Cheap Non-standard Analysis and Computability

Olivier Bournez
Ecole Polytechnique, LIX, 91128 Palaiseau Cedex, France
bournez@lix.polytechnique.fr

Sabrina Ouazzani
LACL, Université Paris-Est Créteil, 61 avenue du Général de Gaulle, 94010 Créteil, France
sabrina.ouazzani@lacl.fr

Abstract
Non standard analysis is an area of Mathematics dealing with notions of infinitesimal and infinitely large numbers, in which many statements from classical analysis can be expressed very naturally. Cheap non-standard analysis introduced by Terence Tao in 2012 is based on the idea that considering that a property holds eventually is sufficient to give the essence of many of its statements. This provides constructivity but at some (acceptable) price.

We consider computability in cheap non-standard analysis. We prove that many concepts from computable analysis as well as several concepts from computability can be very elegantly and alternatively presented in this framework. Our statements provide the bases for dual views and dual proofs to several statements already known in these fields.

2012 ACM Subject Classification Theory of computation/Computability, Mathematics of computing/Numerical analysis, Models of Computation/Continuous functions.

Keywords and phrases Computability Theory, Computable Analysis, Non-Standard Analysis

1 Introduction
While historically reasonings in mathematics were often based on the used of infinitesimals, in order to avoid paradoxes and debates about the validity of some of the arguments, this was later abandoned in favor of epsilon-delta based definitions such as today’s classical definition of continuity for functions over the reals.

Non standard analysis (NSA) originated from the work of Abraham Robinson in the 1960’s who came with a formal construction of non-standard models of the reals and of the integers [15]. Many statements from classical analysis can be expressed very elegantly, using concepts such as infinitesimals or infinitely large numbers in NSA: See e.g. [15] [6] [10] [11]. It not only have interests for understanding historical arguments and the way we came to some of today’s notions, but also clear interests for pedagogy and providing results that have not been obtained before in Mathematics. See e.g. [10] for nice presentations of NSA, or [11] for an undergraduate level book presenting in a very natural way the whole mathematical calculus, based on Abraham Robinson’s infinitesimals approach. See [9] for recent instructive pedagogical experiments on its help for teaching mathematical concepts to students.

However, the construction and understanding of concepts from NSA is sometimes hard to grasp. Its models are built using concepts such as ultrafilter that are obtained using non-constructive arguments through the existence of a free ultrafilters whose existence requires

1 ANR PROJECT RACAF
© Author: Please provide a copyright holder; licensed under Creative Commons License CC-BY
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany
to use the axiom of choice. Moreover, the dependance on the choice of this ultrafilter is sometimes not easy to understand (at least for non model-theory experts).

Terence Tao came in 2012 in a post in his blog [17] with a very elegant explanation of the spirit of many of the statements of non-standard analysis using only very simple arguments, that he called cheap non-standard analysis in opposition to classical non-standard analysis. This theory is based on the idea that the asymptotic limit of a sequence given by its value after some finite rank is enough to define non standard objects. Cheap non-standard analysis provides constructivity but this of course comes with some price (e.g. a non-total order on cheap non-standard integers, i.e. some indeterminacy).

Computability theory, classically dealing with finite or discrete structures such as a finite alphabet or the integers, has been far extended in many directions at this date. Various approaches have been considered for formalizing its issues in analysis, but at this stage the most standard approaches for dealing with computations over the reals are from computable analysis [19] and computability for real functions [12]. For other approaches for modeling computations over the reals and how they relate, see e.g. [6] or the appendices of [19].

In this paper, we explore how computability mixes with cheap non-standard analysis. We prove that many concepts from computable analysis as well as several concepts from computability can be very elegantly and alternatively presented in this framework. In particular, we prove that computable analysis concepts have very nice and simple formulations in this framework. We also obtain alternative, equivalent and nice formulations of many of its concepts in this framework.

Our approach provides an alternative to the usual presentation of computable analysis. In particular, nowadays, a popular approach to formalize computable analysis is based on Type-2 computability, i.e. Turing machines working over representations of objects by infinite sequences of integers: See [10] for a monograph based on this approach. Other presentations include original ones from [18], [7], and [14]. More recently, links have been established between type-2 computability and transfinite computations (see [5] for example) using surreal numbers. NSA has also been used in the context of various applications like systems modeling: See e.g. [13] or [2].

The paper is organized as follows. In Section 2 we recall cheap non-standard analysis. In Section 3, we present the very basics of constructions from NSA, and we state some relations to cheap non-standard analysis. In Section 4 we start to discuss computability issues, and we consider computability of cheap non-standard integers and rational numbers. In Section 5 we discuss some computability issues related to infinitesimals and infinitely large numbers. In Section 6 we discuss computability for real numbers. In Section 7 we go to computability for functions over the reals and we discuss continuity and uniform continuity. In Section 8 we discuss computability of functions over the reals. In Section 9, we discuss some applications illustrating the interest of using our framework. Finally, in Section 10 we discuss our constructions and we discuss some interesting perspectives.

In all what follows, $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ are respectively the set of integers, rational numbers and real numbers. We will sometimes also write $\omega$ for a synonym for $\mathbb{N}$: we will use $\omega$ preferably when talking about indices. In what follows, $\mathcal{A}$ is either $\mathbb{N}$ or $\mathbb{Q}$, and we assume $\mathbb{N} \subset \mathbb{Q}$. By number, we mean either a natural or a rational number. In the current version, we use a color coding to help our reader to visualize the type of each variable. However, the paper can be read without this color coding.
Cheap Non-Standard Analysis

We start by presenting/recalling cheap non-standard analysis [17]. It makes the distinction between two types of mathematical objects: standard objects $x$ and (strictly) non-standard objects $x = x_n$. Cheap non-standard objects are allowed to depend on an asymptotic parameter $n \in \omega$, contrary to standard objects that come from classical analysis. A cheap non-standard “object” $x$ is then defined by a sequence $x_n$, which is studied in the asymptotic limit $n \to \infty$, that is to say for sufficiently large $n$. Every standard “object” $x$ is also considered as a non-standard “object” identified by a constant sequence $x = x_n = x$ having value $x$. The underlying idea is similar to what is for example done in probability theory where an element of $\mathbb{R}$ is also implicitly considered as a probabilistic element of $\mathbb{R}$, depending if it has a measure associated. This is done with “object” referring to any natural mathematical concept such as “natural number”, “rational number”, “real number” “set”, “function” (but this could also be “point”, “graph”, etc. [17]).

The idea is then to consider that all reasonings are done in the asymptotic limit $n \to \infty$, that is to say for sufficiently large $n$. In particular, two cheap non-standard elements $x = x_n$, $y = y_n$ are considered as equal if $x_n = y_n$ after some finite rank. More generally, any standard true relation or predicate is considered to be true for some cheap non-standard object if it is true for all sufficiently large values of the asymptotic parameter, that is to say after some finite rank. Any operation which can be applied to a standard object $x$ can also be applied to a cheap non-standard object $x = x_n$ by applying the standard operation componentwise, that is to say for each choice of the rank parameter $n$. For example, given two cheap non-standard integers $x = x_n$ and $y = y_n$, we say that $x > y$ if one has $x_n > y_n$ after some finite rank. Similarly, we say that the relation is false if it is false after some finite rank. As another example, the sum $x + y$ of two cheap non-standard integers $x = x_n$ and $y = y_n$ is given by $x + y = (x + y)_n := x_n + y_n$. A cheap non-standard set $^\ast X$ is given by $^\ast X = X_n$, where each $X_n$ is a set, and if we write $x = x_n$, we have as expected $x \in ^\ast X$ if after some finite rank $x_n \in X_n$. Similarly, if $f = f_n : X_n \to Y_n$ is a cheap non-standard function from a cheap non-standard set $^\ast X$ to another cheap non-standard set $^\ast Y$, then $f(x)$ is the cheap nonstandard element defined by $f(x) = f(x)_n := f_n(x_n)$. Every standard function is also a non-standard function using all these conventions, as expected.

One key point is that introducing a dependence to a rank parameter leads to the definition of fully new concepts: infinitely small and large numbers. A cheap non-standard rational $x = x_n$ is infinitesimal if $0 < x \leq i$ for all standard rational number $0 < i$. For example, $x = x_n = 1/n$ is an infinitesimal. A cheap non-standard number can be infinitely large too: as an example, consider $\omega = \omega_n = n$ or $x = x_n = 2^n$, greater than any standard number. Note that the inverse of an infinitely large cheap non-standard number is a cheap non-standard infinitesimal.

