Phase transitions for the Brusselator model

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Dynamic phase transitions of the Brusselator model is carefully analyzed, leading to a rigorous characterization of the types and structure of the phase transitions of the model from basic homogeneous states. The study is based on the dynamic transition theory developed recently by the authors. ©2011 American Institute of Physics. [doi:10.1063/1.3559120]

I. INTRODUCTION

The Belousov–Zhabotinsky (BZ) reactions are now a class of reactions that serve as a classical example of nonequilibrium thermodynamics, resulting in the establishment of a nonlinear chemical oscillator.

The main objective of this article is to study the dynamic phase transitions of the Belousov–Zhabotinsky reactions, focusing on the Brusselator that was first introduced by Prigogine and Lefever.¹² The Brusselator is one of the simplest models in nonlinear chemical systems. It has six components, four of which retain constants, and the other two permit their concentrations vary with time and space. The chemical reaction consists of four irreversible steps, given by

\[\begin{align*}
A &\rightarrow \,[k_1] \,X, \\
B + X &\rightarrow \,[k_2] \,Y + D, \\
2X + Y &\rightarrow \,[k_3] \,3X, \\
X &\rightarrow \,[k_4] \,E,
\end{align*}\]

(1.1)

where \(A\) and \(B\) are constant components, \(D\) and \(E\) are products, and \(X\) and \(Y\) are the two components variable in time and space. Over the years, there have been extensive studies for the Brusselator and related chemical reaction problems; see among many others Refs. 1, 2, 4, 13, and 14, and the references therein.

In this article, we address the dynamic phase transition of the Brusselator model. In particular, we derive a complete characterization of the transition from the homogeneous state. There are two aspects of this characterization. First, our analysis shows that both the transitions to multiple equilibria and to time-periodic solutions (spatiotemporal oscillations) can occur for the Brusselator model, and are precisely determined by the sign of an explicit nondimensional parameter \(\delta_0 - \delta_1\) as defined by (3.5) and (3.6). Then in both transition cases, the dynamic behavior of the transition is classified, and the type of transitions is determined also by the signs of some nondimensional and computable parameters.

The analysis is based on the dynamical transition theory developed recently by the authors.⁵⁻⁷ The main philosophy of the dynamic transition theory is to search for the full set of transition states, giving a complete characterization on stability and transition. The set of transition states is represented by a local attractor. Following this philosophy, the dynamic transition theory is developed to identify the transition states and to classify them both dynamically and physically. With this theory, many long-standing phase transition problems are either solved or become more
accessible, providing new insights to both theoretical and experimental studies for the underlying physical problems.

One important ingredient of the theory is the introduction of a new classification scheme of transitions, with which phase transitions are classified into three types: Type-I, Type-II, and Type-III. In more mathematically intuitive terms, they are called continuous, jump, and mixed transitions, respectively. Basically, as the control parameter passes the critical threshold, the transition states stay in a close neighborhood of the basic state for a Type-I transition, are outside of a neighborhood of the basic state for a Type-II (jump) transition. For a Type-III transition, a neighborhood of the basic state is divided into two open regions with Type-I transition in one region, and Type-II transition in the other.

With this dynamic classification scheme, systematic approaches are then developed to identify the types of transitions and to classify the structure of the transition solutions. Technically, these approaches include the following steps.

The first step is to identify the first unstable modes of the linearized problem, and to examine the principle of exchange of stabilities. As soon as the linear problem loses its stability, the classification theorem ensures that the original problem will always undergo a dynamic transition to one of the three types.

The second step is to reduce the original physical system to the center manifold generated by the first unstable directions/modes. As the center manifold function is highly implicitly defined, the key issue in this step is to find an approximation of the center manifold function up to certain order so that the resulting reduced system is “nondegenerate.”

With the reduced system, following the basic philosophy of searching for the complete set of transition solutions, the next step is to classify the transition types and the transition solutions. The type of transition is essentially determined by the flow structure of the reduced system at the critical threshold, and the structure of transition solutions are derived by detailed analysis of the reduced system. We refer the interested readers to Refs. 5–11 for more details.

This article is organized as follows. Section II introduces the model and its mathematical setup, and Sec. III addresses the principle of exchange of stabilities. Dynamic transitions of the model are addressed in Secs. IV–VI, with physical remarks of the main results given in Sec. VII.

II. THE MODEL AND ITS MATHEMATICAL SETUP

Let \( u_1, u_2, a, \) and \( b \) stand for the concentrations of \( X, Y, A, \) and \( B. \) Then the reaction equations of (1.1) read

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \frac{\sigma_1}{\Delta_1} u_1 + k_1 a - (k_2 b + k_4) u_1 + k_3 u_1^2 u_2, \\
\frac{\partial u_2}{\partial t} &= \frac{\sigma_2}{\Delta_1} u_2 + k_2 b u_1 - k_3 u_1^2 u_2.
\end{align*}
\]  

(2.1)

To get the nondimensional form of (2.1), let

\[
t = k_4^{-1} t', \quad x = l x', \quad u_i = \left( \frac{k_4}{k_3} \right)^{1/2} u_i',
\]

\[
a = (k_4^2/k_3^2 k_4)^{1/2} \alpha, \quad b = (k_4/k_2) \lambda, \quad \sigma_i = l^2 k_4 \mu_i,
\]

for \( i = 1, 2. \) Omitting the primes, the equations (2.1) become

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \mu_1 u_1 + \alpha - (\lambda + 1) u_1 + u_1^2 u_2, \\
\frac{\partial u_2}{\partial t} &= \mu_2 u_2 + \lambda u_1 - u_1^2 u_2.
\end{align*}
\]  

(2.2)

where \( u_1, u_2 \geq 0, \Omega \subset \mathbb{R}^n (1 \leq n \leq 3) \) is a bounded domain, and

\[
\mu_1, \mu_2, \alpha, \lambda > 0.
\]
The equations (2.2) have a constant steady state solution
\[ u_0 = (\alpha, \lambda/\alpha). \]

Make the translation
\[ u_1 = \alpha + v_1, \quad u_2 = \frac{\lambda}{\alpha} + v_2, \]
then the equations (2.2) are written as
\[
\begin{align*}
\frac{\partial v_1}{\partial t} &= \mu_1 \Delta v_1 + (\lambda - 1)v_1 + \alpha^2 v_2 + \frac{2\lambda}{\alpha} v_1^2 + 2\alpha v_1 v_2 + v_1^2 v_2, \\
\frac{\partial v_2}{\partial t} &= \mu_2 \Delta v_2 - \lambda v_1 - \alpha^2 v_2 - \frac{2\lambda}{\alpha} v_1^2 - 2\alpha v_1 v_2 - v_1^2 v_2.
\end{align*}
\]

(2.3)

There are two types of physically sound boundary conditions: the Dirichlet boundary condition
\[ v = (v_1, v_2) = 0 \quad \text{on} \ \partial \Omega, \]
and the Neumann boundary condition
\[ \frac{\partial v}{\partial n} = 0 \quad \text{on} \ \partial \Omega. \]

(2.4)

(2.5)

Define the function spaces
\[ H = L^2(\Omega, \mathbb{R}^2), \]
\[ H_1 = \begin{cases} H^2(\Omega, \mathbb{R}^2) \cap H^1_0(\Omega, \mathbb{R}^2) & \text{for b.c. (2.4)}, \\
\{ v \in H^2(\Omega, \mathbb{R}^2) | \frac{\partial v}{\partial n}|_{\partial \Omega} = 0 \} & \text{for b.c. (2.5)}. \end{cases} \]

Define the operators \( L_\lambda = A + B_\lambda \) and \( G : H_1 \rightarrow H \) by
\[
\begin{align*}
A v &= (\mu_1 \Delta v_1, \mu_2 \Delta v_2), \\
B_\lambda v &= ((\lambda - 1)v_1 + \alpha^2 v_2, -\lambda v_1 - \alpha^2 v_2), \\
G v &= \left( \frac{2\lambda}{\alpha} v_1^2 + 2\alpha v_1 v_2 + v_1^2 v_2, -\frac{2\lambda}{\alpha} v_1^2 - 2\alpha v_1 v_2 - v_1^2 v_2 \right).
\end{align*}
\]

(2.6)

Thus the equations (2.3) with (2.4) or with (2.5) can be written in the following abstract form:
\[ \frac{dv}{dt} = L_\lambda v + G(v, \lambda). \]

(2.7)

III. PRINCIPLE OF EXCHANGE OF STABILITY

Consider the eigenvalue equations of (2.3)
\[
\begin{align*}
\mu_1 \Delta v_1 + (\lambda - 1)v_1 + \alpha^2 v_2 &= \beta v_1, \\
\mu_2 \Delta v_2 - \lambda v_1 - \alpha^2 v_2 &= \beta v_2,
\end{align*}
\]

(3.1)

with the boundary condition (2.4) or (2.5).

Let \( \rho_k \) and \( e_k \) be the \( k \)th eigenvalue and eigenvector of the Laplacian with either the Dirichlet or the Neumann condition:
\[ \Delta e_k = -\rho_k e_k, \]
\[ e_k|_{\partial \Omega} = 0 \quad \text{or} \quad \frac{\partial e_k}{\partial n}|_{\partial \Omega} = 0. \]

(3.2)
Denote by $M_k$ the matrix given by
\[ M_k = \begin{pmatrix} -\mu_1 \rho_k + \lambda - 1 & \alpha^2 \\ -\mu_2 \rho_k - \alpha^2 \\ \end{pmatrix}, \quad k = 1, 2, \ldots. \]

It is clear that all eigenvalues $\beta_k^\pm$ and eigenvectors $\phi_k^\pm$ of (3.1) satisfy the following equations:
\[ \phi_k^\pm = \xi_k^\pm e_k, \]
\[ M_k \xi_k^\pm = \beta_k^\pm \xi_k^\pm, \quad (3.3) \]
where $\xi_k^\pm \in \mathbb{R}^2$ are the eigenvectors of $M_k$, $\beta_k^\pm$ are the eigenvalues of $M_k$, which are expressed as
\[ \beta_k^\pm(\lambda) = \frac{1}{2}[\lambda - (\mu_1 \rho_k + \mu_2 \rho_k + \alpha^2 + 1)] \]
\[ \pm \frac{1}{2}[(\lambda - \mu_1 \rho_k - \mu_2 \rho_k - \alpha^2 - 1)^2 \]
\[ + 4(\lambda \mu_2 \rho_k - (\mu_1 \rho_k + 1)(\mu_2 \rho_k + \alpha^2))]^{1/2}. \quad (3.4) \]

It is clear that $\beta_k^-(\lambda) < \beta_k^+(\lambda) = 0$ if and only if
\[ \lambda = \frac{1}{\mu_2 \rho_k} (\mu_1 \rho_k + 1)(\mu_2 \rho_k + \alpha^2), \]
\[ \lambda < \mu_1 \rho_k + \mu_2 \rho_k + \alpha^2 + 1, \]
and $\beta_k^+(\lambda) = \pm \sigma_k(\lambda)i$ with $\sigma_k \neq 0$ if and only if
\[ \lambda = \mu_1 \rho_k + \mu_2 \rho_k + \alpha^2 + 1, \]
\[ \lambda < \frac{1}{\mu_2 \rho_k} (\mu_1 \rho_k + 1)(\mu_2 \rho_k + \alpha^2). \]

Thus we introduce two critical numbers
\[ \lambda_0 = \min_{\rho_k} \frac{1}{\mu_2 \rho_k} (\mu_1 \rho_k + 1)(\mu_2 \rho_k + \alpha^2), \quad (3.5) \]
\[ \lambda_1 = \mu_1 \rho_1 + \mu_2 \rho_1 + \alpha^2 + 1. \quad (3.6) \]

Obviously, the following lemma holds true.

