PRIME MODULES AND ASSOCIATED PRIMES OF INDUCED MODULOS OVER RINGS GRADED BY UNIQUE PRODUCT MONOIDS

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Abstract. We study prime ideals, prime modules, and associated primes of graded modules over rings $S$ graded by a unique product monoid. We consider two situations in detail: (a) the case where $S$ is strongly group-graded and (b) the case where $S$ is a crossed product and the ideal or module is induced from the identity component $R$ of $S$. We give explicit conditions for ideals and modules of $R$ to induce prime ideals of or prime modules over $S$ in these two cases. We then describe the set of associated prime ideals of an arbitrary induced module.

One of our main interests is to give necessary and sufficient conditions for primeness, and to describe the associated primes, in the crossed product case when the action of the monoid is not an action by automorphisms; this includes the case of a skew polynomial ring $R[x; \sigma]$ where $\sigma$ is an endomorphism of $R$.

At the end, we give some illustrative examples, several of which show the necessity of the various hypotheses in our results.

1. Introduction

Suppose $S$ is a ring graded by a unique product monoid $G$ and $R = S_e$ is the identity component of $S$. Suppose also that $I$ is an ideal of $R$ and $M$ is a right $R$-module, whence the induced right $S$-module $M \otimes_R S$ is a graded right $S$-module. In each of Sections 3, 4, and 5 we give necessary and sufficient conditions for $IS$ to be a prime ideal of $S$, and, more generally, for the induced $S$-module $M \otimes_R S$ to be prime. We then give a description of the annihilators of the prime submodules of $M \otimes_R S$; these are known as the associated primes of $M \otimes_R S$.

The definitions and general results used in the rest of the paper are introduced in Section 2. Assuming the grading monoid to be a unique product monoid allows us to extend arguments used in studying $\mathbb{N}$- and $\mathbb{Z}$-graded rings, such as degree arguments. We are able to conclude that the associated primes of a graded $S$-module must be homogeneous.

In Section 3 we consider the case where $G$ is a group and the grading is strong. In this case, there is a natural notion of $G$-invariant ideal of $R$, and we show that $IS$ is a prime ideal if and only if $I$ is a $G$-invariant ideal such that whenever $I$ contains a product of $G$-invariant ideals, $I$ contains one of them. Let $(I : G)$ denote the largest $G$-invariant ideal contained in the ideal $I$ of $R$. We show that a graded $S$-module is prime if and only if $(\text{ann } N : G) = (\text{ann } M : G)$ for all nonzero $R$-submodules $N$ of the identity component $M$ of the module. We then show that the associated primes of a graded module are of the form

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KS where $K$ is the largest $G$-invariant ideal contained in the annihilator of an $S$-prime submodule of $M$.

In Section 4, we study the case where $S$ is a crossed product over $R$ with an action $r \mapsto r^g$ of $G$ on $R$. When $G$ is not a group and the action of $G$ on $R$ is not by automorphisms, the situation is naturally more complicated. For example, we must distinguish between $G$-stable and $G$-invariant ideals. We give necessary and sufficient conditions for $IS$ or $M \otimes_R S$ to be prime, but the conditions are less symmetric than in Section 3. For example, Proposition 4.9 shows that $M \otimes_R S$ is a prime $S$-module if and only if for any $m \in M, b \in R, g \in G$, if $mR^g(\hat{b}h^g) = 0$ for all $h \in G$, then $m = 0$ or $Mb = 0$. Sometimes nicer conditions can be given for primeness with an extra hypothesis. For example, Corollary 4.6 shows that when each map $r \mapsto r^g$ is surjective, a $G$-invariant ideal $I$ of $R$ induces a prime ideal $IS$ if and only if whenever $A, B$ are ideals of $R$ with $B$ $G$-stable and $AB \subseteq I$, we have $A \subseteq I$ or $B \subseteq I$. This result is also true without the surjectivity hypothesis if $R$ is commutative, but another hypothesis about reversibility of the action of $G$ must then be added. This reversibility hypothesis is a novel feature of considering non-commutative monoids that are not groups; in particular, it has no analog in the study of skew polynomial rings. When $R$ satisfies the a.c.c. on ideals and $G$ acts by automorphisms, conditions just like those in the strongly group-graded case are necessary and sufficient.

We also describe the associated primes of $M \otimes_R S$ as extensions $KS$ where $K$ is the largest $G$-stable ideal contained in the annihilator of an $S$-prime submodule of some twist of $M$. These twists are shown to be necessary in general by examples in Section 6, but may be eliminated when each map $r \mapsto r^g$ is surjective and the reversibility hypothesis mentioned in the previous paragraph holds.

We discuss skew polynomial and skew laurent rings in Section 5; the results are special cases of the results in previous sections but often have a simpler form. One of the motivations for this paper was to understand conditions for the skew polynomial ring $R[x; \sigma]$ to be prime and generalize them to characterize associated primes. Various definitions of a $\sigma$-prime ring have been given over the years: see for example, Goldie-Michler [4], Pearson–Stephenson [13], and Irving [5]. (See the discussion in Section 5 for more details.) All of these definitions give necessary and sufficient conditions for $R[x; \sigma]$ to be prime in special cases, but we are unaware of any published definition that works in complete generality. We give such a definition here.

Even if $\sigma$ is an automorphism, the skew laurent ring $R[x^{\pm 1}; \sigma]$ can be prime when the skew polynomial ring $R[x; \sigma]$ is not. This indicates that we should refer to $S$-primeness rather than $\sigma$-primeness, where $S = R[x; \sigma]$ or $S = R[x^{\pm 1}; \sigma]$ is the extension ring we are concerned with. That is the approach we take in this paper.

As noted above, Section 6 collects various examples illustrating the results of the paper and showing that various hypotheses are necessary. Example 6.1 gives a good picture of our results in a simple setting. It computes the associated primes in the skew polynomial case when $R$ is the coordinate ring of an affine algebraic set and $M$ is the simple module corresponding to a point. The example is still complex enough to exhibit phenomena that do not occur for automorphic skew polynomial rings.
2. Definitions and basic results

Throughout this paper $R$ will be a subring of $S$. We are interested in the connection between primeness of ideals and $R$-modules induced from $R$ to $S$. We give some general definitions and then quickly specialize to the case we will study: $S$ is a $G$-graded ring for a unique product monoid $G$ and $R$ is the identity component of $S$. In this section, we prove that such rings and induced modules possess properties that are similar to properties of polynomial extensions. In particular, we show that it is generally enough to consider graded ideals and graded submodules.

A nonzero $R$-module $M$ is said to be prime if $\text{ann} N = \text{ann} M$ for all nonzero submodules $N$. If $M$ is any $R$-module, an associated ideal of $M$ is the annihilator of a prime submodule of $M$. We denote the set of all associated ideals of $M$ by $\text{Ass} M$. The annihilator of a prime module is always a prime ideal, and so an associated ideal is also called an associated prime. Conversely, an ideal $I$ is prime if and only if the module $R/I$ is prime.

Note that $M$ is prime if and only if for any $m \in M$ and any ideal $J \triangleleft R$ with $mJ = 0$, we have either $m = 0$ or $MJ = 0$. Another equivalent condition for $M$ to be prime is that whenever $m \in M$, $b \in R$ and $mRb = 0$, either $m = 0$ or $Mb = 0$. In each of Sections 3, 4, and 5 we generalize these conditions to equivalent conditions for the primeness of the induced module $M \otimes_R S$ in the setting where $M$ is an $R$-module and $S$ is an overring of $R$.

We say an ideal $I$ of $R$ is right $S$-stable if $SI \subseteq IS$, that is, if $IS$ is an ideal of $S$. We say $I$ is $S$-stable if $IS = SI$. We say an ideal $I$ of $R$ is right $S$-prime if $IS$ is a prime ideal of $S$. Such an $I$ must be right $S$-stable. We say $R$ is an $S$-prime ring if $0$ is a right $S$-prime ideal, i.e., if $S$ is a prime ring.

A similar definition of $S$-stable was given in Montgomery-Schneider [8], although the name they used would be “$G$-stable” in our case. They did not make the analogous definition for $S$-prime.

If $M$ is a right $R$-module, we say $M$ is $S$-prime if the induced $S$-module $M \otimes_R S$ is prime.

Lemma 2.1. Let $I$ be a right $S$-stable ideal of $R$. Then the right $R$-module $R/I$ is $S$-prime if and only if the ideal $I$ is right $S$-prime.

Proof. Since $IS \triangleleft S$, our remarks after the definition of prime module imply $S/IS$ is a prime module if and only if $IS$ is a prime ideal. Since $R/I \otimes_R S \cong S/IS$ as right $S$-modules, this proves the lemma.

A unique product monoid is a monoid $G$ with the property that whenever $X, Y$ are nonempty finite subsets of $G$, there exist $x \in X, y \in Y$ such that $xy \neq x'y'$ for any pair $(x', y') \in X \times Y$ different from $(x, y)$. By choosing the sets $X, Y$ appropriately, we see that a unique product monoid must be cancellative and torsionfree (i.e., if $a$ is not the identity, all powers $a, a^2, \ldots$ are distinct). If a unique product monoid is a group, we call it a unique product group.

Remark. Any orderable monoid is a unique product monoid. Thus, for example, any submonoid of a free group or a torsionfree nilpotent group is a unique product monoid.
The group generated by $x, y$ subject to the relations $x^{-1}y^2x = y^{-2}$ and $y^{-1}x^2y = x^{-2}$ is torsionfree but not a unique product group. For a discussion of this and related examples, see Carter [1].

The monoid generated by $x, y$ subject to the relations $xy = yx$ and $x^2 = y^2$ is commutative, torsionfree, and cancellative but is not a unique product monoid, as can be seen by taking $X = Y = \{x, y\}$.

**Conventions.** Throughout this paper, $G$ will be a unique product monoid or a unique product group with identity $e$. We will use $\mathbb{N}$ to denote the additive monoid of nonnegative integers. Modules will be unital right modules and ideals will be two-sided unless the contrary is explicitly stated.

Recall that an abelian group $A$ is $G$-graded if a decomposition $A = \bigoplus_{g \in G} A_g$ into a direct sum of subgroups is given. The nonzero elements of $A_g$ are said to be homogeneous of degree $g$. For a homogeneous $a \in A$, we write $\partial a$ for the degree of $a$. Note that 0 is not regarded as a homogeneous element, although we will sometimes write “nonzero homogeneous element” for emphasis.

If $a \in A$ is arbitrary, we say an expression $a = a_1 + \cdots + a_k$ is a canonical form for $a$ if the elements $a_i$ are homogeneous and their degrees $\partial a_i$ are distinct. If $a = a_1 + \cdots + a_k$ is a canonical form, the $\partial a_i$ homogeneous component of $a$ is $a_i$. We call $\{\partial a_1, \ldots, \partial a_k\}$ the support of $a$. A subset $B \subseteq A$ is said to be homogeneous if the components of any element of $B$ are all in $B$.

If $A$ is a ring, we say it is a $G$-graded ring if $A_gA_h \subseteq A_{gh}$ for all $g, h \in G$ and $1_A \in A_e$. If $A$ is a $G$-graded ring and $M = \bigoplus_{g \in G} M_g$ is a right $A$-module that is $G$-graded, we say $M$ is a graded module if $M_gA_h \subseteq M_{gh}$ for all $g, h \in G$.

Since $G$ is a cancellation monoid, degrees and homogeneous components are fairly well-behaved. This is exhibited in the following lemma and corollary. Recall that if $X$ is a subset of $R$ or of an $R$-module $M$, then $\text{ann} X = \{r \in R \mid xr = 0 \text{ for all } x \in X\}$. We leave the proofs to the reader.

**Lemma 2.2.** Let $G$ be a cancellation monoid, $S$ a $G$-graded ring, and $M$ a graded right $S$-module.

1. If $m \in M$ is homogeneous and $s = s_1 + \cdots + s_k \in A$ is a canonical form, then $ms = ms_1 + \cdots + ms_k$ becomes a canonical form when we omit any $ms_i$ that are 0. In particular, $ms = 0$ if and only if each $ms_i = 0$.
2. If $s \in S$ is homogeneous and $m = m_1 + \cdots + m_k \in M$ is a canonical form, then $ms = m_1s + \cdots + m ks$ becomes a canonical form when we omit any $m_is$ that are 0. In particular, $ms = 0$ if and only if each $m_is = 0$.

**Corollary 2.3.** Let $G$ be a cancellation monoid, $S$ a $G$-graded ring, and $M$ a graded right $S$-module.

1. If $m \in M$ is homogeneous, then $\text{ann} m$ is a homogeneous right ideal of $S$.
2. $\text{ann} M$ is a homogeneous ideal of $S$.

