The sharp higher-order Lorentz–Poincaré and Lorentz–Sobolev inequalities in the hyperbolic spaces

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Abstract
In this paper, we study the sharp Poincaré inequality and the Sobolev inequalities in the higher-order Lorentz–Sobolev spaces in the hyperbolic spaces. These results generalize the ones obtained in Nguyen VH (J Math Anal Appl, 490(1):124197, 2020) to the higher-order derivatives and seem to be new in the context of the Lorentz–Sobolev spaces defined in the hyperbolic spaces.

Keywords Poincaré inequality · Poincaré–Sobolev inequality · Lorentz–Sobolev space · Hyperbolic space

Mathematics Subject Classification 26D10 · 46E35 · 46E30

1 Introduction
For \( n \geq 2 \), let us denote by \( \mathbb{H}^n \) the hyperbolic space of dimension \( n \), i.e., a complete, simply connected, \( n \)-dimensional Riemannian manifold having constant sectional curvature \( -1 \). The aim in this paper is to generalize the main results obtained by the author in [21] to the higher-order Lorentz–Sobolev spaces in \( \mathbb{H}^n \). Before stating our results, let us fix some notation. Let \( V_g, \nabla_g \) and \( \Delta_g \) denote the volume element, the hyperbolic gradient and the Laplace–Beltrami operator in \( \mathbb{H}^n \) with respect to the metric \( g \), respectively. For higher-order derivatives, we shall adopt the following convention

\[
\nabla^m_g = \begin{cases} 
\Delta_g^{m/2} & \text{if } m \text{ is even}, \\
\nabla_g \left( \Delta_g^{m-1/2} \right) & \text{if } m \text{ is odd}.
\end{cases}
\]

Furthermore, for simplicity, we write \(|\nabla^m_g \cdot | \) instead of \(|\nabla^m_g \cdot |_g \) when \( m \) is odd if no confusion occurs. For \( 1 \leq p, q < \infty \), we denote by \( L^{p,q}(\mathbb{H}^n) \) the Lorentz space in \( \mathbb{H}^n \) and by \( \| \cdot \|_{p,q} \) the Lorentz quasi-norm in \( L^{p,q}(\mathbb{H}^n) \). When \( p = q, \| \cdot \|_{p,p} \) is replaced by \( \| \cdot \|_p \) the...
Lebesgue $L_p$–norm in $\mathbb{H}^n$, i.e., $\|f\|_p = (\int_{\mathbb{H}^n} |f|^p dV_g)^{1/p}$ for a measurable function $f$ on $\mathbb{H}^n$. The Lorentz–Sobolev space $W^mL^{p,q}(\mathbb{H}^n)$ is defined as the completion of $C_0^\infty(\mathbb{H}^n)$ under the Lorentz quasi-norm $\|\nabla^m u\|_{p,q} := \|\nabla^m u\|_{p,q}$. In [21], the author proved the following Poincaré inequality in $W^1L^{p,q}(\mathbb{H}^n)$

$$\|\nabla^m u\|_{p,q}^q \geq \left(\frac{n-1}{p}\right)^q \|u\|_{p,q}^q, \quad \forall u \in W^1L^{p,q}(\mathbb{H}^n).$$

(1.1)

provided $1 < q \leq p$. Furthermore, the constant $\left(\frac{n-1}{p}\right)^q$ in (1.1) is the best possible and is never attained. The inequality (1.1) generalizes the result in [16] to the setting of Lorentz–Sobolev space. The first main result in this paper extends the inequality (1.1) to the higher-order Sobolev space $W^mL^{p,q}(\mathbb{H}^n)$.

**Theorem 1.1** Given $n \geq 2$, $m \geq 1$ and $1 < p < \infty$, let us denote the following constant

$$C(n, m, p) = \begin{cases} (\frac{(n-1)^2}{pp'})^{\frac{n}{2}} & \text{if } m \text{ is even,} \\ \frac{n-1}{p} \left(\frac{(n-1)^2}{pp'}\right)^{\frac{m-1}{2}} & \text{if } m \text{ is odd,} \end{cases}$$

where $p' = \frac{p}{p-1}$. Then, the following Poincaré inequality holds in $W^mL^{p,q}(\mathbb{H}^n)$

$$\|\nabla^m u\|_{p,q}^q \geq C(n, m, p)^q \|u\|_{p,q}^q, \quad u \in W^mL^{p,q}(\mathbb{H}^n)$$

(1.2)

for any $1 < p, q < \infty$ if $m$ is even, or for any $1 < q \leq p < \infty$ if $m$ is odd. Moreover, the constant $C(n, m, p)$ in (1.2) is sharp and is never attained.

Let us give some comments on Theorem 1.1. The Poincaré inequality in the hyperbolic space was proved by Tataru [24]

$$\int_{\mathbb{H}^n} |\nabla^m u|^p dV_g \geq C \int_{\mathbb{H}^n} |u|^p dV_g, \quad u \in C_0^\infty(\mathbb{H}^n)$$

(1.3)

for some constant $C > 0$. The sharp value of constant $C$ in (1.3) is computed by Mancini and Sandeep [14] when $p = 2$ and by Ngo and the author [16] for arbitrary $p$ (see [6] for another proof when $m = 1$). Theorem 1.1 gives an extension of the Poincaré inequality (1.3) with the sharp constant to the higher-order Sobolev spaces $W^mL^{p,q}(\mathbb{H}^n)$. Similar to the case $m = 1$ established in [21], we need an extra condition $q \leq p$ when $m$ is odd to apply the symmetrization argument. The Proof of Theorem 1.1 follows the idea in the proof of Theorem 1.1 in [16] by using the iterate argument. The main step in the proof is to establish the inequality when $m = 2$. The case $m = 1$ was already done in [21].

There have been many improvements of (1.3) with the sharp constant in literature. For examples, the interesting readers may consult the papers [4, 6, 7, 14, 18] for the improvements of (1.3) for $m = 1$ by adding the remainder terms concerning to Hardy weights or to the $L^q$–norms with $p < q \leq \frac{np}{n-p}$. For the higher-order Sobolev spaces, we refer the readers to the papers of Lu and Yang [10, 12, 13, 19]. Especially, in [19, Theorem 1.1] the author established the following improvement of (1.3) for $p = 2$.
\[
\int_{\mathbb{R}^n} |\Delta_g u|^2 dV_g - \frac{(n - 1)^4}{16} \int_{\mathbb{R}^n} |u|^2 dV_g \geq S_{n,2} \left( \int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} dV_g \right)^{\frac{n-4}{n}}, \quad u \in C_0^\infty (\mathbb{H}^n)
\]

provided \( n \geq 5 \) where \( S_{n,k} \) denotes the sharp constant in the Sobolev inequality in Euclidean space \( \mathbb{R}^n \)

\[
\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq S_{n,k} \left( \int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2k}{n}}, \quad u \in C_0^\infty (\mathbb{R}^n)
\]

when \( n > 2k \). The constant \( S_{n,1} \) was found out independently by Talenti [22] and Aubin [2]. The sharp constant \( S_{n,k}, k \geq 2 \) was computed explicitly by Lieb [11] by proving the sharp Hardy–Littlewood–Sobolev inequality which is the dual version of (1.5). In [21] the author proved the following inequality: given \( n \geq 4 \) and \( \frac{2n}{n-1} \leq q \leq p < n \), then for any \( q \leq l \leq \frac{mq}{n-p} \) we have

\[
\| \nabla_g u \|_{p,q}^q - \left( \frac{n-1}{p} \right)^q \| u \|_{p,q}^q \geq S_{n,p,q,l} \| u \|_{p,q}^q, \quad \forall u \in W^1 L^{p,q}(\mathbb{H}^n),
\]

where \( p^* = \frac{np}{n-p} \) and

\[
S_{n,p,q,l} = \begin{cases} 
\left[ n^{1-\frac{q}{np}} \left( \frac{(n-p)(l-q)}{qp} \right)^{\frac{q}{np}} \right]^{\frac{1}{q}} S \left( \frac{lq}{l-p}, q \right) & \text{if } q < l \leq \frac{mq}{n-p}, \\
\frac{1}{2} \left( \frac{n}{p} \right)^{\frac{1}{p}} \sigma_n^{\frac{1}{q}} & \text{if } l = q,
\end{cases}
\]

where \( \sigma_n \) denotes the volume of unit ball in \( \mathbb{R}^n \) and \( S \left( \frac{lq}{l-p}, q \right) \) is the sharp constant in the Sobolev inequality with fractional dimension (see [17]). It is interesting that the constant \( S_{n,p,q,l} \) in (1.6) is sharp and coincides with the sharp constant in the Lorentz–Sobolev type inequality in Euclidean space \( \mathbb{R}^n \),

\[
\| \nabla u \|_{p,q}^q \geq S_{n,p,q,l} \| u \|_{p,q}^q.
\]

The previous inequality was proved by Alvino [1] for \( l = q \leq n \) and by Cassani, Ruf and Tarsi [8] for \( l = q \geq p \). Our next aim is to improve the inequality (1.2) in the spirit of (1.4) and (1.6).

