Abstract

Model-free reinforcement learning algorithms combined with value function approximation have recently achieved impressive performance in a variety of application domains, including games (Silver et al., 2016) and robotics (Kober et al., 2013). However, the theoretical understanding of such algorithms is limited, and existing results are largely focused on episodic or discounted Markov decision processes (MDPs). In this work, we present adaptive approximate policy iteration (AAPI), a learning scheme which enjoys a \( O(T^{2/3}) \) regret bound for undiscounted, continuing learning in uniformly ergodic MDPs. This is an improvement over the best existing bound of \( O(T^{3/4}) \) for the average-reward case with function approximation. Our algorithm and analysis rely on adversarial online learning techniques, where value functions are treated as losses. The main technical novelty is the use of a data-dependent adaptive learning rate coupled with a so-called optimistic prediction of upcoming losses. In addition to theoretical guarantees, we demonstrate the advantages of our approach empirically on several environments.

1. Introduction

Our work focuses on model-free algorithms for learning in infinite-horizon undiscounted continuing Markov decision processes (MDPs), which capture tasks such as game playing, routing, and the control of physical systems. Unlike model-based algorithms, which estimate a model of the environment dynamics and plan based on the estimated model, model-free algorithms directly optimize the expected return of policies, which is the objective of interest. In combination with powerful function approximation, model-free algorithms have recently achieved impressive advances in multiple applications (Mnih et al., 2015; Van Hasselt et al., 2016). Unfortunately, many successful approaches come with few performance guarantees, especially in the infinite-horizon undiscounted, continuing case. Existing theoretical results often only apply to episodic or discounted MDPs, with either tabular representation (Jin et al., 2018) or known special MDP structure (Even-Dar et al., 2009; Neu et al., 2014). In this work, we introduce Adaptive Approximate Policy Iteration (AAPI), a practical model-free learning scheme that can work with function approximation. We analyze its performance in infinite-horizon undiscounted, continuing MDPs in terms of high-probability regret with respect to a fixed policy.

Our approach follows the “online MDP” line of work (Even-Dar et al., 2009; Neu et al., 2014), where the agent iteratively selects policies by running an adversarial online learning algorithm in each state, and the loss fed to each algorithm is the policy Q-function in that state. This results in a variant of approximate policy iteration (API), where the policy improvement step produces a policy optimal in hindsight w.r.t. the sum of all previous Q-functions rather than just the most recent one. The original work of Even-Dar et al. (2009) focused on the tabular case with known dynamics and adversarial reward functions. More recent works (Abbasi-Yadkori et al., 2019a;b) have adapted this approach to the realistic setting of unknown dynamics, stochastic rewards, and value function approximation.

We improve over existing results in this setting (Abbasi-Yadkori et al., 2019a) by exploiting the fact that losses (Q-function estimates) are slow-changing rather than completely adversarial. In particular, our policy improvement step relies on the adaptive optimistic follow-the-regularized-leader (AO-FTRL) update (Mohri & Yang, 2016). The resulting policies are Boltzmann distributions over the sum of past estimated Q-functions, coupled with an optimistic prediction of the upcoming loss and a state-dependent adaptive learning rate (softmax temperature). Intuitively, the adaptive learning rate makes the policy more exploratory in states on which past consecutive Q-estimates disagree.
On the theoretical side, we prove the first $O(T^{2/3})$ regret upper bound in the undiscounted, continuing setting with function approximation. This is an improvement over the best existing $O(T^{3/4})$ bound of Abbasi-Yadkori et al. (2019a) for the same setting, which ignores the slow-changing nature of the estimated Q-functions. We exploit the fact that the change in consecutive Q-function estimates can be bounded by the change in policies. Our analysis relies on the results of Rakhlin & Sridharan (2013), but employs a different regret decomposition, with additional information provided by MDP properties. We emphasize that our learning framework is not limited to a particular function approximation method, and that in practice it serves the purpose of appropriately regularizing the policy improvement step of API.

2. Related work

Undiscounted MDPs. Most no-regret algorithms for infinite-horizon undiscounted MDPs are model-based, and only applicable to tabular representations (Bartlett, 2009; Jaksch et al., 2010; Ouyang et al., 2017; Fruit et al., 2018; Jia et al., 2019; Talebi & Maillard, 2018). For a weakly-communicating MDP with $X$ states, $A$ actions, and diameter $D$, these algorithms achieve the minimax lower bound of $O(\sqrt{DXAT})$ (Jaksch et al., 2010) with high probability up to constant factors (typically $\sqrt{DX}$). The downside of model-based algorithms is that they are memory intensive in large-scale MDPs, as they require $X^2A$ storage, and as well as being difficult to extend to continuous-valued states.

In the model-free tabular setting, Wei et al. (2019) show that optimistic Q-learning achieves $O(s(V_t)(XA)^{1/3}T^{2/3})$ regret in weakly-communicating MDPs, where $s(V_t)$ is the span of the optimal state-value function (upper bounded by the diameter $D$). In the case of uniformly ergodic MDPs, they show a bound of $O(\sqrt{t_{\text{mix}}^3\rho AT})$ on expected regret, where $t_{\text{mix}}$ is the mixing time and $\rho$ is the stationary distribution mismatch coefficient.

In the model-free setting with function approximation, the POLITEX algorithm (Abbasi-Yadkori et al., 2019a), a variant of API, achieves $O(d^{1/2}T^{3/4})$ regret in ergodic MDPs, under the assumption that policy evaluation error scales as $r^{1/2}$ after $r$ transitions. Here $d$ is the size of the compressed state-action space ($XA$ for tabular representation, number of features for linear Q-functions).

Episodic MDPs. In episodic MDPs with horizon $H$, Jin et al. (2018) show an $O(\sqrt{H^3XAT})$ regret bound for Q-learning with UCB exploration with tabular representation. With function approximation, Yang & Wang (2019b); Jin et al. (2019) show an $O(\sqrt{d^3H^3T})$ regret bound for an optimistic version of least-squares value iteration in MDPs by assuming special linear MDP. Their algorithms heavily exploit the linear MDP structure that is hard to be satisfied in practice. The RLSVI algorithm (Osband et al., 2016) performs exploration in the value function parameter space, and therefore can be applied with function approximation, though its worse-case regret bound of $O(\sqrt{H^3X^3AT})$ holds only in the tabular setting (Russo, 2019). In the infinite-horizon discounted setting, single-trajectory regret makes less sense as an objective, and most theoretical results are focused on the sample complexity of exploration in the tabular case (Strehl et al., 2006; Dong et al., 2019).

Conservative PI. API is also similar to the conservative policy iteration works, which attempt to stabilize API by regularizing each policy towards the previous policy (Kakade & Langford, 2002; Schulman et al., 2015; Abdolmaleki et al., 2018; Geist et al., 2019). Our policy improvement step can be seen as regularizing each policy by the KL-divergence to the previous policy; the reduction to online learning offers a principled way to incorporate such regularization.

3. Problem setting and definitions

We first introduce some notation. We use $\Delta_S$ to denote the space of probability distributions defined on the set $S$ and write $[d] = \{1, 2, \ldots, d\}$. For vectors $u,v \in \mathbb{R}^d$, we define the weighted $\ell_2$-norm as $||v||_2^2 = \sum_{i=1}^d u_i v_i^2$ and $\ell_\infty$-norm as $||u||_\infty = \max_{j \in [d]} u_j$. In general, we treat discrete distributions as row vectors.

Infinite-horizon undiscounted MDPs are often characterized by a finite state space $\mathcal{X}$, a finite action space $\mathcal{A}$, a reward function $r : \mathcal{X} \times \mathcal{A} \rightarrow [0,1]$, and a transition probability function $\mathcal{P} : \mathcal{X} \times \mathcal{A} \rightarrow \Delta_{\mathcal{X}}$. The agent does not know the transition probability and the reward function in advance. A policy $\pi : \mathcal{X} \rightarrow \Delta_{\mathcal{A}}$ is a mapping from a state to a distribution over actions. Let $\{(x_t^\pi, a_t^\pi)\}_{t=1}^\infty$ denote the state-action sequence obtained by following policy $\pi$. The expected average reward of policy $\pi$ is defined as

$$\lambda_\pi := \lim_{T \to \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^T r(x_t^\pi, a_t^\pi) \right].$$

The agent interacts with the environment as follows: at each round $t$, the agent observes a state $x_t \in \mathcal{X}$, chooses an action $a_t \sim \pi_t(\cdot|x_t)$, and receives a reward $r(x_t, a_t)$. The environment then transitions to the next state $x_{t+1}$ with probability $\mathcal{P}(x_{t+1}|x_t, a_t)$. The initial state $x_1$ is randomly
generated from some unknown distribution. Let \( \pi^* \) be an unknown fixed policy. The regret of an algorithm with respect to this fixed policy is defined as

\[
R_T = \sum_{t=1}^T \left( \lambda_{\pi^*} - r(x_t, \pi_t(x_t)) \right). \tag{3.1}
\]

The learning goal is to find an algorithm that minimizes the long-term regret \( R_T \).

