Categorical smooth compactifications and generalized Hodge-to-de Rham degeneration

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Abstract We disprove two (unpublished) conjectures of Kontsevich which state generalized versions of categorical Hodge-to-de Rham degeneration for smooth and for proper DG categories (but not smooth and proper, in which case degeneration is proved by Kaledin (in: Algebra, geometry, and physics in the 21st century. Birkhäuser/Springer, Cham, pp 99–129, 2017). In particular, we show that there exists a minimal 10-dimensional $A_\infty$-algebra over a field of characteristic zero, for which the supertrace of $\mu_3$ on the second argument is non-zero. As a byproduct, we obtain an example of a homotopically finitely presented DG category (over a field of characteristic zero) that does not have a smooth categorical compactification, giving a negative answer to a question of Toën. This can be interpreted as a lack of resolution of singularities in the noncommutative setup. We also obtain an example of a proper DG category which does not admit a categorical resolution of singularities in the terminology of Kuznetsov and Lunts (Int Math Res Not 2015(13):4536–4625, 2015) (that is, it cannot be embedded into a smooth and proper DG category).

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0 Introduction

Given a smooth algebraic variety $X$ over a field of characteristic zero, we have the Hodge-to-de Rham spectral sequence $E_1^{p,q} = H^q(X, \Omega^p_X) \Rightarrow H^{p+q}_{DR}(X)$. It is classically known that when $X$ is additionally proper, this spectral sequence degenerates at $E_1$, that is, all differentials vanish. This follows from the classical Hodge theory for compact Kähler manifolds, and can also be proved algebraically [3].

We recall the following fundamental result of Kaledin [11], see also [22] for a different proof.

**Theorem 0.1** [11, Theorem 5.4] Let $A$ be a smooth and proper DG algebra over a field of characteristic zero. Then the Hochschild-to-cyclic spectral sequence degenerates, so that we have an isomorphism $H^p(A) \cong HH^p(A)((u))$.

Here $u$ denotes a variable of degree 2.

When applied to $\text{Perf}(A) \simeq \text{Perf}(X)$ for a smooth and proper variety $X$, Theorem 0.1 gives exactly the classical Hodge-to-de Rham degeneration.

In this paper we study some generalizations of Hodge-to-de-Rham degeneration to DG categories which are not smooth and proper.

Recall that for a proper DG algebra $B$ over $k$, one has a pairing $HH_\bullet(B) \otimes HC_\bullet(B^{op}) \to k$, introduced by Shklyarov [28]. Kontsevich [16] proposed the following generalization of Theorem 0.1.

**Conjecture 0.2** Let $B$ be a proper DG algebra over a field $k$ of characteristic zero. Then the composition map

$$(HH_\bullet(B) \otimes HC_\bullet(B^{op}))[1] \xrightarrow{id \otimes \delta^+} HH_\bullet(B) \otimes HH_\bullet(B^{op}) \to k \quad (0.1)$$

is zero.

Here $\delta^+: HC_\bullet(B^{op})[1] \to HH_\bullet(B^{op})$ denotes the boundary map, see [20, Section 2.2]. This conjecture can also be formulated for finite-dimensional
A∞-algebras (since any DG algebra is quasi-isomorphic to a minimal A∞-algebra). Then the pairing can be given explicitly in terms of supertraces of certain operators involving higher multiplications, see [27, Proposition 5.22]. A special case of Conjecture 0.2 states the vanishing of the supertrace of μ3 on the second argument, for any minimal finite-dimensional A∞-algebra, see Corollary 2.2.

Kontsevich also proposed a “dual” version of Conjecture 0.2 for smooth DG algebras.

**Conjecture 0.3** Let A be a smooth DG algebra over a field of characteristic zero. Then the composition

\[ K_0(A \otimes A^{op}) \xrightarrow{\text{ch}} (HH_\bullet(A) \otimes HH_\bullet(A^{op})_0 \xrightarrow{id \otimes \delta^-} (HH_\bullet(A) \otimes HC^{-\bullet}(A^{op}))_1 \]

vanishes on the class [A] of the diagonal bimodule.

Here \( \delta^- : HH_\bullet(A^{op}) \to HC^{-\bullet}(A^{op})[-1] \) denotes the boundary map, see [5, Section 3].

The motivation for Conjecture 0.2 is the following. Let us assume that the category Perf(B) has a categorical resolution in the terminology of Kuznetsov and Lunts [18]. That is, suppose that there is a quasi-fully-faithful functor Perf(B) \( \hookrightarrow C \), where C is a smooth and proper DG category. By construction of the pairing on the Hochschild homology [28], we have a commutative diagram

\[
\begin{array}{ccc}
HH_\bullet(B) \otimes HH_\bullet(B^{op}) & \longrightarrow & HH_\bullet(C) \otimes HH_\bullet(C^{op}) \\
\downarrow & & \downarrow \\
k & = & k.
\end{array}
\]

By Theorem 0.1, the map \( \delta^+ : HC^{-\bullet}(C^{op}) \to HH_{\bullet+1}(C^{op}) \) equals zero. Now, since the boundary map \( \delta^+ : HC^{-\bullet}(-) \to HH_{\bullet+1}(-) \) is compatible with DG functors, it follows from Theorem 0.1 that the composition (0.1) vanishes. Therefore, existence of a categorical resolution for a given proper DG algebra B implies that Conjecture 0.2 holds for B. See also Proposition 5.1 below.

By [18], for any proper scheme X over a field of characteristic zero k, the category Perf(X) has a categorical resolution of singularities. Together with the Nagata compactification theorem [23], this implies that Conjecture 0.2 holds for any DG algebra of the form \( B = R \text{End}(F) \), where \( F \in \text{Perf}_{\text{prop}}(Y) \) is a perfect complex with proper support on a separated scheme Y of finite type over k.

The motivation for Conjecture 0.3 is “dual”. We first recall the following question of Toën [31].
**Question 0.4** Is it true that any homotopically finitely presented DG category $\mathcal{A}$ over a field is Morita equivalent to a quotient $\mathcal{C}/\mathcal{S}$, where $\mathcal{C}$ is smooth and proper, and $\mathcal{S} \subset \mathcal{C}$ is a full subcategory?

Such a quotient presentation of $\mathcal{A}$ is called a smooth categorical compactification. Assuming the existence of a smooth compactification for the DG category $\text{Perf}(\mathcal{A})$, similar considerations imply that Conjecture 0.3 is valid for $\mathcal{A}$, for details see Proposition 4.1 below. In particular, the results of [6] imply that Conjecture 0.3 holds for smooth DG algebras of the form $\mathbf{R}\text{End}(\mathcal{G})$, where $\mathcal{G} \in D^b_{\text{coh}}(Y)$ is a generator of the category $D^b_{\text{coh}}(Y)$, for any separated scheme $Y$ of finite type over a field $k$ of characteristic zero.

So, it seemed plausible that both conjectures hold in full generality, without any assumptions on existence of categorical resolutions and smooth compactifications. However, in this paper we disprove both Conjectures 0.2 and 0.3. As an application, we give a negative answer to Question 0.4.

The starting point for our counterexamples is to disprove the main conjecture of [5], see Sect. 3. A counterexample to Conjecture 0.3 is obtained in Sect. 4. It is deduced from the results of Sect. 3 by some trick.

Finally, a counterexample to Conjecture 0.2 is obtained in Sect. 5. More precisely, we construct a 10-dimensional minimal $A_\infty$-algebra for which the supertrace of $\mu_3$ on the second argument is non-zero, see Theorem 5.4. The construction is deduced from our new result on nilpotent elements in the cohomology of a DG algebra (Theorem 5.3), which is of independent interest. In particular, we obtain an example of a proper DG algebra $B$ such that the DG category $\text{Perf}(B)$ cannot be fully faithfully embedded into a saturated DG category.

Section 5 can be read independently from Sects. 3 and 4.

### 1 Preliminaries on DG categories and $A_\infty$-algebras

#### 1.1 DG categories

For an introduction to DG categories, we refer the reader to [12]. The references for DG quotients are [4, 13]. For the model structures on DG categories we refer to [29, 30], and for a general introduction on model categories we refer to [10]. Everything will be considered over some base field $k$.

 Mostly we will consider DG categories up to a quasi-equivalence. By a functor between DG categories we sometimes mean a quasi-functor. In some cases it is convenient for us to choose a concrete DG model or a concrete DG functor. By a commutative diagram of functors we usually mean the commutative diagram in the homotopy category $\text{Ho}(\text{dgcat}_k)$. Finally, we denote by $\text{Ho}_M(\text{dgcat}_k)$ the Morita homotopy category of DG categories (with inverted Morita equivalences).
All modules are assumed to be right unless otherwise stated. For a small DG category $\mathcal{C}$ and a $\mathcal{C}$-module $M$, we denote by $M^\vee$ the $\mathcal{C}^{op}$-module $\text{Hom}_{\mathcal{C}}(M, \mathcal{C})$. We denote by $M^*$ the $\mathcal{C}^{op}$-module $\text{Hom}_{\mathcal{C}}(M, k)$.

Given a small DG category $\mathcal{C}$, we denote by $D(\mathcal{C})$ its derived category of DG $\mathcal{C}$-modules. This is a compactly generated triangulated category. We denote by $D_{\text{perf}}(\mathcal{C})$ the full triangulated subcategory of perfect $\mathcal{C}$-modules. It coincides with the subcategory of compact objects.

Recall from [34] that a $\mathcal{C}$-module $M$ is pseudo-perfect if for each $x \in \mathcal{C}$, the complex $M(x)$ is perfect over $k$ (that is, $M(x)$ has finite-dimensional total cohomology). We denote by $D_{\text{pspe}}(\mathcal{C}) \subset D(\mathcal{C})$ the full triangulated subcategory of pseudo-perfect $\mathcal{C}$-modules.

For any DG category $\mathcal{C}$, we denote by $[\mathcal{C}]$ its (non-graded) homotopy category, which has the same objects as $\mathcal{C}$, and the morphisms are given by $[\mathcal{C}](x, y) = H^0(\mathcal{C}(x, y))$. We use the terminology of [34, Definition 2.4] by calling $\mathcal{C}$ triangulated if the Yoneda embedding provides an equivalence $[\mathcal{C}] \xrightarrow{\sim} D_{\text{perf}}(\mathcal{C})$. In this case $[\mathcal{C}]$ is a Karoubi complete triangulated category.

