Topological complexity of generic hyperplane complements

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Abstract. We prove that the topological complexity of (a motion planning algorithm on) the complement of generic complex essential hyperplane arrangement of \( n \) hyperplanes in an \( r \)-dimensional linear space is \( \min\{n + 1, 2r\} \).

1. Introduction

In this paper we continue the theme started in [3] - studying the topological (motion planning) complexity \( \text{TC}(M) \) of the complement \( M \) of a complex hyperplane arrangement. The number \( \text{TC}(X) \) was defined for any path-connected topological space \( X \) by M. Farber in [1, 2]. This number is of fundamental importance for the motion planning problem: \( \text{TC}(X) \) determines character of instabilities for all motion planning algorithms in \( X \).

The main result of this paper can be stated as follows:

**Theorem 1.1.** Let \( M \) be the complement of a complex central essential arrangement of \( n \) hyperplanes in the linear space \( V \) of dimension \( r > 0 \). Then \( \text{TC}(M) = \min\{n + 1, 2r\} \).

2. The motion planning problem

In this section we recall the definitions and results from [1, 2] that we will use later in this paper.

Let \( X \) be a connected topological space \( X \) that is homotopy equivalent to a CW complex. Let \( PX \) be the space of all continuous paths \( \gamma : [0,1] \to X \), equipped with the compact-open topology, and let \( \pi : PX \to X \times X \) be the map assigning the end points to a path: \( \pi(\gamma) = (\gamma(0), \gamma(1)) \). The map \( \pi \) is a fibration whose fiber is the based loop space \( \Omega X \). The topological complexity of \( X \), denoted by \( \text{TC}(X) \), is the smallest number \( k \) such that \( X \times X \) can be covered by open sets \( U_1, \ldots, U_k \), so that for every \( i = 1, \ldots, k \) there exists a continuous section \( s_i : U_i \to PX, \pi \circ s_i = 1 \).

According to [2], a motion planner in \( X \) is defined by finitely many subsets \( F_1, \ldots, F_k \subset X \times X \) and continuous maps \( s_i : F_i \to PX \), where \( i = 1, \ldots, k \), such that:

(a) the sets \( F_1, \ldots, F_k \) are pairwise disjoint (i.e., \( F_i \cap F_j = \emptyset, i \neq j \)), and cover \( X \times X \);

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Now we state Hattori's theorem [4], Theorem 5.21. Denote by $T^m$ the (compact) torus of dimension $m$ and for every $I \subset \mathbb{M}$ put

$$T^m_I = \{(z_1, \ldots, z_m) \in T^m | z_j = 1, \text{ for } j \notin I\}.$$ 

**Theorem 3.1.** Let $n > r > 1$. For any general position arrangement of $n - 1$ affine hyperplanes in $(r - 1)$-dimensional space its complement has the homotopy type of $M_0$ where $M_0$ is the skeleton of dimension $r - 1$ of the canonical CW-complex of $T^{n-1}$, i.e.,

$$M_0 = \bigcup_{|I|=r-1} T^{n-1}_I.$$

**Corollary 3.2.** For any generic arrangement of $n$ linear hyperplanes in $r$-dimensional space its complement $M$ has the homotopy type of $M_0$ where

$$M_0 = S^1 \times \bigcup_{|I|=r-1} T^{n-1}_I.$$

**Proof.** For $n > r$ it follows immediately from Hattori’s theorem. For $n = r$ (in particular for $r = 1$) the arrangement consists of all coordinate hyperplanes whence $M \approx (\mathbb{C}^*)^r \approx T^r = M_0$. \hfill \Box

The property (i) of $\mathbf{TC}(X)$ allows us to focus in the rest of the paper on calculating $\mathbf{TC}(M_0)$. We will always denote by $n$ the number of hyperplanes in the generic central arrangement $A$ we will consider and by $r$ the dimension of the ambivalent space $V$.

### 4. Low bound

In this section we use the definition of $M_0$ to describe $H^*(M_0; \mathbb{C})$ and to exhibit a low bound on $\mathbf{TC}(M_0)$ using the property (iv).

Denote by $E(n) = \bigoplus_{i=0}^n E(n)_i$ the exterior algebra over $\mathbb{C}$ with $n$ generators of degree one. Also for every $k, 0 \leq k \leq n$, put $E(n)^k = E(n)/\bigoplus_{i>k} E(n)_i$ (a truncated exterior algebra).

