Isotropic Scattering in a Flatland Half-Space

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\textbf{ABSTRACT}

We solve the Milne, constant-source, and albedo problems for isotropic scattering in a two-dimensional “Flatland” half-space via the Wiener–Hopf method. The Flatland $H$-function is derived and benchmark values and some identities unique to Flatland are presented. A number of the derivations are supported by Monte Carlo simulation.

\textbf{KEYWORDS}

Wiener–Hopf; Flatland; $H$-function; Milne problem

1. Introduction

The study of linear transport theory (Davison 1957; Chandrasekhar 1960) in lower-dimensional spaces serves a number of purposes. The simplicity of the one-dimensional rod model (Wing 1962) makes it a useful tool for education (Hoogenboom 2008) and occasionally the starting place for exploring new general transport processes (Adams, Larsen, and Pomraning 1989). Most rod model problems can be solved exactly and admit simple closed-form solutions, with diffusion encompassing the entire solution. These properties are attractive, but distance the rod model from the complexity of three-dimensional transport and therefore also limit its utility.

Sandwiched between the rod model and traditional three-dimensional scattering, two-dimensional “Flatland” provides a transport domain with much of the complexity of full 3D scattering, while occasionally admitting simple closed-form solutions that have not been found in 3D (interestingly, the time-resolved Green’s functions for the isotropic point source in infinite media are known exactly for 2D and 4D, but not 3D; Paasschens 1997).

Because Flatland transport research in bounded media has led to insights that improve the efficiency of 3D light transport (albeit so far participating media has not been considered; Jarosz et al. 2012), we solve the classic half space problems in Flatland for isotropic scattering and investigate the form of the Flatland $H$ function and some of its numerical properties. These derivations may aid future studies of this form in many fields. These solutions
may also directly apply to physical processes where the transport is fundamentally two-dimensional (Yang and Deb 2009; Bal et al. 2000; Meylan and Masson 2006; Vynck et al. 2012).

1.1. Related work

Infinite media problems have been well studied in Flatland as well as spaces of general dimension (Pearson 1905; Kluyver 1906; Rayleigh 1919; Grosjean 1953; Watson 1962; Paasschens 1997; Liemert and Kienle 2011; Zoia, Dumonteil, and Mazzolo 2011; d’Eon 2014) and for beams (Asadzadeh and Larsen 2008). Some exact solutions have been presented for bounded (Liemert and Kienle 2012a) and layered (Liemert and Kienle 2012b) media, and the singular eigenfunctions for Flatland have been derived (Machida 2016). Bal et al. (2000), using an asymptotic analysis, have presented solutions to the classic Milne and albedo problems in two dimensions. The solutions make use of the Flatland equivalent of Chandrasekhar’s $H$-function, which is left as the solution to an integral equation. We present a complementary derivation of the Milne and albedo problem solutions using the Wiener–Hopf technique. In addition, we present new solutions and benchmark values for $H$ and provide some Monte Carlo comparisons for the albedo problem.

In the study of energy-dependent neutron transport in three-dimensional volumes with plane symmetry, Stewart, Kuščer, and McCormick (1966) presented a general family of solutions to the Milne problem using the method of singular eigenfunctions. When their variable-$c$ factor $c(\xi)$ takes on the specific quantity (in our notation) $\frac{2c}{\pi \sqrt{1-\xi^2}}$, their energy-dependent 3D solution becomes equivalent to our monoenergetic Flatland solution. Thus, the Flatland $H$-function has, if only inadvertently, been presented long ago.

2. General theory

The Flatland one-speed transport equation can be written (Asadzadeh and Larsen 2008) as

$$\cos \theta \frac{\partial \phi(x, y, \theta)}{\partial x} + \sin \theta \frac{\partial \phi(x, y, \theta)}{\partial y} + \phi(x, y, \theta) = \frac{c}{2\pi} \int_{-\pi}^{\pi} d\theta' \phi(x, y, \theta') + \frac{1}{2\pi} S(x, y),$$

(1)

where $c = \Sigma_t / \Sigma$ and the notation is standard. The two-dimensional analog of angular flux (or radiance) (Jarosz et al. 2012) is denoted $\phi$. If we assume that there is spatial variation in only the $x$-direction we find
\[
\cos \theta \frac{\partial \phi(x, \theta)}{\partial x} + \phi(x, \theta) = \frac{c}{2\pi} \int_{-\pi}^{\pi} d\theta' \phi(x, \theta') + \frac{1}{2\pi} S(x). \tag{2}
\]

We change the angular variable in Equation (2) such that \( \mu = \cos \theta \), which leads to
\[
\left( \mu \frac{\partial}{\partial x} + 1 \right) \phi(x, \mu) = \frac{c}{2\pi} 2 \int_{-1}^{1} \frac{d\mu'}{\sqrt{1-\mu'^2}} \phi(x, \mu') + \frac{1}{2\pi} S(x). \tag{3}
\]

For the sake of completeness, we now convert Equation (3) to integral form for the scalar flux
\[
\phi_0(x) = 2 \int_{-1}^{1} \frac{d\mu'}{\sqrt{1-\mu'^2}} \phi(x, \mu'). \tag{4}
\]

