Quantum Algorithms for Autocorrelation Spectrum

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Abstract
In this paper we design quantum algorithms for studying the autocorrelation spectrum of a Boolean function and its individual coefficients. Informally, the autocorrelation coefficient of a Boolean function $f$ at some point $a$ measures the average correlation among the values $f(x)$ and $f(x \oplus a)$. The Walsh spectrum is a related concept that is well-studied primarily due to its connection to the quantum circuit for the Deutsch-Jozsa problem but the autocorrelation spectrum has not received similar attention that we attempt to deliver in this paper.

We propose efficient probabilistic algorithms for several problems regarding the autocorrelation spectrum. First and foremost, we give an algorithm that samples from the Walsh spectrum of any derivative of $f$; the derivative of a Boolean function is an extension of autocorrelation to correlation among multiple values of $f$. Using a relation between the 1st-order derivative and the autocorrelation coefficients, we design an algorithm to sample the input points according to squares of the autocorrelation coefficients. Then we given a different set of algorithms for estimating the square of a particular coefficient or cumulative sum of their squares. Our last algorithm combines the technique of amplitude estimation and amplification in a novel manner to find points with high values of autocorrelation coefficients.

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1 Introduction

Boolean functions are very important building blocks in cryptology, learning theory and coding theory. Any combinatorial analysis on such functions are of great intellectual interest. Different properties of Boolean functions can be well understood by different spectra; specifically, Walsh and autocorrelation spectra are two most important tools for cryptographic purposes. For a Boolean function $f$, these spectra can be thought as the list of all values of the Walsh transform and autocorrelation transform, respectively, of $f$. We use Walsh coefficients and autocorrelation coefficients to indicate the individual values in those spectra.

Shannon related these spectra to confusion and diffusion of cryptosystems long ago [21]. Confusion of a Boolean function used in a cryptosystem can be characterized by a Walsh spectrum with low absolute values – such functions are known to resist linear cryptanalysis [4]; similarly, functions with less diffusion (high absolute value in the autocorrelation spectrum) may make a cryptosystem vulnerable against differential attacks (see for example [22] and
the references therein). Walsh spectrum (often referred to as Fourier spectra for Boolean functions) has been shown to be useful for learning Boolean functions as well \[17\].

Analyzing these spectra and designing functions with specific spectral properties are therefore important tasks. This problem becomes challenging for large functions, e.g., 80-bit LFSRs/NFSRs (Linear/Nonlinear Feedback Shift Registers), that are used in stream ciphers, turn out be 160-bit Boolean functions! Modelling such a complicated Boolean function by analysing the spectra is clearly elusive \[20\]. In classical domain, for an \(n\)-input 1-output Boolean function, generation of complete Walsh or autocorrelation spectrum requires \(O(2^n)\) space and \(O(n2^n)\) time. Needless to mention that for analysing a cipher or learning a Boolean function, what are really needed are the high coefficients in a spectrum. Thus it makes sense to design techniques for sampling points with high coefficients and estimate the high coefficients in which a Boolean function can be used only as a black-box.

The situation is well settled for the Walsh spectrum. Walsh spectrum of a function \(f : \{0,1\}^n \rightarrow \{0,1\}\) is defined as the following function \[\hat{f}\] from \(\{0,1\}^n\) to \(\mathbb{R}[-1,1]\) in which \(x \cdot y\) stands for the \(0 \rightarrow 1\) valued expression \(\oplus_{i=1}^n x_i y_i:\)

\[
\hat{f}(y) = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)}(-1)^{yx}
\]

The Deutsch-Jozsa algorithm \[6\], even though usually described as solving a different problem, makes only one query to \(U_f\) (a standard unitary implementation of \(f()\)) and essentially generates the state \(\sum_{z \in \mathbb{F}_2^n} \hat{f}(z)|z\rangle\) at the end \[16\]. Measuring this state generates a state \(|z\rangle\) with probability \(\hat{f}(z)^2\) (Walsh coefficients do satisfy \(\sum_z \hat{f}(z)^2 = 1\); thus this algorithm can be considered as an efficient sampling algorithm for Walsh coefficients. So if one can implement a stream cipher (a Boolean function) as a quantum oracle \[8\], then it is possible to sample high points in a Walsh spectrum in constant time with linear number of gates and that enables us to answer several questions related to the spectrum \[23\].

In contrast to the Walsh spectrum, the autocorrelation spectrum is less studied. It is defined as the following transformation \[1\] from \(\{0,1\}^n\) to \(\mathbb{R}[-1,1]\).

\[
\hat{f}(a) = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)}(-1)^{f(x \oplus a)}
\]

The entire autocorrelation spectrum can be obtained by first computing the Walsh spectrum (using the well-known FFT algorithm with complexity \(O(n2^n)\)), squaring each of the coefficients, and then running FFT once more on this squared spectrum. However, a question remains that what can we find out about the autocorrelation spectrum in \(o(2^n)\), preferably polynomial, time. Especially, can we identify the points with high coefficients? Can we estimate a particular coefficient? Counting and sampling often go hand-in-hand, so one would also like to sample from a distribution proportional to the coefficients.

The quantum algorithms we propose in this paper address these questions. We hope that they may be able to expose the weaknesses of a Boolean crypto-function better than classical approaches. There are quite a few important research results related to quantum cryptanalysis of symmetric ciphers \[12\ \[13\ \[3\]. A recent work \[15\] in this direction considered merging the ideas from Grover’s \[9\] and Simon’s \[11\] algorithms. However, there has been no specific attempt to solve concrete problems related to the autocorrelation spectrum. This we present in this paper.

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\[1\] The normalization factor used depends upon the application but has no bearing on properties of interest.
1.1 Summary of Results

The contributions of this paper are several quantum algorithms for various autocorrelation related problems described below. Due to its exponential size ab initio, exact algorithms for problems on this spectrum are expected to have exponential complexity. However, randomization and definitely, quantum, techniques can be expected to produce reliable estimates from the functions values of a relatively fewer number of samples from $\mathbb{F}_2^n$. We explore this direction in this paper and all our estimation algorithms below come with tunable accuracy and confidence parameters ($\delta$ denoting the maximum probability of error).

Our results apply to all $n$-bit Boolean functions but the specific contributions are easier to understand for functions with polynomially many points having polynomially-high autocorrelation coefficients (say, $\Omega(\frac{1}{\text{pol}(n)})$) and the rest of the coefficient being exponentially low – let $\mathcal{F}$ denote the class of such functions. The query complexities of our algorithms are with respect to $U_f$ (a standard unitary operator to compute $f()$), and include a $\log\frac{1}{\delta}$ factor that is not explicitly mentioned.

Autocorrelation sampling [Section 4]: The well-known Deutsch-Jozsa algorithm is able to produce a state $|x\rangle$ with probability $|\tilde{f}(x)|^2$ and this observation forms the backbone of several quantum algorithms on Boolean functions [23][5][10]. Therefore, it is natural to ask whether such an algorithm exists for the autocorrelation spectrum as well. A noticeable difficulty is the fact that $\sum_a |\tilde{f}(a)|^2 \geq 1$ whereas $\sum_a |f(x)|^2 = 1$. It is unclear if such sampling can be done without necessarily computing the entire spectrum (taking $\Theta(n^2)$ time).

