BLOW-UP PROFILE OF NEUTRON STARS
IN THE HARTREE–FOCK–BOGOLIUBOV THEORY

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ABSTRACT. We consider the gravitational collapse for neutron stars in the Hartree–Fock–Bogoliubov theory. We prove that when the number particle becomes large and the gravitational constant is small such that the attractive interaction strength approaches the Chandrasekhar limit, the blow-up profile of the minimers is given by the Lane–Emden solution.

1. Introduction

A neutron star is a relativistic system of identical fermions in $\mathbb{R}^3$ with Newtonian gravitational interaction. From the first principles of quantum mechanics, such a system is typically described by the $N$-particle Hamiltonian

$$H_N = \sum_{i=1}^N \sqrt{-\Delta x_i + m^2} - \kappa \sum_{1 \leq i < j \leq N} |x_i - x_j|^{-1}$$ (1.1)

acting on $\bigwedge^N L^2(\mathbb{R}^3; \mathbb{C}^q)$, the Hilbert space of square-integrable functions which are anti-symmetric under the permutations of space-spin variables $q \geq 1$ ($q = 2$ in nature). Here $m > 0$ is the neutron mass (we choose the unit $\hbar = c = 1$) and $\kappa = Gm^2$ with $G$ the gravitational constant.

It is a fundamental fact that the neutron star collapses (namely $H_N$ is not bounded from below) if the particle number is too big, such that

$$\tau := \kappa N^{2/3} > \tau_c.$$ The critical constant $\tau_c$ was first computed by Chandrasekhar [4] (see also [9, 24, 39, 40]) using an effective semiclassical theory, and then confirmed rigorously by Lieb and Yau [40] using the many-body Schrödinger theory. In fact, $\tau_c$ is the optimal constant in the Hardy–Littlewood–Sobolev inequality

$$K_{cl} \|\rho\|^{2/3}_{L^1} \|\rho\|^{4/3}_{L^{4/3}} \geq \frac{\tau_c}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x - y|} \, dx \, dy, \quad \forall \rho \in L^1 \cap L^{4/3} (\mathbb{R}^3)$$ (1.2)

where $K_{cl} = \frac{2}{3} (6\pi^2/q)^{1/3}$. Numerically, the proportion $\sigma_f := K_{cl} \tau_c^{-1}$ is about 1.092.

1991 Mathematics Subject Classification. 81V17, 35Q55, 49J40.

Key words and phrases. Chandrasekhar limit, concentration compactness, gravitational interaction, Hartree–Fock–Bogoliubov theory, Lane–Emden solution, mass concentration, minimizers, neutron stars.
It is well-known (see [42, Appendix A]) that (1.2) has a minimizer $Q \in L^1 \cap L^{4/3}(\mathbb{R}^3)$ which is unique up to dilations and translations. Such $Q$ can be chosen uniquely to be non-negative symmetric decreasing by rearrangement inequality (see [34, Chapter 3]) and it satisfies

$$\sigma f \int_{\mathbb{R}^3} Q(x)^{4/3} dx = \int_{\mathbb{R}^3} Q(x) dx = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{Q(x)Q(y)}{|x-y|} dxdy = 1.$$  \hspace{1cm} (1.3)

Moreover, $Q$ solves the Lane–Emden equation of order 3,

$$\frac{4}{3}\sigma f Q(x)^{1/3} - (|\cdot|^{-1} * Q)(x) + \frac{2}{3} = 0 \quad \text{if} \quad Q(x) > 0,$$

$$\geq 0 \quad \text{if} \quad Q(x) = 0.$$  \hspace{1cm} (1.4)

The Lane–Emden equation goes back to [25] (see [5, 51] for detailed studies). Note that it can be easily seen from (1.3) that $Q$ has compact support (see [42, Appendix A]).

The critical value $\tau_c$ can be obtained easily from the Chandrasekhar theory, a semi-classical approximation of the full many-body theory. In this effective theory, the ground state energy of a neutron star is given by

$$E_{\tau}^{\text{Ch}}(1) := \inf \left\{ \mathcal{E}_{\tau}^{\text{Ch}}(\rho) : 0 \leq \rho \in L^1 \cap L^{4/3}(\mathbb{R}^3), \int_{\mathbb{R}^3} \rho(x) dx = 1 \right\}$$  \hspace{1cm} (1.5)

where the Chandrasekhar functional is

$$\mathcal{E}_{\tau}^{\text{Ch}}(\rho) = \int_{\mathbb{R}^3} \frac{q}{(2\pi)^3} \int_{|p| < (6\pi^2 \rho(x)/q)^{1/3}} \sqrt{|p|^2 + m^2} dp dx - \frac{\tau}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|} dxdy.$$  \hspace{1cm} (1.6)

It can be easily seen from (1.3) that $E_{\tau}^{\text{Ch}}(1) > -\infty$ if and only if $\tau \leq \tau_c$.

In the seminal paper [40], Lieb and Yau proved that for any fixed $\tau = \kappa N^{2/3} < \tau_c$, the quantum energy converges to the semiclassical energy

$$\lim_{N \to \infty} N^{-1} \inf \text{spec} H_N = E_{\tau}^{\text{Ch}}(1).$$

See also [39] for an earlier related result and [8] for a recent extension to general interaction potentials.

In the present paper we are interested in the ground states of the neutron star in the critical regime, when $\tau = \kappa N^{2/3} \uparrow \tau_c$ simultaneously as $N \to \infty$. It turns out that the many-body theory is very complicated to study: the ground state does not exist due to the translation-invariant, and even if we consider approximate ground states (in appropriate sense) then their behavior is very unstable since the system can easily split into many small pieces without lower much the energy. Therefore, it is reasonable to focus on some effective models where physical properties are easier to observe thanks to non-linear effects.

In the following, we will focus on the blow-up phenomenon of the neutron star in the Hartree–Fock–Bogoliubov (HFB) theory. This is one of the most important approximation methods in quantum mechanics, and it is a generalization of the traditional
Hartree–Fock (HF) theory, taking into account all quasi-free states in Fock space. We refer to Bach, Lieb and Solovej [4] for a general discussion on the derivation of the HFB theory from many-body quantum mechanics (see also Bach, Fröhlich and Jonsson in [3] for a simplification). In this model, we study the HFB energy functional given by

$$E_{\text{HFB}}^{\tau}(\gamma, \alpha) = \text{Tr} \sqrt{-\Delta + m^2 \gamma - \kappa} \frac{\kappa}{2} D(\rho_\gamma, \rho_\gamma) + \frac{\kappa}{2} \text{Ex}(\gamma) - \kappa \mathbb{I} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\alpha(x, y)|^2}{|x - y|} \ dx \ dy \quad (1.7)$$

where we use the subscript $\tau = \kappa N^{2/3}$ and the shorthand notations

$$D(f, g) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(x) g(y) \frac{1}{|x - y|} \ dx \ dy, \quad \text{Ex}(\gamma) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} |\gamma(x, y)|^2 \frac{1}{|x - y|} \ dx \ dy$$

which we refer to as the direct term and the exchange term, respectively. The density matrix $\gamma$ is a self-adjoint, non-negative operator on $L^2(\mathbb{R}^3, \mathbb{C})$ with $\text{Tr}(\gamma) = N$. The pairing density matrix $\alpha$ is a Hilbert–Schmidt operator on $L^2(\mathbb{R}^3, \mathbb{C})$, i.e. $\text{Tr} \alpha^* \alpha < \infty$, and its kernels is a $(2 \times 2)$-matrix which is supposed to be anti-symmetric in the sense $\alpha^T = -\alpha$. The set of HFB states are given by

$$\mathcal{K} = \left\{ (\gamma, \alpha) = (\gamma^*, -\alpha^T) \in \mathcal{X}_{\text{HFB}} : \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} \gamma & \alpha \\ \alpha^* & 1 - \gamma \end{pmatrix} \leq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \quad (1.8)$$

where the Sobolev-type space $\mathcal{X}_{\text{HFB}}$ is defined by

$$\mathcal{X}_{\text{HFB}} := \left\{ (\gamma, \alpha) \in \mathcal{S}_1 \times \mathcal{S}_2 : \| (1 - \Delta)^{1/4} \gamma (1 - \Delta)^{1/4} \|_{\mathcal{S}_1} + \| (1 - \Delta)^{1/4} \alpha \|_{\mathcal{S}_2} < \infty \right\}.$$