Rearranging Equation (3) as
\[
\frac{\partial}{\partial x} \left( \phi(x, \mu)e^{x/\mu} \right) = \frac{1}{2\pi \mu} (c\phi_0(x) + S(x))e^{x/\mu} \tag{5}
\]
and for \( \mu > 0 \) let us integrate from 0 to \( x \), viz:
\[
\phi(x, \mu) = \phi(0, \mu)e^{-x/\mu} + \frac{1}{2\pi \mu} \int_{0}^{x} dx' \left( c\phi_0(x') + S(x') \right)e^{-(x-x')/\mu}; \mu > 0. \tag{6}
\]

Assuming that we have a finite slab of width \( a \) we can now integrate Equation (5) from \( x \) to \( a \) for \( \mu < 0 \), thus
\[
\phi(x, \mu) = \phi(a, \mu)e^{(a-x)/\mu} - \frac{1}{2\pi \mu} \int_{x}^{a} dx' \left( c\phi_0(x') + S(x') \right)e^{(x-x')/\mu}; \mu < 0. \tag{7}
\]

We now find the scalar flux from Equation (4) as
\[
\phi_0(x) = 2 \int_{0}^{1} \frac{d\mu}{\sqrt{1-\mu^2}} \phi(0, \mu)e^{-x/\mu} + \frac{1}{2\pi} \int_{0}^{x} dx' \left( c\phi_0(x') + S(x') \right)2 \int_{-1}^{1} \frac{d\mu'}{\sqrt{1-\mu'^2}} e^{-(x-x')/\mu}
+ 2 \int_{-1}^{0} \frac{d\mu}{\sqrt{1-\mu^2}} \phi(a, \mu)e^{(a-x)/\mu} - \frac{1}{2\pi} \int_{x}^{a} dx' \left( c\phi_0(x') + S(x') \right)2 \int_{0}^{1} \frac{d\mu}{\sqrt{1-\mu^2}} e^{(x-x')/\mu}. \tag{8}
\]

Equation (8) reduces to
\[
\phi_0(x) = 2 \int_{0}^{1} \frac{d\mu}{\sqrt{1-\mu^2}} \phi(0, \mu)e^{-x/\mu} + 2 \int_{0}^{1} \frac{d\mu}{\sqrt{1-\mu^2}} \phi(a, \mu)e^{-(a-x)/\mu}
+ \frac{1}{\pi} \int_{0}^{x} dx' \left( c\phi_0(x') + S(x') \right)\int_{0}^{1} \frac{d\mu}{\mu \sqrt{1-\mu^2}} e^{-|x-x'|/\mu}. \tag{9}
\]
The last integral above reduces to
\[ \int_0^1 \frac{d\mu}{\mu \sqrt{1-\mu^2}} e^{-|x-x'|/\mu} = K_0(|x-x'|). \]  

(10)

In Equation (9), \( \phi(0, \mu) \) and \( \phi(a, \mu) \) are the incident fluxes on the faces 0 and \( a \) respectively and we may write it in general form as
\[ \phi_0(x) = I_0(x) + I_a(x) + \frac{1}{2\pi} \int_0^a dx' \left( c \phi_0(x') + S(x') \right) K_0(|x-x'|). \]  

(11)

We now consider three classic problems in turn; Milne, albedo and constant source.

3. Milne problem

The Milne problem is a special case of the above equations. Namely, when \( a = 1 \), \( I_0(x) = I_a(x) = S(x) = 0 \). However, in order to solve the problem using the Wiener–Hopf technique we will return to the integro-differential Equation (3) with \( S = 0 \) and the boundary condition \( \phi(0, \mu) = 0; \mu > 0 \), i.e., there is no incident current. Also, it is assumed by definition of the Milne problem that there is a continuous supply of particles from infinity that eventually leak out of the surface. The equation to solve is
\[ \left( \mu \frac{\partial}{\partial x} + 1 \right) \phi(x, \mu) = c \frac{\phi_0(x)}{2\pi}; \quad \phi(0, \mu) = 0, \mu > 0 \]  

(12)

We define the Laplace transform with complex argument \( s \) as
\[ \tilde{\phi}(s, \mu) = \int_0^\infty dx e^{-sx} \phi(x, \mu). \]  

(13)

Applying this to Equation (12) we find
\[ -\mu \phi(0, \mu) + (1 + s\mu) \tilde{\phi}(s, \mu) = c \frac{\phi_0(s)}{2\pi}. \]  

(14)

Dividing by \( 1 + s\mu \), multiplying by \( 2/\sqrt{1-\mu^2} \) and integrating over \( \mu(-1, 1) \), we find after some rearrangement
\[ \left[ 1 - \frac{c}{\pi} \int_{-1}^1 \frac{d\mu}{(1 + s\mu) \sqrt{1-\mu^2}} \right] \bar{\phi}_0(s) = 2 \int_{-1}^0 d\mu \frac{\mu \phi(0, \mu)}{(1 + s\mu) \sqrt{1-\mu^2}}; \]  

(15)

where we have used the boundary condition. The integral in the square brackets is
\[ \frac{c}{\pi} \int_{-1}^1 \frac{d\mu}{(1 + s\mu) \sqrt{1-\mu^2}} = \frac{c}{\sqrt{1-s^2}}. \]  

(16)
Thus, we may write Equation (15) as

$$\left[1 - \frac{c}{\sqrt{1-s^2}}\right] \phi_0(s) = 2 \int_{-1}^{0} d\mu \frac{\mu \phi(0, \mu)}{(1+s\mu)\sqrt{1-\mu^2}} \equiv g(s). \tag{17}$$