**Theorem 1.2** Let \( n > m \geq 1 \) be integers, \( 1 < p < \frac{n}{m} \) and \( \frac{2n}{n-1} \leq q < \infty \). Suppose, in addition, that \( q \leq p \) if \( m \) is odd. Then, there holds

\[
\| \nabla^m u \|_{p,q}^q - C(n,m,p)^q \| u \|_{p,q}^q \geq S(n,m,p)^q \| u \|_{p,m,q}^q, \quad u \in W^m L^{p,q}(\mathbb{H}^n),
\]

where \( p_i^* = \frac{np_i}{n-2i} \), \( i = 0, 1, 2, \ldots \), and

\[
S(n,m,p) = \begin{cases} 
\sigma_n \prod_{i=0}^{k-1} \frac{n(n-2i)p_{n-i}}{p_{n-i}} & \text{if } m = 2k, k \geq 1, \\
\frac{1}{2} \sigma_n \frac{n-p}{p} \prod_{i=1}^{k} \frac{n(n-2i-1)}{p_{n-i}^*} & \text{if } m = 2k+1, k \geq 1.
\end{cases}
\]

Furthermore, the constant \( S(n,m,p) \) in (1.7) is sharp.
Comparing with the case \( n \) even, we have a restriction \( q \leq p \) when \( m \) is odd. This restriction comes from the fact that in the proof of the case \( m \) odd, we used a result in [21, Theorem 2.2] in which the condition \( q \leq p \) is crucial to apply the Pólya–Szegő principle in the hyperbolic space (see, [21, Theorem 2.2]). Note that the condition \( q \leq p \) also appears in the Euclidean case (see, [1] for \( m = 1 \)). In the recent paper [8], Cassani, Ruf and Tarsi have extended this condition to \( q \leq p - \frac{1}{p} \).

We conclude this introduction by some comments on the Proof of Theorem 1.2. In order to prove the inequality (1.7), we first prove a special case of (1.7) with \( m = 2 \) (see (2.14) below) and a second-order Sobolev inequality under Lorentz–Sobolev norm in \( \mathbb{H}^n \) (see (2.12) below). Then, by using the inequality (2.14), the same inequality for the first-order derivative proved by the author in [21] and an iterative argument based on the inequality (2.12) we obtain the inequality (1.7). The sharpness of \( S(n, m, p) \) is proved by constructing a sequence of test functions. The main idea is that the Lebesgue measure and the hyperbolic measure are asymptotically proportional near the origin, and that the truncation of the function \( |x|^{-\frac{n}{p}} \) gives the best constant in the Rellich inequality (see [9]). So, our test function is built by truncating the function \( |x|^{-\frac{n}{p}} \) and rescaling the obtained functions such that their support is contained in centered balls with arbitrarily small radius. Using the asymptotically proportion of the Lebesgue measure and the hyperbolic measure near the origin, we can use the Euclidean rearrangement argument instead of the hyperbolic rearrangement argument. This enables us to work only with Euclidean structure which is much easier to handle. The detailed proof is given in Sect. 4 below.

The rest of this paper is devoted to prove Theorems 1.1 and 1.2 and is organized as follows. In Sect. 2, we recall some facts on the hyperbolic spaces and the non-increasing spherically symmetric rearrangement in the hyperbolic spaces. We also prepare some auxiliary results which are important in the proof of our main results. Theorems 1.1 and 1.2 are proved in Sects. 3 and 4, respectively.

## 2 Preliminaries

We start this section by briefly recalling some basis facts on the hyperbolic spaces and the Lorentz–Sobolev space defined in the hyperbolic spaces. Let \( n \geq 2 \), a hyperbolic space of dimension \( n \) (denoted by \( \mathbb{H}^n \)) is a complete, simply connected Riemannian manifold having constant sectional curvature \(-1\). There are several models for the hyperbolic space \( \mathbb{H}^n \) such as the half-space model, the hyperboloid (or Lorentz) model and the Poincaré ball model. Notice that all these models are Riemannian isometry. In this paper, we are interested in the Poincaré ball model of the hyperbolic space since this model is very useful for questions involving rotational symmetry. In the Poincaré ball model, the hyperbolic space \( \mathbb{H}^n \) is the open unit ball \( B_n \subset \mathbb{R}^n \) equipped with the Riemannian metric

\[
g(x) = \left( \frac{2}{1 - |x|^2} \right)^2 \, dx \otimes dx.
\]

The volume element of \( \mathbb{H}^n \) with respect to the metric \( g \) is given by

\[
dV_g(x) = \left( \frac{2}{1 - |x|^2} \right)^n dx,
\]
where $dx$ is the usual Lebesgue measure in $\mathbb{H}^n$. For $x \in B_n$, let $d(0, x)$ denote the geodesic distance between $x$ and the origin, then we have $d(0, x) = \ln(1 + |x|)/(1 - |x|)$. For $\rho > 0$, $B(0, \rho)$ denote the geodesic ball with center at origin and radius $\rho$. If we denote by $\nabla$ and $\Delta$ the Euclidean gradient and Euclidean Laplacian, respectively, as well as $\langle \cdot , \cdot \rangle$ the standard scalar product in $\mathbb{R}^n$, then the hyperbolic gradient $\nabla_g$ and the Laplace–Beltrami operator $\Delta_g$ in $\mathbb{H}^n$ with respect to metric $g$ are given by

$$\nabla_g = \left( \frac{1 - |x|^2}{2} \right)^2 \nabla, \quad \Delta_g = \left( \frac{1 - |x|^2}{2} \right)^2 \Delta + (n - 2) \left( \frac{1 - |x|^2}{2} \right) \langle x, \nabla \rangle,$$

respectively. For a function $u$, we shall denote $\sqrt{g(\nabla_g u, \nabla_g u)}$ by $|\nabla_g u|_g$ for simplifying the notation. Finally, for a radial function $u$ (i.e., the function depends only on $d(0, x)$) we have the following polar coordinate formula

$$\int_{\mathbb{H}^n} u(x)dx = n\sigma_n \int_0^{\infty} u(\rho) \sinh^{n-1}(\rho) d\rho.$$

It is now known that the symmetrization argument works well in the setting of the hyperbolic (see, example [3]). It is the key tool in the proof of several important inequalities such as the Poincaré inequality, the Sobolev inequality, the Moser–Trudinger inequality in $\mathbb{H}^n$. We shall see that this argument is also the key tool to establish the main results in the present paper. Let us recall some facts about the rearrangement argument in the hyperbolic space $\mathbb{H}^n$. A measurable function $u : \mathbb{H}^n \to \mathbb{R}$ is called vanishing at the infinity if for any $t > 0$ the set $\{|u| > t\}$ has finite $V_g$–measure, i.e.,

$$V_g(\{|u| > t\}) = \int_{\{|u| > t\}} dV_g < \infty.$$ 

For such a function $u$, its distribution function is defined by

$$\mu_u(t) = V_g(\{|u| > t\}).$$

Notice that $t \to \mu_u(t)$ is non-increasing and right-continuous. The non-increasing rearrangement function $u^*$ of $u$ is defined by

$$u^*(t) = \sup\{s > 0 : \mu_u(s) > t\}.$$ 

The non-increasing, spherical symmetry, rearrangement function $u^\#$ of $u$ is defined by

$$u^\#(x) = u^*(V_g(B(0, d(0, x)))), \quad x \in \mathbb{H}^n.$$

It is well known that $u$ and $u^\#$ have the same non-increasing rearrangement function (which is $u^*$). Finally, the maximal function $u^{**}$ of $u^*$ is defined by

$$u^{**}(t) = \frac{1}{t} \int_0^t u^*(s)ds.$$

Evidently, $u^*(t) \leq u^{**}(t)$. For $1 \leq p, q < \infty$, the Lorentz space $L^{p,q}(\mathbb{H}^n)$ is defined as the set of all measurable function $u : \mathbb{H}^n \to \mathbb{R}$ satisfying
\[ \|u\|_{L^p(\mathbb{H}^n)} := \left( \int_0^\infty \left( \frac{1}{t} u^+(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty. \]

