THE YAMABE EQUATION IN SMALL CONVEX DOMAINS IN $\mathbb{R}^3$ AND SMALL BALLS IN $\mathbb{R}^n$

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Abstract. We show an iterative method to solve a Dirichlet problem for a Yamabe-type equation in small convex domains in $\mathbb{R}^3$ and small balls in $\mathbb{R}^n$.

Key words: Yamabe equation, Yamabe problem, nonlinear elliptic equations, Dirichlet boundary conditions.

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1. Introduction

Let $(M, g)$ be a Riemannian manifold of dimension $n \geq 3$, with scalar curvature $R_g$. The scalar curvature of any conformal metric to $g$, say $\tilde{g} = e^{2f} \cdot g$, is given by

$$R_{\tilde{g}} = e^{-2f} (R_g + 2(n-1)\Delta_g f - (n-1)(n-2)|\nabla f|^2)$$

in terms of the Laplacian operator $\Delta_g$ and the Levi-Civita connection $\nabla$, with respect to $g$, for some smooth positive function $f$. Therefore the metric $\tilde{g}$ has constant scalar curvature $\lambda$ if and only if $f$ satisfies the Yamabe equation:

$$2(n-1)\Delta_g f - (n-1)(n-2)|\nabla f|^2 + R_g = \lambda \cdot e^{2f}. \tag{1}$$

After an appropriate substitution, the gradient term can be removed. In fact, for $n \geq 3$, if we write $u = \frac{2}{n-2} \log(f)$, the Yamabe equation becomes:

$$-\frac{4(n-1)}{n-2} \Delta_g u + R_g u = \lambda \cdot u^{\frac{n+2}{n-2}}. \tag{2}$$

Therefore, solving (2) is equivalent to solving (1). We are interested in smooth positive solutions.

The latter equation, treated as the standard form of the Yamabe equation, has been extensively studied. The fundamental result is the existence of at least one positive solution on closed Riemannian manifolds. This follows from the works of H. Yamabe [7], N. Trudinger [6], T. Aubin [1] and R. Schoen [5]. Many other settings have been explored.

In this paper we consider the following Yamabe-type Dirichlet problem on a small convex domain $\Omega$:

$$\begin{cases} \Delta f = \frac{1}{2(n-1)} \left[ e^{2f} R(x) + (n-1)(n-2)|\nabla f|^2 \right] + S(x), & \text{in } \Omega \\ f = 0, & \text{on } \partial \Omega, \end{cases} \tag{3}$$

where $R$ and $S$ are smooth functions.

We shall prove that this problem is solvable for $n = 3$ in any sufficiently small convex domain, and for general $n$ in a sufficiently small ball, in the sense that their volumes and their slab diameter are sufficiently small. Recall that for a domain $\Omega \subset \mathbb{R}^n$, the slab diameter is the smallest distance between two parallel hyperplanes such that $\Omega$ is contained in the region in between them. Now state our first result.

**Theorem 1.** Let $\Omega \subset \mathbb{R}^n$, and $R, S : \Omega \to \mathbb{R}$ be smooth functions such that $\|R\|_{\infty} \leq \Lambda$ and $\|S\| \leq \gamma$, for some $\Lambda, \gamma > 0$. Assume that:

1. If $n = 3$, suppose that $\Omega$ is a bounded convex domain with smooth boundary, and with
   a. either
   
   $$\text{Vol}(\Omega) \leq \min \left\{ 1, \left( \frac{8}{4.76 \pi^\frac{4}{3} (4\Lambda + \gamma)} \right)^3 \right\}$$
   
   and
   
   $$\delta \leq \min \left\{ 1, 1.25 \text{Vol}(\Omega)^\frac{1}{3}, \frac{2}{0.75 \Lambda + 1} \right\},$$

   where $\text{Vol}(\Omega)$ is the volume of $\Omega$.
of $\Omega$ at its two endpoints. This estimate for the volume of $\Omega$ allows to apply the result to thin domains with large diameter, this in contrast to the results in [4].

