THREE-DIMENSIONAL GRAPH PRODUCTS WITH UNBOUNDED STACK-NUMBER

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Abstract. We prove that the stack-number of the strong product of three \(n\)-vertex paths is \(\Theta(n^{1/3})\). The best previously known upper bound was \(O(n)\). No non-trivial lower bound was known. This is the first explicit example of a graph family with bounded maximum degree and unbounded stack-number.

The main tool used in our proof of the lower bound is the topological overlap theorem of Gromov. We actually prove a stronger result in terms of so-called triangulations of Cartesian products. We conclude that triangulations of three-dimensional Cartesian products of any sufficiently large connected graphs have large stack-number.

The upper bound is a special case of a more general construction based on families of permutations derived from Hadamard matrices.

The strong product of three paths is also the first example of a bounded degree graph with bounded queue-number and unbounded stack-number. A natural question that follows from our result is to determine the smallest \(\Delta_0\) such that there exist a graph family with unbounded stack-number, bounded queue-number and maximum degree \(\Delta_0\). We show that \(\Delta_0 \in \{6, 7\}\).

1. Introduction

Stack layouts are ubiquitous objects at the intersection of combinatorics, geometry and topology with applications in computational complexity [14, 15, 26, 34], RNA folding [37], graph drawing in two [5, 55] and three dimensions [58], traffic light scheduling [44], and fault-tolerant multiprocessing [19, 54].

For a graph\(^1\) \(G\) and \(s \in \mathbb{N}_0\), an \(s\)-stack layout of \(G\) consists of an ordering \((v_1, \ldots, v_n)\) of \(V(G)\) together with a function \(\phi: E(G) \to \{1, \ldots, s\}\) such that for all edges \(v_i v_j, v_k v_\ell \in E(G)\) with \(i < k < j < \ell\) we have \(\phi(v_i v_j) \neq \phi(v_k v_\ell)\); see Figure 1 for an example. Each set \(\phi^{-1}(k)\) is called a stack. Edges in a stack do not cross with respect to \((v_1, \ldots, v_n)\), and therefore behave in a LIFO manner (hence the name). Stack layouts can also be viewed topologically via embeddings into books (first defined by Ollmann [49]). The stack-number \(\text{sn}(G)\) of a graph \(G\) is the minimum \(s \in \mathbb{N}_0\) for which there exists an \(s\)-stack layout of \(G\) (also known as page-number, book-thickness or fixed outer-thickness).

Figure 1. A 4-stack layout of the strong product \(P_5 \boxtimes P_5\) of two paths.

\(^{1}\)All graphs in this paper are simple and, unless explicitly stated otherwise, undirected and finite. Let \(\mathbb{N} = \{1, 2, \ldots\}\) and \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\).
Stack layouts have been studied for planar graphs [9, 17, 40, 61, 62], graphs of given genus [31, 41, 47], treewidth [29, 30, 35, 57], minor-closed graph classes [12], 1-planar graphs [1, 7, 8, 16], and graphs with a given number of edges [48], amongst other examples.

This paper studies stack layouts of 3-dimensional products. As illustrated in Figure 2, for graphs $G_1$ and $G_2$, the Cartesian product $G_1 \square G_2$ is the graph with vertex-set $V(G_1) \times V(G_2)$ with an edge between two vertices $(x, y)$ and $(x', y')$ if $x = x'$ and $yy' \in E(G_2)$, or $y = y'$ and $xx' \in E(G_1)$. The strong product $G_1 \boxtimes G_2$ is the graph obtained from $G_1 \square G_2$ by adding edges $(x, y)(x', y')$ and $(x, y')(x', y)$ for all edges $xx' \in E(G_1)$ and $yy' \in E(G_2)$.

Since the Cartesian and strong products are associative, we may write $G_1 \square G_2 \square G_3$ and $(G_1 \boxtimes G_2) \boxtimes G_3$ (identifying pairs of the forms $((v_1, v_2), v_3)$ and $(v_1, (v_2, v_3))$ with the triple $(v_1, v_2, v_3)$).

Let $P_n$ denote the $n$-vertex path. Our first main result is the following tight bound on the stack-number of the strong product of three paths (the 3-dimensional grid plus crosses).

**Theorem 1.** $\text{sn}(P_n \boxtimes P_n \boxtimes P_n) \in \Theta(n^{1/3})$.

Note that $(P_n \boxtimes P_n) \square P_n$ and $(P_n \square P_n) \boxtimes P_n$ both have bounded stack-number, as we now explain. Bernhart and Kainen [10] showed that if $G_1$ and $G_2$ are graphs with bounded stack-number and $G_1$ is bipartite with bounded maximum degree, then the stack-number of $G_1 \square G_2$ is bounded. Pupyrev [53] showed that if additionally $G_2$ has bounded pathwidth, then the stack-number of $G_1 \boxtimes G_2$ is also bounded. These results imply that $(P_n \boxtimes P_n) \square P_n$ and $(P_n \square P_n) \boxtimes P_n$ indeed have bounded stack-number. This shows that in Theorem 1, we cannot replace even one ‘strong product’ by a ‘Cartesian product’.

![Figure 2](image-url)
No non-trivial lower bound on $\text{sn}(P_n \boxtimes P_n \boxtimes P_n)$ was previously known. Indeed, Theorem 1 provides the first explicit example of a graph family with bounded maximum degree and unbounded stack-number. Malitz [48] first proved that graphs of maximum degree 3 have unbounded stack-number (using a probabilistic argument). Further motivation for Theorem 1 is provided in Section 2 where we present various connections to related graph parameters, shallow/small minors and growth.

We now discuss the lower bound in Theorem 1. We actually prove a stronger result that depends on the following definitions. For graphs $G_1$ and $G_2$, a triangulation of $G_1 \sqcup G_2$ is any graph obtained from $G_1 \sqcup G_2$ by adding the edge $(x, y)(x', y')$ or $(x, y)(x', y')$ for each $xx' \in E(G_1)$ and $yy' \in E(G_2)$. A triangulation of $G_1 \sqcup G_2 \sqcup G_3$ is any graph obtained by triangulating all subgraphs induced by sets of the form $\{v_1\} \times V(G_2) \times V(G_3)$, $V(G_1) \times \{v_2\} \times V(G_3)$ and $V(G_1) \times V(G_2) \times \{v_3\}$ with $v_i \in V(G_i)$; see Figure 2(c) for an example.

For directed graphs $G_1$ and $G_2$, if $U_3$ is the undirected graph underlying $G_3$, then let $G_1 \boxtimes G_2$ be the triangulation of $U_1 \sqcup U_2$ containing the edge $(x, y)(x', y')$ for all directed edges $(x, x') \in E(G_1)$ and $(y, y') \in E(G_2)$. Similarly, for directed graphs $G_1$, $G_2$, and $G_3$, let $G_1 \boxtimes G_2 \boxtimes G_3$ be the appropriate triangulation of $U_1 \sqcup U_2 \sqcup U_3$. When using the notation $G_1 \boxtimes G_2$ or $G_1 \boxtimes G_2 \boxtimes G_3$, if some $G_i$ is a path $P_n$, then $P_n$ is a directed path.

Every triangulation of $G_1 \sqcup G_2$ is a subgraph of $G_1 \boxtimes G_2$ and every triangulation of $G_1 \sqcup G_2 \sqcup G_3$ is a subgraph of $G_1 \boxtimes G_2 \boxtimes G_3$. So the next result immediately implies the lower bound in Theorem 1.

**Theorem 2.** Let $T_1$, $T_2$ and $T_3$ be $n$-vertex trees with maximum degree $\Delta_1$, $\Delta_2$ and $\Delta_3$ respectively, where $\Delta_1 \geq \Delta_2 \geq \Delta_3$. Then for every triangulation $G$ of $T_1 \sqcup T_2 \sqcup T_3$,

$$\text{sn}(G) \in \Omega\left(\left(\frac{n}{\Delta_1 \Delta_2}\right)^{1/3}\right).$$

Theorem 2 is similar in spirit to a recent result of Dujmović, Eppstein, Hickingbotham, Morin, and Wood [22], who showed that if $S_n$ is the $n$-leaf star, then

$$\text{sn}(S_n \sqcup (P_n \boxtimes P_n)) \in \Omega(\sqrt{\log \log n}).$$

Their proof actually establishes the following result (since the Hex Lemma holds for any triangulation of $P_n \boxtimes P_n$; see [60] for example).

**Theorem 3 ([22]).** For every triangulation $G$ of $P_n \sqcup P_n$,

$$\text{sn}(S_n \sqcup G) \in \Omega(\sqrt{\log \log n}).$$

Theorem 2 has the advantage over Theorem 3 in that it applies to bounded degree graphs (for example when each $T_i$ is a path). Moreover, the lower bound in Theorem 1 (as a function of the number of vertices) is much stronger than the lower bound of Theorem 3.

We prove Theorem 2 by relating the stack-number of a graph to the topological properties of its triangle complex. The triangle complex of a graph $G$, denoted by $\text{Tr}(G)$, is the geometric 2-dimensional simplicial complex with 0-faces corresponding to vertices of $G$, 1-faces corresponding to edges of $G$, and 2-faces corresponding to triangles of $G$. For topological spaces $X$ and $Y$, define

$$\text{overlap}(X, Y) = \min_{f \in C(X, Y)} \max_{p \in Y} |f^{-1}(p)|,$$

where $C(X, Y)$ denotes the space of all continuous functions $f : X \to Y$. In Sections 3 and 4, respectively, we prove the following two lemmas.

**Lemma 4.** For all $n$-vertex trees $T_1$, $T_2$ and $T_3$, and for every triangulation $G$ of $T_1 \sqcup T_2 \sqcup T_3$,

$$\text{overlap}(\text{Tr}(G), \mathbb{R}^2) \in \Omega(n).$$
Corollary 6. For every graph $G$ such that every vertex is in at most $c$ triangles, 
\[
\text{sn}(G) \geq \left( \frac{\text{overlap}(\text{Tr}(G), \mathbb{R}^2)}{3c} \right)^{1/3}
\]

Theorem 2 (and thus the lower bound in Theorem 1) follows from Lemmas 4 and 5 since (using the notation from Theorem 2) each vertex of $G$ is in at most $2(\Delta_1 \Delta_2 + \Delta_1 \Delta_3 + \Delta_2 \Delta_3) \leq 6\Delta_1 \Delta_2$ triangles. So Lemma 5 is applicable with $c = 6\Delta_1 \Delta_2$.

