We study the singularity locus of the sparse resultant of two univariate polynomials, and apply our results to estimate singularities of a coordinate projection of a generic spatial complete intersection curve.

1. Introduction

Given a pair $B = (B_1, B_2)$ of finite sets $B_1$ and $B_2 \subset \mathbb{Z}^1$ with at least two elements in each, one can consider the space of pairs of sparse Laurent polynomials supported at $B_1$ and $B_2$:

$$
\mathbb{C}^{B_1} \times \mathbb{C}^{B_2} = \left\{ \left( \sum_{k \in B_1} c_k x^k, \sum_{k \in B_2} c_k x^k \right) \right\}.
$$

The sparse resultant $R_B \subset \mathbb{C}^{B_1,B_2}$ is the closure of the set

$$
\widehat{M}_1 = \{(f_1, f_2) \mid f_1(x) = f_2(x) = 0 \text{ has at least one non-zero solution}\}.
$$

**Example 1.1.** For $B_i = \{0, 1, 2, \ldots, d_i\}$ the sparse resultant $R_B$ is the usual resultant, that is the hypersurface defined by Sylvester matrix.
This algebraic hypersurface is actively studied starting from the paper [S] and the book [GKZ]. We aim at describing its singular locus sing \( R_B \).

For the classical case \( B_1 = \{0, 1, 2, \ldots, d_i\} \), the singular locus has one irreducible component of codimension 2: it is the closure of the set

\[
M^{1,1}_{1,1} = \{(f_1, f_2) \mid f_1(x) = f_2(x) = 0 \text{ has exactly two non-zero solutions}\}.
\]

Moreover, we know how singular the resultant is at a generic point: its transversal singularity type at a generic point \( (f_1, f_2) \in M^{1,1}_{1,1} \) (i.e. the type of the singularity of the intersection of \( R_B \) with a germ of a 2-dimensional plane transversal to \( M^{1,1}_{1,1} \) at \( (f_1, f_2) \)) is \( A_1 \) (i.e. that of the union of two transversal lines in the plane).

Our main result describes the conditions on \( B_1 \) and \( B_2 \) under which this description holds true.

**Theorem 1.2.** The codimension 2 part of the singular locus of the resultant \( R_B \) is the closure of \( M^{1,1}_{1,1} \) (with the same transversal singularity type \( A_1 \) at a generic point) unless

1. One can shift \( B_1 \) and \( B_2 \) to the same proper sublattice \( k\mathbb{Z} \subset \mathbb{Z} \) with \( k \geq 2 \).
2. One can split one of \( B_i \)'s (say, \( B_1 \)) into \( B' \cup B'' \) so that \( B', B'' \) and \( B_2 \) can be shifted to the same sublattice \( k\mathbb{Z} \subset \mathbb{Z} \) with \( k \geq 3 \).
3. For every \( i = 1, 2 \), the two smallest elements of \( B_i \) differ by more than 1.
4. For every \( i = 1, 2 \), the two greatest elements of \( B_i \) differ by more than 1.

The following examples illustrate what happens to the singular locus once one of the conditions (1-4) takes place.

**Example 1.3.** 1) If condition (1) takes place, then the closure of \( \hat{M}^{1,1}_{1,1} \) equals the resultant \( R_B \) rather than its singular locus, because for every pair \( (f_1(x), f_2(x)) \) of polynomials with a non-zero common root \( x \) there is also another non-zero common root \( x \cdot \epsilon_k \), where \( \epsilon_k \) is the \( k \)-th root of 1, thus \( \hat{M}^{1}_{1} = \hat{M}^{1,1}_{1,1} \), and the same is true for their closures.

In this case, the study of the resultant \( R_B \) can be reduced to the study of a resultant \( R_{B'} \) for a smaller pair of support sets \( B'_1 \) and \( B'_2 \) such that \( B_1 = k \cdot B'_1 + m_1 \) and \( B_2 = k \cdot B'_2 + m_2 \). Indeed, \( \mathbb{C}^{B_1} \times \mathbb{C}^{B_2} = \mathbb{C}^{B'_1} \times \mathbb{C}^{B'_2} \) and the conditions defining \( R_B \subset \mathbb{C}^{B_1} \times \mathbb{C}^{B_2} \) and \( R_{B'} \subset \mathbb{C}^{B'_1} \times \mathbb{C}^{B'_2} \) are the same. Thus \( R_B \cong R_{B'} \) and, moreover, \( \hat{M}^{1}_{B'1} = \hat{M}^{1,1}_{B'1,1} = \ldots = \hat{M}^{1,1}_{B'1;\ldots,1} \cong \hat{M}^{1}_{B'1} \).

2) If condition (2) takes place, then, for some component of the singular locus of the resultant, the transversal singularity type at a generic point has \( k \geq 3 \) components, i.e. differs from \( A_1 \).

For instance, consider \( B_1 = \{0, 1, 3\} \) and \( B_2 = \{0, 3\} \). Let us denote the polynomials by \( ax^3 + bx + c \) and \( dx^3 + e \). Then the resultant is given by the equation

\[
(ax - cd)^3 + b^3d^2e = 0. \quad (*)
\]

This equation is homogeneous in \( (a, b, c) \) and in \( (d, e) \), so let us restrict it to \( a = d = 1 \). Then the singular locus of the restriction is given by the equation \( b = c - e = 0 \), which corresponds to a component of the singular locus of \( (*) \). Choosing \( b = \epsilon_1, \ c = 1 + \epsilon_2, \ e = 1 \) as a transversal plane germ to this component, the transversal singularity of this component is given by \( \epsilon_1^3 - \epsilon_2^3 = 0 \), thus it is the union of three transversal lines intersecting in a common point.
3) If condition (3) takes place, then, for some component of the singular locus of the resultant, the transversal singularity type at a generic point has a singular component, i.e. a component that differs from $A_1$.

For instance, consider the same example $B_1 = \{0, 1, 3\}$ and $B_2 = \{0, 3\}$. There are two more components of the singularity locus which are given by $a = d = 0$ and $d = e = 0$. For the first one, the transversal singularity is the cusp.

4) The case when condition (4) takes place can be reduced to (3) by the change of coordinate $\tilde{x} = x^{-1}$, because the sets $B_1$ and $B_2$ can be replaced by the sets $B'_1 = \{-\tilde{b} : b \in B_1\}$ and $B'_2 = \{-\tilde{b} : b \in B_2\}$.

**Remark 1.4.** As explained above, we can assume with no loss of generality that $B_1$ and $B_2$ cannot be shifted to a proper sublattice of $\mathbb{Z}$. We now assume this once and for all.

**Remark 1.5.** Actually, once the condition (2), (3) or (4) takes place in the setting of the theorem, the singular locus of the resultant always has a component, at whose generic point the transversal singularity type differs from $A_1$. However, the study of these singularity types (and the proof that they differ from $A_1$) is non-trivial and will be done in a separate paper.

**Remark 1.6.** While conditions similar to (1), (3) and (4) are familiar to the experts in tropical geometry and Newton polytopes, the condition (2) is new (to the best of our knowledge).

**Remark 1.7.** It would be important to obtain a version of this theorem for $A$-discriminants: their singular locus is studied e.g. in [E] and [DHT] under the assumption that it has the expected dimension and transversal singularity type, but there are no known criteria for these assumption to hold true even for univariate polynomials so far (except for some sufficient conditions in Section 3.4 of [E]).

**Remark 1.8.** In this article, we prove Theorem 1.2 over the field of complex numbers. This setting is important in our proof of Lemma 4.3 where one of the key steps essentially makes appeal to elementary geometry in the complex line (see Figure 1). We do not know whether it is true in the finite characteristics, it is an interesting question to study.

**Remark 1.9.** We expect that the theorem is valid for the whole singular locus of the resultant, rather than just for its lowest codimension part. Independently of this conjecture, restricting our attention to the lowest codimension part in this paper is not artificial, because the lowest codimension part is most important for enumerative applications of our theorem, such as Theorem 1.10 below.

Most of the paper is devoted to the proof of Theorem 1.2. As an application, we can prove the following theorem, partially answering the question from [Y].

Given two finite sets $A_1$ and $A_2$ in $\mathbb{Z}^3$ and generic complex (or real) Laurent polynomials $f_1$ and $f_2$ supported at these sets, the equations $f_1 = f_2 = 0$ define a smooth spatial algebraic curve in $(\mathbb{C}^*)^3$ (or $(\mathbb{R}^*)^3$). The closure $C$ of its projection to the first coordinate plane is not in general smooth: at least it may have singularities of type $A_1$ at the points having two preimages. Such $A_1$ singularities are stable under local perturbations of the smooth spatial curve, similarly to transversal self-intersections of a knot diagram.
Theorem 1.10. The curve $C$ has no other singularity types, unless the projections $B_1$ and $B_2$ of $A_1$ and $A_2$ to the last coordinate line satisfy one of the four conditions of Theorem 1.2.

Remark 1.11. Even if the sets $A_1$ and $A_2$ in this theorem are the sets of all lattice points in prescribed Newton polytopes, their projections $B_1$ and $B_2$ rarely consist of several consecutive integers. They usually have gaps. This explains why we need Theorem 1.2 for arbitrary support sets, and not just for the classical case $B_i = \{0, 1, 2, \ldots, b_i\}$.

2. The resultant stratification

From now on, we shall identify a polynomial $f(x) = \sum_{b \in B_i} c_b x^b \in \mathbb{C}^{B_i}$ with the section

$$s_f = \sum_{b \in B_i} c_b x^{b-\min B_i} y^{\max B_i - b}$$

of the invertible sheaf $\mathcal{O}(\max B_i - \min B_i)$ on $\mathbb{C}^1$ with homogeneous coordinates $x$ and $y$.

