On $p$-Parabolicity of Riemannian Submersions

Maria Andrade and Pietro da Silva

Abstract. We provide some criteria implying $p$-parabolicity of submerged Riemannian manifolds. In particular, if $N$ is $p$-parabolic and $\pi : M \to N$ is a proper Riemannian submersion with uniformly bounded volume of fibers, then $M$ is also $p$-parabolic. In the case of warped products we characterize $p$-parabolicity in terms of a volume growth condition.

Mathematics Subject Classification. 31C12, 53C43, 58E20.

Keywords. $p$-parabolicity, Riemannian submersions, warped products, volume growth.

1. Introduction

Let $(M, g)$ be a connected Riemannian manifold and consider a pair of subsets $D \subset \Omega \subset M$ with $D$ compact and $\Omega$ a connected domain. Given $p \in (1, \infty)$, the $p$-capacity of $D$ in $\Omega$ is defined by

$$\text{Cap}_p(D, \Omega) := \inf \left\{ \int_{\Omega} |\nabla u|^p : u \in C^1_0(\Omega), \ u \geq 1 \text{ on } D \right\}. \tag{1}$$

If $\Omega = M$, then we write $\text{Cap}_p(D)$ for simplicity. Due to well known properties of $p$-capacity, given a sequence of domains $\{\Omega_t\}$ with $\Omega_t \subset \Omega_{t+1}$ and $\bigcup_{t=1}^{\infty} \Omega_t = M$, we get $\text{Cap}_p(D) = \lim_{t \to \infty} \text{Cap}_p(D, \Omega_t)$. We say that $M$ is $p$-parabolic if $\text{Cap}_p(D) = 0$ for any compact $D \subset M$ and $p$-hyperbolic otherwise. Clearly every compact manifold is $p$-parabolic for any $p > 1$ and it is known that $\mathbb{R}^n$ is $p$-parabolic if, and only if, $p \geq n$ (see [6]).

The $p$-parabolicity is closely related with properties of the $p$-Laplacian operator defined by $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, where $u : M \to \mathbb{R}$. In fact, the $p$-Laplacian is the Euler-Lagrange operator associated to the energy functional
in the right side of (1) (see [7], [3]). The case $p = 2$ has been extensively studied linking several mathematical areas, namely geometry, analysis and probability ([4] provides a deep survey on this topic).

In this paper we deal with Riemannian submersions $\pi : M \to N$. More precisely, our first result reads as follows.

**Theorem 1.** Let $M$ and $N$ be Riemannian manifolds and $\pi : M \to N$ a surjective Riemannian submersion with compact fibers $\mathcal{F}_x = \pi^{-1}(x)$ having uniformly bounded volume, i.e., $\text{Vol}(\mathcal{F}_x) \leq C$. If $N$ is $p$-parabolic, then $M$ is $p$-parabolic, where $1 < p < \infty$.

A related result due to Brandão and Oliveira [2] asserts the 2-parabolicity of a manifold submerged in a parabolic one in such a way that fibers are minimal and compact. In addition to deal with the more general setting of $p$-parabolicity, here we drop conditions over mean curvature to use weaker restrictions on volume of fibers.

In the special case of warped products we are able to claim necessary and sufficient conditions to $p$-parabolicity.

**Theorem 2.** Let $N^n$ be a complete and $L^1$ be a compact Riemannian manifold and consider the warped product $M = N \times_f L$, where $1 < p < \infty$. A sufficient condition to $M$ be $p$-parabolic is

$$\int_1^{\infty} \left( \int_{\partial B_t} f(x)^t \, d\mu'_N(x) \right)^{\frac{1}{1-p}} \, dt = \infty,$$

where $d\mu'_N$ denotes the Riemannian measure of the co-dimension 1 on hypersurfaces in $N$. Furthermore, if $N$ is a model manifold, then condition (2) is also necessary for $p$-parabolicity.

Such a characterization of $p$-parabolicity of warped products generalizes [6] which deals with cylindrical warped manifolds.

