Constraint on the light quark mass $m_q$ from QCD Sum Rules in the $I = 0$ scalar channel

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In this paper, we reanalyze the $I = 0$ scalar channel with the improved Monte-Carlo based QCD sum rules, which combines the rigorous Hölder-inequality-determined sum rule window and a two Breit-Wigner type resonances parametrization for the phenomenological spectral density that satisfies the low-energy theorem for the scalar form factor. Considering the uncertainties of the QCD parameters and the experimental masses and widths of the scalar resonances $\sigma$ and $f_0(980)$, we obtain a prediction for light quark mass $m_q(2\,\text{GeV}) = \frac{1}{2}(m_u(2\,\text{GeV}) + m_d(2\,\text{GeV})) = 4.7^{+0.5}_{-0.7}$ MeV, which is consistent with the PDG (Particle Data Group) value and QCD sum rule determinations in the pseudoscalar channel. This agreement provides a consistent framework connecting QCD sum rules and low-energy hadronic physics. We also obtain the decay constants of $\sigma$ and $f_0(980)$ at 2 GeV, which are approximately 0.64 – 0.83 GeV and 0.40 – 0.48 GeV respectively.

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I. INTRODUCTION

The light quark masses are fundamental parameters in QCD, thus it is important to determine these parameters from different methods. Due to the color confinement, the light quark masses can not be measured from experiments directly. Therefore, their values are determined by relating the light quark masses to other physical quantities which can be obtained from theories or experiments. The main QCD-based methods for determining the light quark masses are lattice QCD (see e.g., Ref. [1] for a review) and QCD sum rules (QCDSR) [2–8].

The pion channel is the most common method to determine the light quark masses from QCDSR. In Ref. [4], Bijnens et al. studied the value of the light quark mass combination $m_u + m_d$ in QCD using both Finite Energy Sum Rules (FESR) and Laplace Sum rules (LSR) for the divergence of the axial current with the quantum numbers of the pion, finding $m_u(1\,\text{GeV}) + m_d(1\,\text{GeV}) = 12 \pm 2.5$ MeV, which leads to a light quark mass $m_q(2\,\text{GeV}) = \frac{1}{2}(m_u(2\,\text{GeV}) + m_d(2\,\text{GeV})) = 4.8 \pm 1.0$ MeV at the Particle Data Group (PDG) standard energy scale 2 GeV. Later, after including five-loop order and higher order quark-mass corrections to the correlation function of the same current, a more accurate result $m_q(2\,\text{GeV}) = 4.1 \pm 0.2$ MeV was found by using FESR [7].

In addition to the divergence of the axial current, one can also relate the light quark masses to other currents. It is clearly important to establish the self-consistency of the quark mass extracted from different channels. In Ref. [8], Cherry et al. used the $I = 0$ scalar current to study this problem. By linking the phenomenological spectral density to the $\pi\pi$ scattering amplitude, they obtained the average light quark mass $m_q(1\,\text{GeV}) = 5.2 \pm 0.6$ MeV. However, the main uncertainty in this analysis is determining the normalization between the theoretical and phenomenological spectral density. As discussed in Ref. [8], it is difficult to assess the hadronic uncertainties in Ref. [8], motivating our alternative approach. In this paper, we will reinvestigate the $I = 0$ scalar channel using the improved Monte-Carlo based QCD sum rule methodology recently proposed in Ref. [9]. After introducing a two Breit-Wigner type resonances parametrization for the phenomenological spectral density normalized by the low-energy theorem, a Monte-Carlo based analysis will be presented for the QCD sum rule master equation with the $I = 0$ scalar current in the rigorous Hölder-inequality-determined sum rule window. Based on this analysis, we will give robust constraint on the light quark mass $m_q$ and predictions for the decay constants of $\sigma$ and $f_0(980)$.

II. QCD SUM RULE FOR $I = 0$ SCALAR CHANNEL

We consider the correlation function

$$\Pi(q^2) = i \int d^4 x e^{iqx} \langle 0 | T j_s(x) j_s^\dagger(0) | 0 \rangle,$$

(1)

where $j_s = m_q \frac{1}{\sqrt{2}} (\bar{u}u + \bar{d}d)$ is the $I = 0$ renormalization group invariant scalar current and $m_q = \frac{1}{2}(m_u + m_d)$ is the average mass of $u$ and $d$ quarks. The theoretical representation of this function has been calculated by using the
operator product expansion (OPE) method \[10\] \[12\], however, it is believed that other nonperturbative contributions to the correlation function must be included, and thus we also should include instanton contribution \(\Pi^{(\text{inst})}(q^2)\) in the theoretical representation of the correlation function \(13\) \[17\].

