CORRIGENDUM TO THE PAPER "THE OPTIMAL RANGE OF THE CALDERON OPERATOR AND ITS APPLICATIONS." [J. FUNCT. ANAL. 277 (2019), NO. 10, 3513–3559.]

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Abstract. We fix a gap in the proof of a result in our earlier paper [3]. In this note we rectify a mistake contained in Lemma 22 in [3, Section 5]. Because of a mistake in the proof of [3, Lemma 22], the main results of such section [3, Theorem 21] is not proved in full generality. We provide an alternative proof of [3, Lemma 22] and show that [3, Theorem 21] still holds.

1. Preliminaries

For convenience, we keep almost all notations as in [3]. So, for undefined notations and notions below, we refer the reader to [3].

Define the triangular truncation (with respect to a continuous chain) as usual: if the operator $A$ is an integral operator on the Hilbert space $L_2(-\pi, \pi)$, with the integral kernel $K$, i.e. $(Af)(t) = \int_{-\pi}^{\pi} K(t, s) f(s) ds$, $t \in (-\pi, \pi)$, $f \in L_2(-\pi, \pi)$, then $T(A)$ is an integral operator with truncated integral kernel

$$(T(A)f)(t) = \int_{-\pi}^{\pi} \text{sgn}(t-s) K(t, s) f(s) ds, \quad t \in (-\pi, \pi), \quad f \in L_2(-\pi, \pi).$$

It is established in [3] that the operator $T$ acts from the ideal $\Lambda_{\log}(B(L_2(-\pi, \pi)))$ into $B(L_2(-\pi, \pi))$, where $B(L_2(-\pi, \pi))$ is the algebra of all linear bounded operators on $L_2(-\pi, \pi)$. It is also claimed there that $\Lambda_{\log}(B(L_2(-\pi, \pi)))$ is the maximal domain of $T$. Unfortunately, the proof of this claim contains a substantial oversight: the statement in [3, Lemma 22] is actually false. Nevertheless, we are able to fix this mistake and ensure that [3, Theorem 21] remains true as stated.

Recall the following auxiliary operators from [3]. If $x \in \Lambda_{\log}(\mathbb{Z})$, then the discrete Hilbert-type transform $H_d$ is defined by the formula

$$(H_d x)(n) = \frac{1}{\pi} \sum_{m \in \mathbb{Z}} \frac{x(m)}{m-n}, \quad n \in \mathbb{Z}.$$

Also, define the discrete version of the Calderón operator $S_d$ as

$$(S_d x)(k) = C_d x + C'_d x = \frac{1}{n+1} \sum_{k=0}^{n} x(k) + \sum_{k=n}^{\infty} \frac{x(k)}{k+1}, \quad x \in \Lambda_{\log}(\mathbb{Z}_+).$$

Key words and phrases. triangular truncation operator, discrete Hilbert transform, discrete Calderón operator, lower distributional estimate.
2. LOWER DISTRIBUTIONAL ESTIMATE FOR TRIANGULAR TRUNCATION

The following result replaces Theorem 21 in [3].

**Theorem 1.** For every \( x \in \Lambda_{\log}(\mathbb{Z}_+) \), there exists an operator \( a \in \mathcal{B}(L_2(-\pi, \pi)) \) such that \( \mu(a) = \mu(x) \) and

\[
\mu(T(a)) \geq \frac{1}{8\pi} S_{\mu}(x).
\]

The proof of Theorem 1 relies on Lemma 3 and Lemma 4 below. The following fact is an easy exercise.

**Fact 2.** If \( x \in \ell_2(\mathbb{Z}) \), then

\[
2i\hat{H}_d x(t) = \text{sgn}(t) \cdot \hat{x}(t), \quad t \in (-\pi, \pi),
\]

where

\[
\hat{x}(t) = \sum_{m \in \mathbb{Z}} x(m)e^{int}, \quad t \in (-\pi, \pi).
\]

**Proof.** For every \( k \in \mathbb{Z} \), we have

\[
\int_{-\pi}^{\pi} \text{sgn}(t) \cdot e^{ikt} dt = \int_{0}^{\pi} e^{ikt} dt - \int_{-\pi}^{0} e^{ikt} dt = \begin{cases} 0, & k \text{ is even} \\ \frac{4i}{k}, & k \text{ is odd} \end{cases}.
\]

