Multiple Treatments with Strategic Interaction

Sukjin Han

University of Texas at Austin

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Model

- Semi-triangular model with multiple binary treatments
  \[ D = (D_1, \ldots, D_S) \text{ with } D_s \in \{0, 1\} \]

  \[
  Y = \mu(D, X, \epsilon_D),
  \]

  \[
  D_1 = 1 \left[ \nu^1(D_{-1}, Z_1, V_1) \geq 0 \right]
  \]

  \[
  \vdots
  \]

  \[
  D_S = 1 \left[ \nu^S(D_{-S}, Z_S, V_S) \geq 0 \right]
  \]

  where \( D_{-s} = (D_1, \ldots, D_{s-1}, D_{s+1}, \ldots, D_S) \) and \( \epsilon_D = \sum_d 1[D = d] \epsilon_d \)

  - simultaneity in the first stage (complete info game)
  - \( D \) is endogenous in \( Y \) equation
  - \( (Z, X) \perp (\epsilon_D, V_1, \ldots, V_S) \)
Parameters of Interest

\[ Y = \mu(D, X, \epsilon_D), \quad D = (D_1, \ldots, D_S) \in \{0, 1\}^S \]
\[ D_s = 1_{\nu^s(D_s, Z_s, V_s) \geq 0}, \quad s = 1, \ldots, S \]

- Parameters of interest: average structural functions (ASF)

\[ ASF(x) \equiv E[Y_d|X = x] = E[\mu(d, x, \epsilon_d)] \]

and functions of them, such as average treatment effects (ATE)

\[ ATE(x) \equiv E[Y_d - Y_{\tilde{d}}|X = x] \]

- e.g., \( d = (1, \ldots, 1) \) vs. \( \tilde{d} = (0, \ldots, 0) \); or more general nonlinear effects
- e.g., \( d = (1, d_{-s}) \) vs. \( \tilde{d} = (0, d_{-s}) \) for given \( d_{-s} \);
- or complementarity: \( E[Y_{11} - Y_{01}] > E[Y_{10} - Y_{00}] \)
Example 1

\[ Y = \mu(D, X, \epsilon_D), \quad D = (D_1, \ldots, D_S) \in \{0, 1\}^S \]
\[ D_s = 1[\nu^s(D_{-s}, Z_s, V_s) \geq 0], \quad s = 1, \ldots, S \]

- "Does media affect political participation or electoral competitiveness?"
  - \( Y \in [0, 1] \) voter turnout, or \( Y \in \{0, 1\} \) whether incumbent is re-elected
  - \( D_s \): market entry decision by local newspaper \( s \)
  - \( E[Y_d - Y_{\bar{d}}] \): effects of newspaper entry on political outcome
  - \( Z_s \): neighborhood counties’ population size and income
  - \( X \): voter ID regulation
Example 2

\[ Y = \mu(D, X, \epsilon_D), \quad D = (D_1, \ldots, D_S) \in \{0, 1\}^S \]
\[ D_s = 1[\nu^s(D_{-s}, Z_s, V_s) \geq 0], \quad s = 1, \ldots, S \]

- "Food deserts"
  - \( Y \in [0, 1] \): diabetes rate
  - \( D_s \): exit decision by supermarket \( s \)
  - \( E[Y_d - Y_{\bar{d}}] \): effects of absence of supermarkets on health
  - \( Z_s \): change in local government’s zoning plans
  - \( X \): region’s health-related variables (num of hospitals, obesity rate)
Example 3

\[ Y = \mu(D, X, \epsilon_D), \quad D = (D_1, \ldots, D_S) \in \{0, 1\}^S \]
\[ D_s = 1[\nu^s(D_{-s}, Z_s, V_s) \geq 0], \quad s = 1, \ldots, S \]

“Does airlines competition affect local air quality or health?”