The lower bound on $\text{sn}(G)$ in Theorem 2 is non-trivial only if $\Delta_1 \Delta_2 \in o(n)$. Nevertheless, we have the following result with no assumption on the maximum degree. Theorems 2 and 3 imply that the stack-numbers of triangulations of $G$ implies that the stack-numbers of triangulations of $P_n \boxplus P_n \boxplus P_n$ and $S_n \boxplus P_n \boxplus P_n$ grow with $n$. Moreover, $S_n \boxplus S_n$ contains a subgraph isomorphic to a 1-subdivision of $K_{n,n}$, so its stack-number grows with $n$ as well (see [13, 32]). Since every sufficiently large connected graph contains a copy of $P_n$ or $S_n$, we deduce the following.

Corollary 6. For every $s \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for all $n$-vertex connected graphs $G_1$, $G_2$ and $G_3$, if $G$ is any triangulation of $G_1 \boxplus G_2 \boxplus G_3$, then $\text{sn}(G) > s$.

The best previously known upper bound on $\text{sn}(P_n \boxplus P_n \boxplus P_n)$ was $O(n)$, which follows from Theorem 1 of Dujmović, Morin, and Yelle [24] or from Corollary 1 of Pupyrev [53]. The upper bound in Theorem 1 follows from a more general result in Section 5, which says that $\text{sn}(G \boxplus P_n) \in O(n^{1/2-\epsilon})$ for some $\epsilon > 0$ whenever the graph $G$ has bounded stack-number and bounded maximum degree. The proof is based on families of permutations derived from Hadamard matrices.

Our final results concern maximum degree. Theorem 2 implies that $(P_n \boxplus P_n \boxplus P_n)_{n \in \mathbb{N}}$ is a family of graphs with maximum degree 12, unbounded stack-number and bounded queue-number (defined in Section 2). It is natural to ask what is the smallest bound on the maximum degree in such a family. We prove the answer is 6 or 7.

Theorem 7. The least integer $\Delta_0$ such that there exists a graph family with maximum degree $\Delta_0$, unbounded stack-number and bounded queue-number satisfies $\Delta_0 \in \{6, 7\}$.

The proof of the upper bound in Theorem 7 uses the same topological machinery used to prove Theorem 2, and is based on the tesselation of $\mathbb{R}^3$ by truncated octahedra. The proof of the lower bound exploits a connection with clustered colouring.

2. Connections

This section provides further motivation for our results by discussing connections with related graph parameters, minors and growth.

Consider a graph $G$. The geometric thickness of $G$ is the minimum $k \in \mathbb{N}_0$ for which there is a straight-line drawing of $G$ and a partition of $E(G)$ into $k$ plane subgraphs; see [4, 21, 29]. A $k$-stack layout of $G$ defines such a drawing and edge-partition (with the vertices drawn on a circle in the order given by the stack layout). Thus the geometric thickness of $G$ is at most its stack-number. The slope-number of $G$ is the minimum $k \in \mathbb{N}_0$ for which there is a straight-line drawing of $G$ with $k$ distinct edge slopes; see [3, 4, 23, 27, 45, 50]. Since edges of the same slope do not cross, the geometric thickness of $G$ is at most its slope-number.

Note that $P_n \boxplus P_n \boxplus P_n$ has slope-number and geometric thickness at most 6 (simply project the natural 3-dimensional representation to the plane). Hence Theorem 1 provides a family of graphs with bounded slope-number, bounded geometric thickness, and unbounded stack-number. Eppstein [32] previously constructed a graph family with bounded geometric thickness and unbounded stack-number, but not with bounded slope-number (since the graphs in question have unbounded maximum degree).

For a graph $G$ and $q \in \mathbb{N}_0$, a $q$-queue layout of $G$ consists of an ordering $(v_1, \ldots, v_n)$ of $V(G)$ together with a function $\phi : E(G) \rightarrow \{1, \ldots, q\}$ such that for all edges $v_i v_j, v_k v_l \in E(G)$ with $i < k < l < j$ we have $\phi(v_i v_j) \neq \phi(v_k v_l)$. Each set $\phi^{-1}(k)$ is called a queue.
Edges in a queue do not nest with respect to \((v_1,\ldots,v_n)\), and therefore behave in a FIFO manner (hence the name). The queue-number \(\text{qn}(G)\) of a graph \(G\) is the minimum \(q \in \mathbb{N}_0\) for which there exists a \(q\)-queue layout of \(G\).

Stack and queue layouts are considered to be dual to each other [42]. However, for many years it was open whether there is a graph family with bounded queue-number and unbounded stack-number, or bounded stack-number and unbounded queue-number. Theorem 3 by Dujmović et al. [22] resolved the first question, since they also showed that \(\text{qn}(S_n \boxtimes (P_n \boxplus P_n)) \leq 4\). Observe that a \(4\)-queue layout of \(P_n \boxplus P_n \boxplus P_n\) can be obtained by taking the lexicographical ordering \((v_1,\ldots,v_n)\) of the vertices and letting \(\phi(v_i,v_j)\) be determined by which of the sets \(\{1\}, \{n,n+1\}, \{n^2,n^2+1\}, \{n^2+n\}\) contains \(|i-j|\). By Theorem 1, \((P_n \boxplus P_n \boxplus P_n)_{n \in \mathbb{N}}\) is therefore another graph family with queue-number at most 4 and unbounded stack-number.

We now discuss the behaviour of stack- and queue-number with respect to taking minors. Let \(G, H\) and \(J\) be graphs. Let \(r, s \in \mathbb{N}_0\) and let \(k \geq 0\) be a half-integer (that is, \(2k \in \mathbb{N}_0\)).

**H** is a **minor** of \(G\) if a graph isomorphic to \(H\) can be obtained from \(G\) by vertex deletions, edge deletions, and edge contractions. A **model** of \(H\) in \(G\) is a function \(\mu\) with domain \(V(H)\) such that: \(\mu(v)\) is a connected subgraph of \(G\); \(\mu(v) \cap \mu(w) = \emptyset\) for all distinct \(v, w \in V(G)\); and \(\mu(v)\) and \(\mu(w)\) are adjacent for every edge \(vw \in E(H)\). Observe that \(H\) is a minor of \(G\) if and only if \(G\) contains a model of \(H\). For \(r \in \mathbb{N}_0\), if there exists a model \(\mu\) of \(H\) in \(G\) such that \(\mu(v)\) has radius at most \(r\) for all \(v \in V(H)\), then \(H\) is an **\(r\)-shallow minor** of \(G\). For \(s \in \mathbb{N}_0\), if there exists a model \(\mu\) of \(H\) in \(G\) such that \(|V(\mu(v))| \leq s\) for all \(v \in V(H)\), then \(H\) is an **\(s\)-small minor** of \(G\). We say that \(J\) is an \((\leq s)\)-subdivision of \(H\) if \(J\) can be obtained from \(H\) by replacing each edge by a path with at most \(s\) internal vertices. If every path that replaces an edge has exactly \(s\) vertices, then \(J\) is the **\(s\)-subdivision** of \(H\).

Blankenship and Oporowski [13] conjectured that stack-number is ‘well-behaved’ under shallow topological minors in the following sense:

**Conjecture 8** ([13]). There exists a function \(f\) such that for every graph \(G\) and half-integer \(k \geq 0\), if \(H\) is any \(k\)-shallow topological-minor of \(G\), then \(\text{sn}(H) \leq f(\text{sn}(G), k)\).

Dujmović et al. [22] disproved Conjecture 8. Their proof used the following lemma by Dujmović and Wood [28].

**Lemma 9** ([28]). For every graph \(G\), if \(s = 1 + 2\lceil \log_2 \text{qn}(G) \rceil\) then the \(s\)-subdivision of \(G\) has a 3-stack layout.

Lemma 9 implies that the 5-subdivision of \(S_n \boxtimes (P_n \boxplus P_n)\) admits a 3-stack layout. Using Theorem 3, Dujmović et al. [22] concluded there exists a graph class \(\mathcal{G}\) with bounded stack-number for which the class of 1-shallow topological minors of graphs in \(\mathcal{G}\) has unbounded stack-number. Thus stack-number is not well-behaved under shallow topological minors. We now prove an analogous result for small minors.

**Theorem 10.** There exists a graph class \(\mathcal{G}\) with bounded stack-number for which the class of 2-small minors of graphs in \(\mathcal{G}\) has unbounded stack-number.

**Proof.** Lemma 9 implies that the 5-subdivision of \(P_n \boxplus P_n \boxplus P_n\) admits a 3-stack layout. \(P_n \boxplus P_n \boxplus P_n\) is a 37-small minor of its 5-subdivision since \(P_n \boxplus P_n \boxplus P_n\) has maximum degree at most 12 and \(12 \lceil \frac{37}{2} \rceil + 1 = 37\). Let \(\mathcal{G}_0\) be the class of graphs with stack-number at most 3, and for each \(i \in \mathbb{N}_0\), let \(\mathcal{G}_{i+1}\) be the class of 2-small minors of graphs in \(\mathcal{G}_i\). Thus, \(\mathcal{G}_{37}\) contains all 37-small minors of graphs in \(\mathcal{G}_0\), including all graphs of the form \(P_n \boxplus P_n \boxplus P_n\). Hence, there exists \(i \in \{0,\ldots,37\}\) such that stack-number is bounded for \(\mathcal{G}_i\) and unbounded for \(\mathcal{G}_{i+1}\). \(\square\)
In contrast to Theorem 10, queue-number is well-behaved under small minors. In fact, the following lemma shows it is even well-behaved under shallow minors, which is a key distinction between these parameters.

