SPHERICAL LOGVINENKO-SEREDA-KOVRIJKINE TYPE INEQUALITY AND NULL-CONTROLLABILITY OF THE HEAT EQUATION ON THE SPHERE

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Abstract. It is shown that the restriction of a polynomial to a sphere satisfies a Logvinenko-Sereda-Kovrijkine type inequality. This implies a spectral inequality for the Laplace-Beltrami operator, which, in turn, yields observability and null-controllability with explicit estimates on the control costs for the spherical heat equation that are sharp in the large and in the small time regime.

1. Introduction and results

In the study of controllability of the heat equation in non-euclidean domains the sphere is a prime example. We derive observability and control cost estimates in this situation based on an uncertainty relation for harmonic polynomials. Let us explain, starting with euclidean geometry, how uncertainty relations enter the picture.

There are many physical and mathematical manifestations of the uncertainty principle. All of them imply that it is impossible for a compactly supported function to have a compactly supported Fourier transform. For applications, quantitative versions of this statement are necessary, e.g., the well known Paley-Wiener theorem. In control theory, a different quantitative version is of interest, namely, estimates of the form

\[ \| f \|_{L^2(\mathbb{R}^d)} \leq C \| f \|_{L^2(S)} \quad \text{for all } f \in L^2(\mathbb{R}^d) \text{ with } \text{supp } \hat{f} \subset \Sigma, \]

for certain sets \( S, \Sigma \subset \mathbb{R}^d \), and where the constant \( C \) depends only on \( S, \Sigma \), and the dimension \( d \). If \( \Sigma = B(0, r) \) for some \( r > 0 \), the sets \( S \) for which (1.1) holds were thoroughly studied by Panejah [21, 22], Kacnel’son [11], and Logvinenko-Sereda [15]. Later and using a different approach, Kovrijkine [13, 12] obtained the optimal constant, improving the bounds derived by Kacnel’son and Logvinenko-Sereda.

Theorem 1.1 (Panejah, Kacnel’son, Logvinenko-Sereda, Kovrijkine). Let \( S \subset \mathbb{R}^d \) be a measurable set, and let \( \Sigma = B_r(0) \) for \( r > 0 \). Then there is \( C \) such that (1.1) holds, if and only if \( |S \cap (x + (0,a)^d)| \geq \gamma a^d \) for some \( \gamma, a > 0 \). In this case, \( C = (c_0^d/\gamma)^d/(\gamma^d a^d) \) with a universal constant \( c_0 > 0 \).

A set \( S \) satisfying the geometric condition of the last theorem is frequently referred to as a thick set. While the interest in Logvinenko-Sereda-Kovrijkine inequalities stems from abstract considerations in functional and Fourier analysis, it was later realized that they immediately imply so-called spectral inequalities for the Laplacian on \( \mathbb{R}^d \) used in control theory, cf., e.g., [8].

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More generally, for a domain $M \subset \mathbb{R}^d$, a non-negative self-adjoint operator $H$ on $L^2(M)$ is said to satisfy a spectral inequality with respect to some measurable set $S \subset M$, if for each energy $E \geq 0$ there is a constant $C(E) > 0$, depending only on $M$, $S$, $H$, and $E$, such that

\begin{equation}
\|f\|_{L^2(M)} \leq C(E) \|f\|_{L^2(S)} \quad \text{for all } f \in \text{Ran } P_H((−\infty, E)).
\end{equation}

Here $P_H((−\infty, E)) := 1_{(−\infty, E)}(H)$ denotes the spectral projection up to energy $E \in \mathbb{R}$ of the operator $H$. Hence, Theorem 1.1 implies that $H = −\Delta$ on $M = \mathbb{R}^d$ satisfies a spectral inequality for all thick sets $S$ (since $f \in \text{Ran } P_{−\Delta}((−\infty, E))$ if and only if $\text{supp } f \subset B(0, \sqrt{E})$) and the resulting constant is best possible. In fact, it is possible to adapt Kovrijkine’s approach and obtain spectral inequalities for several different domains $M$ and operators $H$, e.g.,

- if $M$ is the torus, a cube, or an infinite strip and $H = −\Delta$ is an appropriate self-adjoint realization of the Laplacian [9, 5];
- if $M = \mathbb{R}^d$ and $H = −\Delta + |x|^2$ is the harmonic oscillator [1, 16, 4] (and [3] for a treatment of the partial harmonic oscillator);
- if the domain $M$ admits an appropriate covering and $H$ is an operator such that every function in the range of the spherical projections satisfies a Bernstein-type inequality [7].

