Cost Sharing for Connectivity with Budget

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Abstract
We consider a cost sharing problem to connect all nodes in a weighted undirected graph, where the weight of each edge represents the cost to use the edge for the connectivity and the cost has to be shared among all connected nodes. There is one node called the source to which all the other nodes want to connect and it does not share the costs of the connectivity. As a node may need to go through other nodes to reach the source, the intermediate nodes may behave strategically to block the connection by cutting the edges adjacent to them. To prevent such strategic behavior, we design cost sharing mechanisms to incentivize all nodes not to cut any edge so that we can minimize the total cost for connecting all the nodes.

1 Introduction
Cost allocation problems are popular optimization problems in artificial intelligence, especially in economics and computer science [Moulin, 1999; Moulin and Shenker, 2001; Leonardi and Schäfer, 2004; Bogomolnaia and Moulin, 2010]. We consider a group of agents located at different points and there is also a source. The agents need to connect to the source directly or indirectly in order to get a service or information. The cost of connecting any pair of agents is known and the total cost of connecting them has to be shared among all connected nodes. There is one node called the source to which all the other nodes want to connect and it does not share the costs of the connectivity. As a node may need to go through other nodes to reach the source, the intermediate nodes may behave strategically to block the connection by cutting the edges adjacent to them. To prevent such strategic behavior, we design cost sharing mechanisms to incentivize all nodes not to cut any edge so that we can minimize the total cost for connecting all the nodes.

Our contribution: To incentivize agents to share all their connections in the cost sharing problem, we design a mechanism with unlimited budget that is truthful (i.e. each agent is incentivized to offer all its adjacent edges), budget balanced, cost monotonic and positive (i.e. each agent's cost share is non-negative). This mechanism does not work under limited budget. Thus, we propose another solution for the limited budget that is also truthful, budget feasible (i.e. each agent's budget is larger than its cost share), budget balanced and cost monotonic.

There is also a rich literature for the cost sharing problem, but it has not considered the agents' strategic behaviour to cut their adjacent edges [Bird, 1976]. For example, Kar [2002] provided an allocation rule based on Shapley value and showed that it is budget balanced and cost monotonic. Tijs and Driessen [1986] proposed a cost gap allocation rule based on each agent's marginal contribution and showed that it is budget balanced (i.e. the sum of all agents' cost share equals the total cost) and continuous (i.e. each agent's cost share should be continuous functions of adjacent edges' cost). Bergantiños and Vidal-Puga [2007] and Trudeau [2012] introduced the solutions based on irreducible cost matrix and Shapley value, and showed that they are continuous, cost monotonic (i.e. each agent's cost share should weakly increase if the cost of one of its adjacent edges increases) and population monotonic (i.e. when a new agent shows up, the cost share of each incumbent agent cannot increase).

2 The Model
We consider a cost sharing problem to connect all nodes in a weighted undirected graph $G = (V \cup \{s\}, E)$. The weight of each edge $(i, j) \in E$ denoted by $c_{(i, j)} > 0$ represents the cost to use the edge to connect $i$ with $j$. Node $s$ is the source to which all the other nodes want to connect. The total cost of the connectivity has to be shared among all connected nodes
except for $s$. Each node $i \in V$ has a public budget $B_i > 0$ which is the maximum cost that $i$ can pay.

Consider a real scenario, where $s$ represents a power station and each node $i \in V$ represents a village. A village can have electricity if there exists a path connecting $s$ with the village. All villages require electricity and they need to share the total cost of the connectivity.

Given the graph, the minimum cost of connecting all the nodes is the weight of a minimum spanning tree (we assume that the graph is connected). The question here is how they share the minimum cost. We also consider one natural strategic behaviour that each node except for the source can cut the edges adjacent to it. An edge $(i, j)$ cannot be used for the connectivity if either $i$ or $j$ cuts it. Our goal is to design cost sharing mechanisms to incentivize nodes to share all their adjacent edges so that we can use all the edges to minimize the total cost of the connectivity.

Formally, let $\theta_i = \{i \in V \cup \{s\}\}$ be the set of $i$’s adjacent edges which is called $i$’s type. Let $\theta = (\theta_1, \ldots, \theta_{|V|+1})$ be the type profile of all nodes including the source $s$. We also write $\theta = (\theta_i, \theta_{-i})$, where $\theta_{-i} = (\theta_1, \ldots, \theta_{-i}, \theta_{i+1}, \ldots, \theta_{|V|+1})$ is the type profile of all nodes except for $i$. Let $\Theta_i$ be the type space of $i$ and $\Theta$ be the type profile space of all nodes (which includes all possible graphs containing $V \cup \{s\}$).

