VANISHING CYCLES OF PENCILS OF HYPERSURFACES

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To Dirk Siersma, on the occasion of his 60th anniversary

Abstract. We prove an extended Lefschetz principle for a large class of pencils of
hypersurfaces having isolated singularities, possibly in the axis, and show that the module
of vanishing cycles is generated by the images of certain variation maps.

1. Introduction

In this paper we extend the Lefschetz principle of slicing by pencils to nongeneric pencils
of hypersurfaces on singular non-compact spaces. We started to develop this point of view
in [Ti3] for proving connectivity theorems of Lefschetz type for nongeneric pencils. Here
we go further and introduce global variation maps in order to control vanishing cycles. As
a result, we prove a far reaching extension of the Second Lefschetz Hyperplane Theorem.

To get an idea of the main result, let us first briefly recall the classical Lefschetz
Hyperplane Theorems (see also Note 3.4 for some references). For a projective manifold
Y and some hyperplane section $Y_0$, the First Lefschetz Theorem tells us that the map
(induced by inclusion):

$$H_j(Y_0, Z) \to H_j(Y, Z)$$

is bijective for $j < n - 1$ and surjective for $j = n - 1$. The kernel of the surjection in
dimension $n - 1$ is described by the second Lefschetz theorem, whenever $Y_0$ is a generic
member of a generic pencil, i.e. the pencil has only complex Morse critical points. Loosely
speaking, each such critical point produces a local vanishing cycle and those vanishing
cycles together generate $\ker(H_j(Y_0, Z) \to H_j(Y, Z))$.

We consider here a very general situation: a complex analytic space $X = Y \setminus V$ with
arbitrary singularities, where $Y$ is some compact complex space and $V$ is a complex
analytic subspace. We also consider more general divisors than hyperplanes, namely
pencils of hypersurfaces on $Y$. Taking hypersurfaces instead of hyperplanes considerably
enlarges the class under study. Let us remark that any singular holomorphic function
germ is a local pencil of hypersurfaces (but not of hyperplanes): our approach enfolds
the theory of hypersurface singularities started by Milnor [Mi], in some of its aspects. In
the global setting, the polynomial functions $\mathbb{C}^n \to \mathbb{C}$ with isolated singularities represent
a distinguished class of pencils of hypersurfaces, thus non-hyperplane pencils, which we
shall discuss in [5]. For more details on this viewpoint and for examples we refer to [Ti5]
and [LT].

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gularities of functions on stratified spaces, monodromy, invariant cycle theorem, topology of polynomial
functions.
On the other hand, we weaken the classical genericity condition “the axis of the pencil is in general position in \(Y\)” by allowing that the genericity of the axis fails at a finite number of points. This conceptual extension is introduced and explained in detail in our paper \([Ti3, \S 2]\). Pencils which allow such “isolated singularities in the axis” (see \(\S 2.1\) for the definitions) are natural to consider since isolated singularities of functions on singular spaces are central objects of study in modern singularity theory. (One may refer to the pioneering work, on stratified Morse theory by Goresky and MacPherson \([GM]\), and on the topology of functions on singular spaces by Lê Dũng Tráng \([Lê]\).)

We show in \(\S 2\) that one can define the following variation map around each critical value \(a\) of the pencil:

\[
\text{var}_a : H_*(X_c, (X_a)_{\text{reg}}) \to H_*(X_c),
\]

and that the module of vanishing cycles at \(X_a\), i.e. the kernel of the surjection similar to \((1)\), is generated by the images of these variation maps. Our variation maps can be viewed as global versions of the local variation maps that one defines in singularity theory, see e.g. \([La]\), \([Lo]\), \([Ti1, 4.4]\), \([NN2, \S 2]\). It is well-known that in case of non-isolated singularities, the local variation maps do not exist. This is the main reason why the use of variation maps in our results would not extend to this context. Let us remark that in case of one-dimensional singularities, Siersma \([Si]\) defined other types of variation maps, but their behaviour appears to be much more delicate and has not been exploited yet in the literature.

As for our approach, it starts in the spirit of the Lefschetz method \([Le]\), as presented by Thom in his Princeton talk in 1957 and by Andreotti and Frankel in their paper \([AF]\). This vein has been exploited in relatively few papers ever since; we may mention the interesting ones by Lamotke \([La]\), Chéniot \([Ch1, Ch2]\) and Eyral \([Ey]\). The use, in the statement of our Theorem \(3.2\) of the comparison between the general element of the pencil and the axis comes from Lamotke \([La]\) and may evoke Chéniot’s statements in \textit{loc.cit}. Our setting being far more general, we follow a different strategy and use in a crucial way specific geometric constructions and results of stratified singularity theory.

A highly nongeneric situation is encountered when the axis of the pencil is contained in \(V\). We show that if \(V\) contains a member of the pencil then, surprisingly, Theorem \(3.2\) and its proof still work, with even less restrictive assumptions. Actually, one of the reasons to study such nongeneric pencils is that the polynomial functions on \(\mathbb{C}^n\) constitute a class of examples. We show in \(\S 5\) how Theorem \(4.1\) can be extended to a polynomial function with isolated singularities, but without any condition on the singularities along the axis (which are the so-called “singularities at infinity”). We arrive in this way to results on vanishing cycles of polynomials which have been discovered in somewhat different form by Neumann and Norbury \([NN1, NN2]\), such as an invariant cycle theorem (Corollary \(5.2\)).

In \(\S 4\) we compare the assumptions of our main Theorem \(3.2\) to conditions involving the rectified homological depth (defined by Grothendieck and thoroughly studied in \([HL3]\)), showing that the latter are more restrictive. We also point out how, by relaxing the generality of the setting, one may recover several results in the literature.

This paper is based on our preprint \([Ti2]\) and is a natural continuation of \([Ti3]\). We wish to thank the Newton Institute at Cambridge and the Institute for Advanced Study at Princeton for support during the elaboration of this work. We are also thankful to the anonymous referee for his valuable remarks.
2. Nongeneric pencils and variation maps

Let $Y$ be a compact complex analytic space and let $V \subset Y$ be a complex analytic subspace such that $X := Y \setminus V$ is of dimension $n$, $n \geq 2$.

2.1. Pencils with singularities in the axis. Let us recall some definitions that we already used in [Ti3]. By pencil (or meromorphic function) we mean the ratio of two sections $f$ and $g$ of a holomorphic line bundle $L \to Y$. This defines a holomorphic function $h := f/g$ over the complement $Y \setminus A$ of the axis of the pencil $A := \{ f = g = 0 \}$. A pencil is called generic with respect to $X$ when $Y$ is embedded in some manifold $Z$ and the pencil extends to one over $Z$ which satisfies the following conditions: the axis $A$ of the extended pencil is nonsingular and transversal to some Whitney stratification of the pair $(Y, V)$ and the holomorphic map $h = f/g : Y \setminus A \to \mathbb{P}^1$ has only stratified double points as singularities. Notice that part of those singularities might be on $V$, hence outside $X$.