We denote by $\text{Mod}_\mathcal{C}$ the DG category of cofibrant DG $\mathcal{C}$-modules in the projective model structure (these are exactly the direct summands of semifree DG $\mathcal{C}$-modules). We have $D(\mathcal{C}) \simeq [\text{Mod}_\mathcal{C}]$. We denote by $Y : \mathcal{C} \hookrightarrow \text{Mod}_\mathcal{C}$ the standard Yoneda embedding given by $Y(x) = \mathcal{C}(-, x)$.

We write $\text{Perf}(\mathcal{C}) \subset \text{Mod}_\mathcal{C}$ (resp. $\text{PsPerf}(\mathcal{C}) \subset \text{Mod}_\mathcal{C}$) for the full DG subcategory of perfect (resp. pseudo-perfect) $\mathcal{C}$-modules.

For a DG functor $\Phi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ between small DG categories, we denote by $L\Phi^* : D(\mathcal{C}_1) \rightarrow D(\mathcal{C}_2)$, the derived extension of scalars functor. Its right adjoint functor (restriction of scalars) is denoted by $\Phi_* : D(\mathcal{C}_2) \rightarrow D(\mathcal{C}_1)$.

We also recall from [32, Definition 3.6] that a $\mathcal{C}$-module is called quasi-representable if it is quasi-isomorphic to a representable $\mathcal{C}$-module. For two DG categories $\mathcal{C}, \mathcal{C}'$, a $\mathcal{C} \otimes \mathcal{C}'$-module $M$ is called right quasi-representable if for each object $x \in \mathcal{C}$, the $\mathcal{C}'$-module $M(x, -)$ is quasi-representable.

We denote by $\mathbf{R}\text{Hom}(\mathcal{C}, \mathcal{C}') \subset \text{Mod}_{\mathcal{C}^{op} \otimes \mathcal{C}'}$ the full subcategory of right quasi-representable $\mathcal{C}^{op} \otimes \mathcal{C}'$-modules. By [32, Theorem 6.1], this DG category (considered up to a quasi-equivalence) is actually the internal Hom in the homotopy category of DG categories $\text{Ho}(\mathbf{dgcat}_k)$ (with inverted quasi-equivalences). We have a natural quasi-functor $\text{Fun}(\mathcal{C}, \mathcal{C}') \rightarrow \mathbf{R}\text{Hom}(\mathcal{C}, \mathcal{C}')$, where $\text{Fun}(\mathcal{C}, \mathcal{C}')$ is the naive DG category of DG functors $\mathcal{C} \rightarrow \mathcal{C}'$, as defined in [12]. Moreover, if $\mathcal{C}$ is cofibrant, this functor is essentially surjective on the homotopy categories.

A small DG category $\mathcal{C}$ is called smooth (resp. locally proper) if the diagonal $\mathcal{C}-\mathcal{C}$-bimodule is perfect (resp. pseudo-perfect). Moreover, $\mathcal{C}$ is called proper if it is locally proper and is Morita equivalent to a DG algebra (i.e. the triangulated category $D_{\text{perf}}(\mathcal{C})$ has a classical generator).

We recall the notion of a short exact sequence of DG categories.
**Definition 1.1** A pair of functors $A_1 \xrightarrow{F_1} A_2 \xrightarrow{F_2} A_3$ is said to be a (Morita) short exact sequence of DG categories if the following conditions hold

i) the composition $F_2 F_1$ is homotopic to zero;

ii) the functor $F_1$ is quasi-fully-faithful;

iii) the induced quasi-functor $\overline{F_2} : A_2/F_1(A_1) \to A_3$ is a Morita equivalence.

In particular, a short exact sequence of DG categories induces a long exact sequence of K-groups, where $K_\bullet(A)$ is the Waldhausen K-theory [37] of the Waldhausen category of cofibrant perfect $A$-modules. In this paper we will actually need only the boundary map $K_1(A_3) \to K_0(A_1)$.

**1.2 $A_\infty$-algebras and $A_\infty$-(bi)modules**

All the definitions and constructions regarding DG categories which are invariant under quasi-equivalences can be translated into the world of $A_\infty$-categories. For the introduction on $A_\infty$-categories and algebras see [14,17,19].

It will be sufficient for us to work with $A_\infty$-algebras (that is, $A_\infty$-categories with a single object).

In order to write down the signs in formulas it is convenient to adopt the following

**Notation** For a collection of homogeneous elements $a_0, \ldots, a_n$ of a graded vector space $A$, and $0 \leq p, q \leq n$, we put

$$l^q_p(a) = \begin{cases} |a_p| + \cdots + |a_q| + q - p + 1 & \text{if } p \leq q; \\ |a_p| + \cdots + |a_n| + |a_0| + \cdots + |a_q| + n - p + q & \text{if } p > q. \end{cases}$$

If the collection starts with $a_1$ (and there is no $a_0$) we only use $l^q_p(a)$ for $1 \leq p \leq q \leq n$.

**Definition 1.2** A non-unital $A_\infty$-structure on a graded vector space $A$ is a sequence of multilinear operations $\mu_n = \mu_n^A : A \otimes^n \to A$, where $\deg(\mu_n) = 2 - n$, $n \geq 1$, satisfying the following relations:

$$\sum_{i+j+k=n} (-1)^{l^q_i(a)} \mu_{i+k+1}(a_1, \ldots, a_i, \mu_j(a_{i+1}, \ldots, a_{i+j}), a_{i+j+1}, \ldots, a_n) = 0,$$

(1.1)

for $n \geq 1$. 

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**Definition 1.3** A non-unital $A_{\infty}$-morphism $f : A \to B$ is given by a sequence of linear maps $f_n : A^\otimes n \to B$, where $\deg(f_n) = 1 - n$, $n \geq 1$, satisfying the following relations:

$$
\sum_{i_1+\cdots+i_k=n} \mu^B_{i_1}(f_{i_1}(a_1, \ldots, a_{i_1}), \ldots, f_{i_k}(a_{i_1+\cdots+i_{k-1}+1}, \ldots, a_n)) \\
= \sum_{i+j+k=n} (-1)^{i_1(a)} f_{i+k+1}(a_1, \ldots, a_i, \mu^A_j(a_{i+1}, \ldots, a_{i+j}), a_{i+j+1}, \ldots, a_n),
$$

(1.2)

for $n \geq 1$.

Given an $A_{\infty}$-algebra $A$, one defines the $A_{\infty}$-algebra $A^{\text{op}}$ as follows: it is equal to $A$ as a graded vector space, and we have

$$
\mu^{A_{\text{op}}}_n(a_1, \ldots, a_n) = (-1)^{\sigma} \mu^A_n(a_n, \ldots, a_1),
$$

where $\sigma = \sum_{1 \leq i < j \leq n} (|a_i| + 1)(|a_j| + 1)$.

We now define the notion of an $A_{\infty}$-module.

**Definition 1.4** A right $A_{\infty}$-module $M$ over an $A_{\infty}$-algebra $A$ is a graded vector space with a sequence of operations $\mu^M_n : M \otimes A^\otimes n \to M$, where $n > 0$, $\deg(\mu^M_n) = 2 - n$, and the following relations are satisfied:

$$
\sum_{i+j=n} \mu^M_{i+1,j+1}(\mu^M_{i+1}(m, a_1, \ldots, a_i), a_{i+1}, \ldots, a_n) \\
+ \sum_{i+j+k=n+1} (-1)^{|m|+|a_1|} \mu^M_{i+k+1}(m, a_1, \ldots, a_i, \mu^A_j(a_{i+1}, \ldots, a_{i+j}), a_{i+j+1}, \ldots, a_n) = 0.
$$

(1.3)

We also need $A_{\infty}$-bimodules.

**Definition 1.5** Let $A$ and $B$ be non-unital $A_{\infty}$-algebras. An $A_{\infty}$ $A$-$B$-bimodule $M$ is a graded vector space with a collection of operations $\mu^M_{i,j} : A^\otimes i \otimes M \otimes B^\otimes j \to M$, where $i, j \geq 0$, such that for any $n, m \geq 0$ and homogeneous $a_1, \ldots, a_n \in A, b_1, \ldots, b_m \in B, m \in M$, the following relation is satisfied:

$$
\sum_{i+j+k=n} (-1)^{|a_1|} \mu^M_{i+k+1,1,m}(a_1, \ldots, \mu^A_{i+1}(a_{i+1}, \ldots, a_{i+j}), \ldots, a_n, m, b_1, \ldots, b_m) \\
+ \sum_{1 \leq i \leq n+1; \ 0 \leq j \leq m} (-1)^{|a_1|-1} \mu^M_{i-1,1,m-j}
$$
Remark 1.6 1) In our sign convention, a non-unital DG algebra $B$ can be considered as an $A_\infty$-algebra, with $\mu_1(b) = -d(b)$, $\mu_2(b_1, b_2) = (-1)^{d_1}b_1b_2$, and $\mu_{\geq 3} = 0$.

2) If furthermore $M$ is a right DG $B$-module, then the $A_\infty$ $B$-module structure on $M$ is given by $\mu^M_1(m) = d(m)$, $\mu^M_2(m, a) = (-1)^{|m|+1}ma$, and $\mu^M_{\geq 3} = 0$.

3) If $A$ is another non-unital DG algebra, and $M$ is a DG $A$-$B$-bimodule, then the $A_\infty$ $A$-$B$-bimodule structure on $M$ is given by $\mu^M_{0,0}(m) = d(m)$, $\mu^M_{1,0}(a, m) = am$, $\mu^M_{0,1}(m, b) = (-1)^{|m|+1}mb$, and $\mu^M_{i,j} = 0$ for $i + j \geq 2$.

We now recall the strict unitality.

Definition 1.7 1) A non-unital $A_\infty$-algebra $A$ is called strictly unital if there is a (unique) element $1 = 1_A \in A$ such that $\mu_1(1) = 0$, $\mu_2(1, a) = a = (-1)^{|a|}\mu_2(a, 1)$ for any homogeneous element $a \in A$, and for $n \geq 3$ we have $\mu_n(a_1, \ldots, a_n) = 0$ if at least one of the arguments $a_i$ equals 1.

