Large rainbow matchings in edge-colored graphs

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Abstract
There has been much research on the topic of finding a large rainbow matching (with no two edges having the same color) in a properly edge-colored graph, where a proper edge coloring is a coloring of the edge set such that no same-colored edges are incident. Recently, Gao, Ramadurai, Wanless, and Wormald proved that in every proper edge coloring of a graph with \(q\) colors where each color appears at least \(q + o(q)\) times, there is always a rainbow matching using every color. We strengthen this result by simultaneously relaxing two conditions: (i) we lift the condition on the number of colors and allow any finite number of colors and instead, put a weaker condition requiring the maximum degree of the graph to be at most \(q\), and (ii) we also relax the proper coloring condition and require that the graph induced by each of the colors have bounded degree. This strengthening resolves a natural question inspired by the remarks made by Gao, Ramadurai, Wanless, and Wormald.

As an application of this result, we show that for every proper edge coloring of a graph with \(2q + o(q)\) colors where each color appears at least \(q\) times, there is always a rainbow matching of size \(q\). This can be seen as an asymptotic version of a conjecture of Barát, Gyárfás, and Sárközy restricted on simple graphs. We also provide a construction showing that having \(q + 1\) colors is not enough, disproving a conjecture of Aharoni and Berger. As a by-product of our techniques, we obtain a new asymptotic version of the Brualdi–Ryser–Stein Conjecture, which is one of the central open questions in combinatorics.

1 Introduction

1.1 State of the art

Transversals in Latin squares have been a central topic of study in combinatorics, dating back to the work of Euler \cite{Euler} in the 18th century, who studied conditions on when Latin squares can be decomposed into transversals. For a survey of transversals in Latin squares, see, e.g., \cite{csernai}. One of the central and long-standing conjectures in this field is the following, which is attributed to Brualdi, Ryser, and Stein.

Conjecture 1.1 (Brualdi and Ryser \cite{brualdi}, Ryser \cite{ryser}, and Stein \cite{stein}). Every \(n \times n\) Latin square has a partial transversal of size \(n - 1\).

This conjecture translates into the graph-theoretic statement: “In any proper edge-coloring of the complete bipartite graph \(K_{n,n}\) with \(n\) colors, there is a rainbow matching of size \(n - 1\)”. To see this connection, refer to \cite{hatami} or \cite{shor}. Hatami and Shor \cite{hatami} proved an asymptotic version of Conjecture \ref{conj:brualdi} and the error term was further improved in a recent result of Keevash, Pokrovskiy, Sudakov, and Yepremyan \cite{keevash}.

Theorem 1.2 (Keevash, Pokrovskiy, Sudakov, and Yepremyan \cite{keevash}). Every proper edge-coloring of the complete bipartite graph \(K_{n,n}\) with \(n\) colors contains a rainbow matching of size \(n - O\left(\frac{\log n}{\log \log n}\right)\).
In this paper, we study generalizations (in many different aspects) of Conjecture 1.1 in the setting of rainbow matchings in edge-colored (but not only properly edge-colored) graphs. We advise the interested readers to see [5] for a recent survey on various extensions of this conjecture. Aharoni and Berger conjectured the following generalization of Conjecture 1.1 in [3].

**Conjecture 1.3** (Aharoni and Berger [3]). Let \( G \) be a properly edge-colored bipartite multigraph with \( n \) colors, having at least \( n + 1 \) edges of each color. Then \( G \) has a rainbow matching using every color.

In this paper, multigraphs permit parallel edges but not loops. After subsequent efforts by several authors (see, e.g., [7] [11] [15] [23] [27] [29]), an asymptotic version of Conjecture 1.3 is established in [16] and [28], where the conclusion of the conjecture is shown to hold if there are at least \( n + o(n) \) edges of each color. It is very natural to generalize Conjecture 1.3 for non-bipartite graphs. Gao, Ramadurai, Wanless, and Wormald [20] mentioned the following conjecture, which was also suggested by Aharoni, Berger, Chudnovsky, Howard, and Seymour [4].

**Conjecture 1.4** (Gao, Ramadurai, Wanless, and Wormald [20]). Let \( G \) be a properly edge-colored multigraph with \( n \) colors having at least \( n + 2 \) edges of each color. Then \( G \) has a rainbow matching using every color.

We remark here that one cannot replace \( n + 2 \) by \( n + 1 \) in Conjecture 1.4 (to see this, refer to [16] or [20]). After a couple of notable works progressing towards Conjecture 1.4 by Gao, Ramadurai, Wanless, and Wormald [20], and by Keevash and Yepremyan [28], recently Correia, Pokrovskiy, and Sudakov [16] established an asymptotic version of Conjecture 1.4. Gao, Ramadurai, Wanless, and Wormald proved the following asymptotic version of Conjecture 1.4 for simple graphs.

**Theorem 1.5** (Gao, Ramadurai, Wanless, and Wormald [20]). For every \( \epsilon > 0 \), there exists \( N = N(\epsilon) \) such that whenever \( n \geq N \), for any graph \( G \) that is properly edge-colored with \( n \) colors such that there are at least \((1 + \epsilon)n\) edges of each color, there is a rainbow matching of \( G \) using every color.

In this paper, we provide a generalization of this result of [20], where we allow non-proper edge-colorings and relax the condition of having only \( n \) colors (see Theorem 1.9 in the next section). Yet another motivation for such generalization comes from the following conjecture by Aharoni and Berger [3], which allows the edge-coloring to be non-proper.

**Conjecture 1.6** (Aharoni and Berger [3]). Let \( G \) be a bipartite multigraph, with maximum degree \( \Delta \), whose edges are (not necessarily properly) colored. If every color appears on at least \( \Delta + 1 \) edges, then \( G \) has a rainbow matching using every color.

Gao, Ramadurai, Wanless, and Wormald refuted Conjecture 1.6 in [20] by constructing an example where \( \Delta \) is linear in terms of the number of colors used. In this context, it will be interesting to check if such a statement holds for edge-coloring with the additional assumption of bounded (perhaps in terms of \( \Delta \)) maximum degree in each color class (color class refers to the subgraph formed by the edges of that color). Note that if the maximum degree of a color class is one, that color appears as a matching. So, this will be yet another generalization of Conjecture 1.3. Our generalization (Theorem 1.9 of Theorem 1.5) implies the asymptotic version of Conjecture 1.4 of edge-coloring of not-necessarily-bipartite simple graphs with only a bounded degree assumption on each color class.

We now move on to a related but slightly different problem. All the problems discussed so far focused on the minimum number of edges in each color class to ensure a rainbow matching using all the colors. Alternatively, we can insist that each color class has size exactly \( n \) and discuss how many colors we need to ensure a rainbow matching of size \( n \). In this context, the first result appeared by Drisko [17] in 1998, which some authors later revisited (see, e.g., [3] [9] [10]). On this topic, Barát, Gyárfás, and Sárközy [13] suggested the following conjecture:

**Conjecture 1.7** (Barát, Gyárfás, and Sárközy [13]). Let \( G \) be a properly edge-colored multigraph with \( 2n - t_n \) colors and exactly \( n \) edges of each color, where \( t_n = 0 \) for even \( n \) and \( t_n = 1 \) for odd \( n \). Then \( G \) has a rainbow matching using \( n \) colors.

For the best-known results on this conjecture, the readers are advised to refer to [4] and [6]. Aharoni and Berger considered Conjecture 3.2 in a simple graph, and made the following general conjecture (Conjecture 3.2 in [3]) for simple hypergraphs.
Conjecture 1.8 (Aharoni and Berger [3]). Let $G$ be an properly edge-colored $r$-uniform hypergraph with $2^{r-2}(s-1) + 2$ colors and $t$ edges of each color. Then, $G$ has a matching with $t$ edges on which at least $s$ colors appear.

In this paper, we disprove this conjecture when $r = 2$ and $s = t$ and progress on the upper bound on the required number of matchings, which will be discussed in Section 1.2. To be more precise, this upper bound is obtained by establishing an asymptotic version of Conjecture 1.7 when we restrict ourselves to simple graphs $G$ (see Theorem 1.11 in the next section).

1.2 Main results

We start with our first result, which generalizes Theorem 1.5.

Theorem 1.9. There exists $N$ such that whenever $q \geq N$ and $1 \leq \Delta \leq \frac{\sqrt{q}}{\log q}$, the following holds: Suppose $G$ is a graph with maximum degree at most $q$ that is edge-colored such that there are at least \((1 + \left(\frac{\Delta}{q}\right)^{\frac{1}{2}} \log q)^2\) edges of each color. Suppose that at most $\Delta$ edges of the same color are incident to any vertex (thus, this is not necessarily a proper coloring). Then, there is a rainbow matching in $G$, which uses every color.

In contrast to the result in Theorem 1.5, we do not need any bound on the number of colors; instead, we require a weaker condition that the graph has maximum degree at most $q$ (note that in Theorem 1.5, a proper-edge coloring of $G$ with $n$ colors ensures the maximum degree to be at most $n$). Furthermore, Theorem 1.9 does not require the color classes to be matchings, but the subgraph formed by each color still must have bounded degree. To prove Theorem 1.9, we use a natural randomized algorithm similar to the one used by Gao, Ramadurai, Wanless, and Wormald [20]. However, we make a few subtle changes in their algorithm to facilitate the analysis of the algorithm in our general setting. We analyze our algorithm by differential equation technique combined with the local lemma. Our argument also gives us a slightly better error term in the required number of edges in each color in the case when the edge-coloring is proper (i.e., when $\Delta = 1$) compared to the one obtained in [20]. Incidentally, this had the least error term among the known results progressing towards Conjecture 1.3.

In this paper, we disprove Conjecture 1.8. As mentioned in [3], the $r = 2$ and $s = t$ case of Conjecture 1.8 would have given us a generalization of Conjecture 1.1. But unfortunately, we have found a counterexample to this.

Proposition 1.10. For all even $t$, there exists a graph that is properly edge-colored with $t + 1$ colors such that there are $t$ edges of each color, but no rainbow matching of size $t$.

Thus, it is natural to find the maximum number $f(t)$ that can be written instead of $t + 1$ in the above proposition. Proposition 1.10 proves that $f(t) \geq t + 1$ for all even $t$. We make progress in the upper bound and establish that $f(t) \leq 2t + o(t)$. This provides an asymptotic version of Conjecture 1.7 in the case of simple graphs. Although we believe that the lower bound on $f(t)$ is closer to the truth, there is a natural barrier to improving the upper bound of $2t + o(t)$, which we elaborate on in the concluding remarks. Our next result is the following:

Theorem 1.11. For every $\epsilon > 0$, there exists $N = N(\epsilon)$ such that whenever $q \geq N$, for any graph $G$ that is properly edge-colored with at least $2(1 + \epsilon)q$ colors such that there are at least $q$ edges of each color, there is a rainbow matching of $G$ using $q$ colors.

We next show that Theorem 1.11 is the best possible with respect to the number of edges in each color class. We need each color to appear at least $q$ times because it is possible to have no matching on $q$ edges when each color appears $q - 1$ times, even if we have any number $n$ of colors. To see this, consider the graph that is a disjoint union of $q - 1$ copies of $K_{1,n}$ and color each $K_{1,n}$ using all the colors exactly once. Although each color appears as a matching with $q - 1$ edges, the graph does not contain any matching with $q$ edges. It turns out that this rigid behavior concerning the number of colors in each color class poses some difficulty in approaching this problem with probabilistic methods. Nevertheless, we overcome this difficulty by proving and using the following statement, which is also of independent interest.
Theorem 1.12. For every \( \epsilon > 0 \), there exists \( N = N(\epsilon) \) such that whenever \( q \geq N \) the following holds. Suppose \( G \) is a bipartite graph on the vertex set with bipartition \( A \cup B \), where \( |A| = q \) and every vertex in \( A \) has degree at least \( (1 + \epsilon)q \). Suppose the edges are properly colored. Then, there always is a rainbow matching in \( G \), which uses every vertex in \( A \).

The constant \( 1 \) in \( (1 + \epsilon)q \) of this result is optimal in the above result, because it is easy to construct examples showing that \( \epsilon = 0 \) does not work (to see such an example, check the first part of Section 2). As a straightforward corollary of Theorem 1.12 we next obtain the following asymptotic version of Conjecture 1.8 (the Brualdi–Ryser–Stein conjecture), which is also best possible up to the small error term.

Corollary 1.13. Every proper edge-coloring of the complete bipartite graph \( K_{q,q+o(q)} \) contains a rainbow matching of size \( q \).

In comparison with Theorem 1.12 we relax one of the parts of the bipartite graph to be slightly larger, but obtain a rainbow matching using every vertex of the smaller part. However, our error term of \( o(q) \) in Corollary 1.13 is larger than the error term in Theorem 1.12 and thus, it will be interesting to investigate whether one can obtain a polylogarithmic error term instead.

We use the differential equation method (also known as the dynamic concentration method) to prove Theorem 1.11 and Theorem 1.12. Although our method is motivated by the probabilistic approach in [20] by Gao, Ramadurai, Wanless, and Wormald, due to the nature of the statements of our results, it is required to keep track of more parameters compared to them, which in turn makes the analysis inevitably complicated. To keep the analysis reasonably concise, we take extra care in processing the underlying graph while executing our algorithms.

Organization. This paper is organized as follows. We disprove Conjecture 1.8 in the next section by proving Proposition 1.10. We mention some standard probabilistic tools in Section 3, which we use throughout this paper. For the next few subsequent sections, we concentrate on Theorem 1.9. Section 4 contains the randomized algorithm we use to prove Theorem 1.10, which is referred to as Algorithm everywhere. Later, we consider several similar randomized algorithms with many subtle changes to prove Theorem 1.11 and Theorem 1.12. In Section 5, we give an intuitive analysis to explain why the algorithm in Section 4 is expected to work. In a subsequent couple of sections, we prove Theorem 1.11 rigorously. Then, Theorem 1.12 is proved in Section 8. In Section 9, we establish a weaker version of Theorem 1.11 which we need later to prove it. Section 10 contains the proof of Theorem 1.12. We finish with a few concluding remarks, which include a discussion on why the required number of colors in Theorem 1.11 might be hard to improve using arguments based on only probabilistic methods.

Throughout this paper, for brevity, we systematically avoid the floor and ceiling signs because they do not affect the underlying analysis.

2 Construction for Proposition 1.10

We prove Proposition 1.10 in this section. Fix an even \( t \). Consider a graph \( G \) on the vertex set \( A \cup B \), where \( A \) and \( B \) have \( t \) vertices each, and recognize each of \( A \) and \( B \) by the group \( \mathbb{Z}_t \). For each \( j \in \mathbb{Z}_t \), introduce a color \( j \) with \( t \) edges, where each \( a \in A \) is adjacent to \( a + j \in B \). Thus, we have \( t \) colors, each of which is a matching of size \( t \). First, we prove that there is no rainbow matching using all of these \( t \) colors. For the sake of contradiction, assume that we have such a rainbow matching and fix such a matching. All colors and vertices of \( G \) have to participate in such a matching. Let \( a_j, b_j \) denote the edge in color \( j \in \mathbb{Z}_t \) in the rainbow matching, where \( a_j \in A \) and \( b_j \in B \). Clearly, we have the following:

\[
\sum_{j \in \mathbb{Z}_t} a_j = \sum_{j \in \mathbb{Z}_t} b_j = \sum_{j \in \mathbb{Z}_t} j = \frac{t(t-1)}{2}.
\]

But by the definition of each color class \( E_j \), we should have \( \sum_{j \in \mathbb{Z}_t} (b_j - a_j) = \sum_{j \in \mathbb{Z}_t} j \), which is a contradiction for even \( t \).

Now let us introduce an extra \( (t + 1) \)-st color in \( G \), whose edges are the union of an arbitrary perfect matching in \( A \) and arbitrary perfect matching in \( B \). This clearly has \( t \) edges. Note that all the matchings...
are still disjoint because any matching considered before contains edges between \( A \) and \( B \) only. We claim that even after introducing this extra color, we still cannot have a rainbow matching of size \( t \). For the sake of contradiction, assume that we have such a rainbow \( t \)-matching. Clearly, the extra color has to participate in that because we could not do it before. Without loss of generality, the extra color edge in the rainbow matching is between two vertices \( u, v \in A \). Now it is impossible to have a rainbow matching of size \( t - 1 \) on \( G \setminus \{u, v\} \) using the rest of the colors because each edge uses a distinct vertex from each of \( A \) and \( B \). Hence, we have a contradiction.

## 3 Preliminaries

We state some standard probabilistic tools, including a few large-deviation inequalities, which will be used throughout the paper.

**Theorem 3.1** (Chernoff bound, see [22]). Let \( X = \sum_{i=1}^{n} X_i \), where \( X_i = 1 \) with probability \( p_i \) and \( X_i = 0 \) with probability \( 1 - p_i \), and all \( X_i \) are independent. Let \( \mu = \mathbb{E}(X) = \sum_{i=1}^{n} p_i \). Then for any \( 0 < \lambda < \mu \), we have that
\[
\mathbb{P}(|X - \mu| \geq \lambda) \leq 2e^{-\frac{\lambda^2}{2\mu}}.
\]

We need Azuma-Hoeffding inequality (see, e.g., [2] and [22]) to prove concentration bounds to keep track of the behavior of certain parameters in our randomized algorithms. More specifically, we use applications of Azuma-Hoeffding inequality in the following general setting from Chapter 7 of [12].

For finite sets \( A \) and \( B \), let \( \Omega = A^B \) denote the set of functions \( g : B \rightarrow A \). Now define a probability measure by setting \( \mathbb{P}[g(b) = a] \) where the values of \( g(b) \) are independent for different \( b \). Fix a gradation \( \emptyset = B_0 \subset B_1 \subset \cdots \subset B_N = B \). Let \( L : A^B \rightarrow \mathbb{R} \) be a functional. Now define a martingale \( X_0, X_1, \ldots, X_N \) as the following:
\[
X_j(h) = \mathbb{E}(L(g) | g(b) = h(b) \text{ for all } b \in B_j).
\]
Clearly, \( X_0 \) is the deterministic expected value of \( L \), and \( X_N \) is the random variable \( L \). The values of \( X_j(g) \) approach \( L(g) \) as the values of \( g(b) \) are exposed. We say that the functional \( L \) is \( l \)-Lipschitz relative to the given gradation if for all \( 0 \leq j < m \):
\[
h, h' \text{ only differ on } B_{j+1} \setminus B_j \quad \Rightarrow \quad |L(h) - L(h')| \leq l. \tag{3.1}
\]

**Theorem 3.2.** Let \( L \) satisfies the Lipschitz condition \eqref{eq:3.1} relative to a gradation of length \( m \), and let \( \mu = \mathbb{E}(L(g)) \). Then for all \( \lambda > 0 \),
\[
\mathbb{P}[|L(g) - \mu| \geq \lambda] \leq 2e^{-\frac{\lambda^2}{2\mu}} \tag{3.2}
\]

To keep track of the evolution of certain parameters (in the proof of Theorem 3.9), we will sometimes need to use Talagrand’s inequality. We need a couple of definitions to state it. For a random variable \( X : \Omega \rightarrow \mathbb{R} \) with \( \Omega = \prod_{i=1}^{n} \Omega_i \) being a product probability space, we say:

- \( X \) is \( C \)-Lipschitz, if for every \( \omega \in \Omega \), changing \( \omega \) in any single coordinate affects the value of \( X(\omega) \) by at most \( C \).

- \( X \) is \( r \)-certifiable, if for every \( \omega \in \Omega \) and \( s \in \mathbb{R} \) such that \( X(\omega) \geq s \), the following holds. There exists a set \( I \subseteq \{1, \ldots, n\} \) of size at most \( rs \) such that every \( \omega' \in \Omega \) that agrees with \( \omega \) on the coordinates indexed by \( I \) also satisfies that \( X(\omega') \geq s \).

**Theorem 3.3** (Talagrand’s inequality, see [12]). Suppose that \( X \) is a \( C \)-Lipschitz and \( r \)-certifiable random variable. Then,
\[
\mathbb{P}
\left[
|X - \mathbb{E}(X)| > t + 60C\sqrt{\mathbb{E}(X)}
\right]
\leq 4e^{-\frac{t^2}{120^2C^2\mathbb{E}(X)}}.
\]

Since the number of colors in Theorem 1.9 can be arbitrarily large, simple union bounds often do not work (e.g., to control the number of edges in every color class). Thus, we need the Lovász local lemma to prove this result. We next mention the symmetric version of the standard local lemma.
Theorem 3.4 (Lovász local lemma, see [12]). Let $A_1, \ldots, A_n$ be events in a probability space. Suppose that there exist constants $p$ and $d$ such that all $P[A_i] \leq p$, and each event $A_i$ is mutually independent of all the other events $\{A_j\}$ except at most $d$ of them. If $ep(d + 1) \leq 1$, then $P[\bigcap A_i] > 0$.

We next give a weaker bound for Theorem 1.9, which will be used in its proof.

Proposition 3.5. Suppose $G$ is a graph with maximum degree at most $q$ that is edge-colored, with at least $4eq$ edges of each color. Then, there is a rainbow matching in $G$, which uses every color.

To prove Proposition 3.5 we use the following result of Alon [1] (see [21] for the best possible result), which can be easily proved using the local lemma (i.e., Theorem 3.4).

Proposition 3.6 (See [1]). Let $G$ be a multipartite graph with maximum degree $\Delta$, whose parts $V_1, \ldots, V_r$ all have size $|V_i| \geq 2e\Delta$. Then, $G$ has an independent transversal.