Thus, for every \( n \in \mathbb{Z} \), we have

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \text{sgn}(t) \cdot \hat{x}(t)e^{-int} dt = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} x(m) \int_{-\pi}^{\pi} \text{sgn}(t) \cdot e^{i(m-n)t} dt
\]

\[
= \frac{2i}{\pi} \sum_{\substack{m \in \mathbb{Z} \\ m-n=1 \text{ mod } 2}} x(m) 2i\hat{H}_d x(n).
\]

\[ \square \]

In the next lemma, we consider the von Neumann algebra \( \mathcal{B}(L_2(-\pi, \pi)) \) and identify it with \( \mathcal{B}(L_2(\mathbb{T})) \). The purpose of the lemma below is to identify the mistake in [3, Lemma 22] (this mistake would be alleviated in Lemma 4 below).

Denote by \( \mathcal{D} \) the differential operator \( \mathcal{D} := \frac{d}{dt} \) defined on the Sobolev space \( W^{1,2}(\mathbb{T}) \).

**Lemma 3.** If \( x \in \Lambda_{\log}(\mathbb{Z}) \), \( a = x(\mathcal{D}) \), then

\[
T(a) = 2ip(\mathcal{H}_d x)(\mathcal{D})p + 2iq(\mathcal{H}_d x)(\mathcal{D})q + px(\mathcal{D})q - qx(\mathcal{D})p,
\]

where \( p = M_{\chi_{(0,\pi)}} \) and \( q = M_{\chi_{(-\pi,0)}} \) are multiplication operators on \( L_2(\mathbb{T}) \).

**Proof.** Suppose first that \( x \in \ell_2(\mathbb{Z}) \) and, therefore, \( \hat{x} \in L_2(\mathbb{T}) \). One can write \( a = x(\mathcal{D}) \) as an integral operator of convolution type

\[
(af)(t) = \frac{1}{2\pi} \int_{\mathbb{T}} \hat{x}(t-s) f(s) ds.
\]

By the definition of \( p \) and \( q \), we have

\[
(papf)(t) = \frac{1}{2\pi} \int_{\mathbb{T}} \chi_{(0,\pi)}(t) \chi_{(0,\pi)}(s) \hat{x}(t-s) f(s) ds
\]
and
\[(qaqf)(t) = \frac{1}{2\pi} \int_{\mathbb{T}} \chi_{(-\pi,0)}(t)\chi_{(-\pi,0)}(s)\hat{x}(t-s)f(s)ds.\]

If \(t, s \in (0, \pi)\) or \(t, s \in (-\pi, 0)\), then \(t - s \in (-\pi, \pi)\). Thus, by Fact 2,
\[\text{sgn}(t-s)\hat{x}(t-s) = 2i\hat{\mathcal{H}}\hat{x}(t-s)\]
whenever \(t, s \in (0, \pi)\) or \(t, s \in (-\pi, 0)\). By the definition of triangular truncation operator, we have
\[(T(pap)f)(t) = \frac{2i}{2\pi} \int_{\mathbb{T}} \chi_{(0,\pi)}(t)\chi_{(0,\pi)}(s)\hat{\mathcal{H}}\hat{x}(t-s)f(s)ds\]
and
\[(T(qaq)f)(t) = \frac{2i}{2\pi} \int_{\mathbb{T}} \chi_{(-\pi,0)}(t)\chi_{(-\pi,0)}(s)\hat{\mathcal{H}}\hat{x}(t-s)f(s)ds.\]

In other words,
\[(3) \quad T(pap) = 2ip(\mathcal{H}\hat{x})(\mathcal{D})p, \quad T(qaq) = 2iq(\mathcal{H}\hat{x})(\mathcal{D})q.\]

Similarly,
\[(paqf)(t) = \frac{1}{2\pi} \int_{\mathbb{T}} \chi_{(0,\pi)}(t)\chi_{(-\pi,0)}(s)\hat{x}(t-s)f(s)ds\]
and
\[(qapf)(t) = \frac{1}{2\pi} \int_{\mathbb{T}} \chi_{(-\pi,0)}(t)\chi_{(0,\pi)}(s)\hat{x}(t-s)f(s)ds.\]

