EXPLICIT CONSTRUCTIONS OF QUASI-MONTE CARLO RULES FOR THE NUMERICAL INTEGRATION OF HIGH DIMENSIONAL PERIODIC FUNCTIONS

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Abstract. In this paper we give explicit constructions of point sets in the s dimensional unit cube yielding quasi-Monte Carlo algorithms which achieve the optimal rate of convergence of the worst-case error for numerically integrating high dimensional periodic functions. In the classical measure \( P_\alpha \) of the worst-case error introduced by Korobov the convergence is of \( O(N^{-\min(\alpha,d)}(\log N)^{s\alpha-2}) \) for every even integer \( \alpha \geq 1 \), where \( d \) is a parameter of the construction which can be chosen arbitrarily large and \( N \) is the number of quadrature points. This convergence rate is known to be best possible up to some \( \log N \) factors. We prove the result for the deterministic and also a randomized setting. The construction is based on a suitable extension of digital \((t,m,s)\)-nets over the finite field \( \mathbb{Z}_q \).

Key words. Numerical integration, quasi-Monte Carlo method, digital net, digital sequence, lattice rule

AMS subject classifications. primary: 11K38, 11K45, 65C05; secondary: 65D30, 65D32;

1. Introduction. Korobov [13] and independently Hlawka [11] introduced a quadrature formula which is suited for numerically integrating high dimensional periodic functions. More precisely, we want to approximate the high dimensional integral \( \int_{[0,1]^s} f(x) \, dx \) (where \( f \) is assumed to be periodic with period 1 in each coordinate) by a quasi-Monte Carlo rule, i.e., an equal weight quadrature rule \( Q_{N,s}(f) = N^{-1} \sum_{n=0}^{N-1} f(x_n) \), where \( x_0, \ldots, x_{N-1} \in [0,1]^s \) are the quadrature points. Specifically, Korobov and Hlawka suggested using a quadrature rule of the form \( Q_{N,g,s}(f) = N^{-1} \sum_{n=0}^{N-1} f((ng)/N) \), where for a vector of real numbers \( x = (x_1, \ldots, x_s) \) we define \( \{x\} \) as the fractional part of each component of \( x \), i.e., \( \{x_j\} = x_j - \lfloor x_j \rfloor = x_j \mod 1 \) and where \( g \in \mathbb{Z}^s \) is an integer vector. The quadrature rule \( Q_{N,g,s} \) is called lattice rule and \( g \) is called the generating vector (of the lattice rule). The monographs [12, 14, 19, 27] deal partly or entirely with the approximation of such integrals. (Note that the assumption that the integrand \( f \) is periodic is not really a restriction since there are transformations which transform non-periodic functions into periodic ones such that the smoothness of the integrand is preserved, see for example [27].)

To analyze the properties of a quadrature rule one considers then the worst-case error \( \sup_{f \in B_H} \left| \int_{[0,1]^s} f(x) \, dx - Q_{N,s}(f) \right| \), where \( B_H \) denotes some class of functions.

In the classical theory the class \( \varepsilon_\alpha^s \) of periodic functions has been considered where one demands that the absolute values of the Fourier coefficients of the function decay sufficiently fast (see [12, 14, 19, 27]). This leads us to the classical measure of the quality of lattice rules \( P_\alpha = \sup_{f \in \varepsilon_\alpha^s} \left| \int_{[0,1]^s} f(x) \, dx - Q_{N,s}(f) \right| \), which then for a lattice rule with generating vector \( g = (g_1, \ldots, g_s) \) can also be written as

\[
P_\alpha = P_\alpha(g, N) = \sum_{\substack{h \in \mathbb{Z}^s \setminus \{0\} \\mod N}} |\bar{h}|^{-\alpha},
\]

where \( h = (h_1, \ldots, h_s) \), \( h \cdot g = h_1 g_1 + \cdots + h_s g_s \) and \( |\bar{h}| = \prod_{j=1}^s \max(1, |h_j|) \). (Later on in this paper we prefer to use the more contemporary notation of reproducing...
kernel Hilbert spaces, in our case so-called Korobov spaces, (see Section 2.3, but as is well understood (and as is also shown in Section 2.3, the results also apply to the classical problem.)

By averaging over all generating vectors \( g \) several existence results for good lattice rules which achieve \( P_n = O(N^{-\alpha} (\log N)^{\alpha s}) \) have been shown, see \[12, 13, 14, 20, 19, 27\]. By a lower bound of Sharygin \[26\] this convergence is also known to be essentially best possible, as he showed that the worst-case error is at least of order \( N^{-\alpha} (\log N)^{s-1} \). But, except for dimension \( s = 2 \), no explicit generating vectors \( g \) which yield a small worst-case error are known. For \( s \geq 3 \) one relies on computer search to find good generating vectors \( g \) and many such search algorithms have been introduced and analysed, especially recently, see \[13, 28, 29, 34\].

On the other hand one can of course also use some other quadrature rule \( Q_{N,s}(f) = \sum_{n=0}^{N-1} \omega_n f(x_n) \) to numerically integrate functions in the class \( \varepsilon_n^\alpha \). In this case the worst-case error in the class \( \varepsilon_n^\alpha \) for a quadrature rule with weights \( \omega_0, \ldots, \omega_{N-1} \) and points \{\( x_0, \ldots, x_{N-1} \)\} \( \subset [0,1)^s \) is given by

\[
P_{\alpha}(\{x_0, \ldots, x_{N-1}\}) = \sum_{n,m=0}^{N-1} \omega_n \omega_m \sum_{h \in \mathbb{Z}^s \setminus \{0\}} e^{2\pi i h \cdot (x_n - x_m)} |h|^{-\alpha}. \tag{1.1}
\]

An explicit construction of such point sets was introduced by Niederreiter, see \[16\] Theorem 5.3, and is called Kronecker sequence. Here the idea is to choose the quadrature points of the form \{\( z \cdot k \), \( k = 1, 2, \ldots \)\}, where \( z \) is an \( s \)-dimensional vector of certain irrational numbers (for example one can choose \( z = (\sqrt{p_1}, \ldots, \sqrt{p_s}) \) where \( p_1, \ldots, p_s \) are distinct prime numbers). Depending on the smoothness \( \alpha \) certain points will be used more than once, see \[16\]. In practice, problems can occur because of the finite precision of computers making it impossible to use points whose coordinates are all irrational numbers.

Another construction of quadrature rules is due to Smolyak \[31\] and is nowadays called sparse grid, see also \[3\]. Those quadrature rules are sums over certain products of differences of one-dimensional quadrature rules. In principle any one-dimensional quadrature rule can be chosen as a basis, leading to different quadrature rules. In many cases the weights \( \omega_n \) of such quadrature rules are not known explicitly but can be precomputed. But even if the underlying one-dimensional quadrature rule has only positive weights, it is possible that some weights in Smolyak’s quadrature rules are negative, which can have a negative impact on the stability of the quadrature formula. In general, quadrature formulae for which all weights are equal and \( \sum_{n=0}^{N-1} \omega_n = 1 \), that is, \( \omega_n = N^{-1} \) for all \( n = 0, \ldots, N-1 \), are to be preferred. As mentioned above, such quadrature rules are called quasi-Monte Carlo rules, to which we now switch for the remainder of the paper.

As the weights for quasi-Monte Carlo rules are given by \( N^{-1} \) the focus lies on the choice of the quadrature points. Constructions of quadrature points have been introduced with the aim to distribute the points as evenly as possible over the unit cube. An explicit construction of well distributed point sets in the unit cube has been introduced by Sobol \[32\]. A similar construction was established by Faure \[7\] before Niederreiter \[18\] (see also \[19\]) introduced the general concept of \((t,m,s)\)-nets and \((t,s)\)-sequences and the construction scheme of digital \((t,m,s)\)-nets and digital \((t,s)\)-sequences. For such point sets it has been shown that the star discrepancy (which is a measure of the distribution properties of a point set) is \( O(N^{-\frac{1}{2}} (\log N)^{s-1}) \), see \[19\]. From this result it follows that those point sets yield quasi-Monte Carlo algorithms.
which achieve a convergence of $O(N^{-2} \log N)^{2s-2})$ for functions in the class $\varepsilon_\alpha^s$ for all $\alpha \geq 2$. This result holds in the deterministic and randomized setting.

For smoother functions though, i.e., larger values of $\alpha$ in the class $\varepsilon_\alpha^s$, one can expect higher order convergence. For example, if the partial derivatives up to order two are square integrable then one would expect an integration error of $O(N^{-4} \log N)^{c(\alpha)}$, for some $c(s) > 0$ depending only on $s$, in the function class $\varepsilon_\alpha^s$, and in general, if the mixed partial derivatives up to order $\alpha/2$ exist and are square integrable then one would expect an integration error in $\varepsilon_\alpha^s$ of $O(N^{-\alpha} \log N)^{c(s,\alpha)}$, for some $c(s,\alpha) > 0$ depending only on $s$ and $\alpha$. But until now $(t,m,s)$-nets and $(t,s)$-sequences have only been shown to yield a convergence of at best $O(N^{-2} \log N)^{2s-2})$ (or $O(N^{-3+\delta})$ for any $\delta > 0$ if one uses a randomization method called scrambling, see [24]) in $\varepsilon_\alpha^s$, even if the integrands satisfy stronger smoothness assumptions.

In this paper we show that a modification of digital $(t,m,s)$-nets and digital $(t,s)$-sequences introduced by Niederreiter [18, 19] yields point sets which achieve the optimal rate of convergence of the worst-case error $P_{2\alpha} = O(N^{-2\min(\alpha,d) \log N)^{2s-2}}$ for any integer $\alpha \geq 1$ and where $d \in \mathbb{N}$ is a parameter of the construction which can be chosen arbitrarily large. We too use the digital construction scheme introduced by Niederreiter [18, 19] for the construction of $(t,m,s)$-nets and $(t,s)$-sequences, but our analysis of the worst-case error shows that the $t$-value does not provide enough information about the point set. Hence we generalize the definition of digital $(t,m,s)$-nets and digital $(t,s)$-sequences to suit our needs. This leads us to the definition of digital $(t,\alpha,\beta,m,s)$-nets and digital $(t,\alpha,\beta,s)$-sequences. For $\alpha = \beta = 1$ those definitions reduce to the case introduced by Niederreiter, but are different for $\alpha > 1$. Subsequently we prove that quasi-Monte Carlo rules based on digital $(t,\alpha,\beta,m,s)$-nets and digital $(t,\alpha,\beta,s)$-sequences achieve the optimal rate of convergence. Further we give explicit constructions of digital $(t,\alpha,\min(\alpha,d),m,s)$-nets and digital $(t,\alpha,\min(\alpha,d),s)$-sequences, where $d \in \mathbb{N}$ is a parameter of the construction which can be chosen arbitrarily large.

Digital $(t,2,2,m,s)$-nets and digital $(t,2,2,s)$-sequences over $\mathbb{Z}_b$ (i.e. where $\alpha = \beta = 2$) can also be used for non-periodic function spaces where one uses randomly shifted and then folded point sets using the baker’s transformation (see [3]). Our analysis and error bounds for $\alpha = 2$ here also apply for the case considered in [3] (with different constants though), hence yielding useful constructions also for non-periodic function spaces where one uses the baker’s transformation. Using a digital $(t,\alpha,m,s)$-net with a scrambling algorithm (see [24]) on the other hand does not improve the performance in non-periodic spaces compared to $(t,m,s)$-nets.

In the following we summarize some properties of the quadrature rules:

- The quadrature rules introduced in this paper are equal weight quadrature rules which achieve the optimal rate of convergence up to some log $N$ factors and we show the result for deterministic and randomly digitally shifted quadrature rules. The upper bound for the randomized quadrature rules even improves upon the best known upper bound (more precisely, the power of the log $N$ factor for lattice rules for the worst-case error in $\varepsilon_\alpha^s$ for all dimensions $s \geq 2$ and even integers $\alpha \geq 2$ (compare Corollary 6.5 to Theorem 2 in [20]).
- The construction of the underlying point set is explicit.
- They automatically adjust themselves to the optimal rate of convergence in the class $\varepsilon_{\alpha,0}^s$ as long as $\alpha$ is an integer such that $\alpha \leq d$, where $d$ is a parameter of the construction which can be chosen arbitrarily large.
- The underlying point set is extensible in the dimension as well as in the
number of points, i.e., one can always add some coordinates or points to an existing point set such that the quality of the point set is preserved.

- Tractability and strong tractability results (see [30]) can be obtained for weighted Korobov spaces.

The outline of the paper is as follows. In the next section we introduce the necessary tools, namely Walsh functions, the digital construction scheme upon which the construction of the point set is based on and Korobov spaces. Further we also introduce the worst-case error in those Korobov spaces and we give a representation of this worst-case error for digital nets in terms of the Walsh coefficients of the reproducing kernel. In Section 3 we give the definition of digital \((t, \alpha, \beta, m, s)\)-nets and digital \((t, \alpha, \beta, s)\)-sequences. Further we prove some propagation rules for those digital nets and sequences. In Section 4 we give explicit constructions of digital \((t, \alpha, \beta, m, s)\)-nets and digital \((t, \alpha, \beta, s)\)-sequences and we prove some upper bounds on the \(t\)-value. We then show, Section 5, that quasi-Monte Carlo rules based on those digital nets and sequences achieve the optimal rate of convergence of the worst-case error in the Korobov spaces. The results are based on entirely deterministic point sets. Section 6 finally deals with randomly digitally shifted digital \((t, \alpha, \beta, m, s)\)-nets and \((t, \alpha, \beta, s)\)-sequences and we show similar results for the mean square worst-case error in the Korobov space for this setting. The Appendix is devoted to the analysis of the Walsh coefficients of the Walsh series representation of \(B_{2\alpha}(|x - y|)\), where \(B_{2\alpha}\) is the Bernoulli polynomial of degree \(2\alpha\). In the last section we give a concrete example of a digital \((t, \alpha, \alpha, m, s)\)-net where we compute the \(t\)-value by hand.

2. Preliminaries. In this section we introduce the necessary tools for the analysis of the worst-case error and the construction of the point sets. In the following let \(\mathbb{N}\) denote the set of natural numbers and let \(\mathbb{N}_0\) denote the set of non-negative integers.

2.1. Walsh functions. In the following we define Walsh functions in base \(b \geq 2\), which are the main tool of analyzing the worst-case error. First we give the definition for the one-dimensional case.

**Definition 2.1.** Let \(b \geq 2\) be an integer and represent \(k \in \mathbb{N}_0\) in base \(b\), \(k = \kappa_{a-1} b^{a-1} + \cdots + \kappa_0\) with \(\kappa_i \in \{0, \ldots, b - 1\}\). Further let \(\omega_b = e^{2\pi i/b}\). Then the \(k\)-th Walsh function \(\text{wal}_k : [0, 1) \rightarrow \{1, \omega_b, \ldots, \omega_{b-1}\}\) in base \(b\) is given by

\[
\text{wal}_k(x) = \omega_b^{x_1 \kappa_{a-1} + \cdots + x_a \kappa_0},
\]

for \(x \in [0, 1)\) with base \(b\) representation \(x = x_1 b^{-1} + x_2 b^{-2} + \cdots\) (unique in the sense that infinitely many of the \(x_i\) are different from \(b-1\)).

**Definition 2.2.** For dimension \(s \geq 2\), \(x = (x_1, \ldots, x_s) \in [0, 1)^s\) and \(k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s\) we define \(\text{wal}_k : [0, 1)^s \rightarrow \{1, \omega_b, \ldots, \omega_{b-1}\}\) by

\[
\text{wal}_k(x) = \prod_{j=1}^s \text{wal}_{k_j}(x_j).
\]

As we will always use Walsh functions in base \(b\) we will in the following often write \(\text{wal}\) instead of \(\text{wal}_b\).

We introduce some notation. By \(\oplus\) we denote the digit-wise addition modulo \(b\), i.e., for \(x = \sum_{i=0}^{\infty} x_i b^{-i}\) and \(y = \sum_{i=0}^{\infty} y_i b^{-i}\) we define

\[
x \oplus y = \sum_{i=\infty}^{\infty} z_i b^{-i},
\]
where \( z_i \in \{0, \ldots, b-1\} \) is given by \( z_i \equiv x_i + y_i \pmod{b} \) and let \( \ominus \) denote the digit-wise subtraction modulo \( b \). In the same manner we also define a digit-wise addition and digit-wise subtraction for non-negative integers based on the \( b \)-adic expansion. For vectors in \([0,1)^s\) or \( \mathbb{N}_0^s \) the operations \( \oplus \) and \( \ominus \) are carried out component-wise. Throughout the paper we always use base \( b \) for the operations \( \oplus \) and \( \ominus \). Further we call \( x \in [0,1) \) a \( b \)-adic rational if it can be written in a finite base \( b \) expansion.

In the following proposition we summarize some basic properties of Walsh functions.

**Proposition 2.3.**
1. For all \( k, l \in \mathbb{N}_0 \) and all \( x, y \in [0,1) \), with the restriction that if \( x, y \) are not \( b \)-adic rationals then \( x \oplus y \) is not allowed to be a \( b \)-adic rational, we have

\[
\text{wal}_k(x) \cdot \text{wal}_l(x) = \text{wal}_{k+l}(x), \quad \text{wal}_k(x) \cdot \text{wal}_l(y) = \text{wal}_k(x \oplus y).
\]

2. We have

\[
\int_0^1 \text{wal}_0(x) \, dx = 1 \quad \text{and} \quad \int_0^1 \text{wal}_k(x) \, dx = 0 \quad \text{if} \quad k > 0.
\]

3. For all \( k, l \in \mathbb{N}_0^s \) we have the following orthogonality properties:

\[
\int_{[0,1)^s} \text{wal}_k(x) \text{wal}_l(x) \, dx = \begin{cases} 1, & \text{if} \quad k = l, \\ 0, & \text{otherwise}. \end{cases}
\]

4. For any \( f \in L_2([0,1)^s) \) and any \( \sigma \in [0,1)^s \) we have

\[
\int_{[0,1)^s} f(x \oplus \sigma) \, dx = \int_{[0,1)^s} f(x) \, dx.
\]

5. For any integer \( s \geq 1 \) the system \( \{\text{wal}_k : k = (k_1, \ldots, k_s), k_1, \ldots, k_s \geq 0\} \) is a complete orthonormal system in \( L_2([0,1)^s) \).

