TOTALLY ASYMMETRIC ZERO-RANGE PROCESS
IN THE RAREFACTION FAN

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ABSTRACT. We consider the one-dimensional totally asymmetric zero-range process starting from a step decreasing profile leading in the hydrodynamic limit to the rarefaction fan of the associate hydrodynamic equation. Under that initial condition, we show that the weighted sum of joint probabilities for second class particles sharing the same site, is convergent and we compute its limit. We derive the Law of Large Numbers for the position of a second class particle initially at the origin under the initial state in which all positive sites are empty and all negative sites are occupied and also for a slight perturbation of the invariant state.

1. INTRODUCTION

Interacting particle systems were introduced in the early seventies by Spitzer [16] and in the hydrodynamic limit theory one obtains the partial differential equations governing the evolution of the conserved thermodynamical quantities of the system. Usually, one can take advantage of the known results on the hydrodynamic equation to obtain knowledge on the asymptotic behavior of the underlying particle system. Nevertheless, the study of the particle system has given answers on the qualitative behavior of the solutions of the corresponding hydrodynamic equation. In this paper we continue the work initiated in [5] - namely we study the asymptotic behavior of second class particles added to a particle system and relate them with the characteristics of the corresponding hydrodynamic equation. We are interested in the rarefaction fan setting and we prove that among the infinite characteristics emanating from the position of the second class particle, this particle chooses at random one of them. The randomness is given in terms of the weak solution of the hydrodynamic equation through some sort of renormalization function.

In this paper we consider a well know interacting particle system: the one-dimensional totally asymmetric zero-range process (tazrp) evolving on $\mathbb{Z}$. The dynamics of this process is the following: at each site there is a mean one exponential time clock, after which a particle at $x$ jumps to $x + 1$ at rate one independently from the number of particles at the destination site. After that jump the clock restarts. For this process, the equilibrium measures that are translation invariant are Geometric product measures of parameter $\frac{1}{1+r}$ with $r \in [0, \infty)$, that we denote by $\mu_r$. Since we are restricted to the one-dimensional setting, we can couple this zero-range process with the totally asymmetric simple exclusion process (tasep) that was analyzed in [5]. The dynamics of the latter process is as follows: after a mean one exponential time, a particle at a site $x$ jumps to $x + 1$ at rate one, only if the destination site is empty, otherwise it does not move. The invariant measures for this process are Bernoulli product measures of parameter $\alpha \in [0, 1]$, that we denote by $\nu_\alpha$. By coupling both tazrp and tasep, we will be able to confirm the results that we prove independently for the tazrp, from the results in [5].

Since the work in [14] it is known that, the empirical measures associated to these processes, converge to a deterministic measure whose density is the unique entropy solution of a hyperbolic conservation law with concave flux. The hydrodynamic limit for these processes was set in two different ways: for both processes in [14] using the Entropy method, for a
general set of initial measures associated to a profile \( \rho_0 \) and for the tasep in [8] using the Relative Entropy method, for a more restricted set of initial measures. We notice that it is not difficult to show the hydrodynamic limit for the tazrp invoking the same arguments as in [8] to derive its hydrodynamic limit. With last result in hands, we can also cover the case of initial measures of slowly varying parameter associated to flat profiles as well as the case of step decreasing profiles.

The main goal of this paper is to analyze the asymptotic behavior of the tazrp starting from an initial configuration with particles with different degree of class. We mainly consider first or second class particles, but we define the interaction dynamics between particles for all \( m \) degree of class with \( m = 1, \ldots, \infty \). Initially each particle is labeled as a \( m \)-th class particle and its movement depends on its label, in such a way that a \( m \)-th class particle sees particles with degree of class less (greater) than \( m \) as particles (holes). We can think of a hole as a particle with \( \infty \) degree of class. Now, we describe the interaction dynamics between particles of different degree of class sharing the same site. Suppose that at the site \( x \) there are particles of degree of class \( 1, \ldots, \infty \). If the clock at \( x \) rings, the first class particle has priority to jump and moves to the right site at rate one, while the remaining particles keep the same position. So, if at a site, there is no particle with degree of class less than \( m \), the \( m \)-th class particle jumps to the right at rate one. The higher the degree of class of a particle, the lower is its priority to jump. On the other hand, for the tasep the interaction dynamics between particles with different degree of class is completely different. In the latter process, a \( m \)-th class particle can move forward to the right neighboring site if the destination site is occupied with a particle with degree of class greater than \( m \), and in this case, they interchange positions. Nevertheless, differently from the tazrp, a \( m \)-th class particle can jump backwards if at the left neighboring site there is a particle with degree of class less than \( m \), that attempts to jump to the right. So, in the tasep a \( m \)-th class particle interchanges with a particle with degree of class less than \( m \) and moves to the left with rate one, or interchanges with a particle with degree of class greater than \( m \) and moves to the right at rate one.

Now, we recall some results that have been proved for the tasep, which we will generalize for the tazrp in a similar setting. Fix \( \rho, \lambda \in [0, 1] \) and let \( \nu_{\rho, \lambda} = \nu_{\rho} \delta_{(-\infty, 0]} + \nu_{\lambda} \delta_{(0, \infty)} \). In [5] it was shown, that starting the tasep from \( \nu_{\rho, \lambda} \) with \( \rho > \lambda \) and adding a second class particle at the origin, it holds a Law of Large Numbers (L.L.N.) for the position of this particle and the limit \( \mathcal{U} \) is uniformly distributed on \([1 - 2\rho, 1 - 2\lambda]\). Later, this result was extended to more general partially asymmetric transition rates of exclusion type in [4]. The speed of the second class particle in the partial asymmetric simple exclusion was also studied in [1] by analyzing the invariant measures of the multi-class process. It was proved in [13], that this L.L.N. holds in the strong sense; and for \( \rho = 1 \) and \( \lambda = 0 \) in [6] this result was also derived by mapping the tasep to a last passage percolation model. As in [6] the second-class particle can be viewed as an interface between two random growing clusters, see [4] for details.

For the tasep, the relation between the distribution function \( F_\mathcal{U}(u) \) and the entropy solution \( \rho(t, u) \) of the associate hydrodynamic equation under initial condition \( \rho_{0, \lambda}^\rho(u) := \rho_{1 \{u \leq 0\}} + \lambda 1_{\{u > 0\}} \) is given by \( F_\mathcal{U}(u) = \frac{\rho - \rho_{1 \{u \leq 0\}}}{\rho - \lambda} \). The purpose of this work is to see how far this picture can go in the tazrp.

Let \( \mu_{\rho, \lambda} = \mu_{\rho} \delta_{(-\infty, 0]} + \mu_{\lambda} \delta_{(0, \infty)} \), \( \rho, \lambda \in [0, \infty] \). Distribute initially the tazrp according to \( \mu_{\infty, 0} \). By adding a second class particle at the origin, a L.L.N. can be derived and the limit \( X \) is distributed according to \( F_X(u) = 1 - \phi(\rho(1, u)) \), where \( \phi(\rho) = \frac{\rho - 1}{\rho - \lambda} \) is the mean of the instantaneous current (see section 2.3) and \( \rho(t, u) \) is the unique entropy solution of the hydrodynamic equation (2.1) with initial condition \( \rho_{0, \lambda}^\rho \). Be aware that, in this case the entropy solution does not define a density of a probability distribution function, so that, the relation between \( F_X \) and \( \rho \) is performed by means of a proper function \( \phi \). The idea of the proof of last result, consists in defining the current through a time-dependent bond and applying a coupling argument introduced in [3]. This argument seems robust enough to obtain the L.L.N.
for second class particles for attractive particle systems with hyperbolic hydrodynamic equation with concave flux as in (2.2). On the other hand, considering the tazrp starting from $\mu_{p,\lambda}$ with $\rho > \lambda$ and adding infinite second class particles at the origin at the initial time, we are able to show that the sum of the joint probability of the speeds of these particles, converge and we compute its limit. For the tasep, the joint distribution of the speeds of second class particles was analyzed in [1].