For each policy \( \pi \), we denote \( \mathcal{P}^\pi \in \mathbb{R}^{|\mathcal{X}| \times |\mathcal{X}|} \) to be the transition matrix induced by \( \pi \), where the component \( (\mathcal{P}^\pi)_{x,x'} \) is the transition probability from \( x \) to \( x' \) under \( \pi \), i.e., \( (\mathcal{P}^\pi)_{x,x'} = \sum_{a \in A} \pi(a|x) \mathcal{P}(x'|x,a) \). For a distribution \( \mu \) over \( \mathcal{X} \), we let \( \mu \mathcal{P}^\pi \) be the distribution over \( \mathcal{X} \) that results from executing the policy \( \pi \) for one step after the initial state is sampled from \( \mu \). A stationary distribution \( \mu_\pi \) of a policy \( \pi \) over states satisfies \( \mu_\pi \mathcal{P}^\pi = \mu_\pi \). In addition, the expected reward can also be expressed as

\[
\lambda_\pi = \sum_{x \in \mathcal{X}} \mu_\pi(x) \sum_{a \in A} \pi(a|x) r(x,a).
\]

We assume that all MDPs are ergodic. An MDP is ergodic if the Markov chain induced by any policy \( \pi \) is both irreducible and aperiodic, which means any state is reachable from any other state by following a suitable policy. Learning in ergodic MDPs is generally easier than weakly-communicating MDPs because ergodic MDPs themselves are exploratory. It is well-known that all ergodic MDPs have an unique stationary state distribution, and so \( \mu_\pi \) and \( \lambda_\pi \) are well-defined. In addition, ergodic MDPs have a finite mixing time, defined below.

**Definition 3.1.** The mixing time of ergodic MDPs is defined as \( t_{\text{mix}} := \max_\pi \min \left\{ t \geq 1, \left\| (\mathcal{P}^\pi)^t(x,\cdot) - \mu_\pi \right\|_1 \leq \frac{1}{4}, \forall x \in \mathcal{X} \right\} \), that characterizes how fast MDPs reach stationary distributions from any state under any policy.

In the end, we define the value function under policy \( \pi \) as

\[
V_\pi(x) = \mathbb{E} \left[ \sum_{t=1}^{\infty} (r(x^*_t, a^*_t) - \lambda_\pi)|x^*_1 = x \right].
\]

The state-action value function \( Q_\pi(x,a) \) and \( V_\pi(x) \) are the unique solutions to the Bellman equation:

\[
Q_\pi(x,a) = r(x,a) - \lambda_\pi + \sum_{x'} \mathcal{P}(x'|x,a)V_\pi(x'),
\]

\[
V_\pi(x) = \sum_a \pi(a|x)Q_\pi(x,a). \tag{3.2}
\]

### 4. Algorithm

AAPI is a variant of approximate policy iteration and it proceeds in phases. Suppose the total number of rounds is \( T \). We divide \( T \) into \( K \) phases of length \( \tau = T/K \) and assume \( \tau \) is an integer for simplicity. Within each phase, our algorithm performs two tasks: policy evaluation and policy improvement.

**Policy evaluation.** In each phase \( k \in [K] \), the algorithm executes the current policy \( \pi_k \) for \( \tau \) time steps, and computes an estimate \( \hat{Q}_{\pi_k} \) of the true action-value function \( Q_{\pi_k} \).

We leave unspecified the value function estimation method (See Step 3 in Section 6 for a generic description of AO-FTRL) The terms in Eq. (4.1) are as follows:

\[
\pi_{k+1}(a|x) = \arg\max_{f \in \mathcal{F}} \left\{ \sum_{s=1}^k \hat{Q}_{\pi_s}(x,\cdot) + M_{k+1}(x,\cdot) \right\}
\]

\[-\eta_k(x)R(f). \tag{4.1}\]

(See Step 3 in Section 6 for a generic description of AO-FTRL.) The terms in Eq. (4.1) are as follows:

- The estimates \( \hat{Q}_{\pi_s}(x,\cdot) \in \mathbb{R}^{|A|} \) are the loss functions fed to the AO-FTRL algorithm. \( R(f) \) is the negative entropy regularizer, and \( \mathcal{F} \) is the probability simplex.

- The side-information \( M_{k+1}(x,\cdot) \in \mathbb{R}^{|A|} \) is a vector computable based on past information and being predictive of the next loss \( \hat{Q}_{\pi_{k+1}}(x,\cdot) \). Since the policies are expected to change slowly due to the nature of exponential-weight-average type algorithms, we set \( M_{k+1}(x,\cdot) = \hat{Q}_{\pi_k}(x,\cdot) \) (better guesses such as off-policy estimates can be used if available).

- The choice of learning rate \( \eta_k(x) \) is crucial both theoretically and empirically. In particular, we choose \( \eta_k(x) \) in a data-dependent fashion as

\[
\eta_k(x) = \eta \sqrt{\frac{2 \sum_{s=1}^k \| \hat{Q}_{\pi_s}(x,\cdot) - M_s(x,\cdot) \|_\infty^2}{\tau}}. \tag{4.2}\]

A notable feature of \( \eta_k(x) \) is that it is also state-dependent. Intuitively, for the choice \( M_s(x,\cdot) = R(x,a) \)
Algorithm 1 Adaptive approximate policy iteration (AAPI)

1: Input: phase length \( T \), number of phase \( K \), initial state \( x_0 \), turning parameter \( \eta \), value function estimation algorithm \( \mathcal{G} \).
2: Initialize: \( \pi_1(a|x) = 1/|A|, \forall x, a; \)
3: Repeat:
4: for \( k = 1, \ldots, K \) do
5: Execute \( \pi_k \) for \( T \) time steps and collect dataset \( D_k \).
6: Estimate \( \hat{Q}_{\pi_k} \) from \( D_1, \ldots, D_k \) using \( \mathcal{G} \).
7: Calculate adaptive learning rate:
\[
\eta_k(x) = \eta \sqrt{\frac{2}{\sum_{s=1}^{k} \| \hat{Q}_{\pi_s}(x, \cdot) - M_k(x, \cdot) \|_\infty^2}},
\]
where \( M_k = \hat{Q}_{\pi_{k-1}} \).
8: Calculate
\[
\text{index}(x, a) = \sum_{s=1}^{k} \hat{Q}_{\pi_s}(x, a) + M_{k+1}(x, a).
\]
9: Update next policy as:
\[
\pi_{k+1}(a|x) \propto \exp \left( \eta_k(x)^{-1} \text{index}(x, a) \right),
\]
where \( \eta_k(x) \) is defined in Eq. (4.2).
10: end for
11: Output: \( \pi_{K+1} \)

5. Analysis

To derive a regret bound for Algorithm 1, we decompose the cumulative regret (3.1) as follows,
\[
R_T = \sum_{t=1}^{T} \left( \lambda_{\pi_t} - r(x_t, a_t) \right) + \sum_{t=1}^{T} \left( \lambda_{\pi^*} - \lambda_{\pi_t} \right). \tag{5.1}
\]
The first term captures the sum of differences between observed rewards and their long term averages. If policies are changing slowly, or if they are kept fixed for extended periods of time, we expect this term to capture the noise in the regret. The second term is called pseudo-regret in literature. It measures the difference between the expected reward of a fixed policy and the policies produced by the algorithm.

We first impose a condition on the quality of policy evaluation step at each phase. For a probability distribution \( \mu \) on \( \mathcal{X} \) and a stochastic policy \( \pi \), define \( \mu \otimes \pi \) to be the distribution on \( \mathcal{X} \times \mathcal{A} \) that puts the probability mass \( \mu(x)\pi(a|x) \) on pair \( (x, a) \in \mathcal{X} \times \mathcal{A} \). Recall that \( \mu_{\pi^*} \) is the stationary distribution of \( \pi^* \) over the states.

Condition 5.1. For each phase \( k \in [K] \), denote \( D_{\pi_k} = \hat{Q}_{\pi_k} - Q_{\pi_k} \). We assume the following holds with probability \( 1 - \delta \),
\[
\max \left\{ \| D_{\pi_k} \|_{\mu_{\pi^*} \otimes \pi_k}, \| D_{\pi_k} \|_{\mu_{\pi^*} \otimes \pi_k}, \| D_{\pi_k} \|_\infty \right\} \leq \varepsilon_0 + \tilde{C} \sqrt{\frac{\log(1/\delta)}{T}}, \tag{5.2}
\]
where \( \varepsilon_0 \) is the irreducible approximation error and \( \tilde{C} \) is a problem dependent constant. Additionally, there exists a constant \( b \) such that \( \hat{Q}_{\pi_k}(x, a) \in [b, b + Q_{\max}] \) for any pair \( (x, a) \in \mathcal{X} \times \mathcal{A} \) and \( k \in [K] \).

Remark 5.2. The problem dependent constant \( \tilde{C} \) will in general depend on \( d, t_{\max}, \mu_{\pi^*}, \mu_{\pi_k} \). Here, \( d \) is the dimension of the representation (e.g. \( |\mathcal{X}| |\mathcal{A}| \) for the tabular case, or number of features for the linear value function case).

