Boundary differentiability of solutions to elliptic equations in convex domains in the borderline case

Dharmendra Kumar

Received: 20 September 2021 / Revised: 29 July 2022 / Accepted: 20 September 2022 / Published online: 6 October 2022
© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2022

Abstract
In this work, we consider the following elliptic partial differential equations:

\[
\begin{cases}
-\sum_{i,j=1}^{n} b_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} = g & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega, 
\end{cases}
\]

where the domain \( \Omega \subset \mathbb{R}^n \) is convex, the matrix \((b_{ij})_{n \times n}\) satisfies the uniform ellipticity conditions. For \( g \) in the scaling critical Lorentz space \( L(n,1)(\Omega) \), we establish boundary differentiability of solutions to the above problem. We also prove \( C^{\log-Lip} \) regularity estimate at a boundary point in the case when \( g \in L^n(\Omega) \).

Mathematics Subject Classification Primary 35J25

Contents

1 Introduction ............................................. 2
2 Main results ............................................. 4
3 Preliminary results .......................................... 4
4 Some useful lemmas ......................................... 6
5 Proof of the theorem 2.1 ....................................... 14
6 Proof of the Theorem 2.2 ...................................... 17
References ................................................ 19

Dharmendra Kumar
dharmendra2020@tifrbng.res.in, dharamsambey90@gmail.com

1 Tata Institute of Fundamental Research, Centre For Applicable Mathematics, Post Bag No 6503, GKV Post Office, Sharada Nagar, Chikkabommsandra, Bangalore, Karnataka 560065, India
1 Introduction

In this work, we consider the following elliptic PDEs:

\[
\begin{cases}
- b_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} = g & \text{in } \Omega, \\
\quad w = 0 & \text{on } \partial \Omega, 
\end{cases}
\]  

(1.1)

where the domain \( \Omega \subset \mathbb{R}^n \), the matrix \((b_{ij})_{n \times n}\) is symmetric and

\[
\lambda I \leq (b_{ij})_{n \times n} \leq \frac{1}{\lambda} I, \quad \text{for some constant } 0 < \lambda \leq 1. 
\]  

(1.2)

The aim of this paper is to establish boundary differentiability of strong solution to (1.1) for \( g \) in the scaling critical Lorentz space \( L(n, 1)(\Omega) \).

To put our results in the right perspective, we begin with a classical result of Stein [24] on the following “limiting” case of Sobolev embedding theorem.

**Theorem 1.1** Let \( L(n, 1) \) denote the standard Lorentz space. Then for each Sobolev function \( w \in W^{1,1} \), the following implication holds:

\[ \nabla w \in L(n, 1) \implies w \text{ is continuous.} \]

Theorem 1.1 can be considered as the limiting case of Sobolev-Morrey embedding claiming that:

\[ \nabla w \in L^{n+\varepsilon} \implies w \in C^{0, \frac{\varepsilon}{n+\varepsilon}}. \]

Note that

\[ L^{n+\varepsilon} \subset L(n, 1) \subset L^n, \quad \forall \varepsilon > 0, \]

with all the inclusions being strict.

Now Theorem 1.1 combined with the standard Calderón-Zygmund theory has the following interesting consequence:

**Theorem 1.2**

\[ \Delta w \in L(n, 1) \implies \nabla w \text{ is continuous.} \]

To the best of our knowledge, we are not aware of analogue of Theorem 1.2 for general nonlinear elliptic and parabolic equations occurring in the past through a rather sophisticated and powerful nonlinear potential theory, see for instances [9, 15, 18] and the references therein.
Theorem 1.2 has been extended to quasilinear operators modelled on the $p-$laplacian by Kuusi and Mingione in [16]. Subsequently they generalized their result to $p$-laplacian type systems in [17]. See also [9, 15, 18] and the references therein. After that there has been several generalization of such a result to operators with different type of nonlinearities. Theorem 1.2 has been extended to fully nonlinear elliptic operators [1, 8]. Finally we also refer to a recent work [2] where the borderline gradient continuity has been obtained in the case of game-theoretic normalized $p$-Laplacian operator.

We now mention some other works in the case of boundary regularity that are closely related to this paper. In 1984, Krylov in [13, 14] proved that solution of (1.1) is $C^{1,\alpha}$ along the boundary under the assumption that $\partial \Omega$ is $C^{1,\alpha}$ using boundary harnack principle. Such $C^1$ results were extended to $C^{1,Dini}$ domains in [19]. Now in case of convex domains, we see that standard barrier arguments imply that solutions are Lipschitz continuous up to the boundary. We also mention that Wang in [25] established similar results as in [14] by a somewhat different iterative argument based on ABP type comparison principle. In 2006, Li and Wang [20] adapted such an approach to establish the boundary differentiability of the solution to (1.1) in convex domains. In their work, they assumed that $g$ satisfies

$$\int_0^{r_0} \frac{||g||_{L^p(\Omega \cap Q_{\rho}\times \rho)}}{\rho}d\rho < \infty. \quad (1.3)$$

For nonhomogeneous Dirichlet boundary condition, see [21]. Also they showed that their result is optimal in the sense that only under the assumption that $\Omega$ is convex, continuity of the gradient of the solution to (1.1) along the boundary can not be expected. They demonstrated this by providing two counterexamples. In this note, we sharpen the result in [21] to the case when $g \in L(n, 1)$. It is to be noted that an arbitrary $g \in L(n, 1)$ need not satisfy (1.3) and thus our result is not covered by [21].

In closing, we mention that Ma and Wang in [22] established the gradient continuity of solutions up to the boundary to fully nonlinear uniformly elliptic equations on $C^{1,Dini}$ domain $\Omega$. Their result was extended by Adimurthi and Banerjee in [1] to the borderline case when the right hand side $g \in L(n, 1)$. In fact, the main result of [1, Theorem 1.3] can also be thought of as boundary counterpart of [8, Theorem 1.3].

For $g \in L^n(\Omega)$, we also establish Log-Lipschitz estimate at a boundary point of a convex domain using compactness arguments which are inspired by the works of Caffarelli in [3–5].

