Generalized coherent states related to the cylindrical Bessel functions and Legendre oscillator

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Abstract
The main objective of this paper is to discuss the Paley-Wiener space $PW_1$ in the framework of a set of coherent states related to the cylindrical Bessel function and the Legendre oscillator. Thus, we show that the kernel of the Fourier transform of the $L^2$-functions that are supported in $[-1, 1]$ form a set of coherent states. This also leads to the construction a coherent states integral transform.

1. Introduction

Coherent states was originally introduced by Schrödinger in 1926 as a Gaussian wavepacket to describe the evolution of a harmonic oscillator [1]. The notion of coherence associated with these states of physics was first noticed by Glauber [2, 3] and then introduced by Klauder [4, 5]. The common point here was that all these coherent states were associated to the quantum harmonic oscillator. Because of their important properties these states were then generalized to other systems either from a physical or mathematical point of view. Here, it is necessary to emphasize that quantum coherence of states nowadays pervades many branches of physics such as quantum electrodynamics, solid-state physics, and nuclear and atomic physics from both theoretical and experimental points of view. As the electromagnetic field in free space can be regarded as a superposition of many classical modes, each one governed by the equation of simple harmonic oscillator, the coherent states became significant as the tool for connecting quantum and classical optics. For a review of all of these generalizations see [6–9].

To construct coherent states, four main different approaches are well used in literature, the so-called Schrödinger, Klauder-Perelomov, Barut-Girardello, and Gazeau-Klauder methods. While the second and the third approaches rely directly on the Lie algebra symmetries with their corresponding generators, the first is established only by means of suitable infinite superposition of wave functions associated with the harmonic oscillator, regardless of the Lie algebraic symmetries. Clearly they introduce coherent states as superpositions of Hamiltons eigenvectors which span complete and (bi-)orthogonal Hilbert spaces [7]. Different types of coherent states for quantum mechanics have been discussed by many authors from various perspectives. Thus, in [10] the authors have constructed coherent states for the Legendre oscillator, in [11–13] the authors have introduced new family of coherent states as suitable superposition of the associated Bessel functions and in [14–16] the authors also use the generating function approach to construct new kind coherent states associated with the Hermite polynomials and associated Legendre functions, respectively. Here, the interesting fact is that it is not necessary to use the algebraic and group approaches (Barut-Girardello and Klauder-Perelomov) to construct generalized coherent states.

In the present paper we discuss coherent states associated with a one-dimensional Schrödinger operator found in [10, 17] by following the procedure described in ([18, 19]), then we build a family of coherent states through superpositions of the corresponding eigenstates, say $|\psi_n\rangle$ which are expressed in terms of the Legendre polynomial [10], where the role of coefficients $z^n/\sqrt{n!}$ of the canonical coherent states is played by
where $\xi \in \mathbb{R}$ and $J_{n+\frac{1}{2}}(\cdot)$ denotes the cylindrical Bessel function [20]. We proceed by determining the wavefunctions of these coherent states in a closed form. The latter gives the kernel of the associated coherent states integral transform which makes correspondence between the quantum states Hilbert space $L^2([-1, 1], 2^{-1}dx)$ of the Legendre oscillator and a subspace of a Hilbert space of square integrable functions with respect to a suitable measure on the real line. Then, we discuss the Paley-Wiener space $PW_a$ with $a = 1$, in the framework of a set of coherent states related to the cylindrical Bessel functions for the Legendre oscillator and we show that the kernel of the Fourier transform of the $L^2$-functions that are supported in $[-1, 1]$ form a set of coherent states. Through the Whittaker–Shannon–Kotel’nikov theorem (in Theorem 3, see below), for the following change of the variable $\xi \rightarrow 2\pi \xi$, we interpret the function $f(\xi) := i^2\xi^{-1/2}J_{n+1/2}(2\pi \xi)$ defined in (1.1) as the Band-limited signal and the parameter $a = 1$ as the band limit of $f(\xi)$ and $\Omega = 2$, the corresponding frequency band [21].

The paper is organized as follows: in section 2 we introduce briefly the Paley-Wiener space $PW_a$. Section 3 is devoted to the coherent states formalism we will be using. In section 4, we recall some notions of the Legendre oscillator. In section 5, we construct a family of coherent states related to the cylindrical Bessel functions for the Legendre oscillator. And finally, section 6 is devoted to some concluding remarks on the paper.

2. Preliminary: the Paley-Wiener space $PW_a$

In this section, we summarize the notion in ([22], pp. 45–47) on the Paley-Wiener space.