**Lemma 11** ([43]). For every graph \( G \) and for every \( r \)-shallow minor \( H \) of \( G \),
\[
\text{qn}(H) \leq 2r(2\text{qn}(G))^{2r}.
\]

We now compare stack and queue layouts with respect to growth. The *growth* of a graph \( G \) is the function \( f_{G} : \mathbb{N} \to \mathbb{N} \) such that \( f_{G}(r) \) is the maximum number of vertices in a subgraph of \( G \) with radius at most \( r \). Similarly, the *growth* of a graph class \( \mathcal{G} \) is the function \( f_{\mathcal{G}} : \mathbb{N} \to \mathbb{N} \cup \{\infty\} \) where \( f_{\mathcal{G}}(r) = \sup\{f_{G}(r) : G \in \mathcal{G}\} \). A graph class \( \mathcal{G} \) has *linear/quadratic/cubic/polynomial growth* if \( f_{G}(r) \in O(r)/O(r^{2})/O(r^{3})/O(r^{d}) \) for some \( d \in \mathbb{N} \). Let \( \mathbb{Z}^{d}_{n} \) be the \( d \)-fold strong product \( P_{n} \boxtimes \cdots \boxtimes P_{n} \). Every subgraph in \( \mathbb{Z}^{d}_{n} \) with radius at most \( r \) has at most \((2r+1)^{d}\) vertices. Thus \((P_{n} \boxtimes P_{n} \boxtimes P_{n})_{n \in \mathbb{N}} \) has cubic growth, so Theorem 1 implies that graph classes with cubic growth can have unbounded stack-number. In contrast, graphs with polynomial growth have bounded queue-number. Krauthgamer and Lee [46] established the following characterisation of graphs with polynomial growth.

**Theorem 12** ([46]). If a graph \( G \) has growth \( f_{G}(r) \in O(r^{d}) \), then \( G \) is isomorphic to a subgraph of \( \mathbb{Z}^{n}(d \log d) \) for some \( n \in \mathbb{N} \).

It follows from the upper bound on the queue-number of products by Wood [59] that \( \text{qn}(\mathbb{Z}^{d}_{n}) \leq c3^{d} \) for some constant \( c \). Theorem 12 therefore implies the following.

**Corollary 13.** If a graph \( G \) has growth \( f_{G}(r) \in O(r^{d}) \), then \( \text{qn}(G) \in 2^{O(d \log d)} \).

### 3. Proof of Lemma 4

The topological arguments in this paper exclusively involve finite polyhedral 2-dimensional cell complexes, and so for brevity we refer to a finite polyhedral 2-dimensional cell complex as simply a *complex*\(^{2}\). The proof of Lemma 4 relies on a Topological Overlap Theorem of Gromov [36]. To use it we need a technical variation of the overlap parameter. For a complex \( X \) and topological space \( Y \), define
\[
\text{overlap}_{\Delta}(X, Y) = \min_{f \in C(X,Y)} \max_{p \in Y} |\{F \in X^{=2} | p \in f(F)\}|,
\]
where \( X^{=2} \) denotes the set of 2-dimensional cells of \( X \). We now state the 2-dimensional case of Gromov’s theorem.

**Theorem 14** ([36, p. 419, Topological \( \Delta \)-inequality]). There exists \( \alpha > 0 \) such that
\[
\text{overlap}_{\Delta}(\text{Tr}(K_{n}), \mathbb{R}^{2}) \geq \alpha n^{3}
\]
for every integer \( n \geq 3 \).

We use Theorem 14 in combination with the following straightforward lemma to lower bound \( \text{overlap}(X, \mathbb{R}^{2}) \) for a complex \( X \) in terms of \( \text{overlap}_{\Delta}(\text{Tr}(K_{n}), X) \).

**Lemma 15.** For every complex \( X_{0} \) and for all topological spaces \( X \) and \( Y \),
\[
\text{overlap}(X, Y) \geq \frac{\text{overlap}_{\Delta}(X_{0}, Y)}{\text{overlap}_{\Delta}(X_{0}, X)}.
\]

**Proof.** Let \( f_{0} : X_{0} \to X \) be a continuous function such that
\[
\text{overlap}_{\Delta}(X_{0}, X) = \max_{p \in X} |\{F \in X_{0}^{=2} | p \in f_{0}(F)\}|.
\]

\(^{2}\)See [11, Section 12] for background on finite polyhedral cell complexes, which are referred to there as convex linear cell complexes.
Let $f : X \to Y$ be an arbitrary continuous function. Then $f_0 \circ f : X_0 \to Y$ is continuous and so there exists $p_0 \in Y$ such that
\[
|\{ F \in X_0^{-2} \mid p_0 \in (f_0 \circ f)(F) \}| \geq \text{overlap}_\Delta(X_0, Y).
\]

But
\[
|\{ F \in X_0^{-2} \mid p_0 \in (f_0 \circ f)(F) \}| \leq \sum_{p \in f^{-1}(p_0)} |\{ F \in X_0^{-2} \mid p \in f_0(F) \}|
\leq |f^{-1}(p_0)| \cdot \text{overlap}_\Delta(X_0, Y).
\]

It follows that
\[
|f^{-1}(p_0)| \geq \frac{\text{overlap}_\Delta(X_0, Y)}{\text{overlap}_\Delta(X_0, X)},
\]
as desired. \hfill \square

A family $\mathcal{B}$ of subcomplexes of a complex $X$ is a **bramble over $X^3$** if:
- every $B \in \mathcal{B}$ is non-empty,
- $B_1 \cup B_2$ is connected for every pair of distinct $B_1, B_2 \in \mathcal{B}$,
- $B_1 \cup B_2 \cup B_3$ is simply connected for every triple of distinct $B_1, B_2, B_3 \in \mathcal{B}$.

The **congestion** $\text{cong}(\mathcal{B})$ of a bramble $\mathcal{B}$ is the maximum size of a collection of elements in $\mathcal{B}$ that all share a point in common; that is,
\[
\text{cong}(\mathcal{B}) = \max_{p \in X} | \{ B \in \mathcal{B} \mid p \in B \} |.
\]

**Lemma 16.** There exists $\beta > 0$ such that for every complex $X$ and bramble $\mathcal{B}$ over $X$,
\[
\text{overlap}(X, \mathbb{R}^2) \geq \beta \frac{|\mathcal{B}|}{\text{cong}(\mathcal{B})}.
\]

**Proof.** Let $\mathcal{B} = \{B_1, \ldots, B_n\}$. Let $X_0 = \text{Tr}(K_n)$ be the triangle complex of the complete graph with vertex set $\{v_1, \ldots, v_n\}$. We construct a continuous function $f : X_0 \to X$ as follows.

Let $p_i \in B_i$ be chosen arbitrarily, and set $f(v_i) = p_i$ for every $i \in \{1, \ldots, n\}$. Extend $f$ to the 1-skeleton of $X_0$ by mapping each edge $v_iv_j$ to a path $\pi_{ij} \subseteq B_i \cup B_j$ from $p_i$ to $p_j$.

Finally, we need to continuously extend $f$ to the interior of every 2-simplex $F_{ijk}$ of $X_0$ with vertices $v_i, v_j, v_k$. Since $B_i \cup B_j \cup B_k$ is simply connected, and the boundary of $F_{ijk}$ is mapped to a closed curve in $B_i \cup B_j \cup B_k$, there is such an extension such that $f(F_{ijk}) \subseteq B_i \cup B_j \cup B_k$.

Consider arbitrary $p \in X$, and let $I = \{ i \in \{1, \ldots, n\} \mid p \in B_i \}$. Then $|I| \leq \text{cong}(\mathcal{B})$ and
\[
|\{ F \in X^2 \mid p \in f(F) \}| \leq |\{ i, j, k \in \{1, \ldots, n\} \mid \{ i, j, k \} \cap I \neq \emptyset \}| \leq 3|I| \binom{n}{2}.
\]

It follows that $\text{overlap}_\Delta(X_0, X) \leq \frac{3}{2} \text{cong}(\mathcal{B})n^2$. Thus by Theorem 14 and Lemma 15,
\[
\text{overlap}(X, \mathbb{R}^2) \geq \frac{\text{overlap}_\Delta(X_0, \mathbb{R}^2)}{\text{overlap}_\Delta(X_0, X)} \geq \frac{\alpha n^3}{\frac{3}{2} \text{cong}(\mathcal{B})n^2},
\]
where $\alpha$ is from Theorem 14. It follows that $\beta = \frac{3}{2} \alpha$ satisfies the lemma. \hfill \square

Lemma 16 allows one to lower bound $\text{overlap}(X, \mathbb{R}^2)$ by exhibiting a bramble with size large in comparison to its congestion. To simplify the verification of the conditions in the definition of a bramble we use a simple consequence of van Kampen’s theorem.\footnote{This definition is inspired by the definition of a **bramble** in a graph, which is used in graph minor theory. A bramble of a graph is only required to satisfy the first and second among the conditions we impose.}

\footnote{See [38, Theorem 1.20] for the general statement of van Kampen’s theorem, or [18, Section 2.1.3] for an example of the application of the theorem in the setting generalizing Lemma 17.}
Lemma 17. Let $B_1, B_2$ be subcomplexes of a complex $X$. If $B_1$ and $B_2$ are simply connected, and $B_1 \cap B_2$ is non-empty and connected, then $B_1 \cup B_2$ is simply connected.

Corollary 18. Let $X$ be a complex. Let $\mathcal{B}$ be a family of subcomplexes of $X$ such that:

- every $B \in \mathcal{B}$ is simply connected,
- $B_1 \cap B_2$ is connected for every pair of distinct $B_1, B_2 \in \mathcal{B}$,
- $B_1 \cap B_2 \cap B_3$ is non-empty for every triple of distinct $B_1, B_2, B_3 \in \mathcal{B}$.

Then $\mathcal{B}$ is a bramble over $X$.

Proof. The first condition in the definition of a bramble trivially holds. The second condition holds since $B_1 \cup B_2$ is simply connected for all distinct $B_1, B_2 \in \mathcal{B}$ by Lemma 17.

It remains to show that $B_1 \cup B_2 \cup B_3$ is simply connected for all distinct $B_1, B_2, B_3 \in \mathcal{B}$. Since $B_1 \cup B_2$ and $B_3$ are simply connected, this follows from Lemma 17 as long as $B_3 \cap (B_1 \cup B_2) = (B_3 \cap B_1) \cup (B_3 \cap B_2)$ is non-empty and connected. By the assumptions of the corollary, each of $B_3 \cap B_1$ and $B_3 \cap B_2$ is connected and they share a point in common. Thus their union is connected as desired. □

We are now ready to prove Lemma 4.