As in the case of the tasep, we can also recover a L.L.N. for the position of the second class particle in the tazrp, in which the limit distribution function is given as $F_2(u) = \frac{\rho u^{\lambda}}{\rho + \lambda}$, where $\rho(t, u)$ is the unique entropy solution of (2.1) with initial condition $\rho_0^{\rho,\lambda}$. For that purpose, we consider the tazrp under a slight perturbation of the invariant state $\mu_\lambda$: first we distribute particles according to $\mu_\lambda$ and then we add a first class particle to each negative site and put a second class particle at the origin.

Finally, we notice that by coupling the tazrp with the tasep and invoking Theorem 2.3 of [4], we show that starting the tazrp from a configuration with a second class particle at the origin, a third class particle at the site 1, all negative sites occupied by infinite first class particles and all positive sites empty, the probability that the second class particle overtakes the third class particle is $2/3$. It would be an interesting problem to derive this result for the tazrp without going through the coupling with the tasep.

Here follows an outline of this paper. In the second section, we introduce the totally asymmetric zero-range process, we state its hydrodynamic limit, we compute the characteristics of hyperbolic conservation laws with concave flux and state the main results. On the third section, we compute the limit of the sum of crossing probabilities for the tazrp starting from $\mu_{p,\lambda}$ ($\rho > \lambda$) and with infinite second class particles at the origin. On the fourth section, starting the tazrp from $\mu_{\infty,0}$, we prove the L.L.N. for the position of the second class particle and for the current of first class particles that cross over the second class particle. On the fifth section, starting the tazrp from a perturbation of the invariant state $\mu_p$ (which is still associated to $\rho_0^{\rho,\lambda}$), by adding a second class particle at the origin, we derive its L.L.N. The sixth section, is devoted to the reproof of the results of the fourth section, under the coupling with the tasep and invoking the results in [5]. Here we also discuss the crossing probability of a second class particle to overtake a third class particle in the tazrp as in [4].

2. Statement of results

2.1. The dynamics. The one-dimensional tazrp is a continuous time Markov process $\xi_t$ with state space $\mathbb{N}^\mathbb{Z}$. In this process, after a mean one exponential time, a particle at $x$ jumps to $x + 1$ at rate 1, independently from the number of particles at the destination site. For a configuration $\xi$ and for $x \in \mathbb{Z}$, $\xi(x)$ denotes the number of particles at that site, i.e. if $\xi(x) = k, k = 0, \ldots, \infty$, then there are $k$ particles at the site $x$. Its infinitesimal generator is defined on local functions $f: \mathbb{N}^\mathbb{Z} \to \mathbb{R}$ as

$$
\mathcal{L} f(\xi) \sum_{x \in \mathbb{Z}} 1_{\{\xi(x) \geq 1\}} [f(\xi^{x,x+1}) - f(\xi)],
$$

where

$$
\xi^{x,x+1}(z) = \begin{cases}
\xi(z), & \text{if } z \neq x, x + 1 \\
\xi(x) - 1, & \text{if } z = x \\
\xi(x+1) + 1, & \text{if } z = x + 1
\end{cases}
$$

We will also consider more general totally asymmetric jumps given by $g(k)$, where $k = \xi(x)$ and the rate function $g: \mathbb{N} \to \mathbb{R}_+$ is non-decreasing, $g(0) = 0$, $g(k) > 0$ for $k > 0$ and with bounded variation, i.e. $\sup_k |g(k+1) - g(k)| < +\infty$.

Now we introduce the dynamics of the tasep. Let $\eta_t$ be a continuous time Markov process with space state $\{0, 1\}^\mathbb{Z}$. In this process, particles evolve on $\mathbb{Z}$ according to interacting random walks with an exclusion rule which prevents to have more than a particle per site. The
dynamics is as follows: after a mean one exponential time, a particle at \( x \) jumps to \( x + 1 \) at rate 1, if the destination site is empty otherwise it does not move and the clock restarts. For a configuration \( \eta \) and for \( x \in \mathbb{Z} \), \( \eta(x) \) denotes the quantity of particles at \( x \): \( \eta(x) = 1 \), the site \( x \) is occupied otherwise it is empty. Its infinitesimal generator is given on local functions \( f : \{0,1\}^\mathbb{Z} \rightarrow \mathbb{R} \) by \( \Omega f(\eta) = \sum_{x\in\mathbb{Z}} \eta(x) (1 - \eta(x + 1)) [f(\eta^{x,x+1}) - f(\eta)] \) where \( \eta^{x,x+1} \) is the configuration obtained from \( \eta \) exchanging the variables at site \( x \) and \( x + 1 \). We notice that in both dynamics described above, particles only move in the one-dimensional lattice, so that, the total number of particles is a conserved quantity for both processes.

Now, we recall the notion of attractiveness of the zero-range process. In \( \mathbb{N}^2 \) there is a partial order between configurations as follows: \( \zeta \leq \tilde{\zeta} \) if \( \forall x \in \mathbb{Z}, \zeta(x) \leq \tilde{\zeta}(x) \). This partial order induces the corresponding stochastic order on the distributions of the process: if \( \rho \) and \( \tilde{\rho} \) are probability measures in \( \mathbb{N}^2 \), then there exists a coupling measure \( \bar{\rho} \) in the product space \( \mathbb{N}^2 \times \mathbb{N}^2 \), whose marginals are \( \rho \) and \( \tilde{\rho} \) and supported in \( \{(\zeta,\tilde{\zeta}) : \zeta \leq \tilde{\zeta}\} \). It is well known that the fact of \( g(\cdot) \) begin non-decreasing implies that given \( \zeta \) and \( \tilde{\zeta} \) such that \( \zeta \leq \tilde{\zeta} \), then it is possible to construct a coupling of the zero-range process \( (\zeta_t,\tilde{\zeta}_t) \) starting from \( (\zeta,\tilde{\zeta}) \) such that \( \forall t > 0 \zeta_t \leq \tilde{\zeta}_t \), see [2], [9] or [12]. Since we took \( g(\cdot) \) non-decreasing, we have in hand the attractiveness property for the processes we are considering. Later we will see that this property is crucial for our conclusions.

Now we describe a set of equilibrium measures that we will recall in the sequel. For the tazrp, for each density \( \rho \) there exists an invariant measure \( \mu_\rho \), translation invariant and such that \( E_{\mu_\rho}[\zeta(0)] = \rho \), namely the Geometric product measure of parameter \( \frac{1}{1+\rho} \):

\[
\mu_\rho(\xi : \xi(x) = k) = \left( \frac{\rho}{1+\rho} \right)^k \frac{1}{1+\rho}, \quad k \in 0, \ldots, \infty.
\]

For the tasep, it is well known that for \( 0 \leq \alpha \leq 1 \), \( \nu_\alpha \), the Bernoulli product measure on \( \{0,1\}^\mathbb{Z} \) with density \( \alpha \):

\[
\nu_\alpha(\eta : \eta(x) = 1) = \alpha,
\]

is an invariant measure, translation invariant and also such that \( E_{\nu_\alpha}[\eta(0)] = \alpha \).

