The construction of real Frobenius Lie algebras from non-commutative nilpotent Lie algebras of dimension ≤ 4

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Abstract. In this present paper, we study real Frobenius Lie algebras constructed from non-commutative nilpotent Lie algebras of dimension ≤ 4. The main purpose is to obtain Frobenius Lie algebras of dimension ≤ 6. Particularly, for a given non-commutative nilpotent Lie algebras \( N \) of dimension ≤ 4 we show that there exist commutative subalgebras \( T \) of dimension ≤ 2 such that the semi-direct sums \( \mathfrak{g} = N \oplus T \) is Frobenius Lie algebras. Moreover, \( T \) is called a split torus which is a commutative subalgebra of derivation of \( N \) and it depends on the given \( N \). To obtain this split torus, we apply Ayala’s formulas of a Lie algebra derivation by taking a diagonal matrix of a standard representation matrix of the Lie algebra derivation of \( N \). The discussion of higher dimension of Frobenius Lie algebras obtained from non-commutative nilpotent Lie algebras is still an open problem.

1. Introduction

A Frobenius Lie algebras is a type of a Lie algebra whose index is equal to zero [1]. Furthermore, if \( \mathfrak{g} \) is a Frobenius Lie algebra then its dimension is always even with trivial center. For instance, \( j \)-algebra is the solvable real Frobenius Lie algebra which has important role in the study of bounded homogeneous domains in \( \mathbb{C}^n \). In addition, the Frobenius Lie algebras answer to constant solution of the classical Yang-Baxter equation (CYBE) [2]. Although a semisimple Lie algebra is not a Frobenius Lie algebra but some Lie subalgebras of semisimple Lie algebras like a parabolic and a biparabolic subalgebras are Frobenius Lie algebras [2].

The recent researches of Frobenius Lie algebras have been done by some researchers. For example, we can observe about classifications of Frobenius Lie algebras of dimension ≤ 6 over a field of characteristics \( \neq 2 \) or over an algebraically closed field [1], principal elements of Frobenius Lie algebras [3] and their properties [4]. Moreover, the harmonic analysis of 4-dimensional real Frobenius Lie algebras can be found in [5]. Moreover, Ooms [2] in his paper explained how to compute invariant and semi-invariant polynomials of Frobenius Lie algebras. Besides that, Ooms gave the key words how to construct a Frobenius Lie algebra for any given Lie algebra. This encourages us to study how to construct Frobenius Lie algebras. The construction idea as follows. Let \( N \) be a Lie algebra. Then we should find a commutative subalgebra of derivation of \( N \) which is called a split torus \( T \) such that its semi-direct sum is a Frobenius Lie algebra. We mention here that not all given Lie algebras have a split torus such that its semi-direct sum is a Frobenius Lie algebra.
In this research, we study how to construct real Frobenius Lie algebras from non-commutative nilpotent Lie algebras of dimension \( \leq 4 \). These non-commutative nilpotent Lie algebras were classified by Graaf [6]. This classification can be listed as follows:

1. The 3-dimensional non-commutative nilpotent Lie algebra \( N_3 = \text{Span}\{x_1, x_2, x_3\} \) whose non-zero brackets are given by
   \[ [x_1, x_2] = x_3. \] (1)

2. Two classes of 4-dimensional non-commutative nilpotent Lie algebra which are denoted by \( N_{4,1} \) dan \( N_{4,2} \).

   The first class is \( N_{4,1} = \text{Span}\{x_1, x_2, x_3, x_4\} \) whose non-zero brackets are given by
   \[ [x_1, x_2] = x_3. \] (2)

   The second class is \( N_{4,2} = \text{Span}\{x_1, x_2, x_3, x_4\} \) whose non-zero brackets are given by
   \[ [x_1, x_2] = x_3, \quad [x_1, x_3] = x_4. \] (3)

   Particularly, if given a non-commutative nilpotent Lie \( N \in \{N_3, N_{4,1}, N_{4,2}\} \) then we shall find (if any) a split torus \( T \subset \text{Der} \ N \) such that the semi-direct sum \( g = N \oplus T \) is a Frobenius Lie algebra. We denote by \( \text{Der} \ N \) a set of derivation of \( N \).

