Abstract

I present a new method for specifying and verifying the partial correctness of sequential programs. The key observation is that, in Hoare logic, assertions are used as selectors of states: an assertion specifies the set of program states which satisfy the assertion. Hence, the usual meaning of the partial correctness Hoare triple \( \{ f \} P \{ g \} \): if execution is started in \textit{any of the states that satisfy assertion} \( f \), then, upon termination, the resulting state will be \textit{some state that satisfies assertion} \( g \). There are of course other ways to specify a set of states. I propose to replace assertions by terminating programs: a program \( \alpha \) specifies a set of states as follows: we start \( \alpha \) in any state whatsoever, and all the states that \( \alpha \) may terminate in constitute the specified set. I call this set the \textit{post-states} of \( \alpha \). I introduce the \textit{operational triple} \( [\alpha] P [\beta] \) to mean: if execution of \( P \) is started in any post-state of \( \alpha \), then upon termination, the resulting state will be some post-state of \( \beta \). Here, \( \alpha \) is the \textit{pre-program}, and plays the role of a pre-condition, and \( \beta \) is the \textit{post-program}, and plays the role of a post-condition.

Keywords: Program verification, Hoare logic
1 Introduction

I present a system for verifying partial correctness of deterministic sequential programs. In contrast to Floyd-Hoare logic [5, 9], my system does not use pre-conditions and post-conditions, but rather pre-programs and post-programs. An assertion is essentially a means for defining a set of states: those for which the assertion evaluates to true. Hence the usual Hoare triple \{f\} P \{g\} means that if execution of P is started in any of the states that satisfy assertion f, then, upon termination of P, the resulting state will be some state that satisfies assertion g.

Another method of defining a set of states is with a program \(\alpha\) which starts execution in any state, i.e., with precondition true. The set of states in which \(\alpha\) terminates (taken over all possible starting states) constitutes the set of states that \(\alpha\) defines. I call these the post-states of \(\alpha\).

I introduce the operational triple \([\alpha] P [\beta]\), in which \(\alpha\) and \(\beta\) are terminating sequential programs, and P is the sequential program that is being verified. \(\alpha\) is the pre-program and \(\beta\) is the post-program. The meaning of \([\alpha] P [\beta]\) is as follows. Consider executions of \(\alpha\) that start in any state (i.e., any assignment of values to the variables). From the final state of all such executions, P is executed. Let \(\varphi\) be the set of resulting final states. That is, \(\varphi\) results from executing P from any post-state of \(\alpha\). Also, let \(\psi\) be the set of post-states of \(\beta\), i.e., the set of states that result from executing \(\beta\) starting in any state. Then, \([\alpha] P [\beta]\) is defined to mean \(\varphi \subseteq \psi\). That is, the post-states of \(\alpha\); P are a subset of the post-states of \(\beta\).

The advantages of my approach are as follows. Since the pre-program \(\alpha\) and the program P are constituted from the same elements, namely program statements, it is easy to “trade” between the two, i.e., to move a statement from the program to the pre-program and vice-versa. This tactic is put to good use in in deriving programs from operational specifications, and is illustrated in the examples given in this paper. Since the pre-program \(\alpha\) and post-program \(\beta\) are not actually executed, then can be written without concern for efficiency. In fact, they can refer to any well defined expression, e.g., \(\delta[t]\) for the shortest path distance from a designated source \(s\) to node \(t\).

2 Technical preliminaries

2.1 Syntax of the programming language

I use a basic programming language consisting of standard primitive types (integers, boolean etc), arrays, and reference types, assignments, if statements, while loops, for loops, procedure definition and invocation, class definition, object creation and referencing. The syntax that I use is given by the BNF grammar in Table 1. I omit the definitions for Id (identifier), Num (numeral), Obj (object reference), as these are primarily lexical in nature. I assume as given the grammar classes Primitive for primitive types, and Reference for reference types, i.e., my syntax is parametrized on these definitions.

My syntax is standard and self-explanatory. I also use \([\cdot]\) to denote non-deterministic choice between two commands [5]. For integers \(i, j\) with \(i \leq j\), I use \(x := [i : j]\) as syntactic sugar for \(x := i\) \(\cdots\) \(x := j\), i.e., a random assignment of a value in \(i, \ldots, j\) to \(x\). This plays the role of the range assertion \(i \leq x \leq j\) in Hoare logic. I use \(tt\) for true, and \(ff\) for false.

In the sequel, when I use the term “program”; I will mean a statement (Stat) written in the
Table 1: Syntax of the programming language

I use \texttt{tt} for true, and \texttt{ff} for false.

### 2.2 Semantics of the programming language

I assume the usual semantics for reference types: object identifiers are pointers to the object, and the identity of an object is given by its location in memory, so that two objects are identical if and only if they occupy the same memory. Parameter passing is by value, but as usual a passed array/object reference allows the called procedure to manipulate the original array/object. My proof method relies on (1) the axioms and inference rules introduced in this paper, and (2) an underlying method for establishing program equivalence. Any semantics in which the above are valid can be used. For concreteness, I assume a standard small-step (SOS) operational semantics \cite{23, 25}.

An execution of program \( P \) is a finite sequence \( s_0, s_1, \ldots, s_n \) of states such that (1) \( s_i \) results from a single small step of \( P \) in state \( s_{i-1} \), for all \( i \in 1, \ldots, n \), (2) \( s_0 \) is an initial state of \( P \), and (3) \( s_n \) is a final (terminating) state of \( P \). A behavior of program \( P \) is a pair of states \( (s, t) \) such that (1) \( s_0, s_1, \ldots, s_n \) is an execution of \( P \), \( s = s_0 \), and \( t = s_n \). Write \( \{\|P\|\} \) for the set of behaviors of \( P \).

### 2.3 Hoare Logic

I assume standard first-order logic, with the standard model of arithmetic and standard Tarskian semantics. Hence I take the notation \( s \models f \), where \( s \) is a state and \( f \) is a formula, to have the usual meaning.

I use the standard notation for Hoare-logic partial correctness: \( \{f\} P \{g\} \) means that if
execution of program $P$ starts from a state satisfying formula $f$, then, if $P$ terminates, the resulting final state will satisfy formula $g$. That is, for all $(s, t) \in \|P\|$, if $s \models f$ then $t \models g$.

3 Operational annotations

I use a terminating program to specify a set of states. There is no constraint on the initial states, and the set of all possible final states is specified. If any initialization of variables is required, this must be done explicitly by the program.

Definition 1 (Post-state set) Let $P$ be a program. Then

$$post(P) \doteq \{ t \mid \exists s : (s, t) \in \|P\| \}$$

That is, $post(P)$ is the set of all possible final states of $P$, given any initial state. If one increases the set of states in which a program can start execution, then the set of states in which the program terminates is also possibly increased, and is certainly not decreased. That is, the set of post-states is monotonic in the set of pre-states. Since prefixing a program $\alpha$ with another program $\gamma$ simply restricts the states in which $\alpha$ starts execution, I have the following.

Proposition 1 $\gamma; \alpha \preceq \alpha$.

Proof. Let $s \in post(\gamma; \alpha)$. Then, there is some state $u$ such that $u \in post(\gamma)$ and $(u, s) \in \|\alpha\|$. Hence $s \in post(\alpha)$. So $post(\gamma; \alpha) \subseteq post(\alpha)$, from which $\gamma; \alpha \preceq \alpha$ follows. \qed

The central definition of the paper is that of operational triple $[\alpha]P[\beta]$:

Definition 2 (Operational triple) Let $\alpha$, $P$, and $\beta$ be programs. Then

$$[\alpha]P[\beta] \doteq post(\alpha; P) \subseteq post(\beta).$$

3.1 Program Ordering and Equivalence

The next section presents a deductive system for establishing validity of operational triples. The rules of inference use three kinds of hypotheses: (1) operational triples (over “substatements” as usual), and (2) program ordering assertions $P \preceq Q$, and (3) program equivalence assertions $P \equiv Q$.

Definition 3 (Program ordering) $P \preceq Q \doteq post(P) \subseteq post(Q)$.

Here $P$ is “stronger” than $Q$ since it has fewer post-states (w.r.t., the pre-condition $tt$, i.e., all possible pre-states), and so $P$ produces an output which satisfies, in general, more constraints than the output of $Q$. Note that this is not the same as the usual program refinement relation, since the mapping from pre-states to post-states induced by the execution of $P$ is not considered. Also, the “direction” of the inclusion relation is reversed w.r.t. the usual refinement ordering, where we write $Q \sqsubseteq P$ to denote that “$P$ refines $Q$”, i.e., $P$ satisfies more specifications than $Q$. This is in keeping with the importance of the post-state set in the sequel. Note that, by Definition 3, $[\alpha]P[\beta] \doteq \alpha; P \preceq post(\beta)$.\[3\]
Definition 4 (Program equivalence, ≡) Programs P and Q are equivalent iff they have the same behaviors: \( P \equiv Q \triangleq \{P\} = \{Q\} \).

That is, I take as program equivalence the equality of program behaviors. Note that equivalence is not ordering in both directions. This discrepancy is because ordering is used for weakening/strengthening laws (and so post-state inclusion is sufficient) while equivalence is used for substitution, and so, for programs at least, equality of behaviors is needed.

Any method for establishing program ordering and equivalence is sufficient for my needs. The ordering and equivalence proofs in this paper were informal, and based on obvious concepts such as the commutativity of assignment statements that modify different variables/objects.

Works that provide proof systems for program equivalence include [2, 21, 12, 3, 22]. Some of these are mechanized, and some use bisimulation and circular reasoning. I will look into using these works for formally establishing program equivalence hypotheses needed in my examples (which are then akin to Hoare logic verification conditions), and to adapting these systems to establish program ordering, e.g., replace bisimulation by simulation.

4 A deductive system for operational annotations

Table 2 presents a deductive system for operational annotations. I do not provide a rule for the for loop, since it can be easily turned into a while. for loops quite compact, and so are very convenient for use in pre-programs and post-programs, i.e., as annotations. The following are informal intuition and proofs of soundness for the axioms and inference rules.

4.1 Axioms

Sequence Axiom. If \( P \) executes after pre-program \( \alpha \), the result is identical to post-program \( \alpha; P \), i.e., the sequential composition of \( \alpha \) and \( P \). This gives an easy way to calculate a post-program for given pre-program and program. The corresponding Hoare logic notion, namely the strongest postcondition, is easy to compute (in closed form) only for straight-line code.

