Homogeneous Yang–Baxter deformations as generalized diffeomorphisms

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Abstract
Yang–Baxter (YB) deformations of string sigma model provide deformed target spaces. We propose that homogeneous YB deformations always lead to a certain class of $\beta$-twisted backgrounds and represent the bosonic part of the supergravity fields in terms of the classical $r$-matrix associated with the YB deformation. We then show that various $\beta$-twisted backgrounds can be realized by considering generalized diffeomorphisms in the undeformed background. Our result extends the notable relation between the YB deformations and (non-commuting) TsT transformations. We also discuss more general deformations beyond the YB deformations.

Keywords: superstring theory, double field theory, Yang–Baxter deformation

1. Introduction
A fascinating topic in string theory is the AdS/CFT correspondence [1]. The integrable structure plays an important role behind this gauge/gravity duality, and in order to find the extension away from the well-studied $\text{AdS}_5 \times S^5$ case, integrable deformations of the $\text{AdS}_5 \times S^5$ superstring have been eagerly studied. In particular, by employing the techniques of the Yang–Baxter (YB) deformations [2–5], a framework to study a class of integrable deformations of $\text{AdS}_5 \times S^5$ superstring has been developed in [6–8]. In this paper, we will concentrate on the homogeneous YB deformations, which are based on classical $r$-matrices satisfying the homogeneous classical YB equation (CYBE) [9–25].

Another intriguing topic in string theory is the $T$-duality. Especially, the familiar Abelian $T$-duality [26–28], which is based on an Abelian isometry of a target space, has uncovered
the hidden connection between string theories. On the other hand, its extension, the so-called non-Abelian T-duality [29–38], is still mysterious and needs further investigations. Recently, it was conjectured in [21] and proven in [22] (see also [23]) that a certain class of non-Abelian T-dualities are equivalent to the homogeneous YB deformations. It is then natural to expect that a deeper understanding of the homogeneous YB deformations will facilitate a further development of the non-Abelian T-dualities.

When we focus on the deformation of supergravity backgrounds, there are other useful techniques to obtain the YB-deformed backgrounds. The famous one is the TsT-transformation [39–41], which is a combination of two (Abelian) T-dualities and a linear coordinate change (referred to as ‘shift’). As it has been noticed in [10–15] and clearly shown in [16], all of the homogeneous YB deformations associated with Abelian r-matrices are equivalent to the TsT-transformations. A certain class of non-Abelian YB-deformed backgrounds can also be realized by a generalization of the TsT-transformation [18–20]. In this paper, we develop this type of technique utilizing the framework of the double field theory (DFT) [42–59], which provides a manifestly T-duality-covariant description for the massless sector of string theory.

As it has been observed in [60], some of YB-deformed backgrounds do not satisfy the usual supergravity equations but rather do the generalized supergravity equations (GSE) [61, 62]. The DFT can reproduce both the usual and generalized supergravity from a single action [63, 64], and it provides a unified description of YB-deformed backgrounds. The purpose of this paper is to elucidate that various YB-deformed backgrounds can be realized by performing a certain class of generalized diffeomorphisms in the undeformed background. The generalized diffeomorphisms are the gauge symmetry of DFT, and the resulting deformed backgrounds automatically solve the equations of motion of DFT (as long as the consistency condition, called the strong constraint, is satisfied). This ensures that the deformed background remains to be the string background.

In this paper, we consider YB-deformed backgrounds specified by classical r-matrices, $r = \frac{1}{2} r^{ij} T_i \wedge T_j \{ T_i \}$ (bosonic isometry generators of the undeformed background), satisfying the (homogeneous) CYBE,

$$f_{ijk} r^{ij} r^{kl} + f_{ikl} r^{jk} r^{li} + f_{jlk} r^{ij} r^{lk} = 0,$$

(1.1)

where $r^{ij} = r^{[ij]}$ is constant and skew-symmetric, and $f_{ijk}$ is the structure constant

$$[T_i, T_j] = f_{ijk} T_k.$$

(1.2)

As it has been noticed and proven in [14, 20, 24, 25] for the homogeneous YB deformations of AdS$_5$, classical r-matrices may be interpreted as non-commutative parameters in the open-string description. On the other hand, in terms of the generalized geometry [65] or DFT, the correspondent of the non-commutative parameter is the $\beta$-field [66, 67] defined below. Given the Killing vectors $e_i$ associated with the isometry $T_i$, the associated r-matrix may be identified with the $\beta$-field through the relation:

$$\beta^{mn} \equiv r^{ij} e_i^m e_j^n.$$

(1.3)

Later, we will argue that this identification holds as well for homogeneous YB deformations of Minkowski space (see section 4.3 of [68] as an example that can be uplifted to a ten-dimensional solution), and we anticipate that the identification works for arbitrary undeformed backgrounds.

Our aim here is to find out generalized diffeomorphisms which produce various $\beta$-twists specified by various r-matrices satisfying CYBE. By checking several examples, we provide strong supports for the anticipated identification between $\beta$ twists and classical r-matrices.
As a result, it seems likely that all of the homogeneous YB deformations should be described as $\beta$-twists. However, it should be noted that the space of $\beta$-twists is much larger than that of the homogeneous YB deformations. Therefore, inversely speaking, a certain class of $\beta$-twists can be captured as the YB deformations and hence the YB deformation procedure can be seen as a generation technique of $\beta$-twists.

This $\beta$-twist picture is quite significant because this picture can be regarded as a huge generalization of the well-known TsT transformations and it enables us to capture the global structure of the YB-deformed backgrounds. Unfortunately, at least so far, the global structure of the YB-deformed backgrounds has not been considered at all, though a caution is given in [18]. This global structure issue will be clearly presented in the forthcoming paper [69].

