Integrability of the $N$-body problem in $(2 + 1)$-AdS gravity

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Abstract

We derive a first order formalism for solving the scattering of point sources in $(2+1)$ gravity with negative cosmological constant. We show that their physical motion can be mapped, with a polydromic coordinate transformation, to a trivial motion, in such a way that the point sources move as time-like geodesics (in the case of particles) or as space-like geodesics (in the case of BTZ black holes) of a three-dimensional hypersurface immersed in a four-dimensional Minkowskian space-time, and that the two-body dynamics is solved by two invariant masses, whose difference is simply related to the total angular momentum of the system.
1 Introduction

In this article we would like to study the problem of treating the interaction between particles and $AdS$ gravity, generalizing our previous investigation in $(2+1)$ gravity and its supergravity extension, which has also been considered by many other authors [1]-[2]-[3]-[4]-[5]-[6]-[7]-[8]-[9]-[10]. The problem from a dynamical point of view is quite difficult, because the introduction of an explicit scale parameter (the cosmological constant) in the theory induces a static Newtonian force between point sources, whose absence was one of the main features that led to the solution of the two body problem in $(2+1)$ gravity. Nevertheless one has to expect surprising simplifications in the dynamics, since the Lagrangian of $AdS$ gravity splits in two Chern-Simons theories, as Witten has shown in ref. [2].

Our aim is to clarify in what sense the interaction given by $AdS$ gravity can be considered integrable. First of all, we have asked ourself if the general scheme for solving $(2+1)$ gravity, described in [5]-[6], can be generalized in a continuous way to the case of negative cosmological constant (hereafter indicated by its modulus $\Lambda$). Our proposal is that it is still possible to reduce the solution of the equations of motion to the knowledge of a polydromic mapping $X^A = X^A(x^a)$, and that it exists a generalized Minkowskian system where the motion of the interacting particles is almost free, apart from the property that they are constrained to move on a hypersurface, instead of moving freely in a plane. The price to pay is that we have to introduce this polydromic mapping as an immersion in four dimensions with a quadratic constraint between the coordinates:

$$ds^2 = dX^A dX^B \eta_{AB}, \quad X^A X^B \eta_{AB} = (X^0)^2 + (X^1)^2 - (X^3)^2 - (X^4)^2 = \frac{1}{\Lambda}. \quad (1.1)$$

Due to this constraint, the general allowed polydromy is reduced to the Lorenz subgroup $SO(2,2)$

$$X^A \rightarrow L^A_B \ X^B \quad (1.2)$$

which has the same number of generators of $ISO(2,1)$.

This procedure is not new, because it has already been discussed in several contexts, for example to introduce the $BTZ$ black hole in $(2+1)$ dimensions [11]-[12]. Our proposal is to unify several partial results in a unitary program, analogously to what has been successfully done in the case of $(2+1)$ gravity with point sources [5]. This investigation should shed also light on the scattering of $BTZ$ black holes, which can contain some important features in common with the four dimensional case, and could open new issues for the quantum case [13].
2 Examples of polydromic mappings in $AdS$-gravity

In general, the conical cuts of the particles in $(2 + 1)$ gravity, firstly defined in \[1\], must be substituted by similar cuts related to the group $SO(2, 2)$ instead of $ISO(2, 1)$, and therefore the translation of the center of rotation must also be represented with a boost of $SO(2, 2)$. While the general case will be postponed to the next section, we are going to show explicit examples of polydromic mappings for static bodies, and in that case the polydromy can still be reduced to a rotation.

Let us start with the metric of a single body, which is given in the radial gauge as:

$$ds^2 = ((1 - \mu)^2 + \Lambda r^2)dt^2 - \frac{dr^2}{(1 - \mu)^2 + \Lambda r^2} - r^2d\theta^2. \quad (2.1)$$

This can be expressed as a polydromic mapping

$$
X^t = X^0 + iX^1 = \frac{1}{\sqrt{\Lambda}}\sqrt{1 + \frac{\Lambda r^2}{(1 - \mu)^2}}e^{i\sqrt{\Lambda}(1 - \mu)t} \\
X^z = X^2 + iX^3 = \frac{r e^{i(1 - \mu)\theta}}{(1 - \mu)} \quad (2.2)
$$

of the flat metric (1.1) that respects the quadratic constraint, where the polydromy is a pure rotation

$$X^z \rightarrow e^{-2\pi i\mu}X^z. \quad (2.3)$$

The equation of motion for the particle source in the $X^a$ coordinates is described by the parametric equation

$$X^2 = X^3 = 0 \quad X^0 = \frac{1}{\sqrt{\Lambda}}\cos(\sqrt{\Lambda}s) \quad X^1 = \frac{1}{\sqrt{\Lambda}}\sin(\sqrt{\Lambda}s), \quad (2.4)$$

and by definition it corresponds to a time-like geodesics:

$$\left(\frac{dX^0}{ds}\right)^2 + \left(\frac{dX^1}{ds}\right)^2 = 1 > 0. \quad (2.5)$$

As a second example let us consider the metric of a spinning particle

$$ds^2 = dt^2((1 - \mu)^2 + \Lambda r^2) + Jdtd\theta - r^2d\theta^2 - \frac{dr^2}{(1 - \mu)^2 + \Lambda r^2 + \frac{J^2}{4r^2}} \quad (2.6)$$
which can be obtained by the following mapping

\[
X^t = \frac{1}{\sqrt{\Lambda}} \left( \sqrt{\frac{r^2 + r_+^2}{r_+^2 - r_-^2}} \right) e^{i(\Lambda r_+ t + \sqrt{\Lambda} r_+ \theta)} \\
X^z = \frac{1}{\sqrt{\Lambda}} \left( \sqrt{\frac{r^2 + r_-^2}{r_-^2 - r_+^2}} \right) e^{i(\Lambda r_- t + \sqrt{\Lambda} r_+ \theta)},
\]

