The profile of chiral skyrmions of large radius

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Abstract

We study the profile of an axially symmetric magnetic skyrmion in a ferromagnet with the Dzyaloshinskii-Moriya interaction (DMI). We give exact formulas for the solution of the Landau-Lifshitz equation in the asymptotic limit that the dimensionless DMI parameter approaches a critical value \( \epsilon \to \epsilon_0 \) and the skyrmion radius becomes large and diverges to infinity. We give the profile field at the skyrmion core, the far field and the field of the skyrmion domain wall. An asymptotic formula for the skyrmion radius as a function of the DMI constant is obtained.

Keywords:
Magnetic skyrmion, Micromagnetics, Dzyaloshinskii-Moriya interaction

1. Introduction

Magnetic skyrmions are two-dimensional topological solitons. After their theoretical prediction \([1, 2]\) they have been observed in ferromagnets with the Dzyaloshinskii-Moriya interaction (DMI) and techniques have been developed for individual skyrmions to be created and annihilated in a controlled manner \([3]\). DMI arises from the loss of chiral symmetry induced by the underlying crystal structure or due to thin-film or multilayer geometries. In micromagnetic models, DMI appears in the form of linear combinations of the so-called Lifshitz invariants which are linear in the gradient of the magnetization field and thereby sensitive with respect to reflections in magnetization and coordinate space. Stable skyrmions arise as individual entities when, in combination with the DMI, there is a large enough perpendicular anisotropy in the film or when a large enough external magnetic field is applied or when we have a combination of these two interactions \([2]\).

The skyrmion profile determines to a large extent, and sometimes crucially, the skyrmion properties \([4]\). Its details are thus essential for the manipulation of individual skyrmions. Specifically, it enters in formulas for dynamical phenomena \([5, 6]\), for example, skyrmion translation and rotation modes \([7]\), and it is crucial for quantitative calculations. In recent years, sufficient resolution has been obtained for the observation of the features of the skyrmion profile in great detail \([8, 9, 10, 11, 12, 13, 14]\). It is thus important to obtain a detailed analytical description of the skyrmion profile. Such an analysis, in combination with the
possibility of observing the profile in high resolution will open the way for a wider exploitation of individual 
skyrmions.

Skyrmionic solutions of the Landau-Lifshitz equation in the presence of DMI can be found by various 
numerical methods [2, 10]. Numerical results provide the phase diagram for the existence of skyrmions and 
various features of the skyrmion profile. The existence of skyrmionic solutions as local minimizers of the 
micromagnetic energy has been rigorously proved only for the case of an external field [15, 16].

Analytic formulas for the profile of axisymmetric skyrmions have been derived for the case of skyrmions 
of small radius [17]. In this asymptotic regime where the dimensionless DMI parameter $\epsilon$ defined in Eq. (3), 
i.e. the ratio of the DMI constant divided by (half) the domain wall energy, is small, magnetic skyrmions 
are well approximated on small scales by the classical Belavin-Polyakov soliton. The results in Ref. [17] 
provide a quantitative description of this approximation in terms of asymptotic formulas for the skyrmion 
radius $R \sim \frac{\epsilon}{\ln(\epsilon)}$ and energy $E - 4\pi \sim \frac{\epsilon^2}{\ln(\epsilon)}$ for $\epsilon \ll 1$. The formulas achieved, however, go way beyond 
this asymptotic near field analysis capturing the multiscale nature and decay behaviour of skyrmion profiles 
away from the core.

For skyrmions of large radius, an ad-hoc ansatz based on explicit one-dimensional (1D) domain wall 
profiles [18] has been suggested and is widely used to examine structural and dynamic properties, see, e.g., 
[8, 19, 20].

In this paper we derive formulas for the skyrmion profile in the case of large skyrmion radius by 
employing asymptotic methods that give analytic approximations of the skyrmion solutions for the time-
independent Landau-Lifshitz equation. A large radius is achieved in the case of perpendicular anisotropy 
and when the dimensionless DMI constant approaches a certain value, $\epsilon \rightarrow \epsilon_0$. Our analysis shows the 
multi-scale nature of the skyrmion profile and reveals a decomposition into three regions; the skyrmion 
core, the transition region, and the far field. Different formulas are obtained for the three regions. The 
deviation of the transition region from the profile of the 1D domain wall is derived in detail. The formula 
relating the skyrmion radius $R$ with the dimensionless constant is obtained as

$$
\epsilon = \epsilon_0 + \frac{\epsilon_2}{R^2} + \frac{\epsilon_4}{R^4} + \cdots
$$

and the numerical values of the constants $\epsilon_0, \epsilon_2, \epsilon_4, \ldots$ are calculated. In particular, $\epsilon_0 = \frac{2}{\pi}$ marks the critical 
DMI strength where the skyrmion radius tends to infinity and where the transition from the ferromagnetic 
regime (with the uniform state as the ground state and the skyrmion as an excited state) to the helical regime 
takes place. The results of the present analysis for large radius, taken in combination with the results of 
Ref. [17] for small radius give a reasonably complete description of the skyrmion profile.

The paper is arranged as follows. In Section 2 we derive the asymptotic formulas for the skyrmion 
profile, except in the skyrmion center. In Section 3 we derive an asymptotic formula for the skyrmion core. 
In Section 4 we give a systematic method to obtain an asymptotic series for the skyrmion profile and the 
corresponding series (1) for the parameter $\epsilon$ as a function of the skyrmion radius $R$. Section 5 contains a 
theorem which establishes a fundamental parity property for the skyrmion profile. In Section 6 we apply 
the derived formulas and calculate specific skyrmion profiles and also numerical values for the constants in 
Eq. (1). Finally, Appendix A gives an alternative derivation of the limiting value $\epsilon_0$.

