Topological charge of Center Vortices

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The topological charge of center vortices is discussed in terms of the self-intersection number of the closed vortex surfaces in 4-dimensional Euclidian space-time and in terms of the temporal changes of the writhing number of the time-dependent vortex loops in 3-dimensional space.

1. Introduction

Center vortices provide an appealing picture of confinement [1]. When center vortices are removed from the Yang-Mills ensemble, the string tension [2] and the quark condensate [3] are lost. Therefore one should expect that center vortices can also explain spontaneous breaking of chiral symmetry. The mechanism of spontaneous breaking of chiral symmetry seems to be tied to the topological properties of the gauge fields. My talk is devoted to the topological charge of center vortices and is mainly based on [4].

In a mathematically idealized way center vortices can be defined in a gauge invariant way in $D$-dimensional space-time as $(D-2)$-dimensional closed hypersurfaces (boundaries) $\partial \Sigma$ of electromagnetic flux which contribute a (non-trivial) center element $Z$ to the Wilson loop when they are non-trivially linked to the latter:

$$P \exp \left[ - \oint_{\partial \Sigma} A \right] = Z^{L(C,\partial \Sigma)}.$$  \hspace{1cm} (1)

Here $L(C,\Sigma)$ denotes the linking-number between the loop $C$ and the hypersurfaces $\partial \Sigma$. Below I will consider $D = 4$.

2. Intersection of center vortex surfaces

Depending on the position of the vortex surface $\partial \Sigma$ in the 4-dimensional spacetime manifold the flux of the vortex can be electric or magnetic or both. Generically, the vortex surface $\partial \Sigma$ evolves in time and at a fixed time the center vortex represents a closed loop of magnetic flux, i.e. the $B$-field is tangential to the loop. There are also purely spatial vortex surfaces existing at a single time instant only. These vortex surface carry only electric flux being normal to the spatial surface $\partial \Sigma$. Obviously, a non-zero topological charge $\nu = \frac{1}{16\pi^2} \int d^4 x \mbox{tr}(F \tilde{F}) = \frac{1}{4\pi^2} \int d^4 x \tilde{E}(x) \tilde{B}(x)$ arises when a generic vortex patch (evolving in time) and a purely spatial vortex patch intersect [4]. It is therefore not surprising that the topological charge of a center vortex is given by its self-intersection number $\nu = \frac{1}{4} I(\partial \Sigma,\partial \Sigma)$. The self-intersection number $I(\partial \Sigma,\partial \Sigma)$ receives contributions from two types of singular points: (i) Transversal intersection points arising from the intersection of two different surface patches (see fig. a) and (ii) twisting points occurring on a single surface patch twisting around a point in such a way to produce 4-linearly independent tangent vectors (see fig. b). Transversal

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2Note, by the Bianchi identity $\nabla \times \tilde{E} = -\partial_t \tilde{B}$ a time-dependent $B$-field generates an electric field $\tilde{E}$, which is however, perpendicular to the $B$-field. Hence a time-dependent magnetic flux loop alone will generically not generate a topological charge, unless non-parallel loop segments intersect (see below). Note, that the dual relation $\nabla \times B = \partial_t E$ (which is part of the Yang-Mills equation of motion) is usually not satisfied, so that a time-dependent $E$-field not necessarily generates also a $B$-field.
intersection points yield a contribution $\pm 2$ to the oriented intersection number, where the sign depends on the relative orientation of the two intersecting surface pieces. Twisting points yield always contributions of module smaller than 2.

Figure 1. Intersection (left) and twisting (right) points. The dashed lines represent time evolution of the vortex.

Fig. 2 shows a vortex with a transversal intersection point ($\nu = -\frac{1}{2}$) and two twisting points ($\nu = \frac{1}{8}$) at the front and back, respectively, edges of the configuration at the intermediate time ($n_0 = 2$). Further twisting points ($\nu = \frac{1}{8}$) occur at the initial ($n_0 = 1$) and final ($n_0 = 3$) times, so that the total topological charge of this configuration vanishes.

3. Writhing of center vortex loops

For generic center vortices representing closed magnetic flux loops $C(t)$ evolving in time the topological charge can be expressed as

$$\nu = \frac{1}{4} \int dt \partial_t W(C(t)),$$

where $W(C) = L(C, C)$ is the writhing number defined by the Gaussian linking number $L(C_1, C_2)$ between two loops $C_1$ and $C_2$.

The writhing number $W(C)$ is a continuous function of the shape of the loop $C$. It is not a topological invariant and thus not integer valued. It vanishes for planar curves or for curves possessing a symmetry plane, and in this sense measures the chirality of the loop. Furthermore, it suffers a discontinuity when two non-parallel line-segments of the curve cross.

Fig. 3 shows the time evolution of a closed magnetic vortex loop in ordinary 3-space, which on a 4-dimensional lattice gives rise to the configuration shown in fig. 2 after eliminating purely spatial vortex patches, which can be considered as lattice artifacts due to the discretization of time. For simplicity I have kept the cubistic form of the vortex in $D = 3$ space, so that the loops consist of straight line segments.

Figure 2. Sample lattice vortex surface configuration taken from [6]. At each lattice time $t = n_0 a$ (a-lattice spacing), shaded plaquettes are part of the vortex surface. These plaquettes are furthermore connected to plaquettes running in time direction.

Figure 3. Snap shots at characteristic time instants of the continuum center vortex loop whose lattice realization is shown in fig. 2.
Singular changes of the vortex loop occur at the creation and annihilation of the loop (which correspond to the twisting points at \( n_0 = 1, 3 \) in fig. 2), where the writhing number changes by \( \Delta W = \frac{1}{2} \), and at the intermediate time \( t = \bar{t}_2 \), where the two long line-segments cross. This crossing changes \( W \) by \((-2)\) (i.e. \( \Delta \nu = -\frac{1}{2} \)) and corresponds in \( D = 4 \) to the transversal intersection point at \( n_0 = 2 \), see fig. 2. Furthermore, when the two long loop segments cross at \( t = \bar{t}_2 \) the two short horizontal loop segments at the front and back edges reverse their direction, which can be interpreted as twisting these loop segments by an angle \( \pi \). In the \( D = 4 \) lattice realization of this vortex shown in fig. 2 these twistings correspond to the twisting points at \( n_0 = 2 \) at the front and back edges of the vortex.

It turns out that transversal intersection points do not give rise to a change of the so-called twist of the vortex loop, while twisting points do, which justifies their name.

In view of the fact that center vortices carry spots of topological charge restricted to \( |\nu| \leq \frac{1}{2} \) the recent lattice measurement of the topological charge distribution gives further support for the vortex picture of the QCD vacuum.

4. Quark guides

A non-vanishing total topological charge requires non-oriented vortex surfaces, which carry magnetic monopole loops at the boundary between oppositely oriented vortex patches. By the Atiyah-Singer index theorem, \( \nu = N_L - N_R \), a non-zero topological charge \( \nu \) is connected to the difference between the numbers \( N_{L/R} \) of left and right handed quark zero-modes. Fig. 4 shows the probability density of the zero-modes of the quarks moving in the background of two pairs of intersecting center vortices on the 4-dimensional torus. As one observes the quark zero-modes are concentrated on the center vortex sheets and are in particular localized at the intersection points, the spots of topological charge \( \nu = \frac{1}{2} \). If the quark zero-modes dominate the quark propagator, at low energies the quarks will travel along the center vortex sheets and can move from one vortex to an other through the intersection points. Since the center vortices percolate in the QCD vacuum, we expect also the percolation of the quark trajectories, which will eventually result in a condensation of the quarks.

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