The purpose is to be able to speak of the multiplicities of the root of $f$ at 0 and $\infty \in \mathbb{C}P^1$. Explicitly, if $B_i = \{b_1 \leq b_2 \leq \ldots \leq b_m\}$, to calculate the multiplicity of $c_f$ at 0 or $\infty$ we write down $c_{b_1}, 0, \ldots, 0, c_{b_2}, 0, \ldots, 0, c_{b_m}$, where $c_{b_i}$ is written on $(b_i - \min B_i)$-th position, and count the number of 0 from the left or from the right respectively.

Remark 2.1. If $f$ is a polynomial, not just a Laurent polynomial, then with this identification the multiplicity of the section $s_f$ at 0 is different from the multiplicity of the root of $f$ at 0 as the polynomial. In particular, if we shift $B_i$ by $n$ and multiply $f$ by $x^n$, then the multiplicity of $s_f$ at 0 would not change, and the multiplicity at 0 of $f$ as a polynomial would increase by $n$.

Remark 2.2. Such a multiplicity has good properties of the usual multiplicity even at 0 and $\infty$. For example, if we have a continuous family $f_t(x)$ of polynomials in $\mathbb{C}^B$ parametrized by $t$, and the family of roots $x_t$ of $f_t(x)$ which tends to $x_0$ as $t$ tends to 0, then the multiplicity of $x_0$ is not lesser then the multiplicities of $x_t$ for small $t$ around 0.

For example, consider $B = \{a, b, c\}$ with $a < b < c$ and $f_t(x) = tx^a + x^b + tx^c$. Then for $t \neq 0$ the multiplicity of $f_t$ at 0 and $\infty$ is 0, and for $t = 0$ they are $(b-a)$ and $(c-b)$ respectively.

This identification allows us to split the resultant $R_A$ into the following strata:

Definition 2.3. The stratum

$$M^{j_1, j_2; j_1^1, \ldots, j_k^1, j_1^2, \ldots, j_k^2; j_0^1, j_0^2, j_\infty^1, j_\infty^2; j_1, \ldots, j_k; j_1^1, \ldots, j_k^2}$$

consist of $(f_1, f_2) \in \mathbb{C}^{B_1} \times \mathbb{C}^{B_2}$ such that

1. $f_i$ has a root of multiplicity exactly $j_i^1$ at 0 and exactly $j_i^\infty$ at infinity;
2. $f_1 = f_2 = 0$ has exactly $k$ distinct solutions $x_1, \ldots, x_k$ in $\mathbb{C}^\infty$;
3. at the $m$-th solution $x_m$, the polynomial $f_1$ has a root of multiplicity exactly $j_m^1$ and the polynomial $f_2$ has a root of multiplicity exactly $j_m^2$.

The strata numbering is defined up to a simultaneous permutation of $j_1^1, \ldots, j_k^1$ and $j_1^2, \ldots, j_k^2$. For example, $M^{0, 0; 1, 2} = M^{0, 0; 2, 1}$.

Our interest to these strata comes from their relation to the singular locus of the resultant.
We say that it is Zariski open in it. Equivalently, the singular locus of $R_B$ is contained in the union of all the other strata except for $M_{0:0:1}^{0:0:1}$.

2) Unless $(B_1, B_2)$ satisfy the condition (3) (respectively (4)) of the main theorem 1.2, the stratum $M_{1:0}^{1:0}$ (respectively $M_{0:1}^{0:1}$) is in the smooth part of the resultant.

3) The strata $M_{0:0:1,1}^{0:0:1}$ is contained in the singular locus of $R_B$. More generally, it holds true for the set of $(f_1, f_2) \in \mathbb{C}^{B_1} \times \mathbb{C}^{B_2}$ such that

- (1) $f_1$ has a root of multiplicity exactly $j^1_0$ at 0 and $f_2$ has a root of multiplicity at least $j^2_0$ at 0 (or vice versa);
- (2) $f_1$ has a root of multiplicity exactly $j^1_\infty$ at infinity and $f_2$ has a root of multiplicity at least $j^2_\infty$ at infinity (or vice versa);
- (3) $f_1 = f_2 = 0$ has exactly $k$ distinct solutions $x_1, \ldots, x_k$ in $\mathbb{C}^*$;
- (4) at the $m$-th solution $x_m$, the polynomial $f_1$ has a root of multiplicity exactly $j^1_m$ and the polynomial $f_2$ has a root of multiplicity at least $j^2_m$ (or vice versa).

More specifically, in a small neighborhood of every point of this set, $R_B$ is the union of two smooth transversal hypersurfaces.

One can prove these facts using Thom’s transversality lemma.

To study these strata, we notice that these sets are indeed strata in the sense that they are locally (Zariski) closed. To prove this, we introduce the following partial order $\prec$ on the set $I$ indexing these strata.

**Definition 2.5.** Let us consider two strata

$$M_p = M^{j^1_0; j^2_\infty; \cdots; j^k_\infty}_{j^1_0; j^2_\infty; \cdots; j^k_\infty} \quad \text{and} \quad M_q = M^{g^1_0; g^2_\infty; \cdots; g^l_\infty}_{g^1_0; g^2_\infty; \cdots; g^l_\infty}.$$  

We say that $p \prec q$ if, informally, the common roots of a pair $(h_1, h_2) \in M_q$ in $\mathbb{C}P^1$ can be obtained from the common roots of a pair $(f_1, f_2) \in M_p$ in $\mathbb{C}P^1$ by (1) increasing their multiplicity, (2) gluing some of the roots together, (3) moving some of the roots from $\mathbb{C}^*$ to 0 or $\infty$ (but not moving the root from $\mathbb{C}^*$ to 0 or $\infty$), (4) adding some new common roots.

Formally, for each $i \in \{0, \infty, 1, \ldots, k\}$ there is $r(i) \in \{0, \infty, 1, \ldots, l\}$ such that $r(0) = 0$, $r(\infty) = \infty$ and for any $s \in \{0, \infty, 1, \ldots, l\}$ holds

$$\left( \sum_{i: r(i) = s} j^1_i \right) \leq g^1_s \quad \text{and} \quad \left( \sum_{i: r(i) = s} j^2_i \right) \leq g^2_s.$$  

**Definition 2.6.** We can now define the closed strata to be $\widetilde{M}_p = \bigsqcup_{p \prec q} M_q$.

Based on the compactness of $\mathbb{C}P^1$, we can prove the following.

**Lemma 2.7.** The sets $\widetilde{M}_p = \bigsqcup_{p \prec q} M_q$ are indeed (Zariski) closed.

**Proof.** Assume that a pair $(h_1, h_2)$ is the limit of $(f^t_1, f^t_2) \in M_p$ as $t \to 0$. Let us prove that its common roots and their multiplicities differ from those of $(f^t_1, f^t_2)$ as described in the preceding remark, and thus $(h_1, h_2)$ belongs to some $M_q$, $p \prec q$.

Consider a point $x_0 \in \mathbb{C}P^1$ and let $k_1$ and $k_2$ be its multiplicities as a root of $h_1(x)$ and $h_2(x)$ respectively, maybe 0’s. We know that $h_i$ is the limit of $f^t_i$ as $t \to 0$, thus there is a
small neighborhood $U \subset \mathbb{CP}^1$ containing $x_0$ and a small number $\epsilon > 0$ such that for every $t$
if 0 < |t| < \epsilon then each of $f_i(t)$ has exactly $n_i$ roots $y_1^i(t), \ldots, y_{n_i}^i(t)$ in the neighborhood $U$, these roots are of multiplicities $m_1^i, \ldots, m_{n_i}^i$ and there holds an equation $k_i = \sum_{j=1}^{k_i} m_j^i$. We can also suppose that $U$ contains no other roots of $f_i(t)$ than $x_0$.

Now let us cover $\mathbb{CP}^1$ with such neighborhoods $U(x)$ for each $x \in \mathbb{CP}^1$, choose a finite subcover $U(x_1), \ldots, U(x_n)$ by compactness and fix $\epsilon = \min(\epsilon(x_1), \ldots, \epsilon(x_n))$. Now the condition of the previous paragraph holds globally: for every $t$ if 0 < |t| < $\epsilon$ then each of $f_i(t)$ has exactly $n_i$ roots $y_1^i(t), \ldots, y_{n_i}^i(t)$ in the $\mathbb{CP}^1$, these roots are of multiplicities $m_1^i, \ldots, m_{n_i}^i$ and there holds an equation $k_i = \sum_{j=1}^{k_i} m_j^i$.

Now let us notice that if
\[
\begin{align*}
k_1 &= \sum_{j_1=1}^{k_1} m_{j_1}^1, \quad \text{and} \quad k_2 = \sum_{j_2=1}^{k_2} m_{j_2}^2,
\end{align*}
\]
then
\[
\min(k_1, k_2) \geq \sum_{y_{j_1}^1 = y_{j_2}^2} \min(m_{j_1}^1, m_{j_2}^2),
\]
from which the conditions from the previous remark follow. \hfill \Box

**Corollary 2.8.** The set $M_p = \widehat{M}_p \setminus \bigcup_{p < q} \widehat{M}_q$ is locally closed.

The relation of these strata to the singular locus of the resultant motivates our interest to their codimension.

**Definition 2.9.** The expected codimension of the stratum $M_{j_0^1 \ldots j_0^k; \ldots; j_k^1}$ is the number
\[
\begin{align*}
j_0^1 + j_0^2 + j_\infty^1 + j_\infty^2 + \sum_{m=1}^{k} (j_m^1 + j_m^2 - 1).\end{align*}
\]

**Remark 2.10.** An algebraic subset can consist of several irreducible components of different dimensions, thus we should be accurate when talking about its codimension. When we speak that an algebraic subset has expected codimension, we mean that the codimension of every its irreducible component is equal to the expected one. When we speak that the codimension of an algebraic subset is at most the expected one, we mean that the codimension of every its irreducible component is at most the expected one. In particular, in both cases it may be empty, although some sources consider the empty set to have codimension $\infty$.