Obviously, we have
\[\text{sgn}(t-s) = 1, \quad t \in (0, \pi), \quad s \in (-\pi, 0)\]
and
\[\text{sgn}(t-s) = -1, \quad t \in (-\pi, 0), \quad s \in (0, \pi).\]

By the definition of triangular truncation operator, we get
\[(T(paq)f)(t) = \frac{1}{2\pi} \int_{\mathbb{T}} \chi_{(0,\pi)}(t)\chi_{(-\pi,0)}(s)\hat{x}(t-s)f(s)ds\]
and
\[(T(qap)f)(t) = \frac{1}{2\pi} \int_{\mathbb{T}} \chi_{(-\pi,0)}(t)\chi_{(0,\pi)}(s)\hat{x}(t-s)f(s)ds.\]

In other words,
\[(4) \quad T(paq) = px(\mathcal{D})q, \quad T(qap) = -qx(\mathcal{D})p.\]

Combining (3) and (4), we obtain
\[(5) \quad T(a) = T(pap) + T(qaq) + T(paq) + T(qap)\]
\[= 2ip(\mathcal{H}\hat{x})(\mathcal{D})p + 2iq(\mathcal{H}\hat{x})(\mathcal{D})q + px(\mathcal{D})q - qx(\mathcal{D})p.\]

This proves the assertion for \(x \in \ell_2(\mathbb{Z}).\)

Define the mappings \(L_1, L_2 : A_{\log}(\mathbb{Z}) \to B(L_2(-\pi, \pi))\) by setting
\[L_1 : x \to T(x(\mathcal{D})), \quad x \in A_{\log}(\mathbb{Z})\]
and
\[L_2 : x \to 2ip(\mathcal{H}\hat{x})(\mathcal{D})p + 2iq(\mathcal{H}\hat{x})(\mathcal{D})q + px(\mathcal{D})q - qx(\mathcal{D})p, \quad x \in A_{\log}(\mathbb{Z}).\]
By Theorems 11 and 14 in [3], the operator $T : \Lambda_{\log}(B(L_2(-\pi, \pi))) \to B(L_2(-\pi, \pi))$ is bounded. Obviously,

$$\|L_1 x\|_\infty \leq \|T\|\Lambda_{\log}(B(L_2(-\pi, \pi))) \to B(L_2(-\pi, \pi))\|x\|_{\Lambda_{\log}}, \quad x \in \Lambda_{\log}(\mathbb{Z}).$$

Similarly, by the $\ell_1 \to \ell_1, \infty$ estimate for $\mathcal{H}_d$ and Theorem 14 in [3] we have that $\mathcal{H}_d : \Lambda_{\log}(\mathbb{Z}) \to \ell_1, \infty(\mathbb{Z})$ is a bounded operator. Obviously,

$$\|L_2 x\|_\infty \leq 2\|\mathcal{H}_d x\|_\infty + 2\|x\|_\infty \leq 2(1 + \|\mathcal{H}_d\|\Lambda_{\log} \to \ell_1, \infty)\|x\|_{\Lambda_{\log}}, \quad x \in \Lambda_{\log}(\mathbb{Z}).$$

By [5], we have $L_1 = L_2$ on $\ell_2(\mathbb{Z})$. Since $\ell_2(\mathbb{Z})$ is dense in $\Lambda_{\log}(\mathbb{Z})$ and since both $L_1$ and $L_2$ are bounded, it follows that $L_1 = L_2$. 

The following result replaces [3, Lemma 22].

**Lemma 4.** If $x \in \Lambda_{\log}(\mathbb{Z})$ is such that $x|_{2\mathbb{Z}} = 0$, then

$$\mu(p(\mathcal{H}_d x)(D)p) = \frac{1}{2}\mu(\mathcal{H}_d x),$$

where $p = M_{x(0, \pi)}$ is a multiplication operator.

