac{ZX}{y}, 0\right)
\]
\[
\equiv \left(\frac{Z(y - 2X^3)}{X^5} + \frac{ZX}{y}, 0\right) - \left(\frac{Z(y - 2X^3)}{X^5}, 0\right) - \left(\frac{ZX}{y}, 0\right)
\]
\[
= \left(0, \frac{Z}{X^5} \frac{Z(y - 2X^3)}{y} \frac{Zy^2}{X^5} \frac{(y - 2X^3)^2 + X^6y(y - 2X^3) + X^{12}}{X^{10}y^2}
\]
\[
= \cdots = \left(0, \frac{Z}{X^5} \frac{Z(y - 2X^3)}{y} \frac{Zy^2}{X^5} \frac{(y - 2X^3)^2 + X^6y(y - 2X^3) + X^{12}}{X^{10}y^2}
\]
\[
\equiv \left(0, -\frac{Z}{X^5} \frac{Z(y - 2X^3)}{y} \frac{Zy^2}{X^5} \frac{(y - 2X^3)^2 + X^6y(y - 2X^3) + X^{12}}{X^{10}y^2}
\]
\[
= V(\beta),
\]
where \(\{\ldots\} \in (X^{21}, y^4)\) are negligible and \(\beta := -X^{-1}y^{-1}Z\). Then \(e' := [\beta]\) is a generator of \(H^2_{\text{tg}}(B)[m_B]\) since \(\text{Ann}([\beta]) = (X, Z) = m_B\). \(\square\)

5. The height of K3 surfaces

In this section we recall the definition and properties of the height of K3 surfaces.

Recall that a K3 surface over a field \(k\) is a smooth proper surface \(Y\) satisfying \(H^1(Y, \mathcal{O}_Y) = 0\) and \(\mathcal{O}_Y^2 \cong \mathcal{O}_Y\). We have \(\dim H^2(Y, \mathcal{O}_Y) = 1\). We say that a proper surface \(Y\) is an RDP K3 surface if it has only rational double points as singularities (if any) and its minimal resolution is a K3 surface.

**Lemma 5.1.** Let \(\pi: \tilde{Y} \to Y\) be the minimal resolution of an RDP K3 surface. Then \(H^1(Y, W_n(\mathcal{O}_Y)) = 0\), and \(H^2(Y, W_n(\mathcal{O}_Y)) \to H^2(\tilde{Y}, W_n(\mathcal{O}_\tilde{Y}))\) are isomorphisms for all \(n \geq 1\).

**Proof.** Since \(Y\) has only rational singularities, we have \(R^i\pi_*\mathcal{O}_Y = 0\) for \(i > 0\). Hence \(H^1(Y, \mathcal{O}_Y) = H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = 0\), and \(H^2(Y, \mathcal{O}_Y) \to H^2(\tilde{Y}, \mathcal{O}_{\tilde{Y}})\) is an isomorphism.

For each \(0 \leq n' \leq n\), there is an exact sequence
\[
0 \to W_{n-n'}(\mathcal{O}_Y) \xrightarrow{V^{n-n'}} W_n(\mathcal{O}_Y) \xrightarrow{R^{n-n'}} W_{n'}(\mathcal{O}_Y) \to 0.
\]

Since \(H^1(Y, \mathcal{O}_Y) = 0\), it follows inductively that \(H^1(Y, W_n(\mathcal{O}_Y)) = 0\) for all \(n \geq 1\).

Hence we obtain an exact sequence
\[
0 \to H^2(Y, W_{n-n'}(\mathcal{O}_Y)) \xrightarrow{V^{n-n'}} H^2(Y, W_n(\mathcal{O}_Y)) \xrightarrow{R^{n-n'}} H^2(Y, W_{n'}(\mathcal{O}_Y)) \to 0,
\]
and this is compatible with pullbacks by \(\pi\). It follows inductively that \(H^2(Y, W_n(\mathcal{O}_Y)) \to H^2(\tilde{Y}, W_n(\mathcal{O}_Y))\) are isomorphisms for all \(n \geq 1\). \(\square\)

**Theorem 5.2** (Artin–Mazur \[AM77\, Corollary II.4.2\]). Let \(Y\) be a (smooth) K3 surface. The functor \(\Phi^2: \{\text{local Artinian } k\text{-algebras}\} \to \{\text{abelian groups}\}\) defined by \(S \mapsto \ker(H^2_{\text{et}}(Y \times S, \mathbb{G}_m) \to H^2_{\text{et}}(Y, \mathbb{G}_m))\) is pro-represented by a 1-dimensional formal group.

**Definition 5.3.** This formal group is called the formal Brauer group of \(Y\) and written \(\text{Br}(Y)\). If \(\text{char } k = p > 0\), then its height is called the (Artin–Mazur) height of \(Y\) and written \(\text{ht}(Y)\).

We define the height of an RDP K3 surface to be the height of its minimal resolution.
Here a 1-dimensional commutative formal group in characteristic $p > 0$ is said to be of height $h \in \mathbb{Z}_{>0}$ if $[p](t) = ct^p + \ldots$ for some $c \in k^*$, and of height $\infty$ if $[p](t) = 0$, where $t$ is a uniformizer and $[p]$ is the multiplication-by-$p$ map. It follows from Proposition 5.8 that $ht(Y) \in \{1, 2, \ldots, 10\} \cup \{\infty\}$.

To relate the height of a K3 surface with the properties of non-taut RDPs, we need the following characterization of the height in terms of Frobenius actions on $W_n$-valued cohomology.

**Theorem 5.4** (van der Geer–Katsura [vdGK00, Theorem 5.1]). Let $Y$ be an RDP K3 surface in positive characteristic and $n \geq 1$ an integer. Then $ht(Y) \leq n$ if and only if the Frobenius map on $H^2(Y, W_n(\mathcal{O}_Y))$ is nonzero.

**Proof.** In [vdGK00] this is stated for smooth K3 surfaces. The case of RDP K3 surfaces is reduced to the smooth case by using Lemma 5.1. □

We also recall several properties that can be used to determine or bound the height of (RDP) K3 surfaces.

**Proposition 5.5.** Suppose $\mathbb{P} = \mathbb{P}(n_0, n_1, n_2, n_3) = \text{Proj } k[x_0, x_1, x_2, x_3]$ is a 3-dimensional weighted projective space, and $Y = (f = 0) \subset \mathbb{P}$ is a hypersurface of degree $\deg(f) = \sum n_i$ that is an RDP K3 surface. Then, $Y$ is ordinary (i.e. $ht(Y) = 1$) if and only if the coefficient of $(x_0 x_1 x_2 x_3)^{p-1}$ in $f^{p-1}$ is nonzero.

**Proof.** The proof is standard and applicable to hypersurface Calabi–Yau varieties of arbitrary dimension, see for example [Har77, Proposition IV.4.21] for the 1-dimensional case. We include the proof for the reader’s convenience. Let $d = \deg(f)$. We have canonical isomorphisms

$$H^2(X, \mathcal{O}_X) \cong H^3(\mathbb{P}, f\mathcal{O}_\mathbb{P}(-d)) \cong H^3(\{(U_i)_{i \in I}, f\mathcal{O}_\mathbb{P}(-d)\}) = \text{Coker} \left( \bigoplus_{|J|=3} \Gamma(U_J, f\mathcal{O}(-d)) \rightarrow \Gamma(U_1, f\mathcal{O}(-d)) \right),$$

where $\{(U_i)_{i \in I} \mid I := \{0, 1, 2, 3\}\}$ is the standard affine covering of $\mathbb{P}$ and $U_J := \bigcap_{i \in J} U_i$ for $J \subset I$. This cokernel is 1-dimensional, generated by the class of

$$\frac{f^{p-1}}{(x_0 x_1 x_2 x_3)^{p-1}} \in \Gamma(U_1, f\mathcal{O}(-d)),$

which is nontrivial if and only if the coefficient of $(x_0 x_1 x_2 x_3)^{p-1}$ in $f^{p-1}$ is nonzero. Apply Theorem 5.4. □

**Remark 5.6.** In principle, it is possible to compute the Frobenius map on $H^2(X, W_n(\mathcal{O}_X))$ in terms of $f$.

