Energy-preserving integration of non-canonical Hamiltonian systems by continuous-stage methods

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Abstract

As is well known, energy is generally deemed as one of the most important physical invariants in many conservative problems and hence it is of remarkable interest to consider numerical methods which are able to preserve it. In this paper, we are concerned with the energy-preserving integration of non-canonical Hamiltonian systems by continuous-stage methods. Algebraic conditions in terms of the Butcher coefficients for ensuring the energy preservation, symmetry and quadratic-Casimir preservation respectively are presented. With the presented condition and in use of orthogonal expansion techniques, the construction of energy-preserving integrators is examined. A new class of energy-preserving integrators which is symmetric and of order 2m is constructed. Some numerical results are reported to verify our theoretical analysis and show the effectiveness of our new methods.

Keywords: Non-canonical Hamiltonian systems; Continuous-stage methods; Energy-preserving methods.

1. Introduction

We consider the following first-order system of ordinary differential equations
\[ \dot{y} = S(y) \nabla H(y), \quad y(t_0) = y_0 \in \mathbb{R}^n, \tag{1.1} \]
where \( S(y) \) is an \( n \times n \) skew-symmetric matrix and \( H(y) \) as a scalar function is called the Hamiltonian (energy) of the system. By differentiation one can easily verify that
\[ \frac{d}{dt} H(y) = \nabla H(y)^T \dot{y} = \nabla H(y)^T S(y) \nabla H(y) = 0, \]
which means the Hamiltonian \( H(y) \), along the solution curves, is an invariant or a first integral of the system \([1.1]\). A scalar function denoted by \( C(y) \) is called a Casimir function of \([1.1]\) if \( \nabla C(y)^T S(y) = 0 \) for all \( y \). Obviously, \( C(y) \) is also an invariant of \([1.1]\) (independent of \( H(y) \)) by noticing that
\[ \frac{d}{dt} C(y) = \nabla C(y)^T \dot{y} = \nabla C(y)^T S(y) \nabla H(y) = 0. \]

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Moreover, if $S(y)$ is a structure matrix of a Poisson bracket satisfying the Jacobi identity \[10, 13\], then (1.1) is referred to as a Poisson system or non-canonical Hamiltonian system. In such a case, the exact flow $\varphi_t(y)$ of the system is a Poisson map which says

$$\varphi'_t(y)S(y)\varphi'_t(y)^T = S(\varphi_t(y)),$$

and to construct a Poisson integrator\(^1\) is of interest \[10, 13\] — such type of geometric integrators usually gives an excellent long-time integration of the given system. Particularly, if $S(y)$ is a constant structure matrix, Poisson integrators can be easily constructed \[10, 15\]. An important special case in point is the well-known symplectic integrators preserving the symplecticity of those canonical Hamiltonian systems (i.e., for the case $S(y) = J^{-1}$ with $J$ being a canonical structure matrix), the literatures of which have grown vigorously in the past decades (see \[1, 10, 13, 16, 24\] and references therein). However, when $S(y)$ is a non-constant structure matrix, the relevant researches are rather few, due to the reason that to get Poisson integrators of arbitrarily high order for a general Poisson system is not an easy task \[10, 15\]. On the other hand, for the sake of obtaining “good” long-term integration, an alternative integration technique is the so-called energy-preserving integration which has received increasing attention in recent years \[3, 8, 20, 21, 22, 36\].

As pointed out in \[8\], for the energy-preserving discretization of the non-canonical Hamiltonian system (1.1), it is not necessary to require $S(y)$ to satisfy the Jacobi identity. Here we mention the simplest integrator proposed in \[8\] for solving (1.1), which reads

$$y_1 = y_0 + hS\left(\frac{y_0 + y_1}{2}\right) \int_0^1 \nabla H(y_0 + \tau(y_1 - y_0)) \, d\tau. \tag{1.2}$$

Such an integrator can exactly preserve the energy and quadratic Casimir functions of the system, is invariant with respect to linear transformations, has order 2, and it obviously generalizes the average vector field method for canonical Hamiltonian system \[23\]. One should note that (1.2) treats the factors $S(y)$ and $\nabla H(y)$ of (1.1) in a different manner which admits a theoretical interpretation with partitioned continuous-stage methods \[8\]. Following the idea of \[8\], the author in \[20, 21\] proposes an algebraic condition for energy preservation in terms of Butcher coefficients (given that all the coefficients are polynomials) and further develops the theory of energy-preserving exponentially-fitted integrators for Poisson systems. In fact, the seminal idea of continuous-stage methods can be led back to the pioneering work of Butcher \[4, 5, 6\] and Hairer \[14\], which has been further developed by other authors in more recent years and some relevant applications in geometric numerical integration are explored. Readers who are interested in more details of this subject may refer to \[8, 17, 20, 22, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35\] and references therein.

In this paper, we propose a new type of continuous-stage methods and explore its sufficient condition for energy-preserving integration of (1.1). The idea originates from a different treatment of $S(y)$, and actually the new method can be viewed as an extension of (1.2) with the factor $S\left(\frac{h(y_0 + y_1)}{2}\right)$ replaced by a definite integral. Specifically, the most simple example of our methods reads

$$y_1 = y_0 + h \int_0^1 S(y_0 + \varsigma(y_1 - y_0)) \, d\varsigma \int_0^1 \nabla H(y_0 + \tau(y_1 - y_0)) \, d\tau, \tag{1.3}$$

\(^1\)A one-step method is called a Poisson integrator if the associated numerical flow is also a Poisson map and preserves all the Casimir functions at the same time \[13\].
which is energy-preserving, symmetric and of order 2 for solving (1.1). Obviously, if we use the midpoint rule to approximate the left integral of (1.3), then the method reduces to (1.2). Besides, it is known that the scheme (1.2) and its high-order version presented in [8] can be recast as the form of partitioned continuous-stage methods with the Butcher coefficients expressed by Lagrangian interpolatory polynomials, while for our new methods it will be more conveniently expressed by using orthogonal polynomial expansions.

This paper is organized as follows. In Section 2, we introduce the new continuous-stage methods named “enhanced continuous-stage methods” and present the corresponding energy-preserving condition, the symmetry condition as well as the condition for preserving quadratic Casimir functions. Section 3 is devoted to discussing the construction of energy-preserving integrators on the basis of the presented condition. This is followed by Section 4, where numerical experiments will be reported. At last, we give some concluding remarks to end this paper.

2. Enhanced continuous-stage methods for non-canonical Hamiltonian systems

In this section, we propose the definition of enhanced continuous-stage methods for solving non-canonical Hamiltonian systems. After that, the energy-preserving condition, the symmetry condition and the condition for preserving quadratic Casimir functions in terms of Butcher coefficients will be given.