**Proof.** The following observation is crucial: $\mathcal{H}_d x|_{2\mathbb{Z}+1} = 0$ (it follows from the definition of $\mathcal{H}_d$ and the assumption $x|_{2\mathbb{Z}} = 0$). Equivalently, the distribution $\mathcal{H}_d x$ is $\pi$-periodic. Now define an orthonormal basis $\{f_n\}_{n \in \mathbb{Z}}$ in the Hilbert space $L_2(0, \pi)$ by setting $f_n(t) = \pi^{-\frac{1}{2}}e^{2i\pi nt}, \: t \in (0, \pi)$. We claim that $\{f_n\}_{n \in \mathbb{Z}}$ is an eigenbasis for the operator $p(\mathcal{H}_d x)(D)p$. By linearity and continuity, it suffices to prove this for the case when $x$ is a finitely supported sequence. In this case, $x \in \ell_2(\mathbb{Z})$ and, therefore, $\mathcal{H}_d x$ is a $\pi$-periodic function. Hence, for any $t \in (0, \pi)$, we have

$$(p(\mathcal{H}_d x)(D)p)f_n(t) = \frac{1}{2\pi} \int_0^\pi \mathcal{H}_d x(t - s)f_n(s)ds = \frac{1}{2\pi} f_n(t) \cdot \int_0^\pi \mathcal{H}_d x(-s)f_n(s)ds$$

$$= \frac{1}{2\pi} f_n(t) \cdot \sum_{m \in \mathbb{Z}} (\mathcal{H}_d x)(2m) \int_0^\pi e^{2i(n-m)s}ds = \frac{1}{2}(\mathcal{H}_d x)(2n)f_n(t).$$

which proves the claim. The assertion follows from the claim. 

**Proof of Theorem 4.** As the statement of the theorem depends only on $\mu(x)$, we may assume without loss of generality that

$$x(n) = \begin{cases} \mu\left(\frac{n-1}{2}\right), & n \geq 1 \text{ is odd} \\ 0, & n \text{ is even} \\ 0, & n \leq 0 \end{cases}$$

and let $a = x(D)$. Obviously, $x|_{2\mathbb{Z}} = 0$ and $\mu(a) = \mu(x)$.

On the other hand, it follows from Lemma 3 and from the equalities $q = 1 - p$ that

$$p \cdot T(a) \cdot p = 2ip(\mathcal{H}_d x)p.$$

Thus, by [2] Formulas (2.2) and (2.3), p. 27] we have

$$(6) \quad \mu(T(a)) \geq \mu(p \cdot T(a) \cdot p) \overset{L.3}{=} 2\mu(p(\mathcal{H}_d x)(D)p) \overset{L.4}{=} \mu(\mathcal{H}_d x).$$

Obviously,

$$\frac{1}{2m + 2n + 1} \geq \frac{1}{2(m + n + 2)} \geq \frac{1}{4} \min\left\{ \frac{1}{n+1}, \frac{1}{m+1} \right\}, \quad m, n \geq 0.$$
Then,
\[
(H_{dx})(-2n) = \frac{1}{\pi} \sum_{m \geq 0} \frac{\mu(m, x)}{2m + 2n + 1} \geq \frac{1}{4\pi} \sum_{m \geq 0} \mu(m, x) \min \left\{ \frac{1}{n+1}, \frac{1}{m+1} \right\}, n \in \mathbb{Z}_+.
\]

Thus,
\[
(H_{dx})(-2n) \geq \frac{1}{8\pi} (S_d \mu(x))(n), \quad n \geq 0.
\]

It is now immediate that
\[
(7) \quad \mu(H_{dx}) \geq \frac{1}{8\pi} S_d \mu(x).
\]

Combining (6) and (7), we complete the proof. □

References

[1] Dykema K., Figiel T., Weiss G., Wodzicki M. Commutator structure of operator ideals. Adv. Math. 185 (2004), no. 1, 1–79.
[2] I.C. Gohberg and M.G. Krein, Introduction to the theory of linear non-selfadjoint operators, Transl. Math. Monogr., 18, Amer. Math. Soc., Providence, R.I., 1969.
[3] Sukochev F., Tulenov K., Zanin D. The optimal range of the Calderón operator and its applications. J. Funct. Anal. 277 (2019), no. 10, 3513–3559.

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