Lie families: theory and applications

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Abstract

We analyze the families of non-autonomous systems of first-order ordinary differential equations admitting a \textit{common time-dependent superposition rule}, i.e. a time-dependent map expressing any solution of each of these systems in terms of a generic set of particular solutions of the system and some constants. We next study the relations of these families, called \textit{Lie families}, with the theory of Lie and quasi-Lie systems and apply our theory to provide common time-dependent superposition rules for certain Lie families.

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1. Introduction

The theory of Lie systems [1–7] deals with non-autonomous systems of first-order ordinary differential equations such that all their solutions can be written in terms of generic sets of particular solutions and some constants by means of a time-independent function. Such functions are called \textit{superposition rules} and the systems admitting this mathematical property are called \textit{Lie systems}. Lie succeeded in characterizing systems admitting a superposition rule. His result, now known as the \textit{Lie theorem} [1], states that a non-autonomous system (time-dependent vector field) $X_t$ is a Lie system if and only if there exists a finite-dimensional Lie algebra of vector fields $V_0$ such that $X_t \in V_0$ for all $t$.

Note that a superposition rule can be found explicitly even for systems whose general solution is not known, as in the case of Riccati equations [8], and its knowledge enables us to obtain the general solution out of certain sets of particular solutions in an easier way than directly solving the system.

In the theory of Lie systems various methods have been developed to obtain superposition rules, time-dependent and time-independent constants of motion, exact solutions, integrability conditions and other interesting properties for particular systems [9–15]. Unfortunately, being a Lie system is rather exceptional and, in order to apply the methods of the theory of Lie...
systems to a broader set of non-autonomous systems, some generalizations of this theory have been proposed. The generalized methods are presently used to investigate some partial differential equations [7], a class of second-order differential equations (the so-called SODE Lie systems [10]), certain Schrödinger equations [16], etc.

With the same aim of applying the theory of Lie systems to a broader family of systems, the theory of quasi-Lie schemes and quasi-Lie systems has been recently developed [17–19]. This theory allows us to investigate some non-Lie systems and it can be applied to dealing with certain second- and even higher order systems of differential equations. For example, it enables us to analyze some nonlinear oscillators [17], dissipative Milne–Pinney equations [18], Emden–Fowler equations [19], etc. One of the main results obtained through the quasi-Lie scheme approach is the existence of the so-called time-dependent superposition rules, that is, time-dependent superposition functions expressing the general solution in terms of a generic family of particular solutions of this system.

Note, however, that the concept of time-dependent superposition rule does not make much sense for a single non-autonomous system. This is because, as explained in [17], any single non-autonomous system admits such a superposition rule which, however, can be as difficult as finding the general solution of the system and, therefore, it cannot be generally used to analyze the properties of the system. This is analogous to the fact that each autonomous system is automatically a Lie system, as each single vector field spans a one-dimensional Lie algebra. Therefore, it only makes non-trivial sense to speak about common time-dependent superposition rules for a bigger family of non-autonomous systems.

In this paper, we give a natural generalization of the Lie theorem characterizing Lie systems. This enables us to show that many families of non-autonomous systems of first-order ordinary differential equations are Lie families, that is, they admit common time-dependent superposition rules. Furthermore, we study some Lie families and we obtain common time-dependent superposition rules for all of them.

This paper is organized as follows. Section 2 describes common time-dependent superposition rules in terms of certain horizontal foliations. In section 3 we generalize the Lie theorem to characterize the families of systems admitting common time-dependent superposition rules. In section 4 we posteriorly use this result to analyze the relations between quasi-Lie systems, Lie systems and time-dependent superposition rules. We finally apply all our results to investigate some Lie families throughout section 5.

2. Time-dependent superpositions and foliations

In this section we develop the concept of a common time-dependent superposition rule for a family of non-autonomous systems of first-order ordinary differential equations and relate this concept to certain horizontal foliations. For the sake of simplicity, we investigate these concepts in local coordinates, but our approach can be slightly modified to handle systems on manifolds.

Consider a family, parameterized by elements $\alpha$ of a set $\Lambda$, of non-autonomous systems of first-order ordinary differential equations on $\mathbb{R}^n$ of the form
\[
\frac{dx^i}{dt} = Y_\alpha^i(t, x), \quad i = 1, \ldots, n, \quad \alpha \in \Lambda.
\] (1)

In applications, $\Lambda$ is often a finite subset of $\mathbb{N}$ or $\Lambda = C^\infty(\mathbb{R})$. Solutions of these systems are integral curves of the family $\{Y_\alpha\}_{\alpha \in \Lambda}$ of time-dependent vector fields on $\mathbb{R}^n$ given by
\[
Y_\alpha(t, x) = \sum_{i=1}^{n} Y_\alpha^i(t, x) \frac{\partial}{\partial x^i}, \quad \alpha \in \Lambda.
\] (2)
Note 1. In order to simplify the terminology, we will use $Y_\alpha$ to designate both: a time-dependent vector field of the above family and the non-autonomous system describing its integral curves.

Denote with $\bar{Y}_\alpha$ the autonomization of the time-dependent vector field $Y_\alpha$, that is, the vector field on $\mathbb{R} \times \mathbb{R}^n$ defined by

$$\bar{Y}_\alpha(t, x) = \frac{\partial}{\partial t} + \sum_{i=1}^n Y^i_\alpha(t, x) \frac{\partial}{\partial x^i}.$$ 

Integral curves of (1) can be identified with the trajectories of the vector field (autonomous system) $\bar{Y}_\alpha$. Let us state the fundamental concept studied throughout the paper.

Definition 2. We say that the family of non-autonomous systems (1) admits a common time-dependent superposition rule, if there exists a map $\Phi_1: \mathbb{R} \times \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}^n$,

$$x = \Phi(t, x(1), \ldots, x(m); k_1, \ldots, k_n), \quad (3)$$

such that the general solution $x(t)$ of any system $Y_\alpha$ of the family (1) can be written, at least for sufficiently small $t$, as

$$x(t) = \Phi(t, x(1)(t), \ldots, x(m)(t); k_1, \ldots, k_n),$$

with $\{x(a)(t) | a = 1, \ldots, m\}$ being a generic set of particular solutions of $Y_\alpha$, and $k_1, \ldots, k_n$ being the constants associated with each particular solution. A family of systems (1) admitting a common time-dependent superposition rule is called a Lie family.

Note 3. We do not want to formalize precisely what ‘generic’ means in the above definition, as it is not crucial for our purposes and depends on the context. One can have in mind the following example: for a system of linear homogeneous differential equations ‘generic’ means that the particular solutions are linearly independent.

Given a common time-dependent superposition rule $\Phi: \mathbb{R} \times \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}^n$ of a Lie family $\{Y_\alpha\}_{\alpha \in \Lambda}$, the map $\Phi(t, x(1), \ldots, x(m); \cdot): \mathbb{R}^n \rightarrow \mathbb{R}^n$, $x(0) = \Phi(t, x(1), \ldots, x(m); k)$, is regular for a generic point $(t, x(1), \ldots, x(m)) \in \mathbb{R} \times \mathbb{R}^n$ and, in view of the implicit function theorem, it can be inverted to write

$$k = \Psi(t, x(0), \ldots, x(m)), \quad \text{for a certain map } \Psi: \mathbb{R} \times \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}^n,$$

and $k = (k_1, \ldots, k_n)$ being the only point in $\mathbb{R}^n$ such that

$$x(0) = \Phi(t, x(1), \ldots, x(m); k).$$

Note 4. As a matter of fact, the maps $\Psi$ and $\Phi$ are defined only locally on the open subsets of $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ but, for simplicity, we will write $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ for their domains.

