Symmetric hyperbolic system in the Ashtekar formulation

Gen Yoneda

Department of Mathematical Science, Waseda University, Okubo 3-4-1, Shinjuku, Tokyo 169-8790, Japan

Hisa-aki Shinkai

Department of Physics, Washington University, St. Louis, MO 63130-4899, USA
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We present a first-order symmetric hyperbolic system in the Ashtekar formulation of general relativity for vacuum spacetime. We add terms from the constraint equations to the evolution equations with appropriate combinations, which is the same technique used by Iriondo, Leguizamón and Reula (Phys. Rev. Lett. 79, 4732 (1997)). However, our system is different from theirs in the points that we primarily use Hermiticity of a characteristic matrix of the system to characterize our system symmetric, discuss the consistency of this system with reality condition, and show the characteristic speeds of the system.

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I. INTRODUCTION

Hyperbolic formulation of the Einstein equation is the one of the main research areas in general relativity [1]. This formulation is used in the proof of the existence, uniqueness and stability (well-posedness) of the solutions of the Einstein equation by analytical methods [2]. So far, several first order hyperbolic formulations are proposed; some of them are flux conservative [3], symmetrizable [4], or symmetric hyperbolic system [5-7]. The recent interest in hyperbolic formulation arises from its application to numerical relativity. One of the expected advantages is the existence of the characteristic speeds of the system, with which we may treat the numerical boundary with appropriate condition. Some numerical tests have been reported along this direction [8-10].

Recently, Iriondo, Leguizamón and Reula (ILR) [11] discuss the consistency of this system with reality condition, and show the characteristic speeds of the system. Some of them are flux conservative [3], symmetrizable [4], or symmetric hyperbolic system [5-7]. The recent interest in hyperbolic formulation arises from its application to numerical relativity. One of the expected advantages is the existence of the characteristic speeds of the system, with which we may treat the numerical boundary with appropriate condition. Some numerical tests have been reported along this direction [8-10].

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We, however, think that this does not derive Hermiticity of the characteristic matrix [A below in eq. (3.1)] from the metric $g_{\mu\nu}$ explicitly. We rather use Hermiticity of the characteristic matrix primarily to construct a symmetric hyperbolic system. Second, they did not mention the consistency of their formulation with the reality conditions which are crucial in the study of the Lorentzian dynamics in the Ashtekar variables [9]. Third, they did not discuss the characteristic structure of the system, which should be shown in the normal hyperbolic formulations. Our discussion covers these two matters.

The construction of this paper is as follows. After giving a brief review of Ashtekar’s variables and reality conditions in §2, we present our formulation in §3. The discussion of characteristic speed and summary are in §4.

II. ASHTEKAR’S FORMULATION

The key feature of Ashtekar’s formulation of general relativity [12] is the introduction of a self-dual connection as one of the basic dynamical variables. Let us write the metric $g_{\mu\nu}$ using the tetrad, $e^I_\mu$, and define its inverse, $E_I^\mu$, by $g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}$ and $E_I^\mu := e^I_\rho g^{\nu\rho} \eta_{IJ}$. We use volume forms $\epsilon_{abc}$; $\epsilon_{abc} \epsilon^{abc} = 3!$. We write the metric $g_{\mu\nu}$ using the tetrad, $e^I_\mu$, and define its inverse, $E_I^\mu$, by $g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}$ and $E_I^\mu := e^I_\rho g^{\nu\rho} \eta_{IJ}$. We use volume forms $\epsilon_{abc}$; $\epsilon_{abc} \epsilon^{abc} = 3!$. We use volume forms $\epsilon_{abc}$; $\epsilon_{abc} \epsilon^{abc} = 3!$. We use volume forms $\epsilon_{abc}$; $\epsilon_{abc} \epsilon^{abc} = 3!$.
define SO(3, C) self-dual and anti self-dual connections 
\( A^a := \omega_{ab} + (i/2) e^a_{\mu} \omega_{\mu} \), where \( \omega_{\mu J} \) is a spin connection 1-form (Ricci connection), \( \omega_{\mu J} := E^{\mu} \nabla_{\mu} c^J \). Ashketar's plan is to use only a self-dual part of the connection \( A^a \) and to use its spatial part \( A^a_\mu \) as a dynamical variable. Hereafter, we simply denote \( A^a_\mu \) as \( A^a \).