Remark 5.3. The requirement for the \( \mu_{\pi^*} \otimes \pi_k \)-norm and \( \mu_{\pi^*} \otimes \pi_k \)-norm has been shown to hold, for example, for linear value function approximation using the LSPE algorithm (Bertsekas & Ioffe, 1996), under Assumptions B.1-B.3 given in the Appendix; see Theorem 5.1 in Abbasi-Yadkori et al. (2019a) for further details. Lemma B.4 in the supplementary material shows that the requirement for \( \ell_\infty \)-norm can also be satisfied, for example, for linear value functions, under similar conditions.

Remark 5.4. The estimation error generally depends on the mismatch between distributions \( \mu_{\pi_k} \) and \( \mu_{\pi^*} \). With value
functions linear in features $\phi(x, a) \in \mathbb{R}^d$, this mismatch depends on the spectra of matrices $\mathbb{E}_\nu[\phi(x, a)\phi(x, a)^\top]$ for different distributions $\nu$, and need not scale in the number of state-action pairs.

**Theorem 5.5** (Main result). Consider an ergodic MDP and suppose Condition 5.1 holds. Denote $\mu^* = \min_{\pi} \mu(x; \pi(x) > 0 \mu^\star(x))$. By choosing the phase length $\tau = (\bar{C}/\mu_\min)^{2/3} T^{2/3}$, we have with probability at least $1 - 1/T$,

$$R_T = \mathcal{O}\left(\frac{t_{\max}^2 C^{2/3}}{\mu_{\min}^{4/3}} T^{2/3} + T\xi_0\right),$$

where $\mathcal{O}(\cdot)$ hides universal constants and poly-logarithmic factors.

**Remark 5.6.** It is worth comparing the above result with Abbasi-Yadkori et al. (2019a). Ignoring the irreducible error $\xi_0$, we improve the leading order of their general result (Corollary 4.6 in Abbasi-Yadkori et al. (2019a)) from $\mathcal{O}(T^{3/4})$ to $\mathcal{O}(T^{2/3})$. When specialized to linear value function approximation where $\bar{C}$ scales with $d^{1/2}$ (Theorem 5 in Abbasi-Yadkori et al. (2019a)), we improve their results from $\mathcal{O}(d^{1/2} T^{3/4})$ to $\mathcal{O}(d^{1/3} T^{2/3})$.

### 6. Proof sketch

In this section, we provide a proof sketch for Theorem 5.5. Technical details are deferred to Appendix A in the supplementary material. At a high level, we bound the two terms in the regret decomposition Eq. (5.1) separately. While the first term is bounded by the fast mixing condition, the second term is split into the regret due to value function estimation error and the regret due to online learning reduction.

**Step 1:** fast mixing. To bound the first term in Eq. (5.1), we require the following uniform fast mixing condition, which is used frequently in online MDP literature (Even-Dar et al., 2009; Neu et al., 2014). Note that ergodic MDPs that this paper focuses on automatically satisfy this condition.

**Condition 6.1** (Uniform fast mixing). There exists a number $t_{\max} > 0$ such that for any policy $\pi$ and any pair of distributions $\mu$ and $\mu'$ over $\mathcal{X}$, it holds that

$$\|\mu - \mu'\|_{P, \pi} \leq \exp(-1/t_{\max})\|\mu - \mu'\|_1.$$  

(6.1)

The following lemma provides upper bounds for the first term (see e.g. Lemma 4.4 in Abbasi-Yadkori et al. (2019a) for a proof).

**Lemma 6.2.** Suppose that Condition 6.1 holds. The following inequality holds with probability at least $1 - \delta$,

$$\sum_{t=1}^{T} (\lambda_{\pi_t} - \tau(x_t, a_t)) \leq K t_{\max} + 4\sqrt{2t_{\max}} \sqrt{KT \log(T/\delta)},$$

where $K$ is the number of phases.

**Step 2:** decomposition. We bound the second term (pseudo regret) in Eq. (5.1). Since the policy is only updated at the end of each phase of length $\tau$ (see line 9 in Algorithm 1), we have $\pi_t = \pi_k$ for $t \in \{\tau(k-1), \ldots, \tau k\}$. Thus, the pseudo-regret term can be rewritten as,

$$\sum_{t=1}^{T} (\lambda_{\pi^*} - \lambda_{\pi_t}) = \tau \sum_{k=1}^{K} (\lambda_{\pi^*} - \lambda_{\pi_k}).$$  

(6.2)

We slightly abuse notation by writing $Q_{\pi}(x, \pi') = \sum_{a} \pi'(a|x)Q_{\pi}(x, a)$. In particular, $Q_{\pi}(x, \pi)$ is exactly the value function $V_{\pi}(x)$ by Definition 3.2. Applying the performance difference lemma (Lemma C.1 in the supplementary material), we have

$$\lambda_{\pi^*} - \lambda_{\pi_k} = \left< \mu_{\pi}, Q_{\pi_k} - Q_{\pi_k} \right>. $$

Bridging by empirical estimations, we decompose (6.2) into $R_{1T} + R_{2T}$, where

$$R_{1T} = \tau \sum_{k=1}^{K} \left< \mu_{\pi}, Q_{\pi_k} - Q_{\pi_k} \right>$$

(6.3)

and

$$R_{2T} = \tau \sum_{k=1}^{K} \left< \mu_{\pi^*}, Q_{\pi_k} - Q_{\pi_k} \right>. $$

(6.4)

**Step 3:** estimation error. The term $R_{1T}$ quantifies the regret incurred in the policy evaluation step due to the estimation error and function approximation error of $Q$-function in each phase. It can be bounded as in Theorem 4.1 of Abbasi-Yadkori et al. (2019a) under similar assumptions, which we reproduce here for completeness.

**Lemma 6.3.** Suppose Condition 5.1 holds. Then

$$R_{1T} \leq T \left( \xi_0 + \bar{C} \sqrt{\frac{\log(1/\delta)}{\tau}} \right),$$

(6.4)

with probability at least $1 - \delta$.

**Step 4:** online learning reduction. Minimizing $R_{2T}$ can be cast into an online learning problem (Cesa-Bianchi &
Lugosi, 2006; Shalev-Shwartz et al., 2012), and this observation determines the choice of our algorithm. Previous work (Abbasi-Yadkori et al., 2019a) has tackled this sub-problem using mirror descent, resulting in $O(T^{3/4})$ regret after optimizing $\tau$ ignoring the irreducible error $\varepsilon_0$. Here we instead use the AO-FTRL framework, which allows us to show an improved $O(T^{2/3})$ regret bound. As we show, the reason we can benefit from optimism is that the losses (Q-functions) change slowly, and we carefully transfer this knowledge to the adaptive learning rate. This is the main technical contribution of the paper.

First, we state the framework of AO-FTRL and its regret results. Let $\{f_t\}_{t=1}^T$ be a sequence of loss vectors and let $\{x_t\}_{t=1}^T \subseteq \mathcal{F}$ be a sequence of prediction vectors, where $\mathcal{F}$ is the probability simplex. At the beginning of each round, the algorithm receives a side-information vector $M_t$. In literature, $\{M_t\}_{t=1}^T$ are also called predictable sequences (Rakhlin & Sridharan, 2012), and the algorithm can be seen as a way of utilizing prior knowledge about loss sequences. The goal of this online learning problem is to minimize the cumulative regret with respect to the best action in hindsight $f^*$, defined as $R_T = \sum_{t=1}^T \langle f_t - f^*, x_t \rangle$.

Let $\mathcal{R}: \mathcal{F} \to \mathbb{R}$ be a 1-strongly convex regularizer on $\mathcal{F}$ with respect to some norm $\| \cdot \|$ and denote by $\| \cdot \|_*$ its dual norm. Initialize $f_1 = \arg\min_{f \in \mathcal{F}} \mathcal{R}(f)$. At each round $t$, AO-FTRL has the following form:

$$f_{t+1} = \arg\min_{f \in \mathcal{F}} \left( \sum_{s=1}^{t} q_s + M_{t+1} \right) + \eta_t \mathcal{R}(f)$$

$$\eta_t = \eta \left( \sum_{s=1}^{t} \| q_s - M_s \|_2^2 \right)$$

where $\eta$ is an absolute constant. It’s easy to see that $\eta_t$ is non-decreasing. For simplicity, we assume $M_1 = 0, \eta_0 = 0$. Next lemma provides a generic regret bound for AO-FTRL. The detailed proof is deferred to Appendix A.2 in the supplementary material.