Proof of Proposition 3.5. Let $r$ be the number of colors in $G$, and $E_i$ denote the edge set of $i$-th color. Consider the $r$-partite graph $\Gamma$ where the $i$-th part is $E_i$ (the edges in $G$ are vertices of $\Gamma$). Two vertices in $\Gamma$ are considered adjacent if the corresponding edges in $G$ are incident to each other. Thus, the maximum degree of $\Gamma$ is at most $2q$, and each part contains at least $2e\cdot 2q$ vertices. Thus, by Proposition 3.6, the graph $\Gamma$ has an independent transversal. Observe that an independent transversal in $\Gamma$ corresponds with a rainbow matching in $G$ using every color. This completes the proof of Proposition 3.5. \qed

4 Algorithm

In this section, we give a randomized algorithm that will be primarily used for Theorem 1.9. Since we use some variants of this algorithm for proving Theorems 1.11 and 1.12 this algorithm will be referred to as Algorithm throughout the paper. Assume that we are given a graph $G$ with maximum degree at most $q$ that is edge-colored such that there are $(1 + \epsilon)q$ edges of each color and at most $\Delta$ edges of the same color can be incident to any vertex. We will provide a randomized algorithm, which constructs a rainbow matching using almost all the colors in several iterations. Each iteration is a random procedure, where we can show that we get to the desired state with some positive probability. Then we fix that choice of desirable state to analyze the next iteration. A single step of our algorithm will look like the following.

1. Activate each color class independently with probability $\theta_t = \frac{\delta}{1 - (1 - \epsilon)^m}$ (this probability is picked to ensure that roughly $\delta m$ colors get activated in each iteration).
2. Select independently one edge from each activated color class uniformly at random. Denote by $T$ the set of all selected edges.
3. Delete all the vertices corresponding to the edges in $T$ from $G$. Deleting vertices always also deletes all incident edges.
4. With some probability (to be specified later), independently delete each vertex in $G$. This will ensure that among Steps 3 and 4 combined, every vertex in $G$ gets deleted with the same probability.
5. For each edge $e \in T$, add it to the rainbow matching if $e$ is not incident to any edge $e' \in T$ with $e' \neq e$.
6. For each edge $e$ added to the rainbow matching in Step 5, delete all the edges of the same color as $e$ from $G$.

We next provide a heuristic on what we can expect from the above algorithm. Thus, it is possible to skip the next section if the reader is only interested in a formal proof of Theorem 1.9.
5 Intuitive analysis

We aim to show that if we run this randomized algorithm until we have picked almost all the colors we need, then near the end, each remaining color still has so many edges left (relative to the maximum degree of $G$) that we can complete the rainbow matching using every color via a straightforward application of Proposition 5.

To this end, tracking the evolution of the color class sizes and the degrees of the vertices is useful. We do that by modeling the remaining color class size and vertex degrees using a system of differential equations.

Define $d_{0,v}$ to be the initial degree of $v$ in $G$. We will define two functions $s(x)$ and $g(x)$ such that after the $t$-th iteration of Algorithm, we have the following:

1. Each remaining color class has size around $s_t \approx s(t\delta)(1+\epsilon)q$.
2. Each surviving vertex $v$ has degree, $d_{t,v} \approx (1-t\delta)g(t\delta)d_{0,v}$.

Clearly, $s(0) = 1$ and $g(0) = 1$. Assume that the $t$-th iteration is done, and the properties 1 and 2 are true. We outline a rough analysis of the $(t+1)$-st iteration of Algorithm to determine the functions $s$ and $g$.

Because of Step 4, every vertex of $G$ is deleted with the same probability among Steps 3 and 4 combined. This is done only for convenience in our analysis; thus, we should set that probability (denote it by $a_t$) as low as possible. In other words, we need to find the maximum probability a vertex $v$ can be deleted in Step 3 and set $a_t$ to be that maximum probability. At the $(t+1)$-st iteration, for every edge $e$ of $G$, the probability (denote by $p_e$) that $e$ will be picked in $T$ is exactly $\theta_{t+1}$ divided by the total number of edges of $G$ with the same color as $e$, which is about $\frac{\theta_{t+1}}{s_t}$. Thus, by a simple union bound, the probability that $v$ is deleted in Step 3 is at most $\sum_{e \subseteq v} p_e \approx d_{t,v} \cdot \frac{\theta_{t+1}}{s_t} \approx \frac{\delta g(t\delta)d_{0,v}}{s(t\delta)(1+\epsilon)q}$. Thus, we can define $a_t$ to be about $\frac{\delta g(t\delta)d_{0,v}}{s(t\delta)(1+\epsilon)q} = \gamma \frac{\delta g(t\delta)}{s(t\delta)}$, where $\gamma = \frac{\epsilon}{1+\epsilon}$.

The probability that a single vertex gets deleted among Steps 3 and 4 is $a_t$; thus, the probability that a pair of vertices gets deleted is about $a_t^2$, which we ignore in this intuitive analysis as it is a strictly lower-order term. Hence, the expected number of edges deleted in each color class among Steps 3 and 4 is approximately $2a_t s_t$. This suggests the following behavior:

$$s'(x) = -2\gamma g(x). \tag{5.1}$$

Next, let us estimate the change in $d_{t,v}$ to get a differential equation for $g$. We know that each neighbor of $v$ is deleted among Steps 3 and 4 with probability $a_t$. On the other hand, any edge $e$ incident to $v$ will be deleted in Step 6 by probability $\approx \theta_{t+1}$ because the color class containing $e$ is activated with this probability in Step 1, and there is a relatively low chance of conflicts in Step 5. Thus, neglecting small error terms due to lack of independence, we expect to have the following:

$$d_{t+1,v} - d_{t,v} \approx [a_t + (1-a_t)\theta_{t+1}] d_{t,v}$$

$$(\tau - t - 1)\delta g((t+1)\delta)d_{0,v} - \tau (t)\delta g(t\delta)d_{0,v} \approx -\delta g(t\delta)d_{0,v} - (\tau - t - 1)\delta g(t\delta)d_{0,v}a_t$$

$$\frac{g((t+1)\delta) - g(t\delta)}{\delta} \approx -\gamma \frac{g(t\delta)^2}{s(t\delta)}.$$

The above suggests:

$$g'(x) = -\gamma \frac{g(x)^2}{s(x)}. \tag{5.2}$$

By Equations (5.1) and (5.2), we have that $\frac{dg}{ds} = \frac{g}{c^2}$, whose solution is $s = cg^2$ for some constant $c$. By the initial conditions that $g(0) = s(0) = 1$, we get that $s = g^2$. Now, Equation (5.2) implies that $g'(x) = -\gamma$. Solving this with the initial condition $g(0) = 1$, we obtain that $g(x) = 1 - \gamma x$. Hence, $s(x) = (1 - \gamma x)^2$ because $s = g^2$.

We show in the next section that the degrees of vertices and the sizes of color classes are concentrated throughout the process, implying that we can not get stuck as long as $t \leq \eta \tau$. Moreover, after $\eta \tau$-th iteration, the number of edges in any color class $\approx (1 - \gamma \eta)^2(1+\epsilon)q$, and the maximum degree of $G$ is at most about $(1 - \eta)(1 - \gamma \eta)q$. As long as $(1 - \gamma \eta)^2(1+\epsilon)q \gg 4e(1 - \eta)(1 - \gamma \eta)q$, we can finish the rainbow matching by including an edge from the remaining colors by using Proposition 5. This inequality can be made true by choosing $\eta$ so that $1 - \eta$ is small enough compared to $1 - \gamma$. 

7
6 Formal analysis

Throughout this section (also in the remaining sections), we assume that $q$ is sufficiently large whenever needed. The previous section motivates us to define $s(x) = (1 - \gamma x)^2$ and $g(x) = 1 - \gamma x$, where $\gamma = \frac{1}{1 + \tau}$. From now on, we introduce error terms $\alpha_t$ and $\beta_t$ (which will be explicitly specified in the next section and much less than $\frac{1}{100}$), which will accumulate as we run the process (i.e., as $t$ increases). Thus, throughout this section, we use the inequalities that $\alpha_t < \frac{1}{100}$ and $\beta_t < \frac{1}{100}$ for every $t \leq \eta \tau$.

It will be easier to analyze Algorithm if all colors have the same number of edges at the start of an iteration. We will add now a 7th step in Algorithm to ensure that all the remaining color classes have same number of edges, $s_t = (1 - \alpha_t)s(t\delta)(1 + \epsilon)q$.

7. For each remaining color class, if the number of edges in that color class is more than $s_t$ after Step 6, then delete arbitrary edges of that color class to ensure that it has exactly $s_t$ edges.

We will make sure the following two happen after the $t$-th iteration for $1 \leq t \leq \eta \tau$:

1. The remaining color classes have size exactly:

   $$s_t = (1 - \alpha_t)s(t\delta)(1 + \epsilon)q.$$  

   (6.1)

2. The degree of each survived vertex is at most:

   $$d_t = (1 + \beta_t)(1 - t\delta)g(t\delta)q.$$  

   (6.2)

We fix some $t$ with $0 \leq t < \eta \tau$ for this and the next section and assume that (6.1) and (6.2) hold for $t$ (note that before starting the 1st iteration of Algorithm, we can run Step 7 with $t = 0$ and make sure that (6.1) and (6.2) hold). Our goal is to show that after the $(t + 1)$-st iteration of Algorithm, with positive probability, (6.1) and (6.2) will hold (for $t + 1$). In this section, we show certain concentration bounds that hold in the $(t + 1)$-st iteration, and in the subsequent section, these bounds will be used to achieve our goal. We next wish to specify probabilities for Step 4 for the $(t + 1)$-st iteration, which requires us to estimate the probability a vertex gets deleted in Step 3.

In the $(t + 1)$-st iteration of Algorithm, the probability that a fixed edge will be selected in Step 2 is exactly $\frac{\delta q_{t+1}}{s_t}$ because the corresponding color gets activated with probability $\theta_{t+1}$ and if a color is activated then each of the $s_t$ edges of this color gets selected with the same probability. Let $\mathcal{C}$ denote the set of all colors and for all $\mathcal{C} \in \mathcal{C}$, denote the number of edges incident to a vertex $v$ that are of color $\mathcal{C}$ by $d_{\mathcal{C}}(v)$. Thus, for a fixed vertex $v$, the probability that $v$ is deleted in Step 3 is given by $p'_v = 1 - \prod_{\mathcal{C} \in \mathcal{C}' \setminus \mathcal{C}} \left(1 - d_{\mathcal{C}}(v) \cdot \frac{\alpha_{t+1}}{s_t}\right)$. Since $\sum_{\mathcal{C} \in \mathcal{C}} d_{\mathcal{C}}(v)$ is equal to the degree $d_{t,v}$ of $v$ after the $t$-th iteration, it is easy to see the following:

$$p'_v \leq \frac{d_{t,v} \theta_{t+1}}{s_t}. \quad (6.3)$$

This together with $d_{t,v} \leq d_t$ implies that $p'_v \leq d_t \cdot \frac{\theta_{t+1}}{s_t} = \frac{g(t\delta)(1 + \beta_t)}{s(t\delta)(1 - \alpha_t)(1 + \epsilon)q} = \gamma \frac{g(t\delta)(1 + \beta_t)}{s(t\delta)(1 - \alpha_t)}$ (recall that $\gamma = \frac{1}{1 + \tau}$). Thus, define $\alpha_t$ as follows:

$$a_t = \gamma \frac{g(t\delta)(1 + \beta_t)}{s(t\delta)(1 - \alpha_t)}. \quad (6.4)$$

Next, we specify a probability for Step 4. For a fixed $t$, at the $(t + 1)$-st iteration we delete each vertex $v$ independently with probability $p_v$ such that $p'_v + (1 - p'_v)p_v = a_t$. Thus, $a_t$ is the total probability by which each vertex gets deleted in Steps 3 and 4.

We next give certain values to the parameters $\epsilon$, $\delta$, and $\eta$ in terms of $q$ and $\Delta$, which will be used in this and the next section. Recall that $\Delta \leq \frac{\sqrt{q}}{(\log q)^2}$. Define the following:

$$\epsilon = \left(\frac{\Delta^2}{q}\right)^{\frac{1}{6}} (\log q)^2, \quad \eta = 1 - \left(\frac{\Delta^2}{q}\right)^{\frac{1}{6}} \log q, \quad \delta = 2 \left(\frac{\Delta^2}{q}\right)^{\frac{1}{6}}. \quad (6.5)$$
Our analysis is inspired by the “nibble” technique introduced in Rödl’s pioneering work \[31\], which was later developed by various authors. Our goal is to show that we can make some deterministic choices by some probabilistic arguments so that \((6.1)\) and \((6.2)\) hold after the \((t + 1)\)-st iteration. Next, we state a few lemmas showing that such deterministic choices can be made.

For the convenience of writing, we define a notion “with very high probability” (in short, w.v.h.p.). We say an event \(A_q\) happens w.v.h.p. to mean that the probability that \(A_q\) occurs with probability at least \(1 - e^{-\omega(\log q)}\), where \(\omega(n)\) is any function whose ratio with \(n\) tends to infinity as \(n\) tends to infinity.

**Lemma 6.1.** In the \((t + 1)\)-st iteration, the number of edges deleted from a single color class in Steps 3 and 4 combined is at most \(2a_t s_t + 2\Delta \sqrt{\delta q} \log q\) w.v.h.p.

**Proof.** Fix a color remaining at the start of the \((t + 1)\)-st iteration. Remember that by assumption, the number of edges in that color class before starting the \((t + 1)\)-st iteration is \(s_t\). Let \(S_{t+1}\) be the random variable denoting the number of edges of that color right after Step 4 in the \((t + 1)\)-st iteration. Thus, \(L = s_t - S_{t+1}\) denotes the number of edges of the fixed color that are removed because of Steps 3 and 4. Recall that every vertex gets deleted with probability \(a_t\) in Steps 3 and 4. For convenience, let \(H\) denote the set of edges in the fixed color class before starting the \((t + 1)\)-st iteration.

Observe that \(E(L) = \sum_{e \in H} a_t = 2a_t s_t\). To show concentration of \(L\), we use Talagrand’s inequality (i.e., Theorem \[33\]). The random variable \(L\) can be seen as a function with domain \(\Omega\), where \(\Omega = \prod_{C \in \mathcal{C}} \Omega_C \times \prod_{v \in V(G)} \Omega_v\) with \(\mathcal{C}\) being the set of colors and \(\Omega_C\) (and \(\Omega_v\)) denoting the probability space for the decision made for the color \(C\) in Steps 1 and 2 (and the vertex \(v\) in Step 4). If the decision for one of the \(\Omega_C\)'s or \(\Omega_v\)'s is altered, then \(L\) will be affected by at most \(2\Delta\). Thus, \(L\) is \(2\Delta\)-Lipschitz. Furthermore, observe that \(L\) is 1-certifiable. Indeed, for any edge \(e\) removed with the fixed color, there must either be another edge incident to \(e\) in \(T\) or one of the vertices in \(e\) is deleted in Step 4, and these events certify the removal of \(e\). Hence, using Theorem \[33\] we can conclude that the probability that \(L\) deviates from its mean by \(\lambda = \Delta \sqrt{a_t s_t} \log q \geq \frac{\lambda}{2} + 60 \cdot (2\Delta) \cdot \sqrt{E(L)} = \max(\delta g(t\delta)^2 / \gamma, \delta q) \leq e^{-\omega(\log q)}\).

Now, Lemma \[6.1\] follows by noting the following inequality:

\[
a_t s_t = \frac{\gamma \delta g(t\delta)(1 + \beta_t)}{s(t\delta)(1 - \alpha_t)} \cdot (1 - \alpha_t) s(t\delta)(1 + \epsilon) q \leq 2\delta q.
\]

**Lemma 6.2.** At the end of \((t + 1)\)-st iteration, for each remaining vertex \(v\), w.v.h.p. the degree of \(v\) is at most \(d_t (1 - a_t)(1 - \theta_{t+1}) + 2d_t \theta_{t+1}^2 + 2d_t \delta^2 + 2d_t \delta + 4\Delta \sqrt{\delta q} \log q\).

**Proof.** First of all, Step 7 can only decrease the degrees of any vertex; thus, it is enough to prove Lemma \[6.2\] before Step 7. For each remaining vertex \(v\) after the \(t\)-th iteration, we define \(D_{t+1,v}\) to be the random variable denoting the degree of \(v\) after Step 6 in the \((t + 1)\)-st iteration. Throughout the proof of this lemma, for a simpler analysis, we use an alternative way to execute \textit{Algorithm} that produces exactly the same outcomes (more precisely, the outcome space has the same probability distribution for both processes). We will do the first four steps in a different order, namely Steps 2, 4, 1, and then 3 (this can be done because these steps can essentially be done independently). We next elaborate on the exact process in the \((t + 1)\)-st iteration executed in this way.

1'. Select independently one edge from each remaining color class (instead of just the activated ones as originally described in \textit{Algorithm}) uniformly at random. Denote by \(H\) the set of all selected edges.

2'. Independently select each vertices \(v\) with probability \(p_v\). Denote by \(V\) the set of all selected vertices.

3'. Activate each color class independently with probability \(\theta_{t+1} = \frac{\delta}{\gamma t \theta_{t+1}}\). For each activated color class, pick the edge from \(H\), which has this color. Denote by \(T\) the set of all picked edges.

4'. Delete all the vertices of \(V\) and all the vertices corresponding to the edges in \(T\). Remember that deleting vertices also deletes all incident edges.
Then, Steps 5 and 6 are done as described in the original Algorithm. The added benefit to performing the steps in the above way is that, typically, the edge-set $H$ in Step 1′ induces a graph with small degree. Appropriately conditioned on this, when we activate color classes in Step 3′, we obtain a much better certifiability constant for the application of Talagrand’s inequality.

Fix a vertex $v$ throughout the proof of this lemma. Let $d$ denote the degree of $v$ after the $t$-th iteration of Algorithm. By our assumption, we have $d \leq d_t$. Let $X$ be the random variable denoting the number of neighbors of $v$ selected in $V$ in Step 2′, and $L$ be the random variable denoting the number of edges $uv$ adjacent to $v$ deleted in Steps 4′ and 6 and $u$ is not selected in $V$ in Step 2′. Clearly, $D_{t+1,v} = d - (X + L)$. Thus, we are interested in a good lower bound on the random variable $X + L$.

We start by collecting a few w.v.h.p. events that will be helpful to prove Lemma 6.2. For convenience, we say a vertex $u$ meets a color $C$ if there is an edge incident to $u$ that is colored with $C$. Define $U$ as the set of vertices $u$ such that $u$ and $v$ meet some common color (notice that $N(v) \subseteq U$). Denote by $d_H(u)$ the number of edges in $H$ that are incident to $u$. Define $Z$ as the number of neighbors $u \in N(v)$ such that the edge in $H$ with the same color as $uv$ is incident to $u$. To facilitate the analysis of the random variable $L$, we will consider the random variable $X' = \theta_{t+1} \sum_{u \in N} d_H(u)$, where $N$ denotes the (random) subset of $N(v)$ containing all the vertices which are not selected in Step 2′. In the proof of the following claim, we often use the observation that $d_t \leq 2s_t$.

Claim 6.3. The following events hold w.v.h.p.

- $F_1$: every vertex $u \in U$ satisfies that $d_H(u) \leq \log q$, and $\sum_{u \in N(v)} d_H(u) \leq 3d_t$.
- $F_2$: $Z \leq \Delta \log q$.
- $F_3$: $X \geq \sum_{u \in N(v)} p_u - \sqrt{\delta q \log q}$.
- $F_4$: $X' \geq \left(\theta_{t+1} \sum_{u \in N(v)} \frac{d_t}{s_t} (1 - p_u)\right) - \sqrt{\delta q \log q}$, where $\delta_{t,u}$ denotes the degree of $u$ after the $t$-th iteration.

Proof. Clearly, $|U| \leq 2(1 + \epsilon)q^2$. To see that $F_1$ holds w.v.h.p., observe that each edge is selected in Step 1′ with probability exactly $\frac{1}{s_t}$. Thus, for each vertex $u$, we have $\mathbb{E}(d_H(u)) \leq \frac{d_t}{s_t} \leq 2$. Clearly, $d_H(u)$ is a random variable with domain $\Omega = \prod_{C \in \mathcal{C}} \Omega_C$ where $\Omega_C$ denotes the probability space for the random edge chosen from the color class $C$. It is straightforward to check that $d_H(u)$ is 1-Lipschitz and 1-certifiable. Thus, using Theorem 3.3 we have that $\mathbb{P}(d_H(u) \geq \log q) = e^{-\omega(\log q)}$. Now, by a simple union bound over all vertices in $U$, we conclude that the first part of $F_1$ holds w.v.h.p.

Define the random variable $Y' = \sum_{u \in N(v)} d_H(u)$. Note that $\mathbb{E}(Y') \leq 2d_t$. Clearly, $Y'$ is a random variable with domain $\Omega = \prod_{C \in \mathcal{C}} \Omega_C$. It is easy to check that $Y'$ is 2-Lipschitz and 1-certifiable. Thus, apply Theorem 3.3 to conclude that $Y'$ deviates from its expectation by at least $d_t \geq \frac{d_t}{s_t} + 60 \cdot 2 \cdot \sqrt{\mathbb{E}(Y')} = 4e^{-\frac{\delta q \log q}{2\mathbb{E}(Y')}} = e^{-\omega(\log q)}$. This proves that the second part of $F_1$ holds w.v.h.p.

Note that $\mathbb{E}(Z) \leq \sum_{u \in N(v)} \frac{\Delta}{s_t} d_t \leq \Delta \cdot \frac{d_t}{s_t} \leq 2\Delta$. Clearly, $Z$ is a random variable with domain $\Omega = \prod_{C \in \mathcal{C}} \Omega_C$. It is again easy to check that $Z$ is 1-Lipschitz and 1-certifiable. Thus, apply Theorem 3.3 to conclude that $Z$ deviates from its expectation by at least $\frac{1}{2}\Delta \log q \geq \frac{1}{4}\Delta \log q + 60\sqrt{\mathbb{E}(Z)}$ is at most $4e^{-\frac{\delta q \log q}{6 \mathbb{E}(Y')}} = e^{-\omega(\log q)}$. Thus, w.v.h.p. it holds that $|Z| \leq 2\Delta + \frac{\delta \Delta \log q}{16} \leq \Delta \log q$.

