Abstract. We investigate various ways to define an analogue of BGG category $O$ for the non-semi-simple Takiff extension of the Lie algebra $\mathfrak{sl}_2$. We describe Gabriel quivers for blocks of these analogues of category $O$ and prove extension fullness of one of them in the category of all modules.

1. Introduction and description of the results

The celebrated BGG category $O$, introduced in [BGG], is originally associated to a triangular decomposition of a semi-simple finite dimensional complex Lie algebra. The definition of $O$ is naturally generalized to all Lie algebras admitting some analogue of a triangular decomposition, see [MP]. These include, in particular, Kac-Moody algebras and Virasoro algebra. Category $O$ has a number of spectacular properties and applications to various areas of mathematics, see for example [Hu] and references therein.

The paper [DLMZ] took some first steps in trying to understand structure and properties of an analogue of category $O$ in the case of a non-reductive finite dimensional Lie algebra. The investigation in [DLMZ] focuses on category $O$ for the so-called Schrödinger algebra, which is a central extension of the semi-direct product of $\mathfrak{sl}_2$ with its natural 2-dimensional module. It turned out that, for the Schrödinger algebra, the behavior of blocks of category $O$ with non-zero central charge is exactly the same as the behavior of blocks of category $O$ for the algebra $\mathfrak{sl}_2$. At the same, block with zero central charge turned out to be significantly more difficult. For example, it was shown in [DLMZ] that some blocks of $O$ for the Schrödinger algebra have wild representation type, while all blocks of $O$ for $\mathfrak{sl}_2$ have finite representation type.

In the present paper we look at a different non-reductive extension of the algebra $\mathfrak{sl}_2$, namely, the corresponding Takiff Lie algebra $\mathfrak{g}$ defined as the semi-direct product of $\mathfrak{sl}_2$ with the adjoint representation. Such Lie algebras were defined and studied by Takiff in [Ta] with the primary interest coming from invariant theory. Alternatively, the Takiff Lie algebra $\mathfrak{g}$ can be described as the tensor product $\mathfrak{sl}_2 \otimes_C (\mathbb{C}[x]/(x^2))$. The latter suggests an obvious generalization of the notion of a triangular decomposition for $\mathfrak{g}$ by tensoring the components of a triangular decomposition for $\mathfrak{sl}_2$ with $\mathbb{C}[x]/(x^2)$.

Having defined a triangular decomposition for $\mathfrak{g}$, we can define Verma modules and try to guess a definition for category $O$. The latter turned out to be a subtle task as the most obvious definition of category $O$ does not really work as expected, in particular, it does not contain Verma modules. This forced us to investigate two alternative definitions of category $O$:

- the first one analogous to the definition of the so-called thick category $O$, see, for example, [Söd], in which the action of the Cartan subalgebra is only expected to be locally finite and not necessarily semi-simple as in the classical definition;
• and the second one given by the full subcategory of the thick category $\mathcal{O}$ from the first definition with the additional requirement that the Cartan subalgebra of $\mathfrak{sl}_2$ acts diagonalizably.

The results of this paper fall into the following three categories:

• We describe the linkage between simple objects in both our versions of category $\mathcal{O}$ and in this way explicitly determine all (indecomposable) blocks, see Theorem 11.

• We determine the Gabriel quivers for all blocks, see Corollaries 18, 21, 23, 28.

• We prove that thick category $\mathcal{O}$ is extension full in the category of all $\mathfrak{g}$-modules, see Theorem 6.

For some of the blocks, we also obtain not only a Gabriel quiver, but also a fairly explicit description of the whole block, see Theorem 17. Some of the results are unexpected and look rather surprising. For example, the trivial $\mathfrak{g}$-module exhibits behavior different from the behavior of all other simple finite dimensional $\mathfrak{g}$-modules, compare Lemma 25 and Proposition 26.

The paper is organized as follows: All preliminaries are collected in Section 2, in particular, in this section we define all main protagonists of the paper and describe their basic properties. In Section 3 we prove extension fullness of thick category $\mathcal{O}$ in the category of all $\mathfrak{g}$-modules. Section 4 is devoted to the study of the decomposition of both $\mathcal{O}$ and its thick version $\tilde{\mathcal{O}}$ into indecomposable blocks. As usual, generic Verma modules over $\mathfrak{g}$ are simple. In Section 5 we describe the structure of those Verma modules that are not simple. Finally, in Section 6 we compute first extensions between simple highest weight modules and in this way determine the Gabriel quivers of all block in $\mathcal{O}$ and $\tilde{\mathcal{O}}$.

This paper is a revision, correction and extension of the master thesis [Sod] of the second author written under the supervision of the first author.

2. Takiff $\mathfrak{sl}_2$ and its modules

2.1. Takiff $\mathfrak{sl}_2$. In this paper we work over the field $\mathbb{C}$ of complex numbers. Consider the Lie algebra $\mathfrak{sl}_2$ with the standard basis $\{e, h, f\}$ and the Lie bracket

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$ 

Let $D := \mathbb{C}[x]/(x^2)$ be the algebra of dual numbers. Consider the associated Takiff Lie algebra $\mathfrak{g} = \mathfrak{sl}_2 \otimes_{\mathbb{C}} D$ with the Lie bracket

$$[a \otimes x^i, b \otimes x^j] = [a, b] \otimes x^{i+j},$$

where $a, b \in \mathfrak{sl}_2$ and $i, j \in \{0, 1\}$ with the Lie bracket on the right hand side being the $\mathfrak{sl}_2$-Lie bracket. Set

$$\tau := e \otimes x, \quad \overline{f} := f \otimes x, \quad \overline{h} := h \otimes x.$$ 

Consider the standard triangular decomposition

$$\mathfrak{sl}_2 = n_- \oplus \mathfrak{h} \oplus n_+,$$

where $n_-$ is generated by $f$, $\mathfrak{h}$ is generated by $h$ and $n_+$ is generated by $e$. Let $\mathfrak{n}_-$ be the subalgebra of $\mathfrak{g}$ generated by $e$ and $\tau$, $\overline{\mathfrak{n}}_+$ the subalgebra of $\mathfrak{g}$ generated by $h$ and $\overline{h}$, and $\mathfrak{n}_+$ be the subalgebra of $\mathfrak{g}$ generated by $f$ and $\overline{f}$. The we have the following triangular decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{n}_- \oplus \overline{\mathfrak{h}} \oplus \mathfrak{n}_+.$$
We set $b := \mathfrak{h} \oplus n_+$ and $
abla b := \mathfrak{h} \oplus \mathfrak{n}_+$. For a Lie algebra $a$, we denote by $U(a)$ the corresponding universal enveloping algebra.

The natural projection $\mathfrak{g} \twoheadrightarrow \mathfrak{sl}_2$ induced an inclusion of $\mathfrak{sl}_2$-Mod to $\mathfrak{g}$-Mod, which we denote by $\iota$.

By a direct calculation, it is easy to check that the Casimir element
\begin{equation}
    c := h^2 + 2n + 2f + 2fe
\end{equation}
belongs to the center of $U(\mathfrak{g})$, see Example 1.2 in [Ma].

2.2. (Generalized) weight modules. A $g$-module $M$ is called a generalized weight module provided that the action of $U(\mathfrak{h})$ on $M$ is locally finite. As $U(\mathfrak{h})$ is just the polynomial algebra in $\mathfrak{h}$ and $\mathfrak{g}$, every generalized weight module $M$ admits a decomposition
\begin{equation}
    M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda,
\end{equation}
where $M^\lambda$ denotes the set of all vectors in $M$ killed by some power of the maximal ideal $m_{\lambda}$ of $U(\mathfrak{h})$ corresponding to $\lambda$. We will say that a generalized weight module $M$ is a weight module provided that the action of $h$ on $M$ is diagonalizable. We will say that a weight module $M$ is a strong weight module provided that the action of $\mathfrak{h}$ on $M$ is diagonalizable.

Note that submodules, quotients and extensions of generalized weight modules are generalized weight modules. Also submodules and quotients of (strong) weight modules are (strong) weight modules. From the commutation relations in $\mathfrak{g}$, for any generalized weight modules $M$ and any $\lambda \in \mathfrak{h}^*$, we have
\begin{equation}
    \mathfrak{h}M^\lambda \subset M^\lambda, \quad \mathfrak{p}_+ M^\lambda \subset M^{\lambda + \alpha}, \quad \mathfrak{p}_- M^\lambda \subset M^{\lambda - \alpha},
\end{equation}
where $\alpha \in \mathfrak{h}^*$ is given by $\alpha(h) = 2, \alpha(\mathfrak{h}) = 0$.

