On PMU Data Integrity under GPS Spoofing Attacks: A Sparse Error Correction Framework

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Abstract—Consider the problem of mitigating the impact on data integrity of phasor measurement units (PMUs) given a GPS spoofing attack. We present a sparse error correction framework to treat PMU measurements that are potentially corrupted due to a GPS spoofing attack. We exploit the sparse nature of a GPS spoofing attack, which is that only a small fraction of PMUs are affected by the attack. We first present attack identifiability conditions (in terms of network topology, PMU locations, and the number of spoofed PMUs) under which data manipulation by the spoofing attack is identifiable. The identifiability conditions have important implications on how the locations of PMUs affect their resilience to GPS spoofing attacks. To effectively correct spoofed PMU data, we present a sparse error correction approach wherein computation tasks are decomposed into smaller zones to ensure scalability. We present experimental results obtained from numerical simulations with the IEEE RTS-96 test network to demonstrate the effectiveness of the proposed approach.

Index Terms—Phasor measurement unit, GPS spoofing attack, sparse error correction

I. INTRODUCTION

PHASOR measurement units (PMUs), which are equipped with clocks synchronized by global positioning systems (GPS), or, more broadly, global navigation satellite systems (GNSS), provide direct measurements of voltage and current phasors at a much faster rate than the legacy SCADA system [1]. Due to this enriched measurement quality, there has been a wide interest in developing approaches to leverage PMU measurements for real-time power grid monitoring, protection, and control [2], [3]. While several PMU-based approaches showed improved performance compared to the legacy approaches [4], [5], those promises can be realized only if the data integrity of PMUs can be ensured.

Compared to legacy measurement devices, PMUs are equipped with more sophisticated security protocols, and thus it is considered difficult for adversaries to tamper with the data by directly compromising data authentication protocols of PMUs [6]. Nevertheless, the dependency of PMUs on civilian GPS signals for clock synchronization renders PMU measurements vulnerable to GPS signal spoofing attacks, which can be successfully launched by an adversary with small resource demand. The cyber attackers can easily deploy GPS transmitters to broadcast counterfeit GPS signals, which can manipulate the time estimation at the target PMU’s GPS receiver [7], [8]. Erroneous time reference successively induces errors in phase angle measurements of the tampered PMU. In practice, an attacker with limited resources can spoof only a few PMUs at a time. Hence the impact of spoofing attacks on PMU measurements is sparse in nature, i.e., we can assume the fraction of PMU measurements corrupted by the spoofing attacks to be small.

In this paper, we exploit the sparse nature of the GPS spoofing attack to recover affected PMU measurements. We first derive identifiability conditions for GPS spoofing attacks, under which a spoofing attack is fundamentally identifiable. This identifiability assessment portrays the vulnerability of the PMU network to GPS spoofing attacks and can be used to determine PMU placement that is resilient to GPS spoofing attacks. Then, we develop a sparse error correction algorithm to effectively correct potentially spoofed PMU measurements. The decomposability of the PMU measurement model is leveraged to make major steps of the algorithm performed based on only local models and measurements thereby ensuring the scalability of the algorithm. The experiment results on IEEE RTS-96 test network show that the proposed approach outperforms other benchmark algorithms when the number of spoofed PMUs is moderate.

A. Related work

Several approaches have been proposed in the literature to enhance the resilience of GPS time estimation procedure at a single PMU against spoofing attacks by developing a new GPS receiver architecture [9]–[11] or a new robust time estimation algorithm [12], [13]. Gong et al. in [9] proposed a spoofing detection algorithm by leveraging multiple GPS receivers per PMU, while authors in [10] exploit the networked and static nature of PMUs in close proximity, to propose a robust receiver architecture. Also in [11] the authors leverage the characteristics of a static receiver network to constrain the adversary’s freedom of GPS spoofing. The main focus of all the aforementioned techniques is on designing robust receiver architectures that harden the spoofing attacks. On the other hand, authors in [12], [13] propose robust time estimation techniques such that time estimation in PMUs become resilient to spoofing attacks. Authors in [12] propose a direct time estimation technique using the maximum likelihood approach. Work in [13] couples this time estimation technique with spatially dispersed multiple-receivers to improve the resilience against spoofing attacks. All the strategies in this category either require additional infrastructure in terms of external clocks and multiple GPS receiver antennas or require a network of GPS receivers in the vicinity of the PMU of interest.

The aforementioned works focused on robustifying the time estimation procedure at a single PMU. In the meanwhile, sev-
eral works in the literature [13]–[19] demonstrated that GPS spoofing attacks on PMUs can be more effectively mitigated by leveraging how phasor measurements from different PMUs are correlated and how they are related to the underlying power system state due to the interconnectedness of the grid. Pradhan et al. in [14] present a dynamic state estimation from PMU measurements that is resilient to spoofing attacks, by devising a generalized likelihood-based hypothesis testing to detect the location and magnitude of the spoofing attacks. A major limitation here is that it assumes that an accurate estimate of the time of the attack is known a priori. In [15] the authors mathematically model the spoofed measurements and propose an algorithm to detect and correct GPS spoofing attack on a single PMU in the network. Risbud et al. in [16], [17] leverage a measurement model that accounts for GPS spoofing attacks on multiple PMUs and develop an alternating minimization algorithm for joint estimation of the state and the attack. Similarly the authors of [18], [19] attempt to jointly estimate the states and phase angle errors by solving weighted least squares problem. However, joint estimation of state and phase angle biases, without an additional assumption on the state or angle bias variables, is ill-posed in that there exist many distinct solutions that can fit PMU measurements optimally.

In this paper, we present a sparse correction approach to correct PMU data in the presence of GPS spoofing attacks. We exploit the PMU measurement model information (i.e., the mathematical relation between PMU measurements and the system state) to identify the phase angle biases in PMU measurements introduced by spoofing attacks. In order to avoid the aforementioned ill-posedness issue, we impose a practical constraint on the attack that only a small fraction of PMUs are subject to spoofing attacks at a given time. Our contributions are: (i) formulating PMU data correction under GPS spoofing attacks as a sparse optimization problem; (ii) analyzing the fundamental identifiability of the attacks and present conditions on attack identifiability in terms of network topology, PMU locations and the number of spoofed PMUs; (iii) developing a computationally efficient iterative greedy algorithm to estimate the attack and correct PMU measurements; and (iv) validating the approach by defending the RTS-96 power network against multiple GPS spoofing attacks under both observable and unobservable PMU settings.

II. PROBLEM FORMULATION

Throughout the paper, boldface lowercase letters (e.g., \( x \)) denote vectors, boldface uppercase letters (e.g., \( X \)) denote matrices and script letters (e.g., \( \mathcal{X}, \mathcal{A} \)) denote sets. The \( l_p \) norm of \( x \) is denoted by \( \| x \|_p \). Furthermore, \( | x |, \angle x \) and \( x^* \) denote the absolute value, the angle, and the Hermitian transpose of the complex number \( x \), respectively. Moreover, for a sparse vector \( x \), \( \text{supp}(x) \) denotes the support of \( x \), which is the set of indices of nonzero entries in \( x \). In addition, \( \mathcal{R}(H) \) and \( \mathcal{N}(H) \) denote the range space and the null space of \( H \) respectively.

A. PMU measurement model

A power network topology can be represented by an undirected graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) where \( \mathcal{V} = \{1, 2, ..., N\} \) denotes the set of \( N \) buses in the network and \( \mathcal{E} \) denotes the set of branches (either transmission lines or transformers) interconnecting these buses, specifically, \( \{i, l\} \in \mathcal{E} \) if and only if there exists an energized line connecting bus \( i \) and bus \( l \). PMUs are installed in a selected subset of buses \( \mathcal{T} \subseteq \mathcal{V} \) where total number of PMUs in the network is denoted by \( K \) (i.e., \( |\mathcal{T}| = K \)).

Let \( z \in \mathbb{C}^m \) denote the PMU measurement vector which consists of measurements from all the PMUs deployed in the network and \( x = [x_1, x_2, ..., x_N]^T \in \mathbb{C}^N \) denote the system state vector where \( x_i \) represents the voltage phasor at bus \( i \). If a PMU is installed at bus \( i \) (i.e., \( i \in \mathcal{T} \)), \( z \) would contain the measurement of the voltage phasor at bus \( i \), which we denote by \( z_i \):

\[
z_i = x_i \quad (1)
\]

In addition, PMU at bus \( i \) also provides measurements of outgoing current phasors in a subset of lines incident to bus \( i \). Suppose \( \mathcal{N}_i \) denotes the set of neighbours of bus \( i \) in the topology \( \mathcal{G} \), and \( \mathcal{M}_i \subseteq \mathcal{N}_i \) denotes a subset of neighbours of bus \( i \) such that the current phasor of line \( \{i, l\} \) with \( l \in \mathcal{M}_i \) is measured by PMU at bus \( i \). The phasor measurement of the line current from bus \( i \) to bus \( l \in \mathcal{M}_i \), denoted by \( z_{i,l} \), can be given as below:

\[
z_{i,l} = y_{i,l}(x_i - x_l) + j b_{il}^* x_i \quad (2)
\]

where \( y_{i,l} = \frac{1}{2} \) is the series admittance of the line \( \{i, l\} \), and \( b_{il}^* \) is its line charging susceptance.

From (1) and (2), we can see that each entry of \( z \) is linearly related to the system state \( x \). Therefore, the linear measurement equation, incorporating the measurement noise, can be obtained as follows:

\[
z = Hx + e, \quad (3)
\]

where \( H \in \mathbb{C}^{m \times N} \) is a linear operator determined based on the network topology and line parameters according to (1) and (2), and \( e \in \mathbb{C}^m \) is complex Gaussian noise.

B. Attack model

A GPS spoofing attack on a PMU can shift the time reference of the PMU, which the PMU uses to compute the phase angle measurements. Assuming that the frequencies of voltage and current waveforms are synchronized to the nominal frequency (e.g., 60 Hz in the United States), the bias in the time reference injected by the spoofing attack would cause a common phase angle bias to all the phase angle measurements collected by the PMU [8], [15].

