AUSLANDER-REITEN DUALITY FOR GROTHENDIECK ABELIAN CATEGORIES

HENNING KRAUSE

Abstract. Auslander-Reiten duality for module categories is generalised to Grothendieck abelian categories that have a sufficient supply of finitely presented objects. It is shown that Auslander-Reiten duality amounts to the fact that the functor Ext$^1_C(−, C)$ into modules over the endomorphism ring of $C$ admits a partially defined right adjoint when $C$ is a finitely presented object. This result seems to be new even for module categories. For appropriate schemes over a field, the connection with Serre duality is discussed.

1. Introduction

The cornerstones of Auslander-Reiten duality are the following formulas for modules over an artin algebra $Λ$:

$$D\text{Ext}^1_Λ(−, C) \cong \text{Hom}_Λ(−, D\text{Tr} C) \quad \text{and} \quad \text{Ext}^1_Λ(−, D\text{Tr} C) \cong D\text{Hom}_Λ(C, −).$$

They were established by Auslander and Reiten [4] and later generalised to modules over arbitrary rings [1]. The crucial ingredient is the explicit construction of the Auslander-Reiten translate by taking the dual of the transpose $D\text{Tr} C$ of a finitely presented module $C$.

There are several options to generalise this. A very elegant approach due to Bondal and Kapranov [7] uses the notion of a Serre functor for a triangulated category. In particular, this reveals the close connection between Auslander-Reiten duality and Serre duality.

For abelian categories, Auslander and Reiten established in [3] a generalisation by introducing the concept of a dualising variety. Further approaches include the work of Reiten and Van den Bergh for hereditary abelian categories [22] and that of Lenzing and Zuazua [19].

The aim of this paper is to establish Auslander-Reiten duality more generally for Grothendieck abelian categories that have a sufficient supply of finitely presented objects. We mimic the construction of the Auslander-Reiten translate by invoking the existence of flat covers in certain functor categories. In fact, we show that the Auslander-Reiten translate is the representing object of a specific functor; so our approach is somewhat similar to Neeman’s account of Grothendieck duality via Brown representability [20].

Motivated by the general setting of Grothendieck abelian categories, we obtain a coherent formulation of Auslander-Reiten duality which seems to be new even for module categories. For a finitely presented module we provide an explicit construction of the Auslander-Reiten translate; it is a modification of the original construction due to Auslander and Reiten.

For an abelian category $A$ let $\overline{A}$ denote the stable category modulo injectives, which is obtained from $A$ by identifying two morphisms $φ, φ': X \to Y$ if

$$\text{Ext}^1_A(−, φ) = \text{Ext}^1_A(−, φ').$$

2010 Mathematics Subject Classification. 18E15 (primary), 14F05, 16E30, 18G15.

April 9, 2016.
When \( \mathcal{A} \) has enough injective objects this means \( \phi - \phi' \) factors through an injective object. We write \( \text{Hom}_{\mathcal{A}}(-, -) \) for the morphisms in \( \mathcal{A} \). Analogously, the stable category modulo projectives \( \underline{\mathcal{A}} \) is defined.

**Theorem 1.1.** Let \( \mathcal{A} \) be a Grothendieck abelian category that is locally finitely presented. Fix a finitely presented object \( C \) and set \( \Gamma = \text{End}_{\mathcal{A}}(C) \). Then for every injective \( \Gamma \)-module \( I \) there exists an object \( \tau_C(I) \) in \( \underline{\mathcal{A}} \) and a natural isomorphism

\[
\text{Hom}_\Gamma(\text{Ext}^1_{\mathcal{A}}(C, -), I) \cong \text{Hom}_{\mathcal{A}}(-, \tau_C(I)).
\]

We refer to Theorem 2.1 for the proof of this result and continue with some consequences.

Another version of Auslander-Reiten duality involves the stable category modulo projectives. We say that \( \mathcal{A} \) has enough projective morphisms if every object \( X \) in \( \mathcal{A} \) admits an epimorphism \( \pi : X' \to X \) such that \( \text{Ext}^1_{\mathcal{A}}(\pi, -) = 0 \).

**Corollary 1.2.** For an injective \( \Gamma \)-module \( I \) there is a natural monomorphism

\[
\text{Ext}^1_{\mathcal{A}}(-, \tau_C(I)) \to \text{Hom}_\Gamma(\text{Hom}_{\mathcal{A}}(C, -), I)
\]

which is an isomorphism when \( \mathcal{A} \) has enough projective morphisms.

A necessary and sufficient condition for \( \mathcal{A} \) to be an isomorphism is given in Theorem 2.15. For an intriguing symmetry between \( \mathcal{A} \) and \( \mathcal{A}^{\text{op}} \), see Appendix B. Note that the roles of \( \mathcal{A} \) and \( \mathcal{A}^{\text{op}} \) are quite different. The first one provides the Auslander-Reiten translate as a representing object, while the second seems to be more suitable for applications. For instance, \( \mathcal{A}^{\text{op}} \) is used for constructing almost split sequences. Also, \( \mathcal{A}^{\text{op}} \) identifies with Serre duality for categories of quasi-coherent sheaves over projective schemes.

There is an equivalent formulation of the isomorphism \( \mathcal{A} \) in terms of the defect of an exact sequence \( [1] \). Given an exact sequence

\[
\xi : 0 \to X \to Y \to Z \to 0
\]

in \( \mathcal{A} \), the covariant defect \( \xi \), and the contravariant defect \( \xi^* \) are defined by the exactness of the following sequences:

\[
\begin{align*}
0 & \to \text{Hom}_{\mathcal{A}}(Z, -) \to \text{Hom}_{\mathcal{A}}(Y, -) \to \text{Hom}_{\mathcal{A}}(X, -) \to \xi \to 0 \\
0 & \to \text{Hom}_{\mathcal{A}}(-, X) \to \text{Hom}_{\mathcal{A}}(-, Y) \to \text{Hom}_{\mathcal{A}}(-, Z) \to \xi^* \to 0
\end{align*}
\]

**Corollary 1.3.** For an injective \( \Gamma \)-module \( I \) there is a natural isomorphism

\[
\text{Hom}_\Gamma(\xi^*(C), I) \cong \xi(\tau_C(I)).
\]

In applications one often deals with an abelian category that is \( k \)-linear over a commutative ring \( k \). In that case the isomorphism \( [1] \) gives for any injective \( k \)-module \( I \) a natural isomorphism

\[
\text{Hom}_k(\text{Ext}^1_{\mathcal{A}}(C, -), I) \cong \text{Hom}_{\mathcal{A}}(-, \tau(C, I))
\]

by setting \( \tau(C, I) = \tau_C(\text{Hom}_{\mathcal{A}}(\Gamma, I)) \). For instance, when \( \mathcal{A} \) is the category of modules over a \( k \)-algebra \( \Lambda \) and \( C \) is a finitely presented \( \Lambda \)-module, then

\[
\tau(C, I) = \text{Hom}_k(\text{Tr} C, I).
\]

A case of particular interest is given by a non-singular projective scheme \( X \) of dimension \( d \geq 1 \) over a field \( k \). For the category \( \mathcal{A} \) of quasi-coherent \( \mathcal{O}_X \)-modules and a coherent \( \mathcal{O}_X \)-module \( C \) we have

\[
\tau(C, k) = \Sigma^{d-1}(C \otimes_X \omega_X)
\]

where \( \omega_X \) is the dualising sheaf and \( \Sigma^{d-1} \) denotes the \((d-1)\)st syzygy in a minimal injective resolution. In that case the isomorphism \( [1.3] \) is a variation of Serre
duality [12][23]. In fact, a more familiar form of Serre duality is given by the
natural isomorphism
\[ \text{Ext}^1_X(\cdot, \Sigma^{-d-1}(C \otimes_X \omega_X)) \cong \text{Ext}^d_X(\cdot, C \otimes_X \omega_X) \cong \text{Hom}_X(\text{Hom}_X(C, \cdot), k) \]
which identifies with (1.2) since \( \text{Hom}_X(\cdot, \cdot) \) preserves filtered colimits. This provides a pre-
cise connection between Auslander-Reiten duality and Serre duality.

For an arbitrary Grothendieck abelian category the construction of the Auslander-Reiten translate \( \tau_C \) is far from explicit; it involves the existence of flat covers which was an open problem for about twenty years [6]. When \( A \) is the category of modules over a ring, we provide an explicit description of the Auslander-Reiten translate, making also the connection with the dual of the transpose of Auslander and Reiten; see Definition 3.3 and Theorem 3.4.

