Universality for polynomial invariants on ribbon graphs with flags

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Abstract. In this paper, we analyze the Bollobas and Riordan polynomial for ribbon graphs with flags introduced in arXiv:1301.1987[math.CO] and prove its universality. We also show that this polynomial can be defined on some equivalence classes of ribbon graphs involving flag moves and that the new polynomial is still universal on these classes.

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1. Introduction

The Bollobas-Riordan (BR) graph polynomial [5] is a polynomial in four variables which extends the Tutte polynomial [17, 12] from simple graphs to graphs with additional structures such as ribbon graphs (such graphs arise as neighbourhoods of graphs embedded into surfaces). Both polynomials satisfy a contraction/deletion recurrence rule defined on the associated graphs and, furthermore, are universal polynomial invariants. The universality property of these invariants means that any invariant of graphs satisfying the same relations of contraction and deletion can be calculated from those. Universality can be also of great use, for example, in statistical mechanics [14] and quantum field theory [8, 7, 16].

The BR polynomial is defined on signed ribbon graphs which are ribbon graphs whose edges are marked either by $+1$ or by $-1$. The signs of the edges play an important role in the orientability of the ribbon graphs. Signed ribbon graphs and their polynomial invariants are still under investigations [13, 10, 15, 1]. For example in [10], the authors provide a “recipe theorem” for the BR polynomial very close to the universality property. The proof of the universality of the BR polynomial is mainly based on the fact that the BR polynomial satisfies a contraction/deletion relation. However the proof of that claim relies on several other ingredients. Chord diagrams associated with one-vertex ribbon graphs and canonical diagrams found from these chord diagrams after a sequence of operations called rotations and twists about chords are extremely useful to establish that fact.
Let us discuss in greater detail the polynomial on a new class of ribbon graphs introduced in [2] called ribbon graphs with flags. A flag is simply a ribbon edge incident to a unique vertex without forming a loop. The presence of flags in a ribbon graph have several interesting combinatorial properties as shown in [2]. Flags also allow to introduce a new and enough intuitive operation which is the cut of an edge which differs from the usual edge deletion. The authors of the above work describe the implications that have flags on the BR polynomial. One notes that in the polynomial worked out therein, the orientability of the ribbons is not taken into account. Since this new invariant satisfies a contraction/cut recurrence relation (replacing in this setting the usual contraction/deletion rule), one may wonder if this invariant is universal or not. Answering this question is the purpose of this paper.

We find in this paper an extension of the polynomial found in [2] by adding now a variable for the orientability of the graphs. In the presence of this new variable, the contraction/cut rule still holds. We then prove a main result (Theorem 4) which is the universality property for the BR polynomial on ribbon graphs with flags. The method used to prove this is close to that given in [5] but it is however specific due to the presence of flags. We then reveal the existence of another polynomial invariant defined over classes of ribbon graphs with flags related to a new operation called flag moves. Theorem 5 establishes the universality of that new polynomial which is a second main result of this paper.

The rest of this paper is organized as follows. In section 2, we give an overview of the BR polynomial and its universality property. In section 3, we recall some results on the BR polynomial for ribbon graphs with flags. In section 4, we give our main result which is the proof of the universality theorem of this polynomial. We define a polynomial invariant on classes of ribbon graphs related by moves of flags and we prove the universality property of this latter polynomial in section 5.

2. Overview of the Bollobas-Riordan polynomial and its universality property

In this section, we give an overview of the BR polynomial for ribbon graphs and mainly focus on its universality theorem introduced in [5]. There are several ingredients in the proof of this theorem which will be useful for our subsequent developments and, thus, are worth to be reviewed as well.

Definition 1 (Ribbon graphs [5][11]). A ribbon graph \( \mathcal{G} \) is a (not necessarily orientable) surface with boundary represented as the union of two sets of closed topological discs called vertices \( \mathcal{V} \) and edges \( \mathcal{E} \). These sets satisfy the following properties:

- Vertices and edges intersect in disjoint line segments,
- each such line segment lies on the boundary of precisely one vertex and one edge,
- every edge contains exactly two such line segments.

There are three kinds of edges that can be identified in a ribbon graph. An edge \( e \) of a ribbon graph \( \mathcal{G} \) is called a bridge in \( \mathcal{G} \) if its removal disconnects a component of \( \mathcal{G} \). The edge \( e \) is a self-loop in \( \mathcal{G} \) if the two ends of \( e \) are incident to the same vertex \( v \) of \( \mathcal{G} \) and \( e \) is a regular edge of \( \mathcal{G} \) if it is neither a bridge nor a self-loop. Ribbon edges can be twisted as well (see Figure 1). We say that a self-loop \( e \) at a vertex \( v \) of a ribbon graph \( \mathcal{G} \) is twisted if \( v \cup e \) forms a Möbius band as opposed to an annulus (an untwisted self-loop). A self-loop \( e \) is trivial if there is no cycle in \( \mathcal{G} \) which can be contracted to form a loop \( \ell \) interlaced with \( e \). Introducing twisted edges has some consequences on the orientation of the ribbon graph.

![Figure 1. Untwisted (left) and twisted (right) edge notations.](image)

In addition, there are other topological notions in a ribbon graph that we now describe.
DEFINITION 2 (Faces and orientation [5]). A face is a component of a boundary of \( \mathcal{G} \) considered as a geometric ribbon graph and hence as a surface with boundary. The orientation of \( \mathcal{G} \) is its orientation as a geometric ribbon.

If \( \mathcal{G} \) is regarded as the neighborhood of a graph embedded into a surface, the set of faces is the set of faces of the embedding. A ribbon graph is denoted by \( \mathcal{G}(\mathcal{V}, \mathcal{E}) \).

DEFINITION 3 (Deletion and contraction [5]). Let \( \mathcal{G} \) be a ribbon graph and \( e \) one of its edges.

- We call \( \mathcal{G} - e \) the ribbon graph obtained from \( \mathcal{G} \) by deleting \( e \) and keeping the end vertices as closed discs.
- If \( e \) is not a self-loop, the graph \( \mathcal{G}/e \) obtained by contracting \( e \) is defined from \( \mathcal{G} \) by deleting \( e \) and identifying its end vertices \( v_{1,2} \) into a new vertex which possesses all edges in the same cyclic order as their appeared on \( v_{1,2} \).
- If \( e \) is a trivial twisted self-loop, contraction is deletion: \( \mathcal{G} - e = \mathcal{G}/e \). The contraction of a trivial untwisted self-loop \( e \) is the deletion of the self-loop and the addition of a new connected component vertex \( v_0 \) to the graph \( \mathcal{G} - e \). We write \( \mathcal{G}/e = (\mathcal{G} - e) \sqcup \{v_0\} \).

We recall that the contraction of a (twisted or untwisted) self-loop \( e \) in \( \mathcal{G} \) coincides with an edge deletion in the graph dual of \( \mathcal{G} \).

A spanning subgraph \( A \) of a ribbon graph \( \mathcal{G}(\mathcal{V}, \mathcal{E}) \) is a ribbon graph defined by a subset of edges \( \mathcal{E}(A) \subseteq \mathcal{E} \) which possesses all vertices \( \mathcal{V} \) of \( \mathcal{G} \). We denote it as \( A \subseteq \mathcal{G} \).

