ON THE OPTIMALITY OF THE HYPERCONTRACTIVITY OF THE COMPLEX BOHNENBLUST–HILLE INEQUALITY

J. R. CAMPOS, G.A. MUÑOZ-FERNÁNDEZ, D. PELLEGRINO, AND J.B. SEOANE-SEPÚLVEDA

Abstract. The main motivation of this paper is the following open problem: Is the hypercontractivity of the complex polynomial Bohnenblust–Hille inequality an optimal result? We show that the solution to this problem has a close connection with the searching of the optimal constants for the real polynomial Bohnenblust–Hille inequality. So we are lead to a detailed study of the hypercontractivity constants for real scalars. In fact we study two notions of constants of hypercontractivity: absolute ($H_{a,R}$) and asymptotic ($H_{\infty,R}$). Among other results, our estimates combined with recent results from [3] show that

$$1.5098 < H_{\infty,R} < 2.829 \quad \text{and} \quad 1.6561 < H_{a,R} < 3.296.$$  

1. Introduction

If $E$ is a Banach space, real or complex, we say that $P$ is a homogeneous polynomial on $E$ of degree $m \in \mathbb{N}$ if there exists a symmetric $m$-linear form on $E^m$ such that $P(x) = L(x, \ldots, x)$ for all $x \in E$. We denote by $\mathcal{P}(^mE)$ and $\mathcal{L}^s(^mE)$ the spaces of continuous $m$-homogeneous and continuous symmetric $m$-linear forms on $E$. It is well-known that a homogeneous polynomial is continuous if and only if $P$ is bounded over the unit ball $B_E$ of $E$, and in that case

$$\|P\| = \sup\{|P(x)| : x \in B_E\}$$

defines a norm in $\mathcal{P}(^mE)$. If $P \in \mathcal{P}(^mE)$, we shall refer to $\|P\|$ as the polynomial norm of $P$ in $E$. This norm is very difficult to compute in most cases, for which reason it would be interesting to obtain reasonably good estimates on it. The $\ell_p$ norm of the coefficients of a given polynomial on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) has also been widely used in mathematics and is much easier to handle. Observe that an $m$-homogeneous polynomial in $\mathbb{K}^n$ can be written as

$$P(x) = \sum_{|\alpha| = m} a_{\alpha} x^{\alpha},$$

2010 Mathematics Subject Classification. 46G25, 47L22, 47H60.

Key words and phrases. Absolutely summing operators, Bohnenblust–Hille constants, Quantum Information Theory.

D. Pellegrino was supported by CNPq Grants 301237/2009-3, 477124/2012-7, INCT-Matemática and CAPES-NF. G.A. Muñoz-Fernández and J. B. Seoane-Sepúlveda were supported by MTM2012-34341.
where $x = (x_1,\ldots,x_n) \in \mathbb{K}^n$, $\alpha = (\alpha_1,\ldots,\alpha_n) \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Thus we define the $\ell_p$ norm of $P$, with $p \geq 1$, as

$$|P|_p = \left( \sum_{|\alpha|=m} |a_\alpha|^p \right)^{\frac{1}{p}}.$$ 

If $E$ has finite dimension $n$, then the polynomial norm $\| \cdot \|$ and the $\ell_p$ norm $| \cdot |_p$ ($p \geq 1$) are equivalent, and therefore there exist constants $k(m,n), K(m,n) > 0$ such that

$$k(m,n)|P|_p \leq \|P\| \leq K(m,n)|P|_p,$$

for all $P \in \mathcal{P}(mE)$. The latter inequalities may provide a good estimate on $\|P\|$ as long as we know the exact value of the best possible constants $k(m,n)$ and $K(m,n)$ appearing in (1.1).

The problem presented above is an extension of the the well known polynomial Bohnenblust-Hille inequality. It was proved in [2] that there exists a constant $D_m > 0$ such that for every $P \in \mathcal{P}(m\ell_\infty^n)$ we have

$$|P|_{\frac{2m}{m+1}} \leq D_m\|P\|.$$  

Observe that (1.2) coincides with the first inequality in (1.2) for $p = \frac{2m}{m+1}$ except for the fact that $D_m$ in (1.2) can be chosen in such a way that it is independent from the dimension $n$. Actually Bohnenblust and Hille showed that $\frac{2m}{m+1}$ is optimal in (1.2) in the sense that for $p < \frac{2m}{m+1}$, any constant $D$ fitting in the inequality $|P|_p \leq D\|P\|$, for all $P \in \mathcal{P}(m\ell_\infty^n)$ depends necessarily on $n$.

It was recently shown in [6] that the complex polynomial Bohnenblust–Hille inequality is hypercontractive. For real scalars, in [3], it was proved that the real Bohnenblust–Hille polynomial inequality is hypercontractive and this result cannot be improved. The optimality of the result for complex scalars is still open.

The polynomial and multilinear Bohnenblust–Hille inequalities have important applications in different fields of Mathematics and Physics, such as Operator Theory, Fourier and Harmonic Analysis, Complex Analysis, Analytic Number Theory and Quantum Information Theory. Since its origins in 1931, in the Annals of Mathematics, the (multilinear and polynomial) Bohnenblust–Hille inequalities were overlooked for a long period (see [2]) and were only rediscovered in the last few years with works of A. Defant, L. Frerick, J. Ortega-Cerdá, M. Ounaïes, D. Popa, U. Schwarting, K. Seip, among others (see, e.g., [7, 8, 11, 12, 15, 16]).

The main motivation of this paper is the following open problem: Is the hypercontractivity of the complex polynomial Bohnenblust–Hille inequality an optimal result?

We show that the search of optimal values for $D_{m,R}$ is closely related to the aforementioned problem.
So, we try to determine the value of the best constant in (1.2). This constant depends considerably on whether we consider the real or complex version of \( \ell_n^\infty \), which motivates the following definition

\[
D_{m,\mathbb{R}} := \inf \left\{ D > 0 : |P|_{m+1}^{2m} \leq D_m \|P\| \text{ for all } P \in \mathcal{P}(m \ell_n^\infty) \right\}.
\]

2. How is the real case connected to the optimality of hypercontractivity of the complex Bohnenblust–Hille inequality?

In this section we show how the problem of the sharpness of the hypercontractivity of the complex polynomial Bohnenblust–Hille inequality is related to the search of optimal constants for the case of real scalars. We need to introduce an asymptotic approach to the notion of hypercontractivity constant:

**Definition 2.1** (Asymptotic approach). The asymptotic hypercontractivity constant of the real polynomial Bohnenblust–Hille inequality is

\[
H_{\infty,\mathbb{R}} := \limsup_m \sqrt[m]{D_{\mathbb{R},m}}.
\]

Observe that for every \( \epsilon > 0 \) there exists \( N_\epsilon \in \mathbb{N} \) such that \( D_{\mathbb{R},m} \leq (H_{\infty,\mathbb{R}} + \epsilon)^m \) for all \( m \geq N_\epsilon \). Hence \( H_{\infty,\mathbb{R}} \) is a sharp measure of the asymptotic growth of the constants \( D_{\mathbb{R},m} \).