**Example 2.11.** Here we list the strata up to the switching the rows, changing the order of columns of $x_1, \ldots, x_k$ or switching the columns of 0 and $\infty$.

The strata of the expected codimension 1 are $M_{0;0}^{1;0}$ and $M_{0;0}^{0;1}$.

The strata of the expected codimension 2 are $M_{0;0}^{2;0}$, $M_{0;0}^{1;1}$, $M_{0;1}^{1;0}$, $M_{0;0}^{1;0}$, $M_{0;0}^{1;0}$, $M_{0;0}^{0;2}$ and $M_{0;0}^{0;2}$.

The strata of the expected codimension 3 are $M_{0;0}^{2;0}$, $M_{0;0}^{2;0}$, $M_{0;1}^{2;0}$, $M_{0;0}^{2;0}$, $M_{0;0}^{1;1}$, $M_{0;0}^{1;1}$, $M_{0;0}^{1;1}$, $M_{0;0}^{1;1}$, $M_{0;1}^{0;2}$, $M_{0;0}^{0;2}$, $M_{0;0}^{0;1}$, $M_{0;0}^{0;1}$, $M_{0;0}^{0;1}$, $M_{0;0}^{0;1}$, $M_{0;0}^{0;1}$, $M_{0;0}^{0;1}$.

**Lemma 2.12.** For the classical case $B_i = \{0, 1, 2, \ldots, d_i\}$, the codimension of the stratum $M_p$ (at every its irreducible component) is equal to the expected one.
For the sake of completeness, we provide the proof of this classical fact by parameterizing naturally the set $\hat{M}_p$.

**Proof.** First, let us consider the case of $M_p = M_{0;0; j_1^1, \ldots, j_n^1}$. Consider the open subset

$$P \subset \mathbb{C}^{e_1} \times \mathbb{C}^{e_2} \times \mathbb{C}^k, \quad e_1 = d_1 - \sum_i j_i^1, \quad e_2 = d_2 - \sum_i j_i^2$$

of tuples $(x_1, \ldots, x_{e_1}; y_1, \ldots, y_{e_2}; z_1, \ldots, z_k)$ such that $z_i$ are pairwise different and each $x_i$ and $y_i$ is different from all $z_i$. There is a map $P \to M_p$ of the form

$$(x_1, \ldots, x_{e_1}; y_1, \ldots, y_{e_2}; z_1, \ldots, z_k) \mapsto \left( \prod_i (t - x_i) \prod_i (t - z_i)^{j_i^1}, \prod_i (t - y_i) \prod_i (t - z_i)^{j_i^2} \right),$$

which is surjective and has finite fibers, thus $\dim P = \dim M_p$ and

$$\text{codim } M_p = d_1 + d_2 - \dim M_p = \sum_i (j_i^1 + j_i^2 - 1)$$

is the expected codimension.

Now to prove the general case, let us notice that the stratum

$$M_{0;0; j_1^1, \ldots, j_n^1} \subset \mathbb{C}^{B_1} \times \mathbb{C}^{B_2}$$

with $B_i = \{0, \ldots, d_i\}$ can identified with the stratum

$$M_{0;0; j_1^2, \ldots, j_k^2} \subset \mathbb{C}^{B'_1} \times \mathbb{C}^{B'_2}$$

with $B'_i = \{0, \ldots, d_i - j_0^i - j_{i-1}^i\}$ by the inclusion $\mathbb{C}^{B'_1} \times \mathbb{C}^{B'_2} \hookrightarrow \mathbb{C}^{B_1} \times \mathbb{C}^{B_2}$ induced by the pair of maps $B'_i \to B_i, b \mapsto b + j_0^i$. □

This fact survives as an estimate in the general case.

**Lemma 2.13.** In general, the codimension of the stratum $M_p$ at every its irreducible component is at most the expected one.

Thus $M_p$ can have irreducible components of larger dimension that expected, but can not have irreducible components of smaller dimension.

**Proof.** Equally, we can prove that the codimension of the stratum $M_p$ at every its point is at most the expected one.

Consider the natural embedding $\mathbb{C}^{B_1} \times \mathbb{C}^{B_2} \to \mathbb{C}^{\text{conv } B_1} \times \mathbb{C}^{\text{conv } B_2}$, where $\text{conv } B_i = \{\min B_i, \ldots, \max B_i\}$ is the convex hull of $B_i$. The stratum $M_p$ in $\mathbb{C}^{B_1} \times \mathbb{C}^{B_2}$ is the preimage of the corresponding stratum in $\mathbb{C}^{\text{conv } B_1} \times \mathbb{C}^{\text{conv } B_2}$. Thus at every point of the stratum $M_p$ in $\mathbb{C}^{B_1} \times \mathbb{C}^{B_2}$ its codimension is not greater then the codimension of the corresponding stratum in $\mathbb{C}^{\text{conv } B_1} \times \mathbb{C}^{\text{conv } B_2}$ at the image point, which is equal to the expected one. □

**Remark 2.14.** A stratum may be empty. For example, let us consider $B_1 = \{a,b\}$ and any non-empty $B_2$. A polynomial $f_1 \in \mathbb{C}^{B_1}$ has form $c_1 x^a + c_2 x^b$, and thus can not have two equal roots from $\mathbb{C}^\times$, thus $M_{0;0;2}^{0;0;2}$, which has expected codimension 2 and lies in the space $\mathbb{C}^{B_1} \times \mathbb{C}^{B_2}$ of the dimension greater then 2, is in fact empty.

The inverse estimate is non-trivial.
THEOREM 2.15. Unless \((B_1, B_2)\) satisfy one of the following conditions:

- The condition (1) or (2) of the theorem 1.2 holds.
- For one of the \(i = 1, 2\), the smallest 3 elements of \(B_i\) differ by more than 2.
- For one of the \(i = 1, 2\), the greatest 3 elements of \(B_i\) differ by more than 2.
- One can shift one of \(B_1\)'s (say, \(B_1\)) and \(B_2 \setminus \{ \min B_2 \}\) to the same proper sublattice \(kZ \subset \mathbb{Z}\) with \(k \geq 2\).

\(\) every subset \(\hat{M}\) of expected codimension 1, 2 or 3 has actual codimension exactly 1, 2 or 3 respectively. Consequently, every stratum \(\hat{M}\) of expected codimension 1 or 2 has actual codimension exactly 1 or 2 respectively.

In fact these conditions are only used for some of the strata. The exact details of which conditions are necessary for which strata see in the theorem 3.14.

The proof of this theorem is given in the next section. Using this fact, we can now prove the main theorem 1.2.

Proof of the main theorem 1.2 Here we will use the strata \(M'\) from the definition 3.10.

By the lemma 2.4, part 1, the singular locus of \(R_B\) in contained in \(\hat{M}_{0;0,1,1}^{1;0,1}\).

Consider the space

\[
X = M'_{0;0,1,1} \setminus \left( M'_{1;0,1,1} \cup M'_{0;1,1} \cup M'_{1;1,1} \cup M'_{0;1,1,1} \right).
\]

By the lemma 3.13, the codimension of \(M'_{0;0,1,1}\) is 2. The strata lie inside \(M'_{1;0,1,1}, M'_{0;1,1}\) and \(M'_{0;1,1,1}\) respectively, and by the same theorem the codimensions of these strata are 3, unless the condition (1) or (2) holds, thus \(X\) has codimension 2.

Let us notice that

\[
\hat{M}_{0;0,1,1}^{1;0,1,1} = M'_{0;0,1,1} \cup M'_{1;0,1,1} \cup M'_{0;1,1} \cup M'_{1;1,1} \cup M'_{2;0} \cup M'_{0;2} \cup M'_{1;1,1}.
\]

By the same lemma 3.13, the codimension of \(M'_{1;0,1,1}\) and \(M'_{0;1,1,1}\) is 3, and obviously the codimensions of \(M'_{2;0}, M'_{0;2}\) and \(M'_{1;1,1}\) are 4. Thus \(X\) is open and dense in \(\hat{M}_{0;0,1,1}^{1;0,1,1}\).

By the lemma 2.4, part 3, \(X\) is contained in the singular locus of \(R_B\).

To sum it up, \(X \subset \text{sing} R_B \subset \hat{M}_{0;0,1,1}^{1;0,1,1}\) and the left set is dense in the right set, thus it is dense in the middle set.

\[\square\]

3. PROOF OF THEOREM 2.15

3.1. Reduction to the coranks of matrices. First, let us ignore the points 0 and \(\infty\) and work with \(\mathbb{C}^{\times}\) only. Let us define the subsets of \(\mathbb{C}^{B_1} \times \mathbb{C}^{B_2}\) in such a way that they will be easy to study.

DEFINITION 3.1. The subset

\[
\mathcal{N}_{j_1^1, \ldots, j_k^1, j_1^2, \ldots, j_k^2} \geq 1
\]

consist of \((f_1, f_2) \in \mathbb{C}^{B_1} \times \mathbb{C}^{B_2}\) such that

1. \(f_1 = f_2 = 0\) has at least \(k\) distinct solutions \(x_1, \ldots, x_k\) in \(\mathbb{C}^{\times}\);
2. at the \(m\)-th solution \(x_m\), the polynomial \(f_1\) has a root of multiplicity at least \(j_m^1\) and the polynomial \(f_2\) has a root of multiplicity at least \(j_m^2\).
Remark 3.2. These subsets $N$ are different from $M_p$ not only in the absence of 0 and $\infty$, but also in the fact that the conditions are inequalities, not equalities. But they are also different from $\hat{M}_p$ in the fact that they are not closed not only due to the possibility of a point to tend to 0 or $\infty$, but also due to the possibility of gluing points. For example, $M_{0;0;2}^{0;0;2}$ lies in the closure of $N_{1,1}^{1,1}$, but not in the $N_{1,1}^{1,1}$ itself.

