Latency-Bounded Target Set Selection in Social Networks

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Abstract
Motivated by applications in sociology, economy and medicine, we study variants of the Target Set Selection problem, first proposed by Kempe, Kleinberg and Tardos. In our scenario one is given a graph \(G = (V, E)\), integer values \(t(v)\) for each vertex \(v\) (thresholds), and the objective is to determine a small set of vertices (target set) that activates a given number (or a given subset) of vertices of \(G\) within a prescribed number of rounds.

The activation process in \(G\) proceeds as follows: initially, at round 0, all vertices in the target set are activated; subsequently at each round \(r \geq 1\) every vertex of \(G\) becomes activated if at least \(t(v)\) of its neighbors are already active by round \(r - 1\). It is known that the problem of finding a minimum cardinality Target Set that eventually activates the whole graph \(G\) is hard to approximate to a factor better than \(O(2^{\log^{1+\epsilon}|V|})\). In this paper we give exact polynomial time algorithms to find minimum cardinality Target Sets in graphs of bounded clique-width, and exact linear time algorithms for trees.

1 Introduction
Let \(G = (V, E)\) be a graph, \(S \subseteq V\), and let \(t : V \rightarrow \mathbb{N} = \{1, 2, \ldots\}\) be a function assigning integer thresholds to the vertices of \(G\). An activation process in \(G\) starting at \(S\) is a sequence \(\text{Active}[S, 0] \subseteq \text{Active}[S, 1] \subseteq \ldots \subseteq \text{Active}[S, i] \subseteq \ldots \subseteq V\) of vertex subsets, with \(\text{Active}[S, 0] = S\), and such that for all \(i > 0\),

\[
\text{Active}[S, i] = \text{Active}[S, i - 1] \cup \left\{ u \mid |N(u) \cap \text{Active}[S, i - 1]| \geq t(u) \right\}
\]

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where $N(u)$ is the set of neighbors of $u$. In words, at each round $i$ the set of active nodes is augmented by the set of nodes $u$ that have a number of already activated neighbors greater or equal to $u$’s threshold $t(u)$. The central problem we introduce and study in this paper is defined as follows:

$\textbf{(λ, β, α)-Target Set Selection (}(\lambda, \beta, \alpha)\text{-TSS)}$.

**Instance**: A graph $G = (V, E)$, thresholds $t : V \rightarrow \mathbb{N}$, a latency bound $\lambda \in \mathbb{N}$, a budget $\beta \in \mathbb{N}$ and an activation requirement $\alpha \in \mathbb{N}$.

**Problem**: Find $S \subseteq V$ s.t. $|S| \leq \beta$ and $|\text{Active}[S, \lambda]| \geq \alpha$ (or determine that no such a set exists).

We will be also interested in the case in which a set of nodes that need to be activated (within the given latency bound) is explicitly given as part of the input.

$\textbf{(λ, β, A)-Target Set Selection (}(\lambda, \beta, A)\text{-TSS)}$.

**Instance**: A graph $G = (V, E)$, thresholds $t : V \rightarrow \mathbb{N}$, a latency bound $\lambda \in \mathbb{N}$, a budget $\beta \in \mathbb{N}$ and a set to be activated $A \subseteq V$.

**Problem**: Find a set $S \subseteq V$ such that $|S| \leq \beta$ and $A \subseteq \text{Active}[S, \lambda]$ (or determine that such a set does not exist).

Eliminating any one of the parameters $\lambda$ and $\beta$, one obtains two natural minimization problems. For instance, eliminating $\beta$, one obtains the following problem:

$\textbf{(λ, A)-Target Set Selection (}(\lambda, A)\text{-TSS)}$.

**Instance**: A graph $G = (V, E)$, thresholds $t : V \rightarrow \mathbb{N}$, a latency bound $\lambda \in \mathbb{N}$ and a set $A \subseteq V$.

**Problem**: Find a set $S \subseteq V$ of minimum size such that $A \subseteq \text{Active}[S, \lambda]$.

Notice that in the above problems we may assume without loss of generality that $0 \leq t(u) \leq d(u) + 1$ holds for all nodes $u \in V$ (otherwise, we can set $t(u) = d(u) + 1$ for every node $u$ with threshold exceeding its degree plus one without changing the problem).

The above algorithmic problems have roots in the general study of the spread of influence in Social Networks (see [14] and references quoted therein). For instance, in the area of viral marketing [13, 12] companies wanting to promote products or behaviors might try initially to target and convince a few individuals which, by word-of-mouth effects, can trigger a cascade of influence in the network, leading to an adoption of the products by a much larger number of individuals. It is clear that the $(\lambda, \beta, \alpha)$-TSS problem represents an abstraction of that scenario, once one makes the reasonable assumption that an individual decides to adopt the products if a certain number of his/her friends have adopted said products. Analogously, the $(\lambda, \beta, \alpha)$-TSS problem can describe other diffusion problems arising in sociological, economical and biological networks, again see [14]. Therefore, it comes as no surprise that special cases of our problem (or variants thereof) have recently attracted much attention by the algorithmic community. In this version of the paper we shall limit ourselves to discuss the work which is strictly related to the present paper (we just mention that our results are also relevant to other areas, like dynamic monopolies [15, 20], for instance). The first authors to study problems of
spread of influence in networks from an algorithmic point of view were Kempe et al. [17, 18]. However, they were mostly interested in networks with randomly chosen thresholds. Chen [6] studied the following minimization problem: Given a graph $G$ and fixed thresholds $t(v)$, find a target set of minimum size that eventually activates all (or a fixed fraction of) vertices of $G$. He proved a strong inapproximability result that makes unlikely the existence of an algorithm with approximation factor better than $O(2^\log^{1-\epsilon}|V|)$. Chen’s result stimulated the work [1, 2, 7]. In particular, in [2], Ben-Zwi et al. proved that the $([V|, \beta, \alpha)$-TSS problem can be solved in time $O(t^w|V|)$ where $t$ is the maximum threshold and $w$ is the treewidth of the graph, thus showing that this variant of the problem is fixed-parameter tractable if parameterized w.r.t. both treewidth and the maximum degree of the graph. Paper [7] isolated other interesting cases in which the problems become efficiently tractable.

All the above mentioned papers did not consider the issue of the number of rounds necessary for the activation of the required number of vertices. However, this is a relevant question: In viral marketing, for instance, it is quite important to spread information quickly. It is equally important, before embarking on a possible onerous investment, to try estimating the maximum amount of influence spread that can be guaranteed within a certain amount of time (i.e., for some $\lambda$ fixed in advance), rather than simply knowing that eventually (but maybe too late) the whole market might be covered. These considerations motivate our first generalization of the problem, parameterized on the number of rounds $\lambda$. The practical relevance of parameterizing the problem also with bounds on the initial budget or the final requirement should be equally evident.

For general graphs, Chen’s [6] inapproximability result still holds if one demands that the activation process ends in a bounded number of rounds. We show that the general $(\lambda, \beta, \alpha)$-TSS problem is polynomially solvable in graph of bounded clique-width and constant latency bound $\lambda$ (see Theorem 1 in Section 2). Since graphs of bounded treewidth are also of bounded clique-width [10], this result implies a polynomial solution of the $(\lambda, \beta, \alpha)$-TSS problem with constant $\lambda$ also for graphs of bounded treewidth, complementing the result of [2] showing that for bounded-treewidth graphs, the TSS problem without the latency bound (equivalently, with $\lambda = |V| - 1$) is polynomially solvable. Moreover, the result settles the status of the computational complexity of the VECTOR DOMINATION problem for graphs of bounded tree- or clique-width, that was posed as an open question in [8].

We also consider the instance when $G$ is a tree. For this special case we give an exact linear time algorithm for the $(\lambda, A)$-TSS problem, for any $\lambda$ and $A \subseteq V$. When $\lambda = |V| - 1$ and $A = V$ our result is equivalent to the (optimal) linear time algorithm for the classical TSS problem (i.e., without the latency bound) on trees proposed in [6].

## 2 TSS Problems on Bounded Clique-Width Graphs

In this section, we give an algorithm for the $(\lambda, \beta, \alpha)$-TARGET SET SELECTION problem on graphs $G$ of clique-width at most $k$ given by an irredundant $k$-expression $\sigma$. For the sake of
The clique-width of a graph. A labeled graph is a graph in which every vertex has a label from \( \mathbb{N} \). A labeled graph is a \( k \)-labeled graph if every label is from \( [k] := \{1, 2, \ldots, k\} \). The clique-width of a graph \( G \) is the minimum number of labels needed to construct \( G \) using the following four operations: (i) Creation of a new vertex \( v \) with label \( a \) (denoted by \( a(v) \)); (ii) disjoint union of two labeled graphs \( G \) and \( H \) (denoted by \( G \oplus H \)); (iii) Joining by an edge each vertex with label \( a \) to each vertex with label \( b \) (\( a \neq b \), denoted by \( \eta_{a,b} \)); (iv) renaming label \( a \) to \( b \) (denoted by \( \rho_{a \rightarrow b} \)). Every graph can be defined by an algebraic expression using these four operations. For instance, a chordless path on five consecutive vertices \( u, v, x, y, z \) can be defined as follows:

\[
\eta_{3,2}(3(z) \oplus \rho_{3 \rightarrow 2}(\eta_{3,2}(3(y) \oplus \rho_{3 \rightarrow 2}(\eta_{3,2}(3(x) \oplus \eta_{2,1}(2(v) \oplus 1(u))))))).
\]

Such an expression is called a \( k \)-expression if it uses at most \( k \) different labels. The clique-width of \( G \), denoted \( cw(G) \), is the minimum \( k \) for which there exists a \( k \)-expression defining \( G \). If a graph \( G \) has a clique-width at most \( k \), then a \( (2^{k+1} - 1) \)-expression for it can be computed in time \( O(|V(G)|^3) \) using the rank-width \[16,19\].