We now prove a general result about submodules of graded modules over rings graded by a unique product monoid. This pivotal result is presumably rather old in the cases $G = \mathbb{N}$ and $G = \mathbb{Z}$. The idea and the proof often show up in results like our Corollary 2.6 below.
Lemma 2.4. Let $G$ be a unique product monoid, $S$ a $G$-graded ring, and $M$ a graded right $S$-module. Suppose $N$ is a nonzero submodule of $M$ and $a \in N$ has canonical form $a = a_1 + \cdots + a_k$ with $k$ minimal for a nonzero element of $N$. Then the following statements hold.

1. $\text{ann } a = \text{ann } a_i$ for $i = 1, \ldots, k$.
2. $\text{ann } a$ is homogeneous.
3. $\text{ann } aS$ is homogeneous.

Proof. (1) Let $s = s_1 + \cdots + s_\ell$ be a canonical form for a nonzero $s \in S$ satisfying $as = 0$. We will show by induction on $\ell$ that $a_is_j = 0$ for all $i, j$. First suppose $\ell = 1$, i.e., $s$ is homogenous. By Lemma 2.2(2), $a_iS$ is homogeneous.

Now suppose $\ell > 1$. By the unique product property, there exist subscripts $m, n$ such that the product $\partial a_m \partial s_n$ is unique. Since $as = 0$, the product $a_ms_n$ must equal 0. Thus $a_ms_n$ has smaller support than $a$. By the minimality of $k$, this forces $a_ns_n = 0$, which in turn implies $a_is_n = 0$ for all $i$. This also implies $a(s - s_n) = 0$, so by induction on $\ell$, we have that $a_is_n = 0$ for all $i$ and all $j \neq n$. This finishes the proof of (1).

(2) This is an immediate consequence of (1) and Corollary 2.3(1).

(3) This follows from (2) and Corollary 2.3(2).

Recall that when $G$ is a group, we say a $G$-graded ring $S$ is strongly graded if $SgS = ShS$ for all $g, h \in G$.

Corollary 2.5. Let $G$ be a unique product monoid, $S$ a $G$-graded ring, $M$ a graded right $S$-module, and $N$ a nonzero submodule of $M$. Then $N$ contains an isomorphic copy of $bS$ for some (nonzero) homogeneous $b \in M$. If $G$ is a group and $S$ is strongly graded, then we can take $b$ to have degree $e$.

Proof. Let $a \in N$ be as in Lemma 2.4 with canonical form $a = a_1 + \cdots + a_k$. Then $a_1 \in M$ and $aS \subseteq N$ is isomorphic to the cyclic module $S/\text{ann } a = S/\text{ann } a_1 \cong a_1S$. Thus we may take $b = a_1$.

If $S$ is strongly graded and $a$ has a nonzero component in degree $g \in G$, then $aS_{g-1}$ contains an element $a'$ with nonzero component $b$ in degree $e$. This $a'$ has support no larger than that of $a$ and is still in $N$. By Lemma 2.4, we see that $\text{ann } a' = \text{ann } b$. Thus $bS \cong a'S \subseteq N$.

The following corollary of Lemma 2.4 tells us that the associated prime ideals of a graded module are all homogeneous and can be found by considering only graded submodules. This result for $G = \mathbb{N}$ (and $R$ commutative) is Proposition 3.12 in Eisenbud [3]. Exercise 3.5 in [3] asks the reader to extend the result to gradings by ordered abelian monoids. The corollary is also a key result in the study of associated primes of induced modules for $R[x; \sigma]$ in Nordstrom [9, Proposition 3.1].
Corollary 2.6. Let $G$ be a unique product monoid, $S$ a $G$-graded ring, $M$ a graded right $S$-module, and $N$ a prime submodule of $M$. Then $\text{ann} N$ is a homogenous prime ideal and is equal to $\text{ann} N'$ for some prime, graded submodule $N'$ of $M$.

Proof. This is immediate from Lemma 2.4 and Corollary 2.5. ■

The next result is well-known in the cases $G = \mathbb{N}$ and $G = \mathbb{Z}$. See for example, Năstăsescu-van Oystaeyen [10, Prop. II.1.4].

Lemma 2.7. Let $G$ be a unique product monoid and let $S$ be a $G$-graded ring. Let $I$ be a homogenous ideal of $S$ and let $M$ be a graded $S$-module.

1. $I$ is prime if and only if for any homogeneous $a,b \in S$, if $aSb \subseteq I$ for all $g \in G$, then $a \in I$ or $b \in I$.

2. $M$ is prime if and only if for any homogeneous $m \in M$, $b \in S$, if $mSb = 0$ for all $g \in G$, then $m = 0$ or $Mb = 0$.

Proof. (1) This follows from (2) if we set $M = S/I$.

(2) If $M$ is prime and $mSb = 0$ for all $g \in G$, then $mSb = 0$. Thus $m = 0$ or $b \in \text{ann} M$.

This proves the “only if” direction.

Suppose now that $M$ satisfies the “if” hypothesis and suppose $mSb = 0$ for nonzero $m \in M$, $b \in S$. Set $N = \{ m' \in M \mid m'Sb = 0 \}$; this $N$ is a nonzero submodule of $M$.

By Corollary 2.5 there is a homogeneous element $x \in M$ such that $xS$ is isomorphic to a submodule of $N$. In particular, $xSb = 0$.

Let $b = b_1 + \cdots + b_\ell$ be a canonical form. Since $\text{ann} xS$ is homogenous and $b \in \text{ann} xS$, we have $xSb_j = 0$ for each $j$. By hypothesis, this implies $Mb_j = 0$ for each $j$, and so $Mb = 0$. ■

3. The strongly graded case

In this section, we assume $G$ is a unique product group, $S$ is a strongly $G$-graded ring, and $R = S_e$. We give alternative characterizations of the notion of $S$-prime in this case. For example, we show $IS$ is a prime ideal of $S$ for an $S$-stable ideal $I$ of $R$ if and only if whenever $J,K$ are $S$-stable ideals of $R$ with $JK \subseteq I$, we have $J \subseteq I$ or $K \subseteq I$. This result — in the case $I = 0$, but this limitation is easily removed — was proved for any torsionfree group $G$ by Passman [11, Corollary 4.6] or [12, Corollary 8.5], using more complicated methods.

We also show that the associated primes of a graded $S$-module $N$ are precisely the ideals $IS$ where $I$ is the largest $S$-stable ideal contained in the annihilator of some $S$-prime submodule of the $R$-module $N_e$ of degree $e$ elements of $N$.

If $S$ is strongly $G$-graded and $N$ is a graded $S$-module, then the multiplication map from $N_g \otimes_R S_h$ to $N_{gh}$ is an $R$-module isomorphism for all $g, h \in G$, and the multiplication map from $N_e \otimes_R S$ to $N$ is a graded $S$-module isomorphism. (See Dade [24, Theorem 2.8] and Năstăsescu-van Oystaeyen [10, Theorem 1.3.4 & Proposition 1.3.6].) In particular, $N_gS_h = N_{gh}$ for all $g, h \in G$. If $M$ is an $R$-module, this isomorphism allows us to identify $M \otimes_R S_g$ with the formal product $MS_g$.

If $I \triangleleft R$, $S$ is strongly $G$-graded, and $g \in G$, we define $I^g = S_gIS_g^{-1}$. This is again an ideal of $R$, and it is easy to see that $(I^h)^g = I^{gh}$. Thus the map taking $g$ to $I \mapsto I^g$ is a homomorphism from $G$ to the automorphism group of the lattice of ideals of $R$. 
Some authors define $I^g = S_g^{-1}IS_g$ so that $(I^g)^h = I^{gh}$, but our definition is consistent with the use of the notation in the rest of this paper, where we are required to define $I^g$ without the existence of $g^{-1}$.

We say an ideal $I$ of $R$ is $G$-invariant if $I^g = I$ for all $g \in G$. Since $G$ is a group, it is easy to see that this is equivalent to the assumption that $I^g \subseteq I$ for all $g \in G$.

The next result relates $S$-stability and $G$-invariance, and it shows that in the strongly graded case, we can drop the adjective “right”.

**Lemma 3.1.** Let $G$ be a group and suppose $S$ is a strongly $G$-graded ring with $R = S_e$. If $I$ is an ideal of $R$, then the following statements are equivalent.

1. $I$ is right $S$-stable.
2. $IS = SI$.
3. $I$ is $G$-invariant.

**Proof.** (1) $\iff$ (3) Clearly $IS \triangleleft S$ if and only if $SI \subseteq IS$. This containment holds if and only if $S_gI \subseteq IS_g$ for all $g \in G$, which is the case if and only if $I^g \subseteq I$ for all $g \in G$. By our remark before the lemma, this is equivalent to the statement that $I$ is $G$-invariant.

(3) $\iff$ (2) This follows from (3) $\Rightarrow$ (1) and the “left” version of (3) $\iff$ (1).

If $I \triangleleft R$, we define $(I : G) = \cap_{g \in G}I^g$. It is easy to see that $(I : G)$ is the largest $G$-invariant ideal of $R$ contained in $I$.

We next give explicit conditions for $S$-primeness. We begin with ideals. The next result follows from Proposition 3.2 below, by way of Lemma 2.1. This result is in fact true for any torsionfree group $G$; this follows from Corollary 4.6 in Passman [11] or Corollary 8.5 in Passman [12] after passing to $R/I \subseteq S/IS$.

**Proposition 3.2.** Let $G$ be a unique product group and suppose $S$ is a strongly $G$-graded ring with $R = S_e$. Let $I$ be a $G$-invariant ideal of $R$. Then the following are equivalent.

1. $I$ is $S$-prime.
2. If $a, b \in R$ and $S_gaS_{(h^{-1})b} \subseteq I$ for all $g, h \in G$, then $a \in I$ or $b \in I$.
3. If $A, B$ are $G$-invariant ideals of $R$ and $AB \subseteq I$, then $A \subseteq I$ or $B \subseteq I$.

We now turn to $S$-modules.

**Lemma 3.3.** Let $G$ be a unique product group and suppose $S$ is a strongly $G$-graded ring with $R = S_e$. Let $M$ be an $R$-module with annihilator $J$.

1. $\text{ann}_S(M \otimes RS) = (J : G)S$.
2. If $M$ is $S$-prime, then $(J : G)$ is $S$-prime.

**Proof.** (1) Let $I = (J : G)$. Then $(M \otimes S_g)I \subseteq M \otimes S_gS_{(h^{-1})}JS_g = MJ \otimes S_g = 0$, so $(M \otimes RS)IS = 0$. This shows $IS \subseteq \text{ann}(M \otimes RS)$.

For the reverse inclusion, suppose $(M \otimes RS)s = 0$ for some $s \in S_g$. Then $M \otimes S_{h^{-1}}sS_{g^{-1}}S_h = 0$, whence $S_{h^{-1}}sS_{g^{-1}}S_h \subseteq J$. Thus $sS_{g^{-1}} \subseteq S_hJS_{h^{-1}}$ for all $h \in G$, and so $s \in IS_g$. This shows $\text{ann}(M \otimes RS) \subseteq IS$.

(2) This follows immediately from (1).

**Proposition 3.4.** Let $G$ be a unique product group and suppose $S$ is a strongly $G$-graded ring with $R = S_e$. Let $M$ be a nonzero $R$-module. Then the following conditions are equivalent.
(1) $M$ is $S$-prime.
(2) If $m \in M$, $b \in R$, and $m(S_g b S_{g^{-1}}) = 0$ for all $g \in G$, then $m = 0$ or $Mb = 0$.
(3) If $m \in M$, $B$ is a $G$-invariant ideal, and $mB = 0$, then $m = 0$ or $MB = 0$.
(4) For every nonzero submodule $N$ of $M$, we have $(\text{ann } N : G) = (\text{ann } M : G)$.

**Proof.**

(1) $\implies$ (4) Suppose $M$ is $S$-prime and $N$ is a nonzero submodule of $M$. Then $(\text{ann } N : G)S = \text{ann}(N \otimes_R S) = \text{ann}(M \otimes_R S) = (\text{ann } M : G)S$, so $(\text{ann } N : G) = (\text{ann } M : G)$.

(4) $\implies$ (3) Suppose (4) holds, let $m \in M$ be nonzero, and suppose $mB = 0$ for a $G$-invariant ideal $B$. Then $B \subseteq (\text{ann } mR : G) = (\text{ann } M : G)$. This implies $MB = 0$.

(3) $\implies$ (2) Suppose $m \in M$, $b \in R$, and $m(S_g b S_{g^{-1}}) = 0$ for all $g \in G$. Set $B = \sum_{g \in G} S_g b S_{g^{-1}}$. Then $B$ is a $G$-invariant ideal of $R$ and $mB = 0$. Thus either $m = 0$ or $MB = 0$, and the latter equality implies $Mb = 0$.