From the fact that applying a standard operation to a cheap non-standard number is basically applying it to each possible value of $n$, separately, most of the classical analysis properties on operations can hence be transferred from the standard framework to the cheap non-standard one: for example, commutativity and associativity for addition and multiplication operations.

However this is not always the case when one considers statements on cheap non-standard objects. One typical illustration that transferring properties from standard predicates to cheap non-standard ones is not automatic is the law of excluded middle failure. Repeating [17]:

For instance, the nonstandard real number $x = x_n := (-1)^n$ is neither positive, negative, nor zero, because none of the three statements $(-1)^n > 0$, $(-1)^n < 0$, or $(-1)^n = 0$ are...
true for all sufficiently large $n$. Nevertheless, despite some peculiarities in the manipulation of statements, most of the standard first-order logic statements remain the same when quantified over cheap non-standard objects. We refer to [17] for a very pedagogical and more complete discussions about cheap non-standard concepts and some of its properties.

3 More on NSA: filters and ultrafilters

The classical constructions for non-standard analysis are done using free ultrafilters.

We recall the definition of an ultrafilter over an infinite set $\mathcal{I}$, called the index set. Typically, for us $\mathcal{I} = \omega$.

▶ Definition 1 (Filter). A filter $U$ over $\mathcal{I}$ is a non-empty set of subsets of $\mathcal{I}$ such that:
1. $U$ is closed under superset: if $X \in U$, and $X \subset Y$, then $Y \in U$.
2. $U$ is closed under finite intersections: If $X \in U$ and $Y \in U$, then $X \cap Y \in U$.
3. $\mathcal{I} \in U$, but $\emptyset \not\in U$.

In particular, since $X \cap X^c = \emptyset$, and $\emptyset \not\in U$, one cannot have both $X$ and its complement in $U$.

▶ Lemma 2 (Fréchet filter). The set of all cofinite (i.e. complements of finite) subsets of $\mathcal{I}$ is a filter. It is called the Fréchet filter.

▶ Definition 3 (Ultrafilter). An ultrafilter over $\mathcal{I}$ is a filter $U$ over $\mathcal{I}$ with the additional property that for each $X$, exactly one of the sets $X$ and $\mathcal{I} - X$ belongs to $U$. A free ultrafilter is an ultrafilter $U$ such that no finite set belongs to $U$. In the literature, a free ultrafilter is sometimes called a non-principal ultrafilter (in opposition to principal or fixed ones that thus contain a smallest element, called the principal element).

We just comment in the remaining lines how this relates to NSA. In NSA, one fixes a free ultrafilter $U$. One also considers sequences indexed by $\omega$. Sequences $(x_i)$ and $(y_i)$ are considered equal iff the set of indices $i$ such that $x_i = y_i$ is in the fixed free ultrafilter $U$. Consequently, basically, cheap non-standard analysis corresponds to the case where $U$ is not a free ultrafilter, but the Fréchet filter. One deep interest of the above construction (also called ultraproduct) is that taking $U$ as a ultrafilter provides a transfer theorem (Löb’s theorem) that guarantees that any first order formula is true in the ultraproduct iff the set of indices $i$ when the formula is true belongs to the ultrafilter $U$.

To some extend, cheap non-standard analysis constructions allow to reason on objects independently from the ultrafilter, in the following sense (missing proofs are in appendix or in arXiv.org version of this article\[2\]).

▶ Theorem 4. Two cheap non-standard numbers $a$ and $b$, respectively corresponding to the sequences $(a_i)$ and $(b_i)$, are equal iff for all free ultrafilter $U$ over $\mathbb{N}$ we have $(a_i) =_U (b_i)$.

The following lemma is based on a statement from [10]. For selfcontentness, and completeness we provide its proof, mostly repeating [10] Theorem 1.42 but proving also the required extension, as we need a variation of it.

▶ Lemma 5 (Folklore). For every infinite set $\mathcal{I}$, there exists a free ultrafilter over $\mathcal{I}$. Fix some infinite set $X_0$. There exists a free ultrafilter over $\mathcal{I}$ that contains $X_0$.

\[2\] Current reference submit/2240185.
Proof. The set of all cofinite (complements of finite) subsets of \( I \) is a filter over \( I \) (called the Fréchet filter).

The set of all cofinite subsets \( Y \) of \( I \) and of \( Y \) such that \( Y^c \cap X_0 \) is finite is also a filter.

Let \( A \) be the set of all filters \( F \) over \( I \) such that \( F \) contains all cofinite subsets of \( I \) and all \( Y \) such that \( Y^c \cap X_0 \) is finite.

Then \( A \) is nonempty and \( A \) is closed under unions of chains. By Zorn’s Lemma, \( A \) has a maximal element \( U \) (in fact, infinitely many maximal elements).

\( U \) is a filter and contains no finite set, because \( U \) contains all cofinite sets but \( \emptyset \not\in U \).

(resp. Furthermore, for \( Y \in U \), \( X_0 \cap Y \) is infinite, because \( U \) contains all \( Z = Y^c \) such that \( X_0 \cap Z^c = X_0 \cap Y \) is finite: otherwise \( Y \cap Z = Y \cap Y^c = \emptyset \) but \( \emptyset \not\in U \)).

To show that \( U \) an ultrafilter, we consider an arbitrary set \( X \subseteq I \) and prove that there is a filter \( V \supseteq U \) which contains either \( X \) or \( I - X \), so by maximality, \( X \in U \) or \( I - X \in U \).

Case 1: For all \( Y \in U \), \( X \cap Y \) is infinite. \( X \) and each \( Y \in U \) belong to the set \( V = \{ Z \subseteq I : Z \supseteq X \cap Y \text{ for some } Y \in U \} \).\( V \) is a filter over \( I \), because \( V \) is obviously closed under supersets and finite intersections, and the hypothesis of Case 1 guarantees that each \( Z \in V \) is infinite.

Case 2: For some \( Y \in U \), \( X \cap Y \) is finite. Then for every \( W \in U \), \( (I - X) \cap W \) is infinite, for otherwise \( Y \cap W \in U \) would be finite. Case 1 applies to \( I - X \), so the set \( V = \{ Z \subseteq I : Z \supseteq (I - X) \cap Y \text{ for some } Y \in U \} \) is a filter over \( I \) such that \( V \subset U \), \( I - X \in V \).

We see that \( X \) belongs to \( U \) iff for all \( Y \in U \), \( X \cap Y \) is infinite. In particular, \( X_0 \) belongs to \( U \) iff for all \( Y \in U \), \( X_0 \cap Y \) is infinite. Hence, \( X_0 \in U \). 

We now go to the proof of Theorem 4.

Proof. Fix a free ultrafilter \( U \). Suppose that \( (a_i) \) and \( (b_i) \) represent the same cheap non-standard number. After some finite rank \( a_i = b_i \). Then \( \{ i \mid a_i = b_i \} \) is in \( U \) (as its complement is finite, and hence not inside). So \( (a_i) =_U (b_i) \) for that free ultrafilter.

Conversely, assume that for all rank \( n_0 \), there is a rank \( n \geq n_0 \) with \( a_n \neq b_n \). Then \( X = \{ n \mid a_n \neq b_n \} \) is infinite. By Lemma 5 one can build a free ultrafilter \( U \) with \( X \in U \). Hence \( (a_i) \neq_U (b_i) \) for that free ultrafilter \( U \). 

4 Computability for Integers or Rational Numbers

4.1 Very Basic Notions From Computability

We assume some basic familiarity with computability theory. In computability theory, any integer \( n \in \mathbb{N} \) is computable: there exists some Turing machine \( M \) that writes \( n \) in binary. However, not all total functions \( f : \mathbb{N} \to \mathbb{N} \) are computable (to avoid ambiguities we will say total recursive for “computable” in this context): there does not always exist some Turing machine \( M \) that takes as input \( n \) in binary and outputs \( f(n) \) in binary. An example of total recursive function is \( x \mapsto x + 1 \). An example of a total non recursive function is the function \( \gamma \) which maps \( n \) to \( x_n + 1 \), where \( x_n \) is the output of the \( n \)th Turing machine on input \( n \), for a given (non-assumed computable) enumeration of terminating Turing machines. In what follows \( \gamma \) will denote such a non total recursive function.

We used the wording “Turing machines”, but it is well known that the set of total recursive functions can be defined abstractly without referring to Turing machines: this is the smallest set that contains the constant function 0, the successor function \( s(x) = x + 1 \), projection functions, and closed under composition, primitive recursion, and safe minimization. Safe
minimization is minimization over safe predicates, that is to say predicates \( P(n, m) \) where for all \( n \) there is a \( m \) with \( P(n, m) = 1 \).