The lapse function, \( N \), and shift vector, \( N^i \), are expressed as \( E^0_0 = 1/(N^2 - N^i N^i) \). This allows us to think of \( E^0_0 \) as a normal vector field to \( \Sigma \) spanned by the condition \( t = x^0 = \text{const.} \), which plays the same role as that of ADM. Ashketar's plan is to use only a self-dual part of the connection \( A^a_\mu \) and to use its spatial part \( A^a_\mu \) as a dynamical variable. Hereafter, we simply denote \( A^a_\mu \) as \( A^a \).

In order to construct metric variables from the vari-
ables \( E^a_\mu \) and \( \tilde{E}^a_\mu \), we first prepare tetrad \( E^a_\mu \) as \( E^a_\mu = (1/\epsilon N^i - N^i/\epsilon N) \) and \( E^a_\mu = (0, \tilde{E}^a_\mu / \epsilon) \). Using them, we obtain metric \( g^{\mu\nu} \) such that

\[
g^{\mu\nu} := E^\mu_i E^{\nu j} i^J. \tag{2.7}
\]

Notice that in general the metric (2.7) is not real. To ensure the metric is real-valued, we need to impose real lapse and shift vectors together with two conditions (metric reality condition);

\[
\text{Im}(E^0_i \tilde{E}^{i0}) = 0, \tag{2.8}
\]
\[
\text{Re}(\epsilon^{abc} E^0_a \tilde{E}^a_b \tilde{D}_c \tilde{E}^0_c) = 0, \tag{2.9}
\]

where the latter comes from the secondary condition of reality of the metric \( \text{Im}(\partial_i (E^a_i \tilde{E}^{i0})) = 0 \) [13], and we assume \( \det \tilde{E} > 0 \) (see [14]).

For later convenience, we also prepare stronger reality conditions. These conditions are

\[
\text{Im}(\tilde{E}^a_i) = 0 \tag{2.10}
\]
\[
\text{Im}(\partial_i \tilde{E}^a_i) = 0, \tag{2.11}
\]

and we call them the “primary triad reality condition” and the “secondary triad reality condition”, respectively. Using the equations of motion of \( E^a_i \), the gauge constraint \( (2.4) \), the metric reality conditions \( (2.8), (2.9) \) and the primary condition \( (2.10) \), we see that \( (2.11) \) is equivalent to \( (2.4) \).

\[
\text{Re}(A^0_\mu) = \partial_i (N \tilde{E}^{i0} + (1/2) \epsilon^b \eta^{i0} \partial_j \tilde{E}^{i0} + N^i \text{Re}(A^a_\mu), \tag{2.12}
\]

or with un-densitized variables,

\[
\text{Re}(A^0_\mu) = \partial_i (N E^{i0} + N^i \text{Re}(A^a_\mu). \tag{2.13}
\]

From this expression we see that the second triad reality condition restricts the three components of “triad lapse” vector \( A^0_\mu \). Therefore (2.12) is not a restriction on the dynamical variables \( A^a_\mu \) but on the slicing, which we should impose on each hypersurface. Thus the second triad reality condition does not restrict the dynamical variables any further than the second metric condition does.

### III. HYPERBOLIC FORMULATION

We start from defining hyperbolic system following Friedhöls [14], which is first applied in general relativity by Fischer and Marsden [10]. That is, we say that the system is first-order (quasi-linear) hyperbolic if a certain pair of variables \( u_i \) form a linear system as

\[
\partial_i u_i = A_{ij} (u) \partial_j u_j + B_i (u), \tag{3.1}
\]

where \( A \) is a characteristic matrix-valued function, of which eigenvalues are all real, and \( B \) is a function. We further define that the system is symmetric when \( A \) is a Hermitian matrix [17].

The symmetric system gives us the energy integral inequalities, which are the primary tools for analyzing well-posedness of the system. As was discussed by Geroch...
most physical systems are expressed as symmetric hyperbolic systems.

Ashtekar’s formulation itself is in the first-order form in the sense of (2.11), but not a symmetric hyperbolic form.

We start from writing the principal part of the Ashtekar’s evolution equations as

$$\partial_t \left[ \tilde{E}_a^j \right] = \left[ A_{a bi j}^l \tilde{E}_b^l + B_{a bi j}^l \partial_t \tilde{E}_b^l \right]$$

where $\cong$ means that we extracted only the terms which appear in the principal part of the system. The system is symmetric hyperbolic if

$$0 = A_{abij}^l - \bar{A}_{bajl}^i,$$

$$0 = D_{abij}^l - \bar{D}_{bajl}^i,$$

$$0 = \bar{B}_{abij}^l - \bar{C}_{bajl}^i$$

where bar denotes complex conjugate.\(^\dagger\)