Every graph of clique-width at most \( k \) admits an irredundant \( k \)-expression, that is, a \( k \)-expression such that before any operation of the form \( \eta_{a,b} \) is applied, the graph contains no edges between vertices with label \( a \) and vertices with label \( b \) \[11\]. In particular, this means that every operation \( \eta_{a,b} \) adds at least one edge to the graph \( G \). Each expression \( \sigma \) defines a rooted tree \( T(\sigma) \), that we also call a clique-width tree.

Our result on graphs with bounded clique-width. We describe an algorithm for the \((\lambda, \beta, \alpha)\)-TSS problem on graphs \( G \) of clique-width at most \( k \) given by an irredundant \( k \)-expression \( \sigma \). Denoting by \( n \) the number of vertices of the input graph \( G \), the running time of the algorithm is bounded by \( O(\lambda k|\sigma|(n + 1)^{(3\lambda+2)^k}) \), where \( |\sigma| \) denotes the encoding length of \( \sigma \). For fixed \( k \) and \( \lambda \), this is polynomial in the size of the input. We will first solve the following decision problem naturally associated with the \((\lambda, \beta, \alpha)\)-TARGET SET SELECTION problem:

\((\lambda, \beta, \alpha)\)-TARGET SET SELECTION \((\lambda, \beta, \alpha)\)-TSS.

**Instance:** A graph \( G = (V, E) \), thresholds \( t : V \rightarrow \mathbb{N} \), a latency bound \( \lambda \in \mathbb{N} \), a budget \( \beta \in \mathbb{N} \) and an activation requirement \( \alpha \in \mathbb{N} \).

**Problem:** Determine whether there exists a set \( S \subseteq V \) such that \( |S| \leq \beta \) and \( |\text{Active}[S, \lambda]| \geq \alpha \).

Subsequently, we will argue how to modify the algorithm in order to solve the \((\lambda, \beta, \alpha)\)- and the \((\lambda, \beta, A)\)-TARGET SET SELECTION problems.

Consider an instance \((G, t, \lambda, \beta, \alpha)\) to the \((\lambda, \beta, \alpha)\)-TARGET SET DECISION problem, where \( G = (V, E) \) is a graph of clique-width at most \( k \) given by an irredundant \( k \)-expression \( \sigma \). We will develop a dynamic programming algorithm that will traverse the clique-width tree bottom up and simulate the activation process for the corresponding induced subgraphs of \( G \), keeping track only of the minimal necessary information, that is, of how many vertices of each
label become active in each round. For a bounded number of rounds \( \lambda \), it will be possible to store and analyze the information in polynomial time. In order to compute these values recursively with respect to all the operations in the definition of the clique-width—including operations of the form \( \eta_{a,b} \)—we need to consider not only the original thresholds, but also reduced ones. This is formalized in Definition 1 below. We view \( G \) as a \( k \)-labeled graph defined by \( \sigma \).

Given a \( k \)-labeled graph \( H \) and a label \( \ell \in [k] \), we denote by \( V_\ell(H) \) the set of vertices of \( H \) with label \( \ell \).

**Definition 1.** Given a \( k \)-labeled subgraph \( H \) of \( G \) and a pair of matrices with non-negative integer entries \( (\alpha, r) \) such that \( \alpha \in (\mathbb{Z}^+)^{[0,\lambda] \times [k]} \) (where \( [0, \lambda] := \{0, 1, \ldots, \lambda\} \)) and \( r \in (\mathbb{Z}^+)^{|\lambda| \times [k]} \), an \( (\alpha, r) \)-activation process for \( H \) is a non-decreasing sequence of vertex subsets \( S[0] \subseteq \ldots \subseteq S[\lambda] \subseteq V(H) \) such that the following conditions hold:

1. For every round \( i \in [\lambda] \) and for every label \( \ell \in [k] \), the set of all vertices with label \( \ell \) activated at round \( i \) is obtained with respect to the activation process starting at \( S[0] \) with thresholds \( t(u) \) reduced by \( r[i, \ell] \) for all vertices with label \( \ell \). Formally, for all \( \ell \in [k] \) and all \( i \in [\lambda] \),

\[
(S[i] \setminus S[i-1]) \cap V_\ell(H) = \left\{ u \in V_\ell(H) \setminus S[i-1] : |N_H(u) \cap S[i-1]| \geq t(u) - r[i, \ell] \right\}.
\]

2. For every label \( \ell \in [k] \), there are exactly \( \alpha[0, \ell] \) initially activated vertices with label \( \ell \):

\[
|S[0] \cap V_\ell(H)| = \alpha[0, \ell].
\]

3. For every label \( \ell \in [k] \) and for every round \( i \in [\lambda] \), there are exactly \( \alpha[i, \ell] \) vertices with label \( \ell \) activated at round \( i \): \( |(S[i] \setminus S[i-1]) \cap V_\ell(H)| = \alpha[i, \ell] \).

Let \( A \) denote the set of all matrices of the form \( \alpha = (\alpha[i, \ell] : 0 \leq i \leq \lambda, 1 \leq \ell \leq k) \) where \( \alpha[i, \ell] \in [0, \alpha] \) for all \( 0 \leq i \leq \lambda \) and all \( 1 \leq \ell \leq k \). Notice that \( |A| = (\alpha + 1)^{(\lambda+1)k} = O((n+1)^{(\lambda+1)k}) \). Similarly, let \( R \) denote the set of all matrices of the form \( r = (r[i, \ell] : 1 \leq i \leq \lambda, 1 \leq \ell \leq k) \), where \( r[i, \ell] \in [0, n] \) for all \( 1 \leq i \leq \lambda \) and all \( 1 \leq \ell \leq k \). Then \( |R| = (n+1)^{\lambda k} \).

Every node of the clique-width tree \( T := T(\sigma) \) of the input graph \( G \) corresponds to a \( k \)-labeled subgraph \( H \) of \( G \). To every node of \( T \) (and the corresponding \( k \)-labeled subgraph \( H \) of \( G \)), we associate a Boolean-valued function \( \gamma_H : A \times R \rightarrow \{0, 1\} \) where \( \gamma_H(\alpha, r) = 1 \) if and only if there exists an \( (\alpha, r) \)-activation process for \( H \). Each matrix pair \((\alpha, r) \in A \times R\) can be described with \( O(\lambda k) \) numbers. Hence, the function \( \gamma_H \) can be represented by storing the set of all triples \( \{((\alpha, r, \gamma_H(\alpha, r)) : (\alpha, r) \in A \times R\} \), requiring, in total, space

\[
O(\lambda k) \cdot |A \times R| = O(\lambda k) \cdot O((n+1)^{(\lambda+1)k}) \cdot O((n+1)^{\lambda k}) = O(\lambda k(n+1)^{(2\lambda+1)k}).
\]

Below we will describe how to compute all functions \( \gamma_H \) for all subgraphs \( H \) corresponding to the nodes of the tree \( T \). Assuming all these functions have been computed, we can extract the solution to the \((\lambda, \beta, \alpha)\)-TARGET SET DECISION problem on \( G \) from the root of \( T \) as follows.
**Proposition 1.** There exists a set $S \subseteq V(G)$ such that $|S| \leq \beta$ and $|\text{Active}[S, \lambda]| \geq \alpha$ if and only if there exists a matrix $\alpha \in \mathcal{A}$ with $\gamma_G(\alpha, 0) = 1$ (where $0 \in \mathcal{R}$ denotes the all zero matrix) such that $\sum_{\ell=1}^{k} \alpha[0, \ell] \leq \beta$ and $\sum_{i=0}^{\lambda} \sum_{\ell=1}^{k} \alpha[i, \ell] \geq \alpha$.

**Proof.** The constraint $\sum_{\ell=1}^{k} \alpha[0, \ell] \leq \beta$ specifies that the total number of initially targeted vertices is within the budget $\beta$, and the constraint $\sum_{i=0}^{\lambda} \sum_{\ell=1}^{k} \alpha[i, \ell] \geq \alpha$ specifies that the total number of vertices activated within round $\lambda$ is at least the activation requirement $\alpha$. \quad \square

Here we give a detailed description of how to compute the functions $\gamma_H$ by traversing the tree $T$ bottom up. We consider four cases according to the type of a node $v$ of the clique-width tree $T$.

**Case 1: $v$ is a leaf.**

In this case, the labeled subgraph $H$ of $G$ associated to $v$ is of the form $H = a(u)$ for some vertex $u \in V(G)$ and some label $a \in [k]$. That is, a new vertex $u$ is introduced with label $a$.

Suppose that $(\alpha, r) \in \mathcal{A} \times \mathcal{R}$ is a matrix pair such that there exists an $(\alpha, r)$-activation process $S = (S[0], S[1], \ldots, S[\lambda])$ for $H$. For every $\ell \in [k] \setminus \{a\}$, we have $V_\ell(H) = \emptyset$ and hence $\alpha[i, \ell] = 0$ for all $i \in [0, \lambda]$. Moreover, since $V_a(H) = \{u\}$, we have

$$0 \leq \sum_{i=0}^{\lambda} \alpha[i, a] = |S[0] \cap V_a(H)| + \sum_{i=1}^{\lambda} |(S[i] \setminus S[i-1]) \cap V_a(H)| \leq |V_a(H)| = 1.$$ 

Suppose first that $\sum_{i=0}^{\lambda} \alpha[i, a] = 0$, that is, $\alpha[i, a] = 0$ for all $i$. Then, $S[i] = \emptyset$ for all $i \in [0, \lambda]$, and the defining property (1) of the $(\alpha, r)$-activation process implies that $r[i, a] < t(u)$ for every $i \in [\lambda]$ (otherwise $u$ would belong to $S[i]$).