The HFB minimization problem then reads

$$E_{\tau}^{\text{HFB}}(N) = \inf \left\{ E_{\tau}^{\text{HFB}}(\gamma, \alpha) : (\gamma, \alpha) \in \mathcal{K}, \text{Tr} \gamma = N \right\}. \quad (1.9)$$

In the stable regime, the existence of the minimizers for the variational problem (1.9) has been proved by Lenzmann and Lewin [27]. The HFB energy $E_{\tau}^{\text{HFB}}(N)$ is attained for $0 < N < N_{\text{HFB}}^{\tau}(\kappa)$ and $0 < \kappa < \pi/4$. In physical terms, the finite number $N_{\text{HFB}}^{\tau}(\kappa)$ can be interpreted as the Chandrasekhar limit for the HFB model. It is provided by the asymptotic estimate $N_{\text{HFB}}^{\tau}(\kappa) \sim (\tau_c/\kappa)^{3/2}$ as $\kappa \to 0$. The authors in [27] also proved that, for every $0 < \tau < \tau_c$, we have

$$\lim_{N \to \infty} \frac{E_{\text{HFB}}^{\tau}(N)}{N} = \lim_{\kappa N^{2/3} \to \tau} E_{\tau}^{\text{Ch}}(1).$$

Thus the HFB theory captures correctly the leading order of the many-body theory. Actually, this theory is believed to be a much better approximation to the full many-body Schrödinger theory than the Chandrasekhar theory.

In this paper, we will focus on the case when $N \to \infty$ and $\tau := \tau_N \uparrow \tau_c$ slowly and show that the HFB minimizers develop a universal blow-up profile given by the Lane–Emden solution. Our main result is
Theorem 1 (Collapse of the HFB minimizers). Let \( q \geq 1 \) be given and suppose that \( m > 0 \). Assume that \( 0 < \tau_N = \tau_c - \mathcal{O}(N^{-\beta}) \) with \( 0 < \beta < 1/9 \). Then

\[
N^{-1} E_{\tau_N}^{HFB}(N) = (\tau_c - \tau_N)^{1/2}(2\Lambda + o(1))_{N \to \infty}
\]

where

\[
\Lambda = \frac{3}{4} m \sqrt{\frac{1}{K_{cl}} \int_{\mathbb{R}^3} Q(x)^{2/3} dx}.
\]

with \( Q \) is the unique non-negative radial function satisfying (1.3) – (1.4). Furthermore, if \((\gamma_N, \alpha_N)\) is a minimizer of \( E_{\tau_N}^{HFB}(N) \) and \( \rho_{\gamma_N}(x) = \gamma_N(x, x) \), then there exists a sequence \( \{y_N\} \subset \mathbb{R}^3 \) such that

\[
\lim_{N \to \infty} (\tau_c - \tau_N)^{3/2} \rho_{\gamma_N}((\tau_c - \tau_N)^{1/2} N^{1/3} x + y_N) = \Lambda^3 Q(\Lambda x)
\]

strongly in \( L^r(\mathbb{R}^3) \) for \( 1 \leq r < 4/3 \) and weakly in \( L^{4/3}(\mathbb{R}^3) \).

Remark 2. Attaining the \( L^{4/3}(\mathbb{R}^3) \) convergence in (1.13) would require, at least with our method, to prove the Lieb–Thirring inequality with the sharp constant as conjectured in [40] (see also [38, 30, 7, 32, 13, 10] for related discussions).

Remark 3. The contribution of the pairing term in (1.7) is coupled by the small parameter \( \kappa = \mathcal{O}(N^{-2/3}) \), it does not show up in the leading order of the blow-up profile.

Remark 4. Since the limit in (1.13) is unique, we expect that the density \( \rho_{\gamma_N} \) of the HFB minimizer is unique, at least when \( N \) is sufficient large. Probably this can be proved using techniques in the ground state problem of non-linear Schrödinger functionals (see e.g. [1, 49, 45, 11, 12, 18]), but it seems non-trivial. We hope to come back this issue in the future.

This result is a continuation of our work in [47] on the blow-up problem of the neutron stars in the Chandrasekhar theory (1.5), which is purely semiclassical (see also [48, 46, 17, 53] for discussions on the blow-up profile of boson stars and [16, 29] for related topic). The HFB theory is believed to be much more precise than the Chandrasekhar theory, and the analysis in this case is also significantly more difficult. Our proof of Theorem 1 is based on concentration-compactness argument [44, 43]. In contrast to the classical dichotomy argument, the relative compactness of the sequence of densities of the HFB minimizers is not the consequence of the strict binding inequality, but it comes from the non-existence of minimizer in the variational problem \( E_{\tau_c}^{Ch}(\nu)|_{m=0} = 0 \) with \( 0 < \nu < 1 \). Finally, we remark that the blow-up phenomenon of neutron stars has been also studied in the time-dependent setting (see [13, 19, 20, 21] for rigorous results). This problem, however, is different from the ground state problem that we consider in the present paper.

Organization of the paper. In Section 2 we establish some estimates for the HFB energy (1.9) via the Chandrasekhar energy (1.5), and a moment estimate for the density of HFB minimizers. In Section 3 we prove Theorem 1 which gives the blow-up profile of minimizers for the HFB minimization problem (1.9).
2. Energy estimates

Since the full many-body Schrödinger theory of the neutron star is very complicated, approximate but simpler theories are often used to study the stellar collapse for neutron stars. The most simplest approximate theory is the semiclassical Chandrasekhar theory (1.5). This theory has been rigorously derived from many-body quantum mechanics by Lieb and Yau in [40] (see also [39]). Note that the kinetic energy functional in (1.6) can be calculated explicitly as follow, with \( \eta = \left( \frac{6\pi^2}{q} \right)^{1/3} \),

\[
j_m(\rho) = \frac{q}{(2\pi)^3} \int_{|p|<\left(\frac{6\pi^2}{q}\right)^{1/3}} \sqrt{|p|^2 + m^2} dp = \frac{q}{16\pi^2} \left[ \eta(2\eta^2 + m^2) \sqrt{\eta^2 + m^2} - m^4 \ln \left( \frac{\eta + \sqrt{\eta^2 + m^2}}{m} \right) \right].
\]

For the reader’s convenience, we recall the following results on the existence and uniqueness of the Chandrasekhar minimizer (see [40, Theorem 3]) and the blow-up profile of neutron stars in the Chandrasekhar theory (see [47, Theorem 2]).

**Theorem 5** (Existence of the Chandrasekhar minimizer). Let \( q \geq 1 \) be given and suppose that \( m > 0 \). Then the variational problem \( E^\text{Ch}_\tau(1) \) in (1.5) has the following properties

(i) If \( 0 < \tau < \tau_c \), then \( E^\text{Ch}_\tau(1) \) has a unique minimizer (up to translation). The minimizer can be chosen to be radially symmetric decreasing;

(ii) If \( \tau = \tau_c \), then \( E^\text{Ch}_\tau(1) = 0 \) but it has no minimizer;

(iii) If \( \tau > \tau_c \), then \( E^\text{Ch}_\tau(1) = -\infty \).

