Rough flows

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Abstract. We introduce in this work a concept of rough driver that somehow provides a rough path-like analogue of an enriched object associated with time-dependent vector fields. We use the machinery of approximate flows to build the integration theory of rough drivers and prove well-posedness results for rough differential equations on flows and continuity of the solution flow as a function of the generating rough driver. We show that the theory of semimartingale stochastic flows developed in the 80’s and early 90’s fits nicely in this framework, and obtain as a consequence some new strong approximation results for general semimartingale flows and provide a fresh look at large deviation theorems for ‘Gaussian’ stochastic flows.

Introduction

An elementary construction recipe of flows was recently introduced in [Bai15] and used there to get back the core results of Lyons’ theory of rough differential equations [Lyo98, FV10] in a very short and elementary way. This work emphasizes the fact that it may be worth considering flows of maps as the primary objects from which the individual trajectories can be built, as opposed to the classical point of view that constructs a flow from an uncountable collection of individual trajectories. (Probabilists know how tricky it can be to deal with uncountably many null sets.) As well-recognized now, the main success of Lyons’ theory was to disentangle probability from pure dynamics in the study of stochastic differential equations by showing that the dynamics is a deterministic and continuous function of an enriched signal that is constructed from the noise in the equation by purely probabilistic means. This very clean picture led to new proofs and extensions of foundational results in the theory of stochastic differential equations, such as Stroock and Varadhan support theorem or the basics of Freidlin and Ventzel theory of large deviations for diffusions.

It was realized in the late 70’s that stochastic differential equations not only define individual trajectories, they also define flows of regular homeomorphisms, depending on the regularity of the vector fields involved in the dynamics [Elw78, Bis81, IW81, Kun81b]. This opened the door to the study of stochastic flows of maps for themselves [Har81, Bax80, LJ82, LJ85] and it did not take long time before Le Jan and Watanabe [LJW84] clarified definitely the situation by showing that, in a semimartingale setting, there is a one-to-one correspondence between flows of diffeomorphisms and time-varying stochastic velocity fields, under proper regularity conditions on the objects involved. We offer in the present work an embedding of the theory of semimartingale stochastic flows into the theory of rough flows similar to the embedding of the theory of stochastic differential equations into the theory of rough differential equations. While the acquainted reader will have noticed that the latter framework can be used to deal with Brownian flows by seeing them

1I.B. was partly supported by the ANR project "Retour Post-doctorant", no. 11-PDOC-0025; I.B. also thanks the U.B.O. for their hospitality, part of this work was written there.
as solution flows to some Banach space-valued rough differential equations driven by a Brownian rough path in some space of vector fields, such as done in [Der10, DD12], the situation is not so clear for more general random velocity fields and stochastic flows of maps. Our approach provides a simple setting for dealing with the general case.

It is based on the "approximate flow-to-flow" machinery introduced in [Bai15], which gives body to the following fact. To a 2-index family \((\mu_{ts})_{0 \leq s \leq t \leq T}\) of maps which falls short from being a flow, in a quantitative way, one can associate a unique flow \((\varphi_{ts})_{0 \leq s \leq t \leq T}\) close to \((\mu_{ts})_{0 \leq s \leq t \leq T}\); moreover the flow \(\varphi\) depends continuously on the approximate flow \(\mu\). The point about such a machinery is that approximate flows appear naturally in a number of situations as simplified descriptions of complicated dynamics. The model situation is given by a controlled ordinary differential equation

\[
\dot{x}_t = \sum_{i=1}^{\ell} V_i(x_t) \dot{h}^i_t,
\]

in \(\mathbb{R}^d\), driven by an \(\mathbb{R}^{\ell}\)-valued \(C^1\) control \(h\). The Euler scheme

\[
\mu_{ts}(x) = x + (\dot{h}^i - \dot{h}^i_s)V_i(x)
\]

defines, under proper regularity conditions on the vector fields, an approximate flow whose associated flow is the flow generated by equation (0.1). One step farther, if we are given a weak geometric Hölder \(p\)-rough path \(X\), with \(2 \leq p < 3\), and sufficiently regular vector fields \(F = (V_1, \ldots, V_\ell)\) on \(\mathbb{R}^d\), one can associate to the rough differential equation

\[
dx_t = F(x_t)X(dt),
\]

some maps \(\mu_{ts}\), defined, for each \(0 \leq s \leq t \leq T\), as the time 1 map of an ordinary differential equation involving \(X_{ts}\), the \(V_i\) and their brackets, that have the same Taylor expansion as the awaited Taylor expansion of a solution flow to equation (0.2). They happen to define an approximate flow whose associated flow is the solution flow to equation (0.2).

We shall use a similar approach here, tailor-made to deal later on with the setting of stochastic time-dependent velocity fields. We shall introduce for that purpose a notion of rough driver, that is an enriched version of a time-dependent vector field, that will be given by the additional datum of a time-dependent second order differential operator satisfying some algebraic and analytic conditions. A notion of solution to a differential equation driven by a rough driver will be given, in the line of what was done in [Bai15] for rough differential equations, and the approximate flow-to-flow machinery will be seen to lead to a clean and simple well-posedness result for such equations. As awaited from the above discussion, the main point of this result is that the Itô map, that associates to a rough driver the solution flow to its associated equation, is continuous. This continuity result is the key to deep results in the theory of stochastic flows.

We shall indeed prove that reasonable semimartingale velocity fields can be lifted to rough drivers under some mild boundedness and regularity conditions, and that the solution flow associated to the semimartingale rough driver coincides almost surely with the solution flow to the Kunita-type Stratonovich differential equation driven by the velocity field. As a consequence of the continuity of the Itô map, a Wong-Zakai theorem will be proved for a general class of semimartingale velocity fields.

The rough drivers introduced here are somewhat a dual version of similar objects that were introduced very recently in [BGI15] in the study of the well-posedness of a general
family of linear hyperbolic symmetric systems of equations driven by time-dependent vector fields that are only distributions in the time direction. The latter work deals with evolutions in function spaces and uses functional analytic tools in the setting of controlled paths to make a first step towards a general theory of rough PDEs, in the lines of the classical PDE approach based on duality, a priori estimates and compactness results. The present work does not overlap with the latter.

The setting of rough drivers and rough flows is presented in section 1, together with the approximate flow-to-flow machinery. This is the core of the deterministic machinery and everything that follows elaborates on this material, in a probabilistic setting. Some additional material on random rough drivers is in particular given in section 1.5, where we provide some new variations around the Kolmogorov regularity theorem needed along the way and some sufficient conditions for a process to be bounded, that may be of interest for themselves. We show in section 2 that reasonable semimartingale velocity fields can be lifted to rough drivers under appropriate mild boundedness and regularity assumptions, and prove that the theory of semimartingale stochastic flows of maps is naturally embedded in the theory of rough flows. As an illustration of the continuity of the Itô map, we prove in section 2.5 a Wong-Zakai theorem for stochastic flows of maps, and provide in section 3 a fresh look at large deviation theorems for Brownian stochastic flows.

Notations. We gather here for reference a number of notations that will be used throughout the text.

- We shall exclusively use the letter $E$ to denote a Banach space; we shall denote by $L(E)$ the set of continuous linear maps from $E$ to itself, and for $M \in L(E)$, we shall write $|M|$ for its operator norm. In this setting, differentiability and regularity notions are understood in the sense of Fréchet.
- For functions $x : [0,T] \to E$, we will use the notation $x_{ts} = x_t - x_s$ for increments.
- Whenever useful, vector fields are identified with the first order differential operator they define in a canonical way.
- As is common, we shall use Einstein’s summation convention that $a^i b_i := \sum_i a^i b_i$.

- Last, recall that a flow on $E$ is a family $(\varphi_{ts})_{0 \leq s \leq t \leq T}$ of maps from $E$ to itself such that $\varphi_{tt} = \text{Id}$, for all $0 \leq t \leq T$, and $\varphi_{tu} \circ \varphi_{us} = \varphi_{ts}$, for all $0 \leq s \leq u \leq t \leq T$. The letter $T$, here and below, will always stand for a finite time horizon.

\section{Rough flows}

\subsection{Flows and approximate flows}

We introduced in \cite{bai15} a simple machinery for constructing flows on $E$ from approximate flows that can be understood as a generalization of Lyons’ workhorse \cite{lyo98} for constructing a rough path from an almost rough path; this is the core tool for the construction of the rough integral. Roughly speaking, the "approximate flow to flow" machinery says that if we are given a family of maps $(\mu_{ts})_{0 \leq s \leq t \leq T}$ from $E$ to itself, and if the maps $\mu$ are close to defining a flow, in the sense that $\mu_{tu} \circ \mu_{us} - \mu_{ts}$ is small in a quantitative way, for $s \leq u \leq t$ with $t - s$ small, then there exists a unique flow close to $\mu$. In the rough paths setting, Lyons almost multiplicative functionals involve a family $a = (a_{ts})_{0 \leq s \leq t \leq T}$ of elements of a tensor algebra such that $a_{tu} a_{us} - a_{ts}$ has some given size whenever $s \leq u \leq t$ with $t - s$
small, with the product on the tensor space used here. Despite their similarity, Lyons’ setting differs from the present setting in that multiplication in a tensor algebra satisfies the distributivity property \( ab - ac = a(b - c) \), which obviously does not hold if \( a, b, c \) are maps and the product stands for composition. This seemingly minor point makes a real difference though, so it is fortunate that one can still get an analogue of Lyons’ theorem in a function space setting. This comes at a little price on the regularity of the set of maps \( \mu \) that one can consider. As usual, for \( 0 < r \leq 1 \), we denote by \( C^r \) the space of \( r \)-Hölder functions, with the understanding that they are Lipschitz continuous for \( r = 1 \).

**Definition.** Let \( 0 < r \leq 1 \) be given. A \( C^r \)-approximate flow on \( E \) is a family \( (\mu_{ts})_{0 \leq s \leq t \leq T} \) of \((1 + \rho)\)-Lipschitz maps from \( E \) to itself, for some \( 0 < \rho \leq 1 \), depending continuously on \((s, t)\) in the topology of uniform convergence and enjoying the following two properties.

- **Perturbation of the identity** – There exists a constant \( \alpha \) with
  \[
  0 < 1 - \rho < \alpha < 1,
  \]
  such that one has for any \( 0 \leq s \leq t \leq T \), and any \( x \in E \), the decomposition
  \[
  D_x \mu_{ts} = \text{Id} + A^ts_x + B^ts_x,
  \]
  for some \( L(E) \)-valued \( \rho \)-Lipschitz maps \( A^ts_x \) with \( \rho \)-Lipschitz norm bounded above by \( c|t - s|^{\alpha} \), and some \( L(E) \)-valued \( C^1 \) bounded maps \( B^ts_x \), with \( C^1 \)-norm bounded above by \( o_{t-s}(1) \).

- **\( C^r \)-approximate flow property** – There exists some positive constants \( c_1 \) and \( a > 1 \), such that one has
  \[
  \|\mu_{ts} \circ \mu_{us} - \mu_{ts}\|_{C^r} \leq c_1 |t - s|^{a}
  \]
  for all \( 0 \leq s \leq u \leq t \leq T \).

Given a partition \( \pi_{ts} = \{s = s_0 < s_1 < \cdots < s_{n-1} < s_n = t\} \) of \((s, t) \subset [0, T] \), set
\[
\mu_{\pi_{ts}} = \mu_{s_n s_{n-1}} \circ \cdots \circ \mu_{s_1 s_0}.
\]

**Theorem 1** ([Ba15] Constructing flows on \( E \)). A \( C^r \)-approximate flow, with \( \frac{1}{a} < r \), defines a unique flow \( (\varphi_{ts})_{0 \leq s \leq t \leq T} \) on \( E \) to which one can associate a positive constant \( \delta \) such that the inequality
\[
\|\varphi_{ts} - \mu_{ts}\|_\infty \leq |t - s|^{a}
\]
holds for all \( 0 \leq s \leq t \leq T \) with \( t - s \leq \delta \); this flow satisfies the inequality
\[
\|\varphi_{ts} - \mu_{\pi_{ts}}\|_\infty \leq 2 c_1 T |\pi_{ts}|^{ar-1}
\]
for any partition \( \pi_{ts} \) of any interval \((s, t) \subset [0, T] \), with mesh \( |\pi_{ts}| \leq \delta \). Moreover, the \( C^r \) norm of the maps \( \varphi_{ts} \) is uniformly bounded by a function of the constant \( c_1 \) that appears in (1.2), for all \( 0 \leq s \leq t \leq T \).

(This theorem is stated in [Ba15] for \( C^1 \)-approximate flows; the proof given there works verbatim for \( C^r \)-approximate flows provided \( \frac{1}{a} < r \); a \( C^1 \) map is then understood in that setting as a Lipschitz map.) The crucial point in the above statement is the fact that if \( \mu \) depends continuously in \( C^r \) on some parameter then \( \varphi \) also happens to depend continuously on that parameter, in \( C^0 \), as a direct consequence of estimate (1.3). As made clear in [Ba15], theorem 1 can be seen as the cornerstone of the theory of rough differential equations, with the continuity of the Itô-Lyons solution map given as a consequence of the aforementioned continuity of \( \varphi \) on a parameter.
We shall see in the present work that theorem 1 is all we need to get back and extend the core results of the theory of stochastic flows intensively developed in the 80’s and early 90’s. We shall need for that purpose to introduce a notion of enriched velocity field that will somehow play the role in our setting of weak geometric Hölder $p$-rough paths, with $2 \leq p < 3$, in rough paths theory.

We shall thus pick a regularity exponent $2 \leq p < 3$ here, once and for all. Let us insist here that like for the theory of rough paths, the technical shape of the theory of rough drivers depends on that regularity exponent. Only two objects are needed in the definition of a rough driver when $2 \leq p < 3$; for $3 \leq p < 4$, we would need to introduce an additional object in the definition of a rough driver, that would thus consist of three objects; and so on. There is no real difficulty other than notational in giving a general theory, but as our applications of rough flows to the study of stochastic flows only require to develop the theory in the case where $2 \leq p < 3$, we stick to that setting and invite the reader to make up herself/himself her/his mind about what the general theory look like.

1.2. Rough drivers

Let $(V(\cdot, t))_{0 \leq t \leq T}$ be a time dependent vector field on $E$, with time increments

$$V_{ts}(\cdot) := V(\cdot, t) - V(\cdot, s).$$

To get a hand on the definition of a weak geometric $p$-rough driver given below, think of $V_{ts}$ as given by the formula

$$V_{ts} = VX_{ts},$$

where $V(x) \in \text{L}(\mathbb{R}^\ell, \mathbb{R}^d)$, for all $x \in \mathbb{R}^d$, and $X = (X, \bar{X})$ is a $p$-rough path over $\mathbb{R}^\ell$. Write $V_i$ for the image by $V$ of the $i$th vector in the canonical basis of $\mathbb{R}^\ell$. A solution path $x_{\bullet}$ to the rough differential equation

$$dx_t = V(x_t)X(dt)$$

can be characterized as a path satisfying some uniform Euler-Taylor expansion of the form

$$f(x_t) = f(x_s) + X_{ts}^i(V_tf)(x_s) + X_{ts}^{jk}(V_jV_kf)(x_s) + O\left(|t - s|^\frac{3}{p}\right)$$

for all sufficiently regular real-valued functions $f$ on $\mathbb{R}^d$. The present section will make it clear that the operators $X_{ts}^iV_i = VX_{ts}$ and $X_{ts}^{jk}V_jV_k = (DV)VX_{ts}$ are all we need in this formula to run the theory, with no need to separate their space part, given by $V$ and $(DV)V$, from their time part $X_{ts}$.

Definition. Let $2 \leq p < 3$, and $p - 2 < \rho \leq 1$ be given. A weak geometric $(p, \rho)$-rough driver is a family $(V_{ts})_{0 \leq s \leq t \leq T}$, with

$$V_{ts} := (V_{ts}, \bar{V}_{ts}),$$

and $\bar{V}_{ts}$ a second order differential operator, such that

(i) the vector fields $V_{ts}$ are additive as functions of time

$$V_{ts} = V_{tu} + V_{us}$$

for all $s < u < t$, and each $V_{ts}$ is of class $C^{2+\rho}$ on $\mathbb{R}^d$, with

$$\sup_{0 \leq s < t \leq T} \frac{\|V_{ts}\|_{C^{2+\rho}}}{|t - s|^\frac{3}{p}} < \infty,$$
(ii) the second order differential operators

\[ W_{ts} := V_{ts} - \frac{1}{2} V_{ts} V_{ts}, \]

are actually a vector fields, and

\[ \sup_{0 \leq s < t \leq T} \left\| W_{ts} \right\|_{C^{1+\rho}} < \infty, \]

(iii) we have

\[ V_{ts} = V_{tu} + V_{us} V_{tu} + V_{us}, \]

for any \( 0 \leq s \leq u \leq t \leq T \).

With in mind the model weak geometric \( p \)-rough driver given by formula (1.4), the requirement \( p - 2 < \rho \) appears as a natural assumption to impose, given known well-posedness results on rough differential equations [Dav07]; the first order condition on the operators \( W_{ts} \) justifies that we call \( V \) a \textit{weak geometric} \( p \)-rough driver, and condition (iii) stands for an analogue of Chen’s relation. We shall freely talk about rough drivers rather than weak geometric \((p, \rho)\)-rough drivers in the sequel.

**Definition.** We define the (pseudo-)norm of \( V \) to be

\[ \| V \|_{p, \rho} := \sup_{0 \leq s < t \leq T} \left\{ \left\| V_{ts} \right\|_{C^{2+\rho}} \sqrt[p]{1 + \frac{\rho}{2}} \right\}. \]

and define an associated (pseudo-)metric on the set \( D_{p, \rho} \) of weak geometric \((p, \rho)\)-rough drivers

\[ d_{p, \rho}(V, V') := \| V - V' \|. \]

Like the space of rough paths the space of rough drivers is not a linear space. We will also need the homogeneous metric

\[ d_{p, \rho}(V, V') := \sup_{0 \leq s < t \leq T} \left\{ \left\| \frac{V_{ts} - V'_{ts}}{t - s} \right\|_{C^{2+\rho}} \sqrt[p]{1 + \frac{\rho}{2}} \right\}. \]

We will often drop the subindices \( p, \rho \) when it is clear from the context in which space we are working in. Note that \( d \) and \( d \) induce the same topology in the space of rough drivers.

Note that one should add the \( C^{2+\rho} \)-norm of \( V_0 \) in formula (1.5) to define a proper norm on the space of rough drivers. This has no consequences as rough drivers are only used via their increments. Note also that given a rough driver \( V \) and \( 0 < a \leq T \), one defines another rough driver \( V^a = (V^a, V^a) \), on the time interval \([0, a]\), setting

\[ V^a_{ts} = V_{a-s, a-t}, \]

\[ V^a_{ts} = -V_{a-s, a-t} + V_{a-s, a-t} V_{a-s, a-t}, \]

for all \( 0 \leq s \leq t \leq a \). It is indeed elementary to check that these operators satisfy the algebraic conditions (ii) and (iii), with

\[ W^a_{ts} := V^a_{ts} - \frac{1}{2} V^a_{ts} V^a_{ts} = -W_{a-s, a-t}; \]

that they satisfy the above analytic requirements is obvious. This rough driver is called the \textit{time reversal of the rough driver} \( V \), \textit{from time} \( a \). Note that \( \| V^a \| = \| V \|\).
1.3. Rough flows  We shall adopt below a definition of a solution flow to the equation

\[ d\varphi = V(\varphi; dt) \]

similar to the above definition of a solution path to a rough differential equation. A solution flow will be required to satisfy some uniform Euler-Taylor expansion of the form

\[ f \circ \varphi_{ts} - \{ f + V_{ts}f + \nabla_{ts}f \} = O\left(|t - s|^{\frac{3}{p}}\right), \]

for all sufficiently regular real-valued functions \( f \) on \( \mathbb{R}^d \). It is actually elementary to construct a family of maps \( (\mu_{ts})_{0 \leq s \leq t \leq T} \) which enjoys the above Euler-Taylor expansion property. The key point is that this family will turn out to be a \( C^0 \)-approximate flow, if \( \rho \) is not too small, so we shall get the existence and uniqueness of a solution flow from its very definition and theorem \( \Box \).

Given \( 0 \leq s \leq t \leq T \), consider the ordinary differential equation

\[ \dot{y}_u = V_{ts}(y_u) + W_{ts}(y_u), \quad 0 \leq u \leq 1, \]

and denote by \( \mu_{ts} \) its well-defined time 1 map, associating to any \( x \in \mathbb{E} \) the value at time 1 of the solution of equation (1.10) started from \( x \). Elementary results on ordinary differential equations imply that if one considers \( y_u \) as a function of \( x \), for \( 0 \leq u \leq 1 \), then we have

\[ \| y_u - \text{Id} \|_{C^1} \leq c \| V \| |t - s|^{\frac{3}{p}}, \]

for some universal positive constant \( c \). Proposition \( \Box \) below shows that the maps \( \mu_{ts} \) have the awaited Euler-Taylor expansion expected from a solution flow to equation (1.9).

**Proposition 2.** We have

\[ \left\| f \circ \mu_{ts} - \{ f + V_{ts}f + \nabla_{ts}f \} \right\|_{\infty} \leq c \| f \|_{C^{2+\rho}} |t - s|^{\frac{3}{p}}, \]

for any \( f \in C^{2+\rho} \) and any \( 0 \leq s \leq t \leq T \).

The proof of this statement is straightforward and relies on the the following formula. For all \( x \in \mathbb{E} \) and all \( f \in C^2 \), we have

\[ f(\mu_{ts}(x)) = f(x) + \int_0^1 (V_{ts}f)(y_u) \, du + \int_0^1 (W_{ts}f)(y_u) \, du \]

\[ = f(x) + (V_{ts}f)(x) + (\nabla_{ts}f)(x) + \epsilon_{ts}^f(x) \]

where

\[ \epsilon_{ts}^f(x) := \int_0^1 \int_0^u \left\{ (V_{ts}V_{ts}f)(y_r) - (V_{ts}V_{ts}f)(x) \right\} \, dr \, du + \int_0^1 \int_0^u (W_{ts}V_{ts}f)(y_r) \, dr \, du \]

\[ + \int_0^1 \left\{ (W_{ts}f)(y_u) - (W_{ts}f)(x) \right\} \, du. \]

The inequality

\[ \left\| \epsilon_{ts}^f \right\|_{C^{\rho}} \leq c \left( 1 + \| V \|^3 \right) \| f \|_{C^{2+\rho}} |t - s|^{\frac{3}{p}}, \]

justifies proposition \( \Box \).

**Theorem 3.** The family of maps \( (\mu_{ts})_{0 \leq s \leq t \leq T} \) is a \( C^0 \)-approximate flow which depends continuously on \( ((s, t), V) \) in \( C^0 \) topology.
Proof — The family \((\mu_{ts})_{0 \leq s \leq t \leq T}\) satisfies the regularity assumptions \((\text{1.11})\) as a direct consequence of classical results on the dependence of solutions to ordinary differential equations with respect to parameters, including the initial condition for the equation. These results also imply the continuous dependence of \(\mu_{ts}\) on \((s, t, V)\) in \(C^0\) topology. To show that the family \(\mu\) defines a \(C^0\)-approximate flow, write, for \(0 \leq s \leq u \leq t \leq T\),

\[
\mu_{tu}(\mu_{us}(x)) = \mu_{us}(x) + V_{tu}(\mu_{us}(x)) + (V_{tu}\text{Id})(\mu_{us}(x)) + \epsilon_{tu}^1(\mu_{us}(x)) \\
= x + V_{us}(x) + (V_{us}\text{Id})(x) + \epsilon_{us}^1(x) \\
+ V_{tu}(x) + (V_{us}V_{tu})(x) + (V_{us}V_{tu})(x) + \epsilon_{Vtu}(x) \\
+ (V_{tu}\text{Id})(x) + \left( (V_{tu}\text{Id})(\mu_{us}(x)) - (V_{tu}\text{Id})(x) \right) + \epsilon_{tu}^1(\mu_{us}(x)) \\
= \mu_{ts}(x) + \left\{ (V_{us}V_{tu})(x) + \left( (V_{tu}\text{Id})(\mu_{us}(x)) - (V_{tu}\text{Id})(x) \right) + \epsilon_{Vtu}(x) \\
+ \epsilon_{us}^1(x) + \epsilon_{tu}^1(\mu_{us}(x)) \right\}.
\]

The approximate flow property then follows from the regularity assumptions on \(V_{ts}\) and \(V_{ts}\), and estimate \((\text{1.13})\).  

\[\Box\]

With the notations used in the definition of an approximate flow, the exponent \(a\) that appears here in the approximate flow identity \((\text{1.2})\) is \(a = \frac{3}{p}\).