This paper is organized as follows. In section 2, we provide the basics of DFT and present a brief summary of the main result. In section 3, we present some examples to support the relation between the homogeneous YB deformations and generalized diffeomorphisms. Section 4 is devoted to conclusion and outlook. In appendix, we have summarized our conventions for Ramond–Ramond (R–R) fields.

2. Basics of DFT and brief summary

2.1. The ingredients of DFT

In DFT, we consider a gravitational theory on a doubled space with coordinates

$$ (x^M) = (x^m, \tilde{x}_m) \quad (M = 1, \ldots, 2D; m = 1, \ldots, D), $$

(2.1)

where $x^m$ are the usual coordinates while $\tilde{x}_m$ are the dual coordinates.

A generalized diffeomorphism in the doubled space is generated by the generalized Lie derivative,

$$ \hat{\mathcal{L}}_V W^M \equiv V^N \partial_N W^M - (\partial_N V^M - \partial^M V_N) W^N, $$

(2.2)

where the indices $M, N$ are raised or lowered with the $O(D, D)$ metric $\eta_{MN}$ defined as

$$ (\eta_{MN}) \equiv \begin{pmatrix} 0 & \delta^a_m \\ \delta^a_m & 0 \end{pmatrix}. $$

(2.3)

This generalized diffeomorphism is a gauge symmetry of DFT as long as the diffeomorphism parameter $V^M$ satisfies the weak constraint

$$ \partial_N \partial^N V^M = 0, $$

(2.4)

and the strong constraint

$$ \partial_N V^M \partial^N A = 0, $$

(2.5)

where $A$ represents the parameter $V^M$ or the supergravity fields. A finite generalized diffeomorphism is realized by $e^{\hat{\mathcal{L}}_V}$ [70].

The gauge algebra $[\hat{\mathcal{L}}_{V_1}, \hat{\mathcal{L}}_{V_2}] = \hat{\mathcal{L}}_{[V_1, V_2]}$ is governed by the C-bracket,

$$ [V_1, V_2]_C \equiv \frac{1}{2} \left( \hat{\mathcal{L}}_{V_2} V_1 - \hat{\mathcal{L}}_{V_1} V_2 \right). $$

(2.6)

For the usual vectors $V^M_a = (v^m_a, 0)$ $(a = 1, 2)$ satisfying $\frac{\partial}{\partial x^m} V^N_a = 0$, the C-bracket gives rise to the Lie bracket,
$[V_1, V_2]_C = [v_1, v_2]$.  \hfill (2.7)

In DFT, the bosonic fields consist of the generalized metric,

$$ (\mathcal{H}_{MN}) = \begin{pmatrix} G_{mn} - B_{mk} G^{kl} B_{ln} & B_{mn} G^{ln} \\ -G_{mk} B_{ln} & G_{mn} \end{pmatrix}, \hfill (2.8) $$

which integrates the (closed-string) metric and the Kalb–Ramond $B$-field, and the DFT dilaton $d(x)$, which can be parameterized as $e^{-2d} = \sqrt{|G|} e^{-2\Phi}$ ($\Phi$: usual dilaton), and an $O(D,D)$ spinor of the R–R fields $|A\rangle$ (see appendix and [64] for our conventions).

### 2.2. The main result

As discussed later, for a certain class of parameter $V(r)$ specified by an $r$-matrix satisfying CYBE, the finitely transformed background is given by

$$ \mathcal{H}^{(r)}_{MN} \equiv e^{\hat{V}(r)} \mathcal{H}_{MN} = \left(e^{\hat{V}} \mathcal{H} e^{\hat{V}}\right)_{MN}, \hfill (2.9) $$

where $r_{mn} \equiv r^i e_i^m e_j^n$. Suppose the absence of the $B$-field in the original background, then the deformed background in terms of the usual supergravity fields becomes

$$ (G^{(r)} + B^{(r)})_{mn} = (G^{-1} - r)^{-1} \, [G_{mn}], $$

$$ e^{-2\Phi^{(r)}} = e^{-2\Phi} \sqrt{\det[\delta_{mn} - (G \, r \, G)^{r}_{mn}]}, $$

$$ \hat{F}^{(r)} = e^{-B^{(r)}} e^{-r} \hat{F}, \hfill (2.10) $$

where $r \nabla$ acts as $r \nabla \hat{F} \equiv \frac{1}{2} r_{mn} \epsilon_{mnpq} \hat{F}$. The formula (2.10) suggests that the deformed background can be conveniently described by the dual fields, $(\check{g}_{mn}, \check{\beta}_{mn}, \check{\phi})$ [71], defined through

$$ (\check{g}^{-1} + \check{\beta})^{mn} = [(G - B)^{-1}]^{mn}, \quad \sqrt{|\check{g}|} e^{-2\check{\phi}} = e^{-2d}. \hfill (2.11) $$

In terms of the dual fields, the (open-string) metric $\check{g}^{(r)}_{mn}$ and the dual dilaton $\check{\phi}^{(r)}$ are the same as the original ones $G_{mn}$ and $\Phi$ while the $\beta$-field becomes

$$ \beta^{(r)}_{mn} = r_{mn}. \hfill (2.12) $$

In addition, a criterion whether the deformed background satisfies the usual supergravity equations of motion [19], called the unimodularity, is expressed as

$$ F^m \equiv D_m \beta^{(r)mn} = -\frac{1}{2} r_{ij} \left[e_i, e_j\right]^{(r)mn} \equiv 0. \hfill (2.13) $$