We emphasize that our purpose is to obtain Frobenius Lie algebras of dimension \( \leq 6 \) from non-commutative nilpotent Lie algebras of dimension \( \leq 4 \). We state our main results as follows:

The first result we have

**Proposition 1.** Let \( N_3 \) be a non-commutative nilpotent Lie algebra of dimension 3 whose brackets are given by eqs.(1). Then there exists a 1-dimensional split torus \( T \subset \text{Der} \ N_3 \) spanned by \( t = \text{diag}(\alpha, \beta, \alpha + \beta) \) with \( \alpha, \beta \in \mathbb{R} \) and \( \alpha + \beta \neq 0 \) such that the semi-direct sum \( g = N_3 \oplus T \) is a 4-dimensional Frobenius Lie algebra.

and the second result we get

**Proposition 2.** Let \( N \in \{N_{4,1}, N_{4,2}\} \) be a 4-dimensional non-commutative nilpotent Lie algebra. Then there exists a 2-dimensional split torus \( T_{\{N_{4,1}, N_{4,2}\}} = \langle t_1, t_2 \rangle \subset \text{Der} \ N \) depending on a given \( N \) such that the semi-direct sum \( g = N \oplus T \) is the 6-dimensional Frobenius Lie algebra.

We shall prove our both main results in Result and Discussion Section. We note here that there are no non-commutative nilpotent Lie algebras of dimension \( \leq 2 \).

Let us review briefly some basic notions that shall be used in this paper.

**Definition 3** [7]. A vector space \( g \) is said to be a Lie algebra if \( g \) is equipped with the bracket \([,]\) defined by

\[ g \times g \ni (a, b) \mapsto [a, b] \in g \] (4)

satisfying the following conditions:
1. The bracket \([,]\) is a bilinear map.
2. \([a, b] = -[b, a]\).
3. \([[a, b], c] = [a, [b, c]] + [b, [c, a]]\),

for all \( a, b, c \in g \). Furthermore, the Lie algebra \( g \) is called commutative if \([a, b] = 0\) for all \( a, b \in g \).
Definition 4 [8]. A linear map \( f : \mathfrak{g} \to \mathfrak{g} \) is called a Lie algebra homomorphism if
\[
f([x, y]) = [f(x), f(y)] \quad (\forall x, y \in \mathfrak{g}).
\] (5)

In addition, the linear map \( d : \mathfrak{g} \to \mathfrak{g} \) is said to be a derivation of \( \mathfrak{g} \) if for every \( e_1, e_2 \in \mathfrak{g} \), we have
\[
d([e_1, e_2]) = [d(e_1), e_2] + [e_1, d(e_2)].
\] (6)

A set of all derivation of \( \mathfrak{g} \) is denoted by \( \text{Der} \mathfrak{g} \) and by defining a bracket as follows
\[
[\psi, \psi'] = \psi \circ \psi' - \psi' \circ \psi,
\] (7)
then we get that the set \( \text{Der} \mathfrak{g} \) is a Lie algebra.

Definition 5 [8]. Let \( \mathfrak{g} \) be a Lie algebra and \( x \) be an element of \( \mathfrak{g} \). The linear map \( \text{ad}(x) : \mathfrak{g} \to \mathfrak{g} \) defined by \( \text{ad}(x)y := [x, y] \) is called an inner derivation on \( \mathfrak{g} \). The set of inner derivations on \( \mathfrak{g} \) is denoted by \( \text{ad} \mathfrak{g} \).

Let \( \mathfrak{g} \) be a Lie algebra. We define
\[
\mathfrak{g}^1 = \mathfrak{g}, \quad \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^3 = [\mathfrak{g}, \mathfrak{g}^2], \ldots, \quad \mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^{k-1}], \ldots, \quad \mathfrak{g}^{k+1} = [\mathfrak{g}, \mathfrak{g}^k],
\] (8)
where \( k \) is non-negative integer. We have the following definition

Definisi 6 [8]. A Lie algebra \( \mathfrak{g} \) is called nilpotent if there is a non-negative integer \( n \) with \( \mathfrak{g}^{n+1} = \{0\} \). If \( n \) is minimal then \( n \) is called the nilpotence degree of \( \mathfrak{g} \).