Definition 4. A flow \((\varphi_{ts})_{0 \leq s \leq t \leq T}\) is said to solve the rough differential equation

\[
d\varphi = V(\varphi; dt)
\]

if there exists a possibly \(V\)-dependent positive constant \(\delta\) such that the inequality

\[
\|\varphi_{ts} - \mu_{ts}\|_\infty \leq |t - s|^\frac{3}{p}
\]

holds for all \(0 \leq s \leq t \leq T\) with \(t - s \leq \delta\). Flows solving a differential equation of the form \((\text{1.14})\) are called rough flows. If equation \((\text{1.14})\) is well-posed, the map which associates to a rough driver \(V\) the solution flow to equation \((\text{1.14})\) is called the Itô map.

The following well-posedness result comes as a direct consequence of Theorems \(\text{1}\) and \(\text{3}\). A family of maps is said to be uniformly \(C^0\) is it has uniformly bounded \(C^0\)-norm.

Theorem 5. Assume \(\rho > \frac{2}{3}\). Then the differential equation on flows

\[
d\varphi = V(\varphi; dt)
\]

has a unique solution flow; it takes values in the space of uniformly \(C^0\) homeomorphisms of \(E\), with uniformly \(C^0\) inverses, and depends continuously on \(V\) in the topology of uniform convergence.

Proof — Note that any solution flow to the rough differential equation \((\text{1.14})\) depends, by definition, continuously on \((s, t)\) in the topology of uniform convergence on \(E\) since \(\mu_{ts}\) does. It follows from the proof of theorem \(\text{3}\) that one can choose as a constant \(c_1\) in inequality \((\text{1.2})\) a multiple of \(1 + \|V\|^4\), so we have from theorem \(\text{1}\) the estimate

\[
\|\varphi_{ts} - \mu_{\pi_{ts}}\|_\infty \leq c \left( 1 + \|V\|^4 \right) T |\pi_{ts}|^{\rho \frac{2}{p} - 1},
\]

for any partition \(\pi_{ts}\) of \((s, t) \subset [0, T]\) with mesh \(|\pi_{ts}|\) small enough, say no greater than \(\delta\). Note that the exponent \(\rho \frac{3}{p} - 1\) is positive. As these bounds are uniform in
and for $V$ in a bounded set of the space of rough drivers, and each $\mu_{s,t}$ is a continuous function of $V$, the flow $\varphi$ depends continuously on $(s,t)$. To prove that $\varphi$ is a homeomorphism, note that it follows from \cite{L} that, for $0 \leq a \leq b \leq t$, each $\mu_{a,b}$ is a diffeomorphism with inverse given by the time one map $\mu_{t-a,t-b}^t$ of the ordinary differential equation

$$y_u = -V_{ba}(y_u) - W_{ba}(y_u) = V^t_{t-a,t-b}(y_u) + W^t_{t-a,t-b}(y_u), \quad 0 \leq u \leq 1,$$

associated with the time reversed rough driver $V^t$. As $\mu^t$ has the same properties as $\mu$, the maps

$$(\mu_{s,t})^{-1} = \mu_{s,1}^{-1} \circ \cdots \circ \mu_{s_{n-1},s_n}^{-1} = \mu_{s_{n-1},s_n}^{-1} \circ \cdots \circ \mu_{s_1,s_0}^{-1}$$

converge uniformly to some continuous map $\varphi_{ts}^{-1}$, as the mesh of the partition $\pi_{ts}$ tends to 0; these limit maps $\varphi_{ts}^{-1}$ satisfy by construction $\varphi_{ts} \circ \varphi_{ts}^{-1} = \text{Id}$.

As theorem \cite{BC15} provides a uniform control of the $C^p$ norm of the maps $\varphi_{ts}$, the same control holds for their inverses since $\|V^t\| = \|V\|$. We propagate this control from the set $\{(s,t) \in [0,T]^2; s \leq t, t-s \leq \delta\}$ to the whole set $\{(s,t) \in [0,T]^2; s \leq t\}$ using the flow property of $\varphi$.

Note that the solution flow to the rough differential equation

$$d\psi = V^T(\psi; dt),$$

driven by the time reversal of the rough driver $V$, from time $T$, provides the inverse flow of $\varphi$, in the sense that

$$\varphi_{ts}^{-1} = \psi_{T-s,T-t},$$

for all $0 \leq s \leq t \leq T$. Last, note that it is elementary to adapt the above results to add a globally Lipschitz drift in the dynamics; the above results hold in that setting as well.

**Remarks.**

1. The continuity of the solution map for differential equations on flows will be used in a crucial way in the next sections to investigate in depth the theory of semimartingale stochastic flows of maps. Rough flows will also be used in the coming work \cite{BC15} on homogenization in fast-slow dynamics, with the continuity of the Itô map playing a fundamental role.

2. The results presented in that section can somehow be seen as a ‘rough’ version of similar results obtained recently by Catellier, Chouk and Gubinelli \cite{CG14, CG15} on one side, and Hu and Le \cite{HL} on the other side, that deal with the construction of dynamics associated with a Young-type analogue of rough drivers, for which the velocity field $V_t$ is a $\frac{1}{p}$-Hölder continuous function of time, with $1 \leq p < 2$, and for which there is no need to introduce a second order object. The method of approximate flows used here gives back their results whenever both setting apply.

3. It is straightforward to use the remark after theorem \cite{BC15} on regularity results for flows associated with approximate flows, to see that if $V$ is a $\frac{1}{p}$-Lipschitz, with values in $C^{2+k_0,p}_b$, and $W_{ts}$ is a $\frac{2}{p}$-Lipschitz, with values in $C^{1+k_0,p}_b$, then the solution flow $\varphi$ to equation \cite{L} is a flow of $C^{k_0}$-diffeomorphisms.

4. It should be clear to the reader that the above results hold in the setting of a compact manifold. They extend in a straightforward way to any smooth bundle over such a manifold if the former is equipped with a smooth connection.
1.4. An Itô formula for rough flows  With a view to identifying stochastic and rough flows in section 2.4, we prove here an elementary Itô formula that does not seem to have been largely emphasized in the literature. As a matter of fact, theorem [4] below states that any \( \frac{1}{p} \)-Hölder path in a Banach space satisfies an Itô formula, outside the setting of rough or controlled paths. To state and prove it, we need to recall Feyel and de la Pradelle sewing lemma [FdLP06], that can be seen as a precursor of the construction theorem for flows, theorem [4] and is actually an additive version of this non-commutative result. We denote here by \( E \) a Banach space. Given a partition \( \pi_{ts} = \{ s = s_0 < s_1 < \cdots < s_{n-1} < s_n = t \} \) of an interval \([s, t]\), and an \( E \)-valued 2-index map \( z = (z_{is})_{0 \leq i \leq s \leq t} \), set

\[
\pi_{is} := z_{is} := z_{is} + \cdots + z_{is_{i+1}},
\]

**Theorem 6** ([FdLP06]). Let \((z_{is})_{0 \leq i \leq s \leq t} \) be an \( E \)-valued 2-index continuous map to which one can associate some positive constants \( c_1 \) and \( a > 1 \) such that

\[
\left| (z_{tu} + z_{us}) - z_{ts} \right| \leq c_1 |t - s|^a
\]

holds for all \( 0 \leq s \leq u \leq t \leq T \). Then there exists a unique continuous function \( Z : [0, T] \rightarrow \mathbb{R} \), with increments \( Z_{ts} := Z_t - Z_s \), to which one can associate a positive constant \( \delta \) such that the inequality

\[
|Z_{ts} - z_{ts}| \leq |t - s|^{a-1}
\]

holds for all \( 0 \leq s \leq u \leq t \leq T \), with \( t - s \leq \delta \); this map \( Z \) satisfies the inequality

\[
|Z_{ts} - z_{\pi_{is}}| \leq 2c_1 T |\pi_{is}|^{a-1}
\]

for any partition \( \pi_{is} \) of any interval \([s, t] \subset [0, T]\), with mesh \(|\pi_{is}| \leq \delta \). It follows in particular that \( Z \) depends continuously on any parameter in uniform topology if \( z \) does.

A map \( z \) satisfying condition (1.16) is said to be **almost-additive**, and we write

\[
Z_{ts} := \int_s^t z_{du}.
\]

We equip the tensor product space \( E \otimes E \) with a compatible tensor norm that makes the natural embedding \( L(E, L(E, \mathbb{R})) \subset L(E \otimes E, \mathbb{R}) \) continuous. Given such an choice, one can identify the second differential of a \( C^2 \) real-valued function on \( E \) to an element of \( L(E \otimes E, \mathbb{R}) \) that is symmetric; this is what we do below.

**Theorem 7** (Itô formula). Let \( F : [0, T] \times E \rightarrow \mathbb{R} \) be a \( C^1 \)-function of time with bounded derivative, uniformly in space, which is also of class \( C^3 \) in the sense of Fréchet as a function of its \( E \)-component, with bounded derivatives \( F^{(1)}, F^{(2)}, F^{(3)} \), uniformly in time. Let \((x_t)_{0 \leq s \leq t \leq T} \) be \( \frac{1}{p} \) Hôlder \( E \)-valued map. Then the continuous 2-index map

\[
z_{ts} := F^{(1)}(x_t - x_s) + \frac{1}{2} F^{(2)}(x_t - x_s) \otimes 2
\]

is almost-additive, and we have

\[
F(t, x_t) = F(s, x_s) + \int_s^t (\partial_r F)(r, x_r) dr + \int_s^t z_{du},
\]

for any \( 0 \leq s \leq t \leq T \).
Proof – The proof is a straightforward application of Feyel-de la Pradelle’s sewing lemma, theorem [1]. Given $0 \leq s \leq u \leq u \leq t \leq T$, the algebraic identity

$$z_{tu} + z_{us} = F^{(1)}_{x_j} (x_t - x_s) + \left( F^{(1)}_{x_u} - F^{(1)}_{x_s} \right) (x_t - x_u) + \frac{1}{2} F^{(2)}_{x_u} (x_t - x_u) \otimes (x_t - x_u) + \frac{1}{2} F^{(2)}_{x_s} (x_u - x_s) \otimes (x_u - x_s) \otimes (x_u - x_s),$$

together with the $C^3_b$ character of $F$ and the symmetric character of $F^{(2)}_x$, for any $x \in E$, give

$$z_{tu} + z_{us} = F^{(1)}_{x_j} (x_t - x_s) + F^{(2)}_{x_u} (x_u - x_s) \otimes (x_t - x_u) + O\left( \|x_u - x_s\|^2 \|x_t - x_u\| \right) + \frac{1}{2} F^{(2)}_{x_u} (x_t - x_u) \otimes (x_u - x_s) \otimes (x_u - x_s) \otimes (x_u - x_s) \otimes (x_u - x_s),$$

$$= z_{ts} + O\left( |t - s|^{3/2} \right).$$

Itô’s formula (1.17) follows by noting that we have for all $n \geq 1$

$$F(t, x_t) = \sum_{i=0}^{n-1} \left\{ F(s_{i+1}, x_{s_{i+1}}) - F(s_i, x_{s_i}) \right\}$$

$$= o_n(1) + \sum_{i=0}^{n-1} (s_{i+1} - s_i) (\partial_s F)(s_i, x_{s_{i+1}}) + \sum_{i=0}^{n-1} \left\{ F(s_i, x_{s_i+1}) - F(s_i, x_{s_i}) \right\},$$

with

$$F(s_i, x_{s_i+1}) - F(s_i, x_{s_i}) = F^{(1)}_{x_i} (x_{s_i+1} - x_{s_i}) + \frac{1}{2} F^{(2)}_{x_i} (x_{s_i+1} - x_{s_i}) \otimes (x_{s_i+1} - x_{s_i}) + O\left( \|x_{s_i+1} - x_{s_i}\|^3 \right)$$

$$= z_{s_{i+1}} + O\left( |s_{i+1} - s_i|^{3/2} \right).$$

As an example, consider the solution flow $\varphi$ on $\mathbb{R}^d$ to a rough differential equation

$$d\varphi = V(\varphi; dt).$$

Write $\varphi_t$ for $\varphi_{t_0}$, and consider it as an element of the space $E$ of continuous paths from $[0, T]$ to $C(\mathbb{R}^d, \mathbb{R}^d)$, equipped with the norm of uniform convergence, with $C(\mathbb{R}^d, \mathbb{R}^d)$ endowed with a norm inducing uniform convergence on compact sets. It satisfies by its very definition and proposition [2] the Euler-Taylor expansion

$$\varphi_t = \varphi_s + (V_{ts} \text{Id}) \circ \varphi_s + (V_{ts} \text{Id}) \circ \varphi_s + O\left( |t - s|^{3/2} \right)$$

so it is a $\frac{1}{p}$-Hölder path in that space. Now, given some points $y_1, \ldots, y_k$ in $\mathbb{R}^d$ and a $C^3_b$ real-valued function $f$ on $(\mathbb{R}^d)^k$, one can think of the function

(1.18)

$$F(f) = f(\varphi(y_1), \ldots, \varphi(y_k)), $$

for $\varphi \in E$, as a typical time-independent example of function satisfying the conditions of theorem [3]. One then has

$$F(\varphi_{s_{i+1}}) - F(\varphi_{s_i}) = f(\varphi_{s_{i+1}}(\varphi_{s_i}(y_1)), \ldots, \varphi_{s_{i+1}}(\varphi_{s_i}(y_k))) - f(\varphi_{s_i}(y_1), \ldots, \varphi_{s_i}(y_k))$$

$$= \sum_{m=1}^k \left( (V_{s_{i+1}}^{(m)} + V_{s_{i+1}}^{(m)} \text{Id}) (\varphi_{s_i}(y_1), \ldots, \varphi_{s_i}(y_k)) + O\left( |s_{i+1} - s_i|^{3/2} \right) \right),$$

$$= \sum_{m=1}^k \left( (V_{s_{i+1}}^{(m)} + V_{s_{i+1}}^{(m)} \text{Id}) (\varphi_{s_i}(y_1), \ldots, \varphi_{s_i}(y_k)) + O\left( |s_{i+1} - s_i|^{3/2} \right) \right).$$
where the upper index \( \{m\} \) means that the operators act on the \( m^{th} \) component of \( f \). The above sum defines an almost-additive continuous function \( z_{ts}^f \), taken here at time \( (s_{i+1}, s_i) \), so we have

\[
f(\varphi_t(y_1), \ldots, \varphi_t(y_k)) = f(\varphi_s(y_1), \ldots, \varphi_s(y_k)) + \int_s^t z_{du}^f
\]

for all times \( 0 \leq s \leq t \leq T \).

### 1.5. A Kolmogorov-type regularity theorem

We shall use below the theory of rough drivers in a setting where the drivers are random. Like in the theory of rough paths, the primary object we are given is not the random rough driver itself, or the random rough path, but rather a genuine random vector field, or random path, which needs to be enhanced in a first step into a random rough driver, or random rough path. This first, purely probabilistic, step can typically be done using some Kolmogorov-type continuity arguments. We give in this section some variations on this theme that will be used to enhance vector field-valued martingales into rough drivers in section 2.3; a reader interested only in these applications is advised to skip the technical details and only have a look at Theorem 12; for the other readers, we hope this section contains some material interesting in itself.

The next Lemma gives sufficient conditions for a process defined on a possibly unbounded domain to be bounded.

**Lemma 8.** Let \((E, d)\) be a complete, separable metric space. Let \( D \) be an open subset of \( \mathbb{R}^d \), \( X: D \to (E, d) \) a continuous stochastic process, \( e \in E \) and \( \kappa > 0 \). Set

\[
D_n := \{ x \in D : n-1 \leq |x| < n \}
\]

and \( N := \{ n \in \mathbb{N} : D_n \neq \emptyset \} \). Let \((a_n)_{n \in \mathbb{N}}\) be a sequence of non-negative real numbers and \((x_n)_{n \in \mathbb{N}}\) a sequence of elements in \( D \) such that \( x_n \in D_n \) for every \( n \in \mathbb{N} \).

(i) Assume that there is a \( q > 1 \) and a \( \gamma \in (\frac{1}{q}, 1]\) such that

\[
\sup_{x,y \in D_n} \| d(X(x), X(y)) \|_{L^q} \leq \kappa a_n |x - y|^{\gamma}
\]

and that

\[
\| d(X(x_n), e) \|_{L^q} \leq \kappa a_n
\]

for every \( n \in \mathbb{N} \). Set \((b_n) := (a_n n^{\gamma})\) and assume that \( \| b \|_{N} \leq K < \infty \). Then for every \( q' \in [1, q) \) there is a constant \( C = C(q, q', \gamma, K) \) such that

\[
\sup_{x \in D} \| d(X(x), e) \|_{L^{q'}} \leq C K.
\]

(ii) Let \( \gamma \in (0, 1] \) and assume that for every \( q \geq 1 \) there is a \( c_q \) such that for every \( n \in \mathbb{N} \),

\[
\sup_{x,y \in D_n} \| d(X(x), X(y)) \|_{L^q} \leq \kappa c_q a_n |x - y|^{\gamma}
\]

and that

\[
\| d(X(x_n), e) \|_{L^q} \leq \kappa c_q a_n
\]
where $c_q = O(\sqrt{q})$ when $q \to \infty$. Assume that $a_n = O\left(n^{-\gamma}(1 + \log(n))^{-\frac{1}{2}}\right)$. Then for every $q \geq 1$ there is some constant $C = C(q, \gamma)$ such that

$$\left\| \sup_{x \in D} d(X(x), e) \right\|_{L^q} \leq C\kappa$$

with $C = O(\sqrt{q})$ when $q \to \infty$. In particular, the random variable $\sup_{x \in D} d(X(x), e)$ has Gaussian tails.

**Proof** – Without loss of generality, one may assume $\kappa = 1$, otherwise we consider the metric $\bar{d} = d/\kappa$ instead, and $\mathbf{N} = \mathbb{N}$ – otherwise we add small, disjoint balls to $D$ and define $X$ to be constant and equal to $e$ on these balls. We first prove claim (i).

Let $\alpha > \frac{d}{q}$ and set $p(u) = u^{|\alpha - \frac{d}{q}|}$. By the Garsia-Rodemich-Rumsey Lemma (cf. e.g. [Sch09 Lemma 2.4 (i)]), for every $x, y \in D_n$, $\dfrac{d(X(x), X(y))}{|x - y|^{\alpha - \frac{d}{q}}} \leq CV_n^\frac{1}{q}$

where

$$V_n = \int_{D_n \times D_n} \frac{|d(X(u), X(v))|^q}{|u - v|^\alpha} \, du \, dv.$$

Thus, by a change of variables,

$$\mathbb{E} \left[ \sup_{x, y \in D_n} \frac{d(X(x), X(y))}{|x - y|^{\alpha - \frac{d}{q}}} \right]^q \leq C_n^q \int_{D_n \times D_n} |u - v|^{|\gamma - \alpha|q - d} \, du \, dv \leq C_n^q \int_{(0,1)^2} |u - v|^{|\gamma - \alpha|q - d} \, du \, dv.$$

Let $\beta \in (0, \gamma - \frac{d}{q})$ and set $\alpha = \frac{d}{q} + \beta < \gamma$. Then the integral is finite, and we have shown that

$$\left\| \sup_{x, y \in D_n} \frac{d(X(x), X(y))}{|x - y|^\beta} \right\|_{L^q} \leq C a_n (\gamma - \beta).$$

By the triangle inequality,

$$\left\| \sup_{x \in D} d(X(x), e) \right\|_{L^q} \leq C a_n (n^\gamma + 1) \leq 2Cb_n$$

therefore

$$\mathbb{P} \left( \sup_{x \in D} d(X(x), e) \geq t \right) \leq \frac{C_n^q b_n^q}{t^q}$$

and

$$\mathbb{P} \left( \sup_{x \in D} d(X(x), e) \geq t \right) \leq \frac{C_n^q b_n^q}{t^q} \sum_{n=1}^{\infty} b_n^q = \frac{C_n^q \|b\|_{L^q}^q}{t^q}.$$

Hence

$$\mathbb{E} \left[ \sup_{x \in D} d(X(x), e) \right]^q \leq 1 + C_n^q \|b\|_{L^q}^q \int_{1}^{\infty} \frac{dt}{b_n^q} \leq 1 + \frac{C_n^q \|b\|_{L^q}^q}{1 - \frac{d}{q}}.$$
Now we prove claim (ii). Note that the constant in the Garsia-Rodemich-Rumsey Lemma may be chosen non-increasing in $q$. Therefore, we can argue similarly as before to see that for every $q \geq 1$ and $n \in \mathbb{N}$,

$$\left\| \sup_{x \in D_n} d(X(x), e) \right\|_{L^q} \leq C_q b_n$$

where $C_q = \mathcal{O}(\sqrt{q})$. This shows that the random variable has Gaussian tails, i.e. there is some constant $C$ such that for every $n \in \mathbb{N}$

$$P\left( \sup_{x \in D_n} d(X(x), e) \geq t \right) \leq C \exp \left( -\frac{t^2}{Cb_n^2} \right)$$

for every $t \geq 0$. Hence

$$P\left( \sup_{x \in D} d(X(x), e) \geq t \right) \leq C \sum_{n=1}^{\infty} \exp \left( -\frac{t^2}{Cb_n^2} \right) \leq C \sum_{n=1}^{\infty} \exp \left( -\frac{t^2}{C} (1 + \log(n)) \right)$$

and the sum is finite for $t$ large enough. This proves that $\sup_{x \in D} d(X(x), e)$ has Gaussian tails which is equivalent to say that its $L^q$ norm grows at most like $\sqrt{q}$ when $q \to \infty$.

**Corollary 9.** Let $D$ be an open subset of $\mathbb{R}^d$, $(E, \| \cdot \|)$ a separable Banach space, $X : D \to E$ a continuous stochastic process and $q > 1$.

(i) Assume that there is a constant $\kappa > 0$ and $\gamma \in (0, 1]$ such that for every $x, y \in D$ with $0 < |x - y| \leq 1$,

$$(1.19) \quad \| X(x) - X(y) \|_{L^q} \leq \kappa |x - y|^{\gamma}.$$  

If $X$ has compact support, assume that $q > d/\gamma$. If $X$ does not have compact support, assume that there is an $\eta \in (0, \infty)$ such that for every $x \in D$,

$$(1.20) \quad \| X(x) \|_{L^q} \leq \frac{\kappa}{1 + \eta^{\gamma}}$$

with $q$ sufficiently large such that

$$q > \frac{1}{\eta} + d \left( \frac{1}{\eta} + \frac{1}{\gamma} \right).$$

Then the random variable $\sup_{x \in D} \| X(x) \|$ is almost surely finite. Moreover, if $X$ has compact support, there is a constant $C = C(\gamma, q)$ such that

$$(1.21) \quad \left\| \sup_{x \in D} \| X(x) \| \right\|_{L^q} \leq C \kappa.$$  

If $X$ does not have compact support, for every $q' \in [1, q)$ there is a constant $C' = C'(\gamma, \eta, q, q')$ such that (1.21) holds with $q$ and $C$ replaced by $q'$ and $C'$.

(ii) Assume that (1.19) holds for every $q \geq 1$ with $\kappa = \kappa(q) \leq \sqrt{q} \kappa$. Moreover, assume that $X$ has compact support or that (1.20) holds for every $q \geq 1$ with $\kappa = \kappa(q) \leq \sqrt{q} \kappa$ and some $\eta \in (0, \infty)$.
Then \( \sup_{x \in D} \| X(x) \| \) has Gaussian tails and there is a constant \( C = C(\gamma, \eta) \) such that
\[
\left\| \sup_{x \in D} \| X(x) \| \right\| \leq C \sqrt[q]{q^\gamma}
\]
for every \( q \geq 1 \).