2) A non-unital $A_\infty$-morphism $f : A \to B$ between strictly unital $A_\infty$-algebras is called strictly unital if $f_1(1_A) = 1_B$, and for $n \geq 2$ we have $f_n(a_1, \ldots, a_n) = 0$ if at least one of the arguments $a_i$ equals 1.

3) Given a strictly unital $A_\infty$-algebra $A$, an $A_\infty$ $A$-module $M$ is called strictly unital if $\mu^M_2(m, 1) = (-1)^{|m|+1}m$, and for $n \geq 3$ we have $\mu^M_n(m, a_1, \ldots, a_{n-1}) = 0$ if at least one of $a_i$’s equals 1.

4) Given strictly unital $A_\infty$-algebras $A$, $B$, an $A_\infty$ $A$-$B$-bimodule is called strictly unital if $\mu^M_{1,0}(1_A, m) = m$, $\mu^M_{0,1}(m, 1_B) = (-1)^{|m|+1}m$, and for $k + l \geq 2$ we have $\mu^M_{k,l}(a_1, \ldots, a_k, m, b_1, \ldots, b_l) = 0$ if at least one of $a_i$’s equals $1_A$ or at least one of $b_j$’s equals $1_B$.

For a strictly unital $A_\infty$-algebra $A$ we put $\overline{A} := A/(k \cdot 1_A)$. From now on in this paper, all $A_\infty$-algebras and (bi)modules will be strictly unital.

Given an $A_\infty$-algebra $A$, we define the DG category $\text{Mod}^\infty - A$ whose objects are $A_\infty$-modules and the morphisms are defined as follows. Given $M, N \in \text{Mod}^\infty - A$, we put

$$
\text{Hom}^\infty_A(M, N)^{gr} := \prod_{n \geq 0} \text{Hom}_k(M \otimes \overline{A}[1]^\otimes n, N),
$$

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and the differential is given by

\[ d(\varphi)_n(m, a_1, \ldots, a_n) = \sum_{i=0}^{n} \mu_{n-i+1}^N(\varphi_i(m, a_1, \ldots, a_i), a_{i+1}, \ldots, a_n) \]

\[ - \sum_{i=0}^{n} (-1)^{|\varphi|} \varphi_{n-i}(\mu_{i+1}^M(m, a_1, \ldots, a_i), a_{i+1}, \ldots, a_n) \]

\[ - \sum_{1 \leq i \leq j \leq n} (-1)^{|\varphi|+|m|+l_{i,j}^{-1}(a)} \varphi_{n+i-j}(m, a_1, \ldots, \mu_{j-i+1}^A(a_i, \ldots, a_j), \ldots, a_n). \]

The composition is given by

\[ (\varphi \psi)_n(m, a_1, \ldots, a_n) = \sum_{i=0}^{n} \varphi_{n-i}(\psi_i(m, a_1, \ldots, a_i), a_{i+1}, \ldots, a_n). \]

We denote by \( \text{Perf}^\infty(A) \) the full DG subcategory of perfect \( A_\infty \)-modules.

Given a unital DG algebra \( B \), we denote by \( \text{PsPerf}(B) \subset \text{Mod}^\infty_{\infty}-B \) the full DG subcategory formed by pseudo-perfect DG modules. We have \([\text{PsPerf}(B)] \simeq D_{\text{pspe}}(B)\).

We also need a DG category of \( A_\infty \)-bimodules. Given \( A_\infty \)-algebras \( A \) and \( B \), we define the DG category \( \text{A-Mod}^\infty_{\infty}-B \) whose objects are \( A_\infty \)-bimodules and the morphism are the following. Given \( M, N \in A_\infty \- \text{Mod}^\infty_{\infty}-B \), we put

\[ \text{Hom}_{\text{A-Mod}}^\infty(M, N)^{gr} := \prod_{r,s \geq 0} \text{Hom}_k(\overline{A}[1]^\otimes r \otimes M \otimes \overline{B}[1]^\otimes s, N), \]

and the differential is given by

\[ d(\varphi)_{r,s}(a_1, \ldots, a_r, m, b_1, \ldots, b_s) \]

\[ = \sum_{1 \leq i \leq r+1; \ 0 \leq j \leq s} (-1)^{l_{i,j}^{-1}(a)} \varphi_i^N(\mu_{i-1,s-j}^N(a_1, \ldots, a_{i-1}, \varphi_{r+1-i,j} \varphi_{r+1-i,j}^N(a_{i+1}, \ldots, a_r, m, b_1, \ldots, b_s), \ldots, a_r, m, b_1, \ldots, b_s) \]

\[ - \sum_{i+j+k=r} (-1)^{|\varphi|+l_{i,j}^A} \varphi_{i+k+1,s}(a_1, \ldots, \mu_{j}^A(a_{i+1}, \ldots, a_{i+j}), \ldots, a_r, m, b_1, \ldots, b_s) \]

\[ - \sum_{1 \leq i \leq r+1; \ 0 \leq j \leq s} (-1)^{|\varphi|+l_{i,j}^{-1}(a)} \varphi_{i-1,s-j}(a_1, \ldots, a_{i-1}, \mu_{r+1-i,j}^M(a_{i+1}, \ldots, a_r, m, b_1, \ldots, b_s), \ldots, a_r, m, b_1, \ldots, b_s). \]
\[ - \sum_{i+j+k=s} (-1)^{|\varphi|+i_1^l(a)+i_1^r(b)+|m|} \varphi_{r,i+k+1} \\
(a_1, \ldots, a_r, m, b_1, \ldots, \mu_j B(b_{i+1}, \ldots, b_{i+j}), \ldots, b_s). \]

The composition is given by
\[
(\varphi \psi)_{r,s}(a_1, \ldots, a_r, m, b_1, \ldots, b_s) = \sum_{0 \leq i \leq r; 0 \leq j \leq s} (-1)^{i_1^l(a)} \varphi_{i,s-j} \\
(a_1, \ldots, a_i, \psi_{r-i,j}(a_{i+1}, \ldots, a_r, m, b_1, \ldots, b_j), b_{j+1}, \ldots, b_s).
\]

Given \(A_{\infty}\)-morphisms \(f : A \rightarrow A', g : B \rightarrow B'\), we have the restriction of scalars DG functor \((f, g)_* : A' - \text{Mod}^\infty - B' \rightarrow A' - \text{Mod}^\infty - B\), given on objects as follows: \((f, g)_* M\) equals \(M\) as a graded vector space, and we have
\[
\mu_{i,s}^{(f, g)_* M}(a_1, \ldots, a_r, m, b_1, \ldots, b_s) = \sum_{i_1 + \cdots + i_p = r; j_1 + \cdots + j_q = r} \mu_{p,q}^M(f_{i_1}(a_1, \ldots, a_{i_1}), \ldots, \\
f_{i_p}(a_{i_1+\cdots+i_{p-1}+1}, \ldots, a_r), m, g_{j_1}(b_1, \ldots, b_{j_1}), \\
\ldots, g_{j_q}(b_{j_1+\cdots+j_{q-1}+1}, \ldots, b_s)).
\]

If both \(f\) and \(g\) are quasi-isomorphisms, then \((f, g)_*\) is a quasi-equivalence. Moreover, if \(A\) and \(B\) are DG algebras, then we have \([A - \text{Mod}^\infty - B] \simeq D(A \otimes B^{op})\).

**Remark 1.8** Let \(A, B\) be \(A_{\infty}\)-algebras. An \(A_{\infty}\) \(A-B\)-bimodule structure on a graded vector space \(M\) is equivalent to the following data:

- the right \(A_{\infty}\) \(B\)-module structure on \(M\);
- the \(A_{\infty}\)-morphism \(f : A \rightarrow \text{End}_{B}^{\infty}(M)\).

Namely, given an \(A_{\infty}\)-bimodule \(M\), the induced \(B\)-module structure is given by \(\mu_n^M = \mu_{0,n-1}^M\), and the \(A_{\infty}\)-morphism is given by \(f_n(a_1, \ldots, a_n)(m, b_1, \ldots, b_l) = \mu_{n,1}(a_1, \ldots, a_n, m, b_1, \ldots, b_l)\).

The diagonal \(A_{\infty}\) \(A-A\)-bimodule is given by \(A\) as a graded vector space, and we have
\[
\mu_{i,j}(a_1, \ldots, a_i, b, c_1, \ldots, c_j) = (-1)^{i_1^l(a)+1} \mu_{i+j+1}^{A}(a_1, \ldots, a_i, b, c_1, \ldots, c_j).
\]

Finally, we mention the gluing of \(A_{\infty}\)-algebras. Let \(M\) be an \(A_{\infty}\) \(A-B\)-bimodule. We denote by \(\begin{pmatrix} B & 0 \\ M & A \end{pmatrix}\) the \(A_{\infty}\)-algebra \(C\) which equals \(A \oplus B \oplus M\).
as a graded vector space, so that the non-zero components of $\mu_n^C$ are given by $\mu_n^A$, $\mu_n^B$, and

$$(-1)^{l_i(a)+1}\mu_{i,j}(a_1, \ldots, a_i, m, b_1, \ldots, b_j), \quad i + j + 1 = n,$$

where $a_1, \ldots, a_i \in A$, $b_1, \ldots, b_j \in B$.

2 Preliminaries on the Hochschild complex, pairings and copairings

In this section all $A_\infty$-algebras are strictly unital. As in the previous section, for an $A_\infty$-algebra $A$, we put $\overline{A} := A/k \cdot 1_A$.

The mixed Hochschild complex (see [13,17]) $(C_\bullet(A), b, B)$ of an $A_\infty$-algebra $A$ is given as a graded vector space by

$$C_\bullet(A) := \bigoplus_{n \geq 0} A \otimes (\overline{A}[1]) \otimes^n.$$

For convenience we write $(a_0, \ldots, a_n)$ instead of $a_0 \otimes \cdots \otimes a_n \in C_\bullet(A)$.

