Time Asymmetric Boundary Conditions and the Definition of Mass and Width for Relativistic Resonances

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The definition of mass and width of relativistic resonances and in particular of the Z-boson is discussed. For this we use the theory based on time asymmetric boundary conditions given by Hardy class spaces Φ− and Φ+ for prepared in-states and detected out-states respectively, rather than time symmetric Hilbert space theory. This Hardy class boundary condition is a mathematically rigorous form of the singular Lippmann-Schwinger equation. In addition to the rigorous definition of the Lippmann-Schwinger kets \(|j, s⟩_±\) as functionals on the spaces Φ∓, one obtains Gamow kets \(|j, s_R⟩\) with complex centre-of-mass energy value \(s_R = (M_R - iΓ_R/2)^2\). The Gamow kets have an exponential time evolution given by \(\exp(-iM_Rt - Γ_Rt/2)\) which suggests that \((M_R, Γ_R)\) is the right definition of the mass and width of a resonance. This is different from the two definitions of the Z-boson mass and width used in the Particle Data Table and leads to a numerical value of \(M_R = (91.1626 ± 0.0031)\) GeV from the Z-boson lineshape data.

11.80.-m; 11.30.Cp; 14.70.Hp; 13.38.Dg

I. INTRODUCTION

The Review of Particle Properties [1] gives two definitions of the mass and width of the Z-boson and lists two different values which are obtained from the fit of two different formulas for the lineshape to the same experimental data. The value \(M_Z\) is obtained from the fit to the “relativistic Breit-Wigner with energy dependent width” of the on-shell renormalisation scheme

\[
a_j^{om}(s) = \frac{-\sqrt{s} \sqrt{Γ_e(s)Γ_f(s)}}{s-M_Z^2+i\sqrt{s}Γ_Z(s)} \approx \frac{-M_Z B_{ej} Γ_Z}{s-M_Z^2+i\frac{s}{M_Z^2}Γ_Z} = \frac{R_Z}{s-M_Z^2+i\frac{s}{M_Z^2}Γ_Z}, \quad m_0^2 \leq s < ∞. \tag{1.1}
\]

The value \(\bar{M}_Z\) is obtained from the relativistic Breit-Wigner of the S-matrix pole

\[
a_j^{BW}(s) = \frac{R_Z}{s-S_R} = \frac{R_Z}{s-M_Z^2+iM_ZΓ_Z} = \frac{R_Z}{s-(M_R - iΓ_R/2)^2}, \quad m_0^2 \leq s < ∞. \tag{1.2}
\]

Both lineshape formulas \((1.1)\) and \((1.2)\) fit the experimental data equally well \([2,3]\). But they lead to values of the mass parameters \(M_Z\) and \(\bar{M}_Z\) which differ from each other by about 10 times the experimental error. From \(a_j^{om}(s)\) the fit gives the values

\[
M_Z = (91.1871 ± 0.0021) \text{ GeV}, \quad Γ_Z = (2.4945 ± 0.0024) \text{ GeV}. \tag{1.3}
\]

From \(a_j^{BW}(s)\) one obtains the values

\[
M_R = (91.1626 ± 0.0031) \text{ GeV}, \quad Γ_R = (2.4934 ± 0.0024) \text{ GeV}. \tag{1.4}
\]

Numerically this means

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\[ M_R = M_Z - 0.026 \text{ GeV} = M_Z - 10 \times \Delta m_{\text{exp}}, \quad \Gamma_R = \Gamma_Z - 1.2 \text{ MeV}, \] (1.5)

or

\[ \bar{M}_Z = M_Z - 34 \text{ MeV}. \] (1.6)

The question thus is: What is the right definition of the Z-boson mass and width and therefore the right numerical value of the mass of the Z-boson?

Even if one chooses the S-matrix definition (1.2) because it is gauge invariant [4], the complex parameter \( s_R \) in Eq. (1.3) can be expressed in terms of the real parameters mass and width in many different ways leading to many more arbitrary definitions of the Z-boson mass. Some of these mentioned in the literature are

1. \( (\bar{M}_Z, \bar{\Gamma}_Z) \) (also called \( (m_2, \Gamma_2) \) [4])

\[ s_R = \bar{M}_Z^2 - i\bar{M}_Z\bar{\Gamma}_Z. \] (1.7)

2. \( (M_R, \Gamma_R) \) [5] which is often used but not dictated by the analytic S-matrix theory (see Ref. [8], in particular p. 248)

\[ s_R = \left( M_R - i\frac{\Gamma_R}{2} \right)^2. \] (1.8)

This is related to \( (\bar{M}_Z, \bar{\Gamma}_Z) \) by the algebraic identity

\[ \bar{M}_Z = M_R \left( 1 - \frac{1}{4} \left( \frac{\Gamma_R}{M_R} \right)^2 \right)^{\frac{1}{2}} \quad \text{and} \quad \bar{\Gamma}_Z = \Gamma_R \left( 1 - \frac{1}{4} \left( \frac{\Gamma_R}{M_R} \right)^2 \right)^{-\frac{1}{2}}. \] (1.9)

3. \( (m_1, \Gamma_1) \) [5] which is numerically very close to the conventional Standard Model values \( (M_Z, \Gamma_Z) \). It can be defined in terms of \( \bar{M}_Z \) and \( \bar{\Gamma}_Z \) by:

\[ \bar{M}_Z^2 = m_1^2 \left( 1 + \left( \frac{\Gamma_1}{m_1} \right)^2 \right)^{-1} \quad \text{and} \quad \bar{\Gamma}_Z^2 = \Gamma_1^2 \left( 1 + \left( \frac{\Gamma_1}{m_1} \right)^2 \right)^{-1}, \] (1.10)

such that \( m_1 \) is numerically the same as \( M_Z \) of (1.3), if one identifies the maximum of \( |a_j^{\text{em}}(s)|^2 \), which is \( M_Z \left( 1 + (\Gamma_Z/M_Z)^2 \right)^{-1} \), with the maximum of \( |a_j^{\text{EM}}(s)|^2 \), which is \( \bar{M}_Z^2 \),

\[ m_1 \approx M_Z, \quad \Gamma_1 \approx \Gamma_Z. \] (1.11)

Before we give an answer to the question of how to define the Z-boson mass and width, we should like to make the following remarks: With the present data it is totally irrelevant which definition of the Z-boson mass one uses for a global fit of the Standard Model, since other Standard Model parameters such as \( M_W \) or \( 1/\alpha_{\text{EM}}(M_Z) \) have errors which far exceed the differences (1.5) and (1.6) [5]. The global fit is good \( (\chi^2 \approx 12 \text{ for 12 degrees of freedom}) \), confirming to the present level of accuracy the experimental data and also the Standard Model as a theory, including electroweak radiative corrections. The exact definition of the Z-boson mass (and other resonance masses) is not a central issue of the Standard Model, and the prevailing opinion is that there are many ways to parameterize the experimental data of electroweak physics [5] and \( m_1 \) may be as good a value for the Z-boson mass as \( \bar{M}_Z \) and \( M_R \). However, though not presently needed for the Standard Model fits, the extraordinary accuracy of the Z-lineshape data, obtained with great expense of time and effort, allowed for the first time to discuss the problem of the definition of mass and width for an unstable relativistic particle and go beyond the level of precision given by the Weisskopf-Wigner approximation and one-loop effects. This has opened up a new tier of inquiries, as witnessed by [6–7], and led us to the question: what is a relativistic unstable particle?

Our answer to this question is given, in analogy to the case of stable relativistic particles, in terms of representations of transformations of relativistic space-time (Poincaré transformations). The states of stable elementary particles are vectors of an irreducible representation space \([m^2, j] \) of the Poincaré group \( \mathcal{P} \) [10] from which one can define the fields [11]. This should not be restricted to interaction free, asymptotic states, but apply also to the exact states and to
Poincaré transformations generated by the (interaction-incorporating) “exact generators” \( P_0 = H = H_0 + V, \ P^\mu, \ J^{\mu\nu} \).

It has been claimed [1] that the exact in-states \(|\alpha^+\rangle\) and the out-states \(|\alpha^-\rangle\) (which fulfill the Lippmann-Schwinger equation) transform under the Poincaré group in the same way as the free states. This is not quite correct but under a precisely defined mathematical hypothesis, which replaces the Hilbert space axiom of standard quantum mechanics, one can show that the in- and out-states \(|\alpha^+\rangle\) and \(|\alpha^-\rangle\) of the Lippmann-Schwinger equation span an irreducible representation space of Poincaré-semigroup transformations into the backward and forward light cone, respectively [2]. The semigroup representations into the forward light cone are characterized by the angular momentum \( j \) (of the partial wave) and the centre-of-mass energy square \( s \) which we will just call energy in the following) where \( m^2 \leq s < \infty \). Thus we denote \(|\alpha^\pm\rangle = |s, j, b^-\rangle\) where \( b \) are some degeneracy parameters (e.g. \( j_3 \) and momentum \( p \); as in Wigner’s basis vectors for unitary representations \([m^2, j]\) of the Poincaré group.) However, one can also take the 4-velocities \( p = \frac{\sqrt{s}}{\sqrt{2}} = \gamma v; \ \gamma = \sqrt[4]{1 + p^2} \). The kets \(|s, j, b = p^-\rangle\), and therewith the semigroup representations \([s, j]\), will be analytically continued into the lower half plane, second Riemann sheet of the partial S-matrix \( S_j(s) \) to the resonance pole position \( s = s_R = (M_R - i\Gamma_R/2)^2 \). The resulting semigroup representations \([s_R, j]\), obtained by integrating around the resonance pole, are the resonance analogues of Wigner’s stable particle representations \([m^2, j]\), and we use \([s_R, j]\) as definition of the relativistic unstable particle in the same way as \([m^2, j]\) serves as definition of the relativistic stable particle. From the transformation property of the \([|s_R, j\rangle, b^-\rangle\) under the Poincaré semigroup transformations it follows that the decay constant of the unstable particle is \( \tau = \Gamma_R = -2\text{Im}\sqrt{s_R}. \) Therefore if we want the lifetime = inverse width relation to hold for relativistic unstable particles, then \([1,2]\) is the right function for the relativistic Breit-Wigner amplitude and \((M_R, \Gamma_R)\) is the right parametrization in terms of “mass” and “width”. In other words, if we define a state vector of a resonance with width \( \Gamma_R \) by integrating the Lippmann-Schwinger kets with a Breit-Wigner energy distribution around the pole position at \( s = s_R \), then one can derive that the lifetime of the exponential decay of these states is \( \tau = 1/\Gamma R \). In addition this is a semigroup (i.e., irreversible) decay.

To discuss this result and to formulate the new axiom of quantum theory from which this result follows, is the subject of this paper. In section II we conjecture the new mathematical hypothesis by which we replace the Hilbert space axiom of quantum mechanics. In section III we obtain the vector description of a resonance in terms of the semigroup representations \([s_R, j]\) from the resonance pole of the S-matrix. In section IV we consider the general case of \( N \) interfering resonances in the \( j \)th partial wave. We establish the connection between each Breit-Wigner amplitude of the S-matrix \( S_j(s) \) and the corresponding exponentially decaying Gamow vector in a “complex” basis vector expansion. To the well known background amplitude corresponds a non-exponential background vector in this complex basis vector expansion, which is usually not considered (Weisskopf-Wigner approximation).

