Lie theory and cohomology of relative Rota–Baxter operators

Jun Jiang¹ | Yunhe Sheng¹ | Chenchang Zhu²

¹Department of Mathematics, Jilin University, Changchun, Jilin, China
²Mathematics Institute, Georg-August-University Gottingen, Gottingen, Germany

Abstract
In this paper, we establish a local Lie theory for relative Rota–Baxter operators of weight 1. First we recall the category of relative Rota–Baxter operators of weight 1 on Lie algebras and construct a cohomology theory for them. We use the second cohomology group to study infinitesimal deformations of relative Rota–Baxter operators and modified $r$-matrices. Then we introduce a cohomology theory of relative Rota–Baxter operators on a Lie group. We construct the differentiation functor from the category of relative Rota–Baxter operators on Lie groups to that on Lie algebras, and extend it to the cohomology level by proving the Van Est theorem between the two cohomology theories. We integrate a relative Rota–Baxter operator of weight 1 on a Lie algebra to a local relative Rota–Baxter operator on the corresponding Lie group, and show that the local integration and differentiation are adjoint to each other. Finally, we give two applications of our integration of Rota–Baxter operators: one is to give an explicit formula for the factorization problem, and the other is to provide an integration for matched pairs.

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INTRODUCTION

The notion of Rota–Baxter operators on associative algebras was introduced by G. Baxter and they are applied in the Connes-Kreimer’s algebraic approach to renormalization of quantum field theory [7, 16]. In the corresponding semiclassical world, the notion of relative Rota–Baxter operators (also called $\mathcal{O}$-operators) on Lie algebras was introduced in [21], and they are closely related to various classical Yang–Baxter equations. Moreover, from a more algebraic approach, a relative Rota–Baxter operator naturally gives rise to a pre-Lie or a post-Lie algebra, and play important roles in mathematical physics [2, 5].

A Rota–Baxter operator of weight 0 on a Lie algebra is naturally the operator form of a classical skew-symmetric $r$-matrix [33]. Rota–Baxter operators of weight 1 are in one-to-one correspondence with solutions of the modified Yang–Baxter equation, and give rise to factorizations of Lie algebras. Integrating such factorizations locally to the level of Lie groups provides natural solutions of a certain Hamiltonian system, which is of great interest for the people coming from integrable systems [33]. In a recent work [17], the notion of a Rota–Baxter operator on a Lie group was introduced. In particular, one can obtain a Rota–Baxter operator of weight 1 on a Lie algebra by taking the differentiation of a Rota–Baxter operator on a Lie group.

Rota–Baxter operators are special cases of relative Rota–Baxter operators, where we distinguish the source and the target of the operator (see Definitions 2.1 and 3.1). In this article, we further study the differentiation and integration problem for relative Rota–Baxter operators, and
explore their cohomology theory. We find such a separation of the source and the target makes the cohomology theory and Lie theory actually much more clear. Differentiation on the level of cohomology, namely a version of Van Est theorem, is also given. We show that differentiation and local integration work for relative Rota–Baxter operators. Moreover, these two procedures are adjoint to each other. Our results in particular imply that we may integrate a Rota–Baxter operator on a Lie algebra to a local Rota–Baxter operator on a Lie group in a functorial way. This seems to provide some global geometrical insight to the above mentioned problem in integrable systems. Moreover, via our integration, we are able to provide an explicit formula for the factorization problem on the level of Lie groups.

Differentiation works without much obstruction, and follows from adapted Lie theoretical calculations. Integration, even local integration, is however, a bit tricky: the relative Rota–Baxter operator is not a Lie algebra or a Lie group homomorphism. To integrate such a map, we turn to its graph, which has a relative good Lie theoretical property but not completely compatible with exponential maps. Nevertheless, a bit luck happens, and the noncompatibleness somehow falls in the kernel of the operator and the local integration works. To really achieve the global integration, we expect some topological obstruction. In fact, we have already faced this when we try to integrate an infinite-dimensional Lie algebra [28, 40], where the obstruction lies in second cohomology groups; and when we try to integrate a Lie algebroid [9] to a Lie groupoid. On the other hand, we might also use Malcev’s method to extend local structures to global ones [26]. As therein, an obstruction in algebraic terms would show up. Nevertheless, it turns out that the algebraic obstruction therein is actually topological in nature [14]. Thus, the local integration and some topological invariants obtained in this article should be good ingredients for the global integration in the next step.

We notice that the cohomology theory of relative Rota–Baxter operators of weight 0 on Lie algebras, together with those on associative algebras have been much studied recently [10–12, 20, 23, 35]. Also there are some further progresses on Rota–Baxter operators on groups in recent works [3, 4, 15].

A byproduct of our cohomology theory is to provide a classification of deformation of modified \( r \)-matrices, which seems to be easier to carry out from the operator point of view.

Now we describe our results in the time order: we first show that a relative Rota–Baxter operator \( B : \mathfrak{h} \to \mathfrak{g} \) of weight 1 on a Lie algebra \( \mathfrak{g} \) with respect to an action \( \phi : \mathfrak{g} \to \text{Der}(\mathfrak{h}) \) gives rise to a representation of the descendent Lie algebra \((\mathfrak{h}, [\cdot, \cdot]_B)\) on the vector space \( \mathfrak{g} \). It turns out that the Chevalley–Eilenberg cochain complex of the descendent Lie algebra gives rise to the controlling algebra of the relative Rota–Baxter operator \( B \) of weight 1 [36]. In principal, the controlling algebra of an algebraic structure together with a differential coming from the algebraic structure itself provides a good cohomology theory for this algebraic structure. We thus define the cohomology of the relative Rota–Baxter operator \( B \) to be the Chevalley–Eilenberg cohomology of the corresponding descendent Lie algebra. As an application or verification of this cohomology theory, we apply the second cohomology group to study infinitesimal deformations of relative Rota–Baxter operators of weight 1 on Lie algebras and that of modified \( r \)-matrices. Then parallely, we show that a relative Rota–Baxter operator \( B : H \to G \) on a Lie group \( G \) with respect to an action \( \Phi : G \to \text{Aut}(H) \) gives rise to a representation of the descendent Lie group \((H, \star)\) on the vector space \( \mathfrak{g} \). Analogously, the cohomology of a relative Rota–Baxter operator \( B \) is taken to be the cohomology of the corresponding descendent Lie group. Moreover, we establish the Van Est map from the cochain complex of the relative Rota–Baxter operator \( B \) on a Lie group \( G \) to the cochain complex of the relative Rota–Baxter operator \( B \) on the corresponding Lie algebra \( \mathfrak{g} \). Van Est theorems hold, and this might also
be viewed as a justification of our cohomology theory for relative Rota–Baxter operators on Lie groups.

In Section 2, we give the cohomology of a relative Rota–Baxter operator of weight 1 on a Lie algebra \( \mathfrak{g} \) with respect to an action \( \phi : \mathfrak{g} \to \text{Der}(\mathfrak{h}) \) using the Chevalley–Eilenberg cohomology of the descendent Lie algebra \( (\mathfrak{h}, [\cdot, \cdot]_\mathfrak{g}) \). As applications, we study infinitesimal deformations of a relative Rota–Baxter operator of weight 1 and modified \( r \)-matrices using the second cohomology group.

In Section 3, we give the cohomology of a relative Rota–Baxter operator on a Lie group \( G \) with respect to an action \( \Phi : G \to \text{Aut}(H) \) using the cohomology of the descendent Lie group \( (H, \star) \).

In Section 4, we first show that the Lie algebra of the descendent Lie group \( (H, \star) \) is the descendent Lie algebra \( (\mathfrak{h}, [\cdot, \cdot]_\mathfrak{g}) \), and the differentiation of the representation \( \Theta : H \to \mathfrak{gl}(\mathfrak{g}) \) of the descendent Lie group \( (H, \star) \) is the representation \( \theta : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}) \) of the descendent Lie algebra \( (\mathfrak{h}, [\cdot, \cdot]_\mathfrak{g}) \). Then we achieve our differentiation operator and consequently, we establish the Van Est map from the cochain complex of the relative Rota–Baxter operator \( B \) on a Lie group \( G \) to the cochain complex of the relative Rota–Baxter operator \( B \) of weight 1 on the corresponding Lie algebra \( \mathfrak{g} \).

In Section 5, we introduce the notion of a local relative Rota–Baxter operator on a Lie group \( G \) and show that a relative Rota–Baxter operator \( B : \mathfrak{h} \to \mathfrak{g} \) of weight 1 on a Lie algebra can be integrated to a local relative Rota–Baxter operator on the corresponding connected and simply connected Lie group \( G \).

In Section 6, we give two applications of the integration of Rota–Baxter operators. One is to provide an explicit formula of local factorization problem and the other is to provide an integration for some special matched pairs.

2 RELATIVE ROTA–BAXTER OPERATORS ON LIE ALGEBRAS AND THEIR COHOMOLOGY THEORY

In this section, we define the cohomology of relative Rota–Baxter operators of weight 1 on Lie algebras. As applications, we classify infinitesimal deformations of relative Rota–Baxter operators of weight 1 on Lie algebras using the second cohomology group.

2.1 The category \( \text{RB}_{\mathfrak{g}}^\mathfrak{h} \) of relative Rota–Baxter operators on Lie algebras

In the sequel, \( (\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}) \) and \( (\mathfrak{h}, [\cdot, \cdot]_\mathfrak{g}) \) are Lie algebras. Let \( \phi : \mathfrak{g} \to \text{Der}(\mathfrak{h}) \) be a Lie algebra homomorphism. We call \( \phi \) an action of \( \mathfrak{g} \) on \( \mathfrak{h} \).

Definition 2.1. Let \( \phi : \mathfrak{g} \to \text{Der}(\mathfrak{h}) \) be an action of a Lie algebra \( (\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}) \) on a Lie algebra \( (\mathfrak{h}, [\cdot, \cdot]_\mathfrak{g}) \). A linear map \( B : \mathfrak{h} \to \mathfrak{g} \) is called a relative Rota–Baxter operator of weight 1 on \( \mathfrak{g} \) with respect to \( (\mathfrak{h}; \phi) \) if

\[
[B(u), B(v)]_\mathfrak{g} = B\left(\phi(B(u))v - \phi(B(v))u + [u, v]_\mathfrak{h}\right), \quad \forall u, v \in \mathfrak{h}.
\]
In particular, if $\mathfrak{g} = \mathfrak{h}$ and the action is the adjoint representation of $\mathfrak{g}$ on itself, then $B$ is called a Rota–Baxter operator of weight $1$.

**Remark 2.2.** In the context of Lie algebras, usually one considers relative Rota–Baxter operators of weight $\lambda$, where $\lambda$ is a parameter in the front of $[,]_\mathfrak{h}$ in (1). However, in the context of Lie groups, one can only consider relative Rota–Baxter operators of weight $\pm 1$ due to the fact that there is no linear structure on Lie groups in general. Thus, in this paper we consider only relative Rota–Baxter operators of weight $1$, and call them simply relative Rota–Baxter operators.

**Example 2.3.** Let $\mathfrak{g} = \text{up}(2; \mathbb{R})$, where $\text{up}(2; \mathbb{R})$ is the set of upper triangular matrices and $\mathfrak{h} = \mathbb{R}$. Define $B : \mathbb{R} \to \text{up}(2; \mathbb{R})$ and $\phi : \text{up}(2; \mathbb{R}) \to \text{Der}(\mathbb{R})$ by

$$B(r) = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}, \quad \phi \left( \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \right) = x r, \quad \forall \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in \text{up}(2; \mathbb{R}), r \in \mathbb{R}.$$

Then $B$ is a relative Rota–Baxter operator on the Lie algebra $\mathfrak{g}$ with respect to the action $(\mathbb{R}; \phi)$.

Let $\phi : \mathfrak{g} \to \text{Der}(\mathfrak{h})$ be an action of a Lie algebra $(\mathfrak{g}, [, ,]_\mathfrak{g})$ on a Lie algebra $(\mathfrak{h}, [, ,]_\mathfrak{h})$. Define a skewsymmetric bracket operation $[,]_\phi$ on $\mathfrak{g} \oplus \mathfrak{h}$ by

$$[x + u, y + v]_\phi = [x, y]_\mathfrak{g} + \phi(x) v - \phi(y) u + [u, v]_\mathfrak{h}, \quad \forall x, y \in \mathfrak{g}, u, v \in \mathfrak{h}. \quad (2)$$

Then it is straightforward to see that $(\mathfrak{g} \oplus \mathfrak{h}, [, ,]_\phi)$ is a Lie algebra, and denoted by $\mathfrak{g} \prec \phi \mathfrak{h}$.

**Proposition 2.4.** Let $\phi : \mathfrak{g} \to \text{Der}(\mathfrak{h})$ be an action of a Lie algebra $(\mathfrak{g}, [, ,]_\mathfrak{g})$ on a Lie algebra $(\mathfrak{h}, [, ,]_\mathfrak{h})$. Then a linear map $B : \mathfrak{h} \to \mathfrak{g}$ is a relative Rota–Baxter operator if and only if the graph $\text{Gr}(B) = \{B(u) + u | u \in \mathfrak{h}\}$ is a subalgebra of the Lie algebra $\mathfrak{g} \prec \phi \mathfrak{h}$.

**Proof.** Let $B : \mathfrak{h} \to \mathfrak{g}$ be a linear map. For all $u, v \in \mathfrak{h}$, we have

$$[B(u) + u, B(v) + v]_\phi = [B(u), B(v)]_\mathfrak{g} + \phi(B(u)) v - \phi(B(v)) u + [u, v]_\mathfrak{h},$$

which implies that the graph $\text{Gr}(B) = \{B(u) + u | u \in \mathfrak{h}\}$ is a subalgebra of the Lie algebra $\mathfrak{g} \prec \phi \mathfrak{h}$ if and only if $B$ satisfies

$$[B(u), B(v)]_\mathfrak{g} = B(\phi(B(u)) v - \phi(B(v)) u + [u, v]_\mathfrak{h}),$$

which means that $B$ is a relative Rota–Baxter operator. \qed

As $\mathfrak{h}$ and the graph $\text{Gr}(B)$ are isomorphic as vector spaces, we have the following result.

**Corollary 2.5.** Let $B : \mathfrak{h} \to \mathfrak{g}$ be a relative Rota–Baxter operator on $\mathfrak{g}$ with respect to an action $(\mathfrak{h}; \phi)$. Then $(\mathfrak{h}, [, ,]_\mathfrak{h})$ is a Lie algebra, called the descendent Lie algebra of $B$, where

$$[u, v]_B = \phi(B(u)) v - \phi(B(v)) u + [u, v]_\mathfrak{h}, \quad \forall u, v \in \mathfrak{h}. \quad (3)$$

Moreover, $B$ is a Lie algebra homomorphism from $(\mathfrak{h}, [, ,]_\mathfrak{h})$ to $(\mathfrak{g}, [, ,]_\mathfrak{g})$. 

