ON CERTAIN EQUIDIMENSIONAL POLYMATROIDAL IDEALS

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Abstract. The class of equidimensional polymatroidal ideals are studied. In particular, we show that an unmixed polymatroidal ideal is connected in codimension one if and only if it is Cohen-Macaulay. Especially a matroidal ideal is connected in codimension one precisely when it is a squarefree Veronese ideal. As a consequence we indicate that for polymatroidal ideals, the Serre’s condition \((S_n)\) for some \(n \geq 2\) is equivalent to Cohen-Macaulay property. We also give a classification of generalized Cohen-Macaulay polymatroidal ideals.

1. Introduction

Throughout we consider monomial ideals of the polynomial ring \(S = k[x_1, \ldots, x_n]\) over a field \(k\) and \(m = (x_1, \ldots, x_n)\) denotes the unique homogenous maximal ideal. The Cohen-Macaulay polymatroidal ideals are classified by Herzog and Hibi [6], into the principal ideals, the Veronese ideals, and the squarefree Veronese ideals. As mentioned in [6], it is natural and interesting to classify all unmixed polymatroidal ideals. Recall that an ideal \(I\) is called unmixed if all prime ideals in \(\text{Ass}(S/I)\) have the same height. If all minimal prime ideals of \(I\) have the same height, then \(I\) is called equidimensional. Obviously an unmixed ideal is equidimensional and the converse holds precisely when \(\text{Min}(S/I) = \text{Ass}(S/I)\). In particular a squarefree monomial ideal is equidimensional if and only if it is unmixed.

In this paper we study certain classes of equidimensional polymatroidal ideals. After giving some preliminary concepts and results in Section 2, we study the polymatroidal ideals connected in codimension one, in Section 3. Consider the Zarisky topology on \(\text{Spec}(S/I)\) for a monomial ideal \(I\). \(\text{Spec}(S/I)\) is a connected space with this topology. The ideal \(I\) is called connected in codimension one, if \(\text{Spec}(S/I)\) remains connected after removing closed subsets with codimension bigger than one [5]. This property can be expressed in the sense of minimal prime ideals of \(I\), which implies that \(I\) is equidimensional (see Remark 3.2). In combinatorial point...
of view, a squarefree monomial ideal is connected in codimension one, if it is the Stanley-Reisner ideal of a strongly connected simplicial complex.

As mentioned in Remark 3.4, Cohen-Macaulay ideals are connected in codimension one. The aim of this section is to find when the converse holds true for polymatroidal ideals. Theorem 3.6 states that matroidal ideals connected in codimension one, are precisely squarefree Veronese ideals and so they are Cohen-Macaulay. We extend this result to unmixed polymatroidal ideals in Theorem 3.9 as an essential result in this section. The main consequence of this result is Corollary 3.11 which asserts that the Serre's condition \( (S_n) \) for some \( n \geq 2 \), is equivalent to Cohen-Macaulay property for all polymatroidal ideals.

The unmixed polymatroidal ideals have been also studied by Văduvă in [13]. He shows that an ideal of Veronese type is unmixed if and only if it is Cohen-Macaulay. Our second target is to find equidimensional polymatroidal ideals which are not Cohen-Macaulay, in Section 4. We show that a polymatroidal ideal generated in degree 2, is equidimensional if and only if it is generalized Cohen-Macaulay (see Proposition 4.2 and Example 4.9(iii)) is a non-Cohen-Macaulay ideal in this class. An unmixed polymatroidal ideal generated in degree \( d > 2 \), is not necessarily generalized Cohen-Macaulay (see Example 4.3). In the case of matroidal ideals, Theorem 4.5 states that generalized Cohen-Macaulay matroidal ideals generated in degree \( d > 2 \), are precisely Cohen-Macaulay matroidal ideals. The classification of generalized Cohen-Macaulay polymatroidal ideals, stated in Theorem 4.8 indicates that a fully supported monomial ideal \( I = J \cap m^s \) generated in degree \( d \) with \( s \in \{0, d\} \), is a generalized Cohen-Macaulay ideal if and only if one of the following statements holds true:

a) \( J \) is a Cohen-Macaulay polymatroidal ideal i.e. \( J \) is either a principal ideal, a Veronese ideal, or a squarefree Veronese ideal.

b) \( J = p_1^{a_1} \cap \cdots \cap p_r^{a_r} \) is equidimensional and \( p_i + p_j = m \) for all \( i \neq j \).

c) \( J \) is an unmixed matroidal ideal of degree 2.

There are examples illustrating the significance of each of the items in the above characterization and showing that none of them can be removed, see examples 4.9 and 4.10.