We will several times use the following easy remark: If a function is computable for all arguments above a certain rank, then this function is computable. More formally:

▶ **Theorem 6 (Computability for all indices).** For any total function \( f : \mathbb{N} \to \mathbb{N} \), for any finite \( n_0 \), if there is some total recursive function \( g : \mathbb{N} \to \mathbb{N} \) such that \( g(n) = f(n) \) for all \( n \geq n_0 \), then there is a total recursive function \( h : \mathbb{N} \to \mathbb{N} \) such that \( h(n) = f(n) \) for all \( n \in \mathbb{N} \).

### 4.2 Computable Cheap Non-Standard Numbers

In the literature, no discussions exist about the computability of numbers: any standard number \( n \in \mathcal{A} \) is computable. But here cheap non-standard integer or cheap non-standard rational numbers may not be computable:

▶ **Definition 7 (Computable cheap non-standard number).** A cheap non-standard number \( x = x_n \) is computable if \( x_n \) seen as a sequence from \( \omega \) to \( \mathcal{A} \) is total recursive.

For example, \( x = x_n = \gamma(n) \) is not a computable cheap non-standard integer. Computable cheap non-standard integers include \( \omega = \omega_n = n \).

Our purpose is first to understand to what corresponds the subset of the computable cheap non-standard numbers among all cheap non-standard numbers: can it be defined abstractly, i.e. taking cheap non-standard analysis as a basis (i.e. in the spirit of [11] that presents mathematical calculus taking NSA as a basis)?

The following facts are easy: As usual \( \oplus(x, y) \) denotes \( \max(0, x - y) \).

▶ **Theorem 8 (Stability by total recursive functions).** For any cheap non-standard computable numbers \( x_1, x_2, \ldots, x_k \), for any standard total recursive function \( f : \mathcal{A}^k \to \mathcal{A} \), we have that \( f(x_1, x_2, \ldots, x_k) \) is a computable cheap non-standard number.

▶ **Theorem 9 (Basic properties).** The set of cheap non-standard computable natural numbers is a semiring: In particular, it is stable by +, ◦, ·. The set of cheap non-standard computable rational numbers is a ring: In particular, it is stable by +, −, inverse, ·.

▶ **Theorem 10 (First characterization).** The set of cheap non-standard computable numbers is the smallest set that contains \( \omega \) and that is stable by standard total recursive \( f : \mathbb{N} \to \mathcal{A} \).

= The set of cheap non-standard computable numbers is also the set of \( f(\omega) \) for total recursive standard \( f : \mathbb{N} \to \mathcal{A} \).

**Proof.** The Cheap non-standard integer \( \omega \) is computable. When \( x \) is computable and \( f \) is a standard total recursive function, then, \( f(x) \) is computable. Now, from definitions \( x = x_n \) is computable, if \( x_n = f(n) \) for some standard total recursive \( f \), hence \( x = f(\omega) \). First item follows.

Second item is a direct corollary of above reasoning. ◼

### 4.3 Shift Operation and Preservation Property

The previous properties can also be stated in another alternative way: Consider the following operation \( \textbf{shift} \) that maps cheap non-standard numbers to cheap non-standard numbers:

▶ **Definition 11 (Shift operation).** Whenever \( x = x_n \), \( x^+ \) is defined by \( x^+ = (x^+)_n = x_{n+1} \).
Notice that \( n^+ = n \) for all standard integer \( n \). However, \( x^+ \) is not necessarily \( x \) for a cheap non-standard \( x \). In particular, \( \omega^+ = \omega + 1 \). In other words, using a non-standard analysis inspired vocabulary, \( \text{shift} \) is not an internal operation.

\[ \text{Theorem 12.} \quad \text{The set of computable cheap non-standard numbers is the smallest set that contains all solutions of } x^+ = f(x) \text{ for } f \text{ standard total recursive, and that is stable by standard total recursive } f : \mathbb{N} \to \mathcal{A}. \]

**Proof.** Cheap non-standard \( \omega \) integer can be obtained as a solution of \( \omega^+ = \omega + 1 \). Hence this class contains all computable cheap non-standard numbers from above statements. We only need to state that cheap non-standard numbers are stable by such a \( \text{shift} \) equation: Assume that \( x = x_n \) is solution of \( x^+ = f(x) \). Then after some finite rank \( n_0 \), we must have \( x_{n_0+k} = f^{[k]}(x_{n_0}) \), where \( f^{[k]} \) denotes \( k \)th iteration of \( f \) (computability of \( x_{n_0} \) follows from Theorem 6). This yields computability for indices \( n \geq n_0 \). And hence, this yields computability for all \( n \) by Theorem 6.

A key remark is that the unary \( \text{shift} \) operation can actually be extended to a binary operation. A cheap non-standard element of \( \omega \) is called a cheap non-standard index.

\[ \text{Definition 13 (Shift).} \quad \text{Given some cheap non-standard number } x = x_n \text{ and some cheap non-standard index } y = y_n, \text{ let } x+y \text{ be defined by } (x+y)_n = x_n+y_n. \]

It can be checked that this is a valid definition: its value is independent of the representative. It can also be checked that it satisfies \( x^0 = x \), \( x^1 = x^+ \), \( x^{(y+1)} = (x^y)^+ \), \( x^{(y+z)} = (x^y)^{+z} \) for any cheap non-standard number \( x \) and cheap indices \( y \) and \( z \).

\[ \text{Theorem 14.} \quad \text{Assume that } x \text{ and } y \text{ are computable. Then } x+y \text{ is computable.} \]

From previous definitions, we derive easily the following preservation property.

\[ \text{Theorem 15 (Preservation property).} \quad \text{Let } P \text{ be some standard property over the numbers. If } P(n_1, \ldots, n_k) \text{ holds for non standard numbers } n_1, \ldots, n_k, \text{ then } P(n_1^n, \ldots, n_k^n) \text{ holds. More generally, } P(n_1^{+n}, \ldots, n_k^{+n}) \text{ holds for all cheap non-standard index } n. \]

In some axiomatic view, computability of cheap non-standard numbers can be summarized as follows:

\[ \text{Theorem 16.} \quad \text{Not all cheap non-standard numbers are computable.} \]

- Computable cheap non-standard numbers include all standard numbers. The image of a computable cheap non-standard number by a standard total recursive function is computable.
- The infinitely large cheap non-standard number \( \omega \) satisfying \( \omega = \omega_n = n \) is among computable numbers.
- Computable cheap non-standard numbers are exactly those that can be obtained by above rules.
- There exists some operation \( (\cdot)^+ \) over cheap non-standard numbers, that preserves standard numbers, and that satisfies preservation property (Theorem 15).

## 5 Infinitesimals and Infinitely Large Numbers

Any cheap non-standard rational number \( x \) is of the form \( p/q \) for some cheap non-standard integers \( p \) and \( q \). It is computable iff it is of the form \( p/q \) with \( p \) and \( q \) computable.
Cheap non-standard integers as well as cheap non-standard rational numbers can be infinitely large. Cheap non-standard rational numbers can also be infinitesimals. For example, $\frac{1}{\gamma}$ and $2^{-\omega}$ are computable infinitesimals. Cheap non-standard rationals $x = x_n = \frac{1}{\gamma(n)}$ as well as $x = x_n = 2^{-\gamma(n)}$ are non-computable infinitesimals.

Definition 17 (Infinitely large and infinitesimal numbers). A cheap non-standard number $x$ is infinitely large iff for all standard number $y$, one has $x \geq y$. A cheap non-standard rational number $0 < x$ is infinitesimal iff for all standard rational number $0 < y$, one has $0 < x \leq y$.

One key point in the above concept is that this involves a quantification over all standard number $y$, which is weaker than over all cheap non-standard $y$. Actually, we however have the following phenomenon:

Theorem 18. Let $x$ (respectively: $0 < x$) be some cheap non-standard number that is infinitely large (resp. infinitesimal). For any cheap non-standard number $y$ (resp. $0 < y$) there exists some cheap non-standard number $x'$, of the form $x' = x + n$ for some cheap non-standard finite index $n$, with $x' \geq y$ (resp. $0 < x' \leq y$).

Proof. Let $y$ some cheap non-standard number. Write $y = y_i$.

Consider some $i \in \mathbb{N}$. As $x = x_n$ is infinitely large (respectively: infinitesimal), there must exists some finite rank $n_0$ such that for all $n \geq n_0$, we have $x_n \geq y_i$ (resp. $0 < x_n \leq y_i$).