Consequently, the map $\Psi$ determines locally an $n$-codimensional foliation $\tilde{\mathcal{F}}$ of the manifold $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ into the level sets of $\Psi$. Moreover, as the fundamental property of the map $\Psi$ proves that $\Psi(t, x_0(t), \ldots, x_m(t))$ is constant for any $(m+1)$-tuple of particular solutions of any system of the family (1), the foliation determined by $\Psi$ is invariant under the permutation of its $(m+1)$ arguments $\{x(a) | a = 0, \ldots, m\}$, and differentiating $\Psi(t, x(0)(t), \ldots, x(m)(t))$ with respect to $t$, we get

$$\frac{\partial \Psi^j}{\partial t} + \sum_{a=0}^m \sum_{i=1}^n Y^i_a(t, x(a)(t)) \frac{\partial \Psi^j}{\partial x^i(a)} = 0, \quad j = 1, \ldots, n, \quad \alpha \in \Lambda, \quad (4)$$

where $\Psi = (\Psi^1, \ldots, \Psi^n)$. 

3
Definition 5. Given a time-dependent vector field \( Y = \sum_{i=1}^{n} Y^i(t, x) \partial / \partial x^i \) on \( \mathbb{R}^n \), we define its prolongation to \( \mathbb{R} \times \mathbb{R}^{n(m+1)} \) as the vector field on \( \mathbb{R} \times \mathbb{R}^{m(m+1)} \) given by
\[
\hat{Y}(t, x(0), \ldots, x(m)) = \sum_{a=0}^{m} \sum_{i=1}^{n} Y^i(t, x(\alpha)) \frac{\partial}{\partial x^i(\alpha)},
\]
and its time prolongation to \( \mathbb{R} \times \mathbb{R}^{n(m+1)} \) as the vector field on \( \mathbb{R} \times \mathbb{R}^{n(m+1)} \) of the form
\[
\tilde{Y}(t, x(0), \ldots, x(m)) = \frac{\partial}{\partial t} + \sum_{a=0}^{m} \sum_{i=1}^{n} Y^i(t, x(\alpha)) \frac{\partial}{\partial x^i(\alpha)}.
\]

Equalities (4) show that the functions \{\( \Psi^i \mid i = 1, \ldots, n \)\} are first integrals for the vector fields \{\( \tilde{Y}_\alpha \)\}_{\alpha \in \Lambda}, that is, \( \tilde{Y}_\alpha \Psi^i = 0 \) for \( i = 1, \ldots, n \) and \( \alpha \in \Lambda \). Therefore, the vector fields \( \tilde{Y}_\alpha \) are tangent to the leaves of \( \mathcal{F} \).

The foliation \( \mathcal{F} \) has another important property. If the leaf \( \mathcal{F}_k \) is the level set of \( \Psi \) corresponding to a certain \( k = (k_1, \ldots, k_n) \in \mathbb{R}^n \), and given \( (t, x_{(1)}, \ldots, x_{(m)}) \in \mathbb{R} \times \mathbb{R}^{nm} \), there is only one point \( x(0) \in \mathbb{R}^n \) such that \( (t, x(0), x_{(1)}, \ldots, x_{(m)}) \in \mathcal{F}_k \). Thus, the projection onto the last \( m \cdot n \) coordinates and the time
\[
\pi : (t, x_{(0)}, \ldots, x_{(m)}) \in \mathbb{R} \times \mathbb{R}^{n(m+1)} \rightarrow (t, x_{(1)}, \ldots, x_{(m)}) \in \mathbb{R} \times \mathbb{R}^{nm},
\]
induces a local diffeomorphism from the leaf \( \mathcal{F}_k \) of \( \mathcal{F} \) into \( \mathbb{R} \times \mathbb{R}^{nm} \). We will say that the foliation \( \mathcal{F} \) is horizontal with respect to the projection \( \pi \).

On the other hand, the horizontal foliation defines the common time-dependent superposition rule without referring to the map \( \Psi \). Indeed, if we take a point \( x_{(0)} \) and \( m \) particular solutions, \( x_{(1)}(t), \ldots, x_{(m)}(t) \), for a system of the family, then \( x_{(0)}(t) \) is the unique curve in \( \mathbb{R}^n \) such that the points of the curve
\[
(t, x_{(0)}(t), x_{(1)}(t), \ldots, x_{(m)}(t)) \subset \mathbb{R} \times \mathbb{R}^{nm}
\]
belong to the same leaf as the point \( (0, x_{(0)}(0), x_{(1)}(0), \ldots, x_{(m)}(0)) \). Thus, it is only the horizontal foliation \( \mathcal{F} \) that really matters when the common time-dependent superposition rule is concerned. It is in a sense obvious, as composing \( \Psi \) with a diffeomorphism on \( \mathbb{R}^n \) changes the superposition function (rearranges the level sets) but yields the same superposition rule. This proves the following (cf [7]).

Proposition 6. Given a common time-dependent superposition rule (3) for a Lie family (1) is equivalent to giving a foliation which is horizontal with respect to the projection \( \pi : \mathbb{R} \times \mathbb{R}^{(m+1)n} \rightarrow \mathbb{R} \times \mathbb{R}^{mn} \), such that the vector fields \( \{\tilde{Y}_\alpha \}_{\alpha \in \Lambda} \) are tangent to their leaves.

3. Generalized Lie theorem

It is generally difficult to determine whether a family (8) admits a common time-dependent superposition rule by means of proposition 6. It is therefore interesting to find a characterization of Lie families by means of a more convenient criterion, e.g. through an easily verifiable condition based on the properties of the time-dependent vector fields \( \{Y_\alpha \}_{\alpha \in \Lambda} \). Finding such a criterion is the main result of this section. It is formulated as a generalized Lie theorem.

We start with three lemmata. The proofs of first two of them are straightforward.

Lemma 7. Given two time-dependent vector fields \( X \) and \( Y \) on \( \mathbb{R}^n \), the commutator \([\hat{X}, \hat{Y}]\) on \( \mathbb{R} \times \mathbb{R}^{n(m+1)} \) is the prolongation of a time-dependent vector field \( \dot{Z} \) on \( \mathbb{R}^n \), \([\hat{X}, \hat{Y}] = \hat{Z} \).
Lemma 8. Given a family of time-dependent vector fields, \( X_1, \ldots, X_r \), on \( \mathbb{R}^n \), their autonomizations satisfy the relations
\[
[\vec{X}_j, \vec{X}_k](t, x) = \sum_{l=1}^r f_{jkl}(t) \vec{X}_l(t, x), \quad j, k = 1, \ldots, r
\]
for some time-dependent functions \( f_{jkl} : \mathbb{R} \to \mathbb{R} \), if and only if their time prolongations to \( \mathbb{R} \times \mathbb{R}^n(m+1) \), \( \tilde{X}_1, \ldots, \tilde{X}_r \), satisfy the analogous relations
\[
[\tilde{X}_j, \tilde{X}_k](t, x) = \sum_{l=1}^r f_{jkl}(t) \tilde{X}_l(t, x), \quad j, k = 1, \ldots, r.
\]
Moreover, \( \sum_{j=1}^r f_{jkl}(t) = 0 \) for all \( j, k = 1, \ldots, r \).

Lemma 9. Consider a family of time-dependent vector fields, \( Y_1, \ldots, Y_r \), with time prolongations to \( \mathbb{R} \times \mathbb{R}^{n(m+1)} \), \( \tilde{Y}_1, \ldots, \tilde{Y}_r \), such that their projections \( \pi_* (\tilde{Y}_j) \) are linearly independent at a generic point in \( \mathbb{R} \times \mathbb{R}^{nm} \). Then,
\[
\sum_{j=1}^r b_j \tilde{Y}_j = \tilde{\mathcal{Y}}(t, x),
\]
where \( \tilde{\mathcal{Y}}(t, x) \) is of the form \( \tilde{Y} \) (resp. \( \tilde{\mathcal{Y}} \)) for a time-dependent vector field \( Y \) on \( \mathbb{R}^n \) if and only if the functions \( b_j \) depend only on the time, that is, \( b_j = b_j(t) \) and \( \sum_{j=1}^r b_j = 0 \) (resp., \( \sum_{j=1}^r b_j = 1 \)).