Let \( \mathfrak{g}^* \) be a dual vector space of \( \mathfrak{g} \) whose elements are defined as linear functionals given by \( f : \mathfrak{g} \to \mathbb{R} \). The value of linear functional \( f \in \mathfrak{g}^* \) at a point \( x \in \mathfrak{g} \) is given by \( \langle f, x \rangle \). For \( \psi \in \mathfrak{g}^* \), we define a stabilizer \( \mathfrak{g}_\psi \) of \( \mathfrak{g} \) by
\[
\mathfrak{g}_\psi = \{ x \in \mathfrak{g} \ ; \ \langle \psi, [x, y] \rangle = 0, \ \forall y \in \mathfrak{g} \}. \tag{9}
\]
We observe that the stabilizer \( \mathfrak{g}_\psi \) is kernel of an alternating bilinear form \( B_\psi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \) defined by \( B_\psi(x, y) = \langle \psi, [x, y] \rangle \). Moreover, the minimum dimension of \( \mathfrak{g}_\psi \) is called index of \( \mathfrak{g} \). Precisely, we have
\[
\text{Index}(\mathfrak{g}) = \min \{ \dim(\mathfrak{g}_\psi) \ ; \ \psi \in \mathfrak{g}^* \}. \tag{10}
\]
Now we define by \( \mathcal{M}(\mathfrak{g}) = ([x_i, x_j])_{i,j=1}^{\dim(\mathfrak{g})} \) a \( \mathfrak{g} \)-matrix whose elements are brackets \( [x_i, x_j] \) with respect to a basis \( S := \{x_1, x_2, \ldots, x_n\} \) of \( \mathfrak{g} \). The determinant of \( \mathcal{M}(\mathfrak{g}) \) is contained in a symmetric algebra \( S(\mathfrak{g}) \). We have the following definition

Definition 7 [9]. Let \( \mathfrak{g} \) be a real Lie algebra with basis \( S = \{x_1, x_2, \ldots, x_n\} \). The Lie algebra \( \mathfrak{g} \) is said to be Frobenius if there exists a linear functional \( \psi_0 \in \mathfrak{g}^* \) such that \( \mathfrak{g}_\psi_0 = \{0\} \). Such \( \psi_0 \) is called a Frobenius functional.
Indeed, the stabilizer \( g_{\psi_0} = \{0\} \) if and only if the index of \( g \) equals zero. Furthermore, we can regard the determinant of the matrix \( M(g) \) a polynomial function defined on \( g^* \). Namely we have, 
\[
(\det M(g))(\psi_0) = \det(\langle \psi_0, [x_i, x_j] \rangle)^{\dim(g)}_{i,j=1}.
\]

**Theorem 8** [9]. The Lie algebra \( g \) with basis \( S=\{x_1,x_2,\ldots,x_n\} \) is a Frobenius Lie algebra if one of the following equivalent conditions is satisfied:

1. The determinant of the matrix \( M(g) \) is not equal to zero.
2. The determinant of the matrix \( \langle \psi_0, [x_i, x_j] \rangle \) is not equal to zero for some \( \psi_0 \in g^* \).

We observe that \((\det M(g))(\psi_0)\) is equal to the determinant of the alternating bilinear form \( B_{\psi_0} : g \times g \to \mathbb{R} \) defined by \( B_{\psi_0}(x, y) = \langle \psi_0, [x, y] \rangle \) \( (\psi_0 \in g^*, x, y \in g) \). Therefore, the Lie algebra \( g \) is Frobenius if and only if \( B_{\psi_0} \) is non-degenerate.

**2. Method**

In this research, we applied the axiomatic method by reviewing some relevant literatures corresponding to nilpotent and Frobenius Lie algebras. Let \( N \) be a non-commutative nilpotent Lie algebra contained in \( \{N_2,N_{4,1},N_{4,2}\} \). We should find a split torus that is a commutative subalgebra of Der \( N \). We determined a matrix derivation using the Ayala’s formula [10] which is stated in the following theorem to find a split torus \( T \) such that the semi-direct sums \( g = N \oplus T \) is a Frobenius Lie algebra.

