Cavity-assisted energy relaxation for quantum many-body simulations

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We propose an energy relaxation mechanism whereby strongly correlated spin systems decay into their ground states. The relaxation is driven by cavity quantum electrodynamics interaction and the decay of cavity photons. It is shown that by applying broadband driving fields, strongly correlated systems can be cooled regardless of the specific details of their energy level profile. The scheme would also have significant implications in other contexts, such as adiabatic quantum computation and steady-state entanglement in dissipative systems.

Introduction.—Advances in laser cooling and trapping technologies have opened up intriguing possibilities of using atomic systems to simulate quantum many-body phenomena in strongly correlated regimes. Concerning such quantum simulation, diverse studies have been carried out both experimentally and theoretically, addressing various fundamental many-body problems. From the theoretical perspective, the properties of the ground states are among the main concerns regarding strongly correlated systems. On the other hand, studies on the experimental realizations have rather placed the main emphasis on how to mimic particular many-body model Hamiltonians, while it is often overlooked that being able to realize a particular Hamiltonian does not mean that one can prepare its energy states.

The most natural way to bridge such a logical gap would be to introduce energy relaxation in simulated strongly correlated systems. First of all, it is important to understand that in strongly correlated regimes no conventional cooling scheme provides a proper relaxation channel. The main reason is because they were devised to cool weakly correlated systems, keeping in mind that cooling individual atoms results in cooling the whole system. This does not hold in strongly correlated regimes. Moreover, while important classes of many-body systems emerge when the atomic internal degrees of freedom (DOFs), i.e., spins, are strongly correlated, cooling of such systems is yet to be considered. In weakly correlated regimes, optical pumping could be regarded as the spin counterpart of the conventional cooling of atomic motions. However, the problem again becomes elusive when the spins are strongly correlated.

Up until now, the adiabatic method is the only available way to prepare the ground state of a given strongly correlated system. The underlying idea is to prepare a known ground state in a weakly correlated regime and transfer it to that in a desired strongly correlated regime by adiabatically changing the system parameter. This method is feasible when the ground-state energy gap is finite throughout the whole process. However, this condition is difficult to achieve because such a transition (from a weakly correlated to a strongly correlated regime) is usually a quantum phase transition, at the critical point of which the energy gap vanishes in the thermodynamic limit. Moreover, this method requires some a priori knowledge about the system, while quantum simulation would be more useful when one has little information about it. As adiabatic dynamics is a slow unitary evolution, the coherence time of the system also limits its performance both during and after the preparation, whereas energy relaxation would continuously stabilize the ground state competing with decoherence.

In this paper, we introduce a cavity-assisted energy relaxation mechanism that directly cools strongly correlated atomic systems. As a concrete example, we discuss cooling of strongly correlated spin systems. Although some schemes have been proposed to prepare particular ground states, there has hitherto been no genuine idea of energy relaxation applicable to general many-body Hamiltonians whose ground states are unknown. Besides the application in quantum simulations, the importance of energy relaxation can also be found in many other contexts. For example, it has a potential to provide a more efficient route to reach the ground state for adiabatic quantum computation. Also, recalling that strongly correlated spin systems have entangled ground states, it would provide a general framework for establishing steady-state entanglement, discussed so far from different perspectives. Our scheme is particularly timely amid huge current interests in placing many-body atomic systems inside cavities with some real experiments.

Physical model.—We consider strongly correlated spin systems as can be simulated in various atomic systems, such as optical lattices, ion traps, and coupled cavities. The common feature of those quantum simulations is that atomic hyperfine ground levels represent spins, while some external DOF (atomic momentum, cavity modes, etc.) mediates their coupling. In so doing, the external DOF is effectively decoupled from the dynamics. Our concern is how to manipulate the spin DOF to lower its respective energy.

