Hilbert geometry of the Siegel disk: The Siegel-Klein disk model

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Abstract

We introduce and study the Hilbert geometry induced by the Siegel disk, an open bounded convex set of complex matrices. This Hilbert geometry naturally yields a generalization of the Klein disk model of hyperbolic geometry, which we term the Siegel-Klein model to differentiate it with the usual Siegel upper plane and Siegel disk domains. In the Siegel-Klein disk, geodesics are by construction always straight, allowing one to build efficient geometric algorithms and data-structures from computational geometry. For example, we show how to approximate the Smallest Enclosing Ball (SEB) of a set of complex matrices in the Siegel domains: We compare two implementations of a generalization of the iterative algorithm of [Badoiu and Clarkson, 2003] in the Siegel-Poincaré disk and in the Siegel-Klein disk. We demonstrate that geometric computing in the Siegel-Klein disk allows one (i) to bypass the time-costly recentering operations to the origin (Siegel translations) required at each iteration of the SEB algorithm in the Siegel-Poincaré disk model, and (ii) to approximate numerically fast the Siegel distance with guaranteed lower and upper bounds.

Keywords: Hyperbolic geometry, symmetric positive-definite matrix manifold, symplectic geometry, Siegel upper space, Siegel disk, Hilbert geometry, Bruhat-Tits space, smallest enclosing ball.

1 Introduction

German mathematician Carl Ludwig Siegel [89] (1896-1981) and Chinese mathematician Loo-Keng Hua [45] (1910-1985) have introduced independently the symplectic geometry in the 1940’s (with a preliminary work of Siegel [88] released in German in 1939). The adjective symplectic stems from the greek “complex”, meaning mathematically the number field $\mathbb{C}$ instead of the ordinary real field $\mathbb{R}$. Symplectic geometry was originally motivated by the study of complex multivariate functions in the two landmark papers of Siegel [89] and Hua [45]. We refer the reader to the thesis [37, 51] for an overview of Siegel domains. More generally, the Siegel domains have been studied and classified in the most general setting of bounded symmetric irreducible homogeneous domains of 6 types by E. Cartan [25] in 1935 (see also [50, 14]).

The Siegel half-space and the Siegel disk provide generalizations of the complex Poincaré upper plane and the complex Poincaré disk to spaces of symmetric square complex matrices. We shall

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1 As we shall see, the naming stems originally from the relationships with the symplectic groups. Nowadays, symplectic geometry is understood as the study of symplectic manifolds which are differentiable manifolds equipped with a closed nondegenerate differential 2-form called the symplectic form.
term them the Siegel-Poincaré upper plane and the Siegel-Poincaré disk in the remainder. The Siegel upper space includes the cone of real symmetric positive-definite (PD) matrices \[35\] (SPD manifold), and the well-known affine-invariant PD Riemannian metric \[40\] can be recovered as a special case of the Siegel metric.

Applications of symplectic geometry are found in radar processing \[8, 9, 11, 10\] specially for Toeplitz matrices \[47, 58\], probability density estimations \[27\] and probability metric distances \[20, 22, 23, 93\], information fusion \[94\], neural network \[52\], theoretical physics \[78, 38, 79\], and image morphology operators \[56\], just to cite a few.

In this paper, we extend the Klein disk model \[83\] of the hyperbolic geometry to the Siegel disk by considering the Hilbert geometry \[43\] induced by the open bounded convex Siegel disk \[80, 59\]. We term this model the **Klein-Siegel model** for short to contrast with the Poincaré-Siegel upper plane and disk models. The main advantages of using the Klein-Siegel disk model instead of the usual Siegel upper plane or the Siegel-Poincaré disk are that the geodesics are always straight and therefore this Siegel-Klein disk model is very well-suited for algorithms and data-structures. Moreover, in the Siegel-Klein disk model, we have an efficient method to approximate with guarantees the calculation of the Siegel distance (specially useful when handling high-dimensional matrices). The algorithmic advantage was already observed for real hyperbolic geometry (included as a special case of the Siegel-Klein model): For example, the hyperbolic Voronoi diagrams can be efficiently computed as an affine power diagram clipped with the boundary ball \[71, 70, 73, 72\]. To demonstrate the advantage of the Siegel-Klein disk over the Siegel-Poincaré disk, we consider approximating the smallest enclosing ball of the a set of matrices in the disk. This problem has potential applications in image morphology \[5, 56\] or anomaly detection of covariance matrices \[95, 29\]. We state the problem as follows:

**Problem 1 (Smallest-radius Enclosing Ball (SEB))** Given a metric space \((X, \rho)\) and a finite set \(\{p_1, \ldots, p_n\}\) of \(n\) points, find the smallest-radius enclosing ball with center \(c^*\) minimizing the following objective function:

\[
\min_{c \in X} \max_{i \in \{1, \ldots, n\}} \rho(c, p_i).
\]

In general, the SEBs may not be unique in a metric space: For example, the SEBs are not unique in a discrete Hamming metric space\[2, 61\]. The SEB is proven unique in the Euclidean metric space \[98\], the hyperbolic geometry \[69\], the Riemannian positive-definite matrix manifold \[5, 55, 68\], and more generally in any Cartan-Hadamard manifold \[6\]. A fast \((1 + \epsilon)\)-approximation algorithm which requires \(\lceil \frac{1}{\epsilon^2} \rceil\) iterations was reported in \[7, 6\]. Since the approximation factor does not depend on the dimension, this SEB approximation algorithm had many applications in machine learning \[96\] (e.g., in Reproducing Kernel Hilbert Spaces \[81\], RKHS).

1.1 Paper outline and contributions

In Section 2 we concisely recall the usual models of the hyperbolic complex plane: The Poincaré upper plane, and the Poincaré and Klein disk models. We then briefly review the Siegel upper plane in \[83\] and the Siegel disk in \[4\]. Section 5 introduces the novel Siegel-Klein model using the Hilbert geometry. To demonstrate the algorithmic advantage of using the Siegel-Klein disk
model, we compare two implementations of the Badoiu and Clarkson’s approximation algorithm \cite{7} extended to Siegel spaces in \cite{6}. Finally, we conclude this work in \cite{7}.

We list our main contributions as follows:

- First, we formulate a generalization of the Klein disk model of hyperbolic geometry to the Siegel disk in Definition \cite{7}. We report the formula of the Siegel-Klein distance to the origin in Theorem \cite{4} (and more generally the Siegel-Klein distance between two points whose line passes through the origin), describe how to convert the Siegel-Poincaré disk to the Siegel-Klein disk and vice versa in Proposition \cite{7}, report an exact algorithm to calculate the Siegel-Klein distance for diagonal matrices in Theorem \cite{4}. In practice, we show how to obtain a fast guaranteed approximation of the Siegel-Klein distance using bisection searches with guaranteed lower and upper bounds detailed in Theorem \cite{5}.

- Second, we report the exact solution to a geodesic cut problem in the Siegel-Poincaré/Siegel-Klein disks in Proposition \cite{179}. This result yields an explicit equation for the geodesic linking the origin of the Siegel disk to any other matrix point (Proposition \cite{3} and Proposition \cite{4}). We then report an implementation of the Badoiu and Clarkson’s iterative algorithm \cite{7} for approximating the smallest enclosing ball tailored to the Siegel disk domains. In particular, we show in \cite{6} that the implementation in the Siegel-Klein model yields a fast algorithm which bypasses the recentering operations required in the Siegel-Poincaré model.

1.2 Matrix spaces and matrix norms

Let $\mathbb{F}$ be a number field considered in the remainder to be either the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$. For a complex number $z = a + ib \in \mathbb{C}$, we denote $\bar{z} = a - ib$ its conjugate, and $|z| = \sqrt{a^2 + b^2}$ its modulus. Let $\text{Re}(z) = a$ and $\text{Im}(z) = b$ denote the real part and the imaginary part of the complex $z$, respectively.

Let $M(d, \mathbb{F})$ be the space of $d \times d$ matrices with coefficients in $\mathbb{F}$, and $\text{GL}(d, \mathbb{F})$ denotes its subspace of invertible matrices. Let $\text{Sym}(d, \mathbb{F})$ denote the space of $d \times d$ symmetric matrices with coefficients in $\mathbb{F}$. The identity matrix is denoted by $I$ (or $I_d$ when we want to emphasize its dimension $d$). The conjugate of a matrix $M = [M_{ij}]$ is $\overline{M} = [\overline{M}_{ij}]$. The conjugate transpose of a matrix $M$ is $M^H = (M^\top)^\top = M^\top$, the adjoint matrix. A complex matrix is said Hermitian when $M^H = M$. Matrix $M M^H$ is Hermitian: $(M M^H)^H = (M^H)^H = M H^H$.

A real matrix $M \in M(d, \mathbb{R})$ is said positive-definite (PD) iff $x^\top M x > 0$ for all $x \in \mathbb{R}^d$ with $x \neq 0$. This positive-definiteness property is written $M \succ 0$, where $\succ$ is the partial Löwner ordering \cite{21}. Let $\text{PD}(d, \mathbb{R}) = \{ P \succ 0 : P \in \text{Sym}(d, \mathbb{R}) \}$ be the space of real symmetric positive-definite matrices \cite{83, 68, 55, 65} of dimension $d \times d$. This space is a cone, i.e., if $P_1, P_2 \in \text{PD}(d, \mathbb{R})$ then $P_1 + \lambda P_2 \in \text{PD}(d, \mathbb{R})$ for all $\lambda > 0$. The boundary of the cone consists of rank-deficient positive semi-definite matrices.

The eigenvalues of a square complex matrix $M$ are ordered such that $|\lambda_1(M)| \geq \ldots \geq |\lambda_d(M)|$, where $| \cdot |$ denotes the complex modulus. The spectrum $\lambda(M)$ of a matrix $M$ is its set of eigenvalues: $\lambda(M) = \{ \lambda_1(M), \ldots, \lambda_d(M) \}$. In general, real matrices may have complex eigenvalues but symmetric matrices (including PD matrices) have always real eigenvalues. The singular values $\sigma_i(M)$ of $M$ are always real:

$$\sigma_i(M) = \sqrt{\lambda_i(M^* M)} = \sqrt{\lambda_i(M M^H)},$$

\footnote{Also denoted by the star operator (i.e., $M^*$) or the dagger (i.e., $M^\dagger$) in the literature.}
and ordered as follows: \( \sigma_1(M) \geq \ldots \geq \sigma_d(M) \) with \( \sigma_{\text{max}}(M) = \sigma_1(M) \) and \( \sigma_{\text{min}}(M) = \sigma_d(M) \). We have \( \sigma_{d-i+1}(M^{-1}) = \frac{1}{\sigma_i(M)} \), and in particular \( \sigma_{d}(M^{-1}) = \frac{1}{\sigma_1(M)} \).

The Fröbenius norm of \( M \) is:

\[
\|M\|_F := \sqrt{\sum_{i,j} |M_{i,j}|^2},
\]

(3)

\[
= \sqrt{\text{tr}(MM^H)} = \sqrt{\text{tr}(M^HM)}.
\]

(4)

The induced Fröbenius distance between two complex matrices \( C_1 \) and \( C_2 \) is \( \rho_E(C_1,C_2) = \|C_1 - C_2\|_F \).

The operator norm or spectral norm of a matrix \( M \) is:

\[
\|M\|_O = \max_{x \neq 0} \frac{\|Mx\|_2}{\|x\|_2},
\]

(5)

\[
= \sqrt{\lambda_{\text{max}}(M^HM)},
\]

(6)

\[
= \sigma_{\text{max}}(M).
\]

(7)

Notice that \( M^HM \) is a Hermitian positive semi-definite matrix. The operator norm coincides with the spectral radius \( \rho(M) = \max_i \{ |\lambda_i(M)| \} \) of the matrix \( M \). Thus calculating the operator norm/spectrum radius requires cubic time in the dimension of the matrix by calculating the Singular Value Decomposition (SVD) of \( M = UDV^H \) where \( D = \text{Diag}(\sigma_1, \ldots, \sigma_d) \) is the diagonal matrix with coefficients being the singular values of \( M \). To calculate the largest singular value, we may use a faster numerical power method of Lanczos [54] or its optimized variants [30]. The operator norm is upper bounded by the Fröbenius norm: \( \|M\|_O \leq \|M\|_F \), and we have \( \|M\|_O \geq \max_{i,j} |M_{i,j}| \). When the dimension \( d = 1 \), the operator norm of \( [M] \) coincides with the complex modulus: \( \|M\|_O = |M| \).

Any matrix norm \( \|\cdot\| \) (including the operator norm) satisfies (i) \( \|M\| \geq 0 \) with equality iff \( M = 0 \) (the matrix with all entries equal to zero), (ii) \( \|\alpha M\| = |\alpha| \|M\| \), (iii) \( \|M_1 + M_2\| \leq \|M_1\| + \|M_2\| \), and (iv) \( \|M_1M_2\| \leq \|M_1\| \|M_2\| \).

2 Hyperbolic geometry in the complex plane: The Poincaré and Klein models

We concisely review the Poincaré upper plane, the Poincaré disk, and the Klein disk models of the hyperbolic plane [24, 39]. In information geometry, the Fisher-Rao geometry of location-scale families amount to hyperbolic geometry [66].

2.1 Poincaré complex upper plane

The Poincaré upper plane domain is

\[
\mathbb{H} = \{ z = a + ib : \ z \in \mathbb{C}, \ b = \text{Im}(z) > 0 \}.
\]

(8)

The Hermitian metric tensor is

\[
ds_U^2 = \frac{d\zeta d\overline{\zeta}}{\text{Im}(z)^2},
\]

(9)
or equivalently the Riemannian metric tensor is:

\[ ds_U^2 = \frac{dx^2 + dy^2}{y^2}, \]  

(10)

Geodesics between \( z_1 \) and \( z_2 \) are arcs of semi-circles whose centers are on the real axis and orthogonal to the real axis, or vertical line segments when \( \text{Im}(z_1) = \text{Im}(z_2) \).

The geodesic distance is

\[ \rho_U(z_1, z_2) := \log \left( \frac{|z_1 - \overline{z}_2| + |z_1 - z_2|}{|z_1 - \overline{z}_2| - |z_1 - z_2|} \right), \]  

(11)

or equivalently

\[ \rho_U(z_1, z_2) = \text{arccosh} \left( \sqrt{\frac{|z_1 - \overline{z}_2|^2}{\text{Im}(z_1)\text{Im}(z_2)}} \right), \]  

(12)

where

\[ \text{arccosh}(x) = \log \left( x + \sqrt{x^2 - 1} \right). \]  

(13)

Equivalent formula can be obtained by using the identity

\[ \log(x) = \text{arccosh} \left( \frac{x^2 + 1}{2x} \right) = \text{artanh} \left( \frac{x^2 - 1}{x^2 + 1} \right), \]  

(14)

where

\[ \text{artanh}(x) = \frac{1}{2} \log \frac{1 + x}{1 - x}, \quad x < 1. \]  

(15)

By interpreting a complex number \( z = x + iy \) as a 2D point with Cartesian coordinates \( (x, y) \), the metric can be rewritten as

\[ ds_U^2 = \frac{dx^2 + dy^2}{y^2} = \frac{1}{y^2} ds_E^2, \]  

(16)

where \( ds_E^2 = dx^2 + dy^2 \) is the Euclidean metric. That is, the Poincaré upper plane metric can be rewritten as a conformal factor times the Euclidean metric. Thus the metric of Eq. (16) shows that the Poincaré upper plane model is a conformal model: That is, the Euclidean angle measurements in the \( (x, y) \) chart coincides with the underlying hyperbolic angles.