**Lemma 6.4.** Choose $\eta = \sqrt{2/\mathcal{R}(f^*)}$ and denote $R_{\max} = \max_{f \in \mathcal{F}} \mathcal{R}(f)$. The cumulative regret for AO-FTRL is upper-bounded by

$$R_T \leq \sqrt{2R_{\max} \sum_{t=1}^{T} \| q_t - M_t \|_*^2}$$

$$- \sum_{t=1}^{T} \frac{\eta_t}{4} \| f_t - f_{t+1} \|_2^2 + \langle M_{t+1}, f^* - f_{T+1} \rangle.$$  \hspace{1cm} (6.5)

**Remark 6.5.** Unlike the AO-FTRL analyses of Rakhlin & Sridharan (2012); Mohri & Yang (2016), but similarly to, e.g., the analysis of Joulani et al. (2017), Eq. (6.5) has a key negative term (at the expense of a slightly larger constant factor in the main positive term). These negative terms, which are retained from a tight regret bound on the forward regret of AO-FTRL (Joulani et al., 2017), track the evolution of the policy $f_t$. With the proper choice of $M_t$, the norm terms $\| q_t - M_t \|_*$ will also be controlled by the evolution of $f_t$ (see Lemma 6.6), and the aforementioned negative terms allow us to greatly reduce the contribution of the norm terms $\| q_t - M_t \|_*$ to the overall regret.

The reason that minimizing $R_{2T}$ can be cast into an online learning problem is as follows. By the definition of $Q_\pi(x, \pi')$ in Step 2, we rewrite $R_{2T}$ in (6.3) as

$$R_{2T} = T \sum_{x \in \mathcal{X}} \mu_\pi(x) \sum_{k=1}^{K} \langle \pi_k(\cdot|x) - \pi_k(x, \cdot), \hat{Q}_{\pi_k}(x, \cdot) \rangle.$$  \hspace{1cm} (6.6)

For each state $x \in \mathcal{X}$, we view $\pi_k(\cdot|x)$ as the prediction vector and $\hat{Q}_{\pi_k}(x, \cdot)$ as the loss vector. The equivalence between $R_{2T}(x)$ and $\hat{R}_T$ enables us to utilize the generic regret bound for AO-FTRL in Lemma 6.4 for each individual state.

Next, we will show that under some conditions, the change in the true $Q$ values can be bounded by the change of policies. This is a unique property of ergodic MDPs that allows us to benefit from the negative term in (6.5). To ensure $Q_\pi$ is unique, we assume $\sum_{x} \mu_\pi(x) V_\pi(x) = 0$.

**Lemma 6.6** (Relative $Q$-function Error). For any two successive policies $\pi_{k-1}$ and $\pi_k$, the following holds for any state-action pair $(x, a)$,

$$\left| Q_{\pi_k}(x, a) - Q_{\pi_{k-1}}(x, a) \right| \leq \frac{t_{\max}^2 \log_2(K) \max_{x} \| \pi_{k-1}(\cdot|x) - \pi_k(\cdot|x) \|_1 + \frac{2}{K^3}.$$

The detailed proof of Lemma 6.6 is deferred to Appendix A.4 in the supplementary material. Combining the result in Lemmas 6.4 and 6.6, we can derive the following lemma.

**Lemma 6.7.** Denote $\mu_{\min} = \min_{x: \mu_\pi(x) > 0} \mu_\pi(x)$. Suppose Condition 5.1 holds. Then the following upper bound holds with probability at least $1 - \delta$,

$$R_{2T} \lesssim \frac{t_{\max}^4 \log_2(K)}{\mu_{\min}^*} + T \left( \frac{C^2 \log(1/\delta)}{\tau} + \varepsilon_0^2 \right).$$  \hspace{1cm} (6.6)
where \( \lesssim \) hides universal constant factors.

The detailed proof of Lemma 6.7 is deferred to Appendix A.3 in the supplementary material. Finally, we optimize \( \tau \) to be \((\tilde{C}_f \mu_{\text{min}}^*/\tau^3)^{1/3} T^{2/3} \) and reach our conclusion.

**Remark 6.8.** Within the upper bound (6.6), \( \tilde{C}^2 \log(1/\delta)/\tau + \varepsilon_0^2 \) stands for the approximation error and estimation error per round. When value functions can be computed exactly (known MDP) and for phase length \( \tau = 1 \), the online learning reduction regret for AAPI scales logarithmically in the number of phases \( K \), while POLITEX (Abbasi-Yadkori et al., 2019a) scales as \( \sqrt{K} \).

This is the main reason why we are able to improve the regret from \( O(T^{3/4}) \) to \( O(T^{2/3}) \) in the end.

### 7. Experiments

**Setup.** In this section, we empirically evaluate the benefits of using adaptive learning rates \( \eta_k(x) \) and side information \( M_{k+1}(x,a) \). In particular, we compare (1) AAPI with \( M_{k+1}(x,a) = \hat{Q}_{\pi_k}(x,a) \) and \( \eta_k(x) \) as in Eq. (4.2); (2) \( k \)-AAPI , similar to AAPI , but approximating the adaptive learning rate by \( \eta_k = \eta_k \sqrt{K} \) for the purpose of computational efficiency, (3) Politex, with \( M_{k+1} = 0 \) and \( \eta_k = \eta_k \sqrt{K} \).

We also evaluate RLSVI ((Osband et al., 2016) Algorithms 1 and 2 with \( \sigma^2 = 1 \) and tuned \( \lambda \)), where policies are greedy w.r.t. a randomized estimate of \( Q_* \). For RLSVI in episodic environments with fixed known horizon \( H \) (DeepSea), we estimate separate parameters for each \( h \in [H] \) and update after each episode, otherwise we share parameters and update every \( \sqrt{T} \) steps.

We approximate all value functions using least-squares Monte Carlo, i.e. linear regression from state-action features to empirical returns. For MDPs with a large or continuous state space \( \mathcal{X} \), updating per-state learning rates can be impractical. Instead, we store the weights of past \( Q \)-functions in memory, and for each state in the trajectory, we compute the learning rate using a subset of \( \eta_k \leq 30 \) randomly-selected past weight vectors (we correct the scale of the estimate by multiplying with \( \sqrt{K}/\eta_k \)).

For Boltzmann policies, we tune the constant \( \eta \) for the learning rate \( \eta_k(x) \) in the range \([0.01, 1] \). For each environment and algorithm we evaluate \( -\sum_{t=1}^T r_t / t \) and plot the mean and standard deviation over 50 runs. We show results on the following three environments.

**Tabular ergodic MDPs.** We consider a simple tabular MDP where \( r(1,a) = 1, r(x,a) = 0 \) for \( x \neq 1 \). On any action in state 1, the environment transitions to a randomly chosen state \( x \neq 1 \). On action 1 in a state \( x \neq 1 \), the environment transitions to state \( x-1 \) with probability 0.9, and to a randomly chosen state with probability 0.1. On all other actions in \( x \neq 1 \), the environment transitions to a randomly chosen state. We represent state-action pairs using one-hot indicator vectors of size \( \mathcal{X} \mathcal{A} \), and experiment with different sizes of the state and action spaces \( \mathcal{X} \) and \( \mathcal{A} \).

**DeepSea** (Osband et al., 2017). In the DeepSea environment, states comprise an \( N \times N \) grid, and there are two actions. The environment transition and costs are deterministic. On action 0 in cell \((i,j)\), the environment transitions to \(((i+1) \mod N, \max(0,j-1))\). On action 1 in cell \((i,j)\), the environment transitions to cell \(((i+1) \mod N, \min(N-1,j+1))\). The agent starts in state \((0,0)\). The reward (negative cost) for being in cell \((N-1, N-1)\) is \( 2N \) for any action. For all other states \( x \), the reward is \( r(x,0) = 0 \) and \( r(x,1) = -1 \). In the infinite horizon setting, an optimal strategy first takes the action 1 \( N \) times (to get to \((N-1, N-1)\)) and then takes an equal number of 0 and 1 actions, and has expected average reward close to 1.5. A simple strategy that always takes action 1 has an average reward 1, and a suboptimal strategy that only takes action 0 has an average reward of 0. We represent states as length-2\(N\) vectors containing one-hot indicators for each grid coordinate, and estimate linear \( Q \)-functions.

**CartPole** (Barto et al., 1983). In the CartPole environment, the goal is to balance an inverted pole attached by an unactuated joint to a cart, which moves along a frictionless rail. There are two actions, corresponding to pushing the cart to the left or right. The observation consists of the position and velocity of the cart, pole angle, and pole velocity at the tip. There is a reward of +1 for every timestep that the pole remains upright. The episodic version of the environment ends if the pole angle is more than 15 degrees from vertical, or if the cart moves more than 2.4 units from the center, or after 200 steps. In the infinite-horizon version, if the episode ends after \( h \) steps, we return a reward of \( h - 200 \) and reset. For this task, in addition to the given observation, we extract multivariate Fourier basis features (Konidaris et al., 2011) of order 4.

**Results and discussion.** The experimental results are shown in Figures 1, 2, and 3. AAPI outperforms other algorithms in the DeepSea and ergodic environment instances. These problems all involve a single high-reward state, and reaching the state requires sufficient exploration. On the other hand, the adaptive per-state learning rate is not helpful in CartPole, possibly because observations are continuous and there is higher generalization across states. In
most of our experiments, AAPI performs better with smaller phase length $\tau$. Our analysis relies on long phases of length $\tau = T^{2/3}$ in part to obtain better side-information $M_k$. However, given that the side-information also depends on $t_{\text{mix}}$ and $\mu_{\tau}$, in some MDPs shorter phases may suffice. This is clear in the DeepSea results, where shorter phases result in better performance for smaller problem instances.