Next, observe that $\mathbb{E}(X) = \sum_{u \in N(v)} p_u \leq a_t d_t$. Observe that $X$ is the sum of at most $d_t$ Bernoulli random variables with probability at most $a_t$. So, by using standard Chernoff bound (Theorem 3.1), the probability that $X$ deviates by more than $\frac{1}{2}\sqrt{a_t d_t \log q} \leq \sqrt{\delta q \log q}$ from its expectation is at most $e^{-\omega(\log q)}$, where we use the following inequality:

$$a_t d_t = \frac{\delta(1 + \beta_1)}{s_t(1 - \alpha_1)} \cdot (1 + \beta_1)(1 - t\delta)g(t\delta)q \leq 2d_t.$$

To show that the event $F_2$ holds w.v.h.p. and condition on the events $F_1$ and $F'$, where $F'$ denotes the event that the random variable $Y = \sum_{u \in N(v)} d_H(u)(1 - p_u)$ satisfies the following:

$$Y \geq \left(\sum_{u \in N(v)} \frac{d_t}{s_t} (1 - p_u)\right) - \sqrt{d_t \log q}.$$
Note that $\mathbb{E}(Y) = \sum_{v \in N(v)} \frac{d_{l,v} u}{s_t} (1 - p_u) \leq d_t^2 / s_t \leq 2d_t$. Clearly, $Y$ is a random variable with domain $\Omega = \prod_{C \in \mathcal{H}} \Omega_C$. It is easy to check that $Y$ is 2-Lipschitz and 1-certifiable. Thus, apply Theorem 3.3 to conclude that $Y$ deviates from its expectation by at least $\sqrt{d_t \log q} \geq \frac{1}{2} \sqrt{d_t \log q} + 60 \cdot 2 \cdot \sqrt{\mathbb{E}(Y)}$ is at most

$$4e^{-\frac{d_t \log q}{2}} = e^{-\omega(\log q)}.$$

Since the events $F_1$ and $F'$ hold w.v.h.p., it is enough to prove that $F_4$ occurs w.v.h.p. conditioned on $F_1 \land F'$. Fix any choice of $H = H'$ in Step 1' such that $F_1$ and $F'$ hold. For each $u \in N(v)$, let the number of edges in $H'$ incident to $u$ be $l(u)$. Thus, we have the following two inequalities:

1. Any vertex $u \in N(v)$ satisfies that $l(u) \leq \log q$.
2. $\sum_{u \in N(v)} l(u) (1 - p_u) \geq \left( \sum_{u \in N(v)} \frac{d_{l,uv}}{s_t} (1 - p_u) \right) - \sqrt{d_t \log q}$.

Observe that $\mathbb{E}(X'|H') = \theta_{t+1} \sum_{u \in N(v)} l(u) (1 - p_u)$. Furthermore, conditioning on $H'$, we can prove a concentration bound on $X'$ using Theorem 3.3 Consider the random variable $X'' = \sum_{u \in N} l(u)$, which is basically same as $\frac{\sum X}{\theta_{t+1}}$ conditioned on $H'$. By Lemma 6.2, we have $\mathbb{E}(X''|X') = \sum_{u \in N(v)} l(u) (1 - p_u) \leq d_t \log q$. The random variable $X''$ can be seen as a function with domain $\Omega = \prod_{u \in N(v)} \Omega_u$, with $\Omega_u$ denoting the probability space for the decision made for the vertex $u$ in Step 1'. If the decision for one of the $\Omega_u$’s is altered, then $X''$ will be affected by at most $\log q$ because of Lemma 6.2. Thus, $X''$ is $(\log q)$-Lipschitz. Furthermore, it is easy to see that $X''$ is 1-certifiable. Hence, using Theorem 3.3 we can conclude that the probability that $X''$ deviates from its mean by $\lambda = \lambda_t \log q \geq \frac{1}{2} \log q + 60 \cdot \sqrt{\mathbb{E}(X'')} \leq 4e^{-\omega(\log q)} = e^{-\omega(\log q)}$ This implies that w.v.h.p. the following holds conditioned on $H'$:

$$X' \geq \theta_{t+1} \sum_{u \in N(v)} l(u) (1 - p_u) - \theta_{t+1} \sqrt{d_t \log q}.$$

This, together with (6.6), gives us the following:

$$\mathbb{P} \left[ X' \leq \left( \theta_{t+1} \sum_{u \in N(v)} \frac{d_{l,uv}}{s_t} (1 - p_u) \right) - 2\theta_{t+1} \sqrt{d_t \log q} \right] = e^{-\omega(\log q)}.$$

Note the following:

$$\theta_{t+1} \sqrt{d_t} = \frac{\delta}{1 - t} \sqrt{(1 + \beta_t)(1 - t\delta) g(t\delta) q} \leq \frac{2\delta \sqrt{q}}{\sqrt{q - \eta}} \leq \frac{2\delta q \log q}{\sqrt{q}}.$$

where the last inequality can be checked by plugging in the values of the parameters set in (6.3). Using (6.7) and the fact that the bound in (6.6) holds for any choice of $H'$ satisfying $F_1$ and $F'$, we have the following conditioned on $F_1 \land F'$:

$$\mathbb{P} \left[ X' \leq \left( \theta_{t+1} \sum_{u \in N(v)} \frac{d_{l,uv}}{s_t} (1 - p_u) \right) - \sqrt{d_t \log q} \right] = e^{-\omega(\log q)}.$$

This finishes the proof of Claim 6.3.

We next show the concentration of $L$ conditioned on all the w.v.h.p. events in Claim 6.3. Formally, let $F$ denote the event that the events $F_1$, $F_2$, $F_3$, and $F_1$ hold. Next, we show the concentration bound on $L$ conditioning on this event $F$, which holds w.v.h.p. Lemma 6.2 follows from the following claim.

Claim 6.4. Conditioned on $F$, at the end of Step 6 of the $(t + 1)$-st iteration, w.v.h.p. the degree of the vertex $v$ after $t$-th iteration is at most $d_t (1 - a_t) (1 - \theta_{t+1}) + 2d_t \theta_{t+1}^2 \log^2 q + 4\Delta \sqrt{d_t \log q}$.

Proof. Fix any choice of $H'$ in Step 1' and $V'$ in Step 1' such that $F$ holds. For each $u \in N(v)$, let the number of edges in $H'$ incident to $u$ be $l(u)$. Suppose that $Z'$ is the set of neighbors of $u \in N(v)$ such that the edge in $H'$ with the same color as $uv$ is incident to $u$. Let $N = N(v) \setminus V'$ and $N' = N(v) \setminus (V' \cup Z')$. 

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Fix a neighbor \( u \in N' \). The probability that one of the edges incident to \( u \) is picked in Step 3’ is exactly \( 1 - (1 - \theta_{t+1})^{l(u)} \geq l(u)\theta_{t+1} - l(u)^2\theta_{t+1}^2 \). Thus, the next task remaining is to calculate the probability that the edge \( uv \) will be deleted in Step 6, and simultaneously no edge incident to the vertex \( u \) gets picked in Step 3’. For the convenience of writing, let \( A \) denote the event that an edge of the same color as \( uv \) is picked in the partial rainbow matching from \( T \) in Step 5. Let \( B \) denote the event that none of the edges incident to \( u \) gets deleted in Step 3’. We are interested in getting a good lower bound on \( \mathbb{P}[A \cap B] \).

Let \( e \) denote edge in \( H' \) that has the same color as \( uv \) (clearly \( e \) is not incident to \( u \) since \( u \notin Z' \)). The event \( A \cap B \) is equivalent to the following event:

(i) the color of \( uv \) is activated in Step 3’ and

(ii) none of the colors corresponding to the edges in \( H' \) incident to \( u \) or \( e \) is activated in Step 3’.

Observe that the event of (i) and the event that none of the edge \( s \) adjacent to \( u \) will be deleted in Step 6, and simultaneously no edge incident to the vertex \( u \) is altered, then

\[
\lambda(u) = \sum_{u \in N} (l(u)\theta_{t+1} - l(u)^2\theta_{t+1}^2 + \theta_{t+1} (1 - \theta_{t+1})^{3\log q})
\]

(6.8) holds because \( l(u) \leq \Delta \log q \) (since the event \( F_1 \) from Claim 6.3 holds). In (6.8), we use \( |Z'| \leq \Delta \log q \) (since the event \( F_2 \) from Claim 6.3 holds). In (6.10), we use the fact that \( \Delta \leq \sqrt{d} \).

Furthermore, conditioning on \( H', V' \), we can prove a concentration bound on \( L \) using Theorem 5.3. Note that the probability of a vertex \( u \) is deleted in Step 4’ at most \( l(u)\theta_{t+1} \) and the probability that an edge with a fixed color is picked in Step 5 is most \( \theta_{t+1} \). Thus, using the fact that \( \sum_{u \in N(v)} l(u) \leq 3d_t \) since \( H' \) satisfies \( F_1 \) from Claim 6.3, we have:

\[
\mathbb{E}(L \mid H', V') \leq \sum_{u \in N(v)} (l(u)\theta_{t+1} + \theta_{t+1}) \leq 4d_t \theta_{t+1} \leq 5\delta q.
\]

The random variable \( L \) conditioned on \( H', V' \) can be seen as a function with domain \( \Omega = \prod_{C \in \Phi} \Omega_C \) with \( \Omega_C \) denoting the probability space for the decision made for color \( C \). If the decision for one of the \( \Omega_C \)’s is altered, then \( L \) will be affected by at most \( \Delta + 2 \leq 3\Delta \). Thus, \( L \) is \( 3\Delta \)-Lipschitz. Furthermore, we claim that \( L \) is \((2\log q)\)-certifiable. Indeed, for any vertex \( u \in N \) removed in Step 4’, there must be an edge \( e \in H' \) incident to \( u \) whose color is activated in Step 3’, and none of the colors (less than \( 2\log q \)) corresponding to the edges in \( H' \) incident to \( e \) is picked. Thus, fixing the choices for these colors along with the color of \( e \) certifies the deletion of \( u \) in Step 4’. Also, for any edge \( uv \) removed in Step 6, the color of \( uv \) is activated in Step 3’ and none of the colors (less than \( 2\log q \)) corresponding to the edges in \( H' \) incident to \( e \) is picked, where \( e \) is the edge in \( H' \) with the same color as \( uv \). Thus, fixing the choices for these colors along with the color of \( uv \) certifies the deletion of \( uv \) in Step 6. This establishes that \( L \) is \((2\log q)\)-certifiable. Hence, using Theorem 5.3, we can conclude that the probability that \( L \) deviates from its mean by \( \lambda = \Delta \sqrt{\delta q} \log q \geq \frac{3}{2} + 60 \cdot 3\Delta \cdot \sqrt{2 \log q} \cdot \mathbb{E}(L \mid H', V') \) is at most \( 4e^{-\frac{(\lambda/2)^2}{4(3\Delta)^2 \cdot 2\log q \cdot \mathbb{E}(L \mid H', V')}} = e^{-\omega(\log q)} \).

Thus, we have the following:

\[
\mathbb{P} \left[ L \geq \mathbb{E}(L \mid H', V') - \Delta \sqrt{\delta q} \log q \mid H', V' \right] = 1 - e^{-\omega(\log q)}.
\]

(6.11)
Conditioning on \( H', V' \) and assuming that the event in (6.11) holds, by (6.10) we have the following:

\[
D_{t+1,v} = |N| - L \leq \sum_{u \in N} (1 - \theta_{t+1} - l(u)\theta_{t+1}) + 2d_i\theta_{t+1}^2 log^2 q + \sqrt{\theta_{t+1}} log q + \Delta \sqrt{\theta_{t+1}} log q. \tag{6.12}
\]

Note that \( \sum_{u \in N} (1 - \theta_{t+1} - l(u)\theta_{t+1}) = (1 - \theta_{t+1})|N| - \theta_{t+1} \sum_{u \in N} l(u) \). Thus, we have:

\[
D_{t+1,v} \leq (1 - \theta_{t+1}) \sum_{u \in N(v)} (1 - p_u) - \theta_{t+1} \sum_{u \in N(v)} \frac{d_{t,u} \theta_{t+1}}{s_t} (1 - p_u) + 2d_i\theta_{t+1}^2 log^2 q + (3 + \Delta) \sqrt{\theta_{t+1}} log q \tag{6.13}
\]

\[
= \sum_{u \in N(v)} (1 - p_u) \left( 1 - \theta_{t+1} - \frac{d_{t,u} \theta_{t+1}}{s_t} \right) (1 - p_u) + 2d_i\theta_{t+1}^2 log^2 q + 4\Delta \sqrt{\theta_{t+1}} log q \leq \sum_{u \in N(v)} (1 - p_u) (1 - p_u') (1 - \theta_{t+1}) + 2d_i\theta_{t+1}^2 log^2 q + 4\Delta \sqrt{\theta_{t+1}} log q \tag{6.14}
\]

\[
= \sum_{u \in N(v)} (1 - a_t) (1 - \theta_{t+1}) + 2d_i\theta_{t+1}^2 log^2 q + 4\Delta \sqrt{\theta_{t+1}} log q, \tag{6.15}
\]

where in (6.13), we use the fact that the choices of \( H' \) and \( V' \) satisfy the events \( F_3 \) and \( F_4 \) from Claim 6.3. In (6.14) and (6.15), we use the inequality at (6.3) and the definition of \( p_u \) (mentioned after 6.3) respectively.

Since the bounds in (6.11), (6.12), and (6.15) hold for any choices of \( H' \) and \( V' \) satisfying \( F \), we have the following conditioned on \( F \):

\[
P \left[ D_{t+1,v} \leq d_t (1 - a_t) (1 - \theta_{t+1}) + 2d_i\theta_{t+1}^2 log^2 q + 4\Delta \sqrt{\theta_{t+1}} log q \right] = 1 - e^{-\omega(\log q)}.
\]

This finishes the proof of Claim 6.4 and Lemma 6.2. \( \square \)

Recall our assumption that after the \( t \)-th iteration, the bounds in (6.1) and (6.2) hold. As long as for the \((t+1)\)-st iteration, there is a positive probability by which the conclusions of Lemma 6.1 for each color class and Lemma 6.2 for each vertex hold, then we can choose a particular possible output satisfying these. Using a simple union bound to show this is impossible since we do not assume any bound on the number of colors or vertices in the graph. However, utilizing the local lemma, we can show that with positive probability, Lemma 6.1 holds for every color, and Lemma 6.2 for each vertex hold, then we can choose a particular possible output satisfying these. Using a simple union bound to show this is impossible since we do not assume any bound on the number of colors or vertices in the graph. However, utilizing the local lemma, we can show that with positive probability, Lemma 6.1 holds for every color, and Lemma 6.2 holds for every vertex simultaneously.

**Lemma 6.5.** The following happens with positive probability in the \((t+1)\)-st iteration.

1. The number of edges deleted from each of the color classes in Steps 3 and 4 combined is at most \( 2a_t s_t + 2\Delta \sqrt{\theta_{t+1}} log q \).
2. The degree of each remaining vertex is at most \( d_t (1 - a_t) (1 - \theta_{t+1}) + 2d_i\theta_{t+1}^2 log^2 q + 4\Delta \sqrt{\theta_{t+1}} log q \).

**Proof.** Let \( \Gamma \) be the graph on \( \mathcal{C} \), the set of all colors, where two colors are joined if a vertex in \( G \) meets both colors. For each \( C \in \mathcal{C} \), define the ‘bad events’ \( B_C \) to be the event that at least one of the following events happens:

- for the color class \( C \) the assertion of (1) does not hold,
- there is a vertex \( v \) that meets the color class \( C \) such that \( v \) violates the assertion of (2).

It is straightforward to see that to prove Lemma 6.3 it is enough to show that with positive probability, none of the events \( B_C \) happen. Note that For each \( C \in \mathcal{C} \), the event \( B_C \) is determined by the random choices involving the color classes \( C' \) with \( d_t(C,C') \leq 2 \) (where \( d_t(C,C') \) denotes the distance between \( C \) and \( C' \) in the graph \( \Gamma \), i.e., the length of the shortest path in between \( C \) and \( C' \)). We claim that the graph \( \Gamma^4 \), defined
by the graph on $\mathcal{G}$ with edges between $\mathcal{C}$ and $\mathcal{C}'$ if and only if $d_{t}(\mathcal{C}, \mathcal{C}') \leq 4$, is a dependency graph for the events $\{B_{\mathcal{C}}\}_{\mathcal{C} \in \mathcal{G}}$. This is because if $\mathcal{C}' \notin \Gamma^{4}$, then the events $B_{\mathcal{C}}$ and $B_{\mathcal{C}'}$ are determined by disjoint random choices. Observe that the maximum degree of $\Gamma$ is at most $4q^{2}$, and consequently, the maximum degree of $\Gamma^{4}$ is at most $256q^{8}$. By Lemmas 6.1 and 6.2 and a simple union bound, for all $\mathcal{C} \in \mathcal{G}$, we have $\mathbb{P}[B_{\mathcal{C}}] = e^{-\omega(\log q)}$. Thus, by a simple application of Lemma 3.4, we conclude that with positive probability, none of the events $B_{\mathcal{C}}$ happen, which is what we desired to show to finish the proof of Lemma 6.5.

Lemma 6.5 implies that there is an output of the $(t + 1)$-st iteration of Algorithm satisfying (1) and (2). Thus, we fix such an outcome and assume (1) and (2) throughout the next section.

7 Estimating the error terms and greedy completion

To estimate the error terms in the parameters throughout Algorithm, let us first estimate the error terms in the ideal expressions from the intuitive analysis. Define the following parameters, which essentially ignore the error terms $\alpha_{t}$ and $\beta_{t}$ in the expressions in (6.1), (6.2), and (6.4).

$$\tilde{s}_{t} = s(t\delta)(1 + \epsilon)q \quad \text{and} \quad \tilde{d}_{t} = (1 - t\delta)g(t\delta)q \quad \text{and} \quad \tilde{a}_{t} = \frac{\delta g(t\delta)}{s(t\delta)}.$$

We have the following couple of lemmas concerning the relations between these three ideal functions.

**Lemma 7.1.** We have that $(1 - 2\tilde{a}_{t})\tilde{s}_{t} = \tilde{s}_{t+1} - \gamma^{2}\delta^{2}(1 + \epsilon)q$.

**Proof.** A routine calculation shows:

$$(1 - 2\tilde{a}_{t})\tilde{s}_{t} = \left(1 - \frac{2\gamma\delta g(t\delta)}{s(t\delta)}\right)s(t\delta)(1 + \epsilon)q = \left((1 - \gamma t\delta)^{2} - 2\gamma\delta(1 - \gamma t\delta)\right)(1 + \epsilon)q = \left((1 - \gamma\delta(t + 1)\right)^{2} - \gamma^{2}\delta^{2}\right)(1 + \epsilon)q = \tilde{s}_{t+1} - \gamma^{2}\delta^{2}(1 + \epsilon)q.$$

\[\square\]

**Lemma 7.2.** We have that $(1 - \tilde{a}_{t})\left(1 - \frac{\tilde{s}_{t}}{1 + \tilde{s}_{t}}\right)\tilde{d}_{t} = \tilde{d}_{t+1}$.

**Proof.** A routine calculation shows:

$$(1 - \tilde{a}_{t})\left(1 - \frac{\tilde{s}_{t}}{1 + \tilde{s}_{t}}\right)\tilde{d}_{t} = \left(1 - \frac{\gamma\delta(1 - \gamma t\delta)}{1 - \gamma t\delta}\right)\left(1 - \frac{\delta}{1 - t\delta}\right)(1 - t\delta)g(t\delta)q = (1 - (t + 1)\gamma\delta)(1 - (t + 1)\delta)q = \tilde{d}_{t+1}.$$

\[\square\]

Now we mention a couple of lemmas relating the error terms $\alpha_{t}$ and $\beta_{t}$ we accumulate for estimating $d_{t}$ and $s_{t}$ as the process goes. In other words, we will specify $y_{t}$ and $z_{t}$ such that $d_{t} \leq \tilde{d}_{t} + y_{t}$ and $s_{t} \geq \tilde{s}_{t} - z_{t}$. We will then define that $\alpha_{t} = \frac{y_{t}}{s_{t}}$ and $\beta_{t} = \frac{z_{t}}{d_{t}}$.

Define $y_{0} = z_{0} = 0$ and recursively define the following sequences for each $0 \leq t < \eta\tau$.

$$y_{t+1} = y_{t} + 2\delta^{2}q\log q \cdot \left(\frac{1 - \gamma t\delta}{1 - t\delta}\right) + 4\Delta \sqrt{\delta q \log q} \quad \text{and} \quad z_{t+1} = z_{t} + \delta^{2}q + 2\Delta \sqrt{\delta q \log q} + \frac{2\delta}{1 - t\delta} \cdot y_{t}.$$ 

Furthermore, define $\alpha_{t} = \frac{y_{t}}{s_{t}}$ and $\beta_{t} = \frac{z_{t}}{d_{t}}$ for each $t$. The following lemma finally shows the validity of (6.1) and (6.2) for the $(t + 1)$-st iteration, which is what we needed to show that we can choose a desirable output from Algorithm for each $t \leq \eta\tau$. 

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Lemma 7.3. At the end of the \((t+1)\)-st iteration, the following two holds:

1. the degree of each survived vertex is at most \(d_{t+1}\).

2. the remaining color classes have size exactly \(s_{t+1}\).