**Remark 6.** The Chandrasekhar minimizer \( \rho \) satisfies the Euler–Lagrange equation, for some Lagrange multiplier \( \mu \),

\[
j_m'(\rho(x)) = (\eta(x)^2 + m^2)^{1/2} = [\tau|x|^{-1} \ast \rho - \mu]_+
\]

where \( [f(x)]_+ = \max\{f(x), 0\} \) and \( \eta = (6\pi^2/\rho/q)^{1/3} \). In fact, this equation is equivalent to the Newtonian limit of the Tolman–Oppenheimer–Volkoff equation (see [52, 50]).

**Theorem 7** (Collapse of the Chandrasekhar minimizer). Let \( q \geq 1 \) be given and suppose that \( m > 0 \). Let \( \rho_\tau \) be the unique minimizer (up to translation) of \( E^\text{Ch}_\tau(1) \) in (1.5) for \( 0 < \tau < \tau_c \). Then for every sequence \( \{\tau_n\} \) with \( \tau_n \uparrow \tau_c \) as \( n \to \infty \), there exists a sequence \( \{y_n\} \subset \mathbb{R}^3 \) such that

\[
\lim_{n \to \infty} (\tau_c - \tau_n)^{3/2} \rho_{\tau_n}((\tau_c - \tau_n)^{1/2}x + y_n) = \Lambda^3 Q(\Lambda x)
\]

strongly in \( L^1 \cap L^{4/3}(\mathbb{R}^3) \). Here, \( \Lambda \) is determined as in (1.12) and \( Q \) is the unique non-negative radial function satisfying (1.3)–(1.4). Furthermore, we have

\[
E^\text{Ch}_{\tau_n}(1) = (\tau_c - \tau_n)^{1/2}(2\Lambda + o(1))_{n \to \infty}.
\]
The aim of this section is to show that the HFB energy $N^{-1}E_{\tau}^{\text{HFB}}(N)$ has the same asymptotic behavior as the Chandrasekhar energy in (2.2) when $N \to \infty$ and $\tau := \tau_N = \kappa N^{2/3} \uparrow \tau_c$ slowly. We note that the operator inequality of HFB states in (1.8) holds on $L^2(\mathbb{R}^3; \mathcal{C}) \oplus L^2(\mathbb{R}^3; \mathcal{C}')$. This guarantees that the pair $(\gamma, \alpha)$ is associated to a unique quasi-free state in Fock space (see [4]). Also, it leads to the operator inequality (see [4])

$$\gamma^2 + \alpha \alpha^* \leq \gamma.$$  \hspace{1cm} (2.3)

The basic fact $0 \leq \gamma \leq 1$ refers to the Pauli’ exclusion principle [35, Theorem 3.2] (see also [33]). Now, we prove the

**Lemma 8** (Collapse of the HFB energy). Let $q \geq 1$ be given and suppose that $m > 0$. Let $0 < \tau_N = \tau_c - \mathcal{O}(N^{-\beta})$ with $0 < \beta < 1/9$. Then we have

$$N^{-1}E_{\tau_N}^{\text{HFB}}(N) = E_{\tau_N}^{\text{Ch}}(1) + o(E_{\tau_N}^{\text{Ch}}(1))_{N \to \infty} = (\tau_c - \tau_N)^{1/2} \left(2\Lambda + o(1)_{N \to \infty}\right).$$ \hspace{1cm} (2.4)

**Proof.** We start with the lower bound. Let $(\gamma_N, \alpha_N)$ be a minimizer of $E_{\tau_N}^{\text{HFB}}(N)$ for $0 < \tau_N = \kappa N^{2/3} < \tau_c$. Applying the Hardy–Kato inequality $|x|^{-1} \leq \frac{\pi}{2} \sqrt{-\Delta}$ (see [22, 23]) in the variable $x$ with $y$ fixed and using (2.3) we obtain

$$\int_{\mathbb{R}^4 \times \mathbb{R}^3} \frac{|\alpha_N(x, y)|^2}{|x - y|} \, dx \, dy \leq \frac{\pi}{2} \text{Tr}(-\Delta \alpha_N \alpha_N^*) \leq \frac{\pi}{2} \text{Tr}(-\Delta \gamma_N).$$ \hspace{1cm} (2.5)

It follows from (2.5) and the non-negativity of the exchange term that

$$E_{\tau_N}^{\text{HFB}}(N) = E_{\tau_N}^{\text{HFB}}(\tau_N, \alpha_N) \geq \left(1 - \frac{\kappa \pi}{4}\right) \text{Tr}(-\Delta + m^2 \gamma_N - \frac{\kappa}{2} D(\rho_{\gamma_N}, \rho_{\gamma_N}).$$ \hspace{1cm} (2.6)

By the arguments in [40, Proof of Theorem 1] we have

$$N^{-1}E_{\tau_N}^{\text{HFB}}(N) \geq E_{\tau_N}^{\text{Ch}}(1) - 2\epsilon m$$ \hspace{1cm} (2.7)

where $\tau'_N = \kappa' N^{2/3} < \tau_c$ with $\kappa' = \kappa(1 - \kappa \pi / 4 - \epsilon) - 1$ and $\epsilon = 1.7 q^{1/3} \kappa^{2/3} N^{1/3} = \mathcal{O}(N^{-1/9})$. We deduce from the asymptotic formula for $E_{\tau_N}^{\text{Ch}}(1)$ in (2.2) that

$$E_{\tau_N}^{\text{Ch}}(1) - E_{\tau_N}^{\text{Ch}}(1) = ((\tau_c - \tau'_N)^{1/2} - (\tau_c - \tau_N)^{1/2}) \left(2\Lambda + o(1)_{N \to \infty}\right)$$

$$\geq -(\tau'_N - \tau_N)^{1/2} \left(2\Lambda + o(1)_{N \to \infty}\right).$$ \hspace{1cm} (2.8)

Since $\kappa = \tau_N N^{-2/3} = \mathcal{O}(N^{-2/3})$ we have $\tau'_N - \tau_N = \mathcal{O}(N^{-1/9})$. Thus, it follows from (2.7), (2.8) and (2.2) that

$$N^{-1}E_{\tau_N}^{\text{HFB}}(N) \geq E_{\tau_N}^{\text{Ch}}(1) \left(1 - \mathcal{O}(N^{-1/18})((\tau_c - \tau_N)^{-1/2})\right).$$

The error term $\mathcal{O}(N^{-1/18})((\tau_c - \tau_N)^{-1/2})$ is of order 1 when $\tau_c - \tau_N = \mathcal{O}(N^{-\beta})$ with $0 < \beta < 1/9$. 

Now we turn to the upper bound. Again, applying the Hardy–Kato inequality (see \textbf{22, 23}) in the variable \(x\) with \(y\) fixed and using (2.3) we have

\[
\text{Ex}(\gamma) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\gamma(x, y)|^2}{|x - y|} \, dx \, dy \leq \frac{\pi}{2} \text{Tr}(\sqrt{-\Delta} \gamma^2) \leq \frac{\pi}{2} \text{Tr}(\sqrt{-\Delta} \gamma). \tag{2.9}
\]

On the other hand, for any \(0 < \tau_N < \tau_c\), let \(\rho^{\text{Ch}}\) be the unique minimizer (up to translation) of \(E_{\tau_N}^{\text{Ch}}(1)\) in (1.5). Since \(D(\rho_N - \rho^{\text{Ch}}(N^{-1/3}), \rho_N - \rho^{\text{Ch}}(N^{-1/3})) \geq 0\), one easily derives that

\[
-D(\rho_N, \rho_N) \leq D(\rho^{\text{Ch}}(N^{-1/3}), \rho^{\text{Ch}}(N^{-1/3})) - 2D(\rho^{\text{Ch}}(N^{-1/3}), \rho_N). \tag{2.10}
\]