**Lemma 3.1:** Let $\lambda_0$ and $\lambda_1$ be the two numbers given by (3.5) and (3.6). Then we have the following assertions:

1. Let $\lambda_0 < \lambda_1$, and $k_0 \geq 1$ be the integer such that the minimum is achieved at $\rho_{k_0}$ in the definition of $\lambda_0$. Then $\beta_k^+(\lambda)$ is the first real eigenvalue of (3.1) near $\lambda = \lambda_0$ satisfying that
\[ \beta_k^+(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_0 \\ = 0 & \text{if } \lambda = \lambda_0 \\ > 0 & \text{if } \lambda > \lambda_0 \end{cases} \quad \forall k \in \mathbb{N} \text{ with } \rho_k = \rho_{k_0}, \quad (3.7) \]

2. Let $\lambda_1 < \lambda_0$. Then $\beta_1^+(\lambda) = \beta_1^-(\lambda)$ are a pair of first complex eigenvalues of (3.1) near $\lambda = \lambda_1$, and
\[ \text{Re} \beta_1^+(\lambda) = \text{Re} \beta_1^-(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_1, \\ = 0 & \text{if } \lambda = \lambda_1, \\ > 0 & \text{if } \lambda > \lambda_1, \end{cases} \quad (3.8) \]
\[ \text{Re} \beta_k^\pm(\lambda_1) < 0 \quad \forall k > 1. \]
IV. TRANSITION FROM REAL EIGENVALUES

Hereafter, we always assume that the eigenvalue $\beta^+_0$ in (3.7) is simple. Based on Lemma 3.1, as $\lambda_0 < \lambda_1$ the transition of (2.7) occurs at $\lambda = \lambda_0$, which is from real eigenvalues. Let $\rho_0$ be as in Lemma 3.1, and $e_0$ the eigenvector of (3.2) corresponding to $\rho_0$ satisfying
\[
\int_\Omega e_0^2 \, dx \neq 0.
\]  

Then, under the condition (4.1), for the system (2.3) with (2.4) or with (2.5) we have the following transition theorem.

Theorem 4.1: Let $\lambda_0 < \lambda_1$. Then the system (2.7) has a transition at $\lambda = \lambda_0$, which is mixed (Type-III). In particular, the system bifurcates on each side of $\lambda = \lambda_0$ to a unique branch $v^\pm$ of steady state solutions, such that the following assertions hold true:

1. On $\lambda < \lambda_0$, the bifurcated solution $v^-$ is a saddle, and the stable manifold $\Gamma^+_1$ of $v^-$ separates the space $\mathcal{H}$ into two disjoint open sets $U^+_1$ and $U^-_1$, such that $v = 0 \in U^+_1$ is an attractor, and the orbits of (2.7) in $U^-_1$ are far from $v = 0$.

2. On $\lambda > \lambda_0$, the stable manifold $\Gamma^-_1$ of $v = 0$ separates the neighborhood $\mathcal{O}$ of $u = 0$ into two disjoint open sets $\mathcal{O}^+_1$ and $\mathcal{O}^-_1$, such that the transition is jump in $\mathcal{O}^+_1$, and is continuous in $\mathcal{O}^-_1$. The bifurcated solution $v^- \in \mathcal{O}^-_1$ is an attractor such that for any $\varphi \in \mathcal{O}^-_2$, 
\[
\lim_{t \to \infty} \|v(t, \varphi) - v^-\|_H = 0,
\]
where $v(t, \varphi)$ is the solution of (2.7) with $v(0, \varphi) = \varphi$.

3. The bifurcated solution $v^+$ can be expressed as 
\[
v^+ = C\beta^+_0(\lambda)e^+_0 + o(|\beta^+_0|),
\]
\[
\xi^+_0 = (-\mu_2 \rho_0, \mu_1 \rho_0 + 1),
\]
\[
C = \frac{(\alpha \mu_2 \rho_0 (\mu_2 \rho_0 + \alpha^2) - \alpha^3 (\mu_1 \rho_0 + 1)) \int_\Omega e_0^2 \, dx}{2 \mu_2^2 \rho^2_0 (\mu_1 \rho_0 + 1) \int_\Omega e_0^2 \, dx},
\]  

Proof: We apply Theorem A.2 in5 to prove this theorem. Let $\Phi$ be the center manifold function of (2.7) at $\lambda = \lambda_0$. We need to simplify the following expression:
\[
g(y) = \frac{1}{\langle \phi^+_0, \phi^{++}_0 \rangle}(G(y \phi^+_0 + \Phi(y)), \phi^{++}_0),
\]  

where $y \in \mathbb{R}^1$, $G$ is the operator defined by (2.6), $\phi^+_0$ is the eigenvector of (2.5) corresponding to $\beta^+_0(\lambda_0) = 0$, and $\phi^{++}_0$ is the conjugate eigenvector. By (3.3),
\[
\phi^+_0 = \xi^+_0 e_0, \quad \phi^{++}_0 = \xi^{++}_0 e_0,
\]
\[
\left(\begin{array}{cc}
\lambda_0 - (\mu_1 \rho_0 + 1) & \alpha^2 \\
-\lambda_0 & -(\mu_2 \rho_0 + \alpha^2)
\end{array}\right)
\left(\begin{array}{c}
\xi^+_0 \\
\xi^{++}_0
\end{array}\right) = 0,
\]  

\[
\left(\begin{array}{cc}
\lambda_0 - (\mu_1 \rho_0 + 1) & -\lambda_0 \\
\alpha^2 & -(\mu_2 \rho_0 + \alpha^2)
\end{array}\right)
\left(\begin{array}{c}
\xi^{++}_0 \\
\xi^{++}_0
\end{array}\right) = 0.
\]
By definition of \( \lambda_0 \) and \( k_0 \), we infer from (4.5) and (4.6) that

\[
\begin{aligned}
\xi^+_0 &= (\xi^+_{k_01}, \xi^+_{k_02}) = (-\mu_2 \rho_{k_0}, \mu_1 \rho_{k_0} + 1), \\
\xi^{++}_0 &= (\xi^{++}_{k_01}, \xi^{++}_{k_02}) = (\mu_2 \rho_{k_0} + \alpha^2, \alpha^2).
\end{aligned}
\]

Denote by \( o(k) = o(|y|^k) \). By \( \Phi(y) = o(1) \), the function \( g(y) \) in (4.3) is rewritten as

\[
g(y) = \frac{1}{(\phi^+_1, \phi^{++}_1)} (G(y \phi^+_1, \phi^{++}_1) + o(2)).
\]

By (4.4) and (4.7) we see that

\[
G(y \phi^+_1, \phi^{++}_1) = \begin{cases}
2y^2 \left( \frac{\lambda_0}{\alpha} \mu^2_2 \rho^2_{k_0} - \alpha \mu_2 \rho_{k_0}(\mu_1 \rho_{k_0} + 1) \right) e^2_k k_0 + o(2), \\
-2y^2 \left( \frac{\lambda_0}{\alpha} \mu^2_2 \rho^2_{k_0} - \alpha \mu_2 \rho_{k_0}(\mu_1 \rho_{k_0} + 1) \right) e^2_k + o(2).
\end{cases}
\]

Thus we deduce from (4.4) and (4.7) and (4.8) that

\[
(\phi^+_1, \phi^{++}_1) = (\alpha^2 (\mu_1 \rho_{k_0} + 1) - \mu_2 \rho_{k_0}(\mu_2 \rho_{k_0} + \alpha^2) \int \Omega e^2_k dx, \alpha^2 \frac{2 \mu^2_2 \rho^2_{k_0}(\mu_1 \rho_{k_0} + 1)}{\alpha} \int \Omega e^2_k dx + o(2)).
\]

Therefore the function (4.3) is given as

\[
g(y) = -\frac{1}{c} y^2 + o(2),
\]

and the theorem follows from Theorem A.2 in Ref. 8. The proof is complete. \( \square \)

**Remark 4.1:** If the domain \( \Omega \neq (0, L) \times D \) with \( D \subset \mathbb{R}^2 \) being a bounded open set, then the condition (4.1) holds true for both the Dirichlet and Neumann boundary conditions (2.4) and (2.5). If \( \Omega = (0, L) \times D \), then (4.1) is not true for the Neumann condition (2.5), and is not true for the Dirichlet condition (2.4) if the number \( m \) in \( \rho_{k_0} = \frac{m^2 \pi^2}{L^2} + \rho'_{k_0} \) is even. Here \( \rho'_{k_0} \) is an eigenvalue of the equation on \( D 
\begin{align*}
-\Delta e &= \rho'_{k_0} e \quad \text{for} \ x \in D, \\
e|_{\partial D} &= 0.
\end{align*}
\]

Now, we consider the case where (4.1) is not true, i.e.,

\[
\int \Omega e^1_{k_0} dx = 0. \tag{4.9}
\]

As mentioned in Remark 4.1, the condition (4.9) may hold true if \( \Omega = (0, L) \times D \). We introduce the following parameter:

\[
b_1 = \left[ \alpha^2 (\mu_1 \rho_{k_0} + 1) - \mu_2 \rho_{k_0}(\mu_2 \rho_{k_0} + \alpha^2) \int \Omega e^2_k dx - 2(\mu_1 \rho_{k_0} + 1)(2 \mu_2 \rho_{k_0} + \alpha^2) \int \Omega \psi^2 e^2_k dx \right]^{-1}.
\]

\[
\begin{align*}
&= \left[ \alpha^2 (\mu_1 \rho_{k_0} + 1) - \mu_2 \rho_{k_0}(\mu_2 \rho_{k_0} + \alpha^2) \right]^{-1}.
\end{align*}
\]
where \( \psi = (\psi_1, \psi_2) \) satisfies
\[
\begin{align*}
\mu_1 \Delta \psi_1 + (\lambda_0 - 1) \psi_1 + \alpha^2 \psi_2 &= -\frac{2 \mu_2^2 \rho_k^2}{\alpha} (\mu_1 \rho_k + 1) e_k^2, \\
\mu_2 \Delta \psi_2 - \lambda_0 \psi_1 - \alpha^2 \psi_2 &= \frac{2 \mu_2^2 \rho_k^2}{\alpha} (\mu_1 \rho_k + 1) e_k^2, \\
\psi|_{\partial \Omega} = 0 \text{ (or } \frac{\partial \psi}{\partial n}|_{\partial \Omega} = 0). 
\end{align*}
\] (4.11)

By the Fredholm Alternative Theorem, under the condition (4.9), the equation (4.11) has a unique solution.