In the present paper we adapt Kovrijkine’s approach to prove a spectral inequality for the Laplace-Beltrami operator on the sphere of radius $R > 0$, that is (1.2) with $H = −\Delta = −\Delta_{S^d_{R, 1}}$ and $M = S^d_{R, 1}$, $d \geq 2$. Put another way, we prove a Logvinenko-Sereda-Kovrijkine type inequality for functions on the sphere having a finite expansion into eigenfunctions of the Laplace-Beltrami operator.

In this setting we introduce the notion of thickness with respect to spherical caps of radius $a$. Since the spherical distance between two points $u, v \in S^d_{R, 1}$ is given by $d_R(u, v) = R \arccos(u \cdot v / R^2)$, these are sets of the form

\[ K(x, a) = \{ y \in S^d_{R, 1} : d_R(x, y) \leq \pi a \}, \]

where $x \in S^d_{R, 1}$. The notion of thickness is then defined as follows.

**Definition 1.2** (Thick sets on $S^d_{R, 1}$). Let $0 < \gamma \leq 1$. A measurable set $S \subset S^d_{R, 1}$ is called $\gamma$-thick, if there exists some radius $a > 0$ such that $|S \cap K| \geq \gamma |K|$ for all spherical caps $K$ of radius $a$.

The eigenfunctions of the Laplace-Beltrami operator on the sphere are given by the spherical harmonics. Since these are restrictions of polynomials on $\mathbb{R}^d$ to the sphere, finite linear combinations $f$ of eigenfunctions are spherical polynomials in the sense that there exists a polynomial $P$ on $\mathbb{R}^d$ such that $f = P|_{S^d_{R, 1}}$. We say that $f$ has degree at most $N \in \mathbb{N}$, if there is a corresponding polynomial $P$ of degree $N$ (or less) on $\mathbb{R}^d$ such that the last equality holds.

The spherical Logvinenko-Sereda-Kovrijkine type inequality then reads as follows. In the formulation of this theorem and throughout the rest of this paper we denote by $c_j > 0$, $j \in \mathbb{N}$, constants depending only on the dimension $d$.

**Theorem 1.3** (Logvinenko-Sereda-Kovrijkine type inequality on $S^d_{R, 1}$). Let $1 \leq q \leq \infty$, let $f$ be a spherical polynomial of degree at most $N \in \mathbb{N}$, and let $S \subset S^d_{R, 1}$ be $\gamma$-thick. Then

\[ \|f\|_{L^q(S^d_{R, 1})} \leq \left( \frac{c_1}{\gamma} \right)^{2N+1/q} \|f\|_{L^q(S)}. \]

**Remark 1.4.** A situation that is closely related to the previous theorem in the case $q = 2$ was studied in [17, Corollary 1.1] where, however, the constant is not explicitly given. In particular, it lacks the explicit dependence on the degree $N$.
of the polynomial, which is crucial for applications in control theory discussed in Theorem 1.6 and Remark 1.7.

With the above theorem at our disposal, we proceed analogously to the case of the Laplacian on \( \mathbb{R}^d \) and show that the Laplace-Beltrami operator on the sphere satisfies a spectral inequality.

**Corollary 1.5** (Spectral inequality). For all \( \gamma \)-thick sets \( S \subset S^{d-1}_R \) with \( \gamma \in (0, 1] \), all \( E \geq 0 \), and all \( f \in \text{Ran} P_{-\Delta}((-\infty, E]) \) we have

\[
\|f\|_{L^2(S^d)} \leq \left( \frac{c_1}{\gamma} \right)^{RE^{1/2}+1/2} \|f\|_{L^2(S)}. \tag{1.3}
\]

The proof of Corollary 1.5 is given in Subsection 2.3 below.

