Clustering dynamics in globally coupled map lattices

Fagen Xie\textsuperscript{1,2} and Gang Hu\textsuperscript{1,3}

\textsuperscript{1} China Center of Advanced Science and Technology, (World Laboratory), P.O. Box 8730, Beijing, 100080, China
\textsuperscript{2} Institute of Theoretical Physics, Academia Sinica, Beijing 100080, China
\textsuperscript{3} Department of Physics, Beijing Normal University, Beijing 100875, China

(Received: November 15, 2018)

Clustering bifurcations are investigated by considering models of globally coupled map lattices. Typical classes of clustering bifurcations are revealed. The clustering bifurcation thresholds of the coupled system are closely related to the bifurcation structures of single map. In particular, cluster-doubling bifurcation induced period-doubling bifurcations and clustering induced chaos are found. At the onset of multiple-cluster states, equal-site-occupation-partition, and consequently, equal-phase-shift states \cite{5} are always identified numerically.

PACS numbers: 05.45.+b

The investigation of globally coupled extended systems has attracted a rapidly growing interest in recent years\textsuperscript{1–6}. They arise naturally in studies of Josephson junctions arrays\textsuperscript{1}, multimode laser\textsuperscript{2}, charge-density wave\textsuperscript{3}, oscillatory neuronal systems\textsuperscript{4} and so on. A number of intriguing and novel high-dimensional features have been revealed in these spatiotemporal systems. For instance, a curious and interesting dynamical state, so called \textit{antiphase} state has been revealed both numerically and experimentally\textsuperscript{1,2}. Such a state is periodic in time, with each element of the system having precisely the same wave form. However, the motion of each element is just shifted by certain phase from its “neighbor”. This state is closely related to the \textit{clustering}, which is numerically studied in the globally coupled chaotic maps\textsuperscript{5}. The antiphase state is a \textit{clustering} with equal occupation elements in each cluster. Up to the date, the mechanism underlying this fascinating phenomenon has not been revealed. In this letter we will thoroughly analyze the bifurcation mechanisms and phase diagrams of clusterizations and reveal the interesting general features of equal-site-occupation-partition (ESOP) and equal-phase-shift (EPS) states at bifurcation points of clustering. (Here we call the antiphase state as EPS state because multiple phases may appear in clustering bifurcations.) We take the following globally coupled map lattice (GCML)

\begin{equation}
 x_{n+1}(i) = (1-\epsilon)f[x_n(i)] + \frac{\epsilon}{L} \sum_{j=1}^{L} f[x_n(j)], \quad i = 1, \cdots, L,
\end{equation}

as our model, where \( n \) denotes the discrete time, \( i \) labels the lattice site with \( L \) system size. \( f(x) \) prescribes the local dynamics, and is chosen as the logistic map \( f(x) = ax(1-x) \). \( \epsilon \) gives the long-range coupling strength. In Refs. 5, Kaneko \textit{et al} presented very rich and interesting behaviors of (1) for positive \( \epsilon \). Negative \( \epsilon \) represents also many practical situations, such as antiferromagnetic coupling\textsuperscript{6} and resistance coupling and so on. Therefore, it is useful to unify the investigations of Eq. (1) for both positive and negative \( \epsilon \).

First, we consider clustering bifurcations from the simplest spatially homogeneous configuration, so called \textit{coherent} state. After some simple algebra, the critical stability condition of this coherent state can be explicitly shown as

\begin{equation}
 \epsilon_c = 1 - e^{-\lambda_0},
\end{equation}

where \( \lambda_0 \) is the Lyapunov exponent of the single logistic map \cite{2}; Eq. (2) is generally valid for any coherent state, whatever periodic or chaotic\textit{). This critical stability boundary is shown in Fig. 1(a) with solid line. As \( \epsilon > \epsilon_c \), the coherent state always exists, and is \textit{locally stable}, while below the solid lines, the coherent state loses its stability and bifurcates to multi-cluster state. From (2) it is clear that, coherent periodic motions are always stable for positive \( \epsilon \), they can lose stability only in the negative coupling regions. However, it should be emphasized that many attractors may coexist with the coherent state in the regime above the solid line for large system size \( L \) and large \( a \).