- \(Y\): pollution or health outcome
- \(D_s\): entry decision by airline \(s\)
- \(E[Y_d - Y_{\bar{d}}]\): effects of competition on pollution or health
- \(Z_s\): cost shifters
- \(X\): factory pollutants emission or registered cars
Example 4

\[ Y = \mu(D, X, \epsilon_D), \quad D = (D_1, \ldots, D_S) \in \{0, 1\}^S \]
\[ D_s = 1[\nu^s(D_{-s}, Z_s, V_s) \geq 0], \quad s = 1, \ldots, S \]

- “How incumbents respond to the threat of entry by competitors?”
  - \( Y \in \{0, 1\} \): whether incumbents respond with price/investment
  - \( D_s \): entry decision by firm \( s \) in “nearby” markets
  - \( E[Y_d - Y_d] \): entry deterrence
  - \( Z_s \): cost shifters
  - \( X \): characteristics of incumbent’s market
    - e.g., in airline entry, distance btw the endpoints of incumbent’s market
Features and Challenges

\[ Y = \mu(D, X, \epsilon_D), \quad D = (D_1, \ldots, D_S) \in \{0, 1\}^S \]
\[ D_s = 1 \left[ \nu^s(D_{-s}, Z_s, V_s) \geq 0 \right], \quad s = 1, \ldots, S \]

1. **Multiple equilibria** in the first stage, thus the model as a whole is *incomplete*

2. **No distributional assumptions** on \((\epsilon_D, V_1, \ldots, V_S)\)
   - arbitrary dependence: players’ correlated type and endogenous treatments

3. Allow **small support** for excluded \(Z_s\) and \(X\)

4. Can allow \(Z_s\) to **fail to be excluded from other players**
   - e.g., allow \(Z_1 = \cdots = Z_S\)
In This Paper

1. **Partial identification** (ID) of ATE with instruments of small support
   - shape restrictions on \( \mu(\cdot) \) and \( \nu^s(\cdot) \) (e.g., monotonicity, symmetry)
   - analytical characterization of equilibrium regions for \( S > 2 \)
     - “monotonic pattern” of regions grants us LATE and MTE frameworks

2. Tighter bounds with additional \( X \) excluded from the first-stage game

3. Sharp bounds with rectangular support for \( (X, Z_1, \ldots, Z_S) \)
Related Literature

- Partial ID of treatment effects
  - Manski (1990): simple bounds with excluded IV (applicable regardless of dim of treatments)
  - Manski (1997), Manski and Pepper (2000): allow multiple treatment but no explicit simultaneity; use shape restrictions (e.g. semi-MTR)
  - **This paper**: tighter bounds for mean by specifying “selection process” with different shape restrictions and existence of $X$

- ID and Partial ID in triangular models with discrete endogenous variables
  - Shaikh and Vytlacil (2011), Chesher (2005): single treatment
  - Heckman, Urzua & Vytlacil (2006), Jun, Pinkse and Xu (2011): allow multiple treatments, but single agent and no simultaneity
  - **This paper**: multi-treatment, multi-agent models with strategic interaction among agents
Related Literature

- Partial ID of treatment response with social interaction
  - Manski (2013): multiple treatments but interaction through agents’ response variables
  - **This paper**: interaction through treatment decisions

- Discrete games with complete information
  - Berry (1992), Tamer (2003), Ciliberto and Tamer (2009): partial ID in general, point ID under large support
  - Ciliberto, Murry and Tamer (2015): entry decision + pricing decision (upon entry)
  - **This paper**: the effect of game’s equilibrium on a second-stage outcome; different approach to solve multiplicity

- Network formation and its effects on outcomes
  - Gilleskie and Zheng (2010), Badev (2013), Goldsmith-Pinkham and Imbens (2013), Hsieh and Lee (2016)
Plan of Talk

- Model and parameters of interest
- Point ID
- Partial ID
  - two-player case
  - many-player case
- Simulation
- Conclusion
Model and Parameters of Interest

- For this talk, $X$ and $Z_s$ scalar, $Y \in \{0, 1\}$, and $\epsilon_d = \epsilon$ for simplicity

\[ Y_d = 1[\mu(d, X) \geq \epsilon] \]
\[ D_s = 1[\nu^s(D_s, Z_s) \geq U_s] \]

where $U_s \sim Unif[0, 1]$ as normalization and $Y = \sum_d 1[D = d]Y_d$

- all the analyses (except sharpness) can be extended to cases of other LDV or continuous $Y$ (Vytlacil and Yildiz (2007))

- rank similarity can be assumed instead

- can allow $Z_1 = \cdots = Z_S$ instead

- when $S = 2$ and $D = (D_1, D_2)$,

\[ Y = 1[\mu(D_1, D_2, X) \geq \epsilon] \]
\[ D_1 = 1[\nu^1(D_2, Z_1) \geq U_1] \]
\[ D_2 = 1[\nu^2(D_1, Z_2) \geq U_2] \]

