THE ENCLOSURE METHOD FOR INVERSE OBSTACLE SCATTERING USING A SINGLE ELECTROMAGNETIC WAVE IN TIME DOMAIN

Masaaru Ikehata
Laboratory of Mathematics, Institute of Engineering
Hiroshima University
Higashi-Hiroshima 739-8527, Japan

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Abstract. In this paper, a time domain enclosure method for an inverse obstacle scattering problem of electromagnetic wave is introduced. The wave as a solution of Maxwell’s equations is generated by an applied volumetric current having an orientation and supported outside an unknown obstacle and observed on the same support over a finite time interval. It is assumed that the obstacle is a perfect conductor. Two types of analytical formulae which employ a single observed wave and explicitly contain information about the geometry of the obstacle are given. In particular, an effect of the orientation of the current is caught in one of two formulae. Two corollaries concerning with the detection of the points on the surface of the obstacle nearest to the centre of the current support and curvatures at the points are also given.

1. Introduction. In this paper, we consider an inverse obstacle scattering problem of a wave whose governing equation is given by Maxwell’s equations. The wave is generated by a source at \( t = 0 \) which is not far away from an unknown obstacle, and we observe a single reflected wave from the obstacle over a finite time interval at the same place as the source. The inverse obstacle scattering problem is to: extract information about the geometry of the obstacle from the observed wave. This is a proto-type of so-called inverse obstacle problem [21] and the solution may have possible applications to radar imaging. Since we consider the data over a finite time interval and thus, this is a time domain inverse problem. Our main interest is to find an analytical method or formula that extracts the geometry of the obstacle from the data by using the governing equation of the wave.

Let us describe the mathematical formulation of the problem. Let \( D \) be a nonempty bounded open subset of \( \mathbb{R}^3 \) with \( C^2 \)-boundary such that \( \mathbb{R}^3 \setminus \overline{D} \) is connected. \( \nu \) denotes the unit normal to \( \partial D \), oriented towards the exterior of \( D \).

Let \( 0 < T < \infty \). We denote by \( E \) and \( H \) the electric field and the magnetic field, respectively. \( \epsilon \) denotes the electric permittivity and \( \mu \) the magnetic permeability assumed to be positive constants.

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We assume that $E$ and $H$ are induced only by the current density $J$ at $t = 0$ and that the obstacle is a perfect conductor. It is well known that the governing equations of $E$ and $H$ take the form

\[
\begin{align*}
\epsilon \frac{\partial E}{\partial t} - \nabla \times H &= J \quad \text{in } (\mathbb{R}^3 \setminus \overline{D}) \times [0, T], \\
\mu \frac{\partial H}{\partial t} + \nabla \times E &= 0 \quad \text{in } (\mathbb{R}^3 \setminus \overline{D}) \times [0, T], \\
\nu \times E &= 0 \quad \text{on } \partial D \times [0, T], \\
E|_{t=0} &= 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \\
H|_{t=0} &= 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}.
\end{align*}
\]

(1)

Now let us describe our problem. Fix a large (to be determined later) $T < \infty$. Let $B$ be the open ball centred at a point $p$ with very small radius $\eta$ and satisfy $B \cap \overline{D} = \emptyset$. There are several choices of the current density $J$ as a model of the antenna ([4, 7]). In this paper, we assume that $J$ takes the form

\[
J(x, t) = f(t) \chi_B(x) a,
\]

where $a \neq 0$ is a constant unit vector, $\chi_B$ denotes the characteristic function of $B$ and $f \in H^1(0, T)$ with $f(0) = 0$. Note that $\chi_B(x)$ has discontinuity across the sphere $\partial B$.

**Problem.** Generate $E$ and $H$ by $J$ and observe $E$ on $B$ over time interval $[0, T]$. Extract information about the geometry of $D$ from the observed data.

This may be the simplified model of the case when the reflected wave is observed at the same place where the source is located. Note that we consider the pair $(E, H)$ the solution of (1) in the sense as described on pages 433-435 in [11] which is based on Stone’s theorem. In particular, we make use of the fact that $(E, H)$ belongs to $C^1([0, T], L^2(\mathbb{R}^3 \setminus \overline{D})^3 \times L^2(\mathbb{R}^3 \setminus \overline{D})^3)$ with $(\nabla \times E(t), \nabla \times H(t)) \in L^2(\mathbb{R}^3 \setminus \overline{D})^3 \times L^2(\mathbb{R}^3 \setminus \overline{D})^3$ and $E(t) \times \nu|_{\partial D} = 0$ for all $t \in [0, T]$.

As far as the author knows there is no result for the problem mentioned above. The point is: the data is taken over a finite time interval and only a single (reflected) wave is employed.

In this paper, we employ the enclosure method for this problem. The origin goes back to a method developed for an inverse boundary value problem in two dimensions for the Laplace equation [13].

The method consists of two tools:

- a special solution $v$ of an elliptic partial differential equation which depends on a large parameter $\tau > 0$ and is independent of unknown obstacles.
- a so-called indicator function of independent variable $\tau$ constructed by using observation data and $v$ above.

Studying the asymptotic behaviour of the indicator function as $\tau \to \infty$ yields some information about the location and shape of unknown discontinuity.

We have already some applications to inverse obstacle scattering problems whose governing equation is given by the classical wave equation in three-space dimensions [14, 15, 16, 17, 18]. The method enables us to extract information about the geometry of unknown obstacle from a single reflected wave over a finite time interval.
However, the governing equation therein is a single partial differential equation and it is not clear that the method can cover also the very important case when the governing equation consists of a system of partial differential equations.

In the following subsection we describe our solution to Problem.

1.1. Statement of the results. We denote by \( H(\text{curl}, \mathbb{R}^3) \) the set of all vector valued-functions \( U \in L^2(\mathbb{R}^3)^3 \) such that \( \nabla \times U \in L^2(\mathbb{R}^3)^3 \). It is a Hilbert space with norm

\[
\|U\|_{H(\text{curl}, \mathbb{R}^3)} = \sqrt{\|U\|_{L^2(\mathbb{R}^3)^3}^2 + \|\nabla \times U\|_{L^2(\mathbb{R}^3)^3}^2}
\]

and \( C_0^\infty(\mathbb{R}^3)^3 \) is dense in \( H(\text{curl}, \mathbb{R}^3) \).

By the Lax-Milgram theorem, we know that given \( f(\cdot, \tau) \in L^2(\mathbb{R}^3)^3 \) there exists a unique \( V \in H(\text{curl}, \mathbb{R}^3) \) such that, for all \( \Psi \in H(\text{curl}, \mathbb{R}^3) \)

\[
\int_{\mathbb{R}^3} \left( \frac{1}{\mu \epsilon} \nabla \times V \cdot \nabla \times \Psi + \tau^2 V \cdot \Psi \right) dx + \int_{\mathbb{R}^3} f(x, \tau) \cdot \Psi dx = 0.
\]

We call this \( V \) the weak solution of

\[
\left(1\right) \frac{1}{\mu \epsilon} \nabla \times \nabla \times V + \tau^2 V + f(x, \tau) = 0 \quad \text{in} \ \mathbb{R}^3.
\]

In this paper, unless otherwise stated, \( f(\cdot, \tau) \) has the form

\[
f(x, \tau) = -\frac{T}{\epsilon} \tilde{f}(\tau) \chi_B(x) a,
\]

where

\[
\tilde{f}(\tau) = \int_0^T e^{-\tau t} f(t) dt.
\]

Note that

\[
\int_0^T e^{-\tau t} J(x, t) dt = -\frac{\epsilon}{\tau} f(x, \tau).
\]

Define

\[
W_e(x, \tau) = \int_0^T e^{-\tau t} E(x, t) dt, \ x \in \mathbb{R}^3 \setminus D.
\]

We call the map defined by

\[
\tau \mapsto \int_B f \cdot (W_e - V) dx
\]

the indicator function. The indicator function can be computed from our observation data \( E \) on \( B \) over time interval \( [0, T] \) since we have (5).

The following results give us some solutions to the problem raised above.

**Theorem 1.1.** Assume that \( \partial D \) is \( C^2 \). Let \( f \) satisfy that there exists \( \gamma \in \mathbb{R} \) such that

\[
\liminf_{\tau \to \infty} \tau^\gamma |\tilde{f}(\tau)| > 0.
\]

If \( T > 2\sqrt{\mu \epsilon \text{dist}(D, B)} \), then, there exists \( \tau_0 > 0 \) such that, for all \( \tau \geq \tau_0 \)

\[
\int_B f \cdot (W_e - V) dx > 0.
\]

Moreover, we have the following formula:

\[
\lim_{\tau \to \infty} \frac{1}{\tau} \log \left| \int_B f \cdot (W_e - V) dx \right| = -2\sqrt{\mu \epsilon \text{dist}(D, B)}.
\]
A remarkable point in this theorem is: there is no restriction on direction \( \mathbf{a} \) in (2). Define \( d_{BD}(p) = \inf_{x \in \partial D} |x - p| \) and \( B_{d_{BD}(p)}(p) = \{x \in \mathbb{R}^3 | |x - p| < d_{BD}(p)\} \). Since we have \( \text{dist} (D, B) = d_{BD}(p) - \eta \), we can find the sphere \( \partial B_{d_{BD}(p)}(p) \) via (7) regardless of the direction of \( \mathbf{a} \) at any time. This sphere is the maximum one whose exterior encloses the unknown obstacle.

As is introduced in the author’s previous papers \([16, 17, 18]\) we denote by \( \Lambda_{BD}(p) \) the set \( \partial D \cap \partial B_{d_{BD}(p)}(p) \). We call this set the first reflector from \( p \) to \( \partial D \) and the points in the first reflector are called the first reflection points, going from \( p \) to \( \partial D \). Using Theorem 1.1, one can also give a criterion for a given direction \( \omega \) in \( S^2 \) whether the point \( p + d_{BD}(p) \omega \) belongs to \( \partial D \) since as pointed out in \([16, 17, 18, 19]\) we have: if \( p + d_{BD}(p) \omega \) belongs to \( \partial D \), then \( d_{BD}(p + s d_{BD}(p) \omega) > d_{BD}(p) - s \); if \( p + d_{BD}(p) \omega \) does not belong to \( \partial D \), then \( d_{BD}(p + s d_{BD}(p) \omega) < d_{BD}(p) - s \). Here \( s \in [0, 1] \) and is fixed. Note that one can always compute \( d_{BD}(p + s d_{BD}(p) \omega) \) via (7) using a suitable input current supported around \( p + s d_{BD}(p) \omega \) and the electric field observed at the same place as the support of the current.

Thus, we obtain the following result which makes use of infinitely many electromagnetic waves corresponding to infinitely many input sources.

**Corollary 1.** Let \( p \in \mathbb{R}^3 \setminus \overline{D} \). Assume that \( d_{BD}(p) \) is known. Fix \( \mathbf{a}, \delta \in [0, d_{BD}(p)] \), \( \eta' \in [0, d_{BD}(p) - \delta] \) and \( f \) satisfying (6) for a \( \gamma \in \mathbb{R} \). Let \( T \) satisfy

\[
T > 2\sqrt{\mu \epsilon} \sup_{p' \in \partial B_{\delta}(p)} \text{dist} (D, B_{\eta'}(p')).
\]

Then, one can extract \( \Lambda_{BD}(p) \) itself from \( E(x, t) \) given at all \( x \in B_{\eta'}(p'), t \in [0, T] \) and \( p' \in \partial B_{\delta}(p) \) for \( J \) given by (2) where \( f \) is as above and \( B \) replaced with \( B_{\eta'}(p') \).

It would be interesting to find a constructive and exact method for extracting \( \Lambda_{BD}(p) \) itself from a single electromagnetic wave, however, at the present time, we have only a positive result for a scalar wave equation with Dirichlet boundary condition on the boundary of the obstacle \([17]\). The point is to make use of the observed data restricted to infinitely many closed balls contained in \( B \) for a fixed initial data supported on \( \overline{B} \), that is the so-called bistatic data.

The condition (6) is a restriction on the strength of the source at \( t = 0 \). Note that we have \( f(t) = O(\tau^{-3/2}) \) as \( \tau \to \infty \) since \( f \in H^1(0, T) \) and \( f(0) = 0 \). Thus, \( \gamma \) in (6) has to satisfy \( \gamma \geq 3/2 \). For example, any \( f \in H^1(0, T) \) such that \( f(t) = t \sin \omega t \) for all \( t \in [0, \epsilon] \) with \( 0 < \epsilon \leq T \) and \( \omega > 0 \), satisfies (6) for \( \gamma = 3 \) since, as \( \tau \to \infty \)

\[
\int_0^\tau e^{-\tau t} \sin \omega t dt = \frac{2\tau \omega}{(\tau^2 + \omega^2)^2} + O(\tau^{-1} e^{-\tau t}).
\]

Let \( q \in \Lambda_{BD}(p) \). Let \( S_q(\partial D) \) and \( S_q(\partial B_{d_{BD}(p)}(p)) \) denote the shape operators (or Weingarten maps) at \( q \) of \( \partial D \) and \( \partial B_{d_{BD}(p)}(p) \) with respect to \( \mathbf{v}_q \) and \( -\mathbf{v}_q \), respectively (see \([26]\) for the notion of the shape operator). These are symmetric linear operators on the common tangent space \( T_q \partial D = T_q \partial B_{d_{BD}(p)}(p) \). We have always \( S_q(\partial B_{d_{BD}(p)}(p)) - S_q(\partial D) \geq 0 \) since \( q \) attains the minimum value of the function \( \partial D \ni y \mapsto |y - p| \). In general, given \( p \) the first reflector from \( p \) to \( \partial D \) can be an infinite set, even more, a continuum. For example, imagine the case when a part of \( \partial D \) coincides with that of \( \partial B_{d_{BD}(p)}(p) \). Note also that, in that case, we have \( S_q(\partial B_{d_{BD}(p)}(p)) = S_q(\partial D) \) the points \( q \) in that part.
Theorem 1.2. Assume that $\partial D$ is $C^4$; $\Lambda_{\partial D}(p)$ is finite and satisfies
\begin{equation}
\det(S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D)) > 0, \forall q \in \Lambda_{\partial D}(p).
\end{equation}
Moreover, assume that
\begin{equation}
\exists q \in \Lambda_{\partial D}(p) \mid a \cdot \nu_q \neq 1.
\end{equation}
Let $f$ satisfy (6) for $a \in \mathbb{R}$. If $T > 2\sqrt{\mu\epsilon}\text{dist}(D, B)$, then we have
\begin{equation}
limit_{\tau \to \infty} \frac{2\tau^2 e^{2\tau\text{dist}(D, B)}}{f(\tau)^2} \int_B f \cdot (W_e - V) dx
\end{equation}
\begin{equation}
= \frac{\pi}{2\epsilon^2} \left( \frac{\eta}{d_{\partial D}(p)} \right)^2 \sum_{q \in \Lambda_{\partial D}(p)} \frac{1 - (a \cdot \nu_q)^2}{\sqrt{\det(S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D))}}.
\end{equation}

(9) is a restriction on the direction of $a$ in (2). Since $|a \cdot \nu_q| = 1$ if and only if $a = \pm \nu_q$, (9) means that there is no first reflection point from $p$ on the straight line passing through $p$ and parallel to $a$. It is clear that if $\Lambda_{\partial D}(p)$ consists of at least three points, then (9) is satisfied.