Definition 2.6. Let $B$ and $B'$ be relative Rota–Baxter operators on a Lie algebra $\mathfrak{g}$ with respect to the action $(\mathfrak{h};\phi)$. A homomorphism from $B'$ to $B$ consists of a Lie algebra homomorphism $\psi_\mathfrak{g} : \mathfrak{g} \rightarrow \mathfrak{g}$ and a Lie algebra homomorphism $\psi_\mathfrak{h} : \mathfrak{h} \rightarrow \mathfrak{h}$ such that

$$\psi_\mathfrak{g} \circ B' = B \circ \psi_\mathfrak{h},$$

(4)

$$\psi_\mathfrak{g}(\phi(x)u) = \phi(\psi_\mathfrak{g}(x))\psi_\mathfrak{g}(u), \quad \forall x \in \mathfrak{g}, u \in \mathfrak{h}. \quad (5)$$

In particular, if both $\psi_\mathfrak{g}$ and $\psi_\mathfrak{h}$ are invertible, $(\psi_\mathfrak{g}, \psi_\mathfrak{h})$ is called an isomorphism from $B'$ to $B$.

In fact, (5) is equivalent that $(\psi_\mathfrak{g}, \psi_\mathfrak{h})$ is an endomorphism of the Lie algebra $\mathfrak{g} \ltimes \phi \mathfrak{h}$.

It is straightforward to see that relative Rota–Baxter operators on a Lie algebra $\mathfrak{g}$ with respect to an action $(\mathfrak{h};\phi)$ together with homomorphisms between them form a category, which we denote by $\mathcal{RB}_{\mathfrak{g}}^\mathfrak{h}$.

2.2 A cohomology theory in $\mathcal{RB}_{\mathfrak{g}}^{\mathfrak{h}}$

We first prove some statements that are important for us to build up our cohomology theory in a functorial way.

Proposition 2.7. Let $B : \mathfrak{h} \rightarrow \mathfrak{g}$ be a relative Rota–Baxter operator on $\mathfrak{g}$ with respect to $(\mathfrak{h};\phi)$. Define a linear map $\partial : \mathfrak{h} \rightarrow \mathfrak{g} \mathfrak{I}(\mathfrak{g})$ by

$$\partial(u)x = B(\phi(x)u) + [B(u),x]_\mathfrak{g}, \quad \forall x \in \mathfrak{g}, u \in \mathfrak{h}. \quad (6)$$

Then $\partial$ is a representation of the descendant Lie algebra $(\mathfrak{h}, [\cdot,\cdot]_B)$ on the vector space $\mathfrak{g}$.

Proof. By the fact that $\phi(x) \in \text{Der}(\mathfrak{h})$ for all $x \in \mathfrak{g}$ and $B : \mathfrak{h} \rightarrow \mathfrak{g}$ is a relative Rota–Baxter operator, for all $u, v \in \mathfrak{h}$, we have

$$[\partial(u), \partial(v)]x = \partial(u)\partial(v)x - \partial(v)\partial(u)x$$

$$= \partial(u)(B(\phi(x)v)) + \partial(u)[B(v),x]_\mathfrak{g} - \partial(v)(B(\phi(x)u)) - \partial(v)[B(u),x]_\mathfrak{g}$$

$$= B(\phi(B(\phi(x)v))u) + [B(u),B(\phi(x)v)]_\mathfrak{g} + B(\phi([B(v),x]_\mathfrak{g})u) + [B(u),[B(v),x]_\mathfrak{g}]_\mathfrak{g}$$

$$- B(B(\phi(\phi(x)u)v) - [B(v),B(\phi(x)v)]_\mathfrak{g} - B(\phi([B(u),x]_\mathfrak{g})v) - [B(v),[B(u),x]_\mathfrak{g}]_\mathfrak{g}$$

$$= B(\phi(B(u))\phi(x)v) - \phi(B(\phi(x)v))u + [u,\phi(x)v]_\mathfrak{h}$$

$$+ B(\phi(B(\phi(x)v))u) + B(\phi([B(v),x]_\mathfrak{g})u) + [B(u),[B(v),x]_\mathfrak{g}]_\mathfrak{g}$$

$$- B(\phi(B(u))\phi(x)u - \phi(B(\phi(x)u))v + [v,\phi(x)u]_\mathfrak{h})$$

$$- B(\phi(B(\phi(x)u)v) - B(\phi([B(u),x]_\mathfrak{g})v) - [B(v),[B(u),x]_\mathfrak{g}]_\mathfrak{g}$$

$$= B(\phi(B(u))\phi(x)v) + B(\phi([B(v),x]_\mathfrak{g})u) - B(\phi(B(u))\phi(x)u) - B(\phi([B(u),x]_\mathfrak{g})v)$$

$$+ B([u,\phi(x)v]_\mathfrak{h}) - B([v,\phi(x)u]_\mathfrak{h}) + [B(u),[B(v),x]_\mathfrak{g}]_\mathfrak{g} - [B(v),[B(u),x]_\mathfrak{g}]_\mathfrak{g}$$
Thus, \( \vartheta \) is a representation of the descendent Lie algebra \((\mathfrak{h}, [\cdot, \cdot])\) on the vector space \(\mathfrak{g} \).

**Proposition 2.8.** Let \( B : \mathfrak{h} \to \mathfrak{g} \) be a relative Rota–Baxter operator on \(\mathfrak{g} \) with respect to \((\mathfrak{h}; \phi)\). Then
\[
\phi(u, v) \vartheta = [\phi(u), \phi(v)] \vartheta + B([u, v]) \vartheta, \quad \forall u, v \in \mathfrak{h}, x \in \mathfrak{g}.
\]

**Proof.** For any \( u, v \in \mathfrak{h}, x \in \mathfrak{g} \), by Proposition 2.7, we obtain
\[
\phi(u, v) \vartheta = [\phi(u), \phi(v)] \vartheta + B([u, v]) \vartheta,
\]
which finishes the proof. \( \square \)

Let \( d^B_{\text{CE}} : \text{Hom}(\wedge^k \mathfrak{h}, \mathfrak{g}) \to \text{Hom}(\wedge^{k+1} \mathfrak{h}, \mathfrak{g}) \) be the corresponding Chevalley–Eilenberg coboundary operator of the descendent Lie algebra \((\mathfrak{h}, [\cdot, \cdot])\) with coefficients in \((\mathfrak{g}; \vartheta)\). More precisely, for all \( f \in \text{Hom}(\wedge^k \mathfrak{h}, \mathfrak{g}) \) and \( u_1, \ldots, u_{k+1} \in \mathfrak{h} \), we have
\[
d^B_{\text{CE}} f(u_1, \ldots, u_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \vartheta(u_i) f(u_1, \ldots, \hat{u_i}, \ldots, u_{k+1}) + \sum_{i<j} (-1)^{i+j} f([u_i, u_j], u_1, \ldots, \hat{u_i}, \ldots, \hat{u_j}, \ldots, u_{k+1}) + \sum_{i=1}^{k+1} (-1)^{i+1} B([u_i], f(u_1, \ldots, \hat{u_i}, \ldots, u_{k+1})) + \sum_{i<j} (-1)^{i+j} f(B(u_i), u_j, \ldots, u_{k+1}) + [u_i, u_j] \vartheta, \quad \forall u, v \in \mathfrak{h}, x \in \mathfrak{g}.
\]

**Definition 2.9.** Let \( B : \mathfrak{h} \to \mathfrak{g} \) be a relative Rota–Baxter operator on the Lie algebra \(\mathfrak{g} \) with respect to an action \((\mathfrak{h}; \phi)\). Define the space of \(k\)-cochains \(C^k(B)\) by \(C^k(B) = \text{Hom}(\wedge^{k-1} \mathfrak{h}, \mathfrak{g})\). Denote by \(C^*(B) = \bigoplus_{k=1}^{\infty} C^k(B)\). The cohomology of the cochain complex \((C^*(B), d^B_{\text{CE}})\) is defined to be the cohomology for the relative Rota–Baxter operator \(B\).

Denote by \(H^k(B)\) the \(k\)th cohomology group.
It implies in particular that \( x \in \mathfrak{g} \) is a 1-cocycle if and only if \( B \circ \phi(x) = \text{ad}_x \circ B \), and \( f \in \text{Hom}(\mathfrak{h}, \mathfrak{g}) \) is a 2-cocycle if and only if

\[
B(\phi(f(u_2))u_1) + [B(u_1), f(u_2)]_{\mathfrak{g}} = B(\phi(f(u_1))u_2) - [B(u_2), f(u_1)]_{\mathfrak{g}}.
\]

\[
f(\phi(B(u_1))u_2 - \phi(B(u_2))u_1 + [u_1, u_2]_{\mathfrak{h}}).
\]

Remark 2.10. The controlling algebra of relative Rota–Baxter operators of weight 1 on Lie algebras was given in [36]. The cohomology theory established here coincides with the one given by the controlling algebra. As inspired by the slogan promoted by Deligne: deformations of a given algebraic or geometric structure are governed by a differential graded Lie algebra (DGLA), or more generally, by an \( L_\infty \)-algebra. This DGLA or \( L_\infty \)-algebra, is called the controlling algebra of the given algebraic or geometric structure. Moreover, given the close relation between the cohomology theory and the deformation theory, we may use the controlling algebra to define a cohomology theory for the given algebraic or geometric structure.

We need the following statement to prove the functoriality of our cohomology theory.

**Proposition 2.11.** Let \( B \) and \( B' \) be relative Rota–Baxter operators on a Lie algebra \( \mathfrak{g} \) with respect to an action \((\mathfrak{h}; \phi)\) and \((\psi_\mathfrak{g}, \psi_\mathfrak{h})\) be a homomorphism from \( B' \) to \( B \).

(i) \( \psi_\mathfrak{h} \) is also a Lie algebra homomorphism from the descendent Lie algebra \((\mathfrak{h}, [\cdot, \cdot]_{B'})\) of \( B' \) to the descendent Lie algebra \((\mathfrak{h}, [\cdot, \cdot]_B)\) of \( B \).

(ii) The induced representation \((\mathfrak{g}, \vartheta)\) of the Lie algebra \((\mathfrak{h}, [\cdot, \cdot]_B)\) and the induced representation \((\mathfrak{g}, \vartheta')\) of the Lie algebra \((\mathfrak{h}, [\cdot, \cdot]_{B'})\) satisfy the following relation:

\[
\psi_\mathfrak{g} \circ \vartheta'(u) = \vartheta(\psi_\mathfrak{h}(u)) \circ \psi_\mathfrak{g}, \quad \forall u \in \mathfrak{h}.
\]

That is, for all \( u \in \mathfrak{h} \), the following diagram commutes:

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\psi_\mathfrak{h}} & \mathfrak{g} \\
\vartheta'(u) \downarrow & & \downarrow \vartheta(\psi_\mathfrak{h}(u)) \\
\mathfrak{g} & \xrightarrow{\psi_\mathfrak{g}} & \mathfrak{g}
\end{array}
\]

**Proof.** By (4), (5), and the fact that \( \psi_\mathfrak{h} \) is a Lie algebra homomorphism, we have

\[
\psi_\mathfrak{h}([u, v]_{B'}) = \psi_\mathfrak{h}(\phi(B'(u))v - \phi(B'(v))u + [u, v]_{\mathfrak{h}}) \\
= \phi(B(\psi_\mathfrak{h}(u)))\psi_\mathfrak{h}(v) - \phi(B(\psi_\mathfrak{h}(v)))\psi_\mathfrak{h}(u) + [\psi_\mathfrak{h}(u), \psi_\mathfrak{h}(v)]_{\mathfrak{h}} \\
= [\psi_\mathfrak{h}(u), \psi_\mathfrak{h}(v)]_{B'}, \quad \forall u, v \in \mathfrak{h},
\]

which implies that \( \psi_\mathfrak{h} \) is a homomorphism between the descendent Lie algebras.

By (4), (5), and (6), for all \( x \in \mathfrak{g}, u \in \mathfrak{h} \), we have

\[
\psi_\mathfrak{g}(\vartheta'(u)x) = \psi_\mathfrak{g}(B'(\phi(x)u)) + [\psi_\mathfrak{g}(B'(u)), \psi_\mathfrak{g}(x)]_{\mathfrak{g}} \\
= B(\phi(\psi_\mathfrak{g}(x))\psi_\mathfrak{g}(u)) + [B(\psi_\mathfrak{g}(u)), \psi_\mathfrak{g}(x)]_{\mathfrak{g}} = \vartheta(\psi_\mathfrak{h}(u))\psi_\mathfrak{g}(x).
\]

We finish the proof. \( \square \)
Let $B$ and $B'$ be relative Rota–Baxter operators on $\mathfrak{g}$ with respect to an action $(\mathfrak{h}; \phi)$, and $(\psi_\mathfrak{g}, \psi_\mathfrak{h})$ a homomorphism from $B'$ to $B$ in which $\psi_\mathfrak{h}$ is invertible. Define a map $p : C^k(B') \to C^k(B)$ by
\[
p(\omega)(u_1, \ldots, u_{k-1}) = \psi_\mathfrak{g}(\omega(\psi_\mathfrak{h}^{-1}(u_1), \ldots, \psi_\mathfrak{h}^{-1}(u_{k-1}))), \quad \forall u_i \in \mathfrak{h}.
\]

**Theorem 2.12.** With above notations, $p$ is a cochain map from the cochain complex $(C^*(B'), d_{CE}^{B'})$ to the cochain complex $(C^*(B), d_{CE}^B)$. Consequently, it induces a homomorphism $p_*$ from the cohomology group $H^k(B')$ to $H^k(B)$ for all $k \geq 1$.

**Proof.** For all $\omega \in C^k(B')$, by Proposition 2.11, we have
\[
d_{CE}^B(p(\omega))(u_1, u_2, \ldots, u_k)
= \sum_{i=1}^{k} (-1)^{i+1} \theta(u_i) p(\omega)(u_1, \ldots, \hat{u}_i, \ldots, u_k)
+ \sum_{i<j} (-1)^{i+j} p(\omega)([u_i, u_j]_B, u_1, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_k)
= \sum_{i=1}^{k} (-1)^{i+1} \theta(u_i) \psi_\mathfrak{g}\left(\omega(\psi_\mathfrak{h}^{-1}(u_1), \ldots, \psi_\mathfrak{h}^{-1}(u_i), \ldots, \psi_\mathfrak{h}^{-1}(u_{k}))\right)
+ \sum_{i<j} (-1)^{i+j} \psi_\mathfrak{g}\left(\omega(\psi_\mathfrak{h}^{-1}([u_i, u_j]_B), \psi_\mathfrak{h}^{-1}(u_1), \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, \psi_\mathfrak{h}^{-1}(u_k))\right)
= \sum_{i=1}^{k} (-1)^{i+1} \theta(u_i) \psi_\mathfrak{g}\left(d_{CE}^{B'} \omega(\psi_\mathfrak{h}^{-1}(u_1), \ldots, \psi_\mathfrak{h}^{-1}(u_k))\right)
+ \sum_{i<j} (-1)^{i+j} \psi_\mathfrak{g}\left(d_{CE}^{B'} \omega([\psi_\mathfrak{h}^{-1}(u_i), \psi_\mathfrak{h}^{-1}(u_j)]_B', \psi_\mathfrak{h}^{-1}(u_1), \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, \psi_\mathfrak{h}^{-1}(u_k))\right)
= p(d_{CE}^{B'} \omega)(u_1, \ldots, u_k), \quad \forall u_i \in \mathfrak{h}.
\]
Thus, $p$ is a cochain map. Consequently, it induces a homomorphism $p_*$ from the cohomology group $H^k(B')$ to $H^k(B)$ for all $k \geq 1$. \hfill $\square$

### 2.3 Cohomology theory and deformations in $\mathcal{RB}_\mathfrak{g}^\mathfrak{h}$

In this subsection, we show that the second cohomology group classifies the infinitesimal deformations of relative Rota–Baxter operators. This may be viewed as a justification of the cohomology theory that we established in the previous subsection.

