Synthetic Likelihood in Misspecified Models: Consequences and Corrections

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Abstract

We analyse the behaviour of the synthetic likelihood (SL) method when the model generating the simulated data differs from the actual data generating process. One of the most common methods to obtain SL-based inferences is via the Bayesian posterior distribution, with this method often referred to as Bayesian synthetic likelihood (BSL). We demonstrate that when the model is misspecified, the BSL posterior can be poorly behaved, placing significant posterior mass on values of the model parameters that do not represent the true features observed in the data. Theoretical results demonstrate that in misspecified models the BSL posterior can display a wide range of behaviours depending on the level of model misspecification, including being asymptotically non-Gaussian. Our results suggest that a recently proposed robust BSL approach can ameliorate this behaviour and deliver reliable posterior inference under model misspecification. We document all theoretical results using a simple running example.

Keywords: likelihood-free inference, approximate Bayesian computation, synthetic likelihood, Bernstein-von Mises, model misspecification

1 Introduction

Over the last two decades, approximate Bayesian methods, sometimes called likelihood-free methods, have become a common approach to conduct Bayesian inference in situations where the likelihood function is intractable. Two of the most prominent methods in this paradigm are approximate Bayesian

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computation (ABC), see Marin et al. (2012) for a review and Sisson et al. (2018) for a handbook treat-
ment, and the method of synthetic likelihood (SL, Wood, 2010). From a Bayesian perspective, SL-based
inference is conducted by placing a prior over the unknown model parameters and using Markov chain
Monte Carlo (MCMC) methods to sample the posterior. Throughout the remainder we refer to such
methods as Bayesian SL (BSL), and refer to Price et al., 2018 for an introduction to BSL.

Both ABC and BSL are predicated on the belief that summary statistics obtained from the observed
data can be matched by the assumed model. The goal of these methods is then to conduct inference on
the unknown model parameters by simulating summary statistics under the assumed model and matching
them to those for the observed data. Values of the parameters that lead to a “good” match, in the sense
that a chosen distance between the observed and simulated summaries is small, are used to estimate
the likelihood of the summaries and ultimately conduct posterior inference on the model unknowns.
While ABC implicitly constructs a nonparametric estimate of the likelihood for the summaries, BSL
approximates the intractable likelihood by assuming the summaries follow a Gaussian distribution with
unknown mean and variance. Repeated model simulation is then used to estimate the unknown mean
and covariance, and the resulting synthetic likelihood (with estimated mean and variance) is used within
an MCMC scheme to conduct inference.

The goal of BSL and ABC methods is to conduct inference in models that are so complicated that
the resulting likelihood is intractable. However, even complex models are only approximations of reality
and correct specification is often unlikely. Hence, for a diverse collection of summary statistics, it is
unlikely that the model can match the features of the data measured by the chosen summaries; with this
problem likely to be exacerbated in early phases of model exploration, design, and formulation. When
the summaries simulated under the assumed model cannot match the observed summaries, for any value
of the unknown parameters, we say that the model is misspecified. This notion of misspecification is
consistent with the notion of model misspecification in the ABC context (see, e.g., Marin et al., 2014,
and Frazier et al., 2020b).

Several authors have discussed the impacts of model misspecification within likelihood-based Bayesian
inference (see, e.g., Kleijn and van der Vaart, 2012, Miller and Dunson, 2019), however, the authors are
unaware of any research that rigorously discusses and characterizes the behavior of BSL in such cases.
While Frazier and Drovandi (2021) explore the application of BSL in simple misspecified models, the
authors do not discuss the general behavior of BSL in misspecified models. In addition, Frazier et al.
(2021) explore the theoretical behavior of BSL, but their analysis is entirely restricted to the case of
“correct” model specification; i.e., where the assumed model can match the observed summary statistics
for some value in the parameter space.\footnote{While Frazier et al. (2021) do allow the variance used in the SL to be misspecified, this “misspecification” only alters the posterior variance and thus does not alter the nature of posterior concentration.}

In this paper, we rigorously characterize the behavior of BSL in cases where the model is misspec-
ified, and demonstrate that the BSL posterior can be highly-sensitive to model misspecification. Further,
we show that under model misspecification the limiting behavior of the BSL and ABC posteriors are significantly different. Indeed, in contrast to ABC, the BSL posterior can concentrate posterior mass on values of the parameters under which the observed and simulated summaries take on very different values. Lastly, we investigate potential approaches for controlling model misspecification in Bayesian inference within the context of BSL. We demonstrate that a popular approach used in likelihood-based Bayesian inference is ineffective for BSL, and obtain new insights into why two existing approaches for BSL can be effective.

Before presenting our general results, we first demonstrate the sensitivity of the BSL posterior to model misspecification in a simple example.

**Example: Moving Average Model**

The researcher believes the observed data $y_{1:n} = (y_1, \ldots, y_n)^\top$ is generated according to a moving average model of order one (MA(1) model)

$$y_t = \epsilon_t + \theta \epsilon_{t-1}, \quad t = 1, \ldots, n,$$

with $\epsilon_t$ independent and identically distributed (iid) standard normal, and where $\theta \in (-1, 1)$ is unknown with our prior beliefs $\pi(\theta)$ uniform over this region. We take as summary statistics the sample autocovariances $S_j(y_{1:n}) = \frac{1}{n} \sum_{t=1}^{n} y_t y_{t-j}$, for $j \in \{0, 1\}$, and let $S_n(y_{1:n}) = (S_0(y_{1:n}), S_1(y_{1:n}))^\top$ denote the observed summaries. Let $z_{1:n} = (z_1, \ldots, z_n)^\top$ denote a data set of length $n$ simulated from (1) under $\theta \sim \pi(\theta)$. SL models the distribution of $S(z_{1:n})|\theta$ as Gaussian with unknown mean and variance. With some effort it is possible to exactly compute the mean and variance of the summaries in this example, so that the SL can be computed exactly, with inference on $\theta$ then conducted using the exact BSL posterior (see Section 2.2 for full details).

While the researcher believes the data is generated according to an MA(1) model, the actual data generating process (DGP) evolves according to the stochastic volatility (SV) model

$$y_t = \exp(h_t/2) u_t, \quad h_t = \omega + \rho h_{t-1} + v_t \sigma_v,$$

with $0 < \rho < 1$, $0 < \sigma_v < 1$, $u_t$ and $v_t$ and both iid standard normal. Under the DGP above, the model is misspecified, however, for any value of $\omega, \rho, \sigma_v$ above, the population auto-covariances are zero. Therefore, a priori we expect the BSL posterior for $\theta$ to have significant mass near $\theta = 0$, as this yields simulated data with no significant autocorrelations, and would most closely “match” the features of the observed data measured by the summaries.

We generate data from the SV model in (2) with parameter values $\omega = -0.736$, $\rho = 0.90$ and $\sigma_v = 0.36$, which produce a series that displays many of the same features as monthly asset returns, and consider three different sample sizes: $n = 100, 500, 1000$. For each sample size and data set, we plot the
resulting exact BSL posterior in Figure 1.

The results in Figure 1 demonstrate that the BSL posterior is bi-modal, with well-separated modes. In addition, the bi-modality does not disappear as the sample size increases, signalling that the BSL posterior will not concentrate onto a single point as \( n \) diverges. Moreover, in this example the normality of the summary statistics is very reasonable: both summaries can be verified to satisfy a central limit theorem under the true DGP.

This behavior is surprising, and worrisome, given that the value of \( \theta \) that (asymptotically) minimizes \( \| S_n(z_{1:n}) - S_n(y_{1:n}) \| \), for \( \| \cdot \| \) denoting the Euclidean norm, is \( \theta = 0 \). While the point \( \theta = 0 \) ensures that the simulated summaries are as close as possible to the observed, in the distance \( \| \cdot \| \), the BSL posterior has little mass near this point. Instead, the BSL posterior gives the impression that we require almost perfect autocorrelation, \( \theta \approx \pm 1 \), to “match” the observed summaries, when in fact the observed data has no autocorrelation.

This behavior of the BSL posterior is in stark contrast to what one would obtain if a simple accept/reject ABC algorithm was applied to conduct posterior inference on \( \theta \). In this case, it is simple to verify that the ABC posterior concentrates its mass on the value \( \theta = 0 \), and is uni-modal.

\[
\begin{align*}
\text{Posterior} & \quad \text{Posterior} & \quad \text{Posterior} \\
n = 100 & \quad n = 500 & \quad n = 1000
\end{align*}
\]

\( \theta \)

\( \theta \)

\( \theta \)

Figure 1: BSL posteriors for \( \theta \) in the misspecified MA(1) model across fifty replicated data sets.

In the remainder of this paper, we elaborate on the above behavior and formally characterize the asymptotic behavior of the BSL posterior when the model generating the simulated data is misspecified. The remainder of the paper is organized as follows. In Section two, we discuss the relevant concept of model misspecification in SL and compare this with the standard notion based on the Kullback-Leibler divergence. In Section three, we characterize the asymptotic behavior of BSL in misspecified models and demonstrate that the BSL posterior can display non-standard asymptotic behavior. Throughout this section, we compare the theoretical behavior of the BSL posterior to that obtained in the case of ABC, and conclude that the two approaches behave very differently when the model is misspecified. In Section four, we obtain new insights into approaches aimed at dealing with model misspecification in our BSL
context. Section five gives an additional example, and Section six concludes.

2 Synthetic likelihood and Model Misspecification

This section gives the general setup and discusses model misspecification in the SL context. Let \( y_{1:n} = (y_1, \ldots, y_n) \) denote the observed data and define \( P_0^{(n)} \) as the true distribution generating \( y_{1:n} \). The map \( S_n : \mathbb{R}^n \to \mathbb{R}^d \) defines the vector of summary statistic used in the analysis. Where there is no confusion, we write \( S_n \) for the mapping or its value at the observed data.

We consider that the observed data is generated from some parametric class of models \( \{ P_\theta^{(n)} : \theta \in \Theta \subset \mathbb{R}^d \} \), with \( d_\theta \leq d \), and for any \( \theta \in \Theta \) we can simulate pseudo-data \( z_{1:n} \sim P_\theta^{(n)} \). Let \( \Pi \) denote the prior measure for \( \theta \) and \( \pi(\theta) \) its corresponding density. The mean and variance of the simulated summary statistics, calculated under \( P_\theta^{(n)} \), are denoted by \( b(\theta) := \mathbb{E}\{S_n(z)\} \) and \( \Sigma_n(\theta) := \text{var}\{S_n(z)\} \). We note that, in general, \( b(\theta) \) may depend on \( n \), however, we suppress this dependence for simplicity.

SL approximates the distribution of \( S_n(z) \) using a normal distribution with mean \( b(\theta) \) and covariance \( \Sigma_n(\theta) \), which we denote throughout by \( N\{b(\theta), \Sigma_n(\theta)\} \). The SL used in the analysis is then given by \( N\{S_n; b_n(\theta), \Sigma_n(\theta)\} \), where \( N(x; \mu, \Sigma) \) denotes the normal density function evaluated at \( x \). In typical applications \( b(\theta) \) and \( \Sigma_n(\theta) \) are unknown, and are estimated using the sample mean \( \bar{b}_n(\theta) \) and sample covariance \( \bar{\Sigma}_n(\theta) \) calculated from \( m \) independent simulated statistics. These sample quantities are depicted as \( n \)-dependent, rather than \( m \)-dependent, as we will later take \( m \) as a function of \( n \).

A common approach for exploring the SL criterion and obtaining point estimates is to sample the BSL posterior using MCMC techniques (Price et al., 2018). Throughout the remainder we carry out our discussion, and the ensuing analysis, within the confines of the BSL posterior.

Exploring the BSL posteriors entails implementing an MCMC scheme where the likelihood is replaced by the estimated SL, which uses the estimated mean and covariance \( \bar{b}_n(\theta) \) and \( \bar{\Sigma}_n(\theta) \). When used within MCMC algorithms, these methods are implicitly based on the estimated SL \( \bar{g}_n(S_n|\theta) := \int N\{S_n; \bar{b}_n(\theta), \bar{\Sigma}_n(\theta)\} \prod_{i=1}^m dP_\theta^{(n)}\{S(z_{1:n}^i)\} dS(z_{1:n}^1) \ldots dS(z_{1:n}^m) \).

The BSL posterior is then stated as

\[
\pi(\theta|S_n) := \frac{\pi(\theta)\bar{g}_n(S_n|\theta)}{\int_\Theta \pi(\theta)\bar{g}_n(S_n|\theta) d\theta},
\]

which is assumed to exist for all \( n \). However, if \( b(\theta) \) and \( \Sigma_n(\theta) \) are known, MCMC can be used to target

\(^2\text{See, e.g., }\text{Frazier et al. (2021) for further discussion of the connection between pseudo-marginal methods and BSL.}\)
the “exact” BSL posterior

\[ \pi(\theta | S_n) := \frac{\pi(\theta) N \{ S_n; b(\theta), \Sigma_n(\theta) \}}{\int_\Theta \pi(\theta) N \{ S_n; b(\theta), \Sigma_n(\theta) \} d\theta}. \] (4)

### 2.1 Model Misspecification in BSL

While BSL is based on a likelihood, it is not a likelihood for the sample \( y_{1:n} \) but for the summary statistics \( S_n \), and this “likelihood” is itself a normal approximation of the sampling distribution for the summaries. As such, interpreting the impact of model misspecification in BSL requires us to consider the loss of information from replacing the data \( y_{1:n} \) by the summaries \( S_n \); and the use of an approximation for the likelihood of the summaries. To cultivate intuition regarding the impact of these two approximations in SL when the model is misspecified, we first explore model misspecification in the case where the mean and covariance of the summaries is known.\(^3\)