The derivative of a Boolean function is yet another important object to study from the cryptanalytic point of view. The $1^{\text{st}}$ derivative of $f(x)$ at a point $\beta$ is defined as $D_{\beta}[f](x) = f(x) \oplus f(x+\beta)$, the $2^{\text{nd}}$ derivative at given points $(\beta,\gamma)$ is defined as the first derivative of $D_{\beta}[f](x)$ at a point $\gamma$ and so on. In this paper we provide an efficient quantum algorithm for sampling from the Walsh spectrum of higher order derivatives of a function and use that to design a probabilistic quantum algorithm that makes constant number of calls to $U_f$ and upon measurement, yields state $|a\rangle$ with probability proportional to $|\tilde{f}(a)|^2$.

\textbf{Corollary 1.} There is a quantum circuit that makes an expected $O(\frac{2^n/2}{\text{pol}(n)} \log\frac{1}{\delta})$ calls to $U_f$ and whose measurement output is a state $|a\rangle$ with probability $|\tilde{f}(a)|^2/S_f$ where $a \in \mathbb{F}_2^n$.

Autocorrelation estimation [Section 5]: Given a function $f()$ as blackbox, we wondered if it is possible to efficiently estimate the value of $|\tilde{f}(a)|^2$ (essentially $|\tilde{f}(a)|$) for a specified $a \in \mathbb{F}_2^n$; exact computation would require $\Theta(2^n)$ queries. It can help us verify cryptographic properties of existing Boolean functions and even construct attacks for existing ones. A generalization of this problem is computation of the sum of square of all autocorrelation coefficients, also known as the second moment of the auto-correlation coefficients: $S_f = \sum_{a \in \mathbb{F}_2^n} |\tilde{f}(a)|^2$

Zhang et al. proposed $S_f$ as the sum-of-square indicator for testing if a cryptographic Boolean function satisfies the global avalanche criteria [25]. Once again the usual classical algorithm for computing $S_f$ consists of computing the entire spectrum and has a complexity $O(n2^n)$. It should be noted that this problem does not exist for the Walsh spectrum since the squares of the Walsh coefficients sum up to 1 by Parseval’s theorem.

We give a quantum algorithm to estimate $|\tilde{f}(a)|^2$, and even the more general $S_f$, that is quite efficient compared to classical algorithms; even those that estimate the values from numerous samples of $f(x)$ would have query complexity $O(\frac{2^n}{\epsilon \text{pol}(n)} \log\frac{1}{\delta})$ for functions in $\mathcal{F}$.

\textbf{Theorem 2.} $S_f = \sum_{a \in \mathbb{F}_2^n} |\tilde{f}(a)|^2$ and $\sum_{a \in \mathbb{F}_2^{n-k}} |\tilde{f}(pa)|^2$ (for any $k \leq n$ and $p \in \mathbb{F}_2^k$) can be estimated with relative accuracy $\epsilon$ and probability of error $\delta$ using $O\left(\frac{1}{\epsilon^2} \log\frac{1}{\delta}\right)$ calls to $U_f$.\n
Our approach also works for estimating $|\hat{f}(x)|$ better than a recently proposed method [23].

**Autocorrelation filtering [Section 6]:** We give an algorithm to approximately compute the list of all points with high autocorrelation coefficients. A such that $|\hat{f}(a)| \geq t$ for a specified $t \in (0, 1)$. Our algorithm adapts to the autocorrelation setting a particular form of the Goldreich-Levin algorithm (GL) for a similar problem, but for Walsh coefficients. To the best of our knowledge, there are two versions of GL, one that samples function values in a pairwise independent manner [7] Ch. 2.5.2 and another which is a recursive algorithm that uses sampling to estimate bits of the points [19] Ch. 3.5. We observed that the former approach really uses certain nice properties of the Walsh spectrum and cannot be easily extended to the autocorrelation spectrum. The latter approach has been earlier used to design a quantum algorithm for Walsh spectrum [18], however, their algorithm has a $\frac{1}{n^2}$ dependency on $t$ and they rightfully claimed that the high complexity is primarily due to the high complexity of estimating sum of squares of a range of autocorrelation coefficients. We use the results stated above to obtain a faster algorithm with a $\frac{1}{n}$ dependence upon $t$.

**Corollary 3.** For the functions in $F$ there exists a quantum algorithm that takes as input some $t = \frac{1}{\text{poly}(n)}$, makes at most $O(2^{n/2} \text{poly}(n))$ queries and returns a list $L$ that, with high probability, contains all $a \in \mathbb{F}_2^n$ such that $|\hat{f}(a)| \geq t$ and contains no $a$ such that $|\hat{f}(a)| < t/4$. In general, the query complexity has a $\frac{1}{n}$ dependency on $t$.

The $2^{n/2}$ term really comes from $\sqrt{\sum_x \hat{f}(x)^2}$ which for Walsh coefficients would be simply 1 (by Parseval’s theorem); our algorithm can be easily adapted to Walsh spectrum and that would lose the $2^{n/2}$ term making it more efficient than the earlier approach [18].

**Amplitude separation problem [Section 3]:** One of the algorithms above require deciding if the probability of observing a good state is above a threshold, say $t \in (0, 1)$ or less than $t/4$. The usual techniques of hypothesis testing lead to a sample complexity of $O(\frac{1}{t})$. In contrast, we give a quantum algorithm that combines two well-known techniques, amplitude amplification and amplitude estimation, with sample complexity of $O(\frac{1}{\sqrt{t}})$. This problem is of general interest and our algorithm may have applications beyond autocorrelation problems.

The rest of the paper deals with arbitrary Boolean functions and not restricted to $F$.

## 2 Background: Amplitude Amplification and Estimation

Our techniques make heavy use of the well-known quantum amplitude amplification and estimation algorithms so we briefly discuss the relevant results along with necessary extensions.

**Amplitude amplification:** Let $A$ be an $n$-qubit unitary operator such that $A|00\ldots0\rangle = |0\rangle|\psi_{\text{good}}\rangle + |1\rangle|\psi_{\text{bad}}\rangle$ and $B$ be a unitary operator that essentially acts on the first qubit and maps $|0\rangle \mapsto -|0\rangle$ and $|1\rangle \mapsto |1\rangle$. Furthermore, suppose that $|\psi_{\text{good}}\rangle$ is known to be some superposition over basis states, i.e., $|\psi_{\text{good}}\rangle = \sum_x \alpha_x |x\rangle$. Let $p$ denote the probability that the output state of $A$ can be observed to be in the state $|0\rangle$ (good state), i.e., $p = \||\psi_{\text{good}}\rangle\|^2$.

First used by Grover for designing a quantum algorithm for unordered search [10] and then studied formally several times, most notably by Brassard et al. [2], amplitude amplification gives an algorithm that calls $A$ and $B$ repeatedly in a black-box manner and whose output state has a much higher probability of being a good state. To explain this, define normalized states $|\tilde{\psi}_{\text{good}}\rangle = \frac{1}{\sqrt{p}} |\psi_{\text{good}}\rangle = \sum_x \frac{\alpha_x}{\sqrt{p}} |x\rangle$ and $|\tilde{\psi}_{\text{bad}}\rangle = \frac{1}{\sqrt{1-p}} |\psi_{\text{bad}}\rangle$. Then we can express the state after applying $A$ as $A|00\ldots0\rangle = \sqrt{p} |0\rangle|\tilde{\psi}_{\text{good}}\rangle + \sqrt{1-p} |1\rangle|\tilde{\psi}_{\text{bad}}\rangle = (\text{say}) |\Phi\rangle$. Amplitude amplification prescribes a way to modify this state to states of the form $\sqrt{q} |0\rangle|\tilde{\psi}_{\text{good}}\rangle + \sqrt{1-q} |1\rangle|\tilde{\psi}_{\text{bad}}\rangle = (\text{say}) |\Phi'\rangle$ where $p \ll q \approx 1$ using very few calls to $A$. 
What should be noted are the states obtained after measurement if $|0\rangle$ is observed in the first qubit. Whether the measurement happens on $|\Phi\rangle$ (before amplification) or on $|\Phi'\rangle$ (after amplification), in both the cases the resultant state turns out to be the same superposition $|0\rangle \psi_{\text{good}} = \sum_x \frac{1}{\sqrt{m}} |0\rangle |x\rangle$.