We first prepare the constraints (2.7) - (2.4) as

$$C_H \cong \pm \epsilon_{abc} \tilde{E}_a^i \partial_t \tilde{A}_j^b = \epsilon_{abc} \tilde{E}_a^i \partial_t \tilde{A}_j^b,$$

$$0 = D_{abij}^l - \bar{D}_{bajl}^i,$$

$$0 = \bar{C}_{bajl}^i - \bar{C}_{bajl}^i$$

where bar denotes complex conjugate.\(^\dagger\)

We apply the same technique with ILR to modify the equation of motion of $\tilde{E}_a^i$ and $\tilde{A}_a^l$ by adding the constraints which weakly produce $C_H = 0, C_{MK} = 0, \text{ and } C_{Ga} = 0.$ With a parametrization for triad lapse $\tilde{A}_a^l$ with $T$ and $S$ as

$$0 = 2D_j (N^j \tilde{E}_a^i) + 2D_j (\pm N^j \tilde{E}_a^i)$$

we write the principal parts of (2.3) and (2.6) as

$$\partial_t \tilde{E}_a^i = -iD_j (\pm N^j \tilde{E}_a^i) + 2D_j (\pm N^j \tilde{E}_a^i)$$

$$\partial_t \tilde{A}_a^l = -iD_j (\pm N^j \tilde{F}_a^i) + 2D_j (\pm N^j \tilde{F}_a^i)$$

\(\dagger\)We think that the reader will not confuse $A_{abij}^l$ and $B_{abij}^l$ with matrix $A$ and $B$ in (2.3).
with reality condition (especially with secondary reality condition). However, since ILR assume $A_{ij} = \mathbb{A}^{a}_{j} N^{j}$, we think that ILR also needs to impose similar restricted lapse condition in order to preserve reality of the system.

The rest of our effort is finished when we specify parameters $P$, $Q$ and $R$. $P$ is given by decomposing \ref{eq:3.16} into real/complex parts:

\begin{align}
0 &= -N^i \gamma^j g^{iab} + N^j \gamma^i \delta^{ba} \\
&\quad + \text{Re}(P)^{iab} \gamma^{ij} - \text{Re}(P)^{jba} \gamma^{li} \\
0 &= -\epsilon^{cba} N^j \gamma^i E^c_j - \epsilon^{abc} N^i \gamma^j E^j_c \\
&\quad + \text{Im}(P)^{iab} \gamma^{ij} + \text{Im}(P)^{jba} \gamma^{li}
\end{align}

By multiplying $\gamma_{ij}$ in these two and taking symmetric and anti-symmetric operation to the index $ab$, we obtain

\begin{equation}
P^{iab} = N^i \delta^{ab} + i N \epsilon^{abc} E^a_c.
\end{equation}

For $Q$ and $R$, we found that a combination of the choice

\begin{align}
Q^{ai} &= e^{-2} N E^{ai} \\
R^{i} = i e^{-2} N \epsilon^{ac} E_{a} E_{c}
\end{align}

satisfies the condition \ref{eq:3.18}.

**IV. DISCUSSION**

In summary, by adding constraint terms with appropriate coefficients, we succeed to construct a symmetric hyperbolic formulation for the Ashtekar’s system. This formulation is consistent with secondary triad reality condition, which requires to impose a constant lapse function for the evolving system.

The characteristic speeds of this system are given by finding eigenvalues of the characteristic matrix $A$ of the right hand side of \ref{eq:3.1}. Since $A$ is a Hermitian, eigenvalues of $A$ are all real. Then it is again clear that this system is symmetric hyperbolic. Actually the eigenvalues of the $18 \times 18$ matrix $A^l$ for $x^l$-direction are: $N^l$ (multiplicity = 6), $N^l \pm 3\sqrt{N}N$ (5 each), and $N^l \pm 3\sqrt{N}N$ (1 each), where we do not take the sum in $\gamma_i$ here. These speeds are independent from the way of taking a triad. We omit to show the related eigen-vectors because of saving space.

As we denoted in §3, our formulation requires triad reality condition. In order to make the system first order, the lapse function is assumed to be constant. Shift vectors and triad lapse $A_{0}^{a}$ should have a relation \ref{eq:3.19}. This can be interpreted that shift is free and triad lapse is determined. This gauge restriction sounds tight, but this arises from our general assumption of \ref{eq:3.9}. There might be a possibility to improve the situation by renormalizing shift and triad lapse terms into left-hand-side of equations of motion like the case of GR. Or this might be because our system is constituted by Ashtekar’s original variables. We are now trying to release this gauge restriction and/or to simplify the characteristic speeds by other gauge possibilities and also by introducing new dynamical variables. This effort will be reported elsewhere.