2. Preliminaries

Throughout \( S = k[x_1, \ldots, x_n] \) is the polynomial ring over a field \( k \) with the unique homogenous maximal ideal \( m = (x_1, \ldots, x_n) \). For a monomial ideal \( I \) of \( S \), the minimal set of monomial generators of \( I \) is denoted by \( G(I) \) and \( \text{supp}(I) := \{x_i; 1 \leq i \leq n, x_i | u \text{ for some } u \in G(I)\} \). We call the monomial ideal \( I \) is fully supported if \( \text{supp}(I) = \{x_1, \ldots, x_n\} \). An ideal \( I \) is said to be unmixed if all associated
prime ideals of $I$ have the same height and is called **equidimensional** if all minimal prime ideals have the same height.

A monomial ideal $I$ is called a **polymatroidal** ideal, if it is generated in a single degree with the exchange property that for any two elements $u, v \in G(I)$ with $\deg_{x_i}(u) > \deg_{x_i}(v)$, there exists an index $j$ with $\deg_{x_j}(u) < \deg_{x_j}(v)$ such that $x_j(u/x_i) \in G(I)$. It is easy to see that a monomial ideal $I$ is polymatroidal if and only if for all monomials $u, v \in G(I)$ with $\deg_{x_i}(u) > \deg_{x_i}(v)$ for some $i$, there exists an integer $j$ such that $\deg_{x_j}(v) > \deg_{x_j}(u)$ and $x_j(u/x_i) \in I$. A squarefree polymatroidal ideal is called a **matroidal** ideal.

Recall that any polymatroidal ideal $I$ has a linear resolution by [3] Lemma 1.3] and [3] Lemma 4.1]. As a consequence the Castelnuovo-Mumford regularity of $I$ is equal to $d$, where $I$ is generated in degree $d$ and we have the following presentation for $I$ which we will use it frequently in our approach.

**Proposition 2.1.** [11] Proposition 2.1] For a polymatroidal ideal $I \subset S$ with $\Ass(S/I) \setminus \{m\} = \{p_1, \ldots, p_r\}$, there are integers $a_i > 0$ and $s \geq 0$ such that $I = p_1^{a_1} \cap \cdots \cap p_r^{a_r} \cap m^s$. Note that when $s > 0$, $I$ is generated in degree $s$.

The following observation shows that an unmixed polymatroidal ideal generated in degree 2, is not very far from a matroidal ideal.

**Lemma 2.2.** Let $I$ be a fully supported polymatroidal ideal of $S$, generated in degree 2. If $I$ is unmixed, then $I$ is a matroidal ideal or $I = m^2$.

**Proof.** If $|\Ass(S/I)| = 1$, then the result is clear. Otherwise, let $I = p_1^{a_1} \cap \cdots \cap p_r^{a_r}$ be the minimal primary decomposition mentioned in Proposition 2.1. Since $\height(p_j) = \height(p_j)$ for all $i \neq j$, there exist $x_i \in p_i \setminus p_j$ and $x_j \in p_j \setminus p_i$. Therefore $x_i^{a_i', x_j^{a_j'}}|u$ for $a_i' \geq a_i \geq 1$, $a_j' \geq a_j \geq 1$ and some $u \in G(I)$. Now, since $\deg(u) = 2$, we have that $a_i = a_j = 1$ for all $i \neq j$ and so $I$ is a matroidal ideal.

The unmixed condition is necessary in the above lemma. For instance consider the equidimensional polymatroidal ideal $I = (x_1^2, x_1 x_2, x_1 x_3, x_2 x_3) = (x_1, x_2) \cap (x_1, x_3) \cap (x_1, x_2, x_3)^2$. The point in this example is that $I$ contains a pure power of a variable $x_1$ but not any other powers $x_2^2$ or $x_3^2$. The following result shows that it cannot happen if the ideal is unmixed.

**Proposition 2.3.** Let $I$ be an unmixed fully supported polymatroidal ideal of $S$, generated in degree $d$. If $x_j^d \in I$ for some $1 \leq j \leq n$, then $I = m^d$.

**Proof.** Let $I = p_1^{a_1 t} \cap \cdots \cap p_r^{a_r t}$ be the minimal primary decomposition. Since $x_j^d \in I$ for some $j$, we have $a_i = d$ for some $i$. So $I(p_i) = p_i^d$ and it follows that $\deg(I(p_i)) = \deg(I)$. Therefore $p_i = m$ and then the claim follows, since $I$ is an unmixed ideal. □
The unmixed polymatroidal ideals appear in the above condition, powers of the maximal ideal \( \mathfrak{m} \), are called veronese ideals. In other words the (squarefree) Veronese ideal of degree \( d \) in the variables \( x_{i_1}, \ldots, x_{i_r} \) is the ideal of \( S \) which is generated by all (squarefree) monomials in \( x_{i_1}, \ldots, x_{i_r} \) of degree \( d \).