(5)

For $n \geq 4$, if $\Omega$ is a ball of radius $d$, with

\[ d \leq \min \left\{ \frac{1}{2}, \frac{n-1}{C_n (2.5\lambda + \gamma)} \right\}. \]

Then the nonlinear Dirichlet problem (5) has a smooth solution.

**Remark** The estimate given for the volume of $\Omega$ and its slab diameter, and of its diameter, are not sharp. It was given mostly to show that this estimate can be obtained explicitly without much effort from the proof of the theorem. For instance, the condition $\delta \leq 1.25\text{Vol}(\Omega)^{\frac{3}{n}}$ holds whenever the largest ball that can be inscribed in $\Omega$ has radius $\delta/2$ and exists a diameter of this ball that hits the boundary of $\Omega$ at its two endpoints. This estimate for the volume of $\Omega$ allows to apply the result to thin domains with large diameter, this in contrast to the results in [4].

The equation above is equivalent to the usual form of the Yamabe equation (2), with boundary condition $u = 1$. In fact, in this case we obtain:

\[
\begin{cases}
\Delta u = Su - \frac{n-2}{2(n-1)} Ru^{\frac{n+2}{n}} , & \text{in } \Omega, \\
u = 1, & \text{on } \partial \Omega.
\end{cases}
\]

As a corollary from Theorem 1, we obtain the following solvability result for a Yamabe-type problem on a convex domain in $\mathbb{R}^3$ and small balls in $\mathbb{R}^n$.

**Theorem 2.** Let $\Omega \subset \mathbb{R}^n$ and $S : \Omega \rightarrow \mathbb{R}$ be a smooth function such that $\|S\|_{\infty} \leq \gamma$. Assume that:

1. If $n = 3$, let $\Omega$ be a bounded convex domain of smooth boundary, with
   
   (a) **either**
   
   \[ \text{Vol}(\Omega) \leq \left( \frac{8}{4.76\pi^\frac{2}{3}(1+\gamma)} \right)^3 \quad \text{and} \quad \delta \leq \min \left\{ 1, 1.25\text{Vol}(\Omega)^{\frac{3}{2}} \right\}, \]
   
   (b) **or**
   
   \[ \text{diam}(\Omega) \leq \min \left\{ 1, \frac{4}{4.76\pi^\frac{2}{3}(0.625 + \gamma)} \right\}. \]

2. For $n \geq 4$, assume $\Omega$ is a ball of radius $d$, with

\[ d \leq \min \left\{ \frac{1}{2}, \frac{n-1}{C_n (0.625 + \gamma)} \right\}. \]

Then for any $-\infty < c < \infty$ there exists a $\lambda \in \mathbb{R}$ such that

\[
\begin{cases}
\Delta f = -\frac{1}{2(n-1)} \left[ \lambda e^{2f} + (n-1)(n-2)|\nabla f|^2 \right] + S(x), & \text{in } \Omega \\
f = c, & \text{on } \partial \Omega,
\end{cases}
\]

has a smooth solution.

As a consequence of the previous result, we obtain that in small enough domain, for any constant $\varphi > 0$, and any $\lambda$ small enough, that there is a $u > 0$ which satisfies:

\[
\begin{cases}
\Delta u = Su - \frac{n-2}{2(n-1)} \lambda u^{\frac{n+2}{n}}, & \text{in } \Omega \\
u = \varphi, & \text{on } \partial \Omega.
\end{cases}
\]

Let us make a few comments on the previous result. First, notice that one of the solvability criteria for the Yamabe problem given above depends on the volume and the slab diameter of the domain, and not on its diameter, in contrast to the requirements in [4]. Thus we can consider, in the case of a convex domain in $\mathbb{R}^3$, as we said before, a very long and thin domain. Also, it is important to notice that our criterion for solvability is independent of $c$ (and hence of $\varphi$), which is not obvious at all, and that to get positivity of $u = e^{2f}$, in the usual form of the Yamabe equation, is trivial in our context.

