A generalization of the Bollobás set pairs inequality

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Submitted: June 7, 2020; Accepted: June 7, 2021; Published: Jul 2, 2021  
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Abstract

The Bollobás set pairs inequality is a fundamental result in extremal set theory with many applications. In this paper, for $n \geq k \geq t \geq 2$, we consider a collection of $k$ families $A_i : 1 \leq i \leq k$ where $A_i = \{A_{i,j} \subset [n] : j \in [n]\}$ so that $A_1 \cup A_2 \cup \cdots \cup A_k \neq \emptyset$ if and only if there are at least $t$ distinct indices $i_1, i_2, \ldots, i_k$. Via a natural connection to a hypergraph covering problem, we give bounds on the maximum size $\beta_k,t(n)$ of the families with ground set $[n]$.

Mathematics Subject Classifications: 05D05, 05D40, 05C65

1 Introduction

A central topic of study in extremal set theory is the maximum size of a family of subsets of an $n$-element set subject to restrictions on their intersections. Classical theorems in the area are discussed in Bollobás [2]. In this paper, we generalize one such theorem, known as the Bollobás set pairs inequality or two families theorem [3]:

**Theorem 1.** (Bollobás) Let $A = \{A_1, A_2, \ldots, A_m\}$ and $B = \{B_1, B_2, \ldots, B_m\}$ be families of finite sets, such that $A_i \cap B_j \neq \emptyset$ if and only if $i, j \in [m]$ are distinct. Then

$$\sum_{i=1}^{m} \left(\frac{|A_i \cup B_i|}{|A_i|}\right)^{-1} \leq 1.$$  

(1)

For convenience, we refer to a pair of families $A$ and $B$ satisfying the conditions of Theorem 1 as a *Bollobás set pair*. The inequality above is tight, as we may take the pairs $(A_i, B_i)$ to be distinct partitions of a set of size $a + b$ with $|A_i| = a$ and $|B_i| = b$ for $1 \leq i \leq \binom{a+b}{a}$.

*Supported by NSF award DMS-1800332.
The latter inequality was proved for \( a = 2 \) by Erdős, Hajnal and Moon [5], and in general has a number of different proofs [11, 12, 14, 17, 18]. A geometric version was proved by Lovász [17, 18], who showed that if \( A_1, A_2, \ldots, A_m \) and \( B_1, B_2, \ldots, B_m \) are respectively \( a \)-dimensional and \( b \)-dimensional subspaces of a linear space and \( \dim(A_i \cap B_j) = 0 \) if and only if \( i, j \in [m] \) are distinct, then \( m \leq \binom{n+a}{a} \).

1.1 Main Theorem

Theorem 1 has been generalized in a number of different directions in the literature [6, 9, 13, 16, 21, 24]. In this paper, we give a generalization of Theorem 1 from the case of two families to \( k \geq 2 \) families of set with conditions on the \( k \)-wise intersections. For \( 2 \leq t \leq k \), a Bollobás \((k, t)\)-tuple is a sequence \((A_1, A_2, \ldots, A_k)\) of set families \( A_j = \{A_{j,i} : 1 \leq i \leq m\} \) where \( \bigcap_{j=1}^{k} A_{j,i} \neq \emptyset \) if and only if at least \( t \) of the indices \( i_1, i_2, \ldots, i_k \) are distinct. We refer to \( m \) as the size of the Bollobás \((k, t)\)-tuple. Let \([m]_{(t)}\) denote the set of sequences of \( t \) distinct elements of \([m]\) and fix a surjection \( \phi : [k] \to [t] \).

For \( \sigma \in [m]_{(t-1)} \), set \( \sigma(t) = \sigma(1) \) and define \( A_{1,\sigma}(\phi) = \bigcap_{j=\phi(t)=1} A_{j,\sigma(1)} \) and, for \( 2 \leq j \leq t \), we define

\[
A_{j,\sigma}(\phi) = \bigcap_{h: \phi(h) = j} A_{h,\sigma(j)} \setminus \bigcup_{h=1}^{j-1} A_{h,\sigma}(\phi).
\]

Using this notation, we generalize (1) as follows:

**Theorem 2.** Let \( k \geq t \geq 2 \) and \( m \geq t \), let \( \phi : [k] \to [t] \) be a surjection, and let \((A_1, A_2, \ldots, A_k)\) be a Bollobás \((k, t)\)-tuple of size \( m \). Then

\[
\sum_{\sigma \in [m]_{(t-1)}} \left( |A_{1,\sigma}(\phi) \cup A_{2,\sigma}(\phi) \cup \cdots \cup A_{t,\sigma}(\phi)| \right)^{-1} \leq 1.
\]

We show in Section 2.1 that this inequality is tight for all \( k \geq t = 2 \), but do not have an example to show that this inequality is tight for any \( t > 2 \).

For \( n \geq k \geq t \geq 2 \), let \( \beta_{k,t}(n) \) denote the maximum \( m \) such that there exists a Bollobás \((k, t)\)-tuple of size \( m \) consisting of subsets of \([n]\). Then (1) gives \( \beta_{2,t}(n) \leq \binom{n}{\lfloor n/2 \rfloor} \) which is tight for all \( n \geq 2 \). Letting \( H(q) = -q \log_2 q - (1-q) \log_2 (1-q) \) denote the standard binary entropy function, we prove the following theorem:

**Theorem 3.** For \( k \geq 3 \) and large enough \( n \),

\[
\frac{1}{k} \leq \frac{\log_2 \beta_{k,2}(n)}{n} \leq H \left( \frac{1}{k} \right) \leq \frac{\log_2 (ke)}{k}.
\]

For \( k \geq t \geq 3 \) and large enough \( n \),

\[
\frac{\log_2 e}{(k,t-1)} \leq \frac{\log_2 \beta_{k,t}(n)}{n} \leq \frac{2}{(k,t-1)}.
\]
This determines $\log_2 \beta_{k,2}(n)$ up to a factor of order $\log_2 k$ and $\log_2 \beta_{k,t}(n)$ up to a factor of order $t^3$. We leave it as an open problem to determine the asymptotic value of $(\log_2 \beta_{k,t}(n))/n$ as $n \to \infty$ for any $k \geq 3$ and $t \geq 2$. A natural source for lower bounds on $\beta_{k,t}(n)$ comes from the probabilistic method – see the random constructions in Section 3.1 which establish the lower bounds in Theorem 3. To prove Theorem 3, we use a natural connection to hypergraph covering problems.

