Family Of Rotating Anisotropic Fluid Solutions which Match to Kerr’s Solution

E. Kyriakopoulos
Department of Physics
National Technical University
157 80 Zografou, Athens, GREECE

Abstract

We present a family of exact anisotropic fluid solutions, which satisfy all energy conditions for certain values of their parameters. The components of the energy-momentum tensor of the solutions and its eigenvalues are explicitly given. All members of the family have the ring singularity of Kerr’s solution and most of them one more singularity. The solutions can be matched to the solution of Kerr on a closed surface, which for small values of a parameter approximates an oblate spheroid. The matching surface, which is a thin shell with positive surface density everywhere, is outside the outer infinite red shift surface of our solutions and also outside the outer event horizon of the exterior solution of Kerr.

PACS number(s): 04.20.-q, 04.20.Jb
Keywords: Exact anisotropic fluid solutions, Matching to Kerr’s solution

Local isotropy is usually assumed in general relativity. However some phenomena seem to lead to local anisotropy. Since the remarkable work of Browers and Liang [1], which refers to the non-rotating case, the influence of local anisotropy in general relativity has been extensively studied [2]. This study was extended to the case of rotation [3].

In astrophysics it is very important to have a metric for the interior of a rotating star, which should match to the solution of Kerr [4], since we usually take the solution of Kerr as exterior solution. Many people tried to find interior solutions but what they found had in general some problem. For example the matching was approximate [5]-[7]. It was found that regular disks as sources of the Kerr metric give energy-momentum tensors which do

---

1E-mail: kyriakop@central.ntua.gr
not satisfy the dominant energy conditions \[8\]. The question if a perfect fluid can be the source of the metric of Kerr is an open one \[9,10\].

In this paper we present a family of rotating anisotropic fluid solutions, which depends on five parameters including the parameters of mass and angular momentum and which for certain values of these parameters satisfy the dominant the strong and of course weak energy conditions. The non-vanishing components of the Riemann tensor \( R_{\mu\nu} \) and the eigenvalues of \( R_\nu^\mu \) are given explicitly. All solutions are singular on a ring lying in the equatorial plane [the ring singularity of Kerr’s solution], while most of them have one more singularity. The infinite red shift surfaces of the solutions are given.

The solutions of the family can be matched to the solution of Kerr on a surface \( S \), which for small values of a parameter approximates an oblate spheroid. Since one component of the second fundamental form is different from zero the surface \( S \) is a thin shell. The surface energy tensor \( S_{ij} \) is computed and from this the surface density is calculated and found to be positive for certain values of a parameter. Finally it is found that the surface \( S \) is outside the outer infinite red shift surface of the family of solutions and also outside the outer horizon of Kerr’s exterior solution for some values of the parameters of the solution.

We consider a metric which in Boyer-Lindquist coordinates [11] has the form

\[
g_{\mu\nu} = \left[ \begin{array}{cccc}
-T, 0, 0, -a(1 - x^2)(1 - T) \\
0, \frac{f(r, x)}{\rho^2 T + a^2 (1 - x^2)}, 0, 0 \\
0, f(r, x), 0, -a(1 - x^2)(1 - T), 0, 0, (1 - x^2)(r^2 + a^2 x^2 + a^2 (1 - x^2)(2 - T)) \\
0, 0, 0, 0, 0, 0, 0
\end{array} \right]
\]

(1)

where \( a \) is an arbitrary constant,

\[
x = \cos \theta, \quad \rho^2 = r^2 + a^2 x^2, \quad T = 1 + \frac{h(r)}{\rho^2}
\]

(2)

and \( h(r) \) and \( f(r, x) \) are functions to be determined. The above form of \( g_{\mu\nu} \) approaches the general form of a metric we have considered before [12] and becomes Kerr’s metric for \( h(r) = -2Mr \) and \( f(r, x) = \rho^2 \). Starting from a metric of the above form we calculate [13] the Ricci tensor \( R_{\mu\nu} \) and the expressions from which the eigenvalues of the matrix \( R_\nu^\mu \) are obtained. To get an anisotropic fluid solution the eigenvalues \( \lambda_\theta \) and \( \lambda_\phi \) of the matrix \( R_\nu^\mu \) must be equal. Imposing the condition \( \lambda_\theta = \lambda_\phi \) we calculate [14] the functions \( h(r) \)
and \( f(r, x) \). We find the family of solutions

\[
\begin{align*}
  h(r) &= -2Mr + M^2 - a^2 + k, \quad f(r, x) = b\rho^2 \left\{ (r - M)^2 + kx^2 \right\}^c \\
\end{align*}
\]  

