THE MOR CRYPTOSYSTEM AND UNITARY GROUP IN ODD CHARACTERISTIC

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Abstract. This paper is a continuation of the work done to understand the security of a MOR cryptosystem over matrix groups defined over a finite field. In this paper we show that in the case of unitary group $U(d,q^2)$ the security of the MOR cryptosystem is similar to the hardness of the discrete logarithm problem in $F_{q^2}$. In our way of developing the MOR cryptosystem, we developed row-column operations for unitary matrices that solves the word problem in the group of unitary matrices. This is similar to row-column operations in special linear groups that write a matrix as a product of elementary transvections.

1. Introduction

This paper is a continuation of a study of the MOR cryptosystem [12,13] over finite groups. In this paper we study the unitary group, which is a twisted Chevalley group, also known as Steinberg groups. Since we were a bit expository in our recent work [14] and gave a detailed introduction to the MOR cryptosystem, we will be brief in this paper. We will often refer to both [12] and [14] in this current work.

Briefly speaking, the MOR cryptosystem is a simple and straightforward generalization of the classic ElGamal cryptosystem put forward by Paeng et. al. [15]. In a MOR cryptosystem one works with the automorphism group rather than the group itself. It provides an interesting change in perspective in public-key cryptography – from finite cyclic groups to finite non-abelian groups.

The description of the MOR cryptosystem is as follows:

Let $G = \langle g_1, g_2, \ldots, g_s \rangle$ be a finite group. Let $\phi$ be a non-identity automorphism.

- Public-key: Let $\{\phi(g_i)\}_{i=1}^s$ and $\{\phi^m(g_i)\}_{i=1}^s$ is public.
- Private-key: The integer $m$ is private.

Encryption:
To encrypt a plaintext $M \in G$, get an arbitrary integer $r \in [1,|\phi|]$ compute $\phi^r$ and $\phi^{rm}$. The ciphertext is $(\phi^r, \phi^{rm}(M))$.

Decryption:
After receiving the ciphertext $(\phi^r, \phi^{rm}(M))$, the user knows the private key $m$. So she computes $\phi^{rm}$ from $\phi^r$ and then computes $M$.

We looked at the MOR cryptosystem over classical Chevalley groups in odd characteristic [14] which was a continuation of an earlier work [12]. There is a certain amount of similarity in studying these groups – groups of classical types. These are essentially
In this paper we show that the security for the MOR cryptosystem is similar to that of the hardness of the discrete logarithm problem in \( \mathbb{F}_{q^d} \) or \( \mathbb{F}_{q^{d^2}} \), here \( \mathbb{F}_q \) is the underlying field and the matrices are of degree \( d \). In the case of the special linear group \( SL(d, q) \) we show that the security of the MOR cryptosystem is the same as the hardness of the discrete logarithm problem in \( \mathbb{F}_{q^d} \). In the case of orthogonal groups \( O(2l + 1, q) \) and \( O(2l, q) \) it is \( \mathbb{F}_{q^d} \). In the case of symplectic groups it is \( \mathbb{F}_{q^{d^2}} \). The obvious question arises, what is the security for the unitary groups? In this paper we show that the security of the MOR cryptosystem over \( U(d, q^2) \) where \( d = 2l \) or \( d = 2l + 1 \) is the hardness of the discrete logarithm problem in \( \mathbb{F}_{q^{2d}} \).

Though in this MOR cryptosystem the security is similar to that of the hardness of a discrete logarithm problem in a finite field, the operational aspect, i.e., how the cryptosystem is implemented is very different. As we saw in the description of the MOR cryptosystem, to encrypt and decrypt we have to compute large powers of \( \zeta \). The usual way to do that is the square-and-multiply algorithm. In the implementation of square-and-multiply we intend to use an algorithm to solve the word problem in unitary group and some replacement formulas. So our cryptosystem is a brand new cryptosystem with a tangible security outcome.

In our way of developing the MOR cryptosystem, we had to develop a fast algorithm to solve the word problem in unitary groups (Section 5). This algorithm is similar to that of row-column operations in special linear groups, which write a matrix as a word in elementary transvections. We developed a similar algorithm in [14] for orthogonal and symplectic groups. This algorithm is of independent interest in computational group theory. The Chevalley groups and Steinberg groups are given by generators obtained from the theory of Lie groups and the Bruhat decomposition gives a way to write every element of \( \mathbb{F}_q \). Cohen et. al. [4, 5] have looked at the word problem in these groups, however their generators were in normal form. Their strategy was based on a collection of algorithms. Our algorithm works directly with these groups in their standard matrix representation.

1.1. Notation. We fix a non-zero \( \varepsilon \in \mathbb{F}_{q^d} \) with \( \bar{\varepsilon} = \varepsilon^q = -\varepsilon \). Then every \( x \in \mathbb{F}_{q^d} \) is of the form \( x = a + \varepsilon b \). The solutions of the equation \( X^q = X \) are the elements of \( \mathbb{F}_q \), i.e., \( b = 0 \). The solutions of the equation \( X^q = -X \) are \( \varepsilon b \) where \( b \in \mathbb{F}_q \). We denote this set as \( \mathbb{F}_q^a = \{ x \in \mathbb{F}_{q^d} \mid \bar{x} = -x \} \). We also denote \( \mathbb{F}_{q^d}^1 = \{ x \in \mathbb{F}_{q^d} \mid x\bar{x} = 1 \} \). We note that the set \( \{ x\bar{x} \mid x \in \mathbb{F}_{q^d}^a \} = \mathbb{F}_q^x \). We fix an element \( \zeta \) which generates the cyclic group \( \mathbb{F}_q^x \). The element \( \zeta^{q+1} \) generates the cyclic subgroup \( \mathbb{F}_q^x \). Also the cyclic subgroup \( \mathbb{F}_q^{1^d} \) is generated by \( \zeta^{q-1} \). A \( d \times d \) matrix \( X \) is called hermitian (skew-hermitian) if \( \bar{X} = X \) \((\bar{T}X = -X)\).

2. Unitary Groups

In this section, we briefly describe unitary groups. Our standard reference for unitary groups are Carter and Grove [2,8]. Let \( V \) be a vector space of dimension \( d \) over a field \( K \). Furthermore suppose \( K \) has an automorphism of order 2 denoted as \( \alpha \mapsto \bar{\alpha} \) and the fixed field is \( k \). Let \( \beta: V \times V \to K \) be a non-degenerate hermitian form, i.e., bar-linear in the first coordinate and linear in the second coordinate satisfying \( \beta(x, y) = \beta(y, x) \). We
fix a basis for $V$ and slightly abuse the notation to denote the matrix of $\beta$ by $\beta$. Thus $\beta$ is a non-singular matrix satisfying $\beta = \overline{\bar{\beta}}$.