(2.7)

where the two roots \( r_\pm \) are defined as

\[
\begin{align*}
  r_+^2 + r_-^2 &= \frac{(1-\mu)^2}{\Lambda} \\
  r_+ r_- &= \frac{J^2}{4\Lambda}.
\end{align*}
\]

(2.8)

This metric realizes the elliptic monodromy

\[
\begin{align*}
  X^t &\rightarrow e^{2\pi \sqrt{\Lambda} r_-} X^t \\
  X^z &\rightarrow e^{2\pi \sqrt{\Lambda} r_+} X^z.
\end{align*}
\]

(2.9)

We characterize with the word elliptic monodromy every monodromy which can be reduced by a similitude transformation to a pure rotation. In this way we distinguish it from a BTZ black hole, which we can call an hyperbolic monodromy, since it can reduced by a similitude transformation to a pure boost. It is useful then to remember how the BTZ black hole has been introduced in refs. [11]-[12]. In the general case of the spinning black hole metric, where the line element is given by

\[
ds^2 = (\Lambda r^2 - M^2)dt^2 + Jdtd\theta - r^2d\theta^2 - \frac{dr^2}{\Lambda r^2 - M^2 + \frac{J^2}{4r^2}},\]

(2.10)

the \( X^A \)-mapping can be written in terms of two radii \( r_\pm \):

\[
r_\pm = \frac{M}{\sqrt{2\Lambda}} \sqrt{1 \pm \sqrt{1 - \frac{\Lambda J^2}{M^4}}}
\]

(2.11)

as follows:

\[
\begin{align*}
  r > r_+ &\quad X_0 \pm X_2 = \frac{1}{\sqrt{\Lambda}} \sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}} e^{\pm \Theta(t,\theta)} \\
  X_1 \pm X_3 &= \frac{\pm 1}{\sqrt{\Lambda}} \sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}} e^{\pm T(t,\theta)}
\end{align*}
\]

3
\[ r_- < r < r_+ \quad X_0 \pm X_2 = \frac{1}{\sqrt{\Lambda}} \sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}} e^{\pm \Theta(t, \theta)} \]

\[ X_1 \pm X_3 = \frac{1}{\sqrt{\Lambda}} \sqrt{\frac{r_+^2 - r^2}{r_+^2 - r_-^2}} e^{\pm T(t, \theta)} \]

\[ 0 < r < r_- \quad X_0 \pm X_2 = \pm \frac{1}{\sqrt{\Lambda}} \sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}} e^{\pm \Theta(t, \theta)} \]

\[ X_1 \pm X_3 = \frac{1}{\sqrt{\Lambda}} \sqrt{\frac{r_+^2 - r^2}{r_+^2 - r_-^2}} e^{\pm T(t, \theta)}, \quad (2.12) \]

where the following functions are defined, as in ref. [12]:

\[ T(t, \theta) = \Lambda r_+ t - \sqrt{\Lambda} r_- \theta = \frac{1}{2} \ln \left( \frac{X_1^3 - X_3^3}{X_1^3 - X_3^3} \left( \theta(r_+ - r) - \theta(r - r_+) \right) \right) \]

\[ \Theta(t, \theta) = \sqrt{\Lambda} r_+ \theta - \Lambda r_- t = \frac{1}{2} \ln \left( \frac{X_0^2 + X_2^2}{X_0^2 - X_2^2} \left( \theta(r - r_+) - \theta(r_+ - r) \right) \right). \quad (2.13) \]

This metric realizes the hyperbolic monodromy:

\[ X^0 \pm X^2 \rightarrow e^{\pm 2\pi \sqrt{\Lambda} r_+} (X_0 \pm X^2) \]

\[ X^1 \pm X^3 \rightarrow e^{\mp 2\pi \sqrt{\Lambda} r_-} (X_1 \pm X^3). \quad (2.14) \]

In the simpler case \( J = 0 \), the \( X^A \)-mapping reduces to

\[ X^0 \pm X^2 = \frac{r}{M} e^{\pm \theta M} \quad X^1 \pm X^3 = \pm \frac{1}{\sqrt{\Lambda}} \left( \sqrt{\frac{\Lambda r^2}{M^2} - 1} \right) e^{\pm \sqrt{\Lambda} M t} \quad (2.15) \]

in the region external to the black hole \( (r > \frac{M}{\sqrt{\Lambda}}) \), while

\[ X^0 \pm X^2 = \frac{r}{M} e^{\pm \theta M} \quad X^1 \pm X^3 = \frac{1}{\sqrt{\Lambda}} \left( \sqrt{1 - \frac{\Lambda r^2}{M^2}} \right) e^{\pm \sqrt{\Lambda} M t} \quad (2.16) \]

in the internal region.

Here the equation of motion for the source in the \( X^a \)-coordinates is described by the parametric equation

\[ X^0 = X^2 = 0 \quad X^1 = \frac{1}{\sqrt{\Lambda}} ch(\sqrt{\Lambda} s) \quad X^3 = \frac{1}{\sqrt{\Lambda}} sh(\sqrt{\Lambda} s) \quad (2.17) \]

and by definition it corresponds to a space-like geodesics:

\[ \left( \frac{dX^1}{ds} \right)^2 - \left( \frac{dX^3}{ds} \right)^2 = -1 < 0. \quad (2.18) \]
To verify the notion that point sources move freely in the Minkowskian coordinate system $X^A$ as geodesics of the $(1.1)$ hypersurface, we are going to derive again a known result, the determination of the geodesics around the spinning black hole [14], making a bridge between our first order formalism, which is different from what physicists call first order or dreibein formalism (see for example ref. [15]), and the standard method based on integrating the geodesic equations.