II. FROM THE LIPPMANN-SCHWINGER EQUATION TO THE HARDY CLASS BOUNDARY CONDITION

The observed out-states \( \psi^- \) and the prepared in-states \( \psi^+ \) are (continuous) superposition of the \(|\alpha^\pm\rangle\):

\[
\psi^\pm = \int \rho(\alpha) d\alpha |\alpha^\pm\rangle \langle \alpha |\psi^\pm\rangle. \tag{2.1} \]

Here \(\alpha\) stands for a whole collection of quantum numbers (eigenvalues of a complete set of commuting observables c.s.c.o.), e.g., 4-momentum, spin, particle species labels,...

\[
\alpha = p^\mu(\alpha), j, j_3, n, \ldots, \tag{2.2}
\]

and the weight function \(\rho(\alpha)\) (or measure \(\rho(\alpha) d\alpha = d\mu(\alpha)\)) is chosen such that

\[
\langle \bar{\psi} \alpha' | \psi^\pm \rangle = \int \rho(\alpha) d\alpha \langle \bar{\psi} \alpha' | \alpha^\pm \rangle \langle \alpha | \psi^\pm \rangle, \tag{2.3}
\]

i.e.,

\footnote{For the conventional quantities we follow here fairly closely the notation of Ref. [1] but omit the letters \(\Psi\) and use Dirac’s notation instead; thus \(|\alpha^\pm\rangle = \Psi^\mp_\alpha\) of \([1]\).}
Equation (2.4) is Dirac’s basis vector expansion or completeness relation (Nuclear Spectral theorem of mathematics). The labels $\mp$ indicate that the $|z\mp\rangle$ are not only eigenvectors (generalized eigenvectors or eigenkets, since the $|z\mp\rangle$ are functionals) of a complete system of commuting observables (c.s.c.o.), but that they also fulfill certain boundary conditions. For a scattering process, like the resonance formation $e^+e^- \rightarrow Z \rightarrow e^+e^-$, the boundary conditions are usually formulated in terms of the Lippmann-Schwinger equations [13] written in its standard form as

$$|z\pm\rangle = |z\rangle + \frac{1}{E(\alpha) - H_0 \pm i\epsilon} V|z\pm\rangle = \left(1 + \frac{V}{E(\alpha) - H \pm i\epsilon}\right)|z\rangle = (\Omega^\pm\rangle z\rangle, \quad (2.5\mp)$$

where $E_\alpha = p^0(\alpha)$. Eqs. (2.5) are highly singular and mathematically ill defined expressions (like the Dirac kets were too, until they were defined as functionals over the Schwartz space $\Phi \subset H$). We, therefore, want to formulate our boundary conditions also in terms of dense subspaces $\Phi_\mp \subset H$ of the Hilbert space $\mathcal{H}$ instead of the singular (integral) equations (2.3). Also, the Lippmann-Schwinger equation singles out one momentum component $E = p^0$ in a non-covariant way; we do not want to use the quantum numbers $p^0 = E(= E_{e^+} + E_{e^-}$ for our resonance scattering process) but the centre of mass energy—or invariant mass squared $s = p_\mu p^\mu$. We therefore choose for the Dirac-Lippmann-Schwinger kets $|\alpha\mp\rangle$ the eigenkets

$$|\alpha\mp\rangle = |\alpha\rangle \pm \frac{1}{E(\alpha) - H_0 \mp i\epsilon} V|\alpha\mp\rangle = \left(1 \mp \frac{V}{E(\alpha) - H \mp i\epsilon}\right)|\alpha\rangle = (\Omega^\alpha\mp\rangle |\alpha\rangle, \quad (2.6\mp)$$

Here $j$ denotes the spin (and parity) and labels the partial amplitude (1.2) or (1.1). For the $Z$-resonance $j = 1$. The quantum number $s$ is the invariant mass squared, and for the resonance formation process $e^+e^- \rightarrow Z \rightarrow e^+e^-$ its “physical” values are $(m_{e^+} + m_{e^-})^2 \equiv s_0 \leq s < \infty$. The continuous quantum numbers $b$ are the degeneracy quantum numbers of $|j,s\rangle$. One usually chooses for $b$ the momentum $p$ and $j_3$, but we will later choose $b = \hat{p} = p/\sqrt{s}$, the space components of the 4-velocity.

We now want to discuss the definition of the kets (2.6). We will start with a single stable particle space. For a fixed isolated value of $s = m^2$ the Dirac kets $|[j,m^2],b\rangle$ span the irreducible unitary representation space of the Poincaré group $\mathcal{P}$ characterized by $[j,m^2],b$,

$$|[j,m^2],b\rangle = \sum_{j_3} \int d\mu(b) |[j,m^2],j_3\rangle f(j_3,b). \quad (2.7)$$

The coordinates $f(j_3,b)$ along the basis vectors $|[j,m^2],j_3\rangle$ (the wave functions) are labelled by the discrete $j_3$ which we will ignore and by continuous quantum numbers $b$ describing, e.g., the momentum resolution if $b = p$ and $d\mu(b) = d^3p/(2p^0)$. If in place of the momentum one chooses the space components of the 4-velocity $b = \hat{p} = p/m$ as degeneracy labels, then $f(j_3,\hat{p} = p/m)$ describes the same momentum resolution since $m$ is a fixed value.

Equation (2.7) is Dirac’s basis vector expansion (Nuclear Spectral Theorem for a Rigged Hilbert Space $\Phi \subset \mathcal{H} \subset \Phi^\times$ see Appendix) for the irreducible representation $[j,m^2]$ of $\mathcal{P}$ and for a fixed value of $j$ and $m^2$ the vectors $|[j,m^2],f\rangle$ describe the state of a relativistic elementary particle with a momentum or 4-velocity distribution given by the function $f(j_3,b)$. For the momentum wave functions $f(j_3,b), b = p$ or $b = \hat{p}$ one usually chooses functions of the Schwartz space $\mathcal{S}(\mathbb{R}^3), \hat{p} \in \mathbb{R}^3$. The vectors $|[j,m^2],f\rangle, (j, m^2=\text{fixed})$ are the elements of the abstract Schwartz space $\Phi$ and the Dirac kets $|[j,m^2],b\rangle$ are elements of its dual $\Phi^\times$ i.e. of the space of antilinear continuous functional on the space $\Phi$ (see Appendix). The Rigged Hilbert space of the function spaces

$$\mathcal{S}(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \subset \Phi^\times(\mathbb{R}^3)$$

are equivalent to, or form a “realization” of, the triplet of abstract spaces

$$\Phi \subset \mathcal{H} \subset \Phi^\times$$

The Dirac kets $|[j,m^2],b\rangle \in \Phi^\times$ are thus defined if one defines $\Phi$ e.g. by its realization in terms of the function space $\mathcal{S}(\mathbb{R}^3)$ of (momentum) wave functions $f(b)$. We now consider the scattering states $|[j,s],b^-\rangle$. Since resonances have a definite value of spin parity (they appear in a partial wave $A_j(s)$), we fix the value of $j$ (e.g., $j = 1$ for $Z$), but we have to consider all “physical” values of $s$, $m_0^2 \leq s < \infty$, for the resonance scattering process (e.g., $e^+e^- \rightarrow Z \rightarrow e^+e^-$). Therefore every out-state vector $\psi^-$ is a continuous linear combination of the Dirac-Lippmann-Schwinger kets $|[j,s],b^-\rangle, (j = \text{fixed}, \text{which we omit})$: 

$$\langle \mp \alpha' | \alpha \mp \rangle = \frac{1}{\rho(\alpha)} \delta(\alpha' - \alpha). \quad (2.4)$$
\[
\psi_f = \int_{m_0^2}^{\infty} ds \langle [j,s], f^- \rangle \psi^-(s) \quad (2.8)
\]
where the set of admitted centre of mass energy wave functions \(\psi^-(s)\) for a fixed \(f\) (fixed momentum distribution \(f(j,b)\)) or in general the set of admitted four-momentum (or \(s, \vec{p} = b\)) wave functions

\[
\langle -b, [j,s] | \psi^- \rangle = \psi^-(j, s, b) = \psi^-(s)
\quad (2.9)
\]
defines the space of vectors \(\psi^-\). The \(\psi^-(j, s, b)\) have already been fixed to be Schwartz space functions of the variable \(b\). We now want to consider the function \(\psi^-(j, s, b)\) as functions of \(s\). Since we are now interested in the variable \(s\), we often ignore the \(b\) dependence and omit \(\sum_b\) over the discrete and/or continuous \(b\). We therefore consider the function \(\psi^-(s)\) for any fixed \(f \in \Phi\) or the function \(\psi^-(j, s, b)\) for any fixed values \(j, b\), which we then also call \(\psi^-(s) = \psi^-(j, s, b)\). In short we will write for \(2.8\) and \(2.9\) as:

\[
\psi^- = \int ds |s^-\rangle \psi^-(s) \quad (2.10)
\]
This is again the Dirac basis vector expansion or Nuclear Spectral theorem, but now only for the energy \(s\), i.e. for the total mass-square operator \(P^\mu P^\mu = (P_{e+} + P_{e-})\mu(P_{e+} + P_{e-})^\mu\) whose eigenvalue is \(s\). Like every Dirac ket, the ket \(|s^-\rangle\) is defined by (1) the eigenvalue equation

\[
(P^\mu P^\mu)^\times |s^-\rangle = s |s^-\rangle \quad (2.11)
\]
and by (2) the boundary condition

\[
|s^-\rangle \in \Phi_+^\times .
\quad (2.12)
\]
Usually one assumes that Dirac kets are Schwartz space functionals like the momentum eigenkets \(|b\rangle\). For the out (and in) scattering states \((2.6\mp)\) that fulfill the Lippmann-Schwinger equations \((2.13\mp)\) we shall make other assumptions. As already indicated by the label \(\mp\) at the kets and \(\pm\) at the spaces \(\Phi_\mp\), we shall assume different boundary conditions for the out and in plane wave states \(|\alpha^\mp\rangle\). To conjecture this boundary conditions we use the heuristic Lippmann-Schwinger equations \((2.3\mp)\) for guidance.

The energy wave function \(\psi^-(s) = \psi^-(j, s, b)\) considered as a function of \(s\) should also be a smooth, well behaved function. That means \(\psi^-(s) \in S(\mathbb{R}_+)\) where \(\mathbb{R}_+ = \{ s : m_0^2 \leq s < \infty \}\). In the relativistic case \(S(\mathbb{R}_+)\) is not exactly the Schwartz space (since we also want \(\sqrt{s} \psi^-(s)\) to be a well behaved function), but it is a closed subspace thereof which is dense in \(L^2(\mathbb{R}_+)\). We will not be concerned with these mathematical details here and refer to [14,15].

However, the functions \(\psi^-(s)\) need to be better than well behaved, because the \(-i\epsilon\) in the Lippmann-Schwinger equation \((2.2\mp)\) indicates that the function \(\langle \psi^- | s^-\rangle = \langle \psi^- | \psi^-(s) \rangle = \psi^+(s)\) also needs to have some meaning when the energy acquires a negative imaginary part. This we generalize by requiring that the \(\psi^-(s)\) can be continued to analytic functions in the lower half complex plane that vanish sufficiently fast at the infinite semicircle. Precisely we assume:

\[
\overline{\psi^-(s)} \in S \cap \mathcal{H}_-^2 ,
\quad (2.14)
\]
where \(\mathcal{H}_-^2\) is the space of Hardy functions \([16]\) of the lower half complex plane, for which we choose the second sheet of the Riemann energy surface for the analytically continued \(j\)-th partial \(S\)-matrix \(S_j(s)\). Thus the Lippmann-Schwinger equation suggests that its complex conjugate \(\overline{\psi^-}(s)\) fulfill:

\[
\psi^-(s) \in S \cap \mathcal{H}_+^2 ,
\quad (2.15)
\]
where \(\mathcal{H}_+^2\) is the space of Hardy functions \([16]\) in the upper half complex plane. One can show that the space of well behaved Hardy functions forms a Rigged Hilbert Space (RHS) \([17]\)

\[
S \cap \mathcal{H}_-^2 \subset L^2(\mathbb{R}_+) \subset (S \cap \mathcal{H}_+^2)^\times
\quad (2.16)
\]
The abstract RHS whose mathematical realization is given by this triplet of function spaces \((2.16)\) we denote by:

\[
\Phi_+ \subset \mathcal{H} \subset \Phi^\times_+.
\quad (2.17)
This means that the vectors $\psi^-$ which have the Dirac basis vector expansion (2.11) (or (2.9)) with the wave functions (coordinates) $\psi^-(s) \in S \cap H_2^+$ are elements $\psi^- \in \Phi_+ \subset \mathcal{H}$. The basis vectors $|s\rangle \equiv |j, s, b^-\rangle$ are then continuous antilinear functionals on the space $\Phi_+$, i.e., $|j, s, b^-\rangle \in \Phi_+^\times$. There are more elements in $\Phi_+^\times$ than the complete system of basis vectors of (2.9), (2.11). The elements that we are particularly interested in are the Gamow kets $|j, s, b^-\rangle$ associated with the resonance pole of the S-matrix $S_j(s)$ at $s = s_R$. To generalize the boundary condition from the Lippmann-Schwinger equation (2.5) with infinitesimal imaginary energy $-i\epsilon$ to the Hardy class spaces $S \cap H^2_+$, (2.14) of the whole lower half complex plane may appear far fetched, but with the resonance pole in mind one does not seem to have another choice.

The Dirac-Lippmann-Schwinger kets $|[j, s], b^-\rangle$ are generalized eigenvectors of the total mass operator $P^\mu P_\mu = (P_+ + P_-)^\mu (P_+ + P_-)^\mu$ with real eigenvalue s,

$$\langle P^\mu P_\mu |j, s, b^-\rangle = s \langle j, s, b^- | j, s, b^- \rangle, \quad \text{for all} \, \psi^- \in \Phi_+. \quad (2.18)$$

This is also written as (2.12), where $(P^\mu P_\mu)^\times$ is the extension of the adjoint operator $(P^\mu P_\mu)^\dagger = (P_\mu P^\mu)^t$ from the space $\mathcal{H}$ to the space $\Phi_+^\times$. In Dirac’s notation the $^\times$ is omitted which we shall also do unless it is needed for clarification.

