Spectral theory for Maxwell’s equations at the interface of a metamaterial. Part I: Generalized Fourier transform

Maxence Cassier\textsuperscript{a,b}, Christophe Hazard\textsuperscript{b}, and Patrick Joly\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, University of Utah, Salt Lake City, UT, USA; \textsuperscript{b}POEMS\textsuperscript{1} Laboratory, Ecole Nationale Supérieure de Techniques Avancées, Palaiseau, France

\section*{ABSTRACT}

We explore the spectral properties of the time-dependent Maxwell’s equations for a plane interface between a metamaterial represented by the Drude model and the vacuum, which fill, respectively, complementary half-spaces. We construct explicitly a generalized Fourier transform which diagonalizes the Hamiltonian that describes the propagation of transverse electric waves. This transform appears as an operator of decomposition on a family of generalized eigenfunctions of the problem. It will be used in a forthcoming paper to prove both limiting absorption and limiting amplitude principles.

\section*{1. Introduction}

In the last years, metamaterials have generated a huge interest among communities of physicists and mathematicians, owing to their extraordinary properties such as negative refraction [35], allowing the design of spectacular devices like the perfect lens [26] or the cylindrical cloak in [25]. Such properties result from the possibility of creating artificially microscopic structures whose macroscopic electromagnetic behavior amounts to negative electric permittivity \(\varepsilon\) and/or negative magnetic permeability \(\mu\) within some frequency range. Such a phenomenon can also be observed in metals in the optical frequency range [20, 22]: in this case one says that this material is a negative material [16]. Thanks to these negative electromagnetic coefficients, waves can propagate at the interface between such a negative material and a usual dielectric material [15]. These waves, often called surface plasmon polaritons, are localized near the interface and allow then to propagate signals in the same way as in an optical fiber, which may lead to numerous physical applications. Mathematicians have so far little explored these negative materials and most studies in this context are devoted to the frequency domain, that is, propagation of time-harmonic waves [3, 4, 8, 24]. In particular, it is now well understood that in the case of a smooth interface between a dielectric and a negative material (both assumed nondissipative), the time-harmonic Maxwell's equations become ill-posed if both ratios of \(\varepsilon\) and \(\mu\) across the interface are equal to \(-1\), which is precisely the conditions required for the perfect lens in [26]. This result raises a fundamental issue which can be seen as the starting point of the present paper.
Indeed, for numerous scattering problems, a time-harmonic wave represents the large time asymptotic behavior of a time-dependent wave resulting from a time-harmonic excitation which is switched on at an initial time. Such a property is referred to as the \textit{limiting amplitude principle} in the context of scattering theory. It has been proved for a large class of physical problems in acoustics, electromagnetism, or elastodynamics [13, 14, 23, 27, 30, 31]. But what can be said about the large time behavior of the time-dependent wave if the frequency of the excitation is such that the time-harmonic problem becomes ill-posed, that is, precisely in the situation described above? What is the effect of the surface plasmons on the large time behavior? Our aim is to give a precise answer to these questions in an elementary situation.

To reach this goal, several approaches are possible. The one we adopt here, which is based on spectral theory, has its own interest because it provides us a very powerful tool to represent time-dependent waves and study their behavior, not only for large time asymptotics. Our aim is to make the spectral analysis of a simple model of interface between a negative material and the vacuum, more precisely to construct a \textit{generalized Fourier transform} for this model, which extends the concept of Fourier transform to inhomogeneous media and is a powerful tool for time–frequency analysis. Indeed, this transform amounts to a \textit{generalized eigenfunction expansion} of any possible state of the system, which yields a representation of time-dependent waves as superpositions of time-harmonic waves. From a mathematical point of view, this transform offers a \textit{diagonal} form of the operator that describes the dynamics of the system. The existence of such a transform is ensured in a very general context [2, 19], but its practical construction highly depends on the considered model.

The situation studied here consists in the basic case of a plane interface between the vacuum and a negative material filling, respectively, two half-spaces. Our negative material is described by a nondissipative Drude model, which is the simplest model of negative material. The technique we use to construct the generalized Fourier transform is inspired by previous studies in the context of stratified media [9, 18, 37, 39]. Compared to these studies, the difficulty of the present work relies in the fact that in the Drude material, $\varepsilon$ and $\mu$ depend on the frequency and become negative for low frequencies. For the sake of simplicity, instead of considering the complete three-dimensional physical problem, we restrict ourselves to the so-called transverse electric (TE) two-dimensional problem, i.e., when the electric field is orthogonal to the plane of propagation. The transverse magnetic (TM) case can be studied similarly. As shown in [17, 37], in stratified media, the spectral theory of the three-dimensional problem follows from both TE and TM cases but is not dealt with here.

The present paper is devoted to the construction of the generalized Fourier transform of the TE Maxwell's equations. It will be used in a forthcoming paper [7] to study the validity of a limiting amplitude principle in our medium, but the results we obtain in the present paper are not limited to this purpose. The generalized Fourier transform is also the main tool to study scattering problems as in [10, 19, 37–39], numerical methods in stratified media as in [18] and has many other applications. Let us mention that both present and forthcoming papers are an advanced version of the preliminary study presented in [6].

The paper is organized as follows. In Section 2, we introduce the above mentioned plane interface problem between a Drude material and the vacuum, more precisely the TE Maxwell's equations. These equations are formulated as a conservative Schrödinger equation in a Hilbert space, which involves a self-adjoint Hamiltonian. We briefly recall some basic notions of spectral theory which are used throughout the paper. Section 3 is the core of our study: we take advantage of the invariance of our medium in the direction of the interface to
reduce the spectral analysis of our Hamiltonian to the analysis of a family of one-dimensional reduced Hamiltonians. We diagonalize each of them by constructing an adapted generalized Fourier transform. We finally bring together in Section 4 this family of results to construct a generalized Fourier transform for our initial Hamiltonian and conclude by a spectral representation of the solution to our Schrödinger equation.

2. Model and method

2.1. The Drude model

We consider a metamaterial filling the half-space \( \mathbb{R}^3_+ := \{ x := (x, y, z) \in \mathbb{R}^3 \mid x > 0 \} \) and whose behavior is described by a Drude model \([21]\) recalled below. The complementary half-space \( \mathbb{R}^3_- := \mathbb{R}_- \times \mathbb{R}^2 \) is composed of vacuum (Figure 1). The triplet \((\varepsilon_x, \varepsilon_y, \varepsilon_z)\) stands for the canonical basis of \( \mathbb{R}^3 \).

We denote, respectively, by \( D \) and \( B \) the electric and magnetic inductions, by \( E \) and \( H \) the electric and magnetic fields. We assume that in the presence of a source current density \( J_s \), the evolution of \((E, D, H, B)\) in the whole space is governed by the macroscopic Maxwell’s equations (in the following, the notation \( \text{Curl} \) refers to the usual 3D curl operator)

\[
\partial_t D - \text{Curl} \, H = -J_s \quad \text{and} \quad \partial_t B + \text{Curl} \, E = 0,
\]

which must be supplemented by the constitutive laws of the material

\[
D = \varepsilon_0 E + P \quad \text{and} \quad B = \mu_0 H + M
\]

involving two additional unknowns, the electric and magnetic polarizations \( P \) and \( M \). The positive constants \( \varepsilon_0 \) and \( \mu_0 \) stand, respectively, for the permittivity and the permeability of the vacuum.

In the vacuum, \( P = M = 0 \) so that Maxwell’s equations become

\[
\varepsilon_0 \partial_t E - \text{Curl} \, H = -J_s \quad \text{and} \quad \mu_0 \partial_t H + \text{Curl} \, E = 0 \quad \text{in} \mathbb{R}^3_-. \tag{1}
\]

On the other hand, for a homogeneous nondissipative Drude material, the fields \( P \) and \( M \) are related to \( E \) and \( H \) through

\[
\partial_t P = J, \quad \partial_t J = \varepsilon_0 \Omega_e^2 E \quad \text{and} \quad \partial_t M = K, \quad \partial_t K = \mu_0 \Omega_m^2 H,
\]

where the two unknowns \( J \) and \( K \) are called usually the induced electric and magnetic currents. Both parameters \( \Omega_e \) and \( \Omega_m \) are positive constants which characterize the behavior

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**Figure 1.** Description of the transmission problem.
of a Drude material. We can eliminate $D$, $B$, $P$, and $M$, which yield the time-dependent Maxwell equations in a Drude material:

$$\begin{align*}
\{ & \varepsilon_0 \partial_t E - \text{Curl} \mathbf{H} + \mathbf{J} = - \mathbf{J}_s, & \partial_t \mathbf{J} = \varepsilon_0 \Omega_c^2 \mathbf{E} & \text{in } \mathbb{R}^3_+, \\
& \mu_0 \partial_t \mathbf{H} + \text{Curl} \mathbf{E} + \mathbf{K} = 0, & \partial_t \mathbf{K} = \mu_0 \Omega_m^2 \mathbf{H} & \text{in } \mathbb{R}^3_-. \end{align*} \tag{2}$$

The above equations in $\mathbb{R}^3_-$ and $\mathbb{R}^3_+$ must be supplemented by the usual transmission conditions

$$[\mathbf{e}_x \times \mathbf{E}]_{x=0} = 0 \quad \text{and} \quad [\mathbf{e}_x \times \mathbf{H}]_{x=0} = 0, \tag{3}$$

which express the continuity of the tangential electric and magnetic fields through the interface $x = 0$ (the notation $[f]_{x=0}$ designates the gap of a quantity $f$ across $x = 0$, i.e., $\lim_{x \to 0^+} f(+) - f(-)$), where the limits must be understood in the trace sense, in the framework of the usual Sobolev spaces adapted to Maxwell’s equations).

The formulation (2) of Maxwell’s equations is local in time. By eliminating the induced currents $\mathbf{J}$ and $\mathbf{K}$, one can obtain an equivalent formulation which is nonlocal in time: the evolution of $\mathbf{E}$ and $\mathbf{H}$ at a given time $t$ depends on their history, i.e., their values at all the preceding instants. The equivalence between both formulations tells us that the knowledge of this history amounts to the knowledge of $\mathbf{E}$, $\mathbf{H}$, $\mathbf{J}$, and $\mathbf{K}$ at a given initial time.

When looking for time-harmonic solutions to these equations for a given (circular) frequency $\omega \in \mathbb{R}$, i.e.,

$$(\mathbf{E}(\mathbf{x}, t), \mathbf{H}(\mathbf{x}, t), \mathbf{J}(\mathbf{x}, t), \mathbf{K}(\mathbf{x}, t)) = (\mathbf{E}_\omega(\mathbf{x}), \mathbf{H}_\omega(\mathbf{x}), \mathbf{J}_\omega(\mathbf{x}), \mathbf{K}_\omega(\mathbf{x})) \ e^{-i\omega t},$$

for a periodic current density $\mathbf{J}_s(\mathbf{x}, t) = \mathbf{J}_{s,\omega}(\mathbf{x}) e^{-i\omega t}$, we can eliminate $\mathbf{J}_s$ and $\mathbf{K}_s$ and obtain the following time-harmonic Maxwell equations:

$$i\omega \varepsilon_\omega(\mathbf{x}) \mathbf{E}_\omega + \text{Curl} \mathbf{H}_\omega = \mathbf{J}_{s,\omega} \quad \text{and} \quad -i\omega \mu_\omega(\mathbf{x}) \mathbf{H}_\omega + \text{Curl} \mathbf{E}_\omega = 0 \quad \text{in } \mathbb{R}^3,$$

where $\varepsilon_\omega(\mathbf{x}) = \varepsilon_0$ and $\mu_\omega(\mathbf{x}) = \mu_0$ if $\mathbf{x} \in \mathbb{R}^3_-$, whereas

$$\varepsilon_\omega(\mathbf{x}) = \varepsilon_\omega^+ := \varepsilon_0 \left( 1 - \frac{\Omega_c^2}{\omega^2} \right) \quad \text{and} \quad \mu_\omega(\mathbf{x}) = \mu_\omega^+ := \mu_0 \left( 1 - \frac{\Omega_m^2}{\omega^2} \right) \quad \text{if } \mathbf{x} \in \mathbb{R}^3_+. \tag{4}$$

Functions $\varepsilon_\omega^+$ and $\mu_\omega^+$ define the frequency-dependent electric permittivity and magnetic permeability of a Drude material (Figure 2). Several observations can be made. First notice that one recovers the permittivity and the permeability of the vacuum if $\Omega_c = \Omega_m = 0$. Then, a Drude material behaves like the vacuum for high frequencies (since $\lim_{|\omega| \to \infty} \varepsilon_\omega^+ = \varepsilon_0$ and $\lim_{|\omega| \to \infty} \mu_\omega^+ = \mu_0$), whereas for low frequencies, it becomes a negative material in the sense that

$$\varepsilon_\omega^+ < 0 \quad \text{for } |\omega| \in (0, \Omega_c) \quad \text{and} \quad \mu_\omega^+ < 0 \quad \text{for } |\omega| \in (0, \Omega_m).$$

Note that if $\Omega_c \neq \Omega_m$, there is a frequency gap $(\min(\Omega_c, \Omega_m), \max(\Omega_c, \Omega_m))$ of width $|\Omega_c - \Omega_m|$ where $\varepsilon_\omega^+$ and $\mu_\omega^+$ have opposite signs. At these frequencies, waves cannot propagate through the material: they are necessarily evanescent. It is precisely what happens in metals at optical frequencies [20]. Finally, there exists a unique frequency for which the relative permittivity $\varepsilon_\omega^+/\varepsilon_0$ (respectively, the relative permeability $\mu_\omega^+/$) is equal to $-1$:

$$\frac{\varepsilon_\omega^+}{\varepsilon_0} = -1 \quad \text{if } |\omega| = \frac{\Omega_c}{\sqrt{2}} \quad \text{and} \quad \frac{\mu_\omega^+}{\mu_0} = -1 \quad \text{if } |\omega| = \frac{\Omega_m}{\sqrt{2}}.$$
Note that both ratios can be simultaneously equal to $-1$ at the same frequency if and only if $\Omega_e = \Omega_m$.

Remark 1. In the physical literature, the Drude model (4) consists in a simple but useful approximation of a metamaterial’s behavior [26, 35]. But one can find more intricate models to express the frequency dependency of $\varepsilon_\omega$ and $\mu_\omega$ in the time-harmonic Maxwell’s equations, for instance, the Lorentz model [15, 16]:

$$
\varepsilon_\omega = \varepsilon_0 \left(1 - \frac{\Omega_e^2}{\omega^2 - \Omega_e^2}\right)
$$

and

$$
\mu_\omega = \mu_0 \left(1 - \frac{\Omega_m^2}{\omega^2 - \Omega_m^2}\right)
$$

where $\omega_e$ and $\omega_m$ are non-negative parameters. For generalized Lorentz materials [34], functions $\varepsilon_\omega$ and $\mu_\omega$ are defined by finite sums of similar terms for various poles $\omega_e$ and $\omega_m$.

2.2. Two-dimensional transmission problem

As mentioned in Section 1, in this paper, we restrict ourselves to the study of the so-called transverse electric equations which result from Eqs. (1)–(3) by assuming that $J_s(x,y,z,t) = J_s(x,y,t) e_z$ and searching for solutions independent of $z$ in the form

$$
\mathbb{E}(x,y,z,t) = E(x,y,t) e_z \quad \text{and} \quad \mathbb{H}(x,y,z,t) = \left(H_x(x,y,t), H_y(x,y,t), 0\right)^T,
$$

$$
\mathbb{J}(x,y,z,t) = J(x,y,t) e_z \quad \text{and} \quad \mathbb{K}(x,y,z,t) = \left(K_x(x,y,t), K_y(x,y,t), 0\right)^T.
$$

Setting $\mathbf{H} := (H_x, H_y)^T$ and $\mathbf{K} := (K_x, K_y)^T$, we obtain a two-dimensional problem for the unknowns $(E, \mathbf{H}, J, \mathbf{K})$, which will be written in the following concise form:

$$
\begin{align*}
\varepsilon_0 \partial_t E - \text{curl} \mathbf{H} + \Pi J &= -J_s \quad \text{in } \mathbb{R}^2, \\
\mu_0 \partial_t \mathbf{H} + \text{curl} E + \Pi \mathbf{K} &= 0 \quad \text{in } \mathbb{R}^2, \\
\partial_t J &= \varepsilon_0 \Omega_e^2 R E \quad \text{in } \mathbb{R}_+^2, \\
\partial_t \mathbf{K} &= \mu_0 \Omega_m^2 R \mathbf{H} \quad \text{in } \mathbb{R}_+^2.
\end{align*}
$$

Figure 2. Relative permeability $\mu_\omega^+ / \mu_0$ as a function of the frequency $\omega$. 
where we have used the 2D curl operators of scalar and vector fields, respectively:

\[
\text{curl } u := (\partial_y u, -\partial_x u)^\top \quad \text{and} \quad \text{curl } \mathbf{u} := \partial_x u_y - \partial_y u_x \text{ where } \mathbf{u} = (u_x, u_y)^\top.
\]

Moreover, \(\Pi\) (respectively, \(\Pi^\top\)) denotes the extension by 0 of a scalar function (respectively, a 2D vectorial field) defined on \(\mathbb{R}^2_+\) to the whole space \(\mathbb{R}^2\), whereas R (respectively, \(\mathbf{R}\)) stands for the restriction to \(\mathbb{R}^2_+\) of a function defined on the whole plane \(\mathbb{R}^2\). Note that in (5) where equations are understood in the sense of distributions, we assume implicitly that the two-dimensional version of the transmission conditions (3) are satisfied, namely,

\[
[E]_{x=0} = 0 \quad \text{and} \quad [H_y]_{x=0} = 0.
\]

The theoretical study of (5) is based on a reformulation of this system as a Schrödinger equation

\[
\frac{d\mathbf{U}}{dt} + i\mathbb{A}\mathbf{U} = \mathbf{G},
\]

where the Hamiltonian \(\mathbb{A}\) is an unbounded operator on the Hilbert space

\[
\mathcal{H}_{2\mathbb{D}} := L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2_+) \times L^2(\mathbb{R}^2)^2.
\]

We assume that this space is equipped with the inner product defined for all \(\mathbf{U} := (\mathbf{E}, \mathbf{H}, J, \mathbf{K})^\top\) and \(\mathbf{U}' := (\mathbf{E}', \mathbf{H}', J', \mathbf{K}')^\top\) by

\[
(\mathbf{U}, \mathbf{U}')_{2\mathbb{D}} := \varepsilon_0 (\mathbf{E}, \mathbf{E}')_{\mathbb{R}^2} + \mu_0 (\mathbf{H}, \mathbf{H}')_{\mathbb{R}^2} + \varepsilon_0^{-1}\Omega_c^{-2} (J, J')_{\mathbb{R}^2_+} + \mu_0^{-1}\Omega_m^{-2} (\mathbf{K}, \mathbf{K}')_{\mathbb{R}^2_+},
\]

where \((u, v)_{\mathcal{O}} := \int_{\mathcal{O}} u \cdot \overline{v} \, dx \, dy\) denotes the usual \(L^2\) inner product, with \(\mathcal{O} = \mathbb{R}^2\) or \(\mathbb{R}^2_+\). We easily verify that (5) writes as the Schrödinger equation (7) with \(\mathbf{G} := (-\varepsilon_0^{-1} J_t, 0, 0, 0)^\top\) if we choose for \(\mathbb{A}\) the operator defined by

\[
\mathbb{A}\mathbf{U} := \mathcal{A}\mathbf{U} \quad \forall \mathbf{U} \in D(\mathcal{A}) := H^1(\mathbb{R}^2) \times \mathbf{H}_{\text{curl}}(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2_+) \times L^2(\mathbb{R}^2)^2 \subset \mathcal{H}_{2\mathbb{D}},
\]

where \(\mathbf{H}_{\text{curl}}(\mathbb{R}^2) := \{\mathbf{u} \in L^2(\mathbb{R}^2)^2 \mid \text{curl } \mathbf{u} \in L^2(\mathbb{R}^2)\}\) and \(\mathcal{A}\) is the following matrix differential operator (all derivatives are understood in the distributional sense):

\[
\mathcal{A} := i \begin{pmatrix}
0 & \varepsilon_0^{-1} \text{curl} & -\varepsilon_0^{-1} \Pi & 0 \\
-\mu_0^{-1} \text{curl} & 0 & 0 & -\mu_0^{-1} \Pi \\
\varepsilon_0 \Omega_c^2 \mathbf{R} & 0 & 0 & 0 \\
\mu_0 \Omega_m^2 \mathbf{R} & 0 & 0 & 0
\end{pmatrix}.
\]

Note that the transmission conditions (6) are satisfied as soon as \((\mathbf{E}, \mathbf{H}) \in H^1(\mathbb{R}^2) \times \mathbf{H}_{\text{curl}}(\mathbb{R}^2)\).

**Proposition 2.** The operator \(\mathbb{A} : D(\mathbb{A}) \subset \mathcal{H}_{2\mathbb{D}} \mapsto \mathcal{H}_{2\mathbb{D}}\) is self-adjoint.