1.2 Covering hypergraphs

Theorem 1 has a wide variety of applications, from saturation problems [3, 19] to covering problems for graphs [11, 20], complexity of 0-1 matrices [23], geometric problems [1], counting cross-intersecting families [7], crosscuts and transversals of hypergraphs [24, 25, 26], hypergraph entropy [15, 22], and perfect hashing [8, 10]. In this section, we give an application of our main results to hypergraph covering problems. For a $k$-uniform hypergraph $H$, let $f(H)$ denote the minimum number of complete $k$-partite $k$-uniform hypergraphs whose union is $H$. In the case of graph covering, a simple connection to the Bollobás set pairs inequality (1) may be described as follows. Let $K_{n,n}\setminus M$ denote the complement of a perfect matching $M = \{x_iy_i: 1 \leq i \leq n\}$ in the complete bipartite graph $K_{n,n}$ with parts $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. If $H_1, H_2, \ldots, H_m$ are complete bipartite graphs in a minimum covering of $K_{n,n}\setminus M$, then let $A_i = \{j: x_i \in V(H_j)\}$ and $B_i = \{j: y_i \in V(H_j)\}$. Setting $A = \{A_i\}_{i \in [m]}$ and $B = \{B_i\}_{i \in [m]}$, it is straightforward to check that $(A, B)$ is a Bollobás set pair, and Theorem 1 applies to give

$$f(K_{n,n}\setminus M) = \min\{m: \left(\frac{m}{\lceil m/2 \rceil}\right) \geq n\}. \quad (5)$$

In a similar way, Theorem 2 applies to covering complete $k$-partite $k$-uniform hypergraphs. Let $K_{n,n,\ldots,n}$ denote the complete $k$-partite $k$-uniform hypergraph with parts $X_i = \{x_{ij}: j \in [n]\}$ for $i \in [k]$. Let $H_{k,t}(n)$ denote the subhypergraph consisting of hyperedges $\{x_{i_1,j_1}, x_{i_2,j_2}, \ldots, x_{i_k,j_k}\}$ such that at least $t$ of the indices $i_1, i_2, \ldots, i_k$ are distinct, and set $f_{k,t}(n) = f(H_{k,t}(n))$. Then there is a one-to-one correspondence between Bollobás $(k,t)$-tuples of subsets of $[m]$ and coverings of $H_{k,t}(n)$ with $m$ complete $k$-partite $k$-graphs. We let $\beta_{k,t}(m)$ be the maximum size of a Bollobás $(k,t)$-tuple of subsets of $[m]$, so that

$$f_{k,t}(n) = \min\{m: \beta_{k,t}(m) \geq n\}. \quad (6)$$

This correspondence together with Theorem 2 will be exploited to prove

$$f_{k,2}(n) \geq \min\{m: \left(\frac{m}{\lceil m/k \rceil}\right) \geq n\} \quad (7)$$

which is partly an analog of (5). More generally, we prove the following theorem:

**Theorem 4.** For $k \geq 3$ and large enough $n$,

$$\frac{k}{\log_2(ke)} \leq 1 \frac{1}{H(\frac{1}{k})} \leq f_{k,2}(n) \leq \frac{k}{\log_2 n} \leq k. \quad (8)$$
For \( k \geq t \geq 3 \) and large enough \( n \),

\[
\binom{k}{t-1} \frac{(t-1)^{t-3}}{2} \leq \frac{f_{k,t}(n)}{\log_2 n} \leq \frac{(t+1)t^{-1}}{\log_2 e} \binom{k}{t-1}.
\]  

(9)

The bounds on \( \beta_{k,t}(n) \) in Theorem 3 follow immediately from this theorem and (6). Equation (9) gives the order of magnitude for each \( t \geq 3 \) as \( k \to \infty \), but for \( t = 2 \), Equation (8) has a gap of order \( \log_2 k \). From (7), we obtain \( \beta_{k,2}(n) \leq \binom{n}{\lfloor n/k \rfloor} \).

It is perhaps unsurprising that the asymptotic value of \( f_{k,t}(n)/\log_2 n \) as \( n \to \infty \) is not known for any \( k > 2 \), since a limiting value of \( f(k^t)/\log_2 n \) is not known for any \( k > 2 \) – see Körner and Marston [15] and Guruswami and Riazanov [10].

1.3 Organization and notation

Given a subset \( A \subset [n] \), let \( A^c := [n] \setminus A \) be the complement of \( A \) in \([n]\). For positive integers \( k \leq n \), let \( (n)_{(k)} = (n)(n-1)\cdots(n-k+1) \) denote the falling factorial. This paper is organized as follows. In Section 2, we prove Theorem 2. In Section 2.1, we construct a Bollobás \((k,2)\)-tuple which achieves equality in Theorem 2 and in Section 2.2, we construct a Bollobás \((k,2)\)-tuple which gives the lower bound in Equation (3). The upper bound on \( f_{k,t}(n) \) in Theorem 4 comes from a probabilistic construction in Section 3.1, and the proof of the lower bound on \( f_{k,t}(n) \) is given in Section 3.3; we prove (7) in Section 3.2.