The Dirac-Lippmann-Schwinger kets at rest $|[j, s], b^-_{\text{rest}}\rangle$ are also eigenkets of the full Hamiltonian $P^0 \equiv H = H_0 + V$:

$$P^0 |[j, s], b^-_{\text{rest}}\rangle = \sqrt{s} |[j, s], b^-_{\text{rest}}\rangle \quad (2.19)$$

and these kets transform under Lorentz transformations $\mathcal{U}(A)$ in the well-known way (like the $|[j, m^2], b^-_{\text{rest}}\rangle$).

In the same way we wrote (2.5) in detail as (2.9) or as (2.11), we will write (2.1+) as continuous linear superpositions of the $|[j, s], b^+\rangle$. Instead of choosing wave functions (2.15) we choose now wave functions $\phi^+(s)$ which fulfill

$$\phi^+(s) \in S \cap H^2_+, \quad (2.20)$$

where $H^2_+$ is the space of Hardy class functions on the lower half plane of the second sheet. Like in the case of (2.16) these spaces of Hardy class functions form another RHS

$$S \cap H^2_- \subset \mathcal{L}^2(\mathbb{R}_+) \subset (S \cap H^2_2)^\times \quad (2.21)$$

where $\mathcal{L}^2(\mathbb{R}_+)$ is the same Hilbert space of Lebesgue integrable functions as in (2.16). The abstract RHS equivalent to the triplet of function spaces (2.22) we call

$$\Phi_- \subset \mathcal{H} \subset \Phi_+^\times \quad (2.22)$$

and $\mathcal{H}$ in (2.23) and in (2.17) are the same Hilbert spaces. We denote the vector whose wave function is $\phi^+(s)$ by $\phi^+$. The Dirac basis vector expansion (2.2+) takes then the mathematically precise form

$$\Phi_- \ni \phi^+ = \int_{m_0^2}^{\infty} ds |[j, s], \phi^+\rangle \phi^+(s) = \int_{m_0^2}^{\infty} ds \sum_b |[j, s], b^+\rangle \langle + b, [j, s]| \phi^+\rangle \quad (2.23)$$

This is the Nuclear Spectral Theorem in the RHS (2.22).

The heuristic Lippmann-Schwinger equations (2.5±) have thus led us to a new pair of RHS (2.17) and (2.22) and a new pair of boundary conditions

$$|[j, s], b^-\rangle \in \Phi_+^\times \quad \text{and} \quad |[j, s], b^+\rangle \in \Phi_+^\times \quad (2.24)$$

for the solutions of the generalized eigenvalue equations for the self-adjoint operator $P^\mu P_\mu$. The condition (2.24) together with the required Hardy class property of the spaces makes the heuristic Lippmann-Schwinger equations (2.5±) mathematically precise. Whether the Hardy class condition is the minimal requirement to accomplish this, we do not want to discuss here. In order to emphasise that the space of the in-states $\{\phi^+\}$ is different from the space of the out-states $\psi^-$, we changed from the notation of (2.1) and denoted the in-state vectors $\phi^+$ by a different letter $\phi$ than the out-state vectors $\psi^-$. We want to distinguish the in-states $\phi^+ \in \Phi_-$ and their basis vectors $|[j, s], b^+\rangle \in \Phi_+^\times$ from the out-states $\psi^- \in \Phi_+$ and their basis vectors $|[j, s], b^-\rangle \in \Phi_+^\times$. The kets $|[j, s], b^+\rangle$ also fulfill the eigenvalue equations (2.18),(2.19) but the two kets fulfill different boundary conditions expressed by the choice of the spaces.
However note that $\Phi_- \cap \Phi_+$ is not empty and it may be even dense in the same Hilbert space $\mathcal{H}_{[S^2]}$. Mathematically the space $\Phi_+$ (or $\Phi_-$) is defined by the choice of function spaces $\mathcal{S} \cap \mathcal{H}^2_s\big|_{[m^2_s, \infty)}$ (or $\mathcal{S} \cap \mathcal{H}^2_s\big|_{[m^2_s, \infty)}$) for the wave functions $\psi^-(s)$ (or $\phi^+(s)$). The abstract vector space $\Phi_\pm$ is mathematically “realized” by the function space $\mathcal{S} \cap \mathcal{H}^2_s\big|_{[m^2_s, \infty)}$ in the same way as the Hilbert space $\mathcal{H}$ is realized by the space of Lebesgue square integrable functions $L^2(\mathbb{R}^+, ds)$ and the abstract Schwartz space $\Phi$ is realized by the space of smooth, infinitely differentiable, rapidly decreasing functions $\mathcal{S}(\mathbb{R})$.

Physically $\{\phi^+\}$ is the set of prepared in-states defined by the preparation apparatus, and $\{\psi^-\}$ is the set of registered out-observables (which are usually also called out-states but) which are defined by the observation device (detector).

Standard Hilbert space quantum mechanics, often including scattering theory, assumes

$$\{\phi^+\} = \{\psi^-\} = \mathcal{H} \quad (2.25)$$

or $\{\phi^+\} = \{\psi^-\} = \mathcal{D}$, a dense subspace of $\mathcal{H}$. If we chose, for instance, $\{\phi^+(s)\} = \{\psi^-(s)\} = \mathcal{S}(\mathbb{R})$ (or $\mathcal{H}$), then $|j, s, b_{\text{rest}}^{-}\rangle$ and $|j, s, b_{\text{rest}}^+\rangle$ could not fulfill the Lippmann-Schwinger equations (2.3). Thus there is some discrepancy between the Hilbert space axiom and the Lippmann-Schwinger equation.

The axiom (2.25) amounts to solving the time-symmetric dynamical equations (such as the Schrödinger equation) with time-symmetric boundary conditions. This is the only possibility within orthodox Hilbert space quantum theory, because the Hilbert space allows only for reversible time evolution given by the unitary group $U^\dagger(t) = \exp(-iHt)$, $-\infty < t < +\infty$. For some idealised physical states, for example the stationary states of the harmonic oscillator, this is an acceptable boundary condition. For scattering states this is not acceptable because the in-state $\phi^+$ must be prepared at times $t'$ before a time $t_0$ ($t' < t_0$), and the out-observable $\psi^-$ can be registered only at times $t''$ after $t_0$ ($t'' > t_0$). Therefore the space of prepared in-states should be distinguished from the space of detected out-observables. With the new mathematical apparatus of RHS’s of Hardy class one can now distinguish between states and observables within the mathematical theory if one replaces the axiom (2.25) by the new hypothesis:

The space of prepared states $|\phi^+\rangle\langle\phi^+|$ defined by the preparation apparatus is $\{\phi^+\} = \Phi_- \subset \mathcal{H}$.

The space of registered observables $|\psi^-\rangle\langle\psi^-|$ defined by the registration apparatus is $\{\psi^-\} = \Phi_+ \subset \mathcal{H}$. (2.26)

This new hypothesis replaces the Hilbert space hypothesis (2.25). The new hypothesis not only allows the continuation of the Lippmann-Schwinger kets $|s, j\rangle b^{-}$ ($|s, j\rangle b^{+}$) into the lower (upper) half complex energy plane for which we choose the second sheet of the $S$-matrix, but it also allows the Gamow kets to be obtained from the resonance poles of the $S$-matrix. This was the original motivation for the introduction of the Hardy functions $\mathcal{H}_{[S^2]}$.

The distinction between prepared in-states $\Phi_-$ and registered observables $\Phi_+$, by the use of different dense (complete in a different topology than the $\mathcal{H}$ topology) subspaces of the same $\mathcal{H}$ is the only modification in the foundation of the theory; the dynamical equations and the algebra of observables, like the commutation relations of the Poincaré transformations, remain the same. All other novel conclusions, like the exact exponential decay law, the precise definition of width and mass of an $S$-matrix pole resonance and the quantum mechanical time asymmetry are mathematical consequences of this new hypothesis (2.26).

### III. FROM THE $S$-MATRIX POLE TO THE GAMOW VECTOR

We define a resonance by a second sheet $S$-matrix pole at the complex value $s = s_R$. It is thus characterized by two real parameters, e.g., by $\text{Re}(s_R) = M^2_R$ and $\text{Im}(s_R) = -M^2_{\Gamma}Z$ as in (1.7) or by $\text{Re}(\sqrt{s_R}) = M_R$ and $\text{Im}(\sqrt{s_R}) = -\Gamma R / 2$ as in (1.8) or in still other ways. The most suitable choice of this parameterization of the complex $s_R$ by two real numbers $(m, \Gamma)$ depends upon the physical interpretation of these numbers. Since Weisskopf-Wigner many physicists believe that resonances and (exponentially) decaying states are the same and, especially for non-relativistic

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2 The inclusion $\subset$ in (2.22) and (2.17) must not be understood like the inclusion of the two-dimensional space in a 3-dimensional space but rather like the inclusion of the rational numbers $\mathbb{Q}$ in the real numbers $\mathbb{R}$; $\mathbb{Q} \subset \mathbb{R}$.
quantum mechanics, a common assumption is that the width $\Gamma$ of the resonance is related to the lifetime of the decay $\tau$ by

$$\frac{\hbar}{\Gamma} = \tau.$$ (3.1)

The width $\Gamma$ is measured as the Breit-Wigner line width in the cross section

$$|a_{BW}(E)|^2 \sim \frac{1}{\left|E - (E_R - i\frac{\Gamma}{2})\right|^2} = \frac{1}{(E - E_R)^2 + \frac{\Gamma^2}{4}},$$ (3.2)

and the lifetime $\tau$ is measured by the exponential law for the counting rate $\dot{N}_\eta(t)$ of the decay products $\eta$ in the decay of the resonance $R \to \eta$:

$$\dot{N}_\eta(t) \equiv \frac{\Delta N_\eta(t)}{\Delta t} \sim e^{-t/\tau}.$$ (3.3)

Though there has not been an exact mathematical proof of (3.1) – since the Weisskopf-Wigner method is an approximation (cf. [21]) – and though there has not been an accurate experimental verification of (3.1), this relation between width and lifetime appears to be favoured by almost everyone in relativistic and non-relativistic physics. The state vector of a quasistable state which we constructed in the non-relativistic theory [22–24] fulfills (3.1). We have called these quasistable state vectors, Gamow vectors $\psi^G(t)$, they cannot be elements of the Hilbert space $\mathcal{H}$, but are kets on Hardy spaces. A vector description is needed if one wants to consider the resonance of a formation process, like $e^+e^- \to Z \to e^+e^-$, also as a decaying particle, like the $K^0_{S,L} \to \pi^+\pi^-$ (where in the past one attributed to it an eigenvector of a complex mass or energy matrix [23]).

According to the fundamental assumption of quantum mechanics, the probability for the decay $R \to \eta$ is given by the Born probability

$$P_\eta(t) = \text{Tr}(\Lambda_\eta |\psi^G(t)\rangle\langle\psi^G(t)|),$$ (3.4)

where $\Lambda_\eta$ is the projection operator on the subspace of the decay products $\eta$. The probability rate, or decay rate into $\eta$, should then be calculated from (3.4)

$$\dot{P}_\eta(t) = \frac{d}{dt} P_\eta(t).$$ (3.5)

Since the experimental counting rate $\dot{N}_\eta(t)$ measures the theoretical $\dot{P}_\eta(t)$, the latter must fulfill

$$\dot{P}_\eta(t) = \Gamma_\eta e^{-t/\tau},$$ (3.6)

in order to ensure agreement with the experimental formula (3.3) with $\tau$ calculated from the property of the decaying state vector $\psi^G$.

We shall construct the relativistic Gamow vector in complete analogy to the non-relativistic case, starting from the resonance pole of the relativistic $S$-matrix at $s = s_R$. We shall see that, if “width” $\Gamma$ and lifetime $\tau$ are to fulfill (3.1), then $\Gamma$ must be chosen as the parameter

$$\Gamma \equiv \Gamma_R = -2\text{Im}(\sqrt{s_R})$$ (3.7)

in the relativistic Breit-Wigner formula (1.2). The real “mass” of the quasistable relativistic state will then be given by the parameter

$$M_R = \text{Re}(\sqrt{s_R}).$$ (3.8)

Constructing a vector that fulfills (3.1) starting from another Breit-Wigner amplitude, e.g. (1.1), appears not possible, since (1.2) plays a very special role in the mathematics (it is the Cauchy kernel) which is needed for the construction.

