Temporal scaling at Feigenbaum points and non-extensive thermodynamics

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We show that recent claims for the non-stationary behaviour of the logistic map at the Feigenbaum point based on non-extensive thermodynamics are either wrong or can be easily deduced from well-known properties of the Feigenbaum attractor. In particular, there is no generalized Pesin identity for this system, the existing “proofs” being based on misconceptions about basic notions of ergodic theory. In deriving several new scaling laws of the Feigenbaum attractor, thorough use is made of its detailed structure, but there is no obvious connection to non-extensive thermodynamics.

During the last decade has appeared a vast literature on a new “non-extensive thermodynamics” (NET), which uses a maximum entropy principle with the Shannon entropy replaced by the Havrda-Charvat \( H(α) \) (“Tsallis”) entropy \( H(x) \). Several versions of NET were proposed by Tsallis and others \( H(α) \), in order to avoid inconsistencies. But as shown most forcefully by Nauenberg \( H(x) \), none of these versions is consistent with thermodynamics, at least for equilibrium systems.

For non-equilibrium systems the situation is not quite so simple, as the standard maximum entropy principle is usually not applicable. As shown by Jaynes \( H(α) \), the formalism of statistical physics can be obtained by Occam’s razor: If all knowledge can be formulated in terms of constraints, then the only rationally justifiable ansatz for the probability distribution is the one which maximizes Shannon entropy (which has to be replaced by Kullback-Leibler information, if some knowledge existed prior to these constraints). The main reason why standard thermodynamics cannot be applied to most non-equilibrium systems is that prior knowledge cannot be cast into the form of a few constraints. But there is no reason for abandoning an information theoretic interpretation of entropy, and Shannon (Kullback-Leibler) entropy is the only consistent probabilistic measure of (relative) information.

The main reason why NET is still vigorously pursued is, it seems, the claim that it is able to make striking predictions that could not be made within a more conventional framework. Typically, these are for non-equilibrium phenomena with distributions showing power laws with heavy tails. In deriving these distributions, not only is Shannon entropy replaced arbitrarily by Tsallis entropy. Also the constraints are modified in a way which has no clear rational motivation – except that one arrives thereby at expressions more easily handled.

A careful study of most – if not all – examples where NET was supposedly successful shows that the success is much less clear than claimed. In a later publication, we will substantiate this further by discussing several such examples. In the present letter, we discuss in depth just one single example, which has been treated in several papers \( H(α) \), and which was claimed to show the success of NET in a particularly clear way.

This example is the non-stationary behaviour of the Feigenbaum attractor \( H(α) \). More precisely, one can study: (1) sensitive dependence on initial conditions (both for finite and infinitesimal perturbations, both on the attractor and in its basin of attraction); (2) scaling of different dynamical (Shannon, Renyi, Tsallis) entropies of various ensembles of trajectories with their time length \( T \); (3) convergence of a typical trajectory (with random initial condition in its vicinity) to the attractor; (4) scaling of “static” (i.e., Boltzmann-Gibbs) entropies of an ensemble of points with time.

Most of these problems have been discussed since the early 1980’s \( H(α) \). Also, it had been realized from the very beginning that some of these questions are subtle due to large fluctuations (“multifractality” of the Feigenbaum attractor). In the first papers based on NET \( H(α) \), this was missed, leading to wrong claims that a single non-extensivity parameter could describe scaling at the onset of chaos. Although this was corrected recently, we shall see that the effect of fluctuations is still not fully appreciated in some of these papers \( H(α) \).

In the following we give theoretical arguments only for the Feigenbaum map \( g(x) \) defined by \( g(g(x)) = −α x + g(x) \) and \( g(x) = 1 - cx^2 + O(x^3) \) for \( x \to 0 \), but we use the logistic map \( x \to x^2 + a \) with \( a = 1.401155189 \ldots \) for numerical calculations. Problems of universality have been discussed e.g. in \( H(α) \).