Proof. Starting with the conclusion of Lemma 6.3 and noting that \(a_t = \frac{1 + \delta_t}{1 - \alpha_t} \geq \alpha_t\), a routine calculation shows the following bound on the degree \(d_{t+1,v}\) of \(v\) after the \((t+1)\)-st iteration.

\[
\begin{align*}
d_{t+1,v} & \leq d_t \cdot (1 - \bar{a}_t) (1 - \theta_{t+1}) + 2d_t \theta_{t+1}^2 \log^2 q + 4\Delta \sqrt{q} \log q \\
& = \left( \tilde{d}_t + y_t \right) (1 - \bar{a}_t) (1 - \theta_{t+1}) + 2\theta_{t+1}^2 \log^2 q + 4\Delta \sqrt{q} \log q \\
& \leq \tilde{d}_t (1 - \bar{a}_t) (1 - \theta_{t+1}) + 2\theta_{t+1}^2 \log^2 q + y_t + 4\Delta \sqrt{q} \log q \\
& = \tilde{d}_t + y_{t+1} = (1 + \beta_{t+1}) \tilde{d}_{t+1} = d_{t+1},
\end{align*}
\]

where in (7.1), we use the following:

\[
(1 - \bar{a}_t) (1 - \theta_{t+1}) + 2\theta_{t+1}^2 \log^2 q \leq 1 - \theta_{t+1} + 2\theta_{t+1}^2 \log^2 q = 1 - \theta_{t+1} (1 - 2\theta_{t+1} \log^2 q)
\]

\[
= 1 - \theta_{t+1} \left( 1 - \frac{2\delta \log^2 q}{1 - t\delta} \right)
\]

\[
\leq 1 - \theta_{t+1} \left( 1 - \frac{2\delta \log^2 q}{1 - \eta} \right) \leq 1.
\]

Next, starting with the conclusion of Lemma 6.3 the size of each remaining color class after Step 4 of the \((t+1)\)-st iteration is at least the following:

\[
(1 - 2\alpha_t) s_t - 2\Delta \sqrt{q} \log q = \left( 1 - 2\bar{a}_t \frac{1 + \beta_t}{1 - \alpha_t} \right) (1 - \alpha_t) \tilde{s}_t - 2\Delta \sqrt{q} \log q
\]

\[
= (1 - 2\bar{a}_t) \tilde{s}_t - \alpha_t \tilde{s}_t - 2\bar{a}_t \beta_t \tilde{s}_t - 2\Delta \sqrt{q} \log q
\]

\[
= \tilde{s}_{t+1} - \gamma^2 \delta^2 (1 + \epsilon) q - z_t - \frac{2\gamma \delta (1 + \epsilon)}{1 - t\delta} \cdot y_t - 2\Delta \sqrt{q} \log q
\]

\[
\geq \tilde{s}_{t+1} - \delta^2 q - z_t - \frac{2\delta}{1 - t\delta} \cdot y_t - 2\Delta \sqrt{q} \log q
\]

\[
= \tilde{s}_{t+1} - z_{t+1} = (1 - \alpha_{t+1}) \tilde{s}_{t+1} = s_{t+1}.
\]

Thus, after Step 7 of the \((t+1)\)-st iteration each remaining color class must have exactly \(s_{t+1}\) edges.

Next, as we promised, we will find that the accumulated error terms are negligible compared to the ideal parameter values, i.e., \(y_t \ll \tilde{d}_t\) and \(z_t \ll \tilde{s}_t\). This, in other words, shows that \(\beta_t < 1/100\) and \(\alpha_t < 1/100\), which were used several times in the previous section.

**Lemma 7.4.** For each \(t \leq \frac{q}{2}\), we have that \(y_t = o \left( \tilde{d}_t \right)\).

Proof. We have the following:

\[
y_t = \sum_{i=0}^{t-1} \left( 2\delta^2 q \log^2 q \cdot \frac{1 - \gamma i \delta}{1 - t\delta} + 4\Delta \sqrt{q} \log q \right) \leq 2\delta q \log^2 q \int_0^{t\delta} \frac{1 - \gamma x}{1 - x} dx + 4t\Delta \sqrt{q} \log q
\]

\[
= 2\delta q \log^2 q \left( (1 - \gamma) \log \left( \frac{1}{1 - t\delta} \right) + \gamma t\delta \right) + 4t\Delta \sqrt{q} \log q
\]

\[
\leq 2\delta q \log^2 q \cdot \left( \log \left( \frac{1}{1 - \eta} \right) \right) + 2t\delta^2 q \log^2 q + 4t\Delta \sqrt{q} \log q.
\]
Plugging in the parameter values in the above inequality, we get the following:

\[ y_t \leq y_{\frac{q}{2}} \leq (\Delta q)^{\frac{3}{2}} (\log q)^{5/2}. \]  

(7.2)

Plugging in the parameters in the expression for \( \tilde{d}_t \), we have the following:

\[ \tilde{d}_t \geq \tilde{d}_{\frac{q}{2}} = (1 - \eta)(1 - \gamma \eta)q \geq \frac{1}{2}(\Delta q)^{\frac{3}{2}} (\log q)^{3/2}. \]  

(7.3)

By Equations (7.2) and (7.3), we conclude that Lemma 7.4 holds.

**Lemma 7.5.** For each \( t \leq \frac{q}{3} \), we have that \( z_t = o(\tilde{s}_t) \).

**Proof.** We have the following:

\[
z_t = \sum_{i=0}^{t-1} \left( 2\delta^2 q + 2\Delta \sqrt{\delta q \log q} + 2\delta \cdot y_i \right) \leq 2\delta^2 q + 2t\Delta \sqrt{\delta q \log q} + 2yt \int_0^{t\delta} \frac{dx}{1 - x} \leq 2\delta^2 q + 2t\Delta \sqrt{\delta q \log q} + 2yt \log \left( \frac{1}{1 - \eta} \right).
\]

Similar to before, plugging in the parameter values in the above and using (7.2), we get the following:

\[ z_t \leq z_{\frac{q}{2}} \leq (\Delta q)^{\frac{3}{2}} (\log q)^{15/4}. \]  

(7.4)

Now, Lemma 7.5 can be seen to be true by noting the following:

\[ \tilde{s}_t \geq \tilde{s}_{\frac{q}{2}} = (1 - \gamma \eta)^2 (1 + \epsilon)q \geq \frac{1}{2}(\Delta q)^{\frac{3}{2}} (\log q)^{4}. \]

(7.5)

Finally, from (6.1) and (6.2), at the end of the \( \frac{q}{3} \)-th iteration, the number of edges in any remaining color is at least \( s_{\frac{q}{2}} \geq \frac{1}{4}(\Delta q)^{\frac{3}{2}} (\log q)^{4} \), and the maximum degree of the graph is at most \( d_{\frac{q}{2}} \leq 2(\Delta q)^{\frac{3}{2}} (\log q)^{3} \).

Now, it is easy to check that \( s_{\frac{q}{2}} > 4ed_{\frac{q}{2}} \). Hence, at this point, we invoke Proposition 5.5 to find a rainbow matching using the remaining colors not yet picked in the partial matching built by Algorithm. This finishes the proof of Theorem 1.9.

**8 Proof of Theorem 1.12**

In this section, we prove Theorem 1.12. We need to adjust Algorithm of Section 4 to apply for this theorem. In this entire section, we will work with the algorithm given below and refer to it.

**8.1 Algorithm**

Assume that we are given a bipartite graph \( G \) with the vertex set partition \( A \cup B \), \(|A| = q \), and all the vertices in \( A \) have degree at least \((1 + \epsilon)q \). Furthermore, \( G \) is edge-colored with some colors such that no two edges of the same color are incident to each other. We will provide a randomized algorithm, which constructs a rainbow matching using almost all the vertices in \( A \) in several iterations. For the convenience of analyzing the algorithm, we ensure that throughout the algorithm, the degree of each of the remaining vertices in \( A \) is the same at the start of each iteration. Thus, we start with deleting arbitrary edges from \( G \) to ensure that every vertex in \( A \) has degree exactly \((1 + \epsilon)q \).

This algorithm is in terms of some parameter \( \delta > 0 \), which will be specified later. Define \( \tau = \frac{1}{\delta} \). We follow the algorithm below for \( \eta \tau \) steps, for some \( \eta < 1 \) to be specified later. For \( 1 \leq t \leq \eta \tau \), the \( t \)-th iteration of the algorithm is given below.

1. Select independently \( \delta q \) edges \( e_1, e_2, \ldots, e_{\delta q} \) uniformly at random with replacement from among the remaining edges. Denote by \( T \) the set of all selected edges.
2. For each edge $e_i \in T$, add it to the rainbow matching if $e_i$ is not incident to any edge $e_j \in T$ for $j < i$ and $e_i$ does not have same color as any edge $e_j \in T$ for $j < i$.

3. Delete all the vertices in $A$ that are incident to the edges added in the rainbow matching in the last step. Additionally, delete all the vertices in $B$ that are incident to some edge in $T$. Deleting vertices always also deletes all incident edges.

4. With some probability (to be specified later), independently delete each vertex in $B$. This will ensure that among Steps 3 and 4 combined, every vertex in $B$ gets deleted with the same probability.

5. For each edge $e \in T$, delete its color class from $G$. Here, deleting a color class means deleting all the remaining edges of that color.

6. With some probability (to be specified later), independently delete each color class. This will ensure that among Steps 5 and 6 combined, every remaining color class gets deleted with the same probability.

7. At this point, every remaining vertex in $A$ will have degree at least $s_t$, where $s_t$ will be specified later. For each remaining vertex $v$ in $A$ after Step 6, delete arbitrary edges incident to $v$ to ensure that $v$ has exactly $s_t$ edges incident to it.

We remark here that we do not have the flexibility of throwing vertices from $A$ without using them because we need to use all of them at the end. As a result, we needed to shift Step 5 of Algorithm in Section 4 to Step 2 here. Furthermore, we needed to change Step 3 to the above form, where we delete vertices only after adding the edges to the rainbow matching in the current iteration. This makes our analysis slightly more complicated than last time.

### 8.2 Intuitive analysis

As before, we aim to show that if we run this randomized algorithm until we have used almost all the vertices in $A$, then near the end, each remaining vertices still has so many edges incident to it (relative to the number of remaining vertices in $A$) that we can conclude via a simple greedy algorithm. Similar to Section 5, we track the evolution of the degrees of the vertices and the color class sizes.

Define $d_{0,v}$ to be the initial degree of $v$ in $G$. Note that $d_{0,v} = (1 + \epsilon)q$ for $v \in A$ and $d_{0,v} \leq q$ for $v \in B$.

We will define three functions $s(x)$, $g_1(x)$, and $g_2(x)$ such that after the $t$-th iteration of the algorithm we have the following:

1. Each surviving vertex in $A$ has degree exactly $s_t$, which is approximately $s(t\delta)(1 + \epsilon)q$.
2. Each surviving vertex $v \in B$ has degree $d_{t,v}$, which is approximately $(1 - t\delta)g_1(t\delta)d_{0,v}$.
3. Each remaining color class has size is at most $d_t$, which is about $(1 - t\delta)g_2(t\delta)q$.

We outline a rough analysis below. Clearly $s(0) = 1$, $g_1(0) = 1$, and $g_2(0) = 1$. Assume that the $t$-th iteration is done, and the above three properties are true after the $t$-th iteration. First, very few edges will be discarded in Step 2 throughout the process. In other words, the number of edges in the partial rainbow matching will be roughly $t\delta q$ after the $t$-th round.

Because of Step 4, every vertex of $B$ is deleted with the same probability among Steps 3 and 4 combined. This is done only for convenience in our analysis; thus, we should set that probability (denote it by $b_t$) as low as possible. In other words, we need to find the maximum probability a vertex $v \in B$ can be deleted in Step 3 and set $b_t$ to be that maximum probability. After the $t$-th iteration, the number of remaining vertices in $A$ is at least $(1 - t\delta)q$, so the number of remaining edges is at least $(1 - t\delta)qs_t$. Now if one picks an edge uniformly at random from $G$, then the probability (denote by $p$) that one of the edges incident to $v \in B$ will be picked is exactly $d_{t,v}$ divided by the total number of edges of $G$, which is at most $\frac{d_{t,v}}{(1 - t\delta)qs_t}$. Thus, the

---

1The readers would expect to see $s_t$ (and analogously $d_t$) being used to track the trajectory of size (and degree). Still, instead, we swap the roles because the trajectory for the degree of vertices in $A$ (and size of color classes) in this section turns out to be the same as ‘size’ (and ‘degree’) of Section 5. This will be clear by the end of this subsection.
probability that \(v\) is not deleted in Step 3 is exactly \((1 - p)^{\delta q} \leq 1 - \frac{\delta g_2(t\delta) d_{0,v}}{s(t\delta)(1 + \epsilon)^\eta q}\). Since \(d_{0,v} \leq q\), we can define \(b_t\) to be about \(\frac{\delta g_1(t\delta) q}{s(t\delta)(1 + \epsilon)^\eta q}\), where \(\gamma = \frac{1}{1 + \epsilon}\).

Next, we similarly provide a probability (denote it by \(a_t\)) with which a single color class will be deleted in Steps 5 and 6 combined. Any color class will be deleted in Step 5 by probability \(\leq \frac{\delta q}{(1 - t\delta)^{q_2}} \approx \gamma^2 \frac{\delta g_2(t\delta)}{s(t\delta)}\), due to a simple union bound using the fact that there are \(\delta q\) edges picked in Step 1, relatively few conflicts in Step 2, and the number of edges remaining in the graph is at least \((1 - t\delta)q_{st}\). Hence, we set \(a_t\) to be about \(\gamma^2 \frac{\delta g_2(t\delta)}{s(t\delta)}\).

We now analyze the change in \(s_t\) to get a differential equation for the function \(s\). Any edge will be deleted in Steps 5 and 6 combined by probability \(a_t\). We also know that for every \(v \in A\), each of its neighbors gets deleted among Steps 3 and 4 with probability \(b_t\). Thus, neglecting small error terms due to lack of independence, we expect to have the following:

\[
\begin{align*}
  s_{t+1} - s_t &\approx -s_t(a_t + b_t) \\
  s'(x) &= -\gamma (g_1(x) + g_2(x)).
\end{align*}
\]

Next, let us estimate the change in \(d_{t,v}\) to get a differential equation for \(g_1\). The probability that a single vertex in \(A\) gets deleted in Step 3 is about \(\frac{\delta q}{(1 - t\delta)^{q_2}} = \frac{\delta}{1 - t\delta}\). Furthermore, each edge will be deleted in Steps 5 and 6 by probability \(a_t\). Thus, neglecting small error terms due to lack of independence, the degree of a vertex \(v \in B\) is decreased by approximately \(\left[\frac{\delta}{1 - t\delta} + \left(1 - \frac{\delta}{1 - t\delta}\right) a_t\right]\). This suggests the following behavior:

\[
\begin{align*}
  d_{t+1,v} - d_{t,v} &\approx -\left[\frac{\delta}{1 - t\delta} + \left(1 - \frac{\delta}{1 - t\delta}\right) a_t\right] d_{t,v} \\
  \left(\tau - t - 1\right)\delta g_1((t + 1)\delta)d_{0,v} - (\tau - t)\delta g_1(t\delta)d_{0,v} &\approx -\delta g_1(t\delta)d_{0,v} - (\tau - t - 1)\delta g_2(t\delta)g_1(t\delta)d_{0,v} \\
  g_1((t + 1)\delta) - g_1(t\delta) &\approx -\frac{\delta g_2(t\delta)}{s(t\delta)} g_1(t\delta)d_{0,v}.
\end{align*}
\]

The above suggests:

\[
\begin{align*}
  g_1'(x) &= -\gamma \cdot \frac{g_1(x)g_2(x)}{s(x)}.
\end{align*}
\]

We next estimate the change in \(d_t\) to obtain a differential equation for \(g_2\). The probability that a single vertex \(v \in A\) gets deleted in Step 3 is about \(\frac{\delta q}{(1 - t\delta)^{q_2}} = \frac{\delta}{1 - t\delta}\), and the probability that a single vertex \(v \in B\) gets deleted among Steps 3 and 4 is \(b_t\), and thus, we can choose \(d_{t+1}\) to be around \(\left(1 - \left[\frac{\delta}{1 - t\delta} + \left(1 - \frac{\delta}{1 - t\delta}\right) b_t\right]\right) d_t\), where the expression is similar to (8.2). Hence, similarly, we expect the following behavior:

\[
\begin{align*}
  d_{t+1} - d_t &\approx -\left[\frac{\delta}{1 - t\delta} + \left(1 - \frac{\delta}{1 - t\delta}\right) b_t\right] d_t \\
  g_2((t + 1)\delta) - g_2(t\delta) &\approx -\frac{g_1(t\delta)g_2(t\delta)}{s(t\delta)}.
\end{align*}
\]

This, in turn, suggests:

\[
\begin{align*}
  g_2'(x) &= -\gamma \cdot \frac{g_1(x)g_2(x)}{s(x)}.
\end{align*}
\]

Equations (8.3) and (8.4) along with the initial conditions \(g_1(0) = g_2(0)\) imply that \(g_1 = g_2 = g\). Combining with Equation (8.1), we have the same equations and initial conditions as in Section 5. Thus, we get \(s(x) = (1 - \gamma x)^2\) and \(g_1(x) = g_2(x) = 1 - \gamma x\).

We show in the next section that the degrees of vertices and the sizes of color classes are concentrated throughout the process, implying that we can not get stuck as long as \(t \leq \eta t\). Moreover, after the \(\eta t\)-th iteration, the degree of any remaining vertex in \(A\) is \(\approx (1 - \eta q)^2(1 + \epsilon)q\), and the number of remaining vertices in \(A\) is about \(q - \eta t\delta q = (1 - \eta)q\). As long as \((1 - \eta q)^2(1 + \epsilon)q \gg 2(1 - \eta)_q\), we can finish the rainbow matching greedily, which can be made true by choosing \(1 - \eta\) small enough compared to \(1 - \gamma\).
8.3 Formal analysis

Throughout this section, we assume that \( q \) is sufficiently large to support our arguments. Similar to Section 6, we define \( s(x) = (1 - \gamma x)^2 \) and \( g(x) = 1 - \gamma x \), where \( \gamma = \frac{1}{10} \). Similar to Section 7, the following are the ideal expressions from the intuitive analysis:

\[
\bar{s}_t = s(t\delta)(1 + \epsilon)q \quad \text{and} \quad \bar{d}_t = (1 - t\delta)g(t\delta)q \quad \text{and} \quad \bar{a}_t = \gamma \frac{\delta g(t\delta)}{s(t\delta)}.
\]

Similarly, Lemmas 7.1 and 7.2 continue to hold. We next introduce the following error term:

\[
\alpha_t = \sqrt{\delta} \left( \left( 1 + \frac{10\delta}{(1 - \gamma\eta)^2} \right)^t - 1 \right).
\]

We will make sure w.v.h.p. (recall the notion from Section 6) the following events happen for \( 1 \leq t \leq \eta\tau \):

1. The survived vertices of \( A \) have degree exactly:

\[
s_t = (1 - \alpha_t) \bar{s}_t = (1 - \alpha_t)s(t\delta)(1 + \epsilon)q.
\]

2. The degree of each survived vertex in \( B \) and the size of each remaining color class both are at most:

\[
d_t = (1 + \alpha_t) \bar{d}_t = (1 + \alpha_t)(1 - t\delta)g(t\delta)q.
\]

We fix some \( t \) with \( 0 \leq t < \eta\tau \) for this subsection and assume that (8.5) and (8.6) hold for \( t \). Our goal is to show that after the \((t + 1)\)-st iteration of the algorithm, w.v.h.p. (8.5) and (8.6) will continue to hold (notice that by a simple union bound, this is enough to establish that w.v.h.p. (8.5) and (8.6) hold for every \( 1 \leq t \leq \eta\tau \)). We will show certain concentration bounds that hold in the \((t + 1)\)-st iteration that will be used to achieve our goal. We start by specifying probabilities for Steps 4 and 6 for the \((t + 1)\)-st iteration.

In the \((t + 1)\)-st iteration, for a fixed remaining vertex \( v \) in \( B \), if one picks an edge uniformly at random from \( G \) in \( B \), then the probability that one of the edges incident to \( v \) will be picked is exactly the degree of \( v \) divided by the total number of edges of \( G \), which is at most \( \frac{d_t}{(1 - t\delta)q} \). Thus, by a simple union bound, the probability \( p_v^t \) that the vertex \( v \) is deleted in Step 3 is at most \( \delta q \cdot \frac{d_t}{(1 - t\delta)q} = \gamma \frac{\delta g(t\delta)(1 + \epsilon)}{s(t\delta)(1 - \alpha_t)} \). Define \( a_t \) to be \( \gamma \frac{\delta g(t\delta)(1 + \epsilon)}{s(t\delta)(1 - \alpha_t)} \). For a fixed \( t \), at the \((t + 1)\)-st iteration we delete each vertex \( v \) in \( B \) in Step 4 independently with probability \( p_v^t \) such that \( p_v^t + (1 - p_v^t)p_v = a_t \). Thus, \( a_t \) is the total probability by which each vertex in \( B \) gets deleted in Steps 3 and 4.

In the \((t + 1)\)-st iteration, for a fixed remaining color class \( C \), if one picks an edge uniformly at random from \( G \) in \( B \), then the probability that one of the edges of \( C \) will be picked is exactly the number of edges of \( C \) divided by the total number of edges of \( G \), which is at most \( \frac{d_t}{(1 - t\delta)q} \). Thus, by a simple union bound, the probability \( p_C^t \) that the color class \( C \) is deleted in Step 5 is at most \( \delta q \cdot \frac{d_t}{(1 - t\delta)q} = a_t \). For a fixed \( t \), at the \((t + 1)\)-st iteration we delete each color class \( C \) in Step 6 independently with probability \( p_C \) such that \( p_C^t + (1 - p_C^t)p_C = a_t \). Thus, each color class gets deleted by a total probability of \( a_t \) in Steps 5 and 6.

We now fix the values of the parameters of this section. Let \( \epsilon \) be a fixed positive number less than \( \frac{1}{10} \) (note that this can be assumed in Theorem 1.12). Recall that \( \gamma = \frac{1}{10} \) as mentioned before. Let \( \eta = 1 - \epsilon^3 \) and \( \theta = \frac{\gamma}{1 - \gamma} \). Note that \( \epsilon, \theta, \gamma, \) and \( \eta \) are all fixed constants, where the latter three constants are fixed once \( \epsilon \) is fixed. On the other hand, \( \delta \) can be made arbitrarily small by picking sufficiently large \( q \). As a consequence, we have \( \alpha_t \leq \frac{1}{100} \) for each \( 0 \leq t \leq \eta\tau \). This will be frequently used in this section.

We next have a few concentration bounds to control the degrees of vertices and the sizes of color classes.

Lemma 8.1. The number of edges discarded in Step 2 at the \((t + 1)\)-st iteration is at most \( \frac{3\delta^2 q}{(1 - t\delta)} \) w.v.h.p.

