Exponent for classical-quantum multiple access channel

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Abstract

In this paper we obtain a lower bound of exponent of average probability of error for classical quantum multiple access channel, which implies that for all rate pairs in the capacity region is achievable by a code with exponential probability of error. Thus we re-obtain the direct coding theorem.

Index Terms

error exponent; multiple access channel; hypothesis testing; classical-quantum channel.

I. INTRODUCTION

Multiple-user information theory, or network information theory, was started by C. E. Shannon, the founder of Information Theory [1]. The only multiple-user channel in classical information theory whose single-letter capacity region is completely known, is the multiple access channel (MAC). MAC is a channel with a single output and two (or more) inputs. Its capacity region was determined by R. Ahlswede [2] and H. Liao [3], independently. Its strong converse coding theorem was proven by G. Dueck [4]. Subsequently its exponential bounds of probability of error were studied by many different authors (e.g., [5]-[10]).

The capacity region of memoryless classical quantum MAC (CQMAC) was determined by A. Winter [11] and its strong converse was proven by R. Ahlswede and N. Cai [12]. The capacity regions of quantum MAC with a classical input and a quantum input ("cq") and the capacity regions of quantum MAC with two quantum inputs ("qq") were determined in [13] and [14]. Quantum MAC with various types of resources were also studied (e.g., [15]-[17]).

However according to our best knowledge the exponential bounds of probability of error for CQMAC was only considered by T. Kubo and H. Nagaoka [18]. They considered a very general (not necessary to be memoryless) model of CQMAC with three different settings at encoders, and obtained a lower bound of probability of error. They applied it to the quantum information spectrum setting, which can be considered as an extension of work...
by T. S. Han from classical MAC to CQMAC. The information spectrum method was introduced by S. Verdú and T. S. Han and extended from classical to quantum by M. Hayashi and H. Nagaoka.

We noticed that no existing paper deals with exponential bounds of probability of error in the direct part of coding theorem for CQMAC. Then we turn to the direct coding theorem of CQMAC and obtain a lower bound to its exponent of average probability of error.

By applying a Hoeffding bound in the quantum hypothesis testing problem due to Ogawa and Hayashi, M. Hayashi and H. Nagaoka obtained a bound to exponent of probability of error for classical quantum point to point channel in [21]. (The details of quantum hypothesis testing problem is reviewed in [25, Chapter 3].) Later on, in [23] M. Hayashi improved the Hoeffding bound in [22] and based on it he obtained a better exponential bound for classical quantum point to point channel than that in [21]. Our exponential bound can be considered as an extension of the bound in [23] from classical quantum point to point channel to CQMAC.

The remainder of this paper is organized as follows. In the next section we shall present our model and previous results related our work. The main result is presented in Section III and proven in Section IV respectively. Finally we conclude the paper by a brief discussion in Section V.

II. Preliminaries

A CQMAC is a communication channel with two classical inputs and a single quantum output for transmission of classical messages from two sets of messages, independently. For a given Hilbert space $\mathcal{H}$ with a finite dimension $d_{\mathcal{H}}$, we denote the set of quantum states, positive semi definite operators with trace one, by $\mathcal{S}(\mathcal{H})$. Then a memoryless CQMAC $W$ defined on $\mathcal{H}$ is specified by a set of quantum states $\{W(x, y), x \in \mathcal{X}, y \in \mathcal{Y}\} \subseteq \mathcal{S}(\mathcal{H})$, indexed by elements of finite sets $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. $\mathcal{X}$ and $\mathcal{Y}$ are called the input alphabets of the channel and $\mathcal{H}$ is called the output space of channel. In general a quantum MAC with input spaces $\mathcal{H}_i, i = 1, 2$ and output space $\mathcal{H}$ is a trace preserving completely positive mapping from $\{\sigma_1 \otimes \sigma_2, \sigma_i \in \mathcal{S}(\mathcal{H}_i), i = 1, 2\}$ to $\mathcal{S}(\mathcal{H})$, where $\otimes$ is the tensor product, and CQMAC is its special case. In this paper we focus at the special case. The inputs of CQMAC are accessed by two senders respectively and its output is accessed by a receiver. The two senders send classical messages from two independent sets of messages, independently. We consider asymptotic bounds for so-called “$n$-shot use the channel”. That is, the quantum state

$$W^\otimes n(x, y) := W(x_1, y_1) \otimes W(x_2, y_2) \otimes \ldots \otimes W(x_n, y_n)$$

is output from the channel, if codewords $x = (x_1, x_2, \ldots, x_n) \in \mathcal{X}$ and $y = (y_1, y_2, \ldots, y_n) \in \mathcal{Y}^n$ are sent to the CQMAC respectively. Sometimes for convenient of proofs, we let $n = 1$ and speak of “one shot using” the channel (use the channel once).