To obtain a QCD sum rule, we first need to Borel-transform the theoretical representation of the correlation function, which gives \[10\] \[17\]

\[
R^{(\text{theo})}(\tau, \hat{m}_q) = \frac{1}{\tau} \hat{B} \Pi^{(\text{OPE})}(q^2) + \frac{1}{\tau} \hat{B} \Pi^{(\text{inst})}(q^2) = m_q^2 (1/\sqrt{\tau}) \cdot \left\{ \frac{3}{8\pi^2} \left( \frac{1 + 14 \alpha_s(1/\tau)}{3} \right) + 3 \langle m_q q \rangle \left( 1 + \frac{13 \alpha_s(1/\tau)}{3} \right) \right\} \]

where \(\hat{B}\) is the Borel transformation operator, \(\alpha_s(1/\tau) = 4\pi/(9 \ln(1/(\tau \Lambda_{\text{QCD}}^2)))\) is the running coupling constant for three flavors at scale \(1/\sqrt{\tau}\) (the QCD scale \(\Lambda_{\text{QCD}} = 0.353 \text{ GeV} \[18\]), \(\kappa\) is the vacuum factorization violation factor which parameterizes the deviation of the four-quark condensate from a product of two-quark condensates, \(\rho\) is the instanton size in the instanton liquid model, and \(K_0\) and \(K_1\) are modified Bessel functions. We have considered the renormalization-group (RG) improvement of the sum rules \[19\] and anomalous dimensions for condensates \[20\] \[21\] in Eq. (2), where \(\mu_0\) is the renormalization scale for condensates, and

\[
m_q(1/\sqrt{\tau}) = \hat{m}_q \cdot \left[ \frac{4\pi}{9 \ln(\tau \Lambda_{\text{QCD}}^2)} \left( 1 - \frac{64 \ln(1/\tau \Lambda_{\text{QCD}}^2)}{81 \ln(1/\tau \Lambda_{\text{QCD}}^2)} \right) \right]^{1/9} \]

is the running light quark mass at scale \(1/\sqrt{\tau}\) where \(\hat{m}_q\) is the RG-invariant light quark mass. In Eq. (2), we also have included the \(\alpha_s\) corrections to dimension-4 operators, which may play an important role in the determination of the QCD sum rule window from the Hölder inequality as in Ref. \[9\].

It is also necessary to construct a phenomenological spectral density model which is related to the correlation function through the dispersion relation integral. Considering the resonance nature of scalar mesons, we insert the lowest two-pion intermediate state \(^{1}\), as part of a complete set, into Eq. (1), i.e., by inserting

\[
\int \frac{d^3 k_1}{(2\pi)^3 2 E_{k_1}} \frac{d^3 k_2}{(2\pi)^3 2 E_{k_2}} \langle \pi^+(k_1)\pi^-(k_2) | \pi^+(k_1)\pi^-(k_2) \rangle + \frac{1}{2} \langle \pi^0(k_1)\pi^0(k_2) | \pi^0(k_1)\pi^0(k_2) \rangle \]

“other intermediate states” for the correlation function of current \(j_s\), and using Cutkosky’s cutting rules \[22\], then the phenomenological expression for \(\text{Im} \Pi(s)\) can be found:

\[
\text{Im} \Pi^{(\text{phen})}(s) = \frac{3}{64\pi} \sqrt{1 - \frac{4m_{\pi}^2}{s}} |F_s(s)|^2 + \text{contributions from excited states and continuum (ESC)},
\]

where \(m_{\pi}\) is the mass of pion, and \(\langle 0|j_s(0)|\pi^+(k_1)\pi^-(k_2) \rangle = \frac{1}{\sqrt{4\pi}} F_s((k_1 + k_2)^2)\) has been used. We have classified all contributions from intermediate states other than two-pion intermediate state, including these from four-pion intermediate state, into contributions from ESC. According to chiral perturbative theory (ChPT), the scalar form factor \(F_s(s)\) will be normalized by a low-energy theorem \(F_s(0) = m_s^2 \[23\]\), so we will constrain our phenomenological spectral density with this condition in the following.

In Ref. \[8\], the phenomenological spectral density for the \(I = 0\) scalar channel is related to the \(\pi\pi\) scattering amplitude via the scalar form factor \(F_s(s)\). However, because of a lack of experimental data which are consistent with ChPT at some energy scale, Cherry et al. introduced multiple assumptions for their phenomenological spectral density, which dominated the uncertainties in their analysis. In this paper, we will perform an independent analysis by parametrizing the phenomenological spectral density with the mass spectrum for the \(I = 0\) scalar channel directly and incorporate the ChPT low-energy theorem.

The \(0^+(0^{++})\) meson spectrum are rather crowded, there are too many particles with quantum numbers \(0^+(0^{++})\) listed in the Review of Particle Physics \[24\] for a single nonet. Many different models have been used to describe

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1 There exist higher intermediate states which contain more particles, e.g., four-pion intermediate state. However, multiple particle intermediate states would be kinetic suppressed by small phase space factors, thus we will classify these intermediate states together with other two particle intermediate states into “other intermediate states” below.
the structures of these scalar mesons in QCDSR, including ordinary $\bar{q}q$ meson, four-quark state, glueball and hybrid [10, 11, 25]. However, the possible mixing between mesons with the same quantum numbers make this problem even more complex, and a widely accepted conclusion of research on the structures of these scalar mesons has not been achieved.

Amongst all these $I = 0$ scalar mesons, we notice that both $\sigma$ and $f_0(980)$ have the two-pion decay mode as their dominant decay mode. Thus we can conjecture that there are contributions from poles of $\sigma$ and $f_0(980)$ in the two-pion scalar form factor, i.e., $F_\pi(s)$ may have two poles at $s = m_\pi - i m_\pi \Gamma_\pi$ and $s = m_{f_0} - i m_{f_0} \Gamma_{f_0}$, where $m_\pi$ and $\Gamma_\pi$ ($m_{f_0}$ and $\Gamma_{f_0}$) are the mass and width of $\sigma$ ($f_0(980)$) meson respectively.

