Triple sets of $\chi^3$-summable sequences of fuzzy numbers defined by an Orlicz function

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Abstract: In this paper we introduce the $\chi^3$ fuzzy numbers defined by an Orlicz function and study some of their properties and inclusion results.

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1. Introduction
A triple sequence (real or complex) can be defined as a function $x: \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers, respectively.

Some initial work on double series is found in Apostol (1978), Alzer, Karayannakis, and Srivastava (2006), Bor, Srivastava, and Sulaiman (2012), Choi and Srivastava (1991), Liu and Srivastava (2006) and double sequence spaces are found in Hardy (1917), Deepmala Subramanian, and Mishra (in press), Deepmala, Mishra, and Subramanian (2016) and many others. Later on some initial work

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PUBLIC INTEREST STATEMENT
In this paper, we introduced the $\chi^3$ fuzzy numbers defined by an Orlicz function and study some of their properties with inclusion results. Furthermore we provided an example of triple sequence of gai which is not symmetric, not solid, not monotone and not convergent free.

Our result unifies the results of several author’s in the case of classical Orlicz spaces. One can extend our results for more general spaces.
on triple sequence spaces is found in Sahiner, Gurdal, and Duden (2007), Esi (2014), Esi and Necdet Catalbas (2014), Esi and Savas (2015), Subramanian and Esi (2015) and many others.

A sequence \( x = (x_{mnk}) \) is said to be triple analytic if \( \sup_{m,n,k}|x_{mnk}|^{1/mnk} < \infty \). The vector space of all triple analytic sequences are usually denoted by \( \Lambda^3 \).

A sequence \( x = (x_{mnk}) \) is called triple entire sequence if \( |x_{mnk}|^{1/mnk} \to 0 \) as \( m, n, k \to \infty \).

A sequence \( x = (x_{mnk}) \) is called triple chi sequence if \( \left((m+n+k)!|x_{mnk}|\right)^{1/mnk} \to 0 \) as \( m, n, k \to \infty \). The triple chi sequences will be denoted by \( \chi^3 \).

This paper deals with introducing the \( \chi^3 \)-fuzzy number defined by an Orlicz function and study some topological properties, inclusion relations and give some examples. Some interesting results may be seen in Alzer et al. (2006), Bor et al. (2012), Choi and Srivastava (1991), Liu and Srivastava (2006).

2. Definitions and preliminaries

**Definition 2.1** An Orlicz function (see Kamthan & Gupta, 1981) is a function \( M: [0, \infty) \to [0, \infty) \) which is continuous, non-decreasing and convex with \( M(0) = 0 \), \( M(x) > 0 \), for \( x > 0 \) and \( M(x) \to \infty \) as \( x \to \infty \). If convexity of Orlicz function \( M \) is replaced by \( M(x+y) \leq M(x) + M(y) \), then this function is called modulus function. Lindenstrauss and Tzafriri (1971) used the idea of Orlicz function to construct Orlicz sequence space.

Throughout a triple sequence is denoted by \( \langle X_{mnk} \rangle \), a triple infinite array of fuzzy real numbers.

Let \( D \) denote the set of all closed and bounded intervals \( X = [a_1, a_2, a_3] \) on the real line \( \mathbb{R} \). For \( X = [a_1, a_2, a_3] \in D \) and \( Y = [b_1, b_2, b_3] \in D \), define

\[
d(x, y) = \max(|a_1 - b_1|, |a_2 - b_2|, |a_3 - b_3|)
\]

It is known that \((D, d)\) is a complete metric space.

A fuzzy real number \( X \) is a fuzzy set on \( \mathbb{R} \), that is, a mapping \( X: \mathbb{R} \times \mathbb{R} \rightarrow I \times I \times I \) associating each real number \( t \) with its grade of membership \( X(t) \).

The \( \alpha \)-level set \( [X]^\alpha \), of the fuzzy real number \( X \), for \( 0 < \alpha \leq 1 \); is defined by

\[
[X]^\alpha = \{ t \in \mathbb{R}: X(t) \geq \alpha \}.
\]

The 0-level set is th closure of the strong 0-cut that is, \( \{ t \in \mathbb{R}: X(t) > 0 \} \).

A fuzzy real number \( X \) is called convex if \( X(t) \geq X(s) \land X(r) \land X(v) = \min(X(s), X(r), X(v)) \), where \( s < t < r < v \). If there exists \( t_0 \in \mathbb{R} \) such that \( X(t_0) = 1 \) then, the fuzzy real number \( X \) is called normal.

