A PROOF OF BAUM-CONNES CONJECTURE OF REAL SEMISIMPLE LIE GROUPS WITH COEFFICIENT ON FLAG VARIETIES

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Abstract. We consider the equivariant K-theory of the real semisimple Lie group which acts on the (complex) flag variety of its complexification group. We construct an assemble map in the framework of KK-theory. Then we prove that it is an isomorphism. The prove relies on a careful study of the orbits of the real group action on the flag variety and then piecing together the orbits. This result can be considered as a special case of the Baum-Connes conjecture with coefficient.

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1. Introduction

Let G be a locally compact topological group and A is a C*-algebra equipped with a continuous action of G by C*-algebra automorphisms. Following [3] section 4, we define the equivariant K-theory of A to be the K-theory of the reduced crossed product algebra:

\[ K^*_G(A) := K^*(C^*_r(G, A)). \]

The equivariant K-theory defined in this way has a useful connection to Baum-Connes conjecture and representation theory. In particular, \( K^*_G(pt) = K^*(C^*_r(G)) \). It is well-known that \( C^*_r(G) \) reflects the tempered unitary dual when G is a reductive Lie group, see [3], 4.1.

Be aware that this is not the same as Kasparov’s definition [10].

When G is compact, this definition coincides with the usual equivariant K-theory, see [9].

Back to general G. For any A and B We can define the equivariant KK-theory \( KK^G(A, B) \) as in [10] 2.4. Now we can state the Baum-Connes conjecture. Let U be its maximal compact subgroup and \( S := G/U \) be the quotient space. We have the assemble map [8]

\[ \mu_{red} : KK^G(S, pt) \to K_G(pt). \]
The Baum-Connes conjecture claims that the assemble map $\mu_{\text{red}}$ is an isomorphism. In 2003, J. Chabert, S. Echterhoff, R. Nest [6] proved this conjecture for almost connected and for linear $p$-adic group $G$.

Moreover, we have the Baum-Connes conjecture conjecture with coefficient in $A$, which claims that

$$\mu_{\text{red},A} : \text{KK}^G(S,A) \to K_G(A)$$

is an isomorphism. This is still an open problem for general $G$ and $A$ and, in fact there are counter examples for some certain $G$ and $A$, see [7].

When $G$ is a real semisimple Lie group, we can also consider the Baum-Connes conjecture with coefficient on flag varieties: Let $G_C$ be the complexification of $G$, we have the flag variety $B$ of $G_C$. The group $G_C$ (hence $G$ and $U$) acts on $B$, so we also have the assemble map

$$\mu_{\text{red},B} : \text{KK}^G(S,B) \to K_G(B).$$  \hspace{1cm} (1)

The main result of this paper the following theorem:

**Theorem 1.1.** For any real semisimple Lie group $G$, the assemble map

$$\mu_{\text{red},B} : \text{KK}^G(S,B) \to K_G(B)$$

is an isomorphism.

The proof of Theorem 1.1 in this paper relies on a careful study of the orbits of the real group action on the flag variety: We first prove the isomorphism on one single orbit of the $G$-action and then piecing together the orbits. The proof does not require the hard techniques in KK-theory and representation theory therefore it can be treat as an elementary proof.

This paper is organized as follows: In Section 2 and 3 we construct the assemble map. In Section 4 we study the assemble map on one of the $G$-orbits of the flag variety. In Section 5 we study the $G$-orbits on $B$ and in Section 6 we prove the Theorem 1.1. In Section 7 we give an example to illustrate the idea of this paper.

This work is inspired by the study of equivariant K-theory in [3] the Matsuki correspondence in [12]. Hopefully it will be useful in the representation theory of real semisimple Lie groups, e.g. in the constructions of discrete series, see [16].

**Acknowledgement.** The author would like to thank his advisor Jonathan Block for introducing me to this topic, for his encouragement and helpful discussions. I am also very grateful to Nigel Higson for his help on representation theory and his introduction to Baum-Connes conjecture. I would like to thank Yuhao Huang, Eric Korman and Shizhuo Zhang for helpful comments about this work.

## 2. Notations and First Constructions

We will use the following notations in this paper

1. Let $G$ be a connected real semisimple Lie group, $U$ be the identity component of a maximal compact subgroup of $G$. In the sequel we fix such a $U$ and call it the maximal compact subgroup of $G$.
2. We denote the space $G/U$ by $S$.
3. Let $G_C$ be the complexification of $G$, $B_C$ be the Borel subgroup of $G_C$ and $B$ be the flag variety. We know that $B \cong G_C/B_C$.
4. Let $T$ be any space with continuous $G$-action. Let $C_0(T)$ be the space of continuous functions on $T$ which vanishes at infinity. If $T$ is compact, then $C_0(T) = C(T)$ is the space of all continuous functions on $X$. We define

$$K_G^*(T) := K^*(C^*_r(G, C_0(T))).$$  \hspace{1cm} (2)
Remark 2. As pointed out in remark 1, we do not require the following isomorphism

Theorem 3.2. The Dirac-Dual Dirac Method and the Assemble Map

In this section we will construct the assemble map

$$\mu_{\text{red},T} : KK^G(S, T) \to K_G(T)$$

for any $G$-space $T$. We work in the framework of Kasparov as in [10].