It is clear that \( L^{p,q}(\mathbb{H}^n) = L^p(\mathbb{H}^n) \). Moreover, the Lorentz spaces are monotone with respect to second exponent, namely

\[ L^{p,q_1}(\mathbb{H}^n) \subseteq L^{p,q_2}(\mathbb{H}^n), \quad 1 \leq q_1 < q_2 < \infty. \]

The functional \( u \rightarrow \|u\|_{L^p(\mathbb{H}^n)} \) is not a norm in \( L^{p,q}(\mathbb{H}^n) \) except the case \( q \leq p \) (see [5, Chapter 4, Theorem 4.3]). In general, it is a quasi-norm which turns out to be equivalent to the norm obtained replacing \( u^* \) by its maximal function \( u^{**} \) in the definition of \( \| \cdot \|_{L^p(\mathbb{H}^n)} \). Moreover, as a consequence of Hardy inequality, we have

**Proposition 2.1** Given \( p \in (1, \infty) \) and \( q \in [1, \infty) \). Then for any function \( u \in L^{p,q}(\mathbb{H}^n) \) it holds

\[ \left( \int_0^\infty \left( \frac{1}{t} u^+(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \leq \frac{p}{p-1} \left( \int_0^\infty \left( \frac{1}{t} u^+(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} = \frac{p}{p-1} \|u\|_{L^p(\mathbb{H}^n)}, \quad (2.1) \]

For \( 1 \leq p, q < \infty \) and an integer \( m \geq 1 \), we define the \( m \)-th order Lorentz–Sobolev space \( W^mL^{p,q}(\mathbb{H}^n) \) by taking the completion of \( C_0^\infty(\mathbb{H}^n) \) under the quasi-norm

\[ \|\nabla_g^m u\|_{p,q} := \|\nabla_g^m u\|_{p,q}. \]

It is obvious that \( W^mL^{p,q}(\mathbb{H}^n) = W^{m,p}(\mathbb{H}^n) \) the \( m \)-th order Sobolev space in \( \mathbb{H}^n \). In [21], the author established the following Pólya–Szegö principle in the first-order Lorenz–Sobolev spaces \( W^1L^{p,q}(\mathbb{H}^n) \) which generalizes the classical Pólya–Szegö principle in the hyperbolic space.

**Theorem 2.2** Let \( n \geq 2, 1 \leq q \leq p < \infty \) and \( u \in W^1L^{p,q}(\mathbb{H}^n) \). Then \( u^r \in W^1L^{p,q}(\mathbb{H}^n) \) and

\[ \|\nabla_g u^r\|_{p,q} \leq \|\nabla_g u\|_{p,q}. \]

For \( r \geq 0 \), define

\[ \Phi(r) = n \int_0^r \sinh^{n-1}(s)ds, \quad r \geq 0, \]

and let \( F \) be the function such that

\[ r = n\sigma_n \int_0^{F(r)} \sinh^{n-1}(s)ds, \quad r \geq 0, \]

i.e., \( F(r) = \Phi^{-1}(r/\sigma_n) \). It was proved in [21, Lemma 2.1] that

\[ \sinh^{\rho(n-1)}(F(t)) \geq \left( \frac{t}{\sigma_n} \right)^{\frac{n-1}{n}} + \left( \frac{n-1}{n} \right) \left( \frac{t}{\sigma_n} \right)^q, \quad t \geq 0, \quad (2.2) \]

provided \( q \geq \frac{2n}{n-1} \). Moreover, we have the following result.
Proposition 2.3. Let $n \geq 2$. Then it holds
\[ \sinh^n(F(t)) > \frac{t}{\sigma_n}, \quad t > 0. \] (2.3)

**Proof** Indeed, for $\rho > 0$, we have
\[ n \int_0^\rho \sinh^{n-1}(s) \, ds < n \int_0^\rho \sinh^{n-1}(s) \cosh(s) \, ds = \sinh^n(\rho). \]
Taking $\rho = F(t), t > 0$ we obtain (2.3). 

Proposition 2.4. Let $n \geq 2$, then the function
\[ \varphi(t) = \frac{t}{\sinh^{n-1}(F(t))} \]
is strictly increasing on $(0, \infty)$, and
\[ \lim_{t \to \infty} \varphi(t) = \frac{n \sigma_n}{n - 1}. \]

**Proof** Since $t \mapsto F(t)$ is strictly increasing function, then it is enough to prove that the function
\[ \eta(\rho) = \int_0^\rho \sinh^{n-1}(s) \, ds \]
is strictly increasing on $(0, \infty)$. Indeed, we have
\[ \eta'(\rho) = 1 - (n - 1) \cosh(\rho) \int_0^\rho \sinh^{n-1}(s) \, ds \]
\[ = \frac{1}{\sinh^n(\rho)} \left( \sinh^n(\rho) - (n - 1) \cosh(\rho) \int_0^\rho \sinh^{n-1}(s) \, ds \right) \]
\[ = \frac{\xi(\rho)}{\sinh^n(\rho)}, \]
and
\[ \xi'(\rho) = \cosh(\rho) \sinh^{n-1}(\rho) - (n - 1) \sinh(\rho) \int_0^\rho \sinh^{n-1}(s) \, ds. \]

For $\rho > 0$, it holds
\[ (n - 1) \int_0^\rho \sinh^{n-1}(s) \, ds < (n - 1) \int_0^\rho \sinh^{n-2}(s) \cosh(s) \, ds = \sinh^{n-1}(\rho), \]
here we use $\cosh(s) > \sinh(s)$ for $s > 0$. Therefore, we get
\[ \xi'(\rho) > \sinh^{n-1}(\rho)(\cosh(\rho) - \sinh(\rho)) > 0, \]
for $\rho > 0$. Consequently, we have $\xi(\rho) > \xi(0) = 0$ for $\rho > 0$. Hence, $\eta'(\rho) > 0$ for $\rho > 0$ which implies that $\eta$ is strictly increasing function on $(0, \infty)$. By L'Hospital rule, we have
\[
\lim_{\rho \to \infty} \eta(\rho) = \lim_{\rho \to \infty} \frac{\sinh^{n-1}(\rho)}{(n-1) \sinh^{n-2}(\rho) \cosh(\rho)} = \frac{1}{n-1}
\]

which yields the desired limit in this proposition. \qed

In the rest of this section, we shall frequently using the following one-dimensional Hardy inequality

**Lemma 2.5** Let \(1 < q < p\). Then for any absolutely continuous function \(u \in (0, \infty)\) such that \(\lim_{t \to \infty} \frac{|u(t)|^p}{t^p} = 0\), it holds

\[
\int_0^\infty |u'(t)|^q t^{p-q-1} \, dt \geq \left( \frac{p-q}{q} \right)^q \int_0^\infty |u(t)|^q t^{p-q-1} \, dt.
\]

**Proof** For any \(0 < a < b\) and \(\alpha > 0\), by integration by parts we have

\[
\int_a^b |u(t)|^q t^{p-q-1} \, dt = \frac{1}{p-q} \int_a^b |u(t)|^q (t^{p-q})' \, dt = \frac{q}{p-q} \int_a^b |u(t)|^{q-2} u(t) (t^{p-q}) \, dt - \frac{1}{p-q} |u(a)|^q a^{p-q}
\]

\[
+ \frac{1}{p-q} |u(b)|^q b^{p-q} \leq \frac{q}{p-q} \int_a^b \left( \alpha |u(t)|^{q-2} u(t) \right) \left( - \frac{1}{\alpha} u'(t) \right) t^{p-q-1} \, dt + \frac{1}{p-q} |u(b)|^q b^{p-q}.
\]

Applying Young’s inequality, we get

\[
\int_a^b \left( \alpha |u(t)|^{q-2} u(t) \right) \left( - \frac{1}{\alpha} u'(t) \right) t^{p-q-1} \, dt \leq \frac{q-1}{q} \int_a^b \frac{1}{\alpha^{q-1}} |u(t)|^q t^{p-q-1} \, dt + \frac{1}{q} \alpha^{q-1} \int_a^b |u'(t)|^q t^{p-q-1} \, dt.
\]

Combining these two estimates, we arrive

\[
(p-q) \left( \alpha^q - \frac{q-1}{p-q} \alpha^{q-1} \right) \int_a^b |u(t)|^q t^{p-q-1} \, dt \leq \int_a^b |u'(t)|^q t^{p-q-1} \, dt + \alpha^q |u(b)|^q b^{p-q}.
\]

for any \(\alpha > 0\). Taking \(\alpha = ((p-q)/q)^{(q-1)/q}\), we get

\[
\left( \frac{p-q}{q} \right)^q \int_a^b |u(t)|^q t^{p-q-1} \, dt \leq \int_a^b |u'(t)|^q t^{p-q-1} \, dt + \left( \frac{p-q}{q} \right)^{q-1} |u(b)|^q b^{p-q}.
\]

Letting \(a \to 0^+\) and \(b \to \infty\), we obtain (2.4). \qed

Let \(u \in C_0^\infty([0,\infty))\) and \(f = -\Delta_y u\). It was proved by Ngo and the author (see [16, Proposition 2.2]) that

\[ \text{Springer} \]
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\[ u^*(t) \leq v(t) := \int_t^\infty \frac{sf^{**}(s)}{(n\sigma_n \sinh^{n-1}(F(s)))^2} \, ds, \quad t > 0. \tag{2.5} \]

The following results are important in the Proof of Theorem 1.1 and Theorem 1.2.