Here we focus on a class of (nongeneric) pencils, namely pencils having at most isolated singularities, possibly in the axis. Let us first explain what we mean by singularities of a pencil.

We define a new space by blowing-up along the base locus $A$. The idea of this construction is due to Thom and was used by Andreotti and Frankel [AF] in case of generic pencils on projective manifolds. So, let:

$$Y := \text{closure}\{ (y, [s : t]) \in Y \times \mathbb{P}^1 \mid sf(y) - tg(y) = 0 \}.$$

This is a hypersurface in $Y \times \mathbb{P}^1$ obtained as a Nash blowing-up of $Y$ along $A$. It is clear that the intersection $Y \cap (Y \setminus A) \times \mathbb{P}^1$ is just the graph of $h$, hence it is isomorphic to $Y \setminus A$. It also follows that the subset $A \times \mathbb{P}^1$ is included into $Y$.

Let us denote $X := Y \cap (X \times \mathbb{P}^1)$. Consider the projection $p : Y \to \mathbb{P}^1$, its restriction $p|_X : X \to \mathbb{P}^1$ and the projection to the first factor $\sigma : Y \to Y$. Notice that the restriction of $p$ to $Y \setminus (A \times \mathbb{P}^1)$ can be identified with $h$.

Now fix a stratification $\mathcal{W}$ on $Y$ such that $V$ is a union of strata. The restriction of $\mathcal{W}$ to the open set $Y \setminus A$ induces a Whitney stratification on $Y \setminus (A \times \mathbb{P}^1)$, via the above mentioned identification. We then denote by $\mathcal{S}$ the coarsest Whitney stratification on $Y$ which coincides over $Y \setminus (A \times \mathbb{P}^1)$ with the one induced by $\mathcal{W}$ on $Y \setminus A$. This stratification exists within a neighbourhood of $A \times \mathbb{P}^1$, by usual arguments (see e.g. [GLPW]), hence such stratification is well defined on $Y$. We call it the canonical stratification of $Y$ generated by the stratification $\mathcal{W}$ of $Y$. The canonical stratification of $X$ will be the restriction of $\mathcal{S}$ to $X$.

**Definition 2.1.** We call the singular locus of $p$ with respect to $\mathcal{S}$ the following closed analytic subset of $Y$:

$$\text{Sing}_{sp} := \bigcup_{s_{\beta} \in \mathcal{S}} \text{Sing} p_{s_{\beta}}.$$

We denote by $\Lambda := p(\text{Sing}_{sp})$ the set of critical values of $p$ with respect to $\mathcal{S}$.

Since $p$ is proper and since $\mathcal{S}$ has finitely many strata, it follows that the set $\Lambda$ is a finite set. By Thom’s Isotopy Lemma [Th], we get that the maps $p : Y \setminus p^{-1}(\Lambda) \to \mathbb{P}^1 \setminus \Lambda$ and $p|_X : X \setminus p^{-1}(\Lambda) \to \mathbb{P}^1 \setminus \Lambda$ are stratified locally trivial fibrations. In particular, $h : Y \setminus (A \cup h^{-1}(\Lambda)) \to \mathbb{P}^1 \setminus \Lambda$ is a locally trivial fibration.
Definition 2.2. We say that the pencil defined by the meromorphic function $h = f/g$ is a pencil with isolated singularities if $\dim \text{Sing}_{sp} \leq 0$.

We shall say that $X$ has the structure of a Lefschetz fibration with isolated singularities if there exists a pencil on $X$ with isolated singularities.

We have pointed out in [13] §2 that in case $Y$ is projective, the condition $\dim \text{Sing}_{sp} \leq 0$ is equivalent to the following condition: the singularies of the function $p$ at the blown-up axis $\hat{\Delta} \times \mathbb{P}^1$ are at most isolated. We have moreover:

Proposition 2.3. [13] Proposition 2.4. Let $Y \subset \mathbb{P}^N$ be a projective variety endowed with some Whitney stratification $\mathcal{W}$ and let $\hat{h} = \hat{f}/\hat{g}$ define a pencil of hypersurfaces in $\mathbb{P}^N$ with axis $\hat{\Delta}$. Let $S$ denote the set of points on $\hat{\Delta} \cap Y$ where some member of the pencil is singular or where $\hat{A}$ is not transversal to $\mathcal{W}$. If $\dim S \leq 0$ and the singular points of $h : Y \setminus S \to \mathbb{P}^1$ with respect to $\mathcal{W}$ are at most isolated, then $\dim \text{Sing}_{sp} \leq 0$. □

2.2. Variation maps. We assume that our pencil defined by $h : Y \dashrightarrow \mathbb{P}^1$ has isolated singularities, as defined in 2.2. Let us fix some notation. For any $M \subset \mathbb{P}^1$, we denote $\mathcal{Y}_M := p^{-1}(M)$, $\mathcal{X}_M := \mathcal{X} \cap \mathcal{Y}_M$, $\mathcal{Y}_M := \sigma(p^{-1}(M))$ and $\mathcal{X}_M := \mathcal{X} \cap \mathcal{Y}_M$. Let $\Lambda = \{a_1, \ldots, a_p\}$. We denote by $a_{ij} \in \mathcal{Y}$ some point of $\text{Sing}_{sp} \cap p^{-1}(a_i)$. We then have $\text{Sing}_{sp} = \cup_{i,j} \{a_{ij}\}$. For $c \in \mathbb{P}^1 \setminus \Lambda$ we say that $\mathcal{Y}_c$, resp. $\mathcal{X}_c$, is a general fiber of $p : Y \to \mathbb{P}^1$, resp. of $p|_\mathcal{X} : \mathcal{X} \to \mathbb{P}^1$. We say that $\mathcal{Y}_c$, resp. $\mathcal{X}_c$, is a general member of the pencil on $Y$, resp. on $\mathcal{X}$.

At some singularity $a_{ij}$, in local coordinates, we take a ball $B_{ij}$ centered at $a_{ij}$. For a small enough radius of $B_{ij}$, this is a “Milnor ball” of the holomorphic function $p$ at $a_{ij}$. Next we may take a small enough disk $D_i \subset \mathbb{P}^1$ at $a_i \in \mathbb{P}^1$, so that $(B_{ij}, D_i)$ is Milnor data for $p$ at $a_{ij}$. Moreover, we may do this for all (finitely many) singularities in the fiber $\mathcal{Y}_{a_i}$, keeping the same disk $D_i$, provided it is small enough.

Now the restriction of $p$ to $\mathcal{Y}_{D_i} \setminus \cup_j B_{ij}$ is a trivial fibration over $D_i$. One may construct a stratified vector field which trivializes this fibration and such that this vector field is tangent to the boundaries of the balls $\mathcal{Y}_{D_i} \cap \partial B_{ij}$. Using this, we may also construct a geometric monodromy of the fibration $p_i : \mathcal{Y}_{\partial D_i} \to \partial \hat{D}_i$ over the circle $\hat{D}_i$, such that this monodromy is the identity on the complement of the balls, $\mathcal{Y}_{\partial D_i} \setminus \cup_j B_{ij}$. The same is then true, when replacing $\mathcal{Y}_{\partial D_i}$ by $\mathcal{X}_{\partial D_i}$.

