Finiteness and Unitarity of 
Lorentz-Covariant Green-Schwarz Superstring Amplitudes

Nathan Berkovits

Math Dept., King’s College, Strand, London, WC2R 2LS, United Kingdom

e-mail: udah101@oak.cc.kcl.ac.uk

KCL-TH-93-6

March 1993

Abstract

In two recent papers, a new method was developed for calculating ten-dimensional superstring amplitudes with an arbitrary number of loops and external massless particles, and for expressing them in manifestly Lorentz-invariant form. By explicitly checking for divergences when the Riemann surface degenerates, these amplitudes are proven to be finite. By choosing light-cone moduli for the surface and comparing with the light-cone Green-Schwarz formalism, these amplitudes are proven to be unitary.
I. Introduction

There are various methods for calculating ten-dimensional superstring amplitudes, each with its advantages and disadvantages. One method is the light-cone Neveu-Schwarz-Ramond formalism,\(^1\) which has the advantage of being manifestly unitary, but the disadvantage of non-trivial operators at the interaction points of the light-cone diagram. Besides complicating considerably the amplitude calculations, the presence of these non-trivial interaction-point operators forces the introduction of higher-order contact terms,\(^2,3\) which are necessary to remove the unphysical divergences when two or more interaction points coincide.

Another method for calculating superstring amplitudes is the manifestly Lorentz-covariant Neveu-Schwarz-Ramond formalism.\(^4\) In place of the interaction-point operators of the light-cone formalism, the covariant Neveu-Schwarz-Ramond formalism contains N=1 fermionic moduli, which after integrating out gives rise to picture-changing operators. These picture-changing operators differ from the light-cone interaction-point operators in that their locations on the surface are not predetermined.

Moving the locations of the picture-changing operators changes the integrand of the scattering amplitude by a total derivative,\(^5\) which gives a surface term contribution due to the presence of a cutoff in the bosonic moduli (the cutoff is necessary in the Neveu-Schwarz-Ramond formalism since before summing over spin structures, the integrand is divergent at points in moduli space where the surface degenerates).\(^6\) Although this surface-term ambiguity can be removed by requiring that the scattering amplitudes agree with those obtained using the manifestly unitary light-cone Neveu-Schwarz-Ramond formalism (this implies that at divergent points in moduli space, the locations of the picture-changing operators should coincide with the interaction points of the corresponding light-cone diagram), the need to impose such a requirement is an unpleasant feature of the covariant Neveu-Schwarz-Ramond formalism. Note that because the light-cone interaction-point locations are not holomorphic functions of the moduli, this requirement can not be imposed independently on the right and left-moving contributions to the amplitude.

It is also possible to calculate superstring amplitudes without integrating out the fermionic moduli, using either the light-cone\(^7,8\) or covariant\(^9\) Neveu-Schwarz-Ramond formalism on supersheets. With either of these two methods, one ends up with a Lorentz-covariant super-integrand which needs to be integrated over a super-moduli space with boundary. The shape of this boundary can be determined by requiring that the bosonic moduli of the corresponding light-cone super-diagram are pure complex numbers with no nilpotent parts, which guarantees that the light-cone supersheet and non-supersheet formalisms give equivalent amplitudes.\(^7\)

Another approach to calculating superstring amplitudes is to use the Green-Schwarz formalism, which is manifestly spacetime-supersymmetric and requires no sum over spin structures. Until recently, the only method available for calculating Green-Schwarz superstring amplitudes was the light-cone method,\(^10,11,12\) which is manifestly unitary but requires non-trivial operators at the interaction points of the light-cone diagram (note that the semi-light-cone method\(^13\) requires the same non-trivial operators when \(\partial_z x^{9+0} = 0\), so it
contains no advantages over the light-cone method\textsuperscript{14}). Because of complications caused by these interaction-point operators, the only manifestly Lorentz-invariant scattering amplitudes that have been calculated using the light-cone Green-Schwarz method have been the tree and one-loop amplitude with four external massless particles. Furthermore, proofs of finiteness and Lorentz invariance using the light-cone Green-Schwarz method are complicated by the presence of non-physical singularities when two or more interaction-point operators collide\textsuperscript{2,3,12}. Although these non-physical singularities can be removed by introducing higher-order contact terms, it has not yet been proven that the resulting scattering amplitudes are Lorentz invariant.\footnote{Restuccia and Taylor have claimed that the heterotic superstring amplitudes are finite and Lorentz-invariant after introducing only fourth-order contact terms\textsuperscript{12}, however their claim is based on mistakenly analyzing the non-physical singularities as a function of the distance between nearby interaction points, \(\nu\), rather than as a function of the Lorentz-invariant positions of the punctures, \(z_r\). Since \(\nu\) is proportional to \(\sqrt{z_r - y}\) for some \(y\) as \(\nu \to 0\textsuperscript{15}\), the behavior \(\nu^{-3} \bar{\nu} d^2 \nu\) of equation (4.60a) in reference 12 becomes \((z_r - y)^{-2} d^2 z_r\), which has the expected holomorphic divergence (when interaction points collide, the right-moving sector of the heterotic superstring has the same regular behavior as the bosonic string). Note that the manifestly Lorentz-invariant amplitudes in equation (3.52) of reference 12 require contact-term contributions similar to those mentioned in Section 3.10 of reference 12.}

An indirect way of proving Lorentz invariance of the light-cone Green-Schwarz formalism is to prove equivalence with the light-cone Neveu-Schwarz-Ramond formalism (after summing over spin structures), which has already been proven equivalent to the manifestly covariant Neveu-Schwarz-Ramond formalism\textsuperscript{9} (after imposing the previously mentioned non-holomorphic condition near divergent points in moduli space). This indirect link also proves finiteness of the covariant Neveu-Schwarz-Ramond formalism, since the only dangerous divergence in the covariant Neveu-Schwarz-Ramond amplitudes comes from the dilaton tadpole,\textsuperscript{7} which is easily seen to vanish in the light-cone Green-Schwarz formalism (after assuming Lorentz invariance).\textsuperscript{12,16} Although it is not difficult to prove the equivalence of the light-cone Green-Schwarz and light-cone Neveu-Schwarz-Ramond formalisms (the three-string vertex is the same in the two formalisms,\textsuperscript{11} and summing over spin structures in the light-cone Neveu-Schwarz-Ramond formalism correctly performs the GSO projection\textsuperscript{3}), it would be preferable to have a more direct proof that Lorentz-covariant superstring amplitudes are finite.

Recently, a new method has been developed for calculating Green-Schwarz superstring amplitudes that does not require light-cone gauge fixing, and therefore does not contain the problems of light-cone interaction-point operators.\textsuperscript{17} Although this formalism is not manifestly Lorentz invariant, it is straightforward to construct covariant vertex operators and the full set of SO(9,1) Lorentz generators, and to write manifestly Lorentz-invariant expressions for scattering amplitudes involving an arbitrary number of loops and external massless particles.\textsuperscript{18}
In this paper, these manifestly Lorentz-invariant Green-Schwarz amplitudes (for external massless bosons) will be proven to be finite by explicitly checking that all possible divergences are absent. By parameterizing the Riemann surface with light-cone moduli, it will then be proven that these superstring amplitudes agree with amplitudes obtained using the light-cone Green-Schwarz formalism. This proves that the finite Lorentz-invariant Green-Schwarz scattering amplitudes are unitary and that the light-cone Green-Schwarz amplitudes are Lorentz invariant. Note that this proof of equivalence with the light-cone formalism will not rely on any assumptions such as those made in reference 17.

Although only the Type IIB Green-Schwarz superstring will be discussed in this paper, it should be straightforward to generalize these results to all other types of Green-Schwarz superstrings.

II. Lorentz Covariant Green-Schwarz Superstring Amplitudes

This section will summarize the results of references 17 and 18 in which Lorentz-covariant Green-Schwarz superstring amplitudes were calculated.