- Parameters of interest: partial ATE, $E[Y_{d^1, D^2} - Y_{\bar{d}^1, D^2}]$ for some partition $d = (d^1, d^2)$ (more in the paper)
Point ID under Full Support

\[ Y_d = 1[\mu(d, X) \geq \epsilon] \]
\[ D_s = 1[\nu^s(D_{-s}, Z_s) \geq U_s] \]

Assumption IN

\((X, Z) \perp (\epsilon_d, U) \ \forall d, \text{ where } U \equiv (U_1, \ldots, U_S).\)

- Under additional exclusion restriction with full support of player-specific instruments, \(E[Y_d|X]\) is point ID’ed
- “ID at infinity” solves multiple equilibria problem and endogeneity, simultaneously
- But we want to avoid full support condition
Partial ID without Full Support

\[ Y_d = 1[\mu(d, X) \geq \epsilon] \]
\[ D_s = 1[\nu^s(D_s, Z_s) \geq U_s] \]

Assumption M

For given \( x \), either \( \mu(1, d_s, x) \geq \mu(0, d_s, x) \ \forall d_s \),

or \( \mu(1, d_s, x) \leq \mu(0, d_s, x) \ \forall d_s \).

- M is mild monotonicity requirement
- No need to know the direction
  - cf. Manski (1997), Manski and Pepper (2000)

Assumption SS (Strategic Substitute)

\( \nu^s(d_s, z_s) \) is strictly decreasing in each element of \( d_s \).
Partial ID without Full Support

Assumption SY (Symmetry)

\[(i) \mu(d, \cdot) = \mu(\tilde{d}, \cdot) \text{ for any permutation } \tilde{d} \text{ of } d; \]
\[(ii) \nu^s(d_{-s}, \cdot) = \nu^s(\tilde{d}_{-s}, \cdot) \text{ for any permutation } \tilde{d}_{-s} \text{ of } d_{-s}. \]

- SY is useful to make the incomplete model tractable
- SY(i) can be relaxed (at the expense of having wider bounds)
- SY(ii) trivially holds in two-player case
- SY(ii) becomes useful with many players; related to “anonymity” assumption in games (e.g., Berry (1992), Kalai (2004), Menzel (2016))
  - still allows heterogeneity via nonseparability in \( \nu^s(d_{-s}, z_s) \)
Theorem

Suppose Assumptions IN, M, SS, and SY hold. The upper and lower bounds of ASF and ATE with \( d, \tilde{d} \in \{0, 1\}^S \) is

\[
L_d(x) \leq E[Y_d|X = x] \leq U_d(x)
\]

and

\[
L_d(x) - U_{\tilde{d}}(x) \leq E[Y_d - Y_{\tilde{d}}|X = x] \leq U_d(x) - L_{\tilde{d}}(x)
\]

where, for some \( d^\dagger \in \{0, 1\}^S \),

\[
U_{d^\dagger}(x) = \inf_z \left\{ \Pr[Y = 1, D = d^\dagger|Z = z, X = x] + \sum_{d' \neq d^\dagger} \inf_{x' \in \mathcal{X}_U(x, d^\dagger, d')} \Pr[Y = 1, D = d'|Z = z, X = x'] \right\}
\]

\[
L_{d^\dagger}(x) = \sup_z \left\{ \Pr[Y = 1, D = d^\dagger|Z = z, X = x] + \sum_{d' \neq d^\dagger} \sup_{x' \in \mathcal{X}_L(x, d^\dagger, d')} \Pr[Y = 1, D = d'|Z = z, X = x'] \right\}
\]
Benchmark: Classic Manski Bounds

- Consider

\[ E[Y_d|X] = E[Y_d|Z, X] = E[Y_d|D = d, Z, X] \Pr[D = d|Z, X] \]

\[ + \sum_{d' \neq d} E[Y_d|D = d', Z, X] \Pr[D = d'|Z, X] \]

- 1st eq. by exclusion restriction of \( Z \)

- \( E[Y_d|D = d', Z, X] = \Pr[Y_d = 1|D = d', Z, X] \) not observed, can be bounded between 0 and 1

- Can narrow the bound by taking \( \max_z \) over LB and \( \min_z \) over UB (Manski (1990))
Proof for Tighter Bounds

- The goal is to derive bounds tighter than above for
  \[ E[Y_d|D = d', Z, X] \]
  by fully exploiting the structure of the model (esp. the “selection process”)

