Asymptotic Signal Detection Rates with 1-bit Array Measurements

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Abstract—This work considers detecting the presence of a band-limited random radio source using an antenna array featuring a low-complexity digitization process with single-bit output resolution. In contrast to high-resolution analog-to-digital conversion, such a direct transformation of the analog radio measurements to a binary representation can be implemented hardware and energy-efficient. For a wireless spectrum monitoring problem, formulated as a binary hypothesis test, we analyze the achievable detection performance when using such 1-bit radio measurements from an array of sensors. To simplify the derivations, we consider asymptotic statistical tests with exponential family models, which results in analytic detection rate expressions without access to the intractable 1-bit array likelihood. As an application, we explore the detection capability of a GPS spectrum monitoring system with different numbers of antennas and different observation intervals. Results show that binary arrays with a moderate amount of sensors are capable of performing fast and reliable spectrum monitoring.

Index Terms—1-bit ADC, analog-to-digital conversion, array processing, exponential family, GPS, Neyman-Pearson test, quantization, signal detection, spectrum monitoring

I. INTRODUCTION

The task of signal detection is of importance in applications like multi-user communication, cognitive radio, and radar. In particular, the advent of the Internet of Things (IoT), where tiny objects feature wireless interfaces, and the increasing dependency of critical infrastructure on synchronization signals from satellite-based radio systems make it necessary to continuously monitor the electromagnetic spectrum reliably and efficiently. Since, in 1965, Gordon E. Moore predicted a doubling in computational capability every two years, chip companies have kept pace with this prognosis and set the foundation for digital computer systems which today allow processing high-rate radio measurements by sophisticated digital algorithms. In conjunction with wireless sensor arrays, they provide advanced signal processing capabilities, see, e.g., [1]. Unfortunately, in the last decades, the advances regarding the analog circuits forming the radio front-end were much slower. Therefore, today cost and power consumption of wireless measurement hardware are more and more becoming an issue. In particular, resource-intensive analog-to-digital conversion with high resolution [2],[3] shows to set constraints on the digitization rate of wide-band applications and the number of antenna elements in massive sensing scenarios.

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In this context, we consider detection of a band-limited source with unknown random structure by a sensor array providing single-bit radio measurements. We formulate the processing task as a binary hypothesis test regarding exponential family models and derive expressions for the asymptotic detection rates. The results are used to determine the design of a low-complexity GPS spectrum monitoring system.

Note that detection with quantized signals has found attention in distributed decision making [4] where data is collected through sensors at different locations and quantization is used to digitally diminish the communication overhead between sensors and a terminal node [5]–[7]. Optimum quantization for detection is the focus of [8]–[11], while [12] considers the detection performance degradation due to hard-limiting. For discussions surrounding the task of symbol detection for communication over quantized channels see, e.g., [13]–[18]. Detection for cognitive radio with single-antenna 1-bit receivers is considered in [19] while array processing with 1-bit measurements is analyzed in [20]–[22].

II. PROBLEM FORMULATION

We consider a receive situation where a narrow-band random wireless signal is impinging on an array of $S$ sensors,

$$y = \gamma A x + \eta.$$  (1)

Each sensor features two outputs (in-phase and quadrature), such that the receive vector $y \in \mathcal{Y} = \mathbb{R}^M$, $M = 2S$, can be decomposed $y = [y_1^T \ y_Q^T]^T$ with $y_1, y_Q \in \mathbb{R}^S$. Likewise, the random source $x = [x_1 \ x_Q]^T \in \mathbb{R}^2$ consists of two zero-mean signal components with covariance matrix

$$R_x = \mathbb{E}_x [xx^T] = I,$$  (2)

where $\mathbb{E}_u [\cdot]$ denotes the expectation concerning the probability distribution function $p(u)$ and $I$ the identity matrix. The steering matrix $A = [A_1^T \ A_Q^T]^T \in \mathbb{R}^{M \times 2}$, with $A_1, A_Q \in \mathbb{R}^{S \times 2}$ models a uniform linear sensor array response (half carrier-wavelength inter-element distance) for a narrow-band signal arriving from direction $\zeta \in \mathbb{R}$, such that

$$A_1 = \begin{bmatrix}
\cos (0) & \sin (0) \\
\cos (\pi \sin (\zeta)) & \sin (\pi \sin (\zeta)) \\
\vdots & \vdots \\
\cos ((S-1) \pi \sin (\zeta)) & \sin ((S-1) \pi \sin (\zeta))
\end{bmatrix}$$  (3)
and
\[ A_Q = \begin{bmatrix}
-\sin (0) & \cos (0) \\
-\sin (\pi \sin (\zeta)) & \cos (\pi \sin (\zeta)) \\
\vdots & \vdots \\
-\sin ((S - 1)\pi \sin (\zeta)) & \cos ((S - 1)\pi \sin (\zeta)) \\
\end{bmatrix}, \]