Now, suppose that $\sum_{i=0}^{\lambda} \alpha[i, a] = 1$. Then, there exists a unique $i^* \in [0, \lambda]$ such that

$$\alpha[i, a] = \begin{cases} 1, & \text{if } i = i^*; \\ 0, & \text{otherwise}. \end{cases}$$

If $i^* = 0$ then $\{u\} = S[0] \subseteq S[1] \subseteq \ldots \subseteq S[\lambda] \subseteq V(H) = \{u\}$, therefore $S[i] = \{u\}$ for all $i \in [0, \lambda]$, independently of $r$. If $i^* \geq 1$ then properties (2) and (3) imply that $S[0] = \ldots = S[i^* - 1] = \emptyset$ and $S[i^*] = S[i^* + 1] = \ldots = S[\lambda] = \{u\}$. Hence, the defining property (1) of the $(\alpha, r)$-activation process implies, on the one hand, that $r[i, a] < t(u)$ for every $i \in \{1, \ldots, i^* - 1\}$ (otherwise $u$ would belong to $S[i]$), while, on the other hand, $r[i^*, a] \geq t(u)$. Hence, $i^* = \min\{i \geq 1 : r[i, a] \geq t(u)\}$. 

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Hence, if there exists an \((\alpha, r)\)-activation process for \(H\), then \((\alpha, r) \in (A \times R)^*\) where

\[
(A \times R)^* = \left\{ (\alpha, r) \in A \times R : (\forall \ell \neq a)(\alpha[i, \ell] = 0) \land \left( \sum_{i=0}^{\lambda} \alpha[i, a] \leq 1 \right) \land \left[ \left( \sum_{i=0}^{\lambda} \alpha[i, a] = 0 \right) \Rightarrow \left( \forall i \right) \left( r[i, a] < t(u) \right) \right] \land \left[ \left( \exists i^* \right) \left( \alpha[i^*, a] = 1 \right) \Rightarrow \left( i^* = 0 \lor i^* = \min\{ i : r[i, a] \geq t(u) \} \right) \right] \right\}.
\]

Conversely, by reversing the above arguments, one can verify that for every \((\alpha, r) \in (A \times R)^*\) there exists an \((\alpha, r)\)-activation process for \(H\). Hence, for every \((\alpha, r) \in A \times R\), we set

\[
\gamma_H(\alpha, r) = \left\{ \begin{array}{ll}
1, & \text{if } (\alpha, r) \in (A \times R)^*; \\
0, & \text{otherwise}.
\end{array} \right.
\]

**Case 2: \(v\) has exactly two children in \(T\).**

In this case, the labeled subgraph \(H\) of \(G\) associated to \(v\) is the disjoint union \(H = H_1 \oplus H_2\), where \(H_1\) and \(H_2\) are the labeled subgraphs of \(G\) associated to the two children of \(v\) in \(T\).

Suppose that \((S[0], \ldots, S[\lambda])\) is an \((\alpha, r)\)-activation process for \(H\). For every round \(i \in [0, \lambda]\) and for every label \(\ell \in [k]\), set

\[
S_1[i] = S[i] \cap V(H_1),
\]

and

\[
\alpha_1[i, \ell] = \begin{cases}
|S_1[i] \cap V_\ell(H_1)|, & \text{if } i = 0; \\
|S_1[i] \setminus S_1[i-1] \cap V_\ell(H_1)|, & \text{otherwise}.
\end{cases}
\]

Then, \((S[0], \ldots, S[\lambda])\) is an \((\alpha_1, r)\)-activation process for \(H_1\). Properties (2) and (3) follow immediately from the definition of \(\alpha_1\). Property (1) follows from the fact that in \(H\) there are no edges between vertices of \(H_1\) and \(H_2\). One can analogously define an \((\alpha_2, r)\)-activation process for \(H_2\). Since \(H\) is the disjoint union of \(H_1\) and \(H_2\), these two processes satisfy the matrix equation \(\alpha_1 + \alpha_2 = \alpha\).

Conversely, suppose that there exist an \((\alpha_1, r)\)-activation process \((S_1[0], \ldots, S_1[\lambda])\) for \(H_1\) and an \((\alpha_2, r)\)-activation process \((S_2[0], \ldots, S_2[\lambda])\) for \(H_2\). Then, defining \(S[i] = S_1[i] \cup S_2[i]\) for all rounds \(i \in [0, \lambda]\), we obtain an \((\alpha, r)\)-activation process \((S[0], \ldots, S[\lambda])\) for \(H\), where \(\alpha = \alpha_1 + \alpha_2\).

Hence, for every \((\alpha, r) \in A \times R\) we set

\[
\gamma_H(\alpha, r) = \left\{ \begin{array}{ll}
1, & \text{if } (\exists \alpha_1, \alpha_2 \in A)(\alpha = \alpha_1 + \alpha_2 \text{ and } \gamma_{H_1}(\alpha_1, r) = \gamma_{H_2}(\alpha_2, r) = 1); \\
0, & \text{otherwise}.
\end{array} \right.
\]

**Case 3: \(v\) has exactly one child in \(T\) and the labeled subgraph \(H\) of \(G\) associated to \(v\) is of the form \(H = \eta_{a,b}(H_1)\).**
In this case, graph $H$ is obtained from $H_1$ by adding all edges between vertices labeled $a$ and vertices labeled $b$. Since the $k$-expression is irredundant, in $H_1$ there are no edges between vertices labeled $a$ and vertices labeled $b$.

Suppose that $S = (S[0], \ldots, S[\lambda])$ is an $(\alpha, r)$-activation process for $H$. For every round $i \in [0, \lambda]$ and for every label $\ell \in [k]$, set

$$r_1[i, \ell] = \begin{cases} 
\min\{n, r[i, a] + \sum_{j<i} \alpha[j, b]\}, & \text{if } \ell = a; \\
\min\{n, r[i, b] + \sum_{j<i} \alpha[j, a]\}, & \text{if } \ell = b; \\
r[i, \ell], & \text{otherwise},
\end{cases}$$

Let us verify that $S$ is an $(\alpha, r_1)$-activation process for $H_1$:

- Defining conditions (2) and (3) are satisfied since the partition of the vertex set $V(H) = V(H_1)$ into label classes is the same in both graphs $H$ and $H_1$.

- To verify condition (1), notice first that for every label $\ell \in [k] \setminus \{a, b\}$ and every vertex $u \in V_i(H_1) = V_i(H)$, we have $N_{H_1}(u) = N_H(u)$. Moreover, for each round $i \in [\lambda]$, it holds that $r_i[i, \ell] = r[i, \ell]$, which implies $|N_{H_1}(u) \cap S[i-1]| \geq t(u) - r_1[i, \ell]$ if and only if $|N_H(u) \cap S[i-1]| \geq t(u) - r[i, \ell]$.

Now consider the case $\ell = a$. (The case $\ell = b$ is analogous.) Since the $k$-expression is irredundant, the $H$-neighborhood of every vertex $u \in V_a(H_1) = V_a(H)$ is equal to the disjoint union $N_H(u) = N_{H_1}(u) \cup V_b(H)$.

Consider an arbitrary round $i \in [\lambda]$. We will show that condition

$$|N_{H_1}(u) \cap S[i-1]| \geq t(u) - r_1[i, a] \quad (1)$$

is equivalent to the condition

$$|N_H(u) \cap S[i-1]| \geq t(u) - r[i, a]. \quad (2)$$

The set $S[i-1]$ can be written as the disjoint union

$$S[i-1] = S[0] \cup \bigcup_{j=1}^{i-1} (S[j] \setminus S[j-1]) ,$$

hence

$$|S[i-1] \cap V_b(H)| = |S[0] \cap V_b(H)| + \sum_{j=1}^{i-1} |(S[j] \setminus S[j-1]) \cap V_b(H)| = \alpha[0, b] + \sum_{j=1}^{i-1} \alpha[j, b] = \sum_{j<i} \alpha[j, b]$$
and consequently

\[ |N_H(u) \cap S[i - 1]| = |N_{H_1}(u) \cap S[i - 1]| + |V_b(H) \cap S[i - 1]| \]
\[ = |N_{H_1}(u) \cap S[i - 1]| + \sum_{j < i} \alpha[j, b]. \]

Suppose first that \( t(u) \leq r_1[i, a] \). Then, condition (1) trivially holds, and condition (2) holds as well:

\[ |N_H(u) \cap S[i - 1]| \geq |S[i - 1] \cap V_b(H)| = \sum_{j < i} \alpha[j, b] \]
\[ = (r[i, a] + \sum_{j < i} \alpha[j, b]) - r[i, a] \]
\[ \geq r_1[i, a] - r[i, a] \geq t(u) - r[i, a]. \]

Suppose now that \( t(u) > r_1[i, a] \). Then, we have \( r_1[i, a] < n \), which implies that

\[ r_1[i, a] = r[i, a] + \sum_{j < i} \alpha[j, b]. \]

Therefore, condition (1),

\[ |N_{H_1}(u) \cap S[i - 1]| \geq t(u) - r_1[i, a], \]

is equivalent to the condition

\[ |N_{H_1}(u) \cap S[i - 1]| \geq t(u) - r[i, a] - \sum_{j < i} \alpha[j, b] \]

which is in turn equivalent to

\[ |N_H(u) \cap S[i - 1]| \geq t(u) - \left( r[i, a] + \sum_{j < i} \alpha[j, b] \right) + \sum_{j < i} \alpha[j, b] \]

which is the same as condition (2),

\[ |N_H(u) \cap S[i - 1]| \geq t(u) - r[i, a]. \]

Putting the two cases together, we have

\[ (S[i] \setminus S[i - 1]) \cap V_a(H_1) = (S[i] \setminus S[i - 1]) \cap V_a(H) \]
\[ = \left\{ u \in V_a(H) \setminus S[i - 1] : |N_H(u) \cap S[i - 1]| \geq t(u) - r[i, a] \right\} \]
\[ = \left\{ u \in V_a(H_1) \setminus S[i - 1] : |N_{H_1}(u) \cap S[i - 1]| \geq t(u) - r_1[i, a] \right\}, \]

and \( \mathcal{S} \) is indeed an \((\alpha, r_1)\)-activation process for \( H_1 \).
Conversely, suppose that \((\alpha, r) \in A \times R\) is such that \(S = (S[0], \ldots, S[\lambda])\) is an \((\alpha, r)\)-activation process for \(H_1\), where

\[
\gamma_1[i, \ell] = \begin{cases} 
\min\{n, r[i, a] + \sum_{j<i} \alpha[j, b]\}, & \text{if } \ell = a; \\
\min\{n, r[i, b] + \sum_{j<i} \alpha[j, a]\}, & \text{if } \ell = b; \\
r[i, \ell], & \text{otherwise},
\end{cases}
\]

Reversing the argument above shows that \(S\) is an \((\alpha, r)\)-activation process for \(H\).