The denominator $\sqrt{\det(S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D))}$ in the right-hand side on (10) is independent of $a$ and numerator $1 - (a \cdot \nu_q)^2$ becomes maximum when $a$ is perpendicular to $\nu_q$; small when $a \times \nu_q \approx 0$. Thus, formula (10) shows us an effect of the directivity of the source term on extracting information about the geometry of the unknown obstacle from the observation data. Someone may think that this fact has similarity to a well known fact in the dipole antenna theory (e.g., [4]), that is, the maximum radiation from the antenna is directed along right angles to the dipole.

Note also that since we have
\begin{equation}
\int_B f \cdot W_e dx = -\frac{\tau}{c} \int_B a \cdot W_e dx,
\end{equation}
from (5) we know that in (7) and (10) instead of all the components of $E$ we need only $a \cdot E$.

It is a due course to deduce the following corollary from Theorem 1.2 (see [16, 18]).

Corollary 2. Assume that $\partial D$ is $C^4$. Let $p \in \mathbb{R}^3 \setminus \overline{D}$ and assume that $q \in \Lambda_{\partial D}(p)$ is known. Let $B_1$ and $B_2$ denote two open balls centred at $p - s_j(p - q)/|p - q|$, $j = 1, 2$, respectively with $0 < s_1 < s_2 < |p - q|$ and satisfy $\overline{B_1} \cup \overline{B_2} \subset \mathbb{R}^3 \setminus \overline{D}$. Let $J_j$ be the $J$ given by (2) in which $B = B_j$ and $f(t) = f_j(t)$ satisfying (6) for $a \gamma = \gamma_j \in \mathbb{R}$; $E_j$ with $j = 1, 2$ denote the corresponding electric fields governed by (1).

If $T > 2\sqrt{\mu\epsilon} \max_{j=1, 2} \text{dist}(D, B_j)$ and $a \times (p - q) \neq 0$, then one can extract the Gauss curvature $K_{\partial D}(q)$ of $\partial D$ at $q$ and mean curvature $H_{\partial D}(q)$ with respect to $\nu_q$ from $a \cdot E_j$ on $B_j$ with $j = 1, 2$ over time interval $[0, T]$.

Note that $\nu_q = (p - q)/|p - q|$ for $q \in \Lambda_{\partial D}(p)$. Thus $\nu_q \neq 0$ and only if $|a \cdot \nu_q| \neq 1$.

Briefly speaking, Corollary 2 says that: one can completely know the Gauss and mean curvatures of the boundary of the obstacle at a known first reflection point $q$, going from a given point $p$ outside the obstacle by observing two reflected electric fields generated by two sources whose centres are placed on the segment connecting $p$ and $q$. Thus, one can know an approximate shape of the boundary of unknown obstacle at a known first reflection point by using two electromagnetic waves.
The concrete procedure in Corollary 2 for extracting both the Gauss and mean curvatures at a known first reflection point consists of the following steps.

(i) Compute \( R_j \) with \( j = 1, 2 \) given by

\[
R_j = \lim_{\tau \to \infty} \frac{\tau^2 e^{2\tau \sqrt{\text{det}(d_{BD}(p) - 2s_j)}}}{f_j(\tau)^2} \int_{B_j} f_j \cdot (W_e^j - V_j)dx,
\]

where \( j \in f, W_e^j \) and \( V_j \) indicates that they are the \( f, W_e \) and \( V \) corresponding to \( f_j, E_j \) and \( J_j \) in a trivial manner.

(ii) Compute \( X_j \) with \( j = 1, 2 \) given by

\[
X_j = \left( \begin{array}{c}
-2\lambda_1 & 1 \\
-2\lambda_2 & 1 
\end{array} \right) \left( \begin{array}{c}
Y_1 \\
Y_2 
\end{array} \right) = \left( \begin{array}{c}
X_1 \\
X_2 
\end{array} \right) - \left( \begin{array}{c}
\lambda_1^2 \\
\lambda_2^2 
\end{array} \right),
\]

where \( \lambda_j = (d_{BD}(p) - s_j)^{-1} \) with \( j = 1, 2 \).

Then, we obtain the Gauss and mean curvatures at \( q \) by the formulae: \( K_{BD}(p) = Y_2 \) and \( H_{BD}(q) = Y_1 \). Note that we have made use of the following trivial facts and formulae as pointed out in [16, 18]:

- \( \Lambda_{BD}(p - s_j \nu_q) = \{q\} \) for \( q \in \Lambda_{BD}(p) \) and \( S_q(\partial B_{d_{BD}(p - s_j \nu_q)}(p - s_j \nu_q)) - S_q(\partial B_{d_{BD}(p)}(p)) \) is positive definite on the common tangent space at \( q \);
- \( \text{dist}(D, B_j) = d_{BD}(p) - 2s_j \) and \( d_{BD}(p - s_j \nu_q) = d_{BD}(p) - s_j \) for \( q \in \Lambda_{BD}(p) \);
- \( \det(\lambda I - S_q(\partial D)) = \lambda^2 - 2\lambda H_{BD}(q) + K_{BD}(q) \);
- \( S_q(\partial B_{d_{BD}(p - s_j \nu_q)}(p - s_j \nu_q)) = \lambda_j I \).

We think that Corollary 2 shows us an advantage of the near field measurement. For this, note that in the third step above \( \lambda_j \to 0 \) as \( d_{BD}(p) \to \infty \) and thus one cannot find \( Y_1 \). Compare also the results with those of [24] where the information about the mean curvature never appear explicitly in the scattering kernel which is the observation data in the context of the Lax-Phillips scattering theory [22].

Note that our result can be applied to a cavity inside a large obstacle which is connected with its exterior by a borehole. This is the case when \( D \) encloses almost \( B \). In this case it is not suitable to use an infinitely extended plane wave as an approximation of the incident wave unlike [24]. See also [3] for some comments on the comparison between incident plane and spherical waves in the frequency domain.

The outline of this paper is as follows. To study the asymptotic behaviour of the indicator function as \( \tau \to \infty \) we need some preliminary facts about \( V \). In Section 2, using the mean value theorem for the modified Helmholtz equation, we give an explicit computation formula for \( V \) outside \( B \). This formula is found in Subsection 2.1 and enables us to study the asymptotic behaviour of the energy integral \( J(\tau) \) of \( V \) over \( D \) as \( \tau \to \infty \) in Subsection 2.2, where

\[
J(\tau) = \frac{1}{\epsilon \mu} \int_D |\nabla \times V|^2 dx + \tau^2 \int_D |V|^2 dx.
\]

However, unlike the previous applications of the enclosure method to scalar wave equations (see e.g., [18]), we need an upper bound of \( L^2 \)-norm of the Jacobian matrix \( V' \) over \( D \) in terms of \( J(\tau) \). This is not trivial and described in Subsection 2.2.
Theorem 1.1 is proved in Section 3. The proof is based on a brief asymptotic formula of the indicator function and the resulted upper and lower bound in terms of $J(\tau)$.

Theorem 1.2 is proved in Section 4. The proof is based on the precise asymptotic formula of the indicator function stated in Theorem 4.1 and the leading profile of $J(\tau)$ obtained in Section 2 via the Laplace method. The precise asymptotic formula is derived from a combination of the brief asymptotic formula of the indicator function and the asymptotic coincidence of $J(\tau)$ with $E(\tau)$ defined by

$$E(\tau) = \frac{1}{\epsilon \mu} \int_{\mathbb{R}^3 \setminus D} |\nabla \times (W_e - V)|^2 dx + \tau^2 \int_{\mathbb{R}^3 \setminus D} |W_e - V|^2 dx.$$  

The proof of the asymptotic coincidence of $E(\tau)$ and $J(\tau)$ is based on the reflection principle across curved surface $\partial D$ for the Maxwell system as described in Propositions 2 and 3 and a representation of $E(\tau) - J(\tau)$ in terms of the reflection which are trivial for a scalar wave equation case. Then, we apply the Lax-Phillips reflection argument [22] to the difference. This story is parallel to the previous scalar wave equation cases [16, 17, 18], however, a proper problem for system of partial differential equations occurs in proving Theorem 4.1. In order to apply their argument, we need an upper bound of the $L^2$-norm of the Jacobian matrix of the so-called reflected solution $W_e - V$ in terms of $E(\tau)$. However, it seems difficult to obtain such an estimate directly and instead, we give the upper bound in terms of $J(\tau)$ directly. This way is different from the original Lax-Phillips reflection argument and makes the argument for the proof of the asymptotic coincidence of $E(\tau)$ with $J(\tau)$ straightforward compared with the scalar wave equation case.

In Appendix we describe some differential identities for the vector fields obtained by the reflection across $\partial D$ and the resulted reflection formula described in Proposition 3 is proved. Note that the regularity assumption that $\partial D$ is $C^4$ in Theorem 1.2 is more restrictive compared with the scalar wave equation case [16, 17, 18] in which the corresponding theorems are valid for $C^3$-smooth boundary. This is coming from the difference of the reflection principle used. Therein only a change of independent variables is used, however, for Maxwell’s equations, the reflection principle involves also a change of dependent variables and this requires a higher regularity.

2. Preliminary facts about $V$. In this section first we give a detailed expression of $V$. Then using the expression we give an asymptotic behaviour of some integrals involving $V$.

2.1. An explicit form of $V$ outside of $B$. Here we give an explicit computation formula of the weak solution of (3) in $\mathbb{R}^3 \setminus B$. First, assume that $V$ has the form

$$V = V_0 + V_1,$$

where $V_0$ and $V_1$ are two vector-valued functions on the whole space.

Write

$$\frac{1}{\mu \epsilon} \nabla \times \nabla \times V + \tau^2 V + f(x, \tau)$$

$$= \left\{-\frac{1}{\mu \epsilon} (\Delta - \mu \epsilon \tau^2) V_0 + f(x, \tau)\right\}$$

$$+ \left(\tau^2 V + \frac{1}{\mu \epsilon} \nabla (\nabla \cdot V_0)\right) + \frac{1}{\mu \epsilon} \nabla \times \nabla \times V_1.$$
From this we see that if
\begin{equation}
- \frac{1}{\mu \epsilon} (\triangle - \mu \epsilon \tau^2) V_0 + f(x, \tau) = 0
\end{equation}
and
\[ \tau^2 V_1 + \frac{1}{\mu \epsilon} \nabla (\nabla \cdot V_0) = 0, \]
then \( \nabla \times V_1 = 0 \) and thus \( V = V_0 + V_1 \) satisfies (3) formally.

From this formal argument we have the following construction of the weak solution of (3) for general \( f(\cdot, \tau) \in L^2(\mathbb{R}^3)^3 \) such that \( \text{supp} f(\cdot, \tau) \subset \overline{B} \).

Let \( V_0 = V_0(\cdot, \tau) \in H^1(\mathbb{R}^3)^3 \) be the unique weak solution of (13). It is well known that \( V_0 \) has the form
\begin{equation}
V_0 = V_0(x, \tau) = - \frac{\mu \epsilon}{4\pi} \int_B e^{-\sqrt{\mu \epsilon} |x-y|} f(y, \tau) dy.
\end{equation}
Then, for each fixed \( \tau \) by the interior regularity or from the expression we see that \( V_0 \in H^2_{\text{loc}}(\mathbb{R}^3)^3 \); \( V_0 \) is smooth outside \( B \); \( V_0 \) together with its all derivatives are exponentially decaying as \( |x| \to \infty \). Thus we have \( V_0 \in H^2(\mathbb{R}^3)^3 \).

Define \( V_1 = V_1(\cdot, \tau) \in L^2(\mathbb{R}^3)^3 \) by the formula
\begin{equation}
V_1 = - \frac{1}{\tau^2 \mu \epsilon} \nabla (\nabla \cdot V_0).
\end{equation}
\( V_1 \) is also smooth outside \( B \) and, for each fixed \( \tau \), \( V_1 \) together with all the derivatives are exponentially decaying as \( |x| \to \infty \); \( \nabla \times V_1 = 0 \) in \( \mathbb{R}^3 \).