2. Large radius asymptotics

We assume a thin film of a ferromagnetic material with exchange, Dzyaloshinskii-Moriya interaction 
(DMI), and anisotropy of the easy-axis type perpendicular to the film which is lying on the $xy$-plane. The 
micromagnetic structure is described via the magnetization vector $\mathbf{m} = \mathbf{m}(x, y)$ with a fixed magnitude
normalized to unity, \( m^2 = 1 \). Static magnetization configurations satisfy the Landau-Lifshitz equation. Its normalized form reads [17]

\[
m \times \left( \partial_\mu \partial_\mu m + m_3 \hat{e}_3 - 2 \epsilon h_{\text{DMI}} \right) = 0.
\]

(2)

A summation over repeated indices \( \mu = 1, 2 \) is assumed and \( \hat{e}_1, \hat{e}_2, \hat{e}_3 \) are the unit vectors for the magnetization in the respective directions. The last term in the parenthesis in Eq. (2) models the DMI. One may choose the bulk DMI form \( h_{\text{DMI}} = \hat{e}_\mu \cdot (\partial_\mu m \times m) \), the interfacial DMI form \( h_{\text{DMI}} = \epsilon_{\mu\nu} \hat{e}_\mu \cdot (\partial_\mu m \times m) \), or a combination of these. In Eq. (2), we measure lengths in units of the domain wall width \( \ell_w = \sqrt{A/K} \), where \( A \) is the exchange and \( K \) the anisotropy constant. The equation contains a single parameter \( \epsilon = \ell_S/\ell_w \) defined via an additional length scale of this model \( \ell_S = D/(2K) \), where \( D \) is the DMI constant. We will sometimes refer to \( \epsilon \) as the dimensionless DMI parameter, but one should keep in mind that it can also be controlled by changing the anisotropy. The lowest energy (ground) state is the spiral for \( \epsilon > 2/\pi \) and the ferromagnetic state for \( \epsilon < 2/\pi \) [2, 21].

Let us consider the angles \( (\Theta, \Phi) \) for the spherical parametrization of the magnetization vector, and the polar coordinates \( (r, \phi) \) for the film plane. We assume an axially symmetric skyrmion with \( \Phi = \phi + \phi_0 \) and \( \Theta = \Theta(r) \). For a bulk DMI term the energy is minimized for \( \phi_0 = \pi/2 \) (Bloch skyrmion) and for interfacial DMI we choose \( \phi_0 = 0 \) (Néel skyrmion). A value \( 0 < \phi_0 < \pi/2 \) should be chosen if the DMI term is a combination of the bulk and interfacial terms.

The skyrmion profile \( \Theta = \Theta(r) \) satisfies the equation

\[
\Theta'' + \frac{\Theta'}{r} - \frac{\sin(2\Theta)}{2r^2} + \frac{\sin(2\Theta)}{2} + 2\epsilon \frac{\sin^2 \Theta}{r} = 0
\]

(4)

with boundary conditions \( \Theta(0) = \pi \) and \( \lim_{r \to \infty} \Theta(r) = 0 \). The same equation applies to all types of skyrmions, e.g., Bloch and Néel skyrmions for the respective DMI terms. We study the case of a large skyrmion radius \( R \); we will see that our results still constitute a fair approximation for relatively small values of the radius. In order to motivate the course of our analysis, we consider the unphysical profile, characterized by a discontinuity at an arbitrary "radius" \( R \), that serves as a crude approximation of a domain wall.

\[
\Theta(r) = \begin{cases} 
\pi, & \text{if } r < R \\
0, & \text{if } R > r.
\end{cases}
\]

(5)

We notice that the unphysical model satisfies all the conditions of the boundary value problem of Eq. (4), when \( r \neq R \). We derive the actual skyrmion profile by making the appropriate adjustments to the unphysical model.

An initial adjustment of the domain wall, to be improved below, arises from neglecting the terms of the equation that have \( r \) in the denominator, when \( r \) is large. We are left with the equation

\[
\Theta'' - \frac{\sin(2\Theta)}{2} = 0.
\]

(6)

The solution of this equation is

\[
\Theta_0 = 2 \arctan(e^{-T})
\]

(7)

where the spatial variable

\[
T = r - R
\]

(8)
is now centered at the domain wall. The constants of integration have been evaluated by requiring \( \Theta_0 = \frac{\pi}{2} \) when \( T = 0 \) and \( \Theta_0 \to 0 \) as \( T \to \infty \). We calculate easily the following quantities that will be used below,

\[
\Theta_0' = -\text{sech} T, \quad \cos(2\Theta_0) = 1 - 2\text{sech}^2 T, \quad \sin(2\Theta_0) = 2 \text{sech} T \tanh T, \quad \sin^2 \Theta_0 = \text{sech}^2 T. \tag{9}
\]

In order to study the broader region outside the domain wall, we make a change of scaling in which the large skyrmion radius becomes the unit of the spatial variable,

\[
r = Rt, \quad \dot{\Theta} = \frac{d\Theta}{dt}. \tag{10}
\]

In the scaled spatial variable, the unphysical profile (5) becomes

\[
\Theta = \begin{cases} 
\pi, & \text{if } t < 1 \\
0, & \text{if } t > 1.
\end{cases} \tag{11}
\]

In the regime \(|T| \gg 1\), the function \( \Theta_0 \) has asymptotics,

\[
\Theta_0 \sim \begin{cases} 
\pi - 2e^tR, & t < 1, \\
2e^{(1-t)R}, & t > 1.
\end{cases} \tag{12}
\]

and the full skyrmion equation becomes

\[
\frac{1}{R^2} \left( \ddot{\Theta} + \frac{\dot{\Theta}}{t} - \frac{\sin(2\Theta)}{2t^2} \right) - \frac{2 \sin(2\Theta)}{R} + \frac{\epsilon \sin^2 \Theta}{t} = 0. \tag{13}
\]

We initially concentrate on the region of \( t \), that excludes the boundary value \( t = 0 \). The third term inside the parenthesis, divided by \( R^2 \), is then dominated by the \( \sin(2\Theta) \) term outside and is neglected. We obtain the equation

\[
\frac{1}{R^2} \left( \ddot{\Theta} + \frac{\dot{\Theta}}{t} \right) - \frac{2 \sin(2\Theta)}{R} + \frac{\epsilon \sin^2 \Theta}{t} = 0, \quad t \neq 0. \tag{14}
\]

We seek asymptotic solutions for which the value of \( \Theta \) is close to \( \pi \) (skyrmion core), or close to zero (far field). The DM term is then clearly negligible when compared to the anisotropy term. Letting \( \Theta = \pi - \theta \) when \( t < 1 \) and approximating the sine function with its argument close to zero, we reduce Eq. (14) to the pair of equations

\[
\frac{1}{R^2} \left( \ddot{\theta} + \frac{\dot{\theta}}{t} \right) - \theta = 0, \quad 0 < t < 1; \quad R \to \infty. \tag{15}
\]

and

\[
\frac{1}{R^2} \left( \ddot{\Theta} + \frac{\dot{\Theta}}{t} \right) - \Theta = 0, \quad t > 1; \quad R \to \infty. \tag{16}
\]