8. Discussion and future work

We have presented AAPI, a model-free learning scheme that can work with function approximation, and enjoys a $O(T^{2/3})$ regret guarantee in infinite-horizon undiscounted, ergodic MDPs. AAPI improves upon previous results for this setting by using the slow-changing property of policies in both theory and practice.

Our result has an undesirable dependence on $1/\mu_{\tau}$. With value function approximation, one general direction for improvement is replacing dependence on $|\mathcal{X}|$ with the size of the compressed representation, such as the minimum eigenvalue of $E_{\mu^*(x)}[\phi(x)\phi(x)^\top]$ in the linear case. Another direction for future work is improving the policy evaluation stage. While we estimate each value function solely using the $\tau$ on-policy transitions, better estimates can potentially be obtained using all data. Using more sophisticated side-information, such as a weighted average of past Q-estimates or an off-policy estimate of the Q-function may also be helpful in practice. Other future work may include practical implementations of the algorithm when trained with neural networks that maintain only a subset of past networks in memory.

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Supplement to “Provably Efficient Adaptive Approximate Policy Iteration”

In Section A, we present the detailed proofs of main results. In Section B, the linear value function approximation is considered. In Section C, some supporting lemmas are included.

A. Proofs of main results

A.1. Proof of Theorem 5.5: main result

We combine the decomposition (5.1), (6.2) and (6.3) together and utilize the results in Lemmas 6.2, 6.3 and 6.7. Then we have

\[ R_T \lesssim Kt \log(T/\delta) + 4Kt \log(T/\delta)^{1/2} + T\varepsilon_0 + \frac{T^4}{\mu_{\min}^2} \log(1/\delta) + \varepsilon_0^2. \]

We choose \( \delta = 1/T \) and ignore any universal constant and logarithmic factor in the following. Since \( K = T/\tau \), it holds that

\[ R_T \lesssim t \log \left( \frac{T}{\delta} \right) + \frac{T^4}{\mu_{\min}^2} \log(1/\delta) + \varepsilon_0^2. \]

with probability at least \( 1 - 1/T \). With a little abuse of notations, we re-define \( \varepsilon_0 = \varepsilon_0^2 + \varepsilon_0 \). By optimizing \( \tau \) such that the first two term above is equal, i.e., \( t \log(1/\delta) = \frac{T^4}{\mu_{\min}^2} \log(1/\delta) + \varepsilon_0^2 \), we choose \( \tau = \frac{\mu_{\min}^2}{tT^2} \). Overall, we reach the final regret bound,

\[ R_T = \tilde{O} \left( \frac{t^2 \log(1/\delta)}{\mu_{\min}^2} T^{2/3} + T \varepsilon_0 \right). \]

This ends the proof.

A.2. Proof of Lemma 6.4: adaptive optimistic FTRL (AO-FTRL)

Lemma 6.4 states that the cumulative regret for AO-FTRL is upper-bounded by

\[ R_T \leq \left( \frac{8}{\eta} + \eta \mathcal{R}(f^*) \right) \left[ \sum_{t=2}^{T} \|q_t - M_t\|^2_2 - \sum_{t=1}^{T} \frac{\eta_t}{4} \|f_t - f_{t+1}\|^2 + g, \right] \]

where \( g = (M_{T+1}, f^* - f_{T+1}) + \|q_1\|^2_2/\eta_1 \).

First, at each round \( t \), AO-FTRL has the form of

\[ f_{t+1} = \arg\min_{f \in F} \left( f \sum_{s=1}^{t} q_s + M_{t+1} \right) + \eta_t \mathcal{R}(f) \]

\[ = \arg\min_{f \in F} \left( f \sum_{s=1}^{t} q_s + \sum_{s=1}^{t} (M_{s+1} - M_s, f) + \eta_t \mathcal{R}(f) \right), \]

where \( \eta_1 \leq \cdots \leq \eta_t \) are data-dependent learning rates. For simplicity, we assume \( \eta_0 = 0 \). For \( s = 1, \ldots, t \), we define

\[ h_s(f) = (M_{s+1} - M_s, f) + (\eta_s - \eta_{s-1}) \mathcal{R}(f). \]
We define $h_0(f) = 0$ for all $f \in \mathcal{F}$ and $h_{1:t}(f) = \sum_{s=1}^{t} h_s(f) = (M_{s+1}, f) + \eta_s R(f)$. Since $R(f)$ is 1-strongly convex with respect to norm $\| \cdot \|$, $h_s(f)$ is $(\eta_s - \eta_{s-1})$-strongly-convex with respect to $\| \cdot \|$. Then we could rewrite the AO-FTRL update as

$$f_{t+1} = \arg\min_{f \in \mathcal{F}} \left( \sum_{s=1}^{t} q_{s} + \sum_{s=1}^{t} h_s(f) \right).$$

Second, let us define the forward linear regret $R_T^+$ as:

$$R_T^+ = \sum_{t=1}^{T} \langle q_t, f_{t+1} - f^* \rangle.$$

One could interpret $R_T^+$ as a cheating regret since it uses prediction $f_{t+1}$ at round $t$. We decompose the cumulative regret based on the forward linear regret as follows,

$$R_T = \sum_{t=1}^{T} \langle q_t, f_t \rangle - \sum_{t=1}^{T} \langle q_t, f^* \rangle = R_T^+ + \sum_{t=1}^{T} \langle q_t, f_t - f_{t+1} \rangle. \quad (A.2)$$

The second term in the right side captures the regret by the algorithms inability to accurately predict the future. We define the Bregman divergence between two vectors induced by a differentiable function $R$ as follows:

$$D_R(w, u) = R(w) - \left( R(u) + \langle \nabla R(u), w - u \rangle \right).$$

Next theorem is used to bound the forward regret.

**Theorem A.1** (Theorem 3 in (Joulani et al., 2017)). For any $f^* \in \mathcal{F}$ and any sequence of linear losses, the forward regret satisfies the following inequality:

$$R_T^+ \leq \sum_{t=1}^{T} \left( h_t(f^*) - h_t(f_{t+1}) \right) - \sum_{t=1}^{T} D_{h_{1:t}}(f_{t+1}, f_t).$$

Recall that $h_{1:t}(f)$ is $\eta_t$-strongly convex. From the definitions of strong convexity and Bregman divergence, we have

$$\sum_{t=1}^{T} D_{h_{1:t}}(f_{t+1}, f_t) \geq \sum_{t=1}^{T} \frac{\eta_t}{2} \| f_{t+1} - f_t \|^2. \quad (A.3)$$

Applying Theorem A.1 and Eq. (A.3), we have

$$R_T^+ \leq \sum_{t=1}^{T} \left( h_t(f^*) - h_t(f_{t+1}) \right) - \sum_{t=1}^{T} \frac{\eta_t}{2} \| f_{t+1} - f_t \|^2. \quad (A.4)$$

To bound the first term in Eq. (A.4), we expand it by the definition of Eq. (A.1),

$$\sum_{t=1}^{T} \left( h_t(f^*) - h_t(f_{t+1}) \right) = \sum_{t=1}^{T} \langle M_{t+1} - M_t, f^* - f_{t+1} \rangle + \sum_{t=1}^{T} \left( \eta_t - \eta_{t-1} \right) (R(f^*) - R(f_{t+1}))$$

$$\leq \sum_{t=1}^{T} \langle M_{t+1} - M_t, f^* \rangle - \sum_{t=1}^{T} \langle M_{t+1} - M_t, f_{t+1} \rangle + \eta_T R(f^*)$$

$$= \langle M_{T+1}, f^* \rangle - \sum_{t=1}^{T} \langle M_{t+1} - M_t, f_{t+1} \rangle + \eta_T R(f^*), \quad (A.5)$$
where the first inequality is due to the fact that $\eta_t$ is non-decreasing and $\eta_0 = 0$. We decompose the second term in Eq. (A.5) as follows,

$$
\sum_{t=1}^{T} (M_{t+1} - M_t, f_{t+1}) = \sum_{t=1}^{T+1} (M_t, f_t) - \sum_{t=1}^{T} (M_t, f_{t+1})
$$

$$
= \sum_{t=1}^{T} (M_t, f_t) - \sum_{t=1}^{T} (M_t, f_{t+1}) + (M_{T+1}, f_{T+1}),
$$

(\text{A.6})

since $M_1 = 0$. Combining Eq. (A.5) and Eq. (A.6) together,

$$
\sum_{t=1}^{T} \left( h_t(f^*) - h_t(f_{t+1}) \right) = -\sum_{t=1}^{T} (M_t, f_t - f_{t+1}) + (M_{T+1}, f^* - f_{T+1}) + \eta_T R(f^*).
$$

(\text{A.7})

Putting Eq. (A.2), Eq. (A.4) and Eq. (A.7) together, we reach

$$
R_T \leq \sum_{t=1}^{T} \langle q_t - M_t, f_t - f_{t+1} \rangle - \sum_{t=1}^{T} \frac{\eta_t}{2} \| f_t - f_{t+1} \|^2 + (M_{T+1}, f^* - f_{T+1}) + \eta_T R(f^*).
$$