3.1. Poincare Duality in KK-theory. We have the Poincare duality isomorphism in KK-theory.

**Theorem 3.1.** ([10] Theorem 4.10, see also [3] Section 4.3) For a $G$-manifold $X$, let $C_\tau(X)$ denote the algebra of continuous sections of the Clifford bundle over $X$ vanishing at infinity. Then we have the following isomorphism

$$KK^G(X, T) \cong K_G(C_0(T) \otimes C_\tau(X)).$$

Let $X = S$. Under the Poincare duality, to get the assemble map, it is sufficient to construct a map

$$K_G(C_0(T) \otimes C_\tau(S)) \to K_G(T).$$

3.2. The Dirac Element. For $G$, $X$ and $C_\tau(X)$ as in Theorem 3.1, Kasparov defines the Dirac element [10] 4.2:

$$d_{G,X} \in KK^G(C_\tau(X), \mathbb{C})$$

**Remark 1.** We do not require that $X$ is spin in the definition of $d_{G,X}$.

Now we want to find the relation between equivariant KK-theory and the K-theory of crossed-product algebras.

First remember we have the map

$$\sigma_T : KK^G(A, B) \to KK^G(A \otimes C_0(T), B \otimes C_0(T)).$$

Apply this to $d_S \in KK^G(C_\tau(S), \mathbb{C})$ we get

$$\sigma_T(d_S) \in KK^G(C_\tau(S) \otimes C_0(T), C_0(T)).$$

Next, we know that Kasparov gives the following definition-theorem (Theorem 3.11 in [10])

**Theorem 3.2.** There is a natural homomorphism

$$j^G_\tau : KK^G(A, B) \to KK(C_\tau^*(G, A), C_\tau^*(G, B))$$

which is compatible with the Kasparov product. Moreover, for $1_A \in KK^G(A, A)$

$$j^G_\tau(1_A) = 1_{C_\tau^*(G, A)} \in KK(C_\tau^*(G, A), C_\tau^*(G, A)).$$

Apply the map $j^G_\tau$ to $\sigma_T(d_S)$ which is in $KK^G(C_\tau(S) \otimes C_0(T), C_0(T))$ we get

$$j^G_\tau(\sigma_T(d_S)) \in KK(C_\tau^*(G, C_\tau(S) \otimes C_0(T)), C_\tau^*(G, C_0(T))).$$

We denote $j^G_\tau(\sigma_T(d_S))$ by $D_{G,S}$ or simply by $D$.

**Definition 3.1** (The assemble map). Let $S = G/U$, for any $T$, the Poincare duality and the Kasparov product in KK-theory gives us the desired map

$$\cdot \otimes D : KK^G(S, T) \cong K_G(C_0(T) \otimes C_\tau(S)) \to K_G(T).$$

**Remark 2.** As pointed out in remark 7, we do not require $S$ to be spin to define the assemble map.
3.3. The Spin Case. Let us look at the spin case and get some intuition.

When $S$ is spin and of even dimension, it is well known that $C^*_\tau(S)$ is strongly Morita equivalent to $C^*_0(S)$. Hence the Poincare duality gives us

$$\text{KK}^G(S, T) \cong K_G(T \times S). \tag{7}$$

In this case, the Dirac element $d_{G,S}$ is exactly the index map of the Dirac operator ([1]) and this justified the name "Dirac element". Therefore the assemble map is given by the index map

$$D : K_G(T \times S) \to K_G(T) \tag{8}$$

We can look at $K_G(T \times S)$ from another viewpoint. Remember that $S = G/U$. In fact we have the following general result

**Lemma 3.3.** Let $T$ be a $G$-space in the above setting. Then $G \times_H T$ is $G$-isomorphic to $G/H \times T$, where $G$ acts on $G/H \times T$ by the diagonal action. Hence

$$K^*_G(G/H \times T) \cong K^*_G(G \times_H T).$$

**Proof:** See [3], Section 2.5. In fact, both sides are quotientspaces of $G \times T$ by right actions of $H$. The point is that the actions are different.

For $G \times_H T$

$$(g, t) \cdot h := (gh, h^{-1} t).$$

For $G/H \times T$

$$(g, t) \circ h := (gh, t).$$

Then we see that the following map

$$G \times_H T \to G/H \times T$$

$$(g, t) \mapsto (g, gt)$$

intertwines the action $\cdot$ and $\circ$. Moreover it is equivariant under the left $G$-action. $\square$

The following isomorphism is very natural, see [14]

**Lemma 3.4** (The induction map). For any group $G$, $H \subset G$ a closed subgroup, and an $H$-space $T$, there is an induction map

$$K^*_H(T) \to K^*_G(G \times_H T)$$

which is an natural isomorphism.