Let \( \mu_{\rho,\lambda} \) be the product measure \( \mu_\rho \delta_{(-\infty,0]} + \mu_\lambda \delta_{(0,\infty)} \) namely for \( k \in 0, \ldots, \infty \):

\[
\mu_{\rho,\lambda}(\xi : \xi(x) = k) = \begin{cases} \left( \frac{\rho}{1+\rho} \right)^k \frac{1}{1+\rho}, & \text{if } x \leq 0 \\ \left( \frac{\lambda}{1+\lambda} \right)^k \frac{1}{1+\lambda}, & \text{if } x > 0 \end{cases},
\]

and denote by \( \nu_{\rho,\lambda} \) the product measure \( \nu_\rho \delta_{(-\infty,0]} + \nu_\lambda \delta_{(0,\infty)} \).

2.2. **Hydrodynamic limit.** Now we state the hydrodynamic limit for the processes introduced above. Fixed \( \xi_n \), let \( \pi^n(\xi, du) \) be the empirical measure given by

\[
\pi^n(\xi, du) = \frac{1}{n} \sum_{x \in \mathbb{Z}} \xi(x) \delta_{\xi(du)}(du),
\]

where \( \delta_u \) denotes the Dirac measure at \( u \) and let \( \pi^n(\xi, du) = \pi^n(\xi_t, du) \).

Since the work of Rezakhanlou [14], it is known that, starting the tazrp from a measure \( \mu_n \) associated to a profile \( \rho_0 \) and some additional hypotheses (for details see [14]), if \( \pi^n_0(\xi, du) \) converges to \( \rho_0(\xi, du) \) in \( \mu_{\rho_0} \)-probability, then \( \pi^n(\xi_t, du) \) converges to \( \rho(t, u) \) in \( \mu_{\rho_0} S_{t\rho_0} \)-probability, where \( S_t \) is the semigroup corresponding to \( \mathcal{L} \) and \( \rho(t, u) \) is the unique entropy solution of the hyperbolic conservation law:

\[
\frac{\partial}{\partial t} \rho(t, u) + \nabla \phi(\rho(t, u)) = 0,
\]

where \( \phi(\rho) = \frac{\rho^2}{1+\rho^2} \). For the process with jump rate \( g(\cdot) \) satisfying the assumptions above, the hydrodynamic limit also holds and in this case, \( \phi(\rho) = \bar{g}(\rho) = E_{\mu_\rho}[g(\eta(0))] \).

In [14] the hydrodynamic limit was derived for the tasep and its hydrodynamic equation is the **inviscid Burgers equation**, i.e. equation (2.1) with \( \phi(\rho) = \rho(1-\rho) \). In [8] the hydrodynamic limit for the tasep was derived, via the Relative Entropy method and the same arguments can
be applied to the tazrp to derive its hydrodynamic limit under this method. As a consequence
the local equilibrium convergence is readily obtained, see Corollary 6.1.3 of [10].

2.3. Characteristics. Now, we go towards a description of characteristics for hyperbolic
conservations law as above, as in [11]. Consider a partial differential equation given by:
\[ \partial_t \rho(t, u) + \nabla F(\rho(t, u)) = 0, \]
where the flux \( F(\cdot) \) is a concave function and suppose it is differentiable. A characteristic
is the trajectory of a point with constant density and denoting by \( v_{\rho_0}(t, u) \) the position at
time \( t \) of a point with density \( \rho_0 = \rho(0, u) \), then \( \rho(v_{\rho_0}(t, u), t) = \rho_0 = \rho(0, u) \). Taking the time
derivative of last expression it follows that \( \partial_t v_{\rho_0}(s, u) = F'(\rho_0) \). Integrating last expression
from time \( 0 \) to time \( t \) and noticing that \( v_{\rho_0}(0, u) = u \), we get to \( v_{\rho_0}(t, u) = u + F'(\rho_0)t \). So, in
this case, characteristics are straight lines with slope \( F'(\rho_0) \).

If the initial condition is a decreasing step function as
\[ \rho_{\rho,\lambda}^0(u) = \rho_1_{\{u \leq 0\}}(u) + \lambda \rho_1_{\{u > 0\}}(u), \]
with \( \rho > \lambda \), then the solution of (2.2) can be explicitly computed and it is given by
\[ \rho(t, u) = \begin{cases} 
\rho, & \text{if } u < F'(\rho)t \\
\lambda, & \text{if } u > F'(\lambda)t \\
\psi\left(\frac{u}{t}\right), & \text{otherwise}
\end{cases} \]
The function \( \psi \) can be easily computed and reads as \( \psi(v) = (F')^{-1}(v) \), the inverse of \( F' \).

Now we derive these solutions for the processes under consideration. We notice that \( \phi(\rho) \)
corresponds to mean with respect to the invariant state of the process of the instantaneous
current at a bond \([x, x + 1]\). The instantaneous current at the bond \([x, x + 1]\) is defined as the
difference between the jump rate to the right neighboring site and the jump rate to the left
neighboring site. In our cases, since jumps are totally asymmetric, it coincides with the jump
rate to the right. For the tazrp, the instantaneous current at the bond \([x, x + 1]\) is \( 1_{\{\xi(x) \geq 1\}} \),
which in turns gives \( \phi(\rho) = \frac{\xi - \rho}{1 - \rho} \). So, under the initial condition \( \rho_{\rho,\lambda}^0 \), the solution at time \( t \) of
(2.1) is given by as above with \( F'(\cdot) := \phi'(\cdot) \) and \( \psi(u/t) := \frac{\sqrt{1 - \rho} - \sqrt{u}}{\sqrt{\rho}} \).

For the zero-range process with jump rate given by \( g(\cdot) \), the instantaneous current at
the bond \([x, x + 1]\) is \( g(\xi(x)) \) and the solution of the corresponding hydrodynamic equation
under initial condition \( \rho_{\rho,\lambda}^0 \), is given on \( u \in [(g'(\rho))^{-1}t, (g'(\lambda))^{-1}t] \) by \( \tilde{g}'(\cdot)^{-1} \), where \( \tilde{g}(\rho) = E_{\rho}[g(\eta(0))] \) and outside of this interval is given as above.

2.4. Discrepancy. Now we introduce the notion of discrepancy between two copies of a
same attractive process. For that purpose, let \( \eta \) and \( \xi \) be two copies of a process such that
\( \eta_0(x) = \xi_0(x) \) for all \( x \neq 0 \) and for example at the site zero take \( \eta_0(0) = \xi_0(0) + 1 \). So, at
zero time there is only one discrepancy between \( \eta \) and \( \xi \) that is located at the origin. By
the attractiveness property and the conservation of the number of particles, at each time \( t \)
there is still a unique discrepancy between \( \eta \) and \( \xi \). This discrepancy is called second class
component. As mentioned in the introduction, the dynamics of second class particles is very
different in zero-range and exclusion type dynamics. On the tazrp, when first and second
class particles share the same site, the first class particles have priority to jump, a second
class particle jumps to the right neighboring site if and only if there is no first class particles
at the departure point. On the other hand, for the exclusion type dynamics, the second class
particle jumps to the right neighboring site if it is empty, but if a first class particle attempts
to jump to the site occupied by the second class particle, they exchange positions. So, in
the exclusion type dynamics a second class particle can jump backwards, contrarily to the tazrp
in which it only moves to the right. Throughout the paper we denote by \( X_2(t) \) the position
of the second class particle on the tazrp at time \( t \). Naturally, that copying two tazrp, we
do not have necessarily one discrepancy between them, e.g. if we take in the copy above
\( \eta_0(0) = \xi_0(0) + k \), with \( k \in \mathbb{N} \), then there are \( k \) second class particles at the origin at time \( 0 \).
The dynamics between second class particles sharing the same site is fully described below.
(see the proof of Theorem 2.1), but the idea is simple: second class particles at the same site are labeled from the bottom to the top, the first to jump to the right neighboring site is the one at the bottom and it only jumps if there is no first class particles at the departure point. For the tazrp we show that the sum of joint distributions of these discrepancies converge and we identify its limit. The joint distribution of the speeds of second class particles for the tasep was studied in [1]. Here we provide the first result on joint distributions of these speeds for the tazrp.

