Gravitational radiation for extreme mass ratio inspirals to the 14th post-Newtonian order

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We derive gravitational waveforms needed to compute the 14th post-Newtonian (14PN) order energy flux for a particle in circular orbit around a Schwarzschild black hole, i.e. $v^{28}$ beyond the leading Newtonian approximation where $v$ is the orbital velocity of a test particle. We investigate the convergence of the energy flux in the PN expansion and suggest a fitting formula which can be used to extract unknown higher order PN coefficients from accurate numerical data for more general orbits around a Kerr black hole. The phase difference between the 14PN waveforms and numerical waveforms after two years inspiral is shown to be about $10^{-7}$ for $\mu/M = 10^{-4}$ and $10^{-3}$ for $\mu/M = 10^{-5}$ where $\mu$ is the mass of a compact object and $M$ the mass of the central supermassive black hole. In first order black hole perturbation theory, for extreme mass ratio inspirals which are one of the main targets of Laser Interferometer Space Antenna, the 14PN expressions will lead to the data analysis accuracies comparable to the ones resulting from high precision numerical waveforms.

Introduction. Gravitational waves emitted from a stellar mass compact object of mass $\mu(\sim 1 - 100 M_\odot)$ orbiting a supermassive black hole of mass $M(\sim 10^5 - 10^7 M_\odot)$ at the centers of galaxies are one of the main astrophysical sources for the Laser Interferometer Space Antenna (LISA). The expected number of events of these extreme mass ratio inspirals (EMRIs) detected by LISA during its mission will be $10 - 100^1$ (see also a recent work on the capture rate by the 2.5PN $N$-body simulations$^2$). Observation of gravitational waves from EMRIs will provide information of the source, such as the masses, the spin of the black hole and the distribution of compact objects in the centers of galaxies. To extract such information by data analysis, the accumulated difference of phase between templates and the true signal during the observation should be less than one radian. The total number of wave cycles detected by LISA is $10^5 - 10^6$ since LISA has maximum sensitivity around $10^{-3}$Hz and its mission time is a few years. Thus, for LISA parameter estimation of EMRIs, the accuracy of theoretical waveforms should be better than $10^{-5}$.

Since the mass ratio is extreme, EMRIs can be described by black hole perturbation theory, which uses the mass ratio $\mu/M$ as an expansion parameter (see Ref.$^3$ for fully nonlinear numerical simulations of $\mu/M = 1/100$). At the lowest order, the small body traces a geodesic in the black hole geometry. Over time scales of order $\sim M^2/\mu$ (we use units $G = c = 1$) however, the orbit evolves adiabatically due to the small body’s interaction with its own gravitational field, i.e. gravitational self-force$^4,5$. This means that the gravitational self-force has to be taken into account for LISA parameter estimation of EMRIs. Second order black hole perturbation theory may also be required for the parameter estimation and is of great interest when the mass ratio is large.
Although black hole perturbation theory is powerful enough that one can compute gravitational waves accurately even in the strong field, the computational cost is very high to perform calculations which cover all the parameter space of EMRIs. This motivates us to use analytic modeling of gravitational waves by the post-Newtonian (PN) theory, which uses the orbital velocity of a compact object \( v = \sqrt{\frac{M}{r_0}} \), where \( r_0 \) is the orbital radius, as an expansion parameter.

