Electrodynamics in a Filled Minkowski Spacetime with Application to Classical Continuum Electrodynamics

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(Dated: February 27, 2009)

Abstract

Minkowski spacetime is a convenient setting for the study of the relativistic dynamics of particles and fields in the vacuum. In order to study events that occur in a dielectric or other linear medium, we adopt the familiar continuum assumption of a linear, isotropic, homogeneous, transparent medium of refractive index n filling all space and seek the principle of relativity that applies in the filled spacetime. Applying the Einstein postulates with c/n as the speed of light, we show how the effective signal velocity results in a scaling of the proper time by the refractive index and examine the consequences for D’Alembert’s principle, the Lagrange equations, and the canonical momentum field. The principles of dynamics in the filled spacetime are then applied to the electromagnetic Lagrangian and we derive equations of motion that are invariant with respect to a material Lorentz transformation. The new representation of the dynamics of macroscopic fields is shown to be consistent with the equal-time commutation relation for quantized macroscopic fields, quantum–classical correspondence, the principle of superposition, and electromagnetic boundary conditions.

I. INTRODUCTION

The basic tenant of classical continuum electrodynamics is that the electrodynamic properties of a material can be characterized by macroscopic parameters. While linear media are not homogeneous on a microscopic scale, most optical phenomena take place on a characteristic scale of many unit cells for which the macroscopic model of a linear medium is a useful abstraction. For the purpose of electrodynamics, a transparent linear medium can be thought of as a collection of atoms or molecules, behaving like simple harmonic oscillators, embedded in the vacuum of Minkowski spacetime. On length scales smaller than the spacing between dipoles, a light signal travels away from an event at the velocity c, defining the null surface of a light cone as \[|\Delta x|^2 = (c \Delta t)^2\] on a four-dimensional Minkowski map [1,2,3]. At substantially longer length scales, the light signal from an event travels at an effective velocity c/n, defining a macroscopic refractive index n. Then the null-surface is an effective light cone that is described by \[|\Delta x|^2 = (c \Delta t/n)^2\]. If we are not interested in the behavior of events on the atomic scale, then we can apply the macroscopic model of a linear medium as a continuum with macroscopic parameters. In this continuum limit, electrodynamics takes place on the background of an effective Minkowski spacetime with a timelike coordinate \(x_0 = ct/n\).

The theory of relativity defines transformations between different inertial reference frames moving at constant velocities [1,2,3]. Einstein showed that the microscopic Maxwell equations are invariant under transformations of the vacuum Lorentz group. Like the microscopic Maxwell equations for free-space, the macroscopic Maxwell equations of continuum electrodynamics need to be invariant under some transformation [4]. We consider a linear isotropic homogeneous medium of refractive index n filling all space and seek the principle of relativity that applies in this spacetime in which light travels at speed c/n. Adopting the Einstein postulates with c/n as the speed of light in the filled spacetime results in a material Lorentz transformation [3] and a scaling of the proper time interval by the refractive index [6]. Here, we derive the field dynamics under these conditions and examine the consequences for continuum electrodynamics.

There are a number of issues that contribute to make this undertaking necessary. i) Macroscopic electromagnetic fields in linear materials, when quantized using current procedures [7,8,9,10], violate quantum–classical correspondence in relation to the electromagnetic boundary conditions [6]. A re-derivation of D’Alembert’s principle and the Lagrange equations of motion for discrete particles in a dielectric-filled Minkowski spacetime was sufficient to repair the violation of the correspondence principle [6]. The appearance of the refractive index in the time-like coordinate of filled spacetime and in the proper time that modifies the dynamics of discrete particles will affect field dynamics, as well. ii) The definition of the linear refractive index in terms of the linear electric and magnetic susceptibilities violates the principle of superposition. Ward, Nelson, and Webb [11] used the Feynman [12] definition of the refractive index in terms of the effect of a refractive medium on a transmitted field to derive the effective index \(n = 1 + \chi_e + \chi_m\) of a magnetodielectric material in terms of the electric susceptibility \(\chi_e\) and the magnetic susceptibility \(\chi_m\). If either the electric or magnetic susceptibility is small compared to unity, which is usually the case, then \(n^2 \approx (1 + \chi_e)(1 + \chi_m) = \varepsilon \mu\) in terms of the permittivity \(\varepsilon\) and permeability \(\mu\). However, the approximation is not valid if the electric and magnetic susceptibilities are not small compared to the unit vacuum susceptibility \(\chi_0 = 1\) and we make the case for accepting the derived result \(n = \chi_0 + \chi_e + \chi_m\), which obeys superposition. The case for the principle of superposition has also been argued from quantum electrodynamics based on the linearity of quantum mechanics [13]. iii) The transformation properties of electromagnetic fields in linear media are a consequence of the continuum assumption, which is requisite for continuum
electrodynamics. In a linear isotropic homogeneous continuous medium, the equations of motion must therefore be constructed for invariance under the material Lorentz group, rather than the vacuum Lorentz group [6].