2.5. Main results.

**Theorem 2.1.** Consider the tazrp starting from $\mu_{\rho,\lambda}$ with $0 \leq \lambda < \rho < \infty$ and suppose that at the initial time there are infinite second class particles and no first class particles at the origin. At the initial time label the second class particles, from the bottom to the top and for $j \in \mathbb{N}$, let $X_j(t)$ be the position at time $t$ of the $j$-th second class particle initially at the origin. Then

$$
\lim_{t \to +\infty} \sum_{j=0}^{+\infty} \left( P(X_{j+2}(t) \geq ut) \left[ \frac{\rho^{j+1}}{(1 + \rho)^j} - \frac{\lambda^{j+1}}{(1 + \lambda)^j} \right] \right) = (1 + \rho + \lambda)[\rho(1, u) - \lambda],
$$

where $\rho(t, u)$ is the entropy solution of (2.1) with initial condition $\rho_{0,\lambda}$ as in (2.3).

In the same spirit as in [5], we establish the L.L.N. for a single second class particle initially at the origin for the tazrp starting from $\mu_{\infty,0}$. This is the content of next theorem.

**Theorem 2.2.** Consider the tazrp starting from $\mu_{\infty,0}$ and at the initial time put a second class particle at the origin and remove the first class particles there. Denote by $X_2(t)$ the position at time $t$ of the second class particle initially at the origin. Then

$$
\lim_{t \to +\infty} \frac{X_2(t)}{t} = X, \quad \text{in distribution}
$$

where $X$ is distributed according to $F_X(u) = \sqrt{u}$, with $u \in J_X = [0, 1]$.

Now we discuss the L.L.N. for the current of first class particles that cross over the second class particle. Let $\xi$ be a configuration distributed according to $\mu_{\infty,0}$ and with a single second class particle at the origin - $\xi$ is represented on the left hand side of figure 2 below. Recall that the position of the second class particle at time $t$ is denoted by $X_2(t)$.

Denote by $J^{sr}_2(t)$ the number of first class particles that cross over the second class particle, i.e. the number of first class particles at the right of $X_2(t)$ at time $t$:

$$
J^{sr}_2(t) = \sum_{x \geq X_2(t)} \xi_t(x).
$$

As a consequence of Theorem 2.2, we conclude that

**Corollary 2.3.** Consider the tazrp starting from $\mu_{\infty,0}$ and at the initial time put a second class particle at the origin and remove the first class particles there. Then

$$
\lim_{t \to +\infty} \frac{J^{sr}_2(t)}{t} = (1 - \sqrt{X})^2, \quad \text{in distribution}
$$

and

$$
\lim_{t \to +\infty} E_{\mu_{\infty,0}} \left[ \frac{J^{sr}_2(t)}{t} \right] = \frac{1}{3}
$$

where $X$ is given in Theorem 2.2.

Now we consider a slight perturbation of the invariant state $\mu_\lambda$ and we derive a L.L.N. for a single second class particle.
Theorem 2.4. Consider the tazrp starting from \( \mu = \mu_\lambda + \delta_{(-\infty,0)} \rho_\lambda \) with \( 0 \leq \lambda < \rho < \infty \) and \( \rho - \lambda \leq 1 \). At the initial time put a second class particle at the origin, let \( X_2(t) \) denote its position at time \( t \) and let \( J^{\tau_2}(t) \) be the number of first class particles that cross over \( X_2(t) \). Then

\[
\lim_{t \to +\infty} \frac{X_2(t)}{t} = Z, \quad \text{in distribution under } \mu^* S_t,
\]

\[
\lim_{t \to +\infty} \frac{J^{\tau_2}(t)}{t} = \left(1 - \sqrt{Z}\right)^2, \quad \text{in distribution under } \mu^* S_t
\]

and

\[
\lim_{t \to +\infty} E_{\mu^* S_t} \left[ \frac{J^{\tau_2}(t)}{t} \right] = 1 + \frac{2}{\rho - \lambda} \log \left(\frac{1 + \lambda}{1 + \rho}\right) + \frac{1}{(1 + \lambda)(1 + \rho)}
\]

where \( \mu^*(\cdot) = \mu(\cdot|X_2(0) = 0) \), \( Z \) has distribution function given by \( F_Z(u) = \frac{\rho - \rho(1,u)}{\rho - \lambda} \), with \( u \in J = [\phi(\rho), \phi(\lambda)] \) and \( \rho(t,u) \) is the unique entropy solution of (2.1) with initial condition \( \rho_0^\lambda \) as in (2.3).

3. PROOF OF THEOREM 2.4

Proof. Fix a configuration \( \xi \in \mathbb{N}^2 \) and denote by \( J_t^u(\xi) \) the current of particles that cross the time dependent bond \( ut \) during the time interval \([0, t]\). Then \( J_t^u(\xi) \) is the number of particles of \( \xi \) that are at left of the origin (including it) at time 0 and are at the right of \( ut \) at time \( t \), minus the number of particles of \( \xi \) that are strictly at the right of the origin at time 0 and are at left of \( ut \) (including it) at time \( t \):

\[
J_t^u(\xi) = \sum_{x \leq 0} \sum_{z = 1}^{\xi(x)} 1\{X^{\tau,\xi}(t) > ut\} - \sum_{x > 0} \sum_{z = 1}^{\xi(x)} 1\{X^{\tau,\xi}(t) \leq ut\},
\]

(3.1)

where for a site \( x \), we label the \( \xi(x) \) particles at this site (the first one being the one at the bottom and the \( \xi(x) \) begin the one at the top) and we denote by \( X^{\tau,\xi}(\cdot) \) the position at time \( t \) of a tagged particle initially at site \( x \) at position \( z \).

For \( x \in \mathbb{Z} \) denote by \( \tau_x \) the space translation by \( x \), so that for a local function \( f \) it holds that \( \tau_x f(\xi) = f(\tau_x \xi) \) and for a probability measure \( \mu \in \mathbb{N}^2 \) we have that \( \int f(\tau_x \xi) \mu(d\xi) = \int f(\tau_x \xi) \mu(d\xi) \).

Now, we compute in two different ways:

\[
\int E[J_t^u(\xi) \mu_{\rho_\lambda}(d\xi)] - \int E[J_t^u(\xi) \tau_{-1} \mu_{\rho_\lambda}(d\xi)].
\]

Let \( \xi_0^0 \) and \( \xi_1^1 \) be two copies of the tazrp starting from \( \xi_0^0 \) and \( \xi_1^1 \), respectively, such that \( \xi_0^0(\xi_1^1) \) is distributed according to \( \mu_{\rho_\lambda}(\cdot | \tau_{-1} \mu_{\rho_\lambda}) \). For any coupling \( \tilde{\mu} \) of \( \mu_{\rho_\lambda} \) and \( \tau_{-1} \mu_{\rho_\lambda} \) and for any coupling \( P \) of \( \xi_0^0 \) and \( \xi_1^1 \), last expression equals to

\[
\int d\tilde{\mu}(\xi_0^0, \xi_1^1) E[J_t^u(\xi_0^0) - J_t^u(\xi_1^1)],
\]