Finally, using our methods we give the following geometric application (as it is, the following theorem can be obtained using the Inverse Function Theorem as well; however, we wanted to give a different proof).
**Theorem 3.** Any bounded subdomain $\Omega \subset \mathbb{R}^n$ of smooth boundary admits a conformal deformation such that the resulting metric has constant scalar curvature, and the deformation can be chosen so that the sign of the scalar curvature is either positive or negative. Furthermore, the deformation can be arranged so that the metric of the boundary remains invariant, that is, $\partial M$ keeps the metric it inherits from $\mathbb{R}^n$.

Recently, by means of an iterative method and a maximum principle for Dirichlet problems we solved the Poisson equation with boundary conditions on bounded domains and some unbounded domains which are bounded in one direction. The preprint can be found in [2]. The present work was inspired by those results and the recent preprint [4], by S. Rosenberg and J. Xu. They considered the Yamabe equation on a Riemannian domain $(\Omega, g)$ of $\mathbb{R}^n$, where $g$ is a Riemannian metric that can be extended smoothly to the boundary of $\Omega$. They proved that if the volume with respect to $g$ and the diameter of $\Omega$ are small enough, then the equation (2) admits at least one solution, with a constant positive boundary condition.

Rosenberg an Xu also proposed an iterative method for solving the Yamabe equation. Nevertheless, their approach is different since they use the equation in its standard form (2). Furthermore, besides proving existence and regularity, the difficulty using this equation is to prove that $u$ is positive, so $u$ can be used as a conformal factor.

If $R \geq 0$ and $S \leq 0$ in the equation above, the positivity of $u$ would be a consequence of the Maximum Principle. However, presented in the form (3) this issue actually disappears without any regard to the signs of the functions $R$ and $S$. Indeed, we think that the arguments in [4] can be greatly simplified: First, by using the gradient form of the Yamabe equation; and second, by using a gradient estimate as the one given in the key Lemma 1 below. In fact, as soon as there is a gradient estimate as the one in Lemma 1 or Lemma 3, our method can be used to prove existence of solutions to a Dirichlet problem for an elliptic operator with some well-behaved nonlinearities in the gradient of the unknown function.

The reader will find the proof of Theorem 1 part (1), where the iteration technique is discussed, in Section 2; the proof of Theorem 1 part (2) is in Section 3; and the proofs of Theorems 2 and 3 are found in Section 4.

### 2. Proof of Theorem 1 part (1)

The present work keep the following notations. The $L^2$-norm on a domain $\Omega \subseteq \mathbb{R}^n$ will be denoted by:

$$\|f\| := \left(\int_{\Omega} f^2 \right)^{1/2},$$

integrating with respect to the Lebesgue measure of $\mathbb{R}^n$. Denote by $L^\infty(\Omega)$ the Banach space of bounded functions with the norm:

$$\|f\|_\infty := \sup_{\Omega} |f|.$$

For the Sobolev spaces $W^{1,p}(\Omega)$ we use the norm:

$$\|f\|_{1,p} = \left(\int_{\Omega} \sum_{|\beta| \leq k} |D^\beta f|^p \right)^{1/p},$$

with respect the Lebesgue measure of $\mathbb{R}^n$, for $p \geq 1$ and any multi-index $\beta = (\beta_1, \beta_2, \ldots, \beta_s)$, where $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_s$. As usual, for $C^1_0(\Omega)$, the space of $C^1$-class functions with compact support in $\Omega$, we will denote by $H^1_0(\Omega)$ its closure in $W^{1,2}(\Omega)$.

We have the following basic estimate.