**Theorem 2.4.** [6, Theorem 4.2] A polymatroidal ideal \( I \) is Cohen-Macaulay if and only if \( I \) is a principal ideal, a Veronese ideal or a squarefree Veronese ideal.

As a generalization of Veronese ideals, if the ideal \( I \) is generated by all monomials \( u \) of degree \( d \) such that \( \deg_{x_{i_1}^{a_1}, \ldots, x_{i_r}^{a_r}}(u) \leq a \) for some integers \( a_i \geq 0 \), the ideal \( I \) is denoted by \( I_d; a_1, \ldots, a_n \) and is called an ideal of Veronese type. Ideals of Veronese type are obviously polymatroidal. If \( I \) is an ideal of Veronese type, then \( \text{Min}(S/I) = \text{Ass}(S/I) \) if and only if \( I \) is unmixed if and only if \( I \) is Cohen-Macaulay, see [13, Theorem 3.4].

Let \( p \) be a prime ideal of \( S \). Then \( p = p_{\{i_1, \ldots, i_t}\} \) where \( \{i_1, \ldots, i_t\} = \{u\} \setminus \text{supp}(p) \) and \( IS_p = JS_p \), where \( J \) is the monomial ideal obtained from \( I \) by the substitution \( x_i \mapsto 1 \) for all \( i = i_1, \ldots, i_t \). The ideal \( J \) is called the monomial localization of \( I \) with respect to \( p \) and is denoted by \( I(p) \). The following easy observation is a crucial point in using monomial localization as an effective tool.

**Remark 2.5.** Let \( I = \cap_{i=1}^r Q_i \) be a primary decomposition of a monomial ideal \( I \).

a) \( I(p_{\{i_1, \ldots, i_t\}}) = \cap_{i \in T} Q_i \) where \( T = \{i; 1 \leq i \leq r, Q_i \cap \{1, \ldots, t\} = \emptyset\} \).

b) If \( I \) is unmixed, then \( I \) is principal if and only if \( I(p) \) is principal for some monomial prime ideal \( p \).

c) If \( I \) is generated in single degree \( d \) and \( I(p_{\{i\}}) \) is single degree in \( d_i \), then \( d_i = d - a_i \) where \( a_i = \max\{\deg_{x_{i_j}^{a_j}}(u); u \in G(I)\} \) and \( G(I(p_{\{i\}})) = \{u_{x_{i_j}^{a_i}}; u \in G(I) \text{ and } x_{i_j}^{a_i}|u\} \).

3. POLYMATROIDAL IDEALS CONNECTED IN CODIMENSION ONE

In this section we study the Cohen-Macaulay property of polymatroidal ideals from topological point of view. Let \( I \) be a monomial ideal of \( S \) and consider the Zarisky topology on \( \text{Spec}(S/I) \). Recall that the closed subsets in this topology are the sets \( V(J) = \{q; q \in \text{Spec}(S) \text{ and } J \subseteq q\} \), where \( J \supseteq I \) is an ideal of \( S \). The irreducible components of \( \text{Spec}(S/I) \) are the closed sets \( V(p) \), where \( p \) is a minimal prime ideal of \( I \). \( \text{Spec}(S/I) \) with this topology is a connected space. The ideal \( I \) is called connected in codimension one, if \( \text{Spec}(S/I) \) remains connected after removing closed subsets with codimension bigger than one [5]. Since the codimension of \( V(p) \) is equal to \( \text{ht}(p) - \text{ht}(I) \) for all prime ideals \( p \supseteq I \), we have the following definition by [5 Proposition 1.1].
Definition 3.1. A monomial ideal \( I \subset S \) with height \( h \), is connected in codimension one, if for any pair of distinct prime ideals \( p, q \in \text{Min}(S/I) \) there exists a sequence of minimal prime ideals \( p = p_1, \ldots, p_r = q \) such that \( |G(p_i + p_{i+1})| = h + 1 \), for all \( 1 \leq i \leq r - 1 \).

Remark 3.2. By the above definition it is clear that a monomial ideal connected in codimension one, is equidimensional and so \( |G(p_i) \cap G(p_{i+1})| = h - 1 \), for all \( 1 \leq i \leq r - 1 \). Since for a squarefree monomial ideal \( I \), all associated prime ideals are minimal, being equidimensional is equivalent to being unmixed. So that if a squarefree monomial ideal \( I \) is connected in codimension one, then \( I \) is unmixed.

Remark 3.3. In the context of Hartshorne [5], an ideal \( I \subset S \) is called locally connected in codimension one if all localizations \( I_p \) is connected in codimension one where \( p \in V(I) \). Since for a monomial ideal \( I \) we have \( I_m = I \), if \( I \) is locally connected in codimension one then \( I \) is connected in codimension one.