3For a given relativistic quasistable particle one either measures $\Gamma$ by (3.2) (for $\Gamma_R/M_R \sim 10^{-1}$) or one measures $\tau$ by (3.3) (for $\Gamma/M \sim 10^{-10}$) and one does not come close in accuracy to an experimental test of (3.1).
We start with the S-matrix element between a prepared in-state \( \phi^+ \) and a detected out-observable \( \psi^- \). We assume the asymmetric boundary conditions \( \phi^+ \in \Phi_- \) and \( \psi^- \in \Phi_+ \) of Section [4]

\[
(\psi^{out}, \phi^{out}) = (\psi^{out}, S\phi^{in}) = (\Omega^+ \psi^{out}, \Omega^+ \phi^{in}) = (\psi^-, \phi^+)
\]

\[
= \sum_{j, j_3, n} \int \frac{d^3 \hat{p}}{2E} ds \sum_{j', j_3', n'} \frac{d^3 \hat{p}'}{2E'} ds' \langle \psi^- | [s, j], n, j_3, \hat{p}^- \rangle \times \langle \hat{p}, j_3, [s, j], n | S | [j', s'], j_3', \hat{p}', n' \rangle (\phi^- \phi^+) .
\]  

(3.9)

In this S-matrix element, \( \phi^{in} \) describes the asymptotically free in-state that is prepared, e.g. by the accelerator, outside the interaction region. This \( \phi^{in} \) becomes the \( \phi^+ \) in the interaction region (of the two beams in \( e^+ e^- \to Z \to f \bar{f} \)), the energy distribution in the beams is described by the wave functions \( \phi^{in}(s) = \phi^+(s) \). The out-state vector \( \psi^{out} \) describes the detected out-particles (e.g., a particular \( f \bar{f} \)) when they are asymptotically free. It comes from the \( \psi^- \) in the interaction region and its wave function \( \psi^{out}(s) = \psi^- (s) \) describes the energy resolution of the detectors. \( \psi^- \) is defined by the registration apparatus (detector)—for which reason \( |\psi^- \rangle \langle \psi^- | \) should be called observable rather than out-state. The kets \([j, s], b^\pm\) are the eigenvectors of the exact energy operator \( P\mu P^\mu \). The kets \([j, s], b\) are the corresponding eigenvectors of the asymptotically free energy operator and

\[
\psi^{out}(s) \equiv \langle b, [j, s] | \psi^{out} \rangle = \langle -b, [j, s] | \psi^- \rangle \equiv \psi^-(s).
\]  

(3.10)

In addition to the property that \(|\psi^{out}(s)|^2 = |\psi^-(s)|^2 \) and \(|\phi^{in}(s)|^2 = |\phi^+(s)|^2 \) be a smooth function of \( s \) (since they describe apparatus resolutions), we also require according to our new hypothesis of Section II that these functions have certain analyticity properties (Hardy class). This is related to causality based on the fact that the in-state \( \phi^+ \) must be prepared first before the out-observable \( \psi^- \) can be detected in it [26]. The S-matrix element \(|(\psi^-, \phi^+)|^2\) describes the probability to detect the observable \( \psi^- \) in the state \( \phi^+ \) (Born probability). This is also expressed by the asymptotically-free quantities \(|(\psi^{out}, \phi^{out})|^2\), where \( \phi^{out} = S\phi^{in} \) is a state, (not an observable like \( \psi^{out} \)), which is defined by the preparation apparatus as a \( \phi^{in} \) and the dynamics described by the S-operator (or by the Hamiltonian \( \mathcal{H} \) if \( S \) is calculated in terms of \( \mathcal{H} = \mathcal{H}_0 + \mathcal{V} \)).

In matrix form \( (\psi^- \phi^+) \) by the r.h.s. of (3.9), we have used for the \( \phi^+ \) and \( \psi^- \) the basis vector expansions (2.3) and (2.23) and chosen for the quantum numbers \( b \) the space components of the 4-velocity, \( b = \hat{b} = \hat{b}/\sqrt{s} \), \( j_3 \) and \( n \), where \( n \) are any additional (e.g., channel) quantum numbers. For the Lorentz invariant integration we choose \( d\mu(b) = d^3\hat{p}/(2\pi^3) \). And we have written the S-matrix as

\[
\langle \hat{p}, j_3, [s, j], n | S | \hat{p}', j_3', [s', j'], n' \rangle = (\Omega^- \hat{p}, j_3, [s, j], n, \Omega^+ | \hat{p}', j_3', [s', j'], n')
\]

\[
= (\hat{p}, j_3, [s, j], n | S | \hat{p}', j_3', [s', j'], n')
\]

(3.11)

From the invariance of the S-operator with respect to Poincaré transformations one can show that the S-matrix element (1.1) can be written as

\[
\langle \hat{p}, j_3, [s, j], n | S | \hat{p}', j_3', [s', j'], n' \rangle = 2E(\hat{p}) \delta^3(\hat{p} - \hat{p}') \delta(s - s') \delta_{j_3 j_3'} \delta_{j j'} \langle n | S_j(s) | n' \rangle ,
\]  

(3.12)

where \( \langle n | S_j(s) | n' \rangle \) is the reduced S-matrix element which depends upon \( j \) (which labels the partial wave; it is the total orbital angular momentum for the case without spins, e.g., \( \pi^+ \pi^- \) system) and the particle species and channel quantum numbers \( n, n' \). For a fixed initial state \( n' \) it is written as

\[
\langle n | S_j(s) | n' \rangle = S_j(s) = \left\{ \begin{array}{ll}
2ia_j(s) + 1 & \text{for elastic scattering } n = n' \\
2ia_j^{(n)}(s) & \text{for reaction from } n' \text{ into the channel } n,
\end{array} \right.
\]  

(3.13)

where \( a_j(s) \) and \( a_j^{(n)}(s) \) are the partial wave amplitudes used in (1.1) and (1.2) (for \( e^+ e^- \to Z \to e^+ e^- \) and \( e^+ e^- \to Z \to \mu \bar{\mu}, \text{etc.} \)). We insert (3.12) and (3.13) into (3.9) and obtain for the S-matrix element (omitting the additional quantum numbers \( n \)):

\[
(\psi^- \phi^+) = \sum_j \int_{m_0^2}^{\infty} ds \sum_{j_3} \int \frac{d^3 \hat{p}}{2E} (\psi^- | [j, s], j_3, \hat{p}^-) S_j(s) (\phi^+) .
\]  

(3.14)

After we have made use of the Poincaré invariance of the S-matrix using the 3-velocities basis vectors \([j, s], b^\pm\) = \([j, s], j_3, \hat{p}^\pm\) we ignore again the degeneracy quantum numbers \( b \) and we consider only the \( j \)-th partial S-matrix element (where \( j \) is the spin-parity of the resonance)
\[(\psi^-, \phi^+)_{ij} = \int_{m_0^2}^{\infty} ds \langle \psi^- | s^- \rangle S_j(s) \langle + s | \phi^+ \rangle, \quad (3.15)\]

where the wave functions are those of (2.9) and (2.23).

\[\psi^-(s) = \langle - s | \psi^- \rangle = \langle - b, [j, s] | \psi^- \rangle, \quad (3.16)\]
\[\phi^+(s) = \langle + s | \phi^+ \rangle = \langle + b, [j, s] | \phi^+ \rangle. \quad (3.17)\]

They have, according to our new hypothesis, the Hardy class property (2.13) and (2.20). \(\phi^+\) describes the prepared in-state (\(e^+e^-\)) and \(|\phi^+(s)|^2 = |\phi^b(s)|^2\) describes the energy distribution of the beam. \(\psi^-\) describes the observed out-observable (\(e^+e^-, \mu\bar{\mu}, \tau\bar{\tau}, \ldots\)) which is registered by the detector, and \(|\psi^-(s)|^2 = |\psi^{\text{out}}(s)|^2\) describes the detector efficiency. Therefore they should be smooth, rapidly decreasing functions. In addition we require by (2.15) and (2.20) certain analyticity properties for them which we conjectured in Section II from the Lippmann-Schwinger equation (in the non-relativistic theory we attributed (2.15) and (2.20) to a causality principle [26]).

In order to be specific we shall consider the case that there are \(N = 2\) resonances in the \(j\)-th partial wave, each described by a first order pole at the position \(s = s_{R_1}\) and \(s = s_{R_2}\) in the second sheet. The integrand in (3.15) is thus analytic in the lower half second sheet except for the two poles at \(s = s_{R_i}\), and we can deform the contour of integration in (3.15) from the positive real line through the cut into the lower half plane of the second sheet. The r.h.s. of (3.15) becomes (dropping the \(j\) notation)

\[\int_{m_0^2}^{\infty} ds \langle \psi^- | s^- \rangle S_{II}(s) \langle + s | \phi^+ \rangle \]
\[= \int_{C_i} ds \langle \psi^- | s^- \rangle S_{II}(s) \langle + s | \phi^+ \rangle + \int_{C_2} ds \langle \psi^- | s^- \rangle S_{II}(s) \langle + s | \phi^+ \rangle, \quad (3.18)\]

where \(C_i\) is the circle around the pole at \(s_{R_i}\), and the first integral extends along the negative real axis in the second sheet (indicated by \(-\infty_{II}\)); the integral along the infinite semicircle is zero due to the assumed property of the integrand. The first term has nothing to do with any of the resonances, it is the non-resonant background term,

\[\int_{m_0^2}^{-\infty_{II}} ds \langle \psi^- | s^- \rangle S_{II}(s) \langle + s | \phi^+ \rangle = \langle \psi^- | \phi^{bg} \rangle \quad (3.19)\]

which we express as the matrix element of \(\psi^-\) with a generalized vector \(\phi^{bg}\) that is defined by it. It will be discussed further in Section IV.

We now use the expansion around the pole \(s_{R_i}\)

\[S(s) = \frac{R^{(i)}}{s - s_{R_i}} + R_0 + R_1(s - s_{R_i}) + \cdots \quad (3.20)\]

for each of the two (or \(N\)) integrals separately. The integrals around the poles, the pole terms, are calculated in the following way:

\[\langle \psi^-, \phi^+ \rangle_{\text{pole term}} = \int_{C_i} ds \langle \psi^- | s^- \rangle S(s) \langle + s | \phi^+ \rangle \]
\[= \int_{C_i} ds \langle \psi^- | s^- \rangle \frac{R^{(i)}}{s - s_{R_i}} \langle + s | \phi^+ \rangle \]
\[= -2\pi i R^{(i)} \langle \psi^- | s_{R_i} \rangle \langle + s_{R_i} | \phi^+ \rangle \quad (3.23)\]
\[= \int_{-\infty_{II}}^{\infty} ds \langle \psi^- | s^- \rangle \langle + s | \phi^+ \rangle \frac{R^{(i)}}{s - s_{R_i}}. \quad (3.24)\]

To get from (3.22) to (3.23), the Cauchy theorem has been applied; to get from (3.22) to (3.24), the contour \(C_i\) of each integral separately has been deformed into the integral along the real axis from \(-\infty_{II} < s < +\infty\) (and an integral along the infinite semicircle, which vanishes, because of the Hardy class property). The equality (3.23) and (3.24) is the Titchmarsh theorem for Hardy class functions.
The integral \((3.24)\) extends from \(s = -\infty\) in the second sheet along the real axis to \(s = 0\) and then from \(s = 0\) to \(s = +\infty\) in either sheet. (It does not matter whether we take the second part of the integral over the physical values of \(s\), \(m_0^2 \leq s < \infty\), immediately below the real axis in the second sheet or in the first sheet immediately above the real axis). The major contribution to the integral comes from the physical values \(m_0^2 \leq s < \infty\), if \(s_{R_i}\) is not too far from the real axis. The integral in \((3.24)\) contains the Breit-Wigner amplitude

\[
a_{j}^{BW}(s) = \frac{R_{ij}}{s - s_{R_i}}, \quad \text{but with } -\infty < s < +\infty.
\]

Unlike the conventional Breit-Wigner of \((1.2)\), for which \(s\) is taken (if one worries about these mathematical details) over \(m_0^2 \leq s < +\infty\), the Breit-Wigner \((3.25)\) is an idealized or exact Breit-Wigner whose domain extends to \(-\infty\) in the second (unphysical) sheet.