**Theorem 5.7.** Let $Y$ be a (smooth) K3 surface. Consider the crystalline cohomology group $H^2_{\text{cycl}}(Y/W(k))$, which is an $F$-crystal. If $h = ht(Y) < \infty$, then $H^2_{\text{cycl}}(Y/W(k))$ has slopes $1 - 1/h$, $1$, and $1 + 1/h$, with respective multiplicity $h$, $22 - 2h$, and $h$. If $ht(Y) = \infty$, then it has slope 1 with multiplicity 22.

**Proof.** By [AM77, Corollary II.4.3], the Dieudonné module of $\widehat{Br}(Y)$ is isomorphic to $H^2(Y, W\mathcal{O}_Y)$. The slope spectral sequence induces an isomorphism $H^2(Y, W\mathcal{O}_Y) \cong H^2_{\text{cycl}}(Y/W(k)) \otimes_{W(k)} K_0$, where $K_0 := \text{Frac } W(k)$ and $-c_1$ denotes the slope $< 1$ part of an $F$-crystal. The assertion follows from this (see [Ill79, Section II.7.2]). □
Proposition 5.8 ([Ill79 Proposition II.5.12]). Suppose $Y$ is a (smooth) K3 surface of height $h$ with Picard number $\rho = \rho(Y) := \text{rank} \text{Pic}(Y)$. If $h < \infty$, then $\rho \leq 22 - 2h$, and if $h = \infty$, then $\rho \leq 22$.

Proof. The subspace of $H^2_{\text{crys}}(Y/W(k))$ generated by the Picard group is of slope 1 with multiplicity $\rho$, which should be at most $22 - 2h$ (resp. 22) if $h < \infty$ (resp. $h = \infty$) by Theorem 5.7.

Corollary 5.9. Suppose $Y$ is an RDP K3 surface of height $h$ with RDPs $z_i$ of type $A_{N_i}$, $D_{N_i}$, or $E_{N_i}$. If $h < \infty$, then $\sum N_i < 22 - 2h$, and if $h = \infty$, then $\sum N_i < 22$.

Proof. The exceptional curves on the minimal resolution $\tilde{Y}$ generate a negative-definite sublattice of $\text{Pic}(Y)$ of rank $\sum N_i$. Since $\text{Pic}(\tilde{Y})$ is of sign $(+1, -(\rho - 1))$, we have $\sum N_i < \rho$.

Proposition 5.10. Suppose $Y$ is an RDP K3 surface defined over a finite field $\mathbb{F}_q$. Define $a(m) \in \mathbb{Q}$ by $|Y(\mathbb{F}_{q^m})| = 1 + (q^m)^2 + a(m)q^m$. Then there exist a family $x_1, \ldots, x_u \in \overline{\mathbb{Q}}$ such that $a(m) = \sum_j x_j^m$. Let $s(j)$ be the $j$-th elementary symmetric polynomial of $x_1, \ldots, x_u$. Then $\text{ht}(Y) > n$ if and only if $s(1), \ldots, s(n) \in (p/q)\mathbb{Z}$.

Note that each $s(j)$ can be expressed as a polynomial of $a(1), \ldots, a(j)$ with coefficients in $\mathbb{Q}$, without knowing the indeterminates $x_i$ or even the number of indeterminates. First few examples are $s(1) = a(1)$, $s(2) = (a(1)^2 - a(2))/2$, and $s(3) = (a(1)^3 - 3a(1)a(2) + 2a(3))/6$.

Proof. We first consider the case $Y$ is smooth. Write $q = p^b$. Then $F^b \in \text{End}(H^2_{\text{crys}}(Y/W(\mathbb{F}_q)))$ is linear (not only semilinear), its characteristic polynomial have coefficients in $\mathbb{Z}$, and this coincides with the characteristic polynomial of the $q$-th power Frobenius on $H^2_{\text{crys}}(Y, \mathbb{Q}_l)$ for any prime $l \neq p$ ([Ill75 3.7.3]). Let $y_i$ $(1 \leq i \leq 22)$ be the eigenvalues, and let $x_i = q^{-1}y_i$. The Weil conjecture (more precisely, the Lefschetz trace formula) asserts that $x_i$ satisfy $|Y(\mathbb{F}_{q^m})| = 1 + (q^m)^2 + q^m \sum x_i^m$ for all $m \geq 1$. Then $a(m) := \sum x_i^m$ and $x_i$ satisfy the required properties. Let $S(j)$ and $s(j)$ be respectively the $j$-th elementary symmetric polynomials of $y_i$ and of $x_i$ ($S(j)$ are the coefficients of the characteristic polynomial). The $p$-adic valuations of $S(j)$ are encoded in the Newton polygon of $H^2_{\text{crys}}(Y/W(k))$, and the Newton polygon is described (Theorem 5.7) in terms of $h = \text{ht}(Y)$ as follows. Let $\text{ord}_p$ be the $p$-adic valuation.

- If $h < \infty$, then $\text{ord}_p(S(j)) \geq j((h-1)/h) \text{ord}_p(q)$ (equivalently $\text{ord}_p(s(j)) \geq -(j/h) \text{ord}_p(q)$) for all $0 \leq j \leq h$, and the equality holds for $j = h$.
- If $h = \infty$, then $\text{ord}_p(S(j)) \geq j \text{ord}_p(q)$ (equivalently $\text{ord}_p(s(j)) \geq 0$) for all $j$.

From this, it follows that $h > n$ if and only if $s(1), \ldots, s(n) \in (p/q)\mathbb{Z}$. We also observe that, for any fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$, the number of $x_i$ with negative valuation is equal to $\text{ht}(Y)$ if $\text{ht}(Y) < \infty$ and to 0 if $\text{ht}(Y) = \infty$.

Now consider an RDP K3 surface $Y$ and its minimal resolution $\pi: \tilde{Y} \to Y$. Let $E_1, \ldots, E_t \subset \tilde{Y}$ be the exceptional curves, and $x_1, \ldots, x_t$ be the eigenvalues of the subspace of $H^2_{\text{crys}}(\tilde{Y}_{\mathbb{F}_q}, \mathbb{Q}_l)$ generated by the Chern classes of $E_i$. Then all $x_i$ are roots of unity. We will show that $|\tilde{Y}(\mathbb{F}_{q^m})| = |Y(\mathbb{F}_{q^m})| + q^m \sum_{i=1}^t x_i^m$. Suppose this for the moment. Then we have $|Y(\mathbb{F}_{q^m})| = 1 + q^{2m} + q^m \sum_{i=1}^{22} x_i^m$. Let $\tilde{s}(j)$ and $s(j)$ be respectively the $j$-th elementary symmetric polynomials of $x_i$ $(1 \leq i \leq 22)$ and
singularity $y$ \quad E

The eigenvalues $x$ of field $\mathbb{K}$

We have the possibilities for $|s|$ are equal for the two families, the height predicted by the sequence $s(j)$ coincides with that by the sequence $\tilde{s}(j)$.