**Lemma 1.** Let $\Omega \subset \mathbb{R}^3$ convex. Let $h$ be a smooth function, and consider the solution to the Dirichlet problem

$$\Delta u = h \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{in} \quad \partial \Omega.$$

There is a constant $C > 0$ such that

$$\|\nabla u\|_\infty \leq CV^{\frac{1}{2}} \|h\|_\infty,$$

where $V$ is the volume of $\Omega$. The constant $C$ can be taken as $4.76\pi^{\frac{1}{2}}$. 
Proof. Let $G$ be the Green’s function of the Dirichlet Laplacian in $\Omega$. Then, by G. C. Evans (Theorem 2 of [3]), we have that
\[ \int_{\Omega} |\nabla G(x, x')| \, dx' \leq 4.76 \left( \pi^2 \text{Vol}(\Omega) \right)^{\frac{1}{2}} \leq 4.76\pi^{\frac{3}{2}} d. \]
From this we obtain that
\[ |\nabla u(x)| \leq \int_{\Omega} |\nabla G(x, x')| \, dx', \]
and hence
\[ ||\nabla u||_\infty \leq 4.76 \left( \pi^2 \text{Vol}(\Omega) \right)^{\frac{1}{2}} ||h||_\infty, \]
which is what we wanted to prove. \hfill \square

2.1. Iteration procedure. To produce a solution to the Yamabe equation, we define the following iteration:
\[ \left\{ \begin{array}{l}
\Delta f_{k+1} = -\frac{1}{2(n-1)} \left[ R \varepsilon^{2f_k} + |\nabla f_k|^2 \right] + S, \quad \text{in} \quad \Omega, \\
f_k = 0, \quad \text{on} \quad \partial \Omega,
\end{array} \right. \tag{6} \]
starting with $f_0 = 0$. For $R$ and $S$ smooth, each of these problems have a smooth solution.

Using Lemma (11) we may now estimate as follows, using the conventions $\Lambda = ||R||_\infty$, $\gamma = ||S||_\infty$ and $A = \text{Vol}(\Omega)^{\frac{1}{2}}$, for each $k \geq 1$:
\[ ||\nabla f_{k+1}||_\infty \leq \frac{CA}{2(n-1)} \left[ \Lambda \varepsilon^{2||f_k||_\infty} + ||\nabla f_k||_\infty^2 + \gamma \right] \leq \frac{CA}{2(n-1)} \left[ \Lambda \varepsilon^{2||\nabla f_k||_\infty} + ||\nabla f_k||_\infty^2 + \gamma \right], \]
where $\delta > 0$ is the slab diameter of $\Omega$. We shall assume that
\[ \delta \leq 2 \left( \frac{\text{Vol}(\Omega)}{\omega_3} \right)^{\frac{1}{3}} \leq 1.25 A, \]
where $\omega_3$ is the volume of the unit ball in $\mathbb{R}^3$. Other restrictions on $\delta > 0$ are possible; we made this choice because it has a geometric meaning as explained in the introduction: it holds whenever the largest ball that can be inscribed in $\Omega$ has $\delta/2$ as its radius and there is a diameter of this ball that hits the boundary of $\Omega$ at its two endpoints.

From this we obtain that if $A$ is small enough, then there exists a $K > 0$ such that for the function
\[ f(t) = \frac{CA}{2(n-1)} \left( \Lambda \varepsilon^{1.25 At} + t^2 + \gamma \right) \]
satisfies that $f(t) \leq K := K(A, \Lambda, \gamma)$ whenever $0 \leq t \leq K$. The proof of this statement is elementary. Take $K$ as the smallest solution to the equation $t = f(t)$, which does exist if $A$ is small enough, and as $f$ is increasing the claim follows. What we can also show, and will be needed in what follows, is that the smallest root of $t = f(t)$ does not increase without bound as $A \to 0$. In fact $K$ is $O(A)$ for $A$ and $\gamma$ fixed. Indeed, we can write
\[ f(t) - t = \frac{CA}{2(n-1)} t^2 + \left( \frac{1.25 CA^2 A}{2(n-1)} - 1 \right) t + \frac{CA}{2(n-1)} (\gamma + \Lambda + O(A^2)), \tag{7} \]
and our assertion is a consequence of the quadratic formula. We shall give a more explicit estimate at the end of this section.