By \((3.24)\) we have associated each resonance at \(s_{R_i}\) to an exact Breit-Wigner \((3.25)\) which we obtain by omitting the integral over the arbitrary function \(\langle -s|\psi^-|s\phi^+ \rangle \in \mathcal{S} \cap \mathcal{H}_R^2\) from \((3.24)\). By \((3.24)\) we have associated each resonance at \(s_{R_i}\) with vectors \(|s_{R_i}\rangle = |j, s_{R_i}|, b^-\rangle\) which we call Gamow kets.

We obtain a representation of the Gamow ket or Gamow vector by using the equality \((3.23) = (3.24)\) and omitting the arbitrary \(\psi^- \in \Phi_+\) (which represents the decay products defined by the detector). For this defining relation of the relativistic Gamow kets or relativistic Gamow vectors we shall use the notation that includes the degeneracy quantum numbers \(b\):

\[
|j, s_{R_i}|, b^-\rangle = \frac{i}{2\pi} \int_{-\infty}^{s_{R_i}} ds \langle |j, s|, b^-\rangle \frac{1}{s - s_{R_i}} \langle +s|\phi^+ \rangle
\]

\[
= \frac{i}{2\pi} \int_{-\infty}^{s_{R_i}} ds \langle |j, s|, b^-\rangle \frac{1}{s - s_{R_i}}.
\]

The Gamow kets \((3.26)\) are a superposition of the exact—not asymptotically free \([27]\)—"out states" \(|j, s|, b^-\rangle\). The degeneracy quantum numbers \(b\) of the Gamow kets \(|j, s_{R_i}|, b^-\rangle\) are the same as the ones chosen for the Dirac-Lippmann-Schwinger kets \(|j, s|, b^-\rangle\). However, whereas for the Dirac-Lippmann-Schwinger kets one can choose for \(b = b_1, \ldots, b_9\), the eigenvectors of any complete set of observables, one does not have the same freedom for the \(b\) in the Gamow kets, since in the contour deformations that one uses to get from \((3.15)\) to \((3.18)\) and ultimately to \((3.21)-(3.24)\) one makes an analytic continuation in the variable \(s\) to complex values. If one chooses for \(b\) quantum numbers that also change when \(s\) is analytically continued, \(b\) could not be kept at one and the same value during this analytic continuation and the Gamow vector on the l.h.s. of \((3.26)\) would be a complicated (continuous) superposition (integral) over different values of \(b\) and not just a superposition over different values of \(s\). For this reason, the momentum \(p\) is not a good choice for the quantum numbers \(b\) in \((3.24)\), because the momentum will also become complex if the energy in the centre of mass rest frame becomes complex. This is also the reason for which we choose the space components of the 4-vector \(p = p/\sqrt{s}\) as the additional quantum numbers \(b\), because then we can impose the condition that \(p\) will become complex in the analytic continuation in such a way that \(p^\gamma = p^\gamma/\sqrt{s}\) will always be real. This condition restricts the arbitrariness of the analytic continuation, it makes the momentum only "minimally complex" and keeps the representations of the Lorentz subgroup of the Poincaré group \(\mathcal{P}\) unitary. Only representations of the space-time translations turn into (causal) semigroup representations. The homogeneous Lorentz transformations \(\mathcal{U}(\Lambda)\) are the same as in \(\Lambda\)'s representations. We will call this subclass of semigroup representations of \(\mathcal{P}\) minimally complex \((3.14)\).

With \((3.25)\) and \((3.26)\) we have obtained for each resonance defined by the pole of the \(j\)-th partial \(S\)-matrix at \(s = s_{R_i}\) an "exact" Breit-Wigner \((3.25)\) and associated to it a set of "exact" Gamow kets \((3.26)\). These Gamow kets \((3.26)\) span, like the Dirac kets \(|j, s|, b\rangle\) in \((2.7)\), the space of an irreducible representation \(|j, s_{R_i}\rangle\) of Poincaré transformations but, unlike the space spanned by the Dirac kets in \((2.7)\), the representation space spanned by the kets of \((3.26)\) is not the representation space of a unitary group. Thus we have the correspondence

\[
\text{Exact Breit-Wigner} \quad \iff \quad \text{Exact Gamow vectors}
\]

\[
a_{j}^{BW}(s) = \frac{R_{ij}}{s - s_{R_i}} \iff \langle |j, s_{R_i}|, f^-\rangle = \int d\mu(b) \langle |j, s_{R_i}|, b^-\rangle f(b)
\]

\[
\text{for } -\infty < s < +\infty \quad \text{for all } f(b) \in \mathcal{S}(\mathbb{R}^3), -j \leq j_3 \leq j.
\]

The Gamow vectors \(|j, s_{R_i}|, f^-\rangle\) have, according to \((3.26)\) and \((2.7)\), the representation

\[
|j, s_{R_i}|, f^-\rangle = \frac{i}{2\pi} \int_{-\infty}^{+\infty} ds \langle |j, s|, f^-\rangle \frac{1}{s - s_{R_i}}.
\]
They are functionals on the Hardy space $Φ_+$, i.e., \([j, s_R], f^− \) \( \in Φ^+ \).

Equation (3.28) is reminiscent of the continuous basis vector expansion (2.8) of $ψ^− \in Φ_+ \subset \mathcal{H}$ with respect to the generalized eigenvectors \([j, s], f^− \) of $P_μ P^μ$ with eigenvalue $s$, where $s$ extends over $m_0^2 ≤ s < ∞$. However in (3.28) the “wave function” $ψ^G(s) \equiv \frac{i}{2π} \frac{1}{s - s_R}$ is not a very well behaved, Hardy class wave function like $ψ^−(s) \in S \cap \mathcal{H}_+^2$ of (2.8). Also in the exact Breit-Wigner “wave function” $ψ^G(s)$ in (3.28), the variable $s$ extends over $−∞_{II} < s < +∞$. Thus the continuous linear superpositions (3.28), which define the relativistic Gamow vectors, are entirely different mathematical entities than the $ψ^−$ of (2.8). The Gamow vectors \([j, s_R], f^− \) and also the Gamow kets \([j, s_R], b^− \) are in addition functionals over the Schwartz space, i.e., $⟨ψ^−|[j, s_R], b^−⟩ = f(b, j_3) ∈ \mathcal{S}(\mathbb{R}^3)$ (for fixed value $s$). The equation (3.28) and (3.26) are functional equations over the space $Φ_+$, and (3.26) can be stated in terms of the smooth Hardy class functions $ψ^−(s) \equiv ⟨ψ^−|[j, s], b^−⟩ \in S \cap \mathcal{H}_+^2$ as

\[
⟨ψ^−|[j, s_R], b^−⟩ \equiv -\frac{1}{2π} \int dσ \langle ψ^−|[j, s], b^−⟩ \frac{1}{s - s_R},
\]

(3.29)

for all $ψ^− ∈ Φ_+$, and similarly for the vectors (3.28).

The first equality (3.29) is again the well known Cauchy formula for the analytic function $\overline{ψ^−}(s) = ⟨ψ^−|[s]⟩$. The second equality (3.30) is the Titchmarsh theorem for the Hardy class function $\overline{ψ^−}(s)$ in the lower half plane of the second sheet. The integration path extends as in (3.24) along the real axis in the second sheet, which is only for physical values $m_0^2 ≤ s < ∞$ the same as the integration along the real axis in the first sheet of (2.9).

The association (3.27) of a space of vectors (3.28) to the Breit-Wigner partial wave amplitude (3.25) requires a very specific property of this amplitude, namely to be a Cauchy kernel, and the definition of the vectors (3.28) and (3.26) will not be possible for other arbitrary functions of $s$ (e.g. not for the amplitude $\phi_{jR}^a(s)$ of (1.1)). Even for the Breit-Wigner (3.24) to (3.29) could we use the Titchmarsh theorem in (3.24), (3.30) and associate to the amplitude $\phi_{jR}^{BW}(s)$ a vector which is defined by this exact Breit-Wigner amplitude. And in order to apply the Titchmarsh theorem we had to restrict the admissible wave functions $ψ^−(s)$ and $φ^+(s)$ in (3.5), (3.14) to be Hardy class in the lower half plane. That means we had to specify the in-state vector $φ^+$ and the out-observable $ψ^−$ that can appear in the $S$-matrix element (3.14) and (3.9) to be in the spaces $Φ_-$ and $Φ_+$, respectively. Only then could we define the Gamow kets \([j, s_R], b^− \) in terms of the Dirac-Lippmann-Schwinger kets \([j, s], b^− \) by e.g. (3.30) as generalized vectors or functionals over the Hardy class space $Φ_+$. The Gamow vectors cannot even be defined as functionals over the Schwartz space $Φ_+$ like the usual Dirac kets. (Similarly we can define another kind of Gamow ket \([j, s_R], b^+ \) \( ∈ Φ^× \) in terms of the Dirac-Lippmann-Schwinger kets \([j, s], b^+ \) for the resonance pole at $s_R = (M_R + iΓ/2)^2$ in the upper half plane of the second sheet). Thus the Hardy spaces $Φ_−$, $Φ_+$, and therewith the new hypothesis of Section I, had to be introduced (as in the non-relativistic theory [13]) in order to be able to construct vectors (3.26), (3.28) and (3.29) with a Breit-Wigner energy distribution.

From these Gamow vectors we can now calculate consequences without any further mathematical assumption. These predictions are that (3.28) and/or (3.21) are generalized eigenvectors of the operators

\[
P_μ P^μ , \quad P^0 = H = H_0 + V , \quad e^{iH t}(e^{-iH t})^*.
\]

(3.31)

We shall now show this for one example $P_μ P^μ$. We consider the vector $ψ^− = P_μ P^μ ψ^−$; this makes sense because $ψ^− ∈ Φ_+$ for any $ψ^− ∈ Φ_+$, because $Φ_+$ is constructed such that all observables are continuous operators with respect to the topology in $Φ_+$ (explicitly given by the space $S$ in (2.13)). We now use (3.30) for $ψ^− = P_μ P^μ ψ^−$:

\[
⟨P_μ P^μ ψ^−|[j, s_R], b^−⟩ = ⟨ψ^−|⟨P_μ P^μ⟩^×[j, s_R], b^−⟩ \frac{1}{s - s_R}
\]

\[
= \frac{i}{2π} \int_{−∞I}^{+∞I} dσ \langle ψ^−|[P_μ P^μ]^×[j, s], b^−⟩ \frac{1}{s - s_R}
\]

\[
= \frac{i}{2π} \int_{−∞I}^{+∞I} dσ \langle ψ^−|[P_μ P^μ]^×[j, s], b^−⟩ \frac{1}{s - s_R}
\]

\[
= s_R ⟨ψ^−|[j, s_R], b^−⟩.
\]

(3.32)
The next to the last step makes use of (2.18) and the last step is again the Titchmarsh theorem as in (3.30) but this time for the function \( \psi^- (s) \equiv \bar{v}^- (s) s \), which is also a Hardy class function \( \psi^- (s) \in \mathcal{S} \cap \mathcal{H}^2_1 \) if \( \psi^- (s) \in \mathcal{S} \cap \mathcal{H}^2_1 \). We write (3.32) as a functional equation over the space \( \Phi^\pm \) omitting the arbitrary \( \psi^- \in \Phi^+ \),

\[
(P^\mu P_\mu)^\times \left| [j, s_R], b, j^- \right\rangle = s_R \left| [j, s_R], b, j^- \right\rangle
\]  
(3.33)

Similarly, we find from (2.18) for the full Hamiltonian \( P^0 = H = H_0 + V \) at rest:

\[
H^\times \left| [j, s_R], b_{\text{rest}}, j^- \right\rangle = \sqrt{s_R} \left| [j, s_R], b_{\text{rest}}, j^- \right\rangle
\]  
(3.34)

and

\[
(P_\mu P^\mu)^\times \left| [j, s_R], f^- \right\rangle = s_R \left| [j, s_R], f^- \right\rangle, \quad \text{for any } f(j_3, b) \in \mathcal{S} (\mathbb{R}^3),
\]  
(3.35)

i.e. for the whole representations space of \( [j, s_R] \) of (3.27). In here \( s_R \) is the resonance pole position. These equations state that the Gamow kets and the whole space of vectors (3.27) spanned by the Gamow kets are generalized eigen-vectors of whether there is any preferred physical significance to one of the parameterisations in (3.36), we shall consider the Poincaré transformations \( (a, \Lambda) \), where \( a \) is the 4-vector of space time translations and \( \Lambda \) is a \( 4 \times 4 \) Lorentz matrix. We shall give here only the action of time translations \( (a = (t, 0, 0, 0), \Lambda = 1) \) on the basis vectors at rest \( \left| [j, s_R], b_{\text{rest}} \right\rangle \). For the general transformation property of the Poincaré semigroup and a detailed proof of the semigroup property we refer to the forthcoming [14].