The free matter fields on a Euclidean worldsheet needed to describe the Type IIB Green-Schwarz superstring consist of ten real spin 0 bosons, $x^\mu$ ($\mu = 0$ to 9), four pairs of right-moving spin $\frac{1}{2}$ fermions, $\Gamma^+ l$ and $\Gamma^- l$ ($l = 1$ to 4), two pairs of right-moving spin 0 and spin 1 fermions, $\psi^\pm$ and $\bar{\psi}^\pm$, one pair of right-moving spin 0 bosons, $h^+$ and $h^-$; and their complex conjugates, four pairs of left-moving spin $\frac{1}{2}$ fermions, $\bar{\Gamma}^+ l$ and $\bar{\Gamma}^- l$, two pairs of left-moving spin 0 and spin 1 fermions, $\bar{\psi}^\pm$ and $\bar{\psi}^\mp$, and one pair of left-moving spin 0 bosons, $\bar{h}^+ \text{ and } \bar{h}^-$. The chiral bosons, $h^\pm \text{ and } \bar{h}^\pm$, all have screening charge $-1$ and take values on a circle of radius 1.

The action for these free matter fields contains an $\mathbb{N}=\left(2,2\right)$ superconformal invariance which can be used to gauge-fix $x^0$, $x^9$, and all $\psi$’s, $\varepsilon$’s, and $h$’s, leaving only the usual light-cone Green-Schwarz fields of eight $x$’s, eight $\Gamma$’s, and eight $\bar{\Gamma}$’s. The generators for the right-moving $\mathbb{N}=2$ superconformal transformations are:

$$T = \Gamma^+ \bar{\Gamma}^- - \partial_z h^+ + \partial_{\bar{z}} h^-,$$
$$G_+ = \partial_z x^\mu \Gamma^+ \bar{\Gamma}^- l + (\varepsilon^+ + \frac{1}{2} \psi^+ \partial_z x^9) e^{-h^-},$$
$$G_- = \partial_{\bar{z}} x^\mu \Gamma^+ \bar{\Gamma}^- l + (\varepsilon^- + \frac{1}{2} \psi^- \partial_{\bar{z}} x^9) e^{-h^+},$$
$$\left(\varepsilon^+ + \frac{1}{2} \partial_z x^9 \psi^-\right) \left((\partial_z x^9 + \frac{1}{2} \psi^+ \partial_z \psi^- + \frac{1}{2} \psi^- \partial_z \psi^+) e^{h^+} + e^{-h^-}\right) - e^{h^+} \left((\partial_z h^+ + \partial_{\bar{z}} h^-) \partial_{\bar{z}} \psi^- + \frac{3}{4} \partial^2 \psi^-\right),$$
$$\left(\varepsilon^- + \frac{1}{2} \partial_z x^9 \psi^+\right) \left((\partial_z x^9 + \frac{1}{2} \psi^+ \partial_z \psi^- + \frac{1}{2} \psi^- \partial_z \psi^+) e^{h^+} + e^{-h^-}\right) - e^{h^+} \left((\partial_z h^- + \partial_{\bar{z}} h^+) \partial_z \psi^- + \frac{3}{4} \partial^2 \psi^-\right),$$
$$L = \partial_z x^{i+} \partial_{\bar{z}} x^{-l} - \frac{1}{2} \left(\Gamma^+ \bar{\Gamma}^- l + \Gamma^- \bar{\Gamma}^+ l\right) \Bigl(\partial_{i} \partial_{\bar{j}} - \delta_{i j}\Bigr) \partial_z \partial_{\bar{z}} \psi^- - \partial_{\bar{z}} \partial_z \psi^+ + \partial_z h^+ \partial_{\bar{z}} h^- + \frac{1}{2} \partial_{i} \partial_{\bar{j}} h^+ + \partial_{i} \partial_{\bar{j}} h^-\Biggr),$$

where $x^{i+} \equiv x^i + i x^{i+4}$, $x^{-l} \equiv x^l - i x^{l+4}$, and $x^9 = \pm x^9$.

The $\mathbb{N}=\left(2,2\right)$ ghost and anti-ghost fields coming from gauge-fixing these invariances consist of a pair of right-moving fermions of spin $-1$ and $+2$, $c$ and $\bar{c}$, two pairs of right-moving bosons of spin $-\frac{3}{2}$ and $+\frac{3}{2}$, $\gamma^\pm$ and $\beta^\pm$, a pair of right-moving fermions of spin 0 and 1, $u$ and $\bar{u}$; and their left-moving complex conjugates, $\bar{c}$, $\bar{b}$, $\bar{\gamma}^\pm$, $\bar{\beta}^\pm$, $\bar{u}$, and $\bar{\bar{u}}$. Since the central charge contribution of the matter fields cancels the contribution of the
ghost fields, a nilpotent BRST charge $Q$ can be constructed in the usual way out of the N=2 stress-energy
tensor and the ghosts.\textsuperscript{20}

It is convenient to bosonize the bosonic ghosts,\textsuperscript{4}

$$
\gamma^+ = e^{\phi^+} \eta^+, \quad \beta^- = e^{-\phi^+} \partial_z \xi^-, \quad \gamma^- = e^{\phi^-} \eta^-, \quad \beta^+ = e^{-\phi^-} \partial_z \xi^+,
$$

where $\eta^+$ and $\xi^\pm$ are a pair of spin 1 and spin 0 fermions, and $\phi^\pm$ are two scalar bosons of screening charge
+2 with negative energy (i.e., $\partial_y \phi^+(y) \partial_z \phi^+(z) \rightarrow -(y-z)^{-2}$).

Picture-changing operators, $F^\pm$, can then be defined in the following way:

$$
F^+ \equiv [Q, \xi^+] = e^{\phi^+} [G^\text{matter} + (b - \frac{1}{2} \partial_z v) \gamma^+ - v \partial_z \gamma^+] + c \partial_z \xi^+,
$$

$$
F^- \equiv [Q, \xi^-] = e^{\phi^+} [G^\text{matter} + (b + \frac{1}{2} \partial_z v) \gamma^- + v \partial_z \gamma^-] + c \partial_z \xi^-,
$$

where $G^\text{matter}$ is defined in equation (II.1).

Instanton-number changing operators, $I$ and $I^{-1}$, can also be defined as

$$
I = e^{\int [Q, v]} = e^{\epsilon_{\mu
\nu} \Gamma^{++} \Gamma^{+\mp} \Gamma^{++\mp} \Gamma^{+-} \Gamma^{++} e^{h^\pm - h^\mp + \phi^+ - \phi^- + ev},
$$

$$
I^{-1} = e^{-\int [Q, v]} = \epsilon_{\mu
\nu} \Gamma^{-\mu} \Gamma^{-\nu} \Gamma^{-p} e^{h^\mp - h^\pm + \phi^+ - \phi^- - ev}.
$$

Just as the ghost-number of an operator is defined by commuting with the ghost charge, $\int dz (eb + uv +
\partial_z \phi^+ + \partial_z \phi^-)$, the instanton-number of an operator is defined by commuting with the instanton charge,
$\frac{1}{2} \int dz (\epsilon^+ \psi^- - \epsilon^- \psi^+ + \partial_z h^- - \partial_z h^+)$). It is easily checked that $F^\pm$ and $I^\pm$ are in the BRST cohomology,
but $\partial_z F^\pm$ and $\partial_z I^\pm$ are BRST trivial.