**Proof** — We first assume that \( X \) does not have compact support. We start with proving (i). Let \( x, y \in D \) such that \( |x - y| \geq 1 \). Then, by (1.20),
\[
\| X(x) - X(y) \|_{L^q} \leq \| X(x) \|_{L^q} + \| X(y) \|_{L^q} \leq 2\kappa \leq 2\kappa |x - y|^{\gamma}
\]
which shows that
\[
(1.22) \quad \| X(x) - X(y) \|_{L^q} \leq 2\kappa |x - y|^{\gamma}
\]
holds for every \( x, y \in D \). Now let \( x, y \in D \) such that \( n - 1 \leq |x|, |y| < n \). Interpolating between the inequality
\[
\| X(x) - X(y) \|_{L^q} \leq \| X(x) \|_{L^q} + \| X(y) \|_{L^q} \leq \frac{2\kappa}{1 + (n - 1)^\eta}
\]
and inequality (1.22), we see that for every \( \lambda \in [0, 1] \),
\[
\| X(x) - X(y) \|_{L^q} \leq C\kappa n^{-(1-\lambda)\eta} |x - y|^{\gamma\lambda}
\]
and
\[
\| X(x) \|_{L^q} \leq C\kappa n^{-(1-\lambda)\eta}
\]
for every \( x, y \in D_n \). Set \( \gamma' := \gamma \lambda \) and \( a_n := n^{-(1-\lambda)\eta} \). In order to obtain \( (a_n n^{\gamma'}) \in \ell^q(\mathbb{N}) \), we must have \( q(\lambda\gamma - (1 - \lambda)\eta) < -1 \) which is equivalent to
\[
\frac{1}{q} < \eta - \lambda(\eta + \gamma).
\]
The condition \( \gamma' > \frac{d}{q} \) is equivalent to
\[
\frac{1}{q} < \frac{\lambda\gamma}{d}.
\]
Choosing \( \lambda^* = \frac{\eta}{\gamma' + \eta + \gamma} \in (0, 1) \), we have
\[
\eta - \lambda^*(\eta + \gamma) = \frac{\lambda^*\gamma}{d} = \frac{\gamma\eta}{\gamma + d(\gamma + \eta)}
\]
which is indeed smaller than \( \frac{1}{q} \) by assumption. Hence we may apply Lemma 8 to conclude (i). The claim (ii) follows by applying Lemma 8 (ii). In the case when \( X \) has compact support, (1.19) implies that
\[
\| X(x) - X(y) \|_{L^q} \leq C\kappa |x - y|^{\gamma}
\]
holds for every \( x, y \in D \) where the constant \( C \) depends on the diameter of the support of \( X \). In (i) and (ii), the claim now follows from the usual Garsia-Rodemich-Rumsey Lemma.
**Example.** Consider the Gaussian process $X: (0, \infty) \to \mathbb{R}$ where $X_t = \frac{B_t}{t^{\alpha/2}}$, for a standard Brownian motion $B$, and $\alpha \in \left(\frac{1}{2}, 1\right)$. Then, if $t > 0$,

$$
\|X_t\|_{L^q} \lesssim \sqrt{q} \|X_t\|_{L^2} \lesssim \frac{\sqrt{q}}{1 + t^{\alpha - \frac{1}{2}}}
$$

and for $s < t$,

$$
\|X_t - X_s\|_{L^2} = \frac{\|B_t(1 + s^\alpha) - B_s(1 + t^\alpha)\|_{L^2}}{(1 + s^\alpha)(1 + t^\alpha)} \leq \frac{\|B_t - B_s\|_{L^2}}{1 + t^\alpha} + \frac{\|B_s\|_{L^2}|t^\alpha - s^\alpha|}{(1 + s^\alpha)(1 + t^\alpha)}
$$

$$
\leq \frac{|t - s|^\frac{1}{2}}{1 + t^\alpha} + \frac{2s^\frac{1}{2}t^\alpha - \frac{1}{2}}{1 + s^\alpha(1 + t^\alpha)}|t - s|^\frac{1}{2}
$$

and

$$
\|X_t - X_s\|_{L^q} \lesssim \sqrt{q} \|X_t - X_s\|_{L^2}.
$$

Applying part (ii) in Corollary 9 shows that the random variable

$$
\sup_{t \in (0, \infty)} |X_t|
$$

is finite and has Gaussian tails. Note that this is sharp in the sense that the law of the iterated logarithm for a Brownian motion implies that it is not possible to choose $\alpha = \frac{1}{2}$.

Next, we apply the same ideas to give conditions for Hölder continuity.

**Lemma 10.** Let $D$ be an open subset of $\mathbb{R}^d$, $X: D \to (E, d)$ a continuous stochastic process and $\kappa > 0$. Set

$$
D_n := \left\{ x \in D : n - 1 \leq |x| < n \right\}
$$

and $N := \{ n \in \mathbb{N} : D_n \neq \emptyset \}$. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of non-negative real numbers.

(i) Assume that there is a $q > 1$ and a $\gamma \in \left(\frac{d}{q}, 1\right]$ such that for every $n \in \mathbb{N}$ and every $x, y \in D_n$ with $0 < |x - y| \leq 4$,

$$
\|d(X(x), X(y))\|_{L^q} \leq \kappa a_n |x - y|^\gamma.
$$

Let $\beta \in (0, \gamma - \frac{d}{q})$. Define the sequence $(b_n) := (a_n n^{\gamma - \beta})$, and assume that $\|b\|_{L^q} \leq K < \infty$. Then for every $q' \in [1, q)$ there is a constant $C = C(q, q', \gamma, K)$ such that

$$
\sup_{0 < |x - y| \leq 1} \mathbb{E} \left[ \left\| \frac{d(X(x), X(y))}{|x - y|^{\beta}} \right\|_{L^{q'}} \right] \leq C \kappa.
$$

(ii) Assume that there is some $\gamma \in (0, 1]$ and that for every $q \geq 1$ there is a $c_q$ such that for every $n \in \mathbb{N}$ and every $x, y \in D_n$ with $0 < |x - y| \leq 4$,

$$
\|d(X(x), X(y))\|_{L^q} \leq \kappa c_q a_n |x - y|^\gamma
$$
where \( c_q = \mathcal{O}(\sqrt{q}) \) when \( q \to \infty \). Let \( \beta \in (0, \gamma) \) and assume that \( a_n = \mathcal{O}\left(n^{-(\gamma-\beta)}(1+\log(n))^{-\frac{1}{2}}\right) \). Then for every \( q \geq 1 \) there is some constant \( C = C(q, \gamma) \) such that

\[
\sup_{0 < |x-y| \leq 1} \frac{d(X(x), X(y))}{|x-y|^{\beta}} \leq C \kappa
\]

with \( C = \mathcal{O}(\sqrt{q}) \) when \( q \to \infty \). In particular, the random variable

\[
\sup_{0 < |x-y| \leq 1} \frac{d(X(x), X(y))}{|x-y|^{\beta}}
\]

has Gaussian tails.

**Proof** – Without loss of generality, one can choose \( \kappa = 1 \) and \( \mathbb{N} = \mathbb{N} \). For \( n \in \mathbb{N} \), set \( \tilde{D}_n := \{ D_n \cup D_{n+1} \cup D_{n+2} \} \). We first prove (i). Fix some \( n \in \mathbb{N} \) and some \( k \in \mathbb{N} \). Let \( \alpha > \frac{d}{q} \) and define

\[
p_k(s) = \begin{cases} 
  s^{\alpha+\frac{d}{q}} & \text{if } s \in [0, 4] \\
  (4^{\alpha q + d} + k(s-4))^\frac{1}{q} & \text{if } s \geq 4.
\end{cases}
\]

Fix \( x, y \in \tilde{D}_n \) with \( 0 < |x-y| \leq 1 \). From the Garsia-Rodemich-Rumsey Lemma,

\[
d(X(x), X(y)) \leq CV_{n,k}^{\frac{1}{q}}|x-y|^\alpha - \frac{d}{q}
\]

where

\[
V_{n,k} = \int_{\tilde{D}_n \times \tilde{D}_n} \frac{d(X(u), X(v))}{p_k(|u-v|)^q} \ du \ dv.
\]

Thus

\[
\mathbb{E}\left[ \sup_{0 < |x-y| \leq 1} \frac{d(X(x), X(y))}{|x-y|^{\alpha-\frac{d}{q}}} \right]^q \leq C^q \int_{\tilde{D}_n \times \tilde{D}_n} \mathbb{E}\left[ \frac{d(X(u), X(v))^q}{p_k(|u-v|)^q} \right] \ du \ dv
\]

\[
\leq C^q \sum_{l=0,1,2} \int_{D_{n+l} \times D_{n+l}} \mathbb{E}\left[ \frac{d(X(u), X(v))^q}{p_k(|u-v|)^q} \right] \ du \ dv.
\]

For every \( m \in \mathbb{N} \), we have

\[
\int_{D_m \times D_m} \mathbb{E}\left[ \frac{d(X(u), X(v))^q}{p_k(|u-v|)^q} \right] \ du \ dv \leq a_m^q \int_{(D_m \times D_m) \cap \{|u-v| \leq 4\}} |u-v|^{(\gamma-\alpha)q-d} \ du \ dv
\]

\[
+ \int_{(D_m \times D_m) \cap \{|u-v| > 4\}} \mathbb{E}\left[ \frac{d(X(u), X(v))^q}{4^{\alpha q + d} + k(|u-v| - 4)} \right] \ du \ dv
\]
Moreover, by a change of variables,
\[ \int_{(D_m \times D_m) \cap \{|u-v| \leq 4\}} |u-v|^{(\gamma-\alpha)q-d} \, du \, dv \leq \int_{D_m \times D_m} |u-v|^{(\gamma-\alpha)q-d} \, du \, dv \]
\[ = m^{d+\gamma-q} \int_{(0,1)^2} |u-v|^{(\gamma-\alpha)q-d} \, du \, dv. \]

Set \( \alpha = \frac{d}{q} + \beta < \gamma \). Then this integral is finite, and sending \( k \to \infty \) shows that
\[ \left\| \sup_{x,y \in D_n} \frac{\int d(X(x), X(y))}{|x-y|^\beta} \right\|_{L^q} \leq C \left( a_{n}(\gamma-\beta) + a_{n+1}(n+1)^{(\gamma-\beta)} + a_{n+2}(n+2)^{(\gamma-\beta)} \right) \]
\[ = C(b_n + b_{n+1} + b_{n+2}). \]

Now take \( x, y \in D \) with \( 0 < |x-y| \leq 1 \) and assume that
\[ \frac{d(X(x), X(y))}{|x-y|^\beta} \geq t. \]

Then there is an \( n \in \mathbb{N} \) such that \( x \in D_n \) and since \( |x-y| \leq 1, y \in \{D_{n-1} \cup D_n \cup D_{n+1}\} \), we have that for every \( t \geq 0 \),
\[ \left\{ \sup_{x,y \in D} \frac{d(X(x), X(y))}{|x-y|^\beta} \geq t \right\} \subseteq \bigcup_{n \in \mathbb{N}} \left\{ \sup_{x,y \in D_n} \frac{d(X(x), X(y))}{|x-y|^\beta} \geq t \right\} \]
and therefore
\[ \mathbb{P} \left( \sup_{x,y \in D} \frac{d(X(x), X(y))}{|x-y|^\beta} \geq t \right) \leq \sum_{n=1}^{\infty} \mathbb{P} \left( \sup_{x,y \in D_n} \frac{d(X(x), X(y))}{|x-y|^\beta} \geq t \right) \]
\[ \leq 3^q C^q \sum_{n=1}^{\infty} b_n^q = \frac{3^q C^q K^q}{t^q}. \]

Hence we can estimate
\[ \mathbb{E} \left| \sup_{x,y \in D} \frac{d(X(x), X(y))}{|x-y|^\beta} \right|^{q'} \leq 1 + 3^q C^q K^q \int_1^{\infty} \frac{dt}{t^\beta} \leq 1 + \frac{3^q C^q K^q}{1 - \frac{\beta}{q}}. \]

Now we prove (ii). Note that the constant in the Garsia-Rodemich-Rumsey Lemma may be chosen non-increasing in \( q \). Therefore, we can argue similarly as before to see that for every \( q \geq 1 \) and \( n \in \mathbb{N} \),
\[ \left\| \sup_{x,y \in D_n} \frac{d(X(x), X(y))}{|x-y|^\beta} \right\|_{L^q} \leq C_q (b_n + b_{n+1} + b_{n+2}) \]
where \( C_q = \mathcal{O}(\sqrt{q}) \). This shows that the random variable has Gaussian tails, i.e. there is some constant \( C \) such that for every \( n \in \mathbb{N} \)

\[
P\left( \sup_{0 < |x-y| \leq 1} \frac{d(X(x), X(y))}{|x-y|^\beta} \geq t \right) \leq C \exp\left( -\frac{t^2}{C(b_n + b_{n+1} + b_{n+2})^2} \right)
\]

for every \( t \geq 0 \). Hence

\[
P\left( \sup_{0 < |x-y| \leq 1} \frac{d(X(x), X(y))}{|x-y|^\beta} \geq t \right) \leq C \sum_{n=1}^{\infty} \exp\left( -\frac{t^2}{C(b_n + b_{n+1} + b_{n+2})^2} \right)
\]

\[
\leq C \sum_{n=1}^{\infty} \exp\left( -\frac{t^2}{C} \right) \frac{1}{n^{t^2/C}}
\]

and the sum is finite for \( t \) large enough. This proves that

\[
\sup_{0 < |x-y| \leq 1} \frac{d(X(x), X(y))}{|x-y|^\beta}
\]

has Gaussian tails which is equivalent to say that its \( L^q \) norm grows like \( \sqrt{q} \) when \( q \to \infty \).

\[\square\]

**Example.** Let \( X : (0, \infty) \to \mathbb{R} \) be the Gaussian process defined as

\[
X_t = \frac{B_t}{\sqrt{t \log(1+t)}},
\]

\( B \) being a standard Brownian motion. Then

\[
\|X_t - X_s\|_{L^2} \leq \frac{|t-s|^{\frac{1}{2}}}{\sqrt{t \log(1+t)}} + \frac{\sqrt{t \log(1+t)} - \sqrt{s \log(1+s)}}{\sqrt{\log(1+s) \sqrt{s \log(1+s)}}}.
\]

By the mean value theorem,

\[
t \log(1+t) - s \log(1+s) \leq (\log(1+t) + 1)(t-s)
\]

and therefore

\[
\sqrt{t \log(1+t)} - \sqrt{s \log(1+s)} \leq \sqrt{t \log(1+t) - s \log(1+s)} \leq \sqrt{(\log(1+t) + 1)(t-s)^{1/2}}.
\]

If \( (n-1) \leq s \leq t \leq n \), we have for any \( q \geq 2 \)

\[
\|X_t - X_s\|_{L^q} \lesssim \sqrt{q} \|X_t - X_s\|_{L^2} \lesssim \sqrt{q} a_n \|t-s\|^{1/2}
\]

with \( a_n = \mathcal{O}\left(n^{1/2}(1 + \log(n))^{-1/2}\right) \). Part (ii) of Lemma 10 shows that for any \( \beta \in (0, 1/2) \), the random variable

\[
\sup_{0 < |t-s| \leq 1} \frac{|X_t - X_s|}{|t-s|^{1/2}}
\]

is finite and has Gaussian tails.
Corollary 11. Let $D$ be an open subset of $\mathbb{R}^d$, and $(B, \| \cdot \|)$ be a separable Banach space. Let $X : D \to B$ be a continuous stochastic process and $q > 1$.

(i) Assume that there is a constant $\kappa > 0$ and $\gamma \in (0, 1]$ such that for every $x, y \in D$ with $0 < |x - y| \leq 1$,

\[(1.23) \quad \|X(x) - X(y)\|_{Lq} \leq \kappa |x - y|^{\gamma}.
\]

Let $\beta \in (0, \gamma)$. If $X$ has compact support, assume that $q > \frac{d}{\gamma - \beta}$. If $X$ does not have compact support, assume that there is a constant $\eta \in (0, \infty)$ such that for every $x \in D$,

\[(1.24) \quad \|X(x)\|_{Lq} \leq \frac{\kappa}{1 + |x|^{\eta}}
\]

where $q > 1$ satisfies

\[q > \frac{\gamma}{\eta(\gamma - \beta)} + d \left( \frac{\gamma}{\eta(\gamma - \beta)} + \frac{1}{\gamma - \beta} \right) \cdot \eta.
\]

Then the random variable

\[
\sup_{x, y \in D \atop 0 < |x - y| \leq 1} \frac{\|X(x) - X(y)\|}{|x - y|^{\beta}}
\]

is almost surely finite. Moreover, if $X$ has compact support, there is a constant $C = C(\gamma, \beta, q)$ such that

\[(1.25) \quad \left\| \sup_{x, y \in D \atop 0 < |x - y| \leq 1} \frac{\|X(x) - X(y)\|}{|x - y|^{\beta}} \right\|_{Lq} \leq C \kappa.
\]

If $X$ does not have compact support, for every $q' \in [1, q)$ there is a constant $C' = C'(\gamma, \eta, \beta, q, q')$ such that (1.25) holds with $q$ and $C$ replaced by $q'$ resp. $C'$.

(ii) Assume that (1.23) holds for every $q \geq 1$ with $\kappa = \kappa(q) \leq \sqrt{q} \hat{\kappa}$. Moreover, assume that $X$ has compact support or that there is some $\eta \in (0, \infty)$ such that (1.24) holds for every $q \geq 1$ with $\kappa = \kappa(q) \leq \sqrt{q} \hat{\kappa}$.

Then for every $\beta \in (0, \gamma)$,

\[
\sup_{x, y \in D \atop 0 < |x - y| \leq 1} \frac{\|X(x) - X(y)\|}{|x - y|^{\beta}}
\]

has Gaussian tails, and there is a constant $C = C(\gamma, \eta, \beta)$ such that

\[
\left\| \sup_{x, y \in D \atop 0 < |x - y| \leq 1} \frac{\|X(x) - X(y)\|}{|x - y|^{\beta}} \right\|_{Lq} \leq C \sqrt{q} \hat{\kappa}
\]

holds for every $q \geq 1$.

Proof – We will only prove the case of $X$ not having compact support, the other case is similar (and even easier). We start with (i). By (1.23) and (1.24),

\[(1.26) \quad \|X(x) - X(y)\|_{Lq} \leq 2\kappa |x - y|^{\gamma}
\]

holds for every $x, y \in D$. By interpolation, for every $\lambda \in [0, 1]$,

\[
\|X(x) - X(y)\|_{Lq} \leq C \kappa \eta^{-1 - \lambda} |x - y|^{\gamma \lambda}
\]
\[ \| X(x) \|_{L^q} \leq C \kappa n^{-(1-\lambda)\eta} \]

for every \( x, y \in D_n \). Set \( \gamma' := \gamma \lambda \) and \( a_n := n^{-(1-\lambda)\eta} \). The condition \( (a_n n^\gamma) \in \ell^2(\mathbb{N}) \) is satisfied when

\[ \frac{1}{q} < \eta + \beta - \lambda(\eta + \gamma). \]

The condition \( \beta < \gamma' - \frac{d}{q} \) is equivalent to

\[ \frac{1}{q} < \frac{\lambda \gamma - \beta}{d}. \]

Choosing \( \lambda^* = (\eta + \beta + \beta/d)/(\gamma/d + \eta + \gamma) \in (0, 1) \), we have

\[ \eta + \beta - \lambda^*(\eta + \gamma) = \frac{\lambda^* \gamma - \beta}{d} = \frac{(\gamma - \beta)\eta}{\gamma + d(\gamma + \eta)} \]

which is smaller than \( \frac{1}{q} \) by assumption, and the claim follows from Lemma 10. (ii) follows similarly. \( \triangleright \)

If \( D \) is an open subset of \( \mathbb{R}^d \), and \( (B, \| \cdot \|) \) is a normed space, \( f: D \to B \) a function and \( \rho \in (0, 1] \), we define

\[ \| f \|_{C^\rho} := \max \left\{ 2 \sup_{x \in D} \| f(x) \|, \sup_{0 < |x-y| \leq 1} \frac{\| f(x) - f(y) \|}{|x-y|^{\rho}} \right\}. \]

Let \( f: D \to \mathbb{R}^{d_1} \) and \( g: D \to \mathbb{R}^{d_2} \). Then we define the function \( (f \circ g): D \to \mathbb{R}^{d_1 \times d_2} \) by setting \( (f \circ g)^{ij}(x) = f^i(x)g^j(x) \). Note that \( \| \cdot \|_{C^\rho} \) is equivalent to \( \| \cdot \|_{C^\rho} \), and with this definition, we have

\[ \| f \circ g \|_{C^\rho} \leq \| f \|_{C^\rho} \| g \|_{C^\rho} \]

provided we equip \( \mathbb{R}^{d_1}, \mathbb{R}^{d_2} \) and \( \mathbb{R}^{d_1 \times d_2} \) with the sup norm. The next theorem is the main result of this section.

**Theorem 12** (Kolmogorov criterion for rough drivers). Let \( I \subset \mathbb{R} \) be a closed interval with \( |I| = T \), \( D \) an open subset of \( \mathbb{R}^d \) and \( \kappa > 0 \). Let \( \gamma_1, \gamma_2 \in (0, 1] \).

(i) Let \( V: D \times I \to \mathbb{R}^{d_1} \) be a stochastic process and \( q > 1 \). Assume that for every \( x, y \in D \) with \( 0 < |x-y| \leq 1 \) and \( s < t \in I \),

\[ \| V_{ts}(x) - V_{ts}(y) \|_{L^q} \leq \kappa |t-s|^{\gamma_1} |x-y|^{\gamma_2}. \]

Assume that there is a compact set in \( \mathbb{R}^d \) such that \( V_{ts} \) is supported on this set almost surely for every \( s < t \in I \), or that there is an \( \eta \in (0, \infty) \) such that and for every \( x \in D \) and \( s < t \in I \),

\[ \| V_{ts}(x) \|_{L^q} \leq \frac{\kappa |t-s|^{\gamma_1}}{1 + |x|^{\eta}}. \]

Let \( \alpha \in (0, \gamma_1), \beta \in (0, \gamma_2) \) and assume that

\[ q > \max \left\{ \frac{\gamma_2}{\eta(\gamma_2 - \beta)}, d \left( \frac{\gamma_2}{\eta(\gamma_2 - \beta)} + \frac{1}{\gamma_2 - \beta} \right), \frac{1}{\gamma_1 - \alpha} \right\} \]
where we set $\eta = \infty$ in the case of $V$ having compact support (in the sense above). Then there is a continuous modification of the process $V$. Moreover, if $V$ has compact support, there is a constant $C = C(\gamma_1, \gamma_2, \alpha, \beta, d, T, q)$ such that

\begin{equation}
\left\| \sup_{s < t \in I} \left\| V_{ts} \right\|_{C^2} \right\|_{L^q} \leq C\kappa.
\end{equation}

If $V$ does not have compact support, for every $q' \in [1, q)$ there is a constant $C' = C'(\gamma_1, \gamma_2, \alpha, \beta, \eta, d, T, q, q')$ such that \((1.31)\) holds with $q$ and $C$ replaced by $q'$ resp. $C'$.

(ii) Let $W : D \times \{0 \leq s \leq t \leq T\} \to \mathbb{R}^{d_1 \times d_2}$ be a stochastic process and $q > 2$. Assume that for every $x, y \in D$ with $0 < |x - y| \leq 1$ and $s < t \in I$,

\begin{equation}
\|W_{ts}(x) - W_{ts}(y)\|_{L^q_{\text{law}}} \leq \kappa^2 |t - s|^{2\gamma_1} |x - y|^{\gamma_2}.
\end{equation}

Assume that $W$ has compact support (in the sense above) or that there is an $\eta \in (0, \infty)$ such that for every $x \in D$ and $s < t \in I$,

\begin{equation}
\|W_{ts}(x)\|_{L^q_{\text{law}}} \leq \frac{\kappa^2 |t - s|^{2\gamma_1}}{1 + |x|^{2\eta}}.
\end{equation}

Let $\alpha \in (0, \gamma_1), \beta \in (0, \gamma_2)$ and assume that $q$ is sufficiently large such that \((1.30)\) holds (again with $\eta = \infty$ in the case of $W$ having compact support). Moreover, assume that for every $s < u < t \in I$ and every $x \in D$,

\begin{equation}
W_{ts}(x) - W_{us}(x) - W_{tu}(x) = V_{us}(x) \circ \tilde{V}_{tu}(x)
\end{equation}

almost surely, where $V : D \times I \to \mathbb{R}^{d_1}$ and $\tilde{V} : D \times I \to \mathbb{R}^{d_2}$ are stochastic processes which satisfy the conditions of part (i), with either all of these processes having compact support or satisfying the respective growth conditions for the same $\eta > 0$. Then there is a continuous modification of the process $W$. Moreover, if $W$ has compact support, there is a constant $C = C(\gamma_1, \gamma_2, \alpha, \beta, d, T, q)$ such that

\begin{equation}
\left\| \sup_{s < t \in I} \left\| W_{ts} \right\|_{C^2} \right\|_{L^q} \leq C^2\kappa^2.
\end{equation}

If $W$ does not have compact support, for every $q' \in [2, q)$ there is a constant $C' = C'(\gamma_1, \gamma_2, \alpha, \beta, \eta, d, T, q, q')$ such that \((1.35)\) holds with $q$ and $C$ replaced by $q'$ resp. $C'$.