**Theorem 4.2:** Let (4.9) hold true, \( \lambda_0 < \lambda_1 \), and \( b_1 \) is the number given by (4.10). Then the transition of (2.7) at \( \lambda = \lambda_0 \) is continuous if \( b_1 < 0 \), and is jump if \( b_1 > 0 \). Moreover, the following assertions hold true:

1. **If** \( b_1 > 0 \), (2.7) **has no bifurcation on** \( \lambda > \lambda_0 \), **and has exact two bifurcated solutions** \( v^+_\lambda \) **and** \( v^-_\lambda \), **which are saddles. Moreover, the stable manifolds** \( \Gamma^+_\lambda \) **and** \( \Gamma^-_\lambda \) **of the two bifurcated solutions divide** \( \mathcal{H} \) **into three disjoint open sets** \( \mathcal{U}^+_\lambda, \mathcal{U}^-_\lambda, \mathcal{U}_0^* \) **such that** \( v = 0 \in \mathcal{U}^*_0 \) **is an attractor, and the orbits in** \( \mathcal{U}^+_\lambda \) **are far from** \( v = 0 \).

2. **If** \( b_1 < 0 \), (2.7) **has no bifurcation on** \( \lambda < \lambda_0 \), **and has exact two bifurcated solutions** \( v^+_\lambda \) **and** \( v^-_\lambda \), **which are attractors. In addition, there is a neighborhood** \( \mathcal{O} \subset \mathcal{H} \) **of** \( v = 0 \), **such that the stable manifold** \( \Gamma \) **of** \( v = 0 \) **divides** \( \mathcal{O} \) **into two disjoint open sets** \( \mathcal{O}^+_\lambda, \mathcal{O}^-_\lambda \) **such that** \( v^+_\lambda \in \mathcal{O}^+_\lambda, v^-_\lambda \in \mathcal{O}^-_\lambda, \) **and** \( v^+_\lambda \) **attracts** \( \mathcal{O}^+ \).

3. **The bifurcated solutions** \( v^\pm_\lambda \) **can be expressed as**
\[
\begin{align*}
v^\pm_\lambda &= \pm C (\beta_{k_0}^\pm(\lambda))^1/2 \xi^\pm_{k_0} e_{k_0} + o(|\beta_{k_0}^\pm|1/2), \\
\xi^\pm_{k_0} &= (-\mu_2 \rho_k, \mu_1 \rho_k + 1), \\
C &= \left[ -\frac{\alpha}{\mu_2 \rho_k b_1} \int_{\mathcal{H}} e_{k_0}^2 dx \right]^{1/2},
\end{align*}
\] (4.12)

**Proof:** We use Theorem A.1 in Ref. 8 to prove this theorem. To get the function \( g(y) \) in (4.3), we need to calculate the center manifold function \( \Phi(y) \). By (A.10) in Ref. 10 \( \Phi(y) \) satisfies
\[
L_{\lambda_0} \Phi = -P_2 G(y \phi_{k_0}^\pm),
\] (4.13)
where \( P_2 : \mathcal{H} \rightarrow E_2 \) is the canonical projection, \( L_{\lambda} \) as in (2.6), \( \phi_{k_0}^+ \) **and** \( \phi_{k_0}^{++} \) **are given by** (4.4), and
\[
E_2 = \{ v \in \mathcal{H}| (v, \phi_{k_0}^{++}) = 0 \}.
\]

We see that
\[
G(y \phi_{k_0}^+) = \left\{ \begin{array}{ll}
2 \alpha^{-1} \mu_2^2 \rho_k^2 (\mu_1 \rho_k + 1) y^2 e_{k_0}^2 + o(2), \\
-2 \alpha^{-1} \mu_2^2 \rho_k^2 (\mu_1 \rho_k + 1) y^2 e_{k_0}^2 + o(2).
\end{array} \right.
\]

Let
\[
\Phi = \psi y^2 + o(2), \quad \psi = (\psi_1, \psi_2) \in \mathcal{H}.
\] (4.14)

By (4.9), \( (e_{k_0}^2, -e_{k_0}^2) \in E_2 \). Hence, it follows from (4.13) and (4.14) that
\[
L_{\lambda_0} \psi = -2 \alpha^{-1} \mu_2^2 \rho_k^2 (\mu_1 \rho_k + 1) (e_{k_0}^2, -e_{k_0}^2),
\] (4.15)
which is an equivalent form of (4.11).
By (4.14), we have
\[ G(y\phi_{k_0}^+ \Phi) = G(y\phi_{k_0}^+ + y^2\psi) + o(3), \]
\[ G(y\phi_{k_0}^+ + y^2\psi) = \begin{cases} 
O(y^3)e_{k_0}^3 + y^3 [\mu_2^2 \rho_{k_0}^2 (\mu_1 \rho_{k_0} + 1)]e_{k_0}^3 - 2\alpha \mu_2 \rho_{k_0} \psi_{k_0} e_{k_0} \\
- 2(\mu_1 \rho_{k_0} + 1)(2\mu_2 \rho_{k_0} + \alpha^2)] + o(3), 
\end{cases} \]
\[ O(y^3)e_{k_0}^3 - y^3 [\mu_2^2 \rho_{k_0}^2 (\mu_1 \rho_{k_0} + 1)]e_{k_0}^3 - 2\alpha \mu_2 \rho_{k_0} \psi_{k_0} e_{k_0} \\
- 2(\mu_1 \rho_{k_0} + 1)(2\mu_2 \rho_{k_0} + \alpha^2)] + o(3). \]

Hence we deduce from (4.4), (4.8), and (4.9) that
\[ (G(y\phi_{k_0}^+ \Phi), \Phi_{k_0}^+) = \frac{\mu_2 \rho_{k_0}^3}{\alpha} \left[ \alpha \mu_2^2 \rho_{k_0}^2 (\mu_1 \rho_{k_0} + 1) \int_{\Omega} e_{k_0}^4 dx \right. \\
\left. - 2(\mu_1 \rho_{k_0} + 1)(2\mu_2 \rho_{k_0} + \alpha^2) \int_{\Omega} \psi_{k_0} \psi_{k_0}^2 dx \right] + o(3). \]

Thus, the function \( g(y) \) in (4.3) can be written as
\[ g(y) = \frac{\mu_2 \rho_{k_0} b_1}{\alpha \int_{\Omega} e_{k_0}^3 dx} y^3 + o(3), \]
where \( b_1 \) is as in (4.10). Hence the theorem follows from Theorem A.1 in Ref. 8.

When the domain \( \Omega \) is a rectangle, i.e., \( \Omega = \prod_{j=1}^n (0, L_j) \), the \( b_1 \) in (4.10) for the Neumann condition can be explicitly expressed in terms of the physical parameters \( \mu_1, \mu_2, \alpha, \) and \( L (1 \leq j \leq n) \). For example, we consider the case where \( \Omega = (0, L) \). The eigenvalues \( \rho_k \) and eigenvectors \( e_k \) of (3.2) are given by
\[ \rho_k = \frac{(k - 1)^2 \pi^2}{L^2}, \quad e_k = \cos \left( \frac{(k - 1)\pi}{L} x \right), \quad k = 1, 2, \ldots. \]

It is clear that \( k_0 \geq 2 \), and (4.9) holds true. We see that
\[ e_{k_0}^2 = \frac{1}{2} \left( 1 + \cos \frac{2(k_0 - 1)\pi}{L} x \right) = \frac{1}{2} [e_1 + e_{2k_0 - 1}]. \]

Hence, by (4.11), we have
\[ \psi = \xi e_1 + \eta e_j \quad \text{with} \quad j = 2k_0 - 1, \quad (4.16) \]
where
\[ \begin{pmatrix} \lambda_0 - 1 & \alpha^2 \\
-\lambda_0 & -\alpha^2 \end{pmatrix} \begin{pmatrix} \xi_1 \\
\xi_2 \end{pmatrix} = \begin{pmatrix} -\frac{\mu_2^2}{\alpha^2} \rho_{k_0}^2 (\mu_1 \rho_{k_0} + 1) \\
\frac{1}{\alpha^2} \mu_2^2 \rho_{k_0}^2 (\mu_1 \rho_{k_0} + 1) \end{pmatrix}, \]
\[ \begin{pmatrix} \lambda_0 - (\mu_1 \rho_j + 1) & \alpha^2 \\
-\lambda_0 & -(\mu_2 \rho_j + \alpha^2) \end{pmatrix} \begin{pmatrix} \eta_1 \\
\eta_2 \end{pmatrix} = \begin{pmatrix} -\frac{\mu_2^2}{\alpha^2} \rho_{k_0}^2 (\mu_1 \rho_{k_0} + 1) \\
\frac{1}{\alpha^2} \mu_2^2 \rho_{k_0}^2 (\mu_1 \rho_{k_0} + 1) \end{pmatrix}, \]
\[ \lambda_0 = \frac{1}{\mu_2 \rho_{k_0}^2} (\mu_1 \rho_{k_0} + 1)(\mu_2 \rho_{k_0} + \alpha^2). \]

It is readily to see that
\[ \xi_1 = 0, \]
\[ \xi_2 = -\alpha^3 \frac{\mu_2^2}{\alpha^2} \rho_{k_0}^2 (\mu_1 \rho_{k_0} + 1), \]
\[ \eta_1 = \frac{\mu_2^2 \rho_j \rho_{k_0}^2 (\mu_1 \rho_{k_0} + 1)}{\alpha [(\mu_2 \rho_j + \alpha^2)(\mu_1 \rho_j + 1) - \mu_2 \rho_j \lambda_0]}, \]
\[ \eta_2 = \frac{-\mu_2^2 \rho_{k_0}^2 (\mu_1 \rho_j + 1)(\mu_1 \rho_{k_0} + 1)}{\alpha [(\mu_2 \rho_j + \alpha^2)(\mu_1 \rho_j + 1) - \mu_2 \rho_j \lambda_0]}\]
Inserting (4.16) and (4.17) into (4.10), we derive that
\[
b_1 = \frac{\mu_2^2 \rho_{k_0}^2 (\mu_1 \rho_{k_0} + 1)}{\alpha} \left[ \mu_2 \rho_{k_0} + \frac{3}{8} \alpha^2 \right] + \frac{\alpha^2 (\mu_1 \rho_j + 1) \mu_2 \rho_{k_0} - \frac{1}{2} (\mu_1 \rho_{k_0} + 1) (2 \mu_2 \rho_{k_0} + \alpha^2) \mu_2 \rho_j}{(\mu_1 \rho_j + 1)(\mu_2 \rho_j + \alpha^2) - \mu_2 \rho_j \lambda_0} \times \left[ \alpha^2 (\mu_1 \rho_{k_0} + 1) - \mu_2 \rho_{k_0} (\mu_2 \rho_{k_0} + \alpha^2) \right]^{-1}.
\] (4.18)

Thus, for the one-dimensional domain \( \Omega = (0, L) \), the number \( b_1 \) in (4.18) can be equivalently rewritten as
\[
b_1 = \left[ \mu_2 \rho_{k_0} + \frac{3}{8} \alpha^2 - \frac{\mu_2 \rho_{k_0} (4 \mu_1 \mu_2 \rho_{k_0}^2 + 4 \mu_2 \rho_{k_0} - 2 \alpha^2 \mu_1 \rho_{k_0} + \alpha^2)}{12 \mu_1 \mu_2 \rho_{k_0}^2 - 3 \alpha^2} \right] \times \left[ \alpha^2 (\mu_1 \rho_{k_0} + 1) - \mu_2 \rho_{k_0} (\mu_2 \rho_{k_0} + \alpha^2) \right]^{-1}.
\] (4.19)

V. TRANSITION FROM COMPLEX EIGENVALUES

As \( \lambda_1 < \lambda_0 \), the transition of (2.7) occurs at \( \lambda = \lambda_1 \), and the system bifurcates to a periodic solution.