(3.2)

where \( E \) is the expectation with respect to \( P \). Obviously that in order to have a difference between the two fluxes, the configurations \( \xi_0^0 \) and \( \xi_1^1 \) must have at least a discrepancy at the origin at time 0, otherwise the difference is zero. Splitting this event by the number of discrepancies between the configurations and conditioning on \( D_k \) which is the event that there are exactly \( k \) discrepancies at the origin between \( \xi_0^0 \) and \( \xi_1^1 \), last expression can be written as

\[
\sum_{k=1}^{\infty} \int d\tilde{\mu}(\xi_0^0, \xi_1^1) E[J_t^u(\xi_0^0) - J_t^u(\xi_1^1)]|D_k| \tilde{\mu}(D_k).
\]

Now, we recall the dynamics of second class particles sharing the same site. Starting from \( D_k \), there are \( k \) discrepancies at the origin which we label from the bottom to the top as \( X_2(0), \ldots, X_{k+1}(0) \). If the clock rings, the first particle to jump is \( X_2(0) \) and then jumps \( X_3(0) \) and so on. For \( j = 2, \ldots, \infty \) let \( X_j(t) \) denote the position at time \( t \) of the \( X_j(0) \) second class particle. Since there are \( k \) discrepancies, the difference between the currents can vary from 1 to \( k \), depending on the relative positions of these second class particles with respect to the
moving bond $ut$. If $j$ is the difference between the fluxes, then we must have $X_{j+1}(t) \geq ut$ but $X_{j+2}(t) < ut$. Notice that having $X_{j+1}(t) \geq ut$, it implies that $X_2(t) \geq ut, \ldots, X_j(t) \geq ut$, since the $X_{j+1}(t)$-second class particle only jumps to the right if the particles with less degree of class are at its right. This is the key point in the proof where we use the total asymmetry of jumps. In the presence of partial asymmetry the second class particles do not preserve their order. This fact is crucial for our conclusions and allow us to write last expression as:

$$\sum_{k=1}^{\infty} \left( \sum_{j=1}^{k-1} jP_{\bar{\mu}}(X_{j+1}(t) \geq ut, X_{j+2}(t) < ut) + kP_{\bar{\mu}}(X_{k+1}(t) \geq ut) \right) \left( \mu(D_k^+) - \mu(D_k^-) \right),$$

(3.3)

where $\mu_k$ is the coupling measure conditioned on $D_k$ and $D_k^+(-) = \{ (\xi^0, \xi^1) : \xi^0(0) = \xi^1(0) + (-)k \}$. In $D_k^+(D_k^-)$ there is a contribution for the current with a positive (negative) sign, since the current can vary from $1 (-1)$ to $k(-k)$ as long as the discrepancies at time $t$ are at the right of $ut$. Since we are dealing with Geometric product measures, it is not hard to show that

$$m_k := \mu(D_k^+) - \mu(D_k^-) = \left( \frac{1}{1 + \rho + \lambda} \right) \left[ \left( \frac{\rho}{1 + \rho} \right)^k - \left( \frac{\lambda}{1 + \lambda} \right)^k \right]$$

(3.4)

and in sake of completeness we prove in Remark 3.1. In order to keep notation simple, write down the probability $P_{\bar{\mu}}(X_{j+1}(t) \geq ut, X_{j+2}(t) < ut)$ as $p_k(j,t)$. Then (3.3) can be written as

$$\sum_{k=1}^{\infty} \left( \sum_{j=1}^{k-1} jP_{\bar{\mu}}(X_{j+1}(t) \geq ut) + kP_{\bar{\mu}}(X_{k+1}(t) \geq ut) \right) m_k.$$

(3.5)

Now, notice that

$$\sum_{j=1}^{k-1} jP_{\bar{\mu}}(X_{j+1}(t) \geq ut) + kP_{\bar{\mu}}(X_{k+1}(t) \geq ut),$$

since by a simple computation $p_k(j,t)$ can be written as $P_{\bar{\mu}}(X_{j+1}(t) \geq ut) - P_{\bar{\mu}}(X_{j+2}(t) \geq ut)$. Last equality follows from the fact that if $\{X_{j+2}(t) \geq ut\}$ then we must have for sure that $\{X_{j+1}(t) \geq ut\}$. Again, total asymmetry of jumps in invoked. If the jumps are partially asymmetric and the difference of currents is $j$, then in $D_k$ $j$ of the $k$ discrepancies are at the right of $ut$. This can happen a number of $C_k^j$ events but we do not have control over the probabilities of each one of them, and for that we cannot get to a similar result. Collecting these facts together, (3.5) can be written as

$$\sum_{k=1}^{\infty} m_k \left( \sum_{j=1}^{k} P_{\bar{\mu}}(X_{j+1}(t) \geq ut) \right).$$

Notice that $P_{\bar{\mu}}(X_{j+1}(t) \geq ut)$ does not depend on $k$, since this is the probability that the $(j + 1)$-th second class particle at the origin (with other $k$ second class particles at the same site), having crossed the line $ut$ at time $t$ and that does only depend on $j$ and $t$. This observation allows us to apply Fubini to change the order of summation in last expression to write it as:

$$\sum_{j=1}^{\infty} \left( P_{\bar{\mu}}(X_{j+1}(t) \geq ut) \sum_{k=j}^{\infty} m_k \right).$$

(3.6)

By (3.4) it holds that

$$\sum_{k=j}^{\infty} m_k = \frac{1}{1 + \rho + \lambda} \left[ \frac{\rho^j}{(1 + \rho)^{j-1}} - \frac{\lambda^j}{(1 + \lambda)^{j-1}} \right].$$

So far we have that:

$$\int d\bar{\mu}(\xi^0, \xi^1) E\left[ J^u_\mu(\xi^0) - J^\mu_\nu(\xi^1) \right] = \sum_{j=1}^{\infty} \left( P_{\bar{\mu}}(X_{j+1}(t) \geq ut) \frac{1}{1 + \rho + \lambda} \left[ \frac{\rho^j}{(1 + \rho)^{j-1}} - \frac{\lambda^j}{(1 + \lambda)^{j-1}} \right] \right).$$
Recall that \( \mu \) and \( \xi \) the expression above in terms of the process by the convergence to local equilibrium. Here we compute Remark 3.1. □

Now, compute (3.2) by coupling \( \mu_{p,\lambda} \) and \( \tau_{-1}\mu_{p,\lambda} \) in such a way that \( \xi^1 = \tau_{-1}\xi^0 \). By the definition of the current (see (3.1)) we write \( J^u(\xi^0) - J^u(\xi^1) \) as

\[
\sum_{x \leq 0} \sum_{z=1}^{\xi^0(x)} 1_{\{X_t^x,z(\xi^0) > ut\}} - \sum_{x < 0} \sum_{z=1}^{\xi^0(x)} 1_{\{X_t^x,z(\xi^0) \leq ut\}} - \sum_{x \leq 0} \sum_{z=1}^{\xi^1(x)} 1_{\{\tilde{X}_t^x,z(\xi^1) > ut\}} + \sum_{x \leq 0} \sum_{z=1}^{\xi^1(x)} 1_{\{\tilde{X}_t^x,z(\xi^1) \leq ut\}},
\]

where \( X_t^x,z(\xi^0) \) and \( \tilde{X}_t^x,z(\xi^1) \) denotes the position of a tagged particle initially at site \( x \) at position \( z \) for the process starting from \( \xi^0 \) and \( \xi^1 \), respectively. Since \( \xi^1 = \tau_{-1}\xi^0 \) and writing the expression above in terms of the process \( \xi^0 \) we get to

\[
\sum_{x \in \mathbb{Z}} \sum_{z=1}^{\xi^0(x)} 1_{\{X_t^x,z(\xi^0) = ut+1\}} - \xi^0(1).
\]

Recall that \( \xi^0 \) is initially distributed according to \( \mu_{p,\lambda} \). Applying expectation with respect to \( \mu_{p,\lambda} \) to last expression, we obtain that (3.2) equals to

\[
E_{\mu_{p,\lambda}}[\xi^0(1)] = E_{\mu_{p,\lambda}}[\xi^0(1)].
\]

Since \( \xi^0(1) \) is distributed according to \( \mu_{p,\lambda} \) it follows that \( E_{\mu_{p,\lambda}}[\xi^0(1)] = \lambda \). On the other hand, by the convergence to local equilibrium (see [3]) it follows that:

\[
\lim_{t \to +\infty} E_{\mu_{p,\lambda}}[\xi^0(ut + 1)] = \rho(1,u),
\]

where \( \rho(t,u) \) is the entropy solution of (2.1) with initial condition \( \rho^0,\lambda \) as in (2.3). Putting together the previous computations the proof ends.