**Proposition 2.6** Let \( n \geq 2, \ p \in (1, n) \) and \( q \in (1, \infty) \). Then it holds

\[ \|\Delta_\delta u\|_{p,q}^q \geq \left( \frac{p - 1}{p} n\sigma_n \right)^{\frac{1}{n}} \int_0^\infty |v'(t)|^q (n\sigma_n \sinh^{n-1}(F(t)))^{\frac{1}{n}} \, dt \tag{2.6} \]

Furthermore, if \( q \geq 2n - 1 \) then we have

\[ \|\Delta_\delta u\|_{p,q}^q \geq \left( \frac{p - 1}{p} n\sigma_n \right)^{\frac{1}{n}} \int_0^\infty |v'(t)|^q (n\sigma_n \sinh^{n-1}(F(t)))^{\frac{1}{n}} \, dt \tag{2.7} \]

**Proof** We have

\[ v'(t) = -\frac{tf^{**}(t)}{(n\sigma_n \sinh^{n-1}(F(t)))^2}, \]

and hence

\[ \int_0^\infty |v'(t)|^q (n\sigma_n \sinh^{n-1}(F(t)))^{\frac{1}{n}} \, dt = \int_0^\infty \frac{(f^{**}(t))^{\frac{q}{n}}}{(n\sigma_n \sinh^{n-1}(F(t)))^{\frac{q}{n}}} \, dt \tag{2.8} \]

Using (2.3) and (2.1) we obtain

\[ \int_0^\infty |v'(t)|^q (n\sigma_n \sinh^{n-1}(F(t)))^{\frac{1}{n}} \, dt \leq \frac{1}{n\sigma_n^{\frac{q}{n}}} \int_0^\infty (f^{**}(t))^{\frac{q}{n}} \, dt \]

\[ \leq \frac{n^{\frac{q}{n}}}{n\sigma_n^{\frac{q}{n}}} \left( \frac{p}{p - 1} \right) \int_0^\infty (f^{**}(t))^q \, dt \]

\[ = \left( \frac{1}{n\sigma_n^{\frac{q}{n}}} \frac{p}{p - 1} \right) \|\Delta_\delta u\|_{p,q}^q, \]

as wanted (2.6).

We next prove (2.7). We notice that

\[ \int_0^\infty |v'(t)|^q (n\sigma_n \sinh^{n-1}(F(t)))^{\frac{1}{n}} \, dt = \int_0^\infty \frac{(f^{**}(t))^{\frac{q}{n}}}{(n\sigma_n \sinh^{n-1}(F(t)))^{\frac{q}{n}}} \, dt \]

This equality together with (2.8), the fact \( q \geq 2n - 1 \) and the inequality (2.1) implies
\[(n - 1)^q \int_0^\infty |v'(t)|^q (n \sigma_n \sinh^{n-1}(F(t)))^q t^{q - 1} \, dt \]
\[+ n^q \sigma_n^q \int_0^\infty |v'(t)|^q (n \sigma_n \sinh^{n-1}(F(t)))^q t^{q(\frac{1}{p} - \frac{1}{n}) - 1} \, dt \]
\[\leq \int_0^\infty (f^{**}(t))^q \left( \frac{t}{\sigma_n \sinh^q(F(t))} \right)^{q - 1} \, dt \]
\[+ \left( \frac{n - 1}{n} \right)^q \int_0^\infty |v'(t)|^q t^{q - 1} \, dt \]
\[\leq \left( \frac{p}{p - 1} \right)^q \int_0^\infty (f^{**}(t))^q t^{q - 1} \, dt \]
\[= \left( \frac{p}{p - 1} \right)^q \|\Delta g u\|_{p,q}^q \]
as wanted (2.7). \qed

**Proposition 2.7** Let \( n \geq 2 \), \( p \in (1, n) \) and \( q \geq \frac{2n}{n - 1} \). Then, we have
\[
\int_0^\infty |v'(t)|^q (n \sigma_n \sinh^{n-1}(F(t)))^q t^{q - 1} \, dt \geq \left( \frac{n - 1}{p} \right)^q \int_0^\infty |v(t)|^q t^{\frac{q}{p} - 1} \, dt \]
\[+ n^q \sigma_n^q \int_0^\infty |v'(t)|^q t^{q(\frac{1}{p} - \frac{1}{n}) - q - 1} \, dt, \tag{2.9} \]
and
\[
\int_0^\infty |v'(t)|^q (n \sigma_n \sinh^{n-1}(F(t)))^q t^{q(\frac{1}{p} - \frac{1}{n}) - 1} \, dt \]
\[\geq \left( \frac{(n - 1)(n - p)}{np} \right)^q \int_0^\infty |v(t)|^q t^{q(\frac{1}{p} - \frac{1}{n}) - 1} \, dt \]
\[+ n^q \sigma_n^q \int_0^\infty |v'(t)|^q t^{q(\frac{1}{p} - \frac{1}{n}) + q - 1} \, dt, \tag{2.10} \]

**Proof** If \( q \geq \frac{2n}{n - 1} \) then by using (2.2) we have
\[
\int_0^\infty |v'(t)|^q (n \sigma_n \sinh^{n-1}(F(t)))^q t^{q - 1} \, dt \geq (n - 1)^q \int_0^\infty |v(t)|^q t^{q + 1} \, dt \]
\[+ n^q \sigma_n^q \int_0^\infty |v'(t)|^q t^{q(\frac{1}{p} - \frac{1}{n}) + q - 1} \, dt, \]
Using the one-dimensional Hardy inequality (2.4), we have
\[
\int_0^\infty |v'(t)|^q t^{q + 1} \, dt \geq \left( \frac{1}{p} \right)^q \int_0^\infty |v(t)|^q t^{q - 1} \, dt.
\]
Combining these two inequalities proves the inequality (2.9).

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Let $q \geq \frac{2n}{n-1}$ then by using again (2.2), we get
\[
\int_0^\infty |v'(t)|^q (n\sigma_n \sinh^{n-1}(F(t)))^{q(\frac{1}{p}-\frac{1}{2})-1} \, dt \geq (n-1)^q \int_0^\infty |v'(t)|^q t^{q(\frac{1}{p}-\frac{1}{2})+q-1} \, dt \\
+ n^q \sigma_n \int_0^\infty |v'(t)|^q t^{q(\frac{1}{p}-\frac{1}{2})+q-1} \, dt.
\]
Using the one-dimensional Hardy inequality (2.4), we have
\[
\int_0^\infty |v'(t)|^q t^{q(\frac{1}{p}-\frac{1}{2})+q-1} \, dt \geq \left( \frac{n-p}{np} \right)^q \int_0^\infty |v(t)|^q t^{q(\frac{1}{p}-\frac{1}{2})-1} \, dt.
\]
Combining these two inequalities proves the inequality (2.10). \hfill \Box