Definition 3.3. The expected codimension of the subset $N_{j_1^{1}, \ldots, j_k^{1}}$ is the number $\sum_{m=1}^{k}(j_m^1 + j_m^2 - 1)$.

Fix $B_1$ and $B_2 \subset \mathbb{Z}$ and the subset $N_{j_1^{1}, \ldots, j_k^{1}}$. Consider the space

$$Z_k = \left\{ (x_1, \ldots, x_k) \in (\mathbb{C}^\times)^k \mid x_i \neq x_j \text{ for } i \neq j \right\}$$

of tuples of different non-zero numbers. We would like to define a stratification on it that is related with the structure of $N_{j_1^{1}, \ldots, j_k^{1}}$.

Let $B_i = \{b_i^1, \ldots, b_i^s\}$. Consider a vector

$$R(B_i; x_m) = \begin{pmatrix} b_i^1 x_m^{b_i^1}, & b_i^2 x_m^{b_i^2}, & \ldots, & b_i^s x_m^{b_i^s} \end{pmatrix}$$

and its derivatives

$$\frac{dR(B_i; x_m)}{dx_m} = \begin{pmatrix} b_i^1 x_m^{b_i^1-1}, & b_i^2 x_m^{b_i^2-1}, & \ldots, & b_i^s x_m^{b_i^s-1} \end{pmatrix},$$

$$\frac{d^2R(B_i; x_m)}{dx_m^2} = \begin{pmatrix} b_i^1 (b_i^1 - 1) x_m^{b_i^1-2}, & b_i^2 (b_i^2 - 1) x_m^{b_i^2-2}, & \ldots, & b_i^s (b_i^s - 1) x_m^{b_i^s-2} \end{pmatrix},$$

$$\ldots$$

$$\frac{d^{j_m^i - 1}R(B_i; x_m)}{dx_m^{j_m^i - 1}} = \begin{pmatrix} b_i^1 \cdots (b_i^1 - j_m^i + 2) x_m^{b_i^1-j_m^i + 1}, & b_i^2 \cdots (b_i^2 - j_m^i + 2) x_m^{b_i^2-j_m^i + 1}, & \ldots, & b_i^s \cdots (b_i^s - j_m^i + 2) x_m^{b_i^s-j_m^i + 1} \end{pmatrix},$$

up to the derivative of order $(j_m^i - 1)$. 
Form them into the matrix $M(B_i; j_1^i, \ldots, j_k^i; x_1, \ldots, x_k) =$

\[
\begin{pmatrix}
R(B_i; x_1) \\
\frac{dR(B_i; x_1)}{dx_1}
\end{pmatrix}
\begin{pmatrix}
\frac{dx_1}{dx_1^1} \\
\frac{dx_2}{dx_1^1} \\
\vdots
\end{pmatrix}
\]

\[= \begin{pmatrix}
x_1^1 \\
\frac{b_1^i}{x_1^{b_1^i - 1}} \\
\frac{b_1^i}{x_1^{b_1^i - 2}} \\
\frac{b_1^i}{x_1^{b_1^i - 3}} \\
\vdots
\end{pmatrix}
\begin{pmatrix}
\frac{dx_1}{dx_1^1} \\
\frac{dx_2}{dx_1^1} \\
\vdots
\end{pmatrix}
\]

For example,

\[
M(1, 2, 3, 4; 1, 1, 2; x, y, z) = \left(\begin{array}{cccc}
R(1, 2, 3, 4; x) \\
R(1, 2, 3, 4; y) \\
R(1, 2, 3, 4; z)
\end{array}\right)
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\]

**Definition 3.4.** For $n_1, n_2 \in \mathbb{N}$ define

\[
S_{n_1} = \left\{(x_1, \ldots, x_k) \in Z_k \mid \text{cork } M(B_1; j_1^1, \ldots, j_k^1; x_1, \ldots, x_k) = n_1 \right\}
\]

\[
S_{n_2} = \left\{(x_1, \ldots, x_k) \in Z_k \mid \text{cork } M(B_2; j_1^2, \ldots, j_k^2; x_1, \ldots, x_k) = n_2 \right\}
\]

\[
S_{n_1,n_2} = \left\{(x_1, \ldots, x_k) \in Z_k \mid \text{cork } M(B_1; j_1^1, \ldots, j_k^1; x_1, \ldots, x_k) = n_1 \right. \\
\left. \text{and } \text{cork } M(B_2; j_1^2, \ldots, j_k^2; x_1, \ldots, x_k) = n_2 \right\} = S_{n_1}^1 \cap S_{n_2}^2.
\]

Here the corank cork $M$ of the matrix $M$ is the number of its rows minus its rank.

To sum it up, we fix $B_1$ and $B_2 \subset Z$, fix a subset $N_{j_1^1, \ldots, j_k^1} \subset \mathbb{C}^{B_1} \times \mathbb{C}^{B_2}$ and define a stratification $Z_k = \bigsqcup_{n_1, n_2} S_{n_1,n_2}^1$.

**Lemma 3.5.** If for each $S_{n_1,n_2}$ the codimension (of every its irreducible component) is at least $n_1 + n_2$, then the codimension of (every irreducible component of) $N_{j_1^1, \ldots, j_k^1}$ is equal to the expected codimension $\sum_{m=1}^k (j_1^1 + j_2^1 - 1)$.

Typically, we would have that codim $S_{n_1}^1 \geq n_1$ and codim $S_{n_2}^2 \geq n_2$ and the difficult part is to restrict the codimension of their intersection from above.
Proof. Consider the subset $\tilde{N}^{j_1,\ldots,j_k}_{j_1,\ldots,j_k}$ consisting of $(f_1, f_2; x_1, \ldots, x_k) \in \mathbb{C}^{B_1} \times \mathbb{C}^{B_2} \times Z_k$ such that at the $x_m$, the polynomial $f_1$ has a root of multiplicity at least $j_m^1$ and the polynomial $f_2$ has a root of multiplicity at least $j_m^2$.

The image of $\tilde{N}^{j_1,\ldots,j_k}_{j_1,\ldots,j_k}$ under the projection $p : \mathbb{C}^{B_1} \times \mathbb{C}^{B_2} \times Z_k \to \mathbb{C}^{B_1} \times \mathbb{C}^{B_2}$ is exactly $N^{j_1,\ldots,j_k}_{j_1,\ldots,j_k}$, and the preimage of any point of $N$ consists of a finite number of points. Thus $\dim N = \dim \tilde{N}$, but $\dim \mathbb{C}^{B_1} \times \mathbb{C}^{B_2} \times Z_k = \dim \mathbb{C}^{B_1} \times \mathbb{C}^{B_2} + k$, thus to prove that $\text{codim } N = \sum_{m=1}^{k}(j_m^1 + j_m^2 - 1)$ it is enough to prove that $\text{codim } \tilde{N} = \sum_{m=1}^{k}(j_m^1 + j_m^2)$.

Now consider the projection $q : \mathbb{C}^{B_1} \times \mathbb{C}^{B_2} \times Z_k \to Z_k$. The fiber over the point $(x_1, \ldots, x_k) \in Z_k$ is a subset in $\mathbb{C}^{B_1} \times \mathbb{C}^{B_2}$ of all pairs of polynomials $(f_1(x), f_2(x))$ such that at the point $x_m$ the polynomial $f_1$ has a root of multiplicity at least $j_m^1$ and the polynomial $f_2$ has a root of multiplicity at least $j_m^2$. It is a direct product of the spaces

$$V_1 = \{ f_1 \in \mathbb{C}^{B_1} \mid \text{ord}_{x_m} f_1 \geq j_m^1 \}$$

and

$$V_2 = \{ f_2 \in \mathbb{C}^{B_2} \mid \text{ord}_{x_m} f_2 \geq j_m^2 \}.$$

The condition that $f_i(x) = \sum_{b \in B_i} c_b x^b$ has a root at $x_m$ is $\sum_{b \in B_i} c_b x^b = 0$, which is a linear equation on the variables $c_b$ with vector $R(B_i; x_m)$. The condition that $f_i(x)$ has a root of order at least $j_m^i$ at $x_m$ is $f_i(x_m) = f_i'(x_m) = f_i''(x_m) = \ldots = f_i^{j_m^i-1}(x_m) = 0$, which is a (homogeneous) system of linear equations. Its matrix has row vectors

$$R(B_i; x_m), \quad \frac{dR(B_i; x_m)}{dx_m}, \quad \frac{d^2R(B_i; x_m)}{dx_m^2}, \quad \ldots, \quad \frac{d^{j_m^i-1}R(B_i; x_m)}{dx_m^{j_m^i-1}},$$

thus $V_i$ is a vector subspace of $\mathbb{C}^{B_i}$ defined by the matrix $M_i = M(B_i; j_1^i, \ldots, j_k^i; x_1, \ldots, x_k)$.

At the point $(x_1, \ldots, x_k) \in S_{n_1,n_2} \subset Z_k$ the codimension of the fiber $q^{-1}(x_1, \ldots, x_k) = V_1 \times V_2 \subset \mathbb{C}^{B_1} \times \mathbb{C}^{B_2}$ is

$$\text{rk } M_1 + \text{rk } M_2 = \left( \sum_{m=1}^{k} j_m^1 - \text{cork } M_1 \right) + \left( \sum_{m=1}^{k} j_m^2 - \text{cork } M_2 \right) = \sum_{m=1}^{k} (j_m^1 + j_m^2) - n_1 - n_2.$$

If the codimension of $S_{n_1,n_2}$ is at least $n_1 + n_2$, that the codimension of $N \cap q^{-1}(S_{n_1,n_2})$ is at least $\sum_{m=1}^{k} (j_m^1 + j_m^2)$, as necessary.