**Definition 2.1.** For the numerical integration of (1.1), the one-step method

\[
Y_\tau = y_0 + h \int_0^1 \int_0^1 A_{\tau, \varsigma, \sigma} S(Y_{\varsigma}) \nabla H(Y_{\sigma}) \, d\varsigma \, d\sigma, \quad \tau \in [0, 1],
\]

\[
y_1 = y_0 + h \int_0^1 \int_0^1 B_{\varsigma, \sigma} S(Y_{\varsigma}) \nabla H(Y_{\sigma}) \, d\varsigma \, d\sigma,
\]

is called an enhanced continuous-stage method, where the Butcher coefficients \(A_{\tau, \varsigma, \sigma}\) and \(B_{\varsigma, \sigma}\) are assumed to be smooth multivariate functions of the variables in the subscripts, and we define the third Butcher coefficient \(C_{\tau}\) by

\[
C_{\tau} = \int_0^1 \int_0^1 A_{\tau, \varsigma, \sigma} \, d\varsigma \, d\sigma, \quad \tau \in [0, 1].
\]  

(2.1)

If \(S(y)\) is a constant matrix, the method (2.1) falls into the category of standard continuous-stage Runge-Kutta methods [14, 26]. Besides, one can easily verify that the method (2.1) is also invariant with respect to linear transformations in the sense that given in [8]. A sufficient condition for the method (2.1) to be energy-preserving is given in the following theorem.

**Theorem 2.1.** The enhanced continuous-stage method (2.1) is energy-preserving if

\[
A_{0, \varsigma, \sigma} = 0 \quad \text{and} \quad B_{\varsigma, \sigma} = A_{1, \varsigma, \sigma}, \quad \text{for} \ \forall \varsigma, \sigma \in [0, 1],
\]

\[
\frac{\partial}{\partial \tau} A_{\tau, \varsigma, \sigma} = \frac{\partial}{\partial \sigma} A_{\sigma, \varsigma, \tau}, \quad \text{for} \ \forall \tau, \varsigma, \sigma \in [0, 1].
\]

(2.3)

**Proof.** It is seen that the first formula of (2.3) implies

\[
Y_0 = y_0, \quad Y_1 = y_1,
\]
Substituting (2.1) into (2.4) and using the second formula of (2.3) yields

\[ H(y_1) - H(y_0) = \int_0^1 \frac{d}{d\tau} H(Y_\tau) d\tau = \int_0^1 \nabla H(Y_\tau)^T Y_\tau' d\tau. \] (2.4)

Substituting (2.1) into (2.4) and using the second formula of (2.3) yields

\[ H(y_1) - H(y_0) = h \int_0^1 \int_0^1 \int_0^1 \frac{\partial}{\partial \tau} A_{\tau,\varsigma,\sigma} \nabla H(Y_\tau)^T S(Y_\varsigma) \nabla H(Y_\sigma) d\tau d\varsigma d\sigma \]

\[ = h \int_0^1 \int_0^1 \int_0^1 \frac{1}{2} \left( \frac{\partial}{\partial \tau} A_{\tau,\varsigma,\sigma} + \frac{\partial}{\partial \sigma} A_{\sigma,\varsigma,\tau} \right) \nabla H(Y_\tau)^T S(Y_\varsigma) \nabla H(Y_\sigma) d\tau d\varsigma d\sigma. \] (2.5)

By exchanging the notations \( \tau \leftrightarrow \sigma \) and using the skew-symmetry of \( S(Y_\varsigma) \), we have

\[ = - \int_0^1 \int_0^1 \int_0^1 \frac{\partial}{\partial \tau} A_{\tau,\varsigma,\sigma} \nabla H(Y_\tau)^T S(Y_\varsigma) \nabla H(Y_\sigma) d\tau d\varsigma d\sigma. \] (2.6)

Inserting (2.6) into (2.5) gives

\[ H(y_1) - H(y_0) = 0. \]

This completes the proof. \( \square \)

From the proof of Theorem 2.1, it is clear that the energy-preserving condition is independent of the treatment of \( \varsigma \), which motivates us to consider the case when the integral with respect to \( \varsigma \) is replaced by a sum associated with numerical integration. Let \( b_i \) and \( c_i \) be the weights and abscissae of the following \( s \)-point interpolatory quadrature formula

\[ \int_0^1 f(x) \, dx \approx \sum_{i=1}^s b_i f(c_i), \quad c_i \in [0, 1], \quad i = 1, \ldots, s, \] (2.7)

where

\[ b_i = \int_0^1 \ell_i(x) \, dx, \quad \ell_i(x) = \prod_{j=1, j \neq i}^s \frac{x - c_j}{c_i - c_j}, \quad i = 1, \ldots, s. \]

By applying (2.7) to approximate the integral with respect to \( \varsigma \), it follows from (2.1) that

\[ Y_{\tau} = y_0 + h \sum_{j=1}^s \int_0^1 b_j A_{\tau,c_j,\sigma} S(Y_{c_j}) \nabla H(Y_\sigma) d\sigma, \quad \tau \in [0, 1], \]

\[ y_1 = y_0 + h \sum_{j=1}^s \int_0^1 b_j B_{c_j,\sigma} S(Y_{c_j}) \nabla H(Y_\sigma) d\sigma, \] (2.8)

which is called a semi-enhanced continuous-stage method.
Theorem 2.2. The semi-enhanced continuous-stage method (2.8) is energy-preserving if
\[ A_{0,c,j,\sigma} = 0 \quad \text{and} \quad B_{c,j,\sigma} = A_{1,c,j,\sigma}, \quad \forall \sigma \in [0,1], \quad j = 1, \ldots, s, \]
\[ \frac{\partial}{\partial \tau} A_{\tau,c,j,\sigma} = \frac{\partial}{\partial \sigma} A_{\sigma,c,j,\tau}, \quad \forall \tau, \sigma \in [0,1], \quad j = 1, \ldots, s. \]  
(2.9)

Proof. Please refer to the proof of Theorem 2.1 since the process of the proof is very the same. □

Remark 2.1. The condition (2.9) is similar to the corresponding result presented in [20, 21]. However, unlike the special treatment in [20, 21], our proof is not based on the polynomial assumption of the Butcher coefficients.

Remark 2.2. The energy-preserving integrators presented in [8] and [21] can be interpreted as the class of semi-enhanced continuous-stage methods in the form (2.8).