**Definition 2.1** (Unitary Group). The unitary group is:

$$U(d, K) = \{ X \in GL(d, K) \mid \overline{\bar{X}}\beta X = \beta \}.$$ 

The special unitary group $SU(d, K)$ consists of matrices of $U(d, K)$ of determinant 1.

We work with finite fields of odd characteristic, i.e., we have $K = F_{q^m}$ and $k = F_q$ with $\bar{\alpha} = \alpha^q$ with $q$ odd. It turns out that over a finite field there is only one non-degenerate hermitian form up to equivalence [8, Corollary 10.4]. Equivalent hermitian forms give conjugate unitary groups. For the convenience of computations we index the basis by $1, \ldots, l, -1, \ldots, -l$ when $d = 2l$ and by $0, 1, \ldots, l, -1, \ldots, -l$ when $d = 2l + 1$; where $l > 1$. We also fix the matrix $\beta$ as follows:

- $d = 2l \text{ fix } \beta = \begin{pmatrix} I_l \\ I_l \end{pmatrix}$
- $d = 2l + 1 \text{ fix } \beta = \begin{pmatrix} 2 & I_l \\ I_l & I_l \end{pmatrix}$

In the above case we denote the unitary group as $U(d, q^2)$ and special unitary group as $SU(d, q^2)$. This is a twisted Chevalley group defined by Steinberg (also called Steinberg group) of type $2A_{d-1}$. We briefly describe Steinberg’s approach in the next section.

Note that the matrix $\beta$ is also symmetric and is same as the $D_l$ and $B_l$ type mentioned in [14, Section 3]. Thus the orthogonal group $O(d, q)$ is a subgroup of $U(d, q^2)$ and also $SO(d, q)$ is a subgroup of $SU(d, q^2)$. This allows us to borrow some ideas and computations from orthogonal groups to develop an algorithm to solve the word problem in unitary groups. For example, we can use the Weyl group elements [14, Lemma 6.6] in this algorithm. The conjugacy classes and some questions related to the unitary groups were studied by Gates et al. [7]. A word of caution on notations used for unitary groups in literature when indicating the field. Note that $U(d, F_q)$, $U(d, q)$ and $U(d, q^2)$ are the same group.

### 3. Twisted Chevalley Groups

Chevalley [3] described a way of constructing groups over an arbitrary field which are now called Chevalley groups. We worked with classical Chevalley groups in our recent paper [14]. Steinberg [16] generalized the idea of Chevalley and introduced twisted Chevalley groups to produce even more new groups. These groups are now called Steinberg groups. These groups can be constructed in those cases where the Dynkin diagram has a symmetry. In this paper we work with the twisted group of type $2A_l$, i.e., unitary groups. The exposition here follows [2, Chapters 13 & 14] and serves as a motivation for choosing the set of generators described below.

Let $K$ be a field of odd characteristic. Let $\mathcal{L}$ be a simple Lie algebra of classical type and $G = \mathcal{L}(K)$ be a Chevalley group of type $\mathcal{L}$ over $K$. Suppose the Dynkin diagram of $\mathcal{L}$ has a non-trivial symmetry $\rho$. Then there is a graph automorphism $\gamma$ of $\mathcal{L}(K)$ corresponding to $\rho$. We can choose a field automorphism $\theta$ such that $\sigma = \gamma \theta$ satisfies $\sigma^n = 1$ where $n$ is the order of $\rho$. Then [2, Proposition 13.4.1] $\sigma(U) = U$, $\sigma(V) = V$, $\sigma(H) = H$ and $\sigma(N) = N$ where $U, V, H$ and $N$ are subgroups of $\mathcal{L}(K)$ described before [14, Section 4]. Denote $U^1 = \{ x \in U \mid \sigma(x) = x \}$ and $V^1 = \{ x \in V \mid \sigma(x) = x \}$. Consider the group $G^1$ generated by $U^1$ and $V^1$. These are called the twisted groups.
In this paper, we work over finite fields $\mathbb{F}_{q^2}$ where $q$ is odd and $L$ is of $A_l$ type. We fix the field automorphism as Frobenius map given by $\alpha \mapsto \alpha^q$. The Lie algebra has a Dynkin diagram symmetry of order two. Let us denote

$$A = \varepsilon \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & \cdot \cdot \cdot \\ \cdot \cdot \cdot & \cdot \cdot \cdot & -1 \end{pmatrix}. $$

Then the map $\sigma$ is given by $X \mapsto A^{-1}T\overline{X}A$ and the group $G^1$ is generated by upper and lower unitriangular matrices fixed by $\sigma$ which is the special unitary group [2, Theorem 14.5.1]. However the unitary group obtained here is with respect to the hermitian form given by the matrix $A$. Over a finite field there is only one hermitian form up to equivalence. For our computation we choose a different hermitian form $\beta$ as described in Section 2.

We begin with a demonstration to construct generators when $l = 2$ using the Steinberg’s construction described above. For an arbitrary $l$, this can be done similarly, the reader can consult the proof of [2, Theorem 14.5.1]. The basis we are working with in this paper is given in Section 2. We denote that by $v_1, v_2, v_{-1}, v_{-2}$. However the unitary group obtained using Steinberg’s construction is with respect to a different basis and we can write that in terms of our basis as $v_1, v_2, \overline{\varepsilon}v_{-2}, \varepsilon v_{-1}$. The change of basis matrix is $P = \begin{pmatrix} 1 & 1 & \varepsilon \\ \end{pmatrix}$.

Let $X = \begin{pmatrix} 1 & t_1 & t_2 & t_3 \\ 1 & t_4 & t_5 & 1 \\ 1 & t_6 & 1 & 1 \end{pmatrix} \in U$ then $X \in U^1$ means $X = \sigma(X) = A^{-1}T\overline{X}A$. We compute $A^{-1}T\overline{X}A = \begin{pmatrix} 1 & -\bar{t}_6 & \bar{t}_5 & -\bar{t}_3 \\ 1 & -\bar{t}_4 & \bar{t}_2 & \bar{t}_1 \\ 1 & -\bar{t}_6 & \bar{t}_5 & -\bar{t}_3 \end{pmatrix}$ and equate with $X$ to get $X = \begin{pmatrix} 1 & t_1 & t_2 & t_3 \\ 1 & t_4 & -\bar{t}_2 & \bar{t}_1 \\ 1 & \bar{t}_2 & 1 & 1 \end{pmatrix}$ where $\bar{t}_i = t_i$ for $i = 3, 4$. Now we need to convert this in our basis. So we compute $P^{-1}XP = \begin{pmatrix} 1 & t_1 & \varepsilon t_3 & \varepsilon t_2 \\ 1 & -\bar{\varepsilon} t_2 & \varepsilon t_4 & 1 \\ 1 & -\bar{t}_1 & 1 \end{pmatrix}$ which is of the form $\begin{pmatrix} 1 & t_1 & T_3 & T_2 \\ 1 & -T_2 & T_4 & 1 \\ 1 & -\bar{t}_1 & 1 \end{pmatrix}$ where $T_3, T_4 \in \mathbb{F}_{q^2}$. This justifies our choice of generators described below.
3.1. Generators for $U(2l, q^2)$. In what follows, $l \geq 2$. Recall that $\ell = t^q$. For $1 \leq i, j \leq l$ and $t \in F_{q^2}$ and $s \in F_{q^2}^*$, the group $U(2l, q^2)$ is generated by the matrices