Here the $G$-action on $G \times_H T$ is the left multiplication on the first component.

**Proof:** Just notice that $C^*_\tau(H, C^*_0(T))$ and $C^*_\tau(G, C^*_0(G \times_H T))$ are Strongly Morita equivalent. $\square$

**Remark 3.** The sprit of Proposition 3.3 and 3.4 will appear later in Lemma 4.2

Now let $H$ be $U$, the maximal compact subgroup. According to Proposition 3.3 and 3.4 the assemble map in Definition 3.1 has the following form

$$D : K_U(T) \to K_G(T). \tag{9}$$

The Connes-Kasparov conjecture claims that the above map is an isomorphism.

**Remark 4.** In the statement of the Connes-Kasparov conjecture we do not require $G/U$ to be spin, see [13].
3.4. The Dual Dirac Element. We are looking for an inverse element of $d_{G,X}$. For this purpose Kasparov introduce the concept of $G$-special manifold in $[10]$, 5.1. For $G$-special manifold $X$, there exists an element, called the dual Dirac element

$$\eta_{G,X} \in KK^G(\mathbb{C}, C_\tau(X))$$

such that the Kasparov product

$$KK^G(C_\tau(X), \mathbb{C}) \times KK^G(\mathbb{C}, C_\tau(X)) \to KK^G(C_\tau(X), C_\tau(X))$$

gives

$$d_{G,X} \otimes \mathbb{C} \eta_{G,X} = 1_{C_\tau(X)}.$$ 

When the group $G$ is obvious, we will denote them by $d_X$ and $\eta_X$.

Kasparov also showed that for the maximal compact subgroup $U$ of $G$, the homogenous space $G/U$ is a $G$-special manifold.

Remark 5. Although $d_{G,X} \otimes \mathbb{C} \eta_{G,X} = 1_{C_\tau(X)}$ for special manifold, the Kasparov product in the other way

$$\eta_X \otimes_{C_\tau(X)} d_X$$

need not to be 1.

We denote $\eta_X \otimes_{C_\tau(X)} d_X$ by $\gamma_X$.

Remark 6. If $\gamma_X = 1$, then $d_X$ and $\eta_X$ are invertible elements under the Kasparov product.

3.5. When is the Dirac Element Invertible? Kasparov proved that $\gamma_X = 1$ in some special cases, which is sufficient for our purpose. To state the result in full generality we need to introduce the concept of restriction homomorphism

Let $f : G_1 \to G_2$ be a homomorphism between groups, the restriction homomorphism

$$r^{G_2,G_1} : KK^{G_1}(A, B) \to KK^{G_2}(A, B)$$

Proposition 3.5. If $X$ is a $G_2$ manifold and $f : G_1 \to G_2$ as above. Then under the map $r^{G_2,G_1}$ we have

$$r^{G_2,G_1}(d_{G_1,X}) = d_{G_2,X}$$

$$r^{G_2,G_1}(\eta_{G_1,X}) = \eta_{G_2,X}$$

$$r^{G_2,G_1}(\gamma_{G_1,X}) = \gamma_{G_2,X}.$$ 

If the manifold $X$ is $G/U$, where $U$ is the maximal compact subgroup of $G$, we denote $d_{G/U}$ by $d_{(G)}$, $\eta_{G/U}$ by $\eta_{(G)}$ and $\gamma_{G/U}$ by $\gamma_{(G)}$.

Theorem 3.6 ($[10]$, 5.9). Let $f : G_1 \to G_2$ be a homomorphism between almost connected groups with the kernel $\ker f$ amenable and the image closed. Then the restriction homomorphism gives us

$$r^{G_2,G_1}((\gamma_{(G_2)})) = (\gamma_{(G_1)}).$$

Corollary 3.7. For any $G$, let $H < G$ be a closed subgroup. Then we have

$$r^{G,H}((\gamma_{(G)})) = (\gamma_{(H)}).$$

Corollary 3.8. $\gamma_{(G)} = 1$ for every amenable almost connected group $G$.

Proof: In Theorem 3.6 let $G_1 = G$ and $G_2$ be the trivial group, we know $\gamma_{(G_2)} = 1$ therefore $\gamma_{(G)} = 1$. □
Remark 7. Remember Remark 6, we know that if $G$ is an amenable almost connected group, then $d_{G/U} = d_G$ and $\eta_{G/U} = \eta_G$ are invertible elements in the $\text{KK}$-groups.

Now we can immediately get a isomorphic result in the almost connected amenable case. The following result is implicitly given in [10],5. 10.

**Theorem 3.9.** If $P$ is an almost connected amenable group, $L$ is the maximal compact subgroup of $P$, $T$ is an $P$-space, then the assemble map

$$\mu_{P,T}: \text{KK}^P(P/L, T) \to K^P(T)$$

is an isomorphism.