2 Proof of Theorem 2

Given a Bollobás set \((k,t)\)-tuple \((A_1,\ldots,A_k)\) with \( A_j = \{A_{j,i} : 1 \leq i \leq m\} \) and a surjection \( \phi : [k] \to [t] \), consider \( A_{\ell,\phi} : 1 \leq \ell \leq t \) where \( A_{\ell,\phi} = \{A_{\ell,i} : 1 \leq i \leq m\} \) and

\[
A_{\ell,\phi} = \bigcap_{h : \phi(h) = \ell} A_{h,i}.
\]

It follows that \((A_{1,\phi},\ldots,A_{t,\phi})\) is a Bollobás set \((t,t)\)-tuple and hence it suffices to prove Theorem 2 in the case where \( t = k \). In this setting, surjections \( \phi : [k] \to [k] \) simply permute the \( k \) families and as such we suppress the notation of \( \phi \) for the remainder of this section. One of the proofs of Theorem 1, given a Bollobás set pair, defines a collection of chains \( C_i \) for \( i \in [m] \) and shows that these chains are necessarily disjoint. Similarly, given a Bollobás set \((k,k)\)-tuple, we will define a collection of chains \( C_\sigma \) for every ordered collection \( \sigma \) of \((k-1)\) distinct indices of \([m]\) and show these chains are pairwise disjoint.

Let \((A_1,\ldots,A_k)\) with \( A_j = \{A_{j,i} : 1 \leq i \leq m\} \) be a Bollobás set \((k,k)\)-tuple, and set

\[
X = \bigcup_{i=1}^m (A_{1,i} \cup A_{2,i} \cup \cdots \cup A_{k,i})
\]
with $|X| = n$. For $\sigma \in [m]_{(k-1)}$, define a subset $C_\sigma$ of permutations $\pi : X \to [n]$ by

$$C_\sigma := \left\{ \pi : X \to [n] : \max_{x \in A_1, \sigma} \pi(x) < \min_{y \in A_2, \sigma} \pi(y) \leq \max_{y \in A_2, \sigma} \pi(y) < \cdots < \min_{z \in A_k, \sigma} \pi(z) \right\}.$$

Letting $U_\sigma := A_{1,\sigma} \cup \cdots \cup A_{k,\sigma}$, elementary counting methods give

$$|C_\sigma| = \frac{n}{|U_\sigma|} |A_{1,\sigma}|! \cdots |A_{k,\sigma}|!(n - |U_\sigma|)! = n! \cdot \left(\frac{|U_\sigma|}{|A_{1,\sigma}|! \cdots |A_{k,\sigma}|!}\right)^{-1}. \quad (10)$$

We will now prove a lemma which states that $\{C_\sigma\}_{\sigma \in [m]_{(k-1)}}$ forms a disjoint collection of a permutations. The general proof only works for $k \geq 4$, so we first consider $k = 3$.

**Lemma 5.** If $\sigma_1, \sigma_2 \in [m]_{(2)}$ are distinct, then $C_{\sigma_1} \cap C_{\sigma_2} = \emptyset$.

**Proof.** Seeking a contradiction, suppose there exists $\pi \in C_{\sigma_1} \cap C_{\sigma_2}$. After relabeling, it suffices to consider the following five cases.

1. $\sigma_1 = \{1, 3\}$ and $\sigma_2 = \{2, 4\}$
2. $\sigma_1 = \{1, 3\}$ and $\sigma_2 = \{2, 3\}$
3. $\sigma_1 = \{1, 2\}$ and $\sigma_2 = \{1, 3\}$
4. $\sigma_1 = \{1, 2\}$ and $\sigma_2 = \{2, 3\}$
5. $\sigma_1 = \{1, 2\}$ and $\sigma_2 = \{3, 1\}$

In case (1), without loss of generality, $\max\{\pi(x) : x \in A_{1,1}\} \leq \max\{\pi(x) : x \in A_{1,2}\}$ and thus $\pi \in C_{\sigma_2}$ yields

$$\max_{x \in A_{1,1}} \pi(x) \leq \max_{x \in A_{1,2}} \pi(x) < \min_{y \in A_{2,4} \setminus A_{1,2}} \pi(y).$$

Then as $A_{1,1} \cap A_{2,4} \cap A_{3,2} \neq \emptyset$, there exists $w \in A_{1,1} \cap A_{2,4} \cap A_{3,2}$. It follows that $w \notin A_{1,2}$ since if $w \in A_{1,2}$, then $w \in A_{1,2} \cap A_{2,4} \cap A_{3,2} \neq \emptyset$; a contradiction. But this yields a contradiction as

$$\pi(w) \leq \max_{x \in A_{1,1}} \pi(x) \leq \max_{x \in A_{1,2}} \pi(x) < \min_{y \in A_{2,4} \setminus A_{1,2}} \pi(y) \leq \pi(w).$$

In case (2), without loss of generality, $\max\{\pi(x) : x \in A_{1,1}\} \leq \max\{\pi(x) : x \in A_{1,2}\}$ and we recover a similar contradiction as case (1) by noting that there exists $w \in A_{1,1} \cap A_{2,3} \cap A_{3,2}$ with $w \notin A_{1,2}$.

In case (3) we may assume $\max\{\pi(x) : x \in A_{2,2} \setminus A_{1,1}\} \leq \max\{\pi(x) : x \in A_{2,3} \setminus A_{1,1}\}$ and $\pi \in C_{1,3}$ yields $\max\{\pi(x) : x \in A_{2,3} \setminus A_{1,1}\} < \min\{\pi(x) : x \in A_{3,1} \setminus (A_{1,1} \cup A_{2,3})\}$. Thus

$$\max\{\pi(x) : x \in A_{2,2} \setminus A_{1,1}\} < \min\{\pi(x) : x \in A_{3,1} \setminus (A_{1,1} \cup A_{2,3})\}$$

and there exists $w \in A_{1,3} \cap A_{2,2} \cap A_{3,1}$ with $w \notin A_{1,1}$ and $w \notin A_{2,3}$. It follows that $\pi(w) < \pi(w)$, a contradiction.