From the $X^A$-mapping (2.12)-(2.13) it is easy to read directly the solution to the geodesic equations:

$$
\begin{align*}
    r^2(\tau) &= r^2_+ + \Lambda(r^2_+ - r^2) \left[ (X^0)^2 - (X^2)^2 \right](\tau) \\
    \theta(\tau) &= \frac{r_+ \Theta(\tau) + r_- T(\tau)}{\sqrt{\Lambda(r^2_+ - r^2_+)}} \\
    t(\tau) &= \frac{r_+ T(\tau) + r_- \Theta(\tau)}{\Lambda(r^2_+ - r^2_+)}.
\end{align*}
$$

We simply need to complete eq. (2.19) with the general parameterization describing the motion of a test body $X^A = X^A(\tau)$ on the $X^A X^A = 1/\Lambda$ hypersurface and satisfying

$$\ddot{X}^A(\tau) = -\Lambda m X^A(\tau),$$

where $m = \dot{X}^A \dot{X}_A = 1, 0, -1$ for time-like, null and space-like geodesics.

A general parameterization of a geodesic is given by:

$$
X^A(\tau) = c^A_0 \cos(\sqrt{\Lambda} \tau) + c^A_1 \sin(\sqrt{\Lambda} \tau) \\
= c^A_0 + c^A_1 \sqrt{\Lambda} \tau \quad \text{if } m = 1 \\
= c^A_0 \cosh(\sqrt{\Lambda} \tau) + c^A_1 \sinh(\sqrt{\Lambda} \tau) \quad \text{if } m = 0 \\
= c^A_0 \cosh(\sqrt{\Lambda} \tau) + c^A_1 \sinh(\sqrt{\Lambda} \tau) \quad \text{if } m = -1,
$$

where the vectors $(c^A_0, c^A_1)$, constants of motion, have to satisfy:

$$
c^A_0 c^A_0 = \frac{1}{\Lambda} \quad c^A_0 c^A_1 = 0 \quad c^A_1 c^A_1 = \frac{m}{\Lambda}.
$$

By using the rescaled variables

$$
\begin{align*}
    r' &= \frac{\sqrt{\Lambda}}{M} r \\
    r'_\pm &= \frac{\sqrt{\Lambda}}{M} r_\pm \\
    \theta' &= M \theta \\
    t' &= \sqrt{\Lambda} M t \\
    r' &= \sqrt{\Lambda} \tau \\
    E' &= \frac{E}{M} \\
    L' &= \frac{\sqrt{\Lambda}}{M^2} L \\
    J' &= \frac{\sqrt{\Lambda}}{M^2} J,
\end{align*}
$$

neglecting the primes from now on, we can compare these total integrals (eqs. (2.19) and (2.21) ) with the first-integrals found in ref. [14]:

$$
\left( \frac{dr}{d\tau} \right)^2 = -m(r^2 - 1 + \frac{J^2}{4r^2}) + E^2 - L^2 + \frac{L^2 - JEL}{r^2}.
$$
\[
\frac{d\theta}{d\tau} = \frac{(r^2 - 1)L + \frac{1}{2}JE}{(r^2 - r_+^2)(r^2 - r_-^2)}
\]
\[
\frac{dt}{d\tau} = \frac{Er^2 - \frac{1}{2}JL}{(r^2 - r_+^2)(r^2 - r_-^2)}.
\]

From eq. (2.20) we notice that each of the following six combinations \( P^{AB} = \Lambda(\dot{X}^A X^B - \dot{X}^B X^A) \) (dot means \( \frac{d}{d\tau} \) which has been relabelled \( \frac{d}{d\tau} \) after eq. (2.23) ) represents a constant of motion along a geodesics. Between them only two are globally defined, i.e. they are invariant under the intrinsic polydromies of the \( X^A \)-coordinates, and correspond to

\[
P^{20} = \Lambda(\dot{\xi}^2 \xi^0 - \dot{\xi}^0 \xi^2) = \Lambda(c_0^0 c_1^2 - c_1^0 c_0^2) = \frac{1}{\sqrt{1 - J^2}} \left( Lr_+ - \frac{1}{2}JE \right)
\]
\[
P^{13} = \Lambda(\dot{\xi}^1 \xi^3 - \dot{\xi}^3 \xi^1) = \Lambda(c_0^3 c_1^1 - c_1^3 c_0^1) = \frac{1}{\sqrt{1 - J^2}} \left( \frac{1}{2}JE - Lr_- \right),
\]

where \( L \) is the angular momentum and \( E \) is the energy as defined in [14].

By developing eqs. (2.19) and (2.21) one arrives at the following complete integrals*:

\[
\begin{align*}
\frac{1}{2}[A + C \sin 2(\tau - \tau_0)] & \quad \text{if } m = 1 \\
A(\tau - \tau_0)^2 - \frac{B}{A} & \quad \text{if } m = 0 \text{ and } A \neq 0 \\
2\sqrt{B}(\tau - \tau_0) & \quad \text{if } m = 0, \ A = 0 \text{ and } B \neq 0 \\
\frac{1}{2}[-A + C \cosh 2(\tau - \tau_0)] & \quad \text{if } m = -1
\end{align*}
\]