Seeking solutions of the form \( t^\lambda e^{\pm Rt} \), we find

\[
\theta \sim \text{constant} \times t^{-\frac{1}{2}} e^{(-t-1)R}, \quad 0 < t < 1; \quad \Theta \sim \text{constant} \times t^{\frac{1}{2}} e^{(1-t)R}, \quad t > 1 \tag{17}
\]

or equivalently,

\[
\Theta \sim \begin{cases} 
\pi - c_- t^{-\frac{1}{2}} e^{(-t-1)R}, & \text{skyrmion core, } 0 < t < 1 \\
c_+ t^{\frac{1}{2}} e^{(1-t)R}, & \text{far field, } t > 1,
\end{cases} \quad R \to \infty. \tag{18}
\]
These expressions do not extend into the region of the domain wall, since the approximation used to derive Eqs. (15) and (16) does not hold in the layer. They also do not extend in the region near \( t = 0 \), because of the zeros in the denominator at \( r = 0 \) in Eq. (13).

We now concentrate in the region \( t \sim 1, \ T \gg 1 \), namely, the margins (not the core) of the domain wall. Selecting the constants \( c_- = c_+ = 2 \), matches this expression of \( \Theta \) with the corresponding expression of \( \Theta_0 \) obtained in Eq. (12). We have obtained the following expression of the skyrmion not too close to its center in the asymptotic limit \( \frac{1}{R} \rightarrow 0 \),

\[
\Theta \sim \begin{cases} 
\pi - 2t^{-\frac{1}{2}}e^{(1-t)R}, & \text{skyrmion core, } \ 0 < t < 1, \\
2\arctan\left(e^{(1-t)R}\right), & \text{domain wall,} \\
2t^{-\frac{1}{2}}e^{(1-t)R}, & \text{far field, } t > 1.
\end{cases}
\]

In the original variable \( r \), Eq. (19) translates to

\[
\Theta \sim \begin{cases} 
\pi - 2\left(\frac{r}{R}\right)^{-\frac{1}{2}}e^{-R}, & \text{skyrmion outer core, } r < R, \ 1 \ll R - r \ll R, \\
2\arctan\left(e^{R-r}\right), & \text{domain wall (exhibits thin overlap with the other two regions),} \\
2\left(\frac{r}{R}\right)^{-\frac{1}{2}}e^{R-r}, & \text{far field, } r > R \gg 1, \ r - R \gg 1.
\end{cases}
\]

3. Uniform formula for the inner and outer core field

Letting, as before, \( \theta(r) = \pi - \Theta \), the core consists of the region in which \( \theta \ll 1 \), hence, \( \sin \theta \sim \theta \). The DM term is clearly subdominant (\( \theta^2 \ll \theta \)) and is neglected. The core equation is then to leading order

\[
\theta'' + \frac{\theta'}{r} - \frac{\theta}{r^2} - \theta = 0, \quad \theta(0) = 0.
\]

The solution of this is given by the series

\[
\theta = \frac{C}{s} \sum_{n=1}^{\infty} \frac{s^{2n}}{(n-1)!n!},
\]

where \( C \) is a constant, determined below by matching this field with the field of the domain wall. The series has infinite radius of convergence and exhibits exponential growth, which is controlled in the core by the exponentially small \( C \).

3.1. Leading asymptotic approximation of the series for large \( s \).

The leading contribution to the series at large \( s \) occurs at large values of \( n \). The Stirling approximation for the factorial \( n! \sim n^ne^{-n}\sqrt{2\pi n} \) obtains

\[
(n-1)!n! = \frac{(n!)^2}{n} \sim 2\pi n^{2n}e^{-2n} = 2\pi \exp(2n \ln n - 2n).
\]

Inserting this into the series, we obtain

\[
s\theta \sim \frac{C}{2\pi} \sum_{n=1}^{\infty} \exp[2n(\ln s - \ln n + 1)].
\]
We introduce the atomic measure

\[ \Delta_{1/s}(x) = \sum_{n \in \mathbb{N}} \delta(x - n/s), \]

usually referred to as a scaled Dirac comb, (it has unit mass at each point where \( sx = n, \ i.e. \) where \( sx \) is a positive integer), in order to express the series as an integral,

\[ s\theta \sim C \frac{2\pi}{\sqrt{8\pi R}} \int_0^{\infty} \exp[2sx(ln x - ln x + 1)] \Delta_{1/s}(dx) = C \frac{2\pi}{\sqrt{8\pi R}} \int_0^{\infty} \exp[2s(-x \ln x + x)] \Delta_{1/s}(dx). \quad (25) \]

By the sampling property of the scaled Dirac comb, we may replace \( \Delta_{1/s}(dx) \) by \( s \, dx \) in the asymptotic limit of large \( s \). Cancelling \( s \) from both sides, yields

\[ \theta \sim C \frac{2\pi}{\sqrt{8\pi R}} \int_0^{\infty} \exp[2s(-x \ln x + x)] \, dx. \quad (26) \]

We now apply the Laplace method for the asymptotic evaluation of the integral, in which the exponent is replaced by its quadratic Taylor approximation, centered at the maximizer of the exponent, \( x = 1 \). Calculating the gaussian integral and recalling that \( s = r/2 \), we obtain

\[ \pi - \Theta = \theta \sim C e^{-r/2\sqrt{2\pi R}}, \quad r \gg 1, \quad (27) \]

This expression matches the one in Eq. (20), when

\[ C = e^{-R/2\sqrt{8\pi R}}. \quad (28) \]

The core profile, for large radius, is given by the equation

\[ \Theta = \pi - e^{-R/\sqrt{8\pi R}} \sum_{n=1}^{\infty} \frac{\left( \frac{1}{2} \right)^{2n-1}}{(n-1)!n!}. \quad (29) \]

The slope of the skyrmion profile at its center \( r = 0 \) is

\[ \frac{d\Theta}{dr}(r = 0) = -e^{-R/2\sqrt{2\pi R}}. \quad (30) \]

Our main result so far is that we have obtained the leading approximation to the skyrmion profile for the entire region \( r \geq 0 \), using perturbation theory. It is noteworthy that

1. the skyrmion radius \( R \) still remains a free variable;
2. the DM constant \( \epsilon \) is absent from the leading order.