**Proof.** We first check that \(\mathbb{A}\) is symmetric. From our choice (9) of an inner product, we see that for all \(\mathbf{U} := (\mathbf{E}, \mathbf{H}, J, \mathbf{K})^\top\) and \(\mathbf{U}' := (\mathbf{E}', \mathbf{H}', J', \mathbf{K}')^\top\) in \(D(\mathbb{A})\), we have

\[
(\mathbb{A}\mathbf{U}, \mathbf{U}')_{2\mathbb{D}} = i \left\{ \varepsilon_0 (\varepsilon_0^{-1} \text{curl } \mathbf{H} - \varepsilon_0^{-1} \Pi J, E')_{\mathbb{R}^2} + \mu_0 \left( -\mu_0^{-1} \text{curl } E - \mu_0^{-1} \Pi \mathbf{K}, \mathbf{H}' \right)_{\mathbb{R}^2} \\
+ \varepsilon_0^{-1}\Omega_c^{-2} \left( \varepsilon_0 \Omega_c^2 \mathbf{R} E, J' \right)_{\mathbb{R}^2_+} + \mu_0^{-1}\Omega_m^{-2} \left( \mu_0 \Omega_m^2 \mathbf{R} \mathbf{H}, \mathbf{K}' \right)_{\mathbb{R}^2_+} \right\}
\]

\[
= i \left\{ \left( \text{curl } \mathbf{H}, E' \right)_{\mathbb{R}^2} - \left( \text{curl } E, \mathbf{H}' \right)_{\mathbb{R}^2} + \left( \mathbf{R} E, J' \right)_{\mathbb{R}^2_+} - \left( \Pi J, E' \right)_{\mathbb{R}^2} \\
+ \left( \mathbf{R} \mathbf{H}, \mathbf{K}' \right)_{\mathbb{R}^2_+} - \left( \Pi \mathbf{K}, \mathbf{H}' \right)_{\mathbb{R}^2} \right\}.
\]
Using the fact that the operators of each of the pairs \((\text{curl}, \text{curl}), (\text{R}, \text{R}), \text{Pi}_1, \text{Pi}_1)\) are adjoint to each other, we deduce the symmetry property \((\hat{A} U, U')_{2D} = (U, \hat{A} U')_{2D}\).

Besides, the domain of the adjoint of \(\hat{A}\) is defined by
\[
D(\hat{A}^*) := \{U \in H_{2D} | \exists V \in H_{2D}, \forall U' \in D(\hat{A}), (U, \hat{A} U')_{2D} = (V, U')_{2D}\}.
\]

In this definition, by a density argument, one can choose equivalently \(U'\) smooth and compactly supported. This shows that \(U\) belongs to \(D(\hat{A}^*)\) if and only if there exists some \(V \in H_{2D}\) such that \(\hat{A} U = V\) in the distributional sense, which implies that \(D(\hat{A}^*) = D(\hat{A})\).

By virtue of the Hille–Yosida theorem [5, Theorem 7.10], Proposition 2 implies that the Schrödinger equation (7) is well posed, hence also the evolution system (5). More precisely, we have the following result.

**Corollary 3.** If \(G \in C^1(\mathbb{R}^+, H_{2D})\), then the Schrödinger equation (7) with zero initial condition \(U(0) = 0\) admits a unique solution \(U \in C^1(\mathbb{R}^+, H_{2D}) \cap C^0(\mathbb{R}^+, D(A))\) given by the Duhamel integral formula:
\[
U(t) = \int_0^t e^{-i\#(t-s)} G(s) \, ds, \quad \forall t \geq 0,
\]
where \(e^{-i\#t}\) is the group of unitary operators generated by the self-adjoint operator \(\hat{A}\).

As a consequence of the Duhamel formula (12), we see that if \(t \mapsto \|G(t)\|_{2D}\) is bounded on \(\mathbb{R}^+\) (for instance a time-harmonic source), then \(\|U(t)\|_{2D}\) increases at most linearly in time. More precisely, as \(e^{-i\#(t-s)}\) is unitary, we have
\[
\|U(t)\|_{2D} \leq t \sup_{s \in \mathbb{R}^+} \|G(s)\|_{2D}, \quad \forall t \geq 0.
\]

### 2.3. Method of analysis: Spectral decomposition of the Hamiltonian

By spectral decomposition of the operator \(\hat{A}\), we mean its diagonalization with generalized eigenfunctions, which extends the usual diagonalization of matrices in the sense that
\[
\hat{A} = F^* \hat{\mathcal{A}} \, F,
\]
where \(F\) is a unitary transformation from the physical space \(H_{2D}\) to a spectral space \(\hat{\mathcal{H}}\) and \(\hat{\mathcal{A}}\) is a multiplication operator in this spectral space (more precisely, the multiplication by the spectral variable). The operator \(F\) is often called a generalized Fourier transform for \(\hat{A}\). The above decomposition of \(\hat{A}\) will lead to a modal representation of the solution \(U\) to (12).

This spectral decomposition of \(\hat{A}\) relies on general results on spectral theory of self-adjoint operators [29, 32], mainly the so-called spectral theorem which roughly says that any self-adjoint operator is diagonalizable.

For nonexpert readers, we collect below some basic materials about elementary spectral theory which allow to understand its statement, using elementary measure theory. The starting point is the notion of spectral measure (also called projection valued measure or resolution of the identity).
Definition 4. A spectral measure on a Hilbert space $\mathcal{H}$ is a mapping $\mathcal{E}$ from all Borel subsets of $\mathbb{R}$ into the set of orthogonal projections on $\mathcal{H}$ which satisfies the following properties:

1. $\mathcal{E}(\mathbb{R}) = \text{Id},$
2. $\mathcal{E}\left(\bigcup_{n=0}^{\infty} \lambda_n\right) u = \sum_{n=0}^{\infty} \mathcal{E}(\lambda_n)u$ for any $u \in \mathcal{H}$ and any sequence $(\lambda_n)_{n \in \mathbb{N}}$ of disjoint Borel sets, where the convergence of the series holds in the space $\mathcal{H}.$ Property 2 is known as $\sigma$-additivity property. Note that 1 and 2 imply $\mathcal{E}(\emptyset) = 0$ and $\mathcal{E}(\Lambda_1 \cap \Lambda_2) = \mathcal{E}(\Lambda_1) \mathcal{E}(\Lambda_2)$ for any Borel sets $\Lambda_1$ and $\Lambda_2.$

Suppose that we know some such $\mathcal{E}$, choose some $u \in \mathcal{H}$ and define

$$\mu_u(\Lambda) := (\mathcal{E}(\Lambda)u, u) = \|\mathcal{E}(\Lambda)u\|^2$$

for any measurable function $f : \mathbb{R} \rightarrow \mathbb{C}$ and any

$$u \in \mathcal{V}_f := \left\{ u \in \mathcal{H} \mid \int_{\mathbb{R}} |f(\lambda)| \, d\|\mathcal{E}(\lambda)u\|^2 < \infty \right\}.$$

Measure theory provides the limiting process which yields such integrals starting from the case of simple functions:

$$\int_{\mathbb{R}} f(\lambda) \, d\mu_u(\lambda) = \int_{\mathbb{R}} f(\lambda) \, d\|\mathcal{E}(\lambda)u\|^2$$

for any measurable function $f : \mathbb{R} \rightarrow \mathbb{C}$ and any

$$u \in \mathcal{V}_f := \left\{ u \in \mathcal{H} \mid \int_{\mathbb{R}} |f(\lambda)| \, d\|\mathcal{E}(\lambda)u\|^2 < \infty \right\}.$$

Go further, choose now two elements $u$ and $v$ in $\mathcal{H}$ and define $\mu_{u,v}(\Lambda) := (\mathcal{E}(\Lambda)u, v),$ which is no longer positive. Integrals of the form $\int_{\mathbb{R}} f(\lambda) \, d\mu_{u,v}(\lambda) = \int_{\mathbb{R}} f(\lambda) \, d(\mathcal{E}(\lambda)u, v)$ can nevertheless be defined for $u, v \in \mathcal{V}_f.$ They are simply deduced from the previous ones thanks to the polarization identity

$$4 \mu_{u,v} = \mu_{u+v} - \mu_{u-v} + i \mu_{u+iv} - i \mu_{u-iv}.$$

Consider then the subspace $D_f$ of $\mathcal{H}$ defined by:

$$D_f := \left\{ u \in \mathcal{H} \mid \int_{\mathbb{R}} |f(\lambda)|^2 \, d\|\mathcal{E}(\lambda)u\|^2 < \infty \right\}.$$

By the Cauchy–Schwarz inequality, this subspace is included in $\mathcal{V}_f$ and one can prove that it is dense in $\mathcal{H}.$ The key point is that for $u \in D_f,$ the linear form $v \mapsto \int_{\mathbb{R}} f(\lambda) \, d(\mathcal{E}(\lambda)u, v)$ is continuous in $\mathcal{H}.$ Thus, by Riesz theorem, we can define an operator denoted $\int_{\mathbb{R}} f(\lambda) \, d(\mathcal{E}(\lambda)u, v)$ with domain $D_f$ by

$$\forall u \in D_f, \forall v \in \mathcal{H}, \quad \left( \int_{\mathbb{R}} f(\lambda) \, d(\mathcal{E}(\lambda)u, v) \right) . v = \int_{\mathbb{R}} f(\lambda) \, d(\mathcal{E}(\lambda)u, v).$$
The operator usually associated to the spectral measure $E$ corresponds to the function $f(\lambda) = \lambda$. This operator is shown to be self-adjoint. If we choose to denote it $A$, one has

$$A := \int_{\mathbb{R}} \lambda \, dE(\lambda). \quad (14)$$

The above construction provides us a \textit{functional calculus}, i.e., a way to construct functions of $A$ defined by

$$f(A) := \int_{\mathbb{R}} f(\lambda) \, dE(\lambda) \quad \text{with domain } D(f(A)) \equiv D_f. \quad (15)$$

These operators satisfy elementary rules of composition, adjoint, and normalization:

$$f(A) g(A) = (fg)(A) = g(A) f(A), \quad f(A)^* = \overline{f(A)}$$

and

$$\|f(A) u\|^2 = \int_{\mathbb{R}} |f(\lambda)|^2 \|\lambda u\|^2.$$ 

The first rule confirms in particular that this functional calculus is consistent with composition and inversion, that is, the case of rational functions of $A$. The second one tells us that $f(A)$ is self-adjoint as soon as $f$ is real-valued. The third one tells us that $f(A)$ is bounded if $f$ is bounded on the support of $E$, whereas it becomes unbounded if $f$ is unbounded. The functions which play an essential role in this paper are the functions $r_\zeta(\lambda)$:

$$r_\zeta(\lambda) := (\lambda - \zeta)^{-1}$$

associated with the \textit{resolvent} of $A$, that is, $R(\zeta) := (A - \zeta)^{-1} = r_\zeta(A)$ for $\zeta \in \mathbb{C} \setminus \mathbb{R}$, exponential functions $\exp(\lambda t)$ which appears in the solution to Schrödinger equations and the indicator function $1_\Lambda$ of an interval $\Lambda$ for which we have by construction

$$1_\Lambda(A) = E(\Lambda). \quad (16)$$

We have shown above that every spectral measure give rise to a self-adjoint operator. The \textit{spectral theorem} tells us that the converse statement holds true.

\textbf{Theorem 5.} For any self-adjoint operator $A$ on a Hilbert space $\mathcal{H}$, there exists a spectral measure $E$ which diagonalizes $A$ in the sense of (14) and (15).

\textbf{Remark 6.} The support the spectral measure $E$ is defined as the smallest closed Borel set $\Lambda$ of $\mathbb{R}$ such that $E(\Lambda) = \text{Id}$. One can show that the spectrum $\sigma(A)$ of $A$ coincides with the support of $E$. Moreover the point spectrum $\sigma_p(A)$ is the set $\{\lambda \in \mathbb{R} | E(\{\lambda\}) \neq 0\}$.

Theorem 5 does not answer the crucial issue: how can we find $E$ if we know $A$? A common way to answer is to use the following Stone’s formulas.

\textbf{Theorem 7.} Let $A$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Its associated spectral measure $E$ is constructed as follows, for all $u \in \mathcal{H}$ :

$$\begin{align*}
\text{if } a < b : & \quad \left\| \frac{1}{2} \left( E((a, b)) + E([a, b]) \right) u \right\|^2 = \lim_{\eta \searrow 0} \frac{1}{\pi} \int_a^b \text{Im}(R(\lambda + i\eta) u, u) d\lambda, \quad (17) \\
\text{if } a \in \mathbb{R} : & \quad \left\| E(\{a\}) u \right\|^2 = \lim_{\eta \searrow 0} \eta \text{Im}(R(a + i\eta) u, u). \quad (18)
\end{align*}$$

Note that formulas (17) and (18) are sufficient by $\sigma$-additivity to know the spectral measure $E$ on all Borel sets. According to Remark 6, formula (18) permits to characterize the point
spectrum \( \sigma_p(\mathcal{A}) \), whereas (17) enables us to determine the whole spectrum \( \sigma(\mathcal{A}) \) and thus its continuous spectrum.

3. Spectral theory of the reduced Hamiltonian

The invariance of our medium in the \( y \)-direction allows us to reduce the spectral theory of our operator \( \mathcal{A} \) defined in (10) to the spectral theory of a family of self-adjoint operators \( (\mathcal{A}_k)_{k \in \mathbb{R}} \) defined on functions which depend only on the variable \( x \). In the present section, we introduce this family and perform the spectral analysis of each operator \( \mathcal{A}_k \). In Section 4, we collect all these results to obtain the spectral decomposition of \( \mathcal{A} \).

3.1. Reduced Hamiltonian \( \mathcal{A}_k \)

Let \( \mathcal{F} \) be the Fourier transform in the \( y \)-direction defined by

\[
\mathcal{F}u(k) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(y) e^{-iky} \, dy \quad \forall u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),
\]

which extends to a unitary transformation from \( L^2(\mathbb{R},y) \) to \( L^2(\mathbb{R},k) \). For functions of both variables \( x \) and \( y \), we still denote by \( \mathcal{F} \) be the partial Fourier transform in the \( y \)-direction. In particular, the partial Fourier transform of an element \( U \in \mathcal{H}_{2D} \) is such that

\[
\mathcal{F}U(\cdot,k) \in \mathcal{H}_{1D} := L^2(\mathbb{R}) \times L^2(\mathbb{R})^2 \times L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+)^2 \quad \text{for a.e. } k \in \mathbb{R},
\]

where the Hilbert space \( \mathcal{H}_{1D} \) is endowed with the inner product \( (\cdot,\cdot)_{1D} \) defined by the same expression (9) as \( (\cdot,\cdot)_{2D} \) except that \( L^2 \) inner products are now defined on one-dimensional domains.

Applying \( \mathcal{F} \) to our transmission problem (5) leads us to introduce a family of operators \( (\mathcal{A}_k)_{k \in \mathbb{R}} \) in \( \mathcal{H}_{1D} \) related to \( \mathcal{A} \) (defined in (10)) by the relation

\[
\mathcal{F}(\mathcal{A}U)(\cdot,k) = \mathcal{A}_k \mathcal{F}U(\cdot,k) \quad \text{for a.e. } k \in \mathbb{R}.
\]

Therefore, \( \mathcal{A}_k \) is deduced from the definition of \( \mathcal{A} \) by replacing the \( y \)-derivative by the product by \( ik \), i.e.,

\[
\mathcal{A}_k \mathcal{U} := \mathcal{A}_k \mathcal{U}, \quad \forall \mathcal{U} \in \text{D}(\mathcal{A}_k) := H^1(\mathbb{R}) \times \mathcal{H}_{\text{curl}_k}(\mathbb{R}) \times L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+)^2 \subset \mathcal{H}_{1D},
\]

where

\[
\mathcal{A}_k := i \begin{pmatrix} 0 & \varepsilon_0^{-1} \text{curl}_k & -\varepsilon_0^{-1} \Pi & 0 \\ -\mu_0^{-1} \text{curl}_k & 0 & 0 & -\mu_0^{-1} \Pi \\ \varepsilon_0 \Omega_c^2 R & 0 & 0 & 0 \\ 0 & \mu_0 \Omega_m^2 R & 0 & 0 \end{pmatrix},
\]

\[
\text{curl}_k u := \left( ik u_x - \frac{du_y}{dx} \right)^\top, \quad \text{curl}_k u := \frac{du_y}{dx} - iku_x \text{ for } u := (u_x,u_y)^\top,
\]

and the operators \( \Pi, \Pi, R, \) and \( R \) are defined as in (11) but for functions of the variable \( x \) only. Finally,

\[
\mathcal{H}_{\text{curl}_k}(\mathbb{R}) := \{ u \in L^2(\mathbb{R})^2 \mid \text{curl}_k u \in L^2(\mathbb{R}) \} = L^2(\mathbb{R}) \times H^1(\mathbb{R}).
\]
Note again that the transmission conditions (6) are satisfied as soon as \((E, H) \in H^1(\mathbb{R}) \times H_{\text{curl}}(\mathbb{R})\).

As in Proposition 2, it is readily seen that \(A_k : D(A_k) \subset H_{1\text{D}} \to H_{1\text{D}}\) is self-adjoint for all \(k \in \mathbb{R}\). The following proposition shows the particular role of the values 0 and \(\pm \Omega_m\) in the spectrum of \(A_k\).

**Proposition 8.** For all \(k \in \mathbb{R}^+\), the values 0 and \(\pm \Omega_m\) are eigenvalues of infinite multiplicity of \(A_k\) whose respective associated eigenspaces Ker\((A_k)\) and Ker\((A_k \mp \Omega_m)\) are given by

\[
\text{Ker}(A_k) = \{(0, \hat{\nabla} k \phi, 0, 0)^\top \mid \phi \in H^1_0(\mathbb{R}_-)\} \quad \text{and} \quad \text{Ker}(A_k \mp \Omega_m) = \{(0, \nabla k \phi, 0, \pm i\mu_0 \Omega_m \nabla k \phi)^\top \mid \phi \in H^1_0(\mathbb{R}_+)\},
\]

where \(\nabla k \phi = (d\phi/dx, ik\phi)^\top\), \(\hat{\Pi}\) is the extension by 0 of a 2D vector field defined on \(\mathbb{R}_-\) to the whole line \(\mathbb{R}\) and \(H^1_0(\mathbb{R}_+) := \{\phi \in H^1(\mathbb{R}_+) \mid \phi(0) = 0\}\). Moreover, the orthogonal complement of the direct sum of these three eigenspaces, i.e., \((\text{Ker} A_k \oplus \text{Ker} (A_k + \Omega_m) \oplus \text{Ker} (A_k - \Omega_m))^\perp\), is

\[
\text{H}_{1\text{D}}(\text{div} k 0) := \{(E, H, J, K) \in \text{H}_{1\text{D}} \mid \text{div}_k H = 0 \text{ in } \mathbb{R}_\pm \text{ and div}_k K = 0 \text{ in } \mathbb{R}_+\} \quad (23)
\]

where \(\text{div}_k u = du_k/dx + iku_y\).

**Proof.** We detail the proof only for \(\pm \Omega_m\). The case of the eigenvalue 0 can be dealt with in the same way. Suppose that \(U := (E, H, J, K)^\top \in D(A_k)\) satisfies \(A_k U = \pm \Omega_m U\), which is equivalent to

\[
\begin{cases}
i \varepsilon_0^{-1}(\text{curl}_k H - \Pi J) = \pm \Omega_m E, \\-i \mu_0^{-1}(\text{curl}_k E + \Pi K) = \pm \Omega_m H, \\
i \varepsilon_0 \Omega_m^2 RE = \pm \Omega_m J, \\
i \mu_0 \Omega_m^2 RH = \pm \Omega_m K,
\end{cases}
\]

thanks to the above definition of \(A_k\). Using (26) and (27), we can eliminate the unknowns \(J\) and \(K\) in (24) and (25), which become

\[
\text{curl}_k H = \mp i \varepsilon_0 \left(\Omega_m - \frac{\Omega_m^2}{\Omega_m^2 (1 - I_{\mathbb{R}_+})}\right) E \quad \text{and} \quad \text{curl}_k E = \pm i \mu_0 \Omega_m (1 - I_{\mathbb{R}_+}) H, \quad (28)
\]

where \(I_{\mathbb{R}_+}\) denotes the indicator function of \(\mathbb{R}_+\). In particular, we have \(\text{curl}_k E = 0 \text{ in } \mathbb{R}_+\), thus \(E|_{\mathbb{R}_+} = 0\) (by definition of \(\text{curl}_k\)), so \(J = 0\) by (26). In \(\mathbb{R}_-\), we can eliminate \(H\) between the two equations of (28), which yields

\[
-\frac{d^2 E}{dx^2} + (k^2 - \varepsilon_0 \mu_0 \Omega_m^2) E = 0 \text{ in } \mathbb{R}_- \quad \text{and} \quad E(0) = 0,
\]

where the last condition follows from (6) and \(E|_{\mathbb{R}_+} = 0\). Obviously the only solution in \(H^1(\mathbb{R}_-)\) is \(E = 0\). Hence \(E\) vanishes on both sides \(\mathbb{R}_+\) and \(\mathbb{R}_-\). The second equation of (28) then tells us that \(H|_{\mathbb{R}_-} = 0\), whereas the first one (together with (6)) shows that \(\text{curl}_k H = 0 \text{ in } \mathbb{R}_+\) and \(H_y(0) = 0\), which implies that \(H|_{\mathbb{R}_+} = \nabla k \phi\) where \(\phi := -i k^{-1} H_y \in H^1_0(\mathbb{R}_+),\) hence \(K = \pm i \mu_0 \Omega_m \nabla k \phi\) by (27).
Conversely, for all $\phi \in H^1_0(\mathbb{R}_+)$, the vector $(0, \Pi \nabla_k \phi, 0, \pm i\mu_0 \Omega_m \nabla_k \phi)^\top$ belongs to $D(\mathcal{A}_k)$ and satisfies (24–27).

Using these characterizations of $\text{Ker}(\mathcal{A}_k \pm \Omega_m)$ and $\text{Ker}\mathcal{A}_k$, we finally identify the orthogonal complement of their direct sum, or equivalently, the intersection of their respective orthogonal complements. We have

$$(E, H, J, K) \in \text{Ker}(\mathcal{A}_k) \perp \iff \int_{\mathbb{R}_-} H \cdot \nabla_k \phi \, dx = 0, \forall \phi \in H^1_0(\mathbb{R}_-)$$

$$\iff \text{div}_k H = 0 \quad \text{in } H^{-1}(\mathbb{R}_-).$$

In the same way,

$$(E, H, J, K) \in \text{Ker}(\mathcal{A}_k \pm \Omega_m) \perp \iff \text{div}_k (\mu_0 H \mp i\Omega_m^{-1} K) = 0 \quad \text{in } H^{-1}(\mathbb{R}_+).$$

This yields the definition (23) of $\mathcal{H}_{1D}(\text{div}_k 0)$.