The proofs of 1.-3. are straightforward and for a proof of the remaining items see [2] or [33] for more information.

**2.2. The digital construction scheme.** The construction of the point set used here is based on the digital construction scheme introduced by Niederreiter, see [19].

**Definition 2.4.** Let integers \( m, s \geq 1 \) and \( b \geq 2 \) be given. Let \( R_b \) be a commutative ring with identity such that \( |R_b| = b \) and let \( \mathbb{Z}_b = \{0, \ldots, b-1\} \). Let \( \mathcal{C}_1, \ldots, \mathcal{C}_s \in R_b^{m \times m} \) with \( C_j = (c_{j,k,l})_{1 \leq k, l \leq m} \). Further, let \( \psi_l : \mathbb{Z}_b \to \mathbb{Z}_b \) for \( l = 0, \ldots, m-1 \) and \( \mu_{j,k} : R_b \to \mathbb{Z}_b \) for \( j = 1, \ldots, s \) and \( k = 1, \ldots, m \) be bijections.

For \( n = 0, \ldots, b^m - 1 \) let \( n = \sum_{l=0}^{m-1} a_l(n) b^l \) with all \( a_l(n) \in \mathbb{Z}_b \), be the base \( b \) digit expansion of \( n \). Let \( \tilde{n} = (\psi_0(a_0(n)), \ldots, \psi_{m-1}(a_{m-1}(n)))^T \) and let \( \tilde{y}_j = (y_{j,1}, \ldots, y_{j,m})^T = C_j \tilde{n} \) for \( j = 1, \ldots, s \). Then we define \( x_{j,n} = \mu_{j,1}(y_{j,1}) b^{-1} + \cdots + \mu_{j,m}(y_{j,m}) b^{-m} \) for \( j = 1, \ldots, s \) and \( n = 0, \ldots, b^m - 1 \) and the \( n \)-th point \( \mathbf{x}_n \) is then given by \( \mathbf{x}_n = (x_{1,n}, \ldots, x_{s,n}) \). The point set \( \{\mathbf{x}_0, \ldots, \mathbf{x}_{b^m-1}\} \) is called a digital net (over \( R_b \)) with generating matrices \( \mathcal{C}_1, \ldots, \mathcal{C}_s \).

For \( m = \infty \) we obtain a sequence \( \{\mathbf{x}_0, \mathbf{x}_1, \ldots\} \), which is called a digital sequence (over \( R_b \)) with generating matrices \( \mathcal{C}_1, \ldots, \mathcal{C}_s \).

Niederreiter’s concept of a digital \((t, m, s)\)-net and a digital \((t, s)\)-sequence will appear as a special case in Section 4. Apart from Section 4 and Section 3, where we state the results using Definition 2.4 in the general form, we use only a special case of Definition 2.4 where we assume that \( b \) is a prime number, we choose \( R_b \) the finite
field \( \mathbb{Z}_b \) and the bijections \( \psi_l \) and \( \mu_{j,k} \) from \( \mathbb{Z}_b \) to \( \mathbb{Z}_b \) are all chosen to be the identity map.

We remark that throughout the paper when Walsh functions \( \text{wal} \), digit-wise addition \( \oplus \), digit-wise subtraction \( \ominus \) or digital nets are used in conjunction with each other we always use the same base \( b \) for each of those operations.

### 2.3. Korobov space.

Historically the function class \( \varepsilon^*_\alpha \) has been used. In this paper we use a more contemporary notation by replacing the function class \( \varepsilon^*_\alpha \) with a reproducing kernel Hilbert space \( \mathcal{H}_\alpha \) called Korobov space. The worst-case error expression (1.1) will almost be the same for both function classes and hence the results apply for both cases.

A reproducing kernel Hilbert space \( \mathcal{H} \) over \([0,1)^s\) is a Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) which allows a function \( K : [0,1)^s \to \mathbb{R} \) such that \( K(x,y) = K(y,x) \) and \( \langle f, K(\cdot,y) \rangle = f(y) \) for all \( x, y \in [0,1)^s \) and all \( f \in \mathcal{H} \). For more information on reproducing kernel Hilbert spaces see [1], for more information on reproducing kernel Hilbert spaces in the context of numerical integration see for example [5, 30].

The Korobov space \( \mathcal{H}_\alpha \) is a reproducing kernel Hilbert space of periodic functions. Its reproducing kernel is given by

\[
K_\alpha(x,y) = \sum_{h \in \mathbb{Z}^s} \frac{e^{2\pi i h \cdot (x-y)}}{|h|^{2\alpha}},
\]

where \( \alpha > 1/2 \) and \( \bar{h} = \prod_{j=1}^s \max(1,|h_j|) \). The inner product in the space \( \mathcal{H}_\alpha \) is given by

\[
\langle f, g \rangle_\alpha = \sum_{h \in \mathbb{Z}^s} \bar{h}^{2\alpha} \hat{f}(h) \hat{g}(h),
\]

(2.1)

where

\[
\hat{f}(h) = \int_{[0,1)^s} f(x) e^{-2\pi i h \cdot x} \, dx
\]

are the Fourier coefficients of \( f \). The norm is given by \( \|f\|_\alpha = \langle f, f \rangle_\alpha^{1/2} \).

Note that for \( \alpha \) a natural number and any \( x \in (0,1) \) we have

\[
B_{2\alpha}(x) = \frac{(-1)^{\alpha+1}(2\alpha)!}{(2\pi)^{2\alpha}} \sum_{h \neq 0} \frac{e^{2\pi i h x}}{|h|^{2\alpha}},
\]

where \( B_{2\alpha} \) is the Bernoulli polynomial of degree \( 2\alpha \). Hence, for \( \alpha \) a natural number we can write

\[
K_\alpha(x,y) = \prod_{j=1}^s \left( 1 + \sum_{h \neq 0} \frac{e^{2\pi i h(x_j-y_j)}}{|h|^{2\alpha}} \right) = \prod_{j=1}^s \left( 1 - (-1)^\alpha \frac{(2\pi)^{2\alpha}}{(2\alpha)!} B_{2\alpha}(|x_j - y_j|) \right).
\]

Let now

\[
K_\alpha(x,y) = 1 + \sum_{h \neq 0} \frac{e^{2\pi i h(x-y)}}{|h|^{2\alpha}} = 1 - (-1)^\alpha \frac{(2\pi)^{2\alpha}}{(2\alpha)!} B_{2\alpha}(|x - y|). \tag{2.2}
\]
Then we have

\[ K_\alpha(x, y) = \prod_{j=1}^s K_\alpha(x_j, y_j), \]

where \( x = (x_1, \ldots, x_s) \) and \( y = (y_1, \ldots, y_s) \). Hence the Korobov space is a tensor product of one-dimensional reproducing kernel Hilbert spaces.

Though \( \alpha > 1/2 \) can in general be any real number we restrict ourselves to integers \( \alpha \geq 1 \) for most of this paper. The bounds on the integration error for \( H_\alpha \) with \( \alpha \geq 1 \) a real number still apply when one replaces \( \alpha \) with \( \lfloor \alpha \rfloor \), as in this case the unit ball of \( H_\alpha \) given by \( \{ f \in H_\alpha : \|f\|_\alpha \leq 1 \} \) is contained in the unit ball \( \{ f \in H_{\lfloor \alpha \rfloor} : \|f\|_{\lfloor \alpha \rfloor} \leq 1 \} \) of \( H_{\lfloor \alpha \rfloor} \) as \( \|f\|_{\lfloor \alpha \rfloor} \leq \|f\|_\alpha \). Hence it follows that integration in the space \( H_\alpha \) is easier than integration in the space \( H_{\lfloor \alpha \rfloor} \).

In general, the worst-case error \( e(P, H) \) for multivariate integration in a normed space \( H \) over \([0,1]^s\) with norm \( \| \cdot \| \) using a point set \( P \) is given by

\[ e(P, H) = \sup_{f \in H, \|f\| \leq 1} \left| \int_{[0,1]^s} f(x) \, dx - Q_P(f) \right|, \]

where \( Q_P(f) = N^{-1} \sum_{x \in P} f(x) \) and \( N = |P| \) is the number of points in \( P \). If \( H \) is a reproducing kernel Hilbert space with reproducing kernel \( K \) we will write \( e(P, K) \) instead of \( e(P, H) \). It is known that (see for example [30])

\[ e^2(P, K) = \int_{[0,1]^s} K(x, y) \, dx \, dy - 2 \sum_{n=0}^{N-1} \int_{[0,1]^s} K(x_n, y) \, dy + \frac{1}{N^2} \sum_{n,l=0}^{N-1} K(x_n, x_l), \]

where \( P = \{x_0, \ldots, x_{N-1}\} \). Hence for the Korobov space \( H_\alpha \) we obtain

\[ e^2(P, K_\alpha) = -1 + \frac{1}{N^2} \sum_{n,h=0}^{N-1} K_\alpha(x_n, x_h). \]

Therefore it follows that \( e^2(P, K_\alpha) = P_{2\alpha} \) and hence our results also apply to the classical setting introduced by Korobov [13].

It follows from Proposition 2.3 that \( K_\alpha \) can be represented by a Walsh series, i.e., let

\[ K_\alpha(x, y) = \sum_{k,l \in \mathbb{N}_0} r_{b,\alpha}(k, l) \overline{\text{wal}_k(x)} \overline{\text{wal}_l(y)}, \]

where

\[ r_{b,\alpha}(k, l) = \int_{[0,1]^s} K_\alpha(x, y) \overline{\text{wal}_k(x)} \overline{\text{wal}_l(y)} \, dx \, dy. \]

As the kernel \( K_\alpha \) is a product of one-dimensional kernels it follows that \( r_{b,\alpha}(k, l) = \prod_{j=1}^s r_{b,\alpha}(k_j, l_j) \), where \( k = (k_1, \ldots, k_s) \) and \( l = (l_1, \ldots, l_s) \) and

\[ r_{b,\alpha}(k, l) = \int_0^1 \int_0^1 K_\alpha(x, y) \overline{\text{wal}_k(x)} \overline{\text{wal}_l(y)} \, dx \, dy. \]
For a digital net with generating matrices $C_1, \ldots, C_s$ let $D = D(C_1, \ldots, C_s)$ be the dual net given by

$$D = \{ k \in \mathbb{N}_0^s \setminus \{0\} : C_1^T \bar{k}_1 + \cdots + C_s^T \bar{k}_s = 0 \},$$

where for $k = (k_1, \ldots, k_s)$ with $k_j = \kappa_{j,0} + \kappa_{j,1} b + \cdots$ we set $\bar{k}_j = (\kappa_{j,0}, \ldots, \kappa_{j,m-1})^T$. Further, for $\emptyset \neq u \subseteq \{1, \ldots, s\}$ let $D_u = D((C_j)_{j \in u})$. We have the following theorem.

**Theorem 2.5.** Let $C_1, \ldots, C_s \in \mathbb{Z}_{\mathbb{N}}^{m \times m}$ be the generating matrices of a digital net $P_{b,m}$ and let $D$ denote the dual net. Then for any $\alpha > 1/2$ the square worst-case error in $H_\alpha$ is given by

$$e^2(P_{b,m}, K_\alpha) = \sum_{k,l \in D} r_{b,\alpha}(k,l).$$

**Proof.** From (2.4) and (2.5) it follows that

$$e^2(P_{b,m}, K_\alpha) = -1 + \sum_{k,l \in \mathbb{N}_0^s} r_{b,\alpha}(k,l) \frac{1}{b^{2m}} \sum_{x,y \in P_{b,m}} \text{wal}_k(x) \overline{\text{wal}_l(y)}.$$

In [5] it was shown that

$$\frac{1}{b^m} \sum_{x \in P_{b,m}} \text{wal}_k(x) = \begin{cases} 1 & \text{if } k \in D \cup \{0\}, \\ 0 & \text{otherwise}. \end{cases}$$

Hence we have

$$e^2(P_{b,m}, K_\alpha) = -1 + \sum_{k,l \in D \cup \{0\}} r_{b,\alpha}(k,l).$$

In the following we will show that $r_{b,\alpha}(0,0) = 1$ and $r_{b,\alpha}(0,k) = r_{b,\alpha}(k,0) = 0$ if $k \neq 0$ from which the result then follows. Note that it is enough to show those identities for the one dimensional case. We have $\text{wal}_0(x) = 1$ for all $x \in [0,1)$ and hence

$$r_{b,\alpha}(0,k) = \int_0^1 \int_0^1 (1 + \sum_{h \in \mathbb{Z} \setminus \{0\}} |h|^{-2\alpha} e^{2\pi i h(x-y)}) \text{wal}_k(y) \, dx \, dy$$

$$= \int_0^1 \text{wal}_k(y) \, dy + \int_0^1 \sum_{h \in \mathbb{Z} \setminus \{0\}} |h|^{-2\alpha} \int_0^1 e^{2\pi i hx} \, dx \, e^{-2\pi i hy} \text{wal}_k(y) \, dy$$

$$= \int_0^1 \text{wal}_k(y) \, dy.$$

It now follows from Proposition (2.3) that $r_{b,\alpha}(0,0) = 1$ and $r_{b,\alpha}(0,k) = 0$ for $k > 0$. The result for $r_{b,\alpha}(k,0)$ can be obtained in the same manner. Hence the result follows.

In the following lemma we obtain a formula for the Walsh coefficients $r_{b,\alpha}$. Let $b \geq 2$ be an integer and let $\alpha > 1/2$ be a real number. The Walsh coefficients $r_{b,\alpha}(k,l)$ for $k, l \in \mathbb{N}$ are given by

$$r_{b,\alpha}(k,l) = \sum_{h \in \mathbb{Z} \setminus \{0\}} \frac{\beta_{b,k} \beta_{b,l}}{|h|^{2\alpha}},$$
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where \( \beta_{h,k} = \int_0^1 e^{-2\pi i h x} \text{wal}_k(x) \, dx \).

**Proof.** We have

\[
 r_{b,\alpha}(k,l) = \int_0^1 \int_0^1 \sum_{h \in \mathbb{Z}\setminus\{0\}} |h|^{-2\alpha} e^{2\pi i h (x-y)} \text{wal}_k(x) \text{wal}_l(y) \, dx \, dy
\]

\[
 = \sum_{h \in \mathbb{Z}\setminus\{0\}} |h|^{-2\alpha} \int_0^1 e^{2\pi i h x} \text{wal}_k(x) \, dx \int_0^1 e^{-2\pi i h y} \text{wal}_l(y) \, dy.
\]

The result follows. \( \square \)

It is difficult to calculate the exact value of \( r_{b,\alpha}(k,l) \) in general, but for our purposes it is enough to obtain an upper bound. Note that \( r_{b,\alpha}(k,k) \) is a non-negative real number.

**Lemma 2.7.** Let \( b \geq 2 \) be an integer and let \( \alpha > 1/2 \) be a real number. The Walsh coefficients \( r_{b,\alpha}(k,l) \) for \( k,l \in \mathbb{N} \) are bounded by

\[
|r_{b,\alpha}(k,l)|^2 \leq r_{b,\alpha}(k,k)^{\frac{1}{2}} r_{b,\alpha}(l,l)^{\frac{1}{2}}.
\]

**Proof.** Using Lemma 2.6 we obtain

\[
|r_{b,\alpha}(k,l)|^2 \leq \left( \sum_{h \in \mathbb{Z}\setminus\{0\}} |\beta_{h,k}| |\beta_{h,l}|\right)^2 \leq \sum_{h \in \mathbb{Z}\setminus\{0\}} |\beta_{h,k}|^2 \sum_{h \in \mathbb{Z}\setminus\{0\}} |\beta_{h,l}|^2 \leq r_{b,\alpha}(k,k)^{\frac{1}{2}} r_{b,\alpha}(l,l)^{\frac{1}{2}}.
\]

The result follows. \( \square \)

In the following we will write \( r_{b,\alpha}(k) \) instead of \( r_{b,\alpha}(k,k) \) and also \( r_{b,\alpha}(k) \) instead of \( r_{b,\alpha}(k,k) \).

**Lemma 2.8.** Let \( C_1, \ldots, C_s \in \mathbb{Z}^{m \times m}_b \) be the generating matrices of a digital net \( P_{b^m} \) and let \( D \) denote the dual net. Then for any natural number \( \alpha \) the worst-case error in \( H_\alpha \) is bounded by

\[
e(P_{b^m}, K_\alpha) \leq \sum_{k \in D} \sqrt{r_{b,\alpha}(k)}.
\]

**Proof.** From Theorem 2.4 and Lemma 2.7 it follows that

\[
e^2(P_{b^m}, K_\alpha) \leq \sum_{k,l \in D} |r_{b,\alpha}(k,l)| \leq \left( \sum_{k \in D} \sqrt{r_{b,\alpha}(k)} \right)^2
\]

and hence the result follows. \( \square \)

For \( \alpha \geq 1 \) a natural number we can write the reproducing kernel in terms of Bernoulli polynomials of degree \( 2\alpha \). Then for \( k \geq 1 \) we have

\[
r_{b,\alpha}(k) = (-1)^{\alpha+1} \frac{(2\pi)^{2\alpha}}{(2\alpha)!} \int_0^1 \int_0^1 B_{2\alpha}(|x-y|) \text{wal}_k(x) \text{wal}_k(y) \, dx \, dy.
\]

Note that the Bernoulli polynomials of even degree \( 2\alpha \) are of the form

\[
B_{2\alpha}(x) = c_\alpha x^{2\alpha} + c_{\alpha-1} x^{2(\alpha-1)} + \cdots + c_0 + c x^{2\alpha-1},
\]
for some rational numbers \( c_1, \ldots, c_0, c \) with \( c_\alpha, c \neq 0 \). Let

\[
I_\alpha(k) = \int_0^1 \int_0^1 |x - y|^2 w_k(x) w_k(y) \, dx \, dy.
\]  

(2.6)

As mentioned above, \( r_{b,\alpha}(k) \) is a real number such that \( r_{b,\alpha}(k) \geq 0 \) for all \( k \geq 1 \) and \( \alpha > 1/2 \), hence it follows that for any natural number \( \alpha \) we have

\[
r_{b,\alpha}(k) \leq \frac{(2\pi)^{2\alpha}}{(2\alpha)!} \left( |c_\alpha I_{2a}(k)| + |c_{\alpha-1} I_{2(\alpha-1)}(k)| + \cdots + |c_0 I_0(k)| + |c I_{2\alpha-1}| \right).
\]

Using Lemma 8.2 and Lemma 8.3 from the Appendix we obtain the following lemma.

