Integral methods of solving boundary-value problems of nonstationary heat conduction and their comparative analysis

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Abstract. The modern state of approximate integral methods used in applications, where the processes of heat conduction and heat and mass transfer are of first importance, is considered. Integral methods have found a wide utility in different fields of knowledge: problems of heat conduction with different heat-exchange conditions, simulation of thermal protection, Stefan-type problems, microwave heating of a substance, problems on a boundary layer, simulation of a fluid flow in a channel, thermal explosion, laser and plasma treatment of materials, simulation of the formation and melting of ice, inverse heat problems, temperature and thermal definition of nanoparticles and nanoliquids, and others. Moreover, polynomial solutions are of interest because the determination of a temperature (concentration) field is an intermediate stage in the mathematical description of any other process. The following main methods were investigated on the basis of the error norms: the Tsoi and Postol’nik methods, the method of integral relations, the Gudman integral method of heat balance, the improved Volkov integral method, the matched integral method, the modified Hristov method, the Mayer integral method, the Kudinov method of additional boundary conditions, the Fedorov boundary method, the method of weighted temperature function, the integral method of boundary characteristics. It was established that the two last-mentioned methods are characterized by high convergence and frequently give solutions whose accuracy is not worse that the accuracy of numerical solutions.

1. Introduction

The modern state of approximate integral methods used in applications, where the processes of heat conduction and heat and mass transfer are of first importance, is considered. Integral methods have found a wide utility in different fields of knowledge: problems of heat conduction with different heat-exchange conditions, simulation of thermal protection, Stefan-type problems, microwave heating of a substance, problems on a boundary layer, simulation of a fluid flow in a channel, thermal explosion, laser and plasma treatment of materials, simulation of the formation and melting of ice, inverse heat problems, temperature and thermal definition of nanoparticles and nanoliquids, and others. These methods are handy in numerical calculations where the formulation of the initial conditions is critical. Here, with the use of integral methods, one can prescribe adequate initial conditions for super-small times. Integral methods are also inviting in the case of problems that have no exact solutions or have solutions so complex in form that these solutions become practically useless. At the same time, the polynomial description of a temperature field guarantees excellent possibilities for solving inverse problems on it and simple analysis of the solutions obtained. In the present work, boundary-value problems for the generalized equations of nonstationary heat conduction in a lengthy plate and in a bounded space were considered.
2. Integral methods of weighted temperature function. Integral method of boundary characteristics

By way of example, we consider two boundary-value problems on the nonstationary heat conduction in a bounded space $x \in [R_1, R_2]$ and in a semi-bounded space $x \in [R_1, \infty]$. 

2.1. Nonstationary heat conduction in a bounded space

Mathematical formulations of the boundary-value problems for a plate ($m = 0$), a hollow cylinder ($m = 1$), and a hollow sphere ($m = 2$) have the form

$$ c(x)\rho(x)\frac{\partial T}{\partial t} = \frac{1}{x^m} \frac{\partial}{\partial x} \left( x^m \lambda(x) \frac{\partial T}{\partial x} \right) + Q(x,t), \quad (R_1, R_2) \times (0, \infty), \quad m = 0, 1, 2, \quad (1) $$

$$ \left( a_i \frac{\partial T}{\partial x} + \beta_i T \right)_{|x_i} = \gamma_i(t), \quad \left( a_j \frac{\partial T}{\partial x} + \beta_j T \right)_{|x_j} = \gamma_j(t), \quad a_i^2 + \beta_i^2 \neq 0, \quad a_j^2 + \beta_j^2 \neq 0, \quad (2) $$

$$ T(x, 0) = \Phi(x). \quad (3) $$

Here $\Phi(x)$ is the initial temperature distribution, $x = R_1$ and $x = R_2$ are the bounding surfaces of the segment $[R_1, R_2]$, $x$ is a coordinate, $t$ is time, $a(x) = \lambda(x) / (c(x)\rho(x)) = \lambda(x), c(x), \rho(x)$ are the coordinate-dependent thermal diffusivity, heat-conduction coefficient, specific heat capacity, and density, respectively, and $Q(x,t)$ is the power of the volume heat sources. It is assumed that the functions $c(x), \rho(x), \lambda(x)$ for $x \in [R_1, R_2]$ are continuous (the positivity of these functions follows from the content of the problem) and the integral $\int_{R_1}^{R_2} (x^m \lambda(x))^{-1} dx$ converges.