Take some point $c_i \in \partial \hat{D}_i$. We have the geometric monodromy representation:

$$\rho_i : \pi_1(\partial \hat{D}_i, c_i) \to \text{Iso}(\mathcal{X}_{c_i}, \mathcal{X}_{c_i} \setminus \cup_j B_{ij}),$$

where Iso(,,,) denotes the group of relative isotopy classes of stratified homeomorphisms (which are $C^\infty$ along each stratum). It follows that the geometric monodromy restricted to $\mathcal{X}_{c_i} \setminus \cup_j B_{ij}$ is the identity.

As shown above, we may identify, in the trivial fibration over $D_i$, the fiber $\mathcal{X}_{c_i} \setminus \cup_j B_{ij}$ to the fiber $\mathcal{X}_{a_i} \setminus \cup_j B_{ij}$. Furthermore, in local coordinates at $a_{ij}$, $\mathcal{X}_{a_i}$ is a germ of a complex analytic space; hence, for a small enough ball $B_{ij}$, the set $B_{ij} \cap \mathcal{X}_{a_i} \setminus \cup_j a_{ij}$ retracts to $\partial B_{ij} \cap \mathcal{X}_{a_i}$ by the local conical structure of analytic sets [17]. Therefore $\mathcal{X}_{a_i}^* := \mathcal{X}_{a_i} \setminus \cup_j a_{ij}$ is homotopy equivalent, by retraction, to $\mathcal{X}_{a_i} \setminus \cup_j B_{ij}$.

Notation. From now on, we shall freely use $X_{a_i}^*$ as notation for $\mathcal{X}_{c_i} \setminus \cup_j B_{ij}$ whenever we consider the pair $(X_{c_i}, X_{a_i}^*)$, having in mind the homotopy equivalence between the two spaces.
It then follows that the geometric monodromy induces an algebraic monodromy, in any dimension $q$:

$$\nu_i : H_q(X_{c_i}, X_{c_i}^*; \mathbb{Z}) \to H_q(X_{c_i}, X_{a_i}^*; \mathbb{Z}),$$

such that the restriction $\nu_i : H_q(X_{a_i}^*) \to H_q(X_{a_i}^*)$ is the identity.

Consequently, any relative cycle $\delta \in H_q(X_{c_i}, X_{a_i}^*; \mathbb{Z})$ is sent by the morphism $\nu_i - \text{id}$ to an absolute cycle. In this way we define a variation map, for any $q \geq 0$:

$$\text{var}_i : H_q(X_{c_i}, X_{a_i}^*; \mathbb{Z}) \to H_q(X_{c_i}; \mathbb{Z}).$$

This enters, as a diagonal morphism, in the following diagram:

$$\begin{array}{ccc}
H_q(X_{c_i}) & \xrightarrow{\nu_i - \text{id}} & H_q(X_{a_i}) \\
\downarrow j_* & & \downarrow j_* \\
H_q(X_{c_i}, X_{a_i}^*) & \xrightarrow{\nu_i - \text{id}} & H_q(X_{c_i}, X_{a_i}^*)
\end{array}$$

where $j_*$ is induced by inclusion.

Variation morphisms enter traditionally in the description of global and local fibrations of holomorphic functions at singular fibers, see e.g. [Mi], [La], [Si], [NN2, §2]. In dimension 2, already Zariski used $\nu_i - \text{id}$ in his theorem for the fundamental group. Chêniot [Ch2] also works with a kind of a variation map, different from ours. Our definition is a direct extension of the local variation maps (see e.g. [La, Lz]) to the global setting.

3. The Main Theorem

Let us recall the definition of the homological depth of a topological space at a point.

**Definition 3.1.** For a discrete subset $\Phi \subset X$, we denote by $\text{Hd}_q X$ the homological depth of $X$ at $\Phi$. We say that $\text{Hd}_q X \geq q + 1$ if, at any point $\alpha \in \Phi$, there is an arbitrarily small neighbourhood $N$ of $\alpha$ such that $H_i(N, N \setminus \{\alpha\}) = 0$, for $i \leq q$.

For a manifold $M$, at some point $\alpha$, we have $\text{Hd}_0 M \geq \dim_X M$. Complex $V$-manifolds are rational homology manifolds. So the homological depth measures the defect of being a homology manifold (for certain coefficients). For stratified complex spaces, Grothendieck [G] introduced the rectified homotopical depth, respectively the rectified homological depth, denoted $\text{rHd}$. This were later investigated by Hamm and Lê [HL3], who proved several of Grothendieck’s conjectures regarding them. See Proposition I.2.2 for more details and results involving $\text{rHd}$.

We may now state our principal result, using the notations in [§2]. The homology is with coefficients in $\mathbb{Z}$.

**Theorem 3.2.** Let $h : Y \to \mathbb{P}^1$ define a Lefschetz fibration on $X = Y \setminus V$ with isolated singularities (cf Definition [22]). Let the axis $A$ be not included in $V$. For some $k \geq 0$, suppose that the following conditions are fulfilled:

1. $H_q(X_c, X_c \cap A) = 0$ for $q \leq k$.
2. $H_q(X_c, X_{a_i}^*) = 0$ for $q \leq k$ and for all $i$.
3. $\text{Hd}_{X \setminus \text{Sing}_p} X \geq k + 3$.

Then $H_q(X, X_c) = 0$ for $q \leq k + 1$ and the kernel of the surjection $H_{k+1}(X_c) \to H_{k+1}(X)$ is generated by the images of the variation maps $\text{var}_i$, for $i = 1, p$. 
Note 3.3. For the annulation of the relative homology we need in fact a weaker condition than (C3), namely the following:

\[ \text{(C3i)} \quad \text{Hd}_{X \cap \text{Sing}_S} X \geq k + 2. \]

This will be clear from the proof, since (C3) is used (with \( k + 3 \)) only in Corollary 3.8 and Proposition 3.9(b). See also Proposition 4.3 for what become conditions (C2) and (C3) in special cases, and Proposition 4.2 for comparison to the rectified homological depth condition. For instance, it is well-known from [HL3] that, in case \( X \) is a complete intersection, then \( \text{rHd} \geq \text{dim}_C X \). This implies (see e.g. the proof of Proposition 4.2) that condition (C3) is satisfied in this case for \( k \leq \text{dim}_C X - 3 \).

In §4, we derive the form of this result in special cases, such as in case \( \text{Sing}_S \cap (A \times \mathbb{P}^1) \cap X = \emptyset \) (i.e. “no singularities in the axis”), in case the Lefschetz structure of the space \( X \) is hereditary on slices and also in the complementary case \( A \subset V \).

Note 3.4. During the time, Lefschetz hyperplane theorems have been generalized in several directions, giving rise to an extended literature, which the limited space does not allow us to cite here. May we just refer to Fulton’s general overview [Fu], Lamotke’s “classical” modern presentation of Lefschetz theorems [La] and to Goresky-MacPherson’s book [GM] which covers a lot of material.