In order to prove Lorentz invariance, one needs to construct a generalization of the SO(9,1) generators,
$m^{\mu\nu} = \int dz (x^\mu \partial_z x^\nu) - \int d\bar{z} (x^\mu \partial_{\bar{z}} x^\nu)$, which is BRST invariant. This generalization is

$$
M^{\mu\nu} = \int dz \{G^+ [G^-, x^\mu, x^n]\} - \int d\bar{z} \{\bar{G}^+ [\bar{G}^-, \bar{x}^\mu, \bar{x}^n]\},
$$

where $x^\mu_{\pm} = x^\mu - \frac{1}{2} \psi^\pm \psi^-$, $x^\mu_{\pm} = x^\mu + \epsilon^\pm h^\pm + \epsilon^\mp h^\mp$, $x^\pm_{\mp} = x^\pm - \epsilon^\pm \psi^\mp \Gamma^\pm_{\mp}$, $x^\mp_{\pm} = x^\mp - \epsilon^\pm \psi^\mp \Gamma^\pm_{\pm}$,

$$
\begin{align*}
x^\pm_{\pm} &= x^\pm + \frac{1}{2} \psi^\pm \psi^-,
\end{align*}
$$

and $G^\pm$ is defined in equation (II.1) (note that $\{G^+, x^\mu\} = \{G^-, x^\mu\} = 0$). It is straightforward to check
that $M^{\mu\nu}$ generates an SO(9,1) algebra, that $x^\mu_{\pm \pm} \equiv x^\mu_{\pm} + \bar{x}^\mu_{\pm} - x^\mu_{\mp}$ and $x^\mu_{\pm \mp} \equiv x^\mu_{\pm} + \bar{x}^\mu_{\mp} - x^\mu_{\mp}$ transform as
SO(9,1) vectors, and that $\psi^+, \psi^-, \psi^\mp, \psi\overline{\psi}$ transform as one component of sixteen-component SO(9,1) Weyl
spinors, $\theta^a_\alpha$, $\bar{\theta}_\alpha^a$, $\bar{\theta}_\alpha^a$, $\bar{\theta}_\alpha^a$ (see reference 18 for their explicit form).

The covariant vertex operators with ghost-number $(-1, -1)$ and instanton-number $(m, \bar{m})$ for the massless particles are:

$$
V_{m\bar{m}}(z, \bar{z}) = p_{AB} \quad e^{-\phi^+ + \phi^- + \bar{\phi}^+ + \bar{\phi}^-} \quad V^A_{m\bar{m}} e^{-ik_{\mu} x^\mu},
$$

(II.6)
where $A$ and $B$ are either SO(9,1) vector-indices $\mu$ or sixteen-component SO(9,1) Weyl spinors $\alpha$, $p_{AB}$ is the polarization tensor satisfying $k^\mu p_{\mu B} = k^\mu p_{A\mu} = 0$, and $V_0^\mu = \theta_{\alpha}^{\beta} \epsilon_{\alpha\beta} x^\nu$, $V_1^\mu = \theta_{\alpha}^{\beta} \epsilon_{\beta} x^\nu$.

On a genus $g$ surface with period matrix $\tau$ and zero instanton number, the scattering amplitude of $N$ external massless states is:

$$A = \prod_{i=1}^{3g-3+N} \int dm_i^T \prod_{j=1}^g \int dm_j^{U(1)} |^2$$

$$\int (\prod_{\mu=0}^9 D x^\mu)(\prod_{l=0}^4 D \gamma^l D \gamma^{-l}) D \psi^+ D \psi^- D \varepsilon^+ D \varepsilon^- D h^+ D h^- D c D b D \gamma^+ D \gamma^- D \beta^+ D \beta^-|^2$$

$$| \prod_{i=1}^{3g-3+N} \int d^2 y_i (M^i(y_i) b(y_i)) (\prod_{k=1}^{2g-2+N} F^+(w_k^+) F^-(w_k^-)) Z_1(\tau) |^2 e^{-S} \prod_{r=1}^N V_m, m_r(z_r, \bar{z_r})$$

where $m_i^T$ and $M^i$ are the usual bosonic Teichmüller parameters and their Beltrami differentials, $m_j^{U(1)}$ are the $U(1)$ moduli which range over the Jacobian variety $C^g / (Z^g + \tau Z^g)$ and measure the change in phase $e^{im_j^{U(1)}}$ when a field of $U(1)$ charge $r$ goes around the $j^{th}$ $b$-cycle of the surface, $F^\pm$ are the picture-changing operators defined in equation (II.3) which come from integrating over the fermionic moduli, $w_k^\pm$ are arbitrary points on the surface chosen independently of the bosonic moduli, the term $Z_1(\tau)$ comes from performing the path integral over the $(u, v)$ ghosts (this path integral is trivial after inserting a zero-mode for $u$, since no $u$ fields appear in $F^\pm$ or $V$), the vertex operators are chosen such that $\sum_{r=1}^n m_r = \sum_{r=1}^n \bar{m}_r = 0$, and $S$ is the Wick-rotated free-field action.

The path integrals over the fermions can be performed by bosonizing and using the formula:

$$\int D(e^\phi) D(e^{-\phi}) e^{-S(\phi)} \prod_{i=1}^n \exp(c_i \phi(z_i)) = Z(\sum_{i=1}^n c_i z_i; q, r, \tau)$$

$$= \delta_{q(\bar{q}-1), \Sigma c_i} \prod_{i<j} E(z_i, z_j)^{c_i c_j} \prod_{i=1}^n \sigma(z_i)^{c_i} (Z_1(\tau))^{-\frac{1}{2}} \Theta(\sum_{i=1}^n c_i z_i - q\Delta) - r m^{U(1)}(\tau),$$

where $q$ and $r$ are the screening charge and the $U(1)$ charge for $e^\phi$, $\Delta$ is the Riemann class, $\tau$ is the period matrix of the surface and is a complex symmetric $g \times g$ matrix with positive-definite imaginary part, $Z_1(\tau)$ is a normalization for $Z$ such that $Z(\sum_{i=1}^n z_i - y_1; 1, 0; \tau) = Z_1(\tau) \det \omega_i(z_j)$, $\omega_i$ are the $g$ canonical holomorphic one-forms satisfying $\int_{a_j} \omega_i = \delta_{i,j}$ and $\int_{b_j} \omega_i = \tau_{i,j}$. $E(y, z)$ is the prime form, $\frac{E(y, z)}{E(z, y)} |_{(y, z)}$ for arbitrary $p_i$ (the final amplitudes will contain equal powers of $\sigma$ in the numerator and denominator because of the vanishing conformal anomaly), $[\sum_{i=1}^n (y_i - z_i)]_j = \sum_{i=1}^n \int_{z_i}^{y_i} \omega_j$ is an element in the Jacobian variety $C^g / (Z^g + \tau Z^g)$, and

$$\Theta(z, \tau) \equiv \sum_{n \in \mathbb{Z}^g} \exp(i \pi n_j \tau_{jk} n_k + 2 \pi i n_j z_j)$$

which satisfies $\Theta(z + \tau n + m, \tau) = \exp(-i \pi n_j \tau_{jk} n_k - 2 \pi i n_j z_j) \Theta(z, \tau)$ for $z \in C^g$ and $n, m \in \mathbb{Z}^g$. For a brief but sufficient review of these objects, see Chapter 3 of reference 21.
The formulas for the path integrals over the bosonic fields are:

\[
\int D\phi e^{-S(\phi)} \prod_{i=1}^{\infty} \exp(i\phi_\mu x_\mu(z_i)) = \delta(\sum_{j=1}^{n} x_j)(\det \text{Im } \tau)^{-\frac{1}{2}}|Z_1(\tau)|^{-\frac{1}{2}} \prod_{j<k} F(z_j, z_k)^{p_j p_k}, \tag{II.10}
\]

where \( F(y, z) = \exp(-2\pi \text{Im } [y - z]) \text{Im } |y - z)|E(y, z)|^2, \]

\[
\int D\phi^+ D\phi^- e^{-S(\phi^+, \phi^-)} \prod_{i=1}^{m} \xi^+(x_i) \prod_{j=1}^{n} \eta^-(y_j) \prod_{k=1}^{p} \exp(c_k \phi^-(z_k)) = \tag{II.11}
\]

\[
\prod_{i=1}^{n} Z(-yi + \sum_{j=0}^{m} x_i - \sum_{j=1}^{n} y_j + \sum_{k=1}^{p} c_k z_k ; 2, -1; \tau) \prod_{i=1}^{n} Z(-xi + \sum_{j=0}^{m} y_i - \sum_{j=1}^{n} y_j + \sum_{k=1}^{p} c_k z_k ; 2, -1; \tau),
\]

where the location of the \( \xi^+ \) zero-mode, \( x_0 \), can be chosen anywhere on the surface, and the constraint imposed on the \( U(1) \) moduli can be understood as coming from the global constraint, \( \Omega \), of equation (II.4) in reference 17. Note that both \( (\beta^\pm, \gamma^\mp) \) and \( h^\pm \) contain fields with negative energy (i.e., \( \partial_\rho \phi^\pm(y) \partial_\sigma \phi^\pm(z) \) and \( \frac{1}{2}(\partial_\rho h^+(y) - \partial_\rho h^-(y))(\partial_\sigma h^+(z) - \partial_\sigma h^-(z)) \) go like \(- (y - z)^2 \) as \( y \to z \), which can cause unphysical poles in the path integrals. For the \( (\beta^\pm, \gamma^\mp) \) path integral, the residues of these poles are BRST trivial; for the \( (h^+, h^-) \) path integral, these unphysical poles are avoided by the \( \delta \)-function restriction on the \( U(1) \) moduli.