Proposition 2 (Sequence Axiom) is valid

Proof. By Definition 2 \([\alpha] P [\alpha; P] \) is \( post(\alpha; P) \subseteq post(\alpha; P) \), which is immediate. □

Empty pre-program. Follows from (Sequence Axiom) by replacing \( \alpha \) by the empty program skip. Program \( P \) is “doing all the work”, and so the resulting post-program is also \( P \). Having skip as a pre-program is similar to having \( tt \) as a precondition in Hoare logic.

Empty Program. Follows from (Sequence Axiom) by replacing \( P \) by the empty program skip. This is analogous to the axiom for skip in Hoare logic: \( \{f\} skip \{f\} \), since skip has no effect on the program state.
\[\alpha P[\alpha; P]\] 
\((\text{Sequence Axiom})\)

\[\text{skip} P[P]\] 
\((\text{Empty Pre-program})\)

\[\alpha \text{ skip} \alpha\] 
\((\text{Empty Program})\)

\[\alpha P1; P2 [\beta] \text{ iff } \alpha; P1 P2 [\beta]\] 
\((\text{Trading})\)

\[\alpha P[\beta] \quad \alpha P[\beta; \gamma] \quad \alpha P[\beta; \gamma] \quad \alpha P[\beta; \gamma]\] 
\((\text{Append})\)

\[\alpha \equiv \alpha' P \equiv P' \quad \beta \equiv \beta' [\alpha] P[\beta]\] 
\((\text{Substitution})\)

\[\alpha \preccurlyeq \alpha' [\alpha'] P[\beta]\] 
\((\text{Pre-program Strengthening})\)

\[\alpha P[\beta'] \quad \beta' \preccurlyeq \beta\] 
\((\text{Post-program Weakening})\)

\[\alpha P1[\beta] \quad \beta P2[\gamma]\] 
\((\text{Sequential Composition})\)

\[\alpha P1[\beta] \quad \beta P2[\gamma]\] 
\((\text{Sequential Composition})\)

\[\alpha' \equiv (\alpha, B), \beta \equiv (\alpha, \neg B)\] 
\((\text{While})\)

\[\alpha' \equiv (\alpha, B), \beta \equiv (\alpha, \neg B)\] 
\((\text{While Consequence})\)

\[\alpha' \equiv (\alpha, B), \beta \equiv (\alpha, \neg B)\] 
\((\text{If})\)

\[\alpha' \equiv (\alpha, B), \beta \equiv (\alpha, \neg B)\] 
\((\text{One-way If})\)

Table 2: Axioms and rules of inference
4.2 Rules of Inference

**Trading Rule.** Sequential composition is associative: \( \alpha; (P1; P2) \equiv (\alpha; P1); P2 \). By Definitions\( \text{[4] [4]} \) post(\( \alpha; (P1; P2) \)) = post((\( \alpha; P1 \)); P2). Hence, if the program is a sequential composition \( P1; P2 \), I can take \( P1 \) and add it to the end of the pre-program \( \alpha \). I can also go in the reverse direction, so technically there are two rules of inference here. I will refer to both rules as (Trading). This seamless transfer between program and pre-program has no analogue in Hoare logic, and provides a major tactic for the derivation of programs from operational specifications.

**Proposition 3** (Trading) is sound, that is, each side holds iff the other does.

**Proof.** By Definition\( \text{[2]} \) \( [\alpha] P1; P2 [\beta] \) is post(\( \alpha; P1; P2 \)) \( \subseteq \) post(\( \beta \)), and \( [\alpha; P1] P2 [\beta] \) is also post(\( \alpha; P1; P2 \)) \( \subseteq \) post(\( \beta \)). Hence each holds iff the other does. \( \square \)

**Append Rule.** Appending the same program \( \gamma \) to the program and the post-program preserves the validity of an operational triple. This is useful for appending new code into both the program and the post-program.

**Proposition 4** (Append) is sound, that is, if the hypothesis holds, then so does the conclusion.

**Proof.** I must show \( [\alpha] P1; \gamma [\beta; \gamma] \), which by Definition\( \text{[2]} \) is post(\( \alpha; P; \gamma \)) \( \subseteq \) post(\( \beta; \gamma \)). Let \( s \in \text{post}(\alpha; P; \gamma) \). Hence there is some state \( t \) such that \( t \in \text{post}(\alpha; P) \) and \( (t, s) \in \{ \gamma \} \). By assumption, \( [\alpha] P [\beta] \), which by Definition\( \text{[2]} \) is post(\( \alpha; P \)) \( \subseteq \) post(\( \beta \)). Since \( t \in \text{post}(\alpha; P) \), I have \( t \in \text{post}(\beta) \). Since \( (t, s) \in \{ \gamma \} \), I also have \( s \in \text{post}(\beta; \gamma) \). Since \( s \) was chosen arbitrarily, I conclude post(\( \alpha; P; \gamma \)) \( \subseteq \) post(\( \beta; \gamma \)). \( \square \)

**Substitution Rule.** Since the definition of operational annotation refers only to the behavior of a program, it follows that one equivalent program can be replaced by another. This rule is useful for performing equivalence-preserving transformations, such as loop unwinding.

**Proposition 5** (Substitution) is sound, that is, if the hypothesis holds, then so does the conclusion.

**Proof.** \( \alpha \equiv \alpha’ \) implies that post(\( \alpha \)) = post(\( \alpha’ \)). \( P \equiv P’ \) means that \( \| P \| = \| P’ \| \). Hence post(\( \alpha; P \)) = post(\( \alpha’; P’ \)). \( \beta \equiv \beta’ \) implies that post(\( \beta \)) = post(\( \beta’ \)). Hence post(\( \alpha; P \)) \( \subseteq \) post(\( \beta \)) iff post(\( \alpha’; P’ \)) \( \subseteq \) post(\( \beta’ \)). Hence \( [\alpha] P [\beta] \) iff \( [\alpha’] P’ [\beta’] \). \( \square \)

4.3 Rules of inference that are analogues of Hoare logic laws

I now present syntax-based rules, which are straightforward analogues of the corresponding Hoare logic rules.

**Pre-program strengthening.** Reducing the set of post-states of the pre-program cannot invalidate an operational triple.

**Proposition 6** (Pre-program Strengthening) is sound.
Proof. Let \( t \in post(\alpha; P) \). Then there exists a state \( s \) such that \( s \in post(\alpha) \) and \((s, t) \in \{P\} \). By assumption, \( \alpha \preceq \alpha' \), and so \( post(\alpha) \subseteq post(\alpha') \). Hence \( s \in post(\alpha') \). From this and \((s, t) \in \{P\} \), I conclude \( t \in post(\alpha'; P) \). Since \( t \) is arbitrarily chosen, I have \( post(\alpha; P) \subseteq post(\alpha'; P) \).

Hypothesis \([\alpha'] P [\beta]\) means \( post(\alpha'; P) \subseteq post(\beta) \). Hence \( post(\alpha; P) \subseteq post(\beta) \), and so \([\alpha] P [\beta]\) by Definition 2.

**Post-program weakening.** Enlarging the set of post-states of the post-program cannot invalidate an operational triple.

**Proposition 7** *(Post-program Weakening) is sound.*

**Proof.** Hypothesis \([\alpha] P [\beta']\) means \( post(\alpha; P) \subseteq post(\beta') \). Hypothesis \( \beta' \preceq \beta \) means that \( post(\beta') \subseteq post(\beta) \). Hence \( post(\alpha; P) \subseteq post(\beta) \), and so \([\alpha] P [\beta]\) by Definition 2.

**Sequential composition.** The post-state set of \( \beta \) serves as the intermediate state-set in the execution of \( P_1; P_2 \): it characterizes the possible states after \( P_1 \) executes and before \( P_2 \) executes. That is, execution of \( P_1 \) starting from a post-state of \( \alpha \) yields a post-state of \( \beta \). Then execution of \( P_2 \) starting from a post-state of \( \beta \) yields a post-state of \( \gamma \).

**Proposition 8** *(Sequential Composition) is sound.*

**Proof.** Let \( t \in post(\alpha; P_1; P_2) \). Then there is some \( s \) such that \( s \in post(\alpha; P_1) \) and \((s, t) \in \{P_2\} \). From hypothesis \([\alpha] P_1 [\beta] \), I have \( post(\alpha; P_1) \subseteq post(\beta) \). Hence \( s \in post(\beta) \). From this and \((s, t) \in \{P_2\} \), I have \( t \in post(\beta; P_2) \). From hypothesis \([\beta] P_2 [\gamma] \), I have \( post(\beta; P_2) \subseteq post(\gamma) \). From this and \( t \in post(\beta; P_2) \), I have \( t \in post(\gamma) \). Since \( t \) is chosen arbitrarily, I conclude \( post(\alpha; P_1; P_2) \subseteq post(\gamma) \). Hence \([\alpha] P_1; P_2 [\gamma]\) by Definition 2.

**While Rule.** Given \([\alpha] P [\alpha] \), I wish to conclude \([\alpha] \textbf{while} (B) P [\beta]\) where \( \beta \) is a “conjunction” of \( \alpha \) and \( \neg B \), i.e., the post-states of \( \beta \) are those that are post-states of \( \alpha \), and also that satisfy assertion \( \neg B \), the negation of the looping condition. Also, I wish to weaken the hypothesis of the rule from \([\alpha] P [\alpha]\) to \([\alpha'] P [\alpha]\), where \( \alpha' \) is a “conjunction” of \( \alpha \) and \( B \), i.e., the post-states of \( \alpha' \) are those that are post-states of \( \alpha \) and that also satisfy assertion \( B \), the looping condition. I therefore define the “conjunction” of a program and an assertion as follows.

**Definition 5** *(Conjunction of program and condition)* Let \( \alpha', \alpha \) be programs and \( B \) a Boolean expression. Then define

\[
\alpha' \cong (\alpha, B) \triangleq post(\alpha') = post(\alpha) \cap \{s \mid s(B) = tt\}.
\]

Note that this definition does not produce a unique result, and so is really a relation rather than a mapping. The construction of \( \alpha' \) is not straightforward, in general, for arbitrary assertion \( B \). Fortunately, most looping conditions are simple, typically a loop counter reaching a limit. I therefore define the needed program \( \alpha' \) by the semantic condition given above, and leave the problem of deriving \( \alpha' \) from \( \alpha \) and \( B \) to another occasion.

