Symmetries of Massive and Massless Neutrinos

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Abstract

Wigner’s little groups are subgroups of the Lorentz group dictating the internal space-time symmetries of massive and massless particles. These little groups are like O(3) and E(2) for massive and massless particles respectively. While the geometry of the O(3) symmetry is familiar to us, the geometry of the flat plane cannot explain the E(2)-like symmetry for massless particles. However, the geometry of a circular cylinder can explain the symmetry with the helicity and gauge degrees of freedom. It is shown further that the symmetry of the massless particle can be obtained as a zero-mass limit of O(3)-like symmetry for massive particles. It is shown further that the polarization of massless neutrinos is a consequence of gauge invariance, while the symmetry of massive neutrinos is still like O(3).

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1 Introduction

In his 1939 paper [1], Wigner considered the subgroups of the Lorentz group whose transformations leave the four-momentum of a given particle invariant. These subgroups are called Wigner’s little groups and dictate the internal space-time symmetries in the Lorentz-covariant world. He observed first that a massive particle at rest has three rotational degree of freedom leading to the concept of spin. Thus the little group for the massive particle is like O(3).

Wigner observed also that a massless particle cannot be brought to its rest frame, but he showed that the little group for the massless particle also has three degrees of freedom, and that this little is locally isomorphic to the group E(2) or the two-dimensional Euclidean group. This means that the generators of this little group share the same set of closed commutation relations with that for two-dimensional Euclidean group with one rotational and two translational degrees of freedom.

It is not difficult to associate the rotational degree of freedom of E(2) to the helicity of the massless particle. However, what is the physics of the those two translational degrees of freedom? Wigner did not provide the answer to this question in his 1939 paper [1]. Indeed, this question has a stormy history and the issue was not completely settled until 1990 [2], fifty one years after 1939.

In this report, it is noted first that the Lorentz group has six generators. Among them, three of them generate the rotation subgroup. It is also possible to construct three generators which constitute a closed set of commutations relations identical to that for the E(2) group. However, it is also possible to construct the cylindrical group with one rotational degree of freedom and two degrees freedom both leading to up-down translational degrees freedom. These two translational degrees freedom correspond to one gauge degree of freedom for the massless particle [3].

While the O(3)-like and E(2)-like little groups are different, it is possible to derive the latter as a Lorentz-boosted O(3)-like little group in the infinite-momentum limit. It is shown then that the two rotational degrees of freedom perpendicular momentum become one gauge degree of freedom [4].

It is noted that the E(2)-like symmetry for the massless spin-1 particle leads to its helicity and gauge degree of freedom. Likewise, there is a gauge degree of freedom for the massless spin-1/2 particle. However, the requirement of gauge invariance leads to the polarization of massless neutrinos [5, 6, 7].

In Sec. 2 we introduce Wigner’s little groups for massive and massless particles. In Sec. 3, it is shown that the E(2)-like little group for massless particles can be obtained as the infinite-momentum limit of the O(3)-like little group. In Sec. 4 the same logic is developed for spin-half particles. It is shown that the polarization of massless neutrinos is a consequence of gauge invariance.
Figure 1: Transformations of the $E(2)$ group and the cylindrical group. They share the same Lie algebra, but only the cylindrical group leads to a geometrical interpretation of the gauge transformation.

2 Wigner’s little groups

If we use the four-vector convention $x^\mu = (x, y, z, t)$, the generators of rotations around and boosts along the $z$ axis take the form

$$J_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

respectively. We can also write the four-by-four matrices for $J_1$ and $J_2$ for the rotations around the $x$ and $y$ directions, as well as $K_1$ and $K_2$ for Lorentz boosts along the $x$ and $y$ directions respectively [6]. These six generators satisfy the following set of commutation relations.

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k. \quad (2)$$

This closed set of commutation relations is called the Lie algebra of the Lorentz group. The three $J_i$ operators constitute a closed subset of this Lie algebra. Thus, the rotation group is a subgroup of the Lorentz group.

In addition, Wigner in 1939 [1] considered a group generated by

$$J_3, \quad N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1. \quad (3)$$

These generators satisfy the closed set of commutation relations

$$[N_1, N_2] = 0, \quad [J_3, N_1] = iN_2, \quad [J_3, N_2] = -iN_1. \quad (4)$$

As Wigner observed in 1939 [1], this set of commutation relations is just like that for the generators of the two-dimensional Euclidean group with one rotation and two
translation generators, as illustrated in Fig. 1. However, the question is what aspect of the massless particle can be explained in terms of this two-dimensional geometry.

Indeed, this question has a stormy history, and was not answered until 1987. In their paper of 1987 [3], Kim and Wigner considered the surface of a circular cylinder as shown in Fig. 1. For this cylinder, rotations are possible around the \( z \) axis. It is also possible to make translations along the \( z \) axis as shown in Fig. 1. We can write these generators as

\[
L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix},
\]

applicable to the three-dimensional space of \((x, y, z)\). They then satisfy the closed set of commutation relations

\[
[Q_1, Q_2] = 0, \quad [L_3, Q_1] = iQ_2, \quad [L_3, Q_2] = -iQ_1.
\]

which becomes that of Eq.(4) when \( Q_1, Q_2, \) and \( L_3 \) are replaced by \( N_1, N_2, \) and \( J_3 \) of Eq.(3) respectively. Indeed, this cylindrical group is locally isomorphic to Wigner’s little group for massless particles.