In this letter, we investigate the extent of which PN order waveforms improve the solution for LISA parameter estimation of EMRIs by using the first order black hole perturbation theory. Currently, the highest available PN order for gravitational waveforms for EMRIs is 5.5PN \((v^{11} \text{ beyond Newtonian order})\) for a test particle in a circular orbit around a Schwarzschild black hole. However, the convergence of the PN expansion becomes worse when \( v \) is larger and the phase difference between the PN and numerical waveforms exceeds one radian after two years inspiral\(^7,\)\(^6\). In this work, we improve on the accuracy of the energy flux to 14PN and exhibit clearly its closeness to a high precision numerical computation of the energy flux. We would like to point out that such high PN order computations have not been performed up till now, since the number of terms necessary to derive the PN waveforms grows exponentially when the PN order becomes higher. For instance, our current code uses \( 70, 3.3 \times 10^2, 9.0 \times 10^2, 1.9 \times 10^3, 5.6 \times 10^3 \) and \( 1.1 \times 10^4 \) MBytes memory, taking seconds to half an hour, to compute multipolar waveforms for \( \ell = m = 2 \) mode at 6PN, 10PN, 12PN, 14PN, 16PN and 18PN respectively. Thus, it will be difficult to obtain 19PN or higher order expressions with reasonable time on a personal computer by using our current code. We also suggest the fitting formula of the energy flux which can be used for more general orbits around a Kerr black hole. One of the important consequence of this work for LISA is our demonstration that the phase difference after two years inspiral between our new 14PN waveforms and high precision numerical waveforms is negligible. This indicates that the 14PN expressions will lead to accuracies in LISA data analysis for EMRIs over two years comparable to accuracies resulting from high accuracy numerical waveforms.

**The method.** The fundamental equation of black hole perturbation is the Teukolsky equation, which describes the gravitational perturbation of a black hole in terms of the Weyl scalar \( \Psi_4 \). \( \Psi_4 \) is related to the gravitational wave polarizations at infinity as \( \Psi_4 \rightarrow (\dot{h}_+ - i \dot{h}_\times)/2 \) where dot, \( \dot{\cdot} \), denotes time derivative, \( d/dt \). The Teukolsky equation can be separated if we expand \( \Psi_4 \) in the Fourier domain using the \(-2\) spin-weighted spheroidal harmonics. For the case of Schwarzschild black hole, the radial Teukolsky equation reads

\[
\left[ \Delta^2 \frac{d}{dr} \left( \frac{1}{\Delta} \frac{d}{dr} \right) + U(r) \right] R_{\ell m \omega}(r) = T_{\ell m \omega}(r),
\]

where \( \Delta = r(r - 2M) \), \( U(r) \) is the potential and \( T_{\ell m \omega} \) is the source term depending on the particle’s orbit.

The radial Teukolsky equation can be solved by the Green function method. In this work, we use a formalism developed by Mano, Suzuki and Takasugi (MST) to obtain homogeneous solutions of the Teukolsky equation\(^9\). In this formalism, the homogeneous solutions of the Teukolsky equation are expressed in a series of hy-
pergeometric functions and Coulomb wave functions, which converge at the horizon and infinity respectively. The formalism is very powerful for the PN expansion of the Teukolsky equation since the series expansion is closely related to the low frequency expansion. The expansion coefficients of the two series \( a_n^\nu \) are the same and satisfy the three-term recurrence relation

\[
\alpha_n^\nu a_{n+1}^\nu + \beta_n^\nu a_n^\nu + \gamma_n^\nu a_{n-1}^\nu = 0,
\]

where \( \nu = \ell + O(\omega^2) \), \( \alpha_n^\nu = O(\omega) \), \( \beta_n^\nu = O(1) \) and \( \gamma_n^\nu = O(\omega) \). We note that the parameter \( \nu \) does not exist in the original Teukolsky equation and is determined so that the series converges and represents a solution of the Teukolsky equation. One can derive the low frequency expansion of \( a_n^\nu \) by solving Eq. (2) iteratively. Thus, if we derive \( a_n^\nu \) up to a required order, we automatically obtain the PN expansion formulas up to the required order. See Ref. [10] for details of the formalism.

Using the MST formalism, the energy flux absorbed into the horizon was derived up to relative 4PN (i.e. 6.5PN beyond Newtonian order) for a test particle in a circular and equatorial orbit around a Kerr black hole. Gravitational wave flux to infinity was also computed up to 2.5PN for a test particle in slightly eccentric and inclined orbits around a Kerr black hole. More recently, the MST formalism was applied to obtain the 5.5PN waveforms for a test particle in a circular orbit around a Schwarzschild black hole and the 4PN waveforms for a test particle in a circular and equatorial orbit around a Kerr black hole, confirming the 5.5PN energy flux in Ref. [15] and the 4PN energy flux in Ref. [16] respectively.