At this point we use the Wiener–Hopf method by defining

$$V(s) = 1 - \frac{c}{\sqrt{1-s^2}}, \tag{18}$$

which has zeroes $s = \pm \sqrt{1-c^2} \equiv \pm \nu$. We now define the function

$$\tau(s) = \frac{s^2-1}{s^2-\nu^2} V(s) = \frac{\tau_+(s)}{\tau_-(s)}. \tag{19}$$

The functions $\tau_{\pm}(s)$ are defined such that $\tau_+$ is analytic in the half-space $\Re(s) < \gamma$ and $\tau_-$ in $\Re(s) > -\gamma$, where constant $\gamma$ is in the range $0 < \gamma < 1$. They are defined by

$$\log \tau(s) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} du \frac{\log \tau(u)}{u-s} - \frac{1}{2\pi i} \int_{-\gamma-i\infty}^{-\gamma+i\infty} du \frac{\log \tau(u)}{u-s} \tag{20}$$

and we choose the branch of the logarithm such that $\log(1) = 0$. We may further write

$$\log \tau(s) = \log \tau_+(s) - \log \tau_-(s) \tag{21}$$

whence

$$\tau(s) = \frac{\tau_+(s)}{\tau_-(s)}. \tag{22}$$

From this we note that

$$\tau_+(s) \tau_-(-s) = 1. \tag{23}$$

In this section, and in the other problems, we follow the method described in Chapter 7 of Williams (1971). Thus, we can write the decomposition as

$$V(s) = \frac{(s^2-\nu^2)\tau_+(s)}{(s^2-1)\tau_-(s)}. \tag{24}$$

We also have the relation (used later)

$$\tau_-(s) \tau_-(-s) = \frac{s^2-\nu^2}{(s^2-1)V(s)}. \tag{25}$$

We may also relate $\tau_-(s)$ to the conventional $H$ function (Chandrasekhar 1960), namely
\[ H(\mu) = \frac{(1 + \mu)}{(1 + \nu \mu)} \tau_-(\frac{1}{\mu}), \quad H\left(\frac{1}{s}\right) = \frac{s + 1}{s + \nu} \tau_-(s) \text{ and } V(s) = \frac{1}{H(1/s)H(-1/s)}. \tag{26} \]

We now use \( V(s) \) from Equation (24) in Equation (17) and rearrange so that

\[ \frac{(s^2 - \nu^2)}{s + 1} \frac{\ddot{\phi}_0(s)}{\tau_-(s)} = \frac{\nu - 1}{\tau_+(s)} g(s). \tag{27} \]

As \(|s| \to \infty\), the left and right hand sides of Equation (27) tend to a constant that we call \( A \). This fulfills the conditions of Liouville’s theorem. Thus, we have

\[ \ddot{\phi}_0(s) = A \frac{(s + 1) \tau_-(s)}{s^2 - \nu^2} = A \frac{H(1/s)}{s - \nu} \tag{28} \]

and

\[ 2 \frac{(s - 1)}{\tau_+(s)} \int_{-1}^{0} \frac{d \mu \mu \phi(0, \mu)}{(1 + s \mu) \sqrt{1 - \mu^2}} = A. \tag{29} \]

Setting \( s = 0 \) in Equation (23), we find

\[ \frac{2}{\tau_+(0)} \int_{0}^{1} \frac{d \mu \mu \phi(0, -\mu)}{\sqrt{1 - \mu^2}} = A. \tag{30} \]

The values of \( \tau_\pm(0) \) are found via the method described in (Williams 1971) yielding

\[ \tau_+(0) \tau_-(0) = 1, \quad \tau_-(0) = 1/\sqrt{\tau(0)} = \frac{\nu}{\sqrt{1 - c}}. \tag{31} \]

Thus

\[ 2 \frac{\nu}{\sqrt{1 - c}} \int_{0}^{1} \frac{d \mu \mu \phi(0, -\mu)}{\sqrt{1 - \mu^2}} = A. \tag{32} \]

Now, the current \( J(x) \) at any point in the half space is defined as

\[ J(x) = -\int_{-\pi}^{\pi} d\theta \cos \theta \phi(x, \theta) = -2 \int_{-1}^{1} d \mu \frac{\mu}{\sqrt{1 - \mu^2}} \phi(x, \mu) \tag{33} \]

from which

\[ J(0) = 2 \int_{0}^{1} d \mu \frac{\mu}{\sqrt{1 - \mu^2}} \phi(0, -\mu). \tag{34} \]
and so

\[ A = \frac{\nu}{\sqrt{1-c}} J(0). \quad (35) \]

We also note that from Equation (7) with \( a = \infty \), that we may write

\[ \phi(0, -\mu) = \frac{c}{2\pi \mu} \phi_0 \left( \frac{1}{\mu} \right) \quad (36) \]

i.e. the emergent angular distribution can be obtained from the Laplace transform. From Equation (28) and Equation (35) we find

\[ \phi(0, -\mu) = \frac{c}{2\pi \sqrt{1-c}} J(0) \frac{(1 + \mu) \tau_-(1/\mu)}{1 - \nu^2 \mu^2} = \frac{c}{2\pi \sqrt{1-c}} J(0) \frac{H(\mu)}{1 - \nu \mu} \quad (37) \]

where the new \( H \) function is

\[ H(\mu) = \frac{(1 + \mu) \tau_-(1/\mu)}{1 + \nu \mu}. \quad (38) \]