In order to make the Lorentz invariance of the amplitude manifest as a function of the momenta \( k^\mu \) (\( r = 1 \) to \( N \)) and the polarizations \( p^\nu_{AB} \), it is convenient to introduce a null real \( SO(9,1) \) vector, \( m^\mu \), and a pure complex \( SO(9,1) \) spinor, \( v^\alpha \), satisfying \( v^{\alpha} \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\epsilon v^\beta = \gamma^\epsilon \gamma^\alpha \gamma^\beta \gamma^\sigma \gamma^\nu \gamma^\mu v^\beta = m^\mu m^\nu = 0 \) and \( v^{\alpha} \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\epsilon m^\mu = 1 \). By using this vector and spinor to break \( SO(9,1) \) down to \( SU(4) \) (e.g. \( k^{\sigma \nu} = (v^\gamma \bar{v}^\rho) k^{\rho \sigma} \)), the amplitude \( A \) can be written as a manifestly Lorentz-covariant function of \( m^\mu, v^\alpha, \bar{v}^\alpha \), and \( p^\nu_{AB} \), where \( A \) is a polynomial in \( m^\mu, v^\alpha, \bar{v}^\alpha \), and \( \bar{v}^\alpha \) (note that \( m^\mu, v^\alpha, \) and \( \bar{v}^\alpha \) appear only in the nilpotent part of \( k^\mu x^\mu \), since the non-nilpotent part of \( x^\mu \) is simply \( x^\mu \)).

Since \( A \) is guaranteed to also be a Lorentz-covariant function of just the \( k^\mu \)'s and \( p^\nu_{AB} \)'s (\( M^{\mu \nu} \) is BRST invariant and the vertex operators transform covariantly), all monomials of \( m^\mu, v^\alpha, \) and \( \bar{v}^\alpha \) can be replaced by their Lorentz-invariant component. For example, \( m^\mu v^\alpha \bar{v}^\beta \to \frac{1}{1600} (\gamma^\mu \gamma^\alpha \gamma^\beta \gamma^\epsilon), \) and \( m^\mu m^\nu v^\alpha \bar{v}^\beta \bar{v}^\gamma \bar{v}^\delta \to \frac{1}{31360} (\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\rho \gamma^\beta \gamma^\epsilon \gamma^\delta + \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\rho \gamma^\beta \gamma^\sigma \gamma^\epsilon \gamma^\delta + \gamma^\mu \gamma^\beta \gamma^\nu \gamma^\alpha \gamma^\gamma \gamma^\epsilon \gamma^\delta + \gamma^\mu \gamma^\beta \gamma^\nu \gamma^\alpha \gamma^\gamma \gamma^\delta + \gamma^\mu \gamma^\beta \gamma^\nu \gamma^\alpha \gamma^\gamma \gamma^\delta - \gamma^\nu \gamma^\beta \gamma^\nu \gamma^\alpha \gamma^\gamma \gamma^\delta - \gamma^\mu \gamma^\beta \gamma^\nu \gamma^\alpha \gamma^\gamma \gamma^\delta)). \) In this way, the amplitude \( A \) can be written as a manifestly Lorentz-invariant function of just the \( k^\mu \)'s and \( p^\nu_{AB} \)'s.
III. Finiteness

In order to check for possible divergences in $A$ of equation (II.7), it is convenient to choose the “tree-with-tadpoles” parameterization for the genus $g$ Riemann surface with $N$ external states. This parameterization consists of sewing together $2g - 2 + N$ spheres, $S_i$, each having three punctures, $P_{i,r}$ ($i = 1$ to $2g - 2 + N$, $r = 1$ to 3).

First sew $P_{i,3}$ to $P_{i+1,1}$ for $i = 1$ to $g - 1$ where the radius of the sewed puncture is $B_i$ ($B_i$ is a complex number whose phase rotates one puncture with respect to the other). Next sew $P_{i+g-1,3}$ to $P_{i+g,1}$ for $i = 1$ to $N - 1$ where the radius of the sewed puncture is $L_i$. Now sew $P_{i+g+N,3}$ to $P_{i+1,2}$ for $i = 1$ to $g - 2$ where the radius of the puncture is $H_i$, and glue $P_{g+N-1,3}$ to $P_{g+N,1}$ where the radius of the puncture is $H_{g-1}$. At this point, spheres $S_1...S_{2+N}$ are sewn together, and sphere $S_{i+g+N}$ is sewn to sphere $S_{i+1}$ for $i = 1$ to $g - 2$. Finally, sew $P_{i+g+N,1}$ to $P_{i+g+N,2}$ with radius $K_i$ for $i = 1$ to $g - 2$, sew $P_{g+N,2}$ to $P_{g+N,3}$ with radius $K_{g-1}$, sew $P_{1,1}$ to $P_{1,2}$ with radius $K_g$, and insert the $N$ external states on the remaining unsewed punctures $P_{g,2}...P_{g+N-1,2}$.

The $3g - 3 + N$ complex bosonic Teichmuller parameters for this parameterization are given by the complex radii, $B_i$ ($i = 1$ to $g - 1$), $L_i$ ($i = 1$ to $N - 1$), $H_i$ ($i = 1$ to $g - 1$), and $K_i$ ($i = 1$ to $g$). For this choice of moduli, the contribution from the Beltrami differentials is

$$| \prod_{i=1}^{3g-3+N} \int_{C_i} \frac{y_i b(y_i)}{R_i} dy_i |^2, \quad (III.1)$$

where $C_i$ is a closed loop surrounding the sewed punctures that have the radius $R_i$, and $y_i = 0$ at the center of these sewed punctures. The locations of the picture-changing operators will be chosen such that one $F^+, F^-, \bar{F}^+$, and $\bar{F}^-$ sits on each of the $2g - 2 + N$ spheres, but are otherwise arbitrary. Choose the extra zero mode of the $\xi^\pm$ and $\bar{\xi}^\pm$ fields to sit on $S_{g+N-1}$.

Since the path integral over the $h^\pm$ fields determines the values of the U(1) moduli $m_j^{U(1)}$, the only possible divergence in $A$ can arise at limiting points of the radii $B_i$, $L_i$, $H_i$, and $K_i$.† Because of modular invariance (as was shown in Section IV.A. of reference 17, the integrand of $A$ is independent of the spin-structure chosen for the fields of $\frac{1}{2}$-integer conformal weight), one only needs to check for divergences when the radii approach zero.24

† The non-physical poles coming from the zeroes of the $\Theta$-functions in equation (II.11) have residues which are total derivatives in the Teichmuller parameters (this is clear since by shifting the picture-changing operators by a BRST trivial quantity, $\{Q, \xi^\pm\}$, which changes the integrand of $A$ by a total derivative, the locations of these non-physical poles can be altered).5 So if the amplitude is finite near the limiting points of the radii, there is no need to introduce a cutoff for the Teichmuller parameters, the moduli space has no boundary, and these poles are harmless.
When $K_i \to 0$, which corresponds to the $i^{th}$ $a$-cycle on the surface being shrunk to zero, $\tau_{ii} \to \frac{1}{\pi \log K_i}$ and $\Delta_i \to -\frac{1}{2}\tau_{ii}$. From the definition of the $\Theta$-function in equation (III.9) and from the fact that $m_i^{U(1)} = [\sum_{j=1}^{n} c_{ij} z_j + \Delta_j]$, one finds that $Z(q, r; \tau)$ diverges like $(K_i^{\frac{-1}{2}(q+r-1)} + 1)$ and that $Im \tau$ diverges like $\log |K_i|$. So the path integral over $(b, c)$ diverges like $K_i^{-1}$, the path integral over $(\beta^-, \gamma^+)$ converges like $K_i$, the ten path integrals over $x^\mu$ each converge like $(\log |K_i|)^{-\frac{1}{2}}$, and all other path integrals are regular. After combining with the $|K_i|^{-2}$ dependence from the Beltrami differentials, one finds that

$$A \to (\log |K_i|)^{-5}|K_i|^{-2}d^2K_i,$$

(III.2)

which is not divergent as $K_i \to 0$ (for $y = (\log |K_i|)^{-1}$, $A \to y^3dy$ as $y \to 0$).