Let us go back to the generators of Eq.(3). The role of \( J_3 \) is well known. It is generates rotations around the momentum and corresponds to the helicity of the massless particle. The \( N_1 \) and \( N_2 \) matrices take the form [6]

\[
N_1 = \begin{pmatrix} 0 & 0 & -i & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & i \\ 0 & i & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}.
\]

The transformation matrix is

\[
D(u, v) = \exp\{-i(uN_1 + vN_2)\}
\]

\[
= \begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & -v & v \\ u & v & 1 - (u^2 + v^2)/2 & (u^2 + v^2)/2 \\ u & v & -(u^2 + v^2)/2 & 1 + (u^2 + v^2)/2 \end{pmatrix}.
\]

If this matrix is applied to the electromagnetic wave propagating along the \( z \) direction,

\[
A^\mu(z, t) = (A_1, A_2, A_3, A_0)e^{i\omega(z-t)},
\]

which satisfies the Lorentz condition \( A_3 = A_0 \), the \( D(u, v) \) matrix can be reduced to [5]

\[
D(u, v) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ u & v & 1 & 0 \\ u & v & 0 & 1 \end{pmatrix}.
\]
If $A_3 = A_0$, the four-vector $(A_1, A_2, A_3, A_3)$ can be written as
\[(A_1, A_2, A_3, A_0) = (A_1, A_2, 0, 0) + \lambda(0, 0, \omega, \omega),\] (11)
with $A_3 = \lambda \omega$. The four-vector $(0, 0, \omega, \omega)$ represents the four-momentum. If the $D$ matrix of Eq.(10) is applied to the above four vector, the result is
\[(A_1, A_2, A_3, A_0) = (A_1, A_2, 0, 0) + \lambda'(0, 0, \omega, \omega),\] (12)
with $\lambda' = \lambda + (1/\omega) (uA_1 + vA_3)$. Thus the $D$ matrix performs a gauge transformation when applied to the electromagnetic wave propagating along the $z$ direction \[2, 5, 6\].

3 Massless particle as a limiting case of massive particle

From the generators of the Lorentz group, it is possible to construct the four-by-four matrices for rotations around the $y$ axis and Lorentz boosts along the $z$ axis as \[6\]
\[R(\theta) = \exp (-i\theta J_2), \quad \text{and} \quad B(\eta) = \exp (-i\eta K_3),\] (13)
respectively. The Lorentz-boosted rotation matrix is $B(\eta)R(\theta)B(-\eta)$ which can be written as
\[
\begin{pmatrix}
\cos \theta & 0 & (\sin \theta) \cosh \eta & -(\sin \theta) \sinh \eta \\
0 & 1 & 0 & 0 \\
-(\sin \theta) \cosh \eta & 0 & \cos \theta - (1 - \cos \theta) \sinh^2 \eta & (1 - \cos \theta)(\cosh \eta) \sinh \eta \\
-(\sin \theta) \cosh \eta & 0 & -(1 - \cos \theta)(\cosh \eta) \sinh \eta & \cos \theta + (1 - \cos \theta) \cosh^2 \eta
\end{pmatrix}.
\] (14)

While $\tanh \eta = v/c$, this boosted rotation matrix becomes a transformation matrix for a massless particle when $\eta$ becomes infinite. On the other hand, it the matrix is to be finite in this limit, the angle $\theta$ has to become small. Let us set
\[\gamma = \frac{1}{2} \theta e^\eta.\] (15)

Then this four-by-four matrix becomes
\[
\begin{pmatrix}
1 & 0 & \gamma & -\gamma \\
0 & 1 & 0 & 0 \\
-\gamma & 0 & 1 - \gamma^2 / 2 & \gamma^2 / 2 \\
-\gamma & 0 & -\gamma^2 / 2 & 1 + \gamma^2 / 2
\end{pmatrix}.
\] (16)

This is the Lorentz-boosted rotation matrix around the $y$ axis. However, we can rotate this $y$ axis around the $z$ axis by $\alpha$. Then the matrix becomes
\[
\begin{pmatrix}
1 & 0 & \gamma \cos \alpha & -\gamma \cos \alpha \\
0 & 1 & \gamma \sin \alpha & -\gamma \sin \alpha \\
-\gamma \cos \alpha & -\gamma \sin \alpha & 1 - \gamma^2 / 2 & \gamma^2 / 2 \\
-\gamma \cos \alpha & -\gamma \sin \alpha & -\gamma^2 / 2 & 1 + \gamma^2 / 2
\end{pmatrix}.
\] (17)
this matrix becomes $D(u, v)$ of Eq. (8), if we let
\[ u = -\gamma \cos \alpha, \quad \text{and} \quad v = -\gamma \sin \alpha, \quad (18) \]