In the following we place special emphasis on the enhanced continuous-stage method (2.1) and consider rephrasing the condition (2.3) by virtue of orthogonal expansions. For this sake, we introduce the shifted Legendre polynomial \( P_j(x) \) of degree \( j \), which can be obtained by using the Rodrigues’ formula
\[ P_0(x) = 1, \quad P_j(x) = \frac{\sqrt{2j + 1}}{j!} \frac{d^j}{dx^j} \left( x^j (x - 1)^j \right), \quad j = 1, 2, 3, \ldots. \]  
(2.10)

Note that these polynomials are normalized and orthogonal in \([0,1]\)
\[ \int_0^1 P_j(x) P_k(x) \, dx = \delta_{jk}, \quad j, k = 0, 1, 2, \ldots, \]  
(2.11)

and possess the following integration properties
\[ \int_0^\tau P_j(x) \, dx = \xi_{j+1} P_{j+1}(\tau) - \xi_j P_{j+1+\delta_{j0}}(\tau), \quad j = 0, 1, 2, \ldots, \]  
\[ \int_\tau^1 P_j(x) \, dx = \xi_{j0} - \xi_{j+1} P_{j+1}(\tau) + \xi_j P_{j+1+\delta_{j0}}(\tau), \quad j = 0, 1, 2, \ldots, \]  
(2.12)

where \( \delta_{jk} \) is the Kronecker delta and
\[ \xi_j = \begin{cases} \frac{1}{2\sqrt{4j^2 - 1}}, & j \geq 1; \\ -\frac{1}{2}, & j = 0. \end{cases} \]

Note that \( \{ P_i(\tau) P_j(\sigma) : i, j \geq 0 \} \) constitutes a complete orthogonal set in \( L^2([0,1] \times [0,1]) \) (Hilbert space), for each fixed \( \varsigma \), we consider the following expansion for (2.3)
\[ \frac{\partial}{\partial \tau} A_{\tau,\varsigma,\sigma} = \frac{\partial}{\partial \sigma} A_{\sigma,\varsigma,\tau} = \sum_{i,j \geq 0} \alpha_{(i,j)}(\varsigma) P_i(\tau) P_j(\sigma), \]  
(2.13)

where \( \alpha_{(i,j)}(\varsigma) \) are real functions of \( \varsigma \). By taking integrals with respect to \( \tau \) and \( \sigma \) respectively, it gives
\[ A_{\tau,\varsigma,\sigma} = \sum_{i,j \geq 0} \alpha_{(i,j)}(\varsigma) \int_0^\tau P_i(x) \, dx P_j(\sigma) + \psi(\varsigma, \sigma), \]
\[ A_{\sigma,\varsigma,\tau} = \sum_{i,j \geq 0} \alpha_{(i,j)}(\varsigma) P_i(\tau) \int_0^\sigma P_j(x) \, dx + \phi(\varsigma, \tau), \]
where \( \psi, \phi \) are arbitrary functions. Noticing \( A_{0,\varsigma,\tau} = 0, A_{0,\varsigma,\tau} = 0 \) (from (2.3)), we have \( \psi \equiv 0, \phi \equiv 0 \), which then gives rise to

\[
A_{\tau,\varsigma,\sigma} = \sum_{i,j \geq 0} \alpha_{(i,j)}(\varsigma) \int_{\tau}^{\sigma} P_i(x) P_j(\sigma) \, dx,
\]

\[\text{(2.14)}\]

By exchanging the notations \( \tau \leftrightarrow \sigma \) and \( i \leftrightarrow j \) of the second formula above, it follows

\[
A_{\tau,\varsigma,\sigma} = \sum_{i,j \geq 0} \alpha_{(j,i)}(\varsigma) \int_{\tau}^{\sigma} P_i(x) P_j(\sigma) \, dx.
\]

\[\text{(2.15)}\]

Comparing the first formula of (2.14) with (2.15) gives the symmetric relation

\[
\alpha_{(i,j)}(\varsigma) = \alpha_{(j,i)}(\varsigma), \quad \varsigma \in [0,1], \quad \forall \ i, j \geq 0.
\]

\[\text{(2.16)}\]

Besides, by (2.3) and noticing that

\[
\int_{0}^{1} P_i(x) \, dx = \delta_{i0}, \quad i = 0, 1, \cdots
\]

\[\text{(2.17)}\]

we have

\[
B_{\varsigma,\sigma} = A_{1,\varsigma,\sigma} = \sum_{i,j \geq 0} \alpha_{(i,j)}(\varsigma) \int_{0}^{1} P_i(x) P_j(\sigma) = \sum_{j \geq 0} \alpha_{(0,j)}(\varsigma) P_j(\sigma).
\]

\[\text{(2.18)}\]

In summary, we have the following result which is a modified version of Theorem 2.1.

**Theorem 2.3.** If the coefficients of the enhanced continuous-stage method (2.1) are in the form

\[
A_{\tau,\varsigma,\sigma} = \sum_{i,j \geq 0} \alpha_{(i,j)}(\varsigma) \int_{\tau}^{\sigma} P_i(x) P_j(\sigma) \, dx
\]

\[\text{for } \forall \ \tau, \varsigma, \sigma \in [0,1], \text{ (2.19a)}\]

\[
B_{\varsigma,\sigma} = \sum_{j \geq 0} \alpha_{(0,j)}(\varsigma) P_j(\sigma)
\]

\[\text{for } \forall \ \varsigma, \sigma \in [0,1], \text{ (2.19b)}\]

where \( P_i(x) \) are the shifted Legendre polynomials and \( \alpha_{(i,j)}(\varsigma) \) satisfy the symmetric relation (2.16), then the method is energy-preserving for solving (1.1).

**Remark 2.3.** By replacing \( \varsigma \) with \( c_j, j = 1, \cdots, s \) in (2.19), it gives a rephrasing form of (2.9) in Theorem 2.2.

We are also interested in the symmetry of the method (2.1). For a one-step method \( y_1 = \Phi_h(y_0) \), it is symmetric if and only if \( \Phi_{-h}^{-1} = \Phi_h \) [13]. Therefore, by definition if the formula of a one-step method is left unaltered after exchanging \( y_1 \leftrightarrow y_0 \) and \( h \leftrightarrow -h \), then the method is symmetric. An attractive property of a symmetric integrator is that the method always possesses an even order [13]. In what follows we give the algebraic condition for (2.1) to be a symmetric integrator.

**Theorem 2.4.** If the coefficients of the method (2.1) satisfy

\[
A_{\tau,\varsigma,\sigma} + A_{1-\tau,1-\varsigma,1-\sigma} = B_{\varsigma,\sigma}, \quad \forall \varsigma, \sigma \in [0,1],
\]

\[\text{(2.20)}\]

then the method is symmetric.
Proof. Please refer to Theorem 3.4 of [30] for deriving a similar proof.

In what follows we consider in which situation the method (2.1) can preserve all quadratic Casimir functions of the system (1.1).

**Theorem 2.5.** Let \( C(y) = y^T D y \) (with a symmetric constant matrix \( D \)) be a Casimir function of the system (1.1). If the Butcher coefficients of the method (2.1) satisfy
\[
B_{\rho, \tau} A_{\rho, \varsigma, \sigma} + B_{\varsigma, \sigma} A_{\varsigma, \rho, \tau} = B_{\rho, \tau} B_{\varsigma, \sigma}, \quad \forall \rho, \tau, \varsigma, \sigma \in [0, 1],
\]
then the method preserves this Casimir.

**Proof.** Note that for the quadratic Casimir function, the constraint \( \nabla C(y)^T S(y) = 0 \), \( \forall y \) by definition becomes
\[
y^T D S(y) = 0, \quad \text{for} \quad \forall y,
\]
and then the proof is quite similar to that of Theorem 2.2 in [13] (page 101).