3.2. Resolvent of the reduced Hamiltonian

To apply Stone's formulas (17) and (18) to $\mathcal{A}_k$, we first derive an integral representation of its resolvent $R_k(\zeta) := (\mathcal{A}_k - \zeta)^{-1}$. We begin by showing how to reduce the computation of $R_k(\zeta)$ to a scalar Sturm–Liouville equation, then we give an integral representation of the solution of the latter and we finally conclude by a convenient expression of $R_k(\zeta)$.

3.2.1. Reduction to a scalar equation

Let $\zeta \in \mathbb{C} \setminus \mathbb{R}$. Suppose that $U = R_k(\zeta) F$ for some $F \in \mathcal{H}_{1D}$ or equivalently that

$$(\mathcal{A}_k - \zeta) U = F.$$

Setting $U = (E, H, J, K)^\top$, $F = (f_E, f_H, f_J, f_K)^\top$ and using definition (22), the latter equation can be rewritten as:

$$\begin{align*}
&i \varepsilon_0^{-1} (\text{curl}_k H - \Pi J) - \zeta \, E = f_E, \\
&-i \mu_0^{-1} (\text{curl}_k E + \Pi K) - \zeta \, H = f_H, \\
&i \varepsilon_0 \Omega_m^2 \mu_0 \, R - \zeta \, J = f_J, \\
&i \mu_0 \Omega_m^2 \mu_0 \, R - \zeta \, K = f_K.
\end{align*}$$

The last two equations provide us expressions of $J$ and $K$ that can be substituted in the first two which become a system for both unknowns $H$ and $E$. We can then eliminate $H$ and obtain a Sturm–Liouville equation for $E$:

$$- \frac{d}{dx} \left( \frac{1}{\mu_\zeta} \frac{dE}{dx} \right) + \frac{\Theta_{k,\zeta}}{\mu_\zeta} E = f,$$

where

$$f := \zeta \left( \varepsilon_0 f_E + \frac{i\mu_0}{\zeta} \text{curl}_k \left( \frac{f_H}{\mu_\zeta} \right) - \frac{i}{\zeta} \Pi f_J + \frac{1}{\zeta^2} \text{curl}_k \left( \frac{\Pi f_K}{\mu_\zeta} \right) \right)$$

(30)
and the following notations are used hereafter:

\[
\varepsilon_\zeta(x) := \begin{cases} 
\varepsilon^-_\zeta := \varepsilon_0 & \text{if } x < 0, \\
\varepsilon^+_\zeta := \varepsilon_0 \left(1 - \frac{\Omega^2_\zeta}{\zeta^2}\right) & \text{if } x > 0,
\end{cases} \tag{31}
\]

\[
\mu_\zeta(x) := \begin{cases} 
\mu^-_\zeta := \mu_0 & \text{if } x < 0, \\
\mu^+_\zeta := \mu_0 \left(1 - \frac{\Omega^2_\zeta}{\zeta^2}\right) & \text{if } x > 0,
\end{cases} \tag{32}
\]

\[
\Theta_{k,\zeta}(x) := k^2 - \varepsilon_\zeta(x) \mu_\zeta(x) \zeta^2 = \begin{cases} 
\Theta^-_{k,\zeta} := k^2 - \varepsilon_0 \mu_0 \zeta^2 & \text{if } x < 0, \\
\Theta^+_{k,\zeta} := k^2 - \varepsilon^+_\zeta \mu^+_\zeta \zeta^2 & \text{if } x > 0.
\end{cases} \tag{33}
\]

The eliminated unknowns \( \mathbf{H}, \mathbf{J}, \) and \( \mathbf{K} \) can finally be deduced from \( E \) by the relations

\[
\mathbf{H} = -\frac{i}{\mu_\zeta \zeta} \text{curl}_k E - \frac{\mu_0}{\mu_\zeta \zeta} \mathbf{f}_H + \frac{i}{\mu_\zeta \zeta^2} \Pi \mathbf{f}_K,
\]

\[
\mathbf{J} = \frac{i \varepsilon_0 \Omega^2_\zeta}{\zeta} \mathbf{R} E - \frac{1}{\zeta} \mathbf{f}_J,
\]

\[
\mathbf{K} = \frac{\mu_0 \Omega^2_\zeta}{\mu_\zeta \zeta^2} \mathbf{R} \text{curl}_k E - \frac{i \mu_0 \Omega^2_\zeta}{\mu^+_\zeta \zeta^2} \mathbf{R} \mathbf{f}_H - \frac{\mu_0}{\mu^+_\zeta \zeta^2} \mathbf{f}_K. \tag{34}
\]

We can write these results in a condensed form by introducing several operators. First, we denote by \( \mathcal{C}_{k,\zeta} \) the operator which maps the right-hand side \( f \) of the Sturm–Liouville equation \( (29) \) to its solution: \( E = \mathcal{C}_{k,\zeta} f. \) By the Lax–Milgram theorem, it is easily seen that \( \mathcal{C}_{k,\zeta} \) is continuous from \( H^{-1}(\mathbb{R}) \) to \( H^1(\mathbb{R}) \) (where \( H^{-1}(\mathbb{R}) \) denotes the dual space of \( H^1(\mathbb{R}) \)). Next, associated to the expression \( (30) \) for the right-hand side of the Sturm–Liouville equation \( (29) \), we define

\[
\mathcal{S}_{k,\zeta} \mathbf{F} := \varepsilon_0 \mathbf{f}_E + \frac{i \mu_0}{\zeta} \text{curl}_k \left( \frac{\mathbf{f}_H}{\mu_\zeta} \right) - \frac{i}{\zeta} \Pi \mathbf{f}_J + \frac{1}{\zeta^2} \text{curl}_k \left( \frac{\Pi \mathbf{f}_K}{\mu_\zeta} \right), \tag{35}
\]

The operator \( \mathcal{S}_{k,\zeta} \) is a “scalarizer” since it maps the vector datum \( \mathbf{F} \) to a scalar quantity. It is clearly continuous from \( \mathcal{H}_{1\text{D}} \) to \( H^{-1}(\mathbb{R}) \). Finally, associated to the two columns of the right-hand side of \( (34) \) which distinguish the role of the electrical field \( E \) from the one of the vector datum \( \mathbf{F} \), we define

\[
\mathcal{V}_{k,\zeta} E := \left(E, -\frac{i}{\mu_\zeta \zeta} \text{curl}_k E, \frac{i \varepsilon_0 \Omega^2_\zeta}{\zeta} \mathbf{R} E, \frac{\mu_0 \Omega^2_\zeta}{\mu^+_\zeta \zeta^2} \mathbf{R} \text{curl}_k E\right)^T, \tag{36}
\]

\[
\mathcal{T}_{k,\zeta} \mathbf{F} := \left(0, -\frac{\mu_0}{\mu_\zeta \zeta} \mathbf{f}_H + \frac{i}{\mu_\zeta \zeta^2} \Pi \mathbf{f}_K, -\frac{1}{\zeta} \mathbf{f}_J, -\frac{i \mu^2_0 \Omega^2_\zeta}{\mu^+_\zeta \zeta^2} \mathbf{R} \mathbf{f}_H - \frac{\mu_0}{\mu^+_\zeta \zeta^2} \mathbf{f}_K\right)^T. \tag{37}
\]

The operator \( \mathcal{V}_{k,\zeta} \) is a “vectorizer” since it maps the scalar field \( E \) to a vector field of \( \mathcal{H}_{1\text{D}} \). It is continuous from \( H^1(\mathbb{R}) \) to \( \mathcal{H}_{1\text{D}} \). Finally, \( \mathcal{T}_{k,\zeta} \) maps the vector datum \( \mathbf{F} \) to a vector field of \( \mathcal{H}_{1\text{D}} \) and is continuous from \( \mathcal{H}_{1\text{D}} \) to \( \mathcal{H}_{1\text{D}} \). The solution of our Sturm–Liouville equation
Proposition 9. Let $k \in \mathbb{R}$. The resolvent of the self-adjoint operator $A_k$ can be expressed as:

$$R_k(\zeta) = T_{k,\zeta} + \zeta V_{k,\zeta} C_{k,\zeta} S_{k,\zeta}, \quad \forall \zeta \in \mathbb{C} \setminus \mathbb{R},$$

where $C_{k,\zeta} f$ is the solution to the Sturm–Liouville equation (29) and the operators $S_{k,\zeta}$, $T_{k,\zeta}$, and $V_{k,\zeta}$ are, respectively, defined in (35), (36), and (37).

It is readily seen that the respective adjoints of the above operators satisfy the following relations:

$$C_{k,\zeta}^* = C_{k,\bar{\zeta}}, \quad S_{k,\zeta}^* = V_{k,\zeta}, \quad V_{k,\zeta}^* = S_{k,\bar{\zeta}} \quad \text{and} \quad T_{k,\zeta}^* = T_{k,\bar{\zeta}},$$

from which we retrieve the usual formula $R_k(\zeta)^* = R_k(\bar{\zeta})$ which is valid for any self-adjoint operator.

Remark 10. Notice that in comparison with previous studies on stratified media which inspire our approach [9, 18, 37, 39], the essential difference lies in the fact that our Sturm–Liouville equation (29) depends nonlinearly on the spectral variable $\zeta$, which is a consequence of the frequency dispersion in a Drude material. This dependence considerably complicates the spectral analysis of $A_k$.

3.2.2. Solution of the Sturm–Liouville equation

To use the expression of $R_k(\zeta)$ given by Proposition 9 in Stone’s formulas, we need an explicit expression of $C_{k,\zeta}$. We recall here some classical results about the solution of a Sturm–Liouville equation, which provides us an integral representation of $C_{k,\zeta}$:

$$C_{k,\zeta} f(x') = \int_{\mathbb{R}} g_{k,\zeta}(x, x') f(x) \, dx, \quad \forall f \in L^2(\mathbb{R}),$$

where the kernel $g_{k,\zeta}$ is the Green’s function of the Sturm–Liouville equation (29). For all $x' \in \mathbb{R}$, function $g_{k,\zeta}(\cdot, x')$ is defined as the unique solution in $H^1(\mathbb{R})$ to

$$-\frac{\partial}{\partial x} \left( \frac{1}{\mu_x} \frac{\partial}{\partial x} g_{k,\zeta}(\cdot, x') \right) + \frac{\Theta_{k,\zeta}}{\mu_x} g_{k,\zeta}(\cdot, x') = \delta_{x'},$$

where $\delta_{x'} \in H^{-1}(\mathbb{R})$ is the Dirac measure at $x'$. Note that formula (39) is only valid for $f \in L^2(\mathbb{R})$. If $f \in H^{-1}(\mathbb{R})$, we just have to replace the integral by a duality product between $H^{-1}(\mathbb{R})$ and $H^1(\mathbb{R})$.

To express $g_{k,\zeta}$, we first introduce the following basis of the solutions to the homogeneous Sturm–Liouville equation associated to (29), i.e., for $f = 0$:

$$c_{k,\zeta}(x) := \cosh \left( \theta_{k,\zeta}(x) x \right) \quad \text{and} \quad s_{k,\zeta}(x) := \mu_x(x) \frac{\sinh \left( \theta_{k,\zeta}(x) x \right)}{\theta_{k,\zeta}(x)},$$

where

$$\theta_{k,\zeta}(x) := \sqrt{\Theta_{k,\zeta}(x)} = \begin{cases} \theta_{k,\zeta}^- := \sqrt{k^2 - \varepsilon_0 \mu_0 \zeta^2} & \text{if } x < 0, \\ \theta_{k,\zeta}^+ := \sqrt{k^2 - \varepsilon_+ \mu_+ \zeta^2} & \text{if } x > 0, \end{cases}$$

(39) can now be expressed as $E = \zeta C_{k,\zeta} S_{k,\zeta} F$, so that $U = (T_{k,\zeta} + \zeta V_{k,\zeta} C_{k,\zeta} S_{k,\zeta}) F$. To sum up, we have the following proposition.
and \( \sqrt{\cdot} \) denotes the principal determination of the complex square root, i.e.,
\[
\sqrt{z} = |z|^{\frac{1}{2}} e^{i \text{arg} z / 2} \quad \text{for} \quad \text{arg} z \in (-\pi, \pi).
\]

The special feature of the above basis is that both \( c_{k, \zeta}(x) \) and \( s_{k, \zeta}(x) \) are analytic functions of \( \zeta \in \mathbb{C} \setminus \{0\} \) for all \( x \) (since they can be expanded as power series of \( \Theta_{k, \zeta}(x) = \theta_{k, \zeta}(x)^2 \)). In particular, they do not depend on the choice of the determination of \( \sqrt{\cdot} \), whereas \( \theta_{k, \zeta} \) depends on \( \zeta \). Note that this definition of \( \theta_{k, \zeta} \) makes sense since \( \Theta_{k, \zeta}^\pm \in \mathbb{C} \setminus \mathbb{R}^- \) for all \( \zeta \in \mathbb{C} \setminus \mathbb{R} \) and \( k \in \mathbb{R} \). This is obvious for \( \Theta_{k, \zeta}^- \), since \( \zeta \notin \mathbb{R} \) implies \( \zeta^2 \notin \mathbb{R}^+ \), hence
\[
\Theta_{k, \zeta}^- = k^2 - \varepsilon_0 \mu_0 \zeta^2 \notin (-\infty, k^2].
\]

On the other hand, by (31), (32), and (33), we have
\[
\Theta_{k, \zeta}^+ = k^2 - \varepsilon_0 \mu_0 (\zeta - \Omega_2^\zeta / \zeta)(\zeta - \Omega_m^2 / \zeta),
\]
which cannot belong to \((-\infty, k^2]\) for the same reasons, since the imaginary parts of both quantities \( \zeta - \Omega_2^\zeta / \zeta \) and \( \zeta - \Omega_m^2 / \zeta \) have the same sign as \( \text{Im} \zeta \).

**Proposition 11.** Let \( \zeta \in \mathbb{C} \setminus \mathbb{R} \) and \( k \in \mathbb{R} \). For all \( f \in L^2(\mathbb{R}) \), the function \( C_{k, \zeta} f \) satisfies the integral representation (39) where the Green’s function \( g_{k, \zeta} \) of the Sturm–Liouville equation (29) is given by
\[
g_{k, \zeta}(x, x') = \frac{1}{W_{k, \zeta}} \psi_{k, \zeta, -1}(\min(x, x')) \psi_{k, \zeta, +1}(\max(x, x')),
\]
where
\[
\psi_{k, \zeta, \pm 1}(x) := c_{k, \zeta}(x) \mp \frac{\theta_{k, \zeta}^\pm}{\mu_\zeta} s_{k, \zeta}(x) \forall x \in \mathbb{R} \quad \text{and} \quad W_{k, \zeta} := \frac{\theta_{k, \zeta}^-}{\mu_\zeta} + \frac{\theta_{k, \zeta}^+}{\mu_\zeta}
\]
is the (constant) Wronskian of \( \psi_{k, \zeta, -1} \) and \( \psi_{k, \zeta, +1} \), i.e., \( \mu_\zeta^{-1}(\psi_{k, \zeta, +1} \psi_{k, \zeta, -1} - \psi_{k, \zeta, -1} \psi_{k, \zeta, +1}) \).

We omit the proof of this classical result (see, e.g., [33, Lemma 9.7]). The expression (42) of \( g_{k, \zeta} \) involves another basis \( \{ \psi_{k, \zeta, -1}, \psi_{k, \zeta, +1} \} \) of the solutions to the homogeneous Sturm–Liouville equation associated to (29) which has the special feature to be evanescent as \( x \) tends either to \(-\infty\) or \(+\infty\). More precisely,
\[
\psi_{k, \zeta, -1}(x) = \begin{cases} 
  e^{+\theta_{k, \zeta}^- x} & \text{if } x < 0, \\
  A_{k, \zeta, -1} e^{+\theta_{k, \zeta}^- x} + B_{k, \zeta, -1} e^{-\theta_{k, \zeta}^+ x} & \text{if } x > 0,
\end{cases}
\]
\[
\psi_{k, \zeta, +1}(x) = \begin{cases} 
  e^{-\theta_{k, \zeta}^- x} & \text{if } x < 0, \\
  A_{k, \zeta, +1} e^{-\theta_{k, \zeta}^- x} + B_{k, \zeta, +1} e^{+\theta_{k, \zeta}^+ x} & \text{if } x > 0,
\end{cases}
\]
where
\[
A_{k, \zeta, \pm 1} := \frac{1}{2} \left( 1 + \frac{\theta_{k, \zeta}^\pm / \mu_\zeta^\pm}{\theta_{k, \zeta}^\pm / \mu_\zeta^\pm} \right) \quad B_{k, \zeta, \pm 1} := \frac{1}{2} \left( 1 - \frac{\theta_{k, \zeta}^\pm / \mu_\zeta^\pm}{\theta_{k, \zeta}^\pm / \mu_\zeta^\pm} \right).
\]
Formulas (44) and (45) show that \( \psi_{k, \zeta, \pm 1} \) decreases exponentially when \( x \to \pm \infty \), since \( \text{Re}(\theta_{k, \zeta}^\pm) > 0 \). Note that the Wronskian \( W_{k, \zeta} \) cannot vanish for \( \zeta \in \mathbb{C} \setminus \mathbb{R} \), otherwise the
resolvent $R_k(\zeta)$ would be singular, which is impossible, because it is analytic in $\mathbb{C} \setminus \mathbb{R}$ since $A_k$ is self-adjoint.

### 3.2.3. Integral expression of the resolvent
We are now able to give an explicit expression of $R_k(\zeta)$, more precisely of the quantity $(R_k(\zeta)U, U)_{1D}$ involved in Stone’s formulas (17) and (18). Thanks to Proposition 9 and (38), we can rewrite this quantity as

$$
(R_k(\zeta)U, U)_{1D} = \left(\mathbb{T}_{k,\zeta} U, U\right)_{1D} + \left\{\zeta \in \mathbb{C} \ni S_{k,\zeta} U, S_{k,\zeta}^\dagger U\right\}, \quad \forall U \in \mathcal{H}_{1D},
$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ denotes the duality product between $H^1(\mathbb{R})$ and $H^{-1}(\mathbb{R})$. To express this duality product as an integral (and to avoid other difficulties which occur when applying Stone’s formulas), we restrict ourselves to particular $U$ chosen in the dense subspace of $\mathcal{H}_{1D}$ defined by

$$
\mathcal{D}_{1D} := \left\{U = (E, H, J, K) \in D(\mathbb{R}) \times D(\mathbb{R})^2 \times D(\mathbb{R}_+) \times D(\mathbb{R}_+)^2 \mid H_j(0) = 0 \right\},
$$

where $D(O)$, for $O = \mathbb{R}$ or $\mathbb{R}_+$, denotes the space of bump functions in $O$ (compactly supported in $O$ and smooth) and the condition $H_j(0) = 0$ ensures that $S_{k,\zeta}(\mathcal{D}_{1D}) \subset L^2(\mathbb{R})$.

Using the integral representation (39), the above formula becomes

$$
(R_k(\zeta)U, U)_{1D} = \left(\mathbb{T}_{k,\zeta} U, U\right)_{1D} + \int_{\mathbb{R}^2} \zeta g_{k,\zeta}(x, x') S_{k,\zeta} U(x) \overline{S_{k,\zeta}^\dagger U(x')} \, dx \, dx', \quad \forall U \in \mathcal{D}_{1D}.
$$

### 3.3. Boundary values on the spectral real axis
Before applying Stone’s formulas (17) and (18), we have to make clear the behavior of the resolvent $R_k(\zeta)$ as $\zeta$ tends to the real axis, hence the behavior of all quantities involved in the integral representation (47), in particular the Green’s function $g_{k,\zeta}$. This is the subject of this subsection.

In the following, for any quantity $f_\zeta$ depending on the parameter $\zeta \in \mathbb{C} \setminus \mathbb{R}$, we choose to denote $f_\lambda$ the one-sided limit (if it exists) of $f_\zeta$ when $\zeta$ tends to $\lambda \in \mathbb{R}$ from the upper half-plane, i.e.,

$$
f_\lambda := \lim_{\eta \searrow 0} f_{\lambda + i\eta}.
$$

Notice that $S_{k,\zeta} \forall_{k,\zeta}$, and $\mathbb{T}_{k,\zeta}$ defined in (35), (36), and (37) have obviously one-sided limits $S_{k,\lambda} \forall_{k,\lambda}$, and $\mathbb{T}_{k,\lambda}$ if $\lambda$ differs from 0 and $\pm \Omega_m$ (they actually depend analytically on $\zeta$ outside these three values).

