The analysis of FETI-DP preconditioner for full DG discretization of elliptic problems

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Abstract In this paper a discretization based on discontinuous Galerkin (DG) method for an elliptic two-dimensional problem with discontinuous coefficients is considered. The problem is posed on a polygonal region $\Omega$ which is a union of $N$ disjoint polygonal subdomains $\Omega_i$ of diameter $O(H_i)$. The discontinuities of the coefficients, possibly very large, are assumed to occur only across the subdomain interfaces $\partial\Omega_i$. In each $\Omega_i$ a conforming quasiuniform triangulation with parameters $h_i$ is constructed. We assume that the resulting triangulation in $\Omega$ is also conforming, i.e., the meshes are assumed to match across the subdomain interfaces. On the fine triangulation the problem is discretized by a DG method. For solving the resulting discrete system, a FETI-DP type method is proposed and analyzed. It is established that the condition number of the preconditioned linear system is estimated by $C(1 + \max_i \log H_i/h_i)^2$ with a constant $C$ independent of $h_i$, $H_i$ and the jumps of coefficients. The method is well suited for parallel computations and it can be extended to three-dimensional problems. This result is an extension, to the case of full fine-grid DG discretization, of the previous result [SIAM J. Numer. Anal., 51 (2013), pp. 400–422] where it was considered a conforming finite element method.
inside the subdomains and a discontinuous Galerkin method only across the subdomain interfaces. Numerical results are presented to validate the theory.

**Keywords** Interior penalty discretization · discontinuous Galerkin · elliptic problems with discontinuous coefficients · finite element method · FETI-DP algorithms · preconditioners · AMS: 65F10, 65N20, 65N30

1 Introduction

In this paper we consider a boundary value problem for elliptic second order partial differential equations with highly discontinuous coefficients and homogeneous Dirichlet boundary condition. The problem is posed on a polygonal region $\Omega$ which is a union of disjoint two-dimensional polygonal subdomains $\Omega_i$ of diameter $O(H_i)$. We assume that this partition $\{\Omega_i\}_{i=1}^N$ is geometrically conforming, i.e., for all $i$ and $j$ with $i \neq j$, the intersection $\partial \Omega_i \cap \partial \Omega_j$ is either empty, a common corner or a common edge of $\Omega_i$ and $\Omega_j$. We consider the case where the discontinuities of the coefficients are assumed to occur only across $\partial \Omega_i$. The problem is approximated by the symmetric interior penalty discontinuous Galerkin (SIPDG) method inside each $\Omega_i$, with $h_i$ as a mesh parameter. The meshes are assumed to match across the interfaces $\partial \Omega_i \cap \partial \Omega_j$. In order to deal with the nonconformity of the DG spaces across $\partial \Omega_i$, a discrete problem is further formulated using the SIPDG method on the $\partial \Omega_i$; see [1,2,3,14,18]. The main goal of this paper is to develop the FETI-DP methodology for this DG discretization. We show that the designed FETI-DP method is almost optimal with rate of convergence independent of the coefficients jumps. The analysis of the discussed preconditioner is based on the analysis used in [10].

For that we construct refinement of the fine meshes on each $\Omega_i$ and introduce special interpolation operators which allow to switch from spaces of piecewise linear functions defined on the fine mesh on $\Omega_i$ to spaces of piecewise linear and continuous functions defined on the refinement of the fine mesh on $\Omega_i$ and vice-versa. The interpolation operators used in this paper are similar to those introduced [4,19] and also used in [5]. Properties of the interpolations operators are proved in this paper. In this paper we extend our results published in [10] to the full DG discretization of the considered problem. In [10] a FETI-DP preconditioner was designed and analyzed for the problem discretized by composed finite element and DG methods, i.e., by continuous Finite Element Method (FEM) inside each $\Omega_i$ and the symmetric interior penalty DG method on the interfaces $\partial \Omega_i$ only, while here in this paper a FETI-DP preconditioner for the case full DG discretization of the problem is designed and analyzed. The proposed FETI-DP algorithm is essentially algebraic and does not require any interpolation operator for the design of the algorithm, therefore, it can be naturally extended to three-dimensional problems and for high-order and discretizations elasticity, Stokes and Maxwell. An interpolation operator is only required for the analysis, and to the best of our knowledge it is the first complete analysis ever developed for any Neumann-Neumann type of discretization such as FETI, FETI-DP, BDD, BDDC and NN, and for any of the
classical interior penalty DG discretizations considered in [1][2][3]. We expect that the analysis developed here can be extended to more general problems as the one mentioned above. We also remark that a new technique based on two interpolation operators were introduced to avoid a condition on the number of elements that can touch a corner of a substructure interface, therefore, the algorithm developed here can applied also for subdomains generated from graph partitioners. Furthermore, we believe that such technique can be extended to other types of full DG discretizations beyond [1][2][3].

For the developing of FETI-DP methods for the continuous FEM, see the Introduction of [10] and the references therein. See also [12,13,15,16,17].

In this paper, the full DG discrete problem is reduced to the Schur complement problem with respect to unknowns on the interfaces of the subdomains $\Omega_i$. For that, discrete harmonic functions defined in a special way, i.e., in the DG sense, are used. We note that there are unknowns on both sides of the interfaces $\partial \Omega_i$. This means that the unknowns on both sides of the interfaces should be kept as degrees of freedom of the linear Schur complement system to be solved. These issues characterize some of the main difficulties on designing and analyzing FETI-DP type methods for full DG discretizations. Distinctively from the classical conforming FEM discretizations, here a double layer of Lagrange multipliers are needed on interfaces rather than a single layer of Lagrange multipliers as normally is seen in FETI-DP for conforming FEMs. Despite the fact we follow the FETI-DP abstract approach, which was aimed to single layer of Lagrange multipliers, see for example [20], in this paper we successfully overcome this difficulty.

The algorithm we develop in this paper is as follows. Let $\Gamma'_i$ be the union of all edges $E_{ij}$ and $E_{ji}$ which are common to $\Omega_i$ and $\Omega_j$, where $E_{ij}$ and $E_{ji}$ refer to the $\Omega_i$ and $\Omega_j$ sides, respectively, see Figure 1. We note that each $\Gamma'_i$ has interface unknowns (degrees of freedom) corresponding to nodal points which are the endpoint of edges of fine triangulation belonging to $\partial \Omega_i \setminus \partial \Omega$ and $E_{ji} \subset \partial \Omega_j$. Unknowns corresponding to vertices of fine triangles which intersect $E_{ij} \subset \partial \Omega_i$ and $E_{ji} \subset \partial \Omega_j$ by only one vertex are treated as interior unknowns. We now need to couple $\Gamma'_i$ with the other side of the interface $\Gamma''_i$. We first impose continuity at the interface unknowns at the corners of $\Gamma'_i$ (which are corners of $\Gamma_i$ and common endpoints of $E_{ij}$) with the interface unknowns at corners of the $\Gamma''_i$, see Figure 3. These unknowns are called primal. The remaining interface unknowns on $\Gamma'_i$ and $\Gamma''_i$ are called dual and have jumps, hence, Lagrange multipliers are introduced to eliminate these jumps, see Figure 2. For the dual system with Lagrange multipliers, a special block diagonal preconditioner is designed. It leads to independent local problems on $\Gamma'_i$ for $1 \leq i \leq N$. It is proved that the proposed method is almost optimal with a condition number estimate bounded by $C(1 + \max_i \log H_i/h_i)^2$, where $C$ does not depend on $h_i, H_i$, the number of subdomains $\Omega_i$ and the jumps in the coefficients.

The method can be extended to full DG discretizations of three-dimensional problems.
The paper is organized as follows. In Section 2 the differential problem and a full DG discretization are formulated. In Section 3, the Schur complement problem is derived using discrete harmonic functions defined in a special way (in the DG sense). In Section 4, the so-called FETI-DP method is introduced, i.e., the Schur complement problem is reformulated by imposing continuity for the primal variables and by using Lagrange multipliers at the dual variables, and a special block diagonal preconditioner is defined. The main results of the paper are Theorem 1 and Lemma 4. Section 5 is devoted to some technical tools and auxiliary lemmas to analyze the FETI-DP preconditioner. The proofs of these results are given in the Appendix A and B. In Section 6 numerical tests are reported which confirm the theoretical results.

2 Differential and discrete problems

In this section we discuss the continuous problem and its DG discretization. The resulting discrete problem is taken into consideration for preconditioning.

2.1 Differential problem

Consider the following problem: Find $u^*_{ex} \in H^1_0(\Omega)$ such that

$$a(u^*_{ex}, v) = f(v) \quad \text{for all } v \in H^1_0(\Omega),$$

(1)

where $\rho_i > 0$ is a constant, $f \in L^2(\Omega)$ and

$$a(u, v) := \sum_{i=1}^N \int_{\Omega_i} \rho_i(x) \nabla u \cdot \nabla v \, dx \quad \text{and} \quad f(v) := \int_{\Omega} f v \, dx.$$

We assume that $\Omega = \bigcup_{i=1}^N \Omega_i$ and the substructures $\Omega_i$ are disjoint shaped regular polygonal subregions of diameter $O(H_i)$. We assume that the partition $\{\Omega_i\}_{i=1}^N$ is geometrically conforming, i.e., for all $i$ and $j$ with $i \neq j$, the intersection $\partial \Omega_i \cap \partial \Omega_j$ is either empty, a common corner or a common edge of $\Omega_i$ and $\Omega_j$. For clarity we stress that here and below the identifier edge means a curve of continuous intervals and its two endpoints are called corners. The collection of these corners on $\partial \Omega_i$ are referred as the set of corners of $\Omega_i$. Let us denote $\bar{E}_{ij}$ as an edge of $\partial \Omega_i$ which belongs to the global boundary $\partial \Omega$, let us introduce a set of indices $J^\partial_i$ to refer these edges. The set of indices of all edges of $\Omega_i$ is denoted by $J^i = J^{i,0}_i \cup J^{i,\partial}_i$. Moreover, if $\Omega_j$ has a common edge $E_{ji}$ with $\Omega_i$, we take into account edges of $\Omega_i$ which belong to the global boundary $\partial \Omega$, let us introduce a set of indices $J^{\partial,0}_i$ to refer these edges. The set of indices of all edges of $\Omega_i$ is denoted by $J^i = J^{i,0}_i \cup J^{i,\partial}_i$.\)
2.2 Discrete problem

Let us introduce a shape regular and quasiuniform triangulation (with triangular elements) \( T_h \) on \( \Omega \) and let \( h_i \) represent its mesh size. The resulting triangulation on \( \Omega \) is matching across \( \partial \Omega \). Let

\[
X_i(\Omega_i) := \prod_{\tau \in T_h^i} X_{\tau}
\]

be the product space of finite element (FE) spaces \( X_{\tau} \) which consist of linear functions on the element \( \tau \) belonging to \( T_h^i \). We note that a function \( u_i \in X_i(\Omega_i) \) allows discontinuities across elements of \( T_h^i \). We also note that we do not assume that functions in \( X_i(\Omega_i) \) vanish on \( \partial \Omega \).