Combining Propositions 2.6 and 2.7, we obtain

**Theorem 2.8** Let $n \geq 2$. If $p \in (1, n)$ and $q \in (1, \infty)$. For any $u \in C_0^\infty(\mathbb{H}^n)$ we define $v$ by (2.5). Then we have
\[
\|\Delta_g u\|_{p,q}^q \geq \left( \frac{n^2 \sigma_n}{p'} \right)^q \int_0^\infty |v'(t)|^q t^{q(\frac{1}{p}-\frac{1}{2})+q-1} \, dt,
\]
where $p' = p/(p-1)$. In particular, if $p \in (1, \frac{n}{2})$ then it holds
\[
\|\Delta_g u\|_{p,q}^q \geq \left( \frac{n(n-2p)}{pp'} \sigma_n \right)^q \|u\|_{p^*,q'}^q.
\]
Furthermore, if $p \in (1, n)$ and $q \geq \frac{2n}{n-1}$ then we have
\[
\|\Delta_g u\|_{p,q}^q - C(n, 2,p)^q \|u\|_{p,q}^q \geq \left( \frac{n^2 \sigma_n}{p'} \right)^q \int_0^\infty |v'(t)|^q t^{q(\frac{1}{p}-\frac{1}{2})+q-1} \, dt.
\]
In particular, if $p \in (1, \frac{n}{2})$ and $q \geq \frac{2n}{n-1}$ then we have
\[
\|\Delta_g u\|_{p,q}^q - C(n, 2,p)^q \|u\|_{p,q}^q \geq \left( \frac{n(n-2p)}{pp'} \sigma_n \right)^q \|u\|_{p^*,q'}^q, \quad u \in C_0^\infty(\mathbb{H}^n).
\]

It is worthy to mention here that in the Euclidean space $\mathbb{E}^n$, an analogue of the inequality (2.12) was proved by Tarsi (see [23, Theorem 2]). Using this inequality and iterative method, she proved the higher-order Sobolev inequalities under Lorentz–Sobolev norm in the Euclidean space (see also [15, Proposition 4.4]).

**Proof** Let $u \in C_0^\infty(\mathbb{H}^n)$ and $v$ be defined by (2.5). We know that $u^* \leq v$, then
\[
\|u\|_{p,q}^q \leq \int_0^\infty v(t)^{\frac{q}{p}-1} \, dt, \quad \text{and} \quad \|u\|_{p^*,q'}^q \leq \int_0^\infty v(t)^{\frac{q}{p}-1} \, dt.
\]

The inequality (2.11) is a consequence of (2.6) and (2.2). The inequality (2.12) is consequence of (2.11), the one-dimensional Hardy inequality (2.4)
\[ \int_0^\infty |v'(t)|^q t^{-\frac{n}{p} - \frac{2}{q} + q - 1} \, dt \geq \left( \frac{n - 2p}{np} \right)^q \int_0^\infty |v(t)|^q t^{\frac{1}{p} - \frac{2}{q} + q - 1} \, dt. \]

and the second inequality in (2.15).

To prove (2.13), we first notice by the first inequality in (2.15) that

\[ \|\Delta_s u\|^q_{p,q} - C(n, 2, p)^q \|u\|^q_{p,q} \geq \|\Delta_s u\|^q_{p,q} - C(n, 2, p)^q \int_0^\infty v(t)^{\frac{2n}{n-1}} \, dt. \]

Hence, it holds

\[ \|\Delta_s u\|^q_{p,q} - C(n, 2, p)^q \|u\|^q_{p,q} \geq \|\Delta_s u\|^q_{p,q} - \left( \frac{n - 1}{p'} \right)^q \int_0^\infty |v'(t)|^q (n\sigma_n \sinh^{n-1}(F(t)))^q t^{\frac{q}{n-1}} \, dt \]

\[ \quad + \left( \frac{n - 1}{p} \right)^q \int_0^\infty v(t)^q t^{\frac{q}{n-1}} \, dt \]

\[ \quad - \left( \frac{n - 1}{p} \right)^q \int_0^\infty |v'(t)|^q (n\sigma_n \sinh^{n-1}(F(t)))^q t^{\frac{q}{n-1}} \, dt, \]

here we use \( q \geq \frac{2n}{n-1} \) and the inequality (2.9). Using again the assumption \( q \geq \frac{2n}{n-1} \) and the inequalities (2.7) and (2.10), we obtain

\[ \|\Delta_s u\|^q_{p,q} - \left( \frac{n - 1}{p'} \right)^q \int_0^\infty |v'(t)|^q (n\sigma_n \sinh^{n-1}(F(t)))^q t^{\frac{q}{n-1}} \, dt \]

\[ \quad \geq \left( n^2 \sigma_n^{\frac{z}{p'}} \right)^q \int_0^\infty |v'(t)|^q t^{\frac{q}{p'} - \frac{n}{2} + q - 1} \, dt. \]

Combining the estimates (2.16) and (2.17) proves (2.13). The inequality (2.14) follows from (2.13) and the one-dimensional Hardy inequality (2.4)

\[ \int_0^\infty |v'(t)|^q t^{\frac{q}{p'} - \frac{2}{q} + q - 1} \, dt \geq \left( \frac{n - 2p}{np} \right)^q \int_0^\infty |v(t)|^q t^{\frac{1}{p'} - \frac{2}{q} + q - 1} \, dt. \]

\[ \square \]

### 3 Proof of Theorem 1.1

In this section, we prove Theorems 1.1.

**Proof of Theorem 1.1** In the case \( m = 1 \), Theorem 1.1 was already proved in [21]. So, we will only consider the case \( m \geq 2 \). We divide the proof into three cases as follows.

**Case 1** \( m = 2 \). If \( p \in (1, n) \) and \( q \geq \frac{2n}{n-1} \) then (1.2) follows from (2.13). In the following, we will give a proof of (1.2) for any \( p, q \in (0, \infty) \). By the density, it is enough to prove...
(1.2) for function $u \in C^\infty_0(\mathbb{H}^n)$, $u \not\equiv 0$. Let $f = -\Delta_g u$ and $v$ be defined by (2.5). We first notice that
\[
\int_0^\infty |v'(t)|q t^{\frac{2}{q} + q - 1} \, dt = \int_0^\infty \left( \frac{t}{n\sigma_n \sinh^{n-1}(F(t))} \right)^{2q} f^{**}(t) t^{\frac{2}{q} - 1} \, dt.
\]
By Lemma 2.4, we have
\[
\frac{t}{n\sigma_n \sinh^{n-1}(F(t))} < \frac{1}{n - 1}, \quad t > 0.
\]
This inequality together with the inequality (2.1) implies
\[
\int_0^\infty \left( \frac{t}{n\sigma_n \sinh^{n-1}(F(t))} \right)^{2q} f^{**}(t) t^{\frac{2}{q} - 1} \, dt \leq \left( \frac{p'}{(n - 1)^2} \right)^q \int_0^\infty f^{**}(t) t^{\frac{2}{q} - 1} \, dt = \left( \frac{p'}{(n - 1)^2} \right)^q \|\Delta_g u\|_{p,q}^q.
\]
By the one-dimensional Hardy inequality and the first inequality in (2.15), we have
\[
\int_0^\infty |v'(t)|q t^{\frac{2}{q} + q - 1} \, dt \geq \frac{1}{p^q} \int_0^\infty v(t)q t^{\frac{2}{q} - 1} \, dt \geq \frac{1}{p^q} \|u\|_{p,q}^q.
\]
Combining two previous inequalities, we obtain (1.2).

Case 2 $m = 2k$, $k \geq 1$. This case follows from the Case 1 and the iteration argument.

Case 3 $m = 2k + 1$, $k \geq 1$. Since $q \leq p$, then it was proved in [21, Theorem 1.1] that
\[
\|\nabla_g \Delta_k u\|_{p,q}^q \geq \left( \frac{n - 1}{p} \right)^q \|\Delta_k u\|_{p,q}^q.
\]
We now apply the Case 2 to obtain the desired result.

We next check the sharpness of the constant $C(n, m, p)$ in (1.2). From Proposition 2.4, we see that for any $\epsilon > 0$ there exists $a > 0$ such that
\[(n - 1)s < n\sigma_n \sinh^{n-1}(F(s)) \leq (1 + \epsilon)(n - 1)s,
\]
for any $s \geq a$. For $R > a$, let us define the function
\[
f_R(s) = \begin{cases} a^{-\frac{1}{r}} & \text{if } s \in (0, a), \\ s^{-\frac{1}{r}} & \text{if } s \in [a, R), \\ R^{-\frac{1}{r}} \max\{2 - s/R, 0\} & \text{if } s \geq R. \end{cases}
\]
Notice that $f_R$ is a nonnegative, continuous, non-increasing function. Following [16, Section 2.2], we define two sequences of functions $\{v_{R,i}\}_{i \geq 0}$ and $\{g_{R,i}\}_{i \geq 1}$ as follows:

(i) First, we set $v_{R,0} = f_R$.