In case (4), if $\max\{\pi(x) : x \in A_{1,1}\} \leq \max\{\pi(x) : x \in A_{1,2}\}$, then using $w \in A_{1,1} \cap A_{2,3} \cap A_{3,2}$ and noting $w \notin A_{1,2}$, we get a contradiction. Thus, we may assume otherwise and $\pi \in C_{1,2}$ gives

$$\max_{x \in A_{1,2}} \pi(x) < \max_{x \in A_{1,1}} \pi(x) < \min_{z \in A_{3,1} \setminus (A_{1,1} \cup A_{2,2})} \pi(z).$$
This is a contradiction as there exists \( w \in A_{1,2} \cap A_{2,3} \cap A_{3,1} \) with \( w \notin A_{1,1} \) and \( w \notin A_{2,2} \).

In case (5), if \( \max\{\pi(x) : x \in A_{1,1}\} \leq \max\{\pi(x) : x \in A_{1,3}\} \), then we may proceed as in the latter part of case (4) using \( w \in A_{1,1} \cap A_{2,2} \cap A_{3,3} \) and \( w \notin A_{2,1} \) and \( w \notin A_{1,3} \) to get a contradiction. Otherwise, proceeding as in case (1) and noting there exists \( w \in A_{1,3} \cap A_{2,2} \cap A_{3,1} \), but \( w \notin A_{1,1} \) yields a contradiction. \( \square \)

A similar argument yields the analog of Lemma 5 to the case where \( k \geq 4 \).

**Lemma 6.** Let \( k \geq 4 \). If \( \sigma_1, \sigma_2 \in [m]_{(k-1)} \) are distinct, then \( C_{\sigma_1} \cap C_{\sigma_2} = \emptyset \).

**Proof.** Since \( \sigma_1, \sigma_2 \in [m]_{(k-1)} \) are distinct, there exists minimal \( h \in [k-1] \) so that \( \sigma_1(h) \neq \sigma_2(h) \). Seeking a contradiction, suppose there exists a \( \pi \in C_{\sigma_1} \cap C_{\sigma_2} \). Without loss of generality,

\[
\max\{\pi(x) : x \in A_{h,\sigma_1}\} \leq \max\{\pi(x) : x \in A_{h,\sigma_2}\} < \min\{\pi(z) : z \in A_{k,\sigma_2}\}.
\]

Now, consider a bijection \( \tau : [k-1] \setminus \{h\} \to [k-1] \setminus \{1\} \) which has no fixed points. As in Lemma 5, we want to show that there exists a \( w \in A_{h,\sigma_1} \cap A_{k,\sigma_2} \) and consider two separate cases.

First, suppose that \( \sigma_1(h) \neq \sigma_2([k-1]) \). As \( |\{\sigma_1(h), \sigma_2(1), \ldots, \sigma_2(k-1)\}| = k \), there exists

\[
w \in A_{h,\sigma_1(h)} \cap A_{k,\sigma_2(1)} \cap \bigcap_{l \in [k-1] \setminus \{h\}} A_{l,\sigma_2(\tau(l))}.
\]

Next, suppose that \( \sigma_1(h) = \sigma_2(x) \) for some \( x \). We now claim that \( x \neq 1 \). If \( h = 1 \), then this is trivial. If \( h > 1 \), then \( \sigma_1(1) = \sigma_2(1) \), so \( \sigma_1(h) \neq \sigma_2(1) \) since \( \sigma_1(1) \neq \sigma_1(1) \).

For \( \tau \) as above, there exists \( y \in [k-1] \setminus \{h\} \) so that \( \tau(y) = x \). Taking \( \gamma \) distinct from \( \{\sigma_2(1), \ldots, \sigma_2(k-1)\} \setminus \{\sigma_2(x)\} \), \( |\{\sigma_1(h), \gamma, \sigma_2(1), \ldots, \sigma_2(k-1)\} \setminus \{\sigma_2(x)\}| = k \) and hence there exists

\[
w \in A_{h,\sigma_1(h)} \cap A_{k,\sigma_2(1)} \cap A_{y,\gamma} \cap \bigcap_{l \in [k-1] \setminus \{y,h\}} A_{l,\sigma_2(\tau(l))}.
\]

By construction, \( w \in A_{k,\sigma_1(h)} \cap A_{k,\sigma_2(1)} \). Suppose there exists a \( t \in [k-1] \setminus \{h\} \) so that \( w \in A_{t,\sigma_2(\tau(t))} \). As \( \tau \) has no fixed points, replacing the set in the \( k \)-wise intersection corresponding to \( A_t \) with \( A_{t,\sigma_2(\tau(t))} \) in either (11) or (12), \( w \) is an element of this new \( k \)-wise intersection with \( (k-1) \) distinct indices; a contradiction. If \( w \in A_{h,\sigma_2(h)} \), then we may similarly replace \( A_{h,\sigma_1(h)} \) with \( A_{h,\sigma_2(h)} \) in the \( k \)-wise intersection in either (11) or (12) to get a contradiction. Thus, \( w \notin A_{1,\sigma_2(1)} \cup \cdots \cup A_{k-1,\sigma_2(k-1)} \) and hence \( w \in A_{h,\sigma_1} \cap A_{k,\sigma_2} \) so that \( \pi(w) < \pi(w) \); a contradiction. \( \square \)

Using Equation (10), Lemma 5, and Lemma 6, we are now able to prove Theorem 2 in the case where \( t = k \). There are \( n! \) total permutations, and Lemma 5 and Lemma 6 yield that each of which appears in at most one of the sets \( C_{\sigma} \) for \( \sigma \in [m]_{(k-1)} \). Hence, using \( |C_{\sigma}| \) in Equation (10),

\[
\sum_{\sigma \in [m]_{(k-1)}} |C_{\sigma}| = \sum_{\sigma \in [m]_{(k-1)}} n! \cdot \left( \left| A_{1,\sigma} \cup \cdots \cup A_{k,\sigma} \right| \right)^{-1} \leq n!
\]

and thus the result follows by dividing through by \( n! \).
\section{2.1 Sharpness of Theorem 2}