We know that in the stable particle representations \( [j, m^2] \) the time translation is represented by the operator \( U((t, 0), 1) \equiv U(t) = e^{itH} \) in the Hilbert space \( \mathcal{H}(j, m^2) \) and therefore also in the space \( \mathcal{H} \) of (2.17). The time translation in the subspace \( \Phi^\pm \) will therefore be the restriction \( U_+ (t) = U(t) \mid \Phi^+ = e^{itH} \mid \Phi^+ \) to this subspace. The time translation in the space \( \Phi^\times \) will be the extension of the unitary group operator \( U_1(t) = e^{-itH} \subset U_+^\times (t) \), whenever it can be defined. In order that \( (e^{itH})^\times = U_+^\times (t) \) can be defined the operator \( U_+ (t) \) needs to be a continuous operator with respect to the topology in \( \Phi^+ \) (cf. Appendix). Therefore the question is: for which value of the parameter \( t \) is \( U_+ (t) \) (and therewith also \( U_+^\times (t) \)) a continuous operator in \( \Phi^+ \),

\[
\Phi^+ \ni \psi^- \rightarrow U_+ (t) \psi^- \equiv \psi'^- \in \Phi^+.
\]  
(3.37)

We show below that (3.37) holds only for \( t \geq 0 \) so that also \( U_+^\times (t) \) can only be defined for \( t \geq 0 \). For those values of \( t \) for which \( U_+ (t) \) and \( U_+^\times (t) \) is defined, we have for all \( m^2 \leq s < \infty \) and then also for all real \( s \)

\[
\langle \psi'^- | [j, s], b^- \rangle = \langle U_+ (t) \psi^- | [j, s], b^- \rangle = \langle \psi^- | U_+^\times (t) [j, s], b^- \rangle = \langle e^{itH} \psi^- | [j, s], b^- \rangle = e^{-i\sqrt{2} t} \langle \psi^- | [j, s], b^- \rangle \text{ for all } \psi^- \in \Phi^+.
\]  
(3.38)

Omitting the arbitrary \( \psi^- \in \Phi^+ \) we write this as a functional equation:

\[
e^{-iH^\times_+ t} [j, s], b^- \rangle = e^{-i\sqrt{2} t} [j, s], b^- \rangle , \quad \text{for } t \geq 0 \text{ only},
\]  
(3.39)

where

\[
e^{-iH^\times_+ t} \equiv U_+^\times (t) = (e^{itH})^\times , \quad t \geq 0.
\]  
(3.40)

\(^4\)Using the van Winter theorem for Hardy class functions, cf. Appendix A2 of [22]
The first term in \((3.40)\) is so far only a definition, but one can show that \(H^\times_+ \supset H^\dagger\), the conjugate operator of \(H|_{\Phi_+} = H_+\), and extension of the operator \(H^\dagger = H\) to \(\Phi^\times_+\) (cf. Appendix) is indeed the generator of the semigroup \(U^\times_+(t)\) \([2]\).

The solutions of the other Lippmann-Schwinger equation with \(+i\epsilon\), \([|j, s\rangle, b^+\) \(\in \Phi^\times_+\), have a time evolution given by the other semigroup \(U^\times_-(t)\) with \(t \leq 0\),

\[
e^{-iH^\times_-t} |[j, s\rangle, b^+_\text{rest}\rangle = e^{-iv\sqrt{s}} |[j, s\rangle, b^+_\text{rest}\rangle \quad \text{for} \quad t \leq 0 \quad \text{only.}\]  

(3.41)

In here \(U^\times_-(t) = e^{-iH^\times_-t}\) (and \(H^\times\)) are analogously defined in the RHS \((2.22)\) as the extension of the \(U^\dagger(t)\) (and of \(H^\dagger\)) to the space \(\Phi^\times\).

{Brief justification of the semigroup condition \(t \geq 0\): The complex conjugate of \((3.38)\) can be written in the notation of \((2.10), (2.13)\)

\[
\psi^- - (s) = e^{i\sqrt{s}t} \psi^-(s) \quad (3.42)
\]

The question: For which \(t\) in \((3.34)\) is \(\psi^- - (s) \in \Phi_+\) if \(\psi^- - (s) \in \Phi_+\), can thus be formulated: for which \(t\) in \((3.42)\) is \(\psi^- - (s) \in \mathcal{S} \cap \mathcal{H}^2_+\) if \(\psi^- (s) \in \mathcal{S} \cap \mathcal{H}^2_+\) is Hardy class only if \(t \geq 0\) will \(e^{i\sqrt{s}t} \psi^- (s)\) decrease sufficiently fast in the upper half plane so that \(\psi^- (s)\) is also Hardy class. For the detailed proof of the semigroup property for the general Poincaré transformation we refer to \([23]\).

We obtain now the action of \(e^{-iH^\times t}\) on the Gamow ket at rest using \((3.30)\) for the vector \(e^{iHt}\psi^- \equiv \psi^-(t)\), where \(t\) is the time in the rest frame of the resonance:

\[
\langle \psi^- | e^{-iH^\times t} | [j, s_R]\rangle b^+_{\text{rest}}\rangle = \langle e^{iHt} \psi^- | [j, s_R]\rangle b^+_{\text{rest}}\rangle \frac{1}{s - s_R}
\]

\[
\quad = \int_\inf^{+}\inf^{+} ds \langle e^{iHt} \psi^- | [j, s]\rangle b^+_{\text{rest}}\rangle \frac{1}{s - s_R}
\]

\[
\quad = \int_\inf^{+}\inf^{+} ds \langle \psi^- | e^{-iH^\times t} | [j, s]\rangle b^+_{\text{rest}}\rangle \frac{1}{s - s_R}
\]

\[
\quad = \int_\inf^{+}\inf^{+} ds \langle \psi^- | [j, s]\rangle b^+_{\text{rest}}\rangle e^{-i\sqrt{s}} \frac{1}{s - s_R} \quad \text{for} \quad t \geq 0\]  

(3.43)

The next to the last step makes use of \((3.39)\) and the last step is again the Titchmarsh theorem as in \((3.30)\) or \((3.32)\) but this time for the function \(\psi^- (s) \equiv \psi^-(s)e^{-i\sqrt{s}} \in \mathcal{S} \cap \mathcal{H}^2_+\), which is also a Hardy class function if \(\psi^- (s)\) is; however \(\psi^- (s)\) is Hardy class only if \(t \geq 0\). For \(t < 0\) the time evolution operator \(e^{-iH^\times t}\) on \([|j, s\rangle, b^-\rangle\) and on \([|j, s_R\rangle, b^-\rangle\) does not exist.

We write \((3.43)\) as a functional equation over the space \(\Phi_+\) omitting the arbitrary \(\psi^- \in \Phi_+\),

\[
|\psi^-_{j, s_R} (t)\rangle \equiv e^{-iH^\times t} |[j, s_R]\rangle b^+_{\text{rest}}\rangle = e^{-iHt} e^{-\frac{i\Gamma_R t}{2}} |[j, s_R]\rangle b^+_{\text{rest}}\rangle, \quad t \geq 0\]  

(3.44)

In \((3.44)\) we have used the most suitable parameterisation for \(\sqrt{s_R}\) of which a few other popular parameterisations are also given in \((3.36)\).

The result \((3.44)\) is a mathematical consequence of the new hypothesis of \((2.26)\). It has two physical consequences:

1. The time evolution of the Gamow vectors has a preferred direction of time, i.e., it is not reversible.

2. The time evolution of the relativistic Gamow vectors is exponential with the decay constant (inverse lifetime) \(\Gamma_R\).

The time asymmetry, \(t \geq 0\), has been discussed in detail elsewhere \([20]\) and also in \([18, 29]\) and is not the main interest of this paper. Though the “profoundly irreversible” character of a quantum decay has been mentioned in the past \([30]\), it usually has been attributed to other principles (effect of the environment, collapse of the wave function, decoherence), probably because of the misconception that time evolution in quantum mechanics must be given by the reversible unitary group as is dictated by the mathematics of the Hilbert space. The irreversibility on the microphysical level expressed by the time asymmetry \((3.44)\) was the first unintended and the most surprising result of our time asymmetric quantum theory \([14]\). It is a consequence of the time asymmetric boundary conditions \((2.26)\).
and has nothing to do with violation of time reversal invariance (which is a statement about the $T$-transformation property of the Hamiltonian $H$) and it is not known to us whether it can be connected to entropy increase $^{[31]}$.

In retrospect this time asymmetry is no longer shocking, since Maxwell’s theory and general relativity theory also use time asymmetric boundary conditions for time symmetric dynamical equations. Time asymmetric (purely outgoing) boundary conditions in place of the Hilbert space assumption have also been suggested in the past for quantum mechanics $^{[32]}$. The germ of this time asymmetry is already inherent in the Lippmann-Schwinger equations $^{[13]}$ and in the Feynman rules, by the infinitesimal $i\epsilon$ mechanics $^{[33]}$. But surprisingly, none of those papers arrived at a semigroup evolution like $^{(3.39)}$ and $^{(3.41)}$. This semigroup evolution of the in- and out-planewave states is a manifestation of a fundamental time asymmetry in quantum scattering – independently of whether resonance formation is involved or not.

It can be shown that the time evolution semigroup $^{(3.44)}$ is the rest frame version of a causal semigroup $P_\eta = (a_+, \Lambda)$ of Poincaré transformations where $\Lambda$ is a proper orthochronous Lorentz transformation and $a_+ = (a_0, \mathbf{a})$ is a 4-vector that fulfills $\mathbf{b} \cdot \mathbf{a} = \sqrt{1 + \mathbf{b}^2 a_0} - \mathbf{b} \cdot \mathbf{a} \geq 0$ for any $\mathbf{b} \in \mathbb{R}^3$. It means that the relativistic Gamow vectors can only undergo Poincaré transformations into the forward light cone $^{[12][14]}$.

The second consequence of $^{(3.44)}$, the exponential time evolution of the Gamow vectors, is not a surprise, because it was the intended property for which the Gamow vectors of the non-relativistic theory were originally constructed. However that the exponential decay constant is $\Gamma_R$ and not any of the other $\Gamma$’s of $^{(3.36)}$ is a new prediction of the relativistic Gamow vector. In the non-relativistic theory $^{[22][24]}$ the exponential time evolution of the non-relativistic Gamow vectors,

$$\psi^G(t) \equiv e^{-iH^*t}\psi^G = e^{-iE_R t}e^{-\Gamma_R / 2t}\psi^G,$$  \hspace{1cm} \text{(3.45)}

was the property needed to calculate from the Born probabilities $P_\eta(t)$ $^{(3.4)}$ the exponential law for the partial decay rates $^{[33]}$

$$\hat{P}_\eta(t) = e^{-\Gamma_R t}\Gamma_\eta,$$  \hspace{1cm} \text{(3.46)}

and the exact Golden Rule

$$\Gamma_\eta = 2\pi \sum_{b'=b_\eta} \int_0^\infty dE \ |\langle b, E|V|\psi^G\rangle|^2 \frac{\Gamma_R}{2\pi} \left( \frac{E - E_R}{2\Gamma_R} \right)^2,$$  \hspace{1cm} \text{(3.47)}

for the partial initial decay rate (also called partial width for the decay into the channel $\eta$). For $t \to 0$ and in the Born approximation, defined by

$$\psi^G \to f^D, \quad \frac{\Gamma_R}{M_R} \to 0, \quad \text{where} \quad H_0 f^D = E_D f^D, \quad H_0 = H - V,$$  \hspace{1cm} \text{(3.48)}

one obtains from the “exact Golden Rule” $^{(3.47)}$ the Fermi-Dirac’s Golden Rule, for the initial decay rate

$$\Gamma_\eta = 2\pi \sum_{b'=b_\eta} \int_0^\infty dE \ |\langle b, E|V|f^D\rangle|^2 \delta(E - E_D).$$  \hspace{1cm} \text{(3.49)}

The sum in $^{(3.47)}$ and $^{(3.49)}$ extends over all values of the quantum numbers $b$ which characterize the channel $\eta$. If one also sums over all channels $\eta$ (using the condition $P(0) \equiv \sum_\eta P_\eta(0) = 0$, which means the probability to find any decay product $\eta$ at $t = 0$, is zero $^{[23]}$), one obtains:

$$\sum_\eta \Gamma_\eta = \sum_\eta \hat{P}_\eta(0) = \Gamma_R.$$  \hspace{1cm} \text{(3.50)}

This means that the sum over all partial initial decay rates is the width of the Breit-Wigner, which according to the exponential law $^{(3.40)}$ is also the inverse lifetime.