Hence, for every \((\alpha, r) \in A \times R\) we define the integer-valued matrix \(r_1\) by setting

\[
r_1[i, \ell] = \begin{cases} 
\min\{n, r[i, a] + \sum_{j<i} \alpha[j, b]\}, & \text{if } \ell = a; \\
\min\{n, r[i, b] + \sum_{j<i} \alpha[j, a]\}, & \text{if } \ell = b; \\
r[i, \ell], & \text{otherwise},
\end{cases}
\]

for every round \(i \in [0, \lambda]\) and for every label \(\ell \in [k]\). Then, we set, for all \((\alpha, r) \in A \times R\),

\[
\gamma_H(\alpha, r) = \gamma_H_1(\alpha, r_1).
\]

**Case 4:** \(v\) has exactly one child in \(T\) and the labeled subgraph \(H\) of \(G\) associated to \(v\) is of the form \(H = \rho_{a \rightarrow b}(H_1)\).

Suppose that \((\alpha, r) \in A \times R\) is such that there exists an \((\alpha, r)\)-activation process \(S = (S[0], \ldots, S[\lambda])\) for \(H\). For every round \(i \in [0, \lambda]\) and for every label \(\ell \in [k]\), set

\[
\alpha_1[i, \ell] = \begin{cases} 
|S[0] \cap V_\ell(H_1)|, & \text{if } i = 0; \\
|(S[i] \setminus S[i-1]) \cap V_\ell(H_1)|, & \text{otherwise};
\end{cases}
\]

and, for every round \(i \in [\lambda]\) and for every label \(\ell \in [k]\), set

\[
r_1[i, \ell] = \begin{cases} 
r[i, a], & \text{if } \ell = a; \\
r[i, \ell], & \text{otherwise}.
\end{cases}
\]

Then, \(S\) is an \((\alpha_1, r_1)\)-activation process for \(H_1\): Properties (2) and (3) follow immediately from the definition of \(\alpha_1\). To verify property (1), let \(i \in [\lambda]\) and \(\ell \in [k]\). If \(\ell \notin \{a, b\}\) then \(V_\ell(H_1) = V_\ell(H)\) and \(r_1[i, \ell] = r[i, \ell]\), hence the condition in property (3) holds in this case.

If \(\ell \in \{a, b\}\) then, since \(V_\ell(H_1) \subseteq V_\ell(H)\), we have

\[
(S[i] \setminus S[i-1]) \cap V_\ell(H_1) = \left((S[i] \setminus S[i-1]) \cap V_\ell(H)\right) \cap V_\ell(H_1)
\]

\[
= \left\{ u \in V_\ell(H) \setminus S[i-1] : |N_H(u) \cap S[i-1]| \geq t(u) - r[i, b] \right\} \cap V_\ell(H_1)
\]

\[
= \left\{ u \in V_\ell(H_1) \setminus S[i-1] : |N_{H_1}(u) \cap S[i-1]| \geq t(u) - r_1[i, \ell] \right\},
\]
so again the condition holds. Notice that the matrices $\alpha$ and $\alpha_1$ are related as follows: For every round $i \in [0, \lambda]$ and for every label $\ell \in [k]$, we have

$$
\alpha[i, \ell] = \begin{cases} 
0, & \text{if } \ell = a; \\
\alpha_1[i, a] + \alpha_1[i, b], & \text{if } \ell = b; \\
\alpha_1[i, \ell], & \text{otherwise.}
\end{cases}
$$

Conversely, suppose that $(\alpha, r) \in A \times R$ is such that there exists an $(\alpha_1, r_1)$-activation process $\bar{S} = (S[0], \ldots, S[\lambda])$ for $H_1$, where for every round $i \in [\lambda]$ and for every label $\ell \in [k]$, we have

$$
r_1[i, \ell] = \begin{cases} 
\bar{r}[i, b], & \text{if } \ell = a; \\
\bar{r}[i, \ell], & \text{otherwise.}
\end{cases}
$$

and for every $i \in [0, \lambda]$ and for every label $\ell \in [k]$, we have

$$
\alpha[i, \ell] = \begin{cases} 
0, & \text{if } \ell = a; \\
\alpha_1[i, a] + \alpha_1[i, b], & \text{if } \ell = b; \\
\alpha_1[i, \ell], & \text{otherwise.}
\end{cases}
$$

Then, it can be verified that $\bar{S}$ is an $(\alpha, r)$-activation process for $H$.

Hence, for every $(\alpha, r) \in A \times R$ we set $\gamma_H(\alpha, r) = 1$ if and only if there exists $(\alpha_1, r_1) \in A \times R$ such that $\gamma_{H_1}(\alpha_1, r_1) = 1$, where for every round $i \in [\lambda]$ and for every label $\ell \in [k]$, we have

$$
r_1[i, \ell] = \begin{cases} 
\bar{r}[i, b], & \text{if } \ell = a; \\
\bar{r}[i, \ell], & \text{otherwise.}
\end{cases}
$$

and for every $i \in [0, \lambda]$ and for every label $\ell \in [k]$, we have

$$
\alpha[i, \ell] = \begin{cases} 
0, & \text{if } \ell = a; \\
\alpha_1[i, a] + \alpha_1[i, b], & \text{if } \ell = b; \\
\alpha_1[i, \ell], & \text{otherwise.}
\end{cases}
$$

This completes the description of the four cases and with it the description of the algorithm.

**Correctness and time complexity.** Correctness of the algorithm follows from the derivation of the recursive formulas. We now analyze the algorithm’s time complexity. Given an irredundant $k$-expression $\sigma$ of $G$, the clique-width tree $T$ can be computed from $\sigma$ in linear time. The algorithm computes the sets $A$ and $R$ in time $|A| = O((n + 1)^{(\lambda+1)k})$ and $|R| = O((n + 1)^{\lambda k})$, respectively.

The algorithm then traverses the clique-width tree bottom-up. At each leaf of $T$ and for each $(\alpha, r) \in A \times R$, it can be verified in time $O(\lambda k)$ whether $(\alpha, r) \in (A \times R)^*$. Hence, the function $\gamma_H$ at each leaf can be computed in time $O(\lambda k(n + 1)^{(2\lambda+1)k})$.

At an internal node corresponding to Case 2, the value of $\gamma_H(\alpha, r)$ for a given $(\alpha, r) \in A \times R$ can be computed in time $O(|A|\lambda k)$ by iterating over all $\alpha_1 \in A$, verifying whether
\( \alpha_2 := \alpha - \alpha_1 \in \mathcal{A} \) and looking up the values of \( \gamma_{H_1}(\alpha_1, r) \) and \( \gamma_{H_2}(\alpha_2, r) \). Hence, the total time spent at an internal node corresponding to Case 2 is

\[
O(|\mathcal{A}| \lambda k) \cdot O((n + 1)^{(2 \lambda + 1)k}) = O(\lambda k(n + 1)^{(3 \lambda + 2)k}).
\]

At an internal node corresponding to Case 3 or Case 4, the value of \( \gamma_H(\alpha, r) \) for a given \( (\alpha, r) \in \mathcal{A} \times \mathcal{R} \) can be computed in time \( O(\lambda k) \). Hence, the total time spent at any such node is \( O(\lambda k(n + 1)^{(2 \lambda + 1)k}) \).

The overall time complexity is \( O(\lambda k|\sigma|(n + 1)^{(3 \lambda + 2)k}) \). For fixed \( k \) and \( \lambda \), this is polynomial in the size of the input.

Given the above algorithm for the \((\lambda, \beta, \alpha)\)-\textsc{Target Set Decision} problem on graphs of bounded clique-width, finding a set \( S \) that solves the \((\lambda, \beta, \alpha)\)-\textsc{Target Set Selection} problem can be done by standard backtracking techniques. We only need to extend the above algorithm so that at every node of the clique-width tree \( T \) (and the corresponding labeled subgraph \( H \) of \( G \)) and every \( (\alpha, r) \in \mathcal{A} \times \mathcal{R} \) such that \( \gamma_H(\alpha, r) = 1 \), the algorithm also keeps track of an \((\alpha, r)\)-activation process for \( H \). As shown in the above analysis of Cases 1–4, this can be computed in polynomial time using the recursively computed \((\alpha, r)\)-activation processes. Hence, we have the following theorem.

**Theorem 1.** For every fixed \( k \) and \( \lambda \), the \((\lambda, \beta, \alpha)\)-\textsc{Target Set Selection} problem can be solved in polynomial time on graphs of clique-width at most \( k \).

When \( \lambda = 1 \) and \( \alpha = |V(G)| \), the \((\lambda, \beta, \alpha)\)-\textsc{Target Set Selection} problem coincides with the \textsc{Vector Domination} problem (see, e.g., [3]). Hence, Theorem 1 answers a question from [3] regarding the complexity status of \textsc{Vector Domination} for graphs of bounded treewidth or bounded clique-width.