\begin{align*}
x_{i,j}(t) &= I + te_{i,j} - t\bar{e}_{j,-i} & \text{for } i \neq j, \\
x_{i,-j}(t) &= I + te_{i,-j} - t\bar{e}_{j,-i} & \text{for } i < j, \\
x_{-i,j}(t) &= I + te_{-i,j} - t\bar{e}_{j,-i} & \text{for } i < j, \\
x_{i,-i}(s) &= I + se_{i,-i}, \\
x_{-i,i}(s) &= I + se_{-i,i}, \\
x_{i,0}(t) &= I + 2t\bar{e}_{i,0} - t\bar{e}_{0,-i} - t\bar{e}_{i,-i}, \\
x_{0,i}(t) &= I + t\bar{e}_{0,i} - 2t\bar{e}_{-i,0} - t\bar{e}_{i,-i}, \\
w_t &= I - e_{t,l} - e_{-l,-i} - e_{i,-l} - e_{-l,l}, \\
d(\zeta) &= \text{diag}(1, \ldots, 1, \zeta, 1, \ldots, 1, \zeta^{-1}) & \text{for fixed } \zeta.
\end{align*}

Note that if we restrict $t \in F_q$ we get a set of generators of $O(2l, q)$.

3.2. Generators for $U(2l + 1, q^2)$. For $1 \leq i, j \leq l$ and $t \in F_{q^2}$ and $s \in F_{q^2}^*$, the group $U(2l + 1, q^2)$ is generated by the matrices

\begin{align*}
x_{i,j}(t) &= I + te_{i,j} - t\bar{e}_{j,-i} & \text{for } i \neq j, \\
x_{i,-j}(t) &= I + te_{i,-j} - t\bar{e}_{j,-i} & \text{for } i < j, \\
x_{-i,j}(t) &= I + te_{-i,j} - t\bar{e}_{j,-i} & \text{for } i < j, \\
x_{i,-i}(s) &= I + se_{i,-i}, \\
x_{-i,i}(s) &= I + se_{-i,i}, \\
x_{i,0}(t) &= I + 2l\bar{e}_{i,0} - t\bar{e}_{0,-i} - t\bar{e}_{i,-i}, \\
x_{0,i}(t) &= I + t\bar{e}_{0,i} - 2l\bar{e}_{-i,0} - t\bar{e}_{i,-i}, \\
w_t &= I - e_{t,l} - e_{-l,-i} - e_{i,-l} - e_{-l,l}, \\
d(\zeta) &= \text{diag}(1, \ldots, 1, \zeta, 1, \ldots, 1, \zeta^{-1}) & \text{for fixed } \zeta, \\
d(\alpha) &= \text{diag}(\alpha, 1, \ldots, 1) & \text{where } \alpha = \zeta^{q-1}.
\end{align*}

Recall that $\ell = t^q$. Note that if we restrict $t \in F_q$ we get a set of generators of $O(2l + 1, q)$. The generators $d(\zeta)$ and $d(\alpha)$ will be used at the last step of our algorithm to identify certain elements and do not play much role in the actual computation.

4. Automorphism Group of the Unitary Group

In a MOR cryptosystem we work with the automorphism group of the unitary group. We describe those automorphisms in this section. First we define the similitude group. We need this group to define diagonal automorphisms

**Definition 4.1** (Unitary similitude group). The unitary similitude group is defined as:

$$GU(d, q^2) = \{ X \in GL(d, q^2) \mid X^t \beta X = \mu \beta, \text{for some } \mu \in F_q^* \}.$$

Note that the multiplier $\mu$ defines a group homomorphism from $GU(d, q^2)$ to $F_q^*$ with kernel the unitary group.

**Conjugation Automorphisms**: The conjugation maps $g \mapsto ngn^{-1}$ for $n \in GU(d, q^2)$ are called conjugation automorphisms. Furthermore, they are composition of two types of automorphisms – inner automorphisms given as conjugation by elements of $U(d, q^2)$ and diagonal automorphisms given as conjugation by diagonals of $GU(d, q^2)$.
Central Automorphisms: Let \( \chi: U(d, q^2) \to \mathbb{F}_q^1 \) be a group homomorphism. Then the central automorphism \( c_\chi \) is given by \( g \mapsto \chi(g)g \). Since \([U(d, q^2), U(d, q^2)] = [SU(d, q^2), SU(d, q^2)] = SU(d, q^2) \) \( \text{[8, Theorem 11.22]} \), any \( \chi \) is equivalent to a group homomorphism from \( U(d, q^2)/SU(d, q^2) \) to \( \mathbb{F}_q^1 \). There are at most \( q + 1 \) such maps.

Field Automorphisms: For any automorphism \( \sigma \) of the field \( \mathbb{F}_{q^2} \), replacing all entries of a matrix by their image under \( \sigma \) give us a field automorphism.

The following theorem, due to Dieudonné \( [6, \text{Theorem 25}] \), describes all automorphisms:

**Theorem 4.1.** Let \( q \) be odd and \( d \geq 4 \). Then any automorphism \( \phi \) of the unitary group \( U(d, q^2) \) is written as \( c_\chi \iota \delta \sigma \) where \( c_\chi \) is a central automorphism, \( \iota \) is an inner automorphism, \( \delta \) is a diagonal automorphism and \( \sigma \) is a field automorphism.

5. **Solving the word problem in Unitary Group**

Solving the word problem in any group is of interest in computational group theory. For many groups, it is a very hard problem. In this paper we present a fast, cubic-time solution to the word problem in unitary groups. The generators for the unitary group are the ones we defined in Section 3.1. The motivation for the word problem was the row-column operations in special linear groups that reduce a matrix as a product of elementary transvections. In \( [14] \) we developed a similar algorithm for classical Chevalley groups. It is well known that every element of an unitary group can be written in a normal form \( [2, \text{Proposition 13.5.3}] \) using Bruhat decomposition. In \([4,5] \) Cohen et. al. used this idea to solve the word problem for elements in normal form for these groups. However, here we work directly with elements of the unitary group written as matrices, i.e., we are working with unitary groups in their standard representation.

