On the Quantization of the GS String on $AdS_5 \times S^5$

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April 14, 2021

Abstract

We discuss the quantization of the Green-Schwarz string action on $AdS_5 \times S^5$. We construct consistent, globally well-defined, gauge fixing choices for kappa symmetry and worldsheet diffeomorphism invariance. We then proceed to quantize the theory in a perturbation series in the inverse radius of curvature, in a background field expansion. We discuss vertex operators and correlation functions, and demonstrate agreement with supergravity results in the appropriate limit.

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1 Introduction

The AdS/CFT correspondence [1, 2, 3] relates string theory on an $AdS_{d+1} \times S^p$ space to a $d$-dimensional conformal field theory. In particular, type IIB string theory on $AdS_5 \times S^5$ is conjectured to be dual to $N=4$ $SU(N)$ Yang-Mills theory in 3+1 dimensions. In view of this conjecture, it is important to understand the formulation of string theory on these spaces. Such a formulation would provide a nonperturbative solution of supersymmetric Yang-Mills theory.

However, formulating string theory on these backgrounds is not a straightforward exercise. The basic reason for this is that these backgrounds typically involve a Ramond-Ramond field strength, and such backgrounds are difficult to describe in the usual RNS formulation of perturbative string theory. An exception is the case of $AdS_3 \times S^3$, where there exists a solution with a background NS-NS field strength. This background has been treated in the RNS formulation in [4, 5].

Another approach to string quantization in $AdS_{d+1} \times S^p$ backgrounds is the Green-Schwarz (GS) formalism [6]. Indeed, actions which represent the classical GS string in these backgrounds have been found. These actions can be written for both the $AdS_5 \times S^5$ background [7, 8, 9, 10, 11] and the $AdS_3 \times S^3$ background [12, 13, 14]. The reason that the GS approach may be appropriate is that spacetime supersymmetry is manifest, and so there is no fundamental difference between NS-NS and R-R backgrounds.

These actions are, however, only classical, and need to be quantized. In particular, there are various gauge symmetries, of reparametrization and kappa-symmetry, that need to be fixed. In the case of the flat space GS action, this is most effectively done in light-cone gauge [6]. However, the backgrounds under consideration have no globally defined light-like direction, and therefore this gauge is invalid. Much of our discussion therefore, revolves about choosing a gauge-fixing. This is described in section 3.

After the gauge fixing, the action does not simplify very much; it is still a complicated sigma-model with several interaction terms. It can nevertheless be treated in sigma-model perturbation theory, since (as we show) there is a well defined expansion in powers of the curvature of the background. This is related by the AdS/CFT correspondence to a perturbation series in $\frac{1}{g^2 N}$ in the dual Yang-Mills theory.

We present leading order expressions for vertex operators in the theory. These expressions provide a starting point for constructing the vertex op-
erators in powers of the curvature. Similarly, correlation functions can be computed in a perturbation expansion. We show that the leading order correlation functions reduce to previously computed results. In particular we recover the supergravity procedure of calculating the CFT correlation functions \[2, 3\].

One should note that we do not prove that the sigma-model we consider defines a spacetime conformal field theory. There have been arguments that this is the case \[15, 7\], but a direct proof is still lacking. The consistency of our results is, however, an encouraging sign.

2 The Green-Schwarz String action

We review here the Green-Schwarz string action constructed in \[7\]. We concentrate on the case of type IIB string theory on \(AdS_5 \times S^5\), but the discussion can also be applied to similar string backgrounds, such as the Green-Schwarz string on \(AdS_3 \times S^3\), discussed in \[12, 13, 14\].

We shall use the metric of \(AdS\) space in the form

\[
ds^2 = -(1 + e^{2r^2})dt^2 + \frac{dr^2}{1 + e^{2r^2}} + r^2 d\Omega_3^2
\]

where \(e^2\) is the cosmological constant in string units, related to gauge theory parameters as \(\frac{1}{e^2} = N g_s^2 Y_M\).

The action is constructed as a \(\sigma\)-model action with the target space being the coset \(SU(2,2|4) / SO(4,1) \times SO(5)\). The action is written in terms of the supervielbeins \(L^a, L^I\), and is given as :

\[
S = -\frac{1}{2} \int_{\partial M_3} d^2\sigma \sqrt{g} g^{ij} L_i^a L_j^\alpha + i \int_{M_3} s^{IJ} L^\alpha \wedge \bar{L}^\beta \wedge L^J ,
\]

where \(\partial M_3\) is the string worldsheet.