3.2. Coranks of matrices for one variable. Fix a subset $N^{j_1,\ldots,j_k}_{j_1,\ldots,j_k} \subset \mathbb{C}^{B_1} \times \mathbb{C}^{B_2}$ and consider the stratification $Z_k = \bigsqcup_{i_1} S_{i_1}$. We wonder about how its structure depends on $J_i = (j_1^i, \ldots, j_k^i)$.

**Definition 3.6.** For $B = \{b_1, \ldots, b_k\}$ define

$$\phi(B) := (b_2 - b_1, b_3 - b_2, \ldots, b_k - b_{k-1})$$

to be the greatest common divisor of the differences.

**Lemma 3.7.** Suppose that $|B_i| \geq \sum_{m=1}^{k} j_m^i$ (thus for the matrix $M_i$ the number of columns is not less than the number of rows). Then

- For $J_i = (1)$ holds $Z_i = S_0$.
- For $J_i = (2)$ holds $Z_2 = S_0$. 

Here we will ignore index $i$ and, for example, write $B$ instead of $B_i$.

- For $J_i = (1, 1)$ holds $Z_2 = \begin{cases} S_0 \cup S_1 \text{ with codim } S_1 \text{ non-empty of codim 1 if } \phi(B_i) \geq 2 \\ S_0 \text{ otherwise} \end{cases}$
- For $J_i = (3)$ holds $Z_1 = S_0$.
- For $J_i = (2, 1)$ holds $Z_2 = S_0 \cup S_1$ with codim $S_1 \geq 1$.
- For $J_i = (1, 1, 1)$ holds $Z_3 = \begin{cases} S_0 \cup S_1 \cup S_2 \text{ with codim } S_1 \geq 1 \\ \text{and } S_2 \text{ non-empty of codim 2 if } \phi(B_i) \geq 2 \\ S_0 \cup S_1 \text{ with codim } S_1 \geq 1 \text{ otherwise} \end{cases}$

Here we list those $J_i$ which can be found in the strata $N_{j_1, \ldots, j_k}^{b_1, \ldots, b_k}$ of expected codimension 1, 2 or 3.

**Proof.** Here we will ignore index $i$ and, for example, write $B$ instead of $B_i$.

- For $J_i = (1)$ the matrix is
  \[ M(B; 1; x) = (R(B; x)) = \begin{pmatrix} x^{b_1} & x^{b_2} & x^{b_3} & x^{b_4} & \ldots \end{pmatrix}. \]
  But $x \in Z_1$ is non-zero, thus cork $M = 0$ and $Z_1 = S_0$.

- For $J_i = (2)$ the matrix is
  \[ M(B; 2; x) = \left( \frac{R(B; x)}{dR(B; x)} \right) = \begin{pmatrix} x^{b_1} & x^{b_2} & x^{b_3} & x^{b_4} & \ldots \\ b_2 x^{b_1-1} & b_2 x^{b_2-1} & b_3 x^{b_3-1} & b_4 x^{b_4-1} & \ldots \end{pmatrix}. \]
  But $x \in Z_1$ is non-zero, thus $M$ can be transformed by the multiplication of rows and columns on non-zero numbers into an equivalent matrix
  \[ \begin{pmatrix} 1 & 1 & 1 & 1 & \ldots \\ b_1 & b_2 & b_3 & b_4 & \ldots \end{pmatrix}. \]
  The numbers $b_i$ are pairwise different, thus $2 \times 2$-minors are non-degenerate, cork $M = 0$ and $Z_1 = S_0$.

- For $J_i = (1, 1)$ the matrix is
  \[ M(B; 1, 1; x, y) = \left( \frac{R(B; x)}{R(B; y)} \right) = \begin{pmatrix} x^{b_1} & x^{b_2} & x^{b_3} & x^{b_4} & \ldots \\ y^{b_1} & y^{b_2} & y^{b_3} & y^{b_4} & \ldots \end{pmatrix}. \]
  But for $(x, y) \in Z_2$ holds $x \neq 0$, thus $M$ can be transformed into an equivalent matrix
  \[ \left( \begin{pmatrix} y \right)^{b_1} & \begin{pmatrix} y \right)^{b_2} & \begin{pmatrix} y \right)^{b_3} & \begin{pmatrix} y \right)^{b_4} & \ldots \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \ldots \\ t^{b_1} & t^{b_2} & t^{b_3} & t^{b_4} & \ldots \end{pmatrix}, \right. \]
  where $t = \frac{y}{x}$. If cork $M > 0$, then $t^{b_1} = t^{b_2} = t^{b_3} = t^{b_4} = \ldots$. But $t \neq 0$, thus $t$ is $k$-th root of 1 for some $k$ and all $(b_q - b_p)$ should be divisible by $k$. Moreover, for $(x, y) \in Z_2$ holds $x \neq y$, thus $t \neq 1$, thus $t$ is $k$-th root of 1 with $k \geq 2$. To sum it up, $\phi(B) \geq 2$.
  Vice versa, if $\phi(B) = k \geq 2$, then for any $(x, y) \in Z_2$ such that $y/x$ is $k$-th root of 1 the corank of $M$ is 1, thus the subset $S_1$ is non-empty.

- For $J_i = (3)$ the matrix can be transformed into an equivalent matrix
  \[ \begin{pmatrix} 1 & 1 & 1 & 1 & \ldots \\ b_1 & b_1 & b_3 & b_4 & \ldots \end{pmatrix}. \]
The numbers $b_i$ are pairwise different, thus $3 \times 3$-minors, which are Vandermonde matrices, are non-degenerate, cork $M$ = 0 and $Z_1 = S_0$.

- For $J_i = (2,1)$ the matrix can be transformed into an equivalent matrix

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & \\
\ell b_1 & \ell b_2 & \ell b_3 & \ell b_4 & \\
\end{pmatrix}.
$$

The numbers $b_i$ are pairwise different, thus $2 \times 2$-minors of the first 2 rows are non-degenerate and $S_2$ is empty. Moreover, $3 \times 3$-minors as polynomials in $t$ are not zero, thus codim $S_1 \geq 1$.

Moreover, for non-zero number $c$ and a stratum $S_k$ holds $(x,y) \in S_k$ if and only if holds $(cx, cy) \in S_k$, thus we can replace

$$
Z_k = \{(x_1, \ldots, x_k) \subset (\mathbb{C}^*)^k \mid x_i \neq x_j \text{ for } i \neq j\}
$$

by

$$
\mathbb{P}Z_k = \{(x_1 : \ldots : x_k) \subset \mathbb{C}P^{k-1} \mid x_i \neq 0 \text{ and } x_i \neq x_j \text{ for } i \neq j\}
$$

and strata $S_k$ by their projectivizations $\mathbb{P}S_k$. Now $\mathbb{P}S_1$ is a subset of 1-dimensional $\mathbb{P}Z_2$, thus (non-empty) $S_1$ can not have codimension more than 1. We do not study here the question when $S_1$ is in fact non-empty.

- For $J_i = (1,1,1)$ the matrix can be transformed into an equivalent matrix

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & \\
\ell b_1 & \ell b_2 & \ell b_3 & \ell b_4 & \\
\end{pmatrix},
$$

with $t = y/x$ and $u = z/x$. We have $(1 : t : u) \in \mathbb{P}Z_3$, that is $t \neq 0, u \neq 0, t \neq 1, t \neq 1, t \neq u$.

If cork $> 1$, then $2 \times 2$-minors are zero, thus $\ell b_1 = \ell b_2 = \ell b_3 = \ell b_4 = \ldots$ and $u b_1 = u b_2 = u b_3 = u b_4 = \ldots$. But $t \neq 0$, thus $t$ is $k_1$-th root of 1 for some $k_1$ and all $(b_q - b_p)$ should be divisible by $k_1$. Similarly, $u$ is $k_2$-th root of 1. Now $t$ and $u$ are both $k$-th roots of 1 for the greatest common divisor $k = (k_1, k_2)$, and all $(b_q - b_p)$ should be divisible by $k$. Moreover, $t \neq u, t \neq 1$ and $u \neq 1$, thus there are at least three different $k$-th root of 1, thus $k \geq 3$. To sum it up, $\phi(B) \geq 3$.

Vice versa, if $\phi(B) = k \geq 3$, then we can choose three $k$-th roots of 1, namely 1, $t$ and $u$, and obtain $(1 : t : u) \in \mathbb{P}S_2$, thus the subset $S_2$ is non-empty.

Now $3 \times 3$-minors as polynomials in $t$ and $u$ are not zero, thus codim $S_1 \geq 1$. We do not study here the questions whether codim $S_1 = 1$ and whether $S_1$ is in fact non-empty.

\[\square\]

3.3. **Coranks of matrices for two variables.** Fix a subset $N_{j_1^1, \ldots, j_k^1} \subset \mathbb{C}^{B_1} \times \mathbb{C}^{B_2}$ and consider the stratification $Z_k = \bigsqcup_{n_1,n_2} S_{n_1,n_2}$. We wonder about how its structure depends on $N_{j_1^1, \ldots, j_k^1}$.