An $(n, M_X, M_Y)$ code $\Gamma$ for CQMAC $W$ consists of encoders $\phi_X$ and $\phi_Y$, a pair of mappings from the message sets $M_X := \{1, 2, \ldots, M_X\}$ and $M_Y := \{1, 2, \ldots, M_Y\}$ to the input alphabets $\mathcal{X}^n$ and $\mathcal{Y}^n$ respectively, and decoding measurement $\{Z_{i,j}, i \in M_X, j \in M_Y\}$, a POVM on the output Hilbert space $\mathcal{H}$ such that $\sum_{i,j} Z_{i,j} = I$, where $I$ is the identity on output Hilbert space $\mathcal{H}$. $R_X = \frac{1}{n} \log M_X$ and $R_Y = \frac{1}{n} \log M_Y$ are called the rates of the
Here and throughout of the paper, the base of logarithm and base of exponent are $e$. The average probability of error is defined as

$$P_e(\Gamma) = \frac{1}{M_X} \frac{1}{M_Y} \sum_{i \in M_X} \sum_{j \in M_Y} \text{Tr}[(I - Z_{i,j})W^{\otimes n}(\phi_X(i), \phi_Y(j))]. \quad (1)$$

A pair of non-negative of real numbers $(r_X, r_Y)$ is achievable for codes for CQMAC $\mathcal{W}$, if for all $\epsilon, \lambda > 0$ and sufficiently large $n$, there exists an $(n, M_X, M_Y)$ code $\Gamma$ for the CQMAC such that $R_X > r_X - \epsilon$, $R_Y > r_Y - \epsilon$ and $P_e(\Gamma) < \lambda$. The set of achievable pairs are called the capacity region of the channel.

Let $\{|x\rangle, x \in X\}$ and $\{|y\rangle, y \in Y\}$ be orthonormal bases of $|X|$- and $|Y|$- dimensional Hilbert spaces respectively, $P_X$ and $P_Y$ be arbitrary probability distributions on $X$ and $Y$ respectively. Then, we define the state

$$T_{XYZ} := \sum_{x,y} P_X(x)P_Y(y)|x,y\rangle\langle x,y| \otimes W(x,y). \quad (2)$$

It was proven by A. Winter that the capacity region of CQMAC is

$$\text{cl}\left\{ \bigcup_{P_X, P_Y} \{(R_X, R_Y)|R_X \leq I(X;Z|Y)_T, R_Y \leq I(Y;Z|X)_T, R_X + R_Y \leq I(XY;Z)_T\} \right\}, \quad (3)$$

where $\text{cl}$ is the convex closure operator [11].

For a given CQMAC and a pair $(R_1, R_2)$ of rates, we define the exponent of probability of error for a CQMAC as

$$E(R_X, R_Y) := -\lim_{n \to \infty} \max_{\Gamma} \frac{1}{n} \log P_e(\Gamma). \quad (4)$$

the maximum is taken over all codes with length $n$ and rates not larger than $R_X$ and $R_Y$. We are interested in the value of $E(R_X, R_Y)$, in particular for memoryless CQMAC.

The exponent of probability of error for classical quantum point to point channel $E(R)$ is defined in a similar way. In [23] M. Hayashi improved a lower bound in [21] and obtained the following bound to the exponent of probability of error for classical quantum point to point channel $W = \{W(x), x \in X\}$, for all $s \in [0, 1]$

$$E(R) \geq \phi_{W,P}(s) - sR, \quad (5)$$

where $\phi_{W,P}(s) := -\log \text{Tr}[\sum_x P(x)W(x)^{1-s}W_P^s]$ and $W_P := \sum_x P(x)W(x)$.

T. Kubo and H. Nagaoka considered general CQMAC (not necessary to be memoryless) and dialoged with three settings of encoders that is, two encoders randomly correlatively choose codewords; randomly independently choose codewords and deterministically choose codewords. They had a lower bound for probability of error. Then they applied it to have an outer bound of the achievable pairs of rates of codes with probability of error upper bounded by a given number, for a sequence of CQMAC [21].

$^1$Since $-\max_{\Gamma} \frac{1}{n} \log P_e(\Gamma)$ is subadditive, the limit exists.
### III. The Main Result

Our result is presented in this section and proven in the next section.

For given CQMAC \( \mathcal{W} = \{W(x, y), x \in \mathcal{X}, y \in \mathcal{Y}\} \) and input distributions \( P_X \) and \( P_Y \), let

\[
W_{P_X}(y) := \sum_x P_X(x)W(x, y),
\]

\[
W_{P_Y}(x) := \sum_y P_Y(y)W(x, y) \quad \text{and}
\]

\[
W_{P_X,P_Y} := \sum_{x,y} P_X(x)P_Y(y)W(x, y).
\]  

(6)

**Theorem 1:** For a given CQMAC \( \mathcal{W} \), input distributions \( P_X \) and \( P_Y \) and sufficiently large \( n \), there exists an \( (n, M_X, M_Y) \) code \( \Gamma \) for the CQMAC with rates \( \frac{1}{n} \log M_X = R_X, \frac{1}{n} \log M_Y = R_Y \) and

\[
\frac{1}{n} \log P_e(\Gamma) \geq \min \left\{ \max_{s \in [0,1]} \left[ \Psi_{W,P_X,P_Y}(s) - s(R_X + R_Y) \right], \max_{s \in [0,1]} \left[ \Psi_{W,P_X|P_Y}(s) - sR_X \right], \max_{s \in [0,1]} \left[ \Psi_{W,P_Y|P_X}(s) - sR_Y \right] \right\},
\]