\[
\begin{align*}
r^2(\tau) &= \begin{cases} 
\frac{1}{2}[A + C \sin 2(\tau - \tau_0)] & \text{if } m = 1 \\ 
A(\tau - \tau_0)^2 - \frac{B}{A} & \text{if } m = 0 \text{ and } A \neq 0 \\ 
2\sqrt{B}(\tau - \tau_0) & \text{if } m = 0, \ A = 0 \text{ and } B \neq 0 \\ 
\frac{1}{2}[-A + C \cosh 2(\tau - \tau_0)] & \text{if } m = -1
\end{cases}
\end{align*}
\]

\[
A = E^2 - L^2 + m \quad B = L^2 - JEL - \frac{1}{4}mJ^2
\]

\[
C = \sqrt{A^2 + 4mB} = \sqrt{F_+^2 + 4mG_+^2}
\]

\[
F_\pm = E^2 - L^2 + m(1 - 2r_\pm^2) \quad G_\pm = r_\pm \left( E - \frac{JL}{2r_\pm^2} \right)
\]

\[
\begin{align*}
\theta(\tau) &= \frac{r_+ \Theta(\tau) + r_- T(\tau)}{r_+^2 - r_-^2} \\
t(\tau) &= \frac{r_- \Theta(\tau) + r_+ T(\tau)}{r_+^2 - r_-^2} \\
\Theta(\tau) &= \frac{1}{2} \ln \left[ \frac{(c_0^0 + c_0^1) f_m^\prime(\tau) + (c_1^0 + c_1^1) f_m(\tau)}{(c_0^0 - c_0^1) f_m^\prime(\tau) + (c_1^0 - c_1^1) f_m(\tau)} \right]
\end{align*}
\]

\[
= -\frac{1}{2} \ln \left[ \frac{(r^2 - r_-^2)}{2G_+^2 + F_-(r^2 - r_-^2) + 2G_-\sqrt{G_+^2 + F_-(r^2 - r_-^2) - m(r^2 - r_-^2)^2}} \right] + \Theta_0
\]

*In the rescaled variables \( r_\pm = \frac{1}{\sqrt{2}} \sqrt{1 \pm \sqrt{1 - J^2}} \).
\[ T(\tau) = \frac{1}{2} \ln \left[ \frac{(c_0^1 + c_0^2)f_m'(\tau) + (c_1^1 + c_1^2)f_m(\tau)}{(c_0^1 - c_0^2)f_m'(\tau) + (c_1^1 - c_1^2)f_m(\tau)} \right] \left( \theta(r_+ - r(\tau)) - \theta(r(\tau) - r_+) \right) \]

\[ = \frac{1}{2} \ln \left\{ \frac{(r^2 - r_+^2)}{2G^2_+ + F_+ (r^2 - r_+^2) + 2G \sqrt{G^2_+ + F_+ (r^2 - r_+^2) - m(r^2 - r_+^2)^2}} \right\} + T_0, \]

(2.27)

where \( f_m(\tau) = (\sin(\tau), \tau, \sinh(\tau)) \) if \( m = (1, 0, -1) \), and use has been made of the formulas given in the Appendix.

Therefore this method reproduces all the previous results [14], and, in our opinion, is more flexible to be generalized to the \( N \)-body problem, discussed also in ref. [16].

### 3 N body -problem

Now we are going to define the integrability of a system of \( N \) point sources interacting with \( AdS \) gravity, as a completely non-interacting system in the Minkowskian \( X^A \)-coordinates, apart from the fact that their motion has to respect the constraint \( X^A X_A = \frac{1}{\Lambda} \). We have just shown that, at least for the one body case, the effect of a point source, particle or \( BTZ \) black hole, is to eliminate a portion of the hypersurface, while a test particle moves as a geodesic of it. In the interacting case we can suppose that each moving point source carries its deficit angle and that it doesn’t scatter until it reaches the extremity of the cut of another one. Therefore we expect that their scattering is again topological, i.e. it can be reduced to a composition of \( SO(2, 2) \) cuts. In this section we will show how to use this global information in the context of the Minkowskian four-dimensional space-time \( X^A \).

Let us first consider the case of a particle, where the geodetic motion is parameterized by (2.21). In the case \( m = 1 \), we can solve the constraints (2.22) by choosing the following parameterization

\[
c_0^A = \frac{1}{\sqrt{\Lambda}} (ch\lambda_1 cos\alpha, -ch\lambda_1 sin\alpha, -sh\lambda_1 cos\beta, sh\lambda_1 sin\beta) \]

\[
c_1^A = \frac{1}{\sqrt{\Lambda}} (ch\lambda_2 sin\alpha, ch\lambda_2 cos\alpha, -sh\lambda_2 sin\beta, -sh\lambda_2 cos\beta). \]

(3.1)

Then, we can recast the equations of motion (2.24) in the following form where the time evolution has been eliminated, by using the parameterization (3.1):

\[
X^ze^{i\beta} + \overline{X}^ze^{-i\beta} + th\lambda_1 (X'^e^{i\alpha} + \overline{X}'e^{-i\alpha}) = 0
\]

\[
X^ze^{i\beta} - \overline{X}^ze^{-i\beta} + th\lambda_2 (X'^e^{i\alpha} - \overline{X}'e^{-i\alpha}) = 0. \]

(3.2)
These equations generalize the well known free motion on the plane, characterized by the equation $Z = VT$.