**Proof:** By Theorem 3.8 we know that $\gamma(P) = 1$ hence $d_P$ is invertible. Since $D$ in the definition of the assemble map (Definition 3.1) is obtained from $d_P$, and remember Theorem 3.2 invertible elements go to invertible elements. So $\mu_{P,T}$ is an isomorphism. $\square$

4. The Assemble Map on a Single $G$-Orbit of the Flag Variety

According to [12], there are finitely many $G$-orbits on $B$. Let us denote $\alpha^+$ to be one of them. Let $H$ be the isotropy group of $G$ at a critical point (see [12] for the definition of critical points) $x \in \alpha^+$.

**Remark 8.** This notation will be justified in Section 5.

We want to prove

**Proposition 4.1.** The assemble map

$$\mu_{\text{red}, \alpha^+}: \text{KK}^G(S, \alpha^+) \to K^G(\alpha^+).$$

is an isomorphism.

**Remark 9.** Proposition 4.1 is the building block of the main theorem of this paper-Theorem 1.1. We will piece together the blocks in Section 5.

The proof of Proposition 4.1 consists of several steps. First we prove the lemma

**Lemma 4.2** (Interchange subgroups). There is an isomorphism:

$$\text{KK}^G(S, \alpha^+) \cong \text{KK}^H(S, pt).$$

**Proof:** First by Poincare duality

$$\text{KK}^G(S, \alpha^+) \cong K^G(C_0(\alpha^+) \otimes C_r(S)).$$

Then notice that $\alpha^+$ can by identified with $G/H$. By a strong Morita equivalence argument similar to Lemma 3.4 we have

$$K^G(C_0(\alpha^+) \otimes C_r(S)) \cong K^H(C_r(S)).$$

Finally by Poincare duality again we have

$$K^H(C_r(S)) \cong \text{KK}^H(S, pt).$$

We get our result. $\square$

Now we are ready to obtain the following result

**Proposition 4.3.** We have the following commuting diagram:

$$\begin{array}{ccc}
\text{KK}^G(S, \alpha^+) & \sim & \text{KK}^H(S, pt) \\
\mu_{\text{red}, \alpha^+} & & \mu_{\text{red}, \text{pt}} \\
K^G(\alpha^+) & \sim & K^H(\text{pt})
\end{array}$$

where the vertical maps are the assemble maps and the horizontal isomorphisms are given in Lemma 3.4 and Proposition 4.3.
Proof: To prove the proposition we need to investigate the maps. First we look at the right vertical map. At the beginning we have the Dirac element \[ d_{G,S} \in KK^G(C_\tau(S), \mathbb{C}) \]
apply the restriction homomorphism \( r^{G,H} \) we get
\[ r^{G,H}(d_{G,S}) \in KK^H(C_\tau(S), \mathbb{C}). \]
Nevertheless we have the Dirac element \[ d_{H,S} \in KK^H(C_\tau(S), \mathbb{C}). \]
In fact from the definition it is easy to see that they are equal:
\[ r^{G,H}(d_{G,S}) = d_{H,S}. \]
Then we apply the map \( j^H_H \):
\[ KK^H(C_\tau(S), \mathbb{C}) \rightarrow KK^H(C_\tau^*(H, C_\tau(S)), C_\tau^*(H)). \]
we get
\[ j^H_H(d_{H,S}) \in KK(C_\tau^*(H, C_\tau(S)), C_\tau^*(H)) \]
and we denote it by \( D_H \). Right multiplication of \( D_H \) gives the vertical map on the right in the diagram
\[ KK^H(S, \text{pt}) \mu_{\text{red,pt}} \rightarrow K_H(\text{pt}). \]
On the other hand we have the map
\[ \sigma_{\alpha^+} : KK^G(C_\tau(S), \mathbb{C}) \rightarrow KK^G(C_\tau(S) \otimes C_0(\alpha^+), C_0(\alpha^+)) \]
so we get
\[ \sigma_{\alpha^+}(d_{G,S}) \in KK^G(C_\tau(S) \otimes C_0(\alpha^+), C_0(\alpha^+)) \]
then via \( j^G_G \) we get
\[ j^G_G(\sigma_{\alpha^+}(d_{G,S})) \in KK(C_\tau^*(G, C_\tau(S) \otimes C_0(\alpha^+)), C_\tau^*(G, C_0(\alpha^+))) \]
which we denote by \( D_{G,\alpha^+} \). Right multiplication of \( D_{G,\alpha^+} \) gives the other vertical map
\[ KK^G(S, \alpha^+) \mu_{\text{red,\alpha^+}} \rightarrow K_G(\alpha^+). \]
The horizontal maps in the diagram are given by Strongly Morita equivalence. Now, under the Strongly Morita equivalence, \( D_{G,\alpha^+} \cong D_H \), so the diagram commutes. \( \square \)

According to Proposition 4.3 in order to prove Proposition 4.1 i.e.
\[ \mu_{\text{red,\alpha^+}} : KK^G(S, \alpha^+) \rightarrow K_G(\alpha^+) \]
is an isomorphism, it is sufficient to prove the following proposition

**Proposition 4.4.**

\[ \mu_{\text{red,pt}} : KK^H(S, \text{pt}) \rightarrow K_H(\text{pt}) \]

is an isomorphism.

