On Solutions of Tetrahedron Equations based on Korepanov Mechanism

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Abstract

We have examined solutions of tetrahedron equations from the elliptic free fermion model by using Korepanov mechanism based on tetrahedral Zamolodchikov algebras. As a byproduct, we have found a new integrable 2-dim. lattice model. We have also studied the relation between tetrahedral Zamolodchikov algebras and tetrahedron equations.

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§1. Introduction

In last two decades, much understanding has been made in 2-dim. integrable statistical models with beautiful mathematical structures [1], and recently much attention has turned to 3-dim. integrable statistical models.

Tetrahedron equations are 3-dim. generalizations of Yang-Baxter equations in 2-dim., and Zamolodchikov first found solutions of tetrahedron equations [2], and Baxter confirmed Zamolodchikov’s result [3]. After these pioneering works, little progress [4] [5] has been made until recently because of difficulties of the problem. After Bazhanov-Baxter [6] have obtained solutions of tetrahedron equations for \( N \) colors model, by generalizing Zamolodchikov’s solutions of 2 colors model, many interesting papers have published on solutions of tetrahedron equations [7, 8, 9, 10, 11, 12].

In this paper, we take Korepanov’s approach and examine solutions of tetrahedron equations from the elliptic free fermion model. In section 2, we review Korepanov’s approach and examine the relation between tetrahedral Zamolodchikov algebras and tetrahedron equations. In section 3, we give explicit solutions of tetrahedron equations from the elliptic free fermion model.

§2. Relation between Tetrahedral Zamolodchikov Algebras and Tetrahedron Equations - Korepanov Mechanism -

We follow the notation of Korepanov, and denote the standard \( R \)-matrices as \( R^0 \) and the related non-symmetric \( R \)-matrices as \( R^1 \) [10, 11, 12]. For the 8-vertex case, \( R^0 \) and \( R^1 \) are written in the forms

\[
R^0_{ij}(u_i, u_j) = f_0(u_i, u_j) \begin{pmatrix} a_{ij} & 0 & 0 & d_{ij} \\ b_{ij} & c_{ij} & 0 \\ 0 & c_{ij} & b_{ij} & 0 \\ d_{ij} & 0 & 0 & a_{ij} \end{pmatrix},
\]

\[
R^1_{ij}(u_i, u_j) = f_1(u_i, u_j) \begin{pmatrix} -a'_{ij} & 0 & 0 & d'_{ij} \\ 0 & -b'_{ij} & c'_{ij} & 0 \\ 0 & -c_{ij} & b_{ij} & 0 \\ -d'_{ij} & 0 & 0 & a'_{ij} \end{pmatrix},
\]

where

\[
a_{ij} = \text{sn}(\lambda - u_i + u_j), \quad b_{ij} = \text{sn}(u_i - u_j),
\]

\[
c_{ij} = \text{sn}(\lambda), \quad d_{ij} = k\text{sn}(\lambda)\text{sn}(u_i - u_j)\text{sn}(\lambda - u_i + u_j),
\]

\[
a'_{ij}, b'_{ij}, c'_{ij}, d'_{ij}
\]
\[ a_{ij}' = \text{sn}(\lambda - u_i - u_j), \quad b_{ij}' = \text{sn}(u_i + u_j), \]
\[ c_{ij}' = \text{sn}(\lambda), \quad d_{ij}' = k\text{sn}(\lambda)\text{sn}(u_i + u_j)\text{sn}(\lambda - u_i - u_j), \]  
(4)

where \( \text{sn}(u) \) is one of the Jacobi’s elliptic functions, \( k \) is the modulus of the elliptic function, and \( \lambda \) is an arbitrary constant parameter. The functions \( f_0(u_i, u_j) \) and \( f_1(u_i, u_j) \) are arbitrary here, and we choose the special functional forms in section 3. We may use short-hand notations \( R_{ij}^a \) instead of full notations \( R_{ij}^a(u_i, u_j) \).

Using the above parametrization, \( R \)-matrices satisfy the following Yang-Baxter equations

\[
R_{12}^0(u_1, u_2)R_{13}^0(u_1, u_3)R_{23}^0(u_2, u_3) = R_{23}^0(u_2, u_3)R_{13}^0(u_1, u_3)R_{12}^0(u_1, u_2),
\]
(5)

where \( R_{ij}^a \) act as \( \text{End}(V^{\otimes 3}) \). Operators \( R_{12}^a \), for example, acts on \( V_1 \otimes V_2 \) as non-trivial \( R \)-matrices and on \( V_3 \) as identity.