When a source has mass, its geodetic motion, described by (3.2), is followed by a generalized deficit angle or conical cut which can be obtained by looking at the static cone, defined in the $X\theta$-coordinates by the cut $X_\theta^z \to e^{-2\pi i \mu} X_\theta^z$, in a different coordinate system, related by a $SO(2,2)$ linear transformation to the $X\lambda$ one:

\[
\begin{align*}
\tilde{X}_0^t &= ch(\lambda_0) \tilde{X}_+^t + sh(\lambda_0) \tilde{X}_-^t \\
\tilde{X}_0^- &= ch(\lambda_0) \tilde{X}_-^t + sh(\lambda_0) \tilde{X}_+^t \\
\tilde{X}_0^z &= ch(\lambda_0) \tilde{X}_+^z + sh(\lambda_0) \tilde{X}_-^z,
\end{align*}
\]

(3.3)

where we have defined the following combinations:

\[
\begin{align*}
\tilde{X}_0^t &= X_t^0 e^{i\alpha} \pm \bar{X}_0^t e^{-i\alpha}, \quad \tilde{X}_0^z &= X_0^z e^{i\beta} \pm \bar{X}_0^z e^{-i\beta} \\
\tilde{X}_0^t &= X_0^t e^{i\alpha_0} \pm \bar{X}_0^t e^{-i\alpha_0}, \quad \tilde{X}_0^z &= X_0^z e^{i\beta_0} \pm \bar{X}_0^z e^{-i\beta_0}.
\end{align*}
\]

(3.4)

Therefore we obtain as definition of the cut corresponding to the geodesic motion (3.2), for arbitrary constants of motion $(\alpha, \beta, \lambda_1, \lambda_2)$, the following linear transformation

\[
\begin{align*}
\tilde{X}_+^t &\to (ch^2 \lambda_1 - \cos 2\pi \mu \ sh^2 \lambda_1) \tilde{X}_+^t + sh \lambda_1 ch \lambda_1 (1 - \cos 2\pi \mu) \tilde{X}_+^z \\
&+ isin 2\pi \mu sh \lambda_1 sh \lambda_2 \tilde{X}_-^t + isin 2\pi \mu sh \lambda_1 ch \lambda_2 \tilde{X}_-^z \\
\tilde{X}_-^t &\to (ch^2 \lambda_2 - \cos 2\pi \mu \ sh^2 \lambda_2) \tilde{X}_-^t + sh \lambda_2 ch \lambda_2 (1 - \cos 2\pi \mu) \tilde{X}_-^z \\
&+ isin 2\pi \mu ch \lambda_1 sh \lambda_2 \tilde{X}_+^t + isin 2\pi \mu sh \lambda_1 sh \lambda_2 \tilde{X}_+^z \\
\tilde{X}_+^z &\to (ch^2 \lambda_1 \cos 2\pi \mu - \sin 2\lambda_1) \tilde{X}_+^z + sh \lambda_1 ch \lambda_1 (\cos 2\pi \mu - 1) \tilde{X}_+^t \\
&- isin 2\pi \mu ch \lambda_1 ch \lambda_2 \tilde{X}_-^z - isin 2\pi \mu ch \lambda_1 sh \lambda_2 \tilde{X}_-^t \\
\tilde{X}_-^z &\to (ch^2 \lambda_2 \cos 2\pi \mu - \sin 2\lambda_2) \tilde{X}_-^z + sh \lambda_2 ch \lambda_2 (\cos 2\pi \mu - 1) \tilde{X}_-^t \\
&- isin 2\pi \mu ch \lambda_1 ch \lambda_2 \tilde{X}_+^z - isin 2\pi \mu sh \lambda_1 ch \lambda_2 \tilde{X}_+^t.
\end{align*}
\]

(3.5)

A more convenient way to write these monodromy conditions is to apply the bispinorial formalism, adapted to the group $SO(2,2) \sim SU(1,1) \otimes SU(1,1)$. It is not difficult to show that the transformation

\[
\begin{pmatrix}
X^t & X^z \\
X^\bar{z} & \bar{X}^t
\end{pmatrix}
\to
\begin{pmatrix}
A_1 & B_1 \\
\bar{B}_1 & \bar{A}_1
\end{pmatrix}
\begin{pmatrix}
X^t & X^z \\
X^\bar{z} & \bar{X}^t
\end{pmatrix}
\begin{pmatrix}
A_2 & B_2 \\
\bar{B}_2 & \bar{A}_2
\end{pmatrix}
\]

(3.6)
is a transformation of $SO(2,2)$.

For a spinless particle, whose static cut

\[ X^t \rightarrow X^t \]
\[ X^z \rightarrow e^{-2i\pi \mu} X^z \] (3.7)

can be decomposed as $A_1 = \overline{A}_2 = e^{-i\pi \mu}$, in general the cut can be defined by the invariant conditions

\[ A_1 + \overline{A}_1 = A_2 + \overline{A}_2 = 2 \cos \pi \mu, \] (3.8)

that can be solved in such a way to reproduce exactly the Lorentz transformation (3.5) applying the law (3.6):

\[ A_1 = \cos \pi \mu - ich(\lambda_1 - \lambda_2) \sin \pi \mu \]
\[ B_1 = -ie^{-i(\alpha + \beta)} sh(\lambda_1 - \lambda_2) \sin \pi \mu \]
\[ A_2 = \cos \pi \mu + ich(\lambda_1 + \lambda_2) \sin \pi \mu \]
\[ B_2 = -ie^{i(\alpha - \beta)} sh(\lambda_1 + \lambda_2) \sin \pi \mu. \] (3.9)