We design a cost sharing mechanism that asks each node to report the set of its adjacent edges that can be used for the connectivity. Let $\theta_i' \subseteq \theta_i$ be the report of $i$, and $\theta' = (\theta_1', \ldots, \theta'_{|V|+1})$ be the report profile of all nodes. Given a report profile $\theta' \in \Theta$, the graph induced by $\theta'$ is denoted by $G_{\theta'} = (V \cup \{s\}, E_{\theta'}) \subseteq (V \cup \{s\}, E)$, where $E_{\theta'} = \{(i, j)|(i, j) \in (\theta'_i \cup \{s\})\}$.

**Definition 1.** A cost sharing mechanism consists of a node selection policy $g : \Theta \rightarrow V$, an edge selection policy $f : \Theta \rightarrow E$ and a cost sharing policy $x = (x_i)_{i \in V}$, where $x_i : \Theta \rightarrow \mathbb{R}$. Given a report profile $\theta' \in \Theta$, $g(\theta') \subseteq V$ selects the nodes to be connected, $f(\theta') \subseteq E_{\theta'}$ selects the edges to connect the selected nodes $g(\theta')$, and $x_i(\theta')$ is the cost that $i$ shares, which is zero if $i \notin g(\theta')$.

For simplicity, we use $(g, f, x)$ to denote the cost sharing mechanism. In the following, we introduce the desirable properties of the mechanism.

A mechanism is truthful if for each node (note that the source does not behave strategically in this setting), its cost share for the connectivity does not decrease by cutting its adjacent edges (i.e. offering all its adjacent edges minimizes its cost share).

**Definition 2.** A cost sharing mechanism $(g, f, x)$ is truthful if $x_i((\theta_i, \theta'_{-i})) \leq x_i((\theta'_i, \theta'_{-i}))$, for all $i \in V$, $\theta_i \in \Theta_i$, and all $\theta' \in \Theta$, $\theta'_{-i} \in \Theta_{-i}$ where $i \notin g(\theta_i, \theta'_{-i})$ and $i \in g(\theta'_i, \theta'_{-i})$.

A mechanism is budget feasible if each node’s cost share is not over its budget.

**Definition 3.** A cost sharing mechanism $(g, f, x)$ is budget feasible (BF) if $x_i(\theta') \leq B_i$, for all $i \in V$, for all $\theta' \in \Theta$.

For a selected node, its cost share will weakly increase, if the cost of one of its adjacent edges increases under the same report profile.

**Definition 4.** A cost sharing mechanism $(g, f, x)$ is cost monotonic (CM) if $x_i(\theta') \leq x'_i(\theta')$ for all $\theta' \in \Theta$ and all $i \in g(\theta')$, where $x'_i(\theta')$ is $i$’s cost share when the cost of one edge $(i, j) \in \theta'$ increases.

A mechanism is budget balanced if the sum of all nodes’ cost share equals the total cost of the selected edges for all report profiles. That is, the mechanism has no profit or loss.

**Definition 5.** A cost sharing mechanism $(g, f, x)$ is budget balanced (BB) if $\sum_{i \in E} x_i(\theta') = \sum_{(i,j) \in f(\theta')} c(i,j)$ for all $\theta' \in \Theta$.

Finally, each node’s cost share should be non-negative.

**Definition 6.** A cost sharing mechanism $(g, f, x)$ is positive if $x_i(\theta') \geq 0$ for all $i \in V$, for all $\theta' \in \Theta$.

In the rest of the paper, we design cost sharing mechanisms to satisfy the above properties.

### 3 Cost Sharing with Unlimited Budget

In this section, we first consider the case where each agent’s budget is unlimited, i.e. they can always afford the cost share. We propose a mechanism using Shapley value that is truthful, budget balanced, cost monotonic and positive.

The key ideas of the mechanism are as follows. We compute the minimum cost of connecting any subset of agents to the source. Then we can get the marginal cost of adding each agent to a subset of other agents. The final cost share of each agent is the average marginal cost by considering all possible joining sequences, similar to what Shapley value does. The mechanism is formally described in Algorithm 1. A running example is given in Example 1.

