DIMENSION RIGIDITY FOR COOKIE-CUTTER CANTOR SETS

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Abstract. We show that two cookie-cutter Cantor sets with the same symbolic coding are differentiably equivalent if and only if their Hausdorff dimensions are equal.

1. Introduction

Cookie-cutter Cantor sets are a well-studied class of fractal invariant sets of dynamical systems. They arise as limit sets of iterated function systems in the line, with a separation condition stronger than the usual open set condition, and with a regularity assumption on the maps. Alternatively, they can be described as conformal expanding repellers, invariant sets for an expanding map of the interval. Let $\Sigma_p = \{0, \ldots, p-1\}^\mathbb{N}$. It can be shown that these Cantor sets are homeomorphic to $\Sigma_p$ and that the expanding map is conjugate to the shift on $\Sigma_p$ via this homeomorphism. We say such a cookie-cutter Cantor set is modeled by $\Sigma_p$. For an introduction, see [1] and [5].

Motivated by the phenomenon of Feigenbaum universality, Sullivan [6] initiated an investigation of these sets and introduced the scaling function, which describes the contraction ratio in the fractal at arbitrarily small scales. This is a function of a dual Cantor set, defined symbolically.

There are two natural notions of equivalence between such Cantor sets. We consider two cookie-cutter Cantor sets geometrically equivalent if they have the same scaling functions. An a priori more restrictive notion is differential equivalence, a $C^{1+\alpha}$-change of coordinates between the Cantor sets.

Sullivan showed that the scaling function is a complete invariant of the differential structure, so that these notions of equivalence coincide for cookie-cutter Cantor sets. The fact that the geometric structure determines the differential structure is a rigidity statement. Proofs of Sullivan’s results appeared later in [4], [5], and [2].

Cookie-cutter Cantor sets have a rich dimension theory that is closely related to the thermodynamic formalism defined on the symbolic coding space. This relation is Bowen’s theorem, which expresses the Hausdorff dimension as the zero of a pressure function. By dualizing this theory and relating the pressure to the scaling function on the dual, we obtain a rigidity statement about the dimension:

Theorem. Two cookie-cutter Cantor sets modeled by $\Sigma_p$ are $C^{1+\alpha}$-equivalent for some $\alpha > 0$ if and only if their Hausdorff dimensions are equal.

A $C^1$ map of a compact space preserves Hausdorff dimension, so the forward implication is trivial. To prove the reverse, by Sullivan’s theorem it suffices to show that if the sets have the same Hausdorff dimension, their scaling functions are equal. We show this in Section 4.
2. Cookie-cutters

2.1. Preliminaries. Let $\Sigma_p = \{0, \ldots, p-1\}^\mathbb{N}$ be the one-sided infinite words on $p$ symbols, and

$$
\Sigma_{p,n} = \{ (\omega_1, \ldots, \omega_n) : \omega_i \in \{0, \ldots, p-1\} \}
$$

the words of length $n \geq 1$.

For each $\omega = (\omega_1, \omega_2, \ldots) \in \Sigma_p$, define $\bar{\omega} = (\ldots, \omega_2, \omega_1)$. We say that $\bar{\omega}$ is the word dual to $\omega$. The collection of all such words is $\bar{\Sigma}_p$, the Cantor set dual to $\Sigma_p$. The set of dual words of length $n$ is

$$
\bar{\Sigma}_{p,n} = \{ (\omega_n, \ldots, \omega_1) : \omega_i \in \{0, \ldots, p-1\} \}.
$$

2.2. Cookie-cutter Cantor sets. Let $I = [0,1]$, choose sequences $\{x_i\}_{i=0}^{p-1}$ and $\{y_i\}_{i=1}^{p-1}$ satisfying

$$
0 \leq x_0 < y_0 < x_1 < y_1 < \cdots < x_{p-1} < y_{p-1} \leq 1,
$$

and for $i = 0, \ldots, p-1$ define $I_i = [x_i, y_i]$. Consider a map

$$
S : \bigcup_{i=0}^{p-1} I_i \to I,
$$

such that $S|_{I_i}$ maps $I_i$ injectively onto $I$ for each $i = 0, \ldots, p-1$. Furthermore, assume that $S$ is $C^{1+\alpha}$ for some $\alpha > 0$, and that $|DS(x)| > 1$ for all $x \in \bigcup_{i=0}^{p-1} I_i$. Such a map $S$ is called a cookie-cutter map.