Let $g: \mathbb{N} \to \mathbb{N}$ be the function that maps $i$ to the corresponding $n_0$ for all $i \in \mathbb{N}$. Consider cheap non-standard index $n$ defined by $n = n_i = g(i)$.

From definitions $x' = x + n$ is such that $x' = x'_i = x_{i+g(i)}$ and hence satisfy $x'_i \geq y_i$ (resp. $0 < x'_i \leq y_i$) for all $i$. The conclusion follows.$\blacksquare$

Fix some computable infinitesimal $0 < y$. We have that for any computable infinitesimal $0 < \epsilon$, there always exists some cheap non-standard finite index $n$ with $0 < \epsilon^n \leq y$. However, this $n$ can be non-computable. Therefore, it is natural to consider the following notion:

Definition 19 (Effectiveness). We say that some computable infinitesimal $0 < \epsilon$ is effective with respect to computable $0 < y$ iff there exists some cheap non-standard computable index $n$ with $\epsilon^n \leq y$.

This is clearly a reflexive relation as $\epsilon^0 = \epsilon$. It is also transitive:

Theorem 20 (Transitivity of computably bounded relation). Let $\epsilon, \epsilon', y$ be some non zero positive computable infinitesimals. If $\epsilon$ is effective with respect to $\epsilon'$ and $\epsilon'$ effective with respect to $y$, then $\epsilon$ is effective with respect to $y$.

Proof. If $\epsilon$ is effective with respect to $\epsilon'$ and $\epsilon'$ effective with respect to $y$ then there exists some cheap non-standard computable finite index $n_j$ with $\epsilon'^{+n_j} \leq \epsilon'$, and some cheap non-standard computable finite index $n_j$ with $\epsilon'^{+n_j} \leq y$. But then $\epsilon'^{+n_j + n_j} \leq y$: Indeed, apply $(\cdot)^{+n_j}$ to members of $\epsilon'^{+n_j} \leq \epsilon'$ to get

$$\epsilon'^{+(n_j + n_j)} = (\epsilon'^{+n_j})^{+n_j} \leq \epsilon'^{+n_j} \leq y$$

from previously stated properties of shift operation.$\blacksquare$

As a consequence, the following notion is natural and provides an equivalence relation:

Definition 21. We say that two computable infinitesimals $0 < \epsilon$ and $0 < \epsilon'$ are computably equivalent iff $0 < \epsilon$ is effective with respect to $0 < \epsilon'$ and conversely.
Theorem 22. \( \frac{1}{2^{n+1}} \) is effective with respect to any computable infinitesimal \( 0 < \epsilon \).

Proof. Consider computable cheap non-standard index \( m \) given by \( m = g(n) \) where standard function \( g(n) = \lfloor 1/n \rfloor - 1 \) for \( n \geq n_0 \), and say \( g(0) = 1 \) (as \( 0 < \epsilon \), its components are non-zero after some rank). Then \( \left( \frac{1}{2^{n+1}} \right)^{+m} < \epsilon \).

A computable infinitesimal \( 0 < \epsilon \) is said to be monotone if \( \epsilon = \epsilon_n \) with \( \epsilon_{n+1} \leq \epsilon_n \) for all \( n \). Monotone computable infinitesimals include \( \frac{1}{2^n} \) and \( 2^{-\omega} \).

Theorem 23. Any monotone computable infinitesimal \( 0 < \epsilon \) is effective with respect to \( \frac{1}{2^{n+1}} \). All monotone computable infinitesimals are computably equivalent.

Proof. Consider \( \epsilon = \epsilon_n = f(n) \) be some computable monotone infinitesimal, i.e. with \( f(n) \) total recursive and decreasing. From Theorem 18, there exists for some cheap non-standard finite index \( m \), with \( \epsilon^m \leq \frac{1}{2^{n+1}} \). We get that predicate \( P(n, m) \) given by \( f(n + m) \leq \frac{1}{2^{n+1}} \) is safe. It follows that \( \mu m P(n, m) \) is computable. Consider \( m' = m_0 = \mu m P(n, m) \), hence computable. We have \( \epsilon^{+m} \leq \frac{1}{2^{n+1}} \).

First statement follows.

Second statement is a clear corollary.

We say that some computable infinitesimal \( 0 < \epsilon \) is effective if it belongs to the above class: it is monotone or computably equivalent to some monotone computable infinitesimal.

Corollary 24. A computable infinitesimal \( 0 < \epsilon \) is effective iff it is effective with respect to \( \frac{1}{2^{n+1}} \). Any effective computable infinitesimal is effective with respect to any computable \( 0 < \epsilon \).

6 Computability for Real Numbers

Functions from the reals to the reals are the main studied functions in computable analysis. That’s why after having studied computability for cheap non-standard integer and rational numbers, we now go to computability for cheap non-standard real numbers.

Definition 25 (Computability for real numbers). Fix some effective computable infinitesimal \( \epsilon \). A standard real \( x \) is said to be computable if there exists some cheap non-standard computable rational \( \frac{p}{q} \) such that \( |x - \frac{p}{q}| \leq \epsilon \).

Theorem 26. The previous definition is not depending on \( \epsilon \) if this holds for an effective infinitesimal \( 0 < \epsilon \), then it holds for any other effective computable infinitesimal \( 0 < \epsilon' \).

Proof. Infinitesimal \( \epsilon \) is effective with respect to \( 1/u \) for \( u = 2^{\lfloor \log(2/\epsilon') \rfloor} \). Let \( \tau = \epsilon^{+n} \) the corresponding infinitesimal: \( \tau \leq 1/u \), that is \( u\tau < 1 \). Let \( \frac{p}{q} = p^{+n} \) and \( \frac{q'}{u} = q^{+n} \). We know (using preservation Theorem 15) that \( |x - \frac{p}{q}| \leq \tau \). Consider then \( \frac{p'}{q'} = \frac{u\frac{p}{q}}{\frac{q'}{u}} \). This guarantees \( |x - \frac{p'}{q'}| \leq \epsilon' \). Indeed, \( |ux - u\frac{p}{q}| = |u||x - \frac{p}{q}| \leq u\tau < 1 \), implies \( |ux - u\frac{p}{q}| \leq |u\frac{p}{q} - u\frac{p}{q}| + |ux - u\frac{p}{q}| \leq 1 + 1 = 2 \) using definition of what integer part is, and then \( \frac{|u\frac{p}{q}}{u} = 2^{1-\lfloor \log(2/\epsilon') \rfloor} \leq \epsilon' \).

Two cheap non-standard reals are said to be infinitely close (respectively: effectively infinitely close) if the absolute value of their difference is less than some (resp. effective) computable infinitesimal \( 0 < \epsilon \).
Definition 27 (Left and Right-Computability for real numbers). A standard real \( x \) is said to be left-computable (respectively: right-computable) if it is infinitely close to some cheap non-standard computable rational \( \frac{p}{q} \) with \( \frac{p}{q} \leq x \) (resp. \( \frac{p}{q} \geq x \)).

Theorem 28. A standard real \( x \) is computable iff it is effectively infinitely close to some cheap non-standard computable rational \( \frac{p}{q} \). A standard real \( x \) is computable iff it is right-computable and left-computable.

Proof. First statement is just a restatement of the definition. Concerning second statement: Direction from left to right of second item is trivial. Direction from right to left is the following. Assume that \( x \) is right and left-computable. There exists some \( p = p_n = p(n) \), \( q = q_n = q(n) \), \( p' = p'_n = p'(n) \), \( q' = q'_n = q'(n) \) such that \( \frac{p}{q} \leq x \leq \frac{p'}{q'} \) and \( x - \frac{p}{q} \) and \( x - \frac{p'}{q'} \) both infinitesimal. This must hold componentwise for \( n \geq n_0 \) for some \( n_0 \). Replacing if needed the values of functions for indices less than \( n_0 \), we can assume without loss of generality that \( n_0 = 1 \).