#### 3.3.1. Spectral zones and one-sided limit of $\Theta_{k,\zeta}(\cdot)$
The determination of the one-sided limit of $\Theta_{k,\zeta}(x)$ defined in (40) requires us to identify the zones, in the $(k, \lambda)$ plane, where $\Theta_{k,\lambda}^+$ or $\Theta_{k,\lambda}^-$ (defined in (33)) is located on the branch cut $\mathbb{R}^-$ of the complex square root (41). Using (31) and (32), one computes that

$$
\Theta_{k,\lambda}(x) = \begin{cases} 
\Theta_{k,\lambda}^- = k^2 - \varepsilon_0 \mu_0 \lambda^2 & \text{if } x < 0, \\
\Theta_{k,\lambda}^+ = -\varepsilon_0 \mu_0 \lambda^4 + (k^2 + \varepsilon_0 \mu_0 (\Omega_\varepsilon^2 + \Omega_m^2) \lambda^2 - \varepsilon_0 \mu_0 \Omega_\varepsilon^2 \Omega_m^2) / \lambda^2 & \text{if } x > 0.
\end{cases}
$$
We denote by $\lambda_0(k)$ (respectively, $\lambda_\text{D}(k)$ and $\lambda_1(k)$, with $0 < \lambda_1(k) \leq \lambda_\text{D}(k)$) the non-negative values of $\lambda$ for which $\Theta^{-\lambda}_{k,\lambda}$ (respectively, $\Theta^{+\lambda}_{k,\lambda}$) vanishes, i.e.,

$$
\lambda_0(k) := |k|/\sqrt{\varepsilon_0\mu_0},
$$

$$
\lambda_\text{D}(k) := \sqrt{\frac{k^2}{2\varepsilon_0\mu_0} + \frac{\Omega_e^2 + \Omega_m^2}{2} + \sqrt{\left(\frac{k^2}{2\varepsilon_0\mu_0} + \frac{\Omega_e^2 - \Omega_m^2}{2}\right)^2 + \frac{k^2\Omega_m^2}{\varepsilon_0\mu_0}}},
$$

$$
\lambda_1(k) := \sqrt{\frac{k^2}{2\varepsilon_0\mu_0} + \frac{\Omega_e^2 + \Omega_m^2}{2} - \sqrt{\left(\frac{k^2}{2\varepsilon_0\mu_0} + \frac{\Omega_e^2 - \Omega_m^2}{2}\right)^2 + \frac{k^2\Omega_m^2}{\varepsilon_0\mu_0}}},
$$

where $\lambda_1(k)$ and $\lambda_\text{D}(k)$ are related by $\lambda_1(k) \lambda_\text{D}(k) = \Omega_e\Omega_m$. In the $(k,\lambda)$-plane, the graphs of these functions are curves through which the sign of $\Theta_{k,\lambda}$ or $\Theta_{k,\lambda}^+$ changes. The main properties of these functions are summarized in the following lemma, whose proof is obvious.

**Lemma 12.** For all $(k,\lambda) \in \mathbb{R} \times \mathbb{R}$, we have

$$
(\Theta^{-\lambda}_{k,\lambda} < 0 \iff |\lambda| > \lambda_0(k)) \quad \text{and} \quad \text{for } \lambda \neq 0 \quad (\Theta^{+\lambda}_{k,\lambda} < 0 \iff |\lambda| \notin [\lambda_1(k), \lambda_\text{D}(k)]).
$$

The function $k \mapsto \lambda_1(k)$ is a $C^\infty$ strictly decreasing function on $\mathbb{R}^+$ whose range is $(0, \min(\Omega_e, \Omega_m)]$, whereas $k \mapsto \lambda_\text{D}(k)$ is a $C^\infty$ strictly increasing function on $\mathbb{R}^+$ whose range is $[\max(\Omega_e, \Omega_m), \infty)$ and $\lambda_0(k) = \lambda_0(k) + O(k^{-1})$ as $k \to +\infty$. Moreover, denoting $k_c$ the unique non-negative value of $k$ for which $\lambda_0(k) = \lambda_1(k)$, that is to say,

$$
k_c = \sqrt{\varepsilon_0\mu_0} \lambda_c \quad \text{where} \quad \lambda_c := \Omega_e\Omega_m/\sqrt{\Omega_e^2 + \Omega_m^2} \quad (\equiv \lambda_0(k_c) = \lambda_1(k_c)),
$$

one has

$$
0 < \lambda_0(k) < \lambda_1(k) < \lambda_\text{D}(k) \text{ for } 0 < k < k_c \text{ and } 0 < \lambda_1(k) < \lambda_0(k) < \lambda_\text{D}(k) \text{ for } k > k_c.
$$

Finally, $\lambda_1(k) = \lambda_\text{D}(k)$ if and only if $k = 0$ and $\Omega_e = \Omega_m$.

Figure 3 illustrates this lemma in the quarter $(k,\lambda)$-plane $\mathbb{R}^+ \times \mathbb{R}^+$ (all zones are symmetric with respect to both $k$ and $\lambda$ axes, since $\Theta_{k,\lambda}^{\pm\lambda}$ are even functions of $k$ and $\lambda$). The signs of $\Theta_{k,\lambda}^{+\lambda}$ and $\Theta_{k,\lambda}^{-\lambda}$ indicate the regime of vibration in the vacuum and in the Drude material: propagative or evanescent. As it will be made clear in Section 3.3.2, two kinds of propagative waves occur in the Drude material, which will be called direct and inverse, whereas propagative waves can only be direct in the vacuum. This explains the choice of the indices D, I, and E used hereafter, which mean, respectively, direct, inverse, and evanescent.

Figure 4, constructed as the superposition of the left and right graphics of Figure 3, then shows the coupling of both half-spaces. This leads us divide the $(k,\lambda)$-plane in several spectral zones defined as follows (see Section 3.3.2 for the justification of the notations):

$$
\Lambda_{\text{DD}} := \{(k,\lambda) \in \mathbb{R}^2 \mid |\lambda| > \lambda_\text{D}(k)\},
$$

$$
\Lambda_{\text{DE}} := \{(k,\lambda) \in \mathbb{R}^2 \mid |\lambda| \neq \Omega_m \text{ and } \max(\lambda_0(k),\lambda_1(k)) < |\lambda| < \lambda_\text{D}(k)\},
$$

$$
\Lambda_{\text{DI}} := \{(k,\lambda) \in \mathbb{R}^2 \mid |k| < k_c \text{ and } \lambda_0(k) < |\lambda| < \lambda_1(k)\},
$$

$$
\Lambda_{\text{EI}} := \{(k,\lambda) \in \mathbb{R}^2 \mid 0 < |\lambda| < \min(\lambda_0(k),\lambda_1(k))\}.
$$
Figure 3. Spectral zones associated to the vacuum (left) and the Drude material (right, case where $\Omega_\epsilon < \Omega_m$). Grey color, vertical and horizontal hatched lines mean, respectively, propagation in the vacuum, direct propagation and inverse propagation in the Drude material.

Figure 4. Spectral zones associated to our medium for $\Omega_\epsilon < \Omega_m$ (left) and $\Omega_\epsilon = \Omega_m$ (right). See caption of Figure 3.

Note that the eigenvalues $\lambda = 0$ and $\lambda = \pm \Omega_m$ of $A_k$ (Proposition 8) are excluded from these zones.

With our choice (41) for the complex square root, we notice that if $\Theta_{k,\lambda}^\pm := \lim_{\eta \to 0} \Theta_{k,\lambda + i\eta}^\pm$ is positive, then $\theta_{k,\lambda}^\pm$ is a positive real number (more precisely, $\theta_{k,\lambda}^\pm = |\Theta_{k,\lambda}^\pm|^{1/2}$). On the other hand, if $\Theta_{k,\lambda}^\pm$ is negative, then $\theta_{k,\lambda}^\pm$ is purely imaginary and its sign coincides with the sign of $\text{Im}(\Theta_{k,\lambda + i\eta}^\pm)$ for small positive $\eta$. From (31), (32), and (33), one computes that

$$\text{Im} \Theta_{k,\xi}^- = -\varepsilon_0 \mu_0 \text{Im}(\xi^2) \quad \text{and} \quad \text{Im} \Theta_{k,\xi}^+ = -\frac{\varepsilon_0 \mu_0}{|\xi|^4} \text{Im}(\xi^2) \left(|\xi|^4 - \Omega_\epsilon^2 \Omega_m^2 \right).$$

Note that for $\xi = \lambda + i\eta$, $\text{Im}(\xi^2)$ has the same sign as $\lambda$, as a consequence $\text{Im}(\Theta_{k,\lambda + i\eta}^\pm)$ has a sign opposite to the sign of $\lambda$. Moreover, the sign of $\lambda^4 - \Omega_\epsilon^2 \Omega_m^2$, the limit as $\eta \to 0$ of $|\xi|^4 - \Omega_\epsilon^2 \Omega_m^2$ when $\xi = \lambda + i\eta$, depends on the position of $|\lambda|$ with respect $\sqrt{\Omega_\epsilon \Omega_m}$. 
From Lemma 12, we deduce that
\[ \lambda_1(k) \leq \min(\Omega_e, \Omega_m) \leq \sqrt{\Omega_e \Omega_m} \leq \max(\Omega_e, \Omega_m) \leq \lambda_0(k), \]
thus the sign of \( \lambda^2 - \Omega^2_c \Omega^2_m \) is positive in the spectral zone \( \Lambda_{\text{DD}} \) (where \( \lambda > \lambda_{\text{DD}}(k) \)) and negative (where \( \lambda < \lambda_{1}(k) \)) in the spectral zones \( \Lambda_{\text{DI}} \) and \( \Lambda_{\text{EI}} \). Finally, we obtain
\[ \theta_{k,\lambda}^- = \begin{cases} -\text{sgn}(\lambda) |\Theta_{k,\lambda}^-|^{1/2} & \text{if } (k, \lambda) \in \Lambda_{\text{DI}} \cup \Lambda_{\text{DE}} \cup \Lambda_{\text{DD}}, \\ |\Theta_{k,\lambda}^-|^{1/2} & \text{otherwise}, \end{cases} \]
\[ \theta_{k,\lambda}^+ = \begin{cases} +\text{sgn}(\lambda) |\Theta_{k,\lambda}^+|^{1/2} & \text{if } (k, \lambda) \in \Lambda_{\text{EI}} \cup \Lambda_{\text{DI}}, \\ -\text{sgn}(\lambda) |\Theta_{k,\lambda}^+|^{1/2} & \text{if } (k, \lambda) \in \Lambda_{\text{DD}}, \\ |\Theta_{k,\lambda}^+|^{1/2} & \text{otherwise}. \end{cases} \]

In addition to the spectral zones defined above, we have to investigate the possible singularities of the Green's function in the \( (k, \lambda) \)-plane, that is, the pairs \((k, \lambda)\) for which the one-sided limit \( \mathcal{W}_k,\lambda \) of \( \mathcal{W}_{k,\lambda,\lambda_0} \) (see (43)) vanishes. Hence we have to solve the following dispersion equation, seen as an equation in the \( (k, \lambda) \)-plane:
\[ \frac{\theta_{k,\lambda}^-}{\mu_{k,\lambda}} + \frac{\theta_{k,\lambda}^+}{\mu_{k,\lambda}} = 0. \] (52)

**Lemma 13.** Define \( K := \varepsilon_0 \mu_0 (\Omega^2_m - \Omega^2_c) \). The solutions \((k, \lambda)\) to (52) are given by
\[ \Lambda_{\text{EE}} = \Lambda_{\text{EE}} \cup \{(k_c, \lambda_c), (k_c, -\lambda_c), (-k_c, \lambda_c), (-k_c, -\lambda_c)\} \]
where (see Figure 4 for an illustration)
\[ \Lambda_{\text{EE}} := \{(k, \lambda) \in \mathbb{R}^2 \mid |k| > k_c \text{ and } |\lambda| = \lambda_{\text{EE}}(k)\}. \] (53)
The critical value \( k_c \) has been defined in Lemma 12 and
\[ \lambda_{\text{EE}}(k) := \begin{cases} \Omega_m \sqrt{\frac{1}{2} + \frac{k^2}{K}} - \text{sgn}(K) \sqrt{\frac{1}{4} + \frac{k^4}{K^2}} & \text{if } K \neq 0, \\ \Omega_m \sqrt{\frac{1}{2}} & \text{if } K = 0. \end{cases} \]
The function \( k \mapsto \lambda_{\text{EE}}(k) \) is strictly decreasing on \([k_c, +\infty)\) if \( K < 0 \), i.e., \( \Omega_m < \Omega_c \) (respectively, strictly increasing if \( K > 0 \), i.e., \( \Omega_m > \Omega_c \)). Moreover \( \lambda_{\text{EE}}(k_c) = \lambda_c := \lambda_0(k_c) = \lambda_1(k_c) \) and \( \lambda_{\text{EE}}(k) = \Omega_m/\sqrt{2} + O(k^{-2}) \) as \( k \to +\infty \).

**Proof.** First notice that if \((k, \lambda)\) is a solution to (52), then
\[ \left( \frac{\theta_{k,\lambda}^-}{\mu_{\lambda}} + \frac{\theta_{k,\lambda}^+}{\mu_{\lambda}} \right) \left( \frac{\theta_{k,\lambda}^-}{\mu_{\lambda}} - \frac{\theta_{k,\lambda}^+}{\mu_{\lambda}} \right) = 0 \iff \frac{\Theta_{k,\lambda}^-}{(\mu_{\lambda})^2} = \frac{\Theta_{k,\lambda}^+}{(\mu_{\lambda})^2}. \] (54)
The proof is made of two steps.

**Step 1.** We first show that \( \Lambda_{\text{EE}} \subset \mathbb{R}^2 \setminus (\Lambda_{\text{DD}} \cup \Lambda_{\text{DE}} \cup \Lambda_{\text{DI}} \cup \Lambda_{\text{EI}}) \).
Indeed, (54) implies that $\Theta_{k,\lambda}^-$ and $\Theta_{k,\lambda}^+$ either have the same sign or vanish simultaneously. Hence

- $\Lambda_{\text{DE}}$ and $\Lambda_{\text{DI}}$ contain no solution since $\Theta_{k,\lambda}^-$ and $\Theta_{k,\lambda}^+$ have opposite signs in these zones (first statement of Lemma 12).
- If $(k, \lambda) \in \Lambda_{\text{DD}}, \theta_{k,\lambda}^\pm = -i \text{sgn}(\lambda) |\Theta_{k,\lambda}^\pm|^{1/2}$ (from (50) and (51)) and $\mu_{\lambda}^\pm > 0$ (since $|\lambda| > \Omega_m$). This shows that $(k, \lambda)$ cannot satisfy (52). A similar argument works for $\Lambda_{\text{DI}} (\mu_{\lambda}^- \text{ and } \mu_{\lambda}^+ \text{ have opposite signs in this zone, but } \text{Im}(\theta_{k,\lambda}^+)) \text{ and } \text{Im}(\theta_{k,\lambda}^-) \text{ also have opposite signs).}

On the other hand, the curves $\lambda = \lambda_0(k), \lambda = \lambda_0(k)$, and $\lambda = \lambda_1(k)$ contain no solution to (52), except the critical points $(k_c, \lambda_c), (-k_c, \lambda_c), (k_c, -\lambda_c)$, and $(-k_c, -\lambda_c)$ where both $\Theta_{k,\lambda}^\pm$ vanish. To sum up, apart from the critical points, the possible solutions to (52) are located outside the closure of all previously defined zones, that is, in the “white area” of Figure 4.

**Step 2.** In this “white area,” $\theta_{k,\lambda}^\pm > 0$, so solutions may occur only if $\mu_{\lambda}^- \text{ and } \mu_{\lambda}^+ \text{ have opposite signs, that is, in the subarea located in } |\lambda| < \Omega_m$. In this subarea, (52) and (54) are equivalent since $\theta_{k,\lambda}^-/\mu_{\lambda}^- + \theta_{k,\lambda}^+/\mu_{\lambda}^+ \neq 0$. Using (31), (32), and (33), (54) can be rewritten as:

$$Q_k(\lambda^2) = 0 \quad \text{where} \quad Q_k(X) := K X^2 - \Omega_m^2 (2k^2 + K) X + k^2 \Omega_m^4,$$

from which we infer that

$$\lambda^2 = X_k^\pm := \Omega_m^2 \left( \frac{1}{2} K + \frac{k^2}{4} \right) \sqrt{\frac{1}{4} + \frac{k^4}{K^2}}.$$

If $K < 0$, the only positive root is $X_k^+$ which is strictly decreasing on $[k_c, +\infty)$ with range $(\Omega_m^2/2, \lambda_c^2]$. Moreover, the curve $\lambda = (X_k^+)^{1/2}$ crosses $\lambda = \lambda_1(k)$ only at $(k_c, \lambda_c)$, which shows that $\lambda_1$ is the located in the abovementioned subarea. On the other hand, if $K > 0$, both roots $X_k^-$ and $X_k^+$ are positive, but $X_k^+ > \Omega_m^2$, so the curve $\lambda = (X_k^+)^{1/2}$ is now outside the subarea. The other root yields the only solution in the admissible subarea and it is strictly increasing on $[k_c, +\infty)$ with range $[\lambda_c^2, \Omega_m^2/2]$. Finally, if $K=0$, the only root of $Q_k(X)$ is $X = \Omega_m^2/2 = \lambda_c^2$.

### 3.3.2. Physical interpretation of the spectral zones

The various zones introduced above are related to various types of waves in both media, which can be either propagative or evanescent. As already mentioned, the indices D, I, and ε we chose to qualify these zones stand, respectively, for direct, inverse, and evanescent. The first two, D and I, are related to propagative waves which can be either direct or inverse waves (in the Drude medium), whereas ε means evanescent, that is, absence of propagation. As shown below, for each pair of indices characterizing the various zones $\Lambda_{\text{DD}}, \Lambda_{\text{DE}}, \Lambda_{\text{DI}}, \Lambda_{\text{EI}},$ and $\Lambda_{\text{EE}},$ the former indicates the behavior of the vacuum ($x < 0$) and the latter, that of the Drude material ($x > 0$).

To see this, substitute formulas (50) and (51) in (44) and (45), which yields the one-sided limits $\psi_{k,\lambda,\pm 1}$ of $\psi_{k,\lambda,\pm}$ in the sense of (48). The aim of this section is to show the physical interpretation of these functions as superpositions of elementary waves. For simplicity, we restrict ourselves to $\lambda > 0$ when interpreting their direction of propagation.

First notice that in each half line $x < 0$ and $x > 0$, function $e^{-i\lambda t} \psi_{k,\lambda,\pm 1}$ is a linear combination of terms of the form $u_k = e^{i(kx - \lambda t)}$, with $\kappa \in \mathbb{R}$ or $\kappa \in i \mathbb{R}$, where for a fixed $k$, $\kappa$ is a certain function of $\lambda$, denoted $\kappa = \kappa(\lambda)$.

- When $\kappa$ is purely imaginary, $u_k$ is an evanescent wave in the direction $x > 0$ or $x < 0$. 
When $\kappa$ is real, $u_\kappa$ is a propagative wave whose phase velocity is given by

$$v_\phi(\lambda) := \frac{\lambda}{\kappa}. \quad (56)$$

Locally (if $d\kappa(\lambda)/d\lambda \neq 0$), $\kappa = \kappa(\lambda)$ defines $\lambda$ as a function of $\kappa$ and the group velocity of $u_\kappa$ is defined by

$$v_g(\kappa) := \frac{d\lambda(\kappa)}{d\kappa} = \left(\frac{d\kappa(\lambda)}{d\lambda}\right)^{-1}. \quad (57)$$

If the product $v_g \cdot v_\phi$ is positive, (i.e., when the group and phase velocities point in the same direction), one says that $u_\kappa$ is a direct propagative wave. If the product $v_g \cdot v_\phi$ is negative, (i.e., when the group and phase velocities point in opposite directions), one says that $u_\kappa$ is an inverse propagative wave (also called backward wave in the physical literature).

One uses also the notion of group velocity, which characterizes the direction of propagation (the direction of the energy transport), to distinguish between incoming and outgoing propagative waves. One says that a propagative wave $u_\kappa$ is incoming (respectively, outgoing) in the region $\pm x > 0$ if its group velocity points toward the interface (respectively, toward $\pm \infty$).

In our case, for a real $\kappa$, one denotes by $v_\phi^\pm$ and $v_g^\pm$, the phase and group velocities of a propagative wave $u_\kappa$ in the region $\pm x > 0$. Let us derive a general formula for the product $v_g^\pm \cdot v_\phi^\pm$. In our case, the dispersion relation (33) can be rewritten as:

$$-\kappa^\pm(\lambda)^2 = \Theta^\pm_{\lambda,k} = k^2 - \lambda^2 \epsilon^\pm_\lambda \mu^\pm_\lambda.$$

By differentiating this latter expression with respect to $\lambda$, one gets

$$2\kappa^\pm \frac{d\kappa^\pm}{d\lambda} = -\frac{\partial \Theta^\pm_{\lambda,k}}{\partial \lambda}.$$

From (56) and (57), this yields

$$v_\phi^\pm v_g^\pm = -2\lambda \left(\frac{\partial \Theta^\pm_{\lambda,k}}{\partial \lambda}\right)^{-1}. \quad (58)$$

In the vacuum, one easily check that formula (58) leads to the classical relation

$$v_\phi^- v_g^- = (\epsilon_0 \mu_0)^{-1} > 0.$$

Thus, all the propagative waves are direct propagative. In the Drude material, one has

$$\frac{\partial \Theta^+_{\lambda,k}}{\partial \lambda} = -\frac{d(\lambda \epsilon^+_{\lambda})}{d\lambda} \mu^+_{\lambda} - \frac{d(\lambda \mu^+_{\lambda})}{d\lambda} \epsilon^+_{\lambda}$$

and by expressing the derivatives in the latter expression, one can rewrite (58) as

$$v_\phi^+ v_g^+ = 2\lambda \left[\mu_0 \left(1 + \frac{\Omega_m^2}{\lambda^2}\right) \epsilon^+_{\lambda} + \epsilon_0 \left(1 + \frac{\Omega_e^2}{\lambda^2}\right) \mu^+_{\lambda}\right]^{-1}. \quad (59)$$

One sees with this last expression that the sign of $v_\phi^+ v_g^+$ depends on the sign of $\epsilon^+_{\lambda}$ and $\mu^+_{\lambda}$.

Let us look at what happens in the different spectral zones. This study is summarized in the tables of Figures 5 and 6.

- Suppose first that $(k, \lambda) \in \Lambda_{DD}$. In this spectral zone, $\Theta^\pm_{k,\lambda} = -i |\Theta^\pm_{k,\lambda}|^{1/2}$ thus the corresponding waves $u_\kappa$ are propagative on both sides of the interface. Moreover, as $\epsilon^+_{\lambda} > 0$ and $\mu^+_{\lambda} > 0$, the product (59) is positive: the propagative waves are direct propagative.
Figure 5. Description of the functions $\psi_{k,\lambda,\pm 1}$ in the different spectral zones $\Lambda_{DD}$, $\Lambda_{DI}$, $\Lambda_{DE}$, and $\Lambda_{EI}$. Propagative waves are represented by an oscillating function and the larger and smaller arrow indicate, respectively, the direction of the group velocity and the phase velocity. Evanescent waves are represented with decreasing exponential. The “unphysical” functions $\psi_{k,\lambda,\pm 1}$ which contain increasing exponential behavior are colored in gray.