Let \( G_0^{(n)} \) denote the probability measure for the summary statistics \( S_n(y_{1:n}) \) under \( P_0^{(n)} \), with corresponding density function \( g_n(\cdot) \). We analyze the impact of model misspecification in BSL through the KL divergence. However, since the only data we observe in BSL is the summary \( S_n(y_{1:n}) \), we analyze the KL divergence between the SL, \( N \{ S; b(\theta), \Sigma_n(\theta) \} \), and the density for the summaries, \( g_n \):

\[
\text{KL}[G_0^{(n)} \| N \{ b(\theta), \Sigma_n(\theta) \}] = -\int \log \left\{ \frac{N \{ s; b(\theta), \Sigma_n(\theta) \}}{g_n(s)} \right\} g_n(s) ds
= \frac{1}{2} \log \{ |\Sigma_n(\theta)| \} + \frac{1}{2} \int \{ s - b(\theta) \}^\top \Sigma_n^{-1}(\theta) \{ s - b(\theta) \} g_n(s) ds + C,
\]

where \( C \) is a constant that does not depend on \( \theta \). For \( b_0 = \int s g_n(s) ds \) and \( V_0 = \int (s - b_0) (s - b_0)^\top g_n(s) ds \), using properties of quadratic forms,

\[
\text{KL}[G_0^{(n)} \| N \{ b(\theta), \Sigma_n(\theta) \}] = \frac{1}{2} \log \{ |\Sigma_n(\theta)| \} + \frac{1}{2} \text{tr} \{ \Sigma_n^{-1}(\theta) V_0 \} + \frac{1}{2} \{ b(\theta) - b_0 \}^\top \Sigma_n^{-1}(\theta) \{ b(\theta) - b_0 \} + C.
\]

This illustrates that, outside of cases where \( G_0^{(n)} \) only depends on the mean and covariance of the summaries, the SL is always misspecified in the sense that

\[ \inf_{\theta \in \Theta} \text{KL}[G_0^{(n)} \| N \{ b(\theta), \Sigma_n(\theta) \}] > 0. \]

The key term in determining the behavior of the KL divergence is the quadratic form

\[ \frac{1}{2} \{ b(\theta) - b_0 \}^\top \Sigma_n^{-1}(\theta) \{ b(\theta) - b_0 \}. \]

\(^3\)The same general conclusions will follow in the case where the SL is estimated, but the additional technicalities are not insightful here.
Since the summaries are generally an average, $\Sigma^{-1}_n(\theta)$ is generally of order $n$, so that, under regularity conditions, for some constants $0 < c_1 \leq c_2 < \infty$,

$$c_1\|\sqrt{n}\{b(\theta) - b_0\}\|^2 \leq \|\Sigma^{-1/2}_n(\theta)\{b(\theta) - b_0\}\|^2 \leq c_2\|\sqrt{n}\{b(\theta) - b_0\}\|^2.$$ 

Therefore, if there exists no $\theta \in \Theta$ such that $b(\theta) = b_0$, then

$$\lim_{n \to \infty} \inf_{\theta \in \Theta} \text{KL} [G_0^{(n)} \| N\{b(\theta), \Sigma_n(\theta)\}] \to \infty.$$  

Consequently, the meaningful concept of model misspecification in BSL is that there does not exist any $\theta \in \Theta$ such that $b(\theta) = b_0$. This condition is precisely the notion of model incompatibility proposed in Marin et al. (2014), which also features in the literature on model misspecification in ABC (Frazier et al., 2020b,a). We then say that the model is \textit{misspecified in the BSL sense} if

$$\lim_{n \to \infty} \inf_{\theta \in \Theta} \{b(\theta) - b_0\}^T [n\Sigma_n(\theta)]^{-1} \{b(\theta) - b_0\} > 0. \quad (5)$$

Throughout the remainder, when we reference the notion of model misspecification, it is meant in the \textit{BSL sense} given by equation (5).

### 2.2 Consequences of Model Misspecification

We now return to the simple MA(1) example given in the introduction to demonstrate that, depending on the level of model misspecification, the BSL posterior can display Gaussian-like posterior concentration, concentration onto a finite, or dense, set of values, or concentration onto the boundary of the parameter space.

**Example: Moving Average model**

The researcher believes $y_{1:n}$ is generated according to an MA(1) model, see equation (1), and our prior beliefs are uniform over $(-1, 1)$. The summary statistics are $S_j(y_{1:n}) = \frac{1}{n} \sum_{t=1}^{n-j} y_t y_{t-j}$, for $j \in \{0, 1\}$, and $S_n(y_{1:n}) = (S_0(y_{1:n}), S_1(y_{1:n}))^T$. In this example, the mean and variance of the summaries can be calculated exactly, with these quantities then used to construct the exact BSL posterior. The mean of the summaries is simple to obtain and is given by

$$b(\theta) = \mathbb{E}[S_n(z_{1:n})|\theta] = (1 + \theta^2, \theta)^T.$$ 

The variance of the summaries also has a closed-form, and can be derived using the results of De Gooijer (1981) on the variance and covariance of sample autocorrelations in autoregressive integrated moving average (ARIMA) models.

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Partitioning $\Sigma_n(\theta)$ as
\[\Sigma_n(\theta) = \begin{pmatrix} \Sigma_{11,n}(\theta) & \Sigma_{12,n}(\theta) \\ \Sigma_{12,n}(\theta) & \Sigma_{22,n}(\theta) \end{pmatrix},\]
the leading terms in the components of $\Sigma_n(\theta)$ are as follows:\(^4\)
\[
\begin{align*}
\Sigma_{11,n}(\theta) &= (2/n^4) \left[ n^3 \cdot (1 + \theta^2)^2 + 2 \cdot n^2 \cdot (n - 1) \cdot \theta_1^2 \right] + O(n^{-2}) \\
\Sigma_{22,n}(\theta) &= (1/n^2) \left[ (n - 1) \cdot ((1 + \theta^2)^2 + \theta_1^2) + 2 \cdot (n - 2) \cdot \theta_1^2 \right] + O(n^{-2}) \\
\Sigma_{12,n}(\theta) &= (2/n^4) \left[ n^2 \cdot ((n - 1) \cdot (2 \cdot (1 + \theta^2) \cdot \theta_1)) \right] + O(n^{-2})
\end{align*}
\]

From this representation, it is clear that each term has a dominant $O(n^{-1})$ term, and that $n\Sigma_n(\theta)$ is positive-definite for all $\theta \in (-1, 1)$ when $n \geq 2$ (neglecting the $O(n^{-2})$ term).

Recall that the actual DGP for $y_{1:n}$ evolves according to the stochastic volatility (SV) model in equation (2). Under this DGP the summaries $S_n(y_{1:n})$ converge in probability towards
\[b_0 = \left( \exp \left( \frac{\omega}{1 - \rho} + \frac{1}{2} \frac{\sigma_v^2}{1 - \rho^2} \right), \ 0 \right)^\top.\]

Therefore, if for given values of $\omega, \sigma_v, \rho$ there does not exist a value of $\theta$ such that
\[\exp \left\{ \omega/(1 - \rho) + \frac{1}{2} \sigma_v^2/(1 - \rho^2) \right\} = 1 + \theta^2,\]
we cannot match the first summary, and the assumed model is misspecified. Asymptotically, the unique minimum of $\|b(\theta) - b_0\|$ is achieved at $\theta = 0$, and it is this value onto which we would hope the BSL posterior would concentrate. However, as we have already seen from Figure 1, for certain values of $\omega, \sigma_v, \rho$ this is not the case, with the BSL posterior concentrating near $\theta \approx \pm 1$.

To help explain this phenomena, we analyze the BSL posterior across various levels of model misspecification by fixing the value of the observed summaries $S_n(y_{1:n})$. To this end, we plot the BSL posteriors for three values of $n = 100, 500, 1000$, and across six different values of the first summary statistic $S_0(y_{1:n}) \in \{.01, .1, .25, .5, .75, .99\}$. These values of $S_0(y_{1:n})$ represent a situation of significant misspecification, at $S_0(y_{1:n}) = 0.01$, tending towards no misspecification, $S_0(y_{1:n}) = 0.99$. We plot the resulting posteriors graphically in Figure 2. The results demonstrate that the behavior of the BSL posterior varies as the level of model misspecification changes. Surprisingly, the posterior can display: bi-modality, with the modal values occurring on the boundary of the parameter space ($S_0(y_{1:n}) = 0.01$);\(^5\) bi-modality with values in the interior of the parameters space ($S_0(y_{1:n}) \in \{0.1, 0.25\}$); a region of

\(^4\)The precise formulas are too long to state analytically. The interested reader is referred to the supplementary material where it is given in full detail.

\(^5\)Note that, for the parameter values $\omega = -0.736, \rho = 0.90$ and $\sigma_v = 0.36$, the resulting value of $b_0$, i.e., the value onto which $S_0(y_{1:n})$ is concentrating, is less than 0.001.
flateness \( S_0(y_{1:n}) = 0.5 \); and approximate Gaussianity \( S_0(y_{1:n}) \in \{0.74, 0.99\} \).\(^6\)

\[ b_0 = 0.01 \]

\[ b_0 = 0.1 \]

\[ b_0 = 0.25 \]

\[ b_0 = 0.5 \]

\[ b_0 = 0.75 \]

\[ b_0 = 0.99 \]

Figure 2: Comparison of “exact” synthetic likelihood posterior under different levels of model misspecification. The solid line corresponds to \( n = 100 \), the dashed line to \( n = 500 \) and the dotted line to \( n = 1000 \).

Critically, at larger levels of model misspecification, the values onto which the exact BSL posterior is concentrating are not at all related to the values of \( \theta \) under which \( \|b(\theta) - b_0\| \) is small. In comparison, if one were to apply ABC based on \( \| \cdot \| \) in the same example, the resulting ABC posterior would be uni-modal and have the majority of its mass near the origin (\( \theta = 0 \)).

3 Asymptotic Behavior of BSL

This section theoretically characterizes the behavior of the BSL posterior when the assumed model is misspecified. We define some notation to make the results easier to state and follow. For \( x \in \mathbb{R}, \quad |x| \) denotes the absolute value of \( x \), and for \( x \in \mathbb{R}^p, \quad \|x\| \) denote the Euclidean norm of \( x \). For \( A \) denoting an \( m \times m \) matrix, we abuse notation and let \( |A| \) denote the determinant of \( A \) and \( \|A\| \) any convenient matrix norm. Throughout, \( C \) denotes a generic positive constant that can change with each usage. For real-valued sequences \( \{a_n\}_{n \geq 1} \) and \( \{b_n\}_{n \geq 1} \): \( a_n \lesssim b_n \) denotes \( a_n \leq Cb_n \) for some finite

\(^6\)In Appendix A.2.1 we expand on the mechanisms causing this posterior behavior in the MA(1) model.
Assumption 3.2. There exists a vector \( b \in \mathbb{R}^d \) with \( d \geq d_0 \), such that \( \| S_n - b \| = o_p(1) \), and there exists a covariance matrix \( V \) such that \( \sqrt{n} (S_n - b) \Rightarrow N(0, V) \), under \( P_0^n \).

Assumption 3.3. The set \( \Theta^* \) is non-empty and finite. For some \( \delta > 0 \), at least one \( \theta \in \Theta^* \) satisfies

\[
0 < \delta \leq \lambda_{\min} \{-H(\theta)\} \leq 1/\delta.
\]

Remark 3.1. Assumption 3.3 restricts the SL to have at most a finite collection of local maxima, all of which lie in the interior of \( \Theta \). Importantly, and as illustrated in the simple running example, there is no reason to suspect that values in \( \Theta^* \) deliver small values of \( \| b(\theta) - b_0 \| \). The behavior of the BSL posterior when a root is on or near the boundary of the parameter space can be quite complicated, and we leave a detailed study of this situation for future research.
Lemma 3.1. If Assumptions 3.1, 3.2 and 3.3 are satisfied, then there exists at least one strict local maximum of \( \ln g_n(S_n|\theta) \) in \( \Theta \) that solves \( 0 = M_n(\theta) \) and denoted by \( \theta_\ast \). Furthermore, \( \|\theta_n - \theta_\ast\| = o_p(1) \) for some \( \theta_\ast \in \Theta_\ast \).

Consider the case where \( M(\theta) \) has two unique zeros \( \theta_1^1 \) and \( \theta_1^2 \), and assume that \( -H(\theta_1^1) \) and \( -H(\theta_1^2) \) are positive-definite. Since both values satisfy the sufficient conditions in Lemma 3.1, it follows that \( \theta_n^1 = \theta_1^1 + o_p(1) \) and \( \theta_n^2 = \theta_1^2 + o_p(1) \). Consequently, the BSL posterior will assign non-vanishing probability mass to both points.

The above discussion clarifies that in order to theoretically analyze the behavior of the BSL posterior in misspecified models, we must restrict our attention to a local region around a given root of \( 0 = M_n(\theta) \). We analyze the behavior of the BSL posterior under the following regularity additional conditions.

Assumption 3.4. For any \( \theta_\ast \in \Theta_\ast \), and some \( \delta > 0 \), for all \( \|\theta - \theta_\ast\| \leq \delta \), there exists a \( K > 0 \) such that

\[
\|\partial^2 \ln g_n(S_n|\theta)/\partial \theta_i \partial \theta_j - \partial^2 \ln g_n(S_n|\theta_\ast)/\partial \theta_i \partial \theta_j\| \leq K\|\theta - \theta_\ast\|, \quad i, j = 1, \ldots, d_\theta.
\]

Assumption 3.5. Let \( A_n(\theta) \) denote either \( \Sigma_n(\theta) \) or \( \Sigma_n(\theta) \). For some \( \delta > 0 \), any \( \theta_\ast \in \Theta_\ast \), and all \( \|\theta - \theta_\ast\| \leq \delta \), the sequence of matrices \( A_n(\theta) \) satisfy: (i) for all \( n \) large enough, there exists constants \( c_1, c_2 \), such that \( 0 < c_1 \leq \|nA_n(\theta)\| \leq c_2 < \infty \); (ii) there exists a matrix function \( \Sigma(\theta) \), continuous and positive-definite for all \( \theta \in \Theta \), such that \( \sup_{\theta \in \Theta} \|nA_n(\theta) - \Sigma(\theta)\| = o_p(1) \).

Assumption 3.6. For \( \theta \in \Theta_\ast \), \( \pi(\theta) \geq 0 \), and \( \pi(\cdot) \) is continuous on \( \|\theta - \theta_\ast\| \leq \delta \), for some \( \delta > 0 \) and all \( \theta_\ast \in \Theta_\ast \).

Assumption 3.7. For all \( \theta \in \Theta \), \( \mathbb{E} \left[ N\{S_n; \bar{b}_n(\theta), \Sigma_n(\theta)\} \mid \theta, S_n \right] = g_n(S_n|\theta) \{1 + O(m^{-1})\} \).