This paper has three parts. First we deal with the general case of a Grothendieck abelian category, then we consider the Auslander-Reiten translate for module categories, and the final section is devoted to Auslander-Reiten duality for categories of quasi-coherent sheaves.

2. Auslander-Reiten duality for Grothendieck abelian categories

In this section we introduce the Auslander-Reiten translate for Grothendieck abelian categories that are locally finitely presented (Theorem 2.9).

Following [8], a Grothendieck abelian category \( A \) is **locally finitely presented** if the finitely presented objects generate \( A \). Recall that an object \( X \) in \( A \) is **finitely presented** if the functor \( \text{Hom}_A(\cdot, X) \) preserves filtered colimits. We denote by \( \text{fp}_A \) the full subcategory of finitely presented objects in \( A \). Note that the isomorphism classes of finitely presented objects form a set when \( A \) is locally finitely presented.

We begin with some preparations.

**Modules.** Let \( A \) be an additive category. We write \((A^{\text{op}}, \text{Ab})\) for the category of additive functors \( A^{\text{op}} \to \text{Ab} \) where \( \text{Ab} \) denotes the category of abelian groups. The morphisms between functors are the natural transformations and we obtain an abelian category. Note that (co)kernels and (co)products are computed pointwise: for instance, a sequence \( X \to Y \to Z \) of morphisms in \((A^{\text{op}}, \text{Ab})\) is exact if and only if the sequence \( X(A) \to Y(A) \to Z(A) \) is exact in \( \text{Ab} \) for all \( A \) in \( A \). When \( A \) is essentially small, then the morphisms between two functors in \((A^{\text{op}}, \text{Ab})\) form a set.

Now fix a set \( C \) of objects in \( A \) and view \( C \) as a full subcategory of \( A \). We set \( \text{Mod}C = (C^{\text{op}}, \text{Ab}) \) and call the objects \( C \)-modules. For example, if \( C \) consists of one object \( C \), then \( \text{Mod}C \) is the category of modules over the endomorphism ring of \( C \).

**Restriction and coinduction.** Let \( A \) be an additive category. For a full subcategory \( C \subseteq A \) there is the **restriction functor**
\[ (A^{\text{op}}, \text{Ab}) \to (C^{\text{op}}, \text{Ab}), \quad F \mapsto F|_C \]
and its right adjoint, the **coinduction functor**
\[ \text{coind}_C : (C^{\text{op}}, \text{Ab}) \to (A^{\text{op}}, \text{Ab}) \]
given by
\[ \text{coind}_C I(X) = \text{Hom}(\text{Hom}_A(-, X)|_C, I) \quad \text{for} \ I \in (C^{\text{op}}, \text{Ab}), \ X \in A. \]

\[1\]There is an alternative proof of Theorem 1.1 which obtains the Auslander-Reiten translate from Brown representability, using the fact that the homotopy category of complexes of injective objects \( K(\text{Inj}A) \) is a compactly generated triangulated category [17][23].
Thus for $F \in (\mathcal{A}^{\text{op}}, \text{Ab})$ and $I \in (\mathcal{C}^{\text{op}}, \text{Ab})$, there is an isomorphism
\begin{equation}
(2.1) \quad \text{Hom}(F_{|\mathcal{C}}, I) \cong \text{Hom}(F, \text{coind}_{\mathcal{C}} I).
\end{equation}

**Lemma 2.1.** The functor $\text{coind}_{\mathcal{C}}$ preserves injectivity.

**Proof.** The restriction functor is exact, and any right adjoint of an exact functor preserves injectivity. \qed

**Finitely presented functors.** Let $\mathcal{A}$ be an additive category. We denote by $\text{Fp}(\mathcal{A}^{\text{op}}, \text{Ab})$ the category of finitely presented functors $F: \mathcal{A}^{\text{op}} \to \text{Ab}$. Recall that $F$ is finitely presented (or coherent) if it fits into an exact sequence
\[
\text{Hom}_{A}(\mathcal{C}, X) \to \text{Hom}_{A}(\mathcal{C}, Y) \to F \to 0.
\]
Note that $\text{Fp}(\mathcal{A}^{\text{op}}, \text{Ab})$ is an abelian category when $\mathcal{A}$ admits kernels. Then the assignment $X \mapsto \text{Hom}_{A}(\mathcal{C}, X)$ identifies $\mathcal{A}$ with the full subcategory of projective objects in $\text{Fp}(\mathcal{A}^{\text{op}}, \text{Ab})$ by Yoneda’s lemma.

**Flat functors and flat covers.** Let $\mathcal{C}$ be an essentially small additive category. We consider the category $(\mathcal{C}^{\text{op}}, \text{Ab})$ of additive functors $F: \mathcal{C}^{\text{op}} \to \text{Ab}$. Recall that $F$ is flat if it can be written as a filtered colimit of representable functors.

The following result describes the connection between locally finitely presented categories and categories of flat functors.

**Theorem 2.2** (Breitsprecher [8]). Let $\mathcal{A}$ be a locally finitely presented Grothendieck abelian category. Then the functor $\mathcal{A} \longrightarrow ((\text{fpA})^{\text{op}}, \text{Ab}), \quad X \mapsto \text{Hom}_{A}(\mathcal{C}, X)|_{\text{fpA}}$

identifies $\mathcal{A}$ with the full subcategory of flat functors $(\text{fpA})^{\text{op}} \to \text{Ab}$. Moreover, the functor admits an exact left adjoint. \qed

A morphism $\pi: F \to G$ in $(\mathcal{C}^{\text{op}}, \text{Ab})$ is a flat cover of $G$ if the following holds:

1. $F$ is flat and every morphism $F' \to G$ with $F'$ flat factors through $\pi$.
2. $\pi$ is right minimal, that is, an endomorphism $\phi: F \to F$ satisfying $\pi \phi = \pi$ is invertible.

A minimal flat presentation of $G$ is an exact sequence
\[
F_1 \longrightarrow F_0 \longrightarrow G \longrightarrow 0
\]
such that $F_0 \to G$ and $F_1 \to \text{Ker} \pi$ are flat covers. A projective cover and a minimal projective presentation are defined analogously, replacing the term flat by projective.\footnote{This definition of a projective cover is equivalent to the usual one which requires the kernel to be superfluous.}

**Theorem 2.3** (Bican–El Bashir–Enochs [6]). Every additive functor $\mathcal{C}^{\text{op}} \to \text{Ab}$ admits a flat cover. \qed

The following consequence is straightforward; see [16, Theorem 2.2].

**Corollary 2.4.** Let $\mathcal{A}$ be a locally finitely presented Grothendieck abelian category. Then every functor $F: \mathcal{A}^{\text{op}} \to \text{Ab}$ that preserves filtered colimits belongs to $\text{Fp}(\mathcal{A}^{\text{op}}, \text{Ab})$ and admits a minimal projective presentation.

**Proof.** Choose a minimal flat presentation of $F|_{\text{fpA}}$ and apply Theorem 2.2. \qed

The next lemma will be needed to identify injective objects in a locally finitely presented Grothendieck abelian category.

**Lemma 2.5.** Let $\mathcal{C}$ be an essentially small additive category and consider the following conditions in $(\mathcal{C}^{\text{op}}, \text{Ab})$.\footnote{This definition of a projective cover is equivalent to the usual one which requires the kernel to be superfluous.}
(1) Given a minimal injective copresentation $0 \to G \to I^0 \to I^1$ such that $G$ is flat, then $I^0$ and $I^1$ are flat.

(2) Given a minimal flat presentation $F_1 \to F_0 \to G \to 0$ such that $G$ is injective, then $F_0$ and $F_1$ are injective.

Then (1) implies (2).