Proof. For \( i < j \), let \( X_{i,j} \) be the indicator random variable for the event that \( e_i \in T \) and \( e_j \in T \) are adjacent or have the same color. Clearly, the total number of edges discarded in Step 2 is given by \( X \leq \sum_{1 \leq i < j \leq n} X_{i,j} \).

Now for \( i < j \), the probability of the event that \( X_{i,j} = 1 \) is at most \( \max_{e \in G} f(e) \), where \( f(e) \) is the probability that a uniformly selected edge in \( G \) is adjacent to \( e \) or have the same color as \( e \). For any fixed edge \( e \), the
number of edges adjacent to \( e \) is at most \( s_t + d_t \) and the number of edges which has the same color as \( e \) is at most \( d_t \). Thus, the probability that \( X_{i,j} = 1 \) is at most \( \frac{s_t + 2d_t}{(1-t\delta)q^2} \). By using the linearity of expectation, we get the following:

\[
\mathbb{E}(X) \leq \mathbb{E}\left( \sum_{1 \leq i < j \leq \delta^q} X_{i,j} \right) = \mathbb{E}\left( \sum_{1 \leq i < j \leq \delta^q} \mathbb{P}[X_{i,j} = 1] \right) \leq \frac{(\delta^q)^2}{2} \cdot \frac{s_t + 2d_t}{(1-t\delta)qs_t}. \tag{8.7}
\]

Observe the following:

\[
\frac{d_t}{s_t} = \frac{(1+\alpha_t)(1-t\delta)q(t\delta)q}{(1-\alpha_t)s(t\delta)(1+\epsilon)q} \leq \frac{3}{2}. \tag{8.8}
\]

Thus, from (8.7) and (8.8), we have \( \mathbb{E}(X) \leq \frac{2\delta^2 q}{1-t\delta} \).

We now use Theorem 3.2 to show that the random variable \( X \) is not much greater than its expectation w.v.h.p. In the setting mentioned right before Theorem 3.2, let \( A \) be the set of all remaining edges of \( G \) at the start of \((t+1)\)-st iteration, \( B = \{1, 2, \ldots, \delta^q\} \), and \( \mathbb{P}[g(j) = e] \) is same for all edge \( e \). The random variable \( X \) is same as the functional \( L : A^B \to \mathbb{R} \) and the corresponding martingale is \( X = E(L), X_1, \ldots, X_{\delta^q} \), with respect to the gradation \( \emptyset \subset [1] \subset [2] \subset \cdots \subset [\delta^q] \) (where \( [n] \) denotes the set \( \{1, 2, \ldots, n\} \)). If one edge is replaced by some other edge in any one of the \( \delta q \) coordinates, then \( L \) will be affected by at most 4; in other words, 4 is an upper bound on the Lipschitz constant of the martingale. By Theorem 3.2, we can conclude that the probability that \( L \) deviates from its mean by \( \lambda = \delta^2 q \) is at most \( 2e^{-\frac{\lambda^2}{4\eta^2}} = 2e^{-\frac{\delta^4 q}{4\eta^2}} = e^{-\omega(\log q)} \) (remember that \( \delta = \frac{1}{\log q} \)). Thus, we conclude that w.v.h.p. it holds that

\[
X \leq \mathbb{E}(X) + \delta^2 q \leq \frac{2\delta^2 q}{1-t\delta} + \delta^2 q \leq \frac{3\delta^2 q}{1-t\delta}. \tag{8.9}
\]

**Corollary 8.2.** The total number of edges not added in the rainbow matching in Step 2 throughout the algorithm is at most \( 3\delta^2 q \log \left( \frac{1}{1-\eta} \right) \) w.v.h.p. Consequently, the number of vertices left in \( A \) after the \( t \)-th iteration is at most \( \left( 1 - t\delta + 3\delta \log \left( \frac{1}{1-\eta} \right) \right) q \).

**Proof.** By using Lemma 8.1, the total number is at most the following:

\[
\sum_{t=0}^{\delta^2 q - 3} \frac{3\delta^2 q}{1-t\delta} \leq 3\delta^2 q \int_{0}^{1} \frac{dx}{1-x} = 3\delta^2 q \log \left( \frac{1}{1-\eta} \right).
\]

Next, we bound the degree of each survived vertex in \( A \) throughout the algorithm.

**Lemma 8.3.** In the \((t+1)\)-st iteration, the number of deleted edges incident to each vertex in \( A \) in Steps 3, 4, 5, and 6 combined is at most \( 2\alpha_t s_t + 4\delta^2 q \) w.v.h.p.

**Proof.** Fix a vertex \( v \in A \) at the start of the \((t+1)\)-st iteration. By our assumption, the number of edges incident to \( v \) before starting the \((t+1)\)-st iteration is \( s_t \). Let \( S_{t+1} \) denote the number of edges right after Step 6 in the \((t+1)\)-st iteration. Thus, \( s_t - S_{t+1} \leq L + X \), where \( L \) is the random variable denoting the number of edges incident to \( v \) that got removed because of Steps 3 and 5, and \( X \) is the same number because of Steps 4 and 6. We bound \( L \) and \( X \) separately.

For a fixed neighbor \( w \in B \) of \( v \), the probabilities that an edge \( vw \) gets removed due to the deletion of \( w \) in Steps 3 and 4 are \( p'_w \) and \( p_w \), where \( p'_w + p_w = a_t + p'_q p_w \leq a_t + a_t^2 \). Furthermore, for a fixed neighbor \( w \in B \) of \( v \), the probabilities that an edge \( vw \) gets deleted in Steps 5 and 6 are \( p'_w \) and \( p_w \), where \( C(vw) \) denotes the color class containing the edge \( vw \). We also have \( p'_w + p_w = a_t + p'_w p_w \leq a_t + a_t^2 \).

Firstly, we have \( \mathbb{E}(L) \leq \sum_{w \in N(v)} \left( p'_w + p'_w \right) \), because the probability that an edge \( vw \) gets deleted due to Steps 3 and 5 combined is at most \( p'_w + p'_w \) by a simple union bound. To show the concentration of \( L \), we use the same sets \( A \), \( B \), and the same distribution of \( g \) as in the proof of Lemma 8.1. For the functional \( L \), we consider the martingale considered in Theorem 3.2. If one edge is replaced by another in any of the \( \delta q \) coordinates, then \( L \) will be affected by at most 2. Thus, 2 is an upper bound on the Lipschitz
constant of the underlying martingale. Hence, using Theorem 3.2, we can conclude that the probability that \( L \) deviates from its mean by \( \lambda = \sqrt{\frac{\log q}{\delta}} \) is at most \( 2e^{\frac{-t^2}{2\delta^2}} = e^{-\omega(\log q)} \).

We analyze the behavior of \( X \) by stochastically dominating it with a simpler random variable which counts the number of edges incident to the vertex \( v \) that are deleted in Steps 4 and 6 without deleting anything in Steps 3 and 5 (this is done to avoid conditioning on \( L \)). Clearly, \( X \) is upper bounded by 
\[ Y = \sum_{w \in N(v)} \left( Y_w + Y_{C(vw)} \right), \]
where \( Y_w \) is the indicator random variable of the event that the vertex \( w \) gets deleted in Step 4, and \( Y_{C(vw)} \) is the indicator random variable of the event that the color class \( C(vw) \) gets deleted in Step 6. It is clear that 
\[ \mathbb{E}(Y) = \sum_{w \in N(v)} \left( p_w + p_{C(vw)} \right) \leq 2a_t s_t. \]
Thus, by using Theorem 3.1, the probability that \( Y \) deviates by more than \( \lambda = \sqrt{\frac{\log q}{\delta}}s_t \) from its expectation is less than \( e^{-\omega(\log q)} \).

Observe that 
\[ \mathbb{E}(L) + \mathbb{E}(Y) \leq \sum_{w \in N(v)} \left( p'_w + p_w + p_{C(vw)} + p_{C(vw)} \right) \leq 2(a_t + a_t^2)s_t. \]
Hence, using triangle inequality we conclude that the random variable \( L + X \) does not exceed from \( 2(a_t + a_t^2)s_t \) by more than 
\[ \sqrt{\delta q \log q} + \sqrt{\delta q \log q} \]
and Theorem 3.1 follows by noting the following three inequalities:

\[ a_t s_t = \gamma \frac{\delta q (t \delta)(1 + \alpha_t)}{s(t \delta)(1 - \alpha_t)} \cdot (1 - \alpha_t)s(t \delta)(1 + \epsilon)q \leq 2\delta q. \]

\[ \sqrt{\delta q \log q} + \sqrt{\delta q \log q} \leq 3\sqrt{\delta q \log q} \leq \delta^2 q. \]

\[ a_t^2 s_t = \gamma^2 \frac{\delta^2 q (t \delta)^2(1 + \alpha_t)^2}{s(t \delta)^2(1 - \alpha_t)^2} \cdot (1 - \alpha_t)s(t \delta)(1 + \epsilon)q \leq \frac{3}{2} \delta^2 q. \]

\[ \square \]

**Corollary 8.4.** After the \((t + 1)\)-st iteration, the survived vertices of \( A \) have degree exactly \( s_{t+1} \) w.v.h.p.

**Proof.** By Lemma 8.3 and Lemma 7.1 after Step 6 of the \((t + 1)\)-st iteration, w.v.h.p. every survived vertices of \( A \) has degree at least

\[
(1 - 2a_t)s_t - 4\delta^2 q = \left(1 - 2\tilde{a}_t \frac{1 + \alpha_t}{1 - \alpha_t}\right)(1 - \alpha_t)s_t - 4\delta^2 q = (1 - \alpha_t)(1 - 2\tilde{a}_t)s_t - 4\alpha_t\tilde{a}_t s_t - 4\delta^2 q
\]

\[
= (1 - \alpha_t)(s_{t+1} - \gamma^2 \delta^2(1 + \epsilon)q - 4\alpha_t\tilde{a}_t s_t - 4\delta^2 q
\]

\[
\geq (1 - \alpha_t s_{t+1} - 5\delta^2 q - 5\alpha_t\tilde{a}_t s_t. \quad (8.9)
\]

Next, note the following:

\[
5\delta^2 q + 5\alpha_t\tilde{a}_t s_t = 5\delta^2 q + 5\alpha_t \cdot \frac{\gamma \delta q (t \delta)}{s(t \delta)} \cdot s(t \delta)(1 + \epsilon)q \leq 5\delta^2 q + 5\sqrt{\delta} \left(1 + \frac{10\delta}{(1 - \gamma\eta)^2} \right)^t - 1 \cdot \delta q
\]

\[
\leq 10\delta^{3/2} q \left(1 + \frac{10\delta}{(1 - \gamma\eta)^2} \right)^t
\]

\[
= (\alpha_{t+1} - \alpha_t)(1 - \gamma\eta)^2 q
\]

\[
\leq (\alpha_{t+1} - \alpha_t)s_{t+1}. \quad (8.10)
\]

Thus, combining (8.9) and (8.10), we obtain that after Step 6 of the \((t + 1)\)-st iteration, w.v.h.p. every survived vertices of \( A \) has degree has degree at least \((1 - \alpha_{t+1} s_{t+1} = s_{t+1} \). Thus, after Step 7, w.v.h.p. every survived vertices of \( A \) has degree has degree exactly \( s_{t+1} \).

\[ \square \]

Next, we bound the degree of each survived vertex in \( B \) throughout the algorithm.

**Lemma 8.5.** At the end of \((t + 1)\)-st iteration, w.v.h.p. the degree of each remaining vertex \( v \in B \) is at most \( d_t(1 - a_t) \left(1 - \frac{\delta}{1-\alpha_t}\right) + C'\delta^2 q \), where \( C' \) is a constant depending on \( \epsilon \).
Proof. First, Step 7 can only decrease the degrees of any vertex in $B$; thus, it is enough to prove Lemma 8.5 before Step 7. For each remaining vertex $v \in B$ after the $t$-th iteration, we define $D_{t+1, v}$ to be the random variable denoting the degree of $v$ after Step 6 in the $(t + 1)$-st iteration. To simplify the analysis, we make a slight change in Step 6 which will not alter the output of the original algorithm. Instead of the original Step 6, we delete each color class $C$ (even the ones which get deleted in Step 5) with the same probability $p_C$ as desired in the algorithm. Deleting a color class in Step 6 that was already deleted in Step 5 does not do anything further. This alternative way essentially makes Step 6 in dependent of every other step.

Fix a vertex $v \in B$. Let $d$ denote the degree of $v$ after the $t$-th iteration of the algorithm. Clearly, $d \leq d_t$. Let $X$ be the random variable denoting the number of neighbors of $v$ deleted in Step 6, and $L$ be the random variable denoting the number of edges $uv$ adjacent to $v$ deleted in Steps 3 and 5, and $uv$ is not deleted in Step 6. Clearly, $D_{t+1, v} = d - (X + L)$. Thus, we are interested in a good lower bound on $X + L$.

Observe that $\mathbb{E}(X) = \sum_{u \in N(v)} p_{C(uv)}$, where $C(uv)$ denotes the color class containing the edge $uv$. Clearly, $X$ is the sum of at most $d_t$ independent Bernoulli random variables with probability at most $a_t$. Thus, by using Chernoff bound (Theorem 3.1), the probability that $X$ deviates by more than $\frac{1}{2} \sqrt{\alpha d_t \log q} \leq \sqrt{t} \log q$ from its expectation is at most $e^{-\omega(\log q)}$.

Now to facilitate the analysis of the random variable $L$, we introduce the random variable $X' = \sum_{u \in N(v)} p_{C(uv)}' 1_{u \in N'}$, where $N'$ denote the (random) subset of $N(v)$ containing all the vertices $u$ such that the edge $uv$ is not deleted in Step 6. Clearly, $\mathbb{E}(X') = \sum_{u \in N(v)} p_{C(uv)}' (1 - p_{C(uv)})$. Again by Chernoff bound (Theorem 3.1), the probability that $X'$ deviates by more than $\sqrt{t} \log q$ from its expectation is at most $e^{-\omega(\log q)}$. Let $F$ denote the event that both $X$ and $X'$ are within a gap of $\sqrt{t} \log q$ from their corresponding expected values. Next, we show the concentration bound on $L$ conditioning on this event $F$, which holds w.v.h.p. Lemma 8.5 follows from the following claim.

Claim 8.6. Conditioned on $F$, at the end of Step 6 of the $(t + 1)$-st iteration, w.v.h.p. the degree of each remaining vertex $v \in B$ is at most $d_t \cdot (1 - a_t) \cdot \left(1 - \frac{\Delta}{1 - \delta}\right) + C' \delta q^2$, where $C'$ is a constant depending on $\epsilon$.

Proof. Fix a vertex $v \in B$ that survived after the $t$-th iteration. For any neighbor $u \in A$ of $v$, the probability that the edge $uv$ is deleted in Step 5 is exactly $p_{C(uv)}'$. Thus, the next task remaining is to calculate the probability that the vertex $u$ is deleted in Step 3, conditioned on the fact that the edge $uv$ does not get deleted in Step 5. For the convenience of writing, let $Y$ denote the event that the vertex $u$ is deleted in Step 3. Let $Z$ denote the event that the edge $uv$ does not get deleted in Step 5. We are interested in getting a good lower bound on $P[Z|Y]$.

The event $Z$ is the same as the event that none of the edges uniformly picked in Step 1 has the same color as the edge $uv$. Hence, conditioning on the event $Z$ essentially is equivalent to considering the uniform measure in Step 1, excluding the edges in the color class $C(uv)$. Also, note that the event $Y'$ that one edge $e$ incident to the vertex $u$ is picked in Step 1, and no other edges incident to $e$ or with the same color as $e$ is picked in Step 1. At this point, we can calculate this probability for each fixed edge adjacent to $u$ and sum the probability. So, we have the following:

$$P[Y|Z] \geq P[Y'|Z] \geq \frac{\delta q (s_t - 1)}{(1 - t\delta + 3\delta \log \left(\frac{1}{1 - \eta}\right)) q s_t} \left(1 - \frac{2d_t + s_t}{(1 - t\delta)qs_t - d_t}\right) \delta q^{-1},$$

because the probability to pick an edge $e$ incident to the vertex $u$ if we choose an edge uniformly at random among the edges which are not in $C(uv)$ is at least $\frac{s_t - 1}{(1 - t\delta + 3\delta \log \left(\frac{1}{1 - \eta}\right)) q s_t}$ (which follows from Corollary 8.2). Now, conditioning on the fact that such an $e$ is picked, the probability that each of the remaining $\delta q-1$ edges is not incident to $e$ and is not of the same color, is at least $\left(1 - \frac{2d_t + s_t}{(1 - t\delta)qs_t - d_t}\right)^{\delta q - 1}$. Hence, we can write the following:

$$P[Y|Z] \geq \frac{\delta q (s_t - 1)}{(1 - t\delta + 3\delta \log \left(\frac{1}{1 - \eta}\right)) q s_t} \left(1 - \frac{2d_t + s_t}{(1 - t\delta)qs_t - d_t}\right) \delta q$$

$$\geq \frac{\delta}{1 - t\delta + 3\delta \log \left(\frac{1}{1 - \eta}\right)} \frac{s_t - 1}{s_t} \left(1 - \frac{\delta q \cdot 4s_t}{2(1 - t\delta)qs_t}\right).$$

(8.11)
the number of deletions of edges incident to partial rainbow matching in Step 2 can change the number of neighbors deleted in Step 3 by at most 1, and from its mean by at least a constant of the martingale. Hence, using Theorem 3.2, we can conclude that the probability that \( \delta q \) deviates from its mean by at least \( \eta \), which shows that (1 - \( t \delta \))qs - \( d_t \) > \( \eta \) for all \( t \leq \eta \), as we desired. In (8.12), we have used the fact that \( \frac{3\eta \log \left( \frac{1}{\eta} \right)}{1 - \frac{1}{s_t}} \), and \( \frac{2\delta}{1 - t \delta} \) are all equal to \( O(\delta) \) for all \( t \leq \eta \), which can be verified by plugging in the values of the parameters \( \epsilon, \delta, \) and \( \eta \). Hence, conditioned on \( F \), we have the following:

\[
\mathbb{E}(D_{t+1,v}) \leq \sum_{u \in N^v} \left( 1 - p^*_v \right) \left( 1 - \frac{\delta}{1 - t \delta} + O(\delta^2) \right).
\]

(8.13)

Note that \( \sum_{u \in N^v} p^*_u \geq \sum_{u \in N^v} p^*_u (1 - p \) - \( \sqrt{q} \) log \( q \). Thus, we have the following:

\[
\sum_{u \in N^v} \left( 1 - p^*_u \right) = (d - X) - \sum_{u \in N^v} p^*_u \leq \sum_{u \in N^v} (1 - p) + \sqrt{q} \) log \( q \ - \sum_{u \in N^v} p^*_u (1 - p) + \sqrt{q} \) log \( q \)
\]

\[
\leq d(1 - a_t) + \delta^2 q,
\]

which we use in (8.13) to get the following conditioned on \( F \):

\[
\mathbb{E}(D_{t+1,v}) \leq d_t (1 - a_t) \left( 1 - \frac{\delta}{1 - t \delta} \right) + O(\delta^2) \cdot q.
\]

(8.14)

Furthermore, conditioning on \( F \), we can prove a concentration bound on \( L \) using Theorem 3.2. We use the same sets \( A, B, \) and the same probability distribution as in the proof of Lemma 8.1. For the functional \( L \), we consider the martingale considered in Theorem 3.2. Suppose one edge is replaced by another in any of the \( \delta q \) coordinates. In that case, \( L \) will be affected by at most 2 because picking a different color in the partial rainbow matching in Step 2 can change the number of neighbors deleted in Step 3 by at most 1, and the number of deletions of edges incident to \( v \) in Step 5 by at most 1. Thus, 2 is a bound on the Lipschitz constant of the martingale. Hence, using Theorem 3.2, we can conclude that the probability that \( L \) deviates from its mean by at least \( \lambda = \delta^2 q \) is at most \( 2e^{-\frac{\lambda^2}{\delta^2 q}} = e^{-\omega(\log q)} \). This, together with (8.14), finishes the proof of Claim 8.6, which in turn completes the proof of Lemma 8.5.

Corollary 8.7. After the \( (t+1) \)-st iteration, the survived vertices of \( B \) have degree at most \( d_{t+1} \) w.v.h.p.

Proof. By Lemma 8.6, Lemma 7.2, and the fact that \( a_t \geq \tilde{a}_t \), after the \( (t+1) \)-st iteration, w.v.h.p. every survived vertices of \( B \) has degree at least at most

\[
d_t \cdot (1 - \tilde{a}_t) \left( 1 - \frac{\delta}{1 - t \delta} \right) + C' \delta^2 q = (1 + \alpha_t) \tilde{a}_t (1 - \tilde{a}_t) \left( 1 - \frac{\delta}{1 - t \delta} \right) + C' \delta^2 q = (1 + \alpha_t) \tilde{a}_{t+1} + C' \delta^2 q.
\]

(8.15)
Recalling that $C'$ is a constant depending only on $\epsilon$, we get the following:

$$C'\delta^2 q \leq 10\delta^{3/2} q \left( \frac{1 - \eta}{1 - \gamma \eta} \right) \left( 1 + \frac{10\delta}{(1 - \gamma \eta)^2} \right)^t = (\alpha_{t+1} - \alpha_t)\tilde{d}_{t+1}. \quad (8.16)$$

Thus, combining (8.15) and (8.16), we obtain that after the $(t + 1)$-st iteration, w.v.h.p. every survived vertices of $B$ has degree has degree at most $(1 + \alpha_{t+1})\tilde{d}_{t+1} = d_{t+1}$.

We next control the size of each color class throughout the algorithm. The structure of the proof of the following lemma is very similar to the proof of Lemma 8.5. Thus, we omit some details identical to the arguments used to prove Lemma 8.5.

**Lemma 8.8.** At the end of $(t + 1)$-st iteration, w.v.h.p. the number of edges in each of the remaining color class is at most $d_t(1 - a_t) \left( 1 - \frac{\delta}{1 - \epsilon \gamma} \right) + C'\delta^2 q$, where $C'$ is a constant depending on $\epsilon$.