Remark 3.2. With the exclusion of Assumptions 3.1 and 3.3, the Assumptions are similar to those employed by Frazier et al. (2021) to deduce a Bernstein-von Mises result for the BSL posterior in correctly specified models. The strengthening of these assumption seems necessary to simplify the technicalities that arise in the case of model misspecification. Assumption 3.7 requires that the estimated SL \( N\{\cdot; \bar{b}_n(\theta), \Sigma_n(\theta)\} \) is an asymptotically unbiased estimator of the exact SL \( g_n(\cdot|\theta) \) as \( m \) diverges. This condition is stronger than the assumption required by Frazier et al. (2020b) to demonstrate concentration of the ABC posterior in misspecification models.\(^7\)

To simply state the main result of this section, let \( \Delta := \{-H(\theta_\ast)\}^{-1} \), and \( t := \sqrt{n}(\theta - \theta_n) \).

Theorem 3.1 (Asymptotic shape of the posterior). Under Assumptions 3.1-3.7, for any finite \( \gamma > 0 \), and for some density function \( q_\gamma(t) \propto N\{t; 0; \Delta\} \), for \( m \to \infty \) as \( n \to \infty \),

\[
\left| \int_{\|t\| \leq \gamma} \pi(t|S_n) - \int_{\|t\| \leq \gamma} q_\gamma(t)dt \right| = O_p(1/m).
\]

\(^7\)The results that follow are likely to be satisfied under weaker assumptions, but would require more technical arguments and do not necessarily lead to any further interesting implications.
Theorem 3.1 demonstrates that around the mode $\theta_*$, on sets of the form $\{ t : a \leq t \leq b \}$, where $a, b \in \Theta$, $a \leq b$ (element-wise), the BSL posterior measure for $t$ is proportional to the normal probability $\int_a^b N\{ t; 0; \Delta \} \, dt$ in large samples. The constant of proportionality that equates the two probabilities depends on the behavior of $\pi(\theta_*|S_n)$. Consider again that $\Theta_*$ contains only $\theta_1^*$ and $\theta_2^*$. Under the above assumptions, as shown in Lemma A.2 in the appendix, $C_\pi^{ij} := \text{plim} \pi(\theta_*|S_n)/\sqrt{n}$ exists and satisfies $0 < C_\pi^{ij} \leq (2\pi)^{-d_0/2}|\Delta_j|^{-1/2}$, where $\Delta_j := \{-H(\theta_*^j)\}^{-1}$ for $j = 1, 2$. In this case, for $n$ large, in shrinking neighborhoods of $\Theta$ that contain $\theta_1^*$ (resp., $\theta_2^*$) the BSL posterior density resembles $C_\pi^{1i} N\{ t; 0; \Delta_1 \}$ (resp., $C_\pi^{2i} N\{ t; 0; \Delta_2 \}$).

**Remark 3.3.** The “fractional” normality result in Theorem 3.1 is a consequence of the way the SL measures the discrepancy between the observed and simulated summary statistics. In particular, BSL measures the discrepancy between $S_n$ and $\bar{b}_n(\theta)$ in a relative fashion, where relative is defined in terms of $\bar{S}_n(\theta)$-units, so that high probability mass is assigned to values of $\theta$ that make $\| \bar{S}_n(\theta)^{-1/2} \{ S_n - \bar{b}_n(\theta) \} \|^2$ small. Given this feature, there is no reason to suspect that the BSL posterior will concentrate onto a value of $\theta$ that minimizes the absolute magnitude $\| S_n - b(\theta) \|$. This is in contrast to ABC, where Frazier et al. (2020b) demonstrate that if the model is misspecified, so long as the tolerance sequence in ABC is chosen in a reasonable fashion, then the ABC posterior will concentrate onto the value of $\theta \in \Theta$ that asymptotically minimizes $\| S_n - b(\theta) \|$.

### 3.2 Asymptotic Behavior: Single Mode

In contrast to the asymptotic behavior discussed above, if $\theta_*$ is the unique solution to the limiting score equations $0 = M(\theta)$ that satisfies $\lambda_{\min}\{-H(\theta_*)\} > 0$, then the BSL posterior will be approximately Gaussian in large samples. To demonstrate this result, we reinforce Assumption 3.3 and impose the following additional assumption.

**Assumption 3.3’.** $\Theta_* = \{ \theta_* \}$, and for some $\delta > 0$, $0 < \delta \leq \lambda_{\min}\{-H(\theta_*)\} \leq 1/\delta$.

**Assumption 3.8.** For $\theta_* \in \Theta_*$, and $W_* := \{ \nabla b(\theta_*)^T \Sigma^{-1}(\theta_*) V \Sigma^{-1}(\theta_*) \nabla b(\theta_*) \}$, $\sqrt{n} M_*(\theta_*) \Rightarrow N(0, W_*)$.

Define the set $T_n = \{ t : t = \sqrt{n}(\theta - \theta_0), \theta \in \Theta \}$.

**Proposition 3.1** (Bernstein-von Mises). Assume Assumptions 3.1-3.3’, and Assumptions 3.4-3.8 are satisfied, then, for $m \to \infty$ as $n \to \infty$,

$$\int_{T_n} \| t \| \, |\pi(t|S_n) - N\{ t; 0, \Delta \} | \, dt = O_p(1/m).$$

Theorem 3.1 demonstrates that even though the model is misspecified, if the SL has a single mode, then the BSL posterior density resembles a shrinking Gaussian density in large samples. Surprisingly, this behavior of the BSL posterior is in contrast to the behavior exhibited by the ABC posterior under model misspecification. Let $\theta_0$ denote the minimizer of $\| b(\theta) - b_0 \|$ and let $\epsilon_0 = \| b(\theta_0) - b_0 \|$. If the
ABC tolerance \( \epsilon_n \) satisfies \( \sqrt{n}(\epsilon_n - \epsilon_0) \to 0 \), then Theorem 2 in Frazier et al. (2020b) demonstrates that the ABC posterior converges to the following density: for \( x = n^{1/4}(\theta - \theta_0) \), and \( A = \Sigma(\theta_0)^{-1/2} \),

\[
q(x) \propto N\left\{ -\frac{[A\sqrt{n}(S_n - b_0)]^T [A\{b(\theta_0) - b_0]\epsilon_0]}{\|A\{b(\theta_0) - b_0]\epsilon_0\|} - \frac{x^T L(\theta_0)x}{4 \|A\{b(\theta_0) - b_0]\epsilon_0\|}; 0, I_{d_0}\right\},
\]

where \( L(\theta_0) \) denotes the Hessian of \( \|b(\theta) - b_0\|^2 \) evaluated at \( \theta = \theta_0 \). Comparing the limiting ABC and BSL posteriors, it is abundantly clear that the two methods produce significantly different inferences in misspecified models. This divergence between the BSL and ABC posterior in misspecified models is in contrast to the case of correctly specified models, where the two methods display the same behavior in large samples (Frazier et al., 2021).

The following result presents the asymptotic behavior of the BSL posterior mean in the case where \( \Theta_* \) is a singleton.

**Corollary 3.1.** Let \( \overline{\theta}_n \) be the Bayesian synthetic likelihood posterior mean based on \( \overline{b}_n(\theta) \) and \( \overline{n}(\theta) \). If the conditions in Proposition 3.1 are satisfied, then

\[
\sqrt{n}(\overline{\theta}_n - \theta_*) \Rightarrow N(0, \Delta W_* \Delta^\top), \text{ under } P_0^{(n)}.
\]

The proof of Corollary 3.1 demonstrates that the BSL posterior mean is approximately Gaussian in large samples. However, it is important to note that a similar result will not be in evidence when the posterior does not concentrate onto a single point.

**Remark 3.4.** The result of Theorem 3.1 implies that the width of posterior credible sets is determined by \( \Delta \). In contrast, Corollary 3.1 implies that the asymptotic variance of the BSL posterior mean is \( \Delta W_* \Delta^\top \). Moreover, the matrix \( \Delta \) directly depends on the level of model misspecification, via \( \{b(\theta_*) - b_0\} \), and so if \( b(\theta_*) \neq b_0 \), we can immediately conclude that

\[
\int_{\mathbb{T}_n} t t^\top \overline{\pi}(t|S_n) dt = \Delta + o_p(1) \neq \text{Var}\{\sqrt{n}(\overline{\theta}_n - \theta_*)\} = \Delta W_* \Delta^\top.
\]

Consequently, the BSL posterior does not deliver asymptotically valid uncertainty quantification for \( \theta_* \) in misspecified models.

### 4 Robust BSL

In this section, we compare different approaches for ameliorating the performance of BSL in misspecified models.
4.1 Tempered/Coarsened BSL

To obtain robustness to possible model misspecification, several authors, including, Grünwald et al. (2017), Bissiri et al. (2016), and Miller and Dunson (2019), have proposed to temper or coarsen the likelihood used within Bayesian inference. Given that the SL is based on a Gaussian likelihood approximation, it is tempting to consider the application of such a strategy to correct the behavior of BSL under model misspecification.

Let $\alpha \geq 0$ denote some (potentially unknown) positive constant. Then the common approach to tempering would consider the following version of the SL

$$g_{n}^{\alpha}(S_n|\theta) := \int N\{S_n; \bar{b}_{n}(\theta), \bar{\Sigma}_{n}(\theta)\}^{\alpha} \prod_{i=1}^{m} dP^{(\alpha)}\{S(z_{i})\} dS(z_{1:n}) \ldots dS(z_{m:n}),$$

which yields the posterior distribution associated to $g_{n}^{\alpha}(\cdot|\theta)$:

$$\pi_{\alpha}(\theta|S_n) = \frac{g_{n}^{\alpha}(S_n|\theta)\pi(\theta)}{\int_{\Theta} g_{n}^{\alpha}(S_n|\theta)\pi(\theta)d\theta}.$$

The results of Bhattacharya et al. (2019) suggest that, in the case of a genuine likelihood, so long as $\alpha \in (0, 1)$, the tempered likelihood will still display posterior concentration. However, as the following example demonstrates, while such behavior may be valid for a genuine likelihood, the same is not true for a SL.

Example: Moving Average Model

We return to the moving average example and examine the behavior of the tempered BSL posterior in the MA(1) model. Since the BSL posterior can be computed exactly, so can the tempered version. We apply the tempered version of BSL using a fixed tempering schedule with $\alpha = 1/2$ for each value of $n$. Following the introductory example, we plot the tempered BSL posterior for $n = 100, 500, 1000$ and compare the results to those obtain in Figure 1.\(^8\)

Figure 3 demonstrates that the tempered BSL posterior displays similar behavior to the exact BSL posterior in Figure 1, and does not lead to any noticeable increase of posterior mass in the region of $\theta = 0$, the point under which $\|b(\theta) - b_0\|$ is smallest. This result is perhaps unsurprising considering that the SL is Gaussian, and so tempering only changes the scaling of the posterior and does not alter either its modes or overall shape.

\(^8\)The choice of $\alpha = 1/2$ is in accordance with the theoretical results of Bhattacharya et al. (2019), however, the results displayed in Figure 3 are not overly sensitive to this choice.
4.2 Robustifying BSL

As the running example has concretely illustrated, in cases where the model is significantly misspecified, and due to the nature of \( \ln g_n(S_n|\theta) \), the inference problem can become ill-posed: the population nonlinear SL score equations

\[
\frac{\partial}{\partial \theta} \| \Sigma(\theta)^{-1/2} \{ b(\theta) - b_0 \} \|^2 = 0,
\]

can exhibit multiple solutions. Critically, these solutions need not coincide with the global minimizer of \( \| b(\theta) - b_0 \| \). This can result in a multi-modal posterior that places mass in regions of \( \Theta \) where \( b(\theta) \) and \( b_0 \) are significantly different.

This problem exists because the BSL posterior assigns high probability to values of \( \theta \) that ensure the relative difference between \( S_n \) and \( \hat{b}_n(\theta) \), measured in \( \Sigma_n(\theta) \)-units, is small. This can be seen by noting that, for \( n \) large, the dominant term in \( -\ln g_n(S_n|\theta) \) is the quadratic form \( \| \Sigma_n(\theta)^{-1/2} \{ \hat{b}_n(\theta) - S_n \} \|^2 \), which is a Mahalanobis distance. Measuring differences between summary adequacy using a “relative” distance, rather than an absolute distance such as \( \| \hat{b}_n(\theta) - S_n \| \), means that there can exist values of \( \theta \) such that \( \| \hat{b}_n(\theta) - S_n \| \) is large, while \( \| \Sigma_n(\theta)^{-1/2} \{ \hat{b}_n(\theta) - S_n \} \|^2 \) is small. With the above realization, there are several approaches for correcting this behavior. For brevity, we focus on two, leaving a detailed comparison and discussion on alternative approaches for future research.

4.2.1 Robust BSL

The first approach we detail for correcting the issues with BSL in misspecified models is the robust BSL (r-BSL) approach presented in Frazier and Drovandi (2021).\(^9\) This approach seeks to account for model misspecification by sufficiently altering the weighting matrix used in BSL to ensure that the magnitude

\(^9\)For simplicity, we only focus on the variance adjustment approach detailed in Frazier and Drovandi (2021), but note that the mean adjustment could also be used.
of $\|\tilde{b}_n(\theta) - S_n\|$ is properly taken into account. For $\Gamma = (\gamma_1, \ldots, \gamma_d)'$ denoting a $d$-dimensional random vector with support $\mathcal{G}$, define the regularized BSL covariance matrix

$$\Sigma_n(\theta, \Gamma) := \Sigma_n(\theta) + \Sigma_n^{1/2}(\theta)\text{diag}\{\gamma_1, \ldots, \gamma_d\}\Sigma_n^{1/2}(\theta).$$

Let

$$\mathcal{g}_n(S_n|\theta, \Gamma) := \int N\{S_n; \tilde{b}_n(\theta), \Sigma_n(\theta, \Gamma)\} \prod_{i=1}^m dP_{\theta}(n) d\{S_n(z_{1:n}^i)\} \ldots d\{S_n(z_{1:n}^m)\}$$

denote the SL based on $\Sigma_n(\theta, \Gamma)$. For $\pi(\Gamma)$ denoting the prior density of $\Gamma$, Frazier and Drovandi (2021) use independent exponential priors, the joint BSL posterior is

$$\pi(\theta, \Gamma|S_n) = \frac{\pi(\theta)\pi(\Gamma)\mathcal{g}_n(S_n|\theta, \Gamma)}{\int_{\Theta \times \mathcal{G}} \mathcal{g}_n(S_n|\theta, \Gamma)\pi(\theta)\pi(\Gamma) d\theta d\Gamma},$$

and MCMC methods can be used to sample $\pi(\theta, \Gamma|S_n)$. As well as delivering robust inference in the context of model misspecification, Frazier and Drovandi (2021) demonstrate that this r-BSL approach allows the user to disentangle which summaries cannot be matched by the assumed model.