**Lemma 3.8.** Suppose that $|B_i| \geq \sum_{m=1}^k j_i^m$ (thus for the matrix $M_i$ the number of columns is not less than the number of rows). Then
• For $N^1_1$ holds $Z_1 = S_{00}$.
• For $N^3_2$ holds $Z_1 = S_{00}$.
• For $N^{1,1}_{1,1}$ holds $Z_2 = S_{00} \cup S_{01} \cup S_{10} \cup S_{11}$, where
  (1) codim $S_{01} = \operatorname{codim} S_{10} = 1$ (and they may be empty),
  (2) if $B_1$ and $B_2$ can be shifted to the same proper sublattice $k\mathbb{Z} \subset \mathbb{Z}$ with $k \geq 2$, then
      \[ \operatorname{codim} S_{11} = 1 \] and it is non-empty, otherwise it is empty.
• For $N^5_2$ holds $Z_1 = S_{00}$.
• For $N^3_2$ holds $Z_1 = S_{00}$.
• For $N^{2,1}_{1,1}$ holds $Z_2 = S_{00} \cup S_{01} \cup S_{10} \cup S_{11}$, where
  (1) codim $S_{01} = \operatorname{codim} S_{10} = 1$ (and they may be empty),
  (2) if $B_1$ and $B_2$ can be shifted to the same proper sublattice $k\mathbb{Z} \subset \mathbb{Z}$ with $k \geq 2$, then
      \[ \operatorname{codim} S_{11} = 1 \] and it is non-empty, otherwise it is empty.
• For $N^{1,1,1}_{1,1,1}$ holds $Z_3 = S_{00} \cup S_{01} \cup S_{10} \cup S_{11} \cup S_{02} \cup S_{20} \cup S_{12} \cup S_{21} \cup S_{22}$, where
  (1) codim $S_{01}$ and codim $S_{10} \geq 1$,
  (2) codim $S_{02}$ and codim $S_{20} \geq 2$,
  (3) if $B_1$ and $B_2$ cannot be shifted to the same proper sublattice $k\mathbb{Z} \subset \mathbb{Z}$ with $k \geq 2$, then
      \[ \operatorname{codim} S_{11} = 1 \]
  (4) if $B_1$ can be split into $B' \sqcup B''$ so that $B'$, $B''$ and $B_2$ can be shifted to the same
      sublattice $k\mathbb{Z} \subset \mathbb{Z}$ with $k \geq 3$, then \[ \operatorname{codim} S_{12} = 2 \] and it is non-empty, otherwise it is empty (and symmetrically with $S_{21}$),
  (5) if $B_1$ and $B_2$ can be shifted to the same proper sublattice $k\mathbb{Z} \subset \mathbb{Z}$ with $k \geq 2$, then
      \[ \operatorname{codim} S_{22} = 2 \] and it is non-empty, otherwise it is empty, and all the strata can be empty if not mentioned otherwise.

Here we list $N^{j_1,j_2,...,j_k}_{1,2,3}$ of expected codimension 1, 2 or 3.

Proof. Most of the cases trivially follow from the lemma 3.7.

The case of $S_{11}$ for $N^{1,1}_{1,1}$ is proven as follows. If $S_{11}$ is non-empty, then by lemma 3.7 for the element $(1 : t) \in \mathbb{P}S_{11}$ the number $t$ is $k$-th root of 1 and each of $B_i$ can be shifted to the proper sublattice $k\mathbb{Z} \subset \mathbb{Z}$ with $k \geq 2$. Vice versa, if $B_1$ and $B_2$ can be shifted to the proper sublattice $k\mathbb{Z} \subset \mathbb{Z}$ with $k \geq 2$, we can take a number $t$ which is not equal to 1 and is $k$-th root of 1, then $(1 : t) \in \mathbb{P}S_{11}$ and thus \[ \operatorname{codim} S_{11} = 1 \].

The case of $S_{12}$ (and $S_{21}$) for $N^{1,1,1}_{1,1,1}$ is the most difficult part of the lemma. It follows from the theorem 4.2.

The case of $S_{11}$ for $N^{2,1}_{1,1}$ can be proven using the analogue of the lemma 4.3 for $M(B_1; 2, 1; 1, t)$ instead of $M(B_1; 1, 1, 1; 1, t, u)$, which is proven similarly, and the logic from the case of $S_{11}$ for $N^{1,1}_{1,1}$.

The case of $S_{11}$ for $N^{1,1,1}_{1,1,1}$ can be proven using Newton polytopes and we only sketch the proof here. There are two curves $\mathbb{P}S_1^1$ and $\mathbb{P}S_2^1$ in the two-dimensional space $\mathbb{P}Z_3$. The condition \[ \operatorname{codim} S_{11} \geq 2 \] means that the intersection $\mathbb{P}S_{11} = \mathbb{P}S_1^1 \cap \mathbb{P}S_2^1$ of these curves is a finite set of points, not another curve. We can suppose towards contradiction that this intersection contain a common irreducible component defined by a polynomial $f(x)$, thus $f(x)$ in some sense is a common divisor of all $3 \times 3$-minors that define $\mathbb{P}S_1^1$ and $\mathbb{P}S_2^1$. But then they should have
common faces, from which we can deduce that $B_1$ and $B_2$ can be shifted to the same proper sublattice $k\mathbb{Z} \subset \mathbb{Z}$ with $k \geq 2$.

Now we have a following corollary of the lemmas 3.5 and 3.8.

**Corollary 3.9.** Consider the conditions from the main theorem 1.2:

1. One can shift $B_1$ and $B_2$ to the same proper sublattice $k\mathbb{Z} \subset \mathbb{Z}$ with $k \geq 2$.
2. One can split one of $B_i$'s (say, $B_1$) into $B' \sqcup B''$ so that $B'$, $B''$ and $B_2$ can be shifted to the same sublattice $k\mathbb{Z} \subset \mathbb{Z}$ with $k \geq 3$.

For the strata $N$ of expected codimension 1, 2 and 3 the following statements hold (we do not list here the strata obtained by changing the order of rows or columns):

- $N_{1}^{1}$ has codimension 1.
- $N_{2}^{1}$ has codimension 2.
- $N_{1,1}^{1}$ has codimension 2, unless the condition (1) holds.
- $N_{3}^{2}$ has codimension 3.
- $N_{4}^{2}$ has codimension 3.
- $N_{1,1}^{2}$ has codimension 3, unless the condition (1) holds.
- $N_{1,1,1}^{1}$ has codimension 3, unless the condition (1) or (2) holds.

### 3.4. The reduction of to the toric case.

Here we repeat the constructions from the subsection 3.1 but taking into account 0 and $\infty$. Let us define the subsets of $\mathbb{C}^{B_1} \times \mathbb{C}^{B_2}$ corresponding to the subsets from the definition 3.1.

**Definition 3.10.** The stratum

$$M'^{j_0; j_{\infty}; j_1^{1}; \ldots; j_k^{1}}_{j_0; j_{\infty}; j_1^{2}; \ldots; j_k^{2}}, \quad j_0, j_{\infty}, j_1^{1}, j_2^{1}, j_1^{2}, j_2^{2} \geq 0, \quad j_1^{1}, \ldots, j_k^{1}, j_1^{2}, \ldots, j_k^{2} \geq 1$$

consist of $(f_1, f_2) \in \mathbb{C}^{B_1} \times \mathbb{C}^{B_2}$ such that

1. $f_1$ has a root of multiplicity at least $j_0^1$ at 0 and at least $j_1^i$ at infinity;
2. $f_1 = f_2 = 0$ has at least $k$ distinct solutions $x_1, \ldots, x_k$ in $\mathbb{C}^\times$;
3. at the $m$-th solution $x_m$, the polynomial $f_1$ has a root of multiplicity at least $j_m^1$ and the polynomial $f_2$ has a root of multiplicity at least $j_m^2$.

**Remark 3.11.** These subsets $M'_p$ are different from $M_p$ in the fact that the conditions are inequalities, not equalities. But they are also different from $\hat{M}_p$, because in $\hat{M}_p$ one can move points to 0 or $\infty$ and glue points. For example, $M'^{0;0;2}_{0;0;1}$ and $M^{1;0;1}_{1;0;1}$ lie in $\hat{M'^{0;0;1}_{0;0;1}}$, but not in the $M'^{0;0;1}_{0;0;1}$.

**Definition 3.12.** Just like for $M^{j_0; j_{\infty}; j_1^{1}; \ldots; j_k^{1}}_{j_0; j_{\infty}; j_1^{2}; \ldots; j_k^{2}}$, the expected codimension of the subset $M'^{j_0; j_{\infty}; j_1^{1}; \ldots; j_k^{1}}_{j_0; j_{\infty}; j_1^{2}; \ldots; j_k^{2}}$ is the number $j_0^1 + j_2^2 + j_{\infty}^1 + j_{\infty}^2 + \sum_{m=1}^{k}(j_m^1 + j_m^2 - 1)$.

**Lemma 3.13.** Consider the conditions from the main theorem 1.2:

1. One can shift $B_1$ and $B_2$ to the same proper sublattice $k\mathbb{Z} \subset \mathbb{Z}$ with $k \geq 2$.
2. One can split one of $B_i$'s (say, $B_1$) into $B' \sqcup B''$ so that $B'$, $B''$ and $B_2$ can be shifted to the same sublattice $k\mathbb{Z} \subset \mathbb{Z}$ with $k \geq 3$.
3. For every $i = 1, 2$, the two smallest elements of $B_i$ differ by more than 1.
(4) For every $i = 1, 2$, the two greatest elements of $B_i$ differ by more than 1.

For the strata $M'$ of expected codimension 1, 2 and 3 the following statements hold (here we list the strata up to the switching the rows, changing the order of columns of $x_1, \ldots, x_k$ or switching the columns of 0 and $\infty$):

- $M_{1;0}^{1:0}$ has codimension 1.
- $M_{0;0}^{0;0}$ has codimension 1.
- $M_{2;0}^{2:0}$ has codimension 2, unless the condition (3) holds.
- $M_{1;1}^{1:1}$ has codimension 2.
- $M_{1;0}^{1:0}$ has codimension 2.
- $M_{1;0}^{1:0}$ has codimension 2.
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- $M_{1;0}^{1:0}$ has codimension 2.
- $M_{1;0}^{1:0}$ has codimension 2.
- $M_{1;0}^{1:0}$ has codimension 2.