An important feature of the above inequality is the particular form of the constant on the right hand side of (1.3). Clearly, with \( d_0 = (c_1/\gamma)^{1/2} \) and \( d_1 = R\log(c_1/\gamma) \) we have

\[
\left( \frac{c_1}{\gamma} \right)^{RE^{1/2}+1/2} \leq d_0 e^{d_1 E^{1/2}}.
\]

Since the right hand side depends on the energy only through the factor \( E^{1/2} \) in the exponent, the spectral inequality can be combined with [19, Theorem 2.8] to obtain an observability estimate for the heat semigroup on the sphere with explicit control on the constant.

**Theorem 1.6** (Observability). Let \( S \subset S^{d-1}_R \) be a \( \gamma \)-thick set. Then

\[
\|e^{-t\Delta} g\|_{L^2(S^d)} \leq C_{\text{obs}} \int_0^T \|e^{-t\Delta} g\|_{L^2(S)}^2 \, dt
\]

for all \( g \in L^2(S^d) \) and all \( T > 0 \) where

\[
C_{\text{obs}} := \frac{c_2}{T} \exp \left( c_2 \left( \frac{R^2 |\log \gamma|^2}{T} + |\log \gamma| \right) \right).
\]

**Remark 1.7.** For background in control theory suitable for our context we refer the reader for instance to [6, 14]. In particular it is well known that by duality between observability and null-controllability, Theorem 1.6 implies null-controllability of the spherical heat equation

\[
\partial_t u - \Delta u = 1_S f, \quad u(0) = u_0 \in L^2(S^{d-1}_R), \tag{1.4}
\]

in time \( T > 0 \) with an explicit estimate for the control costs if \( S \) is \( \gamma \)-thick. More precisely, for any given initial datum \( u_0 \in L^2(S^{d-1}_R) \) there is \( f \in L^2((0, T); L^2(S^{d-1}_R)) \) such that \( \|f\|^2 \leq C_{\text{obs}} \|u_0\|^2 \) and the mild solution \( u \) of (1.4) satisfies \( u(T) = 0 \).

In case where \( S \) is a non-empty open set, observability of the heat semigroup on spheres has already been studied in [14]. Since this approach is based on Carleman estimates, it does not seem possible to generalize it to measurable sets \( S \). Our result is, in contrast, applicable to any measurable set \( S \subset S^{d-1}_R \) with strictly positive Lebesgue measure, since such sets are \( \gamma \)-thick with \( \gamma = |S|/|S^{d-1}_R| \). In the large time regime our constant \( C_{\text{obs}} \) decays with \( 1/T \), which is optimal according to [19, Theorem 2.13], and which has not been established before to our best knowledge. Concerning the small time regime, [14] gives the best possible control cost estimate for open \( S \). Our Theorem 1.6 extends this upper bound to all \( S \) of positive measure.

In order to understand the small time asymptotics we consider the situation of a small spherical ball.
Example 1.8. Let \( S = K(x_0, r) \) for some \( r \in (0, 1) \) and some \( x_0 \in S^{d-1} \). Then \( S \) is \( \gamma \)-thick with \( \gamma = |S|/|S^{d-1}| \). For simplicity, we assume that \( \gamma \leq 1/e \), so that \( |\log \gamma| \geq 1 \). For \( T < 1 \), this shows that the constant \( C_{\text{obs}} \) in Theorem 1.6 (with \( R = 1 \)) satisfies

\[
C_{\text{obs}}^2 \leq \frac{c_1}{T} \exp \left( \frac{2c_2|\log \gamma|^2}{T} \right).
\]

Comparing this inequality to the lower bound (valid for sufficiently small \( r \), resp. \( \gamma \))

\[
C_{\text{obs}}^2 \geq C' \exp \left( \frac{C|\log \gamma|^2}{T} \right), \quad C, C' > 0,
\]

given in [14, Theorem 1.2], we see that our bound is best possible in this regime.