For simplicity, we present our idea with $s = \frac{1}{2}$ spin systems. Two ground levels of an atom represent spin-down and up states, as shown in Fig. 1. To begin with, let us write the Hamiltonian of the quantum simulator as
\[ H_0 = \sum_{\mu=0}^{N-1} E_{\mu} |\Psi_\mu\rangle \langle \Psi_\mu|, \]  
where \( E_{\mu} \) is the \( \mu \)th eigenenergy with \( E_\mu \leq E_\nu \) for \( \mu < \nu \), \( |\Psi_\mu\rangle \) is the \( \mu \)th eigenstate, and \( N \) is the number of spins. We assume that the ground state \( |\Psi_0\rangle \) is nondegenerate. Our aim is to figure out a process that allows cooling \( (|\Psi_\mu\rangle \rightarrow |\Psi_\nu\rangle) \) for \( \mu > \nu \) but forbids heating \( (\mu < \nu) \). First, we note that in the aforementioned simulations, spontaneous emission is not feasible as a relaxation channel for three main reasons: (i) As the spontaneous emission rate is larger than the characteristic energy scale of the simulator, the energy levels cannot be resolved due to the line broadening. (ii) The spontaneous emission destroys the atomic coherence, hence the coherence within the many-body states. (iii) As atomic excited levels are out of the spin subspace in consideration, the energy of the simulator is not clearly defined when the atom is excited. For these reasons, we need to introduce some ancillary DOF so that energy is first transferred coherently to it and then relaxed from it. Let us denote by \( |\Psi_\mu, n\rangle \) the state of the system along with the ancillary mode in the \( n \)th excited state. Our strategy is to find such a transition as \( |\Psi_\mu, n\rangle \leftrightarrow |\Psi_\mu, n+1\rangle \rightarrow |\Psi_\mu, n\rangle \) for \( \mu > \nu \). Note that the irreversible transition in the internal DOF is attained by destroying the coherence of the external DOF. It is worth comparing this with the sideband cooling that works in an opposite way. Denoting by \( |g\rangle \) and \( |e\rangle \) the atomic ground and the excited states, respectively, the sideband cooling can be represented as \( |g, n\rangle \leftrightarrow |e, n - 1\rangle \rightarrow |g, n - 1\rangle \).

It can be seen that the cavity field is best suited for the above purpose. Among various possibilities, we consider a situation where one cavity is coupled to the quantum simulator, as depicted in Fig. 1. Two transitions \( |j\rangle_2 \leftrightarrow |\bar{\nu}\rangle \) and \( |j\rangle_1 \leftrightarrow |\bar{e}\rangle \) of the \( j \)th atom are coupled, respectively, to the two orthogonally polarized cavity modes, whose annihilation operators are denoted by \( a_{1j} \) and \( a_{2j} \), with coupling rates \( g_{1j} \) and \( g_{2j} \) and detunings \( \Delta_1 \) and \( \Delta_2 \). The classical field with Rabi frequency \( \Omega_{xj} \) \((x = 1, 2)\) is also applied with detuning \( \Delta_x + \delta_x \). We take a large detuning regime where \( \Delta_x, \Delta_1 - \Delta_2 \gg g_{xj}, \Omega_{xj}, \delta_x \).

In this regime, the atomic excitation is suppressed and the conventional adiabatic elimination method can be used to yield the effective Hamiltonian (1)

\[ H_1 = H_d + H_t, \]

\[ H_d = -\epsilon_z - \epsilon_1 a_1^\dagger a_1 - \epsilon_2 a_2^\dagger a_2, \]

\[ H_t = -a_1^\dagger \Gamma_+ e^{-i\delta t} + a_2^\dagger \Gamma_- e^{i\delta t} + H.c., \]

where \( \epsilon_z = \sum_j \left( \frac{|\Omega_{xj}|^2}{\Delta_2^2} - \frac{|\Omega_{xj}|^2}{\Delta_1^2} \right) s_j^z, \epsilon_1 = \sum_j \frac{|\Omega_{xj}|^2}{\Delta_1^2} s_j^- s_j^+, \epsilon_2 = \sum_j \frac{|\Omega_{xj}|^2}{\Delta_2^2} s_j^+ s_j^-, \Gamma_+ = \sum_j \frac{g_{1j}|\Omega_{xj}|^2}{\Delta_1^2} s_j^-, \) and \( \Gamma_- = \sum_j \frac{g_{2j}|\Omega_{xj}|^2}{\Delta_2^2} s_j^+ \).