The Hochschild differential is given by

$$b(a_0, \ldots, a_n) = \sum_{0 \leq i \leq j \leq n} (-1)^{l_i(a)+1}(a_0, \ldots, \mu_{j-i+1}(a_i, \ldots, a_j), \ldots, a_n)$$

$$+ \sum_{0 \leq p < q \leq n} (-1)^{l_0(a)(l_q(a)+1)}(\mu_{n+p+2-q}(a_q, \ldots, a_n, a_0, \ldots, a_p), a_{p+1}, \ldots, a_{q-1}).$$

(2.1)

The Connes-Tsygan differential $B$ (see [2,8,9,35,36]) is given by

$$B(a_0, a_1, \ldots, a_n) = \sum_{0 \leq i \leq n} (-1)^{l_0(a)(l_i^n(a)+1)}(1, a_i, \ldots, a_n, a_0, \ldots, a_{i-1}).$$

The Hochschild complex can be more generally defined for $A_\infty$-categories, and is Morita invariant [17]. We refer to [17] for the definition of cyclic homology $HC_\bullet$, negative cyclic homology $HC^-_\bullet$ and $HP_\bullet$. In this paper we will in fact deal only with the first differential of the Hochschild-to-cyclic spectral sequence, which is the map $B : HH_n(A) \rightarrow HH_{n+1}(A)$ induced by the Connes-Tsygan differential.

We recall the natural pairings and co-pairings on $HH_\bullet(A)$. Let us restrict ourselves to DG algebras for a moment. Given a DG algebra $A$, we have
a Chern character $\text{ch} : K_n(A) \to HH_n(A)$ (see [1]; the Chern character naturally lifts to $HC^{-}(A)$, but we will not need this).

In particular, given DG algebras $A$, $B$ and an object $M \in D_{\text{perf}}(A \otimes B)$, we have a copairing

$$\text{ch}(M) \in (HH_{\bullet}(A) \otimes HH_{\bullet}(B))_0 \cong HH_0(A \otimes B).$$

This copairing is used in the formulation of Conjecture 0.3 for $A = B^{op}$ being smooth, and $M = A$.

Dually [28], if we have DG algebras $A$ and $B$, and an object $M \in D_{\text{pspe}}(A^{op} \otimes B^{op})$, then we have a pairing (of degree zero)

$$\beta_M : HH_{\bullet}(A) \otimes HH_{\bullet}(B) \to HH_{\bullet}(A \otimes B) \to HH_{\bullet}(\text{End}_k(M)) \to k$$

(2.2)

(the last map is an isomorphism if and only if $M$ is not acyclic). In the formulation of Conjecture 0.2 this pairing is used for $A = B^{op}$ proper, and $M = A$. In this case we denote the pairing by $\langle \cdot, \cdot \rangle$.

We refer the reader to [27, Section 5] for a detailed discussion of pairings on Hochschild homology in the $A_{\infty}$-setup. Here we just mention that for $A_{\infty}$-algebras $A$ and $B$, and an $A_{\infty}$ pseudo-perfect $A$-$B$-bimodule $M$, the pairing (2.2) can be obtained by choosing $A_{\infty}$-quasi-isomorphisms $A \to C$ and $B \to D$, where $C$ and $D$ are DG algebras, and choosing a DG $C$-$D$-bimodule $N$, such that $M$ is quasi-isomorphic to (the restriction of scalars of) $N$ in the category of $A_{\infty}$ $A$-$B$-bimodules. Then we simply define $\beta_M$ using isomorphisms $HH_{\bullet}(A) \cong HH_{\bullet}(C)$ and $HH_{\bullet}(B) \cong HH_{\bullet}(D)$ and the pairing $\beta_N$. Here we need only a special case of an explicit formula for $\beta_M$ (Proposition 2.1 below), for which we give a proof for completeness.

Let us recall that for a graded totally finite dimensional vector space $V$, and an endomorphism $\phi : V \to V$, homogeneous of degree zero, the supertrace of $\phi$ is defined by the formula

$$\text{str}_V(\phi) = \sum_{n \in \mathbb{Z}} (-1)^n \text{tr}_{V^n}(\phi).$$

If moreover $V$ is a complex, then the natural map $HH_{\bullet}(\text{End}_k(V)) \to k$ (which is an isomorphism if and only if $V$ is not acyclic) is given by the following morphism of complexes $C_{\bullet}(\text{End}_k(V)) \to k$:

$$(a_0, \ldots, a_k) \mapsto \begin{cases} \text{str}_V(a_0) & \text{for } k = 0, \ |a_0| = 0; \\ 0 & \text{otherwise.} \end{cases}$$

(2.3)
We also recall the formula for the shuffle product (Eilenberg-Zilber map) for DG algebras (see [20, Section 4.2] for usual associative algebras). Let us recall that a \((p, q)\)-shuffle is a permutation \(\sigma \in S_{p+q}\) such that
\[
\sigma(1) < \cdots < \sigma(p), \quad \text{and} \quad \sigma(p + 1) < \cdots < \sigma(p + q).
\]
For a DG algebra \(A\), let us define an action of \(S_n\) on \(A \otimes \tilde{A}[1] \otimes n\) by the formula
\[
\sigma(a_0, \ldots, a_n) = (-1)^{\varepsilon(\sigma)}(a_0, a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)}),
\]
where
\[
\varepsilon(\sigma) = \sum_{i < j, \sigma(i) > \sigma(j)} (-1)^{|a_i|+1|a_j|+1}.
\]
Now, given DG algebras \(A\) and \(B\), the map
\[
EZ : C_\bullet(A) \otimes C_\bullet(B) \to C_\bullet(A \otimes B)
\]
is given by the formula
\[
EZ((a_0, \ldots, a_p) \otimes (b_0, \ldots, b_q)) = \sum_{\sigma} (-1)^{l_\sigma(a)\varepsilon(\sigma)} a_0 \otimes b_0, a_1 \otimes 1, \ldots, a_p \otimes 1, 1 \otimes b_1, \ldots, 1 \otimes b_q,
\]
where the sum is over all \((p, q)\)-shuffles. The map \(EZ\) induces the Künneth isomorphism in Hochschild homology.

**Proposition 2.1** Let \(A_1\) and \(A_2\) be strictly unital \(A_\infty\)-algebras, and \(M\) a finite dimensional strictly unital \(A_\infty\) \(A_1\)-\(A_2\)-bimodule. Take some closed homogeneous elements \(a \in A_1\), \(b \in A_2\), such that \(|a| + |b| = 1\). If we consider \(a\) and \(b\) as classes in \(HH_\bullet(A_1)\) and \(HH_\bullet(A_2^{op})\) respectively, then
\[
\beta_M(a, B(b)) = str_M(v \mapsto (-1)^{|v|(|b|+1)} \mu^M_{1,1}(a, v, b)). \tag{2.4}
\]

**Proof** Let us choose \(A_\infty\)-quasi-isomorphisms \(f : A_1 \to C_1\), \(g : A_2 \to C_2\), where \(C_1\), \(C_2\) are DG algebras, and a DG \(C_1\)-\(C_2\)-bimodule \(N\) with an \(A_\infty\)-quasi-isomorphism \(\varphi : M \to (f, g)_*N\) of \(A_1\)-\(A_2\)-bimodules. Then we have
\[
\beta_M(a, B(b)) = \beta_N(f_1(a), B(g_1(b))).
\]
Now, we have
\[
EZ(f_1(a) \otimes B(g_1(b))) = -(f_1(a) \otimes 1, 1 \otimes g_1(b)).
Clearly, the map $C_\bullet(C_1 \otimes C_2^{op}) \to C_\bullet(\text{End}_k(N))$ takes $-(f_1(a) \otimes 1, 1 \otimes g_1(b))$ to $-(L_{f_1(a)}, R_{g_1(b)})$. Here for $x \in C_1$, $y \in C_2$, $z \in N$ we put $L_x(z) = xz$, $R_y(z) = (-1)^{|v||z|}zy$. Now, let us take the DG algebra $\mathcal{E}$ which as a graded algebra is given by the gluing

$$
\mathcal{E}^{gr} = \begin{pmatrix}
\text{End}_k(M)^{gr} & 0 \\
\text{Hom}_k(M, N)^{gr}[-1] & \text{End}_k(N)^{gr}
\end{pmatrix},
$$

and for $u \in \text{End}_k(M)^n$, $v \in \text{End}_k(N)^n$, $w \in \text{Hom}_k(M, N)^{n-1}$ we have

$$d(u, v, w) = (d(u), d(v), -d(w) + \varphi_{0,0}u - v\varphi_{0,0}).$$

Also, the multiplication is given by

$$(u, v, w) \cdot (u', v', w') = (uu', vv', uw' + (-1)^{|(u,v,w)|}wu').$$

The natural projections $\text{pr}_1 : \mathcal{E} \to \text{End}_k(M)$ and $\text{pr}_2 : \mathcal{E} \to \text{End}_k(N)$ are quasi-isomorphisms of DG algebras (see for example [7, Proof of Proposition 8.3]). We consider the following linear maps (in $v \in M$):

$$
\mu_{1,1}^M(a, -, b)(v) = (-1)^{|v|(|b|+1)}\mu_{1,1}^M(a, v, b), \quad \mu_{1,0}^M(a, -(v) = \mu_{1,0}^M(a, v),
$$

$$
\mu_{0,1}^M(\varphi_{1,1}(a, v, b), \varphi_{1,1}(a, v, b)) = (-1)^{|v|(|b|+1)}\varphi_{1,1}(a, v, b), \quad \varphi_{0,1}(-, b)(v) = (-1)^{|v|(|b|+1)}\varphi_{0,1}(b, v).
$$

Now, it is straightforward to check that the following Hochschild chain $\gamma$ is a cycle:

$$
\gamma = (\mu_{1,1}^M(a, -, b), 0, \varphi_{1,1}(a, -, b))
$$

$$
+((\mu_{1,0}^M(a, -), L_{f_1(a)}, \varphi_{1,0}(a, -)), (\mu_{0,1}^M(-, b), -R_{g_1(b)}, \varphi_{0,1}(-, b)) \in C_\bullet(\mathcal{E}).
$$

Clearly, $\text{pr}_2(\gamma) = -(L_{f_1(a)}, R_{g_1(b)})$. Further, we have

$$
\text{pr}_1(\gamma) = \mu_{1,1}^M(a, -, b) + (\mu_{1,0}(a, -, b), \mu_{0,1}(-, b)).
$$

Applying the map (2.3) to $\text{pr}_1(\gamma)$, we obtain the formula (2.4). $\Box$

We need the following corollary.