Suppose that spoofing attack introduces a phase angle bias \( \alpha_k \) to all of the phase angle measurements from PMU \( k \) installed at bus \( i \). Then, we can model the spoofed voltage and current measurements, denoted by \( \tilde{z}_i \) and \( \tilde{z}_{i,l} \), respectively, as follows:

\[
\tilde{z}_i = e^{j\alpha_k} z_i, \quad (4a)
\]

\[
\tilde{z}_{i,l} = e^{j\alpha_k} z_{i,l}, \quad l \in \mathcal{M}_i \quad (4b)
\]

If we use \( z_k \in \mathbb{C}^{m_k} \) and \( z_k \in \mathbb{C}^{m_k} \) to denote the intact measurements and the spoofed measurements from the PMU
spoofing attacks and therein can be leveraged to assess the
and PMU locations affect fundamental identifiability of GPS
spoofing attacks. The results explain how the network topology
Section III and present a rigorous identifiability analysis of
correction regime. We formalize the attack identifiability in
might not be fundamentally identifiable in the sparse error
the original measurement vector
\( \bar{z}_k = e^{j\alpha_k}z_k \) (5)
This can be generalized to model PMU measurements from the
entire PMU network as shown below,
\[
\begin{bmatrix}
\bar{z}_1 \\
\bar{z}_2 \\
\vdots \\
\bar{z}_K
\end{bmatrix}
= 
\begin{bmatrix}
e^{j\alpha_1}I_{m_1} & 0 & \cdots & 0 \\
0 & e^{j\alpha_2}I_{m_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & e^{j\alpha_K}I_{m_K}
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
\vdots \\
z_K
\end{bmatrix},
\]
or equivalently,
\[ \bar{z} = \Phi(\alpha)z, \] (6)
where the diagonal matrix \( \Phi(\alpha) \) denotes the attack structure
using \( \alpha \triangleq [\alpha_1, \alpha_2, \ldots, \alpha_K]^T \) and \( I_{m_k} \in \mathbb{R}^{m_k \times m_k} \) denotes
the identity matrix. The angle shift \( \alpha_k \) is nonzero if the mea-
surements from PMU \( k \) are spoofed, and it is zero otherwise.
This model is equivalent to the model employed in [10].

By combining (3) with (6) we can obtain the model for PMU
measurements potentially subject to a GPS spoofing attack:
\[ \bar{z} = \Phi(\alpha)(Hx + e) \] (7)

C. Problem statement

Given a potentially spoofed measurement vector \( \bar{z} \) from the
measurement model (7), we aim to estimate \( \alpha \) such that we
can recover the original measurement vector \( z \). Unfortunately,
many \( \alpha \)'s are fundamentally not identifiable from \( \bar{z} \). In par-
ticular, given noiseless PMU measurements \( z \) generated from
some attack \( \alpha \) and state \( x \), there can exist some \( \tilde{\alpha} \neq \alpha \) and
\( x \) satisfying:
\[ \bar{z} = \Phi(\tilde{\alpha})Hx, \text{ or equivalently, } \Phi^{-1}(\tilde{\alpha})\bar{z} \in \mathbb{R}(H), \] (8)
i.e., the measurements might be consistent with another attack
scenario described by \( \tilde{\alpha} \). Relying solely on the spoofed
measurements \( \bar{z} \), it is impossible to detect which one is the
true attack among the consistent attack scenarios.

In order to alleviate this challenge and accommodate effective-
attack identification, we employ a practical assumption that only a few PMUs in \( \mathcal{J} \) are spoofed by the adversary.
In other words, \( \alpha \) is a sparse vector, or equivalently, only a few \( \alpha_k \)'s are nonzero. By leveraging the sparse attack
assumption, we aim to obtain an accurate estimate of \( \alpha \) based on
observation of a potentially spoofed measurement vector \( \bar{z} \),
which follows the measurement model (7). Note that the state
vector \( x \) is unknown. Once we obtain an estimate of \( \alpha \), we
may use it in conjunction with the attack model (6) to recover
the original measurement vector \( z \).

Note that if the attack vector \( \alpha \) is not sparse enough
and its entries are designed in an elaborate manner, then \( \alpha \)
might not be fundamentally identifiable in the sparse error
correction regime. We formalize the attack identifiability in
Section III and present a rigorous identifiability analysis of
spoofing attacks. The results explain how the network topology
and PMU locations affect fundamental identifiability of GPS
spoofing attacks and therein can be leveraged to assess the
vulnerability of power grid to GPS spoofing attacks by simply
analyzing the grid topology and PMU locations (e.g., what is
the minimum number of PMUs an attacker needs to spoof to
be able to launch an unidentifiable attack). In Section [V], we
present a sparse error correction algorithm that can be used to
effectively estimate spoofing attacks that are identifiable.

III. IDENTIFIABILITY OF SPARSE SPOOFING ATTACKS

Suppose there exist \( \alpha \) and \( \tilde{\alpha} \) such that they are consistent
with the noiseless PMU measurements, and \( \tilde{\alpha} \) has a fewer
number of nonzero entries than \( \alpha \). Since the sparse recovery
algorithms inherently pick the most sparse solution to a prob-
lem [20], identifying the true sparse attack vector \( \alpha \) becomes
fundamentally impossible in such a situation. Based on this
intuition, the attack identifiability can be defined as below:

Definition III.1. An attack \( \alpha \) is said to be identifiable for a
state \( x \) if there does not exist \( \tilde{\alpha} \neq \alpha \) such that
(i) \( \|\tilde{\alpha}\|_0 \leq \|\alpha\|_0 \), and
(ii) \( \Phi(\alpha)Hx = \Phi(\tilde{\alpha})Hx \), for some \( \tilde{x} \), or equivalently,
\( \Phi^{-1}(\tilde{\alpha})(\Phi(\alpha)Hx) \in \mathbb{R}(H) \)

Having formally defined identifiability, we perform the
attack identifiability analysis to characterize the conditions
for attack identifiability in terms of network topology, PMU
locations and spoofed PMU locations. A major challenge in
the identifiability analysis is that the spoofed measurement
model (7) is nonlinear and involves complex-valued variables.
To circumvent this challenge, we first introduce an alternative
measurement vector \( \tilde{w} \) that can be obtained by applying a trans-
formation \( T(\cdot) \) to \( \bar{z} \) and is linearly related to the
voltage phase angles and the attack vector \( \alpha \). In the following
proposition, we define the transformation \( T(\cdot) \).

Proposition III.1. Let \( \tilde{w} \) be a real-valued vector consisting
of the following quantities:
\[ \{ \angle \tilde{V}_i, \Delta \theta_{il}, \text{ for all } i \in \mathcal{J} \text{ and } l \in \mathcal{M}_i \}, \]
where \( \Delta \theta_{il} = (\theta_i - \theta_l) \) and \( \theta_i \) denotes the voltage state angle
at bus \( i \). Then, there exists a mapping \( T(\cdot) \) such that \( \tilde{w} = T(\bar{z}) \).
Specifically, the entries of \( \tilde{w} \) can be obtained from \( \bar{z} \) as follows:
\[ \tilde{w}\angle V_i = \angle V_i, \]
\[ \tilde{w}\Delta \theta_{il} = \angle(\frac{y_{il}^* - j\frac{y_{il}}{2}}{y_{il}^*})\|\tilde{V}_i\|^2 - \tilde{V}_i^*\tilde{V}_{il}^* \]
(9b)

\[ \text{Proof. See Appendix II of the supplementary material.} \]

Unlike the original measurement vector \( \bar{z} \), the alternative
measurement vector \( \tilde{w} = T(\bar{z}) \) can be shown to be linearly
related to the voltage phase angle vector \( \theta \in \mathbb{R}^N \) which
comprises all \( \theta_i \), \( \forall i \in \mathcal{V} \), and the attack vector \( \alpha \). First, we
can use (1) and (4) to derive that,
\[ \tilde{w}\angle V_i = \angle V_i = \angle(e^{j\alpha_k}z_{V_i}) = \angle|x_i|e^{j(\theta_i + \alpha_k)} = \alpha_k + \theta_i, \]
(10)
where \( \alpha_k \) is the phase angle bias introduced by the attack on
PMU \( k \) installed in bus \( i \). By concatenating \( \tilde{w}\angle V_i \) for all the
buses with PMUs, we obtain the vector \( \tilde{w}\angle V \), which is the
vector of voltage phase angle measurements in the presence of
a spoofing attack:
\[ \tilde{w}\angle V = H\angle V \theta + \alpha. \] (11)
The matrix $\mathbf{H}_{\Delta V} \in \mathbb{R}^{K \times N}$ is determined using (10), where each row corresponds to a voltage angle measurement, and each column corresponds to a particular bus in the network. Suppose that row $j$ of $\mathbf{H}_{\Delta V}$ corresponds to the measurement $\bar{w}_{\Delta V}$ from the PMU in bus $i$. Then entry $(j, i)$ in $\mathbf{H}_{\Delta V}$ is set to one and the rest of the entries in row $j$ are set to zeros.

Similarly, from the definition of $\bar{w}_{\Delta \theta l}$ in Proposition III.1
\begin{equation}
\bar{w}_{\Delta \theta l} = \theta_l - \theta_l
\end{equation}
Thus, concatenating $\bar{w}_{\Delta \theta l}$ for all $i \in \mathcal{T}$ and $l \in \mathcal{M}_l$, we obtain the vector $\mathbf{w}_{\Delta}$ consisting of voltage angle differences across all the lines measured by PMUs, which can be mathematically written as below:
\begin{equation}
\mathbf{w}_{\Delta} = \mathbf{H}_{\Delta} \mathbf{\theta}
\end{equation}
Each row of $\mathbf{H}_{\Delta} \in \mathbb{R}^{(m-K) \times N}$ corresponds to voltage angle difference measurements computed using (12) and each column corresponds to a particular bus in the network. Suppose that row $j$ of $\mathbf{H}_{\Delta}$ corresponds to $\bar{w}_{\Delta \theta l}$ for some bus $i \in \mathcal{T}$ and $l \in \mathcal{M}_l$. Then all the entries of row $j$ are set to zeros except for $(j, i)$ and $(j, l)$ entries of the matrix $\mathbf{H}_{\Delta}$, which are set to one and negative one, respectively.