Results and Analysis. Once we compute the homogeneous solutions of the Teukolsky equation, using the Green function method we build the solution of the Teukolsky equation Eq. (1), which is purely outgoing at infinity and ingoing at the horizon. For the case of a test particle in a circular orbit around a Schwarzschild black hole, the gravitational wave flux to infinity is given by

\[
\frac{dE}{dt} = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \frac{|Z_{\ell m \omega}|^2}{4\pi \omega^2},
\]

where \( \omega = m\Omega \) is the frequency of gravitational waves, \( \Omega = \sqrt{M/r_0^3} \) is the angular frequency of the orbit and \( Z_{\ell m \omega} \) is derived from the asymptotic behavior of the solution of the Teukolsky equation at infinity.

We compute gravitational waveforms which are necessary for computing the 14PN energy flux to infinity for a test particle in a circular orbit around a Schwarzschild black hole. We show only the next new 6PN terms of the energy flux, \( dE^{(12)}/dt \), because of space restrictions.
Letters

\[ \begin{align*}
&\quad + \frac{1465472}{11025} \left( \ln(v) \right)^2 - \frac{27392}{315} \pi^2 \gamma + \frac{2930944}{11025} \ln(v) \gamma + \frac{5861888}{11025} \ln(2) \gamma \\
&\quad + \frac{1465472}{11025} \gamma^2 - \frac{437114506833}{789268480} \ln(3) - \frac{37744140625}{260941824} \ln(5),
\end{align*} \]

(4)

where \( \zeta(n) \) is the Zeta function. We note that \( \left( \ln(v) \right)^2 \) term appears at 6PN. We also note that \( \left( \ln(v) \right)^3 \) and \( \left( \ln(v) \right)^4 \) terms appear from 9PN and 12PN respectively. One of the reasons is that \( \left( \ln(v) \right)^n \) terms are produced by the PN expansion of \( z^\nu \) in the homogeneous solution of the Teukolsky equation, where \( z = \omega r \) and \( \nu = \ell + O(v^k) \). These \( \left( \ln(v) \right)^n \) terms agree with Eq. (44) in Ref. [17], which predicts \( \left( \ln(v) \right)^n \) terms at 3n-PN using the renormalization group equations. Our present work suggests that to obtain unknown PN terms by fits to numerical results one must include \( \left( \ln(v) \right)^n \) terms starting from 3n-PN with the structure

\[ \frac{dE}{dt} = \sum_{k=0}^{\infty} \sum_{p=0}^{[k/6]} \frac{dE^{(k,p)}}{dt} \left( \ln(v) \right)^p v^k, \]

(5)

where \([ \cdots ]\) is the floor function. Note that one can use the fitting formula Eq. (5) for more general orbits around a Kerr black hole, which are known at most up to 4PN [19].

We use the factorized multipolar waveforms introduced in Ref. [18] to compute the gravitational waveforms. The multipolar waveforms are decomposed into five factors as

\[ h_{\ell m} = h^{(N,\epsilon_p)} \tilde{S}_{\ell m}^\text{eff} T_{\ell m} e^{i\delta_{\ell m} (\rho_{\ell m})^\ell}, \]

(6)

where \( h^{(N,\epsilon_p)} \) is the Newtonian contribution to waveforms and \( \epsilon_p \) denotes the parity of the multipolar waveforms. In the case of circular orbits, \( \epsilon_p = 0 \) (\( \epsilon_p = 1 \)) when \( \ell + m \) is even (odd). The \( \ell \)-th root of the amplitude \( \rho_{\ell m} \) improves the convergence of the PN expansion by dealing with the linear term of \( \ell \) in the 1PN terms of the amplitudes. See Ref. [18] for the definitions of the other terms.

The factorized multipolar waveform is the simplest and the most efficient resummation technique so far. The energy flux computed from the factorized resummed waveforms agrees better with the numerical results than the ones from Taylor-expanded waveforms and other resummed waveforms as Padé approximation [19].