In this article, the relativity of events in a space-filling linear medium is used in the derivation of Hamilton’s equations of motion for fields. The Lagrangian dynamics of discrete particles in a filled Minkowski spacetime that was presented in Ref. [6] is reviewed. The field theory in a linear medium is then developed as a generalization of the discrete theory. The canonical momentum field is found to depend on the presence of the refractive index in the timelike coordinate of filled Minkowski spacetime through differentiation with respect to the proper time. Using the field theory, we derive macroscopic equations of motion

\[ \nabla \times B = -\frac{\partial \Pi}{\partial x_0} \]  
\[ (1.1a) \]

\[ \nabla \times \Pi = \frac{\partial B}{\partial x_0} \]  
\[ (1.1b) \]

for the electromagnetic field in a linear medium, where \( B = \nabla \times A \) is the magnetic field, \( \Pi = \partial A / \partial x_0 \) is the conjugate momentum field, and \( x_0 = ct/n \) is the timelike coordinate of filled Minkowski spacetime. Before dismissing this derived result \textit{proctor hoc} with respect to the macroscopic Maxwell equations, it should be noted that the use of the timelike coordinate of filled Minkowski spacetime \( x_0 = ct/n \), rather than the time \( t \) or the timelike coordinate \( x_0 = ct \), of vacuum Minkowski spacetime is mandated by Lorentz invariance under the material Lorentz group. As shown here, the canonical momentum field in the macroscopic Maxwell equations leads to a violation of quantum–classical correspondence between the Fresnel boundary conditions and the equal-time commutation relation for quantized fields. Using the Feynman model of the refractive index, we show that the vacuum, electric, and magnetic susceptibilities obey the principle of superposition in the new dynamical theory. We also show that the application of Stokes’ theorem and conservation of energy to the macroscopic equations of motion \( 1.1a \) can be used to derive the Fresnel boundary conditions. Then the principle result of this work is the set of dynamical equations \( 1.1a \) for electromagnetic fields in linear media that are consistent with Lorentz transformations and special relativity, quantum–classical correspondence and equal-time commutation relations, the superposition principle, and electromagnetic boundary conditions.

**II. RELATIVISTIC PARTICLE DYNAMICS IN A FILLED SPACETIME**

A distribution of particles is, in the continuum approximation, regarded as a continuous medium and a property of the particles can be represented by a property density that is a continuous function of the spatial and temporal coordinates. In continuum electrodynamics, a linear medium is a uniform region of space in which light travels from a source to an observer at a constant speed of \( c/n \). In Ref. [6], we considered space to be entirely filled with an isotropic homogeneous continuous linear medium and derived the characteristics of spacetime and relativity for the case in which the speed of light is \( c/n \). The purpose of the current work is to generalize the dynamics of discrete particles to derive the dynamics of electromagnetic fields in a filled Minkowski spacetime.

Consider two inertial reference frames, \( S(t, x, y, z) \) and \( S'(t', x', y', z') \), in a standard configuration in which \( S' \) translates at a constant velocity \( v \) in the direction of the positive \( x \) axis and the origins of the two systems coincide at time \( t = t' = 0 \). If a light pulse is emitted from the common origin at time \( t = 0 \), then

\[ (x')^2 + (y')^2 + (z')^2 - \left( \frac{ct'}{n} \right)^2 = 0 \]  
\[ (2.1) \]

describes wavefronts in the \( S' \) system and

\[ (x)^2 + (y)^2 + (z)^2 - \left( \frac{ct}{n} \right)^2 = 0 \]  
\[ (2.2) \]

describes wavefronts in the \( S' \) system. Position vectors in \( S \) are denoted by \( \mathbf{x} = (x, y, z) \), or by \( \mathbf{x} = (x_1, x_2, x_3) \) with an obvious change in notation. Writing time as a spatial coordinate \( ct/n \), the four-vector \( 2.1 \)

\[ \mathbf{X} = (ct/n, \mathbf{x}) = (x_0, x_1, x_2, x_3) \]  
\[ (2.3) \]

represents the position of a point in a filled Minkowski spacetime. In the modified Minkowski spacetime, the square of the invariant spatial interval \( \Delta s \) is \( 2.1 \)

\[ (\Delta s)^2 = (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 - (\Delta x_0)^2 \]  
\[ (2.4) \]

from which we obtain the interval of proper time

\[ d\tau = \frac{dx_0}{c\gamma} \]  
\[ (2.5) \]

where

\[ \gamma = \frac{1}{\sqrt{1 - n^2 v^2/c^2}} \]  
\[ (2.6) \]