5.1. **Row-Column operations for** \( U(2l, q^2) \). First we deal with \( U(2l, q^2) \). The Chevalley generators are described in Section 3. In general, we have three kind of Chevalley generators. For \( 1 \leq i, j \leq l \) and \( t \in \mathbb{F}_{q^2}, s \in \mathbb{F}_{q^2}^* \):

- **CG1:** \( \begin{pmatrix} R & T_{R^{-1}} \end{pmatrix} \) where \( R = I + te_{i,j}; i \neq j \).
- **CG2:** \( \begin{pmatrix} I & R \\ I \end{pmatrix} \) where \( R \) is either \( te_{i,j} - \bar{t}e_{j,i}; i < j \) or \( se_{i,i} \).
- **CG3:** \( \begin{pmatrix} I & R \\ R \end{pmatrix} \) where \( R \) is either \( te_{i,j} - \bar{t}e_{j,i}; i < j \) or \( se_{i,i} \).

Let \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) be a \( 2l \times 2l \) matrix written in block form of size \( l \times l \). Let us note the effect of multiplying \( g \) by elements from above.

**CG1:**
\[
\begin{pmatrix} R & T_{R^{-1}} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} RA & RB \\ T_{R^{-1}}C & T_{R^{-1}}D \end{pmatrix}
\]

**CG2:**
\[
\begin{pmatrix} I & R \\ I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A + RC & B + RD \\ C & D \end{pmatrix}
\]
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & R \\ I \end{pmatrix} = \begin{pmatrix} A & AR + B \\ C & CR + D \end{pmatrix}
\]
CG3 : \[
\begin{pmatrix}
    I & R & I \\
    A & B & C & D \\
    I & R & I
\end{pmatrix}
\begin{pmatrix}
    A & B \\
    C & D
\end{pmatrix}
= \begin{pmatrix}
    A & B \\
    RA + C & RB + D
\end{pmatrix}.
\]

We will mention the use of other generators $d(\zeta)$ and $w_1$ when required.

5.1.1. The algorithm for even case.

Step 1: **Input**: Matrix $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ which belongs to $\text{U}(2l, q^2)$.

**Output**: Matrix $g_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ which is one of the following kind:

a: The matrix $C_1$ is diag$(1, 1, \ldots, 1, \lambda)$ and $A_1$ is \[
\begin{pmatrix}
    A_{11} & -\lambda^T \bar{A}_{21} \\
    A_{21} & a_{22}
\end{pmatrix}
\] where $A_{11}$ is skew-hermitian of size $l - 1$.

b: The matrix $C_1$ is diag$(1, 1, \ldots, 1, 0, \ldots, 0)$ with number of 1s equal to $m < l$ and $A_1$ is of the form \[
\begin{pmatrix}
    A_{11} & 0 \\
    A_{21} & A_{22}
\end{pmatrix}
\] where $A_{11}$ is an $m \times m$ skew-hermitian matrix.

**Justification**: Observe that the effect of CG1 on $C$ is the usual row-column operations. Thus we can reduce $C$ to the diagonal form and Corollary 5.2 makes sure that $A$ has required form.

Step 2: **Input**: Matrix $g_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$.

**Output**: Matrix $g_2 = \begin{pmatrix} A_2 & B_2 \\ 0 & T \bar{A}_2^{-1} \end{pmatrix}$; $A_2$ is diag$(1, 1, \ldots, 1, \lambda)$.

**Justification**: Observe the effect of CG2. It changes $A_1$ by $A_1 + RC_1$. Using Lemma 5.5 we can make the matrix $A_1$ the zero matrix in the first case and $A_{11}$ the zero matrix in the second case. After that we make use of Weyl group elements of $O(2l, q)$ which is a subgroup of $U(2l, q^2)$ to interchange the rows so that we get zero matrix at the place of $C_1$. If required use CG1 to make $A_1$ a diagonal matrix. The Lemma 5.4 ensures that $D_1$ becomes $T \bar{A}_2^{-1}$.

Step 3: **Input**: Matrix $g_2 = \begin{pmatrix} A_2 & B_2 \\ 0 & \bar{A}_2^{-1} \end{pmatrix}$; $A_2$ is diag$(1, 1, \ldots, 1, \lambda)$.

**Output**: Matrix $g_3 = \begin{pmatrix} A_2 & 0 \\ 0 & \bar{A}_2^{-1} \end{pmatrix}$; $A_2$ is diag$(1, 1, \ldots, 1, \lambda)$.

**Justification**: Using Corollary 5.3 we see that the matrix $B_2$ has certain form. We can use CG2 to make the matrix $B_2$ a zero matrix because of Lemma 5.5.

Step 4: **Input**: matrix $g_3 = \text{diag}(1, 1, \ldots, 1, \lambda, 1, \ldots, 1, \bar{\lambda}^{-1})$.

**Output**: Identity matrix

**Justification**: From Lemma 5.6 we can produce such matrices when $\lambda \in F_q^\times$. Remaining ones we have added in the generating set.

5.2. Some useful lemmas.

**Lemma 5.1.** Let $Y = \text{diag}(1, 1, \ldots, 1, \lambda, \ldots, \lambda)$ of size $l$ with number of 1s equal to $m < l$. Let $X$ be a matrix such that $YX$ is skew-hermitian then $X$ is of the form \[
\begin{pmatrix}
    X_{11} & -\bar{\lambda}^T \bar{X}_{21} \\
    X_{21} & X_{22}
\end{pmatrix}
\] where $X_{11}$ is skew-hermitian and so is $X_{22}$ if $\lambda \neq 0$. 

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Proof. We observe that the matrix $YX = \begin{pmatrix} X_{11} & X_{12} \\ \lambda X_{21} & \lambda X_{22} \end{pmatrix}$. The condition that $YX$ is skew-hermitian implies $X_{11}$ (and $X_{22}$ if $\lambda \neq 0$) is skew-hermitian and $X_{12} = -\lambda^T \bar{X}_{21}$. 

**Corollary 5.2.** Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be in $U(2l, q^2)$.

1. If $C$ is a diagonal matrix $\text{diag}(1,1,\ldots,1,\lambda)$ with $\lambda \neq 0$ then the matrix $A$ is of the form $\begin{pmatrix} A_{11} & -\lambda^T \bar{A}_{21} \\ A_{21} & A_{22} \end{pmatrix}$ where $A_{11}$ is an $(l-1) \times (l-1)$ skew-hermitian and $A_{22} \in \mathbb{F}_{q^2}^\times$.

2. If $C$ is a diagonal matrix $\text{diag}(1,1,\ldots,1,\lambda,0,\ldots,0)$ with number of $1$s equal to $m < l$ then the matrix $A$ is of the form $\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$ where $A_{11}$ is an $m \times m$ skew-hermitian.