Proof. Fix a minimal flat presentation $F_1 \to F_0 \to G \to 0$ such that $G$ is injective. Let $F_0 \to E(F_0)$ be an injective envelope. Then $\pi$ factors through this since $G$ is injective. On the other hand, the morphism $E(F_0) \to G$ factors through $\pi$ since $E(F_0)$ is flat. The minimality of $\pi$ implies that $F_0$ is a direct summand of $E(F_0)$, and therefore $F_0$ is injective. Now choose a minimal injective copresentation $0 \to F_1 \to I^0 \to I^1$. This gives rise to the following commutative diagram with exact rows.

\[
\begin{array}{cccccc}
F_1 & \to & F_0 & \to & G & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
I^0 & \to & F_0 \oplus I^1 & \to & H & \to & 0
\end{array}
\]

The vertical morphisms are monomorphism, and therefore $G \to H$ splits. The inverse $H \to G$ induces the following commutative diagram with exact rows since $I^0$ and $I^1$ are flat.

\[
\begin{array}{cccccc}
I^0 & \to & F_0 \oplus I^1 & \to & H & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_1 & \to & F_0 & \to & G & \to & 0
\end{array}
\]

Now the minimality of the flat presentation implies that the composition $F_1 \to I^0 \to F_1$ is invertible. Thus $F_1$ is injective. □

Let $\mathcal{A}$ be a locally finitely presented Grothendieck abelian category. Set $\mathcal{C} = \text{fp} \mathcal{A}$ and consider the functor $h: \mathcal{A} \to (\mathcal{C}^{\text{op}}, \text{Ab})$ from Theorem 2.2. Then $X \in \mathcal{A}$ is injective if and only if $h(X)$ is injective, since $h$ has an exact left adjoint. In fact, $h$ takes an injective copresentation in $\mathcal{A}$ to one in $(\mathcal{C}^{\text{op}}, \text{Ab})$. Thus $\mathcal{C}$ satisfies condition (1) in Lemma 2.5.

Corollary 2.6. Let $\mathcal{A}$ be a locally finitely presented Grothendieck abelian category and consider a minimal projective presentation

$$
\text{Hom}_\mathcal{A}(-, X_1) \to \text{Hom}_\mathcal{A}(-, X_0) \to F \to 0
$$

of a functor $F$ in $\text{Fp}(\mathcal{A}^{\text{op}}, \text{Ab})$. If $F|_{\text{fp} \mathcal{A}}$ is an injective object in $(\text{fp} \mathcal{A})^{\text{op}}, \text{Ab}$, then $X_0$ and $X_1$ are injective objects in $\mathcal{A}$.

Proof. The sequence

$$
\text{Hom}_\mathcal{A}(-, X_1)|_{\text{fp} \mathcal{A}} \to \text{Hom}_\mathcal{A}(-, X_0)|_{\text{fp} \mathcal{A}} \to F|_{\text{fp} \mathcal{A}} \to 0
$$

is a minimal flat presentation of $F|_{\text{fp} \mathcal{A}}$ in $(\text{fp} \mathcal{A})^{\text{op}}, \text{Ab})$. Thus the assertion follows from Lemma 2.5. □

The defect of an exact sequence. We recall the following notion from [1, II.3]. Let $\mathcal{A}$ be an abelian category. For an exact sequence

$$
\xi: 0 \to X \to Y \to Z \to 0
$$

The defect of $\xi$ is the difference between the ranks of $X$ and $Y$.
in \( \mathcal{A} \) the \textit{covariant defect} \( \xi_* \) and the \textit{contravariant defect} \( \xi^* \) are defined by the exactness of the following sequences:

\[
0 \longrightarrow \text{Hom}_\mathcal{A}(Z, -) \longrightarrow \text{Hom}_\mathcal{A}(Y, -) \longrightarrow \text{Hom}_\mathcal{A}(X, -) \longrightarrow \xi_* \longrightarrow 0
\]

\[
0 \longrightarrow \text{Hom}_\mathcal{A}(-, X) \longrightarrow \text{Hom}_\mathcal{A}(-, Y) \longrightarrow \text{Hom}_\mathcal{A}(-, Z) \longrightarrow \xi^* \longrightarrow 0
\]

The functors \( \xi_* : \mathcal{A} \rightarrow \text{Ab} \) given by the exact sequences \( \xi \) in \( \mathcal{A} \) form an abelian category, with morphisms the natural transformations. We denote this category by \( \text{Eff}(\mathcal{A}, \text{Ab}) \), because the objects are precisely the finitely presented functors \( F : \mathcal{A} \rightarrow \text{Ab} \) that are \textit{locally effaceable} \( \mathcal{I} \), that is, for each \( x \in F(X) \) there exists a monomorphism \( \phi : X \rightarrow Y \) such that \( F(\phi)(x) = 0 \). The assignment \( F \mapsto \mathcal{D}(F) \) given by

\[
\mathcal{D}(F)(X) = \text{Ext}^2(F, \text{Hom}_\mathcal{A}(X, -))
\]

yields an equivalence

\[
\mathcal{D} : \text{Eff}(\mathcal{A}, \text{Ab})^{\text{op}} \sim \text{Eff}(\mathcal{A}^{\text{op}}, \text{Ab}),
\]

where \( \text{Ext}^2(-, -) \) is computed in the abelian category \( \text{Fp}(\mathcal{A}, \text{Ab}) \) and the inverse is given by \( \mathcal{D}^{\text{op}} \). Note that \( \mathcal{D}(\xi_*) = \xi^* \) and \( \mathcal{D}^{\text{op}}(\xi^*) = \xi_* \). When the context is clear we write \( D \) instead of \( \mathcal{D} \).

We continue with some further properties of locally effaceable functors. For a discussion of the following result, see also \( \mathcal{I} \).

**Lemma 2.7.** Suppose that every object \( X \in \mathcal{A} \) admits a monomorphism \( \iota : X \rightarrow Y \) such that \( \text{Ext}^1_\mathcal{A}(-, \iota) = 0 \). Then the assignment \( \phi \mapsto \text{Ext}^1_\mathcal{A}(-, \phi) \) induces for all objects \( X, X' \in \mathcal{A} \) an isomorphism

\[
\text{Hom}_\mathcal{A}(X, X') \sim \text{Hom}(\text{Ext}^1_\mathcal{A}(-, X), \text{Ext}^1_\mathcal{A}(-, X')).
\]

Clearly, the assumption on \( \mathcal{A} \) is satisfied when \( \mathcal{A} \) has enough injective objects.

**Proof.** Let \( \xi : 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \) be an exact sequence such that \( \text{Ext}^1_\mathcal{A}(-, \iota) = 0 \). Then we have

\[
\xi^* \cong \text{Ext}^1_\mathcal{A}(-, X) \quad \text{and} \quad \xi_* \cong \text{Hom}_\mathcal{A}(X, -).
\]

Now apply the duality \( \mathcal{I} \).

**Lemma 2.8.** The inclusion \( \text{Eff}(\mathcal{A}^{\text{op}}, \text{Ab}) \rightarrow \text{Fp}(\mathcal{A}^{\text{op}}, \text{Ab}) \) admits a right adjoint

\[
\text{eff} : \text{Fp}(\mathcal{A}^{\text{op}}, \text{Ab}) \rightarrow \text{Eff}(\mathcal{A}^{\text{op}}, \text{Ab}).
\]

**Proof.** The right adjoint takes a functor \( F = \text{Coker} \text{Hom}_\mathcal{A}(-, \phi) \) given by a morphism \( \phi : X \rightarrow Y \) in \( \mathcal{A} \) to \( \text{Coker} \text{Hom}_\mathcal{A}(-, \phi') \) where \( \phi' : X \rightarrow \text{Im} \phi \) is the morphism induced by \( \phi \).

**Auslander-Reiten duality.** Let \( \mathcal{A} \) be an abelian category and \( \mathcal{C} \) a set of objects in \( \mathcal{A} \). Then every object \( X \in \mathcal{A} \) gives rise to a \( \mathcal{C} \)-module

\[
\text{Ext}^1_\mathcal{A}(\mathcal{C}, X) := \text{Ext}^1_\mathcal{A}(-, X)|_{\mathcal{C}}.
\]

**Theorem 2.9.** Let \( \mathcal{A} \) be a locally finitely presented Grothendieck abelian category and \( \mathcal{C} \) a set of finitely presented objects in \( \mathcal{A} \). Then for every injective \( \mathcal{C} \)-module \( I \) there exists an object \( \tau_\mathcal{C}(I) \) in \( \mathcal{A} \) and a natural isomorphism

\[
\text{Hom}(\text{Ext}^1_\mathcal{A}(\mathcal{C}, -), I) \cong \text{Hom}_\mathcal{A}(-, \tau_\mathcal{C}(I)).
\]

The special case that \( \mathcal{C} \) consists of a single object is precisely Theorem \( \mathcal{I} \) from the introduction. The theorem suggests the following definition.

