Evolution and Dynamics of a Matter creation model

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In a flat Friedmann-Lemaître-Robertson-Walker (FLRW) geometry, we consider the expansion of the universe powered by the gravitationally induced ‘adiabatic’ matter creation. To demonstrate how matter creation works well with the expanding universe, we have considered a general creation rate and analyzed this rate in the framework of dynamical analysis. The dynamical analysis hints the presence of a non-singular universe (without the big bang singularity) with two successive accelerated phases, one at the very early phase of the universe (i.e. inflation), and the other one describes the current accelerating universe, where this early, late accelerated phases are associated with an unstable fixed point (i.e. repeller) and a stable fixed (attractor) points, respectively. We have described this phenomena by analytic solutions of the Hubble function and the scale factor of the FLRW universe. Using Jacobi Last multiplier method, we have found a Lagrangian for this matter creation rate describing this scenario of the universe. To match with our early physics results, we introduce an equivalent dynamics driven by a single scalar field and discussed the associated observable parameters compared them with the latest PLANCK data sets. Finally, introducing the teleparallel modified gravity, we have established an equivalent gravitational theory in the framework of matter creation.

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1. INTRODUCTION

No doubt, cosmology is one of the biggest and fascinating topics in science. However, at the late 90’s, a dramatic change appeared in its history when it was discovered that the universe is going through a phase of accelerated expansion [1, 2]. After that, several independent observations [3–9] confirmed this accelerating expansion. As a result, comprehending this late-accelerating phase, has become an attracting research field in modern cosmology since the end of 90’s. There are mainly two distinct approaches we use in order to describe this accelerating phase. First of all, if we consider that gravity is correctly described by Einstein’s theory, then there must have some matter component with large negative pressure entitled ‘dark energy’ with equation of state \( w < -1/3 \), in order to start this acceleration. As a result, cosmologists brought back the presence of a non-zero cosmological constant \( \Lambda \) (equation of state: \( w = -1 \)) which fuels this current acceleration. Subsequently, ‘\( \Lambda \)-cold-dark-matter’ (ΛCDM) was proposed to describe the current accelerating phase, and it was found that the model agrees with a large number of astronomical data. However, Λ-cosmology has two fundamental problems: Observations demand that, a very small energy density of Λ is enough to power this accelerating universe whereas the prediction from quantum theory of fields claim that, its energy density should be so large, leading to a discrepancy between them of order 10^{121}. This is known as cosmological constant problem [10]. On the other hand, it is not understandable “why did our Universe begin to accelerate just now \((z \sim 1)\) where both the matter and the cosmological constant evolve differently with the evolution of the universe” – known as the cosmic coincidence problem [11]. As a result, some alternatives to ΛCDM were proposed, such as, quintessence, K-essence, phantom, tachyons and others (for a review of dark energy candidates, see [12, 13]). Also, it has been argued that, modifications in the Einstein gravity can describe the current acceleration (the models are sometimes called as geometrical dark energy) [14–16].

However, besides these two distinct approaches, very recently, another alternative to describe the current accelerating universe has been attracted a special attention. The approach is the gravitationally induced ‘adiabatic’ matter creation, a non-equilibrium thermodynamical process. Long time ago, around 1960-1980, Parker and his collaborators [17, 22] and in Russia Zeldovich and others [23, 24], were investigating on the material content of the universe. Following Schrödinger’s ideas presented in [25], they proposed that, as the universe is expanding, the gravitational field of this expanding universe is acting on the quantum vacuum, which results in a continuous creation of radiation and matter particles are going on, and the produced particles have their
mass, momentum, and the energy. The idea was really fascinating, and even today it is, as we do not know how the universe came into its present position after qualifying its previous stages. On the other hand, while dealing with this matter creation process, there is another point which we need to address. It is a real mystery that the ratio of baryon to entropy in our Universe is approximately $9.2 \times 10^{-11}$ [29], and it still remains an unsolved problem why this baryon to entropy ratio exists in our Universe. However, we have an answer from the Sakharov criteria [30], which states that the baryon asymmetry in our Universe can occur if the thermodynamical processes in our expanding universe are non-equilibrium in nature, that means matter creation can take place. So, it is fine that, we have strong motivation behind the material content of our Universe. Now, the main question is how the particle productions play an effective role with the evolution of the universe. It was Prigogine and his group [31] who thought that, as the Einstein’s field equations are the background equations to understand the evolution of the universe, so there must be some way out in order to calculate the evolution equations. And hence the conservation equation gets modified as

$$N^\mu_{;\mu} = n_{;\mu} u^\mu + \Theta n = n \Gamma \iff N_{;\mu} u^\mu = \Gamma N, \quad (1)$$

where $\Gamma$ stands for the rate of change of the particle number in a physical volume $V$ containing $N$ number of particles, $N^\mu = n u^\mu$ represents particle flow vector; $u^\mu$ is the usual particle four velocity; $n = N/V$, is the particle number density, $\Theta = u_{;\mu} u^\mu$, denotes the fluid expansion. The new quantity $\Gamma$ has a special meaning. It is the rate of the produced particles, and the most interesting thing is that, it is completely unknown to us. But, we have one constrain over $\Gamma$, which comes from the validity of the generalized second law of thermodynamics leading to $\Gamma \geq 0$. However, still, we have an open question about the nature of created particles by this gravitational field. One may ask what kind of particles are created by the gravitational field and what are their physical properties. We can not properly say, but there are some justifications over this puzzle. It has been shown that the kind of particles created by this process are much limited by the local gravity constraints [32, 33], and practically radiation has no effect or impact on the late-time accelerated expansion of the universe, whereas dark matter is one of the dominant sources after the unknown “dark energy” component. In what follows, we may assume that the produced particles by this gravitational field are simply the cold dark matter particles. Following this motivation, it has been argued that the models for different particle creation rates can mimic $\Lambda$CDM cosmology [35-39]. In particular, the constant matter creation rate can explain the big bang singularity, as well as intermediate phases ending at the final de Sitter regime [40]. Further, recently, Nunes and Pavón [41] showed that the matter creation models can explain the phantom behavior of our Universe [42-46] without invoking any phantom fields [47]. Subsequently, the cosmological consequences of the matter creation models realizing this phantom behavior have been investigated [48]. Moreover, particle productions in modified gravity theories have attracted by several authors at recent time, for instance, through a nonminimal curvature-matter coupling in modified theories of gravity, particle productions by the gravitational field have been discussed in [49]. Also, in the context of $f(R)$ gravity, the aspects of particle productions have been investigated in [50].

On the other hand, particle production scenario took a novel attempt in order to explain the early accelerated expansion (known as inflation). In the background of the particle creation process, using the energy-momentum tensor of the created particles and their creation rate [23, 24], inflation as a result of this phenomenon was first investigated in [51]. However, it was appeared that such a model with a small number of non-conformal fields cannot produce a sufficiently low curvature during inflation and a graceful exit from it. Soon after that, a viable inflationary model was proposed in [52], where dissipation and creation of particles occurred just after the end of inflation. However, in the same context, it has been discussed earlier that the particle creation of light nonminimally coupled scalar fields due to the changing geometry of a spacetime could drive the early inflationary phase [53]. Also, quantum particle productions in Einstein-Cartan-Sciama-Kibble theory of gravity could also result in an inflationary scenario [54]. Furthermore, very recently, a connection between early and late accelerated universes by the mechanism of particle productions have been pointed out by Nunes [55].

In the present work, we have considered a generalized matter creation model in order to produce a clear image about the matter creation models as a third alternative for current accelerating universe aiming to realize the early physics and its compatibility with the current astronomical data, as well as, the stability of the matter creation models. Hence, we explicitly wrote down the Friedmann, Raychaudhuri equations in the framework of matter creation. The field equations form an autonomous system of differential equations, where the Friedmann equation constrains the dynamics of the universe and the Raychaudhuri equation essentially describes its evolution. Now, considering the Raychaudhuri equation for the matter creation model, we have found the fixed points of the model which are the functions of the model parameters. As the model parameters are simply real numbers, so we have divided the whole phase space into several sub phase spaces, which opens some new possibilities in order to understand the possible dynamics of the universe with respect to the behavior of the fixed points. The fixed points analysis provides a non-singular model of our Universe with two successive accelerating phases, one at very early evolution of the
universe which is unstable in nature, and the other one is the present accelerating phase with stable in nature. We have presented an analytic description for this said evolution of the universe. Further, we apply the Jacobi Last multiplier method in matter creation which eventually provides an equivalent Lagrangian for this creation mechanism. Moreover, as we are also interested to investigate the early physics scenario extracted from matter creation models, so, we introduced a scalar field dynamics, where we found that, it is possible to find an analytic scalar field solutions mimicking the evolution of the universe. Then we have introduced a modification to the Einstein’s gravitational theory, namely, \( f(T) \), the teleparallel equivalent of General Relativity, where we have established that a perfect fluid in addition to matter creation can lead us to an exact expression for \( f(T) \) which can be considered as an equivalent gravitational theory for this dynamical description.