(7)

where

\[
\Psi_{W,P_X,P_Y}(s) := -\log \sum_{x,y} P_X(x)P_Y(y)\text{Tr}[W(x, y)^{1-s}W_{P_X,P_Y}^s],
\]

(8)

\[
\Psi_{W,P_X|P_Y}(s) := -\log \sum_{x,y} P_X(x)P_Y(y)\text{Tr}[W(x, y)^{1-s}W_{P_X}(y)^s],
\]

(9)

and

\[
\Psi_{W,P_Y|P_X}(s) := -\log \sum_{x,y} P_X(x)P_Y(y)\text{Tr}[W(x, y)^{1-s}W_{P_Y}(x)^s].
\]

(10)

Consequently we have that

\[
E(R_X, R_Y) \geq \max_{P_X,P_Y} \min \left\{ \max_{s \in [0,1]} \left[ \Psi_{W,P_X,P_Y}(s) - s(R_X + R_Y) \right], \max_{s \in [0,1]} \left[ \Psi_{W,P_X|P_Y}(s) - sR_X \right], \max_{s \in [0,1]} \left[ \Psi_{W,P_Y|P_X}(s) - sR_Y \right] \right\}.
\]

By taking derivatives of \( \Psi_{W,P_X,P_Y}, \Psi_{W,P_X|P_Y} \) and \( \Psi_{W,P_Y|P_X} \) with respect to \( s \) at 0 we have that

\[
\Psi'_{W,P_X,P_Y}(0) = \text{Tr} \left[ \sum_{x,y} P_X(x)P_Y(y)W(x, y)(\log W(x, y) - \log W_{P_X,P_Y}) \right] = I(XY; Z)_T,
\]

(11)

\[
\Psi'_{W,P_X|P_Y}(0) = \sum_y P_Y(y)\text{Tr} \left[ \sum_x P_X(x)W(x, y)(\log W(x, y) - \log W_{P_X}(y)) \right] = I(X; Z|Y)_T
\]

(12)

and

\[
\Psi'_{W,P_Y|P_X}(0) = \sum_x P_X(x)\text{Tr} \left[ \sum_y P_Y(y)W(x, y)(\log W(x, y) - \log W_{P_Y}(x)) \right] = I(Y; Z|X)_T,
\]

(13)

respectively, for the state \( T_{XYZ} \) defined in (2). Moreover, as \( \Psi_{W,P_X,P_Y}(0) = \Psi_{W,P_X|P_Y}(0) = \Psi_{W,P_Y|P_X}(0) = 0 \), we have that

\[
\lim_{s \to 0} \frac{\Psi_{W,P_X,P_Y}(s)}{s} = I(XY; Z)_T,
\]

\[
\lim_{s \to 0} \frac{\Psi_{W,P_X|P_Y}(s)}{s} = I(X; Z|Y)_T
\]
and
\[ \lim_{s \to 0} \frac{\Psi_{W, P_Y | P_X}(s)}{s} = I(Y; Z|X)_T. \]

Consequently by (7) the average probability of error of the optimal codes is upper bounded by \( e^{-na} \) for a constant \( a \), if \( R_X + R_Y < I(X; Y)_{TR}, R_X < I(X; Z|Y)_{TR} \) and \( R_Y < I(Y; Z|X)_T \). Thus we re-obtain the direct part of capacity region in (3).

Readers may notice the similarity of our bound and the bound (5) of [23]. In fact our bound can be considered as its extension from point to point channel to MAC in this sense.

IV. PROOF OF THEOREM [1]

A. Construction of the Code

We begin with one shot setting in the proof and then apply the quantum Chernoff distance to extend it to \( n \)-shot memoryless using the channel. Let us consider a CQMAC and a pair of input distributions \( P_X \) and \( P_Y \). Let \( \mathcal{M}_X = \{1, 2, \ldots, M_X\} \) and \( \mathcal{M}_Y = \{1, 2, \ldots, M_Y\} \) be two message sets and \( \mathcal{M} = \mathcal{M}_X \times \mathcal{M}_Y \). For all \( i \in \mathcal{M}_X \) and \( j \in \mathcal{M}_Y \), we let the random encoders \( \phi_X(i) \) and \( \phi_Y(j) \) randomly and independently choose codewords from the input alphabets \( \mathcal{X} \) and \( \mathcal{Y} \), with distributions \( P_X \) and \( P_Y \), respectively. We define the projection

\[ Q(x, y) := \{ W(x, y) > 2M_XW_{P_X}(y) + 2M_YW_{P_Y}(x) + 2M_XM_YW_{P_X, P_Y} \}, \tag{14} \]

where \( W_{P_X}, W_{P_Y} \) and \( W_{P_X, P_Y} \) are defined in (5), and for each outcome of the random encoding coding \( \{(\phi_X(i), \phi_Y(j)), (i, j) \in \mathcal{M}\} \), we define the decoding measurement \( \{Z_{i,j} \mid (i, j) \in \mathcal{M}\} \) as

\[ Z_{i', j'} := \sqrt{\sum_{(i, j) \in \mathcal{M}} Q(\phi_X(i), \phi_Y(j))^{-1} Q(\phi_X(i'), \phi_Y(j'))^{-1} \sum_{(i, j) \in \mathcal{M}} Q(\phi_X(i), \phi_Y(j))} \tag{15} \]