**Definition 2.10.** The object \( \tau_\mathcal{C}(I) \) is called the \textit{Auslander-Reiten translate} with respect to \( \mathcal{C} \subseteq \text{fp} \mathcal{A} \) and \( I \in \text{Mod} \mathcal{C} \). The assignment \( I \mapsto \tau_\mathcal{C}(I) \) yields a functor

\[
\tau_\mathcal{C} : \text{Inj Mod} \mathcal{C} \rightarrow \mathcal{A}.
\]
Proof of Theorem 2.9. We fix \(C \subseteq \text{fp} \mathcal{A}\) and \(I \in \text{Mod} \mathcal{C}\). The functor \(\text{coind}_C(I)\) in \((\mathcal{A}^{op}, \text{Ab})\) preserves filtered colimits and admits therefore a minimal presentation

\[0 \to \text{Hom}_\mathcal{A}(-, T_2) \to \text{Hom}_\mathcal{A}(-, T_1) \to \text{Hom}_\mathcal{A}(-, T_0) \to \text{coind}_C(I) \to 0\]

by Corollary 2.4. In \(\mathcal{A}\) this gives rise to an exact sequence

\[\eta: 0 \to T_2 \to T_1 \to \bar{T}_0 \to 0.\]

We set \(\tau_C(I) := T_2\).

Note that \(T_1\) is an injective object when \(I\) is injective. This follows from Corollary 2.6, because \(\text{coind}_C(I)\) is an injective object in \(((\text{fp} \mathcal{A})^{op}, \text{Ab})\) by Lemma 2.1.

Thus

\[(2.3) \quad \text{eff coind}_C(I) \cong \eta^* \cong \text{Ext}_\mathcal{A}^1(-, \tau_C(I)).\]

Now fix an object \(X \in \mathcal{A}\). Then we have

\[\text{Hom}(\text{Ext}_\mathcal{A}^1(-, X)|_C, I) \cong \text{Hom}(\text{Ext}_\mathcal{A}^1(-, X), \text{coind}_C(I))\]

\[\cong \text{Hom}(\text{Ext}_\mathcal{A}^1(-, X), \text{eff coind}_C(I))\]

\[\cong \text{Hom}(\text{Ext}_\mathcal{A}^1(-, X), \text{Ext}_\mathcal{A}^1(-, \tau_C(I)))\]

\[\cong \text{Hom}_\mathcal{A}(X, \tau_C(I)).\]

Let us label the \(n\)th isomorphism by \((n)\). Then (1) and (2) are adjunctions, given by (2.1) and Lemma 2.8, (3) uses (2.3), and (4) follows from Lemma 2.7. This completes the proof.

□

One may wonder whether the functor \(\text{Ext}_\mathcal{A}^1(C, -): \mathcal{A} \to \text{Mod} \mathcal{C}\) admits a right adjoint. In fact, the proof of Theorem 2.9 provides for an arbitrary \(C\)-module \(I\) an object \(\tau_C(I)\) and a natural monomorphism

\[\text{Hom}(\text{Ext}_\mathcal{A}^1(C, -), I) \to \text{Hom}_\mathcal{A}(-, \tau_C(I)).\]

It is easily seen that this is not invertible in general.

Remark 2.11. Let \(C \subseteq D \subseteq \text{fp} \mathcal{A}\) and denote by \(i: C \to D\) the inclusion. Then we have \(\tau_C = \tau_D \circ i_*\) where \(i_*\) is the right adjoint of restriction \(\text{Mod} D \to \text{Mod} C\).

Remark 2.12. Consider the stable category modulo projectives \(\mathcal{A}_p\). For a set of objects \(C \subseteq \mathcal{A}\) let \(\mathcal{C}_C\) denote the corresponding subcategory of \(\mathcal{A}_p\). Then we have

\[\text{Ext}_\mathcal{A}^1(C, X) \in \text{Mod} \mathcal{C}_C \subseteq \text{Mod} \mathcal{C}\]

for all \(X \in \mathcal{A}\) and one can replace \(\text{Mod} \mathcal{C}\) by \(\text{Mod} \mathcal{C}_C\) in Theorem 2.9.

Auslander-Reiten duality relative to a base. Let \(k\) be a commutative ring. Suppose that \(\mathcal{A}\) is a \(k\)-linear and locally finitely presented Grothendieck abelian category.

Corollary 2.13. There is a functor

\[\text{fp} \mathcal{A} \times \text{Inj Mod} k \to \mathcal{A}, \quad (C, I) \mapsto \tau(C, I)\]

and a natural isomorphism

\[\text{Hom}_k(\text{Ext}_\mathcal{A}^1(C, -), I) \cong \text{Hom}_\mathcal{A}(-, \tau(C, I)).\]

Proof. Fix \(C \in \text{fp} \mathcal{A}\) and set \(\Gamma = \text{End}_\mathcal{A}(C)\). Observe that \(\text{Hom}_k(\Gamma, -)\) induces a functor \(\text{Mod} k \to \text{Mod} \Gamma\) that takes injectives to injectives. Then we obtain \(\tau(C, -)\) from the Auslander-Reiten translate \(\tau_C\) by setting

\[\tau(C, -) = \tau_C(\text{Hom}_k(\Gamma, -)).\]

□
A formula for the defect. There is a reformulation of the isomorphism in Theorem 2.9 in terms of the defect of an exact sequence.

**Theorem 2.14.** Let $\mathcal{A}$ be a locally finitely presented Grothendieck abelian category. Fix a set $\mathcal{C}$ of finitely presented objects and an exact sequence $\xi: 0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$. Then for every injective $\mathcal{C}$-module $I$ there is a natural isomorphism

$$\text{Hom}(\xi^*|_{\mathcal{C}}, I) \cong \xi_*(\tau_{\mathcal{C}}(I)).$$

The above isomorphism can be rewritten using the duality (2.2). Thus we obtain for $F$ in $\text{Eff}(\mathcal{A}^{\text{op}}, \text{Ab})$ a natural isomorphism

$$\text{Hom}(F|_{\mathcal{C}}, I) \cong D(F)(\tau_{\mathcal{C}}(I)).$$

**Proof of Theorem 2.14.** Fix $\xi: 0 \to X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \to 0$. It is not hard to see that $\xi^* \cong \text{Coker} \text{Hom}_{\mathcal{A}}(\phi, -)$ and $\xi^* \cong \text{Ker} \text{Ext}^1_{\mathcal{A}}(-, \phi)$. Combining this with the isomorphism in Theorem 2.9 we get

$$\text{Hom}(\xi^*|_{\mathcal{C}}, I) \cong \text{Hom}(\text{Ker} \text{Ext}^1_{\mathcal{A}}(\mathcal{C}, \phi), I)$$

$$\cong \text{Coker} \text{Hom}(\text{Ext}^1_{\mathcal{A}}(\mathcal{C}, \phi), I)$$

$$\cong \text{Coker} \text{Hom}_{\mathcal{A}}(\phi, \tau_{\mathcal{C}}(I))$$

$$\cong \xi_*(\tau_{\mathcal{C}}(I)).$$

Auslander-Reiten duality modulo projectives. Let $\mathcal{A}$ be an abelian category. Recall that the stable category modulo projectives $\underline{\mathcal{A}}$ is obtained from $\mathcal{A}$ by identifying two morphisms $\phi, \phi': X \to Y$ if $\text{Ext}^1_{\mathcal{A}}(\phi, -) = \text{Ext}^1_{\mathcal{A}}(\phi', -)$. We write $\text{Hom}_{\mathcal{A}}(-, -)$ for the morphisms in $\underline{\mathcal{A}}$.

The following result is a refinement of Corollary 1.2 from the introduction.

**Theorem 2.15.** Let $\mathcal{A}$ be a locally finitely presented Grothendieck abelian category and $\mathcal{C}$ a set of finitely presented objects in $\mathcal{A}$. Then for every injective $\mathcal{C}$-module $I$ there is a natural monomorphism $\alpha: \text{Ext}^1_{\mathcal{A}}(-, \tau_{\mathcal{C}}(I)) \to \text{Hom}(\text{Hom}_{\mathcal{A}}(-, X), I)$.

This is an isomorphism for all $I$ if and only if for every object $X \in \mathcal{A}$ there exists an epimorphism $\pi: X_{\mathcal{C}} \to X$ such that for every morphism $\phi: C \to X$ with $C \in \mathcal{C}$ $\phi$ factors through $\pi$ if and only if $\text{Ext}^1_{\mathcal{A}}(\phi, -) = 0$.

The condition for $\alpha$ to be an isomorphism expresses the fact that $\mathcal{A}$ has locally enough projective morphisms. Clearly, the condition is satisfied when $\mathcal{A}$ has enough projective objects, but there are also other examples; see Proposition 4.5.