A fuzzy real number \( X \) is said to be upper-semi continuous if, for each \( \varepsilon > 0 \), \( X^{-1}(\{0, \alpha + \varepsilon\}) \) is open in the usual topology of \( \mathbb{R} \) for all \( \alpha \in I \).

The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by \( L(\mathbb{R}) \).

The absolute value, \(|X|\) of \( X \in L(\mathbb{R}) \) is defined by

\[
|X|(t) = \begin{cases} 
\max(X(t), X(-t)), & \text{if } t \geq 0; \\
0, & \text{if } t < 0 
\end{cases}
\]
Let \( d: L(R) \times L(R) \times L(R) \rightarrow R \times R \times R \) be defined by
\[
\vartheta(X,Y) = \sup_{0 \leq t \leq 1} d([X]^t, [Y]^t).
\]
Then, \( d \) defines a metric on \( L(R) \) and it is well-known that \( (L(R), d) \) is a complete metric space.

A sequence \( \{X_{mnk}\} \subseteq L(R) \) is said to be null if \( d(X_{mnk}, 0) = 0 \).

A triple sequence \( \{X_{mnk}\} \) of fuzzy real numbers is said to be gai in Pringsheim’s sense to a fuzzy number \( 0 \) if \( \lim_{m,n,k \rightarrow \infty} \{(m + n + k)X_{mnk}\}^{1/m+n+k} = 0 \).

A triple sequence \( \{X_{mnk}\} \) is said to \( \chi \) regularly if it converges in the Pringsheim’s sense and the following limits hold:
\[
\lim_{m,n,k \rightarrow \infty} \{(m + n + k)X_{mnk}\}^{1/m+n+k} = 0 \text{ for each } m, n, k \in \mathbb{N}.
\]

A fuzzy real-valued double sequence space \( E^f \) is said to be solid if \( \{Y_{mnk}\} \in E^f \) whenever \( \{X_{mnk}\} \in E^f \) and \( |Y_{mnk}| \leq |X_{mnk}| \) for all \( m, n, k \in \mathbb{N} \).

Let \( K = \{(m, n, k) \in \mathbb{N}; m_1 < m_2 < m_3 \ldots \text{ and } n_1 < n_2 < n_3 \ldots \text{ and } k_1 < k_2 < k_3 < \ldots \} \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} \) and \( E^f_{X} \) be a triple sequence space. A \( K \)-step space of \( E^f \) is a sequence space \( E^f_{X} = \{X_{mnk} \in w^{3f}; \{X_{mnk}\} \in E^f \} \).

A canonical pre-image of a sequence \( \{X_{mnk}\} \in E^f \) is a sequence \( \{Y_{mnk}\} \) defined as follows:
\[
Y_{mnk} = \begin{cases} 
X_{mnk}, & \text{if } (m,n,k) \in K, \\
0, & \text{otherwise}.
\end{cases}
\]

A canonical pre-image of a step space \( E^f_{X} \) is a set of canonical pre-images of all elements in \( E^f_{X} \).

A sequence set \( E^f \) is said to be monotone if \( E^f \) contains the canonical pre-images of all its step spaces.

A sequence set \( E^f \) is said to be symmetric if \( \{X_{\pi(m,n,k)}\} \in E^f \) whenever \( \{X_{mnk}\} \in E^f \), where \( \pi \) is a permutation of \( \mathbb{N} \times \mathbb{N} \times \mathbb{N} \).

A fuzzy real-valued sequence set \( E^f \) is said to be convergent free if \( \{Y_{mnk}\} \in E^f \) whenever \( \{X_{mnk}\} \in E^f \) and \( X_{mnk} = 0 \) implies \( Y_{mnk} = 0 \).