**Algorithm 1** Average Marginal Cost (AMC)

**Input:** A report profile $\theta' \in \Theta$

**Output:** The node selection $g(\theta')$, the edge selection $f(\theta')$, the cost sharing $x(\theta')$

1: Generate $G_{\theta'}$;
2: Starting from $s$, use Prim’s algorithm to compute a minimum spanning tree (MST);
3: Set $g(\theta')$ to be all the nodes in the MST and $f(\theta')$ to be all the edges of the MST;
4: for each subset $S \subseteq g(\theta')$ do
5: Compute $\nu(S)$, which is the minimum cost to connect $S$ to $s$;
6: end for
7: for $i \in g(\theta')$ do
8: Compute $x_i(\theta')$ using Shapley value on $\nu$, i.e.
9: $x_i(\theta') = \sum_{S \subseteq g(\theta') \setminus \{i\}} \frac{|S|!}{|g(\theta')|!} \cdot \nu(S \cup \{i\}) - \nu(S)$;
10: end for
11: $x_i(\theta') = 0$;
12: end for
13: return $g(\theta')$, $f(\theta')$, $x(\theta')$

**Example 1.** The graph $G_{\theta'}$ generated by a report profile $\theta' \in \Theta$ is shown in Figure 1. We can get $g(\theta') = \{A, B, C, D\}$ and $f(\theta') = \{(s, B), (A, B), (A, C), (A, D)\}$ as shown in
3.1 Properties of AMC

Now we analyze some nice properties of the average marginal cost mechanism.

**Theorem 1.** The average marginal cost is truthful.

**Proof.** The cost share of $i$ when $i$ truthfully reports is

\[ x_i(\theta^i) = \sum_{S \subseteq \Theta \setminus \{\theta^i\}} \frac{|S|!}{|\{g(\theta^i)\}||S|!} \cdot (v(S \cup \{i\}) - v(S)), \]

where $\theta^i = (\theta_1^i, \theta_2^i)$.

The cost share of $i$ when $i$ misreports is

\[ x_i(\theta^{i\prime}) = \sum_{S \subseteq \Theta \setminus \{\theta^{i\prime}\}} \frac{|S|!}{|\{g(\theta^{i\prime})\}||S|!} \cdot (v(S \cup \{i\}) - v(S)), \]

where $\theta^{i\prime} = (\theta_1^{i\prime}, \theta_2^{i\prime})$ and $v(S')$ is the value function of $S$ when $i$ misreports.

There are two cases if $i$ misreports: $g(\theta^i) = g(\theta^{i\prime})$ or $g(\theta^i) \neq g(\theta^{i\prime})$. When $g(\theta^i) = g(\theta^{i\prime})$, for any given set $S \subseteq V$, we have $\frac{|S|!}{|\{g(\theta^i)\}||S|!} = \frac{|S|!}{|\{g(\theta^{i\prime})\}||S|!}$. There are two cases when $i$ reports truthfully.

- The MST connecting $S$ does not connect $i$. Then $i$’s misreporting (i.e. $\theta^i \neq \theta_i^i$) cannot change the value function of $S$ since it does not depend on the report of $i$. That is, $v(S) = v(S')$. The value function of $S \cup \{i\}$ will increase or be unchanged because of $i$’s misreporting, i.e. $v'(S \cup \{i\}) - v(S) \geq v(S \cup \{i\}) - v(S)$.

- The MST connecting $S$ connects $i$. Then we have $v(S \cup \{i\}) - v(S) = 0$. If $i$ misreports, there are two cases.
  - The MST connecting $S$ still connects $i$. We get $v'(S \cup \{i\}) - v(S) = v(S \cup \{i\}) - v(S) = 0$.
  - The MST connecting $S$ does not connect $i$. We have $v'(S \cup \{i\}) - v(S) \geq 0$, i.e. $v'(S \cup \{i\}) - v'(S) \geq v(S \cup \{i\}) - v(S)$.

Hence, when $g(\theta^i) = g(\theta^{i\prime})$, we have $x_i(\theta^{i\prime}) \geq x_i(\theta^i)$.