Proof. We shall only detail the proof of the above claim for \( \sum_{j=1}^r b_j \tilde{Y}_j = \tilde{\mathcal{Y}} \), as the proof of the other case is completely analogous. Let us write in coordinates
\[
\tilde{\mathcal{Y}}_j = \frac{\partial}{\partial t} + \sum_{a=0}^m \sum_{i=1}^n A_j^i(t, x(a)) \frac{\partial}{\partial x_i(a)}, \quad j = 1, \ldots, r.
\]
Then,
\[
\sum_{j=1}^r b_j(t, x(0), \ldots, x(m)) \tilde{\mathcal{Y}}_j = \sum_{j=1}^r \sum_{a=0}^m \sum_{i=1}^n b_j(t, x(0), \ldots, x(m)) A_j^i(t, x(a)) \frac{\partial}{\partial x_i(a)} + \sum_{j=1}^r b_j(t, x(0), \ldots, x(m)) \frac{\partial}{\partial t},
\]
which is a prolongation if and only if there are functions \( B^i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \), with \( i = 1, \ldots, n \), such that
\[
\begin{cases}
\sum_{j=1}^r b_j(t, x(0), \ldots, x(m)) A_j^i(t, x(a)) = B^i(t, x(a)), \\
\sum_{j=1}^r b_j(t, x(0), \ldots, x(m)) = 0,
\end{cases}
\]
for \( a = 0, \ldots, m \), \( i = 1, \ldots, n \).

If the functions \( b_1, \ldots, b_r \) are time-dependent only and \( \sum_{j=1}^r b_j = 0 \), the above conditions hold and \( \sum_{j=1}^r b_j \tilde{Y}_j \) is the prolongation to \( \mathbb{R} \times \mathbb{R}^{n(m+1)} \) of the time-dependent vector field \( Y = \sum_{i=1}^n B^i(t, x) \partial / \partial x^i \).

Conversely, suppose that \( \sum_{j=1}^r b_j \tilde{Y}_j \) is a prolongation for a time-dependent vector field on \( \mathbb{R}^n \). In this case, the functions \( b_j(t, x(0), \ldots, x(m)) \) solve the following system of linear equations in the unknown variables \( u_i \):
\[
\begin{cases}
\sum_{j=1}^r u_{j} A_j^i(t, x(a)) = B^i(t, x(a)), \\
\sum_{j=1}^r u_j = 0,
\end{cases}
\]
where \( a = 1, \ldots, m \) and \( i = 1, \ldots, n \). This is a system of \( m \cdot n + 1 \) equations and, as \( \pi_a(\tilde{Y}_j) \), with \( j = 1, \ldots, r \), are linearly independent by assumption, the solutions \( u_a \) are uniquely determined by the variables \( \{ t, x_1(\cdot), \ldots, x_{(m)}(\cdot) \} \) and therefore they do not depend on \( x_{(0)} \). Since time prolongations are invariant with respect to the symmetry group \( S_{m+1} \) acting on \( \mathbb{R}^{n(m+1)} = (\mathbb{R}^n)^{m+1} \), in the obvious way, the functions \( b_j(t, x_1(\cdot), \ldots, x_{(m)}(\cdot)) \), with \( j = 1, \ldots, r \), must satisfy such a symmetry. Hence, as they do not depend on \( x_{(0)} \), they cannot depend on the variables \( \{ x_1(\cdot), \ldots, x_{(m)}(\cdot) \} \) and they are the functions depending only on the time. \( \square \)

**Theorem 10** (Generalized Lie theorem). The family of systems (1) admits a common time-dependent superposition rule if and only if the vector fields \( \{ \tilde{Y}_a \}_{a \in \Lambda} \) can be written in the form

\[
\tilde{Y}_a(t, x) = \sum_{j=1}^r b_{aj}(t) \bar{X}_j(t, x), \quad a \in \Lambda, \tag{6}
\]

where \( b_{aj} \) are the functions of the time only, \( \sum_{j=1}^r b_{aj} = 1 \), and \( \bar{X}_1, \ldots, \bar{X}_r \) are the time-dependent vector fields such that

\[
[\bar{X}_j, \bar{X}_k](t, x) = \sum_{l=1}^r f_{jkl}(t) \bar{X}_l(t, x), \quad j, k = 1, \ldots, r \tag{7}
\]

for some functions \( f_{jkl} : \mathbb{R} \to \mathbb{R} \), with \( j, k, l = 1, \ldots, r \). We call the family of autonomizations, \( \bar{X}_1, \ldots, \bar{X}_r \), a system of generators of the Lie family.

**Proof.** First suppose that the family of systems (1) admits a common time-dependent superposition rule and let \( \bar{Y} \) be the corresponding \( n \)-codimensional horizontal foliation. The vector fields \( \{ \bar{Y}_a \}_{a \in \Lambda} \) are tangent to the leaves of the foliation \( \bar{F} \) and span a distribution \( \mathcal{D}_0 \) on \( \mathbb{R} \times \mathbb{R}^{n(m+1)} \). Such a distribution need not be involutive, see examples in section 5. Nevertheless, we can enlarge the family \( \{ \bar{Y}_a \}_{a \in \Lambda} \) to the Lie algebra of vector fields generated by such a family. This Lie algebra is spanned by \( \{ \bar{Y}_a \}_{a \in \Lambda} \) and all their possible Lie brackets, i.e.

\[
\bar{Y}_a, [\bar{Y}_a, \bar{Y}_b], [\bar{Y}_a, [\bar{Y}_b, \bar{Y}_c]], [\bar{Y}_a, [\bar{Y}_b, [\bar{Y}_c, \bar{Y}_d]]], \ldots \quad \alpha, \beta, \gamma, \delta, \ldots \in \Lambda. \tag{8}
\]

All the above vector fields are tangent to the leaves of the foliation \( \bar{F} \) and therefore there are up to \( m \cdot n + 1 \) linearly independent ones at a generic point of \( \mathbb{R} \times \mathbb{R}^{n(m+1)} \). Consequently, they span an involutive generalized distribution \( \mathcal{D} \) with leaves of dimension \( r \leq m \cdot n + 1 \). In a neighborhood of a regular point of this foliation, take now a finite basis of vector fields from the elements of the family (8) spanning the distribution. By construction, at least one of them must be of the form \( \bar{X}_1 \) for a certain time-dependent vector field \( X_1 \) on \( \mathbb{R}^r \) and, in view of lemma 9 and the form of the family (8), those not being time prolongations are just prolongations. Therefore, if we add \( \bar{X}_1 \) to those elements of the basis being prolongations, we get a new basis of the distribution \( \mathcal{D} \) made up by certain \( r \) time prolongations \( \bar{X}_1, \ldots, \bar{X}_r \). In other words, the distribution \( \mathcal{D} \) is locally spanned, near regular points, by time prolongations, say \( \bar{X}_1, \ldots, \bar{X}_r \). As the generalized distribution \( \mathcal{D} \) is involutive, there exist \( r^3 \) real functions \( f_{jkl} \), with \( j, k, l = 1, \ldots, r \), on \( \mathbb{R} \times \mathbb{R}^{n(m+1)} \) such that

\[
[\bar{X}_j, \bar{X}_k] = \sum_{l=1}^r f_{jkl} \bar{X}_l, \quad j, k = 1, \ldots, r, \tag{9}
\]
and as the left-hand side of the above equalities are prolongations, we get that, in view of lemma 9, all the functions \(f_{jkl}\) depend only on the time and \(\sum_{l=1}^{r} f_{jkl} = 0\). Finally, taking into account lemma 8, we have

\[
[X_j, X_k](t, x) = \sum_{l=1}^{r} f_{jkl}(t) \tilde{X}_l(t, x), \quad j, k = 1, \ldots, r.
\]

Note that as the vector fields \(\{\tilde{Y}_\alpha\}_{\alpha \in \Lambda}\) are contained in the distribution \(\mathcal{D}\), there exist some functions \(b_{\alpha j} \in C^\infty(\mathbb{R}^{n(m+1)})\) such that \(\tilde{Y}_\alpha = \sum_{j=1}^{r} b_{\alpha j}(t) \tilde{X}_j\) for every \(\alpha \in \Lambda\). As a consequence, again according to lemma 9, the functions \(b_{\alpha j}\) depend only on the time, i.e. \(b_{\alpha j} = b_{\alpha j}(t)\). Therefore, we get that

\[
\tilde{Y}_\alpha = \sum_{j=1}^{r} b_{\alpha j}(t) \tilde{X}_j(t, x), \quad \alpha \in \Lambda.
\]