**Proof of Theorem 2.15.** For $X \in \mathcal{A}$ we have

$$\text{Ext}^1_{\mathcal{A}}(X, \tau_{\mathcal{C}}(I)) \cong \text{Hom}(\text{Hom}_{\mathcal{A}}(-, X), \text{Ext}^1_{\mathcal{A}}(-, \tau_{\mathcal{C}}(I)))$$

$$\cong \text{Hom}(\text{Hom}_{\mathcal{A}}(-, X), \text{eff coind}_{\mathcal{C}}(I))$$

$$\cong \text{Hom}(\text{Hom}_{\mathcal{A}}(-, X), \text{coind}_{\mathcal{C}}(I))$$

$$\cong \text{Hom}(\text{Hom}_{\mathcal{A}}(-, X)|_{\mathcal{C}}, I).$$

Let us label the $n$th isomorphism by $(n)$. Then (1) uses Yoneda’s lemma, (2) follows from (2.8), and (3) is the adjunction (2.4).

Now assume that there exists an epimorphism $\pi: X_{\mathcal{C}} \to X$ such that for $F = \text{Coker} \text{Hom}_{\mathcal{A}}(-, \pi)$
we have $F|_C \cong \text{Hom}_A(\cdot, X)|_C$. Then $F$ is effaceable and we can apply Lemma 2.8.

Thus

$$\text{Hom}(\text{Hom}_A(\cdot, X)|_C, I) \cong \text{Hom}(F|_C, I)$$
$\cong \text{Hom}(F, \text{coind}_C(I))$
$\cong \text{Hom}(F, \text{eff coind}_C(I))$
$\subseteq \text{Hom}(\text{Hom}_A(\cdot, X), \text{Ext}^1_A(\cdot, \tau_C(I)))$
$\cong \text{Ext}^1_A(X, \tau_C(I))$

and we conclude that $\alpha$ is an isomorphism.

Finally, assume that $\alpha$ is an isomorphism for all $I$. An injective envelope $\eta: \text{Hom}_A(C, X) \to I$ corresponds under $\alpha$ to an extension

$$0 \to \tau_C(J) \to X \to \pi \to 0.$$

From the functoriality of $\alpha$ one sees that any morphism $\phi: C \to X$ factors through $\pi$ iff the composition of $\text{Hom}_A(C, \phi)$ with $\eta$ equals zero. The latter condition means $\text{Ext}^1_A(\phi, \cdot) = 0$. □

**Remark 2.16.** For $X \in A$ there exists an essentially unique morphism $\pi: X_C \to X$ having the following properties:

1. A morphism $\phi: C \to X$ with $C \in C$ factors through $\pi$ iff $\text{Ext}^1_A(\phi, -) = 0$.
2. Every morphism $X' \to X$ satisfying (1) factors through $\pi$.
3. Every endomorphism $\varepsilon: X_C \to X_C$ satisfying $\pi \varepsilon = \pi$ is invertible.

This follows from [16, Theorem 1.1]. Thus the crucial issue for Theorem 2.15 is the property of $\pi$ to be an epimorphism.

3. Auslander-Reiten duality for module categories

This section is devoted to giving an explicit construction of the Auslander-Reiten translate for module categories (Definition 3.3). This is closely related to the original construction of Auslander and Reiten [1, 3] but it is not the same.

**Stable module categories and the transpose.** Let $\Lambda$ be a ring. Given $\Lambda$-modules $X$ and $Y$, we set

$$\text{Hom}_\Lambda(X, Y) = \text{Hom}_\Lambda(X, Y)/\{\phi \mid \phi \text{ factors through a projective module}\}$$

and

$$\text{Hom}_\Lambda(X, Y) = \text{Hom}_\Lambda(X, Y)/\{\phi \mid \phi \text{ factors through an injective module}\}.$$

Changing not the objects but the morphisms in $\text{Mod } \Lambda$, we obtain the **stable category modulo projectives** $\text{Mod } \Lambda$. Let $\text{mod } \Lambda$ denote the full subcategory of finitely presented $\Lambda$-modules. Analogously, the **stable category modulo injectives** $\text{Mod } \Lambda$ is defined.

For a finitely presented $\Lambda$-module $X$ having a projective presentation

$$P_1 \to P_0 \to X \to 0$$

the transpose $\text{Tr } X$ is defined by the exactness of the following sequence of $\Lambda^{\text{op}}$-modules

$$P_0^* \to P_1^* \to \text{Tr } X \to 0$$

where $P^* = \text{Hom}_\Lambda(P, \Lambda)$.

**Lemma 3.1.** The transpose induces mutually inverse equivalences

$$(\text{mod } \Lambda)^{\text{op}} \sim \text{mod } (\Lambda^{\text{op}})$$

and

$$(\text{mod } \Lambda^{\text{op}}) \sim (\text{mod } \Lambda)^{\text{op}}.$$
Injective modules over quotient rings. Let $\pi: \Gamma \rightarrow \bar{\Gamma}$ be a surjective ring homomorphism. Then restriction of scalars along $\pi$ yields a functor $\text{Mod} \bar{\Gamma} \rightarrow \text{Mod} \Gamma$ that is fully faithful. This functor has a right adjoint which takes a $\Gamma$-module $I$ to $\bar{I} = \text{Hom}_\Gamma(\bar{\Gamma}, I)$. Thus $\pi$ induces a monomorphism $\varepsilon_I: \bar{I} \rightarrow I$ in $\text{Mod} \Gamma$ which identifies $\bar{I}$ with the largest submodule of $I$ that is annihilated by $\text{Ker} \pi$ and gives the isomorphism

$$\text{Hom}_\Gamma(-, I) \cong \text{Hom}_\Gamma(-, \bar{I}).$$

Let $\bar{I} \rightarrow E(\bar{I})$ denote an injective envelope in $\text{Mod} \Gamma$.

Lemma 3.2. For an injective $\Gamma$-module $I$, we have in $\text{Mod} \bar{\Gamma}$

$$\bar{I} \cong \text{Hom}_{\bar{\Gamma}}(\bar{\Gamma}, I) \cong \text{Hom}_\Gamma(\bar{\Gamma}, E(\bar{I})).$$

Proof. The first isomorphism is given by $\varepsilon_I$. The functor $\text{Hom}_\Gamma(\bar{\Gamma}, -)$ preserves injectivity since it is right adjoint to an exact functor. Thus $\bar{I} \rightarrow \text{Hom}_\Gamma(\bar{\Gamma}, E(\bar{I}))$ is a split monomorphism, and it is an isomorphism, because the composition with $\varepsilon_{E(\bar{I})}$ is an injective envelope. □

The dual of the transpose. Let $\Lambda$ be a ring and fix a finitely presented $\Lambda$-module $C$. Set $\Gamma = \text{End}_\Lambda(C)$ and $\bar{\Gamma} = \text{End}_{\Lambda^{op}}(C)$. The transpose $\text{Tr} C$ is a $\Lambda^{op}$-module and there is an isomorphism

$$\gamma: \text{End}_{\Lambda^{op}}(C) \cong \text{End}_{\Lambda^{op}}(\text{Tr} C)^{op}$$

by Lemma 3.1. For an injective $\Gamma$-module $I$ we view $\bar{I} = \text{Hom}_\Gamma(\bar{\Gamma}, I)$ as an $\text{End}_{\Lambda^{op}}(\text{Tr} C)^{op}$-module via $\gamma$ and denote by $E(\bar{I})$ an injective envelope. This yields the assignment

$$\text{Inj Mod} \text{End}_\Lambda(C) \rightarrow \text{Inj Mod} \text{End}_{\Lambda^{op}}(\text{Tr} C)^{op}, \quad I \mapsto E(\bar{I}).$$

Definition 3.3. The dual of the transpose (or Auslander-Reiten translate) of $C$ with respect to $I$ is the $\Lambda$-module

$$\tau_C(I) := \text{Hom}_{\text{End}_{\Lambda^{op}}(\text{Tr} C)^{op}}(\text{Tr} C, E(\bar{I})).$$

In Corollary 3.5 we will see that this definition is consistent with the previous Definition [2, 10]. In particular, the definition does not depend on any choice when $\tau_C(I)$ is viewed as an object in $\text{Mod} \Lambda$.