We derive below a relation between \( R \) and \( \epsilon \), by performing a higher order analysis of the domain wall profile.

4. Higher order skyrmion profile analysis

We will obtain a formula for the skyrmion radius, as \( \epsilon \) approaches a critical value \( \epsilon_0 \) asymptotically from below. In order to prepare for achieving this, we take the following steps.

- We express the parameter \( \epsilon \) as

\[ \epsilon = \epsilon_0 + \frac{\epsilon_1}{R} + \frac{\epsilon_2}{R^2} + \frac{\epsilon_3}{R^3} + \cdots. \quad (31) \]
• We recenter the function $\Theta$ to the center of the domain wall, more precisely at the point $r = R$ at which $\Theta = \frac{\pi}{2}$; this requires introducing the expansions

$$\frac{1}{r} = \frac{1}{R} \left(1 - \frac{T}{R} + \frac{T^2}{R^2} + \cdots\right), \quad \frac{1}{r^2} = \frac{1}{R^2} \left(1 - 2\frac{T}{R} + \cdots\right). \tag{32}$$

• We express the recentered $\Theta$ as an asymptotic series for large $R$,

$$\Theta = \Theta_0 + \tilde{\Theta}, \quad \tilde{\Theta} = \Theta_1 \frac{T}{R} + \Theta_2 \frac{T^2}{R^2} + \Theta_3 \frac{T^3}{R^3} + \cdots. \tag{33}$$

where $\Theta_0$ is given by Eq. (7) and $\Theta_1, \Theta_2, \Theta_3, \cdots$ are functions of $T$.

Inserting the series (33) for $\Theta$ into Eq. (4), applying the identities of trigonometric addition and writing the Taylor series compactly as

$$\cos(2\tilde{\Theta}) = 1 + C(2\tilde{\Theta}), \quad \sin(2\tilde{\Theta}) = 2\tilde{\Theta} + S(2\tilde{\Theta}), \tag{34}$$

obtains

$$\tilde{\Theta}'' - \cos(2\Theta_0) \tilde{\Theta} = \tilde{g} \tag{35}$$

where the prime denotes differentiation with respect to $T$, and

$$\tilde{g} = \frac{g_1}{R} + \frac{g_2}{R^2} + \frac{g_3}{R^3} + \cdots. \tag{36}$$

The explicit form of $\tilde{g}$ is given in Eq. (54). The hierarchy of linear nonhomogeneous equations for the functions $\Theta_n$ is obtained directly from Eq. (35),

$$\Theta_n'' - (\cos 2\Theta_0) \Theta_n = g_n, \quad n = 1, 2, 3, \cdots. \tag{37}$$

The forcing term $g_n$ of the equation for $\Theta_n$ may depend only on the functions $\Theta_l$ with $l \leq n - 1$. All equations have the same homogeneous part. All equations are given the initial condition $\Theta_n(T = 0) = 0$.

The homogeneous equation corresponding to the hierarchy (37) is

$$\Theta_H'' - (1 - 2 \text{sech}^2 T) \Theta_H = 0. \tag{38}$$

This equation describes the motion of a quantum mechanical particle in a potential well (see, e.g., Ref. [22], page 73). The potential equaling negative $\text{sech}^2 T$ is one of the Bargmann reflectionless potentials, a class of potentials of the one-dimensional Schrödinger operator having bound states with negative energy and zero reflection coefficient for all positive energies [23].

Eq. (38) has the explicit basis solutions

$$H_1 = \text{sech} T, \quad H_2 = \sinh T + T \text{sech} T. \tag{39}$$

Their Wronskian is given by

$$\det \begin{pmatrix} H_1 & H_2 \\ H_1' & H_2' \end{pmatrix} = 2. \tag{40}$$

Using the formula of the variation of constants, we obtain the general solution

$$\Theta_n = -\frac{1}{2} H_1(T) \int_0^T g_n(\tau) H_2(\tau) d\tau + \frac{1}{2} H_2(T) \int_0^T g_n(\tau) H_1(\tau) d\tau + c_{n,1} H_2(T), \quad n = 1, 2, 3, \cdots. \tag{41}$$

A term $c_{n,1} H_1(T)$ does not appear in this formula because the initial condition $\Theta_n(T = 0) = 0$ forces $c_{n,1} = 0$ for all $n$. The constants $c_{n,2}$ will be determined in Sec. 6 through the requirement that $\Theta_n$ remain bounded for all $T$. 


4.1. Calculation of $\Theta_1$ and $\epsilon_0$.

The equation for $\Theta_1$ is

$$\Theta_1'' - \cos(2\Theta_0)\Theta_1 = -\Theta_0' - 2\epsilon_0 \sin^2 \Theta_0.$$  \hspace{1cm} (42)

Using Eqs. (9) we obtain

$$\Theta_1'' - (1 - 2 \text{sech}^2 T)\Theta_1 = \text{sech} T - 2\epsilon_0 \text{sech}^2 T$$  \hspace{1cm} (43)

whose solution is

$$\Theta_1 = -\frac{1}{2}H_1(T) \int_0^T g_1(\tau)H_2(\tau) \, d\tau + \frac{1}{2}H_2(T) \int_0^T g_1(\tau)H_1(\tau) \, d\tau  \hspace{1cm} (44)$$

where

$$g_1 = \text{sech} T - 2\epsilon_0 \text{sech}^2 T.$$  \hspace{1cm} (45)

The term $c_{1,2}H_2(T)$ has been set to zero, because $H_2$ is an odd function that diverges in the limits $T \to \pm\infty$. This divergence cannot be canceled by the second term on the right hand side of Eq. (44) because this term is an even function. Finally, the first term of $\Theta_1$ in Eq. (44) converges to zero as $T \to \pm\infty$. In order that the second term also converges to zero, it is required that

$$\int_0^\infty g_1(\tau)H_1(\tau) \, d\tau = 0,  \hspace{1cm} (46)$$

which obtains

$$\epsilon_0 = \frac{1}{2} \int_0^\infty \text{sech}^3 \tau \, d\tau - \frac{1}{2} \int_0^\infty \text{sech}^3 \tau \, d\tau = \frac{\pi}{2}.$$  \hspace{1cm} (47)

The arguments of this paragraph are explained in more detail in the beginning of Sec. 6 where a generalization of Eq. (46) is obtained as Eq. (60). The value of $\epsilon_0$ in Eq. (47) is also obtained by a different method in 70.