Let us first discuss the dependence on infinitesimal changes in the initial conditions, i.e. the behaviour of \( Λ_n(x_0) = \ln \left| dx_n/dx_0 \right| \), where \( x_{i+1} = g(x_i) \). As pointed out in \( H(α) \), \( Λ_n(x_0) \) fluctuates very strongly with \( n \) and \( x_0 \). Different ways of averaging over \( x_0 \) give therefore rise to different scalings with \( n \). If we take arithmetic averages over \( Λ_n \) (i.e., geometric averages over \( |dx_n/dx_0| \)), we get

\[
\int dx_0 w_0(x_0) \ln |dx_n/dx_0| \sim γ \ln n \tag{1}
\]

with \( γ = 0.599 \pm 0.003 \) for all smooth initial distributions \( w_0(x) \), at least when we also do an additional averaging over \( n \) to damp out remaining oscillations \( H(α) \). On the other hand, as proven rigorously in \( H(α) \), arithmetic averages over \( |dx_n/dx_0| \) give

\[
\int dx_0 w_0(x_0)|dx_n/dx_0| \sim n^{\text{const} \ln n}. \tag{2}
\]
Without averaging over \( x_0 \) one can of course obtain completely different behaviour. For \( x_0 = 1 \), e.g., one finds \( |dx_n/dx_0| \to \text{const} \) for \( n = 2^k, k \to \infty \), while
\[
|dx_n/dx_0| = \alpha^k = (n + 1)^{\log_2 \alpha}
\] (exactly) for \( n = 2^k - 1 \). Tsallis et al. prefer to write Eq. (3) as \( |dx_n/dx_0| = (1 + \lambda(1 - q)n)^{1/(1-q)} \) with \( \lambda = 1/(1-q) = \log_2 \alpha \), and call the r.h.s. a \( q \)-exponential.

Notice that Eq. (3) is not a scaling law, since it holds only for special values of \( n \). But one obtains a scaling law by taking averages over \( n \) (see Fig. 1). \[
\bar{\Lambda}_n(x_0) \equiv n^{-1} \sum_{i=1}^{n} \Lambda_i(x_0) \sim \beta \ln n \quad \text{for} \quad x_0 = 1
\] (4) with \( \beta = 0.71339380 \ldots \) Notice that the constants in Eqs. (1), (3), and (4) are not directly related.

Defining \( \bar{\Lambda}_n(x) \) for any \( x \) on the attractor as in Eq. (4) gives the natural generalization of the Lyapunov exponent. It fluctuates strongly with \( x \) and for many applications like Pesin’s identity one needs averages over \( x \) with respect to the natural measure \( \mu(x) \). In the present case, one can easily see that the average
\[
\langle \bar{\Lambda}_n \rangle = \int dx \mu(x) \bar{\Lambda}_n(x)
\] (5) is identically zero for all \( n \), which indicates that there cannot exist a close analogy to Pesin’s identity. The identities suggested and verified in [28, 31] are trivialities where the authors started with an initial distribution \( w_0(x) \) narrowly localized around \( x_0 \) and followed the evolution only up to values for which \( w_n(x) \) is still smooth and described by the flow linearized around the trajectory starting at \( x_0 \). In addition they considered, instead of the Kolmogorov-Sinai (KS) entropy, just the difference \( S_n - S_0 \) between two static “Boltzmann-Gibbs” entropies
\[
S_n = -\int dx w_n(x) \ln w_n(x).
\] (6) In this case one has of course \( S_n - S_0 = \Lambda_n(x_0) \), but this has no connection to any (generalized) Pesin identity.

Eqs. (3) and (4) apply to trajectories starting on the attractor. In order to understand the origin of Eqs. (1) and (2) one has to study how trajectories starting in its vicinity approach the attractor. For this one has to use the detailed triadic Cantor structure of the Feigenbaum attractor [24]. Associated to this structure is a set of open disjoint intervals \( I_k, i \), the closure of which covers the interval \([−1/\alpha, 1] \) which contains the attractor [20]. \( I(0, 1) \) is the hole cut out from the middle of the Cantor set in the first stage of construction; \( I(1, 1) \) and \( I(1, 2) \) are the second generation holes, \( I(2, 1) \) to \( I(2, 4) \) are the holes cut out in the third step, etc. Each \( I_k, i \) contains exactly one point on the unstable periodic orbit of period \( 2^k \). Let us define \( I_k \equiv \bigcup I_{k, i} \), i.e. \( I_k \) is all that is cut out during the \( k \)-th step. The evolution of a typical trajectory can then be viewed as a tumbling through the \( I_k \)‘s, with \( k \) never decreasing with \( n \). The average increase of \( k \) for large \( n \) exactly given by
\[
\langle k \rangle = \text{const} + \log_2 n.
\] (7)

As shown in [20], Eq. (1) follows (up to the precise value of \( \gamma \)) from Eq. (7) and from the scaling behaviour proven by Feigenbaum.