For a spinning particle, whose static cut

\[ X^t \rightarrow e^{2i\pi \sqrt{\Lambda} r_+} X^t \]
\[ X^z \rightarrow e^{2i\pi \sqrt{\Lambda} r_-} X^z \] (3.10)

can be decomposed as $A_1 = -e^{i\pi \sqrt{\Lambda}(r_+ + r_-)}$ and $\overline{A}_2 = -e^{i\pi \sqrt{\Lambda}(r_+ - r_-)}$, the general cut can be defined by the invariant conditions

\[ A_1 + \overline{A}_1 = 2 \cos \pi (1 - \sqrt{\Lambda}(r_+ + r_-)) \quad A_2 + \overline{A}_2 = 2 \cos \pi (1 - \sqrt{\Lambda}(r_+ - r_-)), \] (3.11)

that can be solved similarly to eq. (3.9):

\[ A_1 = \cos \pi (1 - \sqrt{\Lambda}(r_+ + r_-)) + ich(\lambda_1 - \lambda_2) \sin \pi \sqrt{\Lambda}(r_+ + r_-) \]
\[ B_1 = -ie^{-i(\alpha + \beta)} sh(\lambda_1 - \lambda_2) \sin \pi \sqrt{\Lambda}(r_+ + r_-) \]
\[ A_2 = \cos \pi (1 - \sqrt{\Lambda}(r_+ - r_-)) - ich(\lambda_1 + \lambda_2) \sin \pi \sqrt{\Lambda}(r_+ - r_-) \]
\[ B_2 = -ie^{i(\alpha - \beta)} sh(\lambda_1 + \lambda_2) \sin \pi \sqrt{\Lambda}(r_+ - r_-). \] (3.12)

In the case of hyperbolic monodromies, the general invariants relations (3.11), defining implicitly two masses, must be substituted with analogous relations, defining instead two rapidities. For example, in the case of a spinning black hole, whose static cut

\[ X^0 \pm X^2 \rightarrow e^{\pm 2\pi \sqrt{\Lambda} r_+} (X^0 \pm X^2) \]
\[ X^1 \pm X^3 \rightarrow e^{\mp 2\pi \sqrt{\Lambda} r_-} (X^1 \pm X^3) \] (3.13)
can be obtained with the position

\begin{align*}
A_1 &= \cosh \pi \sqrt{\Lambda}(r_+ + r_-) \\
B_1 &= \sinh \pi \sqrt{\Lambda}(r_+ + r_-)
\end{align*}

\begin{align*}
A_2 &= \cosh \pi \sqrt{\Lambda}(r_+ - r_-) \\
B_2 &= \sinh \pi \sqrt{\Lambda}(r_+ - r_-),
\end{align*}

(3.14)

the general cut can be introduced with the condition:

\begin{align*}
A_1 + \bar{A}_1 &= 2\cosh \pi \sqrt{\Lambda}(r_+ + r_-) \\
A_2 + \bar{A}_2 &= 2\cosh \pi \sqrt{\Lambda}(r_+ - r_-)
\end{align*}

(3.15)

and it is enough to choose the following parameterization:

\begin{align*}
A_1 &= \cosh \pi \sqrt{\Lambda}(r_+ + r_-) - i\sinh(\lambda_1 - \lambda_2)\sinh \pi \sqrt{\Lambda}(r_+ + r_-) \\
B_1 &= e^{-i(\alpha + \beta)}\cosh(\lambda_1 - \lambda_2)\sinh \pi \sqrt{\Lambda}(r_+ + r_-) \\
A_2 &= \cosh \pi \sqrt{\Lambda}(r_+ - r_-) + i\sinh(\lambda_1 + \lambda_2)\sinh \pi \sqrt{\Lambda}(r_+ - r_-) \\
B_2 &= e^{i(\alpha - \beta)}\cosh(\lambda_1 + \lambda_2)\sinh \pi \sqrt{\Lambda}(r_+ - r_-).
\end{align*}

(3.16)

Now we can ask ourselves what is the solution of the two-body problem in global terms. The result of the composition of two monodromies, in the case of particles, is of course of the type:

\[
\begin{pmatrix}
X' \cdot t & X' \cdot z \\
X \cdot t & X \cdot z
\end{pmatrix} = M_L \begin{pmatrix}
X' \cdot t & X' \cdot z \\
X \cdot t & X \cdot z
\end{pmatrix} M_R,
\]

(3.17)

where to the $M_L, M_R$ matrices it is possible to associate the corresponding invariant masses $\sqrt{\lambda}$:

\begin{align*}
\cos(\pi \mathcal{M}_L) &= \cos(\pi \mu_1)\cos(\pi \mu_2) - \frac{P^L_1 \cdot P^L_2}{m_1 m_2} \sin(\pi \mu_1)\sin(\pi \mu_2) \\
\cos(\pi \mathcal{M}_R) &= \cos(\pi \mu_1)\cos(\pi \mu_2) - \frac{P^R_1 \cdot P^R_2}{m_1 m_2} \sin(\pi \mu_1)\sin(\pi \mu_2)
\end{align*}

(3.18)

and we have defined the following vectors, constants of motion:

\begin{align*}
P^L_1 &= m_1 \gamma^L_1(1, v^L_1) & \gamma^L_1 &= \cosh(\lambda_1 - \lambda_2) & \gamma^L_1 v^L_1 &= e^{-i(\alpha + \beta)}\sinh(\lambda_1 - \lambda_2) \\
P^R_1 &= m_1 \gamma^R_1(1, v^R_1) & \gamma^R_1 &= \cosh(\lambda_1 + \lambda_2) & \gamma^R_1 v^R_1 &= e^{-i(\alpha - \beta)}\sinh(\lambda_1 + \lambda_2).
\end{align*}

(3.19)

For generic values of the constants of motions, the left-invariant mass $\mathcal{M}_L$ will be different from the right-invariant mass $\mathcal{M}_R$ and therefore the composed system has spin, other
than invariant mass. In fact, by comparing the cut of two-body with that of a spinning particle we obtain that the total mass $\mu_{\text{tot}}$ and $J$ are defined by solving the conditions $(1 - \sqrt{\Lambda}(r_+ + r_-))^2 = \mathcal{M}_L^2$ and $(1 - \sqrt{\Lambda}(r_+ - r_-))^2 = \mathcal{M}_R^2$.