The leading asymptotic behavior of $\Theta_1$ as $T \to \pm\infty$ is determined from the first term of Eq. (44). The second term decays faster by a factor of $|T|$. By calculating the integral explicitly (where only the dominant term of $H_2$ is used), we obtain

$$\Theta_1 \sim -|T| e^{-|T|}, \text{ when } |T| \gg 1.$$  \hspace{1cm} (48)

Thus,

$$\Theta \sim \Theta_0 + \frac{\Theta_1}{R} \sim 2 \arctan(e^{-T}) - \frac{|T|}{R} e^{-T}, \hspace{1cm} 1 \ll R.$$  \hspace{1cm} (49)

In the region $1 \ll T \ll R$ (recall that $\arctan a + \arctan a^{-1} = \frac{\pi}{2}$)

$$\Theta \sim \Theta_0 + \frac{\Theta_1}{R} \sim \frac{\pi}{2} + \begin{cases} \left( \frac{\pi}{2} - 2e^{-|T|} \right) - \frac{|T|}{R} e^{-|T|}, & T < 0 \\ -\left( \frac{\pi}{2} - 2e^{-|T|} \right) - \frac{|T|}{R} e^{-|T|}, & T > 0. \end{cases}  \hspace{1cm} (50)$$

This agrees with Eq. (19), through the relation

$$t^{-\frac{1}{2}} = \left( 1 + \frac{T}{R} \right)^{-\frac{1}{2}} \sim 1 - \frac{T}{2R}.$$  \hspace{1cm} (51)
We finally note that almost all integrals in Eq. (44) can be evaluated and we find
\[ \Theta_1 = \epsilon_0 \sech T \left( 1 - \frac{1}{2} \sech T + \int_0^T \tau \sech^3 \tau \, d\tau \right) \]
\[ - \frac{1}{2} \epsilon_0 + \frac{1}{2} \sinh T \left[ \tanh T - 2 \epsilon_0 \left( \arctan e^T - \frac{\pi}{4} \right) \right] \]  
\[ - \epsilon_0 T \sech T \left( \frac{1}{2} \tanh T \sech T + \arctan e^T - \frac{\pi}{4} \right). \] (52)

5. Parity theory

Proceeding to the calculation at higher orders of $1/R$ we determine the formula for $\tilde{g}$ utilizing Eqs. (34). It will be convenient in our calculations to introduce the alternative notation for $1/r$ (see Eq. (32))
\[ \frac{1}{r} = \frac{p}{T}. \] (53)

The motivation for this notation is that $p$ is a power series of the ratio $T/R$. We obtain
\[ \tilde{g} = -p \frac{\Theta_0'}{T} - p \epsilon \frac{1 - \cos(2\Theta_0)}{T} + p^2 \frac{\sin(2\Theta_0)}{2T^2} - p \frac{\Theta'}{T} + \left( -p \frac{\sin(2\Theta_0)}{T} + p^2 \frac{\cos(2\Theta_0)}{2T^2} \right) 2\Theta \]
\[ + \left( \frac{1}{2} \sin(2\Theta_0) + p \epsilon \frac{\cos(2\Theta_0)}{T} + p^2 \frac{\sin(2\Theta_0)}{2T^2} \right) C(2\Theta) + \left( \frac{1}{2} \cos(2\Theta_0) - p \epsilon \frac{\sin(2\Theta_0)}{T} + p^2 \frac{\cos(2\Theta_0)}{2T^2} \right) S(2\Theta). \] (54)

The calculation of the solutions of Eqs. (37) is facilitated by the following theorem. The hypotheses of the theorem turn out to be necessary conditions for the existence of bounded solutions $\Theta_n$.

**Theorem 1.** Let $\epsilon_{2i-1} = 0$ for $i = 1, 2, 3, \ldots$. Then, the following parity conditions hold.

1. For all $n \geq 1$, the functions $g_n = g_n(T)$ are even if $n$ is odd and they are odd if $n$ is even.
2. The same is true for the functions $\Theta_n = \Theta_n(T)$.

**Proof.** It suffices to prove the theorem for the $g_n$, the statement for the $\Theta_n$ then follows from Eq. (41), which implies that $\Theta_n$ has the same parity as $g_n$. We already know that the theorem is true for $n = 1$.

Clearly, only terms $\Theta_j$ with $j < n$ appear in the expression for $g_n$, so we can truncate $\tilde{\Theta}$ accordingly. We make the inductive assumption that the functions $g_1, g_2, g_3, \ldots, g_{n-1}$ and hence the functions $\Theta_1, \Theta_2, \Theta_3, \ldots, \Theta_{n-1}$ alternate in parity, with $g_1$ and $\Theta_1$ being even functions of $T$. We prove that the theorem is then true for $g_1, g_2, g_3, \ldots, g_n$ and hence for $\Theta_1, \Theta_2, \Theta_3, \ldots, \Theta_n$. We recall that $\Theta_0$ is an odd function of $T$. We also recall that $\epsilon$ is an even power series in $\delta$ and we observe that $p$ is a power series of $(\delta T)$.

We show that $g_n$ satisfies the parity condition term by term. The condition is true for the first term, namely, $p\Theta_0'/T$. Indeed, $p$ must be represented by $(\delta T)^n$ for the term to be of order $\delta^n$. The term becomes $T^{n-1}\Theta_0'$, which satisfies the parity condition since $\Theta_0'$ is even. In a similar way, the parity law is satisfied for the remaining two terms in which no $\Theta_j$ appears. The verification for the terms in which only one $\Theta_j$ appears is equally straightforward. For example, if in the term before the first parenthesis, $p$ is represented by $(\delta T)^m$, the term is $T^{m-1}\Theta_{n-m}^\delta$. The parity requirement is clearly satisfied for $m = 1$, since taking the derivative changes the parity. Increasing the value of $m$ by $k$ introduces $k$ factors $T$ and also shifts the index of $\Theta$ backwards by $k$ positions. According to our inductive assumption, the parity of the term is preserved.
More work is required to show that the parity condition holds for the terms that have the factor \( C(\tilde{\Theta}) \) or \( S(\Theta) \). According to the inductive assumption, the truncated

\[
\tilde{\Theta} = \delta \Theta_1 + \delta^2 \Theta_2 + \delta^3 \Theta_3 + \ldots + \delta^{n-1} \Theta_{n-1}
\]  

has even functions of \( T \) multiplying the odd powers of \( \delta \) and odd functions of \( T \) multiplying the even powers of \( \delta \). The general term of the expansions of \( C(\tilde{\Theta}) \) and \( S(\tilde{\Theta}) \) is represented by

\[
\tilde{\Theta}^k = (2\delta)^k q \Theta_1^{k_1} \Theta_2^{k_2} \ldots \Theta_{n-1}^{k_{n-1}},
\]

where \( k = (k_1, k_2, k_3, \ldots k_{n-1}) \) with \( k_i \in \{0, 1, 2, 3, \ldots\} \), and where \( q = (1, 2, 3, \ldots, n-1) \). The term \( \tilde{\Theta}^k \) is either even or odd, since the factors \( \Theta_j \) are even or odd.