**Corollary 2.2** Let $A$ be a finite-dimensional strictly unital $A_\infty$-algebra, and $a, b \in A$ are closed homogeneous elements such that $|a| + |b| = 1$. If we consider $a$ and $b$ as classes in $HH_\bullet(A)$ and $HH_\bullet(A^{op})$ respectively, then

$$
\langle a, B(b) \rangle = (-1)^{|a|} \text{str}_A(v) \mapsto (-1)^{|(b|+1)|v|} \mu_3(a, v, b)).
$$
Proof. This is the special case of Proposition 2.1 with $A_1 = A_2 = A$ and $M = A$. \qed

3 A counterexample to the generalized degeneration conjecture

Starting from this section, the field $k$ is assumed to be of characteristic zero. We recall the main conjecture of [5].

**Conjecture 3.1** [5, Conjecture 1.3 for $n = 0$] Let $B$ and $C$ be small DG categories over $k$. Then the composition map

$$
\varphi_0 : K_0(B \otimes C) \xrightarrow{\text{ch}} (H^*_B(B) \otimes H^*_C(C))_0 \xrightarrow{\text{id} \otimes \delta^-} (H^*_B(B) \otimes HC^-_C(C))_1
$$

is zero.

Recall that $\delta^-$ denotes the boundary map from $HH^*_B(C) \to HC^-_C(C)$, see [5, Section 3]. It is the boundary map in the long exact sequence

$$
\cdots \to HC^-_{n+2}(C) \xrightarrow{\mu} HC^-_n(C) \to HH^*_n(C) \xrightarrow{\delta^-} HC^-_{n+1}(C) \to \cdots
$$

Let us notice that the composition $HH^*_n(C) \xrightarrow{\delta^-} HC^-_{n+1}(C) \to HH^*_{n+1}(C)$ is exactly the Connes-Tsygan differential $B : HH^*_n(C) \to HH^*_{n+1}(C)$.

In this section we construct a counterexample to Conjecture 3.1. We put $\Lambda_1 = k\langle \xi \rangle/\xi^2$, where $|\xi| = 1$, and (automatically) $d\xi = 0$. We have a quasi-equivalence $\text{Perf}(\Lambda_1) \simeq \text{Perf}_{(0)}(A^1_k)$ (the free $\Lambda_1$-module of rank 1 corresponds to the skyscraper sheaf $O_0$). In particular, we have a short exact sequence

$$
0 \to \text{Perf}(\Lambda_1) \to \text{Perf}(A^1) \to \text{Perf}(\mathbb{G}_m) \to 0
$$

(3.2)

We also denote by $k[\varepsilon] := k[t]/t^2$ the algebra of dual numbers ($|\varepsilon| = 0$, $d\varepsilon = 0$). Let us denote by $x$ the coordinate on $A^1$, and put $T := \text{Spec}(k[\varepsilon])$. Tensoring (3.2) by $\text{Perf}(k[\varepsilon])$ in the symmetric monoidal structure on triangulated DG categories, and using that $\text{Perf}(\cdot)$ is a symmetric monoidal functor from the homotopy category of DG categories to the homotopy category of triangulated DG categories (see [33, Exercise 4.4.12]), we obtain another short exact sequence:

$$
0 \to \text{Perf}(\Lambda_1 \otimes k[\varepsilon]) \to \text{Perf}(A^1 \times T) \to \text{Perf}(\mathbb{G}_m \times T) \to 0.
$$

(3.3)

Now let us take the Cartier divisor $D := \{x + \varepsilon = 0\} \subset A^1 \times T$. This is well-defined since $x + \varepsilon$ is not a zero divisor in $k[x] \otimes k[\varepsilon]$. Moreover, we
have \( D \cap (\mathbb{G}_m \times T) = \emptyset \), since \( x + \varepsilon \) is invertible in \( k[x^{\pm 1}] \otimes k[\varepsilon] \): we have 
\[(x + \varepsilon)(x^{-1} - x^{-2}\varepsilon) = 1.\] Therefore, by (3.3), we may and will consider \( O_D \) as an object of \( \text{Perf}(\Lambda_1 \otimes k[\varepsilon]) \).

We recall the following well-known facts summarized in the following proposition.

**Proposition 3.2**  
1) Let \( R \) be a commutative \( k \)-algebra. Then the Chern character \( ch : K_1(R) \to \text{HH}^1(R) \cong \Omega^1(R) \) induces a map \( d \log : R^\times \to \Omega^1(R) \). That is, for any invertible element \( r \in R^\times \), we have \( ch(r) = \frac{d\log r}{r} \).

2) Let \( A \xrightarrow{F} B \xrightarrow{G} C \) be a short exact sequence of DG categories, and \( X \in B \) is an object, with an endomorphism \( \varphi : X \to X \), closed of degree zero. If \( G(\varphi) \) is a quasi-isomorphism (hence giving a class in \( K_1(C) \)), then the boundary map \( K_1(C) \to K_0(A) \) takes \([G(\varphi)]\) to \([\text{Cone}(\varphi)]\) \( \in K_0(A) \). Here we consider \( \text{Cone}(\varphi) \) as an object of (the essential image of) \( D_{\text{perf}}(A) \).

**Proof**  
1) The first statement is well-known. By functoriality of the Chern character, it follows from the special case \( R = k[x^{\pm 1}], r = x \), see for example [26, Theorem 6.2.16].

2) This also follows from the “universal” special case. Namely, let us for simplicity assume that \( A \) and \( B \) are (weakly) pre-triangulated, and take a DG quasi-functor \( \text{Perf}(\mathbb{A}^1) \to B \), coming from a morphism of DG algebras \( k[x] \to B(X, X), x \mapsto \varphi \). Then it takes the skyscraper sheaf \( O_{\{0\}} \in D_{\text{perf}}(\mathbb{A}^1) \) to the essential image of \( D_{\text{perf}}(A) \). Hence, by functoriality of the boundary map, we may replace our short exact sequence by another sequence

\[
\text{Perf}_{\{0\}}(\mathbb{A}^1) \to \text{Perf}(\mathbb{A}^1) \to \text{Perf}(\mathbb{G}_m),
\]

and assume that \( X = O_{\mathbb{A}^1} \), and \( \varphi = x \). In this case our assertion is that the map \( K_1(\mathbb{G}_m) \to K_0(\text{Perf}_{\{0\}}(\mathbb{A}^1)) \cong \mathbb{Z} \) sends \( x \in K_1(\mathbb{G}_m) \) to \([O_{\{0\}}]\). This is standard and goes back to Quillen [24].

**Theorem 3.3** Conjecture 3.1 does not hold for the DG algebras \( \Lambda_1 \) and \( k[\varepsilon] \). Namely, we have \( \varphi_0([O_D]) \neq 0 \), where \( \varphi_0 \) is defined in (3.1).

**Proof** We will prove a stronger statement: \( \psi_0([O_D]) \neq 0 \), where \( \psi_0 \) is the composition

\[
K_0(\Lambda_1 \otimes k[\varepsilon]) \xrightarrow{ch} (HH_* (\Lambda_1) \otimes HH_* (k[\varepsilon]))_0 \\
\xrightarrow{id \otimes B} (HH_* (\Lambda_1) \otimes HH_* (k[\varepsilon]))_1.
\]

(this is indeed sufficient since \( B : HH_* (k[\varepsilon]) \to HH_{*+1} (k[\varepsilon]) \) factors through \( \delta^- \).)
We use the notation $d_{dR}$ for the de Rham differential in order to avoid confusion with differentials in DG algebras.

First let us identify the Hochschild homology of $/Lambda_1$.

Applying the long exact sequence in Hochschild homology to (3.2), we see that

$$HH_{-1}(/Lambda_1) = k[x^{\pm 1}] / k[x], \quad HH_0( /Lambda_1) = k[x^{\pm 1}] d_{dR} x / k[x] d_{dR} x,$$

and $HH_i( /Lambda_1) = 0$ for $i \notin \{-1, 0\}$.

Further, for any commutative $k$-algebra $R$ we have $HH_0(R) = R$, and $HH_1(R) = \Omega^1_{R/k}$, (and the Connes differential $B : HH_0(R) \to HH_1(R)$ is given by the de Rham differential). In particular, we have $HH_0(k[\epsilon]) = k[\epsilon]$, and $HH_1(k[\epsilon]) = k \cdot d_{dR} \epsilon$ (and we do not need $HH_{\geq 2}(k[\epsilon])$ for our considerations).

**Claim** With the above notation, we have $ch(\mathcal{O}_D) = \frac{d_{dR} x}{x} \otimes 1 - \frac{d_{dR} x}{x^2} \otimes \epsilon + \frac{1}{x} \otimes d_{dR} \epsilon$.

**Proof** As we already mentioned, the function $x + \epsilon$ is invertible on $G_m \times T$, hence it gives an element $\alpha \in K_1( G_m \times T)$. By Proposition 3.2 1), the boundary map

$$K_1( G_m \times T) \to K_0( /Lambda_1 \otimes k[\epsilon])$$

sends $\alpha$ to $[\mathcal{O}_D]$. By Proposition 3.2 2), we have

$$ch(\alpha) = d_{dR} \log(x + \epsilon) \in \Omega^1_{G_m \times T} = HH_1(G_m \times T).$$

Explicitly, we have

$$d_{dR} \log(x + \epsilon) = (x^{-1} - x^{-2} \epsilon) d_{dR} (x + \epsilon) = \frac{d_{dR} x}{x} - \frac{\epsilon d_{dR} x}{x^2} + \frac{d_{dR} \epsilon}{x}.$$

Applying the boundary map

$$(HH_\bullet(G_m) \otimes HH_\bullet(k[\epsilon]))_1 \cong HH_1(G_m \times T) \to HH_0( /Lambda_1 \otimes k[\epsilon])$$

$$\cong (HH_\bullet( /Lambda_1) \otimes HH_\bullet(k[\epsilon]))_0,$$

we obtain the desired formula for $ch(\mathcal{O}_D)$.

It follows from Claim that

$$(id \otimes B)(ch([\mathcal{O}_D])) = -\frac{d_{dR} x}{x^2} \otimes d_{dR} \epsilon \neq 0.$$

This proves the theorem.
4 A counterexample to Conjecture 0.3

In this section we disprove Conjecture 0.3.

**Proposition 4.1** Let $B$ be a smooth DG algebra and $F : \text{Perf}(A) \rightarrow \text{Perf}(B)$ a localization functor, where $A$ is a smooth and proper DG algebra. Then Conjecture 0.3 holds for $B$.