A class of interesting states are multi-cluster states with ESOP (i.e., \( k \) clusters \( N_1 = \cdots = N_k \), with \( N_i, i = 1, \cdots, k \) being the occupation numbers of \( i \)th cluster) , and in case of periodic motion, each cluster may have the same motion except some EPS. It will be shown that this kind of states appear naturally at bifurcation thresholds. Afterwards, a period-\( m \) state with \( k \) clusters (1) will be called \textit{TmCk} state. It often happens that \( m = k \), the evolving dynamics of the \textit{TkCk} state can be much reduced as

\begin{equation}
 x_{n+1} = (1-\epsilon)f(x_n) + \frac{\epsilon}{k} \sum_{j=1}^{k} f(x_j).
\end{equation}

For \( k = 2 \), the solutions of Eq. (3) read

\begin{equation}
 x_{1,2} = \frac{1 + a - a\epsilon \pm \sqrt{(1 + a - a\epsilon)^2 - 2(2 - \epsilon)(1 + a - a\epsilon)}}{2a(1 - \epsilon)}.
\end{equation}
The stability conditions of the ESOP TkCk state can be analytically given by computing the products of Jacobi matrices of system (3). After certain matrix operations, the stability analysis of system (1) can be much simplified to a block-diagonal form of the linear matrix of (3) as

$$J = \begin{pmatrix} \prod_{n=0}^{k} M_n & 0 \\ 0 & M' \end{pmatrix}$$

where

$$M_n = \begin{pmatrix} (1 - \frac{(k-1)\epsilon}{k})f_n^1 & \ldots & \ldots & \ldots \\ \frac{1}{k} f_n^1 & (1 - \frac{(k-1)\epsilon}{k})f_n^2 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \frac{1}{k} f_n^1 & \ldots & \ldots & (1 - \frac{(k-1)\epsilon}{k})f_n^k \end{pmatrix}$$

with $f_n^i = ax_{n+i}(1 - x_{n+i})$, $i = 1, 2, \ldots, k$, and $x_{n+k} = x_n$. $M' = (1 - \epsilon)ck \prod_{n=0}^{k} x_n(1 - x_n)I$, $I$ is the $(L - k) \times (L - k)$ unit matrix. Therefore, we obtain $L - k$ degenerate eigenvalues $\lambda = (1 - \epsilon)ck \prod_{n=0}^{k} x_n(1 - x_n)$, and other $k$ eigenvalues. If the absolute values of all eigenvalues of $J$ are less than one, the reference $k$-cluster state is stable. For $k = 2$, The stability boundaries can be given explicitly. Increasing $a$ from small value, the T2C2 state appears from the spatially homogeneous state via saddle-node bifurcation (SN) and pitchfork bifurcation (PK) for $\epsilon > 0$ and $\epsilon < 0$, respectively, at the following critical thresholds

$$\text{SN: } a_c = 1 + \sqrt{1 + \frac{3}{(1 - \epsilon)^2}}, \quad \epsilon > 0,$$

$$\text{PK: } a_c = 2 + \frac{1}{1 - \epsilon}, \quad \epsilon < 0,$$

and loses its stability via Hopf bifurcation (HF) and pitchfork bifurcation at

$$\text{HF: } a_c = 1 + \sqrt{1 + \frac{5 - 3\epsilon}{(1 - \epsilon)^2}}, \quad \epsilon > 0,$$

$$\text{PK: } a_c = 1 + \sqrt{\frac{2 + (\epsilon - 2)^2}{1 - \epsilon}}, \quad \epsilon < 0,$$