For each $i = 0, \ldots, p-1$ let $\phi_i : I \to I_i$ be the inverse to $S|_{I_i}$. For each $\omega \in \Sigma_{p,n}$, denote

$$
I_{\omega} = \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n}(I).
$$

Because $|DS(x)| > 1$ for all $x \in \bigcup_{i=0}^{p-1} I_i$ and $\phi_i$ is inverse to $S$, we know that $|D\phi_i(x)| < 1$ for each $i$ and $x \in I_i$. Thus there exists $\lambda < 1$ such that $|D\phi_i| \leq \lambda$. By the mean value theorem, if $\omega \in \Sigma_{p,n}$, then $|I_{\omega}| \leq \lambda^n$.

Then for any $(\omega_1, \omega_2, \ldots, \omega_n) \in \Sigma_p$,

$$
\lim_{n \to \infty} |I_{\omega_1, \ldots, \omega_n}| = 0,
$$

and we have the nesting property

$$
I_{\omega_1, \ldots, \omega_n} \supset I_{\omega_1, \ldots, \omega_{n+1}}.
$$

From these facts we deduce that $\bigcap_{n=1}^{\infty} I_{\omega_1, \ldots, \omega_n}$ is a unique point, which defines a coding map $\pi : \Sigma_p \to I$ given by

$$
\pi(\omega) = \bigcap_{n=1}^{\infty} I_{\omega_1, \ldots, \omega_n}.
$$

The map $\pi$ is a homeomorphism, and the $S$-invariant Cantor set $\Lambda = \pi(\Sigma_p)$ is called a cookie-cutter Cantor set. It is easy to see that

$$
\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{\omega \in \Sigma_{p,n}} I_{\omega}.
$$

Each point $x \in \Lambda$ corresponds via $\pi^{-1}$ to a unique word $\omega \in \Sigma_p$. 
2.3. **Bounded distortion.** Our assumption of $C^{1+\alpha}$ regularity on $S$ implies a strong restriction on the distortion of $DS$.

**Proposition 2.1.** There exists a constant $A > 0$ such that for all $n, m \geq 1$, $\omega \in \Sigma_{p,n+m}$, and $x, y \in I_\omega$,

$$e^{-A\lambda^{n\alpha}} < \frac{|DS^m(x)|}{|DS^m(y)|} < e^{A\lambda^{n\alpha}}.$$ 

**Proof.** First, let $f : I \to \mathbb{R}$ be Hölder continuous with exponent $\alpha$, i.e., for any $x, y \in I$ there exists $C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha.$$ 

Now define

$$S_m f(x) = \sum_{i=0}^{m-1} f(S^i x).$$

Because $x, y \in I_\omega = I_{\omega_1, \ldots, \omega_{n+m}}$, for each $0 \leq i \leq n - 1$ we have $S^i(x), S^i(y) \in I_{\omega_{i+1}, \ldots, \omega_{n+m}}$.

$$|S_m f(x) - S_m f(y)| \leq \sum_{i=0}^{m-1} |f(S^i x) - f(S^i y)|$$

$$\leq \sum_{i=0}^{m-1} C|S^i x - S^i y|^\alpha$$

$$\leq C \sum_{i=0}^{m-1} \lambda^{(n+m-i)\alpha}$$

$$< C\lambda^n \sum_{i=0}^{\infty} \lambda^{i\alpha} = \frac{C\lambda^n}{1 - \lambda^\alpha}.$$ 

Now choose $f(x) = -\log |DS(x)|$. Because $DS$ is Hölder continuous and bounded away from zero, so is $f$. By the chain rule, $S_m f(x) = -\log |DS^m(x)|$. Setting $A = \frac{C}{1 - \lambda^\alpha}$ yields the claim. \qed

3. **Ratio geometry**

We will study the fractal geometry of $\Lambda = \bigcap_n \cup_i I_\omega$ by measuring the contraction rate of $|I_{\omega_{n+1}}|$ relative to $|I_{\omega_n}|$ as follows. Fix $\omega = (\ldots, \omega_2, \omega_1)$ in the dual $\tilde{\Sigma}_p$. For each $i = 0, \ldots, p - 1$ we have a sequence of ratios

$$a_n(\omega, i) = \frac{|I_{\omega_n, \ldots, \omega_1, i}|}{|I_{\omega_n, \ldots, \omega_1}|}.$$ 

The sequence is called the *ratio geometry* along the word $\omega$. The following property shows the ratio geometry along $\omega$ is eventually bounded, and the bound is independent of $\omega$.