Let us emphasize that in the small time regime upper and lower bounds (of different type) for the constant \( C_{\text{obs}} \) were already derived in [18].

2. Proofs

In contrast to previous proofs implementing Kovrijkine’s approach, we are not dealing with an Euclidean setting here. We therefore adapt the geometric constructions to the sphere, as already seen in Definition 1.2. More precisely, we replace Euclidean balls (or rectangles) by spherical caps and Euclidean line segments by spherical line segments. A spherical line segment starting at a point \( p \in S^{d-1} \) in direction \( v \in S^{d-1} \) with \( p \cdot v = 0 \) is a set \( I \subset S^{d-1} \) that is the trace of the restriction of the curve

\[
\gamma: [0, 2\pi] \to S^{d-1}, \quad t \mapsto \cos(t)p + \sin(t)v
\]

to an interval \([0, l]\), \( l > 0 \); in other words \( I = \gamma([0,l]) \). Given a spherical line segment and a measurable set \( M \subset S^{d-1} \), it is natural to define the arc length measure of the set \( I \cap M \) by

\[
\sigma(I \cap M) = \int_0^l 1_{I \cap M}(\gamma(t))|\gamma'(t)| \, dt = R \int_0^l 1_{I \cap M}(\gamma(t)) \, dt.
\]

We use spherical line segments to explicitly construct a variant of spherical polar coordinates centered at some point \( p \) on the sphere. To this end, note that there is a one-to-one correspondence \( \Phi: S^{d-2} \to \{ v \in S^{d-1} : p \cdot v = 0 \} \) between the possible directions of spherical line segments starting at \( p \) and the sphere \( S^{d-2} \). Hence, if we let \( \gamma_v \) be the curve \( \gamma \) from (2.1) starting at \( p \) in direction \( \Phi(v) \), integrating over all those possible directions yields the spherical polar coordinates formula

\[
\int_{S^{d-2}_R} f \, d\sigma^{d-1}_R = \frac{R^2}{2} \int_{S^{d-2}} \int_0^{2\pi} |f(\gamma_v(t))| \sin(t)^{d-2} \, dt \, d\sigma^{d-2}_R(v),
\]

where \( \sigma^{d-k}_R \) denotes the surface measure of \( S^{d-k}_R \), \( k < d \).

2.1. Reduction to spherical line segments. We now reduce our considerations to spherical line segments. Our first lemma is based on the classic result [20, Theorem 1], see also [23]. We recall it for convenience.

Lemma 2.1. For all \( n \in \mathbb{N} \), all coefficients \( \beta_1, \ldots, \beta_n \subset \mathbb{C} \), \( \lambda_1, \ldots, \lambda_n \subset \mathbb{R} \), and all measurable sets \( A \subset [0,1] \) with \( |A| > 0 \) the function \( r(x) = \sum_{k=1}^n \beta_k e^{\lambda_k x} \) satisfies

\[
\|r\|_{L_\infty([0,1])} \leq \left( \frac{316}{|A|} \right)^{n-1} \|r\|_{L_\infty(A)}.
\]

The proof of our first lemma is based on the observation that restrictions of spherical polynomials to spherical line segments satisfy the assumptions of Lemma 2.1.
Lemma 2.2. Let \( 1 \leq q \leq \infty \), let \( K \) be a spherical cap of radius \( a > 0 \) and let \( f \) be a spherical polynomial of degree at most \( \mathcal{N} \in \mathbb{N} \). Then there exists a point \( p \in K \) such that \( |f(p)|^q \geq \|f\|_{L^q(K)}^q / |K| \) and for all spherical line segments \( I \subset K \) starting at \( p \) and all measurable subsets \( M \subset S_{d-1}^d \) such that \( \sigma(M \cap I) > 0 \) we have

\[
\|f\|_{L^q(K)} \leq |K|^{1/q} \left( \frac{316 \cdot \sigma(I)}{\sigma(I \cap M)} \right)^{2N} \sup_{M \cap I} |f|.
\]  

Proof. A simple proof by contradiction shows the existence of the point \( p \in K \).