When $g(\theta^i) \neq g(\theta^{i\prime})$, $i$’s misreporting consists of two parts. In part one, $i$’s children under $\theta^i$ will be connected to other nodes in the new generated MST. In this case, we have $g(\theta^{i\prime}) = g(\theta^{i\prime\prime})$. In part two, the connections of $i$’s children in MST under $\theta^i$ will be blocked. So we have $g(\theta^{i\prime}) > g(\theta^{i\prime\prime})$. Then $i$’s marginal cost increases or does not change. Therefore, $i$’s average marginal cost weakly increases and we have $x_i((\theta_i, \theta^i_{-i})) \leq x_i((\theta_i, \theta^{i\prime}_{-i}))$.

\[ \Box \]

**Theorem 2.** The average marginal cost is budget balanced.

**Proof.** Given $\theta^i \in \Theta$, the sum of all nodes’ cost share under our mechanism equals the value function of $g(\theta^i)$, i.e.

\[ \sum_{i \in g(\theta^i)} x_i(\theta^i) = v(g(\theta^i)) \]

In addition, we know that $v(g(\theta^i))$ equals the total cost of MST of the induced graph by $g(\theta^i) \cup \{s\}$, i.e. $v(g(\theta^i)) = \sum_{i,j \in f(\theta^i)} c(i,j)$. Thus we have $x_i(\theta^i) = \sum_{i,j \in f(\theta^i)} c(i,j)$, So the proposed mechanism is budget balanced.

\[ \Box \]

**Theorem 3.** The average marginal cost is cost monotonic.

**Proof.** Given nodes $i,j \in g(\theta^i)$, the edge $(i,j) \in E$ and $\theta^i \in \Theta$, for a set $S \subseteq g(\theta^i)$, $i$’s cost share is $x_i(\theta^i) = \max \{(S \cup \{i\}) - v(S), 0\}$. There are two cases.

- If the MST connecting $S$ does not connect $i$ and $c(i,j)$ increases, $v(S)$ does not change and $v(S \cup \{i\})$ will increase or be unchanged. Thus we get $v(S \cup \{i\}) - v(S)$ will increase or be unchanged.

- If the MST connecting $S$ connects $i$ and $c(i,j)$ increases, there are two cases.
  - The MST connecting $S$ connects $i$. In this case, $v(S \cup \{i\}) - v(S)$ does not change.
  - The MST connecting $S$ does not connect $i$. In this case, $v(S \cup \{i\}) - v(S)$ increases.

According to the analysis above, we know for each given set $S$, $v(S \cup \{i\}) - v(S)$ increases or is unchanged with the increment of $c(i,j)$. Obviously, $v(S)$ is unchanged with the increment of $c(i,j)$. Therefore, each node’s cost share should weakly increase if the cost of one of its adjacent edges increases.

\[ \Box \]

**Theorem 4.** The average marginal cost is positive.

**Proof.** Given a report profile $\theta^i \in \Theta$, for each node in $V \setminus g(\theta^i)$, we have $x_i(\theta^i) = 0$. For each node in $g(\theta^i)$, by Algorithm 1 and the definition of value function, we can get that $v(S \cup \{s\}) - v(S) \geq 0$ for a set $S \subseteq g(\theta^i)$. Therefore, each node’s cost share $x_i(\theta^i)$ is non-negative.

Note that the computation cost of the Shapley value is very high, we could just choose one random joining sequence to compute their marginal cost to reduce the computation, which will not affect the properties.

In the following, we give an intuitive example of our mechanism on trees.

**Example 2.** In Figure 2, given $\theta^i \in \Theta$, using our mechanism, we can get $g(\theta^i) = \{A, B, C\}$. Then we have $v(\{A\}) = 6$, $v(\{B\}) = 10$, $v(\{C\}) = 11$, $v(\{A, B\}) = 10$, $v(\{A, C\}) = 11$, $v(\{B, C\}) = 15$, $v(\{A, B, C\}) = 15$, $v(\emptyset) = 0$. So
we know $x_A(\theta') = 2$, $x_B(\theta') = 6$, $x_C(\theta') = 7$. Next, we compute each node’s cost share in another way. For node $B$, it should pay the cost of the edge $(A, B)$. In addition, it also should pay $\frac{1}{3}$ of the cost of the edge $(s, A)$. Therefore, the total cost share of $B$ is $x_B(\theta') = 4 + \frac{1}{3} \cdot 6 = 6$. Similarly, the cost share of $A$ is $x_A(\theta') = \frac{1}{3} \cdot 6 = 2$ and the cost share of $C$ is $x_C(\theta') = 5 + \frac{1}{3} \cdot 6 = 7$. Actually, the two methods of computing cost share are equivalent on trees.