This paper is organized in this way: Sect. 2 states basics definitions and notations to be used in the following; Sect. 3 presents examples to situate our results in the literature; proofs are in Sect. 4.

### 2. Preliminaries

**Definition 1 (Riemannian Submersion).** A smooth map $\pi : (M, g) \to (N, h)$ is a Riemannian submersion if $\pi_*$ is surjective and satisfies the following property:

$$g_x(v, w) = h_{\pi(x)}(\pi_*v, \pi_*w)$$

for any $v, w$ tangent vectors in $T_xM$ and perpendicular to the kernel of $\pi_*$. If $N$ and $L$ are Riemannian manifolds, then the projection $\pi : N \times L \to N$ is a Riemannian submersion, since for each point $y \in L$, the map $\pi|_{N \times \{y\}}$ is
an isometry on $N$. Examples of manifolds which admits natural Riemannian submersions are the warped product manifolds (see [1] and [5]).

**Definition 2** (*Warped Product*). Let $(N, g)$ and $(L, h)$ be Riemannian manifolds and $f : N \to (0, \infty)$ a differentiable function, the *warped product* $M = N \times_f L$ is the manifold $N \times L$ furnished with the Riemannian metric $\bar{g}_{(x,y)} = g_x + f(x)h_y$.

We call $N$ and $L$ basis (or leaves) and fibers, respectively. In the following we denote a typical point of $M$ by $\xi = (x, y)$ where $x \in N$ and $y \in L$.

Consider the canonical projections $\pi : M \to N$ and $\eta : M \to L$. While for each $y \in L$ the projection $\pi|_{N \times \{y\}}$ is an isometry on $N$, for each $x \in N$ the projection $\eta|_{\{x\} \times L}$ is a dilation on $L$, with scale factor $1/f(x)$. In particular, Theorem 1 implies that, if $L$ is compact, boundedness of the warp function $f$ is a sufficient condition to $p$-parabolicity of $M$.

A *model manifold* is a spherically symmetric manifold which in the case of a complete and non compact one is a warped manifold $M = N \times_f L$ where $N = [0, \infty)$, $L = S^{n-1}$, $f(0) = 0$ and $f'(0) = 1$. A simple example of model manifold is $\mathbb{R}^n$. We refer to [4] for precise definition and properties.

### 3. Examples

Warped products are a powerful tool to produce examples. The Theorems 1 and 2 may be used to generate relevant examples in the Potential Theory on manifolds. In this section we show some examples to situate the range of our results in the literature. Before going on, recall that $\mathbb{R}^n$ is $p$-parabolic for $p \geq n$ as consequence of its decomposition in $B_1 \cup ([1, \infty) \times_f S^{n-1})$, where $B_1$ is the closed unitary ball in $\mathbb{R}^n$ and $f(x) = x$.

**Example 1.** Consider $L = S^l$, $N = \mathbb{R}^n$ and $f(x) = e^{-|x|^2}$ (see Fig. 1). Since the projection $\pi : M = \mathbb{R}^n \times_f S^l \to N$ is a Riemannian submersion, $L$ is compact and $f$ is a bounded function, it follows by the Theorem 1 that $M$ is

![Figure 1. Warped product with $N = \mathbb{R}$, $L = S^1$ and $f(x) = e^{-x^2}$](image)
$p$-parabolic for $p \geq n$. Note that our result generalizes the Theorem 2.3 in [2] since in this example the fibers are not minimal.

**Example 2.** Consider $L = S^\ell$, $N = \mathbb{R}$, $B_t = (-t, t)$ and $f(x) = \sqrt{1 + x^2}$.

Following the Theorem 2, the $p$-parabolicity of the manifold $M$ is equivalent to

$$\int_1^\infty \left( \int_{\partial B_t} (1 + x^2)^{\frac{\ell}{2}} d\mu'_{N}(x) \right)^{\frac{1}{1-p}} dt = \int_1^\infty \left( (1 + t^2)^{\frac{\ell}{2}} \right)^{\frac{1}{1-p}} dt = \infty$$

which occurs if, and only if, $p \geq \ell + 1$. The case $\ell = 1$ returns the hyperboloid of one sheet (see Fig. 2) that is 2-parabolic.