**Remark 2.4.** By using the same technique presented in Theorem 1.5 of [13] (page 99), we find that the method (2.1) automatically (without extra conditions) preserves all linear Casimir functions \( C(y) = d^T y \) (with a constant vector \( d \)) of the system (1.1).

3. Construction of energy-preserving integrators

Now we are in the position to study the construction of energy-preserving integrators on the basis of Theorem 2.3. For this sake, we have to recast the method (2.1) as a continuous-stage partitioned Runge-Kutta method by following the similar idea of [8].

Firstly, let us consider the following partitioned system of ordinary differential equations [8]
\[
\dot{y} = S(z) \nabla H(y), \quad y(t_0) = y_0, \\
\dot{z} = S(z) \nabla H(y), \quad z(t_0) = z_0.
\]
(3.1)

When \( z_0 = y_0 \), the exact solutions of both systems (1.1) and (3.1) coincide with each other [8].

By appending two identities to (2.1) and introducing the notations \( Y_{\rho, \tau} := Y_\tau, Z_{\rho, \tau} := Y_\rho \), we get an equivalent scheme of (2.1), which reads
\[
Y_{\rho, \tau} = y_0 + h \int_0^1 \int_0^1 a_{\rho, \tau; \varsigma, \sigma} S(Z_{\varsigma, \sigma}) \nabla H(Y_{\varsigma, \sigma}) \, d\varsigma d\sigma, \quad \rho, \tau \in [0, 1],
\]
\[
Z_{\rho, \tau} = z_0 + h \int_0^1 \int_0^1 \tilde{a}_{\rho, \tau; \varsigma, \sigma} S(Z_{\varsigma, \sigma}) \nabla H(Y_{\varsigma, \sigma}) \, d\varsigma d\sigma, \quad \rho, \tau \in [0, 1],
\]
\[
y_1 = y_0 + h \int_0^1 \int_0^1 b_{\rho, \tau} S(Z_{\rho, \tau}) \nabla H(Y_{\rho, \tau}) \, dp d\tau,
\]
\[
z_1 = z_0 + h \int_0^1 \int_0^1 \tilde{b}_{\rho, \tau} S(Z_{\rho, \tau}) \nabla H(Y_{\rho, \tau}) \, dp d\tau,
\]
where \( z_0 = y_0 \) and
\[
a_{\rho, \tau; \varsigma, \sigma} = A_{\tau; \varsigma, \sigma}, \quad \tilde{a}_{\rho, \tau; \varsigma, \sigma} = A_{\rho; \varsigma, \sigma}, \quad b_{\rho, \tau} = B_{\rho, \tau}, \quad \tilde{b}_{\rho, \tau} = B_{\rho, \tau}.
\]
(3.2)
In order to analyze the order of the method later, we define $c_{\rho, \tau}$ and $\tilde{c}_{\rho, \tau}$ as
\[
c_{\rho, \tau} = \int_0^1 \int_0^1 a_{\rho, \tau; \varsigma, \sigma} \, d\varsigma \, d\sigma, \quad \tilde{c}_{\rho, \tau} = \int_0^1 \int_0^1 \tilde{a}_{\rho, \tau; \varsigma, \sigma} \, d\varsigma \, d\sigma.
\] (3.4)

By using (2.2), it yields
\[
c_{\rho, \tau} = C_\tau \quad \text{and} \quad \tilde{c}_{\rho, \tau} = C_\rho.
\] (3.5)

Remark that the second formula of (3.2) is equivalent to
\[
Y_\rho = y_0 + h \int_0^1 \int_0^1 A_{\rho, \varsigma, \sigma} S(Y_\varsigma) \nabla H(Y_\sigma) \, d\varsigma \, d\sigma, \quad \rho \in [0, 1],
\]
which is obtained by replacing $\tau$ with $\rho$ in (2.1). Besides, by noticing $z_0 = y_0$ it gives $z_1 = y_1$, and hence the last formula of (3.2) is the same as the third one. Therefore, via (3.2) we see that (2.1) essentially amounts to a continuous-stage partitioned Runge-Kutta method applied to the partitioned system (3.1).

For the sake of deriving an energy-preserving integrator with a certain order, we consider using the standard order theory of partitioned Runge-Kutta methods. A natural idea for constructing the integrators is to substitute (2.19) into the order conditions so as to determine the coefficients $\alpha_{(i,j)}(\varsigma)$. We give the following result as an illustrative example.

**Theorem 3.1.** If and only if
\[
\int_0^1 \alpha_{(0,0)}(\varsigma) \, d\varsigma = 1,
\]
the enhanced continuous-stage method (2.1) with coefficients (2.19) is at least of order 1.

**Proof.** By the standard order theory of partitioned Runge-Kutta methods, the method is at least of order 1, if and only if the following order conditions are satisfied
\[
\int_0^1 \int_0^1 b_{\rho, \tau} \, d\rho \, d\tau = 1, \quad \int_0^1 \int_0^1 \tilde{b}_{\rho, \tau} \, d\rho \, d\tau = 1.
\]

In our case, by using (3.3), this means
\[
\int_0^1 \int_0^1 B_{\rho, \tau} \, d\rho \, d\tau = 1.
\]

Substituting (2.19) into the order condition above and using (2.17) gives
\[
\sum_{j \geq 0} \int_0^1 \alpha_{(0,j)}(\varsigma) \, d\varsigma \int_0^1 P_j(\sigma) \, d\sigma = \sum_{j \geq 0} \int_0^1 \alpha_{(0,j)}(\varsigma) \, d\varsigma \delta_{j,0} = \int_0^1 \alpha_{(0,0)}(\varsigma) \, d\varsigma = 1.
\]

However, generally it is not easy to construct high-order methods by using the standard order conditions, since the number of order conditions increases very fast when the order becomes higher and higher [13]. An alternative approach is to use the simplifying assumptions of order conditions as stated below.

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2The order conditions can be conveniently obtained by P-series theory (e.g., see [13], page 67).
Theorem 3.2. If the partitioned method (3.2) with $b_{p, \tau} = \hat{b}_{p, \tau}$ for all $\rho, \tau \in [0, 1]$ satisfies the following simplifying assumptions (with $k \geq 1$, $l \geq 0$)

$$B(\xi) : \int_0^1 \int_0^1 b_{p, \tau} \rho_{p, \tau}^{k-1} \rho_{p, \tau} d\rho d\tau = \frac{1}{k+l}, \quad 1 \leq k+l \leq \xi,$$

$$C(\eta) : \int_0^1 \int_0^1 a_{p, \tau; \zeta, \sigma} \rho_{p, \tau}^{k-1} \rho_{p, \tau} \rho \zeta \rho \sigma d\rho d\sigma = \frac{c_{p, \tau}^{k+l}}{k+l}, \quad 1 \leq k+l \leq \eta, \quad \rho, \sigma \in [0, 1],$$