We give a simple construction establishing the sharpness of Theorem 2 for \( k \geq t = 2 \). Let \( n \geq 4k \) and using addition modulo \( n \), define \( A_{1,i} = \{ i \}^c \), \( A_{j,i} = \{ i - (j - 1), i + (j - 1) \}^c \) for \( j \in [2,k - 1] \), and \( A_{k,i} = \{ i - k + 2, i - k + 3, \ldots, i + k - 2 \} \). Letting \( \mathcal{A}_j = \{ A_{j,i} \}_{i \in [n]} \) for all \( j \in [k] \), we will show \( (A_1, \ldots, A_k) \) is a Bollobás \((k,2)\)-tuple. Since \( |A_{1,i}| = n - 1 \) and \( |A_{2,i} \cap \cdots \cap A_{k,i}| = 1 \), Theorem 2 with \( t = 2 \) and surjection \( \phi : [k] \to [2] \) with \( \phi(1) = 1 \) and \( \phi(i) = 2 \) for \( i \neq 1 \) gives

\[
1 \geq \sum_{i=1}^{n} \left( \frac{|A_{1,i}| + |A_{2,i} \cap \cdots \cap A_{k,i}|}{|A_{1,i}|} \right)^{-1} = \sum_{i=1}^{n} \frac{1}{n} = 1.
\]

By construction, for all \( i \in [n], A_{1,i} \cap A_{2,i} \cap \cdots \cap A_{k,i} = \varnothing \). It thus suffices to show these are the only empty \( k \)-wise intersections. To this end, for \( i = (i_1, \ldots, i_{k-1}) \), define

\[
A(i) := A_{1,i_1} \cap \cdots \cap A_{k-1,i_{k-1}}.
\]

\begin{lemma}
Let \( i = (i_1, \ldots, i_{k-1}) \). If \( A(i)^c = A_{k,i_k} \), then \( i_1 = \cdots = i_k \).
\end{lemma}

\begin{proof}
We proceed by induction on \( k \) where the result is trivial when \( k = 2 \). In the case where \( k > 2, i_{k-1} - k + 2 = i_k + x \) for some \( x \) such that \(- (k - 2) \leq x \leq (k - 2) \) and thus \( i_{k-1} + (k - 2) = i_k - 1 - (k - 2) + (2k - 2) = i_k + x + (2k - 4) \).

Next, there is a \( y \) such that \(- (k - 2) \leq y \leq (k - 2) \) with \( i_{k-1} + (k - 2) = i_k + y \), and since \( n \geq 4k, x + 2k - 4 = y \) with equality over \( \mathbb{Z} \) and moreover \( i_{k-1} + (k - 2) = i_k + (k - 2) \) over \( \mathbb{Z} \) and hence \( i_k = i_{k-1} \). Removing these elements from each set, the result then follows by induction.
\end{proof}

If \( A_{1,i_1} \cap \cdots \cap A_{k,i_k} = \varnothing \), then as \( A(i) = A_{1,i_1} \cap A_{2,i_2} \cap \cdots \cap A_{k-1,i_{k-1}} \),

\[
\varnothing = A_{1,i_1} \cap A_{2,i_2} \cap \cdots \cap A_{k-1,i_{k-1}} \cap A_{k,i_k} = A(i) \cap A_{k,i_k}.
\]

The result follows by noting \( |A(i)| \geq n - (2k - 3) \), \( |A_{k,i_k}| = 2k - 3 \), and using Lemma 7.

\section{2.2 An Explicit Construction}

Let \( k \geq 3 \). An explicit construction of a Bollobás \((k,2)\)-tuple \((A_1, A_2, \ldots, A_k)\) where \( |A_i| = 2^n \) and each \( A_i \) consists of subsets of \( X \) for \( |X| = kn \) may be described as follows. Let \( I_j := \{ x_{j,1}, x_{j,2}, \ldots, x_{j,k} \} \) and consider \( X = I_1 \sqcup \cdots \sqcup I_n \). Now, for each \( f : [n] \to [2] \) and \( j \in [k] \), define

\[
A_{j,f} := \{ x_{1,f(1)+j-1}, \ldots, x_{n,f(n)+j-1} \}^c
\]

where we work modulo \( k \) within the subscripts of \( I_j \). It is straightforward to check that \((A_1, A_2, \ldots, A_k)\) is a Bollobás \((k,2)\)-tuple. This establishes the lower bound on \( \beta_{k,2}(n) \) in Equation (3) and hence the upper bound on \( f_{k,2}(n) \) in Equation (8).
3 Proof of Theorem 4

3.1 Upper bound on $f_{k,t}(n)$

We wish to find a covering of $H_{k,t}(n)$ with complete $k$-partite $k$-graphs and assume the parts of $H_{k,t}(n)$ are $X_1, X_2, \ldots, X_k$. For each subset $T$ of $[k]$ of size $t$, consider the uniformly random coloring $\chi_T : [n] \to T$. Given such a $\chi_T$, let $Y_i \subseteq X_i$ be the vertices of color $i$ for $i \in T$; that is $Y_i := \{x_{ij} : \chi(j) = i\}$ and $Y_i = X_i$ for $i \notin T$. Denote by $H(T, \chi)$ the (random) complete $k$-partite hypergraph with parts $Y_1, Y_2, \ldots, Y_k$, and note that $H(T, \chi) \subset H_{k,t}(n)$. We place each $H(T, \chi)$ a total of $N$ times independently and randomly where

$$N = \left\lceil \frac{(t+1)^t \log_2 n}{(k-t+1) \log_2 e} \right\rceil$$

and produce $\binom{k}{t} N$ random subgraphs $H(T, \chi)$. For a set partition $\pi$ of $[k]$, let $|\pi|$ denote the number of parts in the partition and index the parts by $[|\pi|]$. Given a set partition $\pi = (P_1, P_2, \ldots, P_s)$, let

$$f(\pi, t) = \sum_{T \in [s](t)} \prod_{\pi \in T} |P_r|.$$

If $U$ is the number of edges of $H_{k,t}(n)$ not in any of these subgraphs, then

$$\mathbb{E}(U) \leq \sum_{|\pi| \geq t} n^{|\pi|} (1-t^{-t})^{Nf(\pi, t)} = \sum_{t \leq s \leq k} n^s \sum_{|\pi| = s} (1-t^{-t})^{Nf(\pi, t)}. \tag{13}$$