The global DG finite element space we consider is defined by

\[
X(\Omega) = \bigoplus_{i=1}^{N} X_i(\Omega_i) \equiv X_1(\Omega_1) \times X_2(\Omega_2) \times \cdots \times X_N(\Omega_N). \tag{2}
\]

We define \( E_h^{i,0} \) as the set of edges of the triangulation \( T_h^i \) which are inside \( \Omega_i \), and by \( E_h^{i,j} \), for \( j \in J_h^i \), the set of edges of the triangulation \( T_h^i \) which are on \( E_{ij} \). An edge \( e \in E_h^{i,0} \) is shared by two elements denoted by \( \tau_+ \) and \( \tau_- \) of \( T_h^i \) with outward unit normal vectors \( n_+ \) and \( n_- \), respectively, and denote

\[
\{ \rho \nabla u \} = \frac{1}{2} (\rho_{\tau_+} \nabla u_{\tau_+} + \rho_{\tau_-} \nabla u_{\tau_-}) \quad \text{and} \quad [u] = u_{\tau_+} n_+ + u_{\tau_-} n_-.
\]

An edge \( e \in E_h^{i,j} \) is shared by one element denoted by \( \tau \) with outward unit normal vectors \( n \), and denote

\[
\{ \rho \nabla u \} = \rho_{\tau} \nabla u_{\tau} \quad \text{and} \quad [u] = u_{\tau} n.
\]

The discrete problem we consider by the DG method is of the form: Find \( u^* = \{ u_i^* \}_{i=1}^{N} \in X(\Omega) \) where \( u_i^* \in X_i(\Omega_i) \), such that

\[
 a_h(u^*, v) = f(v) \quad \text{for all} \quad v = \{ v_i \}_{i=1}^{N} \in X(\Omega), \tag{3}
\]

where the global bilinear form \( a_h \) and the right hand side \( f \) are assembled as

\[
a_h(u, v) := \sum_{i=1}^{N} a_i'(u, v) \quad \text{and} \quad f(v) := \sum_{i=1}^{N} \int_{\Omega_i} f v_i \, dx. \tag{4}
\]

Here, the local bilinear forms \( a_i' \), \( i = 1, \ldots, N \), are defined as

\[
a_i'(u, v) := a_i(u_i, v_i) + s_{i,0}(u_i, v_i) + p_{i,0}(u, v) + s_{i,0}(u, v) + p_{i,0}(u, v) \tag{5}
\]

where \( a_i \) is the local energy bilinear form,

\[
a_i(u_i, v_i) := \sum_{\tau \in T_h^i} \int_{\tau} \rho_{\tau} \nabla u_{\tau_i} \cdot \nabla v_i \, dx. \tag{6}
\]
The (interior edges) symmetrized bilinear form \( s_{i,0} \) is defined by

\[
s_{i,0}(u, v) := -\sum_{e \in E_{i,0}^h} \int_e \left\{ \rho_i \nabla u_i \cdot [v_i] + \{ \rho_i \nabla v_i \} \cdot [u_i] \right\} ds,
\]  
(7)

and the (interior edges) penalty bilinear form is given by

\[
p_{i,0}(u, v) := \sum_{e \in E_{i,0}^h} \int_e \delta \frac{\rho_i}{h_e} [u_i], [v_i] ds.
\]  
(8)

The corresponding symmetric and penalty form over the local interface edges are given by

\[
s_{i,\partial}(u, v) := \sum_{j \in J_{i,0}} \sum_{e \in E_{i,j}^h} \int_e \frac{1}{l_{ij}} \left( \rho_{ij} \frac{\partial v_j}{\partial n} (v_j - v_i) + \rho_{ij} \frac{\partial u_i}{\partial n} (u_j - u_i) \right) ds
\]  
(9)

and

\[
p_{i,\partial}(u, v) := \sum_{j \in J_{i,0}} \sum_{e \in E_{i,j}^h} \int_e \delta \frac{\rho_{ij}}{h_e} (u_i - u_j)(v_i - v_j) ds
\]  
(10)

respectively. Here and above, \( h_e \) denotes the length of the edge \( e \). When \( j \in J_{H,0} \) we set \( l_{ij} = 2 \), while when \( j \in J_{H,\partial} \) we denote the boundary edges \( E_{ij} \subset \partial \Omega_i \) by \( E_{i\partial} \) and set \( l_{i\partial} = 1 \), and on the artificial edge \( E_{i\partial} \equiv E_{i\partial} \) we set \( u_{i\partial} = 0 \) and \( v_{i\partial} = 0 \). The partial derivative \( \frac{\partial}{\partial n} \) denotes the outward normal derivative on \( \partial \Omega_i \) and \( \delta \) is the penalty positive parameter.

**Remark 1** The discrete formulation used here is useful for the adequate formulation of our FETI-DP method. We note that (9) and (10) can also be written in the same form as in (7) and (8) without the term \( l_{ij} \) (since now the edges are counted once). We note also that others DG formulations for discontinuous coefficients can also be considered [3, 6]. We note that the design of FETI-DP methods for those formulations are the same, and the analysis are simple modifications of the proofs we present here in this paper. See for instance [7, 9, 10] where a formulation based on harmonic averages of the coefficients is studied. Some details for these case as well as numerical experiments will be presented elsewhere. We note that three-dimensional versions can also be formulated and analyzed by extending naturally some ideas from this paper and from [11].

For \( u = \{u_i\}_{i=1}^N \), \( v = \{v_i\}_{i=1}^N \in X(\Omega) \), let us introduce the local positive bilinear forms

\[
d_i(u, v) := a_i(u, v) + p_{i,0}(u, v) + p_{i,\partial}(u, v)
\]  
(11)

and the global positive bilinear form assembled as

\[
d_h(u, v) := \sum_{i=1}^N d_i(u, v).
\]  
(12)
Note that the norm defined by \( d_h(\cdot, \cdot) \) is a broken norm in \( X(\Omega) \) with weights given by \( \rho_i, \delta \frac{\rho_i}{h_i} \), and \( \frac{\rho_i}{h_i} \). For \( u = \{u_i\}_{i=1}^N \in X(\Omega) \), this discrete norm is defined by \( ||u||_h^2 = d_h(u, u) \).

It is known that there exists a \( \delta_0 = O(1) > 0 \) and a positive constant \( c \), which does not depend on \( \rho_i, H_i, h_i \) and \( u_i \), such that for every \( \delta \geq \delta_0 \) we obtain \( |s_i(u, u)| \leq cd_i(u, u) \) and therefore, the following lemma is valid.

**Lemma 1** There exists \( \delta_0 > 0 \) such that for \( \delta \geq \delta_0 \) and for all \( u \in X(\Omega) \) we have, in each subdomain,

\[
\gamma_0 d_i(u, u) \leq a'_i(u, u) \leq \gamma_1 d_i(u, u), \quad 1 \leq i \leq N,
\]

and also we have the following global bilinear forms inequality

\[
\gamma_0 d_h(u, u) \leq a_h(u, u) \leq \gamma_1 d_h(u, u).
\]

Here, \( \gamma_0 \) and \( \gamma_1 \) are positive constants independent of the \( \rho_i, h_i, H_i \) and \( u_i \).

The proof of this lemma is a modification of the proof of [7, Lemma 2.1], [9] or [6, Theorem 4.1], therefore it is omitted.

**Lemma 1** implies that the discrete problem (3) is elliptic and continuous, therefore, the solution exists and it is unique and stable. An optimal a priori error estimate of this method was established in [1,2] for the continuous coefficient case. We mention here that Lemma 1 together with Lemma 7, see below, are going to be fundamental for establishing condition number estimates for the FETI-DP preconditioned system developed in the remaining sections.

### 3 Schur complement matrices and discrete harmonic extensions

The first step of many iterative substructuring solvers, such as the FETI-DP method that we consider in this paper, requires the elimination of unknowns interior to the subdomains. In this section, we describe this step for DG discretizations.

We introduce some notation and then formulate (3) as a variational problem with constraints. Associated to a subdomain \( \Omega_i \), we define the extended subdomain \( \Omega'_i \) by

\[
\Omega'_i := \overline{\Omega_i} \cup \{\cup_{j \in \mathcal{J}_i^0} \overline{E_{ji}}\}
\]

i.e., it is the union of \( \overline{\Omega_i} \) and the \( \overline{E_{ji}} \subset \partial \Omega_j \) such that \( j \in \mathcal{J}_i^0 \), and the local interfaces \( \Gamma'_i \) by

\[
\Gamma'_i := \overline{\partial \Omega'_i \setminus \partial \Omega} \quad \text{and} \quad \Gamma_i := \Gamma'_i \cup \{\cup_{j \in \mathcal{J}_i^0} \overline{E_{ji}}\}.
\]
We also introduce the sets

\[ \Gamma := \bigcup_{i=1}^{N} \Gamma_i, \quad \Gamma' := \prod_{i=1}^{N} \Gamma'_i, \quad I_i := \Omega'_i \setminus \Gamma'_i \quad \text{and} \quad I := \prod_{i=1}^{N} I_i. \] (15)

Associated to these sets, we classify degrees of freedom (DG nodal values) on \( \Omega'_i \). We illustrate this classification (along with further classifications to be introduced later) in Figure 1:

- \( \Gamma_i \)-degrees of freedom: The nodal points which are endpoints of edges of \( \mathcal{E}_{h}^{i,j} \) for \( j \in \mathcal{J}_H^{i,0} \).
- Degrees \( \mathcal{V}_i \): The nodal points which are both an endpoint of an edges of \( \mathcal{E}_{h}^{i,j} \) for \( j \in \mathcal{J}_H^{i,0} \) and a corner of \( \Omega_i \).
- \( \Gamma'_i \)-degrees of freedom: It is the union of the degrees \( \Gamma_i \) and the nodal points which are endpoints of edges of \( \mathcal{E}_{h}^{i,j} \) for \( j \in \mathcal{J}_H^{i,0} \).
- \( \mathcal{V}'_i \)-degrees of freedom: It is the union of the degrees \( \mathcal{V}_i \), and the nodal points which are both an endpoint of an edge of \( \mathcal{E}_{h}^{i,j} \) for \( j \in \mathcal{J}_H^{i,0} \) and a corner of \( \Omega_j \).
- \( I_i \)-degrees of freedom: The nodal points which are vertices of elements \( \tau \) of \( \mathcal{T}_h^i \) and are not endpoints of an edge of \( \tau \) in \( \mathcal{E}_{h}^{i,j} \) for \( j \in \mathcal{J}_H^{i,0} \).
- \( \Omega'_{i,j} \)-degrees of freedom: The union of \( \Gamma_i \) and \( I_i \) degrees of freedom.
- \( \Omega'_{i,j} \)-degrees of freedom: The union of \( \Gamma'_i \) and \( I_i \) degrees of freedom.
Remark 2 Nodal points of elements in $\mathcal{T}_h^i$ which intersect $E_{ij}$ by only one vertex are treated as $I_i$ degrees of freedom. The trace of the basis DG functions associated to these nodal points are zero almost everywhere on $\Gamma_i$, hence, the analysis shows that it is convenient to treat these nodes as as $I_i$-degrees of freedom as defined above. For the same reason, nodal points of elements of $\mathcal{T}_h^j$ which intersect $E_{ij}$ by vertex only for $j \in J_H^{i,0}$ are not included as $\Omega_i'$-degrees of freedom. We point out, however, that we could have considered these nodes points as $I_i$ and $\Omega_i'$-degrees of freedom, respectively as well, and by introducing also Lagrange multipliers to force continuity at these type of nodal values, and then eliminating these Lagrange multipliers and also these degrees of freedom. This approach would be equivalent to the approach we consider and analyze here in this paper.

We mention that if we refer to the classical FETI-DP design, i.e., for continuous FEM, the set $I$ corresponds to the set of interior degrees of freedom (that are block-uncoupled) and can be eliminated in a first step. Moreover, the $I'$ corresponds to the global interface with the original coupling and $I''$ corresponds to the (block-wise) torn interface. We recall that the classical FETI-DP design extends the original problem to a problem in the torn interface space and then eliminating these Lagrange multipliers and also these degrees of freedom. We now describe these steps for DG discretization in detail. First we introduce the corresponding functions spaces.

Let $W_i(\Omega_i')$ be the FE space of functions defined by values on $\Omega_i'$

$$W_i(\Omega_i') = W_i(\overline{\Omega}_i) \times \prod_{j \in J_H^{i,0}} W_i(E_{ji}),$$

where $W_i(\overline{\Omega}_i) := X_i(\Omega_i)$ and $W_i(E_{ji})$ is the trace of the DG space $X_j(\Omega_j)$ on $E_{ji} \subset \partial \Omega_j$ for all $j \in J_H^{i,0}$. A function $u'_i \in W_i(\Omega_i')$ is defined by the $\Omega_i'$-degrees of freedom. Below, we denote $u'_i$ by $u_i$ if it is not confused with functions of $X_i(\Omega_i)$. A function $u_i \in W_i(\Omega_i')$ is represented as

$$u_i = \{(u_i)_{j}, \{(u_i)_{j} \}_{j \in J_H^{i,0}}\},$$

where $(u_i)_{j} := u_{i|\overline{\Omega}_j}$ ($u_i$ restricted to $\overline{\Omega}_j$) and $(u_i)_{j} := u_{i|E_{ji}}$ ($u_i$ restricted to $E_{ji}$). Here and below we use the same notation to identify both DG functions and their vector representations. Note that $a'_i(\cdot, \cdot)$, see [5], is defined on $W_i(\Omega_i') \times W_i(\Omega_i')$ with corresponding stiffness matrix $A'_i$ defined by

$$a'_i(u_i, v_i) = \langle A'_i u_i, v_i \rangle \quad u_i, v_i \in W_i(\Omega_i'),$$

where $\langle u_i, v_i \rangle$ denotes the $l^2$ inner product of nodal values associated to the vector space in consideration. We also represent $u_i \in W_i(\Omega_i')$ as $u_i = (u_i, u_{i, r})$ where $u_{i, r}$ represents values of $u_i$ at nodal points on $I'_i$ and $u_{i, I}$ at the interior nodal points in $I_i$, see [15]. Hence, let us represent $W_i(\Omega_i')$ as
the vector spaces $W_i(I_i) \times W_i(I'_i)$. Using the representation $u_i = (u_{i,I}, u_{i,I'})$, the matrix $A'_i$ can be represented as

$$A'_i = \begin{pmatrix} A'_{i,II} & A'_{i,IG'} \\ A'_{i,IG} & A'_{i,GG'} \end{pmatrix},$$

(18)

where the block rows and columns correspond to the nodal points of $I_i$ and $I'_i$, respectively.