If $M$ is a generalized weight module, then the set of all $\lambda$ for which $M^\lambda \neq 0$ is called the support of $M$ and denoted $\text{supp}(M)$.

2.3. Verma modules. For a fixed $\lambda \in \mathfrak{h}^*$, we have the corresponding simple $U(\mathfrak{h})$-module $\mathbb{C}_\lambda := U(\mathfrak{h})/m_{\lambda}$. Setting $\mathfrak{p}_+ \mathbb{C}_\lambda = 0$ defines on $\mathbb{C}_\lambda$ the structure of a $\mathfrak{b}$-module. The $g$-module
\begin{equation}
    \Delta(\lambda) := \text{Ind}_{U(\mathfrak{h})}^{U(\mathfrak{g})} (\mathbb{C}_\lambda) \cong U(\mathfrak{g}) \bigotimes_{U(\mathfrak{h})} \mathbb{C}_\lambda
\end{equation}
is called the Verma module associated to $\lambda$. The standard argument, see Proposition 7.1.11 in [D], shows that $\Delta(\lambda)$ has a unique simple quotient. We denote this simple quotient of $\Delta(\lambda)$ by $L(\lambda)$. From the PBW Theorem and formula (2), it follows that
\begin{equation}
    \text{supp}(\Delta(\lambda)) = \{\lambda - n\alpha : n \in \mathbb{Z}_{\geq 0}\}.
\end{equation}
In fact, from the PBW Theorem, it follows that, for $n \in \mathbb{Z}_{\geq 0}$, we have
\begin{equation}
    \dim (\Delta(\lambda)^{\lambda - n\alpha}) = n + 1
\end{equation}
as the elements $\{f \mathfrak{f}^{i-1} v_\lambda : i = 0, 1, \ldots, n\}$, where $v_\lambda$ denotes the canonical generator of $\Delta(\lambda)$, form a basis of $\Delta(\lambda)^{\lambda - n\alpha}$. The weight $\lambda$ is the highest weight of $\Delta(\lambda)$.

The following simplicity criterion for $\Delta(\lambda)$ can be deduced from the main result of [Wi], however, we include a short proof for the sake of completeness.
Proposition 1. The module $\Delta(\lambda)$ is simple if and only if $\lambda(\mathfrak{h}) \neq 0$.

Proof. Let $v_\lambda$ be the canonical generator of $\Delta(\lambda)$. Assume first that $\lambda(\mathfrak{h}) = 0$ and consider the element $w = \sum v_\lambda$. Then we have $e w = \sum w = 0$ and hence, from the PBW Theorem and (2), it follows that the submodule $N$ in $\Delta(\lambda)$ generated by $w$ satisfies $N^\lambda = 0$ and thus is a non-zero proper submodule. Therefore $\Delta(\lambda)$ is reducible in this case.

Now assume that $\lambda(\mathfrak{h}) \neq 0$. We need to show that any non-zero submodule $N$ of $\Delta(\lambda)$ contains $v_\lambda$. If $N^\lambda = 0$, then the fact that $v_\lambda \in N$ is clear. Assume now that $N^\lambda = 0$ and let $n \in \mathbb{Z}_{>0}$ be minimal such that $N^{\lambda - n\alpha} \neq 0$. Let $w \in N^{\lambda - n\alpha}$ be a non-zero element. Using the PBW Theorem, we may write

$$w = \sum_{i=0}^n c_i f_i v_\lambda.$$ 

Denote the maximal value of $i$ such that $c_i \neq 0$ by $k$. Then it is easy to check that $(\mathfrak{h} - \lambda(\mathfrak{h}))^k w$ equals $f_i v_\lambda$ up to a non-zero scalar. In particular, $N$ contains $f_i v_\lambda$. But then it is easy to check that $e f_i v_\lambda$ equals $f_i^{n-1} v_\lambda$, up to a non-zero scalar. In particular, $N^{\lambda - n\alpha - \alpha} \neq 0$. The obtained contradiction proves that $N^\lambda \neq 0$ and the claim of the proposition follows. \(\square\)

For $\mu \in \mathfrak{h}^*$, we denote by $\Delta^{sl_2}(\mu)$ the $sl_2$-Verma module with highest weight $\mu$ and by $L^{sl_2}(\mu)$ the unique simple quotient of $\Delta^{sl_2}(\mu)$. From Proposition $\|$ we obtain the following corollary.

Corollary 2. For $\lambda \in \mathfrak{h}^*$, we have

$$L(\lambda) \cong \begin{cases} \Delta(\lambda), & \text{if } \lambda(\mathfrak{h}) \neq 0; \\ \iota(L^{sl_2}(\lambda|_{\mathfrak{h}})), & \text{if } \lambda(\mathfrak{h}) = 0. \end{cases}$$

Proof. If $\lambda(\mathfrak{h}) \neq 0$, then the claim is just a part of Proposition $\|$. If $\lambda(\mathfrak{h}) = 0$, then the unique up to scalar non-zero vector in $\iota(L^{sl_2}(\lambda|_{\mathfrak{h}}))^\lambda$ generates a $\mathfrak{h}$-submodule of $\iota(L^{sl_2}(\lambda|_{\mathfrak{h}}))$ isomorphic to $\mathbb{C}_\lambda$. By adjunction, we obtain a non-zero homomorphism from $\Delta(\lambda)$ to $\iota(L^{sl_2}(\lambda|_{\mathfrak{h}}))$ which must be surjective as the latter module is simple. Consequently, $\iota(L^{sl_2}(\lambda|_{\mathfrak{h}}))$ must be isomorphic to $L(\lambda)$ by the definition of $L(\lambda)$. \(\square\)

2.4. (Thick) category $\mathcal{O}$. We define thick category $\mathcal{O}$, denoted $\tilde{\mathcal{O}}$, as the full subcategory of the category of all finitely generated $\mathfrak{g}$-modules consisting of all $\mathfrak{g}$-modules the action of $U(\mathfrak{b})$ on which is locally finite. Note that, by definition, all modules in $\mathcal{O}$ are generalized weight modules.

We define classical category $\mathcal{O}$, denoted $\mathcal{O}$, as the full subcategory of $\tilde{\mathcal{O}}$ consisting of all weight modules. Finally, we define strong category $\tilde{\mathcal{O}}$, denoted $\tilde{\mathcal{O}}$, as the full subcategory of $\mathcal{O}$ consisting of all strong weight modules.

As $U(\mathfrak{g})$ is noetherian, the categories $\tilde{\mathcal{O}}$, $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are abelian categories closed under taking submodules, quotients and finite direct sums. Directly from the definition, it also follows that $\tilde{\mathcal{O}}$ is closed under taking extensions, in particular, $\tilde{\mathcal{O}}$ is a Serre subcategory of $\mathfrak{g}$-mod.

Proposition 3. For each $\lambda \in \mathfrak{h}^*$, the module $\Delta(\lambda)$ belongs to both $\tilde{\mathcal{O}}$ and $\mathcal{O}$. However, $\Delta(\lambda)$ does not belong to $\tilde{\mathcal{O}}$. 
Proof. That \( \Delta(\lambda) \in \tilde{O} \) follows from [4]. That \( \Delta(\lambda) \in O \) follows by combining the fact that \( \Delta(\lambda) \in \tilde{O} \) and that the adjoint action of \( h \) on \( U(g) \) is diagonalizable (implying that the action of \( h \) on \( \Delta(\lambda) \) is diagonalizable).

That \( \Delta(\lambda) \not\in \mathcal{O} \) follows from the fact that the matrix of the action of \( \mathfrak{h} \) in the basis \( \{ f v_\lambda, f \bar{v}_\lambda \} \) of \( \Delta(\lambda)^{-\alpha} \), where \( v_\lambda \) is the canonical generator of \( \Delta(\lambda) \), has the form
\[
\begin{pmatrix}
\lambda(\mathfrak{h}) & 0 \\
-2 & \lambda(\mathfrak{h})
\end{pmatrix}
\]
and hence is not diagonalizable. \( \Box \)

Proposition 4.
(a) The set \( \{ L(\lambda) : \lambda \in \mathfrak{h} \} \) is a complete and irredundant list of representatives of isomorphism classes of simple objects in \( \tilde{O} \).
(b) The set \( \{ L(\lambda) : \lambda \in \mathfrak{h} \} \) is a complete and irredundant list of representatives of isomorphism classes of simple objects in \( O \).
(c) The set \( \{ L^{\mathfrak{e}}(\mu) : \mu \in \mathfrak{h}^* \} \) is a complete and irredundant list of representatives of isomorphism classes of simple objects in \( \mathcal{O} \).