**Proof** This is actually explained in [5, proof of Theorem 4.6]. We explain the argument for completeness. The localization assumption implies that $(F \otimes F^{op})^*(I_A) \cong I_B$. In particular, the map $HH_\bullet(A) \otimes HH_\bullet(A^{op}) \rightarrow HH_\bullet(B) \otimes HH_\bullet(B^{op})$ takes $ch(I_A)$ to $ch(I_B)$. It remains to apply the commutative diagram

$$
\begin{array}{ccc}
HH_\bullet(A) \otimes HH_\bullet(A^{op}) & \longrightarrow & HH_\bullet(A) \otimes HC_\bullet^-(A^{op})[-1] \\
\downarrow & & \downarrow \\
HH_\bullet(B) \otimes HH_\bullet(B^{op}) & \longrightarrow & HH_\bullet(B) \otimes HC_\bullet^-(B^{op})[1],
\end{array}
$$

and Theorem 0.1 applied to $A$. \hfill \Box

Let us recall that by [21, Theorem 6.3], for any separated scheme of finite type $X$ over $k$, the (standard DG enhancement of the) category $D^b_{coh}(X)$ is smooth. Based on the construction of categorical resolution of singularities due to Kuznetsov and Lunts [18], the author [6] proved a stronger result: the category $D^b_{coh}(X)$ is homotopically finitely presented. We have the following corollary, mentioned in the introduction.

**Corollary 4.2** Let $X$ be a separated scheme of finite type over $k$, and $\mathcal{G} \in D^b_{coh}(X)$ – a generator. Then Conjecture 0.3 holds for the smooth DG algebra $A = R \text{End}(\mathcal{G})$.

**Proof** Indeed, by [6, Theorem 1.8 1)], there is a localization functor of the form $D^b_{coh}(Y) \rightarrow D^b_{coh}(X)$, where $Y$ is a smooth projective algebraic variety over $k$. The result follows by Proposition 4.1. Note that here we don’t even need to apply Theorem 0.1 since we only use the classical Hodge-to-de Rham degeneration for $Y$. \hfill \Box

**Remark 4.3** 1) In fact, in the formulation of Proposition 4.1 we could weaken the assumption on the functor $F$ to be a localization, requiring it only to be a homological epimorphism, which means that the extension of scalars functor $D(A) \rightarrow D(B)$ a localization, see [6, Section 3]. Indeed, we only need the isomorphism $(F \otimes F^{op})^*(I_A) = I_B$, which is by [6, Proposition 3.4] equivalent to the property of $F$ being a homological epimorphism. Then
in the proof of Corollary 4.2 we can apply the corresponding weakened version of [6, Theorem 1.8 1]): we have a homological epimorphism of the form $D^b_{coh}(Y) \to D^b_{coh}(X)$, where $Y$ is a smooth projective algebraic variety over $k$.

2) Proving that some DG functor $F : \text{Perf}(A) \to \text{Perf}(B)$ is a homological epimorphism is much easier than to prove that it is a localization; it is sufficient to show that the restriction of scalars $D(B) \to D(A)$ is fully faithful. To prove the localization on the level of perfect complexes one needs to know that moreover the kernel of the extension of scalars $D(A) \to D(B)$ is compactly generated (see [6, Corollary 3.8]), which makes [6, Theorem 1.8 1)] a much more difficult statement.

Clearly, Conjecture 0.3 is a special case of Conjecture 3.1. On the other hand, it was proved in [5] that Conjectures 3.1 and 0.3 are actually equivalent (more precisely, this follows from the proof of [5, Theorem 4.6]). However, deducing an explicit counterexample to Conjecture 0.3 along the lines of [5] would require some computations, which we wish to avoid. Instead, we use some trick.

Let us take some elliptic curve $E$ over $k$, with a $k$-rational point $p \in E(k)$. Choosing a local parameter $x \in \mathcal{O}_{E,p}$, we get an identification $\text{Perf}(\Lambda_1) \cong \text{Perf}(E)$. Let us choose some generator $F \in \text{Perf}(E)$ (e.g. $F = \mathcal{O}_E \oplus \mathcal{O}_p$), and put $B_E = \mathbb{R} \text{ End}(F)$, so that $\text{Perf}(B_E) \cong \text{Perf}(E)$. We denote by $F : \text{Perf}(\Lambda_1) \hookrightarrow \text{Perf}(B_E)$ the resulting embedding.

Let us denote by $HH_F : HH_* (\Lambda_1) \to HH_* (E)$ the map on Hochschild homology induced by the functor $F : \text{Perf}(\Lambda_1) \to \text{Perf}(B_E) \cong \text{Perf}(E)$.

**Proposition 4.4** 1) The morphism $HH_F : HH_0 (\Lambda_1) \to HH_0 (E) = H^0(\mathcal{O}_E) \oplus H^1(\omega_E) \cong k \oplus k$ is given by

$$
\frac{d_d R x}{x^n} \mapsto \begin{cases} 
(0, 1) & \text{for } n = 1; \\
0 & \text{for } n > 1.
\end{cases}
$$

2) The morphism $HH_F : HH_{-1} (\Lambda_1) \to HH_{-1} (E) = H^1(\mathcal{O}_E)$ does not vanish on $x^{-1}$.

**Proof** 1) First, the composition $HH_0 (\Lambda_1) \to HH_0 (E) \to H^0(\mathcal{O}_E)$ equals zero, since $H^0(\mathcal{O}_E)$ injects into $H^0(\mathcal{O}_{E-\{p\}}) \cong H^0(H_0(E - \{p\})$, and the composition of the functor $F$ with the pullback functor $\text{Perf}(E) \to \text{Perf}(E - \{p\})$ is isomorphic to zero. Further, we can use the identifications $HH_0(\text{Perf}(E)) \cong H^1(\mathcal{O}_E) \cong H^1(\mathcal{O}_{E,p}) / H^1(\mathcal{O}_{E,p}) / k$ (HKR with support [15, Proposition 5.7]). The linear functional $H^1(\mathcal{O}_E) \to H^1(\mathcal{O}_{E,p})$ is
simply the residue at \( p \). It remains to recall that
\[
\text{res}_p \left( \frac{dx}{x^n} \right) = \begin{cases} 
1 & \text{for } n = 1; \\
0 & \text{otherwise.}
\end{cases}
\]

2) Similarly, let us use identifications \( HH_{-1}(\text{Perf}_{(p)}(E)) \cong H^1_{(p)}(\mathcal{O}_E) \cong k(E)/\mathcal{O}_{E,p} \) (HKR with support). It is now sufficient to note that there is no rational function on \( E \) with a single simple pole at \( p \), hence \( x^{-1} \) is not in the image of \( H^0(\mathcal{O}_{E-(p)}) \to H^1_{(p)}(\mathcal{O}_E) \).

Further, we denote by \( C \) the semi-free DG algebra \( k\langle t_1, t_2 \rangle \), with \( |t_1| = 0 \), \( |t_2| = -1 \), \( dt_1 = 0 \), and \( dt_2 = t_1^2 \). By definition, \( C \) is homotopically finitely presented.

We take the object \( M \in \text{Perf}(\Lambda_1 \otimes C \otimes C) \) whose image in \( \text{Perf}(k[x] \otimes C \otimes C) \) is given by
\[
\text{Cone}(k[x] \otimes C^\otimes 2 \xrightarrow{x \otimes 1^\otimes 2 + 1 \otimes t_1^\otimes 2} k[x] \otimes C^\otimes 2).
\]

As in the previous section, we see that \( M \) is well-defined since the element
\[
x \otimes 1^\otimes 2 + 1 \otimes t_1^\otimes 2 \in H^0(k[x^{\pm 1}] \otimes C \otimes C) = k[x^{\pm 1}] \otimes k[\varepsilon] \otimes k[\varepsilon]
\]
is invertible.

Finally, we put \( N := (F \otimes \text{id}_C^\otimes 2)^*(M) \in \text{Perf}(B_E \otimes C \otimes C) \).

**Theorem 4.5** 1) With the above notation, the dg algebra
\[
A := \begin{pmatrix}
B_E \otimes C & 0 \\
N & C^\text{op}
\end{pmatrix}
\]
is homotopically finitely presented (hence smooth), but it does not satisfy Conjecture 0.3.

2) The DG category \( \text{Perf}(A) \) gives a negative answer to Question 0.4.

**Proof** First, by Proposition 4.1 we see that 2) reduces to 1).

We now prove 1). The homotopy finiteness of \( A \) follows from [6, Proposition 5.14] (gluing of homotopically finite DG algebras by a perfect bimodule is again homotopically finite).

Let us denote the class \( HH_F(x^{-1}) \in HH_{-1}(E) \) by \( [x^{-1}] \). By Proposition 4.4 2), we have \( [x^{-1}] \neq 0 \).

To prove 1), it suffices to show that \( (\text{id} \otimes \text{id} \otimes B)(\text{ch}(N)) \in (HH_\bullet(E) \otimes HH_\bullet(C)^\otimes 2)_1 \) is non-zero. Indeed, as in [5, Proof of Theorem 4.6], we can use
the additivity of the mixed Hochschild complex in semi-orthogonal decompositions, and decompose the category \( \text{Perf}(A \otimes A^{op}) \) into four components, one of which is \( \text{Perf}(B_E \otimes C \otimes C) \). The corresponding component of the diagonal \( A-A \)-bimodule is \( N[1] \), hence the corresponding component of \( \text{ch}(A) \) is \( -\text{ch}(N) \in (HH_\bullet(E) \otimes HH_\bullet(C)^{\otimes 2})_0 \). For details, see “Step 2” in the proof of [5, Theorem 4.6].

We have a natural projection \( \pi : C \to H^0(C) \cong k[\varepsilon] \). Let us put \( \bar{N} := (id \otimes \pi^* \otimes \pi^*)(N) \in \text{Perf}(E \times T \times T) \). Then \( \bar{N} \) is naturally isomorphic to \( O_{D'} \), where \( D' \subset E \times T \times T \) is a Cartier divisor, set-theoretically contained in \( \{p\} \times T \times T \), and given locally by the equation \( x \otimes 1^{\otimes 2} + 1 \otimes \varepsilon^{\otimes 2} = 0 \). The computation from Sect. 3, together with Proposition 4.4 1), implies that \( \text{ch}(\bar{N}) = (0, 1) \otimes 1^{\otimes 2} + [x^{-1}] \otimes d_{dR}\varepsilon \otimes \varepsilon + [x^{-1}] \otimes \varepsilon \otimes d_{dR}\varepsilon \).

