ON A SDE DRIVEN BY A FRACTIONAL BROWNIAN MOTION AND WITH MONOTONE DRIFT

BRAHIM BOUFOUSSI
Cadi Ayyad University FSSM, Department of Mathematics, P.B.O 2390 Marrakesh, Morocco.
email: boufoussi@ucam.ac.ma

YOUSSEF OUKNINE
Cadi Ayyad University FSSM, Department of Mathematics P.B.O 2390 Marrakesh, Morocco.
email: ouknine@ucam.ac.ma

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Abstract
Let \( B^H \) be a fractional Brownian motion with Hurst parameter \( H > \frac{1}{2} \). We prove the existence of a weak solution for a stochastic differential equation of the form \( X_t = x + B^H_t + \int_0^t (b_1(s, X_s) + b_2(s, X_s)) \, ds \), where \( b_1(s, x) \) is a Hölder continuous function of order strictly larger than \( 1 - \frac{1}{2H} \) in \( x \) and than \( H - \frac{1}{2} \) in time and \( b_2 \) is a real bounded nondecreasing and left (or right) continuous function.

1 Introduction
Let \( B^H = \{ B^H_t, t \in [0, T] \} \) be a fractional Brownian motion with Hurst parameter \( H \in (0, 1) \). That is, \( B^H \) is a centered Gaussian process with covariance
\[
R_H(t, s) = E(B^H_t B^H_s) = \frac{1}{2} \left( |t|^{2H} + |s|^{2H} - |t-s|^{2H} \right).
\]
If \( H = \frac{1}{2} \) the process \( B^H \) is a standard Brownian motion. Consider the following stochastic differential equation
\[
X_t = x + B^H_t + \int_0^t (b_1(s, X_s) + b_2(s, X_s)) \, ds,
\]
where \( b_1, b_2 : [0, T] \times \mathbb{R} \to \mathbb{R} \) are Borel functions. The purpose of this paper is to prove, by approximation arguments, the existence of a weak solution to this equation if \( H > \frac{1}{2} \), under the following weak regularity assumptions on the coefficients:

1RESEARCH SUPPORTED BY PSR PROGRAM 2001, CADI AYYAD UNIVERSITY.
2RESEARCH SUPPORTED BY PSR PROGRAM 2001, CADI AYYAD UNIVERSITY.
(H₁) $b_1$ is Hölder continuous of order $1 > \alpha > 1 - \frac{1}{2H}$ in $x$ and of order $\gamma > H - \frac{1}{2}$ in time:

$$|b_1(t, x) - b_1(s, y)| \leq C (|x - y|^{\alpha} + |t - s|^{\gamma}).$$  \hspace{1cm} (1.2)

(H₂) $\sup_{s \in [0, T]} \sup_{x \in \mathbb{R}} |b_2(s, x)| \leq M < \infty.$

(H₃) $\forall s \in [0, T], \ b_2(s, .)$ is a nondecreasing and left (or right) continuous function.

The same approximation arguments can be used to consider the case where $b_2$ satisfies the following assumptions:

(H₂') $\sup_{s \in [0, T]} \sup_{x \in \mathbb{R}} |b_2(s, x)| \leq M(1 + |x|)$

(H₃') for all $s \in [0, T], \ b_2(s, .)$ is a nonincreasing and continuous function

If $b_2 \equiv 0$ and $H = \frac{1}{2}$ (the process $B^H$ is a standard Brownian motion), the existence of a strong solution is well-known by the results of Zvonkin [18], Veretennikov [16] and Bahlali [2]. See also the work by Nakao [11] and its generalization by Ouknine [14]. In the case of Equation (1.1) driven by the fractional Brownian motion with $b_2 \equiv 0$, the weak existence and uniqueness are established in [13] using a suitable version of Girsanov theorem; the existence of a strong solution could be deduced from an extension of Yamada-Watanabe’s theorem or by a direct arguments.

In the general case $H > 1/2$, to establish existence and uniqueness result, a Hölder type space-time condition is imposed on the drift. Recently, Mishura and Nualart [9] gave an existence and uniqueness result for one discontinuous function namely the $sgn$ function. Their approach relies on the Novikov criterion and it is valid for $\frac{1+\sqrt{2}}{2} > H > 1/2$.

Our aim is to establish existence and uniqueness result for general monotone function including $sgn$ function and $H > 1/2$.

The paper is organized as follows. In Section 2 we give some preliminaries on fractional calculus and fractional Brownian motion. In Section 3 we formulate a Girsanov theorem and show the existence of a weak solution to Equation (1.1). As a consequence we deduce the uniqueness in law and the pathwise uniqueness. Finally Section 4 discusses the existence of a strong solution.

2 Preliminaries

2.1 Fractional calculus

An exhaustive survey on classical fractional calculus can be found in [15]. We recall some basic definitions and results.