The additional parameters $\Gamma$ in the r-BSL posterior allow for larger variances in the SL than are permitted in the standard case, and compensates for the fact that there may be no value in $\Theta$ under which $\|\tilde{b}_n(\theta) - S_n\|$ can be made small. In these cases, the “adjustment parameters” $\Gamma$ turn on and enlarge the covariance matrix used in BSL so that the overall weighted norm can still be made small.

**Example: Moving Average Model**

We now compare the behavior of the r-BSL posterior under different levels of model misspecification in the simple MA(1) example. Following the example in Section 2.2, we consider three sample sizes of $n = 100, 500, 1000$ and obtain the r-BSL posterior via the slice sampling MCMC approach presented in Frazier and Drovandi (2021).\(^\text{10}\) The procedure is implemented using the BSL package in \textit{R} (An et al., 2019), with the default prior choice for $\Gamma$. We plot the r-BSL posterior across these values in Figure 4, and compare these results with those obtained for the BSL posterior in Figure 2.

Figure 4 demonstrates that the posteriors are roughly Gaussian and concentrating around the posterior mode of $\theta = 0$, with the r-BSL posterior being insensitive to the level of model misspecification. This behavior is due to the regularization of the covariance matrix, which ensures the SL criterion is globally concave and achieves its maximum at $\theta = 0$.

\(^\text{10}\) We start the sampler at $\theta = 0$ and retain all resulting draws. In addition, we run the sampler for 50,000 iterations and use 10 synthetic datasets for each replication. These choices are fixed across the different sample size and misspecification combinations. The acceptance rates for the resulting procedure are reasonable, and between 20% and 60% across all combinations.
4.2.2 A Robust Adjustment Approach

While Frazier and Drovandi (2021) demonstrate that the r-BSL approach delivers reliable inference even in highly-misspecified models, it requires conducting posterior inference over $d_{\theta} + d$ (where $d \geq d_{\theta}$) elements, which can become cumbersome in cases where either $\theta$ or $S_n$ is high-dimensional. However, the key insight of Frazier and Drovandi (2021) in regards to misspecification is that it can be handled by sufficiently altering the structure of the SL.

An alternative approach to deal with model misspecification in the case of high-dimensional summaries, or parameters, is to replace the SL variance matrix $\Sigma_n(\theta)$ with a naive but fixed version $\Delta_n$. Replacing $\Sigma_n(\theta)$ by the fixed matrix $\Delta_n$ means that the log SL is roughly a quadratic form based on a fixed weighting matrix, and thus will generally produce a uni-modal posterior. As Proposition 3.1 demonstrates, if this naive posterior is indeed uni-modal, then it will be approximately Gaussian in large samples, but with a covariance matrix that depends on the choice of $\Delta_n$. However, the posterior variance can be adjusted to ensure it attains the correct level of frequentist coverage.

The coverage of this naive BSL posterior can be adjusted using the procedure developed in Frazier et al. (2021). We can describe such an adjustment approach using the following steps.

1. Take $\Sigma_n(\theta) = \Delta_n$, for all $\theta \in \Theta$, as the covariance matrix in BSL and obtain the corresponding naive BSL posterior mean, $\bar{\theta}_n$, and its covariance $\Omega_n$.

2. For $\theta^j$, $j = 1, \ldots, N$, denoting a sample from the above naive BSL posterior, adjust the values of
\( \theta^j \) according to

\[
\hat{\theta}^j = \hat{\theta}_n + \overline{\Omega}_n \hat{W}_n \Sigma_n^{-1/2} (\theta^j - \hat{\theta}_n),
\]

where \( \hat{W}_n \) is any consistent estimator of the asymptotic variance of \( \lim \text{Var}[\sqrt{n} M_n(\theta_0)]. \)

Using the naive BSL posterior ensures that the posterior concentrates mass on values in \( \Theta \) under which \( \| b(\theta) - S_n \| \) is small. However, this resulting posterior will not have valid frequentist coverage; the credible sets may over or under-cover the pseudo-true value, i.e., the value of \( \theta \in \Theta \) that minimizes \( \| b(\theta) - b_0 \| \). Therefore, in the second step we adjust the coverage of this posterior by adjusting the corresponding posterior draws. Since the model is misspecified, the most reliable estimator of \( \lim \text{Var}[\sqrt{n} M_n(\theta_0)] \), would likely be one obtained via a bootstrapping approach. In this way, we can interpret the above adjusted BSL posterior as being similar to the “BayesBag” posteriors of Huggins and Miller (2019), but in the specific context of BSL. Moreover, unlike the BayesBag approach, this adjustment does not require the user to re-estimate the posterior for every bootstrap/bagged sample, but only the variance of the summary statistics. From a practical standpoint, this simplification is crucial since re-estimating the posterior in complex models is cumbersome and time consuming, whereas estimating the variance of the summaries via bootstrap is much simpler given their lack of dependence on the parameters.

**Example: Moving Average Model**

We now compare the behavior of the adjusted BSL posterior described above in the misspecified MA(1) example. However, before doing so we note that the adjusted posterior will not significantly shift location as the level of model misspecification changes. Therefore, to examine the behavior of the adjusted BSL posterior, we consider a repeated sampling approach to ascertain the level of coverage and behavior of the adjustment approach across a large degree of model misspecification. For the naive BSL posterior, we set \( \Delta_n = n^{-1} I_d \) and compute the naive posterior analytically using our previous results for the standard BSL posterior.

In particular, we generate five hundred data sets from the SV model (2), where the parameter values are \( \omega = -0.736, \rho = 0.90 \) and \( \sigma_v = 0.36 \), and with \( n = 100, 500, 1000 \) observations. The adjusted posterior is obtained by calculating the exact naive BSL posterior, with identity covariance matrix, and then adjusting 10,000 samples from the naive BSL posterior. The variance term \( \hat{W}_n \) used in the adjustment is estimated via the block bootstrap with a block size of 10 and using 1,000 bootstrap samples.

In Table 1 we record the mean, variance and Monte Carlo coverage for both the adjusted and naive approaches, and across each of the three samples sizes. For presentation purposes, the reported means

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11 Frazier et al. (2021) propose several approaches to estimate \( \lim \text{Var}[\sqrt{n} M_n(\theta_0)] \), including in situations where the assumed model may not be correctly specified.

12 This follows since the location of the adjusted BSL posterior is constructed to coincide with that of the naive BSL posterior, which is determined by \( \| b(\theta) - b_0 \| \). Moreover, it can be easily verified that the argument minimizer of \( \| b(\theta) - b_0 \| \) remains constant as the level of model misspecification changes.
have been multiplied by $10^3$. The results demonstrate that both approaches are precise estimators for the location of the pseudo-true value, $\theta = 0$, with the adjusted approach having a smaller posterior variance across all sample sizes. In terms of Monte Carlo coverage, both procedures display over-coverage for the unknown pseudo-true value. Therefore, given the tighter posteriors for the adjusted approach, and similar posterior means, we conclude that the adjusted approach is more accurate than the naive BSL approach.

Table 1: Summary measures of posterior accuracy, calculated as averages across the replications. Mean - posterior mean multiplied by $10^3$, Var - posterior variance, COV - Monte Carlo coverage. n-BSL refers to naive BSL, and a-BSL refers to adjusted BSL.

|          | n=100 | n=500 | n=1,000 |
|----------|-------|-------|---------|
| a-BSL    | -0.5691 | -0.1955 | -0.9916 |
| n-BSL    | 0.0863 | 0.0191 | 0.0191 |
| Mean     | 0.3322 | 0.0197 | 0.0107 |
| Var      | 0.0107 | 0.0334 | 0.0334 |

\[
Q(p; A, B, g, k) = A + B \left\{ 1 + c \tanh[gz(p)/2] \right\} z(p) \left[ 1 + z(p)^2 \right]^k, \quad p \in (0, 1),
\]

where $z(p)$ is the quantile function of the standard normal distribution, and the constant $c$ is conventionally fixed at $0.8$, which results in the constraint $k > -0.5$. Bayesian inference for the $g$-and-$k$ model was considered in Allingham et al. (2009), where it was noted that the closed form quantile function allows easy simulation from the model using the inversion method, making likelihood-free inference methods attractive.

In this example a dataset modelled using the $g$-and-$k$ distribution by Prangle (2020) is considered. The data are available in the R package Ecdat (Croissant and Graves, 2020) and following Prangle (2020) we consider the daily log returns for exchange rates of the US versus Canadian dollar. There are 1867 observations over the period 1980 to 1987. We fix $k = 0$ in the $g$-and-$k$ model to induce clear misspecification, with the lack of kurtosis resulting in the inability to capture the heavy-tailed behaviour of the real returns data.

Our initial focus is to explore a suggestion made by Müller (2013) in the context of sandwich-type variance adjustments for Bayesian inference under misspecification. Müller (2013) considers adjustments in which, under correct model specification, a sandwich-type variance estimate and the asymptotic
posterior variance estimate should be approximately equal. Section 4.4 of Müller suggests using some summary of the difference between estimates as a diagnostic for misspecification. We do something similar. Consider the adjusted BSL method of Section 4.2.2, but fixing the summary statistic covariance matrix to be the value for the estimated posterior mean for $\theta$ under standard BSL. Under correct specification, and from the results of Corollary 3.1, using this fixed covariance matrix estimate, the adjusted BSL should result in the same inferences asymptotically as the ordinary BSL. So a large adjustment could be considered evidence for misspecification.

Let $Q_1, Q_2, Q_3$ be the three quartiles of the data $y = (y_1, \ldots, y_n)$. We define summary statistics $S_1 = Q_2$, $S_2 = (Q_3 - Q_1)$ and $S_3 = (Q_3 - 2Q_2 + Q_1)/S_2$. These are three of four robust summary statistics considered in Drovandi and Pettitt (2011) for the $g$-and-$k$ model. We also define a fourth summary statistic $S_4$ as the lower $1$ percentage quantile of the data, which captures the extreme negative returns.

Since we fix $k = 0$, we have a three parameter family of distributions which has flexible behaviour in terms of location, scale and skewness. We use priors which are independent and uniform over ranges $[-1, 1], [0, 1]$ and $[-5, 5]$ for $A$, $B$ and $g$ respectively. Using the summary statistics $S^{(1)} = (S_1, S_2, S_3)$, there is no incompatibility, but with summary statistics $S^{(2)} = (S_1, S_2, S_3, S_4)$ there is. The reason is that capturing the location, scale and skewness evident in the first three summary statistics while simultaneously matching the lower tail behaviour specified through $S_4$ is not possible when $k = 0$.

Figure 5 shows the BSL posterior estimates based on summary statistics $S^{(1)}$, together with the adjusted BSL posterior estimates. The top row shows univariate posterior marginals, and the bottom row shows bivariate posterior marginals. The values denoted KLDN above the plots of the univariate marginals are the Kullback-Leibler divergence between normal approximations to the unadjusted and adjusted posterior densities, where the normal approximations are based on posterior means and standard deviations for each method. Precisely,

$$
\text{KLDN} = \log \frac{\sigma_A}{\sigma_S} + \frac{\sigma_S^2 + (\mu_S - \mu_A)^2}{2\sigma_A^2} - \frac{1}{2},
$$

where $\mu_A, \sigma_A$ and $\mu_S, \sigma_S$ are the estimated mean and standard deviation for the adjusted BSL and standard BSL respectively. The KLDN allows us to measure the overall change of the posterior marginals after adjustment.

The BSL and adjusted BSL estimates are based on 80,000 iterations of a random walk Metropolis algorithm with 10,000 burn-in and $m = 60$ simulations per likelihood estimate, with 1,000 samples retained after thinning. Even though incompatibility is not an issue for the summary statistics $S^{(1)}$, there is a substantial adjustment to the posterior marginal distributions. This seems to be due to the misspecification of the synthetic likelihood variance.

To confirm this finding, we estimate the variance of the summaries for the observed data using the bootstrap and compare this estimate against the bootstrap estimate of the variance that results from 1,000 posterior predictive replicates of the data based on the standard BSL posterior. The bootstrap estimates
Figure 5: Estimated BSL posterior densities for the summary statistic vector $S^{(1)}$ using standard BSL and adjusted BSL for the US-Canadian exchange rate data and the $g$-and-$k$ model with $k = 0$. The top row shows univariate marginals (black=standard BSL, green=adjusted BSL). The KLDN values are described in the text and summarize how much the posterior changes after adjustment. The bottom row shows bivariate marginals. The contours show the standard BSL and the points are adjusted BSL sample values based on 1,000 samples.

of variance for the summary statistics for the observed data are much smaller than the bootstrap variance estimates for the posterior predictive replicates, particularly for $S_3$. This indicates that the variance used in the BSL posterior can not reasonably accommodate the actual variance of the summaries.

Figure 6 shows the BSL posterior estimates based on the summary statistics $S^{(2)}$, which is the case of incompatibility where the summary statistic $S_4$ capturing the tail behaviour is added to $S^{(1)}$. There is a large change in the estimated posterior for $g$, and larger adjustments are being made for both $A$ and $g$ compared to the previous case, which can be seen both graphically and from the KLDN values. The large change again suggests possible misspecification. In Appendix A.2.2 posterior predictive checks using the summary statistics themselves as the discrepancy are considered, and demonstrate the incompatibility of the summary statistics in this case. Although checks based on near-sufficient summary statistics can lead to conservative checks, the model misspecification is quite evident in this example.

The above analysis demonstrates that if the model is not correct, large adjustments can arise from either summary statistic incompatibility (i.e., misspecification), or from misspecification of the summary statistic covariance under the assumed model. Indeed, in Appendix A.2.2 we demonstrate that the differences between the standard and adjusted BSL posteriors are much less stark if the observed data is simulated from the $g$-and-$k$ model. However, as is true for standard BSL, even if the model is correctly specified, the adjustment to the synthetic likelihood could also be large if the summary statistics are non-Gaussian. In addition, since the adjustment is based on asymptotic arguments, large differences could also be observed if the adjustments are unreliable in finite samples. We conclude that while the existence

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The results are presented graphically in Appendix A.2.2.
of a large adjustment is suggestive of misspecification, other diagnostics may be needed to diagnose specific features of the model that may be misspecified.