**The \((\lambda, \beta, A)\)-TSS problem on graphs of small clique-width.** The approach to solve the \((\lambda, \beta, A)\)-\textsc{Target Set Selection} problem on graphs of bounded clique-width is similar to the one above. First, we consider the decision problem naturally associated with the \((\lambda, \beta, A)\)-TSS problem, the \((\lambda, \beta, A)\)-\textsc{Target Set Decision} problem \((\lambda, \beta, A)\)-\textsc{TDS} for short). Consider an instance \((G, t, \lambda, \beta, A)\) to the \((\lambda, \beta, A)\)-\textsc{TSD} problem, where \( G = (V, E) \) is a graph of clique-width at most \( k \) given by an irredundant \( k \)-expression \( \sigma \). First, we construct a \( 2k \)-expression \( \sigma' \) in such a way that every labeled vertex \( a(u) \) with \( u \in A \) changes to \((a + k)(u)\). Moreover, every operation of the form \( \eta_{i,j} \) is replaced with a sequence of four composed operations \( \eta_{i,j} \circ \eta_{i,j+k} \circ \eta_{i+k,j} \circ \eta_{i+k,j+k} \), and every operation of the form \( \rho_{i,j} \circ \rho_{i+k,j+k} \). The so defined expression \( \sigma' \) can be obtained from \( \sigma \) in linear time, and defines a labeled graph isomorphic to \( G \) such that the set \( A \) contains precisely the vertices with labels strictly greater than \( k \). Using the same notation as above (with respect to \( \sigma' \)), we obtain the following

**Proposition 2.** There exists a set \( S \subseteq V(G) \) such that \( |S| \leq \beta \) and \( \text{Active}[S, \lambda] \supseteq A \) if and only if there exists a matrix \( \alpha \in \mathcal{A} \) with \( \gamma_G(\alpha, 0) = 1 \) such that \( \sum_{\ell=1}^{k} \alpha[0, \ell] \leq \beta \) and \( \sum_{i=0}^{2k} \sum_{\ell=k+1}^{\lambda} \alpha[i, \ell] = |A| \).
Hence, the same approach as above can be used to solve first the \((\lambda, \beta, A)\)-TARGET SET DECISION problem, and then the \((\lambda, \beta, A)\)-TARGET SET SELECTION problem itself.

**Theorem 2.** For every fixed \(k\) and \(\lambda\), the \((\lambda, \beta, A)\)-TARGET SET SELECTION problem can be solved in polynomial time on graphs of clique-width at most \(k\).

**Remark 1.** The dependency on \(\lambda\) and \(k\) in Theorems 1 and 2 is exponential. Since the Vector Dominating Set problem (a special case of \((\lambda, \beta, \alpha)\)-TARGET SET SELECTION problem) is \(W[1]\)-hard with respect to the parameter treewidth \([3]\), the exponential dependency on \(k\) is most likely unavoidable. We leave open the question whether the \((\lambda, \beta, \alpha)\)- and \((\lambda, \beta, A)\)-TARGET SET SELECTION problems are FPT (or even polynomial) with respect to parameter \(\lambda\) for graphs of bounded treewidth or clique-width.

### 3 \((\lambda, A)\)-TSS on Trees

Since trees are graphs of clique-width at most 3, results of Section 2 imply that the \((\lambda, \beta, \alpha)\)- and \((\lambda, \beta, A)\)-TSS problems are solvable in polynomial time on trees when \(\lambda\) is constant. In this section we improve on this latter result by giving a linear time algorithm for the \((\lambda, A)\)-TSS PROBLEM, for arbitrary values of \(\lambda\). Our result also extends the linear time solution for the classical TSS problem (i.e., without the latency bound) on trees proposed in [6]. Like the solution in [6], we will assume that the tree is rooted at some node \(r\). Then, once such rooting is fixed, for any node \(v\) we will denote by \(T(v)\) the subtree rooted at \(v\), by \(C(v)\) the set of children of \(v\) and, for \(v \neq r\), by \(p(v)\) the parent of \(v\).

In the following we assume that \(\forall v \in V, 1 \leq t(v) \leq d(v)\). The more general case (without these assumptions) can be handled with minor changes to the proposed algorithm.

The algorithm \((\lambda, A)\)-TSS on Trees on p. 15 considers each node for being included in the target set \(S\) in a bottom-up fashion. Each node is considered after all its children. Leaves are never added to \(S\) because there is always an optimal solution in which the target set consists of internal nodes only. Indeed, since all leaves have thresholds equal to 1, starting from any target set containing some leaves we can get a solution of at most the same size by substituting each targeted leaf by its parent.

Thereafter, for each non-leaf node \(v\), the algorithm checks whether the partial solution \(S\) constructed so far allows to activate all the nodes in \(T(v) \cap A\) (where \(A\) is the set of nodes which must be activated) within round \(\lambda\): the algorithm computes the round \(\tau = \lambda - \text{maxPath}(v)\) by which \(v\) has to be activated (line 12 of the pseudocode), where \(\text{maxPath}(v)\) denotes the maximum length of a path from \(v\) to one of its descendants which requires \(v\)'s influence to become active by round \(\lambda\). Notice that \(\tau < \lambda\) when there exists a vertex in the subtree \(T(v)\) which has to be activated by time \(\lambda\), and this can happen only if \(v\) is activated by time \(\tau\). Then the algorithm computes the set \(\text{Act}(v)\) consisting of those \(v\)'s children which are activated at round \(\tau - 1\) (line 13). The algorithm is based on the following three observations \((a), (b), \text{ and } (c)\) (assuming that \(v\) is in the set of nodes which must be activated):
(a) $v$ must be included in the target set solution $S$ whenever the nodes belonging to $Act(v) \cup \{p(v)\}$ do not suffice to activate $v$, i.e., the current partial solution is such that at most $t(v) - 2$ children of $v$ can be active at round $\tau - 1$.

(b) $v$ must be included in $S$ if $\tau = 0$ (i.e., $\lambda = \maxPath(v)$). Indeed, in this case, there exists a vertex in $T(v)$, at distance $\lambda$ from $v$, which requires $v$’s influence to be activated, and this can only happen if $v$ is activated at round 0.

These two cases for the activation of $v$ are taken care by lines 19-21 of the pseudocode. If neither (a) nor (b) is verified, then $v$ is not activated. However, it might be that the algorithm has to guarantee the activation of some other node in the subtree $T(v)$. To deal with such a case, when

(c) the size of the set $Act(v)$ is $t(v) - 1$, then the algorithm puts $p(v)$ in the set $A$ of nodes to be activated; moreover, the value of the parameter $path(v)$ is updated coherently in such a way to correctly compute the value of $\maxPath(p(v))$ which assures that $p(v)$ gets active within round $\lambda - \maxPath(p(v))$ (see lines 22-24).

For the root of the tree, which has no parent, case (c) is managed as case (a) (see lines 26-28).

In order to keep track of the above cases while traversing the tree bottom-up, the algorithm uses the following parameters:

- $round(v)$ assume value equal to the round (of the activation process with target set $S$) in which $v$ would be activated only thanks to its children and irrespectively of the status of its parent. Namely, $round(v) = \infty$ if $v$ is a leaf, $round(v) = 0$ if $v \in S$, and $round(v) = 1 + \min_{C(v)} \{\min_{u \in C(v)} \{round(u) \mid u \in C\}(v)\}$ otherwise. Here $\min_{C(v)}$ denotes the $t(v)$–th smallest element in the set $C$.

- $path(v)$ assume value equal to $-1$ in case $v$’s parent is not among the activators of $v$; otherwise, assume value equal to the maximum length of a path from $v$ to one of its descendants which (during the activation process with target set $S$) requires $v$’s influence in order to become active. It will be shown that during the activation process with target set $S$, for each node $v \in A$ we have $v \in \text{Active}[S, \min\{\lambda - \max_{u \in C(v)} \{path(u) - 1, round(v)\}\}]$, for each node $v \in A$. Moreover, the algorithm maintains a set $A' \supseteq A$ of nodes to be activated. Initially $A' = A$, the set $A'$ can be enlarged when the algorithm decides not to include in $S$ the node $v$ under consideration but to use $p(v)$ for $v$’s activation, like in the case (c) above.

In the rest of the section, we prove Theorem 3.

**Theorem 3.** Algorithm $(\lambda, A)$–**TSS on Tree** computes, in time $O(|V|)$, an optimal solution for the $(\lambda, A)$–**TARGET SET SELECTION problem on a tree.**

**Time complexity.** The initialization (line 1-10) requires time $O(|V|)$. The order in which nodes have to be considered is determined using a BFS which requires time $O(|V|)$ on a tree. The forall (line 11) considers all the internal nodes: the algorithm analyzes each internal node $v$ in time $O(|C(v)|)$. We notice that the computation in line 15 can be executed in $O(|C(v)|)$ by using an algorithm that solve the selection problem in linear time (see for instance [2]). Overall the complexity of the algorithm is $O(|V|) + \sum_{v \in V} O(|C(v)|) = O(|V|)$. 


Algorithm 1: $(\lambda, A)-TSS$ on Trees

**Input:** A tree $T = (V, E)$, thresholds function $v : V \rightarrow \mathbb{N}$, a latency bound $\lambda \in \mathbb{N}$ and a set to be activated $A \subseteq V$.