To show the equality on the number of rational points, it suffices to show $|Y((\mathbb{F}_{q^m})| = |Y(\mathbb{F}_{q^m})| + q^m \sum_{i=1}^{t'} x_i^m$, where $\pi': Y' \to Y$ is the blowup at one singular point $y \in Y$ and $x_i$ $(1 \leq i \leq t')$ are the eigenvalues corresponding to $\pi'$-exceptional curves. Let $k(y) = \mathbb{F}_{q^r}$. Let $z \in Y_{(y)}$ be a point above $y$ and $E_1, \ldots, E_s \subset Y_{(y)}^\prime$ be the exceptional curves above $z$. Since $z$ is a double point, the possibilities for $s$ and the components $E_1, \ldots, E_s$ are as follows.

1. $s = 1$, $E_1$ is a smooth conic (with an $k(y)$-rational point),
2. $s = 2$, $E_1$ and $E_2$ are two distinct lines.
3. $s = 1$, $E_1$ is a non-smooth conic that splits into two distinct lines over the quadratic extension $\mathbb{F}_{q^r}$ of $k(y)$.
4. $s = 1$, $E_1$ is a double line.

We have $t' = 2sr$ in case (3), and $t' = sr$ in all other cases. In cases (1), (3), and (4), the eigenvalues $x_1, \ldots, x_s$ are all $r$-th roots of unity, each appearing twice. In each case, it is straightforward to check the required equality. \hfill $\square$

**Theorem 5.11** (Ito [Ito18 Theorem 1.1]). Let $X$ be a K3 surface in characteristic $0$, having complex multiplication (CM) by a CM-field $E$, and defined over a number field $K$ containing $E$. Suppose $X$ has good reduction $X_v$ at a prime $v$ of $K$. Let $p$, $q$, and $p$ be respectively the primes of $E$, $F$, and $\mathbb{Q}$ below $v$, where $F$ is the maximal totally-real subfield of $E$.

- If $q$ splits in $E$, then $X_v$ is of height $[E_p: \mathbb{Q}_p] < \infty$.
- If $q$ does not split (in other words, if it ramifies or is inert) in $E$, then $X_v$ is supersingular (i.e., of height $\infty$).

In this paper, we use this theorem only in the following situations.

- If $X$ has Picard number 20, then $X$ has CM by $E = \mathbb{Q}(\sqrt{-\operatorname{disc} T(X)})$ (see [Huy16 Remark 3.3.10]), where $T(X) = \operatorname{Pic}(X_C)^{\perp} \subset H^2(X_C, \mathbb{Z})$ is the transcendental lattice. In this case $F = \mathbb{Q}$.

  In this case Theorem 5.11 is proved by Shimada [Shi09 Theorem 1] for all but finitely many $p$ not dividing 2 disc $\operatorname{Pic}(X)$ (for each $X$). However we use Theorem 5.11 for $p = 2$.

- If $X$ admits an automorphism acting on $H^0(X, \Omega^2)$ by a primitive $m$-th root $\zeta_m$ of unity and if $\operatorname{rank} T(X) = \phi(m)$, then $X$ has CM by the cyclotomic field $E = \mathbb{Q}(\zeta_m)$. In this case $F = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$.

  Jang [Jan16 Corollary 4.3] proved the following related result. Suppose $Y$ is a K3 surface in characteristic $p > 2$ that admits an automorphism acting on $H^0(Y, \Omega^2)$ by a primitive $m$-th root of unity, and assume $22 - \phi(m) \leq \rho(Y)$. If $p^a = 1 (\mod m)$ for some $n$ then $Y$ is supersingular, and otherwise $ht(Y)$ is equal to the order of $p$ in $(\mathbb{Z}/m\mathbb{Z})^\ast$. However we use Theorem 5.11 for $p = 2$. 

$x_i$ $(1 \leq i \leq 22)$. Then, since the number of elements with negative valuation are equal for the two families, the height predicted by the sequence $s(j)$ coincides with that by the sequence $\tilde{s}(j)$.
6. Height of K3 surfaces and gap of morphisms between RDP K3 surfaces

In this section we prove the main results of the paper: Theorems 1.2 and 1.3 (Theorems 6.5, 6.9 and 6.13).

The height of a K3 surface $Y$ is characterized by the Frobenius action on $W_n$-valued cohomology $H^2(W_n(O_Y))$. We regard this as a property of the Frobenius morphism and generalize it to a property of morphisms between RDP K3 surfaces, which we call the gap of such morphisms. Proposition 6.5 shows how to compute the gap from the action on local cohomology groups. Using this, we derive main theorems from the local computations done in Section 4.

In this section everything is in characteristic $p > 0$.

6.1. Gap of morphisms between RDP K3 surfaces.

**Definition 6.1.** Let $\pi: X \to Y$ be a morphism between RDP K3 surfaces. We define the gap $\delta(\pi)$ of $\pi$ to be the minimum integer $n \geq 0$ such that the morphism $\pi^*: H^2(Y, W_{n+1}(O_Y)) \to H^2(X, W_{n+1}(O_X))$ is nonzero if such $n$ exists, and to be $\infty$ if there is no such $n$.

**Remark 6.2.** By Theorem 5.4 we have $ht(Y) = \delta(Frob_Y) + 1$.

We have the following.

**Lemma 6.3.** Let $\pi: X \to Y$ be a morphism between RDP K3 surfaces.

1. The image of $\pi^*: H^2(Y, W_n(O_Y)) \to H^2(X, W_n(O_X))$ is equal to $V^{\delta(\pi)}(H^2(X, W_{n-\delta(\pi)}(O_X)))$ if $n > \delta(\pi)$ and to 0 if $n \leq \delta(\pi)$.

2. If $\pi': U \to X$ is another morphism between RDP K3 surfaces, then $\delta(\pi \circ \pi') = \delta(\pi) + \delta(\pi')$ (under the natural convention that $\infty + n = \infty + \infty = \infty$).

3. If $\pi$ is birational, then $\delta(\pi) = 0$.

4. More generally, if $\pi$ is dominant and separable, then $\delta(\pi) = 0$.

5. If $\pi$ is a $\mu_n$-quotient morphism, then $\delta(\pi) = 0$.

**Proof.** We will use the exact sequences in the proof of Lemma 5.1.

1. For $n \leq \delta(\pi)$, this follows from the definition of $\delta(\pi)$. Suppose $n > \delta(\pi)$. Write $\delta = \delta(\pi)$ and $\pi_n^* = \pi^*: H^2(W_n(O_Y)) \to H^2(W_n(O_X))$. Since $\pi_0^* = 0$, the morphism $\pi_n^*$ induces $\pi_n^*: H^2(W_n(O_Y)) \to H^2(W_{n-\delta}(O_X))$ from the exact sequence. We will show by induction on $n \geq \delta$ that $\pi_n^*$ is surjective. This is clear if $n = \delta$. Suppose it is true for $n - 1 (\geq \delta)$. Then since $\text{Im}(\pi_{n-1}^*) = V(H^2(W_{n-1-\delta}(O_X)))$ and $H^2(W_{n-\delta}(O_X))/V(H^2(W_{n-1-\delta}(O_X)))$ is of length 1, either $\pi_n^*$ is surjective or $\text{Im}(\pi_n^*) = V(H^2(W_{n-1-\delta}(O_X)))$. The latter would imply from the exact sequence that $\pi_{n+1}^*$ is zero, contradicting the definition of $\delta$.