We deduce from (2.9), (2.10) and the non-negativity of the pairing term that

\[
E_{\tau_N}^{\text{HFB}}(N) \leq \inf_{(\gamma, 0) \in K, \text{Tr} \gamma = N} \left\{ \text{Tr} \left[ \left( 1 + \frac{\kappa \pi}{4} \right) \sqrt{-\Delta + m^2} - \kappa \rho^{\text{Ch}}(N^{-1/3}) \ast | \cdot |^{-1} \right] \gamma \right\} + \frac{\kappa}{2} D(\rho^{\text{Ch}}(N^{-1/3}), \rho^{\text{Ch}}(N^{-1/3})). \tag{2.11}
\]

By taking the trial state \(\gamma := 1(\sqrt{-\Delta} \leq \eta^{\text{Ch}})\) with \(\eta^{\text{Ch}}(x) = (6\pi^2 \rho^{\text{Ch}}(N^{-1/3}x)/q)^{1/3}\) we obtain

\[
E_{\tau_N}^{\text{HFB}}(N) \leq \frac{q}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{|p| \leq \eta^{\text{Ch}}(x)} \left( 1 + \frac{\kappa \pi}{4} \right) \sqrt{|p|^2 + m^2} - \kappa \rho^{\text{Ch}}(N^{-1/3}) \ast | \cdot |^{-1}(x) \, dp \, dx
+ \frac{\kappa}{2} D(\rho^{\text{Ch}}(N^{-1/3}), \rho^{\text{Ch}}(N^{-1/3}))
= \left( 1 + \frac{\kappa \pi}{4} \right) \int_{\mathbb{R}^3} \tilde{j}_m(\rho^{\text{Ch}}(N^{-1/3}x)) \, dx - \frac{\kappa}{2} D(\rho^{\text{Ch}}(N^{-1/3}), \rho^{\text{Ch}}(N^{-1/3}))
= N \left( E_{\tau_N}^{\text{Ch}}(1) + \frac{\kappa \pi}{4} \int_{\mathbb{R}^3} \tilde{j}_m(\rho^{\text{Ch}}(x)) \, dx \right). \tag{2.12}
\]

We deduce from (2.12) and the asymptotic formula of \(E_{\tau_N}^{\text{Ch}}(1)\) in (2.2) that

\[
N^{-1} E_{\tau_N}^{\text{HFB}}(N) \leq E_{\tau_N}^{\text{Ch}}(1) + \mathcal{O}(N^{-2/3}) = E_{\tau_N}^{\text{Ch}}(1)(1 + \mathcal{O}(N^{-2/3})(\tau_c - \tau_N)^{-1/2}). \tag{2.13}
\]

The error term \(\mathcal{O}(N^{-2/3})(\tau_c - \tau_N)^{-1/2}\) is of order 1 when \(\tau_c - \tau_N = \mathcal{O}(N^{-\beta})\) with \(0 < \beta < 4/3\).

We now collect a moment estimate for the density of the HFB minimizers by using (2.13). This estimate will be useful for the proof of Theorem \(\text{I}\) in the next section.

**Lemma 9** (Moment estimate). Let \(q \geq 1\) be given and suppose that \(m > 0\). Assume that \(0 < \tau_N = \tau_c - \mathcal{O}(N^{-\beta})\) with \(0 < \beta < 1/9\). Let \((\gamma_N, \alpha_N)\) be a minimizer of \(E_{\tau_N}^{\text{HFB}}(N)\). Then we have

\[
N^{-1} \text{Tr} \sqrt{-\Delta + m^2} \gamma_N \leq C(\tau_c - \tau_N)^{-1/2}. \tag{2.14}
\]
Proof. Let \( \tau'_N = \tau_N + \mathcal{O}(N^{-1/9}) < \tau_c \). For any \( 0 < \epsilon < 1 \) we have

\[
E_{\tau_N}^{\text{Ch}}(1) + \mathcal{O}(N^{-2/3}) \geq N^{-1} E_{\tau_N}^{\text{HFB}}(N) = N^{-1} E_{\tau_N}^{\text{HFB}}(\gamma_N, \alpha_N)
\]

\[
\geq \epsilon N^{-1} \text{Tr} \sqrt{-\Delta + m^2 \gamma_N} + (1 - \epsilon) N^{-1} E_{\tau_N}^{\text{HFB}}(N)
\]

\[
\geq \epsilon N^{-1} \text{Tr} \sqrt{-\Delta + m^2 \gamma_N} + (1 - \epsilon) E_{\tau_N}^{\text{Ch}}(1) - \mathcal{O}(N^{-1/9}).
\] (2.15)

Here we have used (2.13) for the first estimate and (2.7) for the last estimate. Since \( \tau_N = \tau_c - \mathcal{O}(N^{-\beta}) \) with \( 0 < \beta < 1/9 \), we can choose \( \epsilon \) small such that \( (1 - \epsilon)^{-1} \tau'_N < \tau_c \). Indeed, we can choose \( 2\epsilon = \tau_c^{-1}(\tau_c - \tau_N) \). This implies that \( E_{\tau_N}^{\text{Ch}}(1) \geq 0 \). Thus, (2.14) is obtained from (2.15) and the asymptotic formula for \( E_{\tau_N}^{\text{Ch}}(1) \) in (2.2).

\[ \square \]

Remark 10. It follows from (2.14) and the Daubechies’s inequality [7] that

\[
\int_{\mathbb{R}^3} \rho_{\gamma_N}(N^{1/3}x)^{4/3} dx = N^{-1} \int_{\mathbb{R}^3} \rho_{\gamma_N}(x)^{4/3} dx
\]

\[
\leq 1.6^{-1} q^{1/3} N^{-1} \text{Tr} \sqrt{-\Delta} \gamma_N \leq C(\tau_c - \tau_N)^{-1/2}.
\]

3. Blow-up of the HFB Minimizers

In this section, we prove the convergence of the sequence of densities of the HFB minimizers in Theorem [1]. We will need the following two lemmas

Lemma 11. Let \( q \geq 1 \) be given and suppose that \( m \geq 0 \). Let \( g \in H^{1/2}(\mathbb{R}^3) \) with \( \|g\|_{L^2} = 1 \). Then for any positive semi-definite operator \( 0 \leq \gamma \leq 1 \) and \( \rho_{\gamma}(x) = \gamma(x, x) \) we have

\[
\text{Tr} \sqrt{-\Delta + m^2 \gamma} \geq \int_{\mathbb{R}^3} j_m((\rho_{\gamma} \ast |g|^2)(x)) dx - \langle g, \sqrt{-\Delta} g \rangle \text{Tr} \gamma.
\]

Proof. The proof of this Lemma can be found in [40, Proof of Lemma B.3].

Lemma 12. Let \( q \geq 1 \) be given and let \( 0 \leq \gamma_N \leq 1 \) be a sequence of density matrix as a trace class operator such that the dilation of the density \( \rho_{\gamma_N}(N^{1/3}) \) converges to \( \rho \) weakly in \( L^{4/3}(\mathbb{R}^3) \). Then we have

\[
\liminf_{N \to \infty} \frac{1}{N} \text{Tr}(\sqrt{-\Delta} \gamma_N) \geq K_{cl} \int_{\mathbb{R}^3} \rho(x)^{4/3} dx.
\] (3.1)

Proof. Let \( \tilde{\gamma}_N(x, y) = N \gamma_N(N^{1/3}x, N^{1/3}y) \). For every function \( 0 \leq V \in L^4(\mathbb{R}^3) \) we write

\[
\text{Tr}(\sqrt{-\Delta} \gamma_N) = N^{-1/3} \text{Tr}(\sqrt{-\Delta} \tilde{\gamma}_N) = N^{-1/3} \text{Tr}[(\sqrt{-\Delta} - N^{1/3}V)\tilde{\gamma}_N] + \text{Tr}(V\tilde{\gamma}_N).
\] (3.2)