The group of orientation-preserving isometries (i.e., without reflections) is the real projective special group \( \text{PSL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R})/\{\pm I\} \), where \( \text{SL}(2, \mathbb{R}) \) is the special linear group of matrices with unit determinant:

\[ \text{Isom}^+(\mathbb{H}) \cong \text{PSL}(2, \mathbb{R}). \]  

(17)

The left group action is a fractional linear transformation (also called a Möbius transformation):

\[ g.z = \frac{az + b}{cz + d}, \quad g = \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right], \quad ab - cd \neq 0. \]  

(18)

The condition \( ab - cd \neq 0 \) is to ensure that the Möbius transformation is not constant. The set of Möbius transformations form a group \( \text{Moeb}(\mathbb{R}, 2) \). The elements of the Möbius group can be represented by corresponding matrices of \( \text{PSL}(2, \mathbb{R}) \):

\[ \left\{ \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right], \quad ab - cd \neq 0 \right\}. \]  

(19)
The neutral element $e$ is encoded by the identity matrix.

The fractional linear transformations
\[ w(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc \neq 0 \tag{20} \]
are the analytic mappings $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ of the Poincaré upper plane onto itself.

The action is transitive (i.e., $\forall z_1, z_2 \in \mathbb{H}, \exists g$ such that $g.z_1 = z_2$) and faithful (i.e., if $g.z = z \forall z$ then $g = e$). The stabilizer of $i$ is the rotation group:
\[ \text{SO}(2) = \left\{ \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} : \theta \in \mathbb{R} \right\}. \tag{21} \]

The unit speed geodesic anchored at $i$ and going up (geodesic with initial condition) is:
\[ \gamma(t) = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} \times i = ie^t. \tag{22} \]

Since the other geodesics are obtained by the action of $\text{PSL}(2, \mathbb{R})$, it follows that the geodesics in $\mathbb{H}$ are parameterized by:
\[ \gamma(t) = \frac{aie^t + b}{cie^t + d}. \tag{23} \]

### 2.2 Poincaré Disk

The Poincaré unit disk is
\[ \mathbb{D} = \{ \overline{w} \overline{w} < 1 : w \in \mathbb{C} \}. \tag{24} \]

The metric tensor is
\[ ds^2_\mathbb{D} = \frac{4dwd\overline{w}}{(1-|w|^2)^2}. \tag{25} \]

Since $ds^2_\mathbb{D} = \left( \frac{2}{1-|z|^2} \right)^2 ds^2_E$, we deduce that the metric is conformal. The geodesics between $w_1$ and $w_2$ are either arcs of circles intersecting orthogonally the disk boundary $\partial \mathbb{D}$, or straight lines passing through the origin 0 of the disk and clipped to the disk domain.

The geodesic distance in the Poincaré disk is
\[ \rho_\mathbb{D}(w_1, w_2) = \text{arccosh} \left( \sqrt{\frac{|w_1\overline{w}_2 - 1|^2}{(1-|w_1|^2)(1-|w_2|^2)}} \right), \tag{26} \]
\[ = 2\text{arctanh} \left( \frac{|w_2 - w_1|}{1 - \overline{w}_1w_2} \right). \tag{27} \]

The group of orientation preserving isometry is the complex projective special group $\text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\{\pm I\}$ where $\text{SL}(2, \mathbb{C})$ is the group of $2 \times 2$ complex matrices with unit determinant.

In the Poincaré disk model, the transformation
\[ T_{z_0, \theta}(z) = e^{i\theta} \frac{z - z_0}{1 - \overline{z}_0z} \tag{28} \]
corresponds to a hyperbolic motion (a Möbius transformation [82]) which moves point $z_0$ to the origin 0, and then makes a rotation of angle $\theta$. The group of such transformations is the automorphism group of the disk, $\text{Aut}(\mathbb{D})$, and transformation $T_{z_0, \theta}$ is called a biholomorphic automorphism (a one-to-one conformal mapping of the disk onto itself).
2.2.1 Klein disk

The Klein disk model \cite{24,83} (or Klein-Beltrami model) is defined on the unit disk as the Poincaré disk model. The metric is

\[
\begin{align*}
\text{d}s^2_K &= \left( \frac{\text{d}s^2_E}{1 - \|x\|^2_E} + \frac{\langle x, \text{d}x \rangle_E}{(1 - \|x\|^2_E)^2} \right). \\
(29)
\end{align*}
\]

It is not a conformal metric (except at the origin), and therefore the Euclidean angles in the \((x,y)\) chart do not correspond to the underlying hyperbolic angles.

The Klein distance between two points \(k_1 = (x_1, y_1)\) and \(k_2 = (x_2, y_2)\) is

\[
\rho_K(k_1, k_2) = \arccosh \left( \frac{1 - (x_1 x_2 + y_1 y_2)}{\sqrt{(1 - \|k_1\|^2)(1 - \|k_2\|^2)}} \right).
\]

(30)

(An equivalent formula will be reported in page \cite{21} in a more general Theorem \cite{1}.)

The advantage of the Klein disk over the Poincaré disk is that geodesics are straight Euclidean lines clipped to the unit disk domain. Therefore this model is well adapted to implement computational geometric algorithms and data structures, see for example \cite{71,48}.

The group of isometries in the Klein model are projective maps \(\mathbb{RP}^2\) preserving the disc.

2.3 Poincaré and Klein distances to the disk origin and conversions

In the Poincaré disk, the distance of a point \(w\) to the origin 0 is

\[
\rho_D(0, w) = \log \left( \frac{1 + |w|}{1 - |w|} \right).
\]

(31)

Since the Poincaré disk model is conformal (and Möbius transformations are conformal maps), Eq. \ref{eq:31} shows that Poincaré disks have Euclidean disk shapes (with displaced centers).

In the Klein disk, the distance of a point \(k\) to the origin is

\[
\rho_K(0, k) = \frac{1}{2} \log \left( \frac{1 + |k|}{1 - |k|} \right) = \frac{1}{2} \rho_D(0, k).
\]

(32)

Observe the multiplicative factor of \(\frac{1}{2}\) in Eq. \ref{eq:32}.

Thus we can easily convert a point \(p \in \mathbb{C}\) in the Poincaré disk to a point \(k \in \mathbb{C}\) in the Klein disk, and vice-versa as follows:

\[
\begin{align*}
w &= \frac{1}{1 + \sqrt{1 - |k|^2}} k, \\
k &= \frac{2}{1 + |w|^2} w.
\end{align*}
\]

(33)

(34)

Let \(C_{K \to D}(k)\) and \(C_{D \to K}(w)\) denote these conversion functions with

\[
\begin{align*}
C_{K \to D}(k) &= \frac{1}{1 + \sqrt{1 - |k|^2}} k, \\
C_{D \to K}(w) &= \frac{2}{1 + |w|^2} w.
\end{align*}
\]

(35)

(36)
We can write $C_{K \to D}(k) = \alpha(k)k$ and $C_{D \to K}(w) = \beta(w)w$, so that $\alpha(k) > 1$ is an expansion factor, and $\beta(w) < 1$ is a contraction factor.

The conversion functions are Möbius transformations represented by the matrices:

$$M_{K \to D}(k) = \begin{bmatrix} \alpha(k) & 0 \\ 0 & 1 \end{bmatrix},$$  \hspace{1cm} (37)

$$M_{D \to K}(w) = \begin{bmatrix} \beta(w) & 0 \\ 0 & 1 \end{bmatrix}. $$  \hspace{1cm} (38)

For sanity check, let $w = r + 0i$ be a point in the Poincaré disk with equivalent point $k = \frac{2}{1+r}r + 0i$ in Poincaré disk. Then we have:

$$\rho_K(0, k) = \frac{1}{2} \log \frac{1 + |k|}{1 - |k|},$$  \hspace{1cm} (39)

$$= \frac{1}{2} \log \frac{1 + \frac{2}{1+r}r}{1 - \frac{2}{1+r}r},$$  \hspace{1cm} (40)

$$= \frac{1}{2} \log \frac{1 + r^2 + 2r}{1 + r^2 - 2r},$$  \hspace{1cm} (41)

$$= \frac{1}{2} \log \frac{(1+r)^2}{(1-r)^2},$$  \hspace{1cm} (42)

$$= \log \frac{1 + r}{1 - r} = \rho_D(0, w).$$  \hspace{1cm} (43)

We can convert a point $z$ in the Poincaré upper plane to a corresponding point $w$ in the Poincaré disk, or vice versa, using the following Möbius transformations:

$$w = \frac{z - i}{z + i},$$  \hspace{1cm} (44)

$$z = \frac{i(1 + w)}{1 - w}. $$  \hspace{1cm} (45)

Notice that we compose Möbius transformations by multiplying their matrix representations.

3 The Siegel upper space

The Siegel upper space $\mathbb{SH}(d)$ is defined as the space of symmetric complex matrices of size $d \times d$ which have positive-definite imaginary part:

$$\mathbb{SH}(d) := \{ Z = X + iY : X \in \text{Sym}(d, \mathbb{R}), Y \in \text{PD}(d, \mathbb{R}) \}. $$  \hspace{1cm} (46)

The space $\mathbb{SH}(d)$ is a tube domain of dimension $d(d+1)$. We can extract the components $X$ and $Y$ from $Z$ as $X = \frac{1}{2}(Z + \bar{Z})$ and $Y = \frac{1}{2i}(Z - \bar{Z}) = -\frac{i}{2}(Z - \bar{Z})$. The pair $(X, Y)$ belongs to the Cartesian product of a vector space with the SPD cone: $(X, Y) \in \text{Sym}(d, \mathbb{R}) \times \text{PD}(d, \mathbb{R})$. When $d = 1$, the Siegel upper space coincides with the Poincaré upper plane: $\mathbb{SH}(1) = \mathbb{H}$. The geometry of the Siegel upper space was studied independently by Siegel [89] and Hua [45] from different viewpoints in the late 1930’s-1940’s. Historically, these class of complex matrices $Z \in \mathbb{SH}(d)$ were
first studied by Riemann [55], and later eponymously named Riemann matrices. Riemann matrices are used to define Riemann theta functions [84, 92, 2, 1].

The Siegel distance in the upper plane is induced by the following metric tensor:
\[ d_s^2(U) = 2\text{tr} \left( Y^{-1}dZ \ Y^{-1}d\bar{Z} \right). \] (47)

The formula for the Siegel upper distance between \( Z_1 \) and \( Z_2 \in \mathbb{H}(d) \) was calculated in Siegel’s paper [89] as follows:
\[ \rho_U(Z_1, Z_2) = \sqrt{\sum_{i=1}^{d} \log^2 \left( \frac{1 + \sqrt{r_i}}{1 - \sqrt{r_i}} \right)} , \] (48)
where
\[ r_i = \lambda_i (R(Z_1, Z_2)) , \] (49)
with \( R(Z_1, Z_2) \) denoting the matrix generalization of the cross-ratio:
\[ R(Z_1, Z_2) := (Z_1 - Z_2)(Z_1 - \bar{Z}_2)^{-1}(\bar{Z}_1 - \bar{Z}_2)(Z_1 - Z_2)^{-1} , \] (50)
and \( \lambda_i(M) \) denotes the \( i \)-th largest (real) eigenvalue of (complex) matrix \( M \).

This Siegel distance in the upper plane is a smooth spectral distance function: That is, \( \rho_U(Z_1, Z_2) = f \circ \Lambda(R(Z_1, Z_2)) \), where \( \Lambda(\cdot) \) is the eigenvalue map, and \( f \) is the following symmetric function (i.e., invariant under parameter permutations):
\[ f(x_1, \ldots, x_d) = \sqrt{\sum_{i=1}^{d} \log^2 \left( \frac{1 + \sqrt{x_i}}{1 - \sqrt{x_i}} \right)} . \] (52)

A remarkable property is that all eigenvalues of \( R(Z_1, Z_2) \) are positive (see [89]) although \( R \) may not necessarily be a Hermitian matrix. Thus calculating the Siegel distance on the upper plane requires cubic time, i.e., cost of computing the eigenvalue decomposition.

This Siegel distance in the upper plane \( \mathbb{H}(d) \) generalizes several distances:

- When \( Z_1 = iY_1 \) and \( Z_2 = iY_2 \), we have
\[ \rho_U(Z_1, Z_2) = \rho_{PD}(Y_1, Y_2) , \] (51)
the Riemannian distance between \( Y_1 \) and \( Y_2 \) on the symmetric positive-definite manifold [35, 63]:
\[ \rho_{PD}(Y_1, Y_2) = \| \text{Log}(Y_1^{-1}Y_2) \|_F \] (52)
\[ = \sqrt{\sum_{i=1}^{d} \log^2 (\lambda_i(Y_1^{-1}Y_2))} . \] (53)

In that case, the Siegel upper metric for \( Z = iY \) becomes the affine-invariant metric:
\[ d_s^2(U)(Z) = \text{tr} \left( (Y^{-1}dY)^2 \right) = d_{PD}(Y) , \] (54)
Indeed, we have \( \rho_{PD}(C^\top Y_1 C, C^\top Y_2 C) = \rho_{PD}(Y_1, Y_2) \) for any \( C \in \text{GL}(d, \mathbb{R}) \) and \( \rho_{PD}(Y_1^{-1}, Y_2^{-1}) = \rho_{PD}(Y_1, Y_2) \).

5The mnemonic ‘R’ stands for ratio.