When any other radius $R$ is shrunk to zero, the genus $g$ surface with $N$ punctures degenerates into a genus $g_1$ surface with $N_1$ punctures and a genus $g_2$ surface with $N_2$ punctures. These two surfaces, $G_1$ and $G_2$, have period-matrices, $\tau_1$ and $\tau_2$, such that the original period matrix, $\tau$, decomposes into the direct-sum of $\tau_1$ and $\tau_2$. As was shown in reference 21, this implies that

$$E(x, y) \to R^{-\frac{3}{2}}E_1(x, p_1)E_2(y, p_2), \quad \sigma(x) \to R^{\frac{3}{2}g_2} \sigma_1(x)\sigma_2(p_2)^{g_2} E_1(x, p_1)^{-g_2},$$

(III.3)

$$Z(\sum_{i=1}^{m} c_i x_i + \sum_{j=1}^{n} d_j y_j; q, r; \tau) \to R^{-\frac{3}{2}q_1 q_2} Z(\sum_{i=1}^{m} c_i x_i - q_1 p_1 ; q, r; \tau) Z(\sum_{j=1}^{n} d_j y_j - q_2 p_2 ; q, r; \tau),$$

where $x$ is on $G_1$, $y$ is on $G_2$, $p_i$ is the location of the sewed puncture with radius $R$ on $G_i$, $a_i = \frac{q_1 - 1}{g-1}$, $q_1 = \sum_{i=1}^{m} c_i - q(g_1 - 1)$, $q_2 = \sum_{i=1}^{n} d_j - q(g_2 - 1)$, and $q_1 + q_2 = q$.

For a given $R$, choose $G_2$ to be the surface containing $S_{g+N-1}$, and push the loop containing the Beltrami differential for $R$ onto $G_2$. Since there are $3g_1 - 2 + N_1$ $b$ fields coming from Beltrami differentials on $G_1$ and $N_1$ $c$ fields coming from vertex operators on $G_1$, the $(b, c)$ path integral behaves as $R^{\frac{3}{2}n_b(n_b-3)}$, where $n_b - 1$ is the number of $b$’s minus the number of the original $c$’s coming from the $2(2g_1 - 1 + N)$ picture-changing operators $F^\pm$ on $G_1$. Similarly, the $(\beta^+, \gamma^\mp)$ path integrals each behave as $R^{\frac{3}{2}(1-n_{\xi^\pm})}$, where $n_{\xi^\pm}$ is the number of $\xi^\pm$’s minus the number of $\eta^\mp$’s coming from $G_1$. Since $n_{\xi^+} + n_{\xi^-} = 1 - n_b$, the combined path integral behaves as $R^{\frac{3}{2}(n_b-1)^2}$.

The path integral over the $(h^+, h^-)$ fields behaves like $R^{\frac{3}{2}((n_{h^+} + n_{h^-} - 2)(n_{h^+} + n_{h^-}) - (n_{h^+} - n_{h^-})^2)}$, and the path integral over the $(\psi^\pm, \varepsilon^\mp)$ behaves like $R^{\frac{3}{2}((n_{\psi^+} + n_{\psi^-} - 2)(n_{\psi^+} + n_{\psi^-}) + (n_{\psi^+} - n_{\psi^-})^2)}$, where $(n_{h^+} + 1 - g)$ is the number of $e^{h^\pm}$ terms on $G_1$ and $(n_{\psi^+} + 1 - g)$ is the number of $\psi^\pm$’s minus the number of $\varepsilon^\mp$’s on $G_1$. Since all operators on the surface have zero instanton charge (it has been assumed that all vertex operators are boson-boson vertex operators of instanton-number $(0,0)$), $n_{\psi^+} - n_{\psi^-} = n_{h^+} - n_{h^-}$, so the combined path integral behaves like

$$R^{\frac{3}{2}((n_{h^+} + n_{h^-} - 2)(n_{h^+} + n_{h^-}) + (n_{\psi^+} + n_{\psi^-} - 2)(n_{\psi^+} + n_{\psi^-})]}.$$

(III.4)

Finally, the path integrals over the $(\Gamma^{-1}, \Gamma^{+1})$ fields behave like $R^{\frac{3}{2}n_{\tau}}$ where $n_i$ is the number of $\Gamma^{-i}$ fields minus $\Gamma^{+i}$ fields on $G_1$, and the path integrals over the $x^\mu$ fields behave like $|R|^{k^\mu k^\mu}$ where $k^\mu$ is the momentum crossing from $G_1$ to $G_2$. 

9
The only way that the expression in equation (III.4) can diverge is if \( n_{h^+} + n_{h^-} = n_{\psi^+} + n_{\psi^-} = -1 \), in which case it behaves like \( R^{-\frac{5}{2}} \) (note that because of zero instanton charge, \( n_{h^+} + n_{h^-} \) must be odd if \( n_{\psi^+} + n_{\psi^-} \) is odd). However, if \( n_{h^+} + n_{h^-} \) is odd, then \( n_{h^+} - n_{h^-} \) must also be odd, implying by U(1) conservation that either \( n_{\xi^+} - n_{\xi^-} \) is odd (which implies that \( n_{\xi} \) is not equal to one) or at least one of the \( n_i \)'s is non-zero. In either case, the other path integrals behave at worst like \( R^{\frac{3}{2}} \).

So after combining these results with the \( d^2R/|R|^2 \) dependence of the Beltrami differentials, one finds that as \( R \to 0 \),

\[
A \to A_1A_2|R|^{k \mu_k} d^2R, \tag{III.5}
\]

where \( k^\mu = \sum_{r=1}^{N_1} k^\mu_r = -\sum_{r=N_1+1}^N k^\mu_r \), and \( A_1 \) is the amplitude on \( G_1 \) with \( 4(2g-1+1) \) picture-changing operators, \( 3g-2+N_1 \) loops from Beltrami differentials, \( N_1 \) vertex operators, and an operator sitting at the location of the shrunk puncture \( p_1 \) of the form:

\[
|e^{n_{\xi^+}+n_{\xi^-}}(\gamma^-)^{n_{\xi^+}}(\gamma^+)^{n_{\xi^-}}(\varepsilon^-)^{n_{\psi^+}}(\varepsilon^+)^{n_{\psi^-}}e^{-n_{h^+}+h^-}e^{-n_{h^-}+h^-}\prod_{l=1}^4(\varepsilon^{+l})^n|2e^{ik_nu}. \tag{III.6}
\]

If \( R \) is \( L_i \), then \( N_1 = i \), \( g_1 = g - 1 \), and the scattering amplitude has the expected massless pole when \( k^\mu k_\mu = 0 \). If \( R \) is not one of the \( L_i \)'s, then \( N_1 = k^\mu = 0 \), and it will now be shown that this implies \( A_1 \) is zero.