We define the following classes of sequences:
\[
X^f_{f} = \left\{ \{X_{mnk}\}; \sup_{mnk} f \left( \vartheta \left( X_{mnk}, 0 \right) \right) < \infty, X_{mnk} \in L(R) \right\}.
\]
\[
X^{3f}_{f} = \left\{ \{X_{mnk}\}; \lim_{mnk \rightarrow \infty} f \left( \vartheta \left( \{(m + n + k)X_{mnk}\}^{1/m+n+k}, 0 \right) \right) = 0 \right\}.
\]

Also, we define the classes of sequences \( X^{3f}_{f} \) as follows:

A sequence \( \{X_{mnk}\} \in X^{3f}_{f} \) if \( \{X_{mnk}\} \in X^f_{f} \) and the following limits hold:
\[
\lim_{m \rightarrow \infty} f \left( \vartheta \left( \{(m + n + k)X_{mnk}\}^{1/m+n+k}, 0 \right) \right) = 0 \text{ for each } m \in \mathbb{N}.
\]
\[
\lim_{n \rightarrow \infty} f \left( \vartheta \left( \{(m + n + k)X_{mnk}\}^{1/m+n+k}, 0 \right) \right) = 0 \text{ for each } n \in \mathbb{N}.
\]
\[
\lim_{k \rightarrow \infty} f \left( \vartheta \left( \{(m + n + k)X_{mnk}\}^{1/m+n+k}, 0 \right) \right) = 0 \text{ for each } k \in \mathbb{N}.
\]
3. Main results

**Theorem 3.1** Let \( N = \min \left\{ N_0; \sup_{m,n,k \geq 0} f \left( d \left( ((m + n + k) - X_{mnk}) Y_{mnk} \right)^{1/m+n+k}, 0 \right) \right\} \)

\[ N = \min \left\{ N_0; \sup_{m,n,k \geq 0} P_{mnk} < \infty \right\} \text{ and } N = \max \left( N_1, N_2, N_3 \right). \]

(i) \( X_{s}^{sp} \) is not a paranormed space with

\[ g(X) = \lim_{N \to \infty} \sup_{n \geq N} f \left( d \left( ((m + n + k) - X_{mnk}) Y_{mnk} \right)^{1/m+n+k}, 0 \right) \]

(ii) \( X_{s}^{sp} \) is complete with the paranorm (3.1).

**Proof**

(i) **Necessity**: Let \( X_{s}^{sp} \) be a paranormed space with (3.1) and suppose that \( \mu = 0 \). Then \( a = \inf_{m,n,k \geq 0} P_{mnk} = 0 \) for all \( N \in \mathbb{N} \) and \( g(\lambda X) = \lim_{N \to \infty} \sup_{m,n,k \geq N} |\lambda|^{p_{mnk}} = 1 \) for all \( \lambda \in (0, 1] \), where \( X = \langle a \rangle \in X_{s}^{sp} \) whence \( \lambda \to 0 \) does not imply \( \lambda X \to \theta \), when \( X \) is fixed. But this contradicts to (3.1) to be a paranorm.

(ii) **Sufficiency**: Let \( \mu > 0 \). It is trivial that \( g(0) = 0, g(-X) = g(X) \) and \( g(X + Y + Z, \theta) \leq g(X, \theta) + g(Y, \theta) + g(Z, \theta) \). Since \( \mu > 0 \) there exists a positive number \( \beta \) such that \( P_{mnk} > \beta \) for sufficiently large positive integer \( m, n, k \). Hence for any \( \lambda \in \mathbb{C} \), we may write \( |\lambda|^{p_{mnk}} \leq \max(|\lambda|^{n}, |\lambda|^{k}) \) for sufficiently large positive integers \( m, n, k \) \( \geq N \). Therefore, we obtain \( g(\lambda X, \theta) \leq \max(|\lambda|^{n}, |\lambda|^{k}) g(X) \). Using this, one can prove that \( \lambda X \to \theta \), whenever \( X \) is fixed and \( \lambda \to 0 \) or \( \lambda \to x \) and \( X \to \theta \), or \( \lambda \) is fixed and \( X \to \theta \).

Because a paranormed space is a vector space, \( X_{s}^{sp} \) is a set of sequences of fuzzy numbers. But the set \( w_{f} = \{ (X_{mnk}) \}_{X_{mnk} \in \mathcal{F}(R)} \) of all sequences of fuzzy numbers is not a vector space. That is, why, in order to say that \( X_{s}^{sp} \) is a subspace (that is a sequence space) it is not sufficient to show that \( X_{s}^{sp} \) is closed under addition and scalar multiplication. Consequently since \( w_{f} \) is not a vector space, then \( X_{s}^{sp} \) is not a vector subspace so that it is not a sequence space. Therefore it cannot be a paranormed space.