**Proof.** First, Steps 5, 6, and 7 can only decrease the number of edges in a color class; thus, it is enough to prove Lemma 8.8 before Step 5. For each remaining color class $C$ after the $t$-th iteration, we define $S_{t+1,C}$ to be the random variable denoting the number of edges in $C$ after Step 4 in the $(t + 1)$-st iteration. To simplify the analysis, we make a slight change in Step 4 which will not alter the output of the original algorithm. Instead of the actual Step 4, we delete each vertex $v \in B$ (even the ones which get deleted in Step 3) with the same probability $p_v$ as desired in the algorithm. Deleting a vertex in Step 4, which was already deleted in Step 3, does not do anything further. This alternative way essentially makes Step 4 independent of every other step.

Fix a remaining color class $C$. Let $s$ denote the number of edges in $C$ after the $t$-th iteration of the algorithm. Clearly, $s \leq d_t$. Let $X$ be the random variable denoting the number of edges in $C$ deleted in Step 4, and $L$ be the random variable denoting the number of edges $e$ in $C$ deleted in Step 3, and $e$ is not deleted in Step 4. Clearly, $S_{t+1,C} = s - (X + L)$. Thus, we want a good lower bound on $X + L$.

Observe that $\mathbb{E}(X) = \sum_{v \in B_C} p_v$, where $B_C$ denotes the set of all vertices $v \in B$ that are incident to an edge in $C$. Clearly, $X$ is the sum of at most $d_t$ independent Bernoulli random variables with probability at most $a_t$. Thus, using standard Chernoff bound (Theorem 3.1), the probability that $X$ deviates by more than $\frac{1}{\sqrt{a_t d_t}} \log q \leq \sqrt{\delta} \log q$ from its expectation is at most $e^{-\omega(\log q)}$.

Now to facilitate the analysis of the random variable $L$, we introduce the random variable $X' = \sum_{v \in B_C} p'_v 1_{v \in N'}$, where $N'$ denote the (random) subset of $B_C$ containing all the vertices $v$ that are not deleted in Step 4. Clearly, $\mathbb{E}(X') = \sum_{v \in B_C} p'_v (1 - p_v)$. Again by Chernoff bound (Theorem 3.1), the probability that $X'$ deviates by more than $\sqrt{\delta} \log q$ from its expectation is at most $e^{-\omega(\log q)}$. Let $F$ denote the event that both $X$ and $X'$ are within a gap of $\sqrt{\delta} \log q$ from their corresponding expected values. Next, we show the concentration bound on $L$ conditioning on this event $F$, which holds w.v.h.p. Lemma 8.8 follows from the following claim.

**Claim 8.9.** Conditioned on $F$, after Step 6 of the $(t + 1)$-st iteration, w.v.h.p. the number of edges in each of the remaining color class is at most $d_t(1 - a_t) \left( 1 - \frac{\delta}{1 - \epsilon \gamma} \right) + C'\delta^2 q$, where $C'$ is a constant depending on $\epsilon$.

**Proof.** Fix an edge $uv \in C$ with $u \in A, v \in B$ that survived after the $t$-th iteration. The probability that $v$ is deleted in Step 3 is exactly $p'_v$. Thus, the next task remaining is to calculate the probability that the vertex $u$ is deleted in Step 3, conditioned on the fact that the vertex $v$ does not get deleted in the same step. For the convenience of writing, let $Y$ denote the event that the vertex $u$ gets deleted in Step 3. Let $Z$ denote the event that the vertex $v$ does not get deleted in Step 3. We want a good lower bound on $\mathbb{P}[Y|Z]$.

The event $Z$ is the same as the event that none of the edges uniformly picked in Step 1 is incident to $v$. Hence, conditioning on the event $Z$ essentially is equivalent to considering the uniform measure in Step 1, excluding the edges incident to $v$. Also, note that the event $Y$ contains the event $Y'$ that one edge $e$ incident to the vertex $u$ is picked in Step 1, and no other edges incident to $e$ or with the same color as $e$ is picked in Step 1. By the same reasoning as in Claim 8.6, we can calculate this probability for each fixed edge adjacent to $u$ and sum the probability. Thus, we have the following:

$$\mathbb{P}[Y|Z] \geq \mathbb{P}[Y'|Z] \geq \frac{\delta q (s_t - 1)}{1 - t\delta + 3\delta \log \left( \frac{1}{1 - \gamma \eta} \right)} \left( 1 - \frac{2d_t + s_t}{(1 - t\delta)qs_t - d_t} \right) \delta^{-1}q^{-1}. \quad (8.17)$$
Again by the same calculation as in Claim 8.6, we have the following:

\[
P(Y|Z) \geq \frac{\delta}{1 - t\delta} - O_{\epsilon}(\delta^2).
\]  

(8.17)

Hence, conditioned on \( F \), we have the following:

\[
E(S_{t+1, c}) \leq \sum_{v \in N'} (1 - p'_v) \left( 1 - \frac{\delta}{1 - t\delta} + O_{\epsilon}(\delta^2) \right).
\]

(8.18)

Note that \( \sum_{v \in N'} p'_v = X' \geq \sum_{v \in B_c} p'_v (1 - p_v) - \sqrt{q \log q} \). Thus, we have the following:

\[
\sum_{v \in N'} (1 - p'_v) = (d - X) - \sum_{v \in N'} p'_v \leq \sum_{v \in B_c} (1 - p_v) + \sqrt{q \log q} - \sum_{v \in B_c} p'_v (1 - p_v) + \sqrt{q \log q}
\]

\[
= \sum_{v \in B_c} (1 - p'_v) (1 - p_v) + 2\sqrt{q \log q}
\]

\[
\leq d(1 - a_t) + \delta^2 q,
\]

which we use in (8.18) to get the following conditioned on \( F \):

\[
E(S_{t+1, c}) \leq d_t (1 - a_t) \left( 1 - \frac{\delta}{1 - t\delta} \right) + O_{\epsilon}(\delta^2) \cdot q.
\]

(8.19)

Finally, we can finish the proof by showing the concentration of the random variable \( S_{t+1, c} \) exactly as in the proof of Claim 8.6 using Theorem 3.2.

\[ \square \]

The same calculations as in Corollary 8.7 obtains the following:

**Corollary 8.10.** After the \((t + 1)\)-st iteration, the size of each remaining color class is at most \( d_t + 1 \) w.v.h.p.

Since the numbers of both colors and vertices in the graph are bounded by \( 2q^2 \), a simple union bound already shows that w.v.h.p. the conclusions of Corollaries 8.2, 8.4, 8.7, and 8.10 hold simultaneously (note that we do not need any local lemma like in the proof of Lemma 6.5). This proves that w.v.h.p. 8.8 and 8.9 hold for every \( 1 \leq t \leq \eta r \).

Hence, at the end of the \( \eta r \)-th iteration, by Corollary 8.4, the degree of any remaining vertex in \( A \) is at least \( \frac{1}{\sqrt{1 + \epsilon}} (1 - \gamma \eta)^2 (1 + \epsilon) \eta q \), and by Corollary 8.2, the number of remaining vertices in \( A \) is at most \( q - \eta \delta q + 3\delta q \log \left( \frac{1}{1 - \eta \gamma} \right) \leq \sqrt{1 + \epsilon} (1 - \eta \gamma) \log (\frac{1}{1 - \gamma}) \leq \sqrt{1 + \epsilon} (1 - \eta \gamma) \) (here, any constant greater than 1 would work in the places of the \( \sqrt{1 + \epsilon} \) factors, this particular choice was made to make the final inequality easier). The straightforward greedy algorithm will work here as long as \( \sqrt{1 + \epsilon} (1 - \gamma \eta)^2 (1 + \epsilon) \eta q > 2 \sqrt{1 + \epsilon} (1 - \eta \gamma) \), which is equivalent to \( (1 - \frac{1 - \epsilon^3}{1 + \epsilon})^2 > 2\epsilon^3 \) (remember that \( \eta = 1 - \epsilon^3 \)). This can be easily checked to be true for \( \epsilon < \frac{1}{10} \). Hence, the partial rainbow matching obtained from running the algorithm can be completed to a rainbow matching using all vertices in \( A \), proving Theorem 1.12.

## 9  A weaker bound on Theorem 1.11

We need some easy upper bound on the number of colors required in Theorem 1.11 to prove the actual statement of it. We prove the following by a somewhat direct application of 1.9.

**Proposition 9.1.** There exists \( N \) such that whenever \( q \geq N \), for any graph \( G \) that is properly colored with at least \( 4q \) colors such that there are at least \( q \) edges of each color, there is a rainbow matching of \( G \) using \( q \) colors.
Proof. To prove this proposition, we start by showing an easy bound of $2q^2$ instead of $4q$. Suppose we have a properly colored graph $G$ with $2q^2$ colors where each color appears $q$ times. Consider the maximum size rainbow matching $M = \{e_1, e_2, \ldots, e_m\}$ of $G$. Assume for the sake of contradiction that $|M| = m \leq q - 1$. Let $C$ denote all the colors used to color the edges in any edge connecting two vertices participating in $M$. Clearly, $|C| \leq \binom{2m}{2} < 2q^2 - 2q$. So, there are $2q$ colors $c_1, c_2, \ldots, c_{2q}$ which do not appear in any of the edges between two vertices in $M$. If there is an edge with any of these $2q$ colors which does not use any of the vertices used in $M$, then we are already done because we can obtain a larger rainbow matching by adding that edge to $M$. So, all the edges using colors from $\{c_1, c_2, \ldots, c_{2q}\}$ uses exactly one vertex in $M$. By a simple application of pigeonhole principle, for every color $c$ from $\{c_1, c_2, \ldots, c_{2q}\}$, there is an edge $e \in M$ such that there are two edges with color $c$, which are incident to $e$. By another application of the pigeonhole principle, there exists an edge $e \in M$ such that there are three colors (without loss of generality, say $c_1, c_2$, and $c_3$) with the property that there are two edges with each color in $\{c_1, c_2, c_3\}$ such that all six of these edges are incident to $e$. Now, it is easy to find two edges among these six edges such that they are not incident to each other and have distinct colors. By observing that one obtains a larger matching by adding these two edges and subtracting $e$ from $M$, we establish the desired upper bound of $2q^2$.

Next, we show that $4q$ colors suffice for all sufficiently large $q$, as promised. Let $q$ be such that $q \geq N^2$, where $N$ comes from Theorem 1.9. We split the proof into two cases based on the degree sequence $d_1 \geq d_2 \geq \cdots$ of $G$. Define $k$ to be the minimum integer such that $d_k \leq 3(q - k)$, if it exists.

Case 1: If $k > q - \sqrt{q}$ or $k$ does not exist, then remove $q - \sqrt{q}$ highest degree-vertices from $G$. In the remaining graph there are at least $\sqrt{q}$ edges of each color and the number of colors is $4q \geq 2(\sqrt{q})^2$. Thus, using the bound proven in the last paragraph, we can find a rainbow matching using $\sqrt{q}$ colors. Indeed, we can greedily add an edge adjacent to each of the removed vertices to the rainbow matching to get a rainbow matching of size $q$. This can be done in the following way. Let $v_i$ denotes the vertex with degree $d_i$. The number of colors used so far is $\sqrt{q}$, and the number of vertices used in the rainbow matching is $2\sqrt{q}$. So, one can pick a neighbor of $v_i - \sqrt{q}$ out of more than $3\sqrt{q}$ neighbors so that neither the color of that edge nor the neighbor is used in the rainbow matching built yet. We can continue this process greedily for $i = \sqrt{q}, \sqrt{q} - 1, \ldots, 1$ respectively to find a rainbow matching of size $q$.

Case 2: If $k \leq q - \sqrt{q}$, then similarly remove $k$ highest degree-vertices from $G$. Like last time, at least $q - k$ edges of each color are in the remaining graph, and the maximum degree is at most $3(q - k)$. Now merge four colors simultaneously to ensure that each color has $4(q - k)$ edges. Thus, the total number of colors is now $q$, and each color class has maximum degree at most four. Due to the choice of $N$, we can apply Theorem 1.9 on $G$ to obtain a rainbow matching using these $q$ colors. \[\square\]

10 Proof of Theorem 1.11

If we directly apply an appropriate randomized algorithm similar to the ones of Sections 4 or 8, then we can only obtain a matching of size $q - o(q)$, which is not enough for Theorem 1.11. As we have discussed in the introduction, the rigid behavior in terms of the number of edges used in each color class is somewhat responsible for this type of failure of the randomized algorithm used before. It turns out that we need to be extra careful about the vertices with small degree. To do so, we first need to show that there cannot be too many vertices with degree close to the number of colors (note that the maximum degree can be at most the number of colors because we have a proper coloring this time).

To prove Theorem 1.11 without loss of generality, we assume that $0 < \epsilon < \frac{1}{10}$. We start with a simple graph $G$ that is properly edge-colored with $2(1 + \epsilon)q$ colors such that there are at least $q$ edges of each color, and $G$ does not have a rainbow matching of size $q$. Define $\theta = \frac{\epsilon}{2}$. We first apply Theorem 1.12 to show that the number of vertices with degree more than $2(1 + \theta)q$ is at most $(1 - \theta)q$. If not, then select a set $A$ of size exactly $(1 - \theta)q$ such that the degree of any vertex in $A$ is at least $2(1 + \theta)q$. Let $G'$ denote the graph after deleting all the edges incident to $A$. We have at least $\theta q$ edges in each color in $G'$. Thus, by applying Proposition 1.14 established in the last section, we can find a rainbow matching $T$ of size $\theta q$ in $G'$. Now in $G$, discard all the vertices incident to $T$ and all the edges colored with a color used in $T$. For a fixed $v \in A$, we discard at most $3\theta q$ edges incident to $v$. Hence, $v$ still has at least $2(1 + \theta)q - 3\theta q - (1 - \theta)q = q$ neighbors outside of $A$. Now, with a straightforward application of Theorem 1.12 on the graph $A \cup B$ (where $B = V(G) \setminus A$), we can find a rainbow matching $T'$ using all the vertices in $A$. As we have already discarded
the vertices and colors used in $T$, the edge set $T \cup T'$ gives us a rainbow matching of size $q$. Hence, from now on, we assume that the number of vertices with degree more than $2(1 + \theta)q$ is at most $(1 - \theta)q$.

10.1 Algorithm

Assume that we are given a simple graph $G$ that is properly edge-colored with $2(1 + \epsilon)q$ colors such that there are $q$ edges of each color. Additionally, assume that there exists $A \subseteq V(G)$ such that $|A| \leq (1 - \theta)q$ and the vertices outside $A$ have degree at most $2(1 + \theta)q$, where $\theta = \frac{\epsilon}{2}$. Like before, we will provide a randomized algorithm, which constructs a rainbow matching using $q$ colors in several iterations.

This algorithm is in terms of some parameter $\delta > 0$, which will be specified later. Define $\tau = \frac{1}{2}$. We follow the algorithm below for $\eta \tau$ steps, for some $\eta < \frac{1}{2}$ to be specified later.

1. Select independently $\delta \cdot 2(1 + \epsilon)q$ edges $e_1, e_2, \ldots, e_{2\delta(1+\epsilon)q}$ uniformly at random with replacement from among the remaining edges. Denote by $T$ the set of all selected edges.

2. Delete all the vertices corresponding to the edges in $T$ from $G$. Deleting vertices always also deletes all incident edges.

3. With some probability (to be specified later), independently delete each vertex in $G$. This will make sure that among Steps 2 and 3 combined, every vertex in $A$ gets deleted with the same probability, and every vertex outside of $A$ gets deleted with the same probability (possibly different).

4. For each edge $e_i \in T$, add it to the rainbow matching if $e_i$ is not incident to any edge $e_j \in T$ for $j < i$ and $e_i$ does not have same color as any edge $e_j \in T$ for $j < i$.

5. For each edge $e$ added to the rainbow matching in Step 4, delete all the edges of the same color as $e$ from $G$.

6. For each remaining color class, if the number of edges in that color class is more than $s_i$ after Step 5, then delete edges of that color class in the following way to make sure that it has precisely $s_i$ edges, where $s_i$ will be specified later. If the number of edges that need to be deleted is at most the number of edges containing at least one vertex from $A$, then delete arbitrarily from those edges. Otherwise, delete all the edges with at least one vertex from $A$, and then delete the rest arbitrarily from the rest of the edges of the same color.

Compared to Algorithm of Section 4, we need to be more careful here in deleting the vertices with smaller degree, and thus, we are treating the vertices outside $A$ differently in Steps 3 and 6.

10.2 Intuitive analysis

We aim to show that if we run this randomized algorithm, we get a rainbow matching with $q$ edges. Like last time, define $d_{0,v}$ to be the initial degree of $v$ in $G$. We will define two functions $s(x)$ and $g(x)$ such that after the $t$-th iteration of the algorithm, we have the following:

1. Each remaining color class has size exactly $s_t$, which is at least about $s(t\delta)q$.

2. Each surviving vertex $v$ has degree at most $d_{t,v}$, which is approximately $(1 - t\delta)g(t\delta)d_{0,v}$.

3. For each remaining color class, the fraction of number of vertices remaining in $A$ is at most $\frac{1-d_{t}}{d_{t}}$ (i.e., for each color $c$, if $V_c$ denotes the set of all vertices incident to an edge of color $c$, then we have $\frac{|V_c \cap A|}{|V_c|} \leq \frac{1-d_{t}}{d_{t}}$).

We outline a rough analysis below. Clearly $s(0) = 1$ and $g(0) = 1$. Assume that the $t$-th iteration is done, and the above properties are true. First, very few edges will be discarded in Step 4 throughout the process. In other words, the number of edges in the partial rainbow matching will be roughly $2(1 + \epsilon)t\delta q$ after the $t$-th round.

Step 3 is there just to make the analysis more convenient. It ensures that every vertex of $A$ is deleted with the same probability (call it $a_t$), and every vertex outside of $A$ is also deleted with the same probability
Thus, neglecting small error terms due to lack of independence, we can choose to pick v, and thus the number of remaining edges is at least 2(1 − tδ)(1 + ε)q, which is at most \(\frac{d_{t,v}}{2(1-t\delta)(1+\varepsilon)q}\) (i.e., \(p \leq \frac{d_{t,v}}{2(1-t\delta)(1+\varepsilon)q}\)). As a result, the probability that v is not deleted in Step 2 is exactly \((1-p)^{2(1+\varepsilon)q}\), which is at least about 1 − \(\frac{\delta g(t\delta)2(1+\varepsilon)q}{s(t\delta)q}\). Thus, we can define \(a_t\) to be about \(2(1+\varepsilon)\delta g(t\delta)/(s(t\delta)q)\) (note that \(d_{0,v} \leq 2(1+\varepsilon)q\)). Similarly, we can define \(b_t\) to be about \(2(1+\varepsilon)\gamma \delta g(t\delta)/(s(t\delta)q)\), where \(\gamma = \frac{1+\varepsilon}{1+\tau}\).

The probability that a single vertex gets deleted among Steps 2 and 3 is \(a_t\) or \(b_t\). Thus, by a simple union bound and using the expected number of edges deleted in each color class among Steps 2 and 3 is at most \(2s_t \left(\frac{1-\theta}{2} \cdot a_t + \frac{1+\theta}{2} \cdot b_t\right)\). This suggests defining \(s(x)\) with the following behavior:

\[
s'(x) = -2(1+\varepsilon) \left[(1-\theta) + (1+\theta)\gamma\right] g(x). \tag{10.1}
\]

Next, let us estimate the change in \(d_{t,v}\) to get a differential equation for \(g\). We know that each neighbor of \(v\) is deleted among Steps 2 and 3 with probability \(a_t\) or \(b_t\), which is at least \(b_t\). Moreover, any edge incident to \(v\) will be deleted in Step 5 by probability \(\approx \frac{1}{1-t\delta}\), because there are about \((1+\varepsilon)q\) colors picked in Step 1, relatively few conflicts in Step 4, and the number of colors remaining is about \(2(1+\varepsilon)(1-t\delta)q\).

It will be deleted in Step 5 by probability \(\approx \frac{1}{1-t\delta}\), because there are about \((1+\varepsilon)q\) colors picked in Step 1, relatively few conflicts in Step 4, and the number of colors remaining is about \(2(1+\varepsilon)(1-t\delta)q\).

Thus, neglecting small error terms due to lack of independence, we can choose to pick \(d_{t,v}\) with the following behavior:

\[
d_{t+1,v} - d_{t,v} \approx -\left[b_t - \frac{1}{1-t\delta} \left(1 + b_t\right)\right] d_{t,v} t\delta - (t-1)\delta g((t+1)\delta) d_v - (t-\delta)\delta g(t\delta) d_v \approx -\frac{\delta g(t\delta)}{s(t\delta)} \approx -2(1+\varepsilon)\gamma \cdot \frac{g(t\delta)^2}{s(t\delta)}. \tag{10.2}
\]

Equation (10.2) suggests: \(g'(x) = -2(1+\varepsilon)\gamma \frac{g(x)^2}{s(x)}\). Combining with Equation (10.1), we have that \(\frac{dg}{ds} = \frac{(1-\theta)}{(1-\theta) + (1+\theta)} \cdot \frac{g(x)}{s(x)}\), whose solution is \(s = cg^M\) with \(M = \frac{(1-\theta) + (1+\theta)\gamma}{\gamma}\). Similar to Section 5, by using the initial conditions that \(g(0) = s(0) = 1\), we get \(s(x) = (1 - 2(1+\varepsilon) \cdot (1 - \theta + \theta\gamma)x)^{\frac{1}{\gamma}}\) and \(g(x) = (1 - 2(1+\varepsilon) \cdot (1 - \theta + \theta\gamma)x)^{\frac{1}{\gamma}}\).

We show in the next section that the degrees of vertices and the sizes of color classes are concentrated in some sense throughout the process, which implies that we can not get stuck as long as \(t \leq \eta\tau\) with \(\eta < \frac{1}{2(1+\varepsilon)}\). (note that we desire a reasonable gap between \(\eta\) and \(\frac{1}{2(1+\varepsilon)}\) to make room for the error terms arising from concentration inequalities). Moreover, after \(\eta\tau\)-th iteration for \(\eta > \frac{1}{2(1+\varepsilon)}\) (with a reasonable gap), the number of edges in the rainbow matching will exceed \(q\), even after accounting for the discarded edges in Step 2 throughout the process. Thus, we need to ensure that \(\frac{1}{2(1+\varepsilon)} < \eta < \frac{1}{2(1+\varepsilon)(1-\theta+\theta\gamma)}\).