**Definition 2.13.** Let $B : \mathfrak{h} \to \mathfrak{g}$ be a relative Rota–Baxter operator on $\mathfrak{g}$ with respect to an action $(\mathfrak{h}; \phi)$ and $\hat{B} : \mathfrak{h} \to \mathfrak{g}$ be a linear map. If $B_t = B + t\hat{B}$ is still a relative Rota–Baxter operator on the Lie algebra $\mathfrak{g}$ with respect to the action $(\mathfrak{h}; \phi)$ for all $t$, we say that $\hat{B}$ generates a one-parameter infinitesimal deformation of the relative Rota–Baxter operator $B$. 
As $B_t = B + t \hat{B}$ is a relative Rota–Baxter operator on the Lie algebra $\mathfrak{g}$ with respect to the representation $(\mathfrak{h}; \phi)$ for all $t$, we have

$$[\hat{B}(u), \hat{B}(v)]_{\mathfrak{g}} = \hat{B}(\phi(\hat{B}(u))v) - \hat{B}(\phi(\hat{B}(v))u),$$

(8)

and

$$[\hat{B}(u), B(v)]_{\mathfrak{g}} + [B(u), \hat{B}(v)]_{\mathfrak{g}}$$

$$= B(\phi(B(u))v) + B(\phi(\hat{B}(u))v) - B(\phi(\hat{B}(v))u) - B(\phi(B(v))u) + B([u, v]_{\mathfrak{h}}).$$

Note that Equation (8) states that $\hat{B}$ is a relative Rota–Baxter operator of weight 0 on the Lie algebra $\mathfrak{g}$ with respect to the representation $(\mathfrak{h}; \phi)$. Equation (9) states that $\hat{B}$ is a 2-cocycle of the relative Rota–Baxter operator $B$.

**Definition 2.14.** Let $B : \mathfrak{h} \to \mathfrak{g}$ be a relative Rota–Baxter operator on $\mathfrak{g}$ with respect to $(\mathfrak{h}; \phi)$. Two one-parameter infinitesimal deformations $B_1^t = B + t \hat{B}_1$ and $B_2^t = B + t \hat{B}_2$ are said to be equivalent if there exists an $x \in \mathfrak{g}$ such that $(\text{Id}_\mathfrak{g} + t \text{ad} x, \text{Id}_\mathfrak{h} + t \phi(x))$ is a homomorphism from $B_1^t$ to $B_2^t$. In particular, a one-parameter infinitesimal deformation $B_t$ of a relative Rota–Baxter operator $B$ is said to be trivial if there exists an $x \in \mathfrak{g}$ such that $(\text{Id}_\mathfrak{g} + t \text{ad} x, \text{Id}_\mathfrak{h} + t \phi(x))$ is a homomorphism from $B_t$ to $B$.

Let $(\text{Id}_\mathfrak{g} + t \text{ad} x, \text{Id}_\mathfrak{h} + t \phi(x))$ be a homomorphism from $B_1^t$ to $B_2^t$. Then we have

$$(\text{Id}_\mathfrak{g} + t \text{ad} x)(B + t \hat{B}_1)u = (B + t \hat{B}_2)(\text{Id}_\mathfrak{h} + t \phi(x))u, \quad \forall u \in \mathfrak{h},$$

which implies

$$[x, \hat{B}_1(u)]_{\mathfrak{g}} = \hat{B}_2(\phi(x)u),$$

$$\hat{B}_1(u) - \hat{B}_2(u) = B(\phi(x)u) + [B(u), x]_{\mathfrak{g}} \quad \forall x \in \mathfrak{g}, \; u \in \mathfrak{h}. \quad (10)$$

By (10), we have

$$\hat{B}_1 - \hat{B}_2 = d^{B}_{\text{CE}} x,$$

where $d^{B}_{\text{CE}}$ is given by (7). Thus, we have the following.

**Theorem 2.15.** Let $B : \mathfrak{h} \to \mathfrak{g}$ be a relative Rota–Baxter operator on $\mathfrak{g}$ with respect to $(\mathfrak{h}; \phi)$. If two one-parameter infinitesimal deformations $B_1^t = B + t \hat{B}_1$ and $B_2^t = B + t \hat{B}_2$ are equivalent, then $\hat{B}_1$ and $\hat{B}_2$ are in the same cohomology class of $H^2(B)$ defined in Definition 2.9.

### 2.4 Infinitesimal deformations of modified $r$-matrices

A Rota–Baxter operator $B$ on $\mathfrak{g}$ one-to-one corresponds to a solution of the modified Yang–Baxter equation, which in turn has important applications in integrable systems. We now recall some facts from [19, 32–34]. Let $R : \mathfrak{g} \to \mathfrak{g}$ be a solution of the modified Yang–Baxter equation:

$$[R(u), R(v)]_{\mathfrak{g}} = R([R(u), v]_{\mathfrak{g}}) + R([u, R(v)]_{\mathfrak{g}}) - [u, v]_{\mathfrak{g}}, \quad \forall u, v \in \mathfrak{g}. \quad (11)$$

Each such a solution (which we call a modified $r$-matrix) gives rise to a so-called infinitesimal factorization of the Lie algebra $\mathfrak{g}$. The authors integrated this infinitesimal factorization locally into a factorization of the Lie group $G$ of $\mathfrak{g}$. The integrated factorization in turn gives an elegant
solution of a certain hamiltonian system for small time $t$. The factorization itself may also be viewed as some version of Riemann–Hilbert problem.

Under the transformation

$$R = \text{Id} + 2B,$$

the operator $R$ satisfies the modified Yang–Baxter equation if and only if the operator $B$ is a Rota–Baxter operator of weight 1. Based on this observation, we show that our cohomology theory can also control the infinitesimal deformations of modified $r$-matrices.

**Definition 2.16.** Given a Lie algebra $\mathfrak{g}$, let $R$ be a modified $r$-matrix satisfying Equation (11). If $R_t = R + t\hat{R}$ is still a modified $r$-matrix for all $t$, we say that $\hat{R}$ generates a one-parameter infinitesimal deformation of $R$.

**Definition 2.17.** Let $R^1_t = R + t\hat{R}_1$ and $R^2_t = R + t\hat{R}_2$ be two infinitesimal deformations of a modified $r$-matrix $R$. They are said to be equivalent if there exists an $x \in \mathfrak{g}$ such that $\text{Id}_\mathfrak{g} + t\text{ad}_x$ is a homomorphism from $R^1_t$ to $R^2_t$, that is, $\text{Id}_\mathfrak{g} + t\text{ad}_x$ is a Lie algebra endomorphism and

$$(\text{Id}_\mathfrak{g} + t\text{ad}_x) \circ R^1_t = R^2_t \circ (\text{Id}_\mathfrak{g} + t\text{ad}_x).$$

**Theorem 2.18.** Let $R^1_t = R + t\hat{R}_1$ and $R^2_t = R + t\hat{R}_2$ be two equivalent infinitesimal deformations of a modified $r$-matrix $R$. Then $\hat{R}_1$ and $\hat{R}_2$ are in the same cohomology class of $H^2(B)$ defined in Definition 2.9 for $B = \frac{1}{2}(R - \text{Id})$.

**Proof.** It is obvious that an infinitesimal deformation of a Rota–Baxter operator $B_t = B + t\hat{B}$ gives rise to an infinitesimal deformation of a modified $r$-matrix $R_t = R + t(2\hat{B})$, with $R = \text{Id} + 2B$. Conversely, an infinitesimal deformation of a modified $r$-matrix $R_t = R + t\hat{R}$ gives rise to an infinitesimal deformation of a Rota–Baxter operator $B_t = B + t(B_{CE}^B)$, with $B = \frac{1}{2}(R - \text{Id})$. Therefore $\hat{R}$ is a 2-cocycle in $(C^*(B),\text{d}_B_{CE})$ by Theorem 2.15. Similarly, $R^1_t = R + t\hat{R}_1$ and $R^2_t = R + t\hat{R}_2$ are equivalent infinitesimal deformations if and only if their corresponding Rota–Baxter operators are equivalent. Therefore by Theorem 2.15, both $\hat{R}_1$ and $\hat{R}_2$ are cocycles in $(C^*(B),\text{d}_B_{CE})$ and they are differed by a coboundary therein. □

### 3 RELATIVE ROTA–BAXTER OPERATORS ON LIE GROUPS AND THEIR COHOMOLOGY THEORY

In this section, we introduce the cohomology theory of relative Rota–Baxter operators on Lie groups. In the sequel, $(G, e_G, \cdot_G)$ and $(H, e_H, \cdot_H)$ are always Lie groups.

#### 3.1 The category $\mathbf{RB}_G^H$ of relative Rota–Baxter operators on Lie groups

**Definition 3.1.** Let $\Phi : G \to \text{Aut}(H)$ be an action of $G$ on $H$. A smooth map $B : H \to G$ is called a relative Rota–Baxter operator if

$$B(h_1) \cdot_G B(h_2) = B(h_1) \cdot_H \Phi(B(h_1))h_2, \quad \forall h_1, h_2 \in H.$$  (12)
In particular, if the action $\Phi$ is the adjoint action $\text{Ad}$ of the Lie group $G$ on itself, it reduces to a Rota–Baxter operator introduced in [17].

**Example 3.2** [17]. Let $G$ be a Lie group and $G_+, G_-$ be two subgroups such that $G = G_+ G_-$ and $G_+ \cap G_- = \{ e \}$. Define $B : G \to G$ by

$$B(g) = g_+^{-1}, \quad \forall \ g = g_+ g_-, \quad \text{where} \ g_+ \in G_+, \ g_- \in G_-.$$  

Then $B$ is a Rota–Baxter operator.

Let $G$ be a connected compact real Lie group and $(\mathbb{R}, \Phi)$ be a one-dimensional representation. Thus for any $g \in G$, it follows that $|\Phi(g)| = 1$. Moreover, $G$ is a connected Lie group, thus $\Phi(g) = 1$ for any $g \in G$. Thus, a smooth map $B : \mathbb{R} \to G$ is a relative Rota–Baxter operator with respect to the action $\Phi$ if and only if $B$ is a group homomorphism from $\mathbb{R}$ to $G$. This sort of phenomenon also extends to the noncompact case in the following example:

**Example 3.3.** Let $(G, e_G, \cdot_G) = (U P (2; \mathbb{R}), I, \cdot)$, where $U P (2; \mathbb{R})$ is the set of invertible upper triangular matrices and $H = \mathbb{R}$. Define $B : \mathbb{R} \to U P (2; \mathbb{R})$ and $\Phi : U P (2; \mathbb{R}) \to \text{Aut}(\mathbb{R})$ by

$$B(r) = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \quad \Phi \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = ar, \quad \forall \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in U P (2; \mathbb{R}), r \in \mathbb{R}.$$  

Then $B$ is a relative Rota–Baxter operator on the Lie group $G$ with respect to the action $(\mathbb{R}; \Phi)$.

Given an action $\Phi : G \to \text{Aut}(H)$, we have the semidirect product Lie group $G \ltimes_\Phi H$, with multiplication $\cdot_\Phi$ given by

$$(g_1, h_1) \cdot_\Phi (g_2, h_2) = (g_1 \cdot_G g_2, h_1 \cdot_H \Phi(g_1)h_2), \quad \forall g_i \in G, h_i \in H, i = 1, 2.$$  

**Proposition 3.4.** Let $\Phi : G \to \text{Aut}(H)$ be an action of a Lie group $G$ on a Lie group $H$. Then a smooth map $B : H \to G$ is a relative Rota–Baxter operator if and only if the graph $\text{Gr}(B) = \{(B(h), h) | h \in H\}$ is a Lie subgroup of the Lie group $G \ltimes_\Phi H$.

**Proof.** Let $B : H \to G$ be a smooth map. For all $h_1, h_2 \in H$, we have

$$(B(h_1), h_1) \cdot_\Phi (B(h_2), h_2) = (B(h_1) \cdot_G B(h_2), h_1 \cdot_H \Phi(B(h_1))h_2),$$  

which implies that the graph $\text{Gr}(B)$ is a Lie subgroup of the Lie group $G \ltimes_\Phi H$ if and only if $B$ satisfies

$$B(h_1) \cdot_G B(h_2) = B(h_1) \cdot_H \Phi(B(h_1))h_2),$$  

which means that $B$ is a relative Rota–Baxter operator. \qed

For any $g \in G$, we have $\Phi(g)e_H = \Phi(g)(e_H \cdot_H e_H) = \Phi(g)e_H \cdot_H \Phi(g)e_H$, thus $\Phi(g)e_H = e_H$.

**Proposition 3.5.** Let $B : H \to G$ be a relative Rota–Baxter operator on $G$ with respect to an action $(H; \Phi)$. Then $(H,e_H,\ast)$ is a Lie group, called the descendent Lie group of $B$ and denoted by $H_B$. 

where
\[ h_1 \star h_2 = h_1 \cdot_H \Phi(B(h_1))h_2, \quad \forall h_1, h_2 \in H. \]

For \( h \in H \), its inverse \( h^\dagger \) with respect to the multiplication \( \star \) is given by
\[ h^\dagger = \Phi(B(h^{-1}))h^{-1}, \]
(14)

where \( h^{-1} \) is the inverse element of \( h \) in \((H, e_H, \cdot_H)\).

Moreover, \( B : (H, e_H, \star) \to (G, e_G, \cdot_G) \) is a Lie group homomorphism.

**Proof.** It is obvious that \( e_H \star h = \Phi(e_H)h = h \) and \( h \star e_H = h \cdot_H \Phi(B(h))e_H = h \). Thus, \( e_H \) is the unit in \((H, e_H, \star)\).

For any \( h \in H \), by (12), we have \( B(h)^{-1} = B(\Phi(B(h)^{-1}))h^{-1} \). By (13), we have
\[ \Phi(B(h)^{-1})h^{-1} \star h = \Phi(B(h)^{-1})h^{-1} \cdot_H \Phi(\Phi(B(h)^{-1})h^{-1})h \]
\[ = \Phi(B(\Phi(B(h)^{-1})h^{-1}))h^{-1} \cdot_H \Phi(B(h)^{-1)h^{-1})h \]
\[ = \Phi(\Phi(B(h)^{-1}))h^{-1})(h^{-1} \cdot_H h) \]
\[ = e_H. \]

In a similar way, we have \( h \star \Phi(B(h)^{-1})h^{-1} = e_H \). Thus, \( \Phi(B(h)^{-1})h^{-1} \) is the inverse element of \( h \) in \((H, e_H, \star)\).