In combinatorial point of view, a pure simplicial complex \( \Delta \) is said to be strongly connected or connected in codimension one, if for any two facets \( F \) and \( G \), there is a sequence of facets \( F = F_1, F_2, \ldots, F_r = G \) such that \( \dim (F_i \cap F_{i+1}) = \dim \Delta - 1 \) or equivalently \( \dim (F_i \cup F_{i+1}) = \dim \Delta + 1 \), for each \( 1 \leq i \leq r - 1 \). A squarefree monomial ideal is connected in codimension one, if it is the Stanley-Reisner ideal of a strongly connected simplicial complex.

Remark 3.4. Let \( I \) be a Cohen-Macaulay monomial ideal. Then \( I \) is connected in codimension one, by [6, Corollary 2.4] and Remark 5.5. Another way to see this fact is observing that \( \sqrt{I} \) is also Cohen-Macaulay by [10, Theorem 2.6]. So according to [7, Lemma 9.1.12], \( I \) is connected in codimension one, since \( \text{Min}(S/I) = \text{Min}(S/\sqrt{I}) \).

Obviously an unmixed principal ideal is connected in codimension one. As an easy way to construct a monomial ideal connected in codimension one, we may consider \( I \) as the intersection of all prime ideals generated by \( h = \text{ht}(I) \) variables. It is indeed the squarefree Veronese ideal generated in degree \( d = n - h + 1 \) [2, Theorem 3.4]. From another point of view, \( I \) is Cohen-Macaulay by Theorem 2.4 and hence remark \( I \) is connected in codimension one by the above.

In Theorem 3.6, we show that all matroidal ideals connected in codimension one, are precisely the squarefree Veronese ideals. As a key point of our proof, we need the following simple characterization which in the case that \( t = 2 \), it is also proved by a different method in [2, Lemma 2.3].
Lemma 3.5. Let I be a matroidal ideal and $T = \{x_1, \ldots, x_t\} \subseteq \text{supp}(I)$. If for all $t - 1$ elements $x_{j_1}, \ldots, x_{j_{t-1}}$ of $T$, $x_{j_1} \cdots x_{j_{t-1}}|u$ for some $u \in G(I)$. Then the following statements are equivalent.

a) $x_1 \cdots x_t \nmid u$ for all $u \in G(I)$.

b) $I(p_{\{1, \ldots, t\}}) = I(p_{\{j_1, \ldots, j_{t-1}\}})$ for all $\{j_1, \ldots, j_{t-1}\} \subseteq T$.

c) $|p \cap \{x_1, \ldots, x_t\}| \neq 1$ for all $p \in \text{Ass}(S/I)$.

Proof. (a) $\Rightarrow$ (b): By [8, Corollary 3.2] any monomial localization of $I$ is again matroidal and so it is a single degree. Since $x_{j_1} \cdots x_{j_{t-1}}|u$ for some $u \in G(I)$, we have $I_j = I(p_{\{j_1, \ldots, j_{t-1}\}})$ is generated in degree $d - t + 1$ where $d$ is the degree of generators of $I$. Indeed

$$G(I_j) = \{\frac{u}{x_{j_1} \cdots x_{j_{t-1}}}; u \in G(I) \text{ and } x_{j_1} \cdots x_{j_{t-1}}|u\}.$$

On the other hand $x_1 \cdots x_t \nmid u$ for all $u \in G(I)$, therefore $x \notin \text{supp}(I_j)$ for $x \in T \setminus \{x_1, \ldots, x_{j_{t-1}}\}$, and it follows (b).

(b) $\Rightarrow$ (c): Assume that $x_i \in p$ for some $p \in \text{Ass}(S/I)$ and $1 \leq i \leq t$. Therefore $p \notin \text{Ass}(S/I(p_{\{1, \ldots, t\}})) = \text{Ass}(S/I(p_{\{1, \ldots, t-1, t+1, \ldots, t\}}))$, that is $x_j \in p$ for some $1 \leq j \neq i \leq t$.

c) $\Rightarrow$ (a): Assume contrary that $x_1 \cdots x_t|u$ for some $u \in G(I)$. Then $x_t \in \text{supp}(I(p_{\{1, \ldots, t-1\}}))$ and so there exists a prime ideal $p \in \text{Ass}(S/I(p_{\{1, \ldots, t-1\}}))$ such that $x_t \in p$. Now (c) implies that $x_i \in p$ for some $1 \leq i \leq t - 1$ which is contradiction. $\square$

Now we are able to classify all matroidal ideals connected in codimension one.

Theorem 3.6. Let I be a monomial ideal. Then I is matroidal ideal connected in codimension one if and only if I is a squarefree Veronese ideal.