**Definition 1.** Let $F$ be an entire function. Then, $F$ is an entire function of exponential type if there exists constants $A, B > 0$ such that, for all $z \in \mathbb{C}$

$$|F(z)| \leq Ae^{B|z|}. \quad (2.1)$$

If $F$ is entire function of exponential type, we call $a$ the type of $F$ where

$$a = \lim_{r \to +\infty} \sup_{|z|=r} \frac{\log M(r)}{r} \quad (2.2)$$

and where $M(r) = \sup_{|z|=r} |F(z)|$. The following conditions on an entire function $F$:

1. for every $\varepsilon > 0$ there exists $C_\varepsilon$ such that

$$|F(z)| \leq C_\varepsilon e^{(a+\varepsilon)|z|};$$

2. there exists $C > 0$ such that

$$|F(z)| \leq Ce^{a|z|};$$

3. as $|z| \to +\infty$

$$|F(z)| = o(e^{a|z|}).$$

Then clearly, (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow F$ is of exponential type at most $a$.

**Definition 2.** Let $a > 0$ and $1 \leq p < \infty$. The Paley-Wiener space $PW_a^p$ is defined as

$$PW_a^p = \left\{ f : f(x) = \int_{-a}^{a} g(y)e^{-iyx}dy, \text{ where } g \in L^p(-a, a) \right\} \quad (2.3)$$

and we set

$$\|f\|_{PW_a^p} = 2\pi \|g\|_{L^p}. \quad (2.4)$$

In other words, $PW_a^p$ is the image via the Fourier transform of the $L^p$-function that are supported in $[-a, a]$. We will essentially concentrate on the case $p = 2$, in which $PW_a$ to denote the Paley-Wiener space $PW_a^2$. Notice that, it follows from the Plancherel formula that

$$\|f\|_{PW_a^2} = \|g\|_{PW_a^2} = 2\pi \|g\|_{L^2} = \|\hat{f}\|_{L^2} = \|f\|_{L^2}. \quad (2.5)$$
Hence, by polarization, for \( f, \varphi \in PW_a \),
\[
\langle f, \varphi \rangle_{PW} = \langle f, \varphi \rangle_{L^2}.
\] (2.6)

**Theorem 1** ([22], p.66). Let \( F \) be an entire function and \( a > 0 \). Then the following are equivalent

1. \( F \in L^2(\mathbb{R}) \) and
   \[
   |F(z)| = o(e^{rz}) \quad \text{as} \quad |z| \to +\infty,
   \] (2.7)
2. there exists \( f \in L^2(\mathbb{R}) \) with \( \text{supp} \hat{f} \subseteq [-a, a] \) such that
   \[
   F(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi)e^{iz\xi}d\xi.
   \] (2.8)

Furthermore, we remark that a function \( f \) is in \( PW_a \) if and only if \( |F(z)| = o(e^{rz}) \) for \( |z| \to +\infty \).

**Theorem 2** ([22], p.69). The Paley-Wiener space \( PW_a \) is a Hilbert space with reproducing kernel w.r.t the inner product (2.6). Its reproducing kernel is the function
\[
K(x, y) = \frac{a}{\pi} \text{sinc}(a(x - y)),
\] (2.9)
where \( \text{sinc} t = \sin t / t \). Hence, for every \( f \in PW_a \)
\[
f(x) = \frac{a}{\pi} \int_{\mathbb{R}} f(y) \text{sinc}(a(x - y))dy,
\] (2.10)
where \( x \in \mathbb{R} \).

### 3. Generalized coherent states formalism

In this section we follow the generalization of the canonical coherent states according to the procedure defined in ([18, 19]).

Let \( (\mathcal{X}, \mu) \) be a measure space and let \( \mathcal{F} \subseteq L^2(\mathcal{X}, \mu) \) be a closed subspace of infinite dimension. Let \( \{ \Phi_n \}_{n=0}^{\infty} \) be an orthogonal basis of \( \mathcal{F} \) satisfying, for arbitrary \( x \in \mathcal{X} \)
\[
\omega(x) := \sum_{n=0}^{\infty} \rho_n^{-1}|\Phi_n(x)|^2 < +\infty
\] (3.1)
where \( \rho_n := ||\Phi_n||_{L^2(\mathcal{X})}^2 \). Define
\[
K(x, y) := \sum_{n=0}^{\infty} \rho_n^{-1}\Phi_n(x)\overline{\Phi_n(y)}, \quad x, y \in \mathcal{X}.
\] (3.2)
Then, the expression \( K(x, y) \) is a reproducing kernel, \( \mathcal{F} \) is the corresponding kernel Hilbert space and \( \omega(x) := K(x, x), \quad x \in \mathcal{X} \).