Proof of Lemma 4. Let $G$ be any triangulation of $T_1 \sqcup T_2 \sqcup T_3$, where $T_1, T_2$ and $T_3$ are $n$-vertex trees. To simplify our notation we assume that $T_1, T_2$ and $T_3$ are vertex-disjoint. We denote by $\hat{T}_i$ the geometric 1-dimensional complex corresponding to $T_i$, to avoid confusion between discrete and topological objects.

For $u \in V(T_1)$, let $G(u)$ be the subgraph of $G$ induced by all the vertices of $G$ with the first coordinate $u$. That is, $V(G(u)) = \{(u, u_2, u_3) \mid u_2 \in V(T_2), u_3 \in V(T_3)\}$. So $G(u)$ is isomorphic to a triangulation of $T_2 \sqcup T_3$. Let $X(u) = \text{Tr}(G(u))$. As a topological space, $X(u)$ is homeomorphic to $\hat{T}_2 \times \hat{T}_3$. Define $G(u)$ and $X(u)$ for $u \in V(T_2) \cup V(T_3)$ analogously.

For $(u_1, u_2, u_3) \in V(G)$, let $B(u_1, u_2, u_3) = X(u_1) \cup X(u_2) \cup X(u_3)$, and let $\mathcal{B} = \{B(u_1, u_2, u_3) \mid (u_1, u_2, u_3) \in V(G)\}$. We claim that $\mathcal{B}$ is a bramble over $\text{Tr}(G)$. It suffices to check that it satisfies the conditions of Corollary 18.

To verify the first condition for $\mathcal{B}$, we use Corollary 18 to show that $\{X(u_1), X(u_2), X(u_3)\}$ is a bramble over $\text{Tr}(G)$ for every $(u_1, u_2, u_3) \in V(G)$. Note that each $\hat{T}_i$ is simply connected. As the product of simply connected spaces is simply connected, it follows that $X(u)$ is simply connected for every $u \in V(T_1) \cup V(T_2) \cup V(T_3)$. Consider now $(u_1, u_2, u_3) \in V(G)$. Then $X(u_1) \cap X(u_2)$ is homeomorphic to $\hat{T}_3$ and is connected. Similarly, $X(u_1) \cap X(u_3)$ and $X(u_2) \cap X(u_3)$ are connected. Finally, $X(u_1) \cap X(u_2) \cap X(u_3)$ consists of a single point (and is connected). By Corollary 18 the set $\{X(u_1), X(u_2), X(u_3)\}$ is a bramble. In particular, $B(u_1, u_2, u_3) = X(u_1) \cup X(u_2) \cup X(u_3)$ is simply connected. Thus the first condition in Corollary 18 for $\mathcal{B}$ holds.

For the second condition, consider distinct $B(u_1, u_2, u_3), B(v_1, v_2, v_3) \in \mathcal{B}$. Let $R = B(u_1, u_2, u_3) \cap B(v_1, v_2, v_3)$ for brevity. Assume first, for simplicity, that $u_i \neq v_i$ for $i \in \{1, 2, 3\}$. Then

$$R = \bigcup_{i,j \in \{1, 2, 3\}, i \neq j} X(u_i) \cap X(v_j).$$

Each set $X(u_i) \cap X(v_j)$ in this decomposition is connected, since it is homeomorphic to $\hat{T}_k$ for $\{k\} = \{1, 2, 3\} \setminus \{i, j\}$. Moreover, when ordering these sets,

$$X(u_1) \cap X(v_2), X(u_3) \cap X(v_2), X(u_3) \cap X(v_1),$$

$$X(u_2) \cap X(v_1), X(u_2) \cap X(v_3), X(u_1) \cap X(v_3),$$

each pair of consecutive sets share a point, for example, $(u_1, v_2, u_3) \in (X(u_1) \cap X(v_2)) \cap (X(v_3) \cap X(v_2))$. It follows that their union is connected.
It remains to consider the case $u_i = v_i$ for some $i \in \{1, 2, 3\}$. Assume $i = 1$ without loss of generality. If $u_2 \neq v_2$ and $u_3 \neq v_3$ then $R = X(u_1) \cup (X(u_2) \cap X(v_3)) \cup (X(v_2) \cap X(u_3))$. Each of the sets in this decomposition is again connected and the second and third sets intersect the first, implying that their union is connected. Finally, if say $u_2 = v_2$ then $R = X(u_1) \cup X(u_2)$ and is again a union of connected intersecting sets. This verifies the second condition of Corollary 18 for $\mathcal{B}$.

For the last condition, for any $B(u_1, u_2, u_3), B(v_1, v_2, v_3), B(w_1, w_2, w_3) \in \mathcal{B}$,

$$(u_1, v_2, w_3) \in B(u_1, u_2, u_3) \cap B(v_1, v_2, v_3) \cap B(w_1, w_2, w_3),$$

and so $B(u_1, u_2, u_3) \cap B(v_1, v_2, v_3) \cap B(w_1, w_2, w_3) \neq \emptyset$. By Corollary 18, $\mathcal{B}$ is a bramble.

As noted above, $|\mathcal{B}| = n^3$. Moreover, for every $p \in \text{Tr}(G)$ and every $i \in \{1, 2, 3\}$ there exists at most one $v_i \in V(T_i)$ such that $p \in X(v_i)$. Thus $\text{cong}(\mathcal{B}) \leq 3n^2$, and by Lemma 16,

$$\text{overlap}(\text{Tr}(G), \mathbb{R}^2) \geq \beta \frac{|\mathcal{B}|}{\text{cong}(\mathcal{B})} \geq \frac{\beta}{3} n,$$

as desired. \qed

Note that $\text{Tr}(P_n \boxtimes P_n \boxtimes P_n)$ has a natural affine embedding into $\mathbb{R}^3$ with vertices mapped to points in $\{1, \ldots, n\}^3$. Since any line in a direction sufficiently close to an axis direction intersects this complex in $n + O(1)$ points, projecting along such a direction we obtain an affine map $\text{Tr}(G) \to \mathbb{R}^2$ that covers every point at most $n + O(1)$ times. Thus $\text{overlap}(\text{Tr}(P_n \boxtimes P_n \boxtimes P_n), \mathbb{R}^2) \leq n + O(1)$ and so the bound in Lemma 4 can not be substantially improved.

4. PROOF OF LEMMA 5

The proof of Lemma 5 depends on the following result.

**Lemma 19.** Let $T_1, \ldots, T_m$ be pairwise vertex-disjoint pairwise intersecting triangles in $\mathbb{R}^2$ with all the vertices on a circle $S$. Assume that the edges of $T_1, \ldots, T_m$ can be partitioned into $k$ non-crossing sets. Then $m \leq k^3$.

**Proof.** Number the vertices of $T_1, \ldots, T_m$ by $1, \ldots, 3m$ in clockwise order starting at an arbitrary point on $S$. By assumption there is a function $\phi : \bigcup_i E(T_i) \to \{1, \ldots, k\}$ such that $\phi(e_1) \neq \phi(e_2)$ for all crossing edges $e_1, e_2 \in \bigcup_i E(T_i)$. Say the vertices of $T_i$ are $(a_i, b_i, c_i)$ with $a_i < b_i < c_i$. Define the function $f : \{1, \ldots, m\} \to \{1, \ldots, k\}^3$ by $f(i) = (\phi(a_i), \phi(a_i, c_i), \phi(b_i, c_i))$. Suppose that $f(i) = f(j)$ for distinct $i, j \in \{1, \ldots, m\}$. Thus $a_i b_i$ does not cross $a_j b_j$, and $a_i c_i$ does not cross $a_j c_j$, and $b_i c_i$ does not cross $b_j c_j$. Without loss of generality, $a_i < a_j$. If $c_i < c_j$ then $a_i c_i$ crosses $a_j c_j$, so $a_i < a_j < b_i < c_j < c_i$. Now consider $b_i$. If $a_i < b_i < a_j$ or $c_j < b_i < c_i$, then $T_i$ and $T_j$ do not intersect. So $a_i < b_i < c_j$. If $a_j < b_i < b_j$, then $a_i b_i$ crosses $a_j b_j$. Otherwise, $b_i < b_j < c_j$ implying $b_j c_j$ crosses $a_j c_j$. This contradiction shows that $f(i) \neq f(j)$ for distinct $i, j \in \{1, \ldots, m\}$. Hence $m \leq k^3$. \qed

**Proof of Lemma 5.** Let $k = \text{sn}(G)$, and let $(v_1, \ldots, v_n)$ together with a function $\phi : E(G) \to \{1, \ldots, k\}$ be a $k$-stack layout of $G$. Let $p_1, p_2, \ldots, p_n$ be pairwise distinct points chosen on a circle $S$ in the $x-y$ plane, numbered in cyclic order around $S$. Let $X = \text{Tr}(G)$. Define a continuous function $f : X \to \mathbb{R}^2$ by setting $f(v_i) = p_i$ for all $i \in \{1, \ldots, n\}$, extending $f$ affinely to 2-simplices of $X$ (triangles of $G$), and, for simplicity, mapping every edge of $G$ that does not belong to a triangle continuously to a curve internally disjoint from a circle bounded by $S$, so that every point of $\mathbb{R}^2$ belongs to at most two such curves.

Let $m' = \text{overlap}(X, \mathbb{R}^2)$. Then there exists $p \in \mathbb{R}^2$ such that $|f^{-1}(p)| \geq m'$. If $m' \leq 2$ the lemma trivially holds, and so we assume $m' > 2$. Since the restriction of $f$ to each simplex of $X$ is injective, there exist triangles $T_1, T_2, \ldots, T_m' \subseteq \mathbb{R}^2$ corresponding to images of distinct 2-simplices of $X$ so that $p \in \bigcap_{i=1}^{m'} T_i$. Let $m = \lceil \frac{m'}{3} \rceil > \frac{m'}{3}$. Since every triangle shares a vertex with at most $3c - 3$ others, we may assume that $T_1, T_2, \ldots, T_m$ are
Let \( m \leq k^3 \). Thus
\[
\text{sn}(G) = k \geq m^{1/3} \geq \left( \frac{m'}{3c} \right)^{1/3} = \left( \frac{\text{overlap}(X, \mathbb{R}^2)}{3c} \right)^{1/3}.
\]

This lemma completes the proof of Theorems 1 and 2.