DEFINITION 4 (BR polynomial [5]). Let \( \mathcal{G} \) be a ribbon graph. We define the ribbon graph polynomial of \( \mathcal{G} \) to be

\[
R_\mathcal{G}(X, Y, Z, W) = \sum_{A \subseteq \mathcal{G}} (X - 1)^{r(A) - r(\mathcal{G})} (Y - 1)^{n(A)} Z^{k(A) - F(\mathcal{A}) + n(A)} W^{t(A)}
\]

considered as an element of the quotient of \( \mathbb{Z}[X, Y, Z, W] \) by the ideal generated by \( W^2 - W \) and where \( r(A), n(A), k(A), F(\mathcal{A}) \text{ and } t(A) \) are, respectively, the rank, the nullity, the number of connected components, the number of faces and the parameter which characterizes the orientability of \( A \) as a surface. If \( A \) is orientable, then \( t(A) = 0 \), otherwise, \( t(A) = 1 \). By definition, \( r(\mathcal{G}) = |\mathcal{V}| - k(\mathcal{G}) \text{ and } n(\mathcal{G}) = |\mathcal{E}(\mathcal{G})| - r(\mathcal{G}) \).

In the following, we use the variable \((Y - 1)\) for parameterizing the nullity of the subgraphs. This convention differs from the one in [5] which rather uses \( Y \). From a simple change of variable at any moment \((Y \rightarrow Y + 1)\), one can recover the convention used therein. Moreover, putting \( W = 1 = Z \), one recovers the Tutte polynomial for \( \mathcal{G} \) seen as a simple graph. After introducing terminal forms, the choice \((Y - 1)\) will be discussed.

The BR polynomial obeys a contraction and deletion rule.

THEOREM 1 (Contraction and deletion [5]). Let \( \mathcal{G} \) be a ribbon graph. If \( e \) is a regular edge, then

\[
R_\mathcal{G} = R_{\mathcal{G}/e} + R_{\mathcal{G} - e},
\]

for a bridge \( e \) of \( \mathcal{G} \), one has

\[
R_\mathcal{G} = X R_{\mathcal{G}/e},
\]

for a trivial untwisted self-loop \( e \),

\[
R_\mathcal{G} = Y R_{\mathcal{G} - e},
\]

and for a trivial twisted self-loop \( e \), the following holds

\[
R_\mathcal{G} = (1 + (Y - 1)ZW) R_{\mathcal{G} - e}.
\]

The relations \((3) \text{ and } (5)\) are useful for the evaluation of the terminal forms (ribbon graphs which only possess edges which are not regular). For a graph \( \mathcal{G} \) with only \( n \) bridges, \( m \) untwisted trivial self-loops and \( p \) twisted trivial self-loops, the polynomial of \( \mathcal{G} \) is \( X^n Y^m (1 + (Y - 1)ZW)^p \). Note that, in [1], the list of terminal forms has been further extended to specific one-vertex graphs called flowers so that one can complete the above with other contributions (the interested reader is referred to that paper).
Let us discuss in more details the universality of the BR polynomial for ribbon graphs \([5]\). It is shown that the polynomial \(R\) is the universal invariant for connected ribbon graphs satisfying \([2]\) and \([3]\) and any other invariant satisfying the same relations can be calculated from \(R\). First, one must understand that the knowledge of \(R\) can be reduced to one-vertex ribbon graphs also simply called rosettes.

Specifically, we obtain a rosette ribbon graph after a contraction of a spanning tree in a connected ribbon graph \(G\). To achieve the proof of the universality of their polynomial, Bollobas and Riordan used another representation of one-vertex ribbon graphs called “signed chord diagrams” (chord diagrams are also related to Vassiliev invariants \([3, 4]\)). A chord diagram \(D\) is a construction related to a one-vertex ribbon graph \(G\) such that if \(G\) has \(n\) edges, \(D\) is constructed by putting on a circle \(2n\) distinct points paired off by \(n\) chords. In the case of a ribbon graph with twisted and untwisted edges, \(D\) is called a signed chord diagram, if we put an assignment of sign “t” or “unt” to each chord according to the fact that this chord corresponds to a twisted or negative edge or untwisted or positive edge, respectively.

We shall write \(n(D)\) for the number of chords of \(D\) which is also the nullity of \(G\) (each chord corresponds to an edge in a rosette or a cycle generator). Using the “doubling operation” which consists in replacing each chord of \(D\) by two edges joining the parts of the circle on each side of each end of the chord, \(F(D)\) denotes the number of components of the resulting figure. We have \(F(D) = F(G)\) and \(t(D)\) stands for \(t(G)\) which is equal to 0 if all chords of \(D\) have a positive sign (or untwisted) and 1 otherwise.

A subdiagram of a signed chord diagram \(D\) is a signed chord diagram \(D'\) obtained from \(D\) by deleting a subset of chords of \(D\). For a one-vertex ribbon graph \(G\), looked as a signed chord diagram \(D\), the BR polynomial summation is defined over the spanning subdiagrams \(D' \in D\) as:

\[
R(D) = \sum_{D' \in D} (Y - 1)^{n(D')} Z^{1 - F(D') + n(D')} W^{t(D')}.
\]

Later this summation is written as:

\[
R(D) = \sum_{i,j,k} R_{ijk}(D)(Y - 1)^i Z^j W^k,
\]

where \(R_{ijk}(D)\), the coefficient of \((Y - 1)^i Z^j W^k\) in \((7)\), counts the number of subdiagrams \(D' \in D\) which have \(i\) chords, \(j = 1 - F(D') + n(D')\) and \(k = t(D')\) in \((6)\). Consider \(G^*\) the set of isomorphism classes of connected ribbon graphs \([5]\), then, the theorem of universality is given by the following statement:

**Theorem 2 (Universality of Bollobas-Riordan polynomial \([5]\)).** Let \(\mathcal{R}\) be a commutative ring, \(x\) an element of \(\mathcal{R}\), and \(\phi\) a map from \(G^*\) to \(\mathcal{R}\) satisfying

\[
\phi(G) = \begin{cases} 
\phi(G - e) + \phi(G/e) & \text{if } e \text{ is regular,} \\
x \phi(G/e) & \text{if } e \text{ is a bridge.}
\end{cases}
\]

Then there are elements \(\lambda_{ijk}\), \(i \geq 0, 0 \leq j \leq i, 0 \leq k \leq 1, \) such that

\[
\phi(G) = \sum_{i,j,k} \lambda_{ijk} R_{ijk}(x).
\]

The main point of the universality theorem is the determination of the \(\lambda_{ijk}\). The coefficients \(\lambda_{ijk}\) are directly found by the evaluation of \(\phi\) on the so-called “canonical diagrams”. For a one-

\[\text{Figure 2. The canonical chord diagram } D_{5,1,1}.\]
vertex ribbon graph $G$ seen as a chord diagram $D$, a sequence of rotations and twists about chords in $D$ provides a simple diagram called canonical. Given canonical diagrams $D_{i,j,k}$, consisting of $i - 2j - k$ positive chords intersecting no other chords, $j$ pairs of intersecting positive chords, and $k$ negative chords $0 \leq k \leq 2$, intersecting no other chords (see an example in Figure 2), the evaluation of $\lambda_{ijk}$ is directly related to some $\phi(D_{i,j',k'})$. This is proved by a recurrence relation on the number of chords $i$, given the initial value $\lambda_{000}$ for the value of $\phi$ on a bare vertex. The same result holds for any connected ribbon graph using the relations (8) (the case of several connected components can be simply inferred from this point).