**Theorem 9** [10]. Let \( g \) be a Lie algebra with basis \( \{x_1,x_2,\ldots,x_n\} \) and \( d \) be a linear map on \( g \) whose standard representation matrix is given by \( A := [d_{ij}] \) where \([d_{ij}]^T\) is a matrix transpose of matrix \([d_{ij}]\). Then the linear map \( d \) is a derivation on \( g \) if and only if the following equation is satisfied
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}^k d_{kp} = \sum_{i=1}^{n} (d_{ik} C_{kj}^p + d_{jk} C_{ik}^p),
\]
where \( 1 \leq i, j, p \leq n \) and \( C_{ij}^k \in \mathbb{R} \) is a structure constant of the bracket \( [x_i, x_j] = \sum_{k=1}^{n} C_{ij}^k x_k \).

**3. Result and Discussion**

**Proof of Proposition 1.** There is only one the non-commutative nilpotent Lie algebra of dimension three (see [6]), namely \( N_3 \) whose basis is \( \{x_1,x_2,x_3\} \) with non zero bracket is given in eqs.(1) that is \( [x_1,x_2] = x_3 \). This Lie algebra \( N_3 \) is well known as the Heisenberg Lie algebra which is nilpotent. To show that there exists a 1-dimensional split torus \( T \), we should find a representation matrix of derivation of \( N \). Since the bracket of the Lie algebra \( N_3 \) is \( [x_1,x_2] = x_3 \) then its structure constant \( C_{ij}^{12} \) is equal to 1 and 0 elsewhere. Using eqs.(11) then we get
\[
\sum_{k=1}^{3} C_{ij}^k d_{kp} = \sum_{k=1}^{3} (d_{ik} C_{kj}^p + d_{jk} C_{ik}^p),
\]
with \( 1 \leq i, j \leq 3 \) and \( 1 \leq p \leq 3 \). In other words, we have
\[
C_{ij}^1 d_{1p} + C_{ij}^2 d_{2p} + C_{ij}^3 d_{3p} = \left(d_{i1} C_{1j}^p + d_{j1} C_{1i}^p\right) + \left(d_{i2} C_{2j}^p + d_{j2} C_{2i}^p\right) + \left(d_{i3} C_{3j}^p + d_{j3} C_{3i}^p\right).
\]

The eqs. (13) is noting but a linear equation system in variables \( d_{ij} \) with \( 1 \leq i, j \leq 3 \). Choosing a pair \((i,j) \in \{(1,2),(1,3),(2,3)\}\) and fixing \( p = 1,2,3 \) for each \((i,j)\), then we obtain the solution \( d_{ij} \) for this system. More precisely, we have that for the pair \((i,j) = (1,2)\) then the eqs. (13) become
\[
C_{12}^1 d_{1p} + C_{12}^2 d_{2p} + C_{12}^3 d_{3p} = \left(d_{11} C_{12}^p + d_{21} C_{11}^p\right) + \left(d_{12} C_{22}^p + d_{22} C_{22}^p\right) + \left(d_{13} C_{32}^p + d_{23} C_{32}^p\right).
\]
Taking $p = 1$, then we get

$$C_{12}^{1}d_{11} + C_{12}^{2}d_{21} + C_{12}^{3}d_{31} = (d_{11}C_{12}^{1} + d_{21}C_{12}^{1} + d_{31}C_{12}^{1}) + (d_{12}C_{12}^{2} + d_{22}C_{12}^{2} + d_{32}C_{12}^{2}) + (d_{13}C_{12}^{3} + d_{23}C_{12}^{3}). \quad (15)$$

Since the structure constant $C_{12}^{3} = 1$, then we conclude $d_{31} = 0$. Continuing these computations for another $p$ and other $(i, j)$, then we obtain the standard representation matrix of the following form

$$A = \begin{pmatrix} \alpha & y_1 & 0 \\ y_2 & \beta & 0 \\ y_3 & \alpha + \beta \end{pmatrix}, \quad (16)$$

with $\alpha, \beta, y_{i}(i = 1, 2, 3, 4) \in \mathbb{R}$. Then we choose a split torus $T := \langle t \rangle$ with $t := \text{diag}(\alpha, \beta, \alpha + \beta)$ such that we have the semi-direct sum Lie algebra of the form $g := N_{3} \oplus T$ with non-zero brackets are given in the following form.