The above discussions can be seen in a flowchart describing as: Perfect fluid in \( f(T) \) gravity \( \iff \) Matter creation + perfect fluid \( \iff \) Scalar field dynamics. Next, we introduce the cosmology of decaying vacuum energy and its equivalence with gravitationally induced matter creation, which essentially tells us that, there is a one-to-one correspondence between these models. But, we observed that the equivalence not always gives a one-to-one correspondence. The paper is organized as follows.

In section 2 we derived the field equations for matter creation in the flat FLRW space-time. Then introducing a generalized model of matter creation in section 3 we have analyzed its dynamical stability and analytic solutions in the subsection 3.1 and further, we have introduced Jacobi Last multiplier in subsection 3.2 and discussed the cosmological features. Section 4 contains an equivalent field theoretic description for the present model where we have associated its corresponding early physics scenario in subsection 4.1. Furthermore, we have associated a short description on \( f(T) \) gravity in the framework of matter creation in section 4.3. Finally, in section 4 we have summarized our results.

We note that, throughout the text, we have used matter creation and particle creation synonymously.

2. THE FIELD EQUATIONS IN MATTER CREATION

At this stage, it has been verified that, our Universe is perfectly homogeneous and isotropic on the largest scale, and this information gives us a space-time metric known as Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

\[
\text{ds}^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]
\]

(2)

where \( a(t) \) is the scale factor of the universe, the curvature scalar \( k = 0, +1, -1 \), stand for flat, closed and open universes respectively. Furthermore from the cosmological data (Planck collaboration 2014a) the value of the spatial curvature is very close to zero, hence, we set \( k = 0 \).

For the co-moving observer, \( w^\mu = \delta^\mu_0 \), in which \( w^\mu u_\mu = -1 \), and for the line element (2) the fluid expansion becomes, \( \Theta = 3H \), where \( H = \dot{a}/a \) is the Hubble parameter. Hence, the conservation equation (1) becomes

\[
N^\mu_\nu = n_{,\nu} u^\mu + 3Hn = n\Gamma,
\]

(3)

where now the co-moving volume is \( V = a^3 \). Clearly, \( \Gamma > 0 \) indicates the creation of particles while \( \Gamma < 0 \) stands for particle annihilation.

From Gibb’s equation it follows \( \frac{\rho}{n} \) and with the use of equation (1), we have

\[
\frac{\rho V}{n} = \dot{\rho} + 3H \left( 1 - \frac{\Gamma}{3H} \right) (\rho + p),
\]

(5)

where \( T \) indicates the fluid temperature, and “\( s \)” is the specific entropy (i.e. entropy per particle). Now, by assuming that the creation happens under “adiabatic” conditions (see for instance [58, 59]), the specific entropy does not change, i.e. \( \dot{s} = 0 \), and from Eq. (5) one obtains the conservation equation

\[
\dot{\rho} + 3H (\rho + p) = \Gamma (\rho + p).
\]

(6)

Then from conservation equation (6) and taking the derivative of the Friedmann equation, which is nothing else as the first Friedmann’s equation

\[
3H^2 = \rho,
\]

(7)

one gets the Raychaudhuri equation

\[
\dot{H} = -\frac{1}{2} \left( 1 - \frac{\Gamma}{3H} \right) (\rho + p),
\]

(8)

where for a perfect fluid with a lineal Equation of State (EoS) of the form \( p = (\gamma - 1)\rho \), that is the case we will consider throughout the paper, the latter becomes

\[
\dot{H} = -\frac{3\gamma}{2} H^2 \left( 1 - \frac{\Gamma}{3H} \right).
\]

(9)

Thus, the cosmological scenario can be described after we specify the particle creation rate \( \Gamma \), and the equation
of state $\gamma$. We see that under the condition $\Gamma \ll 3H$, we have the standard Raychaudhuri equation without any particle creation process. Further, if one specifies the equation of state $\gamma$ to be constant, the standard evolution equation $a \propto t^{2/3\gamma}$ is retrieved. So, the mechanism of particle creation deviates the standard physical laws, but can be recovered under the condition $\frac{\dot{H}}{3H} \ll 1$. However, the deceleration parameter, $q$, a measurement of state of acceleration/deceleration of the universe, is defined as

$$q \equiv - \left(1 + \frac{\dot{H}}{H^2}\right) = -1 + \frac{3\gamma}{2} \left(1 - \frac{\Gamma}{3H}\right).$$

Further, the effective equation of state (EoS) parameter is given by

$$\omega_{e,ff} = -1 - \frac{2\dot{H}}{3H^2} = -1 + \gamma \left(1 - \frac{\Gamma}{3H}\right),$$

which represents quintessence era for $\Gamma < 3H$, and phantom era for $\Gamma > 3H$. Also, $\Gamma = 3H$ indicates the cosmological constant, i.e.

Perfect fluid + ($\Gamma = 3H$) $\equiv$ cosmological constant.

An equivalent way to see the derivation of the field equations (11–13) is to consider the energy-momentum tensor in the Einstein field equations as a total energy momentum tensor $T_{\mu\nu}^{(f)} = T_{\mu\nu}^{(c)} + T_{\mu\nu}^{(g)}$, where $T_{\mu\nu}^{(c)}$ is the energy-momentum tensor for the fluid with equation of state parameter, $p = (\gamma - 1) \rho$, i.e.

$$T_{\mu\nu}^{(c)} = (\rho + p) u_{\mu}u_{\nu} + pg_{\mu\nu},$$

and $T_{\mu\nu}^{(g)}$, is the energy-momentum tensor which corresponds to the matter creation term. Hence, $T_{\mu\nu}^{(g)}$ has the following form

$$T_{\mu\nu}^{(g)} = P_c (g_{\mu\nu} + u_{\mu}u_{\nu}),$$

the latter energy-momentum tensor provides us with the matter creation pressure $\rho_c$. Therefore, the Einstein field equations are

$$G_{\mu\nu} = T_{\mu\nu}^{(f)} = T_{\mu\nu}^{(c)} + T_{\mu\nu}^{(g)}.$$  

Since the two fluids are interacting, the Bianchi identity gives

$$g^{\nu\sigma} \left(T_{\mu\nu}^{(g)} + T_{\mu\nu}^{(c)}\right)_{,\sigma} = 0,$$

or equivalently,\(^1\)

$$\dot{\rho} + 3H(\rho + p + P_c) = 0.$$  

where with the use of Gibb’s equation (5), we find that

$$P_c = -\frac{\Gamma}{3H} (\rho + p),$$  

or,

$$P_c = -\frac{2}{3H} \Gamma \rho.$$  

Since $\rho > 0$, and for $H > 0$, i.e. $a > 0$, from the latter we have that $P_c < 0$, when $\Gamma > 0$, and $P_c > 0$ when $\Gamma < 0$. Furthermore, from (14) we find the following system

$$3H^2 = \rho,$$

$$2\dot{H} + 3H^2 = -p - P_c,$$

where if we substitute (18) and (19) in (20), we derive the Raychaudhuri equation (9).

In the present model, the cosmic history is characterized by the fundamental physical quantities, namely, the expansion rate $H$, and the energy density which can define in a natural way a gravitational radiation rate $\Gamma$. From a thermodynamic notion, $\Gamma$ should be greater than $H$ in the very early universe to consider the created radiation as a thermalized heat bath. So, the simplest choice of $\Gamma$ should be $\Gamma \propto H^2$ (i.e. $\Gamma \propto \rho$) at the very early epoch. The corresponding cosmological solution (58, 59, 63, 64) shows a smooth transition from inflationary stage to radiation phase and for this “adiabatic” production of relativistic particles, the energy density scales as $\rho_c \sim T^4$ (black body radiation, for details see Ref. 63). Further, $\Gamma \propto H$ (63) explains the decelerated matter dominated era, and $\Gamma \propto 1/H$ has some accelerating feature of the universe (65).