We finish this section by showing that Lemma 19 is best possible up to a constant factor.

**Proposition 20.** For infinitely many \( k \in \mathbb{N} \), there is a set of \( \left( \frac{k}{3} \right)^3 \) pairwise intersecting and pairwise vertex-disjoint triangles with vertices on a circle in \( \mathbb{R}^2 \), such that the edges of the triangles can be \( k \)-coloured with crossing edges assigned distinct colours.

This result is implied by the following (with \( k = 3 \cdot 2^\ell \)).

**Lemma 21.** Let \( S \) be a circle in \( \mathbb{R}^2 \) partitioned into three pairwise disjoint arcs \( A, B, C \). For every \( \ell \in \mathbb{N} \) there is a set \( T \) of \( 8^\ell \) triangles, each with one vertex in each of \( A, B, C \), such that the \( AB \)-edges can be \( 2^\ell \)-coloured with crossing edges assigned distinct colours, the \( BC \)-edges can be \( 2^\ell \)-coloured with crossing edges assigned distinct colours, and the \( CA \)-edges can be \( 2^\ell \)-coloured with crossing edges assigned distinct colours.

**Proof.** We proceed by induction on \( \ell \). The claim is trivial with \( \ell = 0 \). Assume that for some integer \( \ell \geq 0 \), there is a set \( T \) of \( 8^\ell \) triangles satisfying the claim. Let \( X \) be the set of \( 2^\ell \) colours used for the \( AB \)-edges. Let \( Y \) be the set of \( 2^\ell \) colours used for the \( BC \)-edges. Let \( Z \) be the set of \( 2^\ell \) colours used for the \( CA \)-edges. Let \( X' = \{ x', x'' : x \in X \} \) be a set of \( 2^{\ell+1} \) colours. Let \( Y' = \{ y', y'' : y \in Y \} \) be a set of \( 2^{\ell+1} \) colours. Let \( Z' = \{ z', z'' : z \in Z \} \) be a set of \( 2^{\ell+1} \) colours. Let \( T' \) be the set of triangles obtained by replacing each \( T \in T \) by 8 triangles as follows. Say the vertices of \( T \) are \( u, v, w \) where \( u \in A, v \in B \) and \( w \in C \). Replace \( u \) by \( u_1, \ldots, u_8 \) in clockwise order in \( A \), replace \( v \) by \( v_1, \ldots, v_8 \) in clockwise order in \( B \), and replace \( w \) by \( w_1, \ldots, w_8 \) in clockwise order in \( C \). By this we mean that if \( p, q \) are consecutive vertices in the original ordering, then \( p_1, \ldots, p_8, q_1, \ldots, q_8 \) are consecutive vertices in the enlarged ordering. Add the triangles \( u_1v_4w_6, u_2v_3w_5, u_3v_2w_8, u_4v_1w_7, u_5v_8w_2, u_6v_7w_1, u_7v_6w_4, u_8v_5w_3 \) to \( T' \). Say \( uv \) is coloured \( x \in X \), \( vw \) is coloured \( y \in Y \), and \( uw \) is coloured \( z \in Z \). Colour each of \( u_1v_4, u_2v_3, u_3v_2, u_4v_1 \) by \( x' \in X' \), and colour each of \( u_5v_8, u_6v_7, u_7v_6, u_8v_5 \) by \( x'' \in X' \). Colour each of \( v_1w_7, v_2w_5, v_3w_3, v_7w_1 \) by \( y' \in Y' \), and colour each of \( v_2w_8, v_3w_6, v_5w_4, v_8w_2 \) by \( y'' \in Y' \). Colour each of \( w_1u_6, w_2u_5, u_5u_2, w_6u_1 \) by \( z' \in Z' \), and colour each of \( u_3w_8, w_4u_7, w_7u_4, w_8u_3 \) by \( z'' \in Z' \). As illustrated in Figure 3, crossing edges are assigned distinct colours. Thus \( T' \) is the desired set of \( 8^{\ell+1} \) triangles. \( \square \)

![Figure 3. Construction in the proof of Lemma 21.](image-url)
5. Upper Bound

This section proves the upper bound in Theorem 1 showing that $P_n \boxtimes P_n \boxtimes P_n$ has an $O(n^{1/3})$-stack layout. Assume $V(P_n) = \{1, \ldots, n\}$ and $E(P_n) = \{i(i + 1) : i \in \{1, \ldots, n - 1\}\}$. We start with a sketch of the construction. Take a particular $O(1)$-stack layout and a proper 4-colouring of $P_n \boxtimes P_n$. In the corresponding ordering of $V(P_n) \boxtimes P_n$ replace each vertex $(x, y)$ by vertices $((x, y, z_1), \ldots, (x, y, z_n))$ where $(z_1, \ldots, z_n)$ is a permutation of $\{1, \ldots, n\}$ determined by the colour of $(x, y)$. An appropriate choice of the permutations ensures that the edges of $P_n \boxtimes P_n \boxtimes P_n$ can be partitioned into $O(n^{1/3})$ stacks. In particular, there can only be $O(n^{1/3})$ pairwise crossing edges with respect to the ordering, so the length of a longest common subsequence of any two of the permutations is $O(n^{1/3})$. See Figure 4 for an illustration of the case $n = 8$.

![Figure 4](image_url)

**Figure 4.** Four permutations of $\{1, \ldots, 8\}$, no two of which have a common subsequence of length greater than 2.

We actually prove a more general result, Theorem 22 below, which relies on the following definition from the literature [2, 10, 26, 34, 53]. An $s$-stack layout $((v_1, \ldots, v_n), \psi)$ of a graph $G$ is dispersable (also called pushdown) if $\psi^{-1}(k)$ is a matching in $G$ for each $k \in \{1, \ldots, s\}$. The dispersable stack-number $dsn(G)$ is the minimum $s \in \mathbb{N}_0$ for which there exists a dispersable $s$-stack layout of $G$. For example, $dsn(P_n \boxtimes P_n) \leq 8$ since in the 4-stack layout of $P_n \boxtimes P_n$ illustrated in Figure 1, each stack is a linear forest; putting alternative edges from each path in distinct stacks produces a dispersable 8-stack layout. In general, $\Delta(G) \leq dsn(G) \leq (\Delta(G) + 1) sn(G)$ by Vizing’s Theorem. So a graph family has bounded dispersable stack-number if and only if it has bounded stack-number and bounded maximum degree.

**Theorem 22.** Let $G$ be a graph with chromatic number $\chi$ and dispersable stack-number $d$. Let $n \in \mathbb{N}$ and $p = \max\{2^{[\log_2 \chi]}, 2\}$. Then

$$sn(G \boxtimes P_n) \leq 2^{p/2-1}d(2p - 1) \cdot n^{1/2-1/(2p-2)} + 2p - 3.$$ 

Since $\chi(P_n \boxtimes P_n) \leq 4$ and $dsn(P_n \boxtimes P_n) \leq 8$, Theorem 22 implies that

$$sn(P_n \boxtimes P_n \boxtimes P_n) \leq 2 \cdot 8 \cdot 7n^{1/3} + 5 = 112n^{1/3} + 5.$$ 

Furthermore, $\chi(G) \leq \Delta(G) + 1 \leq dsn(G) + 1$. So Theorem 22 shows that graphs $G$ with bounded dispersable stack-number satisfy $sn(G \boxtimes P_n) \in O(n^{1/2-\epsilon})$ for some $\epsilon > 0$.

For an even positive integer $p$, a Hadamard matrix of order $p$ is a $p \times p$ matrix $H$ with all entries in $\{+1, -1\}$ such that every pair of distinct rows differs in exactly $p/2$ entries. An example of a Hadamard matrix of order 4 is

$$\begin{bmatrix} +1 & +1 & +1 & +1 \\ -1 & -1 & +1 & +1 \\ -1 & +1 & -1 & +1 \\ +1 & -1 & -1 & +1 \end{bmatrix}.$$
Sylvester \cite{Sylvester:1867} proved that the order of a Hadamard matrix is 2 or is divisible by 4. He also constructed a Hadamard matrix of order any power of 2. Paley \cite{Paley:1933} constructed a Hadamard matrix of order \( q + 1 \) for any prime power \( q \equiv 3 \pmod{4} \), and a Hadamard matrix of order \( 2(q + 1) \) for any prime power \( q \equiv 1 \pmod{4} \). The Hadamard Conjecture proposes that there exists a Hadamard matrix of order \( p \) whenever \( p \) is divisible by 4. This conjecture has been verified for numerous values of \( p \); see \cite{Seberry:1992} for example.

The following lemma captures a property of the construction of permutations without long common subsequences by Beame, Blais, and Huynh-Ngoc \cite{Beame:2020} that is crucial in the proof of Theorem \ref{thm:main}. 

**Lemma 23.** Assume there exists a Hadamard matrix of order \( p \). Let \( m \in \mathbb{N} \) and \( n = mp^{-1} \).

Then there exist permutations \( \pi_1, \ldots, \pi_p : \{1, \ldots, n\} \to \{1, \ldots, n\} \) such that for all distinct \( k, \ell \in \{1, \ldots, p\} \),

\[
|\{\pi_k(i) + \pi_\ell(j) : i, j \in \{1, \ldots, n\}, |i - j| \leq 1\}| \leq (2p - 1)n^{1/2-1/(2p-2)},
\]

and for each \( k \in \{1, \ldots, p\} \) there exists a \((2p - 3)\)-stack layout of \( P_n \) using the vertex ordering \((\pi_k^{-1}(1), \ldots, \pi_k^{-1}(n))\).

**Proof.** Since there exists a Hadamard matrix of order \( p \), there exist \( 1 \)-vectors \( h_1, \ldots, h_p \), any two of which differ in exactly \( p/2 \) entries. Assume \( h_k = (h_k(0), \ldots, h_k(p - 1)) \) for each \( k \in \{1, \ldots, p\} \). By possibly negating all entries in some of these vectors, we may assume \( h_k(p - 1) = +1 \) for all \( k \in \{1, \ldots, p\} \).