(ii) then in terms of $v_{R,i}$, we define $g_{R,i+1}$ as the maximal function of $v_{R,i}$, i.e.,
\[ g_{R,i+1}(t) = \frac{1}{t} \int_0^t v_{R,i}(s) \, ds, \]

(iii) and finally in terms of \( g_{R,i+1} \), we define \( v_{R,i+1} \) as follows

\[ v_{R,i+1}(t) = \int_t^\infty \frac{s g_{R,i+1}(s)}{(n\sigma_n \sinh^{n-1}(F(s)))^2} \, ds, \]

for \( i = 0, 1, 2, \ldots \)

Note that \( v_{R,i} \) and \( g_{R,i} \) are positive, non-increasing functions. Following the proof of [16, Proposition 2.1], we can prove the following result. \( \square \)

**Proposition 3.1** For any \( i \geq 1 \), there exist function \( h_{R,i} \) and \( w_{R,i} \) such that

\[ v_{R,i} = h_{R,i} + w_{R,i}, \quad \int_0^\infty |w_{R,i}|^{q t^{q-1}} \, dt \leq C \]

and

\[ \frac{1}{(1 + \epsilon)^{2i}} \left( \frac{pp'}{(n-1)^2} \right)^i f_R \leq h_{R,i} \leq \left( \frac{pp'}{(n-1)^2} \right)^i, \]

where we use \( C \) to denote various constants which are independent of \( R \).

**Proof** Let us define the operator \( T \) acting on functions \( v \) on \([0, \infty)\) by

\[ T(v)(t) = \int_t^\infty \frac{s}{(n\sigma_n \sinh(F(s)))^2} \left( \frac{1}{s} \int_0^s v(r) \, dr \right) \, ds. \]

We shall prove that

\[ \int_0^\infty |T(v)(t)|^{q t^{q-1}} \, dt \leq \left( \frac{pp'}{(n-1)^2} \right)^q \int_0^\infty |v(t)|^{q t^{q-1}} \, dt. \]  

(3.1)

Indeed, it is enough to prove (3.1) for nonnegative function \( v \) such that

\[ \int_0^\infty |v(t)|^{q t^{q-1}} \, dt < \infty. \]

We claim that

\[ \lim_{t \to 0^+} T(v)(t)^{\frac{1}{t}} = 0 = \lim_{t \to \infty} T(v)(t)^{\frac{1}{t}}. \]  

(3.2)

For any \( \epsilon > 0 \) there exists \( t_0 > 0 \) such that \( \int_0^{t_0} |v(t)|^{q t^{q-1}} \, dt \leq \epsilon^q \). For \( s \leq t_0 \), by using Hölder inequality, we have

\[ \frac{1}{s} \int_0^s v(r) \, dr \leq \frac{1}{s} \left( \int_0^s v(r) r^{q t^{q-1}} \, dr \right)^{\frac{1}{q}} \left( \int_0^s r^{q-1} \, dr \right)^{\frac{q-1}{q}} \leq C \epsilon^{-\frac{1}{q}}. \]

This together with the inequality \( n\sigma_n \sinh^{n-1}(F(s)) > (n-1)s \) implies for \( t \leq t_0 \) that
\[ T(v)(t) = \left( \int_{t_0}^{t_0 + \infty} + \int_{t_0}^{\infty} \right) \frac{s}{(n\sigma_n (\sinh(F(s))))^2} \left( \frac{1}{s} \int_0^s v(r) dr \right) ds \]

\[ \leq C e \int_t^{t_0} s^{-1 - \frac{1}{p}} ds + C \left( \int_0^{\infty} v(r)^q r^{\frac{q-1}{p}} dr \right)^{\frac{1}{q}} \int_0^{\infty} s^{-1 - \frac{1}{p}} ds \]

\[ \leq C e (t^{-\frac{1}{p}} - t_0^{-\frac{1}{p}}) + C t_0^{-\frac{1}{p}} \left( \int_0^{\infty} v(r)^q r^{\frac{q-1}{p}} dr \right)^{\frac{1}{q}}. \]

This estimate yields

\[ \limsup_{t \to 0} T(v)(t) t^{\frac{1}{p}} \leq C e. \]

Since \( \epsilon > 0 \) is arbitrary, then the first limit in (3.2) is proved.

Similarly, for any \( \epsilon > 0 \) there exists \( t_1 > 0 \) such that \( \int_{t_1}^{\infty} |v(t)|^q t^{\frac{q-1}{p}} dt < \epsilon^q \). Hence, for \( s \geq t_1 \), by using Hölder inequality we get

\[ \int_0^s v(r) dr = \int_0^{t_1} v(r) dr + \int_{t_1}^s v(r) dr \leq C \left( \int_0^{\infty} |v(t)|^q t^{\frac{q-1}{p}} dt \right)^{\frac{1}{q}} t_1^{\frac{1}{r} - \frac{1}{p}} + C s s^{-\frac{1}{p}}. \]

Consequently, for any \( t \geq t_1 \) we get

\[ T(v)(t) \leq C \int_0^t \left( \left( \int_0^{\infty} |v(t)|^q t^{\frac{q-1}{p}} dt \right)^{\frac{1}{q}} t_1^{\frac{1}{r} - \frac{1}{p}} s^{-2} + \epsilon s^{-\frac{1}{p}} \right) ds \]

\[ \leq C \left( \int_0^{\infty} |v(t)|^q t^{\frac{q-1}{p}} dt \right)^{\frac{1}{q}} t_1^{\frac{1}{r} - \frac{1}{p}} t^{\frac{1}{p}} + C e t^{-\frac{1}{p}}. \]

This estimate implies

\[ \limsup_{t \to \infty} T(v)(t) t^{\frac{1}{p}} \leq C e. \]

Since \( \epsilon > 0 \) is arbitrary, then the second limit in (3.2) is proved.

Using the integration by parts, the claim (3.2) and the inequality \( n\sigma_n \sinh^{n-1}(F(t)) > (n-1)t \), we have

\[ \int_0^{\infty} T(v)(t)^q t^{\frac{q}{r} - 1} dt = \frac{p}{q} \int_0^{\infty} T(v)(t)^q (t^{\frac{q}{r}})' dt \]

\[ = p \int_0^{\infty} T(v)(t)^{q-1} \left( \frac{t}{n\sigma_n \sinh^{n-1}(F(t))} \right)^2 \left( \frac{1}{t} \int_0^t v(s) ds \right) t^{\frac{q}{r} - 1} dt \]

\[ \leq \frac{p}{(n-1)^2} \int_0^{\infty} T(v)(t)^{q-1} \left( \frac{1}{t} \int_0^t v(s) ds \right) t^{\frac{q}{r} - 1} dt. \]

An easy application of Hölder inequality implies

\[ \int_0^{\infty} T(v)(t)^q t^{\frac{q}{r} - 1} dt \leq \left( \frac{p}{(n-1)^2} \right)^q \int_0^{\infty} \left( \frac{1}{t} \int_0^t v(s) ds \right)^q t^{\frac{q}{r} - 1} dt. \]
The inequality (3.1) follows from the previous inequality and the Hardy inequality (2.1).

Thus, with the help of (3.1), we can using the induction argument to prove this proposition by establishing the result for $v_{R,1}$. In fact, the decomposition for $v_{R,1}$ is already proved in the proof of Proposition 2.1 in [16]. The estimate

$$\int_0^\infty |w_{R,1}|^{\frac{q}{r} - 1} dt \leq C,$$

is proved by the same way of the estimate $\int_0^\infty |w_{R,1}|^p dt \leq C$. □

We are now ready to check the sharpness of $C(n, m, p)$. The case $m = 1$ was done in [21]. Hence, we only consider the case $m \geq 2$. We first consider the case $m = 2k, k \geq 1$. Define

$$u_R(x) = v_{R,k}(V_g(B(0, d(0,x)))].$$

It is clear that $(-\Delta_g)^k u_R(x) = f_R(V_g(B(0, d(0,x)))].$ Hence, there hold

$$\| \Delta_g^k u_R \|_{p,q}^q = \int_0^\infty \int_0^\infty \frac{v^q}{\text{uni} \varphi} \left| f_R(t) \right|^q t^{\frac{q}{r} - 1} dt = \frac{p}{q} + \ln \frac{R}{a} + \int_1^2 (2-s)^{s^{\frac{q}{r} - 1}} ds,$$

and

$$\| u_R \|_{p,q} = \left( \int_0^\infty \frac{v^q}{\text{uni} \varphi} \right)^{\frac{1}{q}} \left( \int_0^\infty \left| w_{R,k} \right|^{\frac{q}{r} - 1} dt \right)^{\frac{1}{q}} \geq \frac{1}{(1 + \epsilon)^{2k}} \left( \frac{pp'q}{(n-1)^2} \right)^k \left( \int_0^\infty \left| f_R(t) \right|^q t^{\frac{q}{r} - 1} dt \right)^{\frac{1}{q}} - C.$$