(2) $\implies$ (1) Suppose (2) holds and let $x \in M \otimes_R S$ be (nonzero) homogeneous of degree $g$. By Lemma [2.7] we need to show that if $xSs = 0$ for a homogeneous $s \in S$, then $(M \otimes_R S)s = 0$. Let $h$ be the degree of $s$. Then $0 = xSsS_{(gih)^{-1}} = xS_{g^{-1}} S_{g^{-1}} S_{h^{-1}} S_{(gih)^{-1}}$ for all $i \in G$. If $xS_{g^{-1}} = 0$, then $x = xR = xS_{g^{-1}} S_g = 0$. This is impossible, so there is a nonzero $m \in M$ with $m \otimes 1 \in xS_{g^{-1}}$. We can make $gi$ arbitrary, whence $m(S_j S_{g^{-1}} S_{j^{-1}}) = 0$ for all $j \in G$. It follows from (2) that $MSS_{h^{-1}} = 0$. Since elements of $S_g S_{g^{-1}}$ are also homogeneous, the same argument implies $MS_g S_{(h g)^{-1}} = 0$ for all $g \in G$. Thus $M \otimes S_g s = 0$ for all $g$, whence $(M \otimes_R S)s = 0$.

We can use this result to describe the set of associated primes of induced modules.

**Proposition 3.5.** Let $G$ be a unique product group and suppose $S$ is a strongly $G$-graded ring with $R = S_e$. Let $M$ be an $R$-module. Then

$$\text{Ass}(M \otimes_R S) = \{ (J : G)S \mid J \text{ is the annihilator of an } S \text{-prime submodule of } M \}.$$ 

If $R$ satisfies the a.c.c. on ideals, then $\text{Ass}(M \otimes_R S) = \{ (J : G)S \mid J \in \text{Ass } M \}$. 

**Proof.** Let $M$ be an $R$-module and $N$ be a prime submodule of $M \otimes_R S$. Since $S$ is strongly graded, Corollary [2.5] implies there is a nonzero $a \in M$ such that $(a \otimes 1)S = aR \otimes_R S$ is isomorphic to a submodule of $N$. Set $\text{ann}_R aR = J$; since $N$ is prime, Lemma [3.3] tells us $\text{ann } N = \text{ann}(a \otimes 1)S = (J : G)S$.

If $R$ satisfies the a.c.c. on ideals, the module $aR$ contains a nonzero submodule $K$ with maximal annihilator. In particular, $K$ is prime and $\text{ann } K \in \text{Ass } M$. By Proposition [3.4] $(J : G) = (\text{ann } K : G)$.

Since all graded modules are induced, we could re-write Proposition [3.5] as follows. If $N$ is a graded $S$-module, then $\text{Ass } N = \{ (J : G)S \mid J \text{ is the annihilator of an } S \text{-prime submodule of } N_e \}$.

**4. The crossed product case**

In this section, we assume $S = R \ast G$ is a crossed product. We will give conditions for primeness of $IS$ and $M \otimes_R S$, and we will show that prime submodules of $M \otimes_R S$ contain submodules of the form $L \otimes_R S$, but in the case $S = R \ast G$, the subset $L$ is not necessarily a submodule of $M$. Instead, it might be a twist of a submodule over a subring $R^g$ of $R$. If $\sigma$ is pointwise surjective, we can take $L$ to be an $S$-prime submodule of $M$ in the presence of an extra reversibility hypothesis.
The crossed product case is the most complicated case considered in this paper, and there are many subsidiary hypotheses one can make to get better results. Among these assumptions are commutativity of $R$, surjectivity of the maps $\sigma(g)$, the a.c.c. on ideals of $R$, and a reversibility condition on $G$ and $\sigma$ defined below (which holds, e.g., if $G$ or the image $\sigma(G)$ is commutative). In each of the three parts of the chapter (primeness of ideals, primeness of modules, associated primes), we prove a general statement with no extra hypotheses and then we prove statements in the presence of some combination of extra hypotheses. In Section 4 we give examples showing that (some) of our results fail without these extra hypotheses.

For example, Proposition 4.9 gives necessary and sufficient conditions for the induced module $M \star G = M \otimes_R S$ to be a prime $S$-module. One condition is an elementwise condition and another is an idealwise condition. In Corollary 4.10 we show that primeness implies other conditions that are then shown in Corollary 4.11 to be equivalent in the presence of reversibility when $R$ is commutative or $\sigma$ is pointwise surjective. In Corollary 4.12 essentially these same conditions are shown to be equivalent when $R$ satisfies the a.c.c. on ideals and $\sigma$ is pointwise surjective.

A crossed product $S = R \star G$ is a ring that is determined as follows by a 4-tuple $(R, G, \sigma, c)$, where $R$ is a ring, $G$ is a monoid with identity $e$, and $\sigma: G \to \text{End}_R(R)$ and $c: G \times G \to GL_1(R)$ are functions. We denote $\sigma(g)(r)$ by $r^g$.

1. The ring $S = \oplus_{g \in G} R\overline{g}G$ is a free left $R$-module with basis $\{\overline{g} \mid g \in G\}$;
2. Multiplication is determined by the relations $\overline{g}r = r^g\overline{g}$ and $\overline{g}h = c(g, h)\overline{gh}$;
3. The following conditions are satisfied for all $r \in R$, $g, h, i \in G$:
   a. $r^e = r$;
   b. $c(g, e) = c(e, g) = 1$;
   c. $(r^h)^g c(g, h) = c(g, h) r^{gh}$;
   d. $c(g, h) c(gh, i) = c(h, i)^g c(g, hi)$.

Conditions (c) and (d) guarantee that the multiplication in $S$ is associative and conditions (a) and (b) guarantee that $1_{\overline{e}}$ is the multiplicative identity of $S$. The condition that the values $c(g, h)$ are units is not required for $S$ to be a ring with identity, but this condition is customary and is necessary for our later results to hold. See Passman [12] pp. 1–3 for more details on crossed products, but note that Passman writes coefficients on the right of basis elements, that is, he has $S = \oplus_{g \in G} \overline{g}R$. Thus his formulas are generally the “reverse” of ours.

If $M$ is a right $R$-module, the induced $S$-module $M \otimes_R S$ can be identified with $\oplus_{g \in G} M\overline{g}$ with multiplication $m\overline{g} \cdot r\overline{h} = mr^g c(g, h)\overline{gh}$. We will denote this $S$-module by $M \star G$.

Condition (c) above can be stated as $\sigma(g) \circ \sigma(h) = \tau_{g, h} \circ \sigma(gh)$ for all $g, h \in G$, where $\tau_{g, h}$ is the inner automorphism defined as conjugation by $c(g, h)$.

For any $X \subseteq R$, let us define $X^g = \{x^g \mid x \in X\}$. We say $\sigma$ is pointwise surjective if $\sigma(g)$ is surjective for all $g \in G$. Note that $\sigma$ is pointwise surjective if and only if $I^g \triangleleft R$ whenever $I \triangleleft R$. Since an ideal is invariant under inner automorphisms (and inner automorphisms are trivial when $R$ is commutative), we get the simpler statement $(I^h)^g = I^{gh}$ for ideals $I$ when either $\sigma$ is pointwise surjective or $R$ is commutative.

For $g, h \in G$, define $g \equiv h$ if there is an inner automorphism $\tau$ such that $\sigma(g) = \tau \circ \sigma(h)$. The relation $\equiv$ is a congruence, that is, it is an equivalence relation and whenever $g \equiv h$,
If either $R$ is commutative or $\sigma$ is pointwise surjective, we can eliminate the conjugation when operating on ideals and thus if $(G, \sigma)$ is reversible up to conjugation, then for all $g, h \in G$ and all $I \triangleleft R$, there exist $h_1, h_2 \in G$ such that $I^{gh} = I^{h_1g}$ and $I^{hg} = I^{gh_2}$. If $\sigma(h)$ is always the identity, e.g., in the monoid ring case, then $(G, \sigma)$ is reversible up to conjugation. The reversibility condition is also satisfied if $gG = Gg$ for all $g$, which occurs if $G$ is a group or $G$ is commutative. Another example of a monoid $G$ with $gG = Gg$ for all $g \in G$ is the monoid generated by $x, y, z, z^{-1}$ subject to the relations that $z$ is central and $yx = xyz$. This monoid embeds in a torsionfree nilpotent group, so it is a unique product monoid. However, if we leave out $z^{-1}$ in the definition of $G$, we get a unique product monoid that does not satisfy $Gg = gG$.

The terminology used to describe whether a map takes an ideal into itself or does something stronger is rather variable in the literature. We will adopt the following terminology. We say an ideal $I$ of $R$ is $G$-stable if $I^g \subseteq I$ for all $g \in G$, and we say $I$ is $G$-invariant if $\sigma(g)^{-1}(I) = I$ for all $g \in G$.

**Lemma 4.1.** Let $I$ be an ideal of $R$ and $S = R \ast G$. Then $I$ is right $S$-stable if and only if $I$ is $G$-stable.

**Proof.** If we look at terms of degree $g$, we see $SI \subseteq IS$ if and only if $I^g \overline{g} = \overline{g}I \subseteq I\overline{g}$ for all $g \in G$. This last containment holds if and only if $I^g \subseteq I$ for all $g \in G$. $\blacksquare$

If $X \subseteq R$, we define

$$(X : G) = \cap_{g \in G} \sigma(g)^{-1}(X) = \{ r \in R \mid r^g \in X \text{ for all } g \in G \}. $$

If $X$ is an ideal of $R$, so is $(X : G)$. It is not hard to see that $(X : G)$ is the largest $G$-stable subset of $R$ contained in $X$.

We now state two results that tell us certain $G$-stable ideals are necessarily $G$-invariant. The first result is a standard application of the ascending chain condition.

**Lemma 4.2.** Suppose $S = R \ast G$, $\sigma$ is pointwise surjective, and $R$ satisfies the a.c.c. on ideals. If $I$ is a $G$-stable ideal of $R$, then $I$ is $G$-invariant and $I^g = I$ for all $g \in G$. $\blacksquare$

If $I$ is $S$-prime, Lemma 4.1 implies $I$ is $G$-stable. The next lemma shows that often an $S$-prime ideal is necessarily $G$-invariant. Examples 6.5 and 6.6 show that some condition along the lines of the reversibility condition must be imposed to make this true.

**Lemma 4.3.** Suppose $S = R \ast G$ and suppose that $(G, \sigma)$ is reversible up to conjugation.

1. If $I$ is a right $S$-prime ideal of $R$, then $I$ is $G$-invariant.
2. If $M$ is an $S$-prime $R$-module, then $(\text{ann } M : G)$ is $G$-invariant.
Proof. (2) Let $I = (\text{ann } M : G)$ and suppose $r^g \in I$ for some $g \in G, r \in R$. Set $A = \sum_{h,i \in G} R^{hg_i}$. It is easy to see that $A \triangleleft S$, and $A \not\subseteq IS$ since $1 \notin I$.

By the reversibility condition, there exists $i' \in G$ such that $gi \equiv i'g$. Thus there exists a unit $u$ such that $hgi r = r^{hgi}hgi = u(r^g)^{h'u^{-1}gh_i}$. The coefficient $u(r^g)^{h'u^{-1} \in I}$ because $I$ is $G$-stable. It follows that $Ar \subseteq IS$. Since $IS$ is prime, this implies $r \in I$.

(1) This follows from (2). \qed

The next lemma that shows another way in which the reversibility condition can be useful.

Lemma 4.4. Suppose $S = R \ast G$, $I$ is a $G$-stable ideal of $R$, $g \in G$, and $(G, \sigma)$ is reversible up to conjugation. Then the ideals $RI^gR$ and $\sigma(g)^{-1}(I)$ are $G$-stable.

Proof. Set $A = RI^gR$ and $B = \sigma(g)^{-1}(I)$. Suppose $h \in G$ and let $h' \in G$ satisfy $hg \equiv gh'$. Then for some unit $u \in R$, we have $A^h = R^h(I^g)^hR^h = R^h u(I^{h'})^{g^{-1}h^{-1}R^h} \subseteq RI^gR = A$, where the containment holds because $I$ is $G$-stable.

Suppose that $b \in B$, so $b^g \in I$, and $h \in G$. Let $h' \in G$ satisfy $gh \equiv h'g$. Then for some unit $u \in R$, we have $(b^g)^h = u(b^g)^{h'u^{-1}} \subseteq uI^{h'}u^{-1} \subseteq I$, where the containment holds because $I$ is $G$-stable. Thus $b^h \in B$. \qed

We now give explicit conditions for $S$-primeness.