**Lemma 2.9.** Let \( b, \alpha \in \mathbb{N} \) with \( b \geq 2 \). For \( k \in \mathbb{N} \) with \( k = \kappa_1 b^{a_1-1} + \cdots + \kappa_r b^{a_r-1} \) where \( \nu \geq 1 \), \( \kappa_1, \ldots, \kappa_r \in \{1, \ldots, b-1\} \) and \( 1 \leq a_\nu < \cdots < a_1 \) let \( q_{b,\alpha}(k) = b^{-a_1 - \cdots - a_{\min(\nu, \alpha)}}. \) Then for any natural number \( \alpha \) and any natural number \( b \geq 2 \) there exists a constant \( C_{b,\alpha} > 0 \) which depends only on \( b \) and \( \alpha \) such that

\[
r_{b,\alpha}(k) \leq C_{b,\alpha}^2 q_{b,\alpha}(k) \quad \text{for all } k \geq 1.
\]

Let now \( q_{b,\alpha}(0) = 1 \). For \( k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s \) we define \( q_{b,\alpha}(k) = \prod_{j=1}^s q_{b,\alpha}(k_j) \).

We have the following lemma.

**Lemma 2.10.** Let \( m \geq 1 \), \( b \geq 2 \) and \( \alpha \geq 2 \) be natural numbers and let \( D_{b,m,u} = D_u \cap \{1, \ldots, b^m-1\}^u \). Then we have

\[
\sum_{k \in D} \sqrt{r_{b,\alpha}(k)} \leq \sum_{\emptyset \neq u \subseteq \{1, \ldots, s\}} \left( 1 + b^{-\alpha m} C_{\alpha} (\alpha + b^{-2})^{s-u} |C_{b,\alpha}^{|u|} (1 + \alpha + b^{-2})^{|u|} Q_{b,m,u,\alpha}^*(C_1, \ldots, C_s) \right. \nonumber
\]

\[
+ \left. (1 + b^{-\alpha m} C_{\alpha} (\alpha + b^{-2})^s - 1, \right)
\]

where \( C_{\alpha} \) is the constant from Lemma 2.9 and where

\[
Q_{b,m,u,\alpha}^*(C_1, \ldots, C_s) = \sum_{k \in D_{b,m,u}} q_{b,\alpha}(k).
\]

**Proof.** Every \( k \in \mathbb{N}_0^n \) can be uniquely written in the form \( k = h + b^m l \) with \( h \in \{0, \ldots, b^m-1\}^s \) and \( l \in \mathbb{N}_0^m \). Let \( D_{b,m} = D \cap \{0, \ldots, b^m-1\}^s \). Then we have

\[
\sum_{k \in D} \sqrt{r_{b,\alpha}(k)} = \sum_{l \in \mathbb{N}_0^m} \sqrt{r_{b,\alpha}(b^m l)} + \sum_{h \in D_{b,m}} \sum_{l \in \mathbb{N}_0^m} \sqrt{r_{b,\alpha}(h + b^m l)}.
\]

For the first sum we have

\[
\sum_{l \in \mathbb{N}_0^m} \sqrt{r_{b,\alpha}(b^m l)} = -1 + \sum_{l \in \mathbb{N}_0^m} \sqrt{r_{b,\alpha}(b^m l)} = -1 + \left( \sum_{l=0}^{\infty} \sqrt{r_{b,\alpha}(b^m l)} \right)^s.
\]

By using Lemma 8.8 from the Appendix and Lemma 2.9 we obtain that

\[
\sum_{l=0}^{\infty} \sqrt{r_{b,\alpha}(b^m l)} = 1 + b^{-\alpha m} \sum_{l=1}^{\infty} \sqrt{r_{b,\alpha}(l)} \leq 1 + b^{-\alpha m} C_{b,\alpha} \sum_{l=1}^{\infty} q_{b,\alpha}(l).
\]
We need to show that \( \sum_{l=1}^{\infty} q_{b,\alpha}(l) \leq \alpha + b^{-2} \). Let \( l = l_1b^{c_1-1} + \cdots + l_{\nu}b^{c_{\nu}-1} \) for some \( \nu \geq 1 \) with \( 1 \leq c_1 < \cdots < c_{\nu} \) and \( l_1, \ldots, l_{\nu} \in \{1, \ldots, b-1\} \). First we consider the sum over all those \( l \) for which \( 1 \leq \nu \leq \alpha \). This part of the sum is bounded by

\[
\sum_{\nu=1}^{\alpha} (b-1)^\nu \sum_{c_1=\nu}^{\infty} \sum_{c_2=\nu}^{c_1-1} \cdots \sum_{c_{\nu}=1}^{c_{\nu-1}-1} b^{-c_1-\cdots-c_{\nu}} \leq \sum_{\nu=1}^{\alpha} (b-1)^\nu (\sum_{c=1}^{\infty} b^{-c})^\nu = \alpha.
\]

If \( \nu > \alpha \) we have \( q_{b,\alpha}(l) = q_{b,\alpha}(l') \) for \( l = l_1b^{c_1-1} + \cdots + l_{\nu}b^{c_{\nu}-1} \) and where \( l' = l'(l) = l_1b^{c_1-1} + \cdots + l_{\alpha}b^{c_{\alpha}-1} \). Thus we only need to sum over all \( l' \) (i.e. natural numbers with exactly \( \alpha \) digits) and for given \( l' \) multiplying it with the number of \( l \) which yield the same \( l' \), which is \( b^{c_{\alpha}-1} - 1 \) (and which we bound in the following by \( b^{c_{\alpha}-1} \)). We have

\[
(b-1)^\alpha \sum_{c_1=\alpha+1}^{\infty} \sum_{c_2=\alpha}^{c_1-1} \cdots \sum_{c_{\alpha}=2}^{c_{\alpha-1}-1} b^{-c_1-\cdots-c_{\alpha}} \leq b^{-1}(b-1)^\alpha \sum_{c_1=\alpha+1}^{\infty} \sum_{c_2=\alpha}^{c_1-1} \cdots \sum_{c_{\alpha}=1}^{c_{\alpha-1}-1} (c_{\alpha-1} - 2) b^{-c_1-\cdots-c_{\alpha}} = \frac{1}{b^2}.
\]

Thus we obtain \( \sum_{l=1}^{\infty} q_{b,\alpha}(l) \leq \alpha + b^{-2} \).

Further we have

\[
\sum_{h \in D_{b,\alpha}} \sum_{l \in \mathbb{N}_0} \sqrt{r_{b,\alpha}(h + b^m l)} = \sum_{h \in D_{b,\alpha}} \prod_{j=1}^{s} \sum_{l=0}^{\infty} \sqrt{r_{b,\alpha}(h_j + b^m l)},
\]

where \( h = (h_1, \ldots, h_s) \). By using Lemma 8.8 from the Appendix and Lemma 2.9 we obtain

\[
\sum_{l=0}^{\infty} \sqrt{r_{b,\alpha}(b^m l)} = 1 + b^{-\alpha m} C_{b,\alpha} \sum_{l=1}^{\infty} q_{b,\alpha}(l) \leq 1 + b^{-\alpha m} C_{b,\alpha} (\alpha + b^{-2}).
\]

Let now \( 0 < h_j < b^m \). From Lemma 2.9 we obtain

\[
\sqrt{r_{b,\alpha}(h_j + b^m l)} \leq C_{b,\alpha} q_{b,\alpha}(h_j + b^m l) \leq C_{b,\alpha} q_{b,\alpha}(h_j) q_{b,\alpha}(l).
\]

From above we have \( \sum_{l=0}^{\infty} q_{b,\alpha}(l) \leq 1 + \alpha + b^{-2} \) and hence

\[
\sum_{l=0}^{\infty} \sqrt{r_{b,\alpha}(h_j + b^m l)} \leq q_{b,\alpha}(h_j) C_{b,\alpha} \sum_{l=0}^{\infty} q_{b,\alpha}(l) \leq C_{b,\alpha} (1 + \alpha + b^{-2}) q_{b,\alpha}(h_j).
\]
Thus we obtain
\[
\sum_{h \in D_{b,m}^s} \sum_{t \in \mathbb{N}_0^s} \sqrt{r_{b,\alpha}(h + b^m t)} = \sum_{\emptyset \neq u \subseteq \{1, \ldots, s\}} \sum_{h_u \in D_{b,m,u}^s} \prod_{j \in u} \sum_{l = 0}^{\infty} \sqrt{r_{b,\alpha}(h_j + b^m l)} \prod_{j \notin u} \sum_{l = 0}^{\infty} \sqrt{r_{b,\alpha}(b^m l)} \leq \sum_{\emptyset \neq u \subseteq \{1, \ldots, s\}} (1 + b^{-\alpha m} C_{b,\alpha}(\alpha + b^{-2}))^{s-|u|} |u|^{1+\alpha+b^{-2}} |u| \sum_{h_u \in D_{b,m,u}^s} \prod_{j \in u} q_{b,\alpha}(h_j),
\]
where \(h_u = (h_j)_{j \in u}\). The result follows. \(\square\)

In [26] it was shown that the square worst-case error for numerical integration in the Korobov space can at best be of \(O(N^{-2\alpha}(\log N)^{s-1})\), where \(N\) is the number of quadrature points. Hence Lemma 2.10 shows that it is enough to consider only \(Q_{b,m,u,\alpha}^s(C_1, \ldots, C_s)\) in order to investigate the convergence rate of digitally shifted digital nets.

3. \((t, \alpha, \beta, \gamma, \delta, s)\)-nets and \((t, \alpha, \beta, s)\)-sequences. The \(t\) value of a \((t, m, s)\)-net is a quality parameter for the distribution properties of the net. A low \(t\) value yields well distributed point sets and it has been shown, see for example [5, 19], that a small \(t\) value also guarantees a small worst-case error for integration in Sobolev spaces for which the partial first derivatives are square integrable.

In the following we will show how the definition of the \(t\) value needs to be modified in order to obtain faster convergence rates for periodic Sobolev spaces for which the partial derivatives up to order \(\alpha\) are square integrable. It is the aim of this definition to translate the problem of minimizing the worst-case error into an algebraical problem concerning the generating matrices. (This definition can therefore also be used in a computer search algorithm, where one could for example search for the polynomial lattice with the smallest \(t(\alpha)\) value which in turn yields a small worst-case error for integration of periodic functions.)

For natural numbers \(\alpha \geq 1\), Lemma 2.9 suggests to define the following metric \(\mu_{b,\alpha}(k, l) = \mu_{b,\alpha}(k \odot l)\) on \(\mathbb{N}_0^s\) which is an extension of the metric introduced in [17], see also [24] (for \(\alpha = 1\) we basically obtain the metric in [17, 23]). Here \(\mu_{b,\alpha}(0) = 0\) and for \(k \in \mathbb{N}\) with \(k = k_1 b^{\alpha_1-1} + \cdots + k_s b^{\alpha_s-1}\) where \(1 \leq \alpha_1 < \cdots < \alpha_s\) and \(k_i \in \{1, \ldots, b-1\}\) let \(\mu_{b,\alpha}(k) = a_1 + \cdots + a_{\min(\alpha, \nu)}\). For a \(k \in \mathbb{N}_0^s\) with \(k = (k_1, \ldots, k_s)\) let \(\mu_{b,\alpha}(k) = \mu_{b,\alpha}(k_1) + \cdots + \mu_{b,\alpha}(k_s)\). Then we have \(q_{b,\alpha}(k) = b^{-\mu_{b,\alpha}(k)}\). Hence in order to obtain a small worst-case error in the Korobov space \(\mathcal{H}_\alpha\), we need digital nets for which \(\min \{\mu_{b,\alpha}(k) : k \in D\}\) is large. We can translate this property into a linear independence property of the row vectors of the generating matrices \(C_1, \ldots, C_s\). We have the following definition.

**Definition 3.1.** Let \(m, \alpha \geq 1\) be natural numbers, let \(0 < \beta \leq \alpha\) be a real number and let \(0 \leq t \leq \beta m\) be a natural number. Let \(R_b\) be a ring with \(b\) elements and let \(C_1, \ldots, C_s \in R_b^{m \times m}\) with \(C_j = (c_{j,1}, \ldots, c_{j,m})^T\). If for all \(1 \leq i_j, \nu_j < \cdots < i_{j,1} \leq m\), where \(0 \leq \nu_j \leq m\) for all \(j = 1, \ldots, s\), with

\[
i_{1,1} + \cdots + i_{1,\min(\nu_1, \alpha)} + \cdots + i_{s,1} + \cdots + i_{s,\min(\nu_s, \alpha)} \leq \beta m - t
\]

the vectors

\[
C_{1,i_1,\nu_1}, \ldots, C_{1,i_{1,\nu_1}}, \ldots, C_{s,i_s,\nu_s}, \ldots, C_{s,i_{s,\nu_s}}
\]
are linearly independent over \( R_b \) then the digital net which has generating matrices \( C_1, \ldots, C_s \) is called a digital \((t, \alpha, \beta, m, s)\)-net over \( R_b \). Further we call a digital \((t, \alpha, m, s)\)-net over \( R_b \) a digital \((t, m, s)\)-net over \( R_b \).

If \( t \) is the smallest non-negative integer such that the digital net generated by \( C_1, \ldots, C_s \) is a digital \((t, \alpha, \beta, m, s)\)-net, then we call the digital net a strict digital \((t, \alpha, \beta, m, s)\)-net or a strict digital \((t, \alpha, m, s)\)-net if \( \alpha = \beta \).

A concrete example of a digital \((t, \alpha, \beta, m, s)\)-net, where we also calculate the exact \( t \)-value by hand, is given in Section 4.

Remark 1. Using duality theory (see [21]) it follows that for every digital \((t, \alpha, \beta, m, s)\)-net we have \( \min_{k \in D} \mu_{b, \alpha}(k) > \beta m - t \) and for a strict digital \((t, \alpha, \beta, m, s)\)-net we have \( \min_{k \in D} \mu_{b, \alpha}(k) = \beta m - t + 1 \). Hence digital \((t, \alpha, \beta, m, s)\)-nets with high quality have a large value of \( \beta m - t \).

Definition 3.2. Let \( \alpha \geq 1 \) and \( t \geq 0 \) be integers and let \( 0 < \beta < \alpha \) be a real number. Let \( R_b \) be a ring with \( b \) elements and let \( C_1, \ldots, C_s \in R_b^{\infty \times \infty} \) with \( C_j = (c_{j,1}, c_{j,2}, \ldots)^T \). Further let \( C_{j,m} \) denote the left upper \( m \times m \) submatrix of \( C_j \). If for all \( m > t/\beta \) the matrices \( C_{1,m}, \ldots, C_{s,m} \) generate a digital \((t, \alpha, \beta, m, s)\)-net then the digital sequence with generating matrices \( C_1, \ldots, C_s \) is called a digital \((t, \alpha, \beta, s)\)-sequence over \( R_b \). Further we call a digital \((t, \alpha, \alpha, s)\)-sequence over \( R_b \) a digital \((t, \alpha, s)\)-sequence over \( R_b \).

If \( t \) is the smallest non-negative integer such that the digital sequence generated by \( C_1, \ldots, C_s \) is a digital \((t, \alpha, \beta, s)\)-sequence, then we call the digital sequence a strict digital \((t, \alpha, \beta, s)\)-sequence or a strict digital \((t, \alpha, s)\)-sequence if \( \alpha = \beta \).

Remark 2. Note that the definition of a digital \((t,1, m, s)\)-net coincides with the definition of a digital \((t, m, s)\)-net and the definition of a digital \((t, 1, s)\)-sequence coincides with the definition of a digital \((t, s)\)-sequence as defined by Niederreiter [19]. Further note that the \( t \)-value depends on \( \alpha \) and \( \beta \), i.e., \( t = t(\alpha, \beta) \) or \( t = t(\alpha) \) if \( \alpha = \beta \).

In the following theorem we establish some propagation rules.

Theorem 3.3. Let \( P \) be a digital \((t, \alpha, \beta, m, s)\)-net over a ring \( R_b \) and let \( S \) be a digital \((t, \alpha, \beta, s)\)-sequence over a ring \( R_b \). Then we have:

(i) \( P \) is a digital \((t', \alpha', \beta', m, s)\)-net for all \( 1 \leq \beta' \leq \beta \) and all \( t \leq t' \leq \beta'm \) and \( S \) is a digital \((t', \alpha', \beta', s)\)-sequence for all \( 1 \leq \beta' \leq \beta \) and all \( t \leq t' \).

(ii) \( P \) is a digital \((t', \alpha', \beta', m, s)\)-net for all \( 1 \leq \alpha' \leq m \) and \( S \) is a digital \((t', \alpha', \beta', s)\)-sequence for all \( \alpha' \geq 1 \), where \( \beta' = \beta \min(\alpha, \alpha')/\alpha \) and \( t' = \lceil t \min(\alpha, \alpha')/\alpha \rceil \).