Proof: It is sufficient to prove
\[ D_H = j^H_H(d_{H,S}) \in KK(C_\tau^*(H, C_\tau(S)), C_\tau^*(H)) \]
is invertible. In fact, we can prove that \( d_{H,S} \in KK^H(C_\tau(S), \mathbb{C}) \) is invertible. This follows from the fact that \( H \) is almost connected amenable together with some formal arguments.

As in the construction in Section 3, we have the dual Dirac element \( \eta_{H,S} \in KK^H(C, C_\tau(S)) \)
and
\[ d_{H,S} \otimes \eta_{H,S} = 1 \in \text{KK}^H(C_\tau(S), C_\tau(S)), \]
\[ \eta_{H,S} \otimes C_\tau(H)d_{H,S} = \gamma_{H,S} \in \text{KK}^H(\mathbb{C}, \mathbb{C}). \]

We want to prove \( \gamma_{H,S} = 1 \). Remember that \( H \) is an almost connected amenable group and we have Theorem 3.8 which claims that
\[ \gamma(H) = 1. \]
where by definition \( \gamma(H) = \gamma_{H,H/U \cap H} \). We need to prove \( \gamma(H) \) is equal to \( \gamma_{H,S} \).

Notice that \( \gamma_{H,S} \) is nothing but the image of the element \( \gamma_{G,S} \) under the restriction homomorphism
\[ r^{G,H} : \text{KK}^G(\mathbb{C}, \mathbb{C}) \rightarrow \text{KK}^H(\mathbb{C}, \mathbb{C}). \]
(19)
i.e.
\[ r^{G,H}(\gamma_{G,S}) = \gamma_{H,S}. \]
(20)

Other hand, in the notation of Theorem 3.8, \( \gamma_{G,H} \) is nothing but \( \gamma(G) \), and again by Theorem 3.8 we have
\[ r^{G,H}(\gamma(G)) = \gamma(H) \]
Compare the last two identity we get
\[ \gamma_{H,S} = \gamma(H) \]
(21)
so
\[ \gamma_{H,S} = 1. \]
(22)

Now we proved that \( d_{H,S} \) hence \( D_H \), is invertible, as a result
\[ \mu_{\text{red}, \text{pt}} : \text{KK}^H(\mathbb{C}, \mathbb{C}) \rightarrow \text{K}_H(\mathbb{C}, \mathbb{C}) \]
is an isomorphism. \( \Box \)

Proof of Proposition 4.1: Combine Proposition 4.3 and 4.4 we know get
\[ \mu_{\text{red}, \alpha^+} : \text{KK}^G(S, \alpha^+) \rightarrow \text{K}_G(\alpha^+) \]
is an isomorphism, which finishes the prove of Proposition 4.1. \( \Box \)

5. The G-orbits on the Flag Variety

We have proved the isomorphism on one orbit of \( G \). Now we need to study the \( G \)-orbits on \( B \) and in the next section we will “piece together orbits”.

The result on the \( G \)-orbits in [12] is important to our purpose, so we summarize their result here.

**Theorem 5.1** ([12] 1.2, 3.8). On the flag variety \( B \) there exists a real value function \( f \) such that

1. \( f \) is a Morse-Bott function on \( B \).
2. \( f \) is \( U \) invariant, hence the gradient flow \( \phi : \mathbb{R} \times B \rightarrow B \) is also \( U \) invariant.
3. The critical point set \( \mathcal{C} \) consists of finitely many \( U \)-orbits \( \alpha \). The flow preserves the orbits of \( G \).
4. The limits \( \lim_{t \rightarrow \pm \infty} \phi_t(x) \) exist for any \( x \in B \). For \( \alpha \) a critical \( U \)-orbit, the stable set
   \[ \alpha^+ = (\pi^+)^{-1}(\alpha) \]
is an \( G \)-orbit, and the unstable set
   \[ \alpha^- = (\pi^-)^{-1}(\alpha) \]
is an \( U_\mathbb{C} \)-orbit, where \( U_\mathbb{C} \) is the complexification of \( U \) in \( G_\mathbb{C} \).
5. \( \alpha^+ \cap \alpha^- = \alpha \).