Here, the Killing property $D_m e_i^m = 0$ is used and $D_m$ is the covariant derivative associated with the metric $G_{mn} = \check{g}_{mn}$. As we will discuss later, the formula (2.10) is applicable for both the unimodular and non-unimodular cases, but in the latter case, the background follows the GSE with the extra vector $F^m$ given by (2.13) as observed in [24, 25]. It is also interesting to note that, in terms of the dual fields, CYBE (1.1) can also be expressed as

$$ R \equiv [\check{\beta}^{(r)}, \check{\beta}^{(r)}]_8 = 0. \hfill (2.14) $$

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Here, \([,]_S\) denotes the Schouten bracket that is defined for a \(p\)-vector and a \(q\)-vector by (see section 3 of [65])

\[
[a_1 \wedge \cdots \wedge a_p, b_1 \wedge \cdots \wedge b_q]_S \\
\equiv \sum_{ij} (-1)^{ij} [a_i, b_j] \wedge a_1 \wedge \cdots \wedge \hat{a}_i \cdots \wedge a_p \wedge b_1 \wedge \cdots \wedge \hat{b}_j \cdots \wedge b_q,
\]  
(2.15)

where the check \(\hat{a}_i\) denotes the omission of \(a_i\). Namely, for the homogeneous YB-deformed backgrounds, the non-geometric \(R\)-flux [72] has to vanish.

3. YB deformations as generalized diffeomorphisms

In this section, we will discuss that homogeneous YB deformations can be expressed as generalized diffeomorphisms.

3.1. Almost Abelian twists

Now, it is useful to explain the classification of the homogeneous YB deformations. An \(r\)-matrix

\[
r = \frac{1}{2} r^{ij} T_i \wedge T_j
\]

(3.1)
is called Abelian if it consists of a set of generators which commute with each other \([T_i, T_j] = 0\), and otherwise called non-Abelian. Most of homogeneous YB deformations studied in the literature are based on Abelian \(r\)-matrices. The classification of non-Abelian \(r\)-matrices is very complicated in general, and it is mainly classified by the unimodularity condition (2.13) (The Abelian \(r\)-matrices are obviously unimodular). If the rank of an \(r\)-matrix is defined as the number of generators contained in \(r\), non-Abelian unimodular \(r\)-matrices with lower rank are classified well. Obviously, the rank-2 unimodular \(r\)-matrix is Abelian and the rank-4 unimodular \(r\)-matrix for the bosonic isometry of AdS5 has been classified in [19]. We here consider a class of unimodular \(r\)-matrices, called the almost Abelian \(r\)-matrices [20], which covers most of the rank-4 and rank-6 examples studied in [19]. By using constant deformation parameters \(\eta_i (i = 1, \ldots, N)\), it takes the form, \(r = r_N\) with

\[
r_k \equiv \sum_{j=1}^{k} \eta_j T_{2j-1} \wedge T_{2j}, \quad [T_{2k-1}, T_{2k}] = 0,
\]

(3.2)

and obviously satisfies the unimodularity condition (2.13). The almost Abelian condition can be expressed as

\[
[e_{2k-1}, \beta_{(k-1)}]_S = 0, \quad [e_{2k}, \beta_{(k-1)}]_S = 0, \quad (1 \leq k \leq N).
\]

(3.3)

This condition ensures CYBE. As argued in [14, 20], this class of YB deformations can be realized as a sequence of non-commuting TsT-transformations (see [19] for the explicit form in the rank-4 examples), which consists of the usual TsT-transformations and diffeomorphisms that make the Killing vectors as coordinate basis. In principle, for a given almost Abelian \(r\)-matrix, it is possible to find non-commuting TsT-transformations and determine the resulting deformed background. However, it is a tough task in general, and in the following, by considering a specific type of generalized diffeomorphisms, we find the simple formula (2.10) for \(\beta\)-twisted backgrounds associated with the almost Abelian \(r\)-matrix.
3.2. Generalized diffeomorphism

Let us introduce the generalized Killing vectors $E_i$ associated with $T_i$. For the usual isometries, $E_i$ take the form $(E_i^M) = (e_i^m, 0)$ and satisfy
\[ \eta_{MN} E_i^M E_i^N = 0. \] (3.4)

In addition, they are independent of the dual coordinates $\tilde{x}_i$.

For an Abelian $r$-matrix,
\[ r_1 = \eta_1 T_1 \wedge T_2, \] (3.5)
a coordinate system can always be found so that $e_2 = e_2^m \partial_m$ ($e_2^m$; constant) is realized. In such coordinates, we consider
\[ V_1 = \eta_1 e_2^m \tilde{x}_m E_1. \] (3.6)

Thanks to $[e_1, e_2] = 0$ and the Killing property of $e_2$, $V_1$ satisfies the weak constraint $\partial_M \partial^M V_1 = 0$ and the strong constraint $\partial^M V_1^N \partial_M A = 0$ where $A$ denotes $V_1$ or supergravity fields. Then, it is easy to show that
\[ \hat{L}_{V_1} \mathcal{H}_{MN} = (r_1^T \mathcal{H} + \mathcal{H} r_1)_{MN}, \quad \hat{L}_{V_1} d = 0, \] (3.7)
where $r_1^{MN} \equiv 2 \eta_1 E_1^M E_2^N$. The R–R potentials $C$ and the field strengths $F$ are invariant. The finite transformation $e^{\hat{L}_{V_1}}$ gives (2.9) with $r$ replaced with $r_1$.

We then consider a further twist,
\[ r_2 = r_1 + \eta_2 T_3 \wedge T_4. \] (3.8)

From the almost Abelian property, we can again find a coordinate system where $e_4 = e_4^m \partial_m$ ($e_4^m$; constant) is realized, and perform a transformation $e^{\hat{L}_{V_2}}$ with
\[ V_2 = \eta_2 e_4^m \tilde{x}_m E_3. \] (3.9)

Repeating this procedure, we obtain the $\beta$-twisted background associated with the almost Abelian $r$-matrix, $r = r_N$.