All these bifurcation lines and other clustering bifurcation lines in the period-doubling region are shown in Fig. 1(b). It is remarkable that in Fig. 1(b) one can find a clear rule to describe the entire clustering bifurcation structure in $\epsilon - a$ plane from the bifurcation points in the $a$ axis at $\epsilon = 0$. Actually, all bifurcation points for a single cell (at $\epsilon = 0$) are multidimension bifurcation points in the $\epsilon - a$ parameter plane. For $\epsilon > 0$ one can find first order saddle-node bifurcation and second order Hopf bifurcation curves, while for $\epsilon < 0$ one can find second order pitch-fork bifurcation and first order subcritical bifurcation (SC) curves, all these bifurcation curves intersect with the $a$ axis ($\epsilon = 0$) at the critical points for single cell. These beautiful bifurcation trees can be also found in chaos region associated with each periodic window. The enlarged regions of the rectangles in Fig. 1(a) for the bifurcations to ESOP three-cluster, and five-cluster states are shown in Fig.1 (c) and (d), respectively. The bifurcation figures are similar to Fig. 1(b). A difference of these figures from the two-cluster state is that for $\epsilon < 0$ these multi-cluster states appear from chaotic motions via saddle-node bifurcation rather than pitch-fork bifurcation.

To give clear pictures about the clustering bifurcations we show the asymptotic states of system (3) in Figs. 2(a) and (b), where all stable homogeneous states are plotted by diamonds, stable multicluster states by solid lines, unstable states (both homogeneous and inhomogeneous) by dashed lines. The black regions represent stable quasiperiod motion. In Fig. 2(a) we fix $\epsilon = 0.2$, the system has only the coherent state below $a_c \approx 3.386$ (T1C1 for $a < 3$ and T2C1 for $a > 3$), the ESOP T2C2 state occurs via a saddle-node bifurcation at $a_c$, then the two states (T2C1 and T2C2) coexist in certain $a$ interval. As $a$ continuously increases, the coherent state undergoes a series of period doubling bifurcations leading to chaos, and then loses coherence at $a \approx 3.640$. The T2C2 state subjects to Hopf bifurcation at $a \approx 3.808$. In Fig. 2(b) we fix $\epsilon = -0.15$, the bifurcations are essentially different from those of (a). The T1C1 state first undergoes a cluster-period-doubling bifurcation at $a \approx 2.870$ to create a stable T2C2 state. After $a > 3.07$, the coherent T2C1 state turns to be stable via subcritical bifurcation. In a large interval $3.4 > a > 3.07$ the coherent T2C1 state coexists with multicluster state, while the T2C2 state undergoes further cluster-period-doubling bifurcation and Hopf bifurcation leading to chaos. At $a \approx 3.4$ the T2C1 state bifurcates via cluster-period-doubling to form a T4C2 state. Fig. 2(b) is extremely interesting due to the following novel features. First, we find a cluster-doubling induced period-doubling. The value $a \approx 2.870$ is far below the period-doubling condition for a single map. Global coupling leads to cluster doubling at this parameter, that induces period doubling in time. Second, we find a cluster-doubling sequence 1-2-4 (and the induced period-doubling sequence). We expect that this cluster doubling cascade may proceed to a very large numbers of cluster. In our case this cascade is stopped at $k = 4$ by Hopf bifurcation at $a \approx 3.316$. Nevertheless, the tendency of cluster increasing bifurcations leading to chaos can be still seen in Fig. 2(c) for $a < 3.4$, where we plot the number of clusters vs. $a$ for the states of Fig. 2(b). It is found in Fig. 2(b) that chaos can appear for $a < 3.4$, where the nonlinear parameter $a$ is far below the value for chaotic motion for the single map. In the same time in Fig. 2(c) we find the number of clusters diverges (to the order of $L$) in this chaos region. Then we conclude this chaos is made possible by clusterization,
and such is clustering induced chaos. It is remarkable that all analytical predictions in Fig. 1(b) are perfectly confirmed by numerical simulations of (1) in Figs. 2.

