A Symmetry Property of Momentum Distribution Functions in the
Nonequilibrium Steady State of Lattice Thermal Conduction

Akira Ueda\(^1\) * and Shinji Takesue\(^2\) †

\(^1\)Department of Mathematical Sciences, Osaka Prefecture University, Sakai, 699-8531, Japan
\(^2\)Department of Physics, Kyoto University, Sakyo-ku, Kyoto, 606-8501, Japan

We study a symmetry property of momentum distribution functions in the steady state of heat conduction. When the equation of motion is symmetric under change of signs for all dynamical variables, the distribution function is also symmetric. This symmetry can be broken by introduction of an asymmetric term in the interaction potential or the on-site potential, or employing the thermal walls as heat reservoirs. We numerically find differences of behavior of the models with and without the on-site potential.

KEYWORDS: Lattice dynamical systems, heat conduction, nonequilibrium steady state, Maxwellian distribution, symmetry

1. Introduction

Recently, considerable progress has been made in statistical mechanics of nonequilibrium steady states beyond the linear-response regime. First, new formulas such as the fluctuation theorem\(^1\) and the Jarzynski equality\(^2\) have been found. These formulas can be used as a basis for arguments on transport phenomena in small systems, such as energy transfer in motor proteins or heat and electric transport in nano-scale devices. Second, Sekimoto proposed energetics on Langevin systems, namely systems in the scale where fluctuations become important.\(^3\) It was further developed and some interesting results were obtained\(^4\) Lastly, some exactly solvable models were found,\(^5,6\) which give us important insights to the problem.

Lattice thermal conduction is a typical example of the nonequilibrium steady states. It may be more fundamental than stochastic systems because it is described by a Hamiltonian. Study of thermal conductivity has a long history, but many important results were obtained only recently. It was established already in 1960s that integrable systems do not have temperature gradient and heat flux is proportional to not temperature gradient but temperature difference between the ends. For example, heat conduction in the harmonic chain was solved in 1958.\(^7\) However, it was found in 1997 that the thermal conductivity in the Fermi-Pasta-Ulam (FPU) chain diverges like \(\kappa \sim N^{\alpha} (\alpha \simeq 0.37)\) as the system size \(N\) is increased.\(^8\) Since then, a lot of researches were carried out and it was clarified that the divergence is caused by a long-time tail in current autocorrelation function, which is originated from momentum

\*E-mail: ueda@ms.osakafu-u.ac.jp

\†E-mail: takesue@phys.h.kyoto-u.ac.jp
On the contrary, if the system includes a potential force from substrate the thermal conductivity is convergent. The $\phi^4$ system, the ding-a-ling model and the ding-dong model belong to the class of systems. However, we must mention that exceptions exist for this rule and that proposed theories have some discrepancy concerning the value of exponent $\alpha$. Thus, we are still far from complete understanding.

As we see in the above, the study of lattice thermal conduction has been restricted to the Fourier heat law and existence or nonexistence of thermal conductivity. In fact, there are few studies on distribution functions that describe the nonequilibrium steady state of heat conduction. To construct nonequilibrium statistical mechanics, however, properties of the distribution function should be studied. Thus, in this paper we investigate characteristics of the nonequilibrium steady states.

Analogously with the Gibbs ensemble in equilibrium statistical mechanics, there should be a distribution function on the phase space that describes a nonequilibrium steady state. However, the phase space is high-dimensional and numerically intractable. So, we focus on momentum distribution of a single particle. In equilibrium, it is a Maxwellian distribution at some temperature. We study how it deviates from the Maxwellian distribution in nonequilibrium steady states.

In particular, we focus on a symmetry property. In equilibrium systems, the momentum distribution does not depend on the potentials. On the other hand, symmetry of the potentials can affect the momentum distribution in nonequilibrium states. We numerically investigate what happens if the symmetry is broken by introducing a small asymmetric term into the on-site potential or the interaction potential or employing the thermal wall as the heat reservoir. As a result, differences are found in the behavior of the models with and without on-site potentials. This may be relevant with the reported behavior of heat conduction and might be useful for full understanding of the problem.