2. By 1, the image of $\pi^* \circ \pi^*: H^2(Y, W_n(O_Y)) \to H^2(U, W_n(O_U))$ is

\[
\begin{cases}
0 & (n \leq \delta(\pi) + \delta(\pi')) \\
V^{\delta(\pi) + \delta(\pi')}((H^2(U, W_{n-\delta(\pi) - \delta(\pi')}(O_U))/0 & (n > \delta(\pi) + \delta(\pi')).
\end{cases}
\]

3. By 2, it suffices to consider the case where $\pi$ is the minimal resolution $\tilde{Y} \to Y$, and this follows from Lemma 5.1.

4. For a general closed point $x \in X$, the induced morphism $O_{Y, y} \to O_{X, x}$ is an isomorphism, where $y := \pi(x)$. Apply Proposition 6.5.1 to a generator of $H^2_{y}(O_{Y, y})[m_y]$ with $n = 1$. 


Given $\gamma$ of RDP K3 surfaces with suitable singularities. It turns out, surprisingly, that

$$H^2(Y,O_Y) \to H^2(Y,\pi_*O_X) = H^2(X,O_X)$$

is injective, hence nonzero. □

**Corollary 6.4.** If a morphism $\pi: X \to Y$ between RDP K3 surfaces has finite gap (e.g. if it is dominant and separable), then $ht(X) = ht(Y)$.

**Proof.** Apply Lemma 6.3(2) to $Frob_Y \circ \pi = \pi \circ Frob_X$. □

In some cases we can bound $\delta(\pi)$ using behaviors of $\pi^*$ on local cohomology groups.

**Proposition 6.5.** Let $\pi: X \to Y$ be a morphism between RDP K3 surfaces. Let $x \in X$ be a point, $y = \pi(x)$, $m_y \subset O_{Y,y}$ the maximal ideal, and $I \subset O_{Y,y}$ an $m_y$-primary ideal. Let $n \geq 1$ be an integer. Consider an element $e \in H^2_y(W_n(O_{Y,y}))[I]$ and its image by the morphism

$$\pi^*: H^2_y(W_n(O_{Y,y}))[I] \to H^2_x(W_n(O_{X,x}))[I] =: H^2_x(O_{X,x})$$

(1) Suppose $\pi^*(e) = V^{n-1}(e')$ for some generator $e' \in H^2_x(O_{X,x})[m_x]$. Then $\delta(\pi) < n$.

(2) Suppose $I = m_y$ and $R^{n-1}(e) \in H^2_y(O_{Y,y})[m_y]$ is a generator. If $\pi^*(e) = 0$, then $\delta(\pi) \geq n$.

(3) Suppose $I = m_y$, $R^{n-1}(e) \in H^2_y(O_{Y,y})[m_y]$ is a generator, and $\pi^*(e) = V^{n-1}(e')$ for some generator $e' \in H^2_x(O_{X,x})[m_x]$. Then $\delta(\pi) = n - 1$.

**Proof.** Given $I \subset O_{Y,y}$ and $n$ as in the statement, we consider the map $\gamma = \gamma_{I,n}: H^2_y(W_n(O_{Y,y}))[I] \to H^2(Y, W_n(O_Y))$ defined by

$$H^2_y(W_n(O_{Y,y}))[I] \cong Ext^2(O_{Y,y}/I, W_n(O_{Y,y})) \cong Ext^2(O_Y/I, W_n(O_Y)) \to H^2(Y, W_n(O_Y)),$$

where $I := Ker(O_Y \to O_{Y,y}/I) \subset O_Y$ (so $\text{Supp}_Y/I = \{y\}$ and $O_Y/I \cong O_{Y,y}/I$). This $\gamma = \gamma_{I,n}$ commutes with inclusions of ideals $I' \subset I$, with $V$, and with $\pi^*$ (with $\pi$ as in the statement).

If $n = 1$ and $I = m_y$, then $\gamma_{m_y,1}$ is an isomorphism, since the final map in the diagram is, by Serre duality, the dual of the isomorphism $H^0(Y, O_Y) \cong H^0(Y, k(y))$.

(1) Applying $\gamma$ to $\pi^*(e) = V^{n-1}(e')$, we obtain $\pi^*(\gamma(e)) = V^{n-1}(\gamma(e'))$. As mentioned above, $\gamma_{m_y,1}$ is an isomorphism, hence $V^{n-1}(\gamma(e'))$ is nonzero. Hence $\delta(\pi) \leq n - 1$.

(2) By applying the induction hypothesis to $R(e)$ if $n > 1$, we obtain $\delta(\pi) \geq n - 1$. Since $\gamma_{m_y,1}$ is an isomorphism, $R^{n-1}(\gamma(e)) \in H^2(O_Y)$ is a generator. Hence $H^2(W_n(O_Y))$ is generated by $\gamma(e)$ and $V(H^2(W_{n-1}(O_Y)))$, and $\pi^*$ annihilates both since $\pi^*(e) = 0$ and $\delta(\pi) \geq n - 1$.

(3) Applying (2) to $R(e)$ if $n > 1$, we obtain $\delta(\pi) \geq n - 1$. Applying (1) to $e$, we obtain $\delta(\pi) < n$. □

From this proposition we deduce bounds, or moreover exact values, of the height of RDP K3 surfaces with suitable singularities. It turns out, surprisingly, that every non-taut RDP is suitable.

**6.2. Gap of Frobenius maps and non-taut RDPs.** Combining Proposition 6.5 with the computation on Frobenius maps on local RDPs given in Section 1.2, we prove the following relation between the isomorphism class of a non-taut RDP on an RDP K3 surface and the height of the surface. This is trivially true when $r_{\text{max}}(p, S) = 0$ (recall Convention 1.3).
Theorem 6.6 (Precise form of Theorem 1.2). Let $S$ be a Dynkin diagram and $p \geq 0$ be a characteristic. Let $r_{\text{max}} = r_{\text{max}}(p, S)$ be the integer defined in Section 4. Define a subsequence $(r_1, r_2, \ldots, r_l)$ of $(r_{\text{max}}(p, S), \ldots, 2, 1)$ as follows.

- If $(p, S) = (2, D_N)$, $N \geq 8$ ($r_{\text{max}} = \lfloor N/2 \rfloor - 1$):
  - If $8 \leq N < 9$: $(r_1, r_2) = (\lfloor N/2 \rfloor - 1, \lfloor N/2 \rfloor - 2)$.
  - If $10 \leq N$: $(r_1, r_2, r_3) = (\lfloor N/2 \rfloor - 1, \lfloor N/2 \rfloor - 2, \lfloor N/2 \rfloor - 4)$.
- $(p, S) = (2, E_8)$ ($r_{\text{max}} = 4$): $(r_1, r_2, r_3) = (4, 3, 2)$.
- all other cases: $(r_1, \ldots, r_l)$ is the entire sequence $(r_{\text{max}}(p, S), \ldots, 2, 1)$.

Then we have the following. Suppose an RDP K3 surface $Y$ admits an RDP of type $S^r$.

- If $r > 0$, then $ht(Y) \leq l$ and $r = r_{ht(Y)}$.
- If $r = 0$, then $ht(Y) > l$.

Corollary 6.7. On RDP K3 surfaces in characteristic 2, RDPs of type $D_N^r$ ($r > 0$ and $\lfloor N/2 \rfloor - r \notin \{1, 2, 4\}$) and $E_8^r$ do not occur.