**Proposition 3.1.** There exists a constant $A > 0$ such that for all $\omega \in \tilde{\Sigma}_p$, $n, m \geq 1$, and $i = 0, \ldots, p - 1$,

$$e^{-A\lambda^{n\alpha}} < \frac{a_{n+m}(\omega, i)}{a_n(\omega, i)} < e^{A\lambda^{n\alpha}}.$$
Proof. By the mean value theorem there exist \( x \in I_{\omega_n+\ldots,\omega_1} \) and \( y \in I_{\omega_n+\ldots,\omega_1,i} \) such that

\[
|I_{\omega_n,\ldots,\omega_1}| = |DS^m(x)||I_{\omega_n+\ldots,\omega_1,1}|, \quad \text{and} \quad |I_{\omega_n,\ldots,\omega_1,i}| = |DS^m(y)||I_{\omega_n+\ldots,\omega_1,i}|.
\]

Combining these equations yields

\[
\frac{a_{n+m}(\omega, i)}{a_n(\omega, i)} = \frac{|DS^m(x)|}{|DS^m(y)|}.
\]

Because \( I_{\omega_n+\ldots,\omega_1,i} \subset I_{\omega_n+\ldots,\omega_1} \), the desired inequality now follows from Proposition 2.1. \( \square \)

3.1. The scaling function. As \( n \to \infty \), the ratio geometry sequence \( a_n(\omega, i) \) measures the contraction rate at arbitrarily small scales. Along dual words \( \omega \), this rate approaches a constant:

(2) \[
a(\omega, i) = \lim_{n \to \infty} a_n(\omega, i).
\]

The function \( a : \tilde{\Sigma}_p \to \mathbb{R}^p \) defined by \( \omega \mapsto \{a(\omega, i)\}_{i=0}^{p-1} \) is called the scaling function, defined on the dual Cantor set \( \tilde{\Sigma}_p \).

**Proposition 3.2.** For each \( \omega \in \tilde{\Sigma}_p \) and \( i = 0, \ldots, p - 1 \), the limit in Equation (2) exists, and the convergence is exponentially fast in \( n \).

For the proof (and for later proofs) we will require an auxiliary lemma.

**Lemma 3.3.** For \( C, A, \delta > 0 \) and \( 0 < t < 1 \), the sequences \( \log(1 + Ce^{-\delta n}) \), \( \log(1 - Ce^{-\delta n}) \), and \( At^n \) are all asymptotically equivalent; i.e. given \( C, \delta > 0 \) there exist \( A > 0 \) and \( 0 < t < 1 \) such that \( \log(1 + Ce^{-\delta n}) \leq At^n \) for all \( n \geq 1 \), and there are identical statements comparing all pairs of these three sequences.

Proof. The proof follows easily from the Taylor expansion of \( \log(1 \pm x) \) about \( x = 0 \). \( \square \)

**Proof of Proposition 3.2** By Proposition 3.1

\[
\log \left( \frac{a_{n+m}(\omega, i)}{a_n(\omega, i)} \right) < A\lambda^{na}
\]

for all \( n, m \geq 1 \). Because \( \lambda < 1 \), this shows that the sequence \( \log a_n(\omega, i) \) is Cauchy. Because \( a_n(\omega, i) \) is bounded away from zero, the limit \( a(\omega, i) \) exists.

