A short proof of the phase transition for the vacant set of random interlacements

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Abstract

The vacant set of random interlacements at level $u > 0$, introduced in [8], is a percolation model on $\mathbb{Z}^d$, $d \geq 3$ which arises as the set of sites avoided by a Poissonian cloud of doubly infinite trajectories, where $u$ is a parameter controlling the density of the cloud. It was proved in [6, 8] that for any $d \geq 3$ there exists a positive and finite threshold $u_*$ such that if $u < u_*$ then the vacant set percolates and if $u > u_*$ then the vacant set does not percolate. We give an elementary proof of these facts. Our method also gives simple upper and lower bounds on the value of $u_*$ for any $d \geq 3$.

1 Introduction

The model of random interlacements was introduced in [8]. The interlacement $I^u$ at level $u > 0$ is a random subset of $\mathbb{Z}^d$, $d \geq 3$ that arises as the local limit as $N \to \infty$ of the range of the first $\lfloor uN^d \rfloor$ steps of a simple random walk on the discrete torus $(\mathbb{Z}/N\mathbb{Z})^d$, $d \geq 3$, see [14]. The law of $I^u$ is characterized by

$$\mathbb{P}[I^u \cap K = \emptyset] = e^{-u \cdot \text{cap}(K)}, \quad \text{for any finite } K \subseteq \mathbb{Z}^d, \quad (1.1)$$

where $\text{cap}(K)$ denotes the discrete capacity of $K$, see [25]. The vacant set of random interlacements $V^u$ at level $u$ is defined as the complement of $I^u$ at level $u$:

$$V^u = \mathbb{Z}^d \setminus I^u, \quad u > 0. \quad (1.2)$$

By [8] (1.68)] the correlations of $V^u$ decay polynomially for any $u > 0$:

$$\mathbb{P}[x, y \in V^u] - \mathbb{P}[x \in V^u] \cdot \mathbb{P}[y \in V^u] \asymp (|x - y| \lor 1)^{2-d}, \quad x, y \in \mathbb{Z}^d. \quad (1.3)$$

One is interested in the connectivity properties of the subgraphs of the nearest-neighbour lattice $\mathbb{Z}^d$ spanned by the above random sets. For any $u > 0$, $I^u$ is a $\mathbb{P}$-a.s. connected random subset of $\mathbb{Z}^d$ (see [8] (2.21)), but $V^u$ exhibits a percolation phase transition: there exists $u_* \in (0, \infty)$ such that

(i) for any $u > u_*$, $\mathbb{P}$-a.s. all connected components of $V^u$ are finite, and

(ii) for any $u < u_*$, $\mathbb{P}$-a.s. $V^u$ contains an infinite connected component.

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The fact that \( u_\ast < \infty \) was proved in \([8]\) Section 3], and the positivity of \( u_\ast \) was established in \([8]\) Section 4] when \( d \geq 7 \), and later in \([6]\) for all \( d \geq 3 \).

There is no reason to believe that an exact formula for the value of the critical threshold \( u_\ast = u_\ast(d) \) exists. However, it is proved in \([9, 10]\) that

\[
\lim_{d \to \infty} \frac{u_\ast(d)}{\ln(d)} = 1, \tag{1.4}
\]

in agreement with the principal asymptotic behaviour of the critical threshold of random interlacements on \( 2d \)-regular trees, which is explicitly computed in \([12]\) Proposition 5.2].

The aim of this paper is to give a short proof of the non-triviality of phase transition of \( \mathcal{V}^u \) and to provide simple explicit upper and lower bounds on the value of \( u_\ast = u_\ast(d), d \geq 3 \).

For any \( d \geq 3 \) let us denote by \( 0 < c_g = c_g(d) \) and \( C_g = C_g(d) < +\infty \) the best constants such that the inequalities

\[
c_g \cdot (|x - y| + 1)^{2-d} \leq g(x, y) \leq C_g \cdot (|x - y| + 1)^{2-d}, \quad x, y \in \mathbb{Z}^d \tag{1.5}
\]

hold, where \(| \cdot |\) is the \( \ell^\infty \)-norm on \( \mathbb{Z}^d \) and \( g(\cdot, \cdot) \) is the Green function of simple random walk on \( \mathbb{Z}^d \), see \([2, 3]\). The positivity of \( c_g \) and \( C_g < +\infty \) follow from \([4]\) Theorem 1.5.4]

**Theorem 1.1.** For any \( d \geq 3 \), we have

\[
\frac{c_g}{L_0 C_2} 2^{-d(d+5)} \leq u_\ast \leq \frac{5}{2} C_g \ln(C_d), \tag{1.6}
\]

where

\[
C_d = (13^d - 11^d)(25^d - 23^d), \quad d \geq 2, \tag{1.7}
\]

and

\[
L_0 = \begin{cases} 
\exp \left( 48 \frac{C_u}{c_g} C_2 \right) & \text{if } d = 3, \\
\left( 48 \frac{C_u}{c_g} C_2 \right)^{\frac{d-3}{d-1}} & \text{if } d \geq 4.
\end{cases} \tag{1.8}
\]