A technical challenge in the initial amplitude amplification techniques was the requirement of knowing $p$ to decide how many times $A$ needs to be called (and calling more times can actually worsen this probability). A series of algorithms called “fixed-point search algorithms” were later devised in more repetitions would not worsen the probability, so, $p$ need not be known in advance. A recent fixed-point search algorithm also improved the running time so that $O\left(\frac{1}{p^2} \log \frac{1}{\delta} \right)$ calls to $A$ are enough to increase the probability of observing a good state to $(1 - \delta)$ [23]. The above observation holds true even for the fixed-point amplitude amplification algorithms.

Amplitude estimation: Now we discuss the technique of amplitude estimation, i.e., estimating the value of $p$ above. Let $k$ and $m$ be some parameters that we shall fix later. A quantum amplitude estimation algorithm (say, named as $\text{AmpEst}$) was proposed by Brassard et al. [2] that acts on two registers of $m$ and $n$ qubits, makes $2^m$ calls to controlled-$A$ and outputs a $\hat{p} \in [0, 1]$ that is a good approximation of $p$ in the following sense.

\[ \Pr[|p - \hat{p}| \geq \epsilon] \leq \frac{\frac{1}{2} \log \frac{1}{\delta}}{2^m} \]

The $\text{AmpEst}$ algorithm can be used to estimate $p$ with desired accuracy and error. Suppose we want error to be at most $\delta$ (for some $\delta \leq 1/2$) and confidence interval $\epsilon$. Let $M$ denote $2^m$. Setting $\frac{1}{2^m} = \delta$, we get an upper bound of $k \leq \frac{1}{\delta}$. Therefore, we can solve

\[ 2\pi \sqrt{\frac{2(1-p)}{M}} + \pi^2 \frac{k^2}{2M} \leq \epsilon \] for $M$ and obtain that $2^m = O\left(\frac{1}{\epsilon^2} \frac{1}{\epsilon} \frac{1}{\delta} \right)$ to show that $|p - \hat{p}| \leq \epsilon \delta$ holds with probability at least $1 - \delta$. However it is possible to reduce the number of calls to $A$ to a great extent. For example, $\frac{1}{\epsilon^2} \ln \frac{1}{\delta} \frac{1}{\delta}$ samples are sufficient if the mean of $\ln \frac{1}{\delta} \frac{1}{\delta}$ estimates of $p$ is reported (for individual estimates of $p$ use $k = 1$ and $2^m = \frac{\sqrt{2}}{\epsilon^2 \delta}$). One can use even fewer $\frac{8}{\epsilon^2 \delta} \log \frac{1}{\delta}$ samples if the median of $\log \frac{1}{\delta}$ estimates of $p$ (obtained using $k = 1$ and $2^m = \frac{8}{\epsilon^2 \delta}$) is reported as $\hat{p}$ [2]. We will simply state the last result as a corollary to be used later.

\[ \Pr[|p - \hat{p}| \geq \epsilon] \leq \frac{\frac{1}{2} \log \frac{1}{\delta}}{2^m} \]

It should be noted that an algorithm is already known with the same query complexity [14] Corollary 8.3.3 but, in our opinion, that uses a more complicated wrapper over $\text{AmpEst}$ and a more complex analysis than what is presented above. We now present an additive version of the above corollary that will also be used later. The number of queries required for additive-accuracy estimation, as specified in the corollary below, will be denoted by $M^+_{\epsilon, \delta}$.

\[ \Pr[|p - \hat{p}| \geq \epsilon] \leq \frac{\frac{1}{2} \log \frac{1}{\delta}}{2^m} \]

\[ \text{Corollary 6. Let } \text{AmpEst quantum algorithm be run } 7 \left(\ln \frac{1}{\delta}\right)^{1/3} = O\left(\ln \frac{1}{\delta}\right) \text{ times using } \]

\[ k = 1 \text{ and } 2^m \geq \frac{\sqrt{2}}{\epsilon^2 \delta} \text{ and the median of the obtained estimates of } p \text{ is returned as } \hat{p}. \text{ Then we obtain an estimate } \hat{p} \text{ such that } \Pr[|p - \hat{p}| \geq \epsilon] \leq \frac{\frac{1}{2} \log \frac{1}{\delta}}{2^m} \text{ for any } \epsilon \leq \frac{1}{\delta} \text{ using a total of } \Theta\left(\frac{\sqrt{2}}{\epsilon^2 \delta} \log \frac{1}{\delta}\right) \text{ calls to } A. \]

\[ ^2 \text{ All of these are straight forward to prove using Chernoff bound and can also be found, along with many other cases, in a recent paper [11].} \]
Proof. Let $M$ denote $2^n$. If we require that $2\pi \sqrt{\frac{p(1-p)}{M}} + \frac{\pi^2}{3 \sqrt{2}} \leq \epsilon$, it suffices to take $M \geq \frac{3}{\pi^2} \left[ \sqrt{p(1-p)} + \sqrt{p(1-p)} + \epsilon \right]$. Since $1/4 \geq p(1-p)$, $rac{1}{4} e^{2\epsilon} \geq \sqrt{\frac{1}{4} + \epsilon}$ (since $\epsilon \leq 1/4$) and $3 \geq 1 + e^{2\epsilon}$, it suffices to take $M \geq \frac{3}{\pi^2}$.

Therefore, for $k = 1$ and using the above value of $M$ we obtain that each individual estimate $p_i$ satisfies $\Pr[|p - p_i| \leq \epsilon] \geq \frac{3}{\pi^2}$. This success probability can be increased to $1 - \delta$ for any arbitrary $\delta$ if we use median of $\Theta(\log \frac{1}{\delta})$ values by standard Chernoff bounds.

### 3 Amplitude separation problem

Suppose we have an algorithm $A$ such that $A[0^n] = \sin \theta |\psi_{good}\rangle + |\psi_{bad}\rangle$. Here $|\psi_{good}\rangle$ denotes a normalized “good state” (or a normalized superposition of “good states”) whose probability we are interested in. Let $p = \sin^2 \theta$ denote this probability of observing a good state upon measuring $A[0^n]$. We also have access to $A^\dagger$ and to an oracle to identify the good states (e.g., in the following manner: $O|x\rangle \mapsto -|x\rangle$ if $|x\rangle$ is a good state and $O|x\rangle \mapsto |x\rangle$ otherwise) — essentially all ingredients required for amplitude amplification and estimation.

In the amplitude separation problem we are also given a promise that either $p \geq t$ or $p \leq \frac{1}{16} t$ for some known $t \in (0, 1]$ and we have to decide which case it is.

Let $\tau$ denote the angle $\sin^{-1} \sqrt{t}$. We follow a strategy that is partly similar to the one used by Chakraborty et al. \cite{5} with some important modifications \cite{8}. On a high level, we first amplify the amplitude of $|0^n\rangle$ and then apply amplitude estimation since amplified probabilities have a larger gap and are easier to separate. This allows us to solve the problem with a number of queries to $A$ that scales as $O(\frac{1}{\sqrt{t}}) \approx O(\frac{1}{\tau})$. Contrast this to the strategy of making multiple observations of $A[0^n]$ and deciding based on the number of times $|0^n\rangle$ is observed — the number of required queries there can be obtained using Chernoff bound and scales as $O(\frac{1}{\tau})$. Another possibility would have been to use the amplitude estimation methods. If we use the additive-accuracy estimation (Corollary \cite{6}), then too the number of queries scales as in the previous case. Finally, using the relative-accuracy estimation (Corollary \cite{5}) with a fixed small $\epsilon$ would require knowing a lower-bound on the success-probability of $A$ (i.e., $\sin^2 \theta$) to obtain an upper-bound on the number of queries to use during estimation.