Given \( n \), consider \( \phi(n) = \max_{1 \leq i \leq n} \frac{p(i)}{q(i)} \), and \( \phi'(n) = \min_{1 \leq i \leq n} \frac{p(i)}{q(i)} \). \( \phi \) (respectively: \( \phi' \)) is an increasing (resp. decreasing) function converging to \( x \). We have

\[ \phi(n) \leq x \leq \phi'(n). \]

The predicate \( P(n, m) \) given by \( \phi'(m) - \phi(m) \leq \frac{1}{2^n} \) is safe. Consequently, \( \mu m P(n, m) \) is computable, and \( \phi(\mu m P(n, m)) \) is a computable sequence of rational numbers proving that \( x \) is computable, since considering \( \frac{p}{q} = \left( \frac{p}{q} \right)_{n} = \phi(\mu m P(n, m)) \), we get

\[ \left| \frac{p}{q} - x \right| \leq \frac{1}{2^n}. \]

Let \( D = \{ x \in \mathbb{Q} | r = \frac{p}{q} \text{ for integers } n, m \} \): these are the rationals with finite binary representation. They are sometimes also called dyadic rationals.

Theorem 29. We can always assume \( \frac{p}{q} \in *D \) in previous statements, i.e. \( q \) to be of the form \( 2^m \) for some cheap non-standard integer \( m \).

Proof. This is the case in the proof of Theorem 26. The case of left and right-computability is similar.

One important theorem is that this corresponds to the classical definition of computability for reals (in the sense of computable analysis): Formally, according to classical definitions and statements from [19, 12, 1], this is equivalent to say that the following holds:

Theorem 30. A standard real \( x \) is computable iff there exist some total recursive functions \( p(n) \) and \( q(n) \) such that \( |x - \frac{p(n)}{q(n)}| \leq \frac{1}{2^n} \) for all integer \( n \). A standard real \( x \) is left-computable (resp. right-computable) iff there exist some total recursive functions \( p(n) \) and \( q(n) \) such that \( x = \sup_{n} \frac{p(n)}{q(n)} \) (resp. \( x = \inf_{n} \frac{p(n)}{q(n)} \)).

Proof. Consider monotone computable infinitesimal \( \epsilon = \epsilon_n = \frac{1}{2^n} \). There must exist some computable cheap non-standard integers \( p \) and \( q \) such that \( |\frac{p}{q} - x| \leq \epsilon \). The total recursive functions \( p \) and \( q \) such that \( p = p_n = p(n) \) and \( q = q_n = q(n) \) satisfies the above property after some finite rank \( n_0 \). They can be fixed to \( p(n_0) \) and \( q(n_0) \) on the finitely many \( n \) before \( n_0 \) so that this holds for all \( n \).
Conversely, if this holds, \( \epsilon = \epsilon_n = \frac{1}{2^n} \) is a monotone computable infinitesimal, and \( p \) and \( q \) such that \( p = p_n = p(n) \) and \( q = q_n = q(n) \) are computable cheap non-standard integers such that \( \left| \frac{p}{q} - x \right| \leq \epsilon \).

The statements for right and left-computability are obtained in a similar fashion.

7 Continuity and Effective Uniform Continuity for Real Functions

The following theorem is left as an exercise in [17].

\( \textbf{Theorem 31.} \) A function \( f : X \subset \mathbb{R} \rightarrow \mathbb{R} \) is continuous iff for all standard element \( x \) of \( X \), and for all cheap non-standard element \( y \) infinitely close to \( x \), then \( f(y) \) is infinitesimally close to \( f(x) \).

We provide here the proof for completeness:

\( \textbf{Proof.} \) Function \( f \) is continuous in \( x \) iff for all \( \epsilon \) there exists some \( \delta \) such that whenever \( |x - y| \leq \delta \) we have \( |f(x) - f(y)| \leq \epsilon \).

For the right to the left direction: Assume \( f \) is continuous. Consider some standard \( x \) of \( X \), and some cheap non-standard element \( y \) infinitely close to \( x \). Consider some standard \( 0 < \epsilon \), and the corresponding standard \( \delta \). As \( x - y \) is infinitesimal, writing \( y = y_n \), there is some \( n_0 \) such that for all \( n \geq n_0 \), we have \( |x - y_n| \leq \delta \). Consequently, \( |f(x) - f(y_n)| \leq \epsilon \), that is to say we have \( |f(x) - f(y)| \leq \epsilon \). As this holds for all standard \( \epsilon \), \( f(y) \) is infinitely close to \( f(x) \).

For the left to the right direction: Assume \( f \) is not continuous. That means that there exists some \( \epsilon \) such that for all \( \delta \), say \( \delta(n) = \frac{1}{n} \), there exists some \( y(n) \) with \( |x - y(n)| \leq \delta(n) \) and \( |f(x) - f(y(n))| > \epsilon \). That means that \( y = y_n = y(n) \) is some cheap non-standard element infinitely close to \( x \) but with \( f(y) \) not infinitely close to \( f(x) \).

Similarly, the following can be established:

\( \textbf{Theorem 32.} \) A function \( f : X \subset \mathbb{R} \rightarrow \mathbb{R} \) is uniformly continuous if for all cheap non-standard element \( x \) of \( X \), and for all cheap non-standard element \( y \) infinitely close to \( x \), then \( f(y) \) is infinitesimally close to \( f(x) \).

For example standard function \( x \mapsto x^2 \) with domain \( \mathbb{R} \) is not-uniformly continuous as \( (x + 1/x)^2 = x^2 + 2 + \frac{1}{x^2} \) is not infinitely close to \( x^2 \) when \( x \) is infinite large. However, it is uniformly-continuous (and hence continuous) on \([0, 1]\) as for \( y \) infinitesimal, \((x + y)^2 = x^2 + 2xy + y^2 \) is always infinitely close to \( x^2 \) when \( x \in [0, 1] \) (hence is bounded).

Notice that above theorems are defining concepts of continuity and uniform continuity very elegantly: there is no alternance of quantifiers compared to the classical \( \epsilon-\delta \) definition. Refer to [9] for practical measurements on the benefits of NSA concepts in teaching.

8 Computability For Real Functions

We now go to computability issues:

\( \textbf{Definition 33.} \) Fix some effective computable infinitesimal \( 0 < \epsilon \). A function \( f : X \subset \mathbb{R} \rightarrow \mathbb{R} \) has an effective modulus of continuity iff there exists some computable nonstandard \( \delta \) such that for all standard \( x \) and \( y \), if \( |x - y| \leq \delta \) then \( |f(x) - f(y)| \leq \epsilon \).
Obviously, such a function is uniformly continuous, and hence continuous. More fundamentally:

**Theorem 34.** The previous concept is not depending on \( 0 < \epsilon \): if this holds for an effective computable infinitesimal \( 0 < \epsilon \), then it holds for any other effective computable infinitesimal \( 0 < \epsilon' \).

**Proof.** Infinitesimal \( \epsilon \) is effective with respect to \( \epsilon' \). Let \( \tau = \epsilon^{+m} \) the corresponding infinitesimal: that is to say \( \tau \leq \epsilon' \). But then \( \delta = \epsilon^{+m} \) provides the property for \( \epsilon' \) as we know using Theorem 15 that

\[
|\frac{f(x) - f(y)}{|x - y|}| \leq \epsilon \leq \epsilon'.
\]

\(\triangleright\)

**Theorem 35.** We can always assume \( \delta \in \ast D \) in previous statements, i.e. \( \delta \) to be of the form \( \frac{p}{2^m} \) for some cheap non-standard integer \( m \).

**Proof.** This follows from the proof of previous theorem as applying the sense from left to right, and then left to right provides a \( \delta \) of that form. \(\triangleright\)

This still corresponds to the classical notion from computable analysis. Formally, according to classical definitions and statements from [19, 12, 1], this is equivalent to say that the following holds:

**Theorem 36.** A function \( f : X \subset \mathbb{R} \to \mathbb{R} \) has an effective modulus of continuity iff there exists some total recursive \( m : \mathbb{N} \to \mathbb{N} \) such that if \( |x - y| \leq 2^{-m(n)} \) then \( |f(x) - f(y)| \leq 2^{-n} \) for all standard \( x, y \in X \).

**Proof.** Assume there is such a recursive \( m \). Consider monotone computable cheap non-standard infinitesimal \( 0 < \epsilon \) given by \( \epsilon = \epsilon_n = 2^{-n} \). Consider computable cheap non-standard rational \( \delta \) given by \( \delta = \delta_n = 2^{-m(n)} \). Then

\[
|\frac{f(x) - f(y)}{|x - y|}| \leq \delta \text{ then } |f(x) - f(y)| \leq \epsilon
\]

holds.

