On the effect of a parallel resistor in the Chua’s circuit

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Abstract. We report a numerical bifurcation study on the Chua’s circuit with parallel resistor. Through the largest Lyapunov exponent, we constructed a two-dimensional parameter space of the model. We also implemented the experimental circuit to show the similarities between the model and the experimental data. With that modification we discuss the effect of a parallel resistor in the dynamics of a Chua’s circuit.

1. Introduction
Chua’s circuit is a paradigm of chaotic behavior [1] and is extensively studied, with many experimental and/or theoretical works reporting important properties of this system [2, 3, 4, 5, 6, 7, 8].

In this work, we study the dynamical behavior of a Chua’s circuit with parallel resistor. Our studies are based on a recent work of Braga et al. [9], where the authors analytically studied the governing nonlinear equations, where the nonlinearity of the diode is a cubic function, and the intrinsic inductor resistance was omitted. In our studies, we numerically study the complete set of equations, based on the Chua’s circuit model [5, 6], with a piecewise-linear function for the nonlinearity, and adding a resistor in parallel with the inductor. Our aim is to obtain the global bifurcation behavior constructing two-dimensional parameter spaces of the model with the largest Lyapunov exponent method [5, 6, 10, 11, 12]. We also realized the experimental circuit, and we obtained the experimental phase portraits (attractors) for three parameters of the system.

In Fig. 1, we present the Chua’s circuit with parallel resistor $R_P$. The dynamical variables are $i_L$, the current across the inductor, $v_1$ and $v_2$, the voltages across the capacitors $C_1$ and $C_2$, respectively. The set of equations that describes the circuit in Fig. 1 is given by

\begin{equation}
\begin{align*}
\dot{v}_1 &= \frac{dv_1}{dt} = \frac{v_2 - v_1}{RC_1} - \frac{i_d(v_1)}{C_1}, \\
\dot{v}_2 &= \frac{dv_2}{dt} = \frac{v_1 - \alpha v_2}{RC_2} + \frac{i_L}{C_2}, \\
\dot{i}_L &= \frac{di_L}{dt} = -\frac{v_2}{L} - \frac{i_L(r_L/L)}{L},
\end{align*}
\end{equation}
where

\[ i_d(v_1) = m_0 v_1 + \frac{1}{2} (m_1 - m_0) (|v_1 + b| - |v_1 - b|) , \]

and \( \alpha = (R + R_P) / R_P \). \( R_P \) is the parallel resistor which modifies the standard Chua’s circuit \( (\alpha = 1) \) [2, 5, 6].

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This paper is organized as follows: Sec. 2 is devoted to the results obtained from the numerical studies of Eqs. (1), Sec. 3 is devoted to the results obtained from the experimental studies, and the summary is given in Sec. 4.

2. Numerical Results

The numerical study carried out in this work consists of to calculate the largest Lyapunov exponent, numerically solving the Eqs. (1) using the fourth-order Runge-Kutta method with fixed time step for each pair of parameters \((R, r_L)\). We tested the time step from \(10^{-4}\) up to \(10^{-1}\) and the system behavior remained the same, but to decrease the asymptotic convergence time to the attractor, we used the time step equal to \(10^{-1}\). The ranges of parameter values were discretized in a mesh of \(500 \times 500\) points equally spaced. For each pair of parameters \((R, r_L)\) we used the same initial condition \((x, y, z) = (0.1, 0.1, 0.1)\), and to calculate the largest Lyapunov exponent we firstly tested the convergence time of the exponent for many pairs \((R, r_L)\), looking at the exponent behavior with the iteration time. The tests showed that \(5 \times 10^5\) iterations are the optimal convergence time to estimate the largest Lyapunov exponent in most cases. We identify for each largest Lyapunov exponent a color, varying continuously from black (zero exponent), up to yellow (positive exponent). Blue color identifies the divergence cases of Eqs. (1).

Figure 2 shows the two-dimensional parameter space for the parameters \((R, r_L)\) in Eqs. (1), with \( \alpha \neq 1 \), i.e., \( R_P = 1600 \Omega \). Black regions represent stable behaviors (fixed-points or periodic), and the yellowish regions represent unstable behaviors (chaotic). The blue color represents the divergence of Eqs. (1), and due to be large the convergence time of the largest Lyapunov exponent in some regions of the parameter space (Fig. 2), the black region marked with \( FP \) is also a fixed-point region. The white color represents fixed points for regions with small convergence time of the Lyapunov exponent. Inside the chaotic regions, we can observe the existence of immersed periodic structures, represented by the black regions inside of the yellowish region. Those periodic structures have a different accumulation boundary, where the periodic structures accumulate in a fixed-point region (marked with \( FP \) in Fig. 2) as the periodicity increases.
by period-adding cascade towards that region. Recent works [11, 12] reported accumulation boundaries characterized by periodic regions with periods equal to the period-adding factor, i.e., if the period-adding cascade increases the periodicity of the structures by a factor 2, the accumulation boundary is a period-2 region. However, in our case, the accumulation boundary is a fixed-point region.

**Figure 2.** Parameter space diagram associating colors to different $\lambda$ ranges (see the text). The numbers are the lowest periodicity of the structures. Here, $R$ and $r_L$ are in resistance units, $\Omega$.

**Figure 3.** Parameter space of system (1), with $\alpha = 1$, associating intensity colors to different $\lambda$ ranges (see the text). The numbers 4 and 6 are lowest periodicity of the structures. $R$ and $r_L$ are in resistance units, $\Omega$. The periodic structures organize themselves in a single spiral structure that coils up around the focal point $F$, as reported in Ref. [6].