Proof. If I is squarefree veronese ideal, Then I is connected in codimension one by the explanation after Remark 3.4. Assume that I is a matroidal ideal generated in degree $d$ and is connected in codimension one. We use induction on $i, 1 \leq i \leq d$ to show that for any set $\{x_1, \ldots, x_i\} \subseteq \text{supp}(I)$, there exists $u \in G(I)$ such that $x_1 \cdots x_{i-1}x_i|u$.

Our claim is trivial for $i = 1$. Assume that it’s true for $i = t - 1$ and assume contrary that $t \leq d$ and $\{x_1, \ldots, x_t\} \subseteq \text{supp}(I)$ and $x_1 \cdots x_t \nmid u$ for all $u \in G(I)$. By induction assumption, for any subset $\{x_1, \ldots, x_{i-1}\}$ of $t - 1$ elements of $\{x_1, \ldots, x_t\}$, $x_1 \cdots x_{i-1}|u$ for some $u \in G(I)$. Note that by Lemma 3.5 $I(p_{\{1, \ldots, t\}}) = I(p_{\{1, \ldots, t-1\}})$ and $I(p_{\{1, \ldots, t-1\}}) \neq S$, since $t - 1 < d$ and $x_1 \cdots x_{t-1}|u$ for some $u \in G(I)$. Hence there exists $q \in \text{Ass}(S/I)$ such that $\{x_1, \ldots, x_t\} \cap q = \emptyset$. Let $p \in \text{Ass}(S/I)$ with $x_1 \in p$. Since I is connected in codimension one by Remark 3.2, there exits a chain $p = p_1, \ldots, p_r = q$ of associated prime ideals of I such that
Example 3.7. The ideal $I = (x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2x_3, x_1x_2^2) = (x_1) \cap (x_1, x_2)^2 \cap (x_1, x_2, x_3)^3$ is polymatroidal which is clearly connected in codimension one, but it is not unmixed.

In our main result Theorem 3.9, we show that a connected in codimension one polymatroidal ideal is unmixed if and only if it is Cohen-Macaulay. We will use the following easy lemma, in our proof.

Lemma 3.8. Let $I \subset k[x_1, \ldots, x_n]$ be an unmixed fully supported polymatroidal ideal with $\text{ht} (I) > 1$. If $I$ is not squarefree, then $\text{ht} (I) \neq n - 1$.

Proof. Assume contrary that $\text{ht} (I) = n - 1$. Then $S/I$ is not Cohen-Macaulay by Theorem 2.4 and $\text{dim} (S/I) = 1$. Therefore $\text{depth} (S/I) = 0$ and so $m \in \text{Ass}(S/I)$ which contradicts $\text{ht} (I) = n - 1$ and assumption that $I$ is unmixed. \hfill \Box

Now, we present the main result of this section that

Theorem 3.9. Let $I$ be an unmixed polymatroidal ideal. Then $I$ is connected in codimension one if and only if $I$ is Cohen-Macaulay.

Proof. If $I$ is Cohen-Macaulay, then $I$ is connected in codimension one by Remark 3.4. Now let $I = \mathfrak{p}_1^{a_1} \cap \cdots \cap \mathfrak{p}_r^{a_r}$ is connected in codimension one. We may assume that $I$ is fully supported with $\text{ht} (I) > 1$ and it is not squarefree, by Theorem 2.4 and Theorem 5.6. Therefore to prove that $I$ is Cohen-Macaulay, according to Theorem 2.4, we must show that $r = 1$. We use induction on $d$, which is the common degree of monomial generators of $I$. For $d = 2$, the result follows by Lemma 2.8. Let $d > 2$ and $a_i > 1$ for some $1 \leq i \leq r$ and assume contrary that $r > 1$. Since $I$ is connected in codimension one by Remark 3.2, there exist $1 \leq j \neq i \leq r$ such that $|G(p_i) \cap G(p_j)| = \text{ht} (I) - 1$. Note that $\text{ht} (I) \neq n - 1$ by Lemma 5.6 and so $\mathfrak{p} := \mathfrak{p}_i + \mathfrak{p}_j \neq \mathfrak{m}$. On the other hand $\text{supp} (I(\mathfrak{p})) = G(p_i) \cup \{x\}$ for some variable $x \in p_j \setminus p_i$. Now let $q \in \text{Ass}(S/I(\mathfrak{p}))$ and $q \neq \mathfrak{p}$. Then $G(q) \subseteq G(p_i) \cup \{x\}$. Since $\text{ht} (p_i) = \text{ht} (q)$, then $q = (G(p_i) \setminus \{y_q\}, x)$ for some variable $y_q$. Hence $I(\mathfrak{p})$ is a polymatroidal ideal connected in codimension one which is generated in degree less than $d$. Now, induction assumption implies that $|\text{Ass}(S/I(\mathfrak{p}))| = 1$ which is a contradiction. \hfill \Box
Corollary 3.10. Let $I \subset S$ be an unmixed fully supported polymatroidal ideal and connected in codimension one. Then $\text{supp}(I(p_{\{i\}}))$ is either an empty set or is equal to $\{x_1, \ldots, x_n\} \setminus \{x_i\}$ for each $i = 1, \ldots, n$.