We observe that the leaves in the Example 2 do not have uniformly limited volume. In particular, we can not apply Theorem 1 in this case.

### 4. Proofs

To prove the Theorem 1 we need the following result.

**Proposition 1** (Troyanov [6]). The domain $\Omega$ is $p$-parabolic if, and only if, there exists a sequence of functions $u_j \in C^0_0(\Omega)$ such that $0 \leq u_j \leq 1$, $u_j \to 1$ uniformly on every compact subsets of $\Omega$ and

$$\int_\Omega |\nabla u_j|^p \to 0.$$
Proof of Theorem 1. Since \( N \) is \( p \)-parabolic, by Proposition 1 there exists a sequence of functions \( u_j \in C^1_0(N) \) such that \( 0 \leq u_j \leq 1 \), \( u_j \to 1 \) uniformly on every compact of \( N \) and
\[
\int_N |\nabla u_j|^p \to 0.
\]

Consider \( \tilde{u}_j = u_j \circ \pi \). Note that \( 0 \leq \tilde{u}_j \leq 1 \), since \( \pi \) is a proper submersion. \( \{\tilde{u}_j\} \subseteq C^1_0(M) \) and if \( K \subseteq M \) is compact with \( y \in K \), then \( \pi(y) \in \pi(K) \), a compact subset of \( N \), so \( |\tilde{u}_j(y) - 1| = |u_j(\pi(y)) - 1| \to 0 \). Moreover, \( |\pi_*| \leq 1 \) implies that
\[
\int_M |\nabla \tilde{u}_j|^p d\mu_M = \int_M |\nabla u_j|^p |\pi_*|^p d\mu_M
\leq \int_M |\nabla u_j|^p d\mu_M
= \int_N \int_{\mathcal{F}_x} |\nabla u_j|^p d\mu_{\mathcal{F}_x} d\mu_N
= \int_N Vol(\mathcal{F}_x) |\nabla u_j|^p d\mu_N
\leq C \int_N |\nabla u_j|^p d\mu_N \xrightarrow{j \to \infty} 0.
\]

So, by Proposition 1 \( M \) is \( p \)-parabolic. \( \square \)

To prove Theorem 2 we need the definition of \( p \)-flux of a function \( h \).

Definition 3. To each pair \( D \subset \subset \Omega \subset M \) let \( \Lambda(D, \Omega) \) be the class of functions \( h : \Omega \to \mathbb{R} \) such that

(i) \( h \) is continuous, locally Lipschitz, non constant and bounded below;
(ii) \( D \subset \{ \xi \in \Omega : h(\xi) = r_0 := \min h \} \);
(iii) if \( r < r_1 := \sup h \in \mathbb{R} \cup \{ \infty \} \), then \( \{ \xi \in \Omega : h(\xi) \leq r \} \) is compact.

The \( p \)-flux of a function \( h \in \Lambda(D, \Omega) \) is the function \( \Phi_{h,p} : [r_0, r_1) \to \mathbb{R} \) defined by
\[
\Phi_{h,p}(r) = \int_{\partial \Omega_r} |\nabla h(\xi)|^{p-1} d\mu_M(\xi),
\]
where \( \Omega_r := \{ \xi \in \Omega : h(\xi) < r \} \).

We use the next result that shows a relation between \( p \)-capacity and \( p \)-flux to prove Theorem 2.