\( \Box \)

We will use the following corollary in [12]:
Corollary 5.2 ([12] 1.4). Let $\alpha$ and $\beta$ be two critical U-orbits. Then the closure $\alpha^+ \supset \beta^+$ if and only if $\alpha^+ \cap \beta^- \neq \emptyset$.

\[\square\]

From this we can get

Corollary 5.3. Let $\alpha$ and $\beta$ be two different critical U-orbits, i.e. $\alpha \neq \beta$. Then $\alpha^+ \supset \beta^+$ implies that the Morse-Bott function $f$ has values

\[f(\alpha) > f(\beta)\]

Proof: By the previous corollary,

\[\alpha^+ \cap \beta^- \neq \emptyset.\]

so there exists an $x \in \alpha^+ \cap \beta^-$. Since $\lim_{t \to +\infty} \phi_t(x) \in \alpha$, we have

\[f(\alpha) \geq f(x),\]

similarly

\[f(x) \geq f(\beta).\]

On the other hand since $\alpha \neq \beta$ we get $\alpha \not\subset \beta^-$ and $\beta \not\subset \alpha^+$. So

\[x \not\in \alpha, x \not\in \beta\]

so

\[f(x) \neq f(\alpha), f(x) \neq f(\beta).\]

So we have

\[f(\alpha) > f(\beta).\]

\[\square\]

We can now give a partial order on the set of $G$-orbits of $B$.

Definition 5.1. If $f(\alpha) > f(\beta)$, we say that $\alpha^+ > \beta^+$.

If $f(\alpha) = f(\beta)$, we choose an arbitrary partial order on them.

Now let us list all $G$-orbits in $B$ in ascending order, keep in mind that there are finitely many of them:

\[\alpha_1^+ < \alpha_2^+ < \ldots < \alpha_k^+.\] (23)

From the definition we can easily get

Corollary 5.4. For any $G$-orbits $\alpha_i^+$, the union

\[Z_i := \bigcup_{\alpha_j^+ \subseteq \alpha_i^+} \alpha_j^+\]

is a closed subset of $B$. Notice that $\alpha_i^+ \subset Z_i$.

Proof: It is sufficient to prove that $Z_i$ contains all its limit points, which is a direct corollary of Definition 5.1 and Corollary 5.3.

\[\square\]

Remark 10. Corollary 5.2, Definition 5.1 and Corollary 5.4 are not given in [12].
6. The Baum-Connes Conjecture on Flag Varieties

With the construction in the last section, we can piece together the orbits

**Proposition 6.1.** For $1 \leq i \leq k - 1$ we have a short exact sequence of crossed product algebras:

$$0 \to C^*_r(G, C_0(\alpha_{i+1}^+)) \to C^*_r(G, C(Z_{i+1})) \to C^*_r(G, C(Z_i)) \to 0.$$

Proof: From the construction we also get

$$Z_i \subset Z_{i+1}, \quad \alpha_{i+1}^+ \subset Z_{i+1},$$

and $Z_i$ is closed in $Z_{i+1}$, $\alpha_{i+1}^+$ is open in $Z_{i+1}$.

Since $B$ is a compact manifold, we get that $Z_i$ and $Z_{i+1}$ are both compact.

The inclusion gives a short exact sequence:

$$0 \to C_0(\alpha_{i+1}^+) \to C(Z_{i+1}) \to C(Z_i) \to 0. \quad (24)$$

Now we need to go to the reduced crossed-product $C^*$-algebras. The following technique result will help us:

**Theorem 6.2** ([11] Theorem 6.8). Let $G$ be a locally compact group and

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of $G$-$C^*$ algebra. Then we have a short exact sequence:

$$0 \to C^*_r(G, A) \to C^*_r(G, B) \to C^*_r(G, C) \to 0.$$ 

Since we have

$$0 \to C_0(\alpha_{i+1}^+) \to C(Z_{i+1}) \to C(Z_i) \to 0.$$

exact, Theorem 6.2 gives the short exact sequence

$$0 \to C^*_r(G, C_0(\alpha_{i+1}^+)) \to C^*_r(G, C(Z_{i+1})) \to C^*_r(G, C(Z_i)) \to 0. \quad (25)$$

This finishes the proof of Proposition 6.1. □

From Proposition 6.1 we have the well-known six-term long exact sequence

$$K^*(C^*_r(G, C_0(\alpha_{i+1}^+))) \to K^*(C^*_r(G, C(Z_{i+1}))) \to K^*(C^*_r(G, C(Z_i)))$$

$$\downarrow$$

$$K^{*+1}(C^*_r(G, C(Z_{i+1})) \to K^{*+1}(C^*_r(G, C(Z_i))) \to K^{*+1}(C^*_r(G, C_0(\alpha_{i+1}^+))).$$

i.e.

$$K^*_G(\alpha_{i+1}^+) \to K^*_G(Z_{i+1}) \to K^*_G(Z_i) \quad (26)$$

Similarly we have

$$K^*_G(C_0(\alpha_{i+1}^+ \otimes C_r(S)) \to K^*_G(C(Z_{i+1}) \otimes C_r(S)) \to K^*_G(C(Z_i) \otimes C_r(S))$$

$$\downarrow$$

$$K^{*+1}_G(C(Z_{i+1}) \otimes C_r(S)) \to K^{*+1}_G(C(Z_i) \otimes C_r(S)) \to K^{*+1}_G(C_0(\alpha_{i+1}^+ \otimes C_r(S)). \quad (27)$$