An analogous remark can in principle be made for the composition of hyperbolic monodromies, describing the scattering of BTZ black holes, however we have no control that such a solution exists, free of some fancy extra singularity, while in the particle case at least in the limit $\Lambda \to 0$ a physically acceptable solution is known. To approach the question of the solution for the scattering of black holes we believe that one has to learn how to give an intrinsic, coordinate independent, meaning to the horizons.

To make these monodromies more explicit, we are going to solve the constraint $X^A X_A = \frac{1}{\Lambda}$ with a parameterization which carries spinorial representations of each $SU(1,1)$:

$$f \to \frac{A_1 f + B_1}{B_1 f + A_1}, \quad g \to \frac{A_2 g + B_2}{B_2 g + A_2}.$$  \hspace{1cm} (3.20)

Let us choose the following parameterization:

$$\begin{pmatrix} X^t & X^z \\ X^z & X^t \end{pmatrix} = h \begin{pmatrix} f \\ 1 \end{pmatrix} (\bar{g} \quad 1) + \bar{h} \begin{pmatrix} \frac{1}{f} \\ 0 \end{pmatrix} (1 \quad g).$$ \hspace{1cm} (3.21)

The condition of the constraint (1.1) implies that

$$|h|^2 (1 - f \bar{f})(1 - g \bar{g}) = \frac{1}{\Lambda}.$$ \hspace{1cm} (3.22)

The condition of representing the monodromies implies that:

$$h \sqrt{\partial_z f \partial_{\bar{z}} g} = H(z, \bar{z}, t),$$ \hspace{1cm} (3.23)

where $H$ is a field, invariant under $SU(1,1) \otimes SU(1,1)$, which can be chosen as

$$H = \frac{e^{-iT}}{\sqrt{\lambda}} \frac{(\partial_z f \partial_{\bar{z}} \bar{f} \partial_x g \partial_{\bar{x}} \bar{g})}{(1 - f \bar{f})(1 - g \bar{g})}.$$ \hspace{1cm} (3.24)

where we have added an extra phase, the Minkowskian time $T$, which can play the role of introducing an explicit time evolution of the $X^A$-mapping.

We end up with the following solution

$$\begin{pmatrix} X^t \\ X^z \\ X^x \end{pmatrix} = \frac{1}{\sqrt{\lambda}} \left( \frac{\partial_x g \partial_{\bar{z}} \bar{f}}{\partial_z f \partial_{\bar{z}} g} \right)^{1/4} \frac{e^{-iT}}{\sqrt{(1 - f \bar{f})(1 - g \bar{g})}} \begin{pmatrix} f \bar{g} & f \\ \bar{g} & 1 \end{pmatrix} + \text{...}$$
\[ \frac{1}{\sqrt{\Lambda}} \left( \partial_z f \partial_z g \right)^{\frac{1}{4}} \frac{e^{iT}}{\sqrt{(1 - f f)(1 - g g)}} \left( \frac{1}{f} \frac{g}{f g} \right). \] (3.25)

The analogous case of the static solution of (2+1)-gravity is obtained by choosing the constants of motion to be \( M_L = M_R \), therefore \( f = g = Z \) and in this case the parameterization (3.25) reduces to (in the notation \( X^A = (X^0, X^1, X^z, X^\pi) \))

\[ X^A = \frac{1}{\sqrt{\Lambda}} \left( \cos T \frac{1 + ZZ}{1 - ZZ}, \sin T, \cos T \frac{2Z}{1 - ZZ}, \cos T \frac{2Z}{1 - ZZ} \right). \] (3.26)

A similar mapping has already been introduced in [17].

Let us first discuss the parameterization (3.26) in the case of one body. To make contact with the non-perturbative \( N \)-body solution of (2+1)-gravity [6], it is convenient to introduce the spatial conformal gauge \( g_{zz} = 0 \), which is obtained by (2.2) with a radial mapping

\[ r \to \frac{(1 - \mu)r^{(1-\mu)}}{1 - \frac{\Lambda}{4} r^{2(1-\mu)}}. \] (3.27)

Then the \( X^A \)-mapping (2.2) becomes

\[ X^t = \frac{1}{\sqrt{\Lambda}} \frac{1 + \frac{\Lambda}{4} r^{2(1-\mu)}}{1 - \frac{\Lambda}{4} r^{2(1-\mu)}} e^{i\sqrt{\Lambda}(1-\mu)t} \]
\[ X^z = \frac{z^{1-\mu}}{1 - \frac{\Lambda}{4} r^{2(1-\mu)}}, \] (3.28)

and in the variables \( T, Z \) can be rewritten as :

\[ \sin T = a \frac{1 + \frac{\Lambda}{4} r^{2(1-\mu)}}{1 - \frac{\Lambda}{4} r^{2(1-\mu)}} a = \sin(\sqrt{\Lambda}(1-\mu)t) \]
\[ Z = \frac{\sqrt{\Lambda}}{4} \sqrt{1 - a^2} z^{1-\mu} \left[ 1 + \frac{4}{\Lambda} r^{-2(1-\mu)} \left( 1 - \sqrt{1 - \frac{\Lambda}{2} \left( \frac{1 + a^2}{1 - a^2} \right) r^{2(1-\mu)} + \frac{\Lambda^2}{16} r^{4(1-\mu)} \right) \right]. \] (3.29)

So we can understand how it is difficult to extend such a mapping in the \( N \)-body problem. The polydromic part \( z^{(1-\mu)} \) is only a small piece of the complete solution. Instead in the case of the \( N \)-body solution of (2 + 1) gravity [3-7], the polydromic part is enough to solve completely the fields. This involution of the solution is also to be expected because there is no \( N \)-body static solution. The introduction of an explicit scale parameter allows a
static Newtonian force between point sources. We will show afterwards that a pure analytic solution to the mapping problem \( X^A \) is not physically consistent.