Theorem 3 (Troyanov [6]). Let \( D \subset \subset \Omega \subset M \) and \( p > 1 \), then
\[
Cap_p(D, \Omega) = \inf_{h \in \Lambda(D, \Omega)} \left( \int_{r_0}^{r_1} \Phi_{h,p}(r) \frac{1}{r^{1-p}} dr \right)^{1-p}.
\]
Proof of Theorem 2. If $N$ is compact, then the product $N \times_f L$ is compact, hence $p$-parabolic. Otherwise, if $N$ is not compact we are going to construct subsets $D, D_R$ of $M$ such that $D \subset D_R \subset D_R'$ for $R < R'$ and $M = \bigcup_{R>1} D_R$ attaining the $p$-capacity in the form

$$Cap_p(D, D_R) \leq \text{vol}(L) \left[ \int_1^R \left( \int_{\partial B_r} f(x)^{\ell} d\mu_N'(x) \right)^{\frac{1}{1-p}} dt \right]^{1-p}.$$  \hspace{1cm} (3)$$

Therefore, the sufficiency of the condition (2) will follow from

$$Cap_p(D) = \lim_{R \to \infty} Cap_p(D, D_R).$$

Take an exhaustion of $N$ by closed geodesic balls $(B_t)_{t>0}$ centered at the same point $x_0$. Set $D = B_1 \times_f L$ and $D_R = B_R \times_f L$ for $R > 1$ and define $h : M \to [1, \infty)$ by taking $h|_D \equiv 1$ and $h(\xi) = \rho(\pi(\xi))$ if $\xi \notin D$, where $\rho$ represents the distance to $x_0$. Note that $h \in \Lambda(D, D_R)$ for every $R > 1$, and $|\nabla h(\xi)| \leq |\pi_*(\nabla \rho(\pi(\xi)))| \leq 1$.

Thus, the $p$-flux $\Phi_{h,p} : [1, R) \to \mathbb{R}$ satisfies

$$\Phi_{h,p}(r) = \int_{\partial D_r} |\nabla h(\xi)|^{p-1} d\mu_M'(\xi)$$

$$\leq \int_{\partial D_r} d\mu_M'(\xi)$$

$$= \int_{\partial B_r} \left( \int_{\partial B_r} f(x)^{\ell} d\mu_N'(x) \right) d\mu_M'(\xi)$$

$$= \int_{\partial B_r} \text{vol}(L) f(x)^{\ell} d\mu_N'(x).$$

Therefore,

$$Cap_p(D, D_R) = \inf_{\tilde{h} \in \Lambda(D, D_R)} \left( \int_1^R \Phi_{\tilde{h},p}(r)^{\frac{1}{1-p}} dr \right)^{1-p}$$

$$\leq \left( \int_1^R \Phi_{h,p}(r)^{\frac{1}{1-p}} dr \right)^{1-p}$$

$$\leq \text{vol}(L) \left[ \int_1^R \left( \int_{\partial B_r} f(x)^{\ell} d\mu_N'(x) \right)^{\frac{1}{1-p}} dr \right]^{1-p}.$$}

To attain the converse inequality in (3), and so the necessity of condition (2) for $p$-parabolicity, suppose $N$ is a model manifold and consider a polar coordinate system on $N$ with origin in $x_0$, then a point $x \in N$ will be viewed

\footnote{Indeed, by definition $D \subset \subset D_R \subset M$, $h$ is Lipschitz continuous (because $|\nabla h| \leq 1$), non constant and bounded below, $h^{-1}(1) = D$ and if $1 < r < \infty$, $h^{-1}(r) = D_r$ that are compact subsets of $M$.}
as a pair \((t, x')\) where \(t = \rho(x)\) and \(x' \in \partial B_t\), whereas \(\xi \in M\) is a triple \((t, x', y)\).

Let us take an arbitrary test function \(u \in C^1_0(D_R)\) such that \(u|_D \equiv 1\). Then we have

\[
1 = \left| \int_1^R \frac{\partial u(t, x', y)}{\partial t} dt \right|
\leq \int_1^R |\nabla u(t, x', y)| dt
= \int_1^R |\nabla u(t, x', y)| \left( \int_{\partial B_t} f(x)^{\ell} d\mu'_N(x) \right)^{\frac{1}{p}} \left( \int_{\partial B_t} f(x)^{\ell} d\mu'_N(x) \right)^{-\frac{1}{p}} dt
\leq \left[ \int_1^R \left( |\nabla u(t, x', y)|^p \int_{\partial B_t} f(x)^{\ell} d\mu'_N(x) \right) dt \right]^{\frac{1}{p}} \cdot \left[ \int_1^R \left( \int_{\partial B_t} f(x)^{\ell} d\mu'_N(x) \right)^{\frac{1}{1-p}} dt \right]^{\frac{p-1}{p}}
\]