To this end, we demonstrate that r-BSL can be used as a diagnostic to pinpoint which features of the observed data cannot be matched by the assumed model. For the prior on $\Gamma$, we use an exponential distribution with a mean of 0.5 on each component and assume the components are independent, as suggested by Frazier and Drovandi (2021). We apply standard BSL with $S^{(1)}$ and $S^{(2)}$, and r-BSL with $S^{(2)}$. As above, we use the BSL R package of An et al. (2019) for running the BSL methods. We use 100,000 iterations of MCMC for each run of BSL, and use a starting value with good support under each approximate posterior to avoid the need for a burn-in.

Figure 7a shows the posterior predictive distribution for each component of $S^{(1)}$ when fitted with standard BSL. As suggested earlier, the model is compatible with these three statistics. The corresponding plot for standard BSL with $S^{(2)}$ is shown in Figure 7b. It is evident that the model is not able to recover the four statistics. By trying to match the four statistics simultaneously, the model is unable to recover any of the statistics with high accuracy, particularly $S_3$.

The posterior distribution of $\Gamma$ when using r-BSL is shown in Figure 8. The r-BSL method suggests that the misspecification/incompatibility is due to the models inability to recover $S_3$. This is evident by the large departure in the posterior distribution of $\gamma_3$ compared to its prior. The posterior predictive distribution of the summary statistics obtained with r-BSL is shown in Figure 7c. By allowing for incompatibility, r-BSL produces a posterior distribution that is able to recover $S_1$, $S_2$ and $S_4$ accurately, whilst placing little emphasis on $S_3$.

Furthermore, there is a computational benefit of r-BSL. Using only $m = 30$, r-BSL produces an MCMC acceptance rate of 21%. In contrast, standard BSL applied to $S^{(2)}$ using $m = 300$ gives an acceptance rate of only 14% with a carefully tuned random walk covariance matrix. This is due to the fact that the observed statistic always lies in tail of the model summary statistic distribution regardless of

Figure 6: Estimated BSL and adjusted BSL posterior densities for the summary statistic vector $S^{(2)}$. Please see Figure 5 for further details.
Figure 7: Posterior predictive distribution of the summary statistics when applying standard and robust BSL to the US-Canadian exchange rate data. The dots show the observed values of the summary statistics.
the value of $\theta$. In contrast, r-BSL introduces variance inflation to adjust the model to be compatible even when it is not.

The univariate posterior distributions of $\theta$ produced from BSL and r-BSL based on $S^{(2)}$ are shown in Figure 9. There is a substantial difference between the posterior distributions. The r-BSL method produces a fit to the data where the model is compatible with $S_1$, $S_2$ and $S_4$, whilst largely ignoring $S_3$. The posterior variance of $g$ is substantially larger with r-BSL, and is consistent with the BSL adjustment results. However, unlike BSL with adjustment, r-BSL can shift the location of the posteriors.

Figure 9: Estimated posterior distributions for the components of $\theta$ when applying BSL (dash) and r-BSL (solid) to the US-Canadian exchange rate data based on $S^{(4)}$.

6 Discussion

Over the last decade, approximate Bayesian methods, such as ABC and BSL, have gained acceptance in the statistical community for their ability to produce meaningful inferences in complex models. The ease with which these methods can be applied has also led to their use in diverse fields of research; see Sisson et al. (2018) for examples.
While the initial impetus for these methods was one of practicality, recent research has begun to focus on the theoretical behavior of these methods. In the context of BSL, Frazier et al. (2021) demonstrate that BSL posteriors are well-behaved in large samples, and can deliver inferences that are just as reliable as those obtained by ABC, assuming the model is correctly specified.

The important message delivered in this paper is that if the assumed model is misspecified, then BSL inference can be unreliable, and the BSL and ABC posteriors can be significantly different. In particular, if the model is misspecified, the BSL posterior can display a wide variety of behavior, e.g., multi-modality, uni-modality, and concentration onto a boundary point of the parameter space. Critically, the type of behavior exhibited by the BSL posterior is intimately related to the form and degree of model misspecification, which cannot be reliably measured without first conducting some form of inference.

While our results have only focused on the most commonly applied variant of the BSL posterior, it is highly likely that recently proposed variations of BSL, such as the semiparametric BSL approach of An et al. (2020) or the whitening BSL approach of Priddle et al. (2019), will exhibit similar behavior.

However, we have also demonstrated that there exist versions of BSL that deliver reliable inferences under model misspecification. The first method we discuss is the robust BSL approach of Frazier and Drovandi (2021), which augments the BSL posterior with adjustment parameters that “soak-up” the model misspecification, and can deliver reliable inferences on the parameters of interest. The second approach is a new two-step BSL approach that resembles the adjustment procedure to BSL inference described in Frazier et al. (2021), but which is specifically targeted at dealing with the issue of model misspecification. While both approaches deliver reliable performance in the examples considered in this paper, a more extensive comparison is needed to determine which method performs best across large classes of examples.

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A Technical Details

**Proof of Lemma 3.1.** The proof of this result is given as an intermediate result in Lemma A.1. The reader is referred to the proof of Lemma A.1 for details.

**Proof of Theorem 3.1.** Define $C_\pi = \operatorname{plim} \pi(\theta_n | S_n) / \sqrt{n}$, which exists and satisfies $C_\pi \geq 0$ by Lemma A.2, and define $Q_*(\gamma) := C_\pi \int_{\|t\| \leq \gamma} \exp \left(-\frac{1}{2} t^T \Delta^{-1} t \right) \, dt$. From the triangle inequality,

$$
\left| \int_{\|t\| \leq \gamma} \{\pi(t | S_n) - Q_*(\gamma)\} \, dt \right| \leq \left| \int_{\|t\| \leq \gamma} \{\pi(t | S_n) - \pi(t | S_n)\} \, dt \right| + \left| \int_{\|t\| \leq \gamma} \pi(t | S_n) - Q_*(\gamma) \, dt \right|
$$

By Lemma A.5, $\int_{\|t\| \leq \gamma} |\pi(t | S_n) - \pi(t | S_n)| \, dt = O_p[1/m(n)]$, so that the first term is $o_p(1)$ for $m(n) \to \infty$ as $n \to \infty$. If we can show that

$$
J_n := \int_{\|t\| \leq \gamma} \pi(t | S_n) - Q_*(\gamma) \, dt = o_p(1),
$$

the result then follows.

For $Q_n(\theta) := -\frac{1}{2} \{b(\theta) - S_n\}^T \Sigma_n^{-1}(\theta) \{b(\theta) - S_n\}$, express the exact BSL posterior as

$$
\pi(\theta | S_n) = \frac{\pi(\theta) |\Sigma_n(\theta)|^{-1/2} \exp \{Q_n(\theta)\}}{\int_{\Theta} \pi(\theta) |\Sigma_n(\theta)|^{-1/2} \exp \{Q_n(\theta)\} \, d\theta} = \pi(\theta_n | S_n) [1 / |\Sigma_n(\theta_n)|]^{-1/2} \exp \{Q_n(\theta) - Q_n(\theta_n)\} \pi(\theta) / \pi(\theta_n).
$$

(6)

To simplify notation, denote

$$
f_n(\theta) = |n\Sigma_n(\theta)|^{-1/2} \text{ and } f(\theta) = |\Sigma(\theta)|^{-1/2}.
$$

For any $\gamma_n := \gamma / \sqrt{n} = o(1)$ define $N_{\gamma} := N(\theta_n; \gamma_n) = \{\theta \in \Theta : \|\theta - \theta_n\| \leq \gamma_n\}$, and decompose the posterior probability over this set as

$$
\int_{N_{\gamma}} \pi(\theta | S_n) \, d\theta = \pi(\theta_n | S_n) \int_{N_{\gamma}} \exp \{Q_n(\theta) - Q_n(\theta_n)\} \, d\theta \\
+ \pi(\theta_n | S_n) \int_{N_{\gamma}} \exp \{Q_n(\theta) - Q_n(\theta_n)\} \left[ \frac{\pi(\theta)}{\pi(\theta_n)} - 1 \right] \, d\theta \\
+ \pi(\theta_n | S_n) \int_{N_{\gamma}} \exp \{Q_n(\theta) - Q_n(\theta_n)\} \left[ f_n(\theta) / f_n(\theta_n) - 1 \right] \, d\theta \\
+ \pi(\theta_n | S_n) \int_{N_{\gamma}} \exp \{Q_n(\theta) - Q_n(\theta_n)\} \left[ f_n(\theta) / f_n(\theta_n) - 1 \right] \left[ \frac{\pi(\theta)}{\pi(\theta_n)} - 1 \right] \, d\theta \\
= \pi(\theta_n | S_n) \int_{N_{\gamma}} \exp \{Q_n(\theta) - Q_n(\theta_n)\} d\theta + C_1 + C_2 + C_3 + C_4
$$

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From the consistency of $\theta_n$ for $\theta_*$, we have $\pi(\theta_n) = \pi(\theta_*) + o_p(1)$, with $\pi(\theta_*) > 0$ by Assumption 3.6. Also, by Assumption 3.5, $0 < f_n(\theta) < \infty$ for $n$ large enough. Using this, we can upper bound $C_{jn}$, $j = 1, 2, 3$, as follows:

\[
\begin{align*}
C_{1n} & \leq C \int_{N_\gamma} \pi(\theta_n|S_n) \exp \{Q_n(\theta) - Q_n(\theta_n)\} \, d\theta \int_{N_\gamma} \sup_{\|\theta - \theta_n\| \leq \gamma_n} |\pi(\theta) - \pi(\theta_n)| \, d\theta \\
C_{2n} & \leq C \int_{N_\gamma} \pi(\theta_n|S_n) \exp \{Q_n(\theta) - Q_n(\theta_n)\} \, d\theta \int_{N_\gamma} \sup_{\|\theta - \theta_n\| \leq \gamma_n} |f_n(\theta) - f(\theta)| \, d\theta \\
& \quad + C \int_{N_\gamma} \pi(\theta_n|S_n) \exp \{Q_n(\theta) - Q_n(\theta_n)\} \, d\theta \int_{N_\gamma} \sup_{\|\theta - \theta_n\| \leq \gamma_n} |f(\theta) - f(\theta_n)| \, d\theta \\
C_{3n} & \leq C \int_{N_\gamma} \pi(\theta_n|S_n) \exp \{Q_n(\theta) - Q_n(\theta_n)\} \, d\theta \int_{N_\gamma} \sup_{\|\theta - \theta_n\| \leq \gamma_n} |\pi(\theta) - \pi(\theta_n)||f_n(\theta) - f(\theta)| \, d\theta \\
& \quad + C \int_{N_\gamma} \pi(\theta_n|S_n) \exp \{Q_n(\theta) - Q_n(\theta_n)\} \, d\theta \int_{N_\gamma} \sup_{\|\theta - \theta_n\| \leq \gamma_n} |\pi(\theta) - \pi(\theta_n)||f(\theta) - f(\theta_n)| \, d\theta
\end{align*}
\]

for a given constant $C$ that can change line-by-line.

Over the set $N_\gamma$, by Assumptions 3.5 and the continuous mapping theorem,

\[
\sup_{\theta \in N_\gamma} |f_n(\theta) - f(\theta)| = o_p(1). \tag{7}
\]

By Assumption 3.4, for each $j = 1, \ldots, d_\theta$, $\partial \Sigma(\theta)/\partial \theta_j$ is continuous so that, over the compact set $N_\gamma$, there exists a finite $C$ and an intermediate value $\bar{\theta}$ such that

\[
\sup_{\|\theta - \theta_n\| \leq \gamma_n} |f(\theta) - f(\theta_n)| \leq \sup_{\|\bar{\theta} - \theta_n\| \leq \gamma_n} \|\partial f(\bar{\theta})/\partial \gamma_n\| \leq C_{\gamma_n} = o(1). \tag{8}
\]

Likewise, by continuity of $\pi(\theta)$ and compactness

\[
\sup_{\|\theta - \theta_n\| \leq \gamma_n} \|\pi(\theta) - \pi(\theta_n)\| = o_p(1). \tag{9}
\]

Since $\pi(\theta), f(\theta)$ are continuous, they are bounded over $N_\gamma$ and, by Assumption 3.5, $f_n(\theta)$ is bounded for $n$ large enough. Therefore, equations (7)-(9) imply that

\[
C_{jn} = o_p \left[ \pi(\theta_n|S_n) \int_{N_\gamma} \exp \{Q_n(\theta) - Q_n(\theta_n)\} \, d\theta \right],
\]

so that

\[
\int_{N_\gamma} \pi(\theta|S_n) \, d\theta = \pi(\theta_n|S_n) \{1 + o_p(1)\} \int_{N_\gamma} \exp \{Q_n(\theta) - Q_n(\theta_n)\} \, d\theta. \tag{10}
\]

Now, consider the change of variables $\theta \mapsto t = \sqrt{n}(\theta - \theta_n)$, and note that $\mathcal{N}(\theta_n; \gamma_n) \equiv \{t : \|t\| \leq \gamma\}$,
which yields
\[
\int_{\mathcal{N}(\theta_n; \gamma_n)} \pi(\theta \mid S_n) \, d\theta = \int_{\|t\| \leq \gamma} \pi(\theta_n + t/\sqrt{n} \mid S_n) \frac{1}{\sqrt{n}} \, dt \\
= \frac{\pi(\theta_n \mid S_n)}{\sqrt{n}} \{1 + o_p(1)\} \int_{\|t\| \leq \gamma} \exp \{Q_n(\theta_n + t/\sqrt{n}) - Q_n(\theta_n)\} \, dt,
\]
where the second equality follows from equation (10). Using the expression in (6), we have
\[
J_n \leq \frac{\pi(\theta_n \mid S_n)}{\sqrt{n}} \{1 + o_p(1)\} \left\| \int_{\|t\| \leq \gamma} \left[ \exp \{Q_n(\theta_n + t/\sqrt{n}) - Q_n(\theta_n)\} - \exp (-t^T \Delta^{-1} t/2) \right] \, dt \right\| \\
+ \frac{\pi(\theta_n \mid S_n)}{\sqrt{n}} - C \pi \left\| \int_{\|t\| \leq \gamma} \exp (-t^T \Delta^{-1} t/2) \, dt \right\|.
\]
The integral of the second term is finite for any \( \gamma > 0 \), so that by Lemma A.2, the second term can be dropped from the analysis.