The analytic form of the function can be obtained by deforming the integration contour to the imaginary axis and transforming the range of integration [see Section VII of Placzek and Seidel (1947)] and we find

\[ \log \tau_-(s) = -\frac{s}{\pi} \int_0^\infty \frac{dt}{s^2 + t^2} \log \left[ \frac{t^2 + 1}{t^2 + \nu^2} \left( 1 - \frac{c}{\sqrt{1 + t^2}} \right) \right] \quad (39) \]

whence

\[ \tau_-(1/\mu) = \exp \left( \frac{-\mu}{\pi} \int_0^\infty \frac{dt}{1 + \mu^2 t^2} \log \left[ \frac{t^2 + 1}{t^2 + \nu^2} \left( 1 - \frac{c}{\sqrt{1 + t^2}} \right) \right] \right). \quad (40) \]

We may simplify this integral by writing

\[ \int_0^\infty \frac{dt}{1 + \mu^2 t^2} \log \left[ \frac{t^2 + 1}{t^2 + \nu^2} \left( 1 - \frac{c}{\sqrt{1 + t^2}} \right) \right] = \frac{1}{\mu} \int_0^\infty \log \left[ \frac{t^2 + 1}{t^2 + \nu^2} \left( 1 - \frac{c}{\sqrt{1 + t^2}} \right) \right] d(\tan^{-1}(t \mu)), \quad (41) \]

which upon integration by parts becomes

\[ \frac{1}{\mu} \left[ \tan^{-1}(t \mu) \log \left[ \frac{t^2 + 1}{t^2 + \nu^2} \left( 1 - \frac{c}{\sqrt{1 + t^2}} \right) \right] \right]_0^\infty - \frac{1}{\mu} \int_0^\infty \tan^{-1}(t \mu) \frac{d}{dt} \log \left[ \frac{t^2 + 1}{t^2 + \nu^2} \left( 1 - \frac{c}{\sqrt{1 + t^2}} \right) \right]. \quad (42) \]
The quantity in the square brackets is zero and we find that
\[
\frac{d}{dt} \log \left( \frac{t^2 + 1 - \frac{c}{\sqrt{1 + t^2}}}{t^2 + \nu^2} \right) = \frac{ct}{(t^2 + 1)(c + \sqrt{1 + t^2})}.
\]
(43)

Therefore
\[
\tau_-(\frac{1}{\mu}) = \exp \left( \frac{c}{\sqrt{\pi}} \int_0^\infty \frac{t \tan^{-1}(\mu t)}{(t^2 + 1)(c + \sqrt{1 + t^2})} dt \right).
\]
(44)

We also have from Equation (13)
\[
\bar{\phi}_0(s) = \int_0^\infty dx e^{-sx} \phi_0(x).
\]
(45)

Note, from this and Equation (28) that
\[
\lim_{s \to \infty} \bar{\phi}_0(s) = \phi_0(0) = A = \frac{\nu}{\sqrt{1 - c}} J(0).
\]
(46)

3.1. Spatial variation
We may obtain the spatial variation of the scalar flux by inverting the Laplace transform
\[
\phi_0(x) = A \frac{1}{2\pi i} \int ds \frac{e^{sx}(s + 1)\tau_-(s)}{s^2 - \nu^2}.
\]
(47)

The integrand has two poles at \( s = \pm \nu \) and, as we will see through the structure of \( V(s) \), there is a branch point at \( s = -1 \). The pole contribution, which we denote \( \phi_{asy}(x) \), is given by
\[
\phi_{asy}(x) = A \frac{2}{2\nu} \left( (1 + \nu)\tau_-(\nu)e^{\nu x} - (1 - \nu)\tau_-(\nu)e^{-\nu x} \right).
\]
(48)

We rewrite this expression in the form
\[
\phi_{asy}(x) = A \frac{2}{2\nu} B \sinh[\nu(x + Z_0)],
\]
(49)

where \( Z_0 \) is the extrapolated endpoint, i.e. the distance into the region \( x < 0 \) where the asymptotic flux, mathematically, goes to zero. By expanding Equation (49) and comparing terms with Equation (48) we find
\[
Z_0 = \frac{1}{2\nu} \log \left( \frac{(1 + \nu)\tau_-(\nu)}{(1 - \nu)\tau_-(\nu)} \right) = \frac{1}{2\nu} \log \left( \frac{1 + \nu}{1 - \nu} \right) + \frac{1}{2\nu} \log \left( \frac{\tau_-(\nu)}{\tau_-(\nu)} \right)
\]
(50)
\[ B = 2\sqrt{1-v^2}[\tau_-(\nu)\tau_-(-\nu)]^{1/2} = 2c[\tau_-(\nu)\tau_-(-\nu)]^{1/2} = 2\sqrt{2}c. \] (51)

The latter reduction follows from
\[ \tau_-(\nu)\tau_-(-\nu) = \frac{1}{\tau(\nu)} = \lim_{s \to \nu} \frac{s^2-\nu^2}{s^2 - 1} \frac{1}{V(s)} = 2. \] (52)

Hence
\[ \phi_{asy}(x) = \sqrt{2\frac{A}{\nu}} c \sinh[\nu(x + Z_0)]. \] (53)

But we also know from Williams (1971) that
\[ \frac{\tau_-(\nu)\tau_+(\nu)}{\tau_+(-\nu)} = \frac{\tau_-(\nu)}{\tau_-(-\nu)}. \] (54)

Also
\[ \frac{1}{2\nu} \log \left[ \frac{\tau_-(\nu)}{\tau_-(-\nu)} \right] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\log \tau(u)}{u^2 - \nu^2} du. \] (55)