**Claim.**

1. All the terms in the expansion of \( C(\tilde{\Theta}) \) are even at even powers of \( \delta \) and odd at odd powers of \( \delta \).
2. All the terms in the expansion of \( S(\tilde{\Theta}) \) are odd at even powers of \( \delta \) and even at odd powers of \( \delta \).

In order to prove the claim, we utilize the notion of the parity of a number or a function. The parity equals zero in the case of evenness and it equals unity in the case of oddness of the number or function. We calculate the following three parities.

1. The parity of the exponent \( n \) in the order \( \delta^n \) of a term \( \tilde{\Theta}^k \).
2. The parity of the product \( \Theta_1^{k_1} \Theta_2^{k_2} \ldots \Theta_{n-1}^{k_{n-1}} \) in \( \tilde{\Theta}^k \).
3. The parity of the number of the factors \( \Theta_j \) in \( \tilde{\Theta}^k \), counting multiplicities.

We obtain the following.

1. The exponent of \( \delta \) equals \( n = k \cdot q \). For the calculation of the parity of \( n \), we set \( k_j q_j = 0 \) if \( q_j \) is even (eliminates all the odd factors \( \Theta_j \)) or if \( k_j \) is even (eliminates all factors \( \Theta_j \) with even multiplicity).

   We are thus, keeping only the even factors with odd multiplicity. The parity is thus

   \[
   N_{even} \equiv \# \text{ of factors } \Theta_j \text{ that are even functions with odd multiplicity mod}(2).
   \]  

2. The second parity, which we denote by \( N_{odd} \), equals

   \[
   N_{odd} \equiv \# \text{ of odd factors with odd multiplicity mod}(2).
   \]

3. The parity of the number of the factors \( \Theta_j \) in \( \tilde{\Theta}^k \), counting multiplicities is given by the sum \( k_1 + k_2 + \ldots + k_{n-1} \). It is an even number for the terms of \( C(\tilde{\Theta}) \) and an odd number for the terms of \( S(\tilde{\Theta}) \). The third parity, which we denote by \( N_{total} \), equals

   \[
   N_{total} \equiv \# \text{ of all factors of odd multiplicity mod}(2).
   \]

Clearly, \( N_{even} + N_{odd} \equiv N_{total} \mod(2) \). Hence, \( N_{total} \equiv 0 \) in the case of \( C(\tilde{\Theta}) \) and \( N_{total} \equiv 1 \) for \( S(\tilde{\Theta}) \).

Parities 1 and 2 agree with each other in the terms of \( C(\tilde{\Theta}) \) and differ from each other in the terms of \( S(\tilde{\Theta}) \). This proves the claim.

Now that the parity of the terms of \( C \) and \( S \) are understood, the correctness of the theorem for the terms involving these is verified similarly to the previous terms.
6. Skyrmion profiles and the $\epsilon$ versus $R$ relation

We proceed to a calculation of the functions $\Theta_n(T)$ based on the general formula given in Eq. (41). The procedure will also produce the numerical values of the coefficients $\epsilon_n$ in Eq. (31). The strategy will be based on the observation that the two last terms on the right hand side of Eq. (41) may diverge as $T \to \pm \infty$ unless certain conditions are imposed, while the first term remains finite in the same limit.

The value of $c_{n,2}$ in formula (41) needs to be chosen so that $\Theta_n$ remains bounded as $T \to \pm \infty$. This implies that $c_{n,2} = 0$ for $n = 1, 3, 5, \ldots$ where $g_n$ is even, according to Theorem 1. Indeed, $H_2(T)$ is an odd function that diverges in the limits $T \to \pm \infty$. This divergence of the third term on the right hand side of Eq. (41) cannot be eliminated by any of the other terms since they are all even functions. Therefore, the formula reduces to

$$\Theta_n = -\frac{1}{2} H_1(T) \int_0^T g_n(\tau) H_2(\tau) \, d\tau + \frac{1}{2} H_2(T) \int_0^T g_n(\tau) H_1(\tau) \, d\tau, \quad n = 1, 3, 5, \cdots \quad (60)$$

The second term on the right hand side itself diverges, unless, the integral is zero in the limit $T \to \infty$. This provides the constraint

$$\int_0^\infty g_n(\tau) H_1(\tau) \, d\tau = 0, \quad n = 1, 3, 5, \cdots \quad (61)$$

As in the case of $n = 1$, the coefficient of $\epsilon_{n-1}$ in the expression of $g_n$ equals $-2 \text{sech}^2 T$ when $n$ is odd, thus, the constraint yields

$$\epsilon_{n-1} = \frac{2}{\pi} \int_0^\infty (\text{sech} \tau) g_n(\tau)|_{\tau = 0} \, d\tau. \quad n = 1, 3, 5, \cdots \quad (62)$$

For $n = 2, 4, 6, \cdots$, $g_n$ is odd, according to Theorem 1 and, therefore, the second and third terms on the right side of Eq. (41) are both odd. Their sum should converge to zero in the limits $T \to \pm \infty$. This gives the value for $c_{n,2}$ as

$$c_{n,2} = -\frac{1}{2} \int_0^{\pm \infty} g_n(\tau) H_1(\tau) \, d\tau, \quad n = 2, 4, 6, \cdots \quad (63)$$

