Weighted local Hardy spaces and their applications

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Abstract In this paper, we study weighted local Hardy spaces $h^p_\omega(\mathbb{R}^n)$ associated with local weights which include the classical Muckenhoupt weights. This setting includes the classical local Hardy space theory of Goldberg [10], and the weighted Hardy spaces of Bui [3].

1. Introduction

The theory of local Hardy space plays an important role in various fields of analysis and partial differential equations; see [15, 17, 20, 14]. In particular, pseudodifferential operators are bounded on local Hardy spaces $h^p$ for $0 < p \leq 1$, but they are not bounded on Hardy spaces $H^p$ for $0 < p \leq 1$; see [10].

On the other hand, Bui [1] studied the weighted version $h^p_\omega$ of the local Hardy space $h^p$ considered by Goldberg [10], where the weight $\omega$ is assumed to satisfy the condition $(A_\infty)$ of Muckenhoupt. Recently, Rychkov [14] introduced and studied some properties of the weighted Besov-Lipschitz and Triebel-Lizorkin spaces with weights that are locally in $A_p$ but may grow or decrease exponentially. Recently, Rychkov [14] studied the class of Triebel-Lizorkin $F^s_{p,q}$ spaces, which includes Hardy spaces as its part. In fact, Rychkov explicitly identifies weighted local Hardy space $h^p_\omega$ with $F^0_{p,2}(\omega)$ in Theorem 2.25 of [14]. In particular, Rychkov [14] extended a part of theory of $A_\infty$-weighted local Hardy spaces developed in Bui [3] to the $A_{loc}^{\infty}$ weights, where $A_{loc}^{\infty}$ weights denote local $A_{\infty}$-weights which are non-doubling weights, and the $A_{loc}^{\infty}$ weights include the $A_{\infty}$-weights.

The main purpose of this paper is twofold. The first goal is to establish weighted atomic decomposition characterizations of weighted local Hardy space $h^p_\omega$ with local weights. The second goal is to show that strong singular integrals and Pseudodifferential operators and their commutators are bounded on weighted local Hardy spaces.

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The paper is organized as follows. In Section 2, we first recall some notation and definitions concerning local weights and grand maximal function; and we then obtain a basic approximation of the identity result and the grand maximal function characterization for $L^q_\omega$ with $q \in (q_\omega, \infty]$, where $q_\omega$ is the critical of $\omega$. In Section 3, we introduce weighted local Hardy spaces $h^{p,q,s}_\omega(N)$ via grand maximal functions and weighted atomic local Hardy spaces $h^{p,q,s}_\omega(R^n)$ for any admissible triplet $(p,q,s)$, and study some properties of these spaces. In Section 4, we establish the Calderón-Zygmund decomposition associated with the grand maximal function. In Section 5, we prove that for any admissible triplet $(p,q,s)$, $h^{p,q,s}_\omega(N) = h^{p,q,s}_\omega(R^n)$ with equivalent norms. Moreover, we prove that $\| \cdot \|_{h^{p,q,s}_\omega(N)}$ and $\| \cdot \|_{H^p_\omega(R^n)}$ are equivalent quasi-norms on $h^{p,q,s}_\omega(R^n)$ with $q < \infty$, and we obtain criterions for boundedness of sublinear operators in $h^{p,q,s}_\omega$ in Section 6. Finally, in Section 7, we show that strong singular integrals and Pseudodifferential operators and their commutators are bounded on weighted local Hardy spaces by using weighted atomic decompositions.

It is worth pointing out that we can not adapt the methods in [3] and [10], if $\omega$ is a local weight. In fact, adapting the same idea of (global)weighted Hardy spaces([1,2,16,18]), we give a direct proof for weighted atomic decompositions of weighted local Hardy spaces.

Throughout this paper, $C$ denotes the constants that are independent of the main parameters involved but whose value may differ from line to line. Denote by $\mathbb{N}$ the set $\{1,2,\cdots\}$ and by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $1/C \leq A/B \leq C$.

2. Preliminaries

We first introduce weight classes $A^{loc}_p$ from [14].

Let $Q$ run through all cubes in $\mathbb{R}^n$ (here and below only cubes with sides parallel to the coordinate axes are considered), and let $|Q|$ denote the volume of $Q$. We define the weight class $A^{loc}_p(1 < p < \infty)$ to consists of all nonnegative locally integral functions $\omega$ on $\mathbb{R}^n$ for which

$$A^{loc}_p(\omega) = \sup_{|Q| \leq 1} \frac{1}{|Q|^p} \int_Q \omega(x)dx \left( \int_Q \omega^{-p'/p}(x)dx \right)^{p'/p} < \infty, \quad 1/p + 1/p' = 1. \quad (2.1)$$

The function $\omega$ is said to belong to the weight class of $A^{loc}_1$ on $\mathbb{R}^n$ for which

$$A^{loc}_1(\omega) = \sup_{|Q| \leq 1} \frac{1}{|Q|} \int_Q \omega(x)dx \left( \sup_{y \in Q} \omega(y) \right)^{1} < \infty. \quad (2.2)$$

Remark: For any $C > 0$ we could have replaced $|Q| \leq 1$ by $|Q| \leq C$ in (2.1) and (2.2).

In what follows, $Q(x,t)$ denotes the cube centered at $x$ and of the sidelength $t$. Similarly, given $Q = Q(x,t)$ and $\lambda > 0$, we will write $\lambda Q$ for the $\lambda$-dilate cube,
which is the cube with the same center $x$ and with sidelength $\lambda t$. Given a Lebesgue measurable set $E$ and a weight $\omega$, let $\omega(E) = \int_E \omega \, dx$. For any $\omega \in A_{\infty}^{\text{loc}}$, $L_{\omega}^p$ with $p \in (0, \infty)$ denotes the set of all measurable functions $f$ such that

$$\|f\|_{L_{\omega}^p} \equiv \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx \right)^{1/p} < \infty$$

and $L_{\omega}^\infty = L^\infty$. We define the local Hardy-Littlewood maximal operator by

$$M_{\text{loc}}^p f(x) = \sup_{x \in Q: |Q| < 1} \frac{1}{|Q|} \int_Q |f(y)| \, dy.$$ 

Similar to the classical $A_p$ Muckenhoupt weights, we give some properties for weights $\omega \in A_{\text{loc}}^{\infty} := \bigcup_{1 \leq p < \infty} A_{\text{loc}}^p$.

**Lemma 2.1.** Let $1 \leq p < \infty$, $\omega \in A_{\text{loc}}^p$, and $Q$ be a unit cube, i.e. $|Q| = 1$. Then there exists a $\bar{\omega} \in A_p$ so that $\bar{\omega} = \omega$ on $Q$ and

(i) $A_p(\bar{\omega}) \leq C A_{\text{loc}}^p(\omega)$.

(ii) if $\omega \in A_{\text{loc}}^p$, then there exists $\epsilon > 0$ such that $\omega \in A_{p-\epsilon}^{\text{loc}}(\omega)$ for $p > 1$.

(iii) If $1 \leq p_1 < p_2 < \infty$, then $A_{p_1}^{\text{loc}} \subset A_{p_2}^{\text{loc}}$.

(iv) $\omega \in A_{p}^{\text{loc}}$ if and only if $\omega^{-\frac{1}{p-1}} \in A_{p'}^{\text{loc}}$.

(v) If $\omega \in A_{p}^{\text{loc}}$ for $1 \leq p < \infty$, then

$$\omega(tQ) \leq \exp(c \omega t) \omega(Q) \quad (t \geq 1, |Q| = 1).$$

(vi) the local Hardy-Littlewood maximal operator $M_{\text{loc}}^p$ is bounded on $L_{\omega}^p$ if $\omega \in A_{p}^{\text{loc}}$ with $p \in (1, \infty)$.

(vii) $M_{\text{loc}}^p$ is bounded from $L_{\omega}^1$ to $L_{\omega}^{1, \infty}$ if $\omega \in A_1^{\text{loc}}$.

Proof: (i)-(vi) have been proved in [14]. (vii) can be proved by the standard method.

We remark that Lemma 2.1 is also true for $|Q| > 1$ with $c$ depending now on the size of $Q$. In addition, it is easy to see that $A_p \subset A_{\text{loc}}^p$ for $p \geq 1$ and $e^{c|x|}$, $(1 + |x| \ln^\alpha(2 + |x|))^\beta \in A_{\text{loc}}^1$ with $\alpha \geq 0$, $\beta \in \mathbb{R}$ and $c \in \mathbb{R}$.

As a consequent of Lemma 2.1, we have following result.

**Corollary 2.1.** If $\omega \in A_{\infty}^{\text{loc}}$, then there exists a constant $C > 0$ such that

$$\omega(2Q) \leq \omega(Q)$$

if $|Q| < 1$, and

$$\omega(Q(x_0, r + 1)) \leq C \omega(Q(x_0, r))$$

if $|Q(x_0, r)| \geq 1$. 


Obviously, \( q_\omega \in [1, \infty) \). If \( q_\omega \in (1, \infty) \), then \( \omega \not\in A_p^{loc} \).

The symbols \( D(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n) \), \( D'(\mathbb{R}^n) \) is the dual space of \( D(\mathbb{R}^n) \). The multi-index notation is usual: for \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n} \).

**Lemma 2.2.** Let \( \omega \in A_p^{loc} \), \( q_\omega \) be as in (2.3), and \( p \in (q_\omega, \infty] \). Then

1. if \( 1/p + 1/p' = 1 \), then \( D(\mathbb{R}^n) \subset L_{\omega_1/p-1}^{p'}(\mathbb{R}^n) \);
2. \( L_p^\omega(\mathbb{R}^n) \subset D'(\mathbb{R}^n) \) and the inclusion is continuous.

Proof. We only prove the case \( p < \infty \). The proof for the case \( p = \infty \) is easier and we omit the details. Since \( p \in (q_\omega, \infty) \), then \( \omega \in A_p^{loc} \). Therefore, by the definition of \( A_p^{loc} \), for all ball \( B = B(0, r) \) with radius \( r \) and centered at 0, we have

\[
\int_B [\omega(x)]^{1/(p-1)} dx \leq C[\omega(B)]^{-1/(p-1)} |B|^{p'} < \infty.
\]

From this, for any \( \varphi \in D(\mathbb{R}^n) \) and \( \text{supp} \varphi \subset B \), we obtain

\[
\|\varphi\|_{L_{\omega_1/(p-1)}^{p'}(\mathbb{R}^n)} \leq C \int_B [\omega(x)]^{-1/(p-1)} dx < \infty. \tag{2.4}
\]

For (ii), if \( f \in L_p^\omega(\mathbb{R}^n) \) and \( \varphi \in D(\mathbb{R}^n) \), by Hölder inequality and (2.4), we have

\[
|< f, \varphi >| \leq \|f\|_{L_\omega^p(\mathbb{R}^n)} \left( \int_{\mathbb{R}^n} |\varphi(x)|^{p'} [\omega(x)]^{-1/(p-1)} dx \right)^{1/p'} \leq C\|f\|_{L_\omega^p(\mathbb{R}^n)}.
\]

Thus, Lemma 2.2 is proved.

For \( \varphi \in D(\mathbb{R}^n) \) and \( t > 0 \), set

\[
\varphi_t(x) = t^{-n} \varphi \left( \frac{x}{t} \right).
\]

It is easy to see that we have the following results.

**Proposition 2.1.** Let \( \varphi \in D(\mathbb{R}^n) \) and \( \int_{\mathbb{R}^n} \varphi(x) dx = 1 \).

1. For any \( \Phi \in D(\mathbb{R}^n) \) and \( f \in D'(\mathbb{R}^n) \), \( \Phi * \varphi_t \to \Phi \) in \( D(\mathbb{R}^n) \) as \( t \to 0 \) and \( f * \varphi_t \to f \) in \( D'(\mathbb{R}^n) \) as \( t \to 0 \).
2. Let \( \omega \in A_\infty^\omega \) and \( q_\omega \) be as in (2.3). If \( q \in (q_\omega, \infty) \), then for any \( f \in L_q^\omega(\mathbb{R}^n) \), \( f * \varphi_t \to f \) in \( L_q^\omega(\mathbb{R}^n) \) as \( t \to 0 \).
Let \( N \in \mathbb{N}_0 \) and

\[
\mathcal{M}_N^0 f(x) = \sup \{ |\varphi_t \ast f(x)| : 0 < t < 1, \varphi \in \mathcal{D}(\mathbb{R}^n), \int \varphi \neq 0, \\
supp \varphi \subset B(0,1), \|D^\alpha \varphi\|_\infty \leq 1 \ |\alpha| \leq N + 1 \}.
\]

\[
\mathcal{M}_N^0 f(x) = \sup \{ |\varphi_t \ast f(x)| : 0 < t < 1, \varphi \in \mathcal{D}(\mathbb{R}^n), \int \varphi \neq 0, \\
supp \varphi \subset B(0,2^{2(10+n)}), \|D^\alpha \varphi\|_\infty \leq 1 \ |\alpha| \leq N + 1 \}.
\]

and

\[
\mathcal{M}_N f(x) = \sup \{ |\varphi_t \ast f(z)| : |z - x| < t < 1, \varphi \in \mathcal{D}(\mathbb{R}^n), \int \varphi \neq 0, \\
supp \varphi \subset B(0,2^{2(10+n)}), \|D^\alpha \varphi\|_\infty \leq 1 \ |\alpha| \leq N + 1 \}.
\]

For any \( N \in \mathbb{N}_0 \) and \( x \in \mathbb{R}^n \), obviously,

\[
\mathcal{M}_N^0 f(x) \leq \mathcal{M}_N^0 f(x) \leq \mathcal{M}_N f(x).
\]

For convenience, we write

\[
\mathcal{D}_N^0 = \{ \varphi \in \mathcal{D} : \supp \varphi \subset B(0,1), \int \varphi \neq 0, \|D^\alpha \varphi\|_\infty \leq 1 \ |\alpha| \leq N + 1 \},
\]

and

\[
\mathcal{D}_N = \{ \varphi \in \mathcal{D} : \supp \varphi \subset B(0,2^{2(10+n)}), \int \varphi \neq 0, \|D^\alpha \varphi\|_\infty \leq 1 \ |\alpha| \leq N + 1 \}.
\]

**Proposition 2.2.** Let \( N \geq 2 \). Then

(i) There exists a positive \( C \) such that for all \( f \in (L^1_{\text{loc}}(\mathbb{R}^n)) \cap \mathcal{D}'(\mathbb{R}^n) \) and almost everywhere \( x \in \mathbb{R}^n \), \( |f(x)| \leq M_N^0 f(x) \leq CM_N^0 f(x) \).

(ii) If \( \omega \in A_{p}^{\text{loc}} \) with \( p \in (1, \infty) \), then \( f \in L^p_\omega(\mathbb{R}^n) \) if and only if \( f \in \mathcal{D}'(\mathbb{R}^n) \) and \( M_N^0 f \in L^p_\omega \); moreover, \( \|f\|_{L^p_\omega} \sim \|M_N^0 f\|_{L^p_\omega} \).

(iii) If \( \omega \in A_1^{\text{loc}} \), then \( M_N^0 \) is bounded from \( L^1_\omega(\mathbb{R}^n) \) to \( L^1_\omega(\mathbb{R}^n) \).

The proof of (i) and (iii) is obvious, (ii) has been proved in [14], we omit the details here.

3. The grand maximal function definition of Hardy spaces
In this section, we introduce weighted local Hardy spaces via grand maximal functions and weighted local Hardy spaces. Moreover, we study some properties of these spaces.

Let \( p \in (0, 1] \), \( \omega \in A_{\infty}^{loc} \), and \( q_\omega \) be as in (2.3). Set

\[
N_{p, \omega} = \max\{0, \lfloor n \left( \frac{q_\omega}{p} - 1 \right) \rfloor + 2.\]

For each \( N \geq N_{p, \omega} \), the weighted local Hardy space is defined by

\[
h_{\omega, N}^p(\mathbb{R}^n) \equiv \left\{ f \in \mathcal{D}'(\mathbb{R}^n) : M_N^p(f) \in L_p^0(\mathbb{R}^n) \right\}.
\]

Moreover, we define \( \|f\|_{h_{\omega, N}^p(\mathbb{R}^n)} \equiv \|M_N^p(f)\|_{L_p^0(\mathbb{R}^n)} \). From Theorem 2.24 in [14], we know that \( \|M_N^p(f)\|_{L_p^0(\mathbb{R}^n)} \sim \|M_N^q(f)\|_{L_q^0(\mathbb{R}^n)} \sim \|M_N(f)\|_{L_1^0(\mathbb{R}^n)} \).

For any integer \( N, \bar{N} \) with \( N_{p, \omega} \leq N \leq \bar{N} \), we have

\[
h_{\omega, N_{p, \omega}}^p(\mathbb{R}^n) \subset h_{\omega, N}^p(\mathbb{R}^n) \subset h_{\omega, \bar{N}}^p(\mathbb{R}^n)
\]

and the inclusions are continuous.

Notice that if \( p \in (q_\omega, \infty] \) and \( N \geq N_{p, \omega} = 2 \), then by Proposition 2.2 (ii), we have \( h_{\omega, N}^p(\mathbb{R}^n) = L_p^0(\mathbb{R}^n) \) with equivalent norms. However, if \( p \in (1, q_\omega) \), the element of \( h_{\omega, N}^p(\mathbb{R}^n) \) may be a distribution, and hence, \( h_{\omega, N}^p(\mathbb{R}^n) \neq L_p^0(\mathbb{R}^n) \). But, \( (h_{\omega, N}^p(\mathbb{R}^n)) \cap L_{loc}^1(\mathbb{R}^n) \subset L_p^0(\mathbb{R}^n) \). For applications considered in this paper, we concentrate only on \( h_{\omega, N}^p(\mathbb{R}^n) \) with \( p \in (0, 1] \).