(iii) Any digital \((t, \alpha, m, s)\)-net is a digital \((\lceil t\alpha'/\alpha \rceil, \alpha', m, s)\)-net for all \( 1 \leq \alpha' \leq \alpha \) and every digital \((t, \alpha, s)\)-sequence is a digital \((\lceil t\alpha'/\alpha \rceil, \alpha', s)\)-sequence for all \( 1 \leq \alpha' \leq \alpha \).

Proof. Note that it follows from Definition 3.2 that we need to prove the result only for digital nets.

The first part follows trivially. To prove the second part choose an \( \alpha' \) such that \( \alpha' \geq 1 \). Then choose arbitrary \( 1 \leq i_{j, \nu} < \cdots < i_{j, 1} \leq m \) with \( 0 \leq \nu_j \leq m \) such that
\[
i_{1,1} + \cdots + i_{1, \min(\nu_1, \alpha')} + \cdots + i_{s,1} + \cdots + i_{s, \min(\nu_s, \alpha')} \leq m \beta \frac{\min(\alpha, \alpha')}{\alpha} - \lceil \frac{t \min(\alpha, \alpha')}{\alpha} \rceil.
\]

We need to show that the vectors
\[
c_1, i_{1, \nu}, \ldots, c_1, i_{1, 1}, \ldots, c_s, i_{s, \nu}, \ldots, c_s, i_{s, 1}
\]
are linearly independent over \( R_b \). This is certainly the case as long as
\[
i_{1,1} + \cdots + i_{1, \min(\nu_1, \alpha')} + \cdots + i_{s,1} + \cdots + i_{s, \min(\nu_s, \alpha')} \leq \beta m - t.
\]
Indeed we have
\[
i_{1,1} + \cdots + i_{1,\min(\nu,\alpha)} + \cdots + i_s,1 + \cdots + i_{s,\min(\nu,\alpha)} \\
\leq \frac{\alpha}{\min(\alpha,\alpha')} (i_{1,1} + \cdots + i_{1,\min(\nu,\alpha')} + \cdots + i_s,1 + \cdots + i_{s,\min(\nu,\alpha')}) \\
\leq m\beta - \frac{\alpha}{\min(\alpha,\alpha')} \left[ \frac{t_{\min(\alpha,\alpha')}}{\alpha} \right] \\
\leq m\beta - t,
\]
and hence the second part follows. The third part is just a special case of the second part. \(\square\)

**Remark 3.** Note by choosing \(\alpha' = 1\) in part (iii) of Theorem 3.3 it follows that digital \((t,\alpha,\beta,m,s)\)-nets and digital \((t,\alpha,s)\)-sequences are also well distributed point sets if the value of \(t\) is small, see [19].

4. **Explicit constructions of digital \((t,\alpha,\beta,m,s)\)-nets and digital \((t,\alpha,\beta,s)\)-sequences.** In this section we show how suitable digital \((t,\alpha,\beta,m,s)\)-nets and digital \((t,\alpha,\beta,s)\)-sequences can be constructed.

Let \(d \geq 1\) and let \(C_1,\ldots,C_{sd}\) be the generating matrices of a digital \((t,m,sd)\)-net. Note that many explicit examples of such generating matrices are known, see for example [7, 19, 22, 32] and the references therein. For the construction of a \((t,\alpha,\beta,m,s)\)-net any of the above mentioned explicit constructions can be used, but as will be shown below the quality of the \((t,\alpha,\beta,m,s)\)-net obtained depends on the quality of the underlying digital \((t,m,sd)\)-net on which our construction is based on.

Let \(C_j = (c_{j,1},\ldots,c_{j,m})^T\) for \(j = 1,\ldots,sd\), i.e., \(c_{j,l}\) are the row vectors of \(C_j\). Now let the matrix \(C_j^{(d)}\) be made of the first \(d\) rows of the matrices \(C_{(j-1)d+1},\ldots,C_{jd}\), then the second rows of \(C_{(j-1)d+1},\ldots,C_{jd}\) and so on till \(C_j^{(d)}\) is an \(m \times m\) matrix, i.e., \(C_j^{(d)} = (c_{j,1},\ldots,c_{j,m})^T\) where \(c_{j,l} = c_{u,v}\) with \(l = (v - j)d + u\), \(1 \leq v \leq m\) and \((j-1)d < u \leq jd\) for \(l = 1,\ldots,m\) and \(j = 1,\ldots,s\). In the following we will show that the matrices \(C_1^{(d)},\ldots,C_s^{(d)}\) are the generating matrices of a digital \((t,\alpha,\min(\alpha,d),m,s)\)-net.

**Theorem 4.1.** Let \(d \geq 1\) be a natural number and let \(C_1,\ldots,C_{sd}\) be the generating matrices of a digital \((t,m,sd)\)-net over some ring \(R_b\) with \(b\) elements. Let \(C_1^{(d)},\ldots,C_s^{(d)}\) be defined as above. Then for any \(\alpha \geq 1\) the matrices \(C_1^{(d)},\ldots,C_s^{(d)}\) are generating matrices of a digital \((t,\alpha,\min(\alpha,d),m,s)\)-net over \(R_b\) with

\[
t = \min(\alpha,d) t' + \left[ \frac{s(d-1) \min(\alpha,d)}{2} \right].
\]

**Proof.** Let \(C_j^{(d)} = (c_{j,1}^{(d)},\ldots,c_{j,m}^{(d)})^T\) for \(j = 1,\ldots,s\) and further let the integers \(i_{1,1},\ldots,i_{1,\nu_1},\ldots,i_{s,1},\ldots,i_{s,\nu_s}\) be such that \(1 \leq i_{j,\nu_j} < \cdots < i_{j,1} \leq m\) and \(i_{1,1} + \cdots + i_{1,\min(\nu_1,\alpha)} + \cdots + i_{s,1} + \cdots + i_{s,\min(\nu_s,\alpha)} \leq \min(\alpha,d) m - t\).

We need to show that the vectors
\[
c_{1,i_{1,1}},\ldots,c_{1,i_{1,\nu_1}},\ldots,c_{s,i_{s,1}},\ldots,c_{s,i_{s,\nu_s}}^{(d)}
\]
are linearly independent over \(R_b\). For \(j = 1,\ldots,s\) let \(U_j = \{c_{j,i_{j,1}},\ldots,c_{j,i_{j,\nu_j}}^{(d)}\}\). The vectors in the set \(U_j\) stem from the matrices \(C_{(j-1)d+1},\ldots,C_{jd}\). For \(j = 1,\ldots,s\) and
Thus it follows that $i_d = (j-1)d + 1, \ldots, jd$ let $e_d$ denote the largest index such that $(e_d - j)d + d_j \in \{i_j, \nu, \ldots, i_j, 1\}$ and if for some $d_j$ there is no such $e_d$, we set $e_d = 0$ (basically this means $e_d$ is the largest integer such that $c_d, e_d \in U_d$).

Let $d \leq \alpha$, then we have $d((e_d - j)d + 1) + \cdots + (e_d - 1) + \sum_{l=1}^{L_j} l \leq i_j, 1 + \cdots + i_j, \min(\nu, d)$ where $(x)_+ = \max(x, 0)$ and $L_j = |\{(j-1)d+1 \leq d_j \leq jd : e_d > 0\}|$. Hence we have

$$
\begin{align*}
&d((e_d - j)d + 1) + \cdots + (e_d - 1) + \sum_{l=1}^{L_j} l \\
&= d((e_d - j)d + 1) + \cdots + e_d - L_jd + L_j(L_j + 1)/2 \\
&\geq d((e_d - j)d + 1) + \cdots + e_d - \frac{d(d-1)}{2}.
\end{align*}
$$

Thus it follows that

$$
d(e_1 + \cdots + e_{sd}) \leq \sum_{j=1}^{s} (i_j, 1 + \cdots + i_j, \min(\nu, \alpha)) + s \frac{d(d-1)}{2} \leq dm - t + s \frac{d(d-1)}{2}
$$

and therefore

$$
e_1 + \cdots + e_{sd} \leq m - \frac{t}{d} + s \frac{d-1}{2} \leq m - t'.
$$

Thus it follows from the $(t', m, sd)$-net property of the digital net generated by $C_1, \ldots, C_{sd}$ that the vectors $c_{1, i_1, 1}^{(d)}, \ldots, c_{1, i_1, \nu_1}^{(d)}, \ldots, c_{s, i_s, 1}^{(d)}, \ldots, c_{s, i_s, \nu_s}^{(d)}$ are linearly independent.

Let now $d > \alpha$. Then we have $d((e_d - j)d + 1) + \cdots + (e_d - 1) + \sum_{l=1}^{L_j} l \leq i_j, 1 + \cdots + i_j, \min(\nu, \alpha) + (d - \alpha)i_j, \min(\nu, \alpha)$, where again $L_j = |\{(j-1)d+1 \leq d_j \leq jd : e_d > 0\}|$. Hence we can use inequality again. Note that $i_1, \min(\nu, \alpha) + \cdots + i_s, \min(\nu, \alpha) \leq m - t/\alpha$ and hence we have

$$
\sum_{j=1}^{s} (i_j, 1 + \cdots + i_j, \min(\nu, \alpha)) + (d - \alpha)i_j, \min(\nu, \alpha) \leq \alpha m - t + (d - \alpha)(m - t/\alpha) = dm - dt/\alpha.
$$

Thus it follows that

$$
d(e_1 + \cdots + e_{sd}) \leq \sum_{j=1}^{s} (i_j, 1 + \cdots + i_j, \min(\nu, \alpha)) + (d - \alpha)i_j, \min(\nu, \alpha)) + s \frac{d(d-1)}{2}
$$

and therefore

$$
e_1 + \cdots + e_{sd} \leq m - \frac{dt}{\alpha} + s \frac{d-1}{2} \leq m - t'.
$$

Thus it follows from the $(t', m, sd)$-net property of the digital net generated by $C_1, \ldots, C_{sd}$ that the vectors $c_{1, i_1, 1}^{(d)}, \ldots, c_{1, i_1, \nu_1}^{(d)}, \ldots, c_{s, i_s, 1}^{(d)}, \ldots, c_{s, i_s, \nu_s}^{(d)}$ are linearly independent and hence the result follows.

In Section 7 we use this construction method to construct a digital $(3, 2, 4, 2)$-net over $\mathbb{Z}_2$. 
Note that the construction and Theorem 4.1 can easily be extended to \((t, \alpha, \beta, s)\)-sequences. Indeed, let \(d \geq 1\) and let \(C_1, \ldots, C_{sd}\) be the generating matrices of a digital \((t, sd)\)-sequence. Again many explicit generating matrices are known, see for example [11, 18, 22, 32]. Let \(C_j = (c_{j,1}, c_{j,2}, \ldots)\) for \(j = 1, \ldots, sd\), i.e., \(c_{j,t}\) are the row vectors of \(C_j\). Now let the matrix \(C_j^{(d)}\) be made of the first rows of the matrices \(C_{(j-1)d+1}, \ldots, C_{jd}\), then the second rows of \(C_{(j-1)d+1}, \ldots, C_{jd}\) and so on, i.e.,

\[
C_j^{(d)} = (c_{(j-1)d+1,1}, \ldots, c_{jd,1}, c_{(j-1)d+1,2}, \ldots, c_{jd,2}, \ldots)^T.
\]

The following theorem states that the matrices \(C_1^{(d)}, \ldots, C_s^{(d)}\) are the generating matrices of a digital \((t, \alpha, \min(\alpha, d), s)\)-sequence.

**Theorem 4.2.** Let \(d \geq 1\) be a natural number and let \(C_1, \ldots, C_{sd}\) be the generating matrices of a digital \((t', sd)\)-sequence over some ring \(R_b\) with \(b\) elements. Let \(C_1^{(d)}, \ldots, C_s^{(d)}\) be defined as above. Then for any \(\alpha \geq 1\) the matrices \(C_1^{(d)}, \ldots, C_s^{(d)}\) are generating matrices of a digital \((t, \alpha, \min(\alpha, d), s)\)-sequence over \(R_b\) with

\[
|t - \min(\alpha, d) t' + \frac{s(d-1) \min(\alpha, d)}{2}|.
\]

The last result shows that \((t, \alpha, \beta, m, s)\)-nets indeed exist for any \(0 < \beta \leq \alpha\) and for \(m\) arbitrarily large. We have even shown that digital \((t, \alpha, \beta, m, s)\)-nets exist which are extensible in \(m\) and \(s\). This can be achieved by using an underlying \((t', sd)\)-sequence which is itself extensible in \(m\) and \(s\). If the \(t'\) value of the original \((t', m, s)\)-net or \((t', s)\)-sequence is known explicitly then we also know the \(t\) value of the digital \((t, \alpha, \beta, m, s)\)-net or \((t, \alpha, \beta, s)\)-sequence. Furthermore it has also been shown how such digital nets can be constructed in practise.

In the following we investigate for which values of \(t, \alpha, s, b\) digital \((t, \alpha, s)\)-sequences over \(Z_b\) exist. We need some further notation (see also [23], Definition 8.2.15).

**Definition 4.3.** For given integers \(s, \alpha \geq 1\) and prime number \(b\) let \(d_b(s, \alpha)\) be the smallest value of \(t\) such that a \((t, \alpha, s)\)-sequence over \(Z_b\) exists. We have the following bound on \(d_b(s, \alpha)\).

**Corollary 4.4.** Let \(s, \alpha \geq 1\) be integers and \(b\) be a prime number. Then we have

\[
\alpha \left( \frac{s}{b} - 1 - \log_b \frac{(b-1)s + b + 1}{2} \right) + 1 \\
\leq d_b(s, \alpha) \leq \alpha(s-1) \frac{3b-1}{b-1} - \alpha \frac{2b+4} \sqrt{b^2-1} + 2\alpha + s \frac{\alpha(\alpha-1)}{2}.
\]

**Proof.** The lower bound follows from part (iii) of Theorem 3.3 by choosing \(\alpha' = 1\) and using a lower bound on the \(t\)-value for \((t, s)\)-sequences (see [22]). The upper bound follows from Theorem 4.2 by choosing \(d = \alpha\) and using Theorem 8.4.4 of [23].\(\Box\)

**5. A bound on the worst-case error in \(H_\alpha\) for digital \((t, \alpha, \beta, m, s)\)-nets and digital \((t, \alpha, \beta, s)\)-sequences.** In this section we prove an upper bound on the worst-case error for integration in the Korobov space \(H_\alpha\) using digital \((t, \alpha, \beta, m, s)\)-nets and \((t, \alpha, \beta, s)\)-sequences.

**Lemma 5.1.** Let \(\alpha \geq 2\) be a natural number, let \(b\) be prime and let \(C_1, \ldots, C_s \in Z_b^{m \times m}\) be the generating matrices of a digital \((t, \alpha, \beta, m, s)\)-net over \(Z_b\) with \(m > t/\beta\). Theorem 4.1
Then we have

\[ Q_{b,m,u,\alpha}^*(C_1, \ldots, C_s) \leq 2b^{j_u |\alpha b^{-\beta m + t} (\beta m + 2)^{|\alpha|-1}}, \]

where \( Q_{b,m,u,\alpha}^* \) is defined in Lemma 2.10.

**Proof.** We obtain a bound on \( Q_{b,m,\{1,\ldots,s\},\alpha}^* \) for all other subsets \( u \) the bound can be obtained using the same arguments.

We first partition the set \( D_{b,m,\{1,\ldots,s\}}^* \) into parts where the highest digits of \( k_j \) are prescribed and we count the number of solutions of \( C_1^T \bar{k}_1 + \cdots + C_s^T \bar{k}_s = \bar{0} \). For \( j = 1, \ldots, s \) let now \( i_{j,\alpha} < \cdots < i_{j,1} \leq m \) with \( i_{j,1} \geq 1 \). Note that we now allow \( i_{j,l} < 1 \), in which case the contributions of those \( i_{j,l} \) are to be ignored. This notation is adopted in order to avoid considering many special cases. Now we define

\[
D_{b,m,\{1,\ldots,s\}}^*(i_{1,1}, \ldots, i_{1,\alpha}, \ldots, i_{s,1}, \ldots, i_{s,\alpha}) = \{ k \in D_{b,m,\{1,\ldots,s\}}^* : k_j = [\kappa_{j,1} b^{j_1-1} \cdots + \kappa_{j,\alpha} b^{j_\alpha-1} + l_j] \text{ with } 0 \leq l_j < b^{j_\alpha-1} \text{ and } 1 \leq \kappa_{j,l} < b \text{ for } j = 1, \ldots, s, \}
\]

where \([\cdot]\) just means that the contributions of \( i_{j,l} < 1 \) are to be ignored. Then we have

\[
Q_{b,m,\{1,\ldots,s\},\alpha}^*(C_1, \ldots, C_s) = \sum_{i_{1,1}=1}^{m} \cdots \sum_{i_{1,\alpha}=1}^{m} \cdots \sum_{i_{s,1}=1}^{m} \cdots \sum_{i_{s,\alpha}=1}^{m} \frac{|D_{b,m,\{1,\ldots,s\}}^*(i_{1,1}, \ldots, i_{1,\alpha}, \ldots, i_{s,1}, \ldots, i_{s,\alpha})|}{b^{i_{1,1}+\cdots+i_{1,\alpha}+\cdots+i_{s,1}+\cdots+i_{s,\alpha}}}
\]

Some of the sums above can be empty in which case we just set the corresponding summation index \( i_{j,l} = 0 \).