Working principle.—Let us take two of the energy eigenstates \( |\Psi_\mu\rangle \) and \( |\Psi_\nu\rangle \) of the spin system. In what follows, we adopt the convention that \( \langle \Psi_\mu|O|\Psi_\nu\rangle \) for any operator \( O \) is denoted by \( (O)_{\mu\nu} \). Our plan is to treat the Hamiltonian as a time-dependent perturbation. For this, we impose \( |E_{\mu\nu}| > (\epsilon_z)_{\mu\nu}, (\epsilon_1)_{\nu\nu}, (\epsilon_2)_{\nu\nu} \), and similarly for \( \epsilon_1, \epsilon_2, \epsilon_3, \) and \( \Gamma_+, \Gamma_- \), where \( E_{\mu\nu} = E_\mu - E_\nu \).

The Hamiltonian \( \mathbf{I} \) comprises the diagonal term \( H_d \) and the off-diagonal term \( H_t \). The former makes a small correction to the energy levels, which we take into account later. The essential part is the latter that induces transition between the energy levels. Let us denote by \( |\Psi_\mu, n_1, n_2\rangle \) the state with \( n_1 \) and \( n_2 \) photons in the \( a_1 \) and \( a_2 \) modes, respectively. If the cavity has initially no photon, the transition occurs creating a cavity photon. The transition rate is thus determined by the matrix elements \( \langle \Psi_\mu, 0, 0|H_I|\Psi_\nu, 0, 0\rangle = (\Gamma_-)_{\mu\nu} e^{i(E_{\mu\nu} + \delta_{1,2})t} \) and \( \langle \Psi_\mu, 0, 1|H_I|\Psi_\nu, 0, 0\rangle = (\Gamma_)_{\mu\nu} e^{i(E_{\mu\nu} + \delta_{1,2})t} \), where \( H_I = e^{i\Omega H}H_I e^{-i\Omega H} \) is the interaction Hamiltonian. This expression indicates that the transition occurs dominantly when the resonance condition \( E_{\mu\nu} + \delta_{1,2} = 0 \) is satisfied. Furthermore, if \( \delta_{1,2} \) is chosen to be positive, the resonance condition is satisfied only when \( \mu < \nu \), which means that the transition occurs in favor of decreasing the energy of the quantum simulator. \( \delta_{1,2} \) being positive implies that the two-photon Raman transition takes place in such a way that an atom absorbs a lower-energy photon from the classical field and emits a higher-energy photon into the cavity mode. The net result is thus that the amount of energy \( \delta_{1,2} \) is transferred from the quantum simulator to the cavity mode. The subsequent decay of the cavity photon then completes one cycle of the energy relaxation. The correction by \( H_d \) in Hamiltonian \( \mathbf{I} \), albeit small, can be taken into account as follows. \( H_d \) shifts the energy of state \( |\Psi_\mu, n_1, n_2\rangle \) as much as \( -(\epsilon_z + n_1 \epsilon_1 + n_2 \epsilon_2)_{\mu\nu} \), which modifies the resonance condition as \( E_{\mu\nu} - (\epsilon_z + \epsilon_1, \epsilon_2)_{\nu\nu} + (\epsilon_1)_{\mu\nu} + (\epsilon_2)_{\nu\nu} + \delta_{1,2} = 0 \). This term also modifies the energy eigenstates themselves.

The infidelity of the state for \( n_1 = n_2 = 0 \) due to this correction is of order \( (|\epsilon_z|/|E_{\mu\nu}|)^2 \).

We impose the following additional conditions. First, we require \( |E_{\mu\nu}| \gg \kappa \), where \( \kappa \) is the cavity decay rate. Otherwise, the energy levels can not be resolved.
due to the line broadening, which results in heating of the quantum simulator. Second, for the lower energy state, \((\uparrow_\mu)_\mu \ll \kappa\) should be satisfied, since otherwise the cavity photon can be generated without a transition, resulting in heating, e.g., through transition \(|\Psi_\nu,1,0\rangle \rightarrow |\Psi_\nu,0,0\rangle\) that is allowed for \(\mu < \nu\). Third, for a different reason, there should be no thermal photon coming into the cavity from the environment.