**Definition 3.** Let \( \mathcal{H} \) be a Hilbert space with \( \dim \mathcal{H} = +\infty \) and \( \{ \phi_n \}_{n=0}^{\infty} \) be an orthonormal basis of \( \mathcal{H} \). The coherent states labeled by point \( x \in \mathcal{X} \) are defined as the ket-vector \( \psi_x \equiv |x\rangle \in \mathcal{H} \), such that
\[
\psi_x \equiv |x\rangle := (\omega(x))^{-1/2} \sum_{n=0}^{\infty} \frac{\Phi_n(x)}{\sqrt{\rho_n}} \phi_n.
\] (3.3)
By definition, it is straightforward to show that \( \langle \psi_x, \psi_{y}\rangle_{\mathcal{H}} = 1 \).

**Definition 4.** The coherent states transform associated to the set of coherent states \( \{ \psi_x \}_{x \in \mathcal{X}} \) is the isometric map
\[
\mathcal{W}[\phi](x) := (\omega(x))^{1/2}\langle \phi|x\rangle_{L^2(\mathcal{X})},
\] (3.4)
Thereby, we have a resolution of the identity of \( \mathcal{H} \) which can be expressed in Dirac’s bra-ket notation as
\[
1_{\mathcal{H}} = \int_{\mathcal{X}} d\mu(x)\omega(x)|x\rangle\langle x|
\] (3.5)
and where \( \omega(x) \) appears as a weight function.
Remark 1. Note that the formula (3.3) can be considered as generalization of the series expansion of the canonical coherent states [23]

\[ \psi_z \equiv |z\rangle := e^{-\frac{1}{2}|z|^2} \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{k!}} \phi_k, \quad z \in \mathbb{C} \]  

(3.6)

with \( \{ \phi_k \}_{k=0}^{\infty} \) being an orthonormal basis of eigenstates of the quantum harmonic oscillator. Here the space \( \mathcal{F}(\mathbb{C}) \) is the Fock space and \( \omega(z) = \pi^{-\frac{1}{2}|z|^2}, \quad z \in \mathbb{C} \).

4. The Legendre oscillator

In this section, we summarize the work of Borzov and Demaskinsky ([10, 17]) on the Legendre Hamiltonian

\[ H = X^2 + P^2 = a^+ a^- + a^a a^+, \]  

(4.1)

where \( X \) and \( P \) denotes respectively the position and momentum operators, \( a^+ \) and \( a^- \) denotes respectively the creation and annihilation operators (see bellow). The eigenvalues of operators \( H \) are equal to

\[ \lambda_0 = \frac{2}{3}, \quad \lambda_n = \frac{n(n+1) - \frac{1}{2}}{(n + \frac{1}{2})(n - \frac{1}{2})}, \quad n = 1, 2, 3, \ldots, \]  

(4.2)

and the corresponding eigenfunctions

\[ \psi_n(x) = \sqrt{2n+1} P_n(x), \quad n = 0, 1, 2, 3, \ldots, \]  

(4.3)

which form an orthonormal basis \( \{ |\psi_n\rangle \equiv |n\rangle \}_{n=0}^{\infty} \) in the Hilbert space \( \mathcal{H} := L^2([-1, 1], dx) \). These functions fulfill the recurrence relations

\[ x \psi_n(x) = b_{n-1} \psi_{n-1}(x) + b_n \psi_{n+1}(x), \quad \psi_{-1}(x) = 0, \quad \psi_0(x) = 1, \]  

(4.4)

with coefficients

\[ b_n = \sqrt{\frac{(n+1)^2}{(2n+1)(2n+3)}}, \quad n \geq 0. \]  

(4.5)

The generalized position operator on the Hilbert space \( \mathcal{H} \) connected with the Legendre polynomials \( P_n(x) \) is an operator of multiplication by argument

\[ X|n\rangle = x|n\rangle. \]  

(4.6)

Taking into account of the relation (4.4), then

\[ X|\psi_n\rangle = b_n \psi_{n+1}(x) + b_{n-1} \psi_{n-1}(x), \]  