The supervielbeins, viewed as 1-forms over the coset manifold, are restricted to satisfy the Maurer-Cartan equations, given explicitly in \[7\]. The solution of these equations was found in \[7\]. The expressions for the supervielbeins \(L^a, L^I\) are given as :

\[
L^I = \left[ \left( \frac{\sinh \mathcal{M}}{\mathcal{M}} \right) D\theta \right]^I
\]

2
\[ L^a = e^a_\mu(x)dx^\mu - 4i\bar{\theta}^{\dot{a}}\gamma^\dot{a}\left(\frac{\sinh^2 \frac{M}{2}}{M^2}D\theta\right)^I \] (4)

where \(M^2\) is given by

\[ (M^2)_L^I = e^2[\epsilon^{IJ}(-\gamma^a\theta^J(x)\bar{\theta}^L(x)\gamma^a + \gamma^a'\theta^J(x)\bar{\theta}^L(x)\gamma^a')] + \frac{1}{2}\epsilon^{KL}(\gamma^{ab}\theta^I(x)\bar{\theta}^K(x)\gamma^{ab} + \gamma^{ab'}\theta^I(x)\bar{\theta}^K(x)\gamma^{ab'})] \] (5)

Here,

\[ (D\theta)^I = \left(d + \frac{1}{4}(\omega^{ab}\gamma_{ab} + \omega^{ab'}\gamma_{ab'})\right)\theta^I - \frac{e}{2}\epsilon^{IJ}(\delta^J_{\dot{a}}\gamma_{\dot{a}} + ie^{ab}\gamma_{ab'})\theta^J \] (6)

Our notation is that of [7]; \(a = 0, 1, 2, 3, 4\) are tangent space indices for \(AdS_5\) and \(a' = 5, 6, 7, 8, 9\) the corresponding indices for \(S^5\). The index \(\dot{a}\) runs over \(0, 1..9\), and \(e^{\dot{a}}, \omega^{\dot{a}\dot{b}}\) are the bosonic vielbein and the spin connection respectively.

We work in a Euclidean worldsheet, and fix the worldsheet metric to be \(g_{ij} = \delta_{ij}\). In this gauge, the equations of motion for the sigma model are [7]

\[ \delta^{ij}(\partial_i L^a_j + L^a_{ib} L^b_j) + i\epsilon^{ij} s^{IJ} L^I_i \gamma^a L^J_j = 0, \] (7)

\[ \delta^{ij}(\partial_i L^a_{j'} + L^a_{ib'} L^b_{j'}) - i\epsilon^{ij} s^{IJ} L^I_i \gamma^a L^J_j = 0, \] (8)

\[ (\gamma^a L^a_i + i\epsilon^a L^a_{i'}) (\delta^{ij} \delta^{JJ} - \epsilon^{ij} s^{JJ}) L^a_j = 0, \] (9)

As a result of the gauge fixing, these relations should be supplemented by the standard Virasoro constraints

\[ L^a_i L^a_j + L^a_{i'} L^a_{j'} = \frac{1}{2}\delta_{ij}(L^a_i L^a_j + L^a_{i'} L^a_{j'}) , \] (10)

In this paper we are interested in quantizing the action in leading order in the cosmological constant \(e^2\). The supervielbeins reduce in the leading order to:

\[ L^\ddot{a} = e^\ddot{a} + i\theta^I \Gamma^\ddot{a} D\theta^I \]

\[ L^I = D\theta^I \] (11)

These expressions are the covariant extensions of the corresponding expressions of the flat space Green-Schwarz string. In particular, the interaction
with the background curvature, and the interaction with the RR background, are both included in the leading order action.