We first consider the Neumann boundary condition. In this case, \( \lambda_1 = \alpha^2 + 1 \), and we have the following theorem.

**Theorem 5.1:** For the problem (2.3) with (2.5), when \( \lambda_1 < \lambda_0 \), the transition at \( \lambda = \lambda_1 \) is continuous, and the problem bifurcates on \( \lambda > \lambda_1 \) to one periodic solution which is an attractor. Moreover, the bifurcated periodic solution \( v^\lambda = (v_1^\lambda, v_2^\lambda) \) can be expressed as
\[
v_1^\lambda = 2 (C \beta_0 (\lambda))^{1/2} \alpha \sin(\alpha t + \pi/4) + o(|\beta_0|^{1/2}),
\]
\[
v_2^\lambda = [C \beta_0 (\lambda)]^{1/2} \alpha (\alpha - 1) \cos \alpha t - (\alpha + 1) \sin \alpha t + o(|\beta_0|^{1/2}),
\] (5.1)

where \( \beta_0 = \frac{1}{2} (\lambda - \lambda_1) \), and \( C = (2 \pi \alpha^2 + \frac{3}{2} \pi \alpha^4)^{-1} \).

**Proof:** We shall verify this theorem by using Theorem A.3 in Ref. 3. The eigenvalue \( \beta_1^\pm(\lambda) \) in (3.4) are given by
\[
\beta_1^\pm(\lambda) = \frac{1}{2} (\lambda - \lambda_1) \pm \frac{i}{2} \sqrt{4 \alpha^2 - (\lambda - \lambda_1)^2}, \quad (\lambda_1 = \alpha^2 + 1).
\]
Namely, for \( \lambda \) near \( \lambda_1 \),
\[
\beta_0 (\lambda) = \text{Re} \beta_1^\pm(\lambda) = \frac{1}{2} (\lambda - \lambda_1),
\]
\[
\beta_1^\pm(\lambda_1) = \pm i \alpha.
\]

The eigenvectors \( \xi \) and \( \eta \) corresponding to \( \beta_1^\pm(\lambda_1) \) satisfy
\[
\begin{pmatrix}
\alpha^2 & \alpha^2 \\
-(\alpha^2 + 1) & -\alpha^2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= \alpha
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix},
\]
\[
\begin{pmatrix}
\alpha^2 & \alpha^2 \\
-(\alpha^2 + 1) & -\alpha^2
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}
= -\alpha
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}.
\]
It is readily to check that
\[
\xi = (\xi_1, \xi_2) = (\alpha^2, \alpha(1 - \alpha)), \quad \eta = (\eta_1, \eta_2) = (\alpha^2, -\alpha(\alpha + 1)).
\] (5.2)

The conjugate eigenvectors \(\xi^*\) and \(\eta^*\) satisfy
\[
\begin{pmatrix}
\alpha^2 - (\alpha^2 + 1) \\
\alpha^2 - \alpha^2
\end{pmatrix}
\begin{pmatrix}
\xi_1^* \\
\xi_2^*
\end{pmatrix}
= \alpha
\begin{pmatrix}
\eta_1^* \\
\eta_2^*
\end{pmatrix},
\]
leading to
\[
\xi^* = (\xi_1^*, \xi_2^*) = (\alpha(\alpha + 1), \alpha^2), \\
\eta^* = (\eta_1^*, \eta_2^*) = (\alpha(\alpha - 1), \alpha^2).
\] (5.3)

It is readily to check that
\[
(\xi, \xi^*) = -(\eta, \eta^*) = 2\alpha^3 \int_{\Omega} e_1^3 dx = 2\alpha^3 |\Omega|,
\]
\[
(\xi, \eta^*) = (\eta, \xi^*) = 0.
\] (5.4)

For the operator \(G\) defined by (2.6), we deduce from (5.2) that for \(x, y \in \mathbb{R}^1\),
\[
G(x\xi + y\eta) = \begin{cases}
\frac{2\lambda_1}{\alpha}(x\xi_1 + y\eta_1)^2 + 2\alpha(x\xi_1 + y\eta_1)(x\xi_2 + y\eta_2) + (x\xi_1 + y\eta_1)^2(x\xi_2 + y\eta_2), \\
- \frac{2\lambda_1}{\alpha}(x\xi_1 + y\eta_1)^2 - 2\alpha(x\xi_1 + y\eta_1)(x\xi_2 + y\eta_2) - (x\xi_1 + y\eta_1)^2(x\xi_2 + y\eta_2),
\end{cases}
\]
\[
= 2\alpha^3[(x + y)^2 + \alpha(x^2 - y^2)] + \alpha^2[(x + y)(x^2 - y^2) - \alpha(x + y)^2],
\]
\[
- 2\alpha^3[(x + y)^2 + \alpha(x^2 - y^2)] - \alpha^2[(x + y)(x^2 - y^2) - \alpha(x + y)^2].
\]

Because the first eigenvector space \(E_1 = \text{span}\{\xi, \eta\}\) of (3.1) with (2.5) is invariant for the equations (2.3) with (2.5), the center manifold function \(\Phi\) vanishes, i.e.,
\[
\Phi(x, y) \equiv 0.
\]

Therefore, we derive from (5.2) to (5.5) that
\[
\begin{aligned}
\frac{(G(x\xi + y\eta + \Phi), \xi^*)}{(\xi, \xi^*)} &= a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\
\frac{(G(x\xi + y\eta + \Phi), \eta^*)}{(\eta, \eta^*)} &= b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3,
\end{aligned}
\]
where
\[
\begin{aligned}
a_{20} &= \alpha(\alpha + 1), & a_{02} &= \alpha(1 - \alpha), \\
b_{20} &= \alpha(\alpha + 1), & b_{02} &= \alpha(1 - \alpha), \\
a_{11} &= 2\alpha, & b_{11} &= 2\alpha, \\
a_{30} &= \frac{1}{2}\alpha^3(1 - \alpha), & b_{03} &= -\frac{1}{2}\alpha^3(1 + \alpha), \\
a_{12} &= -\frac{1}{2}\alpha^3(1 + 3\alpha), & b_{21} &= \frac{1}{2}\alpha^3(1 - 3\alpha).
\end{aligned}
\]
Then, the parameter $b$ in Theorem A.3 in Ref. 3 is
\[
    b = \frac{\pi}{2 a} (a_{02} b_{02} - a_{20} b_{20}) + \frac{\pi}{4 a} (a_{11} a_{20} + a_{11} a_{02} - b_{11} b_{20} - b_{11} b_{02}) + \frac{3 \pi}{4} (a_{30} + b_{03}) + \frac{\pi}{4} (a_{12} + b_{21}) = -\pi a^2 (2 + \frac{3}{2} a^2).
\]

Namely, $b < 0$. Hence, by Theorem A.3 in Ref. 3 the system (2.7) bifurcates from $(\nu, \lambda) = (0, \lambda_1)$ to a periodic solution on $\lambda > \lambda_1$, which is an attractor. The proof of the expression (5.1) is classical. Thus the theorem is proved.

Now, we consider the Dirichlet boundary condition. In this case, $\rho_1 > 0$ and $\lambda_1 = (\mu_1 + \mu_2) \rho_1 + \alpha^2 + 1$. By (3.5) and (3.6) it is easy to see that as $\lambda_1 < \lambda_0$ we have
\[
    \mu_2 \rho_1 (\mu_2 \rho_1 + \alpha^2) < \alpha^2 (\mu_1 \rho_1 + 1).
\]

Then we define the following parameter:
\[
b_1 = \frac{2 \pi a^2 \int_{\Omega} e_i^2 dx}{\sigma_0^2 \int_{\Omega} e_i^2 dx} (\mu_1 \rho_1 + 1)(\mu_2 \rho_1^2 + 2 \mu_2 \rho_1 + \mu_1 \rho_1 + 1 - \sigma_0^2)

\]
\[
- \frac{\pi a^2}{2} \int_{\Omega} e_i^2 dx (2 (\mu_2 \rho_1 + 3 \alpha^2)

\]
\[
+ \frac{2 \pi a^2}{2} \int_{\Omega} e_i^2 dx [(3 \mu_1 \rho_1 + \mu_2 \rho_1 + 3) A_1 + (\mu_1 \mu_2 \rho_1^2 + \mu_2 \rho_1 + \delta_0) B_1]

\]
\[
- 8 \pi a^2 \sigma_0^2 \int_{\Omega} e_i^2 dx [(\mu_1 \rho_1 + \mu_2 \rho_1 + 1) A_2 - (\mu_1 \mu_2 \rho_1^2 + \mu_2 \rho_1 - \sigma_0) B_2]

\]
\[
- \frac{4 \pi a^2}{2} \int_{\Omega} e_i^2 dx [(\mu_1 \mu_2 \rho_1^2 + \mu_2 \rho_1 - \sigma_0^2) A_3 + \sigma_0^2 (\mu_1 \rho_1 + \mu_2 \rho_1 + 1) B_3],
\]

where
\[
\sigma_0 = [\alpha^2 (\mu_1 \rho_1 + 1) - \mu_2 \rho_1 (\mu_2 \rho_1 + \alpha^2)]^{1/2},
\]
\[
A_i = (2 \lambda_1 - \mu_2 \rho_1 - \alpha^2) B_i + \alpha^2 C_i \quad \text{for} \ i = 1, 2, 3,
\]
\[
B_1 = \sum_{k>1} \frac{\mu_2 \rho_k [\int_{\Omega} e_i^2 e_k dx]^2}{\det M_k \int_{\Omega} e_i^2 dx},
\]
\[
B_2 = \sum_{k>1} \frac{[\int_{\Omega} e_i^2 e_k dx]^2}{\det M_k \det (M_k^2 + 4 \sigma_0^2) \int_{\Omega} e_i^2 dx} [(\mu_2 \rho_k + \alpha^2) D_k + \alpha^2 Q_k],
\]
\[
B_3 = \sum_{k>1} \frac{[\int_{\Omega} e_i^2 e_k dx]^2}{\det (M_k^2 + 4 \sigma_0^2) \int_{\Omega} e_i^2 dx} D_k,
\]
\[
C_1 = - \sum_{k>1} \frac{(\mu_1 \rho_1 + 1) [\int_{\Omega} e_i^2 e_k dx]^2}{\det M_k \int_{\Omega} e_i^2 dx},
\]
\[
C_2 = \sum_{k>1} \frac{[\int_{\Omega} e_i^2 e_k dx]^2}{\det M_k \det (M_k^2 + 4 \sigma_0^2) \int_{\Omega} e_i^2 dx} [(\mu_1 \rho_k + 1 - \lambda_1) Q_k - \lambda_1 D_k],
\]
\[
C_3 = \sum_{k>1} \frac{[\int_{\Omega} e_i^2 e_k dx]^2}{\det (M_k^2 + 4 \sigma_0^2) \int_{\Omega} e_i^2 dx} Q_k,
\]
The bifurcated periodic solution

It is easy to see that

Then, by (3.3) the eigenvectors of (3.1) corresponding to \( \lambda_1 \) are determined by the matrices \( M_k \) given by (5.7).