In this letter, we show only the next new 6PN terms of \( \rho_{2,2} \), which was earlier computed up to 5.5PN [19]. The complete expressions will be shown elsewhere [19].

\[ \rho_{2,2}^{(12)} = \frac{313425353036319023287}{1132319812111488000} - \frac{6848}{105} \zeta(3) - \frac{91592}{11025} \pi^2 \\
- \frac{24177319107}{3208936500} \text{eulerlog}(2, v) + \frac{91592}{11025} (\text{eulerlog}(2, v))^2, \]

(7)

where \( \rho_{2,2}^{(n)} \) is \( O(v^n) \) coefficient of \( \rho_{2,2} \), \( \text{eulerlog}(2, v) \equiv \gamma + \ln(4v) \) and \( \gamma \) is the Euler constant. Again, we note that \( \left( \ln(v) \right)^2 \) term appears at 6PN.
Fig. 1. Absolute values of the difference of energy flux to infinity between numerical results and Taylor-expanded (left) or factorized (right) PN approximation as a function of orbital velocity. Note that 14PN flux converges very well even around ISCO, \( v = 1/\sqrt{6} = 0.40825 \).

Fig. 1 shows the comparison of energy flux to infinity between the PN approximations and the numerical calculation. To demonstrate the efficiency of the factorized waveforms, we show the results of the PN energy flux using the Taylor-expanded waveforms (left panel) and the factorized waveforms (right panel) in Fig. 1. The numerical energy flux is obtained using the high precision code in Ref. 20). The accuracy of the numerical calculation is better than \( 10^{-10} \) if we set the maximum value of \( \ell \) to 20 in Eq. (5). The \( n \)-PN flux needs \( \ell \) up to \( n + 2 \). The agreement of the total energy flux between the PN and the numerical results becomes better when the PN order is higher even around the innermost stable circular orbit (ISCO) (see Ref. 21) for the calculation beyond ISCO).

To investigate quantitatively the PN order needed for LISA parameter estimation of EMRI, we compare the phase difference during two years quasi-circular inspiral between the factorized PN waveforms and the numerical waveforms. Following Ref. 17, we examine two systems, which are named System-I and System-II. System-I has masses \((M, \mu) = (10^5, 10) \text{M}_\odot\) and inspirals from \( r_0 \approx 29.34 \text{M} \) to \( r_0 \approx 16.1 \text{M} \), whose frequency sweeps from \( f_{\text{GW}} \approx 4 \times 10^{-3} \text{Hz} \) to \( f_{\text{GW}} \approx 10^{-2} \text{Hz} \). System-II has masses \((M, \mu) = (10^6, 10) \text{M}_\odot\) and starts inspiral from \( r_0 \approx 10.6 \text{M} \) to \( r_0 \approx 6.0 \text{M} \), whose frequency sweeps from \( f_{\text{GW}} \approx 1.8 \times 10^{-4} \text{Hz} \) to \( f_{\text{GW}} \approx 4.4 \times 10^{-3} \text{Hz} \). System-I (II) has \( \approx 1 \times 10^6 (\approx 5 \times 10^5) \text{ rads of orbital phase during its inspiral and corresponds to the early (late) inspiral phase of an EMRI in the frequency band of LISA.} \)

Fig. 2 shows the comparison of the phase between the dominant mode \( h_{2,2} \) of the factorized PN and the numerical calculation (see Ref. 22) for the calculation of the phase). For System-I (II), the absolute values of the phase difference between the factorized PN waveforms and the numerical waveforms after two years inspiral are about \( 3.6 (4.0 \times 10^2) \), \( 0.38 (1.2 \times 10^2) \), \( 5.4 \times 10^{-3} (10) \), \( 3.4 \times 10^{-5} (0.57) \), \( 1.8 \times 10^{-6} (0.11) \) and \( 2.5 \times 10^{-8} (8.9 \times 10^{-4}) \) rads for 5.5PN, 6PN, 8PN, 10PN, 12PN and 14PN respectively (The relative error of the amplitude between the factorized 14PN and the numerical waveforms is \( 6.6 \times 10^{-13} (7.1 \times 10^{-7}) \) for System-I (II)).