Proof. If $I$ is a squarefree Veronese ideal in variables $x_1, \ldots, x_n$, then for all $1 \leq i, j \leq n$, $x_ix_j|u$ for some $u \in G(I)$. Hence the result is clear by Theorem 3.9 and Theorem 2.4.

Corollary 3.11. Let $I$ be a polymatroidal ideal. Then $I$ satisfies the Serre’s condition $(S_n)$ for some $n \geq 2$ if and only if $I$ is Cohen-Macaulay.

Proof. Assume that $I$ satisfies the Serre’s condition $(S_n)$ for some $n \geq 2$. Then $I$ is connected in codimension one by [5, Corollary 2.4] and Remark 3.3. On the other hand $I$ is unmixed since it is $(S_1)$. Now the result follows by Theorem 3.9.

4. Generalized Cohen-Macaulay polymatroidal ideals

A finitely generated module $M$ over a local ring $(R, \mathfrak{n})$ is called generalized Cohen-Macaulay, whenever each local cohomology module $\text{H}^i_{\mathfrak{n}}(M)$ has finite length for all $i < \dim M$. It is known that if $M$ is generalized Cohen-Macaulay, then $M_{\mathfrak{p}}$ is Cohen-Macaulay for all prime ideals $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{n}\}$ and the converse holds if $R$ is universally catenary and all it’s formal fibres are Cohen-Macaulay [1, Exercises 9.5.7 and 9.6.8]. In the following we consider graded generalized Cohen-Macaulay modules over the $\ast$local graded polynomial ring $(S, \mathfrak{m})$. We call an ideal $I$ is generalized Cohen-Macaulay whenever the $i$th cohomology module $\text{H}^i_{\mathfrak{m}}(S/I)$ is of finite length for all $i < \dim S/I$.

Lemma 4.1. The following statements are equivalent for a monomial ideal $I$.

a) $I$ is generalized Cohen-Macaulay.

b) $I$ is equidimensional and $I(\mathfrak{p})$ is Cohen-Macaulay for all monomial prime ideals $\mathfrak{p} \neq \mathfrak{m}$

Proof. Note that homogeneous prime ideals of $S$ in multigraded structure are precisely monomial ideals. So that all minimal elements of the non-Macaulay locus of $S/I$ are monomial, by [12, Corollary 3.7]. Now the result follows by [11, Exercises 9.5.7].

Proposition 4.2. Let $I$ be a polymatroidal ideal generated in degree 2. Then the following statements are equivalent:

a) $I$ is equidimensional.

b) $I$ is generalized Cohen-Macaulay.
Proposition 4.4. Let $I \subset k[x_1, \ldots, x_n]$ be a fully supported matroidal ideal generated in degree $d > 2$. If $I$ is generalized Cohen-Macaulay, then $\text{supp}(I(p_{\{i\}})) = \{x_1, \ldots, x_n\} \setminus \{x_i\}$ for each $i = 1, \ldots, n$.

Proof. If $I$ is principal, then there is nothing to prove. Now let $I$ is not principal and for convenience we show that $\text{supp}(I(p_{\{1\}})) = \{x_2, \ldots, x_n\}$. Let $I(p_{\{1\}})$ be fully supported in $K[x_{t+1}, \ldots, x_n]$ for some $t \geq 1$. It is enough to show that $t = 1$. Since $I(p_{\{1\}})$ is a squarefree veronese ideal in variables $x_{t+1}, \ldots, x_n$ of degree $d-1$, where $d$ is the common degree of generators of $I$, it follows that

$$h = \text{ht}(I) = (n-t)-(d-1)+1 = n-t-d+2.$$  

Since $d > 2$, we have that for each $j = t+2, \ldots, n$, there exists $u_j \in G(I)$ such that $x_{t+1}x_j|u_j$ and so

$$\{x_{t+2}, \ldots, x_n\} \subseteq \text{supp}(I(p_{\{t+1\}})).$$

On the other hand since $\text{supp}(I(p_{\{1\}})) = \{x_{t+1}, \ldots, x_n\}$, it follows that $x_1x_j | u$ for each $j = 2, \ldots, t$ and any $u \in G(I)$. So Lemma 3.5 implies that $I(p_{\{1\}}) = I(p_{\{j\}})$ for $j = 2, \ldots, t$. Now again since $\text{supp}(I(p_{\{j\}})) = \{x_{t+1}, \ldots, n\}$ for $j = 1, \ldots, t$, we have that for each $j = 1, \ldots, t$, there exists $u_j \in G(I)$ such that $x_jx_{t+1}|u_j$. So that