Conversely, assume that function \( f \) has an effective modulus of continuity. Consider monotone computable infinitesimal \( \epsilon = 2^-m \). There must exists some computable cheap non-standard \( \delta \) such that

\[
|\frac{f(x) - f(y)}{|x - y|}| \leq \delta \text{ then } |f(x) - f(y)| \leq \epsilon
\]

That means that there exists some finite rank \( n_0 \) such that the properties holds componentwise for \( n \geq n_0 \). Write \( \delta = \delta_n = f(n) \) for some total recursive \( f \). Consider \( m(n) = \lceil -\log(f(n)) \rceil \) for \( n \geq n_0 \) and \( m(n) = m(n_0) \) for \( n < n_0 \). This provides the expected property for all \( n \).

\(\triangleright\)

Consider an indexed family of cheap non-standard numbers: to some parameter \( i \), is associated some cheap non-standard number \( x(i) = x(i)_n \). We say that the family is uniformly computable in \( i \) if there exists some standard total recursive function \( f(i, n) \) such that \( x(i)_n = f(i, n) \).

**Definition 37 (Computability for functions over the reals).** Fix some effective computable infinitesimal \( \epsilon \). We say that \( f : [a, b] \subset \mathbb{R} \to \mathbb{R} \) (standard domain) is computable iff
1. \textbf{[discretization property]}: there exists some computable \( \delta \) such that
   \[
   \text{if } |x - y| \leq \delta \text{ then } |f(x) - f(y)| \leq \epsilon.
   \]

2. \textbf{[it has some uniform approximation function over the rationals]}: There exists some indexed family of cheap non-standard rationals \( \psi(q) \), uniformly computable in \( q \), such that
   \[
   |\psi(q) - f(q)| \leq \epsilon
   \]
   for all \( q \in \mathbb{Q} \cap [a, b] \)

Before going to the statement and proof that this corresponds to the classical notion of computability for functions over the reals, notice that one main interest of the above definition is that it sounds more natural and easier to grasp than classical ones\(^3\) in particular, item 1. is a very natural discretization property\(^4\).

\textbf{Theorem 38.} \textit{The previous definition is not depending on \( \epsilon \).}

\textbf{Proof.} This follows from Theorems 34 for item 1. Item 2. holds for some effective \( \epsilon \) iff it holds for any effective \( \epsilon \) using a reasoning similar to Theorem 34. \( \square \)

\textbf{Theorem 39.} \textit{Without loss of generality, we can always assume \( \delta \) and \( q \) in above definiton to be in \( \ast \mathbb{D} \) and \( \mathbb{D} \), i.e. to be of the form \( 2^m \) for some cheap non-standard integer \( m \) or \( m \) instead of being rational numbers.}

\textbf{Proof.} The fact that this is true for item 1. is Theorem 35. Now, if 1. holds, then \( \psi(q) \) can be replaced by \( \psi(q') \) where \( q' \in \ast \mathbb{D} \) is approximating cheap non-standard rational \( q \) at precision \( \delta \). The error would then be at most \( 2\epsilon \) instead of \( \epsilon \). But as this is equivalent to hold for \( \epsilon \) by above Theorem, we get the statement. \( \square \)

It turns out that our definition is equivalent to the classical notion of computability in computable analysis. Formally, according to classical definitions and statements from \cite{19, 12, 11}, this is equivalent to say that the following holds:

\textbf{Theorem 40.} A real function \( f : [a, b] \to \mathbb{R} \) is computable iff it is computable in the sense of computable analysis.

\textbf{Proof.} It is proved for example in \cite{12, Corollary 2.14} that \( f \) as above is computable in the sense of computable analysis iff

1. \textbf{[it has an effective modulus of continuity]} there exists some total recursive \( m : \mathbb{N} \to \mathbb{N} \) such that if \( |x - y| < 2^{-m(n)} \) then \( |f(x) - f(y)| < 2^{-n} \) for all standard \( x, y \in X \).

2. \textbf{[it has some computable approximation function]}: there exists some total recursive \( \psi : \mathbb{D} \cap [a, b] \times \mathbb{N} \to \mathbb{D} \) such that for all standard rational \( d \in \mathbb{D} \), standard integer \( n \)
   \[
   |\psi(d, n) - f(d)| \leq 2^{-n}.
   \]

\(^3\) See statement of Theorem 10 for example of a classical definition.

\(^4\) It follows from the proofs that the discretization property is equivalent to the existence of an effective modulus of continuity. However, we believe the latter concept is harder to grasp, as basically talking about the dependence of a \( \delta \) from \( \epsilon \), whose meanings is not so natural.
XX:14 Cheap Non-standard Analysis and Computability

Now, Item 1. is equivalent to our Item 1. by Theorem 39. Concerning Item 2. Using Theorem 39 and Theorem 38 considering infinitesimal $2^{-2}$, suppose there exists some $\psi'$ such that

$$|\psi'(d) - f(d)| \leq \epsilon = 2^{-2}$$

for all cheap non-standard number $d \in D \cap [a,b]$. Write $\psi'(d) = \psi'(d)_n = \psi(d)_n$. Above inequality yields item 2 above.

Conversely, assume we have $|\psi'(d, n) - f(d)| \leq 2^{-n}$ for some $\psi'$. Consider $\psi(d) = \psi(d)_n = \psi'(d, n)$. This yields item 2. of our definition.

9 Examples of Applications

Hence, as expected, results known about computable functions are true in this framework, and conversely, however our framework can present alternative ways to establish proofs.

Notice that cheap non-standard analysis is however distinct from NSA, and some of NSA statements and concepts needs to be adapted. As an example, a cheap non-standard number $x = x_n$ is said to be limited if there exists some standard real $y$ such that $|x| \leq y$. In NSA, to every limited non-standard number $x$ is associated some unique standard real number $st(x)$, called its standard-part, such that $x - st(x)$ is infinitesimal. This is not possible in cheap non-standard analysis since for example $x = x_n = (-1)^n$ is clearly non infinitely close to any standard real number. We can however talk about the following:

► **Definition 41 (Standard part $st^+(x)$ and $st^-(x)$)**. Assume $x = x_n$ is limited. We write $st^-(x)$ (respectively: $st^+(x)$) for the limit inf (resp. limit sup) of $n \mapsto x_n$.

We write $inf(x)$ (respectively: $sup(x)$) for the inf (resp. sup) of $n \mapsto x_n$. The following is easy to establish:

► **Lemma 42.** Assume $x = x_n = f(n)$ is limited, where $f$ is some standard function.

Some standard $y$ is some accumulation point of $n \mapsto f(n)$ iff there exists some infinitely large cheap non-standard index $N = N_n$, monotone (i.e. $N_{n+1} \geq N_n$), such that $|f(N) - y|$ is infinitesimal. Consequently, $st^-(x)$ and $st^+(x)$ are respectively the least and largest such $y$, i.e. infinitely close to some $f(N^-)$ and $f(N^+)$.

The following two statements have very elegant classical proofs in NSA: See e.g. [15] [11] This can be adapted to a proof using cheap non-standard arguments.

► **Theorem 43 (Intermediate Value Theorem).** Every continuous standard function $f : [a,b] \rightarrow \mathbb{R}$ with $f(a) \cdot f(b) < 0$ has a zero at some standard $x$.

We need the following whose proof is easy.

► **Lemma 44.** Assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is some continuous function. Assume that $x$ is limited. If $f(x) > 0$ then $f(st^-(x)) \geq 0$. If $f(x) < 0$ then $f(st^-(x)) \leq 0$. Similarly for $st^+$. If $f(x) > 0$ then $f(inf(x)) \geq 0$. If $f(x) < 0$ then $f(sup(x)) \leq 0$. Similarly for $st^+$.

We now prove Theorem 43

**Proof.** Assume w.l.o.g that $[a,b] = [0,1]$ and that $f(0) < 0$ and $f(1) > 0$. Take infinitely large cheap non-standard integer $N$. The idea is to consider the $x(k)$ of the form $k \cdot \frac{1}{N}$ for cheap non-standard integer $0 \leq k \leq N$. 

To do so, consider $k^- = \min(\ast S^+)$ where $\ast S^+ = \ast S^+_n$ and

$$\ast S^+_n = \{0 \leq k \leq N_n \text{ and } f(k \cdot \frac{1}{N_n}) \geq 0\}.$$  

This latter set is non-empty as $f(1) > 0$ and does not contain 0 since $f(0) < 0$.

Function $f$ is continuous, hence uniformly continuous on its domain. Since $x(k^-)$ and $x(k^-) - \frac{1}{N}$ are infinitely close, necessarily $f(x(k^-))$ and $f(x(k^-) - \frac{1}{N})$ must be infinitely close by Theorem 32.