Again, using methods described in Williams (1971), we find
\[ \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\log \tau(u)}{u^2 - \nu^2} du = -\frac{1}{\pi} \int_0^1 \frac{dt}{1 - \nu^2 t^2} \tan^{-1} \left( \frac{ct}{\sqrt{1-t^2}} \right) \] (56)

and so
\[ Z_0 = \frac{1}{2\nu} \log \left[ \frac{1 + \nu}{1 - \nu} \right] - \frac{1}{\pi} \int_0^1 \frac{dt}{1 - \nu^2 t^2} \tan^{-1} \left( \frac{ct}{\sqrt{1-t^2}} \right). \] (57)

Note that for the purely scattering case \( c = 1 \), we find \( Z_0 = \frac{1}{2} + \frac{1}{\pi} = 0.818309886... \), which differs from the well-known value of 0.7104 \ldots for transport in three dimensions. To the best of our knowledge, this constant has not previously appeared in the literature. To obtain the contribution from the branch point we return to Equation (28) and use Equation (19) to get
\[ \phi_0(x) = A \frac{1}{2\pi i} \int_L ds \frac{e^{sx}}{(s-1)\tau_-(s)} \frac{\sqrt{1-s^2}}{\sqrt{1-s^2 - c}}. \] (58)

The contour \( L \) may be deformed so that it encloses the poles and hence leads to the asymptotic part of the solution, but also it is wrapped around the branch cut, which runs from \(-1\) to \(-\infty\). After some algebra and noting that
\[ \sqrt{1-s^2} = \sqrt{|1-s^2|e^{+i\pi/2}}, \]  

(59)

where “+” refers to the upper side of the cut and “−” to the lower side, we get

\[
\phi_{\text{trans}}(x) = -A \frac{c}{\pi} \int_0^1 dt \frac{\sqrt{1-t^2}e^{-x/t}}{(1+t)(1-\nu^2t^2)\tau_-(1/t)} 
= -\frac{\nu}{\sqrt{1-c}} J(0) \frac{c}{\pi} \int_0^1 dt \frac{\sqrt{1-t^2}e^{-x/t}}{(1-\nu^2t^2)(1+\nu t)H(t)}. \tag{61}
\]

The complete spatial variation of the flux is

\[ \phi_0(x) = \phi_{\text{asy}}(x) + \phi_{\text{trans}}(x). \tag{62} \]

4. The Flatland \(H\)-function

There are some useful identities that may be obtained between the \(H\) functions. To get these let us return to Equation (17) and use Equation (26) in the form

\[
\frac{1}{H(1/s)H(-1/s)} \bar{\phi}_0(s) = -2 \int_0^1 d\mu \frac{\mu \phi(0,-\mu)}{(1-s\mu)\sqrt{1-\mu^2}}. \tag{63}
\]

Inserting Equation (28) for \(\bar{\phi}_0(s)\) and Equation (37) for \(\phi(0,-\mu)\), we find

\[
\frac{1}{(s-\nu)H(-1/s)} = -\frac{c}{\pi} \int_0^1 d\mu' \frac{\mu' H(\mu')}{\sqrt{1-\mu'^2(1-s\mu')(1-\nu \mu')}}. \tag{64}
\]

But we may write

\[ \frac{\mu'}{(1-s\mu')(1-\nu \mu')} = \frac{1}{s-\nu} \left( \frac{1}{1-s\mu'} - \frac{1}{1-\nu \mu'} \right), \tag{65} \]

which leads to

\[
\frac{1}{H(-1/s)} = -\frac{c}{\pi} \int_0^1 d\mu' \frac{H(\mu')}{\sqrt{1-\mu'^2(1-s\mu')(1-\nu \mu')}} + \frac{c}{\pi} \int_0^1 d\mu' \frac{H(\mu')}{\sqrt{1-\mu'^2(1-\nu \mu')}}. \tag{66}
\]

Now let \(s \to \infty\) to get, with \(H(0) = 1,\)

\[
\frac{c}{\pi} \int_0^1 d\mu' \frac{H(\mu')}{\sqrt{1-\mu'^2(1-\nu \mu')}} = 1. \tag{67}
\]
Set $s = 0$ in Equation (66) and use $H(-\infty) = 1/\sqrt{1-c}$ to get

$$\frac{c}{\pi} \int_0^1 d\mu' \frac{H(\mu')}{\sqrt{1-\mu'^2}} = 1 - \sqrt{1-c}. \quad (68)$$

Finally, setting $s = -1/\mu$, in Equation (66) we get

$$H(\mu) = 1 + \frac{c}{\pi} \mu H(\mu) \int_0^1 d\mu' \frac{H(\mu')}{\sqrt{1-\mu'^2}(\mu + \mu')} , \quad (69)$$

which is the Flatland equivalent to Chandrasekhar’s equation for conventional radiative transfer. Figure 1 compares this Flatland $H$-function to the traditional 3D form for several absorption levels. In Appendix A we provide benchmark values, new identities and additional forms for numerical evaluation of $H$.