The definition of $\tau_C(I)$ is a variation of Auslander’s definition of the dual of the transpose in [1, I.3]. To be precise, Auslander starts with an injective module over $\text{End}_{\Lambda^{op}}(\text{Tr} C)^{op}$ whereas the above definition takes as input an injective module over $\text{End}_\Lambda(C)$. Keeping this difference in mind, the following is the analogue of Theorem III.4.1 in [1].

Theorem 3.4. Let $\xi: 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ be an exact sequence of $\Lambda$-modules and $C$ a finitely presented $\Lambda$-module. For an injective $\text{End}_\Lambda(C)$-module $I$, there is a natural isomorphism

$$\text{Hom}_{\text{End}_\Lambda(C)}(\xi^*(C), I) \cong \xi_*(\tau_C(I)).$$

Proof. Set $\Sigma = \text{End}_{\Lambda^{op}}(\text{Tr} C)^{op}$ and $\bar{\Sigma} = \text{End}_{\Lambda^{op}}(\text{Tr} C)^{op}$. Note that $\bar{\Gamma} \cong \bar{\Sigma}$ by Lemma 3.1. In the following we use that $\text{Hom}_\Gamma(\Sigma, E(\bar{I})) \cong \bar{I}$ by Lemma 3.2. Also, the isomorphism 3.1 is used for $\Gamma$ and $\Sigma$.

We fix an exact sequence

$$\xi: 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0.$$
A projective presentation $P_1 \to P_0 \to C \to 0$ induces the following commutative diagram with exact rows.

$$
\begin{array}{ccc}
X \otimes_{\Lambda} P_0^* & \rightarrow & X \otimes_{\Lambda} P_1^* \\
\downarrow^i & & \downarrow^i \\
0 & \rightarrow & \text{Hom}_{\Lambda}(C, X) \rightarrow \text{Hom}_{\Lambda}(P_0, X) \rightarrow \text{Hom}_{\Lambda}(P_1, X)
\end{array}
$$

Therefore $\xi$ induces the following commutative diagram with exact rows and columns.

$$
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{Hom}_{\Lambda}(C, X) \rightarrow \text{Hom}_{\Lambda}(P_0, X) \rightarrow \text{Hom}_{\Lambda}(P_1, X) \rightarrow X \otimes_{\Lambda} \text{Tr} C \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \text{Hom}_{\Lambda}(C, Y) \rightarrow \text{Hom}_{\Lambda}(P_0, Y) \rightarrow \text{Hom}_{\Lambda}(P_1, Y) \rightarrow Y \otimes_{\Lambda} \text{Tr} C \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \text{Hom}_{\Lambda}(C, Z) \rightarrow \text{Hom}_{\Lambda}(P_0, Z) \rightarrow \text{Hom}_{\Lambda}(P_1, Z) \rightarrow Z \otimes_{\Lambda} \text{Tr} C \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
& & & & & 0 & 0 & 0
\end{array}
$$

Now we apply the snake lemma and use adjunctions plus the isomorphism from Lemma 3.2, as explained above. This yields

$$
\text{Hom}_{\Gamma}(\xi^*(C), I) = \text{Hom}_{\Gamma}(\text{Coker} \text{Hom}_{\Lambda}(C, \psi), I)
\cong \text{Hom}_{\Gamma}(\text{Coker} \text{Hom}_{\Lambda}(C, C^*), I)
\cong \text{Hom}_{\Gamma}(\text{Coker} \text{Hom}_{\Lambda}(C, C^*), I)
\cong \text{Hom}_{\Gamma}(\text{Ker}(\phi \otimes_{\Lambda} \text{Tr} C), I)
\cong \text{Hom}_{\Gamma}(\text{Ker}(\phi \otimes_{\Lambda} \text{Tr} C, E(I)))
\cong \text{Coker} \text{Hom}_{\Gamma}(\phi \otimes_{\Lambda} \text{Tr} C, E(I))
\cong \text{Coker} \text{Hom}_{\Gamma}(\phi, \text{Hom}_{\Gamma}(\text{Tr} C, E(I)))
\cong \xi^*(\tau_C(I))
$$

and the proof is complete. $\square$

The following result says that the dual of the transpose $\tau_C(I)$ is unique up to morphisms that factor through an injective module; it is the representing object of the functor

$$
\text{Hom}_{\text{End}_{\Lambda}(C)}(\text{Ext}_{\Lambda}^1(C, -), I).
$$

For the analogues in Auslander’s work [1], see Proposition I.3.4 and Corollary III.4.3.

**Corollary 3.5.** Let $\Lambda$ be a ring and $C$ a finitely presented $\Lambda$-module. Sending an injective $\text{End}_{\Lambda}(C)$-module $I$ to $\tau_C(I)$ gives a functor

$$
\tau_C : \text{Inj Mod End}_{\Lambda}(C) \rightarrow \text{Mod} \Lambda
$$

such that

$$
\text{Hom}_{\text{End}_{\Lambda}(C)}(\text{Ext}_{\Lambda}^1(C, -), I) \cong \text{Hom}_{\Lambda}(-, \tau_C(I))
$$

and

$$
\text{Hom}_{\text{End}_{\Lambda}(C)}(\text{Hom}_{\Lambda}(C, -), I) \cong \text{Ext}_{\Lambda}^1(-, \tau_C(I)).
$$
Proof. Both isomorphisms follow from the isomorphism in Theorem 3.4 by choosing an appropriate exact sequence $\xi: 0 \to X \to Y \to Z \to 0$. In the first case choose $Y$ to be injective. Then $\xi_\ast \cong \text{Hom}_\Lambda(X, -)$ and $\xi_\ast = \text{Ext}_1^\Lambda(-, X)$. In the second case choose $Y$ to be projective. Then $\xi_\ast \cong \text{Hom}_\Lambda(-, Z)$ and $\xi_\ast = \text{Ext}_1^\Lambda(Z, -)$. In particular, the assignment $I \mapsto \tau_C(I)$ is functorial. \qed

We have identified the Auslander-Reiten translate $\tau_C(I)$ as the representing object of a specific functor. The following remark suggests that $\tau_C(I)$ is the 'universal kernel' of certain epimorphisms that are right determined by $C$.

Remark 3.6. Fix a finitely presented $\Lambda$-module $C$ and recall from [1] that a morphism $\alpha: X \to Y$ is right $C$-determined if for any morphism $\alpha': X' \to Y$ we have

$$\text{Im} \text{Hom}_\Lambda(C, \alpha') \subseteq \text{Im} \text{Hom}_\Lambda(C, \alpha) \iff \alpha' \text{ factors through } \alpha.$$ 

Now let

$$\xi: 0 \to \tau_C(I) \to X \overset{\alpha}{\to} Y \to 0$$

be an exact sequence of $\Lambda$-modules and $H$ the kernel of an $\text{End}_\Lambda(C)$-linear map

$$\text{Hom}_\Lambda(C, Y) \to \text{Hom}_\Lambda(C, -) \overset{\alpha}{\to} I.$$

Then $\xi$ corresponds to $\eta$ under the isomorphism 3.3 if and only if $\alpha$ is right $C$-determined with $H = \text{Im} \text{Hom}_\Lambda(C, \alpha)$. This follows from the functoriality of 3.3.

Remark 3.7. In Theorem 3.4 one can replace the finitely presented $\Lambda$-module $C$ by a set of finitely presented modules, as in Theorem 2.14.

Examples. (1) Let $\Lambda$ be a $k$-algebra over a commutative ring $k$. For a finitely presented $\Lambda$-module $C$ and an injective $k$-module $I$ we have $\tau(C, I) = \text{Hom}_k(\text{Tr} C, I)$ and therefore

$$\text{Hom}_k(\text{Ext}_1^\Lambda(C, -), I) \cong \text{Hom}_\Lambda(-, \tau(C, I)).$$

For an elementary proof see [15].

(2) Let $\Lambda$ be a noetherian algebra over a complete local ring $k$. Then Matlis duality over $k$ composed with the transpose $\text{Tr}$ identifies the noetherian $\Lambda$-modules (modulo projectives) with the artinian $\Lambda$-modules (modulo injectives); see [1].

(3) Let $\Lambda$ be a Dedekind domain. Then the Auslander-Reiten translate identifies the noetherian $\Lambda$-modules (modulo projectives) with the artinian $\Lambda$-modules (modulo injectives).

4. Auslander-Reiten duality for quasi-coherent sheaves

This section is devoted to giving an explicit construction of the Auslander-Reiten translate for the category of quasi-coherent modules over a scheme (Definition 4.3). In particular, we explain the connection to Serre duality for a non-singular projective scheme over a field.