(iii) Assume that \((1.28)\) and \((1.32)\) hold for every $q \geq 2$, that $\kappa \leq \sqrt{q} \tilde{\kappa}$ for some constant $\tilde{\kappa} > 0$ and that the algebraic relation \((1.34)\) holds. Assume that either all random variables have compact support or that the growth conditions \((1.29)\) and \((1.33)\) are satisfied for some $\eta \in (0, \infty)$ with $\kappa \leq \sqrt{q} \tilde{\kappa}$ for some constant $\tilde{\kappa} > 0$. Then for every $\alpha \in (0, \gamma_1)$ and $\beta \in (0, \gamma_2)$, the random variables

\[\sup_{s < t \in I} \left\| V_{ts} \right\|_{C^2} \quad \text{and} \quad \sqrt{\sup_{s < t \in I} \left\| V_{ts} \right\|_{C^2}}\]

have Gaussian tails, and the $L^q$-estimates \((1.31)\) and \((1.35)\) hold for every $q \geq 1$ with $\kappa$ replaced by $\tilde{\kappa}$, and the constant $C$ depends on $q$ in such a way that $C(q) = O(\sqrt{q})$. 

\[\frac{\kappa^2 |t - s|^{2\gamma_1}}{1 + |x|^{2\eta}}\]
Proof – Without loss of generality we may assume \( \kappa = 1 \), otherwise we can replace \( V \) and \( W \) by \( V/\kappa \) resp. \( W/\kappa^2 \). Furthermore, we will prove the result for the \( \| \cdot \|_{C^2} \) norm, claimed results follow by equivalence of norms.

As in the previous proofs, we will only consider the case of \( V \) and \( W \) not having compact support, the other case is analogous. We start with proving (i). W.l.o.g. \( q' \in (\frac{1}{\gamma_1 - \alpha}, q) \). Fix \( s < t \). Using (1.23) and the classical Kolmogorov theorem (cf. [RY99] (2.1) Theorem on p. 26]), there is a continuous modification of the process \( x \mapsto V_{ts}(x) \) on every bounded hypercube contained in \( D \) with vertices in \( D \cap \mathbb{Q}^d \), and since \( D \) is the countable union of such hypercubes also on \( D \) itself. The estimates (1.28) and (1.29) and Corollary 11 imply that

\[
\left\| \sup_{x,y \in D} \frac{|V_{ts}(x) - V_{ts}(y)|}{|x - y|^\beta} \right\|_{L^{q'}} \leq C|t - s|^{\gamma_1}
\]

and Corollary 9 gives

\[
\left\| \sup_{x \in D} |V_{ts}(x)| \right\|_{L^{q'}} \leq C|t - s|^{\gamma_1}.
\]

Note in particular that the constant on the right hand side of both equations is independent of \( s \) and \( t \). We can repeat this procedure for every \( s < t \) and obtain a process \( t \mapsto V_t \) which, for every \( t \in [0, T] \), takes values in \( C^2_b \) almost surely, and for which

\[
(1.36) \quad \left\| \|V_t - V_s\|_{C^2} \right\|_{L^{q'}} \leq C|t - s|^{\gamma_1}
\]

holds for every \( s < t \). Applying again the Kolmogorov theorem for Banach space valued processes gives the claim.

We proceed with (ii). Again, we may assume that \( q' \in (\frac{1}{\gamma_1 - \alpha}, q) \). As in (i), for every \( s < t \) there are modifications of the process \( x \mapsto W_{ts}(x) \) such that

\[
\left\| \|W_{ts}\|_{C^2} \right\|_{L^{q'}} \leq C|t - s|^{2\gamma_1}.
\]

Using the algebraic relation (1.34), the estimate (1.36) for \( V \) and \( \tilde{V} \) and the compatibility of the \( \| \cdot \|_{C^2} \) norms given in (1.27), we can mimic the proof of the Kolmogorov criterion for rough paths ([FH14, Theorem 3.1]) to conclude.

Assertion (iii) follows similarly by using part (ii) in the Corollaries 11 and 9.

Finally, we give a Kolmogorov criterion for the distance between rough drivers.

**Theorem 13** (Kolmogorov criterion for rough driver distance). Let \( I \subset \mathbb{R} \) be a closed interval with \( |I| = T \), \( D \) an open subset of \( \mathbb{R}^d \) and \( \kappa > 0 \). Let \( \gamma_1, \gamma_2 \in (0, 1] \) and \( q > 2 \). Let \( (V, W) \) and \( (\tilde{V}, \tilde{W}) \) be processes as in Theorem 12. Set \( \Delta V := V - \tilde{V} \) and \( \Delta W := W - \tilde{W} \).

(i) Assume that \( (V, W) \) and \( (\tilde{V}, \tilde{W}) \) satisfy the same moment conditions as in Theorem 12 with \( q \) sufficiently large as in (1.30). Moreover, assume that there is an \( \varepsilon > 0 \) such that

\[
(1.37) \quad \left\| \Delta V_{ts}(x) - \Delta V_{ts}(y) \right\|_{L^q} \leq \varepsilon \kappa |t - s|^{\gamma_1}|x - y|^{\gamma_2}
\]

and

\[
(1.38) \quad \left\| \Delta W_{ts}(x) - \Delta W_{ts}(y) \right\|_{L^{q'}} \leq \varepsilon \kappa^2 |t - s|^{2\gamma_1}|x - y|^{\gamma_2}
\]

Theorem 13 will be proved in the next section.
for every \( x, y \in D \) such that \( 0 < |x - y| \leq 1 \) and every \( s < t \in I \). If \( (V, W) \) and \( (\hat{V}, \hat{W}) \) do not have compact support, assume that also

\[
\| \Delta V_{ts}(x) \|_{L^q} \leq \varepsilon \kappa |t - s|^{\gamma_1} \frac{1}{1 + |x|^{\gamma_2}}
\]

and

\[
\| \Delta W_{ts}(x) \|_{L^q} \leq \varepsilon \kappa^2 |t - s|^{2\gamma_1} \frac{1}{1 + |x|^{2\gamma_2}}
\]

hold for every \( x \in D \) and every \( s < t \in I \). Then there are continuous modifications of the processes \( (V, W) \) and \( (\hat{V}, \hat{W}) \). If the processes have compact support, there is a constant \( C = C(\gamma_1, \gamma_2, \alpha, \beta, d, T, q) \) such that

\[
\left\| \sup_{s < t \in I} \frac{\| V_{ts} - \hat{V}_{ts} \|_{C^a}}{|t - s|^{\alpha}} \right\|_{L^q} \leq \varepsilon C \kappa
\]

and

\[
\left\| \sup_{s < t \in I} \frac{\| W_{ts} - \hat{W}_{ts} \|_{C^a}}{|t - s|^{2\alpha}} \right\|_{L^q} \leq \varepsilon C^2 \kappa^2.
\]

If the processes do not have compact support, for every \( q' \in [1, q) \) there is some constant \( C' = C'(\gamma_1, \gamma_2, \alpha, \beta, \eta, d, T, q, q') \) such that \((1.41)\) and \((1.42)\) hold with \( q \) and \( C \) replaced by \( q' \) resp. \( C' \).

(ii) Assume that \( (V, W) \) and \( (\hat{V}, \hat{W}) \) satisfy the same moment conditions as in Theorem 12 for every \( q \geq 2 \) with \( \kappa \leq \sqrt{q} \). Moreover, assume that \((1.37), (1.38)\) and, in the case of not having compact support, \((1.39)\) and \((1.40)\) hold. Then there is a constant \( C = C(\gamma_1, \gamma_2, \alpha, \beta, \eta, d, T) \) such that

\[
\varepsilon^{-1} \left\| \sup_{s < t \in I} \frac{\| V_{ts} - \hat{V}_{ts} \|_{C^a}}{|t - s|^{\alpha}} \right\|_{L^q} + \varepsilon^{-\frac{1}{2}} \left\| \sup_{s < t \in I} \frac{\| W_{ts} - \hat{W}_{ts} \|_{C^a}}{|t - s|^{2\alpha}} \right\|_{L^q} \leq \sqrt{q} \kappa C
\]

for every \( q \geq 2 \).

**Proof** – The proof is very similar to the one of Theorem 12 using the Kolmogorov criterion for rough path distance [FH14, Theorem 3.3]. We leave the details to the reader. ▷

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### Stochastic and rough flows

The theory of stochastic flows grew out of the pioneering works of the Russian school [BF61, IA72] on the dependence of solutions to stochastic differential equations with respect to parameters and the proof by Bismut [Bis81] and Kunita [Kun81c] that stochastic differential equations generate continuous flows of diffeomorphisms under proper regularity conditions on the driving vector fields. The Brownian character of these random flows, that is the fact that they are continuous with stationary and independent increments, was inherited from the Brownian character of their driving noise. The next natural step consisted in the study of Brownian flows for themselves. After the works of Harris [Har81], Baxendale [Bax80] and Le Jan [LJ82], they appeared to be generated by stochastic differential
equations driven by infinitely many Brownian motions, or better, to be in one-to-one correspondence with vector field-valued Brownian motions. A probabilistic integration theory of such random time-varying velocity fields was developed to establish that correspondence, and it was extended by Le Jan and Watanabe [LJW84] to a large class of continuous semimartingale flows and continuous semimartingale velocity fields. Kunita [Kun81a, Kun86] studied the problem of convergence of stochastic flows, with applications to averaging and homogenization results, and promoted the use of stochastic flows to implement a version of the characteristic method in the setting of first and second order stochastic partial differential equations, notably those coming from the nonlinear filtering theory.

We shall show in this section that the theory of semimartingale stochastic flows can be embedded into the theory of rough flows developed in section 2.2. We review in section 2.2 the basics of the theory of stochastic flows and show in section 2.3 that sufficiently regular (semi)martingale velocity fields can be lifted to rough drivers. The identification of (semi)martingale flows generated by (semi)martingale velocity fields to rough flows associated with the corresponding rough driver is done through the Itô formula, on which one can read the local characteristics of a semimartingale flow.

2.1. Notations for function spaces

The study of stochastic flows classically requires the introduction of a number of function spaces, that we recall here.

Let $E$ and $F$ be Banach spaces. The derivative of a function $f$ from $E$ to $F$ is understood in the Fréchet sense. We shall equip tensor products of Banach spaces with a compatible tensor norm which makes the canonical embedding

$$\mathcal{L}(E, \mathcal{L}(E, F)) \hookrightarrow \mathcal{L}(E \otimes E, F)$$

continuous. Note that the $n$-th derivative of $f$ can be seen as a function $D^n f : E \to \mathcal{L}(E^{\otimes n}, F)$. For $n \in \mathbb{N}_0$ and $\rho \in (0, 1]$, we define

$$\|f\|_n := \|f\|_{C^n} := \sup_{x \in E} \|f(x)\| + \sum_{i=1}^n \sup_{x \in E} \|D^i f(x)\|$$

and

$$\|f\|_{n+\rho} := \|f\|_{C^{n+\rho}} := \|f\|_{C^n} + \sup_{0 < \|x-y\| \leq 1} \frac{\|D^n f(x) - D^n f(y)\|}{\|x-y\|^\rho}.$$

We define $C_b^{n,\rho}(E, F)$ to be the space of $n$-times continuously differentiable functions $f : E \to F$ such that $\|f\|_{C^{n+\rho}} < \infty$. We will often just write $C_b^n(E, F)$ for $C_b^{n,0}(E, F)$ when the domain and codomain of the function space is clear from the context.

Next, we consider the finite dimensional case. Let be $D$ be a domain of $\mathbb{R}^d$, $A$ a subset of $D$, $n \in \mathbb{N}_0$ and $\rho \in (0, 1]$. For a function $f : D \to \mathbb{R}^k$ set

$$\|f\|_{n;A} := \sup_{x \in A} |f(x)| + \sum_{1 \leq |a| \leq n} \sup_{x \in A} |D^a f(x)|,$$

$$\|f\|_{n+\rho;A} := \|f\|_{n;A} + \sum_{|a| = n} \sup_{x,y \in A, \rho < |x-y| \leq 1} \left| \frac{|D^a f(x) - D^a f(y)|}{|x-y|^\rho} \right|.$$
of $D$. Note that although the (semi-)norms we defined here differ slightly from those used by Kunita in his book [Kun90], they are actually equivalent on compact sets, hence the spaces coincide. We also define

$$C^n_b(D, \mathbb{R}^k) := \left\{ f \in C^n(D, \mathbb{R}^k) : \| f \|_{n,\rho} < \infty \right\}.$$  

For a function $g : D \times D \to \mathbb{R}^k$, we similarly define

$$\| g \|_{n;A}^\wedge := \sup_{x,y \in A} |g(x,y)| + \sum_{1 \leq |\alpha| \leq n} \sup_{x,y \in A} |D^\alpha_x D^\alpha_y g(x,y)|$$

and

$$\| g \|_{n+\rho,A}^\wedge := \| g \|_{n;A}^\wedge + \sum_{1 \leq |\alpha| \leq n} \sup_{x,y \in A} \sup_{x',y' \in A} \left| D^\alpha_x D^\alpha_y g(x,y) - D^\alpha_x D^\alpha_y g(x',y) - D^\alpha_x D^\alpha_y g(x,y') + D^\alpha_x D^\alpha_y g(x',y') \right| \left| x - x' \right|^\rho \left| y - y' \right|^\rho.$$  

As above, set $\| g \|_{n;D}^\wedge := \| g \|_{n;D}^\wedge$ resp. $\| g \|_{n+\rho,D}^\wedge := \| g \|_{n+\rho,D}^\wedge$. We denote by $\hat{C}^n(D \times D, \mathbb{R}^k)$ the space of functions $g : D \times D \to \mathbb{R}^k$ which are $n$-times continuously differentiable with respect to each $x$ and $y$ and for which $\| g \|_{n+\rho,K}^\wedge < \infty$ for every compact subset $K$ of $D$. Set $\hat{C}^n(D \times D, \mathbb{R})$ and

$$\hat{C}_b^n(D \times D, \mathbb{R}) := \left\{ g \in \hat{C}^n(D \times D, \mathbb{R}) : \| g \|_{n+\rho}^\wedge < \infty \right\}.$$  

2.2. Semimartingale stochastic flows  

We describe in this section the basics of the theory of semimartingale stochastic flows, and refer the reader to [LJW84] or [Kun90] for a complete account; we refer to Kunita’s book for precise regularity and growth assumptions on the different objects involved.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a filtered probability space; denote by $\text{Diff}$, resp. $\hat{\text{F}}$, the complete separable metric spaces of $C^{k_0}$ diffeomorphisms of $\mathbb{R}^d$, resp. $C^{k_0}$ vector fields on $\mathbb{R}^d$, for some integer $k_0 \geq 2$.

**Definition.** A $\text{Diff}$-valued continuous $(\mathcal{F}_t)_{0 \leq t \leq T}$-adapted random process $(\phi_t)_{0 \leq t \leq T}$ is called a $\text{Diff}$-valued semimartingale stochastic flow of maps if the real-valued processes $f(\phi_t(x))$ are real-valued $(\mathcal{F}_t)_{0 \leq t \leq T}$-semimartingales for all $x \in \mathbb{R}^d$, and all $f \in C_\infty^c(\mathbb{R}^d)$. Such a $\text{Diff}$-valued semimartingale is said to be regular if for every $x, y \in \mathbb{R}^d$, and $f \in C_\infty^c(\mathbb{R}^d)$, the bounded variation part of $f(\phi_t(x))$ and the bracket $(f(\phi_t(x)), g(\phi_t(y)))$ are absolutely continuous with respect to Lebesgue measure $dt$.

Their densities $w^f_t(x)$ and $\{f,g\}_t(x,y)$ can be chosen to be jointly measurable and continuous in $f, g$ in the $C^2$-norm [LJW84]. Set

$$(\mathcal{L}_t f)(x) := w^f_t(\phi_t^{-1}(x)), \quad (f,g)_t(x,y) := \{f,g\}_t(\phi_t^{-1}(x), \phi_t^{-1}(y)),$$

so that the processes

$$M^f_t(x) := f(\phi_t(x)) - f(\phi_0(x)) - \int_0^t (\mathcal{L}_s f)(\phi_s(x)) ds, \quad x \in \mathbb{R}^d, f \in C_\infty^c(\mathbb{R}^d)$$
are continuous \((\mathcal{F}_t)_{0 \leq t \leq T}\)-local martingales with bracket
\[
\langle M^f(x), M^g(y) \rangle_t = \int_0^t \langle f, g \rangle_t(x, y) \, ds.
\]
We have
\[
(\mathcal{L}_t f)(x) = \lim_{h \to 0} \mathbb{E} \left[ \frac{f(\phi_{t+h,t}(x)) - f(x)}{h} \bigg| \mathcal{F}_t \right]
\]
and
\[
\langle f, g \rangle_s(x, y) = \lim_{h \to 0} \frac{1}{h} \mathbb{E} \left[ \{ f(\phi_{t+h,t}(x)) - f(x) \} \{ g(\phi_{t+h,t}(y)) - g(y) \} \bigg| \mathcal{F}_t \right],
\]
with limits in \(L^1\) whenever they exist. Under proper regularity conditions \([_{L,JW81}]\), the operators \(\langle f, g \rangle_s(x, y)\) can be seen to be random differential operators of the form
\[
\langle f, g \rangle_s(x, y) = A_{sij}(x, y) \partial_{x_i y_j}^2,
\]
for some process \(A_s(x, y)\) with values in the space \(\text{Symm}(d)\) of symmetric \(d \times d\) matrices. The operators \(\mathcal{L}_s\) can moreover be expressed in terms of \(A_s\) and its differential with respect to the space variables, so that the data of the processes \(A_s\) and \(\mathcal{L}_s\) is equivalent to the data of the process \(A_s\) and an \(\mathbb{R}\)-valued process \(b_s\). The family of random operators \(\langle \cdot, \cdot \rangle_t\) and the drift \(b_t\) are called the **local characteristics** of the \(\mathcal{D} \mathit{iff}\)-valued semimartingale \(\phi_*\). As an example, for the semimartingale flow generated by a stochastic differential equation of the form
\[
dx_t = V_t(x_t) \circ dB_t^i,
\]
driven by an \(\ell\)-dimensional Brownian motion, we have
\[
\mathcal{L}_t f = \frac{1}{2} \sum_{i=1}^\ell V_i^2 f
\]
and
\[
\langle f, g \rangle_s(x, y) = (V_i f)(x) (V_j g)(y),
\]
and the drift \(b_s\) in the local characteristic is given here by the time-independent vector field
\[
b_s(x) = \frac{1}{2} (V_i V_i)(x) = \frac{1}{2} (D_x V_i) V_i(x).
\]

The infinitesimal counterpart of a \(\mathcal{D} \mathit{iff}\)-valued semimartingale is given by the following notion.

**Definition.** A **semimartingale velocity field** is an \(\mathbb{R}\)-valued process \((V_t)_{0 \leq t \leq T}\) such that the processes \((V_t f)(x)\) are real-valued semimartingale for all \(x \in \mathbb{R}^d\) and all \(f \in C^\infty(\mathbb{R}^d)\). It is called **regular** if one can write
\[
V_t = M_t + \int_0^t v_s \, ds
\]
for a vector field-valued adapted process \(v_*\), and an \(\mathbb{R}\)-valued local martingale \(M_*\) for which there exists a \(\text{Symm}(d)\)-valued process \(a_s(x, y)\) with
\[
\langle \partial_x^\alpha M_*(x,y), \partial_y^\beta M_*(y) \rangle_t = \int_0^t \partial_x^\alpha \partial_y^\beta a_s(x, y) \, ds
\]
for a definite range of multiindices \(\alpha, \beta\). The pair \((a_*, v_*)\) is called the **local characteristics** of the semimartingale velocity field \(V_*\).
A theory of Stratonovich integration can be constructed for making sense of integrals of
the form
\[ \int_0^t V_{ds}(x_s), \]
for any adapted process \( x_\bullet \) and some regular semimartingale velocity field \( V_\bullet \), as a limit in probability of symmetric Riemann sums. This requires some almost sure regularity properties on the local characteristics \((a_\bullet, v_\bullet)\) of \( V_\bullet \). Under these conditions, the integral
\[ x_t = x_0 + \int_0^t V_{ds}(x_s) \]
can be seen to have a unique solution started from any point \( x_0 \in \mathbb{R}^d \).

**Theorem 14 ([LJW84])**. These solutions can be gathered into a semimartingale stochastic
flow whose local characteristics are \((a_\bullet, v_\bullet + c_\bullet)\), where the time-dependent vector field \( c_\bullet \) has coordinates
\[ c^j_s(x) := \frac{1}{2} \sum_{j=1}^d \partial_y a^{ij}_s(x, y) \bigg|_{y=x} \]
in the canonical basis of \( \mathbb{R}^d \). Conversely, one can associate to any regular stochastic flow of
diffeomorphisms \( \phi_\bullet \) a semimartingale velocity field \( V_\bullet \), with the same local characteristics as \( \phi_\bullet \), and such that \( \phi_\bullet \) coincides with the stochastic flow generated by the Stratonovich equation
\[ x_t = x_0 + \int_0^t V_{ds}(x_s) - \int_0^t c_s(x_s) \, ds. \]

The optimal regularity assumptions on the velocity fields and stochastic flows of maps
are given in theorems 4.4.1 and 4.5.1 of Kunita’s book. We shall use the full strength of
these two statements in section 2.4 to identify semimartingale stochastic flows of maps and
the rough flows associated with the lift of the semimartingale velocity fields into a rough
driver.

The correspondence between semimartingale stochastic flows and semimartingale velocity
fields via an *Itô equation* of the form
\[ x_t = x_0 + \int_0^t V_{ds}(x_s) \]
is exact, with no need to add the drift \( \int_0^t c_s \, ds \). We state it here under the above form as
we shall see below that rough flows are naturally associated with Stratonovich differential
equations.

The main difficulty in this business is to deal with the local martingale part of the
dynamics, which is where probability theory is really needed. As a consequence, we shall concentrate our efforts on local martingale velocity fields in the sequel, the remaining changes to deal with regular semimartingale velocity fields being essentially cosmetic. As above, we shall freely identify in the sequel vector fields with first order differential operators. In order to keep consistent notations, we shall also denote by
\[ \int_0^t \alpha_s \, m_{ds} \quad \text{and} \quad \int_0^t \alpha_s \, m_{odds} \]
the Itô and Stratonovich integrals of an adapted process \( \alpha_s \) with respect to a local martingale \( m_s \).
2.3. Local martingale rough drivers

Let $D$ be an open connected subset of $\mathbb{R}^d$ and let $M$ stand for a local martingale velocity field. We prove in this section that such a field can be lifted to a rough driver $\mathbf{M} = (M_{ts}, M_{ts})_{0 \leq s \leq t \leq T}$, with $M_{ts} := M_t - M_s$, under regularity and boundedness assumptions on the local characteristic of $M$. At a heuristic level, if $M$ is differentiable in space, the second level operator $M_{ts}$ associated with $M_{ts} := M_t - M_s$ is given by the formula

$$M_{ts} = \int_s^t M_{us} M_{odu} \, du$$

with obvious notations for the operators $\partial_k$ and $\partial_{jk}^2$. In the following, we will use the notation

$$(M_{ts}, M_{ts}) := (DM_{ts})(M_{ts}).$$

As the classical rules of Stratonovich integration give

$$(\int_s^t M_{us}^j M_{odu}^k) \partial_{jk}^2 = \frac{1}{2} M_{ts}^i M_{ts}^j \partial_{jk}^2 = \frac{1}{2} M_{ts} M_{ts} - \frac{1}{2} (M_{ts}, M_{ts}),$$

we see that the operators $M_{ts}$ can be decomposed as

$$M_{ts} = W_{ts} + \frac{1}{2} M_{ts} M_{ts}, \quad (2.2)$$

where

$$W_{ts} = \left( \int_s^t M_{us}^i \partial_i M_{odu}^k \right) \partial_k - \frac{1}{2} (M_{ts}, M_{ts})$$

is a martingale velocity field defined pointwisely by an Itô integral. The proof that the process $\mathbf{M} := (M_{ts}, M_{ts})$ has a modification that is a $p$-rough driver for every $2 < p < 3$ will require two elementary intermediate results that heavily rest on a now classical modified version of Kolmogorov’s regularity criterion that we recall here for the reader’s convenience – this is different from the content of the above section 1.5; they can be found in section 3 of Kunita’s book [Kun90].