In fact, Aoki and Kusnezov discussed similar deviation from the Maxwellian distribution and derived a scaling form for the fourth-order cumulants. However, they were limited to symmetric systems and symmetric deviations. In this paper, we extend to asymmetric models and asymmetric deviations.

In Section 2, we describe the models and heat reservoirs employed in our simulations. In Section 3, we demonstrate a relation between the symmetry in the equation of motion and that in the momentum distribution functions. In Section 4, we show our numerical results when a weak asymmetry is introduced to a symmetric model. Section 5 is devoted to summary and discussion.
2. Models and heat reservoirs

We consider one-dimensional systems composed of \( N \) particles of unit mass with a Hamiltonian of the form

\[
H(\{q_n\}, \{p_n\}) = \sum_{n=1}^{N} \left[ \frac{p_n^2}{2} + U(q_n) \right] + \sum_{n=0}^{N} V(q_{n+1} - q_n),
\]

where \( q_n \) and \( p_n \) are the displacement and the momentum of particle \( n \in \{1, 2, \ldots, N\} \), \( U(q) \) is an on-site potential, and \( V(q) \) is an interaction potential between nearest-neighbor particles. We impose the fixed boundary condition, which is represented by setting \( q_0 = q_{N+1} = 0 \) in Eq. (1). Various models are generated by varying \( U \) and \( V \). Typical examples are (a) the Fermi-Pasta-Ulam (FPU) model \((U(x) = 0, \text{ and } V(x) = \frac{x^2}{2} + \frac{x^4}{4})\), (b) the \( \phi^4 \) model \((U(x) = \frac{x^2}{2} + \frac{x^4}{4})\), and \( V(x) = \frac{x^2}{2})\), and (c) the Toda model \((U(x) = 0 \text{ and } V(x) = \exp(x) - x)\).

The particles at the ends of the chain are in contact with two heat reservoirs: one at temperature \( T_L \) on particle 1 and the other at temperature \( T_R \) on particle \( N \). If \( T_L = T_R \), the system goes to equilibrium, and if \( T_L \neq T_R \), heat conduction occurs. Thus, for \( 2 \leq n \leq N-1 \), the equations of motions are given as

\[
\dot{q}_n = p_n, \quad \dot{p}_n = -U'(q_n) + V'(q_{n+1} - q_n) - V'(q_n - q_{n-1}),
\]

while those for particles 1 and \( N \) are modified from the Hamiltonian form. In this paper, we consider the following three kinds of reservoirs to study what differences are generated by the types of heat reservoirs. The first is the Langevin reservoir that is given by adding a dissipation term and fluctuating force to the equation of the motion as

\[
\dot{p}_1 = -U'(q_1) + V'(-q_1) - V'(q_1 - q_2) - \gamma p_1 + \xi_L(t) \tag{3}
\]

and

\[
\dot{p}_N = -U'(q_N) + V'(q_{N-1} - q_N) - V'(q_N - q_{N-1}) - \gamma p_N + \xi_R(t) \tag{4}
\]

In the above equations, \( \xi_L(t) \) and \( \xi_R(t) \) are Gaussian white noise with correlation

\[
\langle \xi_{\alpha}(t)\xi_{\beta}(t') \rangle = 2\gamma k_B T_{\alpha} B_{\alpha\beta} \delta(t-t')
\]

where \( \alpha, \beta = L \text{ or } R \). The second is the Nose-Hoover reservoir,\(^{26}\) which is a kind of deterministic reservoir and widely used in the literature. In this reservoir, the equations of motion for particles 1 and \( N \) are modified as

\[
\dot{p}_1 = -U'(q_1) + V'(q_2 - q_1) - V'(q_1) - \xi_{LP_1}, \quad \dot{\xi}_L = \frac{1}{\Theta} \left( \frac{p_1^2}{T_L} - 1 \right), \tag{5}
\]