The Schur complement of $A'_i$ with respect to $u_{i,I'}$ is of the form

$$S'_i := A'_{i,II} - A'_{i,IG'} (A'_{i,II})^{-1} A'_{i,IG'},$$

(19)

and introduce the block diagonal matrix $S = \text{diag}(S'_i)_{i=1}^N$. Note that $S'_i$ satisfies

$$(S'_i u_{i,I'}, u_{i,I'}) = \min a'_i(w_i, w_i),$$

(20)

where the minimum is taken over $w_i = (w_{i,I}, w_{i,I'}) \in W_i(\Omega'_i)$ such that $w_{i,I'} = u_{i,I'}$ on $I'_i$. The bilinear form $a'_i(\cdot, \cdot)$ is symmetric and nonnegative, see Lemma [1]. The minimizing function satisfying (20) is called discrete harmonic in the sense of $a'_i(\cdot, \cdot)$ or in the sense of $\mathcal{H}'_i$. An equivalent definition of the minimizing function $\mathcal{H}'_i u_{i,I'} \in W_i(\Omega'_i)$ is given by the solution of

$$a'_i(\mathcal{H}'_i u_{i,I'}, v_{i,I'}) = 0 \quad v_{i,I'} \in \bar{W}_i(\Omega'_i),$$

(21)

$$\mathcal{H}'_i u_{i,I'} = u_{i,I'} \quad \text{on } I'_i,$$

(22)

where $\bar{W}_i(\Omega'_i)$ is the subspace of $W_i(\Omega'_i)$ of functions which vanish on $I'_i$. We note that for substructures $\Omega_i$ which intersect $\partial \Omega$ by edges, the nodal values of $W_i(\Omega'_i)$ on $\partial \Omega_i \setminus I_i \subset \partial \Omega$ are treated as unknowns and belong to $I_i$.

Let us introduce the product space

$$W(\Omega') := \prod_{i=1}^N W_i(\Omega'_i),$$

(23)

i.e., $u \in W(\Omega')$ means that $u = \{u_i\}_{i=1}^N$ where $u_i \in W_i(\Omega'_i)$; see [16] for the definition of $W_i(\Omega'_i)$. Recall that we write $(u_i)_i = u_i|_{\overline{\Omega}_i}$ $(u_i$ restricted to $\overline{\Omega}_i)$ and $(u_i)_i = u_i|_{\overline{E}_i}$ $(u_i$ restricted to $\overline{E}_i$). Using the representation $u_i = (u_{i,I}, u_{i,I'})$ where $u_{i,I} \in W_i(I_i)$ and $u_{i,I'} \in W_i(I'_i)$, see [18], let us introduce the product space

$$W(I') := \prod_{i=1}^N W_i(I'_i),$$

(24)

i.e., $u_{I'} \in W(I')$ means that $u_{I'} = \{u_i_{I'}\}_{i=1}^N$ where $u_i_{I'} \in W_i(I'_i)$. The space $W(I')$ which was defined on $I'$ only, is also interpreted below as the subspace of $W(\Omega')$ of functions which are discrete harmonic in the sense of $\mathcal{H}'_i$. 


in each $\Omega_i$.

We now consider the subspaces $\hat{W}(\Omega') \subset W(\Omega')$ and $\hat{W}(\Gamma') \subset W(\Gamma')$ as the space of functions which are continuous on $\Gamma$ in the sense of the following definition (for notation see (15)).

![Fig. 2 Illustration of the continuity on $\Gamma$.](image)

**Definition 1 (Spaces $\hat{W}(\Omega')$ and $\hat{W}(\Gamma')$)** We say that $u = \{u_i\}_{i=1}^{N} \in W(\Omega')$ is continuous on $\Gamma$ if for all $i$, $1 \leq i \leq N$, we have

$$
(u_i)_i(x) = (u_j)_i(x) \quad \text{for all } x \in \bar{E}_{ij} \text{ for all } j \in \mathcal{J}_{H}^{i,0} \quad (25)
$$

and

$$
(u_i)_j(x) = (u_j)_j(x) \quad \text{for all } x \in \bar{E}_{ji} \text{ for all } j \in \mathcal{J}_{H}^{i,0}. \quad (26)
$$

In Figure 2, we illustrate this continuity by assigning the same nodal value for nodes connected by a line. The subspace of $\hat{W}(\Omega')$ of continuous functions on $\Gamma$ is denoted by $\hat{W}(\Omega')$, and the subspace of $\hat{W}(\Omega')$ of functions which are discrete harmonic in the sense of $\mathcal{H}'_i$ in each $\Omega_i$ is denoted by $\hat{W}(\Gamma')$.

Note that there is a one-to-one correspondence between vectors in the spaces $X(\Omega)$ and $\hat{W}(\Omega')$. Indeed, let us introduce the restriction matrices $R_{\Omega'} : X(\Omega) \rightarrow W_i(\Omega_i')$ which assign uniquely the vector values of $u = \{u_i\}_{i=1}^{N} \in W(\Omega')$. 

\{u_i\}_{i=1}^N \in X(Ω) where \(u_i \in X_i(Ω)\), see \([2]\), to \(v_i \in W_i(Ω')\) defined by
\((v_i)_i = u_i\) on \(Ω_i\) and \((v_i)_j = u_j\) on \(E_{ij}\) for all \(j \in J^0_{tj}\). It is easy to see that
\(v = \{v_i\}_{i=1}^N\) where \(v_i := R_{tj}u\) belongs to \(W(Ω)\). And vice-versa, for each \(v = \{v_i\}_{i=1}^N \in \hat{W}(Ω)\), we can define uniquely \(u = \{u_i\}_{i=1}^N \in X(Ω)\) by setting \(u_i = (v_i)_i\). Since \(u\) and \(v\) have identical nodal values, we refer sometimes \(u \in \hat{W}(Ω')\) or \(u \in X(Ω)\). For instance, the solution \(u^*\) of \([3]\) can be interpreted as a function in \(\hat{W}(Ω')\) or in \(X(Ω)\).

Note that the discrete problem \([3]\) can be written as a system of algebraic equations
\[ \hat{A}u^* = f \]  
for \(u^* \in X(Ω)\) using the standard FE basis functions, and \(f = \{f_i\}_{i=1}^N \in X(Ω)\), where \(f_i\) is the load vector associated with individual subdomains \(Ω_i\), i.e., is \(\int_Ω f v_i\) when \(v_i\) are the canonical basis functions of \(X_i(Ω)\). The stiffness matrix \(\hat{A}\) can be obtained by assembling the matrices \(A'_i\) from \([17]\), \(W(Ω')\) to \(X(Ω)\) as
\[ \hat{A} = \sum_{i=1}^N R_{tj}^T A'_i R_{tj}. \]

Note that the matrix \(\hat{A}\) is not block diagonal since there are couplings between substructures due to the continuity on \(Γ\), see Definition \([1]\).

Note also that \(X(Ω)\) can be componentwise represented by \(X(I) \times X(Γ)\), denoted also by \(X(Ω)\), where \(X(I) := \prod_{i=1}^N X_i(I_i)\) is the vector space of functions defined by nodal values on \(I_i\), and \(X(Γ) := \prod_{i=1}^N X_i(Γ_i)\) by the nodal values on \(Γ_i\), see \([15]\) and Figure \([1]\). Hence, we can represent \(u \in X(Ω)\) as \(u = \{u_I, u_Γ\}\) with \(u_I \in X(I)\) and \(u_Γ \in X(Γ)\). We introduce the restriction \(R_{Γ} : X(Γ) \rightarrow W_i(Γ'_i)\) by assigning values \(u_Γ \in X(Γ)\) into \(u_i \in W_i(Γ'_i)\) at the nodes of \(Γ'_i\). By eliminating the variable \(u_I' = \{u_I'\}_{i=1}^N\) of \(u^* = \{u_I', u_Γ\}\) from \([27]\), see \([18]\) and \([19]\), it is easy to see that
\[ \hat{S}u'_I = g_Γ \]  
where
\[ \hat{S} = \sum_{i=1}^N R_{tj}^T S'_i R_{tj} \quad \text{and} \quad g_Γ = f_Γ - \sum_{i=1}^N R_{tj}^T A'_i R_{tj}^{-1} f_i. \]  
with \(f_Γ := \{f_i,Γ\}_{i=1}^N\) and \(f_i,Γ \in X_i(Γ_i)\). Here, the load vector \(f_i,Γ\) is defined by \(\int_{Ω_i} f v_i,Γ\) when \(v_i,Γ\) are the canonical basis functions of \(X_i(Γ_i)\) associated to nodes on \(Γ_i\). It is also easy to see from \([21]\) and \([22]\) that
\[ \begin{pmatrix} v_i,Γ \cr v_i,Γ' \end{pmatrix}^T \begin{pmatrix} A'_i,Γ' \cr A'_i,Γ' \end{pmatrix} \begin{pmatrix} \mathcal{H}'_i u_i,Γ' \cr u_i,Γ' \end{pmatrix} = \langle S'_i u_i,Γ', v_i,Γ' \rangle. \]  
Note that \(W(Γ')\) is the natural space for defining \(\langle \cdot, \cdot \rangle\) due to \([29]\), \([30]\) and the continuity of \(\hat{W}(Γ')\) on \(Γ\), see Definition \([1]\).
4 FETI-DP with corner constraints

We now design a FETI-DP method for solving (28). We follow the abstract approach described in pages 160-167 in [20].

For $1 \leq i \leq N$, we introduce the nodal points associated to the corner unknowns, see Figure 1, by

\[ V'_i := V_i \bigcup \{ \bigcup_{j \in J_{i0}^H} \partial E_{ij} \} \quad \text{where} \quad V_i := \{ \bigcup_{j \in J_{i0}^H} \partial E_{ij} \}. \quad (31) \]

We now consider the subspace $\tilde{W}(\Omega') \subset W(\Omega')$ and $\tilde{W}(\Gamma') \subset W(\Gamma')$ as the space of functions which are continuous on all the $V_i$ in the sense of the following definition:

**Definition 2 (Subspaces $\tilde{W}(\Omega')$ and $\tilde{W}(\Gamma')$)** We say that $u = \{ u_i \}_{i=1}^N \in W(\Omega')$ is continuous at the corners $V'_i$ if for $1 \leq i \leq N$ we have

\[(u_i)_i(x) = (u_j)_j(x) \quad \text{at} \quad x \in \partial E_{ij} \quad \text{for all} \quad j \in J_{i0}^H \quad (32)\]

and

\[(u_i)_j(x) = (u_j)_j(x) \quad \text{at} \quad x \in \partial E_{ji} \quad \text{for all} \quad j \in J_{i0}^H. \quad (33)\]

In Figure 3 we illustrate this continuity by assigning the same nodal value at nodes (corners) connected by a line. The subspace of $W(\Omega')$ of continuous
functions at the corners $V_i'$ for all $1 \leq i \leq N$ is denoted by $\tilde{W}(\Omega')$, and the subspace of $\tilde{W}(\Omega')$ of functions which are discrete harmonic in the sense of $\mathcal{H}'_i$ is denoted by $W(\Gamma')$.