Hence we can write the alternative measurement model using (11) and (13) as follows:
\begin{equation}
T(\bar{z}) = \bar{w} = \begin{bmatrix} w_{\Delta V} \\ w_{\Delta} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{\Delta V} \\ \mathbf{H}_{\Delta} \end{bmatrix} \mathbf{\theta} + \begin{bmatrix} \alpha \\ 0 \end{bmatrix}
\end{equation}

A. Identifiability analysis

In this section, we perform identifiability analysis based on the alternative measurement model derived above. Recall that identifiability is defined by Definition III.1 which states that if a GPS spoofing attack $\alpha$ is not identifiable from measurements $\bar{z}$ generated by $(\alpha, x)$, then there exists $\bar{\alpha} \neq \alpha$ such that $\|\bar{\alpha}\|_0 \leq \|\alpha\|_0$ and,
\begin{equation}
\bar{z} = \Phi(\alpha) \mathbf{x} = \Phi(\bar{\alpha}) \mathbf{Hx},
\end{equation}
for some $\bar{x}$. Hence, by applying transformation $T(\cdot)$ defined in Proposition III.1 on (15), we can see that,
\begin{equation}
T(\Phi(\alpha) \mathbf{Hx}) = T(\Phi(\bar{\alpha}) \mathbf{Hx}),
\end{equation}
or equivalently,
\begin{equation}
\begin{bmatrix} \mathbf{H}_{\Delta V} \\ \mathbf{H}_{\Delta} \end{bmatrix} \mathbf{\theta} + \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{\Delta V} \\ \mathbf{H}_{\Delta} \end{bmatrix} \bar{\theta} + \begin{bmatrix} \bar{\alpha} \\ 0 \end{bmatrix},
\end{equation}
where $\bar{\theta}$ denotes the angles of the state $\bar{x}$, precisely $\bar{\theta}_l = \angle \bar{x}_l$. This implies that if the attack $\alpha$ is not identifiable, then there exists $\bar{\alpha} \neq \alpha$ such that $\|\bar{\alpha}\|_0 \leq \|\alpha\|_0$ and
\begin{equation}
\begin{bmatrix} \mathbf{H}_{\Delta V} \\ \mathbf{H}_{\Delta} \end{bmatrix} (\bar{\theta} - \theta) = \begin{bmatrix} \alpha - \bar{\alpha} \\ 0 \end{bmatrix},
\end{equation}
for some $\theta$ and $\bar{\theta}$.

Since this implies that $(\bar{\theta} - \theta)$ is in the null space of $\mathbf{H}_{\Delta}$, (17) is equivalent to,
\begin{equation}
\alpha - \bar{\alpha} \in \mathcal{R} (\mathbf{H}_{\Delta} B_{\Delta}),
\end{equation}
where $B_{\Delta}$ forms a basis for $\mathcal{N}(\mathbf{H}_{\Delta})$. The contrapositive of this statement directly induces the following proposition.

**Proposition III.2.** An attack $\alpha$ is identifiable for any state $x$ if there does not exist $\bar{\alpha} \neq \alpha$ such that

(i) $\|\bar{\alpha}\|_0 \leq \|\alpha\|_0$
(ii) $\alpha - \bar{\alpha} \in \mathcal{R}(\mathbf{H}_{\Delta} B_{\Delta})$

This proposition provides a sufficient condition for identifiability of $\alpha$ in terms of the column space of $\mathbf{H}_{\Delta} B_{\Delta}$. Candès and Tao presented in [21] a theoretical result that can be used to further simplify the sufficient condition in Proposition III.2 to a condition in terms of the sparsity of $\alpha$. By applying Lemma 1.7 in [21] to the conditions in Proposition III.2 we can obtain the following lemma.

**Lemma III.1.** An attack $\alpha$ is identifiable for any state $x$ if,
\begin{equation}
\|\alpha\|_0 < \frac{1}{2} \text{Cospark}(\mathbf{H}_{\Delta V} B_{\Delta}),
\end{equation}
where cospark of a matrix $A$ is defined as,
\begin{equation}
\text{Cospark}(A) = \min_{\mathbf{h} \in \mathcal{R}(A), \|\mathbf{h}\|_0} \|\mathbf{h}\|_0
\end{equation}

**Proof.** See Appendix III-A of the supplementary material □

Even though finding the cospark of a matrix is generally an NP-hard problem, the special structures of the matrices $\mathbf{H}_{\Delta}$ and $\mathbf{H}_{\Delta V}$ make it possible to derive the cospark of $\mathbf{H}_{\Delta V} B_{\Delta}$ exactly in terms of the locations and the number of PMUs in the network. In order to understand this underlying structure of the matrices, we first define the concept of a zone in the power network, which we identify as a region of the network measured by a subset of PMUs whose measurements are correlated via sharing of some common latent state variables.

**Zones in a Power network:** For each $i \in \mathcal{T}$, we define the graph $\mathcal{G}_i = (V_i, E_i)$ such that $V_i = \{i\} \cup \mathcal{M}_i$ and $E_i \subseteq \mathcal{E}$ consists of the edges $\{i, l\}, \forall l \in \mathcal{M}_i$. In other words, $\mathcal{G}_i$ is the subgraph of the topology consisting of all the lines measured by the PMU located at bus $i$. Then, we use $\mathcal{G}_\mathcal{T}$ to denote the union of all $\mathcal{G}_i$’s with $i \in \mathcal{T}$, i.e., $\mathcal{G}_\mathcal{T} = (V_T, E_T) = (\bigcup_{j \in \mathcal{T}} V_j, \bigcup_{j \in \mathcal{T}} E_j)$. Since $\mathcal{G}_\mathcal{T}$ depicts the region in the power network measured by the PMUs placed in the set of buses $\mathcal{T}$, we refer $\mathcal{G}_\mathcal{T}$ as the measurement graph of PMUs in $\mathcal{T}$. Figure 1a and Figure 1b illustrate a simple power network with five buses, four branches, and three PMUs, and the measurement graph $\mathcal{G}_\mathcal{T}$ where $\mathcal{T} = \{2, 4, 5\}$, respectively.

Given the measurement graph $\mathcal{G}_\mathcal{T}$, we define zones as connected components of graph $\mathcal{G}_\mathcal{T}$ as follows. We say that bus
i and bus j in $G_T$ are reachable from each other if there exists a path in $G_T$ that has $i$ and $j$ as the end points. We partition the vertex set $V_T$ into $V_T^{(1)}, \ldots, V_T^{(I)}$ such that two vertices $i$ and $j$ are in the same $V_T^{(g)}$ if and only if they are reachable from each other. We refer to the set of buses in $V_T^{(g)}$ as Zone $\gamma$. For instance, in Figure 1b we have two zones $V_T^{(1)}$ and $V_T^{(2)}$; each of them corresponds to one of the two connected components of $G_T$. This partitioning in the vertex domain naturally partition the set of PMUs $T$ into $T^{(1)}, \ldots, T^{(I)}$, where we use $T^{(g)}$ to denote the set of buses with PMUs in Zone $\gamma$, i.e., $T^{(g)} = T \cap V_T^{(g)}$. Note that the concept of zone defined in this paper is different from the more popular concept of observable island $\bar{V}$; specifically, a single observable island can contain multiple zones. For instance, the entire network in Figure 1a is observable based on PMU measurements, but as Figure 1b shows this observable island contains two zones.

From the PMU measurement model defined in (1) and (2), the PMU measurements from Zone $\gamma$ denoted by $z^{(\gamma)}$, depend only on state variables associated with buses in Zone $\gamma$, which we denote by $x^{(\gamma)}$. Therefore, the spoofed PMU measurement model \cite{6} and (7) has the following block structure:

$$
\begin{align*}
\begin{bmatrix}
z^{(1)} \\
\vdots \\
z^{(I)}
\end{bmatrix} &= \begin{bmatrix}
\Phi_1(\alpha^{(1)}) & \cdots & 0 \\
\vdots \\
\Phi_I(\alpha^{(I)}) & \cdots & 0
\end{bmatrix} \begin{bmatrix}
H^{(1)} & \cdots & 0 \\
0 & \cdots & H^{(I)}
\end{bmatrix}
\begin{bmatrix}
x^{(1)} \\
\vdots \\
x^{(I)}
\end{bmatrix},
\end{align*}
$$

(19)

where $\alpha^{(\gamma)}$ denotes the sub-vector of $\alpha$ that represents the angle biases introduced to the PMUs in Zone $\gamma$. Moreover, $\Phi_\gamma(\alpha^{(\gamma)})$ is a diagonal matrix which denotes the submatrix of $\Phi(\alpha)$ corresponding to Zone $\gamma$, and $H^{(\gamma)}$ denotes the submatrix of $H$ that represents the linear relation between PMU measurements and states in Zone $\gamma$. The decomposed spoofed measurement model per zone can be given as below:

$$
\tilde{z}^{(\gamma)} = \Phi_\gamma(\alpha^{(\gamma)})H^{(\gamma)}x^{(\gamma)}.
$$

(20)

Similarly, the linear model \cite{14} of the alternative measurements can be decomposed in to zones as follows:

$$
\begin{align*}
\begin{bmatrix}
\tilde{w}^{(1)}_L \\
\vdots \\
\tilde{w}^{(I)}_L
\end{bmatrix} &= \begin{bmatrix}
H^{(1)} & \cdots & 0 \\
0 & \cdots & H^{(I)}
\end{bmatrix}
\begin{bmatrix}
\theta^{(1)} \\
\vdots \\
\theta^{(I)}
\end{bmatrix} + \begin{bmatrix}
\alpha^{(1)} \\
\vdots \\
\alpha^{(I)}
\end{bmatrix},
\end{align*}
$$

(21)

where $H^{(\gamma)}_L \in \mathbb{R}^{K^{(\gamma)} \times N^{(\gamma)}}$ and $H^{(\gamma)}_A \in \mathbb{R}^{(m^{(\gamma)} - K^{(\gamma)}) \times N^{(\gamma)}}$ denote the submatrices of $H_L$ and $H_A$ that represent the linear relation of alternative measurements $\tilde{w}^{(\gamma)}_L$ and $\tilde{w}^{(\gamma)}_A$ with voltage state angles in Zone $\gamma$, respectively. Note that $m^{(\gamma)}$, $K^{(\gamma)}$ and $N^{(\gamma)}$ represents the number of measurements in Zone $\gamma$, the number of PMUs in Zone $\gamma$ and the number of busses in Zone $\gamma$ respectively. The decomposed alternative measurement model \cite{21} implies that $H_L$ and $H_A$ are block matrices with off-diagonal blocks equal to zero and each diagonal block corresponds to a zone in the network.