We introduce the following weighted atoms.

Let \( \omega \in A_{\infty}^{loc} \) and \( q_\omega \) be as in (2.3). A triplet \((p, q, s)_{\omega}\) is called to be admissible, if \( p \in (0, 1] \), \( q \in (q_\omega, \infty] \) and \( s \in \mathbb{N} \) with \( s \geq \lfloor n \left( \frac{q_\omega}{p} - 1 \right) \rfloor \). A function \( a \) on \( \mathbb{R}^n \) is said to be a \((p, q, s)_{\omega} - \text{atom}\) if

(i) \( \text{supp } a \subset Q \),

(ii) \( \|a\|_{L_q^0(\mathbb{R}^n)} \leq [\omega(Q)]^{1/q-1/p} \),

(iii) \( \int_{\mathbb{R}^n} a(x)x^\alpha dx = 0 \) for \( \alpha \in (\mathbb{N}_0)^n \) with \( |\alpha| \leq s \), if \( |Q| < 1 \).

Moreover, we call \( a \) is a \((p, q, s)_{\omega} - \text{single atom}\) if \( \|a\|_{L_q^0(\mathbb{R}^n)} \leq [\omega(\mathbb{R}^n)]^{1/q-1/p} \).

Let \( \omega \in A_{\infty}^{loc} \) and \((p, q, s)_{\omega}\) be an admissible triplet. The weighted atomic local Hardy space \( h_{\omega}^{p,q,s}(\mathbb{R}^n) \) is defined to be the set of all \( f \in \mathcal{D}'(\mathbb{R}^n) \) satisfying that \( f = \sum_{i=0}^{\infty} \lambda_i a_i \) in \( \mathcal{D}'(\mathbb{R}^n) \), where \( \{\lambda_i\}_{i=0}^{\infty} \subset \mathbb{C}, \sum_{i=0}^{\infty} |\lambda_i|^p < \infty \) and \( \{a_i\}_{i=\mathbb{N}} \) are \((p, q, s)_{\omega}-\text{atom}\) and \( a_0 \) is a \((p, q, s)_{\omega} - \text{single atom}\). Moreover, the quasi-norm of \( f \in h_{\omega}^{p,q,s}(\mathbb{R}^n) \) is defined by

\[
\|f\|_{h_{\omega}^{p,q,s}(\mathbb{R}^n)} \equiv \inf \left\{ \left[ \sum_{i=0}^{\infty} |\lambda_i|^p \right]^{1/p} \right\}.
\]
where the infimum is taken over all the decompositions of \( f \) as above.

It is easy to see that if the triplets \((p, q, s)\) and \((p, \tilde{q}, \tilde{s})\) are admissible and satisfy \( \tilde{q} \leq q \) and \( \tilde{s} \leq s \), then \((p, q, s)\)-atoms are \((p, \tilde{q}, \tilde{s})\)-atoms, which further implies that \( h_{p,q,s}^{w}(\mathbb{R}^{n}) \subset h_{p,\tilde{q},\tilde{s}}^{w}(\mathbb{R}^{n}) \) and the inclusion is continuous.

Next we give some basic properties of \( h_{\omega,N}^{p}(\mathbb{R}^{n}) \) and \( h_{p,q,s}^{w}(\mathbb{R}^{n}) \)

**Proposition 3.1.** Let \( \omega \in A_{\infty}^{loc} \). If \( p \in (0,1] \) and \( N \geq N_{p,\omega} \), then the inclusion \( h_{\omega,N}^{p}(\mathbb{R}^{n}) \hookrightarrow \mathcal{S}'(\mathbb{R}^{n}) \) is continuous.

Proof. Let \( f \in h_{\omega,N}^{p}(\mathbb{R}^{n}) \). For any \( \varphi \in \mathcal{D}_{N}^{0}(\mathbb{R}^{n}) \), and \( \text{supp} \varphi \subset B_{0} = B(0,1) \), we have

\[
|<f, \varphi>| = |f * \varphi(0)| \leq \| \varphi \|_{\mathcal{D}_{N}} \inf_{x \in B_{0}} \mathcal{M}_{N}^{p}(f)(x)
\]
\[
\leq [\omega(B_{0})]^{-1/p} \| \varphi \|_{\mathcal{D}_{N}^{0}(\mathbb{R}^{n})} \| f \|_{h_{\omega,N}^{p}(\mathbb{R}^{n})},
\]
where \( \varphi(x) = \varphi(-x) \). This implies \( f \in \mathcal{D}'(\mathbb{R}^{n}) \) and the inclusion is continuous. The proof is finished.

**Proposition 3.2.** Let \( \omega \in A_{\infty}^{loc} \). If \( p \in (0,1] \) and \( N \geq [(n(q_{\omega}/p - 1)] + 2 \), the space \( h_{\omega}^{p}(\mathbb{R}^{n}) \) is complete.

Proof. For every \( \varphi \in \mathcal{D}_{N}^{0}(\mathbb{R}^{n}) \) and every sequence \( \{f_{i}\}_{i \in \mathbb{N}} \) in \( \mathcal{D}'(\mathbb{R}^{n}) \) such that \( \sum_{i} f_{i} \) converges in \( \mathcal{D}' \) to the distribution \( f \), the series \( \sum_{i} f_{i} * \varphi(x) \) converges pointwise to \( f * \varphi(x) \) for each \( x \in \mathbb{R}^{n} \). Thus,

\[
\mathcal{M}_{N}^{p}f(x)^{p} \leq \left( \sum_{i} \mathcal{M}_{N}^{p}f_{i}(x) \right)^{p} \leq \sum_{i} (\mathcal{M}_{N}^{p}f_{i}(x))^{p} \quad \text{for all } x \in \mathbb{R}^{n},
\]

and hence \( \| f \|_{h_{\omega,N}^{p}(\mathbb{R}^{n})} \leq \sum_{i} \| f_{i} \|_{h_{\omega,N}^{p}(\mathbb{R}^{n})} \).

To prove that \( h_{\omega,N}^{p}(\mathbb{R}^{n}) \) is complete, it suffices to show that for every sequence \( \{f_{j}\}_{j \in \mathbb{N}} \) with \( \| f_{j} \|_{h_{\omega,N}^{p}(\mathbb{R}^{n})} < 2^{-j} \) for any \( j \in \mathbb{N} \), the series \( \sum_{j \in \mathbb{N}} f_{j} \) convergence in \( h_{\omega,N}^{p}(\mathbb{R}^{n}) \). Since \( \sum_{i=1}^{j} f_{i} \) are Cauchy sequences in \( h_{\omega,N}^{p}(\mathbb{R}^{n}) \), by Proposition 3.1 and the completeness of \( \mathcal{D}'(\mathbb{R}^{n}) \), \( \sum_{i=1}^{j} f_{i} \) are also Cauchy sequences in \( \mathcal{D}'(\mathbb{R}^{n}) \) and thus converge to some \( f \in \mathcal{D}'(\mathbb{R}^{n}) \). Therefore,

\[
\| f - \sum_{i=1}^{j} f_{i} \|_{h_{\omega,N}^{p}(\mathbb{R}^{n})} = \| \sum_{i=j+1}^{\infty} f_{i} \|_{h_{\omega,N}^{p}(\mathbb{R}^{n})} \leq \sum_{i=j+1}^{\infty} 2^{-ip} \to 0
\]
as \( j \to \infty \). This finishes the proof.

**Theorem 3.1.** Let \( \omega \in A_{\infty}^{loc} \). If \((p,q,s)\) is an admissible triplet and \( N \geq N_{p,\omega} \), then \( h_{p,q,s}^{w}(\mathbb{R}^{n}) \subset h_{p,q,s}^{w}(\mathbb{R}^{n}) \subset h_{p,q,s}^{w}(\mathbb{R}^{n}) \), and moreover, there exists a positive constant \( C \) such that for all \( f \in h_{p,q,s}^{w}(\mathbb{R}^{n}) \),

\[
\| f \|_{h_{\omega,N}^{p}(\mathbb{R}^{n})} \leq \| f \|_{h_{\omega,N}^{p,q,s}(\mathbb{R}^{n})} \leq C \| f \|_{h_{\omega,N}^{p,q,s}(\mathbb{R}^{n})}.
\]
Proof. Obviously, we only need to prove $h_{w,q,s}^{p} \subset h_{w,N_{p},\omega}^{p}(\mathbb{R}^{n})$ for all $f \in h_{w,q,s}^{p}(\mathbb{R}^{n})$, $\|f\|_{h_{w,N_{p},\omega}^{p}(\mathbb{R}^{n})} \leq \|f\|_{h_{w,q,s}^{p}(\mathbb{R}^{n})}$. To this end, it suffice to prove that there exists a positive constant $C$ such that

$$
\|M_{N_{p},\omega}^{0}(a)\|_{L_{p}^{q}(\mathbb{R}^{n})} \leq C \text{ for all } (p,q,s)_{\omega} - \text{atoms } a, \tag{3.1}
$$

and

$$
\|M_{N_{p},\omega}^{0}(a)\|_{L_{q}^{p}(\mathbb{R}^{n})} \leq C \text{ for a } (p,q)_{\omega} \text{ single atoms } a. \tag{3.2}
$$

Since $q \in (q_{\omega}, \infty]$, so $\omega \in A_{q}^{\text{loc}}$. We first prove (3.2). Let $a$ is a $(p,q)_{\omega}$ single atom. Using the Hölder inequality, the $L_{q}^{p}(\mathbb{R}^{n})$-boundedness of $M_{N_{p},\omega}^{0}$ and $\omega \in A_{q}^{\text{loc}}$ together with Proposition 2.2 (i), we have

$$
\|M_{N_{p},\omega}^{0}(a)\|_{L_{q}^{p}(\mathbb{R}^{n})} \leq C\|a\|_{L_{q}^{p}(\mathbb{R}^{n})}\omega(\mathbb{R}^{n})]^{1-p/q} \leq C.
$$

It remains to prove (3.1). Let $a$ be a $(p,q,s)_{\omega}$-atom supported in $Q = Q(x_{0}, r)$. The first case is when $|Q| < 1$. Then if $Q$ is the double of $Q$,

$$
\int_{\mathbb{R}^{n}}[M_{N_{p},\omega}^{0}(a)(x)]^{p}\omega(x)dx = \int_{Q}[M_{N_{p},\omega}^{0}(a)(x)]^{p}\omega(x)dx + \int_{Q^{c}}[M_{N_{p},\omega}^{0}(a)(x)]^{p}\omega(x)dx \\
:= I_{1} + I_{2}.
$$

For $I_{1}$, by the properties of $A_{q}^{\text{loc}}$ (see Lemma 2.1), we have

$$
I_{1} \leq C\|a\|_{L_{q}^{p}(\mathbb{R}^{n})}\omega(Q)\]^{1-p/q} \leq C.
$$

To estimate $I_{2}$, we claim that for $x \in \bar{Q}^{c}$

$$
M_{N_{p},\omega}^{0}(a)(x) \leq C|x - x_{0}|^{s_{0} + 1 + n}|Q|^{s_{0}/n}[\omega(Q)]^{-1/p}\chi_{\{|x - x_{0}| < 4n\}}(x), \tag{3.3}
$$

where $s_{0} = [n(q_{\omega}/p - 1)]$. Indeed, let $P$ be the Taylor expansion of $\varphi$ at the point $(x - x_{0})/t$ of order $s_{0}$. Thus, by the Taylor remainder theorem, note that $0 < t < 1$, we then have

$$
|(a * \varphi_{t})(x)(x)| = \left| t^{-n} \int_{\mathbb{R}^{n}}a(y)\left(\varphi\left(\frac{x - y}{t}\right) - P\left(\frac{x_{0} - y}{t}\right)\right)dy\right| \\
\leq C\chi_{\{|x - x_{0}| < 4n\}}(x)|x - x_{0}|^{-(s_{0} + n + 1)}\int_{B}|a(y)||y|^{s_{0} + 1}dy \\
\leq C|\chi_{\{|x - x_{0}| < 4n\}}(x)|^{(s_{0} + 1)/n}[\omega(Q)]^{-1/p}\chi_{\{|x - x_{0}| < 4n\}}(x) \\
\leq C|x - x_{0}|^{s_{0} + 1 + n}|Q|^{(s_{0} + 1)/n}[\omega(Q)]^{-1/p}\chi_{\{|x - x_{0}| < 4n\}}(x).
$$

Hence, (3.3) holds. Choose $\eta > 0$ such that, then by $\omega \in A_{q_{\omega} + \eta}^{\text{loc}}$ and Proposition 2.2 (i), we have

$$
I_{2} \leq C|Q|^{p(n + s_{0}/n)[\omega(Q)]^{-1}}\int_{2r < |x - x_{0}| < 4n}|x - x_{0}|^{p(s_{0} + 1 + n)}\omega(x)dx \leq C.
$$
To deal with the case when $|Q| \geq 1$, the proof is simple. In fact, let $Q^* = Q(x_0, r+n)$, by Corollary 2.1, we get

$$
\int_{\mathbb{R}^n} [M_{N_p,\omega}^0 (a)(x)]^p \omega(x) dx = \int_{Q^*} [M_{N_p,\omega}^0 (a)(x)]^p \omega(x) dx \\
\leq C \|a\|_{p, L_q (\mathbb{R}^n)} [\omega(Q^*)]^{1/p/q} \\
\leq C \|a\|_{p, L_q (\mathbb{R}^n)} [\omega(Q)]^{1/p/q} \\
\leq C.
$$

Thus, Proposition 3.2 is proved.

4. Calderón-Zygmund decompositions

In this section, we establish the Calderón-Zygmund decompositions associated with grand maximal functions on weighted $\mathbb{R}^n$. We follow the constructions in [16], [1] and [2].

Throughout this section, we consider a distribution $f$ so that for all $\lambda > 0$,

$$
\omega(\{x \in \mathbb{R}^n : \mathcal{M}_N (f) > \lambda\}) < \infty,
$$

where $N \geq 2$ is some fixed integer. Later with regard to the weighted local Hardy space $h_{\omega_N}^p (\mathbb{R}^n)$ with $p \in (0, 1]$, we restrict to

$$
N > [nq/p].
$$

For a given $\lambda > \inf_{x \in \mathbb{R}^n} \mathcal{M}_N f(x)$, we set

$$
\Omega \equiv \{x \in \mathbb{R}^n : \mathcal{M}_N (f)(x) > \lambda\},
$$

which implies $\Omega$ is a proper subset of $\mathbb{R}^n$. As in [17], we give the usual Whitney decomposition of $\Omega$. Thus we can find closed cubes $Q_k$ whose interiors distance from $\Omega^c$, with $\Omega = \cup_k Q_k$ and

$$
diam(Q_k) \leq 2^{-(6+n)} dist(Q_k, \Omega) \leq 4 \text{diam}(Q_k).
$$

Next, fix $a = 1 + 2^{-(11+n)}$ and $b = 1 + 2^{-(10+n)}$; if $Q_k = aQ_k, Q_k^* = bQ_k$, then $Q_k \subset Q_k^* \subset Q_k^*$. Also, $\cup Q_k^* = \Omega$, and the $\{Q_k^*\}$ have the bounded interior property: every point is contained in at most a fixed number of the $\{Q_k\}$.

Fix a positive smooth function $\xi$ that equal 1 in the cube of side length 1 centered at the origin and vanishes outside the concentric cube of side length $a$. We set $\xi_k(x) = \xi((x-x_k)/l_k)$, where $x_k$ is the center of the cube $Q_k$ and $l_k$ is its side length. Obviously, for any $x \in \mathbb{R}^n$, we have $1 \leq \sum_k \xi_k(x) \leq L$. Write
\( \eta_k = \xi_k / (\sum_j \xi_j) \). The \( \eta_k \) form a partition of unity for the set \( \Omega \) subordinate to the locally finite cover \( \{ Q_k \} \) of \( Q \); that is to say, \( \chi_{\Omega} = \sum \eta_k \) with each \( \eta_k \) supported in the cube \( Q_k \).

Let \( s \in \mathbb{N}_0 \) be some fixed integers and \( \mathcal{P}_s(\mathbb{R}^n) \) denote the linear space of polynomials in \( n \) variables of degrees no more than \( s \). For each \( i \) and \( P \in \mathcal{P}_s(\mathbb{R}^n) \), set

\[
\| P \|_i \equiv \left[ \frac{1}{\int_{\mathbb{R}^n} \eta_i(x)dx} \int_{\mathbb{R}^n} |P(x)|^2 \eta_i(x)dx \right]^{1/2}.
\]

Then \( (\mathcal{P}_s(\mathbb{R}^n), \| \cdot \|_i) \) is a finite dimensional Hilbert space. Let \( f \in \mathcal{D}'(\mathbb{R}^n) \). Since \( f \) induces a linear functional on \( \mathcal{P}_s(\mathbb{R}^n) \) via \( Q \rightarrow 1/\int_{\mathbb{R}^n} \eta_i(x)dx < f, Q \eta_i > \), by the Riesz lemma, there exists a unique polynomial \( P_i \in \mathcal{P}_s(\mathbb{R}^n) \) for each \( i \) such that for all \( Q \in \mathcal{P}_s(\mathbb{R}^n) \),

\[
\frac{1}{\int_{\mathbb{R}^n} \eta_i(x)dx} < f, Q \eta_i > = \frac{1}{\int_{\mathbb{R}^n} \eta_i(x)dx} < P_i, Q \eta_i > = \frac{1}{\int_{\mathbb{R}^n} \eta_i(x)dx} \int_{\mathbb{R}^n} P_i(x)Q(x)\eta_i(x)dx.
\]

For every \( i \), define distribution \( b_i = (f - P_i)\eta_i \) if \( l_i < 1 \), we set \( b_i = f \eta_i \) if \( l_i \geq 1 \).