For $f \in L^1((a, b))$ and $\alpha > 0$ the left fractional Riemann-Liouville integral of $f$ of order $\alpha$ on $(a, b)$ is given at almost all $x$ by

$$I_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - y)^{\alpha - 1} f(y) dy,$$

where $\Gamma$ denotes the Euler function.

This integral extends the usual $n$-order iterated integrals of $f$ for $\alpha = n \in \mathbb{N}$. We have the first composition formula
The fractional derivative can be introduced as inverse operation. We assume \( 0 < \alpha < 1 \) and \( p > 1 \). We denote by \( I_{a+}^{\alpha}(L^p) \) the image of \( L^p([a,b]) \) by the operator \( I_{a+}^{\alpha} \). If \( f \in I_{a+}^{\alpha}(L^p) \), the function \( \phi \) such that \( f = I_{a+}^{\alpha}\phi \) is unique in \( L^p \) and it agrees with the left-sided Riemann-Liouville derivative of \( f \) of order \( \alpha \) defined by

\[
D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x-y)^\alpha} dy.
\]

The derivative of \( f \) has the following Weil representation:

\[
D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy \right) 1_{[a,b]}(x),
\]

(2.1)

where the convergence of the integrals at the singularity \( x = y \) holds in \( L^p \)-sense.

When \( \alpha p > 1 \) any function in \( I_{a+}^{\alpha}(L^p) \) is \( (\alpha - \frac{1}{p})\)-Hölder continuous. On the other hand, any Hölder continuous function of order \( \beta > \alpha \) has fractional derivative of order \( \alpha \). That is, \( C^\beta([a,b]) \subset I_{a+}^{\alpha}(L^p) \) for all \( p > 1 \).

Recall that by construction for \( f \in I_{a+}^{\alpha}(L^p) \),

\[
I_{a+}^{\alpha}(D_{a+}^\alpha f) = f
\]

and for general \( f \in L^1([a,b]) \) we have

\[
D_{a+}^\alpha(I_{a+}^{\alpha} f) = f.
\]

If \( f \in I_{a+}^{\alpha+\beta}(L^1) \), \( \alpha \geq 0, \beta \geq 0, \alpha + \beta \leq 1 \) we have the second composition formula

\[
D_{a+}^\alpha(D_{a+}^\beta f) = D_{a+}^{\alpha+\beta} f.
\]

### 2.2 Fractional Brownian motion

Let \( B^H = \{ B_t^H, t \in [0,T] \} \) be a fractional Brownian motion with Hurst parameter \( 0 < H < 1 \) defined on the probability space \( (\Omega, \mathcal{F}, P) \). For each \( t \in [0,T] \) we denote by \( \mathcal{F}_t^H \) the \( \sigma \)-field generated by the random variables \( B_s^H, s \in [0,t] \) and the sets of probability zero.

We denote by \( \mathcal{E} \) the set of step functions on \( [0,T] \). Let \( \mathcal{H} \) be the Hilbert space defined as the closure of \( \mathcal{E} \) with respect to the scalar product

\[
\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s).
\]

The mapping \( 1_{[0,t]} \longrightarrow B_t^H \) can be extended to an isometry between \( \mathcal{H} \) and the Gaussian space \( H_1(B_t^H) \) associated with \( B_t^H \). We will denote this isometry by \( \varphi \longrightarrow B^H(\varphi) \).

The covariance kernel \( R_H(t,s) \) can be written as

\[
R_H(t,s) = \int_0^{t \wedge s} K_H(t,r)K_H(s,r)dr,
\]

where \( K_H \) is a square integrable kernel given by (see [3]):

\[
K_H(t,s) = \Gamma(H + \frac{1}{2})^{-1}(t-s)^{H-\frac{1}{2}} F(H - \frac{1}{2}, \frac{1}{2}; -H, H + \frac{1}{2}, 1 - \frac{t}{s}),
\]
On a SDE driven by a fractional Brownian motion

Let \( F(a, b, c, z) \) be the Gauss hypergeometric function. Consider the linear operator \( K_H^* \) from \( E \) to \( L^2([0, T]) \) defined by

\[
(K_H^* \varphi)(s) = K_H(T, s) \varphi(s) + \int_s^T (\varphi(r) - \varphi(s)) \frac{\partial K_H}{\partial r}(r, s) dr.
\]

For any pair of step functions \( \varphi \) and \( \psi \) in \( E \) we have (see [1])

\[
\langle K_H^* \varphi, K_H^* \psi \rangle_{L^2([0, T])} = \langle \varphi, \psi \rangle_{\mathcal{H}}.
\]

As a consequence, the operator \( K_H^* \) provides an isometry between the Hilbert spaces \( \mathcal{H} \) and \( L^2([0, T]) \). Hence, the process \( W = \{W_t, t \in [0, T]\} \) defined by

\[
W_t = B^H((K_H^*)^{-1}(\mathbf{1}_{[0, t]}))
\]

is a Wiener process, and the process \( B^H \) has an integral representation of the form

\[
B^H_t = \int_0^t K_H(t, s) dW_s,
\]

because \( (K_H^* \mathbf{1}_{[0, t]})(s) = K_H(t, s) \mathbf{1}_{[0, t]}(s) \).