6In practice, when calculating numerically the eigenvalues of the complex matrix \( R(Z_1, Z_2) \), we obtain very small imaginary parts which shall be rounded to zero.
• In 1D, the Siegel upper distance \( \rho_U(Z_1, Z_2) \) between \( Z_1 = [z_1] \) and \( Z_2 = [z_2] \) (with \( z_1 \) and \( z_2 \) in \( \mathbb{C} \)) amounts to the hyperbolic distance on the Poincaré upper plane \( \mathbb{H} \):

\[
\rho_U(Z_1, Z_2) = \rho_U(z_1, z_2),
\]

where

\[
\rho_U(z_1, z_2) := \log \frac{|z_1 - \overline{z_2}| + |z_1 - z_2|}{|z_1 - \overline{z_2}| - |z_1 - z_2|}.
\]

• The Siegel distance between two diagonal matrices \( Z = \text{diag}(z_1, \ldots, z_d) \) and \( Z' = \text{diag}(z'_1, \ldots, z'_d) \) is:

\[
\rho_U(Z, Z') = \sqrt{\sum_{i=1}^{d} \rho_U^2(z_i, z'_i)}.
\]

Observe that the Siegel distance is a non-separable metric distance, but its squared distance is separable when the matrices are diagonal:

\[
\rho_U^2(Z, Z') = \sum_{i=1}^{d} \rho_U^2(z_i, z'_i).
\]

The Siegel metric in the upper plane is invariant by generalized matrix Möbius transformations (also called linear fractional transformations or rational transformation):

\[
\phi_S(Z) := (AZ + B)(CZ + D)^{-1},
\]

where \( S \in M(2d, \mathbb{R}) \) is the following \( 2d \times 2d \) block matrix:

\[
S = \begin{bmatrix} A & B \\ C & D \end{bmatrix},
\]

which satisfies

\[
AB^\top = BA^\top, \quad CD^\top = DC^\top, \quad AD^\top - BC^\top = I.
\]

The map \( \phi_S(\cdot) = \phi(S, \cdot) \) is called a symplectic map.

The set of matrices \( S \) encoding the symplectic maps forms a group called the real symplectic group \( \text{Sp}(d, \mathbb{R}) \) \(^{37}\) (the group of Siegel motion):

\[
\text{Sp}(d, \mathbb{R}) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \ A, B, C, D \in M(d, \mathbb{R}) : AB^\top = BA^\top, \ CD^\top = DC^\top, \ AD^\top - BC^\top = I \right\}.
\]

It can be shown that symplectic matrices have unit determinant \(^{60, 86}\), and therefore \( \text{Sp}(d, \mathbb{R}) \) is a subgroup of \( \text{SL}(2d, \mathbb{R}) \), the special group of real invertible matrices with unit determinant. We also check that if \( M \in \text{Sp}(d, \mathbb{R}) \) then \( M^\top \in \text{Sp}(d, \mathbb{R}) \).

Matrix \( S \) denotes the representation of the group element \( g_S \). The symplectic group operation corresponds to matrix multiplications of their representations, the neutral element is encoded by \( E = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \), and the group inverse of \( g_S \) with \( S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) is encoded by the matrix

\[
S^{(-1)} = \begin{bmatrix} D^\top & -B^\top \\ -C^\top & A^\top \end{bmatrix}.
\]
The Siegel upper plane is isomorphic to $\text{Sp}(2d, \mathbb{R})$. Informally speaking, the elements of $\text{SpO}(2d)$, conversely the action $S$ of the matrix inverse usually matrix inverse $S$, we get $\phi = \text{SpO}(2d)$. The action $\phi$ fixes $\mathbb{R}$. Therefore, by taking the group inverse

$$S^{-1}(Z) = \begin{pmatrix} (B^2)^\top & 0 \\ -(AB^{-1})^\top & (B^{-1})^\top \end{pmatrix}, \quad (64)$$

we get

$$\phi_{S^{-1}(Z)}(Z) = iI. \quad (65)$$

The action $\phi(Z)$ can be interpreted as a “Siegel translation” moving matrix $iI$ to matrix $Z$, and conversely the action $\phi_{S^{-1}(Z)}$ as moving $Z$ to $iI$.

The stabilizer group of $Z = iI$ (also called isotropy group, the set of group elements $S \in \text{Sp}(2d, \mathbb{R})$ whose action fixes $Z$) is the subgroup of symplectic orthogonal matrices $\text{SpO}(2d, \mathbb{R})$:

$$\text{SpO}(2d, \mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A^\top A + B^\top B = I, A^\top B \in \text{Sym}(d, \mathbb{R}) \right\}. \quad (66)$$

We have $\text{SpO}(2d, \mathbb{R}) = \text{Sp}(2d, \mathbb{R}) \cap O(2d)$, where $O(2d)$ is the group of orthogonal $2d \times 2d$ matrices:

$$O(2d) := \left\{ R \in M(2d, \mathbb{R}) : RR^\top = R^\top R = I \right\}. \quad (67)$$

Informally speaking, the elements of $\text{SpO}(2d, \mathbb{R})$ represent the “Siegel rotations” in the upper plane. The Siegel upper plane is isomorphic to $\text{Sp}(2d, \mathbb{R})/O_d(\mathbb{R})$.

A pair of matrices $(Z_1, Z_2)$ can be transformed into another pair of matrices $(Z_1', Z_2')$ of $\mathbb{SH}(d)$ if and only if $\lambda(R(Z_1, Z_2)) = \lambda(R(Z_1', Z_2'))$, where $\lambda(M) := \{\lambda_1(M), \ldots, \lambda_d(M)\}$ is the spectrum of matrix $M$.

The orientation-preserving isometry group of the Siegel upper plane is the projective symplectic group $\text{PSp}(d, \mathbb{F}) = \text{Sp}(d, \mathbb{F})/\{I_{2d}\}$ (generalizing $\text{PSL}(2, \mathbb{R})$ obtained when $d = 1$).

The geodesics in the Siegel upper space can be obtained by applying symplectic transformations to the geodesics of the positive-definite manifold (also known as SPD manifold) which is a totally geodesic submanifold of $\text{SU}(d)$. Let $Z_1 = iP_1$ and $Z_2 = iP_2$. Then the geodesic $Z_{12}(t)$ with $Z_{12}(0) = Z_1$ and $Z_{12}(1) = Z_2$ is expressed as:

$$Z_{12}(t) = iP_1^{\frac{1}{2}} \text{Exp}(t\log(P_1^{-\frac{1}{2}}P_2P_1^{-\frac{1}{2}}))P_1^{\frac{1}{2}}, \quad (68)$$

where $\text{Exp}(M)$ denotes the matrix exponential:

$$\text{Exp}(M) = \sum_{i=0}^{\infty} \frac{1}{i!} M^i, \quad (69)$$

and $\text{Log}(M)$ is the principal matrix logarithm, unique when $M$ has all positive eigenvalues.

The equation of the geodesic emanating from $P$ with tangent vector $S \in T_P$ (symmetric matrix) on the SPD manifold is:

$$\gamma_P, S(t) = P^{\frac{1}{2}} \text{Exp}(tP^{-\frac{1}{2}}SP^{-\frac{1}{2}})P_2^{\frac{1}{2}}, \quad (70)$$
Both the exponential and the principal logarithm of a matrix $M$ can be calculated in cubic time when the matrices are diagonalizable: Let $V$ denote the matrix of eigenvectors so that

$$M = V \text{diag}(\lambda_1, \ldots, \lambda_d)V^{-1}, \quad (71)$$

where $\lambda_1, \ldots, \lambda_d$ are the corresponding eigenvalues. Then for a scalar function $f$ (e.g., $f(u) = \exp(u)$ or $f(u) = \log u$), we define the corresponding matrix function $f(M)$ as

$$f(M) := V \text{diag}(f(\lambda_1), \ldots, f(\lambda_d))V^{-1}. \quad (72)$$

## 4 Siegel disk

The Siegel disk\(^7\)\(^8\) is an open convex complex domain defined by

$$\mathbb{D}(d) := \{ W \in \text{Sym}(d, \mathbb{C}) : I - WW^* > 0 \}. \quad (73)$$

The Siegel disk can be written equivalently as $\mathbb{D}(d) := \{ W \in \text{Sym}(d, \mathbb{C}) : I - W^* W > 0 \}$.

When $d = 1$, the Siegel disk\(^7\) coincides with the Poincaré disk: $\mathbb{D}(1) = \mathbb{D}$.

The boundary of the Siegel disk is called the Shilov boundary \(^{28, 37, 36}\). The Shilov boundary is a stratified manifold where each stratum is defined as a space of constant rank-deficient matrices \(^{12}\).

The metric in the Siegel disk is:

$$ds_D^2 = \text{tr} \left( (I - W W^*)^{-1} dW (I - W W^*)^{-1} d\bar{W} \right). \quad (74)$$

When $d = 1$, we recover $ds_D^2 = \frac{1}{1 - |w|^2} dw d\bar{w}$ which is the usual metric in the Poincaré disk (up to a missing factor of 4, see Eq. 25).

This Siegel metric induces a Kähler geometry \(^9\) with Kähler potential:

$$K(W) = -\text{tr} (\text{Log} (I - W^H W)). \quad (75)$$

The distance between $W_1$ and $W_2$ in $\mathbb{D}(d)$ is calculated as follows:

$$\rho_D(W_1, W_2) = \log \left( \frac{1 + \|\Phi_{W_1}(W_2)\|_O}{1 - \|\Phi_{W_1}(W_2)\|_O} \right), \quad (76)$$

where

$$\Phi_{W_1}(W_2) = (I - W_1 W_1^*)^{-\frac{1}{2}} (W_2 - W_1) (I - W_1 W_2)^{-1} (I - W_1 W_1^*)^{\frac{1}{2}}, \quad (77)$$

is a Siegel translation which moves $W_1$ to the origin $O$ of the disk: We have $\Phi_W(W) = 0$. Notice that the Siegel disk distance, although a spectral distance function via the operator norm, is not smooth because of it uses the maximum singular value (recall that the Siegel upper plane distance uses all eigenvalues of a matrix cross-ratio $R$).

---

\(^7\)In the Cartan classification \(^{25}\), the Siegel disk is a Siegel domain of type III.

\(^8\)The Siegel disk was described by Hua in his 1948’s paper \(^{16}\) (page 205) on the geometries of matrices \(^{97}\). Siegel’s paper \(^{89}\) in 1943 only considered the Siegel upper plane. Here, the Siegel (complex matrix) disk is not to be confused with the other notion of Siegel disk in complex dynamics which is a connected component in the Fatou set.
It follows that the cost of calculating a Siegel distance in the Siegel disk is cubic: We require to compute a symmetric matrix square root \[91\] in Eq. 77, and then compute the largest singular value for the operator norm in Eq. 76.

Notice that when \(d = 1\), the “1d” scalar matrices commute, and we have:

\[
\Phi_{w_1}(w_2) = (1 - w_1 \overline{w_1})^{-\frac{1}{2}} (w_2 - w_1)(1 - \overline{w_1} w_2)^{-1} (1 - \overline{w_1} w_1)^{\frac{1}{2}},
\]

\[
= \frac{w_2 - w_1}{1 - \overline{w_1} w_2}.
\]

(78)

(79)

This corresponds to a translation of \(w_1\) to 0 (see Eq. 28).

Let us observe the following special cases of the Siegel-Poincaré distance:

- **Distance to the origin:** When \(W_1 = 0\) and \(W_2 = W\), we have \(\Phi_0(W) = W\), and therefore the Siegel distance in the disk between a matrix \(W\) and the origin 0 is:

\[
\rho_D(0, W) = \log \left( \frac{1 + \|W\|_O}{1 - \|W\|_O} \right).
\]

(80)

In particular, when \(d = 1\), we recover the formula of Eq. 31:

\[
\rho_D(0, w) = \log \left( \frac{1 + |w|}{1 - |w|} \right).
\]

- **When \(d = 1\), we have \(W_1 = [w_1]\) and \(W_2 = [w_2]\), and**

\[
\rho_D(W_1, W_2) = \rho_D(w_1, w_2).
\]

(81)

- **Consider diagonal matrices \(W = \text{diag}(w_1, \ldots, w_d) \in \mathbb{SD}(d)\) and \(W' = \text{diag}(w'_1, \ldots, w'_d) \in \mathbb{SD}(d)\). We have \(|w_i| \leq 1\) for \(i \in \{1, \ldots, d\}\). Thus the diagonal matrices belong to the polydisk domain. Then we have**

\[
\rho_D(W_1, W_2) = \sqrt{\sum_{i=1}^{d} \rho_D^2(w_i, w'_i)}.
\]

(82)

Notice that the polydisk domain is a Cartesian product of 1D complex disk domains, but it is not the unit \(d\)-dimensional complex ball \(\{z \in \mathbb{C}^d : \sum_{i=1}^{d} z_i \overline{z_i} = 1\}\).

We can convert a matrix \(Z\) in the Siegel upper space to an equivalent matrix \(W\) in the Siegel disk by using the following matrix Cayley transformation for \(Z \in \mathbb{SH}_d\):

\[
W_{U \rightarrow D}(Z) := (Z - iI)(Z + iI)^{-1} \in \mathbb{SD}(d).
\]

(83)

Notice that the imaginary positive-definite matrices \(iP\) of the upper plane (vertical axis) are mapped to

\[
W_{U \rightarrow D}(iP) := (P - I)(P + I)^{-1} \in \mathbb{SD}(d),
\]

(84)

i.e., the real symmetric matrices belonging to the horizontal-axis of the disk.

The inverse transformation for a matrix \(W\) in the Siegel disk is

\[
Z_{D \rightarrow U}(W) = i (I + W) (I - W)^{-1} \in \mathbb{SH}(d),
\]

a matrix in the Siegel upper space.
With those mappings, the origin of the disk $0 \in \mathbb{SD}(d)$ coincides with $iI \in \mathbb{SH}(d)$.

A key property is that the geodesics passing through the matrix origin $0$ are expressed by straight line segments in the Siegel disk. We can check that

$$\rho_D(0, W) = \rho_D(0, \alpha W) + \rho_D(\alpha W, W),$$

(86)

for any $\alpha \in [0, 1]$.

To describe the geodesics between $W_1$ and $W_2$, we first move $W_1$ to $0$ and $W_2$ to $\Phi_{W_1}(W_2)$. Then the geodesic between $0$ and $\Phi_{W_1}(W_2)$ is a straight line segment, and we map back this geodesic via $\Phi^{-1}W_1(\cdot)$. The inverse of a symplectic map is a symplectic map which corresponds to the action of an element of the complex symplectic group.

The complex symplectic group is

$$\text{Sp}(d, \mathbb{C}) = \left\{ M^\top J M = J, M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M(2d, \mathbb{C}) \right\},$$

(87)

with

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

(88)

for the $d \times d$ identity matrix $I$. Notice that the condition $M^\top J M = J$ amounts to check that

$$AB^\top = BA^\top, \quad CD^\top = DC^\top, \quad AD^\top - BC^\top = I.$$  

(89)

The conversions between the Siegel upper plane to the Siegel disk (and vice versa) can be expressed using complex symplectic transformations associated to the matrices:

$$W(Z) = \begin{bmatrix} I & -iI \\ iI & iI \end{bmatrix} \cdot Z = (Z - iI)(Z + iI)^{-1},$$

(90)

$$Z(W) = \begin{bmatrix} iI & iI \\ -iI & iI \end{bmatrix} \cdot W = i(I + W)(I - W)^{-1}.$$

(91)

Figure 1 depicts the conversion of the upper plane to the disk, and vice versa.