The strongest EMRI events detected by LISA may have the signal to noise ratio...
Fig. 2. Absolute values of the phase difference due to two years inspiral between the factorized PN and the numerical waveforms for $h_{2,2}$ as a function of time in month. The left panel shows the dephase for $(M, \mu) = (10^{5}, 10) M_\odot$, which starts from $r_0 \simeq 29.34 M$ to $r_0 \simeq 16.1 M$ and sweeps frequencies $f_{GW} \in [4 \times 10^{-3}, 10^{-2}]$ Hz. The right panel shows the dephase for $(M, \mu) = (10^{6}, 10) M_\odot$ from $r_0 \simeq 10.6 M$ to $r_0 \simeq 6.0 M$ with associated frequencies $f_{GW} \in [1.8 \times 10^{-3}, 4.4 \times 10^{-3}]$ Hz. The left (right) panel represents the early (late) inspiral phase in the LISA band. If the dephase is less than 10 milliradians, the PN waveforms will provide the parameter estimation of EMRI comparable to the one resulting from numerical waveforms.

up to $\rho \sim 100$. Thus, LISA can measure phase difference of the order of $1/\rho \sim 10$ milliradians by matched filtering. This suggests that the 8PN (14PN) waveforms will lead to LISA parameter estimation of System-I (II) comparable to the one using numerical waveforms. Since System-II represents the inspiral in the most strong-field of a Schwarzschild black hole, the 14PN waveforms will provide LISA parameter estimation of EMRI comparable to the one resulting from numerical waveforms.

**Conclusion and Discussion.** Using the first order black hole perturbation theory, we have computed gravitational waveforms consistent with the 14PN energy flux for a test particle in a circular orbit around a Schwarzschild black hole. The high PN order computation has been performed systematically using the MST formalism. We provide a fitting formula of the energy flux, Eq. (5), for more general orbits around a Kerr black hole. Comparing the energy flux with a high precision numerical computation, we investigated the approach of the PN expansion towards the numerical results and found that the PN expansion converges well even at ISCO. The phase difference between our new analytic waveforms and the numerical waveforms after two years of inspiral is so negligibly small that it does not lead to any discrepancy in LISA data analysis. Thus, using first order black hole perturbation theory, one can derive sufficiently high PN order waveforms analytically and build the template banks to span the parameter space efficiently without recourse to a more expensive numerical computation.

The EMRIs contain more generic systems such as eccentric and inclined orbits of a test particle around a Kerr black hole (see Ref. 24) for an application to a spinning particle). The orbital velocity at ISCO can be larger when the black hole is rotating. If the orbits become more generic, one needs a larger number of wave modes, which will contain higher frequency contributions than the case in this work. Thus, for
more generic orbits around a Kerr black hole we have to compute higher PN orders than 14PN in order to achieve good agreement with the numerical waveforms. Since the formalism adopted in this work is systematic and does not have theoretical issues to compute higher PN order, in principle one can compute sufficiently high PN order waveforms even for generic orbits. It will be possible to perform sufficiently high PN order computation for the case of a test particle in slightly eccentric and inclined orbits around a Kerr black hole. However, it may be difficult to compute them for the case of a test particle in large eccentric and inclined orbits around a Kerr black hole in reasonable time. Moreover, conservative effects of self-force, which are not included in this work, may contribute $\sim 20$ radians to the phase of waveforms during inspiral\(^7\) (however see Table II in Ref. \[^{26}\], which suggests that conservative effects up to 9PN is sufficient to reduce the phase error to 10 milliradians by investigating 18PN scalar self-force). Another approach to discuss generic orbits may be the effective-one-body formalism which can determine unknown terms in PN approximation and self-force effects by calibrating them with numerical calculation.\(^7\)\(^{27}\)

The present work has implications for this approach too. For the calibration, we recommend the use of our proposed fitting formula Eq. (5) dealing with $(\ln(v))^n$ terms appearing from $3n$-PN.