By the assumption we have \( 0 \leq \gamma_N \leq 1 \) and hence \( 0 \leq \tilde{\gamma}_N \leq 1 \). We may apply the min-max principle [34, Theorem 12.1] and Weyl’s law on the sum of negative eigenvalues of Schrödinger operators (see [35, Chapter 4]) to get the following estimate

\[
\text{Tr}[(\sqrt{-\Delta} - N^{1/3}V)\tilde{\gamma}_N] \geq \text{Tr}[\sqrt{-\Delta} - N^{1/3}V]_-
\]

\[
= -\frac{N^{4/3}q B_{\mathbb{R}^3}(0, 1)}{12(2\pi)^3} \left( \int_{\mathbb{R}^3} V(x)^4 dx + o(1)_{N \to \infty} \right).
\] (3.3)
On the other hand, it follows from the weak convergence $\rho_{\gamma_N}(N^{1/3}, \cdot) \rightharpoonup \rho$ in $L^{4/3}(\mathbb{R}^3)$ that

$$
\lim_{N \to \infty} \frac{1}{N} \text{Tr}(V\tilde{\gamma}_N) = \lim_{N \to \infty} \int_{\mathbb{R}^3} V(x)\rho_{\gamma_N}(N^{1/3}x)dx = \int_{\mathbb{R}^3} V(x)\rho(x)dx. 
$$

We deduce from (3.2), (3.3) and (3.4) that

$$
V
$$

for all $R > 0$. From now on, we will denote $\mathcal{L}_N = \Lambda(\tau_c - \tau_N)^{-1/2}$. Let $\tilde{\gamma}_N(x, y) = \mathcal{L}_N^{-3}\gamma_N(\mathcal{L}_N^{-1}x, \mathcal{L}_N^{-1}y)$ and $\rho_{\tilde{\gamma}_N}(x) = \tilde{\gamma}_N(x, x)$. Setting $w_N(x) := \rho_{\tilde{\gamma}_N}(N^{1/3}x)$ then $w_N$ is bounded uniformly in $L^{4/3}(\mathbb{R}^3)$, by Remark 10. The proof of (1.13) in Theorem 1 is divided into several steps as follows

**Step 1: No vanishing.** We first rule out the vanishing of the sequence $w_N$. By vanishing we mean that

$$
\limsup_{N \to \infty} \left( \sup_{y \in \mathbb{R}^3} \int_{|x-y| \leq R} w_N(x)dx \right) = 0
$$

for all $R > 0$. By the arguments in [44, p.124] (see also [27, 28]) we obtain

$$
\lim_{N \to \infty} D(w_N, w_N) = 0. 
$$

(3.5)

On the other hand, for any $\tau_N'$ such that $0 < \tau_N' < \tau_N < \tau_c$, we have

$$
E^{\text{HFB}}_{\tau_N'}(N) \leq E^{\text{HFB}}_{\tau_N'}(w_N) = E^{\text{HFB}}_{\tau_N'}(N) + \frac{\tau_N - \tau_N'}{2N^{2/3}}(D(\rho_{\gamma_N}, \rho_{\gamma_N}) + CN\lambda_N).
$$

(3.6)

Here we have used the non-negativity of the exchange term and the estimates using (2.5), (2.14) for the paring term. Now we recall the following energy estimate in the Chandrasekhar theory (see [47, Lemma 7])

$$
M_1(\tau_c - \tau_N)^{1/2} \leq E^{\text{Ch}}_{\tau_N'}(1) \leq M_2(\tau_c - \tau_N)^{1/2}
$$

(3.7)

for some positive constants $M_1 < M_2$. We deduce from (3.6) and (3.7) that

$$
\frac{1}{2}D(w_N, w_N) + \mathcal{O}(N^{-2/3}) \geq \frac{N^{-1}E^{\text{HFB}}_{\tau_N'}(N) - N^{-1}E^{\text{HFB}}_{\tau_N'}(N)}{\mathcal{L}_N(\tau_N - \tau_N')}
$$

$$
\geq \frac{E^{\text{Ch}}_{\tau_N'} + \mathcal{O}(N^{-1/9})(1) - E^{\text{Ch}}_{\tau_N'}(1) - \mathcal{O}(N^{-2/3}) - \mathcal{O}(N^{-1/9})}{\mathcal{L}_N(\tau_N - \tau_N')}
$$

$$
\geq \frac{M_1(\tau_c - \tau_N)^{1/2} - M_2(\tau_c - \tau_N)^{1/2} - \mathcal{O}(N^{-1/9})}{\mathcal{L}_N(\tau_N - \tau_N')}
$$

$$
\geq \frac{M_1(\tau_c - \tau_N)^{1/2} - M_2(\tau_c - \tau_N)^{1/2} - \mathcal{O}(N^{-1/18})}{\mathcal{L}_N(\tau_N - \tau_N')}.
$$

□
Here we have used the estimates for the HFB energy as in the proof of Lemma 8. Now, choosing $\tau'_{N} = \tau_{N} - \delta(\tau_{c} - \tau_{N})$ with $\delta > 0$ and recalling $\ell_{N} = \Lambda(\tau_{c} - \tau_{N})^{-1/2}$, we arrive at

$$
\frac{1}{2}D(w_{N}, w_{N}) + \mathcal{O}(N^{-2/3}) \geq \frac{\Lambda^{-1}}{\delta}(M_{1}(1 + \delta)^{1/2} - M_{2} - (\tau_{c} - \tau_{N})^{-1/2}\mathcal{O}(N^{-1/18})). \quad (3.8)
$$

The last term is strictly positive for $\delta$ large enough. For $0 < \tau_{N} = \tau_{c} - \mathcal{O}(N^{-\beta})$ with $0 < \beta < 1/9$ and $N$ sufficient large, we infer from (3.8) that there exists a positive constant $K$ such that

$$
D(w_{N}, w_{N}) \geq K > 0.
$$

This contradicts to (3.5). Hence vanishing does not occurs.

**Step 2: Dichotomy does not occur.** Recall that the minimizing sequence $\{w_{N}\}$ is bounded uniformly in $L^{4/3}(\mathbb{R}^{3})$, by Remark 10. In this step, we assume that $\{w_{N}\}$ is not relatively compact in $L^{1}(\mathbb{R}^{3})$ (up to translations). We now use the following adaptation of classical dichotomy result [44, 43].

**Lemma 13** (Strong local convergence). There exist a function $w^{(1)} \in L^{1} \cap L^{4/3}(\mathbb{R}^{3})$ with $\int_{\mathbb{R}^{3}} w^{(1)}(x)dx = \nu \in (0, 1)$, and sequences $\{R_{N}\}_{N \in \mathbb{N}} \subset \mathbb{R}_{+}$ with $R_{N} \to \infty$ and $\{y_{N}\}_{N \in \mathbb{N}} \subset \mathbb{R}^{3}$ such that, up to extraction of a subsequence,

$$
\lim_{N \to \infty} \int_{|x - y_{N}| \leq R_{N}} w_{N}(x)dx = \int_{\mathbb{R}^{3}} w^{(1)}(x)dx, \quad \lim_{N \to \infty} \int_{R_{N} \leq |x - y_{N}| \leq 6R_{N}} w_{N}(x)dx = 0.
$$

**Proof.** We do not detail the proof of this lemma which uses concentration functions in the spirit of Lions [44, 43] as well as the strong local compactness of $w_{N}$. For instance, a similar argument has been detailed in [44] Lemma 3.1] (see also [26, 27]).