Note that
\[ \tilde{W}(\Gamma') \subset W(\Gamma') \subset W(\Gamma'). \]  
(34)

Let $\tilde{A}$ be the stiffness matrix which is obtained by assembling the matrices $A_i'$ for $1 \leq i \leq N$, from $W(\Omega')$ to $\tilde{W}(\Omega')$. Note that the matrix $\tilde{A}$ is no longer block diagonal since there are couplings between variables at the corners $V_i'$ for $1 \leq i \leq N$. We will represent $u \in \tilde{W}(\Omega')$ as $u = (u_I, u_H, u_\Delta)$ where the subscript $I$ refers to the interior degrees of freedom at nodal points $I = \prod_{i=1}^{N} I_i$, the $H$ refers to the corners $V_i$, for all $1 \leq i \leq N$, and the $\Delta$ refers to the remaining nodal points, i.e., the nodal points $V_i \setminus V_i'$, for all $1 \leq i \leq N$. The vector $u = (u_I, u_H, u_\Delta) \in \tilde{W}(\Omega')$ is obtained from the vector $u = \{u_i\}_{i=1}^{N} \in W(\Omega')$ using the equations (32), (33), i.e., the continuity of $u$ on $V_i'$ for all $1 \leq i \leq N$. Using the decomposition $u = (u_I, u_H, u_\Delta) \in \tilde{W}(\Omega')$ we can partition $\tilde{A}$ as
\[ \tilde{A} = \begin{pmatrix}
A_{II} & A_{IH} & A_{I\Delta} \\
A_{HI} & A_{HH} & A_{H\Delta} \\
A_{I\Delta} & A_{H\Delta} & A_{\Delta\Delta}
\end{pmatrix}. \]  
(35)

We note that the only couplings across subdomains are through the variables $H$ where the matrix $\tilde{A}$ is subassembled.

A Schur complement of $\tilde{A}$ with respect to the $\Delta$-unknowns (eliminating the $I$- and the $H$-unknowns) is of the form
\[ \tilde{S} := A_{\Delta\Delta}' - (A_{\Delta I}' A_{\Delta H})^{-1} (A_{I\Delta}' A_{H\Delta}). \]  
(36)

A vector $u \in \tilde{W}(\Gamma')$ can uniquely be represented by $u = (u_H, u_\Delta)$, therefore, we can represent $\tilde{W}(\Gamma') = \tilde{W}_H(\Gamma') \times W_\Delta(\Gamma')$, where $\tilde{W}_H(\Gamma')$ refers to the $H$-degrees of freedom of $\tilde{W}(\Gamma')$ while $W_\Delta(\Gamma')$ to the $\Delta$-degrees of freedom of $\tilde{W}(\Gamma')$. The vector space $W_\Delta(\Gamma')$ can be decomposed as
\[ W_\Delta(\Gamma') = \prod_{i=1}^{N} W_{i,\Delta}(\Gamma'_i) \]  
(37)

where the local space $W_{i,\Delta}(\Gamma'_i)$ refers to the degrees of freedom associated to the nodes of $\Gamma'_i \setminus V'_i$ for $i = 1 \leq i \leq N$. Hence, a vector $u \in \tilde{W}(\Gamma')$ can be represented as $u = (u_H, u_\Delta)$ with $u_H \in \tilde{W}_H(\Gamma')$ and $u_\Delta = \{u_{i,\Delta}\}_{i=1}^{N} \in W_\Delta(\Gamma')$ where $u_{i,\Delta} \in W_{i,\Delta}(\Gamma'_i)$. Note that $\tilde{S}$, see (36), is defined on the vector space $W_\Delta(\Gamma')$, and the following lemma follows (cf. Lemma 6.22 in [20] and Lemma 4.2 in [17]):
Lemma 2 Let \( u_\triangle \in W_\triangle(I') \) and let \( \bar{S} \) and \( \bar{A} \), defined in [36] and [33], respectively. Then,

\[
(\bar{S}u_\triangle, u_\triangle) = \min(\bar{A}w, w)
\]

where the minimum is taken over \( w = (w_I, w_H, w_\triangle) \in \bar{W}(\Omega) \) such that \( w_\triangle = u_\triangle \).

Let us take \( u \in \bar{W}(I') \) as \( u = (u_H, u_\triangle) \) with \( u_H \in \bar{W}_H(I') \) and \( u_\triangle \in W_\triangle(I') \), where \( u_\triangle = \{u_{i,\triangle}\}_{i=1}^N \) with \( u_{i,\triangle} \in W_{i,\triangle}(I'_i) \). The vector \( u_{i,\triangle} \in W_{i,\triangle}(I'_i) \) can be partitioned as

\[
u_i, \triangle = \{(u_{i,\triangle})_i, \{u_{i,\triangle})_j\}_{j \in J_H^0}\}
\]

where

\[
(u_{i,\triangle})_i = u_{i,\triangle}|_{(\gamma_i)} \quad \text{and} \quad (u_{i,\triangle})_j = u_{i,\triangle}|_{(e_{ij})}.
\]

In order to measure the jump of \( u_\triangle \in W_\triangle(I') \) across the \( \triangle \)-nodes let us introduce the space \( \bar{W}_\triangle(I') \) defined by

\[
\bar{W}_\triangle(I') = \prod_{i=1}^N X_i(I'_i \setminus V_i),
\]

where \( X_i(I'_i \setminus V_i) \) is the restriction of \( X_i(\Omega) \) to \( I'_i \setminus V_i \), see Figure 1. To define the jumping matrix \( B_\triangle : W_\triangle(I') \to \bar{W}_\triangle(I') \), let \( u_\triangle = \{u_{i,\triangle}\}_{i=1}^N \in W_\triangle(I') \) and let \( v \equiv B_\triangle u \) where \( v = \{v_i\}_{i=1}^N \in \bar{W}_\triangle(I') \) is defined by

\[
v_i = (u_{i,\triangle})_i - (u_{j,\triangle})_i \quad \text{on} \quad E_{ijh} \quad \text{for all} \quad j \in J_H^{i,0}.
\]

The jumping matrix \( B_\triangle \) can be written as

\[
B_\triangle = (B_\triangle^{(1)}, B_\triangle^{(2)}, \ldots, B_\triangle^{(N)}),
\]

where the rectangular matrix \( B_\triangle^{(i)} \) consists of columns of \( B_\triangle \) attributed to the \( (i) \) components of functions of \( W_{i,\triangle}(I'_i) \) of the product space \( W_\triangle(I') \), see [37]. The entries of the rectangular matrix consist of values of \{0, 1, -1\}. It is easy to see that the Range \( B_\triangle = \bar{W}_\triangle(I') \), so \( B_\triangle \) is full rank. In addition, if \( u = (u_H, u_\triangle) \in \bar{W}(I') \) and \( B_\triangle u_\triangle = 0 \) then \( u \in \bar{W}(I') \).

We can reformulate the problem [28] as the variational problem with constraints in \( W_\triangle(I') \) space: Find \( u_\triangle^* \in W_\triangle(I') \) such that

\[
J(u_\triangle^*) = \min J(v_\triangle)
\]

subject to \( v_\triangle \in W_\triangle(I') \) with constraints \( B_\triangle v_\triangle = 0 \), where

\[
J(v_\triangle) := 1/2(\bar{S}v_\triangle, v_\triangle) - (\bar{g}_\triangle, v_\triangle)
\]
where $\tilde{S}$ is defined in (36) and
\[
\tilde{g}_\Delta := f_\Delta - (A'_{\Delta I} \ A'_{\Delta \Pi}) \left( \begin{array}{cc} A'_{II} & A'_{I\Pi} \\ A'_{\Pi I} & A'_{\Pi\Pi} \end{array} \right)^{-1} \left( \begin{array}{c} f_I \\ f_\Pi \end{array} \right). \tag{43}
\]

We note that $f = \{f_I\}_{I=1}^N \in \mathbb{X}(\Omega)$ was defined in (27) and it can be represented as $f = \{f_I, f_\Pi, f_{\Gamma I}\in\mathbb{X}(\Gamma')$. It remains to define $f_\Delta$ in (43). The forcing term $f_\Delta \in \mathbb{W}_\Delta(\Gamma')$ is defined by $f_\Delta = \{f_i, f_\Pi\}_{i=1}^N$ where $f_i, f_\Pi$ are the load vectors associated with the individual subdomains $\Omega_i$, i.e., the entries $f_i, f_\Pi$ are defined as $\int_{\Omega_i} f_i(v_i, \Delta) dx$ when $v_i, \Delta$ are the canonical basis functions of $W_i(\Gamma_i)$. Note that $\tilde{S}$ is symmetric and positive definite since $\tilde{A}$ has these properties; see also Lemma 2. Introducing Lagrange multipliers $\lambda \in \mathbb{W}_\Delta(\Gamma')$, the problem (41) reduces to the saddle point problem of the form:
\[
\begin{aligned}
\tilde{S} u_{\Delta}^* + B_T^\Delta \lambda^* &= \tilde{g}_\Delta \\
B_\Delta u_{\Delta}^* &= 0.
\end{aligned} \tag{44}
\]

Hence, (44) reduces to
\[
F \lambda^* = g \tag{45}
\]
where
\[
F := B_\Delta \tilde{S}^{-1} B_T^\Delta, \quad g := B_\Delta \tilde{S}^{-1} \tilde{g}_\Delta. \tag{46}
\]

When $\lambda^*$ is computed, $u_{\Delta}^*$ can be found by solving the problem
\[
\tilde{S} u_{\Delta}^* = \tilde{g}_\Delta - B_T^\Delta \lambda^*. \tag{47}
\]

### 4.1 Dirichlet Preconditioner

We now define the FETI-DP preconditioner for $F$, see (46). Let $S'_i, \Delta$ be the Schur complement of $S'_i$, see (19), restricted to $W_i(\Gamma_i') \subset W_i(\Gamma_i)$, i.e., taken $S'_i$ on functions in $W_i(\Gamma_i')$ which vanish on $\mathcal{V}_i$. Let
\[
S'_\Delta := \text{diag}\{S'_i, \Delta\}_{i=1}^N. \tag{48}
\]

In other words, $S'_i, \Delta$ is obtained from $S'_i$ by deleting rows and columns corresponding to nodal values at nodal points of $\mathcal{V}_i \subset \Gamma_i'$.

Let us introduce diagonal scaling matrices $D^{(i)}_\Delta : W_{i,\Delta}(\Gamma_i') \rightarrow W_{i,\Delta}(\Gamma_i')$, for $1 \leq i \leq N$. The diagonal entry of $D^{(i)}_\Delta$ associated to a node $x \in \Gamma_i' \setminus \mathcal{V}_i'$, which we denote by $D^{(i)}_\Delta(x)$, is defined by
\[
D^{(i)}_\Delta(x) = \frac{\rho_i^\beta}{\rho_i^\beta + \rho_j^\beta} \text{ for } x \in \{E_{ijh} \cup E_{jih}\} \text{ for } j \in J_H^{i,0}, \tag{49}
\]
for $\beta \in [1/2, \infty)$, see [19], and define

$$B_{D, \triangle} = (B_{\triangle}^{(1)} D_{\triangle}^{(1)}, \ldots, B_{\triangle}^{(N)} D_{\triangle}^{(N)}).$$

(50)

Let

$$P_\triangle := B_{D, \triangle}^T B_{D, \triangle}$$

(51)

which maps $W_\triangle(\Gamma')$ into itself. It is easy to check that for $w_\triangle = \{w_{i, \triangle}\}_{i=1}^N \in W_\triangle(\Gamma')$ and $v_\triangle := P_\triangle w_\triangle$, the following equalities hold:

$$(v_{i, \triangle})_i(x) = \frac{\rho^\beta_i}{\rho^\beta_i + \rho^\beta_j} [(w_{i, \triangle})_i(x) - (w_{j, \triangle})_i(x)], \quad x \in E_{ijh},$$

(52)

$$(v_{i, \triangle})_j(x) = \frac{\rho^\beta_j}{\rho^\beta_i + \rho^\beta_j} [(w_{i, \triangle})_j(x) - (w_{j, \triangle})_j(x)], \quad x \in E_{ijh}.$$  

(53)

Note that

$$(v_{j, \triangle})_j(x) = \frac{\rho^\beta_i}{\rho^\beta_i + \rho^\beta_j} [(w_{j, \triangle})_j(x) - (w_{i, \triangle})_j(x)], \quad x \in E_{ijh},$$

(54)

$$(v_{j, \triangle})_i(x) = \frac{\rho^\beta_j}{\rho^\beta_i + \rho^\beta_j} [(w_{j, \triangle})_i(x) - (w_{i, \triangle})_i(x)], \quad x \in E_{ijh}.$$  

(55)

By subtracting (55) from (52) and (53) from (54) we see that $P_\triangle$ preserves jumps in the sense that

$$B_{D} P_\triangle = B_{D}.$$  

(56)

From this follows that $P_\triangle$ is a projection ($P_\triangle^2 = P_\triangle$).

In the FETI-DP method, the preconditioner $M^{-1}$ is defined as follows:

$$M^{-1} = B_{D, \triangle} S_{\triangle}^T B_{D, \triangle} = \sum_{i=1}^N B_{\triangle}^{(i)} D_{\triangle}^{(i)} S_{\triangle}^{(i)} D_{\triangle}^{(i)} (B_{\triangle}^{(i)})^T.$$  

(57)

Note that $M^{-1}$ is a block-diagonal matrix, and each block is invertible since $S_{\triangle}^{(i)}$ and $D_{\triangle}^{(i)}$ are invertible and $B_{\triangle}^{(i)}$ is a full rank matrix. The following theorem holds.