Finally, we prove that \( \star \) is associative. For all \( h_1, h_2 \) and \( h_3 \in H \), by (12) and (13), we have
\[ (h_1 \star h_2) \star h_3 = (h_1 \cdot_H \Phi(B(h_1))h_2) \star h_3 \]
\[ = h_1 \cdot_H \Phi(B(h_1))h_2 \cdot_H \Phi(\Phi(B(h_1))h_2))h_3 \]
\[ = h_1 \cdot_H \Phi(B(h_1))h_2 \cdot_H \Phi(B(h_1) \cdot_G B(h_2))h_3 \]
\[ = h_1 \cdot_H \Phi(B(h_1))h_2 \cdot_H \Phi(B(h_2))h_3 \]
\[ = h_1 \star (h_2 \star h_3). \]

Thus, \((H, e_H, \star)\) is a Lie group. By (12) and \( B(e_H) = e_G \), \( B \) is a Lie group homomorphism from \((H, e_H, \star)\) to \((G, e_G, \cdot_G)\).

**Definition 3.6.** Let \( B \) and \( B' \) be relative Rota–Baxter operators on a Lie group \( G \) with respect to an action \((H, \Phi)\). A **homomorphism** from \( B' \) to \( B \) consists of a Lie group homomorphism \( \Psi_G : G \to G \) and a Lie group homomorphism \( \Psi_H : H \to H \) such that
\[ B \circ \Psi_H = \Psi_G \circ B'. \]
(17)
\[ \Psi_H(\Phi(g)h) = \Phi(\Psi_G(g))\Psi_H(h), \quad \forall g \in G, h \in H. \]
(18)

In particular, if both \( \Psi_G \) and \( \Psi_H \) are invertible, \((\Psi_G, \Psi_H)\) is called an isomorphism from \( B' \) to \( B \).

In fact, (18) is equivalent to the fact that \((\Psi_G, \Psi_H)\) is an endomorphism of the Lie group \( G \kappa_{e_H} H \).
It is clear that relative Rota–Baxter operators on a Lie group $G$ with respect to an action $(H; \Phi)$ together with homomorphisms between them form a category, which we denote by $\mathbf{RB}^H_G$.

### 3.2 A cohomology theory in $\mathbf{RB}^H_G$

To establish a parallel cohomology theory in $\mathbf{RB}^H_G$, we need to first find an action of the descendent Lie group on the vector space $\mathfrak{g}$.

**Lemma 3.7.** Let $\Phi : G \to \text{Aut}(H)$ be an action of $G$ on $H$. Then for all $h_1, h_2 \in H$, we have

$$\Phi(g)(h_1 \star h_2) = \Phi(g)h_1 \cdot_B \Phi(g \cdot_G B(h_1))h_2.$$  

**Proof.** It is straightforward. ☐

**Theorem 3.8.** Let $B$ be a relative Rota–Baxter operator on $G$ with respect to an action $(H; \Phi)$. Define $\Theta : H \to \text{Diff}(G)$ by

$$\Theta(h)g = (B(\Phi(g)h^\dagger))^{-1} \cdot_G g \cdot_G B(h^\dagger), \quad \forall g \in G, h \in H.$$ (19)

Then $\Theta$ is an action of the descendent Lie group $(H, e_H, \star)$ on the manifold $G$.

**Proof.** For all $h_1, h_2 \in H$ and $g \in G$, by Lemma 3.7, we have

$$\Theta(h_1)\Theta(h_2)g$$

$$= \Theta(h_1) \left( B(\Phi(g)h_2^\dagger) \cdot_G g \cdot_G B(h_2^\dagger) \right)$$

$$= B(\Phi(g)h_2^\dagger) \cdot_G \Phi \left( B(\Phi(g)h_2^\dagger) \cdot_G g \cdot_G B(h_2^\dagger) \right) h_1^\dagger$$

$$= B(\Phi(g)h_2^\dagger) \cdot_G \Phi \left( B(\Phi(g)h_2^\dagger) \cdot_G g \cdot_G B(h_2^\dagger) \right) h_1^\dagger$$

$$= B(\Phi(g)h_2^\dagger) \cdot_G \Phi \left( B(\Phi(g)h_2^\dagger) \cdot_G g \cdot_G B(h_2^\dagger) \right) h_1^\dagger$$

$$= B(\Phi(g)(h_2^\dagger \star h_1^\dagger))^{-1} \cdot_G g \cdot_G B(h_2^\dagger \star h_1^\dagger)$$

Thus, $\Theta$ is an action of the descendent Lie group $(H, e_H, \star)$ on the manifold $G$. ☐

As $\Theta(h) \in \text{Diff}(G)$ for all $h \in H$ and $\Theta(h)e_G = e_G$, then $\Theta(h)_{*e_G} : \mathfrak{g} \to \mathfrak{g}$ is an isomorphism of vector spaces. By $\Theta(h_1 \star h_2) = \Theta(h_1)\Theta(h_2)$, we have $\Theta(h_1 \star h_2)_{*e_G} = \Theta(h_1)_{*e_G} \Theta(h_2)_{*e_G}$. Thus, we obtain a Lie group homomorphism from the descendent Lie group $(H, e_H, \star)$ to $GL(\mathfrak{g})$, which is also denoted by $\Theta : H \to GL(\mathfrak{g})$. Consequently, we have the following result that plays important roles in our definition of the cohomology of relative Rota–Baxter operators on Lie groups.
Lemma 3.9. With the above notations, \( \Theta : H \to GL(\mathfrak{g}) \) is an action of the descendent Lie group \((H, e_H, \star)\) on the vector space \(\mathfrak{g}\).

Now we are ready to define a cohomology theory for relative Rota–Baxter operators on Lie groups. First we recall a standard version of the smooth cohomology of a Lie group \(G\) with coefficients in a \(G\)-module \(A\) with a \(G\)-action \(\Pi\) (see, e.g., [38]).

An \(n\)-cochain is a smooth map
\[
\alpha_n : G \times \cdots \times G \to A.
\]

The set of \(n\)-cochains forms an abelian group, which will be denoted by \(C^n(G, A)\). The differential \(d : C^n(G, A) \to C^{n+1}(G, A)\) is defined by
\[
d(\alpha_n)(g_1, \ldots, g_n, g_{n+1}) = \Pi(g_1)\alpha_n(g_2, \ldots, g_n, g_{n+1}) + (-1)^{n+1}\alpha_n(g_1, \ldots, g_n) + \sum_{i=1}^{n}(-1)^i\alpha_n(g_1, \ldots, g_{i-1}, g_i \cdot G g_{i+1}, g_{i+2}, \ldots, g_{n+1}).
\]

We consider the Lie group \((H, e_H, \star)\) and its module \(\mathfrak{g}\) via the action \(\Theta : H \to GL(\mathfrak{g})\) given in Lemma 3.9. Denote by \(C^k(B) = C^{k-1}(H, \mathfrak{g})\) and by \(d^B\) the group cohomology differential. Parallel to Definition 2.9, we have the following.

Definition 3.10. Let \(B\) be a relative Rota–Baxter operator on \((G, e_G, \cdot G)\) with respect to an action \((H; \Phi)\). The cohomology of the cochain complex \((C^*(B) = \bigoplus_{k=1}^{+\infty} C^k(B), d^B)\) is defined to be the cohomology for the relative Rota–Baxter operator \(B\).

Denote by \(H^k(B)\) the \(k\)th cohomology group.

Remark 3.11. Our cohomology theory in \(\mathcal{RB}_8^G\) is well-rooted via the theory of controlling algebras and is tested and justified via the deformation theory. The above cohomology theory in \(\mathcal{RB}_G^G\) on the level of Lie groups will be justified via the Van Est theory, which is the content of the next section.

The following statements are important for the functoriality of our cohomology theory.

Proposition 3.12. Let \(B\) and \(B'\) be relative Rota–Baxter operators on a Lie group \(G\) with respect to an action \((H; \Phi)\) and \((\Psi_G, \Psi_H)\) be a homomorphism from \(B'\) to \(B\). Then \(\Psi_H\) is a Lie group homomorphism from the descendent Lie group \((H, e_H, \star_{H})\) of \(B'\) to the descendent Lie group \((H, e_H, \star_{H})\) of \(B\).

Proof. By (17), (18), and the fact that \(\Psi_H\) is a Lie group homomorphism, we have
\[
\Psi_H(h_1 \star_{H} h_2) = \Psi_H(h_1 \cdot_H B'(\Phi(h_1))h_2) = \Psi_H(h_1) \cdot_H \Psi(B'(\Phi(h_1))h_2)
\]
\[
= \Psi_H(h_1) \cdot_H \Phi(\Psi_B(\Psi_H(h_1)))\Psi_H(h_2) = \Psi_H(h_1) \star_{H} \Psi_H(h_2),
\]
which implies that \(\Psi_H\) is a homomorphism between the descendent Lie groups. \(\Box\)
Proposition 3.13. Let $B$ and $B'$ be relative Rota–Baxter operators on $G$ with respect to an action $(H; \Phi)$ and $(\Psi_G, \Psi_H)$ be a homomorphism from $B'$ to $B$. Then the induced action $\Theta'$ of the Lie group $(H, \star_{B'})$ on $G$ and the induced action $\Theta$ of the Lie group $(H, \star_B)$ on $G$ satisfy the following relation:

$$\Psi_G \circ \Theta'(h) = \Theta(\Psi_H(h)) \circ \Psi_G, \quad \forall h \in H.$$ 

That is for all $h \in H$, the following diagram commutes:

\[ \begin{array}{ccc}
G & \xrightarrow{\Psi_G} & G \\
\Theta'(h) \downarrow & & \downarrow \Theta(\Psi_H(h)) \\
G & \xrightarrow{\Psi_G} & G
\end{array} \]

Proof. By (17), (18), (19), and Proposition 3.12, for all $g \in G$, $h \in H$, we have

$$\begin{align*}
\Psi_G(\Theta'(h)g) &= \Psi_G((B'(\Phi(g)h^{\dagger}))^{-1} \cdot_G g \cdot_G B'(h^{\dagger})) \\
&= \Psi_G((B'(\Phi(g)h^{\dagger}))^{-1} \cdot_G \Psi_G(g) \cdot_G \Psi_G(B'(h^{\dagger})) \\
&= \Psi_G(B'(\Phi(g)h^{\dagger}))^{-1} \cdot_G \Psi_G(g) \cdot_G B(\Psi_H(h^{\dagger})) \\
&= B(\Phi(\Psi_G(g))((\Psi_H(h))^{\dagger})^{-1} \cdot_G \Psi_G(g) \cdot_G B((\Psi_H(h))^{\dagger}) \\
&= \Theta(\Psi_H(h))\Psi_G(g).
\end{align*}$$

We finish the proof. $\square$

Taking differentiation, we get the following.

Corollary 3.14. Let $B$ and $B'$ be relative Rota–Baxter operators on $G$ with respect to an action $(H; \Phi)$, and $(\Psi_G, \Psi_H)$ be a homomorphism from $B'$ to $B$. Then the action $\Theta'$ of the descendent Lie group $(H, e_H, \star_{B'})$ on $\mathfrak{g}$ and the action $\Theta$ of the descendent Lie group $(H, e_H, \star_B)$ on $\mathfrak{g}$ satisfy the following relation:

$$\Theta(\Psi_H(h)) \circ (\Psi_G)_{e_G} = (\Psi_G)_{e_G} \circ \Theta'(h), \quad \forall h \in H.$$ 

Now we prove the functoriality of this cohomology theory. Let $B$ and $B'$ be relative Rota–Baxter operators on $G$ with respect to an action $(H; \Phi)$. Let $(\Psi_G, \Psi_H)$ be a homomorphism from $B'$ to $B$ in which $\Psi_H$ is invertible. Define a map $P : C^k(B') \to C^k(B)$ by

$$P(\omega)(h_1, \ldots, h_{k-1}) = (\Psi_G)_{e_G} \omega(\Psi_H^{-1}(h_1), \ldots, \Psi_H^{-1}(h_{k-1})), \quad \forall h_i \in H.$$ 

Theorem 3.15. With the above notations, $P$ is cochain map from the cochain complex $(C^*(B'), d^{B'})$ to $(C^*(B), d^B)$, which induces a homomorphism $P_*$ from the cohomology group $H^k(B')$ to $H^k(B)$ for all $k \geq 1$.

Proof. For all $\omega \in C^k(B')$, by Corollary 3.14, we have

$$d^B(P(\omega))(h_1, h_2, \ldots, h_k)$$

$$= \Theta(h_1)(\Psi_G)_{e_G} \omega(\Psi_H^{-1}(h_2), \ldots, \Psi_H^{-1}(h_k))$$
\[ + \sum_{i=1}^{k-1} (-1)^i P(\omega)(h_1, \ldots, h_{i-1}, h_i \star_B h_{i+1}, h_{i+2}, \ldots, h_k) \]
\[ + (-1)^k P(\omega)(h_1, \ldots, h_{k-1}) \]
\[ = (\Psi_G)_{\ast e_G} \Theta'(\Psi^{-1}_H(h_1)) \omega(\Psi^{-1}_H(h_2), \ldots, \Psi^{-1}_H(h_k)) \]
\[ + \sum_{i=1}^{k-1} (-1)^i (\Psi_G)_{\ast e_G} \omega(\Psi^{-1}_H(h_1), \ldots, \Psi^{-1}_H(h_{i-1}), \Psi^{-1}_H(h_i) \star_B \Psi^{-1}_H(h_{i+1}), \Psi^{-1}_H(h_{i+2}), \ldots, \Psi^{-1}_H(h_k)) \]
\[ + (-1)^k (\Psi_G)_{\ast e_G} \omega(\Psi^{-1}_H(h_1), \ldots, \Psi^{-1}_H(h_{k-1})) \]
\[ = (\Psi_G)_{\ast e_G} d^{B'} \omega(\Psi^{-1}_H(h_1), \ldots, \Psi^{-1}_H(h_k)) \]
\[ = P(d^{B'} \omega)(h_1, \ldots, h_k), \quad \forall h_i \in H. \]

Thus, \( P \) is a cochain map. Consequently it induces a homomorphism \( P \) from the cohomology group \( H^k(B') \) to \( H^k(B) \) for all \( k \geq 1 \).

4 | DIFFERENTIATION AND VAN EST THEORY

In this section, we establish the differentiation functor for relative Rota–Baxter operators and the Van Est homomorphism on the cohomology level. The classical Van Est isomorphism \([37]\) gives the relation between the smooth cohomology of Lie groups and the cohomology of Lie algebras. See \([1, 6, 8, 18, 24, 27, 30, 39]\) for various Van Est type theorems.

4.1 | The differentiation functor for relative Rota–Baxter operators

Let \( \Phi \) be an action of a Lie group \((G, e_G, \cdot_G)\) on a manifold \( M \). Define \( \Delta : G \times M \to M \) by \( \Delta(g, m) = \Phi(g)m. \) Then \( \Delta(e_G, m) = m \) and \( \Delta(g_1 \cdot_G g_2, m) = \Delta(g_1, \Delta(g_2, m)) \), for all \( g_1, g_2 \in G, m \in M \). Moreover, we have the following Leibniz rule,

\[
\frac{d}{dt} \bigg|_{t=0} \Delta(\gamma_1(t), \gamma_2(t)) = \frac{d}{dt} \bigg|_{t=0} \Delta(\gamma_1(0), \gamma_2(t)) + \frac{d}{dt} \bigg|_{t=0} \Delta(\gamma_1(t), \gamma_2(0))
\]
\[
= \frac{d}{dt} \bigg|_{t=0} \Phi(\gamma_1(t))\gamma_2(t)
\]
\[
= \frac{d}{dt} \bigg|_{t=0} \Phi(\gamma_1(0))\gamma_2(t) + \frac{d}{dt} \bigg|_{t=0} \Phi(\gamma_1(t))\gamma_2(0),
\]

where \( \gamma_1(t) \) is a path in \( G \) with \( \gamma_1(0) = g \) and \( \gamma_2(t) \) is a path in \( M \) with \( \gamma_2(0) = m \).