According to the method of weighted temperature function (WTFM) [1], the following system of functionals is introduced into consideration:

$$ L_n(T) = (T, K_n) = \int_{R_1}^{R_2} T K_n dx, \quad \Phi_n = (\Phi, K_n) = \int_{R_1}^{R_2} \Phi K_n dx, \quad q_n = (q, K_n) = \int_{R_1}^{R_2} q K_n dx, \quad n \in \mathbb{Z}_+, \quad (4) $$

where $K_n = K_n(x) = c(x)\rho(x) x^m M_n(x)$ are weight functions (kernels) representing solutions of the second-order differential equations

$$ \frac{d}{dx} \left( x^m \lambda(x) \frac{dM_n^{(j)}}{dx} \right) = 0, \quad \frac{d}{dx} \left( x^m \lambda(x) \frac{dM_n^{(j)}}{dx} \right) = K_n^{(j)}, \quad n = 2, 3, \ldots $$

with the boundary conditions following from (2), which are selected starting from the following four variants for the boundary function $\varphi(t)$ [1]:

$$ (I.A) \rightarrow \left. \frac{\partial T}{\partial x} \right|_{R_1}, \quad (II.A) \rightarrow \left. \frac{\partial T}{\partial x} \right|_{R_2}, \quad (I.B) \rightarrow T|_{R_1}, \quad (II.B) \rightarrow T|_{R_2}. \quad (5) $$

As a result, we obtain the sequence of identical equalities [1]

$$ L_n(T) = \sum_{i=0}^{n} \int_{t_{i-1}}^{t_i} \left( P_i + \bar{q}_i \right) dt + \frac{t_{n-i}^{n-i} \bar{q}_i}{(n-i)!}, \quad \forall n \in \mathbb{Z}_+, \quad (6) $$

where $P_i (i = 1, n)$ are complex functions accounting for the boundary conditions (2) and involving the boundary function $\varphi(t)$ in the form of multiple integrals with respect to time $t$. 

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As an illustration of the use of the WTFM for solving problems of nonstationary heat conduction, we consider such a boundary-value problem for a lengthy plate with a heat-conduction coefficient \( \lambda(x) = (1 + x/2)^{-2} \) for the case where the surfaces \( x = 0 \) and \( x = 1 \) are held at constant temperatures \( T(0,t) = 1 \) and \( T(1,t) = 0 \). The representation of the temperature profile in the form of the fifth-degree polynomial \( T = 1 + \sum_{j=1}^{5} a_j(t) x^j \) allows one to determine, on the basis of three equations of (6), the boundary condition \( T(1,t) = 0 \), and the condition \( \partial T(1,t) / \partial x = \varphi(t) \), the polynomial coefficients \( a_j(t) \). The boundary function \( \varphi(t) \) can be determined on the basis of the heat-balance integral \([2, 3]\) or from the fourth-order equation (6). Figure 1 presents the temperature profiles constructed on the basis of the approximate solution and the exact solution \([4, 5]\). We call the reader’s attention to the fact that the approximate and exact profiles are practically completely coincident.

![Figure 1](image1.png)

**Figure 1.** Temperature profiles constructed for the plate on the basis of the exact solution (solid lines) and approximate solution by the WTFM at \( N = 5 \) (dotted lines) at \( \lambda(x) = (1 + x/2)^{-2} \).