These estimates imply

$$\limsup_{R \to \infty} \frac{\| \Delta_g^k u_R \|_{p,q}^q}{\| u_R \|_{p,q}^q} \leq (1 + \epsilon)^{2k} C(n, 2k, p)^k,$$

for any $\epsilon > 0$. This proves the sharpness of $C(n, 2k, p)$. We next consider the case $m = 2k + 1, k \geq 1$. Define

$$u_R(x) = v_{R,k}(V_g(B(0, d(0,x)))].$$

It is clear that $(-\Delta_g)^k u_R(x) = f_R(V_g(B(0, d(0,x)))].$ It was shown in the proof of Theorem 1.1 in [21] (the sharpness of $C(n, 1, p)$) that

$$\limsup_{R \to \infty} \frac{\| \Delta_g^k u_R \|_{p,q}^q}{\| \Delta_g^k u_R \|_{p,q}^q} \leq (1 + \epsilon)^{q} \frac{(n-1)^q}{p^q}.$$

This together with the estimate in the case $m = 2k$ implies

\[ \text{Springer} \]
\[
\limsup_{R \to \infty} \frac{\| \nabla \Delta_g^k u_R \|_{p,q}^q}{\| u_R \|_{p,q}^q} \leq \limsup_{R \to \infty} \frac{\| \nabla \Delta_g^k u \|_{p,q}^q}{\| u \|_{p,q}^q} \leq (1 + \epsilon)^{(2k+1)q} C(n, 2k + 1, p)^q,
\]
for any \( \epsilon > 0 \). This proves the sharpness of \( C(n, 2k + 1, p) \).

The Proof of Theorem 1.1 is then completely finished. \( \square \)

4 Proof of Theorem 1.2

This section is addressed to prove Theorem 1.2. The proof uses the results from Theorem 2.8 and [21, Theorem 1.2].

Proof of Theorem 1.2 The case \( m = 1 \) was already proved in [21]. So, we only consider the case \( m \geq 2 \). We divide the proof into two cases as follows.

Case 1: \( m = 2k, k \geq 1 \). The case \( k = 1 \) follows from (2.14). For \( k \geq 2 \), by using Theorem 1.1, we have

\[
\| \Delta_g^k u \|_{p,q}^q - C(n, 2k, p)^q \| u \|_{p,q}^q \geq \| \Delta_g^k u \|_{p,q}^q - C(n, 2, p)^q \| \Delta_g^{k-1} u \|_{p,q}^q.
\]

Applying the inequality (2.14) to the right hand side of the previous inequality, we obtain

\[
\| \Delta_g^k u \|_{p,q}^q - C(n, 2k, p)^q \| u \|_{p,q}^q \geq \left( \frac{n(n - 2p)}{pp' \sigma_n^{\frac{2}{p}} \frac{n}{p} \frac{n}{p'}} \prod_{i=0}^{k-1} \frac{n(2n - 2p_i)}{p_i^2 (p_i^2)^{\frac{1}{p}}} \right) \| \Delta_g^{k-1} u \|_{p,q}^q.
\]

By iterating the inequality (2.12), we then have

\[
\| \Delta_g^k u \|_{p,q}^q - C(n, 2k, p)^q \| u \|_{p,q}^q \geq \left( \frac{n(n - 1)}{p} \sigma_n^{\frac{2}{p}} \right) \| \Delta_g^{k-1} u \|_{p,q}^q,
\]
as wanted (1.7).

Case 1: \( m = 2k + 1, k \geq 1 \). In this case, we have

\[
\| \nabla \Delta_g^k u \|_{p,q}^q - C(n, 2k + 1, p)^q \| u \|_{p,q}^q \geq \| \nabla \Delta_g^k u \|_{p,q}^q - \left( \frac{n - 1}{p} \right)^q \| \Delta_g^k u \|_{p,q}^q.
\]

Since \( q \leq p \) we then have from Theorem 1.2 in [21] that

\[
\| \nabla \Delta_g^k u \|_{p,q}^q - \left( \frac{n - 1}{p} \right)^q \| \Delta_g^k u \|_{p,q}^q \geq \left( \frac{n - p}{p} \sigma_n^{\frac{2}{p}} \right)^q \| \Delta_g^k u \|_{p,q}^q.
\]

Hence, it holds

\[
\| \nabla \Delta_g^k u \|_{p,q}^q - C(n, 2k + 1, p)^q \| u \|_{p,q}^q \geq \left( \frac{n - p}{p} \sigma_n^{\frac{2}{p}} \right)^q \| \Delta_g^k u \|_{p,q}^q.
\]

By iterating the inequality (2.12), we then have
\[ \| \nabla^k g \|_{p,q}^q - C(n, 2k + 1, p) \| u \|_{p,q}^q \geq \left( \frac{\alpha_i}{\sigma_n} \right)^{\frac{k}{p}} \prod_{i=1}^{k} \left( \frac{\alpha_i}{\sigma_n} \right)^{\frac{n}{p2i-1}} \right)^q \| u \|_{p^q_2, q'}^q \]

as desired (1.7).

It remains to check the sharpness of constant \( S(n, m, p) \). To do this, we modify the construction in the proof of Theorem 1.1 in [20] where the limiting case of the Sobolev inequalities in the hyperbolic space under the Lorentz–Sobolev norm was considered. More precisely, we construct a sequence of test functions as follows: denote

\[ C_{m,i} = \frac{1}{2i!} \prod_{k=0}^{i-1} \left( \frac{n}{p} - m + k \right), \quad 1 \leq 1 \leq m - 1, \]

and for \( j \geq 2 \), let us define

\[
v_j(x) = \begin{cases} 
\frac{1}{j^+} - \frac{1}{j^0}\sum_{i=1}^{m-1} C_{m,i}(1 - \frac{m}{2}|x|^2)^i & \text{if } 0 \leq |x| \leq j^{-\frac{1}{2}}, \\
|x|^{-\frac{1}{p} + m} & \text{if } j^{-\frac{1}{2}} < |x| \leq 1, \\
\xi(x) & \text{if } 1 < |x| \leq 2, 
\end{cases}
\]

where \( \xi \in C^\infty_0(2\mathbb{B}_n) \) are radial function chosen such that \( \xi = 1 \) on \( \partial\mathbb{B}^n \) and for \( i = 1, \ldots, m - 1 \)

\[
\frac{\partial^i \xi}{\partial r^i} \bigg|_{\partial\mathbb{B}^n} = (-1)^i \prod_{k=0}^{i-1} \left( \frac{n}{p} - m + k \right).
\]

The choice of \( C_{m,i} \) and \( \xi \) implies \( v_j \in C^{m-1}_0(2\mathbb{B}_n) \). For \( \epsilon \in (0, 1/3) \) let us define \( u_{\epsilon,j}(x) = v_j(x/\epsilon) \). It is not hard to see that \( u_{\epsilon,j} \in W^{m,p}(\mathbb{H}^n) \) and its support is contained in \( \{|x| \leq 2\epsilon\} \). By induction argument, we can show that

\[
\nabla_g^m = \left( \frac{1 - |x|^2}{2} \right)^m \left( \nabla^m + \sum_{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, 0 < \alpha_1 + \cdots + \alpha_n < m} P_{\alpha}(x) \partial^\alpha \right),
\]

where \( P_{\alpha} \in C^\infty(B_n), \partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \) and

\[
\nabla^m = \begin{cases} 
\Delta^m & \text{if } m \text{ is even}, \\
\nabla \Delta^{m-1/2} & \text{if } m \text{ is odd}.
\end{cases}
\]

Using this formula, it is easy to check that

\[
|\nabla_g^m u_{\epsilon,j}(x)| \leq \left( \frac{1 - |x|^2}{2} \right)^m C(\epsilon^{-1} j^{\frac{1}{2}})^m j^{\frac{1}{2} - \frac{m}{2}} \leq C2^{-m}(\epsilon^{-1} j^{\frac{1}{2}})^m j^{\frac{1}{2} - \frac{m}{2}}
\]

for \( |x| \leq \epsilon j^{-\frac{1}{2}} \), and

\[
|\nabla_g^m u_{\epsilon,j}(x)| \leq C \epsilon^{-m} \left( \frac{1 - |x|^2}{2} \right)^m \leq C2^{-m} \epsilon^{-m}
\]