Note that by the \((t,\alpha,\beta,m,s)\)-net property we have

\[ |D_{b,m,\{1,\ldots,s\}}^*(i_{1,1}, \ldots, i_{1,\alpha}, \ldots, i_{s,1}, \ldots, i_{s,\alpha})| = 0 \]

as long as \( i_{1,1} + \cdots + i_{1,\alpha} + \cdots + i_{s,1} + \cdots + i_{s,\alpha} \leq \beta m - t \). Hence let now \( 0 \leq i_{1,1}, \ldots, i_{s,\alpha} \leq m \) be given such that \( i_{1,1}, \ldots, i_{s,1} \geq 1 \), \( i_{j,\alpha} < \cdots < i_{j,1} \leq m \) for \( j = 1, \ldots, s \) and where if \( i_{j,l} < 1 \) we set \( i_{j,l} = 0 \) and \( i_{1,1} + \cdots + i_{1,\alpha} + \cdots + i_{s,1} + \cdots + i_{s,\alpha} > \beta m - t \). We now need to estimate \( |D_{b,m,\{1,\ldots,s\}}^*(i_{1,1}, \ldots, i_{1,\alpha}, \ldots, i_{s,1}, \ldots, i_{s,\alpha})| \), that is we need to count the number of \( k \in D_{b,m,\{1,\ldots,s\}}^* \) with \( k_j = [\kappa_{j,1} b^{j_1-1} + \cdots + \kappa_{j,\alpha} b^{j_\alpha-1} + l_j] \) such that \( C_1^T \bar{k}_1 + \cdots + C_s^T \bar{k}_s = \bar{0} \).

There are at most \((b-1)^{\alpha s}\) choices for \( \kappa_{1,1}, \ldots, \kappa_{s,\alpha} \) (we write at most because if \( i_{j,l} < 1 \) then the corresponding \( \kappa_{j,l} \) does not have any effect and therefore need not to be included). Let now \( 1 \leq \kappa_{1,1}, \ldots, \kappa_{s,\alpha} < b \) be given and define

\[
\bar{g} = \kappa_{1,1} c_{1,i_{1,1}}^T + \cdots + \kappa_{1,\alpha} c_{1,i_{1,\alpha}}^T + \cdots + \kappa_{s,1} c_{s,i_{s,1}}^T + \cdots + \kappa_{s,\alpha} c_{s,i_{s,\alpha}}^T,
\]

where we set \( c_{j,l}^T = 0 \) if \( l < 1 \). Further let

\[
B = (c_{1,1}^T, \cdots, c_{1,i_{1,\alpha}-1}^T, c_{1,i_{1,\alpha}}^T, \cdots, c_{s,i_{s,\alpha}-1}^T).
\]

Now the task is to count the number of solutions \( \bar{I} \) of \( B\bar{l} = \bar{g} \). As long as the columns of \( B \) are linearly independent the number of solutions can at most be 1. By the
\((t, \alpha, \beta, m, s)\)-net property this is certainly the case if (we write \((x)_+ = \max(x, 0)\))

\[
(i_{1, \alpha} - 1)_+ + \cdots + (i_{1, \alpha} - \alpha)_+ + \cdots + (i_{s, \alpha} - 1)_+ + \cdots + (i_{s, \alpha} - \alpha)_+ \\
\leq \alpha (i_{1, \alpha} + \cdots + i_{s, \alpha}) \\
\leq \beta m - t,
\]

that is, as long as

\[
i_{1, \alpha} + \cdots + i_{s, \alpha} \leq \frac{\beta m - t}{\alpha}.
\]

Let now \(i_{1, \alpha} + \cdots + i_{s, \alpha} > \frac{\beta m - t}{\alpha}\). Then by considering the rank of the matrix \(B\) and the dimension of the space of solutions of \(B^\ell = \vec{0}\) it follows the number of solutions of \(B^\ell = \vec{g}\) is smaller or equal to \(b^{s_{1, \alpha} + \cdots + s_{s, \alpha} - \lfloor (\beta m - t)/\alpha \rfloor}\). Thus we have

\[
|D_\alpha^s\{1, \ldots, n\}\{i_{1, \alpha}, \ldots, i_{s, \alpha}\}| \leq \begin{cases} 
0 & \text{if } \sum_{j=1}^s \sum_{l=1}^n i_{j, l} \leq \beta m - t, \\
(b - 1)^{\alpha s} & \text{if } \sum_{j=1}^s \sum_{l=1}^n i_{j, l} > \beta m - t \\
\frac{1}{b^\ell} & \text{if } \sum_{j=1}^s \sum_{l=1}^n i_{j, l} \leq \frac{\beta m - t}{\alpha}
\end{cases}
\]

We estimate the sum \((5.1)\) now. Let \(S_1\) be the sum in \((5.1)\) where \(i_{1, \alpha} + \cdots + i_{s, \alpha} > \beta m - t)\) and \(i_{1, \alpha} + \cdots + i_{s, \alpha} \leq \frac{\beta m - t}{\alpha}\). For an \(l > \beta m - t\) let \(A_1(l)\) denote the number of admissible choices of \(i_{1, \alpha}, \ldots, i_{s, \alpha}\) such that \(l = i_{1, \alpha} + \cdots + i_{s, \alpha}\). Then we have

\[
S_1 = (b - 1)^{\alpha s} \sum_{l=\beta m - t + 1}^{\beta m - t + 1} A_1(l) \frac{1}{b^\ell}.
\]

We have \(A_1(l) \leq \binom{l + s_{1, \alpha} - 1}{s_{1, \alpha} - 1}\) and hence we obtain

\[
S_1 \leq (b - 1)^{\alpha s} \sum_{l=\beta m - t + 1}^{\beta m - t + 1} \left( l + s_{1, \alpha} - 1 \right) \frac{1}{b^\ell} \leq b^{s_{1, \alpha} - \beta m + t - 1} \left( \beta m - t + s_{1, \alpha} - 1 \right).
\]

where the last inequality follows from a result by Matoušek [13, Lemma 2.18], see also [6, Lemma 6].

Let \(S_2\) be the part of \((5.1)\) for which \(i_{1, \alpha} + \cdots + i_{s, \alpha} > \beta m - t)\) and \(i_{1, \alpha} + \cdots + i_{s, \alpha} > \frac{\beta m - t}{\alpha}\), i.e., we have

\[
S_2 = (b - 1)^{\alpha s} \sum_{i_{1, \alpha} = 1}^{m} \cdots \sum_{i_{1, \alpha - 1} = 1}^{m} \cdots \sum_{i_{s, \alpha - 1} = 1}^{m} \sum_{i_{s, \alpha} = 1}^{m} \frac{1}{b^{i_{1, \alpha} + \cdots + i_{s, \alpha}}} \\
\leq \frac{m^s (b - 1)^{\alpha s}}{b^{\lfloor (\beta m - t)/\alpha \rfloor}} \sum_{i_{1, \alpha} = 1}^{m} \cdots \sum_{i_{1, \alpha - 1} = 1}^{m} \cdots \sum_{i_{s, \alpha - 1} = 1}^{m} \sum_{i_{s, \alpha} = 1}^{m} \frac{1}{b^{i_{1, \alpha} + \cdots + i_{s, \alpha}}},
\]

where in the first line above we have the additional conditions \(i_{1, \alpha} + \cdots + i_{s, \alpha} > \beta m - t)\) and \(i_{1, \alpha} + \cdots + i_{s, \alpha} > \frac{\beta m - t}{\alpha}\). From the last inequality and \(i_{1, \alpha} + \cdots + i_{s, \alpha} > i_{1, \alpha} + \cdots + i_{s, \alpha}\) for \(l = 1, \ldots, \alpha - 1\) it follows that \(i_{1, \alpha} + \cdots + i_{1, \alpha} + \cdots + i_{s, \alpha} + \cdots + i_{s, \alpha}\)
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\[ i_{s\alpha-1} \geq \lceil (\beta m - t)(1 - \alpha^{-1}) \rceil + 1. \]

Let \( A_2(l) \) denote the number of admissible choices of \( i_{11}, \ldots, i_{1\alpha-1}, i_{s1}, \ldots, i_{s\alpha-1} \) such that \( l = i_{11} + \cdots + i_{1\alpha-1} + \cdots + i_{s1} + \cdots + i_{s\alpha-1} \). Note that we have \( A_2(l) \leq \binom{l + s(\alpha-1)-1}{s(\alpha-1)-1} \). Then we have

\[
S_2 \leq \frac{m^s (b-1)^{\alpha}}{b(\beta m - t) / \alpha} \sum_{l = \lceil (\beta m - t)(1 - \alpha^{-1}) \rceil + 1}^{\infty} \frac{(l + s(\alpha - 1) - 1)(1)}{s(\alpha - 1) - 1} \beta^l
\]

\[
\leq \frac{m^s (b-1)^{\alpha}}{b(\beta m - t) / \alpha} \frac{b(\beta m - t) / \alpha}{(1 - b^{1}) s(\alpha - 1) b^{\beta m - t + 1}} \frac{(l + s(\alpha - 1) - 1)(1 + s(\alpha - 1))}{s(\alpha - 1) - 1},
\]

where the last inequality follows again from a result by Matoušek [15, Lemma 2.18], see also [6 Lemma 6]. Hence we have

\[
S_2 \leq m^s b^{\alpha} b^{-\beta m + t} \frac{(l + s(\alpha - 1) - 1)(1 + s(\alpha - 1))}{s(\alpha - 1) - 1}.
\]

Note that we have \( Q^*_b, m, \alpha, \{1, \ldots, s\} (C_1, \ldots, C_s) = S_1 + S_2 \). Let \( a \geq 1 \) and \( b \geq 0 \) be integers then we have

\[
\left( \frac{a + b}{b} \right) = \prod_{i=1}^{b} \left( 1 + \frac{a}{i} \right) \leq (1 + a)^b.
\]

Therefore we obtain \( S_1 \leq b^{s\alpha} b^{-\beta m + t - 1}(\beta m - t + 2)^{s\alpha} - 1 \) and \( S_2 \leq b^{s\alpha} b^{-\beta m + t} m^s (\beta m - t + 2)^{s(\alpha-1)} - 1 \). Thus we have

\[
Q^*_b, m, \alpha, \{1, \ldots, s\} (C_1, \ldots, C_s) \leq 2 b^{s\alpha} b^{-\beta m + t} (\beta m + 2)^{s\alpha} - 1,
\]

from which the result follows.

Remark 4. By the lower bound of Sharygin [26] we have that the worst-case error in the Korobov space \( \mathcal{H}_\alpha \) is at most \( \mathcal{O}(N^{-\alpha}(\log N)^{s-1}) \). Hence it follows from Theorem 5.2 that for a digital \((t, \alpha, \beta, m, s)\)-net with \( \beta > \alpha \) we must have \( t = \mathcal{O}(\beta - \alpha) m \). Thus in order to avoid having a \( t \)-value which grows with \( m \) we added the restriction \( \beta \leq \alpha \) in Definition 5.1. Further, this also implies that a digital \((t, \alpha, \beta, s)\)-sequence with \( t < \infty \) cannot exist if \( \beta > \alpha \), hence \( \beta \leq \alpha \) is in this case a consequence of the definition rather than a restriction.

Remark 5. Lemma 2.8 also holds for digital nets which are digitally shifted by an arbitrary digital shift \( \sigma \in [0, 1]^s \) and hence it follows that Theorem 5.2 also holds in a more general form, namely for all digital \((t, \alpha, \beta, m, s)\)-net which are digitally shifted.


Theorem 5.2 shows that we can obtain the optimal convergence rate for natural numbers \( \alpha \geq 2 \) by using a digital \((t, \alpha, m, s)\)-net. The constructions previously proposed (for example by Sobol, Faure, Niederreiter or Niederreiter-Xing) have only been shown to be \((t, 1, m, s)\)-nets and it has been proven that they achieve a convergence of the worst-case error of \( O(N^{-1}(\log N)^{s-1}) \).

We can use Theorem 5.2 to obtain the following corollary.

**Corollary 5.3.** Let \( b \) be prime and let \( C_1^{(d)}, \ldots, C_s^{(d)} \in \mathbb{Z}_b^{\infty \times \infty} \) be the generating matrices of a digital \((t(a), a, \min(a, d), s)\)-sequence over \( \mathbb{Z}_b \) for any integer \( a \geq 1 \). Then for any real \( \alpha \geq 1 \) there is a constant \( C_{b,s,a} > 0 \), depending only on \( b, s, a \), such that the worst-case error in the Korobov space \( \mathcal{H}_\alpha \) using the first \( N = b^m \) points of \( S \) is bounded by

\[
e_{b,m,a}(C_1^{(d)}, \ldots, C_s^{(d)}) \leq C_{b,s,d} b^{t(\alpha)} \frac{(\log N)^{s|\alpha| - 1}}{N^{\min(\alpha, d)}}.
\]

**Remark 6.** The above corollary shows that digital \((t, \alpha, \min(a, d), s)\)-sequences constructed in Section 4 achieve the optimal convergence (apart from maybe some log \( N \) factor) of \( P_\alpha \) of \( O(N^{-2\alpha}(\log N)^{2s\alpha - 2}) \) as long as \( \alpha \) is an integer such that \( 1 \leq \alpha \leq d \). If \( \alpha > d \) we obtain a convergence of \( O(N^{-2d}(\log N)^{2s\alpha - 2}) \).

6. **A bound on the mean square worst-case error in \( \mathcal{H}_\alpha \) for digital \((t, \alpha, \beta, m, s)\)-nets and digital \((t, \alpha, \beta, s)\)-sequences.** To combine the advantages of random quadrature points with those of deterministic quadrature points one sometimes uses a combination of those two methods, see for example [6, 10, 15, 24]. The idea is to use a random element which preserves the essential properties of a deterministic point set. We call the expectation value of the square worst-case error of such randomized point sets the mean square worst-case error.

6.1. **Randomization.** In the following we introduce a randomization scheme called digital shift (see [5, 15]). Let \( P_N = \{x_0, \ldots, x_{N-1}\} \subseteq \{0, 1\}^s \) with \( x_n = (x_{1,n}, \ldots, x_{s,n}) \) and \( x_{j,n} = x_{j,n,1} b^{-1} + x_{j,n,2} b^{-2} + \cdots \) for \( n = 0, \ldots, N - 1 \) and \( j = 1, \ldots, s \). Let \( \sigma_{j,1}, \sigma_{j,2}, \ldots \in \{0, 1\} \) be i.i.d. for \( j = 1, \ldots, s \). Then the randomly digitally shifted point set \( P_{N,\sigma} = \{z_0, \ldots, z_{N-1}\} \), \( z_n = (z_{1,n}, \ldots, z_{s,n}) \) using a digital shift, is then given by

\[
z_{j,n} = (x_{j,n,1} \oplus \sigma_{j,1}) b^{-1} + (x_{j,n,2} \oplus \sigma_{j,2}) b^{-2} + \cdots
\]

for \( j = 1, \ldots, s \) and \( n = 0, \ldots, N - 1 \), where \( x_{j,n,k} \oplus \sigma_{j,n} = x_{j,n,k} + \sigma_{j,n} \pmod{b} \) (note that all additions of the digits are carried out in the finite field \( \mathbb{Z}_b \)). Subsequently let \( P_N = \{x_0, \ldots, x_{N-1}\} \) and let \( P_{N,\sigma} \) be the digitally shifted point set \( P_N \) using the randomization just described.

6.2. **The mean square worst-case error in the Korobov space.** In this section we will analyze the expectation value of \( \tilde{e}^2(P_{N,\sigma}, K_\alpha) \), which we denote by \( \mathbb{E}[e^2(P_{N,\sigma}, K_\alpha)] \), with respect to the random digital shift described above. We call \( \tilde{e}^2(P_N, K_\alpha) \) the mean square worst-case error.

From [2.1] and the linearity of the expectation operator we have

\[
\mathbb{E}[e^2(P_{N,\sigma}, K_\alpha)] = -1 + \frac{1}{N^2} \sum_{n,l=0}^{N-1} \prod_{j=1}^{s} \mathbb{E}[K_\alpha(z_{j,n}, z_{j,l})].
\]
In order to compute $E[K_\alpha(z_{j,n}, z_{j,l})]$ we need the following lemma, which, in a very similar form, was already shown in \[6\], Lemma 3. Hence we omit a proof.

**Lemma 6.1.** Let $x_1, x_2 \in [0,1)$ and let $z_1, z_2 \in [0,1)$ be the points obtained after applying an i.i.d. random digital shift to $x_1$ and $x_2$. Then we have

$$E[\text{wal}_k(z_1)\text{wal}_l(z_2)] = \begin{cases} \text{wal}_k(x_1)\text{wal}_l(x_2) & \text{if } k = l, \\ 0 & \text{otherwise}. \end{cases}$$

Recall that

$$K_\alpha(x_1, x_2) = \sum_{k, l=0}^{\infty} r_{b,\alpha}(k, l)\text{wal}_k(x_1)\text{wal}_l(x_2),$$

where

$$r_{b,\alpha}(k, l) = \int_0^1 \int_0^1 K_\alpha(x_1, x_2)\text{wal}_k(x_1)\text{wal}_l(x_2) \, dx_1 \, dx_2.$$  

Let $z_1, z_2$ be obtained by applying an i.i.d. random digital shift to $x_1, x_2$. Using Lemma 6.1 and the linearity of expectation we obtain

$$E[K_\alpha(z_1, z_2)] = \sum_{k=0}^{\infty} r_{b,\alpha}(k)\text{wal}_k(x_1)\text{wal}_k(x_2),$$

where $r_{b,\alpha}(k) = r_{b,\alpha}(k, k)$ and $r_{b,\alpha}(0) = 1$.

Therefore we obtain

$$E[e^2(P_N, \sigma, K_\alpha)] = -1 + \frac{1}{N^2} \sum_{n,l=0}^{N-1} s \prod_{j=1}^{s} \sum_{k=0}^{\infty} r_{b,\alpha}(k)\text{wal}_k(x_n \ominus x_l).$$

Further we have

$$\prod_{j=1}^{s} \sum_{k=0}^{b^m-1} r_{b,\alpha}(k)\text{wal}_k(x_{j,n})\text{wal}_k(x_{j,l}) = 1 + \sum_{k \in \{0, \ldots, b^m-1\} \setminus \{0\}} r_{b,\alpha}(k)\text{wal}_k(x_n \ominus x_l),$$

where we write $r_{b,\alpha}(k) = \prod_{j=1}^{s} r_{b,\alpha}(k_j)$ for $k = (k_1, \ldots, k_s)$. We have shown the following theorem.