Considering the normalization of the form factor $|F_\pi(0)|^2 = m_\pi^2$ from ChPT, we can construct a two Breit-Wigner type resonances model for the phenomenological spectral density which meets the above requirements as follows \(^2\)

\[
\frac{1}{\pi} \text{Im} \Pi^{(\text{resonance})}(s) = \frac{3}{64\pi^2} \left| F_\pi(s) \right|^2 = \frac{3}{64\pi^2} m_\pi^4 \left( \beta \cdot \frac{m_\pi^4 + m_{f_0}^2 \Gamma_{f_0}^2}{(s-m_{f_0}^2)^2 + m_{f_0}^2 \Gamma_{f_0}^2} + (1-\beta) \cdot \frac{m_{f_0}^4 + m_{f_0}^2 \Gamma_{f_0}^2}{(s-m_{f_0}^2)^2 + m_{f_0}^2 \Gamma_{f_0}^2} \right),
\]

where we have omitted the small mass of pion ($m_\pi = 0.139$ GeV [24]) in the square root in Eq. (4). The parameter $\beta$ ($0 < \beta < 1$) describes the relative contribution of $\sigma$ and $f_0(980)$ to the phenomenological spectral density in our model.

For the ESC contributions in the phenomenological spectral density, we still use the traditional ESC model, i.e.,

\[
\frac{1}{\pi} \text{Im} \Pi^{(\text{ESC})}(s) = m_q^2 (1/\sqrt{\tau}) \cdot \left( \frac{3}{8\pi^2} \left( 1 + \frac{17}{3} \alpha_s \right) \frac{\alpha_s}{\pi} \right) s - \frac{3}{4\pi^2} \frac{\alpha_s}{\pi} \ln(s) - \frac{3}{4\pi} s J_1(\sqrt{s}) Y_1(\sqrt{s}) \right) \theta(s-s_0),
\]

where $s_0$ is the continuum threshold separating the contributions from excited states and continuum, $J_1$ and $Y_1$ are Bessel function of the first and second kind respectively.

Collecting Eq. (5) and (6) together, we can obtain our phenomenological spectral density as follows

\[
\frac{1}{\pi} \text{Im} \Pi^{(\text{phen})}(s) = \frac{1}{\pi} \text{Im} \Pi^{(\text{resonance})}(s) + \frac{1}{\pi} \text{Im} \Pi^{(\text{ESC})}(s).
\]

Then the phenomenological representation for the Borel-transformed correlation function can be obtained by using the dispersion relation:

\[
R^{(\text{phen})}(\tau, s_0, \beta, \hat{m}_q) = \frac{1}{\pi} \int_0^\infty e^{-s\tau} \text{Im} \Pi^{(\text{phen})}(s) \, ds = R^{(\text{resonance})}(\tau, \beta) + R^{(\text{ESC})}(\tau, s_0, \hat{m}_q).
\]

Finally, the master equation for QCD sum rule can be obtained by demanding the equivalence between Eq. (2) and (8):

\[
R^{(\text{theo})}(\tau, \hat{m}_q) = R^{(\text{phen})}(\tau, s_0, \beta, \hat{m}_q),
\]

which can be used to obtain the predictions for $s_0$, $\beta$ and $\hat{m}_q$ providing we take the condensates and instanton size in the theoretical side as well as the physical parameters for $\sigma$ and $f_0(980)$ in the phenomenological side as input parameters \(^3\).

Obviously, because of the truncation of OPE and the simplicity of the phenomenological spectral density, Eq. (9) can not be valid for all $\tau$, thus one requires a sum rule window in which the validity of the master equation can be established. Benmerrouche et al. presented a method based on the Hölder inequality which provides fundamental

\(^2\) Notice that our model does not exclude other $0^+(0++)$ mesons from having $\bar{q}q$-component, however, the contributions to the two-pion scalar form factor originate from heavier scalar mesons should be negligible because of the exponential suppression factor in the Borel-transformed dispersion relation integral and the form factor will be suppressed by the small branching ratio of the two-pion decay mode.

\(^3\) We can use Eq. (4) to obtain predictions for resonance parameters in our phenomenological spectral density as in Ref. [3] in principle. However, because the theoretical side of Eq. (4) is proportional to the square of the light quark mass $m_q$, the master equation is sensitive with the value of $m_q$, thus stable match between the two sides of the master equation is difficult to establish providing different input $m_q$. Conversely, by taking the resonance parameters as input parameters, we can use Eq. (4) to constrain the value of $m_q$ effectively.
constraints on QCD sum rules [30]. By placing the excited states and continuum contributions on the theoretical side, we obtain

$$R^{(\text{theo-ESC})}(\tau, s_0, \tilde{m}_q) = \frac{1}{\pi} \int_{s_0} e^{-\tau \omega} \text{Im} \Pi^{(\text{phen})}(s) \, ds,$$

(10)

then the Hölder inequality for QCD sum rules can be written as

$$R^{(\text{theo-ESC})}(\tau_1 + (1 - \omega)\tau_2, s_0, \tilde{m}_q) \leq \left[R^{(\text{theo-ESC})}(\tau_1, s_0, \tilde{m}_q)\right]^{1-\omega} \left[R^{(\text{theo-ESC})}(\tau_2, s_0, \tilde{m}_q)\right]^\omega,$$

(11)

where 0 ≤ ω ≤ 1 and for parameters τ_1 and τ_2 we demand τ_1 < τ_2. Notice that different value of \( \tilde{m}_q \) does not change the allowed (τ, s_0) region from the Hölder inequality, thus we can set any value for \( \tilde{m}_q \) in Eq. (11). Following Ref. [30], we will perform a local analysis on Eq. (11) with \( \tau_1 - \tau_2 = \delta \tau = 0.01 \, \text{GeV}^{-2} \).