(4)

The parameter \( \gamma \in \mathbb{R} \) characterizes the source strength in relation to the additive zero-mean sensor noise \( \eta \in \mathbb{R}^M \) with
\[ R_\eta = E_\eta [\eta \eta^T] = I. \]

(5)

Due to the properties of the source and noise signals, the receive data (1) can be modeled by a Gaussian distribution
\[ y \sim p_y(y; \gamma) = \frac{\exp \left( -\frac{1}{2} y^T R_y^{-1}(\gamma)y \right)}{\sqrt{(2\pi)^M \det (R_y(\gamma))}}, \]

(6)

with covariance matrix
\[ R_y(\gamma) = E_{y;\gamma} [yy^T] = \gamma^2 A A^T + I. \]

(7)

Based on the likelihood (6), the problem of signal detection can be stated as the decision about which of the two models
\[ \mathcal{H}_0 : y \sim p_y(y; \gamma_0), \quad \mathcal{H}_1 : y \sim p_y(y; \gamma_1) \]

(8)

has generated the data \((K\text{ independent array snapshots})\)
\[ Y = \begin{bmatrix} y_1 & y_2 & \ldots & y_K \end{bmatrix} \in \mathcal{Y}^K. \]

(9)

In wireless spectrum monitoring, the case \( \gamma_0 = 0, \gamma_1 \neq 0 \), is of particular interest, i.e., to distinguish between the absence and presence of a radio source. Using model (1) in practical applications implies that a high-resolution analog-to-digital converter (ADC) is available for each array output channel. As hardware complexity and power consumption of ADCs grow exponentially with the number of bits, significant savings are possible when using a converter with 1-bit output resolution. Such a receiver can be modeled by an ideal sampling device with infinite resolution followed by a hard-limiter
\[ z = \text{sign} (y), \]

(10)

where \( \text{sign} (u) \) is the element-wise signum function, i.e.,
\[ [z]_i = \begin{cases} +1 & \text{if } [y]_i \geq 0 \\ -1 & \text{if } [y]_i < 0. \end{cases} \]

(11)

Note, that (10) characterizes a low-complexity digitization process without feedback and is, therefore, distinct from sigma-delta conversion [23], where a single fast comparator with feedback is used to mimic a high-resolution ADC.

Modeling the output of (10) by its exact parametric probability distribution function, requires computing the integral
\[ p_z(z; \gamma) = \int_{\mathcal{Y}(z)} p_y(y; \gamma)dy \]

(12)

for all \( 2^M \) points in \( Z = \mathbb{B}^M \), where \( \mathcal{Y}(z) \) characterizes the subset in \( \mathcal{Y} \) which by (10) is transformed to \( z \). Additionally, (12) requires the multivariate Gaussian orthant probability in closed-form, which for \( M > 4 \) (or \( S > 2 \)) is an open mathematical problem. As the multivariate Bernoulli model resulting from (10) is part of the exponential family like (6), in the following we resort to discussing the considered signal processing task for generic data models within this broad class. Without exactly specifying the quantized likelihood (12), this will allow us to rigorously analyze the asymptotically achievable detection rates with low-complexity 1-bit array measurements.

III. DECISIONS IN THE EXPONENTIAL FAMILY

Consider the multivariate parametric exponential family
\[ p_z(z; \theta) = \exp \left( \beta^T(\theta) \phi(z) - \lambda(\theta) + \kappa(z) \right), \]

(13)

where \( \theta \in \mathbb{R}^D \) constitute its parameters, \( \beta(\theta) : \mathbb{R}^D \to \mathbb{R}^L \) the natural parameters, \( \phi(z) : \mathbb{R}^M \to \mathbb{R}^L \) the sufficient statistics, \( \lambda(\theta) : \mathbb{R}^D \to \mathbb{R} \) the log-normalizer and \( \kappa(z) : \mathbb{R}^M \to \mathbb{R} \) the carrier measure. Given a data set \( Z \in \mathbb{Z}^K \) of the form (9), the simple binary hypothesis test between
\[ \mathcal{H}_0 : z \sim p_z(z; \theta_0), \quad \mathcal{H}_1 : z \sim p_z(z; \theta_1) \]

(14)

is to be performed. To this end, we assign a critical region \( \mathcal{C} \subset \mathbb{Z}^K \) and decide in favor of \( \mathcal{H}_1 \) if the observed data satisfies \( Z \in \mathcal{C} \). The probability of erroneously deciding for \( \mathcal{H}_1 \) while \( \mathcal{H}_0 \) is the true data-generating model is calculated
\[ P_{F_A} = \int_{\mathcal{C}} p_z(Z; \theta_0) dZ, \]