As an example, we analyze in Fig. 2 the energy relaxation process of a two spin system with Hamiltonian \(H_0 = B(s_1^+ + s_2^+) + J\hat{s}_1 \cdot \hat{s}_2\) and \(B = 2J > 0\). In this case \(E_{\mu+1} - E_\mu = B\) for every \(\mu\). The parameters are chosen as \(\Delta_{(1,2)} / 100 = \Omega_{1j} / 10 = g_{(1,2)} / 7\kappa\) and \(\delta_{(1,2)} = 10\kappa = B\). \(\Omega_{2j}\) is chosen to be \(-\Omega_{1j}\) to optimize the process. \(\epsilon_2\) is taken to be zero. The top panel of Fig. 2(b) shows the population of each energy level with respect to time and the middle and the bottom panels show, respectively, the detection rates of the photons in mode \(a_1\) and \(a_2\) averaged over the corresponding time bins by using the quantum trajectory method \([15]\). The population, initialized in the highest energy level to show the cascade-like transition structure, eventually trapped in the ground level, during which the detection eventually ceases, indicating the completion of the relaxation. We show in Fig. 2(a) the ground state population for various mean thermal photon number \(\bar{n}\). As discussed before, the asymptotic fidelity decreases as \(\bar{n}\) increases.

Note that this simple model already illustrates a novel application of the present mechanism. The ground state, which is a maximally entangled state, is prepared as a steady state without any initialization of the system or time control of the fields. For a small number of atoms, one can apply several resonant fields in the same manner to prepare the entangled ground state. The effective spontaneous emission rate \(\sim \sum_j \gamma \Omega_j / \Delta^2\), with \(\gamma\) the intrinsic spontaneous emission rate, can be made arbitrarily negligible compared to the Raman transition frequency \(\sim \sum_j |g_j \Omega_j / \Delta|\) by increasing \(\Delta\).

General scheme.—The above analysis leads us to the reasoning that if the classical fields are broadband with \(\delta_{1,2} > 0\), many-body systems would tend to cool even if they have many energy levels with different energy gaps, hence many different resonance conditions. Let us first simplify the situation by considering spectrally incoherent broadband fields. We denote the spectral density of the broadband field by \(I(\delta)\), which is normalized as \(\int I(\delta) d\delta = 1\) (the subscripts 1 and 2 are omitted for brevity). The spectral incoherence can be reflected by replacing \(e^{i\delta}\) with \(\int d\delta f(\delta) e^{i\delta}\) and \(\int d\delta f(\delta) \delta^*(\delta')\) explicitly with \(\int d\delta d\delta' I(\delta) \delta(\delta - \delta')\), where \(\delta(x)\) is the Dirac delta function. The standard treatment of the time-dependent perturbation theory then yields the Fermigolden-rule-like transition rate \(2\pi n_1 |\langle \Gamma_\mu \rangle\|^2 I(E_\mu)\) for transition \(|\Psi_\nu, n_1 - 1, n_2\rangle \rightarrow |\Psi_\mu, n_1, n_2\rangle\) (similarly for \(n_2\)). \(H_0\) is invariant under the above change, making the same contribution.

Now that the transition is understood to be Markovian, it can be represented by a Markov chain, where each node represents an energy level (including the cavity state). For each pair of the energy levels, the transition is established if the broadband field contains the spectral component \(\delta\) corresponding to the energy gap. The cavity decay \(|\Psi_\mu, n_1, n_2\rangle \rightarrow |\Psi_\mu, n_1 - 1, n_2\rangle\) is also included with rate \(n_1 \kappa\) (similarly for \(n_2\)).

The detailed relaxation process depends on the specific many-body systems at hand. In what follows, we discuss the relaxation process of the isotropic Heisenberg spin chain with Hamiltonian \(H_0 = J \sum_{j=1}^{N-1} \hat{s}_j \cdot \hat{s}_{j+1}\), where \(J > 0\) and \(N\) is even for \(|\Psi_0\rangle\) to be nondegenerate. Those arguments below, however, mostly hold when \(H_0\) commutes with the total spin operator \(S_z = \sum_j s_j^z\), as is the case for any Heisenberg spin system without XY anisotropy. Most of the spin systems simulable by earlier schemes fall into this class. Several remarks are in order. (i) All the conditions mentioned above are inherited. However, we strictly impose them only between \(|\Psi_\mu\rangle\) and every \(|\Psi_\nu\rangle\) with \(\mu \neq 0\), since this is important for the ground state to remain stable. Even if the conditions are not met for some higher levels, the overall effect still pushes the population downwards and once it reaches the ground state, any further transition is suppressed. (ii) We lift most degeneracies by applying a magnetic field \(BS_z\), which can be done by an additional field along with the \(\epsilon_2\) term. As \([H_0, S_z]\), the eigenstates of \(H_0\) are now the simultaneous eigenstates of \(S_z\). \(|\Psi_0\rangle\) remains unchanged. (iii) We break the spatial coherence by applying spatially incoherent fields, or alternatively by breaking the symmetry, e.g., by tilting, focusing, or displacing the fields and applying those complementary field configurations in turn. In what follows, we simply assume \(|\langle \Gamma_\mu \rangle\|\|^2 = \sum_j |\langle \Gamma_j \rangle\|^2 |\langle \hat{s}_j^z \rangle\|^2\).