Taking the derivative of the position four-vector \( 2.3 \) with respect to the proper time, we obtain the four-velocity \( 2.5 \)

\[ U = \frac{d\mathbf{X}}{d\tau} = \frac{d\mathbf{X}}{dx_0} \frac{dx_0}{d\tau} = c\gamma \left( \frac{dx_1}{dx_0}, \frac{dx_2}{dx_0}, \frac{dx_3}{dx_0} \right) \]  
\[ (2.7) \]

and the four-momentum \( 2.5 \)

\[ \mathbf{P} = m_0 U = mc \left( \frac{dx_1}{dx_0}, \frac{dx_2}{dx_0}, \frac{dx_3}{dx_0} \right) \]  
\[ (2.8) \]
where \( m = \gamma m_0 \) is the relativistic mass. In the nonrelativistic limit, \( \gamma = 1 \), \( \tau = t/n \), the three-velocity

\[
u = n \mathbf{x} = c \left( \frac{dx_1}{dx_0}, \frac{dx_2}{dx_0}, \frac{dx_3}{dx_0} \right)
\]  

(2.9)

and three-momentum

\[
p = nm_0 \mathbf{x} = m_0 c \left( \frac{dx_1}{dx_0}, \frac{dx_2}{dx_0}, \frac{dx_3}{dx_0} \right)
\]  

(2.10)

are re-defined in a region of reduced light velocity \([6]\).

For a system of particles, the transformation of the position vector \( \mathbf{x}_i \) of the \( i \)th particle to \( J \) independent generalized coordinates is

\[
\mathbf{x}_i = \mathbf{x}_i(\tau; q_1, q_2, \ldots, q_J),
\]  

(2.11)

where \( \tau = t/n \). Applying the chain rule, we obtain the virtual displacement

\[
\delta \mathbf{x}_i = \sum_{j=1}^{J} \frac{\partial \mathbf{x}_i}{\partial q_j} \delta q_j
\]  

(2.12)

and the velocity

\[
\mathbf{u}_i = \frac{d \mathbf{x}_i}{d \tau} = \sum_{j=1}^{J} \frac{\partial \mathbf{x}_i}{\partial q_j} \frac{d q_j}{d \tau} + \frac{\partial \mathbf{x}_i}{\partial \tau}
\]  

(2.13)

of the \( i \)th particle in the new coordinate system. Substitution of

\[
\frac{\partial \mathbf{u}_i}{\partial (dq_j/d\tau)} = \frac{\partial \mathbf{x}_i}{\partial q_j}
\]  

(2.14)

into the identity

\[
\frac{d}{d \tau} \left( \mathbf{u}_i \cdot \frac{\partial \mathbf{x}_i}{\partial q_j} \right) = m \frac{d \mathbf{u}_i}{d \tau} \frac{\partial \mathbf{x}_i}{\partial q_j} + m \mathbf{u}_i \frac{d}{d \tau} \left( \frac{\partial \mathbf{x}_i}{\partial q_j} \right)
\]  

(2.15)

yields

\[
\frac{d \mathbf{p}_i}{d \tau} \cdot \frac{\partial \mathbf{x}_i}{\partial q_j} = \frac{d}{d \tau} \left( \frac{\partial}{\partial (dq_j/d\tau)} \left( \frac{1}{2} m \mathbf{u}_i^2 \right) \right) - \frac{\partial}{\partial q_j} \left( \frac{1}{2} m \mathbf{u}_i^2 \right).
\]  

(2.16)

For a system of particles in equilibrium, the virtual work of the applied forces \( \mathbf{f}_i \), vanishes and the virtual work on each particle vanishes leading to the principle of virtual work

\[
\sum_i \mathbf{f}_i \cdot \delta \mathbf{x}_i = 0
\]  

(2.17)

and D’Alembert’s principle

\[
\sum_i \left( \mathbf{f}_i - \frac{d \mathbf{p}_i}{d \tau} \right) \cdot \delta \mathbf{x}_i = 0.
\]  

(2.18)

Using Eqs. (2.12) and (2.16) and the kinetic energy of the \( i \)th particle

\[
T_i = \frac{1}{2} m \mathbf{u}_i^2,
\]  