**Proof**

(ii) Let \( \langle X^{kr} \rangle \) be a Cauchy sequence in \( X_{s}^{sp} \), where \( X^{kr} = \langle X^{kr}_{mnk} \rangle_{m,n,k \in \mathbb{N}} \). Then for every \( \varepsilon > 0 \), \( 0 < \varepsilon < 1 \), there exists a positive integer \( s_{0} \) such that

\[ g(X^{kr} - X^{\tau r}) = \lim_{N \to \infty} \sup_{n \geq N} f \left( d \left( ((m + n + k) - X^{kr}_{mnk}) Y^{\tau r}_{mnk} \right)^{1/m+n+k}, 0 \right) \]

\[ P_{mnk} < \frac{\varepsilon}{2} \] (3.2)

for all \( k, r, t > s_{0} \).

By (3.2) there exists a positive integer \( n_{0} \) such that

\[ \sup_{n \geq N} f \left( d \left( ((m + n + k) - X^{kr}_{mnk}) Y^{\tau r}_{mnk} \right)^{1/m+n+k}, 0 \right) \]

\[ P_{mnk} < \frac{\varepsilon}{2} \] (3.3)

for all \( k, r, t > s_{0} \) and for \( N > n_{0} \). Hence we obtain

\[ f \left( d \left( ((m + n + k) - X^{kr}_{mnk}) Y^{\tau r}_{mnk} \right)^{1/m+n+k}, 0 \right) \]

\[ P_{mnk} < \frac{\varepsilon}{2} < 1 \] (3.4)

so that

\[ f \left( d \left( ((m + n + k) - X^{kr}_{mnk}) Y^{\tau r}_{mnk} \right)^{1/m+n+k}, 0 \right) \]

\[ f \left( d \left( ((m + n + k) - X^{kr}_{mnk}) Y^{\tau r}_{mnk} \right)^{1/m+n+k}, 0 \right) \]

\[ P_{mnk} < \frac{\varepsilon}{2} \] (3.5)
for all $k, \ell, t > s_0$. This implies that $\left\{ x_{mnk}^{k/\ell} \right\}_{k/\ell \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for each fixed $m, n, k \geq n_0$. Hence the sequence $\left\{ x_{mnk}^{k/\ell} \right\}_{k/\ell \in \mathbb{N}}$ is convergent to $x_{mnk}$ say,

$$\lim_{k/\ell \to \infty} x_{mnk}^{k/\ell} = x_{mnk} \quad \text{for each fixed } m, n, k > n_0. \quad (3.6)$$

Getting $x_{mnk}$, we define $X = \left\langle x_{mnk} \right\rangle$. From (3.2) we obtain

$$g(\lambda - X) = \limsup_{N \to \infty} f\left( \left( \frac{\left( (m + n + k) \right)^{1/(m + n + k)}}{\left( m + n + k \right)^{1/(m + n + k)}} \right)^{\frac{1}{m + n + k}} , \hat{0} \right) < \frac{\epsilon}{2} \quad (3.7)$$

as $r, t \to \infty$, for all $k, \ell, r, t > s_0$. By (3.6). This implies that $\lim_{k/\ell \to \infty} x_{mnk}^{k/\ell} = X$. Now we show that $X = \left\langle x_{mnk} \right\rangle \in \Lambda_1^{\mathbb{Z}}$. Since $x_{mnk}^{k/\ell} \in \Lambda_1^{\mathbb{Z}}$ for each $(k,1) \in \mathbb{N} \times \mathbb{N}$ for every $\epsilon > 0 (0 < \epsilon < 1)$ there exists a positive integer $n_1 \in \mathbb{N}$ such that

$$f\left( \left( (m + n + k) \right)^{1/(m + n + k)} , \hat{0} \right) < \frac{\epsilon}{2} \quad \text{for every } m, n, k > n_1. \quad (3.8)$$

By (3.6) and (3.7) we obtain

$$f\left( \left( (m + n + k) \right)^{1/(m + n + k)} , \hat{0} \right) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for } k, \ell > 1 \quad \text{and } m, n, k > \max (n_0, n_1).$$

This implies that $X \in \Lambda_1^{\mathbb{Z}}$.

**Proposition 3.2** The class of sequences $\Lambda_1^{\mathbb{Z}}$ is symmetric but the classes of sequences $\Lambda_1^{\mathbb{Z}}$ and $\Lambda_1^{\mathbb{Z}}$ are not symmetric.

**Proof** Obviously the class of sequences $\Lambda_1^{\mathbb{Z}}$ is symmetric. For the other classes of sequences, consider the following example.