In order to demonstrate the relation to the usual TsT-transformation, let us consider an Abelian $r$-matrix and choose a coordinate system where $e_i = \partial_i$ are realized. Then, our diffeomorphism parameter becomes
\[ V = \sum_{i=1}^{N} \eta_i \tilde{x}_{2i} \partial_{2i-1}, \] (3.10)
and it generates a generalized diffeomorphism,
\[ x^M \rightarrow x'^M = e^x x^M, \] (3.11)
or more explicitly,
\[ x'^{2i-1} = x^{2i-1} + \eta_i \tilde{x}_{2i}. \] (3.12)

This is nothing but the TsT-transformation in the DFT language.

As a non-trivial example, let us consider a deformation of $\text{AdS}_5 \times S^5$ background with the Poincaré metric,
\[ ds^2 = \frac{dz^2 - 2 dx^+ dx^- + (dx^2)^2 + (dx^3)^2}{z^2} + ds^2_{S^5}. \] (3.13)
We denote the translation, Lorentz, and dilatation generators by \( P_\mu, M_{\mu\nu}, \) and \( D \) \((\mu, \nu = +, -, 2, 3)\), respectively, and consider a rank-4 \( r \)-matrix, \( r = r_2 \), with
\[
T_1 = M_{4+2}, \quad T_2 = P_3, \quad T_3 = D - M_{3-}, \quad T_4 = P_+. \tag{3.14}
\]
These satisfy \([T_3, T_1] = T_1\) and \([T_3, T_2] = -T_2\), and constitute an almost Abelian \( r \)-matrix. This case, we consider a sequence of finite transformations \( e^{\tilde{E}_{v_1}}e^{\tilde{E}_{v_2}} \) with
\[
V_1 \equiv \eta_1 \tilde{\eta}_1 \tilde{M}_{4+2}, \quad V_2 \equiv \eta_2 \tilde{\eta}_2 (\tilde{D} - \tilde{M}_{3-}), \tag{3.15}
\]
where hatted quantities like \( \tilde{M}_{4+2} \) denote the generalized Killing vectors associated with the unhatted generators. These finite transformations produce the \( \beta \)-field,
\[
\beta_{(r_2)} = \eta_1 (x^2 \partial_+ + x^- \partial_2) \land \partial_1 + \eta_2 (z \partial_2 + 2x^- \partial_- + x^2 \partial_2 + x^3 \partial_3) \land \partial_+, \tag{3.16}
\]
and the deformed background (2.10) is indeed a solution of type IIB supergravity. If one prefers to combine the transformations as a single one, \( e^{\tilde{E}_{v_1}}e^{\tilde{E}_{v_2}} = e^{\tilde{E}_{v_3}} \), the Baker–Campbell–Hausdorff formula [70] would be useful.

3.3. A more general class
Let us consider a wider class of unimodular \( r \)-matrices, \( r = r_N \) with (3.2) satisfying
\[
[e_{2k-1}, [\beta_{(\alpha, -)}]_S] = 0, \quad e_{2k} \land [e_{2k}, \beta_{(\alpha, -)}]_S = 0, \tag{3.17}
\]
for \( 1 \leq k \leq N \), which covers all of the rank-4 unimodular \( r \)-matrices of ADs [19], including the example where any TsT-like transformation has not been found. We explain a subtle issue in this class by considering the rank-4 example \((N = 2)\) where \([e_1, \beta_{(\alpha, -)}]_S = 0\) but \([e_2, \beta_{(\alpha, -)}]_S \neq 0\). Similar to the almost Abelian case, in coordinates where \( e_2 = \partial_2 \), we first consider a finite transformation
\[
H^{(1)}_{MN} = e^{\tilde{E}_{v_1}}H_{MN} \quad \text{with} \quad V_1 = \eta_1 \tilde{\eta}_1 \tilde{E}_1. \tag{3.18}
\]
Then, in coordinates where \( e_4 = \partial_4 \), we perform the second transformation
\[
H^{(2)}_{MN} = e^{\tilde{E}_{v_2}}H^{(1)}_{MN} \quad \text{with} \quad V_2 = \eta_2 \tilde{\eta}_2 \tilde{E}_3. \tag{3.19}
\]
According to \([e_4, \beta_{(\alpha, -)}]_S \neq 0\), \( H^{(1)}_{MN} \) depends on the \( x^4 \) coordinate and hence the second transformation breaks the strong constraint;
\[
\partial_4 V_2 \partial^4 H^{(1)}_{MN} \neq 0. \tag{3.20}
\]
In fact, the formula (2.9) itself does not require the strong constraint, and indeed \( e^{\tilde{E}_{v_1}}e^{\tilde{E}_{v_2}} \) provides the desired background. The problem is that a generic strong-constraint-violating generalized diffeomorphism is not a gauge symmetry of DFT. Therefore, the deformed background may not be a solution of DFT. Interestingly, for all examples in the list presented in [19] (which cover all inequivalent rank-4 deformations of ADs), one can check that the equations of motion transform covariantly under the diffeomorphisms. At the present stage, we are not aware of the clear reason why such diffeomorphisms are allowed. A more general formulation of DFT [73–75], where the strong constraint is rather relaxed, may help us to answer the question.
3.4. Non-unimodular cases

The last type of homogeneous YB deformations is the non-unimodular one. For simplicity, we will here focus upon the rank-2 Jordanian \( r \)-matrix,

\[
r = \eta T_1 \wedge T_2 \quad \text{with} \quad [T_1, T_2] = T_1.
\]