Proof. Let \( L \) be a simple module in \( \tilde{O} \) and \( v \) a non-zero element in \( L \). Since the vector space \( U(\mathfrak{h})v \) is finite dimensional, it contains a non-zero element \( w \) such that \( \Pi\mathfrak{h}\lambda w = 0, b \mathfrak{h}w = \lambda(h)w \) and \( \Pi\mathfrak{h}w = \lambda(\mathfrak{h})w \), for some \( \lambda \in \mathfrak{h} \). Then \( \mathfrak{c}w \), is isomorphic, as a \( \mathfrak{h} \)-module, to \( \mathfrak{c}_\lambda \). By Proposition 3 we have \( \Delta(\lambda) \in \tilde{O} \). By adjunction, we obtain a non-zero homomorphism from \( \Delta(\lambda) \) to \( L \). This implies \( L \cong L(\lambda) \) and proves claim (a).

To prove claim (b), we can use claim (a) and hence just need to check, for which \( \lambda \in \mathfrak{h} \), the module \( L(\lambda) \) belongs to \( \mathcal{O} \). If \( \lambda(\mathfrak{h}) \neq 0 \), then \( L(\lambda) = \Delta(\lambda) \) by Corollary 2 and hence \( L(\lambda) \notin \mathcal{O} \) by Proposition 3. If \( \lambda(\mathfrak{h}) = 0 \), then \( L(\lambda) = \iota(L^{\mathfrak{e}}(\lambda)_{\mathfrak{h}}) \) by Corollary 2 and \( \iota(L^{\mathfrak{e}}(\lambda)_{\mathfrak{h}}) \notin \mathcal{O} \) since the action of \( \mathfrak{h} \) on \( \iota(L^{\mathfrak{e}}(\lambda)_{\mathfrak{h}}) \) is zero and thus diagonalizable. This completes the proof. \( \Box \)

Proposition 4 has the following consequence.

Corollary 5. The functor \( \iota \) induces an equivalence between the category \( \mathcal{O} \) for \( \mathfrak{sl}_2 \) and the category \( \tilde{O} \).

Proof. By construction, the functor \( \iota \) is full and faithful and maps the category \( \mathcal{O} \) for \( \mathfrak{sl}_2 \) to the category \( \tilde{O} \). Hence, what we need to prove is that this restriction of \( \iota \) is dense. By Proposition 4(c), \( \iota \) hits all isomorphism classes of simple objects in \( \tilde{O} \). In particular, \( \mathfrak{h} \) annihilates all simple objects in \( \mathcal{O} \). Since, by the definition of \( \mathcal{O} \), the action of \( \mathfrak{h} \) on any object in \( \mathcal{O} \) is semi-simple, it follows that \( \mathfrak{h} \) annihilates all objects in \( \mathcal{O} \).

Since the ideal of \( \mathfrak{g} \) generated by \( \mathfrak{h} \) contains both \( \mathfrak{r} \) and \( \mathfrak{t} \), it follows that the latter two elements annihilate all object in \( \mathcal{O} \). This yields that every object in \( \mathcal{O} \) is, in fact, isomorphic to an object in the image of \( \iota \). The claim follows. \( \Box \)

Due to Corollary 5, the category \( \mathcal{O} \) is fairly well-understood, see e.g. [Ma] for a very detailed description. Therefore, in what follows, we focus on studying the categories \( \tilde{O} \) and \( O \).
3. Extension fullness of $\tilde{\mathcal{O}}$ in $g$-Mod

The inclusion functor $\Phi : \tilde{\mathcal{O}} \hookrightarrow g$-Mod is exact and hence induces, for each $M, N \in \tilde{\mathcal{O}}$ and $i \geq 0$, homomorphisms

$$\varphi_{M,N}^{(i)} : \text{Ext}^i_{\tilde{\mathcal{O}}}(M, N) \to \text{Ext}^i_{g\text{-Mod}}(M, N)$$

of abelian groups. As $\tilde{\mathcal{O}}$ is a full subcategory of $g$-Mod, all $\varphi_{M,N}^{(0)}$ are isomorphisms. As $\tilde{\mathcal{O}}$ is a Serre subcategory of $g$-Mod, all $\varphi_{M,N}^{(1)}$ are isomorphisms. The main result of this section is the following statement.

**Theorem 6.** The category $\tilde{\mathcal{O}}$ is extension full in $g$-Mod in the sense that all $\varphi_{M,N}^{(i)}$ are isomorphisms.

Theorem 6 is a generalization of Theorem 16 in [CM2] to our setup. We refer the reader to [CM1] and [CM2] for more details on extension full subcategories.

**Proof.** We follow the proof of Theorem 2 in [CM1]. Denote by $\hat{\tilde{\mathcal{O}}}$ the full subcategory of $g$-Mod consisting of all modules, the action of $U(\hat{b})$ on which is locally finite. The difference between $\hat{\tilde{\mathcal{O}}}$ and $\tilde{\mathcal{O}}$ is that, in the case of $\hat{\tilde{\mathcal{O}}}$, we drop the condition on modules to be finitely generated.

First we note that $\tilde{\mathcal{O}}$ is extension full in $\hat{\tilde{\mathcal{O}}}$. Indeed, if $M \in \hat{\tilde{\mathcal{O}}}$, $N \in \tilde{\mathcal{O}}$ and $\alpha : M \to N$ is a surjective homomorphism, we can use that $N$ is finitely generated to claim that $N$ is in the image of a finitely generated submodule of $M$. Therefore the fact that $\tilde{\mathcal{O}}$ is extension full in $\hat{\tilde{\mathcal{O}}}$ follows from Proposition 3 in [CM1] (applied in the situation $B = \hat{\tilde{\mathcal{O}}}$ and $A = \tilde{\mathcal{O}}$).

To complete the proof of the theorem, it remains to prove that $\hat{\tilde{\mathcal{O}}}$ is extension full in $g$-Mod. For a locally finite dimensional $U(\hat{b})$-module $V$, denote by $M(V)$ the induced module $\text{Ind}_{U(\hat{b})}(M(V))$. Note that, by Theorem 6 in [SM], the action of $U(\hat{b})$ on $M(V)$ is locally finite. The same computation as in the proof of Lemma 3 in [CM1] shows that, for any $V$ as above, any $N \in \hat{\tilde{\mathcal{O}}}$ and any $i \geq 0$, the natural map

$$\text{Ext}^i_{\hat{\tilde{\mathcal{O}}}}(M(V), N) \to \text{Ext}^i_{g\text{-Mod}}(M(V), N)$$

is an isomorphism. Therefore the extension fullness of $\hat{\tilde{\mathcal{O}}}$ in $g$-Mod follows from Proposition 1 in [CM1] (applied in the situation $A = g$-Mod, $B = \hat{\tilde{\mathcal{O}}}$ and $B_0$ consisting of modules of the form $M(V)$, for $V$ as above). This completes the proof. $\square$

4. Description of blocks

4.1. Characters and composition multiplicities.

**Lemma 7.** Let $\mathcal{X} \in \{O, \tilde{\mathcal{O}}\}$ and $M \in \mathcal{X}$. There exist $k \in \mathbb{Z}_{>0}$ and $\lambda_1, \lambda_2, \ldots, \lambda_k \in \hat{\mathfrak{h}}^*$ such that

$$\text{supp}(M) \subset \bigcup_{i=1}^{k} \{\lambda_i - \mathbb{Z}_{\geq 0}\alpha\},$$

moreover, for each $\mu \in \text{supp}(M)$, the space $M^\mu$ is finite dimensional.
Proof. If two modules $M_1$ and $M_2$ have the properties described in the formulation of the lemma, then any extension of $M_1$ and $M_2$ also has similar properties. By definition, $M$ is finitely generated, and hence, taking the first sentence into account, without loss of generality we may assume that $M$ is generated by one element $v \in M'$, for some $\nu \in \tilde{\mathfrak{n}}$.