$$[x_{1}, x_{2}] = x_{3}, \quad [t, x_{1}] = \alpha x_{1}, \quad [t, x_{2}] = \beta x_{2}, \quad [t, x_{3}] = (\alpha + \beta)x_{3}. \quad (17)$$

Furthermore, for $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta \neq 0$, we claim that $g$ is the 4-dimensional real Frobenius Lie algebra. To prove our claim, let $M(g) = \{[x_{i}, x_{j}]\}_{i,j=1}^{4}$ be the $4 \times 4$ matrix whose elements are brackets $[x_{i}, x_{j}]$ with respect to a basis $S := \{x_{1}, x_{2}, x_{3}, t\}$ of $g$. Namely we have

$$M(g) = \begin{pmatrix} 0 & \alpha x_{1} & \beta x_{2} & (\alpha + \beta)x_{3} \\ -\alpha x_{1} & 0 & x_{3} & 0 \\ -\beta x_{2} & -x_{3} & 0 & 0 \\ -(\alpha + \beta)x_{3} & 0 & 0 & 0 \end{pmatrix}, \quad (18)$$

with determinant of $M(g)$ is given by

$$\det(M(g)) = (\alpha + \beta)^{2}x_{3}^{4}. \quad (19)$$

Since $\alpha + \beta \neq 0$, using Definition (7) part. 3, then our claim is true, that is, $g := N_{3} \oplus T$ is the 4-dimensional Frobenius Lie algebra. In the other hand, from the eqs.(19), let us choose $\psi_{0} = x_{3}^{*} \in g^{*}$ as the linear functional on $g$ and we define $\langle \psi_{0}, x_{i} \rangle = 1$ if $i = 3$ and 0 elsewhere. We obtain that

$$(\det M(g)) (\psi_{0}) = \det(\langle \psi_{0}, [x_{i}, x_{j}] \rangle)_{i,j=1}^{4} = (\alpha + \beta)^{2} \neq 0. \quad (20)$$

In other words, there exists a Frobenius functional $\psi_{0} = x_{3}^{*} \in g^{*}$ such that the alternating bilinear form $B_{x_{3}}$ is non-degenerate. Therefore, this show again that $g := N_{3} \oplus T$ is the 4-dimensional real Frobenius Lie algebra as desired.

Remark 9. In our result if we take $\alpha = \beta = 1/2$, then the Lie algebra $g$ whose brackets are given by eqs. (17) is equal to Csikós and L. Verhóczki’s result of the first class of 4-dimensional Frobenius Lie algebra over fields of characteristics $\neq 2$ (see [1], page. 433).
Proof of Proposition 2. There are 2 classes of non-commutative nilpotent Lie algebras of dimension 4 (see [6]). We are going to prove the first case \( N_{4,1} \) whose non-zero brackets are given in eqs. (2). The structure constant \( C_{ij}^k \) is equals 1 and 0 elsewhere. To find a split torus \( T \subset \text{Der} \ N_{4,1} \), we follow our computations before. Let us concern to the linear equation system given by

\[
\sum_{k=1}^{4} C_{ij}^k d_{kp} = \sum_{k=1}^{4} \left( d_{ik} C_{kj}^p + d_{jk} C_{ki}^p \right),
\]

with \( 1 \leq i,j \leq 4 \) and \( 1 \leq p \leq 4 \). We compute the eqs. (21) for any pair \((i,j) \in \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}\) and \( p = 1,2,3 \) to consider all \( d_{ij} \). Following Ayala’s formulas and our computations before, then we get the standard matrix representation of derivation of \( N_{4,1} \) which is written in the form

\[
B := \begin{pmatrix}
\alpha & y_1 & y_2 & y_3 \\
y_4 & \beta & y_5 & y_6 \\
0 & 0 & \alpha + \beta & 0 \\
0 & 0 & 0 & \gamma
\end{pmatrix},
\]

with \( \alpha, \beta, \gamma, y_i \in \mathbb{R}, \ i = 1,2,3,...,7 \). We choose a 2-dimensional split torus \( T := \{ t_1, t_2 \} \) \( t_1 = \text{diag}(\alpha, \alpha + \beta, 0) \) and \( t_2 = \text{diag}(0,0,0,\gamma) \) such that \( g = N_{4,1} \oplus T \) is the 6-dimensional Lie algebra whose non zero brackets are given in the following formulas

\[
[x_1, x_2] = x_3, \ [t_1, x_2] = \alpha x_1, \ [t_1, x_3] = (\alpha + \beta) x_3, \ [t_2, x_4] = \gamma x_4.
\]