**Algorithm 2 Selection Function $g^*(\theta', S)$**

**Input:** A report profile $\theta' \in \Theta$, a set $S \subseteq V$ of $S$

**Output:** A subset of $S$ $E_i = \{(p, q)|\, p \in N, q \in S\}$

1. Initialize $N = \{s\}$
2. Set $E_i = \{(p, q)|\, p \in N, q \in S\}$
3. while $E_i$ is not empty do
4. Find the edge $(i, j)$ in $E_i$ with the smallest weight;
5. Find the end node of $(i, j)$ that is in $S \setminus N$ and denote it by $k$;
6. if $B_k \geq c_{i,j}$ then
7. $N = N \cup \{k\}$
8. $E_i = \{(p, q)|\, p \in N, q \in S \setminus N\}$
9. else
10. $E_i = E_i \setminus \{(i, j)\}$
11. end if
12. end while
13. return $N \setminus \{s\}$

4 Cost Sharing with Limited Budget

Now we consider the situation that each node has a limited budget. We first show that the AMC defined in Algorithm 1 cannot be budget feasible. Consider the example in Figure 3, for nodes $C$ and $D$, we have $x_C(\theta') = \frac{55}{6} > 7 = B_C$ and $x_D(\theta') = \frac{49}{6} > 6 = B_D$. So in this section, we propose another budget feasible cost sharing mechanism.

Before introducing the mechanism, we need to design a special node selection mechanism defined in Algorithm 2. As their budget has to cover the cost, it is likely that some nodes’ budget is too small to cover their cost share. Then, we have to do some filtering to remove some nodes from the final connection. The idea is that whenever an agent is added to a group of agents who are connected to the source, its budget should at least cover the cost of the edge connecting to the group. Following this principle, starting from $s$, we gradually add nodes into the connected group until no more nodes can be added, which is what Algorithm 2 does.

After choosing which nodes can be connected, can we simply apply the AMC to compute their cost to meet their budget? The answer is no. A counter-example is also given in Figure 3, where the selection function of Algorithm 2 will simply choose all nodes, but AMC will ask $C$ and $D$ to be added, which is what Algorithm 2 does.

To consider their budgets in the cost calculation, we compute the Shapley value for each node $i$ in $g(\theta')$ based on value function $v$, i.e.
\[
\phi_i = \sum_{S \subseteq g(\theta') \setminus \{i\}} \frac{|S|!(|g(\theta')| - |S| - 1)!}{|g(\theta')|!} \cdot (v(S \cup \{i\}) - v(S));
\]

10. for $i \in V \setminus g(\theta')$ do
11. $x_i(\theta') = 0$
12. end for
13. Set $f(\theta') = E_{\text{MST}}^{\theta'}$, where $E_{\text{MST}}^{\theta'}$ is the set of edges of the MST of the graph induced by $g(\theta') \cup \{s\}$
14. return $g(\theta'), f(\theta'), x(\theta')$