For \( Z_n := \sqrt{n} \{b_0 - S_n\} \), let \( \Omega_n := \{Z_n : \|Z_n\| \leq M_n/2\} \), for \( M_n \to \infty \) and \( M_n = O(\sqrt{n}) \), and note that \( \text{pr}(\Omega_n) \to 1 \) by Assumption 3.2. For \( \|t\| \leq \gamma \), on the set \( \Omega_n \) the expansion in Lemma A.1 becomes
\[
Q_n(\theta_n + t/\sqrt{n}) - Q_n(\theta_n) = \frac{1}{2} t^T H(\theta_n) t + O(\|t\|^3/\sqrt{n}) + O(\|t\|^2 \|Z_n/\sqrt{n}\|) \\
= \frac{1}{2} t^T H(\theta_n) t + o(1),
\]
since \( \text{pr}(\Omega_n) \to 1 \). On the set \( \{t : \|t\| \leq \gamma\} \cap \Omega_n \),
\[
Q_n(\theta_n + t/\sqrt{n}) - Q_n(\theta_n) = -\frac{1}{2} t^T (I + V_n) \Delta^{-1} t + o(1),
\]
for \( V_n = [-H(\theta_n) - \Delta^{-1}] \Delta \). For some \( K > 0 \), \( \|V_n\| \leq K \|\Delta\| \|\theta_n - \theta_*\| \) by Assumption 3.4. Define the matrix \( A_n = K \Delta \|\theta_n - \theta_*\| \). By Assumption 3.3, \( \lambda_{\text{max}}(\Delta) \) is finite. Conclude that \( A_n \) is positive semi-definite with maximal eigenvalue
\[
\lambda_{\text{max}}(A_n) = K \|\theta_n - \theta_*\| \lambda_{\text{max}}(\Delta) \geq 0,
\]
which converges to zero as \( n \to \infty \).

From the consistency of \( \theta_n \) for \( \theta_* \) in Lemma 3.1, for any \( \delta > 0 \) there exists an \( N := N_\delta \) such that for all \( n > N, \theta_n \in \mathcal{N}(\theta_*; \delta) \) (with probability converging to one).\(^{14}\) Since \( \delta \) is arbitrary, this result is satisfied for \( \delta = \gamma/\sqrt{n} \) so that for any \( t \) such that \( \|t\| \leq \gamma \),
\[
- \|A_n\| \leq \|V_n\| \leq \|A_n\|.
\]
\(^{14}\)In what follows, we let wpct denote the phrase “with \( P_0^{(n)} \) - probability converging to one.”
Apply the inequality in (13) into (12), to obtain
\[-t^\top \Delta^{-1} t/2 - t^\top A_n \Delta^{-1} t/2 \leq \left\{ Q_n(\theta_n + t/\sqrt{n}) - Q_n(\theta_n) \right\} \leq -t^\top \Delta^{-1} t/2 + t^\top A_n \Delta^{-1} t/2.\]

For sets \( N^\pm_\gamma := \{ t : ||t|| \leq |I \pm A_n|^{1/2} \gamma \} \), we have \( N^\pm_\gamma \subseteq N_\delta \subseteq N^\pm_\gamma \) and the posterior over \( N_\gamma \) can be bounded above and below as follows

\[
\frac{\pi(\theta_n|S_n)}{\sqrt{n}} \int_{N_\gamma} \exp \left\{ -\frac{1}{2} t^\top (I + V_n) \Delta^{-1} t \right\} \, dt
\leq \frac{\pi(\theta_n|S_n)}{\sqrt{n}} \int_{N_\delta} |(I - A_n) \Delta^{-1}|^{-1/2} \exp\{-t^\top (I - A_n) \Delta^{-1} t/2\} \, dt
\geq \frac{\pi(\theta_n|S_n)}{\sqrt{n}} \int_{N^\pm_\gamma} |(I + A_n) \Delta^{-1}|^{-1/2} \exp\{-t^\top (I + A_n) \Delta^{-1} t/2\} \, dt,
\]

As \( n \to \infty, |I \pm A_n| \to 1 \), and \( N^\pm_\gamma \) converges to \( N_\gamma \). This convergence, Lemma A.2, and the dominated convergence theorem allow us to deduce that

\[
o(1) + Q_*(\gamma) \leq \frac{\pi(\theta_n|S_n)}{\sqrt{n}} \int_{N_\gamma} \exp \left\{ -\frac{1}{2} t^\top (I + V_n) \Delta^{-1} t \right\} \, dt \leq Q_*(\gamma) + o(1),\]

where we recall that \( Q_*(\gamma) = C_\gamma \int_{N_\gamma} \exp \left\{ -\frac{1}{2} t^\top \Delta^{-1} t \right\} \, dt \). We can then conclude that \( J_n \to 0 \) in probability.

\( \square \)

Proof of Proposition 3.1. From the triangle inequality

\[
\int_{T_n} ||t|| ||\pi(t|S_n) - N\{ t; 0, \Delta \}|| \, dt \leq \int_{T_n} ||t|| ||\pi(t|S_n) - \pi(t|S_n)|| \, dt + \int_{T_n} ||t|| ||\pi(t|S_n) - N\{ t; 0, \Delta \}|| \, dt
\]

From Lemma A.4, the first term satisfies

\[
\int_{T_n} ||t|| ||\pi(t|S_n) - \pi(t|S_n)|| \, dt = O_p(1/m),
\]

while from Lemma A.3, the second term is \( o_p(1) \).

\( \square \)

Proof of Corollary 3.1. Define \( \bar{\theta}_n := \int \theta \pi(\theta|S_n) \, d\theta \) as the posterior mean of \( \theta \). The change of variables \( \theta \mapsto t := \sqrt{n}(\theta - \theta_n) \) yields

\[
\bar{\theta}_n = \int_{\Theta} \theta \pi(\theta|S_n) \, d\theta = \int_{T_n} (t/\sqrt{n} + \theta_n) \pi(t|S_n) \, dt = \frac{1}{\sqrt{n}} \int_{T_n} t \pi(t|S_n) \, dt + \theta_n,
\]

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so that
\[
\sqrt{n}(\bar{\theta}_n - \theta_n) = \int_{T_n} t\pi(t | S_n) \, dt = \int_{T_n} t \left[ \pi(t | S_n) - N\{ t; 0, \Delta \} \right] \, dt + \int_{T_n} t N\{ t; 0, \Delta \} \, dt. \tag{15}
\]

The second term on the right-hand side of (15) is zero by definition. Therefore, by Proposition 3.1,
\[
\left\| \sqrt{n}(\bar{\theta}_n - \theta_n) \right\| = \left| \int t \left\{ \pi(t | S_n) - N\{ t; 0, \Delta \} \right\} \, dt \right| \leq \int \left| t \right| \left\| \pi(t | S_n) - N\{ t; 0, \Delta \} \right\| \, dt = o_p(1).
\]

Under Assumption 3.8 and the expansion in Lemma A.1, we obtain
\[
\sqrt{n}(\theta_n - \theta_\ast) = \Delta \sqrt{n} M_n(\theta_\ast) + o_p(1) \Rightarrow N(0, \Delta W_\ast \Delta^\top).
\]

The two display equations together yield the stated result. \hfill \Box

### A.1 Lemmas

The following section gives lemmas that are used in the proofs of the main results.

Before presenting the results, we first recall some key definitions. The “exact” SL criterion is given by \( \ln g_n(S_n | \theta) \), which, neglecting constants that do not depend on \( \theta \), can be written as
\[
\log \{ g_n(S_n | \theta) \} = - \log(\left| \Sigma_n(\theta) \right|) - n \left\{ b(\theta) - S_n \right\} \left( n \Sigma_n(\theta) \right)^{-1} \left\{ b(\theta) - S_n \right\} / 2
\]
\[
= - \log(\left| \Sigma_n(\theta) \right|) + Q_n(\theta).
\]

For \( M_n(\theta) := n^{-1} \partial \ln g_n(S_n | \theta) / \partial \theta \), by Lemma 3.1 there exists at least one strict local maximum of \( \ln g_n(S_n | \theta) \) in \( \Theta \), denoted generically by \( \theta_n \), that satisfies
\[
\left\| M_n(\theta_n) \right\| = o_p(1 / \sqrt{n}). \tag{16}
\]

Using the fact that \( n \Sigma_n(\theta) = \Sigma(\theta) + o_p(1) \), uniformly over \( \Theta \) (Assumption 3.5), \( M_n(\theta) \) can be stated as (up to \( o_p(1) \) terms)
\[
M_n(\theta_n) = - \left( \text{tr} \left\{ \Sigma^{-1}(\theta) \Lambda_1(\theta) \right\} \right) / n + \left\{ (\partial / \partial \theta^\top) b(\theta) \right\} \Sigma^{-1}(\theta) \left\{ b(\theta) - S_n \right\}
\]
\[- \left\{ (\partial / \partial \theta^\top) \text{Vec} [\Sigma(\theta)] \right\} \left[ \Sigma^{-1}(\theta) \otimes \Sigma^{-1}(\theta) \right] \times \text{Vec} \left[ \left\{ S_n - b(\theta) \right\} \left\{ S_n - b(\theta) \right\}^\top \right]. \tag{17}
\]

In the case of scalar \( \theta \), \( M_n(\theta) \) has the more analytically useful representation
\[
M_n(\theta_n) = - \text{tr} \left\{ \Sigma^{-1}(\theta) \Lambda(\theta) \right\} - \left\{ (\partial / \partial \theta^\top) b(\theta) \right\} \Sigma^{-1}(\theta) \left\{ b(\theta) - S_n \right\} + \left\{ S_n - b(\theta) \right\} \Sigma^{-1}(\theta) \Lambda(\theta) \Sigma^{-1}(\theta) \left\{ S_n - b(\theta) \right\}.
\]

where \( \Lambda(\theta) := d\Sigma(\theta) / d\theta \). Using the scalar representation of \( M_n(\theta) \), the Hessian matrix \( H_n(\theta) := \)
\[ \frac{\partial M_n(\theta)}{\partial \theta} \], can be constructed by concatenating the following partial derivatives \( \frac{\partial M_n(\theta)}{\partial \theta_j} \), \( j = 1, \ldots, d_\theta \), column-wise; i.e.,

\[
H_n(\theta) = \left( \frac{\partial M_n(\theta)}{\partial \theta_1} \ldots \frac{\partial M_n(\theta)}{\partial \theta_{d_\theta}} \right).
\]

The following result collects several frequentist properties of \( \theta_n \), and \( Q_n(\theta) \) that are used to prove the main results.

**Lemma A.1.** Under Assumptions 3.1-3.7, the following are satisfied.

1. For some \( \theta_\ast \in \Theta_\ast \), the estimator \( \theta_n \) exists and satisfies \( \|\theta_n - \theta_\ast\| = o_p(1) \).

2. If in addition to Assumptions 3.1-3.7, Assumption 3.8 is satisfied, then \( \|\theta_n - \theta_\ast\| = O_p(n^{-1/2}) \).

3. For any constant \( \delta > 0 \), \( T_n := \{ \theta \in \Theta, \theta_n \in \Theta_\ast : \|\theta - \theta_\ast\| \leq \delta / \sqrt{n} \} \), and \( t = \sqrt{n}(\theta - \theta_\ast) \), \( \theta \in T_n \),

\[
Q_n(\theta_\ast + t / \sqrt{n}) - Q_n(\theta_\ast) = t^T \sqrt{n} M_n(\theta_\ast) + t^T H(\theta_\ast) t + O_p(||t||^3 / \sqrt{n}).
\]

**Proof.** **Part 1.** The result follows from verifying the sufficient conditions in Theorem 2 of Yuan and Jennrich (1998). Firstly, from the definition of \( M_n(\theta) \) given in equation (17), and the definition of \( \theta_\ast \), it is not hard to show that \( M_n(\theta_\ast) = o_p(1) \). Secondly, from Assumptions 3.1, \( M_n(\theta) \) is continuously differentiable for all \( \|\theta - \theta_\ast\| \leq \delta \), and some \( \delta > 0 \). Moreover, from the definition of \( H_n(\theta) \) and Assumption 3.2, we conclude that \( ||H_n(\theta) - H(\theta)|| \leq O_p(||Z_n / \sqrt{n}||) = o_p(1) \), for all \( \|\theta - \theta_\ast\| \leq \delta \). Moreover, \( H(\theta_\ast) \) is non-singular by Assumption 3.1. This verifies the sufficient conditions in Yuan and Jennrich (1998) and we can conclude that: 1) \( \theta_n \) exists for \( n \) large enough; 2) \( \theta_n \) satisfies, \( \|\theta_n - \theta_\ast\| = o_p(1) \). Lemma 3.1 in the main text follows.

**Part 2.** To simplify the derivation we do so in the case of scalar \( \theta \), and note that the result can be extended by applying the same argument dimension-by-dimension.

Firstly, from consistency there exists some positive \( \delta_n = o(1) \) such that \( \Pr \{ \|\theta_n - \theta_\ast\| \geq \delta_n \} = o(1) \). With \( P_0^{(n)} \) - probability converging to one for this sequence, we first show that

\[
\sup_{\|\theta - \theta_\ast\| \leq \delta_n} \|M_n(\theta) - M(\theta) - M_n(\theta_\ast)\| = o_p(n^{-1/2})
\]

From the definition of \( M_n(\theta) \) and \( M(\theta) \) in the univariate case, for \( G(\theta) := \frac{db(\theta)}{d\theta}, \)

\[
M_n(\theta) = M(\theta) + [G(\theta)^T \Sigma(\theta)^{-1} + 2\{b(\theta) - b_0\}^T \Sigma(\theta)^{-1} \Lambda(\theta) \Sigma(\theta)^{-1}] \{b_0 - S_n\} \\
- \{b_0 - S_n\}^T \Sigma(\theta)^{-1} \Lambda(\theta) \Sigma^{-1}(\theta)\{b_0 - S_n\} \\
= M(\theta) + O_p(1 / \sqrt{n}) + O_p(1 / n).
\]
In particular, using $M(\theta_*) = 0$,

$$M_n(\theta_*) = \left[ G(\theta_*)^\top \Sigma(\theta_*)^{-1} + 2\{b(\theta_*) - b_0\}^\top \Sigma(\theta_*)^{-1} \Lambda(\theta_*) \Sigma(\theta_*)^{-1} \right] \{b_0 - S_n\} - \{b_0 - S_n\}^\top \Sigma(\theta_*)^{-1} \Lambda(\theta_*) \Sigma(\theta_*)^{-1} \{b_0 - S_n\} = \left[ G(\theta_*)^\top \Sigma(\theta_*)^{-1} + 2\{b(\theta_*) - b_0\}^\top \Sigma(\theta_*)^{-1} \Lambda(\theta_*) \Sigma(\theta_*)^{-1} \right] \{b_0 - S_n\} + O_p(1/n).$$

Therefore, for $B(\theta) = \Sigma^{-1}(\theta) \Lambda(\theta) \Sigma^{-1}(\theta), e(\theta) = b(\theta) - b_0$, and $X(\theta) = G(\theta)^\top \Sigma^{-1}(\theta)$,

$$\|M_n(\theta) - M(\theta) - M_n(\theta_*)\| \leq \|Z_n/\sqrt{n}\| \|B(\theta) - B(\theta_*)\| + \|Z_n/\sqrt{n}\| \|[X(\theta) + 2e(\theta)^\top B(\theta)] - [X(\theta_*) + 2e(\theta_*)^\top B(\theta_*)]\|.$$

By Assumption 3.4, $B(\theta), e(\theta)$, and $X(\theta)$ are Lipschitz in a neighborhood of $\theta_*$. Therefore, for $\delta_n$ as above, all $\|\theta - \theta_*\| \leq \delta_n$, and some $C > 0$,

$$\|M_n(\theta) - M(\theta) - M_n(\theta_*)\| \leq C\|Z_n/\sqrt{n}\| \|\theta - \theta_*\| \{1 + \|Z_n/\sqrt{n}\|\}$$

Applying the fact that $Z_n/\sqrt{n} = O_p(1/\sqrt{n})$, this proves (18).