The bounds \((1.6)\) are not at all sharp, especially if we compare them with \((1.3)\) as \( d \to \infty \). This shortcoming of Theorem 1.1 is counterbalanced by the fact that its proof is very simple. In particular, our self-contained proof does not use the “sprinkling” technique and decoupling inequalities usually applied in order to overcome the long-range correlations \((1.3)\) present in the model. The proof of \( u_\ast(d) > 0 \) for \( d \geq 7 \) in \([8]\) Section 4] does not use “sprinkling”, but the proof of \( u_\ast(d) < +\infty \) for any \( d \geq 3 \) in \([8]\) Section 3] and the proof of \( u_\ast(d) > 0 \) for \( 3 \leq d \leq 7 \) in \([6]\) does. Various forms of decoupling inequalities have been subsequently developed to study the connectivity properties of \( \mathcal{V}^u \) in the subcritical \([5, 7, 11]\) and supercritical \([2, 13]\) phases. These techniques are very useful once they are available, but the elementary method of our paper seems to be easier to adapt to other percolation models with long-range correlations, e.g., branching interlacements \([1]\).

Let us briefly describe the idea of the proof of Theorem 1.1. We employ multi-scale renormalization. In order to prove \( u_\ast < +\infty \) we show that if \( \mathcal{V}^u \) crosses an annulus at scale \( L_n = 6^n \) then this vacant crossing contains a set \( \mathcal{X}_T \) of \( 2^n \) well-separated vertices which arise as the image of leaves under an embedding \( T \) of the dyadic tree of depth \( n \) (this method already appears in \([11]\)). By construction, the number of possible embeddings is less than \( C_d^{2^n} \) (c.f. \([1.7]\)), so we only need to show that \( \text{cap}(\mathcal{X}_T) \propto 2^n \) if we want to use \((1.1)\) to to show that crossing of the annulus by \( \mathcal{V}^u \) is unlikely when \( u \) is big enough. This is indeed the case, because by construction the embedding \( T \) is “spread-out on all scales”, thus the cardinality and the capacity of \( \mathcal{X}_T \) are comparable.
In order to prove $u_* > 0$, we restrict our attention to a plane inside $\mathbb{Z}^d$. By planar duality we only need to show that a $*$-connected crossing of a planar annulus at scale $L_n = L_0 \cdot 6^n$ by $\mathcal{I}^n$ is unlikely. We show that such a crossing must intersect $2^n$ “frames”, where each frame is the union of four “sticks” of length $2L_0 - 1$. Such a collection of frames again arises from a spread-out embedding of the dyadic tree of depth $n$. We use that $I^u$ can be written as the union of the ranges of a Poissonian cloud of independent random walks and the fact that random walks tend to avoid sticks if $L_0$ is large enough (c.f. (1.8)) to arrive at a large deviation estimate on the probability that the number of frames that intersect $I^u$ is $2^n$ which is strong enough to beat the combinatorial complexity term $C_{2^n}$. This stick-based approach to $u_* > 0$ is already present in [6, Section 3] and our large deviation estimate resembles the one in the proof of [8, Theorem 2.4].

The rest of this paper is organized as follows. In Section 2 we introduce further notation and recall some useful facts related to the notion of capacity and random interlacements. In Section 3 we define the notion of a proper embedding of a dyadic tree into $\mathbb{Z}^d$ and derive some facts about such embeddings. In Sections 4 and 5 we prove the upper and lower bounds on $u_*$ stated in Theorem 1.1.

2 Preliminaries

For a set $K$, we denote by $|K|$ its cardinality. We denote by $K \subset\subset \mathbb{Z}^d$ the fact that $K$ is a finite subset of $\mathbb{Z}^d$. We denote by $|x|$ the $\ell^\infty$-norm of $x \in \mathbb{Z}^d$ and by $S(x, R)$ the $\ell^\infty$-sphere of radius $R$ about $x$ in $\mathbb{Z}^d$:

$$ S(x, R) = \{ y \in \mathbb{Z}^d : |y - x| = R \}. $$

(2.1)

For $x \in \mathbb{Z}^d$, denote by $P_x$ the law of simple random walk $(X_n)_{n=0}^\infty$ on $\mathbb{Z}^d$ starting at $X_0 = x$. If $m$ is a probability measure on $\mathbb{Z}^d$, we denote by

$$ P_m = \sum_{x \in \mathbb{Z}^d} m(x) P_x $$

(2.2)

the law of simple random walk with initial distribution $m$ and by $E_m$ the corresponding expectation. The Green function of simple random walk on $\mathbb{Z}^d$ is defined by

$$ g(x, y) = \sum_{n=0}^\infty P_{x}[X_n = y], \quad x, y \in \mathbb{Z}^d. $$

(2.3)