For sufficiently large $n$, we claim that $\mathbb{E}(U) < 1$, which implies there exists a covering of $H_{k,t}(n)$ with at most $\binom{k}{t} N$ complete $k$-partite $k$-graphs, as required. The following technical lemma states that $f$ is a decreasing function in the set partition lattice, and that $f(\pi, t)$ increases when we merge all but one element of a smaller part of $\pi$ with a larger part of $\pi$:

**Lemma 8.** Let $k \geq s \geq t \geq 2$, and let $\pi = (P_1, P_2, \ldots, P_s)$ be a partition of $[k]$.

- (i) If $\pi'$ is a refinement of $\pi$ with $|\pi'| = s + 1$, then $f(\pi, t) \leq f(\pi', t)$.
- (ii) If $|P_1| \geq |P_2| \geq 2$ and $a \in P_2$, and $\pi'$ is the partition $(P'_1, P'_2, \ldots, P'_s)$ of $[k]$ with $P'_1 = P_1 \cup P_2 \setminus \{a\}$ and $P'_2 = \{a\}$ and with $P'_i = P_i$ for $3 \leq i \leq s$, then $f(\pi', t) \leq f(\pi, t)$.

The proof of Lemma 8 part (i) is in Appendix A and the proof of (ii) is similar to the proof of (i). By Lemma 8, a set partition of $[k]$ into $s$ parts which minimizes $f(\pi, t)$ consists of one part of size $k - s + 1$ and $s - 1$ singleton parts and hence

$$\min \{f(\pi, t) : |\pi| = s\} = (k-s+1) \binom{s-1}{t-1} + \binom{s-1}{t}. \tag{14}$$

In what follows, we denote a set partition of $[k]$ into $s$ parts which minimizes $f(\pi, t)$ by $\pi_s$. 

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For $n$ large enough, and all $s$ where $t \leq s \leq k$, we will show
\[
\frac{\sum_{|\pi|=t}(1-t^{-t})^N f(\pi,t)}{\sum_{|\pi|=s}(1-t^{-t})^N f(\pi,t)} \geq n^{s-t}.
\]
Replacing the numerator with its largest term and each term in denominator with its largest term,
\[
\frac{\sum_{|\pi|=t}(1-t^{-t})^N f(\pi,t)}{\sum_{|\pi|=s}(1-t^{-t})^N f(\pi,t)} \geq \frac{(1-t^{-t})^N f(\pi,t)}{S(k,s)(1-t^{-t})^N f(\pi,t)} = \frac{1}{S(k,s)}(1-t^{-t})^{N(f(\pi,t)-f(\pi,t))}
\]
where $S(k,s)$ is the Stirling number of the second kind. Taking $n \geq S(k,s)$, we will show in Appendix B that
\[
\frac{1}{S(k,s)}(1-t^{-t})^{N(f(\pi,t)-f(\pi,t))} \geq n^{s-t}.
\]
Therefore, the index $s = t$ maximizes the right hand side of Equation (13), and hence
\[
E[U] \leq (k-t+1)(n^t) \sum_{|\pi|=t} (1-t^{-t})^N f(\pi,t) < (k-t+1)n^t S(k,t)(1-t^{-t})^{N(k-t+1)} < 1
\]
for our choice of $N$ provided $n \geq kS(k,t)$. Thus,
\[
f_{k,t}(n) \leq \binom{k}{t} \frac{(t+1)t^t \log_2 n}{(k-t+1)\log_2 e} = \frac{(t+1)t^{t-1}}{\log_2 e} \left( \frac{k}{t-1} \right) \log_2 n.
\]

### 3.2 Lower bound on $f_{k,2}(n)$

In this section, we show
\[
f_{k,2}(n) \geq \min\{m : \left\lfloor \frac{m}{\lceil m/k \rceil} \right\rfloor \geq n\}.
\]
Let $\{H_1, H_2, \ldots, H_n\}$ be a covering of $H_{k,2}(n)$ with $m = f_{k,2}(n)$ complete $k$-partite $k$-graphs. We recall $H_{k,2}(n) = K_{n,n,n,\ldots,n} \setminus M$, where $M$ is a perfect matching of $K_{n,n,\ldots,n}$. For $i \in [k]$ and $j \in [n]$, define $A_{i,j} = \{H_r : x_{ij} \in V(H_r)\}$ and $A_i = \{A_{i,j} : 1 \leq j \leq n\}$. As in (6), $(A_1, A_2, \ldots, A_k)$ is a Bollobás $(k,2)$-tuple of size $n$. For convenience, for each $i \in [k]$, let $\phi_i : [k] \to [2]$ be so that $\phi_i^{-1}(1) = \{i\}$. Taking the sum of inequality from Theorem 2 with $t = 2$ over all $i \in [k]$,
\[
\sum_{i=1}^k \sum_{j=1}^n \left( \frac{|A_{1,j}(\phi_i) \cup A_{2,j}(\phi_i)|}{|A_{1,j}(\phi_i)|} \right)^{-1} \leq k.
\]
We use this inequality to give a lower bound on $f_{k,2}(n) = m$. First we observe
\[
\sum_{r=1}^m |V(H_r)| = \sum_{j=1}^n \sum_{i=1}^k |A_{i,j}| = \sum_{j=1}^n \sum_{i=1}^k |A_{1,j}(\phi_i)|.
\]
Let $\partial H$ denote the set of $(k - 1)$-tuples of vertices contained in some edge of a hypergraph $H$. Then
\[ \sum_{r=1}^{m} |\partial H_r \cap \partial M| = \sum_{j=1}^{n} \sum_{i=1}^{k} |A_{2,j}(\phi_i)|. \] (19)