(\text{A.8})

To bound the first term in Eq. (A.8), we first use Hölder’s inequality such that

$$
\langle q_t - M_t, f_t - f_{t+1} \rangle = \frac{2}{\eta_t} \langle q_t - M_t \rangle^\top \eta_t f_t - f_{t+1}) \\
\leq \frac{2}{\eta_t} \| q_t - M_t \| \| f_t - f_{t+1} \| \\
\leq \frac{1}{\eta_t} \| q_t - M_t \|^2 + \frac{\eta_t}{4} \| f_t - f_{t+1} \|^2,
$$

where the last inequality is due to $2ab \leq a^2 + b^2$. Thus we have

$$
R_T \leq \sum_{t=1}^{T} \frac{1}{\eta_t} \| q_t - M_t \|^2 + \sum_{t=1}^{T} \frac{\eta_t}{4} \| f_t - f_{t+1} \|^2 - \sum_{t=1}^{T} \frac{\eta_t}{2} \| f_t - f_{t+1} \|^2 \\
+ (M_{T+1}, f^* - f_{T+1}) + \eta_T R(f^*) \\
= \sum_{t=1}^{T} \frac{1}{\eta_t} \| q_t - M_t \|^2 + \sum_{t=1}^{T} \frac{\eta_t}{4} \| f_t - f_{t+1} \|^2 + (M_{T+1}, f^* - f_{T+1}) + \eta_T R(f^*).
$$

By choosing $\eta_t = \eta \sqrt{\sum_{s=1}^{t} \| q_s - M_s \|^2}$ for some absolute constant $\eta$, we have

$$
R_T \leq \sum_{t=1}^{T} \frac{\| q_t - M_t \|^2}{\eta \sqrt{\sum_{s=1}^{t} \| q_s - M_s \|^2}} + \eta \sqrt{\sum_{t=1}^{T} \| q_t - M_t \|^2 R(f^*)}
$$

$$
- \sum_{t=1}^{T} \frac{\eta_t}{4} \| f_t - f_{t+1} \|^2 + (M_{T+1}, f^* - f_{T+1}).
$$

(\text{A.9})

Lemma 4 in McMahan (2017) states that for any non-negative real numbers $a_1, \ldots, a_T$,

$$
\sum_{t=1}^{T} \frac{a_t}{\sqrt{\sum_{s=1}^{t} a_s}} \leq 2 \sqrt{T} \sum_{t=1}^{T} a_t.
$$

Applying this inequality to the first term in Eq. (A.9) with $a_t = \| q_t - M_t \|^2$, we have

$$
R_T \leq \left( \frac{2}{\eta} + \eta R(f^*) \right) \sqrt{T} \sum_{t=1}^{T} \| q_t - M_t \|^2 - \sum_{t=1}^{T} \frac{\eta_t}{4} \| f_t - f_{t+1} \|^2 + (M_{T+1}, f^* - f_{T+1}).
$$
Letting $\eta = \sqrt{2f/R(f^*)}$ and $R_{\max} = \max_f R(f)$, this concludes the proof.

---

A.3. Proof of Lemma 6.7: online learning reduction

**Step 1.** We utilize Lemma 6.4 for each individual state $x$. Recall that

$$R_{2T} = \tau \sum_{k=1}^{K} \left( \mu_{\pi^*} Q_{\pi_k}(\cdot, \pi^*) - \hat{Q}_{\pi_k}(\cdot, \pi_k) \right)$$

$$= \tau \sum_{x \in X} \mu_{\pi^*}(x) \sum_{k=1}^{K} \left( \hat{Q}_{\pi_k}(x, \cdot) - \hat{Q}_{\pi_k}(x, \cdot) \right).$$

Applying Lemma 6.4 with $f_k = \pi_k(\cdot|x)$, $q_k = \hat{Q}_{\pi_k}(x, \cdot)$ and $M_k = \hat{Q}_{\pi_k-1}(x, \cdot)$, we have

$$R_{2T} \leq \tau \sum_{x \in X} \mu_{\pi^*}(x) \left( \sqrt{2R_{\max}} \sum_{k=1}^{K} \| \hat{Q}_{\pi_k}(x, \cdot) - \hat{Q}_{\pi_k-1}(x, \cdot) \|_{\infty}^{2} 
- \sum_{k=1}^{K} \eta_k(x) \| \pi_k(\cdot|x) - \pi_{k+1}(\cdot|x) \|_{1}^{2} + 2(b + Q_{\max}) \right),$$

(A.10)

since $\hat{Q}_{\pi_K}(x, a) \in [b, b + Q_{\max}]$ from Condition 5.1. Here, $\eta_k(x) = \eta \sqrt{\sum_{s=1}^{K} \| \hat{Q}_{\pi_s}(x, \cdot) - \hat{Q}_{\pi_{s-1}}(x, \cdot) \|_{2,\infty}^{2}}$.

**Step 2.** It remains to bound the cumulative change of estimated $Q$-values in Eq. (A.10). We first decompose it by substracting the true $Q$-function and using the triangle inequality and $2ab \leq a^2 + b^2$:

$$\sum_{k=1}^{K} \| \hat{Q}_{\pi_k}(x, \cdot) - \hat{Q}_{\pi_k-1}(x, \cdot) \|_{\infty}^{2} \leq 2 \sum_{k=1}^{K} \| \hat{Q}_{\pi_k}(x, \cdot) - Q_{\pi_k}(x, \cdot) \|_{\infty}^{2} + 2 \sum_{k=1}^{K} \| Q_{\pi_k-1}(x, \cdot) - \hat{Q}_{\pi_k-1}(x, \cdot) \|_{\infty}^{2}$$

(A.11)

$$+ \sum_{k=1}^{K} 2 \| Q_{\pi_k}(x, \cdot) - Q_{\pi_k-1}(x, \cdot) \|_{\infty}^{2}.$$  

The first two terms in Eq. (A.11) measure the estimation error. By Condition 5.1, we have,

$$\| \hat{Q}_{\pi_k}(x, \cdot) - Q_{\pi_k}(x, \cdot) \|_{\infty}^{2} \leq \frac{2C^2 \log(1/\delta)}{\tau} + 2\varepsilon_0^2,$$

(A.12)

with probability at least $1 - \delta$ for each $k \in [K]$ and for problem-dependent constants $C$. Putting Eq. (A.11), Eq. (A.12) and Lemma 6.6 together, the following holds with probability $1 - K\delta$,

$$\sum_{k=1}^{K} \| \hat{Q}_{\pi_k}(x, \cdot) - \hat{Q}_{\pi_k-1}(x, \cdot) \|_{\infty}^{2}$$

$$\leq \frac{8C^2K \log(1/\delta)}{\tau} + 8K\varepsilon_0^2 + 2d_{\max} \log^4(K \sum_{k=1}^{K} \max_x \| \pi_k(\cdot|x) - \pi_{k-1}(\cdot|x) \|_{1}^{2} + \frac{4K}{K^6}.$$

(A.13)

**Step 3.** Finally, by our choice of the data-dependent learning rate $\eta_k(x)$, we are able to cancel out the positive term in Eq. (A.10) such that the regret is greatly sharpened. For notation simplicity, we denote $d_k(x) = \| \pi_k(\cdot|x) - \pi_{k-1}(\cdot|x) \|_{1}$. Putting Eq. (A.10) and Eq. (A.13) together, with a union bound, we have

$$\frac{R_{2T}}{\tau} \leq C_1 \sum_{x \in X} \mu_{\pi^*}(x) \left( \sqrt{R_{\max}} \sum_{k=1}^{K} \max_x d_k^2(x) + \frac{C^2 K \log(K \mu_{\pi^*}(x) \delta^{-1})}{\tau} + K\varepsilon_0^2 
- \sum_{k=1}^{K} \eta_k(x) \| d_{k+1}(x) + 2(b + Q_{\max}) \right).$$

(A.14)
holds with probability at least $1 - \delta$. Assuming $\eta_0(x) = \eta_1(x)$, we have

$$\sum_{k=1}^{K} \frac{\eta_k(x)}{4} d_{k+1}^2(x) \geq \sum_{k=1}^{K} \frac{\eta_{k-1}(x)}{4} d_k^2(x).$$

Moreover, we denote $\mu_{\min}^* = \min_{x: \mu_\pi(x) > 0} \mu_\pi(x)$ and

$$g_1 = R_{\max}^t \max_{\pi} \log^4(K)$$
$$g_2 = \tilde{C}^2 R_{\max} K \log(K\mu_{\min}^*)^{-1}/\tau + K \tilde{C}^2 R_{\max}$$
$$g_3 = 2(b + Q_{\max}).$$