(4.7)

the coefficients \( b_n \) are defined by the relation (4.5). Because \( \sum_{k=0}^{\infty} \frac{1}{k} \) is \( \infty \), the operator \( X \) is a self-adjoint operator in the space \( \mathcal{H} \) (see [24–26]). The generalized momentum operator \( P \) by the way described in ([17], p.126) acts on the basis elements in \( \mathcal{H} \) by the following formula

\[ P|n\rangle = i(b_n n + 1) - b_{n-1} (n - 1) \). \]  

(4.8)

Calculating the usual commutator of operator \( X \) and \( P \) on the basis elements, we obtain

\[ [X, P]|n\rangle = 2i(b_n^2 - b_{n-1}^2)|n\rangle = \frac{2i}{(2n - 1)(2n + 1)(2n + 3)}|n\rangle. \]  

(4.9)

The creation and annihilation operators defined in (4.1) are define by the standard relations

\[ a^+ = \frac{1}{\sqrt{2}}(X - iP); \quad a^- = \frac{1}{\sqrt{2}}(X + iP). \]  

(4.10)

On the basis element in \( \mathcal{H} \) these operators act by the rule

\[ a^+|n\rangle = \sqrt{2} b_n|n+1\rangle; \quad a^-|n\rangle = \sqrt{2} b_{n-1}|n\rangle, \]  

(4.11)

they satisfy the commutation relations

\[ [a^-, a^+] = \frac{1}{i}[X, P]. \]  

(4.12)

Remark 2. In the given research the Legendre polynomials \( P_n(x) \) and Legendre function \( \psi_n(x) \) play the same role as the Hermite polynomials and the Hermite function play in the standard quantum mechanics. For more details on the mathematical analysis of the Hamiltonian (4.1) (see [17], p.126).
5. Generalized coherent states for the Legendre Hamiltonian

We now construct a class of coherent states indexed by point \( \xi \in \mathbb{R} \) by replacing the coefficients \( z^n/\sqrt{n!} \) of the canonical coherent states by the cylindrical Bessel functions as mentioned in the introduction.

**Definition 5.** Define the set of states labeled by points \( \xi \in \mathbb{R} \) by the following superposition

\[
|\xi\rangle = \mathcal{N}(\xi)^{-1/2} \sum_{n=0}^{\infty} \frac{\Phi_n(\xi)}{\rho_n} |\psi_n\rangle, \tag{5.1}
\]

here \( \mathcal{N}(\xi) \) is a normalization factor, the function

\[
\Phi_n(\xi) = i^n \sqrt{\frac{\pi}{2\xi}} J_{n+\frac{1}{2}}(\xi), \tag{5.2}
\]

where \( J_{n+1/2}(\cdot) \) is the cylindrical Bessel function ([20], p. 626):

\[
J_{n+\frac{1}{2}}(z) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s + n + 1/2)!} \left( \frac{z}{2} \right)^{2s+n+\frac{1}{2}}, \quad z \in \mathbb{C} \tag{5.3}
\]

and \( \rho_n \) are positive numbers given by

\[
\rho_n = \frac{1}{2n + 1}, \quad n = 0, 1, 2, \ldots, \tag{5.4}
\]

and \( \{ |\psi_n\rangle \} \) is an orthonormal basis of the Hilbert space \( \mathcal{H} = L^2([-1, 1], 2^{-1}dx) \) defined in (4.3).

In the following results we give some properties verified by coherent states (5.1).

**Proposition 1.** The normalization factor defined by coherent states (5.1) reads

\[
\mathcal{N}(\xi) = 1, \tag{5.5}
\]

for every \( \xi \in \mathbb{R} \).

**Proof.** According to (5.1) and using the orthonormality relation of basis elements \( \{ |\psi_n\rangle \}_{n=0}^{\infty} \) defined in (4.3) it follows that

\[
\langle \xi|\xi\rangle = \pi (\xi \mathcal{N}(\xi))^{-1} \sum_{n=0}^{\infty} \left( n + \frac{1}{2} \right) J_{n+\frac{1}{2}}(\xi) J_{n+\frac{1}{2}}(\xi). \tag{5.6}
\]

In order to identify the above series, we make appeal to the formula ([27], p. 591):

\[
\sum_{n=0}^{\infty} \left( n + \frac{1}{2} \right) J_{n+\frac{1}{2}}(\xi) J_{n+\frac{1}{2}}(\xi) = \pi^{-1} \xi, \tag{5.7}
\]

we obtain the announced result by using the condition \( \langle \xi|\xi\rangle = 1 \). \( \square \)

**Proposition 2.** The coherent states defined in (5.1) satisfy the following resolution of the identity

\[
\int_{\mathbb{R}} \langle \xi|\xi\rangle \ d\mu(\xi) = 1_{\mathcal{H}}, \tag{5.8}
\]

in terms of an acceptable measure

\[
d\mu(\xi) = \frac{1}{\pi} \ d\xi, \tag{5.9}
\]

where \( d\xi \) the Lebesgue’s measure on \( \mathbb{R} \).