Let us denote by $\{X\} \subseteq \mathbb{Z}^d$ the range of the random walk:

$$ \{X\} = \bigcup_{n=0}^\infty \{X_n\} $$

(2.4)

2.1 Potential theory

If $K \subset\subset \mathbb{Z}^d$, we define the equilibrium measure $e_K(\cdot)$ of $K$ by

$$ e_K(x) = P_x[X_n \notin K \text{ for any } n \geq 1], \quad x \in K. $$

The total mass of the equilibrium measure is called the capacity of $K$:

$$ \text{cap}(K) = \sum_{x \in K} e_K(x). $$

(2.5)

One defines the normalized equilibrium measure $\tilde{e}_K(\cdot)$ of $K$ by

$$ \tilde{e}_K(x) = \frac{e_K(x)}{\text{cap}(K)}. $$

(2.6)
Let us now collect some facts about capacity that we will use in the sequel. The proofs of the properties (2.7)-(2.10) below can be found in, e.g., [3, Section 1.3].

For any \( x \in \mathbb{Z}^d \) and any \( K \subset \subset \mathbb{Z}^d \),

\[
P_x \{ \{ X \} \cap K \neq \emptyset \} = \sum_{y \in K} g(x, y) e_K(y) \leq \text{cap}(K) \max_{y \in K} g(x, y). \quad (2.7)
\]

For any \( K_1, K_2 \subset \subset \mathbb{Z}^d \),

\[
\text{cap}(K_1 \cup K_2) \leq \text{cap}(K_1) + \text{cap}(K_2). \quad (2.8)
\]

For any \( K \subseteq K' \subset \subset \mathbb{Z}^d \),

\[
\text{cap}(K) \leq \text{cap}(K'). \quad (2.9)
\]

For any \( K \subset \subset \mathbb{Z}^d \),

\[
\frac{|K|}{\max_{x \in K} \sum_{y \in K} g(x, y)} \leq \text{cap}(K) \leq \frac{|K|}{\min_{x \in K} \sum_{y \in K} g(x, y)}. \quad (2.10)
\]

Let us denote by \( F \) the plane

\[
F = \mathbb{Z}^2 \times \{0\}^{d-2} \subseteq \mathbb{Z}^d. \quad (2.11)
\]

For any \( y \in F \) and \( L \geq 1 \) let us define the frame \( \Box_y^L \subseteq F \) by

\[
\Box_y^L \equiv S(y, L-1) \cap F. \quad (2.12)
\]

The next lemma gives an explicit upper bound on the capacity of a frame. The bounds of (2.13) are actually sharp up to a dimension-dependent constant factor, but we will only use the upper bounds. The stronger bound for \( d = 3 \) is crucial to showing that random walks tend to avoid frames in \( \mathbb{Z}^3 \). The extra \( \ln(L) \) makes the parameter \( p \) defined in (5.6) small, which is necessary for our proof of \( u_*(3) > 0 \). Recall the notion of \( c_g \) from (1.5).

**Lemma 2.1.** For any \( L \geq 1 \) we have

\[
\text{cap}(\Box_y^L) \leq \begin{cases} 8 \frac{L}{c_g} & \text{if } d \geq 4, \\ 8 \frac{L}{c_g (1 + \ln(L))} & \text{if } d = 3. \end{cases} \quad (2.13)
\]

**Proof.** Denote by \( S_\ell = \{1, \ldots, \ell\} \times \{0\}^{d-1} \subseteq \mathbb{Z}^d \) the stick of length \( \ell \). We will use (2.10) to bound \( \text{cap}(S_\ell) \). If \( x \in S_\ell \) then \( x = \{i\} \times \{0\}^{d-1} \) for some \( 1 \leq i \leq \ell \) and

\[
\sum_{y \in S_\ell} g(x, y) \geq \sum_{j=1}^\ell c_g \cdot (|j-i| \lor 1)^{2-d} \geq \sum_{j=1}^\ell c_g \cdot (|j-1| \lor 1)^{2-d} = c_g \left( 1 + \sum_{k=1}^{\ell-1} k^{2-d} \right) \geq c_g \left( c_g \cdot \left( 1 + \int_1^\ell \frac{1}{s} \, ds \right) = c_g \cdot (1 + \ln(\ell)) \right) \quad \text{if } d \geq 4,
\]

\[
\text{and } \frac{\ell}{c_g (1 + \ln(\ell))} \quad \text{if } d = 3.
\]

Using these bounds, (2.10) and \( |S_\ell| = \ell \) we obtain that \( \text{cap}(S_\ell) \leq \ell / c_g \) if \( d \geq 4 \) and \( \text{cap}(S_\ell) \leq \ell / (c_g \cdot (1 + \ln(\ell))) \) if \( d = 3 \). Now the frame \( \Box_y^L \) is the union of four sticks of length \( 2L - 1 \), thus (2.13) follows from the above bounds and (2.8), (2.9).
2.2 Constructive definition of random interlacements

The definition of the interlacement $I^u$ at level $u$ by the formula (1.1) is short, but it is not constructive. The construction of [8, Section 1] involves a Poisson point process with intensity measure $u \cdot \nu$, where $\nu$ is a sigma-finite measure on the space of equivalence classes of doubly infinite trajectories modulo time-shift. The union of the ranges of trajectories which are contained in the support of this Poisson point process is denoted by $I^u$, and this random subset of $\mathbb{Z}^d$ indeed satisfies (1.1).