**Output:** $S \subseteq V$ of minimum size such that $A \subseteq \text{Active}[S, \lambda]$.

1. $S = \emptyset$;
2. $A' = A$;
3. Fix a root $r \in V$ // $T(r)$ denotes the tree $T$ rooted at $r$
4. for all $v$ in the set of $T(r)$ leaves do
   5. $\text{round}(v) = \infty$
   6. if $v \in A'$ then // $v$ belongs to the set of nodes to be activated
      7. $A' = A' \cup \{p(v)\}$ // $p(v)$ denotes $v$'s parent
      8. $\text{path}(v) = 0$
   else
   10. $\text{path}(v) = -1$
5. forall $v$ in the set of $T(r)$ internal nodes, listed in reverse order with respect to the time they are visited by a breadth-first traversal from $r$ do
12. $\text{maxPath}(v) = 1 + \max_{u \in C(v)} \text{path}(u)$ // $C(v)$ is the set of $v$'s children
13. $\text{Act}(v) = \{u \in C(v) \mid \text{round}(u) < \lambda - \text{maxPath}(v)\}$
14. $\text{path}(v) = -1$
15. $\text{round}(v) = 1 + \min_{t(v)} \{\text{round}(u) \mid u \in C(v)\}$
16. if $v \in A'$ then // $v$ has to be activated
   17. if $v \neq r$ then
      18. switch do
         19. case $(|\text{Act}(v)| \leq t(v) - 2) \ OR (\text{maxPath}(v) = \lambda)$
            20. $S = S \cup \{v\}$ // $v$ has to be in the target set
            21. $\text{round}(v) = 0$
         22. case $(|\text{Act}(v)| = t(v) - 1) \ AND (\text{maxPath}(v) < \lambda)$
            23. $A' = A' \cup \{p(v)\}$ // $v$ will be activated thanks to its parent $p(v)$
            24. $\text{path}(v) = \text{maxPath}(v)$
      25. else // $v$ is the root
         26. if $(|\text{Act}(v)| \leq t(v) - 1)$ then
            27. $S = S \cup \{v\}$
            28. $\text{round}(v) = 0$
      29. return $(S)$
Figure 1: An example of execution of the algorithm \((\lambda, A) - \text{TSS on Tree}\): (left) a subtree rooted in \(v_5\) (subscripts describe the order in which nodes are analyzed by the algorithm), each node is depicted as a circle and its threshold is given inside the circle. Circles having a solid border represent nodes in the set \(A\); (right) the first 6 steps of the algorithm are shown in the table; (bottom) the activation process is shown. Activated nodes are shaded. At round 0, \(S = \{v_1, v_5\}\).

**Correctness.** Consider the computed solution \(S\). Let \(\text{Active}[S, 0] = S\) and \(\text{Active}[S, i]\) be the sets of nodes which become active within the \(i\)-th round of the activation process.

**Lemma 1.** Algorithm \((\lambda, A) - \text{TSS on Tree}\) outputs a solution for the \((\lambda, A) - \text{TARGET SET SELECTION}\) problem on \(T = (V, E)\).

**Proof.** Given a node \(v \in V\), let \(a(v) = \min\{\lambda - \text{maxPath}(v), \text{round}(v)\}\); for a leaf node we assume \(\text{maxPath}(v) = 0\). We prove, by induction on \(a = 0, 1, \ldots\), that for each \(v \in A',\) s.t. \(a(v) = a\) we have

\[
v \in \text{Active}[S, a]
\]

For \(a = 0\), let \(v\) be a node such that \(a(v) = 0\). This implies that \(\text{round}(v) = 0\) or \(\lambda - \text{maxPath}(v) = 0\); therefore, \(v \in S = \text{Active}[S, 0]\).

Now fix \(a > 0\) and assume that \(w \in \text{Active}[S, a(w)]\) for any node \(w\) with \(a(w) \leq a - 1\). We will prove that \(v \in \text{Active}[S, a(v)]\) holds for any node \(v\) with \(a(v) = a\).

Let \(v\) be such that \(a(v) = a\). When \(v\) is processed, there are three possible cases:

- **CASE** \(v\) is added to the target set \(S\).
  - Actually, this case cannot occur under the standing hypothesis that \(a > 0\) since, if \(v \in S\) then \(v \in \text{Active}[S, 0]\) which would imply \(a(v) = 0 < a\).

- **CASE** \(|\text{Act}(v)| \geq t(v)|\).
  - We know that for each \(u \in \text{Act}(v)\) it holds \(\text{round}(u) < \lambda - \text{maxPath}(v)\).
In case \( a = \lambda - \maxPath(v) \), we have \( a(u) \leq \text{round}(u) \leq \lambda - \maxPath(v) - 1 = a - 1 \).

Analogously, if \( a = \text{round}(v) \). The algorithm poses \( \text{round}(v) \geq \text{round}(u) + 1 \) for each \( u \in \Act(v) \). Therefore, \( a(u) \leq \text{round}(u) \leq a - 1 \).

In both the above cases the inductive hypothesis applies to each \( u \in \Act(v) \), that is \( \Act(v) \subseteq \Act[S, a - 1] \). Since \( |\Act(v)| \geq t(v) \) we have \( v \in \Act[S, a] \).

- **CASE** \( |\Act(v)| = t(v) - 1, v \neq r \).
  In such a case the algorithm sets \( \text{round}(v) = 1 + \min t(v) \{ \text{round}(u) \mid u \in C(v) \} \) where \( \min t(v) C \) denotes the \( t(v) \)-th smallest element in the set \( C \). Since \( |\Act(v)| = t(v) - 1 \), we have that \( \text{round}(v) > \lambda - \maxPath(v) \), hence \( a(v) = \lambda - \maxPath(v) \).
  Recalling that for each \( u \in \Act(v) \) it holds \( \text{round}(u) < \lambda - \maxPath(v) \), as above we have that the inductive hypothesis applies to each \( u \in \Act(v) \), that is \( \Act(v) \subseteq \Act[S, a - 1] \).
  Consider now the parent \( p(v) \) of \( v \). The algorithm implies \( \maxPath(p(v)) \geq \maxPath(v) + 1 \). Hence \( \lambda - \maxPath(p(v)) \leq \lambda - \maxPath(v) - 1 = a - 1 \) and the inductive hypothesis applies also to \( p(v) \). Therefore, \( \{ p(v) \} \cup \Act(v) \) is a subset of size \( t(v) \) of \( \Act[S, a - 1] \) and \( v \in \Act[S, a] \).

We finally notice that \( a(v) = \min \{ \lambda - \maxPath(v), \text{round}(v) \} \leq \lambda \) for each \( v \in A' \). Indeed, the smallest possible value of \( \text{path}() \) is \(-1\), which implies that \( \maxPath(v) \geq 0 \) for any \( v \). \( \square \)

Let \( T(r) = (V, E) \) be a tree rooted at a \( r \in V \), and let \( X \subseteq V \) be a target set such that \( \Act[X, \lambda] \supseteq A \). Let \( T(v) \) be the subtree of \( T(r) \) rooted at a node \( v \). Henceforth let \( \Act[X, i, T(v)] \) be the set of nodes that is active at round \( i \) by targeting \( X \cap T(v) \) in the subtree \( T(v) \). Notice that while \( X \) is a target set for \( T(r) \) this not necessarily means that \( X \cap T(v) \) is a target set for \( T(v) \).

\[
\text{round}_X(v) = \begin{cases} 
  i & \text{if } v \in \Act[X, i, T(v)] \setminus \Act[X, i - 1, T(v)] \\
  0 & \text{if } v \in X \\
  \infty & \text{otherwise}
\end{cases}
\]

\[
\text{path}_X(v) = \begin{cases} 
  0 & \text{if } v \in A \text{ is a leaf AND } v \notin X \\
  i & \text{if } (\maxPath_X(v) = i < \lambda) \\
  & \text{AND } (|\Act[X, \lambda - i - 1, T(v)] \cap C(v)| = t(v) - 1) \\
  & \text{AND } (v \in A \text{ OR } \maxPath_X(v) > 0) \text{ AND } (v \notin X)
\end{cases}
\]

\[
\maxPath_X(v) = \begin{cases} 
  1 + \max_{u \in C(v)} \text{path}_X(u) & \text{if } v \text{ is an internal node } (C(v) \neq \emptyset) \\
  0 & \text{if } v \text{ is a leaf}.
\end{cases}
\]
When $X = S$ then the values $\text{round}(v)$ and $\text{path}(v)$, computed by the algorithm, correspond to the values defined above.

**Lemma 2.** If $X = S$ then for each node $v \in V$, $\text{round}(v) = \text{round}_S(v)$, $\text{path}(v) = \text{path}_S(v)$ and $\text{Act}(v) = \text{Active}[S, \lambda - \text{maxPath}_S(v) - 1, T(v)] \cap C(v)$.

**Proof.** First we show by induction on the height of $v$ that $\text{round}(v) = \text{round}_S(v)$.

**Induction Basis:** For each leaf $v$ we have $T(v) = v$. Since $v \notin S$, we have that $v \notin \text{Active}[S, j, v]$ for any value of $j$. Hence $\text{round}_S(v) = \text{round}(v) = \infty$ (line 5).

**Induction Step:** Let $v$ an internal node and suppose that the claim is true for any children of $v$. If $v \in S$ the claim is trivially true, $\text{round}_S(v) = 0 = \text{round}(v)$ (lines 21 and 28). Otherwise, let $\text{round}(v) = i = 1 + \min^t(v) \{ \text{round}(u) \mid u \in C(v) \}$ where $\min^t(v) C$ denotes the $t(v)$-th smallest element in the set $C$. By induction $i = 1 + \min^t(v) \{ \text{round}_S(u) \mid u \in C(v) \}$. There is a set $C_S(v) \subseteq C(v)$ such that $\mid C_S(v) \mid = t(v)$, $\forall w \in C_S(v)$, $\text{round}_S(w) \leq i - 1$ and $\exists u \in C_S(v)$ such that $\text{round}_S(u) = i - 1$. Hence, $\forall w \in C_S(v), v \in \text{Active}[S, i - 1, T(w)] \text{ and } \exists u \in C_S(v)$ such that $u \in \text{Active}[S, i - 1, T(u)] \setminus \text{Active}[S, i - 2, T(u)]$ and we have $v \in \text{Active}[S, i, T(v)] \setminus \text{Active}[S, i - 1, T(v)]$ which means that $\text{round}_S(v) = i$.

Now we show that $\text{path}(v) = \text{path}_S(v)$ and $\text{Act}(v) = \text{Active}[S, \lambda - \text{maxPath}_S(v) - 1, T(v)] \cap C(v)$. Again, we argue by induction on the height of $v$.

**Induction Basis.** For each leaf $v$, if $v \in A$ then $\text{path}_S(v) = \text{path}(v) = 0$ (line 8). On the other hand if $v \notin A$ then $\text{path}(v) = -1$ (line 10). Moreover, since $v \notin A$ and has no children $\text{path}_S(v) \neq i$. Hence $\text{path}_S(v) = -1$.