On the other hand, the operator \( K_H \) on \( L^2([0, T]) \) associated with the kernel \( K_H \) is an isomorphism from \( L^2([0, T]) \) onto \( L^{H+1/2}([0, T]) \) and it can be expressed in terms of fractional integrals as follows (see [3]):

\[
(K_H h)(s) = \mathcal{I}^{2H}_{0^+} s^{\frac{1}{2} - H} \mathcal{I}^{1-H}_{0^+} s^{H - \frac{1}{2}} h, \text{ if } H \leq 1/2,
\]

\[
(K_H h)(s) = \mathcal{I}^{1-H}_{0^+} s^{\frac{1}{2} - H} \mathcal{I}^{H - \frac{1}{2}}_{0^+} s^{\frac{1}{2} - H} h, \text{ if } H \geq 1/2,
\]

where \( h \in L^2([0, T]) \).

We will make use of the following definition of \( \mathcal{F}_t \)-fractional Brownian motion.

**2.1 Definition.** Let \( \{\mathcal{F}_t, t \in [0, T]\} \) be a right-continuous increasing family of \( \sigma \)-fields on \( (\Omega, \mathcal{F}, P) \) such that \( \mathcal{F}_0 \) contains the sets of probability zero. A fractional Brownian motion \( B^H = \{B^H_t, t \in [0, T]\} \) is called an \( \mathcal{F}_t \)-fractional Brownian motion if the process \( W \) defined in (2.2) is an \( \mathcal{F}_t \)-Wiener process.

### 3 Existence of strong solution for SDE with monotone drift.

In this section we are interested by the special case \( b_1 \equiv 0 \). We will prove by approximation arguments that there is a strong solution of equation (1.1). We will discuss two cases:

1) \( b_2(s, \cdot) \) satisfies \( (H_2) \) and \( (H_3) \).
2) \( b_2(s, \cdot) \) satisfies \( (H'_2) \) and \( (H'_3) \).

**1– The first case:**

To treat the first situation, let us suppose that \( b_2(s, \cdot) \) is nondecreasing and left continuous function. We will use the following approximation lemma:
3.1 Lemma. Let $b : [0, T] \times \mathbb{R} \to \mathbb{R}$, a bounded measurable function such that for any $s \in [0, T]$, $b(s, \cdot)$ is a nondecreasing and left continuous function. Then there exists a family of measurable functions
\[
\{b_n(s, x) ; \ n \geq 1, \ s \in [0, T], x \in \mathbb{R}\}
\]
such that
\[
\begin{aligned}
&\bullet \text{ For any sequence } x_n \text{ increasing to } x \in \mathbb{R}, \text{ we have } \\
&\quad\quad\lim_{n \to \infty} b_n(s, x_n) = b(s, x), \quad ds \text{ a.e.} \\
&\bullet \ x \mapsto b_n(s, x) \text{ is nondecreasing, for all } n \geq 1, \ s \in [0, T] \\
&\bullet \ n \mapsto b_n(s, x) \text{ is nondecreasing, for all } x \in \mathbb{R}, \ s \in [0, T] \\
&\bullet \ |b_n(s, x) - b_n(s, y)| \leq 2n M |x - y| \text{ for all } n \geq 1, \ s \in [0, T] \\
&\bullet \sup_{n \geq 1} \sup_{s \in [0, T]} \sup_{x \in \mathbb{R}} |b_n(s, x)| \leq M.
\end{aligned}
\]