The orientation-preserving isometries in the Siegel disk is the projective complex symplectic group $\text{PSp}(d, \mathbb{C}) = \text{Sp}(d, \mathbb{C})/\{\pm I_{2d}\}$.

It can be shown that

$$\text{Sp}(d, \mathbb{C}) = \left\{ M = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \in M(2d, \mathbb{C}) \right\},$$

(92)

with

$$A^\top B - B^H A = 0, \quad A^\top \bar{A} - B^H B = I.$$  

(93)

(94)

and the left action of $g \in \text{Sp}(d, \mathbb{C})$ is

$$g.W = (AW + B)(\bar{A}W + \bar{B})^{-1}.$$  

(95)
Figure 1: Illustrating the properties and conversion between the Siegel upper plane and the Siegel disk.

The isotropy group at the origin 0 is

$$\{ \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix} : A \in U(d) \}$$, (96)

where \( U(d) \) is the unitary group: \( U(d) = \{ U \in \text{GL}(d, \mathbb{C}) : U^H U = U U^H = I \} \).

Thus we can “rotate” a matrix \( W \) with respect to the origin so that its imaginary part becomes 0: There exists \( A \) such that \( \text{Re}(A W W^{-1} \overline{A}^{-1}) = 0 \).

More generally, we can define a Siegel rotation \([62]\) in the disk with respect to a center \( W_0 \in \mathbb{SD}(d) \) as follows:

$$R_{W_0}(W) = (A W - A W_0)(B - B W_0 W)^{-1},$$ (97)

where

$$\overline{A} A = (I - W_0 W_0)^{-1},$$ (98)

$$\overline{B} B = (I - W_0 W_0)^{-1},$$ (99)

$$\overline{A} A W_0 = W_0 B B.$$ (100)

In 1D, the Poincaré disk can be embedded non-diagonally onto the Siegel upper plane \([57]\).

5 Siegel-Klein distance

5.1 Background on Hilbert geometry

Consider a normed vector space \((V, \| \cdot \|)\), and define the Hilbert distance \([43]\) as follows:

\[
\rho_U(Z_1, Z_2) = \sqrt{\sum_{i=1}^d \log^2 \left( \frac{1 + r_i}{1 - r_i} \right)}
\]

where

\[
r_i = \lambda_i(R(Z_1, Z_2))
\]

\[
R(Z_1, Z_2) := (Z_1 - Z_2)(Z_1 - \overline{Z}_2)^{-1}(Z_1 - Z_2)(Z_1 - Z_2)^{-1}
\]
Figure 2: Hilbert distance induced by a bounded open convex domain $\Omega$. 

**Definition 1 (Hilbert distance)** The Hilbert distance is defined for any open bounded convex domain $\Omega$ and a prescribed positive factor $\kappa > 0$ by

$$
H_{\Omega,\kappa}(p, q) := \begin{cases} 
\kappa \log |\text{CR}(\bar{p}, p; q, \bar{q})|, & p \neq q, \\
0 & p = q.
\end{cases}
$$

(101)

where $\bar{p}$ and $\bar{q}$ are the unique two intersection points of the line $(pq)$ with the boundary $\partial \Omega$ of the domain $\Omega$ as depicted in Figure 2 and $\text{CR}$ denotes the cross-ratio of four points (a projective invariant):

$$
\text{CR}(a, b; c, d) = \frac{\|a - c\|\|b - d\|}{\|a - d\|\|b - c\|}.
$$

(102)

When $p \neq q$, we have:

$$
H_{\Omega,\kappa}(p, q) := \kappa \log \frac{\|\bar{q} - p\|\|\bar{p} - q\|}{\|q - q\|\|\bar{p} - p\|}.
$$

(103)

The Hilbert distance is a *metric distance* which does not depend on the underlying norm of the vector space:

**Proposition 1 (Formula of Hilbert distance)** The Hilbert distance between two points $p$ and $q$ of an open bounded convex domain $\Omega$ is

$$
H_{\Omega,\kappa}(p, q) = \begin{cases} 
\kappa \log \left| \frac{\alpha_+(1 - \alpha_-)}{\alpha_-(\alpha_+ - 1)} \right|, & p \neq q, \\
0 & p = q.
\end{cases}
$$

(104)

where $\bar{p} = p + \alpha^-(q - p)$ and $\bar{q} = p + \alpha^+(q - p)$ are the two intersection points of the line $(pq)$ with the boundary $\partial \Omega$ of the domain $\Omega$.

**Proof:** For distinct points $p$ and $q$ of $\Omega$, let $\alpha^+ > 1$ be such that $\bar{q} = p + \alpha^+(q - p)$, and $\alpha_- < 0$ such that $\bar{p} = p + \alpha^-(q - p)$. Then we have $\|\bar{q} - p\| = \alpha_+\|q - p\|$, $\|\bar{p} - p\| = |\alpha_-|\|q - p\|$, $\|q - \bar{q}\| = (\alpha_+ - 1)\|p - q\|$ and $\|\bar{p} - q\| = (1 - \alpha_-)\|p - q\|$. Thus we get

$$
H_{\Omega,\kappa}(p, q) = \kappa \log \frac{\|\bar{q} - p\|\|\bar{p} - q\|}{\|q - q\|\|\bar{p} - p\|},
$$

(105)

$$
= \kappa \log \frac{\alpha_+(1 - \alpha_-)}{\alpha_-(\alpha_+ - 1)},
$$

(106)
and \( H_\Omega(p, q) = 0 \) if and only if \( p = q \).

We may also write the source points \( p \) and \( q \) as linear interpolations of the extremal points \( \bar{p} \) and \( \bar{q} \) on the boundary: \( p = (1 - \alpha_p)\bar{p} + \alpha_p\bar{q} \) and \( q = (1 - \alpha_q)\bar{p} + \alpha_q\bar{q} \) with \( 0 < \alpha_p < \alpha_q < 1 \). In that case, the Hilbert distance can be written as

\[
H_{\Omega, \kappa}(p, q) = \begin{cases} 
\kappa \log \left( \frac{1 - \alpha_p}{\alpha_p} \frac{\alpha_q}{1 - \alpha_q} \right) & \alpha_p \neq \alpha_q, \\
0 & \alpha_p = \alpha_q.
\end{cases}
\]  

The space \((\Omega, H_\Omega)\) is a metric space. Notice that the above formula has demonstrated that \( H_{\Omega, \kappa}(p, q) = H_{\Omega \cap (pq), \kappa}(p, q) \). That is, the Hilbert distance between two points of a \( d \)-dimensional domain is equivalent to the Hilbert distance between the two points on the 1D domain defined by \( \Omega \) restricted to the line \((pq)\) passing through the points \( p \) and \( q \).

Notice that the boundary \( \partial \Omega \) of the domain may not be smooth (e.g., \( \Omega \) may be a simplex or a polytope). The Hilbert geometry for a unit disk with \( \kappa = \frac{1}{2} \) yields the Klein model [49] (or Klein-Beltrami model [13]) of hyperbolic geometry. The Hilbert geometry for an ellipsoid yields the Cayley-Klein hyperbolic model [26, 83, 70] generalizing the Klein model. The Hilbert geometry for a simplicial polytope is isometric to a normed vector space [32, 76]. We refer to the handbook [80] for a survey with recent results on Hilbert geometry. The Hilbert geometry of the elliptope (i.e., space of correlation matrices) was studied in [76]. Hilbert geometry may be studied from the viewpoint of Finslerian geometry which is Riemannian if and only if the domain \( \Omega \) is an ellipsoid (i.e., Klein and Cayley-Klein hyperbolic geometries).

### 5.2 Hilbert geometry of the Siegel disk domain

Let us consider the Siegel-Klein disk model which is defined as the Hilbert geometry for the domain \( \Omega = \mathbb{S}D(d) \) as depicted in Figure 3 with \( \kappa = \frac{1}{2} \).

**Definition 2 (Siegel-Klein geometry)** The Siegel-Klein disk model is the Hilbert geometry for the open bounded convex domain \( \Omega = \mathbb{S}D(d) \) with constant \( \kappa = \frac{1}{2} \). The Siegel-Klein distance is \( \rho_K(K_1, K_2) := H_{\mathbb{S}D(d), \frac{1}{2}}(K_1, K_2) \).
When \( d = 1 \), the Siegel-Klein disk is the Klein disk model of hyperbolic geometry, and the Klein distance \([71]\) between two any points \( k_1 \in \mathbb{C} \) and \( k_2 \in \mathbb{C} \) restricted to the unit disk is

\[
\rho_K(k_1, k_2) = \arccosh \left( \frac{1 - (\text{Re}(k_1)\text{Re}(k_2) + \text{Im}(k_1)\text{Im}(k_2))}{\sqrt{(1 - |k_1|)(1 - |k_2|)}} \right),
\]

where

\[
\arccosh(x) = \log \left( x + \sqrt{x^2 - 1} \right).
\]

This formula can be derived from the Hilbert distance induced by the Klein unit disk \([83]\).

### 5.3 Calculating the Siegel-Klein distance

The Siegel disk domain \( \mathbb{S} \mathbb{D}(d) = \{ W \in \text{Sym}(d, \mathbb{C}) : I - W > 0 \} \) can be rewritten using the operator norm as

\[
\mathbb{S} \mathbb{D}(d) = \{ W \in \text{Sym}(d, \mathbb{C}) : \|W\|_O < 1 \}. \tag{110}
\]

Let \( \{ K_1 + \alpha(K_2 - K_1), \alpha \in \mathbb{R} \} \) denote the line passing through (matrix) points \( K_1 \) and \( K_2 \). That line intersects the Shilov boundary when

\[
\|K_1 + \alpha(K_2 - K_1)\|_O = 1. \tag{111}
\]

When \( K_1 \neq K_2 \), there are two unique solutions\(^9\). Let one solution be \( \alpha_+ \) with \( \alpha_+ > 1 \), and the other solution be \( \alpha_- \) with \( \alpha_- < 0 \). The Siegel-Klein distance is then defined as

\[
\rho_K(K_1, K_2) = \frac{1}{2} \log \left( \frac{\alpha_+(1 - \alpha_-)}{|\alpha_-|(\alpha_+ - 1)} \right). \tag{112}
\]

Let \( \bar{K}_1 = K_1 + \alpha_-(K_2 - K_1) \) and \( \bar{K}_2 = K_1 + \alpha_+(K_2 - K_1) \) be the extremal matrices (rank deficient belonging to the Shilov boundary).

In practice, we may perform a bisection search on the line \( (K_1K_2) \) to approximate these two extremal points \( \bar{K}_1 \) and \( \bar{K}_2 \) (such that these matrices are ordered along the line as \( \bar{K}_1, K_1, K_2, \bar{K}_2 \)). We may find a lower bound for \( \alpha_- \) and an upper bound for \( \alpha_+ \) as follows: We seek \( \alpha \) on the line \( (K_1K_2) \) such that \( K_1 + \alpha(K_2 - K_1) \) falls outside the Siegel disk:

\[
1 < \|K_1 + \alpha(K_2 - K_1)\|_O. \tag{113}
\]

Since \( \| \cdot \|_O \) is a matrix norm, we have

\[
1 < \|K_1 + \alpha(K_2 - K_1)\|_O \leq \|K_1\|_O + |\alpha|\|(K_2 - K_1)\|_O. \tag{114}
\]

Thus we deduce that

\[
|\alpha| > \frac{1 - \|K_1\|_O}{\|(K_2 - K_1)\|_O}. \tag{115}
\]

\(^9\) A line intersects the boundary of a bounded open convex domain in at most two points.
5.4 Siegel-Klein distance to the origin

When \( K_1 = 0 \) (the 0 matrix denoting the origin of the disk), and \( K_2 = K \in \mathbb{SD}(d) \), it is easy to solve:

\[
\| \alpha K \|_O = 1.
\]  

(116)

We have \( |\alpha| = \frac{1}{\|K\|_O} \), that is,

\[
\alpha_+ = \frac{1}{\|K\|_O} > 1, \quad \alpha_- = -\frac{1}{\|K\|_O} < 0.
\]  

(117)

(118)

In that case, the Siegel-Klein distance of Eq. 112 is expressed as:

\[
\rho_K(0, K) = \frac{1}{2} \log \left( \frac{1 + \|K\|_O}{1 - \|K\|_O} \right),
\]  

(119)

\[
= \frac{1}{2} \log \left( \frac{1 + \|K\|_O}{1 - \|K\|_O} \right),
\]  

(120)

\[
= 2 \rho_D(0, K),
\]  

(121)

where \( \rho_D(0, W) \) is defined in Eq. 80.

Theorem 1 (Siegel-Klein distance to the origin) The Siegel-Klein distance of matrix \( K \in \mathbb{SD}(d) \) to the origin \( O \) is

\[
\rho_K(0, K) = \frac{1}{2} \log \left( \frac{1 + \|K\|_O}{1 - \|K\|_O} \right).
\]  

(122)

The constant \( \kappa = \frac{1}{2} \) is chosen in order to ensure that when \( d = 1 \) the corresponding Klein disk has negative unit curvature.

The result can be easily extended to the case of the Siegel-Klein distance between \( K_1 \) and \( K_2 \) where the origin \( O \) belongs to the line \((K_1 K_2)\). In that case, \( K_2 = \lambda K_1 \) for some \( \lambda \in \mathbb{R} \) (e.g., \( \lambda = \frac{\text{tr}(K_2)}{\text{tr}(K_1)} \)). It follows that

\[
\| K_1 + \alpha(K_2 - K_1) \|_O = 1,
\]  

(123)

\[
|1 + \alpha(\lambda - 1)| = \frac{1}{\|K_1\|_O}.
\]  

(124)

Thus we get the two values defining the intersection of \((K_1 K_2)\) with the Shilov boundary:

\[
\alpha' = \frac{1}{\lambda - 1} \left( \frac{1}{\|K_1\|_O} - 1 \right),
\]  

(125)

\[
\alpha'' = \frac{1}{1 - \lambda} \left( 1 + \frac{1}{\|K_1\|_O} \right).
\]  

(126)

We then apply formula Eq. 112:

\[
\rho_K(K_1, K_2) = \frac{1}{2} \left| \log \left( \frac{\alpha'(1 - \alpha'')}{\alpha''(\alpha' - 1)} \right) \right|,
\]  

(127)

\[
= \frac{1}{2} \left| \log \left( \frac{1 - \|K_1\|_O}{1 + \|K_1\|_O} \right) \right|.
\]  

(128)
Theorem 2 The Siegel-Klein distance between two points $K_1 \neq 0$ and $K_2$ on a line $(K_1K_2)$ passing through the origin is

$$\rho_K(K_1, K_2) = \frac{1}{2} \log \frac{1 - \|K_1\|_O \|K_1\|_O (1 - \lambda) - (1 + \|K_1\|_O)}{1 + \|K_1\|_O \|K_1\|_O (\lambda - 1) - (1 - \|K_1\|_O)},$$

where $\lambda = \frac{\text{tr}(K_2)}{\text{tr}(K_1)}$.