Proposition 4.5. Let $G$ be a unique product monoid, $I$ an ideal of $R$, and $S = R \ast G$. If $I$ is $G$-stable, then the following conditions are equivalent.

(1) $I$ is right $S$-prime.
(2) If $a, b \in R$, $g \in G$, and $aR^g(b^g) \subseteq I$ for all $h \in G$, then $a \in I$ or $b^h \in I$ for all $h \in G$.
(3) If $A$ is a left ideal of $R$, $B$ is a $G$-stable ideal of $R$, and $AB^g \subseteq I$ for some $g \in G$, then $A \subseteq I$ or $B \subseteq I$.

If $R$ is commutative or $\sigma$ is pointwise surjective, we can replace “left ideal” by “ideal” in (3).

Proof. This follows from Proposition 4.9 below, by way of Lemma 2.1 \qed

The first corollary below gives a more appealing condition for an ideal to be $S$-prime when $\sigma$ is pointwise surjective or $R$ is commutative and the reversibility condition holds. Example 6.2 shows that the corollary does not hold for an arbitrary crossed product, even when $I$ is $G$-invariant and the reversibility condition holds. Example 6.3 shows that the corollary fails without the assumption that $I$ is $G$-invariant. Example 6.7 shows that the corollary fails in the commutative case without the reversibility condition. This corollary is unique among our results in that we do not have a module analog when $\sigma$ is pointwise surjective.

Corollary 4.6. Let $G$ be a unique product monoid, $I$ an ideal of $R$, and $S = R \ast G$. Suppose $I$ is $G$-invariant and suppose that either (a) $R$ is commutative and $(G, \sigma)$ is reversible up to conjugation or (b) $\sigma$ is pointwise surjective. Then $I$ is $S$-prime if and only if $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$ whenever $A, B$ are ideals of $R$ with $B$ $G$-stable.

Proof. Under the assumption of the reversibility condition, this result follows from Corollary 4.11 below, by way of Lemma 2.1.
Let us drop the reversibility condition assumption and suppose that \( \sigma \) is pointwise surjective and that whenever \( A, B \) are ideals of \( R \) with \( B \) \( G \)-stable and \( AB \subseteq I \), we have \( A \subseteq I \) or \( B \subseteq I \). Suppose that \( A \) is an ideal of \( R \), \( B \) is a \( G \)-stable ideal of \( R \), and \( AB^g \subseteq I \) for some \( g \in G \). Since \( \sigma(g) \) is surjective, if we set \( \hat{A} = \sigma(g)^{-1}(A) \), then \( \hat{A}^g = A \). Certainly, \( (\hat{A}B)^g \subseteq I \); since \( I \) is \( G \)-invariant, this implies \( \hat{A}B \subseteq I \). By our hypothesis, this implies \( \hat{A} \subseteq I \) or \( B \subseteq I \). If the former containment holds, then \( A = \hat{A}^g \subseteq I^g \subseteq I \). Thus by Proposition 4.5, \( I \) is \( S \)-prime.

The next corollary gives situations where the reversibility condition hypothesis is not needed.

**Corollary 4.7.** Let \( G \) be a unique product monoid, \( I \) a \( G \)-invariant ideal of \( R \), and \( S = R^*G \). Suppose \( R \) satisfies the a.c.c. on ideals and \( \sigma \) is pointwise surjective. Then \( I \) is \( S \)-prime if and only if whenever \( J, K \) are \( G \)-invariant ideals of \( R \) with \( JK \subseteq I \), we have \( J \subseteq I \) or \( K \subseteq I \).

**Proof.** This result follows from Corollary 4.12 below, by way of Lemma 2.1.

We now turn to modules.

**Lemma 4.8.** Let \( G \) be a unique product monoid and \( S = R^*G \). Let \( M \) be an \( R \)-module with annihilator \( J \).

1. \( \text{ann}_S M * G = (J : G) * G \).
2. If \( M \) is \( S \)-prime, then \( (J : G) \) is right \( S \)-prime.

**Proof.** (1) Let \( I = (J : G) \). Then \( (M\overline{g})I \subseteq MI^g \overline{g} \subseteq MJ\overline{g} = 0 \), so \( (M * G)(J : G) = 0 \). This shows \( (J : G) \subseteq \text{ann} M * G \).

For the reverse inclusion, suppose \( (\text{ann} M : G)r\overline{h} = 0 \) for some \( r \in R, h \in G \). Then \( Mr^g(c, g, h)\overline{gh} = (M\overline{g})r\overline{h} = 0 \), whence \( Mr^g = 0 \) and \( r^g \in J \) for all \( g \in G \). This shows \( \text{ann} M * G \subseteq (J : G) \).

(2) This follows immediately from (1).

We next give conditions for \( S \)-primeness of a module when \( S = R^*G \). We begin with two results in the general case and then proceed to results with extra hypotheses.

**Proposition 4.9.** Let \( G \) be a unique product monoid and \( S = R^*G \). Let \( M \) be a nonzero \( R \)-module. The following conditions are equivalent.

1. \( M \) is \( S \)-prime.
2. If \( m \in M, b \in R, g \in G \), and \( mR^g(b^g)^g = 0 \) for all \( h \in G \), then either \( m = 0 \) or \( Mb^h = 0 \) for all \( h \in G \).
3. If \( m \in M, B \) is a \( G \)-stable ideal of \( R \), and \( mB^g = 0 \) for some \( g \in G \), then \( m = 0 \) or \( MB = 0 \).

If \( R \) is commutative or \( \sigma \) is pointwise surjective, we can add the following equivalent condition:

4. If \( N \) is a submodule of \( M \), \( B \) is a \( G \)-stable ideal of \( R \), and \( NB^g = 0 \) for some \( g \in G \), then \( N = 0 \) or \( MB = 0 \).
Proof. (1) \implies (3) Suppose \( M \) is \( S \)-prime and \( mB^g = 0 \) for some nonzero \( m \in M \), some \( g \in G \), and some \( G \)-stable ideal \( B \). Then \( BS \) is an ideal of \( S \) and \( (m \overline{g} S)BS = m \overline{g} BS = mB^g \overline{g} S = 0 \). Since \( M \ast G \) is prime, \( \text{ann} m \overline{g} S = \text{ann} M \ast G \). Thus \( MB = 0 \).

(3) \implies (2) Suppose \( m \in M \), \( b \in R \), \( g \in G \), and \( mR^g(h^g) = 0 \) for all \( h \in G \). Set \( B = \sum_{h \in G} Rb^h R \). Then \( B \) is a \( G \)-stable ideal of \( R \) and \( mB^g = 0 \). Thus either \( m = 0 \) or \( MB = 0 \), and the latter equality implies \( MB^h = 0 \) for all \( h \in G \).

(2) \implies (1) Suppose (2) holds and let \( m \overline{g} \in M \ast G \) be a (nonzero) homogeneous element. By Lemma 2.7, we need to show that if \( m \overline{g} Sb^\overline{g} = 0 \) for \( b \in R \), \( i \in G \), then \( M \ast Gb^\overline{g} = 0 \). Suppose \( m \overline{g} Sb^\overline{g} = 0 \); this implies that for any \( r \in R \), \( h \in G \) we have

\[
0 = m \overline{g} r h b^\overline{g} = mr^g \overline{g} h b^h h^\overline{g} = mr^g(b^h)^c(h, i)^c(g, hi)g hi.
\]

Thus \( mR^g(h^g)^c = 0 \) for all \( h \in G \) since \( c \) always yields units. Thus by (2), we can conclude that \( MB^h = 0 \) for all \( h \in G \). This implies that \( (M \ast G)b = 0 \), and this proves \( M \ast G \) is prime.

(3) \implies (4) This is clearly true without any extra hypotheses.

Assume \( R \) is commutative or \( \sigma \) is pointwise surjective, and suppose (4) holds. We prove (3). Suppose \( m \in M \), \( B \) is a \( G \)-stable ideal of \( R \), and \( mB^g = 0 \) for some \( g \in G \). If \( R \) is commutative, \( mRB^g = mB^g R = 0 \), while if \( \sigma \) is pointwise surjective, \( B^g \trianglelefteq I \), so \( mRB^g = mB^g = 0 \). Thus if we set \( N = mR \), we have \( NB^g = 0 \), and so we may apply condition (4) to obtain \( m = 0 \) or \( MB = 0 \).

Corollary 4.10. Let \( G \) be a unique product monoid and \( S = R \ast G \). Let \( M \) be a nonzero \( R \)-module. Then (1) \implies (2) \iff (3) \iff (4) for the following conditions.

1. \( M \) is \( S \)-prime.
2. If \( m \in M \), \( b \in R \), and \( mRB^h = 0 \) for all \( h \in G \), then either \( m = 0 \) or \( Mb = 0 \).
3. If \( m \in M \) and \( B \) is a \( G \)-stable ideal of \( R \) with \( mB = 0 \), then \( m = 0 \) or \( MB = 0 \).
4. \( \text{ann} N : G = (\text{ann} M : G) \) for every nonzero submodule \( N \) of \( M \).

Proof. (1) \implies (3) This follows from (1) \implies (3) in Proposition 4.9 by setting \( g = e \).

(3) \implies (4) Let \( N \) be a nonzero submodule of \( M \), let \( n \in N \) be nonzero, and set \( B = (\text{ann} N : G) \). Then \( nB = 0 \) and \( B \) is \( G \)-stable, and so by (2), \( MB = 0 \). Thus \( (\text{ann} N : G) \subseteq (\text{ann} M : G) \subseteq (\text{ann} N : G) \); this proves (3).

(4) \implies (2) Let \( m \in M \setminus \{0\} \) and \( b \in R \) satisfy \( mRB^h = 0 \) for all \( h \in G \). Set \( B = \sum_{h \in G} Rb^h R \), so that \( B \) is a \( G \)-stable ideal of \( R \). Clearly \( mB = 0 \). Thus \( B \subseteq (\text{ann} mR : G) = (\text{ann} M : G) \), whence \( MB = 0 \) and so \( Mb = 0 \).

(2) \implies (3) Let \( m \in M \) and \( B \) be a \( G \)-stable ideal of \( R \) with \( mB = 0 \). If \( b \in B \), then \( mRB^h = 0 \) for all \( h \in G \). By (2), \( Mb = 0 \). This holds for any \( b \in B \), whence \( MB = 0 \).

The next corollary states that the four conditions in Corollary 4.10 are equivalent when several extra conditions hold. Example 6.2 shows that the corollary does not hold for an arbitrary crossed product, even when \( \text{ann} M : G \) is \( G \)-invariant and the reversibility condition holds. Example 6.3 shows that the hypothesis \( \text{\"(ann} M : G \) is \( G \)-invariant\} \) is necessary in parts (2) and (3) of the corollary. Example 6.7 shows that the corollary fails without the reversibility condition, at least in the commutative case.
Corollary 4.11. Let $G$ be a unique product monoid and $S = R \ast G$. Let $M$ be a nonzero $R$-module. Assume that $(\text{ann} \, M : G)$ is $G$-invariant and $(G, \sigma)$ is reversible up to conjugation. If either $\sigma$ is pointwise surjective or $R$ is commutative, then the four conditions in Corollary 4.10 are equivalent.

Proof. By Corollary 4.10 it is enough to prove $(3) \implies (1)$. To do so, suppose $m \in M$ is nonzero, $B$ is a $G$-stable ideal, and $mB^g = 0$. Set $B' = R B^g R$. By Lemma 4.4, $B'$ is $G$-stable. If $\sigma(g)$ is surjective, then $B' = B^g$ and if $R$ is commutative, $B' = B^g R$, whence $mB' = 0$ in either case. Thus by (3), $M B' = 0$, and so $B' \subseteq \text{ann} \, M$. As $B'$ is $G$-stable, this implies $B^g \subseteq B' \subseteq (\text{ann} \, M : G)$. Since $(\text{ann} \, M : G)$ is $G$-invariant, this in turn implies $B \subseteq (\text{ann} \, M : G)$, whence $MB = 0$. This proves $M$ is $S$-prime by Proposition 4.9. 

We get the same result when $R$ satisfies the a.c.c. on ideals and $\sigma$ is pointwise surjective; reversibility is not required, nor must we assume anything about $(\text{ann} \, M : G)$.

Corollary 4.12. Let $G$ be a unique product monoid and $S = R \ast G$. Let $M$ be a nonzero $R$-module. Suppose that $R$ satisfies the a.c.c. on ideals and $\sigma$ is pointwise surjective. Then the following conditions are equivalent.

1. $M$ is $S$-prime.
2. If $m \in M$, $b \in R$, and $mR b^h = 0$ for all $h \in G$, then either $m = 0$ or $Mb = 0$.
3. If $m \in M$ and $B$ is a $G$-invariant ideal of $R$ with $mB = 0$, then $m = 0$ or $MB = 0$.
4. $(\text{ann} \, N : G) = (\text{ann} \, M : G)$ for every nonzero submodule $N$ of $M$.