On the other hand, the description of the kernel of the surjection stated above has been considered in a few papers only. The most recent results are for generic pencils of hyperplanes on quasi-projective manifolds, by Chéniot [Ch2], and on complements in \( \mathbb{P}^n \) of hypersurfaces with isolated singularities and for higher homotopy groups, by Libgober [Li]. The extension of Theorem 3.2 to homotopy groups is investigated in the preprints [Ty2, Ty3]; see also [CL].

In §4 we compare the conditions of our Theorem 3.2 (and show that they are significantly less restrictive) to the conditions used by some other authors in more particular settings than ours: the rectified homology depth condition used by Hamm and Lê [HL1, HL2, HL3] (see §4.2), respectively conditions used by Chéniot and Eyral [Ch1, Ch2, Ey] (see §4.3).

3.1. Proof of Theorem 3.2. Let \( K \subset \mathbb{P}^1 \) be a closed disk with \( K \cap \Lambda = \emptyset \) and let \( \mathcal{D} \) denote the closure of its complement in \( \mathbb{P}^1 \). We denote by \( S := K \cap \mathcal{D} \) the common boundary, which is a circle, and take a point \( c \in S \). Then take standard paths \( \gamma_i \subset \mathcal{D} \setminus \cup_i D_i \) (non self-intersecting, non mutually intersecting) from \( c \) to \( c_i \in \partial D_i \). The configuration \( \cup_i (D_i \cup \gamma_i) \) is a deformation retract of \( \mathcal{D} \). We shall also identify all fibers \( X_{c_i} \) to the fiber \( X_c \), by parallel transport along the paths \( \gamma_i \).

We denote \( A' := A \cap X_c \). Since \( A \not\subset V \), we have that \( A' \neq \emptyset \).

Proposition 3.5. If \( H_q(X_c, A') = 0 \) for \( q \leq k \), then the morphism induced by inclusion:

\[ H_q(X_D, X_c) \rightarrow H_q(X, X_c) \]

is an isomorphism for \( q \leq k + 1 \) and an epimorphism for \( q = k + 2 \).

Proof. We claim that, if \( H_q(X_c, A') = 0 \), for \( q \leq k \), then \( H_q(X_S, X_c) = 0 \) for \( q \leq k + 1 \). Note first that \( X_S \) is homotopy equivalent to the subset \( \mathbb{X}_S \cup (A' \times K) \) of \( \mathbb{X}_K \). Let \( I \) and \( J \) be two arcs which cover \( S \). We have the homotopy equivalence \( (X_S, X_c) \simeq \mathbb{X}_S \cup (A' \times K) \).
(X_I ∪ (A' × K) ∪ X_J ∪ (A' × K), X_J ∪ (A' × K)). Then, by excision, we have the isomorphism:

\[ H_*(X_S, X_c) \simeq H_*(X_I \cup (A' \times K), X_{\partial I} \cup (A' \times K)). \]

Furthermore, we have the homotopy equivalences of pairs: (X_I ∪ (A' × K), X_{\partial I} \cup (A' × K)) \overset{ht}{\simeq} (X_c \times I, X_c \times \partial I \cup A' \times I) and the latter is just the product of pairs (X_c, A') × (I, \partial I).

Our claim follows.

Next, by examining the exact sequence of the triple \((X_D, X_S, X_c)\) and by using the annulation of \(H_q(X_S, X_c)\) proved above, we see that \((X_D, X_c) \rightarrow (X_D, X_c)\) gives, in homology, an isomorphism in dimensions \(q \leq k + 1\) and an epimorphism in \(q = k + 2\). To end our proof, we just combine this with the isomorphism \(H_*(X_D, X_S) \simeq H_*(X, X_K)\), obtained by excision.

\[ \square \]

Since the kernel of the map \(H_{k+1}(X_c) \rightarrow H_{k+1}(X)\) is equal to the image of the boundary map \(H_{k+2}(X, X_c) \rightarrow H_{k+1}(X_c)\), we focus on the latter. Consider the commutative diagram:

\[
\begin{array}{ccc}
H_{k+2}(X_D, X_c) & \xrightarrow{i_*} & H_{k+2}(X, X_c) \\
\partial_1 \downarrow & & \downarrow \partial \\
H_{k+1}(X_c)
\end{array}
\]

where \(\partial\) and \(\partial_1\) are boundary morphisms. Since Proposition 3.5 shows that \(i_*\) is an epimorphism, we get:

**Corollary 3.6.** If \(H_q(X_c, A') = 0\) for \(q \leq k\) then, in the diagram (3), we have \(\text{im} \, \partial = \text{im} \, \partial_1\). \(\square\)

Notice that, for any \(M \subset \mathbb{P}^1\), \(X_M\) is homotopy equivalent to \(X_M\) to which one attaches, along \(A' \times M\), the product \(A' \times \text{Cone}(M)\). Since \(D\) is contractible, it follows that \(X_D \overset{ht}{\simeq} X_D\). Hence the pair \((X_D, X_c)\) is homotopy equivalent to \((X_D, X_c)\) and we may identify the boundary morphism \(H_{k+2}(X_D, X_c) \overset{\partial_1}{\rightarrow} H_{k+1}(X_c)\) to the boundary morphism \(H_{k+2}(X_D, X_c) \overset{\partial_1}{\rightarrow} H_{k+1}(X_c)\).

Remark also that we have the excision \(H_*(\bigcup_i X_{D_i}, \bigcup_i X_{c_i}) \overset{\sim}{\rightarrow} H_*(X_D, X_c)\) which gives a decomposition of the homology \(H_*(X_D, X_c)\) into the direct sum \(\oplus_i H_*(X_{D_i}, X_{c_i})\). Then the boundary map \(\partial_1\) is identified to the boundary map \(\partial_2\) obtained as sum of the boundary maps \(\partial_i : H_{k+2}(X_{D_i}, X_{c_i}) \rightarrow H_{k+1}(X_{c_i})\), where \(X_{c_i}\) is identified with \(X_c\) by parallel transport along the path \(\gamma_i\).

With these identifications, we have the following commutative diagram:

\[
\begin{array}{ccc}
\oplus_i H_{k+2}(X_{D_i}, X_{c_i}) & \simeq & H_{k+2}(\bigcup_i X_{D_i}, \bigcup_i X_{c_i}) \\
\text{exc} \downarrow & & \downarrow \partial_2 \\
H_{k+2}(X_D, X_c) & \xrightarrow{\partial_1} & H_{k+1}(X_c).
\end{array}
\]

It then follows that

\[
\text{im} \, \partial_1 = \sum_i \text{im} \, \partial_i.
\]
Our theorem will be proved if we do the following:

(i). Prove that $H_q(X_{D_i}, X_{a_i}) = 0$, for $q \leq k + 1$ and all $i$.

(ii). Find the image of the map $\partial_i : H_{k+2}(X_{D_i}, X_{c_i}) \to H_{k+1}(X_{c_i})$, for all $i$.