The vector space $U(\tilde{\mathfrak{n}})v$ is finite dimensional and $\tilde{\mathfrak{n}}$-stable. Hence the $\tilde{\mathfrak{n}}$-module $U(\tilde{\mathfrak{n}})v$ has finite support, say $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$. By the PBW Theorem, we have the decomposition $U(\mathfrak{g}) = U(\tilde{\mathfrak{n}})U(\mathfrak{f})$. Hence $M = U(\tilde{\mathfrak{n}})(U(\tilde{\mathfrak{n}})v)$, implying Formula (5). Moreover, since, considered as an adjoint $\tilde{\mathfrak{n}}$-module, all generalized weight spaces of $U(\tilde{\mathfrak{n}})$ are finite dimensional, it follows that all $M^\mu$ are finite dimensional. This completes the proof. \qed

For a finite subset $\mu \subset \tilde{\mathfrak{n}}$, set

$$\overline{\mu} = \bigcup_{\nu \in \mu} \{\mu - Z_{\geq 0} \alpha\}.$$

We write $\mu \preceq \nu$ provided that $\overline{\mu} \subset \overline{\nu}$.

Consider the set $F$ of all functions $\chi : \tilde{\mathfrak{n}} \to \mathbb{Z}_{\geq 0}$ having the property that the support $\{\lambda \in \tilde{\mathfrak{n}} : \chi(\lambda) \neq 0\}$ of $\chi$ belongs to $\overline{\mu}$, for some $\mu$ as above. The set $F$ has the natural structure of an additive monoid with respect to the pointwise addition of functions. The neutral element of this monoid is the zero function.

Let $X \in \{O, \tilde{O}\}$. Given $M \in X$, we define the character $ch(M)$ as the function from $\tilde{\mathfrak{n}}$ to $\mathbb{Z}_{\geq 0}$ sending $\lambda$ to $\dim(M^\lambda)$. By Lemma 7, we have $ch(M) \in F$. Clearly, characters are additive on short exact sequences, that is, for any short exact sequence $0 \to K \to M \to N \to O$ in $X$, we have $ch(M) = ch(K) + ch(N)$.

Proposition 8. Let $X \in \{O, \tilde{O}\}$.

(a) For any $M \in X$, there are uniquely determined $k_\lambda(M) \in \mathbb{Z}_{\geq 0}$, where $\lambda \in \tilde{\mathfrak{n}}$, such that

$$ch(M) = \sum_{\lambda \in \tilde{\mathfrak{n}}} k_\lambda(M) ch(L(\lambda)).$$

(b) For every $\lambda \in \tilde{\mathfrak{n}}$, the function $k_\lambda : Ob(X) \to \mathbb{Z}_{\geq 0}$ has the following properties:

(i) $k_\lambda(L(\lambda)) = 1$;

(ii) $k_\lambda(L(\mu)) = 0$, if $\mu \neq \lambda$;

(iii) $k_\lambda(M) = 0$, if $\lambda \notin supp(M)$;

(iv) $k_\lambda(M)$ is additive on short exact sequences.

Proof. Clearly, $k_\lambda(M) = 0$ if $\lambda \notin supp(M)$, and hence the sum in (a) can be taken over $supp(M)$ instead of the whole $\tilde{\mathfrak{n}}$.

Assume first that, for $M \in X$, we have

$$ch(M) = \sum_{\lambda \in supp(M)} a_\lambda ch(L(\lambda)) = \sum_{\lambda \in supp(M)} b_\lambda ch(L(\lambda)),$$

where all $a_\lambda$ and $b_\lambda$ are in $\mathbb{Z}_{\geq 0}$. Assume that there is some $\lambda$ such that $a_\lambda \neq b_\lambda$. Let $X := \{\lambda : a_\lambda > b_\lambda\}$ and $Y := supp(M) \setminus X$. Then we have

$$\chi := \sum_{\lambda \in X} (a_\lambda - b_\lambda) ch(L(\lambda)) = \sum_{\mu \in Y} (b_\mu - a_\mu) ch(L(\mu)).$$
By our assumptions, $\chi \in \Gamma$ is non-zero. Then there exists $\nu \in \overline{\mathbb{N}}$ such that $\chi(\nu) \neq 0$ but $\chi(\nu + ma) = 0$, for all $m \in \mathbb{Z}_{>0}$. If $\nu \in X$, then $\nu \notin Y$ and from the property $\chi(\nu + ma) = 0$, for all $m \in \mathbb{Z}_{>0}$, we see that $\chi(\nu) \neq 0$ is not possible if we compute $\chi$ using the second expression. Similarly, if $\nu \in Y$, then $\nu \notin X$ and we get that $\chi(\nu) \neq 0$ is not possible if we compute $\chi$ using the first expression. The obtained contradiction shows that, if a decomposition of the form as in (3) exists, then it is unique.

Let us now prove existence of (2). If $M = 0$, we set $k_\lambda(M) = 0$, for all $\lambda$. Let $M \in X$ be non-zero and $\lambda \in \text{supp}(M)$ be such that $\lambda + ma \notin \text{supp}(M)$, for all $m \in \mathbb{Z}_{>0}$. Let $v \in M^\lambda$ be a non-zero element which is an eigenvector for both $h$ and $\overline{h}$ and set $K := U_q v \subset M$ and $N := M/K$. By adjunction, there is a non-zero epimorphism from $\Delta(\lambda)$ to $K$ sending the canonical generator of $\Delta(\lambda)$ to $v$. Let $K'$ denote the image, under this epimorphism, of the unique maximal submodule of $\Delta(\lambda)$. By construction, we have two short exact sequences:

$$0 \to K' \to K \to L(\lambda) \to 0 \quad \text{and} \quad 0 \to K \to M \to N \to 0.$$  

For each $\mu \in \text{supp}(M)$, define

$$(6) \quad k_\mu(M) = \begin{cases} k_\mu(K') + k_\mu(N), & \text{if } \mu \neq \lambda; \\ k_\mu(K') + k_\mu(N) + 1, & \text{if } \mu = \lambda. \end{cases}$$

Note that $\text{supp}(N) \subset \text{supp}(M)$ and $\dim(N^\lambda) < \dim(M^\lambda)$, moreover, we also have $\text{supp}(K') \subset \text{supp}(M)$ and $\lambda \notin \text{supp}(K')$. Therefore, thanks to Lemma 7, Formula (6) gives an iterative procedure which, after a finite number of iterations, completely determines $k_\mu(M)$ such that (3) holds by construction.

It remains to check that $k_\mu(M)$ defined above have all the properties listed in (3). Properties (i)-(vii) follow directly from the definition in the previous paragraph. Property (viii) follows from the equality in (3) and the fact that characters are additive on short exact sequences.

The number $k_\mu(M)$ will be called the composition multiplicity of $L(\mu)$ in $M$.

4.2. Some first extensions between simple objects.

**Proposition 9.** Let $\lambda, \mu \in \overline{\mathbb{N}}$ be such that $\lambda \neq \mu$ and $\lambda(\overline{\lambda}) \neq 0$. Then, for any $X \in \{\mathcal{O}, \mathcal{O}^\vee\}$, we have

$$\text{Ext}^1_X(L(\lambda), L(\mu)) = \text{Ext}^1_X(L(\mu), L(\lambda)) = 0.$$  

**Proof.** We prove that $\text{Ext}^1_X(L(\lambda), L(\mu)) = 0$, the second claim is similar. Assume that (7) is a short exact sequence in $X$. Note that

$$\text{supp}(M) = \text{supp}(L(\lambda)) \bigcup \text{supp}(L(\mu)) \subset \{\lambda - Z_{>0} \alpha\} \bigcup \{\mu - Z_{>0} \alpha\}$$

by (2). If $\mu \notin \lambda + Z\alpha$, then $M^{\lambda+\alpha} = 0$ and hence $\pi^+M^\lambda = 0$. By adjunction, this gives as a non-zero homomorphism from $\Delta(\lambda) = L(\lambda)$ to $M$ which splits (7). This implies the necessary claim in case $\mu \notin \lambda + Z\alpha$.

If $\mu \in \lambda + Z\alpha$, then $\mu(\overline{\lambda}) = \lambda(\overline{\mu}) \neq 0$. By applying the Casimir element $c$, see (11), to the highest weight elements in $L(\lambda)$ and $L(\mu)$, we see that $c$ acts as the scalar $\lambda(\overline{\lambda})(\lambda(h)+2)$ on $L(\lambda)$ and as the scalar $\mu(\overline{\mu})(\mu(h)+2)$ on $L(\mu)$. As $\mu(\overline{\lambda}) = \lambda(\overline{\mu}) \neq 0$ but $\mu \neq \lambda$, we obtain

$$\lambda(\overline{\lambda})(\lambda(h)+2) \neq \mu(\overline{\lambda})(\mu(h)+2).$$

This means that $L(\lambda)$ and $L(\mu)$ have different central characters and hence (7) splits. The claim of the proposition follows.