(2.19)

we can write D’Alembert’s principle, Eq. (2.18), as

\[
\sum_i \left[ \left( \frac{d}{d \tau} \left( \frac{\partial T}{\partial (dq_j/d\tau)} \right) - \frac{\partial T}{\partial q_j} \right) \delta q_j \right] = 0, \quad (2.20)
\]

where

\[
Q_j = \sum_i \mathbf{f}_i \cdot \frac{\partial x_i}{\partial q_j}.
\]  

(2.21)

If the generalized forces come from a generalized scalar potential function \( V \), then we can write Lagrange equations of motion

\[
\frac{d}{d \tau} \left( \frac{\partial L}{\partial (dq_j/d\tau)} \right) - \frac{\partial L}{\partial q_j} = 0,
\]  

(2.22)

where \( L = T - V \) is the Lagrangian. The canonical momentum is therefore

\[
p_j = \frac{\partial L}{\partial (dq_j/d\tau)} = \frac{1}{c} \frac{\partial L}{\partial (dq_j/dx_0)}
\]  

(2.23)

in a linear medium. This formula for the canonical momentum differs from the usual canonical momentum formula

\[
p_j = \frac{\partial L}{\partial (dq_j/dt)}
\]  

(2.24)

by a factor of \( n \).

We can provide two examples that support the use of the new formulation of the canonical momentum in a region of space in which light travels at a speed of \( c/n \). First, the momentum formula, Eq. (2.23), applied to a free particle traveling unimpeded in an arbitrarily large linear medium yields \( p = nm_0 \mathbf{x} \) in agreement with the result of special relativity, Eq. (2.10). Second, it was shown in Ref. \([6]\) that application of the new momentum formula (2.23) to the quantization of the electromagnetic field in a dielectric repairs a violation of quantum–classical correspondence caused by the vacuum formulation (2.24) of the canonical momentum.

The field theory \([13, 14]\) is based on a generalization of the discrete case in which the dynamics are derived from a Lagrangian density \( \mathcal{L} \) instead of the Lagrangian

\[
L = \int dv \mathcal{L}.
\]  

(2.25)

The generalization of the Lagrange equation (2.22) for fields in a linear medium is

\[
\frac{\partial}{\partial x_0} \frac{\partial \mathcal{L}}{\partial (\partial A_\nu/\partial x_0)} = \frac{\partial \mathcal{L}}{\partial A_\nu} - \sum_i \partial_i \frac{\partial \mathcal{L}}{\partial (\partial A_\nu)}.
\]  

(2.26)

where \( x_0 = ct/n \) is the time-like coordinate in the filled Minkowski spacetime and \( x_1, x_2, \) and \( x_3 \) correspond to the respective \( x, y \) and \( z \) coordinates. We adopt the typical conventions that Roman indices run from one to
three, Greek indices run from zero to three, and $\partial_i$ represents the operator $\partial/\partial x_i$. The conjugate momentum field
\[ \Pi_\nu = \frac{\partial L}{\partial (\partial A_\nu/\partial x_0)} \] (2.27)
is used to construct the Hamiltonian density
\[ H = \sum_\nu \Pi_\nu \frac{\partial A_\nu}{\partial x_0} - L \] (2.28)
from which Hamilton’s equations of motion
\[ \frac{\partial A_\nu}{\partial x_0} = \frac{\partial H}{\partial \Pi_\nu} \] (2.29a)
\[ \frac{\partial \Pi_\nu}{\partial x_0} = -\frac{\partial H}{\partial A_\nu} + \sum_i \partial_i \frac{\partial H}{\partial (\partial_i A_\nu)} \] (2.29b)
are derived for an arbitrarily large region of space in which the velocity of light is $c/n$.

For the purpose of comparison, we quote the usual Hamilton’s equations of motion in free space,
\[ \frac{1}{c} \frac{\partial A_\nu}{\partial t} = \frac{\partial H}{\partial \Pi_\nu} \] (3.1a)
\[ \frac{1}{c} \frac{\partial \Pi_\nu}{\partial t} = \frac{\partial H}{\partial A_\nu} + \sum_i \partial_i \frac{\partial H}{\partial (\partial_i A_\nu)} \] (3.1b)
where
\[ \Pi_\nu = c \frac{\partial L}{\partial (\partial A_\nu/\partial t)} \] (3.2a)
\[ H = \sum_\nu \frac{1}{c} \Pi_\nu \frac{\partial A_\nu}{\partial t} - L. \] (3.2b)