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1) J. R. Gair, L. Barack, T. Creighton, C. Cutler, S. L. Larson, E. S. Phinney and M. Vallisneri, Class. Quantum Grav. \textbf{21}, S1595 (2004).
2) D. Merritt, T. Alexander, S. Mikkola and C. Will, Phys. Rev. D \textbf{84}, 044024 (2011).
3) C. O. Lousto and Y. Zlochower, Phys. Rev. Lett. \textbf{106}, 041101 (2011).
4) E. Poisson, A. Pound and I. Vega, Living Rev. Relativity \textbf{14}, 7 (2011), \url{http://www.livingreviews.org/lrr-2011-7}.
5) J. Thornburg, \url{arXiv:1102.2857}.
6) R. Fujita and B. R. Iyer, Phys. Rev. D \textbf{82}, 044051 (2010).
7) N. Yunes, A. Buonanno, S. A. Hughes, M. C. Miller and Y. Pan, Phys. Rev. Lett. \textbf{104}, 091102 (2010).
8) S. A. Teukolsky, Astrophys. J. \textbf{185}, 635 (1973).
9) S. Mano, H. Suzuki and E. Takasugi, Prog. Theor. Phys. \textbf{95}, 1079 (1996); S. Mano and E. Takasugi, Prog. Theor. Phys. \textbf{97}, 213 (1997).
10) M. Sasaki and H. Tagoshi, Living Rev. Relativity \textbf{6}, 6 (2003), \url{http://relativity.livingreviews.org/Articles/lrr-2003-6}.
11) H. Tagoshi, S. Mano and E. Takasugi, Prog. Theor. Phys. \textbf{98}, 829 (1997).
12) N. Sago, T. Tanaka, W. Hikida, K. Ganz and H. Nakano, Prog. Theor. Phys. \textbf{115}, 873 (2006).
13) K. Ganz, W. Hikida, H. Nakano, N. Sago and T. Tanaka, Prog. Theor. Phys. \textbf{117}, 1041 (2007).
14) Y. Pan, A. Buonanno, R. Fujita, E. Racine and H. Tagoshi, Phys. Rev. D \textbf{83}, 064003 (2011).
15) T. Tanaka, H. Tagoshi and M. Sasaki, Prog. Theor. Phys. \textbf{96}, 1087 (1996).
16) H. Tagoshi, M. Shibata, T. Tanaka and M. Sasaki, Phys. Rev. D \textbf{54}, 1439 (1996).
17) W. D. Goldberger and A. Ross, Phys. Rev. D \textbf{81}, 124015 (2010).
18) T. Damour, B. R. Iyer and A. Nagar, Phys. Rev. D \textbf{79}, 064004 (2009).
19) R. Fujita, in preparation.
20) R. Fujita and H. Tagoshi, Prog. Theor. Phys. \textbf{112}, 415 (2004); \textbf{113}, 1165 (2005).
21) A. Ori and K. S. Thorne, Phys. Rev. D \textbf{62}, 124022 (2000).
22) S. A. Hughes, Phys. Rev. D \textbf{64}, 064004 (2001).
23) P. Amaro-Seoane, J. R. Gair, M. Freitag, M. C. Miller, I. Mandel, C. J. Cutler and S. Babak, Class. Quantum Grav. 24, R113 (2007).
24) Takahiro Tanaka, Yasushi Mino, Misao Sasaki and Masaru Shibata, Phys. Rev. D 54, 3762 (1996).
25) E. A. Huerta, and J. R. Gair, Phys. Rev. D 79, 084021 (2009).
26) W. Hikida, S. Jhingan, H. Nakano, N. Sago, M. Sasaki and T. Tanaka, Prog. Theor. Phys. 113, 283 (2005).
27) N. Yunes, A. Buonanno, S. A. Hughes, Y. Pan, E. Barausse, M. C. Miller and W. Throwe, Phys. Rev. D 83, 044044 (2011).