From the description of $M_0$ in Corollary 3.2 we have

$$H^*(M_0, \mathbb{C}) = E(1) \otimes E(n-1)^{r-1}$$

where the tensor product is taken in the category of graded algebras. In particular we have the following lemma.

Denote by $e_0$ a generator of $H^*(S^1) = E(1)$ and by $e_1, \ldots, e_{n-1}$ the generators of $H^*(M_0) = E(n-1)^{r-1}$. Also for every $I = \{i_1 < i_2 < \cdots < i_k\} \subset n-1$ put $e_I = e_{i_1} \cdots e_{i_k}$.

**Lemma 4.1.** The set $\{e_0 e_I | I \subset n-1, |I| = r-1\}$ is a basis of the linear space $H^r(M_0, \mathbb{C})$.

Now we define the elements in the ideal of zero divisors of $H^*(M_0) \otimes H^*(M_0)$ corresponding to the generators. Namely put $\overline{e_i} = 1 \otimes e_i - e_i \otimes 1$ for every $i = 0, 1, \ldots, n-1$.

**Proposition 4.2.** Let $k = \min\{n-1, 2r-2\}$ and $J \subset n-1$ with $|J| = k$. Then $\pi = \overline{e_0} \prod_{i \in J} \overline{e_i} \neq 0$. 


Proof. The linear space $H^*(M_0) \otimes H^*(M_0)$ is double graded by the subspaces $H^s \otimes H^t$, $0 \leq s, t \leq r$. It suffices to prove that $(r, k + 1 - r)$-component $\pi_{r,k+1-r}$ of $\pi$ does not vanish. Clearly this component is

$$\pi_{r,k+1-r} = \sum_{I \subseteq J, |I|=r-1} \pm e_0 e_I \otimes e_{J \setminus I}.$$ 

Since $|J\setminus I| = k+1-r \leq r-1$ and $H^*(\overline{M_0}) = E(n-1)^{r-1} \subset H^*(M_0)$ all monomials $e_{J \setminus I}$ belong to a basis of $H^{k+1-r}(M_0)$. The monomials $e_0 e_I$ belong to a basis of $H^{r-1}(M_0)$ by Lemma [L.1]. Hence all the summands of $\pi_{r,k+1-r}$ belong to a basis of $H^*(M_0) \otimes H^*(M_0)$ whence $\pi_{r,k-r} \neq 0$. This completes the proof. □

Now the property (iv) of $\text{TC}(X)$ immediately implies the following.

**Corollary 4.3.**

$$\text{TC}(M) = \text{TC}(M_0) \geq \min\{n + 1, 2r\}.$$ 

5. Motion planning

In this section we prove that the upper bound for $\text{TC}(M_0)$ coincides with the low bound from the previous section.

First since $M_0 \cong \overline{M_0} \times S^1$ we have by property (iii)

$$\text{TC}(M_0) \leq \text{TC}(\overline{M_0}) + 1.$$ 

Now suppose $n + 1 \geq 2r$. Since $\dim \overline{M_0} = r - 1$ we have using property (ii) that $\text{TC}(\overline{M_0}) \leq 2r - 1$ whence

$$\text{TC}(M_0) \leq 2r = \min\{n + 1, 2r\}.$$ 

Thus we have to consider only the case $n + 1 < 2r$. To find the upper bound in this case we construct an explicit motion planning for $\overline{M_0}$ with $n$ rules.

**Theorem 5.1.** For arbitrary $r \leq n$ there exists a motion planning for $\overline{M_0}$ with $n$ rules.

Proof. First for every $J \subset \overline{n-1}$ we define the close subset $F'_j$ of $T^{n-1} \times T^{n-1}$ via

$$F'_j = \{ (u, u') | u_j = u'_j \text{ if and only if } j \in J \}$$

and put $F_J = F'_j \cap (M_0 \times \overline{M_0})$. Then we put $F_i = \bigcup_{|I|=i} F_j$ for every $i = 0, 1, \ldots, n - 1$. The sets $F_i$ are pairwise disjoint and cover $\overline{M_0} \times \overline{M_0}$ whence we can take them as the local domains of the motion planning we are constructing. Since the sets $F_j$ are also pairwise disjoint it suffices now to construct local rules on them, i.e., (continuous) sections $s_J : F_J \rightarrow \overline{M_0}$.