It will turn out that, unlike before, we do not need to use any greedy algorithm at the end to finish.

10.3 Formal analysis

Throughout this section, we assume \(q\) to be sufficiently large to support our arguments. Let \(\gamma = \frac{1+\theta}{1+\varepsilon}\) and \(M = \frac{(1-\theta) + (1+\theta)\gamma}{\gamma}\). Observe that \(M > 2\), which will be used often in our analysis. Define the following two functions:

\[
s(x) = (1 - 2(1+\varepsilon) \cdot (1 - \theta + \theta\gamma)x)^{\frac{1}{\gamma}} \quad \text{and} \quad g(x) = (1 - 2(1+\varepsilon) \cdot (1 - \theta + \theta\gamma)x)^{\frac{1}{\gamma}}.
\]

Similar to Section 6, we introduce error terms \(\alpha_t\) and \(\beta_t\) which will be explicitly specified later and much less than \(\frac{1}{\gamma}\).

We will make sure that the following events happen after the \(t\)-th iteration for \(1 \leq t \leq \eta\tau\):

1. The remaining color classes have size exactly:

\[
s_t = (1 - \alpha_t)s(t\delta)q. \tag{10.3}
\]
2. The degree of each survived vertex in $A$ is at most:

$$d_i = 2(1 + \beta_i)(1 - t\delta)g(t\delta)(1 + \epsilon)q.$$  \hfill (10.4)

3. The degree of each survived vertex outside $A$ is at most:

$$d'_i = 2(1 + \beta_i)(1 - t\delta)g(t\delta)(1 + \theta)q.$$  \hfill (10.5)

4. For each color $C$, if $V_C$ denotes the set of all vertices incident to an edge of color $C$, then $|V_C \cap A| \leq \frac{1 - \theta}{2}$.

We fix some $t$ with $0 \leq t < \eta \tau$ for this subsection and assume that the above properties hold for $t$. Our goal is to show that after the $(t+1)$-st iteration of the algorithm, w.v.h.p. the above properties will continue to hold (notice that by a simple union bound, this is enough to establish that w.v.h.p. these properties hold for every $1 \leq t \leq \eta \tau$). In this subsection, we show certain concentration bounds that hold in the $(t+1)$-st iteration, and in the subsequent subsection, these bounds will be used to achieve our goal.

Next, we explicitly define $a_t$ and $b_t$. The similar analysis as done in Sections 6 or 8 leads us to define the following:

$$a_t = 2(1 + \epsilon)\delta \cdot \frac{g(t\delta)(1 + \beta_i)}{s(t\delta)(1 - \alpha_i)} \quad \text{and} \quad b_t = 2(1 + \theta)\delta \cdot \frac{g(t\delta)(1 + \beta_i)}{s(t\delta)(1 - \alpha_i)}$$

Similar to before, we explicitly assign a probability for Step 3. For a fixed $t$, at the $(t+1)$-st iteration we delete each vertex $v \in A$ independently with probability $p_v$ such that $p'_v + (1 - p'_v)p_v = a_t$, where $p'_v$ is the probability that $v$ gets deleted in Step 2. If $v \notin A$, then we pick $p_v$ such that $p'_v + (1 - p'_v)p_v = b_t$.

We now finally pick the values of the parameters. Recall that $\epsilon$ is a fixed positive number less than $\frac{1}{10}$. Recall $\theta = \frac{\epsilon}{2}$ and $\gamma = \frac{1 + \theta}{1 + \epsilon}$ as introduced before. As mentioned in the intuitive analysis, we want $\eta$ such that $\frac{1}{2(1 + \epsilon)} < \eta < \frac{1}{2(1 + \epsilon)(1 - \theta + \theta\gamma)}$. Thus, explicitly let us define $\eta = \frac{1}{2(1 + \epsilon)} \cdot \frac{1 - \theta + \theta\gamma}{1 - \theta - \theta\gamma}$. Let $\delta = \frac{1}{\log q}$. Note that $\epsilon$, $\theta$, $\gamma$, and $\eta$ are all fixed constants, where the latter three constants are fixed once $\epsilon$ is fixed. On the other hand, $\delta$ can be made arbitrarily small by picking sufficiently large $q$.

Next, we have a few concentration bounds similar to the ones in Sections 6 and 8.

**Lemma 10.1.** In Step 4 of the $(t+1)$-st iteration, at most $\frac{8s^2q}{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)t\delta}$ edges are discarded w.v.h.p.

**Proof.** For $i < j$, let $X_{i,j}$ be the indicator random variable as defined in the proof of Lemma 8.1. Clearly, the total number of edges discarded in Step 4 is given by $X = \sum_{1 \leq i < j \leq 2(1 + \epsilon)q} X_{i,j}$. By a similar argument as in the proof of Lemma 8.1, the probability that $X_{i,j} = 1$ is at most $\frac{2d_k + s_t}{2(1 + \epsilon)(1 - t\delta)qs_t}$. Hence, we have:

$$\mathbb{E}(X) \leq \mathbb{E} \left( \sum_{1 \leq i < j \leq 2(1 + \epsilon)q} X_{i,j} \right) = \sum_{1 \leq i < j \leq 2(1 + \epsilon)q} \mathbb{P}[X_{i,j} = 1]$$

$$\leq \frac{(2(1 + \epsilon)\delta q)^2}{2} \cdot 2d_k + s_t$$

$$\leq \frac{4(1 + \beta_i)(1 - t\delta)g(t\delta)(1 + \epsilon)q}{(1 + \epsilon)(1 - t\delta)(1 - \alpha_i)s(t\delta)q} + \frac{1}{1 - (1 + \epsilon)(1 - t\delta)}$$

$$\leq \frac{5}{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)t\delta}$$

$$\leq \frac{8s^2q}{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)q}.$$  

Similar to the proof of Lemma 8.1 use Theorem 8.3 to conclude that the probability that $X$ deviates from its mean by $\lambda = \delta^2 q$ is at most $2e^{-\frac{8s^2q}{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)q}} = e^{-\omega(\log q)}$ (remember that $\delta = \frac{1}{\log q}$). Thus, we conclude that w.v.h.p. it holds that $X \leq \mathbb{E}(X) + \delta^2 q \leq \frac{8s^2q}{1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)q}$. \hfill $\square$
Corollary 10.2. The total number of edges not added in the rainbow matching in Step 4 throughout the algorithm until the $\eta t$-th iteration is at most $8\delta q \log \left( \frac{1}{1-2(1+\epsilon)(1-\theta+\theta\gamma)\eta} \right)$ w.v.h.p. Consequently, the number of colors left after the $t$-th iteration is at most the following:
\[
\left(1 - t\delta + 4\delta \log \left( \frac{1}{1-2(1+\epsilon)(1-\theta+\theta\gamma)\eta} \right) \right) \cdot 2(1+\epsilon)q.
\]

Proof. By using Lemma 10.1 the total number is at most the following:
\[
\sum_{t=0}^{\frac{\eta}{2}-1} \frac{8\delta^2 q}{1-2(1+\epsilon)(1-\theta+\theta\gamma)t\delta} \leq 8\delta q \int_0^{\eta} \frac{dx}{1-2(1+\epsilon)(1-\theta+\theta\gamma)x} = \frac{8\delta q}{2(1+\epsilon)(1-\theta+\theta\gamma)} \log \left( \frac{1}{1-2(1+\epsilon)(1-\theta+\theta\gamma)\eta} \right).
\]

Lemma 10.3. In the $(t+1)$-st iteration, the number of edges deleted from a single color class in Steps 2 and 3 combined is at most $2 \left( \frac{2\theta}{3} \cdot a_t + \frac{4\theta}{3} \cdot b_t \right) s_t + \sqrt{q \log q}$ w.v.h.p.

Proof. Fix a color remaining at the start of the iteration $t+1$. Let $s_t$ denote the number of edges in that color class before starting the $(t+1)$-st iteration, and $S_{t+1}$ denote the number of edges right after Steps 2 and 3 in the $t$-th iteration. We have that $\mathbb{E}(S_{t+1}) - s_t \geq -2 \left( \frac{2\theta}{3} \cdot a_t + \frac{4\theta}{3} \cdot b_t \right) s_t$, because the probability that an edge gets deleted due to the deletion of one of its vertices in Steps 2 and 3 combined is at most the sum of the probabilities that they get deleted, and the fraction of number of vertices in the color class which is in $A$ is at most $\frac{1-\theta}{2}\theta$.

Let $L$ be the random variable denoting the number of edges of the fixed color removed because of Steps 2 and 3. In the setting mentioned right before Theorem 3.2 let $A = A' \cup \{0,1\}$ where $A'$ is the set of all remaining edges of $G$ at the start of $(t+1)$-st iteration, and $B = [\delta q] \cup V$ where $V$ is the set of remaining vertices of $G$ that are incident to the fixed color. For $j \in [\delta q]$, the probability $\mathbb{P}[g(j) = e]$ is same for all edge $e$ and $\mathbb{P}[g(j) \in \{0,1\}] = 0$, and for $v \in V$, the probability $\mathbb{P}[g(v) = 1]$ is same as the probability with which the vertex $v$ gets deleted in Step 3, and $\mathbb{P}[g(v) = 0] = 1 - \mathbb{P}[g(v) = 1]$. Consider the functional $L : \mathcal{A}^g \rightarrow \mathbb{R}$ and the corresponding martingale is $X_0 = \mathbb{E}(L), X_1, \ldots, X_{2(1+\epsilon)\delta q}, X_{v_1}, X_{v_2}, \ldots, X_{v_k}$, with respect to the gradation $\emptyset \subset [1] \subset [2] \subset \ldots \subset [2(1+\epsilon)\delta q] \subset [2(1+\epsilon)\delta q] \cup \{v_1\} \subset \ldots \subset [2(1+\epsilon)\delta q] \cup V$ (where $V = \{v_1, v_2, \ldots, v_k\}$). If one edge is replaced by some other edge in any one of the $\delta q$ coordinates or the decision of deleting a vertex in Step 3 is reversed, then $L$ will be affected by at most 2, in other words, 2 is an upper bound on the Lipschitz constant of the martingale. Use Theorem 3.2 to conclude that the probability that $L$ deviates from its mean by $\lambda = \sqrt{q \log q}$ is at most $2e^{-\frac{\lambda^2}{2(1+\epsilon)}} = e^{-\omega(\log q)}$.

Lemma 10.4. At the end of $(t+1)$-st iteration, w.v.h.p. the degree of each remaining vertex in $A$ is at most $d_t(1-b_t) \left(1 - \frac{\delta}{1+\theta} \right) + C'\delta^2 q$, and the degree of each remaining vertex outside $A$ is at most $d_t'(1-b_t') \left(1 - \frac{\delta}{1+\theta} \right) + C'\delta^2 q$, where $C'$ is a constant depending on $\epsilon$.

Proof. We only deal with the vertices in $A$; the same treatment also works for other vertices. First of all, Step 6 can only decrease the degrees of any vertex, so it is enough to prove Lemma 10.4 before Step 6. For each remaining vertex $v$ after the $t$-th iteration, we define $D_{t+1,v}$ to be the random variable denoting the degree of $v$ after Step 5 in the $(t+1)$-st iteration. To simplify the analysis, we make a slight change in Step 3 which will not alter the output of the original algorithm. Instead of the actual Step 3, we delete each vertex $v$ (even the ones which get deleted in Step 2) with the same probability $p_v$ as desired in the algorithm. Deleting a vertex in Step 3, which was already deleted in Step 2, does not do anything further. This alternative way essentially makes Step 3 independent of every other step.

Fix a vertex $v \in A$. Let $d$ denote the degree of $v$ after the $t$-th iteration of the algorithm. Clearly, $d \leq d_t$. Let $X$ be the random variable denoting the number of neighbors of $v$ deleted in Step 3, and $L$ be the random
variable denoting the number of edges $uv$ adjacent to $v$ deleted in Steps 2 and 5, and $u$ is not deleted in Step 3. Note that $D_{t+1,v} = d - (X + L)$. Thus, we are interested in a good lower bound on $X + L$.

Observe that $E(X) = \sum_{u \in N(v)} p_u$. Clearly, $X$ is the sum of at most $d_t$ Bernoulli random variables with probability at most $\omega_t$. Thus, by using Chernoff bound (Theorem 3.1), the probability that $X$ deviates by more than $\frac{1}{2} \sqrt{\omega_t d_t \log q} \leq \sqrt{q} \log q$ from its expectation is at most $e^{-\omega(\log q)}$.

Now to facilitate the analysis of the random variable $L$, we introduce the random variable $X' = \sum_{u \in N(v)} p'_u$, where $N'$ denote the (random) subset of $N(v)$ containing all the vertices which are not deleted in Step 3. Clearly, $E(X') = \sum_{u \in N(v)} p'_u (1 - p_u)$. Again by Chernoff bound (Theorem 3.1), the probability that $X'$ deviates by more than $\sqrt{q} \log q$ from its expectation is at most $e^{-\omega(\log q)}$. Let $F$ denote the event that both $X$ and $X'$ are within a gap of $\sqrt{q} \log q$ from their corresponding expected values. Next, we show the concentration bound on $L$ conditioning on this event $F$, which holds w.v.h.p. Lemma 10.4 follows from the following claim.

**Claim 10.5.** Conditioned on $F$, at the end of Step 5 of the $(t+1)$-st iteration, w.v.h.p. the degree of each remaining vertex $v \in A$ is at most $d_t (1 - b_t) \left(1 - \frac{1}{1 - \theta} \right) + C' \delta^2 q$, where $C'$ is a constant depending on $\epsilon$.

**Proof.** Fix a vertex $v \in A$ that survived after the $t$-th iteration. The probability that any neighbor $u$ of $v$ is deleted in Step 2 is exactly $p'_u$. Thus, the next task remaining is to calculate the probability that an edge $uv$ will be deleted in Step 5, conditioned on the fact that the vertex $u$ does not get deleted in Step 2. For the convenience of writing, let $Y$ denote the event that an edge of the same color as $uv$ is picked in the partial rainbow matching from $T$ in Step 4. Let $Z$ denote the event that $u$ does not get deleted in Step 2. We are interested in getting a good lower-bound on $P[Z|Y]$.

The event $Z$ is same as the event that none of the edges uniformly picked in Step 1 is adjacent to the vertex $u$. Hence, conditioning on the event $Z$ essentially is same as considering the uniform measure in Step 1, excluding the edges adjacent to $u$. Also, note that the event $Y$ contains the event $Y'$ that exactly one edge $e$ with the same color as $uv$ is picked in Step 1 and no other edges adjacent to $e$ is picked in Step 1. Clearly, we can just calculate this probability for each fixed edge with the same color as $uv$ and sum the probability. Thus, we have the following:

\[
P[Z|Y] \geq P[Z'|Y]
\]

because, the probability to pick an edge $e$ with the same color as $uv$ if we choose an edge uniformly at random among the edges which are not incident to $u$ at least

\[
\left(1 - \frac{2(1+ \epsilon) \delta q (s_t - 1)}{2(1+ \epsilon) \delta q s_t - d_t}\right) \sum_{u \in N'} p'_u (1 - p_u) - \sqrt{q} \log q.
\]

which follows from Corollary 10.2. Now, conditioning on the fact that such an $e$ is picked, the probability that each of the remaining $2(1+ \epsilon) \delta q - 1$ edges is not incident to $e$ and is not of the same color, is at least

\[
\left(1 - \frac{2d_t + s_t}{2(1+ \epsilon) \delta q s_t - d_t}\right)^{2(1+ \epsilon) \delta q - 1}.
\]

Now, by the same calculation as in Claim 8.6 we have the following:

\[
P[Z|Y] \geq \frac{\delta}{1 - \theta} - O_\epsilon(\delta^2).
\]  

Hence, conditioned on $F$, we have the following:

\[
E(D_{t+1,v}) \leq \sum_{u \in N'} (1 - p'_u) \left(1 - \frac{\delta}{1 - \theta} + O_\epsilon(\delta^2)\right).
\]  

Note that $\sum_{u \in N'} p'_u = X' \geq \sum_{u \in N(v)} p'_u (1 - p_u) - \sqrt{q} \log q$. Thus, we have the following:

\[
\sum_{u \in N'} (1 - p'_u) = (d - X) - \sum_{u \in N'} p'_u \leq \sum_{u \in N(v)} (1 - p_u) + \sqrt{q} \log q - \sum_{u \in N(v)} p'_u (1 - p_u) + \sqrt{q} \log q
\]
\[
\sum_{u \in N(v)} (1 - p'_u)(1 - p_u) + 2\sqrt{q} \log q \\
\leq d(1 - b_t) + \delta^2 q,
\]

where we use two facts: (1) for every \( u \), we know that \( (1 - p'_u)(1 - p_u) \) is either \( a_t \) or \( b_t \), and (2) \( b_t \leq a_t \).

The above, together with (10.7), yields the following conditioned on \( F \):

\[
\mathbb{E}(D_{t+1,v}) \leq d_t(1 - b_t) \left( 1 - \frac{\delta}{1 - t\delta} \right) + O(\delta^2) \cdot q. \tag{10.8}
\]

Finally, we can finish the proof by showing the concentration of the random variable \( D_{t+1,v} \) exactly as in the proof of Claim 8.6 using Theorem 3.2.

For the convenience of showing that (10) holds at the end of the \((t+1)\)-st iteration, we call a vertex \( v \) to be in a color class \( C \) if there is an \( C \)-colored edge adjacent to \( v \).

**Lemma 10.6.** At the end of the \((t+1)\)-st iteration, for each color class, the fraction of vertices that are in \( A \) is at most \( \frac{1 - \delta}{2} \) w.v.h.p.

**Proof.** Fix a color class. After completing the \( t \)-th iteration, let \( \rho \) denote the ratio between the number of vertices of \( A \) in this color class to the total number of vertices in this color class. By our assumption, \( \rho \leq \frac{1 - \delta}{2} \). First consider the case that \( \rho \leq \frac{1}{8} \). In this case, at the end of the \((t+1)\)-st iteration, the fraction of vertices in \( A \) is at most the following:

\[
\rho \cdot \frac{2s_t}{2s_{t+1}} \leq \frac{\rho(1 - \alpha_t)(1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)(1 - \theta + \theta\gamma)t\delta)}{(1 - \alpha_{t+1})(1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma)(1 - \theta + \theta\gamma)(1 + \delta))} \leq \frac{1/8}{1 - \alpha_{t+1}} \left( 1 + \frac{2(1 + \epsilon)(1 - \theta + \theta\gamma)}{(1 - 2(1 + \epsilon)(1 - \theta + \theta\gamma))} \right) \leq \frac{1}{\log q} \leq 1 - \frac{1 - \theta}{2},
\]

where \( C(\epsilon) \) is a constant depending on \( \epsilon \) (recall that \( \theta, \eta, \) and \( \gamma \) were defined in terms of \( \epsilon \)).

Thus, we can assume that \( \rho > \frac{1}{8} \). For convenience, let \( X \) be the number of vertices from that color class that are removed in Steps 2 and 3 of the \((t+1)\)-st iteration. Similarly, let \( X_A \) be the number of vertices from that color class, which are in \( A \) and are removed in Steps 2 and 3 of the \((t+1)\)-st iteration. In order to prove Lemma 10.6, it is enough to prove that \( \rho X \leq X_A \) w.v.h.p. (it is easy to check that after Step 6, we still satisfy the statement of Lemma 10.6). The same proof of Lemma 10.3 obtains the following w.v.h.p. (note that we multiplied the expression in Lemma 10.6 by 2 because the number of vertices deleted is exactly 2 times the number of edges deleted):

\[
X \leq 4(\rho \cdot a_t + (1 - \rho) b_t) s_t + 2\sqrt{q} \log q. \tag{10.9}
\]

Next, we find the expected value of \( X_A \). Fix a vertex \( u \in A \) such that there is an alive edge \( uv \) in the fixed color class for some vertex \( v \). We compute the probability that \( u \) will be removed from this color class after Steps 2 and 3. This is exactly the same as the probability of the event in which at least one of the vertices in \( \{u, v\} \) gets deleted in Steps 2 and 3 combined. Denote the event that \( u \) gets deleted in Steps 2 and 3 by \( U \) (similarly, for \( v \), denote the same event by \( V \)). It is clear that

\[
P[U \cup V] = P[U] + P[U^c] \cdot P[V | U^c] = a_t + (1 - a_t) \cdot P[V | U^c]. \tag{10.10}
\]

After the completion of the \( t \)-th iteration, let \( d_{a,v} \) be the degree of the vertex \( v \) and \( |E| \) be the number of edges in \( G \). Recall that \( p'_v \) and \( p_v \) denote the probability that \( v \) gets deleted in Step 2 and Step 3 respectively. Clearly, we have that \( 1 - p'_v = \left( 1 - \frac{d_{a,v}}{2\theta + \epsilon q} \right)^{28(1 + \epsilon)q} \). Unfortunately, we cannot use \( p'_v \) directly
to estimate $\mathbb{P}[V|U^c]$ because the probability we care about is the probability with which $v$ gets deleted in Steps 2 and 3 conditioned on the event that $u$ is not deleted in Steps 2 and 3. For the convenience of estimating that probability, denote by $W$ the event that $u$ is not deleted in Step 2. Since Step 3 can be done independently of the remaining steps (see the first paragraph of the proof of Lemma 10.4), it follows that

$$\mathbb{P}[V|U^c] = \mathbb{P}[V|W].$$

(10.11)

The event $W$ is the same as the event that none of the edges uniformly picked in Step 1 is adjacent to the vertex $u$. Hence, conditioning on the event $W$ essentially is the same as considering the uniform measure in Step 1, excluding the edges adjacent to $u$. Let $q'_v$ denote the probability that $v$ gets deleted in Step 2 conditioned on $W$. Then, we have that $1 - q'_v = \left(1 - \frac{d_{t,v} - 1}{|E| - d_{t,u}}\right)^{2\delta(1 + \varepsilon)q}$. Hence, we have the following:

$$q'_v - p'_v = \left(1 - \frac{d_{t,v} - 1}{|E|}\right)^{2\delta(1 + \varepsilon)q} - \left(1 - \frac{d_{t,v} - 1}{|E| - d_{t,u}}\right)^{2\delta(1 + \varepsilon)q} \geq \left(1 - \frac{1}{|E|}\right)^{2\delta(1 + \varepsilon)q} - 1 \geq -\frac{2\delta(1 + \varepsilon)q}{|E|}$$

$$\geq \frac{2\delta(1 + \varepsilon)q}{2(1 - t\delta)(1 + \varepsilon)s_t}$$

$$\geq \frac{d}{(1 - t\delta) \cdot s_t},$$

(10.12)

where in (10.12), we have used the following fact, which can be proved using basic calculus. For $m > 1$ and $\zeta < 1$, the function $f(x) = (1 - x)^m - (1 + \zeta - x)^m$ is increasing in the range $x \in [\zeta, 1]$. Then, using Equation (10.13), we have the following:

$$\mathbb{P}[V|W] = q'_v + (1 - q'_v)p_v = q'_v(1 - p_v) + p_v \geq p'_v(1 - p_v) + p_v - \frac{\delta}{(1 - t\delta) \cdot s_t} \geq b_t - \frac{\delta}{(1 - t\delta) \cdot s_t},$$

(10.14)

where we have used the definition that $p'_v + p_v - p'_vp_v = a_t$ or $b_t$ depending on the fact if $v \in A$ or not.