Proof. That $(1) \implies (2) \iff (3) \iff (4)$ is the content of Corollary 4.10 (since $G$-stable and $G$-invariant are equivalent by Lemma 4.2).

$(3) \implies (1)$ Suppose $m \in M$ is nonzero, $B$ is a $G$-stable ideal, and $mB^g = 0$. By Lemma 4.2, we know $B^g = B$, and so by (3), we conclude $MB = 0$. Thus $M$ is $S$-prime by Proposition 4.9.

It is of some interest to know whether prime implies $S$-prime. The next corollary gives us a result in this direction. It will be used in several examples in Section 6.

Corollary 4.13. Let $G$ be a unique product monoid and $S = R \ast G$. Let $M$ be a prime $R$-module with annihilator $I$. Suppose that $R$ is commutative or $\sigma$ is pointwise surjective. Then the following conditions are equivalent.

1. $M$ is $S$-prime.
2. If $b \in R$, $g \in G$, and $(b^h)^g \in I$ for all $h \in G$, then $b \in I$.
3. If $B$ is a $G$-stable ideal of $R$ and $B^g \subseteq I$ for some $g \in G$, then $B \subseteq I$.

Proof. $(1) \implies (2)$ Suppose $(b^h)^g \in I$ for all $h \in G$. Then $R^g (b^h)^g \subseteq I$ for all $h$, whence $mR^g (b^h)^g = 0$ for all $h$, for any $m \in M$. We may choose $m \neq 0$ and apply Proposition 4.9 to obtain $M B^h = 0$ for all $h \in G$. In particular, $b \in \text{ann} \, M = I$.

$(2) \implies (3)$ Suppose $B$ is a $G$-stable ideal of $R$ and $B^g \subseteq I$ for some $g \in G$. For any $b \in B$ and $h \in G$, we have $(b^h)^g \in B^g \subseteq I$. By (2), this implies $b \in I$, so $B \subseteq I$.

$(3) \implies (1)$ We will prove condition (4) of Proposition 4.9 holds, whence $M$ is $S$-prime. Thus we suppose $N$ is a nonzero submodule of $M$, $B$ is a $G$-stable ideal of $R$, $g \in G$, and $NB^g = 0$. Since $M$ is prime, this implies $B^g \subseteq \text{ann} \, N = \text{ann} \, M = I$. Thus by (3), $B \subseteq I$, which in turn implies $MB = 0$. 

\[\square\]
We might hope to prove the analog of Proposition 3.5 for \( S = R \ast G \), namely that the associated prime ideals of \( M \ast G \) are of the form \((J : G) \ast G\) where \( J \) is the annihilator of an \( S \)-prime submodule of \( M \). Unfortunately, this statement is not true in general; see Example 6.2 or Example 6.1. The result becomes true when we replace “submodule of \( M \)” with “submodule of some twist \( M_{\sigma(g)}\)”. The result is true as stated if \( \sigma \) is pointwise surjective and either the reversibility condition holds or \( R \) satisfies the a.c.c. on ideals.

Recall that if \( M \) is an \( R \)-module and \( \phi : R \to R \) is a ring endomorphism, we can define a new \( R \)-module \( M_\phi \) as follows: \( M_\phi \) is the same additive group as \( M \), but the \( R \)-action is twisted according to the rule \( m \ast r = m\phi(r) \). It is easy to see that there is a bijection between \( \phi(R) \)-submodules of \( M \) (regarded as \( \phi(R) \)-modules) and \( R \)-submodules of \( M_\phi \), given by \( L \mapsto L_\phi \). Clearly we have \( \text{ann} L_\phi = \phi^{-1}(\text{ann} L) \). When \( \phi \) is not surjective, there may be \( \phi(R) \)-submodules \( L \) of \( M \) that are not \( R \)-submodules, that is, that are not closed under the action of \( R \). (See Example 6.2 for instance.)

If \( L \) is an \( R^g \)-submodule of \( M \), then \( L\overline{g} \) is an \( R \)-submodule of \( M \ast G \). The map \( a \mapsto a\overline{g} \) is an \( R \)-module isomorphism from \( L_{\sigma(g)} \) to \( L\overline{g} \).

We are now ready to describe the associated prime ideals of \( M \ast G \).

**Proposition 4.14.** Let \( G \) be a unique product monoid and \( S = R \ast G \). Let \( M \) be an \( R \)-module and \( N \) be a prime submodule of \( M \ast G \). Then there is an \( i \in G \) and an \( R^i \)-submodule \( L \) of \( M \) such that \( L_{\sigma(i)} \) is \( S \)-prime. Moreover, if we set \( J = \text{ann} L_{\sigma(i)} = \sigma(i)^{-1}(\text{ann} R \cdot L) \) and \( I = (J : G) \), then \( I \) is \( S \)-prime and \( \text{ann} N = IS \).

**Proof.** Let \( a \in N \setminus \{0\} \) be as in Lemma 2.4 that is, \( a = r_1\overline{g_1} + \cdots + r_k\overline{g_k} \) is a canonical form for \( a \) with \( k \) minimal among nonzero elements of \( N \). Thus \( \text{ann} a \) is homogeneous and \( \text{ann} a = \text{ann} r_1\overline{g_1} \), so \( aS \cong r_1\overline{g_1}S \). If we set \( L = r_1R\overline{g_1} \), then \( r_1\overline{g_1}S \cong L_{\sigma(g_1)} \ast G \).

Set \( J = \text{ann}_R(L_{\sigma(g_1)}) \). Then \( \text{ann} N = \text{ann} aS = \text{ann} L_{\sigma(g_1)} \ast G = (J : G)S \). \( \blacksquare \)

The next lemma shows that when the reversibility condition holds and \( \sigma \) is pointwise surjective, we do not need to consider the twists of \( M \).

**Lemma 4.15.** Let \( G \) be a unique product monoid and \( S = R \ast G \). Let \( M \) be an \( R \)-module and let \( g \in G \). Suppose that \((G, \sigma)\) is reversible up to conjugation and that \( \sigma \) is pointwise surjective. If \( L \) is an \( S \)-prime submodule of \( M_{\sigma(g)} \), then \( L \) is an \( S \)-prime \( R \)-submodule of \( M \).

**Proof.** We first note that since \( \sigma(g) \) is surjective, \( L \) is closed under multiplication by elements of \( R \) and so is an \( R \)-submodule of \( M \). Suppose that \( B \) is a \( G \)-stable ideal of \( R \), \( m \in L \), \( m \neq 0 \), and \( mB = 0 \). Let \( C = \sigma(g)^{-1}(B) \). By Lemma 4.4 \( C \) is \( G \)-stable. Moreover \( m \ast C = mC^g = mB = 0 \). Since \( L_{\sigma(g)} \) is \( S \)-prime, Corollary 4.11\( (2) \) implies \( L \ast C = LB = 0 \). \( \blacksquare \)

The next proposition sums up our results on associated primes in the crossed product case.

**Proposition 4.16.** Let \( G \) be a unique product monoid and \( S = R \ast G \). Let \( M \) be an \( R \)-module.

1. \( \text{Ass} M \ast G = \{ (\sigma(g)^{-1}(J) : G) \ast G \mid g \in G \text{ and } J \text{ is the annihilator of an } S \text{-prime submodule of } M_{\sigma(g)} \} \).
(2) If $\sigma$ is pointwise surjective and $(G, \sigma)$ is reversible up to conjugation, then $\text{Ass} M * G = \{ (J : G) * G \mid J$ is the annihilator of an $S$-prime submodule of $M \}$.

(3) If $R$ satisfies the a.c.c. on ideals and $\sigma$ is pointwise surjective, then $\text{Ass} M * G = \{ (P : G) * G \mid P \in \text{Ass} M \}$.

Proof. (1) This follows from Proposition 4.14.

(2) This follows from Proposition 4.14 and Lemma 4.15.

(3) First suppose $P \in \text{Ass} M$, so $P = \text{ann} L$ for some prime submodule $L$ of $M$. The ideal $(P : G)$ is $G$-stable and $(P : G) = (\text{ann} K : G)$ for all nonzero submodules $K$ of $L$. Thus $L$ is $S$-prime by Corollary 4.12, and so $(P : G) * G \in \text{Ass} M * G$.

Next, let $(J : G) * G \in \text{Ass} M * G$, where $J = \text{ann} N$ for an $S$-prime submodule $N$ of $M$. Let $L$ be a nonzero submodule of $N$ such that $P = \text{ann} L$ is as large as possible. It is well-known that in this case, $L$ is a prime module, so $P \in \text{Ass} M$. By Corollary 4.12, $(J : G) = (P : G)$. ■

Proposition 4.16(1) describes $\text{Ass} M * G$ in general, but it requires us to check a great many modules. Lemma 4.18 below shows that once we find a twist that is prime, any twist of that is automatically prime as well, so we don’t need to check twists of twists. First, we need another lemma.

Lemma 4.17. Let $G$ be a unique product monoid, $S = R * G$, $\phi$ be an endomorphism of $R$, and $\tau$ be an inner automorphism of $R$. Then the following conditions are equivalent.

(1) $M_{\phi}$ is $S$-prime.

(2) $M_{\tau \phi}$ is $S$-prime.

(3) $M_{\phi \tau}$ is $S$-prime.

Proof. Note that if $\psi$ is an endomorphism of $R$, then $M_{\psi \phi} = (M_{\psi})_{\phi}$. Since $\tau^{-1}$ is also inner, this observation implies that if we prove $(1) \implies (2)$ for arbitrary $\tau$, then $(2) \implies (1)$. Applying $(2)$ with $\phi = \text{id}$, we see that any $M_{\tau}$ is prime whenever $M$ is $S$-prime. Replacing $M$ by $M_{\phi}$ and using $M_{\phi \tau} = (M_{\phi})_{\tau}$, we see that $(1) \implies (3)$. We then get $(3) \implies (1)$ as we get $(2) \implies (1)$.

Thus we only need to prove $(1) \implies (2)$. Let $*$ denote the action in $M_{\phi}$, that is $m * r = m \phi(r)$ and let $**$ denote the action in $M_{\tau \phi}$, that is $m ** r = m \tau(\phi(r))$.

Suppose that $M_{\phi}$ is $S$-prime, $m$ is a nonzero element of $M$, $B$ is a $G$-stable ideal of $R$, and $g \in G$. Let $u$ be a unit of $R$ such that $\tau(r) = uru^{-1}$ and suppose that $m ** B^g = 0$. Then $0 = m \tau(\phi(B^g)) = mu \phi(B^g) u^{-1}$. Thus $0 = mu \phi(B^g) = mu *(B^g)$. Since $mu \neq 0$ and $M_{\phi}$ is $S$-prime, we conclude that $0 = M * B = M \phi(B) = Mu \phi(B)$. This implies $0 = Mu \phi(B) u^{-1} = M * B$. This proves $M_{\phi \tau}$ is $S$-prime. ■

Lemma 4.18. Let $G$ be a unique product monoid and $S = R * G$. Let $M$ be an $R$-module. If $M$ is $S$-prime, then $M_{\sigma(h)}$ is $S$-prime for all $h \in G$. More generally, if $M_{\sigma(g)}$ is $S$-prime, then $M_{\sigma(hg)}$ is $S$-prime for all $h \in G$.

Proof. Suppose $M$ is $S$-prime, $m \in M$ is nonzero, $g, h \in G$, and $B$ is a $G$-stable ideal of $R$ with $m * B^h = 0$. Then there is a unit $u \in R$ with $0 = m(B^h)^g = muB^ghu^{-1}$. Thus $(mu)B^h = 0$ and $mu \neq 0$, so by primeness, $MB = 0$. Since $B$ is $G$-stable, we have $M * B = MB^g \subseteq MB = 0$. This proves $M_{\sigma(g)}$ is $S$-prime.
Now suppose $M_{\sigma(g)}$ is $S$-prime and $h \in G$. Then $(M_{\sigma(g)})_{\sigma(h)} = M_{\sigma(g)\circ \sigma(h)} = (M_{\tau})_{\sigma(hg)}$, where $\tau$ is conjugation by $c(h, g)$. Since $(M_{\sigma(g)})_{\sigma(h)}$ is $S$-prime by the first part of the proof, we can conclude from Lemma 4.17 that $M_{\sigma(hg)}$ is $S$-prime.