Given a while loop \textbf{while} (B) P eliwh and pre-program \( \alpha \), let \( \alpha' \) be a program such that \( \alpha' \cong (\alpha, B) \), and let \( \beta \) be a program such that \( \beta \cong (\alpha, \neg B) \). The hypothesis of the rule is:
execute $\alpha$ and restrict the set of post-states to those in which $B$ holds. That is, have $\alpha'$ as a pre-program for the loop body $P$. Then, after $P$ is executed, the total resulting effect must be the same as executing just $\alpha$. So, $\alpha$ is a kind of “operational invariant”. Given that this holds, and taking $\alpha$ as a pre-program for while $(B)\ P\ elihw$, then upon termination, we have $\alpha$ as a post-program. On the last iteration, $B$ is false, and the operational invariant $\alpha$ still holds. Hence I can assert $\beta$ as a post-program for the while loop.

**Theorem 9** (While) is sound, i.e., if the hypothesis is true, then so is the conclusion.

**Proof.** I establish, by induction on $i$, the following claim:

Assume the hypothesis $[\alpha'] P [\alpha]$, and that execution of the loop starts in a state $s_0 \in post(\alpha)$. Then if the loop executes for at least $i$ iterations, the state $s_i$ at the end of the $i$'th iteration is in $post(\alpha)$.

Base case is for $i = 0$: The state at the end of the 0'th iteration is just the start state $s_0$, which is in $post(\alpha)$ by assumption.

Induction step for $i > 0$: By the induction hypothesis, $s_{i-1} \in post(\alpha)$. Now $s_{i-1}(B) = tt$, since otherwise the $i$'th iteration would not have been executed. Hence $s_{i-1} \in post(\alpha')$. By the hypothesis $[\alpha'] P [\alpha]$, I have $s_i \in post(\alpha)$.

Hence the claim is established. I now show that $[\alpha] while (B) P elihw [\beta]$ holds. Assume execution starts in an arbitrary $s \in post(\alpha)$ and that the loop terminates in some state $t$. By the above claim, $t \in post(\alpha)$. Also, $t(B) = ff$ since otherwise the loop cannot terminate in state $t$. Hence $t \in post(\beta)$, and so $[\alpha] while (B) P elihw [\beta]$ is established. Hence (While) is sound.

**While Rule with Consequence.** By applying Proposition 1 and (Post-program Weakening) to (While), I obtain (While Consequence), which states that the operational invariant can be a “suffix” of the actual post-program of the loop body. This is often convenient, in practice.

**Theorem 10** (While Consequence) is sound, i.e., if the hypothesis is true, then so is the conclusion.

**Proof.** Assume the hypothesis $[\alpha'] P [\gamma; \alpha]$. By Proposition 1 $\gamma; \alpha \preccurlyeq \alpha$. Hence by (Post-program Weakening), $[\alpha'] P [\alpha]$. Hence by (While), $[\alpha] while (B) Pelihw [\beta]$.

**If Rule.** Let $\alpha$ be the pre-program. Assume that execution of $P1$ with pre-program $\alpha' \cong (\alpha, B)$ leads to post-program $\beta$, and that execution of $P2$ with pre-program $\alpha'' \cong (\alpha, \neg B)$ also leads to post-program $\beta$. Then, execution of if $B$ then $P1$ else $P2$ fi with pre-program $\alpha$ leads to post-program $\beta$.

**Theorem 11** (If) is sound, i.e., if both hypotheses are true, then so is the conclusion.

**Proof.** Assume execution starts in an arbitrary $s \in post(\alpha)$. Suppose that $B$ holds in $s$. Then, $s \in post(\alpha')$. Also, the if branch will be taken, so that $P1$ is executed. Let $t$ be any resulting state. From the hypothesis $[\alpha'] P1 [\beta]$, I have $t \in post(\beta)$. Now suppose that $B$ does not hold in $s$. Then $s \in post(\alpha'')$. Also, the else branch will be taken, so that $P2$ is executed. Let $t$ be any resulting state. From the hypothesis $[\alpha''] P2 [\beta]$, I have $t \in post(\beta)$. By Definition 2 $[\alpha] if \ B \ then \ P1 \ else \ P2 \ fi [\beta]$ is valid.
One-way If Rule. Assume that execution of $P$ with pre-program $\alpha' \equiv (\alpha, B)$ leads to post-program $\beta$. Assume that any post-state of $\alpha'' \equiv (\alpha, \neg B)$ is also a post-state of $\beta$. Then execution of \textit{if} $B$ \textit{then} $P$ \textit{fi} with pre-program $\alpha$ always leads to post-program $\beta$.

Theorem 12 (One-way If) is sound, i.e., if both hypotheses are true, then so is the conclusion.

Proof. Assume execution starts in an arbitrary $s \in \text{post}(\alpha)$. Suppose that $B$ holds in $s$. Then, $s \in \text{post}(\alpha')$. Also, the \textit{if} branch will be taken, so that $P$ is executed. Let $t$ be any resulting state. From the hypothesis $[\alpha'] P [\beta]$, I have $t \in \text{post}(\beta)$. Now suppose that $B$ does not hold in $s$. Then $s \in \text{post}(\alpha'')$. Now $\alpha'' \preceq \beta$, and so $s \in \text{post}(\beta)$ by Def. 3. Since the \textit{if} branch is not taken, there is no change of state, and so the resulting state $t$ is the same as $s$. Hence $t \in \text{post}(\beta)$. By Definition 2, $[\alpha] \text{if} B \text{then} P \text{fi} [\beta]$.

5 Examples

I now illustrate program verification with operational triples by means of several examples. Throughout, I use an informally justified notion of program equivalence, based on well-known transformations such as eliminating the empty program \textit{skip}, and unwinding the last iteration of a \textit{for} loop. Specifications programs are written in bold red italics, and regular programs are written in typewriter.

5.1 Example: selection sort

The input is an array $a$ and its size $n$, with $a$ indexed from 0 to $n - 1$. The assignment $i := [a : b]$ (with $a \leq b$) nondeterministically chooses a value between $a$ and $b$ inclusive, and assigns it to $i$. The following holds by (Empty Program). I will develop this into the loop body for selection sort, which will be iterated for $i$ from 0 to $n - 2$. \textit{findRank}(a, I) returns the index of the rank $I$ element in array $a$. In case of duplicates, I take the element with lower index to also be of lower rank. This breaks ties for duplicates, and also ensures that the resulting sort is stable. We can specify a non-stable sort by assigning the rank randomly amongst duplicate values (within the appropriate range).

$$i := [0 : n - 2];$$
$$\text{for } (I = 0 \text{ to } i) J := \text{findRank}(a, I); a[I] \leftrightarrow a[J] \text{rof}$$
$$\text{skip}$$
$$i := [0 : n - 2];$$
$$\text{for } (I = 0 \text{ to } i) J := \text{findRank}(a, I); a[I] \leftrightarrow a[J] \text{rof}$$

Now unwind the last iteration of the loop in the pre-program:

$$i := [0 : n - 2];$$
$$\text{for } (I = 0 \text{ to } i) J := \text{findRank}(a, I); a[I] \leftrightarrow a[J] \text{rof}$$
$$J := \text{findRank}(a, i); a[i] \leftrightarrow a[J]$$
$$\text{skip}$$
$$i := [0 : n - 2];$$
$$\text{for } (I = 0 \text{ to } i) J := \text{findRank}(a, I); a[I] \leftrightarrow a[J] \text{rof}$$
Now use \((\text{Trading})\) to move the third line of the pre-program into the program. Then remove the \(\text{skip}\), as it is no longer needed.

\[
i := [0 : n - 2];
\]
\[
\text{for } (I = 0 \text{ to } i - 1) J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\rof
\]
\[
J := \text{findRank}(a, i); a[i] \leftrightarrow a[J]
\]
\[
i := [0 : n - 2];
\]
\[
\text{for } (I = 0 \text{ to } i) J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\rof
\]

Now I compute \(\text{findRank}(a, i)\) explicitly using an inner loop. \(\text{findMin}(a, \varphi)\) computes the location of the lowest minimum element in array \(a\), excluding the consideration of elements whose indices are in the set \(\varphi\).

\[
i := [0 : n - 2];
\]
\[
\text{for } (I = 0 \text{ to } i - 1) J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\rof
\]
\[
\varphi := \emptyset; \text{for } (k = 0 \text{ to } i - 1) \varphi := \varphi \cup \text{findMin}(a, \varphi)\rof
\]
\[
J := \text{findMin}(a, \varphi)
\]
\[
a[i] \leftrightarrow a[J]
\]
\[
i := [0 : n - 2];
\]
\[
\text{for } (I = 0 \text{ to } i) J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\rof
\]

Now use \((\text{Trading})\) to trade into the pre-program:

\[
i := [0 : n - 2];
\]
\[
\text{for } (I = 0 \text{ to } i - 1) J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\rof
\]
\[
\varphi := \emptyset; \text{for } (k = 0 \text{ to } i - 1) \varphi := \varphi \cup \text{findMin}(a, \varphi)\rof
\]
\[
J := \text{findMin}(a, \varphi)
\]
\[
a[i] \leftrightarrow a[J]
\]
\[
i := [0 : n - 2];
\]
\[
\text{for } (I = 0 \text{ to } i) J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\rof
\]

Now construct a sequence of equivalences on the pre-program

\[
\text{for } (I = 0 \text{ to } i - 1) J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\rof
\]
\[
\varphi := \emptyset; \text{for } (k = 0 \text{ to } i - 1) \varphi := \varphi \cup \text{findMin}(a, \varphi)\rof
\]
\[
\varphi := \emptyset;
\]
\[
\text{for } (I = 0 \text{ to } i - 1) J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]; \varphi := \varphi \cup \text{findMin}(a, \varphi)\rof
\]
\[
\text{for } (I = 0 \text{ to } i - 1) J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]; \varphi := \varphi \cup I\rof
\]
\[
\text{for } (I = 0 \text{ to } i - 1) J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\rof;
\]
\[
\varphi := \{0, \ldots, i - 1\};
\]

I therefore have, using \((\text{Substitution})\)

\[
i := [0 : n - 2];
\]
\[\text{for } (I = 0 \text{ to } i - 1) \ J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\text{rof};\]
\[\varphi := \{0, \ldots, i - 1\};\]
\[J := \text{findMin}(a, \varphi)\]
a[i] \leftrightarrow a[J]
\[i := [0 : n - 2];\]
\[\text{for } (I = 0 \text{ to } i) \ J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\text{rof}\]