To see that the convergence is exponential, take \( m \to \infty \) in Proposition 3.1 which yields

\[
e^{-A\lambda^{na}} < \frac{a_n(\omega, i)}{a(\omega, i)} < e^{A\lambda^{na}}.
\]

By Lemma 3.3 there exist constants \( C, \delta > 0 \) such that \( A\lambda^{na} \leq \log(1+Ce^{-\delta n}) \), and \(-A\lambda^{na} \geq \log(1-Ce^{-\delta n})\). Letting \( K = C a(\omega, i) \), we obtain

\[
|a_n(\omega, i) - a(\omega, i)| \leq Ke^{-\delta n}.
\]

\( \square \)
3.2. Geometric and differential equivalence. Let $\Lambda$ and $\Xi$ be two cookie-cutter Cantor sets modeled by $\Sigma_p$. Suppose that $\pi, \rho : \Sigma_p \to I$ are the coding maps for $\Lambda, \Xi$, respectively, and that $a, b : \Sigma_p \to \mathbb{R}^2$ are their respective scaling functions. We say that $\Lambda$ and $\Xi$ are *geometrically equivalent* if their scaling functions are equal, i.e. $a(\omega) = b(\omega)$ for all $\omega \in \Sigma_p$. We say that $\Lambda$ and $\Xi$ are $C^{1+\alpha}$-equivalent if there exists a $C^{1+\alpha}$ diffeomorphism $\Phi : I \to I$ such that the following diagram commutes.

These two notions of equivalence are related by Sullivan’s theorem.

**Theorem 3.4** (Sullivan [6]). Two cookie-cutter Cantor sets modeled by $\Sigma_p$ are geometrically equivalent if and only if they are $C^{1+\alpha}$-equivalent for some $\alpha > 0$.

For an exposition and proof of Sullivan’s theorem, see [5], [2].

4. Pressure

Let $\Lambda = \bigcap_n \bigcup_{\omega} I_\omega$ be a cookie-cutter Cantor set modeled by $\Sigma_p$. For any $t > 0$ and $n \geq 1$, define

$$p_n(t) = \sum_{\omega \in \Sigma_{p,n}} |I_\omega|^t.$$  

(3)

It can be shown that there exists a constant $K > 0$ such that $K^{-1} p_n(t) p_m(t) \leq p_{n+m}(t) \leq K p_n(t) p_m(t)$, so that the following limit exists

$$p(t) = \lim_{n \to \infty} \frac{1}{n} \log p_n(t),$$

called the *pressure* of $\Lambda$. Furthermore, $p$ is strictly decreasing and convex, with a unique zero on $(0,1)$. For an introduction to pressure with proofs of these properties see [3]. The pressure is related to the Hausdorff dimension by Bowen’s theorem (see [1]).

**Theorem 4.1** (Bowen). Let $\Lambda$ be a cookie-cutter Cantor set with pressure $p(t)$. The Hausdorff dimension of $\Lambda$ is the unique solution to $p(t) = 0$.

4.1. Pressure and ratio geometry. Consider the function $p_n(t)$ defined in Equation 3. The sum of all $|I_\omega|^t$ is the same whether $\omega$ ranges over $\Sigma_{p,n}$ or the dual $\tilde{\Sigma}_{p,n}$, so we re-index and obtain

$$p_n(t) = \sum_{\omega \in \Sigma_{p,n}} |I_\omega|^t.$$
This allows us to relate the pressure to the ratio geometry on the dual.

\begin{equation}
    p(t) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{(\omega_n, \ldots, \omega_1) \in \Sigma_{p,n}} \prod_{i=1}^{n} \left( \frac{|I_{\omega_n, \ldots, \omega_1}|}{|I_{\omega_n, \ldots, \omega_1+1}|} \right)^t.
\end{equation}

\textbf{Theorem 4.2.} If \( \Lambda \) and \( \Xi \) are cookie-cutter Cantor sets modeled by \( \Sigma_p \) that are not geometrically equivalent, then their pressures are bounded away from each other.