The only starting point of the Hölder inequality is that \( \text{Im} \Pi^{(\text{phen})}(s) \) should be positive because of its relation to physical spectral functions, thus Eq. (11) must be satisfied if sum rules are to consistently describe integrated physical spectral functions. In this paper, we will use the same iterative procedure to determine the sum rule window from the Hölder inequality rigorously as in Ref. [9], i.e., by choosing the maximally allowed region \([\tau_{\text{min}}, \tau_{\text{max}}]\) of the Hölder inequality which is consistent with fitted \( s_0 \), where \( \tau_{\text{min}} \) and \( \tau_{\text{max}} \) are respectively the lower bound and upper bound of the allowed \( \tau \) region.

In order to match the two sides of the master equation (6) in the sum rule window, a weighted-least-squares method [31] will be used in this paper. By randomly generating 200 sets of Gaussian distributed phenomenological input QCD parameters with given uncertainties (10% in this paper, which is the typical uncertainty in QCDSR) at \( \tau_j = \tau_{\text{min}} + (\tau_{\text{max}} - \tau_{\text{min}}) \times (j - 1)/(n_B - 1) \), where \( n_B = 21 \), we can estimate the standard deviation \( \sigma^{\text{theo}}(\tau_j) \) for \( R^{(\text{theo})}(\tau_j, \tilde{m}_q) \). Then, the phenomenological output parameters \( s_0, \beta \) and \( \tilde{m}_q \) can be obtained by minimizing

$$\chi^2 = \sum_{j=1}^{n_B} \frac{(R^{(\text{theo})}(\tau_j, \tilde{m}_q) - R^{(\text{theo})}(\tau_j, s_0, \beta, \tilde{m}_q))^2}{\sigma^{\text{theo}}(\tau_j)}.$$

(12)

### III. NUMERICAL RESULTS

In the numerical analysis, we use the central values of input QCD parameters (at \( \mu_0 = 1 \, \text{GeV} \)) as follows [32, 33]

$$\langle \alpha_s G^2 \rangle = 0.07 \, \text{GeV}^4, \quad \langle m_{\bar{q}q} \rangle = -(0.1 \, \text{GeV})^4,$$

$$\kappa \alpha_s \langle \bar{q}q \rangle^2 = \kappa \times 1.49 \times 10^{-4} \, \text{GeV}^6, \quad \rho = 1/0.6 \, \text{GeV}^{-1}.$$

(13)

The size of \( \kappa \) has been observed in different channels to be 2–4 [18, 34, 35]. Based on our previous study, \( \kappa = 2.8 \) is the favored result in the vector channel with a traditional ESC model [6]. Although the factorization violation effect may differ between channels, it is still reasonable to assume the value of \( \kappa \) is in the region of 2–3 in the scalar channel, too. Thus we consider \( \kappa = 2.0 \) and \( \kappa = 3.0 \) in our analysis, and as outlined below, we demonstrate that \( \kappa \sim 2 \) leads to greater agreement between our light quark mass predictions and the PDG value. In this paper, we will minimize the \( \chi^2 \) with 1000 sets of Gaussian distributed input QCD parameters listed in Eq. (13) with 10% uncertainties. Based on these 1000 fitting samples, we can obtain the median and the asymmetric standard deviations from the median for all output parameters, thus we obtain the uncertainty originating from uncertainties of QCD parameters for \( s_0, \beta \) and \( \tilde{m}_q \).*

In FIG. 1 we plot the allowed region for (τ, s_0) by the Hölder inequality for \( \kappa = 2.0 \) and \( \kappa = 3.0 \) respectively. From this figure, we find that the \( \alpha_s \) corrections to \( \langle \alpha_s G^2 \rangle \) and \( \langle m_{\bar{q}q} \rangle \) extend the allowed region to a higher τ region and lower \( s_0 \) region as in the ρ channel [3], and the instanton contribution extends the allowed region further more.

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4 In practice, we will divide \( R^{(\text{theo})} \) by \( \tilde{m}_q^2 \) in order to remove the to-be-fitted parameter from the theoretical side, i.e., we estimate the standard deviation for \( R^{(\text{theo})}(\tau_j, \tilde{m}_q)/\tilde{m}_q^2 \).

5 The mass and width of \( \sigma \) and \( \rho_0(980) \) will be considered as fixed input parameters in each fit. However, we will input different combination of parameters for resonances based on experiment to estimate the uncertainties for output parameters which originate from parameters of resonances in the following.
FIG. 1: The region allowed by the Hölder inequality for $\kappa = 2.0$ (a) and $\kappa = 3.0$ (b). The region with (blue) dot or (red) line is allowed for sum rule with or without instanton contribution respectively. The region with (green) asterisk is allowed for sum rule without both $\alpha_s$ corrections to dimension-4 operators and instanton contribution.