We will show that for suitable choices of \( s \) and \( N \), the series \( \sum_i b_i \) converges in \( \mathcal{D}'(\mathbb{R}^n) \), and in this case, we define \( g = f - \sum_i b_i \) in \( \mathcal{D}'(\mathbb{R}^n) \).

The representation \( f = g + \sum_i b_i \), where \( g \) and \( b_i \) are as above, is said to be a Calderón-Zygmund decomposition of degree \( s \) and the height \( \lambda \) associated with \( \mathcal{M}_N(f) \).

The rest of this section consists of series of lemmas. In Lemma 4.1 and Lemma 4.2, we give some properties of the smooth partition of unity \( \{ \eta_i \} \). In Lemmas 4.3 through 4.6, we derive some estimates for the bad parts \( \{ b_i \} \). Lemma 4.7 and Lemma 4.8 give controls over the good part \( g \). Finally, Corollary 4.1 shows the density of \( L^q(\mathbb{R}^n) \cap h^p_{\omega, N}(\mathbb{R}^n) \) in \( h^p_{\omega, N}(\mathbb{R}^n) \), where \( q \in (q_\omega, \infty) \).

**Lemma 4.1.** There exists a positive constant \( C_1 \), depending only on \( N \), such that for all \( i \) and \( l \leq l_i \),

\[
\sup_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} |\partial^\alpha \eta_i(lx)| \leq C_1.
\]

Lemma 4.1 is essentially Lemma 5.2 in [1].

**Lemma 4.2.** If \( l_i < 1 \), then there exists a constant a constant \( C_2 > 0 \) independent of \( f \in \mathcal{D}'(\mathbb{R}^n) \), \( l_i \) and \( \lambda > 0 \) so that

\[
\sup_{y \in \mathbb{R}^n} |P_i(y)\eta_i(y)| \leq C_2 \lambda.
\]
Proof. As in the proof of Lemma 5.3 in [1]. Let \(\pi_l, \cdots, \pi_m\) \((m = \text{dim} \mathcal{P}_s)\) be an orthonormal basis of \(\mathcal{P}_s\) with respect to the norm (4.1). we have

\[
P_i = \sum_{k=1}^{m} \left( \frac{1}{\eta_i} \int f(x) \pi_k(x) \eta_i(x) dx \right) \overline{\pi}_k,
\]

where the integral is understood as \(<f, \pi_k \eta_i>\). Hence

\[
1 = \frac{1}{\eta_i} \int |\pi_k(x)|^2 \eta_i(x) dx \geq \frac{2^{-n}}{|Q_k|} \int_{Q_k} |\pi_k(x)|^2 \eta_i(x) dx
\]

\[
\geq \frac{2^{-n}}{|Q_k|} \int_{Q_k} |\pi_k(x)|^2 dx = 2^{-n} \int_{Q^0} |\pi_k(x)|^2 dx,
\]

where \(\pi_k(x) = \pi_k(x_i + l_i x)\) and \(Q^0\) denotes the cube of side length 1 centered at the origin.

Since \(\mathcal{P}_s\) is finite dimensional all norms on \(\mathcal{P}_s\) are equivalent, there exists \(A_1 > 0\) such that for all \(p \in \mathcal{P}_s\)

\[
\sup_{|\alpha| \leq s} \sup_{z \in bQ^0} |\partial^\alpha P(z)| \leq A_1 \left( \int_{Q^0} |P(z)|^2 dz \right)^{1/2}.
\]

From this and (4.3), for \(k = 1, \cdots, m\), we have

\[
\sup_{|\alpha| \leq s} \sup_{z \in bQ^0} |\partial^\alpha \pi_k(z)| \leq A_1.
\]

For \(k = 1, \cdots, m\) define

\[
\Phi_k(y) = \frac{l_i}{\eta_i} \pi_k(z - l_i y) \eta_i(z - l_i y),
\]

where \(z\) is some point in \(2^{n+\alpha} n Q_k \cap \Omega^c\).

It is easy to see that \(\text{supp} \Phi_k \subset B_n := B(0, 2^{3(10+n)})\) and \(\|\Phi_k\|_{\mathcal{D}_N} \leq A_2\) by Lemma 4.1.

Note that

\[
\frac{1}{\eta_i} \int f(x) \pi_k(x) \eta_i(x) dx = (f * (\Phi_k)_{l_i})(z),
\]

since \(l_i < 1\), we then have

\[
\frac{1}{\eta_i} \int f(x) \pi_k(x) \eta_i(x) dx \leq M_N f(z) \|\Phi_k\|_{\mathcal{D}_N} \leq A_2 \lambda.
\]

By (4.2), (4.4) and above estimate

\[
\sup_{z \in Q^*} |P_i(z)| \leq mA_1 A_2 \lambda.
\]

Thus,

\[
\sup_{z \in \mathbb{R}^n} |P_i(z) \eta_i(z)| \leq C_2 \lambda.
\]

The proof is complete.
Lemma 4.3. There exists a constant $C_3 > 0$ such that
\[ \mathcal{M}_N^0 b_i(x) \leq C_3 \mathcal{M}_N f(x) \quad \text{for } x \in Q^*. \] (4.5)

Proof. Take $\varphi \in \mathcal{D}_N^0$, and $x \in Q_i^*$. 
Case I. For $t \leq l_i$, we write
\[ (b_t \ast \varphi_t)(x) = (f \ast \Phi_t)(x) - ((P_t \eta_t) \ast \varphi_t)(x), \]
where $\Phi(z) := \varphi(z) \eta_t(x-tz)$. Define $\bar{\eta}_t(z) = \eta_t(x-l_i z)$. Obviously, $\text{supp} \Phi \subset B_n$. By Lemma 4.1, there exists a positive constant $C$ such that
\[ \|\Phi\|_{\mathcal{D}_N} \leq A_1 C. \]
Note that for $N \geq 2$ there is a constant $C > 0$ so that $\|\varphi\|_{L^1(\mathbb{R}^n)} \leq C$ for all $\varphi \in \mathcal{D}_N^0$. Therefore, by Lemma 4.2 and (4.5), we have
\[ |b_t \ast \varphi_t(x)| \leq \|\Phi\|_{\mathcal{D}_N} \mathcal{M}_N f(x) + C_2 \lambda \|\varphi\|_{L^1(\mathbb{R}^n)} \leq C_3 \mathcal{M}_N f(x), \]
since $\mathcal{M}_N f(x) > \lambda$ for $x \in \Omega$.
Case II. For $l_i < t < 1$ by a simple calculation we can write
\[ (b_t \ast \varphi_t)(x) = \frac{l_i}{t} (f \ast \Phi_{l_i})(x) - ((P_t \eta_t) \ast \varphi_t)(x), \]
where $\Phi(z) = \varphi(l_i z/t) \eta_t(x-l_i z)$. Define $\bar{\varphi}(z) := \varphi(l_i z/t)$ and $\bar{\eta}_i(z) = \eta_t(x-l_i z)$. It is easy to see that $\text{supp} \Phi \subset B_n$. By Lemma 4.1, we can find a positive constant $C$ independent of $1 > t > l_i$ so that
\[ \sup_{|\alpha| \leq N} \sup_{z \in \mathbb{R}^n} |\partial^\alpha \bar{\varphi}(z)| \leq C, \quad \sup_{|\alpha| \leq N} \sup_{z \in \mathbb{R}^n} |\partial^\alpha \bar{\eta}_i(z)| \leq A_1. \]
Hence, there exists a positive constant $C$ such that $\|\Phi\|_{\mathcal{D}_N} \leq C$, and $\|\varphi\|_{L^1(\mathbb{R}^n)} \leq C$ for $\varphi \in \mathcal{D}_N$ for $N \geq 2$. As in the case I
\[ |(b_t \ast \varphi_t)(x)| \leq \|\Phi\|_{\mathcal{D}_N} \mathcal{M}_N f(x) + C_2 \lambda \|\varphi\|_{L^1(\mathbb{R}^n)} \leq C \mathcal{M}_N f(x). \]
By combining both cases, we can obtain the desired result.

Lemma 4.4. Suppose $Q \subset \mathbb{R}^n$ is bounded, convex, and $0 \in Q$, and $N$ is a positive integer. Then there is a constant $C$ depending only on $Q$ and $N$ such that for every $\phi \in \mathcal{D}(\mathbb{R}^n)$ and every integer $s, 0 \leq s < N$ we have
\[ \sup_{x \in Q} \sup_{|\alpha| \leq N} |\partial^\alpha R_y(z)| \leq C \sup_{x \in Q} \sup_{0 \leq |\alpha| \leq N} |\partial^\alpha \phi(z)|, \]
where $R_y$ is the remainder of the Taylor expansion of $\phi$ of order $s$ at the point $y \in \mathbb{R}^n$. 

Lemma 4.4 is Lemma 5.5 in [1].

**Lemma 4.5.** Suppose $0 \leq s < N$. Then there exist positive constants $C_3, C_4$ so that for $i \in \mathbb{N}$,

$$
\mathcal{M}_N^0(b_i)(x) \leq C \frac{l_i^{n+s+1}}{(l_i + |x - x_i|)^{n+s+1}} \chi_{\{|x-x_i|<C_3\}}(x) \quad \text{if} \quad x \notin Q_i^r.
$$

Moreover,

$$
\mathcal{M}_N^0(b_i)(x) = 0, \quad \text{if} \quad x \notin Q_i^r \quad \text{and} \quad l_i \geq C_4.
$$

Proof. Take $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Recall that $\eta_i$ is supported in the cube $Q_i$, and we have taken $Q_i$ to be strictly contained in $Q_i^r$. Thus if $x \notin Q_i^r$ and $\eta_i(y) \neq 0$, then there exists a positive constant $C_3$ such that $|x - y| \leq |x - x_i| \leq C_3|x - y|$, and the support property of $\Phi$ requires that $1 \leq t \geq |x - y| \leq 2^{11-n}l_i$. Hence, $|x - x_i| \leq C_3 t$ and $l_i < 2^{11+n} := C_4$ and $l_i < C_4 t$. Pick some $w \in (2^{8+n}Q_i) \cap \Omega^c$.

Case I. If $1 \leq l_i < C_4$ and $\varphi \in \mathcal{D}_N^0$, where define $\phi(z) = \varphi(l_iz/t)$ and $l_i = l_i/C_4 < 1$. We have

$$
(b \ast \varphi_i)(x) = t^{-n} \int b_i \varphi((x - z)/t)dz = t^{-n} \int b_i \phi((x - z)/l_i)dz = t^{-n} \int b_i \phi(x-w)/l_i((x-z)/l_i)dz = \frac{l_i^n}{t^n}(f \ast \Phi_l_i)(w),
$$

where

$$
\Phi(z) := \phi_{(x-w)/l_i}(z)\eta_i(l_i - l_i z), \quad \phi_{(x-w)/l_i}(z) = \phi(z + (x - w)/l_i).
$$

Obviously, $\text{supp} \Phi \subset B_n$. Note that $l_i < tC_4$ and $|x - x_i| \leq C_3 t$, we obtain

$$
|(b \ast \varphi_i)(x)| \leq C \frac{l_i^n}{t^n} \mathcal{M}_N f(w) \leq C \frac{l_i^n}{t^n} \leq C \lambda \frac{l_i^{n+s+1}}{(l_i + |x - x_i|)^{n+s+1}}.
$$

(4.7)

Case II. If $l_i < 1$ and $\varphi \in \mathcal{D}_N^0$ define $\phi(z) = \varphi(l_iz/t)$. Consider the Taylor expansion of $\phi$ of order $s$ at the point $y := (x - w)/l_i$,

$$
\phi(y + z) = \sum_{|\alpha| \leq s} \frac{\partial^\alpha \phi(y)}{\alpha!} z^\alpha + R_y(z),
$$

where $R_y$ denotes the remainder.
Thus,

\[(b \ast \varphi_i)(x) = t^{-n} \int b_i \varphi((x - z)/t)dz\]

\[= t^{-n} \int b_i \phi((x - z)/l_i)dz\]

\[= t^{-n} \int b_i R_{(x-w)/l_i}((w - z)/l_i)dz\]

\[= \frac{t^n}{l_i} (f \ast \Phi_{l_i})(w) + t^{-n} \int P_i(z)\eta_i(z)R_{(x-w)/l_i}((w - z)/l_i)dz,\]

where

\[\Phi(z) := R_{(x-w)/l_i}(z)\eta_i(w - l_i z).\]

Obviously, supp\(\Phi \subset B_n\). Apply Lemma 4.4 to \(\phi(z) = \varphi(l_i z/t), \ y = (x - w)/l_i \) and \(Q = B_n\). We have

\[\sup_{z \in B_n} \sup_{|a| \leq N} |\partial^a R_y(z)| \leq C \sup_{z \in y + B_n} \sup_{s+1 \leq |a| \leq N} |\partial^a \phi(z)|\]

\[\leq C \sup_{z \in y + B_n} \left(\frac{l_i}{t}\right)^{-s} \sup_{s+1 \leq |a| \leq N} |\partial^a \varphi(l_i z/t)|\]

\[\leq C \left(\frac{l_i}{t}\right)^{-s}.\]

Note that \(l_i < tC_4 \) and \(|x - x_i| \leq C_3 t\), therefore by (4.8), we have

\[(b \ast \varphi_i)(x) \leq \frac{t^n}{l_i} (f \ast \Phi_{l_i})(w) + t^{-n} \int |P_i(z)\eta_i(z)R_{(x-w)/l_i}((w - z)/l_i)|dz\]

\[\leq C \left(\frac{l_i}{t^n} \mathcal{M}_N f(w) \|\Phi\|_{D_N} + \lambda \sup_{z \in B_n} \sup_{|a| \leq N} |\partial^a R_y(z)|\right)\]

\[\leq C \lambda \left(\frac{l_i}{t} + |x - x_i|\right)^{n+s+1}.\]

Combining (4.7) and (4.9), we obtain (4.6).

**Lemma 4.6.** Let \(\omega \in A_{\infty}^{loc}\) and \(q_{\omega}\) be as in (2.3). If \(p \in (0, 1]\), \(s \geq [nq_{\omega}/p]\) and \(N > s\), there exists a positive constant \(C_5\) such that for all \(f \in h^p_{\omega, N}(\mathbb{R}^n)\), 

\[
\int_{\mathbb{R}^n} [\mathcal{M}^0_N(b_i)(x)]^p \omega(x)dx \leq C_5 \int_{Q^*_i} [\mathcal{M}_N(f)(x)]^p \omega(x)dx. \tag{4.10}
\]

Moreover the series \(\sum_i b_i\) converges in \(h^p_{\omega, N}(\mathbb{R}^n)\) and

\[
\int_{\mathbb{R}^n} [\mathcal{M}^0_N(\sum b_i)(x)]^p \omega(x)dx \leq C_5 \int_{\Omega} [\mathcal{M}_N(f)(x)]^p \omega(x)dx. \tag{4.11}
\]
Proof. By Lemma 4.4, we have

$$
\int_{\mathbb{R}^n} [\mathcal{M}_N^0(b_i)(x)]^p \omega(x)dx \leq \int_{Q_i^*} [\mathcal{M}_N^0(b_i)(x)]^p \omega(x)dx \\
+ \int_{C_3 Q_i^0 \setminus Q_i^*} [\mathcal{M}_N^0(b_i)(x)]^p \omega(x)dx,
$$

where $Q_i^0 = Q(x_i, 1)$. Note that $s \geq \lceil n q_\omega/p \rceil$ implies $2^{-n(q_\omega+\eta)/2(s+n+1)p} > 1$ for sufficient small $\eta > 0$. Using Lemma 2.1 (iii) with $\omega \in A^\text{loc}_{q_\omega+\eta}$, Lemma 4.5 and the fact that $M_N(f)(x) > \lambda$ for all $x \in Q_i^*$, we have

$$
\int_{C_3 Q_i^0 \setminus Q_i^*} [\mathcal{M}_N^0(b_i)(x)]^p \omega(x)dx \leq \sum_{k=0}^{k_0} \int_{2^k Q_i^0 \setminus 2^{k-1} Q_i^*} [\mathcal{M}_N^0(b_i)(x)]^p \omega(x)dx \\
\leq \lambda^p \omega(Q_i^*) \sum_{k=0}^{k_0} [2^{-n(q_\omega+\eta)+(s+n+1)p}]^{-k} \\
\leq C \int_{Q_i^*} [\mathcal{M}_N f(x)]^p \omega(x)dx,
$$

where $k_0 \in \mathbb{Z}$ such that $2^{k_0-1} \leq C_3 < 2^{k_0}$.

Combining (4.12) and (4.13), then (4.10) holds. By (4.10), we have

$$
\int_{\mathbb{R}^n} [\mathcal{M}_N^0(b_i)(x)]^p \omega(x)dx \leq C \sum_i \int_{Q_i^*} [\mathcal{M}_N f(x)]^p \omega(x)dx \leq C \int_{\Omega} [\mathcal{M}_N f(x)]^p \omega(x)dx,
$$

which together with complete of $h^p_{\omega,N}$ (see Proposition 3.2) implies that $\sum_i b_i$ converges in $h^p_{\omega,N}$. So by Proposition 3.1, the series $\sum_i b_i$ converges in $\mathcal{D}'(\mathbb{R}^n)$, and therefore $\mathcal{M}_N^0(\sum_i b_i)(x) \leq \sum_i \mathcal{M}_N^0(b_i)(x)$, which gives (4.11). Thus, Lemma 4.6 is proved.