**Notations:**

$B_r :=$ the open ball of radius $r$ and center 0.

$B_r(y) := y + B_r.$

$\mathbb{R}^n_+ := \{ (x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > 0 \}.$

$T_r := B_r \cap \partial \mathbb{R}^n_+.$

$Q[c \times d] := T_c \times (0, d) \subset \mathbb{R}^n$ for any real number $c > 0$, $d > 0$.

$\int_{\Omega} f(x)dx := f$ over the positive measure set $\Omega$. 


The organization of the paper is as follows. In Sect. 2, we state our main result. In Sect. 3, we shall recall some definitions and characterization of relevant function spaces that will be used in the subsequent sections. Now given that the blow-up of a convex set at each boundary point is cone, in Sect. 4, we first examine the boundary differentiability of the solutions to (1.1), when $\Omega$ is a cone. This is starting point of the proof of Theorem 2.1. In the same section, we also establish a basic comparison lemma that is relevant to the proof of Theorem 2.2 below. In Sect. 5, we prove Theorem 2.1. Furthermore by compactness arguments, we also prove Theorem 2.2.

2 Main results

Main results of this paper are the following:

**Theorem 2.1** Assume that $\Omega$ is convex and $g \in L(n, 1)(\Omega)$. Let $w$ be a strong solution of (1.1) for $q \in (n - n_0, n)$, where $n_0$ is a small universal constant depends only on $n$ and ellipticity constants. Then $w$ is differentiable at each $x \in \partial \Omega$.

**Theorem 2.2** Assume that $\Omega$ is convex and $g \in L^n(\Omega)$. Let $w$ be a strong solution of (1.1), then $w$ is Log-Lipschitz at any $x \in \partial \Omega$.

3 Preliminary results

In this section, we shall recall some definitions and characterization of relevant function spaces that will be used in the subsequent sections.

**Definition 3.1** A set $X \subset \mathbb{R}^n$ is called a cone if the following two conditions are satisfied: (i) $X + X \subset X$ (ii) $\mathbb{R}_+ \cdot X \subset X$.

**Definition 3.2** A continuous function $w$ in $\Omega$ is called $L^q$- strong solution if $w \in W^{2,q}_{loc}(\Omega)$ and satisfies

$$-b_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} = g \text{ almost everywhere in } \Omega.$$ 

Relevant Function Spaces:

Let us also recall some basic maximal-type characterization of Lorentz spaces, see for instance [8, 12].

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $1 \leq p < \infty$, $0 < q \leq \infty$.

The Lorentz space $L(p, q)(\Omega)$ is defined by prescribing all those measurable functions $f$ satisfying

$$\int_0^{\infty} \left( \lambda^p \{ x \in \Omega : |f(x)| > \lambda \} \right)^{q/p} \frac{d\lambda}{\lambda} < \infty \text{ when } q < \infty; \quad (3.1)$$
and

\[
\sup_{\lambda > 0} \lambda^p \left| \{ x \in \Omega : |f(x)| > \lambda \} \right| \leq \infty, \quad \text{when } q = \infty. \tag{3.2}
\]

We denote \( L(p, \infty)(\Omega) \equiv \mathcal{M}^p(\Omega) \). This is known as Marcinkiewicz space. Using above definition, it follows that

\[
f \in L(p, q) \implies |f|^r \in L\left(\frac{p}{r}, \frac{q}{r}\right) \text{ for } r \leq p. \tag{3.3}
\]

First we introduce the non-increasing rearrangement

\[
f^* : [0, \infty) \longrightarrow [0, \infty)
\]

of a measurable function \( f \) defined by

\[
f^*(s) := \sup \left\{ t \geq 0 : \left| \{ x \in \mathbb{R}^n : |f(x)| > t \} \right| > s \right\}.
\]

And set

\[
f^{**}(\varrho) := \varrho^{-1} \int_{0}^{\varrho} f^*(t)dt.
\]

For \( p > 1, q > 0 \), maximal-type characterization of \( L(p, q) \) claims that

\[
f \in L(p, q) \iff \int_{0}^{\infty} \left( f^{**}(\varrho)\varrho^{1/p}\right)^q \frac{d\varrho}{\varrho} < \infty.
\]

We also fix an exponent \( q \in (n - n_0, n) \). Here \( n_0 \) is a small universal constant depends only on \( n \) and ellipticity constants such that the following generalized maximum principle as in \([7, (2.3) – (2.4)]\) holds for all \( q \in (n - n_0, n] \), see also \([10, \text{Theorem 1}]\).

**Lemma 3.3** [Generalized maximum principle] The following estimates hold for \( L^q \)-strong solutions of \((1.1)\)

\[
\sup_{\Omega} u \leq \sup_{\partial \Omega} u + C \|g^+\|_{L^q(\Omega)}
\]

with a constant \( C = C(\lambda, n, \Omega) > 0 \).

The scaled version of above estimate is

\[
\sup_{B_r(x)} u \leq \sup_{\partial B_r(x)} u + Cr^{2-n/q} \|g^+\|_{L^q(\Omega)}
\]

with a constant \( C = C(\lambda, n, \Omega) > 0 \) is independent of \( r \) for \( r \leq 1 \).
4 Some useful lemmas

Since the blow-up of a convex set at any boundary point is cone, we should first consider Theorem 2.1 in the case when the domain is a cone.

Assume that $X \subset \mathbb{R}^n_+$ is an open cone with nonempty interior. For $X \neq \mathbb{R}^n_+$, we define

$$v = \inf \{ r > 0 : (e_n + T_r) \cap \partial X \neq \emptyset \}. \quad (4.1)$$

Then it is easy to see that $v < \infty$. Let $r_0 > 0$ and $w$ be a solution of the following elliptic PDEs:

$$\begin{cases}
-b_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} = g & \text{in } X \cap Q[r_0 \times r_0], \\
w = 0 & \text{on } \partial X \cap Q[r_0 \times r_0].
\end{cases} \quad (4.2)$$

**Lemma 4.1** If $X = \mathbb{R}^n_+$, then $\exists$ positive constants $C$, $\beta$ and $\Lambda$ depending only on $\lambda$, $n$ and $v$, and $\exists$ a constant $a$ depending on $w$ such that

$$|w(x) - ax_n| \leq C \left( \frac{r^\beta}{r_0^{1+\beta}} \|w\|_{L^\infty(X \cap Q[r_0 \times r_0])} + \frac{r^\beta}{r_0^\beta} \|g\|_{L^q(X \cap Q[r_0 \times r_0])} \right),$$

for any $x \in Q[r \times r] \cap X$ and $r \leq r_0^{\frac{n}{\lambda}}$.  