The next result shows the precise connection between the real and complex cases:

**Proposition 2.2.** If \( H_{\infty,\mathbb{R}} > 2 \) then the hypercontractivity of the complex polynomial Bohnenblust–Hille inequality is optimal.

*Proof.* Let \( C \) be a real number such that

\[
H_{\infty,\mathbb{R}} > C > 2.
\]

Then there is a sequence \( m_1 < m_2 < \cdots \) of positive integers so that

\[
D_{m_j,\mathbb{R}} > C^{m_j}
\]

for all \( j \). Let \( \epsilon > 0 \) be a positive integer such that

\[
C - \epsilon > 2.
\]

Thus, for all such \( m_j \) there is an \( m_j \)-homogeneous polynomial

\[
P_{m_j} = \sum_{|\alpha| = m_j} a_{\alpha} x^\alpha
\]

so that

\[
\left( \sum_{|\alpha| = m_j} |a_{\alpha}|^{2m_j} \right)^{m_j+1} \left( \frac{m_j+1}{2m_j} \right) \geq (C - \epsilon)^{m_j}.
\]

If \( P_{\mathbb{C},m_j} \) denotes the complexification of \( P_{m_j} \) then, from [3] (using a result due to C. Visser [13]), we know that

\[
\|P_{\mathbb{C},m_j}\| \leq 2^{m_j-1} \|P_{m_j}\|.
\]
4 CAMPOS, MUÑOZ, PELLEGRINO, AND SEOANE

We thus have

\[
\left( \sum_{|\alpha|=m_j} |a_\alpha|^{2m_j+1 \over 2m_j} \right)^{m_j+1 \over 2m_j} \geq \left( \sum_{|\alpha|=m_j} |a_\alpha|^{2m_j+1 \over 2m_j} \right)^{m_j+1 \over 2m_j} \gtrsim (C - \varepsilon)^{m_j} \gtrsim (C - \varepsilon)^{m_j}.
\]

The above result is the motivation of the rest of the paper: to look for lower bounds for the constants of the real polynomial Bohnenblust–Hille inequality.

3. LOWER BOUNDS FOR THE POLYNOMIAL BOHNENBLUST-HILLE CONSTANT

If \( P : \ell_\infty^n (\mathbb{R}) \to \mathbb{R} \) is given by

\[
P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha,
\]

then

\[
(3.1) \quad D_{\mathbb{R},m} \geq \left( \sum_{|\alpha|=m} |a_\alpha|^{2m+1 \over m+1} \right)^{m+1 \over 2m} \geq \left( (a_\alpha)_{|\alpha|=m} \right)^{2m \over m+1} \geq \left( (a_\alpha)_{|\alpha|=m} \right)^{2m \over m+1},
\]

for all choices of the \( a_\alpha \)'s, where \( | \cdot |_p \) denotes the usual \( \ell_p \) norm. The inequality (3.1) would provide an easy way to obtain lower bounds for \( D_{\mathbb{R},m} \) if we knew a simple way to compute \( \| P \| \) for special \( P \).

Besides the notion of asymptotic hypercontractivity constant we consider the following alternative approach:

**Definition 3.1 (Absolute approach).** The hypercontractivity constant of the real polynomial Bohnenblust–Hille inequality is

\[
H_{a,\mathbb{R}} := \inf \{ H > 0 : D_{\mathbb{R},m} \leq H^m \text{ for all } m \}.
\]

We prove that (combining our results with [3])

\[
\begin{cases} 
1.5098 < H_{\infty,\mathbb{R}} < 2.829 \\
1.6561 < H_{a,\mathbb{R}} < 3.296
\end{cases}
\]

and we provide numerical evidence that

\[
1.6561 < H_{\infty,\mathbb{R}}.
\]
3.1. The case $m = 2$. Our notation follows [4]. The value of the constant $D_{R,2}(\mathcal{P}(2\ell_\infty^2))$ can be obtained using the geometry of the unit ball of $\mathcal{P}(\ell_\infty^2)$ described in [1]. We state the result we need for completeness:

**Theorem 3.2.** [Choi, Kim [4]] The extreme points of the unit ball of $\mathcal{P}(2\ell_\infty^2)$ are the polynomials of the form

$$\pm x^2, \pm y^2, \pm (tx^2 - ty^2 \pm 2\sqrt{t(1-t)}xy),$$

with $t \in [1/2, 1]$.

As a consequence of the previous result, we obtain the following:

**Theorem 3.3.**

$$D_{R,2}(\mathcal{P}(2\ell_\infty^2)) = \sup_{t \in [1/2,1]} \left[ 2t^4 + (2\sqrt{t(1-t)})^4 \right]^{1/4} \approx 1.8374.$$

Moreover, the following polynomials are extreme for this problem:

$$P(x, y) = \pm (0.8678352808x^2 - 0.8678352808y^2 \pm 0.6773395196xy).$$

**Proof.** Let

$$f(t) = \left[ 2t^4 + (2\sqrt{t(1-t)})^4 \right]^{1/4}.$$

We just have to notice that due to the convexity of the $\ell_p$-norms and Theorem 3.2 we have

$$D_{R,2}(\mathcal{P}(2\ell_\infty^2)) = \sup\{|a|_4^4 : a \in B_{\mathcal{P}(2\ell_\infty^2)}\} = \sup\{|a|_4^4 : a \in \text{ext } B_{\mathcal{P}(2\ell_\infty^2)}\} = \sup_{t \in [1/2,1]} f(t).$$

Observe also that the last supremum is attained at $t \approx 0.8678352808$, concluding the proof. \qed

**Corollary 3.4.**

$$D_{R,2} \geq 1.8374.$$

3.2. The case $m = 3$. To the authors’ knowledge the calculation of $\|P\|$ is, in general, far from being easy. However there is a way to compute $\|P\|$ for specific cases. For instance Grecu, Muñoz and Seoane prove in [10, Lemma 3.12] the following formula:

**Lemma 3.5.** If for every $a, b \in \mathbb{R}$ we define $P_{a,b}(x, y) = ax^3 + bx^2y + bxy^2 + ay^3$ then

$$\|P_{a,b}\| = \begin{cases} 
\left| a - \frac{b^2}{3a} + \frac{2b^3}{27a^2} + \frac{2a}{27} \left(-\frac{3b}{a} + \frac{b^2}{a^2}\right)^{\frac{1}{3}} \right| & \text{if } a \neq 0 \text{ and } b_1 < \frac{b}{a} < 3 - 2\sqrt{3}, \\
|2a + 2b| & \text{otherwise},
\end{cases}$$

where

$$b_1 = \frac{3}{7} \left( 3 - \frac{2\sqrt{9}}{\sqrt{-12 + 7\sqrt{3}}} + 2\sqrt{3-36 + 21\sqrt{3}} \right) \approx -1.6692.$$
A combination of Lemma 3.5 and (3.1) provide the following sharp polynomial Bohnenblust-Hille type constant:

**Theorem 3.6.** Let $P_{a,b}(x,y) = ax^3 + bx^2y + bxy^2 + ay^3$ for $a, b \in \mathbb{R}$ and consider the subset of $\mathcal{P}(3\ell_2^\infty)$ given by $E = \{P_{a,b} : a, b \in \mathbb{R}\}$. Then

$$\frac{|(a,b,b,a)|^{\frac{3}{2}}}{\|P_{a,b}\|} = \begin{cases} 
\frac{27a^2(2|a|^\frac{3}{2}+2|b|^\frac{3}{2})^{\frac{4}{5}}}{27a^3-9ab^2+2b^3+2\text{sign}(a)(-3ab+b^2)^{\frac{3}{2}}} & \text{if } a \neq 0 \text{ and } b_1 < \frac{b}{a} < 3 - 2\sqrt{3}, \\
\frac{(2|a|^\frac{3}{2}+2|b|^\frac{3}{2})^{\frac{4}{5}}}{2|a+b|^{\frac{3}{2}}} & \text{otherwise}
\end{cases}$$

where $b_1$ was defined in Lemma 3.5. Moreover, the above function attains its maximum when $\frac{b}{a} = b_1$, which implies that

$$D_{R,3}(E) = \left(\frac{2 + 2|b_1|^{\frac{3}{2}}}{2|1+b_1|}\right)^{\frac{4}{5}} \approx 2.5525$$

**Corollary 3.7.**

$$D_{R,3} \geq 2.5525.$$
The authors have numerical evidence to state that
\[ D_{\mathbb{R},3}(P(3, \ell_\infty^2)) = D_{\mathbb{R},3}(E). \]
Moreover, one polynomial for which \( D_{\mathbb{R},3}(P(3, \ell_\infty^2)) \) would be attained is approximated by
\[ P_3(x, y) = x^3 - 1.6692x^2y - 1.6692xy^2 + y^3. \]
One might think that the powers of this polynomial would improve the lower estimates for the real polynomial Bohnenblust-Hille inequality found in [3], but it is not true. Indeed, Notice that
\[ P_3(x, y)^4 = x^{12} - 6.676x^{11}y + 10.041x^{10}y^2 + 18.832x^9y^3 - 51.359x^8y^4 \\
- 11.353x^7y^5 + 82.242x^6y^6 - 11353x^5y^7 - 51.359x^4y^8 + 18.832x^3y^9 \\
+ 10.041x^2y^{10} - 6.676xy^{11} + y^{12} \]
Therefore, if \( a \) is the vector of the coefficients of \( P_3(x, y)^4 \) and we use the fact that \( \|P_3\| \approx 3.2088 \), by (3.1) we conclude that
\[ D_{\mathbb{R},12} \geq \frac{|a|_{12}}{\|P_3\|^4} \approx 38.1. \]
The latter estimate is much worse than the estimate \( D_{\mathbb{R},12} \geq 66.39 \) found in [3, Theorem 6.3].

Despite the powers of \( P_3 \) do not provide an improvement on the lower estimates obtained on \( D_{\mathbb{R},m} \), we will see in the next section that those estimates can be improved by finding \( D_{\mathbb{R},4} \) numerically.

3.3. The case \( m = 4n \).

**Theorem 3.8.** If \( n \in \mathbb{N} \), then
\[ D_{\mathbb{R},4n} \geq D_{\mathbb{R},4n}(P(4n, \ell_\infty^2)) \geq \frac{\sum_{k=0}^{n} \binom{n}{k} \frac{8n}{4n+1} \frac{8n}{3n}}{\left(\frac{2\sqrt{3}}{9}\right)^n} \geq \frac{(2n)!}{\left(\frac{2\sqrt{3}}{9}\right)^n n!}. \]

**Proof.** Consider the 4-homogeneous polynomial given by
\[ P_4(x, y) = x^3y - xy^3 = xy(x^2 - y^2). \]
A straightforward calculation shows that \( P_4 \) attains its norm at \( \pm(\pm \frac{1}{\sqrt{3}}, 1) \) and \( \pm(1, \pm \frac{1}{\sqrt{3}}) \) and that \( \|P_4\| = \frac{2\sqrt{3}}{9} \). Therefore
\[ D_4 \geq \frac{2\sqrt{3}}{2\sqrt{3}} \approx 4.006. \]
On the other hand \( \| P_4^n \| = \left( \frac{2\sqrt{3}}{9} \right)^n \) and
\[
P_4(x, y)^n = x^n y^n \sum_{k=0}^{n} \binom{n}{k} x^{2k} y^{2(n-k)}.
\]

Hence, if \( \mathbf{a} \) is the vector of the coefficients of \( P_4 \), by (3.1) and using the fact that \( | \cdot |_{\frac{s_n}{4n+1}} \geq | \cdot |_2 \) (notice that here \( | \cdot |_2 \) is the euclidean norm), we have
\[
D_{R,4n} \geq \frac{| \mathbf{a} |_{\frac{s_n}{4n+1}}}{\| P_4 \|^n} = \frac{\left( \sum_{k=0}^{n} \binom{n}{k} (\frac{s_n}{4n+1}) \right)^{\frac{1}{2}}}{\left( \frac{2\sqrt{3}}{9} \right)^n} = \frac{\sqrt{\frac{2n}{4n+1}}}{\left( \frac{2\sqrt{3}}{9} \right)^n} n!.
\]

(3.3)

Above we have used the well known formula
\[
\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.
\]

Remark 3.9. The following table shows estimates on \( D_{\mathbb{R}, m} \) for \( m = 4n \), \( n \in \mathbb{N} \) based on (3.1):

| \( m \) | \( D_{\mathbb{R}, 8n} \geq \) | \( m \) | \( D_{\mathbb{R}, 8n} \geq \) |
|---|---|---|---|
| 8 | 17.4817 | 80 | 7.3769 \times 10^{13} |
| 12 | 81.8865 | 120 | 9.5448 \times 10^{20} |
| 16 | 395.178 | 160 | 1.2730 \times 10^{28} |
| 20 | 1938.6 | 200 | 1.7261 \times 10^{45} |
| 24 | 9610.8 | 240 | 2.3650 \times 10^{42} |
| 28 | 4799.2 | 280 | 3.2638 \times 10^{49} |
| 32 | 24093 \times 10^6 | 320 | 4.5279 \times 10^{56} |
| 36 | 12145 \times 10^6 | 360 | 6.3068 \times 10^{63} |
| 40 | 61418 \times 10^6 | 400 | 8.8123 \times 10^{70} |

Table 1. Estimates on \( D_{\mathbb{R}, m} \) for some multiples of 4.

Observe that the estimates presented in Table 1 certainly improve those in [3], however the improvement may not be asymptotically spectacular since, according to Table 1 \( D_{\mathbb{R}, 400} \geq (1.5044)^{400} \), whereas \( D_{\mathbb{R}, 400} \geq (1.4896)^{400} \) according to [3] Table 1.

The constant 1.5044 appearing in the estimate \( D_{\mathbb{R}, 400} \geq (1.5044)^{400} \) found in the previous remark is quite accurate as the following result shows:
Theorem 3.10. If $m = 4n$ for $n$ large enough, then

$$D_{\mathbb{R},m} \geq \left( \mathcal{P}(^{4n}e^2) \right) \geq 4 \sqrt{\frac{4}{m\pi}} \left( \sqrt[4]{27} \right)^m.$$

Observe that $\sqrt[4]{27} \approx 1.5098$.