The fact is that Formulae 26 and 27 together form a commuting diagram.
Proposition 6.3. We have the following commuting diagram:

\[
\begin{array}{cccc}
K^*_G(C_0(\alpha_{i+1}^+) \otimes C_\tau(S)) & \rightarrow & K^*_G(C(Z_{i+1}) \otimes C_\tau(S)) & \rightarrow \\
K^*_G(C_0(\alpha_{i+1}^+) \otimes C_\tau(S)) & \rightarrow & K^*_G(C(Z_i) \otimes C_\tau(S)) & \rightarrow \\
\vdots & \vdots & \vdots & \vdots \\
K^*_G(C_0(\alpha_{i+1}^+) \otimes C_\tau(S)) & \rightarrow & K^*_G(C(Z_{i+1}) \otimes C_\tau(S)) & \rightarrow \\
K^*_G(C_0(\alpha_{i+1}^+) \otimes C_\tau(S)) & \rightarrow & K^*_G(C(Z_i) \otimes C_\tau(S)) & \rightarrow
\end{array}
\]

where the top and bottom are the six-term exact sequences and the vertical arrows are assemble maps \(\mu\).

Proof: The diagram commutes because all the vertical maps \(\mu\) come from the same element \(d_{G,S} \in KK^G(C_\tau(S), C)\) as in Section 4.

After all this work we can prove Theorem 1.1.

Proof of Theorem 1.1 We use induction on the \(Z_i\)'s. First, for \(Z_1 = \alpha_1^+\), by Proposition 4.1

\[
K^*_G(C(Z_1) \otimes C_\tau(S)) \xrightarrow{\mu} K^*_G(Z_1)
\]

is an isomorphism.

Assume that for \(Z_i\),

\[
K^*_G(C(Z_i) \otimes C_\tau(S)) \xrightarrow{\mu} K^*_G(Z_i)
\]

is an isomorphism.

By Proposition 4.1 the vertical maps on the left face of Commuting Diagram 28 are isomorphisms. Moreover by induction we can get that the vertical maps on the right face are isomorphism too, hence by a 5-lemma-argument we get the middle vertical maps are also isomorphisms, i.e. for \(Z_{i+1}\),

\[
K^*_G(C(Z_{i+1}) \otimes C_\tau(S)) \xrightarrow{\mu} K^*_G(Z_{i+1})
\]

is an isomorphism.

There are finitely many orbits and let \(\alpha_k^+\) be the largest orbit, it follows that \(Z_k = \bigcup_{\text{all orbits}} \alpha_i^+ = B\)

\[
Z_k = \bigcup_{\text{all orbits}} \alpha_i^+ = B
\]

hence

\[
\mu_{\text{red,B}} : KK^G(S, B) \rightarrow K^*_G(B)
\]

is an isomorphism. we finished the proof Theorem 1.1.

7. An Example

We look at the case when \(G = \text{SL}(2, \mathbb{R})\) and \(G_\mathbb{C} = \text{SL}(2, \mathbb{C})\). Hence

\[
B_\mathbb{C} = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{C}^*, B \in \mathbb{C} \right\}
\]