We now describe our strategy. Divide the range of angles $[\tau, \frac{\pi}{4}]$ into a series of $1 + s$ ranges; the factor of 3 is related to the increase of angles by 3 in amplitude amplification.

$$R_0 = [\tau, \frac{1}{3\sqrt{2}} \cdot \frac{\pi}{4}], \ R_1 = [\frac{1}{3\sqrt{2}} \cdot \frac{1}{2} \cdot \frac{\pi}{4}, \frac{1}{2} \cdot \frac{\pi}{4}], \ R_{s-1} = [\frac{1}{3\sqrt{2}} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{4}, \ R_2 = \frac{1}{2} \cdot \frac{\pi}{4}, \frac{3}{4} \cdot \frac{\pi}{4}], \ R_1 = [\frac{3}{4} \cdot \frac{\pi}{4}, \frac{5}{4} \cdot \frac{\pi}{4}], \ \text{and } R_0 = [\frac{5}{4} \cdot \frac{\pi}{4}]$$

First, observe that $s = \lfloor \log_3 \frac{1}{\tau} \rfloor$. Secondly, any $\theta \in [\tau, \frac{\pi}{2}]$ lies in some $R_i$. Define $k_i = \frac{1}{2}(3^i - 1)$. The third observation is that for $\theta \in R_i$, $(2k_i + 1)\theta \in [\frac{\pi}{4}, \frac{3\pi}{4}]$ which implies that $\sin^2((2k_i + 1)\theta) \geq \sin^2 \frac{\pi}{4} = \frac{1}{2}$. Finally, $k_0 < k_1 < \ldots < k_s$ and for all $i$, $\frac{\pi}{4} \leq k_i + 1 \leq \frac{3\pi}{4}$.

The decision algorithm is presented in Algorithm \cite{8} in which we use $M^\theta_{\epsilon, \delta}$, that is defined in Section \cite{2}. The behaviour of the above algorithm is summarized in the following lemma. Note that the statement is void of the promise nature — we simply leave the behaviour undefined if $p$ happens to lie between $t$ and $\frac{1}{15} t$.

\textbf{Lemma 7.} Suppose we given an algorithm $A$ (and its inverse $A^\dagger$) and suppose the probability of observing some “good state” upon measuring $A[0^n]$ is $p$. Given $\delta \in (0, 1)$ and a threshold $\epsilon$...
We first show how to sample from the Walsh spectrum of derivative of a function and then use that approach for autocorrelation sampling.

## Autocorrelation sampling

We first show how to sample from the Walsh spectrum of derivative of a function and then use that approach for autocorrelation sampling.
4.1 Walsh transform of derivative

Suppose \( f() \) denotes an \( n \)-bit Boolean function. For some (possibly small) integer \( k, A = \{a_1, a_2, \ldots, a_k\} \) is a multiset of \( k \) \( n \)-bit strings. For any subset \( S \subseteq A \) (including \( S = \emptyset \)), \( f(x \oplus S) \) indicates the expression \( f(x \oplus \bigoplus_{a_i \in S} a_i) \). The \( k \)-th derivative of \( f() \) is defined as the \( n \)-bit Boolean function \( D^k_A(x) = \bigoplus_{S \subseteq A} f(x \oplus S) \). For example, if \( k = 0 \) then \( A = \{\} \) and \( D^0_{\{\}}(x) \) is simply \( f(x) \) itself. For \( k = 1 \), \( A \) is of the form \( \{a\} \) for \( a \in \{0,1\}^n \) and \( D^1_{\{a\}}(x) = f(x) \oplus f(x \oplus a) \). For \( k = 2 \), \( A \) is of the form \( \{a,b\} \) for \( a \in \{0,1\}^n, b \in \{0,1\}^n \) and \( D^2_{\{a,b\}}(x) \) is defined as \( f(x) \oplus f(x \oplus a) \oplus f(x \oplus b) \oplus f(x \oplus a \oplus b) = \bigoplus_{S \subseteq \{a,b\}} f(x \oplus S) \). We use \( \Delta(x) \) to denote \((-1)^{D(x)}\) and refer to it as derivative in the rest of this section.

It is straightforward to construct a quantum circuit that generates the Walsh-spectrum of the \( k \)-derivative of \( f() \). As is the norm in quantum circuits, we will use \( \hat{U} \) to denote the operation \(|x\rangle \rightarrow (-1)^{f(x)}|x\rangle\) for all \( x \in \{0,1\}^n \). We refer to the circuit as \( HoDJ^n_k \) (\( HoDJ \) stands for “Higher-order Deutsch-Jozsa”); note that \( HoDJ^n_0 \) is a circuit for the Walsh-spectrum of \( f() \). The circuit for \( HoDJ^n_k \) acts on \( k + 2 \) registers, \( R_1, R_2, \ldots, R_{k+1}, R_{k+2} \).

- \( R_1 \) has one qubit that is initialized to \(|1\rangle\),
- \( R_2 \) consists of \( n \)-qubits that is initialized to \(|0^n\rangle\),
- and each of \( R_3 \ldots R_{k+2} \) consists of \( n \)-qubits in which \( R_{2+t} \) is initialized to \( a_t \) of \( A \).

\[ \begin{array}{cccc}
R_1 & |1\rangle & \hat{U} & \hat{H} \\
R_2 & |0^n\rangle & H^\otimes n & U_f \\
R_{3a} & |a\rangle & H^\otimes n & U_f \\
R_{3b} & |b\rangle & H^\otimes n & U_f
\end{array} \]

**Figure 1** Circuit for sampling according to the Walsh spectrum of \( 2^n \)-order derivative

For the ease of explanation, we illustrate a construction of the circuit for \( k = 2 \) in Figure 1. The circuit can be easily generalized to higher values of \( k \). A detailed step-by-step analysis of the circuit presented below.

Initial state \( = |1\rangle |0^n\rangle |a\rangle |b\rangle \)

\[
\begin{align*}
H^\otimes n & \rightarrow \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |\Delta(x)|a\rangle |b\rangle \\
\hat{U}^f & \rightarrow \frac{1}{\sqrt{2^n}} \sum_{x} (-1)^{f(x)} |x, a, b\rangle \\
\hat{U}^f.CNOT & \rightarrow \frac{1}{\sqrt{2^n}} \sum_{x} (-1)^{f(x) \oplus f(x \oplus a)} |x \oplus a\rangle |a, b\rangle \\
\hat{U}^f.CNOT & \rightarrow \frac{1}{\sqrt{2^n}} \sum_{x} (-1)^{f(x) \oplus f(x \oplus a \oplus b)} |x \oplus a \oplus b\rangle |a, b\rangle \\
\hat{U}^f.CNOT & \rightarrow \frac{1}{\sqrt{2^n}} \sum_{x} (-1)^{f(x \oplus S)} |x \oplus b\rangle |a, b\rangle \\
H^\otimes CNOT & \rightarrow \frac{1}{\sqrt{2^n}} \sum_{x} (-1)^{\bigoplus_{S \subseteq \{a,b\}} f(x \oplus S)} |1\rangle |a, b\rangle \\
H^\otimes & \rightarrow |1\rangle \sum_y \left[ \frac{1}{2^n} \sum_{x} (-1)^{f(y \oplus S)} \bigoplus_{S \subseteq \{a,b\}} f(x \oplus S) \right] |y\rangle |a, b\rangle
\end{align*}
\]
It is evident that measuring $R2$ at the end will collapse it into $|y\rangle$ for some $y \in \{0, 1\}^n$ with probability $\Pr_y[y] = \left(\frac{1}{2^n} \sum_{x}(−1)^{xy} \Delta^2_{A_b}(x)\right)^2 = \Delta^2_{A_b}(y)$ that is the square of the Walsh coefficient of $\Delta^2_{A_b}(\cdot)$ (2nd-order derivative function) at the point $y$. For any function $g()$, suppose $\hat{g}(y)$ denotes the superposition representing the Walsh spectrum of $g()$, i.e., $\hat{g}(y) = \frac{1}{2^n} \sum_x (-1)^{g(x)y}$. The following theorem generalizes this result to any $k$-th order derivative; we ignore the first register which contains an ancillary qubit that is reset to its initial states at the end of the computation.