**Theorem 1** For any $\lambda \in \hat{W}_{\triangle}(\Gamma)$ it holds that

$$\langle M \lambda, \lambda \rangle \leq \langle F \lambda, \lambda \rangle \leq C (1 + \log \frac{H}{h})^2 \langle M \lambda, \lambda \rangle$$  

(58)

where $C$ is a positive constant independent of $h_i$, $h_i/h_j$, $H_i$, $\lambda$ and the jumps of $\rho_i$. Here and below, $\log \frac{H}{h} := \max_{i=1}^N \log \frac{H_i}{h_i}$. 
Proof We follow to the general abstract theory for FETI-DP methods developed in Theorem 6.35 of [20]. This abstract theory relies only on duality and linear algebra arguments, and properties such as that $B_\triangle$ is full rank, $P_\triangle$ is a projection, $\tilde{S}$ is invertible and the subspace inclusion $\tilde{\triangle}$. Using the same abstract arguments, the proof of the theorem follows by checking the Lemma 3 and Lemma 4 see below. The proof of Lemma 4 is not algebraic and it depends on the problem. The proofs of these two lemmas are presented separately below.

Lemma 3 For $u_\triangle \in W_\triangle(I')$ it follows that
\begin{equation}
\langle \tilde{S}u_\triangle, u_\triangle \rangle \leq \langle S_\triangle' u_\triangle, u_\triangle \rangle. \tag{59}
\end{equation}

Proof The proof follows from Lemma 2 and from
\begin{equation}
\langle \tilde{S}u_\triangle, u_\triangle \rangle = \min \langle \tilde{A}w, w \rangle \leq \min \langle \tilde{A}v, v \rangle = \langle S_\triangle' u_\triangle, u_\triangle \rangle \tag{60}
\end{equation}
where the minima are taken over $w = (w_I, w_{\Pi}, w_\triangle) \in \tilde{W}(\Omega')$ such that $w_\triangle = u_\triangle$, and $v = (v_I, v_{\Pi}, v_\triangle) \in \tilde{W}(\Omega')$ such that $v_{\Pi} = 0$ and $v_\triangle = u_\triangle$.

Lemma 4 For any $u_\triangle \in W_\triangle(I')$ it holds that
\begin{equation}
\| P_\triangle u_\triangle \|_S^2 \leq C(1 + \log \frac{H}{h})^2 \| u_\triangle \|_{S}^2 \tag{61}
\end{equation}
where $C$ is a positive constant independent of $h$ and $H$, $u_\triangle$ and the jumps of $\rho_\prime$.

The proof of this lemma is presented in the second part of next section.

5 Refinement and Interpolations

In this section we introduce some technical tools to analyze the FETI-DP preconditioner.

Definition 3 (The triangulation $T_h^e(\Omega_i)$) Let us introduce the triangulation $T_h^e(\Omega_i)$, which is a refinement of $T_h^e(\Omega_i)$ as follows. We refine an element $\tau \in T_h^e(\Omega_i)$ by considering four cases:

- No edge of $\tau$ belongs to $\partial \Omega_i$: Let $V_1$, $V_2$ and $V_3$ be the vertices of $\tau$ and let us denote $(\lambda_1(x), \lambda_2(x), \lambda_3(x))$ as the barycenter coordinates of a point $x \in \tau$ such that $V_1 = (1,0,0)$, $V_2 = (0,1,0)$ and $V_3 = (0,0,1)$. Let $M_1 = (0,1/2,1/2)$, $M_2 = (1/2,0,1/2)$, $M_3 = (1/2,1/2,0)$ and $C_1 = (2/3,1/6,1/6)$, $C_2 = (1/6,2/3,1/6)$, $C_3 = (1/6,1/6,2/3)$. The refinement of $\tau$ is defined by the triangles $C_1C_2C_3$, $V_1M_3C_1$, $M_3C_2C_1$, $M_3V_2C_2$, $V_2M_1C_2$, $M_1V_3C_2$, $M_1V_3C_3$, $V_3M_2C_3$, $M_2C_1C_3$ and $M_2V_1C_1$. See Figure 4 upper-left picture.
- Only one edge of \( \tau \) belongs to \( \partial \Omega_l \): Let \( V_1, V_2 \) and \( V_3 \) be the vertices of \( \tau \) and let us assume that the edge opposed to \( V_1 \) belongs to \( \partial \Omega_l \). Put \( V_1 = (1, 0, 0), V_2 = (0, 1, 0), V_3 = (0, 0, 1) \) and let \( M_2 = (1/2, 0, 1/2) \), \( M_3 = (1/2, 1/2, 0) \), \( C_1 = (1/3, 1/3, 1/3) \), \( C_2 = (0, 2/3, 1/3) \) and \( C_3 = (0, 1/3, 2/3) \). The refinement of \( \tau \) is defined by the triangles \( V_1M_2C_1, M_3V_2C_1, V_2C_2C_1, C_3V_3C_1, V_3M_2C_1 \) and \( M_2V_1C_1 \). See Figure \( \text{[FETI-DP for full DG]} \) upper-right picture.
- Exactly two edges of \( \tau \) belong to \( \partial \Omega_l \): Let \( C_1, V_2 \) and \( V_3 \) be the vertices of \( \tau \) and let us assume the edges opposed to \( V_2 \) and \( V_3 \) belong to \( \partial \Omega_l \). We have \( C_1 = (1, 0, 0), V_2 = (0, 1, 0), V_3 = (0, 0, 1) \), and define \( M_1 = (0, 1/2, 1/2) \) and \( C_2 = (1/2, 0, 1/2) \). The refinement of \( \tau \) is defined by the triangles \( C_1C_2C_2, C_3V_2M_1, M_1C_3C_2 \) and \( M_1V_3C_3 \). See Figure \( \text{[FETI-DP for full DG]} \) lower-left picture.
- All three edges of \( \tau \) belong to \( \partial \Omega_l \): No refinement is needed, let \( C_1, C_2 \) and \( C_3 \) be the vertices of \( \tau \).

We distinguish the above nodes by saying that they are nodes of type \( C \), type \( M \) and type \( V \), respectively. It is easy to see that \( \mathcal{T}_h \), the refinement of \( \mathcal{T}_h \), is a geometrically conforming, shape regular and quasi-uniform triangulation of \( \Omega \). Therefore, we denote by \( \overline{W}_i(\Omega_l) \) as the space of piecewise linear and continuous functions on \( \mathcal{T}_h \).

**Definition 4 (The space \( \overline{W}_i(\Omega_l) \))** Let \( W_i(E_{ji}) \) be the space \( \overline{W}_j(\Omega_j) \) restricted to \( E_{ji} \), for \( j \in J_{ji}^0 \). We define \( \overline{W}_i(\Omega_l) \) by

\[
\overline{W}_i(\Omega_l) = W_i(\Omega_i) \times \prod_{j \in J_{ji}^0} W_i(E_{ji}).
\]

(62)

A function \( u_i \in \overline{W}_i(\Omega_l) \) is represented by

\[
u_i = \{(u_i)_i, \{(u_i)_j\}_{j \in J_{ji}^0}\},\]

where \((u_i)_i := u_i|_{\overline{\Omega}_i} (u_i \text{ restricted to } \overline{\Omega}_i)\) and \((u_i)_j := u_i|_{E_{ji}} (u_i \text{ restricted to } E_{ji})\). Note that \((u_i)_i \text{ and } (u_i)_j \text{ are continuous on } \overline{\Omega}_i \text{ and } E_{ji} \text{, respectively.}\)

**Definition 5 (The interpolators \( \mathcal{I}_h \))** Given \( u_i = \{(u_i)_i, \{(u_i)_j\}_{j \in J_{ji}^0}\} \in \overline{W}_i(\Omega_l) \), we construct \n
\[
\{(u_i)_i, \{(u_i)_j\}_{j \in J_{ji}^0}\} = \mathcal{I}_h u_i \in \overline{W}_i(\Omega_l)
\]

as follows:

- Define \( (u_i)_i \in \overline{W}_i(\Omega_l) \) by assigning values at the nodes of type \( C, M, V \in \mathcal{T}_h(\overline{\Omega}_l) \):
    - Nodes of type \( C \): define \( (u_i)_i(C) = (u_i)_i(C) \).
    - Nodes of type \( M \) (with \( M \) shared by \( \tau_- \), \( \tau_+ \in \mathcal{T}_h(\Omega_l) \)): define \( (u_i)_i(M) = 1/2(u_{\tau_-}(M) + u_{\tau_+}(M)) \).
    - Nodes of type \( V \notin \partial \Omega_l \) (with \( V \) shared by the elements \( \tau_1, \cdots, \tau_p \in \mathcal{T}_h(\Omega_l) \)): define \( (u_i)_i(V) = 1/p (u_{\tau_1}(V) + \cdots + u_{\tau_p}(V)) \).
Fig. 4 Illustration of refinement of an element according to its relative position with respect to $\partial \Omega_i$.

- Nodes of type $V \in \partial \Omega_i$ (with $V$ shared by two element edges $e_r, e_\ell \subset \partial \Omega_i$): define $(u_i)_i(V) = 1/2 \left( u_{\tau_r}(V) + u_{\tau_\ell}(V) \right)$, where the elements $\tau_r$ and $\tau_\ell$ share the edges $e_r$ and $e_\ell$, respectively.
- For $j \in J_H^{j,0}$ define $(u_i)_j \in W_h(E_{ji})$ by assigning nodal values at the nodes of type $C, V$ on $E_{ji}$:
  - Nodes of type $C$ define $(u_i)_j(C) = (u_i)_j(C)$
  - Nodes of type $V$ define $(u_i)_j(V)$ according to
    - If $V \in \partial E_{ji}$, define $(u_i)_j(V) = \lim_{x \in E_{ji} \to V} (u_i)_j(x)$.
    - If $V \notin \partial E_{ji}$, define $(u_i)_j(V) = 1/2 \left( u_{e_r}(V) + u_{e_\ell}(V) \right)$, where $e_r$ and $e_\ell$ are the two edges of $E_{ji}$ sharing $V$.

Note that from the definition of $u_i = I_h u_i$, the value of $(u_i)_i$ on $\partial \Omega_i$ depends only on the value of $(u_i)_i$ on $\partial \Omega_i$. In addition, the values of $(u_i)_j$ depends only on the value of $(u_i)_j$ on $E_{ji}$ for any $j \in J_H^{j,0}$. We note however that the value of $(u_i)_j$ on an edge $E_{ij}$ does not necessary depend only on the values of $(u_i)_i$ on $E_{ij}$ due to the way that the nodal values at the subdomain corners are assigned.
For a fixed edge $j \in J_H^*$, we next modify $\mathcal{I}_h^j$ (denoted by $\mathcal{I}_h^{ij}$) so that $u_i := \mathcal{I}_h^{ij} u_i$ on $E_{ij}$ depends only on the value of $(u_i)_i$ on $E_{ij}$. We note however that $(\bar{u}_i)_i$ on $E_{ik}$, for $k \in J_H^*$ and $k \neq j$, does not depend on the values of $(u_i)_i$ on $E_{ik}$. The other properties described above for $\mathcal{I}_h^j$ also holds for $\mathcal{I}_h^{ij}$. We note that in the case of Figure 4, lower-left picture, the operator $\mathcal{I}_h^j$ would be enough in order to perform the analysis of the FETI-DP method. The operator $\mathcal{I}_h^{ij}$ is needed only in cases where two or more elements touch simultaneously a corner of $\Gamma_1$.

**Definition 6 (The interpolator $\mathcal{I}_h^{ij}$)** We introduce the interpolator $\mathcal{I}_h^{ij} : W_i(\Omega_i') \rightarrow \overline{W'_i}(\Omega_i')$, by modifying $\mathcal{I}_h^j$ only for the definition of $(u_i)_i$ at the two endpoints of $\partial E_{ij}$. For $V$ an endpoint of $\partial E_{ij}$ define $(\bar{u}_i)_i(V) = \lim_{x \in E_{ij}} - V(u_i)_i(x)$.