The latter relation should hold true either when the upper limit is $+\infty$ or $-\infty$. The formula for $\Theta_n(T)$ can be written as

$$\Theta_n = -\frac{1}{2} H_1(T) \int_0^T g_n(\tau) H_2(\tau) \, d\tau + \frac{1}{2} H_2(T) \int_0^T g_n(\tau) H_1(\tau) \, d\tau, \quad n = 2, 4, 6, \cdots \quad (64)$$

In order to be explicit, we write the non-homogeneous terms of Eq. (37) for $n = 1, 2, 3$ by applying Eq. (54). We include the odd indexed $\epsilon_n$ in order to explain below why they must be equal to zero,

$$g_1 = -(\Theta'_0 + 2\epsilon_0 \sin^2 \Theta_0) \quad (65a)$$

$$g_2 = T(\Theta'_0 + 2\epsilon_0 \sin^2 \Theta_0) - 2\epsilon_1 \sin^2 \Theta_0 + \sin 2\Theta_0 \left( \frac{1}{2} - 2\epsilon_0 \Theta_1 - \Theta_1^2 \right) - \Theta'_1 \quad (65b)$$

$$g_3 = -T^2 (\Theta'_0 + 2\epsilon_0 \sin^2 \Theta_0) + 2(T \epsilon_1 - \epsilon_2) \sin^2 \Theta_0 + T \Theta'_1 - \Theta'_2 \quad (65c)$$

$$+ \sin 2\Theta_0 \left[ -T + (T \epsilon_0 - \epsilon_1) \Theta_1 - 2\epsilon_0 \Theta_2 - 2\Theta_1 \Theta_2 \right] + \cos 2\Theta_0 \left( \Theta_1 - 2\epsilon_0 \Theta_1^2 - \frac{2}{3} \Theta_1^3 \right) .$$

In the expression for $g_2$, the term containing $\epsilon_1$ is even while all other terms are odd. This even term will give a contribution to $C_{n,2}$ in Eq. (63) which has opposite sign in the two limits $\pm \infty$. We thus need to set
Figure 1: The functions (a) $\Theta_1(T)$, $\Theta_2(T)$ (b) $\Theta_3(T)$, $\Theta_4(T)$ and (c) $\Theta_5$, $\Theta_6$ calculated by numerical evaluation of the integrals in Eqs. (60) and (64). As seen by the change of scale of the vertical axis in the three entries, the values of the functions $\Theta_n$ increase fast with increasing index $n$. Note the even parity of the odd indexed functions and the odd parity of the even indexed ones with respect to the variable $T = r - R$.

$\epsilon_1 = 0$. The same phenomenon occurs for all even indexed $g_n$ enforcing all odd indexed $\epsilon_n$ to be equal to zero. The hypothesis of Theorem 1 is thus justified. The relation of the dimensionless DMI constant with the skyrmion radius (31) is simplified to

$$\epsilon = \epsilon_0 + \frac{\epsilon_2}{R^2} + \frac{\epsilon_4}{R^4} + \frac{\epsilon_6}{R^6} + \cdots.$$  

(66)

We now apply Eqs. (60) and (64) to calculate the functions $\Theta_n$. For $g_2$ we need the expressions for $\Theta_1, \Theta_4$ found earlier while $\Theta_1'$ is found by finite differences. The function $\Theta_2$ is calculated from Eq. (64) for $n = 2$, by numerical integration. Fig. 1a shows the functions $\Theta_1(r), \Theta_2(r)$. They have opposite parity as expected from Theorem 1.

Eq. (62) for $n = 3$ gives, by a numerical evaluation of the integral,

$$\epsilon_2 \approx -0.3057.$$  

(67)

If we insert this value in Eq. (66), ignore higher order terms, and invert the relation we obtain the skyrmion radius versus the parameter $\epsilon$,

$$R = \frac{0.553}{\sqrt{\frac{2}{\pi}} - \epsilon}.$$  

(68)

This can be compared with the formula $R = 1/ \sqrt{\pi (\epsilon_0 - \epsilon)} \approx 0.564/ \sqrt{\epsilon_0 - \epsilon}$ which has been derived in Ref. [24] by an energy minimization argument.

We find $\Theta_3, \Theta_4, \Theta_5, \Theta_6$ by applying Eqs. (60) and (64) for $n = 3, 4, 5, 6$. In the process, we need the expressions for $g_n$ which are quite long and they have been derived using the mathematics software system SageMath [25]. Fig. 1 shows the functions $\Theta_1$ through $\Theta_6$. As expected from Theorem 1, odd indexed $\Theta_n$'s are even functions of $T$ and even indexed ones are odd. Functions $\Theta_n$ with higher index take higher values and they take significant values over larger intervals of $T$. These features have consequences for the quality of the approximation, especially for small $R$, as we shall see in the following.

We have calculated the skyrmion profiles, which are solutions of Eq. (4), for various values of the parameter $\epsilon$. This was done by a shooting method which solves the equation for the skyrmion profile (4) as an initial value problem. The profiles are shown by small circles in Fig. 2 for two values of the parameter $\epsilon$. In Fig. 2a, we have $\epsilon = 0.60$ which gives a skyrmion of radius $R = 3.29 \ell_w$. The profile at the skyrmion
Figure 2: The profile of a skyrmion $\Theta(r)$ is shown by small circles for two values of the parameter $\epsilon$, obtained numerically by solving the original Eq. (4) using a shooting method. The blue line shows $\Theta_0$, which is the one-dimensional domain wall profile. The orange line shows the series solution (29) for the skyrmion profile at the core, obtained by neglecting the DMI term and linearizing the original equation about $\Theta = \pi$. (a) For $\epsilon = 0.60$ the skyrmion has a radius $R = 3.29 \ell_w$, obtained by the shooting method. The red line shows the series (33) summed up to the term $\Theta_6$. (b) For $\epsilon = 0.55$ the skyrmion has a radius $R = 2.02 \ell_w$. For this smaller value of $R$ the optimal point of truncation of the series occurs at the term $\Theta_2$. The green line shows the series (33) summed up to and including the term $\Theta_2$. The core is approximated very well by the Frobenius series solution given in Eq. (29), and is shown as an orange dashed line in the figure. The blue solid line shows the one-dimensional domain wall profile $\Theta_0$, shown in Eq. (7), centred at the skyrmion radius position $r = R$. The red line shows the series solution (33) for $\Theta$ up to the term $O(1/R^2)$. The approximation of the skyrmion profile is excellent for all $r$ except near the skyrmion center $r = 0$. In Fig. 2b, we have $\epsilon = 0.55$ and a smaller skyrmion radius $R = 2.02 \ell_w$. The Frobenius series (29) still gives an excellent approximation at the skyrmion core. The green line shows the series solution (33) for $\Theta$ up to the term $O(1/R^2)$ and obtains a good approximation of the profile, especially around the skyrmion radius and for $r > R$. The terms of the series (33) of order higher than $O(1/R^2)$ cannot be used to improve the approximation, and they rather give larger deviations from the true profile if added to the series. This phenomenon could have already been anticipated given the form of the $\Theta_n$’s shown in Fig. 1 and the observation that an increasing index $n$ gives $\Theta_n$’s with rapidly increasing values. When a term $\Theta_n/R^n$ is larger than the previous term in the series the series should be truncated omitting this term.