**Theorem 15.**

1. Let $M_\bullet(x), x \in D$, be a family of continuous local martingales started from 0, such that the joint quadratic variation $\langle M_\bullet(x), M_\bullet(y) \rangle$ has a continuous modification in $C^{0,\delta}$. Then the process $M$ has a modification that is a continuous process with values in $C^{0,\epsilon}$, for every $\epsilon < \delta$.

2. Let $M$ be a local martingale velocity field started from 0, with local characteristic the random field $a_t(x, y)$. Assume that $a$ has a continuous modification that belongs almost surely to $\widehat{C}^{m,\delta}$, for some integer $m \geq 1$, and $0 < \delta \leq 1$. Then, for every $0 < \epsilon < \delta$, the velocity field $M$ has a modification that is a continuous process with values in $C^{m,\epsilon}$. 


we still denote it by $M$. Furthermore, for each multi-index $\alpha$, with $|\alpha| \leq m$, the time varying random field $\partial_\alpha^a M$ is a local martingale velocity field with quadratic variation

$$d\langle \partial_\alpha^a M_\bullet(x), \partial_\alpha^a M_\bullet(y) \rangle_t = \partial_\alpha^a \partial_\alpha^a a_t(x, y) \, dt.$$ 

(3) Let here $M$ and $N$ be two local martingale velocity fields with values in $C^{m,\delta}$. Then their joint quadratic variation

$$\langle M_\bullet(x), N_\bullet(y) \rangle_t$$

has a continuous modification taking values in $\tilde{C}^{m,\epsilon}$ for every $\epsilon < \delta$. Furthermore, if $m \geq 1$, this modification satisfies the identity

$$\partial_\alpha^a \partial_\beta^\beta \langle M_\bullet(x), N_\bullet(y) \rangle_t = \langle \partial_\alpha^a M_\bullet(x), \partial_\beta^\beta N_\bullet(y) \rangle_t,$$

for all $|\alpha|, |\beta| \leq m$.

**Proof** – Cf. Theorem 3.1.1, Theorem 3.1.2 and Theorem 3.1.3 in [Kun90]. ▷

These regularity results will be instrumental in the proof of the following intermediate result. Note that $N$ in equation (2.3) below is seen as a vector field, not a differential operator, so $(MN)(x) = (D_x N)(M(x))$.

**Proposition 16.** Let $M, N: D \times [0,T] \to \mathbb{R}$ be continuous $C^{m,\delta}$-valued local martingale fields, for $m \in \mathbb{N}_0$ and $\delta \in (0,1]$. Assume $M$ is adapted to the filtration generated by $N$. Then the pointwisely defined Itô integral

$$(2.3) \quad t \mapsto \int_0^t (M_s N ds)(x)$$

has a continuous modification taking values in $C^{m,\alpha}$ process for every $\alpha < \delta$. Moreover, if $m \geq 1$, the derivative is almost surely given by the formula

$$(2.4) \quad \partial_{x_i} \left( \int_0^t M_s N ds \right) = \int_0^t \partial_{x_i} M_s N ds + \int_0^t M_s \partial_{x_i} N ds.$$ 

Both assertions also hold for the Stratonovich integral.

The proof of this result is somewhat lengthy but rests on classical considerations based on the regularization theorem [15].

**Proof** – Set

$$U_t(x) := \left( \int_0^t M_s N ds \right)(x).$$

This is a continuous local martingale field with joint quadratic variation given by

$$(2.5) \quad \langle U_\bullet(x), U_\bullet(y) \rangle_t = \int_0^t M_s(x) M_s(y) d\langle N_\bullet(x), N_\bullet(y) \rangle_s.$$ 

Fix $t \in [0,T]$, some compact set $K \subset D$ and set

$$g(x, y) := \langle U_\bullet(x), U_\bullet(y) \rangle_t.$$
We first consider the case $m = 0$. Choose $\alpha < \delta' < \delta$. Then, for $x, x', y, y' \in K$, we have
\[
\left| g(x, y) - g(x', y) - g(x, y') + g(x', y') \right| = \left| \langle U_\bullet(x) - U_\bullet(x'), U_\bullet(y) - U_\bullet(y') \rangle \right| \leq \left| \int_0^t (M_s(x) - M_s(x')) (M_s(y) - M_s(y')) d\langle N_\bullet(x), N_\bullet(y) \rangle_s \right| \\
+ \left| \int_0^t (M_s(x) - M_s(x')) M_s(x') d\langle N_\bullet(x), N_\bullet(y) - N_\bullet(y') \rangle_s \right| \\
+ \left| \int_0^t M_s(x') (M_s(y) - M_s(y')) d\langle N_\bullet(x) - N_\bullet(x'), N_\bullet(y) \rangle_s \right| \\
+ \left| \int_0^t M_s(x') M_s(y') d\langle N_\bullet(x) - N_\bullet(x'), N_\bullet(y) - N_\bullet(y') \rangle_s \right|.
\]
For the first integral, we use Kunita’s extended Cauchy-Schwarz inequality, as stated in [Kun90] Theorem 2.2.13, to see that
\[
\left| \int_0^t (M_s(x) - M_s(x')) (M_s(y) - M_s(y')) d\langle N_\bullet(x), N_\bullet(y) \rangle_s \right| \\
\leq \left( \int_0^t (M_s(x) - M_s(x'))^2 d\langle N_\bullet(x) \rangle_s \right)^{\frac{1}{2}} \left( \int_0^t (M_s(y) - M_s(y'))^2 d\langle N_\bullet(y) \rangle_s \right)^{\frac{1}{2}} \\
\leq |x - x'|^\delta |y - y'|^\delta' \sup_{s \in [0, T]} \| M_s \|_{\infty; K}^2 \| N_\bullet(x) \|_{\delta'; K} \| N_\bullet(y) \|_{\delta'; K}.
\]
Similarly, for the second integral,
\[
\left| \int_0^t (M_s(x) - M_s(x')) M_s(x') d\langle N_\bullet(x), N_\bullet(y) - N_\bullet(y') \rangle_s \right| \\
\leq \left( \int_0^t (M_s(x) - M_s(x'))^2 d\langle N_\bullet(x) \rangle_s \right)^{\frac{1}{2}} \left( \int_0^t M_s(x')^2 d\langle N_\bullet(y) - N_\bullet(y') \rangle_s \right)^{\frac{1}{2}} \\
\leq |x - x'|^\delta |y - y'|^\delta' \sup_{s \in [0, T]} \| M_s \|_{\infty; K} \| N_\bullet(x) \|_{\delta'; K} \| N_\bullet(y) \|_{\delta'; K}.
\]
From point 3 in theorem 15, we know that there is a version of the joint quadratic variation of $N$ such that $\| \langle N_\bullet, N_\bullet \rangle_T \|_{\delta'; K} < \infty$. The other integrals are estimated similarly. This shows that
\[
\sup_{x, x', y, y' \in K \atop x \neq x', y \neq y'} \left| \frac{g(x, y) - g(x', y) - g(x, y') + g(x', y')}{|x - x'|^\delta |y - y'|^\delta'} \right| < \infty.
\]
Clearly $\| g \|_{\infty; K} < \infty$, thus we have shown that the joint quadratic variation of $U$ has a modification of a continuous $C_{0, \delta'}$-process. Point 1 in theorem 15 shows that $U$ has a modification of a continuous $C_{0, \alpha}$-process. Now let $m \geq 1$. From point 3 in theorem 15 we may deduce that the joint quadratic variation of $N$ has a modification of a continuous $C_{m, \delta'}$-process with
\[
\partial_x^\beta \partial_y^\gamma \langle N_\bullet(x), N_\bullet(y) \rangle = \langle \partial_x^\beta N_\bullet(x), \partial_y^\gamma N_\bullet(y) \rangle.
\]
for every $|\beta|, |\gamma| \leq m$. We may apply Proposition 30 in Appendix iteratively in equation (2.5) to show that $\langle U_\bullet(x), U_\bullet(y) \rangle_t$ has a modification which is $m$-times differentiable with respect to $x$ and $y$, and we can calculate the derivatives using the product rule stated in Proposition 30. As above, one can show that the $m$-th derivative has the claimed Hölder regularity, and we can conclude with point 2 of theorem 15 that $U$ has a modification of a continuous $C^{m,\alpha}$-process. The Itô-Stratonovich conversion formula

$$
\int_0^t M_s \circ N ds = \int_0^t M_s N ds + \frac{1}{2} \langle M_\bullet, N_\bullet \rangle_t
$$

and point 3 in theorem 15 show that the same is true for the Stratonovich integral.

We now come to equation (2.4). In the following, we use $\| \cdot \|_{L^1}$ for the $L^1$-norm with respect to $\mathbb{P}$. For $n \in \mathbb{N}$, set

$$
\tau_n = \inf \left\{ t \in [0, T] : \| M_t \|_{C^{1,\delta}} + \| N_t \|_{C^{1,\delta}} \geq n \right\}.
$$

The random times $\tau_n$ define an increasing sequence of stopping times such that $\mathbb{P}(\tau_n < T) \to 0$ for $n \to 0$. Let $x \in D$, $t > 0$ and choose $h$ such that $x + he_i \in D$. Then

$$
\left\| \frac{1}{h} \left\{ \int_0^{t \wedge \tau_n} (M_{s} \circ N_{ds})(x + he_i) - \int_0^{t \wedge \tau_n} (M_{s} \circ N_{ds})(x) \right\} \right\|_{L^1}
$$

$$
= \left\| \int_0^{t \wedge \tau_n} M_s(x + he_i) - M_s(x) \frac{N_{ds}(x + he_i) - N_{ds}(x)}{h} \right\|_{L^1}
$$

$$
+ \left\| \int_0^{t \wedge \tau_n} M_s(x) \frac{N_{ds}(x + he_i) - N_{ds}(x)}{h} \right\|_{L^1}
$$

$$
\leq \left\| \int_0^{t \wedge \tau_n} \frac{M_s(x + he_i) - M_s(x)}{h} \right\|_{L^1} \left\| N_{ds}(x + he_i) - N_{ds}(x) \right\|_{L^1}
$$

$$
+ \left\| \int_0^{t \wedge \tau_n} \frac{M_s(x + he_i) - M_s(x)}{h} \right\|_{L^1} \left\| \frac{N_{ds}(x + he_i) - N_{ds}(x)}{h} - \partial_t N_{ds}(x) \right\|_{L^1}.
$$

We aim to show that the integrals on the right hand side vanish for $h \to 0$, using the Burkholder-Davis-Gundy inequality. We start with the first integral. Note that since $N$ is a $C^{1,\delta}$ process, also the stopped process is a $C^{1,\delta}$ process and its joint quadratic variation has a modification of a $\tilde{C}^{1,\delta'}$ process for any $\delta' < \delta$. In particular,

$$
\langle N_\bullet(x + he_i) - N_\bullet(x) \rangle_{t \wedge \tau_n} \to 0
$$

almost surely for $h \to 0$. From the Burkholder-Davis-Gundy inequality,

$$
\mathbb{E} \left| \langle N_\bullet(x + he_i) - N_\bullet(x) \rangle_{t \wedge \tau_n} \right|^{p/2} \leq C_p \mathbb{E} \sup_{s \in [0, t \wedge \tau_n]} \left| N_s(x + he_i) - N_s(x) \right|^p \leq C 2^p n^p
$$
which shows that \( \langle N_t(x + he_i) - N_s(x) \rangle_{t \wedge \tau_n} \to 0 \) in \( L^p \) for any \( p \geq 1 \). Using the Burkholder-Davis-Gundy inequality for the first integral gives
\[
\mathbb{E} \left| \int_0^{t \wedge \tau_n} \left( \frac{M_s(x + he_i) - M_s(x)}{h} \right) \left( N_{ds}(x + he_i) - N_{ds}(x) \right) \right| \leq C \mathbb{E} \left| \int_0^{t \wedge \tau_n} \left( \frac{M_s(x + he_i) - M_s(x)}{h} \right)^2 d\langle N_s(x + he_i) - N_s(x) \rangle_s \right|^{\frac{1}{2}}
\]

\[
\leq C \mathbb{E} \sup_{s \in [0, t \wedge \tau_n]} \| \partial_x M_s \|_{\infty}^2 \left| \langle N_s(x + he_i) - N_s(x) \rangle_{t \wedge \tau_n} \right| \leq Cn \mathbb{E} \left| \langle N_s(x + he_i) - N_s(x) \rangle_{t \wedge \tau_n} \right|^{\frac{1}{2}} 
\]

for \( h \to 0 \). For the second integral, the Burkholder-Davis-Gundy inequality gives
\[
\mathbb{E} \left| \int_0^{t \wedge \tau_n} \left( \frac{M_s(x + he_i) - M_s(x)}{h} - \partial_t M_s(x) \right) N_{ds}(x) \right| \leq C \mathbb{E} \left| \int_0^{t \wedge \tau_n} \left( \frac{M_s(x + he_i) - M_s(x)}{h} - \partial_t M_s(x) \right)^2 d\langle N_s(x) \rangle_s \right|^{\frac{1}{2}}
\]

For \( h \to 0 \), we can use dominated convergence twice for the expectation and the Lebesgue-Stieltjes integral to see that this term indeed converges to 0. Now we come to the third integral. Set
\[
N_t^h(x) := \frac{N_t(x + he_i) - N_t(x)}{h}.
\]

As before, we can use the Burkholder-Davis-Gundy inequality to see that
\[
\mathbb{E} \left| \int_0^{t \wedge \tau_n} M_s(x) \left( N_{ds}(x) - \partial_t N_{ds}(x) \right) \right| \leq Cn \mathbb{E} \left| \langle N_t^h(x) - \partial_t N_s(x) \rangle_{t \wedge \tau_n} \right|^{\frac{1}{2}}
\]

Taking a modification of a \( \tilde{\mathcal{U}}^1, \delta \) process of the joint quadratic variation of \( N \) gives, using point 3 of theorem 15 that
\[
\langle N_t^h(x) - \partial_t N_s(x) \rangle_{t \wedge \tau_n} \to 0
\]

almost surely as \( h \to 0 \). Using the Burkholder-Davis-Gundy inequality as above shows that the convergence also holds in \( L^p \) for any \( p \geq 1 \). To conclude, we have shown that
\[
\frac{1}{h} \left( \int_0^{t \wedge \tau_n} M_s(x + he_i) N_{ds}(x + he_i) - \int_0^{t \wedge \tau_n} M_s(x) N_{ds}(x) \right)
\]

\[
\to \int_0^{t \wedge \tau_n} \partial_{x_i} M_s(x) N_{ds}(x) + \int_0^{t \wedge \tau_n} M_s(x) \partial_{x_i} N_{ds}(x)
\]

in \( L^1 \) for \( h \to 0 \). Since we already know that the convergence holds almost surely, the limits coincide and we have shown that
\[
(2.6) \quad \partial_{x_i} \left( \int_0^{t \wedge \tau_n} M_s(x) N_{ds}(x) \right) = \int_0^{t \wedge \tau_n} \partial_{x_i} M_s(x) N_{ds}(x) + \int_0^{t \wedge \tau_n} M_s(x) \partial_{x_i} N_{ds}(x)
\]
holds almost surely for every $n \in \mathbb{N}$. Now,

$$\mathbb{P}\left(\partial_{x_i} \left( \int_0^t M_s(x) N_{ds} (x) \right) \neq \partial_{x_i} \left( \int_0^{t \wedge \tau_n} M_s(x) N_{ds} (x) \right) \right) \leq \mathbb{P}(\tau_n < T) \to 0$$

as $n \to \infty$, therefore

$$\partial_{x_i} \left( \int_0^{t \wedge \tau_n} M_s(x) N_{ds} (x) \right) \to \partial_{x_i} \left( \int_0^t M_s(x) N_{ds} (x) \right)$$

in probability as $n \to \infty$. The same is true for the other two integrals on the right hand side of (2.6) which shows (2.4) for the Itô integral. For the Stratonovich integral, the assertion follows from the Itô-Stratonovich conversion formula and the equality

$$\partial_{x_i} \langle M \bullet (x), N \bullet (x) \rangle_t = \langle \partial_{x_i} M \bullet (x), N \bullet (x) \rangle_t + \langle M \bullet (x), \partial_{x_i} N \bullet (x) \rangle_t.$$  \hfill \Box

Recall the definition of the space $\tilde{C}_b^{n,\delta}$.

**Proposition 17.** Let $M, N : D \times [0, T] \to \mathbb{R}$ be continuous local martingale fields adapted to the same filtration. Assume that the quadratic variation of the processes is given by

$$d\langle M \bullet (x), M \bullet (y) \rangle_t = a_t(x, y) dt \quad \text{resp.}$$

$$d\langle N \bullet (x), N \bullet (y) \rangle_t = b_t(x, y) dt$$

for every $x, y \in D$ and every $t \in [0, T]$. Moreover, assume that there is a $\delta \in (0, 1]$ such that $a$ and $b$ have continuous modifications in the space $\tilde{C}_b^{0,\delta}$. Let $p > 2$ and $\rho \in (0, \delta)$.

(i) Assume either that there is some compact set $K$ in $D$ such that $a_t$ is supported on $K \times K$ for every $t \in [0, T]$ almost surely, or that there is a constant $\kappa > 0$ and some $\eta \in (0, \infty)$ such that

$$\sup_{u \in [0, T]} \left\| \sqrt{a_u(x, x)} \right\|_{L^q(\Omega)} \leq \frac{\kappa}{1 + |x|^\eta}$$

holds for every $x \in D$ where

$$q \geq \max \left\{ \frac{\delta}{\eta (\delta - \rho)} + \frac{1}{\frac{\delta}{\eta (\delta - \rho)} + \frac{1}{\frac{\delta}{\eta (\delta - \rho)}}}, \frac{1}{2 - \frac{1}{p}} \right\}. \quad (2.7)$$

Then the process $M$ has a modification which satisfies

$$\sup_{0 \leq s < t \leq T} \frac{\| M_{ts} \|_{C^0}}{|t - s|^{\frac{1}{p}}} < \infty$$

almost surely.

(ii) Set

$$U_{ts}(x) := \int_s^t (M_{us} N_{du})(x).$$

Assume either that $a$ and $b$ are compactly supported in the sense above, or that there is an $\eta > 0$ and a constant $\kappa > 0$ such that

$$\sup_{u \in [0, T]} \left\| \sqrt{a_u(x, x)} \right\|_{L^q(\Omega)} + \sup_{u \in [0, T]} \left\| \sqrt{b_u(x, x)} \right\|_{L^q(\Omega)} \leq \frac{\kappa}{1 + |x|^\eta} \quad (2.8)$$

for some $\eta \in (0, \infty)$ and every $x \in D$ with $q$ sufficiently large as in (2.7).
Then the process $U$ has a modification which satisfies

$$\sup_{0 \leq s < t \leq T} \frac{\|U_{ts}\|_{C^p}}{|t - s|^{\frac{p}{2}}} < \infty$$

almost surely.

(iii) Assume that $\sup_{t \in [0,T]} \|a_t\|_\delta^\delta$ and $\sup_{t \in [0,T]} \|b_t\|_\delta^\delta$ are almost surely bounded random variables and that either both processes are compactly supported or that (2.8) holds for some $\eta \in (0, \infty)$ and for $q = \infty$.

Then the random variables

$$\sup_{0 \leq s < t \leq T} \frac{\|M_{ts}\|_{C^p}}{|t - s|^{\frac{p}{2}}} \quad \text{and} \quad \sqrt{\sup_{0 \leq s < t \leq T} \frac{\|U_{ts}\|_{C^p}}{|t - s|^{\frac{p}{2}}}}$$

have Gaussian tails.

**Proof** — We start with (i). We assume in a first step that $\sup_{t \in [0,T]} \|a_t\|_\delta^\delta$ is a bounded random variable. Under this assumption, we aim to show that $M$ has a modification such that

(2.9)

$$\sup_{0 \leq s < t \leq T} \frac{\|M_{ts}\|_{C^p}}{|t - s|^{\frac{p}{2}}}$$

is finite and has Gaussian tails. Set

(2.10)

$$\alpha_1 := \left\| \sup_{u \in [0,T]} \|a_u\|_{\delta} \right\|_{L^\infty}.$$

Let $s < t$ and $x, y \in D$ with $0 < |x - y| \leq 1$. By the Burkholder Davis Gundy inequality with optimal constants (cf. [CK91] Theorem A), for every $q \geq 2$,

$$\mathbb{E}|M_{ts}(x) - M_{ts}(y)|^q \leq C^q q^{q/2} \mathbb{E} \left| \int_s^t a_u(x, x) - a_u(x, y) - a_u(y, x) + a_u(y, y) \, du \right|^{q/2}$$

$$\leq C^q q^{q/2} |t - s|^{q/2} \alpha_1^{q/2} |x - y|^{q\delta}$$

where we used the estimate

$$\sup_{u \in [0,T]} |a_u(x, x) - a_u(x, y) - a_u(y, x) + a_u(y, y)| \leq \alpha_1 |x - y|^{2\delta}$$

in the last step. If $a_t$ is supported on some compact set $K$ for every $t \in [0,T]$, we can use the Burkholder Davis Gundy inequality to show that $M_t(x) = 0$ for every $t \in [0,T]$ and every $x \in D \setminus K$ on a set of full measure which, a priori, depends on $t$ and $x$. By a standard density argument, using continuity of $M$, we can deduce that there is a set of full measure on which $M_t$ is supported on $K$ for every $t \in [0,T]$. If $a$ does not have compact support, for $s < t$ and $x \in D$,

$$\mathbb{E}|M_{ts}(x)|^q \leq C^q q^{q/2} \left( \int_s^t a_u(x, x) \, du \right)^{q/2} \leq C^q q^{q/2} |t - s|^{q/2} \left( \frac{k}{1 + |x|^{\eta}} \right)^q.$$

In both cases, we can can conclude with Theorem 12 (iii) that $M$ has a modification such that (2.9) is finite and has Gaussian tails.
Now we drop the assumption that $\alpha_1 < \infty$. Consider the stopped process $M^n_t := M_{t \wedge \tau_n}$ where
\[ \tau_n = \inf \left\{ t \in [0, T] : \sup_{u \in [0,t]} \|a_u\|^\lambda_n \geq n \right\}, \]
hence
\[ \langle M^n_t(x), M^n_t(y) \rangle_t = \int_0^{t \wedge \tau_n} a_u(x, y) \, du. \]
Fix $n \in \mathbb{N}$. As before, we have the estimates
\[ \|M^n_s(x) - M^n_s(y)\|_{L^q} \leq C \sqrt{n} \sqrt{q} |t - s|^{\frac{\lambda}{2}} |x - y|^\delta \]
for every $x, y \in D$, $s < t$ and $q \geq 1$. If $a$ is compactly supported, also $M$ and $M^n$ are compactly supported. Otherwise we have the estimate
\[ \mathbb{E}|M^n_t(x)|^q \leq C \sqrt{q} |t - s|^{\frac{\delta}{2}} \frac{\kappa}{1 + |x|^\eta} \]
for every $x \in D$ and $s < t$ and with $q$ as in (2.7). In both cases, Theorem 12 implies that there is a modification of $M^n$ such that
\[ \sup_{0 \leq s < t \leq T} \frac{\|M^n_t\|_{C^q}}{|t - s|^{\frac{\lambda}{2}}} \]
is finite almost surely for every $n \in \mathbb{N}$. Now,
\[ \mathbb{P} \left( \sup_{0 \leq s < t \leq T} \frac{\|M^n_t\|_{C^q}}{|t - s|^{\frac{\lambda}{2}}} = \infty \right) \leq \mathbb{P} \left( \sup_{0 \leq s < t \leq T} \frac{\|M^n_t\|_{C^q}}{|t - s|^{\frac{\lambda}{2}}} \neq \sup_{0 \leq s < t \leq T} \frac{\|M^n_t\|_{C^q}}{|t - s|^{\frac{\lambda}{2}}} \right) \leq \mathbb{P}(\tau_n < T) \to 0 \]
as $n \to \infty$, which shows that indeed
\[ \sup_{0 \leq s < t \leq T} \frac{\|M^n_t\|_{C^q}}{|t - s|^{\frac{\lambda}{2}}} < \infty \]
almost surely. This shows (i) and the first part of (iii).