$$\tilde{C}(\eta) : \int_0^1 \int_0^1 \tilde{a}_{p, \tau; \zeta, \sigma} \rho_{p, \tau}^{k-1} \rho_{p, \tau} \rho \zeta \rho \sigma d\rho d\sigma = \frac{\tilde{c}_{p, \tau}^{k+l}}{k+l}, \quad 1 \leq k+l \leq \eta, \quad \rho, \sigma \in [0, 1],$$

$$D(\xi) : \int_0^1 \int_0^1 b_{p, \tau} \rho_{p, \tau}^{k-1} \rho_{p, \tau} \rho \sigma \rho d\rho d\sigma = \frac{b_{p, \rho} (1 - c_{p, \sigma}^{k+l})}{k+l}, \quad 1 \leq k+l \leq \xi, \quad \zeta, \sigma \in [0, 1],$$

$$\tilde{D}(\xi) : \int_0^1 \int_0^1 \tilde{b}_{p, \tau} \rho_{p, \tau}^{k-1} \rho_{p, \tau} \rho \sigma \rho d\rho d\sigma = \frac{\tilde{b}_{p, \rho} (1 - \tilde{c}_{p, \sigma}^{k+l})}{k+l}, \quad 1 \leq k+l \leq \xi, \quad \zeta, \sigma \in [0, 1],$$

then the method is at least of order

$$p = \min\{\xi, 2\eta + 2, \zeta + \eta + 1\}.$$  

Proof. Please refer to Theorem 4.1 and Lemma 4.2 of [8], which provides a similar idea for the proof. 

For simplicity, hereafter we assume that $C_{\tau} = \tau$, and thus (3.5) becomes

$$c_{p, \tau} = \tau \quad \text{and} \quad \tilde{c}_{p, \tau} = \rho.$$  

By substituting (3.3) and (3.7) into (3.6), then it gives (with $k \geq 1$, $l \geq 0$)

$$B(\xi) : \int_0^1 \int_0^1 B_{p, \tau} \tau^{k-1} \rho \sigma d\rho d\tau = \frac{1}{k+l}, \quad 1 \leq k+l \leq \xi,$$

$$C(\eta) : \int_0^1 \int_0^1 A_{\tau, \zeta, \sigma} \rho^{k-1} \rho \zeta \rho \sigma d\rho d\sigma = \frac{\tau^{k+l}}{k+l}, \quad 1 \leq k+l \leq \eta, \quad \tau \in [0, 1],$$

$$\tilde{C}(\eta) : \int_0^1 \int_0^1 A_{\rho, \zeta, \sigma} \rho^{k-1} \rho \zeta \rho \sigma d\rho d\sigma = \frac{\tilde{\tau}^{k+l}}{k+l}, \quad 1 \leq k+l \leq \eta, \quad \rho \in [0, 1],$$

$$D(\xi) : \int_0^1 \int_0^1 B_{p, \tau} \tau^{k+l-1} A_{\tau, \zeta, \sigma} d\rho d\sigma = \frac{B_{\zeta, \sigma} (1 - s_{\zeta, \sigma}^{k+l})}{k+l}, \quad 1 \leq k+l \leq \xi, \quad \zeta, \sigma \in [0, 1],$$

$$\tilde{D}(\xi) : \int_0^1 \int_0^1 \tilde{B}_{p, \tau} \tau^{k+l-1} A_{\rho, \zeta, \sigma} d\rho d\sigma = \frac{\tilde{B}_{\zeta, \sigma} (1 - \tilde{s}_{\zeta, \sigma}^{k+l})}{k+l}, \quad 1 \leq k+l \leq \xi, \quad \zeta, \sigma \in [0, 1].$$

We point out that (2.2) is equivalent to $C(\eta)$ with $\eta = 1$. Observing that $C(\eta)$ is the same as $\tilde{C}(\eta)$ except for the difference in notations, we can remove the condition $\tilde{C}(\eta)$. However, it is not an easy task to consider the most general case. For the sake of simplicity, we resort to the following two assumptions with $m \geq 1$:

$$(A1) \quad \text{Assume} \ B_{p, \tau} \text{ satisfies}$$

$$\int_0^1 B_{p, \tau} \tau^{k-1} d\tau = \rho^{k-1}, \quad k = 1, \cdots, m.$$
Assume there exists a bivariate function $\tilde{A}_{\tau, \varsigma}$ such that
\[
\int_0^1 A_{\tau, \varsigma, \sigma} \sigma^{k-1} d\sigma = \tilde{A}_{\tau, \varsigma} \varsigma^{k-1}, \quad k = 1, \ldots, m.
\] (3.10)

Remark that from (2.3), we have $\tilde{A}_{0, \varsigma} = 0$ and by combining (3.9) and (3.10) it gives $\tilde{A}_{1, \varsigma} = 1$. The existence of $\tilde{A}_{\tau, \varsigma}$ will be seen later.

Lemma 3.1. The formulas (3.9) and (3.10) in the assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent to, respectively,
\[
\int_0^1 B_{\rho, \tau} \phi(\tau) d\tau = \phi(\rho), \quad \text{for } \deg(\phi) \leq m - 1,
\] (3.11)
\[
\int_0^1 A_{\tau, \varsigma, \sigma} \phi(\sigma) d\sigma = \tilde{A}_{\tau, \varsigma} \phi(\varsigma), \quad \text{for } \deg(\phi) \leq m - 1,
\] (3.12)
where $\phi(x) \in \text{span}\{1, x, \ldots, x^{m-1}\}$ and $\deg(\phi)$ represents the degree of $\phi$.

Particularly, we can use the shifted Legendre polynomials $P_j(\rho)$ with $j = 0, \ldots, m - 1$ to take the place of $\phi$ in Lemma 3.1. Consequently, by inserting (2.19b) (with $\varsigma, \sigma$ replaced by $\rho, \tau$) into (3.11) we get
\[
\alpha_{(0,j)}(\rho) = \alpha_{(j,0)}(\rho) = P_j(\rho), \quad j = 0, \ldots, m - 1,
\] (3.13)
and thus,
\[
B_{\rho, \tau} = \sum_{j=0}^{m-1} P_j(\rho)P_j(\tau) + \sum_{j=m}^{m-1} \alpha_{(0,j)}(\rho)P_j(\tau), \quad \text{for } \forall \rho, \tau \in [0,1].
\] (3.14)

Next, we consider how to devise the Butcher coefficient $A_{\tau, \varsigma, \sigma}$ of the method (2.1) for energy-preserving integration. Taking partial differentiation of (3.10) with respect to $\tau$ yields
\[
\int_0^1 \frac{\partial}{\partial \tau} A_{\tau, \varsigma, \sigma} \sigma^{k-1} d\sigma = \frac{\partial}{\partial \tau} \tilde{A}_{\tau, \varsigma} \varsigma^{k-1}, \quad k = 1, \ldots, m,
\] (3.15)
which is equivalent to
\[
\int_0^1 \frac{\partial}{\partial \tau} A_{\tau, \varsigma, \sigma} P_j(\sigma) d\sigma = \frac{\partial}{\partial \tau} \tilde{A}_{\tau, \varsigma} P_j(\varsigma), \quad j = 0, \ldots, m - 1.
\] (3.16)