III. HAMILTON’S EQUATIONS FOR FIELDS

Before deriving the equations of motion of electromagnetic fields in linear media, we review the microscopic Maxwell equations so that they and their derivation may serve as a basis for comparison with the present work. The Lagrangian for the electromagnetic field in free space is typically
\[ L = \int dv \frac{1}{2} \left( \frac{1}{c^2} \left( \frac{\partial A}{\partial t} \right)^2 - (\nabla \times A)^2 \right). \] (3.3)

Applying Eqs. (3.3) to the Lagrangian density, we obtain the canonical momentum field in vacuum
\[ \Pi = \frac{1}{c} \frac{\partial A}{\partial t} \] (3.4)
and the Hamiltonian density
\[ H = \frac{1}{2} (\Pi^2 + (\nabla \times A)^2). \] (3.5)

Then Hamilton’s equations of motion, Eqs. (3.5), become
\[ \frac{1}{c} \frac{\partial A}{\partial t} = \Pi \] (3.6a)
\[ \frac{\partial \Pi}{\partial t} = -\nabla \times \nabla \times A. \] (3.6b)

Substituting the definition of the magnetic field into Eqs. (3.6), we obtain the microscopic Faraday and Maxwell–Ampère laws
\[ \nabla \times \Pi = \frac{\partial B}{\partial x_0} \] (3.7a)
\[ \nabla \times B = -\frac{\partial \Pi}{\partial x_0} \] (3.7b)
in terms of the canonical momentum field $\Pi = -E$ and timelike coordinate $x_0 = ct$ of Minkowski spacetime.

The filled Minkowski spacetime is the setting for the study of continuum electrodynamics in which the linear medium acts like a region of space in which the speed of light is $c/n$. We take the Lagrangian of the electromagnetic field in the medium to be
\[ L = \int dv \frac{1}{2} \left( \frac{1}{c^2} \left( \frac{\partial A}{\partial x_0} \right)^2 - (\nabla \times A)^2 \right), \] (3.8)
where $x_0 = ct/n$. Applying Eq. (2.27), the canonical momentum field
\[ \Pi = \frac{\partial A}{\partial x_0} \] (3.9a)
is used to construct the Hamiltonian density
\[ H = \frac{1}{2} (\Pi^2 + (\nabla \times A)^2). \] (3.9b)

Hamilton’s equations of motion in the filled spacetime
\[ \frac{\partial A}{\partial x_0} = \Pi \] (3.10a)
\[ \frac{\partial \Pi}{\partial x_0} = -\nabla \times \nabla \times A \] (3.10b)
are obtained from the Hamiltonian density, Eq. (3.9), using Eqs. (2.29).
As an alternative, the universe can be modeled as a vacuum Minkowski spacetime. In that case, the equations of motion for electromagnetic fields in linear media are derived using Eqs. (2.30), instead of Eqs. (2.29). The Lagrangian of the electromagnetic field in a linear medium of refractive index $n$, Eq. (3.7) can be written as

\[ L = \int dv \frac{1}{2} \left( \frac{n^2}{c^2} \left( \frac{\partial A}{\partial t} \right)^2 - (\nabla \times A)^2 \right). \tag{3.11} \]

In this case, the canonical momentum field

\[ \Pi_v = \frac{n^2}{c^2} \frac{\partial A}{\partial t} \tag{3.12} \]

is the same as the canonical momentum field that is used by Hillery and Mlodinow \[16\] and other authors. The canonical momentum of Garrison and Chiao \[17\] is similar in name, but is a variant of the Minkowski electromagnetic momentum. In this case, the Hamiltonian density is

\[ \mathcal{H} = \frac{1}{2} \left( \frac{\Pi_v^2}{n^2} + (\nabla \times A)^2 \right), \tag{3.13} \]

from which Hamilton’s equations of motion

\[
\begin{align*}
\frac{1}{c} \frac{\partial A}{\partial t} &= \frac{c}{n^2} \Pi_v & \tag{3.14a} \\
\frac{1}{c} \frac{\partial \Pi_v}{\partial t} &= -\nabla \times \nabla \times A & \tag{3.14b}
\end{align*}
\]

are derived using Eqs. (2.30).

### IV. COMMUTATION RELATIONS

Two different representations of equations of motion for the macroscopic electromagnetic field in a linear medium have been derived from the same Lagrangian. In one case, the canonical momentum field was derived in a filled Minkowski spacetime in which light travels at $c/n$. In the other case, the canonical momentum field has its roots in the special relativity of the vacuum. Although the two representations do not differ much, we show that the relativity of a filled Minkowski spacetime, rather than that of a vacuum Minkowski spacetime, leads to satisfaction of quantum–classical correspondence.