The same result one expects also for the relativistic case, this means from the exponential time evolution $^{(3.44)}$ of the relativistic Gamow vectors at rest, follows the exponential decay law $^{(3.46)}$ with $t$ being the time in the rest frame of the decaying state $R$, (i.e., for $b_{\text{rest}} = p_{\text{rest}} = 0$ and therefore $p_{\text{rest}} = \sqrt{\gamma_R}p_{\text{rest}} = 0$). Therefore we conclude from the results $^{(3.43)}$, $^{(3.44)}$ that

$$\Gamma_R = \frac{1}{\tau}.$$  \hspace{1cm} \text{(3.51)}

15
where $\tau$ is the lifetime of the relativistic resonance $R$ (average lifetime in the rest frame).

The parameter lifetime $\tau$ of a relativistic decaying state is measured by the counting rate \( \frac{\Delta N_R(t)}{\Delta t} \) as a function of time $t$ in the rest frame or (practically always) as a function of the distance $d = \hat{p} t = \gamma \nu t$ which the quasistable particle with space components of the 4-velocity $\hat{p} = \gamma \nu = (1 - v^2)^{-1/2}$ travels in the lab frame:

$$\frac{\Delta N_R(t)}{\Delta t} = N_R(0) e^{-t/\tau} = e^{-\frac{d}{\gamma \nu \tau}} = e^{-\frac{\gamma \nu \tau}{\tau} \frac{1}{\gamma \nu}}. \quad (3.52)$$

For the $Z$-boson (and for hadron resonances) $h/\tau$ is too small to be measured in this way and one can only measure $\Gamma$ from the lineshape \( [1.1] \). For other weakly decaying particles \([24]\), like the $K^0$, one measures the lifetime $\tau$ using \([3.52]\) but one cannot resolve the lineshape \( [1.1] \) because $h/\tau$ is too small. Thus there may be no way to test the relation \( [3.51] \) ever.

The width-lifetime relation \( [3.51] \) is a result for the relativistic Gamow vectors in the same way as \( [1.1] \) is a result for the non-relativistic Gamow vectors. Without Gamow vectors neither \( [3.46] \), \( [3.47] \) nor \( [3.51] \) can be proven as an exact relation. The difference between the relativistic and non-relativistic case is that in the latter there was never any doubt regarding the meaning of the width, it is the full width at half-maximum of the non-relativistic Breit-Wigner \( [3.2] \). For the relativistic resonance one was not sure which part of the $j$-th partial wave amplitude \( [3.13] \) one should assign to the resonance per se, i.e., whether \( [1.1] \) or \( [1.2] \) or any other part of $a(s)$ describes the resonance \( [3.30] \). Neither was one sure, that even when \( [3.2] \) was chosen (using the $S$-matrix pole definition), which of the parameterisations \( [1.7], [1.8], [1.10] \) should be used to define the “width” and the real mass of the resonance. The result \( [3.44] \) for the relativistic Gamow vector fixes this: the mass is $M_R = \text{Re}(\sqrt{s_{16}})$ (the coefficient of the phase for the time evolution \( [3.41] \)) and the width is $\Gamma_R = -2\text{Im}(\sqrt{s_{16}})$ (the coefficient of the exponential decay). Then \( [3.51] \) holds universally. This means that if for every resonance and every decaying state one wants \( [3.1] \) to relation \( [3.51] \) ever.

IV. FROM A SINGLE BREIT-WIGNER TO THEIR SUPERPOSITIONS

The exact Breit-Wigner amplitude $a_j^{BW}(s)$ of \( [3.27] \) was one particular part of the $j$-th partial $S$-matrix \( [3.20] \) namely the one that was obtained from the pole of $S_j(s)$ and which we considered separately in Section \( [11] \). With our mathematical hypothesis of Section \( [4] \) it was completely natural to treat each integral around a resonance pole at $s_{R_i}$ separately and assign to each a Breit-Wigner amplitude \( [2.29] \) and a corresponding Gamow vector \( [3.27] \). In an experiment it is not that easy to separate the resonance term(s) from the remainder of the $S$-matrix, since the $j$-th partial cross section does not only contain the resonance part but contains also the non-resonant background of the scattering process in the amplitude $a_j(s)$. We now want to consider the whole $S$-matrix element \( [8.18] \). Inserting \( [3.13] \) into \( [8.18] \) and using the definition \( [8.19] \) we can write the $j$-th partial $S$-matrix element $(\psi^-, \phi^+)$ as a discrete sum over Gamow vectors and the background term,

$$\langle \psi^-, \phi^+ \rangle = \langle \psi^- | \phi^{bg} \rangle + \sum_i \langle \psi^- | s_{R_i}^\dagger \rangle (2\pi/i) (s_{R_i}^\dagger | \phi^+) \cdot \quad (4.1)$$

Omitting the arbitrary $\psi^- \in \Phi_+$ (the observable) one writes Eq. \( [4.1] \) as a functional equation in the space $\Phi_+$ and obtains the following expansion of the prepared in-state $\phi^+ \in \Phi_-$:

$$\phi^+ = \phi^{bg} + \sum_i | s_{R_i}^\dagger \rangle c_{R_i} \cdot \quad (4.2)$$

In this way the in-state $\phi^+$ has been decomposed into a vector representing the non-resonant part $\phi^{bg}$ and a sum over the Gamow vectors representing resonance states. The complex eigenvalue resolution \( [4.2] \) is an alternative generalized eigenvector expansion to Dirac’s eigenvector expansion \( [2.23] \),

$$\Phi_+ \ni \phi^+ = \int_0^\infty ds | s^+ \rangle \langle s^+ | \phi^+ \rangle \cdot | s^+ \rangle \in \Phi_+. \quad (4.3)$$

While Eq. \( [4.3] \) expresses the in-state $\phi^+$ in terms of the Lippmann-Schwinger kets $| s^+ \rangle \in \Phi_+$, which are generalized eigenvectors of the Hamiltonian $H$ with real eigenvalue $\sqrt{s}$, Eq. \( [4.2] \) is an expansion of $\phi^+ \in \Phi_+^\dagger$ in terms of eigenkets.
The term \( \phi^{bg} \) is defined by (3.19) and is therefore an element of \( \Phi_{\gamma}^\times \). We want to rewrite (3.19) into a more familiar form. According to the van Winter theorem (34), a Hardy class function on the negative real axis is uniquely determined by its values on the real positive axis (cf. Appendix A2 of (32)). Therefore one can use the Mellin transform to rewrite the integral on the l.h.s. of (3.19) into an integral over the interval \( m_0^2 \leq s < \infty \) and obtain

\[
\langle \psi^- | \phi^{bg} \rangle = \int_{m_0^2}^{-\infty} dB \langle \psi^- | s^- \rangle S_j(s) \langle ^+s| \phi^+ \rangle = \int_{m_0^2}^{\infty} ds \langle \psi^- | s^- \rangle b_j(s) \langle ^+s| \phi^+ \rangle ,
\]

where \( b_j(s) \) is uniquely defined by the values of \( S_j(s) \) on the negative real axis. Without more specific information about \( S_j(s) \), we cannot be certain about the energy dependence of the background \( b_j(s) \). If there are no further poles or singularities besides those included in the sum, then \( b_j(s) \) is likely to be a slowly varying function of \( s \) (38).

Omitting the arbitrary \( \psi^- \in \Phi_+ \), we write the expansion for the non-resonant background part \( \phi^{bg} \) of the prepared in-state vector \( \phi^+ \) as:

\[
| \phi^{bg} \rangle = \int_{m_0^2}^{\infty} ds | s^- \rangle \langle ^+s| \phi^+ \rangle b_j(s) ,
\]

Inserting (4.3) into (4.2) we obtain the basis vector expansion of every \( \phi^+ \in \Phi_+ \),

\[
\phi^+ = \sum_i | s^{-}_{R_i} \rangle c_{R_i} + \int_{m_0^2}^{\infty} ds | s^- \rangle \langle ^+s| \phi^+ \rangle b_j(s) ; \quad | s^{-}_{R_i} \rangle , | s^- \rangle \in \Phi_{\gamma}^\times
\]

(We have assumed as in (4.3) that there are no bound states of \( H \). Otherwise one would have in addition to the r.h.s. of (1.3) and (1.4) the discrete sum over the bound states, which are orthogonal to the rest).

The vector (4.3) does not have an exponential time evolution, but all the Gamow vectors \( | s^{-}_{R_i} \rangle \) in the basis vector expansion (4.2) evolve exponentially. The much debated deviation from the exponential decay law (21) has its origin in (4.3) and (4.6) the discrete sum over the bound states, which are orthogonal to the rest).

According to the van Winter theorem (35), a Hardy class function on the negative real axis is uniquely determined by its values on the real positive axis (cf. Appendix A2 of (32)). Therefore one can use the Mellin transform to rewrite the integral on the l.h.s. of (3.19) into an integral over the interval \( m_0^2 \leq s < \infty \) and obtain

\[
\langle \psi^- | \phi^{bg} \rangle = \int_{m_0^2}^{-\infty} dB \langle \psi^- | s^- \rangle S_j(s) \langle ^+s| \phi^+ \rangle = \int_{m_0^2}^{\infty} ds \langle \psi^- | s^- \rangle b_j(s) \langle ^+s| \phi^+ \rangle ,
\]

where \( b_j(s) \) is uniquely defined by the values of \( S_j(s) \) on the negative real axis. Without more specific information about \( S_j(s) \), we cannot be certain about the energy dependence of the background \( b_j(s) \). If there are no further poles or singularities besides those included in the sum, then \( b_j(s) \) is likely to be a slowly varying function of \( s \) (38).

Omitting the arbitrary \( \psi^- \in \Phi_+ \), we write the expansion for the non-resonant background part \( \phi^{bg} \) of the prepared in-state vector \( \phi^+ \) as:

\[
| \phi^{bg} \rangle = \int_{m_0^2}^{\infty} ds | s^- \rangle \langle ^+s| \phi^+ \rangle b_j(s) ,
\]

Inserting (4.3) into (4.2) we obtain the basis vector expansion of every \( \phi^+ \in \Phi_+ \),

\[
\phi^+ = \sum_i | s^{-}_{R_i} \rangle c_{R_i} + \int_{m_0^2}^{\infty} ds | s^- \rangle \langle ^+s| \phi^+ \rangle b_j(s) ; \quad | s^{-}_{R_i} \rangle , | s^- \rangle \in \Phi_{\gamma}^\times
\]

(We have assumed as in (4.3) that there are no bound states of \( H \). Otherwise one would have in addition to the r.h.s. of (1.3) and (1.4) the discrete sum over the bound states, which are orthogonal to the rest).

The vector (4.3) does not have an exponential time evolution, but all the Gamow vectors \( | s^{-}_{R_i} \rangle \) in the basis vector expansion (4.2) evolve exponentially. The much debated deviation from the exponential decay law (21) has its origin in (4.3) and (4.6) the discrete sum over the bound states, which are orthogonal to the rest).

The experimental ingenuity in establishing the exponential decay law for each Gamow state is to suppress or exclude as much as possible this background and the effect of the other interfering Gamow vectors in the analysis of the experimental data (33).

We now rewrite (3.18) in a different form. In place of (\( \psi^- , \phi^+ \)) on the l.h.s. of (3.18), we write the r.h.s. of (3.15), and on the r.h.s. of (3.18) we use (3.21) and (4.4). Then we obtain

\[
\int_{m_0^2}^{\infty} ds \langle \psi^- | s^- \rangle S_j(s) \langle ^+s| \phi^+ \rangle = \int_{m_0^2}^{\infty} ds \langle \psi^- | s^- \rangle b_j(s) \langle ^+s| \phi^+ \rangle + \sum_i \int_{-\infty}^{+\infty} ds \langle \psi^- | s^- \rangle \langle ^+s| \phi^+ \rangle \frac{R(i)}{s - s_{R_i}} .
\]

This equality holds for the whole space of functions \( \langle \psi^- | s^- \rangle \langle ^+s| \phi^+ \rangle \in S \cap H^2 \). Therefore we can omit these arbitrary energy wave functions

\[
\langle \psi^- | s^- \rangle \langle ^+s| \phi^+ \rangle = \langle \psi^{out} | s| \phi^{in} \rangle \in S \cap H^2 ,
\]

which describe the resolution of the preparation apparatus and of the registration apparatus and write equation (4.7) as an equation between distributions over the function space \( S \cap H^2 \):

\[
\theta(s - m_0^2)S_j(s) = \theta(s - m_0^2)b_j(s) + \sum_i \frac{R(i)}{s - s_{R_i}} .
\]

Though one likes to represent the “physics” in this apparatus independent way, what one measures in each experiment contains of course always the convolution with an apparatus resolution so that (4.9) really means (4.7) in its applications to a particular experiment. A corresponding equation can be written for the partial wave amplitudes (3.13) (by dividing (1.9) by 2i and subtracting 1 on both sides for \( n = n' \))

\[
\theta(s - m_0^2)a_j(s) = \theta(s - m_0^2)B_j(s) + \sum_i \frac{\tilde{R}(i)}{s - s_{R_i}} ,
\]

considered as a functional equation in the space of distributions \( (S \cap H^2)^\times \). Here \( B_j(s) \) like \( b_j(s) \) describes an ever present slowly varying non-resonant background. For instance, if there is only one resonance pole at \( s_{R} \) in the \( j \)-th
partial wave, then the cross section contains in addition to the resonance part (Breit-Wigner) also interference terms with the unknown background

\[ |a_j(s)|^2 = \left| B_j(s) + \frac{\hat{R}_j(s)}{s - s_R} \right|^2 = |B_j(s) + a_{j\text{resonance}}(s)|^2. \]  

(4.11)

This shows that it is difficult to distinguish phenomenologically between two alternative functions for the resonance part of the amplitude like (1.1) and (1.2). The formula (4.11), or the formula (4.10) for a single resonance, is one of the frequently used phenomenological formulas. It also has a theoretical justification under the usual analyticity assumption for the S-matrix \( S_j(s) \) (Laurent expansion).