First, we have found that \( R_{ij}^0 \) and \( R_{ij}^1 \) satisfy the following equations

\[
R_{12}^0(u_1, u_2)R_{13}^1(u_1, u_3)R_{23}^1(u_2, u_3) = R_{23}^1(u_2, u_3)R_{13}^1(u_1, u_3)R_{12}^0(u_1, u_2),
\]
(6)

which reflect the symmetry of \( R_{ij}^0 \) and \( R_{ij}^1 \). Eq. (6) gives us a new example of integrable 2-dim. lattice models with Boltzmann weights \( R_{ij}^1 \). In order that transfer matrices which are composed of \( R_{ij}^1 \) commute, it is sufficient to exist the invertible \( R_{ij}^0 \) which satisfy \( R_{12}^0 R_{13}^1 R_{23}^0 = R_{23}^1 R_{13}^1 R_{12}^0 \). Thus Eq. (6) gives us a new class of integrable 2-dim. lattice models, though its Boltzmann weights do not satisfy the physical positivity condition. Though this new integrable model is unphysical, it will be helpful to study new mathematics behind this type of integrable models.

Next we consider Korepanov mechanism to construct solutions of tetrahedron equations (Zamolodchikov equations). Korepanov consider the following tetrahedral Zamolodchikov algebras, which represent scattering relations

\[
R_{12}^a(u_1, u_2)R_{13}^b(u_1, u_3)R_{23}^c(u_2, u_3) = \sum_{d,e,f=0} S(u_1, u_2, u_3)^{abc}_{\text{def}} R_{23}^f(u_2, u_3)R_{13}^e(u_1, u_3)R_{12}^d(u_1, u_2),
\]
(7)

where \( S^{abc}_{\text{def}} \) are the \( S \)-matrices of particles \( R_{ij}^a \). Here Korepanov have taken the very special forms, that is, 2 dim. integrable \( R \)-matrices \( R_{ij}^0 \) and their related \( R_{ij}^1 \) matrices, for particles.

Tetrahedral Zamolodchikov algebras (7) are overdeterministic in a sense that, there are 64 equations with only 8 variables \( S^{abc}_{\text{def}} \) for fixed \( \{a, b, c\} \) in general, because \( R_{12}^a R_{13}^b R_{23}^c \)
and \( R_{ij}^k R_{kl}^{ij} \) are \( 8 \times 8 \) matrices, while \( S_{abc}^{def} \) are simple numbers for given \( \{a, b, c, d, e, f\} \). Because of the special symmetry of these \( R_{ij}^k \) and \( R_{ij}^l \), only 20 equations are not automatically satisfied, while the number of the variables \( S_{abc}^{def} \) to be solved is 8 for the fixed \( \{a, b, c\} \). Tetrahedral Zamolodchikov algebras make it quite promising that these \( S_{abc}^{def} \) become 4 in the tetrahedral Zamolodchikov algebras.

We denote the above equations in the following short-hand way:

\[
\sum_{a', b', c', d', e', f'} S(u_1, u_2, u_3)_{a'b'c'} S(u_1, u_2, u_4)_{a'd'e'}
\times S(u_2, u_3, u_4)_{b'd'f'} S(u_2, u_3, u_4)_{c'e'f'}
= \sum_{a', b', c', d', e', f'} S(u_2, u_3, u_4)_{a'b'c'} S(u_1, u_2, u_4)_{a'd'e'}
\times S(u_1, u_2, u_3)_{a'b'c'}.
\]

This mechanism to find solutions of tetrahedron equations by using solutions of tetrahedral Zamolodchikov algebras, we call Korepanov mechanism.

We denote the above equations in the following short-hand way:

\[
S_{123}(u_1, u_2, u_3) S_{145}(u_1, u_2, u_4) S_{246}(u_1, u_3, u_4) S_{356}(u_2, u_3, u_4)
= S_{356}(u_2, u_3, u_4) S_{246}(u_1, u_3, u_4) S_{145}(u_1, u_2, u_4) S_{123}(u_1, u_2, u_3),
\]

where \( S_{abc}^{def} \) act as \( \text{End}(V^\otimes 6) \). Operators \( S_{123} \), for example, acts on \( V_1 \otimes V_2 \otimes V_3 \) as \( S_{abc}^{def} \) and on the rest as identity.