Then, \( V = V_0 + V_1 \in L^2(\mathbb{R}^3)^3 \) satisfies \( \nabla \times V = \nabla \times V_0 \in L^2(\mathbb{R}^3)^3 \). Thus we have \( V \in H(\text{curl}, \mathbb{R}^3) \). It is easy to see that this \( V \) satisfies (3) in the weak sense. Thus, by the uniqueness of the weak solution of (3) we conclude that the weak solution of (3) has the expression
\begin{equation}
V = V_0 + V_1,
\end{equation}
where \( V_0 \) and \( V_1 \) are given by (14) and (15), respectively. Note that this argument for the construction of \( V \) is based on the form of the fundamental solution for the operator \((1/\mu \epsilon) \nabla \times \nabla \times \cdot - k^2 \cdot \) with \( k > 0 \) (e.g., see \[1\]).

Let \( x \in \mathbb{R}^3 \setminus \overline{B} \). In what follows, we omit to indicate the dependence of several functions of \( x \) on the parameter \( \tau \). By the mean value theorem for the modified Helmholtz equation \[9\], we know that
\[ \frac{1}{4\pi} \int_B e^{-\tau \sqrt{\mu \epsilon} |x-y|} dy = \frac{\varphi(\tau \sqrt{\mu \epsilon} \eta)}{(\tau \sqrt{\mu \epsilon})^3} \frac{e^{-\tau \sqrt{\mu \epsilon} |x-p|}}{|x-p|}, \]
where \( \varphi(\xi) = \xi \cosh \xi - \sinh \xi \). Thus \( V_0 \) given by (14) takes the form
\begin{equation}
V_0(x) = K(\tau) \bar{f}(\tau) v(x) a,
\end{equation}
where \( \bar{f}(\tau) \) is given by (4),
\[ v(x) = \frac{e^{-\tau \sqrt{\mu \epsilon} |x-p|}}{|x-p|} \]
and
\[ K(\tau) = \frac{\mu \epsilon \varphi(\tau \sqrt{\mu \epsilon} \eta)}{(\tau \sqrt{\mu \epsilon})^3}. \]

A straightforward computation gives
\[ \nabla v(x) = - \left( \frac{\hat{r}}{|x-p|} + \frac{1}{|x-p|^2} \right) (x-p) v(x) \]
Moreover, since $\nabla \times (x)$ we have $K$ and its derivatives over lemmas are concerned with the asymptotic behaviour of some integrals involving $V$. Substituting this and (17) into (16), we obtain the following explicit formula of $(i)$ We have Lemma 2.1.

$$V_1(x) = - \frac{K(\tau)\hat{f}(\tau)}{\hat{x}^2} v(x) \left\{ \left( \frac{\hat{x}^2}{|x-p|^2} + \frac{3\hat{x}}{|x-p|^3} + \frac{3}{|x-p|^4} \right) (x-p) \otimes (x-p) - \left( \frac{\hat{x}}{|x-p|} + \frac{1}{|x-p|^2} \right) I_3 \right\} a,$$

where $\hat{x} = \tau \sqrt{|x|}$. Since $\nabla(\nabla \cdot V) = K(\tau)\hat{f}(\tau)(\nabla v)\hat{a}$, it follows from (15) that

$$V_1(x) = - \frac{K(\tau)\hat{f}(\tau)}{\hat{x}^2} v(x) \left\{ \left( \frac{\hat{x}^2}{|x-p|^2} + \frac{3\hat{x}}{|x-p|^3} + \frac{3}{|x-p|^4} \right) (x-p) \otimes (x-p) - \left( \frac{\hat{x}}{|x-p|} + \frac{1}{|x-p|^2} \right) I_3 \right\} a.$$

Substituting this and (17) into (16), we obtain the following explicit formula of $V$ outside $B$:

$$V(x) = K(\tau)\hat{f}(\tau)v(x)M(x;p)a,$$

where

$$M(x;p) = A I_3 - B \frac{x-p}{|x-p|} \otimes \frac{x-p}{|x-p|},$$

$$A = A(x,\tau) = 1 + \frac{1}{\tau \sqrt{|x|}} \left( \frac{1}{|x-p|} + \frac{1}{\tau \sqrt{|x-p|^2}} \right)$$

and

$$B = B(x,\tau) = 1 + \frac{3}{\tau \sqrt{|x|}} \left( \frac{1}{|x-p|} + \frac{1}{\tau \sqrt{|x-p|^2}} \right).$$

2.2. Two basic lemmas about $J(\tau)$ and $V'$. Let $\overline{B} \subset \mathbb{R}^3 \setminus \overline{D}$. The following two lemmas are concerned with the asymptotic behaviour of some integrals involving $V$ and its derivatives over $D$ which is one of the key points in this paper.

Lemma 2.1. (i) We have

$$\limsup_{\tau \to \infty} \tau^3 e^{2\tau \sqrt{|x|}} \text{dist}(D, B) J(\tau) < \infty.$$  

(ii) Assume that $\partial D$ is Lipschitz. Let $f$ satisfy (6) for a $\gamma \in \mathbb{R}$. We have

$$\liminf_{\tau \to \infty} \tau^{5+2\gamma} e^{2\tau \sqrt{|x|}} \text{dist}(D, B) J(\tau) > 0.$$  

Proof. For convenience we introduce $\hat{x} = \tau \sqrt{|x|}$. Since $|\hat{f}(\tau)| = O(\tau^{-3/2})$ and

$$\varphi(\hat{x}) = \frac{\hat{x} \eta e^{\hat{x} \eta}}{2} (1 + O(\tau^{-1})), $$

we have $K(\tau)\hat{f}(\tau) = O(\tau^{-5/2} e^{\hat{x} \eta})$. Thus, it follows from (18) and (19) that, for all $x \in D$

$$|V(x)| \leq C \tau^{-5/2} e^{\hat{x} \eta} v(x). $$

Moreover, since $\nabla \times V = 0$, from (17) we have

$$\nabla \times V = -\hat{x} K(\tau)\hat{f}(\tau)v(x) \left( 1 + \frac{1}{\hat{x} |x-p|} \right) \frac{x-p}{|x-p|} \times a$$
and thus, for all $x \in D$

$$|\nabla \times V(x)| \leq C\tau^{-3/2}e^{\tilde{\tau}\eta}v(x).$$

Therefore one gets

$$\tau^3 J(\tau) \leq Ce^{2\tilde{\tau}\eta}\int_D |v(x)|^2dx.$$ 

Now (20) is clear since we have

$$\int_D |v(x)|^2dx \leq \frac{1}{\delta_{\partial D}(p)^2} \int_D e^{-2\tilde{\tau}|x-p|}dx = O(e^{-2\tilde{\tau}d_{\partial D}(p)}).$$

and $d_{\partial D}(p) - \eta = \text{dist}(D, B)$.

Next we give a proof of (21). From (19) we have

$$|M(x; p)a|^2 = M(x; p)^2 a \cdot a$$

$$= A^2|a|^2 + (B^2 - 2AB) \left|a \cdot \frac{x-p}{|x-p|}\right|^2$$

$$= \left(1 - \left|a \cdot \frac{x-p}{|x-p|}\right|^2\right) A^2 + (A^2 + B^2 - 2AB) \left|a \cdot \frac{x-p}{|x-p|}\right|^2$$

$$= A^2 \left|a \times \frac{x-p}{|x-p|}\right|^2 + (A - B)^2 \left|a \cdot \frac{x-p}{|x-p|}\right|^2$$

$$\geq \frac{1}{\tau^2 \mu |x-p|^2} \left|a \times \frac{x-p}{|x-p|}\right|^2 + \frac{4}{\tau^2 \mu |x-p|^2} \left|a \cdot \frac{x-p}{|x-p|}\right|^2$$

$$\geq \frac{1}{\tau^2 \mu |x-p|^2}.$$

Thus (18) gives

$$|V(x)|^2 \geq \tau^{-2}K(\tau)^2 \tilde{f}(\tau)^2 e^{2\tilde{\tau}\eta}v(x)^2.$$ 

By (22) we have $K(\tau) \sim \tau^{-1}e^{\tilde{\tau}\eta}/(2\epsilon)$ as $\tau \to \infty$. Then, it follows from (24) that there exist positive constants $C''$ and $\tau_0$ such that, for all $x \in D$ and $\tau \geq \tau_0$

$$|V(x)|^2 \geq C''\tau^{-4} \tilde{f}(\tau)^2 e^{2\tilde{\tau}\eta}v(x)^2$$

and thus

$$J(\tau) \geq C''\tau^{-2} \tilde{f}(\tau)^2 e^{2\tilde{\tau}\eta} \int_D |v(x)|^2dx.$$ 

A standard technique [19] yields

$$\liminf_{\tau \to \infty} \tau^3 e^{2\tilde{\tau}d_{\partial D}(p)} \int_D |v(x)|^2dx > 0.$$ 

Thus rewriting (25) as

$$e^{2\tilde{\tau}\text{dist}(D, B)}\tau^{3+2+2\gamma} J(\tau) \geq C\tau^{2\gamma} \tilde{f}(\tau)^2 \times \tau^3 e^{2\tilde{\tau}d_{\partial D}(p)} \int_D |v(x)|^2dx,$$
Moreover, if (9) is also satisfied, then, as \( \tau \rightarrow \infty \),

**Lemma 2.2.** Let \( f \) satisfy: there exists a positive constant \( \tau_0 \) such that, for all \( \tau \geq \tau_0 \) \( f(\tau) \neq 0 \). Assume that \( \Lambda_{\partial D}(p) \) is finite and satisfies (8). Then, we have

\[
\lim_{\tau \to \infty} \frac{\tau^2 e^{2\tau \sqrt{\pi \text{dist}(D, B)}}}{f(\tau)^2} J(\tau) = \frac{\pi}{4e^2} \left( \frac{\eta}{d_{\partial D}(p)} \right)^2 \sum_{q \in \Lambda_{\partial D}(p)} \frac{1 - (a \cdot \nu_q)^2}{\sqrt{\det(S_q(\partial B_{\partial D}(p)) - S_q(\partial D))}}.
\]

Moreover, if (9) is also satisfied, then, as \( \tau \rightarrow \infty \)

\[
\int_D |\nabla f|^2 dx = O(J(\tau)).
\]

**Proof.** Using (3), (11) and Integration by parts, we obtain

\[
J(\tau) = \frac{1}{\mu} \int_{\partial D} (\nu \times V) \cdot \nabla \times V dS.
\]

Then, the identity \( (\nu \times V) \cdot \nabla \times V = -\nu \cdot (\nabla \times V) \times V \) on \( \partial D \) yields another expression

\[
J(\tau) = -\int_{\partial D} \nu \cdot (\nabla \times V) \times V dS.
\]

Let \( x \in \partial D \) and \( \omega_x = (x - p)/|x - p| \).

We have

\[
(\omega_x \times a) \times (a - (\omega_x \cdot a)\omega_x) = a(\omega_x \cdot a) - \omega_x(a \cdot a) + (\omega_x \cdot a)^2 \omega_x - (\omega_x \cdot a)a
\]

\[
= -\omega_x(a \cdot a) + (\omega_x \cdot a)^2 \omega_x.
\]

Thus one can write

\[
(\omega_x \times a) \times (a - (\omega_x \cdot a)\omega_x) \cdot \nu_x = (m(x; p)\nu_x) a \cdot a,
\]

where

\[
m(x; p)\nu_x = (\omega_x \cdot \nu_x)(\omega_x \otimes \omega_x - I_3).
\]

Let \( q \in \Lambda_{\partial D}(p) \). We have \( \nu_q = -\omega_q \). Then \( m(q; p)\nu_q = I_3 - \nu_q \otimes \nu_q \) and hence

\[
(m(q; p)\nu_q) a \cdot a = 1 - (a \cdot \nu_q)^2.
\]

Therefore, \( |a \cdot \nu_q| \neq 1 \) if and only if \( (m(q; p)\nu_q)a \cdot a \neq 0 \).

From (18) and (23) we have

\[
(\nabla \times V) \times V
\]

\[
= -\frac{\tau K(\tau)^2 f(\tau)^2 v(x)^2}{\tau |x - p|} (\omega_x \times a) \times (M(x; p)a),
\]

where \( \tau = \sqrt{\mu \epsilon} \).