(3.21)

In this case, the formula (2.13) indicates that the unimodularity is broken:

\[
\text{Im} = -\eta e_1^m \neq 0.
\]

(3.22)

For some non-unimodular cases, TsT-like transformations have been employed in [18] to reproduce the YB-deformed backgrounds on a case-by-case basis. Instead, we will here stick to our general strategy. In the present case, \([e_1, e_2] \neq 0\) introduces the \( x^2 \) dependence into \( e_1 \) and the parameter \( V = \tilde{\eta} x_2 E_1 \) breaks even the weak constraint:

\[
\partial_N \partial^N V^M \neq 0.
\]

(3.23)

However, the formula (2.9) still works due to the generalized Killing property of \( E_1 \) and the Jordanian property, \([e_1, e_2] = e_1\). A subtle issue is again the covariance of the equations of motion, and, as we show below, they are not transformed covariantly in the non-unimodular case.

In DFT, the generalized connection \( \Gamma_{MK}^{MN} \) is supposed to transform as

\[
\delta_V \Gamma_{MNK} = \hat{\zeta} V \Gamma_{MNK} - 2 \partial_M \partial_N V_K.
\]

(3.24)

At the same time, it is defined to satisfy the condition

\[
\nabla_M d \equiv \partial_M d + \frac{1}{2} \Gamma_K^{KMN} = 0.
\]

(3.25)

By the consistency,

\[
\delta_V \nabla_M d = \hat{\zeta} V \nabla_M d + \partial^K V_M \partial_K d - \frac{1}{2} \partial_N \partial^N V_M,
\]

(3.26)

must vanish. It indeed vanishes if the strong constraint is satisfied. In the present case, the first two terms on the right-hand side vanish but the last term does not vanish because \( V^M \) breaks the weak constraint. A short calculation shows that

\[
\delta_V \nabla^M d = \eta [E_1, E_2]^M \equiv -X^M.
\]

(3.27)

From the Jordanian property, the finite transformation gives rise to

\[
e^{\delta_V} \nabla^M d = -X^M.
\]

(3.28)

Namely, after performing the deformation, \( \nabla^M d \) does not vanish but becomes (minus) the null generalized Killing vector,

\[
X^M = -\eta E_1^M = (I^m, 0).
\]

(3.29)

This is the situation of the modified DFT (mDFT) [63], where the generalized connection is deformed by a null generalized Killing vector \( X^M \). In the R–R sector, we suppose that the potential \( |A⟩ \) and the field strength \( |F⟩ \) transform covariantly (see [64] for our conventions):

\[
|A^{(\gamma)}⟩ = e^{\hat{\zeta}} |A⟩ \quad \text{and} \quad |F^{(\gamma)}⟩ = e^{\hat{\zeta}} |F⟩.
\]

(3.30)

However, the relation \( |F⟩ = \hat{\partial} |A⟩ \) is deformed under the weak-constraint-violating generalized diffeomorphism as
which is again the same relation as the one known in mDFT [64]. The Bianchi identities (or the equations of motion) for the $R - R$ fields are also deformed in a similar manner. It is tough to evaluate the deviation of the generalized Ricci tensors, $(\delta V - \hat{\mathcal{L}} V) S_{MN}$ and $(\delta V - \hat{\mathcal{L}} V) S$, under the weak-constraint-violating generalized diffeomorphisms. In fact, they do not vanish. For all of the rank-2 examples listed in [18], we have checked that the following relations are satisfied:

$$\left(e^r S e^r\right)_{MN} = \hat{S}^{(r)}_{MN}, \quad S = \hat{S}^{(r)}.$$  (3.32)

Here, $\hat{S}^{(r)}_{MN}$ and $\hat{S}^{(r)}$ are modified generalized Ricci tensors [63] in the deformed background. Then, since the stress-energy tensor obviously transforms covariantly,

$$\left(e^r \mathcal{E} e^r\right)_{MN} = \mathcal{E}^{(r)}_{MN},$$  (3.33)

the deformed background is a solution of mDFT. In fact, all solutions of mDFT can be mapped to solutions of DFT via a field redefinition [64], and the deformed background is still a solution of DFT.

To clearly see that the deformed background is indeed a solution of DFT, let us examine another route. For the example of $r = \eta P_\perp \eta D$ [18], instead of the weak-constraint-violating generalized diffeomorphism, we can find another generalized coordinate transformation which does not break the weak/strong constraint,

$$z' = (1 + \eta \tilde{x}_-) z, \quad x'^+ = (1 + \eta \tilde{x}_-) x^+, \quad \rho' = (1 + \eta \tilde{x}_-) \rho, \quad \tilde{x}_- = \eta^{-1} \log(1 + \eta \tilde{x}_-).$$  (3.34)

Then, by employing Hohm and Zwiebach’s finite transformation law [70], this transformation generates the same deformed background from the original AdS$_5 \times$ S$^5$. In this route, instead of $X^M$, a linear $\tilde{x}_-$ dependence is introduced into the DFT-dilaton. This result is compatible with the (m)DFT picture discussed in [64].

### 3.5. Twists by $\gamma$-fields

So far, we have discussed only the homogeneous YB deformations, which always provide $\beta$-twists specified by the associated classical $r$-matrices. In the $U$-duality-covariant extension of DFT, called the exceptional field theory (EFT) [76–88], we can consider a more general twisting via the $\gamma$-fields [89–96]. They are $p$-vectors dual to the $R$–$R$ $p$-form potentials, and in particular, the bi-vector $\gamma_m^n$ in type IIB theory is the $S$-dual of the $\beta_m^n$ (see [96] for the duality rules for these fields).