$$\{x_1, \ldots, x_t\} \subseteq \text{supp}(I(p_{\{t+1\}})).$$

Hence by (2) and (3), we have that $\text{supp}(I(p_{\{t+1\}})) = \{x_1, \ldots, x_n\} \setminus \{x_{t+1}\}$. Therefore since $I(p_{\{t+1\}})$ is a squarefree veronese ideal, it follows that $h = (n-1)-(d-1)+1 = n-d+1$. Hence from (1), $n-t-d+2 = n-d+1$. So $t = 1$. \qedsymbol
Theorem 4.5. Let $I$ be a matroidal ideal generated in degree $d > 2$. Then $I$ is generalized Cohen-Macaulay if and only if $I$ is Cohen-Macaulay.

Proof. By Proposition 4.4, $I(p_i)$ is a squarefree Veronese ideal in the variables $\{x_1, \ldots, x_n\} \setminus \{x_i\}$, for all $1 \leq i \leq n$. Now, since $I = \sum_{i=1}^{n} x_i I(p_i)$, the result is clear.

The following lemma will be used in the classification of generalized Cohen-Macaulay polymatroidal ideals in Theorem 4.8.

Lemma 4.6. Let $I = J \cap m^d$ be a polymatroidal ideal generated in degree $d$ where $J$ is a squarefree monomial ideal. If $\deg(u) > 1$ for all $u \in G(J)$, then $J$ is a matroidal ideal.

Proof. Let $u, v \in G(J)$ such that $x_i | u$ and $x_i \nmid v$. Then $x_i | v$ and $x_i \nmid u$ for some $l \neq i$. By assumption there exists $h \neq i$ such that $x_h | u$. Now $u' = x_h^{d-s} u$ and $v' = x_i^{d-r} v$ belong to $G(I)$ where, $r = \deg(v)$ and $s = \deg(u)$. Since $\deg_{x_i}(u') > \deg_{x_i}(v')$ and $I$ is polymatroidal ideal, there exists $1 \leq j \neq i \leq n$ such that $\deg_{x_j}(u') < \deg_{x_j}(v')$ and $x_j u'/x_i \in G(I)$. Hence $x_j u'/x_i \in J$. Note that $J$ is squarefree, $x_h | u$ and $h \neq i$, so that $x_j u'/x_i \in J$ and also $\deg_{x_j}(u) < \deg_{x_j}(v)$.

Lemma 4.7. Let $I = J \cap m^d$ be a monomial ideal generated in degree $d$ where $J$ is a monomial ideal generated in degree $t \leq d$. Then $I = J m^{d-t}$.

Proof. It is clear that $J m^{d-t} \subseteq I$. Now let $u \in G(I)$, so there exists $v \in G(J)$ such that $v | u$. So since $\deg(v) = t \leq d = \deg(u)$, there exists a monomial $w$ of degree $d-t$ such that $u = vw$. Hence $u \in J m^{d-t}$.

Theorem 4.8. Let $I = J \cap m^s$ be a fully supported monomial ideal in $S = K[x_1, \ldots, x_n]$ and generated in degree $d$, where $s \in \{0, d\}$. Then $I$ is polymatroidal generalized Cohen-Macaulay ideal if and only if one of the following statements holds:

a) $J$ is a polymatroidal Cohen-Macaulay ideal i.e. $J$ is either a principal ideal, a Veronese ideal, or a squarefree Veronese ideal.

b) $J = p_i^{a_i} \cap \cdots \cap p_j^{a_j}$ is equidimensional and $p_i + p_j = m$ for all $i \neq j$.

c) $J$ is an unmixed matroidal ideal of degree 2.

Proof. By the Lemma 4.7, each of statements (a) or (c) implies that $I$ is polymatroidal. Since $I$ is generated in a single degree, the statement (b) follows that $I$ is polymatroidal by [4, Theorem 3.1].

Whenever (a) holds, $I$ is equidimensional, since $J$ is unmixed. On the other hand for all monomial prime $p \neq m$, $I(p) = J(p)$ is Cohen-Macaulay.
Let (b) holds and \( q \in V(I) \setminus \{ m \} \) be a monomial prime ideal. Since \( p_i + p_j = m \)
for all \( i \neq j \) and \( q \neq m \), we get \( I(q) = p_k^{a_k} \) for some \( k \), \( 1 \leq k \leq r \) or \( I(q) = S \).

Let (c) holds. By Proposition 4.2, \( J \) is generalized Cohen-Macaulay. So that for all monomial prime \( p \neq m \), \( I(p) = J(p) \) is Cohen-Macaulay.