To proceed, we next consider
\[ \Delta (f_{k+1} - f_k) = -\frac{1}{2(n-1)} \left[ R \left( e^{2f_k} - e^{2f_{k-1}} \right) + |\nabla f_k|^2 - |\nabla f_{k-1}|^2 \right], \]
with the boundary condition $f_{k+1} - f_k = 0$.

On the other hand, using the mean value theorem, we can estimate
\[ |e^{2f_k} - e^{2f_{k-1}}| \leq e^{2 \max\{||f_k||_\infty, ||f_{k-1}||_\infty\}} |f_k - f_{k-1}| \]
\[ \leq e^{\delta \max\{||\nabla f_k||_\infty, ||\nabla f_{k-1}||_\infty\}} |f_k - f_{k-1}| \leq e^{\delta K} |f_k - f_{k-1}|, \]
where we have used that for all \( k \), \( \| \nabla f_k \|_\infty \leq K \), which can be proved by induction starting from the fact that \( f_0 = 0 \). Next, we can estimate
\[
|\nabla f_k|^2 - |\nabla f_{k-1}|^2 = |\nabla f_k + \nabla f_{k-1}| \| \nabla f_k - \nabla f_{k-1} | \\
\leq 2K \| \nabla f_k - \nabla f_{k-1} |.
\]

Then, multiplying the expression for \( \Delta (f_{k+1} - f_k) \) by \( f_{k+1} - f_k \), integrating, and using the Cauchy-Schwartz inequality yields
\[
\| \nabla f_{k+1} - \nabla f_k \|^2 \leq \frac{1}{2(n-1)} \left( \Lambda e^{K\delta} \| f_{k+1} - f_k \| \| f_k - f_{k-1} \| + 2K \| f_{k+1} - f_k \| \| \nabla f_k - \nabla f_{k-1} \| \right).
\]

Using the Poincaré inequality,
\[
\| f_j - f_k \| \leq \frac{\delta}{\sqrt{2}} \| \nabla f_j - \nabla f_{k-1} \|,
\]
where \( \delta \), as before, stands for the slab diameter of the domain, we get
\[
\| \nabla f_{k+1} - \nabla f_k \|^2 \leq \frac{1}{2(n-1)} \left( \frac{\delta^2 \Lambda e^{K\delta}}{2} \| \nabla f_{k+1} - \nabla f_k \| \| \nabla f_k - \nabla f_{k-1} \| + \frac{2K\delta}{\sqrt{2}} \| \nabla f_{k+1} - \nabla f_k \| \| \nabla f_k - \nabla f_{k-1} \| \right),
\]
which gives after simplification
\[
\| \nabla f_{k+1} - \nabla f_k \| \leq \frac{1}{2(n-1)} \left( \frac{\delta^2 \Lambda e^{K\delta}}{2} + \sqrt{2}K\delta \right) \| \nabla f_k - \nabla f_{k-1} \|. \tag{8}
\]

Thus we have obtained the following:

**Proposition 2.** Consider the sequence \( \{ f_k \} \) produced by the iteration process (6). Let \( A = \text{Vol}(\Omega)^{\frac{1}{2}} \) and \( \delta \) the slab diameter of \( \Omega \). If \( A \) and \( \delta \) are small enough the following holds:

1. At each step \( f_k \) is smooth.
2. \( f_k \) is uniformly in \( W^{1,\infty}(\Omega) \).
3. The sequence \( f_k \) converges in \( H^1_0(\Omega) \).

Hence, there is a uniform limit \( f \), and actually, this \( f_k \rightarrow f \) in every Hölder space \( C^\alpha(\Omega) \), for any \( 0 < \alpha < 1 \) and the limit is actually Lipschitz. To prove that \( f \) is smooth, define \( u \) by the relation \( u^\frac{1}{A} = e^f \). Then we obtain equation (4), and as \( u \) is Lipschitz, as \( f \) is, by a standard bootstrapping argument we obtain that \( u \) is smooth, and so is \( f \).