B. Evaluation of performance

Our proof is based on two important results in quantum information theory. The first one is known as Hayashi-Nagaoka inequality, which is widely used in proofs of direct coding theorem for quantum channels (Chapter 4, [25]).

Proposition 1: When two Hermitian matrices \( S \) and \( T \) satisfy \( I \geq S \geq 0 \) and \( T \geq 0 \), the matrix inequality

\[ I - \sqrt{S + T^{-1}}S\sqrt{S + T^{-1}} \leq 2(I - S) + 4T \]

holds.

The second one was due to Ke Li [28], concerning problem of testing multiple quantum hypotheses. To state it, we need to a few definitions. Let us consider to discriminate outcomes \( \{A_i, i = 1, 2, \ldots, r\} \) of a random quantum source, by performing a POVM measurement \( \{M_i, i = 1, 2, \ldots, r\} \). Namely, suppose the quantum source outputs a quantum state \( A_i \) with probability \( P(i) \) for a prior probability distribution \( P \), and one declares the test state is \( A_j \) if the outcome of the measurement is \( j \), for \( j = 1, 2, \ldots, r \). Naturally, the average probability of error is defined as

\[ P_e \{ P(1)A_1, P(2)A_2, \ldots, P(r)A_r; M_1, M_2, \ldots, M_r \} := \sum_{i=1}^{r} \text{Tr} [P(i)A_i M_i] \tag{16} \]
M. Nussbaum and A. Szkoła defined the multiple quantum Chernoff distance among a set of quantum states \( \{ \rho_i, i = 1, 2, \ldots, r \} \), as

\[
C(\rho_1, \rho_2, \ldots, \rho_r) := \min_{i,j,i \neq j} \max_{s \in [0,1]} -\log \text{Tr} \left[ \rho_i^{1-s} \rho_j^s \right]
\]

(17)

and conjectured that it is the optimal asymptotic exponent of error probability, in discriminating quantum states \( \{ \rho_i^\otimes n, i = 1, 2, \ldots, r \} \). K. Li proved the conjecture in [28]:

Proposition 2: Let a memoryless quantum source generate quantum states \( \{ \rho_i^\otimes n, i = 1, 2, \ldots, r \} \), with an arbitrary prior probability \( P \). Then the asymptotic exponent of error probability

\[
-\lim_{n \to \infty} \frac{1}{n} \log P_e \{ P(1)\rho_1^\otimes n, P(2)\rho_2^\otimes n, \ldots, P(r)\rho_r^\otimes n; M_1, M_2, \ldots, M_r \}
\]

for the optimal POVM measurement \( \{ M_i, i = 1, 2, \ldots r \} \) is given by the multiple quantum Chernoff distance \( C(\rho_1, \rho_2, \ldots, \rho_r) \).

K. Li had upper bound of probability of error in one shot sitting for discriminating positive semi definite matrices in the same paper as well.

Here, we notice that Proposition 2 is still valid even when \( \rho_j \) are unnormalized states.

Now, we consider the discrimination between the state \( P(1)\rho_1^\otimes n \) and the mixture state \( P(2)\rho_2^\otimes n + P(3)\rho_3^\otimes n + P(4)\rho_4^\otimes n \). The optimal discriminating POVM is given by \( \{ P_n, I - P_n \} \) with \( P_n := \{ P(1)\rho_1^\otimes n \geq P(2)\rho_2^\otimes n + P(3)\rho_3^\otimes n + P(4)\rho_4^\otimes n \} \). Hence, using this proposition, we have the following corollary.

Corollary 1:

\[
-\lim_{n \to \infty} \frac{1}{n} \log P_e \{ P(1)\rho_1^\otimes n, P(2)\rho_2^\otimes n + P(3)\rho_3^\otimes n + P(4)\rho_4^\otimes n; P_n, I - P_n \}
\]

\[
= \min_{j=2,3,4} \max_{s \in [0,1]} -\log \text{Tr} [\rho_0^{1-s} \rho_j^s].
\]

Again, we notice that Corollary 1 is still valid even when \( \rho_j \) are unnormalized states.