Proof. First assume that $b(\cdot, \cdot)$ is left continuous and let us choose for any $n \geq 1$
\[
b_n(s, x) = n \int_{x - \frac{1}{n}}^{x} b(s, y) \, dy.
\]
Since $b(\cdot, \cdot)$ is nondecreasing then $b_n(\cdot, \cdot)$ is also a nondecreasing function for any fixed $n \geq 1$. Let $x, y \in \mathbb{R}$, we clearly have for any $n \geq 1$,
\[
|b_n(s, x) - b_n(s, y)| \leq 2n M |x - y|.
\]
(3.1)
Obviously, we get that $b_n$ is uniformly bounded by the constant $M$. Let $n < m$, $s \in [0, T]$ and $x \in \mathbb{R}$, we have
\[
\begin{aligned}
b_m(s, x) - b_n(s, x) &= (m - n) \int_{x - \frac{1}{n}}^{x} b(s, y) \, dy - n \int_{x - \frac{1}{m}}^{x - \frac{1}{n}} b(s, y) \, dy \\
&\geq (m - n) \int_{x - \frac{1}{n}}^{x} b(s, y) \, dy - \frac{m - n}{m} b(s, x - \frac{1}{m}), \\
&= (m - n) \int_{x - \frac{1}{n}}^{x} \left(b(s, y) - b(s, x - \frac{1}{m})\right) \, dy \geq 0.
\end{aligned}
\]
Now let $x_0 \in \mathbb{R}$ and take an increasing sequence of real numbers $x_n$ converging to $x_0$. We want to show that for any $s \in [0, T]$, \(\lim_{n \to \infty} b_n(s, x_n) = b(s, x_0)\). It is enough to prove that there exists a subsequence $b_{\varphi(n)}(s, x_{\varphi(n)})$ which converges to $b(s, x_0)$. To do this, remark first that since $b(\cdot, \cdot)$ is left continuous we have \(\lim_{n \to \infty} b_n(s, x_0) = b(s, x_0)\). Now let us consider any strictly increasing sequence $x'_n$ converging to $x_0$ such that $x_0 - x'_n = o(\frac{1}{n})$. We clearly get by (3.1)
\[
\forall s \in [0, T], \lim_{n \to \infty} b_n(s, x'_n) = b(s, x_0).
\]
(3.2)
We may choose a sequence $\varphi(n) \geq n$ such that $x'_n \leq x_{\varphi(n)}$. Since $(b_n(s, x))_{n \geq 1}$ is increasing and for any fixed $n \geq 1$ the function $b_n(s, \cdot)$ is nondecreasing, we have
\[
b_n(s, x'_n) \leq b_{\varphi(n)}(s, x_{\varphi(n)}) \leq b(s, x_{\varphi(n)}).
\]
(3.3)
We deduce by (3.2) and the left continuity of \( b(s,\cdot) \),

\[
\lim_{n \to \infty} b_{\varphi(n)}(s, x_{\varphi(n)}) = b(s, x_0) .
\]

Which ends the proof.

Let \((B^H)_t\) a fractional Brownian motion with Hurst parameter \( H \in (\frac{1}{2}, 1) \). We consider the following SDE

\[
X_t = x + B^H_t + \int_0^t b_2(s, X_s) \, ds, \quad 0 \leq t \leq T .
\]

3.2 Theorem. Suppose that \( b_2 \) satisfies the assumptions \((H_2)\) and \((H_3)\). Then there exists a strong solution to the equation (3.4).

Proof. Assume that \( b_2 : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) is a measurable and bounded function which is nondecreasing and left continuous with respect to the space variable \( x \). For \( n \geq 1 \), let \( b_n \) be as in lemma 3.1 and consider the following SDE

\[
X^n_t = x + B^H_t + \int_0^t b_n(s, X^n_s) \, ds, \quad 0 \leq t \leq T .
\]

By standard Picard’s iteration argument, one may show that for any \( n \geq 1 \), the equation (3.5) has a strong solution which we denote by \( X^n_t \).

Let \( n > m \), we denote by \( \Delta_t = X^n_t - X^m_t \). Using the monotony argument on \( b_n \), we have

\[
\Delta_t \geq \int_0^t \left( b_m(s, X^n_s) - b_m(s, X^m_s) \right) ds ,
\]

\[
\geq \int_0^t \left( b_m(s, X^n_s) - b_m(s, X^m_s) \right) I_{\{ \Delta_s \leq 0 \}} ds ,
\]

\[
\geq 2 m M \int_0^t \Delta_s I_{\{ \Delta_s \leq 0 \}} ds \geq -2 m M \int_0^t \Delta_s^- ds .
\]

We then get

\[
\Delta_t^- \leq 2 m M \int_0^t \Delta_s^- ds .
\]

By Gronwall’s lemma, we have for almost all \( w \) and for any \( t \in [0, T] \), the sequence \((X^n_t(w))\) is a nondecreasing function of \( n \) which is bounded since \( b_n \) is. Therefore it has a limit when \( n \to \infty \) and we set

\[
\lim_{n \to \infty} X^n_t(\omega) = X_t(\omega) ,
\]

which entails in particular that \( X \) is \( \mathcal{F}^H_t \) adapted. Applying the convergence result in Lemma 3.1 and the boundedness of \( b_n \) we get by Lebesgue’s dominated convergence theorem,

\[
X_t = x + B^H_t + \int_0^t b_2(s, X_s) \, ds .
\]
3.1 Remark. To show that Equation (3.4) has a weak solution, a continuity condition is imposed on the drift in [13]. Here, the function $b_2$ may have a countable set of discontinuity points. The solution constructed in Theorem 3.2 is the minimal one.