Moreover since $C(v) = \emptyset$ we have $\text{Act}(v) = \text{Active}[S, \lambda - \text{maxPath}_S(v) - 1, T(v)] \cap C(v) = \emptyset$.

**Induction Step.** Let $v \neq r$ be an internal node and suppose that the claim is true for any children of $v$. Hence, $\forall u \in C(v)$, $\text{path}(u) = \text{path}_S(u)$ and we have $\max\text{Path}_S(v) = 1 + \max_{u \in C(v)} \text{path}(u) = \max\text{Path}(v)$. Notice that $\max\text{Path}(v) = 1 + \max_{u \in C(v)} \text{path}(u) \geq 0$.

We are going to show that $\text{Act}(v) = \text{Active}[S, \lambda - \text{maxPath}_S(v) - 1, T(v)] \cap C(v)$.

Let $u \in \text{Act}(v)$. Hence $u \in C(v)$ and $\text{round}(u) < \lambda - \text{maxPath}(v)$. Since $\text{round}(u) = \text{round}_S(u)$ and by induction $\max\text{Path}(v) = \text{maxPath}_S(v)$ we have $\text{round}_S(u) \leq \lambda - \max\text{Path}(v) - 1$. Hence $u \in \text{Active}[S, \lambda - \max\text{Path}(v) - 1, T(u)]$ and therefore $u \in \text{Active}[S, \lambda - \max\text{Path}(v) - 1, T(v)] \cap C(v)$. Hence $\text{Act}(v) \subseteq \text{Active}[S, \lambda - \max\text{Path}(v) - 1, T(v)] \cap C(v)$.

Let $u \in \text{Active}[S, \lambda - \max\text{Path}(v) - 1, T(v)] \cap C(v)$. Hence $u \in C(v)$ and $\text{round}_S(u) \leq \lambda - \max\text{Path}(v) - 1$. Since $\text{round}(u) = \text{round}_S(u)$ and by induction $\max\text{Path}(v) = \text{maxPath}_S(v)$ we have $\text{round}(u) < \lambda - \max\text{Path}(v)$ and therefore $u \in \text{Act}(v)$. Hence $\text{Act}(v) \supseteq \text{Active}[S, \lambda - \max\text{Path}(v) - 1, T(v)] \cap C(v)$.

Since $v$ is an internal node, in order to show that $\text{path}(v) = \text{path}_S(v)$ two cases have to be considered: $\text{path}_S(u) = \max\text{Path}(v)$ or $\text{path}_S(u) = -1$. 

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case \((\text{path}(v) = \text{maxPath}(v))\): According to the algorithm this case happens when

(a) \(v \in A' \text{ AND} \)

(b) \(\text{maxPath}(v) < \lambda \text{ AND} \)

(c) \(|\text{Act}(v)| = t(v) - 1\)

Moreover, when this case occur \(v\) is not added to \(S\) (i.e., \(v \notin S\)). Thanks to (a) we have that \(v \in A'\) if either \(v \in A\) (line 2) or \(v\) has a children \(u\) such that \(\text{path}(u) = \text{maxPath}(u) \geq 0\) (line 23-24) or \(v\) has a children \(u\) such that \(u \in A\) and \(u\) is a leaf, that is \(\text{path}(u) = 0\) (line 7-8). Hence we have \(v \in A'\) iff \(v \in A\) OR \(\text{maxPath}_S(v) > 0\).

Thanks to (b) and (c) we have that \(\text{maxPath}_S(v) < \lambda\) and \(|\text{Act}[S, \lambda - \text{maxPath}(v) - 1, T(v)] \cap C(v)| = t(v) - 1\). Hence using (a), (b) and (c) and the fact that \(v \notin S\) we have \(\text{path}_S(v) = \text{maxPath}(v)\).

case \((\text{path}(v) = -1)\): In this case one of the above requirement is not satisfied and we have \(\text{path}_S(v) = -1\).

Similar reasoning can be used to show that \(\text{path}(r) = \text{path}_S(r)\) and \(\text{Act}(r) = \text{Act}[S, \lambda - \text{maxPath}_S(r) - 1, T(r)] \cap C(r)\).

Let \(X\) be a target set solutions (i.e., \(\text{Act}[X, \lambda] \supseteq A\)). For an edge \((v, u)\) we say that \(v\) activates \(u\) and write \(v \rightarrow u\) if \(v \in \text{Act}[X, i - 1]\) and \(u \in \text{Act}[X, i] \setminus \text{Act}[X, i - 1]\), for some \(1 \leq i \leq \lambda\). An activation path \(v \rightarrow u\) from \(v\) to \(u\) is a path in \(T\) such that \(v = x_0 \rightarrow x_1 \rightarrow \ldots \rightarrow x_k = u\) with \(x_j \in \text{Act}[X, i_j] \setminus \text{Act}[X, i_j - 1]\) for \(0 \leq i_1 < i_2 < \ldots < i_k \leq \lambda\). In other words \(x_i\) is activated before \(x_{i+1}\), for \(i = 0, \ldots, k - 1\).

Lemma 3. Let \(X\) be a target set solutions (i.e., \(\text{Act}[X, \lambda] \supseteq A\) and \(v \in V\). If \(\text{maxPath}_X(v) = i\) then there is an activation path of length \(i\) in \(T(v)\) starting at \(v\) and ending at a node \(u \in A\).

Proof. Since \(\text{maxPath}_X(v) = i\) then there is a path in \(T(v)\) from \(v\) to a node \(u\) such that \(v = x_i \rightarrow x_{i-1} \rightarrow \ldots \rightarrow x_0 = u\) where for each \(i = 0, 1, \ldots, i - 1\), \(\text{path}_X(x_i) = i\). We are able to show by induction that for each \(j = 0, 1, \ldots, i - 1\), \(x_j\) is activated after \(x_{j+1}\).

Induction basis: \(j = 0\). Hence, \(\text{path}_X(x_0) = 0\) which means that \(x_0 \notin X\). There are two case to consider:

\((x_0\text{ is a leaf})\) hence \(x_0 \in A\). Moreover since \(x_0\) has no children we have that \(x_0\) will be activated after its parent \(x_1\).

\((x_0\text{ is an internal node})\) since \(\text{path}_X(x_0) = 0\) we have that \(\text{maxPath}_X(x_0) = 0\) hence \(x_0 \in A\). Moreover, since \(|\text{Act}[X, \lambda - 1, T(x_0)] \cap C(x_0)| = t(x_0) - 1\) we have that \(x_0\) will be activated after its parent \(x_1\). Otherwise \(x_0\) will not be activated by round \(\lambda - 1\).
**Induction step:** $j = i$. Hence, $\text{path}_X(v_j) = \max \text{Path}_X(x_j) = j$ which means that $x_i \notin X$. Moreover, by induction, we know that $\forall j < i$, $x_j$ is activated after $x_{j+1}$. Hence in order to activate $x_0$ by round $\lambda$, $x_j$ has to be activated by round $\lambda - j - 1$. Since $|\text{Active}[X, \lambda - j - 1, T(x_j)] \cap C(x_j)| = t(x_j) - 1$ we have that $x_j$ will be activated after its parent $x_{j+1}$. Otherwise $x_j$ will not be activated by round $\lambda - j - 1$. \hfill \Box

Let $X$ and $Y$ be two target set solutions and $v \in V$. The following properties hold:

**Property 1.** If $\text{round}_X(v) > \text{round}_Y(v)$ and $v \notin Y$ then there exists $u \in C(v)$ such that $\text{round}_X(u) > \text{round}_Y(u)$.

**Proof.** Let $\text{round}_Y(v) = i$ we have that $v \in \text{Active}[Y, i, T(v)] \setminus \text{Active}[Y, i-1, T(v)]$. Hence, there is a set $C_Y \subseteq C(v)$ such that $|C_Y| = t(v)$ and $\forall u \in C_Y, u \in \text{Active}[Y, i-1, T(v)]$ (i.e., $\forall u \in C_Y, \text{round}_Y(u) \leq i - 1$). On the other hand, since $\text{round}_X(v) > i$ the size of the set $C_X = \{u \in C(v) | \text{round}_X(u) \leq i - 1\}$ is at most $t(v) - 1$. Hence there is at least a vertex $u \in C(v)$ such that $\text{round}_X(u) > \text{round}_Y(u)$. \hfill \Box

**Property 2.** If $v \notin X$ then $\max \text{Path}_X(v) < \lambda$ AND $|\text{Active}[X, \lambda - \max \text{Path}_X(v) - 1, T(v)] \cap C(v)| > t(v) - 2$.

**Proof.** In the following we show that if either $\max \text{Path}_X(v) \geq \lambda$ or $|\text{Active}[X, \lambda - \max \text{Path}_X(v) - 1, T(v)] \cap C(v)| \leq t(v) - 2$ then $v \in X$.

When $\max \text{Path}_X(v) \geq \lambda$ then by Lemma 3 there is an activation path of length at least $\lambda$ in $T(v)$ starting at $v$ and ending at a node $u \in A$ and we have that $v$ has to be active at round 0. When $|\text{Active}[X, \lambda - \max \text{Path}_X(v) - 1, T(v)] \cap C(v)| \leq t(v) - 2$, since by Lemma 3 there is an activation path of length $\max \text{Path}_X(v)$ starting at $v$ and ending at a node $u \in A$, we have that $v$ has to be active at round $\lambda - \max \text{Path}_X(v)$ (i.e., $v$ should belong to $\text{Active}[X, \lambda - \max \text{Path}_X(v)]$). Since $|\text{Active}[X, \lambda - \max \text{Path}_X(v) - 1, T(v)] \cap C(v)| \leq t(v) - 2$ then $v$ will not be activated (even considering its parent) at round $\lambda - \max \text{Path}_X(v)$. Hence $v$ has to be in $X$. \hfill \Box

**Lemma 4.** Algorithm $(\lambda, A) - \text{TSS on Tree}$ outputs an optimal solution for the $(\lambda, A) - \text{TGSET SELECTION problem on } T = (V, E)$.