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\[\text{Electronic address: yoneda@mn.waseda.ac.jp} \]  
\[\text{Electronic address: shinkai@null.wustl.edu} \]  
\[\text{Current review is given by O. A. Reula, Living Rev. Relativity, 98-3, (1998).} \]  
\[\text{Y. Choquet-Bruhat and J.W. York Jr., in General Relativity and Gravitation, vol. 1, ed. by Held,} \] (Plenum, New York, 1980).  
\[\text{C. Bona, J. Massó, E. Seidel, J. Stela, Phys. Rev. Lett. 75, 600 (1995); Phys. Rev. D56, 3405 (1997).} \]  
\[\text{Y. Choquet-Bruhat and J.W. York, C. R. Acad. Sci. Paris, t. 321, Série I, 1089 (1995); A. Abrahams, A. Anderson, Y. Choquet-Bruhat and J.W. York, Phys. Rev. Lett. 75, 3377 (1995).} \]  
\[\text{A. E. Fischer and J. E. Marsden, Commun. Math. Phys. 28, p1-38 (1972).} \]  
\[\text{H. Friedrich, Class. Quantum Grav. 13, 1451 (1996).} \]  
\[\text{S. Frittelli, O.A. Reula, Phys. Rev. Lett. 76, 4667 (1996).} \]  
\[\text{M. Alcubierre, Phys. Rev. D55, 5981 (1997).} \]  
\[\text{M.A. Scheel, T.W. Baumbarite, G.B. Cook, S.L. Shapiro, and S.A. Teukolsky, Phys. Rev. D56 6320 (1997); ibid. D58, 044020 (1998).} \]  
\[\text{C. Bona, J. Massó, E. Seidel and P. Walker, gr-qc/9803077} \]  
\[\text{Recent review is given by O. A. Reula, Living Rev. Relativity, 98-3, (1998).} \]  
\[\text{Y. Choquet-Bruhat and J.W. York Jr., in General Relativity and Gravitation, vol. 1, ed. by Held,} \] (Plenum, New York, 1980).  
\[\text{C. Bona, J. Massó, E. Seidel, J. Stela, Phys. Rev. Lett. 75, 600 (1995); Phys. Rev. D56, 3405 (1997).} \]  
\[\text{Y. Choquet-Bruhat and J.W. York, C. R. Acad. Sci. Paris, t. 321, Série I, 1089 (1995); A. Abrahams, A. Anderson, Y. Choquet-Bruhat and J.W. York, Phys. Rev. Lett. 75, 3377 (1995).} \]  
\[\text{A. E. Fischer and J. E. Marsden, Commun. Math. Phys. 28, p1-38 (1972).} \]  
\[\text{H. Friedrich, Class. Quantum Grav. 13, 1451 (1996).} \]  
\[\text{S. Frittelli, O.A. Reula, Phys. Rev. Lett. 76, 4667 (1996).} \]  
\[\text{M. Alcubierre, Phys. Rev. D55, 5981 (1997).} \]  
\[\text{M.A. Scheel, T.W. Baumbarite, G.B. Cook, S.L. Shapiro, and S.A. Teukolsky, Phys. Rev. D56 6320 (1997); ibid. D58, 044020 (1998).} \]  
\[\text{C. Bona, J. Massó, E. Seidel and P. Walker, gr-qc/9804052} \]  
\[\text{M.S. Iriondo, E.O. Leguizamón, O.A. Reula, Phys. Rev. Lett. 79, 4732 (1997).} \]  
\[\text{A. Ashtekar, Phys. Rev. Lett. 57, 2244 (1986); Phys. Rev. D36, 1587 (1987); Lectures on Non-Perturbative Canonical Gravity (Singapore, World Scientific, 1991).} \]  
\[\text{A. Ashtekar, J. D. Romano and R. S. Tate, Phys. Rev. D40, 2572 (1989).} \]  
\[\text{G. Yoneda and H. Shinkai, Class. Quantum Grav. 13, 783 (1996).} \]  
\[\text{G. Yoneda, H. Shinkai and A. Nakamichi, Phys. Rev. D56, 2086 (1997).} \]  
\[\text{K. O. Friedrichs, Comm. Pure and Appl. Math., 7, 345 (1954).} \]  
\[\text{R. Courant and D. Hilbert, Methods of Mathematical Physics, Volume II, (John Willey & Sons, 1962) } \]  
\[\text{R. Geroch, Partial Differential Equations in Physics, gr-qc/9602055} \]  
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