5. The skew polynomial and skew laurent cases

In this section, $\sigma$ is a ring endomorphism of $R$ and $S = R[x; \sigma]$, or $\sigma$ is an automorphism and $S = R[x^{\pm 1}; \sigma]$. Skew polynomial rings form a special case of the crossed products considered in Section 4 with $G = \mathbb{N}$, $\sigma(g) = \sigma^g$, and $c(g, h)$ identically 1. Note also that $\sigma$ is pointwise surjective if and only if $\sigma$ is surjective. Skew laurent rings form a special case of the strongly graded rings considered in Section 3. In the skew laurent case $S_1 = Rx$, so $S_1IS_{-1} = RxIRx^{-1} = \{ \sigma(i) \mid i \in I \}$ for $I \subseteq R$. Thus our previous use of the notation $\sigma(I)$ does not conflict with our new use of $\sigma$. Likewise, $\sigma^{-1}(I)$ is the same set under either meaning of the notation.

We will switch to the more common terminology of “$\sigma$-stable” and “$\sigma$-invariant” in place of “$G$-stable” and “$G$-invariant” and the notation $(I : \sigma)$ in place of $(I : G)$. Thus we say a subset $X$ of $R$ is $\sigma$-stable if $\sigma(X) \subseteq X$, and we say $X$ is $\sigma$-invariant if $\sigma^{-1}(X) = X$.

When $S = R[x; \sigma]$, the induced module $M \otimes_R S$ can be identified with $M[x; \sigma] = \oplus_{n=0}^{\infty} Mx^n$. In case $S = R[x^{\pm 1}; \sigma]$, we identify $M \otimes_R S$ with $M[x^{\pm 1}; \sigma] = \oplus_{n=-\infty}^{\infty} Mx^n$. We write $M[x; \sigma]x^k$ for the $S$-module $\oplus_{n=k}^{\infty} Mx^n$, and we will write $I[x; \sigma]$ or $I[x^{\pm 1}; \sigma]$ in place of $IS$ for an ideal $I$ of $R$.

As one would expect, the results in this section correspond to the results in Sections 4 and 3, generally with some simplifications.

Lemma 5.1. Let $I \triangleleft R$ and let $\sigma$ be an endomorphism of $R$.

1. If $S = R[x; \sigma]$, then $I$ is right $S$-stable if and only if $I$ is $\sigma$-stable.
2. If $S = R[x; \sigma]$ and $I$ is right $S$-prime, then $I$ is $\sigma$-invariant.
3. If $\sigma$ is an automorphism and $S = R[x^{\pm 1}; \sigma]$, then $I$ is right $S$-stable if and only if $I$ is $\sigma$-invariant.

Proof. (1) & (2) These follow from Lemma 4.1.

(3) This follows from Lemma 3.1. 

If $X \subseteq R$, we define

$$(X : \sigma) = \cap_{n=0}^{\infty} \sigma^{-n}(X) = \{ r \in R \mid \sigma^n(r) \in X \text{ for all } n \in \mathbb{N} \} \text{ and } (X : \sigma^{\pm}) = \cap_{n=-\infty}^{\infty} \sigma^n(X).$$

If $X$ is an ideal of $R$, so is $(X : \sigma)$, and if $\sigma$ is surjective and $X$ is an ideal of $R$, so is $(X : \sigma^{\pm})$. Clearly $(X : \sigma)$ is the largest $\sigma$-stable subset of $R$ contained in $X$, and if $\sigma$ is an automorphism, then $(X : \sigma^{\pm})$ is the largest $\sigma$-invariant subset of $R$ contained in $X$.

We now give explicit conditions for $S$-primeness in the skew polynomial and skew laurent cases. As noted in the introduction, finding such conditions for skew polynomial rings was one of our motivations. In the case $S = R[x; \sigma]$, definitions of $\sigma$-prime ideal seem to have been given first in Goldie-Michler [4], assuming $\sigma$ is an automorphism and $R$ is right Noetherian, in Pearson-Stephenson [13], assuming $\sigma$ is an automorphism, and in Irving [5], assuming $R$ is commutative. None of these definitions guarantee that $I[x; \sigma]$ is prime in the general case.
The Goldie-Michler definition is the following: a \( \sigma \)-invariant ideal \( I \) is \( \sigma \)-prime if whenever \( J, K \) are \( \sigma \)-invariant ideals and \( JK \subseteq I \), either \( J \subseteq I \) or \( K \subseteq I \). (The definition of \( H \)-prime in Montgomery-Schneider \cite{MontgomerySchneider} is virtually the same, except that they use “stable” in place of “invariant”.) This is certainly the most aesthetically pleasing definition, and it guarantees that \( I[x^{\pm 1}; \sigma] \) is prime in the skew Laurent extension \( S = R[x^{\pm 1}; \sigma] \). Unfortunately, it does not guarantee that \( I[x; \sigma] \) is prime in \( S = R[x; \sigma] \), even if \( \sigma \) is an automorphism.

The following lemma gives equivalent conditions generalizing the Irving and Pearson-Stephenson conditions. Since our conditions, as well as those in the rest of this section, are shaped by the assumption that coefficients of elements of \( S = R[x; \sigma] \) are written on the left and \( xr = \sigma(r)x \), they may be left–right reversed from the conditions stated in parts of the literature.

**Proposition 5.2.** Let \( \sigma \) be an endomorphism of \( R \) and \( S = R[x; \sigma] \). Let \( I \) be a \( \sigma \)-stable ideal of \( R \). Then the following conditions are equivalent.

1. \( I \) is right \( S \)-prime.
2. If \( a, b \in R \), \( p \in \mathbb{N} \), and \( a\sigma^p(R)\sigma^q(b) \subseteq I \) for all \( q \geq p \), then \( a \in I \) or \( b \in I \).
3. If \( A \) is a left ideal of \( R \), \( B \) is a \( \sigma \)-stable ideal of \( R \), and \( A\sigma^p(B) \subseteq I \) for some \( p \in \mathbb{N} \), then \( A \subseteq I \) or \( B \subseteq I \).

If \( I \) is \( \sigma \)-invariant, and either \( R \) is commutative or \( \sigma \) is surjective, then all of the above conditions are equivalent to:

4. If \( A, B \) are ideals of \( R \) with \( B \) \( \sigma \)-stable and \( AB \subseteq I \), then \( A \subseteq I \) or \( B \subseteq I \).

*Proof.* This is a special case of Proposition 4.5 and Corollary 4.6.

The conditions for being \( S \)-prime when \( S = R[x^{\pm 1}; \sigma] \) are a special case of those in Proposition 3.2, here we state them in slightly different terms.

**Proposition 5.3.** Let \( \sigma \) be an automorphism of \( R \) and \( S = R[x^{\pm 1}; \sigma] \). Let \( I \) be a \( \sigma \)-invariant ideal of \( R \). Then the following conditions are equivalent.

1. \( I \) is right \( S \)-prime.
2. If \( a, b \in R \) and \( \sigma^p(a)R\sigma^q(b) \subseteq I \) for all \( p, q \in \mathbb{Z} \), then \( a \in I \) or \( b \in I \).
3. If \( A, B \) are \( \sigma \)-invariant ideals of \( R \) with \( AB \subseteq I \), then \( A \subseteq I \) or \( B \subseteq I \).

We next give conditions for \( S \)-primeness of a module when \( S = R[x; \sigma] \). The following result is a special case of Proposition 4.9 and Corollaries 4.10 and 4.11.

Example 6.2 shows that conditions (4) and (5) in Proposition 5.3 below are not equivalent to the others for an arbitrary \( \sigma \), even when \( \text{ann} M : \sigma \) is \( \sigma \)-invariant. Example 6.3 shows that the hypothesis “\( \text{ann} M : \sigma \) is \( \sigma \)-invariant” is necessary for the equivalence of (4) and (5) to the other conditions even if we assume \( R \) is commutative and \( \sigma \) is an automorphism. (If \( R \) is an integral domain and \( \sigma : R \to R \) is a non-injective ring homomorphism, then \( R \) is not \( S \)-prime but it does satisfy conditions (4) and (5). This is another example that the hypothesis “\( \text{ann} M : \sigma \) is \( \sigma \)-invariant” is necessary for the equivalence of (4) and (5) to the other conditions.)

**Proposition 5.4.** Let \( S = R[x; \sigma] \) where \( \sigma \) is an endomorphism of \( R \) and let \( M \) be a nonzero \( R \)-module. Consider the following conditions.

The implications \((1) \iff (2) \iff (3) \implies (4) \iff (5)\) are always true.
If (ann\(M : \sigma\)) is \(\sigma\)-invariant and either \(\sigma\) is surjective or \(R\) is commutative, then all five conditions are equivalent.

1. \(M\) is \(S\)-prime.
2. If \(m \in M\), \(b \in R\), \(p \in \mathbb{N}\), and \(m\sigma^p(R)\sigma^q(b) = 0\) for all \(q \geq p\), then \(m = 0\) or \(M\sigma^k(b) = 0\) for all \(k \in \mathbb{N}\).
3. If \(m \in M\), \(B\) is a \(\sigma\)-stable ideal of \(R\), and \(m\sigma^p(B) = 0\) for some \(p \in \mathbb{N}\), then \(m = 0\) or \(MB = 0\).
4. If \(m \in M\) and \(B\) is a \(\sigma\)-stable ideal of \(R\) with \(mB = 0\), then \(m = 0\) or \(MB = 0\).
5. If \(N\) is a nonzero submodule of \(M\), then \((\text{ann } N : \sigma) = (\text{ann } M : \sigma)\).

The next result follows from Corollary 4.12.

**Corollary 5.5.** Let \(S = R[x; \sigma]\) where \(\sigma\) is an automorphism of \(R\) and let \(M\) be a nonzero \(R\)-module. If \(R\) satisfies the a.c.c. on ideals, then the following conditions are equivalent.

1. \(M\) is \(S\)-prime.
2. If \(m \in M\) and \(B\) is a \(\sigma\)-invariant ideal with \(mB = 0\), then \(m = 0\) or \(MB = 0\).
3. If \(N\) is a nonzero submodule of \(M\), then \((\text{ann } N : \sigma) = (\text{ann } M : \sigma)\).

The next corollary is the analog of Corollary 4.13 in the skew polynomial case; it relates primeness and \(S\)-primeness. It will be used in some examples in Section 6.

**Corollary 5.6.** Let \(M\) be a prime \(R\)-module with annihilator \(I\) and let \(S = R[x; \sigma]\) where \(\sigma\) is an endomorphism of \(R\). Suppose that \(k \in \mathbb{N}\) and either \(R\) is commutative or \(\sigma\) is surjective.

1. \(M\) is \(S\)-prime if and only if \(\cap_{q \geq p} \sigma^{-q}(I) = (I : \sigma)\) for all \(p \in \mathbb{N}\).
2. \(M_{\sigma^k}\) is \(S\)-prime if and only if \(\cap_{q \geq p} \sigma^{-q}(I) = (\sigma^{-k}(I) : \sigma)\) for all \(p \in \mathbb{N}\).

**Proof.** When \(R\) is commutative or \(\sigma\) is surjective, condition (2) in Proposition 5.3 becomes: if \(m \neq 0\) and \(m\sigma^q(b) = 0\) for all \(q \geq p\), then \(b \in I\). Since \(M\) is prime, \(m\sigma^q(b) = 0\) if and only if \(\sigma^q(b) \in I\). Thus \(M\) is \(S\)-prime if and only if \(\cap_{q \geq p} \sigma^{-q}(I) \subseteq I\) for all \(p\). Likewise, \(M_{\sigma^k}\) is \(S\)-prime if and only if \(\cap_{q \geq p} \sigma^{-q}(I) \subseteq \sigma^{-k}(I)\) for all \(p\).

We now turn to the case \(S = R[x^{\pm 1}; \sigma]\). The following is a special case of the corresponding result for strongly graded rings, Proposition 3.4.

**Proposition 5.7.** Let \(S = R[x^{\pm 1}; \sigma]\) where \(\sigma\) is an automorphism of \(R\) and let \(M\) be an \(R\)-module. Then the following conditions are equivalent.

1. \(M\) is \(S\)-prime.
2. If \(m \in M\), \(b \in R\), and \(m\sigma^p(b) = 0\) for all \(p \in \mathbb{Z}\), then \(m = 0\) or \(Mb = 0\).
3. If \(m \in M\), \(B\) is a \(\sigma\)-invariant ideal of \(R\), and \(mB = 0\), then \(m = 0\) or \(MB = 0\).
4. For every nonzero submodule \(N\) of \(M\), we have \((\text{ann } N : \sigma^\pm) = (\text{ann } M : \sigma^\pm)\).

As in Section 4 for an \(R\)-module \(N\), \(N_{\sigma^k}\) is the module with \(R\)-action \(n \ast r = n\sigma^k(r)\).

If \(L\) is a \(\sigma^k(R)\)-submodule of \(M\) and we regard \(Lx^k\) as a subset of \(M[x; \sigma]\), then it is an \(R\)-submodule and the map \(a \mapsto ax^k\) is an \(R\)-module isomorphism from \(L_{\sigma^k}\) to \(Lx^k\).