Proof. Using Stirling’s approximation formula $n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n$ in (3.3) we have, for $m = 4n$

$$D_{\mathbb{R},4n} \geq \frac{\sqrt{(2n)!}}{(2\sqrt{\pi})^n n!} \sim \frac{\sqrt{2\sqrt{n\pi} \left( \frac{2n}{e} \right)^{2n}}}{\sqrt{2\pi n} \left( \frac{n}{e} \right)^n} = 4 \sqrt{\frac{4}{m\pi}} \left( \sqrt[4]{27} \right)^m.$$

□

As a consequence of the latter result and [3, Theorem 6.4] we have:

Corollary 3.11. $1.5098 < \sqrt[4]{27} \leq H_{\infty,\mathbb{R}} \leq 2\sqrt{2} < 2.829$.

The above result will be improved later.

The following corollary, although simple, is stressed because, as it will be seen later on Theorem 6.1 it highlights a rupture between the real and complex cases.

Corollary 3.12. For all fixed positive integer $n$, the hypercontractivity of the real polynomial Bohnenblust-Hille inequality for polynomials on $\mathbb{R}^n$ can not be improved.

3.4. The case $m = 5n$. Let us define the polynomial

$$P_5(x, y) = ax^5 - bx^4y - cx^3y^2 + cx^2y^3 + bxy^4 - ay^5,$$

with

$$a = 0.194627836350, \quad b = 0.660089997037, \quad c = 0.978333058512.$$  

Using elementary calculus it can be seen that

$$\|P_5\| = 0.286170950359,$$

up to 10 decimal places. Then if $a_n$ is the vector of the coefficients of $P_5^n$ for each $n \in \mathbb{N}$, then we know that

$$D_{\mathbb{R},5n} \geq \frac{|a_n|^{\frac{5n}{\frac{5n}{n}}}}{\|P_5\|^n},$$

in particular one can check numerically that

$$D_{\mathbb{R},5} \geq 6.835918785877.$$

The authors have numerical evidence showing that $D_{\mathbb{R},5}(\mathcal{P}(^5e^2)) = 6.8359$ up to 4 decimal places.
A table with estimates on $D_{R,m}$ obtained by using (3.4) for different values of $n \in \mathbb{N}$ can be found in Table 2.

| $m = 10$ | $D_{R,10} \geq 48.03065$ | $m = 80$ | $D_{R,80} \geq 9.90603 \times 10^{11}$ |
| $m = 15$ | $D_{R,15} \geq 399.007$ | $m = 120$ | $D_{R,120} \geq 2.83620 \times 10^{22}$ |
| $m = 20$ | $D_{R,20} \geq 3271.54$ | $m = 160$ | $D_{R,160} \geq 1.19496 \times 10^{39}$ |
| $m = 25$ | $D_{R,25} \geq 28308.7$ | $m = 200$ | $D_{R,200} \geq 5.11958 \times 10^{47}$ |
| $m = 30$ | $D_{R,30} \geq 2.41034 \times 10^9$ | $m = 240$ | $D_{R,240} \geq 2.21659 \times 10^{58}$ |
| $m = 35$ | $D_{R,35} \geq 2.11695 \times 10^9$ | $m = 280$ | $D_{R,280} \geq 9.66672 \times 10^{62}$ |
| $m = 40$ | $D_{R,40} \geq 1.83355 \times 10^9$ | $m = 320$ | $D_{R,320} \geq 4.23805 \times 10^{69}$ |
| $m = 45$ | $D_{R,45} \geq 1.62275 \times 10^{10}$ | $m = 360$ | $D_{R,360} \geq 1.86553 \times 10^{78}$ |
| $m = 50$ | $D_{R,50} \geq 1.41925 \times 10^{10}$ | $m = 400$ | $D_{R,400} \geq 8.23785 \times 10^{89}$ |

Table 2. Estimates on $D_{R,m}$ for some multiples of 5.

**Remark 3.13.** Observe that using the estimates in Table 2 we obtain

$$D_{R,400} \geq (1.5480)^{400},$$

whereas using the estimates in table 1 we have

$$D_{R,400} \geq (1.5044)^{400}.$$

It is interesting to notice too that $\sqrt[3]{27} \approx 1.5098 < 1.5491$, where is the lower bound on $H_R$ found in Corollary 3.11. Hence, the results that have appeared in this section suggest that we have the following:

(3.5)\[ H_{\infty, R} \geq 1.5480. \]

Concerning the absolute hypercontractivity constant we have (combining our results with [3]):

**Corollary 3.14.**

$$1.5480 \leq H_{a,R} < 3.296$$

3.5. **The case** $m = 6n$. The authors have numerical evidence pointing to the fact that the extreme polynomial in the Bohnenblust-Hille inequality for polynomials in $P(6\ell_2^2)$ is of the form

$$Q_{a,b}(x, y) = ax^5y + bx^3y^3 + axy^5.$$

This motivates a deeper study of this type of polynomials, which we do in the following result.
Theorem 3.15. Let \( Q_{a,b}(x,y) = ax^5y + bx^3y^3 + axy^5 \) for \( a, b \in \mathbb{R} \) and consider the subspace of \( \mathcal{P}(\mathcal{L}^2_{\infty}) \) given by \( F = \{ Q_{a,b} : a, b \in \mathbb{R} \} \). Suppose \( \lambda_0 < \lambda_1 \) are the only two roots of the equation
\[
\frac{|3\lambda^2 - 20 + \lambda \sqrt{9\lambda^2 - 20}|}{25} \sqrt{|-3\lambda - \sqrt{9\lambda^2 - 20}|} = |2 + \lambda|.
\]

Then if \( \lambda = \frac{b}{a} \) we have
\[
\frac{|(0, a, 0, b, 0, a, 0)|}_{25} = \begin{cases}
\frac{25 \sqrt{10} (2 + |\lambda|^{\frac{12}{7}})}{|3\lambda^2 - 20 + \lambda \sqrt{9\lambda^2 - 20}| \sqrt{-3\lambda - \sqrt{9\lambda^2 - 20}}}, & \text{if } a \neq 0 \text{ and } \lambda_0 < \frac{b}{a} < \lambda_1, \\
\frac{(2 + |\lambda|^{\frac{12}{7}})}{|2 + \lambda|}, & \text{otherwise}.
\end{cases}
\]

Observe that \( \lambda_0 \approx -2.2654, \lambda_1 \approx -1.6779 \) and the above function attains its maximum when \( \frac{b}{a} = \lambda_0 \) (see Figure 3), which implies that
\[
D_{\mathbb{R}, 6}(F) = \frac{(2 + |\lambda_0|^{\frac{12}{7}})}{|2 + \lambda_0|} \approx 10.7809.
\]

Proof. We do not lose generality by considering only polynomials of the form \( Q_{1, \lambda} \), in which case
\[
\|Q_{1, \lambda}\| = \sup \{|x^5 + \lambda x^3 + x| : x \in [0, 1]\}.
\]