We will not use the full definition of random interlacements, only a corollary of it, which allows one to construct a set with the same law as $I^u \cap K$ for any $K \subseteq \mathbb{Z}^d$.

Recall the notion of $P_m$ from (2.2), $\{X\}$ from (2.4) and $\tilde{e}_K(\cdot)$ from (2.6).

Claim 2.2. Let $d \geq 3$, $K \subseteq \mathbb{Z}^d$, $N_K$ be a Poisson random variable with parameter $u \cdot \text{cap}(K)$, and $(X_j)_{j \geq 1}$ i.i.d. simple random walks with distribution $P_{\tilde{e}_K}$ and independent from $N_K$. Then $K \cap \bigcup_{j=1}^{N_K} \{X_j\}$ has the same distribution as $I^u \cap K$.

This explicit “local representation” of $I^u$ follows from the very construction of the sigma-finite measure $\nu$, which is obtained by patching together certain explicit measures $Q_K$, $K \subseteq \mathbb{Z}^d$ in a consistent manner in [8, Theorem 1.1]. The above representation of $I^u \cap K$ is obtained from the Poisson point process with intensity measure $uQ_K$.

3 Renormalization

For $n \geq 0$, let $T_n = \{1, 2\}^n$ (in particular, $T_0 = \emptyset$). Denote by

$$T_n = \bigcup_{k=0}^{n} T_{(k)}$$

the dyadic tree of depth $n$. For $0 \leq k < n$ and $m \in T_{(k)}$, $m = (\xi_1, \ldots, \xi_k)$, we denote by

$$m_1 = (\xi_1, \ldots, \xi_k, 1) \quad \text{and} \quad m_2 = (\xi_1, \ldots, \xi_k, 2)$$

the two children of $m$ in $T_{(k+1)}$. Given some $L_0 \geq 1$ we define the sequence of scales

$$L_n := L_0 \cdot 6^n, \quad n \geq 0.$$

For $n \geq 0$, we denote by $L_n = L_n \mathbb{Z}^d$ the lattice $\mathbb{Z}^d$ renormalized by $L_n$.

Definition 3.1. $T : T_n \to \mathbb{Z}^d$ is a proper embedding of $T_n$ with root at $x \in L_n$ if

1. $T(\emptyset) = x$;
2. for all $0 \leq k \leq n$ and $m \in T_{(k)}$ we have $T(m) \in L_{n-k}$;
3. for all $0 \leq k < n$ and $m \in T_{(k)}$ we have

$$|T(m_1) - T(m)| = L_{n-k}, \quad |T(m_2) - T(m)| = 2L_{n-k}. \quad (3.3)$$

We denote by $\Lambda_{n,x}$ the set of proper embeddings of $T_n$ into $\mathbb{Z}^d$ with root at $x$.

Lemma 3.2. For any $L_0 \geq 1$, $n \geq 0$ and $x \in L_n$ the number of proper embeddings of $T_n$ into $\mathbb{Z}^d$ with root at $x$ is equal to

$$|\Lambda_{n,x}| = \mathcal{C}_d^{2^n-1}. \quad (3.4)$$
Proof. The claim is trivially true for \( n = 0 \). If \( n \geq 1 \), \( x \in \mathcal{L}_n \) and \( \mathcal{T} \in \Lambda_{n,x} \), we denote by \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) the two embeddings of \( T_{n-1} \) which arise from \( \mathcal{T} \) as the embeddings of the descendants of the two children of the root, i.e., for any \( 0 \leq k \leq n - 1 \) and \( m = (\xi_1, \ldots, \xi_k) \in T(k) \) let \( \mathcal{T}_k(m) = \mathcal{T}(\xi, \xi_2, \ldots, \xi_k) \) for \( \xi \in \{1, 2\} \). By Definition 3.1 we have \( \mathcal{T}_k \in \Lambda_{n-1,T(\xi)} \) for \( \xi \in \{1, 2\} \), thus we obtain (3.4) by induction on \( n \):

\[
|\Lambda_{n,x}| = |\mathcal{L}(x, L_n) \cap \mathcal{L}_{n-1}| \cdot |\mathcal{L}(x, 2L_n) \cap \mathcal{L}_{n-1}| \cdot |\Lambda_{n-1,T(1)}| \cdot |\Lambda_{n-1,T(2)}| = |\mathcal{L}(0, 6)| \cdot |\mathcal{L}(0, 12)| \cdot |\Lambda_{n-1,T(1)}| \cdot |\Lambda_{n-1,T(2)}| = C_d \cdot C_d^{2^{n-1} - 1} \cdot C_d^{2^{n-1} - 1} = C_d^{2^n},
\]

where in (*) we used the induction hypothesis. \( \square \)