Proof. We use the condition that $g$ satisfies $^t\bar{C}g = \beta$.

$$^t\bar{C} = \begin{pmatrix} ^tA & ^tC \\ ^tB & ^tD \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} ^tC & ^tA \\ ^tD & ^tB \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} ^tC A + ^tAC & ^t \ast \\ ^tD B & ^t \ast \end{pmatrix}$$

This gives $^tC A + ^tAC = 0$ which means $CA$ is skew-hermitian (note $C = ^tC$ as $C$ is diagonal). The Lemma 5.1 gives the required form for $A$. 

**Corollary 5.3.** Let $g = \begin{pmatrix} A & B \\ 0 & A^{-1} \end{pmatrix}$ where $A = \text{diag}(1,\ldots,1,\lambda)$ be an element of $U(2l, q^2)$ then the matrix $B$ is of the form $\begin{pmatrix} B_{11} & -\lambda^{-1}^T \bar{B}_{21} \\ B_{21} & B_{22} \end{pmatrix}$ where $B_{11}$ is a skew-hermitian matrix of size $l-1$.

Proof. Yet again, we use the condition that $g$ satisfies $^t\bar{C}g = \beta$ and $A = ^tA$. 

**Lemma 5.4.** A matrix $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ belongs to $U(2l, q^2)$ if and only if $D = ^t\bar{A}^{-1}$ and $A^{-1}B$ is skew-hermitian.

Proof. The proof is simple computation. 

**Lemma 5.5.** Let $Y = \text{diag}(1,1,\ldots,1,\lambda)$ be of size $l$ where $\lambda \neq 0$ and $X = (x_{ij})$ be a matrix such that $YX$ is skew-hermitian. Then $X = (R_1 + R_2 + \ldots)Y$ where each $R_m$ is of the form $te_{i,j} - \bar{e}_{j,i}$ for some $i < j$ or of the form $se_{i,i}$ for some $i$.

Proof. Since $YX$ is skew-hermitian, the matrix $X$ is of the following form (see Lemma 5.1): $\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & x \end{pmatrix}$ where $X_{11}$ is skew-hermitian of size $(l-1) \times (l-1)$ and $X_{21}$ is a row of size $l-1$ and $X_{12} = -\lambda^T \bar{X}_{21}$ and $x$ is a scalar. Clearly any such matrix is sum of the matrices of the form $R_m Y$.

**Lemma 5.6.**

1. In the case of $U(2l, q^2)$, the element $\text{diag}(1,\ldots,1,\lambda,1,\ldots,1,\lambda^{-1})$ is a product of generators where $\lambda \in \mathbb{F}_{q^2}^\times$.

2. In the case of $U(2l+1, q^2)$ diagonal elements $\text{diag}(1,\ldots,1,\lambda,1,\ldots,1,\lambda^{-1})$ where $\lambda \in \mathbb{F}_{q^2}$ and $\text{diag}(-1,1,\ldots,1)$ are a product of generators.
Proof. We compute \( w_{l-1,l}(t) = (I + te_{l-1,l} - \bar{e}_{l-1,l})(I + \bar{e}_{l-1,l})^{-1}(I - t^{-1}e_{l-1,l})(I + te_{l-1,l} - \bar{e}_{l-1,l})(I + \bar{e}_{l-1,l})^{-1} \). Similarly we compute \( \sigma_{l-1,l}(t) = (I + te_{l-1,l} - \bar{e}_{l-1,l})(I - t^{-1}e_{l-1,l} + \bar{e}_{l-1,l})(I + te_{l-1,l} - \bar{e}_{l-1,l})(I - e_{l-1,l} - \bar{e}_{l-1,l} + \bar{e}_{l-1,l}) \). Thus multiplying \( h_{l-1,l}(t) \) with \( h_{l-1,1}(t^{-1}) \) we get diag\([1, \ldots, 1, \bar{t}, 1, \ldots, 1, t^{-1}, \bar{t}^{-1}]\). However the elements of the form \( \bar{t} \bar{t} \) cover whole set \( \mathbb{F}_q^\times \).

Lemma 5.7. Let \( g = \begin{pmatrix} \alpha & X & \ast \\ \ast & A & \ast \\ \ast & C & \ast \end{pmatrix} \) be in \( U(2l + 1, q^2) \).

1. If \( C = \text{diag}(1, \ldots, 1, \lambda) \) and \( X = 0 \) then \( A \) is of the form \( \begin{pmatrix} A_{11} & -\lambda \bar{A}_{21} \\ A_{21} & a \end{pmatrix} \) with \( A_{11} \) skew-hermitian and \( a \in \mathbb{F}_q^\times \).
2. If \( C = \text{diag}(1, \ldots, 1, 0, \ldots, 0) \) with number of \( 1 \)s equal \( m < l \) and \( X \) has first \( m \) entries 0 then \( A \) is of the form \( \begin{pmatrix} A_{11} & 0 \\ \ast & \ast \end{pmatrix} \) with \( A_{11} \) skew-hermitian and \( X = 0 \).

Proof. We use the equation \( TgTg = \beta \) and get \( 2TX = -(\bar{C}A + T\bar{A}C) \). In the first case \( X = 0 \), so we can use Corollary 5.2 to get the required form for \( A \). In the second case we write \( A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \) then the equation implies: \( \begin{pmatrix} A_{11} + T\bar{A}_{11} & A_{12} \\ T\bar{A}_{12} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2TMM \end{pmatrix} \) where \( M = (x_{m+1}, \ldots, x_l) \). This gives the required result.

Lemma 5.8. Let \( g = \begin{pmatrix} \alpha & X & Y \\ \ast & A & \ast \\ \ast & 0 & D \end{pmatrix} \) be in \( U(2l + 1, q^2) \) then \( X = 0 \) and \( D = T\bar{A}^{-1} \).

Proof. We compute \( TgTg = \beta \) and get \( 2TX = 0 \) and \( 2TX + T\bar{A}D = I \). This gives the required result.

Lemma 5.9. Let \( g = \begin{pmatrix} \alpha & 0 & Y \\ 0 & A & B \\ F & 0 & D \end{pmatrix}, \) with \( A \) an invertible diagonal matrix, be in \( U(2l + 1, q^2) \) then \( \alpha \bar{\alpha} = 1, F = 0 = Y, \) \( D = \bar{A}^{-1} \) and \( TDB + T\bar{B}D = 0 \).