**Theorem 8.** $|0^n\rangle|a_1\rangle\ldots|a_k\rangle \xrightarrow{HoDJ} \sum_y \Delta^k_{A_b}(y)|y\rangle|a_1\rangle\ldots|a_k\rangle$

A quick observation is that $\text{HoDJ}_n^k$ essentially generates $\sum_y \hat{f}(y)|y\rangle$ that is exactly the same output as that of the Deutsch-Jozsa circuit and in fact, the circuit for $\text{HoDJ}_n^0$ is exactly same as that of the Deutsch-Jozsa circuit for $n$-bit functions.

### 4.2 Circuit for sampling

Now we describe a quantum circuit that returns samples from a distribution that is proportional to the autocorrelation coefficients of a function, rather their squares (partly because the autocorrelation coefficients can be both positive and negative). Our circuit is based on the observation that $\hat{f}(a) = \frac{1}{2^n} \sum_x (-1)^{f(x)}(-1)^{f(x \oplus a)} = \frac{1}{2} \sum_x \Delta^1(x) = \Delta^1(0)$ (using the identity $\hat{g}(0) = \frac{1}{2^n} \sum_x (-1)^{g(x)}$ for any Boolean function $g()$) and Theorem 8.

![Figure 2 Circuit for partial autocorrelation sampling](image)

Algorithm 2 (the corresponding circuit is illustrated in Figure 2) actually solves a more general problem that will be useful later. Given some $a \in \{0, 1\}^n$ of length at most $n$, it outputs samples only among points with $a$ as the common prefix. To analyse the algorithm, we need to define a partial sum of autocorrelation coefficients with a common prefix.

$$\text{for } k \leq n, a \in \mathbb{F}_2^k \quad v(a) = \sum_{b \in \mathbb{F}_2^{n-k}} |f(ab)|^2$$

The algorithm takes as input $|a\rangle$ with $k$ qubits as register $R_{3a}$. We would like to note that $a$ can be set to empty (denoted by $\lambda$), i.e., $k = 0$, in which case $v(a) = S_f$. So, Algorithm 2 can be used to sample from the complete autocorrelation spectrum by not having any qubits in $R_{3a}$ and $R_{2a}$ and using $|0^n\rangle$ in $R_{3b}$ and $R_{2b}$. The following lemma summarizes the behaviour of this algorithm.

**Lemma 9.** The circuit in Algorithm 2 calls $U_f$ twice and either outputs “FAIL” or outputs some $ab$ where $a \in \mathbb{F}_2^k$ is the input and $b \in \mathbb{F}_2^{n-k}$. Probability that “FAIL” is not output is $v(a)/2^{n-k}$ and probability that a particular $ab$ is output is $|\hat{f}(ab)|^2/2^{n-k}$. If $a = \lambda$ and $k = 0$,
Algorithm 2 Algorithm for partial autocorrelation sampling

Require: $a \in \mathbb{F}_2^n$
1: Start with five registers initialized as $|1\rangle$, $|0^k\rangle$, $|0^{n-k}\rangle$, $|a\rangle$ and $|0^{n-k}\rangle$.
2: Apply $H_{n-k}$ to $R_{ab}$ to generate the state $\frac{1}{\sqrt{2^n}} \sum_{b \in \mathbb{F}_2^n} |1\rangle |0^k\rangle |0^{n-k}\rangle |a\rangle |b\rangle$.
3: Let $R_2$ denote the combination of registers $R_{2a}$ and $R_{2b}$.
4: Let $R_3$ denote the combination of registers $R_{3a}$ and $R_{3b}$.
5: Apply $\text{Hod}\text{Fred}^1_0$ on the registers $R_1$, $R_2$ and $R_3$ to generate the state
   $$|\Phi\rangle = \frac{1}{\sqrt{2^{n-k}}} |1\rangle \sum_{b \in \mathbb{F}_2^{n-k}} \sum_{y \in \mathbb{F}_2} \Delta^1_{ab}(y) |y\rangle |a\rangle |b\rangle.$$ 
6: Note that $\sum_b \sum_y \Delta^1_{ab}(y) |y\rangle |ab\rangle = |0^n\rangle \sum_b \hat{f}(ab) |ab\rangle + \sum_b \sum_{y \neq 0} \Delta^1_{ab}(y) |y\rangle |ab\rangle$
7: Measure $R_2$ and $R_3$.
8: If $|0^n\rangle$ is not observed in $R_2$, output “FAIL”.
9: If $|0^n\rangle$ is observed in $R_2$, output the observed state of $R_3$.

then the probability that “FAIL” is not output is $S_f/2^n$ and probability that a particular
$c \in \mathbb{F}_2^n$ is output is $|\hat{f}(c)|^2/2^n$.

It is immediate that Algorithm 2 can be used for sampling from the autocorrelation
spectrum, i.e., the distribution of $n$-bit strings with probability $p(a) = |\hat{f}(a)|^2$. Simply,
repeatedly call the algorithm (by setting $k = 0$ and using the empty string for $a$ and stop when it does not output “FAIL”.
The expected number of calls to the algorithm to obtain a sample is $\frac{2^n}{S_f}$; the number of calls to $U_f$ is of the same order.

However, we can obtain a sample using far lesser calls to $U_f$ using the technique of fixed-point amplitude amplification. This technique can be applied on Algorithm 2 essentially increasing the probability of observing $|0\rangle$ in $R_2$ of $|\Phi\rangle$, from $S_f/2^n$ to any desired $1 - \delta$. For that it will suffice to call the circuit in Figure 2 and hence $U_f$, $O(\frac{2^{n/2}}{\sqrt{S_f}} \log \frac{3}{\delta})$ times.

We discussed an important observation in Section 2 that the output state after observing $|0\rangle$ post-amplification remains identical to the state that would be obtained after observing $|0\rangle$ pre-identification. Therefore, the amplified version of Algorithm 2 will output a sample according to the autocorrelation spectrum with probability $1 - \delta$. Hence, the expected number of calls to $U_f$ to obtain a sample is now reduced to $O(\frac{2^{n/2}}{(1-\delta)\sqrt{S_f}} \log \frac{3}{\delta})$ which is substantial if $\delta \rightarrow 0$. Corollary 1 is obtained by setting $\delta = 1/10$.

5 Autocorrelation estimation

The main problem here is to estimate, with high accuracy and small error (if any), the value of $|\hat{f}(a)|$ for any particular $a \in \{0,1\}^n$; this is identical to estimating $|\hat{f}(a)|^2$.