**Example** Consider the class of sequences $\Lambda_1^{\mathbb{Z}}$. Let $f(X) = X$ and consider the sequence $\left\langle x_{mnk} \right\rangle$ be defined by

$$X_{1ns}(t) = \begin{cases} \frac{(-1)^{m+n+k}}{t} & \text{for } t = -1, \\ \frac{(-1)^{m+n+k}}{t} & \text{for } t = 1, \\ 0 & \text{otherwise.} \end{cases}$$

and for $m > 1$,

$$X_{mnk}(t) = \begin{cases} \frac{(-1)^{m+n+k}}{t} & \text{for } t = -2, \\ \frac{(-1)^{m+n+k}}{t} & \text{for } t = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\left\langle y_{mnk} \right\rangle$ be a rearrangement of $\left\langle x_{mnk} \right\rangle$ defined by

$$y_{mnk}(t) = \begin{cases} \frac{(-1)^{m+n+k}}{t} & \text{for } t = -1, \\ \frac{(-1)^{m+n+k}}{t} & \text{for } t = 1, \\ 0 & \text{otherwise.} \end{cases}$$
and for $m \neq n \neq k$,

$$Y_{mnk}(t) = \begin{cases} \frac{(t+2)^{m+n+k}}{(m+n+k)!}, & \text{for } t = -2, \\ \frac{(t-1)^{m+n+k}}{(m+n+k)!}, & \text{for } t = -1, \\ 0, & \text{otherwise}. \end{cases}$$

Then, $\langle X_{mnk} \rangle \in \mathcal{X}^F_J$ but $\langle Y_{mnk} \rangle \notin \mathcal{X}^F_J$. Hence, $X_{mnk}^F$ is not symmetric. Similarly other sequences are also not symmetric.

**Proposition 3.3** The classes of sequences $\Lambda^F_J, \mathcal{X}^F_J$ and $\mathcal{X}^{3F}_J$ are solid.

**Proof** Consider the class of sequences $\mathcal{X}^F_J$. Let $\langle X_{mnk} \rangle$ and $\langle Y_{mnk} \rangle \in \mathcal{X}^F_J$ be such that $d\left( \left( (m+n+k)!Y_{mnk} \right)^{1/(m+n+k)}, 0 \right) \leq d\left( (m+n+k)!X_{mnk}, 0 \right)$. As $f$ is non-decreasing, we have

$$\lim_{n \to \infty} f\left( d\left( (m+n+k)!Y_{mnk} \right)^{1/(m+n+k)}, 0 \right) \leq \lim_{n \to \infty} f\left( d\left( (m+n+k)!X_{mnk}, 0 \right) \right)$$

Hence, the class of sequence $\mathcal{X}^F_J$ is solid. Similarly it can be shown that the other classes of sequences are also solid.

**Proposition 3.4** The classes of sequences $\mathcal{X}^{3F}_J$ and $\mathcal{X}^{3F}_J$ are not monotone and hence not solid.

**Proof** The result follows from the following example.

**Example** Consider the class of sequences $\mathcal{X}^{3F}_J$ and $\mathcal{X}^{3F}_J$. Let $J = \{(m, n, k) : m \geq n \geq k \} \subseteq N \times N \times N$. Let $\langle X_{mnk} \rangle$ be defined by

$$X_{mnk}(t) = \begin{cases} \frac{(t+3)^{m+n+k}}{(m+n+k)!}, & \text{for } -3 < t \leq -2, \\ \frac{(3m)^{m+n+k}}{(m+n+k)!}, & \text{for } -2 \leq t \leq -1 + \frac{1}{m}, \\ 0, & \text{otherwise}. \end{cases}$$

for all $m, n, k \in N$.

Then $\langle X_{mnk} \rangle \in \mathcal{X}^{3F}_J$. Let $\langle Y_{mnk} \rangle$ be the canonical pre-image of $\langle X_{mnk} \rangle$ for the subsequence $J$ of $N \times N \times N$. Then

$$Y_{mnk} = \begin{cases} X_{mnk}, & \text{for } (m, n, k) \in J, \\ 0, & \text{otherwise}. \end{cases}$$

Then, $\langle Y_{mnk} \rangle \notin \mathcal{X}^{3F}_J$. Hence $\mathcal{X}^{3F}_J$ is not monotone. Similarly, it can be shown that the other classes of sequences are also not monotone. Hence, the classes of sequences $\mathcal{X}^F_J$ and $\mathcal{X}^{3F}_J$ are not solid.