**Theorem 6.2.** Let $b \geq 2$ be a natural number and let $\alpha > 1/2$ be a real number. Then the mean square worst-case error for integration in the Korobov space $H_\alpha$ using the point set $P_N$ randomized by a digital shift is given by

$$E[e^2(P_N, \sigma, K_\alpha)] = \sum_{k \in \mathbb{N} \setminus \{0\}} r_{b,\alpha}(k)\frac{1}{N^2} \sum_{n,l=0}^{N-1} \text{wal}_k(x_n \ominus x_l).$$

In the following we closer investigate the mean square worst-case error for digital nets randomized with a digital shift.

Subsequently we will often write $e^2_{b,m,\alpha}(C_1, \ldots, C_s)$ to denote the mean square worst-case error $E[e(P_{b^m, \sigma}, K_\alpha)]$, where $P_{b^m}$ is a digital net with generating matrices.
$C_1, \ldots, C_s$ and $b^m$ points and $P^m_{b, \sigma}$ is the digital net $P^m_{b, \sigma}$ randomized with a digital shift.

**Theorem 6.3.** Let $m \geq 1$, $b$ be a prime number and $\alpha > 1/2$ be a real number. The mean square worst-case error in the Korobov space $\mathcal{H}_\alpha$ using a randomly digitally shifted digital net over $\mathbb{Z}_b$ with generating matrices $C_1, \ldots, C_s \in \mathbb{Z}^{m \times m}$ is given by

$$\tilde{e}_{b,m,\alpha}^2(C_1, \ldots, C_s) = \sum_{k \in D} r_{b,\alpha}(k).$$

**Proof.** In [5] it was shown that

$$\frac{1}{b^{2m}} \sum_{n,l=0}^{b^m-1} \text{val}_k(x_n \oplus x_l) = \frac{1}{b^{m}} \sum_{n=0}^{b^m-1} \text{val}_k(x_n) = \begin{cases} 1 & \text{if } k \in D \cup \{0\}, \\ 0 & \text{otherwise}. \end{cases}$$

Hence the result follows from Theorem 6.2.

**Remark 7.** Theorem 2.5 and Theorem 6.3 now imply that

$$\tilde{e}_{b,m,\alpha}^2(C_1, \ldots, C_s) = \sqrt{\sum_{k \in D} r_{b,\alpha}(k)} \leq \sqrt{\sum_{k,l \in D} r_{b,\alpha}(k, l)} = e(P^m_{b, K_\alpha}),$$

i.e. the root mean square worst-case error is always smaller than the worst-case error, see also Remark 5.

Remark 5 and also the above Remark imply that the bounds on the worst-case error also hold for the root mean square worst-case error. On the other hand, following the proofs for the bound on the worst-case error using the criterion for the root mean square worst-case error yields a better bound. We outline the results subsequently.

Following the proof of Lemma 2.10 we obtain

$$\sum_{k \in D} r_{b,\alpha}(k) \leq \sum_{\emptyset \neq u \subseteq \{1, \ldots, s\}} (1 + b^{-2\alpha m}(C_{b,\alpha}^2)^s - |u|(C_{b,\alpha}^2 + \bar{C}_{b,\alpha}^2)^{|u|}) \sum_{k \in D_{b,m,u}} q_{b,\alpha}^2 C_{b,\alpha}^2 (k)$$

\[+(1 + b^{-2\alpha m}(C_{b,\alpha}^2)^s - 1),\]

where $C_{b,\alpha}$ is the constant from Lemma 2.9 and

$$\bar{C}_{b,\alpha} = C_{b,\alpha} \left( b^{-1} + (b^2 - b)^{-1} \prod_{c=3}^{\alpha+1} (b^2 c - b^2(c-1))^{-1} \right).$$

(6.1)

The sum $\sum_{k \in D_{b,m,u}} q_{b,\alpha}^2 C_{b,\alpha}^2 (k)$ can now be bounded using almost the same arguments as in the proof of Lemma 5.1. Doing this one can obtain that for a digital $(t, \alpha, \beta, m, s)$-net we have

$$\sum_{k \in D_{b,m,u}} q_{b,\alpha}^2 C_{b,\alpha}^2 (k) \leq (2b)^{|u|\alpha} b^{-2(\beta m - t) + (\beta m - t + 1)} |u|^{-1}.$$

Hence we obtain the following theorem.

**Theorem 6.4.** Let $b$ be prime, $\alpha \geq 1$ an integer and let $C_1, \ldots, C_s \in \mathbb{Z}^{m \times m}_{b}$ be the generating matrices of a digital $(t, \alpha, \beta, m, s)$-net over $\mathbb{Z}_b$ with $m > t/\beta$. Then the
mean square worst-case error in the Korobov space \( \mathcal{H}_\alpha \) is bounded by

\[
\tilde{e}_{b,m,\alpha}(C_1, \ldots, C_s) \\
\leq \left( 1 + b^{-2am\tilde{C}_{b,\alpha}^2} + (2b)^\alpha (C_{b,\alpha}^2 + \tilde{C}_{b,\alpha}^2) (\beta m - t + 1)^\alpha \right)^{s} - (1 + b^{-2am\tilde{C}_{b,\alpha}^2})^{s} \\
+ (1 + b^{-2am\tilde{C}_{b,\alpha}^2})^{s} - 1,
\]

where \( C_{b,\alpha} > 0 \) is the constant in Lemma 2.9 and the constant \( \tilde{C}_{b,\alpha} > 0 \) is given by (6.4).

We can use Theorem 6.4 to obtain the following corollary.

**Corollary 6.5.** Let \( b \) be prime and let \( C_1^{(d)}, \ldots, C_s^{(d)} \in \mathbb{Z}_b^{\infty \times \infty} \) be the generating matrices of a digital \( (t, a, \min(a, d), s) \)-sequence \( S \) over \( \mathbb{Z}_b \) for any integer \( a \geq 1 \). Then for any real \( \alpha \geq 1 \) there is a constant \( C'_{b,s,\alpha} > 0 \), depending only on \( b, s \) and \( \alpha \), such that the root mean square worst-case error in the Korobov space \( \mathcal{H}_\alpha \) using the first \( N = b^m \) points of \( S \) is bounded by

\[
\tilde{e}_{b,m,\alpha}(C_1^{(d)}, \ldots, C_s^{(d)}) \leq C'_{b,s,\alpha} b^{t(\lceil a \rceil)} \frac{(\log N)^{(s\lfloor a \rfloor - 1)/2}}{N^{\min(\lfloor a \rfloor, \delta)}}.
\]

**Remark 8.** The above corollary shows that the digital \( (t, \alpha, \min(\alpha, d), s) \)-sequences constructed in Section 4 achieve the optimal convergence of \( P_{2\alpha} \) of \( \mathcal{O}(N^{-2\alpha} (\log N)^{s\alpha - 1}) \) as long as \( \alpha \) is an integer such that \( 1 \leq \alpha \leq d \). (This convergence is best possible for \( \alpha = 1 \) by the lower bound in [20].) If \( \alpha > d \) we obtain a convergence of \( \mathcal{O}(N^{-2d} (\log N)^{s\alpha - 1}) \).

Using the construction of Theorem 4.1 or Theorem 4.2 it follows that \( t(\alpha) \) also depends on the choice of \( d \). Hence choosing a large value of \( d \) also increases the constant factor \( b^{t(\lfloor a \rfloor)} \) in Corollary 5.3 and Corollary 6.5.

**7. Some examples of digital \( (t, \alpha, m, s) \)-nets over \( \mathbb{Z}_2 \).** In this section we give a simple example to show how the nets described in this paper can be constructed. We use the construction method outlined in Section 4.

**7.1. Example of a digital \( (0, 2, m, 1) \)-net over \( \mathbb{Z}_2 \).** First we use the so-called Hammersley net as the underlying digital net, which is a \( (0, m, 2) \)-net over \( \mathbb{Z}_2 \). The generating matrices for this net are given by

\[
C_1 = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & \ldots & 0 & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \vdots \\ 1 & 0 & \ldots & 0 \end{pmatrix}.
\]

(7.1)

Now we use the construction method of Section 4 to construct the matrix \( C_1^{(2)} \), i.e. \( d = 2 \) in this case. The first row of \( C_1^{(2)} \) is the first row of \( C_1 \), the second row of \( C_1^{(2)} \) is the first row of \( C_2 \), the third row of \( C_1^{(2)} \) is the second row of \( C_1 \), the fourth row of \( C_1^{(2)} \) is the second row of \( C_2 \) and so on. Assume that \( C_1, C_2 \) are \( m \times m \) matrices
where \( m \) is even. Then we obtain
\[
C^{(2)}_1 = \begin{pmatrix}
1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & 1 \\
0 & 1 & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & 0 & 1 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 0 & 1 & 0 & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & 0 & 1 & 0 & \ldots & 0
\end{pmatrix}.
\]

So for example if \( m = 4 \) we obtain
\[
C^{(2)}_1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\] (7.2)

The matrix \( C^{(2)}_1 \) is of course non-singular and therefore the point set one obtains are just equidistant points starting with 0.

Assume that \( m \) is even. Then the digital net which one obtains from \( C^{(2)}_1 \) is a digital \((0, 1, m, 1)\)-net over \( \mathbb{Z}_2 \) and, at the same time, it is also a digital \((0, 2, m, 1)\)-net. Note that using the bound from Theorem 4.1 we obtain a \( t \)-value of 1, but by closer investigation using Definition 3.1 one can see that the properties also hold for \( t = 0 \). Hence the \( t \)-value obtained from Theorem 4.1 is not necessarily strict even if the value of the underlying digital net is strict.

### 7.2. Example of a digital \((t, 2, 4, 2)\)-net over \( \mathbb{Z}_2 \)

Consider the digital \((1, 4, 4)\)-net over \( \mathbb{Z}_2 \) with generating matrices given by \( C_1, C_2 \) above and
\[
C_3 = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
and
\[
C_4 = \begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

Then \( C^{(2)}_1 \) is given by (7.2) and \( C^{(2)}_2 \) is given by
\[
C^{(2)}_2 = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1
\end{pmatrix}.
\]

Using the digital construction scheme we obtain the points
\[
(0, 0), \left( \frac{1}{2}, \frac{1}{16} \right), \left( \frac{1}{2}, \frac{1}{16} \right), \left( \frac{1}{16}, \frac{1}{2} \right), \left( \frac{1}{16}, \frac{1}{2} \right), \left( \frac{1}{16}, \frac{1}{16} \right), \left( \frac{1}{16}, \frac{1}{16} \right),
\]
\[
\left( \frac{1}{16}, \frac{1}{16} \right), \left( \frac{1}{2}, \frac{1}{16} \right), \left( \frac{1}{16}, \frac{1}{16} \right), \left( \frac{1}{16}, \frac{1}{16} \right), \left( \frac{1}{16}, \frac{1}{16} \right), \left( \frac{1}{16}, \frac{1}{16} \right), \left( \frac{1}{16}, \frac{1}{16} \right).
\]

It can be checked that this digital net is a digital \((1, 1, 4, 2)\)-net, i.e. a digital \((1, 4, 2)\)-net (the first two rows of \( C^{(2)}_1 \) and the first two rows of \( C^{(2)}_2 \) are linearly dependent, so the \( t \)-value cannot be 0 when \( \alpha = 1 \)).

Now we investigate the \( t \)-value when \( \alpha = 2 \). First note that Theorem 4.1 yields a \( t \)-value of 4 for \( \alpha = 2 \) (\( d = s = 2 \)). Further the \( t \)-value cannot be 2 in this case: we need
to consider all cases where \( i_{1,1} + i_{1,\min(\nu_1, 2)} + i_{2,1} + i_{2,\min(\nu_2, 2)} \leq \alpha m - t = 2 \cdot 4 - 2 = 6 \) with \( 0 \leq \nu_1, \nu_2 \leq 4 \). But by choosing \( i_{1,1} = i_{2,1} = 2 \) and \( i_{1,2} = i_{2,2} = 1 \) we obtain the first two rows of \( C_1 \) and the first two rows of \( C_2 \), and as those 4 rows are linearly dependent it follows that the \( t \)-value cannot be 2. Now let us check whether a \( t \)-value of 3 is possible: we need to have \( i_{1,1} + i_{1,\min(\nu_1, 2)} + i_{2,1} + i_{2,\min(\nu_2, 2)} \leq 5 \), hence \( \nu_1, \nu_2 \geq 2 \) is not possible (because then we would have \( i_{1,1} + i_{2,1} + i_{2,1} + i_{2,2} \geq 2 + 1 + 2 + 1 > 5 \)).

Further the conditions are satisfied if either \( \nu_1 = 0 \) or \( \nu_2 = 0 \) as the matrices \( C_1^{(2)} \) and \( C_2^{(2)} \) are non-singular. If \( \nu_1 > 2 \) then \( i_{1,1} \geq 3 \) and \( i_{1,2} \geq 2 \) and hence \( i_{1,1} + i_{1,2} \geq 5 \) and we can only get \( i_{1,1} + i_{1,\min(\nu_1, 2)} + i_{2,1} + i_{2,\min(\nu_2, 2)} \leq 5 \) if \( \nu_2 = 0 \). Hence if either \( \nu_1 > 2 \) or \( \nu_2 > 2 \) the properties are also satisfied. Thus we are left with the following three cases: \((\nu_1, \nu_2) = (1, 1), (\nu_1, \nu_2) = (1,2)\) and \((\nu_1, \nu_2) = (2, 1)\).

Now let \( \nu_1 = \nu_2 = 1 \). Then we need to take one row of each matrix \( C_1^{(2)} \) and \( C_2^{(2)} \) such that the sum of their row indices is smaller or equal to 5 and check whether those two rows are linearly independent. It can be checked that this is always the case: let \( C_j^{(2)} = (c_{j,1}^T, c_{j,2}^T, c_{j,3}^T, c_{j,4}^T) \), i.e. \( c_{j,k} \) denotes the \( k \)-th row of \( C_j^{(2)} \). Then the pairs of vectors \((c_{1,k}, c_{2,l})\) where \( k + l \leq 5 \) are always linearly independent for all admissible choices of \( k \) and \( l \) (i.e. \( c_{1,k} \neq c_{2,l} \)).

Consider now \( \nu_1 = 1 \) and \( \nu_2 = 2 \), i.e. we take one row from \( C_1^{(2)} \) and two rows from \( C_2^{(2)} \) such that the sum of the row indices does not exceed 5. Note that \( i_{2,2} \) has to be 1 otherwise \( i_{2,1} + i_{2,2} \geq 5 \) and \( i_{1,1} \) cannot even be 1. As \( i_{1,1} \geq 1 \) and \( i_{2,1} \geq 2 \) the only choices left are \( i_{1,1} = 1 \) and \( i_{2,1} = 2,3 \) and \( i_{1,1} = 2 \) and \( i_{2,1} = 2 \). So we need to check whether the triplets \((c_{1,1}, c_{2,1}, c_{2,2}), (c_{1,1}, c_{2,1}, c_{2,3})\) and \((c_{1,2}, c_{2,1}, c_{2,2})\) are all linearly independent, which upon inspection can be seen to be the case.

The case \( \nu = 2 \) and \( \nu = 1 \) can also be checked as the previous case. In this case all the relevant sets of vectors are also always linearly independent, hence a \( t \)-value of 3 is possible for \( \alpha = 2 \), i.e. the digital net above is a (strict) digital \((3, 2, 4, 2)\)-net.

The classical \( t \)-value (i.e. \( \alpha = 1 \)) of this digital net is not as good as for example the \( t \)-value of the Hammersley net (which is 0). On the other hand it can be checked that for \( \alpha = 2 \) the \( t \)-value of the Hammersley net where \( m = 4 \) is 4 and hence for this case it is worse than the \( t \)-value of the digital net constructed above.

As a last example let us consider the Hammersley net again for arbitrary \( m \geq 1 \), i.e. with the \( m \times m \) generating matrices given by \((\underline{1})\). As for example the first row of \( C_1 \) and the last row of \( C_2 \) are the same (and therefore linearly dependent) we must have \( \beta m - t < m + 1 \) for all \( \alpha \geq 1 \) (for \( \alpha = 1 \) we can still choose \( \beta = 1 \) and \( t = 0 \) and hence the Hammersley net achieves the optimal \( t \)-value, but for \( \alpha > 1 \) we have seen in Section \((\underline{3})\) that there are better constructions). It is sensible to choose \( \beta \) such that we can have a \( t \)-value which is independent of \( m \) (for example this is the case when one considers sequences and which is also the motivation for introducing those parameters; for digital nets it would of course also make sense to just state the value of \( \beta m - t \) and \( m \) instead of \( t, \beta \) and \( m \)). This means that \( \beta \leq 1 \), and as \( \beta \) indicates the convergence rate one can obtain it follows that one cannot expect to obtain a convergence rate beyond \((b^m)^{-1+\delta}\) (for an arbitrary small \( \delta > 0 \)) when using a Hammersley net.