We are now ready to describe the associated prime ideals of \(M[x; \sigma]\). (The skew laurent case is already covered in Proposition 3.5.) Proposition 4.14 tells us that if \(M\) is an \(R\)-module and \(N\) is a prime submodule of \(M[x; \sigma]\), then there is a \(k \in \mathbb{N}\) and a \(\sigma^k(R)\)-submodule \(L\).
of $M$ such that $L_{\sigma^k}$ is $S$-prime and such that if we set $J = \sigma^{-k}(\text{ann } L)$ and $I = (J : \sigma)$, then $I$ is $S$-prime and $\text{ann } N = I[x; \sigma]$. This, together with Lemma 4.15, implies the next result, which is a special case of Proposition 5.6. Part (2) of the proposition is Theorem 1.2 in Nordstrom [9] and part (3) is Corollary 1.5 in that paper.

**Proposition 5.8.** Let $S = R[x; \sigma]$ where $\sigma$ is an endomorphism of $R$ and let $M$ be an $R$-module.

1. $\text{Ass } M[x; \sigma] = \{ (\sigma^{-k}(J) : \sigma)[x; \sigma] \mid k \in \mathbb{N} \text{ and } J \text{ is the annihilator of an } S \text{-prime submodule of } M_{\sigma^k} \}.$

2. If $\sigma$ is surjective, then $\text{Ass } M[x; \sigma] = \{ (J : \sigma)[x; \sigma] \mid J \text{ is the annihilator of an } S \text{-prime submodule of } M \}.$

3. If $R$ satisfies the a.c.c. on ideals and $\sigma$ is an automorphism, then $\text{Ass } M[x; \sigma] = \{ (P : \sigma)[x; \sigma] \mid P \in \text{Ass } M \}.$

In the general case of Proposition 5.8 we may have to check all submodules of each twist $M_{\sigma^k}$. We can reduce this work a little bit in some cases.

**Corollary 5.9.** Let $S = R[x; \sigma]$ where $\sigma$ is an endomorphism of $R$ and let $M$ be an $R$-module such that $M_{\sigma^n}$ is $S$-prime for some $n \in \mathbb{N}$. Then there is a nonnegative integer $k$ such that $M_{\sigma^k}$ is $S$-prime if and only if $n \geq k$. In this case, $\text{Ass } M[x; \sigma] = \{ (\sigma^{-k}(\text{ann } M) : \sigma)[x; \sigma] \}.$

**Proof.** This follows from Lemma 4.18 and Proposition 5.8.

6. Examples

In this section we give several examples, organized into subsections. In the first subsection, Example 6.1 discusses the case where $R$ is the coordinate ring of an affine algebraic set and $M$ is the simple module corresponding to a point. In the second subsection, we give several examples where associated primes come from submodules of twists of $M$. Example 6.2 is of a prime ring $R$ that is not $S$-prime when $S = R[x; \sigma]$; it also provides an example showing that conditions (4) and (5) in Proposition 5.4 and conditions (2) and (3) in Corollary 4.9 are not always equivalent to $S$-primeness. Examples 6.3 and 6.4 show that one of $M$, $M[x; \sigma]$ may have associated prime ideals while the other does not. In the third subsection, we consider the reversibility condition. Examples 6.5 and 6.6 show that without the reversibility condition, an $R \ast G$-prime ideal need not be $G$-invariant. Example 6.7 shows that the weaker conditions for $S$-primeness in the commutative case generally fail without the reversibility condition.

**An example from algebraic geometry**

Our first example describes the case where $R = \mathcal{O}(X)$ is the coordinate ring of an affine algebraic set, $\sigma$ corresponds to a regular map from $X$ to itself, and $M$ is the simple $R$-module corresponding to a point $a \in X$. The induced module $M[x; \sigma]$ always has a unique associated prime, and it is always the annihilator of some twist $M_{\sigma^k}$. The $S$-primeness or twisted $S$-primeness of $M$, as well as $\text{Ass } M[x; \sigma]$ are controlled by the sequence $\{ \phi^n(a) \}_{n \in \mathbb{N}}$. If the sequence repeats, the starting point of the repeating segment is the $k$ above and the length of the repeating part determines the annihilator. If the sequence never repeats, the
value of \( k \) and the annihilator are determined by the Zariski closures of the tails of the sequence. We close with a discussion of the special case \( R = F[t], \sigma(t) = t^p \).

Example 6.1. Let \( F \) be an algebraically closed field, let \( X \) be an affine algebraic set over \( F \), let \( R = O(X) \) be the coordinate ring of \( X \), and let \( \phi : X \to X \) be a regular map. There is an endomorphism \( \sigma \) of the \( F \)-algebra \( R \) corresponding to \( \phi \). The maximal ideals of \( R \) have the form \( \mathfrak{m}_a \) for points \( a \in X \), and \( \sigma^{-1}(\mathfrak{m}_a) = \mathfrak{m}_{\phi(a)} \).

Let \( a \in X \) and set \( M = R/\mathfrak{m}_a \). Define a sequence of points of \( X \) by \( a_0 = a \) and \( a_n = \phi^n(a) \). The claims below follow from Corollary 5.6

If the sequence \( a_0, a_1, \ldots \) is finite, then it eventually becomes periodic of period \( \ell \), starting at some \( a_k \), so it has the form \( a_0, a_k, a_{k+1}, \ldots, a_{k+l-1}, a_k, \ldots \). Let \( J \) be the intersection of the ideals \( \mathfrak{m}_{a_0}, \mathfrak{m}_{a_{k+1}}, \ldots, \mathfrak{m}_{a_{k+l-1}} \). Then \( M_{a^k} \) is \( S \)-prime, and \( \text{ann} M[x; \sigma] x^k = J[x; \sigma] \). By Corollary 5.9 Ass \( M[x; \sigma] = \{ J[x; \sigma] \} \).

If the sequence is infinite, let \( Y_n \) be the Zariski-closure of \( \{ a_n, a_{n+1}, \ldots \} \) for each \( n \in \mathbb{N} \) and let \( Y = \bigcap_{n=0}^\infty Y_n \). Let \( J_n \) be the ideal determined by the algebraic set \( Y_n \), that is, \( J_n = \cap_{q \geq n} \mathfrak{m}_{a_q} \). Then \( \text{ann} M[x; \sigma] x^n = J_n[x; \sigma] \) for all \( n \in \mathbb{N} \).

The module \( M \) is \( S \)-prime if and only if \( J_n = J_0 \) for all \( n \). This is true if and only if the original sequence is contained in \( Y \). Likewise, \( M_{a^n} \) is \( S \)-prime if and only if \( a_n, a_{n+1}, \ldots \in Y \).

If \( R \) is noetherian, there is an \( n \) such that \( Y = Y_n \). Let \( k \) be the smallest such integer \( n \). If \( k = 0 \), then \( M \) is \( S \)-prime. If \( k > 0 \), then \( M \) is not \( S \)-prime, but \( M_{a^k} \) is \( S \)-prime. By Corollary 5.9 Ass \( M[x; \sigma] = \{ J_k[x; \sigma] \} \).

For a concrete example, let \( p \) be a prime number, \( X = \mathbb{A}^1 \), and \( \phi(x) = x^p \) for \( x \in X \). Then \( R \) is the polynomial ring \( F[t] \) with endomorphism \( \sigma(t) = t^p \) and \( S = R[x; \sigma] \). Let \( a \in X \) and set \( M = R/(t - a) \).

Four distinct cases occur, unless \( p \) divides the characteristic of \( F \), when case (c) below cannot occur. We introduce the following notation for use in cases (b) and (c). Let \( r \) be a positive integer, \( \omega \) a primitive \( r \)-th root of unity, and \( p \) a prime that does not divide \( r \). Let \( \ell \) be the order of \( p \) as an element of \( GL_1(\mathbb{Z}_r) \). We define \( f_{a,p} \in F[t] \) by \( f_{a,p} = \prod_{i=0}^{\ell-1}(t - \omega^{ip}) \).

If \( p \) is a primitive root modulo \( r \), then \( f_{a,p} \) is the \( r \)-th cyclotomic polynomial.

In each case, \( \text{Ass} M[x; \sigma] \) is a singleton, containing the annihilator ideal we describe.

(a) Suppose \( a = 0 \). The sequence determined by \( a, \phi \) is \( 0, 0, \ldots \). Thus \( M \) is \( S \)-prime with \( \text{ann} M[x; \sigma] = tS \).

(b) Suppose \( a \) is a primitive \( r \)-th root of unity where \( r \) is not divisible by \( p \), and let \( \ell \) be the order of \( p \) as an element of \( GL_1(\mathbb{Z}_r) \). The sequence determined by \( a, \phi \) is \( a, a^p, a^{p^2}, \ldots, a^{p^{\ell-1}}, a, \ldots \). Thus \( M \) is \( S \)-prime with \( \text{ann} M[x; \sigma] = f_{a,p} S \), where \( f_{a,p} \) is defined above.

(c) Suppose \( a \) is a primitive \( p^k r \)-th root of unity where \( k \geq 1 \) and \( r \) is not divisible by \( p \), and let \( \ell \) be the order of \( p \) as an element of \( GL_1(\mathbb{Z}_r) \). Let \( b = a^{p^k} \), so \( b \) is a primitive \( r \)-th root of unity. The sequence determined by \( a, \phi \) is \( a, a^p, a^{p^2}, \ldots, a^{p^{k-1}}, b, b^p, b^{p^2}, \ldots, b^{p^{\ell-1}}, b, \ldots \). Thus \( M \) is not \( S \)-prime but \( M_{a^k} \) is \( S \)-prime with \( \text{ann} M[x; \sigma] x^k = f_{b,p} S \), where \( f_{b,p} \) is defined above.

(d) Suppose \( a \neq 0 \) is not a root of unity. The sequence determined by \( a, \phi \) is \( a, a^p, a^{p^2}, \ldots \), which has no repetition. Thus every subsequence is infinite and hence Zariski-dense. In this case, \( M \) is \( S \)-prime with \( \text{ann} M[x; \sigma] = 0 \).
Twisted submodules and the lack of a relationship between prime and $S$-prime

Our first example shows that even a prime ideal need not be $S$-prime in the skew polynomial case. It includes an instance of an $S$-prime submodule of $M_\sigma$ that is not an $R$-submodule. The example also shows that an $R$-module $M$ can satisfy conditions (4) and (5) of Proposition 5.4 or conditions (2) and (3) of Corollary 4.11 without being $S$-prime.

**Example 6.2.** Let $F$ be a field and let $R$ be the $F$-algebra generated by variables $s_0, s_1, \ldots, t_0, t_1, \ldots$ subject to the relations $s_i s_j = 0$ if $i \leq j$ and $s_i t_j = 0$ if $i < j$. (We can also allow $t_i t_j = t_j t_i$ for all $i, j$ without affecting any of the results below.)

The ring $R$ is prime. To see this, let $m_1, m_2$ be (nonzero) monomials with $a$ the final letter of $m_1$ and $b$ the initial letter of $m_2$, and suppose $m_1 R m_2 = 0$. In particular $m_1 t_0 m_2 = 0$. Since the relations of $R$ are of degree 2, this implies that either $a t_0 = 0$ or $t_0 b = 0$. But $t_0$ does not annihilate any generator on either side, so this is impossible.

Now define $\sigma : R \to R$ by $\sigma(s_i) = s_{i+1}$ and $\sigma(t_i) = t_{i+1}$ for all $i$. This map preserves the relations, so it yields a well-defined ring endomorphism. It is easy to see that $\sigma$ is injective. We will show $R$ is not $S$-prime for $S = R[x; \sigma]$. As a matter of fact, $S$ is not even semiprime.

To see this, note that $s_0 \sigma(R) = F s_0$, so $s_0 \sigma(R) \sigma^q(s_0) = 0$ for all $q > 1$. This shows $R$ is not $S$-prime, and one easily sees that $(S s_0 x S)^2 = 0$.

Note that $L = F s_0$ is not a right ideal, but $L$ is a $\sigma(R)$-submodule of $R$. The $R$-module $L_\sigma$ is $S$-prime, since $l \sigma^p(R) = l \sigma^{p+1}(R) = L$ for all nonzero $l \in L$ and all $p \geq 0$. If $I$ is the ideal generated by all the $s$’s and $t$’s, then $\text{ann}L_\sigma = I$. Thus even though $R_R$ contains no $S$-prime submodules, $S = R[x; \sigma]$ contains the prime submodule $s_0 x S$, and hence $S_S$ has $I[x; \sigma]$ as an associated ideal.