Let \( I \subset K \) be a spherical line segment starting at \( p \), such that \( \sigma(M \cap I) > 0 \). If \( v \in S_{d-1}^d \) denotes the direction of \( I \), then the curve

\[
\kappa(t) = \cos(\sigma(I)t/R)p + \sin(\sigma(I)t/R)v, \quad t \in [0, 1],
\]

parameterizes \( I \). Since \( p = \kappa(0) \), we clearly have

\[
\|f \circ \kappa\|_{L^q([0,1])} \geq |f \circ \kappa|(0) = |f(p)| \geq \|f\|_{L^q(K)} / |K|^{1/q}.
\]

Let \( A = \{ t \in [0,1] \colon \kappa(t) \in M \} \) be the set of all parameters for which \( \kappa \) takes values in \( M \), so that

\[
\sup_{M \cap I} |f| \leq \|f \circ \kappa\|_{L^q(A)}.
\]

Since the set \( A \subset [0,1] \) satisfies

\[
|A| = \int_0^1 1_M(\kappa(t)) \, dt = \frac{1}{\sigma(I)} \int_0^1 1_M(\kappa(t))|\kappa'(t)| \, dt = \sigma(I \cap M) / \sigma(I) > 0,
\]

the assumptions of Lemma 2.1 are satisfied for \( r = f \circ \kappa \), if there are finite sequences \((\beta_k)_k\) and \((\lambda_k)_k\) such that for some \( n \in \mathbb{N} \) we have

\[
(f \circ \kappa)(t) = \sum_{k=1}^n \beta_k e^{i\lambda_k t}.
\]

Since \( f \) is the restriction to the sphere of a polynomial of \( P \) of degree at most \( \mathcal{N} \), there are coefficients \((b_\alpha)_{|\alpha| \leq \mathcal{N}}\) such that \( P(x) = \sum_{|\alpha| \leq \mathcal{N}} b_\alpha x^\alpha \) satisfies \( P = f \) on \( S_{d-1}^d \). We set \( v_j = v \cdot e_j \) and \( p_j = p \cdot e_j \) and write

\[
\prod_{j=1}^d (\kappa_j(t))^{\alpha_j} = \prod_{j=1}^d [p_j \cos(\sigma(I)t/R) + v_j \sin(\sigma(I)t/R)]^{\alpha_j}
\]

leading to

\[
(f \circ \kappa)(t) = \sum_{|\alpha| \leq \mathcal{N}} c_\alpha \prod_{j=1}^d [p_j \cos(\sigma(I)t/R) + v_j \sin(\sigma(I)t/R)]^{\alpha_j}.
\]

Writing

\[
p_j \cos(\sigma(I)t/R) + v_j \sin(\sigma(I)t/R) = p_j \text{ Re}(e^{i\sigma(I)t/R}) + v_j \text{ Im}(e^{i\sigma(I)t/R})
\]

and using the binomial theorem there exist finite sequences for which equation (2.6) holds with \( n = 2\mathcal{N} + 1 \). (In fact, all \( \lambda_k \) are integer multiples of \( \sigma(I)t/R \).) Thus, applying Lemma 2.1 with \( A \) as above shows

\[
\|f \circ \kappa\|_{L^q([0,1])} \leq \left( \frac{316}{|A|} \right)^{2\mathcal{N}} \|f \circ \kappa\|_{L^q(A)} = \left( \frac{316 \sigma(I)}{\sigma(I \cap M)} \right)^{2\mathcal{N}} \|f \circ \kappa\|_{L^q(A)}.
\]

Combining this with (2.4) and (2.5) finishes the proof. \( \square \)

The above lemma holds for all possible directions of the spherical line segments \( I \). Using the polar coordinates formula (2.2) we optimize the right hand side of (2.3), i.e., choose the direction of \( I \) such that the quotient \( \sigma(I)/\sigma(I \cap M) \) is small.
Lemma 2.3. Let $K$ be a spherical cap, let $p \in K$, and let $M \subset K$ be a measurable set satisfying $|M| > 0$. Then there is a spherical line segment $I \subset K$ such that
\[ \frac{\sigma(I)}{\sigma(I \cap M)} \leq c_4 \cdot \frac{|K|}{|K \cap M|} \]