- Suppose \( S = 2 \) for simplicity

- Suppose we know the direction of monotonicity in \( M \). Then e.g., we have tighter UB

  \[
  \Pr[Y_{00} = 1|D = (1, 0), Z = z, X = x] \\
  = Pr[\mu(0, 0, x) \geq \epsilon|D = (1, 0), Z = z, X = x] \\
  \leq Pr[\mu(1, 0, x) \geq \epsilon|D = (1, 0), Z = z, X = x] \\
  = \Pr[Y_{10} = 1|D = (1, 0), Z = z, X = x] \leq 1
  \]

- Below, we construct quantities from which we can identify the direction of the monotonicity
Key Lemma: Using Variation of $Z$

For $z \neq z'$, define

\[
\begin{align*}
    h_{11}(z, z', x) &\equiv \Pr[Y = 1, D = (1, 1)|Z = z, X = x] \\
                   &- \Pr[Y = 1, D = (1, 1)|Z = z', X = x] \\
    h_{00}(z, z', x) &\equiv \Pr[Y = 1, D = (0, 0)|Z = z, X = x] \\
                   &- \Pr[Y = 1, D = (0, 0)|Z = z', X = x] \\
    h_M(z, z', x) &\equiv \Pr[Y = 1, D \in \{(1, 0), (0, 1)\}|Z = z, X = x] \\
                 &- \Pr[Y = 1, D \in \{(1, 0), (0, 1)\}|Z = z', X = x]
\end{align*}
\]

- by Assumption SS, $\{(1, 0), (0, 1)\}$ are values of $D$ possibly predicted by multiple equilibria

- $h_{11}$, $h_{00}$, and $h_M$ can be directly recovered from data

- $Z$ changes the equilibrium behavior in the first stage without directly affecting $Y$
Key Lemma: Using Variation of $Z$

- Define
  
  $$h(z, z', x) \equiv \Pr[Y = 1|Z = z, X = x] - \Pr[Y = 1|Z = z', X = x]$$

  which satisfies $h \equiv h_{11} + h_{00} + h_M$

- Also define
  
  $$h_{11}^D(z, z) \equiv \Pr[D = (1, 1)|Z = z] - \Pr[D = (1, 1)|Z = z']$$
  
  and
  
  $$h_{00}^D(z, z') \equiv \Pr[D = (0, 0)|Z = z] - \Pr[D = (0, 0)|Z = z']$$

- $h_{11}^D$ and $h_{00}^D$ can also be recovered from data

- can determine volume change of the equilibrium regions at $Z$
Key Lemma: Using Variation of $Z$

- Let $\mu_{d_1d_2}(x) \equiv \mu(d_1, d_2, x)$ for simplicity

Lemma

Under Assumptions of Theorem, for any $(z, z', x)$ such that $h_{11}^D(z, z')$ and $h_{00}^D(z, z')$ are nonzero,

$$\text{sgn} \left\{ h(z, z', x) \right\} = \text{sgn} \left\{ h_{11}^D(z, z') \right\} \cdot \text{sgn} \left\{ \mu_{11}(x) - \mu_{01}(x) \right\}$$

$$= \text{sgn} \left\{ -h_{00}^D(z, z') \right\} \cdot \text{sgn} \left\{ \mu_{10}(x) - \mu_{00}(x) \right\}.$$ 

- Given the lemma, we can derive tighter bounds
  - e.g., $h \geq 0$ and $-h_{00}^D > 0$ imply $\mu_{10}(x) \geq \mu_{00}(x)$, thus we can use the bound previously shown
  - more generally, can exploit full variation of $Z$
  - extending the idea of Shaikh and Vytlacil (2011) and Vytlacil and Yildiz (2007) to multi-agent, incomplete model setup
Proof of Key Lemma

- Let $R_{11}(z)$, $R_{10}(z)$, $R_{01}(z)$ and $R_{00}(z)$ denote equilibrium regions of $U \equiv (U_1, U_2)$

- By IN and the model, can show that

$$h_{11}(z, z', x) \equiv \Pr[Y = 1, D = (1, 1)|Z = z, X = x]$$

$$- \Pr[Y = 1, D = (1, 1)|Z = z', X = x]$$

$$= \Pr[\epsilon \leq \mu_{11}(x), U \in R_{11}(z)]$$

$$- \Pr[\epsilon \leq \mu_{11}(x), U \in R_{11}(z')]$$

- Below,
  - let $\nu^s(d_{-s}, z_s) \equiv \nu^s_{d_{-s}}(z_s)$ for simplicity
  - suppose $h^D_{11}(z, z') > 0$ and $-h^D_{00}(z, z') > 0$ for expositional purpose
Proof of Key Lemma