The polynomial \( q_{\lambda}(x) := x^5 + \lambda x^3 + x \) has no critical points if \( \lambda > -\frac{2 \sqrt{5}}{3} \), otherwise it has the following critical points in \([0, 1]\):
\[
x_0 := \sqrt{-\frac{3\lambda - \sqrt{9\lambda^2 - 20}}{10}} \quad \text{and} \quad x_1 := \sqrt{-\frac{3\lambda + \sqrt{9\lambda^2 - 20}}{10}},
\]
and \( x_0 \) if \( \lambda \leq -2 \). Notice that
\[
q_{\lambda}(x_0) = -\frac{3\lambda^2 + 20 - \lambda \sqrt{9\lambda^2 - 20}}{20} x_0,
\]
\[
q_{\lambda}(x_1) = -\frac{3\lambda^2 + 20 + \lambda \sqrt{9\lambda^2 - 20}}{20} x_1.
\]

It is easy to check that \( |q_{\lambda}(x_0)| \geq |q_{\lambda}(x_1)| \) for \( -2 \leq \lambda \leq -\frac{2 \sqrt{5}}{3} \), which implies that
\[
\|Q_{1, \lambda}\| = \begin{cases}
\max \{|2 + \lambda|, |q_{\lambda}(x_0)|\} & \text{if } -2 \leq \lambda \leq -\frac{2 \sqrt{5}}{3}, \\
|2 + \lambda| & \text{otherwise}.
\end{cases}
\]

The equation \( |2 + \lambda| = |q_{\lambda}(x_0)| \) turns out to have only two roots, namely \( \lambda_0 \approx -2.2654 \) and \( \lambda_1 \approx -1.6779 \). By continuity, it is easy to prove that \( |2 + \lambda| \leq |q_{\lambda}(x_0)| \) only if \( -2 \leq \lambda \leq -\frac{2 \sqrt{5}}{3} \), which concludes the proof. \( \square \)
Corollary 3.16.

\[ D_{\mathbb{R}, 6} \geq 10.7809. \]

As we did in the previous cases, it would be interesting to know if we can improve our best lower bound on \( D_{\mathbb{R}, m} \) by considering powers of

\[ P_6(x, y) = Q_{1, \lambda_0}(x, y) = x^5 y + \lambda_0 x^3 y^3 + xy^5, \]

with \( \lambda_0 = -2.2654 \).

If \( a_n \) is the vector of the coefficients of \( P_6^n \) for each \( n \in \mathbb{N} \), then we know that

\[ D_{\mathbb{R}, 6n} \geq \left| a_n \right| \frac{12n}{5n+1}. \]

Using (3.6) we can construct a table (see Table 3 of lower estimates for \( D_{\mathbb{R}, 6m} \) that improves the results obtained in tables 1 and 2.

Remark 3.17. Observe that using the estimates in Table 3 we obtain

\[ D_{\mathbb{R}, 420} \geq (1.5828)^{420}, \]

whereas using the polynomial \( P_5 \) as in the construction of Table 2 we would have

\[ D_{\mathbb{R}, 420} \geq (1.5483)^{420}. \]
The last inequality suggests that

$$H_{\infty, R} \geq 1.5828.$$  

Concerning the absolute hypercontractivity constant we have (combining our results with [3]):

**Corollary 3.18.**

$$1.5828 \leq H_{\alpha, R} < 3.296$$

### 3.6. The case \( m = 2^n \) and \( n > 1 \). For \( m = 4 \) let us consider

$$P_4(x_1, ..., x_4) = P_2(x_1, x_2)^2 - P_2(x_3, x_4)^2 = (x_1^2 - x_2^2)^2 - (x_3^2 - x_4^2)^2.$$  

We thus have

$$D_4 \geq \frac{(4 + 2 \times 2^{8/5})^{5/8}}{1} \geq 4.2335 = (1.4344)^4$$

Using induction we consider

$$P_{2^n}(x_1, ..., x_{2^n}) = P_{2^{n-1}}(x_1, ..., x_{2^{n-1}})^2 - P_{2^{n-1}}(x_{2^{n-1}+1}, x_{2^n})^2.$$  

For \( n = 3, 4, 5 \) we were able to compute

$$D_8 \geq (1.5241)^8$$

$$D_{16} \geq (1.59527998)^{16}$$

$$D_{32} \geq (1.65617484)^{32}$$

and these results are quite better than those from the previous sections when \( m = 2^n \). We thus conclude that

$$H_{\alpha, R} \geq 1.65617484$$

and we have numerical evidence that in fact (3.5) can be improved to

$$H_{\infty, R} \geq 1.65617484.$$
The following table summarizes our estimates for special values of $m$:

| $m$ | $D_{\mathbb{R},m}$ |
|-----|-------------------|
| 2   | $1.8374 = (1.3555)^2$ |
| 3   | $2.5525 = (1.3666)^3$ |
| 4   | $4.2335 = (1.4344)^4$ |
| 5   | $6.8359 = (1.4688)^5$ |
| 6   | $10.7809 = (1.4863)^6$ |
| 8   | $29.1209 = (1.5241)^8$ |
| 16  | $1.59527998^{16}$ |
| 32  | $1.65617484^{32}$ |

The above table induces us to believe that it is quite likely that by computing $D_m$ for $m = 2^n$ and big values of $m$ we will substantially improve the value $1.65617484^{4}$.

4. VARIANTS OF THE POLYNOMIAL BOHNENBLUST-HILLE INEQUALITIES

4.1. A result for both complex and real scalars. The optimality of the power \( \frac{2m}{m+1} \) means that for \( r < \frac{2m}{m+1} \) the factor \( D_{\mathbb{C}, m} \) will appear multiplied by a factor depending on \( n \). The next result shows that this factor is precisely \( n \left( \frac{m}{r} - \frac{m+1}{2} \right) \). The power \( \frac{2m}{m+1} \) is the precise value where the dependence on \( n \) disappears:

**Proposition 4.1.** Let \( r \in [1, \frac{2m}{m+1}] \), \( m, n \) be positive integers and \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). There is an universal constant \( L_{\mathbb{K}} > 0 \) such that

\[
\| P \|_r \leq L_{\mathbb{K}}^m \cdot n \left( \frac{m}{r} - \frac{m+1}{2} \right) \| P \|_{\infty}
\]

for all \( m \)-homogeneous polynomials \( P : \ell^n_{\mathbb{K}} \rightarrow \mathbb{K} \). Moreover, the power \( \frac{m}{r} - \frac{m+1}{2} \) is optimal.

**Proof.** Let \( r \in [1, \frac{2m}{m+1}] \). The proof of (4.1) for complex scalars is easily obtained by using the Holder Inequality in combination with the case \( \frac{2m}{m+1} \); we learned this argument from [5]. For real scalars, according to [3], if \( P : \ell^n_{\infty} (\mathbb{R}) \rightarrow \mathbb{R} \) is an \( m \)-homogeneous polynomial, then (as in Proposition 2.2),

\[
\| P_{\mathbb{C}} \|_{\infty} \leq 2^{m-1} \| P \|_{\infty},
\]

where \( P_{\mathbb{C}} \) is the complexification of \( P \). So, we obtain (4.1) for real scalars. It also simple to see that the constant \( L_{\mathbb{K}} \) can be chosen independent of \( m, r, n \). Now let us prove the optimality of the exponent \( \frac{m}{r} - \frac{m+1}{2} \); for this task let us suppose that the result holds for a power \( q < \frac{m}{r} - \frac{m+1}{2} \).