For two (or more) resonances we have obtained (1.9), which contains in addition the interference between two (or more) resonances. The formula (4.13) cannot be derived using the usual analyticity assumption of \( S_j(s) \) only. We needed for its derivation the Hardy class hypothesis (2.26) to obtain (3.24) for each pole term separately.

The interference of resonances as predicted by (1.9), (4.14) has been established experimentally for the non-relativistic case in nuclear physics and for the relativistic case in the \( p-\omega \) interference [21]. Since its derivation required the use of the assumption \( \psi^- \in \Phi_+ \) and \( \phi^+ \in \Phi_- \) of Section I, the phenomenological success of formulas like (1.9) for two interfering resonances is another argument in favour of our new hypothesis of Section II.

With (1.9) and (1.10) we have now established a term-by-term correspondence between the complex eigenvalue resolution (1.8) of the prepared in-state vector \( \phi^+ \) and the representation (1.10) for the partial wave amplitude (or also (4.4) for the S-matrix). To each Breit-Wigner in (1.10) corresponds a Gamow vector in (1.2) or (1.4) and to the background amplitude \( B_j(s) \) in (1.10) corresponds the vector \( \phi^{bg} \). This establishes a unique correspondence between the vector description of quasiparticle and the S-matrix description of resonances. The vector description is used for instance in the effective theories with complex Hamiltonian matrix (like the Lee-Oehme-Yang theory of \( K_0^2 \) and \( K_0^2 \) or the finite dimensional models of nuclear physics [22]) only that these finite dimensional models omit the background vectors \( \phi^{bg} \) (4.5) which span an infinite dimensional space. This is the typical feature of the Weisskopf-Wigner approximations. In the S-matrix description of the resonance by the phenomenological ansatz (1.10) one usually does not omit the background amplitude \( B(s) \), but often includes in \( B(s) \) also the contribution of a second distant resonance which according to our prediction (4.10) belongs into the sum of the Breit-Wigner amplitude. The correspondence between (4.10) and (4.10) unifies the theory of resonance scattering and the theory of particle decay.

V. SUMMARY

The mathematical axioms of standard quantum mechanics in the Hilbert space are not completely in agreement with the calculational tools (like for instance Dirac kets) that physicists use in their practical calculations. As a consequence of von Neumann’s mathematical assumptions, standard quantum mechanics is time symmetric, because the Hilbert Space does not allow time asymmetric boundary conditions for the time symmetric dynamical equations, and the time evolution and all other symmetry transformations of non-relativistic and relativistic space time are described by unitary group transformations. In practice, however, one uses properties of time asymmetry, as is e.g. expressed by the \( \pm \)ie of the Dirac kets in the Lippmann-Schwinger equation. Though orthodox quantum theory would require that the set of all in-states \( \{\phi^+\} \) and the set of all out-states \( \{\psi^-\} \) is the same Hilbert space \( \mathcal{H} \), the Lippmann-Schwinger equation suggests a different hypothesis. If one modifies the Hilbert Space axioms slightly and replaces it by the new hypothesis that the set of in-states defined by the preparation apparatus (accelerator) and the set of outstates or observables defined by the registration apparatus (detector) form different dense subspaces of the Hilbert Space, \( \Phi^- \) and \( \Phi^+ \), respectively, one obtains asymmetric boundary conditions. Guided by the Lippmann-Schwinger equations we require that the space of states \( \Phi^- = \{\phi^+\} \) is a Hardy class space of the lower complex energy half-plane, and the space of observables \( \Phi^+ = \{\psi^-\} \) is a Hardy space of the upper complex energy half-plane. This assumption allows a mathematically rigorous definition of the Dirac-Lippmann-Schwinger kets as continuous antilinear functionals, \( |p, \alpha^+\rangle \in \Phi^2 \) and \( |p, \alpha^-\rangle \in \Phi^1_+ \). The Hardy class boundary condition thus makes the singular Lippmann-Schwinger equation mathematically rigorous. But it does more, because in addition to the Dirac-Lippmann-Schwinger kets for real \( s, |s| \in \Phi^1_+ \), the space \( \Phi^1_+ \) contains also Gamow kets \( |s_B^j\rangle \), which are generalized eigenvectors of the invariant mass operator \( P_L P^\nu \) with complex eigenvalue \( s_B = (M_B - i\Gamma_B/2)^2 \). In addition to the unitary group representations \( |j, m\rangle \) of the Poincaré transformation for the time symmetric stationary states one obtains also semigroup representations \( |j, s_R\rangle \) of Poincaré transformations which distinguish a preferred direction of time. Thus the new hypothesis (2.28), leads to a time asymmetric relativistic quantum theory based, like the time symmetric theory for stable particle, on representations of relativistic space-time transformations.

18
The Gamow kets, spanning the irreducible representation space \([j, s_R]\), can be obtained from the pole position \(s_R = (M_R - i\Gamma_R/2)^2\) of the \(j\)-th partial \(S\)-matrix \(S_j(s)\) (or scattering amplitude \(a_j(s)\)) and correspond to the Breit-Wigner part \(a^{BW}_j(s)\) of the scattering amplitude.

From the Poincaré semigroup transformation of the Gamow kets, or more specifically from the time evolution of the Gamow kets in the rest frame, one obtains the exponential decay law \(e^{-\Gamma_R t}\) for the Gamow states \(|s_R\rangle\). This means that the parameter of exponential time evolution, the inverse lifetime, is predicted to be \(\hbar/\tau = \Gamma_R = -2\text{Im}(\sqrt{s_R})\). The Gamow vector unites the notion of resonance described by the Breit-Wigner amplitude and the notion of decaying particle described by the exponential time evolution into one and the same entity with \(\Gamma\) which are used for the \(Z\)-boson in the particle data table [1].

In summary, a relativistic time asymmetric quantum theory that is consistent with the Lippmann-Schwinger equation leads to a definition of resonance mass \(M_R\) and resonance width \(\Gamma_R\) for which \(1/\Gamma_R\) is the lifetime and for which the \(Z\)-boson mass is given by the value \(M_R = (91.1626 \pm 0.0031)\) GeV, which is different from either of the two values quoted in the Particle Data Table [1] for the \(Z\)-boson mass.

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APPENDIX: RIGGED HILBERT SPACES

Let \(\Phi_{\text{alg}}\) be a linear space with scalar product \((\cdot, \cdot)\) (this is the space which most physicists call Hilbert space, but which mathematicians call a pre-Hilbert space). Then the triplet of spaces that form a Rigid Hilbert Space \(\Phi\) are obtained by the completion of \(\Phi_{\text{alg}}\) with respect to three different topologies (meanings of convergence):

\[ \phi \in \Phi \subset \mathcal{H} \subset \Phi^\times \ni |F\rangle. \]

The space \(\mathcal{H}\) is the mathematician’s Hilbert space and it is complete with respect to the norm convergence \(\tau_n\). The space \(\Phi\) is complete with respect to a stronger convergence \(\tau_{\Phi}\) (test function space). The space \(\Phi^\times\) is the space of \(\tau_{\Phi}\)-continuous, antilinear functionals on the space \(\Phi\): \(|F\rangle \in \Phi^\times\) maps \(\phi \in \Phi\) into \(\mathbb{C}\), i.e. \(F(\phi) = (\phi | F\rangle \in \mathbb{C}\). The space \(\mathcal{H}^\times\) of \(\tau_{\mathcal{H}}\)-continuous functionals \(f : h \in \mathcal{H} \rightarrow f(h) \in \mathbb{C}\) can be identified with \(\mathcal{H}\) by \(f(h) = (h, f)\) and then \((\cdot, \cdot)\) is the extension of \((\cdot, \cdot)\) to those \(F\) which are not elements of \(\mathcal{H}\).

For a \(\tau_{\Phi}\)-continuous operator \(A\) there is also a triplet:

\[ A^\dagger |\Phi\rangle \subset A^\dagger \subset A^\times. \]

Here \(A^\dagger\) is the Hilbert space adjoint operator and \(A^\times\) is the conjugate operator. \(A^\times\) is defined by:

\[ \langle A\phi | F\rangle = \langle \phi | A^\times | F\rangle, \quad \forall \phi \in \Phi \quad \forall F \in \Phi^\times. \]

A vector \(|F\rangle \in \Phi^\times\) is a generalized eigenvector of the \(\tau_{\Phi}\)-continuous operator \(A\) if for some complex number \(\omega\)

\[ \langle A\phi | F\rangle = \langle \phi | A^\times | F\rangle = \omega \langle \phi | F\rangle, \quad \forall \phi \in \Phi. \]

This is also written as \(A^\times | F\rangle = \omega | F\rangle\), or \(A | F\rangle = \omega | F\rangle\) if \(A^\dagger\) is self-adjoint. An example is the Dirac ket

\[ H^\times | E\rangle = E | E\rangle, \quad E \geq 0. \]

In the Rigid Hilbert Space one can prove the Nuclear Spectral Theorem (referring to the nuclearity of the topological spaces \(\Phi\)) which is the most important property for quantum theory and another Golden Rule of Dirac: For every self-adjoint operator \(H^\dagger\) there exists a complete set of generalized eigenvectors \(|E\rangle \in \Phi^\times\) such that every vector \(\phi \in \Phi\) can be written as

\[ \phi = \int \rho(E) dE | E\rangle \langle E | \phi \rangle \]
where the (wave) functions $\langle E | \phi \rangle \equiv \phi(E)$ are elements of a space of well-behaved functions (e.g. the Schwartz space of infinitely differentiable rapidly decreasing functions). $\rho(E)$ is an integrable function and determines the normalisation of the eigenvectors

$$\langle E' | E \rangle = \rho^{-1}(E) \delta(E' - E).$$

The integration can – in all known cases important for physics – be chosen to be Riemann integration, and the set of wave functions $\phi(E)$ define the space of vectors $\phi \in \Phi$ and vice-versa. This theorem generalizes the basis vector expansion in 3- or $N$-dimensional spaces

$$\vec{x} = \sum_{i=1}^{N} \vec{e}_i \cdot \hat{x}, \quad \vec{e}_i \cdot \vec{x} = x_i$$

to continuously infinite dimensions and is therefore also called Dirac’s basis vector expansion. Note that it represents $\phi \in \Phi$ as a continuous linear superposition of basis vectors $|E\rangle \in \Phi \times$ which are not elements of $\Phi$. This may appear counter intuitive but has its justification in the choice of a set of very special functions $\phi(E)$ as possible “coordinates” for a vector $\phi \in \Phi$.

Depending on the choice of the space $\Phi$, one can also have generalized eigenvectors with complex eigenvalues for an operator $H$ that is essentially self-adjoint (and bounded from below, which we always assume). The Gamow vectors are an example:

$$H^\times |E_R - i\Gamma_R/2^-\rangle = (E_R - i\Gamma_R/2)|E_R - i\Gamma_R/2^-\rangle.$$
In the conventional Hilbert space theory one proves the theorem that the survival probability and thus the partial decay rate $P_\eta(t)$ (which is measured as the counting rate $\dot{N}_\eta(t)$) must have some deviations from the exponential law. Cf. L.A. Khalfin, JETP Lett. 15, 388 (1972). However, in practical calculations we use many concepts, like the Lippmann-Schwinger equations, that lie outside the Hilbert Space, so that this mathematical theorem need not worry us.

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