Motivated by the above studies, a more generalized particle creation rate, $\Gamma = \Gamma_0 + 1H^2 + mH + n/H$, was considered in order to explain the whole cosmic evolution (66). Later on, it was established in Ref. 10 that, $\Gamma = \Gamma_0$, a constant, can predict the initial big bang singularity, subsequent intermediate phases, and finally describes the late de Sitter phase. Further, it has been noticed that the effective equation of state of the cosmic substratum could go beyond $-1$ without introducing any kind of phantom fields (11, 12). So, $\Gamma$ plays an important role to elucidate the cosmic evolution. Thus, it is clear that, we can produce any arbitrary $\Gamma$ as a function of $H$ from which we can develop the dynamics of the universe analytically (if possible), or numerically (if analytic solutions are not found). But, the dynamics could be stable or unstable which may lead to some discrepancies in the dynamical behavior of the model.

Keeping all these in mind, the present paper aims to study a generalized model for matter creation in order to study their viability to describe the current accelerating phase of the universe, and also, to check their limit of extension to trace back the early physics scenario as well.

\(^1\) Recall that for a Killing vector field $X$, of the metric tensor $g_{\mu\nu}$, i.e. $L_X g_{\mu\nu} = 0$, holds $L_X G_{\mu\nu} = 0$, consequently we have that $\rho, p, \text{ and } P_c$, are functions of “t” only.
3. COSMOLOGICAL SOLUTIONS:

In this section, we will study the solutions of the Raychaudhuri equation (9) for the following matter creation rate
\[ \Gamma(H) = -\Gamma_0 + mH + n/H, \] (21)
where we have chosen the negative sign in \( \Gamma_0 \) for convenience. Note that, the choice (21) is a generalized one which could cover different matter creation rate, for instance, \( \Gamma \propto H, \Gamma = \text{constant}, \Gamma \propto 1/H \), and some other combinations. However, in that case, the dynamical equation becomes
\[ \dot{H} = -\frac{\gamma}{2} ((3-m)H^2 + \Gamma_0 H - n). \] (22)

Since the equation (9) or equivalently (22) is a one dimensional first order differential equation, hence, the dynamics is obtained from the study of its critical points (or, fixed points).

The fixed points of the Eq. (9) are obtained by \( \dot{H} = 0 \). Thus, if \( H = H_* \) be the fixed point of Eq. (9), then
\[ \dot{H} = 0 \Rightarrow H_* = 0, \text{ or, } \Gamma(H_*) = 3H_. \] (23)

Now, at the fixed points, in which \( H_* \neq 0 \), the FLRW metric (2) describes a de Sitter universe.

Let \( \dot{H} = F(H) \) be the general form of (9). Now, if at the fixed point, \( \frac{dF}{dH}(H_*) < 0 \), then the fixed point is asymptotically stable (attractor), and on the other hand, if we have \( \frac{dF}{dH}(H_*) > 0 \), then the fixed point is unstable in nature (repeller). The repeller point is suitable for early universe, since it can describe the inflationary epoch, whereas the attractor point is stable for late-time accelerating phase.

For the simplest case in which the particle creation rate is \( \Gamma = n/H \) with \( n > 0 \), solving Eq. (9) for the fixed points, we have \( H_* = \pm \sqrt{\frac{n}{3}} \). Now, for the above choice for \( \Gamma \), one has \( F(H) = -\frac{3}{2} (H^2 - \frac{n}{3}) \) and thus, \( \frac{dF}{dH}(H_*) = \mp \sqrt{3n} \) which means that \( \sqrt{3} \) is an attractor and \(-\sqrt{3} \) is a repeller.

If \( \Gamma(H) \) is a polynomial function of \( H \), then the fixed point condition, (23) for \( H_* \neq 0 \), is a polynomial equation which has as many solutions (not necessary real solutions) as is the higher power of the polynomial \( \Gamma(H_*) = 3H_* \).

Hence, for (21) we have the following second-order polynomial equation
\[ F(H_*) \equiv (m-3) H_*^2 - \Gamma_0 H_* + n = 0 \] (24)
where in order to find two critical points, as many as the inflationary phases of the universe, we are interested in the case when \( m \neq 3 \), and \( n \neq \frac{1}{4(m-3)} \).

3.1. Dynamical study

For our model, the matter creation rate is: \( \Gamma(H) = -\Gamma_0 + mH + n/H \). Now, solving (24) for our model, the critical points are found to be
\[ H_\pm = \frac{\Gamma_0}{2(m-3)} \left( 1 \pm \sqrt{1 + \frac{4(3-m)n}{\Gamma_0^2}} \right). \]

To perform the dynamical analysis, we start with the case \( \Gamma_0 > 0 \), then we have to divide the plane \( (m,n) \) in six different regions:

1. \( \Omega_1 = \{(m,n) : m-3 < 0, n \geq 0\} \), where \( H_+ < 0 \) and \( H_- > 0 \). \( H_+ \) is a repeller and \( H_- \) an attractor.
2. \( \Omega_2 = \{(m,n) : m-3 > 0, n \geq 0, \frac{4(3-m)n}{\Gamma_0^2} > -1\} \), where \( H_+ > H_- > 0 \). \( H_+ \) is a repeller and \( H_- \) an attractor.
3. \( \Omega_3 = \{(m,n) : m-3 > 0, n > 0, \frac{4(3-m)n}{\Gamma_0^2} < -1\} \), where \( H_\pm \) are complex numbers. \( \dot{H} \) is always positive.
4. \( \Omega_4 = \{(m,n) : m-3 < 0, n < 0, \frac{4(3-m)n}{\Gamma_0^2} < -1\} \), where \( H_\pm \) are complex numbers. \( \dot{H} \) is always negative.
5. \( \Omega_5 = \{(m,n) : m-3 < 0, n < 0, \frac{4(3-m)n}{\Gamma_0^2} > -1\} \), where \( H_+ < H_- < 0 \). \( H_+ \) is a repeller and \( H_- \) an attractor.
6. \( \Omega_6 = \{(m,n) : m-3 > 0, n < 0\} \), where \( H_+ > 0 \) and \( H_- < 0 \). \( H_+ \) is a repeller and \( H_- \) an attractor.

On the other hand, for \( \Gamma_0 < 0 \), we have
e1. \( \Omega_7 = \{(m,n) : m-3 < 0, n \geq 0\} \), where \( H_+ > 0 \) and \( H_- < 0 \). \( H_+ \) is an attractor and \( H_- \) a repeller.
2. \( \Omega_8 = \{(m,n) : m-3 > 0, n \geq 0, \frac{4(3-m)n}{\Gamma_0^2} > -1\} \), where \( H_+ < H_- < 0 \). \( H_+ \) is an attractor and \( H_- \) a repeller.
3. \( \Omega_9 = \{(m,n) : m-3 > 0, n < 0, \frac{4(3-m)n}{\Gamma_0^2} > -1\} \), where \( H_\pm \) are complex numbers. \( \dot{H} \) is always positive.
4. \( \Omega_{10} = \{(m,n) : m-3 < 0, n < 0, \frac{4(3-m)n}{\Gamma_0^2} < -1\} \), where \( H_\pm \) are complex numbers. \( \dot{H} \) is always negative.
5. \( \Omega_{11} = \{(m,n) : m-3 < 0, n < 0, \frac{4(3-m)n}{\Gamma_0^2} > -1\} \), where \( H_+ > H_- > 0 \). \( H_+ \) is an attractor and \( H_- \) a repeller.
6. \( \Omega_{12} = \{(m,n) : m-3 > 0, n < 0\} \), where \( H_+ < 0 \) and \( H_- > 0 \). \( H_+ \) is an attractor and \( H_- \) a repeller.
The case $m = 3$ is special, in the sense that, there is only one critical point given\(^2\) by $H_\gamma = \frac{\ell_0}{\Gamma_0}$ which is always an attractor for $\Gamma_0 > 0$, and a repeller for $\Gamma_0 < 0$.