Now use (Trading) to trade into the program

\[i := [0 : n - 2];\]
\[\text{for } (I = 0 \text{ to } i - 1) \ J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\text{rof};\]
\[\varphi := \{0, \ldots, i - 1\};\]
\[J := \text{findMin}(a, \varphi)\]
a[i] \leftrightarrow a[J]
\[i := [0 : n - 2];\]
\[\text{for } (I = 0 \text{ to } i) \ J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\text{rof}\]

I now replace \(\varphi := \{0, \ldots, i - 1\}; J := \text{findMin}(a, \varphi)\) by the equivalent \(J := \text{findMin}(a, [0 : i - 1])\), so that \(\text{findMin}(a, [0 : i - 1])\) finds a minimum element in array \(a\) in the range \(i\) to \(n - 1\), since the indices in \([0 : i - 1]\) are excluded. Hence, by (Substitution)

\[i := [0 : n - 2];\]
\[\text{for } (I = 0 \text{ to } i - 1) \ J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\text{rof};\]
\[J := \text{findMin}(a, [0 : i - 1]);\]
a[i] \leftrightarrow a[J]
\[i := [0 : n - 2];\]
\[\text{for } (I = 0 \text{ to } i) \ J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\text{rof}\]

I now add the increment of \(i\) at the end of the loop:

\[i := [0 : n - 2];\]
\[\text{for } (I = 0 \text{ to } i - 1) \ J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\text{rof};\]
\[J := \text{findMin}(a, [0 : i - 1]);\]
a[i] \leftrightarrow a[J]
\[i := [0 : n - 2];\]
\[\text{for } (I = 0 \text{ to } i) \ J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\text{rof}\]
\[i := i + 1\]
\[i := [1 : n - 1];\]
\[\text{for } (I = 0 \text{ to } i - 1) \ J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\text{rof}\]

which gives the body of selection sort. I finish up the example by including the outer loop

\[\text{skip}\]
\[i := 0\]
\[i := 0\]
\[\text{for } (I = 0 \text{ to } i - 1) \ J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\text{rof}\]
\[\text{while } (i \neq n - 1)\]
\[i := [0 : n - 2];\]
\[\text{for } (I = 0 \text{ to } i - 1) \ J := \text{findRank}(a, I); a[I] \leftrightarrow a[J]\text{rof}\]
J := findMin(a,[0 : i - 1]);
a[i] ↔ a[J]
i := [0 : n - 2];
for (I = 0 to i) J := findRank(a,I); a[I] ↔ a[J]rof
i := i + 1
i := [1 : n - 1];
for (I = 0 to i - 1) J := findRank(a,I); a[I] ↔ a[J]rof;

elihw
i := n - 1
for (I = 0 to i) J := findRank(a,I); a[I] ↔ a[J]rof

By applying (Post-program Weakening), I obtain

skip
i := 0
i := 0
for (I = 0 to i - 1) J := findRank(a,I); a[I] ↔ a[J]rof
while (i ≠ n - 1)
i := [0 : n - 2];
for (I = 0 to i - 1) J := findRank(a,I); a[I] ↔ a[J]rof
J := findMin(a,[0 : i - 1]);
a[i] ↔ a[J]
i := [0 : n - 2];
for (I = 0 to i) J := findRank(a,I); a[I] ↔ a[J]rof
i := i + 1
i := [1 : n - 1];
for (I = 0 to i - 1) J := findRank(a,I); a[I] ↔ a[J]rof;

elihw
for (I = 0 to n - 1) J := findRank(a,I); a[I] ↔ a[J]rof

The post-program states that the element at index I has rank I, that is, array a is sorted. I omit the development of findMin.

5.2 Example: Dijkstra’s shortest paths algorithm

The input consists of the following:

- a directed graph $G = (V,E)$ with node set $V$ and edge set $E$, together with a weight function $w : E \rightarrow \mathbb{R}^{\geq 0}$, since Dijkstra’s algorithm requires that edge weights are non-negative.
- a distinguished vertex $s$: the source.
- for each node $t$, a real number $t.d$, which records the current estimate of the shortest path distance from the source $s$ to $t$

I use $\delta[t]$ to denote the shortest path distance from the source $s$ to node $t$, and $u \rightarrow v$ to mean $(u,v) \in E$, i.e., there is an edge in $G$ from $u$ to $v$. 
Dijkstra's algorithm is incremental, each iteration of the main loop computes the shortest path distance of some node \( m \). Vertices are colored black (shortest path distance from the source \( s \) is known), grey (have an incoming edge form a black node), and white (neither black nor grey). The algorithm maintains the sets \( B \) and \( G \) of black and grey nodes.

I start with an instance of (Empty Program). The assignment \( B := ? \) nondeterministically sets \( B \) to any subset of \( V \) which contains the source \( s \). It can be implemented using the nondeterministic choice operator \( [] \) (Section 2). The pre- and post-programs simply assign the correct shortest path distances to the \( t.d \) variable for each node \( t \) in \( B \).

\[
\begin{align*}
B & := \emptyset \\
for (t \in B) & \ t.d := \delta[t] \rof; \for (t \notin B) & \ t.d := +\infty \rof \\
skip \\
B & := \emptyset \\
for (t \in B) & \ t.d := \delta[t] \rof; \for (t \notin B) & \ t.d := +\infty \rof \\
\end{align*}
\]

Now append to both the pre- and post-programs a statement which relaxes all of the grey nodes. This is still an instance of (Empty Program).

\[
\begin{align*}
B & := \emptyset \\
for (t \in B) & \ t.d := \delta[t] \rof; \for (t \notin B) & \ t.d := +\infty \rof \\
\for (g \in G) & \ \for (b \in B \land b \to g) \relax(b,g) \rof \rof \\
skip \\
B & := \emptyset \\
for (t \in B) & \ t.d := \delta[t] \rof; \for (t \notin B) & \ t.d := +\infty \rof \\
\for (g \in G) & \ \for (b \in B \land b \to g) \relax(b,g) \rof \rof \\
\end{align*}
\]

Apply (Append) to add “\( m := \{ g \mid g.d = (\MIN h \in G : h.d); m.d := (\MIN \pi \in \text{paths}(s,m) : |\pi|) \} \)” to both program and post-program, and also remove the \( \text{skip} \), as it is no longer needed. \( |\pi| \) is the total cost of path \( \pi \), and \( \text{paths}(s,m) \) is the set of all simple paths from \( s \) to \( m \).

\[
\begin{align*}
B & := \emptyset \\
\for (t \in B) & \ t.d := \delta[t] \rof; \for (t \notin B) & \ t.d := +\infty \rof \\
\for (g \in G) & \ \for (b \in B \land b \to g) \relax(b,g) \rof \rof \\
m & := \{ g \mid g.d = (\MIN h \in G : h.d) \} \\
m.d & := (\MIN \pi \in \text{paths}(s,m) : |\pi|) \\
B & := \emptyset \\
\for (t \in B) & \ t.d := \delta[t] \rof; \for (t \notin B) & \ t.d := +\infty \rof \\
\for (g \in G) & \ \for (b \in B \land b \to g) \relax(b,g) \rof \rof \\
m & := \{ g \mid g.d = (\MIN h \in G : h.d) \} \\
m.d & := (\MIN \pi \in \text{paths}(s,m) : |\pi|) \\
\end{align*}
\]

Now construct a sequence of equivalences as follows:

\[
\begin{align*}
\for (g \in G) & \ \for (b \in B \land b \to g) \relax(b,g) \rof \rof \\
\equiv & \quad \text{\footnotesize \( \text{// definition of relax(b,g)} \)} \\
\for (g \in G) & \ \for (b \in B \land b \to g) \ g.d := g.d \min b.d + w(b,g) \rof \rof \\
\equiv & \quad \text{\footnotesize \( \text{// min is commutative and associative} \)}
\end{align*}
\]
\textbf{for} (g \in G) \ g.d := g.d \min (\text{MIN} \ b \in B \land b \to g : b.d + w(b, g))
\equiv
\  \  \  // \ use \ the \ operational \ invariant
\textbf{for} (g \in G) \ g.d := g.d \min (\text{MIN} \ b \in B \land b \to g : \delta[b] + w(b, g))
\equiv
\  \  \  // \ s \to g \ denotes \ all \ paths \ from \ s \ to \ g \ containing \ only \ black \ nodes \ except \ g
\textbf{for} (g \in G) \ g.d := g.d \min (\text{MIN} \ \pi \in s \xrightarrow{B} g : |\pi|)
\equiv
\  \  \  // \ s \to g \ denotes \ all \ paths \ from \ s \ to \ g
\textbf{for} (g \in G) \ g.d := g.d \min (\text{MIN} \ \pi \in s \xrightarrow{B} g : |\pi|) \min (\text{MIN} \ \pi \in s \to g : |\pi|)
\equiv
\  \  \  // \ definition \ of \ \delta
\textbf{for} (g \in G) \ g.d := g.d \min (\text{MIN} \ \pi \in s \xrightarrow{B} g : |\pi|)
\textbf{for} (g \in M) \ g.d := \delta[g]

The above equivalence allows me to conclude the following:

\begin{align*}
B & :=? \\
\textbf{for} (t \in B) \ t.d := \delta[t] & \text{raf} & \textbf{for} (t \notin B) \ t.d := +\infty & \text{raf} \\
\textbf{for} (g \in G) \textbf{ for} (b \in B \land b \to g) \ relax(b, g) & \text{raf} & \text{raf}
\end{align*}

\text{m} := \\{g \mid g.d = (\text{MIN} \ h \in G : h.d)\}
\text{m.d} := (\text{MIN} \ \pi \in \text{paths}(s, m) : |\pi|)

is equivalent to

\begin{align*}
B & :=? \\
\textbf{for} (t \in B) \ t.d := \delta[t] & \text{raf} & \textbf{for} (t \notin B) \ t.d := +\infty & \text{raf} \\
\textbf{for} (g \in G) \textbf{ for} (b \in B \land b \to g) \ relax(b, g) & \text{raf} & \text{raf}
\end{align*}