Due to this special block structure of $H_A$ described in \cite{21} and the sparsity pattern of each block in this matrix imposed by (13), we can derive the basis for the null space of $H_A$ as given in the following proposition:

**Proposition III.3.** The basis for $\mathcal{N}(H_A)$ is a block matrix $B_A \in \mathbb{R}^{N \times I}$,

$$
B_A = \begin{bmatrix}
B^{(1)}_A & 0 & \cdots & 0 \\
0 & B^{(2)}_A & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B^{(I)}_A
\end{bmatrix},
$$

where $B^{(\gamma)}_A = 1_{N^{(\gamma)}}$.

Here $1_{N^{(\gamma)}}$ denotes the $N^{(\gamma)}$-dimensional vector with all entries equal to one.

**Proof.** See Appendix II-B of the supplementary material \square

Since both $H_L$ and $B_A$ are matrices with the special block structure, $H_LB_A$ also takes the same block structure, i.e.,

$$
H_LB_A = \begin{bmatrix}
H^{(1)}_LB_1^{(1)} & 0 & \cdots & 0 \\
0 & H^{(2)}_LB_2^{(2)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & H^{(I)}_LB_I^{(I)}
\end{bmatrix},
$$

(22)

where $H^{(\gamma)}L^{(\gamma)}_A$ denotes the diagonal block corresponding to Zone $\gamma$. By leveraging this special block structure of $H_LB_A$, we can derive the following lemma:

**Lemma III.2.**

$$
\text{Cospark}(H_LB_A) = \min_{\gamma \in \{1, 2, \ldots, I\}} \text{Cospark}(H^{(\gamma)}L^{(\gamma)}_A)
$$

**Proof.** See Appendix II-C of the supplementary material \square

Furthermore, using Proposition III.3 together with the structure of $H_LB_A$ imposed by (21), we can obtain the cosparse of $H^{(\gamma)}L^{(\gamma)}_A$ in terms of the number of PMUs in each zone.

**Lemma III.3.** For all zones $\gamma \in \{1, 2, \ldots, I\}$ in the network,

$$
\text{Cospark}(H^{(\gamma)}L^{(\gamma)}_A) = K^{(\gamma)}
$$

**Proof.** See Appendix II-D of the supplementary material \square

Combining Lemma III.1, Lemma III.2 and Lemma III.3, we obtain a sufficient condition for the identifiability of $\alpha$ in terms of $K_{min}$, which is the smallest number of PMUs in any zone of the power network.

**Theorem III.1.** An attack $\alpha$ is identifiable for any state $x$ if,

$$
||\alpha||_0 \leq \left\lfloor \frac{K_{min}}{2} - 1 \right\rfloor,
$$

where $K_{min} = \min_{\gamma \in \{1, 2, \ldots, I\}} K^{(\gamma)}$.

In addition, given any state $x$, there exists an unidentifiable attack $\alpha$ for $x$ with the sparsity level $||\alpha||_0 = \left\lfloor \frac{K_{min}}{2} - 1 \right\rfloor + 1$.

**Proof.** See Appendix II-E of the supplementary material \square

Suppose we measure the size of a zone in the network by the number of PMUs in it. Theorem III.1 implies that if the number of spoofed PMUs in the entire network is less than half the number of PMUs in the smallest zone, such attacks are identifiable. Furthermore, we have proved that if the number
of spoofed PMUs exceed this threshold by 1, then there exists an unidentifiable attack. By leveraging the block structure of \( H_{\Delta V} B_{\Delta} \) and decomposability of the model (21), we were able to derive the following theorem providing us with a more relaxed condition that implies identifiability of the attack:

**Theorem III.2.** An attack \( \alpha \) is identifiable for any state \( x \) if,

\[
\| \alpha^{(\gamma)} \|_0 \leq \left\lceil \frac{K^{(\gamma)}}{2} - 1 \right\rceil \quad \forall \gamma \in \{1, \ldots, \Gamma\}.
\]

In addition, given any state \( x \) there exists an unidentifiable attack \( \alpha \) that has the sparsity level \( \| \alpha^{(\gamma)} \|_0 = \left\lceil \frac{K^{(\gamma)}}{2} - 1 \right\rceil + 1 \), for some \( \gamma \in \{1, 2, \ldots, \Gamma\} \) and \( \| \alpha^{(\gamma)} \|_0 \leq \left\lceil \frac{K^{(\gamma)}}{2} - 1 \right\rceil \) for \( \gamma \in \{1, 2, \ldots, \Gamma\} \setminus \{\xi\} \).

**Proof.** See Appendix II of the supplementary material.

This theorem states that as long as the number of spoofed PMUs in each zone is less than half of the number of PMUs in the zone, the attack is identifiable. Furthermore if there exists at least one zone where the sparsity condition is not satisfied then there exists an unidentifiable attack.

**Leveraging identifiability analysis to improve grid resilience:** Theorem III.1 implies that the smaller \( K_{\text{min}} \), the more vulnerable the grid is to spoofing attacks in that the attacker can launch an unidentifiable spoofing attack by spoiling a smaller number of PMUs. This implies that when we allocate PMUs (or add an additional PMU to the grid), we can improve the grid resilience by maximizing \( K_{\text{min}} \), i.e., the number of PMUs in the zone containing the smallest number of PMUs.

Figure 2 illustrates this idea with an example PMU allocation for the RTS-96 test network. This PMU allocation has 21 PMUs that naturally induces two zones in the network, with 7 and 14 PMUs respectively, where \( K_{\text{min}} = 7 \). Based on Theorem III.1 if the number of spoofed PMUs is less than or equal to three, then the attack is identifiable regardless of the locations of the spoofed PMUs. Suppose that the operator combines Zone 2 with Zone 1 by deploying an additional PMU such that \( K_{\text{min}} \) will be increased to 22. The new PMU placement ensures that the network is resilient to any spoofing attack with less than 11 spoofed PMUs. Thus the operators can significantly reduce the vulnerability of the network to spoofing attacks at a small increase in the cost.

**IV. PMU DATA CORRECTION ALGORITHM**

In this section, we present a sparse error correction algorithm to mitigate GPS spoofing. Note that we can rewrite the spoiled measurement (7) as follows:

\[
\Phi^{-1}(\alpha) \bar{z} = Hx + e.
\]

The above equation implies that with the true \( \alpha \), \( \Phi^{-1}(\alpha) \bar{z} \) would reside very close to the column space of \( H \), where the distance will be due to a small perturbation introduced by the measurement noise. In particular, if we project \( \Phi^{-1}(\alpha) \bar{z} \) onto \( \mathcal{R}(H) \), the projection residue \( r \) can be given as below,

\[
r := (I_m - P_H) \Phi^{-1}(\alpha) \bar{z},
\]

where \( P_H \) denotes the projection operator for projection on to the column space of \( H \). Then, \( r \) is equivalent to the projection of only the measurement noise \( e \) onto the orthogonal complement of \( \mathcal{R}(H) \) because, by plugging (23) in (24),

\[
r = (I_m - P_H)(Hx + e) = (I_m - P_H)e.
\]

We propose to estimate \( \alpha \) by finding the sparsest estimate of \( \alpha \) that makes the squared magnitude of the projection residue (24) no greater than a pre-set threshold \( \tau \):

\[
\hat{\alpha} = \arg \min_{\alpha} \| \alpha \|_0
\quad \text{subject to} \quad \| (I_m - P_H) \Phi^{-1}(\alpha) \bar{z} \|_2^2 \leq \tau,
\]

The threshold \( \tau \) is set such that \( \| (I_m - P_H)e \|_2^2 \leq \tau \) is satisfied with high probability.

We propose a greedy iterative algorithm to solve (26) efficiently, which has a similar structure with existing residue-based greedy algorithms such as orthogonal matching pursuit (24). In each iteration of the algorithm, we evaluate the projection residue (24) based on the estimated set of spoofed PMUs and the attack vector estimate (from the previous iteration), add a PMU with the largest normalized projection residue to the estimated set of spoofed PMU, and update the estimate of the attack vector \( \alpha \) accordingly by solving a nonlinear least squares problem with a support constraint representing the estimated set of spoofed PMUs. The detailed pseudocode is given in Algorithm 1. Note that \( r_i \in \mathbb{C}^m \) denotes the projection residue of measurements from the PMU at bus \( i \in \mathcal{T} \).