Then we simplify Eq. (A.14) as

$$\frac{R_{2T}}{\tau} \leq \sum_{x \in X} \mu_\pi(x) \left( \sqrt{g_1 \sum_{k=1}^{K} \max_x d_k^2(x) + g_2 - \sum_{k=1}^{K} \eta_{k-1}(x) d_k^2(x)} + g_3 \right)$$
$$= \sqrt{g_1 \sum_{k=1}^{K} \max_x d_k^2(x) + g_2 - \sum_{x \in X} \mu_\pi(x) \sum_{k=1}^{K} \eta_{k-1}(x) d_k^2(x) + g_3.}$$

Let us denote $x^*_k = \arg\max_x d_k^2(x)$. Noting that

$$\sum_{x \in X} \mu_\pi(x) \sum_{k=1}^{K} \eta_{k-1}(x) \frac{d_k^2(x)}{4} \geq \sum_{k=1}^{K} \mu_\pi(x_k) \frac{\eta_{k-1}(x_k^*) d_k^2(x_k^*)}{4}, \quad (A.15)$$

we have

$$\frac{R_{2T}}{\tau} \leq \sqrt{g_1 \sum_{k=1}^{K} d_k^2(x_k^*) + g_2 - \sum_{k=1}^{K} \mu_\pi(x_k^*) \frac{\eta_{k-1}(x_k^*) d_k^2(x_k^*)}{4} + g_3}$$
$$= \sqrt{g_1 \sum_{k=1}^{K} \mu_\pi(x_k^*) d_k^2(x_k^*) + g_2 - \sum_{k=1}^{K} \mu_\pi(x_k^*) \frac{\eta_{k-1}(x_k^*) d_k^2(x_k^*)}{4} + g_3}$$
$$\leq \sqrt{\frac{4g_1}{\eta_1(x_k^*) \mu_{\min}} \sum_{k=1}^{K} \mu_\pi(x_k^*) \frac{\eta_{k-1}(x_k^*) d_k^2(x_k^*)}{4} + g_2 - \sum_{k=1}^{K} \mu_\pi(x_k^*) \frac{\eta_{k-1}(x_k^*) d_k^2(x_k^*)}{4} + g_3}$$
$$\leq 2 \sqrt{\frac{g_1}{\eta_1(x_k^*) \mu_{\min}} \left( \sum_{k=1}^{K} \mu_\pi(x_k^*) \frac{\eta_{k-1}(x_k^*) d_k^2(x_k^*)}{4} + \mu_{\min}^* \eta_1(x_k^*) g_2 \right) + g_3} + \frac{\mu_{\min}^* \eta_1(x_k^*) g_2}{4g_1} + g_3,$$

where the second inequality we use the fact that $\eta_k$ is monotone increasing. Letting

$$a = \frac{g_1}{\eta_1(x_k^*) \mu_{\min}}, \quad b = \sum_{k=1}^{K} \mu_\pi(x_k^*) \frac{\eta_{k-1}(x_k^*) d_k^2(x_k^*)}{4} + \mu_{\min}^* \eta_1(x_k^*) g_2 \frac{4g_1}{\eta_1(x_k^*)},$$

and using the fact that $2\sqrt{ab} - b \leq a$, we reach

$$\frac{R_{2T}}{\tau} \leq \frac{g_1}{\eta_1(x_k^*) \mu_{\min}} + \frac{\mu_{\min}^* \eta_1(x_k^*) g_2}{4g_1} + g_3. \quad (A.16)$$
We first introduce a lemma that illustrates the true Q-value can be bounded by the mixing time.

\[ R_{2T} \leq \frac{R_{\text{mix}}^4 t_{\text{mix}}^4 \log^4(K)}{\eta_1(x_1^*) \mu_{\text{min}}^*} + \frac{\eta_1(x_1^*) \tilde{\mathcal{C}}^2 K \log(K \mu_{\text{min}}^* \delta)^{-1} + T_0^2}{4 \mu_{\text{min}}^* \log^2(K)} + 2(b + Q_{\text{max}}). \]  

(A.17)

By definition, \( \eta_1(x_1^*) = \sqrt{2R_{\text{max}} \| \tilde{Q}_1(x_k^*) \|_\infty} \). Since \( \tilde{Q}_1(x, a) \in [b, b + Q_{\text{max}}] \) from Condition 5.1, we have \( \eta_1(x_1^*) \) is lower and upper bounded by some constant. Based on this, we simplify the upper bound (A.17) as

\[ R_{2T} \lesssim \frac{t_{\text{mix}}^4 \log^4(K)}{\mu_{\text{min}}^*} + \tilde{\mathcal{C}}^2 K \log(K / \delta) + T_0^2, \]

where \( \lesssim \) hides constant factors. This ends the proof.

A.4. Proof of Lemma 6.6: relative Q-function error

We first introduce a lemma that illustrates the true Q-value can be bounded by the mixing time.

**Lemma A.2** (Lemma 3 in (Neu et al., 2014)). For any policy \( \pi \) and any state-action pair \((x, a) \in \mathcal{X} \times \mathcal{A}\), for any reward function \( r \in [0, 1] \), we have

\[ |Q_\pi(x, a)| \leq 2t_{\text{mix}} + 3. \]  

(A.18)

From the Bellman equation Eq. (3.2),

\[ Q_{\pi_k}(x, a) - Q_{\pi_{k-1}}(x, a) = \sum_{x'} P(x'|x, a) \left( V_{\pi_k}(x') - V_{\pi_{k-1}}(x') \right) + \lambda_{\pi_{k-1}} - \lambda_{\pi_k}. \]  

(A.19)

We first bound \( \lambda_{\pi_{k-1}} - \lambda_{\pi_k} \). By Lemma C.1 (performance difference lemma),

\[ \lambda_{\pi_{k-1}} - \lambda_{\pi_k} = \sum_x \mu_{\pi_{k-1}}(x) \left( \sum_a (\pi_{k-1}(a|x) - \pi_k(a|x)) \right) Q_{\pi_k}(x, a). \]

We first bound \( \lambda_{\pi_{k-1}} - \lambda_{\pi_k} \) by Lemma A.2, it implies

\[ \lambda_{\pi_{k-1}} - \lambda_{\pi_k} \leq (2t_{\text{mix}} + 3) \max_x \| \pi_{k-1}(|x) - \pi_k(|x) \|_1. \]  

(A.20)

Next we bound \( V_{\pi_k}(x) - V_{\pi_{k-1}}(x) \). In an ergodic MDP, the expected average reward \( \lambda_\pi \) can be written as \( \lambda_\pi = \mu_{\pi}^T r_\pi \), where \( r_\pi(x) = \sum_a \pi(a|x) r(x, a) \). Let \( e_x \) be an indicator vector for state \( x \). For all \( \pi \),

\[ V_\pi(x) = \sum_{t=0}^{\infty} \left( e_x^T (P^\pi)^t - \mu_{\pi}^T \right) r_\pi \]

\[ = \sum_{t=0}^{N-1} \left( e_x^T (P^\pi)^t - \mu_{\pi}^T \right) r_\pi + \sum_{t=N}^{\infty} \left( e_x^T (P^\pi)^t - \mu_{\pi}^T \right) r_\pi, \]  

(A.21)

Corollary 13.2 of Wei et al. (2019) shows that for an ergodic MDP with mixing time \( t_{\text{mix}} \) and \( N = \left\lceil 4t_{\text{mix}} \log_2(K) \right\rceil \), for all \( \pi \),

\[ \sum_{t=N}^{\infty} \| e_x^T (P^\pi)^t - \mu_{\pi}^T \|_1 \leq \sum_{t=N}^{2t_{\text{mix}}} 2^{-t_{\text{mix}}} = \frac{2^{1 - N_{\text{mix}}}}{1 - 2^{-t_{\text{mix}}}} \leq \frac{2t_{\text{mix}}}{\ln 2} 2^{1 - \frac{N_{\text{mix}}}{t_{\text{mix}}}} = \frac{2t_{\text{mix}}}{\ln 2} 2 \frac{2}{K^2} \leq \frac{1}{K^3}. \]

Thus, the second term in Eq. (A.21) can be bounded by

\[ \left| \sum_{t=N}^{\infty} \left( e_x^T (P^\pi)^t - \mu_{\pi}^T \right) r_\pi \right| \leq \sum_{t=N}^{\infty} \| e_x^T (P^\pi)^t - \mu_{\pi}^T \|_1 \leq \frac{1}{K^3}. \]
The following steps are similar to the proof of Lemma 7 in Wei et al. (2019). For the sake of completeness, we present a full proof here. The difference between $V_{\pi_k}(x)$ and $V_{\pi_{k-1}}(x)$ can be bounded by

$$
\left| V_{\pi_k}(x) - V_{\pi_{k-1}}(x) \right| 
= \left| \sum_{t=0}^{N-1} \left( (P^{\pi_k})^t - (P^{\pi_{k-1}})^t \right) r_{\pi_k} + \sum_{t=0}^{N-1} e_x^T (P^{\pi_k})^t (r_{\pi_k} - r_{\pi_{k-1}}) \right|_\infty 
\leq \sum_{t=0}^{N-1} \left| \left( (P^{\pi_k})^t - (P^{\pi_{k-1}})^t \right) r_{\pi_k} \right|_\infty 
+ \sum_{t=0}^{N-1} \left| e_x^T (P^{\pi_k})^t (r_{\pi_k} - r_{\pi_{k-1}}) \right|_\infty 
+ N|\lambda_{\pi_k} - \lambda_{\pi_{k-1}}| + \frac{2}{K^3}.
$$