Here we sketch the reason why solutions of tetrahedral Zamolodchikov algebras are promising candidates to solutions of tetrahedron equations. We start from the following objects

\[
R_{12}^{a_1} R_{13}^{a_2} R_{23}^{a_3} R_{14}^{a_4} R_{24}^{a_5} R_{34}^{a_6},
\]
and we successively use tetrahedral Zamolodchikov algebras (7) for $R_{12}^{a_1} R_{13}^{a_2} R_{23}^{a_3} R_{12}^{b_1} R_{13}^{b_2} R_{23}^{b_3}$, $R_{13}^{b_4} R_{23}^{c_4} R_{34}^{a_4}$, $R_{23}^{c_5} R_{34}^{a_5}$, starting from the above object (10). We then obtain

$$R_{12}^{a_1} R_{13}^{a_2} R_{23}^{a_3} R_{14}^{a_4} R_{24}^{a_5} R_{34}^{a_6} = \sum_{b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5, c_6} S(u_1, u_2, u_3)^{a_1} b_2 b_3 b_4 S(u_1, u_2, u_4)^{a_2} b_5 b_6 S(u_1, u_3, u_4)^{a_3} b_7 b_8 b_9 \times S(u_2, u_3, u_4)^{a_4} b_{10} b_{11} b_{12} R_{34}^{c_6} R_{24}^{c_5} R_{23}^{c_4} R_{13}^{c_3} R_{12}^{c_2} R_{11}^{c_1}. \tag{11}$$

Here we have used the fact that $[R_{ij}^{a}, R_{kl}^{b}] = 0$ for $i \neq k, l$ and $j \neq k, l$, for example $[R_{12}^{a}, R_{34}^{b}] = 0$, from the property of the tensor product.

Next, starting from the same objects (10), we successively use tetrahedral Zamolodchikov algebras for $R_{12}^{a_1} R_{13}^{a_2} R_{23}^{a_3} R_{14}^{a_4} R_{24}^{a_5} R_{34}^{a_6}$, $R_{13}^{a_7} R_{23}^{a_8} R_{34}^{a_9}$, $R_{23}^{a_{10}} R_{34}^{a_{11}} R_{14}^{a_{12}} R_{13}^{a_{13}} R_{12}^{a_{14}} R_{11}^{a_{15}} R_{10}^{a_{16}}$, which is different from the previous ordering in the use of tetrahedral Zamolodchikov algebras. In this ordering, we obtain

$$R_{12}^{a_1} R_{13}^{a_2} R_{23}^{a_3} R_{14}^{a_4} R_{24}^{a_5} R_{34}^{a_6} = \sum_{b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5, c_6} S(u_2, u_3, u_4)^{a_1} b_2 b_3 b_4 S(u_1, u_3, u_4)^{a_2} b_5 b_6 S(u_1, u_2, u_4)^{a_3} b_7 b_8 b_9 \times S(u_1, u_2, u_3)^{a_4} b_{10} b_{11} b_{12} R_{34}^{c_6} R_{24}^{c_5} R_{23}^{c_4} R_{13}^{c_3} R_{12}^{c_2} R_{11}^{c_1}. \tag{12}$$

From Eqs. (11) and (12), we obtain

$$\sum_{b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5, c_6} S(u_1, u_2, u_3)^{a_1} b_2 b_3 b_4 S(u_1, u_2, u_4)^{a_2} b_5 b_6 S(u_1, u_3, u_4)^{a_3} b_7 b_8 b_9 \times S(u_2, u_3, u_4)^{a_4} b_{10} b_{11} b_{12} R_{34}^{c_6} R_{24}^{c_5} R_{23}^{c_4} R_{13}^{c_3} R_{12}^{c_2} R_{11}^{c_1} \tag{13}$$

These relations strongly suggest that the following Zamolodchikov’s tetrahedron equations are satisfied

$$\sum_{b_1, b_2, b_3, b_4, b_5, b_6} S(u_1, u_2, u_3)^{a_1} b_2 b_3 S(u_1, u_2, u_4)^{a_2} b_4 b_5 \times S(u_1, u_4)^{a_3} b_6 b_7 S(u_2, u_3, u_4)^{a_4} b_8 b_9 S(u_2, u_4)^{a_5} b_2 b_3 b_4 b_5 b_6 b_7 b_8 b_9 R_{34}^{c_6} R_{24}^{c_5} R_{23}^{c_4} R_{13}^{c_3} R_{12}^{c_2} R_{11}^{c_1}. \tag{13}$$
which are equivalent to short-hand forms (14).