To have a non-singular universe (without the big bang singularity) with an accelerated phase both at early and late times, one possibility is to have two critical points $H_+ > H_- > 0$, where $H_+$ was a repeller and $H_-$ must be an attractor. If so, in principle, when the universe leaves $H_+$, realizing the inflationary phase, and when it comes asymptotically to $H_-$, it enters into the current accelerated phase. Of course, the viability of the background has to be checked dealing with cosmological perturbations and comparing the theoretical predictions with the observational ones.

For our model, this only happens in the region $\Omega_2$, and when $m = 3$ with $\Gamma_0 > 0$, that is in the region of the space parameters given by

$$W = \{ \Gamma_0, m, n : \Gamma_0 > 0, m \geq 3, n \geq 0, \frac{4(3 - m)n}{\Gamma_0^2} > -1 \}.$$

Note that, in the case $m = 3$ we have $H_+ = +\infty$, but the universe is not singular, because in that case the Raychaudhuri equation becomes $\dot{H} = -\frac{\dot{\Omega}_0 H}{n}$. For large values of $H$, this equation is approximately equal to $\dot{H} = -\frac{\dot{\Omega}_0}{H}$, the solution of which is given by $H(t) = H_0 e^{-\frac{\dot{\Omega}_0}{\dot{\Omega}_0}(t-t_0)}$. Therefore, $H$ only diverges when $t = -\infty$, that is, there is no singularities at finite time.

For the parameters that belong to $W$, the solution of the Raychaudhuri equation is given by:

$$H(t) = \frac{\Gamma_0}{2(m-3)} - \frac{\omega}{2(m-3)} \tanh \left(\frac{\gamma}{4} (t - t_0)\right), \quad (26)$$

for $m > 3$, where $\omega = \sqrt{\Gamma_0^2 + 4(3-m)n}$.

For the completeness of our analysis, for $m = 3$, we have that

$$H(t) = \Gamma_0 e^{-\frac{\dot{\Omega}_0}{2}(t-t_0)} + \frac{n}{\Gamma_0}, \quad (27)$$

Last but not least, when $m \neq 3$ and $n = \frac{\Gamma_0^2}{4(m-3)}$, where $H_+ = H_-$, that is, equation \(^2\) admits one fixed point we find the following analytical solution for the Hubble function

$$H(t) = \frac{\Gamma_0}{2(m-3)} - \frac{1}{\gamma} \frac{1}{(m-3)} (t - t_0), \quad (28)$$

in which for $m > 3$, in order to have $H(t) > 0$, we have $t \in (-\infty, t_0)$.

Note that, this last solution, when the values of the parameters belong in $W$, depicts a phantom universe that starts at the critical point and ends in a Big Rip singularity at $t = t_0$.

From \(^2\), \(^2\) and \(^2\), we can find the solution of the scale factor. Hence, from \(^2\) we have

$$a(t) = a_0 \exp \left[ \frac{\Gamma_0}{2(m-3)} (t - t_0) \right] \frac{\dot{\Omega}_0}{2(m-3)/\gamma} \ln \left( \cosh \left( \frac{\gamma}{4} (t - t_0) \right) \right). \quad (29)$$

Furthermore, from \(^2\) we have

$$a(t) = a_0 \exp \left[ -\frac{2}{\gamma} \left( e^{\frac{\Gamma_0}{2}(t-t_0)} - 1 \right) + \frac{n}{\Gamma_0} (t-t_0) \right]. \quad (30)$$

Finally, from the case $m \neq 3$, and $n = \frac{\Gamma_0^2}{4(m-3)}$, the scale factor becomes

$$a(t) = a_0 \exp \left[ \frac{\Gamma_0}{2(m-3)} (t - t_0) \right] \left( \frac{t}{t_0} \right)^{-\frac{2}{\gamma(m-3)}}. \quad (31)$$

in which for $-\frac{2}{\gamma(m-3)} = \frac{1}{3}$, the last solution describes also the two-scalar field cosmological model in which the scalar fields are interacting in their kinetic parts \(^2\), where it has been showed that the model fits the cosmological data in a similar way with the A-cosmology.

Now, with the use of equation \(^2\), it is possible to determine the effective equation of state parameter. Therefore we have

$$\omega_{\text{eff}} = -1 + \frac{(m-3)}{3} \omega^2 \gamma \left( \Gamma_0 \cosh \left( \frac{\gamma}{4} (t - t_0) \right) \right)^{-1} \gamma \sinh \left( \frac{\gamma}{4} (t - t_0) \right)^2, \quad (32)$$

\(^2\) When $m = 3$, equation \(^2\) is a linear equation which admits only one real solution.
or,
\[ \omega_{\text{eff}} = -1 + \frac{\gamma}{3} \frac{e^{-\frac{2}{3} \Gamma_0 (t-t_0)}}{\left( e^{-\frac{2}{3} \Gamma_0 (t-t_0)} + \frac{n}{\Gamma_0} \right)^2} \]  \hspace{1cm} (33)

and
\[ \omega_{\text{eff}} = -1 - \frac{4(m-3)\gamma}{3(\Gamma_0 \gamma (t-t_0)-2)^2} \]  \hspace{1cm} (34)

for the solutions (26), (27) and (28) respectively.

Consider now the initial condition that at \( t = t_1 \), \( \omega_{\text{eff}} (t_0) = \gamma - 1 \). From the latter we can define a constraint equation between the free parameters of the model, i.e. \( \{ \Gamma_0, m, n \} \). Without any loss of generality, let say that \( t_1 = t_0 \), that is possible since the model is autonomous and invariant under time translations.

Hence, from (32), we find the condition
\[ \Gamma_0^2 = \frac{m-3}{3} \omega^2. \]  \hspace{1cm} (35)

Figure 1 shows the evolution of the effective equation of state parameter (Eq. (32)) for a set of parameters \( \{ \Gamma_0, m, n \} \in W \), describing the early and late de Sitter phases of the universe, where we have shown its evolution for three different values of \( \gamma \), namely, \( \gamma = 4/3, 1, \) and 1.03.

3.2. Particle creation rate from Jacobi Last multiplier

Equation (33) is a first-order differential equation for the Hubble function \( H(t) \), or a second order differential equation for the scale factor \( a(t) \). Apply in (33) the transformation \( a(t) = \exp (\dot{N}(t)) \), i.e. \( H = \dot{N} \), we have the second-order differential equation
\[ \ddot{N} = -\frac{3\gamma}{2} \dot{N}^2 \left( 1 - \frac{\Gamma (\dot{N})}{3 \dot{N}} \right) \]  \hspace{1cm} (36)

which is of the form \( \ddot{x} = F(t, x, \dot{x}) \). One would like to have a geometric method to construct the unknown function \( \Gamma (\dot{N}) \), such as the application of group invariant transformations in scalar field cosmology or in modified theories of gravity\(^3\). In this approach we would like to solve the inverse problem, i.e. to construct a Lagrangian function for equation (36) by using the method of Jacobi Last multiplier. For one-dimensional second-order differential equations if there exist a function \( M(t, x, \dot{x}) \), which satisfy the following condition
\[ \frac{d}{dt} (\ln M) + \frac{\partial F}{\partial \dot{x}} = 0 \]  \hspace{1cm} (37)

then for the second-order differential equation \( \ddot{x} = F(t, x, \dot{x}) \), a Lagrangian can be constructed [71]. For equation (36), we have that \( F = F(\dot{x}) = F(\dot{N}) \), therefore condition (37) gives that
\[ \frac{\partial}{\partial t} (\ln M) + \dot{x} \frac{\partial}{\partial x} (\ln M) + F \frac{\partial}{\partial \dot{x}} (\ln M) = -\frac{\partial F}{\partial \dot{x}} \]  \hspace{1cm} (38)

Then, since for our model we have \( F(\dot{x}) = -\frac{2}{3} \Gamma_0 (3 - m) \dot{x}^2 + \Gamma_0 \dot{x} - n \), we can deduce that
\[ \frac{\partial}{\partial t} (\ln M) = -\frac{\Gamma_0}{2}, \quad \frac{\partial}{\partial x} \ln (M) = -\gamma (3 - m) \]  \hspace{1cm} (39)

