RAPID COMMUNICATION

Pure 4-geometry of quantum magnetic spin matter from Kondo effect

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Abstract. We determine a smooth Euclidean 4-geometry on $\mathbb{R}^4$ from quantum interacting spin matter like in the multichannel Kondo effect. The CFT description of both: the $k$-channel Kondo effect of spin magnetic impurities quantum interacting with spins of conducting electrons and exotic smooth $\mathbb{R}^4_k$, by the level $k$ WZW model on $SU(2)$, indicates the relation between smooth $\mathbb{R}^4$'s and the quantum matter. We propose a model which shows: exotic smooth $\mathbb{R}^4_k$ generates fermionic fields via the topological structure of Casson handles and when this handle is attached to some subspace $A$ of $\mathbb{R}^4$ these fermions represent electrons bounded by the magnetic impurity. Thus the Kondo bound state of $k$ conducting electrons with magnetic impurity of spin $s$ is created like in the low temperature Kondo effect. Then the quantum character of the interactions is encoded in 4-exoticness. The complexity as well the number of Casson handles correspond to the number of channels in the Kondo effect. When the smoothness structure is the standard one, no quantum interactions are carried on by standard $\mathbb{R}^4$.

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Einstein's equation describes the balance of the density of gravity with density of energy and matter in 4-dimensional spacetime. The density of gravity is the curvature of smooth 4-manifold. All kinds of matter and energy have their contributions to $T_{\mu\nu}$ hence affect the metric structure of spacetime. Quantum matter contributes by the averaged energy-momentum tensor. This shows that there is no substantial difference between quantum and classically described matter provided the densities entering $T_{\mu\nu}$ are the same. The metric field remains classical also in the quantum regime of matter. Such a classical universality would be an attractive feature if general relativity (GR) is consistent with quantization of interactions prescribed by the Standard Model (SM) of particle physics. However, this is not the case: quantum field theory (QFT) faces problems even with its formulation on dynamical curved backgrounds. On the other hand quantization of GR is not consistent in dimension 4 with SM. Thus leaving the classical Einstein's equation at the fundamental level is not a solution and the attempts to quantize these theory fail in dimension 4. Since, classical GR equivalently deals with geometry of 4-smooth manifolds, the above problem is the fate of a smooth 4-geometry in a description of quantum regime of gravity or the relation of smooth 4-geometry with quantum matter in general spacetimes. Two major possibilities follow: a quantum spacetime analogue is not a smooth 4-manifold any longer and, because of that, it can be related with quantum matter, or spacetime is still represented by a smooth 4-manifold though this time it has to be related to quantum matter in a different way than by averaged energy densities.

This letter explains how quantum interacting spin matter, as in the multichannel Kondo effect, relates to smooth Euclidean 4-geometry. We propose a model where the quantum interacting spin matter is indeed derivable from the smooth Euclidean 4-manifold structure alone. Thus the classical, smooth geometry of 4-manifolds can naturally overlap with the quantum regime of interacting matter fields. Thus the dichotomy between the classical and the quantum regime (or between the smooth 4-geometry of manifolds and non smooth models) is overpassed because smooth 4-geometry is essentially the same as quantum interacting spin matter. The physical Kondo effect holds in a variety of alloys at low temperatures hence certainly not in the regime of Planck energies or quantum gravity. This kind of relation of smooth 4-manifolds with quantum matter appears as new phenomenon which has never been discussed before. In the following we will understand a 4-geometry as smooth exotic $\mathbb{R}^4$. We refer the reader to the excellent textbooks, physical as well mathematical, relating exotic smooth $\mathbb{R}^4$\cite{7,15}. Every exotic $\mathbb{R}^4$ is a smooth 4-manifold which is topologically the same as (homeomorphic to) the standard smoothness structure as the topological product of the four coordinate axes. However, every exotic $\mathbb{R}^4$ is smoothly distinct from (or nondiffeomorphic to) the standard $\mathbb{R}^4$. Therefore we call these spaces nonstandard or exotic $\mathbb{R}^4$. Every Riemannian 4-manifold is locally described by the standard $\mathbb{R}^4$. It follows that every exotic $\mathbb{R}^4$ can not be covered by a single coordinate patch, which is a standard $\mathbb{R}^4$. Thus a nontrivial metric exists on an exotic $\mathbb{R}^4$ (or one cannot define a constant smooth metric). Hence a form of gravity might be present on this $\mathbb{R}^4$\cite{14,13,21}. The existence of nonstandard smoothness of $\mathbb{R}^n$ is assigned exclusively to dimension $n = 4$. In fact there is a continuum of infinitely many different exotic $\mathbb{R}^4$ but only one smooth standard $\mathbb{R}^n$ for any other dimension $n \neq 4$.