This leading order action, to be discussed in the rest of the paper, is given in [7]. It is the covariantization of the usual flat space GS action:

\[ L_1 = -\frac{1}{2}(\epsilon_i^\hat{a} - i\bar{\theta}^I \gamma^\hat{a} D_I \theta^I)^2 - i\epsilon_{ij} \epsilon_i^\hat{a} s^{IJ} \bar{\theta}^I \gamma_j^\hat{a} D_j \theta^J + \epsilon^{ij}(\bar{\theta}^i \gamma^\hat{a} D_j \theta^j)(\bar{\theta}^2 \gamma^\hat{a} D_i \theta^2). \]  

(12)

We wish to quantize this action in a background field expansion. The background we choose is a worldsheet at a constant spatial position. The gauge we wish to work in is the covariantization of the flat space \( X_0 = \tau \) gauge. In order to perform this quantization we need to define our gauge choices globally, before going to any limit of interest. We turn, then, to discuss the gauge fixing of the action.

### 3 Gauge Fixing

In order to quantize the action we first need to gauge-fix the worldsheet gauge symmetries: reparametrization invariance and kappa-symmetry.

We first change the fermionic basis to the "Killing spinor basis"\(^1\) [9]. These, by definition, satisfy

\[ (\Theta^\alpha)^I = E_\alpha^\hat{a} (\theta^\alpha)^I \]  

(13)

\[ (d\Theta^\alpha)^I = E_\alpha^\hat{a} (D\theta^\alpha)^I \]  

(14)

These equations permit solutions because of the highly symmetric nature of the space \( AdS_5 \times S^5 \), which has 32 Killing spinors. We note also that the covariant derivative used in this definition includes an interaction with the RR background, in addition to the usual coupling to the spin connection.

In this basis, the equations of motion for the spinors simplify

\[ (\delta_{ij} S^{IJ} - \epsilon_{ij} S^{IJ}) \partial_j \Theta^I = 0 \]  

(15)

We can now fix the kappa symmetry. Denote \( \Gamma^+ = \Gamma^0 + \Gamma^r \), and choose:

\[ \Gamma^+ \Theta^I = 0 \]  

(16)

\(^1\)This is a somewhat misleading term, as \( \Theta^I \) are generally not Killing spinors. However, we stick to the established notations.
This gauge is globally well-defined, unlike the gauge choices of [9, 10]. The reason for imposing this gauge on the Killing spinor basis rather than the original basis is that the original fermions pick up a phase factor if the string is transported around a path with nontrivial holonomy. The spinors $\Theta^I$, on the other hand, do not pick up such a factor. This can be seen from the fact that their kinetic term does not contain a coupling to the spin connection or to the RR background. Hence this gauge choice is globally well defined.

We now turn to fix the worldsheet diffeomorphism invariance. We already chose the conformal gauge: $g_{ij} = \delta_{ij}$. The conformal gauge leaves an unbroken subgroup of the worldsheet gauge symmetries, the conformal transformations. A consistent way to fix those residual gauge symmetries is described in [6]. One may choose any free field on the worldsheet and set it to equal one of the worldsheet coordinates, say $\tau$. This consistently forces (almost) all the Fourier modes of the gauge parameter to vanish, leaving $\sigma$ rigid translations, generated by $L_0 - \bar{L}_0$, as the only leftover gauge freedom. This gauge choice is utilized in the flat space string action to fix the lightcone gauge, as discussed in [6].

Similarly, one may utilize an exactly conserved current on the worldsheet to fix the conformal transformation. Being exactly conserved determines the conformal transformation law of the current. Therefore, fixing $J_i$ to be a constant forces almost all of the Fourier modes of the gauge parameter to vanish. The leftover gauge invariance in this case consists of rigid translations on the worldsheet, generated by both $L_0$ and $\bar{L}_0$.

We wish to choose a gauge which reduces to the temporal gauge $X_0 = \Delta \tau$ asymptotically in spacetime (where $\Delta$ is a constant). As a timelike coordinate can be defined globally, unlike a null coordinate, this gauge choice can be made globally well-defined. Moreover, the momentum conjugate to the global time defines the dimension operator in the spacetime CFT. Therefore the constant $\Delta$ has a natural interpretation in that CFT.

The reader may be aware of the notorious problems of string quantization in the temporal gauge, in the flat space case. The interpretation of the quantization procedure in the case of the $AdS$ space is, however, radically different. For example, by constructing vertex operators of the first quantized string one constructs operators in the spacetime CFT. Those operators are not required to have a particle interpretation, and indeed lie generally in non-unitary representations of the spacetime symmetry group. Furthermore, the dual gauge theory is believed to be consistent when treated in the global
time, having a Hamiltonian and unitary time evolution\(^2\).