In this way, if tetrahedron equations (14) are satisfied, integrability conditions (13) of tetrahedral Zamolodchikov algebras are automatically satisfied, but the opposite is not always true. According to Korepanov mechanism, we must first solve tetrahedral Zamolodchikov algebras to find $S_{abc}^{def}$, which are promising candidates to satisfy tetrahedron equations. Next we must check explicitly whether these $S_{abc}^{def}$ really satisfy tetrahedron equations or not.

\section{Solutions of Tetrahedron Equations from Elliptic Free Fermion Model}

Here we start to solve tetrahedral Zamolodchikov algebras. We consider first the sector $\{a = 0, b = 0, c = 0\}$ in tetrahedral Zamolodchikov algebras (14). From Yang-Baxter equations (5), we can see that $S(u_1, u_2, u_3)_{000}^{000} = \delta_{d,0}^{0}\delta_{e,0}^{0}\delta_{f,0}^{0}$ are one of the solutions. In the eight vertex case, solutions become unique, and the above are only unique solutions. The same situation happens for the sector $\{a = 0, b = 1, c = 1\}$, and solutions are uniquely determined to give $S(u_1, u_2, u_3)^{101}_{de,1} = \delta_{d,1}^{1}\delta_{e,1}^{1}\delta_{f,1}^{1}$. For $\{a = 1, b = 0, c = 1\}$ and $\{a = 1, b = 1, c = 0\}$ sectors, some delicate thing happens. From four equations we obtain solutions $S(u_1, u_2, u_3)^{101}_{de,1} = \delta_{d,1}^{1}\delta_{e,1}^{1}\delta_{f,1}^{1}$ and $S(u_1, u_2, u_3)^{110}_{de,1} = \delta_{d,1}^{1}\delta_{e,1}^{1}\delta_{f,0}^{1}$ for $\{a = 1, b = 0, c = 1\}$ and $\{a = 1, b = 1, c = 0\}$ sectors respectively. But because of the overdeterministic property of tetrahedral Zamolodchikov algebras, there are still six equations to be satisfied, and we can see that these six equations are not satisfied in this 8 vertex parametrization, that is, we can conclude that there are no solutions for tetrahedral Zamolodchikov algebras in the 8 vertex parametrization. Then we must consider more restricted cases to find solutions of tetrahedral Zamolodchikov algebras. One of the cases which give solutions of tetrahedral Zamolodchikov algebras is the symmetric free fermion case. The symmetric free fermion case is the special case of the 8 vertex case, that is, it is the case that $\lambda$ in Eqs.(3) and (4) becomes the complete elliptic integral $\lambda = K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$. In this special case, we can easily check that the free fermion condition $a^2 + b^2 = c^2 + d^2$ is satisfied [13, 14]. Choosing $f_0(u_i, u_j) = \frac{\text{dn}(u_i - u_j)}{\text{sn}(u_i - u_j)}$, and $f_1(u_i, u_j) = \frac{1}{\text{cn}(u_i + u_j)}$, $R_{ij}^0$ and $R_{ij}^1$ become in the
forms

\[
R_{ij}^0(u_i, u_j) = \begin{pmatrix}
\frac{\text{cn}(u_i - u_j)}{\text{sn}(u_i - u_j)} & 0 & 0 & k\text{cn}(u_i - u_j) \\
0 & \frac{\text{dn}(u_i - u_j)}{\text{sn}(u_i - u_j)} & \frac{\text{dn}(u_i - u_j)}{\text{sn}(u_i - u_j)} & 0 \\
0 & \frac{\text{dn}(u_i - u_j)}{\text{sn}(u_i - u_j)} & \frac{\text{dn}(u_i - u_j)}{\text{sn}(u_i - u_j)} & 0 \\
k\text{cn}(u_i - u_j) & 0 & 0 & \frac{\text{cn}(u_i - u_j)}{\text{sn}(u_i - u_j)}
\end{pmatrix}
\]