When $Z = z$:
Proof of Key Lemma

When $Z = z$:

\[
\begin{align*}
\nu_1^1(z_1), & \quad \nu_0^2(z_2) \\
\nu_0^0(z_1), & \quad \nu_1^2(z_2) \\
R_{11}(z)
\end{align*}
\]
Proof of Key Lemma

When $Z = z'$:

\[ R_{11}(z') \]

\[ \nu^1_1(z'_1), \nu^2_0(z'_2) \]

\[ \nu^1_0(z'_1), \nu^2_1(z'_2) \]
Proof of Key Lemma

$Z = z$ vs. $Z = z'$:
Proof of Key Lemma

Therefore

\[ h_{11}(z, z', x) = \Pr[\epsilon \leq \mu_{11}(x), U \in R_{11}(z)] \]

\[ - \Pr[\epsilon \leq \mu_{11}(x), U \in R_{11}(z')] \]

\[ = \Pr[\epsilon \leq \mu_{11}(x), U \in \Delta_+ R_{11}] \]

Recall

\[ h_M(z, z', x) \equiv \Pr[Y = 1, D \in \{(1, 0), (0, 1)\} | Z = z, X = x] \]

\[ - \Pr[Y = 1, D \in \{(1, 0), (0, 1)\} | Z = z', X = x] \]

Then, using symmetry that \( \mu_{10} = \mu_{01} \) (SY(i))

\[ h_M(z, z', x) = \Pr[\epsilon \leq \mu_{10}(x), U \in R_{10}(z) \cup R_{01}(z)] \]

\[ - \Pr[\epsilon \leq \mu_{10}(x), U \in R_{10}(z') \cup R_{01}(z')] \]
Proof of Key Lemma

When $Z = z$:

\[ R_{01}(z) \]

\[ R_{10}(z) \]

\[ \nu_0^1(z_1), \nu_0^2(z_2) \]

\[ \nu_1^1(z_1), \nu_1^2(z_2) \]
Proof of Key Lemma

When $Z = z'$:
Proof of Key Lemma

$Z = z$ vs. $Z = z'$:
Proof of Key Lemma

- Therefore,

\[ h_M(z, z', x) = \Pr[\epsilon \leq \mu_{10}(x), U \in R_{10}(z) \cup R_{01}(z)] \]

\[ - \Pr[\epsilon \leq \mu_{10}(x), U \in R_{10}(z') \cup R_{01}(z')] \]

\[ = \Pr[\epsilon \leq \mu_{10}(x), U \in \Delta_{-} R_{00}] \]

\[ - \Pr[\epsilon \leq \mu_{10}(x), U \in \Delta_{+} R_{11}] \]

- Also can show

\[ h_{00}(z, z', x) = - \Pr[\epsilon \leq \mu_{00}(x), \in \Delta_{-} R_{00}] \]
Proof of Key Lemma

Therefore,

\[
h(x) = h_{11}(x) + h_{00}(x) + h_M(x)
\]

\[= \Pr[\epsilon \leq \mu_{11}(x), U \in \Delta_+ R_{11}] + \Pr[\epsilon \leq \mu_{10}(x), U \in \Delta_- R_{00}] - \Pr[\epsilon \leq \mu_{10}(x), U \in \Delta_+ R_{11}] - \Pr[\epsilon \leq \mu_{00}(x), U \in \Delta_- R_{00}]\]

Now, \(\mu_{11}(x) \geq \mu_{10}(x) = \mu_{01}(x)\) and \(\mu_{10}(x) \geq \mu_{00}(x)\) (same directions by M) iff

\[
h(x) = \Pr[\mu_{10}(x) \leq \epsilon \leq \mu_{11}(x), U \in \Delta_+ R_{11}] + \Pr[\mu_{00}(x) \leq \epsilon \leq \mu_{10}(x), U \in \Delta_- R_{00}]\]

which is non-negative (sum of two probs)

- Conversely, \(\mu_{11}(x) < \mu_{10}(x)\) and \(\mu_{10}(x) < \mu_{00}(x)\) iff \(h < 0\)
Discussions