The macroscopic field in a linear medium can be quantized in a manner analogous to quantum electrodynamics. In Ref. [4], the macroscopic field was quantized in terms of discrete modes of a quantization volume. In that work, quantum–classical correspondence with respect to the boundary conditions required a re-derivation of the discrete D’Alembert’s principle and the discrete Lagrange equations of motion for dynamics in a filled Minkowski spacetime. The current work extends the previous result to continuous fields.

Quantization for continuous fields can be achieved by applying the equal-time commutation relation \[16\]

\[ [A_i(x, t), \Pi_j(x', t)] = i\delta_{ij} \delta(x - x') \tag{4.1} \]

to the canonically conjugate fields. For macroscopic fields, which satisfy the limit of large numbers, the boundary conditions on the quantized fields must be of the same form as the classical electromagnetic boundary conditions in which the amplitude of the vector potential inside the material is smaller than the vacuum amplitude by $\sqrt{n}$, absent reflection.

We have two candidates for the canonical momentum field in a linear medium and the issue to be decided is which version satisfies both the equal-time commutation relation and quantum-classical correspondence with respect to the electromagnetic boundary conditions. Relating the vector potential $A$ inside the medium to the vector potential $A^0$ in the vacuum, the commutator

\[
\left[ A_0^0(x, t), \frac{c}{\sqrt{n}} \frac{\partial A^0(x', t)}{\partial (ct/n)} \right] \left[ \frac{1}{\sqrt{n}} \frac{\partial A^0(x', t)}{\partial (ct/n)} \right] = i\delta_{ij} (x - x') \tag{4.2}
\]

is consistent with the equal-time commutation relation and boundary conditions. This commutator is based on the canonical momentum field (3.12) in a filled Minkowski spacetime. In contrast, the commutator

\[
\left[ A_0^0(x, t), \frac{n^2}{\sqrt{n}} \frac{\partial A^0(x', t)}{\partial t} \right] \left[ \frac{1}{\sqrt{n}} \frac{\partial A^0(x', t)}{\partial t} \right] = i\Delta_{ij} (x - x') \tag{4.3}
\]

for the canonical momentum field (3.12) results in a contradiction between the commutation relation and the boundary conditions, disproving Hamilton’s equations (3.14).

The dynamical field theory of the filled Minkowski spacetime, Eq. (3.7)–Eq. (3.10), is consistent with the equal-time commutation relation and the boundary conditions. The substitution of the magnetic field $B = \nabla \times A$ into Hamilton’s equations (3.10) produces equations of motion

\[
\nabla \times \Pi = \frac{\partial B}{\partial x_0} \tag{4.4a}
\]

\[
\nabla \times B = -\frac{\partial \Pi}{\partial x_0} \tag{4.4b}
\]

for the fields.

The field equations of motion that were derived relativistically in a filled Minkowski spacetime with time-like coordinate $x_0 = ct/n$ and canonical momentum field $\Pi = (n/c)\partial A / \partial t$ are isomorphic with the microscopic Faraday and Maxwell–Ampère laws derived in vacuum Minkowski spacetime. On this basis, it is a trivial matter to transform from microscopic electrodynamics in the vacuum to macroscopic electrodynamics in the continuum limit of a linear medium. Every occurrence of $ct$
that appears in free space electrodynamics is replaced by \( x_0 \) and every occurrence of \( \mathbf{E} \) is replaced by \(-\Pi\). Examples of electromagnetic quantities and equations that are valid both microscopically and macroscopically are: Poynting’s theorem
\[
\nabla \cdot (\mathbf{B} \times \Pi) = -\frac{\partial}{\partial x_0} \frac{1}{2} (\Pi^2 + \mathbf{B}^2),
\]
the electromagnetic energy
\[
H = \int dv \frac{1}{2} (\Pi^2 + \mathbf{B}^2),
\]
the electromagnetic momentum
\[
G = \int dv \frac{1}{c} (\mathbf{B} \times \Pi),
\]
the symmetric Maxwell stress tensor
\[
T^{\alpha\beta} = \Pi_\alpha \Pi_\beta + B_\alpha B_\beta - \frac{1}{2} (\Pi \cdot \Pi + \mathbf{B} \cdot \mathbf{B}) \delta_{\alpha\beta},
\]
the Lorentz force
\[
\mathbf{F} = q \left( -\Pi + \frac{dx}{dx_0} \times \mathbf{B} \right),
\]
and the antisymmetric field tensor
\[
F^{\alpha\beta} = \begin{bmatrix}
0 & -\Pi_x & -\Pi_y & -\Pi_z \\
\Pi_x & 0 & -B_z & B_y \\
\Pi_y & B_z & 0 & -B_x \\
\Pi_z & -B_y & B_x & 0
\end{bmatrix}.
\]