Figure 6. Description of the plasmonic wave $\psi_{k,\lambda,1}$ in the spectral zone $\Lambda_{EE}$.

waves in both media and thus their direction of propagation is the sign of their wave number $\kappa$. (44) shows that for $x < 0$, function $\psi_{k,\lambda,-1}$ is an oscillating wave of amplitude 1 and wave number $\kappa = -|\Theta_{k,\lambda}^-|^{1/2}$ which propagates toward $-\infty$, whereas for $x > 0$, it is a superposition of a wave of amplitude $A_{k,\lambda,-1}$ and wave number $\kappa = -|\Theta_{k,\lambda}^-|^{1/2}$ which propagates toward the origin and a wave of amplitude $B_{k,\lambda,-1}$ and wave number $\kappa = |\Theta_{k,\lambda}^-|^{1/2}$ which propagates toward $+\infty$. In other words, $\psi_{k,\lambda,-1}$ can be interpreted as an incoming incident wave of amplitude $A_{k,\lambda,-1}$ propagating from $+\infty$ whose diffraction on the interface $x = 0$ generates two outgoing waves: a reflected wave of amplitude $B_{k,\lambda,-1}$ and a transmitted wave of amplitude 1. Similarly, $\psi_{k,\lambda,+1}$ can be interpreted as an incoming incident wave of amplitude $A_{k,\lambda,+1}$ and wave number $\kappa = |\Theta_{k,\lambda}^+|^{1/2}$ propagating from $-\infty$ whose diffraction on the interface $x = 0$ generates two outgoing waves: a reflected wave of amplitude $B_{k,\lambda,+1}$ and wave number $\kappa = -|\Theta_{k,\lambda}^-|^{1/2}$ and a transmitted wave of amplitude 1 and wave number $\kappa = |\Theta_{k,\lambda}^+|^{1/2}$. 
• If \((k, \lambda) \in \Lambda_{DE}\), we still have \(\theta_{k,\lambda}^- = -i |\Theta_{k,\lambda}^-|^1/2\) but now \(\theta_{k,\lambda}^+ = |\Theta_{k,\lambda}^+|^1/2\), which means that propagative waves are no longer allowed in the Drude material: the only possible waves are exponentially decreasing or increasing. Proceeding as above, we see that on the one hand, \(\psi_{k,\lambda,-1}\) can be interpreted as an incident wave of amplitude \(A_{k,\lambda,-1}\) which is exponentially increasing as \(x \to +\infty\) whose diffraction on the interface \(x = 0\) generates two waves: a reflected evanescent wave of amplitude \(B_{k,\lambda,-1}\) and a transmitted outgoing wave of amplitude 1 and wave number \(\kappa = -|\Theta_{k,\lambda}^-|^1/2\). On the other hand, \(\psi_{k,\lambda,+1}\) can be interpreted as an incoming incident wave of amplitude \(A_{k,\lambda,+1}\) and wave number \(\kappa = |\Theta_{k,\lambda}^+|^1/2\) propagating from \(-\infty\) whose diffraction on the interface \(x = 0\) generates two waves: a reflected outgoing wave of amplitude \(B_{k,\lambda,+1}\) and wave number \(\kappa = -|\Theta_{k,\lambda}^-|^1/2\) and a transmitted evanescent wave of amplitude 1.

• Suppose now that \((k, \lambda) \in \Lambda_{DI}\). We still have \(\theta_{k,\lambda}^- = -i |\Theta_{k,\lambda}^-|^1/2\) but propagative waves are again allowed in the Drude material since \(\theta_{k,\lambda}^+ = +i |\Theta_{k,\lambda}^+|^1/2\). Compared with the case where \((k, \lambda) \in \Lambda_{DD}\), we simply have a change of sign in the expression of \(\theta_{k,\lambda}^+\), which amounts to reversing all sign of the wave numbers \(\kappa\) associated to the propagative waves in the Drude material. Hence for \(x > 0\), one might be tempted to interchange the words incoming and outgoing in the above interpretation of \(\psi_{k,\lambda,\pm1}\) for \((k, \lambda) \in \Lambda_{DD}\). It would be wrong! Indeed, the change of sign of the imaginary part of \(\theta_{k,\lambda}^+\) yields an opposite sign of the phase velocity in the Drude material, but not of the group velocity which characterizes the direction of the energy transport. As we see in (59), since \(\varepsilon_\lambda^+\) and \(\mu_\lambda^+\) are both negative in this spectral zone, the phase and group velocity are pointing in opposite directions. Hence, the waves \(u_k\) are inverse propagative in the Drude material.

• Assuming now that \((k, \lambda) \in \Lambda_{EI}\), we have \(\theta_{k,\lambda}^- = |\Theta_{k,\lambda}^-|^1/2\) and \(\theta_{k,\lambda}^+ = +i |\Theta_{k,\lambda}^+|^1/2\). The Drude material behaves as a negative material as in the previous case (since \(\varepsilon_\lambda^+\) and \(\mu_\lambda^+\) are also both negative in this spectral zone), but propagative waves are no longer allowed in the vacuum.

• Finally, if \((k, \lambda) \in \Lambda_{EE}\), both \(\theta_{k,\lambda}^\pm\) are real and positive, so that waves are evanescent on both sides of the interface \(x = 0\). In this case, functions \(\psi_{k,\lambda,-1}\) and \(\psi_{k,\lambda,+1}\) are equal and real. Such a wave is referred as plasmonic wave in the physical literature [22]. Note that in the particular case \(\Omega_e = \Omega_m\), we have \(\mu_\lambda^+ = -\mu_0\) and \(\varepsilon_\lambda^+ = -\varepsilon_0\) so that \(\theta_{k,\lambda}^+ = \theta_{k,\lambda}^-\). This shows that \(\psi_{k,\lambda,-1} = \psi_{k,\lambda,+1}\) is an even function of \(x\).

Among the various categories of waves described above, some of them may be called “unphysical” since they involve an exponentially increasing behavior at infinity, which occurs for \(\psi_{k,\lambda,-1}\) if \((k, \lambda) \in \Lambda_{DE}\) as well as \(\psi_{k,\lambda,+1}\) if \((k, \lambda) \in \Lambda_{EI}\). Fortunately, these waves will disappear by limiting absorption in the upcoming Proposition 14: only the “physical” ones are needed to describe the spectral behavior of \(A_k\). That is why in Figure 4, all the curves represent spectral cuts since they are the boundaries where some \(\psi_{k,\lambda,\pm1}\) appear or disappear.

3.3.3. Boundary values of the Green’s function

We are now able to express the one-sided limit \(g_{k,\lambda}\) of the Green’s function defined in (42). The following two propositions, which distinguish the case of \(\Lambda_{EE}\) from the other zones, provide us convenient expressions of the quantities related to \(g_{k,\lambda}\) which are needed in Stone’s formulas.
Proposition 14. For all \((k, \lambda) \in \Lambda_z\) with \(z \in \{DD, DI, DE, EI\}\), we have

\[
\frac{1}{\pi} \text{Im}(\lambda g_{k,\lambda}(x, x')) = \sum_{j \in I_z} w_{k,\lambda,j}(x) w_{k,\lambda,j}(x') \quad \text{where}
\]

\[
w_{k,\lambda,\pm 1}(x) := \frac{1}{\abs{\mathcal{W}_{k,\lambda}}} \left| \frac{\lambda \theta_{k,\lambda}^\pm}{\pi \mu_{k,\lambda}^\pm} \right|^{1/2} \psi_{k,\lambda,\pm 1}(x) \quad \text{and} \quad I_z := \begin{cases} 
-1, +1 & \text{if } z = DD \text{ or } DI, \\
+1 & \text{if } z = DE, \\
-1 & \text{if } z = EI.
\end{cases}
\]

Proof. Suppose first that \((k, \lambda) \in \Lambda_{DD}\). If this case, (50) and (51) tell us that \(\theta_{k,\lambda}^\pm = -i \text{sgn}(\lambda) \abs{\theta_{k,\lambda}^\pm}\). To express the imaginary part of the Green's function, we rewrite the expression of Proposition 11 in terms of functions \(c_{k,\lambda}\) and \(s_{k,\lambda}\) which are real-valued. We obtain that \(\text{Im}(\lambda g_{k,\lambda}(x, x'))\) is equal to

\[
\abs{\lambda} \abs{\mathcal{W}_{k,\lambda}}^{-1} \left( c_{k,\lambda} \left( \min(x, x') \right) c_{k,\lambda} \left( \max(x, x') \right) \right.
\]

\[
+ \frac{|\theta_{k,\lambda}^- \theta_{k,\lambda}^+|}{\mu_{k,\lambda}^- \mu_{k,\lambda}^+} s_{k,\lambda} \left( \min(x, x') \right) s_{k,\lambda} \left( \max(x, x') \right).
\]

This expression shows that we can replace \(\min(x, x')\) and \(\max(x, x')\) by \(x\) and \(x'\), respectively. It can be written in matrix form as:

\[
\text{Im}(\lambda g_{k,\lambda}(x, x')) = \abs{\lambda} \abs{\mathcal{W}_{k,\lambda}}^{-1} \begin{pmatrix} c_{k,\lambda}(x) \\
|s_{k,\lambda}(x)| \end{pmatrix} \begin{pmatrix} 1 & 0 \\
0 & |\theta_{k,\lambda}^- \theta_{k,\lambda}^+| / \mu_{k,\lambda}^- \mu_{k,\lambda}^+ \end{pmatrix} \begin{pmatrix} c_{k,\lambda}(x') \\\ns_{k,\lambda}(x') \end{pmatrix},
\]

where the symbol * denotes the conjugate transpose of a matrix. Note that the conjugation could be omitted since \(c_{k,\lambda}\) and \(s_{k,\lambda}\) are real-valued. However, it is useful for the next step which consists in rewriting this expression in terms of \(\psi_{k,\lambda,\pm 1}\) by noticing that

\[
\begin{pmatrix} c_{k,\lambda}(x) \\
|s_{k,\lambda}(x)| \end{pmatrix} = \frac{1}{\mathcal{W}_{k,\lambda}} \begin{pmatrix} \theta_{k,\lambda}^+ / \mu_{k,\lambda}^+ & \theta_{k,\lambda}^- / \mu_{k,\lambda}^- \\
1 & -1 \end{pmatrix} \begin{pmatrix} \psi_{k,\lambda,-1}(x) \\
\psi_{k,\lambda,+1}(x) \end{pmatrix},
\]

which follows from the definition (43) of \(\psi_{k,\lambda,\pm 1}\). Therefore, after some calculations exploiting that \(\theta_{k,\lambda}^\pm\) is purely imaginary, one obtains

\[
\text{Im}(\lambda g_{k,\lambda}(x, x')) = \abs{\lambda} \abs{\mathcal{W}_{k,\lambda}}^{-2} \begin{pmatrix} \psi_{k,\lambda,-1}(x) \\
|s_{k,\lambda}(x)| \end{pmatrix} \begin{pmatrix} |\theta_{k,\lambda}^+| / \mu_{k,\lambda}^+ & 0 \\
0 & |\theta_{k,\lambda}^-| / \mu_{k,\lambda}^- \end{pmatrix} \begin{pmatrix} \psi_{k,\lambda,-1}(x') \\\n\psi_{k,\lambda,+1}(x') \end{pmatrix},
\]

which yields the announced result, for \(\mu_{k,\lambda}^+ > 0\) (since \(\abs{\lambda} > \Omega_m\)).

Suppose now that \((k, \lambda) \in \Lambda_{DI}\). Compared with the previous case, we have now \(\theta_{k,\lambda}^+ = +i \text{sgn}(\lambda) \abs{\theta_{k,\lambda}^+}\) and \(\mu_{k,\lambda}^+\) becomes negative (since \(\abs{\lambda} < \Omega_m\)). It is readily seen that in this case, the calculations above hold true if we replace \(\mu_{k,\lambda}^+\) by \(|\mu_{k,\lambda}^+|\).
When \((k, \lambda) \in \Lambda_{DE}\), we still have \(\theta_{k, \lambda}^- = -i \text{sgn}(\lambda) |\theta_{k, \lambda}^-|\), but \(\theta_{k, \lambda}^+\) is now a positive number, which shows that \(\psi_{k, \lambda, +1}\) is a real-valued function. Following the same steps as above, we obtain

\[
\text{Im}(\lambda, g_{k, \lambda}(x, x')) = |\lambda| |W_{k, \lambda}|^{-2} \begin{pmatrix} \psi_{k, \lambda, -1}(x) \\ \psi_{k, \lambda, +1}(x) \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ 0 & \theta_{k, \lambda}^- / \mu_{k, \lambda}^- \end{pmatrix} \begin{pmatrix} \psi_{k, \lambda, -1}(x') \\ \psi_{k, \lambda, +1}(x') \end{pmatrix}.
\]

Finally, if \((k, \lambda) \in \Lambda_{EE}\), we have \(\theta_{k, \lambda}^- > 0\) and \(\theta_{k, \lambda}^+ = -i \text{sgn}(\lambda) |\theta_{k, \lambda}^+|\). As in \(\Lambda_{DI}\), the Drude material behaves as a negative material \((\mu_{k, \lambda}^+ < 0)\). In this case, we obtain

\[
\text{Im}(\lambda, g_{k, \lambda}(x, x')) = |\lambda| |W_{k, \lambda}|^{-2} \begin{pmatrix} \psi_{k, \lambda, -1}(x) \\ \psi_{k, \lambda, +1}(x) \end{pmatrix}^* \begin{pmatrix} |\theta_{k, \lambda}^+ / \mu_{k, \lambda}^+| & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_{k, \lambda, -1}(x') \\ \psi_{k, \lambda, +1}(x') \end{pmatrix},
\]

which completes the proof.

\[\square\]

**Proposition 15.** For all \((k, \lambda) \in \Lambda_{EE}\), we have

\[
(\lambda + i\eta) g_{k, \lambda} + i\eta(x, x') = \eta^{-1}w_{k, \lambda, 0}(x) W_{k, \lambda, 0}(x') + O(1) \quad \text{as } \eta \searrow 0,
\]

where the real-valued function \(w_{k, \lambda, 0}\) is given by

\[
w_{k, \lambda, 0}(x) := \frac{\lambda^2 |\mu_{k, \lambda}^+|^{1/2}}{\Omega_{m}(4k^4 + (\xi_0 \mu_0)^2(\Omega_{m}^2 - \Omega_{m}^2)^2)^{1/4}} e^{-\theta_{k, \lambda}^- |x|} \quad \text{if } \pm x > 0,
\]

and the remainder \(O(1)\) is uniform in \((x, x')\) on any compact set of \(\mathbb{R}^2\). Finally, if \(|k| = k_c\), then \(g_{k, \lambda}(x, x') = O(|\zeta + \lambda_c|^{-1/2})\) as \(\zeta \rightarrow \pm \lambda_c\) from the upper-half plane uniformly with respect to \((x, x')\) on any compact set of \(\mathbb{R}^2\). As a consequence,

\[
\lim_{\eta \searrow 0} (\lambda + i\eta) g_{k, \lambda} + i\eta(x, x') = 0 \quad \text{for } |k| = k_c \text{ and } |\lambda| = \lambda_c.
\]

**Proof.** Let \((k, \lambda) \in \Lambda_{EE}\). It is readily seen that the definition (43) of \(W_{k, \xi}\) can be written equivalently (see also the proof of Lemma 13)

\[
W_{k, \xi} = \frac{Q_k(\zeta^2)}{\zeta^4 (\mu_{k, \xi}^+)^2 (\theta_{k, \xi}^- / \mu_{k, \xi}^- - \theta_{k, \xi}^+)},
\]

where \(Q_k\) is the polynomial defined in (55). As \(Q_k(\lambda_\pm(k)^2) = 0\) and one can compute that \(Q_k(\lambda_{\pm}(k)^2) = -\Omega_{m} \sqrt{4k^4 + K^2}\), we deduce after some manipulations that, for \(\lambda = \pm \lambda_\pm(k)\),

\[
\frac{1}{W_{k, \lambda, + i\eta}} = \frac{i}{\eta} \frac{\lambda^3 |\mu_{k, \lambda}^+| |\theta_{k, \lambda}^+|}{\Omega_{m}^2 \sqrt{4k^4 + K^2}} + O(1) \quad \text{as } \eta \searrow 0.
\]

(where we used the dispersion relation (52) and the fact that \(\mu_{k, \lambda}^+ = -|\mu_{k, \lambda}^+|\)). The announced result follows from the expression (43) of \(g_{k, \xi}\), since \(\psi_{k, \xi, \pm 1}\) are analytic functions of \(\xi\) near \(\lambda\) which both tend to \(\psi_{k, \lambda, -1} = \psi_{k, \lambda, +1}\) uniformly on any compact set of \(\mathbb{R}\).

Finally, if \(|k| = k_c\) and \(\pm \lambda_c\) are simple zeros of \(\zeta \mapsto \Theta_{k, \lambda}^\pm\), it is clear (see formula (43)) that \(W_{k, \xi} = O(|\xi \mp \lambda_c|^{1/2})\) as \(\xi \rightarrow \pm \lambda_c\) from the upper-half plane. Thus, the conclusion follows again from the expression (43) of \(g_{k, \xi}\), since \(\psi_{k, \xi, \pm 1}\) are here uniformly bounded in \(\xi\), \(x\), and \(x'\) when \(\xi\) belongs to a vicinity of \(\pm \lambda_c\) and \((x, x')\) to any compact set of \(\mathbb{R}^2\). \(\square\)
3.4. Diagonalization of the reduced Hamiltonian

3.4.1. Spectral measure of the reduced Hamiltonian

As we shall see, the spectral zones introduced in Section 3.3.1 actually show us the location of the spectrum of $\mathbb{A}_k$ for each $k \in \mathbb{R}$: we simply have to extract the associated sections of these zones, that is, the sets

$$\Lambda_z(k) := \{ \lambda \in \mathbb{R} \mid (k, \lambda) \in \Lambda_z \} \quad \text{for} \quad z \in \mathcal{Z} := \{ \text{DD, DI, DE, EI, EE} \},$$

which all are unions of symmetric intervals with respect to $\lambda = 0$. This is a by-product of the following proposition which tells us that apart from the three eigenvalues $-\Omega_m, 0, \Omega_m$ (see Proposition 8), the spectrum of $\mathbb{A}_k$ is composed of two parts: an absolutely continuous spectrum defined by $\cup_{z \in \mathcal{E}\setminus\{\text{EE}\}} \Lambda_z(k)$ if $|k| < k_c$ and by $\cup_{z \in \mathcal{Z}\setminus\{\text{DI, EE}\}} \Lambda_z(k)$ if $|k| > k_c$ and a pure point spectrum given by $\Lambda_{\text{EE}}(k) = \{ -\lambda_{\text{EE}}(k), +\lambda_{\text{EE}}(k) \}$ if $|k| > k_c$ (we point out that there is no singularly continuous spectrum).

This proposition yields a convenient expression of the spectral projection $\mathbb{E}_k(\Lambda)$ of $\mathbb{A}_k$ for $\Lambda \subset \mathbb{R} \setminus \{-\Omega_m, 0, \Omega_m\}$. As $\mathbb{E}_k(\Lambda)$ is a projection on an invariant subspace by $\mathbb{A}_k$, the canonical way to express such a projection is to use a spectral basis. Propositions 14 and 15 provide us such a basis: these are the vector fields deduced from the $w_{k,\lambda,j}$'s (see (60) and (61)) by the “vectorizator” defined in (36), i.e.,

$$W_{k,\lambda,j} := \forall_{k,\lambda} w_{k,\lambda,j} \quad \text{for} \quad \lambda \in \Lambda_z(k) \quad \text{and} \quad j \in J_z := \begin{cases} \{ -1, +1 \} & \text{if} \quad z = \text{DD or DI,} \\ \{ +1 \} & \text{if} \quad z = \text{DE,} \\ \{ -1 \} & \text{if} \quad z = \text{EI,} \\ \{ 0 \} & \text{if} \quad z = \text{EE,} \end{cases}$$

(62)

for each zone $z \in \mathcal{Z}$. As we shall see afterward, the knowledge of these vector fields leads to a diagonal form of $\mathbb{A}_k$. These are generalized eigenfunctions of $\mathbb{A}_k$.