Following the proof of Proposition 9 we also obtain the following claim.

**Corollary 10.** Let $\lambda, \mu \in \bar{\mathbb{h}}^+$ be such that $\lambda \notin \mu + \mathbb{Z}\alpha$. Then, for any $X \in \{O, \tilde{O}\}$, we have

$$\text{Ext}^1_X(L(\lambda), L(\mu)) = \text{Ext}^1_X(L(\mu), L(\lambda)) = 0.$$  

**4.3. Easy blocks.** Let $X \in \{O, \tilde{O}\}$. Set

$$\bar{h}_1 := \{\lambda \in \bar{h} : \lambda(\bar{h}) \neq 0\}, \quad \bar{h}_0 := \bar{h} \setminus \bar{h}_1.$$

For $\lambda \in \bar{h}_1$, denote by $X(\lambda)$ the full subcategory of $X$ consisting of all modules with support $\{\lambda - \mathbb{Z}_{\geq 0}\alpha\}$.

**4.4. Difficult blocks.** For $\xi \in \bar{h}_0/\mathbb{Z}\alpha$, denote by $X(\xi)$ the full subcategory of $X$ consisting of all modules whose support is contained in $\xi$.

**4.5. Block decomposition.**

**Theorem 11.** For $X \in \{O, \tilde{O}\}$, we have a decomposition

$$X = \bigoplus_{\lambda \in \bar{h}_1} X(\lambda) \oplus \bigoplus_{\xi \in \bar{h}_0/\mathbb{Z}\alpha} X(\xi)$$

of $X$ into a direct sum of indecomposable abelian subcategories (blocks).

**Proof.** Let $M \in X$ be an indecomposable module. Then there is $\lambda \in \bar{h}$ such that $\text{supp}(M) \subset \xi := \lambda + \mathbb{Z}\alpha$. If $\lambda(\bar{h}) = 0$, then, by definition, $M \subset X(\xi)$. If $\lambda(\bar{h}) \neq 0$, then, by Proposition 9 all composition subquotients of $M$ are isomorphic to some $L(\mu)$, where $\mu \in \xi$. Therefore $M \in X(\mu)$. This implies existence of the direct sum decomposition as in (8).

It remains to prove that all summands in the right hand side of (8) are indecomposable. That each $X(\lambda)$, where $\lambda \in \bar{h}_1$, is indecomposable, is clear as $X(\lambda)$ contains, by construction, only one simple module, up to isomorphism.

Let us argue that each $X(\xi)$, where $\xi \in \bar{h}_0/\mathbb{Z}\alpha$, is indecomposable. For this it is enough to show that, for every $\lambda \in \xi$, there is an indecomposable module $M \in X(\xi)$ such that both $k_\lambda(M)$ and $k_{\lambda - \alpha}(M)$ are non-zero. Take $M = \Delta(\lambda)$. The module $\Delta(\lambda)$ is indecomposable as it has simple top. Moreover, $k_\lambda(\Delta(\lambda)) \neq 0$. Since $\lambda(\bar{h}) = 0$, we have

$$e_f v_\lambda = e_f v_\lambda = 0 \quad \text{and} \quad h_f v_\lambda = (\lambda - \alpha)(h) f v_\lambda.$$

Therefore, by adjunction, mapping $v_{\lambda - \alpha}$ to $Tv_\lambda$, extends to a non-zero homomorphism from $\Delta(\lambda - \alpha)$ to $\Delta(\lambda)$, implying that $k_{\lambda - \alpha}(\Delta(\lambda)) \neq 0$. The claim follows. \qed

**5. Structure of non-simple Verma modules**

**5.1. Easy case.**

**Proposition 12.** Assume that $\lambda \in \bar{h}^+$ is such that $\lambda(\bar{h}) = 0$ and $\lambda(\mu) \notin \mathbb{Z}_{\geq 0}$. Then there is a short exact sequence

$$0 \to \Delta(\lambda - \alpha) \to \Delta(\lambda) \to L(\lambda) \to 0.$$
Proof. Let $v_\lambda$ be the canonical generator of $\Delta(\lambda)$. From $\lambda(\mathfrak{h}) = 0$, it follows that $e^j v_\lambda = f^j v_\lambda = \mathbb{H} f v_\lambda = 0$ and $h^j v_\lambda = (\lambda - \alpha)(h) v_\lambda$. Hence, by adjunction, there is a non-zero homomorphism $\Delta(\lambda - \alpha) \to \Delta(\lambda)$ sending $v_{\lambda - \alpha}$ to $v_\lambda$. By the PBW Theorem, this homomorphism is injective and the quotient $\Delta(\lambda)/\Delta(\lambda - \alpha)$ has a basis of the form \{ $f^j v_\lambda : i \in \mathbb{Z}_{\geq 0}$ \}.

Up to a positive integer, $e^j f^i v_\lambda$ is a multiple of $v_\lambda$ with the coefficient $\prod_{j=0}^{i-1} (\lambda(h) - j)$. As $\lambda(h) \notin \mathbb{Z}_{\geq 0}$, we obtain that the quotient $\Delta(\lambda)/\Delta(\lambda - \alpha)$ is a simple module and hence is isomorphic to $L(\lambda)$. The claim follows.

\[ \text{Corollary 13.} \quad \text{Assume that } \lambda \in \mathfrak{n}^* \text{ is such that } \lambda(\mathfrak{h}) = 0 \text{ and } \lambda(h) \notin \mathbb{Z}_{\geq 0}. \]

(a) Then there is a filtration

\[ \cdots \subset \Delta(\lambda - 2 \alpha) \subset \Delta(\lambda - \alpha) \subset \Delta(\lambda). \]

Moreover, all subquotients in this filtration are simple modules and we also have

\[ \bigcap_{i \in \mathbb{Z}_{\geq 0}} \Delta(\lambda - i \alpha) = 0. \]

(b) The filtration given by (23) is the unique composition series of $\Delta(\lambda)$, in other words, $\Delta(\lambda)$ is a uniserial module.

Note that, under the assumptions of Corollary 13, the module $\Delta(\lambda)$ has infinite length. This emphasizes the difference with the classical $\mathfrak{sl}_2$-situation, see Subsection 3.2 in [Ma] for the latter.

Proof. Existence of such filtration and the claim that all subquotients in this filtration are simple follows directly from Proposition 12. The claim that $\bigcap_{i \in \mathbb{Z}_{\geq 0}} \Delta(\lambda - i \alpha) = 0$ follows from the fact that $\bigcap_{i \in \mathbb{Z}_{\geq 0}} \supp(\Delta(\lambda - i \alpha)) = \emptyset$, which, in turn, is a consequence of (23). This proves claim (a).

To prove claim (b) we only need to show that any non-zero submodule $M$ of $\Delta(\lambda)$ has the form $\Delta(\lambda - i \alpha)$, for some $i$. Choose $i$ such that $\lambda - i \alpha$ is the highest weight of $M$. From (23) it follows that any simple subquotient of $\Delta(\lambda)/\Delta(\lambda - i \alpha)$ has a weight of the form $\lambda - j \alpha$, where $j < i$. Therefore $M \subset \Delta(\lambda - i \alpha)$. That $M = \Delta(\lambda - i \alpha)$ follows from the fact that $\Delta(\lambda - i \alpha)$ is generated by its highest weight vector. This proves claim (b) and completes the proof of the corollary.

\[ \text{5.2. Difficult case.} \]

\[ \text{Lemma 14.} \quad \text{Assume that } \lambda \in \mathfrak{n}^* \text{ is such that } \lambda(\mathfrak{h}) = 0 \text{ and } \lambda(h) = n \in \mathbb{Z}_{\geq 0}. \text{ Then there are short exact sequences} \]

\[ 0 \to \Delta(\lambda - \alpha) \to \Delta(\lambda) \to M \to 0 \]

and

\[ 0 \to L(\lambda - (n + 1) \alpha) \to M \to L(\lambda) \to 0. \]

Proof. Similarly to Proposition 12, the vector $v_\lambda$ generates a submodule of $\Delta(\lambda)$ isomorphic to $\Delta(\lambda - \alpha)$, giving the exact sequence (9), with $M = \Delta(\lambda)/\Delta(\lambda - \alpha)$. The module $M$ is isomorphic to a Verma module for $\mathfrak{sl}_2$ and has simple subquotients as described in (10), see Theorem 3.16 in [Ma].
Lemma 15. Assume that $\lambda \in \mathfrak{h}^*$ is such that $\lambda(\mathfrak{k}) = 0$ and $\lambda(h) = n \in \mathbb{Z}_{\geq 0}$. Then the element $f^{n+1}v_\lambda$ generates a submodule $K(\lambda)$ of $\Delta(\lambda)$ such that the module $M_n := \Delta(\lambda)/K(\lambda)$ is uniserial and has a filtration

$$0 = X_k \subset \cdots \subset X_1 \subset X_0 = M_n,$$

where $k = \left\lceil \frac{n+1}{2} \right\rceil$ and $X_i/X_{i+1} \cong L(\lambda - i\alpha)$, for $i = 0, 1, \ldots, k - 1$.