Therefore, we obtain

\[
(id \otimes id \otimes B)(\text{ch}(\bar{N})) = [x^{-1}] \otimes d_{dR}\varepsilon \otimes d_{dR}\varepsilon \neq 0.
\]

By functoriality, this implies \( (id \otimes id \otimes B)(\text{ch}(N)) \neq 0 \). This proves 1). \( \square \)

5 A counterexample to Conjecture 0.2

In this section we disprove Conjecture 0.2. Let us recall that for a DG algebra \( A \), we denote by \( \delta^+ : HC_\bullet(A)[1] \to HH_\bullet(A) \) the boundary map [20, Section 2.2]. It comes from the long exact sequence for cyclic homology:

\[
\cdots \to HH_{n+2}(A) \to HC_{n+2}(A) \xrightarrow{u} HC_n(A) \xrightarrow{\delta^+} HH_{n+1}(A) \to \cdots.
\]

The composition \( HH_n(A) \to HC_n(A) \xrightarrow{\delta^+} HH_{n+1}(A) \) is exactly the Connes-Tsygan differential \( B : HH_n(A) \to HH_{n+1}(A) \).

More precisely, we will construct an example of a minimal finite-dimensional \( A_\infty \)-algebra \( B \) and two elements \( a, b \in B \), such that \( |a| + |b| = 1 \), and

\[
\text{str}_B(v \mapsto (-1^{(|b|+1)|v|} \mu_3(a, v, b)) \neq 0,
\]

thus disproving Conjecture 0.2 (by Corollary 2.2), since \( B : HH_\bullet(A^{op}) \to HH_{\bullet+1}(A^{op}) \) factors through \( \delta^+ \).

We first mention the following observation, which in fact motivates Conjecture 0.2.
**Proposition 5.1** Let $B$ be a proper DG algebra and $\text{Perf}(B) \hookrightarrow \text{Perf}(A)$ a quasi-fully-faithful functor, where $A$ is a smooth and proper DG algebra. Then Conjecture 0.2 holds for $B$.

**Proof** Indeed this follows from the commutative diagram

\[
\begin{array}{ccc}
HH_\bullet(B) \otimes HC_\bullet(B^{\text{op}})[1] & \xrightarrow{\text{id} \otimes \delta^+} & HH_\bullet(B) \otimes HH_\bullet(B^{\text{op}}) \longrightarrow k; \\
\downarrow & & \downarrow \\
HH_\bullet(A) \otimes HC_\bullet(A^{\text{op}})[1] & \xrightarrow{\text{id} \otimes \delta^+} & HH_\bullet(A) \otimes HH_\bullet(A^{\text{op}}) \longrightarrow k
\end{array}
\]

and Theorem 0.1 applied to $A$.

We have the following corollary, mentioned in the introduction.

**Corollary 5.2** Let $X$ be a separated scheme of finite type over $k$, and $Z \subset X$ a closed proper subscheme. For any object $\mathcal{F} \in \text{Perf}_Z(X)$, Conjecture 0.2 holds for the proper DG algebra $B = R \text{End}(\mathcal{F})$.

**Proof** Choosing some compactification $X \subset \bar{X}$ (which exist by the Nagata’s compactification theorem [23]), we get $\text{Perf}_Z(X) \simeq \text{Perf}_Z(\bar{X})$. Thus, we may and will assume $X = \bar{X} = Z$. Then the result follows by applying Proposition 5.1 with [18, Theorem 6.12]. As in the proof of Corollary 4.2, we only use here the classical Hodge-to-de Rham degeneration.

The crucial point is the following theorem, which is of independent interest.

**Theorem 5.3** 1) Let $A$ be a DG algebra, and $a \in H^0(A)$ a nilpotent element. Then the corresponding morphism $f : k[x] \rightarrow A$ (where $|x| = 0$) in $\text{Ho}(\text{dgAlg}_k)$ factors through $k[x]/x^n$ for a sufficiently large $n$.

2) If moreover $a^2 = 0$ in $H^0(A)$, then it suffices to take $n = 6$.

Before we prove Theorem 5.3, we show how it allows to construct a counterexample to Conjecture 0.2. In fact, in order to construct such counterexample it is sufficient to use Theorem 5.3 1), which is much easier to prove than Theorem 5.3 2) (see below). Part 2) is only used to construct a counterexample for which the total dimension of the cohomology equals 10.

**Theorem 5.4** 1) Let us denote by $y$ the variable of degree 1. Then there exists an $A_\infty$ $k[y]/y^3\cdot k[x]/x^6$-bimodule structure on the 1-dimensional vector space $V = k \cdot z$ (where $|z| = 0$) such that $\mu^V_3(x, z, y) = z$. In particular, in the glued $A_\infty$-algebra

\[
B = \begin{pmatrix}
k[y]/y^3 & 0 \\
V & k[x]/x^6
\end{pmatrix}
\]
we have \( \text{str}(v \mapsto \mu_3(x, v, y)) = 1. \) Therefore, by Corollary 2.2 this \( A_{\infty} \)-algebra (and any quasi-isomorphic DG algebra) does not satisfy Conjecture 0.2.

2) In particular, the proper DG category \( \text{Perf}^\infty(B) \) does not have a categorical resolution of singularities.

**Proof** 1) An easy computation shows that \( \text{Ext}^0_{k[y]/y^3}(k, k) = k[\varepsilon] \) (dual numbers). By Theorem 5.3 2), we have an \( A_{\infty} \)-morphism \( g : k[x]/x^6 \to \text{End}_{k[y]/y^3}^\infty(k) \), such that \( g(1) = \varepsilon \in H^0(\text{End}_{k[y]/y^3}^\infty(k)) \). This gives the desired \( A_{\infty} \)-bimodule structure on \( V \). The other conclusions are clear.

2) follows from 1) and Proposition 5.1.

\[ \square \]

**Proof of Theorem 5.3, part 1)** Let us denote by \( A_f \) the \( k[x] \)-\( A \)-bimodule which equals \( A \) as an \( A \)-module, and whose \( k[x] \)-module structure comes from \( f \). Since the algebra \( k[x] \) is smooth, we have \( A_f \in D_{\text{perf}}(k[x] \otimes A) \).

Since \( a \in H^0(A) \) is nilpotent, we have \( k[x^{\pm 1}] \otimes_{k[x]} A = 0 \). We conclude that \( A_f \) is contained in the essential image of \( D_{\text{perf}}(A_1 \otimes A) \hookrightarrow D_{\text{perf}}(k[x] \otimes A) \).

Now, let us note that in \( \text{Ho}(\text{dgcat}_k) \) we have \( \text{Perf}(A_1) \simeq \text{colim}_n \text{PsPerf}(k[x]/x^n) \). Indeed, recall from Sect. 3 that \( \text{Perf}(\Lambda_1) \simeq \text{Perf}_{[0]}(\Lambda^1) \simeq D_{\text{coh},[0]}^b(\Lambda^1) \), and we have \( D_{\text{coh},[0]}^b(\Lambda^1) \simeq \text{colim}_n D_{\text{coh}}^b \text{Spec } k[x]/x^n \), see for example [6, Lemma 8.15].

It follows that we have an equivalence of triangulated categories

\[ D_{\text{perf}}(A_1 \otimes A) \simeq \text{colim}_n D_{\text{perf}}(\text{PsPerf}(k[x]/x^n) \otimes A). \]

Indeed, we have \( D_{\text{perf}}(A_1 \otimes A) \simeq D_{\text{perf}}(\text{Perf}(A_1) \otimes A) \). It is clear that the symmetric monoidal structure on \( \text{Ho}(\text{dgcat}_k) \) commutes with filtered homotopy colimits. And by [34, Lemma 2.10], we know that \( D_{\text{perf}}(\quad) \) commutes with filtered homotopy colimits.

Therefore, there exists \( n > 0 \) such that \( A_f \) is contained in the essential image of \( D_{\text{perf}}(\text{PsPerf}(k[x]/x^n) \otimes A) \). Let us denote by \( \widetilde{M} \in D_{\text{perf}}(\text{PsPerf}(k[x]/x^n) \otimes A) \) an object whose image is isomorphic to \( A_f \). We have a natural functor

\[ \Phi : \text{PsPerf}(k[x]/x^n) \otimes \text{Perf}(A) \to \mathbf{R}\text{Hom}(k[x]/x^n, \text{Perf}(A)). \]

By construction, the \( k[x]/x^n \)-\( A \)-bimodule \( \Phi(\widetilde{M}) \) is quasi-isomorphic to \( A \) as an \( A \)-module. Using the isomorphism \( \Phi(\widetilde{M}) |_A \simeq A \), we obtain the following composition morphism in \( \text{Ho}(\text{dgalg}_k) \):

\[ g : k[x]/x^n \to \mathbf{R} \text{ End}_A(\Phi(\widetilde{M})) \to A. \]
By construction, \( H^0(g)(x) = a \). Thus, \( g \) factors \( f \) through \( k[x]/x^n \). This proves part 1)

The proof of part 2) of Theorem 5.3 requires some computations which we split into several lemmas.

First, we may replace the abstract algebra \( A \) by the concrete DG algebra \( C \) which was used in Sect. 4. Recall that it is freely generated by the elements \( t_1, t_2 \) with \( |t_1| = 0, |t_2| = -1 \), and \( dt_1 = 0, dt_2 = t_1^2 \). Indeed choosing a representative \( \tilde{a} \in A^0 \) of \( a \), and an element \( h \in A^{-1} \) such that \( dh = \tilde{a}^2 \), we obtain a morphism of DG algebras \( C \to A, t_1 \mapsto \tilde{a}, t_2 \mapsto h \). Thus, we may assume that \( A = C \) and \( a = t_1 \).

It will be very useful to introduce an additional \( \mathbb{Z} \)-grading on \( C \), which can be thought of as a \( \mathbb{G}_m \)-action. We will denote this grading by \( w \), putting \( w(t_1) = 1, w(t_2) = 2 \), and then extend by the rule \( w(uv) = w(u) + w(v) \). Clearly, the differential \( d \) has degree zero with respect to \( w \).