(15)

while the probability of correctly deciding for \( \mathcal{H}_1 \) is given by
\[ P_{D} = \int_{\mathcal{C}} p_z(Z; \theta_1) dZ. \]

(16)

Approaching the decision problem (14) under the desired test size \( P_{F_A} \) and maximum \( P_{D} \), the Neyman-Pearson theorem shows that it is optimum to use the likelihood ratio test [24]
\[ L(Z) = \frac{p_z(Z; \theta_1)}{p_z(Z; \theta_0)} > \xi' \]

(17)

for the assignment of the critical region
\[ \mathcal{C} = \{ Z : L(Z) > \xi' \}, \]

(18)

while the decision threshold \( \xi' \) is determined through
\[ P_{F_A} = \int_{\{Z : L(Z) > \xi'\}} p_z(Z; \theta_0) dZ. \]

(19)

Based on the ratio (17), a test statistic \( T(Z) : \mathbb{Z}^K \to \mathbb{R} \) can be formulated such that the binary decision is performed by
\[ \begin{align*}
& \mathcal{H}_0 & \text{if } T(Z) \leq \xi \\
& \mathcal{H}_1 & \text{if } T(Z) > \xi' 
\end{align*} \]

(20)

To analyze the performance of (20), it is required to characterize the distribution of \( T(Z) \) and evaluate (15) and (16). As the data \( Z \) consists of \( K \) independent samples, the test statistic can be factorized into a sum of independent components
\[ T(Z) = \sum_{k=1}^{K} t(z_k) \]

(21)
such that, by the central limit theorem, the test statistic in the large sample regime follows the normal distribution

$$p(T(Z)|\mathcal{H}_i) \overset{a}{=} \frac{1}{\sqrt{2\pi\sigma_i}} \exp\left(-\frac{(T(Z) - \mu_i)^2}{2\sigma_i^2}\right),$$  

(22)

where by \( \overset{a}{=} \) we denote asymptotic equality. Through the mean and standard deviation of the test statistic

$$\mu_i = \mathbb{E}_{Z,\omega_i}[T(Z)],$$  

$$\sigma_i = \sqrt{\mathbb{E}_{Z,\omega_i}[(T(Z) - \mu_i)^2]},$$  

(23) \hspace{1cm} (24)

the asymptotic performance is then given by

$$P_D = \Pr\{T(Z) > \xi|\mathcal{H}_1\} \overset{a}{=} Q\left(\frac{\xi - \mu_1}{\sigma_1}\right),$$  

(25)

$$P_{FA} = \Pr\{T(Z) > \xi|\mathcal{H}_0\} \overset{a}{=} Q\left(\frac{\xi - \mu_0}{\sigma_0}\right),$$  

(26)

where \( Q(u) \) denotes the Q-function. Consequently, for a desired \( P_{FA} \), the decision threshold is

$$\xi \overset{a}{=} Q^{-1}(P_{FA}) \sigma_0 + \mu_0,$$  

(27)

resulting in the asymptotic probability of detection

$$P_D(P_{FA}) \overset{a}{=} Q\left(Q^{-1}(P_{FA}) \frac{\sigma_0}{\sigma_1} - \frac{\mu_1 - \mu_0}{\sigma_1}\right).$$  

(28)

Writing

$$b = \beta(\theta_1) - \beta(\theta_0),$$  

(29)

the log-likelihood ratio for exponential family models (13) is

$$\ln L(Z) = \sum_{k=1}^{K} b^T \phi(z_k) - K(\lambda(\theta_1) - \lambda(\theta_0)),.$$  

(30)

such that with the empirical mean of the sufficient statistics

$$\bar{\phi} = \frac{1}{K} \sum_{k=1}^{K} \phi(z_k),$$  

(31)

a likelihood-based test statistic is

$$T(Z) = b^T \bar{\phi}.$$  

(32)

The mean and standard deviation of the test are

$$\mu_i = b^T \mu_{\phi}(\theta_i), \quad \sigma_i = \sqrt{\frac{1}{K} b^T R_{\phi}(\theta_i) b},$$  

(33)

where

$$\mu_{\phi}(\theta) = \mathbb{E}_{Z,\theta}[\phi(z)],$$  

(34)

$$R_{\phi}(\theta) = \mathbb{E}_{Z,\theta}\left[\phi(z)\phi^T(z)\right] - \mu_{\phi}(\theta) \mu_{\phi}^T(\theta).$$  

(35)