From (19) we have, as \( \tau \to \infty \) in the compact uniform topology in \( \mathbb{R}^3 \setminus \mathcal{B} \)

\[
M(x; p) = I_3 - \omega_x \otimes \omega_x + O\left(\frac{1}{\tau}\right)
\]
and this yields
\[
\left(1 + \frac{1}{\tau|x-p|}\right) (\omega_x \times a) \times (M(x; p)a)
\]
\[= (\omega_x \times a) \times \{(I_3 - \omega_x \otimes \omega_x) a\} + O\left(\frac{1}{\tau}\right)
\]
uniformly for \(x \in \partial D\). Applying this together with (30) to (32), we obtain, as \(\tau \rightarrow \infty\)
\[-\nu \cdot (\nabla \times V) \times V = \tau K(\tau)^2 \hat{f}(\tau)^2 v(x)^2 \left\{ (m(x; p)\nu_x)a \cdot a + O\left(\frac{1}{\tau}\right) \right\}
\]
ununiformly for \(x \in \partial D\). Thus, we have
\[
-\int_{\partial D} \nu \cdot (\nabla \times V) \times V dS
\]
(34)
\[= \tau K(\tau)^2 \hat{f}(\tau)^2 \left\{ \int_{\partial D} v(x)^2 (m(x; p)\nu_x)a \cdot a dS + O\left(\frac{1}{\tau}\right) \right\} \int_{\partial D} v(x)^2 dS.
\]
Under the finiteness of \(\Lambda_{\alpha D}(p)\) and (8), using the Laplace method [5] we obtain
\[
\lim_{\tau \rightarrow \infty} \tau e^{2\tau d_{\alpha D}(p)} \int_{\partial D} e^{-\frac{2\tau}{|x-p|}} (m(x; p)\nu_x)a \cdot a dS
\]
(35)
\[= \pi \frac{1}{d_{\alpha D}(p)^2} \sum_{q \in \Lambda_{\alpha D}(p)} \frac{(m(q; p)\nu_q)a \cdot a}{\sqrt{\det S_q(\partial B_{d_{\alpha D}(p)(p)}) - S_q(\partial D))}
\]
and
\[
\lim_{\tau \rightarrow \infty} \tau e^{2\tau d_{\alpha D}(p)} \int_{\partial D} e^{-\frac{2\tau}{|x-p|}} dS
\]
(36)
\[= \pi \frac{1}{d_{\alpha D}(p)^2} \sum_{q \in \Lambda_{\alpha D}(p)} \frac{1}{\sqrt{\det S_q(\partial B_{d_{\alpha D}(p)(p)}) - S_q(\partial D))}
\]
Thus, applying (35) and (36) to (34), we obtain
\[
-\int_{\partial D} \nu \cdot (\nabla \times V) \times V dS
\]
\[= \pi \frac{1}{d_{\alpha D}(p)^2} \sum_{q \in \Lambda_{\alpha D}(p)} \frac{(m(q; p)\nu_q)a \cdot a}{\sqrt{\det S_q(\partial B_{d_{\alpha D}(p)(p)}) - S_q(\partial D))}
\]
Thus, from this, (29) and (31) we obtain
\[
\lim_{\tau \rightarrow \infty} e^{2\tau d_{\alpha D}(p)}\int_{\partial D} e^{-\frac{2\tau}{|x-p|}} (m(x; p)\nu_x)a \cdot a dS
\]
(37)
\[= \pi \frac{1}{d_{\alpha D}(p)^2} \sum_{q \in \Lambda_{\alpha D}(p)} \frac{1 - (a \cdot \nu_q)^2}{\sqrt{\det (S_q(\partial B_{d_{\alpha D}(p)(p)}) - S_q(\partial D))}}
\]
Here from (22) we have
\[
\frac{e^{2\tau d_{\alpha D}(p)}}{\tau K(\tau)^2} = \frac{4\pi^2 e^{2\tau d_{\alpha D}(p) - \eta}}{\eta^2 (1 + O(\tau^{-1}))}
\]
And also \(d_{\alpha D}(p) - \eta = \dist(D, B)\). These together with (37) yield (26).
Next we prove (27). Since we have

\[ \int_D |V'|^2 \, dx = \int_D |\nabla \times V|^2 \, dx + \int_{\partial D} \nu \cdot V' V \, dS, \]

from (11) we see that it suffices to study the asymptotic behaviour of the second integral on this right-hand side. Thus, for this purpose we compute \( V' \).

From (19) we have

\[
(M(x;p)a)' = a \otimes \nabla A - \left\{ \left( \frac{x - p}{|x - p|} \otimes \frac{x - p}{|x - p|} \right) a \right\} \otimes \nabla B - B \left\{ \left( \frac{x - p}{|x - p|} \otimes \frac{x - p}{|x - p|} \right) a \right\}.
\]

Since

\[
\left( \frac{x - p}{|x - p|} \right)' = \frac{1}{|x - p|} \left( I_3 - \frac{x - p}{|x - p|} \otimes \frac{x - p}{|x - p|} \right),
\]

one gets

\[
\left\{ \left( \frac{x - p}{|x - p|} \otimes \frac{x - p}{|x - p|} \right) a \right\}'
\]

\[
= \left( \frac{x - p}{|x - p|} \right)' \left( \frac{x - p}{|x - p|} \cdot a \right) + \frac{x - p}{|x - p|} \otimes \left\{ \left( \frac{x - p}{|x - p|} \right)' \right\}^T a
\]

\[
= \frac{1}{|x - p|} \left\{ \left( I_3 - \frac{x - p}{|x - p|} \otimes \frac{x - p}{|x - p|} \right) \left( \frac{x - p}{|x - p|} \cdot a \right)
\]

\[
+ \frac{x - p}{|x - p|} \otimes \left( I_3 - \frac{x - p}{|x - p|} \otimes \frac{x - p}{|x - p|} \right) a \right\}
\]

\[
= \frac{1}{|x - p|} (\omega_x \cdot a I_3 - 2\omega_x \cdot a \omega_x \otimes \omega_x + \omega_x \otimes a).
\]

Inserting this together with the direct computation results of \( \nabla A \) and \( \nabla B \) into the expression above we have

\[
(M(x;p)a)'
\]

\[
= \frac{1}{\tau} \left( \frac{1}{|x - p|^2} + \frac{2}{\tau |x - p|^2} \right) (3\omega_x \otimes \omega_x \omega_x \cdot a - a \otimes \omega_x)
\]

\[
- \left\{ \frac{1}{|x - p|} + \frac{3}{\tau} \left( \frac{1}{|x - p|^2} + \frac{1}{\tau |x - p|^2} \right) \right\}
\]

\[
\times (\omega_x \cdot a I_3 - 2\omega_x \cdot a \omega_x \otimes \omega_x + \omega_x \otimes a).
\]

In particular, we have as \( \tau \to \infty \),

\[ (M(x;p)a)' = -\frac{1}{|x - p|} (\omega_x \cdot a I_3 - 2\omega_x \cdot a \omega_x \otimes \omega_x + \omega_x \otimes a) + O\left( \frac{1}{\tau} \right) \]

uniformly for \( x \in \partial D \).
On the other hand, we have
\[
(M(x; p)a) \otimes \nabla v(x)
\]
\[
= - \left( \frac{\tilde{\tau}}{|x - p|} + \frac{1}{|x - p|^2} \right) v(x)(M(x; p)a) \otimes (x - p)
\]
\[
= - \tilde{\tau}v(x) \left( 1 + \frac{1}{\tilde{\tau}|x - p|} \right) (M(x; p)a) \otimes \omega_x
\]
and thus (33) gives, as \( \tau \to \infty \)
\[
(40) \quad (M(x; p)a) \otimes \nabla v(x) = - \tilde{\tau}v(x) \left\{ (I_3 - \omega_x \otimes \omega_x)a \otimes \omega_x + O \left( \frac{1}{\tau} \right) \right\}
\]
uniformly for \( x \in \partial D \).

From (18) we have
\[
V'(x) = K(\tau) \tilde{f}(\tau) \left\{ v(x)(M(x; p)a)' + (M(x; p)a) \otimes \nabla v(x) \right\}.
\]
This together with (39) and (40) yields, as \( \tau \to \infty \)
\[
(41) \quad V'(x) = - \tilde{\tau}K(\tau) \tilde{f}(\tau)v(x) \left\{ (I_3 - \omega_x \otimes \omega_x)a \otimes \omega_x + O \left( \frac{1}{\tau} \right) \right\}.
\]
On the other hand, from (18) and (33) we obtain
\[
V(x) = K(\tau) \tilde{f}(\tau)v(x) \left\{ (I_3 - \omega_x \otimes \omega_x)a + O \left( \frac{1}{\tau} \right) \right\}.
\]
A combination of this and (41) gives
\[
(42) \quad V'(x)V(x) = - \tilde{\tau}K(\tau)^2 \tilde{f}(\tau)^2 v(x)^2
\]
\[
\times \left\{ (I_3 - \omega_x \otimes \omega_x)a \otimes \omega_x (I_3 - \omega_x \otimes \omega_x)a + O \left( \frac{1}{\tau} \right) \right\}.
\]
Since a direct computation yields
\[
(I_3 - \omega_x \otimes \omega_x)a \otimes \omega_x (I_3 - \omega_x \otimes \omega_x)a = 0,
\]
it follows from (42) that
\[
|V'(x)V(x)| \leq CK(\tau)^2 \tilde{f}(\tau)^2 v(x)^2.
\]
Thus we obtain
\[
\frac{1}{K(\tau)^2 \tilde{f}(\tau)^2} \left| \int_{\partial D} \nu \cdot V'Ve dS \right| \leq C \int_{\partial D} v^2 dS.
\]
Write
\[
\left[ \int_{\partial D} \nu \cdot V'Ve dS \right] J(\tau) \leq \frac{1}{\tilde{\tau}} \frac{C\tilde{\tau}e^{2\tilde{\tau}d_{\partial D}(p)}}{e^{2\tilde{\tau}d_{\partial D}(p)}} \int_{\partial D} v^2 dS \frac{J(\tau)}{K(\tau)^2 \tilde{f}(\tau)^2}.
\]
Then applying (36) and (37) to this right-hand side, we conclude
\[
\int_{\partial D} \nu \cdot V'Ve dS = O \left( \frac{J(\tau)}{\tau} \right).
\]
Now a combination of this and (38) yields (27).
3. **Proof of Theorem 1.1.** The proof of Theorem 1.1 starts with establishing the following brief asymptotic formula of the indicator function.

**Proposition 1.** It holds that, as $\tau \to \infty$

\[
\int_{\mathbb{R}^3 \setminus \overline{D}} f(x, \tau) \cdot (W_e - V) \, dx = J(\tau) + E(\tau) + O(\tau^{-3/2} e^{-\tau T}).
\]

**Proof.** Set $R = W_e - V$. The proof is divided into two steps.

**Step 1.** First we show that

\[
\int_{\mathbb{R}^3 \setminus \overline{D}} f(x, \tau) \cdot R \, dx = J(\tau) + E(\tau) - e^{-\tau T} \int_{\mathbb{R}^3 \setminus \overline{D}} F(x, \tau) \cdot V \, dx,
\]

where

\[
F(x, \tau) = -\left( \frac{\tau}{\epsilon} \nabla E(x, T) + \frac{1}{\epsilon} \nabla \times H(x, T) \right)
\]

and $(E, H)$ is the solution of (1).

Define

\[
W_m(x, \tau) = \int_0^T e^{-\tau t} H(x, t) \, dt, \quad x \in \mathbb{R}^3 \setminus \overline{D}.
\]

It is easy to see that integration by parts yields

\[
\nabla \times W_e + \tau \mu W_m = -e^{-\tau T} \mu H(x, T) \quad \text{in} \ \mathbb{R}^3 \setminus \overline{D},
\]

\[
\nabla \times W_m - \tau \epsilon W_e - \frac{\epsilon}{\tau} f(x, \tau) = e^{-\tau T} \epsilon E(x, T) \quad \text{in} \ \mathbb{R}^3 \setminus \overline{D}
\]

and

\[
\boldsymbol{\nu} \times W_e = 0 \quad \text{on} \ \partial D.
\]

Taking the rotation of (46) and (47), respectively, we obtain the following equation:

\[
\frac{1}{\mu \epsilon} \nabla \times \nabla \times W_e + \tau^2 W_e + f(x, \tau) = e^{-\tau T} F(x, \tau) \quad \text{in} \ \mathbb{R}^3 \setminus \overline{D}.
\]

Integration by parts gives

\[
\int_{\mathbb{R}^3 \setminus \overline{D}} \{(\nabla \times \nabla \times W_e) \cdot V - (\nabla \times \nabla \times V) \cdot W_e\} \, dx
\]

\[
= \int_{\partial D} \{(\boldsymbol{\nu} \times (\nabla \times V)) \cdot W_e - (\boldsymbol{\nu} \times (\nabla \times W_e)) \cdot V\} \, dS.
\]

(48) ensures that the first term on this right-hand side vanishes. And we have

\[(\boldsymbol{\nu} \times (\nabla \times W_e)) \cdot V = (\nabla \times W_e) \times V \cdot \boldsymbol{\nu} = (V \times \boldsymbol{\nu}) \cdot (\nabla \times W_e) = -(\boldsymbol{\nu} \times V) \cdot (\nabla \times W_e).
\]

Thus

\[
\int_{\mathbb{R}^3 \setminus \overline{D}} \{(\nabla \times \nabla \times W_e) \cdot V - (\nabla \times \nabla \times V) \cdot W_e\} \, dx
\]

\[
= \int_{\partial D} (\boldsymbol{\nu} \times V) \cdot (\nabla \times W_e) \, dS.
\]
Substituting (3) and (49) into this, we obtain

\[
\frac{1}{\mu \epsilon} \int_{\partial D} (\nu \times V) \cdot \nabla \times W_e \, dS
\]

\[
= \int_{\mathbb{R}^3 \setminus D} f(x, \tau) \cdot R \, dx + e^{-\tau T} \int_{\mathbb{R}^3 \setminus D} F(x, \tau) \cdot V \, dx.
\]

Write

\[
\frac{1}{\mu \epsilon} \int_{\partial D} (\nu \times V) \cdot \nabla \times W_e \, dS
\]

\[
= \frac{1}{\mu \epsilon} \int_{\partial D} (\nu \times V) \cdot \nabla \times V \, dS + \frac{1}{\mu \epsilon} \int_{\partial D} (\nu \times V) \cdot \nabla \times R \, dS.
\]

Since \( R \) satisfies

\[
\frac{1}{\mu \epsilon} \nabla \times \nabla \times R + \tau^2 R = e^{-\tau T} F(x, \tau) \quad \text{in} \, \mathbb{R}^3 \setminus D
\]

and

\[
\nu \times R = -\nu \times V \quad \text{on} \, \partial D,
\]

integration by parts gives

\[
e^{-\tau T} \int_{\mathbb{R}^3 \setminus D} F(x, \tau) \cdot R \, dx = -\frac{1}{\mu \epsilon} \int_{\partial D} (\nu \times V) \cdot \nabla \times R \, dS + E(\tau),
\]

that is,

\[
\frac{1}{\mu \epsilon} \int_{\partial D} (\nu \times V) \cdot \nabla \times R \, dS = E(\tau) - e^{-\tau T} \int_{\mathbb{R}^3 \setminus D} F(x, \tau) \cdot R \, dx.
\]

Now (44) follows from (11), (12), (50), (51), (54) and (28).