Let us define the following bilinear forms for the space $\overline{W_i}(\Omega_i')$

\[ a_i((u_i)_i, (u_i)_i) = a_i((u_i)_i, (u_i)_i) + p_i, (u_i)_i \]  

and the semi-norm

\[ b_i((u_i)_i, (u_i)_i) = a_i((u_i)_i, (u_i)_i) + p_i, (u_i)_i \]

where

\[ a_i((u_i)_i, (u_i)_i) := \int_{\Omega_i} \rho_i \nabla (u_i)_i \cdot \nabla (u_i)_i \, dx. \]

The following lemma holds. From here on, in order to avoid the proliferation of constants, we use the notation $A \leq B$ to express the fact that there is a constant $C$ independent of $h$ such that $A \leq CA$. Similarly for the symbol $A \asymp B$ which means that $A \leq B$ and also $B \leq A$.

**Lemma 5** For $u_i \in W_i(\Omega_i')$ and let $u_i = \mathcal{I}_h^j u_i$ or $u_i = \mathcal{I}_h^{ij} u_i$. Then

\[ a_i((u_i)_i, (u_i)_i) \asymp a_i((u_i)_i, (u_i)_i), \]  

and

\[ p_{\partial,i}(u_i)_i \leq p_{\partial,i}(u_i)_i + a_i((u_i)_i, (u_i)_i). \]

Additionally, if $u_i = \mathcal{I}_h^{ij} u_i$, then

\[ \| (u_i)_i - (u_i)_i \|_{L^2(E_{ij})} \asymp \| (u_i)_i - (u_i)_i \|_{L^2(E_{ij})}. \]

The proof of this and next lemma are presented in the Appendix.

**Definition 7** The interpolator $I_h^i$ We now introduce the interpolator $I_h^i : \overline{W_i}(\Omega_i') \rightarrow W_i(\Omega_i')$ as follows. Given $u_i = \{(u_i)_i, \{u_i\}_j \in J_H^{ij} \} \in \overline{W_i}(\Omega_i')$ we construct

\[ \{(u_i)_i, \{u_i\}_j \in J_H^{ij} \} := I_h^i u_i \in W_i(\Omega_i') \]  

as follows:
For $\tau \in \mathcal{T}_h^0$, let $C_1, C_2, C_3$ be the nodes of type $C$ in $\mathcal{T}_h^0$ on $\bar{\tau}$. We define $(u_i)_i$ on $\tau$ as the linear extrapolation on $\tau$ of the linear function defined by the nodal values $(u_{i,j})_j(C_1), (u_{i,j})_j(C_2)$ and $(u_{i,j})_j(C_3)$.

For $j \in \mathcal{E}_h^{i,0}$ and $e \in \mathcal{E}_h^{i,1}$, let $C_1$ and $C_2$ be the nodes of type $C$ on element edge $e$. We define $(u_{i,j})_j$ on element edge $e$ as the linear extrapolation on $e$ of the linear function defined by the nodal values $(u_{i,j})_j(C_1)$ and $(u_{i,j})_j(C_2)$.

The following lemma holds.

**Lemma 6** Let $u_i \in W_i(\Omega'_i)$. Then

\[
a_i(I_h^1 u_i, I_h^1 u_i) \leq a_i(u_i, u_i)
\]

and

\[
p_i,\sigma(I_h^1 u_i, I_h^1 u_i) \leq p_i,\sigma(u_i, u_i).
\]

In case $u_i = I_h^1 u_i$ or $u_i = I_h^{i,3} u_i$, we have then

\[
I_h^{i,3} u_i = u_i.
\]

Let us also introduce $H_h^{i,1}, r \in W_i(\Omega'_i)$ as the standard discrete harmonic function of $u_i, r \in W_i(\Omega'_i)$, i.e., $H_h^{i,1} u_i r = u_i r$ on $\Omega'_i$ and discrete harmonic in $\Omega_i$ in the sense of $a_i(\cdot, \cdot)$, see (63), with Dirichlet data on $\Gamma_i$. We note that the extensions $H_h^{i,1}$ and $H_h^{i,1}$ differ from each other not only because they are defined in different space $W_h(\Omega'_i)$ and $W_h(\Omega'_i)$, respectively, but also because $H_h^{i,1} u_i r$ at the interior nodes of $\mathcal{T}_h^0$ depends only on the nodal values of $u_i r$ on $\Gamma_i$, while $H_h^{i,1} u_i r$ depends on the nodal values of $u_i r$ on $\Gamma_i$.

The following lemma shows the equivalence (in the energy form defined by $d_i(\cdot, \cdot)$) between discrete harmonic functions in the sense of $H_h^{i,1}$ and in the sense of $\mathcal{H}^{i,1};$ for the proof see [2]. This equivalence allows us to take advantages of all the discrete Sobolev results known for $H_h^{i,1}$ discrete harmonic extensions.

**Lemma 7** Let $u_i \in W_i(\Omega'_i)$ and $u_i \in W_i(\Omega'_i)$ defined by $u_i = I_h^1 u_i$ or by $u_i = I_h^{i,3} u_i$. Then

\[
d_i(H_h^{i,1} u_i, H_h^{i,1} u_i) \equiv d_i(H_h^{i,1} u_i, H_h^{i,1} u_i)
\]

where $C$ is a positive constant independent of $\delta, h_i, H_i, \rho_i$ and $u_i$.

**Proof** First note by construction that $I_h^1 H_h^{i,1} u_i = I_h^1 I_h^{i,3} u_i = u_i$ on $I'_i$. Using Lemma 6 a minimum $H_h^{i,1}$-energy argument and Lemma 2 again, we obtain

\[
d_i(H_h^{i,1} u_i, H_h^{i,1} u_i) \leq a_i(I_h^1 H_h^{i,1} u_i, I_h^1 H_h^{i,1} u_i) \leq d_i(I_h^1 H_h^{i,1} u_i, I_h^1 H_h^{i,1} u_i) \equiv d_i(I_h^{i,3} H_h^{i,1} u_i, I_h^{i,3} H_h^{i,1} u_i).
\]

By Lemma 6 we have

\[
d_i(I_h^{i,3} H_h^{i,1} u_i, I_h^{i,3} H_h^{i,1} u_i) \leq d_i(H_h^{i,1} u_i, H_h^{i,1} u_i).
\]
The proof of the left inequality of (71) is complete. The proof using the operator $I_{h,ij}^\alpha$ instead of $I_h^\alpha$ is similar.

Now let us prove the left inequality of (71). Note that $\mathcal{H}_h u_i = I_h^\alpha \mathcal{H}_i^\alpha u_i = u_i$ on $\mathcal{I}_h$. Using a minimal $\mathcal{H}_i^\alpha$-energy argument and Lemma 5 we obtain

$$d_i(\mathcal{H}_h u_i, \mathcal{H}_h u_i) = a_i(\mathcal{H}_h u_i, \mathcal{H}_h u_i) \leq a_i(I_h^\alpha \mathcal{H}_i^\alpha u_i, I_h^\alpha \mathcal{H}_i^\alpha u_i) \leq a_i(\mathcal{H}_i^\alpha u_i, \mathcal{H}_i^\alpha u_i),$$

and again Lemma 3 we have

$$p_{i,}\mathcal{H}_h(\mathcal{H}_h u_i, \mathcal{H}_h u_i) = p_{i,}\mathcal{H}_h(I_h^\alpha \mathcal{H}_i^\alpha u_i, I_h^\alpha \mathcal{H}_i^\alpha u_i) \leq d_i(\mathcal{H}_i^\alpha u_i, \mathcal{H}_i^\alpha u_i),$$

therefore, the left inequality of (71) follows. The proof with $I_{h,ij}^\alpha$ is similar.

We are now in position to prove Lemma 4.

**Proof of Lemma 4.**

Proof We first consider the case when the edges $E_{ij}$ are made by a single interval only. Let $u_\Delta \in W_\Delta(I')$ and let $u = (u_H, u_\Delta) \in \tilde{W}(I')$ be the solution of

$$\langle \tilde{S}u_\Delta, u_\Delta \rangle = \min \langle S'w, w \rangle = : \langle S'u, u \rangle, \quad (72)$$

where the minimum is taken over $w = (w_H, w_\Delta) \in \tilde{W}(I')$ such that $w_H \in W_H(I')$ and $w_\Delta = u_\Delta$. Here $S' = \text{diag} [S'_{ij}]_{i=1}^N$ where $S'_{ij}$ is defined in (19). The problem has a unique solution, see (60). Hence, we can replace $\|u_\Delta\|_{\tilde{S}}$ in (61) by $\|u\|_{\tilde{S}}$.

Let us represent the $u$ defined above as $\{u_i\}_{i=1}^N \in W(I')$ where $u_i \in W_i(I')$. Let $I_{E_{ij}}(u_i)_j$ be the linear function on $E_{ij}$ defined by the values of $(u_i)_j$ at $x \in \partial E_{ij}$, and let $I_{E_{ji}}(u_i)_j$ be the linear function on $E_{ji}$ defined by the values of $(u_i)_j$ at $x \in \partial E_{ji}$. Let $\tilde{u} := \{\tilde{u}_i\}_{i=1}^N$ where $\tilde{u}_i \in W_i(I'_i)$ is defined by

$$(\tilde{u}_i)_i = I_{E_{ij}}(u_i)_j \text{ on } E_{ijh} \text{ for all } j \in J_H^{i,0}$$

and

$$(\tilde{u}_i)_j = I_{E_{ji}}(u_i)_j \text{ on } E_{jih} \text{ for all } j \in J_H^{i,0}.$$

Note that $\tilde{u} \in \tilde{W}(I')$, therefore, let us represent $\tilde{u} = (\tilde{u}_H, \tilde{u}_\Delta)$ where $B_{\Delta} \tilde{u}_\Delta = 0$. Using this we have, see (51),

$$P_{\Delta} u_\Delta \equiv B_{\Delta}^T B_{\Delta} u_\Delta = B_{\Delta}^T B_{\Delta} (u_\Delta - \tilde{u}_\Delta) = P_{\Delta} (u_\Delta - \tilde{u}_\Delta).$$

Note that $u - \tilde{u} = 0$ at the $H$-nodes, hence, let us define $v \in W(I')$ to be equal to $P_{\Delta}(u_\Delta - \tilde{u}_\Delta)$ at the $\Delta$-nodes and equal to zero at the $H$-nodes. Let us represent $v = \{v_i\}_{i=1}^N$ where $v_i \in W_i(I'_i)$. We have

$$\| P_{\Delta} u_\Delta \|_{\tilde{S}_\Delta} = \| v \|_{\tilde{S}_\Delta}' = \sum_{i=1}^N \| v_i \|_{\tilde{S}_i}^2$$

(73)
in view of the definition of $S_{i,\omega}', S_{i,\delta}'$, and $S', see (48), (19) and (36), hence, to prove the lemma it remains to show that

$$\sum_{i=1}^{N} \| v_i \|_{S_i}^2 \leq C(1 + \log H/h)^2 \| u \|_{S_i}^2$$  

(74)

since by (72) we obtain (61). By Lemma 1 we need to show

$$\sum_{i=1}^{N} d_i(\mathcal{H} v_i, \mathcal{H} v_i) \leq C(1 + \log H/h)^2 \sum_{i=1}^{N} d_i(\mathcal{H} u_i, \mathcal{H} u_i).$$  

(75)

Define $v_i = \mathcal{I}_i^0 v_i$. Note that because $v_i$ vanishes at the $\Pi$-nodes, $v_i$ also vanishes at the $\Pi$-nodes. Using Lemma 7 we obtain

$$d_i(\mathcal{H} v_i, \mathcal{H} v_i) = d_i(\mathcal{H} v_i, \mathcal{H} v_i).$$

From now on, let us denote $\tilde{v}_i = \mathcal{H}_i \tilde{v}_i$, and $v_j = \mathcal{H}_j v_j$ and so,

$$d_i(\mathcal{H}_i, \tilde{v}_i) = \rho_i \| \nabla(\tilde{v}_i) \|_{L^2(\Omega_i)} + \sum_{j \in \mathcal{J}_H^0} \frac{\rho_i \delta}{h_j} \| (\tilde{v}_i)_i - (v_i)_j \|_{L^2(E_i)}.$$  

(76)

We first estimate the first term of (76). We have

$$\| \nabla(\tilde{v}_i) \|_{L^2(\Omega_i)} \leq C \sum_{j \in \mathcal{J}_H^0} \| (v_i)_j \|_{H^{1/2}_{00}(E_{i,j})}$$  