3.2 Remark. Let $b_2 : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a bounded measurable function, which is nondecreasing and right continuous. In this case we consider a decreasing sequence of Lipschitz continuous functions which approximate the drift. One may take

$$b_n(s, x) = n \int_x^{x + \frac{1}{n}} b_2(s, y) \, dy.$$ 

For any fixed $(s, x) \in [0, T] \times \mathbb{R}$, the sequence $(b_n(s, x))_{n \geq 1}$ is nonincreasing and for any fixed $n \geq 1$ and $s \in [0, T]$ the function $b_n(s, \cdot)$ is nondecreasing. The same arguments as in Lemma 3.1 can be used to prove that for any sequence $(x_n)_{n \geq 1}$ decreasing to $x$, we have

$$\lim_{n \to \infty} b_n(s, x_n) = b_2(s, x).$$ 

This allows us to construct the maximal solution to the equation (3.4).

2. The second case:

In this case we use the following lemma:

3.3 Lemma. Let $b(\cdot, \cdot) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function with linear growth, that is there exists a constant $M < \infty$ such that $\forall (s, x) \in [0, T] \times \mathbb{R}$, $|b(s, x)| \leq M (1 + |x|)$. Then the sequence of functions

$$b_n(s, x) = \sup_{y \in \mathbb{Q}} (b(s, y) - n |x - y|),$$

is well defined for $n \geq M$ and it satisfies

\[
\begin{aligned}
&\quad \lim_{n \to \infty} b_n(s, x_n) = b(s, x), \\
&n \mapsto b_n(s, x) \text{ is nonincreasing, for all } x \in \mathbb{R}, \ s \in [0, T] \\
&|b_n(s, x) - b_n(s, y)| \leq n |x - y| \text{ for all } n \geq M, \ s \in [0, T], \ x, y \in \mathbb{R} \\
&|b_n(s, x)| \leq M(1 + |x|), \text{ for all } (s, x) \in [0, T] \times \mathbb{R}, \ n \geq M.
\end{aligned}
\]

For the proof of this lemma we refer for example to [8].

3.4 Theorem. Assume that $b_2$ satisfies conditions $H^I_2$ and $H^I_3$. Then there exists a unique strong solution to the equation (3.4).

Proof. For any $n \geq 1$, let $b_n$ be as in Lemma 3.3. Since $b_n$ is Lipschitz and linear growth, the result in [13] assures the existence of a strong solution $X^n$ to the equation

$$X^n_t = x + B^H_t + \int_0^t b_n(s, X^n_s) \, ds.$$ 

Since $(b_n)_{n \geq 1}$ is nonincreasing, comparison theorem entails that $(X^n)_{n \geq 1}$ is a.s. nonincreasing. By the linear growth condition on $b_n$ and Gronwall’s lemma we may deduce that $X^n$ converges
a.s to $X$, which is clearly a strong solution to the SDE (3.4). Moreover, if $X^1$ and $X^2$ are two solutions of (3.4), using the fact that $b_2(s,.)$ is nonincreasing, we get by applying Tanaka’s formula to the continuous semi-martingale $X^1 - X^2$,

$$(X_t^1 - X_t^2)^+ = \int_0^t \text{sign}(X_s^1 - X_s^2) \left( b_2(s, X_s^1) - b_2(s, X_s^2) \right) ds \leq 0.$$ 

Then we have the pathwise uniqueness of the solution. \hfill \square

### 4 Existence of a weak solution

#### 4.1 Girsanov transform

As in the previous section, let $B^H$ be a fractional Brownian motion with Hurst parameter $0 < H < 1$ and denote by $\mathcal{F}_t^B$, $t \in [0,T]$ its natural filtration.

Given an adapted process with integrable trajectories $u = \{u_t, t \in [0,T]\}$ and consider the transformation

$$\tilde{B}_t^H = B_t^H + \int_0^t u_s ds.$$  \hfill (4.1)

We can write

$$\tilde{B}_t^H = B_t^H + \int_0^t u_s ds = \int_0^t K_H(t,s)dW_s + \int_0^t u_s ds$$

where

$$\tilde{W}_t = W_t + \int_0^t \left( K_H^{-1} \left( \int_0^r u_s ds \right) (r) \right) dr.$$  \hfill (4.2)

Notice that $K_H^{-1} (\int_0^r u_s ds)$ belongs a.s to $L^2([0,T])$ if and only if $\int_0^T u_s ds \in L_H^{1+1/2}(L^2([0,T]))$.

As a consequence we deduce the following version of the Girsanov theorem for the fractional Brownian motion, which has been obtained in [3, Theorem 4.9]:

**4.1 Theorem.** Consider the shifted process (4.1) defined by a process $u = \{u_t, t \in [0,T]\}$ with integrable trajectories. Assume that:

i) $\int_0^T u_s ds \in L_H^{1+1/2}(L^2([0,T]))$, almost surely.

ii) $E(\xi_T) = 1$, where

$$\xi_T = \exp \left( -\int_0^T \left( K_H^{-1} \int_0^r u_s ds \right) (s)dW_s - \frac{1}{2} \int_0^T \left( K_H^{-1} \int_0^r u_s ds \right)^2 (s)ds \right).$$

Then the shifted process $\tilde{B}^H$ is an $\mathcal{F}_t^B$-fractional Brownian motion with Hurst parameter $H$ under the new probability $\tilde{P}$ defined by $\frac{d\tilde{P}}{dP} = \xi_T$. 