**Step 2.** It follows from the definition of the weak solution of (3) that

\[
\frac{1}{\mu \epsilon} \int_{\mathbb{R}^3} |V|^2 \, dx + \tau^2 \int_{\mathbb{R}^3} \left| V + \frac{f}{2\tau^2} \right|^2 \, dx = \frac{1}{4\tau^2} \int_{\mathbb{R}^3} |f|^2 \, dx.
\]

Since \( \tilde{f}(\tau) = O(\tau^{-3/2}) \), we have

\[
\| f(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} = O(\tau^{-1/2}).
\]

Then, applying the inequality

\[
|A + B|^2 \geq \frac{1}{2} |A|^2 - |B|^2
\]

to the second term in the left-hand side on (55), we obtain, as \( \tau \to \infty \)

\[
\frac{1}{\mu \epsilon} \int_{\mathbb{R}^3} |V|^2 \, dx + \tau^2 \int_{\mathbb{R}^3} |V|^2 \, dx = O(\tau^{-3}).
\]

Next we prove that, as \( \tau \to \infty \)

\[
E(\tau) = O(\tau^{-3}).
\]

Write

\[
\tau^2 |R|^2 - f \cdot R - e^{-\tau T} F \cdot R = \tau^2 \left| R - \frac{f + e^{-\tau T} F}{2\tau^2} \right|^2 - \frac{|f + e^{-\tau T} F|^2}{4\tau^2}.
\]
Substituting this into (44), we obtain
\[
\frac{1}{\mu \epsilon} \int_{\mathbb{R}^3 \setminus \mathcal{D}} |\nabla \times \mathbf{R}|^2 dx + \tau^2 \int_{\mathbb{R}^3 \setminus \mathcal{D}} \left| \mathbf{R} - \frac{\mathbf{f} + e^{-\tau T} \mathbf{F}}{2\tau^2} \right|^2 dx + J(\tau)
\]
(60)
\[
= \frac{1}{4\tau^2} \int_{\mathbb{R}^3 \setminus \mathcal{D}} |\mathbf{f} + e^{-\tau T} \mathbf{F}|^2 dx + e^{-\tau T} \int_{\mathbb{R}^3 \setminus \mathcal{D}} |\mathbf{F} \cdot \mathbf{V}| dx.
\]
Dropping the second and third terms on the left-hand side of (60), we obtain
\[
\frac{1}{\mu \epsilon} \int_{\mathbb{R}^3 \setminus \mathcal{D}} |\nabla \times \mathbf{R}|^2 dx \leq \frac{1}{4\tau^2} \int_{\mathbb{R}^3 \setminus \mathcal{D}} |\mathbf{f} + e^{-\tau T} \mathbf{F}|^2 dx + e^{-\tau T} \int_{\mathbb{R}^3 \setminus \mathcal{D}} |\mathbf{F} \cdot \mathbf{V}| dx.
\]
(61)
By (45) we have
\[
\|\mathbf{F}\|_{L^2(\mathbb{R}^3 \setminus \mathcal{D})} = O(\tau).
\]
(62)
It follows from (58) that
\[
\|\mathbf{V}\|_{L^2(\mathbb{R}^3 \setminus \mathcal{D})} = O(\tau^{-5/2}).
\]
Applying these and (56) to the right-hand side on (61), we obtain
\[
\frac{1}{\mu \epsilon} \int_{\mathbb{R}^3 \setminus \mathcal{D}} |\nabla \times \mathbf{R}|^2 dx = O(\tau^{-3}).
\]
On the other hand, dropping the first and third terms on the left-hand side on (60) and using (57), we obtain
\[
\frac{\tau^2}{2} \int_{\mathbb{R}^3 \setminus \mathcal{D}} |\mathbf{R}|^2 dx \leq \frac{1}{2\tau^2} \int_{\mathbb{R}^3 \setminus \mathcal{D}} |\mathbf{f} + e^{-\tau T} \mathbf{F}|^2 dx + e^{-\tau T} \int_{\mathbb{R}^3 \setminus \mathcal{D}} |\mathbf{F} \cdot \mathbf{V}| dx.
\]
Thus, by the same reason above we obtain
\[
\tau^2 \int_{\mathbb{R}^3 \setminus \mathcal{D}} |\mathbf{R}|^2 dx = O(\tau^{-3}).
\]
This completes the proof of (59).
Finally from (58), (59) and (62) we have
\[
\int_{\mathbb{R}^3 \setminus \mathcal{D}} \mathbf{F}(x, \tau) \cdot \mathbf{R} dx + \int_{\mathbb{R}^3 \setminus \mathcal{D}} \mathbf{F}(x, \tau) \cdot \mathbf{V} dx = O(\tau^{-3/2}).
\]
Thus, a combination of this and (44) yields (43).

**Remark 1.** From (46) and (47) we obtain also the following equation for $\mathbf{W}_m$:
\[
\frac{1}{\mu \epsilon} \nabla \times \nabla \times \mathbf{W}_m + \tau^2 \mathbf{W}_m - \frac{1}{\tau} \nabla \times \mathbf{f}(x, \tau) = e^{-\tau T} \tilde{\mathbf{F}}(x, \tau) \quad \text{in} \ \mathbb{R}^3 \setminus \mathcal{D},
\]
where
\[
\tilde{\mathbf{F}}(x, \tau) = \frac{1}{\mu} \nabla \times \mathbf{E}(x, T) - \tau \mathbf{H}(x, T).
\]
In this paper, we will not make use of this equation.

Next we prove

**Lemma 3.1.** _There exist positive constant $C$ and $\tau_0$ such that, for all $\tau \geq \tau_0$
\[
E(\tau) \leq C(\tau^2 J(\tau) + e^{-2\tau T}).
\]
(63)
Proof. Set $\mathbf{R} = \mathbf{W}_e - \mathbf{V}$. Taking the scalar product of equation (52) with $\mathbf{R}$, integrating over $\mathbb{R}^3 \setminus \overline{\mathcal{D}}$ and using boundary condition (53) on $\partial \mathcal{D}$ and (12), we have

$$E(\tau) = \frac{1}{\mu_e} \int_{\partial \mathcal{D}} \mathbf{\nu} \cdot \nabla \times \mathbf{R} \cdot dS + e^{-\tau T} \int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} \mathbf{F} \cdot \mathbf{R} \, dx.$$

By the trace theorem ([25], p. 209, Theorem 5.4.2.), one can choose a lifting $\mathbf{\tilde{V}}$ of $\mathbf{\nu} \times \mathbf{V}$ on $\partial \mathcal{D}$ in such a way that

$$\|\mathbf{\tilde{V}}\|^2_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} + \|\nabla \times \mathbf{\tilde{V}}\|^2_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} \leq C^2\|\mathbf{\nu} \times \mathbf{V}\|^2_{H^{1/2}_{\text{div}}(\partial \mathcal{D})}.$$

See also p. 191 in [25] for the definition of $H^{1/2}_{\text{div}}(\partial \mathcal{D})$ and the norm. Note that $C$ is a positive constant and independent of $\mathbf{V}$.

Again the trace theorem tells us also that

$$\|\mathbf{\nu} \times \mathbf{V}\|^2_{H^{1/2}_{\text{div}}(\partial \mathcal{D})} \leq (C')^2(\|\mathbf{V}\|^2_{L^2(D)} + \|\nabla \times \mathbf{V}\|^2_{L^2(D)}),$$

where $C'$ is a positive constant and independent of $\mathbf{V}$. Thus, we have

$$\|\mathbf{\tilde{V}}\|^2_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} + \|\nabla \times \mathbf{\tilde{V}}\|^2_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} \leq (CC')^2(\|\mathbf{V}\|^2_{L^2(D)} + \|\nabla \times \mathbf{V}\|^2_{L^2(D)}).$$

Moreover, from equation (52) one gets

$$\frac{1}{\mu_e} \int_{\partial \mathcal{D}} \mathbf{\nu} \times \mathbf{V} \cdot \nabla \times \mathbf{R} \, dS$$

$$= -\frac{1}{\mu_e} \int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} \nabla \times \mathbf{R} \cdot \mathbf{\nu} \times \mathbf{V} \, dS - \tau^2 \int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} \mathbf{R} \cdot \mathbf{\tilde{V}} \, dx + e^{-\tau T} \int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} \mathbf{F} \cdot \mathbf{\tilde{V}} \, dx.$$

Substituting this into (64) and estimating from above, we obtain

$$E(\tau) \leq \frac{1}{\mu_e} \|\nabla \times \mathbf{R}\|_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} \|\mathbf{\tilde{V}}\|_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} + \tau \|\mathbf{R}\|_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} \|\mathbf{\tilde{V}}\|_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} + e^{-\tau T} \|\mathbf{F}\|_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} (\|\mathbf{\tilde{V}}\|_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} + \|\mathbf{R}\|_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})}).$$

Here we make use of the following trivial estimates

$$\|\nabla \times \mathbf{R}\|_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} \leq \sqrt{\mu_e} \sqrt{E(\tau)}$$

and

$$\|\mathbf{R}\|_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} \leq \tau^{-1} \sqrt{E(\tau)}.$$ 

Applying (62), (67) and (68) to the right-hand side on (66), we obtain

$$E(\tau) \leq C_1 \left( \|\nabla \times \mathbf{\tilde{V}}\|_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} + \tau \|\mathbf{\tilde{V}}\|_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} \right) \sqrt{E(\tau)}$$

$$+ C_2 e^{-\tau T} \|\mathbf{\tilde{V}}\|_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} + C_3 e^{-\tau T} \sqrt{E(\tau)}.$$

Then, applying a standard technique to the first and last terms on this right-hand side, we obtain

$$E(\tau) \leq C_4 \left( \|\nabla \times \mathbf{\tilde{V}}\|^2_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} + \tau^2 \|\mathbf{\tilde{V}}\|^2_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} \right) + C_5 e^{-2\tau T}.$$

Then, a combination of this, (65) and trivial inequality

$$\|\mathbf{V}\|^2_{L^2(D)} + \|\nabla \times \mathbf{V}\|^2_{L^2(D)} \leq C(1+\tau^{-2})J(\tau)$$

yields (63). □
Now from (43), (20), (21) and (63) we obtain
\[
\limsup_{\tau \to \infty} \tau^2 e^{2\tau \sqrt{\mu \epsilon} \text{dist}(D, B)} \int_{\mathbb{R}^3 \setminus D} f(x, \tau) \cdot (W_\epsilon - V)dx < \infty
\]
and
\[
\liminf_{\tau \to \infty} \tau^{5+2\gamma} e^{2\tau \sqrt{\mu \epsilon} \text{dist}(D, B)} \int_{\mathbb{R}^3 \setminus D} f(x, \tau) \cdot (W_\epsilon - V)dx > 0
\]
provided \( T > 2\sqrt{\mu \epsilon} \text{dist}(D, B) \). From these we immediately obtain Theorem 1.1.

**Remark 2.** (63) in Lemma 3.1 is *not sharp*, however, for Theorem 1.1 it is enough. For Theorem 1.2 we need more accurate estimate like \( E(\tau) \sim J(\tau) \).

### 4. Proof of Theorem 1.2

First it is easy to see that Theorem 1.2 follows from the following theorem and (26).

**Theorem 4.1.** Assume that \( \Lambda_{\partial D}(p) \) is finite and that (8) and (9) are satisfied. Let \( f \) satisfy (6) for a \( \gamma \in \mathbb{R} \). Let \( T > 2\sqrt{\mu \epsilon} \text{dist}(D, B) \). Then, as \( \tau \to \infty \), we have
\[
\int_{\mathbb{R}^3 \setminus D} f(x, \tau) \cdot (W_\epsilon - V)dx = 2J(\tau)(1 + O(\tau^{-1/2})).
\]

Thus the purpose of this section is to describe the proof of Theorem 4.1. However, note that under the assumption (6) for a \( \gamma \in \mathbb{R} \) it holds that
\[
\frac{\tau^2 e^{2\tau \sqrt{\mu \epsilon} \text{dist}(D, B)}}{f(\tau)^2} \tau^{-1/2} e^{-\tau T} = O(\tau^{2-1/2+2\gamma \epsilon} e^{-\tau(T-2\sqrt{\mu \epsilon} \text{dist}(D, B))}).
\]
Thus, if we have the estimate
\[
E(\tau) = J(\tau)(1 + O(\tau^{-1/2})),
\]
then (43) yields (69). Thus, the proof of (69) is reduced to that of (70) which shows the asymptotic coincidence of \( E(\tau) \) and \( J(\tau) \) as \( \tau \to \infty \).

The proof of (70) employs the Lax-Phillips reflection argument in [22], however, some technical parts are different. Anyway that is based on: a representation formula of \( E(\tau) - J(\tau) \) via a reflection. Thus, the following subsection starts with describing a reflection principle across \( \partial D \) from inside to outside.

#### 4.1. Reflection principle

One can choose a positive number \( \delta_0 \) in such a way that: given \( x \in \mathbb{R}^3 \setminus D(x \in D) \) with \( d_{\partial D}(x) < 2\delta_0 \) there exists a unique \( q = q(x) \in \partial D \) such that \( x = q + d_{\partial D}(x) \nu_q(x) = q - d_{\partial D}(x) \nu_q \). Both \( d_{\partial D}(x) \) and \( q(x) \) are \( C^k \) therein provided \( \partial D \) is \( C^k \) with \( k \geq 2 \). See Lemma 14.16 in [12] for this.

For \( x \) with \( d_{\partial D}(x) < 2\delta_0 \) define \( x^r = 2q(x) - x \), \( \pi(x) = \nu_q(x) \otimes \nu_q(x) \) and \( n(x) = \nu_q(x) \). Note that \( n \) is \( C^3 \) if \( \partial D \) is \( C^4 \).

The reflection principle what we say in this paper consists of two parts summarized as the following propositions.