and

\[
\dot{p}_N = -U'(q_N) + V'(-q_N) - V'(q_N - q_{N-1}) - \xi_{RP_N}, \quad \dot{\xi}_R = \frac{1}{\Theta} \left( \frac{p_N^2}{T_R} - 1 \right), \tag{6}
\]
and time evolution of $\xi_L$ and $\xi_R$ are given by
\[ \dot{\xi}_L = \frac{1}{\Theta} \left( \frac{p^2}{T_L} - 1 \right) \]
and
\[ \dot{\xi}_R = \frac{1}{\Theta} \left( \frac{p^2}{T_R} - 1 \right), \]
where $\Theta$ is a constant which means response time of the thermostat. The last is the thermal wall. When a particle hits the thermal wall, it is reflected with a momentum chosen according to the probability distribution
\[ f(p) = \frac{|p|}{T} \exp \left( -\frac{p^2}{2T} \right) \]
where $T$ denotes the temperature of the wall. For our systems, we place a thermal wall at temperature $T_L$ on the left hand of particle 1 and the other wall at temperature $T_R$ on the right hand of particle $N$.

The three models mentioned in Section 1 are representatives of the types of heat conduction behavior. Namely, the FPU model represents the models where the thermal conductivity diverges in the thermodynamic limit due to a long-time tail in the autocorrelation function of heat flux. The $\phi^4$ model represents the class of models where the thermal conductivity converges as the system size is increased. The Toda model represents the integrable models where temperature gradients are not formed and heat flux is proportional to temperature differences between the reservoirs.

3. Deviation from the Maxwellian distribution

We carry out numerical simulations on the three models with Langevin heat reservoirs at $T_L = 2.0$ and $T_R = 1.0$ and calculate the single-particle momentum distribution functions $P_n(p)$ ($n = 1, \ldots, N$) as follows. The deterministic part of the equation of motion is solved with a mixed use of the sixth-order symplectic method and the fourth-order Runge-Kutta method with a single time step $\Delta t = 0.01$. After computing the deterministic part, we add a Gaussian random noise with variance $2\gamma T_L \Delta t$ to the momentum of particle 1, and one with variance $2\gamma T_R \Delta t$ to the momentum of particle $N$. To compute the distribution functions, simulation was done during $10^{10}$-steps after the system reached a steady state. Figure 1 shows the difference between the computed momentum distribution function and the Maxwellian distribution function
\[ P_M(p; T_n) = \frac{1}{\sqrt{2\pi T_n}} \exp \left( -\frac{p^2}{2T_n} \right), \]
where temperature $T_n$ is determined via average of kinetic energy, namely $T_n = \langle p_n^2 \rangle$.

We find that the deviation is symmetric in the FPU model and the $\phi^4$ model, while it is asymmetric in the Toda chain. This is because the equations of motion are symmetric under the transformation of changing signs of the variables for the former two models, but not for
the Toda chain. In general, the equation of motion (2) is invariant under the transformation of changing the signs of the variables \((q_1, p_1; \ldots; q_N, p_N) \rightarrow (-q_1, -p_1; \ldots; -q_N, -p_N)\) if both the on-site potential \(U(q)\) and the interaction potential \(V(q)\) are even functions, i.e., \(U(-q) = U(q)\) and \(V(-q) = V(q)\). We here note that there is no physical reason that the interaction potential must be even. Denoting lattice constant by \(c\), equality \(V(-q) = V(q)\) means that two particles with distances \(c + q\) and those with distance \(c - q\) have the same interaction energy, which is not expected in general.

Moreover, the Langevin equations for particles 1 and \(N\), (3) and (4), are also invariant under the transformation if the signs of the noise terms \(\xi_L(t)\) and \(\xi_R(t)\) are also changed. The last operation does not change the statistics of the noise terms. Similarly, Nose-Hoover reservoir is also symmetric under operation \((q_1, p_1; \ldots; q_N, p_N; \xi_L, \xi_R) \rightarrow (-q_1, -p_1; \ldots; -q_N, -p_N; -\xi_L, -\xi_R)\). On the other hand, since the thermal wall introduces an asymmetric potential, it breaks the symmetry.