Putting the above identities together,
\[ \sum_{r=1}^{m} |V(H_r)| + \sum_{r=1}^{m} |\partial H_r \cap \partial M| = \sum_{j=1}^{n} \sum_{i=1}^{k} (|A_{1,j}(\phi_i)| + |A_{2,j}(\phi_i)|). \] (20)

We note $|\partial H_r \cap \partial M| \leq |V(H_r)|/(k - 1)$, and therefore
\[ \sum_{r=1}^{m} |\partial H_r \cap \partial M| \leq \frac{1}{k - 1} \sum_{r=1}^{m} |V(H_r)|. \] (21)

It follows that
\[ \sum_{j=1}^{n} \sum_{i=1}^{k} (|A_{1,j}(\phi_i)| + |A_{2,j}(\phi_i)|) \leq \frac{k}{k - 1} \sum_{r=1}^{m} |V(H_r)|. \] (22)

Subject to the linear inequalities (18) and (22), the left side of (17) is minimized when
\[ kn|A_{1,j}(\phi_i)| = \sum_{r=1}^{m} |V(H_r)| \quad \text{and} \quad kn(|A_{1,j}(\phi_i)| + |A_{2,j}(\phi_i)|) = (k - 1)|A_{1,j}(\phi_i)|. \]
Since $|V(H_r)| \leq (k - 1)n$ for all $r \in [m]$, (17) implies $\binom{m}{\lceil m/k \rceil} \geq n$, which gives (16). \hfill \Box

### 3.3 Lower bound on $f_{k,k}(n)$

Let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ be a minimal covering of $H_{k,k}(n)$ with complete $k$-partite $k$-graphs, so $m = f(H_{k,k}(n))$. Given a $k$-partite $k$-graph $H$, consider its 2-shadow $\partial_2(H) = \{R \subset V(H) : |R| = k - 2, R \subset e \text{ for some } e \in E\}$. Let $\partial_2(\mathcal{H}) = \bigcup_{i=1}^{m} \partial_2(H_i)$.

Given $R \in \partial_2(\mathcal{H})$ and $H_i \in \mathcal{H}$, let $H_i(R) := \{e \in \binom{V(H_i)}{2} : e \cup R \in H_i\}$ be the possibly empty link graph of the edge $R$ in the hypergraph $H_i$ and let $V(H_i(R))$ be the set of vertices in the link graph. Observe that double counting yields
\[ \sum_{R \in \partial_2(\mathcal{H})} \left( \sum_{i=1}^{m} |V(H_i(R))| \right) = \sum_{i=1}^{m} \left( \sum_{R \in \partial_2(H_i)} |V(H_i(R))| \right). \] (23)

An optimization argument yields $|\partial_2(H_i)|$ is maximized when the parts of $H_i$ are of equal or nearly equal maximal size. Since $|V(H_i(R))| \leq 2(n - k + 2)$, the right hand side of Equation (23) is bounded above by
\[ \sum_{i=1}^{m} \left( \sum_{R \in \partial_2(H_i)} |V(H_i(R))| \right) \leq m \cdot \binom{k}{2} \cdot \left( \frac{n}{k} \right)^{k-2} \cdot 2(n - k + 2). \] (24)
For a lower bound on the left hand side of Equation (23), fix $R \in \partial_2(\mathcal{H})$ and without loss of generality suppose that $R = \{x_{1,1}, \ldots, x_{k-2,k-2}\}$. Let $Y = [k-1,n]$. Let $K_{Y,Y}$ be the complete bipartite graph with two distinct copies of $Y$ and $\mathcal{M} = \{(x_{k-1,i}, x_{k,i} : i \in Y\}$ be a perfect matching in $K_{Y,Y}$. Then, $\{H_1(R), \ldots, H_m(R)\}$ forms a biclique cover of $K_{Y,Y} \setminus \mathcal{M}$. Applying the convexity result of Tarjan [23, Lemma 5],

$$\sum_{i=1}^{m} |V(H_i(R))| \geq (n-k+2) \log_2(n-k+2).$$

Noting that $|\partial_2(\mathcal{H})| = \binom{k}{2}(n)(k-2)$, the left hand side of Equation (23) is bounded below by

$$\sum_{R \in \partial_2(\mathcal{H})} \left( \sum_{i=1}^{m} |V(H_i(R))| \right) \geq \binom{k}{2}(n)(k-2)(n-k+2) \log_2(n-k+2). \quad (25)$$

Comparing the bounds from Equation (24) and Equation (25),

$$m \geq \frac{(n)(k-2) \log_2(n-k+2)}{2 \left(\frac{n}{k}\right)^{k-2}} \geq \frac{k^{k-2}}{2} \log_2 n$$

provided that $n$ is large enough.

For $t \geq 3$ and $t < k$, the lower bound on $f_{k,t}(n)$ in Theorem 4 is obtained from the lower bounds on $f_{t-1,t-1}(n-1)$ as follows: Let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ be a minimal covering of $H_{k,t}(n)$ with complete $k$-partite $k$-graphs, so $m = f(H_{k,t}(n))$. Given $T \in \binom{[k]}{k-t+1}$, define $H_T \subset H_{k,t}(n)$ by

$$H_T := \{\{x_{1,i_1}, \ldots, x_{k,i_k}\} \in H_{k,t}(n) : i_j = 1 \forall j \in T\}.$$ 

It follows that at least $f_{t-1,t-1}(n-1)$ of the complete $k$-partite $k$-graphs in $\mathcal{H}$ are needed to cover $H_T$. Moreover, for distinct $T, T' \in \binom{[k]}{k-t+1}$, the corresponding complete $k$-partite $k$-graphs from $\mathcal{H}$ are necessarily pairwise disjoint and hence

$$f_{k,t}(n) \geq \binom{k}{k-t+1} f_{t-1,t-1}(n-1) \geq \binom{k}{t-1} \frac{(t-1)^{t-3}}{2} \log_2 n$$

provided that $n$ is large enough.