Proposition 1 and (15) imply that

\[
I - Z_{i',j'} \leq 2(I - Q(\phi_X(i'), \phi_Y(j'))) + 4 \sum_{(i,j) \neq (i',j')} Q(\phi_X(i), \phi_Y(j)).
\]

(18)
Hence we may evaluate the expectation of the average probability of error with respect to the random encoding as

\[
E \frac{1}{|M|} \sum_{(i',j') \in M} \text{Tr} \left[ W(\phi_X(i'), \phi_Y(j')) (I - Z_{i',j'}) \right]
\]

\[
\leq 2 \frac{|M|}{|M|} E \sum_{(i',j') \in M} \text{Tr} \left[ W(\phi_X(i'), \phi_Y(j')) (I - Q(\phi_X(i'), \phi_Y(j'))) \right]
\]

\[
+ 4 \frac{|M|}{|M|} E \sum_{(i',j') \in M} \text{Tr} \left[ W(\phi_X(i'), \phi_Y(j')) \sum_{(i,j) \neq (i',j')} Q(\phi_X(i), \phi_Y(j)) \right]
\]

\[
= 2 \frac{|M|}{|M|} E \sum_{(i,j) \in M} \text{Tr} \left[ W(\phi_X(i), \phi_Y(j)) (I - Q(\phi_X(i), \phi_Y(j))) \right]
\]

\[
+ 4 E \text{Tr} \left[ \sum_{(i,j) \in M} Q(\phi_X(i), \phi_Y(j)) \left( \frac{1}{|M|} \sum_{(i',j') \in M} W(\phi_X(i'), \phi_Y(j')) \right) \right]
\]

\[
= 2 \frac{|M|}{|M|} E \sum_{(i,j) \in M} \text{Tr} \left[ W(\phi_X(i), \phi_Y(j)) (I - Q(\phi_X(i), \phi_Y(j))) \right]
\]

\[
+ 4 E \text{Tr} \left[ \sum_{(i,j) \in M} Q(\phi_X(i), \phi_Y(j)) \frac{1}{|M|} \sum_{i' \neq i} W(\phi_X(i'), \phi_Y(j)) \right]
\]

\[
+ \frac{4}{M_Y} E \text{Tr} \left[ \sum_{(i,j) \in M} Q(\phi_X(i), \phi_Y(j)) \frac{1}{M_X} \sum_{j' \neq j} W(\phi_X(i), \phi_Y(j')) \right]
\]

\[
+ \frac{4}{M_X} E \text{Tr} \left[ \sum_{(i,j) \in M} Q(\phi_X(i), \phi_Y(j)) \frac{1}{M_Y} \sum_{i' \neq i} W(\phi_X(i'), \phi_Y(j)) \right],
\]  

(19)

where \( E \) is the expectation operator with respect to the random encoding.

Now the first term at the right hand side of (19) is

\[
\frac{2}{|M|} \sum_{(i,j) \in M} \sum_{x,y} \text{Pr}(\phi_X(i) = x, \phi_Y(j) = y) \text{Tr} \left[ W(x, y) (I - Q(x, y)) \right]
\]

\[
= 2 \sum_{x,y} P_X(x) P_Y(y) \text{Tr} \left[ W(x, y) (I - Q(x, y)) \right],
\]  

(20)
whereas the second term at the right hand side of (19) is

\[
4 \mathbb{E} \left( \operatorname{Tr} \left[ \sum_{(i,j) \in M} Q(\phi_X(i), \phi_Y(j)) \frac{1}{|M|} \sum_{i' \neq j, i' \neq j} W(\phi_X(i'), \phi_Y(j')) \right] \right)
\]

\[
= 4 \left( \operatorname{Tr} \left[ \sum_{(i,j) \in M} \mathbb{E}_{\phi_X(i), \phi_Y(j)} Q(\phi_X(i), \phi_Y(j)) \frac{1}{|M|} \sum_{i' \neq j, i' \neq j} \mathbb{E}_{\phi_X(i'), \phi_Y(j')} W(\phi_X(i'), \phi_Y(j')) \right] \right)
\]

\[
= 4 \sum_{(i,j) \in M} \sum_{x,y} \Pr(\phi_X(i) = x, \phi_Y(j) = y) \operatorname{Tr} \left[ Q(x, y) \left( \frac{1}{|M|} \sum_{i' \neq j, i' \neq j} \mathbb{E}_{\phi_X(i'), \phi_Y(j')} W(\phi_X(i'), \phi_Y(j')) \right) \right]
\]

\[
= 4 \sum_{(i,j) \in M} \sum_{x,y} P_X(x) P_Y(y) \operatorname{Tr} \left[ Q(x, y) \left( \frac{1}{|M|} \sum_{i' \neq j, i' \neq j} W_{P_X, P_Y} \right) \right]
\]

\[
\leq 4M_X M_Y \sum_{x,y} P_X(x) P_Y(y) \operatorname{Tr} \left[ Q(x, y) W_{P_X, P_Y} \right],
\]

(21)

where \( \mathbb{E}_{\phi_X(i), \phi_Y(j)} \) is the expectation with respect to the random variables \( \phi_X(i), \phi_Y(j) \), and \( \mathbb{E}_{\phi_X(i'), \phi_Y(j')} | \phi_X(i), \phi_Y(j) \) is the conditional expectation given \( \phi_X(i) \) and \( \phi_Y(j) \) with respect to the random variables \( \phi_X(i'), \phi_Y(j') \). Here, the second equality holds because the code \( (\phi_X(i), \phi_Y(j)) \) is independent of \( (\phi_X(i'), \phi_Y(j)) \) for \( i' \neq i, j \neq j' \).