Theorem 3.14. For the strata $M_p$ of expected codimension 1, 2 and 3 the following statements hold (listed with the same restrictions as in the previous lemma):

- $\tilde{M}_{1;0}^{1;0}$ has codimension 1.
- $\tilde{M}_{0;0}^{0;0}$ has codimension 1.
- $\tilde{M}_{2;0}^{2;0}$ has codimension 2, unless the condition (3) holds.
- $\tilde{M}_{1;0}^{1;0}$ has codimension 2.
- $\tilde{M}_{1;0}^{1;0}$ has codimension 2.
• $\hat{M}_{1:0}^{0:1}$ has codimension 2.
• $\hat{M}_{1:0:1}^{0:0:1}$ has codimension 2, unless the condition (3) holds.
• $\hat{M}_{0:0:1}^{0:0:1}$ has codimension 2, unless the condition (3) holds.
• $\hat{M}_{1:0:1}^{0:0:1:1}$ has codimension 2, unless the condition (3) holds.
• $\hat{M}_{1:0}^{0:0:1}$ has codimension 3, unless the smallest 3 elements of $B_1$ differ by more than 2.
• $\hat{M}_{1:0}^{0:0:2}$ has codimension 3, unless the condition (3) holds.
• $\hat{M}_{2:0}^{1:0:1}$ has codimension 3, unless the condition (3) holds.
• $\hat{M}_{3:0}^{1:0:1}$ has codimension 3, unless the condition (3) holds.
• $\hat{M}_{2:0}^{1:0:1}$ has codimension 3, unless the condition (3) holds.
• $\hat{M}_{1:1}^{1:0:1}$ has codimension 3.
• $\hat{M}_{2:0:1}^{0:0:1:1}$ has codimension 3, unless the condition (3) holds.
• $\hat{M}_{1:1:1}^{1:0:1}$ has codimension 3.
• $\hat{M}_{1:0:1}^{1:0:1}$ has codimension 3.
• $\hat{M}_{1:0:1}^{0:0:1}$ has codimension 3, unless the condition (3) holds.
• $\hat{M}_{1:0:1}^{0:0:1}$ has codimension 3, unless the condition (3) holds.
• $\hat{M}_{1:0:1}^{0:0:1}$ has codimension 3, unless the condition (3) holds.
• $\hat{M}_{1:0:11}^{0:0:11:1}$ has codimension 3, unless the condition (3) holds or one can shift $B_1 \setminus \{\min B_1\}$ and $B_2$ to the same proper sublattice $k\mathbb{Z} \subset \mathbb{Z}$ with $k \geq 2$.
• $\hat{M}_{0:0:1}^{0:0:3}$ has codimension 3, unless the condition (3) holds.
• $\hat{M}_{0:0:1}^{0:0:2}$ has codimension 3, unless the condition (3) holds.
• $\hat{M}_{0:0:1}^{0:0:1}$ has codimension 3, unless the condition (1) or (3) holds.
• $\hat{M}_{0:0:1}^{0:0:1}$ has codimension 3, unless the condition (1), (2) or (3) holds.

4. The stratification on the roots space

In this section we will study the strata $S_{1:2}$ for $N_{1:1:1}$, although without referring to it as such. The same ideas can be also used for the strata $S_{1:1}$ for $N_{1:1:1}$. This section is independent of other parts of the text.

**Definition 4.1.** For a finite collection of integer numbers $n_1, n_2, \ldots, n_k$ let us define the function

$$\phi(n_1, n_2, \ldots, n_k) := (n_2 - n_1, n_3 - n_2, \ldots, n_k - n_{k-1})$$

to be the greatest common divisor of their differences.

**Theorem 4.2.** Fix $n \in \mathbb{Z}$. Let $x$ and $y$ be two $n$-th roots of 1, such that $x \neq 1$, $y \neq 1$, $x \neq y$. Let $B = \{a, b, c, d, \ldots\} \subset \mathbb{Z}$ be a finite set of integer numbers. Consider a matrix

$$M(B; x, y) = \begin{pmatrix} 1 & 1 & 1 & 1 & \ldots \\ x^a & x^b & x^c & x^d & \ldots \\ y^a & y^b & y^c & y^d & \ldots \end{pmatrix},$$
where the columns are parametrized by $B$. If all the $3 \times 3$-minors of $M(B,x,y)$ are degenerate, then $B$ can be split into $B = B_1 \sqcup B_2$ such that $\phi(B_1)$, $\phi(B_2)$ and $n$ have a common divisor $k \geq 3$.

This section is devoted to the proof of this theorem. First, let us consider one $3 \times 3$-minor.

**Lemma 4.3.** Let $x$ and $y$ be complex numbers of module 1, let $a$, $b$ and $c$ be integer numbers and consider a matrix

$$M = M(a,b,c;x,y) = \begin{pmatrix} 1 & 1 & 1 \\ x^a & x^b & x^c \\ y^a & y^b & y^c \end{pmatrix}$$

If this matrix is degenerate, then it has either two proportional rows or two proportional columns.

**Proof.** Let us write the equation $\det M = 0$ as

$$(x^a - x^b)(y^a - y^c) = (x^a - x^c)(y^a - y^b).$$

Suppose that $x^a = x^b$. Then either $x^a = x^c$ and $M$ has two proportional rows or $y^a = y^b$ and $M$ has two proportional (even equal) columns.

As the conditions of the lemma are symmetric in $a$, $b$ and $c$, and are symmetric in $x$ and $y$, we can now suppose that the numbers $x^a$, $x^b$, $x^c$ are pairwise different and that the numbers $y^a$, $y^b$, $y^c$ are also pairwise different. Thus we can write the condition above as

$$\frac{x^a - x^b}{x^a - x^c} = \frac{y^a - y^c}{y^a - y^b} \neq 0.$$ 

In particular, the following angles are equal: $\angle(x^b,x^a,x^c) = \angle(y^b,y^a,y^c)$.

But the numbers $x$ and $y$ has module 1, thus the points $x^a$, $x^b$, $x^c$, $y^a$, $y^b$, $y^c$ lie on the unit circle and these angles are the inscribed angles on the arcs $(x^b,x^c)$ and $(y^b,y^c)$. An inscribed angle is half of the central angle on the same arc (see the figure [1]), thus

$$\angle(x^b,0,x^c) = 2\angle(x^b,x^a,x^c) = 2\angle(y^b,y^a,y^c) = \angle(y^b,0,y^c).$$
But \(|x^b| = |x^c| = |y^b| = |y^c| = 1\), thus \(\angle(x^b, 0, x^c) = \angle(y^b, 0, y^c)\) implies \(x^b / x^c = y^b / y^c\). As the conditions of the lemma are symmetric in \(a, b\) and \(c\), the matrix \(M\) has two proportional rows.

**Definition 4.4.** Let us define

\[
R_m = \{(x, y) \in (\mathbb{C}^\times)^2 \mid x^m = y^m = 1, \ x \neq 1, \ y \neq 1, \ x \neq y\}
\]
to be the subset of pairs of \(m\)-th roots of 1 such that the conditions \(x \neq 1, \ y \neq 1\) and \(x \neq y\) from the theorem 4.2 hold.

Then \(R_m \cap R_n = R_{(m,n)}\) where \((m, n)\) is the greatest common divisor of \(m\) and \(n\).

**Definition 4.5.** Let us also define

\[
\begin{align*}
S^1_m &= \{(x, y) \in (\mathbb{C}^\times)^2 \mid x^m = 1, \ x \neq 1, \ y \neq 1, \ x \neq y\}, \\
S^2_m &= \{(x, y) \in (\mathbb{C}^\times)^2 \mid y^m = 1, \ x \neq 1, \ y \neq 1, \ x \neq y\}, \\
S^3_m &= \{(x, y) \in (\mathbb{C}^\times)^2 \mid x^m = y^m, \ x \neq 1, \ y \neq 1, \ x \neq y\}.
\end{align*}
\]

Then \(S^i_m \cap S^j_n = S^i_{(m,n)}\) and \(S^i_m \cap S^j_m = R_m\) for \(i \neq j\). Moreover, \(S^i_{\phi(a,b)} \cap S^i_{\phi(b,c)} = S^i_{\phi(a,b,c)}\), where \(\phi\) is the greatest common divisor of differences.

Now let us use this notation to write down the solutions of the equation \(\det M(a, b, c; x, y) = 0\) from the lemma 4.3.

| Proportional elements | Equations | Solutions (in several equivalent forms) |
|------------------------|-----------|----------------------------------------|
| Columns 1 and 2        | \(x^a = x^b, \ y^a = y^b\) | \(R_{a-b} = R_{\phi(a,b)} = S^1_{a-b} \cap S^2_{a-b} \cap S^3_{a-b}\) |
| Columns 2 and 3        | \(x^b = x^c, \ y^b = y^c\) | \(R_{b-c} = R_{\phi(b,c)} = S^1_{b-c} \cap S^2_{b-c} \cap S^3_{b-c}\) |
| Columns 3 and 1        | \(x^c = x^a, \ y^c = y^a\) | \(R_{c-a} = R_{\phi(c,a)} = S^1_{c-a} \cap S^2_{c-a} \cap S^3_{c-a}\) |
| Rows 1 and 2           | \(x^a = x^b, \ y^c = x^c\) | \(S^1_{(a,c-b)} = S^1_{\phi(a,b,c)} = S^1_{a-b} \cap S^1_{b-c} \cap S^3_{c-a}\) |
| Rows 1 and 3           | \(y^a = y^b = y^c\) | \(S^2_{(b,a-c-b)} = S^2_{\phi(a,b,c)} = S^2_{a-b} \cap S^2_{b-c} \cap S^2_{c-a}\) |
|Rows 2 and 3            | \(x^a / y^b = x^b / y^c\) | \(S^3_{(b,a-c-b)} = S^3_{\phi(a,b,c)} = S^3_{a-b} \cap S^3_{b-c} \cap S^3_{c-a}\) |

**Definition 4.6.** Let us define \(K_n(B) = K_n(a, b, c, d, \ldots) \subset (\mathbb{C}^\times)^2\) to be the set of pairs \((x, y) \in R_m\) such that the conditions of the theorem 4.2 hold, that is such that all the 3 \times 3-minors of \(M(B; x, y)\) are degenerate.