The situation where one needs to truncate a series as described in the precious paragraph is common for asymptotic series. In the present problem, we use the radius $R$ of the skyrmion as our asymptotic parameter in the limit of large values. Our asymptotic series $\Theta(T) = \Theta_R(T)$ is a means of approximating the angle $\Theta$ at a physical point $r = R + T$, where the value of $T$ remains fixed. When $R$ is held fixed at some reasonably high value of interest, the approximation becomes sharper as more terms of the asymptotic series are being considered, up to and including a certain term of optimal truncation of the series. Beyond this term, the error increases and, as is typical with asymptotic series, eventually the series diverges. Which one is the term of the optimal truncation of the series depends on both the value of $R$ and the value of $T$ that are considered. Increasing the value of $R$ and/or decreasing the value of $|T|$, leads to an optimal truncation deeper in the series and improves the approximation. This is observed in Fig. 2 for the skyrmion profile.

Along the process of solving Eqs. (60), (64) for the $\Theta_n$’s we apply the condition shown in Eq. (62) for
The skyrmion radius $R$ found numerically by solving the original equation (4) for various values of $\epsilon$ is shown by open circles. Relation (70) is shown by the colored lines to order $O(1/R^2)$, $O(1/R^4)$, $O(1/R^6)$ as shown in the legend. The dotted blue line is an asymptote and marks the critical value $\epsilon = 2/\pi$. The successive approximations enhance the accuracy as $R$ increases.

(b) A blow-up of the graph for the region of large values of $R$.

$n = 5$ and $n = 7$ and we obtain the results

$$\epsilon_4 \approx -0.8792, \quad \epsilon_6 \approx -5.901.$$  \hfill (69)

We substitute these in Eq. (66) and have

$$\epsilon \approx \frac{2}{\pi} - \frac{0.3057}{R^2} - \frac{0.8792}{R^4} - \frac{5.901}{R^6}, \quad R \gg 1.$$  \hfill (70)

Fig. 3 shows by black dots the skyrmion radius extracted from the calculation of the skyrmion profiles by the shooting method. These data are compared with formula (70) for the successive approximations up to and including order $O(1/R^2)$, $O(1/R^4)$, and $O(1/R^6)$. The approximation is excellent for large $R$ and it is improving as we add higher order terms, as seen in the blow-up in Fig. 3b. For smaller $R$, higher order approximations give larger deviations from the correct result, especially when the term $O(1/R^6)$ is included. This is a consequence of the increasing error in the asymptotic series for small values of $R$ as was discussed in relation to Fig. 2.

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Appendix A. Alternative derivation of the limit value $\epsilon_0$ of the DMI constant $\epsilon$ as $R \to \infty$.

We define
\[ \delta = \frac{1}{R} \]  
and write Eq. (14) as
\[ \delta^2 \left( \ddot{\Theta} + \frac{\dot{\Theta}}{t} \right) - \frac{\sin(2\Theta)}{2} + \delta \epsilon \sin^2 \Theta \frac{t}{t} = 0, \quad t \neq 0. \]  

We multiply Eq. (A.2) by $\dot{\Theta}$ and integrate over the interval $t \in \left(1 - \sqrt{\delta}, 1 + \sqrt{\delta}\right)$, replacing the denominators with unity over this interval of integration (see below). With this simplification, all terms with the exception of $\dot{\Theta}$ reduce to elementary integrations with respect to the variable $\Theta$. With exponentially small error, the domain of the theta integrations maybe taken to be from $\pi/2$ or 0.

\[
\frac{1}{2} \delta^2 \Theta^2 \bigg|_{\pi/2}^{0} + \delta^2 \int_{1-\sqrt{\delta}}^{1+\sqrt{\delta}} \dot{\Theta}^2 dt + \frac{1}{4} \cos(2\Theta) \bigg|_{\pi/2}^{0} + \epsilon \delta \left( \Theta - \frac{1}{2} \sin(2\Theta) \right) \bigg|_{\pi/2}^{0} = 0. \]  

Neglecting the exponential error from the limits of integration, we obtain,
\[ \delta^2 \int_{1-\sqrt{\delta}}^{1+\sqrt{\delta}} \dot{\Theta}^2 dt = \epsilon \delta \pi. \]  

Clearly, in a mechanical analogue, the term on the left represents the energy of dissipation and the term on the right represents the loss of potential energy. We make a leading order evaluation of the dissipation energy by approximating $\Theta$ with its leading expression $\Theta_0$, given by Eq. (7). We obtain
\[ \delta^2 \int_{1-\sqrt{\delta}}^{1+\sqrt{\delta}} \Theta^2 dt = \delta \int_{1-\sqrt{\delta}}^{1+\sqrt{\delta}} \Theta^2 dT = \delta \int_{1-\sqrt{\delta}}^{1+\sqrt{\delta}} \left( \frac{2e^{-T}}{1 + e^{-2T}} \right)^2 dT = \delta \int_{1-\sqrt{\delta}}^{1+\sqrt{\delta}} \text{sech}^2 TdT = \delta \text{tanh} \int_{1-\sqrt{\delta}}^{1+\sqrt{\delta}} \sim 2\delta. \]  

Inserting this result into Eq. (A.4), we obtain the limit value of $\epsilon$ as the skyrmion radius tends to infinity,
\[ \epsilon_0 = \frac{2}{\pi}. \]  

For larger values of $\epsilon$, the drop in potential energy is too big to be absorbed by the dissipation term and there is no skyrmion solution. Decreasing the value of $\epsilon$ from $\epsilon_0$, decreases the skyrmion radius.

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