3. The Bollobas-Riordan polynomial for ribbon graphs with flags

This section introduces a polynomial invariant for ribbon graphs with flags which is a notion studied in [11]. The polynomial which will be discussed extends the invariant found in [2] by adding an orientability variable. It is this general polynomial which turns out to have a universal property as will be shown in the next section.

We first recall some definitions.

DEFINITION 5 (Ribbon flag and external points [2]). A ribbon flag or half-edge or simply flag, is a ribbon incident to a unique vertex by a unique segment and without forming a loop. A flag has two segments one touching a vertex and another free or external segment. The end-points of any free segment are called external points of the flag (see Figure 3).

![Figure 3. A ribbon flag with two end segments (in red): $s'$ touching the vertex and $s$ external; the ends $a$ and $b$ of $s$ are the external points.](image)

DEFINITION 6 (Cut of a ribbon edge [11]). Let $G$ be a ribbon graph and $e$ be an edge in $G$. The cut graph $G \vee e$ is the graph obtained by removing $e$ and let two flags attached at the end vertices of $e$. If $e$ is a self-loop, the two flags are on the same vertex. (See an illustration in Figure 4.)

![Figure 4. Cutting a ribbon edge.](image)

The definition of a ribbon graph with flags may be introduced at this stage.

DEFINITION 7 (Ribbon graph with flags [2]). $\bullet$ A ribbon graph $G$ with flags is a ribbon graph $G(\mathcal{V}, \mathcal{E})$ with a set $\mathcal{F}$ of flags defined by the disjoint union of $\mathcal{F}^{1}$ the set of flags obtained only from the cut of all edges of $G$ and a set $\mathcal{F}^{0}$ of additional flags together with a relation which associates with each additional flag a unique vertex. We denote a ribbon graph with set $\mathcal{F}^{0}$ of additional flags as $G(\mathcal{V}, \mathcal{E}, \mathcal{F}^{0})$. (See Figure 3.)

$\bullet$ A c-subgraph $A$ of $G(\mathcal{V}, \mathcal{E}, \mathcal{F}^{0})$ is defined as a ribbon graph with flags $A(\mathcal{V}_{A}, \mathcal{E}_{A}, \mathcal{F}^{0}_{A})$ the vertex set of which is a subset of $\mathcal{V}$, the edge set of which is a subset of $\mathcal{E}$ together with their end vertices. Call $\mathcal{E}_{A}$ the set of edges incident to the vertices of $A$ and not contained in $\mathcal{E}_{A}$. The flag set of $A$ contains a subset of $\mathcal{F}^{0}$ plus additional flags attached to the vertices of $A$ obtained by cutting all edges in $\mathcal{E}_{A}$. In symbols, $\mathcal{E}_{A} \subseteq \mathcal{E}$ and $\mathcal{V}_{A} \subseteq \mathcal{V}$, $\mathcal{F}^{0}_{A} = \mathcal{F}^{0}_{A} \cup \mathcal{F}^{1}_{A}(\mathcal{E}_{A})$ with $\mathcal{F}^{0}_{A} \subseteq \mathcal{F}^{0}$ and $\mathcal{F}^{1}_{A}(\mathcal{E}_{A}) \subseteq \mathcal{F}^{1}$, where $\mathcal{F}^{1}_{A}(\mathcal{E}_{A})$ is the set of flags obtained by cutting all edges in $\mathcal{E}_{A}$ and incident to vertices of $A$. We write $A \subseteq G$. (See a c-subgraph $A$ illustrated in Figure 3.)
A spanning c-subgraph \(A\) of \(G(V,E,f_0)\) is defined as a c-subgraph \(A(V_A,E_A,f_0_A)\) of \(G\) with all vertices and all additional flags of \(G\). Hence \(E_A \subseteq E\) and \(V_A = V, f_0_A = f_0 \cup f_{01}(E_A)\). (See \(\bar{A}\) in Figure 5.)

![Figure 5. A ribbon graph with flags \(G\), a c-subgraph \(A\) and a spanning c-subgraph \(\bar{A}\).](image)

The notion that we will extensively use is the one of spanning c-subgraph. We can simply explain that notion in the following way: Take a subset of edges of a given graph, cut them all. Consider the spanning subgraph then formed by the resulting graph. The set of flags of this subgraph contains both the set of flags of the initial graph \((f_0)\) plus an additional set induced by the cut of the edges.

Note that cutting an edge of a graph modifies the boundary faces of this graph. There are new boundary faces following the contour of the flags. However combinatorially, we distinguish this new type of faces and the initial ones which follow the boundary of well-formed edges.

**Definition 8 (Closed and open faces [9])**. Consider \(G(V,E,f_0)\) a ribbon graph with flags.
- A closed or internal face is a boundary face component of a ribbon graph (regarded as a geometric ribbon) which never passes through any free segment of additional flags. The set of closed faces is denoted \(F_{\text{int}}\).
- An open or external face is a boundary face component leaving an external point of some flag rejoining another external point. The set of open faces is denoted \(F_{\text{ext}}\).
- The two boundary lines of a ribbon edge or a flag are called strands. Each strand belongs either to a closed or to open face.
- The set of faces \(F\) of a graph is defined by \(F_{\text{int}} \cup F_{\text{ext}}\).
- A graph is said to be open if \(F_{\text{ext}} \neq \emptyset\) i.e. \(f_0 \neq \emptyset\). It is closed otherwise.

Open and closed faces are illustrated in Figure 6.

![Figure 6. A ribbon graph with set of internal faces \(F_{\text{int}} = \{f_0\}\), and set of external faces \(F_{\text{ext}} = \{f_1, f_2, f_3\}\).](image)

**Definition 9 (Boundary graph [9])**. The boundary \(\partial G\) of a ribbon graph \(G(V,E,f_0)\) is a simple graph \(\partial G(V_{\partial},E_{\partial})\) such that \(V_{\partial}\) is one-to-one with \(f_0\) and \(E_{\partial}\) is one-to-one with \(F_{\text{ext}}\).
- The boundary graph of a closed graph is empty.

The boundary graph \(\partial G\) of the graph \(G\) is obtained by inserting a vertex of valence two at each flag, the edges of \(\partial G\) which are external faces are incident to these vertices. \(\partial G\) has only vertices with two incident lines or one incident line if the two sides of the flag defined in fact the same external face (see Figure 7).

The notions of edge contraction and deletion for ribbon graphs with flags keep their meaning as in Definition 3. We are in position to identify a new polynomial invariant.
For ribbon graphs with flags under particular conditions. The following proposition holds.

**Definition 3** (Respects the cyclic order of all edges and flags on the previous vertices). Where \( R \) is given by Definition 10.