Let us prove the converse. Assume that we can write

\[
\tilde{Y}_\alpha(t, x) = \sum_{j=1}^{r} b_{\alpha j}(t) \tilde{X}_j(t, x)
\]

for certain time-dependent vector fields \(X_1, \ldots, X_r\) on \(\mathbb{R}^n\) such that

\[
[X_j, X_k](t, x) = \sum_{l=1}^{r} f_{jkl}(t) \tilde{X}_l(t, x), \quad j, k = 1, \ldots, r.
\]

In view of lemma 8, the vector fields \(\tilde{X}_1, \ldots, \tilde{X}_r\) span an involutive distribution \(\mathcal{D}\) on \(\mathbb{R} \times \mathbb{R}^{n(m+1)}\) for any \(m\). Furthermore, the rank of this distribution is not greater than \(r\) and therefore, for \(m\) big enough, this distribution is at least \(n\)-codimensional and it gives rise to a foliation \(\mathcal{F}_0\) which is horizontal with respect to the projection \(\pi\). Moreover, if the codimension of \(\mathcal{F}_0\) is bigger than \(n\), we can enlarge \(\mathcal{F}_0\) to an \(n\)-codimensional foliation \(\mathcal{F}\), still horizontal with respect to the map \(\pi\) giving rise to a common time-dependent superposition rule for the family \((1)\).

\[\square\]

4. Lie families, quasi-Lie and Lie systems

This section is devoted to recalling the theories of quasi-Lie schemes and Lie systems needed to investigate the relations among these theories and Lie families. A full detailed report on these topics can be found in [7, 17].

The theory of quasi-Lie schemes provides various results on the transformation properties of time-dependent vector fields by a certain kind of time-dependent changes of variables associated with generalized flows.

Each time-dependent vector field \(X\) gives rise to a generalized flow \(g^X\), i.e. a map \(g^X : (t, x) \in \mathbb{R} \times \mathbb{R}^n \rightarrow g^X_t(x) \in \mathbb{R}^n\) (more precisely, defined in a neighborhood of \((0) \times \mathbb{R}^n\) in \(\mathbb{R} \times \mathbb{R}^n\)), with \(g^X_0 = \text{Id}_{\mathbb{R}^n}\), such that the curve \(\gamma^X_{s_0}(t) = g^X_t(s_0)\) is the integral curve of the time-dependent vector field \(X\) starting from the point \(s_0 \in \mathbb{R}^n\), i.e. \(\dot{\gamma}^X_{s_0} = X(t, \gamma^X_{s_0}(t))\) and \(\gamma^X_s(0) = g^X_s(s_0) = x_0\).

Denote with \(\mathcal{X}_t(\mathbb{R}^n)\) the set of all time-dependent vector fields on \(\mathbb{R}^n\). Each generalized flow \(h\) acts on the set of time-dependent vector fields \(\mathcal{X}_t(\mathbb{R}^n)\) transforming each time-dependent vector field \(X \in \mathcal{X}_t(\mathbb{R}^n)\) into a new one, \(h \circ X\), with the generalized flow of the form \(g^{h \circ X} = h \circ g^X\). In terms of autonominations we can write ([17, theorem 3])

\[
(h \circ X)(t, x) = h_t(x) X(t, x).
\]
where \( \hat{h} \) is the natural autonomization of the generalized flow \( h \) to a (local) diffeomorphism of \( \mathbb{R} \times \mathbb{R}^n \), \( \hat{h}(t, x) = (t, h_t(x)) \) and where \( \hat{h}_* \) is the standard action of the diffeomorphism \( \hat{h} \) on vector fields. This, in turn, implies that
\[
[h \circ X, h \circ Y] = \hat{h}_* [X, Y].
\] (9)

Let \( V \) be a finite-dimensional vector space of vector fields on \( \mathbb{R}^n \). We denote with \( V(\mathbb{R}) \) the set of time-dependent vector fields \( X \in \mathfrak{X}(\mathbb{R}^n) \) such that, for every \( t \in \mathbb{R} \), the vector field \( X_t(x) \) belongs to \( V \). In terms of the introduced terminology and notation, the Lie theorem, whose statement can be found for instance in \([1, 7]\), can be reformulated as follows.

**Proposition 11** (Lie theorem). A non-autonomous system \( X \) is a Lie system on \( \mathbb{R}^n \) if and only if there exists a finite-dimensional Lie algebra of vector fields \( V_0 \subset \mathfrak{X}(\mathbb{R}^n) \) such that \( X \in V_0(\mathbb{R}) \).

**Definition 12.** A quasi-Lie scheme \( S(W, V) \) on the manifold \( M \) consists of two finite-dimensional vector spaces of vector fields \( W, V \subset \mathfrak{X}(M) \) such that
- \( W \) is a linear subspace of \( V \),
- \( W \) is a Lie algebra of vector fields, that is, \( [W, W] \subset W \)
- \( W \) normalizes \( V \), i.e. \( [W, V] \subset V \).

It has been proved in \([17]\) that given a quasi-Lie scheme \( S(W, V) \), the space \( V(\mathbb{R}) \) is stable under the action of the infinite-dimensional group \( G(W) \) of generalized flows of vector fields in \( W(\mathbb{R}) \), i.e. \( g \circ X \in V(\mathbb{R}) \), for every \( X \in V(\mathbb{R}) \) and \( g \in G(W) \).

**Definition 13.** Given a quasi-Lie scheme \( S(W, V) \), we say that a time-dependent vector field \( X \in V(\mathbb{R}) \) is a quasi-Lie system with respect to this scheme, if there exist a generalized flow \( g \in G(W) \) and a Lie algebra of the vector fields \( V_0 \subset V \), such that \( g \circ X \in V_0(\mathbb{R}) \).

As for each Lie system \( X \) there exists a Lie algebra of vector fields \( V_0 \subset V \), it is obvious that \( X \in S(V_0, V_0) \) and, consequently, every Lie system is also a quasi-Lie system.

From now on, given a quasi-Lie scheme \( S(W, V) \), a generalized flow \( g \in G(W) \), and a Lie algebra of vector fields \( V_0 \subset V \), we denote with \( S_g(W, V; V_0) \) the set of quasi-Lie systems of the scheme \( S(W, V) \) such that \( g \circ X \in V_0(\mathbb{R}) \).

**Proposition 14.** The family of quasi-Lie systems \( S_g(W, V; V_0) \) is a Lie family admitting the common time-dependent superposition function of the form
\[
\hat{\Phi}(t, x(1), \ldots, x(m), k) = g_t^{-1} \circ \Phi(x(1), \ldots, x(m), k)
\] (10)
for any time-independent superposition function \( \Phi \) associated with the Lie algebra of vector fields \( V_0 \) by the Lie theorem.

**Proof.** Let \( Z_1, \ldots, Z_r \) be a basis in \( V_0 \). Since \( V_0 \) is closed with respect to the Lie bracket,
\[
[Z_j, Z_k] = \sum_{l=0}^{r} c_{jk}^l Z_l
\] (11)
for some constants \( c_{jk}^l \), \( j, k, l = 1, \ldots, r \). For any \( Y \in S_g(W, V; V_0) \), there exist functions \( b_j \) such that
\[
(g \circ Y)_t(x) = \sum_{j=1}^{r} b_j(t) Z_j(x).
\]
Consequently, for the autonomization we can write
\[ g Y(t, x) = \sum_{j=0}^{r} b_j(t) Z_j(t, x), \]  
(12)
where we put \( Z_0 = 0 \) (thus, \( Z_0 = \partial / \partial t \)) and \( b_0(t) = 1 - \sum_{j=1}^{r} b_j(t) \).