GR shows that matter/energy is linked with 4-geometry of smooth manifolds. In our model quantum interacting spin-magnetic matter is also linked with 4-geometry of the smooth, though nonstandard, $\mathbb{R}^4$. We will apply our model to the case of the
multichannel Kondo effect with 2-components spins. Exotic $\mathbb{R}^4$’s are Euclidean 4-manifolds. The existence of Minkowski metric on exotic $\mathbb{R}^4$ is a strong limitation \[21\]. This would suggest that the above mentioned link could have a rather nonperturbative description at the dual limit of some quantum field theory (cf. \[22\]). This important issue, however, will be addressed separately.

**CFT, magnetic spin-$s$ impurities and exotic $\mathbb{R}^4$.** Kondo proposed a simple phenomenological Hamiltonian \[1\]:

$$H = \sum_{k,\alpha} \psi_+^k \psi_-^k \epsilon(k) + \lambda \vec{S} \cdot \sum_{k,k'} \frac{\psi_+^k \sigma_{k,k'}}{2} \psi_-^k. \quad (1)$$

explaining the growth of the resistivity $\rho(T)$ in some metals when the temperature $T$ is lowering below the Kondo temperature $T_K$ in the presence of magnetic spin-$s$ impurity. Here $\psi$ is the anihilation operator for the conduction electron of spin $\alpha$ and momentum $k$, the antiferromagnetic interaction term is that of spin impurity $\vec{S}$ with spins of conducting electrons, at $\vec{x} = 0$. From this Hamiltonian one can derive in the Born approximation $\rho(T) \sim [\lambda + \nu \lambda^2 \ln \frac{T}{T_K} + ...]^2$ where the second term is divergent in $T = 0$. The Hamiltonian \[1\] is derivable from the more microscopic Anderson model \[5\]. The Kondo antiferromagnetic coupling appears as the tunneling of electrons screening the spin impurity, hence it is a pure quantum process (see eg. \[20\]).

The exact low $T$ behavior was insightfully and exactly reformulated by Affleck and Ludwig by the use of boundary conformal field theory (BCFT) techniques \[2, 4, 1, 3, 20\] which is also the base for working out the connection with smooth 4-geometry.

Namely, given an integer class $h = k[\rangle \in H^3(S^3, \mathbb{Z})$ where $\langle$ denotes the generator of $H^3(S^3, \mathbb{Z})$. To this class one can associate the corresponding WZW model on $S^3 = SU(2)$ at integer level $k$. Changing the class $k[\rangle \rightarrow l[\rangle$ results in the change of the level $k$ of the WZW model. The Kac-Moody algebra $SU(2)_k$ is spanned on 3-components currents $\vec{J}^a_n$, $n = ... -2, -1, 0, 1, 2, ...$:

$$[J^a_m, J^b_n] = i\epsilon^{abc} J^c_{m+n} + \frac{1}{2} k n \delta^{ab} \delta_{n,-m}. \quad (2)$$

Next we decompose the currents $\vec{J}_n$ as $\vec{J}_n = \vec{J}_n + \vec{S}$ such that $\vec{J}_n$ obey the same Kac-Moody algebra, i.e. $[J^a_n, J^b_m] = i\epsilon^{abc} J^c_{n+m} + \frac{1}{2} k n \delta^{ab} \delta_{n,-m}$ and usual relations for $\vec{S}$, i.e. $[S^a_n, S^b_m] = i\epsilon^{abc} S^c_{n+m}$, $[S^a_n, J^b_m] = 0$. From the point of view of field theories describing the interacting currents with spins, $\vec{J}_n$ corresponds to the effective infrared fixed point of the theory of interacting spins $\vec{S}$ with $J_n$ where the coupling constant $\lambda$ is taken as $\frac{2}{3}$ for $k = 1$. The interacting Hamiltonian of the theory for $k = 1$ reads:

$$H_s = c \left( \frac{1}{3} \sum_{n = -\infty}^{+\infty} \vec{J}_{-n} \cdot \vec{J}_{n} + \lambda \sum_{n = -\infty}^{+\infty} \vec{J}_{-n} \cdot \vec{S} \right). \quad (3)$$