\text{m} := \\{g \mid g.d = (\text{MIN} \ h \in G : h.d)\}

I also observe that \text{m.d} := (\text{MIN} \ \pi \in \text{paths}(s, m) : |\pi|) is equivalent to \text{m.d} := \delta[m] by definition of \delta[m], the shortest path distance from source \(s\) to node \text{m}. Hence, by (Substitution) I can now write

\begin{align*}
B & :=? \\
\textbf{for} (t \in B) \ t.d := \delta[t] & \text{raf} & \textbf{for} (t \notin B) \ t.d := +\infty & \text{raf} \\
\textbf{for} (g \in G) \textbf{ for} (b \in B \land b \to g) \ relax(b, g) & \text{raf} & \text{raf}
\end{align*}

\text{m} := \\{g \mid g.d = (\text{MIN} \ h \in G : h.d)\}

\text{B} :=?

\begin{align*}
\text{for} (t \in B) \ t.d := \delta[t] & \text{raf} & \textbf{for} (t \notin B) \ t.d := +\infty & \text{raf} \\
\textbf{for} (g \in G) \textbf{ for} (b \in B \land b \to g) \ relax(b, g) & \text{raf} & \text{raf}
\end{align*}

\text{m} := \{g \mid g.d = (\text{MIN} \ h \in G : h.d)\}
\text{m.d} := \delta[m]

Now color \text{m} black. This causes, in general, some nodes to turn grey, and so requires the addition of the line \textbf{for} (g \in G \land m \to g) \ relax[m, g] to preserve the truth of the operational triple. I retain the intermediate specification program between the program statement that selects \text{m}, and the statements which turn \text{m} black and then relax all of \text{m}'s grey neighbors. The assignment \text{B} :=!m nondeterministically sets \text{B} to any subset of \(V\) which contains both \text{s} and \text{m}.
By applying (Post-program Weakening), I obtain

This concludes the development of the loop body. The pre-program consists of making the looping condition true followed by the specification program

\[
\begin{align*}
&\text{for } (t \in B)\ t.d := \delta [t]rof; \text{for } (t \notin B)\ t.d := +\infty rof \\
&\text{for } (g \in G)\ \text{for } (b \in B \land b \rightarrow g)\ relax(b, g)rof rof \\
&m := \{ g \mid g.d = (\text{MIN}\ h \in G : h.d)\} \\
&B := ?
\end{align*}
\]

and the post-program has this specification program as a suffix. Hence this specification program plays the role of an “invariant”, and we have, by (While Consequence), the complete annotated program:

\[
\begin{align*}
&\text{skip} \\
&B := \{ s \}; s.d := 0; \text{for } (t \notin B)\ t.d := +\infty rof \\
&\text{for } (t \in B)\ t.d := \delta [t]rof; \text{for } (t \notin B)\ t.d := +\infty rof \\
&\text{while } (B \neq V) \\
&B := ?
\end{align*}
\]

\[
\begin{align*}
&\text{for } (t \in B)\ t.d := \delta [t]rof; \text{for } (t \notin B)\ t.d := +\infty rof \\
&\text{for } (g \in G)\ \text{for } (b \in B \land b \rightarrow g)\ relax(b, g)rof rof \\
&m := \{ g \mid g.d = (\text{MIN}\ h \in G : h.d)\} \\
&m.d := \delta [m] \\
&B := B \cup \{ m \} \\
&\text{for } (g \in G \land m \rightarrow g)\ relax(m, g) \\
&B := ?' \\
&\text{for } (t \in B)\ t.d := \delta [t]rof; \text{for } (t \notin B)\ t.d := +\infty rof \\
&\text{for } (g \in G)\ \text{for } (b \in B \land b \rightarrow g)\ relax(b, g)rof rof \\
&\text{elihw} \\
&B := V \\
&\text{for } (t \in B)\ t.d := \delta [t]rof; \text{for } (t \notin B)\ t.d := +\infty rof \\
&\text{for } (g \in G)\ \text{for } (b \in B \land b \rightarrow g)\ relax(b, g)rof rof
\end{align*}
\]
there is no aliasing: all elements are distinct, by construction. Array

the nodes and construct the linked list, which is then reversed. An array also ensures that

fields. The use of this array is purely for specification purposes, so that I can go through

5.3 Example: in-place list reversal

I now illustrate the use of operational annotations to derive a correct algorithm for the in-place reversal of a linked list. The input is a size $l > 0$ array $n$ of objects of type $Node$, indexed from 0 to $l - 1$. $Node$ is declared as follows: \texttt{class Node\{Node p; other fields...\}}. Element $i$ is referred to as $n_i$ instead of $n[i]$, and contains a pointer $n_i.p$, and possibly other (omitted) fields. The use of this array is purely for specification purposes, so that I can go through the nodes and construct the linked list, which is then reversed. An array also ensures that there is no aliasing: all elements are distinct, by construction. Array $n$ is created by executing \texttt{Node\[]\ n := new\ Node[l]}. The pre-program and post-program both start with code to declare $Node$, followed by the above line to create array $n$. I omit this code as including it would be repetitive and would add clutter.

I start by applying (Empty Program). The pre (and post) programs do 3 things: (1) construct the linked list, (2) reverse part of the list, up to position $i + 1$, and (3) maintain 3 pointers, into positions $i + 1, i + 2$, and $i + 3$.

\begin{verbatim}
skip
B := \{s\}; s.d := 0; for (t \notin B) t.d := +\infty rof
for (t \in B) t.d := \delta[t] rof; for (t \notin B) t.d := +\infty rof
while (B \neq \emptyset)
B := ?
for (t \in B) t.d := \delta[t] rof; for (t \notin B) t.d := +\infty rof
for (g \in G) for (b \in B \land b \rightarrow g) relax(b, g) rof rof
m := \{g \mid g.d = (\text{MIN } h \in G : h.d)\}
B := ?
for (t \in B) t.d := \delta[t] rof; for (t \notin B) t.d := +\infty rof
for (g \in G) for (b \in B \land b \rightarrow g) relax(b, g) rof rof
m.d := \delta[m]
B := B \cup \{m\}
for (g \in G \land m \rightarrow g) relax(m, g)
B := ?!
for (t \in B) t.d := \delta[t] rof; for (t \notin B) t.d := +\infty rof
for (g \in G) for (b \in B \land b \rightarrow g) relax(b, g) rof rof
\end{verbatim}

The post-program sets the $t.d$ variable for every node $t$ to the correct shortest path value.
Now unwind the last iteration of the second \textit{for} loop of the pre-program. So, by \textit{(Substitution)}

\begin{verbatim}
 i := [0 : l - 3]
 for (j = 0 to l - 1) n_j.p := n_{j+1}.rof
 for (j = i downto 1) n_j.p := n_{j-1}.rof
 n_{i+1}.p := n_i;
 r := n_{i+1}; s := n_{i+2}; t := n_{i+3}
 skip
 i := [0 : l - 3]
 for (j = 0 to l - 1) n_j.p := n_{j+1}.rof
 for (j = i + 1 downto 1) n_j.p := n_{j-1}.rof
 r := n_{i+1}; s := n_{i+2}; t := n_{i+3}
\end{verbatim}

Now introduce the line $r := n_i; s := n_{i+1}; t := n_{i+2}$ into the pre-program. Since $r, s, t$ are subsequently overwritten, and not referenced in the interim, this preserves equivalence of the pro-program with its previous version. So, by \textit{(Substitution)}

\begin{verbatim}
 i := [0 : l - 3]
 for (j = 0 to l - 1) n_j.p := n_{j+1}.rof
 for (j = i downto 1) n_j.p := n_{j-1}.rof
 r := n_i; s := n_{i+1}; t := n_{i+2}
 n_{i+1}.p := n_i;
 r := n_{i+1}; s := n_{i+2}; t := n_{i+3}
 skip
 i := [0 : l - 3]
 for (j = 0 to l - 1) n_j.p := n_{j+1}.rof
 for (j = i + 1 downto 1) n_j.p := n_{j-1}.rof
 r := n_{i+1}; s := n_{i+2}; t := n_{i+3}
\end{verbatim}

Now apply \textit{(Trading)} to trade into the program, and remove the \textit{skip} since it is no longer needed.

\begin{verbatim}
 i := [0 : l - 3]
 for (j = 0 to l - 1) n_j.p := n_{j+1}.rof
 for (j = i downto 1) n_j.p := n_{j-1}.rof
 r := n_i; s := n_{i+1}; t := n_{i+2}
 n_{i+1}.p := n_i;
 r := n_{i+1}; s := n_{i+2}; t := n_{i+3}
 i := [0 : l - 3]
 for (j = 0 to l - 1) n_j.p := n_{j+1}.rof
 for (j = i + 1 downto 1) n_j.p := n_{j-1}.rof
 r := n_{i+1}; s := n_{i+2}; t := n_{i+3}
\end{verbatim}

Since $r := n_i; s := n_{i+1}$ immediately precedes $n_{i+1}.p := n_i$, I can replace $n_{i+1}.p := n_i$ by $s.p := r$ while retaining equivalence. So, by \textit{(Substitution)}

\begin{verbatim}
 i := [0 : l - 3]
 for (j = 0 to l - 1) n_j.p := n_{j+1}.rof
 for (j = i downto 1) n_j.p := n_{j-1}.rof
 r := n_i; s := n_{i+1}; t := n_{i+2}
\end{verbatim}
s.p := r;
\[ r := n_{i+1}; s := n_{i+2}; t := n_{i+3} \]
\[ i := [0 : l - 3] \]
for \( j = 0 \) to \( l - 1 \) \( n_j.p := n_{j+1}.r.o.f \)
for \( j = i \) downto \( 1 \) \( n_j.p := n_{j-1}.r.o.f \)
\[ r := n_{i+1}; s := n_{i+2}; t := n_{i+3} \]

Since \( s := n_{i+1} \) precedes \( r := n_{i+1} \) and \( s \) is not modified in the interim, I can replace \( r := n_{i+1} \) by \( r := s \) while retaining equivalence. So, by (Substitution)

\[ i := [0 : l - 3] \]
for \( j = 0 \) to \( l - 1 \) \( n_j.p := n_{j+1}.r.o.f \)
for \( j = i \) downto \( 1 \) \( n_j.p := n_{j-1}.r.o.f \)
\[ r := n_{i+1}; s := n_{i+2}; t := n_{i+3} \]

In a similar manner, I can replace \( s := n_{i+2} \) by \( s := t \). So, by (Substitution)

\[ i := [0 : l - 3] \]
for \( j = 0 \) to \( l - 1 \) \( n_j.p := n_{i+1}.r.o.f \)
for \( j = i \) downto \( 1 \) \( n_j.p := n_{j-1}.r.o.f \)
\[ r := n_{i+1}; s := n_{i+2}; t := n_{i+3} \]