### 5. Constant source problem

The equation in this case is

$$\left( \mu \frac{\partial}{\partial x} + 1 \right) \phi(x, \mu) = \frac{1}{2\pi} (c\phi_0(x) + S_0); \quad \phi(0, \mu) = 0, \mu > 0. \quad (70)$$

Following the Wiener–Hopf procedure as described above, we may write the solution for the emergent radiation as
\[ \phi(0, -\mu) = \frac{S_0}{2\pi\sqrt{1-c}}H(\mu) \]  

(71)

and the scalar intensity at the surface as

\[ \phi_0(0) = \frac{S_0}{c} \left( \frac{1}{\sqrt{1-c}} - 1 \right). \]  

(72)

6. Albedo problem

The equation in this case is

\[ \left( \mu \frac{\partial}{\partial x} + 1 \right) \phi(x, \mu) = \frac{c}{2\pi} \phi_0(x); \quad \phi(0, \mu) = \delta(\mu - \mu_0), \mu > 0. \]  

(73)

Again, following the procedure described above, we find

\[ \phi(0, \mu) = \frac{c}{2\pi} \frac{\mu_0 H(\mu_0) H(\mu)}{\mu_0 + \mu} \]  

(74)

and

\[ \phi_0(0) = \phi_0(0; \mu_0) = H(\mu_0). \]  

(75)

Thus, the two-dimensional BRDF (Jarosz et al. 2012) for the isotropic Flatland half space is

\[ f_r(\theta_i, \theta_o) = \frac{c}{2\pi} \frac{H(\cos \theta_i) H(\cos \theta_o)}{\cos \theta_i + \cos \theta_o}. \]  

(76)

A Taylor series expansion about \( c = 0 \) of \( f_r \) gives the BRDFs for the singly- and doubly- scattered (and higher order) reflectances

\[ f_1(\theta_i, \theta_o) = \frac{c}{2\pi} \frac{1}{\cos \theta_i + \cos \theta_o} \]  

(77)

and

\[ f_2(\theta_i, \theta_o) = \frac{c^2}{2\pi^2} \frac{\sec^{-1} \left( \cos \theta_i \right) \sec^{-1} \left( \cos \theta_o \right)}{\sqrt{1 - \sec^2 \theta_i} + \sqrt{1 - \sec^2 \theta_o}}, \]  

(78)

respectively. The surface flux can be averaged over all incident directions to give

\[ \int_0^1 \frac{d\mu_0}{\sqrt{1-\mu_0}} \phi_0(0; \mu_0) = \int_0^1 \frac{d\mu_0}{\sqrt{1-\mu_0^2}} H(\mu_0) = \frac{\pi}{c} \left( 1 - \sqrt{1-c} \right). \]  

(79)

Thus, the total albedo from the half space under illumination arriving at cosine \( \mu_i \) is
with singly scattered contribution

$$R_1(c, \mu_i) = c \left( \frac{1}{2} - \frac{\sec^{-1}(\mu_i)}{\pi \sqrt{1 - \frac{1}{\mu_i^2}}} \right)$$

and doubly scattered contribution

$$R_2(c, \mu_i) = c^2 \left( \frac{1}{8} \left( \frac{1-\mu_i}{1+\mu_i} \right) + \frac{\sec^{-1}(\mu_i)}{2\pi \sqrt{1-\frac{1}{\mu_i^2}}} + \frac{\mu_i^2 \sec^{-1}(\mu_i)^2}{2\pi^2 (1 - \mu_i^2)} \right).$$

7. Numerics

We performed Monte Carlo simulation in a two-dimensional domain to test several of the previous derivations. The total albedo from the half space (Equation 80) is validated in Figure 2 for four incidence angles $\mu_i$. The singly and doubly scattered portions of the albedo (Equations 81 and 82) are validated in Figure 3. The BRDF for the half space is validated in Figure 4.
Figure 3. Monte Carlo evaluation for Flatland half-space single-scattering albedo $R_1$ and double-scattering albedo $R_2$.

Figure 4. Monte Carlo evaluation for Flatland half-space BRDF and its single- and double-scattered portions for the case of $c = 0.8$. 
8. Conclusion

We have solved the classic half-space problems of isotropic scattering for transport in a two-dimensional Flatland domain. We found that the solutions correspond closely to Monte Carlo simulation. A variety of forms for numerically evaluating the $H$-function have been provided together with benchmark values.

The Flatland solutions have a similar form to the familiar three-dimensional half space. However, a new $H$-function identity unique to Flatland as well as analytic expressions of the $H$-function in terms of hypergeometric functions have been presented.

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Appendix A: Numerical benchmarks

In this appendix, we derive additional forms of $H$ and compare their performance and accuracy in numerical integration contexts. For the remainder of the article, we assume $0 \leq \mu \leq 1$ and $0 \leq c \leq 1$. Combining Equations (38) and (40) and simplifying the argument of the log, we can write

$$H(\mu) = \frac{(1 + \mu)}{1 + \nu \mu} \exp \left( \frac{\mu}{\pi} \int_0^\infty \frac{dt}{1 + t^2 + \mu^2} \log \left(1 + \frac{c}{\sqrt{1 + t^2}}\right) \right).$$

(A.1)

Expanding the log argument into even and odd functions of $c$

$$\log \left(1 + \frac{c}{\sqrt{1 + t^2}}\right) = \frac{1}{2} \log \left(1 - \frac{c^2}{t^2 + 1}\right) + \tanh^{-1}\left(\frac{c}{\sqrt{t^2 + 1}}\right)$$