After doing the field redefinition,

\[
[x^{9+0} \to x^{9+0} + \frac{1}{2} \psi^+ \psi^-, \varepsilon^+ \to \varepsilon^+ + \frac{1}{2} \psi^+ \partial_z x^{9-0}, \varepsilon^- \to \varepsilon^- - \frac{1}{2} \psi^- \partial_z x^{9-0}], \tag{III.7}
\]

the picture-changing operators, \( F^\pm \), no longer contain the zero mode of \( \psi^+ \) (note that this redefinition process does not affect the measure factor since it preserves the free-field commutation relations). Similarly, after doing the field redefinition,

\[
[x^{9+0} \to x^{9+0} - \frac{1}{2} \psi^+ \psi^-, \varepsilon^+ \to \varepsilon^+ - \frac{1}{2} \psi^+ \partial_z x^{9-0}, \varepsilon^- \to \varepsilon^- + \frac{1}{2} \psi^- \partial_z x^{9-0}], \tag{III.8}
\]

the picture-changing operators, \( F^\pm \), no longer contain the zero mode of \( \psi^- \). Since \( A_1 \) is zero unless there is a zero mode for \( \psi^+ \) and \( \psi^- \) somewhere on the surface, \( n_{\psi^+} \) and \( n_{\psi^-} \) must both be less than zero (these zero modes of \( \psi^\pm \) are related to spacetime supersymmetry since two of the sixteen right-moving spacetime-supersymmetry generators are \( \int dz(\varepsilon^\pm - \frac{1}{2} \psi^\pm \partial_z x^{9-0}) \)).

Also, since for each term in the picture-changing operators, \( (n_{\psi^+} + n_{\psi^-} - n_{h^+} - n_{h^-}) \) is equal to twice the number of \( \partial_z x^{9-0} \)'s minus twice the number of \( \partial_z x^{9+0} \)'s (which must be zero since there are no vertex operators on \( G_1 \)), the only possible values for the \( n_i \)'s which does not remove the divergence is \( n_{\xi^+} = n_{h^-} - 1 = n_{l} = n_{\psi^+} + 1 = n_{h^+} + 1 = 0 \). So the resulting operator at \( p_1 \) is

\[
|\varepsilon e^{\phi^+ - \phi^-} e^{h^+ + h^-} |^{2}, \tag{III.9}
\]
which can be associated with the target-space dilaton in the \((-1, -1)\) ghost picture since \(e^{h^+ + h^-} \psi^+ \psi^-\) is precisely the matter part of the screening charge that couples to the two-dimensional curvature.

Now suppose \(R\) is either \(B_1\) or one of the \(H_i\)'s, so \(g_1 = 1\). Changing the location of a picture-changing operator, \(F^\pm\), replaces it with the integral of \([Q, \partial_z \xi^\pm]\), and after pulling \(Q\) through the Beltrami differential for \(K_i\) (this total derivative is harmless since it has already been shown that the amplitude is finite near \(K_i = 0\)), \(Q\) is left surrounding the operator at the point \(p_1\). Since

\[
[Q, c e^{-\phi^+ - \phi^-} e^{h^+ + h^-} \psi^+ \psi^-] = c (\eta^+ e^{-\phi^+} e^{h^-} \psi^+ + \eta^- e^{-\phi^-} e^{h^+} \psi^-),
\]

III.10

there are no terms with zero modes of both \(\psi^+\) and \(\psi^-\), and therefore \(A_1\) is independent of the locations of the picture-changing operators. So all of the picture-changing operators can be moved to \(p_1\), forcing the zero modes of \(\psi^\pm\) to be cancelled and \(A_1\) to vanish (the \(\psi^\pm\) path integral becomes proportional to \(\Theta([g_1 - 1]p_1 - \Delta_1]\), which vanishes by the Riemann identity).

To prove \(A_1\) is zero for the other \(B_i\) radii, use precisely the same argument inductively in \(i\) (the total derivative one gets when calculating \(A_1\) near \(B_i = 0\) is harmless once \(A\) has been shown to be finite near \(B_{i-1} = 0\)).

So the Green-Schwarz superstring amplitudes of equation (II.7) have been shown to be divergence-free near the limiting points of the Teichmuller parameters, and are therefore finite. Furthermore, it was proven in reference 17 that these amplitudes satisfy the non-renormalization theorem, i.e. all multiloop amplitudes with fewer than four external massless states vanish (this proof relied on the assumption that total derivatives coming from BRST-trivial operators don’t contribute to scattering amplitudes, which has now been verified since finiteness implies that there is no need to introduce a cutoff in the moduli space).

IV. Unitarity

In this section, it will be proven that the scattering amplitude of equation (II.7) for external massless boson-boson states is equivalent to the light-cone Green-Schwarz amplitude, and is therefore unitary. This proof will not rely on any assumptions such as those made in reference 17 concerning the contributions of the non-light-cone parts of the picture-changing operators.

The first step is to choose the usual light-cone moduli for the surface, \(\tilde{\rho}_a - \tilde{\rho}_1\) for \(a = 2\) to \(2g - 2 + N\) (the complex interaction-point locations), \(A_j\) for \(j = 1\) to \(g\) (the light-cone momenta of the internal loops), and \(B_j\) for \(j = 1\) to \(g\) (the twists when going around a loop).\(^15\) These moduli can be understood as coming from the unique meromorphic one-form, \(\partial_z \rho\), which has poles of residue \(\alpha_r\) at the \(N\) punctures (\(\alpha_r\) is the \(k^{9+0}\) momentum of the \(r^{th}\) external particle) and purely imaginary periods, \(A_j\) and \(B_j\), when integrated around the \(j^{th}\) \(a\)-cycle and \(b\)-cycle.

The locations of the picture-changing operators will be chosen such that \(|F^+ F^-|^2\) sits at each of the \(2g - 2 + N\) zeros of the one-form, i.e. at the \(\tilde{\rho}_a\)’s. Note that since there is no cutoff in the moduli space, there
is no contribution from the total derivatives which arise when changing to this light-cone parameterization of the surface.

With this choice of the moduli, the Beltrami differentials contribute\textsuperscript{26}

\[
| \prod_{a=2}^{2g-2+N} \left( \int_{C_a} d\hat{p}(\rho) - \int_{C_1} d\hat{p}(\rho) \right)^2 \prod_{j=1}^g \left( \int_{a_j} d\hat{p}(\rho) - \int_{b_j} d\hat{p}(\rho) \right) (\int_{b_j} d\hat{p}(\rho) - \int_{a_j} d\hat{p}(\rho)), \tag{IV.1}
\]

where \( \hat{b} \equiv (\partial_2 \rho)^{-2} b \), \( C_a \) is a small circle surrounding \( \tilde{p}_a \), and \( a_j, b_j \) are the \( a \)-cycles and \( b \)-cycles.

The vertex operators for the external boson-boson states will be chosen in light-cone gauge, i.e. \( p_{9-,0,\mu} \) and \( p_{\mu,9-0} \) are gauged to zero. In this gauge, the bosonic vertex operator with polarization in the \( 4^{-\frac{1}{2}} \) direction is \( V_{-\frac{1}{2}} = e^{-\phi^+ - \phi^-} (\psi^- e^{h^+ \Gamma^{-\frac{1}{2}}} - \frac{k^+}{G} \epsilon^{ik\phi_\mu} x^\mu) \), and the bosonic vertex operator with polarization in the \( \bar{4}_{-\frac{1}{2}} \) direction is \( V_{+\frac{1}{2}} = e^{\phi^+ - \phi^-} (\psi^+ e^{h^+ \Gamma^{\frac{1}{2}}} - \frac{k^+}{G} \epsilon^{ik\phi_\mu} x^\mu) \).