Note that, as \( Z_k \) are time independent, the autonomizations \( Z_k, k = 0, \ldots, r \), form a Lie algebra:

\[ [\bar{Z}_j, \bar{Z}_k] = \sum_{l=0}^{r} c_{jk} \bar{Z}_l, \]
(13)
where \( c_{0l} = c_{j0l} = 0 \) and \( c_{ijkl} = -\sum_{i=1}^{r} c_{ijkl} \) for \( j, k = 1, \ldots, r \). Hence, according to (9), the autonomizations \( \bar{Z}'_k = g^{-1}(Z_k) \) are also closed with respect to the bracket,

\[ [\bar{Z}'_j, \bar{Z}'_k] = \sum_{l=0}^{r} c_{jk} \bar{Z}'_l, \]
(14)
and, in view of (12), the autonomization of any \( Y \in S_g(W, V; V_0) \) can be written in the form

\[ \bar{Y}(t, x) = \sum_{j=0}^{r} b_j(t) Z'_j(t, x). \]

This means, in view of theorem 10, that \( S_g(W, V; V_0) \) is a Lie family. The form (10) can now be easily derived (see [17, theorem 4]).

In view of the above proposition, every quasi-Lie system and, consequently, every Lie system can be included in a Lie family satisfying theorem 10. This fact justifies once more calling this theorem the generalized Lie theorem.

5. Applications

In this section we will apply common time-dependent superposition rules for studying some first- and second-order differential equations. In this way, we will show how that common time-dependent superposition rules can be used to analyze equations which cannot be studied by means of the usual theory of Lie systems. Additionally, some new results for the study of Abel and Milne–Pinney equations are provided.

5.1. A time-dependent superposition rule for Abel equations

We illustrate here our theory by deriving a common time-dependent superposition rule for a Lie family of Abel equations whose elements do not admit a standard superposition rule except for a few particular instances. In this way, we single out that our theory provides new tools for investigating solutions of non-autonomous systems of differential equations that cannot be analyzed by means of the theory of Lie systems.

With this aim, we analyze the so-called Abel equations of the first type [20, 21], i.e. the differential equations of the form

\[ \frac{dx}{dr} = a_0(t) + a_1(t)x + a_2(t)x^2 + a_3(t)x^3, \]
(15)
with \( a_3(t) \neq 0 \). Abel equations appear in the analysis of several cosmological models [22–24] and other different fields in Physics [25–30]. Additionally, the study of integrability conditions
for Abel equations is a research topic of current interest in Mathematics and multiple studies have been carried out in order to analyze the properties of the solutions of these equations [21, 31–34].

Note that, apart from its inherent mathematical interest, the knowledge of particular solutions of Abel equations allows us to study the properties of those physical systems that such equations describe. Thus, the expressions enabling us to obtain easily new solutions of Abel equations by means of several particular ones, like common time-dependent superposition rules, are interesting to study the solutions of these equations and, therefore, their related physical systems.

Unfortunately, all the expressions describing the general solution of Abel equations presently known can only be applied to study autonomous instances and, moreover, they depend on the families of particular conditions satisfying certain extra conditions, see [31, 32]. Taking this into account, common time-dependent superposition rules represent an improvement with respect to these previous expressions, as they permit one to treat non-autonomous Abel equations and they do not require the use of particular solutions satisfying additional conditions.

Recall that, according to theorem 10, the existence of a common time-dependent superposition rule for a family of time-dependent vector fields (2) requires the existence of a system of generators, i.e. a certain set of time-dependent vector fields, \( X_1, \ldots, X_r \), satisfying relations (7). Conversely, given such a set, the family of time-dependent vector fields \( V \) whose automonizations can be written in the form

\[
\dot{Y}_a(t, x) = \sum_{j=1}^{r} b_j(t) \dot{X}_j(t, x), \quad \sum_{j=1}^{r} b_j(t) = 1,
\]

admits a common time-dependent superposition rule and becomes a Lie family.

Consequently, a Lie family of Abel equations can be determined, for instance, by finding two time-dependent vector fields of the form

\[
X_1(t, x) = (b_0(t) + b_1(t)x + b_2(t)x^2 + b_3(t)x^3) \frac{\partial}{\partial x},
\]

\[
X_2(t, x) = (b'_0(t) + b'_1(t)x + b'_2(t)x^2 + b'_3(t)x^3) \frac{\partial}{\partial x}, \quad b'_3(t) \neq 0,
\]

such that

\[
[\dot{X}_1, \dot{X}_2] = 2(\dot{X}_2 - \dot{X}_1). \tag{17}
\]

Let us analyze the existence of such two time-dependent vector fields \( X_1 \) and \( X_2 \) holding relation (17). In coordinates, the Lie bracket \([\dot{X}_1, \dot{X}_2]\) reads

\[
[(b'_3 - b'_2)b_1 + b'_1 b_3)x^4 + (2b'_3 b_1 - b_2 b'_1 + b'_3) x^3 + (-3b'_0 b_3 - b_0 b'_3) + (b'_2 b_1 - b_2 b'_1) x^2 + (b'_0 b_2 + 2b_0 b'_2 - b_1 + b'_1) x - b'_0 b_1 + b_0 b'_1 - b_0 + b'_0] \frac{\partial}{\partial x}.
\]

Hence, in order to satisfy condition (17), \( b'_3 - b'_2 b_3 = 0 \), e.g. we may fix \( b_2 = b_3 = 0 \). Additionally, for the sake of simplicity, we assume \( b'_2 = 1 \). In this case, the previous expression takes the form

\[
[2b_1 x^3 + (3b_0 + b'_2 b_1 + b'_1) x^2 + (2b_0 b'_2 - b_1 + b'_1) x - b'_0 b_1 + b_0 b'_1 - b_0 + b'_0] \frac{\partial}{\partial x},
\]

and, taking into account the values chosen for \( b_2, b_3 \) and \( b'_3 \), assumption (17) yields \( b_1 = 1 \) and

\[
\begin{align*}
\{ b'_2 &= 3b_0 + b'_1, \\
2(b'_1 - 1) &= 2b_0 b'_2 + b'_1, \\
2(b'_0 - b_0) &= -b'_0 + b_0 b'_1 - b_0 + b'_0.
\end{align*}
\]
As the above system has more variables than equations, we can try to fix some values of the variables in order to simplify it and obtain a particular solution. In this way, taking \( b_0(t) = t \), the above system reads

\[
\begin{aligned}
\dot{b}_2 &= b_2 - 3t, \\
\dot{b}_1 &= 2(b_1 - 1) - 2b_2, \\
\dot{b}_0 &= 2(b_0 - t) + b_0 - tb_1 + 1.
\end{aligned}
\]

This system is integrable by quadratures and it can be verified that it admits the particular solution

\[
b'_2(t) = 3(1 + t), \quad b'_1(t) = 3(1 + t)^2 + 1, \quad b'_0(t) = (1 + t)^3 + t.
\]

Summing up, we have proved that the time-dependent vector fields

\[
\begin{aligned}
X_1(t, x) &= (t + x) \frac{\partial}{\partial x}, \\
X_2(t, x) &= ((1 + t)^3 + t + (3(1 + t)^2 + 1)x + 3(1 + t)x^2 + x^3) \frac{\partial}{\partial x}
\end{aligned}
\]  

satisfy (17) and, therefore, the family of time-dependent vector fields \( Y_{b0}(t, x) = (1 - b(t))X_1(x) + b(t)X_2(x) \) is a Lie family. The corresponding family of Abel equations is

\[
\frac{dx}{dt} = (t + x) + b(t)(1 + t + x)^3.
\]  

According to the results proved in section 3, in order to determine a common time-dependent superposition rule for the above Lie family we have to determine a first-integral for the vector fields of the distribution \( \mathcal{D} \) spanned by the time prolongations \( \tilde{X}_1 \) and \( \tilde{X}_2 \) on \( \mathbb{R} \times \mathbb{R}^{m+1} \) for a certain \( m \) so that the time prolongations of \( X_1 \) and \( X_2 \) to \( \mathbb{R} \times \mathbb{R}^{m} \) become linearly independent at a generic point. Taking into account expressions (18), the prolongations of the vector fields \( X_1 \) and \( X_2 \) to \( \mathbb{R} \times \mathbb{R}^{2} \) are linearly independent at a generic point and, in view of (17), the time prolongations \( \tilde{X}_1 \) and \( \tilde{X}_2 \) to \( \mathbb{R} \times \mathbb{R}^{3} \) span an involutive generalized distribution \( \mathcal{D} \) with leaves of dimension 2 in a dense subset of \( \mathbb{R} \times \mathbb{R}^3 \). Finally, a first integral for the vector fields in the distribution \( \mathcal{D} \) will provide us a common time-dependent superposition rule for the Lie family (19).