For each \( m, n \), let

\[
P_{m,n} : \ell^n_{\infty} (\mathbb{K}) \rightarrow \mathbb{K}
\]

\[
P_{m,n}(w) = \sum_{|\alpha| = m} \varepsilon_{\alpha} w^{\alpha}
\]
be the $m$-homogeneous Bernoulli polynomial satisfying the Kahane–Salem–Zygmund inequality (note that this inequality is also valid for real scalars, see [14]).

The proof follows the lines of [14, Theorem 10.2]; the essence of this argument can be traced back to Boas’ classical paper [1]. We can suppose $n > m$. As in [14], we have

$$\sum_{|\alpha|=m} |\varepsilon_\alpha|^r = p(n) + \frac{m-1}{m!} \prod_{k=0}^{m-1} (n-k),$$

where $p(n) > 0$ is a polynomial of degree $m - 1$. If (4.1) was valid with the power $q$, then there would exist a constant $C_{q,K} > 0$ so that

$$\left( \sum_{|\alpha|=m} |\varepsilon_\alpha|^r \right)^{1/r} \leq C_{q,K}^m n^q \|P_{m,n}\|_\infty \leq C_{q,K}^m n^q \cdot C_{KSZ} \cdot n^{(m+1)/2} \sqrt{\log m},$$

where $C_{KSZ} > 0$ is the universal constant from the Kahane–Salem–Zygmund inequality. Hence

$$C_{q,K}^m C_{KSZ} \geq \frac{1}{n^q \cdot n^{(m+1)/2} \sqrt{\log m}} \left( p(n) + \frac{m-1}{m!} \prod_{k=0}^{m-1} (n-k) \right)^{1/r}$$

for all $n$. Raising both sides to the power of $r$ and letting $n \to \infty$ we obtain

$$\left( C_{q,K}^m C_{KSZ} \right)^r \geq \lim_{n \to \infty} \left( \frac{p(n)}{n^q r \cdot n^{(m+1)/2} \sqrt{\log m}} + \frac{s(n)}{n^{q r} \cdot n^{(m+1)/2} \sqrt{\log m}} \right),$$

with

$$s(n) = \frac{1}{m!} \prod_{k=0}^{m-1} (n-k).$$

Since $q < \frac{m}{r} - \frac{m+1}{2}$, we have $\deg s = m > qr + r(m+1)/2$ and thus the limit above is infinity, a contradiction. \hfill \Box

### 4.2. Another rupture between the cases of real and complex scalars.

If we replace $\frac{2m}{m+1}$ by $q > \frac{2m}{m+1}$ in the polynomial Bohnenblust–Hille inequality it is natural to investigate if, at some point on, the dependence on the factor depending on $m$ disappears. More precisely, we consider

$$\rho_{K} = \inf \left\{ q; \|P\|_q \leq \|P\|_\infty \text{ for all } m\text{-homogeneous } P : \ell_\infty^n (\mathbb{K}) \to \mathbb{K} \right\}.$$

For $q = 2$ and complex scalars, it is well-known (see [17] and details can be found in the proof of the Theorem 6.1) that

$$\|P\|_2 \leq \|P\|_\infty$$

for all $m$ and $n$, and thus

$$\rho_{C} \leq 2.$$
From a recent result of D. Nuñez-Alarcón (13), we know that for all \( m \) there is an \( m \)-homogeneous polynomial so that
\[
\|P\|_{2,m+1} > \|P\|_{\infty}.
\]
More precisely,
\[
D_{C,m} \geq (2 + 2^m)^{m+1}2^m \sqrt{4 + 2^{m+1}} > 1.
\]
We thus have:

**Proposition 4.2.** \( \rho_C = 2 \).

For real scalars, however, from the previous section it is easy to see that
\[ \rho_R = \infty. \]

5. **Bohnenblust–Hille type inequalities and multilinear forms**

The multilinear Bohnenblust–Hille inequality (1931, [2]) asserts that for every positive integer \( m \geq 2 \) there exists a sequence of positive scalars \( C_{C,m} \geq 1 \) such that
\[
(5.1) \quad \left( \sum_{i_1, \ldots, i_m=1}^{N} |U(e_{i_1}, \ldots, e_{i_m})|^{2m/(m+1)} \right)^{m+1}/2^m \leq C_{C,m} \sup_{z_1, \ldots, z_m \in \mathbb{D}^N} |U(z_1, \ldots, z_m)|
\]
for all \( m \)-linear mapping \( U : \mathbb{C}^N \times \cdots \times \mathbb{C}^N \to \mathbb{C} \) and every positive integer \( N \), where \( (e_i)_{i=1}^{N} \) denotes the canonical basis of \( \mathbb{C}^N \).

The result is also valid for real scalars and \( 2^{m+1} \) is optimal in both real and complex cases. As in the polynomial case, the only differences between the complex and real scalars appear in the best constants involved. For example
\[
C_{C,2} \leq \frac{2}{\sqrt{\pi}} < \sqrt{2} = C_{R,2}.
\]

5.1. **A variant for both complex and real scalars.** The next proposition is the multilinear version of Theorem 4.1

**Proposition 5.1.** Let \( r \in [1, 2m/(m+1)] \), \( m, n \) be positive integers and \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). There is an universal constant \( C_m > 0 \) such that
\[
\|T\|_r \leq C_m \cdot n^{(m \cdot 2^m - m+1)/2} \|T\|_{\infty}
\]
for all \( m \)-linear forms \( T : \ell_\infty^n(\mathbb{K}) \times \cdots \times \ell_\infty^n(\mathbb{K}) \to \mathbb{K} \). Moreover, the power \( m/r - m+1/2 \) is optimal.