**Lemma 4.7.** Let $\omega \in A^\text{loc}_0$ and $q_\omega$ be as in (2.3), $s \in \mathbb{N}_0$, and $N \geq 2$. If $q \in (q_\omega, \infty]$ and $f \in L^q_\omega(\mathbb{R}^n)$, then the series $\sum_i b_i$ converges in $L^q_\omega(\mathbb{R}^n)$ and there exists a positive constant $C_6$, independent of $f$ and $\lambda$, such that $\| \sum_i |b_i| \|_{L^q_\omega(\mathbb{R}^n)} \leq C_6 \| f \|_{L^q_\omega(\mathbb{R}^n)}$.

Proof. The proof for $q = \infty$ is similar to that for $q \in (q_\omega, \infty)$. So we only give the proof for $q \in (q_\omega, \infty)$. Set $F_1 = \{ i \in \mathbb{N} : |Q_i| \geq 1 \}$ and $F_2 = \{ i \in \mathbb{N} : |Q_i| < 1 \}$. By lemma 4.3, for $i \in F_2$, we have

$$
\int_{\mathbb{R}^n} |b_i(x)\omega(x)|dx \leq \int_{Q_i^*} |f(x)|^q \omega(x)dx + \int_{Q_i^*} |P_i(x)\eta_i(x)|^q \omega(x)dx \\
\leq \int_{Q_i^*} |f(x)|^q \omega(x)dx + \lambda^q \omega(Q_i^*).
$$
For \( i \in F_1 \), we have
\[
\int_{\mathbb{R}^n} |b_i(x)| \omega(x) dx \leq \int_{Q_i^*} |f(x)|^q \omega(x) dx.
\]
From these, we obtain
\[
\sum_i \int_{\mathbb{R}^n} |b_i(x)| \omega(x) dx = \sum_{i \in F_1} \int_{\mathbb{R}^n} |b_i(x)| \omega(x) dx + \sum_{i \in F_2} \int_{\mathbb{R}^n} |b_i(x)| \omega(x) dx
\leq \int_{Q_i^*} |f(x)|^q \omega(x) dx + \int_{Q_i^*} |P_i(x)\eta_i(x)|^q \omega(x) dx
\leq \sum_i \int_{Q_i^*} |f(x)|^q \omega(x) dx + C \sum_{i \in F_2} \lambda^q \omega(Q_i^*)
\leq \sum_i \int_{Q_i^*} |f(x)|^q \omega(x) dx + C \lambda^q \omega(Q)
\leq C_6 \int_{\mathbb{R}^n} |f(x)|^q \omega(x) dx.
\]
From this and applying \( b_i \) have finite covers, we have
\[
\| \sum_i |b_i| \|_{L^q_\omega(\mathbb{R}^n)} \leq C_6 \| f \|_{L^q_\omega(\mathbb{R}^n)}.
\]
The proof is finished.

**Lemma 4.8.** If \( N > s \geq 0 \) and \( \sum_i b_i \) converges in \( \mathcal{D}'(\mathbb{R}^n) \), then there exists a positive constant \( C_7 \), independent of \( f \) and \( \lambda \), such that for all \( x \in \mathbb{R}^n \),
\[
\mathcal{M}^0_N(g)(x) \leq \mathcal{M}^0_N(f)(x) \chi_{\Omega^c}(x) + C_7 \frac{l_i^{n+s+1}}{(l_i + |x - x_i|)^n + s + 1} \chi_{\{|x - x_i| < C_3\}}(x)
\]
Proof. If \( x \notin \Omega \), since \( \mathcal{M}^0_N(g)(x) \leq \mathcal{M}^0_N(f)(x) + \sum_i \mathcal{M}^0_N(b_i)(x) \), by Lemma 4.5, we obtain
\[
\mathcal{M}^0_N(g)(x) \leq \mathcal{M}^0_N(f)(x) \chi_{\Omega^c}(x) + C \sum_i \frac{l_i^{n+s+1}}{(l_i + |x - x_i|)^n + s + 1} \chi_{\{|x - x_i| < C_3\}}(x).
\]
If \( x \in \Omega \), choose \( k \in \mathbb{N} \) such that \( x \in Q_k^* \). Let \( J := \{ i \in \mathbb{N} : Q_i^* \cap Q_k^* \neq \emptyset \} \). Then the cardinality of \( J \) is bounded by \( L \). By Lemma 4.5, we have
\[
\sum_{i \in J} \mathcal{M}^0_N(b_i)(x) \leq C \lambda \sum_{i \notin J} \frac{l_i^{n+s+1}}{(l_i + |x - x_i|)^n + s + 1} \chi_{\{|x - x_i| < C_3\}}(x).
\]
It suffices to estimate the grand maximal function of \( g + \sum_{i \neq J} b_i = f - \sum_{i \in J} b_i \). Take \( \varphi \in \mathcal{D}_N^0 \) and \( 0 < t < 1 \). We write

\[
(f - \sum_{i \in J} b_i) \ast \varphi_t(x) = (f \xi) \ast \varphi_t + (\sum_{i \in J} P_i \eta_i) \ast \varphi_t
\]

\[
= f \ast \Phi_t(w) + (\sum_{i \in J} P_i \eta_i) \ast \varphi_t,
\]

where \( w \in (2^{8+n}Q_k) \cap \Omega^c \), \( \xi = 1 - \sum_{i \in J} \eta_i \) and

\[
\Phi(z) := \varphi(z + (x - w)/t) \xi(w - tz).
\]

Since for \( N \geq 2 \) there is a constant \( C > 0 \) so that \( \| \varphi \|_{L^1(\mathbb{R}^n)} \leq C \) for all \( \varphi \in \mathcal{D}_N^0 \) and Lemma 4.1, we have

\[
\left| \left( \sum_{i \in J} P_i \eta_i \right) \ast \varphi_t(x) \right| \leq C \lambda.
\]

Finally, we estimate \( f \ast \Phi_t(w) \). There are two cases: If \( t \leq 2^{-(11+n)l_k} \), then \( f \ast \Phi_t(w) = 0 \), because \( \xi \) vanishes in \( Q^*_k \) and \( \varphi_t \) is supported in \( B(0, t) \). On the other hand, if \( t \geq 2^{-(11+n)l_k} \), then there exists a positive constant \( C \) such that \( \text{supp} \Phi \subset B_n \) and \( \| \Phi \|_{D_N} \leq C \). Hence,

\[
|(f \ast \Phi_t)| \leq \mathcal{M}_N f(w) \| \Phi \|_{D_N} \leq C \lambda.
\]

By the above estimates, we have

\[
|(f - \sum_{i \in J} b_i) \ast \varphi_t| \leq C \lambda.
\]

That is,

\[
\mathcal{M}_N^0((f - \sum_{i \in J} b_i))(x) \leq C \lambda.
\]

Thus, Lemma 4.8 is proved.

**Lemma 4.9.** Let \( \omega \in A^\infty_{\infty} \), \( q_\omega \) be as in \( (2.3) \) and \( p \in (0, 1] \).

(i) If \( N > s \geq [nq_\omega/p] \) and \( \mathcal{M}_N(f) \in L^p_\omega(\mathbb{R}^n) \), then \( \mathcal{M}_N(g) \in L^1_\omega(\mathbb{R}^n) \) and there exists a positive constant \( C_8 \), independent of \( f \) and \( \lambda \), such that

\[
\int_{\mathbb{R}^n} [\mathcal{M}_N^0(g)(x)]^q \omega(x)dx \leq C_8 \lambda^{1-p} \int_{\mathbb{R}^n} [\mathcal{M}_N(f)(x)]^p \omega(x)dx.
\]

(ii) If \( N \geq 2 \) and \( f \in L^1_\omega(\mathbb{R}^n) \), then \( g \in L^\infty_\omega(\mathbb{R}^n) \) and there exists a positive constant \( C_9 \), independent of \( f \) and \( \lambda \), such that \( \| g \|_{L^\infty_\omega} \leq C_9 \lambda \).
Proof. Since \( f \in h^{p}_{\omega,N}(\mathbb{R}^{n}) \), by Lemma 4.6, \( \sum b_{i} \) converges in \( h^{p}_{\omega,N}(\mathbb{R}^{n}) \) and there in \( \mathcal{D}'(\mathbb{R}^{n}) \) by proposition 3.1. Observe that \( s \geq \lfloor np_{\omega}/p \rfloor \), by Lemma 4.8, we obtain

\[
\int_{\mathbb{R}^{n}} [\mathcal{M}^{0}_{N}(g)(x)]\omega(x)dx \leq C\lambda \sum_{i} \int_{\mathbb{R}^{n}} \frac{l_{i}^{(n+s+1)}}{(l_{i} + |x - x_{i}|)^{(n+s+1)}} \chi_{|x - x_{i}|<c_{3}}(x)\omega(x)dx
\]

\[
+ \int_{\Omega^{c}} [\mathcal{M}_{N}(f)(x)]\omega(x)dx
\]

\[
\leq C\lambda \omega(\Omega) + C\lambda^{1-p} \int_{\Omega^{c}} [\mathcal{M}_{N}(f)(x)]^{p}\omega(x)dx
\]

\[
\leq C\lambda^{1-p} \int_{\Omega^{c}} [\mathcal{M}_{N}(f)(x)]^{p}\omega(x)dx.
\]

Thus, (i) holds.

Moreover, if \( f \in L^{1}_{\omega}(\mathbb{R}^{n}) \), then \( g \) and \( \{b_{i}\} \) are functions, and Lemma 4.7, \( \sum b_{i} \) converges in \( L^{q}_{\omega}(\mathbb{R}^{n}) \) and thus in \( \mathcal{D}'(\mathbb{R}^{n}) \) by Lemma 2.4. Write

\[
g = f - \sum b_{i} = f(1 - \sum i \eta_{i}) + \sum \eta_{i} P_{i} \eta_{i} = f \chi_{\Omega^{c}} + \sum \eta_{i} P_{i} \eta_{i}.
\]

By Lemma 4.3, we have \( |g(x)| \leq C\lambda \) for all \( x \in \Omega \), and by Proposition 2.3, \( |g(x)| = |f(x)| \leq \mathcal{M}_{N}f(x) \leq \lambda \) for almost everywhere \( x \in \Omega^{c} \), which leads to that \( \|g\|_{L^{\infty}_{\omega}(\mathbb{R}^{n})} \leq C\lambda \) and thus yields (ii). The proof is finished.

**Corollary 4.1.** Let \( \omega \in A^{loc}_{\infty} \) and \( q_{\omega} \) be as in (2.3). If \( q \in (q_{\omega}, \infty), N > [np_{\omega}/p] \) and \( p \in (0,1] \), then \( h^{p}_{\omega,N}(\mathbb{R}^{n}) \cap L^{1}_{\omega}(\mathbb{R}^{n}) \) is dense in \( h^{p}_{\omega,N}(\mathbb{R}^{n}) \).

Proof. Let \( f \in h^{p}_{\omega,N}(\mathbb{R}^{n}) \). For any \( \lambda > \inf_{x \in \mathbb{R}^{n}} \mathcal{M}_{N}f(x) \), let \( f = g^{\lambda} + \sum b_{i}^{\lambda} \) be the Calderón-Zygmund decomposition of \( f \) of degree \( s \) with \( \lfloor np_{\omega}/p \rfloor \leq s < N \) and height \( \lambda \) associated to \( \mathcal{M}_{N}f \). By Lemma 4.6,

\[
\| \sum b_{i}^{\lambda} \|_{h^{p}_{\omega,N}(\mathbb{R}^{n})} \leq C \int_{\{x \in \mathbb{R}^{n}: \mathcal{M}_{N}f(x) > \lambda\}} [\mathcal{M}_{N}f(x)]^{p}\omega(x)dx.
\]

Therefore, \( g^{\lambda} \rightarrow f \) in \( h^{p}_{\omega,N}(\mathbb{R}^{n}) \) as \( \lambda \rightarrow \infty \). But by Lemma 4.9, \( \mathcal{M}_{N}(g^{\lambda}) \in L^{1}_{\omega}(\mathbb{R}^{n}) \), so by Proposition 2.2, \( g^{\lambda} \in L^{1}_{\omega}(\mathbb{R}^{n}) \). Thus, Corollary 4.1 is proved.

**5. Weighted atomic decompositions of \( h^{p}_{\omega,N}(\mathbb{R}^{n}) \)**

We will follow the proof of atomic decomposition as presented by Stein in [16].

In this section, we take \( k_{0} \in \mathbb{Z} \) such that \( 2^{k_{0} - 1} \leq \inf_{x \in \mathbb{R}^{n}} \mathcal{M}_{N}f(x) < 2^{k_{0}} \), if \( \inf_{x \in \mathbb{R}^{n}} \mathcal{M}_{N}f(x) = 0 \), write \( k_{0} = -\infty \). Let \( \omega \in A^{loc}_{\infty}, q_{\omega} \) be as in (2.3), \( p \in (0,1] \)
and \( N > s \equiv \lfloor nq_\omega/p \rfloor \). Let \( f \in h^p_{\omega,N}(\mathbb{R}^n) \). For each integer \( k \geq k_0 \) consider the Calderón-Zygmund decomposition of \( f \) of degree \( s \) and height \( \lambda = 2^k \) associated to \( M_N f \),

\[
f = g^k + \sum_{i \in \mathbb{N}} b_i^k,
\]

where

\[
\Omega^k := \{ x \in \mathbb{R}^n : M_N f(x) > 2^k \}, \quad Q^k_i := Q^k_{i,j}
\]

and \( b_i^k := (f - P^k_i)\eta_i^k \) if \( l_i^k < 1 \) and \( b_i^k := f \eta_i^k \) if \( l_i^k \geq 1 \).

Recall that for fixed \( k \geq k_0 \), \( (x_i = x^k_i)_{i \in \mathbb{N}} \) is a sequence in \( \Omega^k \) and \( (l_i = l_i^k)_{i \in \mathbb{N}} \) for \( \Omega = \Omega^k \), \( \eta_i = \eta_i^k \) given in Section 4 and \( P_i = P^k_i \) is the projection of \( f \) onto \( P_s \) with respect to the norm given in Section 4.

Define a polynomial \( P^{k+1}_{ij} \) as an orthogonal projection of \( (f - P^k_{ij})\eta_i^j \) on \( P_s \) with respect to the norm

\[
\| P \|_2 = \frac{1}{\int_{\mathbb{R}^n} \eta_{ij}^{k+1}} \int_{\mathbb{R}^n} |P(x)|^2 \eta_{ij}^{k+1}(x) dx,
\]

that is \( P^{k+1}_{ij} \) is the unique element of \( P_s \) such that

\[
\int_{\mathbb{R}^n} (f(x) - P^{k+1}_{ij}(x))\eta_i^j(x) Q(x) \eta_{ij}^{-1}(x) dx = \int_{\mathbb{R}^n} P^{k+1}_{ij}(x) Q(x) \eta_{ij}^{k+1}(x) dx.
\]

For convenience we denote \( Q^k_{ij} = (1 + 2^{-(9+n)})Q^k_i \), \( E_1 = \{ i \in \mathbb{N} : |Q_i| \geq 1/(2^4 n) \} \) and \( E_2 = \{ i \in \mathbb{N} : |Q_i| < 1/(2^4 n) \} \), \( F_1 = \{ i \in \mathbb{N} : |Q_i| \geq 1 \} \) and \( F_2 = \{ i \in \mathbb{N} : |Q_i| < 1 \} \).

There are two things we need to know about the polynomials \( P^{k+1}_{ij} \). First, \( P^{k+1}_{ij} \neq 0 \) only if \( Q^k_{ij} \cap Q^{k+1}_{ij} \neq \emptyset \); this follows directly from the the definition of \( P^{k+1}_{ij} \) (since it involves \( \eta_{i}^{k+1} \), which is supported in \( Q^k_{ij} \)). More precisely, we have the following results.

**Lemma 5.1.** Note that \( \Omega^{k+1} \subset \Omega^k \); then

(i) If \( Q^k_{ij} \cap Q^{k+1}_{ij} \neq \emptyset \), then \( l_i^{k+1} \leq 2^4 \sqrt{n} l_i^k \) and \( Q^{k+1}_{ij} \subset 2^6 n Q^k_{ij} \subset \Omega^k \).

(ii) There exists a positive \( L \) such that for each \( i \in \mathbb{N} \) the cardinality of \( \{ i \in \mathbb{N} : Q^k_{ij} \cap Q^{k+1}_{ij} \neq \emptyset \} \) is bounded by \( L \).

**Lemma 5.2.** If \( l_i^{k+1} < 1 \),

\[
\sup_{y \in \mathbb{R}^n} |P^{k+1}_{ij}(y)\eta_{ij}^{k+1}(y)| \leq C 2^{k+1}.
\]

**Lemma 5.3.** For every \( k \in \mathbb{Z} \), \( \sum_{i \in \mathbb{N}} (\sum_{j \in F_2} P^{k+1}_{ij} \eta_{ij}^{k+1}) = 0 \), where the series converges pointwise and in \( D'(\mathbb{R}^n) \).
Lemmas 5.1-5.3 can be proved by the methods in Lemmas 6.1-6.3 in [1].