### 2.2. Estimating \( \text{Vol}(\Omega) \) and \( \text{diam}(\Omega) \)

The \( O(A^2) \) term in (7) is given by
\[
\frac{1,25^2 A^2 e^{1.25 A \xi^2}}{2}, \quad 0 < \xi < t.
\]

We shall assume that \( A \leq 1 \), and thus if \( A \) is small enough, as the roots of the resulting quadratic equation in (7) when we replace the \( O(A) \) by its largest possible value, \( 0.8 A^2 e^{1.25 A \xi^2} \), are less than
\[
\frac{CA}{8} (\Lambda + \gamma + 3\Lambda), \tag{9}
\]
where we are also assuming that \( t \leq 1 \), then the smallest root of \( f(t) - t \) will be bounded above also by (9). We make the assumption \( t \leq 1 \) self-consistent by arranging the previous expression to be smaller than 1. From this we get that by making
\[
A \leq \min \left\{ 1, \frac{8}{C(4\Lambda + \gamma)} \right\},
\]
then we guarantee that the smallest fixed point of \( f \) is less than the expression given by (9), which in turn is less than 1. Since \( \delta \leq 1.25A \), we have that in order to have the constant in the righthand side of (5), we require that
\[
\frac{1}{4} \left( \frac{\delta^2 \Lambda e^\delta}{2} + \sqrt{2}\delta \right) < 1,
\]
where we have used that \( K \leq 1 \). Also, as we shall require that \( \delta \leq 1 \), the previous inequality will hold if
\[
\frac{1}{4} \left( 1.5\delta \Lambda + \sqrt{2}\delta \right) \leq 1,
\]
which in turn will hold if we require that
\[ \frac{1}{4} (1.5\delta\Lambda + 2\delta) \leq 1. \]

From this we obtain that as long as
\[ \delta \leq \min \left\{ 1, \frac{2}{0.75\Lambda + 1} \right\}, \]
the constant in the right-hand side of (8) will be less than 1.

Putting all this together yields that if
\[ \text{Vol}(\Omega) \leq \min \left\{ 1, \frac{8}{4.76\pi^3 (4\Lambda + \gamma)} \right\} \]
and \( \delta \leq \min \left\{ 1, 1.25\text{Vol}(\Omega)^{\frac{1}{2}}, \frac{2}{0.75\Lambda + 1} \right\}, \)
then the Dirichlet problem (8) has a solution.

If we take into account that \( A \leq \text{diam}(\Omega) := d, \) we could have worked out our estimates in terms of \( d. \) We indicate how this can be done. Our first estimate for \( \|\nabla f_{k+1}\|_\infty \) can be replaced by
\[ \|\nabla f_{k+1}\|_\infty \leq \frac{Cd}{2} (n-1) \left[ \Lambda e^{d\|\nabla f_k\|_\infty} + \|\nabla f_k\|_\infty^2 + \gamma \right], \]
and thus we estimate the smallest fixed point of the function
\[ f(t) = \frac{Cd}{4} (e^{dt} + t^2 + \gamma). \]
Hence, reasoning as before, using \( d \) instead of \( A, \) we find that we have that
\[ d \leq \min \left\{ 1, \frac{4}{C (2.5\Lambda + \gamma)} \right\} \]
guarantees that the smallest fixed point of \( f \) is smaller than 1. Estimate (8) in terms of \( d \) reads as
\[ \|\nabla f_{k+1} - \nabla f_k\| \leq \frac{1}{4} \left( \frac{d^2\Lambda e^{Kd}}{2} + \sqrt{2}dK \right) \|\nabla f_k - \nabla f_{k-1}\|, \]
where \( K \) is the smallest fixed point of \( f. \) Thus, if we require \( d \leq 1 \) and the constant in the right-hand side of the previous estimate to be \( < 1, \) it suffices to have
\[ \frac{1}{4} (1.5d\Lambda + \sqrt{2}d) \leq 1, \]
that is, if
\[ d \leq \min \left\{ 1, \frac{4}{C (2.5\Lambda + \gamma)}, \frac{4}{1.5\Lambda + \sqrt{2}} \right\}, \]
then the Dirichlet problem (8) has a solution.