Proof.

\[
TgTg = \begin{pmatrix} \bar{\alpha} & 0 & T\bar{F} \\ 0 & T\bar{A} & 0 \\ T\bar{Y} & T\bar{B} & T\bar{D} \end{pmatrix} \begin{pmatrix} 2 & \ast & \ast \\ \ast & I & \ast \\ \ast & \ast & \ast \end{pmatrix} \begin{pmatrix} \alpha & 0 & Y \\ 0 & A & B \\ F & 0 & D \end{pmatrix}
\]

\[
= \begin{pmatrix} 2\alpha \bar{\alpha} & T\bar{F}A & 2\bar{\alpha}Y + T\bar{F}B \\ 0 & T\bar{A}D \\ T\bar{Y} + T\bar{B}D & T\bar{D}A & 2\bar{\alpha}Y + T\bar{D}B + T\bar{B}D \end{pmatrix}.
\]

Equating this with \( \beta \) we get the required result.
Lemma 5.10. Let \( g = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & A & B \\ 0 & 0 & A^{-1} \end{pmatrix} \in U(2l + 1, q^2) \) where \( A = \text{diag}(1, \ldots, 1, \lambda) \) is invertible then \( B \) is of the form \( \begin{pmatrix} B_{11} & -\bar{\lambda}^{-1}TB_{21} \\ B_{21}^- & b \end{pmatrix} \) where \( B_{11} \) is skew-hermitian and \( b \in \mathbb{F}_{q^2}^* \).

Proof. This follows from the computation in the Lemma 5.9 that \( A^{-1}B + T\bar{B}A^{-1} = 0 \) and Corollary 5.2.

5.3. Row-Column operations for \( U(2l + 1, q^2) \). Here we work with the group \( U(2l + 1, q^2) \). Recall that the basis will be indexed by \( 0, 1, \ldots, l, -1, \ldots, -l \). The Chevalley generators are described in the Section 3. In general, we have four kinds of Chevalley generators. For \( 1 \leq i, j \leq l, t \in \mathbb{F}_q \) and \( s \in \mathbb{F}_{q^2}^* \),

CG1: \( \begin{pmatrix} 1 \\ R \\ T\bar{R}^{-1} \end{pmatrix} \) where \( R = I + te_{i,j}; i \neq j \).

CG2: \( \begin{pmatrix} 1 \\ I \\ R \end{pmatrix} \) where \( R \) is either \( te_{i,j} - \bar{t}e_{j,i}; i < j \) or \( se_{i,i} \).

CG3: \( \begin{pmatrix} 1 \\ I \\ R \end{pmatrix} \) where \( R \) is either \( te_{i,j} - \bar{t}e_{j,i}; i < j \) or \( se_{i,i} \).

CG4: \( I + 2te_{i,0} - t\bar{e}_{i,-i} - \bar{t}e_{-i,0} - te_{0,i} - t\bar{e}_{-i,-i} \).

We note that if we restrict to \( t \in \mathbb{F}_q \) we get \( O(2l+1, q) \subset U(2l+1, q^2) \). We also observe that CG1, CG2 and CG3 generate the subgroup \( U(2l, q^2) \) of \( U(2l+1, q^2) \) given by \( x \mapsto \begin{pmatrix} 1 \\ x \end{pmatrix} \).

Let \( g = \begin{pmatrix} \alpha & X & Y \\ E & A & B \\ F & C & D \end{pmatrix} \) be a \( (2l + 1) \times (2l + 1) \) matrix where \( A, B, C, D \) are \( l \times l \) matrices.

The matrices \( X = (X_1, X_2, \ldots, X_l), Y = (Y_1, Y_2, \ldots, Y_l), E = t(E_1, E_2, \ldots, E_l) \) and \( F = \bar{t}(F_1, F_2, \ldots, F_l) \). Let \( \alpha \in \mathbb{F}_{q^2} \). Let us note the effect of multiplication by elements of one of the types from above.

\[
\text{CG1: } \begin{pmatrix} 1 \\ R \\ T\bar{R}^{-1} \end{pmatrix}\begin{pmatrix} \alpha & X & Y \\ E & A & B \\ F & C & D \end{pmatrix} = \begin{pmatrix} \alpha & X & Y \\ RE & RA & RB \\ \bar{T}\bar{R}^{-1}F & T\bar{R}^{-1}C & T\bar{R}^{-1}D \end{pmatrix}
\]

\[
\begin{pmatrix} \alpha & X & Y \\ E & A & B \\ F & C & D \end{pmatrix} \begin{pmatrix} 1 \\ R \\ T\bar{R}^{-1} \end{pmatrix} = \begin{pmatrix} \alpha & X & Y \\ RE & RA & RB \\ \bar{T}\bar{R}^{-1}F & T\bar{R}^{-1}C & T\bar{R}^{-1}D \end{pmatrix}
\]

\[
\text{CG2: } \begin{pmatrix} 1 \\ I \\ R \end{pmatrix}\begin{pmatrix} \alpha & X & Y \\ E & A & B \\ F & C & D \end{pmatrix} = \begin{pmatrix} \alpha & X & Y \\ E + RF & A + RC & B + RD \\ F & C & D \end{pmatrix}
\]

\[
\begin{pmatrix} \alpha & X & Y \\ E & A & B \\ F & C & D \end{pmatrix} \begin{pmatrix} 1 \\ I \\ R \end{pmatrix} = \begin{pmatrix} \alpha & X & X + Y \\ E & A & AR + B \\ F & C & CR + D \end{pmatrix}
\]
CG3: $\begin{pmatrix}
1 & \alpha & X & Y \\
I & E & A & B \\
R & F & C & D \\
I & E & A & B
\end{pmatrix}
= \begin{pmatrix}
\alpha & X & Y \\
E & A & B \\
RE + F & RA + C & RB + D \\
F & C + DR & D
\end{pmatrix}$.

CG4: We only write equations that we need.

- Let the matrix $g$ has $C = \text{diag}(d_1, \ldots, d_l)$.

$$[(I + te_{0,-i} - 2te_{i,0} - ti\varepsilon_{i,-i})g]_{0,i} = X_i + td_i$$
$$[g(I + te_{0,-i} - 2te_{i,0} - ti\varepsilon_{i,-i})]_{-i,0} = F_i - 2id_i.$$  

- Let the matrix $g$ has $A = \text{diag}(d_1, \ldots, d_l)$.

$$[(I + te_{0,i} - 2te_{-i,0} - ti\varepsilon_{i,-i})g]_{0,i} = X_i + td_i$$
$$[g(I + te_{0,i} - 2te_{-i,0} - ti\varepsilon_{i,-i})]_{i,0} = E_i - 2id_i.$$  

5.3.1. The Algorithm for odd case. An overview of the algorithm is as follows:

**Step 1:** **Input:** matrix $g = \begin{pmatrix}
\alpha & X & Y \\
E & A & B \\
F & C & D
\end{pmatrix}$ which belongs to $U(2l + 1, q^2)$;

**Output:** matrix $g_1 = \begin{pmatrix}
\alpha & X_1 & Y_1 \\
E_1 & A_1 & B_1 \\
F_1 & C_1 & D_1
\end{pmatrix}$ of one of the following kind:

a: $C_1$ is a diagonal matrix $\text{diag}(1, \ldots, 1, \lambda)$ with $\lambda \neq 0$.

b: $C_1$ is a diagonal matrix $\text{diag}(1, \ldots, 1, 0, \ldots, 0)$ with number of $1$s equal to $m$ and $m < l$.