**Proposition 2.** Assume that \( \partial D \) is \( C^4 \). Let \( V \) be a vector field over \( D \) and \( C^2 \) in \( D \). For \( x \in \mathbb{R}^3 \setminus D \) with \( d_{\partial D}(x) < 2\delta_0 \), define
\[
V^*(x) = -A(x^r) + B(x^r) + 2d_{\partial D}(x)n'(x)A(x^r),
\]
where \( A(y) = (I - \pi(y))V(y) \) and \( B(y) = \pi(y)V(y) \) for \( y \in D \) with \( d_{\partial D}(y) < 2\delta_0 \). Then, \( V^* \) satisfies
\[
V^* \times \nu = -V \times \nu \quad \text{on} \ \partial D
\]
and

\( \nu \times (\nabla \times \mathbf{V}^*) = \nu \times (\nabla \times \mathbf{V}) \quad \text{on } \partial D. \)

**Proof.** Define

\( \tilde{\mathbf{V}}(x) = -A(x^r) + B(x^r) \)

and

\( \mathbf{C}(x) = 2d_{\partial D}(x)n'(x)V(x^r). \)

We have

\( \mathbf{V}^*(x) = \tilde{\mathbf{V}}(x) + \mathbf{C}(x). \)

First, we claim that \( \tilde{\mathbf{V}} \) satisfies the following boundary conditions.

**Claim 1.** \( \tilde{\mathbf{V}} \) satisfies the following boundary conditions

\( \tilde{\mathbf{V}} \times \nu = -\mathbf{V} \times \nu \quad \text{on } \partial D; \)

\( \nu_z \times (\nabla \times \tilde{\mathbf{V}}) = \mathbf{V} \times (\nabla \times \mathbf{V}) - 2S_z(\partial D)A \quad \text{on } \partial D, \)

where \( S_z(\partial D) \) denotes the shape operator of \( \partial D \) at \( x \in \partial D \) with respect to \( \nu_z \).

Next we claim

**Claim 2.** We have

\( \nu_z \times (\nabla \times \mathbf{C}) = 2S_z(\partial D)A \quad \text{on } \partial D. \)

Now from trivial identity \( \mathbf{C} = 0 \) on \( \partial D \) and (76) we obtain (72); from (77) and (78) we obtain (73). This completes the proof of Proposition 2. See also Appendix for the proof of Claims 1 and 2.

Since the proof of the following proposition is tedious and so is described in Appendix.

**Proposition 3.** Assume that \( \partial D \) is \( C^4 \). If \( \mathbf{V} \) satisfies

\( \frac{1}{\mu \epsilon} \nabla \times \nabla \times \mathbf{V} + \tau^2 \mathbf{V} = 0 \quad \text{in } D, \)

then, \( \mathbf{V}^* \) defined as (71) satisfies

\( \frac{1}{\mu \epsilon} \nabla \times \nabla \times \mathbf{V}^* + \tau^2 \mathbf{V}^* \)

\( = \text{terms from } \mathbf{V}(x^r) \text{ and } \mathbf{V}'(x^r) + 2d_{\partial D}(x) \times \text{terms from } (\nabla^2 \mathbf{V})(x^r) \)

and all the coefficients in this right-hand side are independent of \( \tau \) and continuous, in particular, the coefficients come from the second order terms are \( C^1 \) in a tubular neighbourhood of \( \partial D \).

**Remark 3.** Note that, for \( y \in D \) with \( d_{\partial D}(y) < 2\delta_0 \) we have the decomposition

\( \mathbf{V}(y) = \mathbf{A}(y) + \mathbf{B}(y). \)

If \( \partial D \) is a plane, then \( n'(x) \equiv 0 \) and the third term in the right-hand side on (71) vanishes. Thus, in this case Propositions 2 and 3 become the reflection principle used in [23] for inverse obstacle scattering for Maxwell’s equations in a frequency domain (replaced \( \tau^2 \) with \(-k^2\)). They employed this principle for a different purpose from us, more precisely, establishing a uniqueness theorem for polygonal obstacles in a single frequency domain. In the curved boundary case, \( n' \neq 0 \) and we need the
The enclosure method using a single electromagnetic wave correction term $2d_{BD}(x)n'(x)A(x')$. For more detailed information about (80) see (117) and (119) in Appendix.

4.2. Proof of the estimate (70). In this subsection we start with describing a representation formula of $E(\tau) - J(\tau)$ in terms of the reflection across $\partial D$.

Let $0 < \delta < \delta_0/2$. Choose a smooth function $\phi = \phi_3$ defined on the whole space in such a way that (i) $0 \leq \phi \leq 1$; (ii) $\phi(x) = 1$ if $d_{BD}(x) < \delta$ and $\phi(x) = 0$ if $d_{BD}(x) > 2\delta$; (iii) $|\nabla \phi(x)| \leq C\delta^{-1}$; $|(\partial^2/\partial x_i \partial x_j) \phi(x)| \leq C\delta^{-2}$ with $i, j = 1, 2, 3$.

Define

$$V^\tau(x) = \phi(x)V^*(x), \ x \in \mathbb{R}^3,$$

where $V^*$ is given by (71).

Set $R = W_e - V$. Since $\nabla \cdot (A \times B) = \nabla \times A \cdot B - A \cdot \nabla \times B$, integration by parts yields

$$\int_{\mathbb{R}^3 \setminus B} R \cdot \nabla \times \nabla \times V^\tau dx$$

(83)
$$= - \int_{\partial D} \nu \cdot ((\nabla \times V^\tau) \times R) dS + \int_{\mathbb{R}^3 \setminus B} \nabla \times V^\tau \cdot \nabla \times R dx$$

and

$$\int_{\mathbb{R}^3 \setminus B} \nabla \times \nabla \times R \cdot V^\tau dx$$

(84)
$$= - \int_{\partial D} \nu \cdot ((\nabla \times R) \times V^\tau) dS + \int_{\mathbb{R}^3 \setminus B} \nabla \times R \cdot \nabla \times V^\tau dx.$$

Taking the difference of (83) from (84) and noting $\phi \equiv 1$ in a neighbourhood of $\partial D$, we obtain

$$\int_{\mathbb{R}^3 \setminus B} (R \cdot \nabla \times \nabla \times V^\tau - \nabla \times \nabla \times R \cdot V^\tau) dx$$

(85)
$$= \int_{\partial D} \nu \cdot ((\nabla \times R) \times V^* - (\nabla \times V^*) \times R) dS.$$

Since $R$ satisfies $R \times \nu = -V \times \nu$ on $\partial D$ (see (53)), we have

$$\nu \cdot ((\nabla \times V^*) \times R) = (\nabla \times V^*) \cdot (R \times \nu) = -(\nabla \times V^*) \cdot (V \times \nu) = -V \cdot \nu \times (\nabla \times V^*).$$

Thus, applying (73) to this, we obtain

$$\nu \cdot ((\nabla \times V^*) \times R) = -V \cdot \nu \times (\nabla \times V) = -\nu \cdot ((\nabla \times V) \times V).$$

Substituting this into (34), we obtain

$$J(\tau) = \frac{1}{\mu \epsilon} \int_{\partial D} \nu \cdot ((\nabla \times V^*) \times R) dS.$$  

Moreover, from (72) we have

$$\nu \cdot ((\nabla \times R) \times V^*) = \nabla \times V \cdot (V^* \times \nu) = -\nabla \times R \cdot (V \times \nu) = \nu \times V \cdot \nabla \times R.$$  

Substituting this into (54), we obtain

$$E(\tau) = \frac{1}{\mu \epsilon} \int_{\partial D} \nu \cdot ((\nabla \times R) \times V^*) dS + e^{-\tau T} \int_{\mathbb{R}^3 \setminus B} F \cdot R dx.$$  

Substituting (86) and (87) into the right-hand side on (85), we obtain

\[ E(\tau) - J(\tau) = \frac{1}{\mu \epsilon} \int_{\mathbb{R}^3 \setminus \overline{D}} (R \cdot \nabla \times \nabla \times V - \nabla \times \nabla \times R \cdot V^r) \, dx \]

\[ + e^{-\tau T} \int_{\mathbb{R}^3 \setminus \overline{D}} F \cdot R \, dx. \]

From this and (52) we obtain

\[ E(\tau) - J(\tau) = \int_{\mathbb{R}^3 \setminus \overline{D}} R \cdot \left( \frac{1}{\mu \epsilon} \nabla \times \nabla \times V^r + \tau^2 V^r \right) \, dx \]

\[ + e^{-\tau T} \left( \int_{\mathbb{R}^3 \setminus \overline{D}} F \cdot R \, dx - \int_{\mathbb{R}^3 \setminus \overline{D}} F \cdot V^r \, dx \right) \]

\[ \equiv I + e^{-\tau T} I. \tag{88} \]

Here we prove

\[ I = O(\tau^{-1/2}) J(\tau) + O(\tau^{-2} e^{-\tau T}). \tag{89} \]

From Proposition 3 we have

\[ \frac{1}{\mu \epsilon} \nabla \times \nabla \times V^r (x) + \tau^2 V^r (x) \]

\[ = \phi(x) \left( \sum_{j,k,l} d_{\partial D} (x) C_{ijkl} (x) \frac{\partial^2 V^j}{\partial x_k \partial x_l} (x^r) \right) \]

\[ + \sum_{j,k,l} D_{ijkl} (x) \frac{\partial V^j}{\partial x_k} (x^r) \frac{\partial \phi}{\partial x_l} (x) \]

\[ + \sum_{j,k,l} E_{ijkl} (x) \frac{\partial^2 \phi}{\partial x_k \partial x_l} (x) \]

where \( C_{ijkl} \) are of class \( C^1 \); \( D_{ijkl} \) and \( E_{ijkl} \) are of class \( C^1 \) and \( C^0 \) in a neighbourhood of \( \partial D \); \( F_{ijkl} \) are constants.

Substituting this into the first term on the right-hand side of (88) and making a change of variables \( x = y^r \), we obtain

\[ I = \int_D R(y^r). \]

\[ \left( \frac{\partial^2 V^j}{\partial y_k \partial y_l} (y) + \text{lower order terms} \right) J(y) \, dy, \]

where \( J(y) \) denotes the Jacobian of the map: \( y \mapsto y^r \). A routine involving an integration by parts and \( d_{\partial D} (y^r) = d_{\partial D} (y) \) yields

\[ \int_D R(y^r) \cdot \left( \frac{\partial^2 V^j}{\partial y_k \partial y_l} (y) \right) J(y) \, dy \]
\[ R(\parallel \text{Lemma 4.2. Assume that } \Lambda_{DD}(p) \text{ is finite and that (8) and (9) are satisfied. Then, there exist positive constants } C \text{ and } \tau_0 \text{ such that, for all } \tau \geq \tau_0 \text{ we have} \]
\[ E(\tau) \leq C(J(\tau) + e^{-2\tau T}) \]
and
\[ \|R\|^2_{L^2((\mathbb{R}^3 \setminus \mathcal{D}), \delta)} \leq C(J(\tau) + e^{-2\tau T}). \]

For the proof see the next subsection. Once we have (92) and (93), we see that (91) becomes
\[ I = (O(\delta) + O((\delta \tau)^{-1}) + O((\delta \tau)^{-2})) J(\tau) + O((\delta \tau)^{-1}) + O((\delta \tau)^{-2}) + \delta) e^{-\tau T} \tau^{-1/2} \sqrt{J(\tau)}. \]

Let \( \theta > 0 \) and choose \( \delta = \tau^{-\theta} \) with \( \tau >> 1 \). Then, this right-hand side becomes
\[ I = O(\tau^{-\theta} + \tau^{-1(1-\theta)} + \tau^{-2(1-\theta)}) J(\tau) + O(\tau^{-1(1-\theta)} + \tau^{-2(1-\theta)} + \tau^{-\theta}) e^{-\tau T} \sqrt{J(\tau)}. \]

Now choosing \( \theta \) in such a way that \( \theta = 1 - \theta \), that is, \( \theta = 1/2 \). Then (89) follows from (94) and (58).

It is easier to obtain the estimate \( e^{-\tau T} II = O(e^{-\tau T}(\sqrt{E(\tau)} + \sqrt{J(\tau)}) \)) than (89). Then (58) and (59) give \( e^{-\tau T} II = O(\tau^{-3/2} e^{-\tau T}) \). A combination of this and (89) yields
\[ |E(\tau) - J(\tau)| = O(\tau^{-1/2}) J(\tau) + O(\tau^{-3/2} e^{-\tau T}). \]

Here note that (21) yields
\[ \frac{e^{-\tau T} \tau^{-3/2}}{J(\tau)} = \frac{5+2\gamma e^{-\tau (T-2\text{dist}(D,B))}}{\tau^{5+2\gamma e^{2\text{dist}(D,B) J(\tau)}}} = O(\tau^{5+2\gamma -3/2} e^{-\tau (T-2\text{dist}(D,B))}). \]

Therefore (95) becomes \( |E(\tau) - J(\tau)| = O(\tau^{-1/2}) J(\tau) \). This completes the proof of (70).
Remark 4. Summing up, we have shown that for the proof of (70) it suffices to have estimates (92) and (93). Note that (92) is sharper than (63). However, we need more restrictive assumptions that $\Lambda_{\partial D}(p)$ is finite and that (8) and (9) are satisfied.

It seems that giving an estimate of $\|R'\|_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})}$ in terms of $E(\tau)$ directly is not trivial unlike the scalar case. Of course now we have (70) and thus, from (93) and (21) we obtain $\|R'\|_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} \leq CE(\tau)$ for $\tau >> 1$ provided $\Lambda_{\partial D}(p)$ is finite and that (8) and (9) are satisfied.

4.3. Proof of Lemma 4.2. Let $\varphi$ be a smooth function on the whole space and satisfy $0 \leq \varphi \leq 1$: $\varphi(x) = 1$ for $x$ with $d_{\partial D}(x) \leq \delta_0/2$ and $\varphi(x) = 0$ for $d_{\partial D}(x) \geq \delta_0$. Here $\delta_0$ is chosen in such a way that given $x$ with $d_{\partial D}(x) < 2\delta_0$ there exists a unique $q = q(x) \in \partial D$ that attains the minimum of the function $\partial D \ni y \mapsto |y - x|$. We assume that $\partial D$ is $C^2$. Then one may think that both $d_{\partial D}(x)$ and $q(x)$ are $C^2$ for $x \in \mathbb{R}^3 \setminus D$ with $d_{\partial D}(x) < 2\delta_0$ (see [12], p.355, Lemma 14.16).