In [15], a YB-deformed background associated with $r = P_+ \wedge (D - M_{+-})$ has been determined including the $R$–$R$ fields, and indeed it is a solution of GSE [18]. Interestingly, a solution of standard supergravity which has the same NS–NS fields but different $R$–$R$ fields has been found in [9]. In fact, the former is twisted by a $\beta$-field while the latter is twisted by a $\gamma$-field. At the supergravity level, the latter can be obtained by a combination of the TsT-transformations and the $S$-duality [97] (where the TsT-transformations generate a $\beta$-field and the last step converts the $\beta$-field to the $\gamma$-field). This deformation can also be realized as a generalized diffeomorphism in EFT. However, according to our current understanding, the YB deformation can produce only the former $\beta$-twisted background, and it is important to invent the extension, for example, by revealing YB deformations of the $S$-duality covariant $(\rho, q)$-string action [98].
4. Conclusion and outlook

In this paper, we have argued that homogeneous YB deformations associated with classical $r$-matrices $r = \frac{1}{2} \rho^i T_i \wedge T_i$ can be regarded as a technique to provide $\beta$-twists with
\[ \beta^{mn} = r^i \epsilon^m_i \epsilon^n_j. \] (4.1)

In this picture, the CYBE is equivalent to the absence of the $R$-flux. For undeformed backgrounds without the $B$-field, we have provided a simple formula (2.10) for a general $\beta$-twisted background. Then, we have found the generalized diffeomorphism parameters which produce various $\beta$-twisted backgrounds, including all rank-4 deformations classified in [19].

The advantage of our approach is that it (i) includes all the non-commuting TsT-transformations as special cases and (ii) can be applied to arbitrary undeformed backgrounds. In some examples, the associated diffeomorphism parameters break the strong or weak constraint, but the resulting deformed backgrounds remain to be solutions of DFT. Interestingly, for a general rank-2 Jordanian $r$-matrix, the generalized diffeomorphism has produced a non-unimodular deformation, and the extra vector $F^i$ appeared in the process of the diffeomorphism. We have also shown the novel relation,
\[ I^m = D_{n}[\beta^{mn}(r)], \] (4.2)

which is consistent with the result in [24, 25].

To date, finite diffeomorphisms in DFT/EFT have not understood clearly [99–111]. Hence we need to do more detailed analysis and clarify why the strong/weak-constraint-breaking diffeomorphism is allowed and how the global structure is deformed. It is also desirable to extend YB deformations and non-Abelian $T$-dualities. The duality covariant formulation, such as the double sigma model and its extensions [112–121], would be helpful along this direction.

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Appendix. Conventions of Ramond–Ramond fields

In our conventions (see [64] for the detail), the R–R potentials $\hat{C}_p$ and the field strengths $\hat{F}_{p+1}$ are related through the formulas
\[ \hat{F}_{p+1} = d\hat{C}_p + H_3 \wedge \hat{C}_{p-2}, \] (A.1)
and the Bianchi identity is given by
\[ d\hat{F}_p + H_3 \wedge \hat{F}_{p-2} = 0. \] (A.2)

We also introduce other types of R–R potentials $(A, \hat{C})$ and the field strengths $(F, \hat{F})$ via the following relations:
\[ e^{B_i} \wedge \hat{C} = A = e^{-\beta \vee} \hat{C}, \quad e^{B_i} \wedge \hat{F} = F = e^{-\beta \vee} \hat{F}. \] (A.3)
References

[1] Maldacena J M 1999 The large $N$ limit of superconformal field theories and supergravity *Int. J. Theor. Phys.* **38** 1113

Maldacena J M 1998 The large $N$ limit of superconformal field theories and supergravity *Adv. Theor. Math. Phys.* **2** 231

[2] Klimcik C 2002 Yang–Baxter sigma models and dS/AdS $T$ duality *J. High Energy Phys.* JHEP12(2003)051

[3] Klimcik C 2009 On integrability of the Yang–Baxter sigma-model *J. Math. Phys.* **50** 043508

[4] Delduc F, Magro M and Vicedo B 2013 On classical $q$-deformations of integrable sigma-models *J. High Energy Phys.* JHEP11(2013)192

[5] Matsumoto T and Yoshida K 2015 Yang–Baxter sigma models based on the CYBE *Nucl. Phys.* B **892** 287

[6] Delduc F, Magro M and Vicedo B 2014 An integrable deformation of the $AdS_5 \times S^5$ superstring action *Phys. Rev. Lett.* **112** 5

[7] Delduc F, Magro M and Vicedo B 2014 Derivation of the action and symmetries of the $q$-deformed $AdS_5 \times S^5$ superstring *J. High Energy Phys.* JHEP10(2014)132

[8] Kawaguchi I, Matsumoto T and Yoshida K 2014 Jordanian deformations of the $AdS_5 \times S^5$ superstring *J. High Energy Phys.* JHEP04(2014)153

[9] Kawaguchi I, Matsumoto T and Yoshida K 2014 A Jordanian deformation of AdS space in type IIB supergravity *J. High Energy Phys.* JHEP06(2014)146

[10] Matsumoto T and Yoshida K 2014 Lunin-Maldacena backgrounds from the classical Yang–Baxter equation—towards the gravity/CYBE correspondence *J. High Energy Phys.* JHEP06(2014)135

[11] Matsumoto T and Yoshida K 2014 Integrability of classical strings dual for noncommutative gauge theories *J. High Energy Phys.* JHEP06(2014)163

[12] Matsumoto T and Yoshida K 2015 Schrödinger geometries arising from Yang–Baxter deformations *J. High Energy Phys.* JHEP04(2015)180