(77)

by the well-known estimate, see (20), and the fact that $(\tilde{v}_i)_i = 0$ at corners of $\Gamma_i$. Note that (77) is also valid for subdomains $\Omega_i$ which intersect $\partial \Omega$ by edges since we use the obvious inequality

$$\| \nabla(\tilde{v}_i) \|_{L^2(\Omega_i)} \leq \| \nabla(\tilde{v}_i) \|_{L^2(\Omega_i)} \leq C \sum_{j \in \mathcal{J}_H^0} \| (v_i)_j \|_{H^{1/2}_{00}(E_{i,j})}$$

where $(\tilde{v}_i)_i$ is the $\mathcal{H}_i$ discrete harmonic extension on $\Omega_i$ with $(\tilde{v}_i)_i = (v_i)_i$ on edges $E_{ij}$ for $j \in \mathcal{J}_H^0$, and $(\tilde{v}_i)_0 = 0$ on edges $E_{ij}$ for $j \in \mathcal{J}_H^0$. For the case $E_{ij}$ such that $j \in \mathcal{J}_H^0$, define $\tilde{u}_i = \mathcal{I}_i^0 u_i$ and $\tilde{u}_j = \mathcal{I}_i^0 u_j$. Note also that $\tilde{u}_i = \mathcal{I}_i^1 \tilde{u}_i = \tilde{u}_i$ on $E_{ij}$ and $\tilde{E}_{ij}$, and also $\tilde{u}_j = \mathcal{I}_i^1 \tilde{u}_j = \tilde{u}_j$. We use (52) to get

$$\rho_i \| (v_i)_i \|_{H^{1/2}_{00}(E_{i,j})} \leq \frac{\rho_i \rho_j}{(\rho_i^2 + \rho_j^2)^{1/2}} \| (u_i - \tilde{u}_i)_i - (u_j - \tilde{u}_j)_j \|_{H^{1/2}_{00}(E_{i,j})} \leq \| (u_i - \tilde{u}_i)_i - (u_j - \tilde{u}_j)_j \|_{H^{1/2}_{00}(E_{i,j})} +$$

(78)

$$+ \frac{\rho_i \rho_j}{(\rho_i^2 + \rho_j^2)^{1/2}} \| (u_i - \tilde{u}_i)_i - (u_j - \tilde{u}_j)_j \|_{H^{1/2}_{00}(E_{i,j})},$$
where we have used that \( \frac{\rho_i \rho_j^\beta}{(\rho_i + \rho_j)^2} \leq \min\{\rho_i, \rho_j\} \) if \( \beta \in [1/2, \infty) \), see [19].

Following the same steps of the proof of Lemma 4.5 in [10] (see there (4.49)-(4.51)), we can bound
\[
\rho_i \| (w_i) \|_{H_{00}^{1/2}(E_{ij})}^2 \leq C(1 + \log \frac{H}{h})^2 \{d_i(\mathcal{H}_i u_i, \mathcal{H}_e u_i) + d_j(\mathcal{H}_j u_j, \mathcal{H}_e u_j)\} \tag{79}
\]
and using Lemma [7] we obtain
\[
\rho_i \| (w_i) \|_{H_{00}^{1/2}(E_{ij})}^2 \leq C(1 + \log \frac{H}{h})^2 \{d_i(\mathcal{H}_i' u_i, \mathcal{H}_e' u_i) + d_j(\mathcal{H}_j' u_j, \mathcal{H}_e' u_j)\}. \tag{80}
\]

It remains to estimate the second term of the right-hand side of (76). The case \( E_{ij} \) where \( j \in J_H^{+0} \) is trivial. For the case \( E_{ij} \) such that \( j \in J_H^{-0} \) using [32] - [33], and similar arguments as in the proof of Lemma 4.5 in [10] (see there (4.45)-(4.51)), we obtain
\[
d_i(\mathcal{H}_i u_i, \mathcal{H}_e u_i) = \frac{\rho_i \rho_j^2 \gamma}{(\rho_i + \rho_j)^2} \times \
\leq C(1 + \log \frac{H}{h})^2 \{d_i(\mathcal{H}_i' u_i, \mathcal{H}_e' u_i) + d_j(\mathcal{H}_j' u_j, \mathcal{H}_e' u_j)\}. \tag{81}
\]
Using Lemma [7] we obtain
\[
d_i(\mathcal{H}_i u_i, \mathcal{H}_e u_i) \leq C(1 + \log \frac{H}{h})^2 \times \
\{d_i(\mathcal{H}_i' u_i, \mathcal{H}_e' u_i) + d_j(\mathcal{H}_j' u_j, \mathcal{H}_e' u_j)\}. \tag{82}
\]

Using the inequalities (80) and (82) in (76), summing the resulting inequality for \( i \) from 1 to \( N \) and noting that the number of edges of each subdomain can be bounded independently of \( N \), we obtain (75) and (74).

The proof also works with minor modifications for the case when \( E_{ij} \) is a continuous curve of intervals. For that, we should consider discrete Sobolev tools for non straight edges, see for instance [15], and interpret \( I_{E_{ij}}(u_i) \) and \( I_{E_{ji}}(u_j) \) as the linear function with respect to parametrized path on the edge defined by the nodal value of \((u_i), (u_j)\) at \( x \in \partial E_{ij} \) and \( \partial E_{ji} \).

6 Numerical experiments

In this section, we present numerical results for solving the linear system (45) with the left preconditioner [57]. We show that the lower and upper bounds of Theorem [1] are reflected in the numerical tests. In particular we show that the constant \( C \) in (58) does not depend on \( h_i, H_i \), and the jumps of \( \rho_i \).

We consider the domain \( \Omega = (0,1)^2 \) and divide into \( N = M \times M \) squares subdomains \( \Omega_i \), of size \( H = 1/M \). Inside each subdomain \( \Omega_i \) we generate a structured triangulation with \( n = m \times m \) subintervals in each coordinate direction and apply the discretization presented in Section 2.2 with penalty
term $\delta = 10$. In the numerical experiments we use a red and black checkerboard type of subdomain partition, where the most bottom-left subdomain has a black color. We solve the second order elliptic problem $-\text{div}(\rho(x)\nabla u^*_x(x)) = 1$ in $\Omega$ with homogeneous Dirichlet boundary conditions $u^* = 0$. In the numerical experiments, we run PCG until the $l_2$ initial residual is reduced by a factor of $10^{10}$.

Table 1  Number of iterations, condition numbers (in parenthesis) for different sizes of coarse and local problems and with constant coefficient $\rho = 1$. Here $\beta = 1$, see (49).

| $M \downarrow m \to$ | 4   | 8   | 16  |
|----------------------|-----|-----|-----|
| 4                    | 13 (2.29) | 13 (2.84) | 13 (3.64) |
| 8                    | 15 (2.50) | 17 (3.16) | 18 (4.01) |
| 16                   | 15 (2.59) | 17 (3.28) | 20 (4.16) |

In the first test of experiments we consider the constant coefficient case $\rho = 1$. We consider different values of $N = M \times M$ coarse partitions and different values of local refinements $n = m \times m$. Table 1 lists the number of PCG iterations and in parenthesis the condition number estimate of the preconditioned system. As expected from the analysis, the condition numbers appear to be independent of the number of subdomains and grow by a two-logarithmically factor when the size of the local problems increases. As expected from the theory, the lower bounds estimates are always very closed to one, therefore, we do not show in the tables.

Table 2  Number of iterations, condition numbers (in parenthesis) for different sizes of coarse and local problems and with constant coefficient $\rho = 1$ in the black substructures and $\rho = 1000$ in the red substructures. Here $\beta = 1$, see (49).

| $M \downarrow m \to$ | 4   | 8   | 16  |
|----------------------|-----|-----|-----|
| 4                    | 5 (1.10) | 5 (1.10) | 5 (1.10) |
| 8                    | 6 (1.10) | 6 (1.12) | 6 (1.16) |
| 16                   | 7 (1.29) | 8 (1.42) | 8 (1.55) |

We now consider the discontinuous coefficients case where we set $\rho_b = 1$ on the black substructures and we vary $\rho_r$ on the red substructures. We first consider different values of $N = M \times M$ coarse partitions and different values of local refinements $n = m \times m$ while we keep $\rho_r = 1000$. The results are shown in Table 2 and are similar to the previous test for continuous coefficient.

For the next experiment the substructures partition is kept fixed to $8 \times 8$. Table 3 lists the results on runs for different values of $\rho_r$ and for different levels of refinements. In Table 4 and Table 5 we repeat the test with two different values of $\beta$, see (49). The performance of the preconditioner is robust with respect to the coefficients and $h$ as predicted.
Table 3 Number of iterations and condition numbers (in parenthesis) for different values of the coefficient $\rho_r$ on the red substructures and local meshes with $n = m \times m$. On black substructures the coefficient $\rho_b = 1$ is kept fixed. The substructure partition is also kept fixed to $N = M \times M = 8 \times 8$. Here $\beta = 1$, see (49).

| $\rho_r \downarrow m \rightarrow$ | 2  | 4  | 8  | 16 |
|-------------------------------|----|----|----|----|
| 10000                        | 5 (1.10) | 5 (1.09) | 5 (1.09) | 5 (1.09) |
| 1000                         | 6 (1.10) | 6 (1.10) | 6 (1.12) | 6 (1.16) |
| 100                          | 7 (1.21) | 7 (1.35) | 8 (1.50) | 9 (1.66) |
| 10                           | 10 (1.50) | 11 (1.79) | 13 (2.15) | 15 (2.55) |
| 1                            | 12 (1.96) | 15 (2.50) | 17 (3.16) | 18 (4.01) |
| 0.1                          | 10 (1.51) | 12 (1.82) | 13 (2.18) | 15 (2.59) |
| 0.01                         | 7 (1.27) | 8 (1.44) | 9 (1.62) | 10 (1.80) |
| 0.001                        | 6 (1.10) | 6 (1.14) | 6 (1.21) | 6 (1.28) |
| 0.0001                       | 5 (1.10) | 5 (1.09) | 5 (1.09) | 5 (1.09) |

Table 4 Number of iterations and condition numbers (in parenthesis) for different values of the coefficient $\rho_r$ and local meshes with $n = m \times m$ on the red substructures. On black substructures the coefficient $\rho_b = 1$ is kept fixed. The substructure partition is also kept fixed to $N = M \times M = 8 \times 8$. Here $\beta = 0.5$, see (49).

| $\rho_r \downarrow m \rightarrow$ | 2  | 4  | 8  | 16 |
|-------------------------------|----|----|----|----|
| 1000                         | 20 (4.21) | 24 (5.37) | 25 (6.58) | 28 (9.68) |
| 1                            | 12 (1.96) | 15 (2.50) | 17 (3.16) | 18 (4.01) |
| 0.001                        | 20 (4.21) | 24 (5.39) | 25 (6.55) | 27 (9.54) |

Table 5 Number of iterations and condition numbers (in parenthesis) for different values of the coefficient $\rho_r$ and local meshes with $n = m \times m$ on the red substructures. On black substructures the coefficient $\rho_b = 1$ is kept fixed. The substructure partition is also kept fixed to $N = M \times M = 8 \times 8$. Here $\beta = 10$, see (49).

| $\rho_r \downarrow m \rightarrow$ | 2  | 4  | 8  | 16 |
|-------------------------------|----|----|----|----|
| 1000                         | 6 (1.10) | 6 (1.10) | 6 (1.12) | 6 (1.16) |
| 1                            | 12 (1.96) | 15 (2.50) | 17 (3.16) | 18 (4.01) |
| 0.001                        | 6 (1.10) | 6 (1.14) | 6 (1.21) | 6 (1.29) |

A Proof of Lemma 5

We first prove the right hand side of the first inequality, the inequality (64). That is, we prove that there exist a constant $C$ such that, for all $u_i \in W_i(\Omega')$ we have

$$a_i((u_i)_i, (u_i)_i) \leq C a_i(u_i, u_i).$$

(83)

First, note that,

$$a_i((u_i)_i, (u_i)_i) = \sum_{\tau \in T^i_h} \rho_i \int_{\tau} |\nabla(u_i)|^2 dx.$$  

(84)

We consider the cases of refined mesh $T^i_h$ listed in Definition 3 and illustrated in Figure 4.

First case (Figure 4, upper-left picture). For the first case, that is, $\tau \in T^i_h$ and let us denote this triangle by $\tau_i$ and its neighbor by $\tau_p$, see Figure 5.