To work in the temporal gauge it is natural to utilize the time translation current on the worldsheet:

\[
J^0_i = e^0_i \left( e^0_{i,0} + i \bar{\theta}^I \gamma^0 D_i \theta^I \right) + O(e^2) \quad (17)
\]

In order to simplify the action it is convenient to add to this current another conserved current which has no action on the zero modes of the fields. It therefore corresponds to a uniform shift of the dimension of each multiplet separately. The shift can be shown to vanish for the supergravity vertex operators constructed below\(^3\).

The conserved current we use to fix the gauge is then:

\[
J_i = J^0_i - i(\Theta^I)^I \partial_i \Theta^I \quad (18)
\]

The extra current is exactly conserved as a consequence of the equation of motion (15). With this gauge fixing taken the fermions have a non-degenerate kinetic term.

Thus, in order to specify the temporal gauge we would like to choose:

\[
J_i = \Delta \delta_{\tau r} \quad (19)
\]

However, this is not possible, as it is inconsistent with the equations of motion\(^4\). Indeed, the current \(J_i\) satisfies:

\[
\partial_0 J_0 + \partial_1 J_1 = 0 \quad (20)
\]

\[
\partial_0 J_1 - \partial_1 J_0 = O(e^2) \quad (21)
\]

This implies:

\[
\left( \partial_0^2 - \partial_1^2 \right) J_i = O(e^2) \quad (22)
\]

which is only consistent with our gauge choice to leading order in \(e\). At higher orders in \(e\) we must correct the gauge choice perturbatively. This can be done by adding the inhomogeneous solution of (22) to the gauge choice above.

The constant \(\Delta\) is to be interpreted as the total dimension, similar to the total light cone momentum \(P^+\) in lightcone quantization. This can be demonstrated for supergravity states as follows.

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\(^2\)We thank Tom Banks for discussions on this point.

\(^3\)The shift is given by a certain number operator for fermionic oscillators.

\(^4\)We thank E. Witten for pointing this out to us.
Define the worldsheet current corresponding to $X^\mu$ translation as $J_\tau^\mu$. Assuming the spacetime superconformal algebra is an exact symmetry of the quantum theory, one can work in states that are representations of the superconformal algebra. Therefore

$$\left( \int d\tau J_\tau^\mu \right) | \text{state} > = \mathcal{R}^\mu | \text{state} >$$

where $\mathcal{R}^\mu$ is a generator in some matrix representation of the spacetime superconformal algebra.

We take now the mini-superspace approximation, setting all $\sigma$ derivatives to zero. This is the particle limit of the string theory, and the states in this approximation are those of the first quantized supergravity. The worldsheet current satisfies:

$$\partial_\tau J_\tau^\mu = 0 \quad (24)$$

In addition one has the Virasoro constraint:

$$T_{00} = G_{\mu\nu} J_\tau^\mu J_\tau^\nu = 0 \quad (25)$$

This constraint sets the second Casimir of the superconformal algebra, in the $\mathcal{R}$ representation, to zero. This is exactly the equation satisfied by the supergravity modes, which determines the dimension of any mode in terms of its other quantum numbers.

In the gauge chosen this equation relates the constant $\Delta$ to the quantum numbers of the state in exactly the same way (up to a trivial shift). This result is independent of the approximation we take next, the background field expansion.

With the same assumption, namely that the quantum theory preserves the spacetime superconformal algebra, one can organize the string states into supersymmetry multiplets. We note that as a consequence of the superconformal algebra, members of any given multiplet carry different dimensions. Specifically, this follows from the commutation relations:

$$[Q, D] = \frac{Q}{2} \quad (26)$$

where $Q$ are the supersymmetry generators, and $D$ is the dimension.

Formally, therefore, different states in the same supersymmetry multiplet belong to separate Hilbert spaces. In the leading order approximation around
flat space, however, the splitting is zero, and multiplets are degenerate. A proof of closure of the spacetime symmetry algebra to the first subleading order would therefore establish the required splitting.

4 Background Field Expansion

Having defined our gauge choices globally, we are ready to use the resulting action. We proceed by expanding the action in a background field expansion around the following worldsheet configuration:

Define $X^\mu = X^\mu_0 + \xi^\mu$, where $X^\mu_0$ are constants, and $\mu = 0, \ldots, 9$ is a curved coordinate index. In order to normalize the fluctuating fields canonically define:

$$\xi^\hat{a} = e^\mu_\mu (X^\mu_0) \xi^\mu$$

(27)

Where $\hat{a} = 0, \ldots, 9$ is a tangent space index.