and

\[
B = G_\mathbb{C} / B_\mathbb{C} = \mathbb{C}P^1 \cong S^2.
\]
It is well-known that the \( G_C \) (hence) \( G \) acts on \( B = \mathbb{CP}^1 \) by fractional linear transform, i.e. using projective coordinate
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} au + bv \\ cu + dv \end{pmatrix}.
\]
Or let \( z = u/v \), then
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}. \tag{34}
\]
From Formula \ref{34} we can see that the action of \( G \) on \( B \) is not transitive. In fact, it has three orbits
\[
\begin{align*}
\alpha_1^+ &= \mathbb{R} \cup \infty \cong S^1 \text{ the equator}, \\
\alpha_2^+ &= \{x + iy | y > 0\} \cong \mathbb{C} \text{ the upper hemisphere}, \\
\alpha_2^- &= \{x + iy | y < 0\} \cong \mathbb{C} \text{ the lower hemisphere}.
\end{align*}
\]
\( \alpha_1^+ \) is a closed orbit with dimension 1; \( \alpha_2^+ \) and \( \alpha_2^- \) are open orbits with dimension 2.
Let’s look at \( \alpha_1^+ \) first. Take the point \( 1 \in \alpha_1^+ \). The isotropy group at 1 is the upper triangular group \( B \) in \( \text{SL}(2, \mathbb{R}) \). So
\[
K^*_G(\alpha_1^+) = K^*_B(pt).
\]
\( B \) is solvable hence amenable and \( \mathbb{Z}/2\mathbb{Z} \) is the maximal compact group of \( B \). By Theorem \ref{3.9}
\[
K^0_B(pt) = R(\mathbb{Z}/2\mathbb{Z})
\]
is the representation ring of the group with two elements and
\[
K^1_B(pt) = 0.
\]
So
\[
K^0_G(\alpha_1^+) = R(\mathbb{Z}/2\mathbb{Z})
\]
and
\[
K^1_G(\alpha_1^+) = 0.
\]
For \( \alpha_2^+ \) and \( \alpha_3^+ \), the isotropy groups are both
\[
T = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right\}
\]
hence by the similar reason to \( \alpha_1^+ \) we have
\[
K^0_G(\alpha_2^+) = K^0_G(\alpha_3^+) = K^0_T(pt) = R(T)
\]
is the representation ring of \( T \) and
\[
K^1_G(\alpha_2^+) = K^1_G(\alpha_3^+) = K^1_T(pt) = 0.
\]
Now \( \alpha_2^+ \cup \alpha_3^+ \) is open in \( B \) so as in the last section we have the short exact sequence
\[
0 \longrightarrow C_0(\alpha_2^+ \cup \alpha_3^+) \longrightarrow C(B) \longrightarrow C(\alpha_1^+) \longrightarrow 0 \tag{35}
\]
and further
\[
0 \longrightarrow C^*_r(G, \alpha_2^+ \cup \alpha_3^+) \longrightarrow C^*_r(G, B) \longrightarrow C^*_r(G, \alpha_1^+) \longrightarrow 0.
\]
i.e.
\[
0 \longrightarrow C^*_r(G, \alpha_2^+) \oplus C^*_r(G, \alpha_3^+) \longrightarrow C^*_r(G, B) \longrightarrow C^*_r(G, \alpha_1^+) \longrightarrow 0. \tag{36}
\]
We get the six-term exact sequence
\[
\begin{array}{ccccccc}
K^0_G(\alpha_2^+) & \oplus & K^0_G(\alpha_3^+) & \longrightarrow & K^0_G(B) & \longrightarrow & K^0_G(\alpha_1^+) \\
\uparrow & & & & \downarrow & & \\
K^1_G(\alpha_1^+) & \leftarrow & K^1_G(B) & \leftarrow & K^1_G(\alpha_2^+) & \oplus & K^1_G(\alpha_3^+). \tag{37}
\end{array}
\]
Combine with the previous calculation we get
\[ 0 \longrightarrow R(T) \oplus R(T) \longrightarrow K^0_G(B) \longrightarrow R(\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0 \] 
(38)
and \( K^1_G(B) = 0 \).

In conclusion we have
\[
K^0_G(B) \approx R(T) \oplus R(T) \oplus R(\mathbb{Z}/2\mathbb{Z}),
\]
(39)
\[
K^1_G(B) = 0.
\]

Then we look at \( K^0_U(B) \) and \( K^1_U(B) \). We know that for \( G = \text{SL}(2, \mathbb{R}) \) the maximal compact subgroup \( U = T \). By Bott periodicity we have
\[
K^0_U(\alpha_2^+) = K^0_U(\alpha_3^+) \cong K^0_U(C) \cong K^0_U(pt) = R(U) = R(T)
\]
(40)
and
\[
K^1_U(\alpha_2^+) = K^1_U(\alpha_3^+) \cong K^1_U(C) \cong K^1_U(pt) = 0.
\]
(41)

As for \( \alpha_1^+ \), we notice that \( U \) acts on \( \alpha_1^+ \cong S^1 \) by "square", so the isotropy group is \( \mathbb{Z}/2\mathbb{Z} \). Hence
\[
K^0_U(\alpha_1^+) = R(\mathbb{Z}/2\mathbb{Z})
\]
and
\[
K^1_U(\alpha_1^+) = 0.
\]
By the six-term long exact sequence we have
\[
K^0_U(B) \approx R(T) \oplus R(T) \oplus R(\mathbb{Z}/2\mathbb{Z}),
\]
(42)
\[
K^1_U(B) = 0.
\]

Compare (39) and (42) we have
\[
K^*_U(B) \approx K^*_G(B).
\]

On the other hand, for \( G = \text{SL}(2, \mathbb{R}), U = T \) and \( S = G/U \), we have \( S \) is spin and \( \dim S = 2 \), then by Baum-Connes conjecture (in fact, Connes-Kasparov conjecture as in Section 3.3) we know that the above \( \approx \) is an isomorphism, i.e
\[
K^*_U(B) \cong K^*_G(B).
\]
(43)

**Remark 11.** Using Bott periodicity theorem we can obtain precisely the algebra structure of \( K^*_G(B) \) as in [17]. Therefore Baum-Connes conjecture will be a powerful tool to investigate \( K_G(B) \) and to study the representation theory of \( G \).

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