Proof of Theorem 6.6. Since $ht(Y) = \delta(Frob_Y) + 1$ by Remark 6.2 it suffices to compute $\delta := \delta(Frob_Y)$. We apply Proposition 6.5 to suitable elements of the local cohomology groups.

Suppose $Y$ is an RDP K3 surface in characteristic $p$ having a non-taut RDP of type $S^r$. If $(p, S^r) \neq (2, D_N^r), (2, E_8^r)$, then the assertion follows from Proposition 6.5 (1) if $r > 0$ and (2) if $r = 0$ applied to the elements $e$ given in Proposition 4.4.

Suppose $(p, S^r) = (2, E_8^4)$. By Proposition 6.5 (1) and (2) applied to the elements given in Propositions 4.4 and 4.7 respectively, we obtain $\delta \leq 0$ and $\delta \geq 3$. Contradiction.

Suppose $(p, S^r) = (2, D_N^r)$. We have $r_{\text{max}} + 1 = \lfloor N/2 \rfloor$. It suffices to show that

- the inequality $\lfloor N/2 \rfloor - r \leq 2^\delta$ holds,
- this inequality is equality if $r > 0$, and
- $\delta \leq 2$ if $r > 0$.

Let $n'$ be the (unique) non-negative integer satisfying $2^{n'-1} < \lfloor N/2 \rfloor - r \leq 2^n$. By applying Proposition 6.5 (2) to the element given in Proposition 4.3 for $(n, j) = (n', 1)$ (in which case $a \geq 0$), we obtain $\delta \leq n'$. Hence $\lfloor N/2 \rfloor - r \leq 2^{n'} \leq 2^\delta$. Suppose moreover $r > 0$. Let $(n, j)$ be the (unique) pair of positive integers with $\lfloor N/2 \rfloor - r = 2^{n-1}(2j - 1)$. By applying Proposition 6.5 (1) to the element given in Proposition 4.3 for this $(n, j)$ (in which case $a = -1$), we obtain $\delta < n$. Hence we have $2^\delta \leq 2^{n-1} \leq 2^{n-1}(2j - 1) = \lfloor N/2 \rfloor - r$, therefore $\lfloor N/2 \rfloor - r = 2^\delta$. Since $N < 22 - 2ht(Y) = 20 - 2\delta$ (Corollary 5.9) and $N \geq 2(\lfloor N/2 \rfloor - r) = 2^{\delta+1}$, we have $\delta \leq 2$. \hfill $\Box$

6.3. Gap of $\mu_p$- and $\alpha_p$-quotient morphisms. Suppose $X$ and $Y$ are RDP K3 surfaces and $\pi: X \to Y$ is a $G$-quotient morphism with $G \in \{\mu_p, \alpha_p\}$. The author proved [Mat22] Theorem 4.3 that the “dual” map $\pi': Y^{(1/p)} \to X$ is also a $G'$-quotient morphism with $G' \in \{\mu_p, \alpha_p\}$. Here, both $G' = G$ and $G' \neq G$ are possible (see [Mat22] Examples 10.2-10.4).

Definition 6.8 ([Mat22] Definition 3.4). We say that a $G$-quotient morphism $\pi: X \to Y$ between RDP K3 surfaces is maximal if there is no point $x \in X$ such that $x$ and $\pi(x)$ are both RDPs.
The author proved \cite{Mat22} Corollary 3.5 that for any $G$-quotient morphism $\pi: X \to Y$ between RDP K3 surfaces there is a maximal $G$-quotient morphism $\pi_1: X_1 \to Y_1$ between RDP K3 surfaces with a birational and $G$-equivariant morphism $X_1 \to X$. Then $Y_1 \to Y$ is also birational, and hence $\delta(\pi) = \delta(\pi_1)$ by Lemma \cite{G23}.

**Theorem 6.9** (Precise form of Theorem 1.3). Let $\pi: X \to Y$ be as above.

1. If $\pi$ is maximal (Definition \cite{G23}), then we have

   $$
   \delta(\pi) = \begin{cases}
   0 & \text{if } G = \mu_p \text{ (in which case } p \leq 7\text{ and } \text{Sing}(Y) = \frac{24}{p+1}A_{p-1}), \\
   1 & \text{if } G = \alpha_p \text{ and } (p, \text{Sing}(Y)) = (2, 2D^1_8), (3, 2E^0_8), (5, 2E^0_8), \\
   2 & \text{if } G = \alpha_p \text{ and } (p, \text{Sing}(Y)) = (2, 1D^0_8), \\
   3 & \text{if } G = \alpha_p \text{ and } (p, \text{Sing}(Y)) = (2, 1E^0_8).
   \end{cases}
   $$

   This covers all possibilities for $G$, $p$, and $\text{Sing}(Y)$ in the maximal case (\cite{Mat22} Theorem 4.6).

2. We have $\text{ht}(X) = \text{ht}(Y) = \delta(\pi) + \delta(\pi') + 1$. In particular, $X$ and $Y$ are of finite height.

**Proof.**

1. Let $y \in Y$ be a singular point. Since $\pi$ is maximal, the inverse image $\pi^{-1}(y)$ of $y$ is smooth, hence $\text{Spec} \mathcal{O}_{X\pi^{-1}(y)} \to \text{Spec} \mathcal{O}_{Y,y}$ is as in Proposition \cite{G20}. Hence we obtain $\delta(\pi)$ from Proposition \cite{G20}. For the case $G = \mu_p$, we can also use Lemma \cite{G20}.

2. Since $\text{Frob}_Y = \pi \circ \pi'$ and $\text{Frob}_X = \pi' \circ \pi^{(1/p)}$, the first assertion follows from Lemma \cite{G20}. To show the finiteness of $\delta(\pi)$ and $\delta(\pi')$, we may assume $\pi$ is maximal, and then $\pi'$ is also maximal, and we can apply (1) to $\pi$ and $\pi'$. \qed 

**Corollary 6.10.** Let $\pi: X \to Y$ be as above.

If $p = 5$, then $(G, G') \neq (\alpha_5, \alpha_5)$.

If $p = 2$ and $\pi$ is maximal, then $(G, G', \text{Sing}(X), \text{Sing}(Y)) \neq (\alpha_2, \alpha_2, 1E^0_8, 1E^0_8)$.

**Proof.** We may suppose $\pi$ is maximal. (By above, this implies that if $p = 5$ and $G = \alpha_5$ then $\text{Sing}(Y) = 2E^0_8$. Then the height of $Y$ asserted in Theorem 6.9 which is 3 or 7 respectively, contradicts Corollary 5.9. \qed

All other $(G, G', \text{Sing}(X), \text{Sing}(Y))$ is realizable (see \cite{Mat22} Examples 10.2–10.5). Hence we have the following.

**Corollary 6.11.** Suppose an RDP K3 surface $X$ in characteristic $p$ admits an action of $\mu_p$ or $\alpha_p$ whose quotient is an RDP K3 surface. Then $\text{ht}(X) \leq 6, 3, 2, 1$ for $p = 2, 3, 5, 7$ respectively, and every such positive integer is realizable.

Furthermore, we have the following criterion.

**Corollary 6.12.** Suppose $X$ is an RDP K3 surface in characteristic $p$ with a nontrivial $G$-action, $G \in \{\mu_p, \alpha_p\}$. Then,

- $\text{ht}(X) < \infty$ if and only if $X/G$ is an RDP K3 surface.
- $\text{ht}(X) = \infty$ if and only if $X/G$ is either an RDP Enriques surface or a rational surface.