5.5 Converting Siegel-Poincaré matrices $\iff$ Siegel-Klein matrices

From Eq. 122, we deduce that we can convert a matrix $K$ in the Siegel-Klein disk to a corresponding matrix $W$ in the Siegel-Poincaré disk, and vice versa, as follows:

- Converting $K$ to $W$: We convert a matrix $K$ in the Siegel-Klein model to an equivalent matrix $W$ in the Siegel-Poincaré model as follows:

$$C_{K \rightarrow D}(K) = \frac{1}{1 + \sqrt{1 - \|K\|_O^2}} K. \quad (129)$$

- Converting $W$ to $K$: We convert a matrix $W$ in the Siegel-Poincaré model to an equivalent matrix $K$ in the Siegel-Klein model as follows:

$$C_{D \rightarrow K}(W) = \frac{2}{1 + \|W\|_O^2} W. \quad (130)$$

Proposition 2 (Conversions Siegel-Poincaré $\leftrightarrow$ Siegel-Klein disk) The conversion of a matrix $K$ of the Siegel-Klein model to its equivalent matrix $W$ in the Siegel-Poincaré model, and vice versa, is done by the following radial expansion and contraction functions: $C_{K \rightarrow D}(K) = \frac{1}{1 + \sqrt{1 - \|K\|_O^2}} K$ and $C_{D \rightarrow K}(W) = \frac{2}{1 + \|W\|_O^2} W$.

By virtue of the cross-ratio property\textsuperscript{10}, the (pre)geodesics\textsuperscript{11} in the Hilbert-Klein disk are Euclidean straight. Thus we can write the pregeodesics as:

$$\gamma_{K_1, K_2}(\alpha) = (1 - \alpha)K_1 + \alpha K_2 = K_1 + \alpha(K_2 - K_1). \quad (131)$$

Another way to get a generic closed-form formula for the Siegel-Klein distance is by using the formula for the Siegel-Poincaré disk after converting the matrices to their equivalent matrices in the Siegel-Poincaré disk. We get the following expression:

$$\rho_K(K_1, K_2) = \rho_D(C_{K \rightarrow D}(K_1), C_{K \rightarrow D}(K_2)), \quad (132)$$

$$= \frac{1}{2} \log \left( \frac{1 + \|\Phi_{C_{K \rightarrow D}(K_1)}(C_{K \rightarrow D}(K_2))\|_O}{1 - \|\Phi_{C_{K \rightarrow D}(K_1)}(C_{K \rightarrow D}(K_2))\|_O} \right). \quad (133)$$

Theorem 3 (Formula for the Siegel-Klein distance) The Siegel-Klein distance between $K_1$ and $K_2$ in the Siegel disk is $\rho_K(K_1, K_2) = \frac{1}{2} \log \left( \frac{1 + \|\Phi_{C_{K \rightarrow D}(K_1)}(C_{K \rightarrow D}(K_2))\|_O}{1 - \|\Phi_{C_{K \rightarrow D}(K_1)}(C_{K \rightarrow D}(K_2))\|_O} \right)$.

The isometries in Hilbert geometry have been studied in [90].

\textsuperscript{10}The cross-ratio $(p, q; P, Q) = \frac{|p - P||q - Q|}{|p - Q||q - P|}$ of four collinear points on a line is such that $(p, q; P, Q) = (p, r; P, Q) \times (r, q; P, Q)$ whenever $r$ belongs to that line.

\textsuperscript{11}Geodesics are paths which minimize locally the distance and are parameterized proportionally to the arc-length. A pregeodesic is a path which minimizes locally the distance but is not necessarily parameterized proportionally to the arc-length.
5.6 Siegel-Klein distance between diagonal matrices

Solving for the general case: Let $K_\alpha = K_1 + \alpha K_{21}$ with $K_{21} = K_2 - K_1$. We seek for the extremal values of $\alpha$ such that

\begin{align*}
I - \overline{K}_\alpha K_\alpha &> 0, \quad (134) \\
I - (\overline{K}_1 + \alpha \overline{K}_{21})(K_1 + \alpha K_{21}) &> 0, \quad (135) \\
I - (\overline{K}_1 K_1 + \alpha (\overline{K}_1 K_{21} + \overline{K}_{21} K_1) + \alpha^2 \overline{K}_{21} K_{21}) &> 0, \quad (136) \\
\overline{K}_1 K_1 + \alpha (\overline{K}_1 K_{21} + \overline{K}_{21} K_1) + \alpha^2 \overline{K}_{21} K_{21} &< I. \quad (137)
\end{align*}

This last equation is reminiscent to a Linear Matrix Inequality \[\text{LMI}, \text{i.e., } \sum_i y_i S_i > 0 \text{ with } y_i \in \mathbb{R} \text{ and } S_i \in \text{Sym}(d, \mathbb{R}) \text{ where the coefficients } y_i \text{ are however linked between them}.\]

Let us consider the special case of diagonal matrices corresponding to the polydisk domain: $K = \text{diag}(k_1, \ldots, k_d)$ and $K' = \text{diag}(k'_1, \ldots, k'_d)$ of the disk.

First, let us start with the simple case $d = 1$, i.e., the Siegel disk $\mathbb{D}(1)$ which is the complex open unit disk $\{k \in \mathbb{C} : \bar{k}k < 1\}$. Let $k_\alpha = (1 - \alpha)k_1 + \alpha k_2 = k_1 + \alpha k_{21}$ with $k_{21} = k_2 - k_1$. We have $\bar{k}_\alpha k_\alpha = a \alpha^2 + b \alpha + c$ with $a = \bar{k}_2 k_{21}$, $b = \bar{k}_1 k_{21} + \overline{k}_{21} k_1$, and $c = \bar{k}_1 k_1$. To find the two intersection points of line $(k_1 k_2)$ with the boundary of $\mathbb{D}(1)$, we need to solve $\bar{k}_\alpha k_\alpha = 1$. This amounts to solve an ordinary quadratic equation since all coefficients $a$, $b$, and $c$ are provably reals. Let $\Delta = b^2 - 4ac$ be the discriminant ($\Delta > 0$ when $k_1 \neq k_2$). We get the two solutions $\alpha_0 = \frac{-b - \sqrt{\Delta}}{2a}$ and $\alpha_M = \frac{-b + \sqrt{\Delta}}{2a}$, and apply the 1D formula for the Hilbert distance:

\[
\rho_K(k_1, k_2) = \frac{1}{2} \log \frac{\alpha_M (1 - \alpha_0)}{\alpha_0 (\alpha_M - 1)}. \quad (138)
\]

Doing so, we obtain a formula equivalent to Eq. \[\text{[30]}\]

For diagonal matrices with $d > 1$, we get the following system of $d$ inequalities:

\[
\alpha^2 (k'_i - \bar{k}_i) (k'_i - k_i) + \alpha (\bar{k}_i (k'_i - k_i) + k_i (\bar{k}'_i - \bar{k}_i)) + \bar{k}_i k_i - 1 \leq 0, \forall i \in \{1, \ldots, d\}. \quad (139)
\]

For each inequality, we solve the quadratic equation as in the 1d case above, yielding two solutions $\alpha_i^-$ and $\alpha_i^+$. Then we satisfy all those constraints by setting

\[
\alpha_- = \max_{i \in \{1, \ldots, d\}} \alpha_i^-, \quad (140) \\
\alpha_+ = \min_{i \in \{1, \ldots, d\}} \alpha_i^+, \quad (141)
\]

and we compute the Hilbert distance:

\[
\rho_K(K_1, K_2) = \frac{1}{2} \log \left( \frac{\alpha_+ (1 - \alpha_-)}{\alpha_- (\alpha_+ - 1)} \right). \quad (142)
\]

**Theorem 4** (Siegel-Klein distance for diagonal matrices) **The Siegel-Klein distance between two diagonal matrices in the Siegel-Klein disk can be calculated exactly in linear time.**

Notice that the proof extends to triangular matrices as well.

When the matrices are non-diagonal, we have to solve analytically the equation:
\[
\max |\alpha|, \quad \text{such that} \quad \alpha^2 S_2 + \alpha S_1 + S_0 \prec 0,
\]
with the following symmetric matrices:
\[
S_2 = \bar{K}_{21} K_{21},
\]
\[
S_1 = \bar{K}_{1} K_{21} + \bar{K}_{21} K_{1},
\]
\[
S_0 = \bar{K}_{1} K_{1} - I.
\]

When \( S_0, S_1 \) and \( S_2 \) are simultaneously diagonalizable via congruence \cite{21}, the optimization problem becomes:
\[
\max |\alpha|, \quad \text{such that} \quad \alpha^2 D_2 + \alpha D_1 \prec -D_0,
\]
where \( D_i = P^\top S_i P \) for some \( P \in \text{GL}(d, \mathbb{C}) \), and we apply Theorem 4. The same result applies for simultaneously diagonalizable matrices \( S_0, S_1 \) and \( S_2 \) via similarity: \( D_i = P^{-1} S_i P \) with \( P \in \text{GL}(d, \mathbb{C}) \).

Notice that the Hilbert distance (or its squared distance) is not a separable distance, even in the case of diagonal matrices. (But recall that the squared Siegel-Poincaré distance in the upper plane is separable for diagonal matrices.)

### 5.7 A fast guaranteed approximation of the Siegel-Klein distance

In the general case, we use the bisection approximation algorithm which is a geometric approximation technique that requires to only calculate operator norms (and not the square root matrices required in the functions \( \Phi(\cdot) \)).

We have the following important property of the Hilbert distance:

**Property 1 (Bounding Hilbert distance)** Let \( \Omega_+ \subset \Omega \subset \Omega_- \) be strictly nested open convex bounded domains. Then we have the following inequality for the corresponding Hilbert distances:
\[
H_{\Omega_+, \kappa}(p, q) \geq H_{\Omega, \kappa}(p, q) \geq H_{\Omega_-, \kappa}(p, q).
\]

See Figure 5 for a visual proof.

Figure 4 illustrates the Property 1 of Hilbert distances corresponding to nested domains. Notice that when \( \Omega_- \) is a large enclosing ball of \( \Omega \) with radius increasing to infinity, we have \( \alpha_- \simeq \alpha_+ \), and therefore the Hilbert distance tends to zero.

Therefore the bisection search for finding the values of \( \alpha_- \) and \( \alpha_+ \) yields lower and upper bounds on the exact Siegel-Klein distance as follows: Let \( \alpha_- \in (l_-, u_-) \) and \( \alpha_+ \in (l_+, u_+) \) where \( l_-, u_- \), \( l_+ \), \( u_+ \) are reals defining the extremities of the intervals. Using Property 1 we get the following theorem:

**Theorem 5 (Lower and upper bounds on the Siegel-Klein distance)** The Siegel-Klein distance between two matrices \( K_1 \) and \( K_2 \) of the Siegel disk is bounded as follows:
\[
\rho_K(l_-, u_+ \leq \rho_K(K_1, K_2) \leq \rho_K(u_-, l_+),
\]

22
\[ H_{\Omega,+}(p,q) \geq H_{\Omega,0}(p,q) \geq H_{\Omega,-}(p,q) \]

Figure 4: Hilbert distances induced by nested bounded open convex domains.

\[ \bar{p} - \bar{p}^+ \quad \bar{p}^+ \quad \partial\Omega_+ \quad \bar{p} \quad \partial\Omega \quad \bar{q} \quad \bar{q}^+ \quad \bar{q}^+ \]

\[ \bar{p} \quad \bar{p}^+ \quad \bar{q} \quad \bar{q}^+ \quad \bar{q}^+ \quad \bar{q}^+ \quad \partial\Omega_+ \quad \partial\Omega \quad \partial\Omega_- \]

Figure 5: Comparison of the Hilbert distances induced by nested open interval domains.

where

\[ \rho_K(\alpha_m,\alpha_M) := \frac{1}{2} \log \left( \frac{\alpha_M(1 - \alpha_m)}{\alpha_m(\alpha_M - 1)} \right). \]  

(152)

Figure 6 depicts the guaranteed lower and upper bounds obtained by performing the bisection search for approximating the point \( \bar{K}_1 \in (\bar{K}_1'',\bar{K}_1') \) and the points \( \bar{K}_2 \in (\bar{K}_2',\bar{K}_2'') \).

We have:

\[ \text{CR}(\bar{K}_1',K_1;K_2',\bar{K}_2') \geq \text{CR}(\bar{K}_1,K_1;K_2,\bar{K}_2) \geq \text{CR}(\bar{K}_1'',K_1;K_2,\bar{K}_2''), \]  

(153)

and hence

\[ H_{\Omega',\frac{1}{2}}(K_1,K_2) \geq \rho_K(K_1,K_2) \geq H_{\Omega'',\frac{1}{2}}(K_1,K_2). \]  

(154)

Notice that the approximation of the Siegel-Klein distance by line bisection requires only to calculate an operator norm \( \|M\|_O \) at each step: This involves calculating the smallest and largest eigenvalues of \( M \), or the largest eigenvalue of \( \bar{M} \bar{M} \). To get a \((1 + \epsilon)\)-approximation, we need to perform \( O(\log \frac{1}{\epsilon}) \) dichotomic steps. This yields a fast method to approximate the Siegel-Klein distance compared with the exact calculation of the Siegel-Klein distance of Eq. 132 which requires to calculate \( \Phi(\cdot) \) functions: This involves the calculation of a square root of a complex matrix. Notice that the operator norm can be numerically approximated using a Lanczos’s power iteration scheme [53, 42] (see also [57]).

5.8 Hilbert-Fröbenius distances and fast simple bounds on the Siegel-Klein distance

Let us notice that although the Hilbert distance does not depend on the chosen norm in the vector space, but the Siegel complex ball \( \mathbb{S}\mathbb{D}(d) \) is defined according to the operator norm. In a finite-dimensional vector space, all norms are equivalent: That is, given two norms \( \| \cdot \|_a \) and \( \| \cdot \|_b \) of
vector space $X$, there exists positive constants $c_1$ and $c_2$ such that
\[ c_1 \|x\|_a \leq \|x\|_b \leq c_2 \|x\|_b, \quad \forall x \in X. \quad (155) \]

In particular, this property holds for the operator norm and Fröbenius norm of complex matrices with positive constants $c_d$, $C_D$, $c_d'$ and $C_D'$ depending on the dimension $d$ of the square matrices:
\[
\begin{align*}
c_d \|M\|_O &\leq \|M\|_F \leq C_D \|M\|_O, \quad \forall M \in M(d, \mathbb{C}), \\
c_d' \|M\|_F &\leq \|M\|_O \leq C_D' \|M\|_F, \quad \forall M \in M(d, \mathbb{C}).
\end{align*}
\]

As mentioned in the introduction, we have $\|M\|_O \leq \|M\|_F$.