Let \((G, e_G, \cdot_G)\) and \((H, e_H, \cdot_H)\) be Lie groups and denote by \( \exp_G \) and \( \exp_H \) the exponential maps for the Lie groups \((G, e_G, \cdot_G)\) and \((H, e_H, \cdot_H)\), respectively. Let \( \Phi : G \to \text{Aut}(H) \) be an action of \( G \) on \( H \). As \( \Phi(g) \in \text{Aut}(H) \) for all \( g \in G \), then \( \Phi(g)_{\ast e_H} : \mathfrak{h} \to \mathfrak{h} \) is a Lie algebra isomorphism. By \( \Phi(g_1 \cdot_G g_2) = \Phi(g_1)\Phi(g_2) \), we have \( \Phi(g_1 \cdot_G g_2)_{\ast e_H} = \Phi(g_1)_{\ast e_H} \Phi(g_2)_{\ast e_H} \). Thus, we obtain a Lie group homomorphism from \( G \) to \( \text{Aut}(\mathfrak{h}) \), which we denote by \( \Phi : G \to \text{Aut}(\mathfrak{h}) \). Then taking the
differential, we obtain a Lie algebra homomorphism \( \phi := \Phi \ast e_G \) from the Lie algebra \( g \) to \( \text{Der}(\mathfrak{h}) \). We call \( \phi \) the differentiated action of \( \Phi \). In fact, Lie II theorem tells us that \( \text{Aut}(H) \cong \text{Aut}(\mathfrak{h}) \), if \( H \) is connected and simply connected. Therefore, when both \( G \) and \( H \) are connected and simply connected, given a Lie algebra action \( \phi : g \to \text{Der}(\mathfrak{h}) \), there is a unique integrated action \( \Phi : G \to \text{Aut}(H) \) whose differentiation is \( \phi \). This procedure can be well-explained by the following diagram:

![Diagram](image_url)

**Theorem 4.1.** Let \( B : H \to G \) be a relative Rota–Baxter operator on \( G \) with respect to an action \( (H; \Phi) \). Define \( B : \mathfrak{h} \to g \) by

\[
B = B \ast e_H,
\]

which is the tangent map of \( B \) at the identity \( e_H \). Then \( B \) is a relative Rota–Baxter operator on \( g \) with respect to the action \( (\mathfrak{h}; \phi) \), where \( \phi \) is the differentiated action of \( \Phi \) defined in (20).

**Proof.** For all \( u, v \in \mathfrak{h} \), we have

\[
[B(u), B(v)]_g = \left. \frac{d^2}{dt ds} \right|_{t,s=0} \exp_G(tB(u)) \cdot_G \exp_G(sB(v)) \cdot_G \exp_G(-tB(u))
\]

\[
= \left. \frac{d^2}{dt ds} \right|_{t,s=0} B(\exp_H(tu)) \cdot_G B(\exp_H(sv)) \cdot_G (B(\exp_H(tu)))^{-1}
\]

\[
= \left. \frac{d^2}{dt ds} \right|_{t,s=0} B(\exp_H(tu)) \cdot_G \Phi(\exp_H(tu)) \cdot_G B(\Phi(\exp_H(tu))^{-1}) \cdot_G \exp_H(-tu)
\]

\[
= \left. \frac{d^2}{dt ds} \right|_{t,s=0} B(\exp_H(tu)) \cdot_H \Phi(\exp_H(tu)) \cdot_H \exp_H(sv)
\]

\[
\cdot_H \Phi(\text{Ad}_{\exp_H(tu)}(\Phi(\exp_H(tu))) \exp_H(sv)) \cdot_H \exp_H(tu)
\]

\[
= B \ast e_G \left. \frac{d^2}{dt ds} \right|_{t,s=0} \text{Ad}_{\exp_H(tu)}(\Phi(\exp_H(tu))) \cdot_H \exp_H(tu)
\]

\[
\cdot_H \Phi(\text{Ad}_{\exp_H(tu)}(\Phi(\exp_H(tu))) \exp_H(sv)) \cdot_H \exp_H(-tu)
\]

\[
= B \ast e_G \left. \frac{d^2}{dt ds} \right|_{t,s=0} \text{Ad}_{\exp_H(tu)}(\Phi(\exp_H(tu))) \exp_H(sv)
\]
Thus, $B$ is a relative Rota–Baxter operator. \hfill \Box

The above result extends to the level of morphisms:

**Proposition 4.2.** Let $B$ and $B'$ be relative Rota–Baxter operators on a Lie group $G$ with respect to an action $(H; \Phi)$, and $(\Psi_G, \Psi_H)$ be a homomorphism from $B'$ to $B$. Then $(\Psi_G)_{se_G}, (\Psi_H)_{se_H}$ is a homomorphism from the relative Rota–Baxter operator $B'_{se_H}$ to $B_{se_H}$.

**Proof.** We denote $(\Psi_G)_{se_G}, (\Psi_H)_{se_H}$ by $(\psi_g, \psi_h)$ and $B'_{se_H} = B', B_{se_H} = B$. As $\Psi_G : G \to G$ and $\Psi_H : H \to H$ are Lie group homomorphisms, thus $\psi_g : g \to g$ and $\psi_h : h \to h$ are Lie algebra homomorphisms. By (17), we have $B \circ \psi_g = \psi_g \circ B'$. Then we have

$$\frac{d}{ds} \frac{d}{dt} \psi_H(\Phi(\exp_G(x)) \exp_H(tu)) = \psi_h \frac{d}{ds} \frac{d}{dt} \Phi(\exp_G(x)) \exp_H(tu) = \psi_h(\Phi(x)u)$$

and

$$\frac{d}{ds} \frac{d}{dt} \Phi(\psi_g(\exp_G(x))) \exp_H(tu) = \frac{d}{ds} \frac{d}{dt} \Phi(\exp_G(s\psi_g(x))) \exp_H(t\psi_h(u)) = \Phi(\psi_g(x)) \psi_h(u), \; \forall x \in g, u \in h.$$

By (18), we have $\psi_h(\Phi(x)u) = \phi(\psi_g(x)) \psi_h(u)$, which implies that $(\Psi_G)_{se_G}, (\Phi_H)_{se_H}$ is a homomorphism from $B'_{se_H}$ to $B_{se_H}$. \hfill \Box

Given Lie groups $G, H$ and their Lie algebras $g, h$, respectively, then the above differentiation procedure gives us a functor

$$\boxdot : \text{RB}_G \to \text{RB}_h$$

(22)
Actually, the set of homomorphisms between relative Rota–Baxter operators on Lie groups and the set of homomorphisms between the differentiated relative Rota–Baxter operators on the corresponding Lie algebras are isomorphic. First we need a lemma:

**Lemma 4.3.** Let $G$ and $H$ be connected Lie groups whose Lie algebras are $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let $\Phi : G \to \text{Aut}(H)$ be an action $G$ on $H$ and $\phi : \mathfrak{g} \to \text{Der}(\mathfrak{h})$ be the induced action of $\mathfrak{g}$ on $\mathfrak{h}$. Let $\Psi_G : G \to G$ and $\Psi_H : H \to H$ be Lie groups homomorphisms and $\psi_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g}$ and $\psi_\mathfrak{h} : \mathfrak{h} \to \mathfrak{h}$ be the induced Lie algebra homomorphisms. If $\psi_\mathfrak{g}$ and $\psi_\mathfrak{h}$ satisfies (5), then $\Psi_G$ and $\Psi_H$ satisfy (18), that is,

$$\Phi(\Psi_G(g))\Psi_H(h) = \Psi_H(\Phi(g)h), \quad \forall g \in G, h \in H.$$  

**Proof.** We first show that (18) holds for $g$ and $h$ in a small neighborhood of identities. In fact, by writing $g = \exp_G x$ and $h = \exp_H u$ and using (5) and Diagram (20), we have

$$\Psi_H(\Phi(\exp_G(x))\exp_H(u)) = \Psi_H(\exp_H(\Phi(\exp_G(x)u))) = \exp_H(\Psi_\mathfrak{h}(\Phi(\psi_\mathfrak{g}(x))\psi_\mathfrak{h}(u))) = \Phi(\Psi_G(\exp_G(x)))\exp_H(\psi_\mathfrak{h}(u)) = \Phi(\Psi_G(\exp_G(x)))\Psi_H(\exp_H(u)).$$

As $G$ and $H$ are connected, we can write $G = \bigcup_{n \in \mathbb{N}} U^n_G$ and $H = \bigcup_{n \in \mathbb{N}} U^n_H$, for some small open set $U_G \subseteq G$ and $U_H \subseteq H$ containing $e_G$ and $e_H$, respectively. Then $\Psi_H$ being a Lie group homomorphism and $\Phi$ being an action of Lie groups imply that for $g = \exp_G x$, and $h = h_1 \cdots h_k$, but each $h_j$ very small, (18) still holds; finally, the same facts imply that for $g = g_1 \cdots g_l$, with each $g_j$ very small, and any $h \in H$, (18) still holds. Thus, we have shown that (18) holds in general. 

**Theorem 4.4.** Let $G$ and $H$ be connected, simply connected Lie groups, $B'$ and $B$ be relative Rota–Baxter operators on $G$ with respect to the action $(H; \Phi)$. Then

$$\text{Hom}_{\text{RB}G}^*(B', B) \cong \text{Hom}_{\text{RB}B}^*(\mathfrak{D}(B'), \mathfrak{D}(B)).$$

**Proof.** For any $(\Psi_G, \Psi_H) \in \text{Hom}_{\text{RB}G}^*(B', B)$, we obtain $((\Psi_G)_{e_G}, (\Psi_H)_{e_H}) \in \text{Hom}_{\text{RB}B}^*(\mathfrak{D}(B'), \mathfrak{D}(B))$ via Proposition 4.2.

Conversely, for any $(\psi_\mathfrak{g}, \psi_\mathfrak{h}) \in \text{Hom}_{\text{RB}B}^*(\mathfrak{D}(B'), \mathfrak{D}(B))$, $(\psi_\mathfrak{g}, \psi_\mathfrak{h})$ is a Lie algebra endomorphism of the semidirect product Lie algebra $\mathfrak{g} \ltimes_{\phi} \mathfrak{h}$ and $(\psi_\mathfrak{g}, \psi_\mathfrak{h}) (\text{Gr}(B')) \subseteq \text{Gr}(B)$. By Lemma 4.3, there exist unique Lie group homomorphisms $\Psi_G : G \to G$ and $\Psi_H : H \to H$ such that $(\Psi_G, \Psi_H)$ is a Lie group endomorphism of the semidirect product Lie group $G \ltimes_{\phi} H$. As the image $(\Psi_G, \Psi_H)\text{Gr}(B')$ is a Lie subgroup of $G \ltimes_{\phi} H$ and its Lie algebra is $(\psi_\mathfrak{g}, \psi_\mathfrak{h})(\text{Gr}(B')) \subseteq \text{Gr}(B)$, it follows$^3$ that $(\Psi_G, \Psi_H)\text{Gr}(B') \subseteq \text{Gr}(B)$. This in turn implies that $\Psi_G \circ B' = B \circ \Psi_H$. Thus $(\Psi_G, \Psi_H) \in \text{Hom}_{\text{RB}G}^*(B', B)$, and $\text{Hom}_{\text{RB}G}^*(B', B) \cong \text{Hom}_{\text{RB}B}^*(\mathfrak{D}(B'), \mathfrak{D}(B)).$ 

---

$^3$ It is obvious that the inclusion happens locally near identity, the global statement follows again using the fact that both $(\Psi_G, \Psi_H)\text{Gr}(B')$ and $\text{Gr}(B)$ are connected and they can be written as products of elements near identity.
4.2 The Van Est homomorphism for the cohomologies

Now we extend the above differentiation to the level of cohomology. For this, let us consider the relation between the descendent Lie group of the relative Rota–Baxter operator $\mathcal{D}(B)$ and the descendent Lie algebra of the relative Rota–Baxter operator $\mathfrak{h}([\cdot, \cdot]_\mathcal{D}(B))$.

**Proposition 4.5.** Let $B$ be a relative Rota–Baxter operator on $G$ with respect to the action $(H; \Phi)$. Then the Lie algebra $(\mathfrak{h}, [\cdot, \cdot]_\star)$ of the descendent Lie group $(H, e_H, \star)$ is the descendent Lie algebra $(\mathfrak{h}, [\cdot, \cdot]_\mathcal{D}(B))$ of the relative Rota–Baxter operator $\mathcal{D}(B)$ on the level of Lie algebras.

**Proof.** We denote by $\exp$ the exponential map for the Lie group $(H, e_H, \star)$. For all $u, v \in \mathfrak{h}$, with the help of Proposition 3.5, we have

$$
[u, v]_\star = \left. \frac{d^2}{dt \, ds} \right|_{t, s = 0} \exp(tu) \star \exp(sv) \star \exp(-tu)
$$

$$
= \left. \frac{d^2}{dt \, ds} \right|_{t, s = 0} (\exp(tu) \cdot_H \Phi(B(\exp(sv)))\exp(sv)) \star \exp(-tu)
$$

$$
= \left. \frac{d^2}{dt \, ds} \right|_{t, s = 0} (\exp(tu) \cdot_H \Phi(B(\exp(sv)))\exp(sv)) \star \Phi(B(e_H(tu)))^{-1}(\exp(tu))^{-1}
$$

$$
= \left. \frac{d^2}{dt \, ds} \right|_{t, s = 0} (\exp(tu) \cdot_H \Phi(B(\exp(sv)))\exp(sv)) \cdot_H (\exp(tu) \cdot_H \Phi(B(\exp(\exp(tu))))(\exp(tu))^{-1})
$$

$$
= \left. \frac{d^2}{dt \, ds} \right|_{t, s = 0} (\exp(tu) \cdot_H \Phi(B(\exp(\exp(tu))))(\exp(tu))^{-1})
$$

$$
= \left. \frac{d^2}{dt \, ds} \right|_{t, s = 0} (\exp(tu) \cdot_H \Phi(B(\exp(\exp(tu))))(\exp(tu))^{-1})
$$

$$
= [u, v]_\mathfrak{h} + \phi(B_{e_H}(u))v + \left. \frac{d^2}{ds \, dt} \right|_{t, s = 0} \Phi(B(\exp(sv)))(\exp(tu))^{-1}
$$

$$
= [u, v]_\mathfrak{h} + \phi(B_{e_H}(u))v - \phi(\Phi(B(e_H(tu))))u
$$

Therefore, $[\cdot, \cdot]_\star = [\cdot, \cdot]_\mathcal{D}(B)$, which implies that the Lie algebra of the descendent Lie group $(H, e_H, \star)$ is the descendent Lie algebra of the relative Rota–Baxter operator $\mathcal{D}(B)$. □
By Theorem 3.8, we know that $\Theta$ is an action of the descendent Lie group $(H, e_H, \star)$ on $G$. By Lemma 3.9, $\Theta : H \to GL(g)$ is an action of the descendent Lie group $(H, e_H, \star)$ on the vector space $g$. Taking the differentiation, we have the following result.