The third term at the right hand side of (19) is

\[
\frac{4}{M_Y} \mathbb{E} \left\{ \operatorname{Tr} \left[ \sum_{(i,j) \in M} Q(\phi_X(i), \phi_Y(j)) \frac{1}{M_X} \sum_{i' \neq j \in M_X} W(\phi_X(i'), \phi_Y(j)) \right] \right\}
\]

\[
= \frac{4}{M_Y} \operatorname{Tr} \left[ \sum_{(i,j) \in M} \mathbb{E}_{\phi_X(i), \phi_Y(j)} Q(\phi_X(i), \phi_Y(j)) \frac{1}{M_X} \sum_{i' \neq j \in M_X} \mathbb{E}_{\phi_X(i'), \phi_Y(j')} W(\phi_X(i'), \phi_Y(j)) | \phi_X(i), \phi_Y(j) \right]
\]

\[
= \frac{4}{M_Y} \sum_{(i,j) \in M} \sum_{x,y} \Pr(\phi_X(i) = x, \phi_Y(j) = y)
\]

\[
\times \operatorname{Tr} \left[ \sum_{(i,j) \in M} Q(x, y) \frac{1}{M_X} \sum_{i' \neq j \in M_X} \mathbb{E}_{\phi_X(i'), \phi_Y(j')} W(\phi_X(i'), y) | \phi_X(i) = x, \phi_Y(j) = y \right]
\]

\[
= \frac{4}{M_Y} \sum_{(i,j) \in M} \sum_{x,y} P_X(x) P_Y(y) \operatorname{Tr} \left[ Q(x, y) \frac{1}{M_X} \sum_{i' \neq j \in M_X} W_{P_X} \right]
\]

\[
= \frac{4}{M_Y} \sum_{(i,j) \in M} \sum_{x,y} P_X(x) P_Y(y) \operatorname{Tr} \left[ Q(x, y) \frac{1}{M_X} (M_X - 1) W_{P_X} \right]
\]

\[
\leq 4M_X \sum_{x,y} P_X(x) P_Y(y) \operatorname{Tr} \left[ Q(x, y) W_{P_X} \right].
\]

(22)

Similarly the last term at the right hand side of (19) is upper bounded by

\[
4M_Y \sum_{x,y} P_X(x) P_Y(y) \operatorname{Tr} \left[ Q(x, y) W_{P_Y} \right].
\]

(23)
Finally (19)-(23) yield an upper bound to expectation of average probability of error
\[
2 \sum_{x,y} P_X(x)P_Y(y) \text{Tr} \left[ W(x,y)(I - Q(x,y)) \right] + 4M_X M_Y \sum_{x,y} P_X(x)P_Y(y) \text{Tr} \left[ Q(x,y)W_{P_X,P_Y} \right] \\
+ 4M_X \sum_{x,y} P_X(x)P_Y(y) \text{Tr} \left[ Q(x,y)W_{P_X} \right] + 4M_Y \sum_{x,y} P_X(x)P_Y(y) \text{Tr} \left[ Q(x,y)W_{P_Y} \right] \\
= 2 \sum_{x,y} P_X(x)P_Y(y) \text{Tr} \left[ (I - Q(x,y))W(x,y) \right] \\
+ Q(x,y) \left( 2M_X M_Y W_{P_X,P_Y} + 2M_X W_{P_X}(y) + 2M_Y W_{P_Y}(x) \right). \tag{24}
\]

To apply quantum Chernoff bound, we define the states
\[
S_1 := \sum_{x,y} P_X(x)P_Y(y)|x,y\rangle\langle x,y| \otimes W_{P_X,P_Y}, \\
S_2 := \sum_{x,y} P_X(x)P_Y(y)|x,y\rangle\langle x,y| \otimes W_{P_X}(y) \quad \text{and} \\
S_3 := \sum_{x,y} P_X(x)P_Y(y)|x,y\rangle\langle x,y| \otimes W_{P_Y}(x). \tag{25}
\]

Then, by considering the non-normalized states, we observe that the right hand of (24) is 2 times of the error probability of the discrimination between the state \( T := T_{XYZ} \) defined in (2) and the unnormalized state \( 2M_X M_Y S_1 + 2M_X S_2 + 2M_Y S_3 \).