We expand the action (12) in powers of $\xi^\hat{a}$, the bosonic fluctuations. Since $r$ determines the magnitude of spacetime derivatives, this is an expansion in $\xi/r$. Thus, this approximation is useful in studying small fluctuations of a worldsheet located asymptotically in spacetime, near the boundary of the AdS space. In particular it is suited for the study of non-normalizable modes, which correspond to the operator algebra of the spacetime CFT.

The action and the gauge choice simplify significantly in the leading order in the background field expansion, becoming essentially the flat space GS action in the $X^0 = \tau$ gauge. However, the treatment of the zero modes, bosonic and fermionic, is sensitive to the curved nature of the space.

The action (12) is constructed from the following ingredients, written to first subleading order in $\epsilon$, the cosmological constant. We use the indices $m,n,\ldots = 1,\ldots,8$ to denote the “transverse” fluctuations. The kappa symmetry gauge choice is imposed, but we delay imposing the diffeomorphism gauge choice till later.

$$e^\hat{a}_i = \partial_i \xi^\hat{a} + B^a_{bc}(X_0) \xi^b \partial_i \xi^c$$

(28)

$$D_i \theta^I = \partial_i \theta^I + \frac{1}{4} (\omega^\hat{a}_i (x_0) \gamma_{\hat{a} \hat{b}} \theta^I - \epsilon_{IJ} e^\hat{a}_i (x_0) \gamma_{\hat{a}} \theta^J) =$$

$$= \partial_i \theta^I - \frac{e}{2} \partial_i \xi^m \gamma_{mn} \theta^I - \frac{\epsilon_{IJ}}{2} (\partial_i \xi^\hat{a} \gamma^\hat{a}) \theta^J$$

(29)

where

$$B^{a}_{bc} = \epsilon^a_{\nu} \epsilon^m_{\mu} \partial_{\nu} e^a_{\mu}$$

(30)
The action to leading order in \( e \), before gauge fixing, is the flat space GS action

\[
\mathcal{L}_1 = -\frac{1}{2} (\partial_i \xi^\hat{a} - i \bar{\theta}^I \gamma^0 \partial_i \theta^I)^2 - i \epsilon_{i j} \partial_i \xi^\hat{a} s^I \bar{\theta}^I \gamma^\hat{a} \partial_j \theta^I + \epsilon^{i j} (\bar{\theta}^1 \gamma^0 \partial_i \theta^1)(\bar{\theta}^2 \gamma^\hat{a} \partial_i \theta^2).
\]

(31)

The gauge fixing is

\[
J_i = e^0_0(x_0)(e^0_i + i \bar{\theta}^I \gamma^0 D_i \theta^I) - i (\Theta^I)^\dagger \partial_i \Theta^I
\]

(32)

In the limit taken, this expression reduces to:

\[
J_i = e^0_0(x_0) \partial_i \xi^0
\]

(33)

Though the gauge choice has to be imposed exactly, approximating \( J_i \) by its leading order changes the action only in subleading terms in the background field expansion. To find the complete action to first subleading order one needs to include the next order action written in [7], use the expansion (28), and expand the gauge choice as well.

Thus we impose in the leading order:

\[
e^0_0(x_0) \partial_i \xi^0 = \delta_i^\tau
\]

(34)

In this limit the action for the fluctuations simplifies. The transverse fluctuations and the fermions obey free field equations, and can be expanded in modes. The zero mode structure is still sensitive to the global structure of the background. We demonstrate this by constructing some simple vertex operators in the next section.

5 Vertex Operators

In order to find the vertex operators one needs to solve the Virasoro constraints. Apart from the zero-mode contribution, this is identical to solving the constraints in flat space, in the \( X^0 = \tau \) gauge.

We use the following notation: we work in worldsheet lightcone coordinates \( \sigma_{\pm} = \sigma \pm \tau \). We concentrate on the left-moving part, and suppress the corresponding indices. In spacetime we use the indices \( a, b, \ldots \) to denote all coordinate labels except for 0, the gauge fixed coordinate.
The fermion $\Theta^1$ is left-moving by its equation of motion, and we denote it by $S$.