**Proposition 4.6.** Let $B : H \to G$ be a relative Rota–Baxter operator on $G$ with respect to an action $(H; \Phi)$. Then the differentiation of the action $\Theta : H \to GL(g)$ of the descendent Lie group $(H, e_H, \star)$ on $g$ is exactly the representation $\tilde{\Theta} : h \to gl(g)$ of the descendent Lie algebra $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{D}(B)})$ on $\mathfrak{g}$ given in Proposition 2.7. That is

$$
\Theta_{e_H}(u)x = \mathfrak{D}(B)(\phi(x)u) + [\mathfrak{D}(B)(u), x]_g, \quad \forall x \in g, u \in \mathfrak{h}.
$$

**Proof.** By the definition of $\Theta$, and using the above notations in Propositions 4.2 and 4.5, we have

$$
\Theta_{e_H}(u)x = \left. \frac{d}{ds} \frac{d}{dt} \right|_{t,s=0} \Theta(e^t) \exp_G(sx)
$$

$$
= \left. \frac{d}{ds} \frac{d}{dt} \right|_{t,s=0} B(\Phi(\exp_G(sx))) \mathfrak{e}(tu) \exp_G(-tu) \cdot G \exp_G(sx) \cdot G B(\mathfrak{e}(tu))
$$

$$
= \left. \frac{d}{ds} \frac{d}{dt} \right|_{t,s=0} B(\mathfrak{e}(tu))^{-1} \cdot G \exp_G(sx) \cdot G B(\mathfrak{e}(tu))
$$

$$
+ \left. \frac{d}{dt} \frac{d}{ds} \right|_{t,s=0} B(\Phi(\exp_G(sx))) \mathfrak{e}(tu) \exp_G(-tu) \cdot G B(\mathfrak{e}(tu))
$$

$$
= [\mathfrak{D}(B)(u), x]_g + \left. \frac{d}{ds} \frac{d}{dt} \right|_{t,s=0} B(\Phi(\exp_G(sx))) \mathfrak{e}(tu) \exp_G(-tu)
$$

$$
= [\mathfrak{D}(B)(u), x]_g + \mathfrak{D}(B)(\phi(x)u),
$$

which implies that $\Theta_{e_H} = \tilde{\Theta}$.\qed

Let $B : H \to G$ be a relative Rota–Baxter operator on $G$ with respect to an action $(H; \Phi)$, we define $\mathcal{V}E : C^k(B) \to C^k(\mathfrak{D}(B))$ by

$$
\mathcal{V}E(F)(u_1, \ldots, u_{k-1}) = \sum_{s \in S(k-1)} (-1)^{|s|} \left. \frac{d}{dt_s(1)} \ldots \frac{d}{dt_s(k-1)} \right|_{t_{s(1)} = t_{s(2)} = \ldots = t_{s(k-1)} = 0} F(\mathfrak{e}(tu_{s(1)}), \ldots, \mathfrak{e}(tu_{s(k-1)})),
$$

for all $F \in C^k(B), u_1, \ldots, u_{k-1} \in \mathfrak{h}$.

**Theorem 4.7.** With the above notations, $\mathcal{V}E$ is a cochain map, that is, we have the following commutative diagram

$$
\cdots \rightarrow c^k(\mathfrak{g}) \xrightarrow{d^g} c^{k+1}(\mathfrak{g}) \xrightarrow{d^g} c^{k+2}(\mathfrak{g}) \xrightarrow{d^g} \cdots
$$

$$
\downarrow \mathcal{V}E \quad \downarrow \mathcal{V}E \quad \downarrow \mathcal{V}E
$$

$$
\cdots \rightarrow c^k(\mathfrak{D}(\mathfrak{g})) \xrightarrow{d^g} c^{k+1}(\mathfrak{D}(\mathfrak{g})) \xrightarrow{d^g} c^{k+2}(\mathfrak{D}(\mathfrak{g})) \xrightarrow{d^g} \cdots
$$

$$
\downarrow \mathcal{V}E \quad \downarrow \mathcal{V}E \quad \downarrow \mathcal{V}E
$$

$$
\cdots \rightarrow c^k(\mathfrak{D}(\mathfrak{g})) \xrightarrow{d^\mathfrak{D}(\mathfrak{g})} c^k(\mathfrak{D}(\mathfrak{g})) \xrightarrow{d^\mathfrak{D}(\mathfrak{g})} c^k(\mathfrak{D}(\mathfrak{g})) \xrightarrow{d^\mathfrak{D}(\mathfrak{g})} \cdots
$$

\text{Diagram}

\text{Diagram}

\text{Diagram}
Consequently, \( \mathcal{V} \mathcal{E} \) induces a homomorphism \( \mathcal{V} \mathcal{E}_* \) from the cohomology group \( H^k(B) \) to \( H^k(\mathfrak{S}(B)) \). The map \( \mathcal{V} \mathcal{E} \) is called the Van Est map.

**Proof.** By Proposition 4.5, the differentiation of the descendant Lie group \((H, e_H, \star)\) is the descendant Lie algebra \((\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{S}(B)})\). By Proposition 4.6, the differentiation of the action \( \Theta \) of the descendant Lie group \((H, e_H, \star)\) is the representation \( \Theta \) of the descendant Lie algebra \((\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{S}(B)})\). As the cochains \( C^k(B) \) and \( C^k(\mathfrak{S}(B)) \) are in fact exactly those of the descendant Lie group \((H, e_H, \star)\) and the descendant Lie algebra \((\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{S}(B)})\), respectively, the conclusion follows from the classical argument for the cohomologies of Lie groups and Lie algebras. □

Just as the classical situation, under certain conditions, the cohomology group \( \mathcal{H}^k(B) \) and \( H^k(\mathfrak{S}(B)) \) are isomorphic.

**Theorem 4.8.** If the Lie group \((H, e_H, \cdot_H)\) is connected and its homotopy groups are trivial in \( 1, 2, \ldots, n \), then for \( 1 \leq k \leq n \), the cohomology group \( \mathcal{H}^k(B) \) of the relative Rota–Baxter operator \( B \) on the Lie group \( G \) is isomorphic to the cohomology group \( H^k(\mathfrak{S}(B)) \) of the relative Rota–Baxter operator on the level of Lie algebras.

**Proof.** As for the previous theorem, the conclusion also follows from the classical argument for the cohomologies of Lie groups and Lie algebras. □

At the end of this section, we give a concrete example to demonstrate the differentiation procedure, and we also calculate a second cohomology group in this example.

**Example 4.9.** Consider the Euclidean Lie group

\[
G = \{ X | X = \begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix}, A \in \text{SO}(n), \alpha \in \mathbb{R}^n \}.
\]

Then we have \( G = G_1 \ast G_2 \), where \( G_1 = \{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} | A \in \text{SO}(n) \} \) and \( G_2 = \{ \begin{pmatrix} I_{n \times n} & \alpha \\ 0 & 1 \end{pmatrix} | \alpha \in \mathbb{R}^n \} \). As any \( \begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix} \) can be written as \( \begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_{n \times n} & \alpha \\ 0 & 1 \end{pmatrix} \), by Example 3.2, the map \( B : G \rightarrow G \) defined by

\[
B \left( \begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} I_{n \times n} & -A^T \alpha \\ 0 & 1 \end{pmatrix}, \quad \forall \begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix} \in G
\]

is a Rota–Baxter operator on \( G \). By Proposition 3.5, the multiplication \( \star \) and the inverse in the descendent Lie group \((G, \star)\) is given by

\[
\begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix} \star \begin{pmatrix} C & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AC & ACA^T \alpha + A\beta \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix}^\dagger = \begin{pmatrix} A^T & -(A^T)^2 \alpha \\ 0 & 1 \end{pmatrix}.
\]

It is straightforward to deduce that \( G \) is the direct product of \( G_1 \) and \( G_2 \), that is, \( G = G_1 \ast G_2 \), where \( G_1 = \{ Y | Y = \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}, C \in \text{SO}(n) \} \) and \( G_2 = \{ Z | Z = \begin{pmatrix} I & \beta \\ 0 & 1 \end{pmatrix}, \beta \in \mathbb{R}^n \} \). By Theorem 3.8,
the action $\Theta$ of the descendent Lie group $(G, \star)$ on the manifold $G$ is given by

$$\Theta\left(\begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} C & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} C & CAC^T \beta \\ 0 & 1 \end{pmatrix}.$$ 

and the induced representation, which we use the same notation $\Theta : G \to GL(\mathfrak{g})$ is given by

$$\Theta\left(\begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x & u \\ 0 & 0 \end{pmatrix} = \frac{d}{dt}\bigg|_{t=0} \Theta\left(\begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} \exp(tx) & tu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & Au \\ 0 & 0 \end{pmatrix}.$$ 

where the Lie algebra $\mathfrak{g}$ of the Euclidean Lie group $G$ is given by

$$\mathfrak{g} = \{ xu \}_{x \in \mathfrak{so}(n), u \in \mathbb{R}^n}.$$ 

Then the differentiation $\mathfrak{D}(B)$ is given by

$$\mathfrak{D}(B)\left(\begin{pmatrix} x & u \\ 0 & 0 \end{pmatrix}\right) = \frac{d}{dt}\bigg|_{t=0} B\left(\begin{pmatrix} \exp(tx) & tu \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & -u \\ 0 & 0 \end{pmatrix}. \forall \begin{pmatrix} x & u \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}.$$ 

By Proposition 4.5, the descendent Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{D}(B)})$ is given by

$$\left[\begin{pmatrix} x & u \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} y & v \\ 0 & 0 \end{pmatrix}\right]_{\mathfrak{D}(B)} = \begin{pmatrix} xy - yx & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Therefore, the descendent Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{D}(B)})$ is the direct sum of the Lie algebra $\mathfrak{so}(n)$ and the abelian Lie algebra $\mathbb{R}^n$, and obviously $\mathfrak{so}(n)$ is an ideal. By Proposition 4.6, the representation $\theta : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ of the descendent Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{D}(B)})$ on $\mathfrak{g}$ is given by

$$\theta\left(\begin{pmatrix} x & u \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} y & v \\ 0 & 0 \end{pmatrix} = \frac{d}{dt}\bigg|_{t=0} \Theta\left(\begin{pmatrix} \exp(tx) & tu \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} y & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & xv \\ 0 & 0 \end{pmatrix}.$$ 

Denote by $\mathfrak{g}^{\mathfrak{so}(n)} = \{ \xi \in \mathfrak{g} | \theta\left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right) = 0, \forall x \in \mathfrak{so}(n) \}$. Then it is obvious that $\mathfrak{g}^{\mathfrak{so}(n)} = \{ \mathfrak{so}(n) \}$. With above preparations, using the exact sequences of low degree terms in the Hochschild–Serre spectral sequence, we obtain

$$0 \to H^1(\mathfrak{g}/\mathfrak{so}(n), \mathfrak{g}^{\mathfrak{so}(n)}) \to H^1(\mathfrak{g}, \mathfrak{g}) \to H^1(\mathfrak{so}(n), \mathfrak{g})^{\mathfrak{g}/\mathfrak{so}(n)} \to H^2(\mathfrak{g}/\mathfrak{so}(n), \mathfrak{g}^{\mathfrak{so}(n)}) \to H^2(\mathfrak{g}, \mathfrak{g}).$$

As $\mathfrak{so}(n)$ is a semisimple Lie algebra, we have $H^1(\mathfrak{so}(n), \mathfrak{g})^{\mathfrak{g}/\mathfrak{so}(n)} = 0$, thus $H^1(\mathfrak{g}/\mathfrak{so}(n), \mathfrak{g}^{\mathfrak{so}(n)}) \cong H^1(\mathfrak{g}, \mathfrak{g})$. As $\mathfrak{g}/\mathfrak{so}(n) = \mathbb{R}^n$ is abelian and the representation is trivial, it follows that

$$H^1(\mathfrak{g}/\mathfrak{so}(n), \mathfrak{g}^{\mathfrak{so}(n)}) = \text{Hom}(\mathbb{R}^n, \mathfrak{so}(n)) \cong \mathbb{R}^{\frac{n^2(n-1)}{2}}.$$ 

Therefore, we have $H^2(\mathfrak{D}(B)) = H^1(\mathfrak{g}, \mathfrak{g}) \cong \mathbb{R}^{\frac{n^2(n-1)}{2}}$. 
5 | LOCAL INTEGRATION OF RELATIVE ROTA–BAXTER OPERATORS ON LIE ALGEBRAS

In this section, we introduce the notion of a local relative Rota–Baxter operator on a Lie group and show that any relative Rota–Baxter operator on a Lie algebra can be integrated to a local relative Rota–Baxter operator on the Lie group integrating the Lie algebra \( \mathfrak{g} \).

**Definition 5.1.** Let \( \Phi : G \rightarrow \text{Aut}(H) \) be an action of \( G \) on \( H \). If there exists an open neighborhood \( U \) of \( e_H \) in \( H \) and a smooth map \( B : U \rightarrow G \) such that

\[
B(h_1 \cdot_G B(h_2)) = B(h_1 \cdot_H \Phi(B(h_1))h_2), \quad h_1, h_2 \in U,
\]

whenever the element \( h_1 \cdot_H \Phi(B(h_1))h_2 \) is also in \( U \), then \( B \) is called a local relative Rota–Baxter operator on \( G \) with respect to the action \( (H; \Phi) \).

In particular, if \( H = G \) and the action is the adjoint action of \( G \) on itself, then we call \( B \) a local Rota–Baxter operator.

Two local relative Rota–Baxter operators \( B : U \rightarrow G \) and \( B' : U' \rightarrow G \) are defined to be equal if there exists an open neighborhood \( \bar{U} \subset U \cap U' \) of \( e_H \), such that \( B|_{\bar{U}} = B'|_{\bar{U}} \).

**Definition 5.2.** Let \( B : U \rightarrow G \) and \( B' : U' \rightarrow G \) be two local relative Rota–Baxter operators on \( G \) with respect to an action \( (H; \Phi) \). A homomorphism from \( B' \) to \( B \) consists of Lie group homomorphisms \( \Psi_G : G \rightarrow G \) and \( \Psi_H : H \rightarrow H \) such that (17) holds in an open neighborhood \( \bar{U} \subset U \cap U' \) of \( e_H \) and (18) holds.

5.1 | Local integration functor and adjointness

It is clear that local relative Rota–Baxter operators on a Lie group \( G \) with respect to an action \( (H; \Phi) \) together with homomorphisms between them also form a category, which we denote by \( \text{LRB}_G^H \). As the differentiation functor established in the previous section depends only on the local information near the identity element, it induces a differentiation functor,

\[
\mathfrak{D} : \text{LRB}_G^H \rightarrow \text{RB}_\mathfrak{g}^\mathfrak{h},
\]

which we use the same letter by abusing the notations.