We say that \( \gamma : \{0, \ldots, l\} \to \mathbb{Z}^d \) is a *-connected path if \( |\gamma(i) - \gamma(i-1)| = 1 \) for any \( 1 \leq i \leq l \). For such a path we denote by \( \{\gamma\} = \{\gamma(1), \ldots, \gamma(l)\} \) the range of \( \gamma \).

Recall the notion of \( S(x, R) \) from (2.1) and note that \( S(x, 0) = \{x\} \).

**Lemma 3.3.** If \( \gamma \) is a *-connected path in \( \mathbb{Z}^d \), \( d \geq 2 \) and \( x \in \mathcal{L}_n \) such that

\[
\{\gamma\} \cap S(x, L_n - 1) \neq \emptyset \quad \text{and} \quad \{\gamma\} \cap S(x, 2L_n) \neq \emptyset
\]

then there exists \( \mathcal{T} \in \Lambda_{n,x} \) such that

\[
\{\gamma\} \cap S(\mathcal{T}(m), L_0 - 1) \neq \emptyset \quad \text{for all} \quad m \in T(m).
\]

**Proof.** We will prove that (3.5) implies that there exists \( \mathcal{T} \in \Lambda_{n,x} \) such that for all \( 0 \leq k \leq n \) we have

\[
\{\gamma\} \cap S(\mathcal{T}(m), L_{n-k} - 1) \neq \emptyset; \quad \{\gamma\} \cap S(\mathcal{T}(m), 2L_{n-k}) \neq \emptyset \quad \text{for all} \quad m \in T(k).
\]

We will construct such a \( \mathcal{T} \in \Lambda_{n,x} \) by induction on \( k \). By \( \mathcal{T}(\emptyset) = x \) we see that the case \( k = 0 \) of (3.7) is just (3.5). Assuming that (3.7) holds for some \( 0 \leq k \leq n - 1 \) we now show that it also holds for \( k + 1 \). If \( m \in T(k) \) then our induction hypothesis (3.7) and the fact that \( \gamma \) is a *-connected path imply

\[
\{\gamma\} \cap S(\mathcal{T}(m), L_{n-k} + L_{n-k-1} - 1) \neq \emptyset, \quad \{\gamma\} \cap S(\mathcal{T}(m), 2L_{n-k} - L_{n-k-1} + 1) \neq \emptyset.
\]

We also have

\[
S(\mathcal{T}(m), L_{n-k} + L_{n-k-1} - 1) \subseteq \bigcup_{y \in S(\mathcal{T}(m), L_{n-k}) \cap \mathcal{L}_{n-k-1}} S(y, L_{n-k-1} - 1),
\]

\[
S(\mathcal{T}(m), 2L_{n-k} - L_{n-k-1} + 1) \subseteq \bigcup_{z \in S(\mathcal{T}(m), 2L_{n-k}) \cap \mathcal{L}_{n-k-1}} S(z, L_{n-k-1} - 1),
\]

thus we can choose

\[
\mathcal{T}(m_1) \in S(\mathcal{T}(m), L_{n-k}) \cap \mathcal{L}_{n-k-1} \quad \text{and} \quad \mathcal{T}(m_2) \in S(\mathcal{T}(m), 2L_{n-k}) \cap \mathcal{L}_{n-k-1}
\]

such that

\[
\{\gamma\} \cap S(\mathcal{T}(m_1), L_{n-(k+1)} - 1) \neq \emptyset, \quad \{\gamma\} \cap S(\mathcal{T}(m_2), L_{n-(k+1)} - 1) \neq \emptyset.
\]

It follows from this, \( |\mathcal{T}(m_1) - \mathcal{T}(m_2)| \geq L_{n-k} = 6L_{n-(k+1)} \) and the fact that \( \gamma \) is a *-connected path that we also have

\[
\{\gamma\} \cap S(\mathcal{T}(m_1), 2L_{n-(k+1)}) \neq \emptyset, \quad \{\gamma\} \cap S(\mathcal{T}(m_2), 2L_{n-(k+1)}) \neq \emptyset.
\]