Proposition 16 below provides us an expression of the spectral measure of $\mathbb{A}_k$ by inner products with the $W_{k,\lambda,j}$'s. However, except for $z = \text{EE}$, these functions do not belong to $\mathcal{H}_{1D}$ since these are oscillating (bounded) functions at infinity (that is why they are called generalized). A simple way to overcome this difficulty is to introduce weighted $L^2$ spaces

$$L^2_s(\mathcal{O}) := \{ u \in L^2_{\text{loc}}(\mathcal{O}) \mid (1 + x^2)^{s/2} u \in L^2(\mathcal{O}) \} \quad \text{for} \quad s \in \mathbb{R} \quad \text{and} \quad \mathcal{O} = \mathbb{R} \quad \text{or} \quad \mathcal{O} = \mathbb{R}^+. $$

We can then define

$$\mathcal{H}_{1D,s} := L^2_s(\mathbb{R}) \times L^2_s(\mathbb{R})^2 \times L^2_s(\mathbb{R}^+) \times L^2_s(\mathbb{R}^+)^2 \quad \text{for} \quad s \in \mathbb{R}. $$

(63)

It is clear that for positive $s$, the spaces $\mathcal{H}_{1D,+s}$ and $\mathcal{H}_{1D,-s}$ are dual to each other if $\mathcal{H}_{1D}$ is identified with its own dual space, which yields the continuous embeddings $\mathcal{H}_{1D,+s} \subset \mathcal{H}_{1D} \subset \mathcal{H}_{1D,-s}$. We denote by $\langle \cdot, \cdot \rangle_{1D,s}$ the duality product between $\mathcal{H}_{1D,+s}$ and $\mathcal{H}_{1D,-s}$, which extends the inner product of $\mathcal{H}_{1D}$ in the sense that

$$\langle U, U' \rangle_{1D,s} = (U, U')_{1D} \quad \text{if} \quad U \in \mathcal{H}_{1D} \quad \text{and} \quad U' \in \mathcal{H}_{1D,+s}. $$

It is readily seen that $L^\infty(\mathbb{R}) \subset L^2_{-s}(\mathbb{R})$ if and only if $s > 1/2$. As the $W_{k,\lambda,j}$'s are bounded functions, they belong to $\mathcal{H}_{1D,-s}$ as soon as $s > 1/2$ and one can define $\langle U, W_{k,\lambda,j} \rangle_{1D,s}$ for any $U \in \mathcal{H}_{1D,s}$. 
\textbf{Proposition 16.} Let \( s > 1/2, k \in \mathbb{R}, \mathbb{E}_k \) the spectral measure of \( \Lambda_k \) and \( U \in \mathcal{D}_{1D} \) (see (46)). For all interval \( \Lambda \subset \Lambda_z(k) \) with \( z \in \mathcal{Z} := \{DD, DI, DE, E1\} \), we have

\[
\|E_k(\Lambda)\ U\|^2_{1D} = \int_\Lambda \sum_{j \in J_z} |\langle U, W_{k,\lambda, j}\rangle_{1D,s}|^2 \ d\lambda \quad \text{and} \quad \|E_k(\pm \lambda_k)\ U\|^2_{1D} = \begin{cases} \|\langle U, W_{k,\pm \lambda_k,0}\rangle_{1D}|^2 & \text{if } |k| > k_c, \\ 0 & \text{if } |k| = k_c. \end{cases}
\]

Moreover, \( \|E_k(\Lambda)\ U\|^2_{1D} = 0 \) for all interval \( \Lambda \subset \mathbb{R} \setminus \bigcup_{z \in \mathcal{Z}} \Lambda_z(k) \), which does not contain 0 or \( \pm \Omega_m \).

\textbf{Proof.} Let \( z \in \mathcal{Z} \setminus \{EE\}, \Lambda := [a, b] \subset \Lambda_z(k) \) with \( a < b \) and \( U \in \mathcal{D}_{1D} \). We show in the second part of the proof that \( E_k([a]) U = E_k([b]) U = 0 \), which implies that \( E_k((a, b)) U = E_k([a, b]) U \). Hence, Stone’s formula (17) together with the integral representation (47) shows that the quantity \( \|E_k(\Lambda)\ U\|^2_{1D} \) is given by the following limit, where \( \zeta := \lambda + \imath \eta : \)

\[
\|E_k(\Lambda)\ U\|^2_{1D} = \lim_{\eta \searrow 0} \frac{1}{\pi} \int_a^b \mathrm{Im} \left( \left( T_{k,\zeta} U, U \right)_{1D} + \int_{\mathbb{R}^2} \zeta \ g_{k,\zeta} (x, x') \ S_{k,\zeta} U(x) \overline{S_{k,\zeta} U(x')} \ dx \ dx' \right) d\lambda.
\]

The first step is to permute the limit and the integrals in this formula thanks to Lebesgue’s dominated convergence theorem. According to the foregoing, as \( \lambda \neq 0 \) and \( \lambda \neq \pm \Omega_m \), the integrand is a continuous function of \( \eta \) for all \( \lambda \in [a, b] \), \( x \) and \( x' \) in the compact support of \( U \) (recall that \( U \in \mathcal{D}_{1D} \)). Moreover, the integrand is dominated by a constant (provided \( \eta \) remains bounded). Therefore, the permutation is justified: in the above formula, we can simply replace \( \zeta \) and \( \bar{\zeta} \) by \( \lambda \).

Formula (38) tells us that \( T_{k,\lambda} \) is self-adjoint, hence \( \mathrm{Im}(T_{k,\lambda} U, U)_{1D} = 0 \). Besides, notice that

\[
\mathrm{Im} \left( \int_{\mathbb{R}^2} \lambda \ g_{k,\lambda} (x, x') \ u(x) \overline{u(x')} \ dx \ dx' \right) = \int_{\mathbb{R}^2} \mathrm{Im} \left( \lambda \ g_{k,\lambda} (x, x') \right) u(x) \overline{u(x')} \ dx \ dx',
\]

where \( u := S_{k,\lambda} U \) (this is easily deduced from the symmetry of \( g_{k,\lambda} \), i.e., \( g_{k,\lambda}(x, x') = g_{k,\lambda}(x', x) \), see (42)). This allows us to use Proposition 14 so that

\[
\|E_k(\Lambda)\ U\|^2_{1D} = \int_a^b \int_{\mathbb{R}} \sum_{j \in J_z} w_{k,\lambda,j}(x) \overline{w_{k,\lambda,j}(x')} S_{k,\lambda} U(x) \overline{S_{k,\lambda} U(x')} \ dx \ dx' \ d\lambda.
\]

By Fubini’s theorem, this expression becomes

\[
\|E_k(\Lambda)\ U\|^2_{1D} = \int_a^b \sum_{j \in J_z} \left( \int_{\mathbb{R}} S_{k,\lambda} U(x) \overline{w_{k,\lambda,j}(x)} \ dx \right)^2 \ d\lambda,
\]

which is nothing but (64), since an integration by parts shows that

\[
\int_{\mathbb{R}} S_{k,\lambda} U(x) \overline{w_{k,\lambda,j}(x)} \ dx = \langle U, W_{k,\lambda,j}\rangle_{1D,s}.
\]

By virtue of the \( \sigma \)-additivity of the spectral measure (see Definition 4), formula (64) holds true for any interval \( \Lambda := (a, b) \subset \Lambda_z(k) \) even if the resolvent is singular. Indeed, this singular
behavior occurs only if \(|k| = k_c\) at \(\lambda = \pm \lambda_c\) where Proposition 15 tells us that in this case, \(g_{k,\xi}(x, x') = O(|\xi + \lambda_c|^{-1/2})\), which is an integrable singularity.

Suppose now that for \(a < b\), \(\Lambda = [a, b]\) is located outside \(\bigcup_{z \in \mathbb{Z}} \Lambda_z(k)\) and does not contain 0 or \(\pm \Omega_m\). In this case, the one-sided limit \(g_{k,\lambda}\) of the Green's function is real-valued for all \(\lambda \in \Lambda\) (since \(\theta_{k,\lambda}^\pm \in \mathbb{R}^+\), see \((50)-(51)) and the same steps as above yield
\[
\|E_k(\Lambda) U\|_{1D}^2 = 0.
\]
Consider finally singletons \(\Lambda = \{a\}\) for \(a \in \mathbb{R} \setminus \{-\Omega_m, 0, +\Omega_m\}\). Stone's formula (18) together with (47) shows that for all \(U \in \mathcal{D}_{1D}\), the quantity \(\|E_k(\{a\}) U\|_{1D}^2\) is given by the following limit, where \(\zeta := a + i\eta:\)
\[
\|E_k(\{a\}) U\|_{1D}^2 = \lim_{\eta \searrow 0} \eta \text{ Im} \left( (T_{k,\xi} U, U)_{1D} + \int_{\mathbb{R}^2} \zeta g_{k,\xi}(x, x') S_{k,\xi} U(x) \overline{S_{k,\xi} U(x')} \, dx \, dx' \right).
\]
This shows that \(E_k(\{a\})\) can be nonzero only if \(a\) is a singularity of \(T_{k,\xi}, S_{k,\xi}\), or \(g_{k,\xi}\). From (35) and (37), the singularities of \(T_{k,\xi}\) and \(S_{k,\xi}\) are \(\zeta = 0\) and \(\zeta = \pm \Omega_m\), but these points are excluded from our study since Proposition 8. Hence, we are only interested in the singularities of \(g_{k,\xi}\), that is, the zeros \(\pm \lambda_c(k)\) of \(W_{k,\xi}\) defined in Lemma 13. Suppose then that \(|k| > k_c\) and \(a = \pm \lambda_c(k)\). As above, using Lebesgue's dominated convergence theorem and Proposition 15, we obtain
\[
\|E_k(\{a\}) U\|_{1D}^2 = \int_{\mathbb{R}^2} \frac{w_{k,a,0}(x)}{w_{k,a,0}(x')} S_{k,a} U(x) \overline{S_{k,a} U(x')} \, dx \, dx' = \left| \int_{\mathbb{R}} S_{k,a} U(x) \frac{w_{k,a,0}(x)}{w_{k,a,0}(x')} \, dx \right|^2.
\]
where the second inequality follows from Fubini's theorem. Integrating by parts, formula (64) follows. On the other hand, if \(|k| = k_c\) and \(|a| = \lambda_c\), the Green's function is singular near \(a\), but Proposition 15 tells us that \(g_{k,a+i\eta}(x, x') = O(\eta^{-1/2})\), which shows that \(\|E_k(\{a\}) U\|_{1D}^2 = 0\).

### 3.4.2. Generalized Fourier transform for the reduced Hamiltonian

The aim of this subsection is to deduce from the knowledge of the spectral measure \(E_k(\cdot)\) a generalized Fourier transform \(\mathbb{F}_k\) for \(A_k\), that is, an operator which provides us a diagonal form of the reduced Hamiltonian \(A_k\) as
\[
A_k = \mathbb{F}_k^* \lambda \mathbb{F}_k.
\]
This transformation \(\mathbb{F}_k\) maps \(\mathcal{H}_{1D}\) to a spectral space \(\hat{\mathcal{H}}_k\) which contains fields that depend on the spectral variable \(\lambda\). In the above diagonal form of \(A_k\), “\(\lambda\)” denotes the operator of multiplication by \(\lambda\) in \(\hat{\mathcal{H}}_k\). In short, \(\mathbb{F}_k\) transforms the action of \(A_k\) in the physical space \(\mathcal{H}_{1D}\) into a linear spectral amplification in the spectral space \(\hat{\mathcal{H}}_k\). We shall see that \(\mathbb{F}_k\) is a partial isometry which becomes unitary if we restrict it to the orthogonal complement of the eigenspaces associated with the three eigenvalues 0 and \(\pm \Omega_m\), that is, the space \(\mathcal{H}_{1D}(\text{div} \, 0)\) defined in Proposition 8.

The definition of \(\mathbb{F}_k\) comes from formulas (64) and (65): for a fixed \(k \in \mathbb{R}\) and all \(U \in \mathcal{H}_{1D,s}\), we denote
\[
\forall z \in \mathbb{Z}, \forall \lambda \in \Lambda_z(k) \text{ and } \forall j \in J_z, \quad \mathbb{F}_k U(\lambda, j) := \langle U, W_{k,\lambda,j} \rangle_{1D,s}, \tag{66}
\]
which represents the “decomposition” of \( U \) on the family of generalized eigenfunctions \((W_{k,\lambda,j})\) of \( A_k \) defined in \((62)\). We show below that the codomain of \( F_k \) is given by

\[
\hat{H}_k = \begin{cases} 
L^2(\Lambda_{dd}(k))^2 + L^2(\Lambda_{de}(k)) + L^2(\Lambda_{di}(k))^2 + L^2(\Lambda_{ei}(k)) & \text{if } |k| \leq k_c \\
L^2(\Lambda_{dd}(k))^2 + L^2(\Lambda_{de}(k)) + L^2(\Lambda_{ei}(k)) + l^2(\Lambda_{ee}(k)) & \text{if } |k| > k_c.
\end{cases}
\]

\((67)\)

We point out that the space \( L^2(\Lambda_{ee}(k)) \) is here isomorphic to \( \mathbb{C}^2 \) since \( \Lambda_{ee}(k) = \{-\lambda_+(k), \lambda_-(k)\} \). We denote by \( \hat{U}(\lambda, j) \) the fields of \( \hat{H}_k \), where it is understood that \( \lambda \in \Lambda_z(k) \) and \( j \in J_z \) for \( z \in \{dd, de, di, ei\} \) if \( |k| \leq |k_c| \) and for \( z \in \{dd, de, ei, ee\} \) if \( |k| > |k_c| \). The Hilbert space \( \hat{H}_k \) is endowed with the norm \( \| \cdot \|_{\hat{H}_k} \) defined by

\[
\| \hat{U} \|^2_{\hat{H}_k} = \begin{cases} 
\sum_{z \in \{dd, de, di, ei\}} \int_{\Lambda_z(k)} \sum_{j \in J_z} |\hat{U}(\cdot, j)|^2 \ d\lambda & \text{if } |k| \leq k_c, \\
\sum_{z \in \{dd, de, ei\}} \int_{\Lambda_z(k)} \sum_{j \in J_z} |\hat{U}(\cdot, j)|^2 \ d\lambda + \sum_{\lambda \in \Lambda_{ee}(k)} |\hat{U}(\lambda, 0)|^2 & \text{if } |k| > k_c.
\end{cases}
\]

\((68)\)

The following theorem expresses the diagonalization of the reduced Hamiltonian. Its proof is classical (see, e.g., \([19, \text{Section 2.4.2}]\)) and consists of two steps. We first deduce from Theorem 5 that \( F_k \) is an isometry from \( \mathcal{H}_{1D}(\text{div}_k 0) \) (defined by \((23)\)) to \( \hat{H}_k \) which diagonalizes \( A_k \). We then prove that \( F_k \) is surjective.

**Theorem 17.** For \( k \in \mathbb{R}^* \), let \( F_k \) denotes the orthogonal projection on the subspace \( \mathcal{H}_{1D}(\text{div}_k 0) \) of \( \mathcal{H}_{1D} \), i.e., \( F_k = \chi \left( A_k \right) \) where \( \chi \) is the indicator function of \( \mathbb{R} \setminus (-\Omega_m, 0, \Omega_m) \). The operator \( F_k \) defined in \((66)\) extends by density to a partial isometry from \( \mathcal{H}_{1D} \) to \( \hat{H}_k \) whose restriction to the range of \( F \) (that is, \( \mathcal{H}_{1D}(\text{div}_k 0) \)) is unitary. Moreover, \( F_k \) diagonalizes the reduced Hamiltonian \( A_k \) in the sense that for any measurable function \( f : \mathbb{R} \to \mathbb{C} \), we have

\[
F_k f(A_k) = f(A_k) F_k = F_k^* f(\lambda) F_k \quad \text{on } \text{D}(f(A_k)),
\]

\((69)\)

where \( f(\lambda) \) stands for the operator of multiplication by the function \( f \) in the spectral space \( \hat{H}_k \).

**Proof.** First, notice that the orthogonal projection \( P_k \) onto \( \mathcal{H}_{1D}(\text{div}_k 0) \) is indeed the spectral projection \( E_k(\mathbb{R} \setminus (-\Omega_m, 0, \Omega_m)) = \chi(A_k) \) by \((16)\) and Proposition 8.

From Proposition 16 and the definition \((66)\) of \( F_k \), one can rewrite the spectral measure of \( A_k \) for any interval \( \Lambda \subset \mathbb{R} \setminus (-\Omega_m, 0, \Omega_m) \) and for all \( U \in \mathcal{D}_{1D} \) as

\[
\| E_k(\Lambda) U \|^2_{1D} = \begin{cases} 
\sum_{z \in \{dd, de, di, ei\}} \int_{\Lambda \cap \Lambda_z(k)} \sum_{j \in J_z} |F_k U(\lambda, j)|^2 \ d\lambda & \text{if } 0 < |k| \leq k_c, \\
\sum_{z \in \{dd, de, ei\}} \int_{\Lambda \cap \Lambda_z(k)} \sum_{j \in J_z} |F_k U(\lambda, j)|^2 \ d\lambda + \sum_{\Lambda \cap \Lambda_{ee}(k)} |F_k U(\lambda, 0)|^2 & \text{if } |k| > k_c.
\end{cases}
\]

In the particular case \( \Lambda = \mathbb{R} \setminus (-\Omega_m, 0, \Omega_m) \), we have \( \Lambda \cap \Lambda_z(k) = \Lambda_z(k) \) for all \( z \in \mathbb{Z} \), thus using the definition \((68)\) of the norm \( \| \cdot \|_{\hat{H}_k} \) leads to the following identity

\[
\| F_k U \|^2_{\mathcal{H}_{1D}} = \| F_k U \|^2_{\hat{H}_k}, \quad \forall U \in \mathcal{D}_{1D}.
\]
Hence, as $\mathcal{D}_{1D}$ is dense in $\mathcal{H}_{1D}$, $F_k$ extends to a bounded operator on $\mathcal{H}_{1D}$ and the latter formula holds for all $U \in \mathcal{H}_{1D}$. Thus, $F_k$ is a partial isometry which satisfies $F_k^* F_k = I$ and its restriction to the range of $F_k$ is an isometry.

In the sequel, as the above expression of the spectral measure depends on $k$, we only detail the case $0 < |k| \leq k_c$. The case $|k| > k_c$ can be dealt with in the same way. Using the polarization identity, the above expression of $\|E_k(A)\|_{1D}^2$ yields that of $(E_k(A) U, V)_{1D}$ and the spectral theorem 5 then shows that

$$
(f(A_k) F_k U, V)_{\mathcal{H}_{1D}} = \sum_{z \in \{\text{dd}, \text{de}, \text{di}, \text{ei}\}} \int_{\Lambda_z(k)} f(\lambda) \sum_{j \in J_z} F_k U(\lambda, j) \overline{F_k V(\lambda, j)} \, d\lambda,
$$

which holds for all $U \in D(f(A_k))$ (note that $f(A_k)$ and $P_k$ commute, thus $P_k U \in D(f(A_k))$ for $U \in D(f(A_k))$) and $V \in \mathcal{H}_{1D}$. Using the definition of the inner product in $\mathcal{H}_k$ (see (68)), this latter formula can be rewritten as:

$$(f(A_k) P_k U, V)_{\mathcal{H}_{1D}} = (f(\lambda) F_k U, F_k V)_{\mathcal{H}_k},$$

which yields (69).

Let us prove now that the isometry $F_k : \mathcal{H}_{1D}(\text{div} 0) \to \mathcal{H}_k$ is unitary, i.e., that $F_k$ is surjective or equivalently that $F_k^*$ is injective. Let $\hat{U} \in \mathcal{H}_k$ such that $F_k^* \hat{U} = 0$, then

$$(\hat{U}, F_k V)_{\mathcal{H}_k} = 0, \quad \forall V \in \mathcal{H}_{1D}. \tag{71}$$

We now choose a spectral zone $z \in \{\text{dd}, \text{de}, \text{di}, \text{ei}\}$. For any interval $\Lambda \subset \Lambda_z(k)$, one denotes $1_\Lambda \in \mathcal{L}(\mathcal{H}_k)$, the orthogonal projection in $\mathcal{H}_k$ corresponding to the multiplication by the indicator function of $\Lambda$. We shall show at the end of the proof the commutation property

$$F_k E_k(\Lambda) = 1_\Lambda F_k \text{ in } \mathcal{L}(\mathcal{H}_{1D}, \mathcal{H}_k). \tag{72}$$

Using this relation and (71) for $E_k(\Lambda)V$ instead of $V$, we get

$$(\hat{U}, F_k E_k(\Lambda)V)_{\mathcal{H}_k} = (\hat{U}, 1_\Lambda F_k V)_{\mathcal{H}_k} = \int_{\Lambda} \sum_{j \in J_z} \hat{U}(\lambda, j) \overline{F_k V(\lambda, j)} \, d\lambda = 0,$$

where we have used the definition of inner product in $\mathcal{H}_k$ (cf (68)). Hence, as the last formula holds for any interval $\Lambda \subset \Lambda_z(k)$, we get

$$\sum_{j \in J_z} \hat{U}(\lambda, j) \overline{F_k V(\lambda, j)} = \sum_{j \in J_z} \hat{U}(\lambda, j) W_{k,\lambda, j} V_{1D, s} = 0, \quad \forall V \in \mathcal{H}_{1D,s} \text{ for a.e. } \lambda \in \Lambda_z(k)$$

and thus $\sum_{j \in J_z} \hat{U}(\lambda, j) W_{k,\lambda, j} = 0$ in $\mathcal{H}_{1D,s}$ for a.e. $\lambda \in \Lambda_z(k)$. The family $(\psi_{k,\lambda,j})_{j \in J_z}$ is clearly linearly independent (see (44) and (45)), so it is also the family $(w_{k,\lambda,j})_{j \in J_z}$ (see (60)). Then it follows from (62) that the $(W_{k,\lambda,j})_{j \in J_z}$ are linearly independent too. Therefore

$$\hat{U}(\lambda, j) = 0 \quad \text{for a.e. } \lambda \in \Lambda_z(k).$$

As it holds for any $z \in \{\text{dd}, \text{de}, \text{di}, \text{ei}\}$, $\hat{U} = 0$. Hence, $F_k^*$ is injective and $F_k$ is surjective.

It remains to prove (72). Using (69) with $f = 1_\Lambda$ leads to $E_k(\Lambda) = F_k^* 1_\Lambda F_k$ and therefore to $F_k E_k(\Lambda) = Q_k 1_\Lambda F_k$ where $Q_k = F_k F_k^*$ is the orthogonal projection onto the (closed) range of $F_k$. To remove $Q_k$, we point out that

$$\|F_k E_k(\Lambda)V\|_{\mathcal{H}_k}^2 = (E_k(\Lambda)V, V)_{\mathcal{H}_{1D}}^2 = \|1_\Lambda F_k V\|_{\mathcal{H}_k}^2, \quad \forall V \in \mathcal{H}_{1D},$$

which yields (69).
where the first equality is an immediate consequence of the relations $F_k^* F_k = F_k$ and $P_k E_k(\Lambda) = E_k(\Lambda)$, whereas the second one is readily deduced from (70) by taking $U = V$ and $f = 1_\Lambda = 1^2_\Lambda$.