Thus both $\alpha_s$ corrections to dimension-4 operators and instanton contribution are important since we adopt the same iterative procedure as described in Ref. [9] to determine the sum rule window from the Hölder-inequality-allowed region rigorously.

Taking the experimental values of mass and width for $\sigma$ and $f_0(980)$ [24]

$$m_\sigma = 400 - 550 \text{ MeV}, \quad \Gamma_\sigma = 400 - 700 \text{ MeV}, \quad m_{f_0} = 990 \pm 20 \text{ MeV}, \quad \Gamma_{f_0} = 10 - 100 \text{ MeV}$$

as our input in the phenomenological spectral density model, we obtain different fitted $s_0$, $\beta$ and $\hat{m}_q$ by minimizing the corresponding $\chi^2$ function. Detailed results are listed in TABLE I where we show the fitted results for $\kappa = 2.0$ and $\kappa = 3.0$ respectively. From this table we find that we can achieve very stable fits with $\kappa = 2.0$, all uncertainties of output parameters are less than 10% providing 10% uncertainties of input QCD parameters. When we set $\kappa = 3.0$, the uncertainty of $\hat{m}_q$ will reach to about 14%-18%, still in the accepted range of uncertainties for QCDSR.

| Inputs          | Outputs          |
|-----------------|------------------|
| $m_\sigma$/MeV, $\Gamma_\sigma$/MeV, $m_{f_0}$/MeV, $\Gamma_{f_0}$/MeV | $s_0$/GeV$^2$, $\beta$, $\hat{m}_q$/MeV, $m_q$(2 GeV)/MeV |
| $\kappa = 2.0$  |                  |
| 400, 400, 990, 100 | 2.77$^{+0.14}_{-0.16}$, 0.941$^{+0.016}_{-0.023}$, 7.02$^{+0.62}_{-0.44}$, 4.0$^{+0.4}_{-0.3}$ |
| 440, 400, 990, 100 | 2.71$^{+0.13}_{-0.15}$, 0.995$^{+0.003}_{-0.002}$, 6.87$^{+0.54}_{-0.40}$, 4.0$^{+0.3}_{-0.2}$ |
| 400, 700, 990, 100 | 2.77$^{+0.14}_{-0.16}$, 0.955$^{+0.013}_{-0.002}$, 6.40$^{+0.58}_{-0.41}$, 3.7$^{+0.3}_{-0.2}$ |
| 440, 700, 990, 100 | 2.71$^{+0.13}_{-0.15}$, 0.996$^{+0.003}_{-0.002}$, 6.25$^{+0.50}_{-0.37}$, 3.6$^{+0.3}_{-0.2}$ |
| 550, 400, 990, 100 | 2.66$^{+0.16}_{-0.20}$, 0.935$^{+0.024}_{-0.033}$, 8.41$^{+0.70}_{-0.51}$, 4.8$^{+0.4}_{-0.3}$ |
| 550, 700, 990, 100 | 2.60$^{+0.15}_{-0.19}$, 0.995$^{+0.002}_{-0.003}$, 8.35$^{+0.67}_{-0.49}$, 4.8$^{+0.4}_{-0.3}$ |
| 550, 700, 990, 100 | 2.73$^{+0.14}_{-0.16}$, 0.958$^{+0.016}_{-0.024}$, 7.16$^{+0.64}_{-0.44}$, 4.1$^{+0.4}_{-0.3}$ |
| 550, 700, 990, 100 | 2.68$^{+0.14}_{-0.16}$, 0.996$^{+0.001}_{-0.002}$, 7.06$^{+0.57}_{-0.42}$, 4.1$^{+0.3}_{-0.2}$ |

| $\kappa = 3.0$  |                  |
| 400, 400, 990, 100 | 3.03$^{+0.09}_{-0.08}$, 0.872$^{+0.030}_{-0.078}$, 9.00$^{+0.60}_{-0.28}$, 5.2$^{+0.9}_{-0.4}$ |
| 440, 400, 990, 100 | 2.94$^{+0.08}_{-0.09}$, 0.990$^{+0.002}_{-0.005}$, 8.56$^{+1.25}_{-0.69}$, 4.9$^{+0.7}_{-0.4}$ |
| 400, 700, 990, 100 | 3.03$^{+0.09}_{-0.08}$, 0.896$^{+0.026}_{-0.071}$, 8.28$^{+1.54}_{-0.75}$, 4.8$^{+0.9}_{-0.4}$ |
| 440, 700, 990, 100 | 2.95$^{+0.08}_{-0.09}$, 0.992$^{+0.002}_{-0.004}$, 7.82$^{+1.14}_{-0.63}$, 4.5$^{+0.7}_{-0.4}$ |
| 550, 400, 990, 100 | 2.99$^{+0.10}_{-0.10}$, 0.835$^{+0.041}_{-0.095}$, 10.7$^{+1.6}_{-0.9}$, 6.2$^{+0.9}_{-0.5}$ |
| 550, 400, 990, 100 | 2.90$^{+0.09}_{-0.10}$, 0.986$^{+0.003}_{-0.006}$, 10.5$^{+1.5}_{-0.8}$, 6.0$^{+0.9}_{-0.5}$ |
| 550, 700, 990, 100 | 3.03$^{+0.09}_{-0.08}$, 0.881$^{+0.032}_{-0.082}$, 9.27$^{+1.61}_{-0.81}$, 5.3$^{+0.9}_{-0.5}$ |
| 550, 700, 990, 100 | 2.95$^{+0.08}_{-0.09}$, 0.991$^{+0.002}_{-0.005}$, 8.91$^{+1.29}_{-0.71}$, 5.1$^{+0.7}_{-0.4}$ |

TABLE I: Fitted results with different choices of the mass and width for the two resonances. All uncertainties of QCD input parameters listed in Eq. (14) are set to 10%.