Let us denote by $\tau$ a generic triangle of $T^i_h$. We have

$$\int_{\tau} |\nabla(u_i)|^2 dx = \sum_{\tau \subset \tau_j} \int_{\tau_j} |\nabla(u_i)|^2 dx.$$
The sum runs over ten triangles listed in the first case of Definition 3. Let \( (u_i)_i \) and \( (\varphi_i)_i \) on \( \tau \) be denoted by \( u_i^{(t)} \) or \( u^{(t)} \) and \( \varphi_i^{(t)} \) or \( \varphi^{(t)} \), respectively. Note that in the triangle \( \Delta^{(t)} := C_1^{(t)} C_2^{(t)} C_3^{(t)} \subset \tau \) we have \( (u_i)_{|\tau} = (u_i^{(t)}) \) and then

\[
\int_{\Delta^{(t)}} |\nabla u_i^{(t)}|^2 dx = \int_{\Delta^{(t)}} |\nabla u_{i}^{(t)}|^2 dx = \int_{\Delta^{(t)}} |\nabla u_i^{(t)}|^2 dx.
\]

Let us consider now the triangle \( \Delta^{(t)} := C_1^{(t)} C_2^{(t)} M_3^{(t)} \) where \( M_3^{(t)} = M_1^{(p)} \). We have (see Figure 5)

\[
I := \int_{\Delta^{(t)}} |\nabla u_i^{(t)}|^2 dx \leq C \left[ |u_i^{(t)}(C_3^{(t)}) - u_i^{(t)}(C_2^{(t)})|^2 + |u_i^{(t)}(C_2^{(t)}) - 0.5(u_i^{(t)}(M_1^{(t)}) + u_i^{(p)}(M_1^{(p)}))|^2 + |u_i^{(t)}(C_1^{(t)}) - 0.5(u_i^{(t)}(M_1^{(t)}) + u_i^{(p)}(M_1^{(p)}))|^2 \right]
\]

\[
:= I_1 + I_2 + I_3.
\]

The first term above, \( I_1 \), it is estimated by \( \|\nabla u^{(t)}\|_{L^2(\Delta^{(t)})}^2 \). The second term, \( I_2 \), it is estimated as follows. We have

\[
I_2 \leq C \left\{ |u_i^{(t)}(C_3^{(t)}) - u_i^{(t)}(M_1^{(t)})|^2 + \frac{1}{2} |u_i^{(t)}(M_1^{(t)}) - u_i^{(p)}(M_1^{(p)})|^2 \right\};
\]

\[
\leq C \left\{ \|\nabla u_i^{(t)}\|_{L^2(\Delta^{(t)})}^2 + \frac{1}{h} |u_i^{(t)} - u_i^{(p)}|^2_{L^2(\Delta^{(t)})} \right\}.
\]  \hfill (85)

where here \( V_2^{(t)} V_3^{(t)} \) denotes the edge of \( \tau \) with the end points \( V_2^{(t)} \) and \( V_3^{(t)} \). In the same way we can estimate the third term, \( I_3 \). Thus,

\[
I \leq C \left\{ \|\nabla u_i^{(t)}\|_{L^2(\tau_\ell)}^2 + \frac{1}{h} |u_i^{(t)} - u_i^{(p)}|^2_{L^2(V_2^{(t)}, V_3^{(t)})} \right\}.
\]  \hfill (86)

Similarly it is possible to estimate the terms involving the triangles \( C_1^{(t)} C_2^{(t)} M_3^{(t)} \) and \( C_1^{(t)} C_3^{(t)} M_2^{(t)} \).
We now estimate the term on $\Delta^e := C_2^{(t)} M_1^{(t)} V_2^{(t)}$. We have then

$$I := \int_{\tau_h} |\nabla u_{(t)}|^2 dx \leq C \left\{ \left| \left( u_{(t)}(C_2^{(t)}) - \frac{1}{2} \left(u_{(t)}(M_3^{(t)}) + u_{(h)}(M_3^{(h)})\right) \right)^2 \right| + \left| \left( u_{(t)}(C_2^{(t)}) - \frac{1}{n_{npk}} \left(u_{(t)}(V_2^{(t)}) + u_{(p)}(V_3^{(t)}) + \cdots + u_{(h)}(V_2^{(h)})\right) \right)^2 \right| + \left| \left( \frac{1}{2} (u_{(t)}(M_3^{(t)}) + u_{(h)}(M_3^{(h)})) \right) - \frac{1}{n_{npk}} \left(u_{(t)}(V_2^{(t)}) + u_{(p)}(V_3^{(t)}) + \cdots + u_{(h)}(V_2^{(h)})\right) \right|^2 \right\} \right.$$  

where the first sum runs over the elements $e$ of $\tau$ which have a common vertex or edge with $\tau_e$, and then second term, $I_2$, are estimated as in (85). The third term, $I_3$, is estimated in a similar way by adding and subtracting the quantity $u_{(t)}(V_2^{(t)}) = u_{(t)}(V_2^{(h)})$, see Figure 5. We proceed as above and using these estimates in (88) we obtain

$$I \leq C \left\{ \|\nabla u_{(t)}\|_{L^2(\tau_e)}^2 + \|\nabla u_{(h)}\|_{L^2(\tau_h)}^2 + \frac{1}{h} \left\{ \|u_{(t)} - u_{(h)}\|_{L^2(\partial \tau_e \cap \partial \tau_h)}^2 + \cdots + \|u_{(t)} - u_{(p)}\|_{L^2(\partial \tau_e \cap \partial \tau_p)}^2 \right\} \right\}. \quad (88)$$

In a similar way are estimated the terms over the remaining triangles of $\tau_e$.

Using the above estimates we show that

$$\int_{\tau_e} |\nabla u_{(e)}|^2 dx \leq C \left\{ \sum_{\tau} \|\nabla (u_{(h)})\|_{L^2(\tau)}^2 + \frac{1}{h} \sum_{\tau} \|(u_{(i)})^+ - (u_{(i)})^-\|_{L^2(\tau)}^2 \right\} \quad (89)$$

where the first sum runs over the elements $\tau$ which intersect $\tau_e$ by an edge and the second sum runs over edges $e$ of $\tau$ which have a common vertex or edge with $\tau_e$.

**Second case (Figure 4, upper-right picture).** We now consider the case when a vertex of $\Omega_1$ is common for two and more triangles of $T_h$. Let us consider the case of two triangles, see Figure 5. This case is estimated similar as the first case.

**Third case (Figure 4, lower-left picture).** This case (see Figure 7) is also estimated similar as the first case.

Adding the above estimates for the three cases we get the estimate (83) for the case $\mathbf{u} = L_h u_1$. For the case $\mathbf{u} = L_h u_2$, we need only some minor modifications of the proof of the second case above, see Figure 6. This finishes the proof of the left hand side inequality of (64).

We now present the proof of the left hand side of the result stated in Lemma 6. We need to show that there exists a constant $C$ such that

$$a_i((u_{(i)})^+, (u_{(i)})^-) \leq C \alpha_i((u_{(i)})^+, (u_{(i)})^-). \quad (90)$$

Note that, on $\tau_e \subset T_h$ with vertices of type C, we have (see Figure 5)

$$\|\nabla (u_{(i)})\|_{L^2(\tau_e)}^2 \leq C \left\{ \left[ u_{(i)}(C_1^{(i)}) - u_{(i)}(C_2^{(i)}) \right]^2 + \left[ u_{(i)}(C_1^{(i)}) - u_{(i)}(C_3^{(i)}) \right]^2 + \left[ u_{(i)}(C_2^{(i)}) - u_{(i)}(C_3^{(i)}) \right]^2 \right\} \leq C \|\nabla (I_h u_{(i)})\|_{L^2(\tau_e)}^2 \leq C \|\nabla \mathbf{u}_{(i)}\|_{L^2(\tau_e)}^2. \quad (91)$$
This is valid for all three cases considered above. Using the estimate (91) we prove (90). The proof of the equivalence (64) is now complete.

We now prove the second inequality of Lemma 5, the inequality (65). We have

\[ p_i,\partial(u_i,u_i) = \sum_{j \in \mathcal{T}_h} \sum_{e \in \mathcal{F}_{ij}^e} \int_e \frac{\delta}{\varepsilon_{ij}} \rho e (u_i - u_j)^2 dS. \] (92)

On the edge $e$ we have (see Figure 8)

\[ \int_e (u_i - u_j) dS = \sum_{2 \in e} \int_e (u_i - u_j)^2 dS \] (93)
where \( \xi \) runs over the edges of \( T_h^b \), \( \xi \subset E_{ij} \). Note that, on \( \xi = [C_2^+, C_3^+] = [C_2^-, C_3^-] \), we can write

\[
\int_\xi (u_i - u_j)^2 ds = \int_\xi (u_i - u_j)^2 ds.
\]

Additionally, on \( \xi = [V_1^+, C_2^+] = [V_1^-, C_2^-] \), we have

\[
\int_\xi (u_i - u_j)^2 ds \leq C \left\{ \left[ u_i(C_2^+) - u_j(C_2^-) \right]^2 + \left[ \frac{1}{2} (u_i(C_2^+) - u_j(C_2^-)) - \frac{1}{2} (u_j(C_2^-) - u_j(C_2^-)) \right]^2 \right\}
\]

where \( C_2^+ \) and \( C_2^- \) are the nodal points on edges \( \tilde{e} \) of the triangles of \( T_h^b \) and \( T_h^J \) on \( E_{ij} \) and \( E_{ji} \) with common nodal points \( V_1^+ \) and \( V_1^- \), respectively. Thus, in this case,

\[
\int_\xi (u_i - u_j)^2 ds \leq C \left\{ \|u_i - u_j\|_{L^2(\xi)}^2 + \|u_i - u_j\|_{L^2(\xi)}^2 \right\}
\]

where \( \tilde{e} \cap e = V_1^+ = V_1^- \). In the case when \( V_1^+ \) or \( V_1^- \) are corners of \( \partial \Omega \), we do the same modification which give \( \|u_i\|_{L^2(\tau)}^2 \) on \( \tau \) with vertices \( V_1^+ \) or \( V_1^- \). Using these in \( 93 \) and the resulting estimate into \( 92 \) we get an estimate of the second inequality of Lemma 5 for the case when \( u_i = I_h u_i \). The case when \( u_i = I_h^3 u_i \) is proved similarly.

Now we prove the third inequality of Lemma 5, the inequality \( 66 \). We have that \( 93 \) still holds if we replace \( u_i \) by \( u_i \) and \( u_j \) by \( u_j \), respectively. Note that (see Figure 4)

\[
\int_\xi (u_i - u_j)^2 ds \leq C \left\{ \left[ u_i(C_1^+) - u_j(C_1^-) \right]^2 + \left[ u_i(C_2^+) - u_j(C_2^-) \right]^2 \right\}
\]

\[
\leq C \int_\xi (u_i - u_j)^2 dx
\]

where \( \xi = (C_1^+, C_2^+) \). Using these estimates we see that the third inequality is valid for \( u_i = I_h u_i \). The case \( u_i = I_h^3 u_i \) is similar.

It remains only to estimate the fourth inequality, inequality \( 67 \). It is proved as in the third inequality for \( u_i = I_h^3 u_i \).

The proof of Lemma 5 is complete.

### B Proof of Lemma 6

For the first inequality, \( 68 \), note that on \( \tau \in T_h^b \) (see Figure 4 upper-left picture)

\[
\int_\tau |\nabla (I_h u_i)|^2 ds \leq C \left\{ \left[ u_i(C_1) - u_i(C_2) \right]^2 + \left[ u_i(C_1) - u_i(C_3) \right]^2 + \left[ u_i(C_2) - u_i(C_3) \right]^2 \right\}
\]

\[
\leq C \|\nabla u_i\|_{L^2(\tau)}^2 \leq C \|\nabla u_i\|_{L^2(\tau)}^2
\]

Summing this for \( \tau \subset T_h \), we get the first inequality.

To prove the second inequality, \( 69 \), note that on \( e \subset \partial \Omega \) (see Figure 8)

\[
\|I_h u_i - I_h u_j\|_{L^2(e)}^2 \leq C \left\{ \left[ u_i(C_2^+) - u_j(C_2^-) \right]^2 + \left[ u_i(C_3^+) - u_j(C_3^-) \right]^2 \right\}
\]

\[
\leq C \frac{1}{h} \|u_i - u_j\|_{L^2(e)}^2 \quad (\text{with } \xi = (C_2^+, C_3^+))
\]

\[
\leq C \frac{1}{h} \|u_i - u_j\|_{L^2(e)}^2.
\]
Summing this estimate over $e \subset \partial \Omega_i$ we get the second inequality.

Fig. 8 Illustration of common edge refinement.

The equality $I_h^i u = u_i$ follows from the definitions of $I_h^i$ and $u_i$. The proof of Lemma 6 is now complete.

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