**Proof.** It is known (\cite{Mat22} Proposition 4.1) that the quotient is either an RDP K3 surface, an RDP Enriques surface, or a rational surface.
We saw in Theorem 6.9 that if \( X/G \) is an RDP K3 surface then \( X \) is of finite height.

If \( X/G \) is a rational surface or an RDP Enriques surface, then \( H^2_{et}(X/G, \mathbb{Q}) \) is generated by algebraic cycles, hence so is \( H^2_{et}(X, \mathbb{Q}) \), hence \( X \) is supersingular. \( \square \)

6.4. The case of \( \mathbb{Z}/p\mathbb{Z} \)-quotients. Suppose \( \pi: X \to Y \) is a \( \mathbb{Z}/p\mathbb{Z} \)-quotient morphism between RDP K3 surfaces. One can replace \( X \) with its minimal resolution, to which the action extends, and this does not change the height of the surfaces.

Assuming \( X \) is smooth, the author determined all possible configurations of singularities on \( Y \) [Mat22, Theorem 7.3(1)]. In each case, the configuration contains a non-taut RDP with \( r > 0 \), hence by Theorem 6.6 we can determine the height of \( Y \). By Corollary 6.14 we have \( \text{ht}(X) = \text{ht}(Y) \). Hence we obtain the following.

**Theorem 6.13** (Precise form of Theorem 1.5). Let \( \pi: X \to Y \) be a \( \mathbb{Z}/p\mathbb{Z} \)-quotient morphism between RDP K3 surfaces. Then \( \text{ht}(X) \) and \( \text{ht}(Y) \) coincide, and are finite. Moreover, if \( X \) is smooth then

\[
\text{ht}(X) = \text{ht}(Y) = \begin{cases} 1 & \text{if } (p, \text{Sing}(Y)) = (2, 2D_1^4), (3, 2E_6^1), (5, 2E_8^1), \\
2 & \text{if } (p, \text{Sing}(Y)) = (2, 1D_8^2), \\
3 & \text{if } (p, \text{Sing}(Y)) = (2, 1E_6^2).
\end{cases}
\]

This covers all possibilities for \( p \) and \( \text{Sing}(Y) \) in the smooth case ([Mat22, Theorem 7.3(1)]).

**Remark 6.14.** In the case of \( \mathbb{Z}/p\mathbb{Z} \)-quotients, we do not have an equivalence as in Corollary 6.12. There is an example of an \( \mathbb{Z}/p\mathbb{Z} \)-action on an ordinary K3 surface \( X \) with rational or Enriques quotient \( Y \), at least in characteristic 2.

7. RDPs realizable on K3 surfaces

We determine which RDPs can occur on K3 surfaces.

7.1. Non-taut RDPs. In the non-taut case, Theorem 6.6 (and Corollary 6.7) and Corollary 6.9 give necessary conditions. We will show in Proposition 7.2 that \( D_{19}^8 \) in characteristic 2 is impossible. It turns out that all remaining RDPs are realizable on RDP K3 surfaces, as we will see in Section 8. Summarizing:

**Theorem 7.1.** Consider a non-taut RDP \( D_{19}^8 \) or \( E_6^8 \) in characteristic \( p > 0 \). Then it occurs on some RDP K3 surface \( Y \) in characteristic \( p \) if and only if it satisfies the following conditions.

- It does not contradict Corollary 6.12 and Proposition 7.2 (i.e. if \( p = 2 \), then it is not \( D_{19}^r \) with \( r > 0 \) and \( \lfloor N/2 \rfloor - r \notin \{1, 2, 4\} \), nor \( D_{19}^8 \), nor \( E_6^4 \)).
- \( N < 22 - 2h \) if \( r > 0 \), where \( h \) is the height predicted in Theorem 6.6. 
- \( N < 22 \) if \( r = 0 \).

**Proposition 7.2.** An RDP K3 surface in characteristic 2 cannot have an RDP of type \( D_{19}^8 \).

**Proof of Proposition 7.2.** Suppose \( z \in Y \) is an RDP of type \( D_{19}^8 \) in characteristic 2 on an RDP K3 surface \( Y \). By Theorem 6.6 \( \text{ht}(Y) = 1 \). Let \( \tilde{Y} \to Y \) be the minimal resolution. Since \( \text{ht}(\tilde{Y}) < \infty \) there exists, by [LM18, Corollary 4.2], a K3 surface \( \mathcal{X} \) over \( \text{Spec} W(k) \) with \( \mathcal{X} \otimes_{W(k)} k \cong \tilde{Y} \) and \( \text{Pic}(\mathcal{X}) \cong \text{Pic}(\tilde{Y}) \). Let \( X_K := \mathcal{X} \otimes_{W(k)} K \) be the generic fiber of \( \mathcal{X} \) over \( K := \text{Frac} W(k) \) and let \( X_C := X_K \otimes_K \mathbb{C} \) for any
embedding $K \to \mathbb{C}$ (which we may assume to exist by replacing $k$). Then we have $\text{Pic}(X_C) \cong \text{Pic}(X') \cong \text{Pic}(Y)$.

Let $L_1$ be the sublattice of $\text{Pic}(Y) \cong \text{Pic}(X_C)$ generated by the exceptional curves above $z$, and $L_2 := L_1^\perp$ be its orthogonal complement. Since $L_1$ is negative definite, $L_2$ is nonzero, and since $\rho(Y) \leq 22 - 2 \text{ht}(Y) = 20$ (Proposition 5.3), we have rank $L_2 = 1$. The transcendental lattice $T = T(X_C) = (\text{Pic}(X_C))^\perp$ in $H^2(X_C, \mathbb{Z})$ of $X_C$ is a rank 2 positive definite lattice, and then $X_C$ has complex multiplication by the imaginary quadratic field $E := \mathbb{Q}(\sqrt{-d})$, $d := \text{disc}(T(X_C))$. By Theorem 5.11 the reduction $\tilde{Y}$ of $Y$ at a prime above 2 being ordinary implies that 2 is split in $E/\mathbb{Q}$. Writing $d = k^2d_0$ with $d_0$ square-free, this means $d_0 \equiv -1 \pmod{8}$. By Lemma 7.3 this is impossible. \hfill \square

**Lemma 7.3.** Suppose $L_1$, $L_2$, and $L_3$ are lattices with
\begin{itemize}
  \item $\text{disc}(L_1) = -4^x$ for some non-negative integer $x$,
  \item $L_2$ is positive definite, rank$(L_2) = 1$,
  \item $L_3$ is positive definite, rank$(L_3) = 2$, disc$(L_3) = k^2d_0$ with $d_0$ square-free and $d_0 \equiv -1 \pmod{8}$.
\end{itemize}

Then $L_1 \oplus L_2 \oplus L_3$ does not admit a unimodular overlattice of finite index.

A non-degenerate lattice $L$ is called unimodular if the natural injection $L \hookrightarrow L^* := \text{Hom}(L, \mathbb{Z})$ is an isomorphism, equivalently if disc$(L) = \pm 1$.