Let \( B : \mathfrak{h} \rightarrow \mathfrak{g} \) be a relative Rota–Baxter operator on \( \mathfrak{g} \) with respect to an action \( (\mathfrak{h}; \phi) \). Let \( G \) and \( H \) be Lie groups (not necessarily connected and simply connected) integrating \( \mathfrak{g} \) and \( \mathfrak{h} \), respectively. If there is an action \( \Phi : G \rightarrow \text{Aut}(H) \) whose differentiation is \( \phi \) (see (20)), and a local relative Rota–Baxter operator \( B \) on \( G \) with respect to \( \Phi \) such that \( \mathfrak{D}(B) = B \), then we call that \( B \) integrates \( \mathfrak{b} \).

**Theorem 5.3.** Let \( B : \mathfrak{h} \rightarrow \mathfrak{g} \) be a relative Rota–Baxter operator on a Lie algebra \( \mathfrak{g} \) with respect to an action \( (\mathfrak{h}; \phi) \). Let \( G \) and \( H \) be the connected and simply connected Lie groups integrating \( \mathfrak{g} \) and \( \mathfrak{h} \), respectively, and \( \Phi : G \rightarrow \text{Aut}(H) \) be the integrated action (see the paragraph above (20)). Then there is a unique local relative Rota–Baxter operator \( B \) on \( G \) with respect to the action \( (H; \Phi) \) integrating \( B \).
Proof. We consider the semidirect product Lie group $G \ltimes_{\Phi} H$, with multiplication $\cdot_{\Phi}$ given by

$$(g_1, h_1) \cdot_{\Phi} (g_2, h_2) = (g_1 \cdot_G g_2, h_1 \cdot_H \Phi(g_1)h_2), \quad \forall g_i \in G, h_i \in H, i = 1, 2.$$ 

Its Lie algebra is the semidirect product Lie algebra $\mathfrak{g} \ltimes_{\phi} \mathfrak{h}$, and by Proposition 2.4, $Gr(B)$ is a Lie subalgebra of $\mathfrak{g} \ltimes_{\Phi} \mathfrak{h}$. Thus, there exists a connected Lie subgroup $E$ of $G \ltimes_{\Phi} H$ such that its Lie algebra is $Gr(B)$.

Define a smooth map $P_H : E \to H$ by $P_H(g, h) = h$ for all $(g, h) \in E$. Then its tangent map at the identity $P_{H*} : Gr(B) \to \mathfrak{h}$ is given by

$$P_{H*}(B(u), u) = u, \quad \forall u \in \mathfrak{h},$$ 

which is an isomorphism from the vector space $Gr(B)$ to the vector space $\mathfrak{h}$. Thus there exists an open set $V \subseteq E$ such that $P_H|_V$ is an isomorphism of manifolds, which implies that there is an open set $U$ of $H$ that contains $e_H$ and a smooth map $B : U \to G$, such that $V \cong Gr(B)$.

For all $h_1, h_2 \in U$, we have

$$(B(h_1), h_1) \cdot_{\Phi} (B(h_2), h_2) = (B(h_1) \cdot_G B(h_2), h_1 \cdot_H \Phi(B(h_1))h_2).$$ 

Therefore, as soon as $h_1 \cdot_H \Phi(B(h_1))h_2 \in U$, we have

$$(B(h_1) \cdot_G B(h_2), h_1 \cdot_H \Phi(B(h_1))h_2) \in Gr(B),$$ 

which implies that $B(h_1) \cdot_G B(h_2) = B(h_1) \cdot_H \Phi(B(h_1))h_2).$ Therefore, $B : U \to G$ is a local relative Rota–Baxter operator. Furthermore, as $Gr(B) \subseteq E$, and the Lie algebra of $E$ is $Gr(B)$, it follows that $B_* = B$, and $B$ is an integration of $B$.

Now we show the uniqueness of $B$ through an explicit formula. Denote by Exp the exponential map for the Lie group $G \ltimes_{\Phi} H$, and by $P_H$ the projection $G \ltimes_{\Phi} H \to H$. For all $x \in \mathfrak{g}, u \in \mathfrak{h}$, it is obvious that

$$Exp(x, u) = (exp_G x, P_H Exp(x, u)).$$ 

As the Lie algebra of the Lie subgroup $E$ is $Gr(B)$, it follows that locally $Exp(B(u), u) \in Gr(B) \subseteq E$. Therefore,

$$B(P_H(Exp(B(u), u))) = \exp_G B(u). \quad (26)$$ 

As $P_H \circ Exp : Gr(B) \to H$ is a local isomorphism in the neighborhood of the identity, the uniqueness of $B$ naturally follows.

Now we extend the integration to the level of morphisms and establish a functor

$$\mathfrak{S} : \text{RB}^B_\mathfrak{g} \to \text{LRB}^H_\mathfrak{G}.$$ 

Theorem 5.4. Let $B$ and $B'$ be relative Rota–Baxter operators on a Lie algebra $\mathfrak{g}$ with respect to an action $(\mathfrak{h}; \Phi)$ and $\psi = (\psi_{\mathfrak{g}}, \psi_{\mathfrak{h}})$ be a homomorphism from $B'$ to $B$. Let $G$ and $H$ be connected and simply connected Lie groups integrating $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and let $B' : U' \to G$ and $B : U \to G$

\[\text{The map itself is also defined locally therein.}\]
be the integrated local relative Rota–Baxter operators of $B'$ and $B$, respectively, as in the previous theorem. Let $\Psi_G$ and $\Psi_H$ be the Lie group homomorphisms integrating the Lie algebra homomorphisms $\psi_\mathfrak{g}$ and $\psi_\mathfrak{h}$, respectively. Then $(\Psi_G, \Psi_H)$ is a homomorphism from $B'$ to $B$. Consequently, we obtain a functor

$$\mathcal{F} : \text{RB}^b_\mathfrak{g} \to \text{LRB}^H_G.$$  \hspace{1cm} (27)

**Proof.** To prove that $(\Psi_G, \Psi_H)$ is a homomorphism from $B'$ to $B$, we need to show Equation (18), and a local version of Equation (17). Equation (18) does not involve relative Rota–Baxter operators and follows directly from Lemma 4.3.

Now to show Equation (17) holds locally, we need to show that $B \circ \Psi_H = \Psi_G \circ B'$ holds in an open neighborhood $U \subset U \cap U'$ of $e_H$. This follows from the explicit formula (26), the corresponding infinitesimal condition (4), and the fact that $(\Psi_G, \Psi_H) : G \ltimes \mathfrak{h} \to G \ltimes \mathfrak{h}$ is the integrated homomorphism of the Lie algebra homomorphism $(\psi_\mathfrak{g}, \psi_\mathfrak{h}) : \mathfrak{g} \ltimes \mathfrak{h} \to \mathfrak{g} \ltimes \mathfrak{h}$ (i.e., Lemma 4.3). More precisely, we have

$$\begin{align*}
\Psi_G \left( B' \left( P_H(\exp(B'(u), u)) \right) \right) &= \Psi_G(\exp_G B'(u)) = \exp_G(\psi_\mathfrak{h} B'(u)) = \exp_G(B(\psi_\mathfrak{h}(u))) \\
&= B(P_H(\exp(B(\psi_\mathfrak{h}(u)), \psi_\mathfrak{h}(u)))) = B(P_H(\exp(B'(u)), \psi_\mathfrak{h}(u)))) \\
&= B(P_H(\Psi_G, \Psi_H)(\exp(B'(u), u))) = B \Psi_H(P_H(\exp(B'(u), u))).
\end{align*}$$

Thus, we obtain

$$\Psi_G \circ B' = B \circ \Psi_H,$$

which implies that $(\Psi_G, \Psi_H)$ is a homomorphism from $B'$ to $B$. Then it is straightforward to see that $\mathcal{F}$ is a functor. \hfill \Box

**Theorem 5.5.** The integration functor $\mathcal{F}$ in (27) and the differentiation functor $\mathcal{D}$ in (25) are adjoint functors. More precisely, $\mathcal{F}$ is left adjoint to $\mathcal{D}$ and $\mathcal{D}$ is right adjoint to $\mathcal{F}$.

**Proof.** We need to show that, there is an isomorphism $\alpha_{B'B}$, such that

$$\alpha_{B'B} : \text{Hom}_{\text{LRB}^b_{\mathfrak{g}}}(\mathcal{F}(B'), B) \cong \text{Hom}_{\text{RB}^b_{\mathfrak{g}}}(B', \mathcal{D}(B)), \quad \forall B' \in \text{RB}^b_{\mathfrak{g}}, B \in \text{LRB}^H_{\mathfrak{g}},$$  \hspace{1cm} (28)

and $\alpha$ is bi-natural in $B'$ and in $B$, that is, when fixing $B$, $\alpha_{-B}$ is a natural isomorphism between functors $\text{Hom}_{\text{LRB}^b_{\mathfrak{g}}}(\mathcal{F}(-), B)$ and $\text{Hom}_{\text{RB}^b_{\mathfrak{g}}}(\mathcal{F}(-), \mathcal{D}(B))$; and when fixing $B'$, $\alpha_{B'\mathcal{F}}$ is a natural isomorphism $\text{Hom}_{\text{LRB}^b_{\mathfrak{g}}}(\mathcal{F}(B'), -) \cong \text{Hom}_{\text{RB}^b_{\mathfrak{g}}}(B', \mathcal{D}(-))$.

We see that by the definition of $\mathcal{F}$ in Theorem 5.4, $G$ and $H$ are required to be connected and simply connected. Given a morphism $\Psi := (\Psi_G, \Psi_H) \in \text{Hom}_{\text{LRB}^b_{\mathfrak{g}}}(\mathcal{F}(B'), B)$, using the construction in Proposition 4.2, we define $\alpha_{B'B}$ by

$$\alpha_{B'B} : \Psi \mapsto \mathcal{D}(\Psi) := \left( (\Psi_G)_* e_G, (\Psi_H)_* e_H \right) \in \text{Hom}_{\text{RB}^b_{\mathfrak{g}}}(B', \mathcal{D}(B)).$$

By Theorem 5.4, $\alpha_{B'B}$ is an isomorphism of sets with the inverse given by

$$\text{Hom}_{\text{RB}^b_{\mathfrak{g}}}(B', \mathcal{D}(B)) \ni (\phi_\mathfrak{g}, \phi_\mathfrak{h}) \mapsto (\Psi_G, \Psi_H) \in \text{Hom}_{\text{LRB}^b_{\mathfrak{g}}}(\mathcal{F}(B'), B),$$

and $\alpha$ is bi-natural in $B'$ and in $B$, that is, when fixing $B$, $\alpha_{-B}$ is a natural isomorphism between functors $\text{Hom}_{\text{LRB}^b_{\mathfrak{g}}}(\mathcal{F}(-), B)$ and $\text{Hom}_{\text{RB}^b_{\mathfrak{g}}}(\mathcal{F}(-), \mathcal{D}(B))$; and when fixing $B'$, $\alpha_{B'\mathcal{F}}$ is a natural isomorphism $\text{Hom}_{\text{LRB}^b_{\mathfrak{g}}}(\mathcal{F}(B'), -) \cong \text{Hom}_{\text{RB}^b_{\mathfrak{g}}}(B', \mathcal{D}(-))$. 

where $\Psi_G$ and $\Psi_H$ are the Lie group homomorphisms integrating $\phi_g$ and $\phi_h$, respectively. The bi-naturality of $\alpha$ basically follows from the fact that $\alpha_{B'B}$ preserves the composition, which in turn follows from the functoriality of $\mathfrak{D}$. We now give a detailed proof for the naturality in $B'$. The naturality in $B$ is similar and we leave it to interested readers. Let $f': B'_1 \to B'_2$ be a morphism in $RB^h_B$. The naturality of $\alpha_{-B}$ follows from the commutativity of the diagram,

\[
\begin{array}{ccc}
\text{Hom}_{LBB^h}(\mathfrak{A}(B'_1), B) & \xrightarrow{\alpha_{B'_1}^h} & \text{Hom}_{RB^h}(B'_1, \mathfrak{D}(B)) \\
\mathfrak{A}(f')^* & & f^* \\
\text{Hom}_{LBB^h}(\mathfrak{A}(B'_2), B) & \xrightarrow{\alpha_{B'_2}^h} & \text{Hom}_{RB^h}(B'_2, \mathfrak{D}(B)),
\end{array}
\]

where $^*$ denotes the precomposition (or pullback). In fact, for all $\Psi \in \text{Hom}_{LBB^h}(\mathfrak{A}(B'_1), B)$, we have

\[
\alpha_{B'_1}^h \mathfrak{A}(f')^*(\Psi) = \alpha_{B'_2}^h (\Psi \circ \mathfrak{A}(f')) = \mathfrak{D}(\Psi \circ \mathfrak{A}(f')) = \mathfrak{D}(\Psi) \circ f' = f'^* \alpha_{B'_2}^h (\Psi),
\]

which implies that $\alpha_{-B}$ is natural. □

6 | GEOMETRIC APPLICATIONS

In this section, we give some geometric applications including the explicit expression of the local factorization of a Lie group given in [33] and the integration of matched pairs of Lie algebras given by Rota–Baxter operators.

6.1 | Local descendent Lie group

As in the global case, a local Rota–Baxter operator also determines a local descendent Lie group. This turns out to be important to our geometric application. Thus, we make this construction here in this subsection.

While the definition of a Lie group is standard, the definition of a local Lie group varies slightly. Thus, we first fix our definition that is taken mostly from the classical one in [31], and however with a slight modification similar to the more recent paper [29].

Definition 6.1 [29, 31]. A local Lie group $L$ is a manifold equipped with an open set $V \subset L$ together with

(i) a multiplication map $m : V \times V \to L$, which satisfies $g_1(g_2g_3) = (g_1g_2)g_3$, as long as $g_1, g_2, g_3, g_1g_2, g_2g_3 \in V$ (here we write $m(g_1, g_2)$ as $g_1g_2$ for short);
(ii) an identity element $e \in V$, which satisfies $eg = ge = g$, for all $g \in V$;
(iii) an inverse map $i : V \to V$, such that $i(V) = V$, and $g^{-1}g = e = gg^{-1}$ (here we write $i(g)$ as $g^{-1}$ for short).

Theorem 6.2. Let $B : U \to G$ be a local Rota–Baxter operator on the Lie group $(G, e, \cdot)$. Then it gives rise to a local Lie group structure on $G$:
the multiplication $\star$ is given by
\begin{equation}
g_1 \star g_2 := g_1 \cdot B(g_1) \cdot g_2 \cdot B(g_1)^{-1}, \tag{31}
\end{equation}

the identity element is $e$, which is the identity element of the Lie group $G$,

the inverse $g^\dagger$ of $g$ is given by
\begin{equation}
g^\dagger := B(g)^{-1} \cdot g^{-1} \cdot B(g). \tag{32}
\end{equation}

Proof. Notice that the inverse is defined from $U$ to $G$, thus we may take $V := U \cap U^\dagger$. Clearly, $V$ is an open set of $G$ containing $e$. Note that $\star : U \times U \rightarrow G$ is well-defined, thus $\star$ is well-defined also on $V \times V$.