**Proof**  **Claim I** Normalization. By translation, rotation and scaling, we shall assume that

$r_0 = 1$, $\|w\|_{L^\infty(X \cap Q[1 \times 1])} + \|g\|_{L^q(X \cap Q[1 \times 1])} \leq 1.$

Since the equation is linear, without loss of generality, we can assume that $w(x) \geq 0$ for $x \in Q[1 \times 1] \cap X$ and $f(x) \geq 0$ for $x \in Q[1 \times 1] \cap X$.

From the above fact, instead of establishing inequality (4.3), it is enough to show

$$|w(x)| \leq C \left( r^\beta + r^\beta \int_r^1 \frac{\|g\|_{L^q(X \cap Q[\rho \times \rho])}}{\rho^{1+\beta}} d\rho + \int_0^{A r} \frac{\|g\|_{L^q(X \cap Q[\rho \times \rho])}}{\rho} d\rho \right). \quad (4.4)$$
for any \( x \in Q[r \times r] \cap X \) and \( r \leq \frac{1}{\Lambda_1} \).

**Claim II** Next we claim that \( \exists \) positive constants \( \sigma, \eta(<1), K_1, K_2 \) and \( K_3 \) depending only on \( \lambda \) and \( n \) such that whenever

\[
m x_n - c \leq w(x) \leq M x_n + c \quad \text{for any} \quad x \in Q[1 \times 1],
\]

for some constants \( c (>0), m \) and \( M \), then this implies that \( \exists \tilde{m} \) and \( \tilde{M} \) such that

\[
\tilde{m} x_n - K_1 \|g\|_{L^\sigma(S)} \leq w(x) \leq \tilde{K} x_n + K_1 \|g\|_{L^\sigma(S)} \quad \text{for any} \quad x \in Q[\sigma \times \sigma],
\]

with

\[
\tilde{M} - \tilde{m} = (1 - \eta)(M - m) + K_2 c
\]

and

\[
|\tilde{M} + \tilde{m} - (M + m)| \leq \eta(M - m) + K_3 c.
\]

Proof of the claim:

Define \( K := \sqrt{n-1} \left( 1 + \frac{2\sqrt{n-1}}{\lambda} \right) \), and choose \( \varepsilon > 0 \) such that

\[
(1 + \varepsilon)(2 + \varepsilon)(K - 1)^\varepsilon \leq 4.
\]

Let

\[
\varphi(x) = \frac{2x_n}{\tilde{\sigma}} - \left( \frac{x_n}{\tilde{\sigma}} \right)^2 + \frac{\lambda^2}{2(n-1)} \sum_{i=1}^{n-1} \left( \left| \frac{x_i}{\tilde{\sigma}} \right| - 1 \right)^{2+\varepsilon},
\]

where \( \tilde{\sigma} = \frac{1}{K} \).

Then \( \varphi \) is differentiable and we have the following:

\[
\begin{cases}
-b_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \geq 0 & \text{in} \ Q[1 \times \tilde{\sigma}], \\
\varphi(x) \geq 1 & \text{if} \ x_n = \tilde{\sigma}; \text{or} \ |x'| = 1 \ \text{and} \ 0 \leq x_n \leq \tilde{\sigma}; \\
\varphi(x) \geq 0 & \text{if} \ x_n = 0.
\end{cases}
\]

From (4.2) and (4.11), we get

\[
\begin{cases}
-b_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \left( w(x) - M x_n - c \varphi(x) \right) \leq g & \text{in} \ Q[1 \times \tilde{\sigma}], \\
w(x) - M x_n - c \varphi(x) \leq 0 & \text{on} \ \partial(Q[1 \times \tilde{\sigma}]).
\end{cases}
\]
By the generalized maximum principle as in Lemma 3.3 for \( q \in (n - n_0, n) \), where \( n_0 \) is a small universal constant depends only on \( n \) and ellipticity constants, we have
\[
w(x) - M x_n - c \varphi(x) \leq C \| g \|_{L^q(X \cap Q[1 \times \tilde{\sigma}])}, \quad \text{for each } x \in Q[1 \times \tilde{\sigma}],
\] (4.13)
with a constant \( C = C(\lambda, n) > 0 \).

Since \( \varphi(x) \leq \frac{2K}{\sigma} = 2K x_n \) for each \( x \in Q[\tilde{\sigma} \times \tilde{\sigma}] \), this implies that
\[
w(x) \leq (M + 2K c) x_n + C \| g \|_{L^q(Q[1 \times 1])} \quad \text{in } Q[\tilde{\sigma} \times \tilde{\sigma}],
\] (4.14)
by considering \( m x_n - c \varphi(x) - w(x) \) instead of \( w(x) - M x_n - c \varphi(x) \).

Hence, we have the following estimate in \( Q[\tilde{\sigma} \times \tilde{\sigma}] \)
\[
(m - 2K c) x_n - C \| g \|_{L^q(Q[1 \times 1])} \leq w(x) \leq (m + 2K c) x_n + C \| g \|_{L^q(Q[1 \times 1])},
\] (4.15)
We define
\[
\sigma = \frac{\tilde{\sigma}}{2K} \text{ and } \Sigma = \{ \sigma e_n + T_{\sigma K} \}.
\]
We shall divide the remaining proof into two cases:
\[
w(\sigma e_n) \geq \frac{1}{2}(M + m)\sigma \quad \text{and} \quad w(\sigma e_n) < \frac{1}{2}(M + m)\sigma.
\]
Let us consider the case: \( w(\sigma e_n) \geq \frac{1}{2}(M + m)\sigma \). We define
\[
u(x) := w(x) - (m - 2K c) x_n + C \| g \|_{L^q(Q[1 \times 1])}.
\] (4.16)
From (4.15), it is easy to see that
\[
u(x) \geq 0, \quad \text{for each } x \in Q[\tilde{\sigma} \times \tilde{\sigma}],
\] (4.17)
Then
\[
u(\sigma e_n) \geq \frac{1}{2}(M - m + 4K c)\sigma + C \| g \|_{L^q(Q[1 \times 1])}.
\] (4.18)
From the Harnack inequality, we conclude that
\[
\inf_{\Sigma} \nu(x) \geq \frac{1}{2C}(M - m + 4K c)\sigma + C \| g \|_{L^q(X \cap Q[1 \times 1])},
\]
with constant $\tilde{C} = \tilde{C}(\lambda, n)$.