From for \( j = 0 \) to \( l - 1 \) \( n_j.p := n_{i+1}.r.o.f \), I have \( n_{i+2}.p := n_{i+2} \), and I observe that \( n_{i+2}.p \) is not subsequently modified. Also I have \( t := n_{i+2} \) occurring before the program, and \( t \) is not modified until \( t := n_{i+3} \), we can replace \( t := n_{i+3} \) by \( t := t.p \). So, by (Substitution)

\[ i := [0 : l - 3] \]
for \( j = 0 \) to \( l - 1 \) \( n_j.p := n_{j+1}.r.o.f \)
for \( j = i \) downto \( 1 \) \( n_j.p := n_{j-1}.r.o.f \)
\[ r := n_{i+1}; s := n_{i+2}; t := n_{i+3} \]
Now apply (While) to obtain the complete program, while also incrementing the loop counter \( i \) at the end of the loop body.

\[
\text{for } (j = 0 \text{ to } l - 1) \ n_j.p := n_{j+1}.rof \\
\text{r} := n_0; \text{s} := n_1; \text{t} := n_2 \\
i := 0 \\
\text{for } (j = 0 \text{ to } l - 1) \ n_j.p := n_{j+1}.rof \\
\text{for } (j = i \text{ downto } 1) \ n_j.p := n_{j-1}; \text{rof} \\
r := n_i; \text{s} := n_{i+1}; \text{t} := n_{i+2} \\
\text{while}(i \neq l - 2) \\
i := [0 : l - 3] \\
\text{for } (j = 0 \text{ to } l - 1) \ n_j.p := n_{j+1}.rof \\
\text{for } (j = i \text{ downto } 1) \ n_j.p := n_{j-1}; \text{rof} \\
r := n_i; \text{s} := n_{i+1}; \text{t} := n_{i+2} \\
s.p := r; \\
r := s; \text{s} := t; \text{t} := t.p \\
i := [0 : l - 3] \\
\text{for } (j = 0 \text{ to } l - 1) \ n_j.p := n_{j+1}.rof \\
\text{for } (j = i + 1 \text{ downto } 1) \ n_j.p := n_{j-1}; \text{rof} \\
r := n_{i+1}; \text{s} := n_{i+2}; \text{t} := n_{i+3} \\
i := i + 1; \\
i := [1 : l - 2] \\
\text{for } (j = 0 \text{ to } l - 1) \ n_j.p := n_{j+1}.rof \\
\text{for } (j = i \text{ downto } 1) \ n_j.p := n_{j-1}; \text{rof} \\
r := n_i; \text{s} := n_{i+1}; \text{t} := n_{i+2} \\
\text{elihw} \\
i := l - 2 \\
r := n_i; \text{s} := n_{i+1}; \text{t} := n_{i+2} \\
\text{for } (j = 0 \text{ to } l - 1) \ n_j.p := n_{j+1}.rof \\
\text{for } (j = i \text{ downto } 1) \ n_j.p := n_{j-1}; \text{rof} \]

The loop must terminate at \( i = l - 2 \) to avoid dereferencing NIL. The post-program of the loop then gives \text{for } (j = l - 2 \text{ downto } 1) \ n_j.p := n_{j-1}; \text{rof}, \) which means that the last node’s pointer is not set to the previous node, since the list ends at index \( l - 1 \). Hence we require a final assignment that is equivalent to \( n_{l-1}.p := n_{l-2}. \)

The post-program of the loop gives \( i := l - 2; r := n_i; s := n_{i+1}; t := n_{i+2} \), which yields \( i := l - 2; r := n_{l-2}; s := n_{l-1}; t := n_l \). Hence the last assignment can be rendered as \( s.p := r \). Also, the loop termination condition can be rewritten as \( t \neq \text{NIL} \), since \( t \) becomes NIL when it is assigned \( n_i \), which happens exactly when \( i \) becomes \( l - 2 \). Hence, using (Substitution) and (Post-program Weakening), I obtain

\[
\text{for } (j = 0 \text{ to } l - 1) \ n_j.p := n_{j+1}.rof \\
\text{r} := n_0; \text{s} := n_1; \text{t} := n_2 \\
i := 0 \\
\text{for } (j = 0 \text{ to } l - 1) \ n_j.p := n_{j+1}.rof \\
\text{for } (j = i \text{ downto } 1) \ n_j.p := n_{j-1}; \text{rof} \\
r := n_i; \text{s} := n_{i+1}; \text{t} := n_{i+2} \\
\text{while}(t \neq \text{NIL}) \\
i := [0 : l - 3] \]
Now, as desired, the “loop counter” $i$ is no longer needed as a program variable, and can be converted to an auxiliary (“ghost”) variable. The result is

$$\begin{align*}
\text{for (} j = 0 \text{ to } l - 1 \text{) } n_j.p &:= n_{j+1}.\text{rof} \\
\text{for (} j = i \text{ downto } 1 \text{) } n_j.p &:= n_j-1;\text{rof} \\
r &:= n_i; s := n_{i+1}; t := n_{i+2} \\
s.p &:= r; \\
r &:= s; s := t; t := t.p \\
i &:= [0 : l - 3] \\
\text{for (} j = 0 \text{ to } l - 1 \text{) } n_j.p &:= n_{j+1}.\text{rof} \\
\text{for (} j = i + 1 \text{ downto } 1 \text{) } n_j.p &:= n_j-1;\text{rof} \\
r &:= n_{i+1}; s := n_{i+2}; t := n_{i+3} \\
i &:= i + 1; \\
i &:= [1 : l - 2] \\
\text{for (} j = 0 \text{ to } l - 1 \text{) } n_j.p &:= n_j+1.\text{rof} \\
\text{for (} j = i \text{ downto } 1 \text{) } n_j.p &:= n_j-1;\text{rof} \\
r &:= n_i; s := n_{i+1}; t := n_{i+2}
\end{align*}$$

\text{elihw}

\begin{align*}
i &:= l - 2 \\
r &:= n_i; s := n_{i+1}; t := n_{i+2} \\
\text{for (} j = 0 \text{ to } l - 1 \text{) } n_j.p &:= n_j+1.\text{rof} \\
\text{for (} j = i \text{ downto } 1 \text{) } n_j.p &:= n_j-1;\text{rof} \\
s.p &:= r; \\
\text{for (} j = 0 \text{ to } l - 1 \text{) } n_j.p &:= n_j+1.\text{rof} \\
\text{for (} j = i \text{ downto } 1 \text{) } n_j.p &:= n_j-1;\text{rof}
\end{align*}

\text{while (} t \neq \text{NIL})

\begin{align*}
i &:= [0 : l - 3] \\
\text{for (} j = 0 \text{ to } l - 1 \text{) } n_j.p &:= n_j+1.\text{rof} \\
\text{for (} j = i \text{ downto } 1 \text{) } n_j.p &:= n_j-1;\text{rof} \\
r &:= n_i; s := n_{i+1}; t := n_{i+2} \\
s.p &:= r; \\
r &:= s; s := t; t := t.p \\
i &:= [1 : l - 2] \\
\text{for (} j = 0 \text{ to } l - 1 \text{) } n_j.p &:= n_j+1.\text{rof} \\
\text{for (} j = i \text{ downto } 1 \text{) } n_j.p &:= n_j-1;\text{rof} \\
r &:= n_i; s := n_{i+1}; t := n_{i+2}
\end{align*}

\text{elihw}

\begin{align*}
i &:= l - 2 \\
r &:= n_i; s := n_{i+1}; t := n_{i+2} \\
\text{for (} j = 0 \text{ to } l - 1 \text{) } n_j.p &:= n_j+1.\text{rof} \\
\text{for (} j = i \text{ downto } 1 \text{) } n_j.p &:= n_j-1;\text{rof}
\end{align*}
Upon termination, \( s \) points to the head of the reversed list. The post-program is quite pleasing: it constructs the list, and then immediately reverses it!

6 Operational annotations with non-recursive procedures

Let \( p \) name be a non-recursive procedure with parameter passing by value, and with body \( p \) body. Let \( a \) denote a list of actual parameters, and let \( f \) denote a list of formal parameters. An actual parameter is either an object identifier or an expression over primitive types, and a formal parameter is either an object identifier or a primitive-type identifier. I handle non-recursive procedures in a similar manner to Hoare logic verification rules for non-recursive procedures [6]: formal parameters are replaced by actual parameters. The most convenient expression of this principle is as an equivalence between a procedure call and an instance of the procedure body with the appropriate assignment of actuals to formals. The equivalence rules for non-recursive procedure calls are:

\[
\begin{align*}
\text{rename}(a) & \equiv f := a; pbody \quad \text{(Equiv Nonrecursive Void)} \\
\text{r := rename}(a) & \equiv f := a; pbody[r := e/\text{return}(e)] \quad \text{(Equiv Nonrecursive)}
\end{align*}
\]

Rule \textit{Equiv Nonrecursive} is for when no value is returned (procedure type is void), or the returned value is not saved by being assigned to a variable (return value is “thrown away”). In this case, the form of the procedure call is \text{rename}(a), i.e., the procedure name followed by the actual parameter list. Rule \textit{Equiv Nonrecursive} then assigns the formals to the actuals (I assume a multiple assignment, with the obvious semantics) and executes the procedure body.