Set $R = W_e - V$. Taking the scalar product of (52) with $\varphi V^*$ and integrating over $\mathbb{R}^3 \setminus \overline{\mathcal{D}}$, we obtain

$$-\frac{1}{\mu^2} \int_{\partial D} \nu \times (\varphi V^*) \cdot \nabla \times R dS,$$

$$= \frac{1}{\mu^2} \int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} \nabla \times R \cdot \nabla \times (\varphi V^*) dx + \tau^2 \int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} R \cdot (\varphi V^*) dx - e^{-\tau T} \int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} F(x, \tau) \cdot (\varphi V^*) dx.$$

Since $V^*$ satisfies (72), from this and (64) we obtain the expression

$$E(\tau) = \frac{1}{\mu^2} \int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} \nabla \times R \cdot \nabla \times (\varphi V^*) dx + \tau^2 \int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} R \cdot (\varphi V^*) dx - e^{-\tau T} \left( \int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} F(x, \tau) \cdot (\varphi V^*) dx - \int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} F \cdot R dx \right).$$

This yields

$$E(\tau) \leq \frac{1}{\mu^2} \int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} \nabla \times R \cdot \nabla \times (\varphi V^*) dx + \tau^2 \int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} R \cdot (\varphi V^*) dx,$$

$$+ e^{-\tau T} \int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} \left( \int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} F(x, \tau) \cdot (\varphi V^*) dx - \int_{\mathbb{R}^3 \setminus \overline{\mathcal{D}}} F \cdot R dx \right).$$

A change of variable $y = x - 2d_{\partial D}(x)u_x$ gives

$$\|\nabla \varphi \times V^*\|_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} + \|\varphi V^*\|_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} \leq C\|V\|_{L^2(D)} \leq C\tau^{-1} \sqrt{J(\tau)}$$

and also

$$\|\varphi \nabla \times V^*\|_{L^2(\mathbb{R}^3 \setminus \overline{\mathcal{D}})} \leq C\|V'\|_{L^2(D)} \leq C' \sqrt{J(\tau)}.$$
Next we give a proof of (93). Define $U = \varphi(R - V^*)$. Since $\nabla \cdot R = e^{-\tau T} \nabla \cdot F / \tau^2$ and $\nabla \cdot F = -\tau \nabla \cdot E(x,T)$, we have $\nabla \cdot R = -e^{-\tau T} \tau^{-1} \nabla \cdot E(x,T)$. However, from the governing equations of $E$ and $H$ in the time domain we have

$$\nabla \cdot E(x,T) = \int_0^T \nabla \cdot J(x,t)dt.$$ So choosing $\delta_0$ in such a way that $\overline{\mathcal{B}} \cap (\mathbb{R}^3 \setminus \overline{\mathcal{D}})_{\delta_0/2} = \emptyset$, we conclude $\nabla \cdot R = 0$ in $(\mathbb{R}^3 \setminus \overline{\mathcal{D}})_{\delta_0/2}$ and hence $\nabla \cdot U = \nabla \varphi \cdot R - \nabla \cdot (\varphi V^*) \in L^2((\mathbb{R}^3 \setminus \overline{\mathcal{D}})_{\delta_0/2})$. Moreover, $U$ together with $\nabla \times U$ belongs to $L^2((\mathbb{R}^3 \setminus \overline{\mathcal{D}})_{\delta_0/2})$; $U$ satisfies $U \times \nu = 0$ on $\partial D$ and $U = 0$ in $(\mathbb{R}^3 \setminus \overline{\mathcal{D}})_{\delta_0/2}$. Therefore, by Corollary 1.1 on p. 212 and (ii) of Remark 2 on p. 213 in [10], we have $U \in H^1((\mathbb{R}^3 \setminus \overline{\mathcal{D}})_{\delta_0/2})$ and

$$\|U\|_{H^1((\mathbb{R}^3 \setminus \overline{\mathcal{D}})_{\delta_0/2})}^2 \leq C \left( \|U\|_{L^2((\mathbb{R}^3 \setminus \overline{\mathcal{D}})_{\delta_0/2})}^2 + \|\nabla \times U\|_{L^2((\mathbb{R}^3 \setminus \overline{\mathcal{D}})_{\delta_0/2})}^2 + \|\nabla \cdot U\|_{L^2((\mathbb{R}^3 \setminus \overline{\mathcal{D}})_{\delta_0/2})}^2 \right).$$

Applying (67) and (68) and a change of variables to this right-hand side, we obtain

$$\|U\|_{H^1((\mathbb{R}^3 \setminus \overline{\mathcal{D}})}^2 \leq C(E(\tau) + \|V\|_{H^1(D)}^2).$$

Since $\varphi R = U + \varphi V^*$, (97) together with the estimate

$$\|\varphi V^*\|_{H^1((\mathbb{R}^3 \setminus \overline{\mathcal{D}})} \leq C \|V\|_{H^1(D)} \leq C' \sqrt{J(\tau)}$$

gives $\|R^*\|^2_{L^2((\mathbb{R}^3 \setminus \overline{\mathcal{D}})_{\delta_0/2})} \leq C(E(\tau) + J(\tau))$. A combination of this and (92) yields (93). This completes the proof of Lemma 4.2.

5. Concluding remarks. In this paper, we employed a simple form (2) as a model of the current density, however, in principle, it may be possible to cover more complicated model of the current density, at least, in the framework of the solution as constructed in [11].

The method presented here can be applied also to an interior problem similar to that considered in [16]. The problem therein aims at extracting information about the geometry of an unknown cavity from the wave in time domain which is produced by the initial data localized inside the cavity and propagates therein.

Single measurement version of the time domain enclosure method also finds an application to an inverse initial boundary value problem for the heat equation in three-space dimensions. For this see Theorem 1.1 in [20] and consult Section 3 in [19] for an open problem in the visco elasticity.

Some of open problems are in order.

- A lot of papers deals with the perfectly conducting obstacle as the first step (see [8] and references therein). It is a typical and important condition as everyone first considers, like the Dirichlet boundary condition for the wave equation. This paper also follows that traditional order and note that the aim of this paper is to introduce a method for inverse electromagnetic obstacle scattering. However, as a next step, it is natural to ask: how about the case when the electromagnetic wave satisfies a more general boundary condition like the Leontovich condition on the surface of the obstacle (see, e.g., [2]). Note that, for the wave equation with the Robin type boundary condition we have [15] and [18] which contain results corresponding to Theorem 1.1 and Theorem 1.2, respectively.
• How about the case when the reflected electromagnetic wave is observed at a different place from the support of the source? We expect that the observed data give us different information about the geometry of unknown obstacle together with a constructive method which yields the location of all the first reflection points from a single observed wave as seen for an acoustic wave case in [17]. It would be interesting to see also [6] for a comparison of monostatic and bistatic radar images.

• There are several other inverse obstacle scattering problems in time domain whose governing equations are systems of partial differential equations. Extend the range of the applications of the method presented here to such systems.

Appendix. Proof of Claim 1. Clearly $\tilde{V}$ satisfies (76). To check (77) we have to compute $\nabla \times \tilde{V}$. Set $B(x) = V(x) \cdot n(x)$.

We have
\begin{equation}
\nabla (B(x^r)) = (2q'(x)^T - I)(\nabla B)(x^r).
\end{equation}
Let $y \in \partial D$. Since $q'(y)\nu_y = 0$, we get
\begin{equation}
\nabla (B(x^r))|_{x=y} \cdot \nu_y = - (\nabla B)(y) \cdot \nu_y.
\end{equation}
On the other hand, we have $q'(y)\nu = \nu$ for all vectors with $\nu \cdot \nu = 0$. Thus (98) gives
\begin{equation}
\nabla (B(x^r))|_{x=y} \cdot \nu = (\nabla B)(y) \cdot \nu.
\end{equation}
From these we obtain
\begin{equation}
\nabla (B(x^r))|_{x=y} = \{(\nabla B)(y) - (\nabla B)(y) \cdot \nu_y \nu_y\} - (\nabla B)(y) \cdot \nu_y.
\end{equation}
Here we note that $n(x) = \nabla (d_{D}(x))$ for $x \in \mathbb{R}^3 \setminus D$ and $n(x) = - \nabla (d_{D}(x))$ for $x \in D$. This gives $\nabla \times n = 0$. Thus, we have $\nabla \times \tilde{B}(x) = \nabla (B(x^r)) \times n(x)$ and $\nabla \times \tilde{B}(x) = (\nabla B)(x) \times n(x)$, where $B(x) = B(x^r)$. Thus, from (99) we obtain
\begin{equation}
\nabla \times \tilde{B} = \nabla \times B \quad \text{on } \partial D.
\end{equation}
Define $\tilde{A}(x) = -A(x^r)$ for $x \in \mathbb{R}^3 \setminus D$. Let $y \in \partial D$. Applying (99) for $B$ replaced with $-A_i$ for each $i = 1, 2, 3$, we have
\begin{equation}
(\nabla \tilde{A}^i)(y) = -(\nabla A^i)(y) + 2((\nabla A^i)(y) \cdot \nu_y)\nu_y.
\end{equation}
Note that
\begin{equation}
\nabla \times \tilde{A} = \sum_{i=1}^{3} \nabla \times (\tilde{A}^i e_i) = \sum_{i=1}^{3} \nabla \tilde{A}^i \times e_i
\end{equation}
and the same for $\nabla \times A$. These yield
\begin{equation}
(\nabla \times \tilde{A})(y) = -(\nabla \times A)(y) + 2 \sum_{i=1}^{3} ((\nabla A^i)(y) \cdot \nu_y)\nu_y \times e_i.
\end{equation}
Write
\begin{equation}
\sum_{i=1}^{3} ((\nabla A^i)(y) \cdot \nu_y)\nu_y \times e_i = \nu_y \sum_{i=1}^{3} ((\nabla A^i)(y) \cdot \nu_y) e_i = \nu_y \times \{A^i(y)\nu_y\}.
\end{equation}
Then, (101) becomes
\begin{equation}
(\nabla \times \tilde{A})(y) = -(\nabla \times A)(y) + 2\nu_y \times \{A^i(y)\nu_y\}.
\end{equation}
Taking the vector product of both sides on (102) with $\nu_y$, we obtain
\begin{equation}
\nu_y \times (\nabla \times \tilde{A})(y) = -\nu_y \times (\nabla \times A)(y) - 2A^i(y)\nu_y.
\end{equation}
Note that, in the derivation of this, we have made use of the identity
\[(104) \quad A \times (B \times C) = B(A \cdot C) - C(A \cdot B);\]
the equation \(A'(x')n(x) \cdot n(x) = 0\) which is an easy consequence of the property \(A(x) \cdot n(x) = 0\).

It is easy to see that, for any vector \(v\), we have
\[(105) \quad v \times (\nabla \times A)(x) = (A'(x)^T - A'(x))v.\]
Rewrite the right-hand side on (103) as
\[\nu_y \times (\nabla \times A)(y) - 2 \{\nu_y \times (\nabla \times A)(y) + A'(y)\nu_y\}.\]
Then applying (105) to the second term of this, we know that (103) becomes
\[(106) \quad \nu_y \times (\nabla \times A)(y) = \nu_y \times (\nabla \times A)(y) - 2(A'(y)^T \nu_y).\]
Let \(y(\sigma)\) be an arbitrary curve on \(\partial D\) with \(y(0) = y\). We have \(A(y(\sigma)) \cdot \nu_y(\sigma) = 0\).

Differentiating this both sides with respect to \(\sigma\), we obtain,
\[A'(y)\frac{\partial y}{\partial \sigma_j} |_{\sigma=0} \cdot \nu_y = -A(y) \cdot \frac{\partial}{\partial \sigma_j} (\nu_y(\sigma)) |_{\sigma=0}.\]
Recalling the definition and symmetry of the shape operator for \(\partial D\) at \(y \in \partial D\) with respect to \(\nu_y\), we have
\[(107) \quad A'(y)^T \nu_y \cdot v = S_y(\partial D)A(y) \cdot v\]
for all tangent vectors \(v\) at \(y \in \partial D\). Since \(S_y(\partial D)A(x)\) is a tangent vector at \(y \in \partial D\) and \(A'(y)^T \nu_y \cdot v = 0\), we know that (107) is valid for all vectors of \(\mathbb{R}^3\). Thus we obtain \(A'(y)^T \nu_y = S_y(\partial D)A(y)\). Now from this and (106) we obtain
\[\nu_x \times (\nabla \times A)(x) = \nu_x \times (\nabla \times A)(x) - 2S_x(\partial D)(A(x)) \text{ on } \partial D.\]
Now from this and (100) we obtain (77).

**Proof of Claim 2.** Since \(\nabla (d_{\partial D}(x)) = n(x)\), we have \(n' = (n')^T\) and thus \((n')(x)n(x) = 0\). These yield \(C(x) = 2d_{\partial D}(x)n'A(x')\) and \(\nabla \times C(x) = 2n \times (n'A(x')) + 2d_{\partial D}(x)\nabla \times (n'A(x'))\). Thus \(\nabla \times C = 2n \times (n'A)\) on \(\partial D\). Using (104) and \((n')^T n = 0\), from this we obtain
\[n \times (\nabla \times C) = 2n \times \{n \times (n'A)\} = 2n(n \cdot n'A) - 2n'A(n \cdot n) = -2n'A.\]
Since \(-n'A = S(\partial D)A\) on \(\partial D\), we obtain (78).

**Proof of Proposition 3.** It is clear that Proposition 3 is a direct consequence of (117) and (119) in the following subsubsections.