Set

$$\mathcal{M} = \left( \frac{1}{2\tilde{C}}(M - m + 4Kc)\sigma + C\|g\|_{L^q(\chi \cap Q(1 \times 1))} \right)^+.$$  

Hence it follows that

$$\inf_{\Sigma} u(x) \geq \mathcal{M}.$$  

(4.19)

Now we consider the following barriers

$$\Phi_1(x) = \frac{1}{2} \left( \frac{x_n}{\sigma} + \left( \frac{x_n}{\sigma} \right)^2 \right) - \frac{\lambda^2}{4(n-1)} \sum_{i=1}^{n-1} \left( \left| \frac{x_i}{\sigma} \right| - 1 \right)^{2+\epsilon}.$$  

(4.20)

Then $\Phi$ is differentiable with the followings:

$$-b_{ij} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} \leq 0 \text{ in } Q[K\sigma \times \sigma],$$

$$\Phi(x) \leq 1 \text{ if } x_n = \sigma;$$

$$\Phi(x) \leq 0 \text{ if } x_n = 0; \text{ or } |x'| = \tilde{\sigma} = K\sigma \text{ and } 0 \leq x_n \leq \sigma.$$  

From (4.2), (4.17) and (4.19), we see that

$$\mathcal{M} \Phi(x) - u(x) \leq C\|g\|_{L^q(\chi \cap Q[K\sigma \times \sigma])}, \text{ for each } x \in Q[K\sigma \times \sigma],$$  

with a constant $C = C(\lambda, n, \nu)$.

Furthermore making use of $\Phi(x) \geq \frac{1}{2\sigma}x_n$ in $Q[\sigma \times \sigma]$, we get

$$\frac{\mathcal{M}}{2\sigma}x_n - u(x) \leq C\|g\|_{L^q(\chi \cap Q[K\sigma \times \sigma])}, \text{ for each } x \in Q[\sigma \times \sigma].$$  

(4.21)

Thus using (4.16) and the second inequality of (4.16), we get (4.6), (4.7) and (4.8).

Claim III:

We will show that if $l \geq 0$ and

$$mx_n - c \leq w(x) \leq Mx_n + c, \text{ for each } x \in Q[\sigma^l \times \sigma^l],$$
for some constants $c > 0$, $M$ and $m$, then $\exists$ constants $\tilde{m}$ and $\tilde{M}$ such that the following bound holds:

$$\tilde{m}x_n - K_1\sigma^l \|g\|_{L^q(Q[\sigma^l \times \sigma^l])} \leq w(x) \leq \tilde{M}x_n + K_1\sigma^l \|g\|_{L^q(Q[\sigma^l \times \sigma^l])},$$

(4.22)

for each $x \in Q[\sigma^{l+1} \times \sigma^{l+1}]$, where

$$\tilde{M} - \tilde{m} = (1 - \eta)(M - m) + K_2 \frac{c}{\sigma^m},$$

(4.23)

and

$$|\tilde{M} + \tilde{m} - (M + m)| \leq \eta(M - m) + K_3 \frac{c}{\sigma^m},$$

(4.24)

for some constants $\eta, \sigma, K_1, K_2$ and $K_3$ are as in Claim II.

Proof of the claim:

Set

$$x = \sigma^l y, \quad a_{ij}(y) = b_{ij}(\sigma^l y), \quad u(y) = \frac{w(\sigma^l y)}{\sigma^l} \quad \text{and} \quad f(y) = \sigma^l g(\sigma^l y),$$

for each $y \in Q[1 \times 1]$.

From Claim II, we have the following estimate in $Q[\sigma \times \sigma]$:

$$\tilde{m}y_n - K_1 \|f\|_{L^2(X \cap Q[1 \times 1])} \leq u(y) \leq \tilde{M}y_n + K_1 \|f\|_{L^2(X \cap Q[1 \times 1])}.$$

It is easy to see that above inequality is equivalent to (4.22). Also $\tilde{M} - \tilde{m}$ and $|\tilde{M} + \tilde{m} - (M + m)|$ satisfy (4.23) and (4.24).

Proof of (4.4):

Set $g_r := \|g\|_{L^2(X \cap Q[r \times r])}$, for $0 < r \leq 1$.

Starting from $-1 \leq w \leq 1$ in $Q[1 \times 1]$ and using Claim III repeatedly, we get the following estimate for $l = 1, 2, \cdots$ in $Q[\sigma^l \times \sigma^l]$:

$$m_l x_n - K_1\sigma^{l-1} g_{\sigma^{l-1}} \leq w(x) \leq M_l x_n + K_1\sigma^{l-1} g_{\sigma^{l-1}},$$

(4.25)

with

$$M_l - m_l = (1 - \eta)^{l-1} K_2 + \frac{K_1 K_2}{\sigma} \sum_{i=0}^{l-2} g^{\sigma^i}(1 - \eta)^{l-2-i}$$

and

$$|M_{l+1} + m_{l+1} - (M_l + m_l)| \leq \eta(M_l - m_l) + \frac{K_1 K_3 f_{\sigma^{l-1}}}{\sigma}. $$
Let us suppose that $1 - \eta = \sigma^\gamma$. Since $g_1 \leq 1$, we have

$$M_l - m_l \leq C(\sigma^{l-1})^\gamma \left( 1 + \int_{\alpha^{l-1}}^{1} \frac{g_r}{r^{1+\gamma}} \right)$$

(4.26)

and

$$\sum_{j=l}^{\infty} \left| (M_{j+1} + m_{j+1}) - (M_j + m_j) \right| \leq \eta \sum_{j=l}^{\infty} (\sigma^{l-1})^\gamma \left( 1 + \int_{\alpha^{l-1}}^{1} \frac{g_r}{r^{1+\gamma}} \right) + \frac{K_1 K_3 \eta}{\sigma} \sum_{j=l}^{\infty} g_{\sigma^{l-1}}.$$

Set $G_r := \int_r^{1} \frac{g_r}{r^{1+\gamma}} \, dt$.