**Proposition 3.5** (i) $\mathcal{X}^{3F}_J \cap \mathcal{X}^{3F}_J \subseteq \mathcal{X}^{3F}_J \cap \mathcal{X}^{3F}_J \subseteq \mathcal{X}^{3F}_J \cap \mathcal{X}^{3F}_J$, (ii) $\mathcal{X}^{3F}_J \cap \mathcal{X}^{3F}_J \subseteq \mathcal{X}^{3F}_J \cap \mathcal{X}^{3F}_J \subseteq \mathcal{X}^{3F}_J \cap \mathcal{X}^{3F}_J$

**Proof** It is easy, so omitted.

**Proposition 3.6** Let $f, f_1$ and $f_2$ be three Orlicz functions, then, (i) $\mathcal{X}^{3F}_J \subseteq \mathcal{X}^{3F}_J$, (ii) $\mathcal{X}^{3F}_J \subseteq \mathcal{X}^{3F}_J$, (iii) $\Lambda^{3F}_J \subseteq \Lambda^{3F}_J$

**Proof** We prove the result for the case $\mathcal{X}^{3F}_J \subseteq \mathcal{X}^{3F}_J$, the other cases are similar. Let $\epsilon > 0$ be given. As $f$ is continuous and non-decreasing, so there exists $\eta > 0$, such that $f(\eta) = \epsilon$. Let $\langle X_{mnk} \rangle \in \mathcal{X}^{3F}_J$. Then, there exist $m_0, n_0, k_0 \in N$, such that
\[ f_1\biggl( d\big((m + n + k)X_{mnk}\big)^{1/m+n+k}, \hat{0}\biggr) < \eta, \quad \text{for all } m \geq m_0, n \geq n_0, k \geq k_0 \]
\[ \Rightarrow f \circ f \circ f_1\biggl( d\big((m + n + k)X_{mnk}\big)^{1/m+n+k}, \hat{0}\biggr) < \varepsilon, \quad \text{for all } m \geq m_0, n \geq n_0, k \geq k_0. \]

Hence, \( \langle X_{mnk} \rangle \in \mathcal{X}_{1f/f/1f} \). Thus, \( \mathcal{X}_{1f} \subseteq \mathcal{X}_{1f/f/1f} \).

**Proposition 3.7** (i) \( \mathcal{X}_{1f} \subseteq \Lambda_{1f}^{3F} \), (ii) \( \mathcal{X}_{1f} \subseteq \Lambda_{1f}^{3F} \), the inclusions are strict.

**Proof** The inclusion (i) \( \mathcal{X}_{1f} \subseteq \Lambda_{1f}^{3F} \) (ii) \( \mathcal{X}_{1f} \subseteq \Lambda_{1f}^{3F} \) is obvious. For establishing that the inclusions are proper, consider the following example.

**Example** We prove the result for the case \( \mathcal{X}_{1f} \subseteq \Lambda_{1f}^{3F} \), the other case similar. Let \( f(X) = X \). Let the sequence \( \langle X_{mnk} \rangle \) be defined by for \( m > n > k \),
\[
X_{mnk}(t) = \begin{cases} 
\frac{m-1}{m+n+k} \left(1-(m+n+k)\right), & \text{for } 1 + \frac{1}{m} \leq t \leq 2, \\
\frac{m-1}{m+n+k} \left(1-(m+n+k)\right), & \text{for } 2 < t \leq 3, \\
0, & \text{otherwise.}
\end{cases}
\]

and for \( m < n < k \),
\[
X_{mnk}(t) = \begin{cases} 
\frac{m-1}{m+n+k} \left(1-(m+n+k)\right), & \text{for } \frac{1}{m} \leq t \leq 1, \\
\frac{m-1}{m+n+k} \left(1-(m+n+k)\right), & \text{for } 1 \leq t \leq 2, \\
0, & \text{otherwise.}
\end{cases}
\]

Then, \( \langle X_{mnk} \rangle \in \Lambda_{1f}^{3F} \) but \( \langle X_{mnk} \rangle \notin \mathcal{X}_{1f} \).

**Proposition 3.8** The classes of sequences \( \Lambda_{1f}^{3F} \), \( \mathcal{X}_{1f}^{3F} \) and \( \mathcal{X}_{1f}^{3F} \) are not convergent free.