**Theorem** \( \text{3 generalize the BR polynomial } R \text{.} \)

The polynomial \( R \text{.} \) extends to ribbon graphs \( \text{5 and to ribbon graphs with flags } \text{2.} \)

**Proposition 1** (Operations on BR polynomials \( \text{2.} \)). Let \( G_1 \) and \( G_2 \) be two disjoint ribbon graphs with flags, then

\[
\mathcal{R}_{G_1 \sqcup G_2} = \mathcal{R}_{G_1} \mathcal{R}_{G_2}, \quad \mathcal{R}'_{G_1 \sqcup G_2} = \mathcal{R}'_{G_1} \mathcal{R}'_{G_2},
\]

for any disjoint vertices \( v_1, v_2 \) in \( G_{1,2} \), respectively.

**Proof**. The proof of Proposition \( \text{1} \) corresponds to that of Proposition 5 in \( \text{2} \) where the sole additional fact concerns the variable \( W \) associated with the orientability. This can be simply achieved by adding the fact that \( W^2 = W \) in the proof of Proposition 5 in \( \text{2} \).

**Theorem 3** (Contraction and cut on BR polynomial). Let \( G(V, E, \psi) \) be a ribbon graph with flags. Then, for a regular edge \( e \),

\[
\mathcal{R}_G = \mathcal{R}_{G \setminus e} + \mathcal{R}_{G/e},
\]

for a bridge \( e \), we have

\[
\mathcal{R}_G = (X - 1)\mathcal{R}_{G \setminus e} + \mathcal{R}_{G/e};
\]

for a trivial twisted self-loop \( e \), the following holds

\[
\mathcal{R}_G = \mathcal{R}_{G \setminus e} + (Y - 1)ZW \mathcal{R}_{G/e},
\]

whereas for a trivial untwisted self-loop \( e \), we have

\[
\mathcal{R}_G = \mathcal{R}_{G \setminus e} + (Y - 1)\mathcal{R}_{G/e}.
\]
Proof. This can be proved in the same lines of Theorem 3 in [2] where the new point associated with the orientability can be recovered from [5]. □

**Corollary 1 (Contraction and cut on BR polynomial \( \mathcal{R}' \)).** Let \( \mathcal{G}(V,E,f^0) \) be a ribbon graph with flags. Then, for a regular edge \( e \),

\[
\mathcal{R}'_G = \mathcal{R}'_{G\vee e} + \mathcal{R}'_{G/e}, \quad \mathcal{R}'_{G\vee e} = T^2 \mathcal{R}'_{G-e} ;
\]

for a bridge \( e \), we have

\[
\mathcal{R}'_{G/e} = \mathcal{R}'_{G-e} = T^{-2} \mathcal{R}'_{G\vee e},
\]

\[
\mathcal{R}'_{G} = [(X-1)T^2 + 1] \mathcal{R}'_{G/e} ;
\]

for a trivial twisted self-loop, \( \mathcal{R}'_{G-e} = \mathcal{R}'_{G/e} = T^2 \mathcal{R}'_{G\vee e} \) and

\[
\mathcal{R}'_{G} = [T^2 + (Y-1)ZW] \mathcal{R}'_{G-e} ;
\]

whereas for a trivial untwisted self-loop, we have

\[
\mathcal{R}'_{G-e} = \mathcal{R}'_{G/e} = (X-1)T^4 + T^2 \mathcal{R}'_{G}.
\]

Proof. The corollary is immediate from Theorem 3 and Corollary 1 in [2]. The new relation (20) can be achieved using a similar identity in [5]. □

The polynomial \( \mathcal{R}' \) is universal. Indeed, from Corollary 1 we have the following relation verified by \( \mathcal{R}' \):

\[
\mathcal{R}'_G = \begin{cases} T^2 \mathcal{R}'(G-e) + \mathcal{R}'(G/e) & \text{if } e \text{ is neither a bridge nor a self-loop,} \\ [(X-1)T^4 + T^2] \mathcal{R}'(G-e) & \text{if } e \text{ is a bridge.} \end{cases}
\]

After a change of variables as:

\[
\begin{align*}
\tilde{X} &= (X-1)T^2 + 1 \\
\tilde{Y} &= Y - 1 + T^2
\end{align*}
\]

and given the fact that, for a given graph \( \mathcal{G}(V,E,f^0) \) and \( A \subseteq \mathcal{G} \),

\[
f(A) = |f^0| + 2(|E| - |E_A|),
\]

we get

\[
\mathcal{R}'_G(X,Y,Z,W,T) = T^{|f^0|T^{2n(G)}} R_G(\tilde{X}, \tilde{Y}, Z, W),
\]

with \( R \) the BR polynomial defined in [10]. The above equation shows that the polynomial \( \mathcal{R}' \) is universal. In the following, our main task is to prove that there exists a universality theorem for \( \mathcal{R} \). [10]

4. Main result: Universality theorem for \( \mathcal{R} \)

4.1. Chord diagrams with flags. The main objective of this sub-section is the determination of a special class of diagrams called canonical which turn out to be necessary for the proof of the universality of the polynomial in [10]. To succeed in this, we need to understand how the operations of rotation and twist about chords make sense on “open” chord diagrams or chord diagrams associated to one-vertex ribbon graphs with flags. After defining open chord diagrams, we will focus on a two-vertex ribbon graph with flags where the distinct ways of contracting the edges lead to some equivalent diagrams.

**Definition 11 (Chord diagrams).** • A flag on a chord diagram is a segment attached to a unique point on its circle.

• An (open) chord diagram is a chord diagram in sense of [5] with a (nonempty) set of flags. In the case where this set is empty, it becomes BR chord diagram.

• A signed (open) chord diagram is an (open) chord diagram with an assignment of a sign “t” or “unt” to each chord.
Remark that in the previous definition of chord diagram $D$, if $D$ has $n$ chords and $l$ flags, there is $2n + l$ distinct and remarkable points on the circle.

If $G$ is a one-vertex ribbon graph with flags and $D$ the corresponding (open) signed chord diagram, the number $n(D)$ of chords of $D$ is equal to the nullity of $G$ and we have $n(D) = e(G) = n(G)$. The doubling operation on $D$ consists of replacing each chord of $D$ by two edges joining the parts of the circle on each side of each end of the chord and each flag of $D$ by two parallel segments. With this operation, the number of components of the resulting figure, is equal to $F_{\text{int}}(D) + C_{\partial}(D)$ where $F_{\text{int}}(D)$ is the number of components which are closed and $C_{\partial}(D)$ is the remaining or boundary components.

The ordinary operations on ribbon graphs simply translate in chord diagrams. In particular, the deletion or the cutting of chords and disjoint union or one-point-joint between two separate diagrams obey the same principles as in ribbon graphs.

Consider a two-vertex ribbon graph with flags $G$ with at least two edges $e$ and $g$ which are not loops. Let us write $a$, $b$, $c$ and $d$ for the sections into which $e$ and $g$ divide the cyclic orders at the vertices of $G$ (some flags may be attached to the vertices as illustrated in Figure 8). The contractions of $e$ or of $g$ give two different one-vertex ribbon graphs with flags. If $e$ and $g$ are positive chords, let $D_1$ be the (open) chord diagram associated to the graph we obtain by contracting $g$ in $G$, $D'_1$ the (open) chord diagram associated to the graph we obtain by contracting $g$ in $G \vee e$, $D_2$ the (open) chord diagram associated to the graph we obtain by contracting $e$ in $G$ and $D'_2$ the one we obtain by contracting $e$ in $G \vee g$ (see Figure 9). If $g$ is negative (without loss of generality), we replace $D_1$, $D_2$, $D'_1$ and $D'_2$, respectively, by $D_3$, $D_4$, $D'_3$ and $D'_4$ in the previous statement (see Figure 10).