Rule \textit{Equiv Nonrecursive} is for when the returned value is assigned to a variable. In this case, the form of the procedure call is \text{r := rename}(a), i.e., an assignment statement in which the value returned by the call \text{rename}(a) is saved in variable \text{r}. Rule \textit{Equiv Nonrecursive} then assigns the formals to the actuals (I assume a multiple assignment, with the obvious semantics) and executes a modified procedure body in which

I now give two examples of application of the above rules: selection sort and linked list reversal. First, consider the selection sort algorithm developed above, packaged as a procedure:

```plaintext
void Procedure sort(int[] a, int n){
    i := 0
    while (i != n - 1)
        J := findMin(i : n - 1);
        a[i] ↔ a[J]
        i := i + 1 
    elihw
}
```

Now consider a call \text{sort}(b, m) where \text{b} is an array of length \text{m}. By (Equiv Nonrecursive Void), I have
\[ \text{sort}(b, m) \equiv \{ \]
\[ a := b; n := m; \]
\[ i := 0 \]
\[ \text{while} (i \neq n - 1) \]
\[ J := \text{findMin}(a, [0 : i - 1]); \]
\[ a[i] \leftrightarrow a[J] \]
\[ i := i + 1 \]
\[ \text{elihw} \]
\[ \} \]

This clearly has the correct effect. Now, consider the linked list reversal algorithm above, also packaged as a procedure:

Node Procedure listRev(Node h) {
    \[ r := h; s := h.next; t := h.next.next; \]
    while \( t \neq \text{NIL} \)
    \[ s.p := r; \]
    \[ r := s; s := t; t := t.p \]
    \[ \text{elihw}; \]
    \[ s.p := r; \]
    return(s)
}

Now consider a call \( v := \text{listRev}(l) \) where \( l \) is the head of a linked list. By (Equiv Nonrecursive), I have:

\[ v := \text{listRev}(l) \equiv \{ \]
\[ h := l; \]
\[ r := h; s := h.next; t := h.next.next; \]
\[ \text{while} (t \neq \text{NIL}) \]
\[ s.p := r; \]
\[ r := s; s := t; t := t.p \]
\[ \text{elihw}; \]
\[ s.p := r; \]
\[ v := s \]
\[ \} \]

Again, this gives the correct effect for the procedure call \( \text{listRev}(l) \), namely that \( v \) points to the head of the reversed list. Note that both primitive and reference types (as parameters) are handled correctly, and need not be distinguished in the above rules.

7 Operational annotations with recursive procedures

For recursive procedures, it is not sufficient to simply replace the call by the body, since the body contains recursive instances of the call. Clearly an inductive proof method is needed. What I use is an inductive rule to establish the equivalence between a sequence of two procedure calls and a third procedure call. These correspond, respectively, to the pre-program, the program, and the post-program. The method is as follows:
1. Let $\alpha, P, \beta$ be recursive procedures which give, respectively, the pre-program, program, and post-program.

2. To establish $\alpha; P \equiv \beta$ I proceed as follows:

   (a) Start with $\alpha; P$, replace the calls by their corresponding bodies, and then use sequence of equivalence-preserving transformations to bring the recursive calls of $\alpha'; P'$ next to each other.

   (b) Use the inductive hypothesis for equivalence of the recursive calls $\alpha'; P' \equiv \beta'$ to replace $\alpha'; P'$ by $\beta'$.

   (c) Use more equivalence-preserving transformations to show that the resulting program is equivalent to $\beta$.

The appropriate rule of inference is as follows:

$$
\alpha'; P' \equiv \beta' \vdash \alpha; P \equiv \beta \quad \text{ (Equiv Recursive)}
$$

where $\vdash$ means “is deducible from”, as usual. This states that if we can prove $\alpha; P \equiv \beta$ (pre-program followed by program is equivalent to post-program) by assuming $\alpha'; P' \equiv \beta'$ (a recursive invocation of the pre-program followed by a recursive invocation of the program is equivalent to a recursive invocation of the post-program), then we can conclude, by induction on recursive calls, that $\alpha; P \equiv \beta$.

I illustrate this approach with an example which verifies the standard recursive algorithm for inserting a node into a binary search tree (BST). Each node $n$ in the tree consists of three fields: $n.key$ gives the key value for node $n$, $n.l$ points to the left child of $n$ (if any), and $n.r$ points to the right child of $N$ (if any). The constructor $\text{Node}(v)$ returns a new node with key value $v$ and null left and right child pointers. I assume that all key values in the tree are unique.

The procedure $\text{insert}(t, v)$ gives the standard recursive algorithm for insertion of key $v$ into a BST with root $t$.

$$
\text{insert}(T, k) ::
\begin{align*}
\quad & \text{if } (T = \text{NIL}) T := \text{new Node}(k); \\
\quad & \text{else if } (k < T.key) \text{ insert}(T.l, k); \\
\quad & \text{else insert}(T.r, k); \quad // k > T.key
\end{align*}
$$

To verify the correctness of $\text{insert}(T, k)$, I define two recursive procedures as follows.

The recursive procedure $\text{ct}(T, \psi)$ takes a set $\psi$ of key values, and constructs a random binary search tree which contains exactly these values, and sets $T$ to point to the root of this tree. The statement $x := \text{select in } \psi$ selects a random value in $\psi$ and assigns it to $x$.

The recursive procedure $\text{cti}(T, \psi, k)$ takes a set $\psi$ of key values and a key value $k \notin \psi$, constructs a random binary search tree which contains exactly the values in $\psi$ together with the key $k$, and where $k$ is a leaf node, and sets $T$ to point to the root of this tree.

$$
\text{ct}(T, \psi) ::
$$
if ($\psi = \emptyset$) $T := \text{NIL}$;
else
\begin{itemize}
  \item $x := \text{select in } \psi$;
  \item $\psi := \psi - x$;
  \item Node $T := \text{new Node}(x)$;
\end{itemize}
$ct(T.l, \{y \mid y \in \psi \land y < x\})$;
$ct(T.r, \{y \mid y \in \psi \land y > x\})$;

c$ti(T, \psi, k)$:
\begin{itemize}
  \item if ($\psi = \emptyset$) $T := \text{new Node}(k)$;
  \item else
\begin{itemize}
  \item $x := \text{select in } \psi$;
  \item $\psi := \psi - x$;
  \item Node $n := \text{new Node}(x)$;
  \item if ($k < x$)
    \begin{itemize}
      \item $cti(T.l, \{y \mid y \in \psi \land y < x\}, k)$;
      \item $ct(T.r, \{y \mid y \in \psi \land y > x\})$
    \end{itemize}
  \item else
    \begin{itemize}
      \item $ct(T.l, \{y \mid y \in \psi \land y < x\})$
      \item $ct(T.r, \{y \mid y \in \psi \land y > x\}, k)$;
    \end{itemize}
\end{itemize}
\end{itemize}

I now verify

\[ [ct(T, \varphi)] \quad \text{insert}(T, k) \quad [ct(T, \varphi \cup k)]. \]

That is, the pre-program creates a random BST with key values in $\varphi$ and sets $T$ to the root, and the post-program creates a random BST with key values in $\varphi \cup k$ and sets $T$ to the root. Hence, the above operational triple states that the result of $T := \text{insert}(T, k)$ is to insert key value $k$ into the BST rooted at $T$. I first establish

\[ cti(T, \varphi, k) \preceq ct(T, \varphi \cup k). \tag{a} \]

Intuitively, this follows since $cti(T, \varphi, k)$ constructs a BST with key values in $\varphi \cup k$, and where $k$ is constrained to be a leaf node, while $ct(T, \varphi \cup k)$ constructs a BST with key values in $\varphi \cup k$, with no constraint of where $k$ can occur. A formal proof proceeds by induction on the length of an arbitrary execution $\pi$ of $cti(T, \varphi, k)$, which shows that $\pi$ is also a possible execution of $ct(T, \varphi \cup k)$. Recall that the use of the random selection statement means that there are, in general, many possible executions for a given input. The details are straightforward and are omitted.

In the sequel, I show that

\[ [ct(T, \varphi)] \quad \text{insert}(T, k) \quad [cti(T, \varphi, k)] \tag{b} \]

is valid. From (a,b) and (Post-program Weakening), I conclude that

\[ [ct(T, \varphi)] \quad \text{insert}(T, k) \quad [ct(T, \varphi \cup k)] \]

is valid, as desired. To establish (b), I show

\[ ct(T, \varphi); \quad \text{insert}(T, k) \equiv cti(T, \varphi, k). \tag{c} \]

from which (b) follows immediately by Definitions 4 and 2.
I establish (c) by using induction on recursive calls. I replace the above calls by the corresponding procedure bodies, and then assume as inductive hypothesis (c) as applied to the recursive calls within the bodies. This is similar to the Hoare logic inference rule for partial correctness of recursive procedures [6].

To be able to apply the inductive hypothesis, I take the sequential composition \( ct(T, \varphi); \ insert(T, k) \), replace each call by the corresponding procedure body, and then I “interleave” the procedure bodies using commutativity of statements. This enables me to bring the recursive calls to \( ct \) and to \( insert \) together, so that the inductive hypothesis can apply to their sequential composition, which can then be replaced by the equivalent recursive call to \( cti \). This results in procedure body that corresponds to a call of \( cti \), which completes the equivalence proof. I first replace \( ct(T, \varphi) \) by the procedure body that results from parameter binding:

\[
\begin{align*}
\textbf{if} \ (\varphi = \emptyset) \ T & := \text{NIL}; \\
\textbf{else} & \\
& \ x := \text{select in } \varphi; \\
& \ \varphi := \varphi - x; \\
& \ \text{Node } T := \text{new Node}(x); \\
& \ ct(T.l, \{ y \mid y \in \varphi \land y < x \}); \\
& \ ct(T.r, \{ y \mid y \in \varphi \land y > x \});
\end{align*}
\]

I now take \( insert(T,k) \) and place it at the end of both the if branch and the else branch, which clearly preserves equivalence with \( ct(T, \varphi); insert(T, k) \):

\[
\begin{align*}
\textbf{if} \ (\varphi = \emptyset) \ T & := \text{NIL}; \ insert(T, k) \\
\textbf{else} & \\
& \ x := \text{select in } \varphi; \\
& \ \varphi := \varphi - x; \\
& \ \text{Node } T := \text{new Node}(x); \\
& \ ct(T.l, \{ y \mid y \in \varphi \land y < x \}); \\
& \ ct(T.r, \{ y \mid y \in \varphi \land y > x \}); \\
& \ insert(T, k)
\end{align*}
\]

I replace \( insert(T, k) \) by the procedure body that results from parameter binding:

\[
\begin{align*}
& \textbf{if} \ (T = \text{NIL}) \ T := \text{new Node}(k); \\
& \textbf{else if} \ (k < T.val) \ insert(T.l,k) \\
& \textbf{else} \ insert(T.r,k) \quad \quad // k > T.val
\end{align*}
\]

In the if branch, I have \( T := \text{NIL} \), and so the body of \( insert(T, k) \) simplifies to

\[
\begin{align*}
T & := \text{new Node}(k).
\end{align*}
\]