We thus have a decomposition of \( C \) as a complex:
\[
C = \bigoplus_{n \geq 0} C_{-n}^n.
\]

It is not immediately obvious how to compute the cohomology \( C \) explicitly. It is easier to pass through its “completion” and use a version of Koszul duality. Namely, let us define \( \hat{C} := \prod_{n \geq 0} C_{-n}^n \). This is also a DG algebra, and we have a map \( C \to \hat{C} \). The homogeneous elements of degree \( -m \) in \( \hat{C} \) are just non-commutative power series in \( t_1, t_2 \) such that in each monomial there are exactly \( m \) copies of \( t_2 \).

**Lemma 5.5** The cohomology algebra \( H^\bullet(\hat{C}) \) is generated by the elements \( u_1 = t_1 \) and \( u_2 = [t_1, t_2] \), with two relations: \( u_2^2 = 0, u_1u_2 + u_2u_1 = 0 \).

**Proof** Let us notice that the DG algebra \( \hat{C} \) is isomorphic to the endomorphism DG algebra \( \text{End}_{k[y]/y^3}^\infty(k) \) on the chain level (that is, the underlying graded algebras are isomorphic, and the isomorphism commutes with the differentials). Indeed, by the definition of the DG category of \( A_\infty \)-modules (Sect. 1.2), for any finite-dimensional (strictly unital) \( A_\infty \)-algebra \( B \) and a finite-dimensional \( A_\infty \)-module \( M \), we have an isomorphism of graded algebras
\[
\text{End}_{B}^\infty(M)_{gr} \cong \hat{T}(\overline{B}^*[-1]) \otimes \text{End}_k(M^{gr}),
\]
where \( \hat{T}(\cdot) \) denotes the complete tensor algebra. In our special case \( B = k[y]/y^3, M = k \), we see that the graded algebra \( \text{End}_{k[y]/y^3}^\infty(k)^{gr} \) is freely topologically generated by the elements \( \xi_1 \) and \( \xi_2 \), where \( |\xi_1| = 0, |\xi_2| = -1 \) (the dual basis of \( \{y, y^2\} \) in \( \overline{B}^*[-1] \)). Obviously, \( d(\xi_1) = 0 \), and it is easy to check that \( d(\xi_2) = -\xi_1^2 \). Therefore, if we take the (adically continuous)
isomorphism \( \hat{C} \xrightarrow{\sim} \text{End}_{k[y]/y^3}^\infty(k)^{gr}, \) sending \( t_1 \) to \( \xi_1 \) and \( t_2 \) to \(-\xi_2\), then this isomorphism commutes with the differentials.

Thus, we have an isomorphism of graded algebras \( H^\bullet(\hat{C}) \cong \text{Ext}_{k[y]/y^3}^\bullet(k, k) \). To compute this Ext-algebra, we take the semi-free resolution \( P \to k \). The underlying graded \( k[y]/y^3 \)-module is defined by

\[
P^{gr} := \bigoplus_{n=0}^{\infty} e_n \cdot k[y]/y^3,
\]

where \( |e_n| = \lfloor \frac{n}{2} \rfloor \). The differential is given by \( d(e_0) = 0 \), and \( d(e_{2k}) = e_{2k}y \), \( d(e_{2k+2}) = e_{2k+1}y^2 \) for \( k \geq 0 \). The morphism \( P \to k \) sends \( e_0 \) to 1, and \( e_n \) to 0 for \( n > 0 \). Clearly, this is a quasi-isomorphism.

We see that \( \text{Ext}_{k[y]/y^3}^\bullet(k, k) \cong \text{Hom}_{k[y]/y^3}(P, k) \), where the last complex has zero differential, and is equipped with the homogeneous basis \( \{ v_n \}_{n \geq 0} \), where \( |v_n| = -\lfloor \frac{n}{2} \rfloor \), and \( v_i(e_j) = \delta_{ij} \). It is easy to see that the elements \( v_1 \) and \( v_2 \) correspond to the classes \( u_1, u_2 \in H^\bullet(\hat{C}) \), mentioned in the formulation of the lemma. Clearly, we have \( u_1^2 = 0 \). It remains to show that \( u_1u_2 = -u_2u_1 \), and \( u_1u_2^k \neq 0 \) for \( k \geq 0 \). Let us choose the lifts \( \tilde{v}_n \in \text{End}_{k[y]/y^3}(P) \) of \( v_n \), putting

\[
\tilde{v}_2k(e_n) = \begin{cases} 
(-1)^{nk} e_{n-2k} & \text{for } n \geq 2k, \\
0 & \text{otherwise};
\end{cases}
\]

\[
\tilde{v}_{2k+1}(e_n) = \begin{cases} 
e_{n-2k-1} & \text{for } n \text{ odd}, n \geq 2k + 1, \\
(-1)^k e_{n-2k-1}y & \text{for } n \text{ even}, n \geq 2k + 2, \\
0 & \text{otherwise}.
\end{cases}
\]

It is easy to check that \( \tilde{v}_n \)'s super-commute with the differential, and that \( \tilde{v}_1 \tilde{v}_2 + \tilde{v}_2 \tilde{v}_1 = 0 \), \( \tilde{v}_1(\tilde{v}_2)^k = (-1)^k \tilde{v}_{2k+1} \). This proves the lemma. \( \square \)

**Lemma 5.6** The natural inclusion \( C \to \hat{C} \) is a quasi-isomorphism.

**Proof** We already know that \( \dim H^n(\hat{C}) < \infty \) for all \( n \in \mathbb{Z} \). It remains to observe the following: for any infinite sequence of complexes of vector spaces \( K_0^\bullet, K_1^\bullet, \ldots \) such that \( \dim H^n(\prod_{n \geq 0} K_n^\bullet) < \infty \) for all \( n \in \mathbb{Z} \), the morphism \( \bigoplus_{n \geq 0} K_n^\bullet \to \prod_{n \geq 0} K_n^\bullet \) is a quasi-isomorphism. Applying this observation to the complexes \( C^\bullet, C \), we conclude the proof. \( \square \)

We now construct a strictly unital \( A_\infty \)-morphism \( k[x]/x^6 \to C \), using obstruction theory. First, we introduce the weight grading (\( \mathbb{G}_m \)-action) on
k[x]/x^6 by putting w(x) = 1. Our $A_\infty$-morphism will be compatible with the $\mathbb{G}_m$-actions, and its component $f_1$ is given by

$$f_1(x^k) = t_i^k \text{ for } 0 \leq k \leq 5.$$ \hfill (5.1)

Note that all the cohomology spaces $H^n(C)$ are $H^0(C)-H^0(C)$-bimodules, hence also $k[x]/x^6$ is $k[x]/x^6$-bimodules (via $f_1$).

Let us also note that for any $k[x]/x^6$-bimodule $M$, equipped with the compatible $\mathbb{G}_m$-action, the Hochschild cohomology $HH^\bullet(k[x]/x^6, M)$ also becomes bigraded; the second grading again corresponds to the $\mathbb{G}_m$-action. For a vector space $V$ equipped with a $\mathbb{G}_m$-action, $V = \bigoplus_{n \in \mathbb{Z}} V^n$, we denote by $V(k)$ the same space with a twisted $\mathbb{G}_m$-action: $V(k)^n = V^{k+n}$.

In the following lemma it is important that we use our assumption $\text{char}(k) = 0$ (more precisely, we need $\text{char}(k) \neq 2$).

**Lemma 5.7** We have $HH^{2k+2}(k[x]/x^6, H^{-2k}(C)) \cong k[\varepsilon](6)$ for $k \geq 0$, and $HH^{2k+3}(k[x]/x^6, H^{-2k-1}(C)) \cong k(4)$ ($\mathbb{G}_m$-equivariant isomorphisms).

**Proof** We have the following $\mathbb{G}_m$-equivariant resolution of the diagonal bimodule:

$$\ldots \xrightarrow{d_3} k[x]/x^6 \otimes k[x]/x^6(-6) \xrightarrow{d_2} k[x]/x^6 \otimes k[x]/x^6(-1) \xrightarrow{d_1} k[x]/x^6 \otimes k[x]/x^6 \xrightarrow{m} k[x]/x^6,$$

where $d_{2k+1} = x \otimes 1 - 1 \otimes x$, and $d_{2k} = x^5 \otimes 1 + x \otimes x^4 + \cdots + 1 \otimes x^5$. Further, by Lemmas 5.5 and 5.6 we know the $\mathbb{G}_m$-equivariant $H^0(C)-H^0(C)$-bimodules $H^n(C)$. Namely, $H^{-2k}(C) \cong k[\varepsilon]((-6k)$ (twisted diagonal bimodule), and $H^{-2k-1}(C) \cong (k[\varepsilon])_{\sigma}(-6k - 3)$ – the twisted anti-diagonal bimodule. For the later, the left and right $H^0(C)$-actions are given respectively by $\varepsilon \cdot 1 = \varepsilon$, $1 \cdot \varepsilon = -\varepsilon$. The result follows by an elementary computation. \hfill $\square$

We are finally able to finish the proof of the theorem.

**Proof of Theorem 5.3, part 2** As we already mentioned, we will construct (or rather show the existence of) a $\mathbb{G}_m$-equivariant strictly unital $A_\infty$-morphism $f : k[x]/x^6 \to C$, where $f_1$ is given by (5.1). Since $H^0(f_1)$ is a homomorphism, we can construct $f_2$ such that the required relation is satisfied. Suppose that we have already constructed $\mathbb{G}_m$-equivariant $f_1, \ldots, f_n$ (where $n \geq 2$) satisfying all the relations for the $A_\infty$-morphism that involve only $f_1, \ldots, f_n$. We want to construct the $(n + 1)$ components $f_1, \ldots, f_n, f_{n+1}$ (again, satisfying all the relevant relations) where only $f_n$ is possibly being replaced by another map $f'_n$. The standard obstruction theory [25, Lemma 7.2.1] tells us that the obstruction to this is given by a class in $HH^{n+1,0}(k[x]/x^6, H^{1-n}(C))$ (the $\mathbb{G}_m$-invariant part). Applying Lemma 5.7, we see that this space vanishes.
Thus, proceeding inductively we can construct the desired $A_{\infty}$-morphism $f$. This proves the theorem. □

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