4 Spin-1/2 particles

Let us go back to the Lie algebra of the Lorentz group given in Eq. (2). It was noted that there are six four-by-four matrices satisfying nine commutation relations. It is possible to construct the same Lie algebra with six two-by-two matrices [6]. They are
\[ J_i = \frac{1}{2} \sigma_i, \quad \text{and} \quad K_i = \frac{i}{2} \sigma_i, \quad (19) \]
where $\sigma_i$ are the Pauli spin matrices. While $J_i$ are Hermitian, $K_i$ are not. They are anti-Hermitian. Since the Lie algebra of Eq. (2) is Hermitian invariant, we can construct the same Lie algebra with
\[ J_i = \frac{1}{2} \sigma_i, \quad \text{and} \quad \dot{K}_i = -\frac{i}{2} \sigma_i. \quad (20) \]
This is the reason why the four-by-four Dirac matrices can explain both the spin-1/2 particle and the anti-particle.

Here again the $J_i$ matrices generate the rotation-like $SU(2)$ subgroup. Here also we can consider the $E(2)$-like subgroup generated by
\[ J_3, \quad N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1. \quad (21) \]
The $N_1$ and $N_2$ matrices take the form
\[ N_1 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (22) \]

On the other hand, in the “dotted” representation,
\[ \dot{N}_1 = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix}, \quad \dot{N}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (23) \]
There are therefore two different $D$ matrices:
\[ D(u, v) = \exp\{-(iuN_1 + ivN_2)\} = \begin{pmatrix} 1 & u + iv \\ 0 & 1 \end{pmatrix}, \quad (24) \]
and
\[ \dot{D}(u, v) = \exp\{-(iu\dot{N}_1 + iv\dot{N}_2)\} = \begin{pmatrix} 1 & 0 \\ u - iv & 1 \end{pmatrix}. \quad (25) \]
These are the gauge transformation matrices applicable to massless spin-1/2 particles [5][6].
We are familiar with the notation
\[ \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \]
(26)

In addition, we need two additional spinors
\[ \dot{\chi}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \dot{\chi}_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]
(27)

Here are talking about the Dirac equation for with four-component spinors. The spinors \( \chi_+ \) and \( \dot{\chi}_- \) are gauge-invariant since
\[ D(u, v) \chi_+ = \chi_+, \quad \text{and} \quad D(u, v) \dot{\chi}_- = \dot{\chi}_-. \]
(28)
As for \( \chi_- \) and \( \dot{\chi}_+ \),
\[ D(u, v) \chi_- = \chi_- + (u - iv)\chi_+, \]
\[ \dot{D}(u, v) \dot{\chi}_+ = \dot{\chi}_+ + (u + iv)\dot{\chi}_-. \]
(29)
They are not gauge invariant. Thus, we can conclude that the polarization of massless neutrinos is a consequence of gauge invariance.

In this two-by-two representation, the Lorentz boost along the positive direction is
\[ B(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}, \]
(30)
the rotation around the \( y \) axis is
\[ R(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \]
(31)
Then, the boosted rotation matrix is
\[ B(\eta)R(\theta)B(-\eta) = \begin{pmatrix} \cos(\theta/2) & -e^\eta \sin(\theta/2) \\ e^{-\eta} \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \]
(32)

If \( \eta \) becomes very large, and this matrix is to remain finite, \( \theta \) has to become very small, and this expression becomes
\[ \begin{pmatrix} 1 - \gamma^2 e^{-2\eta}/2 & -\gamma \\ \gamma e^{-2\eta} & 1 - \gamma^2 e^{-2\eta}/2 \end{pmatrix}, \]
(33)
with \( \gamma = \frac{1}{2} \theta e^\eta \) as given in Eq. (15). This expression becomes
\[ D(\gamma) = \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}. \]
(34)
In this two-by-two representation, the rotation around the $z$ axis is

$$Z(\phi) = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix},$$

respectively. Thus

$$D(u, v) = Z(\phi + 180)D(\gamma)Z^{-1}(\phi + 180),$$

which becomes

$$D(u, v) = \begin{pmatrix} 1 & u + iv \\ 0 & 1 \end{pmatrix},$$

with $u$ and $v$ given in Eq.(18).

We have studied how the massive neutrino with its O(3)-like symmetry to become the massless neutrino in the massless limit. The boost parameter $\eta$ can be derived from the mass and momentum of the massive neutrino from

$$\tanh \eta = \frac{p}{\sqrt{m^2 + p^2}},$$

where $m$ and $p$ are the mass and the momentum of the neutrino respectively. For small values of $m/p$,

$$e^{\eta} = \frac{\sqrt{2}p}{m},$$

which becomes large when $m$ becomes very small.

The question is then the reverse process [7]. Start from the massless neutrino with its gauge degree of freedom. What happens when the particle gains its mass? In that case, the particle can be brought to its rest frame? The matrix of Eq.(32) is one way to bring it the rest frame, and $\gamma$ becomes $\sin \theta$. However, what happens when $\gamma$ is larger than one? This is a challenging future question.

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