For $\lambda = \frac{2}{3}$ one completes the square and the algebra \[2\] for the currents $\vec{J}_n$ follows., Then the new Hamiltonian where $\vec{S}$ is now effectively absent (still for $k = 1$) is given by $H = c' \sum_{n = -\infty}^{+\infty} \left( \vec{J}_{-n} \cdot \vec{J}_{n} - \frac{2}{3} \right)$.\[2\]
A similar procedure holds for arbitrary integer $k$ where the spin part of the Hamiltonian reads: $H_{s,k} = \frac{1}{2\pi(k+2)} \vec{J}^2 + \lambda \vec{S} \cdot \vec{\delta}(x)$ and the infrared effective fixed point is now reached for $k = \frac{2}{2+1}$. The spins $\vec{S}$ reappear as the boundary conditions in a boundary CFT (BCFT) represented by the WZW model on $SU(2)$. Now, the following fusion rules hypothesis [7], explains the creation and nature of the multichannel Kondo states: The infrared fixed point in the $k$-channel spin-$s$ Kondo problem is given by fusion with the spin-$s$ primary for $s \leq k/2$ or with the spin $k/2$ primary for $s > k/2$. The level $k$ Kac-Moody algebra, as in the level $k$ WZW $SU(2)$ model, governs the behavior of the Kondo state in the presence of $k$ channels of conducting electrons and magnetic impurity of spin $s$.

Let us now approach this CFT description from the point of view of 4-dimensional geometry. Exotic smooth $\mathbb{R}^4$’s are the simplest topological 4-manifolds but without a simple local coordinate patch presentation. Therefore mathematics in this case should be substantially different than in other dimensions. In a series of papers the authors have recently proposed a way to obtain some relative results in this field $[8, 9, 10, 11]$. In particular, different (i.e. non-diffeomorphic) so-called small exotic $\mathbb{R}^4$’s correspond to the codimension-1 foliations of the 3-sphere $S^3$ which lies at the boundary of some compact contractable 4-submanifold $A$ in $\mathbb{R}^4$. This compactum $A$ is called Akbulut cork and its embedding, together with the attached Casson handles, is responsible for the exotic smoothness of $\mathbb{R}^4$. The continuous family of small exotic $\mathbb{R}^4$ is defined by a radius function $\rho : \mathbb{R}^4 \rightarrow [0, +\infty)$ (polar coordinates) so that $\mathbb{R}^4 = \rho^{-1}([0, r))$ with $t = 1 - \frac{1}{4}$. Next, each member of the radial family determines a codimension-1 foliation of the homology 3-sphere and these are uniquely determined by a codimension-1 foliation of some ordinary $S^3 \subset \partial A$. However, the codimension-1 foliations of the 3-sphere (especially the foliated cobordism class) are classified by the 3-rd real cohomology classes $H^3(S^3, \mathbb{R})$. The integer classes in $H^3(S^3, \mathbb{Z})$ are a special case. It uses flat $PSL(2, \mathbb{R})$–bundles over the space $(S^2 \setminus \{k \text{ punctures}) \times S^1$ where the gluing of $k$ solid tori produces a 3-sphere (so-called Heegard decomposition). Then one obtains the relation

$$\frac{1}{(4\pi)^2} (GV(\mathcal{F}), [S^3]) = \frac{1}{(4\pi)^2} \int_{S^3} GV(\mathcal{F}) = \pm (2 - k)$$