(15)

where \(\text{cn}(u), \text{dn}(u)\) are also Jacobi’s elliptic functions. Solutions of tetrahedral Zamolodchikov algebras are unique in this case, and we obtain the following solutions (In [12], Korepanov announced that solutions of tetrahedral Zamolodchikov algebras in the elliptic free fermion case really satisfy tetrahedron equations, but did not give explicit expressions of \(S_{\text{def}}^{abc}\));

\[
S_{\text{Sol}}^{000}(u_1, u_2, u_3)_{000} = \frac{g(u_2 - u_3)g(u_2 + u_3)}{g(u_1 - u_3)g(u_1 + u_3)},
\]

\[
S_{\text{Sol}}^{001}(u_1, u_2, u_3)_{001} = (1 - k^2)g(u_1 - u_3)g(u_2 + u_3),
\]

\[
S_{\text{Sol}}^{010}(u_1, u_2, u_3)_{010} = \frac{1}{g(u_1 - u_3)g(u_1 + u_3)},
\]

\[
S_{\text{Sol}}^{011}(u_1, u_2, u_3)_{011} = \frac{g(u_2 - u_3)g(u_2 + u_3)}{(1 - k^2)g(u_1 - u_3)},
\]

\[
S_{\text{Sol}}^{001}(u_1, u_2, u_3)_{001} = \frac{g(u_2 - u_3)g(u_3)g(u_1 + u_3)}{(1 - k^2)g(u_2 - u_3)g(u_2 + u_3)},
\]

\[
S_{\text{Sol}}^{010}(u_1, u_2, u_3)_{010} = \frac{g(u_1 - u_3)g(u_2 + u_3)}{(1 - k^2)g(u_1 + u_3)},
\]

\[
S_{\text{Sol}}^{100}(u_1, u_2, u_3)_{100} = \frac{g(u_1 - u_3)g(u_2 + u_3)}{(1 - k^2)g(u_1 + u_3)},
\]

\[
S_{\text{Sol}}^{101}(u_1, u_2, u_3)_{101} = \frac{g(u_1 - u_3)g(u_2 + u_3)}{(1 - k^2)g(u_1 + u_3)},
\]

\[
S_{\text{Sol}}^{110}(u_1, u_2, u_3)_{110} = \frac{g(u_1 - u_3)g(u_2 + u_3)}{(1 - k^2)g(u_1 + u_3)},
\]

\[
S_{\text{Sol}}^{111}(u_1, u_2, u_3)_{111} = \frac{g(u_1 - u_3)g(u_2 + u_3)}{(1 - k^2)g(u_1 + u_3)},
\]

(17)
\[ S^{\text{Sol.}}(u_1, u_2, u_3)_{010}^{100} = \frac{g(u_1 - u_2)g(u_1 + u_2)}{g(u_1 - u_3)g(u_1 + u_3)} , \]
\[ S^{\text{Sol.}}(u_1, u_2, u_3)_{011}^{100} = -\frac{1}{g(u_1 - u_3)g(u_1 + u_3)} , \]
\[ S^{\text{Sol.}}(u_1, u_2, u_3)_{001}^{111} = g(u_1 - u_2)g(u_1 + u_2) , \]
\[ S^{\text{Sol.}}(u_1, u_2, u_3)_{110}^{010} = -(1-k^2)g(u_1 - u_2)g(u_1 + u_2)g(u_2 - u_3)g(u_2 + u_3) , \]
\[ S^{\text{Sol.}}(u_1, u_2, u_3)_{111}^{010} = -g(u_2 - u_3)g(u_2 + u_3) , \]

and all other components are zeros, where we use the notation \( g(u) = \frac{\text{sn}(u)}{c_0(u)\text{dn}(u)} \).

We have explicitly checked that these solutions of tetrahedral Zamolodchikov algebras really satisfy tetrahedron equations.

Notice that there is the symmetry \( S^{\text{Sol.}}(u_1, u_2, u_3)^{abc} = S^{\text{Sol.}}(u_3, u_2, u_1)^{bca} \), which comes by taking the transpose of tetrahedral Zamolodchikov algebras.