**Remark 3.1.** Here we compute \( \bar{\mu}(D_k^+) \), i.e. the \( \bar{\mu} \) probability of having \( k \) discrepancies between \( \xi^0 \) and \( \xi^1 \), such that \( \xi^0(0) = \xi^1(0) + k \). Recall that \( \xi^0 \) and \( \xi^1 \) are distributed according to \( \mu_{p,\lambda} \) and \( \tau_{-1}\mu_{p,\lambda} \), respectively. Notice that

\[
\bar{\mu}(D_k^+) = \bar{\mu}\left( \bigcup_{n=0}^{\infty} \left\{ \xi^0(0) = n + k, \xi^1(0) = n \right\} \right)
\]

\[
= \sum_{n=0}^{\infty} \mu_p(\xi^0(0) = n + k)\mu_\lambda(\xi^1(0) = n)
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{\rho}{1 + \rho} \right)^{n+k} \frac{1}{1 + \lambda} \left( \frac{\lambda}{1 + \lambda} \right)^n \frac{1}{1 + \lambda}
\]

\[
= \frac{1}{(1 + \rho + \lambda)} \left( \frac{\rho}{1 + \rho} \right)^k.
\]

In the second equality above, we use the fact that \( \bar{\mu} \) is the coupling measure and it is a product measure with marginals \( \mu_{p,\lambda} \) and \( \tau_{-1}\mu_{p,\lambda} \). A similar computation shows that

\[
\bar{\mu}(D_k^-) = \frac{1}{(1 + \rho + \lambda)} \left( \frac{\lambda}{1 + \lambda} \right)^k.
\]

**Remark 3.2.** Last result also holds for totally asymmetric jumps given by \( g(\cdot) \) as in the beginning of section 2.7. This is a consequence of the attractiveness property and the local equilibrium convergence for the process starting from \( \mu_{p,\lambda} \), see [3].

4. TAZRP STARTING FROM \( \mu_{\infty,0} \)

Consider the tazrp starting from \( \mu_{\infty,0} \), put a second class particle at the origin and remove the first class particles there, i.e. the process starts from \( \xi \) introduced before the statement of Corollary 2.3. We prove a L.L.N. for the position of the second class particle and for the current of first class particles that cross over the second class particle. We start by the former.
4.1. Proof of Theorem \[2.2\]

Proof. Let \( \tilde{\xi} \) be the configuration with infinite particles at negative sites, the origin and all positive sites empty:

\[
\tilde{\xi}(z) = \begin{cases} 
\infty, & \text{if } z \leq -1 \\
0, & \text{if } z \geq 0 
\end{cases}
\]

Recall the definition of \( J^u_t(\xi) \) from the proof of Theorem \[2.1\], for \( \tilde{\xi} \), it follows that

\[
J^u_t(\tilde{\xi}) = \sum_{x \geq 1} \tilde{\xi}_t(x + ut),
\]

i.e. \( J^u_t(\tilde{\xi}) \) is the number of first class particles at the right of \( ut \) at time \( t \). By the Kolmogorov’s backwards equation we have that

\[
\frac{d}{dt}E_{\mu_{\infty,0}}[J^u_t(\tilde{\xi})] = E_{\mu_{\infty,0}}[L(J^u_t(\tilde{\xi}))] = E_{\mu_{\infty,0}}[J^u_t(\tilde{\xi}^{-1,1})] - E[J^u_t(\tilde{\xi})].
\] (4.1)

Notice that since at site \(-1\) there are infinite particles, there is only one discrepancy at the origin between \( \tilde{\xi}^{-1,0} \) and \( \tilde{\xi} \). In the figure below the configuration represented on the top is \( \tilde{\xi} \) and the other one is \( \tilde{\xi}^{-1,0} \).

\[
\text{FIGURE 1. One single discrepancy at the origin.}
\]

Now we consider to copies \( \xi^0_t \) and \( \xi^1_t \) of the tazrp starting from \( \tilde{\xi}^{-1,0} \) and \( \tilde{\xi} \), respectively, and we use the basic coupling on them. The basic coupling was introduced in \[12\] and it means that, by attaching a Poisson clock of parameter one to each site of \( \mathbb{Z} \), when the clock rings the particle jumps to the right neighboring site. With this coupling both processes use the same realizations of the clocks. By the attractiveness property together with the conservation of the number of particles, there is still a unique discrepancy between \( \xi^0_t \) and \( \xi^1_t \) at any time \( t > 0 \). Let \( X_2(t) \) denote the position at time \( t \) of this discrepancy and notice that by construction \( X_2(0) = 0 \). Now, since \( J^u_t(\xi^{-1,0}) - J^u_t(\xi) = 1_{(X_2(t) > ut)} \), then (4.1) is equal to

\[
P(X_2(t) > ut).
\]

On the other hand, by writing \( J^u_t(\tilde{\xi}) \) as a martingale plus a compensator:

\[
J^u_t(\tilde{\xi}) = M_t(\tilde{\xi}) + \int_0^t 1_{\{\tilde{\xi}_s(ut) \geq 1\}} ds,
\]

we get that

\[
\frac{d}{dt}E_{\mu_{\infty,0}}[J^u_t(\tilde{\xi})] = E_{\mu_{\infty,0}}[M_t(\tilde{\xi})] + \int_0^t 1_{\{\tilde{\xi}_s(ut) \geq 1\}} ds,
\]

since the mean of martingales is constant and the martingale \( M_t \) vanishes at time 0. Now, by the convergence to local equilibrium it follows that

\[
\lim_{t \to +\infty} E_{\mu_{\infty,0}}[\{X(t) \geq 1\}] = \phi(\rho(1,u)) = 1 - \sqrt{u}.
\]

Here \( \rho(t, u) \) is the unique entropy solution of \[2.1\] starting from \( \rho^0,\lambda \) as in \[2.3\] with \( \rho = \infty \) and \( \lambda = 0 \). Collecting these facts together the proof ends. \( \square \)
Now, we prove the L.L.N. for the current of first class particles that cross over the second class particle.