For that define an auxiliary function $\tau : S^1 \rightarrow [0, 1]$ by treating $S^1$ (in the rest of the proof) as the set of all complex numbers of norm 1 and putting

$$\tau(z) = \begin{cases} \frac{1}{2}(1 - \frac{|z-1|}{\sqrt{2}}) & \text{if } |z-1| \leq \sqrt{2}, \\ 0 & \text{otherwise}. \end{cases}$$ 

Notice that $\tau(1) = \frac{1}{2}$. Also for two points $z \neq z' \in S^1$, $z = \exp[\sqrt{-1}\phi]$, $z' = \exp[\sqrt{-1}\phi']$, where $0 \leq \phi, \phi' < 2\pi$, define the path $\zeta_{z, z'}$ on $S^1$ via $\zeta_{z, z'}(t) = \exp[\sqrt{-1}(t\phi + (1-t)\phi')]$ (i.e., the moving with a constant speed from $z$ to $z'$ along the natural orientation of $C$).
Now for \((u, u') = ((u_1, \ldots, u_{n-1}), (u'_1, \ldots, u'_{n-1})) \in T^{n-1} \times T^{n-1}\) we define \(s_J(t) = (s_{J,j}(t))_{j \in \mathbb{N}}\) via \(s_{J,j}(t) = u_j = u'_j\) for every \(t \in [0, 1]\) if \(j \in J\). If \(j \notin J\) we put
\[
s_{J,j}(t) = \begin{cases} 
  u_j & \text{if } 0 \leq t < \tau(u_j), \\
  \zeta_{u_j,u'_j}(\frac{t-\tau(u_j)}{1-\tau(u_j)-\tau(u'_j)}) & \text{if } \tau(u_j) \leq t \leq 1 - \tau(u'_j), \\
  u'_j & \text{if } 1 - \tau(u'_j) < t \leq 1.
\end{cases}
\]

It is clear from the definition that \(s_J\) is continuous and \(s_J(0) = u, s_J(1) = u'\). Also since \(\tau\) is continuous and \(\zeta_{z,z'}\) depends continuously on \((z, z')\) on \(S^1 \times S^1\) with the diagonal deleted we see that \(s_J\) is continuously depending on \((u, u')\) on \(F_J\). It is left to check only that \(s_J(t) \in \overline{M}_0\) for every \(t \in [0, 1]\). In other words we need to check that for every \(t\) we have \(s_{J,j}(t) = 1\) for at least \(n-r\) values of \(j\).

Suppose that \(u \in T^{n-1}_I\) and \(u' \in T^{n-1}_{I'}\), \(|I| = |I'| = r - 1\). Consider the complements \(\overline{I} = n-1 \setminus I\) and \(\overline{I'} = n-1 \setminus I'\). Put \(I_0 = \overline{I} \cap \overline{I'}\) and fix a bijection \(\phi : \overline{I} \setminus I_0 \to \overline{I'} \setminus I_0\) putting \(j' = \phi(j)\) for every \(j \in \overline{I} \setminus I_0\). Then if \(j \in I_0\) we have \(j \in J\) whence \(s_{J,j}(t) = u_j = u'_j = 1\) for every \(t\). Suppose \(j \in \overline{I} \setminus I_0\). Then \(\tau(u_j) = \tau(1) = \frac{1}{2}\) whence \(s_{J,j} = u_j = 1\) for \(t \leq \frac{1}{2}\). On the other hand, \(\tau(u'_j) = \tau(1) = \frac{1}{2}\) whence \(s_{J,j}(t) = 1\) for \(t \geq \frac{1}{2}\). Collecting this data we see that indeed for arbitrary \(t\) there are \(n-r\) values of \(j\) such that \(s_{J,j}(t) = 1\) which completes the construction of the motion planning whence also the proof.

\textbf{Corollary 5.2.} \(\text{TC}(\overline{M}_0) \leq \min\{n, 2r-1\}\) whence \(\text{TC}(M) = \text{TC}(M_0) \leq \min\{n+1, 2r\}\) and Theorem [11] follows.

In all cases where the topological complexity has been computed for hyperplane arrangement complements it coincides with the low bound given by the zero-divisors-cup-length (see property iv in section 2). This justifies the following conjecture.

\textbf{Conjecture 5.3.} For every complex central hyperplane arrangement with the complement \(M\) the topological complexity \(\text{TC}(M)\) is greater by 1 than the zero-divisors-cup-length of \(H^*(M, \mathbb{C})\).

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