With $P_0^{(n)}$ - probability converging to one for the sequence $\delta_n$, we then have

$$\|M_n(\theta_n) - M(\theta_n) - M_n(\theta_*)\| \leq o_p(n^{-1/2}) \geq \|M(\theta_n)\| - \|M_n(\theta_n)\| - \|M_n(\theta_*)\|.$$

Rearranging terms, and applying Assumption 3.8,

$$\|M(\theta_n)\| \leq o_p(n^{-1/2}) + \|M_n(\theta_*)\| \{1 + o_p(1)\} = O_p(n^{-1/2}).$$

From the differentiability of $M(\theta)$, and the full rank condition on $H(\theta_*)$, there exists $C > 0$ such that

$$C\|\theta_n - \theta_*\| \leq \|M(\theta_n)\| \leq o_p(n^{-1/2}) + O_p(n^{-1/2}).$$

**Part 3.** On the set $T_n$, the result follows from a Taylor expansion of $Q_n(\theta) := -\frac{1}{2}\{b(\theta) - S_n\}^\top \Sigma_n^{-1}(\theta)\{b(\theta) - S_n\}$ around $\theta_*$, which for $M_n(\theta)$ and $H_n(\theta)$ as defined in Section 3.1, gives

$$Q_n(\theta) = Q_n(\theta_*) + \sqrt{n}(\theta - \theta_*)^\top \sqrt{n}M_n(\theta_*) + \sqrt{n}(\theta - \theta_*)^\top H_n(\tilde{\theta}) \sqrt{n}(\theta - \theta_*),$$

for $\tilde{\theta}$ a term-by-term intermediate value such that $\|\tilde{\theta} - \theta_*\| \leq \|\theta - \theta_*\|$, and where $\theta \in T_n$. From the definition of $H(\theta)$, and the twice continuous differentiability hypothesis on $Q_n(\theta), H_n(\theta)$ is Lipschitz in
this neighbourhood and we have that:\footnote{By Assumption 3.5, the map $H_n(\theta) = \partial M_n(\theta)/\partial \theta'$ is continuously differentiable in a neighbourhood of $\theta_*$, for some $\delta_n$, and any $\theta_* \in \Theta_*$. Therefore, for each $\theta_* \in \Theta$, $H_n(\theta)$ is Lipschitz, with (possibly) differing Lipschitz constant, in this neighbourhood.} for $Z_n := \sqrt{n} \{ b_0 - S_n \}$

$$
\| H_n(\tilde{\theta}) - H(\theta_*) \| \leq \| H_n(\tilde{\theta}) - H_n(\theta_*) \| + \| H_n(\theta_*) - H(\theta_*) \| \leq O_p(\| \theta - \theta_* \|) + O_p(\| Z_n / \sqrt{n} \|).
$$

From the above, the change of variables $t := \sqrt{n}(\theta - \theta_*)$, and rearranging terms,

$$
Q_n(\theta_* + t/\sqrt{n}) = Q_n(\theta_*) + t^\top M_n(\theta_*)/\sqrt{n} + t^\top H(\theta_*)t + O_p(\| t \|^2 \| Z_n / \sqrt{n} \|) + O_p(\| t \|^3 / \sqrt{n}).
$$

\[\square\]

**Lemma A.2.** Under Assumptions 3.1-3.7, $\pi(\theta_n | S_n) / \sqrt{n} = O_p(1)$.

**Proof.** The proof is similar to Lemma 2.1 in Chen (1985). For any $\delta_n = o(1)$ with $\delta_n \sqrt{n} \to \infty$, recall $N_\delta := N(\theta_n; \delta_n)$, and apply the expression for the posterior in equation (6) to obtain

$$
\int_{N_\delta} \pi(\theta | S_n) d\theta = \pi(\theta_n | S_n) \int_{N_\delta} [\Sigma_n(\theta_n)/|\Sigma_n(\theta_n)|]^{1/2} \exp \{ Q_n(\theta) - Q_n(\theta_n) \} \pi(\theta) / \pi(\theta_n) d\theta.
$$

Similar arguments to that in the proof of Theorem 3.1 show that, over the set $N_\delta$,

$$
\int_{N_\delta} \pi(\theta | S_n) d\theta = \pi(\theta_n | S_n) \{ 1 + o_p(1) \} \int_{N_\delta} \exp \{ Q_n(\theta) - Q_n(\theta_n) \} d\theta.
$$

From the proof of Theorem 3.1, and for $A_n$ as defined therein,

$$
-t^\top \Delta^{-1}t/2 - t^\top A_n \Delta^{-1}t/2 \leq Q_n(\theta_n + t/\sqrt{n}) - Q_n(\theta_n) \leq -t^\top \Delta^{-1}t/2 + t^\top A_n \Delta^{-1}t/2
$$

Now, applying the change of variables $\theta \mapsto t := \sqrt{n}(\theta - \theta_*)$ we can bound the posterior probability over $N_\delta$ as

$$
\int_{N_\delta} \pi(\theta | S_n) d\theta \leq \frac{\pi(\theta_n | S_n)}{\sqrt{n}} |(I - A_n) \Delta^{-1}|^{-1/2} \int_{T^-_n} \exp \{ -t^\top (I - A_n) \Delta^{-1}t/2 \} dt
\geq \frac{\pi(\theta_n | S_n)}{\sqrt{n}} |(I + A_n) \Delta^{-1}|^{-1/2} \int_{T^+_n} \exp \{ -t^\top (I + A_n) \Delta^{-1}t/2 \} dt
$$

where

$$
T^-_n := \left\{ t : \| t \| \leq \delta_n \sqrt{n} [1 - \lambda_{\min}(A_n)]^{1/2} \lambda_{\min}(\Delta) \right\}, \quad T^+_n := \left\{ t : \| t \| \leq \delta_n \sqrt{n} [1 + \lambda_{\max}(A_n)]^{1/2} \lambda_{\max}(\Delta) \right\}.
$$

By construction, $T^+_n \subseteq N(\theta_n; \delta_n) \subseteq T^-_n$. Under the restriction, $\delta_n \sqrt{n} \to \infty$, $T^+_n$, $T^-_n \to \mathbb{R}^{d_n}$ and we
obtain, for $n \to \infty$, wpc1,
\[
\frac{\pi(\theta_n|S_n)}{\sqrt{n}}(2\pi)^{d_0/2}|\Delta|^{1/2} \leq \left|(I + A_n)\Delta^{-1}\right|^{1/2} \int_{N_0} \pi(\theta|S_n)d\theta \\
\geq \left|(I - A_n)\Delta^{-1}\right|^{1/2} \int_{N_0} \pi_n(\theta|S_n)d\theta.
\]
Since $|I \pm A_n| \to 1$, $|\Delta| > 0$ and $0 \leq \int_{N_0} \pi(\theta|S_n)d\theta \leq 1$, wpc1,
\[
0 \leq \frac{\pi(\theta_n|S_n)}{\sqrt{n}} \leq (2\pi)^{-d_0/2}|\Delta|^{-1/2}.
\]

\[\square\]

**Lemma A.3.** Under the Assumptions of Proposition 3.1,
\[
\int ||t|| \pi(t|S_n) - N\{t; 0, \Delta\}|dt = o_p(1).
\]

**Proof of Lemma A.3.** Recalling $Q_n(\theta) := -n\{b(\theta) - S_n\}^\top[n\Sigma_n^{-1}(\theta)]^{-1}\{b(\theta) - S_n\}/2$, as in the proof of Theorem 3.1, we first rewrite the exact BSL posterior as
\[
\pi(\theta|S_n) = \frac{\pi(\theta_n|S_n)n\Sigma_n(\theta_n)^{1/2}}{\pi(\theta_n)}|n\Sigma_n(\theta)|^{-1/2} \exp\{Q_n(\theta) - Q_n(\theta_n)\}\pi(\theta).
\]

The posterior density of $t := \sqrt{n}(\theta - \theta_n)$, $\pi(t|S_n) := \pi(t/\sqrt{n} + \theta_n|S_n)/\sqrt{n}$, is given by
\[
\pi(t|S_n) = \frac{|n\Sigma_n(\theta_n)|^{1/2}\pi(\theta_n|S_n)/\sqrt{n}}{\pi(\theta_n)}|n\Sigma_n(\theta)|^{-1/2} \exp\{Q_n(\theta_n + t/\sqrt{n}) - Q_n(\theta_n)\}\pi(\theta).
\]

For some $M > 0$ and $\delta = o(1)$
\[
\int_{T_n} ||t|| \pi(t|S_n) - N\{t; 0, \Delta\}|dt = \int_{||t|| \leq M} ||t|| \pi(t|S_n) - N\{t; 0, \Delta\}|dt + \int_{M < ||t|| \leq \sqrt{n}\delta} ||t|| \pi(t|S_n) - N\{t; 0, \Delta\}|dt \\
+ \int_{||t|| > \sqrt{n}\delta} ||t|| \pi(t|S_n) - N\{t; 0, \Delta\}|dt \\
\equiv I_1 + I_2 + I_3
\]

We now show that each of the above terms are $o_p(1)$.

\textbf{I}_1 \textbf{ Term.} For any finite $M$, $||t||$ is finite and the first term in the integral can be ignored. Applying Theorem 3.1 allows us to directly conclude that $\int_{||t|| \leq M} \pi(t|S_n) - q_*(t)|dt = o_p(1)$. 

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**I2 Term.** For $\delta = o(1)$, by Assumptions 3.5 and 3.6, and consistency of $\theta_n$,

$$\sup_{M < \|t\| \leq \delta \sqrt{n}} |\pi(\theta_n + t/\sqrt{n}) - \pi(\theta_0)| = o_p(1), \quad \text{and} \quad \sup_{M < \|t\| \leq \delta \sqrt{n}} \|n\Sigma_n(\theta_n + t/\sqrt{n}) - n\Sigma_n(\theta_n)\| = o_p(1),$$

(19)

so that these terms can be dropped from the computation. Moreover, over $M < \|t\| \leq \delta \sqrt{n}$, the term

$$\int_{M < \|t\| \leq \sqrt{n} \delta} \|t\| N\{0; t, \Delta\} dt$$

can be made arbitrarily small by taking $M$ large enough and $\delta$ small enough. It then suffices to show that, for any $\varepsilon > 0$ there exists an $M$ and $\delta$ such that, for some $n$ large enough,

$$\text{pr}\left[ \int_{M < \|t\| \leq \sqrt{n} \delta} \|t\| \pi(t|S_n) dt < \varepsilon \right] \geq 1 - \varepsilon.$$ 

Now, note that similar computations to those in the proof of Theorem 3.1 demonstrate that for $\delta = o(1)$, over the set $M < \|t\| \leq \delta \sqrt{n}$

$$Q_n(\theta_n + t/\sqrt{n}) - Q_n(\theta_n) \leq -t^T [I - A_n] \Delta^{-1} t/2 + o_p(1),$$

for $A_n$ as defined in the proof of Theorem 3.1. Using the above, the result of Lemma A.2, and the convergence in (19), for some $C_n = O_p(1)$, with $0 \leq C_n \leq (2\pi)^{-d_0/2} |\Delta|^{-1/2}$, for $n$ large enough,

$$\int_{M < \|t\| \leq \sqrt{n} \delta} \|t\| \pi(t|S_n) dt \leq C_n \int_{M < \|t\| \leq \sqrt{n} \delta} \|t\| \exp\{-t^T [I - A_n] \Delta^{-1} t/2\} dt$$

$$\leq (2\pi)^{-d_0/2} |\Delta|^{-1/2} \int_{M < \|t\| \leq \sqrt{n} \delta} \|t\| \exp\{-t^T [I - A_n] \Delta^{-1} t/2\} dt$$

wpc1.

Note that there exists some $M'$ large enough such that for all $M > M'$,

$$\|t\| \exp\{-t^T \Delta^{-1} t/2\} = O(M^{-1}).$$

Therefore, for any $\varepsilon > 0$, there is an $M$ large enough and a $\delta$ small enough such that (wpc1)

$$(2\pi)^{-d_0/2} |\Delta|^{-1/2} \int_{M < \|t\| \leq \sqrt{n} \delta} \|t\| \exp\{-t^T \Delta^{-1} t/2\} dt < \varepsilon.$$ 

Moreover, since $[I - A_n] \to I$, we can conclude that for some $n$ large enough, with probability at least $1 - \varepsilon$,

$$(2\pi)^{-d_0/2} |\Delta|^{-1/2} \int_{M < \|t\| \leq \sqrt{n} \delta} \|t\| \exp\{-t^T [I - A_n] \Delta^{-1} t/2\} dt < \varepsilon.$$
**I3 Term.** Similar to the proof of Region 2, over the set \(|t| \geq \delta \sqrt{n}\),

\[
\int_{|t| \geq \delta \sqrt{n}} ||t|| N\{t; 0, \Delta\} dt
\]
can be made arbitrarily small by taking \(\delta \sqrt{n}\) large. Therefore, it remains to show that \(\int_{|t| \geq \delta \sqrt{n}} ||t|| \pi(t|S_n) = o_p(1)\). From consistency of \(\theta_n\) for \(\theta_0\), and Assumptions 3.5 and 3.6, \(|n\Sigma_n(\theta_n)| = O_p(1)\) and \(\pi(\theta_n) = O_p(1)\), and from Lemma A.2, we can conclude that, for some \(C_n = O_p(1)\),

\[
\int_{||t|| > \delta \sqrt{n}} ||t|| \pi(t|S_n) dt \leq C_n \int_{||t|| > \delta \sqrt{n}} ||t|| \pi(\theta_n + t/\sqrt{n})|n\Sigma_n(\theta_n + t/\sqrt{n})|^{-1/2} \exp\{Q_n(\theta_n + t/\sqrt{n}) - Q_n(\theta_n)\}
\]

Using the change of variables \(t \mapsto \theta\), the integral on the RHS becomes

\[
\{1 + o_p(1)\} C_n n^{1/2} \int_{||\theta - \theta_0|| > \delta} ||\theta - \theta_0|| \pi(\theta)|n\Sigma_n(\theta)|^{-1/2} \exp\{Q_n(\theta) - Q_n(\theta_0)\} d\theta,
\]  
(20)

where the \(o_p(1)\) term follows from the triangle inequality and consistency of \(\theta_n\) for \(\theta_0\).