Proof. Integrating with respect to the spherical polar coordinates centered at $p$, we have
\[ |M \cap K| = \frac{R}{2} \int_{S^d_{R}} \int_{0}^{2\pi} 1_{M \cap K}(\gamma_v(t))|\sin^{d-2}(t)| \, dt \, d\sigma^{d-2}(v) \]
and there exists $v_0 \in \mathbb{S}^{d-2}_{R}$ such that
\[ |M \cap K| \leq \frac{R}{2} \cdot \sigma^{d-2}(\mathbb{S}^{d-2}_{R}) \int_{0}^{2\pi} 1_{M \cap K}(\gamma_v(t))|\sin^{d-2}(t)| \, dt. \]
Denoting the integral on the right hand side by $J$, it is clear that
\[ J = \int_{0}^{\pi} 1_{M \cap K}(\gamma_{v_0}(t))|\sin^{d-2}(t)| \, dt + \int_{0}^{\pi} 1_{M \cap K}(\gamma_{-v_0}(t))|\sin^{d-2}(t)| \, dt \]
\[ =: J_1 + J_2. \]
This shows that there are two possible spherical line segments starting at $p$ and we choose the one that sees a larger part of the set $M$. More precisely, we either have $J \leq 2J_1$ or $J \leq 2J_2$; without loss of generality we suppose $J \leq 2J_1$. Let
\[ l = \sup\{t \in [0,\pi] : \gamma_{v_0}(t) \in K\} \quad \text{and} \quad I = \text{tr}(\gamma_{v_0}|_{[0,l]}). \]
Note that $Rl = d_R(p, \gamma_{v_0}(l)) \leq \text{diam}(K)$ implies $l \leq \text{diam}(K)/R$. Hence, using $\sin(t) \leq t$ and the upper bound for $l$, we obtain
\[ |M \cap K| \leq R\sigma^{d-2}(\mathbb{S}^{d-2}_{R})J_1 \]
\[ \leq R\sigma^{d-2}(\mathbb{S}^{d-2}_{R}) \int_{0}^{l} 1_{M \cap K}(\gamma_{v_0}(t))t^{d-2} \, dt \]
\[ \leq \sigma^{d-2}(\mathbb{S}^{d-2}_{R})(\text{diam}(K)/R)^{d-2}\sigma(I \cap M). \]
Since $\sigma(I) \leq \text{diam}(K)$ and $\text{diam}(K)^{d-1} = c_4 |K|$, we have thus shown
\[ \frac{\sigma(I \cap M)}{\sigma(I)} \geq \frac{|M \cap K|}{c_4 |K|}. \]
\[ \square \]

2.2. Covering argument and the proof of Theorem 1.3. Let $f$ be a spherical polynomial of degree at most $\mathcal{N}$. Combining Lemma 2.2 and 2.3, we have already shown
\[ \|f\|_{L^q(K)} \leq |K|^{1/q} \left( c \cdot \frac{|K|}{|K \cap M|} \right)^{2\mathcal{N}} \sup_{M \cap K} |f|, \quad c = 316c_4, \]
where $M$ is any measurable set satisfying $|K \cap M| > 0$. Here we choose $M$ as the set of points inside the spherical cap $K$ where $|f|$ is small relative to its own $L^q$-norm on $K$ and the measure of $S$, that is,
\[ M = M_{f,S} = \left\{ x \in K : |f(x)| < |K|^{-1/q} \left( \frac{|K \cap S|}{2c|M|} \right)^{2\mathcal{N}} \|f\|_{L^q(K)} \right\}. \]
In what follows, we assume without loss of generality that $M_{f,S} \neq \emptyset$ and since $M_{f,S}$ is an open set, we then have $|M_{f,S} \cap K| = |M_{f,S}| > 0$. In particular, we may apply inequality (2.7) and the definition of $M$ to obtain
\[ \|f\|_{L^q(K)} \leq |K|^{1/q} \left( c \cdot \frac{|K|}{|K \cap M_{f,S}|} \right)^{2\mathcal{N}} \sup_{M} |f| \leq \left( \frac{|K \cap S|}{2|M_{f,S}|} \right)^{2\mathcal{N}} \|f\|_{L^q(K)} \cdot \]
Since $M_{f,S} \neq \emptyset$, we have $\|f\|_{L^q(K)} > 0$ and therefore $|K \cap S| \geq 2|M_{f,S}|$ and, hence, $|(K \cap S) \setminus M_{f,S}| \geq |K \cap S|/2$ by the last inequality. Since $f$ is small on $M_{f,S}, \ldots$
we estimate \( \|f\|_{L^q(K \cap S)} \geq \|f\|_{L^q((K \cap S) \setminus M_{J,S})} \) and use the lower bound for \(|f|\) on \(K \setminus M_{J,S}\). Thereby we obtain
\[
\|f\|_{L^q(K \cap S)} \geq \left( \frac{|K \cap S|}{2e^{|K|}} \right)^{2N+1/q} \|f\|_{L^q(K)}.
\]