![Figure 8. Two-vertex ribbon graph with flags](image)

![Figure 9. Related chords diagrams $D_1$, $D'_1$, $D_2$, $D'_2$](image)

![Figure 10. Related chords diagrams $D_3$, $D'_3$, $D_4$, $D'_4$](image)

In Figure 10, the sector $c'$ is obtained from $c$ after a sequence of two operations: we reverse the order of the endpoints of the flags and chords of $c$ and we change the sign of any chord from $c$ to the rest of the diagram. The same apply to $d'$ obtained from $d$. 
Two signed (open) chord diagrams are related by a rotation about the chord $e$ if they are related as $D_1$ and $D_2$ in Figure 9 and that they are related by a twist about $e$, if they are related as $D_3$ and $D_4$ in Figure 10. Now we can give the definitions of $R$-equivalent diagrams and the sum of two chord diagrams.

**Definition 12 (R-equivalence relation [5])**. Two diagrams or signed diagrams $D_1$ and $D_2$ are $R$-equivalent if and only if they are related by a sequence of rotations and twists. We write $D_1 \sim D_2$.

**Definition 13 (Sum of diagrams [5])**. The sum of two diagrams or signed diagrams $D_1$ and $D_2$ is obtained by choosing a point $p_i$ (not the end-point of a chord or a flag) on the boundary of each $D_i$, joining the boundary circles at these points and then deforming the result until it is again a circle.

By choosing the $p_i$ differently, this sum can be formed in many different ways but we shall show that all of them are $R$-equivalent.

**Lemma 1**. If two diagrams $D$ and $D'$ are both sums of diagrams $D_1$ and $D_2$, then they are $R$-equivalent.

**Proof**. The proof is the same as in [6] since the rotations and twists about chords move only the points $p_1$ or $p_2$ chosen on $D_1$ or $D_2$, respectively. The only fact that one must pay attention is to respect the cyclic order of the flags on the resulting circle. In the case where there are some flags coming before the chord we want to rotate about or twist about, we must rotate or twist the flag about a chord before the next step.

**Canonical chord diagrams**. For $i \geq 0$, $0 \leq 2j \leq i$, $0 \leq k \leq i + 1$, $t \geq 0$ and $0 \leq m \leq 2$, let $D_{i,j,k,(s;1_i,\ldots, l_q),m}$ be the chord diagram consisting of $i$ chords, $j$ pairs of chords intersecting each other, $k$ connected components of the boundary of this diagram, $l$ flags ($l = s + \sum_{p=1}^{q} l_{p}^{s}$) disposed in a specific way and $m$ negative chords (or twisted chords) intersecting no other chords (hence $i - 2j - m$ is the number of positive chords intersecting no other chords). This diagram is drawn in such a way that there is a number $l - s$ of flags partitioned in $(l_{p})_{p=1,\ldots,q}$, positive chords intersecting no other chords (we shall also call these isolated chords) and $s$ is the rest of the flags. We put “$t$” for only twisted chords for simplicity. All these chords and flags are arranged around the circle of the diagram (see an illustration for $D_{4,1,2,(3;1),1}$ and $D_{5,1,2,(0;1,2),1}$):

![Figure 11. Canonical diagrams: $D_{4,1,2,(3;1),1}$ and $D_{5,1,2,(0;1,2),1}$](image)

If there is no flags on the graph, our canonical diagram corresponds exactly to that of Bollobas and Riordan [5]. Consider now a chord diagram $D$ with $l > 0$ flags. Forgetting about the flags for a moment, one performs a sequence of rotations and twists about chords in the same way as [5] and is led to a BR canonical diagram. The flags in $D$ were disposed on open faces (open components) which are preserved under rotations and twists. Therefore, at the end, one adds the flags on the resulting BR canonical diagram in order to obtain the result if the same sequence of rotations and twists about chords was performed on the initial signed chord diagram $D$ considered with flags. The issue here is the disposition of the flags in the BR canonical diagram. We will show however that, from the knowledge of $D$, either we can directly reconstruct the new canonical diagram or find a canonical diagram $R$-equivalent to it.

**Lemma 2**. Any (open) chord diagram $D$ is $R$-equivalent to some $D_{i,j,k,(s;1_i,\ldots, l_q),m}$. 

Proof. Let $D$ be a signed (open) chord diagram with $i$ chords, $k$ connected components of the boundary and $l$ flags.

Suppose $l = 0$. In this case $k = 0$ and $D$ is $R$-equivalent to some $D_{ijm}$ in sense of \[5\]. We denote it as $D_{i,j,0,(0),m}$ since the set of partitions $(s; l_1, \ldots , l_q)$ is empty.

Assume now that $l > 0$. If we forget the flags for a moment and perform a sequence of rotations and twists about chords, we obtain that $D$ is $R$-equivalent to some $D_{ijm}$, a signed chord diagram consisting of $i$ chords, $j$ pairs intersecting positive chords, $i - 2j - m$ isolated positive chords and $m$ ($0 \leq m \leq 2$) negative isolated chords. One can add now the $l$ flags to $D_{ijm}$. Note that there is only one internal face which passes through all the pairs of positive chords intersecting each other and all negative chords. Then inserting flags on this face just leads to only one connected component of the boundary graph. The remaining connected components of the boundary can be formed by putting a number of flags in a certain number of isolated positive chords. Suppose at first that $i - 2j - m > 0$ (there is at least one positive isolated chord). If $k \leq i$, we have two possible cases to arrange the $l$ flags. One way is to arrange the $l$ flags such that they are partitioned in $k$ isolated positive chords and then we obtain the canonical diagram $D_{i,j,k,(0); l_1, \ldots , l_q}, m$ ($l_p > 0$, $\forall p = 1, \ldots , k$). The second way is to arrange $l - s$ ($s > 0$) flags such that they are partitioned in $k - 1$ isolated positive chords and the remaining $s$ flags are not in any chord and then we obtain the canonical diagram $D_{i,j,k,(0), l_1, \ldots , l_q}, m$ ($l_p > 0$, $\forall p = 1, \ldots , k - 1$). By a sequence of rotations and twists about chords we have $D_{i,j,k,(0), l_1, \ldots , l_q}, m \sim D_{i,j,k, (s); l_1, \ldots , l_q}, m$ (see Figure 12). If $k = i + 1$, then all the $i$ chords of $D$ must be positive isolated chords and $l - s$ ($s > 0$) flags of $D$ must be partitioned in the $i$ chords so that $D = D_{i,0,(i+1), l_1, \ldots , l_q}, 0$. If $i - 2j - m = 0$ that means that we do not have any positive isolated chord and $k = 1$ then $D \sim D_{i,j,1,(1),m}$.

\[\square\]

Figure 12. Two $R$-equivalent canonical diagrams: $D_{5,1,2,(0,1,2),1} \sim D_{5,1,2,(2,1),1}$

Given a permutation $\sigma$ in $S_p$ (the permutation group with $p$ elements), $D_{i,j,k,(s); l_1, \ldots , l_q}, m \sim D_{i,j,k, (s); l_1, \ldots , l_q}, m$. This simply means that the order of the sequence $(l_1, \ldots , l_p)$ does not matter. In the following, only the total number of flags of a canonical diagram is relevant for the universality theorem. For simplicity, we use $D_{i,j,k, (s); l_1, \ldots , l_q}, m$ to denote $D_{i,j,k, (s); l_1, \ldots , l_q}, m$.