\[
M_k = \begin{pmatrix}
-\mu_1 \rho_k + \lambda_1 - 1 & \alpha^2 \\
-\lambda_1 & -\mu_2 \rho_k - \alpha^2
\end{pmatrix}
\]  

(5.7)

**Theorem 5.2**: Let \( b_1 \) be the number given by (5.6) and \( \lambda_1 < \lambda_0 \). For the problem (2.3) and (2.4), the following assertions hold true:

1. **(1)** The problem undergoes a dynamic transition at \( \lambda = \lambda_1 \), which is the Hopf bifurcation.
2. **(2)** When \( b_1 < 0 \), the transition is of Type-I and bifurcates to a stable periodic solution on \( \lambda > \lambda_1 \), and when \( b_1 > 0 \) the transition is of Type-II and bifurcates to an unstable periodic solution on \( \lambda < \lambda_1 \).
3. **(3)** The bifurcated periodic solution \( v^\lambda = (v_1^\lambda, v_2^\lambda) \) can be expressed as

\[
v_1^\lambda = 2\alpha^2[-\gamma(\lambda)/b_1]^{1/2}e_1 \sin(\sigma_0 t + \pi/4) + o(|\gamma|^{1/2}),
\]

\[
v_2^\lambda = 2(\sigma_0^2 + (\mu_2 \rho_1 + \alpha^2)^2)[-\frac{\gamma(\lambda)}{b_1}]^{12}e_1 \cos(\sigma_0 t + \theta) + o(|\gamma|^{1/2}),
\]

\[
\theta = \tan^{-1} \frac{\sigma_0 + \mu_2 \rho_1 + \alpha^2}{\sigma_0 - \mu_2 \rho_1 - \alpha^2},
\]

where \( \gamma = (\lambda - \lambda_1)/2 \).

**Proof**: By (3.3) the eigenvalues and eigenvectors of (3.1) with (2.4) at \( \lambda_1 = (\mu_1 + \mu_2)\rho_1 + \alpha^2 + 1 \) are determined by the matrices \( M_k \) given by (5.7). It is clear that \( M_1 \) has a pair of imaginary eigenvalues

\[
\beta_1^{\pm}(\lambda_1) = \pm i \sigma_0 \quad [\sigma_0 \text{ as in } (5.6)].
\]

Let \( \tilde{\xi}, \tilde{\eta} \in \mathbb{R}^2 \) be the eigenvectors of \( M_1 \) satisfying

\[
M_1 \tilde{\xi} = \sigma_0 \tilde{\eta}, \quad M_1 \tilde{\eta} = -\sigma_0 \tilde{\xi}.
\]

Then, by (3.3) the eigenvectors of (3.1) corresponding to \( \beta_1^{\pm}(\lambda_1) \) are given by \( \xi = \tilde{\xi} e_1 \) and \( \eta = \tilde{\eta} e_1 \).

It is readily to check that

\[
\xi = (\xi_1, \xi_2) = (\alpha^2 e_1, (\sigma_0 - \mu_2 \rho_1 - \alpha^2) e_1),
\]

\[
\eta = (\eta_1, \eta_2) = (\alpha^2 e_1, - (\sigma_0 + \mu_2 \rho_1 + \alpha^2) e_1).
\]

(5.9)  

(5.10)

We consider the conjugate eigenvectors \( \xi^* = \tilde{\xi}^* e_1 \) and \( \eta^* = \tilde{\eta}^* e_1 \) with

\[
M_1^* \xi^* = \sigma_0 \eta^*, \quad M_1^* \eta^* = -\sigma_0 \xi^*,
\]

where \( M_1^* \) is the transpose of \( M_1 \). Direct calculation shows that

\[
\xi^* = (\xi_1^*, \xi_2^*) = ((\sigma_0 + \mu_2 \rho_1 + \alpha^2) e_1, \alpha^2 e_1),
\]

\[
\eta^* = (\eta_1^*, \eta_2^*) = ((-\sigma_0 + \mu_2 \rho_1 + \alpha^2) e_1, \alpha^2 e_1).
\]

(5.11)  

(5.12)

It is easy to see that

\[
(\xi, \eta^*) = (\eta, \xi^*) = 0,
\]

\[
(\xi, \xi^*) = -(\eta, \eta^*) = 2\alpha^2 \sigma_0 \int_{\Omega} e_1^2 dx.
\]

(5.13)
Let \( u = x\xi + y\eta + \Phi(x, y) \in H \) be a solution of (2.3) and (2.4) at \( \lambda = \lambda_1 \), and \( \Phi \) be the center manifold function. By (5.13) the reduced equations of (2.3) and (2.4) read
\[
\begin{align*}
\frac{dx}{dt} &= -\sigma_0 y + \frac{1}{(\xi, \xi^*)} (G(x\xi + y\eta + \Phi), \xi^*), \\
\frac{dy}{dt} &= \sigma_0 x + \frac{1}{(\eta, \eta^*)} (G(x\xi + y\eta + \Phi), \eta^*),
\end{align*}
\]
(5.14)
where the operator \( G \) is given by
\[
G(u) = G_2(u) + G_3(u),
\]
(5.15)
and \( G_k \ (k = 2, 3) \) is a \( k \)-multilinear operator defined by
\[
G_2(u, v) = 2 \left( \frac{\lambda_1}{\alpha} u_1 v_1 + au_1 v_2, -\left( \frac{\lambda_1}{\alpha} u_1 v_1 + au_1 v_2 \right) \right),
\]
(5.16)
\[
G_3(u, v, w) = (u_1 v_1 w_2, -u_1 v_1 w_2),
\]
\[
G_2(u) = G_2(u, u),
\]
\[
G_3(u) = G_3(u, u, u).
\]
Based on (5.9)–(5.13) and (5.15)–(5.16), (5.14) are rewritten as
\[
\begin{align*}
\frac{dx}{dt} &= -\sigma_0 y + a_{20} x^2 + a_{11} x y + a_{02} y^2 + a_{30} x^3 + a_{21} x^2 y \\
&\quad + a_{12} x y^2 + a_{03} y^3 + \frac{x}{(\xi, \xi^*)} (G_2(\xi, \Phi) + G_2(\Phi, \xi), \xi^*) \\
&\quad + \frac{y}{(\xi, \xi^*)} (G_2(\eta, \Phi) + G(\Phi, \eta), \eta^*) + o(3), \\
\frac{dy}{dt} &= \sigma_0 x + b_{20} x^2 + b_{11} x y + b_{02} y^2 + b_{30} x^3 + b_{21} x^2 y \\
&\quad + b_{12} x y^2 + b_{03} y^3 + \frac{x}{(\eta, \eta^*)} (G_2(\xi, \Phi) + G_2(\Phi, \xi), \eta^*) \\
&\quad + \frac{y}{(\eta, \eta^*)} (G_2(\eta, \Phi) + G_2(\Phi, \eta), \eta^*) + o(3),
\end{align*}
\]
(5.17)
where
\[
\begin{align*}
a_{20} &= \frac{(G_2(\xi, \xi), \xi^*)}{(\xi, \xi^*)} = \frac{\alpha}{\sigma_0} \frac{\int_{\Omega} \xi_0^1 dx}{\int_{\Omega} \xi_1^1 dx} (\sigma_0 + \mu_2 \rho_1)(\mu_1 \rho_1 + 1 + \sigma_0), \\
a_{11} &= \frac{(G_2(\xi, \eta) + G_2(\eta, \xi), \xi^*)}{(\xi, \xi^*)} = \frac{2\alpha}{\sigma_0} \frac{\int_{\Omega} \xi_0^1 dx}{\int_{\Omega} \xi_1^1 dx} (\sigma_0 + \mu_2 \rho_1)(\mu_1 \rho_1 + 1), \\
a_{02} &= \frac{(G_2(\eta, \eta), \xi^*)}{(\xi, \xi^*)} = \frac{\alpha}{\sigma_0} \frac{\int_{\Omega} \xi_0^1 dx}{\int_{\Omega} \xi_1^1 dx} (\sigma_0 + \mu_2 \rho_1)(\mu_1 \rho_1 + 1 - \sigma_0), \\
a_{30} &= \frac{(G_3(\xi, \xi, \eta), \xi^*)}{(\xi, \xi^*)} = \frac{\alpha^2}{2\sigma_0} \frac{\int_{\Omega} \xi_0^1 dx}{\int_{\Omega} \xi_1^1 dx} (\sigma_0 + \mu_2 \rho_1)(\sigma_0 - \mu_2 \rho_1 - \alpha^2), \\
a_{21} &= \frac{1}{(\xi, \xi^*)} (G_3(\xi, \xi, \eta) + G_3(\xi, \eta, \xi) + (G_3(\eta, \xi, \xi)), \xi^*)
\end{align*}
\]
\[ a_{12} = \frac{1}{(\xi, \xi^*)} \left( G_3(\xi, \eta) + G_3(\eta, \xi) + G_3(\eta, \eta, \xi, \xi^*) \right) = -\frac{\alpha^2 \int_{\Omega} e_1^4 dx}{2\sigma_0 \int_{\Omega} e_1^2 dx} (\sigma_0 + \mu_2 \rho_1)(\sigma_0 + 3\mu_2 \rho_1 + 3\alpha^2), \]
\[ a_{03} = \frac{\int_{\Omega} e_3^4 dx}{(\xi, \xi^*)} = -\frac{\alpha^2 \int_{\Omega} e_1^4 dx}{2\sigma_0 \int_{\Omega} e_1^2 dx} (\sigma_0 + \mu_2 \rho_1)(\sigma_0 + \mu_2 \rho_1 + \alpha^2), \]
and
\[ b_{20} = \frac{(G_2(\xi, \xi), \eta^*)}{(\eta^*)} = \frac{\alpha \int_{\Omega} e_1^4 dx}{\sigma_0 \int_{\Omega} e_1^2 dx} (\sigma_0 - \mu_2 \rho_1)(\mu_1 \rho_1 + 1 + \sigma_0), \]
\[ b_{11} = \frac{(G_2(\xi, \eta) + G(\eta, \xi), \eta^*)}{(\eta^*)} = \frac{2\alpha \int_{\Omega} e_1^4 dx}{\sigma_0 \int_{\Omega} e_1^2 dx} (\sigma_0 - \mu_2 \rho_1)(\mu_1 \rho_1 + 1), \]
\[ b_{02} = \frac{(G_2(\eta, \eta), \eta^*)}{(\eta, \eta^*)} = \frac{\alpha \int_{\Omega} e_1^4 dx}{\sigma_0 \int_{\Omega} e_1^2 dx} (\sigma_0 - \mu_2 \rho_1)(\mu_1 \rho_1 + 1 - \sigma_0), \]
\[ b_{30} = \frac{(G_3(\xi, \xi, \xi), \eta^*)}{(\eta, \eta^*)} = \frac{\alpha^2 \int_{\Omega} e_1^4 dx}{2\sigma_0 \int_{\Omega} e_1^2 dx} (\sigma_0 - \mu_2 \rho_1)(\sigma_0 - \mu_2 \rho_1 - \alpha^2), \]
\[ b_{21} = \frac{(G_3(\xi, \xi, \eta) + G_3(\xi, \eta, \xi) + G(\eta, \xi, \xi), \eta^*)}{(\eta^*)} = \frac{\alpha^2 \int_{\Omega} e_1^4 dx}{2\sigma_0 \int_{\Omega} e_1^2 dx} (\sigma_0 - \mu_2 \rho_1)(\sigma_0 - 3\mu_2 \rho_1 - 3\alpha^2), \]
\[ b_{12} = \frac{(G_3(\eta, \eta, \xi) + G_3(\eta, \xi, \eta) + G_3(\xi, \eta, \eta), \eta^*)}{(\eta, \eta^*)} = -\frac{\alpha^2 \int_{\Omega} e_1^4 dx}{2\sigma_0 \int_{\Omega} e_1^2 dx} (\sigma_0 - \mu_2 \rho_1)(\sigma_0 + 3\mu_2 \rho_1 + 3\alpha^2), \]
\[ b_{03} = \frac{(G_3(\eta, \eta, \eta), \eta^*)}{(\eta, \eta^*)} = -\frac{\alpha^2 \int_{\Omega} e_1^4 dx}{2\sigma_0 \int_{\Omega} e_1^2 dx} (\sigma_0 - \mu_2 \rho_1)(\sigma_0 + \mu_2 \rho_1 + \alpha^2). \]