The following lemma establishes the weighted atomic decompositions for a dense subspace of $h^p_{\omega,N}(\mathbb{R}^n)$.

**Lemma 5.4.** Let $\omega \in A^\text{loc}_\infty$ and $q_\omega$ be as in (2.3). If $p \in (0,1]$, $s \geq [nq_\omega/p]$ and $N > s$, then for any $f \in (L^1(\mathbb{R}^n) \cap h^p_{\omega,N}(\mathbb{R}^n))$, there exists numbers $\lambda_0$ and \{\lambda_i^k\}_{k \in \mathbb{Z},i} \subset \mathbb{C}$, $(p,\infty,s)$-atoms $\{a_i^k\}_{k \in \mathbb{Z},i}$ and single atom $a_0$ such that

$$f = \sum_{k \in \mathbb{Z}} \sum_i \lambda_i^k a_i^k + \lambda_0 a_0,$$

where the series converges almost everywhere and in $\mathcal{D}'(\mathbb{R}^n)$, moreover, there exists a positive $C$, independent of $f$, such that $\sum_{k \in \mathbb{Z},i} |\lambda_i^k|^p + |\lambda_0|^p \leq C\|f\|_{h^p_{\omega,N}(\mathbb{R}^n)}$.

Proof. Let $f \in (L^1(\mathbb{R}^n) \cap h^p_{\omega,N}(\mathbb{R}^n))$. We first consider the case $k_0 = -\infty$. For each $k \in \mathbb{Z}$, $f$ has a Calderón-Zygmund decomposition of degree $s \geq [nq_\omega/p]$ and height $2^k$ associated to $\mathcal{M}_N(f), f = g^k + \sum_i b_i^k$ as above. By Corollary 4.1 and Proposition 3.1, $g^k \to f$ in both $h^p_{\omega,N}(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$ as $k \to \infty$. By Lemma 4.9(ii), $\|g^k\|_{L^p(\mathbb{R}^n)} \to 0$ as $k \to -\infty$, and moreover, by Lemma 2.2 (ii), $g^k \to 0$ in $\mathcal{D}'(\mathbb{R}^n)$ as $k \to -\infty$. Therefore,

$$f = \sum_{k=-\infty}^{\infty} (g^{k+1} - g^k)$$  \hspace{1cm} (5.2)

in $\mathcal{D}'(\mathbb{R}^n)$. Moreover, since $\text{supp}(\sum_i b_i^k) \subset \Omega_k$ and $\omega(\Omega_k) \to 0$ as $k \to \infty$, then $g^k \to f$ almost everywhere as $k \to \infty$. Thus, (5.2) also holds almost everywhere.

By Lemma 5.1 and $\sum_i \eta_i^k b_j^{k+1} = \chi_{\Omega_k} b_j^{k+1} = b_j^{k+1}$ for all $j$, then $\sum_i \eta_i^k b_j^{k+1} = \chi_{\Omega_k} b_j^{k+1} = b_j^{k+1}$ for all $j$,

$$g^{k+1} - g^k = \left( f - \sum_j b_j^{k+1} \right) - \left( f - \sum_i b_i^k \right)$$

$$= \sum_i b_i^k - \sum_j b_j^{k+1}$$

$$= \sum_i \left[ b_i^k - \sum_{j \in F_1^k} b_j^{k+1} + \sum_{j \in F_2^k} b_j^{k+1} \eta_i^k \right]$$

$$= \sum_i h_i^k.$$

where $F_1^k = \{ i \in \mathbb{N} : |Q_i^k| \geq 1 \}$ and $F_2^k = \{ i \in \mathbb{N} : |Q_i^k| < 1 \}$ and the series converges in $\mathcal{D}'(\mathbb{R}^n)$ and almost everywhere. Furthermore, we rewrite $h_i^k$ into

$$h_i^k = f\chi_{(\Omega_{k+1})} \eta_i^k - P_i^k \eta_i^k + \sum_j P_j^{k+1} \eta_i^k \eta_j^{k+1} + \sum_{j \in F_2} P_j^{k+1} \eta_j^{k+1}.$$
By proposition 2.2, $|f(x)| \leq M_N f(x) \leq 2^{k+1}$ for almost everywhere $x \in (\Omega_{k+1})^c$, and by Lemma 4.2 and (5.1),

$$\|h_i^k\|_{L^\infty(\mathbb{R}^n)} \leq C 2^k$$ for $i \in \mathbb{N}$. (5.3)

Next we consider three cases about $i$.

Case I. When $i \in F_1$, we have

$$h_i^k = f \eta_i^k + \sum_{j \in F_1} f \eta_j^{k+1} \eta_i^k + \sum_{j \in F_2} (f - P_j^{k+1}) \eta_j^{k+1} \eta_i^k + \sum_{j \in F_2} P_{ij}^{k+1} \eta_i^{k+1}. $$

Case II. When $i \in E_1 \cap F_2$, we have

$$h_i^k = (f - P_i^k) \eta_i^k + \sum_{j \in F_1} f \eta_j^{k+1} \eta_i^k + \sum_{j \in F_2} (f - P_j^{k+1}) \eta_j^{k+1} \eta_i^k + \sum_{j \in F_2} P_{ij}^{k+1} \eta_i^{k+1}.$$ 

Case III. When $i \in E_2$, if $j \in F_1$, then $l_i^k < l_j^{k+1}/(2^4n)$, so $Q_i^{k*} \cap Q_j^{k+1*} = \emptyset$ by Lemma 5.1 (i). Thus, we have

$$h_i^k = (f - P_i^k) \eta_i^k + \sum_{j \in F_1} f \eta_j^{k+1} \eta_i^k + \sum_{j \in F_2} (f - P_j^{k+1}) \eta_j^{k+1} \eta_i^k + \sum_{j \in F_2} P_{ij}^{k+1} \eta_i^{k+1}.$$ 

We next let $\gamma = 1 + 2^{-12-n}$.

For Cases I and II. Obviously, $h_i^k$ is supported in a cube $Q_i^k$ that contains $Q_i^{k*}$ as well as all the $Q_j^{k+1*}$ that intersect $Q_i^{k*}$. In fact, observe that if $Q_i^{k*} \cap Q_j^{k+1*} \neq \emptyset$, by Lemma 5.1, we have

$$Q_j^{k+1*} \subset 2^n Q_j^{k*} \subset \Omega^k.$$ 

So, if $l_i^k < Ln/(\gamma - 1)$, we set

$$Q_i^k := 2^n Q_j^{k*}.$$ 

On the other hand, note that $l_j^{k+1} > 1$ and $l_i^k \geq 2^{-n-4}$, then $Q_j^{k+1*} \subset Q(x_i^k, l_i^k + Ln)$. So, if $l_i^k \geq Ln/(\gamma - 1)$, we set $Q_i^k = \gamma Q_j^k$. Hence,

$$Q_j^{k+1*} \subset Q(x_i^k, l_i^k + Ln) \subset Q_i^k = \gamma Q_j^k = Q_i^{k*} \subset \Omega^k,$$

if $l_i^k \geq Ln/(\gamma - 1)$.

From these, for Cases I and II, there exists a positive constant $C_{10}$ such that

$$Q_i^k \subset \Omega^k,$$ and $\omega(Q_i^k) \leq C_{10} \omega(Q_i^{k*}).$

But, $h_i^k$ does not satisfy the moment conditions.
For Case III. We claim that \( h_i^k \) is supported in a cube \( Q_i^k \) that contains \( Q_i^{k+1} \) as well as all the \( Q_j^{k+1} \) that intersect \( Q_i^{k+1} \). In fact, observe that if \( Q_i^{k+1} \cap Q_j^{k+1} \neq \emptyset \), by Lemma 5.1, we have

\[
Q_i^{k+1} \subset 2^6 n Q_j^{k+1} \subset \Omega^k.
\]

So, we set \( \bar{Q}_i^k := 2^6 n Q_j^{k+1} \). Note that \( l^j_{k+1} < 1 \) and \( l^j_k < 1 \), then

\[
\bar{Q}_i^k \subset \Omega^k, \quad \text{and} \quad \omega(\bar{Q}_i^k) \leq C_{10} \omega(Q_i^{k+1}).
\]

Moreover, \( h_i^k \) satisfies the moment conditions. This is clear for \( (f - P_i^k) \eta_i^k \) and \( (f - P_i^{k+1}) \eta_i^{k+1} + P_i^{k+1} \eta_i^{k+1} \).

Let \( \lambda_i^k = C_{10} 2^k [\omega(\bar{Q}_i^k)]^{1/p} \) and \( a_i^k = (\lambda_i^k)^{-1} h_i^k \). Moreover, by (5.3) and above Cases I, II and III, we know that \( a_i^k \) is a \((p, \infty, s)_{\omega}\)-atom. By \( \omega \in A_{q^*}^{\infty} \) and Proposition 2.1(i), we have

\[
\sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}} |\lambda_i^k|^p \leq C \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}} 2^{kp} \omega(\bar{Q}_i^k) \leq C \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}} 2^{kp} \omega(Q_i^{k+1})
\]

\[
\leq C \sum_{k \in \mathbb{Z}} 2^{kp} \omega(\Omega_k) \leq C \mathcal{M}_N(f)_p \leq C \|f\|_{h_{\omega,N}^{p}(\mathbb{R}^n)}.
\]

We now consider the case \( k_0 > -\infty \), which together with \( f \in h_{\omega,N}^{p}(\mathbb{R}^n) \) implies \( \omega(\mathbb{R}^n) < \infty \). Adapting the previous arguments, we have

\[
f = \sum_{k = k_0}^{\infty} (g^{k+1} - g^k) + g^{k_0} := \bar{f} + g^{k_0}.
\]

For the function \( \bar{f} \), we have the same \((p, \infty, s)_{\omega}\) atomic decomposition as above and

\[
\sum_{k \geq k_0} \sum_{i \in \mathbb{N}} |\lambda_i^k|^p \leq C \|f\|_{h_{\omega,N}^{p}(\mathbb{R}^n)}.
\]

For the function \( g^{k_0} \), it is easy to see that there exists a positive constant \( C_{11} \) such that

\[
\|g^{k_0}\|_{L^\infty(\mathbb{R}^n)} \leq C_{11} 2^{k_0} \leq 2C_{11} \inf_{x \in \mathbb{R}^n} \mathcal{M}_N f(x).
\]

Let

\[
a_0(x) = g^{k_0}(x) 2^{-k_0} C_{11}^{-1} [\omega(\mathbb{R}^n)]^{-1/p}, \quad \lambda_0 = C_{11} 2^{k_0} [\omega(\mathbb{R}^n)]^{1/p}.
\]

Hence,

\[
|\lambda_0|^p \leq (2C_{11})^p \|f\|_{h_{\omega,N}^{p}(\mathbb{R}^n)}, \quad \text{and} \quad \|a_0\|_{L^\infty(\mathbb{R}^n)} \leq [\omega(\mathbb{R}^n)]^{-1/p}.
\]

Then,

\[
\sum_{k \geq k_0} \sum_{i \in \mathbb{N}} |\lambda_i^k|^p + |\lambda_0|^p \leq C \|f\|_{h_{\omega,N}^{p}(\mathbb{R}^n)}.
\]

The proof of Lemma 5.4 is complete.

Remark 5.1: In fact, from the proof of Lemma 5.4, we can take all \((p, \infty, s)_{\omega}\) atoms with sidelengths \( \leq 2 \) in Lemma 5.4.

The following is one of the main results in this paper.
**Theorem 5.1.** Let $\omega \in A^\text{loc}_\infty$ and $q_\omega$ be as in (2.3). If $q \in (q_\omega, \infty]$, $p \in (0, 1]$, $N \geq N_{p,\omega}$, and $s \geq [n(q_\omega/p - 1)$, then $h^p_{\omega,q,s,\infty}(\mathbb{R}^n) = h^p_{\omega,N}(\mathbb{R}^n) = h^p_{\omega,N_{p,\omega}}(\mathbb{R}^n)$ with equivalent norms.

Proof. It is easy to see that

$$h^p_{\omega,\infty,s}(\mathbb{R}^n) \subset h^p_{\omega,q,s}(\mathbb{R}^n) \subset h^p_{\omega,N_{p,\omega}}(\mathbb{R}^n) \subset h^p_{\omega,N}(\mathbb{R}^n) \subset h^p_{\omega,N}(\mathbb{R}^n),$$

where $\bar{s}$ is an integer no less than $s$ and $\bar{N}$ is an integer larger than $N$, and the inclusions are continuous. Thus, to prove Theorem 5.1, it suffices to prove that for any $N > s \geq [q_\omega/n]$, $h^p_{\omega,N}(\mathbb{R}^n) \subset h^p_{\omega,\infty,s}(\mathbb{R}^n)$, and for all $f \in h^p_{\omega,N}(\mathbb{R}^n)$, $\|f\|_{h^p_{\omega,\infty,s}(\mathbb{R}^n)} \leq C\|f\|_{h^p_{\omega,N}(\mathbb{R}^n)}$.

To this end, let $f \in h^p_{\omega,N}(\mathbb{R}^n)$. By Corollary 4.1, there exists a sequence of functions, $\{f_m\}_{m \in \mathbb{N}} \subset (h^p_{\omega,N}(\mathbb{R}^n) \cap L^p_{\omega}(\mathbb{R}^n))$, such that $\|f_m\|_{h^p_{\omega,N}(\mathbb{R}^n)} \leq 2^{-m} \leq \|f\|_{h^p_{\omega,N}(\mathbb{R}^n)}$ and $f = \sum_{m \in \mathbb{N}} f_m$ in $h^p_{\omega,N}(\mathbb{R}^n)$. By Lemma 5.4, for each $m \in \mathbb{N}$, $f_m$ has an atomic decomposition $f_m = \sum_{i \in \mathbb{N}_0} |\lambda_i^m|^p a_i^m$ in $D'(\mathbb{R}^n)$, where $\sum_{i \in \mathbb{N}_0} |\lambda_i^m|^p \leq C\|f_m\|_{h^p_{\omega,N}(\mathbb{R}^n)}$ and \{a_i^m\}_{i \in \mathbb{N}_0} in $D'(\mathbb{R}^n)$, where $\sum_{i \in \mathbb{N}_0} |\lambda_i^m|^p \leq C\|f_m\|_{h^p_{\omega,N}(\mathbb{R}^n)}$ and \{a_i^m\}_{i \in \mathbb{N}_0} are $(p, \infty, s, \omega)$-atoms. Since

$$\sum_{m \in \mathbb{N}_0} \sum_{i \in \mathbb{N}_0} |\lambda_i^m|^p \leq C \sum_{m \in \mathbb{N}_0} \|f_m\|_{h^p_{\omega,N}(\mathbb{R}^n)}^p \leq C\|f\|_{h^p_{\omega,N}(\mathbb{R}^n)}^p,$$

then $f = \sum_{m \in \mathbb{N}_0} \sum_{i \in \mathbb{N}_0} |\lambda_i^m|^p a_i^m \in h^p_{\omega,\infty,s}(\mathbb{R}^n)$ and $\|f\|_{h^p_{\omega,\infty,s}(\mathbb{R}^n)} \leq C\|f\|_{h^p_{\omega,N}(\mathbb{R}^n)}$. Thus, Theorem 5.1 is proved.

For simplicity, from now on, we denote by $h^p_{\omega}(\mathbb{R}^n)$ the weighted local Hardy space $h^p_{\omega,N}(\mathbb{R}^n)$ associated with $\omega$, where $N \geq N_{p,\omega}$. Moreover, it is easy to see that $h^1_{\omega} \subset L^1_{\omega}(\mathbb{R}^n)$ via weighted atomic decomposition. However, the elements in $h^p_{\omega}(\mathbb{R}^n)$ with $p(0, 1)$ are not necessary functions thus $h^p_{\omega}(\mathbb{R}^n) \neq L^p_{\omega}(\mathbb{R}^n)$. But, for any $q \in (q_\omega, \infty)$, by Lemma 5.4 and pointwise convergence of weighted atomic decompositions, we have $(h^p_{\omega}(\mathbb{R}^n) \cap L^1_{\omega}(\mathbb{R}^n)) \subset L^p_{\omega}(\mathbb{R}^n)$, and for all $f \in (h^p_{\omega}(\mathbb{R}^n) \cap L^1_{\omega}(\mathbb{R}^n))$, $\|f\|_{L^p_{\omega}(\mathbb{R}^n)} \leq \|f\|_{h^p_{\omega}(\mathbb{R}^n)}$.

### 6. Finite atomic decompositions

In this section, we prove that for any given finite linear combination of weighted atoms when $q < \infty$, its norm in $h^p_{\omega}(\mathbb{R}^n)$ can be achieved via all its finite weighted atomic decompositions. This extends the main results in [13] to the setting of weighted local Hardy spaces.

Let $\omega \in A^\text{loc}_\infty$ and $(p, q, s)_\omega$ be an admissible triplet. Denote by $h^{p,q,s}_{\omega,\text{fin}}(\mathbb{R}^n)$ the vector space of all finite linear combination of $(p, q, s)_\omega$-atoms and single atom, and
the norm of $f$ in $h_{\omega,fin}^{p,q,s}(\mathbb{R}^n)$ is defined by
\[
\|f\|_{h_{\omega,fin}^{p,q,s}(\mathbb{R}^n)} = \inf \left\{ \sum_{j=0}^{k} |\lambda_j|^p \right\}^{1/p} : f = \sum_{j=0}^{k} \lambda_j a_j, k \in \mathbb{N}_0, \{a_i\}_{i=1}^k \text{ are } (p,q,s)_\omega- \text{atoms with sidelengths } \leq 2, \text{ and } a_0 \text{ is a } (p,q)_\omega \text{ single atom} \}.
\]

Obviously, for any admissible triplet $(p,q,s)_\omega$ atom and $(p,q)_\omega$ single atom, the set $h_{\omega,fin}^{p,q,s}(\mathbb{R}^n)$ is dense in $h_{\omega}^{p,q,s}(\mathbb{R}^n)$ with respect to the quasi-norm $\| \cdot \|_{h_{\omega}^{p,q,s}(\mathbb{R}^n)}$.