**Proof.** We shall now determine \( \sigma(\xi) \). Let

\[
d\mu(\xi) = \sigma(\xi) \ d\xi, \tag{5.10}
\]

where \( \sigma(\xi) \) is an auxiliary density function. Then, according to (5.1) and by writing

\[
\int_{\mathbb{R}} |\xi\rangle \langle \xi| \ d\mu(\xi) = \sum_{n,m=0}^{\infty} \frac{\pi}{2} (-1)^{m+n} \left( \int_{-\infty}^{\infty} \frac{J_{n+\frac{1}{2}}(\xi) J_{m+\frac{1}{2}}(\xi)}{\sqrt{\rho(n) \rho(m)}} \sigma(\xi) \frac{d\xi}{\xi} \right) |\psi_n\rangle \langle \psi_m|, \tag{5.11}
\]
\[ \sum_{n,m=0}^{\infty} \frac{\pi}{2} (-1)^n i^{n+m} \sqrt{(2n + 1)(2m + 1)} \left( \int_{-\infty}^{\infty} J_{n+\frac{1}{2}}(\xi) J_{m+\frac{1}{2}}(\xi) \frac{d\xi}{\xi} \right) \left| \psi_n \right> \left< \psi_m \right|. \]  

(5.12)

Hence, we need \( \sigma(\xi) \) such that

\[ \int_{-\infty}^{\infty} J_{n+\frac{1}{2}}(\xi) J_{m+\frac{1}{2}}(\xi) \sigma(\xi) \frac{d\xi}{\xi} = \frac{2}{\pi(2n + 1)} \delta_{m,n}. \]  

(5.13)

We make appeal to the integral (27), p.211:

\[ \int_{-\infty}^{\infty} J_{n+\frac{1}{2}}(\xi) J_{m+\frac{1}{2}}(\xi) dy = \frac{2}{2n + 1} \delta_{m,n}, \]  

(5.14)

with condition \( c > 0 \). Then, for parameters \( c = 1 \) and \( y = \xi \), we obtain

\[ \int_{-\infty}^{\infty} \frac{1}{\xi} J_{n+\frac{1}{2}}(\xi) J_{m+\frac{1}{2}}(\xi) d\xi = \frac{2}{2n + 1} \delta_{m,n}. \]  

(5.15)

By comparing (5.15) with (5.13) we obtain finally the desired weight function \( \sigma(\xi) = 1/\pi \). Therefore, the measure (5.10) has the form (5.9). Indeed (5.12) reduces further to

\[ \sum_{n=0}^{\infty} |\psi_n\rangle \langle \psi_n| = I_H, \]  

(5.16)

in other words

\[ \int_{\mathbb{R}} |\xi\rangle \langle \xi| d\mu(z) = I_H, \]  

(5.17)

in terms of Dirac’s bra-ket notation. This ends the proof.

Thus, according to this construction the states \( |\xi\rangle \) form an overcomplete basis in the Hilbert space \( \mathcal{H} \).

The coherent states in (5.1) are defined as vectors in the Hilbert space \( \mathcal{H} \). In the case where these states describe a quantum system, by the usual quantum mechanical convention, the probability of finding the state \( |\psi_n\rangle \) in some normalized state \( |\xi\rangle \) of the state Hilbert space \( \mathcal{H} \) is given by \( \langle \psi_n|\xi\rangle^2 \). Thus for coherent states (5.1) it is given by

\[ n \mapsto \frac{\pi(2n + 1)}{2|\xi|^2} |J_{n+\frac{1}{2}}(\xi)|^2 \]  

(5.18)

for every \( \xi \in \mathbb{R} \).

Now, in the following results we will establish a closed form for the construct coherent states in (5.1) and we will discuss the associated coherent states integral transform

**Proposition 3.** For \( \xi \in [-1, 1] \). Then, the wavefunctions of coherent states defined in (5.1) can be written in a closed form as

\[ \langle x|\xi\rangle = e^{-i\xi x}, \]  

(5.19)

for every \( \xi \in \mathbb{R} \).