Moreover there are other problems. In the conformal gauge, there is a physical limit in which the spatial slice of the universe ends, which is visible in the \( X^A \) coordinates, since the values of some \( X^A \)-coordinates diverges there, but it is not related to the apparent singularity of the parameterization \( (3.26) \) at the particular value \(|Z| = 1\), because \( \cos T \) vanishes also. Therefore apart from the incidental physical limit on the values of the spatial coordinates, depending on the gauge choice, there is another artificial limit on the spatial coordinates, because the parameterization \( (3.26) \) cannot globally extended, but it has to substituted with another one outside its domain of validity.

For more than one body, we must solve the monodromy conditions

\[
\begin{align*}
Z & \to \frac{a_1 Z + b_1}{b_i Z + c_i} \quad \ldots \\
& \to \frac{a_i Z + b_i}{b_i Z + c_i}
\end{align*}
\]

that maintain the constraint \(|Z| = 1\), which defines a well defined closed line in the plane. However this line is purely artificial, since the solution can be continued across it towards the true edge of the spatial slice of the universe.

As in the one-body case, we are convinced that it is not useful to solve these monodromy conditions analytically. Let us suppose to define a gauge choice such that \( Z = Z(z, \xi_i(t)) \) has cuts as defined in eq. \( (3.30) \). Then the geodesics equations of motions for the particles \( \xi_i(t) \), which imply that the values of the \( Z \)-mapping coincide at the particle sites with the fixed points of the \( Z \)-monodromy, are, in the case of an analytic solution, automatically satisfied for an arbitrary motion \( \xi_i(t) \) and one is tempted to conclude that the dynamics is completely arbitrary against any physical intuition and the correspondence with the geodesic limit.

In the case of \((2 + 1)\) gravity, the dynamics was defined not only by the equations of motion but also by the choice of the boundary conditions of the fields at infinity. In the case of the cosmological constant, the choice of the boundary conditions is a harder problem because the fields do not vanish at infinity. Our proposal is to choose a physical gauge, like the one introduced in \((9)\), and require that in the limit \( \Lambda \to 0 \) one recovers the \( N \)-body solution of \((2 + 1)\)-gravity, which automatically imposes some boundary conditions. Then one can consistently check that the spatial field equation for \( Z = Z(z, \tau, t) \) is explicitly time-dependent, and that this property is enough to produce non-trivial solutions to the geodesic equations. Detailed analysis will be presented in a future work.

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A Appendix

In this appendix we show the equivalence between our method of solving the geodesics equation and the traditional method of integrate them. In particular the main point is to rearrange the first order constants of motion \( c_i^A \) in terms of the angular momentum \( L \) and the energy \( E \), by using the following relations:

\[
\sqrt{\Lambda}(X^0 \pm X^2) = \frac{k_0^{\pm 1}}{\sqrt{r_+^2 - r_-^2}} \left\{ \begin{array}{ll}
\pm \frac{1}{\sqrt{F_-}} \left( G_- + F_-(\tau - \tau_0) \right) & \text{if } m = 0, A \neq 0 \\
\pm \left[ \sqrt{\frac{c_0^2}{2}} + G_- \cos(\tau - \tau_0) + \sqrt{\frac{c_0^2}{2}} + G_- \sin(\tau - \tau_0) \right] & \text{if } m = 1
\end{array} \right.
\]

\[
\sqrt{\Lambda}(X^1 \pm X^3) = \frac{k_1^{\pm 1}}{\sqrt{r_+^2 - r_-^2}} \left\{ \begin{array}{ll}
\pm \frac{1}{\sqrt{F_+}} \left( G_+ + F_+(\tau - \tau_0) \right) & \text{if } m = 0, A \neq 0 \\
\pm \left[ \sqrt{\frac{c_1^2}{2}} + G_+ \cos(\tau - \tau_0) + \sqrt{\frac{c_1^2}{2}} + G_+ \sin(\tau - \tau_0) \right] & \text{if } m = 1
\end{array} \right.
\]

\[
k_0 = \sqrt{r_+^2 - r_-^2} \left\{ \begin{array}{ll}
\sqrt{\frac{c_0^2}{2}} + G_- \left( c_0^0 \cos \tau_0 + c_0^0 \sin \tau_0 \right) - \sqrt{\frac{c_0^2}{2}} - G_- \left( c_0^0 \cos \tau_0 - c_0^0 \sin \tau_0 \right) & \text{if } m = 0, A \neq 0 \\
0 & \text{if } m = 1
\end{array} \right.
\]

\[
k_1 = \sqrt{r_+^2 - r_-^2} \left\{ \begin{array}{ll}
\sqrt{\frac{c_1^2}{2}} + G_+ \left( c_1^1 \cos \tau_0 - c_1^1 \sin \tau_0 \right) - \sqrt{\frac{c_1^2}{2}} - G_+ \left( c_1^1 \cos \tau_0 + c_1^1 \sin \tau_0 \right) & \text{if } m = 0, A \neq 0 \\
0 & \text{if } m = 1
\end{array} \right.
\]

The definition of \( \tau_0 \), depending on \( m \), is implicit in the matching with eq. (2.26). In the derivation of eq. (A.1) we have used the fact that these formulas must be compatible with equations (2.25) and (2.26).

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