8. Appendix: Some lemmas. We need the following lemmas.

**Lemma 8.1.** Let \( j \geq 1, a \geq 0, b \geq 2 \) and \( 0 \leq u, v < b^a \) with \( u \neq v \). Then we have

\[
\int_{u/b^a}^{(u+1)/b^a} \int_{u/b^a}^{(u+1)/b^a} |x - y|^j \, dx \, dy = \frac{2}{b^{a(j+2)}(j+1)(j+2)}
\]
Hence we have
\[ \int_{u/b^n}^{(u+1)/b^n} \int_{v/b^n}^{(v+1)/b^n} (x - y)^j \, dx \, dy = \frac{2j!}{b^{j+2}} \sum_{l=0}^{[j/2]} \frac{[u - v]^{j-2l}}{(j-2l)!(2l+2)!}. \]

**Proof.** We have
\[ \int_{u/b^n}^{(u+1)/b^n} \int_{v/b^n}^{(v+1)/b^n} (x - y)^j \, dx \, dy = \int_0^{1/b^n} \int_0^{1/b^n} (x - y)^j \, dx \, dy \]
\[ = \frac{1}{b^{j+2}} \int_0^1 \int_0^1 (x - y)^j \, dx \, dy. \]

We divide the last double integral in two parts, we have
\[ \int_0^1 \int_0^1 (x - y)^j \, dx \, dy = \int_0^1 \int_0^y (y - x)^j \, dx \, dy + \int_0^1 \int_y^1 (x - y)^j \, dx \, dy. \]

We calculate the first part and obtain
\[ \int_0^1 \int_0^y (y - x)^j \, dx \, dy = \frac{1}{j+1} \int_0^1 y^{j+1} \, dy = \frac{1}{(j+1)(j+2)} \]
and the second part is given by
\[ \int_0^1 \int_y^1 (x - y)^j \, dx \, dy = \frac{1}{j+1} \int_0^1 (1 - y)^{j+1} \, dy = \frac{1}{(j+1)(j+2)}. \]

Hence we have
\[ \int_0^1 \int_0^1 (x - y)^j \, dx \, dy = \frac{2}{(j+1)(j+2)}. \]

For the second part we have
\[ \int_{u/b^n}^{(u+1)/b^n} \int_{v/b^n}^{(v+1)/b^n} (x - y)^j \, dx \, dy = \int_0^{1/b^n} \int_{u/v/b^n}^{(u-1)/b^n} (x - y)^j \, dx \, dy \]
\[ = \frac{1}{b^{j+2}} \int_0^1 \int_{|u-v|}^{|u-v|+1} (x - y)^j \, dx \, dy, \]
where now \(|u - v| \geq 1\). We have
\[ \int_0^1 \int_{|u-v|}^{(u-1)/b^n} (x - y)^j \, dx \, dy = \frac{1}{j+1} \int_0^1 \left( (|u - v| + 1 - y)^{j+1} - (|u - v| - y)^{j+1} \right) \, dy \]
\[ = \frac{2(|u - v|)^{j+2} - ((|u - v| + 1)^{j+2} - (|u - v| - 1)^{j+2})}{(j+1)(j+2)}. \]

The result follows by simplifying the sum in the numerator. \( \square \)

**Lemma 8.2.** Let \( k \geq 1 \) be given by \( k = \kappa_{a_1 - 1} b^{a_1 - 1} + \cdots + \kappa_{a_2 - 1} b^{a_2 - 1} \) for some \( \nu \geq 1, \kappa_{a_1 - 1}, \ldots, \kappa_{a_2 - 1} \in \{1, \ldots, b - 1\} \) and \( 1 \leq a_{\nu} < \cdots < a_1 \). For any even \( 0 \leq j < 2\nu \) we have \( I_j(k) = 0 \).
Proof. The result for \( j = 0 \) follows from Proposition 2.3 and (2.4). It was shown in [5], Appendix A, that

\[
x = \frac{1}{2} + \sum_{c=1}^{\infty} \sum_{\tau=1}^{b-1} \frac{1}{b^c(e^{-2\pi i \tau/b} - 1)} \text{wal}_{\tau^{b^c-1}}(x)
\]

and hence

\[
|x - y|^j = \left( \sum_{c=1}^{\infty} \sum_{\tau=1}^{b-1} \frac{1}{b^c(e^{-2\pi i \tau/b} - 1)} \left( \text{wal}_{\tau^{b^c-1}}(y) - \text{wal}_{\tau^{b^c-1}}(x) \right) \right)^j
\]

\[
= \sum_{c_1, \ldots, c_j=1}^{\infty} \frac{1}{b^{c_1 + \cdots + c_j}} \prod_{i=1}^{j} \sum_{\tau=1}^{b-1} \frac{\text{wal}_{\tau^{b^c_i-1}}(y) - \text{wal}_{\tau^{b^c_i-1}}(x)}{e^{-2\pi i \tau/b} - 1}.
\]

Let

\[
A_k(c_1, \ldots, c_j) = \int_0^1 \int_0^1 \prod_{i=1}^{j} \sum_{\tau=1}^{b-1} \frac{\text{wal}_{\tau^{b^c_i-1}}(y) - \text{wal}_{\tau^{b^c_i-1}}(x)}{e^{-2\pi i \tau/b} - 1} \text{wal}_k(x) \text{wal}_k(y) \, dx \, dy.
\]

Then we have

\[
I_j(k) = \sum_{c_1, \ldots, c_j=1}^{\infty} \frac{A_k(c_1, \ldots, c_j)}{b^{c_1 + \cdots + c_j}}.
\]

We have

\[
\prod_{i=1}^{j} \sum_{\tau=1}^{b-1} \frac{\text{wal}_{\tau^{b^c_i-1}}(y) - \text{wal}_{\tau^{b^c_i-1}}(x)}{e^{-2\pi i \tau/b} - 1}
\]

\[
= \sum_{\tau_1, \ldots, \tau_j=1}^{b-1} \prod_{i=1}^{j} (e^{-2\pi i \tau_i/b} - 1)^{-1} \sum_{u \subseteq \{1, \ldots, j\}} (-1)^{|u|} \prod_{i \in u} \text{wal}_{\tau_i^{b^c_i-1}}(y) \prod_{i \notin u} \text{wal}_{\tau_i^{b^c_i-1}}(x)
\]

\[
= \sum_{\tau_1, \ldots, \tau_j=1}^{b-1} \prod_{i=1}^{j} (e^{-2\pi i \tau_i/b} - 1)^{-1} \sum_{u \subseteq \{1, \ldots, j\}} (-1)^{|u|} \text{wal}_{u, \tau}(y) \text{wal}_{(1, \ldots, j) \setminus u, \tau}(x)
\]

where \( C_{u, \tau} = \sum_{i \in u} \tau_i^{b^c-1} \) and hence

\[
A_k(c_1, \ldots, c_j)
\]

\[
= \sum_{\tau_1, \ldots, \tau_j=1}^{b-1} \prod_{i=1}^{j} (e^{-2\pi i \tau_i/b} - 1)^{-1} \sum_{u \subseteq \{1, \ldots, j\}} (-1)^{|u|} \int_0^1 \int_0^1 \text{wal}_{u, \tau}(y) \text{wal}_{(1, \ldots, j) \setminus u, \tau}(x) \text{wal}_k(x) \text{wal}_k(y) \, dx \, dy
\]

\[
= \sum_{\tau_1, \ldots, \tau_j=1}^{b-1} \prod_{i=1}^{j} (e^{-2\pi i \tau_i/b} - 1)^{-1} \sum_{u \subseteq \{1, \ldots, j\}} (-1)^{|u|} \int_0^1 \text{wal}_{u, \tau}(y) \, dy \int_0^1 \text{wal}_{(1, \ldots, j) \setminus u, \tau}(x) \, dx.
\]
Note that if $\nu > j/2$ we either have $C_{\nu, \tau} \oplus k \neq 0$ or $C_{\{1, \ldots, j\} \setminus \nu, \tau} \oplus k \neq 0$ and hence $A_k(c_1, \ldots, c_j) = 0$. The result now follows. \hfill \Box

Let $\sigma_p(n) = \sum_{h=1}^{n-1} b^p$. It is known that

$$\sigma_p(n) = \sum_{h=0}^{p} \frac{B_h}{h!} \frac{p!}{(p+1-h)!} n^{p+1-h}, \quad (8.1)$$

where $B_0, B_1, \ldots$ are the Bernoulli numbers (in particular, $B_0 = 1$, $B_1 = -1/2$ and $B_2 = 1/6$).

**Lemma 8.3.** Let $b \geq 2$, $1 \leq d \leq a$, $k = \kappa_{d-1} b^{d-1} + \cdots + \kappa_0$ where $\kappa_{d-1} \in \{1, \ldots, b-1\}$, $\kappa_{d-2}, \ldots, \kappa_0 \in \{0, \ldots, b-1\}$, $m = m_{a-1} b^{a-1} + \cdots + m_0$ and $n = n_{a-1} b^{a-1} + \cdots + n_0$. Then we have

$$\sum_{n=0}^{b^a-2} \sum_{m=m_{n+1}}^{b^a-1} \text{walk}_k((n \oplus m)/b^a) = b^{2a-d} \left( \frac{1}{2} + \frac{1}{e^{2\pi i \kappa_{d-1}/b} - 1} \right) - \frac{b^a}{2},$$

$$\sum_{n=0}^{b^a-2} \sum_{m=m_{n+1}}^{b^a-1} (m-n) \text{walk}_k((n \oplus m)/b^a) = b^{3a-2d} \left( \frac{1}{6} - \frac{1}{2 \sin^2(\kappa_{d-1} \pi/b)} \right) - \frac{b^a}{6}$$

and

$$\sum_{n=0}^{b^a-2} \sum_{m=m_{n+1}}^{b^a-1} (m-n) = \frac{1}{6} (b^{3a} - b^a).$$

**Proof.** In order to obtain a formula for the first sum, let $m' = m_{a-1} b^{a-1} + \cdots + m_{a-d+1} b^{a-d+1}$, $m'' = m_{a-d} b^{a-d} + \cdots + m_0$, $n' = n_{a-1} b^{a-1} + \cdots + n_{a-d+1} b^{a-d+1}$ and $n'' = n_{a-d} b^{a-d} + \cdots + n_0$. First consider the case where $m' > n'$ and arbitrary $m'', n''$. We have

$$\sum_{n_{a-d}=0}^{b-1} \sum_{m_{a-d}=0}^{b-1} e^{2\pi i (\kappa_0 (n_{a-1} - m_{a-1}) + \cdots + \kappa_{d-1} (n_{a-d} - m_{a-d}))/b} = 0,$$

as $\sum_{m=0}^{b-1} e^{2\pi i \kappa m/b} = 0$ for all $\kappa = 1, \ldots, b-1$. Thus we only need to consider the case where $m' = n'$, for which case we have

$$e^{2\pi i (\kappa_0 (n_{a-1} - m_{a-1}) + \cdots + \kappa_{d-1} (n_{a-d} - m_{a-d}))/b} = e^{2\pi i \kappa_{d-1} (m_{a-d} - n_{a-d})/b}. $$

This part is now given by

$$b^{d-1} \sum_{n''=0}^{b^a-d-1} \sum_{m''=0}^{b^a-d-1} \sum_{n_{a-d}=0}^{b-1} \sum_{m_{a-d}=0}^{b-1} e^{2\pi i \kappa_{d-1} (n_{a-d} - m_{a-d})/b}, \quad (8.2)$$

where we have the additional assumption $m_{a-d} b^{a-d} + m'' > n_{a-d} b^{a-d} + n''$. First
consider the case where \( m_{a-d} > n_{a-d} \). This part of (8.2) is given by

\[ b^d - 1 \sum_{n'=0}^{b^d - 1} \sum_{m'=0}^{b^d - 1} e^{2\pi i \kappa_{d-1}(n_{a-d} - m_{a-d})/b} \]

\[ = b^d - 1 \sum_{n'=0}^{b^d - 1} \sum_{m'=0}^{b^d - 1} e^{2\pi i \kappa_{d-1}(n_{a-d} - m_{a-d})/b} \]

\[ = \frac{b^{2a-d}}{e^{2\pi i \kappa_{d-1}/b - 1}}. \]

Now consider the case where \( m_{a-d} = n_{a-d} \). In this case we have the assumption that \( m'' > n'' \) and hence this part of (8.2) is given by

\[ b^d - 1 \sum_{n'=0}^{b^d - 1} \sum_{m'=n''+1}^{b^d - 1} 1 = \frac{1}{2} (b^{2a-d} - b^a). \]

Thus (8.2) is given by

\[ \frac{b^{2a-d}}{e^{2\pi i \kappa_{d-1}/b - 1}} + \frac{1}{2} (b^{2a-d} - b^a) \]

and the first result follows.

For the second sum let again \( m' = m_{a-1}b^{a-1} + \cdots + m_{a-d+1}b^{a-d+1} \), \( m'' = m_{a-d-1}b^{a-d-1} + \cdots + m_0 \), \( n' = n_{a-1}b^{a-1} + \cdots + n_{a-d+1}b^{a-d+1} \) and also \( n'' = n_{a-d-1}b^{a-d-1} + \cdots + n_0 \). First consider the case where \( m' > n' \) and arbitrary \( m'', n'' \).

We have

\[ \sum_{n_{a-d}=0}^{b-1} \sum_{m_{a-d}=0}^{b-1} (m - n)e^{2\pi i (\kappa_0(n_{a-1} - m_{a-1}) + \cdots + \kappa_{d-1}(n_{a-d} - m_{a-d}))/b} \]

\[ = \sum_{n_{a-d}=0}^{b-1} \sum_{m_{a-d}=0}^{b-1} (m_{a-d} - n_{a-d})e^{2\pi i (\kappa_0(n_{a-1} - m_{a-1}) + \cdots + \kappa_{d-1}(n_{a-d} - m_{a-d}))/b} \]

\[ = 0, \]

as \( \sum_{m=0}^{b-1} e^{2\pi i \kappa m/b} = 0 \) for all \( \kappa = 1, \ldots, b-1 \).

Thus we are left with the case where \( m' = n' \). We have

\[ e^{2\pi i (\kappa_0(n_{a-1} - m_{a-1}) + \cdots + \kappa_{d-1}(n_{a-d} - m_{a-d}))/b} = e^{2\pi i \kappa_{d-1}(m_{a-d} - n_{a-d})/b}. \]

Hence this part is given by

\[ b^d - 1 \sum_{n_{a-d}=0}^{b-1} \sum_{m_{a-d}=0}^{b-1} \sum_{n'=0}^{b^d - 1} \sum_{m'=0}^{b^d - 1} (m'' - n'' + b^{a-d}(m_{a-d} - n_{a-d}))e^{2\pi i \kappa_{d-1}(n_{a-d} - m_{a-d})/b}, \]

where we have the additional assumption \( m_{a-d}b^{a-d} + m'' > n_{a-d}b^{a-d} + n'' \). First
consider the case where \(m_{a-d} > n_{a-d}\). This part of (8.3) is given by

\[
b^{a-1} \sum_{0 \leq n_{a-d} + m_{a-d} < b} \sum_{n'' = 0}^{b^{a-d} - 1} (m'' - n'' + b^{a-d}(m_{a-d} - n_{a-d})) e^{2\pi i \kappa_{a-d - 1}(n_{a-d} - m_{a-d})/b} 
\]

\[
= b^{a-1} b^{(a-d)} \sum_{n_{a-d} = 0}^{b-1} \sum_{m_{a-d} = n_{a-d} + 1}^{b^{a-d} - 1} (m_{a-d} - n_{a-d}) e^{2\pi i \kappa_{a-d - 1}(n_{a-d} - m_{a-d})/b} 
\]

\[
= -\frac{b^{3a-2d}}{2 \sin^2(\kappa_{a-d - 1} \pi / b)}. 
\]

Now consider the case where \(m_{a-d} = n_{a-d}\). In this case we have the assumption that \(m'' > n''\) and hence this part of (8.3) is given by

\[
b^{a-1} \sum_{n'' = 0}^{b^{a-d} - 2} \sum_{m'' = n'' + 1}^{b^{a-d} - 1} (m'' - n'') = \frac{b^d}{6} (b^{3(a-d)} - b^{a-d}). 
\]

(This result can be obtained using (8.1), see the proof of the third part below.) Thus (8.3) is given by

\[-\frac{b^{3a-2d}}{2 \sin^2(\kappa_{a-d - 1} \pi / b)} + \frac{b^d}{6} (b^{3(a-d)} - b^{a-d}) 
\]

and the second result follows.

The third result can easily be verified by using (8.1). Indeed we have

\[
\sum_{m=n+1}^{b^{a}} \sum_{n=0}^{b^{a-2}} (m - n) = \sum_{m=1}^{b^{a-2}} \sum_{n=0}^{b^{a-2}} m = \sum_{n=0}^{b^{a-2}} \sigma_1(b^n - n) = \frac{1}{2} \sum_{n=0}^{b^{a-2}} ((b^n)^2 - (b^n - n)). 
\]

The last sum can be written as \(\frac{1}{2} \sum_{n=1}^{b^n} (n^2 - n) = \frac{1}{4} (\sigma_2(b^n + 1) - \sigma_1(b^n + 1))\) and by using (8.1) again the result follows. \(\blacksquare\)

**Lemma 8.4.** Let \(j \geq 0, \nu \geq 1, 1 \leq a_\nu < \cdots < a_1 \leq a, k = \kappa_{a_1 - 1} b^{a_1 - 1} + \cdots + \kappa_{a_\nu - 1} b^{a_\nu - 1}\) where \(\kappa_{a_1 - 1}, \ldots, \kappa_{a_\nu - 1} \in \{1, \ldots, b - 1\}\). Then we have

\[
\sum_{n=0}^{b^{a-2}} \sum_{m=n+1}^{b^{a-1}} (m-n)^j = b^a \sigma_j(b^a) - \sigma_{j+1}(b^a) \leq \frac{b^{a(j+2)}}{(j+1)(j+2)} 
\]

and

\[
\sum_{n=0}^{b^{a-2}} \sum_{m=n+1}^{b^{a-1}} (m-n)^j \text{val}_k((n \in m)/b^a) \leq C_{b,j} b^{(j+2) a - 2(a_1 + \cdots + a_{\min(j, \nu/2)})} 
\]

for some constant \(C_{b,j} > 0\) which is independent of \(\nu, a\) and \(a_1, \ldots, a_\nu\).

**Proof.** We have

\[
\sum_{n=0}^{b^{a-2}} \sum_{m=n+1}^{b^{a-1}} (m-n)^j = \sum_{n=1}^{b^{a-1}-1} (b^a - n)^j = b^a \sigma_j(b^a) - \sigma_{j+1}(b^a), 
\]
and by using (8.4) it follows that
\[ b^a \sigma_j(b^a) - \sigma_{j+1}(b^a) = b^a(j+2) \left( \sum_{h=0}^{j} B_h \left( \frac{j!}{h!(j+1-h)!} - \frac{(j+1)!}{h!(j+2-h)!} \right) b^{-ah} - B_{j+1} b^{-a(j+1)} \right) \leq b^a(j+2) B_0 \frac{1}{(j+1)(j+2)}, \]
from which the first part follows as $B_0 = 1$.