4.2. Proof of Corollary 2.3

Proof. Recall from Theorem 2.2 that \( X_2(t)/t \) converges in distribution as \( t \to +\infty \) to \( X \) distributed according to \( F_X(u) = \sqrt{u} \) for \( u \in \mathbb{R} \). By the definition of \( J^{\tau_*(t)}(t) \), see (2.4); from the hydrodynamic limit (see section 2.2) and Theorem 2.2, it follows that

\[
\lim_{t \to \infty} \frac{J^{\tau_*(t)}(t)}{t} = \int_{\mathbb{R}} \rho(1,u)du = \int_{\mathbb{R}} \left( \frac{1}{\sqrt{u}} - 1 \right) du = \left( 1 - \sqrt{X} \right)^2,
\]

where \( \rho(t,u) \) is the solution at time \( t \) of (2.1) with initial condition \( \rho_0^\rho,\lambda \) as in (2.2) with \( \rho = \infty \) and \( \lambda = 0 \). This finishes the proof of the first claim. For the second, since \( X \) has distribution given by \( F_X(u) = \sqrt{u} \) for \( u \in [0,1] \), then

\[
\lim_{t \to +\infty} E_{\mu_{\infty,\alpha}} \left[ \frac{J^{\tau_*(t)}(t)}{t} \right] = \int_{0}^{1} (1 - \sqrt{u})^2 \frac{1}{2\sqrt{u}} du = \frac{1}{3}.
\]

\( \square \)

5. Law of Large Numbers More General \( \mu \)

5.1. Proof of Theorem 2.4

Proof. The proof follows the same arguments as in the proof of Theorem 1 of [5]. The main features of the process that we need in order to achieve the result are: the attractiveness property, the existence of at most one discrepancy when applying any coupling to \( \mu \) and \( \tau_{-1}\mu \) and the convergence to local equilibrium (see remark 5.1). It follows that:

As in the proof of Theorem 2.1 fix \( \xi \in \mathbb{N}^2 \) and denote by \( J^\mu_\alpha(\xi) \) the current of first class particles that cross the moving bond \( ut \). We compute in two different ways:

\[
\int E[J^\mu_\alpha(\xi)]\mu(d\xi) - \int E[J^\mu_\alpha(\xi)]\tau_{-1}\mu(d\xi).
\]

Let \( \xi_0^\mu \) and \( \xi_1^\mu \) be two copies of the tazrp starting from \( \xi^0 \) and \( \xi^1 \), respectively, such that \( \xi^0(\xi^1) \) is distributed according to \( \mu(\tau_{-1}\mu) \). For any coupling \( \bar{\mu} \) of \( \mu \) and \( \tau_{-1}\mu \) and any coupling \( \bar{P} \) of \( \xi_0^\mu \) and \( \xi_1^\mu \), last expression equals to:

\[
\int d\bar{\mu}(\xi_0^\mu,\xi_1^\mu)\tilde{E}[J^\mu_\alpha(\xi_0^\mu) - J^\mu_\alpha(\xi_1^\mu)],
\]

where \( \hat{E} \) is the expectation with respect to \( \bar{P} \). The measure \( \bar{\mu} \) induces in \( \bar{\mu} \) the property that \( \xi_0^0(x) = \xi_1^1(x) \) for \( x \neq 0 \) and at the origin \( \xi_0^0 \) has an extra particle relatively to \( \xi_1^1 \), with probability \( \rho - \lambda \). Let \( X_2(t) \) denote the position at time time of this discrepancy. Since there is at most a discrepancy between \( \xi_0^0 \) and \( \xi_1^1 \), the difference between the currents is at most one, and that happens as long as \( X_2(t) \) is at the right of \( ut \) at time \( t \). With this observation (5.1) can be written as

\[
\langle \rho - \lambda \rangle P_{\mu^* S_\epsilon}(X_2(t) > ut),
\]

where \( \mu^* S_\epsilon \) is the distribution of the process at time \( t \) starting from \( \mu^* \).

On the other hand, we couple \( \mu \) and \( \tau_{-1}\mu \) in such a way that \( \xi_1^1 = \tau_{-1}\xi_0^0 \), which allows to rewrite (5.1) as

\[
E_{\mu}[\xi_0^0(ut + 1)] - E_{\mu}[\xi_0^0(1)].
\]

Since under \( \mu \), \( \xi_0^0(1) \) is distributed according to \( \mu_\lambda \), then \( E_{\mu}[\xi_0^0(1)] = \lambda \). Now, by the convergence to local equilibrium (see remark 5.1), it follows that:

\[
\lim_{t \to +\infty} E_{\mu}[\xi_0^0(ut + 1) = 1] = \rho(1,u),
\]

where \( \rho(1,u) \) is the unique entropy solution of (2.1) with initial condition \( \rho_0^\rho,\lambda \). This is enough to conclude the first claim of the Theorem.
Now we derive the L.L.N. for the current of first class particles that jump over the second class particle. Denote by $X_1(t)$ the position at time $t$ of the first class particle initially at site $-1$. Since first class particles do preserve their order, the current of first class particles that cross over the second class particle from time $0$ to time $t$, is equal to the number of first class particles at time $t$ between the sites $X_1(t)$ and $X_2(t)$. Since $X_1(t)$ is the first to jump over the second class particle, then

$$J_{2}^{\tau}(t) = \sum_{x \geq X_2(t)} \xi_t(x).$$

Notice that at positive sites, $\xi_t(x)$ is distributed according to $\mu_\lambda$. So that a L.L.N. for $X_1(t)$ holds (see [15]):

$$\lim_{t \to +\infty} \frac{X_1(t)}{t} = 1 - \frac{\lambda}{1 + \lambda} = \frac{1}{1 + \lambda}.$$

On the other hand, from the hydrodynamic limit and the previous result we get that:

$$\lim_{t \to +\infty} \frac{J_{2}^{\tau}(t)}{t} = \int_{Z} \rho(1,u)du = \int_{Z} \frac{1 - \sqrt{u}}{\sqrt{u}} du + \int_{\frac{1}{1+\lambda}} \lambda du.$$

Now, a simple computation ends the proof. For the second claim, it is enough to notice that

$$\lim_{t \to +\infty} E_{\mu^t \cdot S_t} \left[ \frac{J_{2}^{\tau}(t)}{t} \right] = \int_{\phi'}(\rho) (1 - \sqrt{u})^2 \frac{1}{2(\rho - \lambda)u^{3/2}} du.$$

\[ \square \]

**Remark 5.1.** We recall that the hydrodynamic limit for the tazrp and for more general jumps rates given by $g(\cdot)$ (as defined in section 2.7) starting from $\mu$, can be obtained by the Relative Entropy method following the same arguments as done for the tasep in [8]. As a straightforward consequence we obtain the local equilibrium convergence. For details see chapter 6 of [10]. In possession of last result, we can obtain for the zero-range with jumps rates given by $g(\cdot)$ the same result as in Theorem 2.4 but with $Z$ is distributed according to $F_\rho(u) = \frac{e^{-\rho(\rho(\cdot))^{-1}}}{\rho - \lambda}$, for $u \in [(\tilde{g}(\rho))^{-1}, (\tilde{g}(\lambda))^{-1}]$.

**Remark 5.2.** We remark here that last result also holds for partially asymmetric zero-range processes with jump rate $\gamma_t(\cdot)$ with positive and finite mean: $0 < \gamma := \sum_{x \in Z} p(x) < \infty$. The proof follows the same lines as in the proof of Theorem 2.4. The main difference is that $\rho(t,u)$ is the solution of the corresponding hydrodynamic equation.

6. Coupling with tasep

6.1. **Law of Large Numbers.** In this section we reprove Theorem 2.2 and Corollary 2.3 by coupling the tazrp with the tasep. Recall that we consider the tazrp starting from $\nu_{\rho,0}$, with a second class particle at the origin and no first class particles there. We start by showing a L.L.N. for the position of the second class particle initially at the origin. First we recall a result that we will use in the sequel:

**Theorem 6.1.** ([5])

Consider the tasep starting from $\nu_{\rho,\lambda}$ with $0 \leq \lambda < \rho \leq 1$. At time zero put a second class particle at the origin regardless the value of the configuration at this point and let $Y_2(t)$ denote its position at time $t$. Then

$$\lim_{t \to +\infty} \frac{Y_2(t)}{t} = \mathcal{U}, \text{ in distribution}$$

where $\mathcal{U}$ is uniformly distributed on $[(1 - 2\rho), (1 - 2\lambda)]$.