**Algorithm 3 Saving-based Cost Sharing (SCS)**

**Input:** A report profile $\theta' \in \Theta$

**Output:** The node selection $g(\theta')$, the edge selection $f(\theta')$, the cost sharing $x(\theta')$

1. Set $g(\theta') = g(\theta', V)$
2. for $\forall S \subseteq g(\theta')$ do
3. Let $S' = g(\theta', S)$
4. Set $v(S) = \sum_{i \in S'} B_i - C(S')$, where $C(S')$ is the cost of the MST of the graph induced by $S' \cup \{s\}$
5. end for
6. Compute the Shapley value $\phi_i$ for each node $i$ in $g(\theta')$
7. for $i \in g(\theta')$ do
8. $x_i(\theta') = B_i - \phi_i$
9. end for
10. for $i \in V \setminus g(\theta')$ do
11. $x_i(\theta') = 0$
12. end for
13. Set $f(\theta') = E_{\text{MST}}^{g(\theta')}$, where $E_{\text{MST}}^{g(\theta')}$ is the set of edges of the MST of the graph induced by $g(\theta') \cup \{s\}$
14. return $g(\theta'), f(\theta'), x(\theta')$
Example 3. The graph $G_{\theta'}$ generated by a report profile $\theta' \in \Theta$ is shown in Figure 3(1). According to Algorithm 2, first we compare $B_B$ with $c_{(A,B)}$. Since $B_B > c_{(A,B)}$, node $B$ can be selected by the mechanism. Second, we compare $B_A$ with $c_{(A,B)}$. Since $B_A > c_{(A,B)}$, node $A$ can be selected by the mechanism. Similarly, we can get that nodes $C$ and $D$ can be selected by the mechanism. Then we get the set of nodes which can be selected by the mechanism is $g(\theta') = \{A, B, C, D\}$ (see Figure 3(2)). For each subset $S \subseteq \{A, B, C, D\}$, we can compute $g^*(\theta', S)$ by Algorithm 2. For example, we have $S = \{C\}$, $g^*(\theta', S) = 0$. $S = \{B, C\}$, $g^*(\theta', S) = \{B\}$, $S = \{A, B, C\}$, $g^*(\theta', S) = \{A, B, C, D\}$. $S = \emptyset$, $g^*(\theta', S) = \emptyset$. We can get the value function of each subset. For example, we have $v((B, C)) = \sum_{i \in g^*(\theta', B, C)} w_i - C\left(g^*(\theta', B, C)\right) = 2$. So the Shapley values of $A, B, C, D$ are 4.3, 0.5. Thus the cost share is $x(\theta') = \{8, 6, 6.5, 5.5\}$. The selected edges are $\theta' = \{(s, B), (A, B), (A, C), (A, D)\}$. Figure 3(2) shows the selected edges. The savings allocated to the node is zero. Therefore, each node’s cost share is in the end is not more than its budget.

Next, we give an intuitive example of our mechanism on trees.

Example 4. In Figure 4, given $\theta' \in \Theta$, using our mechanism defined in Algorithm 1, we can get $g(\theta') = \{A, B, C\}$. We have $v(\{A\}) = 2$, $v(\{B\}) = 0$, $v(\{C\}) = 0$, $v(\{A, B\}) = 5$, $v(\{A, C\}) = 3$, $v(\{B, C\}) = 0$, $v(\{A, B, C\}) = 6$, $v(\emptyset) = 0$. So we know $\phi_A = 4$, $\phi_B = 1.5$, $\phi_C = 0.5$, and thus $x_A(\theta') = 4$, $x_B(\theta') = 5.5$, $x_C(\theta') = 5.5$. We then compute each node’s cost share in another way. For node $B$, its savings $B_B - c_{(A,B)}$ is shared among $A$ and $B$ equally. Thus, its cost share is $x_B(\theta') = B_B - \frac{1}{2} \cdot (B_B - c_{(A,B)}) = 7 - \frac{7}{2} = 5.5$. Similarly, we can get $x_C(\theta') = B_C - \frac{1}{2} \cdot (B_C - c_{(A,C)}) = 6 - \frac{6-5}{2} = 5.5$. For node $A$, its savings are not shared among other nodes. So its cost share is $x_A(\theta') = B_A - (B_A - c_{(A,C)}) - \frac{1}{2} \cdot (B_B - c_{(A,B)}) = 8 - (8 - 6) = \frac{6-5}{2} = 4$.

4.1 Properties of SCS

We show the properties of SCS in this section.

Theorem 5. The saving-based cost sharing is truthful.

Proof. When node $i$ truthfully reports its type $\theta_i$, i.e., $\theta' = (\theta_i, \theta'_{-i})$. If node $i \in V \setminus g(\theta')$, it cannot be selected by misreporting. Otherwise, we have: $\phi_i = \sum_{S \subseteq g(\theta') \setminus \{i\}} \frac{|S|}{|\prod_{j \in S} g(\theta'_j)|} (v(S \cup \{i\}) - v(S))$.

When $i$ misreports its type $\theta'_i$, we let $\theta'' = (\theta'_i, \theta'_{-i})$ and then we have:

$\phi'_i = \sum_{S \subseteq g(\theta'_{\theta''}) \setminus \{i\}} \frac{|S|}{|\prod_{j \in S} g(\theta'_j)|} (v(S \cup \{i\}) - v'(S))$,

where $v'(S)$ is the value function of $S$ when $i$ misreports. There are two cases if $i$ misreports: $g(\theta') = g(\theta''')$ and $g(\theta') \neq g(\theta''')$.