(3)

where \( M, k, b \) and \( c \) are arbitrary constants. The nonzero components of \( R_{\mu\nu} \) of this family are the following \[13\]

\[
\begin{align*}
  R_{tt} &= \frac{[(r - M)^2 + a^2(1 - x^2) + k](M^2 - a^2 + k)}{\rho^2}N \\
  R_{rr} &= \frac{2ck\rho^2 - [(r - M)^2 + k](M^2 - a^2 + k)}{\rho^2[(r - M)^2 + k]^2} \\
  R_{\theta\theta} &= \frac{M^2 - a^2 + k}{\rho^2} \\
  R_{\phi\phi} &= \frac{\{(r^2 + a^2)^2 + a^2[(r - M)^2 + k](1 - x^2)](1 - x^2)(M^2 - a^2 + k)}{\rho^2}N \\
  R_{t\phi} &= -\frac{a(r^2 + a^2 + (r - M)^2 + k)(1 - x^2)(M^2 - a^2 + k)}{\rho^2}N
\end{align*}
\]  

(4) - (8)

where

\[
N = \frac{1}{b(\rho^2)^2} \left\{ \frac{(r - M)^2 + kx^2}{(r - M)^2 + k} \right\}^{-c}
\]  

(9)

The eigenvalues of the matrix \( [(R^t_t, R^t_\phi), (R^t_\phi, R^\phi_\phi)] \) are

\[
\lambda_\pm = \pm(M^2 - a^2 + k)N
\]  

(10)

and the other eigenvalues \( \lambda_r \) and \( \lambda_\theta \) of the matrix \( R^\mu_\nu \) are

\[
\begin{align*}
  \lambda_r &= \frac{2ck\rho^2 - [(r - M)^2 + k](M^2 - a^2 + k)}{(r - M)^2 + k}N \\
  \lambda_\theta &= \lambda_+ = (M^2 - a^2 + k)N
\end{align*}
\]  

(11) - (12)

Calculating the eigenvectors of the matrix \( R^\mu_\nu \) we find that the timelike eigenvector \( (u_t)^\mu \) corresponds to the eigenvalue \( \lambda_- \) that is

\[
\lambda_t = \lambda_- = -(M^2 - a^2 + k)N
\]  

(13)
and its normalized form is

\[(u_t)^\mu = \frac{1}{\sqrt{\rho^2((r-M)^2 + k)}}\{(r^2 + a^2)\delta^\mu_t + a\delta^\mu_\phi}\] (14)

(if we choose + as overall sign). Then the eigenvector \((u_\phi)^\mu\) corresponds to the eigenvalue \(\lambda_\phi\), that is

\[\lambda_\phi = \lambda_t = (M^2 - a^2 + k)N\] (15)

and its normalized form is

\[(u_\phi)^\mu = \frac{1}{\sqrt{\rho^2(1 - x^2)}}\{a(1 - x^2)\delta^\mu_t + \delta^\mu_\phi\} \] (16)

Also the normalized eigenvectors \((u_r)^\mu\) and \((u_\theta)^\mu\), which correspond to the eigenvalues \(\lambda_r\) and \(\lambda_\theta\) respectively, are

\[(u_r)^\mu = \sqrt{\left[(r - M)^2 + k\right]\rho^2 N\delta^\mu_r} \text{ and } (u_\theta)^\mu = \sqrt{\rho^2 N\delta^\mu_\theta} \] (17)

The Ricci scalar \(R\) of our solution is [13]

\[R = \frac{2ck\rho^2}{(r - M)^2 + k} N = \frac{2ck[(r - M)^2 + k]^{c-1}}{b\rho^2[(r-M)^2 + kx^2]^c} \] (18)