The proof of last result was given in [5] and extended for more general asymmetric exclusion processes in [4]. By coupling the tazrp with the tasep, we can show a L.L.N. for $X_2(t)$ and show that:
Proposition 6.2. Consider the tazrp starting from \( \mu_{\infty,0} \) and at time zero put a second class particle at the origin and remove the first class particles there. Let \( X_2(t) \) denote the position at time \( t \) of this second class particle. Then

\[
\lim_{t \to +\infty} \frac{X_2(t)}{t} = \left( \frac{1 + \eta}{2} \right)^2, \quad \text{in distribution}
\]  

where \( \eta \) is uniformly distributed on \([-1, 1]\).

Proof. The idea of the proof is to couple the tazrp \( \xi_t \) with the tasep \( \eta_t \) and to establish a relation between the position of the second class particle on the tazrp with some microscopic function on the tasep. Now we explain the coupling we use between the two processes.

Suppose to start the tazrp from \( \xi \) - the configuration distributed according to \( \mu_{\infty,0} \) and with a single second class particle at the origin, see the left hand side of figure 2. On the other hand, start the tasep from \( \nu_{1,0} \), remove the first class particle at the origin if necessary and add there a second class particle. On the tasep, initially label the first class particles by denoting the position of the first class particle at site \(-i\) at time 0 by \( x_i(0) \). Recall that the occupation variables in the tazrp are represented by \( (\xi(x))_{x \in \mathbb{Z}} \). We relate both processes in such a way that in the tasep the distance between two consecutive first class particles minus one, becomes the number of particles at a site in the tasep representation. With this in mind, we define for \( i \geq 1 \): \( \xi_0(i) = x_{i+1}(0) - x_i(0) - 1 \), at the origin both have a second class particle and for \( i \leq -1 \), \( \xi_0(i) = \infty \) since on the tasep there are no other first class particles at the right of \( x_1(0) \). On the left hand side of the figure below, we represent the tazrp, the second class particle is represented by \( \circ \), while first class particles are represented by \( \bullet \) and holes by \( \circ \). On the right hand side, the tasep is represented, first class particles are represented by \( \bullet \) and holes by \( \circ \).

![Figure 2. Coupling between tazrp and tasep](image)

At first notice that in the tazrp, a first class particle can jump to the site where the second class particle stands and the latter does not move. This corresponds to a first class particle jumping over the pair \( \circ \bullet \) on the tasep representation. On the other hand, in the tazrp when the second class particle jumps to the right, this corresponds to a jump of the pair \( \bullet \circ \) over a hole on the tasep. With this relation, the position of second class particle at time \( t \) in the tazrp, corresponds to the number of holes to the left of the pair \( \bullet \circ \) at time \( t \) in the tasep. Now, we also notice that for the tasep, the dynamics of the pair \( \circ \bullet \) in the configuration above is the same as the dynamics of a second class particle at the origin in the configuration with all negative sites occupied and all positive sites empty, see [6] for details on this last correspondence.

In order to summarize the established relations we introduce some notation. Recall that \( X_2(t) \) denotes the position of the second class particle at time \( t \) for the tazrp starting from \( \xi \). Let \( J_2^\circ(t) \) \((J_2^\bullet(t))\) denote the number of first class particles at the right of the second class particle at time \( t \) in the tasep (tazrp) and let \( H_2^\circ(t) \) denote the number of holes to the left of the second class particle in the tasep at time \( t \). For this particular initial conditions, by the relations mentioned above we have that

\[
J_2^\circ(t) = J_2^\bullet(t) \quad \text{and} \quad H_2^\circ(t) = X_2(t).
\]

So, the position of the second class particle in the tazrp starting from \( \xi \), corresponds to the number of holes at the left of the second class particle for the tasep starting from \( \nu_{1,0} \) with a second class particle at the origin. Recall the L.L.N. for \( H_2^\circ(t) \) shown in [7], namely that

\[
\lim_{t \to +\infty} \frac{H_2^\circ(t)}{t} = \left( \frac{1 + \eta}{2} \right)^2, \quad \text{in distribution}
\]
where \( \mathcal{U} \) is a Uniform random variable on \([−1, 1]\). Then, the equality on the right hand side of (6.2) together with last result finishes the proof. So, we identify \( X \) in the statement of Theorem 2.2 as \( \left( \frac{1 + \mathcal{U}}{2} \right)^2 \).

\[ \text{□} \]

**Remark 6.1.** We remark that the convergence in (6.3) in fact takes place almost surely. As a consequence, the convergence of the second class particle in (6.1) also holds almost surely.

Now we reprove Corollary 2.3 by identifying the limit random variable for which \( J_{se}^2(t)/t \) converges.

**Corollary 6.3.** Consider the tazrp starting from \( \mu_{\infty,0} \) and add at time 0 a second class particle at the origin and remove the first class particles there. Then

\[
\lim_{t \to +\infty} \frac{J_{se}^2(t)}{t} = \left( \frac{1 - \mathcal{U}}{2} \right)^2, \quad \text{almost surely}
\]

where \( \mathcal{U} \) is uniformly distributed on \([−1, 1]\).

**Proof.** By (6.2) and since it was shown in [7] that \( J_{se}^2(t)/t \) converges almost surely to \( \left( \frac{1 - \mathcal{U}}{2} \right)^2 \) where \( \mathcal{U} \) is uniformly distributed on \([−1, 1]\), the result follows. \[ \text{□} \]

**6.2. Crossing probabilities for a second and a third class particle on the tazrp.**

**Corollary 6.4.** Consider the tazrp starting from \( \mu_{\infty,0} \) and at time zero add a second class particle at site 0 and a third class particle at site 1 and remove the first class particles there. Denote by \( X_2(t) \) (\( X_3(t) \)) the position at time \( t \) of the second (third) class particle. Then

\[
\lim_{t \to +\infty} P(X_2(t) > X_3(t)) = \frac{2}{3}.
\]

**Proof.** The proof of this result is a consequence of the relation between the tazrp and the tasep that we introduced above and of Theorem 2.3 of [4]. The probability that we want to compute is the same as starting the tazrp from a configuration with all negative sites with infinite particles, a single second class particle at the origin and another single one at site 1 and all positive sites empty (see the left hand side of the figure below), and ask for the probability of the second class particle at the origin to jump over the other second class particle. In the picture bellow, on the left hand side the tazrp is represented and first class particles are denoted by ●, the second class particles by ⊙ and holes by ◦. On the right hand side the tasep is represented and first class particles are represented by ● and holes by ◦.

**Figure 3.** Relation between tazrp and tasep with a second and a third class particle

As explained in the proof of Proposition 6.2, the dynamics of a second class particle in the tazrp corresponds to the dynamics of the pair ● ● in the tasep. In this case, the dynamics of the two second class particles in the tazrp representation corresponds to the dynamics of the two pairs ○ ● ● in the tasep. But, the dynamics of the two pairs in the tasep, is the same as the dynamics of two second class particles for the tasep starting from a configuration with all negative sites occupied, two second class particles (one at the origin and other at site 1) and all positive sites empty, see [6] and [4] for details. Also, notice that this last dynamics corresponds to the dynamics of second and third class particles for the tasep starting from the configuration with all negative sites occupied, all positive sites empty, a second class particle at the origin an a third class particle at site 1, see [4]. As a consequence, the claim above follows straightforwardly from the result in Theorem 2.3 of [4]. \[ \text{□} \]
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