The symmetric deviation seen in the FPU model and the \(\phi^4\) model is in contrast with the antisymmetric deviation seen in particle systems described by the Boltzmann equation.\(^{27}\) In the latter case, the expectation value of heat current is represented as \(\frac{1}{2} \int pp^2 f(p) dp\) with use
of the momentum distribution function $f(p)$. Clearly, it vanishes if $f(p)$ is symmetric. Thus, antisymmetric deviation must exist. On the other hand, in our lattice models the expectation value of heat current is represented as

$$- \int (p_n + p_{n+1}) V'(q_{n+1} - q_n) P_{n,n+1}(q_n, p_n, q_{n+1}, p_{n+1}) dq_n dp_n dq_{n+1} dp_{n+1},$$

where $P_{n,n+1}(q_n, p_n, q_{n+1}, p_{n+1})$ denotes a two-body distribution function in the steady state. Because $V'(q)$ is antisymmetric when $V(q)$ is symmetric, the integral does not vanish if $P_{n,n+1}(q_n, p_n, q_{n+1}, p_{n+1})$ is symmetric. This also indicates the existence of correlation. Namely, if $P_{n,n+1}(q_n, p_n, q_{n+1}, p_{n+1})$ is decomposed into a product form $P_{n,n+1}(q_n, p_n, q_{n+1}, p_{n+1}) = P_{n,n+1}^{(q)}(q_n, q_{n+1}) P_{n,n+1}^{(p)}(p_n, p_{n+1})$ as in the equilibrium case and $P_{n,n+1}^{(q)}$ and/or $P_{n,n+1}^{(p)}$ is symmetric, the heat current vanishes. Thus, to produce nonzero heat current some correlation must exist between momentum and position. The symmetric deviation in the momentum distribution function is a reflection of such correlation.

4. Properties of asymmetric deviation

In this section, we examine how the momentum distribution functions are affected if the symmetry is broken by small modifications to a symmetric system. As we noted in the Introduction, there are three kinds of such modifications, which we describe in the following.

![Fig. 2. The third-order moments $\langle p_n^3 \rangle$ in the system with asymmetric interaction potential (11) and on-site potential $U(q) = 0$. The system size is $N = 16$, parameter values are $a = -0.3, 0, 0.3, 0.5, 0.7, 1.0$. Langevin reservoirs at $T_L = 2.0$ and $T_R = 1.0$ are used.](image)

The first is the case of modifying the interaction potential. Let us consider a system with the following interaction potential,

$$V(q) = \frac{1}{2} q^2 + \frac{a}{3} q^3 + \frac{1}{4} q^4,$$  \hspace{1cm} (11)

with a constant $a$ representing the magnitude of asymmetry. The on-site potential $U(q)$ is set to be zero. We carry out numerical simulations of the system of size $N = 16$ and calculate
the third-order moments

$$\langle p_n^3 \rangle = \int_{-\infty}^{+\infty} p^3 P_n(p) dp \quad (n = 1, ..., N).$$  \hspace{1cm} (12)

The results are shown in Fig. 2, which indicates that asymmetric deviations from the Maxwellian distribution emerge in the whole system. Namely, the moments are in the same order of magnitude except a few particles near the ends. This behavior is not changed if the system size is varied. See Fig. 3. Although there appear relatively large variations near both the ends, changes are smooth in the middle of the system. Figure 4 shows that the third-order moment changes in proportion to $a$ when $a$ is small. But the tendency changes around $a \simeq 2$, where the moment have an extremum. This is because the stability of $q = 0$ is lost at $a = 2$ and two stable points with $q \neq 0$ appear beyond $a = 2$. Then, $q$ loses the meaning of displacement from the stable point in the latter case.

Fig. 3. The third-order moments $\langle p_n^3 \rangle$ in various sizes of the system having the asymmetric potential (11) with $a = 1$. The reservoir temperatures are given as $T_L = 2.0$ and $T_R = 1.0$.