We have $f(x(k^-)) \geq 0$ and $f(x(k^-) - \frac{1}{N}) < 0$ by definition of $k^-$. Consequently, using Lemma 44, necessarily $f(st^+(x(k^-))) \geq 0$ and $f(st^+(x(k^-))) = f(st^+(x(k^-) - \frac{1}{N})) \leq 0$, hence $f(x) = 0$ for standard $x = st^+(x(k^-))$.

Notice that we could have considered $st^-$, or the $\max(\ast S^-)$ defined symmetrically, and this could provide possibly other zeros.

**Theorem 45** *(Extreme Value Theorem).* Every continuous standard function $f : [a, b] \to \mathbb{R}$ attains its maximum at some standard $x$.

We need the following whose proof is easy.

**Lemma 46.** Assume that $f : \mathbb{R} \to \mathbb{R}$ is some continuous function. Assume that $x$ is limited and $m$ is some standard value. If $f(x) \leq m$ then $f(st^-(x)) \leq m$.

We can now go to the proof of Theorem 45.

**Proof.** Assume w.l.o.g that $[a, b] = [0, 1]$. Take infinitely large cheap non-standard integer $N$. The idea is to consider the $x(k)$ of the form $k \cdot \frac{1}{N}$ for cheap non-standard integer $0 \leq k \leq N$.

To do so, consider $k^- = \min(\ast S)$, where $\ast S = \ast S_n$ and

$$\ast S_n = \{0 \leq k \leq N_n \text{ and } f(k \cdot \frac{1}{N_n}) \geq f(k' \cdot \frac{1}{N_n}) \text{ for all } 0 \leq k' \leq N_n\}.$$  

This latter set is non-empty as a finite set always has a maximum.

Function $f$ is continuous, hence uniformly continuous on its domain.

Consider $x = st^-(\frac{1}{N})$, and $m = f(x)$. Then we claim that $f(y) \leq m$ for all standard $y$. Indeed, any $y$ contain at least one $k' \cdot \frac{1}{N}$, $0 \leq k' \leq N$ infinitely close to it: Consider $k' = [N \cdot y]$.

Hence $f(y)$ is infinitely close to $f(k' \cdot \frac{1}{N})$. The latter, is less than $f(k \cdot \frac{1}{N})$ by definition of $k^-$, and hence less than $m$ by Lemma 45.

Notice that we could have considered $st^+$, or the $\max(\ast S^-)$ defined symmetrically, and this could provide possibly other maximums.

We now adapt these proofs to go to computability issues:

**Lemma 47.** Assume that $x$ is limited and computable. Then $\inf(x)$ is right-computable and $\sup(x)$ is left-computable.

**Theorem 48.** Every computable function $f : [a, b] \to \mathbb{R}$ with $f(a) \cdot f(b) < 0$ has a right-computable zero and a left-computable zero.
Proof. Assume w.l.o.g that \([a, b] = [0, 1]\) and that \(f(0) < 0\) and \(f(1) > 0\).

Consider effective infinitesimal \(\epsilon = \epsilon_\alpha\). There must exists some computable \(\delta\) such that if \(|x - y| \leq \delta\) then \(|f(x) - f(y)| \leq \epsilon\). There must also exists some indexed family of cheap non-standard rationals \(\psi(q) = \psi(q)_\alpha = \psi(q,n)\), uniformly computable in \(q\), such that \(|\psi(q) - f(q)| \leq \epsilon\) for all \(q \in \mathbb{Q} \cap [0,1]\).

Consider infinitely large computable cheap non-standard integer \(N = \max(\mathbb{Q}, \frac{1}{\delta})\). The idea is to consider the \(x(k)\) of the form \(k \cdot \frac{1}{N}\) for cheap non-standard integer \(0 \leq k \leq N\).

To do so, consider \(k^- = \min(\langle S^\alpha\rangle)\) where \(\langle S^\alpha\rangle = \langle S^\alpha_n\rangle\) and
\[
\langle S^\alpha_n \rangle = \{0 \leq k \leq N_n\text{ and }\psi(k \cdot \frac{1}{N_n}, n) \geq -\epsilon_n\}.
\]

Function \(f\) is continuous, hence uniformly continuous on its domain. Since \(x(k^-)\) and \(x(k^-) - \frac{1}{N}\) are infinitely close, necessarily \(f(x(k^-))\) and \(f(x(k^-) - \frac{1}{N})\) must be infinitely close by Theorem \(32\).

We have \(f(x(k^-)) \geq -2\epsilon\) and \(f(x(k^-) - \frac{1}{N}) < 0\) by definition of \(k^-\) and \(\psi\).

Consequently
\[
\langle S^\alpha\rangle \text{ necessarily } f(st^*(x(k^-))) \geq 0 \text{ and } f(st^*(x(k^-))) = f(st^*(x(k^-) - \frac{1}{N})) \leq 0, \text{ hence } f(x) = 0 \text{ for standard } x = st^*(x(k^-)).
\]

Furthermore by the property about \(f\) and \(\delta\) and \(\epsilon\), we are sure that \(f(y) < 0\) for all \(y < x\).
Consequently, we also have \(x = \sup(x(k^-))\). From Lemma \(47\) it is right-computable.

Considering \(st^\alpha\), and the \(\max(\langle S^\alpha\rangle)\) defined symmetrically, provides a left-computable zero.

\(\blacktriangleleft\)

\textbf{Corollary 49.} Every computable function \(f : [a, b] \to \mathbb{R}\) with \(f(a) : f(b) < 0\) has a standard zero \(x\). It this zero is isolated (there exists some standard \(\epsilon > 0\) such that \(f\) has no other zero on \([x - \epsilon, x + \epsilon]\)), then it is computable.

\textbf{Proof.} The existence of \(x\) follows from previous theorem (Intermediate Value Theorem). If \(x\) is isolated, then by considering \(f\) on \([x - \epsilon, x + \epsilon]\) in previous Theorem, we get that this (unique) zero \(x\) is left-computable and right-computable. Hence, it is computable. \(\blacktriangleleft\)

All this can be used to prove for example Rice’s theorem.

\textbf{Theorem 50 (Rice’s theorem).} The set of standard computable reals is a real closed field.

\textbf{Proof.} Standard computable reals are closed by addition, subtraction, multiplication and division. We do the proof for multiplication, other proofs are similar. Fix some effective infinitesimal \(0 < \epsilon\). Assume \(x\) and \(y\) are computable. Let \(K\) be some standard constant such that \(|x| \leq K\) and \(|y| \leq K\). Consider effective infinitesimal \(\epsilon' = \frac{\epsilon}{2K + 1}\).

There must exist some cheap non-standard computable rationals \(\frac{p}{q}\) and \(\frac{p'}{q'}\) such that
\[
|x - \frac{p}{q}| \leq \epsilon' \text{ and } |y - \frac{p'}{q'}| \leq \epsilon'.
\]

Then
\[
|x \cdot y - \frac{p}{q} \cdot \frac{p'}{q'}| \leq |x - \frac{p}{q}| \cdot |y| + |\frac{p}{q}| \cdot |y - \frac{p'}{q'}| \leq (2K + 1)\epsilon' = \epsilon,
\]

\(5\) Formally, we are using implicitly Lemma \(14\) which is easy to establish from definitions.
bounding the $e'$ in term $K + e'$ by 1.

Similarly, it is easy to establish that polynomials with coefficients that are standard computable reals are computable. Then given such a polynomial, if it has a real root $x$, then one can always find some standard rational $a, b$ such that $x$ is the only root in interval $[a, b]$.

One can then apply previous theorem (Intermediate Value Theorem) on the polynomial restricted to this interval to get that it must have a computable root. This computable root can only be $x$.

With the same principle, the following can be established:

**Theorem 51.** Every computable function $f : [a, b] \to \mathbb{R}$ attains its maximum in a right-computable standard point and in a left-computable standard point. If a maximum point is isolated, then it is computable.

**Proof.** Assume w.l.o.g that $[a, b] = [0, 1]$.

Consider effective infinitesimal $\epsilon = \epsilon_n$. There must exists some computable $\delta$ such that if $|x - y| \leq \delta$ then $|f(x) - f(y)| \leq \epsilon$. There must also exists some indexed family of cheap non-standard rationals $\psi(q) = \psi(q, n) = \psi(q, n)$, uniformly computable in $q$, such that $|\psi(q) - f(q)| \leq \epsilon$ for all $q \in \mathbb{Q} \cap [0, 1]$.

Consider infinitely large computable cheap non-standard integer $N = \max(\omega, \frac{1}{\epsilon})$. The idea is to consider the $x(k)$ of the form $k \cdot \frac{1}{N}$ for cheap non-standard integer $0 \leq k \leq N$.