**Justification:** Using CG1 we can do row and column operations on $C$.

**Step 2:** **Input:** matrix $g_1 = \begin{pmatrix}
\alpha & X_1 & Y_1 \\
E_1 & A_1 & B_1 \\
F_1 & C_1 & D_1
\end{pmatrix}$.

**Output:** matrix $g_2 = \begin{pmatrix}
\alpha_2 & 0 & Y_2 \\
E_2 & A_2 & B_2 \\
F_2 & C_2 & D_2
\end{pmatrix}$ of one of the following kind:

a: $C_2$ is $\text{diag}(1, 1, \ldots, 1, \lambda)$ with $\lambda \neq 0$, $F_2 = 0$ and $A_2$ is of the form $\begin{pmatrix}
A_{11} & -\lambda^T A_{21} \\
A_{21} & a_{22}
\end{pmatrix}$ where $A_{11}$ is skew-hermitian of size $l - 1$ and $a_{22} \in \mathbb{F}_{q^2}$.

b: $C_2$ is $\text{diag}(1, \ldots, 1, 0, \ldots, 0)$ with number of $1$s equal to $m$; $F_2$ has first $m$ entries 0, and $A_2$ is of the form $\begin{pmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{pmatrix}$ where $A_{11}$ is an $m \times m$ skew-hermitian.

**Justification:** Once we have $C_1$ in diagonal form we use CG4 to change $X_1$ and $F_1$. In the first case they can be made 0, however in the second case we can only make first $m$ entries zero. Then Lemma 5.7 makes sure that $A_1$ has the required form.

**Step 3:** **Input:** matrix $g_2 = \begin{pmatrix}
\alpha_2 & 0 & Y_2 \\
E_2 & A_2 & B_2 \\
F_2 & C_2 & D_2
\end{pmatrix}$.

**Output:**
Step 6: Input:

Step 5: Input: $g_3 = \begin{pmatrix} \alpha & 0 & Y_3 \\ E_3 & A_3 & B_3 \\ F_3 & C_3 & D_3 \end{pmatrix}$ where $C_3$ is diag$(1,1,\ldots,1,\lambda)$.  

Step 4: Input: $g_3 = \begin{pmatrix} \alpha & 0 & Y_3 \\ E_3 & A_3 & B_3 \\ F_3 & C_3 & D_3 \end{pmatrix}$ where $C_3$ is diag$(1,1,\ldots,1,0,\ldots,0)$ with number of 1s equal to $m$; $F_3$ has first $m$ entries 0, and $A_3$ is of the form $\begin{pmatrix} 0 & 0 \\ A_{21} & A_{22} \end{pmatrix}$.

**Justification:** Observe the effect of CG2 and CG4. Then the Lemma 5.10 ensures that $B_3$ is of a certain kind. We can use CG2 to make $B_3 = 0$. The Lemma 5.8 ensures that $A_3$ has full rank. Further we can use CG4 to make $X_3 = 0$ and $E_3 = 0$. The Lemma 5.9 gives the required form.

Output: $g_4 = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & A_4 & B_4 \\ 0 & 0 & \bar{A}_4^{-1} \end{pmatrix}$ with $A_4$ diag$(1, \ldots, 1, \lambda)$ and $\alpha \in \mathbb{F}_{q^2}^1$.

**Justification:** In the first case, interchange rows $i$ and $-i$ for all $1 \leq i \leq l$. Now the matrix is in the form so that we can apply Lemma 5.9 and get the required result. In the second case we interchange $i$ with $-i$ for $1 \leq i \leq m$. This will make $C_3 = 0$. Then if needed we use CG1 on $A_3$ to make it diagonal. The Lemma 5.8 ensures that $A_3$ has full rank. Further we can use CG4 to make $X_3 = 0$ and $E_3 = 0$. The Lemma 5.9 gives the required form.

Step 5: Input: $g_4 = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & A_4 & B_4 \\ 0 & 0 & \bar{A}_4^{-1} \end{pmatrix}$ with $A_4 = \text{diag}(1, \ldots, 1, \lambda)$ and $\alpha \in \mathbb{F}_{q^2}^1$.

Output: $g_5 = \text{diag}(\alpha, 1, \ldots, 1, \lambda, 1, \ldots, 1, \lambda^{-1})$ where $\alpha \in \mathbb{F}_{q^2}^1$.

**Justification:** Lemma 5.10 ensures that $B_4$ is of a certain kind. We can use CG2 to make $B_4 = 0$.

Step 6: Input: matrix diag$(\alpha, 1, \ldots, 1, \lambda, 1, \ldots, 1, \lambda^{-1})$ where $\alpha \in \mathbb{F}_{q^2}^1$.

Output: Identity matrix.

**Justification:** We use Lemma 5.6 to reduce $\lambda$ modulo $\mathbb{F}_q^*$ and use the generators $d(\zeta)$ and $d(\alpha)$ to get the result.

6. SECURITY OF THE PROPOSED MOR CRYPTOSYSTEM

As we saw in Section 4 there are three kind of automorphisms in an unitary group. One is conjugation automorphism, the others are central and field automorphisms. A central automorphism being multiplication by an element of the center, that is a field element. Exponentiation of that will be a discrete logarithm problem in $\mathbb{F}_q$. Similar is the case with a field automorphism. So the only choice for a better MOR cryptosystem is a conjugating automorphism.

The success of any cryptosystem comes from a balance between speed and security. This paper is largely about the security of a proposed MOR cryptosystem. So we will be brief in our study of implementation of this cryptosystem. It is often argued, once the security of a proposed MOR cryptosystem is the hardness of the discrete logarithm problem in $\mathbb{F}_q^*$; where we are working with matrices of size $d$ over a finite field $\mathbb{F}_q$, there is no point working with automorphisms as presented as action on generators. Instead one should move into matrices. We saw that security in $\mathbb{F}_q$ is the same as finding the conjugating element of the automorphism up to a scalar multiple. So instead of presenting
the automorphisms as action on generators, we should present the conjugating matrix and its power as the public key.

We refrain from doing that in this paper. To explain our choice we present how we envision the implementation of exponentiation of automorphisms as presented as action on generators. The algorithm of our choice is the famous square-and-multiply algorithm. Since we do not use any special algorithm for squaring, squaring and multiplying is the same for us. So let us talk about multiplying two automorphisms. As we talked before, we present the automorphisms as action on generator s, i.e., \( \phi(g) \) is a matrix for \( i = 1, 2, \ldots, s \). The first step of the algorithm is to find the word in generators from the matrix\(^\dagger\). So now the automorphism is \( \phi(g_i) = w_i \) where each \( w_i \) is a word in generators. Once that is done then composing with an automorphism is substituting each generator in the word by another word. This can be done fast. The challenging thing is to find the matrix corresponding to the word thus formed. This is not a hard problem, but can be both time and memory intensive. What is the best way to do it is still an open question! However, there are many shortcuts available. One being an obvious time-memory trade off, like storing matrices corresponding to a word in generators. The other being there are many trivial and non-trivial relations among these generators and moreover these generators are sparse matrices. One can use them.