Fig. 4. Variation of $\langle p_8^3 \rangle$ as the parameter $a$ is varied. The system is the same as in Fig. 2

We check this behavior with some conditions modified. One is the introduction of on-site
potential. When we use the nonlinear on-site potential

$$U(q) = \frac{1}{2}q^2 + \frac{1}{4}q^4,$$

the third-order moments become almost constant except for the end particles and this behavior does not change with the system size. The next is the use of Nose-Hoover reservoirs instead of Langevin reservoirs. In this case, the magnitude of the third-order moments is larger than the Langevin case and their spatial variation is rather smooth. No noticeable peaks are formed if the system size is changed. The next is the variation of the boundary temperatures. When the temperatures of the reservoirs are given as $T_L = 60$ and $T_R = 30$, the overall behavior is the same as the low temperature case, only magnifying the absolute values of the moments. The last is insert of the fifth-order term instead of the third-order term into the interaction potential. The use of the fifth-order term leads to little difference from the case of third-order term. In all cases, the overall appearance of the third-order moments are observed.

Fig. 5. Deviations from the Maxwellian distribution for (a) particle 8 and (b) particle 16 in the model with the asymmetric on-site potential $U(q) = \frac{1}{2}q^2 + \frac{1}{3}q^3 + \frac{1}{4}q^4$ and the harmonic interaction potential $V(q) = \frac{1}{2}q^2$. The system size is $N = 16$ and Langevin reservoirs at $T_L = 2.0$ and $T_R = 1.0$ are used. $10^{10}$ iterations are carried out for each of them.

Next, we consider the case of modifying the on-site potential. Then, the third-order term is inserted into the on-site potential of the $\phi^4$ model as

$$U(q) = \frac{1}{2}q^2 + aq^3 + \frac{1}{4}q^4,$$  \hspace{1cm} (13)

and the harmonic interaction potential $V(q) = \frac{1}{2}q^2$ is used. Figures 5 (a) and (b) show the deviation from the Maxwellian distribution for particles 8 and 16 in the system with $a = 1.0$ of system size $N = 16$. Asymmetric deviation is observed for the end particle but invisible in the middle of the chain. Figure 6 shows the third-order moments $\langle p^3_n \rangle$ and its variation with parameter $a$, which clearly indicates that only the particles at the ends of the chain are affected by the asymmetry. Notice that even for the end particles the magnitude of moments is much smaller than that in the case of modifying interaction potential. Namely, asymmetry in the
Fig. 6. The third-order moments $\langle p_n^3 \rangle$ in the system with the asymmetric on-site potential $U(q) = q^2 + a q^3 + b q^4$ and the harmonic interaction potential $V(q) = \frac{1}{2} q^2$. The system size is $N = 16$, Langevin reservoirs at $T_L = 2.0$ and $T_R = 1.0$ are used, and parameter values are $a = -0.3, 0, 0.3, 0.5, 0.7, 1.0$.  

Fig. 7. Variation of $\langle p_1^3 \rangle$, $\langle p_{N/2}^3 \rangle$ and $\langle p_N^3 \rangle$ with parameter $a$. The system and reservoir are the same as in Fig. 6.

on-site potential produces much smaller effects than asymmetry in the interaction potential. Figure 7 shows $\langle p_n^3 \rangle$ vs $a$ for particles at the ends and one in the bulk. The moments change linearly with $a$ for particles at the ends but shows little changes for the particle in the bulk. The linearity implies that this change is certainly caused by the introduced asymmetry. In the middle of the system, the effect of asymmetry is too small to be observed.

When the interaction potential is not a harmonic potential but a nonlinear one as