We shall reduce these problems again, by replacing $X_{D_i}$ by $X'_{D_i} := X_{D_i} \setminus \text{Sing}_S$. For this, we use condition (C3) for (ii), respectively condition (C3i) for (i).

**Lemma 3.7.** If $\text{Hd}_\mathbb{X} \cap \text{Sing}_SP\mathbb{X} \geq s + 1$, then, for all $i$, the map induced by inclusion $H_q(X'_{D_i}, X_{c_i}) \xrightarrow{\sim} H_q(X_{D_i}, X_{c_i})$ is an isomorphism, for $q \leq s - 1$, and an epimorphism, for $q = s$.

**Proof.** Due to the exact sequence of the triple $(X_{D_i}, X'_{D_i}, X_{c_i})$, it will be sufficient to prove, for all $i$, that $H_q(X'_{D_i}, X_{c_i}) = 0$, for $q \leq s$. This is true since the inclusion:

$$(X_{D_i} \cap (\cup_j B_{ij}), X_{D_i} \cap (\cup_j B_{ij} \setminus \{a_{ij}\})) \hookrightarrow (X'_{D_i}, X_{D_i}')$$

is an excision in homology (notice that the unions are disjoint). As usual, $B_{ij} \subset \mathbb{X}$ denotes a Milnor ball centered at the singular point $a_{ij} \in \text{Sing}_S$.

Indeed, the hypothesis $\text{Hd}_\mathbb{X} \cap \text{Sing}_SP\mathbb{X} \geq s + 1$ tells that the homology of each pair $(X_{D_i} \cap B_{ij}, X_{D_i} \cap B_{ij} \setminus \{a_{ij}\})$ annulates up to dimension $s$. \hfill $\square$

**Corollary 3.8.** If $\text{Hd}_\mathbb{X} \cap \text{Sing}_SP\mathbb{X} \geq k + 3$, then, for all $i$:

$$\text{im}(\partial_i : H_{k+2}(X_{D_i}, X_{c_i}) \to H_{k+1}(X_{c_i})) = \text{im}(\partial'_i : H_{k+2}(X'_{D_i}, X_{c_i}) \to H_{k+1}(X_{c_i})).$$

**Proof.** We have that $\partial'_i = \partial_i \circ j_*$, where $j_* : H_{k+2}(X'_{D_i}, X_{c_i}) \to H_{k+2}(X_{D_i}, X_{c_i})$ is induced by the inclusion. By Lemma 3.7, $j_*$ is surjective, hence $\text{im} \partial'_i = \text{im} \partial_i$. \hfill $\square$

The last step in the proof of Theorem 3.2 is the following result, where the variation maps come in:

**Proposition 3.9.** If $H_q(X_{c_i}, X'_{a_i}) = 0$, for $q \leq k$, then:

(a) $H_q(X'_{D_i}, X_{c_i}) = 0$ for $q \leq k + 1$.

(b) $\text{im} \partial'_i = \text{im} (\text{var}_i : H_{k+1}(X_{c_i}, X'_{a_i}) \to H_{k+1}(X_{c_i})).$

**Proof.** Let us take Milnor data $(B_{ij}, D_i)$ at the (stratified) singularities $a_{ij}$. Recall that the radius of $D_i$ is very small in comparison to the radius of $B_{ij}$. We shall give the proof for a fixed index $i$ and therefore we suppress the lower indices $i$ in the following.

(a). Let $D^* = D \setminus \{a\}$. By retraction, we identify $D^*$ to a circle and cover this circle with the union of two arcs $I \cup J$, as follows: for the standard circle $S^1$, we take $I := \{\exp i\pi t \mid t \in [-\frac{1}{2}, 1]\}$, $J := \{\exp i\pi t \mid t \in [\frac{1}{2}, 2]\}$. Then $X_{D_i}^\text{ht} \simeq X_I \cup X_J$ and $X_{c_i}^\text{ht} \simeq X_I \cup X_J \simeq X_c \times J$. With these notations, we have the following isomorphisms induced by homotopy equivalences:

$$H_*(X_{D_i}, X_{c_i}) \simeq H_*(X_{D_i}^\text{ht}, X_{D_i}^\text{ht} \cup X^*_{a_i} \times D, X_{c_i} \cup X^*_{a_i} \times D) \simeq H_*(X_{I} \cup X_{J} \cup X^*_{a_i} \times D, X_{J} \cup X^*_{a_i} \times D),$$

where $X^*_{a_i} \times D$ is a notation for $X_{D_i} \setminus \cup_j B_{ij}$, which is the total space of a trivial fibration over $D$, with fiber $X_{a_i} \setminus \cup_j B_{ij} \simeq X^*_{a_i}$.

We then excise $X_{J} \cup X^*_{a_i} \times D$ from the last pair and get the homology of the pair $(X_{I}, X_{\partial I} \cup X^*_{a_i} \times I)$, which pair is homotopy equivalent to the product $(X_c, X^*_{a_i}) \times (I, \partial I)$. 


Since, by hypothesis, the homology of the pair \((X_c, X_a^*)\) annulates up to dimension \(k\), it follows that the homology of the last product annulates up to dimension \(k + 1\).

(b). In the following commutative diagram, the variation map identifies to the right-hand vertical arrow. This diagram is a Wang type exact sequence, the proof of which is explained by Milnor in [Mi, page 67, Lemma 8.4].

\[
\begin{array}{ccc}
H_{k+2}(X_{D^i}, X_{c_i}) & \xrightarrow{\partial_i^t} & H_{k+1}(X_{c_i}) \\
\text{excision} \uparrow \simeq & & \uparrow \text{var}_i \\
H_{k+2}(X_I, X_\partial \cup X_{a_i}^* \times I) & \simeq & H_{k+1}(X_{c_i}, X_{a_i}^* \otimes H_1(I, \partial I))
\end{array}
\]

This shows that \(\text{im} \partial_i^t = \text{im} \text{var}_i\).

We are now able to conclude the proof of Theorem 3.2. The claim (i) above, and hence the first claim of the theorem, follows from Lemma 3.7 and Proposition 3.9(a).

The second claim of the theorem follows by the sequence of results: Corollary 3.6, equality (4), Corollary 3.8 and Proposition 3.9(b).

4. Further results and particular cases

4.1. The case \(A \subset V\). We discuss in the following the case \(A' = \emptyset\), equivalently, \(A \subset V\), which is complementary to the one we have considered until now. One would be tempted to replace the condition (C1) with “\(H_q(X_c) = 0\), for \(q \leq k\)”, but this appears to be too restricting.

Nevertheless, in case \(h|_X\) is not onto \(\mathbb{P}^1\), the situation becomes more interesting. So let us assume that \(V\) contains a fiber of the pencil \(h : Y \setminus A \to \mathbb{P}^1\). Even if the axis \(A\) is outside the space \(X\), the “singularities in the axis” influence the topology of the pencil. We have the following result on a class of nongeneric pencils, disjoint from the class considered in Theorem 3.2. Let us denote \(\Sigma := \sigma(\text{Sing}_S)\).