Proof. We prove this statement by induction on $n$. The induction step moves $\lambda - \alpha$ to $\lambda$ and hence changes $n - 2$ to $n$. Therefore we have two different cases for the basis of the induction.

Case 1: $n = 0$. In this case $efv_\lambda = \tau f v_\lambda = 0$ and $h f v_\lambda = \overline{v} v_\lambda$. From Lemma 14 we thus get $M_0 \cong L(\lambda)$.

Case 2: $n = 1$. In this case $ef^2v_\lambda = 0$ and $ef^2v_\lambda = 2\overline{v} v_\lambda$. Again, from Lemma 14 we thus get $M_1 \cong L(\lambda)$.

Let us now prove the induction step. Consider $\Delta(\lambda - \alpha)$ as a submodule of $\Delta(\lambda)$ generated by $\overline{f} v_\lambda$. We claim that $K(\lambda) \cap \Delta(\lambda - \alpha) = K(\lambda - \alpha)$. Indeed, we have $\overline{f}^{n+1}v_\lambda = -n(n+1)f^{n-1}f v_\lambda$, and thus $K(\lambda - \alpha) \subset K(\lambda) \cap \Delta(\lambda - \alpha)$. To prove the reverse inclusion, let us analyze the result of applying a monomial $\overline{f}^e f^n \overline{h}^q e^q h^b \in U(g)$ to $f^{n+1}v_\lambda$. As $f^{n+1}v_\lambda$ is a weight element, setting $b = 0$ changes the outcome by a scalar. As $e \cdot f^{n+1}v_\lambda = 0$ by the $\mathfrak{sl}_2$-theory, we may assume $q = 0$. If $x = a = p = 0$, then the elements $f^x, f^{n+1}v_\lambda$ are linearly independent and do not belong to $\Delta(\lambda - \alpha)$. Therefore, if a linear combination of elements of the form

$$\overline{f}^e f^n \overline{h}^q e^q h^b \cdot f^{n+1}v_\lambda$$

is in $\Delta(\lambda - \alpha)$, then at least one of $y, a$ or $q$ must be non-zero. Commuting the corresponding overlined basis element to the right and using $\overline{v} v_\lambda = \overline{h} v_\lambda = 0$, one shows that our linear combination ends up in $K(\lambda - \alpha)$.

Consider now the following diagram:

(12) $\xymatrix{ M_{n-2} \ar[r] & \cdots \ar[r] & M_n \ar[r] & \text{Coker}_M \ar[d] \\
\Delta(\lambda - \alpha) \ar[u] \ar[r] & \Delta(\lambda) \ar[r] & \text{Coker}_\Delta \ar[u] \\
K(\lambda - \alpha) \ar[u] \ar[r] & K(\lambda) \ar[r] & \text{Coker}_K \ar[u] }$

The solid part of this diagram consists of natural inclusions. By the previous paragraph, this solid part is, in fact, a commutative pullback diagram. The vertical dashed arrows are natural projections. The horizontal dashed arrow is induced by the solid part such that the dashed box commutes. The horizontal dashed arrow is injective since the solid part is a pullback. The dotted part of the diagram is given by the Snake Lemma and the whole diagram (12) commutes. By the Snake Lemma, all rows and all columns of (12) are short exact sequences.

From the Second Isomorphism Theorem and definitions, we have

$$\text{Coker}_M \cong \Delta(\lambda)/(K(\lambda) + \Delta(\lambda - \alpha)) \cong L(\lambda).$$

Therefore, the upper row of (12) gives a short exact sequence

$$0 \to M_{n-2} \to M_n \to L(\lambda) \to 0.$$
As $L(\lambda)$ is a unique simple top of $\Delta(\lambda)$, the module $L(\lambda)$ also must be a unique simple top of $M_n$. Now all necessary claims follow by induction. \hfill \Box

As an immediate consequence of the above, we obtain:

**Corollary 16.** Assume that $\lambda \in \overline{\mathfrak{h}}$ is such that $\lambda(\overline{h}) = 0$ and $\lambda(h) = n \in \mathbb{Z}_{>0}$. The Hasse diagram of the partially ordered, by inclusion, set of submodules of $\Delta(\lambda)$ of the form $\Delta(\lambda - i\alpha)$ and $K_i$ is as follows (here $k = \lceil \frac{n-1}{2} \rceil$):

\[
\begin{array}{c}
\Delta(\lambda) \\
K_n \\
\Delta(\lambda - \alpha) \\
K_{n-2} \\
\vdots \\
\Delta(\lambda - k\alpha) \\
K_{n-2k} \\
\Delta(\lambda - (k+1)\alpha) \\
\Delta(\lambda - (k+2)\alpha) \\
\vdots
\end{array}
\]

6. **Gabriel quivers for all blocks**

6.1. **Easy blocks.**

**Theorem 17.** For $\lambda \in \overline{\mathfrak{h}}_1$, we have:

(a) The block $O(\lambda)$ is equivalent to the category of finite dimensional $C[[x]]$-modules.

(b) The block $\bar{O}(\lambda)$ is equivalent to the category of finite dimensional $C[[x, y]]$-modules.

**Proof.** Set $x := \overline{\mathfrak{h}} - \lambda(\overline{h})$. Then, for any $M \in O(\lambda)$, the finite dimensional vector space $M^\lambda$ is naturally a $C[[x]]$-module. Moreover, the functor $F$ sending $M$ to $M^\lambda$ and the parabolic induction functor $G$ are, by the usual hom-tensor adjunction, a pair of adjoint functors between $O(\lambda)$ and the category of finite dimensional $C[[x]]$-modules. From the definitions, it follows immediately that they are each others quasi inverses, proving claim (a).

Claim (b) is proved similarly, with $x := \overline{\mathfrak{h}} - \lambda(\overline{h})$ and $y := h - \lambda(h)$. \hfill \Box

Recall that the **Gabriel quiver** of a block is a directed graph whose
• vertices are isomorphism classes of simple objects in the block;
• the number of arrows from a vertex $L$ to a vertex $S$ equals the dimension of $\text{Ext}^1(L, S)$.

As an immediate consequence of Theorem 17, we obtain:

**Corollary 18.** For $\lambda \in \mathfrak{h}$, we have:

(a) The Gabriel quiver of $\mathcal{O}(\lambda)$ is: 

(b) The Gabriel quiver of $\tilde{\mathcal{O}}(\lambda)$ is: 

6.2. **Partial simple preserving duality.** Denote by $\sigma$ the anti-involution of $\mathfrak{g}$ swapping $e$ with $f$, and $\mathfrak{h}$ with $\mathfrak{f}$. Note that $\sigma(h) = h$.

Let $\mathcal{X} \in \{\tilde{\mathcal{O}}, \mathcal{O}\}$. Denote by $\mathcal{X}_f$ the full subcategory of $\mathcal{X}$ consisting of modules of finite length. For $M \in \mathcal{X}_f$, we can define on $M^* := \bigoplus_{\lambda \in \mathfrak{h}} \text{Hom}_\mathbb{C}(M^\lambda, \mathbb{C})$ the structure of a $\mathfrak{g}$-module via $(a \cdot f)(m) := f(\sigma(a)m)$. Then $M \mapsto M^*$ is a contravariant and involutive self-equivalence of $\mathcal{X}_f$. From $\sigma(h) = h$, it follows that $\text{ch}(M) = \text{ch}(M^*)$. In particular, as simple modules in $\mathcal{X}$ are uniquely determined by their characters, it follows that $L(\lambda)^* \cong L(\lambda)$, for all $\lambda \in \mathfrak{h}$. In other words, the duality $\star$ is simple preserving.