**Observation 1:**

$$\sum_{j=l}^{\infty} (\sigma^{l-1})^\gamma \int_{\alpha^{l-1}}^{1} \frac{g_r}{r^{1+\gamma}} \, dr = \sum_{j=l}^{\infty} (\sigma^{l-1})^\gamma G_{\sigma^{l-1}} \frac{\sigma^l - \sigma^{l+1}}{\sigma^l - \sigma^{l+1}}$$

$$= \frac{1}{(1 - \sigma)\sigma^\gamma} \sum_{j=l}^{\infty} \int_{\alpha^{l+1}}^{\sigma^l} r^{\gamma-1} G_r \, dr$$

$$= \frac{1}{(1 - \sigma)\sigma^\gamma} \left( \int_{0}^{\sigma^{l-1}} \frac{g_r}{r} \, dr + (\sigma^{l-1})^\gamma \int_{\alpha^{l-1}}^{1} \frac{g_r}{r^{1+\gamma}} \, dr \right).$$

**Observation 2:**

$$\sum_{j=l}^{\infty} g_{\sigma^{l-1}} \leq \frac{1}{1 - \sigma} \int_{0}^{\sigma^{l-2}} \frac{g_r}{r} \, dr$$

$$\leq \frac{1}{1 - \sigma} \int_{0}^{\sigma^{l-2}} \frac{1}{r} \left( \int_{B_{\sqrt{n_r}}} |g|^q \, dx \right)^{1/p} \, dr$$

$$\leq C \int_{0}^{\sigma^{l-2}} \left( \int_{B_{\sqrt{n_r}}} |g|^q \, dx \right)^{1/p} \, dr$$

$$\leq C \int_{0}^{\sigma^{l-2}} \left( \int_{B_\rho} |g|^q \, dx \right)^{1/p} \, d\rho$$

$$= C \tilde{r}^q (0, \sigma^{l-2}).$$

with constant $C = C(n, \sigma)$.

Also

$$\sum_{j=l}^{\infty} (\sigma^{j-1})^\alpha = \frac{\sigma^{m-1} \alpha}{1 - \sigma^\gamma}.$$
Observation 3:

\[
\sup_x \tilde{I}_q^g (x, r) \leq \frac{1}{|B_1|^{\frac{1}{n}}} \int_{0}^{|B_{1}|} \left[ g^{**} (\rho) \rho^{\frac{q}{q}} \right]^{\frac{1}{q}} \frac{d \rho}{\rho}.
\] (4.27)

Since \( g \in L(n, 1) \), then this implies that \(|g|^q \in L(n/q, 1/q)\). So right hand side of (4.27) is finite. Also since right hand side of (4.27) tends to zero as \( r \to 0 \), it follows that

\[
\sup_x \tilde{I}_q^g (x, r) \to 0 \text{ uniformly with respect to } x \text{ as } r \to 0.
\]

Observation 4:

\[
\sum_{j=l}^{\infty} \left| (M_{l+1} + m_{l+1}) - (M_l + m_l) \right| \leq C \left\{ (\sigma^{l-1})^\gamma + (\sigma^{l-1})^\gamma \int_{\sigma^{l-1}}^{1} \frac{g_r}{r^{1+\gamma}} dr + \int_{0}^{\sigma^{l-2}} \frac{g_r}{r^2} dr \right\}.
\] (4.28)

Also

\[
(\sigma^{l-1})^\gamma + (\sigma^{l-1})^\gamma \int_{\sigma^{l-1}}^{1} \frac{g_r}{r^{1+\gamma}} dr + \int_{0}^{\sigma^{l-2}} \frac{g_r}{r^2} dr \to 0 \text{ as } l \to \infty.
\]

Therefore, \((M_l + m_l)\) is convergent, say

\[
\lim_{l \to \infty} \frac{(M_l + m_l)}{2} = b.
\]

Now,

\[
\left| b - \frac{(M_l + m_l)}{2} \right| \leq \sum_{j=l}^{\infty} \left| \frac{(M_{l+1} + m_{l+1})}{2} - \frac{(M_l + m_l)}{2} \right| \leq C \left\{ (\sigma^{l-1})^\gamma + (\sigma^{l-1})^\gamma \int_{\sigma^{l-1}}^{1} \frac{g_r}{r^{1+\gamma}} dr + \int_{0}^{\sigma^{l-2}} \frac{g_r}{r^2} dr \right\}.
\]

Thus using (4.26) and (4.28), we get the following estimate in \( Q[\sigma^l \times \sigma^l] \),

\[
|w(x) - b x_n| \leq \left| w(x) - \frac{(M_l + m_l)}{2} x_n \right| + \left| b - \frac{(M_l + m_l)}{2} \right| x_n 
\leq \frac{(M_l - m_l)}{2} x_n + K_1 \sigma^{l-1} g_{\sigma^{l-1}} + \left| b - \frac{(M_l + m_l)}{2} \right| x_n.
\]
\[
\leq C \left\{ (\sigma^{l-1})^\gamma + (\sigma^{l-1})^\gamma \int_{\sigma^{l-1}}^{1} \frac{g_r}{r^{1+\gamma}} dr + \int_{0}^{\sigma^{l-2}} \frac{g_r}{r} dr \right\} x_n
\]

+ \frac{d r}{r} \left( \frac{d r}{r} \right) \sigma^{l-1} f_{\sigma^{l-1}}.

This completes the proof of the Lemma 4.1. \qed

In the next lemma, we will treat the case when \( X \neq \mathbb{R}^n_+ \).