![Figure 2](image2.png)

**Figure 2.** Temperature profiles constructed for the half-space on the basis of the exact solution (solid lines) and the approximate solution by the WTFM at \( N = 7 \) (dotted lines) at \( \lambda(x) = (1 + 5x)^{-1} \).

### 2.2. Nonstationary heat conduction in a semi-bounded space

The problem on the nonstationary heat conduction in the space bounded from the inside by a plane \((m = 0)\), cylindrical \((m = 1)\), or spherical \((m = 2)\) surface is considered in the following formulation:

\[
c(x) \rho(x) \frac{\partial T}{\partial t} = \frac{1}{x^m} \frac{\partial}{\partial x} \left( x^m \lambda(x) \frac{\partial T}{\partial x} \right) + Q(x,t), \quad (R, \infty) \times (0, \infty), \quad m = 0, 1, 2,
\]

\[
\left( \alpha \frac{\partial T}{\partial x} + \beta T \right) \bigg|_{x=R} = \gamma(t), \quad (\alpha^2 + \beta^2 \neq 0), \quad T \bigg|_{x \to \infty} = 0,
\]

\[
T(x,0) = \Phi(x).
\]

It should be noted that, in the integral method of boundary characteristics (BChIM) based on the introduction of the front of a temperature disturbance \( \delta(t) \) into consideration \([6–8]\), the function \( \Phi(x) \) can be zero, linear, or quadratic. Moreover, the integral methods based on the consideration of the temperature disturbance front \( \delta(t) \) involve certain limitations for the heat-source function \( Q(x,t) \).

As an illustration, we consider the case where \( T(0,t) = 1, \lambda = (1 + 5x)^{-1} \), and the temperature profile is defined by the polynomial \( T = 1 + \sum_{j=1}^{6} a_j(t) x^j / \delta(t)^j \). The temperature-disturbance front
δ(t) is determined with the use of the heat-balance integral \[ \frac{d}{dt} \int_0^t \delta(t) T \, dx = -\lambda(0) \frac{\partial T(0,t)}{\partial x} , \]
from which we obtain the differential equation
\[
\sum_{j=0}^{6} \left[ a_j(t) \sigma(t) + \frac{a'_j(t)}{j+1} \right] + a_6(t) = 0,
\]
where \( \sigma(t) = \delta(t)^2 \). The temperature profiles, calculated by the BChIM, are presented in figure 2. It is seen from this figure that the solution of the problem by the BChIM is practically completely coincident with the "exact" (numerical) solution.

3. Comparative analysis of integral methods

3.1. Nonstationary heat conduction in a bounded space
We consider, as a partial case, the boundary-value problem on the symmetric heating of a lengthy plate by a unit heat source. In accordance with [1], we introduce the boundary function \( \varphi(t) = T(0,t) \).

The approximate temperature profile is defined by the power polynomial
\[
T(x,t) = \sum_{j=0}^{n} a_j(t)x^j.
\]
The polynomial coefficients of the temperature profile are determined from the solution of corresponding systems of linear algebraic equations, based on the system of integral equalities obtained [1]
\[
\int_0^t \int_0^y \frac{y^{2n}}{2n!} T(y,t) \, dy = Y_{n+1}(t) - \sum_{i=0}^{n} \frac{\mathcal{F}_{i+1}(t)}{(2n-2i-1)!}
\]
where \( \mathcal{F}_{i+1}(t) = \int_0^t \int_0^t \varphi(t) \, dt^{i+1} \), \( Y_{n+1}(t) = \int_0^t \int_0^t \gamma(t) \, dt^{(n)} \). We supplement this system with boundary conditions and the relation \( \varphi(t) = T(0,t) \). On determination of the polynomial coefficients, we arrive at the solution
\[
T(x,t) = t - \frac{1}{6} + \frac{x^2}{2} + \sum_{i=1}^{4} \Psi_j(x) e^{-\mu_i^2 t},
\]
where \( \Psi_j(x) \) are eigenfunctions (not presented for brevity). Using the constitutive integral equality (12) of the \( N+1 \) order, we obtain
\[
\int_0^t \int_0^y \frac{x^{12}}{12!} T(x,t) \, dx = Y_j(t) - \sum_{i=0}^{6} \frac{\mathcal{L}_{i+1}[\varphi(t)]}{(2n-2i-1)!}.
\]