In the above we focused on the discussion of ESOP and EPS multiple cluster states. On one hand, these states can be easily treated by analyzing Eqs. (3). On the other hand, these kinds of states appear generically and naturally under bifurcation conditions. For instance, in Fig. 2(a) around at \( a \approx 3.386 \) and from arbitrary initial conditions we can get T2C1 or T2C2 states. Whenever we get T2C2 state it must be an ESOP state \((N_1 = N_2 = \frac{L}{4} \text{ if } L \text{ is even, or } N_1 = \frac{L+1}{4}, N_2 = \frac{L-1}{4} \text{ if } L \text{ is odd})\). In Fig. 2(b) we run Eqs. (1) from random initial conditions at \( a = 2.8 \), then compute Eqs. (1) by gradually increasing \( a \) and by using the final state for the previous \( a \) as the initial state for the new \( a \), we can surely get ESOP and ESP states for all cluster-doubling cascade.

Let us take a two-cluster state as an example. Usually, many two-cluster attractors with different \( N_1 \) and \( N_2 \) \((N_1 + N_2 = L)\) may coexist. However, at the onset of two-cluster state we always first find the ESOP state. In Fig. 3(a) and (b) we fix \( \epsilon = 0.2, L = 200 \), and plot the two-cluster states for the occupation numbers \( N_1 = N_2 = 100 \), and \( N_1 = 98, N_2 = 102 \), respectively. It is really striking that in (b) by a slightly breaking the occupation balance the threshold for two-cluster state is considerably raised. In Fig. 3(c) we fix \( \epsilon = -0.3, L = 200 \), and plot \( N_m \) vs. \( a \), where \( N_m \) is the possible smallest occupation number for the two-cluster states of Eqs. (1). Small noises are added to the system sites for wiping out glass states with extremely small basins. It is interesting that \( N_m \) goes to \( \frac{L}{4} \) at both left and right critical points. (This Phenomenon is also found by Kaneko et al for \( \epsilon > 0 \).) We find that the ESOP state naturally appears at both first and second order bifurcation points.

Similar behavior happens also for general \( k \)-cluster bifurcations. In Figs. 4 and 5 we plot three and five cluster states, respectively, arising via saddle-node bifurcation from spatiotemporal chaos. In the two cases, the ESOP states arise much earlier than those with slight deviations from equal occupation partition for both \( \epsilon > 0 \) and \( \epsilon < 0 \). We have also examined many \( k \)-cluster bifurcations and always find the same behavior.

In conclusion we have revealed a detailed bifurcation structure for clusterization and found a cluster-doubling sequence and clustering induced spatiotemporal chaos. At any onset of \( k \)-cluster state one always finds ESOP and EPS state. This is a general extension of the results obtained by Wiesenfeld et al. However, in our case the natures of bifurcations and the links of clustering bifurcations with the bifurcations of single site are clearly shown in Figs. 1 and 2, then these ESOP and EPS states of globally coupled systems can be predicted, based on the bifurcation structure of single site.

In this letter we took the globally coupled map lattices as our models. However, the ideas can be extended to more general globally coupled systems. For instance, we have examined globally coupled Josephson Junctions, the clustering bifurcation features are qualitatively the same as those for the coupled map lattice systems.

This work is supported by the National Natural Science foundation of China and Project of Nonlinear Science.

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![FIG. 1 Bifurcation figure for homogeneous (coherent) state.](image1)

![FIG. 2 Bifurcation sequences for certain couplings.](image2)

![FIG. 3 T2C2 states for L = 200.](image3)
FIG. 4 Three-cluster states, $L = 300$. (a) $\epsilon = 0.05$, $N_{1,2,3} = 100$. (b) $\epsilon = 0.05$, $N_1 = 98, N_2 = 100, N_3 = 102$. (c) $\epsilon = -0.1$, $N_{1,2,3} = 100$. (d) $\epsilon = -0.1$, $N_1 = 98, N_2 = 100, N_3 = 102$. At the onset of three-cluster state one can see only the ESOP state.

FIG. 5 Five-cluster states, $L = 500$, the same behavior as in Fig. 4. (a) $\epsilon = 0.005$, $N_{1,2,3,4,5} = 100$. (b) $\epsilon = 0.005$, $N_1 = 94, N_{2,3,4} = 100, N_5 = 106$. (c) $\epsilon = -0.005$, $N_{1,2,3,4,5} = 100$. (d) $\epsilon = -0.005$, $N_1 = 94, N_{2,3,4} = 100, N_5 = 106$. 