Thus the Siegel ball domain $\mathbb{S}^d(d)$ may be enclosed by an open Fröbenius ball $\mathbb{F}^d(d, \frac{1}{(1+\epsilon)c_d})$ (for any $\epsilon > 0$) with
\[
\mathbb{F}^d(d, r) = \{ M \in M(d, \mathbb{C}) : \|M\|_F < r \}. \quad (158)
\]

Therefore we have
\[
H_{\mathbb{F}^d(d, \frac{1}{(1+\epsilon)c_d})} \left( K_1, K_2 \right) \leq \rho_K(K_1, K_2), \quad (159)
\]

where $H_{\mathbb{F}^d(d, r), \frac{1}{2}}$ denotes the Fröbenius-Klein distance, i.e., the Hilbert distance induced by the Fröbenius balls $\mathbb{F}^d(d, r)$ with constant $\kappa = \frac{1}{2}$.

Now, we can calculate in closed-form the Fröbenius-Klein distance by computing the two intersection points of the line $(K_1 K_2)$ with the Fröbenius ball $\mathbb{F}^d(d, r)$. This amounts to solve an ordinary quadratic equation $\|K_1 + \alpha (K_2 - K_1)\|_F^2 = r$ for parameter $\alpha$:
\[
\|K_2\|_F^2 \alpha^2 + \left( \sum_{i,j} K_2^{i,j} \tilde{K}_1^{i,j} + K_1^{i,j} \tilde{K}_2^{i,j} \right) \alpha + (\|K_1\|_F^2 - r) = 0, \quad (160)
\]
where $K^{i,j}$ denotes the coefficient of matrix $K$ at row $i$ and column $j$. Notice that $\left( \sum_{i,j} K_2^{i,j} \tilde{K}_1^{i,j} + K_1^{i,j} \tilde{K}_2^{i,j} \right)$ is a real. Once $\alpha_-$ and $\alpha_+$ are found, we apply the 1D formula of the Hilbert distance of Eq. 107.

We summarize the result as follows:

Figure 6: Guaranteed lower and upper bounds for the Siegel-Klein distance.
Theorem 6 (Lower bound on Siegel-Klein distance) The Siegel-Klein distance is lower bounded by the Fröbenius-Klein distance for the unit complex Fröbenius ball, and it can be calculated in $O(d^2)$ time.

6 The smallest enclosing ball in the SPD manifold and in the Siegel spaces

The goal of this section is to compare two implementations of a generalization of the Badoiu and Clarkson’s algorithm [7] to approximate the smallest enclosing ball of a set of complex matrices either in the Siegel-Poincaré disk or in the Siegel-Klein disk.

In general, we may encode a pair of features $(S, P) \in \text{Sym}(d, \mathbb{R}) \times \mathbb{P}_{++}(d, \mathbb{R})$ in applications as a Riemann matrix $Z(S, P) := S + iP$, and consider the underlying symplectic geometry. For example, anomaly detection of time-series maybe considered by considering $(\dot{\Sigma}(t), \Sigma(t))$ where $\Sigma(t)$ is the covariance matrix at time $t$ and $\dot{\Sigma}(t) \simeq \frac{1}{dt}(\Sigma(t + dt) - \Sigma(t))$ is the approximation of the derivative of the covariance matrix (a symmetric matrix).

The generic Badoiu and Clarkson’s algorithm [7] (BC algorithm) for a set $\{p_1, \ldots, p_n\}$ of $n$ points in a metric space $(X, \rho)$ is described as follows:

- Initialization: Let $c_1 = p_1$ and $l = 1$

- Repeat $L$ times:
  - Calculate the farthest point: $f_l = \arg \min_{i \in [d]} \rho(c_l, p_i)$.
  - Geodesic cut: Let $c_{l+1} = c_l \#_{t_l} f_l$, where $p \#_{t_l} q$ is the point which satisfies $\rho(p, p \#_{t_l} q) = t_l \rho(p, q)$.
  - $l \leftarrow l + 1$.

This elementary SEB approximation algorithm has been instantiated to various metric spaces with proofs of convergence according to the sequence $\{t_l\}$: see [69] for the case of hyperbolic geometry, [6] for Riemannian geometry with bounded sectional curvatures, [77] for dually flat spaces (a non-metric space equipped with a Bregman divergences), etc. The number of iterations $L$ to get a $(1 + \epsilon)$-approximation of the SEB depends on the underlying geometry and the sequence $\{t_l\}$. For example, in Euclidean geometry, setting $t_l = \frac{1}{L^{1/2}}$ with $L = \frac{1}{\epsilon^2}$ steps yield a $(1 + \epsilon)$-approximation of the SEB [7].

We start by recalling the Riemannian generalization of the BC algorithm, and then consider the Siegel spaces.

6.1 Approximating the SEB in Riemannian spaces

We first instantiate a particular example of Riemannian space, the space of Symmetric Positive-Definite matrix manifold (PD/SPD manifold), and then consider the general case on a Riemannian manifold $(M, g)$.

---

12 In Cartan-Hadamard manifolds [6], we require the series $\sum t_i$ to diverge while the series $\sum t_i^2$ to converge.
6.1.1 Approximating the SEB on the SPD manifold

Given \(n\) positive-definite matrices \(P_1, \ldots, P_n\) of size \(d \times d\), we ask to calculate the SEB with circumcenter \(P^*\) minimizing the following objective function:

\[
\min_{P \in \text{PD}(d)} \max_{i \in \{1, \ldots, n\}} \rho_{\text{PD}}(P, P_i).
\]

This is a minimax optimization problem. The SPD cone is not a complete metric space with respect to the Fröbenius distance, but is a complete metric space with respect to the natural Riemannian distance.

When the minimization is performed with respect to the Fröbenius distance, we can solve this problem using techniques of Euclidean computational geometry \([16, 7]\) by vectorizing the PSD matrices \(P_i\) into corresponding vectors \(v_i = \text{vec}(P_i)\) of \(\mathbb{R}^{d \times d}\) such that \(\|P - P'\|_F = \|\text{vec}(P) - \text{vec}(P')\|_2\), where \(\text{vec}(-) : \text{Sym}(d, \mathbb{R}) \to \mathbb{R}^{d \times d}\) vectorizes a matrix by stacking its column vectors. In fact, since the matrices are symmetric, it is enough to half-vectorize the matrices:

\[
\|P - P'\|_F = \|\text{vec}^+(P) - \text{vec}^+(P')\|_2,
\]

where \(\text{vec}^+(\cdot) : \text{Sym}_+(d, \mathbb{R}) \to \mathbb{R}^{d(d+1)/2}\), see \([74]\).

**Property 2** The smallest enclosing ball of a finite set of positive-definite matrices is unique.

Let us mention the two following proofs:

- The SEB is well-defined and unique since the SPD manifold is a Bruhat-Tits space: That is, a complete metric space enjoying a semiparallelogram law: For any \(P_1, P_2 \in \text{PD}(d)\) and geodesic midpoint \(P_{12} = P_1(P_1^{-1}P_2)^{1/2}\) (see below), we have:

\[
\rho_{\text{PD}}^2(P_1, P_2) + 4\rho_{\text{PD}}^2(P, P_{12}) \leq 2\rho_{\text{PD}}^2(P, P_1) + 2d^2\rho_{\text{PD}}^2(P, P_2), \forall P \in \text{PD}(d).
\]

See \([55]\) page 83 or \([15]\) Chapter 6). In a Bruhat-Tits space, the SEB is guaranteed to be unique \([55, 18]\).

- Another proof of the uniqueness of the SEB on a SPD manifold consists in noticing that the SPD manifold is a Cartan-Hadamard manifold \([6]\), and the SEB on Cartan-Hadamard manifolds are guaranteed to be unique.

We shall use the invariance property of the Riemannian distance by congruence:

\[
\rho_{\text{PD}}(C^T P_1 C, C^T P_2 C) = \rho_{\text{PD}}(P_1, P_2), \quad \forall C \in \text{GL}(d, \mathbb{R}).
\]

In particular, choosing \(C = P_1^{-1/2}\), we get

\[
\rho_{\text{PD}}(P_1, P_2) = \rho(I, P_1^{-1/2}P_2P_1^{-1/2}).
\]

The geodesic from \(I\) to \(P\) is \(\gamma_{I,P}(\alpha) = \text{Exp}(\alpha \text{Log} P) = P^\alpha\). The set \(\{\lambda_i(P^\alpha)\}\) of the \(d\) eigenvalues of \(P^\alpha\) coincide with the set \(\{\lambda_i(P)^\alpha\}\) of eigenvalues of \(P\) raised to the power \(\alpha\) (up to a permutation).

Thus to cut the geodesic \(I \neq P\), we have to solve the following problem:

\[
\rho_{\text{PD}}(I, P^\alpha) = t \times \rho_{\text{PD}}(I, P).
\]
That is
\[ \sqrt{\sum_i \log^2 \lambda_i(P)^\alpha} = t \times \sqrt{\sum_i \log^2 \lambda_i(P)}, \quad (166) \]
\[ \alpha \times \sqrt{\sum_i \log^2 \lambda_i(P)} = t \times \sqrt{\sum_i \log^2 \lambda_i(P)}. \quad (167) \]

The solution is \( \alpha = t \). Thus \( I_{PD}^t t P = P_t \). For arbitrary \( P_1 \) and \( P_2 \), we first apply the congruence transformation with \( C = P_1^{-\frac{1}{2}} \), use the solution \( I_{PD}^t C P C^\top = (C P C^\top)^t \), and apply the inverse congruence transformation with \( C^{-1} = P_1^{\frac{1}{2}} \). It follows the theorem:

**Theorem 7 (Geodesic cut on the SPD manifold)** For any \( t \in (0, 1) \), we have the closed-form expression of the geodesic cut on the manifold of positive-definite matrices:

\[ P_1^\#_{PD} P_2 = P_1^{\frac{1}{2}} \text{Exp} \left( t \log \left( P_1^{-\frac{1}{2}} P_2^{-\frac{1}{2}} P_1 \right) \right) P_1^{\frac{1}{2}}, \quad (168) \]
\[ = P_1^{\frac{1}{2}} \left( P_1^{-\frac{1}{2}} P_2 P_1^{-\frac{1}{2}} \right)^t P_1^{\frac{1}{2}}, \quad (169) \]
\[ = P_1 (P_1^{-1} P_2)^t, \quad (170) \]
\[ = P_2 (P_2^{-1} P_1)^{1-t}. \quad (171) \]

The matrix \( P_1^{-\frac{1}{2}} P_2 P_1^{-\frac{1}{2}} \) can be rewritten using the orthogonal eigendecomposition as \( U D U^\top \), where \( D \) is the diagonal matrix of generalized eigenvalues. Thus the PD geodesic can be rewritten as

\[ P_1^\#_{PD} P_2 = P_1^{\frac{1}{2}} U D U^\top P_1^{\frac{1}{2}}. \quad (172) \]

We instantiate the generic algorithm to positive-definite matrices as follows:

```
Algorithm ApproximatePDSEB(\{P_1, \ldots, P_n\}, L):
- Initialization: Let \( C_1 = P_1 \) and \( l = 1 \)
- Repeat \( L \) times:\n  - Calculate the index of the farthest matrix:
    \[ f_l = \arg \min_{i \in \{1, \ldots, n\}} \rho_{PD}(C_l, P_i). \]
  - Geodesic walk:
    \[ C_{l+1} = C_l^{\frac{1}{2}} \left( C_l^{-\frac{1}{2}} P_{f_l} C_l^{-\frac{1}{2}} \right)^t C_l^{\frac{1}{2}} \]
  - \( l \leftarrow l + 1. \)
```

The complexity of the algorithm is in \( O(d^3 n T) \) where \( T \) is the number of iterations, \( d \) the row dimension of the square matrices \( P_i \) and \( n \) the number of matrices.
Observe that the solution corresponds to the arc-length parameterization of the geodesic with boundary values on the SPD manifold:

\[ \gamma_{p_1,p_2}(t) = P_1^{\frac{1}{2}} \exp(t \log(P_1^{-\frac{1}{2}} P_2 P_1^{-\frac{1}{2}})) P_1^{\frac{1}{2}}. \] (173)

In fact, we have shown the following property:

**Property 3 (Riemannian geodesic cut)** Let \( \gamma_{p,q}(t) \) denote the Riemannian geodesic linking \( p \) and \( q \) on a Riemannian manifold \( (M, g) \) (i.e., parameterized proportionally to the arc-length and with respect to the Levi-Civita connection induced by the metric tensor \( g \)). Then we have

\[ p_1 \#_I p_2 = \gamma_{p_1,p_2}(t) = \gamma_{p_2,p_1}(1 - t). \] (174)

Thus it follows the following generic Riemannian algorithm:

| Algorithm ApproximateRieSEB\((\{p_1, \ldots, p_n\}, g, L)\): |
| --- |
| • Initialization: Let \( c_1 = p_1 \) and \( l = 1 \) |
| • Repeat \( L \) times: |
|   - Calculate the index of the farthest point: |
|     \[ f_l = \arg \min_{i \in \{1, \ldots, d\}} \rho_g(c_l, p_i). \] |
|   - Geodesic walk: |
|     \[ c_{l+1} = \gamma_{c_l, p_{f_l}}(t_l). \] |
|   - \( l \leftarrow l + 1. \) |

Theorem 1 of [6] guarantees the convergence of the ApproximateRieSEB algorithm provided that we have a lower bound and an upper bound on the sectional curvatures of the manifold \( (M, g) \). The sectional curvatures of the PD manifold have been proven to be negative [41]. The SPD manifold is a Cartan-Hadamard manifold with scalar curvature \( \frac{1}{8} d(d+1)(d+2) \) depending on the dimension \( d \) of the matrices. Notice that we can identify \( P \in \text{PD}(d) \) with an element of the quotient space \( \text{GL}(d, \mathbb{R})/O(d) \) since \( O(d) \) is the isotropy subgroup of the \( \text{GL}(d, \mathbb{R}) \) for the action \( P \mapsto C^T PC \) (i.e., \( I \mapsto C^T IC = I \) when \( C \in O(d) \)). Thus we have \( \text{PD}(d) \cong \text{GL}(d, \mathbb{R})/O(d) \).