The left moving Virasoro constraint is found to be:

$$T = (\partial \xi^m)^2 - \left(\frac{\Delta}{\epsilon_0(X_0)} - i\tilde{S}\gamma^0\partial S\right)^2 + \left(\partial \xi^r + i\tilde{S}\gamma^0\partial S\right)^2$$  \hspace{1cm} (35)

The bosonic fluctuations have the mode expansion:

$$\xi^a = e^a_{\mu}(X_0)(\xi^\mu)_0 + \sum_{n \neq 0} \frac{1}{n} \alpha^a_n \exp(in\sigma) + \text{right movers}$$  \hspace{1cm} (36)

The normalizations chosen in this mode expansion need to be clarified. The fundamental variables to be quantized are the unrescaled fluctuations $\xi^\mu$ defined above. For oscillator modes the rescaling in equation (27) has the mere effect of changing their normalization. However, the zero mode of $\xi^r$ does not commute with $e^a_{\mu}(X_0)$, hence we choose to keep the original zero modes $(\xi^\mu)_0$, and their conjugate momenta $P^\nu$, as our canonical variables.

One imposes the following commutation relations:

$$[(\xi^\mu)_0, P^\nu] = -G^{\mu\nu}(X_0)$$

$$[\alpha^a_n, \alpha^b_m] = i\gamma^0 \delta_{m+n,0}$$  \hspace{1cm} (37)

For the fermions one has the mode expansion:

$$S = S_0 + \sum_{n \neq 0} S_n \exp(in\sigma)$$  \hspace{1cm} (38)

and the anti-commutation relations are:

$$\{(S_m)^\dagger, S_n\} = i\gamma^0 \delta_{m+n,0}$$  \hspace{1cm} (39)

Define $L_n$ to be the Fourier modes of $T$. The gauge fixing chosen here fixes almost all the conformal transformations, leaving unfixed the rigid translations generated by $L_0$ (and $\bar{L}_0$). Thus one needs to solve the constraints for all $L_n$, $n \neq 0$. These can be used to express all the modes $\{\alpha^r_n\}$ in terms of the transverse fluctuations $\xi^m$.

In the absence of fermionic oscillators these equations simplify, and are given as:

$$L_n = \sum_{k} \alpha^a_{n-k} \alpha^b_k = 0$$  \hspace{1cm} (40)
These are quadratic equations for the modes \( \{\alpha_n^r\} \), unlike the linear equation in the lightcone gauge quantization of the flat space string. The equation can be formally solved as:

\[
\alpha_r^r = \int d\sigma_+ \exp(-ik\sigma_+) \sqrt{\Delta^2 G^{00} - (\partial \xi^m)^2}
\]  

(41)

The vertex operators represent emission vertices in flat space, and their interpretation in our case is demonstrated by computing their correlation function in the next section. As the vertex operators carry non-zero dimensions, they cannot be interpreted as acting on the Hilbert space of states with a fixed total dimension \( \Delta \), denoted as \( \mathcal{H}_\Delta \). They should be formulated as operators acting in a larger Hilbert space, where the dimension \( \Delta \) is allowed to change.

To formally express this fact we introduce the operators \( O_\delta = \exp(\delta X^0_0) \). Here \( X^0_0 \) is the zero mode of the time coordinate \( X^0 \), unfixed by our gauge choice. These operators act naturally in an enlarged Hilbert space \( \mathcal{H} \) defined as the direct sum of all the Hilbert spaces \( \mathcal{H}_\Delta \) of fixed \( \Delta \).

It is convenient also to introduce an operator \( \hat{\Delta} \), which is proportional to the unit operator in any summand Hilbert space \( \mathcal{H}_\Delta \), and takes the value \( \Delta \) there. Then one has the following commutation relation:

\[
[\hat{\Delta}, O_\delta] = \delta O_\delta
\]  

(42)

The vertex operators are functionals of the transverse fluctuations and their conjugate momenta, as well as \( r_0 \), the radial zero mode, and its conjugate momentum. They include an appropriate operator \( O_\delta \) defined above. As a result of the leftover gauge freedom, the vertex operators are restricted to satisfy the \( L_0 \) constraint:

\[
[L_0, V] = V \quad L_0 = G^{\mu\nu}(X_0)P_\mu P_\nu + G^{00}(X_0)\hat{\Delta}^2 + \sum_k \alpha_{-k}^a \alpha_k^a
\]  

(43)

where in this equation the oscillators \( \alpha_n^r \) are given in (41), as solutions to equation (40).