4.2. Universality of the polynomial $R$. For a ribbon graph with flags, we are in position to prove that the polynomial invariant $R_{G}$ in \[10\] is universal.

Consider the following expansion of $R_{G}$

$$R_{G}(X, Y, Z, S, T, W) = \sum_{i,j,k,l,m} R_{ijklm}(G)(Y - 1)^{i} Z^{j} S^{k} T^{l} W^{m}, \quad (26)$$

where each $R_{ijklm}$ is a map from the set $G^{*}$ of isomorphism classes of connected ribbon graphs with flags to $\mathbb{Z}[X]$. By equating coefficients of $(Y - 1)^{i} Z^{j} S^{k} T^{l} W^{m}$ or performing a straightforward computation from the definition of $R_{ijklm}$, we can see from Theorem 3 that $R_{ijklm}$ satisfies \[14\] and \[15\].

Given a ring $\mathcal{R}$ and an element $x$ of $\mathcal{R}$, for $i,j,k,l,m$, as $R_{ijklm}$ takes values in $\mathbb{Z}[X]$, we compose it with the ring homomorphism from $\mathbb{Z}[X]$ to $\mathcal{R}$ mapping $X$ to $x$, and obtain a map $R_{ijklm}(x)$ or $R_{ijklm}(G; x)$ $(G \in G^{*})$ from $G^{*}$ to $\mathcal{R}$. Infinite sum of these functions is of significance, but in general a finite number are non-vanishing on any given ribbon graph with flags.
Theorem 4 (Universality of $R$). Let $R$ be a commutative ring and $x \in R$. If a function $\phi: G^* \to R$ satisfies

$$
\phi(G) = \begin{cases} 
\phi(G \vee e) + \phi(G/e) & \text{if } e \text{ is regular,} \\
(x-1)\phi(G \vee e) + \phi(G/e) & \text{if } e \text{ is a bridge.}
\end{cases}
$$

Then there are coefficients $\lambda_{ijklm} \in R$, with $i \geq 0$, $0 \leq k \leq i + 1$, $l \geq 0$, $0 \leq m \leq 1$ and $0 \leq j \leq i + 1$ such that

$$
\phi(G) = \sum_{i,j,k,l,m} \lambda_{ijklm} R_{ijklm}(x).
$$

Proof. Let us consider the two-vertex ribbon graph $G$ of Figure 6. Applying equation (27) provides two different expressions for $\phi(G)$; at first, one applies these relations to the positive edge $e$ and then to the positive edge $g$ (if it is not a self-loop), and vice-versa. Equating these expressions shows that

$$
\phi(D_1) - \phi(D'_1) = \phi(D_2) - \phi(D'_2),
$$

where $D_1$, $D'_1$, $D_2$ and $D'_2$ are signed chord diagrams related as illustrated in Figure 9.

Similarly, considering the case where $g$ is negative allows us to get

$$
\phi(D_3) - \phi(D'_3) = \phi(D_4) - \phi(D'_4),
$$

where $D_3$, $D'_3$, $D_4$ and $D'_4$ are signed chord diagrams related as illustrated in Figure 10.

Suppose that $\phi$ satisfies (27) and let us show that it has the form (28). We will define the $\lambda_{ijklm}$ by induction. If $i = 0$, then $m = 0$ and we set $\lambda_{00000}$ for the value of $\phi$ on one-vertex ribbon graph without loops and flags, $\lambda_{01110}$ (for $l > 0$) for the value of $\phi$ on one-vertex ribbon graph without loops but with flags and $\lambda_{0jklm} = 0$ for all other values of $j, k, l, m$.

Assume that $n \geq 1$ and $\phi(G) = \sum_{\ell < n} \lambda_{ijklm} R_{ijklm}(G; x)$ for all one-vertex ribbon graphs with flags $G$ with fewer than $n$ loops. Let us set $\phi' = \phi - \sum_{\ell < n} \lambda_{ijklm} R_{ijklm}(G; x)$. $\phi'$ vanishes on one-vertex graphs with flags with less than $n$ loops and satisfies (27) since $\phi$ and the $R_{ijklm}$ satisfy it. Since $\phi'$ vanishes on chords diagrams with fewer than $n$ chords, then $\phi'(D_1) = \phi'(D_2)$ or $\phi'(D_3) = \phi'(D_4)$ (for related diagrams with $n$ chords) and $\phi'(D)$ depends only on the $R$-equivalence class of $D$. For $j, k, l$ and $m$, there is an $R_{n,j',k',l',m'}$ such that $R_{n,j',k',l',m'}(D_{n,j'k',l',m'})$ is one if $j'' = j$, $k'' = k$, $l'' = l$ and $m'' = m$, and zero otherwise. We can then choose the $\lambda_{ijklm}$ so that (28) holds on the $D_{n,j,k,l,m}$ and hence on all chord diagrams with $n$ chords.

By induction on $n$, there exist $\lambda_{ijklm}$ such that (28) holds for all one-vertex ribbon graphs with flags $G$. The same result follows for all connected ribbon graphs with flags using (27).

Let $\gamma$ be the function defined on the set $\{0, 1, 2\}$ by:

$$
\begin{aligned}
\gamma(0) &= 0, \\
\gamma(1) &= \gamma(2) = 1.
\end{aligned}
$$

The computation of $\phi'$ on a canonical signed chord diagram $D_{n,j',k',[l'],m'}$ gives:

$$
\phi'(D_{n,j',k',[l'],m'}) = \sum_{jklm} \lambda_{ijklm} R_{ijklm}(D_{n,j',k',[l'],m'}; x)
= \sum_{jklm} \lambda_{ijklm} \delta_{j,j'} \delta_{k,k'} \delta_{l,l'} \delta_{m,m'}
= \lambda_n R_{n,j'+k'+m'}.
$$

For some $j, k, l$ and $m$, we can compute explicitly, $\lambda_{ijklm}$:

- If $m = 0$

$$
\lambda_{ijkl0} = \phi'(D_n, (j-k), k, [l], 0).
$$

Then $\lambda_{ijkl0}$ is the value of $\phi'$ on the canonical signed chord diagram $D_n, (j-k), k, [l], 0$, if and only if $j - k \in 2N$ and $j \leq n + 1$. Otherwise, $\lambda_{ijkl0} = 0$. 

\[\square\]
If \( m = 1 \)

\[
\lambda_{nkkl} = \begin{cases} 
\phi'(D_{n,\frac{j}{2}(j-k-1),k,[l],1}) & \text{if } j - k \in 2\mathbb{N} + 1, \\
\phi'(D_{n,\frac{j}{2}(j-k-2),k,[l],2}) & \text{if } j - k \in 2\mathbb{N} + 2.
\end{cases}
\] (34)

Then \( \lambda_{nkkl} \) is the value of \( \phi' \) on the canonical signed chord diagram \( D_{n,\frac{j}{2}(j-k-1),k,[l],1} \) if and only if \( j - k \in 2\mathbb{N} + 1 \) and \( j \leq n + 1 \). It can be also the value of \( \phi' \) on the canonical signed chord diagram \( D_{n,\frac{j}{2}(j-k-2),k,[l],2} \) if and only if \( j - k \in 2\mathbb{N} + 2 \) and \( j \leq n + 1 \). Otherwise, \( \lambda_{nkkl} = 0 \).