**Lemma 4.2** If \( X \neq \mathbb{R}^n_+ \), then \( \exists \) positive constants \( C, \beta \) and \( \Lambda \) depending only on \( \lambda, n \) and \( \nu \) such that

\[
|w(x)| \leq C \left( \frac{r^\beta}{r_0^{1+\beta}} \|w\|_{L^\infty(X \cap Q[r \times r])} + \frac{r^\beta}{r_0^{\beta}} \|g\|_{L^q(X \cap Q[r \times r])} \\
+ r^\beta \int_{r}^{r_0} \frac{\|g\|_{L^q(X \cap Q[\rho \times \rho])}}{\rho^{1+\beta}} d\rho + \|g\|_{L^q(X \cap \Lambda r \times \Lambda r)} \right), \tag{4.29}
\]

for any \( x \in Q[r \times r] \cap X \) and \( r \leq r_0/\Lambda \).

**Proof** Same lines of proof will work as in the proof of [20, Theorem 2.1]. For sake of brevity, we omit the details. \qed

Let us state an important approximation lemma, which will be useful in the proof of the Theorem 2.2.

**Lemma 4.3** Given \( \varepsilon > 0 \), \( \exists \delta > 0 \) such that

\[
\|g\|_{L^\infty(\Omega)} \leq \delta. \tag{4.30}
\]

Then \( \exists \) a function \( v \in C^0(B_1) \) such that

\[
\begin{cases}
-b_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} = 0 & \text{in } \Omega \cap B_1, \\
v = 0 & \text{on } \partial \Omega \cap B_1,
\end{cases} \tag{4.31}
\]

and

\[
\|w - v\|_{L^\infty(B_1 \cap \Omega)} \leq \varepsilon, \tag{4.32}
\]

where \( w \) is the strong solution of (1.1).

**Proof** Let \( v \) be the solution of the following equations

\[
\begin{cases}
-b_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} = 0 & \text{in } B_1 \cap \Omega, \\
v = 0 & \text{on } \partial (B_1 \cap \Omega) \setminus \Omega, \\
v = w & \text{on } \partial (B_1 \cap \Omega) \cap \Omega.
\end{cases}
\]
Then from (1.1), we have
\[
\begin{cases}
-b_{ij} \frac{\partial^2 (w - v)}{\partial x_i \partial x_j} = g \text{ in } B_1 \cap \Omega, \\
w - v = 0 \text{ on } \partial (B_1 \cap \Omega).
\end{cases}
\]
By using the Alexandroff - Bakelman - Pucci maximum principle [6, 11], we have
\[
\|w - v\|_{L^\infty(B_1 \cap \Omega)} \leq C \|g\|_{L^n(B_1 \cap \Omega)},
\]
where \(C\) is a constant depending only on \(n\).

By choosing \(\delta = \frac{\epsilon}{C}\), we get
\[
\|w - v\|_{L^\infty(B_1 \cap \Omega)} \leq \epsilon.
\]

5 Proof of the theorem 2.1

By translation, rotation and scaling, we can assume that \(0 \in \partial \Omega\). Hence we only consider the differentiability at the origin. Since \(\Omega\) is convex, without loss of generality, we can assume that \(\Omega \subset \mathbb{R}^n_+\). We will consider the proof of the Theorem 2.1 into two cases:

Case I: \(X \neq \mathbb{R}^n_+\) and \(X = \mathbb{R}^n_+\).

Case II: \(X = \mathbb{R}^n_+\).
Let us choose positive sequence \(\{r_j\}_{j=0}^\infty\). Also suppose that \(w_j\) be the solution of the following problem:

\[
\begin{cases}
-b_{ij} \frac{\partial^2 w_j}{\partial x_i \partial x_j} = g \text{ in } Q[r_j \times r_j], \\
w_j = 0 \text{ on } \partial Q[r_j \times r_j] \setminus \Omega, \\
w_j = w \text{ on } \partial Q[r_j \times r_j] \cap \Omega.
\end{cases}
\]
(5.1)

By invoking Lemma 4.1, \(\exists\) constants \(C, \Lambda, \beta\) depending only on \(\lambda, n\) and \(\exists\) constants \(a_j\) such that
\[
|w_j(x) - a_j x_n| \leq C G_j(r) r, \quad \forall x \in Q[r \times r] \text{ and } r \leq \frac{r_j}{\Lambda},
\]
(5.2)
where

\[
G_j (r) = \frac{r^\beta}{r_j^{1+\beta}} \| w \|_{L^\infty (X \cap Q[r_j \times r_j])} + \frac{r^\beta}{r_j} \| g \|_{L^q (X \cap Q[r_j \times r_j])} + \int_r^{r_j} \frac{\| g \|_{L^q (X \cap Q[\rho \times \rho])}}{\rho^{1+\beta}} d\rho + \| g \|_{L^q (X \cap Q[\Lambda r \times \Lambda r])}.
\]

(5.3)

By using convexity of \( \Omega \) along with the fact that 0 is a flat point of it, \( \exists \) a convex function

\[ L : \mathbb{R}^{n-1} \rightarrow \mathbb{R} \]

and \( r_0 > 0 \) such that

\[ Q[r_0 \times r_0] \cap \partial \Omega = \left\{ (x', x_n) : x_n = L(x'), x' \in T_{r_0} \right\}. \]

Let \( r \) be any non-negative real number such that \( r \leq r_0 \). Let us define

\[ \tilde{L}(r) = \max \left\{ \frac{L(x')}{|x'|} : x' \in T_r \right\}. \]

With \( G_0(r) \) obtained from (5.2), we define

\[ \tilde{G}_0(r) = \sup \{ G_0(t) : 0 \leq t \leq r \}. \]

Finally, we define

\[ \psi(r) = \max \{ \tilde{L}(r), \tilde{G}_0(r) \}. \]

From the observation that if \( \tilde{L}(r) = 0 \), then

\[ \partial \left( \Omega \cap Q[\tilde{r} \times \tilde{r}] \right) \subset \partial \mathbb{R}_+^n \text{ for some } \tilde{r} < r_0. \]

Therefore, from Lemma 4.1, \( w \) is differentiable at 0.