**Theorem 4.1.** Let \(X = Y \setminus V\) have a structure of Lefschetz fibration with isolated singularities, such that \(V\) contains a member of the pencil. For some fixed \(k \geq 0\), assume that \(H_q(X_c, X_a^*) = 0\) for \(q \leq k\), where \(X_c\) is a general member \(X_c\) and \(X_a^*\) is any atypical one. We have:

(a) If \(H_q(X, X \setminus \Sigma) = 0\) for \(q \leq k + 1\), then \(H_q(X, X_c) = 0\) for \(q \leq k + 1\).

(b) If \(H_q(X, X \setminus \Sigma) = 0\) for \(q \leq k + 2\), then:

\[H_{k+1}(X) \simeq H_{k+1}(X_c)/\sum_i \text{im} \text{var}_i.\]

**Proof.** The proof follows the lines of the proof of Theorem 3.2 and we shall only point out the differences, using the same notations. In our case, the target of the holomorphic function \(h|_X\) is \(\mathbb{P}^1 \setminus \{\alpha\}\) for some \(\alpha \in \mathbb{P}^1\). We have \(D^t \simeq \mathbb{P}^1 \setminus \{\alpha\}\) and therefore \(X_D^t \simeq X\). Examining the proofs of Proposition 3.3 and Corollary 3.8, we see that, under our assumptions, their conclusions hold without any restrictions on \(k\). Hence (C1) does not enter as condition in our proof. On the other hand, by Proposition 4.3(b), we can use \((C3)'\) instead of (C3). Condition (C2) is itself an assumption of the above theorem. \(\square\)
4.2. Comparing to the rHd condition.

**Proposition 4.2.** Theorem 3.2 holds if we replace the conditions (C2) and (C3) by the single condition:

(C4) \( \text{rHd} X \geq k + 3. \)

The first claim of Theorem 3.2 holds with a weaker assumption in place of (C4), namely (see Note 3.3):

(C4i) \( \text{rHd} X \geq k + 2. \)

**Proof.** Indeed, \( \text{rHd} X \geq q \) implies \( \text{rHd} X \geq q \), since \( X \) is a hypersurface in \( X \times \mathbb{P}^1 \) and one can apply the result of Hamm and Lê [HL3, Theorem 3.2.1]. This in turn implies \( \text{Hd}_\alpha X \geq q \), for any point \( \alpha \in X \), by definition.

Next, \( \text{rHd} X \geq q \) implies that the homology of the pair \((X_{D_i}, X_{c_i})\) annulates up to dimension \( q - 1 \), by [Tı3, Proposition 3.4] (where \( \text{rhd} \) is used instead of \( \text{rHd} \), but the proof is the same). This shows that conditions (C1) + (C4i) imply the first claim of Theorem 3.2.

Furthermore, if we assume (C4) instead of (C4i), then, besides the annulation of the homology of \((X_{D_i}, X_{c_i})\) up to \( k + 2 \) (shown just above), it follows that \( H_q(X_{D_i}, X_{c_i}) = 0 \) for \( q \leq k + 1 \), by Lemma 3.7. The proof of Proposition 3.9(a) shows in fact that the annulation of homology of \((X_{c_i}, X_{a_i}^*)\) up to \( k \), which is condition (C2). Now Theorem 3.2 applies. \( \square \)

4.3. Particular cases. From Theorem 3.2 and its proof, one may derive several versions in particular cases, recovering some of the results in the literature. To do that, one has to take into account the following observations (still under the condition \( A \cap X \neq \emptyset \)):

**Proposition 4.3.**

(a) In case \( X \cap \text{Sing}_{SP} = \emptyset \), the condition (C3) is void.

(b) In case \( (A \times \mathbb{P}^1) \cap X \cap \text{Sing}_{SP} = \emptyset \), we may replace condition (C3) by the following more general condition (which is also more global):

(C3’) \( H_q(X, X \setminus \Sigma) = 0 \), for \( q \leq k + 2 \).

(c) In case \( (A \times \mathbb{P}^1) \cap \text{Sing}_{SP} = \emptyset \), if condition (C1) is true, then (C2) is equivalent to the following:

(C2’) \( H_q(X_{a_i}^*, X_{a_i}^* \cap A) = 0 \), for \( q \leq k - 1 \).

**Proof.** (a) is obvious.

(b) By examining the Proof of Theorem 3.2, we see that we have used the homology depth condition only to compare \( X_{D_i} \) to \( X_{D_i}^* \). We may cut off from the proof this comparison (which means Lemma 3.7 and Corollary 3.8) and start from the beginning with the space \( X \setminus \Sigma \) instead of the space \( X \). Taking into account that, under our hypothesis, \( X_{D_i}^* = X_{D_i}^* \setminus \Sigma \), for all \( i \), the effect of this change is that the proof yields the conclusion “\( H_q(X \setminus \Sigma, X_c) = 0 \), for \( q \leq k + 1 \)” and the corresponding statement for the vanishing cycles. At this final stage, condition (C3’) allows one to replace \( X \setminus \Sigma \) by \( X \).
(c) When there are no singularities in the axis, we have $A \cap X^*_i = A \cap X_c$ for any $i$. Then the exact sequence of the triple $(X_c, X^*_i, A \cap X^*_i)$ shows that the boundary morphism
\[ H_q(X_c, X^*_i) \to H_{q-1}(X_c, A \cap X^*_i) \]
is an isomorphism, for $q \leq k$, by condition (C1). This implies our claimed equivalence. □

In case of quasi-projective varieties, we have an abundance of hyperplane pencils, which are moreover generic, in the sense that the axis is transversal to the stratification. It easily follows that such a pencil has no singularities along the axis (see e.g. the proof of [13, Proposition 2.4] for a detailed explanation). We are therefore in the conditions of Proposition 4.3(b) and (c). Another nice aspect of quasi-projective varieties is that the Lefschetz structure is hereditary on slices. Namely, as already observed by Chéniot [Ch1], since the axis $A$ is chosen to be generic, it becomes in turn a generic slice of a hyperplane slice of $X$, and so on.

Condition (C2)' has been used by Chéniot [Ch1, Ch2] and Eyral [Ey] in theorems on generic pencils of hyperplanes, respectively condition (C3)' has been used by C. Eyral in proving a version of the First Lefschetz Hyperplane Theorem (compare to [Ey, Proof of Theorem 2.5]). Therefore, via Proposition 4.3 and the preceding observations in case of quasi-projective varieties, our Theorem 3.2 recovers the results in the cited articles.