The suggested light quark mass at 2 GeV from PDG reads [24]

$$m_q^{PDG}(2 \text{ GeV}) = \frac{1}{2}(m_u + m_d) = 3.5^{+0.7}_{-0.3} \text{ MeV}.$$  \hspace{1cm} (15)

To compare our fitted results with $m_q^{PDG}(2 \text{ GeV})$, we also list the corresponding light quark mass at 2 GeV from our
fitting procedure in TABLE I. Based on these data, we can obtain

$$m_q(2\text{ GeV}) = 4.1 \pm 0.4(\text{resonance})^{0.4\pm0.3}_{-0.3}(\text{QCD}) \text{ MeV} = 4.1^{+0.6}_{-0.2} \text{ MeV}$$

(16)

for $\kappa = 2.0$ and

$$m_q(2\text{ GeV}) = 5.3 \pm 0.6(\text{resonance})^{0.8\pm0.5}_{-0.5}(\text{QCD}) \text{ MeV} = 5.3^{+1.0}_{-0.8} \text{ MeV}$$

(17)

for $\kappa = 3.0$, where we report the average value of $m_q(2\text{ GeV})$ with different resonance parameters, and combine the standard deviation and the asymmetric standard deviation which originate from different resonance parameters and uncertainties of QCD input parameters respectively. Comparison with the PDG tends to favor the smaller value of $\kappa$. However, since an exact value of $\kappa$ not known, we use the average value for $\kappa = 2.0$ and $\kappa = 3.0$ as a conservative determination of our final result

$$m_q(2\text{ GeV}) = 4.7^{+0.8}_{-0.7} \text{ MeV}.$$  

(18)

This central value result is slightly heavier than the PDG value in Eq. (15), however, is still consistent with it. We expect further experimental data on the mass and width for $\sigma$ and $f_0(980)$ would reduce the uncertainty for our prediction.

From TABLE I we also can obtain

$$s_0 = 2.70 \pm 0.06(\text{resonance})^{0.14\pm0.17}_{-0.17}(\text{QCD}) \text{ GeV}^2 = 2.70^{+0.15}_{-0.18} \text{ GeV}^2$$

(19)

for $\kappa = 2.0$ and

$$s_0 = 2.98 \pm 0.05(\text{resonance})^{0.09\pm0.09}_{-0.09}(\text{QCD}) \text{ GeV}^2 = 2.98^{+0.10}_{-0.10} \text{ GeV}^2$$

(20)

for $\kappa = 3.0$.

We notice that the uncertainty of the fitted continuum threshold $s_0$ is astonishing small, especially those originating from different resonance parameters. Krasnikov et al. pointed out that contributions from below the $n$-th resonances and from above the $n + 1$-th resonances in the spectral density can be separated by using $s_0 = \frac{1}{2}(m_n^2 + m_{n+1}^2)$ where $m_n$ and $m_{n+1}$ is the mass of the $n$-th and $n + 1$-th resonance respectively [36], i.e., $s_0$ is determined only by the mass positions of the two nearest resonances in the spectral density which are located at the two sides of $s_0$. If this choice for $s_0$ is also applicable in the present case, then we can give a simple explanation why $s_0$ is not affected by different resonance parameters: although we input different mass and width for $\sigma$ and different width for $f_0(980)$, the mass of $f_0(980)$ is fixed, thus

$$s_0 = \frac{1}{2}(m_3^2 + m_3^2)$$

(21)

will not change significantly during our fitting procedure, where $m_3$ is the next excited state in the present scalar channel which couples with the scalar current $j_s$ strongly. By using $m_3 = 990 \text{ MeV}$ from experiment and Eq. (21), we can estimate the mass for the next resonance, which ranges from $2.10 \text{ GeV}$ ($\kappa = 2.0$) to $2.23 \text{ GeV}$ ($\kappa = 3.0$). Based on the average value of $m_3$ which is about $2.17 \text{ GeV}$, $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$ are sufficiently weakly-coupled to $j_s$ to be negligible. On the other hand, our result favors one resonance in the group of $f_0(2020)$, $f_0(2100)$, $f_0(2200)$ and $f_0(2330)$ (which are all $0^+(0^{++})$ resonances listed in the latest Review of Particle Physics [24]) for an appreciable coupling to $j_s$ and the exponential suppression in the Laplace sum-rule enables inclusion within the continuum.