Let $k$ be a field and $X$ a quasi-compact and quasi-separated scheme over $k$, given by a morphism $f: X \to \mathcal{Y} = \text{Spec} k$. Consider the category $\text{Qcoh} X$ of quasi-coherent $\mathcal{O}_X$-modules and note that every object is a filtered colimit of finitely presented $\mathcal{O}_X$-modules [13, I.6.9.12]. Thus the Grothendieck abelian category $\text{Qcoh} X$ is locally finitely presented.

From now on assume that the scheme $X$ is locally noetherian. Then an $\mathcal{O}_X$-module is coherent if and only if it is finitely presented.

For an abelian category $\mathcal{A}$ let $D(\mathcal{A})$ denote its derived category. In [20], Neeman shows that Grothendieck duality is a formal consequence of Brown representability, that is, the derived direct image functor

$$Rf_* : D(\text{Qcoh} X) \to D(\text{Qcoh} \mathcal{Y})$$
has a right adjoint
\[ f^! : \mathcal{D}(\text{Qcoh } \mathcal{Y}) \to \mathcal{D}(\text{Qcoh } \mathcal{X}). \]

Now fix objects \( C, X \in \text{Qcoh } \mathcal{X} \) and suppose that \( C \) is coherent.

The following lemma may be deduced from a more general statement in SGA 6; see Proposition 3.7 in \([5, \text{Exp. I}]\).

**Lemma 4.1.** There is in \( \mathcal{D}(\text{Qcoh } X) \) a natural isomorphism
\[ \sigma : X \otimes_X \mathcal{R} \text{Hom}_X(C, \mathcal{O}_X) \xrightarrow{\sim} \mathcal{R} \text{Hom}_X(C, X). \]

**Proof.** Given a ring \( \Lambda \) and \( \Lambda \)-modules \( M, N \), there is a natural morphism
\[ N \otimes_\Lambda \text{Hom}_\Lambda(M, \Lambda) \to \text{Hom}_\Lambda(M, N) \]
which is an isomorphism when \( M \) is finitely generated projective. This gives the morphism \( \sigma \) which is an isomorphism because \( C \) is locally isomorphic to a bounded above complex of finitely generated projective modules (since \( X \) is locally noetherian) while \( X \) is a bounded below complex. For the affine case, see also \([18, \text{Lemma 3.1}]\). \( \square \)

**Lemma 4.2.** There is a natural isomorphism
\[ \text{Hom}_k(\mathcal{R} \text{Hom}_{\mathcal{D}(\mathcal{X})}(C, X), k) \cong \text{Hom}_{\mathcal{D}(\mathcal{X})}(X, \mathcal{R} \text{Hom}_X(\mathcal{R} \text{Hom}_X(C, \mathcal{O}_X), f^k)). \]

**Proof.** Combining Grothendieck duality, Lemma 4.1 and tensor-hom adjunction, we obtain the following chain of isomorphisms:
\[ \text{Hom}_k(\mathcal{R} \text{Hom}_{\mathcal{D}(\mathcal{X})}(C, X), k) \cong \text{Hom}_{\mathcal{D}(\mathcal{Y})}(\mathcal{R} f^*, \mathcal{R} \text{Hom}_X(C, X), k) \]
\[ \cong \text{Hom}_{\mathcal{D}(\mathcal{X})}(\mathcal{R} \text{Hom}_X(C, X), f^k) \]
\[ \cong \text{Hom}_{\mathcal{D}(\mathcal{X})}(X \otimes X \mathcal{R} \text{Hom}_X(C, \mathcal{O}_X), f^k) \]
\[ \cong \text{Hom}_{\mathcal{D}(\mathcal{X})}(X, \mathcal{R} \text{Hom}_X(\mathcal{R} \text{Hom}_X(C, \mathcal{O}_X), f^k)). \] \( \square \)

Given a complex \( M \) of \( \mathcal{O}_X \)-modules and \( n \in \mathbb{Z} \), let \( i(M) \) denote a K-injective resolution \([24]\) and set
\[ Z^n(M) = \text{Ker}(M^n \to M^{n+1}). \]

**Definition 4.3.** The **Auslander-Reiten translate** of a coherent \( \mathcal{O}_X \)-module \( C \) is the \( \mathcal{O}_X \)-module
\[ \tau(C, k) := Z^{-1} i(\mathcal{R} \text{Hom}_X(\mathcal{R} \text{Hom}_X(C, \mathcal{O}_X), f^k)). \]

The following result shows that this definition is consistent with the notation from Corollary 4.1.13 In particular, the definition does not depend on any choice when \( \tau(C, k) \) is viewed as an object of the stable category modulo injectives.

**Theorem 4.4.** Let \( X \) be a locally noetherian scheme over a field \( k \). For a coherent \( \mathcal{O}_X \)-module \( C \) there is a natural isomorphism
\[ \mathcal{H} \text{Hom}_X(-, \tau(C, k)) \xrightarrow{\sim} \text{Hom}_k(\text{Ext}^1_X(C, -), k) \]
and a natural monomorphism
\[ \text{Ext}^1_X(-, \tau(C, k)) \hookrightarrow \text{Hom}_k(\text{Hom}_X(C, -), k) \]
which is an isomorphism if and only if \( H^0(\mathcal{R} \text{Hom}_X(\mathcal{R} \text{Hom}_X(C, \mathcal{O}_X), f^k)) = 0 \).

**Proof.** Set \( \mathcal{A} = \text{Qcoh } X \). The assignment \( M \mapsto i(M) \) yields a fully faithful and exact functor \( \mathcal{D}(\mathcal{A}) \to \mathcal{K}(\text{Inj } \mathcal{A}) \); see \([24]\). Thus we can apply Lemma 4.1 and get both morphisms from the isomorphism in Lemma 4.2. \( \square \)
Let \( X \) be a non-singular proper scheme of dimension \( d \geq 1 \) over a field \( k \). Then the above calculation simplifies since

\[
\mathcal{R} \mathcal{H}om_X(\mathcal{R} \mathcal{H}om_X(C, \mathcal{O}_X), f^! k) \cong C \otimes_X f^! k \quad \text{and} \quad f^! k \cong \omega_X[d]
\]

where \( \omega_X \) denotes the dualising sheaf. The first isomorphism is clear since \( C \) is isomorphic to a bounded complex of locally free sheaves, and for the second isomorphism see [14, IV.4]. Thus

\[
\tau(C, k) = Z^{d-1}(C \otimes_X \omega_X).
\]

Moreover, the isomorphism in Lemma 4.2 gives

\[
\text{Hom}_k(\text{Hom}_X(C, -), k) \cong \text{Ext}^1_X(-, C \otimes_X \omega_X) \cong \text{Ext}^1_X(-, \Sigma^{d-1}(C \otimes_X \omega_X))
\]

where \( \Sigma^{d-1} \) denotes the \((d-1)\)st syzygy in a minimal injective resolution. This isomorphism is a variation of Serre duality [12, Exp. XII] and equals the isomorphism from Theorem 2.15. In particular, we have \( \text{Hom}_X(C, -) = \text{Hom}_X(C, -) \).

**Proposition 4.5.** Let \( X \) be a non-singular proper scheme of dimension \( d \geq 1 \) over a field, and fix a pair of \( \mathcal{O}_X \)-modules \( C, X \) such that \( C \) is coherent. Then there exists an epimorphism \( \pi: X_C \to X \) such that for every morphism \( \phi: C \to X \) the following holds:

\[
\phi \text{ factors through } \pi \iff \text{Ext}^1_X(\phi, -) = 0 \iff \phi = 0.
\]

**Proof.** The construction of \( \pi \) is given in the proof of Theorem 2.15. \( \square \)

**Remark 4.6.** (1) There is a canonical choice for \( \pi: X_C \to X \); see Remark 2.16.

(2) When \( X \) is coherent, then there is a choice for \( \pi: X_C \to X \) such that \( X_C \) is coherent.

(3) One may conjecture that the second morphism in Theorem 2.14 induces an isomorphism

\[
\text{Ext}^1_X(-, \tau(C, k)) \cong \text{Hom}_k(\text{Hom}_X(C, -), k).
\]

**Some questions.** Let \( \mathcal{A} \) be a Grothendieck abelian category.

(1) Is there an alternative construction of the Auslander-Reiten translate \( \tau_C \) for a finitely presented object \( C \in \mathcal{A} \) that is not based on the existence of flat covers?