We have thus constructed the embedding \( \mathcal{T} \) up to depth \( k + 1 \) so that Definition 3.1 is satisfied up to depth \( k + 1 \) and (3.7) also holds for \( k + 1 \). Therefore by induction we have constructed \( \mathcal{T} \in \Lambda_{n,x} \) such that (3.7) holds for all \( 0 \leq k \leq n \), which implies (3.6). The proof of Lemma 3.3 is complete. \( \square \)
For $0 \leq k \leq n$ and $m = (\xi_1, \ldots, \xi_n) \in T(n)$ we denote $m|_k = (\xi_1, \ldots, \xi_k) \in T(k)$. Let us denote the lexicographic distance of $m, m' \in T(n)$ by
\[
\rho(m, m') = \min\{k \geq 0 : m|_{n-k} = m'|_{n-k}\}.
\]
For any $m \in T(n)$ and $0 \leq k \leq n$ we define
\[
\mathcal{T}^{m,k}(n) = \{m' \in T(n) : \rho(m, m') = k\},
\]
(3.8) see Figure 1 for an illustration. Note that
\[
|\mathcal{T}^{m,k}(n)| = 2^{k-1}, \quad 1 \leq k \leq n.
\]
(3.9)

The next lemma shows that a proper embedding is “spread-out on all scales.”

**Lemma 3.4.**
\[
\forall \ n \geq 1, \ x \in \mathcal{L}_n, \ T \in \Lambda_{n,x}, \ m \in T(n), \ k \geq 1,
\forall \ m' \in T^{m,k}(n), \ y \in S(T(m), L_0 - 1), \ z \in S(T(m'), L_0 - 1) : 
|y - z| \geq L_{k-1}.
\]
(3.10)

**Proof.** Let $m'' = m|_{n-k} = m'|_{n-k} \in T(n-k)$. Recalling (3.1) we may assume w.l.o.g. that $m|_{n-k+1} = m''|_{n-k+1} \in T(n-k+1)$ and $m'|_{n-k+1} = m''|_{n-k+1} \in T(n-k+1)$. We have
\[
|\mathcal{T}(m'') - \mathcal{T}(m'')| \geq L_k = 6L_{k-1},
\]
(3.11)
moresover
\[
|\mathcal{T}(m'') - y| \leq |\mathcal{T}(m) - y| + \sum_{j=1}^{k-1} |\mathcal{T}(m|_{n-j}) - \mathcal{T}(m|_{n-j+1})| \leq L_0 - 1 + 2\sum_{j=1}^{k-1} 2L_j \leq 2L_{k-1} \sum_{i=0}^{\infty} 6^{-i} = \frac{6}{5} L_{k-1},
\]
and similarly $|\mathcal{T}(m'') - z| \leq \frac{12}{5} L_{k-1}$. Putting these bounds together we obtain (3.10). \(\square\)
4 Upper bound on \( u_* \)

Let us choose \( L_0 = 1 \) in (3.2). For \( n \geq 1 \) let us denote by \( A_n^u \) the event

\[
A_n^u = \left\{ \text{there exists a nearest-neighbour path in } V^u \text{ that connects } S(0, L_n - 1) \to S(0, 2L_n) \right\}.
\]

Recall the definitions of \( C_g \) from (1.8) and \( C_d \) from (1.7).

**Proposition 4.1.** For any \( d \geq 3 \) and

\[
u > \frac{5}{2} C_g \ln(C_d)
\]

there exists \( q = q(d, u) \in (0, 1) \) such that for any \( n \geq 1 \) we have

\[
\mathbb{P}[A_n^u] \leq q^{2^n}.
\]  

**Corollary 4.2.** Proposition 4.1 implies the upper bound of Theorem 1.1, as we now explain. Let us denote by \( \tilde{A}_n^u \) the event that there exists a nearest-neighbour path in \( V^u \) that connects \( S(0, L_n - 1) \to \infty \) and by \( \tilde{A}_\infty^u \) the event that \( V^u \) has an infinite connected component. If \( (4.1) \) holds, then

\[
\mathbb{P}[\tilde{A}_\infty^u] = \lim_{n \to \infty} \mathbb{P}[\tilde{A}_n^u] \leq \lim_{n \to \infty} \mathbb{P}[A_n^u] \leq 0,
\]

where \( (*) \) holds by monotone convergence. Therefore we have \( u_* \leq \frac{5}{2} C_g \ln(C_d) \).

**Proof of Proposition 4.1.** For any \( n \geq 1 \) and \( T \in \Lambda_{n,0} \) we denote \( \mathcal{X}_T = \bigcup_{m \in T(n)} T(m) \). Noting that \( S(T(m), L_0 - 1) = S(T(m), 0) = \{ T(m) \} \) for any \( m \in T(n) \) and that every nearest-neighbour path is also a \(*\)-connected path we can apply Lemma 3.3 to infer

\[
\mathbb{P}[A_n^u] \leq \mathbb{P} \left[ \bigcup_{T \in \Lambda_{n,0}} \{ \mathcal{X}_T \subseteq V^u \} \right] \leq \sum_{T \in \Lambda_{n,0}} \exp(-u \cdot \text{cap}(\mathcal{X}_T)) \leq C_d^{2^n} \max_{T \in \Lambda_{n,0}} \exp(-u \cdot \text{cap}(\mathcal{X}_T)).
\]  