To do so, consider $k^* = \min(\ast S)$ where $\ast S = \ast S_n$ and

$\ast S_n = \{0 \leq k \leq N_n \text{ and } \psi(k \cdot \frac{1}{N_n}, n) = \psi(k', \frac{1}{N_n}, n) - \epsilon_n \text{ for all } 0 \leq k' \leq N\}$.

Function $f$ is continuous, hence uniformly continuous on its domain.

Consider $x = \inf(k^* \cdot \frac{1}{N})$, and $m = f(x)$. Then we claim that $f(y) \leq m$ for all standard $y$. Indeed, any $y$ contains at least one $k' \cdot \frac{1}{N}$, $0 \leq k' \leq N$ infinitely close to it: Consider $k = \lfloor N \cdot y \rfloor$.

Hence $f(y)$ is $\epsilon$ close to $f(k' \cdot \frac{1}{N})$. Now, by construction $f(k' \cdot \frac{1}{N}) \leq f(k^* \cdot \frac{1}{N}) - \epsilon$, hence $f(y) \leq m$.

Considering sup, and the max($\ast S$) defined symmetrically, provides a left-computable zero.

---

**10 Discussions and Perspectives**

Our presentation of concepts of computable analysis is based on cheap non-standard analysis. It may be important to discuss how this relates to other approaches for presenting computable analysis, in particular to Type-2 analysis. Type-2 analysis is based on the concept of notation and representations: A notation of a denumerable set $X$ is a surjective function $\nu$ from a subset of $\Sigma^*$ to $X$, where $\Sigma^*$ is the set of finite words over alphabet $\Sigma$. A representation of a non denumerable set $X$ is a surjective function $\delta$ from a subset of $\Sigma^\omega$ to $X$ where $\Sigma^\omega$ denotes infinite words over alphabet $\Sigma$. Having fixed a representation or a notation for $X$ and for $Y$, a function $f$ from $X$ to $Y$ is then considered as computable if it has a realizer: given any representation of $x \in X$, the machine outputs a representation of $f(x)$. A common representation of $\mathbb{R}$ is Cauchy’s representation: a real $x \in \mathbb{R}$ is represented by a fastly converging sequence $(q_n)_n$ of rationals, that is to say such that $|x - q_n| \leq 2^{-n}$. With such a
representation of $\mathbb{R}$, Type-2 Analysis basically considers machines working over sequences of rationals.

We want to point out that cheap non-standard analysis brings meaning to sequences, as such a sequence of rationals can be read as a cheap non-standard rational number. However, this analogy is not so direct, as in cheap non-standard analysis, sequences are considered as equal if they coincide after some finite rank, contrary to Type-2 analysis where two reals are (considered to be) equal iff they have the same set of representations. Indeed, this does not clearly imply the existence of a formal simple translation from one framework to the other.

Despite these difficulties, we believe that our framework provides a dual view of statements from computable analysis.

We also believe in the pedagogical value of cheap non-standard analysis, in particular when talking about computability. More precisely, through this paper, we exposed that several of the concepts from Analysis and established properties have very nice presentations in this framework, either avoiding quantifier alternations, or relying on simpler to grasp concepts.

While some of the presented results are not new, the intended main interest of the discussed framework is not in establishing new statements but in its elegance. Actually, as most of our computability notions are proved to be similar to classical notions, if something can be proved using our framework, it can be proved using a classical reasoning. This criticism is not a side effect but an advantage we take into account to obtain a richer overview of some major mathematical concepts. It may also be important to put this discussion in the context of the usual criticisms about NSA’s approach: In particular, NSA transfer property basically implies that any result proved in NSA can be proved without NSA, and hence this is sometimes used as an argument against this approach: See e.g. [20] for more deeper discussions and references about arguments against NSA’s approach. It may also be important to do not forget that NSA and cheap non-standard analysis differ, and that cheap non-standard analysis approach is of some help to provide constructivity.

In this paper, we derived our results using $\omega$ as an index set. But it could be some other well-ordered and well-closed set. This would thus provide some alternative views of statements from computability and analysis: In particular, once such a set is fixed, previous constructions provide infinitesimal and infinitely large elements with respect to all elements in that set. This can hence be iterated, using a transfinite induction, to provide richer and richer index sets, providing statements in richer and richer frameworks. Such an approach yields to contexts such like computation models with ordinal times.

As we mentioned before, also notice that links have already been established between type-2 computability and transfinite computations (see [5] for example) using surreal numbers to extend $\mathbb{R}$. More generally, the links between these ideas and already studied models of computations over the ordinals deserve due attention. Models of computation over the ordinals include Sacks’ higher recursion theory [16] or Infinite Time Register Machines [4] or Infinite Time Turing Machines [8].

References

1. M. B. Pour-El and J. I. Richards. *Computability in Analysis and Physics*. Springer-Verlag, 1989.

2. Albert Benveniste, Timothy Bourke, Benoit Caillaud, and Marc Pouzet. Non-standard semantics of hybrid systems modelers. *Journal of Computer and System Sciences*, 78(3):877–910, 2012.
3 Lenore Blum, Felipe Cucker, Michael Shub, and Steve Smale. *Complexity and Real Computation*. Springer-Verlag, 1998.
4 Merlin Carl, Tim Fischbach, Peter Koepke, Russell Miller, Miriam Nasfi, and Gregor Weckbecker. The basic theory of infinite time register machines. *Arch. Math. Log.*, 49(2):249–273, 2010.
5 Lorenzo Galeotti and Hugo Nobrega. Towards computable analysis on the generalised real line. In Jarkko Kari, Florin Manea, and Ion Petre, editors, *Computability in Europe, CIE 2017, Turku, Finland, June 12-16, 2017, Proceedings*, 2017.
6 Robert Goldblatt. *Lectures on the hyperreals: an introduction to nonstandard analysis*, volume 188. Springer Science & Business Media, 2012.
7 A. Grzegorczyk. On the definitions of computable real continuous functions. *Fundamenta Mathematicae*, 44:61–71, 1957.
8 Joel D. Hamkins and Andrew Lewis. Infinite time turing machines. *Journal of Symbolic Logic*, 65(2):567–604, 2000.
9 Mikhail Katz and Luie Polev. From pythagoreans and weierstrassians to true infinitesimal calculus. *Journal of Humanistic Mathematics*, 7(1):87–104, 2017.
10 H Jerome Keisler. *Foundations of infinitesimal calculus*, volume 20. Prindle, Weber & Schmidt Boston, 1976.
11 H Jerome Keisler. *Elementary Calculus: An Infinitesimal Approach*. On-line Edition [http://www.math.wisc.edu/~keisler/calc.html](http://www.math.wisc.edu/~keisler/calc.html) 2017.
12 Ker-I Ko. *Complexity Theory of Real Functions*. Progress in Theoretical Computer Science. Birkhauser, Boston, 1991.
13 Suenaga Kohei and Hasuo Ichiro. Programming with infinitesimals: A while-language for hybrid system modeling. In *Automata, Languages and Programming - 38th international Colloquium, Proceedings*, pages 392–403, 2011.
14 D. Lacombe. Extension de la notion de fonction récursive aux fonctions d’une ou plusieurs variables réelles III. *Comptes Rendus de l’Académie des Sciences Paris*, 241:151–153, 1955.
15 A. Robinson. *Non-standard analysis*. Princeton Univ Pr, 1996.
16 Gerald E. Sacks. *Higher recursion theory*. Perspectives in mathematical logic. Springer-Verlag, 1990.
17 Terence Tao [online]. April 2012. URL: [https://terrytao.wordpress.com/2012/04/02/a-cheap-version-of-nonstandard-analysis/](https://terrytao.wordpress.com/2012/04/02/a-cheap-version-of-nonstandard-analysis/) [cited January 2018].
18 A. M. Turing. On computable numbers, with an application to the Entscheidungsproblem. *Proc. London Math. Soc.*, 2(42):230–265, 1936.
19 K. Weihrauch. *Computable Analysis: an Introduction*. Springer, 2000.
20 Wikipedia. Criticism of non-standard analysis [online]. 2018. URL: [https://en.wikipedia.org/wiki/Criticism_of_non-standard_analysis](https://en.wikipedia.org/wiki/Criticism_of_non-standard_analysis) [cited April 23th 2018].
ACKS 1. This material is based upon work supported by the RACAF Project from Agence National de la Recherche. Olivier Bournez would like to thank Bruno Salvy for having pointed out the post [17] when preparing a subject for an exam about his logic course.