**Scalable spoofing correction algorithm:** Steps 5 and 6 of Algorithm 1 are the most computationally heavy steps in each iteration. We exploit the decomposability of the measurement model to solve these steps in a computationally efficient manner. From the decomposed measurement model (19) we can infer that \( \Phi^{-1}(\alpha) \) and \( (I_m - P_H) \) are block matrices, wherein off-diagonal blocks are equal to zero matrices and each diagonal block corresponds to a zone in the network. Hence Step 5 of Algorithm 1 is equivalent to,

\[
\hat{\alpha} = \arg \min_{\alpha} \sum_{\gamma \in \{1, \ldots, \Gamma\}} \left\| \left( I_m^{(\gamma)} - P_{H^{(\gamma)}} \right) \Phi^{-1}(\alpha^{(\gamma)}) \bar{z}^{(\gamma)} \right\|_2^2.
\]

This optimization can be solved independently per zone, i.e.,

\[
\hat{\alpha}^{(\gamma)} = \arg \min_{\alpha} \left\| \left( I_m^{(\gamma)} - P_{H^{(\gamma)}} \right) \Phi^{-1}(\alpha^{(\gamma)}) \bar{z}^{(\gamma)} \right\|_2^2.
\]
Algorithm 1 Sparse GPS spoofing correction algorithm

\textbf{Init.:} \( \hat{\alpha} = 0 \), \( r[0] = (I_m - P_H) \Phi^{-1}(\hat{\alpha})\hat{z}, A[0] = \emptyset \), \( itr = 1 \)

1: while \( \| r[itr-1] \|_2^2 > \tau \) do
2: \quad Compute normalized residue magnitudes:
3: \quad \hat{r}_i = \frac{|r[i,itr-1]|}{m_i} , \quad \forall i \in \mathcal{I} \setminus A[itr-1]
4: \quad Select the largest normalized residue magnitude:
5: \quad \hat{\alpha} = \arg \max_{\alpha : \text{supp}(\alpha) \subseteq A[itr]} \| (I_m - P_H) \Phi^{-1}(\alpha)\hat{z} \|_2^2.
6: \quad Update the support of \( \hat{\alpha} \):
7: \quad A[itr] \leftarrow A[itr-1] \cup \{ \hat{\alpha} \}
8: \quad Compute the estimate \( \hat{\alpha} \):
9: \quad \hat{\alpha} = \arg \min_{\alpha : |\text{supp}(\alpha)| \leq |A[itr]|} \| (I_m - P_H) \Phi^{-1}(\alpha)\hat{z} \|_2^2.
10: \quad Update the residual:
11: \quad r[itr] = (I_m - P_H) \Phi^{-1}(\hat{\alpha})\hat{z}
12: \quad itr = itr + 1
end while

\textbf{Output:} \( \hat{\alpha} \) and \( \hat{z} \)

where \( A(\gamma)[itr] \) is the subset of the elements of \( A[itr] \) which corresponds to Zone \( \gamma \). Let \( \gamma^* \) be the zone that contains bus \( i^* \) selected in Step 2 of the algorithm. Since \( A(\gamma)[itr] = A(\gamma)[itr-1] \) for \( \gamma \neq \gamma^* \), \( \hat{\alpha}(\gamma) \) at iteration \( itr \) remains unchanged from the previous iteration for all \( \gamma \neq \gamma^* \). The update on \( \hat{\alpha} \) happens only at Zone \( \gamma^* \), as given below:

\[
\hat{\alpha}(\gamma^*) = \arg \min_{\alpha : |\text{supp}(\alpha)| \leq |A(\gamma^*)[itr]|} \| (I_m - P_H(\gamma)) \Phi^{-1}(\alpha(\gamma^*))\hat{z} \|_2^2.
\]

Therefore, Step 5 reduces to solving a least squares problem for only one zone, \( \gamma^* \).

Similarly, since only update of \( \hat{\alpha} \) happens at \( \hat{\alpha}(\gamma^*) \), the residue update at Step 6 of Algorithm 1 simplifies to,

\[
r(\gamma),[itr] = \begin{cases} (I_m - P_H(\gamma)) \Phi^{-1}(\hat{\alpha}(\gamma^*))\hat{z} & \gamma = \gamma^* \\
(\hat{r}(\gamma),[itr-1]) & \forall \gamma \neq \gamma^* 
\end{cases}
\]

where \( r(\gamma),[itr] \) denotes the sub-vector of \( r[itr] \) corresponding to Zone \( \gamma \). Therefore the computations in Steps 5 and 6 of the algorithm are reduced to single-zone updates. This significantly reduces the computational complexity of the algorithm and makes it scalable.

V. EXPERIMENTS

In this section, we demonstrate the efficacy of the proposed PMU data correction algorithm on the IEEE RTS-96 test network [25], which consists of 73 buses and 120 branches.

PMU placement: We evaluate our data correction algorithm on both observable and unobservable PMU placements. Figure 2 illustrates the observable PMU placement setting which consists of 21 PMUs. As described in Figure 2, this placement setting naturally induces two zones in the network, with 7 PMUs in Zone 1 and 14 PMUs in Zone 2. Furthermore, we obtained an unobservable PMU network by removing PMUs at buses 303, 103, and 316 in the observable placement, which results in a network with two zones, having 5 PMUs in Zone 1 and 13 PMUs in Zone 2. This causes around 15% of the buses in the network to become unobservable.

Spoofed measurement generation: We test the proposed data correction algorithm on measurements generated according to (7) by sampling the state \( x \) from a gaussian distribution with mean set to a known snapshot state and standard deviations of voltage magnitudes and phase angles set to 0.01 p.u. and 5.73 degrees, respectively. Given that the percentage of the spoofed PMUs is set to \( A\% \), in each Monte Carlo run, we selected \( A\% \) of PMUs from each zone uniformly at random and manipulated their phase angle measurements according to (7). The magnitude of attack angle shift \( \alpha_k \) for a spoofed PMU \( k \) is sampled from a uniform distribution in the range of \((0.8\mu, 1.2\mu)\) where \( \mu \) is the attack mean, which we set to 20 degrees for our experiments. The sign of the non zero attack angle shift \( \alpha_k \) is selected uniformly at random.

Benchmark algorithms: We compare the performance of the proposed approach with two existing benchmark algorithms, Risbud et al. [16] and Vanfretti et al. [18]. Risbud et al. [16] presents an alternating minimization algorithm for joint state estimation and attack reconstruction. This algorithm is designed to operate on networks that are observable from the PMU measurements [2]. Vanfretti et al. [18] develops a state estimation technique based on PMU measurements by incorporating potential phase bias errors in PMU measurements [2]. This algorithm is designed for decentralized operation wherein it can be independently applied to correct PMU data in observable islands within an unobservable network.

Results: We first present the results for observable PMU placement setting shown in Figure 3 where an equal percentage of PMUs are spoofed from Zone 1 and Zone 2. For a performance metric, we employ the largest magnitude entry of the attack estimation error vector \( (\hat{\alpha} - \alpha) \), i.e., \( \| \hat{\alpha} - \alpha \|_{\infty} = \max_{i=1,\ldots,K} |\hat{\alpha}_i - \alpha_i| \). Table I presents the median, standard deviation, and the maximum value of this metric from 100 Monte Carlo runs, for various percentages of the spoofed PMUs and for an observable PMU placement setting. The medians and the standard deviations indicate that the proposed sparse error correction approach significantly outperforms the benchmarks on average. In the meanwhile, the maximum error metrics observed among 100 Monte Carlo runs imply that our approach is more reliable compared to the benchmarks. For instance, the error metric remains smaller than 2.1 degrees for our approach in all Monte Carlo runs and all experiment scenarios, but for the benchmarks, the error metric can grow even larger than 12 degrees for some worst case attack scenarios. Table II presents the results for an attack carried out on an unobservable PMU network, where the rest of the attack is designed similarly to the experiments with the observable network. The results show the similar trend as the results for the observable case. In all of the above experiments

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3The authors in [18] extend their algorithm to networks unobservable by PMU measurements by augmenting both PMU and SCADA measurements. Since our focus is on PMU data correction, our comparison is only with their algorithm that uses only PMU measurements.

4We set PMU 102 as the “reference bus” defined in the paper [18]. This PMU is assumed to be intact from PMU attacks.
the percentage of spoofed PMUs in each zone remains less than half of the number of PMUs in the zone. Therefore, the attacks we test here are identifiable based on Theorem 3.2 and thus the sparse error correction algorithm can identify and correct them well. In addition, we present similar experiments on IEEE-300 bus test network, in the Appendix III of the supplementary material.

VI. Conclusion

In this paper, we presented a sparse error correction framework for mitigating GPS spoofing attacks on PMUs. Our attack identifiability analysis provides a detailed characterization of how PMU locations affect the grid resilience to spoofing attacks. The proposed error correction algorithm is scalable because it only requires solving a least squares problem for a single zone in each iteration. It outperformed benchmarks in mitigating GPS spoofing attacks on PMUs. Overall, our results imply that we can mitigate spoofing attacks much more effectively by properly leveraging their sparse nature.

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TABLE I

| Spoofed PMU % | Proposed | Risbud et al. [16] | Vanfretti et al. [18] |
|---------------|----------|------------------|---------------------|
| 10%           | 0.290 ± 0.165 (Max.: 1.590) | 1.360 ± 0.189 (Max.: 9.999) | 3.415 ± 1.166 (Max.: 13.588) |
| 20%           | 0.580 ± 0.163 (Max.: 1.533) | 4.393 ± 1.812 (Max.: 20.418) | 3.964 ± 1.359 (Max.: 14.456) |
| 30%           | 0.789 ± 0.165 (Max.: 2.095) | 6.414 ± 2.158 (Max.: 21.504) | 3.733 ± 1.364 (Max.: 12.755) |
| 40%           | 0.853 ± 0.133 (Max.: 1.990) | 6.634 ± 1.519 (Max.: 18.928) | 3.164 ± 1.398 (Max.: 13.693) |

TABLE II

| Spoofed PMU % | Proposed | Vanfretti et al. [18] |
|---------------|----------|---------------------|
| 10%           | 0.218 ± 0.147 (Max.: 1.461) | 4.470 ± 1.967 (Max.: 23.220) |
| 20%           | 0.703 ± 0.210 (Max.: 2.133) | 5.691 ± 2.127 (Max.: 24.788) |
| 30%           | 0.678 ± 0.179 (Max.: 1.839) | 4.511 ± 1.598 (Max.: 18.161) |
| 40%           | 0.809 ± 0.177 (Max.: 1.867) | 4.389 ± 1.378 (Max.: 16.366) |
Appendix I

Alternative Measurement Model

In this section we present the proof for Proposition III.1 stated in Section III.