If $g_1, g_2, g_3, g_1 \star g_2, g_2 \star g_3 \in V$, then the same calculation (16) for proving the associativity in Proposition 3.5 still goes through because all the relevant elements are in $V$ which is within the definition domain of $B$.

By (24), for all $g \in U$, we have $B(g) \cdot B(g^\dagger) = B(g \cdot B(g) \cdot g^\dagger \cdot B(g)^{-1}) = B(e) = e$. This shows that $B(g^\dagger) = B(g)^{-1}$. Thus,
\begin{equation}
(g^\dagger)^\dagger = B(g) \cdot (g^\dagger)^{-1} \cdot B(g)^{-1} = g. \tag{33}
\end{equation}

Therefore, $(U^\dagger)^\dagger = U$, which implies that $(V)^\dagger = V$. Moreover, for all $g \in V, g^\dagger \star g = e = g \star g^\dagger$ follows from the same calculation (15) in Proposition 3.5 because $B(g^\dagger) = B(g)^{-1}$ still holds for $g \in U$. Thus, $(V \subset G, \star, \dagger)$ is a local Lie group. \hfill $\square$

This local Lie group is called the descendent local Lie group of $B$.

Remark 6.3. Similar to the case of the descendent Lie group, it is clear that

• the Lie algebra of the descendent local Lie group $(V \subset G, \star, \dagger)$ is the descendent Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_B)$, where $B = \mathfrak{D}(B)$;
• $B : (V \subset G, \star, \dagger) \rightarrow G$ is a local Lie group homomorphism. This in particular implies that $\operatorname{Im}(B)$ and $G_-$ are locally isomorphic near the identity, where $G_-$ is the Lie group integrating the Lie algebra $\mathfrak{g}_- := \operatorname{Im}(B)$.

6.2 Application in the local factorization problem

As mentioned in Subsection 2.4, Rota–Baxter operators $B$ on a Lie algebra $\mathfrak{g}$ one-to-one correspond to modified $r$-matrices $R$. The study of integration of such modified $r$-matrices gives arise to the Adler–Kostant–Symes (AKS) theory [33]. The AKS theory [22] allows one to construct an explicit integral curve $L(t)$ of certain Hamiltonian function $h$ on $\mathfrak{g}^*$, by
\begin{equation}
L(t) = \operatorname{Ad}^*_{g_+(t)} L_0 = \operatorname{Ad}^*_{g_-(t)} L_0, \tag{34}
\end{equation}

with initial value $L_0 \in \mathfrak{g}^*$, as long as we know the solution $(g_-(t), g_+(t))$ of the local factorization problem, which in turn comes from a split (i.e., an infinitesimal factorization) of $\mathfrak{g}$. However, usually, the solution $(g_-(t), g_+(t))$ is not explicitly given. With the help of the local integration of Rota–Baxter operators, we can give such a local factorization explicitly.
To state this more precisely, let us first recall some facts adapted to the Rota–Baxter operators context from [17, 33], and we refer the readers to the references therein for the original references. Let $B$ be a Rota–Baxter operator on $\mathfrak{g}$. Then it gives rise to an infinitesimal factorization (or a split) [33, Proposition 9] of $\mathfrak{g}$ as follows. Denote by $\mathfrak{g}_+ = \text{Im}(B + \text{Id})$, $\mathfrak{g}_- = \text{Im}(B)$. We further denote by $\mathfrak{f}_+ := \ker(B)$ and $\mathfrak{f}_- := \ker(B + \text{Id})$. As both $B$ and $B + \text{Id}$ are Lie algebra homomorphisms from the descendant Lie algebra $\mathfrak{g}_B$ to $\mathfrak{g}$, $\mathfrak{g}_+\mathfrak{g}_-$, $\mathfrak{f}_+$ and $\mathfrak{f}_-$ are Lie subalgebras. Moreover, $\mathfrak{f}_+ \subset \mathfrak{g}_+$ are Lie ideals, and $\vartheta : \mathfrak{g}_+ / \mathfrak{f}_+ \to \mathfrak{g}_- / \mathfrak{f}_-$ given by $(B + \text{Id})(u) \mapsto B(u)$ on the representatives of equivalence classes is well-defined and is a Lie algebra isomorphism. The map $\vartheta$ is called the Cayley transform of $B$. Define $\mathfrak{g}_\vartheta = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ by $\mathfrak{g}_\vartheta = \{(x, y) \in \mathfrak{g}_+ \oplus \mathfrak{g}_- \text{ such that } \vartheta(x) = y\}$. Then any $x \in \mathfrak{g}$ can be uniquely expressed as $x = x_+ - x_-$ for $(x_+, x_-) \in \mathfrak{g}_\vartheta$.

Let $G$ be a Lie group of $\mathfrak{g}$, and let $G_\pm$ and $K_\pm$ be the Lie subgroups of $G$ integrating the above Lie subalgebras $\mathfrak{g}_\pm$ and $\mathfrak{f}_\pm$ correspondingly. Then the Cayley transform $\vartheta$ integrates also to the group level into a (local) Lie group homomorphism $\Theta : G_+ / K_+ \to G_- / K_-$, by Lie’s II Theorem. Let us denote by $\tilde{g}$ the equivalence class of $g$. Then the solution $(g_+(t), g_-(t)) \in G_+ \times G_-$ of the (local) factorization problem in [33, Theorem 11] is the solution of the following equations, 

$$
\exp 2tX_0 = g_+(t) \cdot g_-(t)^{-1}, \quad \Theta(\tilde{g}_+(t)) = \tilde{g}_-(t), \quad \text{where } X_0 = \frac{dh}{|L_0|} \in \mathfrak{g},
$$

with the initial value $g_\pm(0) = e \in G$, for sufficiently small $t$.

**Theorem 6.4.** Let $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, B)$ be a Rota–Baxter Lie algebra. Then the above solution $(g_+, g_-)$ of the (local) factorization problem has an explicit expression,

$$
g_-(t) = B(\exp 2tX_0), \quad g_+(t) = \exp 2tX_0 \cdot B(\exp 2tX_0),
$$

for sufficiently small $t$, with $B$ given explicitly by (26) locally.

**Proof.** Let $(G, \cdot, B)$ be the integrated local Rota–Baxter Lie group of $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, B)$ given in Theorem 5.3. By Remark 6.3, $\text{Im}(B)$ and $G_-$ are locally isomorphic. Define $B_+ : U \to G$ by $B_+(g) = g \cdot B(g)$.

By the same discussion in [17, Proposition 3.1], $B_+$ is a local Lie group homomorphism from the descendent local Lie group to $G$. As $\text{Im}(B_+)$ and $G_+$ have the same Lie algebra $\mathfrak{g}_+$, they are also locally isomorphic. Obviously, locally we have $g = g \cdot B(g) \cdot (B(g))^{-1}, \quad g \cdot B(g) \in \text{Im}(B_+), \quad B(g) \in \text{Im}(B)$.

Therefore, we have $\exp 2tX_0 = g_+(t) \cdot g_-(t)^{-1}$, where $g_+(t)$ and $g_-(t)$ are given by (35).

6.3 | Application in the integration of matched pairs

A matched pair of Lie algebras consists of a pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$, a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ of $\mathfrak{g}$ on $\mathfrak{h}$ and a representation $\mu : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$ of $\mathfrak{h}$ on $\mathfrak{g}$ such that

$$
\rho(\xi) [\xi, \eta]_\mathfrak{h} = [\rho(\xi) \xi, \eta]_\mathfrak{h} + [\xi, \rho(\eta) \eta]_\mathfrak{h} + \rho((\mu(\eta) x) \xi) - \rho(\mu(\xi) x) \eta,
$$

(36)
\[
\mu(\xi)[x,y]_\mathfrak{g} = [\mu(\xi)x,y]_\mathfrak{g} + [x,\mu(\xi)y]_\mathfrak{g} + \mu(\rho(y)\xi)x - \mu(\rho(x)\xi)y,
\] (37)

for all \(x, y \in \mathfrak{g}\) and \(\xi, \eta \in \mathfrak{h}\).

It is well-known that if there is a Lie algebra \(\mathfrak{f}\) such that both \(\mathfrak{g}\) and \(\mathfrak{h}\) are Lie subalgebras, and \(\mathfrak{f}\) is isomorphic to \(\mathfrak{g} \oplus \mathfrak{h}\) as vector spaces, then \((\mathfrak{g}, \mathfrak{h})\) is a matched pair of Lie algebras.

**Lemma 6.5.** Let \((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, B)\) be a Rota–Baxter Lie algebra. Then \((\mathfrak{g}_B, \mathfrak{g}_{\text{diag}})\) is a matched pair of Lie algebras, where \(\mathfrak{g}_{\text{diag}} = \{(x,x)\mid \forall x \in \mathfrak{g}\}\) and \(\mathfrak{g}_B = \{(B(x), x + B(x))\mid \forall x \in \mathfrak{g}\}\).

**Proof.** Consider the direct sum Lie algebra \(\mathfrak{g} \oplus \mathfrak{g}\), in which the Lie bracket is given by

\[
[(x,y), (x',y')] = ([x,x'], [y,y'])_\mathfrak{g}.
\]

It is obvious that \(\mathfrak{g}_{\text{diag}}\) is a Lie subalgebra of \(\mathfrak{g} \oplus \mathfrak{g}\). As \((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, B)\) is a Rota–Baxter Lie algebra, we have

\[
[(B(x), x + B(x)), (B(y), y + B(y))]
\]

\[
= ([B(x), B(y)]_\mathfrak{g}, [x,y]_\mathfrak{g} + [x, B(y)]_\mathfrak{g} + [B(x), y]_\mathfrak{g} + [B(x), B(y)]_\mathfrak{g})
\]

\[
= (B([x,y]_\mathfrak{g} + [x, B(y)]_\mathfrak{g} + [B(x), y]_\mathfrak{g} + [B(x), B(y)]_\mathfrak{g})
\]

\[
= [x,y]_\mathfrak{g} + [x, B(y)]_\mathfrak{g} + [B(x), y]_\mathfrak{g} + B([x,y]_\mathfrak{g} + [x, B(y)]_\mathfrak{g} + [B(x), y]_\mathfrak{g})
\]

\[
\in \mathfrak{g}_B,
\]

which implies that \(\mathfrak{g}_B\) is a Lie subalgebra of \(\mathfrak{g} \oplus \mathfrak{g}\). Moreover, it is obvious that \(\mathfrak{g}_{\text{diag}} \oplus \mathfrak{g}_B\) is isomorphic to \(\mathfrak{g} \oplus \mathfrak{g}\) as vector spaces. Therefore, \((\mathfrak{g}_B, \mathfrak{g}_{\text{diag}})\) is a matched pair of Lie algebras. \(\square\)

A pair of Lie groups \((P, Q)\) is called a **matched pair of Lie groups** ([25]) if there is a left action of \(P\) on \(Q\) and a right action of \(Q\) on \(P\):

\[
P \times Q \to Q, \quad (p, q) \mapsto p \triangleright q; \quad P \times Q \to P, \quad (p, q) \mapsto p \triangleleft q,
\]

such that

\[
p \triangleright (q_1q_2) = (p \triangleright q_1)(p \triangleleft q_1) \triangleright q_2; \quad (p_1p_2) \triangleleft q = (p_1 \triangleleft (p_2 \triangleright q))(p_2 \triangleleft q).
\]

(38) (39)

Similar to the case of Lie algebras, the following equivalent characterization of matched pairs of Lie groups is well-known.

**Proposition 6.6.** A pair of Lie groups \((P, Q)\) is a matched pair if there exists a Lie group \(K\), and injective Lie group homomorphism \(i_P : P \to K\) and \(i_Q : Q \to K\), such that the map \(P \times Q \to K\) defined by \((p, q) \mapsto i_P(p)i_Q(q)\) is a diffeomorphism.

It is not hard to see that a matched pair of Lie groups differentiates to a matched pair of Lie algebras. However, not every matched pair of Lie algebras integrates to a matched pair of Lie groups (e.g., [13, Example 2.7] gives rise to a counterexample). Some partial result is available
for this integration problem. For example, it is proved in [25] that a matched pair of Lie algebras \((\mathfrak{g}, \mathfrak{h})\) integrates to a matched pair of Lie groups \((G, H)\), if the simply connected Lie groups \(G\) and \(H\) integrating \(\mathfrak{g}\) and \(\mathfrak{h}\), respectively, are compact. In the following, we give another integration result for matched pairs coming from Rota–Baxter operators using the integration of Rota–Baxter operators.

**Theorem 6.7.** Let \((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, B)\) be a Rota–Baxter Lie algebra. Assume that \(B\) is integrable and \((G, \cdot, B)\) is the integrated Rota–Baxter Lie group of \((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, B)\). Then \((G_B, G_{\text{diag}})\) is a matched pair of Lie groups, where \(G_{\text{diag}} = \{(g, g) \mid \forall g \in G\}\) and \(G_B = \{(B(g), g \cdot B(g)) \mid \forall g \in G\}\). Furthermore, the differentiation of \((G_B, G_{\text{diag}})\) is the matched pair \((\mathfrak{g}_B, \mathfrak{g}_{\text{diag}})\).

**Proof.** Consider the direct product Lie group \(G \times G\), in which the group multiplication is given by

\[(g, h)(g', h') = (g \cdot g', h \cdot h').\]

It is obvious that \(G_{\text{diag}}\) is a Lie subgroup. As \((G, \cdot, B)\) is a Rota–Baxter Lie group, for any \(g, h \in G\), we have

\[(B(g), g \cdot B(g))(B(h), h \cdot B(h)) = (B(g) \cdot B(h), g \cdot B(g) \cdot h \cdot B(h)) = (B(g \cdot B(g), h \cdot (B(g))^{-1}, g \cdot B(g) \cdot h \cdot (B(g))^{-1} \cdot B(g \cdot B(g) \cdot h \cdot (B(g))^{-1})) \in G_B\]

and

\[(B(g), g \cdot B(g))^{-1} = (B((B(g))^{-1} \cdot g^{-1} \cdot B(g)), (B(g))^{-1} \cdot g^{-1} \cdot B(g) \cdot B((B(g))^{-1} \cdot g^{-1} \cdot B(g))) \in G_B,\]

which implies that \(G_B\) is a subgroup. Moreover, as \(B\) is a smooth map, it follows that \(G_B\) is a closed set in \(G \times G\). Thus, \(G_B\) is a Lie subgroup. For any \((a, b) \in G \times G\), there is

\[(a, b) = (B(b \cdot a^{-1}), b \cdot a^{-1} \cdot B(b \cdot a^{-1}))(B(b \cdot a^{-1}))^{-1} \cdot a, (B(b \cdot a^{-1}))^{-1} \cdot a).\]

Thus, the map \(G_B \times G_{\text{diag}} \rightarrow G \times G\) defined by \((B(g), g \cdot B(g)), (h, h)) \rightarrow (B(g) \cdot h, g \cdot B(g) \cdot h)\) is a diffeomorphism. Therefore, by Proposition 6.6, \((G_B, G_{\text{diag}})\) is a matched pair of Lie groups. The other conclusion is straightforward.

**Remark 6.8.** Despite there are explicit counterexamples of nonintegrable matched pairs, we do not know yet an explicit example of nonintegrable Rota–Baxter operators.

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