Finally, we can assume that

\[ \tilde{L}(r) > 0 \text{ in } (0, r_0). \]

Set

\[ \sigma_j = \frac{r_0 \psi(r_0)}{2^j}. \]
From [20, Equation (3.4)], for each \( j \), the following equation
\[
 r_j \sqrt{\psi(r_j)} = \sigma_j \tag{5.4}
\]
has unique solution such that
\[
 r_j > r_{j+1} > 0, \quad r_j \to 0 \text{ as } j \to \infty. \tag{5.5}
\]
By reducing \( r_0 \) suitably, without loss of generality, we can assume that \( \sqrt{\psi(r_0)} \leq \frac{1}{\Lambda_1} \).

**Claim I:** For each \( j \), the following estimate holds:
\[
 0 \leq w_j(x) - w(x) \leq C r_j \psi(r_j) \text{ for each } x \in Q[r_j \times r_j] \cap \Omega. \tag{5.6}
\]
Indeed, by applying the generalized maximum principle as in Lemma 3.3 for \( q \in (n - n_0, n) \), where \( n_0 \) is a small universal constant depends only on \( n \) and ellipticity constants, we have
\[
 0 \leq w(x) \leq w_j(x) \text{ for each } x \in Q[r_j \times r_j] \cap \Omega. \tag{5.7}
\]
Using above estimate (5.7) and by applying once again the generalized maximum principle as in Lemma 3.3 for \( q \in (n - n_0, n) \), we have
\[
 0 \leq w(x) \leq w_{j+1}(x) \leq w_j(x) \text{ for each } x \in Q[r_{j+1} \times r_{j+1}] \cap \Omega. \tag{5.8}
\]
From (5.3) and definition of \( \tilde{L}, \tilde{G}_0, \psi \), we have the following estimate for each \( x \in Q[r_j \times r_j] \cap \partial\Omega \):
\[
 w_0(x) \leq a_0 x_n + C \tilde{G}_0(r_j)r_j \leq a_0 r_j h(r_j) + C \tilde{G}_0(r_j)r_j \leq C \psi(r_j)r_j. \tag{5.9}
\]
From (5.8) and (5.9), we get
\[
 w_j(x) - w(x) \leq C \psi(r_j)r_j \text{ on } Q[r_j \times r_j] \cap \partial\Omega.
\]
Now, generalized maximum principle as in Lemma 3.3 for \( q \in (n - n_0, n) \), gives
\[
 w_j(x) - w(x) \leq C \psi(r_j)r_j \text{ in } Q[r_j \times r_j] \cap \Omega. \tag{5.10}
\]
Combining (5.7) and (5.10), observation follows.

**Claim II:**
\[ \exists \text{ a sequence } C_j \text{ such that} \]
\[
 |w_j(x) - a_j x_n| \leq C_j \sigma_j \text{ in } Q[r_j \times r_j] \cap \Omega, \tag{5.11}
\]
and \( C_j \to 0 \) as \( j \to \infty \).
Indeed, let $C_j = G_j(\sigma_j)$. Then (5.11) follows from (5.2). Also using similar lines of proof as in step III of [20, Proof of Theorem 1.1], we have

$$C_j \to 0 \text{ as } j \to \infty.$$  

**Claim III:** $w$ is differentiable at 0.

Indeed, from (5.8) and $a_j := \frac{\partial w_j(0)}{\partial x_n}$, we see that

$$a_j > a_{j+1}, \text{ for } j = 0, 1, 2, \ldots.$$  

So let us suppose that

$$\lim_{j \to \infty} a_j := a.$$  

Using the Claim II, Claim III and (5.4), we get the following estimates for each $x \in Q[r_j \times r_j] \cap \Omega$:

$$|w(x) - ax_n| \leq |w(x) - w_j(x)| + |w_j(x) - a_jx_n| + |a_jx_n - ax_n|$$

$$\leq Cr_j\psi(r_j) + C_j\sigma_j + |a - a_j|\sigma_j$$

$$= \left(C\sqrt{\psi(r_j)} + C_j + |a - a_j|\right)\sigma_j.$$  

(5.12)

Let us choose $j \geq 0$ with $\sigma_{j+1} < r \leq \sigma_j$ for $r \in (0, \sigma_0)$. Set

$$C(r) := \sup \left\{C\sqrt{\psi(r_i)} + C_i + |a - a_i| : j \leq i \right\}.$$  

Using (5.12) and $\frac{\sigma_j}{r} \leq \frac{\sigma_{j+1}}{r_j} = 2$, finally we get the following estimates:

$$|w(x) - ax_n| \leq \left(C\sqrt{\psi(r_i)} + C_j + |a - a_j|\right)\sigma_j \leq 2C(r)r.$$  

It is easy to see that $C(r) \to 0$ as $r \to 0$.

Hence $w$ is differentiable at 0 with $\nabla w(0) = ae_n$. This completes the proof of the lemma.

**6 Proof of the Theorem 2.2**

By translation, rotation and scaling, without loss of generality, we can assume that $0 \in \partial\Omega$ and only consider the Log-Lipschitz of $w$ at 0.

**Case I:** $X \neq \mathbb{R}_+^n$.

From Lemma 4.3, we know that for any $\varepsilon > 0$, $\exists$ constant $\delta$ (depending on $\varepsilon$) and a function $v \in C^0(B_1)$ such that if (4.30) holds, then

$$\|w - v\|_{L_\infty(Omega \cap B_1)} \leq \varepsilon.$$
Since $v$ solves (4.31), by [20, Theorem 2.1], we see that
\[
\|v\|_{L^\infty(B_\lambda \cap \Omega)} \leq C \lambda^{1+\alpha}, \tag{6.1}
\]
where $C = C(n, \text{ellipticity})$ and $\alpha = \alpha(n, \text{ellipticity})$ are positive constant.