First we will show that the spoofed voltage angle measurements of PMU in bus $i$ denoted by $\bar{w}_{\mathcal{L}V_i}$ can be directly obtained from $\bar{z}$, using the definition of $\bar{w}_{\mathcal{L}V_i}$, shown as below:

$$\bar{w}_{\mathcal{L}V_i} = Z \bar{z}_{i}.$$  \hfill (31)

Next we will show that the angle difference of voltage phase angles between bus $i \in \mathcal{I}$ and bus $l \in \mathcal{M}_i$ denoted by $\bar{w}_{\Delta \theta_{il}}$ can be derived using $\bar{z}$.

Using (1) and (2), we can model the spoofed voltage measurement from PMU $k$ in bus $i$ as shown below:

$$\bar{z}_{V_i} = e^{j \alpha_k (|x_i|e^{\beta_i})} = |x_i|e^{j (\alpha_k+\beta_i)}.$$ \hfill (32)

Note that $\alpha_k$ is nonzero if PMU $k$ is under spoofing attack and $\alpha_k$ is zero otherwise.

Using (2) and (4), we can write down the current measurements in line from bus $i \in \mathcal{I}$ to bus $l \in \mathcal{M}_i$ as below,

$$\bar{z}_{I_{il}} = e^{j \alpha_k ((y_{il} + j \frac{b_{i}^*}{2})|x_i|e^{\beta_i} - y_{il}|x_i|e^{\beta_i})} = (y_{il} + j \frac{b_{i}^*}{2})|x_i|e^{j (\alpha_k+\beta_i)} - y_{il}|x_i|e^{j (\alpha_k+\beta_i)}.$$ \hfill (33)

Now using (32) and (33), we can write $\bar{z}_{V_i} \bar{z}_{I_{il}}^*$ as:

$$\bar{z}_{V_i} \bar{z}_{I_{il}}^* = |x_i|e^{j (\alpha_k+\beta_i)}(y_{il}^* - j \frac{b_{i}^*}{2})|x_i|e^{-j (\alpha_k+\beta_i)} - |x_i|e^{-j (\alpha_k+\beta_i)}y_{il}^*|x_i|e^{j (\alpha_k+\beta_i)} = (y_{il}^* - j \frac{b_{i}^*}{2})|x_i|^2 - y_{il}|x_i||x_i|e^{j (\alpha_k+\beta_i)}.$$ \hfill (34)

Since $|x_i|$ is directly measured from PMU at bus $i$, $|x_i| = |z_{V_i}|$. So we can rewrite the above equation as

$$|x_i|e^{j (\alpha_k+\beta_i)} = \frac{(y_{il}^* - j \frac{b_{i}^*}{2})|x_i|^2 - \bar{z}_{V_i} \bar{z}_{I_{il}}^*}{y_{il}^* |z_{V_i}|}$$ \hfill (35)

Now we can write $\theta_i - \theta_l$, the angle difference of voltage phasors between bus $i$ and bus $l$ denoted by $\bar{w}_{\Delta \theta_{il}}$ as

$$\bar{w}_{\Delta \theta_{il}} = \theta_i - \theta_l = \angle \left( \frac{(y_{il}^* - j \frac{b_{i}^*}{2})|x_i|^2 - \bar{z}_{V_i} \bar{z}_{I_{il}}^*}{y_{il}^* |z_{V_i}|} \right)$$ \hfill (36)

Thus we prove that there exists a mapping $T(\cdot)$ such that $\bar{w} = T(\bar{z})$, where $\bar{w}$ is the concatenation of $\bar{w}_{\mathcal{L}V_i}$ and $\bar{w}_{\Delta \theta_{il}}$ for all $i \in \mathcal{I}$ and $l \in \mathcal{M}_i$.

Appendix II

Identifiability Analysis

In this section, we present proofs for the theorems, lemmas and propositions we have presented on identifiability analysis in Section III.

A. Proof for Lemma III.1

Proof. Suppose $\alpha$ satisfies $\|\alpha\|_0 < \frac{1}{2} \text{Cospark}(H_{\mathcal{L}V \Delta})$ and that $\alpha$ is not identifiable. Then by Proposition III.2 there exists $\alpha \neq \bar{\alpha}$ such that $\|\alpha\|_0 \leq \|\bar{\alpha}\|_0$ and,

$$\alpha - \bar{\alpha} \in \mathcal{R}(H_{\mathcal{L}V \Delta}).$$ \hfill (37)

This implies that,

$$\|\alpha - \bar{\alpha}\|_0 \leq \|\alpha\|_0 + \|\bar{\alpha}\|_0 \leq \frac{1}{2} \|\alpha\|_0 < \text{Cospark}(H_{\mathcal{L}V \Delta}).$$

However, since $\alpha - \bar{\alpha} \in \mathcal{R}(H_{\mathcal{L}V \Delta})$ we have $\|\alpha - \bar{\alpha}\|_0 \geq \text{Cospark}(H_{\mathcal{L}V \Delta})$, which contradicts with the above inequality. Therefore, $\alpha$ should be identifiable.

B. Proof of Proposition III.2

Proof. Since $H_{\Delta} \in \mathbb{R}^{(m-K)\times N}$ is a block matrix with the special block structure as follows,

$$H_{\Delta} = \begin{bmatrix} H_{\Delta}^{(1)} & 0 & \ldots & 0 \\ 0 & H_{\Delta}^{(2)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & H_{\Delta}^{(\Gamma)} \end{bmatrix},$$

we can describe its null space as:

$$N(H_{\Delta}) = \{ \beta \in \mathbb{R}^N : H_{\Delta} \beta = 0 \} = \{ \begin{bmatrix} \beta^{(1)} \\ \vdots \\ \beta^{(\Gamma)} \end{bmatrix} \in \mathbb{R}^N : H_{\Delta}^{(\gamma)} \beta^{(\gamma)} = 0, \forall \gamma \}.$$ \hfill (38)

Above equation implies that, $\beta \in N(H_{\Delta}) \iff \beta^{(\gamma)} \in N(H_{\Delta}^{(\gamma)})$, $\forall \gamma \in \{1, \ldots, \Gamma\}$ \hfill (39)

Therefore, we can derive a basis matrix of $N(H_{\Delta})$ denoted by $B_{\Delta}$ with a similar block structure, having zero off diagonal blocks and $\gamma$-th diagonal block $B_{\Delta}^{(\gamma)}$ being the basis matrix of $N(H_{\Delta}^{(\gamma)})$, as shown below:

$$B_{\Delta} = \begin{bmatrix} B_{\Delta}^{(1)} & 0 & \ldots & 0 \\ 0 & B_{\Delta}^{(2)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & B_{\Delta}^{(\Gamma)} \end{bmatrix}.$$ \hfill (40)

Now let us prove that $B_{\Delta}^{(\gamma)}$ is the $N^{(\gamma)}$- dimensional vector with all entries equal to one. Recall that $H_{\Delta}^{(\gamma)} \in \mathbb{R}^{(m^{(\gamma)}-K^{(\gamma)})\times N^{(\gamma)}}$ where $m^{(\gamma)}$, $K^{(\gamma)}$ and $N^{(\gamma)}$ are the number of measurements from Zone $\gamma$, number of PMUs in Zone $\gamma$ and number of buses in Zone $\gamma$, respectively. Suppose that $b \in \mathbb{R}^{N^{(\gamma)}}$ is a vector in $N(H_{\Delta}^{(\gamma)})$, i.e.,

$$H_{\Delta}^{(\gamma)} b = 0.$$ \hfill (41)

According to the definition, each row of $H_{\Delta}^{(\gamma)}$ consists of exactly two nonzero entries which are equal in magnitude but opposite in sign. Each column corresponds to a bus in Zone $\gamma$. Let the nonzero entries of row $i$ of $H_{\Delta}^{(\gamma)}$ are located in column $k$ and column $l$. Then for each row $i$ of the matrix,

$$H_{\Delta}^{(\gamma)}(i, :) b = 0, b \neq 0 \iff b_k = b_l.$$ \hfill (42)
where $H^{(γ)}[i, :]$ denotes the row $i$ of matrix $H_γ$ and $b_k$ denotes the entry $k$ of vector $b$. Furthermore, from the definition of a zone, there exists a path between any two nodes in the subgraph corresponding to a particular zone. This implies that, (40) has to be true for any entry $k$ and $l$ in the vector $b$. Therefore $b = \bar{b} \cdot 1_{N(γ)}$ where $\bar{b}$ is a scalar. Thus we prove that $N(H^{(γ)}_γ)$ has dimension one and the basis for the null space is in fact the following:

$$B_γ^{(γ)} = 1_{N(γ)}.$$

C. Proof of Lemma III.2

Proof. Since both $H_{ζγ}^γ$ and $B_γ$ are block matrices with off diagonal blocks being zero matrices, $H_{ζγ}^γ B_γ$ is also a block matrix as shown below:

$$H_{ζγ}^γ B_γ = \begin{bmatrix} H_{1γ}^{(1)} & 0 & \ldots & 0 \\ 0 & H_{2γ}^{(1)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & H_{N(γ)}^{(1)} \end{bmatrix} \begin{bmatrix} B_{1γ}^{(1)} & 0 & \ldots & 0 \\ 0 & B_{2γ}^{(1)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & B_{N(γ)}^{(1)} \end{bmatrix},$$