1. Exploiting additional variation from $X$: e.g., can show if, for $x \neq x'$

$$h(x') \equiv h_{11}(x') + h_M(x') + h_{00}(x') \geq 0$$
$$h(x, x', x') \equiv h_{11}(x') + h_M(x') + h_{00}(x) < 0$$

then

$$\mu_{00}(x) \geq \mu_{10}(x')$$

2. Sharp bounds with $\text{supp}(X, Z) = \text{supp}(X) \times \prod_{s=1}^{S} \text{supp}(Z_s)$ for binary $Y$

3. $SY(i)$ can be relaxed given knowledge on

$$\mu_{10}(\cdot) \succeq \mu_{01}(\cdot)$$

- if $\mu_{10}(\cdot) = \mu_{01}(\cdot)$ is implausible, then it can be because the sign of the ineq. is known

- the bounds on ATE are not as tight as the ones under symmetry
The General Case of Many Players \((S > 2)\)

- As seen in \(S = 2\) case, need to fully characterize equilibrium regions
- For \(j = 0, \ldots, S - 1\), define
  \[
e^j \equiv (e^j_1, \ldots, e^j_S) \equiv (1, \ldots, 1, 0, \ldots, 0)_{j \to S-j}
  \]
- Given \(d^j = (\sigma(e^j_1), \ldots, \sigma(e^j_S))\) for some permutation function \(\sigma(\cdot)\),
  \[
  R_{d^j}(z) \equiv \left\{ U : (U_{\sigma(1)}, \ldots, U_{\sigma(S)}) \in \prod_{s=1}^{j} \left( 0, \nu_{\tilde{\sigma}_{s-1}}^{\sigma(s)}(Z_{\sigma(s)}) \right) \times \prod_{s=j+1}^{S} \left( \nu_{\tilde{\sigma}_j}^{\sigma(s)}(Z_{\sigma(s)}), 1 \right) \right\}
  \]
- Can similarly define \(R_{d0}\) and \(R_{dS}\)
The General Case of Many Players ($S > 2$)

- The region of all equilibria with $j$ entrants is denoted as
  \[
  \bar{R}_j(z) \equiv \bigcup_{d \in M_j} R_d(z)
  \]
  where $M_j$ is a collection of all equilibria with $j$ entrants
  - with $S = 3$, e.g., $\bar{R}_2 = R_{110} \cup R_{101} \cup R_{011}$
Visual Illustration with $S = 3$

- $R_{000}$
- $R_{111}$
- $U_1$
- $U_2$
- $U_3$
- $(0, 0, 0)$
- $(\nu_{11}^1, \nu_{11}^2)$
- $(\nu_{00}^1, \nu_{00}^2)$
- $\nu_{00}^3$
- $(1, 1, 1)$
Visual Illustration with $S = 3$
Visual Illustration with $S = 3$

\[ \bar{R}_1 = R_{010} \cup R_{100} \cup R_{001} \]
Characterization of Equilibrium Regions

Proposition

Under Assumptions SS and SY(ii), the following holds:
(i) $\bar{R}_j \cap \bar{R}_{j'} = \emptyset$ for $j, j' = 0, \ldots, S$ with $j \neq j'$;
(ii) $\bar{R}_j$ and $\bar{R}_{j-1}$ are neighboring sets for $j = 1, \ldots, S$;
(iii) $\bar{R}_j$ and $\bar{R}_{j-t}$ are not neighboring sets for $t \geq 2$ and $j = t, \ldots, S$;
(iv) $\bigcup_{j=0}^{S} \bar{R}_j = (0, 1)^S$.

- By (i), unique equilibrium in terms of number of entrants
  - similar result as in Berry (1992) but under weaker assumptions
- (ii) and (iii) are important in equating “inflow” and “outflow”
- (i) and (iv) imply that $\bar{R}_j$ for $j = 1, \ldots, S$ partition the entire space $(0, 1)^S$ of $U$
Visual Illustration with $S = 3$

$R_{010} \cup R_{100}$

$(\nu_1^{10}, \nu_1^{10}) (\nu_0^{10}, \nu_0^{10})$

$(\nu_1^{11}, \nu_1^{11})$

$\nu_0^{10} = \nu_0^{11}$

$\nu_0^{3}$

$U_1$

$U_2$

$U_3$
Visual Illustration with $S = 3$
Visual Illustration with $S = 3$

$\bigcup_{j=0}^{3} \bar{R}_j = (0, 1)^3$

$\bar{R}_0 = R_{000}$
$\bar{R}_1 = R_{010} \cup R_{100} \cup R_{001}$
$\bar{R}_2 = R_{011} \cup R_{101} \cup R_{110}$
$\bar{R}_3 = R_{111}$
Multiple equil. w/ 1 entrant
Multiple equil. w/ 2 entrant
Key Lemma: Using Variation of $Z$