Now, we consider the \( n \)-memoryless extension. To this end let \( M_X = e^{nR_X} \) and \( M_Y = e^{nR_Y} \). Thus (24) implies that for this setting, the upper bound of the average error probability in (24) is 2 times of the error probability of the discrimination between \( T_{XYZ}^n \) and \( 2e^{n(R_X+R_Y)} S_1^n + 2e^{nR_X} S_2^n + 2e^{nR_Y} S_3^n \). Therefore by Corollary 1, the error exponent of average probability of error is lower bounded by the minimum value of quantum Chernoff distance between \( T_{XYZ}^n \) and \( 2e^{n(R_X+R_Y)} S_1^n \), quantum Chernoff distance between \( T_{XYZ}^n \) and \( 2e^{nR_X} S_2^n \) and quantum Chernoff distance between \( T_{XYZ}^n \) and \( 2e^{nR_Y} S_3^n \). As Chernoff distance between \( T_{XYZ}^n \) and \( 2e^{n(R_X+R_Y)} S_1^n \) is
\[
- \min_{s \in [0,1]} \left[ \log \text{Tr} \left[ T^{1-s} S_1^s \right] + s(R_X + R_Y) \right] \\
= - \min_{s \in [0,1]} \left[ \log \left( \sum_{x,y} P_X(x)P_Y(y) \text{Tr} \left[ W(x,y)^{1-s} W_{P_X,P_Y}^s \right] \right) + s(R_X + R_Y) \right] \\
= \max_{s \in [0,1]} \left[ \Psi_{W,P_X,P_Y}(s) - s(R_X + R_Y) \right] \quad \text{(c.f. (8))}. \tag{26}
\]

Since Chernoff distance between \( T_{XYZ}^n \) and \( 2e^{nR_X} S_2^n \) is
\[
- \min_{s \in [0,1]} \left( \log \text{Tr} \left[ T^{1-s} S_2^s \right] + sR_X \right) \\
= - \min_{s \in [0,1]} \left[ \log \left( \sum_{x,y} P_X(x)P_Y(y) \text{Tr} \left[ W(x,y)^{1-s} W_{P_X}(y)^s \right] \right) + sR_X \right] \\
= \max_{s \in [0,1]} \left[ \Psi_{W,P_X,P_Y}(s) - sR_X \right] \quad \text{(c.f. (9))}. \tag{27}
\]
and Chernoff distance between $T^\otimes n$ and $2e^{nR_Y}S^\otimes n$ is

$$- \min_{s \in [0,1]} \left( \log \text{Tr} \left[ T^{1-s}S^s \right] + sR_Y \right)$$

$$= - \min_{s \in [0,1]} \left[ \log \left( \sum_{x,y} P_X(x)P_Y(y)\text{Tr} \left[ W(x, y)^{1-s}W_{P_Y}(x)^s \right] \right) + sR_Y \right]$$

$$= \max_{s \in [0,1]} \left[ \Psi_{W,P_X,P_Y}(s) - sR_Y \right], \quad \text{(c.f. (10))}$$

(28)

we have (7), which completes our proof.

V. Discussion

In this paper, we have derived an upper bound of the exponential decreasing rate of the decoding error probability for CQMAC. This bound is based on a quantum multiple access extension of the dependence test bound of channel coding [21, Remark 15][32]. Fortunately, the dependent test bound of channel coding was extended to universal coding [26]. Although existing study [10], [7] derived an exponent of a universal coding in the classical discrete memoryless case of MAC, this type extension might be useful for a universal coding in the classical continuous memoryless case of MAC.

For a quantum universal coding, using this idea, the paper [24] constructed a universal code with a rate $R$ and any probability distribution $P$ on $\mathcal{X}$ for a quantum compound point to point channel, or a set of memoryless quantum point to point channels with common input alphabet $\mathcal{X}$ and output Hilbert space $\mathcal{H}$, such that the probability of error of the code for each channel $W = \{W(x), x \in \mathcal{X}\}$ in the set is smaller than $\exp[-n \max_{t \in [0,1]} \Phi_{W,P}(t) + o(n)]$, where

$$\Phi_{W,P}(t) := -(1-t) \log \text{Tr} \left[ \sum_x P(x)W^{1-t}(x) \right].$$

(29)

An alternative expression of $\Phi_{W,P}(t)$ is

$$\Phi_{W,P}(t) = -\max_{\sigma} \log \text{Tr} \left[ \sum_x P(x)W(x)^{1-t}\sigma^t \right]$$

(30)

(c.f. Chapter 6, [27]) and is smaller than $\phi_{W,P}(t)$. Hence, we can expect that this idea can be used for a universal coding in CQMAC. Extending our method to universal coding in several setting is an interesting future topic.