$$V(q) = \frac{1}{2} q^2 + \frac{1}{4} q^4,$$

the largest deviation of the momentum distribution is observed not for particles 1 and $N$ but particles 2 and $N - 1$. However, the effect of asymmetry is still confined to the vicinity of both the ends. Figure 8 (a) shows the third-order moments near the right end and their system-size dependence. As seen from this figure, the number of particles affected by the asymmetry is
not changed with the system size $N$. If we further replace the Langevin reservoirs with the Nose-Hoover ones, the confinement of the asymmetric effects is not changed, as is shown in Fig. 8 (b). A notable change is that the magnitude of the moments becomes fairly large. The shape of the deviation depends on the interaction potential and the reservoirs. Figure 9 (a) shows the deviation of the momentum distribution for particle $N - 1$, which has the largest moment in the system with the nonlinear interaction potential and Langevin reservoirs. The deviation has similar shape with Fig. 5 (b). However, the deviation changes its sign if the reservoirs are replaced with the Nose-Hoover ones as seen in Figure 9 (b), which represents the deviation for the rightmost particle. In this case, the deviation for particle $N - 1$ and that for particle $N$ have different signs. From these detailed observations, we conclude that the asymmetry effect is confined to the vicinity of the ends in the systems with an asymmetric on-site potential and a symmetric interaction potential. On the other hand, the extent of the confinement and the shape of the deviation depend on the details of the systems.

Fig. 8. The third-order moments $\langle p_3^n \rangle$ near the right end in the system with the asymmetric on-site potential $U(q) = q_1^2 + q_2^2 + q_3^4$ and the interaction potential $V(q) = q_1^2 + q_4^4$. The system size $N = 16, 32, 64, 128, 256$ and 512. Heat reservoirs are (a) Langevin reservoirs and (b) Nose-Hoover reservoirs at $T_L = 2.0$ and $T_R = 1.0$.

Now we turn to the case where the heat reservoirs break the symmetry. As mentioned in Sec. 3, the Langevin reservoirs and the Nose-Hoover reservoirs keep the symmetry but the thermal wall does not. Figure 10 compares the profiles of the third-order moments in (a) the FPU model and (b) the $\phi^4$ model when the thermal walls are employed as heat reservoirs. In the $\phi^4$ model, the influence of the symmetry-breaking is virtually limited to only the four particles, 1, 2, $N - 1$ and $N$. The moments almost vanish for the other particles. On the contrary, in the FPU model, finite moments appear even in the central part of the system, though the magnitude is much smaller than the case of particles near the boundaries. Figures 11 (a) and (b) enlarge the profile of moments near the ends, and show system-size dependence for the FPU model and the $\phi^4$ model, respectively. In the $\phi^4$ model, the number of sites
Fig. 9. Deviations from the Maxwellian distribution in the system of size 16 with the asymmetric on-site potential $U(q) = \frac{1}{2}q^2 + \frac{1}{3}q^3 + \frac{1}{4}q^4$ and the nonlinear interaction potential $V(q) = \frac{1}{2}q^2 + \frac{1}{4}q^4$.

(a) Particle 15 in the system with Langevin reservoirs. (b) Particle 16 in the system with Nose-Hoover reservoirs. The reservoir temperatures are given as $T_L = 2.0$ and $T_R = 1.0$. 10^10 iterations are carried out for each of them.

Fig. 10. The third-order moments $\langle p_3 \rangle$ for (a) the FPU model and (b) the $\phi^4$ model when the thermal walls at $T_L = 2.0$ and $T_R = 1.0$ are used. The system size is $N = 32$.

affected by the asymmetry does not change. In the FPU model, the moments behave as if to decrease exponentially from the boundary into the bulk. Interestingly, however, it actually goes beyond zero and finite moments are obtained in the central region. Recall that the $\phi^4$ system has a bulk limit of the thermal conductivity, while the FPU model does not. Our result also implies that the FPU system has no bulk.

In the dilute gas system between two thermal walls at different temperatures, it is known that asymmetric deviations appear in the whole system. Thus the lattice system is totally different from the gas system in this point.

5. Summary and Discussion

We have numerically studied the single-particle momentum distribution functions in one-dimensional lattice dynamical system in nonequilibrium steady states of heat conduction. We
especially focus on the deviations from the Maxwellian distribution. This deviation reflects a symmetry of the system. Namely, if the system is invariant under the change of signs of the dynamical variables, the momentum distribution function must be even. This symmetry can be broken by the interaction potential or the on-site potential or the heat reservoir. In the first case, the effect of asymmetry extends to the whole system. In the second case, the effect of asymmetry is concentrated near the ends. In the last case, we have found notable differences in systems with and without on-site potentials. Namely, the effect of asymmetry is localized near the ends in the systems with on-site potentials, whereas the effect is extended to the center of the system without on-site potentials. This means there is no bulk limit in the system without on-site potentials, which is consistent with the known results on the convergence or divergence of thermal conductivity.