\textit{Proof.} Let \( \Lambda = \bigcap_n \cup_\omega I_\omega \) and \( \Xi = \bigcap_n \cup_\omega J_\omega \). Let \( p(t) \) be the pressure of \( \Lambda \) and \( q(t) \) the pressure of \( \Xi \). Then \( p(t) = \lim_{n \to \infty} p_n(t) \) and \( q(t) = \lim_{n \to \infty} q_n(t) \), where

\begin{equation}
    p_n(t) = \sum_{\omega \in \Sigma_{p,n}} |I_\omega|^t, \quad \text{and} \quad q_n(t) = \sum_{\omega \in \Sigma_{p,n}} |J_\omega|^t.
\end{equation}

Let \( a, b : \tilde{\Sigma}_p \to \mathbb{R}^p \) be the scaling functions of \( \Lambda \) and \( \Xi \), respectively. By our assumption, there exists a word \( \tau \in \tilde{\Sigma}_p \) such that \( a(\tau, i) \neq b(\tau, i) \). Because \( a(\tau, i) \) and \( b(\tau, i) \) are the limits of the ratio geometry sequences along \( \tau \), we may assume without loss of generality that the quotient of these sequences converges to \( l < 1 \). By Proposition 3.2, the convergence is exponential, so there exists constants \( C, \delta > 0 \) such that

\begin{equation}
    \left| \frac{a_n(\tau, i)}{b_n(\tau, i)} - l \right| \leq Ce^{-n\delta}.
\end{equation}

Define \( C_0 = \tilde{\Sigma}_{p,n} \) and \( C_n = \{(\tau_n, \ldots, \tau_1)\} \), so that \( \#C_0 = p^n \) and \( \#C_n = 1 \). Consistent with this notation, for each \( 1 \leq k \leq n - 1 \), we define the cylinder sets

\( C_k = \{(i_n, \ldots, i_{k+1}, \tau_k, \ldots, \tau_1) : i_n, \ldots, i_{k+1} \in \{0, 1\}\} \).

Note that \( \#C_k = p^{n-k} \).

Now set \( A_n = C_n \) and for \( 0 \leq k \leq n - 1 \) define \( A_k = C_k \setminus C_{k+1} \). Thus \( \#A_k = p^{n-k-1} \) and \( \sum_{k=0}^{n} \#A_k = p^n \), so that \( \{A_k\}_{k=0}^{n} \) forms a disjoint partition of \( \tilde{\Sigma}_{p,n} \). Splitting the sum in Equation (5) over this partition yields

\begin{equation}
    q_n(t) = \sum_{k=0}^{n} \sum_{(\omega_n, \ldots, \omega_1) \in A_k} \prod_{i=1}^{k} \left( \frac{|I_{\omega_n, \ldots, \omega_1}|}{|I_{\omega_n, \ldots, \omega_1+1}|} \right)^t.
\end{equation}

Because \( A_k \subset C_k \), each word \((\omega_n, \ldots, \omega_1) \in A_k\) right-truncated to \( 1 \leq i \leq k \) is of the form \((\omega_n, \ldots, \omega_i) = (i_n, \ldots, i_{k+1}, \tau_k, \ldots, \tau_1)\) with \( i_j \in \{0, 1\}\). By Proposition 3.1 applied to \( \Xi \),

\begin{equation}
    \frac{|J_{i_n, \ldots, i_{k+1}, \tau_k, \ldots, \tau_1}|}{|J_{i_n, \ldots, i_{k+1}, \tau_k, \ldots, \tau_1+1}|} < \frac{|J_{\tau_k, \ldots, \tau_1}|}{|J_{\tau_k, \ldots, \tau_1+1}|} e^{B\eta(k-i)\alpha},
\end{equation}

where \( \eta \) is the constant such that \( |J_\omega| \leq \eta^n \) when \( \omega \in \tilde{\Sigma}_n \), and \( B \) is defined for \( \Xi \) as \( A \) was for \( \Lambda \).

From Equation (6) we have

\begin{equation}
    \frac{|J_{\tau_k, \ldots, \tau_1}|}{|J_{\tau_k, \ldots, \tau_1+1}|} \leq \frac{|I_{\tau_k, \ldots, \tau_1}|}{|I_{\tau_k, \ldots, \tau_1+1}|} (1 + C e^{-(k-i)\delta}).
\end{equation}
Substituting this into Equation 8 yields
\[
\frac{|J_{\tau_1 \ldots \tau_{k+1}, \tau_{k+1} \ldots \tau_1}|}{|J_{\tau_1 \ldots \tau_{k+1}, \tau_{k+1} \ldots \tau_1}|} < \frac{|I_{\tau_1 \ldots \tau_{k+1}, \tau_{k+1} \ldots \tau_1}|}{|I_{\tau_1 \ldots \tau_{k+1}, \tau_{k+1} \ldots \tau_1}|} e^{A\lambda(k-i)\alpha} (l + Ce^{-(k-i)\delta}).
\]