To discuss another effect of the parallel resistor $R_P$ in the Chua’s circuit, we show in Fig. 3 the parameter space of Eqs. (1) for $\alpha = 1$. In that configuration, the circuit, Fig. 1, is the
standard Chua’s circuit [6]. We observe periodic structures organizing themselves in a single spiral structure that coils up around a chaotic focal point in the parameter space. That behavior seems to be a common feature presented in various systems [6, 10] with homoclinic origin, and recently an experimental evidence of spirals in a chaotic electronic circuit was reported [13]. Comparing Figs. 2 and 3, we can observe the effect of the \( R_P \) resistor in the circuit’s dynamics. The spiral structure was destroyed, and the remaining periodic structures reorganized in an accumulation boundary formation, as previously above described.

Figure 4 shows the attractors for parameter values represented by two green marks labeled by the numbers 4 and 6 in Figs. 2 and 3. For each attractor we numerically solved the Eqs. (1) using the fourth-order Runge-Kutta method with fixed time step equal to \( 10^{-1} \), initial condition \((x, y, z) = (0.1, 0.1, 0.1)\), and with \( 5 \times 10^5 \) iterations discarding the first \( 4.5 \times 10^5 \) iterations. We observe similarities regarding periodicity between the pair of attractors belonging to distinct parameter spaces. Although the spiral has disappeared the first structures with lower periods survived.

![Figure 4](image-url)

**Figure 4.** Three-dimensional attractors generated by Eqs. (1) for different values of the parameters \( R \) and \( r_L \). (a) and (b) periodic attractors for the parameter values represented by two green marks labeled by the periods 4 and 6 in Fig. 2. (c) and (d) periodic attractors for the parameter values represented by two green marks labeled by the periods 4 and 6 in Fig. 3. The axes \( v_1 \) and \( v_2 \) are in voltage units (Volts). The axis \( i_L \) is in current units (Amperè).

3. **Experimental results**

We experimentally implemented the circuit shown in Fig. 1 using the standard Chua’s circuit as base platform, and adding a potentiometer \( R_P \) in series with the inductor \( L \). The component values for the standard Chua’s circuit are the same reported in Ref. [2]. For the measurements we use a digital oscilloscope Tektronix TDS 2024B, four channels, 200 Mhz, and 2 GS/s. For the standard Chua’s circuit, we use the *inductorless* implementation [14], where the inductor \( L \) is simulated by a additional circuit. In that configuration, the state variable \( i_L \) can be replaced by another voltage variable, so we measure three voltages as state variables of the circuit. To evaluate the state variable \( i_L \), we use the inverse transformation obtained by the Kirchhoff’s
laws in the simulated inductor. For details about the inductorless implementation and the transformations, please see Refs. [2, 14].

**Figure 5.** Three-dimensional attractors generated by experiment, (a) and (b), and generated by Eqs. (1), (c) and (d), for different values of the parameters $R$, $r_L$, and $R_P$. (a) $R = 1362\Omega$, $r_L = 1.2\Omega$, and $R_P = 2217\Omega$; (b) $R = 1375\Omega$, $r_L = 1.0\Omega$, and $R_P = 2261\Omega$. (c) $R = 1352.22\Omega$, $r_L = 7.24436\Omega$, and $R_P = 1600\Omega$; (d) $R = 1355.88\Omega$, $r_L = 5.33591\Omega$, and $R_P = 1600\Omega$. The axes $v_1$, $v_2$ and $v_L$ are in voltage units (Volts). The axis $i_L$ is in current units (Amperè).

Figure 5 shows two experimental attractors and two corresponding numerical attractors. For the experimental attractors, Figs. 5(a) and 5(b), the parameters are (a) $R = 1362\Omega$, $r_L = 1.2\Omega$, and $R_P = 2217\Omega$; (b) $R = 1375\Omega$, $r_L = 1.0\Omega$, and $R_P = 2261\Omega$. For the numerical attractors, Figs. 5(c) and 5(d), the parameters are (c) $R = 1352.22\Omega$, $r_L = 7.24436\Omega$, and $R_P = 1600\Omega$; (d) $R = 1355.88\Omega$, $r_L = 5.33591\Omega$, and $R_P = 1600\Omega$. The topological similarities for both experimental and numerical attractors, Figs. 5(a) with 5(c), and Figs. 5(b) with 5(d), are evident despite the difference between experimental and numerical parameters.

In Fig. 6 we show evidences of Shilnikov-type homoclinic chaos, with two different global turns, as reported in Ref. [4] for other modified Chua’s circuit. Figs. 6(a) and 6(c) are the 3D attractors, and Figs. 6(b) and 6(d) are the experimental time series for the $v_1$-variable, respectively. These attractors clearly show that a double or a triple spikes repeats with intermediate intervals of large amplitude oscillations.

**4. Summary**

A two-dimensional parameter space, using the largest Lyapunov exponent codified in a continuous range of colors, for the Chua’s circuit with parallel resistor is reported. With that modification, we observe the disappearance of the spiral structure and the emergence of a fixed points region. A new accumulation boundary sequence is reported. In the experimental implementation of the circuit, we observe a good agreement between theoretical and experimental attractors. Two experimental chaotic attractors are observed, showing experimental evidences of Shilnikov-type homoclinic chaos in that circuit.
Figure 6. Three-dimensional attractors and time series generated by experiment for different parameter values $R$, $r_L$, and $R_P$. (a) and (b) for $R = 1384\Omega$, $r_L = 8.0\Omega$, and $R_P = 2218\Omega$. (c) and (d) for $R = 1434\Omega$, $r_L = 1.5\Omega$, and $R_P = 2238\Omega$. The axes are in voltage units (Volts).

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