This is a local variant of the Logvinenko-Sereda-Kovrijine type inequality. Indeed, recall that there is a radius \(a > 0\) such that \(|K \cap S| \geq \gamma|K|\) for all spherical caps \(K\) of radius \(a\). Hence, for all such \(K\) the above inequality implies
\[
\|f\|_{L^q(K \cap S)} \geq \left( \frac{\gamma}{2e} \right)^{2N+1/q} \|f\|_{L^q(K)}.
\]

In order to complete the proof of Theorem 1.3, we choose a finite sequence of points \(\{x_j\}_{j=1}^n \subset \mathbb{S}_{R}^{d-1}\) such that the spherical caps \(K_j = K(x_j, a)\) cover \(\mathbb{S}_{R}^{d-1}\). Moreover, we let
\[
\kappa := \max_{y \in \mathbb{S}_{R}^{d-1}} \#\{j \in \{1, \ldots, m\}: y \in K_j\}
\]
be the maximal multiplicity of the covering, so that every point of the sphere is contained in at most \(\kappa\)-many caps \(K_j\)'s. Using the local inequality (2.8) on every cap \(K_j\), we get
\[
\|f\|_{L^q(S)}^q \geq \frac{1}{\kappa} \sum_{j=1}^m \|f\|_{L^q(S \cap K_j)}^q \geq \frac{1}{\kappa} \left( \frac{\gamma}{2e} \right)^{2N+1/q} \|f\|_{L^q(\mathbb{S}_{R}^{d-1})}^q.
\]

We conclude the proof using the following lemma which is a simple consequence of results from [10] and [2, Theorem 1.1].

**Lemma 2.4.** For any \(d \in \mathbb{N} \setminus \{1\}\) and all \(R, a > 0\) there is a finite sequence of points \(\{x_j\}_{j=1}^n \subset \mathbb{S}_{R}^{d-1}\) such that \(\mathbb{S}_{R}^{d-1} \subset \bigcup_j K(x_j, a)\) and the maximal multiplicity satisfies \(\kappa \leq 400d \log d\).

With this lemma, we finally conclude
\[
\|f\|_{L^q(S)} \geq \left( \frac{\gamma}{C_0} \right)^{2N+1/q} \|f\|_{L^q(\mathbb{S}_{R}^{d-1})},
\]
and this finishes the proof of Theorem 1.3.