**Theorem 6.1.** Let $\omega \in A_{\infty}^{loc}$, $q_\omega$ be as in (2.3), and $(p,q,s)_\omega$ be an admissible triplet with sidelength $\leq 2$. If $q \in (q_\omega, \infty)$, then $\| \cdot \|_{h_{\omega,fin}^{p,q,s}(\mathbb{R}^n)}$ and $\| \cdot \|_{h_{\omega}^{p,q,s}(\mathbb{R}^n)}$ are equivalent quasi-norms on $h_{\omega,fin}^{p,q,s}(\mathbb{R}^n)$.

Proof. Clearly, $\|f\|_{h_{\omega}^{p,q,s}(\mathbb{R}^n)} \leq \|f\|_{h_{\omega,fin}^{p,q,s}(\mathbb{R}^n)}$ for $f \in h_{\omega,fin}^{p,q,s}(\mathbb{R}^n)$ and for $q \in (q_\omega, \infty)$. Thus, we have to show that for every $q$ in $(q_\omega, \infty)$ there exists a constant $C$ such that for all $f \in h_{\omega,fin}^{p,q,s}(\mathbb{R}^n)$
\[
\|f\|_{h_{\omega,fin}^{p,q,s}(\mathbb{R}^n)} \leq C \|f\|_{h_{\omega}^{p,q,s}(\mathbb{R}^n)}.
\]

(6.1) Suppose that $q \in (q_\omega, \infty)$ and that $f$ is in $h_{\omega,fin}^{p,q,s}(\mathbb{R}^n)$ with $\|f\|_{h_{\omega}^{p,q,s}(\mathbb{R}^n)} = 1$. In this section, we take $k_0 \in \mathbb{Z}$ such that $2^{k_0-1} \leq \inf_{x \in \mathbb{R}^n} M_N f(x) < 2^{k_0}$, if $\inf_{x \in \mathbb{R}^n} M_N f(x) = 0$, write $k_0 = -\infty$. For each integer $k \geq k_0$, set
\[
\Omega_k \equiv \{ x \in \mathbb{R}^n : M_N f(x) > 2^k \},
\]
where and in what follows $N = N_{p,\omega}$ we use the same notation as in Lemma 5.4. We first consider the case $k_0 = -\infty$. Since $f \in (h_{\omega}^{p}(\mathbb{R}^n) \cap L_2^q(\mathbb{R}^n))$, by Lemma 5.4, there exists numbers $\{\lambda_i^k\}_{k \in \mathbb{Z}, i \in \mathbb{N}} \subset \mathbb{C}$ and $(p, \infty, s)_\omega$-atoms $\{a_i^k\}_{k, i \in \mathbb{N}}$, $\lambda_0 \subset \mathbb{C}$ such that
\[
f = \sum_k \sum_i \lambda_i^k a_i^k
\]
holds almost everywhere and in $D'(\mathbb{R}^n)$, and (i) and (ii) in Lemma 5.4 hold.

Obviously, $f$ has compact support. Suppose that $\text{supp } f \subset Q(x_0, r_0)$. We write $\bar{Q} = Q(x_0, 2^{3(10^{-n})}r_0 + 2n)$. For $\varphi \in D_N$ and $x \in \mathbb{R}^n \setminus \bar{Q}$, for $0 < t < 1$, we have
\[
\varphi_t \ast f(x) = 0.
\]

Hence, $\text{supp } \sum_k \sum_{i \in \mathbb{N}} \lambda_i^k a_i^k \subset \bar{Q}$.

We claim that the series $\sum_k \sum_{i \in \mathbb{N}} \lambda_i^k a_i^k$ converges to $f$ in $L_2^q(\mathbb{R}^n)$. For any $x \in \mathbb{R}^n$, since $\mathbb{R}^n = \bigcup_{k \in \mathbb{Z}} (\Omega_k \setminus \Omega_{k+1})$, there exists $j \in \mathbb{Z}$ such that $x \in (\Omega_j \setminus \Omega_{j+1})$. Since $\text{supp } a_i^k \subset Q_i^k \subset \Omega_k \subset \Omega_{j+1}$ for $k > j$, then applying Lemmas 5.1, 5.2 and Lemma 5.4, we have
\[
\left| \sum_k \sum_{i \in \mathbb{N}} \lambda_i^k a_i^k \right| \leq C \sum_{k \leq j} 2^k \leq C 2^j \leq C M_N f(x).
\]
Since \( f \in L^q_n(\mathbb{R}^n) \), we have \( \mathcal{M}_K f \in L^q_n(\mathbb{R}^n) \). The Lebesgue dominated convergence theorem now implies that \( \sum_k \sum_{i \in \mathbb{N}} \lambda^i_k a^k_i \) converges to \( f \) in \( L^q_n(\mathbb{R}^n) \), and the claim is proved.

For each positive integer \( K \) we denote by \( F_K = \{(i, k) : k, |i| + |k| \leq K\} \) and \( f_K = \sum_{(i, k) \in F_K} \lambda^i_k a^k_i \). Observing that for any \( \epsilon \in (0, 1) \), if \( K \) is large enough, by \( f \in L^q_n \), we have \( (f - f_K)/\epsilon \) is a \((p, q, s)_\omega\)-atom. Since \( (f - f_K)/\epsilon \in \mathbb{Q} = Q(x_0, 2^{3(10+n)}r_0 + 2n) \), so we can divide \( Q \) into \( N_0 \) (depending only on \( r_0 \) and \( n \)) disjoint cubes \( \{Q_i\}_{i=1}^{N_0} \) with sidelengths \( 1 \leq l_i \leq 2 \). Then, \( (f - f_K)\chi_{Q_i}/\epsilon \) is a \((p, q, s)_\omega\)-atom for \( i = 1, \ldots, N_0 \). Thus, \( f = f_K + \sum_{i=1}^{N_0} (f - f_K)\chi_{Q_i} \) is a linear weighted atom combination of \( f \). Taking \( \epsilon = N_0^{-1/p} \) and by Lemma 5.4, we have

\[
\|f\|_{h^{p,q,s}_{\omega,f in}(\mathbb{R}^n)} \leq \sum_{(i, k) \in F_K} |\lambda^i_k|^p + N_0 \epsilon^p \leq C.
\]

We now consider the case \( k_0 > -\infty \). Since \( f \in (h^p_n(\mathbb{R}^n) \cap L^q_n(\mathbb{R}^n)) \), by Lemma 5.4, there exists numbers \( \{\lambda^i_k\}_{k \in \mathbb{Z}, i \in \mathbb{N}} \subset \mathbb{C} \) and \((p, \infty, s)_\omega\)-atoms \( \{a^i_k\}_{k \geq k_0, i \in \mathbb{N}} \), \( \lambda_0 \subset \mathbb{C} \) and the \((p, \infty)_\omega\) single atom \( a_0 \) such that

\[
f = \sum_{k \geq k_0} \sum_{i \in \mathbb{N}} \lambda^i_k a^k_i + \lambda_0 a_0
\]

holds almost everywhere and in \( \mathcal{D}'(\mathbb{R}^n) \), and (i) and (ii) in Lemma 5.4 hold. As the case \( k_0 = -\infty \), we can prove that the series \( \sum_{k \geq k_0} \sum_{i \in \mathbb{N}} \lambda^i_k a^k_i + \lambda_0 a_0 \) converges to \( f \) in \( L^q_n(\mathbb{R}^n) \).

Finally, for each positive integer \( K \) we denote by \( F_K = \{(i, k) : k \geq k_0, |i| + |k| \leq K\} \) and \( f_K = \sum_{(i, k) \in F_K} \lambda^i_k a^k_i + \lambda_0 a_0 \). If \( K \) is large enough, then \( \|f - f_K\|_{L^q(\omega)} \leq [\omega(\mathbb{R}^n)]^{1/q-1/p} \). So, \( (f - f_K) \) is a \((p, q)_\omega\) single atom. By Lemma 5.4, we have

\[
\|f\|_{h^{p,q,s}_{\omega,f in}(\mathbb{R}^n)} \leq \sum_{(i, k) \in F_K} |\lambda^i_k|^p + \lambda_0^p \leq C.
\]

Thus, (6.1) holds. The proof is finished.

As an application of finite atomic decompositions, we establish boundedness in \( h^p_n(\mathbb{R}^n) \) of quasi- Banach-valued sublinear operators.

As in [2], we recall that a quasi-Banach space \( \mathcal{B} \) is a vector space endowed with a quasi-norm \( \| \cdot \|_\mathcal{B} \) which is nonnegative, non-degenerate (i.e., \( \|f\|_\mathcal{B} = 0 \) if and only if \( f = 0 \)), homogeneous, and obeys the quasi-triangle inequality, i.e., there exists a positive constant \( K \) no less than 1 such that for all \( f, g \in \mathcal{B} \), \( \|f + g\|_\mathcal{B} \leq K(\|f\|_\mathcal{B} + \|g\|_\mathcal{B}) \).

Let \( \beta \in (0, 1] \). A quasi-Banach space \( \mathcal{B}_\beta \) with the quasi-norm \( \| \cdot \|_\mathcal{B}_\beta \) is said to be a \( \beta \)-quasi-Banach space if \( \|f + g\|_\mathcal{B}_\beta \leq \|f\|_\mathcal{B}_\beta + \|g\|_\mathcal{B}_\beta \) for all \( f, g \in \mathcal{B}_\beta \).

Notice that any Banach space is a 1-quasi-Banach space, and the quasi-Banach space \( l^p \), \( L^p_n(\mathbb{R}^n) \) and \( h^p_\omega(\mathbb{R}^n) \) with \( \beta \in (0, 1) \) are typical \( \beta \)-quasi-Banach spaces.
For any given $\beta$-quasi-Banach space $B_\beta$ with $\beta \in (0,1]$ and a linear space $Y$, an operator $T$ from $Y$ to $B_\beta$ is said to be $B_\beta$-sublinear if for any $f, g \in B_\beta$ and $\lambda, \nu \in \mathbb{C}$, we have
\[
\|T(\lambda f + \nu g)\|_{B_\beta} \leq \left( |\lambda| \|T(f)\|_{B_\beta}^{\beta} + |\nu| \|T(g)\|_{B_\beta}^{\beta} \right)^{1/\beta}
\]
and $\|T(f) - T(g)\|_{B_\beta} \leq \|T(f - g)\|_{B_\beta}$.

We remark that if $T$ is linear, then $T$ is $B_\beta$-sublinear. Moreover, if $B_\beta = L^q_{\omega}(\mathbb{R}^n)$, and $T$ is nonnegative and sublinear in the classical sense, then $T$ is also $B_\beta$-sublinear.

**Theorem 6.2.** Let $\omega \in A^\text{loc}_{\infty}, 0 < p \leq \beta \leq 1$, and $B_\beta$ be a $\beta$-quasi-Banach space. Suppose $q \in (q_\omega, \infty)$ and $T : h^{p,q,s}_{\omega,fin}(\mathbb{R}^n) \rightarrow B_\beta$ is a $B_\beta$-sublinear operator such that
\[
S = \{\|T(a)\|_{B_\beta} : a \text{ is any } (p,q,s)_{\omega} - \text{atom with sidelength } \leq 2 \text{ or } (p,q)_{\omega} \text{ single atom} \} < \infty.
\]

Then there exists a unique bounded $B_\beta$-sublinear operator $\tilde{T}$ from $h^p_{\omega}(\mathbb{R}^n)$ to $B_\beta$ which extends $T$.

Proof. For any $f \in h^{p,q,s}_{\omega,fin}(\mathbb{R}^n)$, by Theorem 6.1, there exist numbers $\{\lambda_j\}_{j=0}^l \subset \mathbb{C}$ and $(p,q,s)_{\omega}$-atoms $\{a_j\}_{j=1}^l$ and the $(p,q)_{\omega}$ single atom $a_0$ such that $f = \sum_{j=0}^l \lambda_j a_j$ pointwise and $\sum_{j=0}^l |\lambda_j|^p \leq C \|f\|_{h^p_{\omega}(\mathbb{R}^n)}$. Then by the assumption, we have
\[
\|T(f)\|_{B_\beta} \leq C \left( \sum_{j=0}^l |\lambda_j|^p \right)^{1/p} \leq C \|f\|_{h^p_{\omega}(\mathbb{R}^n)}.
\]

Since $h^{p,q,s}_{\omega,fin}(\mathbb{R}^n)$ is dense in $h^p_{\omega}(\mathbb{R}^n)$, a density argument gives the desired results.

**7. Applications**

In this section, we study weighted $L^p$ inequalities for strongly singular integrals and pseudodifferential operators and their commutators.

Given a real number $\theta > 0$ and a smooth radial cut-off function $v(x)$ supported in the ball $\{x \in \mathbb{R}^n : |x| \leq 2\}$, we consider the strongly singular kernel
\[
k(x) = \frac{e^{i|x|^{-\theta}}}{|x|^n} v(x).
\]

Let us denote by $Tf$ the corresponding strongly singular integral operator:
\[
Tf(x) = p.v \int_{\mathbb{R}^n} k(x - y)f(y)dy.
\]
This operator has been studied by several authors, see [11], [21], [6], [4] and [8]. In particular, S. Chanillo [4] established the weighted $L^p_\omega(\mathbb{R}^n)$ boundedness for strongly singular integrals provided that $\omega \in A_p(\mathbb{R}^n)$ (Muckenhoupt weights) for $1 < p < \infty$. J. García-Cuerva et al [8] obtained weighted $L^p$ estimates with pairs of weights for commutators generated by the strongly singular integrals and the classical $BMO(\mathbb{R}^n)$ functions. We have the following results for the strongly singular integrals.

**Theorem 7.1.** Let $T$ be strongly singular integral operators, then

(i) $\|Tf\|_{L^p_\omega(\mathbb{R}^n)} \leq C_{p,\omega}\|f\|_{L^p_\omega(\mathbb{R}^n)}$ for $1 < p < \infty$ and $\omega \in A_{p\text{loc}}$.

(ii) $\|Tf\|_{L^{1,\infty}_\omega(\mathbb{R}^n)} \leq C_\omega\|f\|_{L^1_\omega(\mathbb{R}^n)}$ for $\omega \in A_{1\text{loc}}$.

(iii) $\|Tf\|_{L^1_\omega(\mathbb{R}^n)} \leq C_\omega\|f\|_{h^1_\omega(\mathbb{R}^n)}$ for $\omega \in A_{1\text{loc}}$.

**Proof.** We first note that for $\omega \in A_p$ the inequality (i) is known to be true, see [4]. For $\omega \in A_{p\text{loc}}$, by Lemma 2.1 (i) for any unit cube $Q$ there is a $\bar{\omega} \in A_p$ so that $\bar{\omega} = \omega$ on $6Q$. Then

$$\|Tf\|_{L^p_\omega(Q)} = \|T(\chi_{6Q}f)\|_{L^p_\omega(Q)} \leq \|T(\chi_{6Q}f)\|_{L^p_{\bar{\omega}}(Q)} \leq C\|\chi_{6Q}f\|_{L^p_{\bar{\omega}}(\mathbb{R}^n)} \leq C\|f\|_{L^p_{\bar{\omega}}(6Q)}.$$ 

Summing over all dyadic unit $I$ gives (i).

For (ii), similar to (i), note that for $\omega \in A_1$ the inequality (ii) is known to be true, see [4]. Since $\omega \in A_{p\text{loc}}$, by Lemma 2.1 (i) for any unit cube $I$ there is a $\bar{\omega} \in A_1$ so that $\bar{\omega} = \omega$ on $6Q$. Then for any $\lambda > 0$

$$\omega(\{x \in Q : |Tf(x)| > \lambda\}) \leq \omega(\{x \in Q : |T(\chi_{6Q}f)(x)| > \lambda\}) = \bar{\omega}(\{x \in Q : |T(\chi_{6Q}f)(x)| > \lambda\}) \leq C_\lambda^{-1}\|\chi_{6Q}f\|_{L^\infty_{\bar{\omega}}(\mathbb{R}^n)} = C_\lambda^{-1}\|f\|_{L^\infty_{\bar{\omega}}(6Q)}.$$ 

Summing over all dyadic unit $Q$ gives (ii).

Finally, to consider (iii). Let $a(x)$ be an atom in $h^1_\omega(\mathbb{R}^n)$, supported in a cube $Q$ centered at $x_0$ and sidelength $\delta \leq 2$ by Remark 5.1, or $a(x)$ is a single atom. To
prove the (iii), by Theorem 6.2, it is enough to show that
\[ \| Ta \|_{L^1_\omega(R^n)} \leq C, \]  
(7.1)
where C is independent of a.

It is easy to see that (7.1) holds while a(x) is a single atom. It remains to consider this kind of atom supported in a cube Q centered at x_0 and sidelength \( \delta \).

In deed, let \( \delta_0 \) be a number satisfying \( 4\delta_0 = \delta_0^{1/(1+\theta)} \). Obviously, \( \delta_0 < 1 \).