5. Vanishing cycles of polynomial functions on $\mathbb{C}^n$

A polynomial function $f : \mathbb{C}^n \to \mathbb{C}$ can naturally be considered as a nongeneric pencil of hypersurfaces on $\mathbb{C}^n$, which is a particular quasi-projective variety. Indeed, this function extends as a meromorphic function on $\mathbb{P}^n$, as follows. If $\deg f = d$, then $h = \tilde{f}/z^d : \mathbb{P}^n \dashrightarrow \mathbb{P}^1$, where $\tilde{f}$ is the homogenized of $f$ with respect to the new variable $z$ and the axis of the pencil is $A = \{f_d = 0\} \subset H_\infty$. Here we have $Y = \mathbb{P}^n$, $V = H_\infty = \{z = 0\} \subset \mathbb{P}^n$. We are in the situation described in [41, namely we have a pencil on $X = \mathbb{C}^n$, where $h|_{\mathbb{C}^n} = f$. In particular, $\Sigma = \text{Sing} f$.

For such a pencil, we may work under more general hypotheses: we assume that the function $f$ has isolated singularities, but we put no condition on singularities in the axis, which may be non-isolated. We show how this can fit in the theory developed before.

Take the complement of a big ball $B \subset \mathbb{C}^n$, centered at the origin of a fixed system of coordinates on $\mathbb{C}^n$. The complement $C_B := \mathbb{C}^n \setminus B$ plays the role of a “uniform” neighbourhood of the whole hyperplane at infinity $H_\infty$ and of all singularities in the axis together. For big enough radius of $B$, we have
\[ X_{a_i} \cap B \cong X_{a_i} \]
for any $i$, since the distance function has a finite set of critical values on the algebraic sets $X_{a_i}$. We claim that $f^{-1}(B_i) \cap B \setminus \bigcup_j B_{ij} \to D_i$ is a trivial fibration, where the $B_{ij}$’s are small Milnor balls around the critical points of $f$ on $X_{a_i}$ and $D_i$ is a small enough disk. Indeed, the fibers of $f$ over $D_i$ are transversal to the boundary of a big ball and transversal to the boundaries of the Milnor balls. Our claim then follows by Ehresmann’s Theorem.

This implies, as in [22], that there is a well defined geometric monodromy representation at each $a_i \in \Lambda \subset \mathbb{C}$, $\rho_i : \pi_1(\partial D_i, c_i) \to \text{Iso}(X_{c_i}, X_{c_i} \setminus (C_B \cup \bigcup_j B_{ij}))$. This induces a variation
map: \[ \text{var}_i : H_k(X_{c_i}, X_{a_i}^*) \rightarrow H_k(X_{c_i}), \]
where \( X_{a_i}^* := X_{a_i} \setminus \text{Sing} f \) is now used as a notation for the subset \( X_{c_i} \setminus (C_B \cup \cup_j B_{ij}) \) of \( X_{c_i} \).
This is justified by the fact that \( X_{a_i} \setminus \text{Sing} f \) is homotopy equivalent to \( X_{a_i} \cap B \setminus \cup_j B_{ij} \),
which in turn can be identified to \( X_{c_i} \setminus (C_B \cup \cup_j B_{ij}) \) as fibers in the above mentioned trivial fibration.

We shall show that Theorem 4.1 holds for a pencil defined by a polynomial function with isolated singularities \( f : \mathbb{C}^n \rightarrow \mathbb{C} \),
without any condition on singularities at the axis at infinity and moreover, that we have a more precise grip on variation maps.

Let us first remark that the boundary map \( H_{*+1}(\mathbb{C}^n, X_c) \overset{\partial}{\rightarrow} \tilde{H}_*(X_c) \) is an isomorphism in any dimension. This follows from the long exact sequence of the pair \((\mathbb{C}^n, X_c)\).

Next, we have by excision: \( H_{*+1}(\mathbb{C}^n, X_c) \simeq \oplus_i H_{*+1}(X_{D_i}, X_c) \).
These show that \( H_*(X_c) \) decomposes into the direct sum of vanishing cycles at each atypical fiber \( X_{a_i} \).
Note that the direct sum decomposition depends on the paths \( \gamma_i \).

We say that \( \text{im}(H_{*+1}(X_{D_i}, X_c) \overset{\partial}{\rightarrow} H_*(X_c)) \) is the module of vanishing cycles at the fiber \( X_{a_i} \).
It has been proved in general that \( H_*(X_c) \) is the direct sum of the modules of vanishing cycles (see [ST] proof of Theorem 3.1, [NN1] Theorem 1.4) regardless of the singularities of \( f \).

It is well-known that in case of a holomorphic function germ with isolated singularity on \( \mathbb{C}^n \),
the variation map of the local monodromy is an isomorphism [NN1]. But in our global case of a polynomial function with isolated singularities,
the variation maps cannot be isomorphisms since the homology of the fiber \( H_*(X_c) \) captures information on
vanishing cycles at all atypical fibers \( X_{a_i} \) together. We may prove the following statement, the part (b) of which being just Neumann-Norbury’s result [NN2] Theorem 2.3 via an identification (by some excision) of our variation map to the local variation maps used in [NN2].

**Proposition 5.1.** Let \( f : \mathbb{C}^n \rightarrow \mathbb{C} \) be a polynomial function with isolated singularities. Then:

(a) If \( H_q(X_c, X_{a_i}^*) = 0 \) for \( q \leq k \) and for any \( i \),
then \( \tilde{H}_q(X_c) = 0 \) for \( q \leq k \).

(b) [NN2] Theorem 2.3 The variation map \( \text{var}_i : H_*(X_{c_i}, X_{a_i}^*) \rightarrow H_*(X_{c_i}) \) is injective,
for any \( i \). In particular, we have \( H_q(X_c) \simeq \sum_i \text{im}(\text{var}_i) \) for the first integer \( q \geq 1 \)
such that \( H_q(X_c) \neq 0 \).

**Proof.** Since the fibers of \( f \) are Stein spaces of dimension \( n-1 \),
their homology groups are trivial in dimensions \( \geq n \).
The condition \((C3)'\) is largely satisfied, since \((\mathbb{C}^n, \mathbb{C}^n \setminus \text{Sing} f)\)
is \((2n-1)-\)connected. Hence part (a) follows from Theorem 4.1.
For part (b), remark first that, by the above arguments, the boundary map \( \partial_i : H_{*+1}(X_{D_i}, X_{c_i}) \rightarrow \tilde{H}_*(X_{c_i}) \)
is injective, for any \( i \). Next, one may replace \( X_{D_i} \) by \( X_{D_i}^* \) since \((X_{D_i}, X_{D_i}^*)\)
is \((2n-1)-\)connected. It follows that the boundary morphism \( \partial_i : H_{*+1}(X_{D_i}^*, X_{c_i}) \rightarrow \tilde{H}_*(X_{c_i}) \)
is injective. As in Proposition 3.9 one may identify \( H_{*+1}(X_{D_i}^*, X_{c_i}) \rightarrow H_*(X_{c_i}, X_{a_i}^*) \),
by excision and \( \partial_i \) can be identified with \( \text{var}_i : H_*(X_{c_i}, X_{a_i}^*) \rightarrow H_*(X_{c_i}). \)

The image of the “pseudo-embedding” \( \iota : X_{a_i}^* \rightarrow X_{a_i} \cap B \setminus \cup_j B_{ij} \rightarrow X_{c_i} \)
plays here the role of the boundary of the Milnor fiber in the local case. We may therefore call \( \text{im} \iota_* \)
the group of “boundary cycles” at \( a_i \). We immediately get the following consequence; it can
also be deduced, by a series of identifications, from the Neumann-Norbury more general result [NN1, Theorem 1.4].