We are now in a position to derive the center manifold function \( \Phi \). By (A.10) in Ref. 10
\[ \Phi = \Phi_1 + \Phi_2 + \Phi_3 + o(2), \quad (5.18) \]
where
\[ -L_{\lambda_1} \Phi_1 = P_2 \left[ G_2(\xi, \xi) x^2 + (G_2(\xi, \eta) + G_2(\eta, \xi))xy + G_2(\eta, \eta) y^2 \right], \]
\[ -L_{\lambda_2} + 4\sigma_0^2 L_{\lambda_2} \Phi_2 = 2\sigma_0 P_2 \left[ (G_2(\xi, \xi) - G_2(\eta, \eta))(y^2 - x^2) - 2(G_2(\xi, \eta) + G_2(\eta, \xi))xy \right], \]
\[ [L_{\lambda_1} + 4\sigma_0^2] \Phi_3 = \sigma_0 P_2 \left[ (G_2(\xi, \eta) + G_2(\eta, \xi))(y^2 - x^2) + 2(G_2(\xi, \xi) - G_2(\eta, \eta))xy \right]. \]

\( L_{\lambda_1} = A + B_{\lambda_1} \) is the linear operator defined by (2.6), \( P_2 : H \rightarrow E_2 \) the canonical projection, and \( E_2 = \{u \in H|u(\xi^*) = 0, (u, \eta^*) = 0\} \) is the complement of \( E_1 = \text{span}\{\xi, \eta\} \) in \( H \). Note that the eigenvectors of \( L_{\lambda_1} \) satisfy
\[ \phi_k = \tilde{\phi}_k e_1, \quad \tilde{\phi}_k \in \mathbb{R}^2, \quad k = 1, 2, \ldots, \]
\[ M_k \tilde{\phi}_k = \beta_k \tilde{\phi}_k \quad [M_k \text{ the matrix as in (5.7)}]. \]
Hence, we obtain from (5.9), (5.10), (5.16) and (5.18) that

\[
\Phi_1 = 2\alpha^3[(\mu_1\rho_1 + \sigma_0 + 1)x^2 + 2(\mu_1\rho_1 + 1)xy + (\mu_1\rho_1 - \sigma_0 + 1)y^2] \\
\times \sum_{k=1}^{\infty} \frac{\int_{\Omega} \sigma_k^2 e_k dx}{\int_{\Omega} e_k^2 dx} (-M_k)^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e_k,
\]

\[
\Phi_2 = 8\alpha^3 \sigma_0^2 [\sigma_0(y^2 - x^2) - 2(\mu_1\rho_1 + 1)xy] \sum_{k=1}^{\infty} \frac{\int_{\Omega} \sigma_k^2 e_k dx}{\int_{\Omega} e_k^2 dx} (-M_k)^{-1}(M_k^2 + 4\sigma_0^2)^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e_k,
\]

\[
\Phi_3 = 4\alpha^3 \sigma_0 [\mu_1\rho_1 + 1)(y^2 - x^2) + 2\sigma_0xy] \sum_{k=1}^{\infty} \frac{\int_{\Omega} \sigma_k^2 e_k dx}{\int_{\Omega} e_k^2 dx} (M_k^2 + 4\sigma_0^2)^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e_k.
\]

Direct calculation shows that

\[
(-M_k)^{-1} = \frac{1}{\det M_k} \begin{pmatrix} \mu_2 \rho_k + \alpha^2 & \alpha^2 \\ -\lambda_1 & \mu_1 \rho_k + 1 - \lambda_1 \end{pmatrix},
\]

\[
(M_k^2 + 4\sigma_0^2)^{-1} = \frac{1}{\det(M_k^2 + 4\sigma_0^2)} \begin{pmatrix} \mu_2 \rho_k + \alpha^2 + 4\sigma_0^2 - \lambda_1 \alpha^2 & \alpha^2(\mu_1 + \mu_2)(\rho_k - \rho_1) \\ -\lambda_1(\mu_1 + \mu_2)(\rho_k - \rho_1) & (\mu_1 \rho_k + 1 - \lambda_1)^2 + 4\sigma_0^2 - \lambda_1 \alpha^2 \end{pmatrix}.
\]

Thus we have

\[
\Phi_1 = 2\alpha^3[(\mu_1\rho_1 + 1 + \sigma_0)x^2 + 2(\mu_1\rho_1 + 1)xy + (\mu_1\rho_1 + 1 - \sigma_0)y^2] \begin{pmatrix} E_1 \\ F_1 \end{pmatrix},
\]

\[
\Phi_2 = 8\alpha^3 \sigma_0^2 [-\sigma_0x^2 - 2(\mu_1\rho_1 + 1)xy + \sigma_0y^2] \begin{pmatrix} E_2 \\ F_2 \end{pmatrix},
\]

\[
\Phi_3 = 4\alpha^3 \sigma_0[-(\mu_1\rho_1 + 1)x^2 + 2\sigma_0xy + (\mu_1\rho_1 + 1)y^2] \begin{pmatrix} E_3 \\ F_3 \end{pmatrix},
\]

where

\[
E_1 = \sum_{k=1}^{\infty} \frac{\int_{\Omega} \sigma_k^2 e_k dx}{\int_{\Omega} e_k^2 dx} \mu_2 \rho_k e_k,
\]

\[
F_1 = -\sum_{k=1}^{\infty} \frac{\int_{\Omega} \sigma_k^2 e_k dx}{\int_{\Omega} e_k^2 dx} (\mu_1 \rho_k + 1) e_k,
\]

\[
E_2 = \sum_{k=1}^{\infty} \frac{\int_{\Omega} \sigma_k^2 e_k dx}{\int_{\Omega} e_k^2 dx} (\mu_2 \rho_k + \alpha^2)D_k + \alpha^2 Q_k e_k,
\]

\[
E_3 = \sum_{k=1}^{\infty} \frac{\int_{\Omega} \sigma_k^2 e_k dx}{\int_{\Omega} e_k^2 dx} (\mu_1 \rho_k + 1 - \lambda_1)Q_k - \lambda_1 D_k e_k,
\]

\[
F_3 = \sum_{k=1}^{\infty} \frac{\int_{\Omega} \sigma_k^2 e_k dx}{\int_{\Omega} e_k^2 dx} Q_k e_k.
\]

Inserting \(\Phi = \Phi_1 + \Phi_2 + \Phi_3 + o(2)\) into (5.17) we derive that

\[
\frac{dx}{dt} = -\sigma_0 y + \sum_{2i+j \leq 3} a_{ij} x^i y^j + \sum_{k+r=3} \tilde{a}_{kr} x^k y^r + o(3),
\]

\[
\frac{dy}{dt} = \sigma_0 x + \sum_{2i+j \leq 3} b_{ij} x^i y^j + \sum_{k+r=3} \tilde{b}_{kr} x^k y^r + o(3),
\]

(5.19)
where $a_{ij}$ and $b_{ij}$ ($0 \leq i, j \leq 3$) are as in (5.17), and

\[
\tilde{a}_{30} = \frac{2a^2(\sigma_0 + \mu_2 \rho_1)}{\sigma_0 \int_{\Omega} e_i^2 dx}[(\mu_1 \rho_1 + 1 + \sigma_0)I_1 - 4\sigma_0^3 I_2 - 2\sigma_0(\mu_1 \rho_1 + 1)J_3],
\]

\[
\tilde{a}_{12} = \frac{2a^2(\sigma_0 + \mu_2 \rho_1)}{\sigma_0 \int_{\Omega} e_i^2 dx}[(\mu_1 \rho_1 + 1 - \sigma_0)I_1 + 4\sigma_0^3 I_2 + 2\sigma_0(\mu_1 \rho_1 + 1)J_3],
\]

\[
+ 2(\mu_1 \rho_1 + 1)J_1 - 8\sigma_0^2(\mu_1 \rho_1 + 1)J_2 + 4\sigma_0^2 J_3],
\]

\[
\tilde{b}_{03} = \frac{2a^2(\sigma_0 - \mu_2 \rho_1)}{\sigma_0 \int_{\Omega} e_i^2 dx}[(\mu_1 \rho_1 + 1 - \sigma_0)J_1 + 4\sigma_0^3 J_2 + 2\sigma_0(\mu_1 \rho_1 + 1)J_3],
\]

\[
\tilde{b}_{21} = \frac{2a^2(\sigma_0 - \mu_2 \rho_1)}{\sigma_0 \int_{\Omega} e_i^2 dx}[2(\mu_1 \rho_1 + 1)I_1 - 8\sigma_0^2(\mu_1 \rho_1 + 1)I_2 + 4\sigma_0^2 I_3]
\]

\[
+ (\mu_1 \rho_1 + 1 + \sigma_0)J_1 - 4\sigma_0^3 J_2 - 2\sigma_0(\mu_1 \rho_1 + 1)J_3],
\]

where

\[
I_i = A_i + \sigma_0 - \int_{\Omega} E_i e_i^2 dx,
\]

\[
J_i = A_i - \sigma_0 \int_{\Omega} E_i e_i^2 dx,
\]

\[
A_i = (2\lambda - \mu_2 \rho_1 - a^2) \int_{\Omega} E_i e_i^2 dx + \alpha^2 \int_{\Omega} F_i e_i^2 dx, \quad i = 1, 2, 3.
\]

Then, the number

\[
b_1 = \frac{\pi}{2a_0}(a_{02} b_{02} - a_{20} b_{20}) + \frac{\pi}{4a_0}(b_{11} a_{02} + a_{11} b_{20} - b_{11} b_{02})
\]

\[
+ \frac{3\pi}{4}(a_{30} + b_{30} + \tilde{a}_{30} + \tilde{b}_{30}) + \frac{\pi}{4}(a_{12} + b_{21} + \tilde{a}_{12} + \tilde{b}_{21})
\]

is the same as in (5.6). Thus Assertions (1)–(2) of this theorem follow from Theorem A.6 in Ref. 11.