The continuum threshold $s_0$ is introduced to separate out the contributions from excited states and continuum in the phenomenological spectral density. This expected purpose is achieved in many works of QCD sum rules under the narrow resonance approximation. However, we deal with resonances with non-zero width in the present case. Thus there is a second possibility that we actually cannot separate the ESC contributions from the first several resonances contributions exactly because of the overlapping contributions from different resonances. If this is the case, then the traditional one parameter (i.e., $s_0$) ESC model is too simple to describe the true physical spectral density. Although a large $s_0$ is obtained during the fitting procedure, which leads to $\sqrt{s_0} = 1.64 - 1.73 \text{ GeV}$, we still cannot conclude that those scalar mesons between 1-2 GeV are excluded from the phenomenological spectral density. But luckily, due to the heavier mass and relative small two-pion decay branching ratio, the contributions from $f_0(1370)$ and $f_0(1500)$ are expected to be very small. For $f_0(1370)$ as an example, if we assume that there is a contribution from $f_0(1370)$ to the scalar form factor $F_s$, which has the same magnitude of contribution as $f_0(980)$ (obviously, the magnitude of $f_0(1370)$ is overestimated because the position of $f_0(1370)$ is further away from the normalization point of $F_s$, i.e., $s = 0$, than $f_0(980)$), then we can estimate a rough relative contribution from $f_0(1370)$ and $f_0(980)$ to the Borel-transformed correlation function in the whole sum rule window, which is about 20-30%. After considering the relative
small two-pion decay branching ratio, the contribution from $f_0(1370)$ to the Borel-transformed correlation function will be at most at the same magnitude of the uncertainty of QCDSR. Thus the fitted light quark mass will not be affected a lot after including these contributions. However, to solve the $s_0$ problem comprehensively and rigorously, a better description of the ESC is deserved, which needs further studies.

By extracting the coefficients for the two standard Breit-Wigner functions in the phenomenological spectral density in Eq. (7), we can define two effective coupling constants which describe the coupling between the scalar current $j_s$ and the two resonances ($\sigma$ and $f_0(980)$) as follows

$$\lambda_\sigma = \beta \frac{3}{64\pi} m^4_\sigma (m^2_\sigma + \Gamma^2_\sigma) m_\sigma \Gamma_\sigma,$$

$$\lambda_{f_0} = (1 - \beta) \frac{3}{64\pi} m^4_{f_0} (m^2_{f_0} + \Gamma^2_{f_0}) m_{f_0} \Gamma_{f_0}$$

(22)

(23)

These two effective coupling constants can be related to other physical quantities. By inserting one-particle intermediate states ($\sigma$ and $f_0(980)$ states) as part of a complete set, $\frac{d^2k}{(2\pi)^3} \langle \sigma(k) | \sigma(k) \rangle + |f_0(980)(k) \rangle |(f_0(980)(k)) + $ "other intermediate states", into the correlation function (1), a traditional phenomenological density can be obtained

$$\frac{1}{\pi} \text{Im} \Pi^{(\text{phen})}(s) = m^2_\sigma f^2_\sigma m^2_\sigma \cdot \frac{1}{\pi} \frac{m_\sigma \Gamma_\sigma}{(s - m^2_\sigma)^2 + m^2_\sigma \Gamma^2_\sigma} + m^2_\sigma f^2_{f_0} m^2_{f_0} \cdot \frac{1}{\pi} \frac{m_{f_0} \Gamma_{f_0}}{(s - m^2_{f_0})^2 + m^2_{f_0} \Gamma^2_{f_0}} + \frac{1}{\pi} \text{Im} \Pi^{(\text{ESC})}(s),$$

(24)

where $\sigma$ and $f_0$ are the decay constants of $\sigma$ and $f_0(980)$ respectively, which satisfy $\langle 0 | \frac{1}{\sqrt{2}} (uu + dd) | \sigma \rangle = f_\sigma m_\sigma$ and $\langle 0 | \frac{1}{\sqrt{2}} (uu + dd) | f_0(980) \rangle = f_{f_0} m_{f_0}$. Comparing Eq. (7) with (24), we can connect our effective coupling constants with $\sigma$ and $f_0$, as follows

$$\lambda_\sigma = m^2_\sigma (\mu) f^2_{\sigma} (\mu) m^2_{\sigma},$$

$$\lambda_{f_0} = m^2_{f_0} (\mu) f^2_{f_0} (\mu) m^2_{f_0},$$

(25)

(26)

where $\mu$ is an energy scale.

| $m_\sigma$/MeV | $\Gamma_\sigma$/MeV | $m_{f_0}$/MeV | $\Gamma_{f_0}$/MeV | $\lambda_\sigma/10^{-3}$GeV$^3$ | $f_{\sigma}(2\text{GeV})$/GeV | $\lambda_{f_0}/10^{-3}$GeV$^3$ | $f_{f_0}(2\text{GeV})$/GeV |
|----------------|------------------|---------------|------------------|-----------------|-------------------|-----------------|-------------------|
| 400            | 400              | 990           | 100              | 1.68            | 0.81              | 3.22            | 0.45              |
| 400            | 400              | 990           | 100              | 1.77            | 0.83              | 2.70            | 0.44              |
| 400            | 700              | 990           | 100              | 1.98            | 0.95              | 2.46            | 0.43              |
| 400            | 700              | 990           | 100              | 2.06            | 1.00              | 2.16            | 0.41              |
| 550            | 400              | 990           | 100              | 3.31            | 0.69              | 3.55            | 0.40              |
| 550            | 400              | 990           | 100              | 3.52            | 0.71              | 2.70            | 0.35              |
| 64 GeV         |                 |               |                  |                 |                   |                 |                   |
| 550            | 700              | 990           | 100              | 3.32            | 0.81              | 2.29            | 0.37              |
| 550            | 700              | 990           | 100              | 3.45            | 0.82              | 2.16            | 0.36              |}

TABLE II: Effective coupling constants and decay constants of $\sigma$ and $f_0(980)$.