**Proof.** Suppose $L_1 \oplus L_2 \oplus L_3$ admits a finite index overlattice $\Lambda$ with disc$(\Lambda) = \pm 1$. Take bases $e_2$ of $L_2$ and $t_1, t_2$ of $L_3$, and let $(m)$ and \( \begin{pmatrix} a & b \\ b & c \end{pmatrix} \) be the Gram matrices (so $m > 0$, $a > 0$, and disc$(L_3) = k^2d_0 = ac - b^2 > 0$). Since $L_1 \oplus L_2 \oplus L_3 \subset \Lambda$ is finite index, its discriminant disc$(L_1 \oplus L_2 \oplus L_3) = \text{disc}(L_1) \cdot \text{disc}(L_2) \cdot \text{disc}(L_3) = -4^x.mk^2d_0$ coincides with disc$(\Lambda) = \pm 1$ up to a square. Hence $m = n^2d_0$.

Let $g = \text{gcd}(a, b, c)$. Then the discriminant group of $L_3$ is isomorphic to $\mathbb{Z}/g\mathbb{Z} \times \mathbb{Z}/gh\mathbb{Z}$, where $h = g^{-2}\text{disc}(L_3) \in \mathbb{Z}$. By lattice theory this is isomorphic to the discriminant group of the primitive closure of $L_1 \oplus L_2$ in $\Lambda$, which is a subquotient of the discriminant group of $L_1 \oplus L_2$. Hence $g$ is a power of 2.

We claim that $a$ is the norm of some ideal of $O_E$, where $E = \mathbb{Q}(\sqrt{-d_0})$. It suffices to show that ord$_l(a)$ is even for any prime $l$ that is inert in $E/\mathbb{Q}$. Suppose ord$_l(a) = 2j - 1$ and $l$ is inert (then $l \neq 2$ since $-d_0 \equiv 1 \pmod{8}$). Then, since $ac = b^2 + k^2d_0$ and since $l$ is inert, we have ord$_l(k^2d_0) \geq 2j$ and ord$_l(b^2) \geq 2j$, hence $l \mid c$, hence $l \mid g$, hence $l = 2$. Contradiction.

Since $\Lambda$ is unimodular, there is an element $v \in \Lambda$ with $e_2 \cdot v = 1$. Write $2^n = v_1 + v_2 + v_3$ with $v_i \in L_i \otimes \mathbb{Q}$, then $v_i \in 2^nL_i^\ast$. We have $v_2 = (2^n/m)e_2$ and hence $v_2^2 = (2^n/m)^2m = 4^x/d_0$. We have $v_1^2 \in \mathbb{Z}$ (since $v_1 \in 2^nL_1^\ast$ and $L_1^\ast \subset \text{disc}(L_1)^{-1}L_1 = 4^{-x}L_1$). We have $\sum v_i^2 = (2^n)^2 = \sum v_i^2 \in \mathbb{Z}$. Hence we obtain $v_i^2 \equiv -4^x/d_0 \pmod{d_0\mathbb{Z}}$, hence $d_0v_i^2 \equiv -4^x \pmod{d_0\mathbb{Z}}$.

Write $v_i = x_1t_1 + x_2t_2$ ($x_i \in \mathbb{Q}$). Then $d_0 = N_{E/\mathbb{Q}}(\sqrt{-d_0})$ and $av_i^2 = a(x_1^2 + 2bx_1x_2 + cx_2^2) = N_{E/\mathbb{Q}}(ax_1 + (b + \sqrt{-k^2d_0})x_2)$ are the norms of elements of $E$. Hence $d_0v_i^2 = d_0 \cdot a^{-1}$, $av_i^2$ is the norm of a fractional ideal of $E$. Therefore $-4^x$ and hence $-1$ are norm residues modulo $d_0$. But $-1$ cannot be a norm residue of an imaginary quadratic field. Contradiction. \hfill \square

### 7.2. Taut RDPs

For the taut case we have the following, which is almost done by Shimada and Shimada-Zhang.
Theorem 7.4. Suppose \( p \geq 0 \). Suppose \( S \) is a Dynkin diagram \((A_N, D_N, E_N)\) for which RDPs of type \( S \) in characteristic \( p \) are taut. Then such an RDP occurs on some RDP K3 surface \( Y \) in characteristic \( p \) if and only if \( p \) satisfies the following respective conditions.

- If \( N \leq 19 \): any \( p \geq 0 \).
- If \( S \) is \( A_20 \): \( p > 0 \) and \( p \) is non-split in \( \mathbb{Q}(\sqrt{21}) \). Equivalently, either \( p \mid 21 \) or \( p \equiv \pm 2, \pm 8, \pm 10 \pmod{21} \).
- If \( S \) is \( A_21 \): \( p = 11 \).
- If \( S \) is \( D_20 \) or \( D_21 \), or \( N \geq 22 \): no \( p \).

Proof. Suppose \( N \leq 19 \) and \( p \neq 2 \). It is known that there exists an elliptic K3 surface with a section and a singular fiber of type \( I_20 \). Then the union of a section and this singular fiber contains a configuration of type \( S \).

Suppose \( N \leq 20 \) and \( p = 2 \). Then \( S \) is a subset of \( D_21 \), which is realized by Theorem 7.3 (Example 8.1).

Suppose \( S \) is \( A_20 \) and \( p \neq 2 \). If \( p \nmid 2 \text{disc}(A_{20}) = 2 \cdot 3 \cdot 7 \), then by [SZ15, Table 1] this is possible if and only if \( \left( \frac{2}{p} \right) = -1 \). If \( p = 7 \), then this is possible by [Shi04, Table RDP]. It remains to show that it is possible if \( p = 3 \). Let \( L \) be the Dynkin lattice of type \( A_{20} \) and \( T \) be the lattice of rank 2 with basis \( t_1, t_2 \) and Gram matrix \( \left( \begin{array}{cc} 2 & 5 \\ 5 & 2 \end{array} \right) \).

Let \( e_1, e_2, \ldots, e_{20} \) be a basis of \( A_{20} \) with \( e_i \cdot e_j = -2, 1, 0 \) if \( |i - j| = 0, 1 \), respectively. We have \( L^*/L \simeq \mathbb{Z}/21\mathbb{Z} \) and \( T^*/T \simeq \mathbb{Z}/21\mathbb{Z} \).

Let \( l = \frac{1}{2} \sum_{i=1}^{20} e_i \in L^* \) and \( t = \frac{1}{2} (t_1 + t_2) \in T^* \). They generate the prime-to-3 parts of \( L^*/L \) and \( T^*/T \) respectively. We have \( t^2 \equiv -l^2 \pmod{22} \) since \( l^2 + t^2 = \left( \frac{1}{2} \right)^2 \cdot (20 \cdot (20 + 1)) + \left( \frac{1}{2} \right)^2 \cdot 14 = -4 \pmod{22} \). We can apply Lemma 7.5 below.

Suppose \( S \) is \( A_21 \). Then \( Y \) is supersingular and, considering the Picard lattice, we must have \( p \mid 22 \). By [Shi04, Table RDP], this is possible for \( p = 11 \) and impossible for \( p = 2 \).

Suppose \( S \) is \( D_{20} \). Since \( \text{disc}(S) = 4 \) is a square, an RDP of type \( S \) can be realized only in characteristic \( p \) dividing \( \text{disc}(S) \) by [DK09, Lemma 3.2], that is, \( p = 2 \). In this case \( S \) is non-taut and is out of the scope of this theorem. \( \square \)

For a finite abelian group \( A \), we write its \( p \)-primary part (resp. prime-to-\( p \) part) by \( A_p \) (resp. \( A_{p'} \)). For a non-degenerate even lattice \( L \), we define a quadratic map \( q_L: L^*/L \to \mathbb{Q}/2\mathbb{Z} \) by \( q_L(\bar{v}) = v^2 \pmod{2\mathbb{Z}} \), where the bar denotes the projection \( L^* \to L^*/L \). The next lemma is a variant of [SZ15, Proposition 2.6].