(A.22)

Next, we will derive a recursive form for the first term as follows:

$$
\left| \left( (P^{\pi_k})^t - (P^{\pi_{k-1}})^t \right) r_{\pi_k} \right|_\infty 
\leq \left| (P^{\pi_k})^t - (P^{\pi_{k-1}})^t \right|_\infty r_{\pi_k} 
+ \left| (P^{\pi_k} - P^{\pi_{k-1}}) (P^{\pi_{k-1}})^t r_{\pi_k} \right|_\infty 
\leq \left| (P^{\pi_k})^t - (P^{\pi_{k-1}})^t \right|_\infty \max_x e_x^T (P^{\pi_k} - P^{\pi_{k-1}}) (P^{\pi_{k-1}})^t \left| r_{\pi_k} \right|_1 
\leq \left| (P^{\pi_k})^t - (P^{\pi_{k-1}})^t \right|_\infty \max_x \left| \sum_{x'} \sum_a \left( \pi_k(a|x) - \pi_{k-1}(a|x) \right) P(x'|x,a) \right|_1 
\leq \left| (P^{\pi_k})^t - (P^{\pi_{k-1}})^t \right|_\infty \max_x \left| \pi_k(a|x) - \pi_{k-1}(a|x) \right|_1.
$$

By induction, it holds that

$$
\left| \left( (P^{\pi_k})^t - (P^{\pi_{k-1}})^t \right) r_{\pi_k} \right|_\infty \leq t \max_x \left| \pi_k(a|x) - \pi_{k-1}(a|x) \right|_1.
$$

Thus,

$$
\sum_{t=0}^{N-1} \left| \left( (P^{\pi_k})^t - (P^{\pi_{k-1}})^t \right) r_{\pi_k} \right|_\infty \leq N^2 \max_x \left| \pi_k(a|x) - \pi_{k-1}(a|x) \right|_1.
$$

(A.23)

In addition,

$$
\sum_{t=0}^{N-1} \left| r_{\pi_k} - r_{\pi_{k-1}} \right|_\infty \leq N \max_x \left| \pi_k(a|x) - \pi_{k-1}(a|x) \right|_1.
$$

(A.24)

Plugging Eq. (A.20), Eq. (A.23) and Eq. (A.24) into Eq. (A.22) yields

$$
\left| V_{\pi_k}(x) - V_{\pi_{k-1}}(x) \right| \leq \left( N^2 + N + (2t_{\text{mix}} + 3)N \right) \max_x \left| \pi_k(a|x) - \pi_{k-1}(a|x) \right|_1 + \frac{2}{K^3}.
$$

(A.25)

where $N = \lceil 4t_{\text{mix}} \log_2(K) \rceil$. Together with Eq. (A.19), we reach the result.

\[\square\]

B. Linear value function approximation

In this section, we show that with linear value function approximation and under similar assumptions as in Abbasi-Yadkori et al. (2019a), the estimation error in each state can be bounded in the $\ell_\infty$ norm. Note that we consider an unrealizable case such that Q-function could be approximated linear represented up to an irreducible approximation error $\varepsilon_0$. This is in contrast of many existing works (Yang & Wang, 2019a;b; Jin et al., 2019) who consider realizable cases.

Suppose $\phi : \mathcal{X} \times \mathcal{A} \to \mathbb{R}^d$ is a feature map chosen by the user. Consider $\hat{Q}_{\pi_k}(x,a) = \phi(x,a)^T \hat{\theta}_k$ be the linear value function estimate where $\hat{\theta}_k$ is the estimated weight vector. Let $\Psi$ be a $|\mathcal{X}||\mathcal{A}| \times d$ feature matrix whose rows correspond
to state-action feature vectors. We make the regularity assumption on $\Psi$ and assume that for all policies $\pi$, the following feature excitation condition holds.

**Assumption B.1** (Linearly independent features). The columns of the matrix $[\Psi, 1]$ are linearly independent.

**Assumption B.2** (Uniformly excited features, Assumption A4 in Abbasi-Yadkori et al. (2019a)). There exists a positive real $\sigma$ such that for any policy $\pi$, $\lambda_{\min}(\Psi^\top \text{diag}(\mu_\pi \otimes \pi) \Psi) \geq \sigma$.

Furthermore, we assume that the following error bound holds.

**Assumption B.3** (Estimation error in $\mu_\pi \otimes \pi$-norm). For all $k \in [K]$, with probability at least $1 - \delta$, the value error is bounded in the $\mu_\pi \otimes \pi$-norm.

$$
\|\hat{Q}_{\pi_k} - Q_{\pi_k}\|_{\mu_\pi \otimes \pi} \leq C_2 \sqrt{\frac{\log(1/\delta)}{\tau}} + \varepsilon_0,
$$

where $C_2$ is a problem-dependent constant and $\varepsilon_0$ is the irreducible approximation error.

The above error Assumption B.3 can be satisfied, for example, by the LSPE algorithm of Bertsekas & Ioffe (1996), as shown in Theorem 5.1 in Abbasi-Yadkori et al. (2019a). The same authors show that Assumptions B.1, B.2 and B.3 also suffice to bound the error in $\mu^* \otimes \pi_k$ and $\mu^* \otimes \pi^*$-norms, as required by our Lemma 6.3. Here we additionally prove that under same assumptions, the error in each state is bounded in the $\ell_\infty$-norm, as required by Lemma B.4.

**Lemma B.4** (Estimation error in $\ell_\infty$-norm). Under Assumptions B.2 and B.3, we have the following holds with probability at least $1 - \delta$,

$$
\|\hat{Q}_{\pi_k}(x, \cdot) - Q_{\pi_k}(x, \cdot)\|_\infty \leq C_\Psi (C_2 \sqrt{\frac{\log(1/\delta)}{\tau}} + \varepsilon_0),
$$

where $C_\Psi = \max_{x, a} \|\psi(x, a)\|_2$.

**Proof.** Note that under Assumption B.2, $\|\Psi(\hat{w}_k - w_k)\|_{\mu_\pi \otimes \pi}^2 \geq \sigma \|\hat{w}_k - w_k\|_2^2$. We have the following:

$$
\|\hat{Q}_{\pi_k}(x, \cdot) - Q_{\pi_k}(x, \cdot)\|_\infty = \max_a |\phi(x, a)^\top (\hat{w}_k - w_k)|
\leq C_\Psi \|\hat{w}_k - w_k\|_2
\leq C_\Psi \|\Psi(\hat{w}_k - w_k)\|_{\mu_\pi \otimes \pi}/\sqrt{\sigma}
= C_\Psi \|\hat{Q}_{\pi_k} - Q_{\pi_k}\|_{\mu_\pi \otimes \pi}/\sqrt{\sigma}
\leq C_\Psi C_2 \sqrt{\log(1/\delta)/(\sigma \tau)} + C_\Psi / \sqrt{\sigma \varepsilon_0}.
$$

C. Supporting lemmas

**Lemma C.1** (Performance difference lemma). Consider an MDP specified by the transition probability kernel $P$ and reward function $r$. For any policy $\pi, \hat{\pi}$, it holds that

$$
\lambda_\pi - \lambda_{\hat{\pi}} = \sum_{x, a} \mu_\pi(x)(\pi(a|x) - \hat{\pi}(a|x)) Q_{\pi}(x, a),
$$

where $\mu_\pi(x)$ is the stationary distribution of a policy $\pi$. 

Proof. Based on average reward Bellman equation, we have
\[
\sum_{x,a} \mu_\pi(x)\pi(a|x)Q_\pi(x, a) = \sum_{x,a} \mu_\pi(x)\pi(a|x)\left[r(x, a) - \lambda_\hat{\pi} + \sum_{x'} P(x'|x, a)V_\hat{\pi}(x')\right]
= \lambda_\pi - \lambda_\hat{\pi} + \sum_x \mu_\pi(x)V_\hat{\pi}(x),
\]
where the second equation is due to \(\sum_{x,a} \mu_\pi(x)\pi(a|x)P(x'|x, a) = \mu_\pi(x')\). Therefore,
\[
\lambda_\pi - \lambda_\hat{\pi} = \sum_{x,a} \mu_\pi(x)\pi(a|x)Q_\pi(x, a) - \sum_x \mu_\pi(x)V_\hat{\pi}(x) = \sum_{x,a} \mu_\pi(x)\left(\pi(a|x)Q_\pi(x, a) - \hat{\pi}(a|x)Q_\hat{\pi}(x, a)\right).
\]
This ends the proof.

\[\blacksquare\]

Lemma C.2 (Lemma 4 in (McMahan, 2017)). For any non-negative real numbers \(a_1, \ldots, a_T\), the following holds
\[
\sum_{t=1}^{T} \frac{a_t}{\sqrt{\sum_{s=1}^{t} a_s}} \leq 2 \sqrt{\sum_{t=1}^{T} a_t}.
\]