Since, in view of (17), the vector fields \( \tilde{X}_1 \) and \( \tilde{X}_2 \) span the distribution \( \mathcal{D} \), a function \( G : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R} \) is a first integral of the vector fields of the distribution \( \mathcal{D} \) if and only if \( G \) is a first integral of \( \tilde{X}_1 \) and \( \tilde{X}_2 - \tilde{X}_1 \), i.e. \( \tilde{X}_1 G = (\tilde{X}_2 - \tilde{X}_1) G = 0 \).

The condition \( \tilde{X}_1 G = 0 \) reads

\[
\frac{\partial G}{\partial t} + (t + x_0) \frac{\partial G}{\partial x_0} + (t + x_1) \frac{\partial G}{\partial x_1} = 0,
\]

and, using the method of characteristics [35], we note that the curves on which \( G \) is constant, the so-called characteristics, are solutions of the system

\[
\frac{dt}{dr} = \frac{dx_0}{t + x_0} = \frac{dx_1}{t + x_1} \quad \Rightarrow \quad \frac{dx_i}{dr} = t + x_i, \quad i = 0, 1,
\]

i.e. \( x_i(t) = \xi_i e^t - t - 1 \), with \( i = 0, 1 \). These solutions are determined by the implicit equations \( \xi_0 = e^{-t} (x_0 + t + 1) \) and \( \xi_1 = e^{-t} (x_1 + t + 1) \), with \( \xi_0, \xi_1 \in \mathbb{R} \). Therefore, there exists a function \( G_2 : \mathbb{R}^2 \rightarrow \mathbb{R} \) such that \( G(t, x_0, x_1) = G_2(\xi_0, \xi_1) \). In other words, each first integral \( G \) of \( \tilde{X}_1 \) depends only on \( \xi_0 \) and \( \xi_1 \).

Taking into account the previous fact, we look for first integrals of the vector field \( \tilde{X}_2 - \tilde{X}_1 \) being also first integrals of \( \tilde{X}_1 \), that is, for the solutions of the equation \( (\tilde{X}_2 - \tilde{X}_1) G = 0 \) with
\( G \) depending on \( \xi_0 \) and \( \xi_1 \). Using the expression of \( \tilde{X}_2 - \tilde{X}_1 \) in the system of coordinates \( \{t, \xi_0, \xi_1\} \), we get that
\[
\xi_0^3 \frac{\partial G}{\partial \xi_0} + \xi_1^3 \frac{\partial G}{\partial \xi_1} = \xi_0^3 \frac{\partial G_2}{\partial \xi_0} + \xi_1^3 \frac{\partial G_2}{\partial \xi_1} = 0,
\]
and, applying again the method of characteristics, we obtain that there exists a function \( G_3 : \mathbb{R} \rightarrow \mathbb{R} \) such that
\[
G(t, x_0, x_1) = G_2(\xi_0, \xi_1) = G_3(\Delta_1),
\]
where \( \Delta_1 = e^{2t}((x_0 + t + 1)^{-2} - (x_1 + t + 1)^{-2}) \). Finally, using this first integral, we get that the common time-dependent superposition rule for the Lie family (19) reads
\[
k = e^{2t}((x_0 + t + 1)^{-2} - (x_1 + t + 1)^{-2}),
\]
with \( k \) being a real constant. Therefore, given any particular solution \( x_1(t) \) of a particular instance of the family of first-order Abel equations (21), the general solution, \( x(t) \), of this instance is
\[
x(t) = ((x_1(t) + t + 1)^{-2} + k e^{-2t})^{-1/2} - t - 1.
\]

Note that our previous procedure can be straightforwardly generalized to derive common time-dependent superposition rules for generalized Abel equations [36], i.e. the differential equations of the form
\[
\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)x^2 + \cdots + a_n(t)x^n, \quad n \geq 3.
\]

Actually, their study can be approached by analyzing the existence of two vector fields of the form
\[
Y_1(t, x) = (b_0(t) + b_1(t)x + \cdots + b_n(t)x^n) \frac{\partial}{\partial x},
\]
\[
Y_2(t, x) = (b'_0(t) + b'_1(t)x + \cdots + b'_n(t)x^n) \frac{\partial}{\partial x}, \quad b'_n(t) \neq 0
\]
satisfying the relation \([\tilde{Y}_1, \tilde{Y}_2] = 2(\tilde{Y}_2 - \tilde{Y}_1)\) and following a procedure similar to the one developed above.

5.2. Lie families and second-order differential equations

Common time-dependent superposition rules describe solutions of non-autonomous systems of first-order differential equations. Nevertheless, we shall now illustrate how these new kinds of superposition rules can be applied to analyze also the families of second-order differential equations. More specifically, we shall derive a common time-dependent superposition rule in order to express the general solution of any instance of a family of Milne–Pinney equations [31, 37, 38] in terms of each generic pair of particular solutions, two constants, and the time.

In this way, we provide a generalization to the setting of dissipative Milne–Pinney equations of the expression previously derived to analyze the solutions of Milne–Pinney equations in [11].

Consider the family of dissipative Milne–Pinney equations [37–40] of the form
\[
\ddot{x} = -\dot{F} \dot{x} + \omega^2 x + e^{-2F} x^{-3},
\]
with a fixed time-dependent function \( F = F(t) \), and parameterized by an arbitrary time-dependent function \( \omega = \omega(t) \). The physical motivation for the study of dissipative Milne–Pinney equations comes from its appearance in dissipative quantum mechanics [41–44], where, for instance, their solutions are used to obtain Gaussian solutions of non-conservative time-dependent quantum oscillators [43]. Moreover, the mathematical properties of the solutions
of dissipative Milne–Pinney equations have been studied by several authors from different points of view as well as for different purposes [11, 17, 18, 37, 38, 45–47]. As relevant instances, consider the works [18, 37] which outline the state-of-the-art of the investigation of dissipative and non-dissipative Milne–Pinney equations. One of the main achievements on this topic (see [37, corollary 5]) is concerned with an expression describing the general solution of a particular class of these equations in terms of a pair of generic particular solutions of a second-order linear differential equations and two constants. Recently, the theory of quasi-Lie schemes and the theory of Lie systems enabled us to recover this latter result and other new ones from a geometric point of view [10, 17].

Note that introducing a new variable \( v \equiv \dot{x} \), we transform the family (20) of second-order differential equations into a family of first-order ones

\[
\begin{align*}
\dot{x} &= v, \\
\dot{v} &= -Fv + \omega^2 x + e^{-2F} x^{-3},
\end{align*}
\]

whose dynamics is described by the following family of time-dependent vector fields on \( \mathbb{R} \times \mathbb{R} \) parameterized by \( \omega \),

\[ Y_\omega = (-Fv + e^{-2F} x^{-3} + \omega^2 x) \frac{\partial}{\partial v} + v \frac{\partial}{\partial x}, \quad \omega \in \Lambda = C^\infty(t). \]

Let us show that the above family is a Lie family whose common superposition rule can be used to analyze the solutions of the family (20).

In view of theorem 10, if the family of systems related to the above family of time-dependent vector fields is a Lie family, that is, it admits a common time-dependent superposition rule in terms of \( m \) particular solutions, then the family of vector fields on \( \mathbb{R} \times \mathbb{R}^{(m+1)} \) given by

\[ \tilde{Y}_\omega, [\tilde{Y}_\omega, \tilde{Y}_\omega], [\tilde{Y}_\omega, [\tilde{Y}_\omega, \tilde{Y}_\omega]], [\tilde{Y}_\omega, [\tilde{Y}_\omega, [\tilde{Y}_\omega, \tilde{Y}_\omega]]], \ldots, \omega, \omega', \omega'', \omega''' \in \Lambda \]

spans an involutive generalized distribution with leaves of rank \( r \leq m + 1 \).