(2) Are there examples when the morphism

\[
\text{Ext}^1_A(-, \tau(C(I))) \to \text{Hom}_A(\text{Hom}_A(C, -), I)
\]

from Theorem 2.15 is not invertible?

(3) Suppose that \( \text{fpA} \) is \( k \)-linear and Ext-finite over a field \( k \). When is the Auslander-Reiten translate \( \tau(C, k) \) finitely presented for all \( C \in \text{fpA} \)? And when does the Auslander-Reiten translate induce an equivalence between the projectively stable and the injectively stable category associated with \( \text{fpA} \)?

An important class where both properties hold are given by categories \( \mathcal{A} \) such that \( \text{fpA} \) is a dualising \( k \)-variety [3]. However, weighted projective lines [10] provide interesting examples where these properties hold but \( \text{fpA} \) is not a dualising \( k \)-variety; see also [9, 19].

(4) Suppose that \( \mathcal{A} \) is locally noetherian and that \( \text{fpA} \) is \( k \)-linear and Ext-finite over a complete local ring \( k \). When does the Auslander-Reiten translate (via Matlis duality over \( k \)) induce an equivalence between the projectively stable category of noetherian objects and the injectively stable category of artinian objects?
Appendix A. Complexes of injectives and the stable category

Let \( A \) be an abelian category and let \( \text{Inj} A \) denote the full subcategory of injective objects. The chain complexes in \( \text{Inj} A \) with morphisms the chain maps up to homotopy are denoted by \( K(\text{Inj} A) \).

Suppose that \( A \) has enough injective objects. Then we denote by
\[
i : A \rightarrow K(\text{Inj} A)
\]
the fully faithful functor that takes an object in \( A \) to an injective resolution.

For an integer \( n \) consider the functor
\[
Z^n : K(\text{Inj} A) \rightarrow A, \quad X \mapsto \ker(X^n \rightarrow X^{n+1}).
\]

Note that for \( X \in K(\text{Inj} A) \) there is a natural morphism \( iZ^0(X) \rightarrow X \).

**Lemma A.1.** For objects \( X \in A \) and \( Y \in K(\text{Inj} A) \) the following holds.

1. If \( \text{Hom}_{K(\text{Inj} A)}(-, Y) \) vanishes on complexes concentrated in degree zero, then \( Z^0 \) induces a natural isomorphism
\[
\text{Hom}_{K(\text{Inj} A)}(i(X), Y) \cong \text{Hom}_{A}(X, Z^0(Y)).
\]

2. There is a natural monomorphism
\[
\text{Ext}^1_{A}(X, Z^{-1}(Y)) \hookrightarrow \text{Hom}_{K(\text{Inj} A)}(i(X), Y)
\]
which is an isomorphism for all \( X \in A \) if and only if \( H^0(Y) = 0 \).

**Proof.** The proof is straightforward. The second morphism is the composition of
\[
\text{Ext}^1_{A}(X, Z^{-1}(Y)) \hookrightarrow \text{Hom}_{K(\text{Inj} A)}(i(X), Y) \overset{iZ^{-1}(Y)[1]}{\rightarrow} X
\]
and the morphism induced by \( iZ^{-1}(Y)[1] \rightarrow Y \).

Appendix B. Natural maps of extension functors

We rewrite the isomorphisms (1.1) and (1.2) from the introduction in terms of natural transformations between extension functors. This is based on Lemma 2.7 and reveals the symmetry between both formulas.

Let \( A \) be a Grothendieck abelian category that is locally finitely presented. Fix a finitely presented object \( C \) and an injective module \( I \) over \( \Gamma = \text{End}_{A}(C) \). Then for \( X \in A \) there is a natural isomorphism
\[
\text{Hom}_{\Gamma}(\text{Hom}(\text{Hom}_{A}(X, -), \text{Ext}^1_{A}(C, -)), I) \cong \text{Hom}(\text{Ext}^1_{A}(-, X), \text{Ext}^1_{A}(-, \tau_{C}(I))).
\]

When \( A \) has enough projective morphisms we have also
\[
\text{Hom}_{\Gamma}(\text{Hom}(\text{Ext}^1_{A}(X, -), \text{Ext}^1_{A}(C, -)), I) \cong \text{Hom}(\text{Hom}_{A}(-, X), \text{Ext}^1_{A}(-, \tau_{C}(I))).
\]

**Acknowledgements.** I am grateful to Amnon Neeman for his help with the algebraic geometry in this paper.

**References**

[1] M. Auslander, Functors and morphisms determined by objects, in *Representation theory of algebras (Proc. Conf., Temple Univ., Philadelphia, Pa., 1976)*, 1–244. Lecture Notes in Pure Appl. Math., 37, Dekker, New York, 1978.

[2] M. Auslander and M. Bridger, *Stable module theory*, Memoirs of the American Mathematical Society, No. 94, Amer. Math. Soc., Providence, RI, 1969.

[3] M. Auslander and I. Reiten, Stable equivalence of dualizing \( R \)-varieties, Advances in Math. **12** (1974), 306–366.

[4] M. Auslander and I. Reiten, Representation theory of Artin algebras. III. Almost split sequences, Comm. Algebra **3** (1975), 239–294.

[5] P. Berthelot, A. Grothendieck, L. Illusie (eds.), *Théorie des intersections et théorème de Riemann-Roch (SGA 6)*, Lecture Notes in Mathematics, Vol. 225, Springer, Berlin, 1971.
[6] L. Bican, R. El Bashir and E. Enochs, All modules have flat covers, Bull. London Math. Soc. 33 (2001), no. 4, 385–390.

[7] A. I. Bondal and M. M. Kapranov, Representable functors, Serre functors, and reconstructions, Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 6, 1183–1205, 1337; translation in Math. USSR-Izv. 35 (1990), no. 3, 519–541.

[8] S. Breitsprecher, Lokal endlich präsentierbare Grothendieck-Kategorien, Mitt. Math. Sem. Giessen Heft 85 (1970), 1–25.

[9] X.-W. Chen and J. Le, A note on morphisms determined by objects, J. Algebra 428 (2015), 138–148.

[10] W. Geigle and H. Lenzing, A class of weighted projective curves arising in representation theory of finite-dimensional algebras, in Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), 265–297, Lecture Notes in Math., 1273, Springer, Berlin, 1987.

[11] A. Grothendieck, Sur quelques points d’algèbre homologique, Tôhoku Math. J. (2) 9 (1957), 119–221.

[12] A. Grothendieck, Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2), North-Holland, Amsterdam, 1968.

[13] A. Grothendieck and J. A. Dieudonné, Éléments de géométrie algébrique. I, Grundlehren der Mathematischen Wissenschaften, 166, Springer, Berlin, 1971.

[14] R. Hartshorne, Residues and duality, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer, Berlin, 1966.

[15] H. Krause, A short proof for Auslander’s defect formula, Linear Algebra Appl. 365 (2003), 267–270.

[16] H. Krause, Morphisms determined by objects and flat covers, Forum Math., doi:10.1515/forum-2014-0115.

[17] H. Krause, Deriving Auslander’s formula, Documenta Math. 20 (2015), 669–688.

[18] H. Krause and J. Le, The Auslander-Reiten formula for complexes of modules, Adv. Math. 207 (2006), no. 1, 133–148.

[19] H. Lenzing and R. Zuazua, Auslander-Reiten duality for abelian categories, Bol. Soc. Mat. Mexicana (3) 10 (2004), no. 2, 169–177.

[20] A. Neeman, The Grothendieck duality theorem via Bousfield’s techniques and Brown representability, J. Amer. Math. Soc. 9 (1996), no. 1, 205–236.

[21] F. Oort, Natural maps of extension functors, Nederl. Akad. Wetensch. Proc. Ser. A 66=Indag. Math. 25 (1963), 559–566.

[22] I. Reiten and M. Van den Bergh, Noetherian hereditary abelian categories satisfying Serre duality, J. Amer. Math. Soc. 15 (2002), no. 2, 295–366.

[23] J.-P. Serre, Faisceaux algébriques cohérents, Ann. of Math. (2) 61 (1955), 197–278.

[24] N. Spaltenstein, Resolutions of unbounded complexes, Compositio Math. 65 (1988), 121–154.

[25] J. Šťovíček, On purity and applications to coderived and singularity categories, arXiv:1412.1615.

Henning Krause, Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany.
E-mail address: hkrause@math.uni-bielefeld.de