Case 1. \( 2 \geq \delta \geq \delta_0 \). This is the trivial case. Let \( Q^* = (10n/\delta_0)Q \). Now
\[
\int_{\mathbb{R}^n} |Ta| \omega(x)dx = \int_{Q^*} |Ta| \omega(x)dx + \int_{\mathbb{R}^n \setminus Q^*} |Ta| \omega(x)dx = \int_{Q^*} |Ta| \omega(x)dx.
\]
Obviously,
\[
\int_{Q^*} |Ta| \omega(x)dx \leq C \left( \int_{\mathbb{R}^n} |Ta|^p \omega(x)dx \right)^{1/p} \left( \int_{Q^*} \omega(x)dx \right)^{1/p'} \leq C.
\]
(7.2)

Case 2. \( \delta < \delta_0 \). We let \( Q^* = 4Q \) and \( \bar{Q} = Q(x_0, \delta_1^{1/(1+\theta)}) \). Then
\[
\int_{\mathbb{R}^n} |Ta| \omega(x)dx \leq \int_{Q^*} |Ta| \omega(x)dx + \int_{\bar{Q} \setminus Q^*} |Ta| \omega(x)dx + \int_{\mathbb{R}^n \setminus \bar{Q}} |Ta| \omega(x)dx
\]
\[
:= I + II + III.
\]
For I, similar to (7.2), we have
\[
I \leq C \left( \int_{\mathbb{R}^n} |Ta|^p \omega(x)dx \right)^{1/p} \left( \int_{Q^*} \omega(x)dx \right)^{1/p'} \leq C.
\]
We now estimate the term III. Clearly, by the mean value theorem,
\[
|Ta(x)| \leq \frac{C\delta}{|x-x_0|^{g+n+1}} \chi_{\{x-x_0<4n\}}(x) \int_Q |a(y)|dy
\]
\[
\leq \frac{C\delta}{|x-x_0|^{g+n+1}} \chi_{\{x-x_0<4n\}}(x)
\]
\[
\times \left( \int_Q |a(x)|^p \omega(x)dx \right)^{1/p} \left( \int_Q [\omega(x)]^{-p'/p}dx \right)^{1/p'}
\]
\[
\leq \frac{C\delta}{|x-x_0|^{g+n+1}} \chi_{\{x-x_0<4n\}}(x) \frac{|Q|}{\omega(Q)}.
\]
Hence, by the properties of $A^I_{loc}$ (see Lemma 2.1), we have

$$III \leq \frac{C\delta|Q|}{\omega(Q)} \int_{|x-x_0| \leq 4n} \frac{\omega(x)}{|x-x_0|^{n+1}} dx$$

$$\leq \frac{C\delta|Q|}{\omega(Q)} \sum_{k=k_0}^{k_1} \frac{1}{(2^k \delta)^{n+1}} \left( \frac{1}{(2^k \delta)^n} \int_{|x-x_0| \leq 2^k \delta} \omega(x) dx \right)$$

$$\leq C,$$

where $k_0$ and $k_1$ are positive integers such that $2^{k_0} \delta \leq \delta^{1/(1+\theta)} \leq 2^{k_0+1} \delta$ and $2^{k_1-1} \leq 4n \leq 2^{k_1}$. We now estimate the term $II$. For $x \in Q \setminus Q^*$

$$Ta(x) = \int_{\mathbb{R}^n} \frac{e^{i|x-y|^\theta} v(x-y)}{|x-y|^{n(2+\theta)/r'}} \left( \frac{1}{|x_0-x|^{n(2+\theta)/r'}} - \frac{1}{|x_0-x|^{n(1-2\theta)/r'}} \right) a(y) dy$$

$$+ \int_{\mathbb{R}^n} \frac{e^{i|x-y|^\theta} v(x-y)}{|x-y|^{n(2+\theta)/r'}} \frac{a(y)}{|x_0-x|^{n(1-2\theta)/r'}} dy$$

$$= A(x) + B(x),$$

where $r'$ is taken so close to 1 to guarantee that $2+\theta < r$. Applying the mean value theorem to the term in brackets in the integrand of $A$, and noting that for $y \in Q$, and $x \in Q \setminus Q^*$, $|x-y| \geq c |x-x_0|$, we have

$$|A(x)| \leq \frac{C \delta}{|x-x_0|^{n+1}} \chi_{\{|x-x_0| < 4n\}}(x) \int_{Q} |a(y)| dy \leq \frac{C|Q|}{|x-x_0|^{n+1}} \chi_{\{|x-x_0| < 4n\}}(x) \frac{|Q|}{\omega(Q)}.$$

Therefore,

$$II \leq \frac{C\delta|Q|}{\omega(Q)} \int_{|x-x_0| \leq 4n} \frac{\omega(x)}{|x-x_0|^{n+1}} dx$$

$$+ C \int_{|x-x_0| < \delta^{1/(1+\theta)}} \frac{\omega(y)}{|x_0-x|^{n(1-2\theta)/r'}} a dx$$

$$\leq C + C \left( \int_{\mathbb{R}^n} |K_{\theta,r} \ast a| dx \right)^{1/r} \left( \int_{\delta^{1/(1+\theta)} < |x-x_0| < \delta^{1/(1+\theta)}} \frac{\omega(y)}{|x_0-x|^{n(1-2\theta)/r'}} dx \right)^{1/r'}$$

$$\leq C + C \|a\|_{L^{r'}(\mathbb{R}^n)}^r \left( \sum_{k=0}^{k_0} (2^k \delta)^{(r'-1)(\theta+1)} \frac{1}{(2^k \delta)^n} \int_{|x-x_0| \leq 2^k \delta} \omega(x) dx \right)^{1/r}$$

$$\leq C,$$

where $2^{k_0-1} \delta < \delta^{1/(1+\theta)} \leq 2^{k_0} \delta$, $K_{\theta,r}(x) := \frac{e^{i|x|^\theta}}{|x|^{(\theta+1)r/r'}},$ and we used the following fact (see [4])

$$\|K_{\theta,r} \ast f\|_{L^r(\mathbb{R}^n)} \leq C_r \|f\|_{L^{r'}(\mathbb{R}^n)}, \quad r > 2 + \theta.$$
Thus, Theorem 7.1 is proved.

We now introduce $BMO_{loc}$ of locally integrable functions with bounded mean oscillation which has an intimate relationship between the $A^p_{loc}$ weights. Namely,

$$\|b\|_{BMO_{loc}} := \sup_{|Q| \leq 1} \frac{1}{|Q|} \int_Q |b - b_Q| \, dx < \infty,$$

where $b_Q = \frac{1}{|Q|} \int_Q f(x) \, dx$.

It is easy to see that, we have the following result.

**Lemma 7.1.** Fix $p > 1$ and let $b \in BMO_{loc}$. Then there exists $\epsilon > 0$, depending upon the $BMO_{loc}$ constant of $b$, such that $e^{xb} \in A^p_{loc}$ for $|x| < \epsilon$.

**Lemma 7.2.** Let $b \in BMO_{loc}$, then there exist positive constants $c_1$ and $c_2$ such that for every cube $Q$ with $|Q| \leq 1$ and every $\lambda > 0$, we have

$$|\{x \in Q : |b(x) - b_Q| > \lambda\}| \leq c_1 |Q| \exp \left\{ -\frac{c_2 \lambda}{\|b\|_{BMO_{loc}(\mathbb{R}^n)}} \right\}.$$

As a consequence of Lemma 7.2 and Lemma 2.1, we have the following result.

**Corollary 7.1.** Let $b \in BMO_{loc}$ and $\omega \in A^\infty_{loc}$, then there exist positive constants $C_3$ and $C_4$ such that for every cube $Q$ with $|Q| \leq 1$ and every $\lambda > 0$, we have

$$\omega(\{x \in Q : |b(x) - b_Q| > \lambda\}) \leq c_3 \omega(Q) \exp \left\{ -\frac{c_4 \lambda}{\|b\|_{BMO_{loc}(\mathbb{R}^n)}} \right\}.$$

As an application of Corollary 7.2, we have

**Proposition 7.1.** Let $b \in BMO_{loc}$, $1 \leq p < \infty$, and $\omega \in A^\infty_{loc}$, then there exists a positive constant $C$ such that for every cube $Q$ with $|Q| \leq 1$

$$\frac{1}{\omega(Q)} \int_Q |b(x) - b_Q|^p \omega(x) \, dx \leq C \|b\|^p_{BMO_{loc}}.$$

We now consider in this paper commutator of Coifman-Rochberg-Weiss $[b, T]$ defined by the formula

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x) = \int_{\mathbb{R}^n} (b(x) - b(y))k(x - y)f(y) \, dy.$$

As in the case of strongly singular integrals, we have

**Theorem 7.2.** Let $b \in BMO_{loc}(\mathbb{R}^n)$ and $T$ be the strongly singular integral operators, then

(i) $\|[b, T]f\|_{L^p(\mathbb{R}^n)} \leq C_{p, \omega} \|b\|_{BMO_{loc}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}$ for $1 < p < \infty$ and $\omega \in A^p_{loc}$.
(ii) \( \|[b, T]f\|_{L^{\infty}(\mathbb{R}^n)} \leq C_\omega \|b\|_{BMO_{loc}(\mathbb{R}^n)} \|f\|_{h_1(\mathbb{R}^n)} \) for \( \omega \in A_{p}^{loc} \).

Proof: By Lemma 7.1, there is \( \eta > 0 \) such that \( \omega^{(1+\eta)} \in A_{p}^{loc} \). Then, we choose \( \delta > 0 \) such that \( \exp(s\delta b(1+\eta)/\eta) \in A_{p}^{loc} \) if \( 0 \leq s(1+\eta)/\eta < \delta \) with uniform constant.

For \( z \in \mathcal{C} \) we define the operator

\[
T_z f = e^{zb}(e^{-zb}f).
\]

We claim that

\[
\|T_z f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}
\]

uniformly on \( |z| \leq s < \delta \eta/(1 + \eta) \).

The function \( z \rightarrow T_z f \) is analytic, and by the Cauchy theorem, if \( s < \delta \eta/(1 + \eta) \),

\[
\frac{d}{dz} T_z f|_{z=0} = \frac{1}{2\pi i} \int_{|z|=s} \frac{T_z f dz}{s^2}.
\]

Observing that

\[
\frac{d}{dz} T_z f|_{z=0} = [b, T]f
\]

and applying the Minkowski inequality to the previous equality, we get

\[
\|[b, T]f\|_{L^p(\mathbb{R}^n)} \leq \frac{1}{2\pi} \int_{|z|=s} \frac{\|T_z f\|_{L^p(\omega)} dz}{s^2} \leq \frac{C}{s} \|f\|_{L^p(\mathbb{R}^n)}.
\]

It remains to prove the claim, which is equivalent to

\[
\left( \int_{\mathbb{R}^n} |T f(x)|^p \exp(\mathcal{R}(z)q b(x)) \omega(x) dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p \exp(\mathcal{R}(z)p b(x)) \omega(x) dx \right)^{1/p}.
\]

We write \( \omega_0 := \exp(\mathcal{R}(z)b(1+\eta)/\eta) \) and \( \omega_1 := \omega^{1+\eta} \). Since \( \omega_0 \) and \( \omega_1 \in A_{p}^{loc} \), we have

\[
\left( \int_{\mathbb{R}^n} |T f(x)|^p \omega_0(x) dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p \omega_0(x) dx \right)^{1/p}
\]

and

\[
\left( \int_{\mathbb{R}^n} |T f(x)|^p \omega_1(x) dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p \omega_1(x) dx \right)^{1/p}.
\]

Now, by Stein-Weiss interpolation theorem, we have

\[
\left( \int_{\mathbb{R}^n} |T f(x)|^p \omega_0^{(1-\beta)} \omega_1^\beta dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p \omega_0^{(1-\beta)} \omega_1^\beta dx \right)^{1/p}
\]

and taking \( \beta = (1 + \eta)^{-1} \), then we obtain (7.3). Thus, (i) of Theorem 7.2 is proved.
For (ii), Let the function $a_j(x)$ is a $h_\omega^1(\mathbb{R}^n)$ atom and supp $a_j \subset Q(x_j, r_j)$, and $a_0$ is a single atom if $\omega(\mathbb{R}^n) < \infty$, we then have

$$\omega(\{x \in \mathbb{R}^n : |[b, T]f(x)| > \lambda\}) = \omega(\{x \in \mathbb{R}^n : |\sum_{j \in N} \lambda_j[b, T]a_j(x)| > \lambda\})$$

$$\leq \omega(\{x \in \mathbb{R}^n : |\sum_{j \in E_1} \lambda_j[b, T]a_j(x)| > \lambda/3\})$$

$$+ \omega(\{x \in \mathbb{R}^n : |\sum_{j \in E_2} \lambda_j[b, T]a_j(x)| > \lambda/3\})$$

$$+ \omega(\{x \in \mathbb{R}^n : |\lambda_0[b, T]a_0(x)| > \lambda/3\})$$

$$:= F_1 + F_2 + F_3,$$

where $E_1 = \{j \in \mathbb{N} : r_j < \delta_0\}$ and $E_2 = \{j \in \mathbb{N} : 2 \geq r_j \geq \delta_0\}$ and $\delta_0$ be a number satisfying $4\delta_0 = \delta_0^{1/(1+\theta)}$. Obviously, $\delta_0 < 1$.

For $F_1$, let $b_j = \frac{1}{|Q_j|} \int_{Q_j} b(y) dy$. Note that

$$\sum_{j \in E_1} \lambda_j[b, T]a_j(x) = \sum_{j \in E_1} \lambda_j[b - b_j, T]a_j(x)$$

$$= \sum_{j \in E_1} \lambda_j(b(x) - b_j)Ta_j(x)\chi_{4nQ_j}(x)$$

$$+ \sum_{j \in E_1} \lambda_j(b(x) - b_j)Ta_j(x)\chi_{(4nQ_j)^c}(x)$$

$$- T(\sum_{j \in E_1} \lambda_j(b(x) - b_j)a_j)(x)$$

$$:= F_{11}(x) + F_{12}(x) + F_{13}(x).$$

Thus, by (i) of Theorem 7.2 and Theorem 7.1, we obtain

$$\omega(\{x \in \mathbb{R}^n : |F_{11}(x)| > \lambda/9\}) \leq \frac{C}{\lambda} \sum_{j \in E_1} |\lambda_j| \|(b - b_j)(Ta_j)\chi_{4nQ_j}\|_{L^1(\mathbb{R}^n)}$$

$$\leq \frac{C}{\lambda} \sum_{j \in E_1} |\lambda_j| \|(b - b_j)\chi_{4nQ_j}\|_{L^1(\mathbb{R}^n)}\|a_j\|_{L^2(\mathbb{R}^n)}$$

$$\leq \frac{C}{\lambda} \sum_{j \in N} |\lambda_j| \|b\|_{BMO^{loc}}$$

$$\leq \frac{C}{\lambda} \|b\|_{BMO^{loc}} \|f\|_{h_\omega^1(\mathbb{R}^n)}.$$
We now estimate the term $\text{III}$ (see Theorem 7.1 (ii)) and Proposition 7.1, we get
\[
\omega(\{x \in \mathbb{R}^n : |F_{13}(x)| > \lambda/9\}) \leq \frac{C}{\lambda} \sum_{j \in E_1} |\lambda_j||b - b_j|a_j|_{L^1(\mathbb{R}^n)}
\leq \frac{C}{\lambda} \sum_{j \in \mathbb{N}} |\lambda_j||b|_{BMO_{loc}}
\leq \frac{C}{\lambda} |b|_{BMO_{loc}} ||f||_{h^1_{\lambda}(\mathbb{R}^n)}.
\]

Now we consider the term $F_{12}(x)$. Obviously,
\[
\omega(\{x \in \mathbb{R}^n : |F_{12}(x)| > \lambda/6\}) \leq \lambda^{-1} \sum_{j \in E_1} |\lambda_j| \int_{\mathbb{R}^n} T((b(x) - b_j)a_j)(x)\omega(x)dx. \quad (7.4)
\]

We claim that
\[
\int_{\mathbb{R}^n} T((b(x) - b_j)a_j)(x)\omega(x)dx \leq C
\]
holds for all atoms $a_j$ for $j \in E_1$. For convenience, we denote $a_j$ by $a$, $Q_j(x_j, r_j)$ by $Q(x_0, \delta)$ and $b_j$ by $b_Q$ for $j \in E_1$.