**Proof** The result follows from the following example.

**Example** Consider the classes of sequences \( \mathcal{X}_{mnk}^{3F} \). Let \( f(X) = X \) and consider the sequence \( \langle X_{mnk} \rangle \) defined by \( (1 + n + k)X_{1nk}^{1/n+k} = 0 \), and for other values,
\[
X_{mnk}(t) = \begin{cases} 
\frac{1}{m+n+k} \left(1-n\right), & \text{for } 0 \leq t \leq 1, \\
\frac{1}{m+n+k} \left(1-n\right), & \text{for } 1 < t \leq 2 + \frac{1}{m}, \\
0, & \text{otherwise.}
\end{cases}
\]

Let the sequence \( \langle Y_{mnk} \rangle \) be defined by \( (1 + n + k)Y_{1nk}^{1/n+k} = 0 \), and for other values,
\[
Y_{mnk}(t) = \begin{cases} 
\frac{1}{m+n+k} \left(1-n\right), & \text{for } 0 \leq t \leq 1, \\
\frac{1}{m+n+k} \left(1-n\right), & \text{for } 1 < t \leq m, \\
0, & \text{otherwise.}
\end{cases}
\]

Then, \( \langle X_{mnk} \rangle \in \mathcal{X}_{1f}^{3F} \) but \( \langle Y_{mnk} \rangle \notin \mathcal{X}_{1f}^{3F} \). Hence, the classes of sequences \( \mathcal{X}_{1f}^{3F} \) are not convergent free. Similarly, the other spaces are also not convergent free.

**4. Conclusion** The \( g^2 \) fuzzy numbers defined by an Orlicz function and discuss inclusion relation. Furthermore, the given example of triple sequence of gai is not symmetric, not solid, not monotone and not convergent free.
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References
Alzer, H., Karayannakis, D., & Srivastava, H. M. (2006). Series representations for some mathematical constants. Journal of Mathematical Analysis and Applications, 320, 145–162.

Apostol, T. (1978). Mathematical analysis. London: Addison-Wesley.

Bor, H., Srivastava, H. M., & Sulaíman, W. T. (2012). A new application of certain generalized power increasing sequences. Filomat, 26, 871–879. doi:10.2298/FIL1204871B

Choi, J., & Srivastava, H. M. (1991). Certain classes of series involving the Zeta function. Journal of Mathematical Analysis and Applications, 151, 111–117.

Deepmala, Mishra, L. N., & Subramanian, N. (2016). Characterization of some Lacunary \( \chi \)-almost \( \chi \)-convergence of order \( p \) with \( p \)-metric defined by sequence of moduli Musielak. Applied Mathematics & Information Sciences Letters, 4(1).

Deepmala, Subramanian, N., & Mishra, V. N. (in press). Double almost \( (\xi,\mu) \) in \( X \)-Riesz space. Southeast Asian Bulletin of Mathematics.

Esi, A. (2014). On some triple almost lacunary sequence spaces defined by Orlicz functions. Research and Reviews: Discrete Mathematical Structures, 1, 16–25.

Esi, A., & Nectad Calabos, M. (2014). Almost convergence of triple sequences. Global Journal of Mathematical Analysis, 2, 6–10.

Esi, A., & Savas, E. (2015). On lacunary statistically convergent triple sequences in probabilistic normed space. Applied Mathematics & Information Sciences, 9, 2529–2534.

Hardy, G. H. (1917). On the convergence of certain multiple series. Proceedings of the Cambridge Philosophical Society, 19, 86–95.

KanTan, P. K., & Gupta, M. (1981). Sequence spaces and series. Lecture notes, Pure and Applied Mathematics. New York, NY: 65 Marcel Dekker Inc.

Lindenstrauss, J., & Tzafiri, L. (1971). On Orlicz sequence spaces. Israel Journal of Mathematics, 10, 379–390.

Liu, G. D., & Srivastava, H. M. (2006). Explicit formulas for the Nörlund polynomials \( B_{\alpha}^{\beta} \) and \( B_{\alpha}^{\beta} \). Computers and Mathematics with Applications, 51, 1377–1384.

Sahiner, A., Gurdal, M., & Duden, F. K. (2007). Triple sequences and their statistical convergence. Selcuk Journal of Applied Mathematics, 8, 49–55.

Subramanian, N., & Esi, A. (2015). Some new semi-normed triple sequence spaces defined by a sequence of moduli. Journal of Analysis & Number Theory, 3, 79–88.