Let us consider the expansion of $\frac{\partial}{\partial \tau} \tilde{A}_{\tau, \varsigma}$ along the basis $\{P_i(\tau)\}$ with $\varsigma$ being fixed
\[
\frac{\partial}{\partial \tau} \tilde{A}_{\tau, \varsigma} = \sum_{i \geq 0} \gamma_i(\varsigma) P_i(\tau),
\] (3.17)
where the expansion coefficients $\gamma_i(\varsigma)$ are to be determined. By inserting (2.19a) and (3.17) into (3.16) it gives
\[
\sum_{i \geq 0} \alpha_{(i,j)}(\varsigma) P_i(\tau) = \sum_{i \geq 0} \left(\gamma_i(\varsigma) P_j(\varsigma)\right) P_i(\tau), \quad j = 0, \ldots, m - 1,
\] and then by comparing the like basis we get
\[
\alpha_{(i,j)}(\varsigma) = \gamma_i(\varsigma) P_j(\varsigma), \quad i \geq 0, \ j = 0, \ldots, m - 1.
\] (3.18)
By using the symmetry relation (2.16), we have
\[ \gamma_i(\varsigma)P_j(\varsigma) = \gamma_j(\varsigma)P_i(\varsigma), \]
which suggests us to take\(^3\)
\[ \gamma_i(\varsigma) = P_i(\varsigma), \quad i \geq 0. \quad (3.19) \]
Therefore, \( \tilde{A}_{\tau, \varsigma} \) exists and by taking integral of (3.17) and using \( \tilde{A}_{0, \varsigma} = 0 \), it can be expressed as
\[ \tilde{A}_{\tau, \varsigma} = \sum_{i \geq 0} P_i(\varsigma) \int_0^\tau P_i(x)dx, \quad (3.20) \]
given that (3.19) is taken. On the basis of the analysis above, we let the Butcher coefficients of the method (2.1) be (with \( m \geq 1 \))
\[ A_{\tau, \varsigma, \sigma} = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} P_i(\varsigma)P_j(\varsigma) \int_0^\tau P_i(x)dxP_j(\sigma), \quad \text{for } \forall \tau, \varsigma, \sigma \in [0, 1], \quad (3.21a) \]
\[ B_{\varsigma, \sigma} = \sum_{j=0}^{m-1} P_j(\varsigma)P_j(\sigma), \quad \text{for } \forall \varsigma, \sigma \in [0, 1], \quad (3.21b) \]
where \( P_i(x) \) are the shifted Legendre polynomials. Particularly, when \( m = 1 \), we have \( A_{\tau, \varsigma, \sigma} = \tau, B_{\varsigma, \sigma} = 1 \) and the resulting method can be formulated as (1.3).

**Theorem 3.3.** The Butcher coefficients given by (3.21) satisfy \( B(2m), C(m), \hat{C}(m), D(m-1) \) and \( \hat{D}(m-1) \).

**Proof.** Firstly, let us verify \( B(\xi) \) with \( \xi = 2m \). In (3.8a), the requirements \( 1 \leq k + l \leq 2m \) and \( k \geq 1, l \geq 0 \) imply that at least one of the following two cases must be happened: (1) \( 1 \leq k \leq m \); (2) \( 0 \leq l \leq m - 1 \). For the case (1): from the derivation of (3.14), it follows that (3.21b) (as a special case of (3.14)) satisfies (3.9), and thus by substituting (3.9) into (3.8a) it leads to
\[ \int_0^1 \int_0^1 (B_{\rho, \tau} \tau^{k-1}) \rho^l d\rho d\tau = \int_0^1 \rho^{k+l-1} d\rho = \frac{1}{k+l}, \quad k = 1, \cdots, m, \forall l \geq 0. \]
For the case (2): from (3.21b) it gives \( B_{\rho, \tau} = B_{\tau, \rho} \), then by using (3.9) we get
\[ \int_0^1 B_{\rho, \tau} \rho^l d\rho = \tau^l, \quad l = 0, \cdots, m - 1, \]
and thus
\[ \int_0^1 \int_0^1 (B_{\rho, \tau} \rho^l) \tau^{k-1} d\rho d\tau = \int_0^1 \tau^{k+l-1} d\tau = \frac{1}{k+l}, \quad l = 0, \cdots, m - 1, \forall k \geq 1. \]
This demonstrates that (3.8a) holds true for all \( k, l \) satisfying \( 1 \leq k + l \leq 2m \) and \( k \geq 1, l \geq 0 \).

\(^3\)Obviously, the choice of \( \gamma_i(\varsigma) \) is not unique, e.g., a more general choice is \( \gamma_i(\varsigma) = \lambda P_i(\varsigma) \) with \( \lambda \) a constant factor.
Secondly, we consider $C(m)$ and $\hat{C}(m)$. Obviously, we need to verify $C(m)$ only. This can be easily verified by substituting (3.10) into (3.8b), noticing that the formula (3.20) is replaced by
\begin{equation}
\tilde{A}_{r,\varsigma} = \sum_{i=0}^{m-1} P_i(\varsigma) \int_0^\tau P_i(x)dx.
\end{equation}

Finally, we verify $D(m - 1)$ and $\hat{D}(m - 1)$. Observing that (with the help of (2.17))
\begin{equation}
\int_0^1 B_{\rho,\tau} d\rho = \sum_{j=0}^{m-1} \int_0^1 P_j(\rho) d\rho P_j(\tau) = P_0(\tau) = 1,
\end{equation}
the left-hand side of (3.8d) becomes
\begin{equation}
\int_0^1 (\int_0^1 B_{\rho,\tau} d\rho) \tau^{k+l-1} A_{r,\varsigma,\sigma} d\tau = \int_0^1 \tau^{k+l-1} A_{r,\varsigma,\sigma} d\tau.
\end{equation}
By using (3.9), the left-hand side of (3.8d) coincides with (3.24), which means we should verify $\mathcal{D}(m - 1)$ only. It is seen that (3.8d) can be recast as
\begin{equation}
\int_0^1 \tau^{k+l-1} A_{r,\varsigma,\sigma} d\tau = B_{r,\varsigma,\sigma} \int_0^{\varsigma} x^{k+l-1} dx, \quad 1 \leq k + l \leq \zeta, \quad \varsigma, \sigma \in [0,1],
\end{equation}
which is equivalent to
\begin{equation}
\int_0^1 A_{r,\varsigma,\sigma} P_k(\tau) d\tau = B_{r,\varsigma,\sigma} \int_0^{\varsigma} P_k(x)dx, \quad k = 0, \ldots, \zeta - 1, \quad \varsigma, \sigma \in [0,1].
\end{equation}
Substituting (3.21) into (3.25), we find that it suffices to prove
\begin{equation}
\sum_{i=0}^{m-1} P_i(\varsigma) \int_0^1 \left( \int_0^\tau P_i(x) dx P_i(\tau) \right) d\tau = \int_0^1 P_i(x)dx, \quad k = 0, \ldots, m - 2,
\end{equation}
which can be verified by using (2.11) and (2.12).