The eigenvalues \(w_i = \lambda_i - R/2, i = t, r, \theta, \phi\) of the energy-momentum tensor \(T^\mu_\nu = R^\mu_\nu - \frac{R}{2}\delta^\mu_\nu\) are calculated from Eqs (11)-(13), (15) and (18). We get

\[w_t = -\frac{(M^2 - a^2 + k)[(r - M)^2 + k] + ck\rho^2}{(r - M)^2 + k} N \] (19)

\[w_r = -\frac{(M^2 - a^2 + k)[(r - M)^2 + k] - ck\rho^2}{(r - M)^2 + k} N \] (20)

\[w_\theta = w_\phi = \frac{(M^2 - a^2 + k)[(r - M)^2 + k] - ck\rho^2}{(r - M)^2 + k} N \] (21)

The energy density \(\mu\) is given by the relation

\[\mu = -w_t = \frac{(M^2 - a^2 + k)[(r - M)^2 + k] + ck\rho^2}{(r - M)^2 + k} N \] (22)
If we define A and B by the relation

\[ A = (M^2 - a^2 + k)N \quad \text{and} \quad B = \frac{ck\rho^2}{(r - M)^2 + k}N \quad (23) \]

we get

\[ \mu = A + B, \quad w_r = -A + B, \quad w_\theta = w_\phi = A - B \quad (24) \]

It is easy to find that if

\[ A \geq 0 \quad \text{and} \quad B \geq 0 \quad (25) \]

the above expressions for \( \mu, w_r, w_\theta \) and \( w_\phi \) satisfy the weak, the dominant and the strong energy conditions [15]. Relations (25) are satisfied, for example if \((r - M)^2 + kx^2 > 0\) and

\[ bck \geq 0 \quad \text{and} \quad b(M^2 - a^2 + k) \geq 0 \quad (26) \]

From Eqs (14)-(16) we can calculate the normalized eigenvectors \((u_t)_\mu, (u_\phi)_\mu, (u_r)_\mu\) and \((u_\theta)_\mu\). We find

\[ (u_t)_\mu = \frac{(r - M)^2 + k}{\sqrt{\rho^2[(r - M)^2 + k]}} \left\{ -\delta_{t\mu} + a(1 - x^2)\delta_{\phi\mu} \right\} \quad (27) \]

\[ (u_\phi)_\mu = \frac{1 - x^2}{\sqrt{\rho^2(1 - x^2)}} \left\{ -a\delta_{t\mu} + (r^2 + a^2)\delta_{\phi\mu} \right\} \quad (28) \]

\[ (u_r)_\mu = \frac{1}{\sqrt{[(r - M)^2 + k]\rho^2 N}} \delta_{r\mu} \quad \text{and} \quad (u_\theta)_\mu = \frac{1}{\sqrt{\rho^2 N}} \delta_{\theta\mu} \quad (29) \]

The energy-momentum tensor \( T_{\mu\nu} \) of the solution (1)-(3) can be calculated from Eqs (4)-(9), (18), (20)-(22) and (27)-(29). This tensor can be written in the form

\[ T_{\mu\nu} = (\mu + w_\perp)(u_t)_\mu(u_t)_\nu + w_\perp g_{\mu\nu} + (w_\parallel - w_\perp)(u_r)_\mu(u_r)_\nu \quad (30) \]

where

\[ w_\perp = w_\theta = w_\phi, \quad \text{and} \quad w_\parallel = w_r \quad (31) \]

The above \( T_{\mu\nu} \) is the energy-momentum tensor of an anisotropic fluid [2].
The infinite red shift surfaces of our family of solutions are obtained from the relation \( g_{tt} = 0 \). From this relation we get

\[
r_{\pm}^{RS} = M \pm \sqrt{a^2(1 - x^2) - k}
\]  

(32)

For \( k < 0 \) the above surfaces are closed and axially symmetric.