(41)

where $H_{ζγ}^{(γ)}$ and $B_γ^{(γ)}$ are the blocks of $H_{ζγ}^γ$ and $B_γ$ corresponding to each zone $γ \in \{1, 2, \ldots, Γ\}$. Now it is easy to see that the sparsity level of the sparsest nonzero vector in $R(H_{ζγ}^γ B_γ)$ is the smallest sparsity level among the sparsest nonzero vectors in $R(H_{ζγ}^{(γ)} B_{γ}^{(γ)})$ of all zones $γ \in \{1, \ldots, Γ\}$. Therefore by the definition of cospark of a matrix, we can prove the statement in the Lemma, i.e.,

$$\text{Cospark}(H_{ζγ}^γ B_γ) = \min_{γ \in \{1, 2, \ldots, Γ\}} \text{Cospark}(H_{ζγ}^{(γ)} B_{γ}^{(γ)}).$$

D. Proof of Lemma III.3

Proof. We directly prove the claim in this Lemma by leveraging Proposition III.3 and the special sparsity structure of $H_{ζγ}^{(γ)} \in \mathbb{R}^{K(γ) \times N(γ)}$. If $y$ is a nonzero vector in $R(H_{ζγ}^{(γ)} B_{γ}^{(γ)})$, then there exists a nonzero $c \in \mathbb{R}$ such that,

$$y = H_{ζγ}^{(γ)} B_{γ}^{(γ)} c = H_{ζγ}^{(γ)} 1_{N(γ)} c,$$

(42)

where the last equation is due to Proposition III.3. Furthermore, by the definition of $H_{ζγ}^{(γ)}$, each row of this matrix corresponds to a particular voltage angle measurement from a PMU deployed in bus $i \in \mathcal{T}(γ)$ from Zone $γ$, and consists of all zeros except for the value one in the column corresponding to bus $i$. Due to this structure of $H_{ζγ}^{(γ)}$, we can say the following:

$$H_{ζγ}^{(γ)} 1_{N(γ)} = 1_{K(γ)}.$$

where $K(γ)$ is the number of PMUs in Zone $γ$. Thus by substituting this in (42) we get,

$$y = 1_{K(γ)} c.$$

This implies that for any nonzero $y \in R(H_{ζγ}^{(γ)} B_{γ}^{(γ)})$, $\|y\|_0 = K(γ)$ Hence, from the definition of cospark,

$$\text{Cospark}(H_{ζγ}^{(γ)} B_{γ}^{(γ)}) = K(γ).$$

E. Proof of Theorem III.7

Proof. First we directly prove the implications of the inequality in the Theorem by leveraging Lemma III.1 and Lemma III.2 and Lemma III.3. Suppose that $α$ satisfies the following:

$$\|α\|_0 < \frac{1}{2} \text{Cospark}(H_{ζγ} B_γ).$$

Then from Lemma III.1, $α$ is identifiable. By applying Lemma III.2 and Lemma III.3

$$\text{Cospark}(H_{ζγ} B_γ) = \min_{\{1, 2, \ldots, Γ\}} \text{Cospark}(H_{ζγ}^{(γ)} B_{γ}^{(γ)}),$$

where $K_{min} = \min_{1, 2, \ldots, Γ} K(γ)$, i.e., the smallest number of PMUs in a zone in the network. Since the sparsity of $α$ only takes integer values, we can rewrite the inequality and state that if,

$$\|α\|_0 \leq \lfloor \frac{K_{min}}{2} - 1 \rfloor,$$

then $α$ is identifiable. Therefore this proves the first statement in Theorem III.1.

Now we will prove the converse statement. Let $x \in \mathbb{C}^N$ be an arbitrary state vector. We will prove that there exists $α$ with $\|α\|_0 = \lfloor \frac{K_{min}}{2} - 1 \rfloor + 1$ that is not identifiable for the state $x$. In particular we explicitly construct $α$ as follows. Without loss of generality we assume that Zone 1 in the network has the smallest number of PMUs. For legibility, let $κ = (\lfloor \frac{K_{min}}{2} - 1 \rfloor + 1)$. Then we define each entry $i$ of $α(1)$ as follows:

$$α(1)[i] = \begin{cases} a & \text{if } i = 1, \ldots, κ \bigg\} \text{if } i = κ + 1, \ldots, K(1) \\ 0 & \end{cases}$$

where $a$ is a nonzero constant. Furthermore, each entry of $α(γ)$ for the rest of the zones $γ \in \{2, \ldots, Z\}$ are set to zero, i.e.:

$$α(γ)[i] = 0 \quad \text{if } γ \in \{2, \ldots, Γ\}.$$

Let noiseless measurements $\bar{z}$, denote the measurements generated by this $α$ and an arbitrary state vector $x$, i.e., $\bar{z} = Φ(α)x$. In order to show that $α$ is not identifiable for $x$, we will prove existence of $\bar{y} \neq α$ and $\bar{x}$ satisfying:

(i) $\|\bar{y}\|_0 \leq \|\bar{α}\|_0$, and

(ii) $Φ(α)\bar{H} x = Φ(\bar{α})\bar{H} x$

We define $\bar{α} \neq α$ by only altering entries of $α$ corresponding to Zone 1. Specifically as below:

$$\bar{α}(γ) = \begin{cases} α(γ) - a \cdot 1_{K_{min}} & \text{if } γ = 1 \\ α(γ) & \text{if } γ \in \{2, \ldots, Γ\} \end{cases}$$

(43)

And define state $\bar{x}$ by only altering entries of $x$ corresponding to Zone 1. Specifically

$$\bar{x}(γ) = \begin{cases} e^{ja} \cdot x(1) & \text{if } γ = 1 \\ x(γ) & \text{if } γ \in \{2, \ldots, Γ\} \end{cases}$$

(44)
Let the measurements generated by $(\bar{x}, \bar{\alpha})$ be $\bar{z}$. Due to the decomposability of PMU measurements \(^{(20)}\), the spoofed measurements in Zone $\gamma$ for $\gamma \in \{2, \ldots, \Gamma\}$, can be given as below,

\[
\bar{z}(\gamma) = \Phi(\bar{x}(\gamma))H(\gamma)\bar{x}(\gamma) = \Phi(\alpha^\gamma)H(\gamma)x(\gamma)
\]

where last equality is due to \(^{(43)}\) and \(^{(44)}\). Therefore,

\[
\bar{z}(\gamma) = z(\gamma) \quad \text{for} \; \gamma \in \{2, \ldots, \Gamma\}.
\]

Now let us analyse $\bar{z}(\gamma)$ generated from attack $\bar{\alpha}$ and state $\bar{x}$ based on \(^{(32)}\) and \(^{(33)}\). For PMU $k$ deployed in bus $i \in T(1)$ and $l \in M_i$:

\[
\bar{z}_{\gamma i} = |x_l|e^{j(\bar{\alpha}_l+\bar{\theta})} = |x_l|e^{j((\alpha_k-a)+(\theta_i+a))} = |x_l|e^{j(\alpha_k+\theta)} = \bar{z}_{\gamma i},
\]

\[
\bar{z}_{\gamma i} = (y_{il} + \frac{b_{il}}{2})|x_l|e^{j(\bar{\theta} + \bar{\alpha}_k)} - y_{il}|x_l|e^{j(\bar{\theta} + \bar{\alpha}_k)}
\]

\[
= (y_{il} + \frac{b_{il}}{2})|x_l|e^{j((\theta_i+a)+(\alpha_k-a))} - y_{il}|x_l|e^{j((\theta_i+a)+(\alpha_k-a))}
\]

\[
= \bar{z}_{\gamma i}.
\]

Thus we have shown that $\bar{z} = \bar{z}$. Furthermore, by the definition of $\bar{\alpha}$:

\[
\bar{\alpha}(\gamma)[i] = \begin{cases} 
0 & \text{if } i \in \{1, \ldots, \kappa\} \\
-a & \text{otherwise}
\end{cases}
\]

and $\bar{\alpha}(\gamma) = 0$ for all $\gamma \in \{2, \ldots, \Gamma\}$. Therefore,

\[
\|\alpha\|_0 = K_{\min} - \kappa
\]

\[
= K_{\min} - \left(\frac{K_{\min}}{2} - 1\right) + 1
\]

\[
\leq \|\alpha\|_0
\]

Hence from Definition \(^{(III.1)}\), this attack $\bar{\alpha}$ is not identifiable. Thus we have proved that given any state $x$ there exists an unidentifiable attack $\alpha$ with sparsity level $\lceil \frac{K_{\min}}{2} - 1 \rceil + 1$.

\[\Box\]

\[\text{F. Proof of Theorem} \; \text{\[^{(III.2)}\]}\]

\[\text{Proof.} \; \text{Let } x \text{ be an arbitrary state vector } x \in \mathbb{C}^N \text{ and } \alpha \text{ be an arbitrary attack vector satisfying},
\]

\[
\|\alpha^\gamma\|_0 \leq \lceil \frac{K(\gamma)}{2} - 1 \rceil,
\]

for all $\gamma \in \{1, \ldots, \Gamma\}$. We will prove that $\alpha$ is identifiable for $x$ using the proof-by-contradiction approach.

Suppose that $\alpha$ is unidentifiable for $x$. Then, then the contrapositive of Proposition \(^{(III.1)}\) implies that there exists $\alpha \neq \alpha$ such that (i) $\alpha \preceq \alpha$ and (ii) $\alpha - \alpha \in \mathbb{R}(H(\gamma)B_{\Delta})$. Because $\|\alpha\|_0 \leq \|\alpha\|_0$ and $\alpha \neq \alpha$, there exists $\bar{\gamma} \in \{1, \ldots, \Gamma\}$ such that $\|\alpha(\bar{\gamma})\|_0 \leq \|\alpha(\gamma)\|_0$ and $\alpha(\bar{\gamma}) \neq \alpha(\gamma)$. Then from the triangle inequality,

\[
\|\alpha(\gamma) - \alpha(\bar{\gamma})\|_0 \leq \|\alpha(\gamma)\|_0 + \|\alpha(\bar{\gamma})\|_0.
\]

Furthermore,

\[
\|\alpha(\bar{\gamma})\|_0 \leq 2\|\alpha(\gamma)\|_0 \leq 2\left(\frac{K(\gamma)}{2} - 1\right),
\]

where the first inequality is due to the existence of $\bar{\gamma} \in \{1, \ldots, \Gamma\}$ such that $\|\alpha(\gamma)\|_0 \leq \|\alpha(\bar{\gamma})\|_0$ and the last inequality is due to the assumption \(^{(45)}\).