It is known that the bifurcated periodic solution near $\lambda = \lambda_1$ takes the form

\[
v^* = x(t)\xi + y(t)\eta + o(|x| + |y|),
\]

(5.20)

where $\xi, \eta$ are as in (5.9) and (5.10), and $(x(t), y(t))$ is the solution of the following equation:

\[
\frac{dx}{dt} = y(\lambda)x - \sigma_0(\lambda)y + \frac{1}{(\xi_\lambda, \xi^*_\lambda)}(G(x\xi_\lambda + y\eta_\lambda + \Phi_\lambda), \xi^*_\lambda),
\]

\[
\frac{dy}{dt} = \sigma_0(\lambda)x + y(\lambda)y + \frac{1}{(\eta_\lambda, \eta^*_\lambda)}(G(x\xi_\lambda + y\eta_\lambda + \Phi_\lambda), \eta^*_\lambda),
\]

where $\xi_\lambda, \eta_\lambda$ are eigenvectors of $L_\lambda$ corresponding to the first complex eigenvalues $\beta_1^\pm = \gamma \pm i\sigma_0$, and $\xi^*_\lambda, \eta^*_\lambda$ the conjugate eigenvectors. This solution $(x(t), y(t))$ near $\lambda_1$ is of the form

\[
x(t) = \left[-\frac{\gamma(\lambda)}{b_1}\right]^{1/2} \cos \sigma_0 t + o(|\gamma(\lambda)|^{1/2}),
\]

\[
y(t) = \left[-\frac{\gamma(\lambda)}{b_1}\right]^{1/2} \sin \sigma_0 t + o(|\gamma(\lambda)|^{1/2}),
\]

(5.21)

where $b_1$ is as in (5.6). Therefore, Assertion (3) follows from (5.20) and (5.21). The proof is complete. □
VI. ONE-DIMENSIONAL CASE

When the containers \( \Omega \) are taken as rectangles, the criteria in Theorems 4.2 and V.2 can be simplified. For simplicity, we consider here only the one-dimensional case:

\[
\Omega = (0, L) \subseteq \mathbb{R}^1.
\] (6.1)

The eigenvalues \( \rho_k \) and corresponding eigenvectors of (3.2) are given by

\[
\rho_k = \begin{cases} 
  k^2 \pi^2 / L^2 & \text{for b.c. (2.4),} \\
  (k - 1)^2 \pi^2 / L^2 & \text{for b.c. (2.5),}
\end{cases} \quad (k = 1, 2, \ldots),
\] (6.2)

\[
e_k = \begin{cases} 
  \sin \frac{k\pi x}{L} & \text{for b.c. (2.4),} \\
  \cos \frac{(k - 1)\pi x}{L} & \text{for b.c. (2.5).}
\end{cases}
\] (6.3)

Thus the two critical numbers \( \lambda_0 \) and \( \lambda_1 \) in (3.5) and (3.6) are given by

\[
\lambda_0 = \min_{\rho_k} \left[ \frac{\mu_1 k^2 \pi^2}{L^2} + \frac{\alpha^2 L^2}{\mu_2 k^2 \pi^2} + \frac{\mu_1 \alpha^2}{\mu_2} + 1 \right],
\] (6.4)

\[
\lambda_1 = \begin{cases} 
  \frac{\pi^2}{L^2} (\mu_1 + \mu_2) + \alpha^2 + 1 & \text{for b.c. (2.4),} \\
  \alpha^2 + 1 & \text{for b.c. (2.5).}
\end{cases}
\] (6.5)

It is known that the criterion \( b_1 \) in Theorem 4.2 is valid only for the free boundary condition, which can be expressed explicitly by (4.19). Likewise, for the number defined by (5.6) we have the following explicit expression:

\[
b_1 = \frac{2 \pi \alpha^2}{L} - b_0,
\] (6.6)

with

\[
b_0 = \frac{4^3 \times L}{9 \sigma_0^2 \pi^2} (\mu_1 \rho_1 + 1)(\mu_2^2 \rho_1^2 + 2 \mu_1 \mu_2 \rho_1^2 + 2 \mu_2 \rho_1 - \sigma_0^2)
\]

\[
- \frac{3L}{16} (2 \mu_2 \rho_1 + 3 \alpha^2) + 2 (3 \mu_1 \rho_1 + \mu_2 \rho_1 + 3) A_1
\]

\[
- 8 \sigma_0^2 (\mu_1 \rho_1 + \mu_2 \rho_1 + 1) A_2 - 4 (\mu_1 \mu_2 \rho_1^2 + \mu_2 \rho_1 - \sigma_0^2) A_3
\]

\[
+ 2 (\mu_1 \mu_2 \rho_1^2 + \mu_2 \rho_1 + \sigma_0) B_1 + 8 \sigma_0^2 (\mu_1 \mu_2 \rho_1^2 + \mu_2 \rho_1 - \sigma_0) B_2
\]

\[
- 4 \sigma_0^2 (\mu_1 \rho_1 + \mu_2 \rho_1 + 1) B_3,
\] (6.7)
where $A_i = (2\mu_i \rho_1 + \mu_2 \rho_1 + \alpha^2 + 2)B_i + \alpha^2 C_i$ ($i = 1, 2, 3$), and

$$B_1 = 32L \sum_{k=1}^{\infty} \frac{\mu_2}{\det M_{2k+1}[(2k+1)^2 - 4]L^2},$$

$$B_2 = 33L \sum_{k=1}^{\infty} \frac{(\mu_2 \rho_{2k+1} + \alpha^2)D_{2k+1} + \alpha^2 Q_{2k+1}}{\det M_{2k+1}\det(M_{2k+1}^2 + 4\sigma_0^2)\pi^2(2k+1)^2[(2k+1)^2 - 4]^2},$$

$$B_3 = 32L \sum_{k=1}^{\infty} \frac{D_{2k+1}}{\det(M_{2k+1}^2 + 4\sigma_0^2)\pi^2(2k+1)^2[(2k+1)^2 - 4]^2},$$

$$C_1 = -32L \sum_{k=1}^{\infty} \frac{\mu_1 \rho_1 + 1}{\det M_{2k+1}\pi^2(2k+1)^2[(2k+1)^2 - 4]^2},$$

$$C_2 = -32L \sum_{k=1}^{\infty} \frac{(\mu_1 \rho_1 + \mu_2 \rho_1 + \alpha^2 + 1, D_{2k+1} - (\mu_1 \rho_1 - \mu_1 \rho_1 - \mu_2 \rho_1 - \alpha^2)Q_{2k+1}}{\det M_{2k+1}\det(M_{2k+1}^2 + 4\sigma_0^2)\pi^2(2k+1)^2[(2k+1)^2 - 4]^2],}$$

$$C_3 = 32L \sum_{k=1}^{\infty} \frac{Q_{2k+1}}{\det(M_{2k+1}^2 + 4\sigma_0^2)\pi^2(2k+1)^2[(2k+1)^2 - 4]^2}.$$

Let $\lambda_0$ in (6.4) achieve its minimum at the integer $k_0^2$, i.e.,

$$\lambda_0 = \frac{\mu_1 k_0^2 \pi^2}{L^2} + \frac{\alpha^2 L^2}{\mu_2 k_0^2 \pi^2} + \frac{\mu_1 - \alpha^2}{2} + 1. \quad (6.8)$$

Then $k_0 \geq 1$ satisfies

$$k_0(k_0 - 1) \leq \frac{\alpha L^2}{\pi^2 \sqrt{\mu_1 \mu_2}} \leq k_0(k_0 + 1). \quad (6.9)$$

To see this, note that the function

$$\lambda(x) = \frac{\mu_1 \pi^2 x}{L^2} + \frac{\alpha^2 L^2}{\mu_2 \pi^2 x} + \frac{\mu_1 - \alpha^2}{2} + 1$$

has its minimum at $x_0 = \alpha L^2/\pi^2 \sqrt{\mu_1 \mu_2}$, and

$$\frac{d\lambda}{dx} \begin{cases} < 0 & \text{if } x < x_0, \\ > 0 & \text{if } x > x_0. \end{cases}$$

It follows that either $k_0 = m$ or $k_0 = m + 1$, such that $m = \sqrt{x_0 - \epsilon}$ for some $0 < \epsilon < 1$; namely,

$$m^2 \leq x_0 \leq (m + 1)^2.$$

It follows that

$$k_0 = \begin{cases} m & \text{if } \lambda(m^2) < \lambda((m + 1)^2), \\ m \text{ and } m + 1 & \text{if } \lambda(m^2) = \lambda((m + 1)^2), \\ m + 1 & \text{if } \lambda(m^2) > \lambda((m + 1)^2). \end{cases} \quad (6.10)$$

We infer from (6.10) that

$$\frac{\alpha L^2}{\pi^2 \sqrt{\mu_1 \mu_2}} \leq m(m + 1) \Rightarrow k_0 = m,$$

$$\frac{\alpha L^2}{\pi^2 \sqrt{\mu_1 \mu_2}} \geq m(m + 1) \Rightarrow k_0 = m + 1,$$

which yield the inequalities (6.9).
In the following, we compare $\lambda_0$ with $\lambda_1$ in terms of the parameters $\mu_1, \mu_2, \alpha$, and $L$. We proceed in two cases.

**THE CASE WHERE $\mu_1 \geq \mu_2$:** Then, from (3.5) and (3.6), we see that

$$\lambda_1 < \lambda_0 \quad \forall L, \alpha > 0 \text{ for b.c. (2.5)}. \quad (6.11)$$

For b.c. (2.4), we can prove that

$$\lambda_0 > \lambda_1 \quad \text{if } 0 < L < L_c,$$

$$\lambda_1 < \lambda_0, \quad \text{if } L > L_c,$$

where

$$L_c = \frac{\pi^2}{2\alpha} \left[ (\mu_1 - \mu_2)^2 \alpha^2 + 4\mu_2^2 - \alpha(\mu_1 - \mu_2) \right]. \quad (6.13)$$

In fact, from $\lambda_0 = \lambda_1$, we derive the critical scale $L_c$ as

$$L_c^2 = \frac{k_0^2 \pi^2}{2\alpha} \left[ (\mu_2 - \mu_1)\alpha \pm \sqrt{(\mu_2 - \mu_1)^2 \alpha^2 - 4\mu_2^2 (k_0^2 \mu_1 - \mu_2)} \right], \quad (6.14)$$

with $L_{c_1}^2 < L_{c_2}^2$.