**Computation of \((1/\mu)\nabla \times \nabla \times \tilde{V} + \tau^2 \tilde{V}\) for \(\tilde{V}\) given by (74).** We have
\[\langle \nabla \cdot \tilde{V} \rangle (x)\]
\[= -\nabla \cdot (A(x')) + \nabla \cdot (B(x'))\]
\[= -(\nabla \cdot A)(x') - 2d_{\partial D}(x)\text{Trace}(A'(x')n' (x))\]
\[+ (\nabla \cdot B)(x') - 2B'(x')n(x) \cdot n(x) - 2d_{\partial D}(x)\text{Trace}(B'(x')n'(x))\]
\[= -(\nabla \cdot V)(x') + 2 ((\nabla \cdot B)(x') - B'(x')n(x) \cdot n(x))\]
\[- 2d_{\partial D}(x)\text{Trace}(V'(x')n'(x)).\]
Since \( n' \cdot n = 0 \) and \( B' = Bn' + n \otimes \nabla B \), we have \( B' n = (\nabla B \cdot n) n \) and thus \( B'(x')n(x) \cdot n(x) = \nabla B(x') \cdot n(x) \). Since \( \nabla \cdot B = \nabla B \cdot n + B \nabla \cdot n \), we obtain
\[
(\nabla \cdot B)(x') - B'(x')n(x) \cdot n(x) = B(x') (\nabla \cdot n)(x').
\]

Since \( V \) satisfies (79), taking the rotation of both sides, one gets
\[
(108) \quad \nabla \cdot V = 0 \quad \text{in } D.
\]

From these we obtain
\[
(\nabla \cdot \tilde{V})(x) = 2B(x') (\nabla \cdot n)(x') - 2d_{\partial D}(x) \text{Trace } (V'(x') n'(x)).
\]

Further a direct computation yields
\[
\nabla \{B(x')(\nabla \cdot n)(x')\}
= (\nabla \cdot n)(x') (I - 2\pi(x)) (\nabla B)(x')
+ B(x') (I - 2\pi(x)) (\nabla (\nabla \cdot n))(x') - 2d_{\partial D}(x) n'(x) \{\nabla (B \nabla \cdot n)\}(x')
\]
and
\[
\nabla \{\text{Trace } (V'(x') n'(x))\}
= R^{2,0}(x) \nabla^2 V(x') + R^{1,0}(x) \nabla V(x') - 2d_{\partial D}(x) R^{2,1}(x) \nabla^2 V(x'),
\]
where
\[
\begin{align*}
R^{2,0}(x) \nabla^2 V(x') &= \left( \sum_{i,k,l} \delta_{ij} - 2n_i n_j \right) \frac{\partial n_k}{\partial x_l}(x) \frac{\partial^2 V^i}{\partial x_l \partial x_k}(x') , \\
R^{2,1}(x) \nabla^2 V(x') &= \left( \sum_{i,k,l} \frac{\partial n_i}{\partial x_j}(x) \frac{\partial n_k}{\partial x_l}(x) \frac{\partial^2 V^i}{\partial x_l \partial x_k}(x') \right) , \\
R^{1,0}(x) \nabla V(x') &= \left( \sum_{i,k} \frac{\partial^2 n_k}{\partial x_j \partial x_i}(x) \frac{\partial V^i}{\partial x_k}(x') \right) .
\end{align*}
\]

From these we obtain
\[
(109) \quad \nabla(\nabla \cdot \tilde{V})(x) \\
= 2(\nabla \cdot n)(x') (I - 2\pi(x)) (\nabla B)(x') \\
+ 2B(x') (I - 2\pi(x)) (\nabla (\nabla \cdot n))(x') \\
- 2n(x) \text{Trace } (V'(x') n'(x)) \\
- 2d_{\partial D}(x) Z(x, \nabla^2 V(x'), \nabla V(x'), V(x')) ,
\]
where
\[
Z(x, \nabla^2 V(x'), \nabla V(x'), V(x')) \\
= 2n(x) \{\nabla (B \nabla \cdot n)\}(x') + R^{2,0}(x) \nabla^2 V(x') + R^{1,0}(x) \nabla V(x') \\
- 2d_{\partial D}(x) R^{2,1}(x) \nabla^2 V(x') .
\]
A direct computation yields
\[
\triangle (B(x^r)) = (\nabla B)(x^r) - 2(\nabla \cdot n)(x)((\nabla B)(x^r) \cdot n(x))n(x) \\
+ 2(n'(x) - n(x'))(\nabla B)(x^r) + B(x')(\triangle n)(x) - (\triangle n)(x') \\
- 2d_{DD}(x)N(x, \nabla^2 B(x^r), \nabla B(x^r)),
\]
where
\[
N(x, \nabla^2 B(x^r), \nabla B(x^r)) = \{\text{Trace} \{(\nabla^2 B)(x^r)n'(x)\} - 2d_{DD}(x)\text{Trace}\{(n'(x))^2(\nabla^2 B)(x^r)\}n(x) \\
+ \{\nabla(\nabla \cdot n)(x) \cdot (\nabla B)(x^r)\}n(x) + 2n'(x)^2(\nabla B)(x^r).
\]
And also
\[
\triangle (A(x^r)) = (\nabla A)(x^r) - 2(\nabla \cdot n)(x)n'(x)A(x^r) \\
- 2d_{DD}(x)T(x, (\nabla^2 A)(x^r), (\nabla A)(x^r)),
\]
where the $i$-th component of $T(x, \nabla^2 A(x^r), \nabla A(x^r))$ is given by
\[
(T(x, (\nabla^2 A)(x^r), (\nabla A)(x^r)))^i = \{\text{Trace} \{(\nabla^2 A)^i(x^r)n'(x)\} + \text{Trace} \{n'(x)(\nabla^2 A^i)(x^r)\} \\
- d_{DD}(x)\text{Trace} \{(n'(x))^2(\nabla^2 A^i)(x^r)\} + (A'(x^r)(\nabla(\nabla \cdot n))(x))^i.
\]
Using (79), (108) and the formula $\nabla \times \nabla \times V = \nabla(\nabla \cdot V) - \triangle V$, we have
\[
-\frac{1}{\mu_\varepsilon} \triangle V + \tau^2 V = 0 \quad \text{in } D.
\]
From this together with (81) and (74) we have
\[
-\frac{1}{\mu_\varepsilon} \triangle \hat{V} + \tau^2 \hat{V} = -\frac{1}{\mu_\varepsilon} (\triangle \hat{V}(x) + (\triangle V)(x^r)) + \tau^2 (\triangle \hat{V}(x) + (\triangle V)(x^r)) \\
= 2 \left( -\frac{1}{\mu_\varepsilon} (\triangle B)(x^r) + \tau^2 B(x^r) \right) + R(x),
\]
where
\[
R(x) = \frac{1}{\mu_\varepsilon} \{(\triangle (A(x^r)) - (\nabla A)(x^r)) - (\triangle (B(x^r)) - (\nabla B)(x^r))\}.
\]
Here we claim the expression
\[
-\frac{1}{\mu_\varepsilon} (\triangle B)(x^r) + \tau^2 B(x^r)
\]
\[
= -\frac{1}{\mu_\varepsilon} \{n'(x^r)(\nabla B)(x^r) + B(x^r)(I - \pi(x))(\triangle n)(x^r)\} \\
- \frac{1}{\mu_\varepsilon} (2\text{Trace} (A'(x^r)n'(x^r)) + A(x^r) \cdot (\nabla n)(x^r))n(x).
\]
This is proved as follows. From (81) and (112), we have
\[
-\frac{1}{\mu_\varepsilon} \triangle B + \tau^2 B = \frac{1}{\mu_\varepsilon} \triangle A - \tau^2 A.
\]
This gives
\[
\left( -\frac{1}{\mu_\varepsilon} (\triangle B)(x^r) + \tau^2 B(x^r) \right) \cdot n(x) = \frac{1}{\mu_\varepsilon} (\triangle A)(x^r) \cdot n(x).
\]
Since $A(x) \cdot n(x) = 0$, we have
\[
\triangle A(x) \cdot n(x) = -2\text{Trace} (A'(x)(n'(x))^T) - A(x) \cdot (\triangle n)(x).
\]
Since $n(x^r) = n(x)$ and $n'(x)^T = n'(x)$, from this we obtain
\[
(\triangle A)(x^r) \cdot n(x) = -2\text{Trace} (A'(x^r)n'(x^r)) - A(x^r) \cdot (\triangle n)(x^r).
\]
Thus, we have
\[
\pi(x) \left( -\frac{1}{\mu\epsilon} (\triangle B)(x^r) + \tau^2 B(x^r) \right)
\]
\[
= -\frac{1}{\mu\epsilon} \left( 2\text{Trace} (A'(x^r)n'(x^r)) + A(x^r) \cdot (\triangle n)(x^r) \right) n(x).
\]
On the other hand, we have
\[
(I - \pi(x)) \left( -\frac{1}{\mu\epsilon} (\triangle B)(x^r) + \tau^2 B(x^r) \right)
\]
\[
= -\frac{1}{\mu\epsilon} \{ n'(x^r)(\nabla B)(x^r) + B(x^r)(I - \pi(x))(\triangle n)(x^r) \}.
\]
Summing up, we obtain (115).

Now from (109), (110), (111), (113), (114), (115) and using the formula $\nabla \times \nabla \times \hat{V} = \nabla (\nabla \cdot \hat{V}) - \triangle \hat{V}$, we obtain
\[
\frac{1}{\mu\epsilon} \nabla \times \nabla \times \hat{V}(x) + \tau^2 \hat{V}(x)
\]
\[
= L(x, \nabla V(x^r), V(x^r)) - \frac{2}{\mu\epsilon} d_{SB}(x) W(x, (\nabla^2 V)(x^r), (\nabla V)(x^r), (V)(x^r)),
\]
where
\[
L(x, \nabla V(x^r), V(x^r))
\]
\[
= \frac{2}{\mu\epsilon} \{ (\nabla \cdot n)(x^r)(I - 2\pi(x))(\nabla B)(x^r) + B(x^r)(I - 2\pi(x))(\nabla (\nabla \cdot n))(x^r) \}
\]
\[
- \frac{2}{\mu\epsilon} n(x)\text{Trace} (V'(x^r)n'(x))
\]
\[
- \frac{2}{\mu\epsilon} \{ n'(x^r)(\nabla B)(x^r) + B(x^r)(I - \pi(x))(\triangle n)(x^r) \}
\]
\[
- \frac{2}{\mu\epsilon} (2\text{Trace} (A'(x^r)n'(x^r)) + A(x^r) \cdot (\triangle n)(x^r)) n(x)
\]
\[
- \frac{2}{\mu\epsilon} (\nabla \cdot n)(x)n'(x)A(x^r)
\]
that this right-hand side consists of at most first order terms. The point is the first term on this right-hand side. From (115) and (116) we see
\[
\frac{2}{\mu \epsilon} (\nabla \cdot n)(x)((\nabla B)(x') \cdot n(x)) n(x)
\]
and
\[
W(x, \nabla^2 V(x'), \nabla V(x'), V(x'))
\]
\[
= Z(x, (\nabla^2 V)(x'), (\nabla V)(x'), (V)(x'))
\]
\[
+ T(x, (\nabla^2 A)(x'), (\nabla A)(x')) - N(x, (\nabla^2 B)(x'), (\nabla B)(x')).
\]

**Computation of** \((1/\mu \epsilon) \nabla \times \nabla \times C + \tau^2 C\) **for** \(C\) **given by** (75). We assume that \(\partial D\) is \(C^4\). Set \(d = d_{\partial D}(x)\).

Since \(\nabla \cdot C = 2d \nabla \cdot (n' A(x'))\), we have
\[
(118) \quad \nabla \cdot C = 2 \nabla \cdot (n' A(x')) + 2d \nabla \{\nabla \cdot (n' A(x'))\}.
\]

On the other hand, we have
\[
(\triangle C)^i = \sum_j \Delta \left( d \frac{\partial n^i}{\partial x_j} \right) A^j(x')
\]
\[
+ 4 \sum_j \nabla \left( d \frac{\partial n^i}{\partial x_j} \right) \cdot \nabla (A^j(x'))
\]
\[
+ 2 \sum_j d \frac{\partial n^i}{\partial x_j} \triangle (A^j(x')).
\]
The third term on this right-hand side is the \(i\)-th component of \(2d n'(x) \triangle (A(x'))\). However, by (111), this is equal to
\[
2d n'(x) (\triangle A)(x') - 4d n'(x) (\nabla \cdot n)(x) A(x') - 4d^2 n'(x) T(x, (\nabla^2 A)(x'), (\nabla A)(x')).
\]
Thus, using formula \(\nabla \times \nabla \times C = \nabla (\nabla \cdot C) - \triangle C\), we obtain
\[
\frac{1}{\mu \epsilon} \nabla \times \nabla \times C + \tau^2 C = -2d n'(x) \left( \frac{1}{\mu \epsilon} (\triangle A)(x') - \tau^2 A(x') \right)
\]
\[
+ \text{(first and zero-th order terms)}
\]
\[
+ 2d \times (\text{second, first and zero-th order terms}).
\]
The point is the first term on this right-hand side. From (115) and (116) we see that this right-hand side consists of at most first order terms.

Summing up, we obtain
\[
\frac{1}{\mu \epsilon} \nabla \times \nabla \times C + \tau^2 C
\]
\[
= \sum_{ijk} Q_{ijk}(x) \frac{\partial A^j}{\partial x_k}(x') + \sum_{ij} Q_{ij}(x) A^j(x')
\]
\[
+ 2d_{\partial D}(x) \sum_{j,k,l} R_{ijkl}(x) \frac{\partial^2 V^j}{\partial x_k \partial x_l}(x') + \sum_{j,k} R_{ijk}(x) \frac{\partial V^j}{\partial x_k}(x')
\]
\[
+ \sum_{j} R_{ij}(x) V^j(x').
\]
Note that all the coefficients are independent of $\tau$ and continuous in a tubular neighbourhood of $\partial D$, in particular, $R_{ijkl}(x)$ which come from the second order terms in (118) is $C^1$. $\square$

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*E-mail address*: ikehata@amath.hiroshima-u.ac.jp