**Corollary 19.** For all $\mathcal{X} \in \{\tilde{\mathcal{O}}, \mathcal{O}\}$ and $\lambda, \mu \in \mathfrak{h}$, we have

\[
\text{Ext}^1_{\mathcal{X}}(L(\lambda), L(\mu)) \cong \text{Ext}^1_{\mathcal{X}}(L(\mu), L(\lambda)).
\]

**Proof.** The left hand side of the equality is obtained from the right hand side by applying the simple preserving duality $\star$. \qed

We note that $\star$ does not extend to the whole of $\mathcal{X}$ as $\star$ messes up the property of being finitely generated. For example, for an infinite length Verma module $\Delta(\lambda) \in \mathcal{X}$ as in Subsection 5.1, the module $\Delta(\lambda)^*$ is not finitely generated and hence does not belong to $\mathcal{X}$.

6.3. **Difficult non-integral blocks.**

**Proposition 20.** Assume that $\lambda \in \mathfrak{h}$ is such that $\lambda(\mathfrak{h}) = 0$ and $\lambda(h) \notin \mathbb{Z}$. Then, for $\mu \in \lambda + \mathbb{Z}\alpha$, we have

\[
\text{Ext}^1_{\mathcal{O}}(L(\lambda), L(\mu)) \cong \begin{cases} 
\mathbb{C}, & \text{if } \mu = \lambda; \\
\mathbb{C}, & \text{if } \mu = \lambda \pm \alpha; \\
0, & \text{otherwise.}
\end{cases}
\]

**Proof.** By Corollary 19 without loss of generality we may assume that $\mu = \lambda - k\alpha$, for some $k \in \mathbb{Z}_{\geq 0}$. Let

\[
0 \to L(\mu) \to M \to L(\lambda) \to 0
\]

be a short exact sequence in $\mathcal{O}$. 

Assume first that $k > 0$ and that (13) does not split. In this case $M$ must be generated by $M^h$ and hence, by adjunction, is a quotient of the Verma module $\Delta(\lambda)$. Under the assumptions $\lambda(h) = 0$ and $\lambda(h) \notin \mathbb{Z}$, all submodules of $\Delta(\lambda)$ are described in Corollary 13. Out of all possible quotients of $\Delta(\lambda)$, only the quotient $\Delta(\lambda)/\Delta(\lambda - 2\alpha)$ has length two. This quotient has composition subquotients $L(\lambda)$ and $L(\lambda - \alpha)$. This implies that

$$\text{Ext}^1_\mathcal{O}(L(\lambda), L(\lambda - \alpha)) \cong \mathbb{C} \quad \text{and} \quad \text{Ext}^1_\mathcal{O}(L(\lambda), L(\lambda - k\alpha)) = 0, \text{ for } k > 1.$$ 

It remains to compute $\text{Ext}^1_\mathcal{O}(L(\lambda), L(\lambda))$. Consider a non-split short exact sequence (13) in $\mathcal{O}$, with $\lambda = \mu$. The vector space $M^h$ is, naturally, a $U(\mathfrak{h})$-module. If this module were semi-simple, by adjunction there would exist two linearly independent homomorphisms from $\Delta(\lambda)$ to $M$ and hence (13) would be split. Therefore $M^h$ must be an indecomposable $U(\mathfrak{h})$-module. As $h$ is supposed to act diagonalizably, such module $M^h$ is unique, up to isomorphism. In particular, there is a basis $\{v, w\}$ of $M^h$ such that the matrix of the action of $h$ in this basis is

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$ 

Consider now the module $\Delta(M^h) := U(\mathfrak{g}) \otimes M^h$, where $\mathfrak{h}M^h = 0$. By adjunction, $\Delta(M^h)$ surjects onto $M$. Hence, we just need to check how many submodules $K$ of $\Delta(M^h)$ have the property that $\Delta(M^h)/K$ has length two with both composition subquotients isomorphic to $L(\lambda)$. We claim that such submodule is unique, which implies that $\text{Ext}^1_\mathcal{O}(L(\lambda), L(\lambda)) \cong \mathbb{C}$. In fact, since $k_\lambda(\Delta(M^h)) = 2$ by construction, the uniqueness of $K$, provided that $K$ exists, is clear.

To prove existence, we consider the submodule $K$ of $\Delta(M^h)$ generated by $fv$ and $\lambda(h)f - fw$ (note that $\lambda(h) \neq 0$ by our assumptions). It is easy to check that both these vectors are annihilated by $e$ and $\mathfrak{f}$. The vector $fv$ generates a submodule of $\Delta(M^h)$ isomorphic to $\Delta(\lambda - \alpha)$. The image of $\lambda(h)f - fw$ in the quotient $\Delta(M^h)/\Delta(\lambda)$ generates in this quotient a submodule isomorphic to $\Delta(\lambda - \alpha)$. Therefore, from Proposition 12 it follows that $\Delta(M^h)/K$ indeed has length two with both simple subquotients isomorphic to $L(\lambda)$. The claim follows.

As an immediate corollary from Proposition 20, we have:

**Corollary 21.** Assume that $\lambda \in \mathfrak{h}^*$ is such that $\lambda(\mathfrak{h}) = 0$ and $\lambda(h) \notin \mathbb{Z}$. Then, for $\xi := \lambda + Z\alpha$, the Gabriel quiver of $\mathcal{O}(\xi)$ has the form:

```
... λ - α λ λ + α ...
```

Now we can proceed to $\tilde{\mathcal{O}}$.

**Proposition 22.** Assume that $\lambda \in \mathfrak{h}^*$ is such that $\lambda(\mathfrak{h}) = 0$ and $\lambda(h) \notin \mathbb{Z}$. Then, for $\mu \in \lambda + Z\alpha$, we have

$$\text{Ext}^{1}_\tilde{\mathcal{O}}(L(\lambda), L(\mu)) \cong \begin{cases} \mathbb{C}^2, & \text{if } \mu = \lambda; \\ \mathbb{C}, & \text{if } \mu = \lambda \pm \alpha; \\ 0, & \text{otherwise}. \end{cases}$$
Proof. The case $\mu \neq \lambda$ is proved by exactly the same arguments as in Proposition \[20\]. The case $\mu = \lambda$ is also similar, but requires some small adjustments which we describe below.

Consider a non-split short exact sequence \[13\] in $\overline{O}$, with $\lambda = \mu$. The vector space $M^\lambda$ is an indecomposable $U(\overline{h})$-module of length two, namely, a self-extension of the simple $U(\overline{h})$-module $C_\lambda$ corresponding to $\lambda$. The space of such self-extensions is two-dimensional (as $\overline{h}$ is two-dimensional). In fact, using the arguments as in the proof of Proposition \[20\], we can show that parabolic induction, followed by taking a canonical quotient, defines a surjective map from $\text{Ext}^1_{U(\overline{h})}(C_\lambda, C_\lambda)$ to $\text{Ext}^1_{\overline{O}}(L(\lambda), L(\lambda))$ which sends isomorphic module to isomorphic and non-isomorphic modules to non-isomorphic (the latter claim is obvious by restricting the action to the generalized $\lambda$-weight space).

This, clearly, implies the necessary claim. Here are the details.

There is a basis $\{v, w\}$ of $M^\lambda$ such that the matrices of the action of $h$ and $\overline{h}$ in this basis are

$$
\begin{pmatrix}
\lambda(h) & 0 \\
p & \lambda(h)
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 0 \\
q & 0
\end{pmatrix},
$$

respectively, where $p$ and $q$ are complex numbers at least one of which is non-zero.