In the following proposition, we give an explicit expression of the adjoint $F_k^*$ of the generalized Fourier transform $F_k$ which is a “recomposition operator” in the sense that its “recomposes” a function $U \in H_{1D}$ from its spectral components $\hat{U}(\lambda, j) \in \hat{H}_k$ which appear as “coordinates” on the spectral basis $(W_{k,\lambda, j})$. As $F_k^*$ bounded in $\hat{H}_k$, if suffices to know it on a dense subspace of $\hat{H}_k$.

**Proposition 18.** Let $\hat{H}_{k,\text{comp}}$ denote the dense subspace of $\hat{H}_k$ composed of compactly supported functions whose supports contain neither the boundaries of the spectral zones $\Lambda_x(k)$ for $z \in \{\text{DD, DE, DI, EI}\}$ (i.e., the points $\pm \lambda_0(k)$, $\pm \lambda_\alpha(k)$ and $\pm \lambda_\beta(k)$), nor the points 0 and $\pm \Omega_m$. For all $\hat{U} \in \hat{H}_{k,\text{comp}}$, we have

$$F_k^* \hat{U} = \begin{cases} \sum_{z \in \{\text{DD, DE, DI, EI}\}} \sum_{j \in J_z} \int_{\Lambda_z(k)} \hat{U}(\lambda, j) W_{k,\lambda, j} \, d\lambda & \text{if } 0 < |k| \leq k_c, \\
\sum_{z \in \{\text{DD, DE, DI, EI}\}} \sum_{j \in J_z} \int_{\Lambda_z(k)} \hat{U}(\lambda, j) W_{k,\lambda, j} \, d\lambda \\
+ \sum_{\lambda \in \Lambda_{z}(k)} \hat{U}(\lambda, 0) W_{k,\lambda, 0} & \text{if } |k| > k_c. \end{cases} \tag{73}$$

where the integrals in the right-hand side of (73) are vector-valued integrals (Bochner integrals [40, Section 5.5]) with values in $H_{1D, -s}$.

**Remark 19.** (i) The reason why we have to restrict ourselves to functions of $\hat{H}_{k,\text{comp}}$ in (73) is that the $H_{1D, -s}$-norm of $W_{k,\lambda, j}$ remains uniformly bounded if $\lambda$ is restricted to vary in a compact set of $\mathbb{R}$ that intersects neither the boundaries of the spectral zones $\Lambda_x(k)$ for $z \in \{\text{DD, DE, DI, EI}\}$, nor the points 0 and $\pm \Omega_m$. On the contrary, this norm tends to $\infty$ as soon as $\lambda$ approaches any of these points. Indeed, as mentioned in Section 3.3.2, the boundaries of the spectral zones are the lines where some $W_{k,\lambda, j}$ appear or disappear; these $W_{k,\lambda, j}$ have a linear growth at infinity on these lines, except on the critical points where $\lambda = \pm \lambda_c$ (see (44)–(45)). Moreover, the expression (60) of $W_{k,\lambda, \pm 1}$ blows up near $\lambda = 0$ (since $\theta_{k,\lambda}^{\pm} \to \infty$, see (40) and (4)), near $\lambda = \pm \Omega_m$ (since $\mu_\lambda^{\pm} \to 0$) as well as near the critical points (since $W_{k,\lambda} \to 0$).

(ii) When $\hat{U} \in \hat{H}_{k,\text{comp}}$, the integrals considered in (73), whose integrands are valued in $H_{1D, -s}$, are Bochner integrals [40] in $H_{1D, -s}$. However, as $F_k^*$ is bounded from $\hat{H}_k$ to $H_{1D}$, the values of these integrals belongs to $H_{1D}$. By virtue of the density of $\hat{H}_{k,\text{comp}}$ in $\hat{H}_k$, the expression of $F_k^* \hat{U}$ for any $\hat{U} \in \hat{H}_k$ follows by approximating $\hat{U}$ by its restrictions to an increasing sequence of compact subsets of $\mathbb{R}$ as in the definition of $\hat{H}_{k,\text{comp}}$. Of course, the limit we obtain belongs to $H_{1D}$ and does not depend on the sequence (note that this is similar to the limiting process used to express the usual Fourier transform of a $L^2$ function). This limit process will be implicitly understood in the sequel.
Proof. We prove this result in the case $0 < |k| \leq k_c$, the case $|k| > k_c$ can be dealt with in the same way. Let $\hat{U} \in \hat{\mathcal{H}}_{k,\text{comp}}$. By definition of the adjoint, for all $V \in \mathcal{H}_{1D,+s}$, one has $(\mathbb{F}_{k}^{*}\hat{U}, V)_{\mathcal{H}_{k}} = (\hat{U}, \mathbb{F}_{k} V)_{\mathcal{H}_{k}}$ and the expression of $\mathbb{F}_{k} V$ yields

$$(\mathbb{F}_{k}^{*}\hat{U}, V)_{\mathcal{H}_{k}} = \sum_{z \in \{\text{DD,DE,DI,EI}\}} \sum_{j \in J_{z}} \int_{\Lambda_{z}(k)} \hat{U}(\lambda,j) \langle W_{k,\lambda,j}, V \rangle_{1D,s} d\lambda.$$ 

One can permute the duality product in $x$ and the Bochner integral in $\lambda$ to obtain

$$(\mathbb{F}_{k}^{*}\hat{U}, V)_{\mathcal{H}_{k}} = \left( \sum_{z \in \{\text{DD,DE,DI,EI}\}} \sum_{j \in J_{z}} \int_{\Lambda_{z}(k)} \hat{U}(\lambda,j) W_{k,\lambda,j} d\lambda, V \right)_{1D,s}.$$ 

As it holds for any $V \in \mathcal{H}_{1D,+s}$, this yields (73). The permutation is here justified by the following arguments: for any $z \in \{\text{DD,DE,DI,EI}\}$ and $j \in J_{z}$, $\hat{U}(\cdot,j)$ is a $L^{1}(\Lambda_{z}(k))$ compactly supported function in $\lambda$ and the generalized eigenfunctions $W_{k,\lambda,j}$ are uniformly bounded for $x \in \mathbb{R}$ and $\lambda$ in the compact support of $\hat{U}(\cdot,j)$. Thus, the left-hand side of the duality product is a finite sum of Bochner integrals, since the considered integrands (which are vector-valued in $\mathcal{H}_{1D,-s}$) are integrable. Hence, the permutation follows from a standard property (Fubini’s like) of Bochner integrals [40, Corollary 2, p.134].

4. Spectral theory of the full Hamiltonian and application to the evolution problem

The hard part of the work is now done: for each fixed nonzero $k$, we have obtained a diagonal form of the reduced Hamiltonian $A_{k}$. It remains to gather this collection of results for $k \in \mathbb{R}^{*}$, which yields a diagonal form of the full Hamiltonian $\hat{A}$. The proper tools to do so are the notions of direct integrals of Hilbert spaces and operators [11, 28, 36]. We first recall briefly their definitions and basic properties.

Direct integrals of Hilbert spaces provide a generalization of the notion of direct sums. For a given continuous family of Hilbert spaces $\{\mathcal{H}_{p} \mid p \in \mathbb{R}\}$ and a measure $d m_{p}$ on $\mathbb{R}$, the direct integral $\mathcal{H} = \int_{\mathbb{R}}^{\oplus} \mathcal{H}_{p} d m_{p}$ is the Hilbert space composed of all the vector-valued functions $u : \mathbb{R} \ni p \mapsto u_{p} \in \mathcal{H}_{p}$ such that $\int_{\mathbb{R}}^{\oplus} \|u_{p}\|_{p}^{2} d m_{p} < \infty$, equipped with the inner product $\int_{\mathbb{R}}^{\oplus} \langle \cdot, \cdot \rangle_{p} d m_{p}$, where $\langle \cdot, \cdot \rangle_{p}$ and $\| \cdot \|_{p}$ denote the inner product and associated norm in $\mathcal{H}_{p}$. The elements of $\mathcal{H}$ are often denoted by $u = \int_{\mathbb{R}}^{\oplus} u_{p} d m_{p}$ (which is just a notation, not an integral).

The simplest example is the case of a constant fiber direct integral, that is, when $\mathcal{H}_{p} = \mathcal{H}_{0}$ does not depend on $p$. In this case, the direct integral is nothing but the space $L^{2}(\mathbb{R}, d m_{p}; \mathcal{H}_{0})$ of square integrable $\mathcal{H}_{0}$-valued functions [11, p.148] or equivalently the tensor product of Hilbert spaces $L^{2}(\mathbb{R}, d m_{p}) \otimes \mathcal{H}_{0}$ [1, Theorem 12.6.1].

A direct integral of operators, also called a decomposable operator, is an operator which acts separately on the various spaces $\mathcal{H}_{p}$. More precisely, if $\{A_{p} \in \mathcal{L}(\mathcal{H}_{p}) \mid p \in \mathbb{R}\}$ is a family of bounded operators such that $\text{ess-sup} \|A_{p}\|_{\mathcal{L}(\mathcal{H}_{p})} < \infty$ (where ess-sup denotes the essential supremum), one can define a bounded operator $A = \int_{\mathbb{R}}^{\oplus} A_{p} d m_{p}$ (which is again just a notation, not an integral) by

$$\int_{\mathbb{R}}^{\oplus} A_{p} d m_{p} \int_{\mathbb{R}}^{\oplus} u_{p} d m_{p} = \int_{\mathbb{R}}^{\oplus} A_{p} u_{p} d m_{p},$$
which can be expressed equivalently as \((A_u)_p = A_p u_p\) for a.e. \(p \in \mathbb{R}\). The norm of \(A\) is exactly \(\text{ess-sup} \|A_p\|_{\mathcal{L}(\mathcal{H}_p)}\). The notion of direct integral of operators also extends to unbounded operators \([28, \text{pp.283–286}]\).

### 4.1. Abstract diagonalization of \(A\)

The first step consists in rewriting the link \((21)\) between \(A\) and \(A_k\) in an abstract form using direct integrals. The diagonal expression of \(A\) will then follow from Theorem 17 using general arguments.

The partial Fourier transform in the \(y\)-direction \(\mathcal{F}\) led us to define \(A_k\) for each fixed \(k\) as an operator in \(\mathcal{H}_{1D}\) (see \((20)\)). Actually, the initial space \(\mathcal{H}_{2D}\) introduced in \((8)\) is nothing but the tensor product of Hilbert spaces \(\mathcal{H}_{1D} \otimes L^2(\mathbb{R}_y)\) or equivalently the (constant fiber) integral

\[
\mathcal{H}_{2D} = \int_{\mathbb{R}}^{\oplus} \mathcal{H}_{1D} \, dy.
\]

As the Fourier transform \(\mathcal{F}\) is a unitary operator from \(L^2(\mathbb{R}_y)\) to \(L^2(\mathbb{R}_k)\), we infer that it defines a unitary operator from \(\mathcal{H}_{2D}\) to

\[
\mathcal{H}_{1D}^{\oplus} := \mathcal{H}_{1D} \otimes L^2(\mathbb{R}_k) = \int_{\mathbb{R}}^{\oplus} \mathcal{H}_{1D} \, dk.
\]

A vector \(U_\oplus \in \mathcal{H}_{1D}^{\oplus}\) is simply a 4-uplet analogous to a \(U \in \mathcal{H}_{2D}\) but depending on the pair of variables \((x, k)\) instead of \((x, y)\). The notation \(U_\oplus = \int_{\mathbb{R}}^{\oplus} U_k \, dk\) amounts to saying that \(U_k = U_\oplus(\cdot, k) \in \mathcal{H}_{1D}\) for a.e. \(k \in \mathbb{R}\). Moreover

\[
\|U_\oplus\|^2_{\mathcal{H}_{1D}^{\oplus}} = \int_{\mathbb{R}} \|U_k\|^2_{\mathcal{H}_{1D}} \, dk.
\]

In this functional framework, we can gather the family of reduced Hamiltonian \(A_k\) for \(k \in \mathbb{R}\) as a direct integral of operators \(A_\oplus\) defined by

\[
A_\oplus := \int_{\mathbb{R}}^{\oplus} A_k \, dk,
\]

which means that for a.e. \(k \in \mathbb{R}\),

\[
(A_\oplus U_\oplus)_k = A_k U_k, \quad \forall U_\oplus \in D(A_\oplus),
\]

where

\[
D(A_\oplus) := \left\{ U_\oplus \in \mathcal{H}_{1D}^{\oplus} \mid U_k \in D(A_k) \text{ for a.e. } k \in \mathbb{R} \text{ and } \int_{\mathbb{R}} \|A_k U_k\|^2_{\mathcal{H}_{1D}} \, dk < \infty \right\}.
\]  \((74)\)

Relation \((21)\) can then be rewritten in the concise form \(\mathcal{F} A = A_\oplus \mathcal{F}\), or equivalently

\[
A = \mathcal{F}^* A_\oplus \mathcal{F}.
\]  \((75)\)

This relation expresses in a synthetic form that the Fourier transform converts the action of the Hamiltonian \(A\) into the collection \(A_\oplus\) of the reduced Hamiltonians \(A_k\) which act independently of each other. As each generalized Fourier transform \(\mathbb{F}_k\) diagonalizes \(A_k\), one can expect that the collection \(\mathbb{F}_\oplus\) of the \(\mathbb{F}_k\) diagonalizes \(A_\oplus\), so that the product \(\mathbb{F}_\oplus \mathcal{F}\) diagonalizes \(A\). Note that we understand here why we need the notion of nonconstant fiber.
Figure 7. Diagonalization diagram of $A$.

direct integral of Hilbert spaces to identify the range of $F\oplus$, since each $F_k$ maps $\mathcal{H}_{1D}$ to the spectral space $\hat{\mathcal{H}}_k$ which depends on $k$.

The following theorem, illustrated by the commutative diagram of Figure 7, provides a rigorous formulation of these intuitive results. Its proof uses only general properties of direct integrals. This theorem is actually the main result of the present paper. It is formulated here in an abstract form which may seem a bit abstruse. It will become clearer in Section 4.2 where we make more explicit the various objects involved in this theorem.

**Theorem 20.** Let $P$ denote the orthogonal projection defined in $\mathcal{H}_{2D}$ by $P := \chi(A)$ where $\chi$ is the indicator function of $\mathbb{R} \setminus \{-\Omega_m, 0, \Omega_m\}$. Consider the direct integral $F\oplus$ of the family of all generalized Fourier transforms $F_k$ for $k \in \mathbb{R}$ (Theorem 17), that is,

$$F\oplus := \int_{\mathbb{R}} F_k dk$$

which maps $\mathcal{H}_{1D}$ to $\hat{\mathcal{H}} := \int_{\mathbb{R}} \hat{\mathcal{H}}_k dk$. (76)

Then, for any measurable function $f : \mathbb{R} \to \mathbb{C}$, we have

$$f(A)P = Pf(A) = F^* f(\lambda) F \text{ in } D(f(A)),$$

(77)

where

$$F : \mathcal{H}_{2D} \to \hat{\mathcal{H}}$$

is defined by $F := F\oplus F$ and satisfies $FF^* = \text{Id}_{\hat{\mathcal{H}}}$. Thus, the restriction of $F$ on the range of $P$ is a unitary operator.

**Proof.** From the definition of $P$, the diagonalization formula (77) amounts to $f\chi(A) = \chi f(A) = F^* f(\lambda) F$. To prove this, we start from formula (75) which shows that $A$ and $A\oplus$ are unitarily equivalent. So the same holds for $f(A)$ and $f(A\oplus)$ for any measurable function $f : \mathbb{R} \to \mathbb{C}$ (see [12, Theorem 10.5.12]). More precisely, using $f\chi$ instead of $f$, we have

$$f\chi(A) = F^* f\chi(A\oplus) F \text{ in } D(f(A)) = F^* (D(f(A\oplus))).$$

(78)
It remains to diagonalize \( f \chi (A \oplus) \). We first use the essential property (see [28, Theorem 13.85])

\[
    f \chi (A \oplus) = \int_{\mathbb{R}} f \chi (A_k) \, dk \quad \text{in } D(f(A \oplus)),
\]

(where the domain \( D(f(A \oplus)) \) is defined as \( D(A \oplus) \) in (74) by replacing \( A \oplus \) and \( A_k \) by \( f(A \oplus) \) and \( f(A_k) \)). Roughly speaking, this latter relation means that the functional calculus “commutes” with direct integrals of operators. We deduce from this relation that for \( U \oplus \in D(f(A \oplus)) \) and \( V \oplus \in H_1^{\oplus} \), one has

\[
    (f \chi (A \oplus) U \oplus, V \oplus)_{H_1^{\oplus}} = \int_{\mathbb{R}} (f \chi (A_k) U_k, V_k)_{H_k} \, dk.
\]

Hence, using the family of diagonalization formulas (69), i.e., \( f \chi (A_k) = P_k f(A_k) = F_k^* f(\lambda) F_k \), we obtain

\[
    (f \chi (A \oplus) U \oplus, V \oplus)_{H_1^{\oplus}} = \int_{\mathbb{R}} (f(\lambda) F_k U_k, F_k V_k)_{H_k} \, dk = (f(\lambda) F_\oplus U \oplus, F_\oplus V \oplus)_{H_1},
\]

which shows that

\[
    f \chi (A \oplus) = F_\oplus^* f(\lambda) F_\oplus,
\]

where \( F_\oplus \) is defined in (76). Note that this operator is bounded since \( \text{ess-sup}_{k \in \mathbb{R}} \|F_k\| = 1 \) because \( \|F_k\| = 1 \) for all \( k \in \mathbb{R}^* \). Combining the latter relation with (78) yields (77).

In the particular case where \( f(\lambda) \equiv 1 \), relation (77) shows that \( \chi (A \oplus) = F_\oplus^* F \), whereas similar arguments as above tell us that

\[
    F_\oplus F_\oplus^* = \int_{\mathbb{R}} F_k F_k^* \, dk = \int_{\mathbb{R}} \text{Id}_{H_k} \, dk = \text{Id}_{H_1},
\]

hence \( F F^* = F_\oplus \mathcal{F} F^* F_\oplus = \text{Id}_{H_1} \), since \( \mathcal{F} F^* = \text{Id}_{H_1} \). This completes the proof.

4.2. Generalized Fourier transform for \( A \)

The aim of this subsection is to identify more explicitly the various objects involved in Theorem 20.

4.2.1. Characterization of the projection \( P \)

By (78), the orthogonal projection \( P := \chi (A \oplus) \) can be equivalently written as:

\[
    P = \mathcal{F}^* \chi (A \oplus) \mathcal{F} \quad \text{where} \quad \chi (A \oplus) = \int_{\mathbb{R}} f \chi (A_k) \, dk = \int_{\mathbb{R}} P_k \, dk,
\]

where the \( P_k \)'s are defined in Theorem 17. This shows in particular that the range of \( P \) is given by

\[
    \text{Im}(P) = \mathcal{F}^* \left( \int_{\mathbb{R}} \text{Im}(P_k) \, dk \right),
\]

where \( \text{Im}(P_k) = H_{1D} (\text{div}_k 0) \) is defined in (23). As \( \mathcal{F} \text{div} = \text{div}_k \mathcal{F} \), we deduce that

\[
    \text{Im}(P) = H_{2D} (\text{div} 0) := \{(E, H, J, K) \in H_{2D} | \text{div} H = 0 \text{ in } \mathbb{R}_+^2 \text{ and } \text{div} K = 0 \text{ in } \mathbb{R}_+^2 \}. \quad (79)
\]
In the same way, \( \text{Ker}(\mathcal{F}) = \text{Ker}(\mathcal{A}) \oplus \text{Ker}(\mathcal{A} - \Omega_m) \oplus \text{Ker}(\mathcal{A} + \Omega_m) \) is described by
\[
\text{Ker}(\mathcal{A}) = \mathcal{F}^*(\int_{\mathbb{R}} \text{Ker}(\mathcal{A}_k) \, dk) \quad \text{and} \quad \text{Ker}(\mathcal{A} \mp \Omega_m) = \mathcal{F}^*(\int_{\mathbb{R}} \text{Ker}(\mathcal{A}_k \mp \Omega_m) \, dk),
\]
where \( \text{Ker}(\mathcal{A}_k) \) and \( \text{Ker}(\mathcal{A}_k \pm \Omega_m) \) are characterized in Proposition 8. As \( \mathcal{F} \nabla = \nabla_k \mathcal{F} \), we have
\[
\text{Ker}(\mathcal{A}) = \{(0, \tilde{\Pi} \nabla \phi, 0, 0)^\top \mid \phi \in H^1_0(\mathbb{R}^2)\}, \\
\text{Ker}(\mathcal{A} \mp \Omega_m) = \{(0, \Pi \nabla \phi, 0, \pm i \mu_0 \Omega_m \nabla \phi)^\top \mid \phi \in H^1_0(\mathbb{R}^2)\},
\]
where \( \tilde{\Pi} \) is the extension by 0 of a 2D vector field defined on \( \mathbb{R}^2 \) to the whole plane \( \mathbb{R}^2 \) and \( H^1_0(\mathbb{R}^2) := \{\phi \in H^1(\mathbb{R}^2) \mid \phi(0, y) = 0 \text{ for a.e. } y \in \mathbb{R}\} \).