Note that the distribution spanned by all \( \tilde{Y}_\omega \) is generated by the vector fields \( \tilde{Y}_1 \) and \( \tilde{Y}_2 \), with

\[ Y_1 = (-Fv + e^{-2F} x^{-3} + x) \frac{\partial}{\partial v} + v \frac{\partial}{\partial x}, \quad Y_2 = (-Fv + e^{-2F} x^{-3}) \frac{\partial}{\partial v} + v \frac{\partial}{\partial x}, \]

since \( \tilde{Y}_\omega = (1 - \omega^2) \tilde{Y}_2 + \omega^2 \tilde{Y}_1 \). It is easy to see that the prolongation \( [\tilde{Y}_1, \tilde{Y}_2] \) is not spanned by \( \tilde{Y}_1 \) and \( \tilde{Y}_2 \) and, so we have to include the prolongation \( \tilde{Y}_3 = [\tilde{Y}_1, \tilde{Y}_2] \) to the picture where

\[ Y_3 = x \frac{\partial}{\partial x} - (v + x \hat{F}) \frac{\partial}{\partial v}. \]

In the case \( m = 0 \), the distribution spanned by the vector fields, \( \tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3, \tilde{Y}_4 \), does not admit a non-trivial first integral. In the case \( m > 0 \), the vector fields \( \tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3 \) do not span all the elements of family (22) and we need to add to them the prolongation \( \tilde{Y}_4 = [\tilde{Y}_1, [\tilde{Y}_1, \tilde{Y}_2]] \) with

\[ Y_4 = (2v + x \hat{F}) \frac{\partial}{\partial x} + (2e^{-2F} x^{-3} - 2x - \hat{F}(v + x \hat{F}) - x \hat{F}) \frac{\partial}{\partial v}. \]

The vector fields \( \tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3, \tilde{Y}_4 \) satisfy the commutation relations

\[
\begin{align*}
[\tilde{Y}_1, \tilde{Y}_2] &= \tilde{Y}_3, \\
[\tilde{Y}_1, \tilde{Y}_3] &= \tilde{Y}_4, \\
[\tilde{Y}_1, \tilde{Y}_4] &= (4 + F^2 + 2F) \tilde{Y}_1 - (F \hat{F} + \hat{F})(\tilde{Y}_1 - \tilde{Y}_2), \\
[\tilde{Y}_2, \tilde{Y}_3] &= 2(\tilde{Y}_1 - \tilde{Y}_2) + \tilde{Y}_4, \\
[\tilde{Y}_2, \tilde{Y}_4] &= (2 + 2F^2 + 2F) \tilde{Y}_1 - (F \hat{F} + \hat{F})(\tilde{Y}_1 - \tilde{Y}_2), \\
[\tilde{Y}_3, \tilde{Y}_4] &= -2 \tilde{Y}_4 - 2(\tilde{Y}_1 - \tilde{Y}_2)(4 + F^2 + 2F).
\end{align*}
\]
Consequently, the vector fields \( \tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3, \tilde{Y}_4 \) span the vector fields of the family (22). Adding \( \tilde{Y}_1 \) to each prolongation of the previous set, that is, considering the vector fields \( \tilde{X}_1 = \tilde{Y}_1, \tilde{X}_2 = \tilde{Y}_2, \tilde{X}_3 = \tilde{Y}_3 + \tilde{Y}_4 \) and \( \tilde{X}_4 = \tilde{Y}_4 \), we get that the family of time prolongations, \( \tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4 \), which spans the vector fields of the family (22). The commutation relations among them read

\[
[X_1, \tilde{X}_3] = \tilde{X}_3 - \tilde{X}_1, \\
[X_1, \tilde{X}_4] = \tilde{X}_4 - \tilde{X}_1, \\
[X_2, \tilde{X}_3] = 2\tilde{X}_1 - 2\tilde{X}_2 - \tilde{X}_3 + \tilde{X}_4, \\
[X_2, \tilde{X}_4] = -(1 + 2F + 4\dot{F})\tilde{X}_1 + (4 + 2\dot{F} + 2\ddot{F})\tilde{X}_3, \\
[X_3, \tilde{X}_4] = -3\tilde{X}_4 + (4 + 2\dot{F} + 2\ddot{F})\tilde{X}_3 + (8 + 4\dot{F} + 2\dddot{F} + 4\dddot{F})\tilde{X}_2 \\
+ (-9 + 3\dot{F}^2 - 6\dot{F} - 2\dddot{F} - \dddot{F})\tilde{X}_1.
\]

As a consequence of lemma 9, we get that the vector fields \( \tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4 \) close on the same commutation relations as the vector fields \( X_1, X_2, X_3, X_4 \). Hence, in view of theorem 10, the family (21) is a Lie family and the knowledge of non-trivial first integrals of the vector fields of the distribution \( D \) spanned by \( X_1, X_2, X_3, X_4 \) provides us with a common time-dependent superposition rule.

Note that, as the vector fields \( \tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4 \) and their Lie brackets span the whole distribution \( D \), a function \( G : \mathbb{R} \times T^3 \rightarrow \mathbb{R} \) is a first integral for the vector fields of the distribution \( D \) if and only if it is a first integral for the vector fields \( \tilde{X}_1 \) and \( \tilde{X}_2 \). Therefore, we can reduce the problem of finding first integrals for the vector fields of the distribution \( D \) to finding common first integrals \( G \) for the vector fields \( \tilde{X}_1 \) and \( \tilde{X}_2 \).

Let us analyze the implications of \( G \) being a first integral of the vector field

\[
\tilde{X}_1 - \tilde{X}_2 = \sum_{i=0}^{2} x_i \frac{\partial}{\partial v_i}.
\]

The characteristics of the above vector field are the solutions of the system

\[
\frac{dx_0}{x_0} = \frac{dx_1}{x_1} = \frac{dx_2}{x_2}, \quad dx_0 = 0, \quad dx_1 = 0, \quad dx_2 = 0, \quad dr = 0,
\]

that is, the solutions are the curves in \( \mathbb{R} \times T^3 \) of the form \( s \mapsto (t, x_0, x_1, x_2, v_0(s), v_1(s), v_2(s)) \), with \( \xi_{02} = x_0v_2(s) - x_2v_0(s) \) and \( \xi_{12} = x_1v_2(s) - x_2v_1(s) \) for two real constants \( \xi_{02} \) and \( \xi_{12} \). Thus, there exists a function \( G_2 : \mathbb{R}^6 \rightarrow \mathbb{R} \) such that \( G(p) = G_2(t, x_0, x_1, x_2, \xi_{02}, \xi_{12}) \), with \( p \in \mathbb{R} \times T^3 \), \( \xi_{02} = x_0v_2 - x_2v_0 \), and \( \xi_{12} = x_1v_2 - x_2v_1 \). In other words, \( G \) is a function of \( t, x_0, x_1, x_2, \xi_{02}, \xi_{12} \).

The function \( G \) also satisfies the condition \( \tilde{X}_1G = 0 \) which, in terms of the coordinate system \( t, x_0, x_1, x_2, \xi_{02}, \xi_{12}, v_2 \), reads

\[
\tilde{X}_1G = \frac{\partial G}{\partial t} + \frac{(x_0v_2 - \xi_{02})}{x_2} \frac{\partial G}{\partial x_0} + \frac{(x_1v_2 - \xi_{12})}{x_2} \frac{\partial G}{\partial x_1} + v_2 \frac{\partial G}{\partial x_2} \\
- \left[ \tilde{F}\xi_{12} + e^{-2\tilde{F}} \left( \frac{x_2}{x_1} - \frac{x_1}{x_2} \right) \right] \frac{\partial G}{\partial \xi_{12}} - \left[ \tilde{F}\xi_{02} + e^{-2\tilde{F}} \left( \frac{x_3}{x_0} - \frac{x_0}{x_3} \right) \right] \frac{\partial G}{\partial \xi_{02}} = 0.
\]

That is, defining the vector fields

\[
\Xi_1 = \frac{\partial}{\partial t} - \frac{\xi_{12}}{x_2} \frac{\partial}{\partial x_1} - \frac{\xi_{02}}{x_2} \frac{\partial}{\partial x_0} + \left[ -\tilde{F}\xi_{12} - e^{-2\tilde{F}} \left( \frac{x_2}{x_1} - \frac{x_1}{x_2} \right) \right] \frac{\partial}{\partial \xi_{12}} \\
+ \left[ -\tilde{F}\xi_{02} - e^{-2\tilde{F}} \left( \frac{x_3}{x_0} - \frac{x_0}{x_3} \right) \right] \frac{\partial}{\partial \xi_{02}},
\]
the condition \( \bar{\lambda}_1 G = 0 \) implies that \( \bar{\Xi}_1 G_2 + v_2 \Xi_2 G_2 = 0 \) and, as \( G_2 \) does not depend on \( v_2 \), the function \( G \) must be simultaneously a first integral for \( \Xi_1 \) and \( \Xi_2 \), i.e. \( \Xi_1 G = 0 \) and \( \Xi_2 G = 0 \).