Under the change of variables from \( u_i \) to new variables \( \varphi_i \) in the form \( \varphi_i = (1-k^2)g(u_i)^2 \), we can prove the relations \( \tanh(\varphi_i - \varphi_j) = (1-k^2)g(u_i+u_j)g(u_i-u_j)^3 \). Then we can rewrite \( S^{\text{Sol.}} \) as the function only of the differences of \( \varphi_i \). If we change the transformation of the Boltzmann weights \( S^{\text{Sol.}}(\varphi_1, \varphi_2, \varphi_3)^{abc} = (1-k^2)^{(a+b+c-d-e-f)/2} S^{\text{Sol.}}(u_1, u_2, u_3)^{abc} \), which keeps tetrahedron eqs. invariant, we obtain the same expression as those of Korepanov \[11\].

\[ \text{[12]} \text{ in the following form;} \]
\[ \tilde{S}^{\text{Sol.}}(\varphi_1, \varphi_2, \varphi_3)_{000}^{000} = \tilde{S}^{\text{Sol.}}(\varphi_1, \varphi_2, \varphi_3)_{011}^{011} = \tilde{S}^{\text{Sol.}}(\varphi_1, \varphi_2, \varphi_3)_{101}^{101} = \tilde{S}^{\text{Sol.}}(\varphi_1, \varphi_2, \varphi_3)_{110}^{110} = 1 , \]
\[ \tilde{S}^{\text{Sol.}}(\varphi_1, \varphi_2, \varphi_3)_{001}^{011} = \frac{\tanh(\varphi_2 - \varphi_3)}{\tanh(\varphi_1 - \varphi_3)} , \]
\[ \tilde{S}^{\text{Sol.}}(\varphi_1, \varphi_2, \varphi_3)_{010}^{101} = \frac{\tanh(\varphi_2 - \varphi_3)}{\tanh(\varphi_1 - \varphi_3)} , \]
\[ \tilde{S}^{\text{Sol.}}(\varphi_1, \varphi_2, \varphi_3)_{011}^{101} = \frac{1}{\tanh(\varphi_1 - \varphi_3)} , \]
\[ \tilde{S}^{\text{Sol.}}(\varphi_1, \varphi_2, \varphi_3)_{010}^{100} = \tilde{S}^{\text{Sol.}}(\varphi_1, \varphi_2, \varphi_3)_{010}^{100} = 1 , \]
\[ \tilde{S}^{\text{Sol.}}(\varphi_1, \varphi_2, \varphi_3)_{011}^{101} = 1 , \]
\[ \tilde{S}^{\text{Sol.}}(\varphi_1, \varphi_2, \varphi_3)_{101}^{010} = -\tanh(\varphi_1 - \varphi_2) , \]
\[ \tilde{S}^{\text{Sol.}}(\varphi_1, \varphi_2, \varphi_3)_{111}^{010} = \frac{\tanh(\varphi_1 - \varphi_2)}{\tanh(\varphi_1 - \varphi_3)} , \]
\[ \tilde{S}^{\text{Sol.}}(\varphi_1, \varphi_2, \varphi_3)_{110}^{010} = \frac{1}{\tanh(\varphi_1 - \varphi_3)} . \]

\[ 3 \text{ We thank Y.G. Stroganov and J. Hietarinta for this observation.} \]
\[ \tilde{S}^\text{Sol.}(\varphi_1, \varphi_2, \varphi_3)_{001}^{111} = \tanh(\varphi_1 - \varphi_2), \]
\[ \tilde{S}^\text{Sol.}(\varphi_1, \varphi_2, \varphi_3)_{010}^{111} = -\tanh(\varphi_1 - \varphi_2) \tanh(\varphi_2 - \varphi_3), \]
\[ \tilde{S}^\text{Sol.}(\varphi_1, \varphi_2, \varphi_3)_{100}^{111} = -\tanh(\varphi_2 - \varphi_3), \]

Though it is quite non-trivial, we thus obtained the same solutions of tetrahedron eqs. by using Korepanov mechanism even though we start from the elliptic case instead of the trigonometric one in the free fermion model. Though we cannot find new solutions of tetrahedron eqs. from the elliptic free fermion model by using Korepanov mechanism, it will be helpful to understand Korepanov’s method and to find new solutions by using Korepanov mechanism.