### 4.2.2. Description of the spectral space \( \hat{\mathcal{H}} \)

Consider now the spectral space \( \hat{\mathcal{H}} \) defined in (76) where each fiber \( \hat{\mathcal{H}}_k \) is given in (67). The elements of \( \hat{\mathcal{H}} \) are then vector fields \( \hat{U} : k \to \hat{U}_k \in \hat{\mathcal{H}}_k \) such that
\[
\|\hat{U}\|_{\mathcal{H}}^2 := \int_{\mathbb{R}} \|\hat{U}_k\|_{\mathcal{H}_k}^2 \, dk < \infty.
\]
Each space \( \hat{\mathcal{H}}_k \) is composed of \( L^2 \)-spaces defined on the various zones \( \Lambda_Z(k) \), which are vertical sections of the spectral zones \( \Lambda_Z \) represented in Figure 4 (and defined in (49) and (53)). The above formula gathers the spaces associated with all sections to create a space of fields defined on the zones \( \Lambda_Z, z \in \mathcal{Z} \). Indeed, by Fubini's theorem, we see that \( \hat{\mathcal{H}} \) can be identified with the following direct sum:
\[
\hat{\mathcal{H}} = \bigoplus_{z \in \mathcal{Z}} L^2(\Lambda_Z)^{\text{card}(\mathcal{J}_z)} = L^2(\Lambda_{DD})^2 \oplus L^2(\Lambda_{DI})^2 \oplus L^2(\Lambda_{EI})^2 \oplus L^2(\Lambda_{EE}).
\]
As we did for the generalized eigenfunctions \( W_{k,\lambda,j} \), we denote somewhat abusively by \( \hat{U}(k, \lambda, j) \) the fields of \( \hat{\mathcal{H}} \), where it is understood that \( j \in J_z \) while \( (k, \lambda) \in \Lambda_Z \) for the various zones \( \Lambda_Z, z \in \mathcal{Z} \). The norm in \( \hat{\mathcal{H}} \) can then be rewritten as:
\[
\|\hat{U}\|_{\mathcal{H}}^2 = \sum_{z \in \mathcal{Z} \setminus \{\mathcal{E}\}} \sum_{j \in J_z} \int_{\Lambda_Z} |\hat{U}(k, \lambda, j)|^2 \, dk \, dk + \sum_{\pm} \int \int_{|k| > k_c} |\hat{U}(k, \pm E(k), 0)|^2 \, dk.
\]

### 4.2.3. Generalized Fourier transform \( \mathcal{F} \) and its adjoint

We show here that, as for the reduced Hamiltonian, the generalized Fourier transform \( \mathcal{F} \) appears as a “decomposition” operator on a family of generalized eigenfunctions of \( \mathcal{A} \), denoted by \( (\mathbb{W}_{k,\lambda,j}) \) and constructed from the generalized eigenfunctions \( (W_{k,\lambda,j}) \) of the reduced Hamiltonian \( \mathcal{A}_k \) (see (62)) via the following formula:
\[
\forall z \in \mathcal{Z}, \; \forall (k, \lambda) \in \Lambda_Z, \; \forall j \in J_z, \; \forall (x, y) \in \mathbb{R}^2, \; \mathbb{W}_{k,\lambda,j}(x, y) := W_{k,\lambda,j}(x) \frac{e^{iky}}{\sqrt{2\pi}}.
\]
As $\mathbb{F}$ (respectively, $\mathbb{F}^*$) is bounded in $\mathcal{H}_{2D}$ (respectively, $\hat{\mathcal{H}}_s$), if suffices to define it on a dense subspace of $\mathcal{H}_{2D}$ (respectively, $\hat{\mathcal{H}}_s$). Consider first the case of the physical space $\mathcal{H}_{2D}$.

In the same way as the 1D-case (see (63)), we define a dense subspace of $\mathcal{H}_{2D}$ understood when applying formulas (84) and (85) for general $s > 1/2$. By de/inition, in each zone $\Lambda_{z_*}$, the curves $\lambda = \pm \lambda_0(k)$, $\lambda = \pm \lambda_0(k)$ and $\lambda = \pm \lambda_1(k)$, nor the lines $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{\pm \Omega_m\}$. For all $\hat{U} \in \mathcal{H}_{\text{comp}}$, we have

\[
\mathbb{F}^* \hat{U} = \sum_{z \in \mathbb{Z} \setminus \{0\}} \sum_{j \in J_z} \int_{\Lambda_{z_*}} \hat{U}(k, \lambda, j) \, \mathbb{W}_{k, \lambda, j} \, d\lambda \, dk + \sum_{\pm} \int_{|k| > k_c} \hat{U}(k, \pm \lambda_k(k), 0) \, \mathbb{W}_{k, \pm \lambda_k, 0} \, dk,
\]

where the integrals are Bochner integrals with values in $\mathcal{H}_{2D,-s}$.

**Proposition 21.** Let $s > 1/2$. The generalized Fourier transform $\mathbb{F}U$ of all $U \in \mathcal{H}_{2D,s}$ is explicitly given in each zone $\Lambda_{z_*}$, $z \in \mathbb{Z}$, by

\[
\mathbb{F}U(k, \lambda, j) = \langle U, \mathbb{W}_{k, \lambda, j} \rangle_{2D,s} \quad \text{for all } (k, \lambda) \in \Lambda_{z_*} \text{ and } j \in J_z,
\]

where the $\mathbb{W}_{k, \lambda, j}$'s are defined in (83).

Let $\hat{\mathcal{H}}_{\text{comp}}$ denotes the dense subspace of $\hat{\mathcal{H}}$ composed of compactly supported functions whose supports contain neither the boundaries of the spectral zones $\Lambda_{z_*}$ for $z \in \{0D, 0E, 0D, 0D, 0E\}$ (i.e., the curves $\lambda = \pm \lambda_0(k)$, $\lambda = \pm \lambda_0(k)$ and $\lambda = \pm \lambda_1(k)$), nor the lines $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{\pm \Omega_m\}$. For all $\hat{U} \in \mathcal{H}_{\text{comp}}$, we have

\[
\mathbb{F}^* \hat{U} = \sum_{z \in \mathbb{Z} \setminus \{0\}} \sum_{j \in J_z} \int_{\Lambda_{z_*}} \hat{U}(k, \lambda, j) \, \mathbb{W}_{k, \lambda, j} \, d\lambda \, dk + \sum_{\pm} \int_{|k| > k_c} \hat{U}(k, \pm \lambda_k(k), 0) \, \mathbb{W}_{k, \pm \lambda_k, 0} \, dk,
\]

where the integrals are Bochner integrals with values in $\mathcal{H}_{2D,-s}$.

**Remark 22.** The content of Remark 19 could be transposed here with obvious changes. In particular, for general $U \in \mathcal{H}_{2D}$ or $\hat{U} \in \hat{\mathcal{H}}_s$, the expressions of $\mathbb{F}U$ or $\mathbb{F}^* \hat{U}$ are deduced from the above ones by a limit process on the domain of integration (exactly as for the usual Fourier transform of a square integrable function). In the sequel, this process will be implicitly understood when applying formulas (84) and (85) for general $U$ and $\hat{U}$.

**Proof.** Let $U \in \mathcal{H}_{2D,s}$ with $s > 1/2$. By definition, $\mathbb{F} := \mathbb{F}_d \mathcal{F}$ where $\mathbb{F}_d$ is defined in (76). Hence, in each zone $\Lambda_{z_*}$, $z \in \mathbb{Z}$, we have

\[
\mathbb{F}U(k, \lambda, j) = \mathbb{F}_d(\mathcal{F}U(\cdot, k))(\lambda, j), \quad \text{for a.e. } (k, \lambda) \in \Lambda_{z_*} \text{ and } j \in J_z.
\]

Note that $\mathcal{H}_{2D,s} = \mathcal{H}_{1D,s} \otimes \mathcal{F}(L_2^{s}(\mathbb{R}))$ and $\mathcal{F}(L_2^{s}(\mathbb{R})) = H^s(\mathbb{R})$ where $H^s(\mathbb{R})$ stands for the Sobolev space of index $s$, thus $\mathcal{F}U \in \mathcal{H}_{1D,s} \otimes H^s(\mathbb{R})$. As $s > 1/2$, $H^s(\mathbb{R})$ is included in $C_0(\mathbb{R})$, the space of continuous function on $\mathbb{R}$, so $\mathcal{F}U \in C_0(\mathbb{R}, \mathcal{H}_{1D,s})$ and one can define $\mathcal{F}U(\cdot, k) \in \mathcal{H}_{1D,s}$ for all real $k$. Hence, using the respective definitions (19) and (66) of $\mathcal{F}$ and $\mathbb{F}_d$ leads to

\[
\mathbb{F}U(k, \lambda, j) = \left\langle \int_{\mathbb{R}} U(\cdot, y) \frac{e^{-i k y}}{\sqrt{2\pi}} \, dy, \mathbb{W}_{k, \lambda, j} \right\rangle_{1D,s}, \quad \text{for a.e. } (k, \lambda) \in \Lambda_{z_*} \text{ and } j \in J_z,
\]

which yields (84) thanks to a Fubini’s like theorem for Bochner integrals (which applies here since $U \in \mathcal{H}_{2D,s}$ and $\mathbb{W}_{k, \lambda, j} \in \mathcal{H}_{2D,-s}$).
We now prove (85). Recall that the adjoint of a direct integral of operators is the direct integral of their adjoints [36, Theorem 2.14.2(1)]. Hence,

$$\mathcal{F}^* = \mathcal{F}^* \mathcal{F}^*$$

where

$$\mathcal{F}^* = \int_{\mathbb{R}} \mathcal{F}^*_k \, dk$$

does maps $\mathcal{H}$ to $\mathcal{H}_{1D}$.

Let $\hat{\mathcal{U}} \in \mathcal{H}_{comp}$. For a.e. $k \in \mathbb{R}$, $\hat{\mathcal{U}}_k := \hat{\mathcal{U}}(k, \cdot, \cdot) \in \mathcal{H}_{comp}$ (where $\mathcal{H}_{comp}$ is defined in Proposition 18). Moreover $\mathcal{F}^*_k \hat{\mathcal{U}}(\cdot, k) \equiv \mathcal{F}^*_k \hat{\mathcal{U}}$ vanishes for $|k|$ large enough, so

$$\mathcal{F}^* \mathcal{U}(x, y) = \mathcal{F}^* \mathcal{F}^* \mathcal{U}(x, y) = \int_{\mathbb{R}} \mathcal{F}^*_k \hat{\mathcal{U}}_k(x) \frac{e^{iky}}{\sqrt{2\pi}} \, dk$$

for a.e. $(x, y) \in \mathbb{R}^2$,

where $\mathcal{F}^*_k \hat{\mathcal{U}}_k$ is given by formula (73) with $\hat{\mathcal{U}}_k$ instead of $\hat{\mathcal{U}}$. Then it suffices to apply Fubini’s theorem again to obtain formula (85), using the fact that the function $(k, \lambda) \mapsto \mathcal{W}_{k,\lambda,j}$ is bounded in $\mathcal{H}_{2D,\cdots}$ when $(k, \lambda)$ varies in the support of $\hat{\mathcal{U}}$.

4.3. Spectrum of $\mathcal{A}$

The preceding results actually show that the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is obtained by superposing the spectra $\sigma(\mathcal{A}_k)$ of $\mathcal{A}_k$ for all $k \in \mathbb{R}$. More precisely, we have the following property.

**Corollary 23.** The spectrum of $\mathcal{A}$ is the whole real line: $\sigma(\mathcal{A}) = \mathbb{R}$. The point spectrum $\sigma_p(\mathcal{A})$ is composed of eigenvalues of infinite multiplicity: $\sigma_p(\mathcal{A}) = \{-\Omega_m, 0, \Omega_m\}$ if $\Omega_e \neq \Omega_m$ and $\sigma_p(\mathcal{A}) = \{-\Omega_m, -\Omega_m/\sqrt{2}, 0, \Omega_m/\sqrt{2}, \Omega_m\}$ if $\Omega_e = \Omega_m$.

**Proof.** It is based on Remark 6. First consider an interval $\Lambda = (a, b) \subset \mathbb{R} \setminus \{-\Omega_m, 0, \Omega_m\}$ with $a < b$. The diagonalization formula (77) applied to the indicator function $f = \mathbf{1}_\Lambda$ shows that the spectral projection $\mathcal{E}(\Lambda)$ is given by $\mathcal{E}(\Lambda) = \mathcal{F}^* \mathbf{1}_\Lambda \mathcal{F}$ (since $\chi \mathbf{1}_\Lambda = \mathbf{1}_\Lambda$). Moreover, from the identification (82) of the spectral space $\mathcal{H}$, we see that the operator of multiplication by $\mathbf{1}_\Lambda(\lambda)$ in $\mathcal{H}$ cannot vanish. Hence $\mathcal{E}(\Lambda) \neq 0$ for all non empty $\Lambda = (a, b) \subset \mathbb{R} \setminus \{-\Omega_m, 0, \Omega_m\}$. As $\sigma(\mathcal{A})$ is closed, we conclude that $\sigma(\mathcal{A}) = \mathbb{R}$.

Suppose now that $\Lambda = \{a\} \subset \mathbb{R} \setminus \{-\Omega_m, 0, \Omega_m\}$. We still have $\mathcal{E}(\Lambda) = \mathcal{F}^* \mathbf{1}_\Lambda \mathcal{F}$ but here, the operator of multiplication by $\mathbf{1}_{\{a\}}(\lambda)$ in $\mathcal{H}$ always vanishes except if $\Omega_e = \Omega_m$ and $a = \pm \Omega_m/\sqrt{2}$. To understand this, first consider a two-dimensional zone $\Lambda_Z$ for $z \in \mathbb{Z} \setminus \{ee\}$ (the reading of the following explanations may be made easier with the help of Figure 4).

As $\mathbb{R} \times \{a\}$ is one-dimensional, its intersection with $\Lambda_Z$ has measure zero in $\Lambda_Z$, so the operator of multiplication by $\mathbf{1}_{\{a\}}(\lambda)$ in $L^2(\Lambda_Z)$ vanishes. On the other hand, for the one-dimensional zone $\Lambda_{ee}$, several situations may occur. If $\Omega_e \neq \Omega_m$, the intersection of $\mathbb{R} \times \{a\}$ and $\Lambda_{ee}$ is either empty or consists of two points (which are symmetric with respect to the $\lambda$-axis), hence this intersection still have measure zero in $\Lambda_{ee}$. If $\Omega_e = \Omega_m$, this intersection is empty when $a \notin (-\Omega_m/\sqrt{2}, -\Omega_m/\sqrt{2})$, whereas it is one half of $\Lambda_{ee}$ when $a = \pm \Omega_m/\sqrt{2}$ (the half located in the half-plane $\pm \lambda > 0$). In the latter case, we see that the range of the projection $\mathcal{E}(\pm \Omega_m/\sqrt{2})$ is isomorphic, through the generalized Fourier transform $\mathcal{F}$, to the infinite-dimensional space $L^2(|k| > k_c)$. To sum up, if $\Omega_e \neq \Omega_m$, there is no eigenvalue of $\mathcal{A}$ in $\mathbb{R} \setminus \{-\Omega_m, 0, \Omega_m\}$, whereas if $\Omega_e = \Omega_m$, the only eigenvalues of $\mathcal{A}$ located in $\mathbb{R} \setminus \{-\Omega_m, 0, \Omega_m\}$ are $\pm \Omega_m/\sqrt{2}$ and they both have infinite multiplicity.
Finally, formulas (80) and (81) show that 0 and ±Ω_m are also eigenvalues of infinite multiplicity.

### 4.4. Generalized eigenfunction expansions for the evolution problem

As an application of the above results, let us express the solution U(t) of our initial time-dependent Maxwell’s equations written as the Schrödinger equation (7).

Consider first the free evolution of the system, that is, when G = 0. In this case, U(t) = e^{-i\lambda t}U_0 for all t > 0, where U_0 = U(0) ∈ H_{2D} is the initial state. Theorem 20 provides us a diagonal expression of e^{-i\lambda t}. More precisely, \( \mathbb{P} e^{-i\lambda t} = e^{-i\lambda t} \mathbb{P} = \mathbb{P}^* e^{-i\lambda t} \mathbb{P} \). Hence, if we restrict ourselves to initial conditions \( U_0 \in H_{2D}(\text{div}0) = \text{Im}(\mathbb{P}) \) (see (79)), we simply have \( U(t) = \mathbb{P}^* e^{-i\lambda t} \mathbb{P} U_0 \). Thanks to (84) and (85), this expression becomes

\[
U(t) = \sum_{z \in \mathcal{Z}} \sum_{j \in J_z} \int_{\Lambda_z} \langle U_0, \mathbb{W}_{k,\lambda,j} \rangle_{2D,s} \mathbb{W}_{k,\lambda,j} e^{-i\lambda t} d\lambda dk.
\]

For simplicity, we have condensed here the various sums in the right-hand side of (85) in a single sum which includes the last term corresponding to \( z = \text{ee} \) (which is of course abusive for this term, since it is a single integral represented here by a double integral). The above expression is a generalized eigenfunction expansion of U(t). It tells us that U(t) can be represented as a superposition of the time-harmonic waves \( \mathbb{W}_{k,\lambda,j} e^{-i\lambda t} \) modulated by the spectral components \( \langle U_0, \mathbb{W}_{k,\lambda,j} \rangle_{2D,s} \) of the initial state. Strictly speaking, the above expression is valid if we are in the context of Proposition 21, i.e., if \( U_0 \in H_{2D,+s} \) with \( s > 1/2 \) and \( \mathbb{F} U_0 \in H_{\text{comp}} \). For general \( U_0 \in H_{2D}(\text{div}0) \), the limit process mentioned in Remark 22 is implicitly understood.

Consider now Eq. (7) with a nonzero excitation \( G \in C^1(\mathbb{R}^+, H_{2D}(\text{div}0)) \). For simplicity, we assume zero initial conditions. In this case, the Duhamel integral formula (12) writes as

\[
U(t) = \int_0^t \mathbb{F}^* e^{-i\lambda(t-s)} \mathbb{F} G(s) ds,
\]

which yields the following generalized eigenfunction expansion:

\[
U(t) = \sum_{z \in \mathcal{Z}} \sum_{j \in J_z} \int_0^t \int_{\Lambda_z} \langle G(s), \mathbb{W}_{k,\lambda,j} \rangle_{2D,s} \mathbb{W}_{k,\lambda,j} e^{-i\lambda(t-s)} d\lambda dk ds.
\]

As mentioned in the introduction, we are especially interested in the case of a time-harmonic excitation which is switched on at an initial time, that is, \( G(t) = H(t) e^{-i\omega t} G_\omega \) for given \( \omega \in \mathbb{R} \) and \( G_\omega \in H_{2D}(\text{div}0) \), where \( H(t) \) denotes the Heaviside step function (i.e., \( H := 1_{\mathbb{R}^+} \)). In this particular situation, Duhamel’s formula simplifies as

\[
U(t) = \phi_{\omega}(\lambda, t) G_\omega = \mathbb{F}^* \phi_{\omega}(\lambda, t) \mathbb{F} G_\omega,
\]

where \( \phi_{\omega}(\cdot, t) \) is the bounded continuous function defined for non negative \( t \) and real \( \lambda \) by

\[
\phi_{\omega}(\lambda, t) := e^{-i\lambda t} \int_0^t e^{-ist(\omega-\lambda)} ds = \begin{cases} \frac{e^{-i\lambda t} - e^{-i\omega t}}{\lambda - \omega} & \text{if } \lambda \neq \omega, \\ t e^{-i\omega t} & \text{if } \lambda = \omega. \end{cases}
\]
The generalized eigenfunction expansion of $U(t)$ then takes the form

$$U(t) = \sum_{z \in \mathbb{Z}} \sum_{j \in J_z} \int_{\Lambda_z} \phi_{\omega}(\lambda, t) \langle G_{\omega}, W_{k,\lambda,j} \rangle_{2D,s} \, W_{k,\lambda,j} \, d\lambda \, dk.$$ 

In the second part [7] of the present paper, we will study the asymptotic behavior of this quantity for large time, in particular the validity of the limiting amplitude principle, that is, the fact that $U(t)$ becomes asymptotically time-harmonic. We can already predict that this principle fails in the particular case $\Omega_z = \Omega_m$ if we choose $\omega = \pm \Omega_m/\sqrt{2}$, since $\pm \Omega_m/\sqrt{2}$ are eigenvalues of infinite multiplicity of $\mathbb{A}$ (see Corollary 23). Indeed, if $G_{\omega}$ is an eigenfunction associated to $\pm \Omega_m/\sqrt{2}$, it is readily seen that the above expression becomes

$$U(t) = \int_{|k| > k_c} \phi_{\omega}(\omega, t) \langle G_{\omega}, W_{k,\omega,0} \rangle_{2D} \, W_{k,\omega,0} \, dk.$$ 

Using the expression of $\phi_{\omega}(\omega, t)$, we obtain

$$U(t) = t e^{-i\omega t} \int_{|k| > k_c} \langle G_{\omega}, W_{k,\omega,0} \rangle_{2D} \, W_{k,\omega,0} \, dk,$$

which shows that the amplitude of $U(t)$ increases linearly in time. This resonance phenomenon is compatible with formula (13) which tells us that $\|U(t)\|_{2D}$ increases at most linearly in time. It is similar to the resonances which can be observed in a bounded electromagnetic cavity filled with a nondissipative dielectric medium, when the frequency of the excitation coincides with one of the eigenfrequencies of the cavity. But it is related here to surface plasmon polaritons which are the waves that propagate at the interface of our Drude material and that are described by the eigenfunctions associated with the eigenvalues $\pm \Omega_m/\sqrt{2}$ of $\mathbb{A}$. Moreover, unlike the eigenfrequencies of the cavity, these eigenvalues are of infinite multiplicity and embedded in the continuous spectrum. In the case of an unbounded stratified medium composed of standard non-dissipative dielectric materials such nonzero eigenvalues do not exist [37].

In [7], we will explore more deeply this phenomenon and we will investigate all the possible behaviors of our system submitted to a periodic excitation. This forthcoming paper is devoted to both limiting absorption and limiting amplitude principles, which yield two different but concurring processes that characterize time-harmonic waves. In the limiting absorption principle, the time-harmonic regime is associated to the existence of one-sided limits of the resolvent of the Hamiltonian on its continuous spectrum, whereas in the limiting amplitude principle, it appears as an asymptotic behavior for large time of the time-dependent regime.

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