In the else branch, I have \( \text{Node } T := \text{new Node}(x) \), and so the body of \( insert(T, k) \) simplifies to

\[
\begin{align*}
& \textbf{if} \ (k < T.val) \ insert(T.l,k) \\
& \textbf{else} \ insert(T.r,k) \quad \quad // k > T.val
\end{align*}
\]
I therefore now have

\[
\begin{align*}
\text{if } (\varphi = \emptyset) & \quad T := \text{NIL}; \quad T := \text{new Node}(k) \\
\text{else} & \\
& \quad x := \text{select in } \varphi; \quad \varphi := \varphi - x; \quad \text{Node } T := \text{new Node}(x); \\
& \quad ct(T.l, \{ y \mid y \in \varphi \land y < x \}); \quad ct(T.r, \{ y \mid y \in \varphi \land y > x \}); \\
& \quad \text{if } (k < T.val) \quad \text{insert}(T.l, k) \quad \text{else } \text{insert}(T.r, k) \quad //k > T.val
\end{align*}
\]

It is immediate that \( T := \text{NIL}; T := \text{new Node}(k) \equiv T := \text{new Node}(k) \). I also move

\[ ct(T.l, \{ y \mid y \in \varphi \land y < x \}); \quad ct(T.r, \{ y \mid y \in \varphi \land y > x \}); \]

down into both branches of the following if statement:

\[
\begin{align*}
\text{if } (\varphi = \emptyset) & \quad T := \text{new Node}(k) \\
\text{else} & \\
& \quad x := \text{select in } \varphi; \quad \varphi := \varphi - x; \quad \text{Node } T := \text{new Node}(x); \\
& \quad \text{if } (k < T.val) \quad \text{insert}(T.l, k) \quad \text{else } \text{insert}(T.r, k) \quad //k > T.val
\end{align*}
\]

Since \( \text{insert}(T.r, k) \) and \( ct(T.l, \{ y \mid y \in \varphi \land y < x \}) \); commute, the above is equivalent to

\[
\begin{align*}
\text{if } (\varphi = \emptyset) & \quad T := \text{new Node}(k) \\
\text{else} & \\
& \quad x := \text{select in } \varphi; \quad \varphi := \varphi - x; \quad \text{Node } T := \text{new Node}(x); \\
& \quad \text{if } (k < T.val) \quad \text{insert}(T.l, k) \quad \text{else } \text{insert}(T.r, k) \quad //k > T.val
\end{align*}
\]
I now apply the inductive hypothesis to conclude that
\[
\text{insert}(T.l,k); \text{ct}(T.l,\{y \mid y \in \varphi \land y < x\}); \equiv \text{cti}(T.l,\{y \mid y \in \varphi \land y < x\}, k);
\]
and that
\[
\text{insert}(T.r,k); \text{ct}(T.r,\{y \mid y \in \varphi \land y > x\}); \equiv \text{cti}(T.r,\{y \mid y \in \varphi \land y > x\}, k);
\]
Making these substitutions results in:

\[
\begin{align*}
\text{if } (\varphi = \emptyset) & \quad T := \text{new Node}(k) \\
\text{else} & \\
& \quad x := \text{select in } \varphi; \\
& \quad \varphi := \varphi - x; \\
& \quad \text{Node } T := \text{new Node}(x); \\
& \quad \text{if } (k < T.val) \\
& \quad \quad \text{cti}(T.l,\{y \mid y \in \varphi \land y < x\}, k); \\
& \quad \quad \text{ct}(T.r,\{y \mid y \in \varphi \land y > x\}); \\
& \quad \text{else} \quad // k > T.val \\
& \quad \quad \text{ct}(T.l,\{y \mid y \in \varphi \land y < x\}); \\
& \quad \quad \text{cti}(T.r,\{y \mid y \in \varphi \land y > x\}, k);
\end{align*}
\]
and the above is seen to be the procedure body corresponding to the call \text{cti}(T,\varphi,k).

The above was formed by starting with \text{ct}(T,\varphi); \text{insert}(T,k), replacing the calls by the procedure bodies, and then performing a sequence of equivalence-preserving transformations. Hence I conclude \text{ct}(T,\varphi); \text{insert}(T,k) \equiv \text{cti}(T,\varphi,k), which is (c) above. This completes the proof. For clarity, I have retained the formatting of pre-/post-program code in red italics and of program code in black typewriter, even as I was mixing these to establish the above equivalence.

### 8 Related Work

The use of assertions to verify programs was introduced by Floyd [5] and Hoare [9]: a precondition \( f \) expresses what can be assumed to hold before execution of a program \( P \), and a postcondition \( g \) expresses what must hold after the statement. The “Hoare triple” \( \{f\} P \{g\} \) thus states that if \( f \) holds when execution of \( P \) starts, then \( g \) will hold upon termination of \( P \). If termination is not assumed, this is known as partial correctness, and if termination is assumed we have total correctness. Both precondition and postcondition are expressed as a formula of a suitable logic, e.g., first order logic.

Subsequently, Dijkstra introduced the weakest precondition predicate transformer [4]: \text{wp}(P,g) is the weakest predicate \( f \) whose truth before execution of \( P \) guarantees \( g \) afterwards, if \( P \) terminates. He then used weakest preconditions to define a method for formally deriving a program from a specification, expressed as a precondition-postcondition pair. Later, Hoare observed that the Hoare triple can be expressed operationally, when he wrote “\( \{p\} q \{r\} \triangleq p; q < r \)” in [11], but he does not seem to have developed this observation into a proof system.

The formalization of both specifications and program correctness has lead to a rich and extensive literature on program verification and refinement. Hoare’s original rules [9] were extended to deal with non-determinism, fair selection, and procedures [6]. Separation logic [24] was introduced to deal with pointer-based structures.
A large body of work deals with the notion of program refinement [1,19]: start with an initial artifact, which serves as a specification, and gradually refine it into an executable and efficient program. This proceeds incrementally, in a sequence of refinement steps, each of which preserves a “refinement ordering” relation $\subseteq$, so that we have $P_0 \subseteq \ldots \subseteq P_n$, where $P_0$ is the initial specification and $P_n$ is the final program. Morgan [19] starts with a pre-condition/post-condition specification and refines it into an executable program using rules that are similar in spirit to Dijkstra’s weakest preconditions [4]. Back and Wright [1] use contracts, which consist of assertions (failure to hold causes a breach of the contract), assumptions (failure to hold causes vacuous satisfaction of the contract), and executable code. As such, contracts subsume both pre-condition/post-condition pairs and executable programs, and so serve as an artifact for the seamless refinement of a pre-condition/post-condition specification into a program.

A related development has been the application of monads to programming [17,18]. A monad is an endofunctor $T$ over a category $C$ together with a unit natural transformation from $1_C$ (the identity functor over $C$) to $T$ and a multiplication natural transformation from $T^2$ to $T$. The Hoare state monad contains Hoare triples (precondition, program, postcondition) [11], and a computation maps an initial state to a pair consisting of a final state and a returned value. The unit is the monadic operation return, which lifts returned values into the state monad, and the multiplication is the monadic operation bind, which composes two computations, passing the resulting state and returned value of the first computation to the second [27]. The Dijkstra monad captures functions from postconditions to preconditions [11,26]. The return operation gives the weakest precondition of a pure computation, and the bind operation gives the weakest precondition for a composition of two computations.

Hoare logic and weakest preconditions are purely assertional proof methods. Monads combine operational and assertional techniques, since they provide operations which return the assertions that are used in the correctness proofs. My approach is purely operational, since it uses no assertions (formula in a suitable logic) but rather pre- and post-programs instead. My approach thus represents the operational endpoint of the assertional–operational continuum, with Hoare logic/weakest preconditions at the other (assertional) endpoint, and monads somewhere in between.

9 Conclusions

I have presented a new method for verifying the correctness of sequential deterministic programs. The method does not use assertions to specify correctness properties, but rather “specification programs”, which define a set of “post states”, and can thus replace an assertion, which also defines a set of states, namely the states that satisfy it. Since specification programs are not executed, they can be inefficient, and can refer to any mathematically well-defined quantity, e.g., shortest path distances in a directed graph. In general, any formula of first/higher order logic can be referenced.

I illustrated my method on three examples: selection sort, Dijkstra’s shortest path algorithm, and in-place list-reversal. My approach has the following advantages, as illustrated by the examples:

- **Code synthesis**: unwinding the outer loop of the pre-program and then trading it into the program can give initial code for a loop body of the program. This technique was illustrated in all three examples.
• **Trading**: trading gives great flexibility in developing both the program and the pre-program, as code can be freely moved between the program and the pre-program. This provides a tactic which is not available in logic-based verification methods such as Floyd-Hoare logic [3, 9] and separation logic [24]. Trading was used in the sorting and list reversal examples.

• **Separation effect**: in the in-place list reversal example, a key requirement is that the reversed part of the list does not link around back to the non-reversed part. This requirement is easily expressed in my framework by the pre-program; in particular by how the pre-program constructs the list and then reverses part of it. Hence, using a pre-program which expresses, in an operational manner, the needed separation in pointer-based data structures, achieves the same effect as logic-based methods such as separation logic [24].

• **Refinement**: use a “coarse” and inefficient specification program and derive a more efficient program. Now iterate by using this program as a specification program to derive a still more efficient program, etc. My approach thus accommodates multi-level refinement.

• **Practical application**: it may be easier for developers to write specifications in code, a formalism that they are already well familiar with.

My approach is, to my knowledge, the first which uses purely operational specifications to verify the correctness of sequential programs, as opposed to pre- and post-conditions and invariants in a logic such as Floyd-Hoare logic and separation logic, or axioms and signatures in algebraic specifications [28]. The use of operational specifications is of course well-established for the specification and verification of concurrent programs. The process-algebra approach [7, 15, 16] starts with a specification written in a process algebra formalism such as CSP, CCS, or the Pi-calculus, and then refines it into an implementation. Equivalence of the implementation and specification is established by showing a bisimulation [15, 20] between the two. The I/O Automata approach [13] starts with a specification given as a single “global property automaton” and shows that a distributed/concurrent implementation respects the global property automaton by establishing a simulation relation [14] from the implementation to the specification.

My approach requires, in some cases, that one establish the equivalence of two programs [2, 21, 12]. Any method for showing program equivalence works, since the equivalence is simply the hypothesis for the (Substitution) rule. The equivalence proofs in this paper were informal and “by inspection”, based on concepts such as the commutativity of assignment statements that modify different variables/objects.

Future work includes more examples and case studies, and in particular examples with pointer-based data structures. I am also extending the operational annotations approach to the verification of concurrent programs.
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