- Based on Proposition, define, for $z \neq z'$,

$$h_j(z, z', x) = \Pr[Y = 1, D \in M_j | Z = z, X = x]$$

$$- \Pr[Y = 1, D \in M_j | Z = z', X = x].$$

- Recall,

$$h(z, z', x) = \Pr[Y = 1 | Z = z, X = x]$$

$$- \Pr[Y = 1 | Z = z', X = x]$$

- since $M_j$ are disjoint, $h(z, z', x) = \sum_{j=0}^{S} h_j(z, z', x)$

- Also define

$$h_j^D(z, z') = \Pr[D \in M_j | Z = z]$$

$$- \Pr[D \in M_j | Z = z'].$$
Key Lemma: Using Variation of $Z$

**Lemma**

*Under Assumptions of Theorem, for any $(z, z', x)$ such that $\sum_{k=j}^{S} h^D_k(z, z') \neq 0$ and $j = 1, \ldots, S$,*

$$\text{sgn} \{ h(z, z', x) \} = \text{sgn} \left\{ \sum_{k=j}^{S} h^D_k(z, z') \right\} \cdot \text{sgn} \{ \mu_{e j}(x) - \mu_{e j-1}(x) \}.$$ 

- In practice, for efficiency, we determine the sign of $\mu_{e j}(x) - \mu_{e j-1}(x)$ by the sign of

$$H_j(x) \equiv E \left[ h(Z, Z', x) \left| \sum_{k=j}^{S} h^D_k(z, z') > 0 \right. \right]$$
Discussions

4. Again, symmetry in $\mu_d(\cdot)$ can be relaxed if $\mu_d(\cdot)$ can be ordered within $d \in M_j$

5. Based on Proposition, versions of LATE (and MTE) can be considered in the spirit of Vytlacil (2002)
   - cf. Angrist & Imbens (1995), Heckman, Urzua & Vytlacil (2006)
   - e.g., IV estimand has LATE interpretation: with a common binary $Z$,
     \[
     \frac{h(z, z')}{\sum_{k=j}^S h_k^D(z, z')} = \frac{\Pr[Y = 1|Z = z] - \Pr[Y = 1|Z = z']}{\Pr[D \in M^\geq j|Z = z] - \Pr[D \in M^\geq j|Z = z']} = E[Y_{M^\geq j} - Y_{M_{j-1}} | D(z) \in M^\geq j, D(z') \in M^{j-1}]
     \]
     e.g., when $S = 2$,
     \[
     E[Y_{11} - Y_{\{(1,0),(0,1),(0,0)\}} | D(z) = (1,1), D(z') \in \{(1,0), (0,1), (0,0)\}]
     \]
LATE subgroup

\( Z = z \) vs. \( Z = z' \):

\[
U_2 \quad \text{0} \quad \Delta + R_{11} \quad U_1 \quad 1
\]
Monte Carlo Simulation

- DGP:

\[
Y_d = 1\{\tilde{\mu}_d + \beta X \geq \epsilon\}
\]

\[
D_1 = 1\{\delta_2 D_2 + \gamma_1 Z_1 \geq V_1\}
\]

\[
D_2 = 1\{\delta_1 D_1 + \gamma_2 Z_2 \geq V_2\}
\]

- \((\epsilon, V_1, V_2)\) drawn from joint normal, mean zero, indep. of \((X, Z)\)
- \(Z \in \{-1, 1\}\) and \(X \in \{-1, 0, 1\}\) or \(X \in \{-1, -\frac{6}{7}, -\frac{5}{7}, \ldots, \frac{5}{7}, \frac{6}{7}, 1\}\)
  drawn from multinomial
- \(\delta_1 < 0\) and \(\delta_2 < 0\) (SS)

- **Design 1:** \(\tilde{\mu}_{11} > \tilde{\mu}_{10} = \tilde{\mu}_{01} > \tilde{\mu}_{00}\) (consistent with M and SY)

- **Design 2:** \(\tilde{\mu}_{11} > \tilde{\mu}_{01}, \tilde{\mu}_{11} > \tilde{\mu}_{10}, \tilde{\mu}_{01} > \tilde{\mu}_{00},\) and \(\tilde{\mu}_{10} > \tilde{\mu}_{00}\), but \(\tilde{\mu}_{10} \neq \tilde{\mu}_{01}\) (consistent with M)