in dependence of the orientation of the fundamental class $[S^3]$. Furthermore we can interpret the Godbillon-Vey invariant as WZW term. For that purpose we use the group structure $SU(2) = S^3$ of the 3-sphere $S^3$ or better we identify $SU(2) = \mathbb{C}^2 \setminus \{0\}$. Let $g \in SU(2)$ be a unitary matrix with $\det g = -1$. The left invariant 1-form $g^{-1}dg$ generates locally the cotangent space connected to the unit. The forms $\omega_k = Tr((g^{-1}dg)^k)$ are complex $k$–forms generating the deRham cohomology of the Lie group. The cohomology classes of the forms $\omega_1, \omega_2$ vanish and $\omega_3 \in H^3(SU(2), \mathbb{R})$ generates the cohomology group. Then we obtain for the integral

$$\frac{1}{8\pi^2} \int_{S^3=SU(2)} \omega_3 = 1$$

of the generator. This integral can be interpreted as winding number of $g$. Now we consider a smooth map $G : S^3 \rightarrow SU(2)$ with 3-form $\Omega_3 = Tr((G^{-1}dG)^3)$ so that the
integral
\[ \frac{1}{8\pi^2} \int_{S^3=SU(2)} \Omega_3 = \frac{1}{8\pi^2} \int_{S^3} \text{Tr}((G^{-1}dG)^3) \in \mathbb{Z} \]
is the winding number of $G$. Every Godbillon-Vey class with integer value like (4) is generated by a 3-form $\Omega_3$. Therefore the Godbillon-Vey class is the WZW term of the $SU(2)$. For integer $H^3(S^3, \mathbb{Z})$ and for the $S^3 \subset \partial A$ as above, one has the correspondence:
\[ H^3(S^3, \mathbb{Z}) \ni h \mapsto \mathbb{R}^4_k. \] (5)

Then we arrive at the correspondence \[8, 19, 10\]: the level $k$ of the WZW model of CFT on $SU(2)$ (determined by $k[\ ] = h \in H^3(S^3, \mathbb{Z})$) corresponds to the exotic smooth $\mathbb{R}^4_k$ when $S^3 = SU(2)$ is the part of the boundary of the Akbulut cork $A$. In that way the exotic $\mathbb{R}^4_k$ determines the fusions in $SU(2)_k$ of boundary spin $s$ impurity with $k$-channel conducting electrons. And conversely, the symmetry of the $k$-channel Kondo effect results in 4-dimensions in the corresponding geometry of exotic $\mathbb{R}^4_k$.

**Spinor fields from exotic $\mathbb{R}^4$.** Formally Einstein’s equation can be written on every smooth 4-manifold. Sładkowski showed \[21\] that Einstein’s equation on the empty (no sources) but exotic smooth $\mathbb{R}^4$ allows solutions which are equivalent to the solutions of the corresponding equation with matter sources on the standard $\mathbb{R}^4$. In this sense the exoticness of the smooth structure generates matter in spacetime. The extension of this correspondence to quantum matter would be an important ingredient of our understanding of the relation of QM and GR and was considered already from various points of view \[17, 16, 12\].

Recent work of one of the authors and Rosé for compact 4-manifolds \[12\] shows how to obtain the action terms of fermions and bosons by using the Einstein-Hilbert action and the Casson handle. The Casson handle consists of towers of kinky handles, i.e., neighborhoods of disks with self-intersections. Every kinky handle can be described by an immersion (injective differential) $D^2 \times D^2 \to M$ into a 4-manifold $M$. Usually the boundary of the immersed disk is a singular knot $K$ and the image of the immersion is a knotted solid torus $T(K) = K \times D^2$ \[12\]. Asselmeyer-Maluga and Rosé \[12\] gave an alternative method to describe the immersion by using a spinor $\Psi$. Then it is possible to express the mean curvature $H_{T(K)}$ of the knotted torus $T(K)$ by $2\nabla \sigma_\mu \Psi$. But the mean curvature will be used in the boundary term of the Einstein-Hilbert action. Let $\psi$ be the extension of $\Psi$ to the 4-manifold then we obtain \[12\] the pure fermionic part of the action:
\[ S(M) = \int_M \left( R + \sum_n (\psi \cdot \partial_D \overline{\psi})_n \right) \sqrt{g} d^4x \] (6)
where $R$ is the scalar curvature on $M$ and $\psi$ the spinor field extended over $M$. Let us note that the calculations relies merely on the Casson handle structure which means that the fermionic field fulfilling the Dirac equation appears also in the open 4-manifold case of exotic $\mathbb{R}^4$. In general there are many Casson handles involved in the description of an exotic $\mathbb{R}^4$ which means that several fermionic fields can be assigned to. Besides, one can decompose a complicated Casson handle into a set of different Casson handles so the number of the resulting fermionic fields is a function of this decomposition and the complexity of the Casson handles. Let us denote these fermionic fields corresponding to the small exotic $\mathbb{R}^4_k$ as $\{\psi^s\}_k$. 