Lemma 7.5. Let \( p \) be an odd prime. Let \( R \) be a formal finite sum of \( A_N, D_N, E_N \), with \( \sum N = 20 \), and let \( L = L(R) \) be the corresponding lattice (of rank 20). Suppose there are an even lattice \( T \) of sign \((+1, -1)\) and a group isomorphism \( \phi: (L^*/L)_{p'} \to (T^*/T)_{p'} \) satisfying \( \phi^*(q_T(\bar{T}^*/T)_{p'}) = -q_L(\bar{(L^*/L)}_{p'}) \) and \( (L^*/L)_{p} \oplus \)

\(^1\)The table is contained only in the preprint version available at Shimada’s website.
(T^*/T)_p \cong (\mathbb{Z}/p\mathbb{Z})^2$. Then there exists an RDP K3 surface $Y$ (supersingular of Artin invariant 1) with $\text{Sing}(Y) = R$.

**Proof.** Let $\Lambda$ be the submodule of $L^* \oplus T^*$ consisting of the elements $(l,t)$ with $l \in (L^*/L)_p'$, $t \in (T^*/T)_p'$, and $\phi(l) = t$. Then $\Lambda$ is an even overlattice of $L \oplus T$ of sign $(+1, -21)$ with $\Lambda^*/\Lambda \cong (\mathbb{Z}/p\mathbb{Z})^2$. This means that $\Lambda$ is isomorphic to the Picard lattice of a supersingular K3 surface of Artin invariant 1. By the argument of [SZ15] Theorem 2.1, $(3) \implies (1)$, we obtain a supersingular RDP K3 surface $Y$ of Artin invariant 1 with $\text{Sing}(Y) = R$. \hfill \Box

8. **Examples**

Examples of maximal $G$-quotient morphisms $X \to Y$ between RDP K3 surfaces with all possible $(G, G', \text{Sing}(X), \text{Sing}(Y))$ are already given in [Mat22] Examples 10.2–10.5.

The non-taut RDPs in Examples 8.1–8.3, together with their partial resolutions, prove the existence part of Theorem 6.6. Let $\tilde{Q}$ and $\tilde{H}$ be elliptic curves, ordinary and supersingular respectively. In Examples 8.1, 8.2, and 8.3, supersingular RDP K3 surfaces with $D_{18}'$ and one of type $E_8$. The respective contractions give RDP K3 surfaces with $D_{18}'$ and $E_8'$. By Theorem 6.6, we have $r = r_{\text{max}}(2, D_{18}) = 8$ and $r' = r_{\text{max}}(2, E_8) = 4$.

**Example 8.1** ($p = 2$).

- Schütt [Sch00 Section 6.2] gave an example of an elliptic K3 surface $y^2 + txy + t^6 y = x^3 + (c^2 t^4 + at^3 + \hat{a}_5) x^2 + ct^8 x + t^{10} \hat{a}_6,$ where $\hat{a}_5, a \in [t]$ is of degree $\leq 2$, with a section and a singular fiber of type $\text{I}_3$. It is of height 1 since the coefficient of $txy$ is nonzero (Proposition 5.5 applied to $\mathbb{P}(6,4,1,1)$). The union of a section and this singular fiber contains a configuration of type $D_{18}'$ and one of type $E_8$. The respective contractions give RDP K3 surfaces with $D_{18}'$ and $E_8'$. By Theorem 6.6, we have $r = r_{\text{max}}(2, D_{18}) = 8$ and $r' = r_{\text{max}}(2, E_8) = 4$.

- Let $E_1$ and $E_2$ be elliptic curves, ordinary and supersingular respectively. Let $X = \text{Km}(E_1 \times E_2)$, i.e. $X$ is the minimal resolution of $(E_1 \times E_2)/\{\pm 1\}$. By [Shi74 Section 6(b)] and [Art75 Examples], $\text{Sing}(X)$ is $2D_8'$. Hence by Theorem 6.6, $\text{lt}(X) = 2$. Consider the elliptic fibrations $f_j : X = \text{Km}(E_1 \times E_2) \to (E_1 \times E_2)/\{\pm 1\} \to E_8/\{\pm 1\} \cong \mathbb{P}^1$. They admit sections and, by [Shi74 Section 4], the singular fibers of $f_1$ and $f_2$ are $2\text{I}_1$ and $2\text{I}_1^*$, respectively. In either case, the union of a section and the singular fiber(s) contains a configuration of type $D_{17}$ and one of type $E_8$.

- Let $X$ be the elliptic RDP K3 surface $y^2 + y^2 x^2 + x^3 + t^5 = 0$, $y^2 + yx^2 + x^3 + u^2 = 0$ (u = t−1, x′ = t−4x, y′ = t−6y). The singular fibers of its minimal resolution are $2\text{I}_1$ and $2\text{I}_1^*$, hence the union of a section and the singular fibers contains a configuration of type $D_{15}$ and one of type $E_8$. We have two proofs for $\text{lt}(X) = 3$. (1) In this case, it is clear that the RDP at $(t = x = y = 0)$ is of type $E_8^*$. Apply Theorem 5.1 (2) Counting $\#(X(F_2))$ (before taking the resolution), we obtain $\#X(F_2) = 1 + 2^2 + 2 - 2$, $\#X(F_4) = 1 + 2^2 + 4 \cdot 2$, and $\#X(F_8) = 45 = 1 + 2^2 + 4 \cdot (-5/2)$, hence $\text{lt}(X) = 3$ by Proposition 5.10 (s(1) = 2, s(2) = 1, s(3) = 3/2). (3) Let $\bar{X}$ be the (smooth) elliptic K3 surface in characteristic 0 defined by the same equation. Since $\bar{X}$ admits an automorphism $((x', y', u) \mapsto (x', y', \zeta^7 u))$ acting on $H^0(\bar{X}, \Omega^2)$ by a primitive 7-th root of unity, $\bar{X}$ has complex multiplication by $\mathbb{Q}(\zeta_7)$. Hence the mod 2 reduction $X$ of $\bar{X}$ has $\text{lt}(X) = 3$ by Theorem 5.11.
The quasi-elliptic K3 surface $y^2 = x^3 + t^2x + t^{11}$ (given by Dolgachev–Kondo [DK03, Theorem 1.1]) admits a fiber of type $I_{16}$ at $t = 0$. The union of a section and the singular fibers contains a configuration of type $D_{21}$ and one of type $E_8$. Since $21 \not\equiv 22 - 2h$ for any $1 \leq h < \infty$, this K3 surface is supersingular.

**Example 8.2** ($p = 3$).

1. $X : y^2 + x^3 + t^2x^2 + t^5 + t^6 + t^7 = 0$. $\text{Sing}(X) = 2E_8^2 + A_2$.
2. $X : y^2 + x^3 + t^3x^2 + t^3 = 0$. $\text{Sing}(X) = E_8^1 + D_8$.
3. $X : y^2 + x^3 + t^5 + t^7 = 0$. $\text{Sing}(X) = 2E_8^1 + 2A_2$.

**Example 8.3** ($p = 5$). As in [Mar22, Example 10.11], $y^2 = x^3 + at^4x + t + t^{11}$ is an RDP K3 surface with $2E_8^1$ (resp. $2E_8^0$) if $a \neq 0$ (resp. $a = 0$).

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