For $j = 0, 1$ the second part immediately follows from Lemma 8.3. Let now $j \geq 2$ and assume the result holds for all $j - 1, \ldots, 1, 0$.

Let $m = m_{a-1} b^{a-1} + \cdots + m_0$ and $n = n_{a-1} b^{a-1} + \cdots + n_0$. In order to obtain a bound on
\[ \sum_{n=0}^{b^{a-1}-1} \sum_{m=m+1}^{b^a-1} (m-n) e^{2\pi i (\kappa_{m-1}(n_{a-1}-m_{a-1}) + \cdots + \kappa_{n_{a-1}}(n_{a-1}-m_{a-1}))/b} \] (8.4)
we first sum over the digits $m_{a-1}$ and $n_{a-1}$.

Let $m' = m_{a-1} b^{a-1} + \cdots + m_{a-1} b^{a-1} + 1, n' = n_{a-1} b^{a-1} + \cdots + n_{a-1} b^{a-1} + 1, m'' = m_{a-1} b^{a-1} + \cdots + m_0$ and $n'' = n_{a-1} b^{a-1} + \cdots + n_0$. We consider two cases, namely where $m' > n'$ and where $m' = n'$.

For $m' = n'$ we either have $m_{a-1} > n_{a-1}$ or $m_{a-1} = n_{a-1}$ and $m'' > n''$, as $m > n$. First let $m_{a-1} > n_{a-1}$. We have $b^{a-1}$ choices for $m' = n'$ and the sum over the digits $m_{a-1}, n_{a-1}$ with $m_{a-1} > n_{a-1}$ can be written as one sum so that the part of (8.4) where $m' = n'$ is given by
\[ b^{a-1} \left| \sum_{n'=0}^{b^{a-1}-1} \sum_{m'=0}^{b^a-1} \sum_{\tau=1}^{b-1} (b-\tau) (\tau b^{a-1} + m'' - n'')^j e^{-2\pi i \kappa_{m-1} \tau / b} \right| \leq b^{a-1} \sum_{n'=0}^{b^{a-1}-1} \sum_{m'=0}^{b^a-1} \sum_{\tau=1}^{b-1} (b-\tau) (\tau b^{a-1} + m'' - n'')^j \leq C_{b,j} b a b^{(j+2)(a-1)}, \]
for some constant $C_{b,j} > 0$ which only depends on $b$ and $j$. Hence this part satisfies the bound. Now let $m_{a-1} = n_{a-1}$, then we have $m'' > n''$ and hence the part of (8.4) where $m' = n'$ and $n_{a-1} = n_{a-1}$ is given by
\[ b^{a-1} \sum_{n'=0}^{b^{a-1}-1} \sum_{m'=n'+1}^{b^a-1} (m'' - n'')^j \leq b^{a-1} b^{(j+2)(a-1)} \frac{(j+1)(j+2)}, \]
where the inequality was already obtained in the first part of this proof. Hence also this part satisfies the bound.
Now we consider the part of (8.3) where \( m' > n' \). We have

\[
\sum_{m_{a_{-1}}, n_{a_{-1}} = 0}^{b-1} (m' - n' + b^{a_{-1}}(m_{a_{-1}} - n_{a_{-1}}) + m'' - n'')^2 e^{2\pi ik_{a_{-1}}-1(n_{a_{-1}}-m_{a_{-1}})/b} \\
= b(m' - n' + m'' - n'')^2 + \sum_{\tau = 1}^{b-1} (b - \tau)[e^{-2\pi ik_{a_{-1}}-1\tau/b}(m' - n' + \tau b^{a_{-1}} + m'' - n'')^2 \\
+ e^{2\pi ik_{a_{-1}}-1\tau/b}(m' - n' - \tau b^{a_{-1}} + m'' - n'')^2] \\
= b(m' - n' + m'' - n'')^2 + \sum_{u = 0}^{j} \binom{j}{u}(m' - n' + m'' - n'')^2 - u b^{a_{-1}} E_u,
\]

(8.5)

where

\[
E_u = \sum_{\tau = 1}^{b-1} (b - \tau)[e^{-2\pi ik_{a_{-1}}-1\tau/b} + e^{2\pi ik_{a_{-1}}-1\tau/b} (-\tau)^u].
\]

It can be checked that \( E_0 = -b \) and \( E_1 = 0 \). Hence (8.5) is given by

\[
\sum_{u = 2}^{j} \binom{j}{u}(m' - n' + m'' - n'')^2 - u b^{a_{-1}} E_u,
\]

and hence the result follows from the induction assumption or the first part. \( \square \)

**Lemma 8.5.** Let \( k \geq 1 \) be given by \( k = \kappa_{a_{-1}} b^{a_{-1}} + \cdots + \kappa_{\nu} b^{a_{\nu}} \) for some \( \nu \geq 1, 1 \leq a_{\nu} < \cdots < a_1 \) and \( \kappa_{a_{-1}}, \ldots, \kappa_{a_{\nu}} \in \{1, \ldots, b-1\} \). Then for \( j \geq 1 \) we have

\[
|I_j(k)| \leq \frac{\hat{C}_{b,j}}{b^{(a_{1} + \cdots + \nu)} / \min(v, (j/2))}
\]

for some constant \( \hat{C}_{b,j} > 0 \) which depends only on \( b \) and \( j \).

**Proof.** Let \( k = \kappa_{a_{-1}} b^{a_{-1}} + \cdots + \kappa_0 \), where now \( a = a_1, u = u_{a-1} b^{a_{-1}} + \cdots + u_0 \) and \( v = u_{a-1} b^{a_{-1}} + \cdots + v_0 \). Then we have

\[
I_j(k) = \int_0^1 \int_0^1 [x - y]^2 \text{wal}_k(x) \text{wal}_k(y) \, dx \, dy \\
= \sum_{u = 0}^{b_{a_{-1}} - b_{a_{-1}}} \sum_{v = 0}^{b_{a_{-1}} - b_{a_{-1}}} e^{2\pi i (\kappa_0 (u_{a-1} - v_{a-1}) + \cdots + \kappa_{a_{-1}} (u_{0} - v_{0}))} \int_{u/b}^{(u+1)/b} \int_{v/b}^{(v+1)/b} |x - y|^2 \, dx \, dy.
\]

For \( u = v \) we have \( e^{2\pi i (\kappa_0 (u_{a-1} - v_{a-1}) + \cdots + \kappa_{a_{-1}} (u_{0} - v_{0}))} = 1 \). Using Lemma 8.1 it follows that this part in the above sum is given by

\[
\frac{2}{b^{a_{(a+1)}(j+1)(j+2)}}.
\]
Hence it remains to calculate
\[
\sum_{a=0}^{b^n-2} \sum_{u=a+1}^{b^n-1} e^{2\pi i (\kappa_0(u_{a-1} - v_{a-1}) + \cdots + \kappa_{a-1}(u_0 - v_0))} \int_{v/b^n}^{(v+1)/b^n} \int_{y/b^n}^{(y+1)/b^n} |x - y|^2 \, dx \, dy
\]

where we used Lemma 8.1. The absolute value of the inner double sum can now be bounded using Lemma 8.3 and hence the result follows. \[\square\]

**Lemma 8.6.** Let \( b \geq 2 \) be an integer and let \( \alpha > 1/2 \) be a real number. Then we have

\[
\sum_{k=1}^{\infty} r_{b,\alpha}(k) = 2\zeta(2\alpha),
\]

where \( \zeta(2\alpha) = \sum_{h=1}^{\infty} h^{-2\alpha} \).

**Proof.** Let \( h \in \mathbb{Z} \setminus \{0\} \) and let \( f_h(x) = e^{2\pi i h x} \). The Walsh coefficients \( \hat{f}_h(k) \) of the function \( f_h \) are then given by \( \hat{f}_h(k) = \int_0^1 f_h(x) \text{wal}_h(x) \, dx \). It follows that \( |\hat{f}_h(k)|^2 = |\beta_{h,k}|^2 \), where \( \beta_{h,k} \) was defined in Lemma 2.6. Using Parseval’s equality we obtain

\[
\sum_{k=1}^{\infty} |\beta_{h,k}|^2 = \sum_{k=1}^{\infty} |\hat{f}_h(k)|^2 = \int_0^1 |f_h(x)|^2 \, dx = \int_0^1 1 \, dx = 1.
\]

Hence we have

\[
\sum_{k=1}^{\infty} r_{b,\alpha}(k) = \sum_{h \in \mathbb{Z} \setminus \{0\}} \frac{1}{|h|^{2\alpha}} \sum_{k=1}^{\infty} |\beta_{h,k}|^2 = \sum_{h \in \mathbb{Z} \setminus \{0\}} \frac{1}{|h|^{2\alpha}} = 2\zeta(2\alpha).
\]

The result follows. \(\square\)

**Lemma 8.7.** Let \( \kappa_{a-1} b^{a-1} + \cdots + \kappa_{a-1} b^{a-1} \) with \( 1 \leq a_\nu < \cdots < a_1 \) and let \( \kappa_{a-1}, \ldots, \kappa_{a_1} - 1 \in \{1, \ldots, b-1\} \). Then

\[
\beta_{h,\kappa_{a-1} b^{a-1} + \cdots + \kappa_{a-1} b^{a-1}} = \sum_{h_1, \ldots, h_b \in \mathbb{Z} \setminus 0, h_1 b^{a-1} + \cdots + h_b b^{a-1}} \frac{b^a}{(2\pi 1)^b} \prod_{l=1}^{b} \frac{1 - e^{2\pi i h_l / b}}{h_l}.
\]

**Proof.** First we consider \( \kappa_{a-1} b^{a-1} \) with \( \kappa_{a-1} \in \{1, \ldots, b-1\} \). Let \( x = \frac{x_1}{b^1} + \frac{x_2}{b^2} + \cdots \), then we have \( \text{wal}_h(x) = e^{2\pi i \kappa_{a-1} x / b} \). Note that \( \text{wal}_h(x) \) is constant in the intervals \([u/b^a, (u+1)/b^a)\) for \( 0 \leq u < b^a \). Let \( u = u_{a-1} b^{a-1} + \cdots + u_0 \). Then for
any \( h \in \mathbb{Z} \setminus \{0\} \) we have
\[
\beta_{h,k} = \sum_{u_0=0}^{b^a-1} e^{2\pi i \kappa_{u_0-1} u_0 / b} \int_{u/b}^{(u+1)/b} e^{-2\pi i h u / b^a} e^{-2\pi i h x} dx
\]
\[
= \sum_{u_0=0}^{b^a-1} e^{2\pi i \kappa_{u_0-1} u_0 / b} \left( \frac{1 - e^{-2\pi i h / b}}{2\pi h} \right) \sum_{u_0=0}^{b-1} \cdots \sum_{u_{a-1}=0}^{b-1} e^{2\pi i \kappa_{u_0-1} u_0 / b} e^{-2\pi i h (u_{a-1} + \cdots + u_0) / b^a}.
\]
Let now \( h \in \mathbb{Z} \setminus \{0\} \) and let \( h = h_c b^c + \cdots + h_0 \) and set \( h_c = h_{c+1} = \cdots = 0 \). If \( h > 0 \) we assume that \( h_1 \in \{0, \ldots, b-1\} \) and if \( h < 0 \) we assume that \( h_1 \in \{-b+1, \ldots, 0\} \) for all \( i \geq 0 \). If \( h_0 \neq 0 \) then \( \sum_{u_0=0}^{b-1} e^{-2\pi i u_0 h / b} = 0 \) and hence \( \beta_{h_{a-1}h^{-a-1}} = 0 \).

If \( h_0 = 0 \) then \( \sum_{u_0=0}^{b-1} e^{-2\pi i u_0 h / b} = \frac{1 - e^{-2\pi i h / b}}{2\pi h} \). In general, if for an \( 0 \leq i < a-1 \) we have \( h_i \neq 0 \) then \( \beta_{h_{a-1}h^{-a-1}} = 0 \). Further, if \( h_i = 0 \) for \( 0 \leq i < a-1 \) then we also have \( \beta_{h_{a-1}h^{-a-1}} = 0 \). Hence, in order to obtain \( \beta_{h_{a-1}h^{-a-1}} \neq 0 \) we must have \( h_0 = \cdots = h_{a-2} = 0 \) and \( \kappa_{a-1} \equiv h_{a-1} \pmod{b} \). In this case we have
\[
\beta_{h_{a-1}h^{-a-1}} = \frac{1 - e^{-2\pi i h_{a-1} / b}}{2\pi h},
\]
where \( h = h_{a-1} b^{a-1} + h_a b^a + \cdots \) with \( h_{a-1} \equiv \kappa_{a-1} \pmod{b} \). We can also write
\[
\beta_{h_{a-1}h^{-a-1}} = \frac{b(1 - e^{-2\pi i h / b})}{2\pi h},
\]
with \( h \in \mathbb{Z} \) such that \( h \equiv \kappa_{a-1} \pmod{b} \).

We can interpret \( \beta_{h,k} = \int_{h}^{b} e^{-2\pi i h x} \text{walk}(x) \, dx \) as the Fourier coefficients of the \( k \)-th Walsh function, hence it follows that
\[
\text{walk}(x) = \sum_{h \in \mathbb{Z}} \beta_{h,k} e^{2\pi i h x}.
\]

Let now \( k = \kappa_{a-1} b^{a-1} + \cdots + \kappa_{a_{\nu-1}} b^{a_{\nu-1}} \) for some \( 1 \leq a_{\nu} < \cdots < a_1 \). Then we have
\[
\text{walk}_{a_{a-1} b^{a-1} + \cdots + \kappa_{a_{\nu-1}} b^{a_{\nu-1}}} (x)
= \sum_{h_{1} \in \mathbb{Z}} \beta_{h_{1}, \kappa_{a-1} b^{a-1} + \cdots + \kappa_{a_{\nu-1}} b^{a_{\nu-1}}} (x) \cdots \sum_{h_{\nu} \in \mathbb{Z}} \beta_{h_{\nu}, \kappa_{a_{\nu-1}} b^{a_{\nu-1}} + \cdots + \kappa_{a_{1}} b^{a_{1}}} (x)
= \sum_{h_{1}, \ldots, h_{\nu} \in \mathbb{Z}} \beta_{h_{1}, \kappa_{a-1} b^{a-1} + \cdots + \kappa_{a_{\nu-1}} b^{a_{\nu-1}}} (x) \cdots \beta_{h_{\nu}, \kappa_{a_{\nu-1}} b^{a_{\nu-1}} + \cdots + \kappa_{a_{1}} b^{a_{1}}} (x) \cdot e^{2\pi i (h_{1} + \cdots + h_{\nu}) x}.
\]
On the other hand we have
\[
\text{walk}_{\kappa_{a-1} b^{a-1} + \cdots + \kappa_{a_{\nu-1}} b^{a_{\nu-1}}} (x) = \sum_{h \in \mathbb{Z} \setminus \{0\}} \beta_{h, \kappa_{a-1} b^{a-1} + \cdots + \kappa_{a_{\nu-1}} b^{a_{\nu-1}}} (x) e^{2\pi i h x}.
\]
On comparing the last two equations we obtain that \( \beta_{h,\kappa_{a_1-1}b^{a_1-1}+\cdots+\kappa_{a_{u-1}}b^{a_{u-1}}} = 0 \) if either \( b^{a_1-1} \mid h \) or \( h \not\equiv \kappa_{a_1-1} \pmod{b^{a_1-1}} \). Now let \( h \in \mathbb{Z} \) such that \( b^{a_1-1}|h \) and \( h \equiv \kappa_{a_1-1} \pmod{b^{a_1-1}} \). Then we have

\[
\begin{align*}
\beta_{h,\kappa_{a_1-1}b^{a_1-1}+\cdots+\kappa_{a_{u-1}}b^{a_{u-1}}} &= \sum_{h_1,\ldots,h_u \equiv 0,1,\ldots,\kappa_{a_1-1} \pmod{b^{a_1-1}}} \beta_{h_1b^{a_1-1},\kappa_{a_1-1}b^{a_1-1}} \cdots \beta_{h_kb^{a_{u-1}},\kappa_{a_{u-1}}b^{a_{u-1}}} \\
&= \sum_{h_1,\ldots,h_u \equiv 0,1,\ldots,\kappa_{a_1-1} \pmod{b^{a_1-1}}} \frac{b^\nu}{(2\pi i)^\nu} \prod_{l=1}^\nu \frac{1 - e^{2\pi i h_l/b}}{h_l}
\end{align*}
\]

and the result follows.  

**Lemma 8.8.** For \( k \geq 1, b \geq 2, m \geq 1 \) and \( \alpha > 1/2 \) we have

\[
r_{b,\alpha}(kb^m) = b^{-2\alpha m} r_{b,\alpha}(k).
\]

**Proof.** First note that \( \beta_{h,\kappa_{a_1-1}b^{m+a_1-1}+\cdots+\kappa_{a_{u-1}}b^{m+a_{u-1}}} = 0 \) if \( b^m \not\mid h \). Further it follows from the previous lemma that

\[
\beta_{h,\kappa_{a_1-1}b^{m+a_1-1}+\cdots+\kappa_{a_{u-1}}b^{m+a_{u-1}}} = \beta_{h,\kappa_{a_1-1}b^{a_1-1}+\cdots+\kappa_{a_{u-1}}b^{a_{u-1}}}
\]

and hence by Lemma 2.6 we have

\[
r_{b,\alpha}(kb^m) = \sum_{h \in \mathbb{Z} \setminus \{0\}} \frac{|\beta_{h,kb^m}|^2}{|hb^m|^2} = b^{-2\alpha m} \sum_{h \in \mathbb{Z} \setminus \{0\}} \frac{|\beta_{h,k}|^2}{|h|^2} = b^{-2\alpha m} r_{b,\alpha}(k).
\]

The result follows.  

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