The model. Fix an Akbulut cork $A$ and Casson handle(s) embedded in $\mathbb{R}^4$. Magnetic spin $s$ is associated with the $SU(2)$ symmetry. Let this $SU(2) = S^3$ lies at the boundary of $A$. Generate spinor fields as above: spin-particles (electrons) $\{\psi's\}$ from Casson handle(s). They are free, in Minkowski spacetime, as far as Casson handles are not attached to $A$ and embedded in $\mathbb{R}^4$, the bound state of electrons and $s$ is thus corresponding to the exoticness of the smooth structure. The impurity refers to a boundary state of BCFT and together with channels for spinors they force the $SU(2)_k$ symmetry as before. This in turn determines the class $k[1] \in H^3(S^3, \mathbb{Z})$ which corresponds to exotic $\mathbb{R}^4_k$. The quantum bound state of electrons and $s$ is the Kondo state. In case of a single Casson handle a single (non-relativistic) electron emerges. One can conjecture that the emerging exotic $\mathbb{R}^4_1$ is the simplest exotic $\mathbb{R}^4$ of Bizaca and Gompf in this case [7, 15]. The Kondo bound state and its interactions with conducting electrons are described by boundary CFT (BCFT) where the impurity is represented by a conformal boundary conditions of WZW $SU(2)$ at $k = 1$.

When we include more than one Casson handle (CH), say $k$, in the description of exotic $\mathbb{R}^4$ (or a single CH though more complicated which can be decomposed into a set of CH’s) the model describes the Kondo bound state where the impurity is bounded with $k$ electrons $\{\psi's\}_k$ from different conducting channels. This is again a necessarily quantum process. The value $k$ is a suitable function of the complexity and decomposition of CH. Thus the geometry of the exotic $\mathbb{R}^4_k$ describes quantum entanglement of $k$ conducting electrons from $k$ symmetric channels with magnetic impurity of spin $s$. When smoothness of $\mathbb{R}^4$ is standard, i.e. $k = 0 \in H^3(S^3, \mathbb{Z})$, the electrons are free and not bounded by $s$. This means that the proposed model recognizes the smooth 4-geometry directly from quantum interacting magnetic matter in spacetime.

Discussion. The presented model assigns the geometry of a 4-dimensional Euclidean smooth $\mathbb{R}^4$ to the quantum interacting spin matter in the $k$ channel Kondo effect. The basic tool is the WZW $SU(2)$ model of CFT at the level $k$ which is derived from an exotic small smooth $\mathbb{R}^4_k$ and is also crucial in the description of the $k$ channel Kondo effect. This agreement allows to interpret fermions generated directly by Casson handles, as in [6], as electrons bounded by the magnetic impurity.

The correlation of 4-geometry and quantum matter occurs at low temperature and the spin-spin and electromagnetic interactions are much stronger at the microscopic scale than gravity. The emerging exotic 4-geometry can be understood as a kind of a relativistic limit for the low energy quantum matter in SM which incorporates also a gravity component, i.e. the non-flatness of the exotic $\mathbb{R}^4$. Here we do not uncover the full physical meaning of the correspondence (as e.g. for cosmology). However this model is the first instance of a possible modification (in a suitable regime) of a spacetime structure by purely quantum interactions as in standard model of particles. The question, whether this modification of a 4-dimensional geometry by quantum spin matter actually holds in reality and under what physical conditions, needs further careful studies and model buildings.

Due to the above discussion, the spacetime should be rather modelled locally by an exotic than a standard $\mathbb{R}^4$. The recognition of the structure of 4-space as replacing quantum interactions, is an important stage in building the final theory of quantum gravity.

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