When $g(\theta') = g(\theta''')$, for any given set $S \subseteq V$, we have $\frac{|S|}{|\prod_{j \in S} g(\theta'_j)|} (v(S \cup \{i\}) - v'(S)) = \frac{|S|}{|\prod_{j \in S} g(\theta'_j)|} (v(S \cup \{i\}) - v(S))$. There are three cases when $i$ reports truthfully:

- $g^*(\theta'', S \cup \{i\}) = g^*(\theta', S) \cup \{i\}$: we have $v(S \cup \{i\}) - v(S) = B_i + C(g^*(\theta', S)) - C(g^*(\theta', S) \cup \{i\})$. In this case, $i$ is selected and there are no more nodes in $S$ being selected with $i$’s participation. For node $i$, there is at least one edge connecting it to $g^*(\theta', S) \cup \{s\}$ and it can afford the cost of the edge. Hence, we have $B_i + C(g^*(\theta', S)) - C(g^*(\theta', S) \cup \{i\}) \geq 0$. If $i$ misreports, there are two cases.

- $g^*(\theta'', S \cup \{i\}) = g^*(\theta', S) \cup \{i\}$: in this case, since $C(g^*(\theta', S))$ cannot use the nodes beyond $g^*(\theta', S)$, it remains unchanged. Then $C(g^*(\theta'', S) \cup \{i\}) \geq C(g^*(\theta', S) \cup \{i\})$ since some edges are removed due to $i$’s misreporting. Hence, we have $v(S \cup \{i\}) - v(S) \geq v'(S \cup \{i\}) - v'(S)$.

- $g^*(\theta'', S \cup \{i\}) = g^*(\theta', S)$: in this case, we have $v(S \cup \{i\}) - v(S) \geq v'(S \cup \{i\}) - v'(S) = 0$.

If $i$ misreports, it still cannot be selected since the set of available edges is reduced. Hence, $v(S \cup \{i\}) - v(S) = v'(S \cup \{i\}) - v'(S) = 0$.

- $g^*(\theta'', S \cup \{i\}) = g^*(\theta', S) \cup S_0 \cup \{i\}$, where $S_0$ denotes the set of nodes that are selected in $g^*(\theta'', S \cup \{i\})$ due to $i$’s participation. We have $v(S \cup \{i\}) - v(S) = \sum_{j \in S \cup \{i\}} B_j + C(g^*(\theta', S)) - C(g^*(\theta', S) \cup S_0 \cup \{i\})$. The cost of the edges connecting $S_0 \cup \{i\}$ of the spanning tree generated by Algorithm 2 is larger than or equal to $C(g^*(\theta'', S) \cup S_0 \cup \{i\}) - C(g^*(\theta', S))$. Hence, we can get $v(S \cup \{i\}) - v(S) \geq 0$. If $i$ misreports, there are two cases.

- $g^*(\theta'', S \cup \{i\}) = g^*(\theta', S)$: in this case, node $i$ cannot be selected by our mechanism. So we get $v(S \cup \{i\}) - v(S) \geq v'(S \cup \{i\}) - v'(S) = 0$.

- $g^*(\theta'', S \cup \{i\}) = g^*(\theta', S) \cup S_0 \cup \{i\}$, where $S_0 \subseteq S_0$: in this case, node $i$ is still selected but its misreporting may reduce the set $S_0$, let $m_i = v(S \cup \{i\}) - v(S) = \sum_{j \in S \cup \{i\}} B_j + C(g^*(\theta', S)) - C(g^*(\theta', S) \cup S_0 \cup \{i\})$ when $i$ reports truthfully.
Let $m'_i = v'(S \cup \{i\}) - v'(S) = \sum_{j \in S'_0 \cup \{i\}} B_j + C(g'(\theta', S)) - C(g^*(\theta', S) \cup S'_0 \cup \{i\})$ when $i$ misreports. We get $m_i - m'_i = \sum_{j \in S_0 \setminus S'_0} B_j - C(g^*(\theta', S) \cup S \cup \{i\}) + C(g^*(\theta', S) \cup S'_0 \cup \{i\}) + C(g^*(\theta', S) \cup S \cup \{i\}) - C(g^*(\theta', S) \cup S'_0 \cup \{i\})$. When node $i$ reports truthfully, the node in $S_0 \setminus S'_0$ can be selected. So we get $\sum_{j \in S_0 \setminus S'_0} B_j + C(g^*(\theta', S) \cup S \cup \{i\}) - C(g^*(\theta', S) \cup S'_0 \cup \{i\}) \geq 0$. For the nodes in $g^*(\theta', S) \cup S'_0 \cup \{i\}$, the cost of the MST of the graph induced by $g^*(\theta', S) \cup S \cup \{i\}$ will increase when node $i$ misreports since the set of available edges is reduced. Then we can get $C(g^*(\theta', S) \cup S \cup \{i\}) - C(g^*(\theta', S) \cup S'_0 \cup \{i\}) \geq 0$. So we have $v(S \cup \{i\}) - v(S) \geq v(S \cup \{i\}) - v'(S)$. 