Equation (14) is easily reduced to the sixth-order ordinary differential equation for the boundary function:
\[
4\varphi^{(6)}(t) + 5130\varphi^{(5)}(t) + 1969110\varphi^{(4)}(t) + 312991965\varphi^{(3)}(t) + 21502681200\varphi''(t) + 565685920800\varphi'(t) + 3771239472000\varphi(t) = 3771239472000t - 62853991200.
\]
The characteristic equation for (15) gives eigenvalues of the boundary-value problem (table 1). As is seen, the eigenvalues obtained on the basis of the WTSM [1] are very close (especially the first of them) to the exact values. The solution obtained is entirely incomparable (in accuracy) with the solutions obtained by the other approximate methods [9–11].
Table 1. Eigenvalues of the problem on the symmetric thermal heating of the plate ($N = 8$).

| Eigenvalues $\mu$ | ABCIM [9, 10] | Tsoi method [11] | WTM [11] | Exact values [4, 5] |
|-------------------|----------------|-------------------|----------|-------------------|
| $\mu_1$           | 9.869139       | 9.869605          | 9.869604401086 | 9.869604401089   |
| $\mu_2$           | 39.46900       | 39.595050         | 39.47842  | 39.47841          |
| $\mu_3$           | -              | 110.5698          | 88.86148  | 88.82643          |
| $\mu_4$           | -              | -                 | 153.9148  | 157.9136          |
| $\mu_5$           | -              | -                 | 233.85734 | 246.7401         |
| $\mu_6$           | -              | -                 | 756.5182  | 355.3057          |

3.2. Nonstationary heat conduction in a semi-bounded space

We now consider the problem on the heat conduction in a semi-bounded space whose initial temperature is equal to zero. Let the surface temperature changes by the known time law $T(0, t) = h(t)$. The BChIM [6–8] involves multiple integration of the heat-conduction equation with respect to the space coordinate over the region $x \in [R, \delta(t)]$ with the use of integral operators, which gives a sequence of entirely determined integral equalities involving, on their right side, corresponding boundary characteristics $H_n(t) = \int_0^\infty \ldots \int_0^\infty h(t) dt^{(n)}$ [6–8]. For example, for a half-space with a variable heat conduction at the first-kind boundary condition, we have a sequence of integral equalities that, in the general form, is \( \{ \mathcal{L}_n T = H_n(t) \} \), $n \in \mathbb{Z}_+$. The temperature-disturbance front $\delta(t)$ is determined from the differential equation following from the heat-balance integral with the zero initial condition [2].

As an example, we consider the case where $T(x, 0) = 0$, $\lambda(x) = \exp(-5x)$. Figures 3 and 4 present the temperature profiles constructed for different instants of time on the basis of the «exact» (numerical) solution and on the solutions obtained by the method of additional boundary conditions (ABCIM) ($N = 8$) [12, 13] and BChIM ($N = 7$) [6–8]. As is seen, the approximate solutions obtained by the ABCIM are almost inadequate. At the same time, the relative error given by the BChIM comprises hundredths of a percent. The temperature profiles calculated by this method are practically completely coincident with the exact ones.

![Figure 3](image1.png)

**Figure 3.** The temperature profiles in the half-space, constructed on the basis of the exact solution (solid lines) and the solution by the ABCIM ($N = 8$) (dash-dot lines) at $\lambda(x) = \exp(-5x)$.