### 6.2 Implementation in the Siegel-Poincaré disk

Given \( n \times d \times d \) complex matrices \( W_1, \ldots, W_n \in \mathbb{S}(d) \), we ask to find the smallest-radius enclosing ball with center \( W^* \) minimizing the following objective function:

\[ \min_{W \in \mathbb{S}(d)} \max_{i \in \{1, \ldots, n\}} \rho_D(W, W_i). \] (175)

This problem may have potential applications in image morphology [4] or anomaly detection [95] of covariance matrices [95].

---

\(^{13}\) We may model the dynamics of a covariance matrix time-series \( \Sigma(t) \) by the representation \( (\Sigma(t), \dot{\Sigma}(t)) \) where \( \dot{\Sigma}(t) = \frac{d}{dt} \Sigma(t) \in \text{Sym}(d, \mathbb{R}) \) and use the Siegel SEB to detect anomalies, see [29] for detection anomaly based on Bregman SEBs.
The Siegel-Poincaré upper plane and disk are not Bruhat-Tits space\textsuperscript{14} but spaces of non-positive curvatures\textsuperscript{31}.

Notice that when \( d = 1 \), the hyperbolic ball in the Poincaré disk have Euclidean shape. This is not true anymore when \( d > 1 \): Indeed, the equation of the ball centered at the origin 0:

\[
\text{Ball}(0, r) = \left\{ W \in \mathbb{SD}(d) : \log \left( \frac{1 + \|W\|_O}{1 - \|W\|_O} \right) \leq r \right\},
\]

amounts to

\[
\text{Ball}(0, r) = \left\{ W \in \mathbb{SD}(d) : \|W\|_O \leq \frac{e^r - 1}{e^r + 1} \right\}. \tag{177}
\]

When \( d = 1 \), \( \|W\|_O = |w| = \|(\text{Re}(w), \text{Im}(w))\|_2 \), and Poincaré balls have Euclidean shapes. Otherwise, when \( d > 1 \), \( \|W\|_O = \sigma_{\text{max}}(W) \) and \( \sigma_{\text{max}}(W) \leq \frac{e^r - 1}{e^r + 1} \) is not a complex Fröbenius ball.

In order to apply the generic algorithm, we need to implement the geodesic cut operation \( W_1 \# t W_2 \). We consider the complex symplectic map \( \Phi_{W_1}(W) \) in the Siegel disk that maps \( W_1 \) to 0 and \( W_2 \) to \( W_2' = \Phi_{W_1}(W_2) \). Then the geodesic between 0 and \( W_2' \) is a straight line.

We need to find \( \alpha(t)W = 0\# t W \) (with \( \alpha(t) > 0 \)) such that \( \rho_D(0, \alpha(t)W) = t \rho_D(0, W) \). That is, we shall solve the following equation:

\[
\log \left( \frac{1 + \alpha(t)\|W\|_O}{1 - \alpha(t)\|W\|_O} \right) = t \times \log \left( \frac{1 + \|W\|_O}{1 - \|W\|_O} \right). \tag{178}
\]

We find the exact solution as

\[
\alpha(t) = \frac{1}{\|W\|_O} \frac{(1 + \|W\|_O)^t - (1 - \|W\|_O)^t}{(1 + \|W\|_O)^t + (1 - \|W\|_O)^t}. \tag{179}
\]

**Proposition 3 (Siegel-Poincaré geodesics from the origin)** The geodesic in the Siegel disk is

\[
\gamma_{0,W}(t) = \alpha(t)W \tag{180}
\]

with

\[
\alpha(t) = \frac{1}{\|W\|_O} \frac{(1 + \|W\|_O)^t - (1 - \|W\|_O)^t}{(1 + \|W\|_O)^t + (1 - \|W\|_O)^t}.
\]

Thus the midpoint \( W_1 \# 1/2 W_2 := W_1 \# 1/2 W_2 \) of \( W_1 \) and \( W_2 \) can be found as follows:

\[
W_1 \# 1/2 W_2 = \Phi_{W_1}^{-1} \left( 0\# 1/2 \Phi_{W_1}(W_2) \right), \tag{181}
\]

where

\[
0\# 1/2 W = \alpha \left( 1/2 \right) W, \quad \alpha \left( 1/2 \right) W = \frac{1}{\|W\|_O} \frac{\sqrt{1 + \|W\|_O} - \sqrt{1 - \|W\|_O}}{\sqrt{1 + \|W\|_O} + \sqrt{1 - \|W\|_O}} W. \tag{182}
\]

To summarize, the algorithm recenters at every step the current center \( C_t \) to the Siegel disk origin 0:

\textsuperscript{14}When \( d = 1 \), the Poincaré disk is not a Bruhat-Space.
Algorithm ApproximateSiegelSEB($\{W_1, \ldots, W_n\}$):

- Initialization: Let $C_1 = 0$ and $l = 1$.
- Compute $W_i' = \Phi_{C_1}(W_i)$ for all $i \in \{1, \ldots, n\}$.
- Repeat $L$ times:
  - Calculate the index of the farthest point: $F_l = \arg \min_{i \in [d]} \rho_D(0, W_i')$.
  - Geodesic cut: Let $C_{l+1} = 0^SE(W_{F_l})$.
  - Recenter $C_{l+1}$ to the origin for the next iteration: Compute $W_i' = \Phi_{C_{l+1}}(W_i)$ for all $i \in \{1, \ldots, n\}$. Set $C_{l+1} = 0$.
  - $l \leftarrow l + 1$.
- Let the approximate circumcenter be mapped back to be consistent with the input:
  $$\tilde{C} = \Phi_{C_1}^{-1}(\Phi_{C_2}^{-1}(\ldots \Phi_{C_L}^{-1}(0)) \ldots).$$ (184)

This amounts to calculate the symplectic map associated to the matrix $S = C_1^{(-1)} \times \ldots \times (C_L^{(-1)}$. Overall it costs $L$ matrix multiplications plus the cost of evaluation of the symplectic map defined by $S$.

The farthest point to the current approximation of the circumcenter can be calculated using the data-structure of the Vantage Point Tree (VPT), see [75].

The Riemannian curvature tensor of the Siegel space is non-positive [89, 44] and the sectional curvatures are non-positive [31] and bounded above by a negative constant. In our implementation, we chose the step sizes $t_l = \frac{1}{l+1}$. Barbaresco [11] also adopted this iterative recentering operation for calculating the median in the Siegel disk. However at the end of his algorithm, he does not map back the median among the source matrix set. Recentering is costly because we need to calculate a square root matrix to calculate $\Phi_C(W)$. A great advantage of Siegel-Klein space is that we have straight geodesics anywhere in the disk so we do not need to perform recentering.

6.3 Fast implementation in the Siegel-Klein disk

The main advantage of implementing the Badoiu and Clarkson’s algorithm [7] in the Siegel-Klein disk is to avoid to perform the costly recentering operations (which require calculation of square root matrices). Moreover, we do not have to roll back our approximate circumcenter at the end of the algorithm.

First, we state the following expression of the geodesics in the Siegel disk:

**Proposition 4 (Siegel-Klein geodesics from the origin)** The geodesic from the origin in the Siegel-Klein disk is expressed

$$\gamma_{0,K}^{SK}(t) = \alpha(t)K$$ (185)

with

$$\alpha(t) = \frac{1}{\|K\|_O} \frac{(1 + \|K\|_O)^t - (1 - \|K\|_O)^t}{(1 + \|K\|_O)^t + (1 - \|K\|_O)^t}. \quad (186)$$

The proof follows straightforwardly from Proposition [3] because we have $\rho_K(0, K) = \frac{1}{2}\rho_D(0, K)$. 

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7 Conclusion and perspectives

In this paper, we have generalized the Klein model of hyperbolic geometry to the the Siegel disk of complex matrices by considering the Hilbert geometry induced by the Siegel disk, an open bounded convex domain. We compared this Siegel-Klein disk model with both the former Siegel-Poincaré disk model and the Siegel-Poincaré upper plane. We show how to convert matrices $W$ of the Siegel-Poincaré disk model into equivalent matrices $K$ of Siegel-Klein disk model and matrices $Z$ in the Siegel-Poincaré upper plane via symplectic maps so that their respective distances is preserved:

$$\rho_D(W_1,W_2) = \rho_K(K_1,K_2) = \rho_U(Z_1,Z_2).$$ (187)

Since the geodesics in the Siegel-Klein disks are by construction straight, this model is well-suited to implement techniques of computational geometry [16]. Furthermore, the calculation of the Siegel distance in the Siegel-Klein disk does not require to recenter one of its arguments to the disk origin, a costly operation. We reported a linear-time algorithm for computing the exact Siegel-Klein distance between diagonal matrices of the disk (Theorem 4), and a fast way to numerically approximate the Siegel distance by bisection searches with guaranteed lower and upper bounds (Theorem 5). Finally, we demonstrated the algorithmic advantage of using the Siegel-Klein disk model instead of the Siegel-Poincaré disk model for approximating the smallest-radius enclosing ball of a finite set of matrices in the Siegel disk. In future work, we shall consider more generally the Hilbert geometry of homogeneous complex domains applications and applications of the Siegel-Klein geometry for radar processing [11], image morphology [56], computer vision, and machine learning [52].

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Notations and main formulas

| Complex matrices: | Real $\mathbb{R}$ or complex $\mathbb{C}$ |
|-------------------|------------------------------------------|
| Number field $F$  | Space of square $d \times d$ matrices in $F$ |
| $M(d,F)$          | Space of real symmetric matrices |
| $0$               | matrix with all coefficients equal to zero (disk origin) |
| Fröbenius norm    | $\|M\|_F = \sqrt{\sum_{i,j} |M_{i,j}|^2}$ |
| Operator norm     | $\|M\|_O = \sigma_{\text{max}}(M) = \max\{\lambda_i(M)\}$ |
| Domains:          | Cone of SPD matrices $PD(d,\mathbb{R}) = \{P \succ 0 : P \in \text{Sym}(d,\mathbb{R})\}$ |
|                   | Siegel-Poincaré upper plane $SH(d) = \{Z = X + iY : X \in \text{Sym}(d,\mathbb{R}), Y \in PD(d,\mathbb{R})\}$ |
|                   | Siegel-Poincaré disk $SD(d) = \{W \in \text{Sym}(d,\mathbb{C}) : I - WW \succ 0\}$ |
| Siegel distances: | Upper plane distance $\rho_U(Z_1,Z_2) = \sqrt{\sum_{i=1}^{d} \log^2 \left( \frac{1 + \sqrt{r_i}}{1 - \sqrt{r_i}} \right)}$ |
|                   | $r_i = \lambda_i(R(Z_1,Z_2))$ |
Upper plane metric

PD distance

PD metric

Disk distance

Translation in the disk

Disk distance to origin

Siegel-Klein distance

Siegel-Klein distance to 0

Symplectic maps and groups:

Symplectic map

Symplectic group

group composition law

group inverse law

Translation in $\mathbb{H}(d)$ of $Z = A + iB$ to $iI$

symplectic orthogonal matrices

(photations in $\mathbb{SH}(d)$)

Translation to 0 in $\mathbb{SD}(d)$

Isom$^+(\mathbb{S})$

Moeb($d$)

Snifflet code

We implemented our software library and smallest enclosing ball algorithms in Java™.

The code below is a snippet written in MAXIMA: A computer algebra system, freely downloadable at [http://maxima.sourceforge.net/](http://maxima.sourceforge.net/)

/* Code in Maxima */
/* Calculate the Siegel metric distance in the Siegel upper space */

load(eigen);

/* symmetric */
S1: matrix( [0.265, 0.5],
            [0.5, -0.085]);

/* positive-definite */
P1: matrix( [0.235, 0.048],
            [0.048, 0.792]);
/* Matrix in the Siegel upper space */
Z1: S1+%i*P1;
S2: matrix([-0.329, -0.2], [-0.2, -0.382]);
P2: matrix([[0.464, 0.289], [0.289, 0.431]]);
Z2: S2+%i*P2;

/* Generalized Moebius transformation */
R(Z1,Z2) := ((Z1-Z2).invert(Z1-conjugate(Z2))).((conjugate(Z1)-conjugate(Z2)).invert(conjugate(Z1)-Z2));

/* R12 is not Hermitian but complex matrix */
R12: ratsimp(R(Z1,Z2));
ratsimp(R12[2][1]-conjugate(R12[1][2]));

/* Retrieve the eigenvalues: They are all reals */
r: float(eivals(R12))[1];

/* Calculate the Siegel distance */
distSiegel: sum(log( (1+sqrt(r[i]))/(1-sqrt(r[i])) )**2, i, 1, 2);

References

[1] Daniele Agostini and Carlos Améndola. Discrete Gaussian distributions via theta functions. *SIAM Journal on Applied Algebra and Geometry*, 3(1):1–30, 2019.

[2] Daniele Agostini and Lynn Chua. Computing Theta functions with Julia. *arXiv preprint arXiv:1906.06507*, 2019.

[3] A Andai. *Information geometry in quantum mechanics*. PhD thesis, Ph. D. dissertation (in Hungarian), BUTE, 2004.

[4] Jésus Angulo. Structure tensor image filtering using Riemannian $L_1$ and $L_\infty$ center-of-mass. *Image Analysis & Stereology*, 33(2):95–105, 2014.

[5] Jésus Angulo and Santiago Velasco-Forero. Morphological processing of univariate gaussian distribution-valued images based on poincaré upper-half plane representation. In *Geometric Theory of Information*, pages 331–366. Springer, 2014.

[6] Marc Arnaudon and Frank Nielsen. On approximating the Riemannian 1-center. *Computational Geometry*, 46(1):93–104, 2013.

[7] Mihai Badoiu and Kenneth L Clarkson. Smaller core-sets for balls. In *Proceedings of the fourteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 801–802. Society for Industrial and Applied Mathematics, 2003.
[8] Frédéric Barbaresco. Innovative tools for radar signal processing based on Cartan’s geometry of SPD matrices & information geometry. In 2008 IEEE Radar Conference, pages 1–6. IEEE, 2008.

[9] Frédéric Barbaresco. Robust statistical radar processing in fréchet metric space: OS-HDR-CFAR and OS-STAP processing in Siegel homogeneous bounded domains. In 12th International Radar Symposium (IRS), pages 639–644. IEEE, 2011.

[10] Frédéric Barbaresco. Information geometry manifold of Toeplitz hermitian positive definite covariance matrices: Mostow/Berger fibration and Berezin quantization of Cartan-Siegel domains. International Journal of Emerging Trends in Signal Processing, 1(3):1–11, 2013.

[11] Frédéric Barbaresco. Information geometry of covariance matrix: Cartan-Siegel homogeneous bounded domains, Mostow/Berger fibration and Fréchet median. In Matrix information geometry, pages 199–255. Springer, 2013.

[12] Giovanni Bassanelli. On horospheres and holomorphic endomorfisms of the Siegel disc. Rendiconti del Seminario Matematico della Università di Padova, 70:147–165, 1983.