Finally, using that the Lagrangian is determined by the relation
\[ \frac{\partial^2 L}{\partial \dot{x}^2} = M, \]  \hspace{1cm} (40)

after comparing with (36) one gets the following Lagrangian for our model
\[ L(\dot{N}, N, t) = e^{\gamma (3-m)\dot{N} + 2\Gamma_0 t} \left( \frac{1}{2} \dot{N}^2 + \frac{n}{2(3-m)} \right). \]  \hspace{1cm} (41)

On the other hand, someone can start with special forms of the Lagrange Multiplier and from condition (37) to determine the creation rate. For instance, consider that \( M = M(x) = M(N) \), hence equation (37) becomes
\[ \frac{d}{dx} \ln (M) = -\frac{1}{\dot{x}} \frac{\partial F}{\partial \dot{x}}, \]  \hspace{1cm} (42)

therefore, the l.h.s. of the latter equation is constant, i.e. \( \frac{\partial}{\partial x} \ln (M) = \gamma (3-m) \), and
\[ \Gamma (H) = mH + \frac{n}{H}. \]  \hspace{1cm} (43)

This is a particular case the one we considered above, i.e. it is expression (21) for \( \Gamma_0 = 0 \). Hence the analytical solution of (36) is
\[ a = a_0 \left[ \sinh \left( \frac{\gamma}{2} \sqrt{n(3-m)} (t-t_0) \right) \right] \frac{\dot{x}}{\sqrt{3-m}} \]  \hspace{1cm} (44)

for \( n \neq 0 \), or
\[ a(t) = a_0 ((t-t_0) \frac{\dot{x}}{\sqrt{(3-m)}}), \]  \hspace{1cm} (45)

for \( n = 0 \). Finally the Lagrangian function for (36), which follows from the Lagrange multiplier \( M \), (38), is,
\[ L(\dot{N}, N) = \frac{\exp (\gamma (m-3)\dot{N})}{2} \left( \dot{N}^2 - \frac{n}{(m-3)} \right). \]  \hspace{1cm} (46)
We can see that the power of the scale factor parameters that we have to determine are $H_0$, $\Omega_{m0}$, and $\bar{m}$. In order to constrain the cosmological parameters, we can write the number of free parameters and the goodness-of-fit statistics.

\[
I_0 = e^{\gamma(m-3)N} \left( \chi^2 + n \right) 
\]

hence

\[
\frac{H^2}{H_0^2} = \Omega_{m0} a(3-m)\gamma + \Omega_\Lambda. \tag{48}
\]

where $\Omega_{m0} = I_0 H_0^2$, and $\Omega_\Lambda = -n H_0^2$, which describes a universe with cosmological constant and a perfect fluid $\bar{\rho} = (\gamma - 1) \bar{\rho}$, in which $\gamma = \frac{m-3}{3}$. We can see that when $m = 6$, $\gamma = 1$, or $(3-m)\gamma = -3$, $\Lambda$-cosmology is recovered, furthermore, $|n| = \rho_\Lambda$. Recall that such an analytical solution have been found recently for a Brans-Dicke cosmological model, in which the term $(m-3)\gamma$, is related with the Brans-Dicke parameter $\bar{\gamma}$. In particular, we found that,

\[
m(\gamma) = 3 + \frac{1}{\gamma} \frac{3\omega_{BD} + 4}{3\omega_{BD} + 1}. \tag{49}
\]

As far as the Hubble function is concerned, we can see that the power of the scale factor $a$ can be written as $(3-m)\gamma = -\bar{m}(m, \gamma)$, that is, the independent parameters that we have to determine are $H_0^2$, $\Omega_{m0}$ and $\bar{m}$. In order to constrain the cosmological parameters, joint likelihood analysis using the Type Ia supernova data set of Union 2.1, the 6dF, SDSS and WiggleZ BAO data, and the 21 one Hubble data of $77$, has been performed. Further, in order to reduce the number of the free variables to two, we select to use the present value of the Hubble function, i.e. $H_0 = 69.6$Km/s/Mpc. Hence, the likelihood function depends on the values of the parameter $\{\Omega_{m0}, \bar{m}(m, \gamma)\}$, and it is given as follows

\[
\mathcal{L}(\Omega_{m0}, \bar{m}(m, \gamma)) = \mathcal{L}_{SN1a} \times \mathcal{L}_{BAO} \times \mathcal{L}_{H(z)}, \tag{50}
\]

where $L_A \propto e^{-\chi^2/2}$, that is $\chi^2 = \chi_{SN1a}^2 + \chi_{BAO}^2 + \chi_H^2(z)$. The results are given in table $1$, and we give the confidence levels $1\sigma$, $2\sigma$, $3\sigma$ for the best fit values. Specifically, figure $2$ compares the constraints SNIa vs. SNIa+BAO data while figure $3$ compares the SNIa+BAO vs. SNIa+BAO+H(z).

Furthermore we note that for the relation $m = \frac{1}{\gamma}(3 + \bar{m})$, for a specific value of $\gamma$, we can determine $m$ from table $1$ and the constant $n$, $I_0$, and $\bar{m}$.

We conclude that the application of the Jacobi Last multiplier gives a function $\Gamma(H)$, which include the terms which explains the decelerated matter dominated era, and the acceleration features of the universe. However, one may study the group invariant transformations of equation $30$ and from the requirement that $30$ is invariant under a specific algebra, the particle creation rate $\Gamma$ might be determined. This would be geometric selection rule, however this analysis is not in the scope of this work.

In the following sections, we study the relation between the particle creation rate with some other cosmological theories.
4. EQUIVALENCE WITH THE DYNAMICS DRIVEN BY A SINGLE SCALAR FIELD

To check the viability of the models one has to verify if they support the observational data, relative to inflation, provided by PLANCK’S team. However, it is not clear at all how hydrodynamical perturbations could provide viable theoretical data, i.e. that fit well with current observational ones, because during the inflationary period one has $p \equiv -\rho$, and thus, the square of the velocity of sound, which appears in the Mukhanov-Sasaki equation [79, 80], could be approximately $c_s^2 \equiv \frac{ho}{p} \equiv -1$, which is negative, leading to a Jeans instability for modes well inside the Hubble radius. However, for a universe filled by an scalar field this problem does not exist because in that case one always has $c_s^2 = 1$. This is an essential reason why we try to mimic the dynamics of an open system, where matter creation is allowed, obtained in the previous section by an scalar field $\phi$ with potential $V(\phi)$. To do that, we use the energy density, namely, $\rho$, and pressure, namely, $p$, of the scalar field given by

$$\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi),$$

$$p = \frac{1}{2} \dot{\phi}^2 - V(\phi),$$

To show the equivalence with our system as described in with EoS $p = (\gamma - 1)\rho$, we perform the replacement

$$\rho \rightarrow \rho_s, \quad p - \frac{\gamma \rho}{3H} \rightarrow p_s, \quad$$

and the Friedmann and Raychaudhuri equations will become

$$3H^2 = \rho, \quad 2\dot{H} = -\dot{\varphi}^2.$$  (54)

Note that, Eq. [54] uses the equations of General Relativity (GR) for a single scalar field, this means that, we are dealing with the equivalence with an open system and the one driven by a single scalar field in the context of GR.

Using the above two equations, we see that the effective EoS parameter is:

$$\omega_{\text{eff}} = -1 + \gamma \left(1 - \frac{\Gamma}{3H}\right) = \omega_s = \frac{\dot{\varphi}^2 - 2V(\phi)}{\dot{\varphi}^2 + 2V(\phi)}.$$  (55)

Note that, the Raychaudhuri equation [55] tells us that $\dot{H} < 0$, which means from [11] that, $\omega_{\text{eff}} > -1$, and thus, one has $\Gamma < 3H$.