Proof. By the standard Girsanov theorem applied to the adapted and square integrable process $K_H^{-1} \left( \int_0^t u_s ds \right)$ we obtain that the process $\tilde{W}$ defined in (4.2) is an $\mathcal{F}_t^{B^H}$-Brownian motion under the probability $\tilde{P}$. Hence, the result follows.

From (2.5) the inverse operator $K_H^{-1}$ is given by

$$K_H^{-1}h = s^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} h', \text{ if } H > 1/2$$

(4.3)

for all $h \in I_0^{H+\frac{1}{2}}(L^2([0,T]))$. Then if $H > \frac{1}{2}$ we need $u \in I_0^{H-1/2}(L^2([0,T]))$, and a sufficient condition for i) is the fact that the trajectories of $u$ are Hölder continuous of order $H - \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$.

4.2 Existence of a weak solution

Consider the stochastic differential equation:

$$X_t = x + B_t^H + \int_0^t \left( b_1(s, X_s) + b_2(s, X_s) \right) ds, \quad 0 \leq t \leq T,$$

(4.4)

where $b_1$ and $b_2$ are Borel functions on $[0, T] \times \mathbb{R}$ satisfying the conditions $\mathbf{H}_1$ for $b_1$ and $\mathbf{H}_2$ and $\mathbf{H}_3$ (resp. $\mathbf{H}_2'$ and $\mathbf{H}_3'$) for $b_2$. By a weak solution to equation (4.4) we mean a couple of adapted continuous processes $(B^H, X)$ on a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t, t \in [0, T]\})$, such that:

i) $B^H$ is an $\mathcal{F}_t$-fractional Brownian motion in the sense of Definition 2.1.

ii) $X$ and $B^H$ satisfy (4.4).

4.2 Theorem. Suppose that $b_1$ and $b_2$ are Borel functions on $[0, T] \times \mathbb{R}$ satisfying the conditions $\mathbf{H}_1$ for $b_1$, $\mathbf{H}_2$ and $\mathbf{H}_3$ (resp. $\mathbf{H}_2'$ and $\mathbf{H}_3'$) for $b_2$. Then Equation (4.4) has a weak solution.

Proof. Let $X^2$ be the strong solution of (3.4) and set $\tilde{B}_t^H = B_t^H - \int_0^t b_1(s, X_s^2) ds$. We claim that the process $u_s = -b_1(s, X_s^2)$ satisfies conditions i) and ii) of Theorem 4.1. If this claim is true, under the probability measure $\tilde{P}$, $\tilde{B}^H$ is an $\mathcal{F}_t^{B^H}$-fractional Brownian motion, and $(\tilde{B}^H, X^2)$ is a weak solution of (4.4) on the filtered probability space $(\Omega, \mathcal{F}, \tilde{P}, \{\mathcal{F}_t^{B^H}, t \in [0, T]\})$.

Set

$$v_s = -K_H^{-1} \left( \int_0^s b_1(r, X_r^2) dr \right)(s).$$

We will show that the process $v$ satisfies conditions i) and ii) of Theorem 4.1. Along the proof $c_H$ will denote a generic constant depending only on $H$. Let $H > \frac{1}{2}$, by (4.3), the process $v$ is clearly adapted and we have

$$v_s = -s^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} b_1(s, X_s^2)$$

$$:= -c_H (\alpha(s) + \beta(s)), $$
where
\[
\alpha(s) = b_1(s, X^2_s) s^{\frac{1}{2} - H} + (H - \frac{1}{2}) s^{H - \frac{3}{2}} b_1(s, X^2_s) \int_0^s \frac{s^{\frac{1}{2} - H} - r^{\frac{1}{2} - H}}{(s - r)^{\frac{1}{2} + H}} dr \\
+ (H - \frac{1}{2}) s^{H - \frac{3}{2}} \int_0^s \frac{b_1(s, X^2_s) - b_1(r, X^2_r)}{(s - r)^{\frac{1}{2} + H}} r^{\frac{1}{2} - H} dr,
\]

and
\[
\beta(s) = (H - \frac{1}{2}) s^{H - \frac{3}{2}} \int_0^s \frac{b_1(r, X^2_s) - b_1(r, X^2_r)}{(s - r)^{\frac{1}{2} + H}} r^{\frac{1}{2} - H} dr.
\]