**Proof.** We start by writing the expression of the wavefunctions of coherent states (5.1) as follows

\[ \langle x|\xi\rangle = \mathcal{N}(\xi)^{-1/2} \sum_{n=0}^{\infty} \frac{\Phi_n(\xi)}{\sqrt{\rho_n}} \langle x|\psi_n\rangle, \]  

(5.20)

we have thus to look for a closed form of the series

\[ \mathcal{G}(x) = \sum_{n=0}^{\infty} \frac{\Phi_n(\xi)}{\sqrt{\rho_n}} \langle x|\psi_n\rangle, \]  

(5.21)

To do this, we start by replacing the function \( \Phi_n(\xi) \) and the positive sequences \( \rho_n \) by their expressions in (5.2) and (5.4) thus Eq.(5.21) reads

\[ \mathcal{G}(x) = \frac{\pi}{2\xi} \sum_{n=0}^{\infty} (-1)^n i^n \sqrt{2n+1} J_{n+\frac{1}{2}}(\xi) \langle x|\psi_n\rangle. \]  

(5.22)

Making use the explicit expression in (4.3) of the eigenstates \( \psi_n(x) \), then the sum (5.22) becomes

\[ \mathcal{G}(x) = \frac{2\pi}{\xi} \sum_{n=0}^{\infty} (-1)^n i^n \left( n + \frac{1}{2} \right) J_{n+\frac{1}{2}}(\xi) P_n(x). \]  

(5.23)
We now appeal to the Gegenbauer’s expansion of the plane wave in Gegenbauer polynomials and Bessel functions ([28], p.116):

\[ e^{i\xi z} = \Gamma(\gamma) \left( \frac{\xi}{2} \right)^{\gamma} \sum_{n=0}^{\infty} i^n (n + \gamma) C_{n+\gamma}^\gamma(y), \]

(5.24)

for the case of parameters \( \gamma = 1/2, y = x \), which by the help of the well known identity \( \Gamma(1/2) = \sqrt{\pi} \), gives the announced result in (5.19).

**Corollary 1.** Let the variable \( \xi \ll 1 \). Then, the coherent states defined in (5.1) transforms to

\[ |\xi| \approx N(\xi)^{-1/2} \sum_{n=0}^{\infty} \frac{\sqrt{2\pi} (-i\xi)^n}{\sqrt{2^{2n+1}(2n + 1)} \Gamma \left( n + \frac{1}{2} \right)} |\psi_n\rangle. \]

(5.25)

**Proof.** The result follows immediately by using the formula ([20], p.647):

\[ \sum_{n=0}^{\infty} \left( n + \frac{1}{2} \right) \frac{h_{n+\frac{1}{2}}(\eta) h_{n+\frac{1}{2}}(\xi)}{\sqrt{\eta}} = \frac{\sqrt{\eta} \xi}{\pi (\eta - \xi)} \sin(\eta - \xi), \]

(5.28)

where

\[ J_n(\xi) = \frac{\sqrt{\eta} \xi}{\sqrt{2\xi}} h_{n+\frac{1}{2}}(\xi), \quad n = 0, 1, 2, \ldots, \]

(5.27)

is the spherical Bessel function [20]. This ends the proof.

The careful reader has certainly recognized in (5.25) the expression of nonlinear coherent states [23].

Note that, in view of the formula ([27], p.667):

\[ \sum_{n=0}^{\infty} \left( n + \frac{1}{2} \right) \frac{h_{n+\frac{1}{2}}(\eta) h_{n+\frac{1}{2}}(\xi)}{\sqrt{\eta}} = \frac{\sin(\eta - \xi)}{\eta - \xi}, \]

(5.29)

which is the Dyson sine kernel, so from the theorem 2, equation (5.29) is the reproducing kernel of the Paley-Wiener Hilbert space \( PW_1 \) (see [22]) as defined in (2.3). Thus, with respect to the measure \( d\mu \) given in (5.9), the family \( \{ \pi (n + 1/2)/\xi^{1/2} h_{n+\frac{1}{2}}(\xi) \}; \quad n \in \mathbb{N}_0 \) form an orthonormal basis of \( PW_1 \) (see [29]).