On the other hand, from the Friedmann and Raychaudhuri equations one easily obtains

$$\dot{\varphi} = \sqrt{-2H} = \sqrt{3\gamma H^2 \left(1 - \frac{\Gamma}{3H}\right)},$$  (56)

and

$$V(\varphi) = \frac{3H^2}{2} \left(2 - \gamma + \frac{\gamma \Gamma}{3H}\right).$$  (57)

The first step is to integrate [50]. Performing the change of variable $dt = \frac{dH}{H}$, we will obtain

$$\varphi = -\int \sqrt{-\left(\frac{2}{H}\right)} \, dH = -\frac{2}{\sqrt{\gamma}} \int \frac{dH}{\sqrt{3H^2 - \Gamma H}}.$$  (58)

In the particular case $\Gamma = -\Gamma_0 + mH + n/H$ one has

$$\varphi = -\frac{2}{\sqrt{\gamma}} \left[\frac{dH}{\sqrt{(3 - m)H^2 + \Gamma_0 H - n}}\right].$$  (59)

This integral could be solved analytically in the region $W$, giving

$$\varphi = \frac{2}{\sqrt{(m - 3)\gamma}} \arcsin \left(\frac{m - 3}{\omega} \left(\frac{\Gamma_0}{m - 3} - 2H\right)\right),$$  (60)

when $m > 3$, and

$$\varphi = -\frac{4}{\sqrt{\gamma} \Gamma_0} \sqrt{\Gamma_0 H - n},$$  (61)

for $m = 3$.

Conversely,

$$H = \frac{1}{2(m - 3)} \left[\Gamma_0 - \omega \sin \left(\frac{\sqrt{(m - 3)\gamma}}{2} \varphi\right)\right], \quad$$

when $m > 3$,  (62)

and

$$H = \frac{n}{\Gamma_0} + \frac{\gamma \Gamma_0}{16} \varphi^2, \quad$$

when $m = 3$.  (63)

On the other hand, for our model, the potential [57] is given by

$$V(\varphi) = \frac{1}{2} ((6 + (m - 3)\gamma)H^2 - \gamma \Gamma_0 H + \gamma n),$$  (64)

then, inserting on it, the values of $H$ given by [62] and [63], one obtains the corresponding potentials. In fact, in the case [62] one gets

$$V(\varphi) = \frac{3}{4(m - 3)^2} \left[\Gamma_0 - \omega \sin \left(\frac{\sqrt{(m - 3)\gamma}}{2} \varphi\right)\right]^2 \quad$$

$$-\frac{\gamma \omega^2}{8(m - 3)} \cos^2 \left(\frac{\sqrt{(m - 3)\gamma}}{2} \varphi\right),$$  (65)

and for [63]

$$V(\varphi) = \frac{\gamma^2 \Gamma_0^2}{256} \varphi^4 + \frac{\gamma}{8} \left(3n - \frac{\gamma \Gamma_0^2}{4}\right) \varphi^2 + \frac{3n^2}{\Gamma_0^2}.$$  (66)
The following remark is in order: In the context of General Relativity driven by scalar field, the backgrounds, that now has to be understood as mere solutions of the Raychaudhuri equation when the universe is filled by a scalar field and not as solutions of an open system, are not viable because they do not contain a mechanism to reheat the universe, because the potential has a minimum when the universe reaches the de Sitter solution $H_+$, that depicts the current cosmic acceleration, but it is clear that, in order to match with the hot Friedmann universe, it has to reheat at higher scales. Then, the simplest solution is to introduce a sudden phase transition that breaks the adiabaticity, and thus, particles could be produced in an enough amount to thermalize the universe.

4.1. A viable model

What we choose is a continuous transition at some scale $H_E$, of the rate of particle production $\Gamma$, of the form:

$$
\Gamma = \begin{cases} 
-\Gamma_0 + 3\Gamma + \frac{\Gamma_0^2}{12\Gamma} & \text{for } H > H_E \\
\Gamma_1 & \text{for } H_E > H > \tilde{H}_- 
\end{cases}
$$

(67)

where $0 < \Gamma_1 \ll \Gamma_0$ and $\tilde{H}_- = \frac{\Gamma_0}{3}$. The continuity requires,

$$
H_E = \frac{\Gamma_0 + \Gamma_1}{6} \left( 1 + \sqrt{1 - \frac{\Gamma_0^2}{(\Gamma_0 + \Gamma_1)^2}} \right) \cong \frac{\Gamma_0}{6}.
$$

(68)

Moreover, we will assume that universe has a deflationary phase, which can be mimicked by an stiff fluid, at the transition phase, since at that moment one has

$$
\omega_{eff} = -1 + \gamma \left( 1 - \frac{\Gamma_1}{H_E} \right) \cong -1 + \gamma
$$

(69)

one has to choose $\gamma = 2$, i.e. the EoS must be $p = \rho$.

Now, to check the viability we have to study the model at early times. We start with the slow roll parameters

$$
\epsilon = -\frac{\dot{H}}{H^2}, \quad \eta = 2\epsilon - \frac{\dot{\epsilon}}{2H\epsilon},
$$

(70)

that allows us to calculate the spectral index ($n_s$), its running ($\alpha_s$) and the ratio of tensor to scalar perturbations ($r$) given by

$$
n_s - 1 = -6\epsilon + 2\eta, \quad \alpha_s = -\frac{Hn_s}{H^2 + H}, \quad r = 16\epsilon.
$$

(71)

At early times, i.e. when $H > H_E$, introducing the notation $x \equiv \frac{1}{H}$, since for our model the Raychaudhuri equation is

$$
\dot{H} = -\Gamma_0 H + \frac{\Gamma_0^2}{12}.
$$

(72)

one will have

$$
\epsilon = x \left( 1 - \frac{x}{12} \right), \quad \eta = \epsilon + \frac{x}{2},
$$

(73)

and as a consequence,

$$
n_s - 1 = -3x + \frac{x^2}{3}. \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
$$

(74)

From recent PLANCK+WP 2013 data (see table 5 of [82]), the spectral index at 1σ Confidence Level (C.L.) is $n_s = 0.9583 \pm 0.0081$, which means that $1 - n_s \cong 5 \times 10^{-2}$. Therefore, and we can apply the results obtained in [40].

Since,

$$
x = \frac{9}{2} \left( 1 - \sqrt{1 - \frac{4(1 - n_s)}{27}} \right),
$$

(75)

at 2σ C.L., one has $0.0085 \leq x \leq 0.0193$, and thus, $0.1344 \leq r = 16\epsilon \leq 0.3072$. Since PLANCK+WP 2013 data provides the constrain $r \leq 0.25$, at 95.5% C.L., then when $0.0085 \leq x \leq 0.0156$, the spectral index belongs to the 1-dimensional marginalized 95.5% C.L., and also $r \leq 0.25$, at 95.5% C.L.

For the running at 1σ C.L., PLANCK+WP 2013 data gives $\alpha_s = -0.021 \pm 0.012$, and our background leads to the theoretical value $\alpha_s \cong -\frac{3\epsilon}{2H\epsilon} \cong -3(-\epsilon)^2$. Consequently, at the scales we are dealing with, $-7 \times 10^{-4} \leq \alpha_s \leq -2 \times 10^{-4}$, and thus, the running also belongs to the 1-dimensional marginalized 95.5% C.L.

Note also that, we have the relation $w_{eff}(H) = -1 + \frac{\dot{\epsilon}}{2\epsilon}$. Therefore, if we assume that the slow-roll ends when $\epsilon = 1$, and let $H_{end}$ be the value of the Hubble parameter when the slow roll ends, then the slow roll will end when $w_{eff}(H_{end}) = -\frac{1}{3}$, i.e. when the universe will start to decelerate.

On the other hand, the number of e-folds from observable scales exiting the Hubble radius to the end of inflation, namely $N(H)$, could be calculated using the formula $N(H) = -\int_{H_{end}}^{H} \frac{H}{H} dH$, leading to

$$
N(x) = \frac{1}{x} - \frac{1}{x_{end}} + \frac{1}{12} \ln \left( \frac{12 - x_{end}}{12 - x} \right),
$$

(76)

where $x_{end} = 6(1 - \sqrt{2/3}) \cong 1.1010$, is the value of the parameter $x$ when inflation ends. For our values of $x$ that allow to fit well with the theoretical value of the spectral index, its running and the tensor/scalar ratio with their observable values, we will obtain $64 \leq N \leq 117$.