### 4 Concluding remarks

- Our main theorem, Theorem 2 is tight for $t = 2$ and $k \geq 2$, as shown in Section 2.1. It would be interesting to generalize this example to $2 < t \leq k$ to determine whether Theorem 2 is tight in general. The first open case is $t = k = 3$.

- A particular case of the Bollobás set pairs inequality occurs when every set in $\mathcal{A}$ has size $a$ and every set in $\mathcal{B}$ has size $b$, and one obtains the tight bound $|\mathcal{A}| \leq \binom{a+b}{b}$. The
generalization to Bollobás \((k,t)\)-tuples for \(k \geq 3\) is equally interesting but wide open, as are potential generalizations to vector spaces – see Lovász [17, 18].

- Orlin [20] proved that the clique cover number \(cc(K_n \setminus M)\) of a complete graph \(K_n\) minus a perfect matching \(M\) is precisely \(\min\{m : 2^{\lceil m/2 \rceil} \geq n\}\). Theorem 4 yields lower bounds on the clique cover number of the complement of a perfect matching \(M\) in the complete \(k\)-uniform hypergraph \(K_n^k\):

**Corollary 9.** Let \(K_n^k \setminus M\) be the complement of a perfect matching in \(K_n^k\). Then

\[
cc(K_n^k \setminus M) \geq \frac{\log_2 \frac{n}{k}}{H(\frac{1}{k})} \geq k \log_2 \frac{n}{k}.
\]

- It would be interesting to prove an analog of Equation (16) for \(t \geq 3\). That is,

\[
f_{k,t}(n) \geq \min\{m : \left(\frac{m}{\alpha_1, \ldots, \alpha_t}\right) \geq n(t-1)\}
\]

for some optimal \(\alpha_1, \ldots, \alpha_t\). The difficulty here lies in determining effective bounds on \(|A_{i,\sigma}(\phi)|\).

**Acknowledgements**

We would like to thank the anonymous referees for their helpful comments.

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A Proof of Lemma 8(i)

Let $k \geq s \geq t \geq 2$, and let $\pi = (P_1, P_2, \ldots, P_s)$ be a partition of $[k]$. In this section, we will show that if $\pi'$ is a refinement of $\pi$ with $|\pi'| = s + 1$, then $f(\pi, t) \leq f(\pi', t)$.

Proof. Let $\pi = P_1|P_2|\cdots|P_s$ and without loss of generality, $\pi' = P_s|P_2|\cdots|P_1$. Setting $T(1) = \{ T \in [s]^{(t)} : 1 \notin T \}$ and $T'(x, y) = \{ T \in \{ x, y, 2, \ldots, s \}^{(t)} : x \notin T, y \notin T \}$, it follows that

$$\sum_{T \in T(1)} \prod_{i \in T} |P_i| = \sum_{T \in T'(x, y)} \prod_{i \in T} |P_i|.$$

Now, letting $T(1) = \{ T \in [s]^{(t)} : 1 \in T \}$ and $T'(x, y) = \{ T \in \{ x, y, 2, \ldots, s \}^{(t)} : x \in T, y \notin T \}$ and $T'(x, y) = \{ T \in \{ x, y, 2, \ldots, s \}^{(t)} : x \notin T, y \in T \}$, we see that

$$\sum_{T \in T(1)} \prod_{i \in T} |P_i| = \sum_{T \in T'(x, y)} \prod_{i \in T} |P_i| + \sum_{T \in T'(x, y)} \prod_{i \in T} |P_i|$$

since $|P_1| = |P_x| + |P_y|$. Thus letting $T'(x, y) = \{ T \in \{ x, y, 2, \ldots, s \}^{(t)} : x \in T, y \in T \},$

$$f(\pi', t) - f(\pi, t) = \sum_{T \in T'(x, y)} \prod_{i \in T} |P_i|$$

and in particular $f(\pi, t) \leq f(\pi', t)$.

B Proof of Equation (15)

Let $S(k, s)$ be the Stirling number of the second kind and $f(\pi)$ be as in Section 3. In this section we will show

$$\frac{1}{S(k, s)} (1 - t^{-t})^N f(\pi, t) \geq n^{s-t}.$$

Proof. First, we recall that

$$N = \left\lceil \frac{(t + 1)t^t \log_2 n}{(k - t + 1) \log_2 e} \right\rceil \quad \text{and} \quad f(\pi, t) = (k - s + 1) \binom{s - 1}{t - 1} + \binom{s - 1}{t}.$$

As a result, when $t \leq s < k$, a calculation yields that

$$f(\pi_{s+1}, t) - f(\pi_s, t) = (k - s) \binom{s - 1}{t - 2}. \quad (26)$$

Letting $n \geq S(k, t)$, after taking $\log_2(\cdot)$ on both sides of (15), it suffices to prove that

$$N \cdot \frac{f(\pi, t) - f(\pi_t, t)}{t^t} \left( - t \log_2 (1 - t^{-t}) \right) \geq (s - t + 1) \log_2(n). \quad (27)$$

Using the fact that $(1 - t^{-t})^t \leq e^{-1}$ and our choice of $N$, it suffices to show that

$$f(\pi, t) - f(\pi_t, t) \geq \frac{(s - t + 1)(k - t + 1)}{t + 1}. \quad (28)$$

The inequality in (28) holds for all $k \geq s > t \geq 3$ by using (26). \qed