Since the picture-changing operators must contribute at least \( 2g - 2 + N \) \( \varepsilon \)'s in order to have a non-zero amplitude, each factor of \( F^+ F^- \) at the interaction points must contribute an average of one more \( \varepsilon \) than \( \psi \). For this to happen, each \( F^+ F^- \) factor must contribute either

\[
f_+ = e^{\phi^+} e^{-h^+} \varepsilon^+ [e^{\phi^-} (\partial_2 x^{-\frac{1}{2}} \Gamma^{\frac{1}{2}} + e^{h^+} \varepsilon^- \partial_2 x^{9+0} + \gamma^+ b) + c \partial_2 \xi^+] \quad \text{or} \quad \tag{IV.2}
\]

\[
f_- = e^{\phi^-} e^{-h^-} \varepsilon^- [e^{\phi^+} (\partial_2 x^{9+0} \Gamma^{\frac{1}{2}} - e^{-h^+} \psi^+ \partial_2 x^{9-0} + \gamma^- b) + c \partial_2 \xi^-],
\]

where normal-ordering needs to be defined for the terms in \( f_\pm \) that involve \( \gamma^{\pm} b \).

Normal-ordering will be defined by \( :F^+ F^-: = \{Q, \xi^+ \{Q, \xi^- \} \} \), which corresponds to first integrating over the fermionic moduli that couple to the fermionic stress-energy tensor \( G^- \), and then integrating over the fermionic moduli that couple to \( G^+ \). With this definition, the term \( :e^{\phi^-} b \gamma^+ (e^{\phi^+} e^{-h^+} \varepsilon^+) \) in \( f_+ \) becomes

\[
\partial_2 (be^{\phi^-} \eta^+) e^{2\phi^+} e^{-h^+} \varepsilon^+ + \frac{1}{2} be^{\phi^-} \eta^+ e^{-h^+} \varepsilon^+ \partial_2 (e^{2\phi^+}), \tag{IV.3}
\]

while the term \( :e^{\phi^-} e^{-h^-} \varepsilon^- (e^{\phi^+} b \gamma^-) \) in \( f_- \) becomes

\[
\partial_2 (e^{-h^-} \varepsilon^-) e^{2\phi^+} b \eta^- e^{\phi^+} + \frac{1}{2} be^{\phi^+} \eta^- e^{-h^-} \varepsilon^- \partial_2 (e^{2\phi^+}). \tag{IV.4}
\]

Note that each \( \partial_2 x^{9+0} \) factor in \( f_+ \) must be balanced with a \( \partial_2 x^{9-0} \) factor in \( f_- \) since \( \partial_2 \rho |_{\tilde{p}_a} = 0 \) implies that contracting \( \partial_2 x^{9+0} \) with the external momentum factors, \( e^{i\alpha \cdot k^{\pm} - \varepsilon} \), gives zero. Also, all \( \psi \) ghosts can be safely ignored since there are no \( u \) ghosts to cancel them, and all \( \psi^+ \psi^- \) factors coming from the vertex operators can be ignored since there are no extra \( \varepsilon \) factors to absorb them.

It is convenient to attach an instanton-number-changing operator, \( I^{-1} \), to each of the \( m \) vertex operators of polarization \( \bar{4}_{-\frac{1}{2}} \), and to attach \( g - 1 + m \) instanton-number-changing operators, \( I^{+1} \), to the interaction-point operators of type \( f_- \) (since the integrand of the amplitude, rather than just the integral, is independent of the locations of the \( I \)'s, there is no problem with moving the \( I \)'s to different locations for different splittings of the \( 2g - 2 + N \) \( F^+ F^- \)'s into \( f_+ \)'s and \( f_- \)'s). As was discussed in reference 17, this addition of a total
instanton number of \( g - 1 \) shifts the conformal weights of all U(1) transforming fields by half of their U(1) charge, so the fields \([\Gamma^+, \Gamma^-, \beta^+, \beta^-, \gamma^+, \gamma^-, e^+, e^-]\) now have conformal weights \([1, 0, 2, 1, 0, -1, 1, 0]\).

It is easy to check that in order to get a non-zero amplitude, there must be precisely \((g - 1 + m)\) \( f_-\) operators. So since \( h^-\) no longer appears anywhere in the integrand, the path integral over the \((h^+, h^-)\) fields is trivial, giving a factor of \((Z_1(\tau))^{-1}\) that cancels the \(Z_1(\tau)\) factor coming from the path integral over the \((u, v)\) ghosts.

After combining with the Beltrami differential contribution at \( \tilde{\rho}_a \) from the loop \( C_a \), the operators at the interaction points become:

\[
\int_{C_a} d\tilde{\rho} I f_+ = \left(\frac{\partial^2 \rho}{dz^2}\right)^{-1} \int d\tilde{\kappa}_a \exp \left[ \lim_{\rho \to \tilde{\rho}_a} \left( \frac{\partial^2 \rho}{dz^2}\right) - \frac{1}{2} \left(\frac{\partial^2 \rho}{dz^2}\right)^{-1} \partial \rho \right] \frac{d\tilde{\rho}}{dz} \exp \left( \int \frac{d\rho}{dz^2}\right) \frac{d\tilde{\rho}}{dz} \exp \left( \int \frac{d\rho}{dz^2}\right)
\]

\[
= \frac{d\tilde{\rho} f_+}{dz^2} = \left(\frac{\partial^2 \rho}{dz^2}\right)^{-1} \int d\tilde{\kappa}_a \exp \left[ \lim_{\rho \to \tilde{\rho}_a} \left( \frac{\partial^2 \rho}{dz^2}\right) - \frac{1}{2} \left(\frac{\partial^2 \rho}{dz^2}\right)^{-1} \partial \rho \right] \frac{d\tilde{\rho}}{dz} \exp \left( \int \frac{d\rho}{dz^2}\right) \frac{d\tilde{\rho}}{dz} \exp \left( \int \frac{d\rho}{dz^2}\right) \left( \begin{array}{c}
\phi^+ \\
\phi^-
\end{array} \right)
\]

\[
= \left(\frac{\partial^2 \rho}{dz^2}\right)^{-1} \left(\frac{\partial^2 \rho}{dz^2}\right) \int d\tilde{\kappa}_a \exp \left[ \lim_{\rho \to \tilde{\rho}_a} \left( \frac{\partial^2 \rho}{dz^2}\right) - \frac{1}{2} \left(\frac{\partial^2 \rho}{dz^2}\right)^{-1} \partial \rho \right] \frac{d\tilde{\rho}}{dz} \exp \left( \int \frac{d\rho}{dz^2}\right) \frac{d\tilde{\rho}}{dz} \exp \left( \int \frac{d\rho}{dz^2}\right) \left( \begin{array}{c}
\phi^+ \\
\phi^-
\end{array} \right)
\]

Note that the \( \tilde{\beta}^+ \partial \rho \tilde{c} \) term in \( f_+ \) can be dropped since there is no \( \gamma^- \) term in \( f_- \) to absorb it. After performing the shift of variables, \( x^{9+0} \rightarrow x^{9+0} - \sum_{a=1}^N k_{9+0} n(z, \tau) \), where \( n \) is the Neumann function on the surface, the \( \exp(ik_{9+0} x^{9+0} + ik_{9-0} x^{9+0}) \) factors in the vertex operators are replaced by \( \exp(\sum_{a=1}^N k_{9-0} x^{9+0}) \). This shift leaves the interaction-point operators unchanged since \( \partial \beta \partial x^{9+0} \) is zero and there are no extra \( \varepsilon^- \)s available to absorb a \( \psi^\prime \) that is unaccompanied by a \( \partial x^{9+0} \) (i.e., all extra \( \varepsilon^- \)s are accompanied by \( \partial x^{9+0} \)).

