Reduced 2-coloured Khovanov Homology detects
the Trefoil

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1. Introduction

The question of which knots are detected by the Jones polynomial and its
coloured variants is still very much open, but recently several weaker questions
have been answered regarding a categorification of the Jones polynomial, first
introduced by Khovanov in [5]. In [6], Khovanov also introduced categorifica-
tions for two versions of the $n$-coloured Jones polynomial, one returning $[n + 1]$ for the unknot, the other returning 1.

The first major detection result concerning these constructions came in [2],
where Grigsby and Wehrli constructed a spectral sequence from the n-coloured
reduced categorification to knot Floer homology.

Theorem 1.1. Let $K \subset S^3$ be an oriented knot, $\bar{K} \subset S^3$, and $K^r$ its orientation
reverse. There is a spectral sequence whose $E^2$ term is $\tilde{Kh}_2(\bar{K})$ and whose $E^\infty$
term is $\tilde{HF}(S^3, K \# K^r)$.

This leads to the easy corollary that the reduced 2-coloured Khovanov homol-
ogy detects the unknot (and in fact it generalises to all $n > 1$). In the
unreduced case for $n = 2$, Hedden proved unknot detection in [3], using a
spectral sequence from the Khovanov homology to the Floer homology of the
branched double cover of a knot, first noted by Ozsváth and Szabó in [9].

It was then shown by Kronheimer and Mrowka in [7] that Khovanov homol-
ogy detects the unknot, and by Baldwin and Sivek in [1] that it detects the
trefoils.

The main result of this note is Theorem 2.1, where we prove that the reduced
2-coloured Khovanov homology also detects the trefoil.

2. Result

Theorem 2.1. Let $K$ be a knot in $S^3$. Then $K$ is the trefoil if and only if
rk $\tilde{Kh}_2(K) = 9$, where $\tilde{Kh}_2(K)$ denotes the reduced 2-coloured Khovanov homol-
ogy of $K$.

Proof. Let $K$ denote the (right-handed) trefoil in $S^3$. A computer calculation
(the code for which is available on GitHub[1]) of the rank of $\tilde{Kh}_2(K)$ gives 9.

https://github.com/robinsongeorge/Reduced-Khovanov-Homology
For the converse implication, suppose that $K \subset S^3$ is a knot such that $\text{rk} \tilde{Kh}_2(K) \leq 24$. In particular, from the spectral sequence in [2], it follows that
\[
\left( \text{rk} \tilde{HF}(K) \right)^2 = \text{rk} \tilde{HF}(S^3, K \# K') \\
\leq \text{rk} \tilde{Kh}_2(K) \\
\leq 24.
\]
Combining this with the fact that $\text{rk} \tilde{HF}(K)$ is odd, we see that $\text{rk} \tilde{HF}(K) = 1$ or 3. As noted in [3], when $\text{rk} \tilde{HF}(K) = 1$ the fact that $\tilde{HF}(K)$ is symmetric in the Alexander grading implies it must be supported in grading 0, and then since $\tilde{HF}(K)$ detects genus, $K$ must be the unknot. In the case $\text{rk} \tilde{HF}(K) = 3$, $K$ must be a trefoil, as shown in [4] by Hedden and Watson.

The fact that $\text{rk} \tilde{Kh}_2(K) = 9$ for the trefoil, and $\text{rk} \tilde{Kh}_2(K) = 25$ for the figure-eight knot suggests that perhaps the spectral sequence always collapses by the $E^2$ page for alternating knots, however this is not true for links since the 2-4 torus link has $\text{rk} \tilde{Kh}_2(L) = 18$.

References

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