For each \( i \in \{1, \ldots, n\} \), let \( d(i, 0), \ldots, d(i, p - 2) \in \{0, \ldots, m - 1\} \) denote the digits of \( i - 1 \) in base \( m \), so that

\[
i = 1 + \sum_{a=0}^{p-2} d(i, a)m^a.
\]

Note that the mapping \( \{1, \ldots, n\} \ni i \mapsto (d(i, 0), \ldots, d(i, p - 2)) \in \{0, \ldots, m - 1\}^{p-1} \) is bijective.

For all \( k \in \{1, \ldots, p\} \), \( i \in \{1, \ldots, n\} \) and \( a \in \{0, \ldots, p - 2\} \), let

\[
d_k(i, a) = \begin{cases} d(i, a) & \text{if } h_k(a) = +1, \\ m - 1 - d(i, a) & \text{if } h_k(a) = -1. \end{cases}
\]

Observe that for any \( k, \ell \in \{1, \ldots, p\} \) and \( a \in \{0, \ldots, p - 2\} \), if \( h_k(a) = -h_\ell(a) \) then \( d_k(i, a) + d_\ell(i, a) = m - 1 \) for all \( i \in \{1, \ldots, n\} \).

For every \( k \in \{1, \ldots, p\} \), define \( \pi_k \) by

\[
\pi_k(i) = 1 + \sum_{a=0}^{p-2} d_k(i, a)m^a.
\]

We claim that \( \pi_1, \ldots, \pi_p \) satisfy the lemma. Fix distinct \( k, \ell \in \{1, \ldots, p\} \). For \( i, j \in \{1, \ldots, n\} \) and \( a \in \{0, \ldots, p - 2\} \), define \( \tau_a(i, j) = d_k(i, a) + d_\ell(j, a) \), and let

\[
\tau(i, j) = (\tau_0(i, j), \ldots, \tau_{p-2}(i, j)).
\]

Since \( \pi_k(i) + \pi_\ell(j) = \sum_{a=0}^{p-2} \tau_a(i, j)m^a \), the value of \( \tau(i, j) \) determines the value of \( \pi_k(i) + \pi_\ell(j) \). As such, to prove the first part of the lemma it suffices to show that

\[
|\{\tau(i, j) : i, j \in \{1, \ldots, n\}, |i - j| \leq 1\}| \leq (2p - 1)n^{1/2-1/(2p-2)}.
\]

Let \( A^+ = \{a \in \{0, \ldots, p - 2\} : h_k(a) = h_\ell(a)\} \) and \( A^- = \{a \in \{0, \ldots, p - 2\} : h_k(a) = -h_\ell(a)\} \). Since \( h_k \) and \( h_\ell \) differ in exactly \( p/2 \) entries and \( h_k(p - 1) = +1 = h_\ell(p - 1) \), we have \(|A^+| = p/2 - 1 \) and \(|A^-| = p/2 \).
Let $i \in \{1, \ldots, n\}$. For every $a \in A^*$, we have $\tau_a(i, i) = m - 1$ since $h_k(a) = -h_\ell(a).$

For every $a \in A^+$ we have $d_k(i, a) = d_\ell(i, a) \in \{0, \ldots, m - 1\}$, and thus $\tau_a(i, i)$ is an even integer between 0 and $2m - 2$. Hence

$$|\{\tau(i, i) : i \in \{1, \ldots, n\}\}| \leq 1^{A^\ast - 1}m^{A^+} = mp^{2-1} = n^{p/2-1/(p-1)} = n^{1/2-1/(2p-2)}.$$

Now let $i, j \in \{1, \ldots, n\}$ be such that $j - i = 1$. We first show that $d(j, a) - d(i, a) \in \{-(m - 1), 1, 0\}$ for all $a \in \{0, \ldots, p - 2\}$. Since $j > 1$, there exists $b \in \{0, \ldots, p - 2\}$ such that $d(j, b) \neq 0$. Let $c_{i,j}$ denote the least element $c \in \{0, \ldots, p - 2\}$ such that $d(j, c) \neq 0$. Observe that whenever $0 \leq a < c_{i,j}$, we have $d(i, a) = m - 1$ and $d(j, a) = 0$. Furthermore, $d(j, c_{i,j}) - d(i, c_{i,j}) = 1$, and $d(j, b) = d(i, b)$ whenever $c_{i,j} < b \leq p - 2$. In particular, the value of $c_{i,j}$ determines the value of $d(j, b) - d(i, b)$ for each $b \in \{0, \ldots, p - 2\}$:

$$d(j, b) - d(i, b) = \begin{cases} - (m - 1) & \text{if } b < c_{i,j}, \\ 1 & \text{if } b = c_{i,j}, \\ 0 & \text{if } b > c_{i,j}. \end{cases}$$

Hence for any $c \in \{0, \ldots, p - 2\}$ and $a \in A^-$, for all pairs $(i, j)$ with $j - i = 1$ and $c_{i,j} = c$, the value of $\tau_a(i, j)$ is determined and is independent of $i$ and $j$: if $h_k(a) = +1$ and $h_\ell(a) = -1$, then $\tau_a(i, j) = m - 1 - (d(j, a) - d(i, a))$, and if $h_k(a) = -1$ and $h_\ell(a) = +1$, then $\tau_a(i, j) = m - 1 + (d(j, a) - d(i, a))$.

Let $a \in A^+$. If $a < c_{i,j}$, then $d(i, a) = m - 1$ and $d(j, a) = 0$ so $\{d_k(i, a), d_\ell(j, a)\} = \{0, m - 1\}$ and thus $\tau_a(i, j) = m - 1$. If $a = c_{i,j}$, then $\{d_k(i, a), d_\ell(j, a)\} = 1$ since $d(j, c_{i,j}) - d(i, c_{i,j}) = 1$, and thus $\tau_a(i, j)$ is an odd integer between 1 and $2m - 3$. If $a > c_{i,j}$, then $d_k(i, a) = d_\ell(j, a)$ since $d(i, a) = d(j, a)$, and thus $\tau_a(i, j)$ is an even integer between 0 and $2m - 2$.

Summarizing, for any $a, c \in \{0, \ldots, p - 2\}$,

$$|\{\tau_a(i, j) : i, j \in \{1, \ldots, n\}, j - i = 1, c(i, j) = c\}| \leq \begin{cases} 1 & \text{if } a \in A^-, \\ m & \text{if } a \in A^+. \end{cases}$$

Hence

$$|\{\tau(i, j) : i, j \in \{1, \ldots, n\}, j - i = 1\}| \leq \sum_{c=0}^{p-2} |\{\tau(i, j) : i, j \in \{1, \ldots, n\}, j - i = 1, c_{i,j} = c\}|$$

$$\leq (p - 1)1^{A^\ast - 1}m^{A^+}$$

$$= (p - 1)m^{p/2-1}$$

$$= (p - 1)n^{1/2-1/(2p-2)}.$$

By a symmetric argument,

$$|\{\tau(i, j) : i, j \in \{1, \ldots, n\}, j - i = -1\}| \leq (p - 1)n^{1/2-1/(2p-2)}.$$

Therefore

$$|\{\tau(i, j) : i, j \in \{1, \ldots, n\}, |j - i| \leq 1\}| \leq (1 + 2(p - 1))n^{1/2-1/(2p-2)} = (2p - 1)n^{1/2-1/(2p-2)},$$

which proves the proof of the first part of the lemma.

It remains to show that for each $k \in \{1, \ldots, p\}$, the set $E(P_n)$ can be partitioned into $2p - 3$ stacks with respect to $(\pi_k^{-1}(1), \ldots, \pi_k^{-1}(m^{p-1}))$. Observe that for each $a \in \{0, \ldots, p - 2\}$ and each $j \in \{1, \ldots, m^{p-1-a}\}$, the integers $(j - 1)m^a + 1, \ldots, jm^a$ forms a block of consecutive elements in $(\pi_k^{-1}(1), \ldots, \pi_k^{-1}(n))$. Furthermore, if $j$ is not divisible by $m$, then the blocks corresponding to $j$ and $j + 1$ are consecutive. Construct a partition of $E(P_n)$ into $2p - 3$ stacks as follows. The first stack consists of the edges $i(i + 1)$ such that $i$ is not divisible by $m$. For each $a \in \{1, \ldots, p - 2\}$, partition the edges $i(i + 1)$ such that $i$ is divisible by $m^a$ but not by $m^{a+1}$ into two stacks: edge $i(i + 1)$ is in the first stack if $i/m^a$
is odd and in the second stack if \( i/m^a \) is even. The resulting two sets of edges are indeed stacks; in fact, no two edges cross or even nest in these sets. Thus \( E(P_n) \) can be partitioned into \((2p - 3)\) stacks.

Theorem 22 is a consequence of the following technical variant.

**Lemma 24.** Assume there exists a Hadamard matrix of order \( p \). Let \( m \in \mathbb{N} \) and \( n = m^{p-1} \). Let \( G \) be a \( p \)-colourable graph with a dispersable \( d \)-stack layout. Then

\[
\text{sn}(G \boxtimes P_n) \leq d(2p - 1)n^{1/2 - 1/(2p-2)} + 2p - 3.
\]

Before proving Lemma 24, we show that it implies Theorem 22.

**Proof of Theorem 22.** Let \( m = \lceil n^{1/(p-1)} \rceil \). Since \( p \) is a power of 2, there exists a Hadamard matrix of order \( p \). Since \( P_n \subseteq P_{m^{p-1}} \), Lemma 24 implies

\[
\text{sn}(G \boxtimes P_n) \leq \text{sn}(G \boxtimes P_{m^{p-1}}) \\
\leq d(2p - 1)m^{(p-1)(1/2 - 1/(2p-2))} + 2p - 3 \\
\leq d(2p - 1)m^{p/2 - 1} + 2p - 3 \\
< d(2p - 1)(n^{1/(p-1)} + 1)^{p/2 - 1} + 2p - 3 \\
\leq d(2p - 1)(2n^{1/(p-1)})^{p/2 - 1} + 2p - 3 \\
= 2^{p/2 - 1}d(2p - 1)n^{1/2 - 1/(2p-2)}. 
\]

**Proof of Lemma 24.** Let \( \rho : V(G) \to \{1, \ldots, p\} \) be a proper colouring of \( G \). Let \( \pi_1, \ldots, \pi_p \) be the permutations of \( \{1, \ldots, n\} \) given by Lemma 23. For each \( v \in V(G) \), let \( P^v \) denote the path in \( G \boxtimes P_n \) induced by \( \{v\} \times \{1, \ldots, n\} \), and let \( \overrightarrow{P^v} = ((v, \pi_{\rho(v)}(1)), \ldots, (v, \pi_{\rho(v)}(n))) \).