Similarly to the proof of Proposition \[20\] one shows that the submodule $K$ of $\Delta(M^\lambda)$ generated by $\overline{f}v$ and $\frac{\lambda(h)}{q}\overline{f}v - fw$, in case $q \neq 0$, or $\overline{f}v$, in case $q = 0$, is the unique submodule of $\Delta(M^\lambda)$ such that $\Delta(M^\lambda)/K$ is isomorphic to $M$. The claim follows. \[\Box\]

As an immediate corollary from Proposition \[22\] we have:

**Corollary 23.** Assume that $\lambda \in \overline{h}$ is such that $\lambda(\overline{h}) = 0$ and $\lambda(h) \notin \mathbb{Z}$. Then, for $\xi := \lambda + Z\alpha$, the Gabriel quiver of $\overline{O}(\xi)$ has the form:

```
  . . .
\lambda - \alpha \quad \lambda \quad \lambda + \alpha
  . . .
```

6.4. Other self-extensions of simples.

**Corollary 24.** Let $\lambda \in \overline{h}_0$ be such that $\lambda(\overline{h}) = 0$ and $\lambda(h) \notin \mathbb{Z}_{>0}$. Then we have

$$
\text{Ext}^1_{\overline{O}}(L(\lambda), L(\lambda)) \cong \mathbb{C} \quad \text{and} \quad \text{Ext}^1_{\overline{O}}(L(\lambda), L(\lambda)) \cong \mathbb{C}^2.
$$

**Proof.** The follows directly from the corresponding parts in the proofs of Proposition \[20\] and Proposition \[22\] \[\Box\]

**Lemma 25.** We have

$$
\text{Ext}^1_{\overline{O}}(L(0), L(0)) = \text{Ext}^1_{\overline{O}}(L(0), L(0)) = 0.
$$

**Proof.** The elements $e$, $f$, $\overline{e}$ and $\overline{f}$ must annihilate any self-extension $M$ of $L(0)$ since $M^{kn} = 0$, for such $M$. As $h = [e, f]$ and $\overline{h} = [\overline{e}, f]$, it follows that both $h$ and $\overline{h}$ must annihilate $M$ as well. Therefore $M$ splits. \[\Box\]

**Proposition 26.** Let $\lambda \in \overline{h}_0$ be such that $\lambda(\overline{h}) = 0$ and $\lambda(h) \in \mathbb{Z}_{>0}$. Then we have

$$
\text{Ext}^1_{\overline{O}}(L(\lambda), L(\lambda)) \cong \text{Ext}^1_{\overline{O}}(L(\lambda), L(\lambda)) \cong \mathbb{C}.
$$
Proof. By Weyl’s complete reducibility theorem, \( h \) acts diagonalizably on any finite dimensional \( g \)-module. Hence any self-extension of \( L(\lambda) \) lives in \( O \). Therefore it is enough to prove that \( \text{Ext}^1_O(L(\lambda), L(\lambda)) \cong \mathbb{C} \).

Let \( M \) be a self-extension of \( L(\lambda) \). Then \( M^\lambda \) is an \( \mathbb{C}[h] \)-module and, similarly to Proposition 20, \( M \) is indecomposable if and only if \( M^\lambda \) is. As \( \mathbb{C}[h] \) is a polynomial algebra in one variable, this implies that \( \text{Ext}^1_O(L(\lambda), L(\lambda)) \) is at most one dimensional.

To prove that \( \text{Ext}^1_O(L(\lambda), L(\lambda)) \) is exactly one-dimensional, it is enough to construct one non-split self-extension of \( L(\lambda) \), which we do below.

Let \( n := \lambda(h) \in \mathbb{Z}_{\geq 0} \). By Example 1.24 in [Ma], \( L(\lambda) \) has a basis \( \{v_0, v_1, \ldots, v_n\} \) such that

\[
eq v_{i-1}, \quad f v_i = (n-i)v_{i+1}, \quad h v_i = (n-2i)v_i, \quad \text{for } i = 0, 1, \ldots, n.
\]

Take another copy \( L(\lambda) \) of \( L(\lambda) \) with basis \( \{\tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_n\} \) and similarly defined action. Consider \( M = L(\lambda) \oplus L(\lambda) \) and define

\[
t v_i = \tilde{v}_{i-1}, \quad f v_i = (n-i)\tilde{v}_{i+1}, \quad \tilde{h} v_i = (n-2i)\tilde{v}_i, \quad \text{for } i = 0, 1, \ldots, n,
\]

and \( \tilde{v}L(\lambda) = f L(\lambda) = hL(\lambda) = 0 \). It is straightforward that this defines on \( M \) the structure of a \( g \)-module. As the action of \( \tilde{h} \) on \( v_0 \) is non-zero (here the condition \( n > 0 \) is crucial!), the module \( M \) is a non-split self-extension of \( L(\lambda) \). This completes the proof. \( \square \)

6.5. Difficult integral blocks.

**Proposition 27.** Let \( \lambda \in \overline{h} \) be such that \( \lambda(\overline{h}) = 0 \).

(a) We have

\[
\text{Ext}^1_O(L(\lambda), L(\lambda - \alpha)) \cong \text{Ext}^1_O(L(\lambda), L(\lambda - \alpha)) \cong \mathbb{C}.
\]

(b) If \( \lambda(h) = n \in \mathbb{Z}_{\geq 0} \), then we have

\[
\text{Ext}^1_O(L(\lambda), L(\lambda - (n+1)\alpha)) \cong \text{Ext}^1_O(L(\lambda), L(\lambda - (n+1)\alpha)) \cong \mathbb{C}.
\]

(c) If \( \lambda(h) \neq n \in \mathbb{Z}_{\geq 0} \), then we have

\[
\text{Ext}^1_O(L(\lambda), L(\lambda - (n+1)\alpha)) = \text{Ext}^1_O(L(\lambda), L(\lambda - (n+1)\alpha)) = 0.
\]

**Proof.** We start with claim (a). Assume that

\[
0 \to L(\lambda - \alpha) \to M \to L(\lambda) \to 0
\]

is a non-split short exact sequence. Then, similarly to Proposition 20, \( M \) must be a quotient of \( \Delta(\lambda) \). If \( \lambda(h) \notin \mathbb{Z}_{\geq 0} \), then from Corollary 13 it follows that \( \Delta(\lambda) \) has a unique quotient with correct composition subquotients. If \( \lambda(h) \in \mathbb{Z}_{\geq 0} \), then from Lemma 14 it follows that \( \Delta(\lambda) \) has a unique quotient with correct composition subquotients. This completes the proof of claim (a).

We proceed with claim (b). Assume that \( \lambda(h) = n \in \mathbb{Z}_{\geq 0} \) and

\[
0 \to L(\lambda - (n+1)\alpha) \to M \to L(\lambda) \to 0
\]

is a non-split short exact sequence. Then, from Lemma 14 it follows that \( \Delta(\lambda) \) has a unique quotient with correct composition subquotients. This completes the proof of claim (b).

Proposition 12, Lemma 14 and Lemma 15 imply that the only socle components possible in length two quotients of \( \Delta(\lambda) \) are \( \Delta(\lambda - \alpha) \) and \( \Delta(\lambda - (n+1)\alpha) \), and the latter one
is only possible under the additional assumption that $\lambda(h) = n \in \mathbb{Z}_{\geq 0}$. This implies claim (2) and completes the proof.

Combining Proposition 27, Corollary 24, Lemma 25, Corollary 19 and Proposition 26 we obtain:

**Corollary 28.**

(a) The Gabriel quiver of $\mathcal{O}(\mathbb{Z} \alpha)$ is:

```
0 ↘ ↘ ↘ ↘ ...
-α ↘ ↘ ↘ ↘ ...
-2α ↘ ↘ ↘ ↘ ...
-3α ↘ ↘ ↘ ↘ ...
```

(b) The Gabriel quiver of $\tilde{\mathcal{O}}(\mathbb{Z} \alpha)$ is:

```
0 ↘ ↘ ↘ ↘ ...
-α ↘ ↘ ↘ ↘ ...
-2α ↘ ↘ ↘ ↘ ...
-3α ↘ ↘ ↘ ↘ ...
```

(c) The Gabriel quiver of $\mathcal{O}(\frac{1}{2} \alpha + \mathbb{Z} \alpha)$ is:

```
\frac{1}{2} \alpha ↘ ↘ ↘ ↘ ...
α - \alpha ↘ ↘ ↘ ↘ ...
α - 2α ↘ ↘ ↘ ↘ ...
α - 3α ↘ ↘ ↘ ↘ ...
α - 4α ↘ ↘ ↘ ↘ ...
```

(d) The Gabriel quiver of $\tilde{\mathcal{O}}(\frac{1}{2} \alpha + \mathbb{Z} \alpha)$ is:

```
\frac{1}{2} \alpha ↘ ↘ ↘ ↘ ...
α - \alpha ↘ ↘ ↘ ↘ ...
α - 2α ↘ ↘ ↘ ↘ ...
α - 3α ↘ ↘ ↘ ↘ ...
α - 4α ↘ ↘ ↘ ↘ ...
```

**Acknowledgements:** This research was partially supported by the Swedish Research Council and Göran Gustafsson Stiftelse. We thank the referee for helpful comments.
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V. M.: Department of Mathematics, Uppsala University, Box. 480, SE-75106, Uppsala, SWEDEN, email: mazor@math.uu.se

C. S.: Department of Mathematics, Uppsala University, Box. 480, SE-75106, Uppsala, SWEDEN, email: christoffer.soderberg@math.uu.se