3. Proof of Theorem II part (2)

For the case of balls of radius \( d \) in \( \mathbb{R}^n, \) we can prove that if \( G_r \) is the Green’s function for the Dirichlet Laplacian for the ball \( B^n_r \) of radius \( r \) centered at the origin then, it is related to the Dirichlet Laplacian of the unit ball by the formula
\[ G_r (x, x') = \frac{1}{r^{n-2}} G_1 \left( \frac{x'}{r'}, \frac{x'}{r} \right). \]
From this scaling property, the following estimate follows
\[ \int_{B^n_r} |\nabla G_r (x, x')| \, dx' \leq r \int_{B^n_1} |\nabla G_1 (x, x')| \, dx' \leq Cr, \]
and thus we have the analogue of Lemma 1.

Lemma 3. Let \( h \) be a smooth function, and consider the solution to the Dirichlet problem
\[ \Delta u = h \quad \text{in} \quad B^n_d, \quad u = 0 \quad \text{in} \quad \partial B_d. \]
There is a constant \( C_n > 0, \) which depends only on the dimension \( n, \) such that
\[ \|\nabla u\|_\infty \leq C_n d \|h\|_\infty. \]
The constant \( C_n \) can be taken as
\[ \sup_{x \in B^n_1} \int_{B^n_1} |\nabla G_1 (x, x')| \, dx'. \]
The rest of the proof is the same as the one given in the case of a bounded convex subdomain of \( \mathbb{R}^3 \), and estimating the size of \( d \) follows along the same lines as in estimating the volume in the case of convex domains in \( \mathbb{R}^3 \), but working with \( d \) instead of working with \( A \) and \( \delta \) (and recall that \( \delta = 2d \)).

4. PROOF OF THEOREMS 2 AND 3

As mentioned before, Theorem 2 is a corollary of Theorem 1. Define \( u = f - c \), for any \( c \in (-\infty, \infty) \), and consider the Dirichlet problem

\[
\begin{align*}
\Delta u &= -\frac{1}{2 (n-1)} \left[ \lambda e^{2c} e^{2u} + (n-1) (n-2) |\nabla u|^2 \right] + S(x), \quad \text{in } \Omega \\
u &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

where we may rescale \( \lambda \) in a way that \( |\lambda| = \frac{1}{4} e^{-2c} \). Therefore Theorem 2 follows from Theorem 1 by taking \( \Lambda = 1/4 \).

It remains to prove Theorem 3. Let \( \Omega \) be any bounded subdomain of \( \mathbb{R}^n \), and we denote its Dirichlet Green’s function by \( G_1(x, x') \). Assume that \( 0 \in \Omega \), and define \( \Omega_d \) by

\[
\Omega_d = d \cdot \Omega := \{ d \cdot x \in \mathbb{R}^n : x \in \Omega \}.
\]

Then the Dirichlet Green’s function of \( \Omega_d \) is given by

\[
G_d(x, x') = \frac{1}{d^{n-2}} G_1 \left( \frac{x}{d}, \frac{x'}{d} \right).
\]

Therefore, reasoning as above, the Dirichlet problem

\[
\begin{align*}
\Delta u &= -\frac{1}{2 (n-1)} \left[ \lambda e^{2u} + (n-1) (n-2) |\nabla u|^2 \right], \quad \text{in } \Omega_d \\
u &= 0, \quad \text{on } \partial \Omega_d,
\end{align*}
\]

has a solution for \( d \) and \( |\lambda| \) small enough. This means, that there is a conformal deformation of the Euclidean metric such that \( \Omega_d \) has constant scalar curvature \( \lambda \), and such that \( \partial \Omega_d \) keeps its metric invariant. But then, by reescaling, i.e., considering \( \Omega = \frac{1}{d} \cdot \Omega_d \), we obtain that \( \Omega \) admits a metric of constant scalar curvature \( \lambda/d^2 \) whereas the metric in \( \partial \Omega \) remains as the metric it inherits from \( \mathbb{R}^n \).

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