First, observe that $\hat{f}(a) = \frac{1}{\sqrt{n}} \sum_x (-1)^{f(x)}(-1)^{f(x \oplus a)} = \mathbb{E}_x[X_a]$ where the $\pm 1$-valued random variable $X_a = (-1)^{f(x)}(-1)^{f(x \oplus a)}$ is defined for $x$ chosen uniformly at random from $\{0,1\}^n$. Therefore, the number of samples needed if we were to classically estimate $\hat{f}(a)$ with accuracy $\epsilon$ and error $\delta$ is $O(\frac{1}{\epsilon^2} \log \frac{4}{\delta})$. In fact, $\mathbb{V}ar[X_a]$ is also not bounded — this leaves out the median-of-mean technique as well.

Now we will focus on a quantum algorithm for the aforementioned task aiming for a better
sample complexity. A naive approach is to use the algorithm for autocorrelation sampling
presented in Section 4. Recall that the output of the circuit is a superposition in which $|0^n\rangle |a\rangle$ appears with amplitude $|\hat{f}(a)|/2^{\frac{n}{2}}$. Therefore, one can use the amplitude estimation
technique of Corollary 5 to estimate $|\tilde{f}(a)|^2/2^n$, and hence, $|\tilde{f}(a)|$. The number of samples necessary is going to be $\Theta\left(\frac{2^n}{|f(a)|^2}\right)$ which is going to be $\Omega(2^n)$ since $|\tilde{f}(a)| \in [1, 1]$.

Figure 3 Swap-gate (left) and quantum circuit for swap-test (right)

In this section we present a better estimation approach using the quantum technique of “swap test”. Our technical objective will be to generate a state with a probability that is related to $|\tilde{f}(a)|$ but much higher than that in the earlier approach.

Figure 4 Circuit for estimation of autocorrelation spectrum at a point

Suppose we have two registers over the same number of qubits that are in states denoted by $|\psi\rangle$ and $|\phi\rangle$. The swap test circuit, denoted by $ST$ and illustrated in Figure 3, uses an additional qubit initialized to $|0\rangle$ and applies a conditional swap-gate in a clever manner such that if the first (single-qubit) register is measured, then $|0\rangle$ is observed with probability $\frac{1}{2}[1 + |\langle\psi|\phi\rangle|^2]$. It is easy to show that the circuit performs the following transformation.

$$|0\rangle|\psi\rangle|\phi\rangle \xrightarrow{ST} |0\rangle \otimes \frac{1}{2} \left[|\psi\rangle|\phi\rangle + |\phi\rangle|\psi\rangle\right] + |1\rangle \otimes \frac{1}{2} \left[|\psi\rangle|\phi\rangle - |\phi\rangle|\psi\rangle\right]$$

Our algorithm for estimation of $|\tilde{f}(a)|^2$ is presented in Algorithm 3 and a circuit diagram is given in Figure 4. Obviously, an accurate estimation of $\frac{1}{2}[1 + |\tilde{f}(a)|^2]$ will automatically lead to an accurate estimation of $|\tilde{f}(a)|^2$. Observe that $\frac{1}{2}[1 + |\tilde{f}(a)|^2] \gg |\tilde{f}(a)|^2/2^n$ and therefore, estimation using Algorithm 3 is more efficient compared to that obtained from autocorrelation sampling (describe earlier in this section).

Theorem 10. Algorithm 3 can estimate $|\tilde{f}(a)|^2$ within $\pm \epsilon$ and with probability at least $1 - \delta$ by making $\Theta\left(\frac{1}{\epsilon^3 \log \frac{1}{\delta}}\right)$ calls to $U_f$.

A minor improvement may be added to this algorithm to handle $\tilde{f}(a) = 0$ coefficients by first applying the previously mentioned technique of applying amplitude estimation (with a larger $\epsilon$) on the output state of sampling algorithm from Section 3. Note that amplitude estimation does not err when the probability it is estimating is 0 (even for few queries). Then run Algorithm 3 as usual and return the minimum of the two estimates. In case $\tilde{f}(a) = 0$, the first amplitude estimation will return 0 as the estimate.

5.1 Estimation of Walsh coefficients

A recent paper proposed a quantum algorithm for estimating Walsh coefficients at a specified point, say $a$. The approach of Xie et al. [23] was to first generate the state $\sum_{a \in F_2} \tilde{f}(a)|a\rangle$ (by
Algorithm 3 Autocorrelation estimation at point $a$

Require: Parameters: $\epsilon$ (confidence), $\delta$ (error)

1. Start with four registers of which $R_1$ is initialized to $|a\rangle$, $R_2$ to $|0\rangle$, and $R_3$, $R_4$ to $|0^n\rangle$.
2. Apply these transformations.
   - $|a\rangle|0^n\rangle|0^n\rangle$ to $|a\rangle|0\rangle\left(\frac{1}{\sqrt{2^n}} \sum_x |x\rangle\right)\left(\frac{1}{\sqrt{2^n}} \sum_x |x\rangle\right)$
   - $\text{CNOT} \quad |a\rangle|0\rangle\left(\frac{1}{\sqrt{2^n}} \sum_x |x\rangle\right)\left(\frac{1}{\sqrt{2^n}} \sum_x |x \oplus a\rangle\right)$
   - $U_f \otimes U_f \quad |a\rangle|0\rangle\left(\frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)}|x\rangle\right)\left(\frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x \oplus a)}|x \oplus a\rangle\right)$ ▷ Uses reusable $|-\rangle$
   - $\text{CNOT} \quad |a\rangle|0\rangle\left(\frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)}|x\rangle\right)\left(\frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x \oplus a)}|x \oplus a\rangle\right) = \text{(say)} \quad |a\rangle|0\rangle|\psi\rangle|\phi\rangle$
   - Observe that $\langle\psi|\phi\rangle = \frac{1}{\sqrt{2}} (-1)^{f(x)} (-1)^{f(x \oplus a)} = \hat{f}(a)$
3. Apply $ST$ on $R_2$, $R_3$ and $R_4$ to obtain $|a\rangle\left[0 \otimes \frac{1}{\sqrt{4}} (|\psi\rangle|\phi\rangle + |\psi\rangle|\phi\rangle) + |1 \otimes \frac{1}{\sqrt{4}} (|\psi\rangle|\phi\rangle - |\phi\rangle|\psi\rangle)\right]$
   - $\frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x \oplus a)}|x\rangle\langle y|$
   - Let $|1\rangle = \frac{1}{\sqrt{2}} (|\psi\rangle|\phi\rangle + \frac{1}{\sqrt{2}} (|\psi\rangle|\phi\rangle)$
   - The state after applying $ST$ can be rephrased as $|a\rangle|0\rangle \otimes |\chi^0_{f,a}\rangle + |a\rangle|1\rangle \otimes |\chi^1_{f,a}\rangle$
4. $\ell \leftarrow \text{estimate} \left(\langle \chi^1_{f,a}\rangle |\chi^0_{f,a}\rangle\right) = \frac{1}{2} + \frac{1}{\sqrt{2}} |\psi\rangle|\phi\rangle|^2 = \frac{1}{2} + \frac{1}{2} |\hat{f}(a)|^2$ with confidence $\epsilon/2$ and error $\delta$ (Corollary [3])
5. Return $2\ell - 1$ as the estimate of $|\hat{f}(a)|^2$

especially using Deutsch-Jozsa algorithm) and then apply amplitude estimation to estimate $|\hat{f}(a)|^2$. The approach outlined above can also be used to estimate $|\hat{f}(a)|^2$. Algorithm 3 already describes how to construct $|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)}|x\rangle$ from $|0^n\rangle$. One has to then generate a new $|\phi\rangle$ in this manner:

$|a\rangle|0^n\rangle \xrightarrow{\text{CNOT}} |a\rangle|a\rangle \xrightarrow{H^n} \frac{1}{\sqrt{2^n}} \sum_x (-1)^a|x\rangle |a\rangle = |\phi\rangle$

Then, applying the swap-test circuit as in Algorithm 3 constructs a state of the form $|a\rangle|0\rangle \otimes |\phi\rangle + |a\rangle|1\rangle \otimes |\chi^1_{f,a}\rangle$ in which $|\langle \chi^0_{f,a}\rangle |\chi^0_{f,a}\rangle\rangle = \frac{1}{2} + \frac{1}{2} |\hat{f}(a)|^2$. Since, $|\hat{f}(a)|^2$ is usually much much less than 1, we can say that $\frac{1}{2} + \frac{1}{2} |\hat{f}(a)|^2 \gg |\hat{f}(a)|^2$. Therefore, amplitude estimation for a fixed relative accuracy and a fixed probability of error will require much less number of calls to $U_f$ in our approach compared to that which was proposed by Xie et al.

### 5.2 Estimation of $S_f$

In this section we consider the problem of estimating $S_f$ and more generally, $v(a)$ for any $a \in \mathbb{F}_2^n$ for any $k \leq n$. We only consider relative accuracy of $\epsilon$ but our methods can be easily adapted to additive accuracy.

First we describe a sampling-based classical approach for estimating $S_f$. Observe that $S_f$ can simplified in the following manner:

$$S_f = \sum_{a \in \mathbb{F}_2^n} \hat{f}(a)^2 = \sum_{a \in \mathbb{F}_2^n} \sum_{b \in \mathbb{F}_2^n} (-1)^{f(b) \oplus f(b \oplus a)}^2$$

$$= \frac{1}{2^n} \sum_{a \in \mathbb{F}_2^n} \sum_{b \in \mathbb{F}_2^n} (-1)^{f(a \oplus b) \oplus f(a \oplus c)} = 1 + \frac{1}{2^n} \sum_{a \in \mathbb{F}_2^n} \sum_{b \in \mathbb{F}_2^n} (-1)^{f(a \oplus b) \oplus f(a \oplus c)}$$
\[= 1 + (2^n - 1)\mathbb{E}_{a, b, c}[X_{a, b, c}]\]

Here, \(a, b, c\)'s are random variables chosen uniformly at random from \(\mathbb{F}_2^n\) such that \(b \neq c\) and \(X_{a, b, c}\) stands for the \pm 1-valued random variable \((-1)^{f(a \oplus b) f(a \oplus c)}\). Note that 
\[\mathbb{E}[X_{a, b, c}] = \frac{S_{a, b, c} - S_f}{2^n} \approx \frac{S_f}{2^n}.\]
One way to estimate \(\mathbb{E}[X_{a, b, c}]\) is to use multiple independent samples of \(a, b, c\). Since each sample of \(X_{a, b, c}\) requires 2 calls to \(f()\), therefore \(O(\frac{2^n}{\epsilon^2} \log \frac{1}{\delta})\) calls to \(f()\) would be sufficient to estimate \(\mathbb{E}[X_{a, b, c}]\) with \(\epsilon\) relative-accuracy and \(\delta\) error. Suppose \(\tilde{X}\) is the estimate that we obtain; then an estimation of \(S_f\) can be obtained by \(1 + (2^n - 1)\tilde{X}\) — this estimate almost satisfies the same \((\epsilon, \delta)\) guarantee as that of \(X_{a, b, c}\).

On the quantum side, the amplitude estimation results from Section 2 can be used to estimate \(S_f\) and \(\upsilon(a)\) by applying those techniques on the output state of Algorithm 2. It directly follows that \(S_f\) and \(\upsilon(a)\) can be estimated with relative accuracy \(\epsilon\) and probability of error \(\delta\) using 
\[O\left(\frac{2^{n/2}}{\epsilon^2} \log \frac{1}{\delta}\right) \text{ and } O\left(\frac{2^{n/2}}{\epsilon^2} \log \frac{1}{\delta}\right)\]
calls, respectively, to \(U_f\).

However, a much better estimation algorithm is possible using Algorithm 3 by passing 
\[\frac{1}{\sqrt{2(n-k)/2}} \sum_{b \in \mathbb{F}_2^{n-k}} |ab\rangle|b\rangle\text{ as } R_1 \text{ instead of } |a\rangle.\]
The state after applying \(ST\) would be:
\[\frac{\hat{\upsilon}}{\sqrt{2(n-k)/2}} \sum_{b \in \mathbb{F}_2^{n-k}} |ab\rangle|0\rangle \otimes |\chi_{f, ab}^0\rangle + \frac{1}{\sqrt{2(n-k)/2}} \sum_{b \in \mathbb{F}_2^{n-k}} |ab\rangle|1\rangle \otimes |\chi_{f, ab}^1\rangle\]

The probability of observing \(|0\rangle\) in \(R_2\) can be shown to be \(\frac{1}{2} + \frac{1}{2} \upsilon(a)\) that is lower-bounded by \(\frac{1}{2}\) and, usually, much larger compared to \(\frac{\upsilon(a)}{2^n}\) that Algorithm 2 generates. It is no wonder that the query complexity of \(S_f = \upsilon(\lambda)\) is vastly reduced if amplitude estimation (Corollary 5) is applied to estimate this probability and that directly leads to Theorem 2.

### 6  Autocorrelation filtering

The question we tackle here is similar to the question involved in learning Boolean functions using their Walsh spectrum [19 Ch. 3.5]: Given an \(n\)-bit Boolean function \(f()\) and a threshold \(\tau \in (0, 1)\), output a list \(L\) of points from \([0, 1]^n\) such that \(x \in L\) iff \(|\hat{f}(x)| \geq \tau\).

The quantum algorithm that we design is illustrated in Algorithm 4 and is motivated by a particular form of the Goldreich-Levin algorithm that solves a similar problem, but for Walsh spectrum. The algorithm computes an approximate list \(L\) with high probability.
- If \(|\hat{f}(x)| \geq \tau\) then \(x \in L\).
- If \(x \in L\) then \(|\hat{f}(x)| \geq \tau/4\).
- Algorithm makes \(O(S_f, \text{poly}(n), \frac{1}{\tau^2}, \log \frac{1}{\delta})\) queries to \(U_f\) in which \(S_f\) denotes \(\sum_a \hat{f}(a)^2\).

The specific query complexity is \(O\left(\frac{1}{\tau^2} \log \frac{2}{\epsilon} + \frac{2^{n/2}S_f}{\tau^2} \log \frac{1 + \epsilon}{\epsilon} \frac{nS_f}{2^n}\right)\).

Unlike Walsh coefficients for which the Parseval’s theorem stipulates that \(\sum_a \hat{f}(a)^2 = 1\), there is no such bound known for autocorrelation coefficients and \(S_f\) can be as large as \(2^n\).

Recall that we used \(\upsilon(a)\) to denote the normalized partial second moment of the autocorrelation coefficients at point \(a \in \mathbb{F}_2^n\), i.e, \(\upsilon(a) = \sum_{b \in \mathbb{F}_2^n} |\hat{f}(ab)|^2\) and Algorithm 3 was designed to estimate these moments with a desired accuracy and error. It can thought of as finding all leaves with high values in a complete binary tree \(T\) using a level-order traversal. The nodes of \(T\) are labeled with binary strings of length at most \(n\) such that all nodes in the same level of \(T\) have labels of the same length — in particular, leaves are labels of length \(n\). The “value” of a node \(a\) is simply \(\upsilon(a)\) and the objective becomes to find all leaves with value \(\upsilon(l) = |\hat{f}(l)|^2 \geq \tau^2\) by computing values of only \(\text{poly}(n)\) many nodes.