We claim that for \( \gamma(\alpha + \beta) \neq 0, g = N_{4,1} \oplus T \) is the 6-dimensional Frobenius Lie algebra. Since the determinan of matrix

\[
\mathcal{M}(g) := \begin{pmatrix}
0 & 0 & \alpha x_1 & \beta x_2 & 0 & x_3 & 0 \\
0 & 0 & 0 & 0 & 0 & \gamma x_4 \\
-\alpha x_1 & 0 & 0 & x_3 & 0 & 0 \\
-\beta x_2 & 0 & -x_3 & 0 & 0 & 0 \\
-\vartheta x_3 & 0 & 0 & 0 & 0 & 0 \\
0 & -\gamma x_4 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

with \( \vartheta := \alpha + \beta \) is of the form

\[
\det(\mathcal{M}(g)) = \gamma^2 (\alpha + \beta)^2 x_3^2 x_4^2,
\]

which is not equal to zero since \( \gamma(\alpha + \beta) \neq 0 \). Then by Definition (7) part. 3 our claim is true, that is, \( g = N_{4,1} \oplus T \) is Frobenius Lie algebra of dimension 6. From the eqs.(25), let us choose a linear functional \( \psi_0 \) as a linear combinations of elements in \( \{x_3^3, x_4^4\} \). In this case, we choose \( f_0 = x_3^3 + x_4^4 \) and we define \( f_0(x_i) = 1 \) if \( i = 3 \) or \( i = 4 \) and 0 elsewhere. Computing the determinant the matrix

\[
\mathcal{M}(g)(f_0) = \langle \{ f_0, [x_i, x_j] \} \rangle \mid_{i,j=1}^{6}
\]

then we obtain

\[
\langle \det(\mathcal{M}(g)) \rangle(f_0) = \det(\langle \{ f_0, [x_i, x_j] \} \rangle) \mid_{i,j=1}^{6} = \gamma^2 (\alpha + \beta)^2 \neq 0.
\]

This proved that there exists the Frobenius functional \( f_0 \in \ g^* \) such the the alternating bilinear form \( B_{f_0} \) is non-degenerate. In other words, we show again that \( g \) is 6-dimensional real Frobenius Lie algebra.
Now we are going to prove for the second case $N_{4,2}$. This Lie algebra is well known as the Filiform Lie algebra. The structure constant can be written $C^2_{12} = C^3_{45} = 1$. We follow the similar computations for $N_{4,1}$ above. Finally, we can choose a 2-dimensional split torus $T := \langle t_1, t_2 \rangle$ in the following form

$$t_k := \text{diag}(\alpha, \beta, \alpha + \beta, 2\alpha + \beta), k = 1, 2$$  \hspace{1cm} (27)

with $\alpha$ or $\beta$ is not equal to zero. Then the semi-direct sum $g = N_{4,2} \oplus T$ is is the 6-dimensional Frobenius Lie algebra.

**Remark 10.** In the case $g = N_{4,2} \oplus T$ if we take $\alpha = 0$ and $\beta = 1$ then we obtain $t_1 = \text{diag}(0,1,1,1)$. In addition, for $\alpha = 1$ and $\beta = -2$ then $t_2 = \text{diag}(1,-2,-1,0)$. This result is equal to Ooms’ work (see [2], page 1303).

We found that the classification of Frobenius Lie algebras has been done just until 6-dimension [1]. Therefore, it is more interesting for the future research to classify the higher dimensional of Frobenius Lie algebras. Our results motivate for higher dimensional classification of Frobenius Lie algebras.

4. Conclusion
The real Frobenius Lie algebras of dimension 4 and 6 can be constructed from non commutative nilpotent Lie algebras of dimension $\leq 4$. The method is by choosing the split torus $T$ such that $g = N \oplus T$ is Frobenius Lie algebras of dimension 4 and 6 depending on a given $N$. The computations of the split torus $T$ based on the computations of derivation of $N$. For future research, it is still an open problem to construct Frobenius Lie algebras for higher dimension for any given Lie algebra types.

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