ON RADÓ’S THEOREM FOR POLYANALYTIC FUNCTIONS

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Abstract. We prove versions of Radó’s theorem for polyanalytic functions in one variable and also on simply connected \( \mathbb{C} \)-convex domains in \( \mathbb{C}^n \). Let \( \Omega \subset \mathbb{C} \) be a bounded, simply connected domain and let \( q \in \mathbb{Z}_+ \). Suppose at least one of the following conditions holds true: (i) \( g \in C^q(\Omega) \). (ii) \( g \in C^\kappa(\Omega) \), for \( \kappa = \min\{1, q-1\} \), such that \( g \) is \( q \)-analytic on \( \Omega \setminus g^{-1}(0) \) and such that \( \text{Re} \ g \) (\( \text{Im} \ g \) respectively) is a solutions to the \( p' \)-Laplace equation (\( p'' \)-Laplace equation respectively) on \( \Omega \setminus g^{-1}(0) \), for some \( p', p'' > 1 \). Then \( g \) agrees (Lebesgue) a.e. with a function that is \( q \)-analytic on \( \Omega \). In the process we give a simple proof of the fact that if \( f \in C^q(\Omega) \) is \( q \)-analytic on \( \Omega \setminus f^{-1}(0) \) then \( f \) is \( q \)-analytic on \( \Omega \). The extensions of the results to several complex variables are straightforward using known techniques.

1. Introduction

Radó’s theorem states that a continuous function on an open subset of \( \mathbb{C}^n \) that is holomorphic off its zero set extends to a holomorphic function on the given open set. For the one-dimensional result see Radó [7], and for a generalization to several variables, see e.g. Cartan [4].

Definition 1.1. Let \( \Omega \subset \mathbb{C} \) be an open subset. A function \( f \) on \( \Omega \) is called polyharmonic of order \( q \) if \( \Delta^q f = 0 \) on \( \Omega \), where \( \Delta \) denotes the Laplace operator.

Definition 1.2. Let \( \Omega \subseteq \mathbb{R}^n \) be an open subset. For a fixed \( p > 1 \), the \( p \)-Laplace operator of a real-valued function \( u \) on \( \Omega \) is defined as

\[
\Delta_p := \text{div}(|\nabla u|^{p-2} \nabla u)
\]

The operator can also be defined for \( p = 1 \) (it is then the negative of the so-called mean curvature operator) and \( p = \infty \) but we shall not concern ourselves with such cases.

Remark 1.3. Note the subtle similarity between the notation for the \( p \)-Laplace operator

\[
\Delta_p = \text{div}(|\nabla u|^{p-2} \nabla u)
\]

and that of the \( p \)th power of the Laplace operator \( \Delta^p \). We have that \( \Delta_2 = \Delta \).

More generally, we have

\[
\Delta_p u = |\nabla u|^{p-4} \left( |\nabla u|^2 \Delta u + (p-2) \sum_{i,j=1}^n \partial_{x_i} u \cdot \partial_{x_j} u \cdot \partial_{x_i} \partial_{x_j} u \right)
\]
Note that $\Delta_p$ is quasilinear. At least they both share the property of being elliptic operators. In the case of $\Delta^p$ this is a direct consequence of the fact that $\Delta$ is a elliptic operator and therefore any finite power is also, in particular the elliptic regularity theorem applies to $\Delta^p$ and to $\Delta_p$, and implies that any real-valued distribution solution $u$ to $\Delta^p u = 0$ (or to $\Delta_p$) on a domain $\Omega \subset \mathbb{R}^n$ is Lebesgue a.e. equal to a $C^\infty$-smooth solution $\tilde{u}$ to $\Delta^p \tilde{u} = 0$ (or to $\Delta_p \tilde{u} = 0$) on $\Omega$.

Kilpeläinen [5] proved the following.

**Theorem 1.4.** If $\omega \subset \mathbb{R}^2$ is a domain and if $u \in C^1(\Omega)$ satisfies the p-Laplace equation $\text{div}(|\nabla|^{p-2} \nabla u) = 0$ on $\Omega \setminus u^{-1}(0)$ then $u$ is a solution to the p-Laplacian on $\Omega$.

We mention that, more recently, Tarkhanov & Ly [6] proved the following related result in higher dimension.

**Theorem 1.5.** Let $\Omega \subset \mathbb{R}^n$ be an open subset. If $u \in C^1(\Omega) \cap C^2(\overline{\Omega})$ such that $\text{div}(|\nabla|^{p-2} \nabla u) = 0$ on $\Omega \setminus u^{-1}(0)$ then this holds true on all of $\Omega$.

We shall use the result of Kilpeläinen [5] in order to prove a natural version of Radó’s theorem for polyanalytic functions. Avanissian & Traoré [1, 2] introduced the following definition of polyanalytic functions of order $\alpha \in \mathbb{Z}_+^n$ in several variables.