The value of $\Gamma_0$, could be established taking into account the theoretical and the observational value of the power spectrum

$$
\mathcal{P} \cong \frac{H^2}{8\pi^2\epsilon} = \frac{\Gamma_0^2}{18\pi^2\epsilon^2} = \frac{4\Gamma_0^2}{9\pi m_{pl}^2\epsilon^2} \cong 2 \times 10^{-9},
$$

(77)

where we have explicitly introduced the Planck’s mass, which in our units is $m_{pl} = \sqrt{8\pi}$. Using the values of $x$ in the range $[0.0085, 0.0156]$, we can conclude that

$$
9 \times 10^{-7} m_{pl} \leq \Gamma_0 \leq 2 \times 10^{-7} m_{pl}.
$$

(78)
4.1.1. Particle production and reheating

We will study the production of massless particles nearly conformally coupled with gravity due to the phase transition in our model. To simplify our reasoning we will choose $\Gamma_1 = 0$, and then $\dot{H}(t_E) = \frac{\Gamma_0}{a}$, thus, after the transition the universe is exactly in a deflationary phase if we choose $\gamma = 2$.

The energy density of the produced particles will be given by [22]
\[
\rho_\xi = \frac{1}{(2\pi a)^2} \int_0^\infty k |\beta_k|^2 d^3k, \tag{79}
\]
where the $\beta$-Bogoliubov coefficient is given by [22, 24]
\[
\beta_k \equiv \frac{i(\xi - \frac{1}{2})}{2k} \int_0^\infty e^{-2ik\tau} a^2(\tau) R(\tau) d\tau, \tag{80}
\]
being $R = 6(\dot{H} + 2H^2)$ is the scalar curvature, $\tau$ the conformal time and $\xi$ the coupling constant. This integral is convergent because at early and late time $a^2(\tau) R(\tau)$ converges to zero fast enough. It is not difficult to show, integrating twice by parts, that $\beta_k \sim O(k^{-1})$ (this is due to the fact that $\dot{H}$ is continuous during the phase transition) and, as we will see, this means that the energy density of produced particles is not ultra-violet divergent. Moreover, $\beta_k = (1 - 6\xi) f\left(\frac{\xi}{a E a_0}\right)$, where $f$ is some function.

Then, taking for instance $1 - 6\xi \sim 10^{-1}$, the energy density of the produced particles is of the order
\[
\rho_\xi \sim 10^{-2} T_0^4 \left(\frac{a E}{a}\right)^4 \frac{1}{2\pi^2} \int_0^\infty s^3 f^2(s) ds \sim 10^{-2} \mathcal{M}_0^4 \left(\frac{a E}{a}\right)^4, \tag{81}
\]
where we have introduced the notation $\mathcal{M} \equiv \frac{1}{2\pi^2} \int_0^\infty s^3 f^2(s) ds$.

Since the sudden transition occurs at $H_E \equiv \frac{\Gamma_0}{a} \sim 10^{-7} m_{pl} \sim 10^{12}$ GeV (the same result was obtained in formula (15) of [81]), one can deduce that the universe preheats, due to the gravitational particle production, at scales
\[
\rho \sim \frac{3H_E^2 m_{pl}^4}{8\pi} \sim 10^{-17} \rho_{pl}, \tag{82}
\]
where $\rho_{pl} = m_{pl}^4$ is the Planck’s energy density. On the other hand, at the transition time the energy density of the produced particles is of the order
\[
\rho_\chi \sim 10^{-30} \mathcal{M} \rho_{pl}, \tag{83}
\]
which is smaller than the energy density of the background.

After the phase transition, first of all, these particles will interact exchanging gauge bosons and constituting a relativistic plasma that thermalises the universe [81, 82] before the universe was radiation dominated. Moreover, in our model, the background is in a deflationary stage, meaning that its energy density decays as $a^{-6}$, and the energy density of the produced particles decreases as $a^{-4}$. Then, eventually the energy density of the produced particles will dominate and the universe will become radiation dominated and matches with the standard hot Friedmann universe. The universe will expand and cool becoming the particles no-relativistic, and thus, the universe enters into a matter dominated regime, essential for the grow of cosmological perturbations, and only at very late time, when the Hubble parameter is of the same order as $\Gamma_1$, the field takes back its role to start the cosmic acceleration.

The reheating temperature, namely $T_R$, is defined as the temperature of the universe when the energy density of the background and the one of the produced particles are of the same order ($\rho \sim \rho_\chi$). Since $\rho_\chi \sim \rho_\chi = 10^{-2} \mathcal{M}_0^4 \left(\frac{a E}{a}\right)^4$ and $\rho = \frac{3H_E^2 m_{pl}^4}{8\pi} \sim 10^{-3} \mathcal{M}_{pl}^4 \left(\frac{a E}{a}\right)^6$ one obtains $\frac{a E}{a (t_R) \sim \sqrt{\mathcal{M} \rho_{pl}}}$, and therefore,
\[
T_R \sim \rho_\chi^{1/4} (t_R) \sim \mathcal{M}_0^2 \frac{T_0^2}{m_{pl}} \sim 10^5 \mathcal{M}^{\frac{1}{2}} GeV. \tag{84}
\]

This reheating temperature is below the GUT scale 10$^{16}$ GeV, which means that the GUT symmetries are not restored preventing a second monopole production stage. Moreover, this guarantees the standard successes with nucleosynthesis, because it requires a reheating temperature below 10$^9$ GeV [86].

Finally, to obtain the temperature when the equilibrium is reached, we will follow the thermalization process depicted in [82] (see also [81]), where it is assumed that the interactions between the produced particles are due to gauge bosons, one might estimate the interaction rate as $\Gamma \sim a^2 T_{eq}$. Then, since thermal equilibrium is achieved when $\Gamma \sim H(t_{eq}) \sim H_E \left(\frac{a_0}{a}\right)^3$ (recall that, in our model, this process is produced in the deflationary phase where $\rho \sim a^{-6}$), and $T_{eq} \sim 10^{-\frac{5}{8}} \mathcal{M}^{\frac{1}{2}} H_E a_0 a_{eq}$ when the equilibrium is reached one has $a_{eq} \sim 10^{-\frac{3}{8}} a \mathcal{M}^{\frac{1}{2}}$, and thus, $T_{eq} \sim 10^{-\frac{5}{8}} \mathcal{M}^{\frac{1}{2}} \alpha H_E$. Therefore, one obtains
\[
T_{eq} \sim 10^{-8} \mathcal{M}^{\frac{1}{2}} a m_{pl} \sim 10^{31} \mathcal{M}^{\frac{1}{2}} \alpha GeV. \tag{85}
\]

And choosing as usual $\alpha \sim (10^{-2} - 10^{-1})$ [81, 82], one obtains the following equilibrium temperature
\[
T_{eq} \sim (10^9 - 10^{10}) \mathcal{M}^{\frac{1}{2}} GeV. \tag{86}
\]

5. $f(T)$-GRAVITY AND PARTICLE CREATION RATE

$f(T)$-gravity has recently gained a lot of attention. The essential properties of this modified theory of gravity
are based on the rather old formulation of the teleparallel equivalent of General Relativity (TEGR) \cite{83,91}. In particular, one utilizes the curvature-less Weitzenböck connection in which the corresponding dynamical fields are the four linearly independent vierbeins rather than the torsion-less Levi-Civita connection of the classical General Relativity. A natural generalization ofTEGR gravity is $f(T)$ gravity which is based on the fact that we allow the gravitational Action integral to be a function of $T$ \cite{92,93}, in a similar way such as $f(R)$ Einstein-Hilbert action. However, $f(T)$ gravity does not coincide with $f(R)$ extension, but it rather consists of a different class of modified gravity. It is interesting to mention that the torsion tensor includes only products of first derivatives of the vierbeins, giving rise to second-order field differential equations in contrast with the $f(R)$ gravity that provides fourth-order equations.

Consider the unholonomic frame $e_i$, in which $g(e_i, e_j) = e_i e_j = \eta_{ij}$, where $\eta_{ij}$ is the Lorentz metric in canonical form, we have $g_{\mu\nu}(x) = \eta_{ij} h^i_{\mu}(x) h^j_{\nu}(x)$, where $e^i(x) = h^i_{\mu}(x) dx^\mu$ is the dual basis. The non-null torsion tensor which flows from the Weitzenböck connection is defined as

$$T^\beta_{\mu\nu} = \hat{\Gamma}^\beta_{\mu\nu} - \Gamma^\beta_{\mu\nu} = h^i_\beta (\partial^i h^\nu_\mu - \partial^\nu h^i_\mu), \quad (87)$$

and the action integral of the gravitational field equations in $f(T)$-gravity is assumed to be

$$A_T = \int d^4x \sqrt{|g|} f(T) + \int d^4x \sqrt{|g|} L_m, \quad (88)$$

where $e = \text{det}(e^i_\mu \cdot e^i_\nu) = \sqrt{-g}$.