V. SUPERPOSITION

For homogeneous media, the equations of motion for the fields, Eqs. (4.1), can be transformed into the macroscopic Faraday and Maxwell–Ampère laws
\[
\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t},
\]
\[
\nabla \times \mathbf{B} = \frac{n^2}{c} \frac{\partial \mathbf{E}}{\partial t},
\]
of classical continuum electrodynamics by defining an electric field \( \mathbf{E} = -\Pi/n \). However, one should be cautious about ascribing any physical meaning to \( n^2 \) as anything other than the square of the factor that scales the timelike coordinate of vacuum Minkowski spacetime in a linear medium. In particular, the disjunction
\[
n^2 = \varepsilon \mu
\]
of the square of the refractive index into an electric permittivity \( \varepsilon \) and a magnetic permeability \( \mu \) violates the principle of superposition.

Feynman [12] defines the refractive index by the effect of a slab of refractive material that is placed between the source of an electromagnetic field and the point at which it is observed. The macroscopic Feynman model can be easily extended to consider the effect of a stack or mixture of refractive materials on the field. Here, we derive the extended Feynman model and relate the result to the microscopic model of Ward, Nelson, and Webb [11] in which the refractive slab of the Feynman model is replaced with thin sheets of microscopic electric and magnetic dipoles.

Consider a monochromatic source field of amplitude \( E_0 \) and frequency \( \omega \) that passes through a transparent plate of thickness \( d \) and macroscopic refractive index \( n \). The electric field that is detected at a location \( z \) is
\[
E_d(z) = e^{-i\omega d(n-1)/c} E_0 e^{i\omega(t-z/c)},
\]
absent reflections that can be neglected if \( \delta n = n - 1 \) is small or if an anti-reflection coating is applied to the surfaces. Expanding the first exponential in a power series, one obtains
\[
E_d(z) = E_0 e^{i\omega(t-z/c)} - \frac{i\omega d(n-1)}{c} E_0 e^{i\omega(t-z/c)}
\]
for small \( d \) [12]. Now consider the slab to be composed of two layers of material of thickness \( d \) and refractive indices of \( n_1 \) and \( n_2 \). The effect of the material on the detected field is
\[
E_d(z) = e^{-i\omega d(\delta n_1 + \delta n_2)/c} E_0 e^{i\omega(t-z/c)},
\]
where an algebraic property of exponentials has been used to write the product of exponentials as a single exponential by adding the exponents. Again expanding the first exponential in a power series, produces
\[
E_d(z) = E_0 e^{i\omega(t-z/c)} - \frac{i\omega d(\delta n_1 + \delta n_2) d}{c} E_0 e^{i\omega(t-z/c)}.
\]
Comparing Eqs. (5.4) and (5.6), one obtains
\[
n = 1 + \delta n_1 + \delta n_2
\]
for the effective macroscopic index of refraction. For small \( \delta n \), we can make the approximation [11]
\[
n^2 \approx (1 + \delta n_1)(1 + \delta n_2).
\]
For a material that responds linearly to electromagnetic radiation, the refractive index must depend linearly on the material parameters. Not only is Eq. (5.8) nonlinear, but there is no requirement that the \( \delta n \) are small because we have other techniques to minimize reflections. Extending the treatment to a stack of materials with layers indexed with \( i \), we obtain
\[
n = 1 + \sum_i \delta n_i
\]
as the definition of the effective macroscopic refractive index.

Now consider the slab as a bounded region of space of refractive index \(n_1\) and thickness \(d\) containing a density of particles. As before,

\[
E_d(z) = e^{-i\omega d(n_1 - 1)/c} E_0 e^{i\omega(t-z/c)}. \quad (5.10)
\]

If the volume emptied and then filled with a second group of particles that create a refractive index of \(n_2\), we get

\[
E_d(z) = e^{-i\omega d(n_2-1)/c} E_0 e^{i\omega(t-z/c)}. \quad (5.11)
\]

Combining the two groups of particles in the same volume, we obtain

\[
E_d(z) = e^{-i\omega d(n_1+\delta n_2)/c} E_0 e^{i\omega(t-z/c)}. \quad (5.12)
\]

by superposition. Comparing Eqs. (5.12) and (5.4), one obtains

\[
n = 1 + \delta n_1 + \delta n_2 \quad (5.13)
\]

for the effective macroscopic index of refraction. The approximation

\[
n^2 \approx (1 + \delta n_1)(1 + \delta n_2) \quad (5.14)
\]

is unnecessarily restrictive because there is no guarantee that the \(\delta n\) are small. The approximation is clearly incorrect if the two groups of particles are the same, since \(n = 1 + 2\delta n\) in that case.