Again by Proposition 3.1,
\[
\frac{|I_{\tau_1 \ldots \tau_{k+1}, \tau_{k+1} \ldots \tau_1}|}{|I_{\tau_1 \ldots \tau_{k+1}, \tau_{k+1} \ldots \tau_1}|} < \frac{|I_{\tau_1 \ldots \tau_{k+1}, \tau_{k+1} \ldots \tau_1}|}{|I_{\tau_1 \ldots \tau_{k+1}, \tau_{k+1} \ldots \tau_1}|} e^{A\lambda(k-i)\alpha}.\]

Substituting this expression into Equation 9, replacing \( \lambda \) by the larger of \( \lambda \) and \( \eta \), and absorbing constants, we obtain
\[
\frac{|J_{\tau_1 \ldots \tau_{k+1}, \tau_{k+1} \ldots \tau_1}|}{|J_{\tau_1 \ldots \tau_{k+1}, \tau_{k+1} \ldots \tau_1}|} < \frac{|I_{\tau_1 \ldots \tau_{k+1}, \tau_{k+1} \ldots \tau_1}|}{|I_{\tau_1 \ldots \tau_{k+1}, \tau_{k+1} \ldots \tau_1}|} e^{A\lambda(k-i)\alpha} (l + Ce^{-(k-i)\delta}).\]

Returning to Equation 7, this shows that
\[
q_n(t) < \frac{n}{n} \sum_{k=0}^{n} \sum_{\omega_n \cdots \omega_1 \in A_{\tau_k \ldots \tau_1}} \prod_{i=1}^{k} \left( \frac{|I_{\omega_n \cdots \omega_1}|}{|I_{\omega_n \cdots \omega_1}|} \right)^t \left( e^{A\lambda(k-i)\alpha} (l + Ce^{-(k-i)\delta}) \right)^t.
\]

A geometric series argument shows that
\[
\prod_{i=1}^{k} e^{A\lambda(k-i)\alpha} < e^{\frac{A}{\lambda}},
\]

And a similar argument shows that
\[
\prod_{i=1}^{k} (l + Ce^{-(k-i)\delta}) \leq Dl^k
\]

for some constant \( D > 0 \); if \( C > 1 \) we take \( D = C(1 - l^{-1}e^{-\delta})^{-1} \), and if \( C < 1 \) we take \( D = (1 - Cl^{-1}e^{-\delta})^{-1} \). By absorbing \( e^{\frac{A}{\lambda}} \) into the constant \( D \), these two computations allow us to improve Equation 10 to
\[
q_n(t) < D^t \sum_{k=0}^{n} \sum_{\omega_n \cdots \omega_1 \in A_{\tau_k \ldots \tau_1}} |I_\omega|^t q_{kt}.
\]

Taking logs, dividing by \( n \), and taking \( n \to \infty \) yields
\[
q(t) < \lim_{n \to \infty} \frac{1}{n} \log \sum_{k=0}^{n} \sum_{\omega_n \cdots \omega_1 \in A_{\tau_k \ldots \tau_1}} |I_\omega|^t q_{kt}.
\]

Recall the pressure \( p \) of \( \Lambda \):
\[
p(t) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in \Sigma_p,n} |I_\omega|^t = \lim_{n \to \infty} \frac{1}{n} \log \sum_{k=0}^{n} \sum_{\omega_n \cdots \omega_1 \in A_{\tau_k \ldots \tau_1}} |I_\omega|^t.
\]

Because \( l < 1 \), this is bounded below by the right side of Equation 11 which yields the claim. \( \square \)
We can now give the proof of the Theorem from Section II.

*Proof.* If Λ and Ξ are not $C^{1+\alpha}$ equivalent for some $\alpha > 0$, then by Theorem 3.4 they are not geometrically equivalent. By Theorem 4.2 their pressures are bounded away from each other, which by Theorem 4.1 implies that their Hausdorff dimensions are not equal. □

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