In order to finish the proof of Proposition 4.1 we only need to show that for any \( T \in \Lambda_{n,0} \) we have

\[
\text{cap}(\mathcal{X}_T) \geq \frac{2}{5} \frac{1}{C_g} 2^n,
\]

because then we indeed obtain

\[
\mathbb{P}[A_n^u] \leq C_d^{2^n} \exp \left(-u \frac{2}{5} \frac{1}{C_g} 2^n \right) = \left( C_d \exp \left(-u \frac{2}{5} \frac{1}{C_g} \right) \right)^{2^n} = q^{2^n}, \quad q < 1.
\]

We will show (4.1) using (2.10). For any \( T \in \Lambda_{n,0} \) and any \( m \in T(n) \) we have

\[
\sum_{m' \in T(n)} g(T(m), T(m')) \leq \sum_{k=0}^{n} \sum_{m' \in T^m_{k-1}(n)} g(T(m), T(m')) \leq \sum_{k=1}^{n} C_g L^{k-1} T_{k-1}^{m-1}(n) \leq C_g \left( 1 + \sum_{k=1}^{n} 2^{(k-1)(2d-2)} 2^{(k-1)} \right) \leq C_g \frac{5}{2} C_g.
\]

Now (4.1) follows from (2.10), (4.5) and the fact that \( |\mathcal{X}_T| = 2^n \). The proof of Proposition 4.1 is complete. \( \square \)
5 Lower bound on \( u_* \)

Let us choose \( L_0 \) according to (1.8) in (3.2). Recall the notion of the plane \( F \) from (2.12). For \( n \geq 1 \) and \( x \in \mathcal{L}_n \cap F \) let us denote by \( B_{n,x}^u \) the event

\[
B_{n,x}^u = \left\{ \text{there exists a } * \text{-connected path in } \mathcal{I}_u \cap F \text{ that connects } S(x, L_n - 1) \text{ to } S(x, 2L_n) \right\}.
\]

Recall the definitions of \( c_g, C_g \) from (1.5) and \( C_d \) from (1.7).

**Proposition 5.1.** For any \( d \geq 3 \) and

\[
u < \frac{c_g 1}{L_0 c_2} 2^{-(d+5)}, \tag{5.1}
\]

for any \( n \geq 1 \) and \( x \in \mathcal{L}_n \cap F \) we have

\[
P[B_{n,x}^u] \leq \left( \frac{3}{4} \right)^2^n. \tag{5.2}
\]

**Corollary 5.2.** Proposition 5.1 implies the lower bound of Theorem 1.1, as we now explain. Let us denote by \( \hat{A}_{n,x}^u \) the event that there exists a nearest-neighbour path in \( \mathcal{V}_u \cap F \) that connects \( S(0, L_n) \) to infinity and by \( \hat{A}_{\infty}^u \) the event that \( \mathcal{V}_u \cap F \) has an infinite connected component. By planar duality the event \( \hat{A}_{n,x}^u \) is equal to the event that there exists a \( * \)-connected path in \( \mathcal{I}_u \cap F \) that surrounds \( S(0, L_n - 1) \), thus if (5.1) holds, then

\[
P[\hat{A}_{n,x}^u] \geq 1 - \sum_{k=n}^{\infty} 25^d \cdot \left( \frac{3}{4} \right)^{2k},
\]

which in turn implies \( P[\hat{A}_{\infty}^u] = \lim_{n \to \infty} P[\hat{A}_{n,x}^u] = 1 \). Therefore we have \( u_* \geq \frac{c_g 1}{L_0 c_2} 2^{-(d+5)} \).

**Proof of Proposition 5.1.** We say that \( T : \mathcal{T}_n \to F \) is a proper embedding of the dyadic tree \( \mathcal{T}_n \) with root at \( x \in \mathcal{L}_n \cap F \) into \( F \) if \( T \in \Lambda_{n,x} \) (see Definition 3.1). We denote by \( \Lambda_{n,x}^F \) the set of proper embeddings of \( \mathcal{T}_n \) into \( F \).

For any \( y \in \mathcal{L}_0 \cap F \) let us define the frame \( \square_y \subseteq F \) by

\[
\square_y \equiv \square_{L_0} = S(y, L_0 - 1) \cap F.
\]

For any \( n \geq 1 \), \( x \in \mathcal{L}_n \cap F \) and \( T \in \Lambda_{n,x}^F \) let us denote by

\[
A_T^{\square} = \bigcup_{m \in T(n)} \square_{T(m)}.
\]