In TABLE II we list the effective coupling constants and the decay constants of $\sigma$ and $f_0(980)$ based on our fitted results listed in TABLE I. For simplicity, we only use the central values of the fitted $\beta$ and $m_\sigma(2\text{GeV})$ to estimate the effective coupling constants and the decay constants, and we do not estimate the uncertainties for these constants. Based on our estimation, we obtain the average value $f_{\sigma}(2\text{GeV}) = 0.83\text{GeV}$ for $\kappa = 2.0$ and $f_{\sigma}(2\text{GeV}) = 0.64\text{GeV}$ for $\kappa = 3.0$, we may conclude that the value of the decay constant of $\sigma$ at 2 GeV is around $0.64 - 0.83\text{GeV}$. In

\footnote{We have extended narrow resonances model with Breit-Wigner resonances model for $\sigma$ and $f_0(980)$.}
Ref. [37], Celenza et al. estimated the value of \( f_\sigma \) by using the Nambu-Jona-Lasinio (NJL) model, their result reads \( f_\sigma (2\text{ GeV}) = 0.42 \text{ GeV}, 0.48 \text{ GeV}, 0.35 \text{ GeV} \) and \( 0.43 \text{ GeV} \) depending on different model parameters.\(^7\) Our result which favors a larger coupling between \( j_s \) and the \( \sigma \) state is more consistent with the result from the linear sigma model (\( \sigma M \)), which gives \( f_\sigma (2\text{ GeV}) = 0.65 - 0.90 \text{ GeV} \)\(^8\). We also obtain \( f_{f_0}(2\text{ GeV}) = 0.40 \text{ GeV} \) for \( \kappa = 2.0 \) and \( f_{f_0}(2\text{ GeV}) = 0.48 \text{ GeV} \) for \( \kappa = 3.0 \), thus the value of the decay constant of \( f_0(980) \) at 2 GeV is about \( 0.40 - 0.48 \text{ GeV} \). It is interesting that our \( f_0(980) \) decay constant agrees with Ref. [39], where \( f_{f_0}(1\text{ GeV}) \simeq 0.35 \text{ GeV} \) and \( f_{f_0}(2.1\text{ GeV}) \simeq 0.41 \text{ GeV} \) considering the differences in our approaches.

We also tried to use a one resonance model, i.e., set \( \beta = 0 \) or \( 1 \) in Eq. (5), to finish our fitting procedure. However, after including the constraint on the phenomenological spectral density from low-energy theorem, i.e., \( |F_\sigma(0)|^2 = m_\sigma^2 \), none of the combination of resonance mass and width based on Eq. (14) would lead to reasonable match between the total contribution from the \( \sigma \) channel with a normalization constrained by the ChPT low-energy theorem, and conducted the sum rule analysis of this channel in the Hölder-inequality-determined sum rule window via the Monte Carlo based fitting procedure. Based on our analysis, we obtain a prediction for the light quark mass determinations confirms the validity of our improved Monte-Carlo based QCD sum rules, which has previously been systematically examined in the \( \rho \) meson channel in Ref. [9]. Our results indicate both \( \sigma \) and \( f_0(980) \) couple to the scalar current \( j_s \) strongly, i.e., both \( \sigma \) and \( f_0(980) \) have \( \bar{q}q \)-component.

The continuum threshold \( s_0 \) obtained from our fitting procedure, seems to exclude scalar mesons between 1-2 GeV from the ESC contributions. There are two possibilities to understand this result. One possibility is that those mesons are weakly-coupled enough to be excluded from the phenomenological spectral density, and we expect the next excited state is in the group of scalar mesons which is heavier than 2 GeV and the exponential suppression in the Laplace sum-rule enables inclusion within the continuum. The other possibility is that the traditional ESC model is too simple to describe the true ESC contributions exactly, and we cannot use one parameter to separate ESC contributions from a spectral density with overlapping resonance contributions, thus a more realistic ESC model includes parameters other than \( s_0 \) is needed to solve this problem comprehensively and rigorously.

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\(^7\) We have converted the value of \( f_\sigma \) at the momentum cutoff in the NJL model into the value of \( f_\sigma \) at 2 GeV.

\(^8\) We use the result \( \langle 0 | m_\sigma (\bar{u}u + \bar{d}d) | \sigma \rangle = f_\sigma m_\sigma^2 \) from the linear sigma model, where \( f_\sigma = 93 \text{ MeV} \) is the pion decay constant, \( m_\sigma^{\text{PDG}}(2\text{ GeV}) \) and the mass of \( \sigma \) from experiment to estimate \( f_{f_0}(2\text{ GeV}) \).
From our analysis, we also obtain the value of the decay constants of $\sigma$ and $f_0(980)$ at 2 GeV, which are respectively around 0.64 – 0.83 GeV and around 0.40 – 0.48 GeV. These two decay constants can be used in further studies on the decays of heavier mesons, e.g., B mesons, which can decay through the $s$-wave two pions state.

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