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References

[1] C. E. Shannon, “Two-way communication channels,” Proc. 4th Berkeley Symp. Math. Statist. and Prob., Univ. of Calif. Press, Berkeley, 1, 611–644, 1961.
[2] R. Ahlswede, “Multi-way communication channels,” in Proc. 2nd Int. Symp. Inf. Theory (Thakadsor, Armenian SSR, Sep. 1971). Budapest, Hungary: Academia Kiado, 1971, 23–52.
[3] H. Liao, Multiple Access Channels, Ph.D. dissertation, Dept. Electr. Eng., University of Hawaii, Honolulu, 1972.
[4] G. Dueck, “The strong converse to the coding theorem for the multiple Cacess channel,” J. Comb. Inf. Syst. Sci., Vol. 6, 187–196, 1981.
[5] E. A. Haroutunian, “Lower bound for the error probability of multiple-access channels,” Problemy Peredachi Informatsii, 11, 23-36, 1975.
[6] J. Pokorney and H. S. Wallmeier, “Random coding bounds and codes produced by permutations for the multiple-access channels,” IEEE Trans. Inf. Theory, 31, 741-750, 1985.
[7] Y. S. Liu and B. L. Hughes, “A new universal random coding bound for the multiple-access channel,” IEEE Trans. Inf. Theory, 42, 376-386, 1999.
[8] A. Nazari, Error exponent for discrete memoryless multiple-access channels, Ph.D. dissertation, Dept. Electr. Eng., Univ. Michigan, Ann Arbor, MI, USA, 2011.
[9] A. Nazari, A. Anastasopoulos, and S. S. Pradhan, “Error exponent for multiple-access channels: Lower bounds,” IEEE Trans. Inf. Theory, 60, 5095-5115, 2014.
[10] A. Nazari, A. Anastasopoulos and S. Pradhan, “A new universal random-coding bound for average probability error exponent for multiple-access channels” Proc. Conf. Inf. Sciences and Systems, 295–300, 2009.
[11] A. Winter, “The capacity of the quantum multiple-access channel,” IEEE Trans. Inf. Theory, 47, 3059–3065, 2001.
[12] R. Ahlswede and N. Cai, “A Strong converse theorem for quantum multiple access channels,” Lecture Notes in Computer Science, Volume 4123, 459-484, Springer, Berlin, 2006.
[13] M. Horodecki, J. Oppenheim, and A. Winter, “Partial quantum information,” Nature, 436, 673–676, 2005.
[14] J. Yard, P. Hayden and I. Devetak, “Capacity theorems for quantum multiple-access channels: classical-quantum and quantum-quantum capacity regions,” IEEE Trans. Inf. Theory, 54, 884-888, 2005.
[15] M.-H. Hsieh, I. Devetak and A. Winter, “Entanglement-Assisted Capacity of Quantum Multiple-Access Channels,” IEEE Trans. Inf. Theory, 54, 3078-3090, 2008.
[16] H. Boche and J. Nötzel, “The classical-quantum multiple access channel with conferencing encoders and with common messages,” Quantum Inf. Processing, 13, 2595-2617, 2014.
[17] H. Boche and J. Nötzel, “Cooperation for the classical-quantum multiple access channel,” IEEE Int. Symp. Inf. Theory, 156-160, 2014.
[18] T. Kubo and H. Nagaoka, “A Fundamental Inequality for Lower-bounding the Error Probability for Classical and Quantum Multiple Access Channels and Its Applications,” IEICE Transactions on Fundamentals of Electronics Communications & Computer Sciences, E98-A, 2376-2383, 2015.
[19] T. S. Han, “An information-spectrum approach to capacity theorems for the general multiple-access channel” IEEE Trans. Inf. Theory, 44, 2773-2795, 1998.
[20] S. Verdú and T. S. Han, “A General formula for channel capacity;” IEEE Trans. Inf. Theory, 40, 1147–1157, 1994.
[21] M. Hayashi and H. Nagaoka, “General formulas for capacity of classical-quantum-channels,” IEEE Trans. Inf. Theory, 49, 1753–1768, 2003.
[22] T. Ogawa and M. Hayashi, “On Error Exponents in Quantum Hypothesis Testing,” IEEE Trans. Inf. Theory, 50, 1368–1372, 2004.
[23] M. Hayashi, “Error exponent in asymmetric quantum hypothesis testing and its application to classical-quantum channel coding,” Physical Review A, 76, 062301 (2007)
[24] M. Hayashi, “Universal coding for classical-quantum-channel,” Communications in Mathematical Physics, 289, 1087-1098 (2009).
[25] M. Hayashi, Quantum Information Theory: Mathematical Foundation, Graduate Texts in Physics, Springer (2017) (Second edition of Quantum Information: An Introduction Springer 2017).
[26] M. Hayashi, “Universal channel coding for general output alphabet,” [arXiv:1502.02218v2](http://arxiv.org/abs/1502.02218) (2016).
[27] M. Hayashi, A Group Theoretic Approach to Quantum Information, Springer, 2017, Originally published in Japanese in 2014.
[28] Ke Li, “Discriminating quantum states: The multiple Chernoff distance,” Ann. Statist. 44, 1661-1679, 2016.
[29] M. Nussbaum and A. Szkoła, “Exponential error rates in multiple state discrimination on a quantum spin chain,” J. Math. Phys. 51, 072203 (2010).
[30] M. Nussbaum and A. Szkoła, “Asymptotically optimal discrimination between multiple pure quantum states,” In Theory of Quantum Computation, Communication and Cryptography. 5th Conference, TQC 2010, Leeds, UK. Lecture Notes in Computer Science 6519 1–8 Springer, Berlin (2011).
[31] M. Nussbaum and A. Szkoła, “An asymptotic error bound for testing multiple quantum hypotheses,” Ann. Statist., 39, 3211–3233, 2011.
[32] Y. Polyanskiy, H.V. Poor, and S. Verdú, “Channel coding rate in the finite blocklength regime,” IEEE Trans. Inf. Theory, 56, 2307 – 2359, 2010.