**Theorem 3.4.** The enhanced continuous-stage method (2.1) with coefficients (3.21) is energy-preserving, symmetric and of order $2m$ ($m \geq 1$) but not preserves the quadratic Casimir function $C(y) = y^TDy$ (with a symmetric constant matrix $D$) of the system (1.1).

**Proof.** The energy-preserving property is straightforward from Theorem 2.3 and the order result is from Theorem 3.3 and Theorem 3.2. The symmetry of the method can be easily obtained by verifying the condition (2.20) with the help of (2.12) and the symmetry relation
\begin{equation}
P_i(1 - x) = (-1)^i P_i(x), \quad \forall i \geq 0,
\end{equation}
for the shifted Legendre polynomials.

\footnote{Remark that, if $m = 1$, then both conditions $\mathcal{D}(\varsigma)$ and $\hat{D}(\varsigma)$ are deemed to be not satisfied, since $\varsigma = 0$ is meaningless in (3.8d) as well as in (3.8e).}
From (3.21), we find that

\[ A_{\tau,\varsigma,\sigma} = \tilde{A}_{\tau,\varsigma} B_{\varsigma,\sigma}, \]  

(3.26)

where \( \tilde{A}_{\tau,\varsigma} \) is given by (3.22). By making difference of the two sides of (2.21) and substituting (3.26) into the resulting formulation, it yields

\[ B_{\rho,\tau} A_{\rho,\varsigma,\sigma} + B_{\varsigma,\sigma} A_{\varsigma,\rho,\tau} - B_{\rho,\tau} B_{\varsigma,\sigma} = B_{\rho,\tau} (\tilde{A}_{\rho,\varsigma} + \tilde{A}_{\varsigma,\rho} - 1). \]

Since \( B_{\rho,\tau} B_{\varsigma,\sigma} \neq 0 \) and \( \tilde{A}_{\rho,\varsigma} + \tilde{A}_{\varsigma,\rho} - 1 \neq 0 \) (by using (2.12)), thus the condition (2.21) is not satisfied.

Remark 3.1. If we use the Gaussian quadrature formula with \( s \) nodes (\( s = m - 1 \) or \( m \)) for approximating the integrals of (2.1), then the resulting method can exactly preserve the quadratic Casimir function. This is because the counterpart of the condition (2.21) for the resulting method becomes

\[ B_{c_i,c_j} A_{c_i,c_k,c_l} + B_{c_k,c_l} A_{c_k,c_i,c_j} = B_{c_i,c_j} B_{c_k,c_l}, \quad \forall \ i,j,k,l = 1, \ldots, s, \]

and one can verify that \( \tilde{A}_{c_i,c_j} + \tilde{A}_{c_j,c_i} - 1 = 0 \) (See the relevant analysis placed before the Remark 5.2 of [25], page 2170).

Corollary 3.1. If \( S(y) = S_0 \) is a constant matrix, then the enhanced continuous-stage method (2.1) with coefficients (3.21) reduces to the standard continuous-stage Runge-Kutta method

\[ Y_{\tau} = y_0 + h \int_{0}^{1} A_{\tau,\varsigma,\sigma} S_0 \nabla H(Y_{\sigma}) \, d\sigma, \quad \tau \in [0,1], \]

\[ y_1 = y_0 + h \int_{0}^{1} B_{\varsigma,\sigma} S_0 \nabla H(Y_{\sigma}) \, d\sigma, \]

where

\[ A_{\tau,\varsigma,\sigma} = \sum_{i=0}^{m-1} \int_{0}^{\tau} P_i(x) \, dx \, P_{1}(\sigma), \quad B_{\varsigma,\sigma} = 1. \]

(3.28)

Proof. When \( S(y) = S_0 \) is a constant matrix, (2.1) becomes

\[ Y_{\tau} = y_0 + h \int_{0}^{1} \left[ \int_{0}^{1} A_{\tau,\varsigma,\sigma} \, d\varsigma \right] S_0 \nabla H(Y_{\sigma}) \, d\sigma, \quad \tau \in [0,1], \]

\[ y_1 = y_0 + h \int_{0}^{1} \left[ \int_{0}^{1} B_{\varsigma,\sigma} \, d\varsigma \right] S_0 \nabla H(Y_{\sigma}) \, d\sigma. \]

Denote

\[ A_{\tau,\varsigma,\sigma} = \int_{0}^{1} A_{\tau,\varsigma,\sigma} \, d\varsigma, \quad B_{\varsigma,\sigma} = \int_{0}^{1} B_{\varsigma,\sigma} \, d\varsigma, \]

and substitute (3.21) into the formulas above, then it gives

\[ A_{\tau,\varsigma,\sigma} = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \int_{0}^{1} P_i(\varsigma) P_j(\varsigma) \, d\varsigma \int_{0}^{\tau} P_i(x) \, dx P_{j}(\sigma) \]

\[ = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \delta_{ij} \int_{0}^{\tau} P_i(x) \, dx P_{j}(\sigma) \]

\[ = \sum_{i=0}^{m-1} \int_{0}^{\tau} P_{i}(x) \, dx P_{i}(\sigma), \]
while $\mathcal{R}_\sigma = 1$ has been previously proved in (3.23).

**Remark 3.2.** The standard continuous-stage Runge-Kutta method (3.27) with coefficients (3.28) coincides with some existing energy-preserving methods for canonical Hamiltonian systems, including $m$-degree continuous time finite element method [25], infinite Hamiltonian boundary value methods HBVM$(\infty, m)$ [2] and energy-preserving collocation methods with the optimal order $2m$ [14]. Particularly, if $m = 1$, the method reduces to the average vector field method with order 2 [23].

Last but not least, for the numerical implementation of the methods given in Theorem 3.4, usually one has to approximate the integrals of (2.1) by a quadrature rule. A natural way is to use the form of standard partitioned Runge-Kutta schemes (3.2) and the numerical solutions can be obtained by iteration. An alternative way is based on the fact that $Y_{\tau}$ is a polynomial of degree $m$, which can be expressed in terms of $y_0$ and the internal stage values denoted by $Y_1, \cdots, Y_m$ ($y_1$ can be included). Therefore, we get a nonlinear system of equations associated with the unknowns $Y_1, \cdots, Y_m$ which can be solved by iteration.

4. Numerical experiments

In this section, we perform some numerical experiments to verify our theoretical analysis and show the numerical behaviors of our new methods. In the following, we apply the second order ($m = 1$, see also (1.3)) and fourth order integrator ($m = 2$) given in Theorem 3.4 for our experiments. Although the two integrals of (2.1) in terms of $\varsigma$ and $\sigma$ can be respectively approximated by using two different quadrature formulas, for simplicity, in the following we use the same quadrature formula for each method in each test problem.