Applying again the method of characteristics to the vector field \( \Xi_2 \), we get that \( F \) can depend just on the variables \( t, \xi_{12}, \Delta_{12} = x_0/x_2 \) and \( \Delta_{12} = x_1/x_2 \), that is, there exists a function \( G_3 : \mathbb{R}^5 \rightarrow \mathbb{R} \) such that \( G(t, x_0, x_1, x_2, v_0, v_1, v_2) = G_3(t, x_0, x_1, x_2, \xi_{12}, \xi_{12}, \Delta_{12}, \Delta_{12}) \).

We are left to check out the implications of the equation \( \Xi_1 G = 0 \). Using the coordinate system \( \{ t, \xi_{12}, \Delta_{12}, \xi_{12}, \Delta_{12} \} \) and taking into account that \( G(t, x_0, x_1, x_2, v_0, v_1, v_2) = G_3(t, \xi_{12}, \Delta_{12}, \Delta_{12}) \), the previous equation can be cast into the form \( \Xi_1 G = \frac{1}{\xi_{12}^2} \Upsilon_1 G_3 + \Upsilon_2 G_3 = 0 \), where

\[
\Upsilon_1 = \sum_{i=0}^{1} \left[ -\xi_{12} \frac{\partial}{\partial \Delta_{12}} - e^{-2F} \left( \Delta_{12}^2 - \Delta_{12} \right) \frac{\partial}{\partial \xi_{12}} \right],
\]

\[
\Upsilon_2 = -F \xi_{12} \frac{\partial}{\partial \xi_{12}} - F \xi_{12} \frac{\partial}{\partial \xi_{12}} + \frac{\partial}{\partial t}.
\]

As \( G_3 \) depends on the variables \( t, \Delta_{12}, \xi_{12}, \Delta_{12} \), \( \Upsilon_1 G = 0 \) and \( \Upsilon_2 G = 0 \). Repeating \textit{mutatis mutandis} the previous procedures in order to determine the implications of being a first integral of \( \Upsilon_1 \) and \( \Upsilon_2 \), we finally get that the first-integrals of the distribution \( \mathcal{D} \) are the functions of \( I_1, I_2 \) and \( I \), with

\[
I_i = e^{2F} (x_0 v_i - x_i v_0)^2 + \left[ \frac{x_0}{x_i} \right]^2 + \left[ \frac{x_i}{x_0} \right]^2, \quad i = 1, 2,
\]

and

\[
I = e^{2F} (x_1 v_2 - x_2 v_1)^2 + \left[ \frac{x_1}{x_2} \right]^2 + \left[ \frac{x_2}{x_1} \right]^2.
\]

Defining \( \bar{v}_2 = e^{-F} v_2, \bar{v}_1 = e^{-F} v_1 \) and \( \bar{v}_0 = e^{F} v_0 \), the above first integrals read

\[
I_i = (x_0 \bar{v}_i - x_i \bar{v}_0)^2 + \left[ \frac{x_0}{x_i} \right]^2 + \left[ \frac{x_i}{x_0} \right]^2, \quad i = 1, 2,
\]

and

\[
I = (x_1 \bar{v}_2 - x_2 \bar{v}_1)^2 + \left[ \frac{x_1}{x_2} \right]^2 + \left[ \frac{x_2}{x_1} \right]^2.
\]

Note that these first integrals have the same form as the ones considered in [10] for \( k = 1 \). Therefore, we can apply the procedure done there to obtain that

\[
x_0 = \sqrt{k_1 x_1^2 + k_2 x_2^2 + 2\sqrt{\lambda_{12}} \left[ -(x_1^4 + x_2^4) + I \right] x_1^2 x_2^2},
\]

with \( \lambda_{12} \) being a function of the form

\[
\lambda_{12}(k_1, k_2, I) = \frac{k_1 k_2 I + (-1 + k_1^2 + k_2^2)}{I^2 - 4},
\]

and where the constants \( k_1 \) and \( k_2 \) satisfy special conditions in order to ensure that \( x_0 \) is real [11].
Expression (23) permits us to determine the general solution, $x(t)$, of any instance of family (20) in the form
\[
x(t) = \sqrt{k_1 x_1^2(t) + k_2 x_2^2(t) + 2\sqrt{\lambda_1 2 \left[ -\left( x_1^4(t) + x_2^4(t) \right) + I x_1^2(t) x_2^2(t) \right]},}
\]
with
\[
I = e^{2F(t)} (x_1(t) \dot{x}_2(t) - x_2(t) \dot{x}_1(t))^2 + \left[ \left( \frac{x_1(t)}{x_2(t)} \right)^2 + \left( \frac{x_2(t)}{x_1(t)} \right)^2 \right],
\]
in terms of two of its particular solutions, $x_1(t)$, $x_2(t)$, its derivatives, the constants $k_1$ and $k_2$, and the time (included in the constant of motion I).

Note that the role of the constant $I$ in expression (24) differs from the roles carried out by $k_1$ and $k_2$. Indeed, the value of $I$ is fixed by the particular solutions $x_1(t)$, $x_2(t)$ and its derivatives, while, for every pair of generic solutions $x_1(t)$ and $x_2(t)$, the values of $k_1$ and $k_2$ range within certain intervals ensuring that $x(t)$ is real.

It is clear that the method illustrated here can also be applied to analyze the solutions of any other family of second-order differential equations related to a Lie family by introducing the new variable $v = \dot{x}$. Additionally, it is worth noting that in the case $F(t) = 0$ the family of dissipative Milne–Pinney equations (20) reduces to a family of Milne–Pinney equations appearing broadly in the literature (see [48] and references therein), and expression (24) takes the form of the expression obtained in [11] for these equations.

6. Conclusions and outlook

We have proposed a generalization of the Lie theorem in order to characterize those families of non-autonomous systems of first-order ordinary differential equations, the so-called Lie families, which admit a common time-dependent superposition rule. We have studied the relations of quasi-Lie systems and Lie families.

In order to illustrate the usefulness of our achievements, we have derived common time-dependent superposition rules for studying dissipative Milne–Pinney equations and Abel equations. In the case of Abel equations, our result expresses the general solution of any particular instance of a Lie family of non-autonomous Abel equations in terms of each generic particular solution, a constant, and the time. In this way, we have initiated a new approach to study the solutions of these equations. Additionally, it is worth noting that the analyzed Lie family of Abel equations contains an autonomous instance admitting a special kind of superposition rule derived by Chiellini [31]. Unlike such a special superposition rule, our common superposition rule does not require the use of particular solutions obeying any kind of extra condition and, therefore, it clearly represents an improvement with respect to Chiellini’s technique.

We have shown how common time-dependent superposition rules can be used to analyze second-order differential equations by means of the study of a family of Milne–Pinney equations. More specifically, we have derived a common-superposition rule allowing us to obtain the general solution of any instance of such a family in terms of a generic pair of its particular solutions, their derivatives in terms of the time, and the time. Such an expression represents an interesting improvement with respect to the previous results and methods, as it generalizes the superposition rule given in [11] for the usual Milne–Pinney equations to the dissipative case.

We hope to get in the future new results on the theory of common time-dependent superposition rules and, additionally, to describe new applications, where our achievements can be used.
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