The scalar $T$ is given from the following expression

$$T = S^\beta_{\mu\nu} T^\beta_{\mu\nu}, \quad (89)$$

where

$$S^\mu_{\rho\nu} = \frac{1}{2} (K^\mu_{\beta\rho} + \delta^\mu_{\beta} T^\theta_{\rho\nu} - \delta^\mu_{\nu} T^\theta_{\rho\beta}), \quad (90)$$

and $K^\mu_{\rho\nu\beta}$ is the contorsion tensor

$$K^\mu_{\rho\nu\beta} = -\frac{1}{2} (T^\mu_{\rho\nu\beta} - T^\nu_{\rho\mu\beta} - T^\nu_{\rho\beta\nu}), \quad (91)$$

which equals the difference of the Levi-Civita connection in the holonomic and the unholonomic frame. We note that, in the special case where $f(T) = \frac{\Gamma}{2}$, then the gravitational field equations are that of General Relativity \cite{95,96}.

For the spatially flat FLRW space-time \cite{2} with a perfect fluid $\dot{\rho}$ minimally coupled to gravity, and for the vierbeins given by the diagonal tensor,

$$h^i_{\mu}(t) = \text{diag}(1, a(t), a(t), a(t)), \quad (92)$$

the modified Friedmann’s equation is \cite{97,98}

$$12H^2 f' + f = \dot{\rho}, \quad (93)$$

while the modified Raychaudhuri equation is as follows

$$48H^2 H f'' - 4(\dot{H} + 3H^2) f' - f = \ddot{\rho}. \quad (94)$$

where $f'(T) = \frac{df(T)}{dT}$, and $T = -6H^2$. Finally, for the perfect fluid from the Bianchi identity, it follows $\dot{\rho} + 3H (\ddot{\rho} + \ddot{p}) = 0$. Obviously, the extra terms which arise from the function $f(T)$, can be seen as an extra fluid. In this work, we are interested in the evolution of the total fluid.

Now, with the use of Eq. \cite{93}, equation \cite{91} becomes

$$\dot{H} = -\frac{3\gamma}{2} \left( \frac{4H^2 f' + \frac{\Gamma}{2}}{2f' - 2AH^2 f''} \right), \quad (95)$$

which is a first-order differential equation on $H$, since $f(T) = f\left(\sqrt{\frac{1}{a} |T|}\right) = f(H)$. It is easy to see that Eq. \cite{93} is same in comparison with Eq. \cite{1} and provides the same solution if and only if

$$\frac{4H^2 f' + \frac{\Gamma}{2}}{2f' - 2AH^2 f''} = H^2 \left(1 - \frac{\Gamma}{3H}\right), \quad (96)$$

or equivalently,

$$H^2 \left(1 - \frac{\Gamma}{3H}\right) \left( \frac{d^2 f}{dH^2} \right) - 2 \left( H \left( \frac{df}{dH} \right) - f \right) = 0. \quad (97)$$

The latter is a linear non-autonomous second-order differential equation. For example, when the particle creation rate is, $\Gamma (H) = mH$, then from Eq. \cite{97} we have the solution

$$f(T) = f_0 \sqrt{|T|} + f_1 T^{\frac{\gamma}{2m}} \quad (98)$$

while for $\Gamma (H)$, given by \cite{91}, $f(T)$ function is given in terms of the Legendre Polynomials. On the other hand starting from a known $f(T)$ model, the solution of the algebraic equation \cite{97} provides us with the function $\Gamma (H)$.

Here, we would like to remark that the evolution of the perfect fluid, with energy density $\dot{\rho}$, will be different with that of the matter creation model with energy density $\dot{\rho}$. However, the total fluid, i.e. the fluid $\rho$, and the fluid components which correspond to $f(T)$-gravity provide us with an effective fluid which has the same evolution with the fluid $\rho$, of the previous sections when Eq. \cite{97} holds.

However, as far as \cite{98} is concerned, since \cite{95} provide us with the same scalar factor with $\Gamma$ for $\Gamma (H) = mH$, or because only the r.h.s of $\cite{98}$ depends only on $m$, then we can say that the constants $f_0, f_1$ are not essential, while $m$ is related with the power of the power-law solution of the scale factor and specifically for $m \neq 3$, it holds that $a(t) = a_0 t^\rho, \quad p = \frac{2}{3 \gamma m} \dot{\rho}$. Of course the equivalence between these two theories is only on the level that
they can provide the same scale factor, which is possible since the two theories have exactly the same degree of freedom, in contrast to \( f(R) \)-gravity which has more degrees of freedom.

6. SUMMARY AND DISCUSSIONS

In the present work, we have addressed several issues concerning the expanding universe powered by adiabatic matter creation. In general, for any cosmological model, the dynamical analysis plays a very important role related to its stability issues. As matter creation models are phenomenological and the literature contains a variety of models, so a generalized model could be a better choice to start with for any study in any context. Hence, in the present work, we have taken a generalized matter creation model as \( \Gamma = -\Gamma_0 + mH + n/H \) (where \( \Gamma_0, m, n \) are real numbers). Then solving the evolution equation described by the Raychaudhuri equation, the model gives ‘two’ fixed points one of which is unstable or repeller in nature (represented by \( H_+ \)) describing the early inflationary phase of the universe, and the other one is a stable or attractor fixed point (represented by \( H_- \)) leading to the present accelerated expansion of the universe asymptotically which is of de Sitter type. In addition two this, the model depicts a non-singular universe. That means it had no big bang singularity in the past. Further, we have shown that, it is possible to find the analytic solutions for such a scenario. Hence, we found a model of a non-singular universe describing two successive accelerated expansions of the universe at early and present times. We then applied the Jacobi Last multiplier method in our framework, and found a Lagrangian which can be taken as an equivalent description to realize such a scenario as we found from the dynamical analysis of the present matter creation model. Also, we have shown that, under a simple condition, Jacobi Last multiplier can give rise to a Lagrangian (see Eq. (10)) which predicts a model of our Universe constituting a cosmological constant and a perfect fluid, which can be realized as a \( \Lambda \)CDM model under certain choice of the parameters involved (see section 3.2). Moreover, we found that the analytic solution for this Lagrangian (Eq. (11)) is an equivalent character with the Brans-Dicke cosmology. Now, performing a joint analysis of Supernovae Type Ia and baryon acoustic oscillation data sets, we constrained the density parameters of the model and hence, the Brans-Dicke parameter.

Now, in order to survey the predicted early accelerated expansion without big bang singularity as produced by our matter creation model, we introduced an equivalent field theoretic description governed by a single scalar field, for the dynamics of the universe supervised by the matter creation mechanism. The prescription established a relation between these two approaches where we were able to produce a complete analytic structure of the field theory, that means it is possible to get explicit analytic expressions for \( \varphi \) and \( V(\varphi) \). Further, introducing the slow roll parameters for this scalar field model, we have calculated the spectral index, its running, and the ratio of tensor to the scalar perturbations, and finally compared with the latest Planck data sets \( \mathcal{S}_2 \) (see table 5) which stay in 95.9\% C.L. Also, we have shown that, it is possible to give a bound on the constant \( \Gamma_0 \) of the matter creation rate that allows us to calculate approximately the reheating and thermalization temperature of the universe.

After that, we have introduced the effects of the teleparallel gravity \( f(T) \) in the matter creation model, and shown that, it is possible to establish an exact functional form of \( f(T) \) for matter creation models.

Finally, one thing it is clear that the present work keeps itself in the domain of cosmology, more specifically in the accelerating cosmology which is a certain plight to understand the evolution of the universe. It seems reasonable that the matter creation mechanism can be studied in several contexts, such as, one can construct an equivalent cosmological theory, for instance, using its equivalence with decaying vacuum models as \( \Lambda(H) = \Gamma H \), established in [60], one may find the corresponding cosmological evolutions driven by decaying vacuum models, and further one may analyze its effect in astrophysical objects, namely, stellar evolution (specifically, in wormhole configuration), gravitational collapse, structure formation of the universe, which can be considered for future works.

7. ACKNOWLEDGMENTS

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