Efficiency.—The dependency of the relaxation time on \(N\) is an important issue. If it grows exponentially with \(N\), the relaxation scheme would be much less useful. Although it is daunting to obtain a general relationship, one can see that it grows at most polynomially with \(N\).
for gapped systems as explained below (note that gapped systems are usually of special interest). The relaxation time can be estimated as the number of energy levels to pass through (in the Markov chain) to reach the ground level, say $N_s$, multiplied by the characteristic transition time $\sim 1/2\pi |(\Gamma_{^+})_{\mu\nu}|^2 I(\delta)$ between two levels. Although the coefficient $|(\Gamma_{^+})_{\mu\nu}|^2$ is bounded by $E_{10}$, it can be kept finite for gapped systems. Besides this, we end up with a trade-off problem: as the fields get broader-band, $N_s$ decreases while $1/I(\delta)$ increases. By noticing that the eigenstates of $H_0$ are also the eigenstates of $S_z$ (ranging from $-N/2$ to $N/2$) and a transition changes $S_z$, by one, we see that if the broadband covers the entire energy spectrum, $N_s$ is at most $2N$ ($N$ for spin flips and $N$ for cavity decays), while $1/I(\delta)$ behaves as $N$.

Spontaneous emission.—Although the atomic excitation is highly suppressed, the spontaneous emission is the prominent source of heating. The spontaneous emission changes the state as $|\Psi\rangle \rightarrow s^+_{j\mu}|\Psi\rangle$ or $s^+_{j\mu} s^+_{j\mu}|\Psi\rangle$, which can be reflected in the Markov chain by adding transitions $|\Psi\rangle \rightarrow |\Psi\rangle$ with rates $\sum_j (\gamma/4)|\Omega_j/\Delta|^2 ||s^+_{j\mu}|^2 + |s^+_{j\mu}|^2 + 2|s^+_{j\mu}|^2||^2$, where the branching ratio is assumed to be 50:50. Note that when spectrally incoherent broadband fields are used, the transition rate is of the same order $\sim |\Omega_j/\Delta|^2$ as the spontaneous emission. In such a case, strong atom-cavity coupling $g_j \gg \kappa$ is required to obtain a high fidelity to the ground state. See the numerical simulation section below. On the other hand, when spectrally coherent broadband fields, i.e., pulses, are used, the spontaneous emission again can be made negligible, in principle, even for weak atom-cavity coupling by increasing $\Delta$, as in the previous case of Fig. 2.

If the spin system is realized with trapped atoms, the spontaneous emission may also excite the atomic momentum. This could be overcome by sympathetic cooling [3].

Numerical simulation.—In Fig. 3b, we plot the asymptotic ground state population with respect to $g_j$ for $N = \{4, 6, 8\}$. We take $\Delta/g_j = g_j/\kappa$ and $\kappa = \gamma = B/10$ with $B = E_{10}/2$. $\Omega_j$ is chosen so that $|(\Gamma_{^+})_{10}| = \kappa$. $I(\delta)$ is constant for $0.5B < \delta - (\epsilon_1)_{10} < 3.5B$ and zero otherwise. The spontaneous emission is included as explained below. In Fig. 3b, we plot the time evolution of the population in the ground state starting from the maximally mixed state for $g_j = 40\kappa$. Note that as explained above, the spectral incoherence, introduced just for ease of analysis, makes the process inefficient, which is why such a strong atom-cavity coupling is required.

This work was supported by the Engineering and Physical Sciences Research Council United Kingdom, the Quantum Information Processing Interdisciplinary Research Centre at Oxford, the Royal Society, the Wolfson Foundation, and the National Research Foundation & Ministry of Education Singapore.