Next, once we have obtained a closed form of the constructed coherent states, we can look for the associated coherent states transform. In view of (5.1), this transform should map the space \( \mathcal{H} = L^2([-1, 1], 2^{-1} dx) \) spanned by eigenstates \( \{ \psi_n \} \) in (4.3) onto \( PW_1 \subset L^2(\mathbb{R}, d\mu) \) in the following manner:

**Proposition 4.** For \( \varphi \in L^2([-1, 1], 2^{-1} dx) \), the coherent states transform is the unitary map

\[ B(L^2([-1, 1], 2^{-1} dx)) = PW_1, \]

(5.31)

defined by means of (5.19) as

\[ B[\varphi](\xi) = \sqrt{N(\xi)} \langle \varphi | \xi \rangle = \int_{-1}^{1} e^{-i\xi \varphi(x)} dx, \]

(5.32)

for all \( \xi \in \mathbb{R} \).

Then, as consequence

**Corollary 2.** The following integral

\[ \frac{(-i)^n}{\sqrt{\xi}} h_{n+\frac{1}{2}}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} P_n(x) e^{-i\xi x} dx, \]

(5.33)

holds for all \( \xi \in \mathbb{R} \).

**Proof.** According to (5.32), the image of the basis vector \( \{ \psi_n \} \) under the transform \( B \) should exactly be the coefficients in (5.1), more precisely
\[ B[\psi_n](\xi) = (-i)^n \sqrt{\frac{\pi(2n+1)}{2\xi}} J_{n+\frac{1}{2}}(\xi). \]  

(5.34)

We start by writing expression in (5.32) as follows

\[ B[\psi_n](\xi) = \int_{-1}^{1} e^{-i\xi \psi_n(x)} \frac{dx}{2}, \]

(5.35)

and replacing \( \psi_n \) by their values given in (4.3), we obtain

\[ B[\psi_n](\xi) = \frac{\sqrt{2n+1}}{2} \int_{-1}^{1} e^{-i\xi \psi_n(x)} \frac{dx}{2}, \]

(5.36)

the integral (5.36) can be evaluated by the help of the following formula ([30], p.456):

\[ \int_{-1}^{1} P_n(x) e^{i\xi x} dx = i^n \sqrt{\frac{2\pi}{\xi}} J_{n+\frac{1}{2}}(\xi), \]

(5.37)

this ends the proof. \( \square \)

Note that in view of ([23], p.29), by considering \( h_n(\xi) := \rho_n^{-1/2} \Phi_n(\xi) \) and coherent states \( \mathcal{K}(\xi, x) = \langle x|\xi \rangle \), then, the basis element \( \psi_n \in L^2([-1, 1], 2^{-1}dx) \) has the integral representation

\[ \psi_n(x) = \int_{-\infty}^{\infty} h_n(\xi) \mathcal{K}(\xi, x) d\mu(\xi) \]

(5.38)

where the function \( \Phi_n(\xi) \) and the positive sequences \( \rho_n \) are given respectively in (5.2) and (5.4), the measure \( d\mu(\xi) \) is given in (5.9), then we find the following expression

\[ P_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-i)^n J_{n+\frac{1}{2}}(\xi) \frac{e^{i\xi x}}{\sqrt{\xi}} d\mu(\xi), \]

(5.39)

which is recognized as the Fourier transform of the spherical Bessel function (5.27) (see [30], p.267):

\[ \int_{-\infty}^{\infty} e^{i\eta t} \mathcal{F}(t) dt = \begin{cases} 
\pi^n P_n(x), & -1 < x < 1 \\
\frac{1}{2} \pi(n+i)^n, & x = \pm 1 \\
0, & x > 1 
\end{cases} \]

(5.40)

where \( P_n \) is the Legendre polynomial [30].

The sampling theorem of band limited functions\(^1\), which is often named after Whittaker-Shannon-Kotel’nikov. This is its classical formulation

**Theorem 3** ([21]). If \( f \in L^1(\mathbb{R}) \) and \( \hat{f} \), the Fourier transform of \( f \), is supported on the interval \([-a, a]\), then

\[ f(\xi) = \int_{-a}^{a} \hat{f}(\tau) e^{2\pi i \tau \xi} d\tau = \sum_{m \in \mathbb{Z}} \hat{f}(\frac{m}{2a}) \text{sinc}\left(2a\pi\left(\xi - \frac{m}{2a}\right)\right) \]

(5.41)

where the equality hold in the \( L^2 \)-sense, that is, the series in RHS of (5.41) converges to \( f \) in \( L^2(\mathbb{R}) \).