As usual one needs to choose operator ordering when writing the expression for \( L_0 \), since the momenta \( P_r \) do not commute with the zero mode \( r_0 \). Note that in this order there is no normal ordering constant in this expression, as we have exact cancellation between free bosonic and fermionic fluctuating modes.
The simplest vertex operators are the supergravity modes. Their oscillator part is identical to the flat space case, and their zero mode wavefunction is determined by the $L_0$ constraint. This simply sets the spacetime covariant Laplacian, which is the second Casimir of the conformal algebra, equal to the ten dimensional mass of the string state, zero in this case.

For example consider the vertex operator of a graviton with transverse polarization. We also set the transverse momenta to zero, as all other cases can be simply obtained from this one. The corresponding vertex operator is then:

$$V^{mn} = \partial X^m \bar{\partial} X^n O_\delta \Psi_{mn}(r_0)$$

The $L_0$ constraint translates then to the following equation on the wave function $\Psi_{mn}(r_0)$:

$$G^{00} \delta^2 \Psi_{mn} + \partial_r (G^{rr} \partial_r \Psi_{mn}) = 0$$

which is identical to the equation satisfied by the corresponding supergravity mode. In order to make correlation functions non-vanishing, one is forced to take the non-normalizable solutions of this equation, as is discussed in the next section.

6 Correlation Functions

The $AdS/CFT$ correspondence includes a prescription for calculating correlation functions of CFT operators in the supergravity limit of the $AdS$ theory[2, 3]. Define the generating functional of the $CFT$ correlation functions as $W[\Phi_0]$. The currents $\Phi_0$ couple to $CFT$ operators, and differentiation of $W[\Phi_0]$ with respect to those currents yields the correlation functions of those operators.

The $AdS/CFT$ correspondence identifies this generating functional with the spacetime partition function with prescribed boundary conditions. In the supergravity limit this is simply:

$$W[\Phi_0] = \exp(S[\Phi, g_{\mu\nu}])$$

The currents $\Phi_0$ enter the supergravity calculation as boundary conditions for the supergravity fields, or equivalently as deformations of these fields by non-fluctuating, non-normalizable modes[16].
We wish to recover this prescription in the present context. In string theory one computes correlation functions of vertex operators as:

\[ \langle \prod V_i \rangle = \int DXD\theta \exp(-S) \prod V_i \]  
(47)

These correlation functions are obtained by differentiation of the following generating functional:

\[ W[J] = \int DXD\theta \exp(-S + \sum_i J_i V_i) \]  
(48)

The expression is readily recognized as the partition function of the string with the string action deformed by the addition of \( \sum_i J_i V_i \). For the gravity modes, this can be interpreted as deformation of the spacetime metric (and its supersymmetric partners). Take for example the graviton vertex operator constructed in the last section, equation (44). Adding it to the string action results in deforming the spacetime metric by \( O_3 \Psi_{mn}(r_0) \).

Therefore we are motivated to identify the string vertex operators with the non-normalizable modes of spacetime fields, and the string generating functional (48) with the CFT generating functional as in (46).

Thus, a natural extension of the supergravity prescription in (46) is to identify the generating functional of the CFT correlation functions with the string partition function, with the string action deformed by the corresponding (1,1) operators.

This leads to the following identification [17]:

\[ \langle \prod O_i \rangle = \langle \prod V_i \rangle \]  
(49)

Here \( V_i \) is the string vertex operator representing the non-normalizable mode which couples to the CFT operator \( O_i \).

For the supergravity modes, in the mini-superspace approximation, this can be shown to reduce to a supergravity calculation. In this limit the correlation of vertex operators is reduced to

\[ \langle \prod V_i \rangle = \int DXDS_0 \exp(- \int d\tau L_0) \prod V_i \]  
(50)

where the string action is (in the minisuperspace approximation)

\[ L_0 = g_{\mu\nu} \partial_\tau X^\mu \partial_\tau X^\nu \]  
(51)
We demonstrate the calculation of correlation functions by calculating the two point function of the graviton vertex operator above, equation (44), which has no angular momentum on either the $S^3$ or the $S^5$. The vertex operator is then:

$$V_{mn} = \dot{X}^m \dot{X}^n \Psi_{mn}(X)$$  \hspace{1cm} (52)

where $\Psi_{mn}(X, x)$ is the bulk-to-boundary propagator [2], corresponding to an operator insertion at point $x$ on the boundary.