As in case of Tutte polynomial and BR polynomial, the condition (27) in Theorem 4 can be replaced by

\[
\phi(G) = \begin{cases} 
\tau \phi(G \lor e) + \sigma \phi(G/e) & \text{if } e \text{ is regular}, \\
(x - 1) \phi(G \lor e) + \sigma \phi(G/e) & \text{if } e \text{ is a bridge},
\end{cases}
\] (35)

with fixed element \( x, \sigma \) and \( \tau \) of \( \mathfrak{R} \). If \( \sigma \) and \( \tau \) are invertible and \( \phi(G) \) satisfies (35), then \( \Phi'(G) = \sigma^{-r(G)} \tau^{-n(G)} \phi(G) \) satisfies (27) with \( (x - 1) \) replaced by \( (x - 1)\sigma^{-1} \) if we want to apply Theorem 4 to this function.

In the proof of the universality of \( R \), we discussed the fact that the set of flags which was partitioned in the canonical diagram \( D_{i,k,s_{1},l_{1},\ldots,l_{n}} \) did not really matter. Only was involved the number of flags \( s + \sum_{p} l_{p} = l \) in that proof. This strongly suggests that there exists another category of ribbon graphs and an associated polynomial invariant for which the universality still holds and only depends on the number of flags. We will investigate such category of ribbon graphs and the associated polynomial in the next section.

5. Polynomial invariant for flag-equivalent ribbon graphs

In order to define the new category of graphs of interest, we must introduce a new equivalence relation on ribbon graphs.

**Definition 14 (Flag move operation).** Let \( G(\mathcal{V}, \mathcal{E}, \mathfrak{F}) \) be a ribbon graph with flags. A flag move in \( G \) consists in removing a flag \( f \in \mathfrak{F} \) from one-vertex \( V \) and placing \( f \) either on \( V \) or on another vertex such that it is called

- a flag displacement if the boundary connected component where \( f \) belongs is not modified (see \( G_{1} \) and \( G_{2} \) in Figure 13);
- a flag jump if the flag is moved from one boundary connected component to another one, provided the former remains a connected boundary component (see \( G_{1} \) and \( G_{3} \) or \( G_{2} \) and \( G_{3} \) in Figure 13).

![Figure 13. Some flag moves](image)

One observes that under flag displacements the boundary graph remains unchanged whereas under flag jumps this graph can be modified. In general, under flag moves, the number of connected components of the boundary graph is not modified. For instance, in Figure 13 the graphs \( G_{2} \) and \( G_{3} \) are obtained from \( G_{1} \) by a flag displacement and a flag jump, respectively.

**Definition 15 (Flag-equivalence relation).** We say that two ribbon graphs with flags \( G \) and \( G' \) are flag-equivalent if they are related by a sequence of flag moves. This relation is denoted by \( G \sim_{F} G' \).
One can check that the flag-equivalence is an equivalence relation. As a consequence of the definition, if \( G \sim_p G' \), then \( V(G) = V(G') \), \( E(G) = E(G') \), \( F_{\text{int}}(G) = F_{\text{int}}(G') \), \( f(G) = f(G') \), \( k(G) = k(G') \), \( t(G) = t(G') \), \( r(G) = r(G') \), \( n(G) = n(G) \) and \( C_0(G) = C_0(G') \). Thus the flag moves only modify the incidence relation between flags and vertices. We denote the flag-equivalence class of \( G \) by \([G]\). Hence the three graphs in Figure 13 are flag-equivalent. For short, we will also use “\( G \) is equivalent to \( G'' \)” if there is no confusion.

Let \([G]\) be a class of a ribbon graph with flags under such relation. We define \( V([G]) = V(G), E([G]) = E(G) \), and \( f([G]) = f(G) \). The number of connected components, the rank, nullity, the number of internal faces and the number of boundary components of \([G]\) are those of \( G \), namely, \( k([G]) = k(G) \), \( r([G]) = r(G) \), \( n([G]) = n(G) \), \( F_{\text{int}}([G]) = F_{\text{int}}(G) \) and \( C_0([G]) = C_0(G) \).

The following statement holds.

**Lemma 3.** If two ribbon graphs with flags \( G \) and \( G' \) are flag-equivalent, then for any edge \( e \) in \( G \) and \( G' \), \( G \lor e \) and \( G' \lor e \) are flag-equivalent.

**Proof.** We shall establish that a single flag move operation commute with cutting an edge \( e \) in \( G \). In order to do so, we must observe that there exists a number of connected components of the boundary graph which may pass through the edge \( e \) and pay attention on how these components get modified under the two processes.

We call \( G' \) the graph obtained from \( G \) after the flag move. A case by case study is required.

(i) Assume that no connected component of the boundary graph passes through \( e \). This means that there is one closed face or there are two closed faces passing through \( e \). Consider in \( G \) a flag move giving \( G' \). The flag cannot visit the closed face(s) passing through \( e \). Then after cutting \( e \), the flag cannot be hooked on the (1 or 2) boundary connected components which are generated in \( G' \lor e \). If we start by cutting \( e \) in \( G \) and perform the same flag move in \( G \lor e \), the flag cannot still visit the boundary components generated by the cut.

(ii) Assuming now that, through \( e \) pass one closed face and one boundary connected component. Cutting \( e \) merges the close face to the boundary component. The reasoning is similar to the above point (i) (in the sense that the flag cannot be hooked to the sector generated by the closed face) and the operations commute.

(iii) Let us consider now that there is no closed face passing through \( e \). Two situations, A and B, might occur:

A) We have a unique boundary component \( C \) passing through \( e \). This case further divides into two possibilities:

A1) The cut of \( e \) generates a unique connected component of the boundary. One easily checks that the flag move commute with the cut.

A2) The cut of \( e \) generates two connected components \( C_1 \) and \( C_2 \) of the boundary graph containing each a flag coming from \( e \).

- Now let us assume that the move is a jump and that the flag come from another boundary component \( C_0 \) and ends on \( C \). After cutting \( e \) that flag must be hooked to a unique \( C_i, i = 1, 2 \). Assuming that we cut \( e \) first, the same flag jump can be performed if and only if \( C_i \) has a flag. This is indeed the case.

- Let then assume that the move is a displacement. Two situations can happen. Either the move is done within a sector \( C_i \) or done from \( C_1 \) to \( C_2 \) (without loss of generality). Then we can cut \( e \). If the displacement was within \( C_i \), one notes that, after cutting \( e \), we can perform the same move within the same \( C_i \) which yields an identical configuration as above. Meanwhile, if the displacement was from \( C_1 \) to \( C_2 \) (as sectors of \( C \)), after cutting \( e \), \( C_1 \) disconnects from \( C_2 \) and the same move cannot be a displacement anymore. It can be however a jump if and only if \( C_1 \) has at least one flag and this is true.