Let \( N = |V(G)| \), and let \( ((v_1, \ldots, v_N), \psi) \) be a dispersable \( d \)-stack layout of \( G \). Let \( \overrightarrow{V} = (\overrightarrow{P^{v_1}}; \overrightarrow{P^{v_2}}; \ldots; \overrightarrow{P^{v_N}}) \) be an ordering of \( V(G) \boxtimes P_n \). By our choice of the permutations \( \pi_1, \ldots, \pi_p \), for each \( v \in V(G) \), the set \( E(P^v) \) can be partitioned into \( 2p - 3 \) stacks. Since the paths \( P^v \) occupy disjoint parts of \( \overrightarrow{V} \), it follows that \( \bigcup_{v \in V(G)} E(P^v) \) admits a partition into \( 2p - 3 \) stacks with respect to \( \overrightarrow{V} \).

We partition the set \( E(G \boxtimes P_n) \setminus \bigcup_{v \in V(G)} E(P^v) \) into sets \( E_{uv} \) indexed by the edges of \( G \). For an edge \( uv \in E(G) \), let \( E_{uv} = \{xy \in E(G \boxtimes P_n) : x \in V(P^u), y \in V(P^v)\} \). For each \( xy \in E_{uv} \), let \( \gamma(xy) = \psi(uv) \). We claim that for all \( k \in \{1, \ldots, d\} \), \( \gamma^{-1}(k) \) can be partitioned into at most \( (2p - 1)n^{1/2 - 1/(2p-2)} \) stacks with respect to \( \overrightarrow{V} \). Since \( \psi \) is a dispersable stack-layout, it suffices to show that for every \( uv \in E(G) \), \( E_{uv} \) can be partitioned into at most \((2p - 1)n^{1/2 - 1/(2p-2)}\) stacks.

Fix an edge \( uv \in E(G) \). For each edge \( e = (u,i)(v,j) \in E_{uv} \), define

\[
\phi(e) = \pi_{\rho(u)}(i) + \pi_{\rho(v)}(j).
\]

By our choice of the permutations \( \pi_1, \ldots, \pi_p \), the size of the image of \( \phi \) is at most \((2p - 1)n^{1/2 - 1/(2p-2)}\). It remains to show that \( \phi \) partitions \( E_{uv} \) into stacks with respect to \( \overrightarrow{V} \). Let \( e = (u,i)(v,j) \) and \( e' = (u,i')(v,j') \) be two edges from \( E_{uv} \) which cross. Without loss of generality, assume that the vertices are in the order \((u,i), (u',i'), (v,j), (v',j')\) in \( \overrightarrow{V} \). This means that \( \pi_{\rho(u)}(i) < \pi_{\rho(u)}(i') \) and \( \pi_{\rho(v)}(j) < \pi_{\rho(v)}(j') \), so \( \phi(e) = \pi_{\rho(u)}(i) + \pi_{\rho(v)}(j) < \pi_{\rho(u)}(i') + \pi_{\rho(v)}(j') = \phi(e') \), so \( \phi(e) \neq \phi(e') \). Hence \( \phi \) partitions \( E_{uv} \) into \((2p - 1)n^{1/2 - 1/(2p-2)}\) stacks, as required.

Note that for values of \( p \) that are not powers of 2 but there exists a Hadamard matrix of order \( p \), Lemma 24 gives a stronger bound than Theorem 22.
6. Smaller maximum degree

This section proves Theorem 7, which says that if $\Delta_0$ is the minimum integer for which there exists a graph family with maximum degree $\Delta_0$, unbounded stack-number and bounded queue-number, then $\Delta_0 \in \{6, 7\}$. These upper and lower bounds are respectively proved in Theorems 25 and 28 below.

Theorem 25. There exists a graph family with maximum degree 7, unbounded stack-number and bounded queue-number.

The construction for Theorem 25 is based on a tessellation of $\mathbb{R}^3$ with truncated octahedra, first studied by Fedorov [33]. Let $Q_0$ denote the convex hull of all points $(x, y, z) \in \mathbb{R}^3$ such that $\{|x|, |y|, |z|\} = \{0, 1, 2\}$; see Figure 5. A truncated octahedron is any polyhedron similar to $Q_0$. At each corner of a truncated octahedron three faces meet: one square and two regular hexagons. Let $Q_0 + (x, y, z)$ be the translation of $Q_0$ by a vector $(x, y, z)$.

Figure 5. The truncated octahedron $Q_0$ inscribed in a cube.

Let $T_{\infty}$ be the family of translations of $Q_0$ defined as

$$T_{\infty} = \{Q_0 + (4x, 4y, 4z) : x, y, z \in \mathbb{Z}\} \cup \{Q_0 + (4x + 2, 4y + 2, 4z + 2) : x, y, z \in \mathbb{Z}\}.$$ 

$T_{\infty}$ is a 3-dimensional tessellation; that is, a family of interior-disjoint polyhedra whose union is $\mathbb{R}^3$. A corner of $T_{\infty}$ is any corner of a truncated octahedron from $T_{\infty}$; edges and faces of $T_{\infty}$ are defined similarly. Every (hexagonal or square) face of $T_{\infty}$ is shared by two truncated octahedra in $T_{\infty}$. At each edge of $T_{\infty}$ a square face and two hexagonal faces meet, and at each corner of $T_{\infty}$ two square and four hexagonal faces meet.

We construct an infinite graph $G_{\infty}$ whose vertices are points in $\mathbb{R}^3$ and edges are line segments between their endpoints. $G_{\infty}$ is the union of copies of the plane graphs $J_3$ and $J_7$ (depicted in Figure 6), where the copies of $J_3$ are placed at the square faces of $T_{\infty}$ and the copies of $J_7$ are placed at the hexagonal faces of $T_{\infty}$. Each copy of $J_3$ or $J_7$ is contained within its corresponding face so that the exterior cycle coincides with the union of the edges contained in that face, black vertices are at the corners of the face, and each edge of the face is split into equal segments by 10 vertices from the exterior face. A vertex of $G_{\infty}$ is called a corner vertex if it coincides with a corner of $T_{\infty}$, an edge vertex if it lies on an edge of $T_{\infty}$ and is not a corner vertex, or a face vertex if it is neither a corner vertex nor an edge vertex.

Lemma 26. The maximum degree of $G_{\infty}$ is 7.

Proof. Let $v \in V(G_{\infty})$. We proceed by case analysis. If $v$ is a face vertex, then its degree is at most 7 because $J_3$ and $J_7$ have maximum degree 7.

Now suppose $v$ is a corner vertex. Since $v$ belongs to four edges of $T_{\infty}$, it is adjacent to four edge vertices in $G_{\infty}$. Furthermore, $v$ is adjacent to a face vertex in each of the two copies of $J_3$ containing $v$ and is not adjacent to any face vertex in a copy of $J_7$. Therefore, $v$ has degree 6.

It remains to consider the case when $v$ is an edge vertex. Let $e$ be the edge of the tessellation $T_{\infty}$ that contains $v$. Let $v_0 \cdots v_{11}$ be the path induced by the vertices of $G_{\infty}$
which lie on $\varepsilon$, so that $v_0$ and $v_{11}$ are corner vertices and $v_1$, $\ldots$, $v_{10}$ are edge vertices (and one of them is $v$). The vertices $v_1$, $\ldots$, $v_{10}$ and all their neighbours lie in one copy of $J_\circ$ and two copies of $J_\circ$. The degrees of the vertices $v_1$, $\ldots$, $v_{10}$ in the copy of $J_\circ$ containing them are $3$, $3$, $5$, $3$, $3$, $5$, $3$, $3$, respectively, and their degrees in each copy of $J_\circ$ containing them are $4$, $4$, $3$, $3$, $4$, $4$, $3$, $3$, $4$, $4$, respectively. Since $3 + 2 = 4 = 5 + 2 = 3 = 11$, for each $i \in \{1, \ldots, 10\}$, the total sum of degrees of $v_i$ in the copy of $J_\circ$ and the two copies of $J_\circ$ is 11. However, we counted the vertices $v_{i-1}$ and $v_{i+1}$ thrice, so the degree of $v_i$ in $G_\infty$ is $11 - 4 = 7$, as required. □

Let $[a, b]$ denote the closed interval $\{x \in \mathbb{R} : a \leq x \leq b\}$. Observe that for every $(i, j, k) \in \mathbb{Z}^3$, the cube $[4i - 2, 4i + 2] \times [4j - 2, 4j + 2] \times [4k - 2, 4k + 2]$ contains exactly $|V(G_\infty) \cap Q_0|$ vertices of $V(G_\infty)$. For every $n \in \mathbb{N}$, let $\mathcal{F}_n$ be the set of all faces of $\mathcal{T}_\infty$ contained in $[4, 4n + 2]^3$ and let $G_n$ be the subgraph of $G_\infty$ induced by the vertices lying on the faces in $\mathcal{F}_n$. Then $|V(G_n)| = \Theta(n^3)$. Furthermore, $(G_n)_{n \in \mathbb{N}}$ has cubic growth as the distance between any pair of adjacent vertices in $G_n$ is $O(1)$. By Corollary 13 and Lemma 26, $(G_n)_{n \in \mathbb{N}}$ has bounded queue-number and maximum degree 7. Thus Theorem 25 follows from the next lemma.

**Lemma 27.** $\text{sn}(G_n) \in \Omega(n^{1/3})$.

**Proof.** $J_\circ$ and $J_\circ$ are plane graphs in which every internal face is a triangle. Therefore, every subgraph induced by a face of $\mathcal{T}_\infty$ has a triangle complex homeomorphic to that face (and to a closed disk). Furthermore, since faces intersect only at their edges and corners, every subgraph of $G_n$ induced by a union of faces from $\mathcal{F}_n$ has a triangle complex homeomorphic to that union.