We remark that our model given by $\mathcal{E}_{\tau_{N}}^{\text{HFB}}$ is invariant under translations. Thus, for the rest of the proof, we may assume that the sequence of translations in Lemma 13 is given by

$$
y_{N} = 0 \text{ for all } N \geq 1.
$$

Let $0 \leq \chi^{(1)} \leq 1$ be a fixed smooth function on $\mathbb{R}^{3}$ such that $\chi^{(1)}(x) \equiv 1$ for $|x| < 1$ and $\chi^{(1)}(x) \equiv 0$ for $|x| \geq 2$. Given the sequence $\{R_{N}\}$ from Lemma 13, we define the functions $\chi^{(1)}_{R_{N}}(x) = \chi^{(1)}(x/R_{N})$ and $\chi^{(2)}_{R_{N}}(x) = \sqrt{1 - \chi^{(1)}_{R_{N}}(x)^{2}}$. Likewise, we define the sequences $\{\tilde{\gamma}_{N}^{(1)}\}_{N \in \mathbb{N}}$ and $\{\tilde{\gamma}_{N}^{(2)}\}_{N \in \mathbb{N}}$ by

$$
\tilde{\gamma}_{N}^{(i)}(x, y) = \chi^{(i)}_{R_{N}N^{1/3}}(x)\gamma_{N}(x, y)\chi^{(i)}_{R_{N}N^{1/3}}(y), \quad i \in \{1, 2\}
$$

which lead to $w^{(i)}_{N}(x) = \chi^{(i)}_{R_{N}}(x)^{2}w_{N}(x)$. We also define $\rho^{(i)}_{N}(x) = \tilde{\gamma}_{N}^{(i)}(x, x)$. The direct term is separated as follows

$$
D(w_{N}, w_{N}) = D(w^{(1)}_{N}, w^{(1)}_{N}) + D(w^{(2)}_{N}, w^{(2)}_{N}) + 2D(w^{(1)}_{N}, w^{(2)}_{N}). \quad (3.9)
$$
To show that the last term in (3.9) is of order 1 we write
\[ \chi^{(2)}_{R_N}(y)^2 = \chi^{(2)}_{R_N}(y)^2 + \chi^{(2)}_{3R_N}(y)^2 - \chi^{(2)}_{3R_N}(y)^2 \]
and remark that \( \chi^{(1)}_{R_N}(x)^2|x - y|^{-1}\chi^{(2)}_{R_N}(y)^2 \leq R_N^{-1} \). So it remains to treat the term with \( \chi^{(1)}_{R_N}(x)^2[\chi^{(2)}_{R_N}(y)^2 - \chi^{(2)}_{3R_N}(y)^2] \), for which we use (1.2) to obtain
\[
\int_\mathbb{R}^3 \int_\mathbb{R}^3 \frac{\chi^{(1)}_{R_N}(x)^2 w_N(x) [\chi^{(2)}_{R_N}(y)^2 - \chi^{(2)}_{3R_N}(y)^2] w_N(y) }{|x - y|} \, dx \, dy \\
\leq \int_\mathbb{R}^3 \int_\mathbb{R}^3 w_N(x) 1(R_N \leq |y| \leq 6R_N) w_N(y) \frac{1}{|x - y|} \, dx \, dy \\
\leq 2\sigma_f \left\| w_N \right\|_{L^{4/3}} \left( \int_{R_N \leq |x| \leq 6R_N} w_N(x) \, dx \right)^{2/3}.
\]
The last term converges to 0 as \( N \to \infty \), thanks to Lemma 13 and the \( L^{4/3} \)-boundedness of \( w_N \), by Remark 10.

Next, we split the kinetic energy by using the IMS-type localization formula (see also [27]) and \( \text{Tr}(\gamma_N) = \text{Tr}(\gamma_N) = N \), we find that
\[
\ell_N^{-1} \frac{1}{N} \text{Tr}(\sqrt{-\Delta + m^2} \gamma_N) \geq \frac{1}{N} \text{Tr}(\sqrt{-\Delta} \gamma_N^{(2)}) \\
\geq \frac{1}{N} \text{Tr}(\sqrt{-\Delta} \gamma_N^{(1)}) + \frac{1}{N} \text{Tr}(\sqrt{-\Delta} \gamma_N^{(2)}) - \frac{C}{R_N N^{1/3}}. 
\]
To deal with the second term on the right hand side in (3.10), we apply Lemma 11 with \( g_t(x) = t^{3/4} \exp(-\pi t|x|^2) \) and \( g_t(\sqrt{-\Delta} g_t) = 2t^{1/2} \); we obtain
\[
\text{Tr} \sqrt{-\Delta} \gamma_N^{(2)} \geq K_{cl} \int_{\mathbb{R}^3} (\rho^{(2)}_{\gamma_N} * g_t^2)(x) \, dx - 2t^{1/2} \int_{\mathbb{R}^3} \rho^{(2)}_{\gamma_N}(x) \, dx \\
\geq N K_{cl} \int_{\mathbb{R}^3} (w^{(2)}_N * g_{t_N}^2)(x) \, dx - 2t^{1/2} N 
\]
where \( t_N = t N^{2/3} \). Using (1.2) with notice that \( \left\| w^{(2)}_N * g_{t_N}^2 \right\|_{L^1} \leq \left\| w^{(2)}_N \right\|_{L^1} \leq \left\| w_N \right\|_{L^1} = 1 \), we get
\[
K_{cl} \int_{\mathbb{R}^3} (w^{(2)}_N * g_{t_N}^2)(x) \, dx \geq \frac{\tau}{2} D(w^{(2)}_N * g_{t_N}^2, w^{(2)}_N * g_{t_N}^2) 
\]
On the other hand, we write
\[
D(w^{(2)}_N, w^{(2)}_N) - D(w^{(2)}_N * g_{t_N}^2, w^{(2)}_N * g_{t_N}^2) = \int_\mathbb{R}^3 \int_\mathbb{R}^3 w^{(2)}_N(x) v_N(x - y) w^{(2)}_N(y) \, dx \, dy 
\]
with \( v_N(x) = |x|^{-1} - (g_{t_N}^2 * |x|^{-1} * g_{t_N}^2)(x) \). The integral in (3.13) can be bounded using Young’s inequality by \( \|v_N\|_{L^2} \|w^{(2)}_N\|_{L^{4/3}} \). By a simple computation we have \( \|v_N\|_{L^2} = Ct_N^{-1/4} = Ct^{-1/4} N^{-1/6} \). Combining (3.11), (3.12) and (3.13) we have
\[
\left(1 - \frac{\kappa \pi}{4}\right) \frac{1}{N} \text{Tr} \sqrt{-\Delta} \gamma_N^{(2)} - \frac{\tau}{2} D(w^{(2)}_N, w^{(2)}_N) \geq R_1 
\]
where $R_1$ the error terms

$$R_1 := -\left(\frac{K\pi}{4} + \epsilon\right) K_{cl} \|w_N^{(2)}\|_{L^{4/3}}^2 + \epsilon N^{-1} \sqrt{\Delta}\gamma_N^{(2)} + C t^{-1/4}N^{-1/6} \|w_N^{(2)}\|_{L^{4/3}}^2 - 2t^{1/2}. \quad (3.15)$$

By the Daubechies’s inequality \[7\] we have

$$\text{Tr}(\sqrt{-\Delta}\gamma_N^{(2)}) \geq 1.6 q^{-1/3} \int_{\mathbb{R}^3} \rho_N^{(2)}(x)^{4/3} \text{d}x = 1.6 q^{-1/3} N \int_{\mathbb{R}^3} w_N^{(2)}(x)^{4/3} \text{d}x. \quad (3.16)$$

Hence, optimizing the last two terms in (3.15) with respect to $t$ and choosing $\epsilon = CN^{-1/9}$ for a suitable constant $C$, we get

$$R_1 \geq -\left(\frac{K\pi}{4} + \epsilon\right) K_{cl} \|w_N^{(2)}\|_{L^{4/3}}^2 \geq -\left(\frac{K\pi}{4} + \epsilon\right) K_{cl} \|w_N^{(2)}\|_{L^{4/3}}^2 = o(1)_{N \to \infty}. \quad (3.17)$$