(A.2)
and noting that
\[
\int_0^\infty dt \frac{1}{t^2 + \mu^2} \frac{1}{2} \log \left( 1 - \frac{c^2}{t^2 + 1} \right) = -\frac{\pi}{2\mu} \log \left( \frac{1 + \mu}{1 + \nu \mu} \right) \tag{A.3}
\]
allows two additional forms of \(H\),
\[
H(\mu) = \exp \left( -\frac{\mu}{\pi} \int_0^\infty \frac{1}{1 + t^2 \mu^2} \log \left( 1 - \frac{c}{\sqrt{1 + t^2}} \right) dt \right) \tag{A.4}
\]
and
\[
H(\mu) = \sqrt{\frac{1 + \mu}{1 + \nu \mu}} \exp(I) \tag{A.5}
\]
where \(I\) is the integral of an odd function of \(c\),
\[
I = \int_0^\infty \mu \tanh^{-1} \left( \frac{c}{\sqrt{1 + t^2}} \right) \frac{dt}{\pi(1 + t^2 \mu^2)} \tag{A.6}
\]
This also leads to a new identity for \(H\) unique to Flatland,
\[
H(\mu, c)H(\mu, -c) = \frac{(1 + \mu)}{1 + \nu \mu}. \tag{A.7}
\]
Following the same approach and using Equation (44) produces
\[
H(\mu) = \exp \left( \int_0^\infty \frac{ct \tan^{-1}(t\mu)}{\pi(t^2 + 1)(\sqrt{1 + t^2} - c)} dt \right) \tag{A.8}
\]
or via Equation (A.5) with
\[
I = \int_0^\infty \frac{ct \tan^{-1}(t\mu)}{\pi \sqrt{t^2 + 1(1 - c^2 + t^2)}} dt. \tag{A.9}
\]
Applying the change of variable \(y = \sqrt{1 + t^2}\), we can write
\[
\int_0^\infty \frac{1}{1 + t^2 \mu^2} \log \left( 1 - \frac{c}{\sqrt{1 + t^2}} \right) dt = \int_1^\infty \frac{y \log \left( 1 - \frac{y}{\sqrt{y^2-1}} \right)}{\sqrt{y^2-1}(\mu^2(y^2 - 1) + 1)} dy. \tag{A.10}
\]
Expanding the integrand
\[
\frac{y \log \left( 1 - \frac{y}{\sqrt{y^2-1}} \right)}{\sqrt{y^2-1}(\mu^2(y^2 - 1) + 1)} = \sum_{j=1}^\infty \binom{c}{y}^j \frac{(-1)^{2j-1}}{j} \frac{y}{\sqrt{y^2-1}(\mu^2(y^2 - 1) + 1)} \tag{A.11}
\]
we note that
\[
\int_1^\infty \binom{c}{y}^j \frac{(-1)^{2j-1}}{j} \frac{y}{\sqrt{y^2-1}(\mu^2(y^2 - 1) + 1)} dy = \frac{\sqrt{\pi}(-1)^{2j+1}c^j \Gamma \left( \frac{j+1}{2} \right)}{2j\mu^2} \tag{A.12}
\]
in terms of the regularized hypergeometric functions \(\tilde{F}_1\) producing the following series expansions for \(I\) over odd powers of \(c\),
\[
I = \sum_{j/\text{odd}>0} \frac{c^j \Gamma \left( \frac{j+1}{2} \right) \tilde{F}_1 \left( 1, \frac{j+1}{2}, \frac{j+2}{2}, 1 - \frac{\mu}{\nu} \right)}{2\sqrt{\pi}j\mu} \tag{A.13}
\]
These produce an analytic form of $H$ for the case of normal incidence ($\mu = 1$)

$$I_{\mu=1} = \sum_{j(\text{odd})>0} \frac{e \Gamma \left( \frac{j+2}{2} \right)}{2 \sqrt{\pi j \Gamma \left( \frac{j+1}{2} \right)}} = \frac{c_5 F_2 \left( \frac{1}{2}, 1; 1; \frac{3}{2}, \frac{3}{2}; \frac{1}{\mu^2} \right)}{\pi}.$$  \hspace{1cm} (A.16)

For conservative ($c = 1$) scattering the analytic forms

$$I_{c=1} = \int_0^\infty \frac{t \arctan \left( t \mu \right)}{\pi (1 + t^2)} \, dt$$

$$= \frac{2 \left( \sqrt{1-\mu^2} + 1 \right) F_2 \left( \frac{1}{2}, 1; \frac{3}{2}, \frac{3}{2}; \frac{1}{\mu^2} \right)}{\pi \mu}$$

$$+ \left( \log \left( \mu + i \sqrt{1-\mu^2} \right) + i \frac{\pi}{2} \right) \text{sec}^{-1} (\mu)$$

$$= \Re \left( \frac{F_2 \left( \frac{1}{2}, 1; \frac{3}{2}, \frac{3}{2}; \frac{1}{\mu^2} \right)}{\pi \mu} \right)$$  \hspace{1cm} (A.17)

agree with those for normal incidence in the case of normal incident conservative scattering giving

$$H(1, c = 1) = \sqrt{2e^{\frac{\pi}{2}}} = 2.53373727948584190958328963404...$$  \hspace{1cm} (A.18)

where $C$ is Catalan’s constant. To simplify numerical quadratures, it is convenient to move to a finite integration domain. Analogous to the approach of Stibbs and Weir (1959) a change of variable $t = \cot \theta$ yields

$$H(\mu) = \frac{(1 + \mu)}{1 + \nu \mu} \exp \left( \int_0^{\pi/2} \frac{\mu \log (1 + c \sin \theta)}{\pi \left( \mu^2 \cos^2 \theta + \sin^2 \theta \right)} \, d\theta \right)$$

$$= \exp \left( \int_0^{\pi/2} \frac{-\mu \log (1- \cos \theta)}{\pi \left( \cos^2 \theta + \sin^2 \theta \right)} \, d\theta \right)$$