![Figure 4](image2.png)

**Figure 4.** The temperature profiles in the half-space, constructed on the basis of the exact solution (solid lines) and the solution by the BChIM ($N = 7$) (dotted lines) at $\lambda(x) = \exp(-5x)$ (dashed line).
The BChIM at $N = 5$ gives the following solution for a uniform half-space:

$$ T = \left( 1 - 0.20367 \frac{x}{\sqrt{t}} \right)^2 \left( 1 - 0.15734 \frac{x}{\sqrt{t}} - 0.10710 \frac{x^2}{t} + 0.02389 \frac{x^3}{t^{3/2}} \right). \quad (16) $$

A comparative analysis of the accuracies of the integral methods was performed with respect to the maximum deviation $E = \left| T - T^* \right|_{\text{max}}$ (table 2). The data presented in table 2 clearly show that the approximation accuracy of the BChIM is much higher as compared to that of other known integral methods, in particular, the heat balance integral method (HBIM) [14, 15], the refined integral method (RIM) [16, 17], and the combined integral method (CIM) [16, 18]. In the works [14, 15], the temperature profile is defined by the $n$-power polynomial. The authors of the works [16] used a more complex profile involving an additional logarithmic function:

$$ T(x, t) = \left( 1 - \frac{x}{\delta(t)} \right) \left[ 1 + b(t) \ln \left( 1 - \frac{1}{\delta(t)} \right) \right]. \quad (17) $$

### Table 2. Maximum deviation of the temperature profiles calculated by different integral methods ($t = 1$) from the exact solution ($E$).

| Calculation method | HBIM [15] | HBIM [14] | RIM [16, 17] | RIM [16] | CIM [16, 18] | BChIM [6, 7] |
|--------------------|-----------|-----------|--------------|---------|--------------|--------------|
| $n$                | 2.121     | 2.008     | 2.074        | 5.5215  | 5.5132       | 5            |
| $E$                | 0.0351    | 0.0330    | 0.0298       | 0.00715 | 0.00721      | 0.00263      | 0.00031      |

### References

[1] Kor V A 2016 J. Eng. Phys. Thermophys. vol 89 pp 192–211
[2] Goodman T R 1964 Advances in Heat Transfer vol 1 (New York: Academic Press) pp 51–122
[3] Wood A S A 2001 Appl. Math. Model. vol 25 pp 815–24
[4] Carslow H S and Jaeger J C 1992 Conduction of Heat in Solids (UK: Oxford University Press)
[5] Luikov A V 1978 Heat Conduction Theory (Moscow: Énergiya)
[6] Kor V A 2016 Heat Transf. Res. vol 47 pp 1035–55
[7] Kor V A 2016 Heat Transf. Res. vol 47 pp 927–44
[8] Kor V A 2015 J. Eng. Phys. Thermophys. vol 88 pp 423–38
[9] Fedorov F M 2000 Boundary Method of Solving Applied Problems of Mathematical Physics (Novosibirsk: Nauka)
[10] Kudinov V A 2004 Izv. Ross. Akad. Nauk ed Énergetika chapter 3 pp 82–101
[11] Tsoi P V 2005 System Methods of Calculating the Boundary-Value Problems of Heat and Mass Transfer (Moscow: Izd. MÉI)
[12] Goodman T R 1961 Trans. ASME ed C chapter 1 pp 83–93
[13] Kudinov V A and Stefanyuk E V 2009 J. Eng. Phys. Thermophys. vol 82 pp 540–58
[14] Myers T G 2009 Int. Commun. Heat Mass Transfer vol 36 pp 143–7
[15] Hristov J 2009 Thermal Sci. vol 13 pp 27–48
[16] Mitchell S L and Myers T G 2010 Int. J. Heat and Mass Transfer vol 53 pp 3540–51
[17] Sadoun N, Si-Ahmed S K and Colinet P 2006 Appl. Math. Model vol 30 pp 531–44
[18] Myers T G and Mitchell S L 2011 Appl. Math. Model. vol 35 pp 4281–94