The solutions have irremovable singularities at the points at which at least one of the invariants \( R \) and \( R^2 = R_{\mu
u\zeta\eta}R^{\mu
u\zeta\eta} \) is singular. The Ricci scalar \( R \) is given by Eq. \[18\], while the curvature scalar \( R^2 \) was computed for certain positive and negative small values of \( c \) \[13\]. For all \( c \) for which \( R^2 \) was computed it was found that all singularities of \( R^2 \) are also singularities of \( R \). Therefore we shall determine the singularities of the solutions from Eq. \[18\]. From this expression it is obvious that for any \( c \) we have an irremovable singularity at

\[
\rho^2 = r^2 + a^2 x^2 = 0
\]

(33)

which is the well known ring singularity of Kerr’s solution \[16\]. For \( c < 0 \) no other singularities exist if \( k > 0 \), while if \( k < 0 \) the solutions are singular also if \( (r - M)^2 + k = 0 \) that is if \( r = M \pm \sqrt{-k} \). If \( c > 0 \) the solutions are in addition singular if \( (r - M)^2 + kx^2 = 0 \), which for \( k > 0 \) is satisfied for \( r = M \) and \( x = 0 \), and for \( k < 0 \) for \( r = M \pm \sqrt{-k}x \).

We shall examine now if our solutions can be matched to the solution of Kerr on a closed surface \( S \) with Kerr’s solution as exterior solution. We express the solution of Kerr in Boyer-Lindquist coordinates \[11\], since our family of solutions is expressed in such coordinates. We assume that the coordinates \( r \) and \( \theta \) of the various points of \( S \) are not independent but that they can be expressed by a single parameter \( \tau \). Therefore \( S \) has the coordinates \( \zeta^i = (t, \tau, \phi) \), while the spacetime on which it is embedded has coordinates \( x^\alpha = (t, r, \theta, \phi) \). Eliminating \( \tau \) we get for \( S \) an equation of the form \( r = R(\theta) \). It is easy to show that the 3-metric \( ^3g_{ij} \) of the surface is connected with the 4-metric \( ^4g_{\alpha\beta} \) of the spacetime by the relation

\[
^3g_{ij} = \frac{\partial x^\alpha}{\partial \zeta^i} \frac{\partial x^\beta}{\partial \zeta^j} ^4g_{\alpha\beta}
\]

(34)

where Latin indices take the values \( (t, \tau, \phi) \) and Greek indices the values \( (t, r, \theta, \phi) \). For any metric dependent quantity \( P \) we must specify the region in which it is calculated. The notation \( P^+(S) \) means that \( P \) is calculated in the exterior (interior) region of \( S \). The notation \( P^+|_S (P^-|_S) \) means that the
quantity $P$ is calculated in the exterior (interior) region and evaluated at the surface, while we use the notation

$$[P] \equiv P^+|_S - P^-|_S$$

which means that $[P]$ denotes the discontinuity of $P$ at the surface.

The Darmois-Israel conditions [17],[18] for the matching of the interior and exterior regions are continuity of the first fundamental form

$$[^3 g_{ij}] = 0$$

and continuity of the extrinsic curvature $K_{ij}$ (second fundamental form)

$$[K_{ij}] = 0$$

If both conditions are satisfied we refer to $S$ as boundary surface. If only condition (36) is satisfied we refer to $S$ as thin shell. If the only off diagonal term of the metrics to be joined is $g_{t\phi}$ condition (36) implies the relations [19]

$$[^3 g_{tt}] = [^4 g_{tt}] = 0, \quad[^3 g_{t\phi}] = [^4 g_{t\phi}] = 0$$

$$[^3 g_{rr}] = \left(\frac{\partial r}{\partial \tau}\right)^2[^4 g_{rr}] + \left(\frac{\partial \theta}{\partial \tau}\right)^2[^4 g_{\theta\theta}] = 0$$