Once the interaction-point operators and vertex operators are in this form, it can easily be shown that the path integrals over the non-light-cone matter fields precisely cancel the path integrals over the ghost fields. This is done by pairing each non-light-cone matter field with a ghost field in the following way:

\[
(x^{9-0}, \tilde{c} + \tilde{\gamma}), \quad (\partial \rho x^{9+0}, \tilde{b}), \quad (\partial \rho x^{9+0}, \tilde{b}), \quad (\psi^+, \tilde{\beta}^-), \quad (\psi^-, \tilde{\beta}^+), \quad (\varepsilon^+, \tilde{\gamma}^-), \quad (\varepsilon^-, \tilde{\gamma}^+)
\]

(this technique can also be used in the Neveu-Schwarz-Ramond formalism, in which case \( \tilde{\Gamma}^{9+0} \) is paired with \( \tilde{\beta} \) and \( \tilde{\Gamma}^{9-0} \) is paired with \( \tilde{\gamma} \)). Because of the Beltrami differential contributions of \( \prod_{j=1}^9 (f_{b_j} d\tilde{b}(\rho) - f_{b_j} d\tilde{b}(\rho)) \), \( \tilde{b} \) and \( \tilde{b} \) have the same periodicity conditions as \( \partial \rho x^{9-0} \) and \( \partial \rho x^{9+0} \).
Since the zero modes of $x^{9+10}$ are absent due to momentum conservation, the zero modes of $\hat{c}$ and $\hat{\tilde{c}}$ must also be absent (removing these zero modes is the same as replacing the Beltrami differential contribution, $|\prod_{a=2}^{g-2+N} (\oint_{C_a} d\rho \hat{b}(\rho) - \oint_{\Gamma_a} d\rho \hat{b}(\rho))|^2$ with $|\prod_{a=1}^{g-2+N} \oint_{C_a} d\rho \hat{b}(\rho)|^2$).

It is easily checked that these matched pairs of non-light-cone matter fields and ghost fields have the same boundary conditions at all of the interaction points and vertex operators. For example, ignoring the $\tilde{\kappa}_a$ dependence, the $\psi^+$ and $\tilde{\beta}^-$ fields behave like $z^{-1}$ near $f_-$ interaction points, like $z$ near vertex operators of polarization $4_+ + 2$, and are regular everywhere else. Since the $\tilde{\kappa}_a$ dependence of these matched pairs is also identical (i.e., after dropping the $\tilde{\beta}^+ \partial_\rho \hat{c}$ term, the matched pairs occur in the same quadratic combinations in the exponential term), and since each matched pair consists of one boson and one fermion, the path integrals for the matched pairs precisely cancel each other.

After performing these path integrals, only the light-cone matter fields remain and it is easy to check that the remaining parts of the vertex and interaction-point operators are precisely the light-cone Green-Schwarz vertex and interaction-point operators (when $SO(8)$ is broken down to $SU(4) \times U(1)$ in such a way that the $SO(8)$ vector splits into $4_+ + \bar{4}$, and when boundary conditions on $\Gamma^+\bar{\Gamma}$ and $\Gamma^-\bar{\Gamma}$ are chosen to correspond to those of fields with conformal weight $+1$ and $0$, the light-cone Green-Schwarz interaction-point operator is simply $f_+ + f_-)^{27}$.

So the two different formalisms give the same scattering amplitudes, thereby proving the unitarity of the Lorentz-invariant amplitudes of equation (II.7). Note that this comparison of scattering amplitudes was only done for light-cone diagrams that did not contain colliding interaction points (it was assumed that $\frac{\partial^2 \rho}{\partial z^2}$ is non-zero at the interaction points). However, since the form of the light-cone contact term is completely determined by the condition that the amplitudes are divergence-free when two or more interaction points collide, and since the amplitudes of equation (II.7) contain no such divergences (the picture-changing operators need not sit on the interaction points), the two formalisms must also give equivalent amplitudes for light-cone diagrams that contain colliding interaction points.

V. Conclusion

In this paper, it was shown that the previously calculated Lorentz-invariant Type IIB Green-Schwarz superstring amplitudes for external massless boson-boson states contain no unphysical divergences, and are therefore finite. Furthermore, it was shown that these amplitudes are unitary since they agree with amplitudes obtained using the light-cone Green-Schwarz formalism (this also proves the finiteness and Lorentz invariance of the light-cone Green-Schwarz formalism).

These proofs required the use of external massless boson-boson states since the other massless states can not be written in a form with zero instanton charge. Nevertheless, it should be possible to generalize the proofs of finiteness and unitarity for amplitudes involving massless fermionic states which carry non-zero instanton charge. It should also be straightforward to construct BRST-invariant vertex operators for
the massive particles out of the covariant $x_\pm^i$ and $\theta_{\pm}^i$ fields, and use them to calculate Lorentz-invariant amplitudes involving external massive states. Although these should be finite, the two and three-point amplitudes are no longer expected to vanish.

One disadvantage of the Lorentz-covariant amplitudes of equation (II.7) is that they are not manifestly N=2 worldsheet supersymmetric. Since the fermionic stress-energy tensor is not a quadratic function of the longitudinal matter fields, it is not obvious how to combine these matter fields into N=2 superfields. Although making the N=2 worldsheet supersymmetry manifest is not necessary, it would be nice to express the Green-Schwarz superstring amplitudes as super-integrals over N=2 bosonic and fermionic moduli.

Acknowledgements

I would like to thank P.DiVecchia, M.Freeman, M.Green, P.Howe, S. Mandelstam, J.Petersen, F. Pezzella, A.Restuccia, J.Sidenius, J.Taylor, A.Tollsten, and P.West for useful discussions, and the SERC for its financial support.

References

(1) Mandelstam,S., Nucl.Phys.B69 (1974), p.77.
(2) Greensite,J. and Klinkhamer,F.R., Nucl.Phys.B291 (1987), p.557.
(3) Mandelstam,S., private communication.
(4) Friedan,D., Martinec,E., and Shenker,S., Nucl.Phys.B271 (1986), p.93.
(5) Verlinde,E. and Verlinde,H., Phys.Lett.B192 (1987), p.95.
(6) Atick,J. and Sen,A., Nucl.Phys.B296 (1988), p.157.
(7) Mandelstam,S., “The n-loop Amplitude: Explicit Formulas, Finiteness, and Absence of Ambiguities”, preprint UCB-PTH-91/53, October 1991.
(8) Berkovits,N., Nucl.Phys.B304 (1988), p.537.
(9) Aoki,K., D’Hoker,E., and Phong,D.H., Nucl.Phys.B342 (1990), p.149.
(10) Green,M.B. and Schwarz,J.H., Nucl.Phys.B243 (1984), p.475.
(11) Mandelstam,S., Prog.Theor.Phys.Suppl.86 (1986), p.163.
(12) Restuccia,A. and Taylor,J.G., Phys.Rep.174 (1989), p.283.
(13) Carlip,S., Nucl.Phys.B284 (1987), p.365.
(14) Gilbert,G. and Johnston,D., Phys.Lett.B205 (1988), p.273.
(15) Mandelstam,S., “The interacting-string picture and functional integration” in 1985 Santa Barbara Workshop on Unified String Theories, eds. M.B. Green and D. Gross (World Scientific, Singapore), p.577.
(16) Kallosh,R. and Morosov,A., Phys.Lett.B207 (1988), p.164.
(17) Berkovits, N., “Calculation of Green-Schwarz Superstring Amplitudes using the N=2 Twistor-String Formalism”, SUNY at Stonybrook preprint ITP-SB-92-42, August 1992, to appear in Nucl.Phys.B, hep-th bulletin board 9208035.

(18) Berkovits, N., Phys.Lett.B300 (1993), p.53, hep-th bulletin board 9211025.

(19) Ademollo, M., Brink, L., D’Adda, A., D’Auria, R., Napolitano, E., Sciuto, S., Del Giudice, E., DiVecchia, P., Ferrara, S., Gliozzi, F., Musto, R., Pettorini, R., and Schwarz, J., Nucl.Phys.B111 (1976), p.77.

(20) Rocek, M., private communication.

(21) Verlinde, E. and Verlinde, H., Nucl.Phys.B288 (1987), p.357.

(22) Lechtenfeld, O., Phys.Lett.B232 (1989), p.193.

(23) Petersen, J.L. and Sidenius, J.R., Nucl.Phys.B301 (1988), p.247.

(24) Martinec, E., Phys.Lett.B171 (1986), p.189.

(25) DiVecchia, P., Pezzella, F., Frau, M., Hornfeck, K., Lerda, A., and Sciuto, S., Nucl.Phys.B322 (1989), p.317.

(26) D’Hoker, E. and Giddings, S., Nucl.Phys.B291 (1987), p.90.

(27) Berkovits, N., Nucl.Phys.B379 (1992), p.96., hep-th bulletin board 9201004.