It should be noted that the materials that provided the refractive index have not been specified. The materials can be dielectric, magnetic, or magnetodielectric and in any combination. Then the effective refractive index obeys superposition and is the sum of the contributions of the various materials. Defining the susceptibility of each component as the contribution of that component to the refractive index, the effective index

\[
n = \sum_{i=0}^{\infty} \chi_i \quad (5.15)
\]

is the linear combination of the susceptibilities of all the components of the material, including the vacuum susceptibility \(\chi_0 = 1\).

**VI. BOUNDARY CONDITIONS**

Boundary conditions determine how fields in different homogeneous media relate to each other. We consider an electromagnetic field normally incident on an interface between a half-space of Minkowski spacetime and a half-space of filled Minkowski spacetime. Propagating wave solutions of the wave equation can be represented by the vector potential

\[
A_f = A_f \cos(-i\omega/c)x_0 + k_z \phi \hat{e}_i \quad (6.1)
\]

for forward traveling waves and by

\[
A_b = A_b \cos(-(i\omega/c)x_0 - k_z + \phi) \hat{e}_i \quad (6.2)
\]

for backward traveling waves. Here, \(\hat{e}_i\) is a unit vector transverse to the direction of propagation, \(k = n_\omega/c\), and \(A_f, A_b\) are temporally and spatially independent in the plane-wave cw regime. The amplitudes of the vector potential for the incident, reflected, and refracted fields are respectively denoted as \(A_i, A_r,\) and \(A_t\).

Conservation of energy provides a boundary condition and, for a flow, the conservation law is represented by a continuity equation for the energy flux. The Hamiltonian density \((\Pi)\) for the field in filled spacetime can be written as the energy density

\[
\rho_e = \frac{1}{2} (\Pi^2 + B^2) \quad (6.3)
\]

The Poynting–Umov vector \(S = \rho_e \hat{x}\) is the continuous energy flux vector that is associated with the energy conservation law. In the plane-wave cw limit,

\[
A_t^2 = A_r^2 + n A_i^2 \quad (6.4)
\]

is obtained from continuity of the Poynting–Umov vector.

A second continuity condition can be derived for the fields by applying Stokes' theorem to a small loop that straddles the boundary. The application of Stokes' theorem to Eq. (6.4)

\[
\nabla \times B = -\frac{\partial \Pi}{\partial x_0} \quad (6.5)
\]

yields a continuity equation for the magnetic field. In terms of amplitudes, continuity of the magnetic field is expressed as

\[
A_i + A_r = n A_t \quad (6.6)
\]

Continuity of the magnetic field provides one factor of Eq. (6.4). The other factor

\[
A_i - A_r = A_t \quad (6.6)
\]

can be interpreted as continuity of a vector potential where the direction of the vector potential is reversed by reflection. The linearly independent continuity equations (6.5) and (6.6) can be combined algebraically to form the Fresnel relations

\[
\frac{A_r}{A_i} = \frac{n - 1}{n + 1} \quad (6.7a)
\]

\[
\frac{A_t}{A_i} = \frac{2}{n + 1} \quad (6.7b)
\]

It is not particularly difficult to derive Fresnel-type boundary conditions for oblique incidence or multiple materials [18].
There is no independent continuity equation that is associated with the canonical momentum field, Eq. (3.8),

\[ \Pi = \frac{\partial A}{\partial x_0}. \]

By the application of Stokes’ law to Eq. (4.4a),

\[ \nabla \times \Pi = \frac{\partial B}{\partial x_0}, \]

we find that the continuity equations for the magnetic and momentum fields are redundant as the conjugate momentum field changes sign upon reflection. There is one continuity equation associated with the conservation of electromagnetic energy, Eq. (6.6), and one continuity equation associated with the fields, Eq. (6.3). In the case of the macroscopic Maxwell equations (5.1), the boundary conditions are over-specified with two independent continuity equations for the field and one continuity equation for the energy flux.

VII. SUMMARY

We derived a theory of special relativity that applies to events that occur in a dielectric or other linear medium that is treated as a continuous material. The relativistic theory was applied to the derivation of a Lagrangian-based theory of particle and field dynamics in linear media that was, in turn, used to develop equations of motion for macroscopic fields. We showed that these equations satisfy basic physical principles.

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