$$= \frac{(1 + \mu) \sqrt{1 + c}}{1 + \nu \mu} \exp \left( -\frac{\mu}{\pi} \cot^{-1} (\mu \cot \theta) \cos \theta \right)$$  \hspace{1cm} (A.19)

$$= \frac{\mu \tanh^{-1} (c \sin \theta)}{\pi \left( \mu^2 \cos^2 \theta + \sin^2 \theta \right)}$$  \hspace{1cm} (A.20)

or via Equation (A.5) with

$$I = \int_0^{\pi/2} \frac{\mu \tan^{-1} (c \sin \theta)}{\pi \left( \mu^2 \cos^2 \theta + \sin^2 \theta \right)} \, d\theta.$$  \hspace{1cm} (A.21)

We evaluated 38 distinct numerical forms for evaluating $H$ and timed and tested the accuracy of these to 13 significant digits using the NIntegrate function in Mathematica 11.0.0.0 with default parameters. Additional forms were formed by expanding the integral $I$
### Table A.1. Benchmark values for the Flatland $H$-function $H(\mu)$.

| $\mu$ | $c = 0.1$ | $c = 0.5$ | $c = 0.75$ | $c = 0.8$ | $c = 0.9$ | $c = 0.95$ | $c = 0.99$ | $c = 0.995$ | $c = 0.999$ | $c = 1.0$ |
|-------|-----------|-----------|------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0.1   | 1.009862372199 | 1.057066398757 | 1.097868083873 | 1.108228229777 | 1.113646865540 | 1.151224305770 | 1.166405350851 | 1.173869067074 | 1.179581033239 | 1.185856067272 |
| 0.2   | 1.015411757078 | 1.092617880873 | 1.164446809190 | 1.183528348032 | 1.232022663846 | 1.267122800699 | 1.298631187074 | 1.314575452073 | 1.325853037905 | 1.340884878881 |
| 0.3   | 1.019540027549 | 1.120236190902 | 1.218958588485 | 1.246004445911 | 1.351605620574 | 1.419082663070 | 1.444487648046 | 1.462699996597 | 1.487302200642 | 1.507432576536 |
| 0.4   | 1.022752041391 | 1.142848076656 | 1.265515213873 | 1.30014198575 | 1.392622989417 | 1.464006928480 | 1.531730871327 | 1.567462480961 | 1.593379507685 | 1.628804711413 |
| 0.5   | 1.025399992000 | 1.161898371891 | 1.306146375095 | 1.347813563309 | 1.461257535169 | 1.551107024075 | 1.638141153671 | 1.685136797587 | 1.719463489647 | 1.766956873763 |
| 0.6   | 1.027525140348 | 1.178259573303 | 1.342115867418 | 1.390389405367 | 1.52408991038 | 1.632349887778 | 1.739814663753 | 1.798403543768 | 1.841787152039 | 1.902350245334 |
| 0.7   | 1.029457908725 | 1.192510693704 | 1.374283324352 | 1.428767171444 | 1.581990734535 | 1.706790148035 | 1.836874033605 | 1.907820540563 | 1.96080751111 | 2.035624885852 |
| 0.8   | 1.031067865338 | 1.205061773552 | 1.403280536600 | 1.463669138315 | 1.63566873300 | 1.780740403939 | 1.929953019201 | 2.013791452251 | 2.07702690721 | 2.16701237191 |
| 0.9   | 1.032473415474 | 1.216215875755 | 1.429590443702 | 1.495425862727 | 1.668543022198 | 1.848788848389 | 2.019405386361 | 2.116593220228 | 2.190325938536 | 2.296707919516 |
| 1.0   | 1.033712591466 | 1.226203880894 | 1.453593356088 | 1.524622994060 | 1.732130621233 | 1.913189719542 | 2.105516087365 | 2.216458389328 | 2.301565481862 | 2.424860908147 |
in Equation (A.5) as a finite sum of powers of $c$ and an integral for the higher powers, for example

$$I = \int_0^\infty \frac{c^3 t \tan^{-1}(t\mu)}{(1 + t^2)^{3/2} (\pi - c^2 \pi + \pi t^2)} dt + \sum_{j=1, \text{odd}}^{I-2} \frac{c^j \Gamma\left(\frac{1+j}{2}\right) \bar{F}_1\left(1, \frac{1+j}{2}; \frac{2+j}{2}; 1 - \frac{1}{\mu^2}\right)}{2j \sqrt{\pi \mu}}. \quad (A.23)$$

We found Equation (A.19) to be the most efficient form, accurate to 12 significant digits. The most efficient form accurate to 13 digits was Equations (A.5) and (A.23) with $I$ simplified to

$$I = \frac{c(3c^4 + 5c^2 U + 15U^2) \sin^{-1}(\sqrt{U}) - c^3 \left((2c^2 + 5) \sqrt{1-U} U^{3/2} + 3c^2 \sqrt{(1-U)U}\right)}{15\pi \mu \sqrt{1-U} U^{5/2}}$$

$$+ \int_0^\infty \frac{c^2 t \tan^{-1}(t\mu)}{(1 + t^2)^{3/2} (\pi - c^2 \pi + \pi t^2)} dt$$

$$\quad (A.24)$$

where $U = 1 - \frac{1}{\mu^2}$. The median values of all 38 numerical evaluations for $H$, which we take to be likely candidates for the correct results, are summarized in Table A.1.