Then, making the change \( \xi \mapsto 2\pi \xi \) in Eq.(5.33) and by taking the parameter \( a = 1 \) in (5.41) we conclude that

\[ \frac{1}{\sqrt{\xi}} J_{n+\frac{1}{2}}(2\pi \xi) = \sum_{m \in \mathbb{Z}} \frac{2}{m} J_{n+\frac{1}{2}}(m\pi) \frac{\sin \pi(2\xi - m)}{\pi(2\xi - m)}, \quad n = 0, 1, 2, 3, \ldots, \]

(5.42)

for all \( \xi \in \mathbb{R} \). For \( n = 0 \), Eq.(5.42) becomes

\[ \frac{\sin(2\pi \xi)}{2\pi \xi} = \sum_{m \in \mathbb{Z}} \frac{\sin(m\pi)}{m\pi} \frac{\sin(\pi(2\xi - m))}{\pi(2\xi - m)}. \]

(5.43)

Thus, the function \( f(\xi) = 2(-i)^n J_n(2\pi \xi) = (-i)^n(\xi)^{-1/2} J_{n+1/2}(2\pi \xi) \) where \( J_n(\cdot) \) denotes the spherical Bessel functions, can be interpreted as the band-limited signal [21]. We see that the spherical Bessel functions expansion may find applications in probability theory, where it can be used to represent classes of characteristic functions and autocorrelation functions [35].

**Remark 3.** Also note that:

\(^1\) \( f \in L^1(\mathbb{R}) \) is bandlimited if there exists \( a \in \mathbb{R} \) such that \( \text{supp} \hat{f} \subseteq [-a, a] \), and \( a \) is a band limite of \( f \) and \( \Omega := 2a \), the corresponding frequency band.
• The usefulness expansion of coherent states in (5.24) was made very clear in a paper authored by Ismail and Zhang, where it was used to solve the eigenvalue problem for the left inverse of the differential operator, on $L^2$-spaces with ultraspherical weights [31, 32].

• The exact differential of coherent states $K(x, \xi) = e^{-ix\xi}$ defined in (5.19) is given by

$$dK(x, \xi) = -iK(x, \xi)(\xi dx + x d\xi)$$

(6.43)

which gives the full information about rates of change of coherent states in the $x$-direction and in the $\xi$-direction, the field of vector is the gradient of $K(x, \xi)$ i.e.

$$V(x, \xi) = \nabla K(x, \xi)$$

(6.44)

for all $x \in [-1, 1]$ and $\xi \in \mathbb{R}$.

• For $x, \xi \in \mathbb{R}$, the function $\varphi_\xi(x) = e^{ix\xi}$ is known as the Gabor’s coherent states introduced in signal theory where the property $\psi_\xi = \hat{T}(\xi)\psi$, with $\psi \in L^2(\mathbb{R})$, and $\hat{T}(\xi)$ the unitary transformation, is obtained by using the standard representation of the Heisenberg group in three dimensions, in $L^2(\mathbb{R})$, for more information (see [33]). The chosen coefficients (1) and eigenfunctions (4.3) have been used in [35, p 1625] where the authors presented a representation for band-limited functions in terms of spherical Bessel functions. The polynomial approximation of the Fourier transform have been used.

6. Concluding remarks

In this work, we have built a class of coherent states related to the cylindrical Bessel functions for the Legendre oscillator while discussing the Paley-Wiener space $PW_1$ in the framework of these coherent states. We found that, the kernel of the Fourier transform of $L^2$-function that are supported in $[-1, 1]$ form a set of coherent states. Furthermore, we were able to define a coherent states transform which maps isometrically $L^2([-1, 1], 2^{-1}dx)$ onto a subspace of a Hilbert space of square integrable functions with respect to a suitable measure; here the space $PW_1$. Clearly, this subspace is spanned by the coefficients (given in terms of Cylindrical Bessel functions) we have chosen to superpose eigenstates of the Legendre oscillator. In fact, it is well known that, in quantum mechanics, canonical coherent and affine coherent states, and in signal analysis, Gabor and wavelet frames have offered an extremely flexible alternative method for describing functions in $L^2(\mathbb{R})$ and in $L^2(\mathbb{R},)$. However, few methods exist for the description of $L^2$-functions on an interval $[a, b]$, see [34]. Thus, here, we were presenting a contribution in this direction. Note that the constructed coherent states (see definition 5) reproduce the nonlinear coherent states (NLCSs) defined in (5.25). So, it is crucial to analyze some properties arising from these NLCSs. This will be the subject of our future work.

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