We let $Q^* = 4Q$ and $Q = Q(x_0, \delta^{1/(1+\theta)})$. Then
\[
\int_{\mathbb{R}^n} |T(b - b)a|\omega(x)dx \leq \int_{Q^*} |T(b - b_Q)a|\omega(x)dx
\]
\[
+ \int_{Q \setminus Q^*} |T(b - b_Q)a|\omega(x)dx + \int_{\mathbb{R}^n \setminus Q} |Ta|\omega(x)dx
\]
\[
:= I + II + III.
\]

For $I$, similar to (7.2), we have
\[
I \leq C \left( \int_{\mathbb{R}^n} |T(b - b_Q)a|^p \omega(x)dx \right)^{1/p} \left( \int_{Q^*} \omega(x)dx \right)^{1/p'} \leq C.
\]

We now estimate the term $III$. Clearly, by the mean value theorem,
\[
|T(b - b_Q)a(x)| \leq \frac{C\delta|b(x) - b_Q|}{|x - x_0|^{\theta + n + 1} \chi_{\{|x - x_0| < 4\}}(x)} \int_Q |a(y)|dy
\leq \frac{C\delta|b(x) - b_Q|}{|x - x_0|^{\theta + n + 1} \chi_{\{|x - x_0| < 4\}}(x)}
\times \left( \int_Q |a(x)|^p \omega(x)dy \right)^{1/p} \left( \int_Q [\omega(x)]^{-p'/p}dx \right)^{1/p'}
\leq \frac{C\delta|b(x) - b_Q|}{|x - x_0|^{\theta + n + 1} \chi_{\{|x - x_0| < 4\}}(x)} \frac{|Q|}{\omega(Q)}.
\]
Hence, by the properties of $A_1^{loc}$ (see Lemma 2.1), we have

$$III \leq \frac{C\delta|Q|}{\omega(Q)} \int_{|x-x_0| \leq 4n} \frac{|b(x) - b_Q|\omega(x)}{|x-x_0|^\theta} dx$$

$$\leq C\delta|Q| \sum_{k=k_0}^{k_1} \frac{1}{(2^k\delta)^{1+\theta}} \int_{|x-x_0| \leq 2^k\delta} \frac{|b(x) - b_Q|\omega(x)}{|x-x_0|^\theta} dx$$

$$\leq C,$$

where $k_0$ and $k_1$ are positive integers such that $2^{k_0}\delta \leq \delta^{1/(1+\theta)} \leq 2^{k_0+1}\delta$ and $2^{k_1-1} \leq 4n \leq 2^{k_1}$. We now estimate the term $II$. For $x \in \bar{Q} \setminus Q^*$

$$Ta(x) = \int_{\mathbb{R}^n} \frac{e^{i|x-y|^{-\theta}}v(x-y)}{|x-y|^{n(2+\theta)/r'}} (b(x) - b_Q) a(y) dy$$

$$+ \int_{\mathbb{R}^n} \frac{e^{i|x-y|^{-\theta}}v(x-y)}{|x-y|^{n(2+\theta)/r'}} \frac{(b(x) - b_Q) a(y)}{|x_0 - x|^{n(1-(2+\theta)/r')}} dy$$

$$= A(x) + B(x),$$

where $r'$ is taken so close to 1 to guarantee that $2 + \theta < r$. Applying the mean value theorem to the term in brackets in the integrand of $A$, and noting that for $y \in Q$, and $x \in \bar{Q} \setminus Q^*$, $|x-y| \geq c|x-x_0|$, we have

$$|A(x)| \leq \frac{C\delta|b(x) - b_Q|}{|x-x_0|^{n+1}} \chi_{\{x-x_0|<4n\}}(x) \int_Q |a(y)| dy$$

$$\leq \frac{C\delta|b(x) - b_Q|}{|x-x_0|^{n+1}} \chi_{\{x-x_0|<4n\}}(x) \frac{|Q|}{\omega(Q)}.$$

Therefore,

$$II \leq \frac{C\delta|Q|}{\omega(Q)} \int_{|x-x_0| \leq 4n} \frac{|b(x) - b_Q|\omega(x)}{|x-x_0|^{n+1}} dx$$

$$+ C \int_{|x-x_0| < \delta^{1/(1+\theta)}} |K_{\theta,r} * a| \frac{|b(x) - b_Q|\omega(x)}{|x-x_0|^{n(1-(2+\theta)/r')}} dx$$

$$\leq C + C \left( \int_{\mathbb{R}^n} |K_{\theta,r} * a|^{r'} dx \right)^{1/r'} \left( \int_{|x-x_0| < \delta^{1/(1+\theta)}} \frac{|b(x) - b_Q|\omega(x)^{r'}}{|x-x_0|^{n(1-(2+\theta)/r')}} dx \right)^{1/r'}$$

$$\leq C.$$

Thus, the claim is proved.
From (7.4), we have
\[ \omega(\{ x \in \mathbb{R}^n : |F_1(x)| > \lambda/9 \}) \leq C \lambda^{-1} \sum_{j \in E_1} |\lambda_j| \leq C \lambda^{-1} \|f\|_{h^1_0(\mathbb{R}^n)}. \]

It remains to consider the term \( F_2 \). In fact, it is very simple. Let \( Q_j^* = \frac{10n}{\delta_0} Q_j \). Note that for any atom \( a_j \)
\[ \int_{\mathbb{R}^n} \omega(x)dx = \int_{Q_j^*} \omega(x)dx. \]
Note that \( \omega(Q_j^*) \leq C \omega(Q_j) \), we then have
\[ F_2 = \omega(\{ x \in \mathbb{R}^n : \sum_{j \in E_2} \lambda_j |[b, T]a_j(x)| > \lambda/3 \}) \]
\[ \leq \lambda^{-1} \sum_{j \in E_2} |\lambda_j| \int_{\mathbb{R}^n} |[b, T]a_j| \omega(x)dx \]
\[ \leq \lambda^{-1} \sum_{j \in E_2} |\lambda_j| \int_{Q_j^*} |[b, T]a_j| \omega(x)dx \]
\[ \leq \lambda^{-1} \sum_{j \in E_2} |\lambda_j| |[b, T]a_j|_{L^2(\mathbb{R}^n)} \omega(Q_j^*)^{1/2} \]
\[ \leq C \|b\|_{BMO^{loc}} \lambda^{-1} \sum_{j \in E_2} |\lambda_j| |a_j|_{L^2(\mathbb{R}^n)} \omega(Q_j^*)^{1/2} \]
\[ \leq C \|b\|_{BMO^{loc}} \lambda^{-1} \sum_{j \in E_2} |\lambda_j|. \]

It remains to estimate the term \( F_3 \).
\[ F_3 \leq C \int_{\mathbb{R}^n} |[b, T]a(x)| \omega(x)dz \]
\[ \leq C \|b\|_{BMO^{loc}} \lambda^{-1} \sum_{j \in E_2} |\lambda_j| |a|_{L^2(\mathbb{R}^n)} \omega(\mathbb{R}^n)^{1/2} \]
\[ \leq C \|b\|_{BMO^{loc}} \lambda^{-1} \sum_{j \in E_2} |\lambda_j| \omega(\mathbb{R}^n)^{1/2} \]
\[ \leq C \frac{\|b\|_{BMO^{loc}} \|f\|_{h^1_0(\mathbb{R}^n)}}{\lambda}. \]

From these, we have
\[ \omega(\{ x \in \mathbb{R}^n : |[b, T]f(x)| > \lambda \}) \leq \sum_{i=1}^3 \omega(\{ x \in \mathbb{R}^n : |F_i(x)| > \lambda/9 \}) \]
\[ + \omega(\{ x \in \mathbb{R}^n : |F_2(x)| > \lambda/3 \}) \]
\[ + \omega(\{ x \in \mathbb{R}^n : |F_3(x)| > \lambda/3 \}) \]
\[ \leq C \|b\|_{BMO^{loc}} \|f\|_{h^1_0(\mathbb{R}^n)}. \]
Thus, the proof of Theorem 7.2 is complete.

Next we show that the pseudodifferential operators are bounded on \( h^p_w(\mathbb{R}^n) \), where the weight \( w \) is in the weight class \( A_p(\varphi) \) which is contained in \( A^{loc}_p \) for \( 1 \leq p < \infty \). Let us first introduce some definitions.

Let \( m \) be real number. Following [19], a symbol in \( S^m_{1,\delta} \) is a smooth function \( \sigma(x, \xi) \) defined on \( \mathbb{R}^n \times \mathbb{R}^n \) such that for all multi-indices \( \alpha \) and \( \beta \) the following estimate holds:

\[
|D_x^\alpha D_\xi^\beta \sigma(x, \xi)| \leq C_{\alpha,\beta} (1 + |\xi|)^{m-|\beta|+|\alpha|},
\]

where \( C_{\alpha,\beta} > 0 \) is independent of \( x \) and \( \xi \).

The operator \( T \) given by

\[
Tf(x) = \int_{\mathbb{R}^n} \sigma(x, \xi) e^{2\pi i x \cdot \xi} \hat{f}(\xi) \, d\xi
\]

is called a pseudo-differential operator with symbol \( \sigma(x, \xi) \in S^m_{1,\delta} \), where \( f \) is a Schwartz function and \( \hat{f} \) denotes the Fourier transform of \( f \).

In the rest of this section, we let \( \varphi(t) = (1 + t)^\alpha \) with \( \alpha > 0 \).

A weight will always mean a positive function which is locally integrable. We say that a weight \( \omega \) belongs to the class \( A_p(\varphi) \) for \( 1 < p < \infty \), if there is a constant \( C \) such that for all cubes \( Q = Q(x, r) \) with center \( x \) and sidelength \( r \)

\[
\left( \frac{1}{\varphi(|Q|)|Q|} \int_Q \omega(y) \, dy \right) \left( \frac{1}{\varphi(|Q|)|Q|} \int_Q \omega^{-\frac{1}{p-1}}(y) \, dy \right)^{p-1} \leq C.
\]

We also say that a nonnegative function \( \omega \) satisfies the \( A_1(\varphi) \) condition if there exists a constant \( C \) for all cubes \( Q \)

\[
M_\varphi(\omega)(x) \leq C \omega(x), \text{ a.e. } x \in \mathbb{R}^n.
\]

where

\[
M_\varphi f(x) = \sup_{x \in Q} \frac{1}{\varphi(|Q|)|Q|} \int_Q |f(y)| \, dy.
\]

Since \( \varphi(|Q|) \geq 1 \), so \( A_p(\mathbb{R}^n) \subset A_p(\varphi) \) for \( 1 \leq p < \infty \), where \( A_p(\mathbb{R}^n) \) denote the classical Muckenhoupt weights; see [9].

**Remark :** It is easy to see that if \( \omega \in A_p(\varphi) \), then \( \omega(x)dx \) may be not a doubling measure. In fact, let \( \alpha > 0 \) and \( 0 \leq \gamma < \alpha \), it is easy to check that \( \omega(x) = (1 + |x| \log(1 + |x|))^{-(n+\gamma)} \notin A_\infty(\mathbb{R}^n) \) and \( \omega(x)dx \) is not a doubling measure, but \( \omega(x) = (1 + |x| \log(1 + |x|))^{-(n+\gamma)} \in A_1(\varphi) \) provided that \( \varphi(r) = (1 + r^{1/n})^\alpha \).

Similar to the classical Muckenhoupt weights, we give some properties for weights \( \omega \in A_\infty(\varphi) = \bigcup_{p \geq 1} A_p(\varphi) \).

**Lemma 7.3.** For any cube \( Q \subset \mathbb{R}^n \), then

(i) If \( 1 \leq p_1 < p_2 < \infty \), then \( A_{p_1}(\varphi) \subset A_{p_2}(\varphi) \).
(ii) $\omega \in A_p(\varphi)$ if and only if $\omega^{1/p} \in A_{p'}(\varphi)$, where $1/p + 1/p' = 1$.

(iii) If $\omega \in A_p$ for $1 \leq p < \infty$, then for any measurable set $E \subset Q$,

$$\frac{|E|}{\varphi(|Q|)|Q|} \leq C \left( \frac{\omega(E)}{\omega(Q)} \right)^{1/p}.$$ 

**Lemma 7.4.** Let $T$ be the $S^0_{1,0}$ pseudodifferential operators, then

$$\|Tf\|_{L^p_\omega(\mathbb{R}^n)} \leq C_{p,\omega} \|f\|_{L^p_\omega(\mathbb{R}^n)}$$

for $1 < p < \infty$ and $\omega \in A_p(\varphi)$. Lemmas 7.3 and 7.4 can be founded in [19]. The following lemma was proved in [10].

**Lemma 7.5.** Let $T$ be the $S^0_{1,0}$ pseudodifferential operators, if $\varphi \in \mathcal{D}$ then $Tf = \varphi \ast Tf$ has a symbol $\sigma_t$ which satisfies $D_x^\beta D_\xi^\alpha \sigma_t(x, \xi) \leq C_{\alpha, \beta}(1 + |\xi|)^{-|\alpha|}$ and a kernel $K_t(x, z) = FT_\xi \sigma_t(x, \xi)$ which satisfies $|D_x^\beta D_\xi^\alpha K_t(x, z)| \leq C_{\alpha, \beta}|z|^{n - |\alpha|}$, where $C_{\alpha, \beta}$ is independent of $t$ if $0 < t < 1$.

**Theorem 7.3.** Let $T$ be the $S^0_{1,0}$ pseudodifferential operators, then

$$\|Tf\|_{h^p_\omega(\mathbb{R}^n)} \leq C_{p,\omega} \|f\|_{h^p_\omega(\mathbb{R}^n)}$$

for $\omega \in A_\infty(\varphi)$ and $0 < p \leq 1$.

Proof. Since $\omega \in A_\infty(\varphi)$, so $\omega \in A_q(\varphi)$ for some $q > 1$. By Theorem 6.2, it suffices to show that for any atom $(p, q, s)_{\omega}$ a supported $Q = Q(x_0, r)$ with $r \leq 2$ and $\|a\|_{L^q_\omega(\mathbb{R}^n)} \leq [\omega(Q)]^{1/p - 1/q}$, such that

$$\|Ta\|_{h^p_\omega(\mathbb{R}^n)} \leq C_{\omega, p}, \quad (7.6)$$

and if $a$ is a single atom, then

$$\|Ta\|_{h^p_\omega(\mathbb{R}^n)} \leq C_{\omega, p}. \quad (7.7)$$

Obviously, (7.7) holds. Now we prove (7.6).

If $Q^* = 2Q$, we then have

$$\int_{Q^*} \sup_{0 < t < 1} |\varphi_t \ast Ta(x)|^q \omega(x) dx \leq \omega(Q^*)^{(q-p)/q} \left( \int_{Q^*} \sup_{0 < t < 1} |\varphi_t \ast Ta(x)|^q \omega(x) dx \right)^{p/q}$$

$$\leq C \omega(Q^*)^{(q-p)/q} \left( \int_{\mathbb{R}^n} |Ta|^q \omega(x) dx \right)^{p/q}$$

$$\leq C \omega(Q^*)^{(q-p)/q} \left( \int_{\mathbb{R}^n} |a|^q \omega(x) dx \right)^{p/q}$$

$$\leq C.$$
To estimate \( \int_{\mathbb{R}^n \setminus Q^*} \sup_{0 < t < 1} |\varphi_t * Ta|^p \), we consider two cases.

The first case is when \( r < 1 \). We expand \( K_t(x, x - z) \) in a Taylor series about \( z = x_0 \) so that

\[
\varphi_t * Ta(x) = \int_{\mathbb{R}^n} K_t(x, x - z)a(z)dx = \int_{\mathbb{R}^n} \sum_{|\alpha| = N+1} D_\alpha^2 K_t(x, x - \xi)z^\alpha a(z)dz,
\]

where \( \xi \) is in \( Q \), and hence by Lemma 7.5,

\[
|\varphi_t * Ta(x)| \leq C|x - x_0|^{-(n+N+1)}\|a\|_1|Q|^{(N+1)/n}.
\]

Taking \( N \) is large enough and \( r < 1 \), by Lemma 7.3 (iii), we then have

\[
\int_{\mathbb{R}^n \setminus Q^*} \sup_{0 < t < 1} |\varphi_t * Ta(x)|^p \omega(x)dx
\]

\[
\leq C|Q|^{p(N+1)/n} \frac{|Q|^p}{\omega(Q)} \int_{\mathbb{R}^n \setminus Q^*} |x - x_0|^{-p(n+N+1)} \omega(x)dx
\]

\[
\leq C|Q|^{p(N+1)/n} \frac{|Q|^p}{\omega(Q)} \sum_{k=1}^{\infty} (2^k r)^{-p(n+N+1)} \int_{|x - x_0| < 2^k r} \omega(x)dx
\]

\[
\leq C \frac{1}{\omega(Q)} \sum_{k=1}^{k_0} (2^k)^{-p(n+N+1)} \int_{|x - x_0| < 2^k r} \omega(x)dx
\]

\[
+ C \frac{1}{\omega(Q)} \sum_{k=k_0}^{\infty} (2^k)^{-(n+N+1)} \int_{|x - x_0| < 2^k r} \omega(x)dx
\]

\[
\leq C \frac{1}{\omega(Q)} \sum_{k=1}^{k_0} 2^{knq} 2^{-kp(n+N+1)} \omega(Q)
\]

\[
+ C \frac{1}{\omega(Q)} \sum_{k=k_0}^{\infty} (2^k r)^{-p(n+N+1)+\alpha} 2^{knq} \omega(Q)
\]

\[
\leq C,
\]

where the integer \( k_0 \) satisfies \( 2^{k_0-1} \leq 1/r < 2^{k_0} \). To estimate with the case when \( 1 < r \leq 2 \), by Lemma 7.5, for all \( M > 0 \), we have

\[
|K_t(x, x - z)| \leq C_M|x - z|^{-M}.
\]

So

\[
|\varphi_t * Ta(x)| \leq \int_Q |K_t(x, x - z)a(z)|dz \leq C_M|x - x_0|^{-M}\|a\|_{L^1(\mathbb{R}^n)}.
\]
Note that $1 < r \leq 2$, we then have

$$\int_{\mathbb{R}^n \setminus Q^*} \sup_{0 < t < 1} |\varphi_t * Ta(x)|^p \omega(x) dx \leq C_M \|a\|^p_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus Q^*} |x - x_0|^{-Mp} \omega(x) dx$$

$$\leq C_M \frac{|Q|^p}{\omega(Q)} \int_{\mathbb{R}^n \setminus Q^*} |x - x_0|^{-M} \omega(x) dx$$

$$\leq C_M \frac{1}{\omega(Q)} \sum_{k=0}^{\infty} (2^k r)^{-M_p + an} 2^{kn} \omega(Q)$$

$$\leq C,$$

if $M$ is large enough. The proof is complete.

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