Conversely, assume that \( I \) is a polymatroidal generalized Cohen-Macaulay ideal, (a) and (b) don’t hold. Note that \( J := p_1^{a_1} \cap \cdots \cap p_r^{a_r} \) is an unmixed ideal. Since (b) doesn’t hold, let for convenience \( q = p_1 + p_2 \neq m \). Then \( I(q) = p_1^{a_1} \cap p_2^{a_2} \cap \cdots \cap p_r^{a_r} \) is Cohen-Macaulay for some \( 2 \leq t \leq r \). So that by Theorem 4.6 and Remark 4.5, we have that \( I(q) \) is squarefree veronese ideal. Therefore \( a_1 = \cdots = a_t = 1 \) and so \( I = p_1 \cap \cdots \cap p_t \cap p_{t+1}^{a_{t+1}} \cap \cdots \cap p_r^{a_r} \cap m^s \).

We claim that \( a_{i+1} = \cdots = a_r = 1 \). Otherwise, there exists \( t + 1 \leq i \leq r \) such that \( a_i \neq 1 \). Since \( p_1 \nsubseteq p_i \), there exists a variable \( x_l \in p_1 \setminus p_i \). Note that \( x_l \notin \bigcap_{j=1}^t \text{supp}(p_j) \), since \( I(q) \) is generated in a single degree and is not a prime ideal. Let \( x_l \notin p_j \) for \( 1 \leq j \leq t \). Then \( I(p_{\{l\}}) = p_j \cap p_i^{a_i} \cap q' \) which is not Cohen-Macaulay. This contradiction implies our claim that \( I = J \cap m^s \) where \( J \) is a squarefree monomial ideal. Since \( I(q) \) is squarefree veronese ideal of height greater than one, \( J \) does not contain any variables since \( J(q) = I(q) \). Now, the result follows by Lemma 4.6 and Theorem 4.5 since \( J \) is not Cohen-Macaulay.

The following examples show that in the above characterization, none of items (a), (b) or (c) can be removed.

**Example 4.9.** (i) The ideal \( I = (x_1x_2^2, x_1^2x_2^2) = (x_1) \cap (x_2^2) \cap (x_1, x_2)^3 \) is polymatroidal which satisfies (a) and (b), but (c) doesn’t hold for it.

(ii) The ideal \( I = (x_1^2x_2, x_1x_2^2, x_1x_2x_3) = (x_1) \cap (x_2) \cap (x_1, x_2, x_3)^3 \) is polymatroidal which satisfies (a) and (c), but (b) doesn’t hold for it.

(iii) The ideal \( I = (x_1, x_2, x_3, x_4) \cap (x_3, x_4, x_5, x_6) \cap (x_1, x_2, x_3, x_6) \) constructed in [3], is matroidal ideal which satisfies (b) and (c), but (a) doesn’t hold for it.

**Example 4.10.** The ideal \( I = (x_1, x_2) \cap (x_2, x_3)^2 \cap (x_1, x_2, x_3)^3 \) is polymatroidal by [1] Theorem 3.1 and generalized Cohen-Macaulay by Theorem 4.5 satisfying condition (b), but \( J = (x_1, x_2) \cap (x_2, x_3)^2 \) is not even single degree.

Note that the above example is connected in codimension one. There exist polymatroidal ideals connected in codimension one, which are not generalized Cohen-Macaulay, see Example 4.7. In this example the localization \( I(p_{\{3\}}) = (x_1) \cap (x_1, x_2)^2 \) is not Cohen-Macaulay.

Polymatroidal ideals which satisfy condition (c) of Theorem 4.8 can be specified by the following lemma.

**Lemma 4.11.** Let \( I \) be a fully supported monomial ideal of degree 2. Then \( I \) is polymatroidal if and only if \( p_i + p_j = m \) for \( i \neq j \) and all \( p_i \in \text{Ass}(S/I) \).
Proof. Let $p_i + p_j = m$ for $i \neq j$ and all $p_i \in \text{Ass}(S/I)$. Since $I$ is a single degree, it follows by [4, Theorem 3.1] that $I$ is polymatroidal. Conversely, Let $I = p_1^{a_1} \cap p_2^{a_2} \cap \cdots \cap p_t^{a_t}$ be polymatroidal and $q = p_i + p_j \neq m$ for some $i \neq j$. Then $I(q) = p_i^{a_i} \cap p_j^{a_j} \cap q'$ for some monomial ideal $q'$. Since $I$ is generated in degree 2 it follows that the ideal $I(q)$ is a monomial prime ideal or is equal to $S$, which is a contradiction. □

By the above lemma, in the case (c) of Theorem 4.8, for any pair of distinct prime ideals $p, q \in \text{Ass}(S/J)$ we have $G(p + q) = \text{supp}(J)$ and $\text{supp}(J)$ is not necessarily equal to the set of all variables. But in the case (b), the same condition holds with the distinctive point that $J$ is fully supported in $\text{supp}(I)$, see Example 4.9(ii).

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