where the last inequality follows by Hölder’s Inequality. Writing in another way

\[
\left[ \int_1^R \left( \int_{\partial B_t} f(x)^{\ell} d\mu'_N(x) \right)^{\frac{1}{1-p}} dt \right]^{1-p}
\leq \int_1^R \left( |\nabla u(t, x', y)|^p \int_{\partial B_t} f(x)^{\ell} d\mu'_N(x) \right) dt
\]

and integrating over \(L\), since \(N\) is a model manifold, we get

\[
vol(L)I(R, p, t) \leq \int_L \left( \int_1^R \left( |\nabla u(t, x', y)|^p \int_{\partial B_t} f(x)^{\ell} d\mu'_N(x) \right) dt \right) d\mu_L(y)
= \int_{D_R} |\nabla u(\xi)|^p d\mu_M(\xi),
\]

where

\[
I(R, p, t) = \left[ \int_1^R \left( \int_{\partial B_t} f(x)^{\ell} d\mu'_N(x) \right)^{\frac{1}{1-p}} dt \right]^{1-p}.
\]

Finally, by taking the infimum among test functions \(u\) we obtain the desired inequality.

Therefore, we can write

\[
Cap_p(D) = \lim_{R \to \infty} vol(L) \left[ \int_1^R \left( \int_{\partial B_t} f(x)^{\ell} d\mu'_N(x) \right)^{\frac{1}{1-p}} dt \right]^{1-p}
\]
and since $1 - p < 0$ the $p$-parabolicity occurs if, and only if,

$$\int_1^\infty \left( \int_{\partial B_t} f(x)^t d\mu'_N(x) \right) = \infty.$$ 

\[\square\]

Acknowledgements

The authors would like to thank the reviewers for their careful reviews and useful comments. The authors are especially grateful to Professor Marcos P. Cavalcante for his helpful suggestions.

References

[1] Bishop, R.L., O’Neill, B.: Manifolds of negative curvature. Trans. Amer. Math. Soc. 145, 1–49 (1969)
[2] Brandão, M.C., Oliveira, J.Q.: Stochastic properties of the Laplacian on Riemannian submersions. Geom. Dedicata 162, 363–374 (2013). doi:10.1007/s10711-012-9732-2
[3] Coulhon, T., Holopainen, I., Saloff-Coste, L.: Harnack inequality and hyperbolicity for subelliptic $p$-Laplacians with applications to Picard type theorems. Geom. Funct. Anal. 11(6), 1139–1191 (2001). doi:10.1007/s00039-001-8227-3
[4] Grigor’yan, A.: Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds. Bull. Amer. Math. Soc. (N.S.) 36(2), 135–249 (1999). doi:10.1090/S0273-0979-99-00776-4
[5] O’Neill, B.: Semi-Riemannian Geometry With Applications to Relativity. Academic press, London (1983)
[6] Troyanov, M.: Parabolicity of manifolds. Sib. Adv. Math. 9(4), 125–150 (1999)
[7] Troyanov, M.: Solving the p-Laplacian on manifolds. Proc. Am. Math. Soc. 128(2), 541–545 (2000). http://www.jstor.org/stable/119920

Maria Andrade and Pietro da Silva
Departamento de Matemática
Cidade Universitária Prof. José Alóisio de Campos
Av. Marechal Rondon, s/n Jardim Rosa Elze
São Cristóvão
SE CEP 49100-000
Brazil
e-mail: pietro@mat.ufs.br

Maria Andrade
e-mail: maria@mat.ufs.br

Received: May 10, 2016.
Accepted: July 18, 2017.