We start the proof of Proposition 5.1 by an application of Lemma 3.3 with \( d = 2 \):

\[
P[B_{n,x}^u] \leq P \left[ \bigcup_{T \in \Lambda_{n,x}^F} \bigcap_{m \in T(n)} \{ \square_{T(m)} \cap \mathcal{I}_u = \emptyset \} \right] \leq \left( \frac{3}{4} \right)^2^n \cdot \max_{T \in \Lambda_{n,x}^F} P \left[ \bigcap_{m \in T(n)} \{ \square_{T(m)} \cap \mathcal{I}_u = \emptyset \} \right], \tag{5.4}
\]

where in (*) we used Lemma 3.2 to infer \( |\Lambda_{n,x}^F| \leq C_2^n \).
In order to bound the probability on the right-hand side of (5.4) let us fix some $T \in \mathcal{A}_{n,x}^F$, recall the constructive definition of random interlacements from Claim 2.2 and denote the probability underlying the random objects (i.e., $N_K$ and $X^j_{\geq 1}$) introduced in that claim by $P$ when $K = X^c_T$. For a simple random walk $X$ let us denote by
\[
\mathcal{N}(X) = \sum_{m \in T(n)} 1\{\{X\} \cap \square_{T(m)} \neq \emptyset\}
\]
the number of frames of form $\square_{T(m)}$, $m \in T(n)$ that $X$ visits. We can bound
\[
P\left[ \bigcap_{m \in T(n)} \{\square_{T(m)} \cap T^u \neq \emptyset\} \right] \leq P\left[ \sum_{j=1}^{N_K} \mathcal{N}(X^j) \geq 2^n \right]. \tag{5.5}
\]

Our next goal is to stochastically bound $\mathcal{N}(X)$. Recall the definitions of $c_g, C_g$ from (1.5) and $L_0$ from (1.8). Let us define
\[
p = \begin{cases} 
12C_g/c_g \cdot L_0^{3-d} & \text{if } d \geq 4, \\
12C_g/c_g \cdot \frac{1}{1+\ln(L_0)} & \text{if } d = 3. 
\end{cases} \tag{5.6}
\]
For any $m \in T(n), y \in \square_{T(m)}$ we have
\[
P_y[\{X\} \cap X^c_T \setminus \square_{T(m)} \neq \emptyset] \leq 10 \sum_{k=1}^{n} \sum_{m' \in T(n)} P_y[\{X\} \cap \square_{T(m')} \neq \emptyset] \leq \sum_{k=1}^{n} \sum_{m' \in T(n)} C_g L_0^{2-d} \text{cap}(\square_{T(m')}) \tag{5.7}
\]
\[
\leq \sum_{k=1}^{n} \sum_{m' \in T(n)} C_g L_0^{2-d} \text{cap}(\square_0) \sum_{k=1}^{\infty} 3^{-k} \leq g (2.8) \leq p. \tag{5.7}
\]

The bound (5.7) together with the strong Markov property of simple random walk imply that $P_{e_K}[\mathcal{N}(X) \geq k] \leq p^{k-1}$ for any $k \geq 1$. In other words, $\mathcal{N}(X)$ is stochastically dominated by a geometric random variable with parameter $1 - p$, which implies $E_{e_K}[z^{\mathcal{N}(X)}] \leq \frac{(1-p)z}{1-pz}$ for any $1 \leq z < \frac{1}{p}$. Recalling from Claim 2.2 that $N_K$ is Poisson with parameter $u \cdot \text{cap}(K) = u \cdot \text{cap}(X^c_T)$, for any $1 \leq z < \frac{1}{p}$ we obtain
\[
E\left[z^{\sum_{j=1}^{N_K} \mathcal{N}(X^j)}\right] = \exp \left(u \cdot \text{cap}(X^c_T) \left(E_{e_K}[z^{\mathcal{N}(X)}] - 1\right)\right) \leq \exp \left(u \cdot \text{cap}(X^c_T) \left(\frac{z - 1}{1 - pz}\right)\right). \tag{\ref{eq:exp_cheb}}
\]

We can thus apply the exponential Chebyshev inequality with $z = \frac{1}{2p}$ to bound
\[
P\left[B_{n,x}^u \right] \leq \left(\frac{1}{2p}\right)^{2^n} \left[1 + \sum_{j=1}^{N_K} \mathcal{N}(X^j)\right] \leq C_2^{2^n} E\left[\left(\frac{1}{2p}\right)^{\sum_{j=1}^{N_K} \mathcal{N}(X^j)}\right] \leq C_2^{2^n} \left(\frac{1}{2p}\right)^{2^n} \leq \exp \left(u \cdot \text{cap}(X^c_T) \left(\frac{1}{2p}\right)^{2^n}\right) \leq \exp \left(u \cdot \text{cap}(X^c_T) \left(\frac{1}{2p}\right)^{2^n}\right) \leq \exp \left(u \cdot \text{cap}(X^c_T) \left(1 + C_2\right)^{2^n}\right) \leq \exp \left(u \cdot \text{cap}(X^c_T) \left(1 + C_2\right)^{2^n}\right) \leq \left(\frac{3}{4}\right)^{2^n} \tag{5.10}
\]
This completes the proof of Proposition 5.1. \qed
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