[13] van Tongeren S J 2015 On classical Yang–Baxter based deformations of the $AdS_5 \times S^5$ superstring *J. High Energy Phys.* JHEP06(2015)048

[14] van Tongeren S J 2016 Yang–Baxter deformations, AdS/CFT, and twist-noncommutative gauge theory *Nucl. Phys.* B **904** 148

[15] Kyono H and Yoshida K 2016 Supercoset construction of Yang–Baxter deformed $AdS_5 \times S^5$ backgrounds *Prog. Theor. Exp. Phys.* **2016** 083B03

[16] Osten D and van Tongeren S J 2017 Abelian Yang–Baxter deformations and TsT transformations *Nucl. Phys.* B **915** 184

[17] Hoare B and van Tongeren S J 2016 On jordanian deformations of $AdS_5$ and supergravity *J. Phys. A: Math. Theor.* **49** 434006

[18] Orlando D, Reffert S, Sakamoto J and Yoshida K 2016 Generalized type IIB supergravity equations and non-Abelian classical $r$-matrices *J. Phys. A: Math. Theor.* **49** 444503

[19] Borsato R and Wulff L 2016 Target space supergeometry of $\eta$ and $\lambda$-deformed strings *J. High Energy Phys.* JHEP10(2016)045

[20] van Tongeren S J 2017 Almost abelian twists and AdS/CFT *Phys. Lett.* B **765** 344

[21] Hoare B and Tseytlin A A 2016 Homogeneous Yang–Baxter deformations as non-abelian duals of the $AdS_5$ sigma-model *J. Phys. A: Math. Theor.* **49** 494001

[22] Borsato R and Wulff L 2016 Integrable deformations of $T$-dual $\sigma$ models *Phys. Rev. Lett.* **117** 251602

[23] Hoare B and Thompson D C 2017 Marginal and non-commutative deformations via non-abelian $T$-duality *J. High Energy Phys.* JHEP02(2017)059

[24] Araujo T, Bakhmatov I, Colgáin E Ó, Sakamoto J, Sheikh-Jabbari M M and Yoshida K 2017 Yang–Baxter $\sigma$-models, conformal twists, and noncommutative Yang–Mills theory *Phys. Rev. D* **95** 105006

[25] Araujo T, Bakhmatov I, Colgáin E Ó, Sakamoto J, Sheikh-Jabbari M M and Yoshida K 2017 Conformal twists, Yang–Baxter $\sigma$-models & holographic noncommutativity (arXiv:1705.02063 [hep-th])

[26] Buscher T H 1987 A symmetry of the string background field equations *Phys. Lett.* B **194** 59