We proceed with (ii). As above, we first assume that
\[ \alpha_2 := \sup_{u \in [0,T]} \|a_u\|_{L^\infty} + \sup_{u \in [0,T]} \|b_u\|_{L^\infty} \]
< $\infty$.

Let $x, y \in D$ such that $0 < |x - y| \leq 1$ and $s < t$. By the triangle inequality, for every $q \geq 2$,
\[ \|U_t(x) - U_t(y)\|_{L^{q/2}} \leq \left\| \int_s^t (M_{us}(x) - M_{us}(y)) N_{du}(x) \right\|_{L^{q/2}} \]
\[ + \left\| \int_s^t M_{us}(y)(N_{du}(x) - N_{du}(y)) \right\|_{L^{q/2}}. \]

Note that
\[ \left\langle \int_0^t (M_u(x) - M_u(y)) N_{du}(x) \right\rangle_t = \int_0^t |M_u(x) - M_u(y)|^2 \langle N_u(x) \rangle_{du} \]
\[ \leq \sup_{u \in [0,t]} |M_u(x) - M_u(y)|^2 \langle N_u(x) \rangle_t. \]
Therefore, applying twice the Burkholder Davis Gundy inequality,
\[
\left\| \int_s^t (M_{us}(x) - M_{us}(y))N_{du}(x) \right\|_{L^{q/2}} \leq C \sqrt{q} \left\| \sqrt{\langle N\star(x) \rangle_{ts}} \sup_{u \in [s,t]} |M_u(x) - M_u(y)| \right\|_{L^{q/2}} \\
\leq C \sqrt{q} \left\| \sqrt{\langle N\star(x) \rangle_{ts}} \right\|_{L^{\infty}} \left\| \sup_{u \in [s,t]} |M_u(x) - M_u(y)| \right\|_{L^{q/2}} \\
\leq C q \left\| \sqrt{\langle N\star(x) \rangle_{ts}} \right\|_{L^{\infty}} \left\| \sqrt{\langle M\star(x) - M\star(y) \rangle_{ts}} \right\|_{L^{q/2}}.
\]
Now we have the estimate
\[
\left\| \sqrt{\langle N\star(x) \rangle_{ts}} \right\|_{L^{\infty}} = \left\| \sqrt{\int_s^t b_u(x,x) \, du} \right\|_{L^{\infty}} \leq \sup_{x \in D} \sup_{t \in [0,T]} \left\| \sqrt{b_u(x,x)} \right\|_{L^{\infty}} |t - s|^{\frac{1}{2}},
\]
and \( \sup_{x \in D} \sup_{t \in [0,T]} \left\| \sqrt{b_u(x,x)} \right\|_{L^{\infty}} \) is finite both in the case of \( b \) having compact support or satisfying the stated growth condition. Furthermore, as seen above,
\[
\left\| \sqrt{\langle M\star(x) - M\star(y) \rangle_{ts}} \right\|_{L^{q/2}} \leq \sqrt{\alpha_2} |t - s|^{\frac{3}{2}} |x - y|^{\delta}
\]
which implies
\[
\left\| \int_s^t (M_{us}(x) - M_{us}(y))N_{du}(x) \right\|_{L^{q/2}} \leq C \alpha_2 q |t - s| |x - y|^\delta.
\]
Similarly
\[
\left\| \int_s^t M_{us}(y)(N_{du}(x) - N_{du}(y)) \right\|_{L^{q/2}} \leq C \alpha_2 q |t - s| |x - y|^\delta,
\]
hence we have shown that
\[
\|U_{ts}(x) - U_{ts}(y)\|_{L^{q/2}} \leq C \alpha_2 q |t - s| |x - y|^\delta
\]
holds for every \( s < t \) and \( x, y \in D \) such that \( 0 < |x - y| \leq 1 \). If \( a \) and \( b \) are compactly supported, it follows (e.g. by the Burkholder Davis Gundy inequality) that also \( U \) is compactly supported. Assume now that \( a \) and \( b \) are not compactly supported, but that the stated growth conditions hold. Similarly as above, for \( x \in D \) and \( s < t \),
\[
\|U_{ts}(x)\|_{L^{q/2}} \leq C q \left\| \sqrt{\langle M\star(x) \rangle_{ts}} \right\|_{L^{\infty}} \left\| \sqrt{\langle N\star(x) \rangle_{ts}} \right\|_{L^{q/2}} \\
\leq C q |t - s|^{\frac{1}{2}} + |x|^{\frac{3}{2}}.
\]
Hence in both cases, we may apply Theorem 12 to show that the process \( U \) has a modification such that
\[
\sqrt{\sup_{0 \leq s < t \leq T} \|U_{ts}\|_{C^p}} \leq \sqrt{\frac{C p}{|t - s|}}
\]
is finite almost surely and has Gaussian tails. It remains to prove (ii) when \( \alpha_2 = \infty \). In this case, consider the stopped processes \( M_{ts}^n := M_{t \wedge \sigma_n} \) and \( N_{ts}^n := N_{t \wedge \sigma_n} \) where
\[
\sigma_n = \inf \left\{ t \in [0,T] : \sup_{u \in [0,t]} \|a_u\|_{L^\beta} + \sup_{u \in [0,t]} \|b_u\|_{L^\beta} \geq n \right\}
\]
and set
\[
U_{ts}^n(x) := \int_s^t (M_{us}^n N_{du}^n)(x).
\]
Using the Cauchy-Schwarz inequality, we can show that for every \( n \in \mathbb{N} \) and \( q \) as in (2.11),
\[
\|U^n_{ts}(x) - U^n_{ts}(y)\|_{L^{q/2}} \leq C|t - s||x - y|^{\delta'}.
\]
 Clearly, if \( a \) and \( b \) are compactly supported, \( U^n \) is compactly supported. Otherwise we have the estimate
\[
\|U^n_{ts}(x)\|_{L^{q/2}} \leq C\|\sqrt{\langle M^1_t(x) \rangle_{ts}}\|_{L^q} \|\sqrt{\langle M^2_t(x) \rangle_{ts}}\|_{L^q} \\
\leq \frac{Cq|t - s|}{1 + |x|^{2\eta}}
\] where the constant \( C \) now depends on \( n \) and \( q \). Theorem \[12\] implies that for every \( n \in \mathbb{N} \), \( U^n \) has a modification such that
\[
\sup_{0 \leq s < t \leq T} \frac{\|U^n_{ts}\|_{C^p}}{|t - s|^{\frac{q}{p}}}
\]
is finite almost surely. We conclude as already seen for the first level process.

**Theorem 18.** Let \( M \) be a continuous local martingale velocity field in \( C^{2,\delta}(D, \mathbb{R}^d) \) with continuous local characteristic \( a \) in \( \partial_{\alpha}^{2,\delta} \) for some \( \delta \in (0, 1] \).

(i) Let \( \rho \in (0, \delta) \) and \( p \in (2, 3) \). Assume that either \( a \) has compact support, or that there is an \( \eta \in (0, \infty) \) and a constant \( \kappa > 0 \) such that
\[
\sum_{0 \leq |\alpha| \leq 2} \sup_{u \in [0, T]} \left\| \sqrt{\partial_x^\alpha \partial_y^\alpha a_u(z, z)} \right\|_{L^q} \leq \frac{\kappa}{1 + |z|^{\eta}}
\]
for every \( z \in D \) where \( q \) satisfies
\[
q > \max \left\{ \frac{\delta}{\eta(\delta - \rho)} + d \left( \frac{\delta}{\eta(\delta - \rho)} + \frac{1}{\delta - \rho} \right), \frac{1}{\frac{\delta}{2} - \frac{1}{p}} \right\}.
\]
Then \( M = (M, \mathbb{M}) \), \( \mathbb{M} \) being defined as in (2.2), has a modification of a weak geometric \((p, \rho)\)-rough driver. We call \( M \) the natural lift of \( M \).

(ii) Assume that \( \sup_{t \in [0, T]} \|a_t\|_{C^{2+\delta}} \) is an almost surely bounded random variable. Assume that either \( a \) has compact support, or that there is an \( \eta \in (0, \infty) \) and a constant \( \kappa > 0 \) such that
\[
\sum_{0 \leq |\alpha| \leq 2} \sup_{u \in [0, T]} \left\| \sqrt{\partial_x^\alpha \partial_y^\alpha a_u(z, z)} \right\|_{L^\infty} \leq \frac{\kappa}{1 + |z|^{\eta}}
\]
holds for every \( z \in D \).

Then for every \( p \in (2, 3) \) and \( \rho \in (0, \delta) \), \( M = (M, \mathbb{M}) \) has a modification of a weak geometric \((p, \rho)\)-rough driver, and the random variables
\[
\sup_{0 \leq s < t \leq T} \frac{\|M_{ts}\|_{C^{2+\rho}}}{|t - s|^{\frac{q}{p}}} \quad \text{and} \quad \sqrt{\sup_{0 \leq s < t \leq T} \frac{\|W_{ts}\|_{C^{1+\rho}}}{|t - s|^{\frac{q}{p}}}},
\]

\( W \) being defined as in (2.2), have Gaussian tails.

**Proof** – The claim for \( M \) follows by applying Proposition \[17\] to \( M \) and its derivatives. For \( W \), we use the product rule in Proposition \[16\] for calculating the derivatives and apply Proposition \[17\] afterwards. The estimates for \( W \) together with \( M \) yield the claimed estimates for \( \mathbb{M} \). We leave the details to the reader. □
2.4. Stochastic and rough flows  We keep in this section the notations of the previous sections, and denote in particular by $(\mathcal{F}_t)_{0 \leq t \leq T}$ a filtration to which the semimartingale velocity field $M$ is adapted. Assume that the local characteristic $a$ of $M$ satisfies the boundedness assumptions of point (i) in Theorem 18. Then we can use Theorem 18 to define the natural lift $\tilde{M}$ of $M$ into a rough driver, and one can make sense of the rough flow $\varphi$ as pathwise solution to the equation

$$d\varphi = \tilde{M}(\varphi; dt)$$

using Theorem 5. It follows from equation (1.3), giving $\varphi_{ts}$ as a limit of compositions of $\mu_{bs}$’s, that $\varphi$ is a semimartingale stochastic flow of homeomorphisms. One can read its local characteristics on the Itô formula that it satisfies. Given $x, y \in \mathbb{R}^d$ and $f, g \in C^3_b$, we have

$$f(\varphi_{ts}(x)) = f(x) + (M_{ts}f)(x) + \frac{1}{2} \left\{ \int_s^t M_{us} \cdot M_{du} - M_{du} \cdot M_{us} \right\} f \right\} (x) + \frac{1}{2} (M_{ts}^2 f)(x)$$

$$+ O(|t-s|^{3/2}),$$

with an $O(\cdot)$ term depending only on $\|M\|$ and $\|f\|_{C^3}$, with a similar formula for $g(\varphi_{ts}(y))$. We read on this identity that

$$\lim_{h \downarrow 0} E \left[ \frac{f(\varphi_{t+h,t}(x)) - f(x)}{h} \bigg| \mathcal{F}_t \right] = \lim_{h \downarrow 0} E \left[ \frac{1}{2} (M_{t+h,t}^2 f)(x) \bigg| \mathcal{F}_t \right],$$

and

$$\lim_{h \downarrow 0} \frac{1}{h} E \left[ \{ f(\varphi_{t+h,t}(x)) - f(x) \} \{ g(\varphi_{t+h,t}(y)) - g(y) \} \bigg| \mathcal{F}_t \right]$$

$$= \lim_{h \downarrow 0} \frac{1}{h} E \left[ (M_{t+h,t} f)(x)(M_{t+h,t} g)(y) \bigg| \mathcal{F}_t \right] = \langle f, g \rangle(t, x, y).$$

So the semimartingale stochastic flow $\varphi$ has the same local characteristics as the semimartingale stochastic flow generated by the Stratonovich differential equation

(2.13)

$$dx_t = M_{cdt}(x_t);$$

they coincide by theorem 14 such as stated in theorems 4.4.1 and 4.5.1 in Kunita’s book, as assumption (2.11) on the local characteristic $a$ of $M$ is clearly stronger than the optimal assumptions of Kunita.

Theorem 19. Let $M$ be a continuous local martingale velocity field in $C^{2,\delta}(\mathbb{R}^d, \mathbb{R}^d)$, for some $\delta \in (\frac{3}{4}, 1]$, with continuous local characteristic $a$ in $C^2_b$. Let $\mathcal{M}$ be the rough driver associated with $M$ by theorem 18. Under the condition that $a$ has compact support or that the growth assumption (2.11) holds, the rough flow solution to the differential equation

$$d\varphi = \mathcal{M}(\varphi; dt)$$

coincides with the stochastic flow generated by the Stratonovich differential equation (2.13).
2.5. Strong approximations. We give in this section an example of use of the continuity of the Itô map, in the setting of rough drivers and rough flows, by proving a Wong-Zakai type theorem for semimartingale stochastic flows of maps. That is, we prove that such flows are limits in probability of flows generated by ordinary differential equations. Granted the continuity of the Itô map, the core of the proof consists in showing that a rough lift of a continuous piecewise linear time interpolation of a semimartingale velocity field \( M \) converges in probability to \( M \) in the topology of rough drivers.

As in the last section, let \( M : [0, T] \to C([D, \mathbb{R}^d]) \) be a continuous local martingale velocity field with quadratic variation
\[
\langle M^i(x), M^j(y) \rangle_t = \int_0^t a_{ij}^s(x, y) ds
\]
and \( \delta \in (0, 1] \). Let \( D = \{0 = t_0 < t_1 < \ldots < t_n = T\} \) be a partition of the interval \([0, T]\) and define the piecewise linear approximation of \( M \) with respect to \( D \) as
\[
M^D_t := M_{t_i} + (t - t_i) \frac{M_{t_{i+1}} - M_{t_i}}{t_{i+1} - t_i} \quad \text{if } t \in [t_i, t_{i+1}].
\]
Note that \( D \mapsto M^D \) commutes with the spatial derivative, i.e.
\[
\partial_x \left( M^D \right) = \left( \partial_x M \right)^D =: \partial_x M^D.
\]
Define the mesh size of the partition by the formula \( |D| := \max_i |t_{i+1} - t_i| \). We define the first order differential operator
\[
W^D_{ts} := \frac{1}{2} \left( \int_s^t M^D_{us} \partial_i M^D_{du} \partial_k M^D_{us} \partial_i M^D_{us} du \right) \partial_k
\]
via usual Riemann-Stieltjes integration. Then we set
\[
M^D_{ts} := W^D_{ts} + \frac{1}{2} M^D_{ts} M^D_{ts}
\]
and
(2.14) \[
M^D := \left( M^D, M^D \right).
\]
Our aim is to prove that \( M^D \) converges towards the natural lift \( M \) of \( M \) when \( |D| \to 0 \) in probability (or even in \( L^p(\Omega) \)) in the topology of rough drivers. Note that
\[
W_{ts} = \frac{1}{2} \int_s^t (M_{us} M_{du} - M_{du} M_{us}) = \frac{1}{2} \int_s^t (M_{us} M_{odu} - M_{odu} M_{us}),
\]
hence it is enough to prove that the Riemann-Stieltjes integrals of the approximated processes converge towards the Stratonovich integrals (in the right topology), and this is what we are going to do.

**Lemma 20.** Let \( M = (M^1, \ldots, M^d) : [0, T] \to \mathbb{R}^d \) be a continuous local martingale and assume that
\[
\| (M)_T \|_{L^2} \leq K < \infty
\]
for some \( q \geq 0 \) and \( K > 0 \). Let \( M \) and \( M^D \) be the associated rough paths lifts to \( M \) and \( M^D \), i.e. \( M = (M, \tilde{M}) \) and \( M^D = (M^D, \tilde{M}^D) \) where

\[
\tilde{M} = \int_s^t M_{us} M_{du} \quad \text{and} \quad \tilde{M}^D = \int_s^t M_{us}^D M_{du}^D
\]

are iterated Stratonovich, resp. Riemann-Stieltjes, integrals. Set

\[
\varepsilon := \left\| \sup_{0 < v - u < |D|} \|M_{vu}\| \right\|_{L^q}
\]

and assume that \( \varepsilon \leq 1 \).

Then for every \( \delta \in (0, 1/3) \) there is a constant \( C = C(\delta, q, K) \) such that

\[
\|M_{ts} - M_{ts}^D\|_{L^q} \leq C\varepsilon^{\frac{q}{2}} \|\langle M \rangle_{ts}\|_{L^2}^{1 - \frac{\delta}{2}} \quad \text{and}
\]

\[
\|M_{ts} - M_{ts}^D\|_{L^q} \leq C\varepsilon^{\frac{q}{2}} \|\langle M \rangle_{ts}\|_{L^2}^{1 - \frac{\delta}{2}}
\]

for every \( s, t \in [0, T] \).

**Proof** — Let \( d \) denote the Carnot-Caratheodory metric on the step-two free nilpotent Lie group \( G^2_d \) over \( \mathbb{R}^d \) [cf. \( \text{[FV10, Chapter 7]} \)]. By interpolation and \( \text{[FV10, Proposition 8.15]} \), for fixed \( s < t \) and \( 2 < p' < p < 3 \),

\[
d(M_{ts}, M_{ts}^D) \leq \left( d_{0-Höld}(M, M^D)^{1 - \frac{q'}{p}} d_{\var\text{-var};[s,t]}(M, M^D)^{\frac{q'}{p}} \right)
\]

\[
\lesssim \left( d_{\infty}(M, M^D)^{1 - \frac{q'}{p}} + d_{\infty}(M, M^D)^{\frac{1}{2} - \frac{q'}{p}} (\|M\|_{30}^{\frac{1}{2} - \frac{q'}{2p}} + \|M^D\|_{30}^{\frac{1}{2} - \frac{q'}{2p}}) \right)
\]

\[
\times \left( \|M\|_{p'-\text{var};[s,t]}^{p'} + \|M^D\|_{p'-\text{var};[s,t]}^{p'} \right).
\]

Taking the \( q \)-th moment, using Hölder’s inequality and Cauchy-Schwarz, we obtain

\[
\mathbb{E}\left[ d(M_{ts}, M_{ts}^D)^q \right]
\]

\[
\lesssim \left( \mathbb{E}\left[ d_{\infty}(M, M^D)^q \right] + \sqrt{\mathbb{E}\left[ d_{\infty}(M, M^D)^q \right]} \left( \mathbb{E}[\|M\|_{30}^q] + \mathbb{E}[\|M^D\|_{30}^q] \right)^{1 - \frac{q'}{p}} \right)^{\frac{q'}{p}}
\]

\[
\times \left( \mathbb{E}\left[ \|M\|_{p'-\text{var};[s,t]}^q \right] + \mathbb{E}\left[ \|M^D\|_{p'-\text{var};[s,t]}^q \right] \right)^{\frac{q'}{p}}.
\]

By \( \text{[FV10, Theorem 14.8 and Theorem 14.15]} \),

\[
\mathbb{E}[\|M\|_{30}^q] \lesssim \mathbb{E}\left[ \langle M \rangle_T^{\frac{q}{2}} \right] \quad \text{and}
\]

\[
\mathbb{E}[\|M^D\|_{30}^q] \lesssim \mathbb{E}\left[ \|M^D\|_{\var\text{-var}}^q \right] \lesssim \mathbb{E}\left[ \langle M \rangle_T^{\frac{q}{2}} \right].
\]

Using the same Theorems, we also have

\[
\mathbb{E}\left[ \|M\|_{p'-\text{var};[s,t]}^q \right] \lesssim \mathbb{E}\left[ \langle M \rangle_{ts}^{\frac{q}{2}} \right] \quad \text{and} \quad \mathbb{E}\left[ \|M^D\|_{p'-\text{var};[s,t]}^q \right] \lesssim \mathbb{E}\left[ \langle M \rangle_{ts}^{\frac{q}{2}} \right].
\]
The estimate [FV10, Equation (14.6) on p. 400] gives
\[
\mathbb{E} \left[ d_{\infty} (\mathbf{M}, \mathbf{M}^D)^q \right] \lesssim \left( \mathbb{E} \left[ \sup_{0 \leq u \leq |D|} \| M_{uv} \|^q \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ (M)_{T}^{\frac{q}{2}} \right] \right)^{\frac{1}{2}} + \mathbb{E} \left[ \sup_{0 \leq v, w \leq |D|} \| M_{vw} \|^q \right] \\
\lesssim \varepsilon^\frac{q}{2} (K^\frac{q}{2} + 1).
\]

Setting \( \delta = 1 - \frac{K^q}{p} \), we have shown that
\[
\left\| d(M_{ts}, M^D_{ts}) \right\|_{L^q} \lesssim \varepsilon^\frac{q}{2} \| (M)_{ts} \|_{L^\frac{q}{2}}.
\]

The result follows by equivalence of homogeneous norms on \( G^2 \) ([FV10, Theorem 7.44]). □

**Proposition 21.** Let \( M, N : D \times [0, T] \to \mathbb{R} \) be continuous local martingale fields adapted to the same filtration. Assume that the quadratic variation of the processes is given by

\[
d(x,y) = \sup_{a \in [0, u], b \in [0, v]} \langle M \rangle_{u,v},
\]

for every \( x, y \in D \) and every \( t \in [0, T] \). Moreover, assume that there is a \( \delta \in (0, 1) \) such that \( a \) and \( b \) have continuous modifications in the space \( C^{0,\delta}_b \). Let \( p > 2 \) and \( \rho \in (0, \delta) \).

(i) Assume either that there is some compact set \( K \) in \( D \) such that \( a_t \) is supported on \( K \times K \) for every \( t \in [0, T] \) almost surely, or that there is an \( \eta \in (0, \infty) \) and a constant \( \kappa > 0 \) such that

\[
\sup_{u \in [0, T]} \left\| \sqrt{a_u(x, x)} \right\|_{L^q} \leq \frac{\kappa}{1 + |x|^{\eta}}
\]

for every \( x \in D \) where

\[
q > \max \left\{ \frac{\delta}{\eta(\delta - \rho)} + d \left( \frac{\delta}{\eta(\delta - \rho)} + \frac{1}{\delta - \rho} \right), \frac{1}{2} - \frac{1}{p} \right\}.
\]

Let \( M \) be the modification of the process given in Proposition 17. Then

\[
\sup_{0 \leq s < t \leq T} \left\| M_{ts} - M^D_{ts} \right\|_{C^p} \to 0
\]

in probability when \( |D| \to 0 \).

(ii) Define

\[
U_{ts}(x) := \int_s^t (M_{us}N_{odu})(x) \quad \text{and} \quad U^D_{ts}(x) := \int_s^t (M^D_{us}N^D_{odu})(x)
\]

as Stratonovich resp. Riemann-Stieltjes integral. Assume either that \( a \) and \( b \) are compactly supported in the sense above, or that there is an \( \eta > 0 \) and a constant \( \kappa > 0 \) such that

\[
\sup_{u \in [0, T]} \left\| \sqrt{a_u(x, x)} \right\|_{L^q} + \sup_{u \in [0, T]} \left\| \sqrt{b_u(x, x)} \right\|_{L^q} \leq \frac{\kappa}{1 + |x|^{\eta}}
\]

for every \( x \in D \) with \( q \) sufficiently large as in (2.15). Let \( U \) be the modification of the process given in Proposition 17.
Then
\[(2.18) \sup_{0 \leq s < t \leq T} \left\| U_{ts} - U_{ts}^{P} \right\|_{C^p} \to 0 \]
in probability when $|D| \to 0$.

(iii) Assume that $\sup_{u \in [0,T]} \|a_u\|_{\delta}^\wedge$ and $\sup_{u \in [0,T]} \|b_u\|_{\delta}^\wedge$ are almost surely bounded random variables and that either both processes are compactly supported or that \((2.17)\) holds for some $\eta \in (0, \infty)$ and for $q = \infty$.