We need to develop a theoretical explanation for our present results and clarify a relation between the deviation and thermal conductivity. Before studying that, it is necessary to specify what parameters are relevant to the deviations. As we mentioned in Sec. 3, deviation of the momentum distribution is related to two-site correlations including heat flux. Thus, heat flux and local temperature, which is the second-order moment of the distribution, are considered to be important parameters. It is a future problem to clarify if they are sufficient to determine the deviations.

Acknowledgment
The authors thank H. Hayakawa, M. M. Sano for valuable discussions and helpful comments. AU thanks H. Nishimori for continuous encouragement. This work is supported by the Grant-in-Aid for the 21st Century COE "Center for Diversity and Universality in Physics" from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan. The numerical computation in this work was carried out at Library & Science Information Center, Osaka Prefecture University.
References

1) D. J. Evans, E. G. D. Cohen, and G. P. Morriss: Phys. Rev. Lett. 71 (1993) 2401.
2) C. Jarzynski: Phys. Rev. Lett. 78 (1997) 2690.
3) K. Sekimoto: J. Phys. Soc. Japan 66 (1997) 1234; K. Sekimoto and S. Sasa, J. Phys. Soc. Japan 66 (1997) 3326; K. Sekimoto, Prog. Theor. Phys. Suppl. 130 (1998) 17.
4) T. Harada and S. Sasa, condmat/0502505.
5) B. Derrida, M. R. Evans, V. Hakim, and V. Pasquier: J. Phys. A 26 (1993) 1493.
6) T. Sasamoto: J. Phys. A 32 (1999) 7109.
7) Z. Rieder, J. L. Lebowitz, and E. Lieb: J. Math. Phys. 8 (1967) 1073.
8) S. Lepri, R. Livi, and A. Politi: Phys. Rev. Lett. 78 (1997) 1896.
9) T. Hatano: Phys. Rev. E 59 (1999) R1.
10) K. Aoki and D. Kusnezov, Ann. Phys. 295 (2002) 50.
11) O. Narayan and S. Ramaswamy, Phys. Rev. Lett. 89 (2002) 200601.
12) D. J. R. Mimnagh and L. E. Ballentine: Phys. Rev. E 56 (1997) 5332.
13) M. Sano and K. Kitahara: Phys. Rev. E 64 (2001) 056111.
14) S. Lepri, Phys. Rev. E 58 (1998) 7165.
15) R. Tehver, F. Toigo, J. Koplik, and J. R. Banavar, Phys. Rev. E 57 (1998) R17.
16) K. Aoki and D. Kusnezov, Phys. Rev. E 70 (2004) 051203.
17) S. Murayama: Physica B 323 (2002) 193.
18) Z. Yao, J. Wang, B. Li, and G. Liu: cond-mat/0402616.
19) G. Zhang and B. Li: cond-mat/0501194.
20) B. Hu, B. Li, and H. Zhao: Phys. Rev. E 57 (1998) 2992.
21) B. Hu, B. Li, and H. Zhao: Phys. Rev. E 61 (2000) 3828.
22) S. Lepri, R. Livi, and A. Politi: Europhys. Lett. 43 (1998) 271.
23) C. Giardiná, R. Livi, A. Politi, and M. Vassali: Phys. Rev. Lett. 84 (2000) 2144; O. V. Gendelman and A. V. Savin, Phys. Rev. Lett. 84 (2000) 2381.
24) S. Lepri, R. Livi, and A. Politi: Phys. Rep. 377 (2003) 1.
25) C. Giardina and J. Kurchan: J. Stat. Mech. (2005) P05009.
26) S. Nose: Prog. Theor. Phys. Suppl. 103 (1991) 1.
27) Kim Hyeon-Deuk and H. Hayakawa: J. Phys. Soc. Japan 72 (2003) 1904; J. Phys. Soc. Japan 73 (2004) 1609.