**Proof.** Let
\[
\|T\|_r := \left( \sum_{i_1, \ldots, i_m=1}^{n} |T(e_{i_1}, \ldots, e_{i_m})|^r \right)^{1/r}.
\]
Then, for $1 \leq r < \frac{2m}{m+1}$,
\[
\left( \sum_{i_1, \ldots, i_m = 1}^{n} |T(e_{i_1}, \ldots, e_{i_m})|^r \right)^{1/r} \leq \left( \left( \sum_{i_1, \ldots, i_m = 1}^{n} |1|^r \right)^{\frac{2m}{2m-m-r}} \left( \sum_{i_1, \ldots, i_m = 1}^{n} (|T(e_{i_1}, \ldots, e_{i_m})|^r)^{\frac{2m}{m+1}} \right)^{\frac{r(m+1)}{2m}} \right)^{\frac{1}{r}}
\leq (n^m)^{\frac{2m-m-r}{2m}} \left( \sum_{i_1, \ldots, i_m = 1}^{n} |T(e_{i_1}, \ldots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{r(m+1)}{2m}}
= n^{\frac{2m-m-r}{2m}} \left( \sum_{i_1, \ldots, i_m = 1}^{n} |T(e_{i_1}, \ldots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{r(m+1)}{2m}}
\]
and thus
\[
\left( \sum_{i_1, \ldots, i_m = 1}^{n} |T(e_{i_1}, \ldots, e_{i_m})|^r \right)^{1/r} \leq n^{\frac{m}{r} \frac{(m+1)}{2}} C_m \|T\|.
\]
Let $P : \ell_\infty^n(\mathbb{K}) \to \mathbb{K}$,
\[
P(z) = \sum_{|\alpha| = m} a_\alpha z^\alpha,
\]
be an $m$-homogeneous polynomial and $T$ be its polar. Then
\[
\sum_{|\alpha| = m} |a_\alpha|^r = \sum_{|\alpha| = m} \left( \frac{m}{\alpha} \right)^r |T(e_1^{\alpha_1}, \ldots, e_n^{\alpha_n})|^r.
\]
However, for every choice of $\alpha$, the term
\[
|T(e_1^{\alpha_1}, \ldots, e_n^{\alpha_n})|^r
\]
is repeated $\binom{m}{\alpha}$ times in the sum
\[
\sum_{i_1, \ldots, i_m = 1}^{n} |T(e_{i_1}, \ldots, e_{i_m})|^r.
\]
Thus
\[
\sum_{|\alpha| = m} \left( \frac{m}{\alpha} \right)^r |T(e_1^{\alpha_1}, \ldots, e_n^{\alpha_n})|^r = \sum_{i_1, \ldots, i_m = 1}^{n} \left( \frac{m}{\alpha} \right)^r \frac{1}{\binom{m}{\alpha}} |T(e_{i_1}, \ldots, e_{i_m})|^r
\]
and, since
\[
\binom{m}{\alpha} \leq m!
\]
we have
\[
\sum_{|\alpha| = m} \left( \frac{m}{\alpha} \right)^r |T(e_1^{\alpha_1}, \ldots, e_n^{\alpha_n})|^r \leq (m!)^{r-1} \sum_{i_1, \ldots, i_m = 1}^{n} |T(e_{i_1}, \ldots, e_{i_m})|^r.
\]
We finally obtain
\[
\left( \sum_{|\alpha| = m} |a_{\alpha}|^r \right)^{1/r} \leq \left( (m!)^{r-1} \sum_{i_1, \ldots, i_m = 1}^n |T(e_{i_1}, \ldots, e_{i_m})|^r \right)^{1/r} \\
\leq (m!)^{r-1} \frac{m}{n^r} \frac{(m+1)}{2} C_m \|T\|.
\]
On the other hand, it is well-known that
\[
\|T\| \leq \frac{m^m}{m!} \|P\|
\]
and hence
\[
\left( \sum_{|\alpha| = m} |a_{\alpha}|^r \right)^{1/r} \leq (m!)^{r-1} \frac{m}{n^r} \frac{(m+1)}{2} C_m \frac{m^m}{m!} \|P\| \\
= C_m \frac{m^m}{(m!)^{1/r}} \frac{m}{n} \frac{(m+1)}{2} \|P\| \\
= C_m \frac{m^m}{(m!)^{1/r}} \frac{m}{n} \frac{(m+1)}{2} \|P\|.
\]
Now the Kahane–Salem–Zygmund as in the proof of Theorem 4.1 asserts that \( \frac{m}{r} - \frac{(m+1)}{2} \) is optimal.

\[\square\]

5.2. Other parameters. Now let us define
\[
\mu_K = \inf \left\{ q; \|T\|_q \leq \|T\|_\infty \ \text{for all} \ m\text{-linear form} \ T : \ell^m_\infty (K) \times \cdots \times \ell^m_\infty (K) \to K \right\}.
\]
For both real and complex scalars it is well-known that
\[
\|T\|_2 \leq \|T\|_\infty
\]
for all \( T : \ell^m_\infty (K) \times \cdots \times \ell^m_\infty (K) \to K \). Hence
\[
\mu_K \leq 2.
\]
However no more information is available for the complex case. In fact no nontrivial lower bounds for the constants of the complex multilinear Bohnenblust–Hille are known. All we know is that
\[
C_{\mathbb{C},m} \geq 1.
\]
For real scalars we have:

**Proposition 5.2.** \( \mu_R = 2. \)
Proof. Let $q < 2$. As in (9), considering $T_2 : \ell_\infty^n (\mathbb{R}) \times \ell_\infty^n (\mathbb{R}) \to \mathbb{R}$ given by

$$T_2(x, y) = x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2$$

we have

$$\|T_2\| = 2$$

and if

$$\left( \sum_{i_1, i_2 = 1}^2 |T_2(e_{i_1}, e_{i_2})|^q \right)^{\frac{1}{q}} \leq C_2 \|T_2\|$$

we get

$$C_2 \geq 2^{\frac{2}{q} - 1} > 1.$$ 

Following the same induction process from (9), we have

$$(4^{m-1})^{\frac{1}{q}} \leq C_m 2^{m-1}$$

and thus

$$C_m \geq 2^{\frac{m(2-q) - (2-q)}{q}} > 1.$$ 

□

6. Contractivity in finite dimensions

In the present section we will prove that the complex polynomial Bohnenblust-Hille constants for polynomials on $\mathbb{C}^n$, with $n \in \mathbb{N}$ fixed, are contractive, and not hypercontractive as it happens for real polynomials on $\mathbb{R}^n$ (Corollary 3.12). Therefore, if we want to prove the sharpness of the hypercontractivity of the complex polynomial Bohnenblust-Hille constants we have to search for polynomials in growing number of variables.

**Theorem 6.1.** For all $n \geq 2$ the complex polynomial Bohnenblust–Hille inequality is contractive in $P(\ell_\infty^n)$. More precisely, for all fixed $n \in \mathbb{N}$, there are constants $D_m$, with $\lim_{m \to \infty} D_m = 1$, so that

$$|P|_{\frac{2m}{m+1}} \leq D_m \|P\|$$

for all $P \in P(\ell_\infty^n)$.

**Proof.** Let $P(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha$ and $f(t) = P(e^{it_1}, \ldots, e^{it_n}) = \sum_{|\alpha|=m} c_\alpha e^{i\alpha t}$, where $t = (t_1, \ldots, t_n) \in \mathbb{R}^n \alpha \in (\mathbb{N} \cup \{0\})^n$ and $\alpha t = \alpha_1 t_1 + \cdots + \alpha_n t_n$. Observe that if $\|f\|$ denotes the sup norm of $f$ on $[-\pi, \pi]$, by the Maximum Modulus Principle $\|f\| = \|P\|$. Also, due to the orthogonality of the system $\{e^{iks} : n \in \mathbb{Z}\}$ in $L^2([-\pi, \pi])$ we have

$$\|P\|^2 = \|f\|^2 \geq \frac{1}{2\pi} \int_{-\pi}^\pi |f(t)|^2 dt = \sum_{|\alpha|=m} |c_\alpha|^2 = |P|_2^2,$$

from which $|P|_2 \leq \|P\|$. On the other hand it is well known that in $\mathbb{R}^d$ we have

$$(6.1) \quad |\cdot|_q \leq |\cdot|_p \leq d^{\frac{1}{p} - \frac{1}{q}} |\cdot|_q,$$
for all $1 \leq p \leq q$. Since the dimension of $\mathcal{P}^{m\ell_n}_{\infty}$ is $\binom{m+n-1}{n-1}$, the result follows from $|P|_2 \leq \|P\|$ by setting in (6.1) $p = \frac{2m}{m+1}$, $q = 2$ and $d = \binom{m+n-1}{n-1}$. So $D_m = \binom{m+n-1}{n-1}\frac{1}{2m}$ and since

$$\lim_{m \to \infty} \left( \frac{m+n-1}{n-1} \right)^{\frac{1}{2m}} = 1$$

the proof is done. $\square$

7. Comparative tables

In this final section we organize the results of the previous sections in comparative tables below (for, respectively, polynomials and multilinear forms). We also add recent results for the constants of the polynomial Bohnenblust–Hille inequality and for the constants of the multilinear Bohnenblust–Hille inequality (see [14]).