More generally, let us define \(K(B) = K(a, b, c, d, \ldots) \subset (\mathbb{C}^\times)^2\) to be the set of pairs \((x, y) \in (\mathbb{C}^\times)^2\) such that \(|x| = |y| = 1, x \neq 1, y \neq 1\) and all the 3 \times 3-minors of \(M(B; x, y)\) are degenerate. Then \(K_n(B) = K(B) \cap R_n\).

**Remark 4.7.** From the lemma 4.3 and the table above we know that

\[
K(a, b, c) = (S^1_{a-b} \cap S^2_{b-c} \cap S^3_{a-b}) \cup (S^1_{a-b} \cap S^2_{a-c} \cap S^3_{b-c}) \cup (S^1_{c-a} \cap S^2_{c-a} \cap S^3_{c-a}) \cup (S^2_{a-b} \cap S^1_{b-c} \cap S^3_{a-b}) \cup (S^3_{a-b} \cap S^2_{b-c} \cap S^3_{a-b}) \cup (S^3_{a-b} \cap S^2_{b-c} \cap S^3_{a-b}).
\]

Moreover,

\[
K(B) = \bigcap_{\{v_1, v_2, v_3\} \subset B} K(v_1, v_2, v_3).
\]
We would like to write down some necessary conditions for the non-emptiness of \( K(B) \). To do this, let us use distributivity of intersection over union to write \( K(B) \) as the union of several intersections of the elements of the form \( S^i_{v_1-v_2} \) for \( v_1, v_2 \in B \) and \( i = 1, 2 \) or 3. We would like to think of \( S^i_{v_1-v_2} \) as of an edge \( v_1v_2 \) in \( i \)-th graph. To formalize it, let us use the following construction.

**Definition 4.8.** Let \( B = \{a, b, c, d, \ldots\} \subset \mathbb{Z} \) be a finite set of integer numbers. Let \( \Gamma_1 = (B, E_1) \), \( \Gamma_2 = (B, E_2) \) and \( \Gamma_3 = (B, E_3) \) be a triple of graphs on the common set of vertices \( B \). Let us define

\[
S(\Gamma_1, \Gamma_2, \Gamma_3) = \bigcap_{(a, b) \in E_1} S^1_{a-b} \cap \bigcap_{(a, b) \in E_2} S^2_{a-b} \cap \bigcap_{(a, b) \in E_3} S^3_{a-b}.
\]

**Example 4.9.** The formula from the remark 4.7 can now be written as

\[
K(a, b, c) = S(\vdash, \vdash, \vdash) \cup S(\vdash, \vdash, \vdash) \cup S(\vdash, \vdash, \vdash) \cup
\]
\[
\cup S(\vdash, \vdash, \vdash) \cup S(\vdash, \vdash, \vdash) \cup S(\vdash, \vdash, \vdash)
\]

As shown above, \( K(B) \) can be written as the union of the sets of the form \( S(\Gamma_1, \Gamma_2, \Gamma_3) \) for several different triples \( (\Gamma_1, \Gamma_2, \Gamma_3) \). We would like to find out for which \( B \) some of \( S(\Gamma_1, \Gamma_2, \Gamma_3) \) constructed above may be non-empty.

The formulas from the remark 4.7 imply the following

**Condition 1.** For every triple of vertices \( v_1, v_2, v_3 \in B \)

1. either there is an edge between them which is present in all three \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \),
2. or one of the graphs \( \Gamma_i \) has all three of the edges \( v_1v_2, v_2v_3, v_1v_3 \).

**Definition 4.10.** Let \( \Gamma_1 = (B, E_1) \), \( \Gamma_2 = (B, E_2) \) and \( \Gamma_3 = (B, E_3) \) be a triple of graphs on the common set of vertices \( B \). Let us define the completion \( C(\Gamma_1, \Gamma_2, \Gamma_3) \) to be the smallest (with respect to inclusion of edge sets) triple of graphs \( \Gamma'_1 = (B, E'_1) \), \( \Gamma'_2 = (B, E'_2) \) and \( \Gamma'_3 = (B, E'_3) \) such that the following conditions hold:

**Condition 2.** Each \( \Gamma'_i \) is transitive as a binary relation, that is if there are edges \( v_1v_2 \) and \( v_2v_3 \) in \( \Gamma'_i \), then there is an edge \( v_1v_3 \) in \( \Gamma'_i \).

**Condition 3.** If there is an edge \( v_1v_2 \) in two of the graphs \( \Gamma'_1, \Gamma'_2, \Gamma'_3 \), there is an edge \( v_1v_2 \) in all three of them.

**Remark 4.11.** For any triple \( (\Gamma_1, \Gamma_2, \Gamma_3) \) holds

\[
S(\Gamma_1, \Gamma_2, \Gamma_3) = S(C(\Gamma_1, \Gamma_2, \Gamma_3)).
\]

Thus we can replace \( (\Gamma_1, \Gamma_2, \Gamma_3) \) by its completion in the expression of \( K(B) \) as a union of intersections. As the triple \( (\Gamma_1, \Gamma_2, \Gamma_3) \) satisfies the condition [1], its completion \( C(\Gamma_1, \Gamma_2, \Gamma_3) \) also satisfies it.

**Lemma 4.12.** Let \( \Gamma_1 = (B, E_1) \), \( \Gamma_2 = (B, E_2) \) and \( \Gamma_3 = (B, E_3) \) be a triple of graphs on the common finite set of vertices \( B \), satisfying the conditions [1], [2] and [3]. Then some \( \Gamma_i \) consists of only one or two connected components.

**Proof.** By the condition [2], the connected components of each \( \Gamma_i \) are cliques.

Suppose that \( \Gamma_1 \) consists of three or more connected components and choose vertices \( a, b \) and \( c \) in different connected components. Consider the condition [1] for the triple \( a, b, c \): the
condition \(\text{(1b)}\) can not hold due to the absence of necessary edges in \(\Gamma_1\), thus the condition \(\text{(1a)}\) holds; w.l.o.g. suppose that it is \(\Gamma_2\) which has all three edges between \(a\), \(b\) and \(c\). Then by the condition \(\text{(3)}\) there is no edges between \(a\), \(b\) and \(c\) in \(\Gamma_3\). Visually edges between \(a\), \(b\) and \(c\) in \(\Gamma_1\), \(\Gamma_2\) and \(\Gamma_3\) look like \\(\vdots\ Δ \vdots\).

Now choose any other vertex \(d\). W.l.o.g. suppose that in \(\Gamma_1\) it is not in the connected components of \(a\) or \(b\), thus there is no edges \(ad\) and \(bd\) in \(\Gamma_1\). Consider the condition \(\text{(1)}\) for the triple \(a, b, d\): the condition \(\text{(1b)}\) can not hold due to the absence of necessary edges in \(\Gamma_1\), thus the condition \(\text{(1a)}\) holds, and it can hold only for \(\Gamma_2\). Thus in \(\Gamma_2\) the vertex \(d\) is in the same connected component as \(a\), \(b\) and \(c\), and thus \(\Gamma_2\) has only one connected component. \(\square\)

Now let us prove theorem 4.2.

Proof. The theorem 4.2 states that for \((x, y) \in R_n\) such that all \(3 \times 3\)-minors of \(M(B; x, y)\) are degenerate, the set \(B\) should be splittable into \(B = B_1 \sqcup B_2\) such that \(\phi(B_1)\), \(\phi(B_2)\) and \(n\) have a common divisor \(k \geq 3\). Equally, the set \(K_n(B)\), consisting of \((x, y) \in R_n\) such that all \(3 \times 3\)-minors of \(M(B; x, y)\) are degenerate, should be non-empty only if \(B\) splits as above.

We know that \(K(B)\) is the union of \(S(\Gamma_1, \Gamma_2, \Gamma_3)\) for some complete triples \((\Gamma_1, \Gamma_2, \Gamma_3)\) satisfying the condition \(\text{(1)}\), and that \(K_n(B) = K(B) \cap R_n\). Thus it is enough to show that for any complete triple \((\Gamma_1, \Gamma_2, \Gamma_3)\) satisfying the condition \(\text{(1)}\), if the set \(S(\Gamma_1, \Gamma_2, \Gamma_3) \cap R_n\) is non-empty, then \(B\) splits as above. The lemma 4.12 shows that \(B = B_1 \sqcup B_2\) and there is \(\Gamma_1\) which is the disjoint union of two cliques, one on \(B_1\) and one on \(B_2\) (one of which may be empty).

Let us notice that the \(i\)-th part of the formula from the definition \(\text{4.8}\) of \(S(\Gamma_1, \Gamma_2, \Gamma_3)\) is

\[
\bigcap_{(a, b) \in E_i} S^i_{a-b} = \bigcap_{a \in B_1, b \in B_1} S^i_{a-b} \cap \bigcap_{a \in B_2, b \in B_2} S^i_{a-b} = S^i_{\phi(B_1)} \cap S^i_{\phi(B_2)}
\]

and it should have non-empty intersection with \(R_n = S^1_n \cap S^2_n \cap S^3_n\). But

\[
S^i_{\phi(B_1)} \cap S^i_{\phi(B_2)} \cap S^i_n = S^i_{\phi(B_1, B_2), n};
\]

and \(S^i_m\) is non-empty only if \(m \geq 3\), thus \(\phi(B_1)\), \(\phi(B_2)\) and \(n\) should have a common divisor \(k \geq 3\). \(\square\)

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