Since $R$ is prime and $\sigma$ is injective, and hence $(\text{ann} R_R : \sigma) = 0$ is $\sigma$-invariant, condition (4) of Proposition 5.2 and conditions (4) and (5) of Proposition 5.4 are certainly satisfied with $M = R_R$, as are conditions (2) and (3) of Corollary 4.11. This shows that the conditions in Propositions 5.2 and 5.4 and Corollary 4.11 are not equivalent in general.

The next example, in the case $n = \infty$, gives an $R$-module $M$ that is prime but has no $S$-prime submodules, with $S = R[x; \sigma]$, where $R$ commutative and $\sigma$ an automorphism. When $n > 0$ is finite, the twist $M_\sigma^n$ is $S$-prime even though $M$ is not $S$-prime. When $n = \infty$, no submodule of any twist of $M$ is $S$-prime and $M[x; \sigma]$ has no associated primes. The case where $n = \infty$ is Example 5.15 in Leroy-Matczuk [7] and essentially the same as Example 2.2 in Nordstrom [9].

**Example 6.3.** Let $F$ be a field and set $R_n = F[t_{-n}, \ldots, t_0, t_1, \ldots]$ for $n \in \mathbb{N}$ and $R_\infty = F[t_{-1}, t_0, t_1, \ldots]$. Define an endomorphism $\sigma$ of $R_\infty$ (including $n = \infty$) by $\sigma(t_i) = t_{i+1}$ for each $i$ and let $I$ be the ideal generated by $t_0, t_1, \ldots$. Then $\sigma^{-j}(I)$ is the ideal generated by $t_{-j}, t_{-j-1}, \ldots$ if $j \leq n$. For $j \geq n$, we have $\sigma^{-j}(I) = \sigma^{-n}(I)$. Let $S = R_n[x; \sigma]$.

Set $M = R_n/I$ and note that $M$ is a prime $R_n$-module. If $n$ is finite, then $\cap_{q \geq p} \sigma^{-q}(I) = \sigma^{-n}(I)$ whenever $p \geq n$. Thus $M$ is $S$-prime if $n = 0$ by Corollary 5.6(1). If $n > 0$, then $M$ is not $S$-prime, but $M_\sigma^n$ is $S$-prime, by Corollary 5.6(2).

If $n = \infty$, then $\sigma$ is an automorphism and the ideals $\sigma^{-q}(I)$ strictly increase as $q$ increases. In particular, $\cap_{q \geq p} \sigma^{-q}(I) = \sigma^{-p}(I)$ for all $p$, and there is no $k$ with $\sigma^{-p}(I) \subset \sigma^{-k}(I)$ for
all \( p \). Thus by Corollary \[5.6\] no \( M_{\sigma^k} \) can be \( S \)-prime. Since \( M \) is prime and the conditions of Corollary \[5.6\] depend only on the annihilator of \( M \), the same considerations apply to all nonzero submodules. Thus by Proposition \[5.8\], \( M[x; \sigma] \) has no associated primes.

Since \( M \) is prime, \( I = (\text{ann } M : \sigma) = (\text{ann } N : \sigma) \) for all nonzero submodules \( N \) of \( M \). Also, if \( mJ = 0 \) for a nonzero \( m \notin M \) and an ideal \( J \) of \( R \), we must have \( MJ = 0 \). However, if \( n > 0 \), the ideal \( I \) is not \( \sigma \)-invariant and \( M \) is not \( S \)-prime. This shows that the hypotheses \( "(\text{ann } M : G) \) is \( G \)-invariant" and \( "(\text{ann } M : \sigma) \) is \( \sigma \)-invariant" are necessary in Corollary \[4.11\] parts (2) and (3) and in Proposition \[5.4\].

The next example gives an \( R \)-module \( M \) that is \( S \)-prime but contains no prime submodules, with \( S = R[x; \sigma] \), \( R \) commutative and \( \sigma \) an automorphism. This is essentially Example 2.3 in Nordstrom \[9\]; we include it for completeness.

**Example 6.4.** Let \( F \) be a field and set \( R = F[\ldots,t_{-1},t_0,t_1,\ldots]/(\ldots,t_{-1}^2,t_0^2,t_1^2,\ldots) \). We will write \( t_i \) and not \( \overline{t}_i \) for elements of \( R \). Define an automorphism \( \sigma \) of \( R \) by \( \sigma(t_i) = t_{i+1} \) for each \( i \) and let \( S = R[x; \sigma] \).

First we note that the module \( R_R \) contains no prime submodules and so \( \text{Ass } R_R = \emptyset \). To see this, suppose \( f \in R \) and suppose the variable \( t_n \) does not occur in \( f \). Then \( t_n \notin \text{ann } f \), but \((ft_n)t_n = 0 \). Thus \( \text{ann } f \nsubseteq \text{ann } ft_n \).

On the other hand, \( R_R \) is \( S \)-prime, i.e., \( S \) is prime and \( \text{Ass } S_S = \{0\} \). To see this, first note that if \( f,g \in R \) are nonzero and have no variables in common, then \( fg \neq 0 \). Given any nonzero \( f,g \), there is a \( p \) such that \( f,\sigma^q(g) \) have no variables in common for all \( q \geq p \). Thus \( f\sigma^q(g) \neq 0 \) for all \( q \geq p \). This proves \( R_R \) is \( S \)-prime by Proposition \[5.4\].

**The reversibility condition**

The next examples concern the reversibility condition that we introduced prior to Lemma \[4.11\]. The first two examples show that Lemma \[4.13\] need not hold when \( G \) is the free monoid on two generators and \( S = R \ast G \), even if the base ring \( R \) is commutative. The first gives an example where \( R \) is noetherian; the second gives an example where each \( \sigma(g) \) is an automorphism. Example \[6.7\] gives an example where \( I \) and \( M = S/I \) are not \( S \)-prime despite the fact that \( R \) is commutative and the conditions of Corollaries \[4.6\] and \[4.11\] hold, except for the reversibility condition.

**Example 6.5.** Let \( G \) be the free monoid on \( v,w \), let \( F \) be a field of characteristic 0, and let \( R = F[t] \). Define \( \sigma : G \to \text{End}_{F-\text{Alg}}(R) \) by \( b(t)^v = b(t+1) \) and \( b(t)^w = b(0) \) for \( b(t) \in F[t] \).

Since every \( \sigma(g) \) is the identity on constants, it is easy to see that for all \( j \in \mathbb{N} \) and \( g \in G \), we have \( (b^g)_g = b(j) \).

Thus if \( (b^h)_g = 0 \) for all \( h \in G \), then \( b \) has infinitely many roots and hence is 0. By Corollary \[4.13\] this implies the ideal 0 of \( R \) is \( S \)-prime. However, 0 is not \( G \)-invariant, since \( \sigma(w) = 1(0) = tR \).

**Example 6.6.** Let \( G \) be the free monoid on \( v,w \), let \( F \) be a field, let \( R = F[\ldots,t_{-1},t_0,t_1,\ldots] \), and let \( I \) be the ideal of \( R \) generated by \( t_0,t_1,\ldots \).
We begin by defining permutations \( \alpha, \beta \) of \( \mathbb{Z} \) by \( \alpha(n) = n + 2 \) and

\[
\beta(n) = \begin{cases} 
  n & \text{if } n \geq 0; \\
  n - 2 & \text{if } n \leq -1 \text{ and } n \text{ is odd}; \\
  -1 & \text{if } n = -2; \\
  n + 2 & \text{if } n \leq -4 \text{ and } n \text{ is even}; 
\end{cases}
\]

In cycle form (with infinite cycles), \( \alpha = (\ldots -3 -113 \ldots)(\ldots -202 \ldots) \) and \( \beta = (\ldots -4 -2 -1 -3 \ldots) \).

Define \( \sigma : G \to \text{Aut}_{F_{-\text{Alg}}}(R) \) by \( (t_n)^v = t_{\alpha(n)} \) and \( (t_n)^w = t_{\beta(n)} \).

Clearly \( \sigma : G \to \text{Aut}_{F_{-\text{Alg}}}(R) \) is pointwise surjective. By Corollary 4.13 it is enough to show that for any \( r \in R, g \in G \), if \( (r^h)^g \in I \) for all \( h \in G \), then \( r \in I \). Because \( I \) is a monomial ideal and each \( \sigma(g) \) takes distinct monomials to distinct monomials, it is enough to show this in the case where \( r \) is a monomial.

Thus we will assume that \( g \in G, r \in R \setminus I \), that \( r \) is a product of indeterminates \( t_n \) with each \( n < 0 \), and that \( (r^h)^g \in I \) for every \( h \in G \). We will show below that for any negative integers \( a, n \), there are integers \( m, b \) such that \( m \) is positive, \( b \) is odd, \( b < a \), and \( (t_n)^{w^m} = t_b \). Suppose we have proven this claim and suppose \( g \) has \( k \) occurrences of \( v \). For each \( t_n \) that occurs in \( r \), let \( m(n) \) be a positive integer as above with \( (t_n)^{w^m(n)} = t_{b(n)} \) where \( b(n) < -2k \) is odd. Suppose \( c < 0 \) is odd. If we apply \( \sigma(v) \) to \( t_c \), we get \( t_{c-2} \). If we apply \( \sigma(v) \) to \( t_c \), we get \( t_{c+2} \). In either case, the subscript is odd and no more than two larger than \( c \). Thus when we apply \( \sigma(g) \) to \( t_{b(n)} \), the result is \( t_{d(n)} \) where \( d(n) < 0 \) is odd. Hence if we let \( m \) be the maximum of the \( m(n) \), we see that \( (r^{w^m})^g \) is a monomial all of whose indeterminate factors have negative subscripts. Thus \( (r^{w^m})^g \notin I \), which contradicts our choice of \( r \).

To finish the proof, we need to verify the claim. That is, we need to show that if \( n, a < 0 \), there exist a positive \( m \) and an odd \( b < a \) such that \( \beta^m(n) = b \). If \( n \) is odd, we may choose \( m = |a| \) and \( b = n - 2m \). If \( n = 2k \) is even, we may choose \( m = |k| + |a| \) and \( b = -1 - 2|a| \).

The next example shows that Corollaries 4.6 and 1.14 fail in the commutative case if the reversibility condition does not hold. We construct an example where \( G \) is the free monoid on two generators, acting on a commutative base ring \( R \), and \( I \) is a \( G \)-invariant ideal of \( R \) that is not \( S \)-prime, but is such that whenever \( A, B \triangleleft R \), and \( B \) is \( G \)-stable, if \( AB \subseteq I \), then either \( A \subseteq I \) or \( B \subseteq I \).

Note that Corollary 4.6 holds in the case where \( \sigma \) is pointwise surjective without the reversibility condition hypothesis. Unfortunately, we do not presently have an example of the necessity of the reversibility condition in the “\( \sigma \) is pointwise surjective” case of Corollary 1.14. Nor do we have an example that Corollaries 4.6 and Corollary 1.14 fail when \( R \) is commutative noetherian.

**Example 6.7.** Let \( G \) be the free monoid on \( v, w \), let \( F \) be a field, let \( R = F[t_0, t_1, \ldots] \), and let \( I \) be the ideal of \( R \) generated by all \( t_it_j \) with \( i \neq j \). (We could also allow all \( t_i^2 \) in \( I \) without affecting the result.)
We begin by defining almost permutations $\alpha, \beta$ of $\mathbb{N}$ by $\alpha(n) = n + 1$ and

$$
\beta(n) = \begin{cases} 
1 & \text{if } n = 0; \\
 n + 2 & \text{if } n \geq 1 \text{ and } n \text{ is odd}; \\
 n - 2 & \text{if } n \geq 2 \text{ and } n \text{ is even}.
\end{cases}
$$

In “cycle form” $\alpha = (0 1 2 \ldots)$ and $\beta = (\ldots 4 2 0 1 3 \ldots)$. Of course $\alpha$ is not a permutation or true cycle; it is injective but not surjective. The map $\beta$ is a bijection.

Define $\sigma : G \to \text{Aut}_F - \text{Alg}(R)$ by $(t_n)^v = t_{\alpha(n)}$ and $(t_n)^w = t_{\beta(n)}$.

We will first show that $I$ is $G$-invariant. It is obvious that $I$ is $G$-stable. Suppose $r$ is a nontrivial monomial and $r^g \in I$ for some $g \in G$. Clearly we cannot have $r = t_n^k$. However, if $r$ is any other nontrivial monomial, then $r$ must have at least two distinct indeterminate factors, and thus $r \in I$. This proves $I$ is $G$-invariant.

Set $g = v$. Then $\alpha(n) > 0$ for all $n \in \mathbb{N}$, so $r^g \in I$ for all indeterminates $r$. If we set $r = t_0$, we have $(r^h)^g \in I$ for all $h \in G$. Since $r \notin I$, Corollary 4.13 shows $I$ is not $S$-prime.

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