Polynomials:

| $\mathbb{K}$ | $\mathbb{R}$ | $\mathbb{C}$ |
|--------------|--------------|--------------|
| Optimal exponent | $\frac{2m}{m+1}$ | $\frac{2m}{m+1}$ |
| Optimal extra factor for $r \in [1, \frac{2m}{m+1}]$ | $n\left(\frac{m}{r} - \frac{m+1}{2}\right)$ | $n\left(\frac{m}{r} - \frac{m+1}{2}\right)$ |
| Hypercontractivity | YES | YES |
| Optimality of the hypercontractivity | YES | ? |
| Contractivity in $\mathbb{K}^n$ | NO | YES |
| $H_{\infty, \mathbb{K}}$ | $\in [1.50, 2.83]$ | $\leq \sqrt{2}$ |
| $p_{\mathbb{K}}$ | $\infty$ | $2$ |

Multilinear forms:

| $\mathbb{K}$ | $\mathbb{R}$ | $\mathbb{C}$ |
|--------------|--------------|--------------|
| Optimal exponent | $\frac{2m}{m+1}$ | $\frac{2m}{m+1}$ |
| Optimal extra factor for $r \in [1, \frac{2m}{m+1}]$ | $n\left(\frac{m}{r} - \frac{m+1}{2}\right)$ | $n\left(\frac{m}{r} - \frac{m+1}{2}\right)$ |
| $\mu_{\mathbb{K}}$ | 2 | $\leq 2$ |
| $C_{\mathbb{K}, m}$ | $< 1.65 \left(n - 1\right)^{0.34} + 0.13$ | $< 1.41 \left(n - 1\right)^{0.34} - 0.04$ |
| $C_{\mathbb{K}, 2}$ | $\geq 2^{\frac{1}{m} - \frac{1}{2}}$ | $\geq 1$ |

References

[1] H. P. Boas, *The football player and the infinite series*, Notices Amer. Math. Soc. **44** (1997), no. 11, 1430–1435.

[2] H. F. Bohnenblust and E. Hille, *On the absolute convergence of Dirichlet series*, Ann. of Math. (2) **32** (1931), no. 3, 600–622.
ON THE HYPERCONTRACTIVITY OF THE COMPLEX BOHNENBLUST–HILLE INEQUALITY

3] J. R. Campos, G. A. Muñoz-Fernández, D. Pellegrino, and J. B. Seoane-Sepúlveda, The Bohnenblust-Hille inequality for real homogeneous polynomials is hypercontractive and this result is optimal, Preprint, 2012.

[4] Y. S. Choi and S. G. Kim, The unit ball of $P(\mathbb{T}_2)$, Arch. Math. (Basel) 71 (1998), no. 6, 472–480.

[5] A. Defant and L. Frerick, Hypercontractivity of the Bohnenblust-Hille inequality for polynomials and multidimensional Bohr radii, arXiv:0903.3395.

[6] A. Defant, L. Frerick, J. Ortega-Cerdà, M. Ounaïs, and K. Seip, The Bohnenblust-Hille inequality for homogeneous polynomials is hypercontractive. Ann. of Math. (2) 174 (2011), no. 1, 485–497.

[7] A. Defant and P. Sevilla-Peris, A new multilinear insight on Littlewood’s $4/3$-inequality, J. Funct. Anal. 256 (2009), no. 5, 1642–1664.

[8] D. Diniz, G. A. Muñoz-Fernández, D. Pellegrino, and J. B. Seoane-Sepúlveda, The asymptotic growth of the constants in the Bohnenblust–Hille inequality is optimal, J. Funct. Anal. 263 (2012), 415–428.

[9] D. Diniz, G. A. Muñoz-Fernández, D. Pellegrino, and J. B. Seoane-Sepúlveda, Lower bounds for the constants in the Bohnenblust–Hille inequality: the case of real scalars, Proc. Amer. Math. Soc., accepted for publication.

[10] B. C. Grecu, G. A. Muñoz-Fernández, and J. B. Seoane-Sepúlveda, Unconditional constants and polynomial inequalities, J. Approx. Theory 161 (2009), no. 2, 706–722.

[11] A. Montanaro, Some applications of hypercontractive inequalities in quantum information theory, J. Math. Physics, accepted for publication.

[12] G. A. Muñoz-Fernández, D. Pellegrino, and J. B. Seoane-Sepúlveda, Estimates for the asymptotic behavior of the constants in the Bohnenblust–Hille inequality, Linear Multilinear Algebra 60 (2012), no. 5, 573–582.

[13] D. Nuñez-Alarcón, A note on the polynomial Bohnenblust–Hille inequality, arXiv:1208.6238.

[14] D. Nuñez-Alarcón, D. Pellegrino, J. B. Seoane-Sepúlveda, and D. M. Serrano-Rodríguez, There exist multilinear Bohnenblust–Hille constants $(C_n)_{n=1}^\infty$ with $\lim_{n \to \infty} (C_{n+1} - C_n) = 0$, J. Funct. Anal. 264 (2013), no. 2, 429–463.

[15] D. Nuñez-Alarcón, D. Pellegrino, and J. B. Seoane-Sepúlveda, On the Bohnenblust–Hille inequality and a variant of Littlewood’s $4/3$ inequality, J. Funct. Anal. 264 (2013), 326–336.

[16] D. Pellegrino and J. B. Seoane-Sepúlveda, New upper bounds for the constants in the Bohnenblust–Hille inequality, J. Math. Anal. Appl. 386 (2012), no. 1, 300–307.

[17] Seip K., Estimates for Dirichlet polynomials, EMS Lecturer, CRM (20), http://www.euro-math-soc.eu/system/files/Seip_CRM.pdf.

[18] C. Visser, A generalization of Tchebychef’s inequality to polynomials in more than one variable, Nederl. Akad. Wetensch., Proc. 49 (1946), 455–456 = Indagationes Math. 8, 310–311 (1946).

Departamento de Ciências Exatas,
Universidade Federal da Paraíba,
58.297-000 - Rio Tinto, Brazil.
E-mail address: jamilson@dce.ufpb.br and jamilsonrc@gmail.com

Departamento de Análisis Matemático,
Facultad de Ciencias Matemáticas,
Plaza de Ciencias 3,
Universidad Complutense de Madrid,
Madrid, 28040, Spain.
E-mail address: gustavo_fernandez@mat.ucm.es

Departamento de Matemática,
Universidade Federal da Paraíba,
58.051-900 - João Pessoa, Brazil.
E-mail address: pellegrino@pq.cnpq.br and dmpellegrino@gmail.com
Departamento de Análisis Matemático,
Facultad de Ciencias Matemáticas,
Plaza de Ciencias 3,
Universidad Complutense de Madrid,
Madrid, 28040, Spain.
E-mail address: jseoane@mat.ucm.es