Here we have used the fact that $w_N$ (and hence $w_N^{(2)}$) is bounded uniformly in $L^{4/3}(\mathbb{R}^3)$ and that $\kappa = \tau_N N^{-1/3} = O(N^{-2/3})$. In summary, from (3.9)–(3.10), (3.14)–(3.17) and (2.6) we have derived the following estimate

$$\ell^{-1}_N E_{HFB, \tau}^{(N)} \geq \left(1 - \frac{\kappa\pi}{4}\right) \frac{1}{N} \text{Tr} \sqrt{-\Delta} + m^2 \gamma_N - \frac{\tau N}{2} D(w_N, w_N) \geq \left(1 - \frac{\kappa\pi}{4}\right) \frac{1}{N} \text{Tr} \sqrt{-\Delta}\gamma_N^{(1)} - \frac{\tau N}{2} D(w_N^{(1)}, w_N^{(1)}) + o(1)_{N \to \infty}. \quad (3.18)$$

We note that the inequality (2.23) implies the Pauli’ exclusion principle \[35\] Theorem 3.2]

$$0 \leq \gamma_N \leq 1.$$

This property is invariant under scaling as well as under restricting on a domain. Recall that $w_N$ (and hence $w_N^{(1)}$) is a bounded sequence in $L^{4/3}(\mathbb{R}^3)$, by Remark [11] we may apply Lemma [12] to the sequence $\gamma_N^{(1)}(x, y, \ell^{-1}x, \ell^{-1}y)$ together with the weak convergence $w_N^{(1)} \rightharpoonup w^{(1)}$ in $L^{4/3}(\mathbb{R}^3)$, we obtain

$$\lim_{N \to \infty} \frac{1}{N} \text{Tr}(\sqrt{-\Delta}\gamma_N^{(1)}) \geq K_{cl} \int_{\mathbb{R}^3} w^{(1)}(x)^{4/3} \text{d}x. \quad (3.19)$$

On the other hand, Lemma [13] implies that $w^{(1)}$ is the weak limit of $w_N$ in $L^{4/3}(\mathbb{R}^3)$ and is the strong limit of $w_N^{(1)}$ in $L^1(\mathbb{R}^3)$. In fact, $w_N^{(1)}$ converges to $w^{(1)}$ strongly in $L^r(\mathbb{R}^3)$ for $1 \leq r < 4/3$ because of $L^{4/3}(\mathbb{R}^3)$-boundedness. By the Hardy–Littlewood–Sobolev inequality (see \[34\] Theorem 4.3) we have

$$\lim_{N \to \infty} D(w_N^{(1)}, w_N^{(1)}) = D(w^{(1)}, w^{(1)}). \quad (3.20)$$

Taking the limit $N \to \infty$ in (3.18) and using (3.19)–(3.20) together with the asymptotic formula for $E_{HFB, \tau}^{(N)}$ in Lemma [8] we obtain

$$0 = \lim_{N \to \infty} \ell^{-1}_N E_{HFB, \tau}^{(N)} \geq \mathcal{E}_{\tau}^{(1)}|_{m=0} \geq \mathcal{E}_{\tau}^{(1)}|_{m=0} = \nu \mathcal{E}_{\tau}^{(1)}|_{m=0} = 0, \quad (3.21)$$
from which it follows that \( w^{(1)} \) is a minimizer for \( E_{\tau_c}^{\text{Ch}}(\nu)|_{m=0} \). But this contradicts to the fact that the variational problem \( E_{\tau_c}^{\text{Ch}}(\nu)|_{m=0} = 0 \) has no minimizer for any \( 0 < \nu < 1 \), which is due to the positivity of the direct term (see [34, Theorem 9.8]). Hence, dichotomy does not occur.

**Step 3: Conclusion.** We conclude that, up to translations, the minimizing sequence \( \{w_N\} \) is relatively compact in \( L^1(\mathbb{R}^3) \). Hence, there exist a subsequence of \( \{w_N\} \) (still denote by \( \{w_N\} \) for simplicity) and a function \( w \in L^1 \cap L^{4/3}(\mathbb{R}^3) \) with \( \int_{\mathbb{R}^3} w(x)dx = 1 \) such that \( w_N \to w \) pointwise almost everywhere, weakly in \( L^{4/3}(\mathbb{R}^3) \) and strongly in \( L^1(\mathbb{R}^3) \). In fact, \( w_N \to w \) strongly in \( L^r(\mathbb{R}^3) \) for \( 1 \leq r < 4/3 \) because of \( L^{4/3}(\mathbb{R}^3) \)-boundedness. Applying Lemma 12 to the sequence \( \tilde{\gamma}_N(x, y) = \ell_N^{-3}\gamma_N(\ell_N^{-1}x, \ell_N^{-1}y) \) and using the Hardy–Littlewood–Sobolev inequality (see [34, Theorem 4.3]), we obtain

\[
0 = \lim_{N \to \infty} \ell_N^{-1} E_{N}^{HFB}(N) \geq E_{\tau_c}^{\text{Ch}}(w)|_{m=0} \geq E_{\tau_c}^{\text{Ch}}(1)|_{m=0} = 0
\]  

(3.22)

where we have used the asymptotic formula for \( E_{N}^{HFB}(N) \) in Lemma 8. This implies that \( w \) is a minimizer for \( E_{\tau_c}^{\text{Ch}}(1)|_{m=0} = 0 \). In other words, \( w \) is an optimizer for \( \mathbb{L} \) with \( \int_{\mathbb{R}^3} w(x)dx = 1 \). We recall that \( \mathbb{L} \) admits a unique (up to translations and dilations) normalized optimizer which satisfies (1.4) (after a suitable scaling). Therefore, we have

\[
w(x) = b^3 Q(bx)
\]

for some \( b > 0 \), and for \( Q \in L^1 \cap L^{4/3}(\mathbb{R}^3) \) the unique non-negative radially symmetric decreasing solution to the equation (1.4). Note that \( \int_{\mathbb{R}^3} Q(x)dx = \int_{\mathbb{R}^3} w(x)dx = 1 \). Hence, we deduce from (1.4) and (3.22) that \( Q \) satisfies (1.3).

We shall show that \( b = 1 \) and hence \( w \equiv Q \). We first apply Lemma 11 with \( g_t(x) = t^{3/4}\exp(-\pi t|x|^2) \) and \( \langle g_t, \sqrt{-\Delta} g_t \rangle = 2t^{1/2} \) to obtain

\[
\text{Tr} \sqrt{-\Delta + m^2} \gamma_N \geq \int_{\mathbb{R}^3} j_m((\rho_{\gamma_N} \ast g_t^2)(x))dx - 2Nt^{1/2}.
\]  

(3.23)

By a simple scaling \( \rho_{\gamma_N}(x) = \ell_N^3 \gamma_N(\ell_N^{-1} x) \) using (2.1) we have

\[
\int_{\mathbb{R}^3} j_m((\rho_{\gamma_N} \ast g_t^2)(x))dx = N \int_{\mathbb{R}^3} j_{mN^{-1}}((w_N \ast g_{tN}^2)(x))dx,
\]  

(3.24)

where \( t_N = t N^{2/3} \ell_N^{-2} \). Now we define the function \( \tilde{j}_m \) by

\[
\tilde{j}_m(\rho) := \frac{q}{(2\pi)^3} \int_{|p| < (6\pi^2 \rho/q)^{1/3}} \frac{1}{\sqrt{|p|^2 + m^2}} dp
\]

\[
= \frac{q}{4\pi^2} \left[ \eta \sqrt{\eta^2 + m^2} - m^2 \ln \left( \frac{\eta + \sqrt{\eta^2 + m^2}}{m} \right) \right]
\]

where \( \eta = (6\pi^2 \rho/q)^{1/3} \). Then we have

\[
\int_{\mathbb{R}^3} j_{mN^{-1}}((w_N \ast g_{tN}^2)(x))dx \geq K_{cl} \int_{\mathbb{R}^3} (w_N \ast g_{tN}^2)(x)^{4/3}dx + \frac{m^2 \ell_N^{-2}}{2} \int_{\mathbb{R}^3} \tilde{j}_{mN^{-1}}((w_N \ast g_{tN}^2)(x))dx
\]  