Using the estimate
\[
|b_1(s, X^2_s)| \leq |b(0, x)| + C \left(|s|^\gamma + |X^2_s|^{\alpha}\right)
\]
and the equality
\[
\int_0^s \frac{r^{\frac{1}{2} - H} - s^{\frac{1}{2} - H}}{(s - r)^{\frac{1}{2} + H}} dr = c_H s^{1-2H},
\]

we obtain
\[
|\alpha(s)| \leq c_H \left(s^{\frac{1}{2} - H} \left(|b_1(0, x)| + C \left(|s|^\gamma + |X^2_s|^{\alpha}\right)\right) + C s^{\gamma + \frac{1}{2} - H}\right)
\]
\[
\leq c_H s^{\frac{1}{2} - H} \left(C \|X^2\|^{\alpha}_\infty + C s^{\gamma} + |b_1(0, x)|\right).
\]

As consequence, taking into account that \(\alpha < 1\), we have for any \(\lambda > 1\)
\[
E \left(\exp \left(\lambda \int_0^T \alpha(s)^2 ds\right)\right) < \infty.
\] (4.5)

In order to estimate the term \(\beta(s)\), we apply the Hölder continuity condition (1.2) and we get
\[
|\beta(s)| \leq c_H s^{H - \frac{3}{2}} \int_0^s \left(\frac{|X^2_r - X^2_s|^{\alpha}}{(s - r)^{H + \frac{3}{2}}} + \frac{|r - s|^\gamma}{(s - r)^{\frac{1}{2} + H}}\right) r^{\frac{1}{2} - H} dr
\]
\[
\leq c_H s^{H - \frac{3}{2}} \int_0^s \left(\frac{|B^H_r - B^H_s|^{\alpha}}{(s - r)^{H + \frac{3}{2}}} + (s - r)^{\alpha - H - \frac{1}{2}} + \frac{|r - s|^\gamma}{(s - r)^{\frac{1}{2} + H}}\right) r^{\frac{1}{2} - H} dr
\]
\[
\leq c_H s^{\frac{1}{2} - H + \alpha(H - \frac{1}{2})} G^\alpha,
\]
where we have fixed \(\varepsilon < H - \frac{1}{2} \alpha (H - \frac{1}{2})\) and we denote
\[
G = \sup_{0 \leq s < r \leq 1} \frac{|B^H_s - B^H_r|}{|s - r|^{H - \varepsilon}}.
\]

By Fernique’s Theorem, taking into account that \(\alpha < 1\), for any \(\lambda > 1\) we have
\[
E \left(\exp \left(\lambda \int_0^T \beta(s)^2 ds\right)\right) < \infty,
\]
and we deduce condition ii) of Theorem 4.1 by means of Novikov criterion. □
4.3 Uniqueness in law and pathwise uniqueness

In this subsection we will prove uniqueness in law of weak solution under the condition $H_1$ for $b_1$, $H'_2$ and $H'_3$ for $b_2$. The main result is

**4.3 Theorem.** Suppose that $b_1$ and $b_2$ are Borel functions on $[0, T] \times \mathbb{R}$, satisfying the conditions $H_1$ for $b_1$, $H'_2$ and $H'_3$ for $b_2$. Then we have the uniqueness in distribution for the solution of Equation (4.4).

**Proof.** It is clear that $X^2$ is pathwise unique, hence the uniqueness in law holds when $b_1 \equiv 0$. Let $(X, B^H)$ be a solution of the stochastic differential equation (4.4) defined in the filtered probability space $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t, t \in [0, T]\})$. Define

$$u_s = \left(K_H^{-1} \int_0^1 b_1(r, X_r)dr\right)(s).$$

Let $\tilde{P}$ defined by

$$\frac{d\tilde{P}}{dP} = \exp\left(-\int_0^T u_s dW_s - \frac{1}{2} \int_0^T u_s^2 ds\right).$$

We claim that the process $u_s$ satisfies conditions i) and ii) of Theorem 4.1. In fact, $u_s$ is an adapted process and taking into account that $X_t$ has the same regularity properties as the fBm we deduce that $\int_0^T u_s^2 ds < \infty$ almost surely. Finally, we can apply again Novikov theorem in order to show that $E\left(\frac{d\tilde{P}}{dP}\right) = 1$, because by Gronwall’s lemma

$$\|X\|_{\infty} \leq (|x| + \|B^H\|_{\infty} + C_1 T) e^{C_2 T},$$

and

$$|X_t - X_s| \leq |B^H_t - B^H_s| + C_3 |t - s|(1 + \|X\|_{\infty})$$

for some constants $C_i$, $i = 1, 2, 3$.