**Definition 1.6.** Let $\Omega \subset \mathbb{C}^n$ be a domain, let $\alpha \in \mathbb{Z}_+^n$ and let $z = x + iy$ denote holomorphic coordinates in $\mathbb{C}^n$. A function $f$ on $\Omega$ is called polyanalytic of order $\alpha$ if in a neighborhood of every point of $\Omega$, \((\frac{\partial}{\partial z_j})^\alpha f(z) = 0, 1 \leq j \leq n\).

**Definition 1.7.** Let $\Omega \subset \mathbb{C}^n$ be an open subset and let $(z_1, \ldots, z_n)$ denote holomorphic coordinates for $\mathbb{C}^n$. A function $f$, on $\Omega$, is said to be separately $C^k$-smooth with respect to the $z_j$-variable, if for any fixed $(c_1, \ldots, c_{n-1}) \in \mathbb{C}^{n-1}$, chosen such that the function

$$z_j \mapsto f(c_1, \ldots, c_{j-1}, z_j, c_j, \ldots, c_{n-1}),$$

is well-defined (i.e. such that $(c_1, \ldots, c_{j-1}, z_j, c_j, \ldots, c_{n-1})$ belongs to the domain of $f$) is $C^k$-smooth with respect to $\text{Re } z_j, \text{Im } z_j$. For $\alpha \in \mathbb{Z}_+^n$ we say that $f$ is separately $\alpha$-smooth if $f$ is separately $C^{\alpha_j}$-smooth with respect to $z_j$ for each $1 \leq j \leq n$.

We shall need the following result.

**Theorem 1.8.** (See [2] Theorem 1.3, p. 264) Let $\Omega \subset \mathbb{C}^n$ be a domain and let $z = (z_1, \ldots, z_n)$, denote holomorphic coordinates in $\mathbb{C}^n$ with $\text{Re } z =: x, \text{Im } z = y$. Let $f$ be a function which, for each $j$, is polyanalytic of order $\alpha_j$ in the variable $z_j = x_j + iy_j$ (in such case we shall simply say that $f$ is separately polyanalytic of order $\alpha$). Then $f$ is jointly smooth with respect to $(x, y)$ on $\Omega$ and furthermore is polyanalytic of order $\alpha = (\alpha_1, \ldots, \alpha_n)$ in the sense of Definition [1,6].

2. Statement and proof of the result

Let us make the following first observation.

**Proposition 2.1.** Let $\Omega \subset \mathbb{C}$ be a simply connected domain, let $q \in \mathbb{Z}_+$ and let $f \in C^q(\Omega)$ be a $q$-analytic function on $\Omega \setminus f^{-1}(0)$. Then $f$ is $q$-analytic on $\Omega$. 
Proof. If \( f \equiv 0 \) then we are done, so assume \( f \not\equiv 0 \). Since \( f \) is \( C^\kappa \)-smooth the function \( \partial_z f \) is continuous. By assumption \( \partial_z f = 0 \) on \( \Omega \setminus f^{-1}(0) \). Set \( Z := (f^{-1}(0))^o \) (\( ^o \) denoting the interior) and \( X := \{ f \neq 0 \} \cup Z \). Now \( f|_Z \) clearly satisfies \( \partial_z f = 0 \). Let \( p \in \partial X \). If \( p \) is an isolated zero of \( f \), then by continuity we have \( \partial_z f(p) = 0 \). Suppose \( p \) is a non-isolated zero. We have for each sufficiently large \( j \in \mathbb{Z}_+ \) that \( \{ |z - p| < 1/j \} \cap X \neq \emptyset \). This implies that there exists a sequence \( \{ z_j \}_{j \in \mathbb{Z}_+} \) of points \( z_j \in X \) such that \( z_j \to p \) as \( j \to \infty \). By continuity we have

\[
\partial_z f(p) = \lim_{j \to \infty} \partial_z f(z_j) = 0
\]

This completes the proof. \( \square \)

**Theorem 2.2.** Let \( \Omega \subset \mathbb{C} \) be a bounded, simply connected domain, let \( q \in \mathbb{Z}_+ \) and let \( f \) be a function \( q \)-analytic on \( \Omega \setminus f^{-1}(0) \). Suppose at least one of the following conditions holds true:

(i) \( f \in C^\kappa(\Omega) \), for \( \kappa = \min\{1, q - 1\} \), and Re \( f \) (Im \( f \) respectively) is a solution to the \( p' \)-Laplace equation (\( p'' \)-Laplace equation respectively) on \( \Omega \setminus f^{-1}(0) \), for some \( p', p'' > 1 \).

(ii) \( f \in C^q(\Omega) \).

Then \( f \) agrees (Lebesgue) a.e. with a function that is \( q \)-analytic on \( \Omega \).