Eqs (38) are satisfied if

$$k = a^2 - M^2$$

while Eq. (39) is satisfied if as matching surface $S$ we chose the surface

$$b\left\{\left(\frac{r^S - M}{r^S - M} + kx^2\right)^\frac{1}{2} - 1\right\} = 0$$

or the surface

$$r^S - M = \sqrt{M^2 - a^2} \cos(\theta - \theta_0)$$

where $\theta_0$ is an arbitrary constant. Assuming that $a^2 < M^2$ and $\sqrt{b} < 1$ and writing

$$y = \frac{a^2}{M^2} < 1 \quad \text{and} \quad v = \sqrt{b} < 1$$

we get from Eq. (41) $r^S - M = \pm M\sqrt{\frac{1-v}{1-v^2}\sqrt{1-vx^2}}$. In the following we shall choose the surface

$$r^S - M = M\sqrt{\frac{1-y}{1-v}\sqrt{1-vx^2}}$$
as matching surface. In Boyer-Lindquist coordinates \([11]\) an equation of the form \(r = \text{constant}\) is an oblate spheroid \([3]\). Therefore if \(v \ll 1\) our matching surface approximates an oblate spheroid.

Condition \((37)\) implies the relations \([19], [20]\)

\[
[K_{tt}] = [g^{rr}]g_{tt,r} - [g^{\theta\theta}]R_{,\theta}g_{tt,\theta} = 0 
\]

\[
[K_{t\phi}] = [g^{rr}]g_{\phi,r} - [g^{\theta\theta}]R_{,\theta}g_{t\phi,\theta} = 0 
\]

\[
[K_{\phi\phi}] = [g^{rr}]g_{\phi,r} - [g^{\theta\theta}]R_{,\theta}g_{\phi,\phi} = 0 
\]

\[
[K_{rr}] = \frac{1}{2}(\frac{\partial r}{\partial r})^2\{[g^{rr}g_{rr,r}] + R_{,\theta}[g^{\theta\theta}g_{rr,\theta}]\} + \frac{\partial r}{\partial r} \frac{\partial \theta}{\partial r} \{[g^{rr}g_{rr,\theta}] - R_{,\theta}[g^{\theta\theta}g_{\theta,\theta}]\}
\]

\[
- \frac{1}{2}(\frac{\partial \theta}{\partial r})^2\{[g^{rr}g_{\theta,\theta}] + R_{,\theta}[g^{\theta\theta}g_{\theta,\theta}]\} = 0 
\]

where all metric components refer to the 4-metric and we have used the notation \(P, a = \frac{\partial P}{\partial x^a}\). Eqs \([K_{tt}] = [K_{t\phi}] = 0\) are identically satisfied. For \(k\) and \(S\) given by Eqs \((40)\) and \((41)\) respectively we find that

\[
[K_{tt}] = [K_{t\phi}] = [K_{\phi\phi}] = 0
\]

Also taking \(\tau = \theta\) we find that

\[
[K_{rr}] = [K_{\theta\theta}] = -\frac{cM\sqrt{(1-y)(1-v)}}{\sqrt{(1-vx^2)^3}}
\]

Therefore \(S\) is a thin shell.

To calculate the surface energy tensor \(S^j_i\) we use the Lanczos relation \([21], [22]\)

\[
8\pi S^j_i = [K_{il}]^3 g^{lj} - \delta^j_l ([K_{ln}]^3 g^{ln})
\]

For \(^3g_{ij}\) and \(^3K_{ij}\) given by Eqs \((34), (49)\) and \((50)\) we find that the only non-vanishing \(S^j_i\) are the following

\[
S^t_t = S^\phi_\phi = -\frac{1}{8\pi}[K_{\theta\theta}]^3 g^{\theta\theta} = -\frac{1}{8\pi}g^{\theta\theta} + \frac{1}{8\pi} \frac{1}{g^{rr}g_{rr,\theta}^2}[K_{\theta\theta}]
\]

The surface density \(\sigma(\theta)\) is defined by the eigenvalue equation

\[
S^b_a u^b = -\sigma u^a \quad \text{with} \quad u_a u^a = -1
\]
From Eqs (52) and (53) we get

\[
\sigma(\theta) = \frac{1}{8\pi} 4 g_{\theta\theta} + 4 g_{rr}(\frac{\partial r}{\partial \theta})^2 [K_{\theta\theta}] = -\frac{cM}{8\pi\rho^2} \sqrt{\frac{(1 - y)(1 - v)}{1 - vx^2}}
\]

\[
= -\frac{c}{8\pi M} \left\{ \left(1 + \frac{(1 - y)(1 - vx^2)}{1 - v}\right)^{\frac{1}{2}} + yx^2 \right\}^{-1} \sqrt{\frac{(1 - y)(1 - v)}{1 - vx^2}} \quad (54)
\]