Here we give examples which satisfy tetrahedral Zamolodchikov algebras but not satisfy tetrahedron equations by considering trigonometric one in the free fermion model. In this case, tetrahedral Zamolodchikov algebras are not uniquely solved \[ [11, 12], \] and general solutions can be obtained by including arbitrary parameters. We denote

\[ S(u_1, u_2, u_3)^{abc}_{\text{def}} = S^\text{Korepa}(u_1, u_2, u_3)^{abc}_{\text{def}} + S^\text{Homo}(u_1, u_2, u_3)^{abc}_{\text{def}}, \]

where \( S^\text{Korepa} = S^\text{Sol.}|_{k=0}, \) then \( S^\text{Homo} \) satisfy homogeneous equations

\[ \sum_{d,e,f=0}^{1} S^\text{Homo}(u_1, u_2, u_3)^{abc}_{\text{def}} R_{23}^{f}(u_2, u_3) R_{13}^{e}(u_1, u_3) R_{12}^{d}(u_1, u_2) = 0, \tag{19} \]

and we obtain solutions \( S^\text{Homo}(u_1, u_2, u_3)^{abc}_{\text{def}}, \) which include arbitrary parameters \( h_{\text{even}} \) and \( h_{\text{odd}} \) of the following forms

\[ S^\text{Homo}(u_1, u_2, u_3)^{abc}_{000} = h_{\text{even}}^{abc}, \]
\[ S^\text{Homo}(u_1, u_2, u_3)^{abc}_{001} = -h_{\text{odd}}^{abc} \tan(u_1 - u_2) \tan(u_1 + u_3), \]
\[ S^\text{Homo}(u_1, u_2, u_3)^{abc}_{010} = -h_{\text{odd}}^{abc} \tan(u_1 - u_2) \tan(u_2 - u_3), \]
\[ S^\text{Homo}(u_1, u_2, u_3)^{abc}_{011} = h_{\text{even}}^{abc} \cot(u_2 - u_3) \cot(u_1 + u_3), \]
\[ S^\text{Homo}(u_1, u_2, u_3)^{abc}_{100} = h_{\text{odd}}^{abc} \tan(u_2 - u_3) \tan(u_1 + u_3), \]
\[ S^\text{Homo}(u_1, u_2, u_3)^{abc}_{101} = -h_{\text{even}}^{abc} \cot(u_1 - u_2) \cot(u_2 - u_3), \]
\[ S^\text{Homo}(u_1, u_2, u_3)^{abc}_{110} = -h_{\text{even}}^{abc} \cot(u_1 - u_2) \cot(u_1 + u_3), \]
\[ S^\text{Homo}(u_1, u_2, u_3)^{abc}_{111} = h_{\text{odd}}^{abc}. \tag{20} \]

These solutions \( S(u_1, u_2, u_3)^{abc}_{\text{def}} = S^\text{Korepa}(u_1, u_2, u_3)^{abc}_{\text{def}} + S^\text{Homo}(u_1, u_2, u_3)^{abc}_{\text{def}} \) satisfy tetrahedral Zamolodchikov algebras, but not satisfy tetrahedron equations if some arbitrary
parameters in $h_{abc}^{\text{even}}$ and/or $h_{abc}^{\text{odd}}$ become non-zero, that is, tetrahedron equations are satisfied only the case $S_{\text{Homo}} = 0$.

§4. Summary

We have examined solutions for tetrahedron equations from the elliptic free fermion model by using Korepanov mechanism. Though it is quite non-trivial, we obtained the same solutions of tetrahedron eqs. by using Korepanov mechanism even though we start from the elliptic case instead of the trigonometric one in the free fermion model. Thus we cannot find new solutions of tetrahedron eqs. from the elliptic free fermion model by using Korepanov mechanism, our results will be helpful to understand the Korepanov’s method and to find new solutions by using Korepanov mechanism. As a byproduct, we have found a new integrable 2-dim. lattice model, where transfer matrices are composed of $R^1$ in Eqs.(2).

We give examples which satisfy tetrahedral Zamolodchikov algebras but not satisfy tetrahedron equations by solving tetrahedral Zamolodchikov algebras in trigonometric symmetric free fermion case.

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