It is easy to see that for the boundary condition (2.4),

$$\lambda_0 < \lambda_1 \quad \text{if } \begin{cases} k_0^2 \alpha^2 (\mu_2 - \mu_1)^2 < 4\mu_2 (k_0^2 \mu_1 - \mu_1 - \mu_2), \\
\text{or } k_0^2 \alpha^2 (\mu_2 - \mu_1)^2 > 4\mu_2 (k_0^2 \mu_1 - \mu_1 - \mu_2) \text{ and } 0 < L^2 < L_{c_1}^2 \text{ or } L_{c_2}^2 < L^2, \end{cases} \quad (6.15)$$

$$\lambda_0 < \lambda_1 \quad \text{if } \begin{cases} L_{c_1}^2 < L^2 < L_{c_2}^2 \quad \text{for } 0 < 4\mu_2 (k_0^2 \mu_1 - \mu_1 - \mu_2) < k_0^2 \alpha^2 (\mu_2 - \mu_1), \\
\text{or } 0 < L^2 < L_{c_2}^2 \text{ for } k_0^2 \mu_1 \leq \mu_1 + \mu_2. \end{cases} \quad (6.16)$$

For the boundary condition (2.5), we obtain two critical scales as

$$l_{c_1}^2 = \frac{k_0^2 \pi^2}{2\alpha} \left[ (\mu_2 - \mu_1)\alpha \pm \sqrt{(\mu_2 - \mu_1)^2 \alpha^2 - 4\mu_1 \mu_2} \right], \quad (6.17)$$

such that

$$\lambda_1 < \lambda_0 \quad \text{if } \begin{cases} (\mu_2 - \mu_1)^2 \alpha^2 < 4\mu_1 \mu_2, \\
\text{or } 0 < L^2 < l_{c_1}^2 \text{ or } l_{c_2}^2 < L^2, \text{ for } (\mu_2 - \mu_1)^2 \alpha^2 > 4\mu_1 \mu_2, \end{cases} \quad (6.18)$$

$$\lambda_0 < \lambda_1 \quad \text{for b.c. (2.5) if } l_{c_1}^2 < L^2 < l_{c_2}^2 \text{ for } (\mu_2 - \mu_1)^2 \alpha^2 > 4\mu_1 \mu_2. \quad (6.19)$$
VII. PHYSICAL REMARKS

We now discuss the phase transition of Brusselator by using Theorem 4.1–5.2 for the one-dimensional case (6.1).

**DIRICHLET BOUNDARY CONDITION.** When \( \mu_1 \geq \mu_2 \), by (6.12), the system (2.3) and (2.4) has a transition to steady states provided \( 0 < \lambda_L < \lambda_c \), and to periodic solutions provided \( \lambda_c < \lambda_L \).

**Physical Conclusion 7.1:** Let \( \mu_1 \geq \mu_2 \). Then for the system (2.3) and (2.4), we have the following conclusions:

1. When \( 0 < \lambda_L < \lambda_c \), the transition of (2.3) and (2.4) at \( \lambda = \lambda_0 \) is of Type-III, and there is a saddle-node bifurcation at some \( 0 < \lambda^* < \lambda_0 \). In other words, the basic state \( u_0 = (\alpha, \lambda/\alpha) \) is stable for \( 0 < \lambda < \lambda^* \), metastable for \( \lambda^* < \lambda < \lambda_0 \), and is unstable for \( \lambda > \lambda_0 \). Moreover, if \( \lambda^* < \lambda \), there are at least two metastable equilibrium states.

2. When \( \lambda_L < \lambda_c \), this system undergoes a dynamic transition at \( \lambda = \lambda_1 \) to periodic solutions. In particular, there exists an \( L_0 > \lambda_c \) such that if \( \lambda_c < \lambda < \lambda_0 \), the transition is of Type-II, and there is a singular separation of periodic solutions at some \( \mu < \mu_1 \). If \( \lambda_0 < \lambda \), the transition is of Type-I, and the system bifurcates from \( u_0 \) to a stable periodic solution on \( \lambda > \lambda_1 \), which is expressed as \( u_1 = (u_1^1, u_1^2) \) with

\[
\begin{align*}
    u_1^1 &= \alpha + 2\alpha^2 \sqrt{\frac{\lambda - \lambda_1}{|b_1|}} \sin \left( \frac{\pi x}{L} \sin(\sigma_0 t + \frac{\pi}{4}) \right) + o(|\lambda - \lambda_1|^{1/2}), \\
    u_1^2 &= \frac{\lambda}{\alpha} + 2(\sigma_0^2 + (\alpha^2 + \frac{\mu\pi^2}{L^2})^2) \sqrt{\frac{\lambda - \lambda_1}{|b_1|}} \sin \left( \frac{\pi x}{L} \cos(\sigma_0 t + \theta) \right) + o(|\lambda - \lambda_1|^{1/2}).
\end{align*}
\]

This periodic solution provides a spatial-temporal oscillation of the Brusselator.

The first conclusion is due to Theorem 4.1, the existence of global attractors, and the fact that \( u_0 = (0, 0) \) is a unique steady state solution of (2.3) and (2.4) at \( \lambda = 0 \).

The second conclusion is based on Theorem 5.2 and the following analysis on the criterion \( b_0 \) given by (6.7). We know that

\[
\lambda_1 - \lambda_0 \to 0 \quad \text{as} \quad L \to \lambda_c,
\]

which implies

\[
\sigma_0 \to 0 \quad \text{as} \quad L \to \lambda_c.
\]

It follows from (6.7) that \( b_0 \to +\infty \) for \( L \to \lambda_c + 0 \). Therefore

\[
b_0 > 0 \quad \forall \lambda_c < L < \lambda_0, \tag{7.1}
\]

for some \( L_0 > \lambda_c \). On the other hand, \( \rho_k \to 0 \) (\( L \to \infty \)). Hence, when \( L \to \infty \),

\[
b_0 \to \frac{64L}{9\pi^2} - \frac{9\alpha^2 L}{16} + 2(3\alpha^2 + 6 + \sigma_0)B_1 - 8\sigma_0^2(\alpha^2 + \sigma_0 + 2)B_2 \tag{7.2}
\]

\[
+ 4\alpha^2(\alpha^2 + 1)B_3 + 6\alpha^2C_1 - 8\sigma_0^2\alpha^2C_2 + 4\alpha^4C_3.
\]

Note that

\[
\begin{align*}
    \sigma_0 &\to \alpha, \quad D_k \to 3\alpha^2, \quad Q_k \to -3\alpha^2, \\
    \det M_k &\to \alpha^2, \quad \det(M_k^2 + 4\sigma_0^2) \to 9\alpha^4, \quad B_1 \to 0, \\
    B_2 &\to 0, \quad B_3 \to \frac{32L}{3\alpha^2}E, \quad C_1 \to -\frac{32L}{\alpha^2}E, \\
    C_2 &\to -\frac{32L}{3\alpha^2}E, \quad C_3 \to -\frac{32L}{3\alpha^2}E.
\end{align*}
\]
for $L \to \infty$, where

$$E = \sum_{k=1}^{\infty} \frac{1}{\pi^2(2k+1)^2[(2k+1)^2 - 4]^2}.$$  

Thus, in view of (7.2),

$$b_0 \to -L\left[\frac{64}{9\pi^2} + \frac{9\alpha^2}{16} + 64E\right] < 0 \quad \text{as} \quad L \to \infty.$$  

Hence

$$b_0 < 0 \quad \forall L_1 < L, \quad (7.3)$$

for some $L_1 \geq L_0$.

From the physical point of view, it is reasonable to consider the case where $b_0$ changes its sign only once in $(L_0, \infty)$. Hence, physically, we have $L_0 = L_1$. Thus, we derive from (7.1) and (7.3) the second conclusion.

Now, we consider the case where $\mu_1 < \mu_2$ by the following two examples. We take $\mu_1 = 2 \times 10^{-3}$, $\mu_2 = 4 \times 10^{-3}$. (7.4)

Example 7.1: Let (7.4) hold true, and $\alpha = 2$, $L = 4$. Then we obtain from (6.9) that $k_0 = 34$, and

$$k_0^2\alpha^2(\mu_2 - \mu_1)^2 < 4\mu_2(k_0^2\mu_1 - \mu_1 - \mu_2).$$

In view of (6.15),

$$\lambda_1 < \lambda_0 \quad \text{for b.c. (2.4)}.$$  

The number $b_0$ in (6.7) is given by

$$b_0 \approx -\left(\frac{64}{9\pi^2} + \frac{9}{4}\right)L + 40B_1 - 256B_2 + 80B_3 + 24C_1 - 128C_2 + 64C_3,$$

and

$$B_1 \approx \frac{2}{25} \times 10^{-3}L, \quad B_2 \approx -\frac{11}{36 \times 25} \times 10^{-3}L,$$

$$B_3 \approx \frac{8}{27 \times 25\pi^2}L, \quad C_1 \approx -\frac{8}{9 \times 25\pi^2}L,$$

$$C_2 \approx \frac{2}{27 \times 25\pi^2}L, \quad C_3 \approx -\frac{8}{27 \times 25\pi^2}L.$$  

It is easy to see that $b_0 < 0$. Then, by Theorem 5.2, the phase transition of (2.3) and (2.4) is of Type-I, and this system undergoes a spatial-temporal oscillation on $\lambda > \lambda_1$.

Example 7.2: Assume (7.4) and $\alpha = 3$, $L = 4$. Then $k_0 = 41$, and

$$k_0^2\alpha^2(\mu_2 - \mu_1)^2 > 4\mu_2(k_0^2\mu_1 - \mu_1 - \mu_2) > 0.$$  

Thus, we can obtain two critical scales $L_{c_1}^2$ and $L_{c_2}^2$ in (6.14) as follows:

$$L_{c_1}^2 = 11.07, \quad L_{c_2}^2 = 22.14.$$  

Hence

$$L_{c_1}^2 < L^2 = 16 < L_{c_2}^2,$$

which leads [by (6.16)] to

$$\lambda_0 < \lambda_1 \quad \text{for b.c. (2.4)}. \quad (7.5)$$
It is clear that
\[ \int_{\Omega} e_{k_0}^3 dx = \int_{0}^{L} \sin^3 \frac{k_0 \pi x}{L} dx \neq 0. \]
By (7.5) and Theorem 4.1, the system (2.3) and (2.4) has a Type-III transition at \( \lambda = \lambda_0 \), and there is a saddle-node bifurcation at some \( \lambda^* (0 < \lambda^* < \lambda_0) \).

**NEUMANN BOUNDARY CONDITION.** Consider the case where \( \mu_1 \geq \mu_2 \). Based on (6.15) and Theorem 5.1, we have the following physical conclusion.

**Physical Conclusion 7.2:** Let \( \mu_1 \geq \mu_2 \). Then the system (2.3) with (2.5) has a Type-I transition to periodic solutions at \( \lambda = \lambda_1 \), i.e., a spatial-temporal oscillation occurs in the Brusselator for \( \lambda > \lambda_1 \).

For the case where \( \mu_1 < \mu_2 \), we have the following example.

**Example 7.3:** Under the same conditions as in Example VII.2, \( k_0 = 41 \) and the two critical scales \( l_{c_1}^2 \) and \( l_{c_2}^2 \) in (6.17) are given by
\[ l_{c_1}^2 = 11.06, \quad l_{c_2}^2 = 22.12. \]
Hence, \( l_{c_1}^2 < l^2 = 16 < l_{c_2}^2 \), which implies, by (6.19), that
\[ \lambda_0 < \lambda_1 \text{ for b.c. (2.5)}. \] (7.6)
On the other hand, by (6.2) and (6.3), we have
\[ \rho_{k_0 + 1} = \frac{k_0^2 \pi^2}{L^2} \approx 10^3, \quad \int_{\Omega} e_{k_0+1}^3 dx = \int_{0}^{L} \cos^3 \frac{k_0 \pi x}{L} dx = 0. \]
Thus, it is easy to check that the number \( b_1 \) in (4.10), which is also given by (4.19), is negative, i.e.,
\[ b_1 < 0 \text{ in (4.10)}. \] (7.7)
By (7.6) and (7.7) and Theorem 4.2, the system (2.3) with (2.5) bifurcates on \( \lambda > \lambda_0 \) to two stable steady states \( v^\pm \) as given by (4.12). It shows that the Brusselator undergoes a transition at \( \lambda_0 = 9.8 \).

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