Therefore the surface density is positive everywhere if \(c < 0\). From Eqs (32), (40), (43) and (44) we find that if \(y < v\) the outer infinite red shift surface of our solution satisfies the relation

\[
\frac{r^S - M}{r^{RS} - M} = \left(\frac{1 - y}{1 - v}\right)^{\frac{1}{2}} \left(\frac{1 - vx^2}{1 - yx^2}\right)^{\frac{1}{2}} \geq 1
\]

(55)

Also for the outer event horizon of Kerr’s solution given by the relation

\[
r^H - M = (M^2 - a^2)^{\frac{1}{2}} = M(1 - y)^{\frac{1}{2}}
\]

we get

\[
\frac{r^S - M}{r^H - M} = \left(\frac{1 - vx^2}{1 - v}\right)^{\frac{1}{2}} \geq 1
\]

(56)

Therefore the matching surface is outside the outer infinite red shift surface of our family of solutions and also outside the outer event horizon of Kerr’s exterior solution. If we had chosen as matching surface the surface \(r^S - M = -M \sqrt{\frac{1 - y}{1 - v}} \sqrt{1 - vx^2}\) or the surface \(r^S - M = \sqrt{M^2 - a^2} \cos(\theta - \theta_0) = M \sqrt{1 - y} \cos(\theta - \theta_0)\), the outer red shift surface and the outer event horizon would be outside the matching surface.

References

[1] Bowers R. W. and Liang E. P. T.: Astrophys. J. 188, 657 (1974)
[2] Herrera L. and Santos N. O.: Phys Rep 286, 53 (1997)
[3] Gurses M. and Gursey F.: J Math Phys 16, 2385 (1975)
[4] Kerr R. P.: Phys. Rev. Lett. 11, 237 (1963)
[5] Cohen J. M.: J. Math. Phys. 8, 1477 (1967)
[6] Florides P. S.: Nuovo Cimento B 13 1 (1973)
[7] Florides P. S.: Nuovo Cimento B 25, 251 (1975)

[8] Hamilty V. H. and Lamberti W.: Gen. Gelativ. Gravit. 19, 917 (1987)

[9] Krasinski A.: Ann. Phys.(NY), 112, 22 (1978). In this paper a concise account of the attempts to construct an interior to the solution of Kerr, which took place before 1976, is given.

[10] McManus D.: Class Quantum Grav 18, 863 (1991)

[11] Boyer R. N. and Lindquist R. W.: J. Math. Phys. 8, 265 (1967)

[12] Kyriakopoulos E.: Gen. Gelativ. Gravit. 44, 157 (2012)

[13] The calculation was done with the help of a program given to me from Dr. S. Bonanos, whom I thank. The program can be found in www.democritos.gr/~sbonano/RGTC/.

[14] Details of these long calculations will be presented elsewhere

[15] S. W. Hawking and G. F. R. Ellis: The Large Scale Structure of Space-Time (Cambridge University Press, Cambridge 1973)

[16] The ring singularity also appears in the papers of Viaggiu S.: Int. J. Mod. Phys. D15, 1441 (2006); Int. J. Mod. Phys. D19, 1783 (2010)

[17] Dormois G.: Memorial de Sciences Mathematiques,Fascicule XXV

[18] Israel W.: Nuovo Cimento B 44 1 (1966) : Erratum Nuovo Cimento B 48, 463 (1967)

[19] Drake S. P. and Turolla R.: Class Quantum Grav 14, 1883 (1997)

[20] Since the right hand side of Eqs (45)-(48) is zero a positive normalization factor of the unit vector, with the covariant derivative of which we define the extrinsic curvature, is omitted. We shall omit this factor everywhere in the paper.

[21] Misner C. W. Thorne K. S. and Wheeler J. A.: Gravitation (W. H. Freeman and Company, San Francisco 1973) p. 553

[22] Lanczos C. Physik Z. 23, 529 (1922); Ann. Physik 74, 518 (1924)