Estimates showed for $v$ implies that
\[
\|w\|_{L^\infty(B_\lambda \cap \Omega)} \leq \lambda.
\]
Indeed,
\[
\|w\|_{L^\infty(B_\lambda \cap \Omega)} = \|w - v\|_{L^\infty(B_\lambda \cap \Omega)} + \|v\|_{L^\infty(B_\lambda \cap \Omega)} \\
\leq \varepsilon + C \lambda^{1+\alpha} \quad \text{(using (4.32) and (6.1)).}
\]
Since $C \lambda^{1+\alpha} \to 0$ faster than $\lambda$ as $\lambda \to 0$. So there exist $\lambda \in (0, \ 1/2)$ such that
\[
C \lambda^{1+\alpha} = \frac{\lambda}{2}.
\]
Now choose $\varepsilon = \frac{\lambda}{2}$ which in turn fixes $\delta$.

Claim:
\[
\|w\|_{L^\infty(B_{\lambda^k} \cap \Omega)} \leq \lambda^k. \tag{6.2}
\]

The case $k = 1$ is precisely (6.1). Now, suppose that we have proved the $k$-th step of induction.

Define
\[
u(x) = \frac{w(\lambda^k x)}{\lambda^k} \quad \text{in} \quad \Omega_{\lambda^k} \cap B_1,
\]
where $\Omega_{\lambda^k} = \{x \in \Omega : \lambda^k x \in \Omega\}$.

Then, it is easy to verifies that $w$ is a strong solution to
\[
-b_{ij}(\lambda^k x) \frac{\partial^2 u}{\partial x_i \partial x_j} = \lambda^k g(\lambda^k x) \quad \text{in} \quad \Omega_{\lambda^k} \cap B_1.
\]

Set
\[
\tilde{g}(x) := \lambda^k g(\lambda^k x),
\]
then it is easy to see that $\|\tilde{g}\|_{L^n(\Omega)} \leq \delta$.

Therefore, by induction assumption,
\[
\|u\|_{(\Omega_{\lambda^k} \cap B_{\lambda^k})} \leq \lambda.
\]
Then, this implies that
\[ \| w \|_{L^\infty(B_{2k+1} \cap \Omega)} \leq \lambda^{k+1}. \]

Finally, in view of (6.2), we get
\[ \| w \|_{B_{2k}} \leq \lambda^k \leq k \lambda^k. \]

This shows that \( w \) is Log-Lipschitz at 0.

Case II: \( X = \mathbb{R}^n_+ \).
In this case, one can argue as in the case of \( C^1 \) domains, using compactness argument and the Krylov type \( C^{1,\alpha} \) boundary decay estimate for the limiting problem
\[
\begin{cases}
  w \in S(\lambda, \frac{1}{\lambda}, 0) & \text{in } \mathbb{R}^n_+,
  \\
  w = 0 & \text{on } \{x_n = 0\},
\end{cases}
\]
as in [23] to get solution is Log-Lipschitz at 0.

Acknowledgements The author would like to thank Agnid Banerjee for discussions and suggestions concerning the preparation of the manuscript. The author is also grateful to TIFR CAM for the financial support. Finally, author thank the editor for the kind handling of our paper and the reviewer for several comments and suggestions that has helped in improving the presentation of the paper.

Data availability statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Adimurthi, K., Banerjee, A.: Borderline regularity for fully nonlinear equations in dini domains (2018), Submitted, arXiv: 1806.07652v2
2. Banerjee, A., Munive, I.: Gradient continuity estimates for the normalized p-Poisson equation. Commun. Contemp. Math. 22(8), 1950069 (2020)
3. Caffarelli, L.A.: Interior estimates for fully nonlinear equations. Ann. Math. 130, 189–213 (1989)
4. Caffarelli, L.A.: Interior \( W^{2,p} \) estimates for solutions of the Monge-Ampère equation. Ann. Math. 131, 135–150 (1990)
5. Caffarelli, L.A.: A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity. Ann. Math. 131, 129–134 (1990)
6. Caffarelli, L.A., Cabre, X.: Fully nonlinear elliptic equations. AMS 43, Providence, RI, pp. 21–28 (1995)
7. Caffarelli, L., Crandall, M.G., Kocan, M., Swiech, A.: On viscosity solutions of fully nonlinear equations with measurable ingredients. Comm. Pure Appl. Math. 49, 365–397 (1996)
8. Daskalopoulos, P., Kuusi, T., Mingione, G.: Borderline estimates for fully nonlinear elliptic equations. Comm. Partial Differ. Equ. 39, 574–590 (2014)
9. Duzaar, F., Mingione, G.: Gradient estimates via non-linear potentials. Amer. J. Math. 133, 1093–1149 (2011)
10. Escauriaza, L.: $W^{2,n}$ a priori estimates for solutions to fully non-linear equations. Indiana Univ. Math. J. 42, 413–423 (1993)
11. Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. Springer, Berlin (1983)
12. Grafakos, L.: Classical and Modern Fourier Analysis. Pearson Education, Upper Saddle River (2004)
13. Krylov, N.V.: Boundedly inhomogeneous elliptic and parabolic equations in a domain. Izvestia Akad. Nauk. SSSR 47, 75–108 (1983) [Russian]. (English translation in Math. USSR Izv. 22, 67–97 (1984)
14. Kuusi, T., Mingione, G.: Universal potential estimates. J. Funct. Anal. 262, 4205–4269 (2012)
15. Kuusi, T., Mingione, G.: Linear potentials in nonlinear potential theory. Arch. Ration. Mech. Anal. 207, 215–246 (2013)
16. Lieberman, G.M.: The Dirichlet problem for quasilinear elliptic equations with continuously differentiable boundary data. Comm. Partial Differ. Equ. 11, 167–229 (1986)
17. Li, D., Wang, L.: Boundary differentiability of solutions of elliptic equations on convex domains. Manuscr. Math. 121, 137–156 (2006)
18. Ma, F., Wang, L.: Boundary first order derivative estimates for fully nonlinear elliptic equations. J. Differ. Equ. 252, 988–1002 (2012)
19. Stein, E.M.: Editor’s note: the differentiability of functions in $\mathbb{R}^n$. Ann. Math. 113, 383–385 (1981)
20. Wang, L.H.: On the regularity theory of fully nonlinear parabolic equations. Ph. D. Dissertation of Courant Institute (1989)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.