Thus, by (10.10), (10.11), and (10.14), the probability that $u \in A$ is removed from the color class is at least $a_t + b_t - a_t b_t - \frac{\delta}{(1 - t\delta) \cdot s_t}$. Hence, the expected number $X_A$ of vertices in $A$ removed from the color class in Steps 2 and 3 is at least $(a_t + b_t - a_t b_t - \frac{\delta}{(1 - t\delta) \cdot s_t}) \cdot 2ps_t$. Similar arguments as in Lemma 10.3 also shows the following w.v.h.p.

$$X_A \geq 2(a_t + b_t - a_t b_t) \cdot \rho s_t - \frac{2\delta \rho}{1 - t\delta} - 2\sqrt{q} \log q.$$  

(10.15)

From the inequalities in (10.9) and (10.15), we have the following w.v.h.p.:

$$X_A - \rho \cdot X \geq 2\rho(1 - 2\rho))(a_t - b_t)s_t - 2a_t b_t \cdot \rho s_t - \frac{2\delta \rho}{1 - t\delta} - 4\sqrt{q} \log q$$

$$\geq \frac{1}{4} \cdot \theta \cdot \frac{g(t\delta)(1 + \beta_t)}{s(t\delta)(1 - \alpha_t)} (1 - \alpha_t)s(t\delta)q - 2(1 + \varepsilon)(1 + \theta)\delta^2 \cdot \frac{g(t\delta)^2(1 + \beta_t)^2}{s(t\delta)^2(1 - \alpha_t)^2} (1 - \alpha_t)s(t\delta)q - \frac{2\delta \rho}{1 - t\delta} - 4\sqrt{q} \log q$$

$$\geq \frac{1}{2} \cdot \frac{g^2(q)}{\log q} \cdot \frac{4q}{\log^2 q} - \frac{2\sqrt{q} \log q}{1 - \eta} - 4\sqrt{q} \log q$$

(10.16)

From (10.16), it is clear that $X_A - \rho \cdot X \geq 0$ for all sufficiently large $q$, which is what we wished to show to complete the proof of Lemma 10.4.

Lemma 10.6 already shows the validity of (4) at the end of the $(t + 1)$-st iteration. We next show the validity of (10.9), (10.11), and (10.15) after the $(t + 1)$-st iteration in the next subsection using the concentration results established in this subsection.
10.4 Estimating the error terms

Similar to Section 7, we estimate the error terms in the parameters throughout the execution of the algorithm in Section 10.1. We first estimate the error terms in the ideal expressions from the intuitive analysis. Define

\[ \tilde{s}_t = s(t\delta)q. \]

\[ \tilde{d}_t = 2(1-t\delta)g(t\delta)(1+\epsilon)q \quad \text{and} \quad \tilde{d}'_t = 2(1-t\delta)g(t\delta)(1+\theta)q. \]

\[ \hat{a}_t = 2(1+\epsilon)\frac{g(t\delta)}{s(t\delta)} \quad \text{and} \quad \hat{b}_t = 2(1+\theta)\frac{g(t\delta)}{s(t\delta)}. \]

We have the following couple of lemmas concerning the relations between these ideal parameters. This time their proofs are a bit technical because the functions \( s(x) \) and \( g(x) \) are not as nice as in Sections 6 or 8.

**Lemma 10.7.** There is some constant \( K \) depending only on \( \epsilon \) such that for all \( t \leq \eta \tau \), we have that

\[ (1 - (1-\theta)\tilde{a}_t - (1+\theta)\tilde{b}_t)\tilde{s}_t \geq \tilde{s}_{t+1} - K\delta^2q. \]

**Proof.** A routine calculation shows:

\[
\tilde{s}_{t+1} - (1 - (1-\theta)\tilde{a}_t - (1+\theta)\tilde{b}_t)\tilde{s}_t \\
= s((t+1)\delta)q - \left(1 - (1-\theta) \cdot 2(1+\epsilon)\frac{g(t\delta)}{s(t\delta)} - (1+\theta) \cdot 2(1+\theta)\frac{g(t\delta)}{s(t\delta)} \right) s(t\delta)q \\
= (s((t+1)\delta) - s(t\delta))q + 2δ((1-\theta)(1+\epsilon) + (1+\theta)^2)g((t+1)\delta)q.
\]  

(10.17)

Using Mean Value Theorem on the function \( s(x) \) in the domain \([0, \eta]\), we obtain that \( s((t+1)\delta) - s(t\delta) = \delta s'(\tilde{y}) \) for some \( \tilde{y} \in [t\delta, (t+1)\delta] \). Note that \( s'(x) = -2((1-\theta)(1+\epsilon) + (1+\theta)^2)g(x) \), which is an increasing function in the domain \([0, \eta]\). Hence, we have \( s((t+1)\delta) - s(t\delta) \leq -2\delta((1-\theta)(1+\epsilon) + (1+\theta)^2)g((t+1)\delta) \).

Using this in (10.17), we get the following:

\[
\tilde{s}_{t+1} - (1 - (1-\theta)\tilde{a}_t - (1+\theta)\tilde{b}_t)\tilde{s}_t \leq -2\delta((1-\theta)(1+\epsilon) + (1+\theta)^2)g((t+1)\delta) - g(t\delta))q. \quad (10.18)
\]

Similarly, using Mean Value Theorem on the function \( g(x) \) in the domain \([0, \eta]\), we obtain that for some \( \tilde{y} \in [t\delta, (t+1)\delta] \), we have \( g((t+1)\delta) - g(t\delta) = \delta g'(\tilde{y}) \). Using this in (10.18) like last time, it is easy to find a constant \( K \) depending on \( \epsilon, \theta, \) and \( \eta \) such that

\[
\tilde{s}_{t+1} - (1 - (1-\theta)\tilde{a}_t - (1+\theta)\tilde{b}_t)\tilde{s}_t \leq K\delta^2q.
\]

\[ \square \]

**Lemma 10.8.** There is some constant \( K' \) depending only on \( \epsilon \) such that for all \( t \leq \eta \tau \), we have that

\[
\left(1 - \tilde{b}_t \right) \left(1 - \frac{1}{\tau - t}\right) \tilde{d}_t \leq \tilde{d}_{t+1} + K'\delta^2q \quad \text{and} \quad \left(1 - \tilde{b}_t \right) \left(1 - \frac{1}{\tau - t}\right) \tilde{d}'_t \leq \tilde{d}'_{t+1} + K'\delta^2q.
\]

**Proof.** We will only show the first one because the second one can be shown the same way. A routine calculation shows:

\[
\tilde{d}_t - \tilde{d}_{t+1} = -2((1 - (1+\delta)g((t+1)\delta) - (1-t\delta)g(t\delta))(1+\epsilon)q. \quad (10.19)
\]

Now using Mean Value Theorem on the function \((1-x)g(x)\) in the domain \([0, \eta]\), we obtain that for some \( \tilde{y} \in [t\delta, (t+1)\delta] \), we have \((1 - (t+1)\delta)g((t+1)\delta) - (1-t\delta)g(t\delta) = \delta(-g(\tilde{y}) + (1-\tilde{y})g'(\tilde{y})) \). Noting that for all \( \tilde{y} \in [t\delta, (t+1)\delta] \), we have that \( g(\tilde{y}) \leq g(t\delta) \) and \((1-\tilde{y})g'(y) \geq (1 - (t+1)\delta)g'((t+1)\delta) \), from (10.19) we get the following:

\[
\left(1 - \tilde{b}_t \right) \left(1 - \frac{1}{\tau - t}\right) \tilde{d}_t - \tilde{d}_{t+1} \leq \tilde{d}_t - \tilde{d}_{t+1} = \frac{\delta}{1-\delta} \tilde{d}_t - \tilde{b}_t \left(1 - \frac{1}{\tau - t}\right) \tilde{d}_t.
\]

\[ 34 \]
\[ \leq 2 \delta [ g(t \delta) - (1 - (t + 1) \delta) g'((t + 1) \delta)] (1 + \epsilon)q - 2 \delta g(t \delta)(1 + \epsilon)q - 2(1 + \theta) \alpha \frac{g(t \delta)}{s(t \delta)} (1 - (t + 1) \delta) \cdot 2g(t \delta)(1 + \epsilon)q \]
\[ \leq 2 \delta (1 - (t + 1) \delta) g'((t + 1) \delta) - g'(t \delta) + (1 + \epsilon)q \leq K' \delta^2 q, \quad (10.20) \]

where \( K' \) depends only on \( \epsilon, \theta, \) and \( \eta, \) and in the last step, we apply Mean Value Theorem on the function \( g'(x) \) like before. In \((10.20), \) we use the fact that \( g'(x) = 2(1 + \theta) \alpha \frac{x^2}{s(x)} \).

Similar to Section 7, we now mention a couple of lemmas relating the error terms we accumulate for estimating \( d_t, d'_t, \) and \( s_t \) as the process goes. In other words, we will specify \( y_t \) and \( z_t \) such that \( d_t \leq d'_t + y_t, \)

\[ d'_t \leq \frac{t}{\alpha t} + y_t, \] and \( s_t \geq \tilde{s}_t - z_t. \) We will then define that \( \alpha_t = \frac{t}{\alpha_t} \) and \( \beta_t = \frac{y_t}{d_t}. \)

For each \( 0 \leq t \leq \eta t, \) define \( y_t = t^3/2q \) and \( z_t = t^6/4q. \) Furthermore, define \( \alpha_t = \frac{t}{\alpha_t} = \frac{y_t}{d_t} \) for each \( t. \) The following lemma finally shows the validity of \((10.3), (10.4), \) and \((10.5)\) for the \((t + 1)-st \) iteration, which is what we needed to show that we can choose a desirable output from Algorithm for each \( t \leq \eta t. \)

Lemma 10.9. At the end of the \((t + 1)-st \) iteration, the following properties hold:

1. the degree of each survived vertex in \( A \) is at most \( d_{t+1} \).
2. the degree of each survived vertex outside \( A \) is at most \( d'_{t+1} \).
3. the remaining color classes have size exactly \( s_{t+1}. \)

Proof. Fix a vertex \( v \in A. \) Starting with the conclusion from Lemma 10.4 and noting that \( b_t = \tilde{b}_t = \frac{t + \theta b_t}{1 - \alpha_t} \geq \tilde{b}_t, \) a routine calculation shows the following bound on the degree \( d_{t+1,v} \) of \( v \) after the \((t + 1)-st \) iteration.

\[ d_{t+1,v} \leq d_t \cdot (1 - \tilde{b}_t) \left( 1 - \frac{\delta}{1 - t \delta} \right) + C' \delta^2 q \leq \left( d_t + y_t \right) \left( 1 - \frac{\tilde{b}_t}{1 - \delta} \right) + C' \delta^2 q \]
\[ \leq \tilde{d}_t (1 - \tilde{b}_t) \left( 1 - \frac{\delta}{1 - t \delta} \right) + y_t + C' \delta^2 q \]
\[ \leq \tilde{d}_{t+1} + K' \delta^2 q + y_t + C' \delta^2 q \]
\[ \leq \tilde{d}_{t+1} + y_t + \delta^3/2q = \tilde{d}_{t+1} + y_{t+1} \leq (1 + \beta_{t+1}) \tilde{d}_{t+1} = d_{t+1}, \quad (10.21) \]


where in \((10.21), \) we use Lemma 10.8. This proves \( 1. \) A similar argument works for \((2); \) thus, we skip it. Next, starting with the conclusion from Lemma 10.3, the size of each remaining color class after Step 3 of the \((t + 1)-st \) iteration is at least the following:

\[ (1 - (1 - \theta) a_t - (1 + \theta) b_t) s_t - \sqrt{q} \log q = \left( 1 - (1 - \theta) \tilde{a}_t \frac{1 + \beta_t}{1 - \alpha_t} - (1 + \theta) \tilde{b}_t \frac{1 + \beta_t}{1 - \alpha_t} \right) (1 - \alpha_t) \tilde{s}_t - \sqrt{q} \log q \]
\[ = \left( 1 - (1 - \theta) \tilde{a}_t - (1 + \theta) \tilde{b}_t \right) \tilde{s}_t - \alpha_t \tilde{s}_t - \left( (1 - \theta) \tilde{a}_t + (1 + \theta) \tilde{b}_t \right) \beta_t \tilde{s}_t - \sqrt{q} \log q \]
\[ \geq \tilde{s}_{t+1} - K' \delta^2 q - z_t - \frac{2 \delta}{1 - t \delta} \cdot y_t - \sqrt{q} \log q \]
\[ \geq \tilde{s}_{t+1} - z_t - \delta^5/4q = \tilde{s}_{t+1} - z_{t+1} = (1 - \alpha_{t+1}) \tilde{s}_{t+1} = s_{t+1}, \quad (10.22) \]

where in \((10.22), \) we use Lemma 10.7 and the fact that \( \left( (1 - \theta) \tilde{a}_t + (1 + \theta) \tilde{b}_t \right) \tilde{s}_t \leq 2 \tilde{a}_t \tilde{s}_t = \frac{2 \delta}{1 - t \delta} \tilde{d}_t. \)

Next, as we promised, we will find that the accumulated error terms are negligible compared to the ideal parameter values, i.e., \( y_t \ll \tilde{d}_t \) and \( z_t \ll \tilde{s}_t. \) This, in other words, shows that \( \beta_t \leq 1/100 \) and \( \alpha_t \leq 1/100, \) which were used several times in the previous subsection.

Lemma 10.10. For each \( t \leq \frac{2}{9}, \) we have that \( y_t = o \left( \tilde{d}_t \right) \) and \( z_t = o \left( \tilde{d}_t \right). \)

Proof. For \( t \leq \eta \tau, \) we have that \( y_t \leq y_{\eta \tau} = O \left( \frac{q}{\log q} \right) \) and \( \tilde{d}_t \geq \tilde{d}_{\eta \tau} = \Omega(q); \) thus, \( y_t = o(\tilde{d}_t). \) Similarly for \( t \leq \eta \tau, \) we have that \( z_t \leq z_{\eta \tau} = O \left( \frac{q}{\log q} \right) \) and \( \tilde{s}_t \geq \tilde{s}_{\eta \tau} = \Omega(q); \) thus, \( z_t = o(\tilde{s}_t). \)  
\( \square \)
Finally, at the end of the \( \eta \tau \)-th iteration, using Corollary 10.2 the number of edges picked in the rainbow matching is at least \( \eta \cdot 2(1 + \epsilon)q - 8\delta q \log \left( \frac{1}{1 - 2(1 + \epsilon)(1 - \theta + \eta \gamma \eta)} \right) > q \), finishing the proof of Theorem 11.1.

### 11 Concluding remarks

Our arguments can be extended to get a generalization of Theorem 1.9 for multigraphs with bounded edge multiplicities. In particular, Theorem 11.1 holds (with a worse error term in the required number of edges in each color) for multigraphs with edge multiplicities at most \( q' \) for some absolute constant \( c > 0 \). Gao, Ramadurai, Wanless, and Wormald have discussed this point in more detail in their paper [20]. Similar to them, extending our arguments to the general multigraphs with no restrictions seems difficult.

We remark here that it was shown in [20] that Theorem 1.9 is true if the number of colors is at most \( q^{1+c} \) colors for some \( 0 < c < 1 \). One of the main contributions of this paper is to remove this condition on the number of colors. In the setting of Theorem 1.9, it will be interesting to investigate further how small the ‘\( o(q) \)’ term can be made in Corollary 1.13. Our proof, with a more careful analysis in the style of the proof of Theorem 1.9, gives some error term of the form \( q' \) for some absolute constant \( 0 < c < 1 \). It will be interesting to make this error term poly-logarithmic.

Finally, we discuss why improving the required number of colors in Theorem 11.1 might be hard using our approach. To demonstrate that, we state an asymptotically tight generalization of Theorem 1.11. We start with a motivation, which was first mentioned in an earlier version of [4]. Consider the setting of Theorem 1.11 where we have some number \( \delta q \) edges to obtain another system of 2 perfect matchings and assign distinct colors corresponding to each of these matchings. From the rest of the 2 perfect matchings of size \( q \), we select any 2 perfect matchings (resulting color classes might not be matching anymore), each of size \( q \). The underlying graph’s maximum degree will still be bounded by the number of colors used. Furthermore, if the scrambling was done randomly, then each class of edges will have bounded maximum degree. The survey [5] contains several interesting results in this ‘scrambling’ setting.

**Theorem 11.1.** For every \( \epsilon > 0 \) and positive integer \( \Delta \), there exists \( N = N(\epsilon, \Delta) \) such that whenever \( q \geq N \), the following holds. Suppose \( G \) is a graph with maximum degree at most \( 2(1 + \epsilon)q \) that is edge-colored with \( 2 \left( 1 + \epsilon - \frac{\theta^2}{(\Delta + \eta)^2} \right) \) colors such that there are at least \( q \) edges of each color and at most \( \Delta \) edges of the same color can be incident to any vertex. Then, there is a rainbow matching in \( G \) using \( q \) colors.

Note that \( \Delta = 1 \) retrieves the original Theorem 11.1 because the condition \( \Delta = 1 \) ensures that the maximum degree of the underlying graph is at most the number of colors used. Next, we show that Theorem 11.1 is tight for \( \Delta \geq 2 \) in the sense that the constant 2 cannot be replaced with a smaller constant in the required number of colors. More precisely, we show that there is a graph \( G \) with maximum degree \( 2q - 2 \) that is edge-colored with \( 2q - 3 \) colors such that each color class has \( q \) edges and maximum degree 2 such that \( G \) does not contain any rainbow matching of size \( q \). To see this, consider a decomposition of the edges of the complete graph \( K_{2q-1} \) into \( 2q - 1 \) perfect matchings (with size \( q - 1 \)). Select any \( 2q - 3 \) such perfect matchings and assign distinct colors corresponding to each of these matchings. From the rest of the \( 2q - 2 \) remaining edges, select any \( 2q - 3 \) edges and color them using the \( 2q - 3 \) colors using each color exactly once. Now, it is easy to check that we do not even have any matching of size \( q \) because we have only \( 2q - 1 \) vertices in the considered graph.

Theorem 11.1 can be proved using the following couple of results which can be established using arguments similar to Sections 9 and 10.

**Theorem 11.2.** For every \( \epsilon > 0 \) and positive integer \( \Delta \), there exists \( N = N(\epsilon, \Delta) \) such that whenever \( q \geq N \), the following holds. Suppose \( G \) is a bipartite graph on the vertex set with bipartition \( A \cup B \), where \( |A| = q \) and every vertex in \( A \) has degree at least \( (1 + \epsilon)q \). Suppose the edges are colored such that there are at most \( q \) edges of each color and at most \( \Delta \) edges of the same color can be incident to any vertex. Then, there always is a rainbow matching in \( G \), which uses every vertex in \( A \).
Theorem 11.3. For every $\epsilon, \theta, \alpha > 0$, and positive integer $\Delta$, there exists $N = N(\epsilon, \theta, \alpha, \Delta)$ such that whenever $q \geq N$ and $\Delta_G \geq (1 + \epsilon)q$, the following holds. Suppose $G$ is a graph with maximum degree at most $\Delta_G$ and the number of vertices with degree at least $(1 - \theta)\Delta_G$ is at most $(1 - \theta)q$. Furthermore, $G$ is edge-colored with $\max((1 + \alpha)q, (1 - \theta^2 + \alpha)\Delta_G)$ colors such that there are at least $q$ edges of each color and at most $\Delta$ edges of the same color can be incident to any vertex. Then, there is a rainbow matching of $G$ using $q$ colors.

Any proof of any significant improvement on the required number of colors in Theorem 1.11 should not work for Theorem 11.1 because Theorem 11.1 is already asymptotically tight. Thus, the above discussion indicates that our probabilistic approach alone cannot yield any such improvement of Theorem 1.11 if it is not tight, and to improve this result, we might need to use some structural arguments taking advantage of the fact that each color appears as a matching.

On a different note, Stein [33] made a conjecture stronger than Conjecture 1.1 which translates into the graph-theoretic statement: “In any $n$-edge-coloring of $K_{n,n}$ with each color appearing exactly $n$ times, there is a rainbow matching of size $n$”. This stronger conjecture is shown to be false in [30] by Pokrovskiy and Sudakov. However, Theorem 11.2 obtains a generalized version of Corollary 1.13 in the style of Stein’s conjecture, where the coloring does not need to be proper, but each color class needs to have bounded degree.

Recently, hypergraph generalizations of some of our main results are obtained by Delcourt and Postle [18]. Moreover, they proved that it is enough to have $q + o(q)$ colors if the underlying graph is bipartite in Theorem 1.11 which addresses a question we suggested in a previous version of this paper.

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