The integral (50) is then

$$\int DXDS_0 \exp \left( - \int d\tau L_0 \right) \dot{X}^m \dot{X}^n \Psi_{mn}(X, x) \dot{X}^m \dot{X}^n \Psi_{mn}(X, y)$$

we can exponentiate the vertex operators to

$$I[J] = \int DXDS_0 \exp \left( - \int d\tau L_0 + J(x) \dot{X}^m \dot{X}^n \Psi_{mn}(X, x) \right)$$  \hspace{1cm} (53)

We can then generate the correlation functions by differentiating with respect to $J(x)$.

If $J(x)$ was zero, then the integral equals (see for example [18])

$$W[0] = \int DXDS_0 \exp \left( - \int d\tau L_0 \right)$$

$$= \int DXDp \exp(p\dot{x}) \exp(-g^{\mu\nu}p_\mu p_\nu)$$

which by standard path integral manipulations is

$$W[0] = \sum_n \exp(-p_n^2 T)$$  \hspace{1cm} (55)

where $p_n$ are the eigenvalues of

$$\partial_i [g^{ij} \partial_j \phi] = p_n^2 \phi$$

We can therefore replace the sum over modes by an integral over field configurations

$$W[0] = \int D\phi \exp (-g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi)$$  \hspace{1cm} (56)
We can redo this calculation in the presence of vertex operators. It is straightforward to show that \[ \int DXDS_0 \exp \left( - \int d\tau L_0 + J(x) \dot{X}^m \dot{X}^n \Psi_{mn}(X,x) \right) \]
\[ = \int D\phi \exp \left( -g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - J g^{\mu\nu} \partial_\mu \phi \partial_\nu \psi \right) \] (59)

To calculate a two point function, we differentiate with respect to \( J \) twice.

At this point, it is clear that we have reduced the computation to the corresponding supergravity calculation, and thus shown the equivalence of the formulae (46) and (48), in the appropriate limit.

In order to compare to the supergravity result we still have to discuss the normalization of the vertex operators.

In order to normalize correctly the vertex operator (44), one needs to consider their CFT interpretation. Usually a vertex operator insertion carries a factor of \( g_{\text{str}} \), the string coupling. This gives, for example, a \( g_{\text{str}} \) independent result for the two point functions calculated on the sphere, as sphere amplitudes are proportional to \( \frac{1}{g_{\text{str}}^2} \). The three point function on the sphere is then proportional to \( g_{\text{str}} \).

In the present context, correlation functions of the graviton are interpreted as correlation functions of the energy-momentum tensor of the spacetime CFT. As such, they obey conformal Ward identities which are \( g_{\text{str}} \) independent. For example, one such identity relates the two-point and 3-point functions as follows:

\[ \partial_\mu \left\langle T^\mu_{\nu(x)} T(y) T(z) \right\rangle = -\delta(x-y) \partial_\nu \left\langle T(y) T(z) \right\rangle - \delta(x-z) \partial_{2\nu} \left\langle T(y) T(z) \right\rangle \] (60)

Thus, the graviton vertex operator cannot carry any factor of \( g_{\text{str}} \). Consequently, all correlation functions on the sphere are proportional to \( \frac{1}{g_{\text{str}}^2} \).

Additional factors of \( e^2 \) may arise from the worldsheet calculation, hence the general correlation function on the sphere is proportional to \( N^b g_{\text{str}}^{b-2} \). The constant \( b \) is determined by dimensional analysis, \( 4b \) being the total conformal dimension of the correlation function.

The two-point function above is then found to be:

\[ \left\langle V_{mn} V_{mn} \right\rangle = N^2 \frac{1}{|x-y|^4} \] (61)

This result agrees with the calculation of the central charge of the gauge theory [3].
7 Acknowledgments

We thank O. Aharony, M. Berkooz, N. Berkovits, M. Douglas, W. Fischler, L. Susskind, E. Witten and especially T. Banks for useful conversations.

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