**Proof.** The case (ii) follows from Proposition 2.1. So suppose (i) holds true. If \( q = 1 \) the theorem is well-known and due to Radó [2], so assume \( q \geq 2 \). Let \( f = u + iv \) where \( u = \text{Re} f \), \( v = \text{Im} f \). Now \( f^{-1}(0) = u^{-1}(0) \cap v^{-1}(0) \), whence \( u \) (and \( v \) respectively) is a solution to the \( p' \)-Laplace equation (\( p'' \)-Laplace equation respectively) on \( \Omega \setminus u^{-1}(0) \) (\( \Omega \setminus v^{-1}(0) \) respectively). If \( f \in C^\kappa(\Omega) \) and \( q \geq 2 \) then \( u \) and \( v \) respectively are at least \( C^1 \)-smooth thus satisfy the conditions of Theorem 1.4. Hence it follows that \( u \) (\( v \) respectively) are solutions to the \( p' \)-Laplace equation (\( p'' \)-Laplace equation respectively) on all of \( \Omega \). By Remark 1.3 (in particular Elliptic regularity) it follows that \( u \) and \( v \) respectively agree (Lebesgue) a.e. on \( \Omega \) with \( C^\infty \)-smooth functions \( \bar{u} \) and \( \bar{v} \) respectively. This implies that the function \( \tilde{f} := \bar{u} + i \bar{v} \) is \( C^\infty \)-smooth on \( \Omega \) and agrees (Lebesgue) a.e. on \( \Omega \) with \( f \). Suppose there exist a point \( p_0 \in \Omega \) such that \( \partial_z \tilde{f}(p_0) \neq 0 \). Set \( Z := (f^{-1}(0))^o \) and \( X := \{ f \neq 0 \} \cup Z \). By continuity there exists an open neighborhood \( U_{p_0} \) of \( p_0 \) in \( \Omega \) such that \( \partial_z \tilde{f} \neq 0 \) on the open subset \( U_{p_0} \cap X \). By the definition of \( \tilde{f} \) there exists a set \( E \) of zero measure such that on \( V_{p_0} := (\Omega \setminus U_{p_0}) \setminus E \) we have that \( \partial_z \tilde{f} \) exists (since \( X \) contains no point of \( f^{-1}(0) \setminus Z \)) and satisfies \( 0 = \partial_z \tilde{f} = \partial_z \tilde{f} \) on \( V_{p_0} \), which could only happen if \( V_{p_0} \) is empty which is impossible since \( E \) cannot possess interior points. We conclude that \( \partial_z \tilde{f} = 0 \) on \( \Omega \). This completes the proof. \( \square \)

**Theorem 2.3** (Radó’s theorem for polyanalytic functions in several complex variables). Let \( \Omega \subset \mathbb{C}^n \) be a bounded \( \mathbb{C} \)-convex domain. Let \( \alpha \in \mathbb{Z}_+^n \). Suppose \( f \) is \( \alpha \)-analytic on \( \Omega \setminus f^{-1}(0) \) such that one of the following conditions hold true:

(i) For each \( j = 1, \ldots, n \), the function \( f \) is separately \( C^{\kappa_j} \)-smooth with respect to \( z_j \) (i.e. for each fixed value of the remaining variables \( z_k \), \( k \neq j \), \( f \) becomes a \( C^{\kappa_j} \)-smooth function of \( z_j \), \( \kappa_j = \min\{1, \alpha_j - 1\} \) and Re \( f \) (Im \( f \) respectively) are solutions to the \( p' \)-Laplace equation (\( p'' \)-Laplace equation respectively) for some \( p', p'' > 1 \).

(ii) For each \( j = 1, \ldots, n \), the function \( f \) is separately \( C^{\alpha_j} \)-smooth with respect to
Then $f$ agrees (Lebesgue) a.e. with a function that is $\alpha$-analytic on $\Omega$.

Proof. Denote for a fixed $c \in \mathbb{C}^{n-1}$, $\Omega_{c,k} := \{z \in \Omega : z_j = c_j, j < k, z_j = c_{j-1}, j > k\}$. Since $\Omega$ is $\mathbb{C}$-convex, $\Omega_{c,k}$ is simply connected. Consider the function $f_c(z_k) := f(c_1, \ldots, c_{k-1}, z_k, c_k, \ldots, c_{n-1})$. Clearly, $f_c$ is $\alpha_k$-analytic on $\Omega_{c,k} \setminus f^{-1}(0)$ for any $c \in \mathbb{C}^{n-1}$. Since $f_c^{-1}(0) \subseteq f^{-1}(0)$, Theorem 2.2 applies to $f_c$ meaning that $f$ agrees a.e. with a function $\tilde{f}$ that is separately polyanalytic of order $\alpha_j$ in the variable $z_j, 1 \leq j \leq n$. By Theorem 1.8 the function $\tilde{f}$ must be polyanalytic of order $\alpha$ on $\Omega$. This completes the proof.

Corollary 2.4. Let $\Omega \subset \mathbb{C}$ be a bounded $\mathbb{C}$-convex domain and let $\alpha \in \mathbb{Z}_n^+$. Suppose $f$ is separately $C^{\alpha_j}$-smooth with respect to $z_j, j = 1, \ldots, n$. If $f$ is $\alpha$-analytic on $\Omega \setminus f^{-1}(0)$, then $f$ agrees (Lebesgue) a.e. with a function that is $\alpha$-analytic on $\Omega$.

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