In [20] the authors studied another average over \( \Lambda_n(x) \), in between Eqs. (1) and (2),
\[
\xi_q(n) \equiv \int dx w_0(x_0)|dx_n/dx_0|^{-1-q}.
\] (8)

From straightforward simulations they concluded that \( \xi_q(n) \) becomes (asymptotically) linear in \( n \), \( \xi_q(n) \sim n \) for \( q = 0.36 \) and called this value \( q_{\text{av}} \). But neither analytic nor numerical estimates of \( \xi_q(n) \) seem easy. A direct numerical estimate as in [21] is prone to large errors, since the integrand of Eq. (5) (with constant \( w_0(x_0) \)) and for \( q = q_{\text{av}} \) is very sharply peaked near the unstable periodic orbits of small periods. Choosing initial values \( x_0 \) at random one will miss these peaks, unless one has extremely high statistics. But it is not clear anyhow why Eq. (5) should be of interest. In contrast to what its name suggests, \( q_{\text{av}} \) does not measure the average sensitivity to initial conditions but just one particular average.

Another consequence of Eq. (7) is that the (geometrically) average distance from the attractor of a point starting randomly in \([−1/\alpha, 1] \) decreases as [20]
\[
d_n \sim n^{-1/D_1}
\] (9) where \( D_1 = 0.517097 \ldots \) is the information dimension of the Feigenbaum attractor.

The same argument can also be immediately used to derive the scaling of the “Boltzmann-Gibbs” entropy with \( n \), after starting with a random ensemble [23]. Assume we have \( N_n \) points distributed randomly with respect to \( w_n(x) \). The distance between point \( x_{n,j} \) and its nearest neighbour is called \( r_j \). The Kozachenko-Leonenko estimator [35] for differential entropies gives
applied the same reasoning to $W_n$ as we did for $S_n$: It scales like the total length of all intervals $I_{k,i}$ with $k = \log_2 n$. Calculating the latter is easy and gives

$$W_n \sim n^{-0.800138194\ldots}.$$  \hfill (12)\hfill

The exponent measured in [11] was quite different ($0.71 \pm 0.01$ instead of 0.80), but this is easily explained by the expected large corrections to scaling, which have even led to claims of non-universality in [23].

The KS entropy for a one-humped map with maximum at $x = 0$ is obtained from the sequences $s_n = (\text{sign}(x_1), \ldots, \text{sign}(x_n))$, as

$$H_{KS} = \lim_{n \to \infty} n^{-1} H_n$$  \hfill (13)\hfill

with $H_n = -\sum_{s_n} p(s_n) \log p(s_n)$. As shown in [27, 30], for the Feigenbaum attractor one has $H_n = \log n + O(1)$. Thus the KS entropy is zero, but there is a logarithmic increase of $H_n$. With the same methods one can prove *exactly*, that all Renyi entropies $H_q = \lim_{n \to \infty} H^n_q$ with $H^n_q = (1 - q)^{-1} \log \left( \sum_{s_n} [p(s_n)]^q \right)$ are equal for this problem, $\lim_{n \to \infty} H^n_q / H_n = 1$ for all $q$ [41]. For Tsallis-type generalized KS entropies, this gives $K^n_q \equiv (1 - q)^{-1} \left( \exp((1 - q)H^n_q) - 1 \right) \sim n^{-1-q}$, at variance with [14].

The $q$-independence of the Renyi entropies is surprising in view of the multifractality of the attractor. For a chaotic attractor, the $H^n_q$ are closely related, via generalizations of Pesin’s identity, to $q$-th moments of $\Lambda(x)$ [42]. This shows again that the onset of chaos is more subtle than expected in the NET literature.