[27] Buscher T H 1988 Path integral derivation of quantum duality in nonlinear sigma models *Phys. Lett.* B **201** 466
[60] Arutyunov G, Borisov R and Frolov S 2015 Puzzles of $\eta$-deformed $AdS_5 \times S^5$ J. High Energy Phys. JHEP12(2015)049
[61] Arutyunov G, Frolov S, Hoare B, Roiban R and Tseytlin A A 2016 Scale invariance of the $\eta$-deformed $AdS_5 \times S^5$ superstring, T-duality and modified type II equations Nucl. Phys. B 903 262
[62] Tseytlin A A and Wulff L 2016 Kappa-symmetry of superstring sigma model and generalized 10d supergravity equations J. High Energy Phys. JHEP06(2016)174
[63] Sakatani Y, Uehara S and Yoshida K 2017 Generalized gravity from modified DFT J. High Energy Phys. JHEP04(2017)123
[64] Sakamoto J, Sakatani Y and Yoshida K 2017 Weyl invariance for generalized supergravity backgrounds from the doubled formalism Prog. Theor. Phys. Exp. Phys. 2017 052A01
[65] Gualtieri M 2004 Generalized complex geometry (arXiv:math/0401221 [math-dg])
[66] Duff M J 1990 Duality rotations in string theory Nucl. Phys. B 335 610
[67] Duff M J and Lu J X 1990 Duality rotations in membrane theory Nucl. Phys. B 347 394
[68] Blumenhagen R, Hassler F and Lüst D 2015 Generalized metric formulation of double field theory J. High Energy Phys. JHEP02(2015)001
[69] Andriot D, Larfors M, Lust D and Patalong P 2011 A ten-dimensional action for non-geometric backgrounds and the first order string sigma model (arXiv:0906.2891 [hep-th])
[70] Berman D S and Perry M J 2011 Generalized geometry and M theory J. High Energy Phys. JHEP06(2011)074
[71] Berman D S, Godazgar H, Perry M J and West P 2012 Duality invariant actions and generalised geometry J. High Energy Phys. JHEP02(2012)108
[72] Berman D S, Godazgar H, Perry M J and West P 2012 Duality invariant actions and generalised geometry J. High Energy Phys. JHEP02(2012)108
[73] Hillmann C 2009 Generalized E(7(7)) coset dynamics and $D = 11$ supergravity J. High Energy Phys. JHEP03(2009)135
[74] Berman D S and Perry M J 2011 Generalized geometry and M theory J. High Energy Phys. JHEP06(2011)074
[75] Berman D S, Godazgar H, Godazgar M and Perry M J 2012 The Local symmetries of M-theory and their formulation in generalised geometry J. High Energy Phys. JHEP01(2012)012
[76] Berman D S, Godazgar H, Perry M J and West P 2012 Duality invariant actions and generalised geometry J. High Energy Phys. JHEP02(2012)108
[77] Berman D S, Godazgar H and Thompson D C 2013 The gauge structure of generalised diffeomorphisms J. High Energy Phys. JHEP01(2013)064
[78] Hohm O and Samtleben H 2013 Exceptional form of $D = 11$ supergravity Phys. Rev. Lett. 111 231601
[79] Hohm O and Samtleben H 2014 Exceptional field theory I: $E_{6(6)}$ covariant form of M-theory and type IIB Phys. Rev. D 89 066016
[80] Hohm O and Samtleben H 2014 Exceptional field theory. II $E_{7(7)}$ Phys. Rev. D 89 066017
[81] Aldazabal G, Grana M, Marqués D and Rosabal J A 2014 The gauge structure of exceptional field theories and the tensor hierarchy J. High Energy Phys. JHEP04(2014)049
[82] Aldazabal G, Grana M, Marqués D and Rosabal J A 2014 The gauge structure of exceptional field theories and the tensor hierarchy J. High Energy Phys. JHEP04(2014)049
[83] Aldazabal G and Samtleben H 2014 Exceptional field theory III. $E_{8(8)}$ Phys. Rev. D 90 066002
[84] Aldazabal G, Andree A, Camara P G and Grana M 2010 U-dual fluxes and generalized geometry J. High Energy Phys. JHEP11(2010)083
[85] Chatzistavrakidis A, Gautason F F, Moutsopoulos G and Zagermann M 2014 Effective actions of nongeometric five-branes Phys. Rev. D 89 066004
[86] Blair C D A, Malek E and Park J H 2014 M-theory and type IIB from a duality manifest action J. High Energy Phys. JHEP01(2014)172
[92] Andriot D and Betz A 2015 Supersymmetry with non-geometric fluxes, or a $\beta$-twist in generalized geometry and dirac operator J. High Energy Phys. JHEP04(2015)006
[93] Blair C D A and Malek E 2015 Geometry and fluxes of SL(5) exceptional field theory J. High Energy Phys. JHEP03(2015)144
[94] Sakatani Y 2015 Exotic branes and non-geometric fluxes J. High Energy Phys. JHEP03(2015)135
[95] Lee K, Rey S J and Sakatani Y 2017 Effective action for non-geometric fluxes from duality covariant actions J. High Energy Phys. JHEP07(2017)075
[96] Sakatani Y and Uehara S 2017 Connecting M-theory and type IIB parameterizations in exceptional field theory Prog. Theor. Exp. Phys. 2017 043B05
[97] Matsumoto T and Yoshida K 2015 Yang–Baxter deformations and string dualities J. High Energy Phys. JHEP03(2015)137
[98] Schwarz J H 1995 An SL(2, Z) multiplet of type IIB superstrings Phys. Lett. B 360 13
[99] Park J H 2013 Comments on double field theory and diffeomorphisms J. High Energy Phys. JHEP06(2013)098
[100] Hohm O, Lüst D and Zwiebach B 2013 The spacetime of double field theory: review, remarks, and outlook Fortsch. Phys. 61 926
[101] Berman D S, Cederwall M and Perry M J 2014 Global aspects of double geometry J. High Energy Phys. JHEP09(2014)066
[102] Cederwall M 2014 The geometry behind double geometry J. High Energy Phys. JHEP09(2014)070
[103] Papadopoulos G 2014 Seeking the balance: patching double and exceptional field theories J. High Energy Phys. JHEP10(2014)089
[104] Hull C M 2015 Finite gauge transformations and geometry in double field theory J. High Energy Phys. JHEP04(2015)109
[105] Cederwall M 2014 T-duality and non-geometric solutions from double geometry Fortsch. Phys. 62 942
[106] Naseer U 2015 A note on large gauge transformations in double field theory J. High Energy Phys. JHEP06(2015)002
[107] Rey S J and Sakatani Y 2015 Finite transformations in doubled and exceptional space (arXiv:1510.06735 [hep-th])
[108] Chaemjumrus N and Hull C M 2016 Finite gauge transformations and geometry in extended field theory Phys. Rev. D 93 086007
[109] du Bosque P, Hassler F, Lüst D and Malek E 2017 A geometric formulation of exceptional field theory J. High Energy Phys. JHEP03(2017)004
[110] Howe P S and Papadopoulos G 2017 Patching DFT, T-duality and gerbes J. High Energy Phys. JHEP04(2017)074
[111] Hassler F 2016 The topology of double field theory (arXiv:1611.07978 [hep-th])
[112] Tseytlin A A 1990 Duality symmetric formulation of string world sheet dynamics Phys. Lett. B 242 163
[113] Tseytlin A A 1991 Duality symmetric closed string theory and interacting chiral scalars Nucl. Phys. B 350 395
[114] Hull C M 2005 A Geometry for non-geometric string backgrounds J. High Energy Phys. JHEP10(2005)065
[115] Hackett-Jones E and Moutsopoulos G 2006 Quantum mechanics of the doubled torus J. High Energy Phys. JHEP10(2006)062
[116] Hull C M 2007 Doubled geometry and T-Folds J. High Energy Phys. JHEP07(2007)080
[117] Copland N B 2012 A double sigma model for double field theory J. High Energy Phys. JHEP04(2012)044
[118] Lee K and Park J H 2014 Covariant action for a string in ‘doubled yet gauged’ spacetime Nucl. Phys. B 880 134
[119] Blair C D A, Malek E and Routh A J 2014 An $O(D,D)$ invariant Hamiltonian action for the superstring Class. Quantum Grav. 31 205011
[120] Sakatani Y and Uehara S 2016 Branes in extended spacetime: brane worldvolume theory based on duality symmetry Phys. Rev. Lett. 117 191601
[121] Park J H 2016 Green-Schwarz superstring on doubled-yet-gauged spacetime J. High Energy Phys. JHEP11(2016)005