Finding the exact value of a leaf is computationally expensive and not even necessary for approximate filtering. Instead of classical sampling (as is done in the proof of the
Goldreich-Levin theorem), we make use of the following lemma that applies Algorithm 1 on Algorithm 2 to decide if value of a node is above or below a threshold. Recall that the probability that $R_2$ is observed in state $|0\rangle$ at the end of Algorithm 2 is $v(a)/2^{n-|a|}$. We would like to note that the Algorithm in Theorem 2 cannot be used in place of Algorithm 2 because the estimates $\frac{1}{2} + \frac{v(a)}{2^{n-a}}$ would not preserve the gap as required below.

Lemma 11. Consider some $a \in \mathbb{F}_2^n$ for some $k \leq n$, some $\delta \in (0, 1)$ and some $t \in (0, 1]$. If Algorithm 2 is run using Algorithm 3 as $A$ and “good state” as observing $|0^n\rangle$ in $R_2$, then we can decide if $v(a)/2^{n-k} \geq t$ or $v(a)/2^{n-k} \leq t/16$ using $\tilde{O}(\frac{1}{\sqrt{\delta}} \log \frac{1}{\delta})$ calls to $U_f$ and with probability at least $1 - \delta$.

Algorithm 4 Find points with high autocorrelation coefficients of an $n$-bit Boolean function

Require: Parameters: $\tau$ (threshold), $\delta$ (error)
1: Set internal parameter $\epsilon'$ (small constant)
2: Initialize FIFO queue $Q$ and list $L$
3: Push $\lambda$ (empty string) into $Q$
4: $\tilde{S}_f = \text{estimate } S_f$ with relative accuracy $\epsilon'$ and error $\delta/2$ using Theorem 2
5: Set $\delta' = \delta \tau^2(1 - \epsilon')/64n\tilde{S}_f$
6: Determine if $v(p0) \geq \tau^2$ ...
7: while $Q$ is not empty do
8: \hspace{1em} $p \in \mathbb{F}_2^n \leftarrow \text{ pop from } Q$
9: \hspace{1em} Decide if $v(p0) \geq \tau^2$ or $v(p0) \leq \tau^2/16$ with error $\delta'$ and $t = \frac{\tau^2}{\tau^2}$ using Lemma 11
10: \hspace{1em} if $v(p0) \geq \tau^2$ then
11: \hspace{1em} \hspace{1em} If $|p| = n$, add $p0$ to $L$
12: \hspace{1em} \hspace{1em} Else add $p0$ to $Q$
13: \hspace{1em} end if
14: \hspace{1em} Decide if $v(p1) \geq \tau^2$ or $v(p1) \leq \tau^2/16$ with error $\delta'$ and $t = \frac{\tau^2}{\tau^2}$ using Lemma 11
15: \hspace{1em} if $v(p1) \geq \tau^2$ then
16: \hspace{1em} \hspace{1em} If $|p| = n$, add $p1$ to $L$
17: \hspace{1em} \hspace{1em} Else add $p1$ to $Q$
18: \hspace{1em} end if
19: end while

Proof of Algorithm 4. The proof of correctness and complexity analysis follows the same line as that used in the Goldreich-Levin algorithm 19. For the sake of completeness, we describe it here along with the necessary modifications. We need to make a few observations first. First, observe that $v(p) = v(p0) + v(p1)$ holds ab initio. Furthermore, recall that $S_f$ denotes $v(\lambda) = \sum_{b \in \mathbb{F}_2^n} |f(b)|^2$ and therefore, for every $k$ it holds that $S_f = \sum_{p \in \mathbb{F}_2^n} v(p)$.

For the next few claims, (temporarily) assume that Lines 9, 10 and 14 of Algorithm 4 are processed without any error. For the forward direction of correctness about $L$, consider any $a \in L$ and any prefix $p$ of $a$. Since $v(a) \geq \tau^2$, $v(p) \geq \tau^2$ as well. Therefore, all such $p$ would be added to $Q$ and finally, $a$ would be added to $L$. For the other direction, consider any $a \in \mathbb{F}_2^n$ such that $|\tilde{f}(a)| \leq \tau/4$, therefore, $v(a) \leq \tau/16$ which implies that $a$ would not be added to $L$. These ensure the approximate nature of $L$.

Since $S_f$ is the total sum of values at each level, there can be at most $S_f/(\frac{\tau^2}{16})$ nodes in each level with value at least $\tau^2/16$ – the others would never be in $Q$. Therefore, $Q$ will contain at most those many nodes from each level and in total, at most $16nS_f/\tau^2$ nodes will be added to $Q$ leading to at most $2 \cdot 16nS_f/\tau^2$ invocations of Lemma 11 (Lines 9 and 14).
To keep the overall error within $\delta$, it suffices to (i) bound the error of estimation of $S_f$ to $\delta/2$ and (ii) combined error of all Lines 9 and 14 to $\delta/2$. For (i), we know from Theorem 2 that $\tilde{S}_f$ can be obtained using $O(\frac{2n/\epsilon'}{\sqrt{S_f}} \log \frac{1}{\delta})$ calls to $U_f$ and it satisfies $\tilde{S}_f/(1+\epsilon') \leq S_f \leq S_f/(1-\epsilon')$ with probability at least $1-\delta/2$. For (ii), we invoke Lemma 11 with error $\delta/(1-\epsilon')\tau^2/64n\tilde{S}_f$ to ensure that the combined error of $2 \cdot 16nS_f/\tau^2$ invocations lead to an overall $\delta/2$ error.

The number of calls to $U_f$ can be similarly broken into two parts: one due to estimation of $S_f$ and other is a combined total of Lemma 11 invocations. The number of calls for the former is $O(\frac{1}{\epsilon'} \log \frac{1}{\delta})$. For the number of latter calls, it will be easier to analyse the number of calls in each level, say, level $k \in [1 \ldots n]$. The number of nodes in this level on the lemma is invoked is $O\left(\frac{32nS_f}{\tau^2}\right)$ and the number of calls in each such invocation is $\tilde{O}\left(\frac{2n/\epsilon'}{\sqrt{2} \tau} \log \frac{1}{\delta'}\right)$. Using the fact that $\frac{1}{\delta'} \leq \frac{64n(1+\epsilon')S_f}{(1-\epsilon')\tau^2}$ and $\sum_{k=1}^{n} 1/\sqrt{2^k} = O(1)$, we arrive at the total complexity across all invocations as $\tilde{O}\left(\frac{S_f}{\tau^2} \frac{2^{n/2}}{\epsilon'} \log \frac{1+\epsilon'}{1-\epsilon'}\frac{n\tilde{S}_f}{\tau^2}\right)$.

Corollary 3 is obtained by replacing $S_f$ and $1/\tau$ by polynomials in $n$.

## 7 Conclusions

In this paper we design several efficient quantum algorithms that analyse the autocorrelation spectrum of a Boolean function given as a black-box. Our algorithms can be used to estimate, with low error and high accuracy, the spectrum value at a desired point and identify all points with high spectral values. Autocorrelation spectrum is a very important tool for designing Boolean functions with good cryptographic properties and also for mounting differential attacks of cryptosystems. We hope that the results of this paper can motivate better cryptosystems that are resistant towards quantum techniques.
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