For Gaussian models (6),

$$\beta(\theta) = -\frac{1}{2} \text{vec}(R_{y}(\theta)^{-1}),$$  

(36)

$$\phi(y) = \text{vec}(yy^T),$$  

(37)

$$\mu_{\phi}(\theta) = \text{vec}(R_y(\theta)),$$  

(38)

and the matrix (35) can be determined through Isserlis’ theorem. If the model (13) is unspecified, one needs to favorably choose \( \phi(z) \) and \( b \). For the analysis with 1-bit measurements, we use

$$\phi(z) = \Phi \text{vec}(zz^T),$$  

(39)

where the matrix \( \Phi \) eliminates duplicate and diagonal statistics. This is potentially suboptimal as (39) does, in general, not contain all sufficient statistics of a multivariate Bernoulli distribution. The missing statistics are absorbed in the carrier measure of (13) and, therefore, do not contribute to the decision process. For the natural parameter difference (29), we use

$$b = R_{\phi}^{-1}(\theta_1) \mu_{\phi}(\theta_1) - R_{\phi}^{-1}(\theta_0) \mu_{\phi}(\theta_0)$$  

(40)

as it maximizes the distance of the asymptotic outcome in (32) under both hypotheses. The mean of the statistics (39),

$$\mu_{\phi}(\theta) = \mathbb{E}_{Z,\theta}[\phi(z)] = \Phi \text{vec}(R_z(\theta)),$$  

(41)

is obtained by the arcsine law [26, pp. 284],

$$R_z(\theta) = \frac{2}{\pi} \text{arcsin}(\Sigma_y(\theta)), \quad \Sigma_y(\theta) = \text{diag}(R_y(\theta))^{-\frac{1}{2}} R_y(\theta) \text{diag}(R_y(\theta))^{-\frac{1}{2}}.$$  

(42)

Further, the evaluation of (35) requires determining the matrix

$$C(\theta) = \mathbb{E}_{Z,\theta}\left[\text{vec}(zz^T) \text{vec}(zz^T)^T\right]$$  

(43)

which is possible [21] by the arcsine law and the orthant probability of the quadrivariate Gaussian distribution [25].

**IV. RESULTS**

We apply the analytic results to discuss the appropriate design of a multi-sensor 1-bit GPS spectrum monitoring system. The task is to check a spectral band with two-sided bandwidth \( B = 2.046 \) Mhz, centered at 1.57 GHz carrier frequency, for a source signal from a specific direction. The receiver is supposed to sample the low-pass filtered baseband receive signal at the Nyquist rate, i.e., \( f_s = B \), such that \( K = 2046 \) array snapshots are available within one millisecond. For this

![Fig. 1. Quality vs. Array Size (K = 2046, ζ = 45°)](image-url)
scenario, we determine the performance for different array sizes and different observation intervals. The upper triangular area under the receiver operating characteristic (ROC) curve, 

\[ \chi = 2 \int_0^1 P_D(u) \, du - 1, \]  

(44)
determines the quality of the system regarding the signal detection task. For different signal-to-noise ratio (SNR) scenarios, Fig. 1 shows the 1-bit system quality for an exemplary setting with \( K = 2046, \zeta = 45^\circ \), versus the number of array elements \( S \). While for very weak signal sources (SNR = \(-24\) dB) more than \( S = 20 \) sensors are required to provide high system performance, at a power level of SNR = \(-15\) dB already \( S = 5 \) antennas suffice to operate close to a perfect monitoring system with \( \chi = 1 \). To determine a favorable observation length, for a scenario with \( S = 8, \zeta = 30^\circ \), Fig. 2 shows the system quality \( \chi \) for different numbers of samples. While the reliable detection of a signal with SNR = \(-24\) dB will require sampling more than 10 ms, the decision on the presence of a source of strength SNR = \(-15\) dB can be made trustworthy within less than 1 ms. Finally, in Fig. 3 we depict the analytic and simulated detection performance (using 10^5 data realizations) with \( S = 8, K = 100, \zeta = 15^\circ \). Already at a moderate sample size, the asymptotic results show a close correspondence with Monte-Carlo simulations.

V. CONCLUSION

We have derived achievable detection rates with a large number of radio measurements obtained with a low-complexity sensor array performing 1-bit analog-to-digital conversion. Discussing the simple binary hypothesis test in the framework of the exponential family enables circumventing the intractability of the 1-bit array likelihood. Its difficult characterization forms a fundamental obstacle to the application of analytic tools in statistical signal and information processing to problems with coarsely quantized correlated data. Using the analytic results to determine the spectrum monitoring capability with wireless multi-sensor receivers shows that, under the right system design, radio frequency measurements from binary arrays are sufficient to perform the processing task of signal detection in a fast and reliable way.

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