This problem, which is of independent interest in computational group theory and is the reason that we insist on automorphisms being presented as generators in this paper. For more information, see [12, Section 8].

6.1. Reduction of security. In this subsection, we show that for unitary groups, the security of the MOR cryptosystem is the hardness of the discrete logarithm problem in \( \mathbb{F}_{q^2} \). This is the same as saying that we can find the conjugating matrix up to a scalar multiple. Let \( \phi \) be an automorphism that works by conjugation, i.e., \( \phi = \iota_g \) for some \( g \) and we try to determine \( g \).

**Step 1:** The automorphism \( \phi \) is presented as action on generators. Thus \( \phi(x_{i,-i}(s)) = g(I + s\epsilon_{i,-i})g^{-1} = I + s\epsilon g_{i,-i}g^{-1} \). This implies that we know \( \epsilon g_{i,-i}g^{-1} \) and similarly \( \epsilon g_{-i,i}g^{-1} \) for fixed \( s = \epsilon \). We first claim that we can determine \( N := gD \) where \( D \) is diagonal.

When \( d = 2l \), write \( g \) in the column form \([G_1, \ldots, G_l, G_{-1}, \ldots, G_{-l}]\). Now,

1. \([G_1, \ldots, G_l, G_{-1}, \ldots, G_{-l}] \epsilon_{e_{i,-i}} = [0, \ldots, 0, \epsilon G_{i}, 0, \ldots, 0] \) where \( G_i \) is at \(-i^{th}\) place.
   - Multiplying this further with \( g^{-1} \) gives us scalar multiple of \( G_i \), say \( d_i \).
2. \([G_1, \ldots, G_l, G_{-1}, \ldots, G_{-l}] \epsilon_{e_{-i,i}} = [0, \ldots, 0, \epsilon G_{-i}, 0, \ldots, 0] \) where \( G_{-i} \) is at \( i^{th}\) place.
   - Multiplying this with \( g^{-1} \) gives us scalar multiple of \( G_{-i} \), say \( d_{-i} \).

Thus we get \( N = gD \) where \( D \) is a diagonal matrix diag\((d_1, \ldots, d_l, d_{-1}, \ldots, d_{-l})\). In the case when \( d = 2l + 1 \) we write \( g = [G_0, G_1, \ldots, G_l, G_{-1}, \ldots, G_{-l}] \) and get scalar multiple of columns \( G_i \) and \( G_{-i} \). Further we use \( x_{i,0}(l) \) and \( x_{0,i}(l) \) to get linear combination of \( G_0 \) with \( G_i \) or \( G_{-i} \), say we get \( \alpha G_0 + \beta G_{-i} \). In this case we get \( N = gD \) where \( D \) is of the

\(^\dagger\)One can also present the automorphisms as word in generators, we choose matrices.
form

\[
\begin{pmatrix}
\alpha & d_1 & \cdots & d_l \\
\beta & d_l & \ddots & \vdots \\
& & \ddots & d_{-1}
\end{pmatrix}
\]

**Step 2:** Now we compute \(N^{-1}\phi(x_r(t))N = D^{-1}g^{-1}(gx_r(t)g^{-1})gD = D^{-1}x_r(t)D\). Substituting various \(x_r(t)\) it amounts to computing \(D^{-1}e_rD\). When \(d = 2l\), we first compute \(D^{-1}(e_{i,j} - e_{-j,-i})D\) and get \(d_i^{-1}d_j, d_{-1}^{-1}d_{-j}\) for \(i \neq j\). Then we compute \(D^{-1}e_{i,-i}D, D^{-1}e_{-i,i}D\) and get \(d_id_{-1}^{-1}, d_{-i}d_i^{-1}\). We form a matrix

\[
\text{diag}(1, d_1^{-1}d_1, \ldots, d_l^{-1}d_l, d_{-1}^{-1}d_{-2}, d_{-2}^{-1}d_2d_1^{-1}d_1, \ldots, d_{-1}^{-1}d_{-1}d_{-1}^{-1}d_1)
\]

and multiply it to \(N = gD\) to get \(d_ig\). Thus we can determine \(g\) up to a scalar multiple and the attack follows [12, Section 7.1.1].

In the case \(d = 2l + 1\), the matrix \(D\) is almost a diagonal matrix except the first column. However while computing \(D^{-1}(e_{12} - e_{-2,-1})D\) we also get \(d_2^{-1}\beta\) and by computing \(D^{-1}(e_{0,1} - 2e_{-1,0} - e_{-1,1})D\) we get \(\alpha^{-1}D\). Thus we can multiply \(\alpha G_0 + \beta G_{-1}\) by \(\beta^{-1}d_1 = \beta^{-1}d_{-2}d_{-2}^{-1}d_{-1}d_{-1}^{-1}d_1\) and get \(\alpha\beta^{-1}d_1G_0 + d_1G_{-1}\). With the computation in even case we can determine \(d_1G_{-1}\) and hence can determine \(\alpha G_0\). Further since we know \(\alpha^{-1}d_1\) we can determine \(d_1G_0\) thus in this case as well we can determine \(d_ig\), i.e., \(g\) up to a scalar multiple.

### 7. Conclusion

In coming years, public-key cryptography might go through a major change. This is because of the new improvements in index-calculus attacks by Joux [110]. If the worst-anticipated situation do occur then we will be left with only one cryptographic primitive – the discrete logarithm problem in the group of rational points of an elliptic curve over a finite fields. This view is somewhat supported by NSA’s move to use only elliptic curves in suite B cryptosystems. The obvious question then is how to build a new cryptosystem. Cryptosystems are not that readily trused by the community as secure. It takes decades, if not longer to get to a marketable cryptosystem. In this paper, we study the MOR cryptosystem with an eye towards the horizon. It can provide us with the means to get to a new cryptosystem. There are some challenges to this idea. The chief being large key sizes. It is true that the classical groups like the orthogonal groups, symplectic groups and unitary groups are generated by two generators [9, Theorem A]. So we can have very small key sizes. On the other hand using Chevalley generators, one can solve the word problem easily. That has the advantage in implementation as we demonstrated in this paper. Now the obvious question comes: how do we go from one set of generators to the other in an efficient way? This is aligned with current research directions in computational group theory [11]. So this line of research is not only beneficial to cryptography but also to computational group theory.

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