Put the analysis above together, when $g(\theta') = g(\theta'')$, we have $\phi_i > \phi'_i$. Since $x_i(\theta') = B_i - \phi_i$ and $x_i(\theta'') = B_i - \phi'_i$, we can get $x_i(\theta') < x_i(\theta'')$. When $g(\theta') \neq g(\theta'')$, $i$'s misreporting can be divided into two parts. In part one, $i$'s children under $\theta'$ will reconnect to other nodes in the new generated MST, which is same as the case $g(\theta') = g(\theta'')$. In the other part, the connections of $i$'s children in MST under $\theta''$ will be blocked, so $g(\theta') > g(\theta'')$. Then $i$'s marginal saving increases or does not change. Therefore, $\phi_i$ weakly decreases and we have $x_i((\theta_i, \theta''_i)) \leq x_i((\theta'_i, \theta''_i)).$ 

Theorem 6. The saving-based cost sharing is budget feasible. 

Proof. According to the edge selection policy, our mechanism outputs a spanning tree of the induced graph of $g(\theta') \cup \{s\}$. According to the cost sharing policy, given a report profile $\theta' \in \Theta$, we have $\sum_{i \in g(\theta')} x_i(\theta') = \sum_{i \in g(\theta')} B_i - \sum_{i \in g(\theta')} \phi_i + \sum_{i \in g(\theta')} B_i - (\sum_{i \in g(\theta')} B_i - C(g(\theta')) = C(g(\theta')) = \sum_{(i,j) \in \delta(f(\theta'))} c_{ij}$. Therefore, our mechanism is budget balanced. 

Theorem 8. The saving-based cost sharing is cost monotonic. 

Proof. Given a node $i \in g(\theta')$, the edge $(i, j) \in E$ and $\theta' \in \Theta$, for a set $S \subseteq g(\theta')$, we have $\phi_i = \sum_{S \subseteq g(\theta') \setminus \{i\}} \frac{\sum_{j \in g(\theta')} |E(\theta')| - |E(S)|}{|E(\theta')|} \cdot (v(S \cup \{i\}) - v(S))$ and $x_i(\theta') = B_i - \phi_i$. So we need to show that $v(S \cup \{i\}) - v(S)$ will decrease if $c_{ij}$ increases under the condition that $g(\theta')$ is unchanged. There are two cases. 

- $i \notin g(\theta')$: in this case, the node $i$ is not selected by the mechanism. Obviously, we have that the statement holds in this case. 
- $i \in g(\theta')$: in this case, node $i$ is selected by the mechanism. If $c_{ij}$ increases, for node $i$, there are two cases. 
  - Node $i$ cannot be selected. Obviously, we have that the statement holds in this case. 
  - Node $i$ is still selected. Given a set $S \subseteq V$, $v(S \cup \{i\}) - v(S)$ will decrease. 

When $g(\theta')$ will change if $c_{ij}$ increases. There are two cases for node $i$ if $c_{ij}$ increases. 

- If $i$ cannot be selected, obviously, we have that the statement holds in this case. 
- If $i$ is still selected, then we can know $j \notin g(\theta')$ before $c_{ij}$ increases. When $c_{ij}$ increases, then we get $j \notin g(\theta')$. We get $j$ connects to the source with $(i, j)$ before $c_{ij}$ increases. Therefore, $i$'s saving will decrease. Thus we know the cost share of $i$ will increase. 

From the analysis above together, we get that saving-based cost sharing is cost monotonic. 

5 Conclusions 

We proposed two cost sharing mechanisms for connecting all nodes in a weighted undirected graph. We considered an important strategic behaviour of a node to cut her adjacent edges to reduce her cost share. Our mechanisms can prevent such behaviour and can also handle the setting where each node has a limited budget to pay the cost. 

One interesting future work is to characterize all possible cost share mechanisms to incentivize them to share their connections. Since we have different solutions for the same problem, one way to evaluate them is some kind of fairness, e.g., an agent far from the source may end up paying a lot.
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