By the classical Girsanov theorem the process

$$\tilde{W}_t = W_t + \int_0^t u_r dr$$

is an $\mathcal{F}_t$-Brownian motion under the probability $\tilde{P}$. In terms of the process $\tilde{W}_t$ we can write

$$X_t = x + \int_0^t K_H(t, s) d\tilde{W}_s + \int_0^t b_2(s, X_s) ds,$$

Set

$$\tilde{B}^H_s = \int_0^t K_H(t, s) d\tilde{W}_s.$$ 

Then $X$ satisfies the following SDE,

$$X_t = x + \tilde{B}^H_t + \int_0^t b_2(s, X_s) ds.$$
As a consequence, the processes $X$ and $X^2$ have the same distribution under the probability $P$. In fact, if $\Psi$ is a bounded measurable functional on $C([0,T])$, we have

$$E_P(\Psi(X)) = \int_{\Omega} \Psi(\xi) \frac{dP}{d\tilde{P}}(\xi) d\tilde{P}$$

$$= E_{\tilde{P}} \left( \Psi(X) \exp \left( \int_0^T \left( K_H^{-1} \int_0^r b_1(s, X_s) ds \right) (s) dW_s \right) \right.$$

$$+ \frac{1}{2} \int_0^T \left( K_H^{-1} \int_0^r b_1(s, X_s) ds \right)^2 (s) ds \left. \right)$$

$$= E_{\tilde{P}} \left( \Psi(X) \exp \left( \int_0^T \left( K_H^{-1} \int_0^r b_1(s, X_s) ds \right) (s) d\tilde{W}_s \right) \right.$$

$$- \frac{1}{2} \int_0^T \left( K_H^{-1} \int_0^r b_1(s, X_s) ds \right)^2 (s) ds \left. \right)$$

$$= E_P \left( \Psi(X^2) \exp \left( \int_0^T \left( K_H^{-1} \int_0^r b_1(s, X^2_s) ds \right) (s) dW_s \right) \right.$$

$$- \frac{1}{2} \int_0^T \left( K_H^{-1} \int_0^r b_1(s, X^2_s) ds \right)^2 (s) ds \left. \right)$$

$$= E_P(\Psi(X^2)).$$

In conclusion we have proved the uniqueness in law, which is equivalent to pathwise uniqueness (see [13] Theorem 5).

4.1 Remark. In the case $H < 1/2$, a deep study is made between stochastic differential equation with continuous coefficient and unit drift and anticipating ones (cf [4]).

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References

[1] E. Alòs, O. Mazet and D. Nualart: Stochastic calculus with respect to Gaussian processes, *Annals of Probability* 29 (2001) 766-801.

[2] K. Bahlali: Flows of homeomorphisms of stochastic differential equations with measurable drift, *Stoch. Stoch. Reports*, Vol. 67 (1999) 53-82.

[3] L. Decreusefond and A. S. Üstunel: Stochastic Analysis of the fractional Brownian Motion. *Potential Analysis* 10 (1999), 177-214.

[4] L. Denis, M. Erraoui and Y. Ouknine: Existence and uniqueness of one dimensional SDE driven by fractional noise. (preprint) (2002).

[5] X. M. Fernique: Regularité des trajectoires de fonctions aléatoires gaussiennes. In: École d’Été de Saint-Flour IV (1974), *Lecture Notes in Mathematics* 480, 2-95.
[6] A. Friedman: *Stochastic differential equations and applications*. Academic Press, 1975.

[7] I. Gyöngy and E. Pardoux: On quasi-linear stochastic partial differential equations. *Probab. Theory Rel. Fields* 94 (1993) 413-425.

[8] J.P. Lepeltier and J. San Martin: Backward stochastic differential equations with continuous coefficient, *Stat. Prob. Letters*, 32 (1997) 425-430.

[9] Yu. Mishura and D. Nualart: Weak solution for stochastic differential equations driven by a fractional Brownian motion with parameter $H > 1/2$ and discontinuous drift. Preprint IMUB N° 319. 2003.

[10] S. Moret and D. Nualart: Onsager-Machlup functional for the fractional Brownian motion. Preprint.

[11] S. Nakao: On the pathwise uniqueness of solutions of one-dimensional stochastic differential equations. *Osaka J. Math.* 9 (1972) 513-518.

[12] I. Norros, E. Valkeila and J. Virtamo: An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motion. *Bernoulli* 5 (1999) 571-587.

[13] D. Nualart and Y. Ouknine: Regularizing differential equations by fractional noise. Stoch. Proc. and their Appl. 102 (2002) 103-116.

[14] Y. Ouknine: Généralisation d’un Lemme de S. Nakao et Applications. *Stochastics* 23 (1988) 149-157.

[15] S. G. Samko, A. A. Kilbas and O. I. Marichev: *Fractional integrals and derivatives*. Gordon and Breach Science, 1993.

[16] A. Ju. Veretennikov: On strong solutions and explicit formulas for solutions of stochastic integral equations. *Math. USSR Sb.* 39 (1981) 387-403.

[17] M. Zähle: Integration with respect to fractal functions and stochastic calculus I. *Prob. Theory Rel. Fields*, 111 (1998) 333-374.

[18] A. K. Zvonkin: A transformation of the phase space of a diffusion process that removes the drift. *Math. USSR Sb.* 22 (1974) 129-149.