For $a \in \{1, 2, 3\}$ and $i \in \{1, \ldots, n\}$, let $X_i^a$ be the union of all faces $F \in \mathcal{F}_n$ such that $4i \leq x_a \leq 4i + 2$ for all $(x_1, x_2, x_3) \in F$. Each set $X_i^a$ is homeomorphic to a closed disk (see Figure 7). Let $L_i^a$ denote the subgraph of $G_n$ induced by $X_i^a$. As observed earlier, $X_i^a$ as a union of faces from $\mathcal{F}_n$ is homeomorphic to $\text{Tr}(L_i^a)$. Hence we identify $X_i^a$ with $\text{Tr}(L_i^a)$.

Observe that there are $(2n - 1)^3$ hexagonal faces in $\mathcal{F}_n$, each of which is contained in a different cube of the form $[2x + 2, 2x + 4] \times [2y + 2, 2y + 4] \times [2z + 2, 2z + 4]$ with $(x, y, z) \in \{1, \ldots, 2n - 1\}^3$. Furthermore, each square face has its centre in a corner of one of these cubes (but not every corner is a centre of a square face). Hence, for $(i, j) \in \{1, \ldots, n\}^2$, the intersection $X_i^1 \cap X_j^2$ is the union of the 2n - 1 hexagonal faces from $\mathcal{F}_n$ contained in $[4i, 4i + 2] \times [4j, 4j + 2] \times \mathbb{R}$. These hexagons form a sequence such that any pair of consecutive hexagons share an edge. The intersection $X_i^1 \cap X_j^2$ is thus connected. Symmetric arguments show that

1. $X_i^a \cap X_j^b$ is connected for $(i, j) \in \{1, \ldots, n\}^2$ and $(a, b) \in \{1, 2, 3\}^2$ such that $a \neq b$.

**Figure 6.** The graphs $J_\circ$ and $J_\circ$. 
Since the maximum degree of $G_n$ is 7, each vertex is contained in at most $\binom{7}{2} = 21$ triangles. Therefore, by Lemma 5,

$$\text{sn}(G_n) \geq \left( \frac{\text{overlap}(\text{Tr}(G_n), \mathbb{R}^2)}{63} \right)^{1/3} \geq \left( \frac{\beta}{189} \right)^{1/3}.$$  

We now prove the lower bound for $\Delta_0$ (defined at the start of the section).
Theorem 28. Every graph class with maximum degree 5 and bounded queue-number has bounded stack-number.

The proof of Theorem 28 depends on the following definitions. For $k, c \in \mathbb{N}$, a graph $G$ is $k$-colourable with clustering $c$ if each vertex of $G$ can be assigned one of $k$ colours such that each monochromatic component has at most $c$ vertices. Here a monochromatic component is a maximal monochromatic connected subgraph. The clustered chromatic number of a graph class $\mathcal{G}$ is the minimum $k \in \mathbb{N}$ such that for some $c \in \mathbb{N}$ every graph in $\mathcal{G}$ is $k$-colourable with clustering $c$. See [60] for a survey on clustered graph colouring. Haxell, Szabó, and Tardos [39] proved that the class of graphs with maximum degree at most 5 has clustered chromatic number 2 (which is best possible, since $(P_n \uplus P_n)_{n \in \mathbb{N}}$ has maximum degree 6 and clustered chromatic number 3 by the Hex Lemma). Thus, Theorem 28 is an immediate consequence of the following result.

Theorem 29. Every graph class $G$ with bounded queue-number and clustered chromatic number at most 2 has bounded stack-number.

The proof of Theorem 29 depends on the following lemmas.

Lemma 30 ([25, 52]). For every bipartite graph $G$,
\[
\text{sn}(G) \leq 2 \text{qn}(G).
\]

Ganley and Heath [35] proved that every graph of treewidth $k$ has stack-number at most $k + 1$. A close inspection of their proof actually shows the following result. A star forest is a forest where each component is isomorphic to a star. In a $k$-vertex component of a star forest with $k \neq 2$, there is a unique vertex of degree $k - 1$ called its centre. In a 2-vertex component, any vertex may be chosen as the centre.

Lemma 31 ([35]). Every graph $G$ of treewidth $k$ has an edge-partition into $k + 1$ spanning star forests $G_1, \ldots, G_{k+1}$, such that each vertex is the centre of a star component of some $G_i$.

Lemma 32. For every graph $G$ and $t \in \mathbb{N}$,
\[
\text{sn}(G \boxtimes K_t) \leq \max\{3t \text{ sn}(G), \lceil \frac{t}{2} \rceil \}.
\]

Proof. We may assume that $G$ is connected and $V(K_t) = \{1, \ldots, t\}$. If $|V(G)| = 1$ then the claim holds since $\text{sn}(G) = 0$ and $\text{sn}(G \boxtimes K_t) = \text{sn}(K_t) \leq \lceil \frac{t}{2} \rceil$; see [10]. Now assume that $|V(G)| \geq 2$ and thus $E(G) \neq \emptyset$. Let $s = \text{sn}(G) \geq 1$. Let $\{v_1, \ldots, v_s\}$ together with $\psi: E(G) \to \{1, \ldots, s\}$ be an $s$-stack layout of $G$. For each $k \in \{1, \ldots, s\}$, let $G_k$ be the spanning subgraph of $G$ with $E(G_k) = \psi^{-1}(k)$. Thus $G_k$ admits a 1-stack layout and is therefore outerplanar. Since outerplanar graphs have treewidth at most 2, it follows by Lemma 31 that $G_k$ has an edge-partition into three spanning star forests $G_{k,1}, G_{k,2}, G_{k,3}$, such that for each vertex $v \in V(G)$, there is some $a \in \{1, 2, 3\}$ such that $v$ is the centre of a star component of $G_{k,a}$.

Define $\phi: E(G \boxtimes K_t) \to \{1, \ldots, s\} \times \{1, 2, 3\} \times \{1, \ldots, t\}$ as follows. Consider an edge $e = (u, i)(v, j)$ of $G \boxtimes K_t$. If $u = v$ then let $(k, a) \in \{1, \ldots, s\} \times \{1, 2, 3\}$ be such that $u$ is the centre of a star component of $G_{k,a}$. Let $\phi(e) = (k, a, i)$. Otherwise $uv \in E(G_{k,a})$ for some $k \in \{1, \ldots, s\}$ and $a \in \{1, 2, 3\}$. Since $G_{k,a}$ is a star-forest, without loss of generality, $u$ is the centre of the star component of $G_{k,a}$ containing $uv$. Let $\phi(e) = (k, a, i)$.

We claim that $\phi$ and the vertex-ordering
\[
((v_1, 1), \ldots, (v_1, t); (v_2, 1), \ldots, (v_2, t); \ldots; (v_n, 1), \ldots, (v_n, t))
\]
define a $3st$-stack layout of $G \boxtimes K_t$ where $\phi^{-1}((k, a, i))$ is a non-crossing star-forest for each $(k, a, i) \in \{1, \ldots, s\} \times \{1, 2, 3\} \times \{1, \ldots, t\}$. Consider edges $e$ and $e'$ of $G \boxtimes K_t$ with $\phi(e) = \phi(e') = (k, a, i)$ for some $(k, a, i) \in \{1, \ldots, s\} \times \{1, 2, 3\} \times \{1, \ldots, t\}$. By construction, $e = (u, i)(v, j)$ and $e' = (x, i)(y, \ell)$ for some $u, v, x, y \in V(G)$ and $j, \ell \in \{1, \ldots, t\}$. If $u \neq v$
then \( uv \) is an edge of some component of \( G_{k,a} \) centred at \( u \). Similarly, if \( x \neq y \) then \( xy \) is an edge of some component of \( G_{k,a} \) centred at \( x \). If \( u \) and \( x \) are in the same component of \( G_{k,a} \), then \( e \) and \( e' \) have a common end-vertex \( (u,i) = (x,i) \) and therefore do not cross. Now assume \( u \) and \( x \) are in distinct components of \( G_{k,a} \). If \( u \neq v \) and \( x \neq y \), then \( \psi(uv) = \psi(xy) = k \) and \( uv \) and \( xy \) do not cross, implying \( e \) and \( e' \) do not cross by the choice of vertex-ordering. Now assume that \( u = v \) or \( x = y \). Since \( \{u,v\} \cap \{x,y\} = \emptyset \), \( e \) and \( e' \) do not cross by the choice of vertex-ordering.

\[ \square \]

**Proof of Theorem 29.** By assumption, there exist \( c, \ell \in \mathbb{N} \) such that for every graph \( G \in \mathcal{G} \), we have \( qn(G) \leq c \) and \( G \) is 2-colourable with each monochromatic component having at most \( \ell \) vertices. Contracting each monochromatic component to a single vertex gives a bipartite \( \ell \)-small minor \( H \) of \( G \). Lemma 11 implies \( qn(H) \leq 2\ell(2c)^{2\ell} \). Lemma 30 implies \( sn(H) \leq 4\ell(2c)^{2\ell} \). By construction, \( G \) is isomorphic to a subgraph of \( H \boxtimes K_\ell \). Thus \( sn(G) \leq sn(H \boxtimes K_\ell) \), which is at most \( 12\ell^2(2c)^{2\ell} \) by Lemma 32. Hence \( \mathcal{G} \) has bounded stack-number.

7. **Open Problems**

We finish with some open problems:

- Does there exist a graph class with bounded stack-number and unbounded queue-number? This is equivalent to the question of whether graphs with stack-number 3 have bounded queue-number [28].
- Do graphs with queue-number 2 (or 3) have bounded stack-number [22]?
- Do graph classes with quadratic (or linear) growth have bounded stack-number?
- Does there exist a graph family with unbounded stack-number, bounded queue-number and maximum degree 6?
- The best known lower bound on the maximum stack-number of \( n \)-vertex graphs with fixed maximum degree \( \Delta \) is \( \Omega(n^{1/2-1/\Delta}) \), proved by Malitz [48] using a probabilistic argument. Is there a constructive proof of this bound?
- The best upper bound on the stack-number of \( n \)-vertex graphs with fixed maximum degree \( \Delta \) is \( O(n^{1/2}) \), also due to Malitz [48]. Closing the gap between the lower and upper bounds is an interesting open problem. For example, the best bounds for graphs of maximum degree 3 are \( \Omega(n^{1/6}) \) and \( O(n^{1/2}) \).

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