In the first papers [8, 9, 10, 11], it was supposed but never substantiated that the parameter $q$ of NET can be obtained, also for the Feigenbaum map, by some maximum entropy principle. Although this was never withdrawn, $q$ is now fixed such that one obtains linear time dependencies, when logarithms are replaced by $q$-deformed logarithms (see e.g. [20]). The reason for this is not clear, since there is no need for all time dependencies to be linear. But even worse, with the proliferation of different (although closely related) scaling laws, one obtains – for the single case of the Feigenbaum map! – a rich zoo of different $q$ values [4, 20]. With the additional scaling laws found in the present paper, there would be even more $q$’s – unless one accepts at last that non-extensive thermodynamics is (at least at the onset of chaos) just a chimera. On the other hand, we consider these new scaling laws as an important part of the present letter.

One reason for preferring “$q$-exponentials” (i.e., generalized Pareto distributions [13]) over power laws could be that the former give correct deviations from pure power laws at small arguments. But in all cases studied in the present paper (except Eq. 3) the small-$n$ limits are neither given by pure powers nor by $q$-exponentials.

Finally, as a last remark: In [4, 13, 24], the Feigenbaum map was chosen as the prototype of a supposedly weakly mixing system. But it is not mixing at all.

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For an invariant measure which is absolutely continuous in all expanding dimensions, Pesin’s identity (see, e.g. [34]) relates the Kolmogorov-Sinai entropy $H_{KS}$ to the Lyapunov spectrum via $H_{KS} = \sum_{\lambda_j > 0} \lambda_j$. For a measure with partial dimensions $D_i < 1$ in the expanding dimensions, this generalizes to $H_{KS} = \sum_{\lambda_j > 0} D_i \lambda_j$. The latter should be further generalized, if one looks for a Pesin type identity for the Feigenbaum map.

In [10], [13], [14], the authors first make a partition of the attractor basin into $W$ equally long bins and take $N$ starting points $x_0^j$, $j = 1 \ldots N$, in one of them. The spatial order-$q$ Tsallis entropy after $n$ iterations, estimated from the points $x_0^j$, is denoted as $S_q(n)$. Then the “generalized KS entropy” of order $q$ is defined as

$$K_q = \lim_{n \to \infty} \lim_{W \to \infty} n^{-1} S_q(n).$$

For $q \to 1$ this is claimed in [11] to coincide with the standard KS entropy, but in the latter the limits $n \to \infty$ and $W \to \infty$ are exchanged, the points are chosen randomly according to the natural invariant measure, and $S_1(n)$ should be replaced by the Shannon entropy for trajectories of length $n$.

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[31] These oscillations and the ones seen in Fig. 1 are due to the Cantor structure of the Feigenbaum attractor and will not be discussed further.
[32] Eq. (14) follows immediately from the Feigenbaum-Cvitanovic relation $\alpha g'(g(x)) = -g(\alpha x)$ and from the well known facts [22] that $g(1) = -1/\alpha, g'(1) = -\alpha,$ and $g'(-1/\alpha) = 1$. The equation for $n = 2^k$ uses in addition the fact that $x_{2^{k-1}} = g^{(2^{k-1})}(1)$ is close to zero, where $g(x)$ has its quadratic maximum. See also [14]. For Feigenbaum type maps $g(x)$ with maxima of order $z$, the same easy derivation gives $|dx_n/dx_0| = \alpha^{(z-1)k}$. For no obvious reasons, this relation was claimed to show the importance of NET in [10].
[33] For an invariant measure which is absolutely continuous in all expanding dimensions, Pesin’s identity (see, e.g. [34]) relates the Kolmogorov-Sinai entropy $H_{KS}$ to the Lyapunov spectrum via $H_{KS} = \sum_{\lambda_j > 0} \lambda_j$. For a measure with partial dimensions $D_i < 1$ in the expanding dimensions, this generalizes to $H_{KS} = \sum_{\lambda_j > 0} D_i \lambda_j$. The latter should be further generalized, if one looks for a Pesin type identity for the Feigenbaum map.
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