(3.25)
which follows from the operator inequality
\[
\sqrt{|p|^2 + m^2 \ell_N^{-2}} \geq |p| + \frac{m \ell_N^{-2}}{2 \sqrt{|p|^2 + m^2 \ell_N^{-2}}}.
\]

Now, using (1.22) with notice that \( \| w_N \ast g^2_{tN} \|_{L^1} \leq \| w_N \|_{L^1} = 1 \) we get
\[
K_{cl} \int_{\mathbb{R}^3} (w_N \ast g^2_{tN})(x)^{4/3} dx \geq \frac{\tau_c}{2} D(w_N \ast g^2_{tN}, w_N \ast g^2_{tN}). \tag{3.26}
\]

On the other hand, we write
\[
D(w_N, w_N) - D(w_N \ast g^2_{tN}, w_N \ast g^2_{tN}) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} w_N(x)v_N(x-y)w_N(y)dxdy \tag{3.27}
\]
with \( v_N(x) = |x|^{-1} - (g^2_{tN} \ast |x|^{-1} \ast g^2_{tN})(x) \). The integral in (3.27) can be bounded using Young’s inequality by \( \| v_N \|_{L^2} \| w_N \|_{L^{4/3}} \). By a simple computation we have \( \| v_N \|_{L^2} = C t_N^{-1/4} = C t^{-1/4} N^{-1/6} \ell_1^{1/2} \). Combining (3.23)–(3.27) together with (2.6) we have
\[
\frac{E_{\text{HFB}}^{\tau_c}(N)}{\mathbb{N}} \geq \left( 1 - \frac{\kappa \pi}{4} \right) \frac{1}{N} \text{Tr} \sqrt{-\Delta + m^2 \gamma_N} - \frac{\kappa}{2} D(p_{\gamma N}, p_{\gamma N}) \geq \left( 1 - \frac{\kappa \pi}{4} - \epsilon \right) \ell_N^{-1} \frac{m^2}{2} \int_{\mathbb{R}^3} j_m \ell^{-1}_N((w_N \ast g^2_{tN})(x)) dx + \ell_N \frac{\tau_c - \tau_N}{2} D(w_N, w_N) + R_2 \tag{3.28}
\]
where \( R_2 \) the remainder terms
\[
R_2 := - \left( \frac{\kappa \pi}{4} + \epsilon \right) \ell_N K_{cl} \| w_N \ast g^2_{tN} \|_{L^{4/3}}^{4/3} \ell_N \| w_N \ast g^2_{tN} \|_{L^{4/3}}^{1/3} - C t^{-1/4} N^{-1/6} \ell_1^{3/2} \ell_N^{-1} \| w_N \ast g^2_{tN} \|_{L^{4/3}}^{2} - 2 t^{1/2}. \tag{3.29}
\]
By the Daubechies’s inequality [7] we have
\[
\text{Tr} \sqrt{-\Delta} \gamma_N \geq 1.6 q^{-1/3} \int_{\mathbb{R}^3} \rho_{\gamma N}(x)^{1/3} dx = 1.6 q^{-1/3} N \ell_N \int_{\mathbb{R}^3} w_N(x)^{1/3} dx.
\]
Optimizing the last two terms in (3.29) with respect to \( t \), whence \( t_N = CN^{4/9} \rightarrow \infty \) as \( N \rightarrow \infty \), and choosing \( \epsilon = C q^{1/3} N^{-1/9} \) we get
\[
R_2 \geq - \left( \frac{\kappa \pi}{4} + \epsilon \right) \ell_N K_{cl} \| w_N \ast g^2_{tN} \|_{L^{4/3}}^{4/3} \geq - \ell_N^{-1} o(1)_{N \rightarrow \infty}. \tag{3.30}
\]
Here we have used the fact that \( w_N \) is bounded uniformly in \( L^{4/3}(\mathbb{R}^3) \) and that \( \kappa = \tau_N N^{-2/3} = \mathcal{O}(N^{-2/3}) \). Putting (3.28) and (3.30) together we obtain
\[
\frac{\ell_N}{\Lambda} \cdot \frac{E_{\tau_c}^{\text{HFB}}(N)}{\mathbb{N}} \geq \left( 1 + o(1)_{N \rightarrow \infty} \right) \frac{m^2}{2 \Lambda} \int_{\mathbb{R}^3} j_m \ell^{-1}_N((w_N \ast g^2_{tN})(x)) dx + \frac{\Lambda}{2} D(w_N, w_N) + o(1)_{N \rightarrow \infty}. \tag{3.31}
\]
Now we note that the strong convergence \( w_N \to w \) in \( L^r(\mathbb{R}^3) \) for \( 1 \leq r < 4/3 \) implies the strong convergence \( w_N * g_{t_N}^2 \to w \) in \( L^r(\mathbb{R}^3) \) for \( 1 \leq r < 4/3 \). This follows from the fact that \( w * g_{t_N}^2 \to w \) strongly in \( L^r(\mathbb{R}^3) \) for \( 1 \leq r < 4/3 \) (recall that \( t_N \to \infty \)) and that

\[
\|w_N * g_{t_N}^2 - w\|_{L^r} \leq \|(w_N - w) * g_{t_N}^2\|_{L^r} + \|w * g_{t_N}^2 - w\|_{L^r}
\]

Here we have used the Minkowski’s inequality and the Young’s inequality. Hence, up to a subsequence, \( w_N * g_{t_N}^2 \to w \) pointwise almost everywhere. By Fatou’s Lemma we have

\[
\liminf_{N \to \infty} \int_{\mathbb{R}^3} \tilde{j}_{mN}^{-1}((w_N * g_{t_N}^2)(x))dx \geq \frac{9}{4\pi^2} \int_{\mathbb{R}^3} \theta(x)^2 dx = \frac{9}{8bK_{cl}} \int_{\mathbb{R}^3} Q(x)^{2/3} dx
\]

where \( \theta = \left( \frac{6\pi^2 w}{q} \right)^{1/3} \). On the other hand, by the Hardy–Littlewood–Sobolev inequality (see [34, Theorem 4.3]) we have

\[
\lim_{N \to \infty} D(w_N, w_N) = D(w, w) = bD(Q, Q) = 2b.
\]

Thus, after passing to the limit \( N \to \infty \) in (3.31) and using the asymptotic formula for \( E_{\tau N}^{HFB} \) in Lemma 8 we obtain

\[
2\Lambda \geq \frac{9m^2}{16bK_{cl}} \int_{\mathbb{R}^3} Q(x)^{2/3} dx + b\Lambda.
\]

It is elementary to check that

\[
\inf_{\lambda > 0} \left( \frac{9m^2}{16\lambda K_{cl}} \int_{\mathbb{R}^3} Q(x)^{2/3} dx + \lambda \right) = 2\Lambda
\]

with the unique optimal value \( \lambda = \Lambda \). Therefore, the equality in (3.34) must occurs and hence \( b = 1 \). We thus have shown that, up to subsequence, \( w_N \) converges to the unique Lane–Emden solution satisfying (1.3)–(1.4). By uniqueness of the limit, we conclude that passing to a subsequence is unnecessary and we have the convergence of the whole family in (1.13). This complete the proof of Theorem 1.

4. Acknowledgements

The manuscript was completed when the author was visiting the Mittag–Leffler Institute for the semester program Spectral Methods in Mathematical Physics. The author would like to thank the organizers for their warm hospitality. He also thanks E. Lenzmann for helpful discussions.
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