- Parameter

\[
ATE(0) = E[Y_{11} - Y_{00} | X = 0]
\]
Design 1: Variation of $Z$

Bounds with $\delta = 0.1$ and $\beta = 1$
Design 1: Additional $X$ with $|\text{supp}(X)| = 3$
Design 1: Additional $X$ with $|\text{supp}(X)| = 15$
Design 2: Additional $X$ with $|\text{supp}(X)| = 15$
Conclusions and Future Works

- Partial ID of ATE in a model for heterogeneous effects of multiple treatments, where the treatments are equilibrium of a complete info game
  - no distributional assumptions and allow arbitrary dependence
  - shape restrictions and excluded variables with small supports
  - symmetry can be relaxed with some knowledge on effectiveness of each treatment

- Inference
  - modification of Andrews and Shi (2013), Chernozhukov, Lee and Rosen (2013), Armstrong and Chan (2015)
  - possible first step: Armstrong and Shen (2015)

- Empirical works: 1. pollution with airline entry; 2. entry deterrence
  - data is collected from Dept. of Transportation and US Environmental Protection Agency
Exploiting Additional Variation from $X$

- Analogously define

$$\mathcal{X}_- \equiv \{ x' : h(x') < 0 \}$$
$$\mathcal{X}_{2+}(x) \equiv \{ x' : h(x, x', x') \geq 0 \}$$
$$\mathcal{X}_{1+}(x) \equiv \{ x' : h(x, x, x') \geq 0 \}$$

- Then, similarly as above, the UB’s are

$$\Pr[Y_{00} = 1 | D = (1, 0), Z, X = x] \leq \inf_{x' \in \mathcal{X}_- \cap \mathcal{X}_{2+}(x)} \Pr[Y_{10} = 1 | D = (1, 0), Z, X = x']$$
$$\Pr[Y_{00} = 1 | D = (0, 1), Z, X = x] \leq \inf_{x' \in \mathcal{X}_- \cap \mathcal{X}_{2+}(x)} \Pr[Y_{01} = 1 | D = (0, 1), Z, X = x']$$
$$\Pr[Y_{00} = 1 | D = (1, 1), Z, X = x] \leq \inf_{x' \in \mathcal{X}_{1+}(x''), x'' \in \mathcal{X}_- \cap \mathcal{X}_{2+}(x)} \Pr[Y_{11} = 1 | D = (1, 1), Z, X = x']$$

- Can analogously derive LB’s and UB’s for $\Pr[Y_{11} = 1 | D = d', Z, X]$ for $d' \in \{(0, 0), (1, 0), (0, 1)\}$

- Lastly, take $\sup_z$ and $\inf_z$
Relaxing Symmetry

Since the equilibrium selection rule is unknown,

$$h_{10} + h_{01} = \Pr[\epsilon \leq \mu_{10}, U \in R_{10}^*(z)] + \Pr[\epsilon \leq \mu_{01}, U \in R_{01}^*(z)]$$

$$- \Pr[\epsilon \leq \mu_{10}, U \in R_{10}^*(z')] - \Pr[\epsilon \leq \mu_{01}, U \in R_{01}^*(z')]$$

which, following Tamer (2003), has a LB of

$$\geq \Pr[\epsilon \leq \mu_{10}, U \in R_{10}^{\min}(z)] + \Pr[\epsilon \leq \mu_{01}, U \in R_{01}^{\min}(z)]$$

$$- \Pr[\epsilon \leq \mu_{10}, U \in R_{10}^{\max}(z')] - \Pr[\epsilon \leq \mu_{01}, U \in R_{01}^{\max}(z')]$$

but, following our approach, has a LB of

$$\geq \Pr[\epsilon \leq \mu_{10} \land \mu_{01}, U \in R_{10}^*(z)] + \Pr[\epsilon \leq \mu_{10} \land \mu_{01}, U \in R_{01}^*(z)]$$

$$- \Pr[\epsilon \leq \mu_{10} \lor \mu_{01}, U \in R_{10}^*(z')] - \Pr[\epsilon \leq \mu_{10} \lor \mu_{01}, U \in R_{01}^*(z')]$$

and $R_{10}^* \cup R_{01}^* = R_{10} \cup R_{01}$