### 2.3. Proof of the spectral inequality in Corollary 1.5.

Recall that the eigenfunctions of the Laplace-Beltrami operator on the unit sphere are given by the spherical harmonics \(Y_{\ell,k}\), \(\ell \in \mathbb{N}, k = 1, \ldots, n_\ell\), where \(\ell\) denotes the degree and \(n_\ell\) the multiplicity. It is well-known that \((Y_{\ell,k})_{\ell,k}\) forms an orthonormal basis of \(L^2(\mathbb{S}_{R}^{d-1})\) and that \(-\Delta Y_{\ell,k} = (\ell + d - 2)Y_{\ell,k}\) for all \(\ell \in \mathbb{N}\) and \(k = 1, \ldots, n_\ell\). We let \(Y_{\ell,k,R}(x) = Y_{\ell,k}(x/R)\) for \(x \in \mathbb{S}_{R}^{d-1}\). Then a simple scaling argument shows that the functions \((Y_{\ell,k,R})_{\ell,k}\) are the eigenfunctions of the Laplace-Beltrami operator on the sphere \(\mathbb{S}_{R}^{d-1}\). More precisely,
\[
-\Delta Y_{\ell,k,R} = R^{-2}(\ell + d - 2)Y_{\ell,k,R} \quad \text{for all} \quad \ell \in \mathbb{N}, \quad k = 1, \ldots, n_\ell,
\]
and the family \((Y_{\ell,k,R})_{\ell \in \mathbb{N}, k = 1, \ldots, n_\ell}\) forms an orthonormal basis of \(L^2(\mathbb{S}_{R}^{d-1})\).

In order to conclude the proof of Corollary 1.5, note that each \((Y_{\ell,k,R})\) is a spherical polynomial of degree at most \(\ell\), so that (2.10) shows that for all \(Y_{\ell,k,R} \in \text{Ran} P_{-\Delta}((-\infty, E])\) we have \(\ell \leq RE^{1/2}\). Hence, \(f \in \text{Ran} P_{-\Delta}((-\infty, E])\) being a finite linear combination of \(Y_{\ell,k,R}\) with \(\ell \leq RE^{1/2}\) is a spherical polynomial of degree at most \(RE^{1/2}\). Therefore, Theorem 1.3 implies
\[
\|f\|_{L^2(\mathbb{S}_{R}^{d-1})} \leq \left( \frac{C_1}{\gamma} \right)^{RE^{1/2}+1/2} \|f\|_{L^2(S)},
\]
i.e., the claim of Corollary 1.5.
3. Discussion of the covering argument

For every \( \gamma_0 > 0 \) and every \( \gamma_0 \)-thick set \( S \subset S^{d-1}_R \) there exists a scale \( a_0 > 0 \) such that \( |S \cap K| \geq \gamma_0 |K| \) for all spherical caps of radius \( a_0 \). On the other hand, we have \( |S| > 0 \) and hence \( \gamma_1 := |S|/|S^{d-1}_R| > 0 \). This results in two competing estimates in Theorem 1.3 with constants

\[
\left( \frac{c_1}{\gamma_0} \right)^{2N+1/q} \quad \text{and} \quad \left( \frac{c_1}{\gamma_1} \right)^{2N+1/q},
\]

respectively. Furthermore, for a covering \((K_j)_j\) of \( S^{d-1}_R \) with spherical caps of radius \( a_0 \) we have

\[
|S^{d-1}_R| \leq \sum_j |K_j| \leq \sum_j \frac{1}{\gamma_0} |S \cap K_j| \leq \frac{\kappa}{\gamma_0} |S \cap \bigcup K_j| = \frac{\kappa}{\gamma_0} |S|,
\]

where \( \kappa \) is the multiplicity from (2.9) (depending only on the dimension). Hence, \( \gamma_1 \leq \kappa/\gamma_0 \) and the second constant in (3.1) can be brought into the form of the first one by increasing \( c_1 \), that is

\[
\left( \frac{c_1}{\gamma_1} \right)^{2N+1/q} \leq \left( \frac{\kappa c_1}{\gamma_0} \right)^{2N+1/q}
\]

and \( c_1 \) is indeed allowed to depend on the dimension (only). This illustrates that our covering argument only improves the constant \( c_1 \) when compared to the situation if we consider exclusively \( K = S^{d-1}_R \).

This stands in contrast to the Euclidean estimate (cf. Theorem 1.1), where thick sets with small scales \( a \) yield a substantially better bound since \( a \) also appears as a (small) factor in the exponent. We are not (yet) able to reproduce this exponential scaling with respect to \( a \) in the spherical setting (cf. Theorem 1.3). For this purpose it would be necessary to replace inequality (2.3) by a bound where the exponent exhibits a factor \( a \).

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