Then the convergences in \((2.16)\) and \((2.18)\) hold in $L^q$ for every $q \geq 1$.

**Proof** – We will only give a proof in the case of $a$ and $b$ having compact support; the other case works analogous, as seen in the proof of Proposition \[17\]. We start with (i).

Assume first that $\sup_{u \in [0,T]} \|a_u\|_{\delta}^\wedge$ is an almost surely bounded random variable. Fix $x, y \in D$ such that $0 < |x - y| \leq 1$ and let $s < t \in [0,T]$. Define the martingale $\hat{M} := M(x) - M(y)$ and let $\hat{M}$ denote its canonical rough path lift (given by $\hat{M}$ and its iterated Stratonovich integrals). From the Burkholder Davis Gundy inequality for enhanced martingales (cf. \[FV10, Theorem 14.8\]), for every $q \geq 2$,
\[
\left\| \sup_{0 < u \leq |D|} \|\hat{M}_{vu}\| \right\|_{L^q} \leq C \left\| \sqrt{\langle \hat{M} \rangle_D} \right\|_{L^q} \leq C|D|^\frac{1}{2}|x - y|\delta
\]
where the constant $C$ depends on the essential supremum of $\sup_{u \in [0,T]} \|a_u\|_{\delta}^\wedge$. We may assume that $|D|$ is sufficiently small such the the right hand side of the equation is smaller than 1. By Lemma \[20\] for every $q \geq 2$ and every $\beta \in (0, 1/3)$,
\[
\left\| M_{ts}(x) - M_{ts}^P(x) - M_{ts}(y) + M_{ts}^P(y) \right\|_{L^q} \leq C|D|^\frac{\beta}{16}|x - y|^{\beta \delta} \left\| \langle M(x) - M(y) \rangle_{ts} \right\| \frac{1}{2}\frac{1}{\beta \delta}
\leq C|D|^\frac{\beta}{16}|x - y|^{\delta(1 - \frac{2}{16})}|t - s|^{\frac{1 - \beta}{2}}.
\]
We have already seen that
\[
\left\| M_{ts}(x) - M_{ts}(y) \right\|_{L^q} \leq C|x - y|^{\delta}|t - s|^{\frac{1}{2}},
\]
and by the triangle inequality
\[
\left\| M_{ts}^P(x) - M_{ts}^P(y) \right\|_{L^q} \leq C|x - y|^{\delta}|t - s|^{\frac{1}{2}}.
\]
Choosing $\beta$ small enough and $q$ large enough, we can apply Theorem \[13\] to see that
\[
\left\| \sup_{s < t} \left\| M_{ts} - M_{ts}^P \right\|_{C^p} \right\| \leq C|D|^\frac{\beta}{16}
\]
which shows the claim. If the essential supremum of $\sup_{u \in [0,T]} \|a_u\|_{\delta}^\wedge$ is not bounded, define the stopping times
\[
\tau_n = \inf \left\{ t \in [0,T] : \sup_{u \in [0,t]} \|a_u\|_{\delta} \geq n \right\}
\]
and set $M^n := M_{t \wedge \tau_n}$. We can repeat the argument above and conclude that

$$\sup_{s \leq t} \frac{\|M^n_{ts} - M^n_{ts}\|_{L^p}}{|t-s|^{1/p}} \to 0$$

in probability as $|D|$ tends to 0, and every $n \in \mathbb{N}$. Fix some $\varepsilon > 0$ and some $n \in \mathbb{N}$. Then

$$\mathbb{P} \left( \left\| \sup_{s < t} \frac{\|M_{ts} - M^n_{ts}\|_{L^p}}{|t-s|^{1/p}} \right\|_{C^p} \geq \varepsilon \right)$$

$$\leq \mathbb{P} \left( \left\| \sup_{s < t} \frac{\|M_{ts} - M^n_{ts}\|_{L^p}}{|t-s|^{1/p}} \right\|_{C^p} - \sup_{s < t} \frac{\|M^n_{ts} - M^n_{ts}\|_{L^p}}{|t-s|^{1/p}} \geq \frac{\varepsilon}{2} \right)$$

$$+ \mathbb{P} \left( \left\| \sup_{s < t} \frac{\|M^n_{ts} - M^n_{ts}\|_{L^p}}{|t-s|^{1/p}} \right\|_{C^p} \geq \frac{\varepsilon}{2} \right)$$

$$\leq \mathbb{P} (\tau_n < T) + \mathbb{P} \left( \left\| \sup_{s < t} \frac{\|M^n_{ts} - M^n_{ts}\|_{L^p}}{|t-s|^{1/p}} \right\|_{C^p} \geq \frac{\varepsilon}{2} \right),$$

and the first term converges to 0 for $n \to \infty$. This shows that indeed

$$\sup_{s < t} \|M^n_{ts} - M^n_{ts}\|_{L^p} \to 0$$

in probability when $|D| \to 0$.

Now we prove (ii). As above, we first assume that $\sup_{u \in [0,T]} \|a_u\|^{\Delta}$ and $\sup_{u \in [0,T]} \|b_u\|^{\Delta}$ are almost surely bounded random variables. Let $x, y \in D$ with $0 < |x - y| \leq 1$ and $s < t$. We have already seen that in case of the Itô integral, for every $q \geq 2$,

$$\left\| \int_s^t (M_{us}N_{du})(x) - \int_s^t (M_{us}N_{du})(y) \right\|_{L^q} \leq C|t-s||x-y|^\delta.$$

Moreover, we have

$$\left\| \langle M(x) - N(x) \rangle ts - \langle M(y) - N(y) \rangle ts \right\|_{L^{2q}}$$

$$\leq \left\| \langle M(x) - M(y) \rangle ts \right\|_{L^{2q}} + \left\| \langle M(y) - N(x) - N(y) \rangle ts \right\|_{L^{2q}}$$

$$\leq \sqrt{\langle M(x) - M(y) \rangle ts} \sqrt{\langle N(x) - N(y) \rangle ts} + \sqrt{\langle M(y) - N(x) - N(y) \rangle ts} \sqrt{\langle M(x) \rangle ts}$$

$$\leq C|t-s||x-y|^\delta.$$
By the triangle inequality,
\[
\left| U^\mathcal{D}_{ts}(x) - U^\mathcal{D}_{ts}(y) \right|_{L^\frac{q}{2}} 
\leq \left| \int_s^t \left( M^\mathcal{D}_{us}(x) - M^\mathcal{D}_{us}(y) \right) N^\mathcal{D}_{du}(x) \right|_{L^\frac{q}{2}} + \left| \int_s^t M^\mathcal{D}_{us}(y) \left( N^\mathcal{D}_{du}(x) - N^\mathcal{D}_{du}(y) \right) \right|_{L^\frac{q}{2}}.
\]
Now we define the martingale \( \hat{M} := \left( \frac{M(x) - M(y)}{|x - y|^\beta}, N(x) \right) \). We can check that for its quadratic variation, we have
\[
\left\langle \langle \hat{M} \rangle \right\rangle_{ts} \leq C|t - s|
\]
where the constant \( C \) does not depend on \( x \) or \( y \). From \cite[Theorem 14.15]{FV10}, it follows that
\[
\left| \int_s^t \left( M^\mathcal{D}_{us}(x) - M^\mathcal{D}_{us}(y) \right) N^\mathcal{D}_{du}(x) \right|_{L^\frac{q}{2}} \leq C|t - s| |x - y|^\delta.
\]
Similarly,
\[
\left| \int_s^t M^\mathcal{D}_{us}(y) \left( N^\mathcal{D}_{du}(x) - N^\mathcal{D}_{du}(y) \right) \right|_{L^\frac{q}{2}} \leq C|t - s| |x - y|^\delta
\]
and hence
\[
\left| U^\mathcal{D}_{ts}(x) - U^\mathcal{D}_{ts}(y) \right|_{L^\frac{q}{2}} \leq C|t - s||x - y|^\delta.
\]
Therefore, by the triangle inequality,
\[
\left| U^\mathcal{D}_{ts}(x) - U^\mathcal{D}_{ts}(x) - U^\mathcal{D}_{ts}(y) + U^\mathcal{D}_{ts}(y) \right|_{L^\frac{q}{2}} \leq C|t - s||x - y|^\delta.
\]
On the other hand, we can apply Lemma \( \ref{lemma} \) to the martingale \( M(x) \), \( x \) fixed, to see that for every \( q \geq 2 \) and every \( \beta \in (0, \frac{1}{3}) \),
\[
\left| U^\mathcal{D}_{ts}(x) - U^\mathcal{D}_{ts}(x) \right|_{L^\frac{q}{2}} \leq C|\mathcal{D}|^\frac{\beta}{\frac{q}{2}} \left\langle \langle M(x) \rangle \right\rangle_{ts} \left\langle \langle M(x) \rangle \right\rangle_{ts} \leq C|\mathcal{D}|^\frac{\beta}{\frac{q}{2}} |t - s|^{-\beta}.
\]
The same estimate holds if we replace \( x \) by \( y \), and by the triangle inequality, we also get
\[
\left| U^\mathcal{D}_{ts}(x) - U^\mathcal{D}_{ts}(x) - U^\mathcal{D}_{ts}(y) + U^\mathcal{D}_{ts}(y) \right|_{L^\frac{q}{2}} \leq C|\mathcal{D}|^\frac{\beta}{\frac{q}{2}} |t - s|^{-\beta}.
\]
Interpolating the two inequalities \( \ref{lemma} \) and \( \ref{lemma} \), we see that for every \( q \geq 2 \), every \( \beta \in (0, 1/3) \) and every \( \lambda \in [0, 1] \), we have the estimate
\[
\left| U^\mathcal{D}_{ts}(x) - U^\mathcal{D}_{ts}(x) - U^\mathcal{D}_{ts}(y) + U^\mathcal{D}_{ts}(y) \right|_{L^\frac{q}{2}} \leq C|\mathcal{D}|^\frac{\lambda\beta}{\frac{q}{2}} |x - y|^{(1-\lambda)\delta} |t - s|^{1-\lambda\beta}.
\]
Choosing \( \lambda > 0 \) and \( \beta > 0 \) small enough and \( q \) large enough, we can again use Theorem \( \ref{thm} \) to see that
\[
\sup_{s < t} \left| U^\mathcal{D}_{ts} - U^\mathcal{D}_{ts} \right|_{L^\frac{q}{2}} \leq C|\mathcal{D}|^\frac{\lambda\beta}{\frac{q}{2}}
\]
which proves the claim if \( \sup_{u \in [0,T]} ||a_u||_{L^\beta} \) and \( \sup_{u \in [0,T]} ||b_u||_{L^\beta} \) are bounded random variables. The general case follows by the same stopping argument as above. \( \triangleright \)

**Theorem 22.** Let \( M \) be a continuous local martingale velocity field in \( C^{2,\delta}(D, \mathbb{R}^d) \), for some \( \delta \in (0, 1] \), with continuous local characteristic \( a \) in \( \mathcal{C}^{2,\delta}_b \).
(i) Let $\rho \in (0, \delta)$ and $p \in (2, 3)$. Assume that either $a$ has compact support, or that there is an $\eta \in (0, \infty)$ and a constant $\kappa > 0$ such that

$$
\sum_{0 \leq |\alpha| \leq 2} \sup_{u \in [0, T]} \left\| \sqrt{\partial_x^\alpha \partial_y^\alpha a_u(z, z)} \right\|_{L^q} \leq \frac{\kappa}{1 + |z|^{\eta}}
$$

for every $z \in D$ where $q$ satisfies

$$
q > \max \left\{ \frac{\delta}{\eta(\delta - \rho)} + d \left( \frac{\delta}{\eta(\delta - \rho)} + \frac{1}{\delta - \rho} \right) \cdot \frac{1}{\frac{2}{p} - \frac{1}{2}} \right\}.
$$

Let $M = (M, M)$ be the weak geometric $(p, \rho)$-rough driver given in Theorem 18. Then

$$
M^D \to M
$$
in probability for $|D| \to 0$, with $M^D$ defined as in $\text{(2.14)}$.

(ii) Assume that $\sup_{u \in [0, T]} \|a_u\|^\wedge_\delta$ is almost surely bounded, and that either $a$ has compact support or that the growth condition $\text{(2.21)}$ holds for some $\eta \in (0, \infty)$, a constant $\kappa > 0$ and $q = \infty$.

Then for every $p \in (2, 3)$ and $\rho \in (0, \delta)$, $M = (M, M)$ has a modification of a weak geometric $(p, \rho)$-rough driver, and the convergence in $\text{(2.22)}$ holds in $L^q$ for every $q \geq 1$.

**Proof** – This follows by applying Proposition 21 to $M, W,$ and $M^D$ and $W^D$ and its derivatives. We use the product rule in Proposition 16 for calculating the derivatives of $W$ and Proposition 30 for the derivatives of $W^D$. The details are left to the reader.

It follows then directly from this statement and the continuity of the Itô solution map, theorem 5 that the solution flow to the equation

$$
d\varphi = M(\varphi; dt)
$$

satisfies a Wong-Zakai theorem. Using that the flow coincides with the one generated by the corresponding Stratonovich SDE, we obtain the following corollary.

**Corollary 23.** Let $M$ be a continuous local martingale velocity field in $C^{2,\delta}(\mathbb{R}^d, \mathbb{R}^d)$, for some $\delta \in (\frac{2}{3}, 1]$, with local characteristic $a$ in $\hat{C}_b^{2,\delta}$. Assume that $a$ has compact support or that the growth assumption $\text{(2.11)}$ holds. Let $\varphi$ be the flow generated by the Stratonovich solution to

$$
d\varphi = M(\varphi; \circ dt)
$$

and $\varphi^D$ be the pathwise solution to

$$
d\varphi^D = M^D(\varphi^D; dt).
$$

Then $\varphi^D \to \varphi$ in the space of $C^\rho$ homeomorphisms uniformly in probability when $|D| \to 0$ for all $\rho \in (0, \delta)$. 

Application: Large deviations

We provide in this section another illustration of use of the continuity of the Itô map by proving a large deviation theorem for Brownian flows. Relatively few works were dedicated to these topics, and we mention [BDM10] and [DD12]. In [BDM10], Dupuis and his co-authors use Dupuis’ weak convergence approach to large deviation principles to prove such a result for Brownian flows of maps, building on a general large deviation criterion proved earlier in [BDM08]. Dereich and Dimitroff’s approach in [DD12] is more in the line of the present work. They consider Brownian flows of maps as solutions to rough differential equations driven by a Banach space valued Brownian rough path, whose construction in a vector field setting was made possible by the previous work [Der10] of Dereich. The support and large deviation theorems for Brownian flows are then inherited from the corresponding results proved in [Der10] for the above mentioned vector field-valued Brownian rough

Let $(E, \mathcal{H}_1, \gamma)$ be a Gaussian Banach space with norm $\| \cdot \|$, i.e. $(E, \| \cdot \|)$ is a separable Banach space, $\gamma$ is a Gaussian measure defined on the Borel $\sigma$-algebra and $\mathcal{H}_1$ the Cameron-Martin Hilbert space (cf. [Bog98] or [Led96, Chapter 4] for the precise definitions and further properties). Recall that $\mathcal{H}_1$ is continuously embedded in $E$, and for every $h \in \mathcal{H}_1$,

$$\| h \| \leq \sigma_\gamma \| h \|_{\mathcal{H}_1} = \sigma_\gamma \sqrt{\langle h, h \rangle_{\mathcal{H}_1}}$$

where

$$\sigma_\gamma^2 := \int_E \| x \|^2 \gamma(dx).$$

A process $X: [0, T] \to E$, defined on some probability space, is called a $E$-valued Wiener process if it has almost surely continuous sample paths starting from 0, has independent increments, and for every $\xi \in E^*$, the distribution of $\langle X_t - X_s, \xi \rangle$ is a centered Gaussian random variable with variance $|t - s| \| \xi \|^2_{\mathcal{H}_1}$ (cf. [LLQ02] and [Der10] for more properties about $E$-valued Wiener processes). The law of $X$ on the space $C([0, T], E)$ is again Gaussian, and one can see that the corresponding Cameron-Martin space $\mathcal{H}$ is given by

$$\mathcal{H} = \left\{ \int_0^* f_s \, ds : f \in L^2([0, T], \mathcal{H}_1) \right\}$$

where the integral is a Bochner-integral. Moreover, if $h_i^t = \int_0^t \dot{h}_i^s \, ds$, $i = 1, 2$, the scalar product is given by

$$\langle h_1^t, h_2^t \rangle_{\mathcal{H}} = \int_0^T \langle h_1^s, h_2^s \rangle_{\mathcal{H}_1} \, ds.$$

Let $p \in [1, \infty)$. In the following, we will use the notion of $p$-variation of a path $h: [0, T] \to E$ which is defined as

$$\| h \|_{p\text{-var}; [s, t]} := \sup_{(t_i) \subset [s, t]} \left( \sum_{t_i} \| h_{t_{i+1}} - h_{t_i} \|^p \right)^{1/p}$$

where the supremum is taken over all finite partitions $(t_i)$ of the interval $[s, t]$. If $\| h \|_{1\text{-var}; [0, T]} < \infty$, we say that $h$ has finite variation.
Lemma 24. For every $h \in \mathcal{H}$ we have
\begin{equation}
\sup_{0 \leq s < t \leq T} \| h_t - h_s \| \leq \sigma \gamma \sqrt{(h,h)_{\mathcal{H}}}
\end{equation}
and for $[s,t] \subseteq [0,T]$,
\begin{equation}
\| h \|_{1-\text{var}[s,t]} \leq \sigma \gamma |t-s|^{\frac{1}{2}} \sqrt{(h,h)_{\mathcal{H}}}.
\end{equation}
In particular, every $h \in \mathcal{H}$ is $\frac{1}{2}$-Hölder continuous and has finite variation on $[0,T]$.

Proof – Clearly, (3.1) follows from (3.2), hence we only prove the second estimate. Let $h \in \mathcal{H}$ with $h(t) = \int_0^t \dot{h}_s \, ds$ and let $(t_i)$ be a partition of some interval $[s,t] \subseteq [0,T]$. Then
\begin{align*}
\sum_i \| h_{t_{i+1}} - h_{t_i} \| &\leq \sum_i \int_{t_i}^{t_{i+1}} \| \dot{h}_u \| \, du = \int_s^t \| \dot{h}_u \| \, du \\
&\leq |t-s|^{\frac{1}{2}} \left( \int_0^T \| \dot{h}_u \|^2 \, du \right)^{\frac{1}{2}} \leq \sigma \gamma |t-s|^{\frac{1}{2}} \sqrt{(h,h)_{\mathcal{H}}}.
\end{align*}
Taking the supremum over all partitions shows the claim.

Let $D$ be a connected subset in $\mathbb{R}^d$, $m \in \mathbb{N}_0$ and $\delta \in (0,1]$. In the following, we would like to take the space $C_b^{m,\delta}(D,\mathbb{R}^d)$ as $E$ and consider a Gaussian measure on this space. However, $C_b^{m,\delta}(D,\mathbb{R}^d)$ is not separable (which is usually the case for Hölder-type spaces). Instead, we define the space $C_b^{m,0,\delta}(D,\mathbb{R}^d)$ as the closure of smooth paths from $D$ to $\mathbb{R}^d$ with respect to the norm $\| \cdot \|_{m+\delta}$. As for Hölder spaces, one can show that these spaces are separable. From now on, let $E = C_b^{m,0,\delta}(D,\mathbb{R}^d)$ and assume that there is a Gaussian Banach space $(E,\mathcal{H}_1,\gamma)$.

If $v$ is a $C_b^{m,\delta}(D,\mathbb{R}^d)$ valued path with finite variation and if $m \geq 1$, we define the pair $S(v)_{ts} = (v_{ts}, u_{ts})$ by setting
\begin{align*}
v_{ts}(x) &= v_t(x) - v_s(x) \quad \text{and} \quad u_{ts} = w_{ts} + \frac{1}{2} v_{ts} v_{ts}
\end{align*}
where $w_{ts}$ is the first order differential operator
\begin{equation*}
w_{ts} = \frac{1}{2} \left( \int_s^t u_{us} \cdot v_{du} - v_{du} \cdot u_{us} \right)
\end{equation*}
and the integral is a Riemann-Stieltjes integral. Note that if $X$ is a Wiener process in $C_b^{m,0,\delta}$, $S(h)$ is always defined for every Cameron-Martin path $h$ since these paths are continuous and have bounded variation by Lemma 24. Moreover, the following holds:

Lemma 25. Let $(E,\mathcal{H}_1,\gamma)$ be a Gaussian Banach space with $E = C_b^{2,0,\delta}(D,\mathbb{R}^d)$ and $\delta \in (0,1]$. Then, for every $h \in \mathcal{H}$, $S(h)$ is a geometric $(2,\delta)$-rough driver, and there is a constant $C$ such that
\begin{align*}
\sup_{s < t} \frac{\| h_t - h_s \|_{c^{2+\delta}}}{|t-s|^{\frac{1}{2}}} &\leq \sigma \gamma \sqrt{(h,h)} \quad \text{and} \quad \sup_{s < t} \frac{\| w_{ts} \|_{c^{1+\delta}}}{|t-s|} \leq C \sigma \gamma \sqrt{(h,h)}
\end{align*}
where
\begin{equation*}
w_{ts} = \frac{1}{2} \left( \int_s^t h_{us} \cdot h_{du} - h_{du} \cdot h_{us} \right).\]
Proof – Let \( h \in \mathcal{H} \) and \( S(h) = (h, \mathbb{H}) \) be defined as above. The claim for \( h \) follows directly from Lemma 24 and the algebraic condition for \( (h, \mathbb{H}) \) follows from well-known identities for Riemann-Stieltjes integrals. Let \( i, k \in \{1, \ldots, d\} \), \( x \in D \) and \( s < t \). Then, by Riemann-Stieltjes estimates and Lemma 24

\[
\left| \int_s^t h^i_{ux}(x) \partial_i h^k_{du}(x) \right| \leq \sup_{u \in [s, t]} |h^i_{ux}(x)| \sup_{(t_j) \subset [s, t]} \sum_j \left| \partial_i h^k_{t_{j+1}}(x) - \partial_i h^k_{t_j}(x) \right|
\]

\[
\leq \sup_{u \in [s, t]} \|h_{us}\|_2 + \delta \sup_{(t_j) \subset [s, t]} \sum_j \|h_{t_{j+1}} - h_{t_j}\|_{2+\delta}
\]

\[
\leq \sigma^2 \|t - s\|(h, h).
\]

One can perform the same estimate for the second term in \( w_{ts} \). By the triangle inequality, this shows that

\[
\sup_{s < t} \frac{\|w_{ts}\|_0}{|t - s|} \leq C(h, h).
\]

We can calculate the derivative of \( w \) using Proposition 30 and perform similar estimates, also for the Hölder norm, to conclude.

3.1. Schilder’s theorem and Freidlin-Ventzel large deviations for stochastic flows

In this section, we will prove a large deviation result for a \((p, \rho)\)-rough driver \( X \) in the case where the underlying vector field \( X \) is a Wiener process. We will do this by using the extended contraction principle (see e.g. [DZ98, Theorem 4.2.23]), a strategy which has proven to be useful in rough paths theory (\[LQZ02\], \[MSS06\], \[FV07\], \[FV10\]). As a corollary, we obtain a Freidlin-Ventzel-type large deviation result for the flow generated by this driver. The key step is to prove that \( X^n := X^{D_n} \) with \( D_n = \{\frac{X^n}{\varepsilon^n} : k = 0, \ldots, n\} \) is an exponentially good approximation to \( X \). This is done in the Lemmata 26 and 27.

For \( \varepsilon > 0 \), set \( \delta_{\varepsilon} X := (\varepsilon X, \delta_{\varepsilon} X) \) where

\[
\delta_{\varepsilon} X_{ts} := \delta_{\varepsilon} W_{ts} + \frac{\varepsilon^2}{2} X_{ts} X_{ts}
\]

\[
:= \frac{1}{2} \left( \int_s^t (\varepsilon X)_{us} (\varepsilon X)_{odu} (\varepsilon X)_{odu} (\varepsilon X)_{us} \right) + \frac{\varepsilon^2}{2} X_{ts} X_{ts}.
\]

We similarly define \( \delta_{\varepsilon} X^n \) with Riemann-Stieltjes integrals. Note that the homogeneous metric \( \mathfrak{d} \) defined in (1.3) enjoys the property that \( \mathfrak{d}(\delta_{\varepsilon} X, \delta_{\varepsilon} X^n) = \varepsilon \mathfrak{d}(X, X^n) \) for every \( \varepsilon > 0 \) (which is the reason why we call it homogeneous metric). Note that since \( X \) is Gaussian, the local characteristic \( \alpha \) is almost surely deterministic. From now on, we make the standing assumption that \( a \) has either compact support or that it decays as in (2.12).