**Proof.** Let $S$ and $O$ be respectively the solutions found by the Algorithm $(\lambda, A) - \text{TSS on Tree}$ and an optimal solution. For each $v \in V$ let $S(v) = S \cap T(v)$ (resp. $O(v) = O \cap T(v)$) be the set of target nodes in $S$ (resp. $O$) which belong to $T(v)$. Let $s(u) = |S(u)|$ and $o(u) = |O(u)|$ be the cardinality of such sets. We will use the following claim.

**Claim 1.** For any vertex $v \in V$, if $\text{path}_S(v) > \text{path}_O(v)$ OR $\text{round}_S(v) > \text{round}_O(v)$ then $s(v) < o(v)$.

**Proof.** We argue by induction on the height of $v$. The claims trivially hold when $v$ is a leaf. Since our algorithm does not target any leaf (i.e. $v \notin S$), two cases need to be analyzed:
case ($v \in O$): then $s(v) = 0 < 1 = o(v)$ and the inequality is satisfied.

case ($v \notin O$): then $\text{round}_S(v) = \text{round}_O(v) = \infty$ and $\text{path}_S(v) = \text{path}_O(v) = 0$ or $-1$ depending whether $v \in A$ or not. Hence none of the two conditions of the if are satisfied then the claim holds.

Now consider any internal vertex $v \in V$. By induction, we have that $\forall u \in C(v), s(u) \leq o(u)$, hence
\[
\sum_{u \in C(v)} s(u) \leq \sum_{u \in C(v)} o(u).
\] (4)

There are four cases to consider:

case ($v \notin S$ and $v \in O$): Using eq. (4) we have $s(v) \leq o(v) - 1 < o(v)$, hence the claim holds.

case ($v \in S$ and $v \in O$): We have $\text{path}_S(v) = \text{path}_O(v) = -1$ and $\text{round}_S(v) = \text{round}_O(v) = 0$. Hence none of the two conditions of the if statement are satisfied and the claim holds.

case ($v \notin S$ and $v \notin O$): if $\text{path}_S(v) > \text{path}_O(v)$ OR $\text{round}_S(v) > \text{round}_O(v)$ then in order to prove the claim we need to find a child $u$ of $v$ such that $s(u) < o(u)$.

In the following we analyze the two cases separately:

if $\text{round}_S(v) > \text{round}_O(v)$ then using Property 1 we have that there is a vertex $u \in C(v)$ such that $\text{round}_S(u) > \text{round}_O(u)$. By induction on the height of $v$ we have that $s(u) < o(u)$.

if $\text{path}_S(v) > \text{path}_O(v)$ then $\text{path}_S(v) \neq -1$, hence by definition of $\text{path}_X(\cdot)$,
\[
\text{maxPath}_S(v) = i < \lambda
\] (5)
\[
\text{AND} \ |\text{Active}[S, \lambda - i - 1, T(v)] \cap C(v)| = t(v) - 1
\] (6)
\[
\text{AND} \ (v \in A \text{ OR } \text{maxPath}_S(v) > 0)
\] (7)

There are two cases to consider:

case ($\text{path}_O(v) \geq 0$): then $\text{path}_O(v) = \text{maxPath}_O(v) < \text{maxPath}_S(v)$ and there is a child $u$ of $v$ such that $\text{path}_S(u) = \text{maxPath}_S(v) - 1 > \text{maxPath}_O(v) - 1 \geq \text{path}_O(u)$ and we have found the desired vertex because by induction we have $s(u) < o(u)$.

case ($\text{path}_O(v) = -1$): Since $v \notin O$ and by Property 2 we know that $\text{maxPath}_O(v) < \lambda$ AND $|\text{Active}[O, \lambda - \text{maxPath}_O(v) - 1, T(v)] \cap C(v)| > t(v) - 2$. Hence $\text{path}_O(v) = -1$ can happen only for two reasons:
case \((v \notin A \text{ and } \maxPath_O(v) = 0)\): Since \(v \notin A\), then by eq. 7 there is a child \(u\) of \(v\) such that \(\path_S(u) \geq 0 > -1 = \path_O(u)\) and we have found the desired vertex because by induction it holds that \(s(u) < o(u)\).

case \(|\text{Active}[O, \lambda - \maxPath_O(O) - 1, T(v)] \cap C(v)| \geq t(v)\): Hence by equation 6 we have

\[|\text{Active}[O, \lambda - j - 1, T(v)] \cap C(v)| > |\text{Active}[S, \lambda - i - 1, T(v)] \cap C(v)|\]

where \(i = \maxPath_S(v)\) and \(j = \maxPath_O(v)\). If \(i = \maxPath_S(v) > \maxPath_O(v) = j\) (i.e., \(1 + \max_{u \in C(v)} \path_S(u) > 1 + \max_{u \in C(v)} \path_O(u)\)) then there is a child \(u\) of \(v\) such that \(\path_S(u) > \path_O(u)\) and we have found the desired vertex because by induction we have \(s(u) < o(u)\). On the other hand if \(i = \maxPath_S(v) \leq \maxPath_O(v) = j\), then there exists a child \(u \in C(v)\) such that, \(u \in \text{Active}[O, \lambda - j - 1, T(u)] \setminus \text{Active}[O, \lambda - j - 2, T(u)]\) and \(u \notin \text{Active}[S, \lambda - i - 1, T(u)]\). Hence \(\round_O(u) = \lambda - j - 1 \leq \lambda - i - 1 < \round_S(u)\). By induction we have that \(s(u) < o(u)\).

In all the cases above we are able to find the desired vertex and the claim holds.

case \((v \in S \text{ and } v \notin O)\): Using eq. 4 we have \(s(v) - 1 \leq o(v)\). For each \(u \in C(v)\) we know by induction that \(s(u) \leq o(u)\). Since \(v \in S\) we have \(\path_S(v) = -1\) and \(\round_S(v) = 0\), hence none of the two requirement of the if is satisfied hence the claim holds true.

We show by induction on the height of the node that \(s(v) \leq o(v)\), for each \(v \in V\).

The inequality trivially holds when \(v\) is a leaf. Since our algorithm does not target any leaf (i.e. \(v \notin S\)), we have \(s(v) = 0\). Since we have \(o(v) = 0\) or \(o(v) = 1\) according to whether \(v\) belongs to the optimal solution, the inequality is always satisfied.

Now consider any internal node \(v\). By induction, \(s(u) \leq o(u)\) for each \(u \in C(v)\); hence

\[
\sum_{u \in C(v)} s(u) \leq \sum_{u \in C(v)} o(u).
\]

It is not hard to see that if \(v \in O\) by 8 we immediately have \(s(v) \leq o(v)\). The same result follows from 8 for the case where \(v\) is neither in \(O\) nor in \(S\). We are left with the case \(v \in S\) and \(v \notin O\) In this case eq. 8 only gives \(s(v) - 1 \leq o(v)\). In order to obtain the desired result we need to find a child \(u\) of \(v\) such that \(s(u) < o(u)\). We distinguish the following two cases:

case \((v \notin A \text{ and } \maxPath_O(v) = 0)\): Since \(v \in S\) we have \(v \in A'\) (that is \(v \in A\) or \(\maxPath_S(v) > 0\)). Since \(v \in A' \setminus A\) then \(\maxPath_S(v) > 0\) and there is a children \(u\) of
v such that \(\text{path}_S(u) \geq 0 > -1 = \text{path}_O(u) = -1\) and we have found the desired vertex because by the Claim above we have \(s(u) < o(u)\).

**Case** \((v \in A \text{ or } \max\text{Path}_O(v) > 0):\) Since \(v \in S\) we have that either \(\max\text{Path}_S(v) = \lambda\) or \(|\text{Act}(v)| = |\text{Act}[S, \lambda - i - 1, T(v)] \cap C(v)| \leq t(v) - 2\) where \(i = \max\text{Path}_S(v)\). We consider the two subcases separately:

- \((\max\text{Path}_S(v) = \lambda):\) Since \(v \notin O\), by Property 2 we have that \(\max\text{Path}_O(v) < \lambda\). Hence, there is a vertex \(u \in C(v)\) such that \(\text{path}_S(u) = \lambda - 1 > \text{path}_O(u)\) and we have found the desired vertex because, by the Claim we have \(o(u) > s(u)\).

- \(|\text{Act}(v)| = |\text{Act}[S, \lambda - \max\text{Path}_S(v) - 1, T(v)] \cap C(v)| \leq t(v) - 2):\) Since \(v \notin O\) by Property 2 we have that \(|\text{Act}[O, \lambda - \max\text{Path}_O(v) - 1, T(v)] \cap C(v)| > t(v) - 2\). Hence, \(|\text{Act}[O, \lambda - j - 1, T(v)] \cap C(v)| > |\text{Act}[S, \lambda - i - 1, T(v)] \cap C(v)|\) where \(i = \max\text{Path}_S(v)\) and \(j = \max\text{Path}_O(v)\). If \(i = \max\text{Path}_S(v) > \max\text{Path}_O(v)\) (i.e., \(1 + \max_{u \in C(v)} \text{path}_S(u) > 1 + \max_{u \in C(v)} \text{path}_O(u)\)) then there is a child \(v\) such that \(\text{path}_S(u) > \text{path}_O(u)\) and we have found the desired vertex because, by the Claim we have \(o(u) > s(u)\). On the other hand if \(i = j\), then there exists a child \(u \in C(v)\) such that, \(u \in \text{Act}[O, \lambda - j - 1, T(u)] \setminus \text{Act}[O, \lambda - j - 2, T(u)]\) and \(u \notin \text{Act}[S, \lambda - i - 1, T(u)]\). Hence \(\text{round}_O(u) = \lambda - j - 1 \leq \lambda - i - 1 < \text{round}_S(u)\). By the Claim we have \(o(u) > s(u)\).

In all cases we have that there is \(u \in C(v)\) with \(s(u) < o(u)\). Hence \(s(v) \leq o(v)\). \(\square\)

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