**Corollary 5.2.** The invariant cycles under the monodromy at \( a_i \) are exactly the boundary cycles, i.e. the following sequence is exact:

\[
H_*(X_{a_i}^*) \xrightarrow{\iota_*} H_*(X_{c_i}) \xrightarrow{\nu_i - \text{id}} H_*(X_{c_i}).
\]

**Proof.** We have the following commutative diagram, where the first row is the exact sequence of the pair \((X_{c_i}, X_{a_i}^*)\):

\[
\begin{array}{c}
H_*(X_{a_i}^*) \xrightarrow{\iota_*} H_*(X_{c_i}) \xrightarrow{j_*} H_*(X_{c_i}, X_{a_i}^*) \\
v_i - \text{id} \searrow \var_\iota \ \ \ \ \upaarrow
\end{array}
\]

We have that \( \text{im} \iota_* = \ker j_* \). Since \( \nu_i - \text{id} = \var_i \circ j_* \), and since \( \var_i \) is injective by Proposition 5.1, our claim follows. \( \Box \)

**Note 5.3.** This result may be considered as a counterpart, in a non-proper situation, of the well-known “invariant cycle theorem” proved by Clemens [Cl]. The later holds for proper holomorphic functions \( g : X \to D \), in cohomology (thus “invariant co-cycle theorem” would be more appropriate), where \( X \) is a Kähler manifold. It sais that the following sequence is exact: \( H^*(X) \xrightarrow{j_*} H^*(X_c) \xrightarrow{h - \text{id}} H^*(X_c) \), where \( h \) denotes the monodromy around the center of the disk \( D \) (assumed to be the single critical value of \( g \)).

Is is natural to ask if an invariant cycle result similar to Corollary 5.2 holds for more general classes of non-proper pencils.

**References**

[AF] A. Andreotti, T. Frankel, *The second Lefschetz theorem on hyperplane sections*, in: Global Analysis, Papers in Honor of K. Kodaira, Princeton Univ. Press 1969, p.1-20.

[Br] S.A. Broughton, *Milnor numbers and the topology of polynomial hypersurfaces*, Invent. Math., 92, 2 (1988), 217–241.

[BV] D. Burghelea, A. Verona, *Local homological properties of analytic sets*, Manuscripta Math., 7 (1972), 55–66.

[Ch1] D. Chéniot, *Topologie du complémentaire d’un ensemble algébrique projectif*, L’Enseign. Math., 37 (1991), 293-402.

[Ch2] D. Chéniot, *Vanishing cycles in a pencil of hyperplane sections of a non-singular quasi-projective variety*, Proc. London Math. Soc. (3), 72, no. 3 (1996), 515–544.

[CL] D. Chéniot, A. Libgober, *Zariski-van Kampen theorem for higher homotopy groups*, math.AG/0203019.

[Cl] C.H. Clemens, *Degeneration of Kähler manifolds*, Duke Math. J., 44 (1977), no. 2, 215–290.

[Ey] C. Eyral, *Tomographie des variétés singulières et théorèmes de Lefschetz*, Proc. London Math. Soc. (3), 83 (2001), 141-175.

[Fu] W. Fulton, *On the topology of algebraic varieties*, in: Algebraic Geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 15–46, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[GLPW] C.G. Gibson, K. Wirthmüller, A.A. du Plessis, E.J.N. Looijenga, *Topological Stability of Smooth Mappings*, Lect. Notes in Math., 552, Springer Verlag 1976.

[GM] M. Goresky, R. MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 14. Springer-Verlag, Berlin, 1988.
A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux* (SGA 2), Séminaire de Géométrie Algébrique du Bois-Marie, 1962. Advanced Studies in Pure Mathematics, Vol. 2. North-Holland Publishing Co., Amsterdam; Masson & Cie, Paris, 1968.

H.A. Hamm, Lê D.T., *Un théorème de Zariski du type de Lefschetz*, Ann. Sci. École Norm. Sup. (4), 6 (1973), 317–355.

H.A. Hamm, Lê D.T., *Local generalizations of Lefschetz-Zariski theorems*, J. Reine Angew. Math., 389 (1988), 157–189.

H.A. Hamm, Lê D.T., *Rectified homotopical depth and Grothendieck conjectures*, The Grothendieck Festschrift, Collect. Artic. in Honor of the 60th Birthday of A. Grothendieck. Vol. II, Prog. Math., 87 (1990), 311–351.

K. Lamotke, *The topology of complex projective varieties after S. Lefschetz*, Topology, 20 (1981), 15–51.

Lê Dũng Tráng, *Le concept de singularité isolée de fonction analytique*, in: Complex analytic singularities, 215–227, Adv. Stud. Pure Math., 8, North-Holland, Amsterdam, 1987.

S. Lefschetz, *L’analysis situs et la géométrie algébrique*, Gauthier-Villars, Paris 1924, (nouveau tirage 1950).

A. Libgober, *Homotopy groups of the complements to singular hypersurfaces. II*, Ann. Math., II. Ser. 139, No.1 (1994), 117–144.

A. Libgober, M. Tibăr, *Homotopy groups of complements and nonisolated singularities*, Int. Math. Res. Not. 2002, no. 17, 871–888.

E.J.N. Looijenga, *Isolated singular points on complete intersections*, London Mathematical Society Lecture Note Series, 77. Cambridge University Press, Cambridge, 1984.

J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies 61, Princeton 1968.

W.D. Neumann, P. Norbury, *Vanishing cycles and monodromy of complex polynomials*, Duke Math. J. 101 (2000), no. 3, 487–497.

W.D. Neumann, P. Norbury, *Unfolding polynomial maps at infinity*, Math. Ann. 318 (2000), no. 1, 149–180.

D. Siersma, *Variation mappings on singularities with a 1-dimensional critical locus*, Topology, 30, no. 3 (1991), 445–469.

D. Siersma, M. Tibăr, *Singularities at infinity and their vanishing cycles*, Duke Math. Journal, 80, 3 (1995), 771-783.

R. Thom, *Ensembles et morphismes stratifiés*, Bull. Amer. Math. Soc., 75 (1969), 249–312.

M. Tibăr, *Topology at infinity of polynomial maps and Thom regularity condition*, Compositio Math., 111, 1 (1998), 89-109.

M. Tibăr, *Topology of Lefschetz fibrations in complex and symplectic geometry*, Newton Institute preprint NI01029, 2001.

M. Tibăr, *Connectivity via nongeneric pencils*, Internat. J. Math., 13, 2 (2002), 111–123.

M. Tibăr, *On higher homotopy groups of pencils*, [math.AG/0207108](https://arxiv.org/abs/math.AG/0207108)

M. Tibăr, *Singularities and topology of meromorphic functions*, in: Trends in singularities, 223–246, Trends Math., Birkhuser, 2002.

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