Furthermore, due to the block structure of $H(\gamma)B_{\Delta}$ described in \(^{(22)}\), $\alpha - \alpha \in \mathbb{R}(H(\gamma)B_{\Delta})$ implies that $\alpha(\gamma) - \alpha(\bar{\gamma})$ is in $\mathbb{R}(H(\gamma)B_{\Delta})$. Therefore,

\[
\alpha(\gamma) - \alpha(\bar{\gamma}) \in \mathbb{R}(H(\gamma)B_{\Delta}).
\]

Since $\alpha(\gamma) - \alpha(\bar{\gamma})$ is a nonzero vector in $\mathbb{R}(H(\gamma)B_{\Delta})$, the definition of Cospark($\mathbb{R}(H(\gamma)B_{\Delta})$) implies that

\[
\|\alpha(\gamma) - \alpha(\bar{\gamma})\|_0 \geq \text{Cospark}(H(\gamma)B_{\Delta}).
\]

From Lemma \(^{(III.3)}\) we have Cospark($\mathbb{R}(H(\gamma)B_{\Delta})$) $= K(\gamma)$. Therefore, the above inequality can be rewritten as follows:

\[
\|\alpha(\gamma) - \alpha(\bar{\gamma})\|_0 \geq K(\gamma).
\]

This contradicts with \(^{(47)}\), thereby proving the theorem statement.

Now we will prove the converse statement. Let $x \in \mathbb{C}^N$ be an arbitrary state vector and $\bar{\gamma} \in \{1, \ldots, \Gamma\}$ be an arbitrary zone. We will prove that there exists $\alpha$ with (i) $\|\alpha(\gamma)\|_0 = \lceil \frac{K(\gamma)}{2} - 1 \rceil + 1$ and $\|\alpha(\gamma)\|_0 \leq \lceil \frac{K(\gamma)}{2} - 1 \rceil$, for $\gamma \in \{1, \ldots, \Gamma\} \setminus \{\bar{\gamma}\}$, and (ii) $\alpha$ is not identifiable for the state $x$. In particular we explicitly construct such $\alpha$ as follows. For legibility, let $k := \lceil \frac{K(\gamma)}{2} - 1 \rceil + 1$. Then we set the entries of $\alpha(\gamma)$ as follows:

\[
\alpha(\gamma)[i] = \begin{cases} 
0 & \text{if } i = 1, \ldots, \kappa \\
-a & \text{if } i = \kappa + 1, \ldots, K(\gamma)
\end{cases}
\]

where $a$ is a nonzero constant. Furthermore, for every $\gamma \in \{1, \ldots, \Gamma\} \setminus \{\bar{\gamma}\}$ we set $\alpha(\gamma)$ to be an arbitrary $K(\gamma)$-dimensional vector satisfying $\|\alpha(\gamma)\|_0 \leq \lceil \frac{K(\gamma)}{2} - 1 \rceil$.

Let noiseless measurements $\bar{z}$ denote the measurements generated by this $\alpha$ and an arbitrary state vector $x$, i.e., $\bar{z} = \Phi(\alpha)Hx$.

In order to show that $\alpha$ is not identifiable for $x$, we will prove existence of $\alpha \neq \alpha$ and $x$ satisfying,

(i) $\|\alpha\|_0 \leq \|\alpha\|_0$, and

(ii) $\Phi(\alpha)Hx = \Phi(\alpha)Hx$.

We define $\bar{\alpha} = [(\alpha(1))^T, \ldots, (\alpha(\Gamma))^T]^T$ by only altering the entries of $\alpha$ corresponding to Zone $\bar{\gamma}$ as below:

\[
\bar{\alpha}(\gamma) = \begin{cases} 
\alpha(\gamma) - a \cdot 1_{K(\gamma)} & \text{if } \gamma = \bar{\gamma} \\
\alpha(\gamma) & \text{if } \gamma \in \{1, \ldots, \Gamma\} \setminus \{\bar{\gamma}\}
\end{cases}
\]

And define state $\bar{x}$ by only altering entries of $x$ corresponding to Zone $\bar{\gamma}$. Specifically,

\[
\bar{x}(\gamma) = \begin{cases} 
e^{j\alpha} \cdot x(\gamma) & \text{if } \gamma = \bar{\gamma} \\
x(\gamma) & \text{if } \gamma \in \{1, \ldots, \Gamma\} \setminus \{\bar{\gamma}\}
\end{cases}
\]

Due to the decomposability of PMU measurements \(^{(20)}\), the spoofed measurements in Zone $\gamma$ for $\gamma \in \{1, \ldots, \Gamma\} \setminus \{\bar{\gamma}\}$, can be given as below,

\[
\bar{z}(\gamma) = \Phi(\bar{\alpha}(\gamma))H(\gamma)\bar{x}(\gamma) = \Phi(\alpha(\gamma))H(\gamma)x(\gamma)
\]

where last equality is due to \(^{(48)}\) and \(^{(49)}\). Therefore,

\[
\bar{z}(\gamma) = \bar{z}(\gamma) \quad \text{for } \gamma \in \{1, \ldots, \Gamma\} \setminus \{\bar{\gamma}\}.
\]
Now let us analyse $\tilde{z}^{(\gamma)}$ generated from attack $\alpha$ and state $\bar{x}$ based on (32) and (33). For PMU $k$ deployed in bus $i \in \mathcal{T}^{(\gamma)}$ and $l \in \mathcal{M}_i$:

$$\tilde{\xi}_{i} = |x_i|e^{j(\alpha_k + \theta_i)} = |x_i|e^{j((\alpha_k - \alpha) + (\theta_i + \alpha))} = |x_i|e^{j(\alpha_k + \theta_i)}$$

$$\tilde{z}_{l,i} = (y_{il} + \frac{b_i^2}{2})|x_i|e^{j(\theta_i + \alpha_k)} - y_{il}|x_i|e^{j(\theta_i + \alpha_k)}$$

$$= (y_{il} + \frac{b_i^2}{2})|x_i|e^{j((\theta_i + \alpha) + (\alpha_k - \alpha))} - y_{il}|x_i|e^{j((\theta_i + \alpha) + (\alpha_k - \alpha))}$$

$$= \tilde{z}_{l,i}$$

Thus we have shown that $\tilde{z} = \bar{z}$. Furthermore, by the definition of $\alpha$:

$$\bar{a}^{(\gamma)}[i] = \begin{cases} 
0 & \text{if } i = 1, \ldots, \kappa \\
-a & \text{if } i = \kappa + 1, \ldots, K^{(\gamma)}
\end{cases}$$

Therefore,

$$\|\bar{a}^{(\gamma)}\|_0 = K^{(\gamma)} - \kappa$$

$$\leq \left(\left\lceil \frac{K^{(\gamma)} - 1}{2} \right\rceil - 1 \right) + 1$$

Due to the above inequality together with the fact that $\bar{a}^{(\gamma)} = a^{(\gamma)}$ for all $\gamma \in \{1, \ldots, \Gamma\} \setminus \{\gamma\}$, implies that

$$\|\bar{a}\|_0 \leq \|a\|_0$$

Hence from Definition III.1, this attack $\alpha$ is unidentifiable.

### APPENDIX III

**SIMULATION RESULTS ON IEEE-300 BUS TEST NETWORK**

In this section we demonstrate the effectiveness of the proposed sparse error correction method on the IEEE-300 bus test network. Figure 3 illustrates the PMU locations of IEEE-300 bus network used for the experimental purpose, when the network is observable from PMU measurements. As described in Figure 3, this placement setting naturally induces six zones in the network, with 20, 11, 20, 13, 23 and 16 PMUs in Zone 1 to Zone 6 respectively. The measurement generation was performed similar to the experiments on IEEE RTS-96 network, as described in Section IV. The Table III presents the median, standard deviation and the maximum value of the maximum absolute attack estimation error $\|\bar{a} - a\|_\infty$ metric from 100 Monte Carlo runs, for various percentages of the spoofed PMUs in an observable PMU placement setting. The results follow a similar trend to the IEEE RTS-96 test network, which further affirms the efficacy of the proposed sparse error correction approach.

![Fig. 3. Observable PMU placement in IEEE-300 test network and the six zones induced by these PMUs.](image)

| Spoofed PMU % | Proposed | Risbud et al. | Vanfretti et al. |
|--------------|-----------|--------------|------------------|
|              | MEDIAN (Max.) | MEDIAN (Max.) | MEDIAN (Max.) |
| 10%          | 0.879 ± 0.145 (1.584) | 6.682 ± 2.625 (23.8574) | 17.590 ± 5.786 (80.920) |
| 20%          | 1.074 ± 0.163 (2.361) | 10.331 ± 3.129 (26.619) | 15.138 ± 4.753 (54.985) |
| 30%          | 1.222 ± 1.923 (22.756) | 15.764 ± 2.754 (25.850) | 17.549 ± 5.484 (76.824) |
| 40%          | 1.471 ± 4.057 (23.734) | 17.391 ± 2.313 (31.350) | 15.361 ± 4.310 (46.991) |

**TABLE III**

**COMPARISON OF (MEDIAN ± STANDARD DEVIATION), AND THE MAXIMUM OF $l_\infty$ NORM OF THE ESTIMATION ERROR OF $\alpha$ FROM 100 MONTE CARLO RUNS, FOR OBSERVABLE PMU PLACEMENT SETTING, IN DEGREES**

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**Fig. 3. Observable PMU placement in IEEE-300 test network and the six zones induced by these PMUs.**