Now, note that for any \(\delta > 0\), and for \(Q(\theta) = -\{b(\theta) - b_0\}^\top \Sigma(\theta)^{-1} \{b(\theta) - b_0\}/2\),

\[
\sup_{||\theta - \theta_0|| > \delta} n^{-1} \{Q_n(\theta) - Q_n(\theta_0)\} \leq 2 \sup_{||\theta - \theta_0|| > \delta} \{Q_n(\theta)/n - Q(\theta)\} + \sup_{||\theta - \theta_0|| > \delta} \{Q(\theta) - Q(\theta_0)\}
\]

\[
\leq \sup_{||\theta - \theta_0|| > \delta} \{Q(\theta) - Q(\theta_0)\} + o_p(1),
\]

where the \(o_p(1)\) term follows from uniform convergence of \(\{Q_n(\theta)/n - Q(\theta)\}\) (guaranteed under Assumption 3.1 and 3.2). Further, from the continuity of \(Q(\theta)\) (Assumption 3.1), and the uniqueness of \(\theta_0\) (Assumption 3.3'), for any \(\delta > 0\) there exists some \(\epsilon > 0\) such that \(\sup_{||\theta - \theta_0|| > \delta} \{Q(\theta) - Q(\theta_0)\} \leq -\epsilon\).

Therefore, for any \(\delta > 0\),

\[
\lim_{n \to \infty} \pr\left[ \sup_{||\theta - \theta_0|| > \delta} \exp\{Q_n(\theta) - Q_n(\theta_0)\} \leq \exp(-\epsilon n) \right] = 1.
\]

Applying the above to the term in equation (20), and dropping the \(o_p(1)\) term, we can conclude

\[
\int_{||t|| > \delta \sqrt{n}} ||t|| \pi(t|S_n) dt \leq C_n n^{1/2} \exp\{-\epsilon n\} \int_{||\theta - \theta_0|| > \delta} ||\theta - \theta_0|| \pi(\theta)|n\Sigma_n(\theta)|^{-1/2} d\theta
\]

wpc1. From Cauchy-Schwarz, upper bound the last term as

\[
C_n n^{1/2} \exp\{-\epsilon n\} \int_{||\theta - \theta_0|| > \delta} ||\theta - \theta_0|| \pi(\theta)|n\Sigma_n(\theta)|^{-1/2} d\theta
\]

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By Assumption 3.5, Lemma A.4. Under the assumptions of Proposition 3.1, under weaker conditions than those maintained in this paper. Hence, we omit a proof for brevity.

Recalling the definition of $Q$, conclusion that

\[
\int \left| \frac{\pi_n(S_n)}{\pi_n(S_n)} - \pi(t|S_n) \right| dt = O_p(1/m).
\]

Lemma A.5. Under the assumptions of Theorem 3.1,

\[
\int \left| \frac{\pi_n(S_n)}{\pi_n(S_n)} - \pi(t|S_n) \right| dt = O_p(1/m).
\]

Proof of Lemma A.5. Recall the definition $Q_n(\theta) := -\|\Sigma_n^{-1/2}(\theta)\{b(\theta) - S_n\}\|^2/2$ and define

\[
\hat{Q}_n(\theta) := -\|\Sigma_n^{-1/2}(\theta)\{b_n(\theta) - S_n\}\|^2/2, \text{ and } \overline{Q}_n(\theta) := -\|\Sigma_n^{-1/2}(\theta)\{b_n(\theta) - S_n\}\|^2/2.
\]

By Assumption 3.5, $\sup_{\theta \in \Theta} \|n\Sigma_n(\theta) - \Sigma(\theta)\| = O_p(1)$ and $\sup_{\theta} \|n\Sigma_n(\theta) - \Sigma(\theta)\| = o_p(1)$, so that

$\sup_{\theta \in \Theta} \|n\Sigma_n(\theta) - n\Sigma_n(\theta)\| = o_p(1)$ for $m := m(n) \to \infty$ as $n \to \infty$. Therefore, uniformly over $\Theta$,

\[
\hat{Q}_n(\theta) = \overline{Q}_n(\theta)\{1 + o_p(1)\}.
\]

By Assumption 3.7, and compactness of $\Theta$,

\[
E \left[ \exp \left\{ \overline{Q}_n(\theta) \right\} \right] = \exp \left\{ Q_n(\theta) \right\} \{1 + O(1/m)\}.
\]

Recalling the definition of $\overline{g}_n(S_n|\theta)$ and $g_n(S_n|\theta)$, apply (21) to obtain

\[
|\overline{g}_n(S_n|\theta) - g_n(S_n|\theta)| \leq g_n(S_n|\theta)O(1/m).
\]

Rewrite the difference of the posteriors over $\mathcal{N}_\gamma$ as follows,

\[
\int |\pi(\theta|S_n) - \pi(\theta|S_n)| d\theta = \int_{\mathcal{N}_\gamma} \left| \frac{\overline{g}_n(S_n|\theta)\pi(\theta)}{\int \overline{g}_n(S_n|\theta)\pi(\theta)} - \frac{g_n(S_n|\theta)\pi(\theta)}{\int g_n(S_n|\theta)\pi(\theta)} \right|
\]

where the last inequality follows the bound for $C_n$ in Lemma 3.1, compactness of $\Theta$, and the continuity of $\pi(\theta)$, $\|\theta\|$ and $n\Sigma_n(\theta)$. Conclude that $\int_{\|t\| > \sqrt{n}d} \|t\| \pi(t|S_n) - N\{t; 0, \Delta\} dt = o_p(1)$.

The proof of the following results was given in the first part of Theorem 1 in Frazier et al. (2021) under weaker conditions than those maintained in this paper. Hence, we omit a proof for brevity.

Lemma A.4. Under the assumptions of Proposition 3.1,

\[
\int \left| \frac{\pi_n(S_n)}{\pi_n(S_n)} - \pi(t|S_n) \right| dt = O_p(1/m).
\]

Lemma A.5. Under the assumptions of Theorem 3.1,

\[
\int \left| \frac{\pi_n(S_n)}{\pi_n(S_n)} - \pi(t|S_n) \right| dt = O_p(1/m).
\]
\[
\begin{align*}
&= \int \left| \frac{\{g_n(S_n|\theta) - \bar{g}_n(S_n|\theta)\} \pi(\theta)}{\bar{g}_n(S_n|\theta) \pi(\theta)} \int g_n(S_n|\theta) \pi(\theta) d\theta \right. \\
&\quad - \left. g_n(S_n|\theta) \pi(\theta) \left( \frac{1}{\bar{g}_n(S_n|\theta) \pi(\theta)} - \frac{1}{\bar{g}_n(S_n|\theta) \pi(\theta)} \right) \right| \\
\end{align*}
\]

and apply the triangle inequality twice to obtain,

\[
\int |\bar{\pi}(\theta|S_n) - \pi(\theta|S_n)| d\theta \leq \int \frac{|\bar{g}_n(S_n|\theta) - g_n(S_n|\theta)| \pi(\theta)}{\bar{g}_n(S_n|\theta) \pi(\theta)} + \int \frac{g_n(S_n|\theta) \pi(\theta) \left( 1 - \frac{1}{\bar{g}_n(S_n|\theta) \pi(\theta)} \right)}{\bar{g}_n(S_n|\theta) \pi(\theta)} \\
\leq \int \frac{|\bar{g}_n(S_n|\theta) - g_n(S_n|\theta)| \pi(\theta)}{\bar{g}_n(S_n|\theta) \pi(\theta)} + \left( 1 - \frac{1}{\bar{g}_n(S_n|\theta) \pi(\theta)} \right)
\]

where the second inequality uses the fact that \(0 \leq \int \pi(\theta|S_n) \leq 1\). Apply equation (22) twice to obtain

\[
\begin{align*}
&= \int \frac{|\bar{g}_n(S_n|\theta) - g_n(S_n|\theta)| \pi(\theta)}{\bar{g}_n(S_n|\theta) \pi(\theta)} + \left( 1 - \frac{1}{\bar{g}_n(S_n|\theta) \pi(\theta)} \right) \\
&= \int \bar{g}_n(S_n|\theta) \pi(\theta) d\theta \left\{ 1 + O(1/m) \right\} + \left( 1 - \frac{1}{\bar{g}_n(S_n|\theta) \pi(\theta)} \right) \\
&= \frac{1}{m} \int \pi(\theta|S_n) + O(1/m).
\end{align*}
\]

By construction, \(0 \leq \int \pi(\theta|S_n) \leq 1\), so pulling all the terms together we have that

\[
\int |\bar{\pi}(\theta|S_n) - \pi(\theta|S_n)| d\theta \leq \frac{1}{m} \int \pi(\theta|S_n) + O(1/m) = O(1/m).
\]

\[\square\]

### A.2 Additional Example Details

#### A.2.1 Moving Average Model

The multi-modal behavior of the BSL posterior, and its lack of mass near the origin, can be traced back to the behavior of the log SL \(\ln \bar{g}_n(S_n|\theta)\) and its Hessian \(H_n(\theta) := \partial^2 \ln \bar{g}_n(S_n|\theta)/\partial^2\). To see this, we plot the SL criterion and the corresponding Hessian in Figure 10, for a single sample size of \(n = 1000\). Values of \(\theta\) in Sub-figure 10a such that \(H_n(\theta) < 0\), and which correspond to \(\partial \ln \bar{g}_n(S_n|\theta)/\partial \theta = 0\) (see, sub-figure 10b), define the local maxima of the SL. These local maxima directly coincide with the points of bi-modality for the posterior in Figure 2. Moreover, from Figure 10b we see that the slight difference in the modes of the criterion function are exacerbated by the BSL posterior, due to the Gaussian kernel. As a result, the posterior modes appear to have significantly different height, even though the modes of the SL are of similar height.

\[\square\]
Figure 10: Behavior of synthetic likelihood Hessian under model misspecification.

(a) Synthetic likelihood Hessian $H_n(\theta)$, across $\Theta$.

(b) Synthetic likelihood Hessian $\ln\{g_n(S_n|\theta)\}$, across $\Theta$. 

$S_n = 0.01, S_n = 0.1, S_n = 0.25$
A.2.2  \textit{g-and-k} Model

Figure 11 compares, for the three summary statistics, $S_1$, $S_2$ and $S_3$, the bootstrap estimates of the summary statistic variance based on the observed data (red vertical lines) together with kernel density estimates of the bootstrap variance estimates for 1,000 posterior predictive replicates of the data. The posterior predictive replicates are based on the standard BSL posterior distribution. The bootstrap estimates of variance are larger than expected for the observed data, particularly for $S_3$.

![Figure 11: Kernel estimates of posterior predictive densities for bootstrap estimated variances for $S_1$, $S_2$ and $S_3$ for the US-Canadian exchange rate data and the $g$-and-$k$ model with $k = 0$. The posterior density was obtained using standard BSL. The red lines indicate the bootstrap estimated variances for the observed data. The kernel estimates are computed from 1,000 posterior predictive samples.](image)

Figure 11: Kernel estimates of posterior predictive densities for bootstrap estimated variances for $S_1$, $S_2$ and $S_3$ for the US-Canadian exchange rate data and the $g$-and-$k$ model with $k = 0$. The posterior density was obtained using standard BSL. The red lines indicate the bootstrap estimated variances for the observed data. The kernel estimates are computed from 1,000 posterior predictive samples.

Figure 12 shows density estimates of posterior predictive replicates for the summary statistics $S_1$, $S_2$, $S_3$ and $S_4$ together with the observed values marked by the red lines. The observed value for the summary statistic $S_3$, which captures the skewness, is far out in the tails of the posterior predictive distribution. Although it is not recommended to perform model checking on summary statistics used to fit the model, since this can lead to conservative checks, the poor model fit is quite clear in this case.

![Figure 12: This figure includes the same information as in Figure 11, but for the summaries $S_1$, $S_2$, $S_3$ and $S_4$. Please refer to Figure 11 for details.](image)

Figure 12: This figure includes the same information as in Figure 11, but for the summaries $S_1$, $S_2$, $S_3$ and $S_4$. Please refer to Figure 11 for details.
As a sanity check we now consider repeating the analyses in Section 5 using a simulated dataset where the parameters \((A, B, g)\) used for the simulation are the estimated BSL posterior mean values. Since the data are simulated, there is no misspecification of the model for the data. Figure 13 shows the analyses based on the summary statistic \(S^{(1)}\). Here there is no meaningful adjustment as we might expect. Figure 14 shows the analyses based on the summary statistic \(S^{(2)}\). Now there is quite a large adjustment for \(g\). The reason for the large adjustment is most likely due to the summary statistic \(S_4\) not being approximately normal. The adjustment process also relies on asymptotic arguments, and this may also result in a deviation from the expected behaviour when there is no misspecification.

![Figure 13](image-url)

Figure 13: Estimated BSL posterior densities for the summary statistic vector \(S^{(1)}\) using standard BSL and adjusted BSL for the simulated US-Canadian exchange rate data and the \(g\)-and-\(k\) model with \(k = 0\). The top row shows univariate marginals (black=standard BSL, green=adjusted BSL). The KLDN values are described in the text and summarize how much the posterior changes after adjustment. The bottom row shows bivariate marginals. The contours show the standard BSL and the points are adjusted BSL sample values based on 1,000 samples.

![Figure 14](image-url)

Figure 14: This figure includes the same information as in Figure 5 but for the case where the BSL and adjusted BSL posterior densities are based on \(S^{(2)}\). Please refer to Figure 5 for details.