Lemma 26. Let \( D \) be a domain in \( \mathbb{R}^d \) and \( X \) be a Wiener process in \( C_b^{2,0,\delta} \) for some \( \delta \in (0, 1] \).

Let \( X = (X, X) \) denote its natural lift to a \((p, \rho)\)-rough driver for some \( \rho \in (0, \delta) \) and \( p \in (2, 3) \). Let \( \eta > 0 \) be fixed. Then the following holds:

\[
\lim_{n \to \infty} \lim_{\varepsilon \to 0} \sup \varepsilon^2 \log \mathbb{P} \left( \mathfrak{d}_{p, \rho}(\delta_{\varepsilon} X, \delta_{\varepsilon} X^n) > \eta \right) = -\infty.
\]

\[
\lim_{n \to \infty} \lim_{\varepsilon \to 0} \sup \varepsilon^2 \log \mathbb{P} \left( \mathfrak{d}_{p, \rho}(\delta_{\varepsilon} X, \delta_{\varepsilon} X^n) > \eta \right) = -\infty.
\]
Proof – Let \( \varepsilon > 0 \) and \( n \in \mathbb{N} \) be fixed. Since \( X \) is Gaussian, the quadratic variation process \( a \) is deterministic, and all estimates for \( X, X^n, X - X^n \) and its iterated integrals in the proof of Proposition \( 21 \) hold for \( q = 2 \). Moreover, \( |X|_{L^q} \lesssim \sqrt{q} |X|_{L^2} \).

The iterated Stratonovich and Riemann-Stieltjes integrals are both elements in the second inhomogeneous Wiener chaos, therefore

\[
\left\| \int \od X \otimes \od X \right\|_{L^q} \lesssim q \left\| \int \od X \otimes \od X \right\|_{L^2} \quad \text{and} \quad \left\| \int dX^n \otimes dX^n \right\|_{L^q} \lesssim q \left\| \int dX^n \otimes dX^n \right\|_{L^2},
\]

for all \( q \geq 2 \), see e.g. \cite{FV10} Theorem D.8, and similar estimates hold for the other quantities. Therefore, we may apply Theorem 13 with \( \kappa \) equal to a constant times \( \sqrt{q} \) which shows that

\[
\left\| \vartheta_{p,\rho}(X, X^n) \right\|_{L^q} = \alpha_n \sqrt{q}
\]

holds for all \( q \geq 2 \) and some constant \( \alpha_n \). Repeating this argument for every \( n \in \mathbb{N} \), we obtain a sequence \( (\alpha_n) \) converging to 0 for \( n \to \infty \). Thus

\[
\mathbb{P} \left( \vartheta_{p,\rho}(\delta_n X, \delta_n X^n) > \eta \right) = \mathbb{P} \left( \vartheta_{p,\rho}(X, X^n) > \eta \right) \\
\leq \left( \frac{\varepsilon}{\eta} \right)^q q^2 \alpha_n^q \\
\leq \exp \left[ q \log \left( \frac{\varepsilon \alpha_n \sqrt{q}}{\eta} \right) \right].
\]

Choosing \( q = \varepsilon^{-2} \) we obtain the inequality

\[
\varepsilon^2 \log \mathbb{P} \left( \vartheta_{p,\rho}(\delta_n X, \delta_n X^n) > \eta \right) \leq \log(\alpha_n / \eta)
\]

from which the claim follows.

If \( \mathcal{H} \) is the Cameron-Martin space of a \( C_b^{2,0,\delta} \)-valued Wiener process and \( v \) is a path with values in \( C_b^{2,0,\delta} \), set

\[
I(v) := \begin{cases} \frac{1}{2} \langle v, v \rangle_{\mathcal{H}}, & \text{if } v \in \mathcal{H} \\ +\infty, & \text{otherwise}. \end{cases}
\]

Lemma 27. Let \( \mathcal{H} \) be the Cameron-Martin space for some \( C_b^{2,0,\delta} \)-valued Wiener process. Choose \( \Lambda > 0 \). Then

\[
\lim_{|D| \to 0} \sup_{|h| \leq \Lambda} \vartheta_{p,\delta} \left( S(h^D), S(h) \right) = 0
\]

for every \( p > 2 \).

Proof – It is easy to check (cf. \cite{FV10} Proposition 5.20 and Lemma 25) that

\[
\sup_{0 \leq s < t \leq T} \left\| h_t^D - h_s^D \right\|_{2+\delta} \leq \sqrt{3} \sup_{0 \leq s < t \leq T} \left\| h_t - h_s \right\|_{2+\delta} \leq \sqrt{3} \sigma_{\gamma} \sqrt{\langle h, h \rangle}.
\]

From Lemma 25, we know that

\[
\sup_{s < t} \frac{\| w_{t,s} \|_{C^{1+\delta}}}{|t-s|} \leq C \sigma_{\gamma}^2 \langle h, h \rangle
\]

Now fix \( j, k \in \{1, \ldots, d\} \), \( s < t \) and \( x \in D \). Then

\[
\left\| \int_s^t \dot{h}^{D,j}_{su}(x) \partial_j h^{D,k}_{du}(x) \right\|_{1-\var:|s,t|} \leq \left\| h^{D,j}(x) \right\|_{1-\var:|s,t|} \left\| \partial_j h^{D,k}(x) \right\|_{1-\var:|s,t|}.
\]
For $u \in [0,T]$, define $u_D := \sup \{ t_i \in \mathcal{D} : t_i \leq u \}$ and $u_D^0 := \inf \{ t_i \in \mathcal{D} : t_i \geq u \}$.

With this notation, using the estimates from Lemma 24,

$$
\left\| h^{D,j}(x) \right\|_{1-\text{var};[s,t]} \leq \left\| h^{D,j}(x) \right\|_{1-\text{var};[s,s^D]} + \left\| h^{D,j}(x) \right\|_{1-\text{var};[s^D,t_D]} + \left\| h^{D,j}(x) \right\|_{1-\text{var};[t_D,t]}
$$

$$
\leq \frac{|s^D - s|}{|s^D - s_D|} \left| h^{D,j}_s(x) - h^{D,j}_{s^D}(x) \right| + \left\| h^j(x) \right\|_{1-\text{var};[s^D,t_D]}
$$

$$
+ \frac{|t - t_D|}{|t^D - t_D|} \left| h^{D,j}_t(x) - h^{D,j}_{t_D}(x) \right|
$$

$$
\leq \frac{|s^D - s|}{|s^D - s_D|} \sup_{0 \leq u < v \leq T} \left\| h_u - h_v \right\|_0 + \sigma_\gamma |t - s|^\frac{1}{2} \sqrt{\langle h, h \rangle}
$$

$$
+ \frac{|t - t_D|}{|t^D - t_D|} \sup_{0 \leq u < v \leq T} \left\| h_u - h_v \right\|_0
$$

$$
\leq 3\sigma_\gamma \sqrt{\langle h, h \rangle} |t - s|^\frac{1}{2}.
$$

A similar estimate holds for $\partial_j h^{D,k}$. Therefore,

$$
\sup_{0 \leq s < t \leq T} \left\| \frac{1}{|t - s|} \int_s^t h^{D,j}_{su}(\cdot) \partial_j h^{D,k}_{du}(\cdot) \right\|_0 \leq 9\sigma_\gamma^2 \langle h, h \rangle.
$$

Let $s < t$ and $x, y \in \mathcal{D}$. We have

$$
\left| \int_s^t h^{D,j}_{su}(x) \partial_j h^{D,k}_{du}(x) - \int_s^t h^{D,j}_{su}(y) \partial_j h^{D,k}_{du}(y) \right| \leq \left\| h^{D,j}(x) - h^{D,j}(y) \right\|_{1-\text{var};[s,t]} \left\| \partial_j h^{D,k} \right\|_{1-\text{var};[s,t]}
$$

$$
+ \left\| h^{D,j} \right\|_{1-\text{var};[s,t]} \left\| \partial_j h^{D,k}(x) - \partial_j h^{D,k}(y) \right\|_{1-\text{var};[s,t]}
$$

Similar to $h^{D,j}(x)$, one can estimate

$$
\left\| h^{D,j}(x) - h^{D,j}(y) \right\|_{1-\text{var};[s,t]} \leq 3\sigma_\gamma \sqrt{\langle h, h \rangle} |t - s|^\frac{1}{2} |x - y|^\delta
$$

and similarly for $\partial_j h^{D,k}(x) - \partial_j h^{D,k}(y)$. Thus

$$
\sup_{0 \leq s < t \leq T} \left\| \frac{1}{|t - s|} \int_s^t h^{D,j}_{su}(\cdot) \partial_j h^{D,k}_{du}(\cdot) \right\|_0 \leq 9\sigma_\gamma^2 \langle h, h \rangle.
$$

Using the product rule (Proposition 30),

$$
\partial_t \left( \int_s^t h^{D,j}_{su}(x) \partial_j h^{D,k}_{du}(x) \right) = \int_s^t \partial_t h^{D,j}_{su}(x) \partial_j h^{D,k}_{du}(x) + \int_s^t h^{D,j}_{su}(x) \partial^2_{i,j} h^{D,k}_{du}(x)
$$

and similar estimates as above, we can show that

$$
\sup_{0 \leq s < t \leq T} \left\| \frac{1}{|t - s|^{1+\delta}} \right\|_0 \leq C\sigma_\gamma^2 \langle h, h \rangle
$$

where

$$
w^{D}_{ts} = \frac{1}{2} \left( \int_s^t h^{D,j}_{us} h^{D,k}_{du} - h^{D,j}_{du} h^{D,k}_{us} \right).
$$
This implies

\[ \sup_{\mathcal{D}} \sup_{h : I(h) \leq \Lambda} \left\| S(h^\mathcal{D}) \right\|_{2, \delta, T} < \infty. \]

Let \( p > 2 \). Then

\[ \sup_{0 \leq s < t \leq T} \frac{\left\| h_{ts}^\mathcal{D} - h_{ts} \right\|_{2+\delta}}{|t - s|^{2p}} \leq \left( \sup_{0 \leq s < t \leq T} \frac{\left\| h_{ts}^\mathcal{D} - h_{ts} \right\|_{2+\delta}}{|t - s|^{2}} \right)^{\frac{2}{p}} \left( \sup_{0 \leq s < t \leq T} \left\| h_{ts}^\mathcal{D} - h_{ts} \right\|_{2+\delta} \right)^{1-\frac{2}{p}} \]

\[ \leq \left( \sup_{0 \leq s < t \leq T} \frac{\left\| h_{ts}^\mathcal{D} \right\|_{2+\delta}}{|t - s|^{2}} + \sup_{0 \leq s < t \leq T} \frac{\left\| h_{ts} \right\|_{2+\delta}}{|t - s|^{2}} \right)^{\frac{2}{p}} \]

\[ \times \left( \sup_{0 \leq s < t \leq T} \left\| h_{ts}^\mathcal{D} - h_{ts} \right\|_{2+\delta} \right)^{1-\frac{2}{p}}. \]

Therefore, the claim follows if we can prove

\[ (3.3) \lim_{|\mathcal{D}| \to 0} \sup_{h : I(h) \leq \Lambda} \sup_{0 \leq s < t \leq T} \left\| h_{ts}^\mathcal{D} - h_{ts} \right\|_{2+\delta} = 0 \]

and

\[ (3.4) \lim_{|\mathcal{D}| \to 0} \sup_{h : I(h) \leq \Lambda} \sup_{0 \leq s < t \leq T} \left\| \int_{s}^{t} h_{su}^\mathcal{D}(\cdot) \partial_{j} h_{du}^\mathcal{D}(\cdot) \right\|_{1+\delta} = 0. \]

Concerning \( (3.3) \), note that

\[ \sup_{0 \leq s < t \leq T} \left\| h_{ts}^\mathcal{D} - h_{ts} \right\|_{2+\delta} \leq 2 \sup_{t \in [0, T]} \left\| h_{t}^\mathcal{D} - h_{t} \right\|_{2+\delta}. \]

If \( t \in [0, T] \) is fixed, using Lemma \( 2.4 \) we have

\[ \left\| h_{t}^\mathcal{D} - h_{t} \right\|_{2+\delta} = \left\| (t - t) (h_{t}^\mathcal{D} - h_{t}) \right\|_{2+\delta} \leq 2 \sup_{|v - u| \leq |\mathcal{D}|} \left\| h_{v} - h_{u} \right\|_{2+\delta} \]

\[ \leq 2 \sqrt{|\mathcal{D}|} \sup_{|v - u| \leq |\mathcal{D}|} \left\| h_{v} - h_{u} \right\|_{2+\delta} \]

\[ \leq 2 \sigma_{\gamma} \sqrt{(h, h)} \sqrt{|\mathcal{D}|}, \]
Proof – The proof is standard, using the large deviation principle for Gaussian measure (e.g. [FV10 Theorem 6.8]),
\[
\| h^{D;j}_{su}(x) \partial_j h^{D;k}_{du}(x) - \int_s^t h^{j}_{su}(x) \partial_j h^{k}_{du}(x) \| \\
\leq \| h^{D;j} - h^j \|_{2-\text{var}:[0,T]} \| \partial_j h^{D;k} \|_{1-\text{var}:[0,T]} + \| h^j \|_{1-\text{var}:[0,T]} \| \partial_j h^{D;k} - \partial_j h^k \|_{2-\text{var}:[0,T]}.
\]
From interpolation for the p-variation [FV10 Proposition 5.5] and our former estimates,
\[
\| h^{D;j} - h^j \|_{2-\text{var}:[0,T]} \leq \left( \sup_{0 \leq u < v \leq T} \| h^{D;j}_{uv} - h^j_{uv} \|_{1+\delta} \right)^{\frac{1}{\delta}} \left( \| h^{D;j} \|_{1-\text{var}:[0,T]} + \| h^j \|_{1-\text{var}:[0,T]} \right)^{\frac{1}{\delta}} \leq C \sigma_\gamma \sqrt{\langle h, h \rangle_\mathcal{H} |D|^\frac{1}{2}}.
\]
A similar estimate holds for \( \| \partial_j h^{D;k} - \partial_j h^k \|_{2-\text{var}:[0,T]} \) and we obtain
\[
\| \int_s^t h^{D;j}_{su}(\cdot) \partial_j h^{D;k}_{du}(\cdot) - \int_s^t h^{j}_{su}(\cdot) \partial_j h^{k}_{du}(\cdot) \|_0 \leq C \sigma_\gamma \langle h, h \rangle_\mathcal{H} |D|^\frac{1}{4}.
\]
As above, one can use the product rule and obtain similar estimate for the Hölder norm of the derivative. This shows that
\[
\sup_{0 \leq s \leq t \leq T} \left\| \int_s^t h^{D;j}_{su}(\cdot) \partial_j h^{D;k}_{du}(\cdot) - \int_s^t h^{j}_{su}(\cdot) \partial_j h^{k}_{du}(\cdot) \right\|_{1+\delta} \leq C \sigma_\gamma \langle h, h \rangle_\mathcal{H} |D|^{\frac{1}{4}}
\]
and (3.3) follows. ▫

**Theorem 28** (Schilder’s theorem for Wiener rough drivers). Let \( X \) be a Wiener process in \( C^{2,0.\delta}_b(D, \mathbb{R}^d) \) with Cameron-Martin space \( \mathcal{H} \). Assume that the local characteristic satisfies the conditions stated in Theorem 18. Denote by \( X \) its natural lift to a \((p,\rho)\)-rough driver. For \( \varepsilon > 0 \), set \( P_\varepsilon := P \circ (\varepsilon X)^{-1} \). Then the family \( \{ P_\varepsilon : \varepsilon > 0 \} \) of probability measures satisfies a large deviation principle on the space of rough drivers with speed \( \varepsilon^{-2} \) and good rate function
\[
J(v) = \begin{cases} \frac{1}{2} \langle v, v \rangle & \text{if } v = (v, \nu) \text{ and } \nu \in \mathcal{H} \\ +\infty & \text{otherwise} \end{cases}.
\]

**Proof** – The proof is standard, using the large deviation principle for Gaussian measure [DS89 Section 3.4], the extended contraction principle [DZ98 Theorem 4.2.23] and the results in the Lemmas 26 and 27 (cf. e.g. [FV10 Theorem 13.42]). ▫

As an immediate corollary, we obtain Freidlin-Ventzel large deviations for a class of stochastic flows.

**Theorem 29.** Let \( X \) be a Wiener process in \( C^{2,0.\delta}_b(\mathbb{R}^d, \mathbb{R}^d) \), for some \( \delta \in \left( \frac{2}{3}, 1 \right] \), with local characteristic satisfying the conditions of Theorem 18. Let \( \varphi^\varepsilon \) be the flow generated by the Stratonovich solution to
\[
d\varphi^\varepsilon = \varepsilon X(\varphi^\varepsilon ; dt).
\]
Let \( \nu^\varepsilon \) denote the law of \( \varphi^\varepsilon \) in the space of \( C^0 \) homeomorphisms, \( \rho \in \left( \frac{2}{3}, \delta \right) \). Then the family \( \{ \nu^\varepsilon : \varepsilon > 0 \} \) of probability measures satisfies a large deviation principle with speed \( \varepsilon^{-2} \) and good rate function
\[
L(\psi) = \inf \left\{ J(v) : d\psi = v(\psi ; dt) \right\}.
\]
Proof – The Stratonovich solution equals the solution generated by the \((p,\rho)\)-rough driver \(X\). Using Theorem 28 and the pathwise continuity \(X \mapsto \varphi\), we can use the usual contraction principle in large deviation theory [DZ98, Theorem 4.2.1] to conclude. 

4

Appendix

We provide in this Appendix an elementary regularity result for integrals depending on a parameter.

Let \(\delta_\varepsilon\) be a standard Dirac sequence. If \(I\) is a closed interval and \(f : I \rightarrow \mathbb{R}\) is a continuous function, let \(\bar{f} : \mathbb{R} \rightarrow \mathbb{R}\) denote the unique continuous extension which coincides with \(f\) on \(I\) and which is constant outside this interval. Set \(f^\varepsilon := \delta_\varepsilon \ast \bar{f}\). If \(D\) is some subset of \(\mathbb{R}^d\) and if \(f : D \times I \rightarrow \mathbb{R}\) is a continuous function in time for every \(x \in D\), set

\[
f^\varepsilon(x, t) := (\delta_\varepsilon \ast f(x, \cdot))(t).
\]

Proposition 30. Let \(D \subset \mathbb{R}^d\) be an open set and let \(f : D \times [0, T] \rightarrow \mathbb{R}\) and \(g : D \times [0, T] \rightarrow \mathbb{R}\) be continuous. Assume that \(f\) and \(g\) are continuously differentiable on \(D\) and that \(f(x, 0) = \partial_x f(x, 0) = 0\) for every \(x \in D\) and every \(i = 1, \ldots, d\). Moreover, assume that there are \(p, q \in [1, \infty)\) with \(\frac{1}{p} + \frac{1}{q} > 1\) such that

\[
\sup_{(t_i)} \sum_{t_{i+1}} ||f(\cdot, t_{i+1}) - f(\cdot, t_i)||_C^p \quad \text{and} \quad \sup_{(t_i)} \sum_{t_{i+1}} ||g(\cdot, t_{i+1}) - g(\cdot, t_i)||_C^q
\]

are finite, where the suprema are taken over all finite partitions of the interval \([0, T]\). Then the Young integral (cf. e.g. [PV10, Chapter 6] for the precise definition) \(\int_0^T f(x, t) g(x, dt)\) exists, is continuously differentiable for all \(x \in D\) and the derivative is given by

\[
\partial_{x_i} \left( \int_0^T f(x, t) g(x, dt) \right) = \int_0^T \partial_{x_i} f(x, t) g(x, dt) + \int_0^T f(x, t) \partial_{x_i} g(x, dt)
\]

for all \(i = 1, \ldots, d\).

Proof – One can suppose without loss of generality that \(i = 1\). Fix \(x \in D\).

\[
\partial_{x_1} \left( \int_0^T f(x, t) g^\varepsilon(x, dt) \right) = \partial_{x_1} \left( \int_0^T f(x, t) \partial_t (g^\varepsilon(x, t)) dt \right)
\]

\[
= \int_0^T \partial_{x_1} f(x, t) \partial_t (g^\varepsilon(x, t)) dt + \int_0^T f(x, t) \partial_t((\partial_{x_1} g(x, \cdot))\varepsilon(t)) dt
\]

\[
= \int_0^T \partial_{x_1} f(x, t) g^\varepsilon(x, dt) + \int_0^T f(x, t)(\partial_{x_1} g(x, \cdot))\varepsilon(dt).
\]


Let $q' > q$ such that $\frac{1}{p} + \frac{1}{q'} > 1$. Let $U$ be some neighbourhood of $x$ and let $y \in U$. From Young estimates and interpolation, we obtain

$$\left| \int_0^T \partial_y f(y, t) g^\varepsilon(y, dt) - \int_0^T \partial_y f(y, t) g(y, dt) \right| \leq C \sup_{(t_i) \subset [0, T]} \left( \sum_{t_i} \left| \partial_{x_1} f(y, t_{i+1}) - \partial_{x_1} f(y, t_i) \right|^p \right)^{\frac{1}{p}}$$

$$\times \sup_{(t_i) \subset [0, T]} \left( \sum_{t_i} \left| g^\varepsilon(y, t_{i+1}) - g(y, t_{i+1}) - g^\varepsilon(y, t_i) \right| q' \right)^{\frac{1}{q'}}$$

$$\leq C \sup_{(t_i) \subset [0, T]} \left( \sum_{t_i} \left\| f(\cdot, t_{i+1}) - f(\cdot, t_i) \right\|_{C^1}^p \right)^{\frac{1}{p}}$$

$$\times \left\{ \sup_{(t_i) \subset [0, T]} \left( \sum_{t_i} \left\| g^\varepsilon(\cdot, t_{i+1}) - g(\cdot, t_i) \right\|_{C^1}^q \right)^{\frac{1}{q}} + \sup_{(t_i) \subset [0, T]} \left( \sum_{t_i} \left\| g(\cdot, t_{i+1}) - g(\cdot, t_i) \right\|_{C^1}^q \right)^{\frac{1}{q}} \right\}^{\frac{2}{q'}}$$

$$\times 2^{1 - \frac{\beta}{q'}} \sup_{0 \leq t \leq T} \left\| g^\varepsilon(\cdot, t) - g(\cdot, t) \right\|_{C^0}^{1 - \frac{\beta}{q'}}.$$

It is easy to check that

$$\sup_{(t_i) \subset [0, T]} \left( \sum_{t_i} \left\| g^\varepsilon(\cdot, t_{i+1}) - g(\cdot, t_i) \right\|_{C^1}^q \right)^{\frac{1}{q'}} \leq \sup_{(t_i) \subset [0, T]} \left( \sum_{t_i} \left\| g(\cdot, t_{i+1}) - g(\cdot, t_i) \right\|_{C^1}^q \right)^{\frac{1}{q'}}.$$

Therefore, we obtain a bound of the form

$$\left| \int_0^T \partial_y f(y, t) g^\varepsilon(y, dt) - \int_0^T \partial_y f(y, t) g(y, dt) \right| \leq C \sup_{0 \leq t \leq T} \left\| g^\varepsilon(\cdot, t) - g(\cdot, t) \right\|_{C^0}^{1 - \beta'/\beta}$$

where $C$ is independent of $y$ and $\varepsilon$. Thus,

$$\int_0^T \partial_{x_1} f(y, t) g^\varepsilon(y, dt) \to \int_0^T \partial_{x_1} f(y, t) g(y, dt)$$

uniformly in a neighbourhood around $x$ when $\varepsilon \to 0$. Similarly,

$$\int_0^T f(y, t)(\partial_{x_1}(g(y, \cdot)))^\varepsilon(dt) \to \int_0^T f(y, t)\partial_{y_1} g(y, dt)$$

uniformly in a neighbourhood around $x$ when $\varepsilon \to 0$. This shows differentiability in $x$ of the integral

$$\int_0^T f(x, t) g(x, dt)$$

and the claimed identity.
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