4.1. Test problem I

Consider the following Euler’s equations [7]

\[
\begin{align*}
\dot{y}_1 &= (\alpha - \beta)y_2y_3, \\
\dot{y}_2 &= (1 - \alpha)y_1y_3, \\
\dot{y}_3 &= (\beta - 1)y_1y_2,
\end{align*}
\]

(4.1)

which describe the motion of a rigid body under no forces. Corresponding to the initial values $y(0) = (0, 1, 1)^T$ and the parameter values $\alpha = 1+1/\sqrt{1.51}$, $\beta = 1-0.51/\sqrt{1.51}$, the exact solution of the system is known as [7]

\[
y(t) = (\sqrt{1.51}\text{sn}(t, 0.51), \text{cn}(t, 0.51), \text{dn}(t, 0.51))^T,
\]

where sn, cn, dn represent the elliptic Jacobi functions. The Euler’s equations can be recast as the system (1.1) with [21]

\[
S(y) = \begin{pmatrix}
0 & \alpha y_3 & -\beta y_2 \\
-\alpha y_3 & 0 & y_1 \\
\beta y_2 & -y_1 & 0
\end{pmatrix},
\]

and

\[
H(y) = \frac{y_1^2 + y_2^2 + y_3^2}{2}.
\]

\footnote{For the sake of obtaining the “practical” energy preservation, it is suggested to use a Gaussian quadrature rule with a high enough degree of precision.}
Besides, one can easily verify that
\[ C(y) = \frac{y_1^2 + \beta y_2^2 + \alpha y_3^2}{2} \]
is a Casimir function of the system. For our experiment, we take the step size as \( h = 0.1 \) for computing 10,000 steps. To compute the integrals of the method (2.1), we have used the 2-point Gaussian quadrature formula. The time evolution of errors in terms of the two invariants \( H(y), C(y) \) are presented in Fig. 4.1 and 4.3 respectively. Clearly, Fig. 4.1 shows the energy-preserving property of our methods and Fig. 4.2 indicates that the numerical solutions are on the manifold of the constant Hamiltonian. Besides, from Fig. 4.3 we see that the quadratic Casimir function is preserved up to round-off along the numerical solutions. For the second-order method with the integrals approximated by Gaussian quadrature, it is equivalent to the scheme (1.2) with the same quadrature formula used for the current problem (4.1) (note that both \( S \) and \( H \) are polynomial functions), which explains the preservation of quadratic Casimir functions. While for the fourth-order method, the theoretical interpretation for the preservation of quadratic Casimir functions has been given in Remark 3.1. We also find that when 1-point Gaussian quadrature (namely midpoint rule) is used, the energy and quadrature Casimir function are also preserved up to round-off (not shown here) which coincides with Remark 3.1, but the order of the fourth order method reduces to 2 (due to the low degree of precision of the quadrature). Fig. 4.4 exhibits a linear growth of the global errors in terms of the numerical solutions.

4.2. Test problem II

Consider the 2-dimensional Lotka-Volterra system [13]
\[ \dot{u} = u(v - 2), \quad \dot{v} = v(1 - u), \] (4.2)
which can be written as the form of (1.1) with
\[ S(u, v) = \begin{pmatrix} 0 & -uv \\ uv & 0 \end{pmatrix}, \]
and
\[ H(u, v) = \ln u - u + 2 \ln v - v. \]
We take the initial values as \( (u(0), v(0)) = (1, 1) \) and use the step size \( h = 0.01 \) for computing 100,000 steps. To compute the integrals of the method (2.1), we have used the 4-point Gaussian quadrature formula. Fig. 4.5 clearly shows the energy-preserving property of our methods.

4.3. Test problem III

Consider the 3-dimensional Lotka-Volterra system [8]
\[ \dot{y}_1 = -\frac{1}{2} y_1 (y_2 - 2y_3 + 3), \quad \dot{y}_2 = y_2 (y_1 - 2y_3 + 2), \quad \dot{y}_3 = y_3 (-y_1 + y_2 + 1), \] (4.3)
which can be written as the form of (1.1) with
\[ S(y) = \begin{pmatrix} 0 & -\frac{1}{2} y_1 y_2 & \frac{1}{2} y_1 y_3 \\ \frac{1}{2} y_1 y_2 & 0 & -y_2 y_3 \\ -\frac{1}{2} y_1 y_3 & y_2 y_3 & 0 \end{pmatrix}, \]
Figure 4.1: Time evolution of the energy (Hamiltonian) errors of two methods for the Euler’s equations (4.1), with step size $h = 0.1$.

Figure 4.2: Time evolution of the numerical solutions along the energy surface by two methods for the Euler’s equations (4.1), with step size $h = 0.1$. 
Figure 4.3: Time evolution of the Casimir errors of two methods for the Euler’s equations (4.1), with step size $h = 0.1$.

Figure 4.4: Global solution errors of two methods for the Euler’s equations (4.1), with step size $h = 0.1$. 


and

\[ H(y) = 2y_1 + y_2 + 2y_3 + \ln y_2 - 2 \ln y_3. \]

Besides, the system possesses a Casimir function

\[ C(y) = 2 \ln y_1 + \ln y_2 + \ln y_3. \]

In our experiment, we take the same initial values as in [8] namely \( y(0) = (1.0, 1.9, 0.5)^T \) and integrate the system for 100,000 steps with step size \( h = 0.01 \). To compute the integrals of the method (2.1), we have used the 6-point Gaussian quadrature formula. The numerical results are presented in Fig. 4.6-4.7 from which we can observe that the energy is preserved up to nearly round-off along the numerical solutions. However, the error of Casimir function shows a linear drift. Therefore, the newly-derived methods are not conjugate to Poisson integrators, since for a poisson integrator all the Casimir functions should be nearly preserved without drift [8].

5. Concluding remarks

This paper deals with the energy-preserving integration for non-canonical Hamiltonian systems by developing a new type of continuous-stage methods which is referred to as the enhanced continuous-stage methods. The conditions for energy preservation of the methods are presented and by combining such conditions with the order conditions, the construction of energy-preserving integrators is examined. For the construction, the orthogonal expansion technique associated with Legendre polynomials is fully utilized. As a result, we derive a new class of integrators which
Figure 4.6: Time evolution of the energy (Hamiltonian) errors of two methods for the 3-dimensional Lotka-Volterra system (4.3), with step size $h = 0.01$.

Figure 4.7: Time evolution of the Casimir errors of two methods for the 3-dimensional Lotka-Volterra system (4.3), with step size $h = 0.01$. 
is energy-preserving, symmetric and of order $2m$. This class of integrators turns out to be an extension of the existing energy-preserving integrators for solving canonical Hamiltonian systems.

It is known that a system of ordinary differential equations $\dot{y} = f(y)$ with a first integral $I$ can be rewritten as a linear-gradient system [18, 19], namely the system in the form (1.1) with $H$ replaced by $I$, therefore the derived methods in this paper can also be used for first-integral-preserving integration of those systems with a known first integral.

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