[13] Eugenio Beltrami. Saggio di interpretazione della geometria non-euclidea. Giornale di Matematiche, IV:pp. 284, 1868.

[14] Felix A Berezin. Quantization in complex symmetric spaces. Mathematics of the USSR-Izvestiya, 9(2):341, 1975.

[15] Rajendra Bhatia. Positive definite matrices, volume 24. Princeton university press, 2009.

[16] Jean-Daniel Boissonnat and Mariette Yvinec. Algorithmic geometry. Cambridge university press, 1998.

[17] Philippe Bougerol. Kalman filtering with random coefficients and contractions. SIAM Journal on Control and Optimization, 31(4):942–959, 1993.

[18] François Bruhat and Jacques Tits. Groupes réductifs sur un corps local: I. Données radicielles valuées. Publications Mathématiques de l’IHÉS, 41:5–251, 1972.

[19] RS Bucy and BG Williams. A matrix cross ratio theorem for the Riccati equation. Computers & Mathematics with Applications, 26(4):9–20, 1993.

[20] Jacob Burbea. Informative geometry of probability spaces. Technical report, Pittsburgh Univ. PA center for multivariate analysis, 1984.

[21] Miguel D Bustamante, Pauline Mellon, and M Velasco. Solving the problem of simultaneous diagonalisation via congruence. arXiv preprint arXiv:1908.04228, 2019.

[22] Miquel Calvo and Josep M Oller. A distance between multivariate normal distributions based in an embedding into the Siegel group. Journal of multivariate analysis, 35(2):223–242, 1990.

[23] Miquel Calvo and Josep M Oller. A distance between elliptical distributions based in an embedding into the Siegel group. Journal of Computational and Applied Mathematics, 145(2):319–334, 2002.

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[24] James W Cannon, William J Floyd, Richard Kenyon, and Walter R Parry. Hyperbolic geometry. *Flavors of geometry*, 31:59–115, 1997.

[25] Élie Cartan. Sur les domaines bornés homogènes de l’espace de n variables complexes. *Abhandlungen aus dem mathematischen Seminar der Universität Hamburg*, 11(1):116–162, 1935.

[26] Arthur Cayley. A sixth memoir upon quantics. *Philosophical Transactions of the Royal Society of London*, 149:61–90, 1859.

[27] Emmanuel Chevallier, Thibault Forget, Frédéric Barbaresco, and Jésus Angulo. Kernel density estimation on the Siegel space with an application to radar processing. *Entropy*, 18(11):396, 2016.

[28] Jean-Louis Clerc et al. Geometry of the Shilov boundary of a bounded symmetric domain. In *Proceedings of the Tenth International Conference on Geometry, Integrability and Quantization*, pages 11–55. Institute of Biophysics and Biomedical Engineering, Bulgarian Academy, 2009.

[29] Arshia Cont, Shlomo Dubnov, and Gérard Assayag. On the information geometry of audio streams with applications to similarity computing. *IEEE Transactions on Audio, Speech, and Language Processing*, 19(4):837–846, 2010.

[30] Jane K Cullum and Ralph A Willoughby. *Lanczos Algorithms for Large Symmetric Eigenvalue Computations: Vol. 1: Theory*, volume 41. Siam, 2002.

[31] JE D’Atri and I Dotti Miatello. A characterization of bounded symmetric domains by curvature. *Transactions of the American Mathematical Society*, 276(2):531–540, 1983.

[32] Pierre De La Harpe. On Hilbert’s metric for simplices. *Geometric group theory*, 1:97–119, 1993.

[33] Laurent El Ghaoui and Silviu-lulian Niculescu. *Advances in linear matrix inequality methods in control*. SIAM, 2000.

[34] P Thomas Fletcher, John Moeller, Jeff M Phillips, and Suresh Venkatasubramanian. Horoball hulls and extents in positive definite space. In *Workshop on Algorithms and Data Structures*, pages 386–398. Springer, 2011.

[35] Wolfgang Förstner and Boudewijn Moonen. A metric for covariance matrices. In *Geodesy—the Challenge of the 3rd Millennium*, pages 299–309. Springer, 2003.

[36] Pedro J Freitas and Shmuel Friedland. Revisiting the siegel upper half plane II. *Linear algebra and its applications*, 376:45–67, 2004.

[37] Pedro Jorge Freitas. *On the action of the symplectic group on the Siegel upper half plane*. PhD thesis, University of Illinois at Chicago, 1999.

[38] Richard Froese, David Hasler, and Wolfgang Spitzer. Transfer matrices, hyperbolic geometry and absolutely continuous spectrum for some discrete Schrödinger operators on graphs. *Journal of Functional Analysis*, 230(1):184–221, 2006.
[39] William Mark Goldman. *Complex hyperbolic geometry*. Oxford University Press, 1999.

[40] Mehrtash T Harandi, Mathieu Salzmann, and Richard Hartley. From manifold to manifold: Geometry-aware dimensionality reduction for SPD matrices. In *European conference on computer vision*, pages 17–32. Springer, 2014.

[41] Sigurður Helgason. *Differential geometry, Lie groups, and symmetric spaces*. Academic press, 1979.

[42] Nicholas J Higham and Awad H Al-Mohy. Computing matrix functions. *Acta Numerica*, 19:159–208, 2010.

[43] David Hilbert. Über die gerade Linie als kürzeste Verbindung zweier Punkte (About the straight line as the shortest connection between two points). *Mathematische Annalen*, 46(1):91–96, 1895.

[44] Lo-Keng Hua. The estimation of the Riemann curvature in several complex variables. *Acta Mathematica Sinica, Chinese Series*, 4:143–170, 1954.

[45] Lo-Keng Hua. On the theory of automorphic functions of a matrix variable I: Geometrical basis. *American Journal of Mathematics*, 66(3):470–488, 1944.

[46] Lo-Keng Hua. Geometries of matrices. II. study of involutions in the geometry of symmetric matrices. *Transactions of the American Mathematical Society*, 61(2):193–228, 1947.

[47] Ben Jeuris and Raf Vandebril. The Kähler mean of block-Toeplitz matrices with Toeplitz structured blocks. *SIAM Journal on Matrix Analysis and Applications*, 37(3):1151–1175, 2016.

[48] Miao Jin, Xianfeng Gu, Ying He, and Yalin Wang. *Conformal Geometry: Computational Algorithms and Engineering Applications*. Springer, 2018.

[49] Felix Klein. Über die sogenannte nicht-euklidische geometrie. *Mathematische Annalen*, 6(2):112–145, 1873.

[50] Jean Louis Koszul. *Exposés sur les espaces homogènes symétriques*. Sociedade de matematica, 1959.

[51] Khalid Koufany. Analyse et géométrie des domaines bornés symétriques. PhD thesis, Université Henri Poincaré - Nancy I, 2006.

[52] Daniel Krefl, Stefano Carrazza, Babak Haghighat, and Jens Kahlen. Riemann-Theta Boltzmann machine. *Neurocomputing*, 2020.

[53] Jacek Kuczyński and Henryk Woźniakowski. Estimating the largest eigenvalue by the power and Lanczos algorithms with a random start. *SIAM journal on matrix analysis and applications*, 13(4):1094–1122, 1992.

[54] Cornelius Lanczos. An iteration method for the solution of the eigenvalue problem of linear differential and integral operators1. *Journal of Research of the National Bureau of Standards*, 45(4), 1950.
[55] Serge Lang. *Math talks for undergraduates*. Springer Science & Business Media, 2012.

[56] Reiner Lenz. Siegel descriptors for image processing. *IEEE Signal Processing Letters*, 23(5):625–628, 2016.

[57] Yi Li and David P Woodruff. Tight bounds for sketching the operator norm, Schatten norms, and subspace embeddings. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2016.

[58] Congwen Liu and Jiajia Si. Positive Toeplitz operators on the Bergman spaces of the Siegel upper half-space. *Communications in Mathematics and Statistics*, pages 1–22, 2019.

[59] Carlangleo Liverani, Maciej P Wojtkowski, et al. Generalization of the Hilbert metric to the space of positive definite matrices. *Pacific Journal of Mathematics*, 166(2):339–355, 1994.

[60] D Steven Mackey and Niloufer Mackey. *On the determinant of symplectic matrices*. Manchester Centre for Computational Mathematics, 2003.

[61] Arya Mazumdar, Yury Polyanskiy, and Barna Saha. On Chebyshev radius of a set in hamming space and the closest string problem. In *2013 IEEE International Symposium on Information Theory*, pages 1401–1405. IEEE, 2013.

[62] Josephine Mitchell. Potential theory in the geometry of matrices. *Transactions of the American Mathematical Society*, 79(2):401–422, 1955.

[63] Maher Moakher. A differential geometric approach to the geometric mean of symmetric positive-definite matrices. *SIAM Journal on Matrix Analysis and Applications*, 26(3):735–747, 2005.

[64] Yukihioko Namikawa. The Siegel upperhalf plane and the symplectic group. In *Toroidal Compactification of Siegel Spaces*, pages 1–6. Springer, 1980.

[65] Constantin Niculescu and Lars-Erik Persson. *Convex functions and their applications: a contemporary approach*. Springer Science & Business Media, 2018. Second edition.

[66] Frank Nielsen. An elementary introduction to information geometry. *arXiv preprint arXiv:1808.08271*, 2018.

[67] Frank Nielsen. Hilbert geometry of the Siegel disk: The Siegel-Klein disk model. *arXiv preprint arXiv:2004.08160*, 2020.

[68] Frank Nielsen and Rajendra Bhatia. *Matrix information geometry*. Springer, 2013.

[69] Frank Nielsen and Gaëtan Hadjeres. Approximating covering and minimum enclosing balls in hyperbolic geometry. In *International Conference on Geometric Science of Information*, pages 586–594. Springer, 2015.

[70] Frank Nielsen, Boris Muzellec, and Richard Nock. Classification with mixtures of curved Mahalanobis metrics. In *2016 IEEE International Conference on Image Processing (ICIP)*, pages 241–245. IEEE, 2016.
[71] Frank Nielsen and Richard Nock. Hyperbolic Voronoi diagrams made easy. In 2010 International Conference on Computational Science and Its Applications, pages 74–80. IEEE, 2010.

[72] Frank Nielsen and Richard Nock. The hyperbolic Voronoi diagram in arbitrary dimension. arXiv preprint arXiv:1210.8234, 2012.

[73] Frank Nielsen and Richard Nock. Visualizing hyperbolic Voronoi diagrams. In Proceedings of the thirtieth annual symposium on Computational geometry, pages 90–91, 2014.

[74] Frank Nielsen and Richard Nock. Fast (1 + ε)-approximation of the Löwner extremal matrices of high-dimensional symmetric matrices. In Computational Information Geometry, pages 121–132. Springer, 2017.

[75] Frank Nielsen, Paolo Piro, and Michel Barlaud. Bregman vantage point trees for efficient nearest neighbor queries. In 2009 IEEE International Conference on Multimedia and Expo, pages 878–881. IEEE, 2009.

[76] Frank Nielsen and Ke Sun. Clustering in Hilberts projective geometry: The case studies of the probability simplex. Geometric Structures of Information, page 297, 2018.

[77] Richard Nock and Frank Nielsen. Fitting the smallest enclosing Bregman ball. In European Conference on Machine Learning, pages 649–656. Springer, 2005.

[78] Tomoki Ohsawa. The Siegel upper half space is a Marsden–Weinstein quotient: Symplectic reduction and Gaussian wave packets. Letters in Mathematical Physics, 105(9):1301–1320, 2015.

[79] Tomoki Ohsawa and Cesare Tronci. Geometry and dynamics of Gaussian wave packets and their Wigner transforms. Journal of Mathematical Physics, 58(9):092105, 2017.

[80] Athanase Papadopoulos and Marc Troyanov. Handbook of Hilbert geometry. European Mathematical Society (EMS), Zürich, 22, 2014. IRMA Lectures in Mathematics and Theoretical Physics.

[81] Vern I Paulsen and Mrinal Raghupathi. An introduction to the theory of reproducing kernel Hilbert spaces, volume 152. Cambridge University Press, 2016.

[82] John G Ratcliffe. Foundations of hyperbolic manifolds, volume 3. Springer, 1994.

[83] Jürgen Richter-Gebert. Perspectives on projective geometry: A guided tour through real and complex geometry. Springer Science & Business Media, 2011.

[84] B Riemann. Über das Verschwinden der Theta-Functionen. Borchardt’s [= Crelle’s] J. für reine und angew. Math, 65:214–224, 1865.

[85] Bernhard Riemann. Theorie der Abel'schen Functionen. Journal fr die reine und angewandte Mathematik, 54:101–155, 1857.

[86] Donsub Rim. An elementary proof that symplectic matrices have determinant one. Advances in Dynamical Systems and Applications (ADSA), 12(1):15–20, 2017.
[87] Feng Rong. Non-diagonal holomorphic isometric embeddings of the Poincaré disk into the Siegel upper half-plane. *Asian Journal of Mathematics*, 22(4):665–672, 2018.

[88] Carl Ludwig Siegel. Einführung in die Theorie der Modulfunktionen n-ten Grades. *Mathematische Annalen*, 116(1):617–657, 1939.

[89] Carl Ludwig Siegel. Symplectic geometry. *American Journal of Mathematics*, 65(1):1–86, 1943.

[90] Timothy Speer. Isometries of the Hilbert metric. *arXiv preprint arXiv:1411.1826*, 2014.

[91] Suvrit Sra. On the matrix square root via geometric optimization. *arXiv preprint arXiv:1507.08366*, 2015.

[92] Christopher Swierczewski and Bernard Deconinck. Computing Riemann Theta functions in Sage with applications. *Mathematics and computers in Simulation*, 127:263–272, 2016.

[93] Mengjiao Tang, Yao Rong, and Jie Zhou. An information geometric viewpoint on the detection of range distributed targets. *Mathematical Problems in Engineering*, 2015, 2015.

[94] Mengjiao Tang, Yao Rong, Jie Zhou, and X Rong Li. Information geometric approach to multisensor estimation fusion. *IEEE Transactions on Signal Processing*, 67(2):279–292, 2018.

[95] Mahbod Tavallaee, Wei Lu, Shah Arif Iqbal, and Ali A Ghorbani. A novel covariance matrix based approach for detecting network anomalies. In *6th Annual Communication Networks and Services Research Conference (cnsr 2008)*, pages 75–81. IEEE, 2008.

[96] IW-H Tsang, JT-Y Kwok, and Jacek M Zurada. Generalized core vector machines. *IEEE Transactions on Neural Networks*, 17(5):1126–1140, 2006.

[97] Zhexian Wan and Luogeng Hua. *Geometry of matrices*. World Scientific, 1996.

[98] Emo Welzl. Smallest enclosing disks (balls and ellipsoids). In *New results and new trends in computer science*, pages 359–370. Springer, 1991.