B) We have exactly two boundary components \( C_1 \) and \( C_2 \) passing through \( e \). Note that the cut of \( e \) generates a unique connected component \( C \) of the boundary. This case divides in two further possibilities:
- The move is a displacement within a sector \( C_i \): there is no difficulty to see that the operations commute in this case.
- The move is a jump. Two further cases must be discussed. Either the jump is from another boundary component \( C_0 \) to \( C_i, i = 1, 2 \), then this case is again easily solved or the jump occurs from the component \( C_1 \) to the component \( C_2 \) (without loss of generality). Then, if we cut first \( e \), and perform the same move, one realizes that this move is simply a displacement within \( C \).

So far, we checked the case where the jump operation was defined by adding a flag to the boundary connected components passing through \( e \). The proof for the converse case when these components loose a flag after flag jump can be done in the totally symmetric way.

Let \([G] \lor e\) be the set obtained by cutting \( e \) in all elements of \([G]\), \([G] - e\) the set obtained by deleting \( e \) in all elements of \([G]\) and \([G]/e\) the set obtained by contracting \( e \) in all elements of \([G]\).

We have:
- \([G \lor e] \supset [G] \lor e\) and \([G - e] \supset [G] - e\).
- If \( e \) is not a self-loop, \([G/e] = [G]/e\).

It might happen that \([G \lor e] \subset [G] \lor e\) and \([G - e] \subset [G] - e\). Thus it is not clear that \([G \lor e] \lor e\) and \([G] - e\) correspond to some equivalence classes of some graphs.

**Lemma 4.** For two flag-equivalent ribbon graphs, \( G \) and \( G' \), \( R(G) = R(G') \) with \( R \) the polynomial defined in (10).

**Proof.** The proof of this lemma uses Lemma 3. The number of monomials in the expansion of \( R(G) \) or \( R(G') \) is the same since \( G \) and \( G' \) have exactly the same set of edges. Each monomial of \( R(G) \) is obtained from the contribution of a spanning subgraph \( A \in G \). Since \( A \) is obtained by cutting a subset \( E' \) of edges in \( G \), we choose also the spanning subgraph \( A' \in G' \) obtained by cutting the same subset of edges in \( G' \). Applying successively Lemma 3 to all elements of \( E' \), \( A \) and \( A' \) are flag-equivalent. Then, the monomial associated with \( A \) in \( R(G) \) is equal to the one associated with \( A' \) in \( R(G') \). This achieves the proof.

We are now ready to define the polynomial \( R \) on flag-equivalence classes.

**Definition 16** (Polynomial for flag-equivalence classes). Let \( G(V, E, f) \) be a ribbon graph with flags and \( [G] \) be its flag-equivalence class. We define the polynomial of \( [G] \) to be

\[
R_{[G]} = R_G.
\]

(36)

The following statement is trivial.

**Proposition 2.** Let \( G \) be a ribbon graph with flags, \( [G] \) its flag-equivalence class and \( e \) one of its edges. The following relations hold \( R_{[G \lor e]} = R_{[G] \lor e} \) and \( R_{[G/c]} = R_{[G]/c} \).

**Corollary 2** (Contraction and cut on BR polynomial). Let \( G(V, E, f) \) be a ribbon graph with flags and \( [G] \) be its flag-equivalence class. Then, for a regular edge \( e \),

\[
R_{[G]} = R_{[G \lor e]} + R_{[G]/e},
\]

(37)

for a bridge \( e \), we have

\[
R_{[G]} = (X - 1)R_{[G \lor e]} + R_{[G]/e},
\]

(38)

for a trivial twisted self-loop \( e \), the following holds

\[
R_{[G]} = R_{[G \lor e]} + (Y - 1)ZW R_{[G]/e},
\]

(39)

whereas for a trivial untwisted self-loop \( e \), we have

\[
R_{[G]} = R_{[G \lor e]} + (Y - 1)R_{[G]/e}.
\]

(40)

**Proof.** The proof of this theorem is immediate using Theorem 3 and Proposition 2.
The polynomial (36) is also universal and the proof of this claim can be achieved in the same way as done for Theorem 4. Consider the following expression:

\[ R_{i,j,k,l,m}(\mathcal{G}) := R_{i,j,k,l,m}(\mathcal{G}) \]  

where \( R_{i,j,k,l,m} \) keeps its meaning of (26).

Consider \( \mathcal{G} \) the set of flag-equivalence classes of isomorphism classes of connected ribbon graphs with flags. This means that we have \( \mathcal{G} = (\mathcal{G}^*/\sim_F) \). Classes of chord diagrams under flag-equivalence relation are naturally well defined. Then the following statement holds.

**Theorem 5 (Universality of \( R \) on classes).** Let \( R \) be a commutative ring and \( x \in R \). If a function \( \phi: \mathcal{G} \rightarrow R \) satisfies

\[
\phi((\mathcal{G} \vee e) + \phi((\mathcal{G} / e)) \quad \text{if } e \text{ is regular,}
\]

\[
(x - 1)\phi((\mathcal{G} \vee e) + \phi((\mathcal{G} / e)) \quad \text{if } e \text{ is a bridge.}
\]

Then there are coefficients \( \lambda_{ijklm} \in R \), with \( i \geq 0, 0 \leq k \leq i + 1, l \geq 0, 0 \leq m \leq 1 \) and \( 0 \leq j \leq i + 1 \) such that

\[ \phi((\mathcal{G})) = \sum_{i,j,k,l,m} \lambda_{ijklm} R_{ijklm}(x). \]  

**Proof.** We simply define the canonical diagram \( D_{i,j,k,l,m} \) to be \([D_{i,j,k,l,m}]\) the flag-equivalence class of the canonical diagram \( D_{i,j,k,l,m} \). Then, we no longer need to track the partition of flags in isolated chords.

As in the proof of Theorem 4 if \( D_1, D'_1, D_2 \) and \( D'_2 \) are signed chord diagrams related as in Figures 9 or 10

\[ \phi(D_1) - \phi(D'_1) = \phi(D_2) - \phi(D'_2), \]  

then

\[ \phi([D_1]) - \phi([D'_1]) = \phi([D_2]) - \phi([D'_2]). \]

Following step by step the proof of Theorem 4 one proves the existence of the coefficients \( \lambda_{ijklm} \) so that (43) holds on the \( D_{i,j,k,l,m} \) and therefore on all chord diagrams with \( i \) chords. The rest of the proof is similar to what was done for Theorem 4.

\[ \square \]

It is natural to find the restricted polynomial \( R' \) over classes of flag-equivalent ribbon graphs and to show its universality. Several other interesting developments can be now undertaken from the polynomial invariants treated in this paper. For instance, the polynomial \( R \) does not satisfy the ordinary factorization property under the one-point-joint operation (see Proposition 1). Therefore, finding a recipe theorem in the sense of 10 becomes a nontrivial task for ribbon graphs with flags. This certainly deserves to be investigated. Furthermore, significant progresses around matroids 7 and Hopf algebra techniques 8 applied to the Tutte polynomial have been recently highlighted. These studies should find as well an extension for the present types of invariants. Finally, combining some ideas of this work and Hopf algebra calculations 13, another important investigation would be to find a universality theorem for polynomial invariants over stranded graphs 2 extending ribbons with with flags.

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