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for $|x| \in (e, 2e)$ with a positive constant $C$ independent of $e < \frac{1}{3}$ and $j$. Furthermore, we can check that

$$|\nabla^m u_{e,j}(x)| \leq \left( \frac{1 - |x|^2}{2} \right)^m e^{\frac{m}{p}} \left( \sigma_n^{-\frac{m}{p}} S(n, m, p) |x|^{\frac{n}{p}} + C |x|^{-\frac{n}{p} + 1} \right)$$

$$\leq 2^m e^{\frac{m}{p}} \left( \sigma_n^{-\frac{m}{p}} S(n, m, p) |x|^{\frac{n}{p}} + C |x|^{-m+1} \right)$$

and

$$|\nabla^m u_{e,j}(x)| \geq \left( \frac{1 - |x|^2}{2} \right)^m e^{\frac{m}{p}} \left( \sigma_n^{-\frac{m}{p}} S(n, m, p) |x|^{\frac{n}{p}} - C |x|^{-\frac{n}{p} + 1} \right)$$

$$\geq \left( \frac{1 - e^2}{2} \right)^m e^{\frac{m}{p}} \left( \sigma_n^{-\frac{m}{p}} S(n, m, p) |x|^{\frac{n}{p}} - C |x|^{-\frac{n}{p} + 1} \right)$$

for $|x| \in (e^{j^{-\frac{1}{2}}}, e)$ with $e > 0$ small enough where $C$ is a positive constant independent of $e$ and $j$, here we use

$$|\nabla^m (|x|^{-\frac{n}{p} + m})| = \sigma_n^{-\frac{m}{p}} S(n, m, p) |x|^{-\frac{n}{p}}.$$

We next define

$$h_1(x) = \begin{cases} 
C2^{-m}(e^{-1} j^\frac{1}{2})^m j^\frac{1}{2} n \frac{m}{p} & \text{if } |x| \leq e^{j^{-\frac{1}{2}}} \\
2^{-m} e^{\frac{m}{p}} \left( \sigma_n^{-\frac{m}{p}} S(n, m, p) |x|^{\frac{n}{p}} + C |x|^{-\frac{n}{p} + 1} \right) & \text{if } |x| \in (e^{j^{-\frac{1}{2}}}, e) \\
2^{-m} e^{\frac{m}{p}} & \text{if } |x| \in (e, 2e) \\
0 & \text{if } |x| \in (2e, 1), 
\end{cases}$$

Then, we have $0 \leq |\nabla^m u| \leq h_1$. Consequently, we get $0 \leq |\nabla^m u|^* \leq h_1^*$. Let us denote by $h_1^{*, e}$ the rearrangement function of $h_1$ with respect to Lebesgue measure. Since the support of $h_1$ is contained in $\{|x| \leq 2e\}$, then we can easy check that

$$h_1^*(t) \leq h_1^{*, e} \left( \frac{1 - 4e^2}{2} \right)^m t.$$

Consequently, we have

$$\|\nabla^m u_{e,j}\|^q_{p,q} \leq \left( \frac{2}{1 - 4e^2} \right)^{\frac{m}{p}} \int_0^\infty h_1^{*, e}(t)^q t^{\frac{2}{p} - 1} dt$$

Enlarging the constant $C$ (which is still independent of $e$ and $j$) if necessary, we can assume that

$$C2^{-m} e^{\frac{m}{p}} \geq h_1 \bigg|_{\{|x|=e\}} = 2^{-m} e^{\frac{m}{p}} \left( \sigma_n^{-\frac{m}{p}} S(n, m, p) + C e \right)$$

and

$$C2^{-m}(e^{-1} j^\frac{1}{2})^m j^\frac{1}{2} n \frac{m}{p} \geq h_1 \bigg|_{\{|x|=e^{j^{-\frac{1}{2}}}\}} = 2^{-m} e^{-1} j^\frac{1}{2} n \frac{m}{p} e^{\frac{m}{p}} \left( \sigma_n^{-\frac{m}{p}} S(n, m, p) + C e^{j^{-\frac{1}{2}}} \right)$$
for $\epsilon > 0$ small enough and for any $j \geq 2$. For $j$ large enough, we can choose $x_0$ with $\epsilon j^{-\frac{1}{2}} < |x_0| \leq \epsilon$ such that $C 2^{-m} e^{-m} = h_1(x_0)$. It is easy to see that $\epsilon e \leq |x_0| \leq C \epsilon$ for constant $C, c > 0$ independent of $\epsilon$ and $j$. We have

$$h_1(x) \leq g(x) := \begin{cases} 
    h_1(x) & \text{if } |x| \leq |x_0| \\
    C 2^{-m} e^{-m} & \text{if } |x| \in (|x_0|, 2\epsilon) \\
    0 & \text{if } |x| \geq 2\epsilon,
\end{cases}$$

which gives $h_*^{*, \epsilon} \leq g_*^{*, \epsilon}$. Notice that $g$ is non-increasing radially symmetric function in $\mathbb{R}^n$. Making the simple computations on $g$, we get

$$\int_0^\infty h_*^{*, \epsilon}(t)^q t^{\frac{q}{q-1}} \, dt \leq \int_0^\infty g_*^{*, \epsilon}(t)^q t^{\frac{q}{q-1}} \, dt \leq 2^{-mq} \epsilon^{-\frac{mp}{p-q}} \sigma_n^{\frac{mp}{q-p}} (n, m, p)^q (\ln j + C).$$

Therefore, we have

$$\|\nabla^m g u_{\epsilon,j}\|_{1, q}^q \leq \left( \frac{1}{1 - 4\epsilon^2} \right) \left( 2\epsilon \right)^{\frac{mq}{1 - 4\epsilon^2}} \sigma_n^{\frac{mp}{q-p}} (n, m, p)^q (\ln j + C). \tag{4.1}$$

Obviously, we have

$$|u_{\epsilon,j}(x)| \geq h_2(x) = \begin{cases} 
    j^{\frac{1}{2} - \frac{mp}{q}} + j^{\frac{1}{2} - \frac{mp}{q}} \sum_{|l|=1}^{m-1} C_{m,l} (1 - j^\frac{1}{2} |e^{-1} x|^2)^l & \text{if } 0 \leq |x| \leq \epsilon j^{-\frac{1}{2}}, \\
    \epsilon^{\frac{mp}{q-p}} |x|^{-\frac{mp}{q-p} + m} & \text{if } \epsilon j^{-\frac{1}{2}} < |x| \leq \epsilon, \\
    0 & \text{if } 1 < |x| < 2,
\end{cases}$$

which implies $u_*^{*, \epsilon} \geq h_*^{*, \epsilon}$. Since the support of $h_2$ is contained in $\{|x| \leq \epsilon\}$, then we can easy check that

$$h_*^{*, \epsilon}(t) \geq h_*^{*, \epsilon}(2^{-n} t).$$

Therefore, it holds

$$\|u_{\epsilon,j}\|_{p_*^{\epsilon}, q}^q \geq \int_0^\infty h_*^{*, \epsilon}(t)^q t^{\frac{q}{q-1}} \, dt \geq 2^{\frac{mq}{p-q}} \int_0^\infty h_*^{*, \epsilon}(t)^q t^{\frac{q}{q-1}} \, dt.$$ 

Note that $h_2$ is non-increasing radially symmetric function in $\mathbb{R}^n$. The direct computations with function $h_2$ imply

$$\|u_{\epsilon,j}\|_{p_*^{\epsilon}, q}^q \geq (2\epsilon)^{\frac{mq}{p-q}} \sigma_n^{\frac{mp}{q-p}} \int_{\sigma_n (\epsilon j^{-\frac{1}{2}})^m} \frac{dt}{t} = (2\epsilon)^{\frac{mq}{p-q}} \sigma_n^{\frac{mp}{q-p}} \ln j. \tag{4.2}$$

Combining (4.1) and (4.2), we obtain

$$\frac{\|\nabla^m g u_{\epsilon,j}\|_{p, q}^q - C(n, m, p)^q \|u_{\epsilon,j}\|_{p, q}^q}{\|u_{\epsilon,j}\|_{p_*^{\epsilon}, q}^q} \leq S(n, m, p)^q \left( \frac{1}{1 - 4\epsilon^2} \right)^{\frac{mq}{1 - 4\epsilon^2}} (1 + C(\ln j)^{-1}).$$

Letting $j \to \infty$ and then letting $\epsilon \to 0^+$ prove the sharpness of $S(n, m, p)$.
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