Observability Certification and Optimal Design of Nonlinear Observation Parameters in the Presence of Measurement Noise and Model Mismatches

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Abstract

This paper addresses the observability analysis and the optimal design of observation parameters in the presence of noisy measurements and model mismatches. The main underlying frameworks are the nonlinear constrained moving horizon observer design and the probabilistic certification via randomized optimization. As the perfect observability concept is not possible under the considered uncertain and noisy context, the notion of almost ε-observability is introduced and a systematic procedure to assess its satisfaction for a given system with a priori known measurement noise statistics and parameter discrepancy is sketched. A nice feature in the proposed framework is that the observability is not necessarily defined as the ability to observe the whole state, rather, the more general concept of observation-target quantities is used so that one can analyze the precision with which specific chosen expressions of the state and the parameters can be reconstructed. The overall framework is exposed and validated through an illustrative example.

Keywords: Nonlinear Observability, Certification, Numerical Algorithms, almost ε-observability, Probabilistic Certification, Observation-target.

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1. Introduction

In nowadays data-focused period, there are still many engineering problems that require the design of state observers in the traditional sense [1], namely those which are based on knowledge-driven models. Indeed, data-driven models only involve those variables for which sensors are available on-board. Engineering problems however, involve quite often formulations that contain references to real-life variables that are not directly accessible via sensors, whether it is for cost killing reasons or because such sensors are not available for the variables under interest. These physically meaningful non measured variables are used in the expression of some constraints and/or performance related computation and therefore need to be reconstructed based on knowledge-based models.

Knowledge-based models condense human scientific discoveries related to almost all life phenomena and should be considered with proud as the signature of our species in her quest to understanding the laws of the universe. But these models that can be written elegantly using well motivated functional terms, generally involve parameters with perfectly identified role but rather loosely known values. The lack of precise knowledge of these values may have drastic consequences on the quality of the state observers that would be designed to dynamically inverse these models. The unavoidable measurement noise affecting the real-life sensors has comparable effects.

The solutions our community worked out to address this uncertainty issue have always been stamped by the not-without-proof syndrom meaning that only provable results in the conventional Lemma-Theorem form can be elected for acceptance in major publication supports.

The consequence of this syndrom is that many academic works simply preferred to put the proof at the center and to adapt the systems being studied (generally by numerical simulation) via ad-hoc assumptions to fit whatever is needed to produce a rigorous proof of convergence. Some examples of such ad-hoc assumptions might be given without pretending to be exhaustive:

- The parameters can be identified on-line since they appear in affine form, moreover there is persistant excitation to ensure a well conditioned information matrix over all possible operating scenarios;

- The system’s model structure enables the observation error to be decoupled from the uncertainty or the associated sensitivity can be made as small as possible by high observer gain;

- The system’s model is such that one can build two lower and upper bounding behaviors so that interval analysis and certification can be operated with rigorously provable worst case errors [2].

- The measurement noise is of zero mean and independent of state noise (if any).
The list can be endlessly continued.

Over the last ten years or so, our community, or at least a part of it, witnessed a slight shift away from the not-without-proof paradigm thanks to the emergence of the probabilistic certification ideas and tools [3, 4, 5]. This paradigm probably came as a reaction to the accumulation of failures of provable solutions in addressing the real-world problems. The fact that the industrial world became more prone than before to welcome the advanced control solutions probably accelerated this shift if not made it unavoidable.

In a nutshell, the probabilistic certification framework gives the minimal number of scenarios one has to randomly draw in order to check whether some statement can be certified with a given probability in an uncertain context. The higher the probability one is seeking to assess is, the larger the number of scenarios to be tested with success should be.

Amazingly enough, the probabilistic certification is not very different from the old and already widely used Monte-Carlo simulations, apart from the fact that it comes with a theorem regarding the number of needed scenarios! This might explain its success in a not-without-proof community even if this theorem simply tells how probable is the failure! and even if the usefulness of this theorem is conditioned by a good knowledge of the statistics of the uncertainties which is rarely available!

Generally speaking the probabilistic certification is a paradigm that can be invoked each time one would like to evaluate the veracity of a statement in a no more binary (True/False) way. The statement regarding observability makes no exception and this is precisely the starting point of the present contribution. More precisely, in the presence of noise and model mismatches, expecting that the observability definition can be checked in a binary way can be questioned. To say it differently, Once we refuse to adapt the problem to our tools and proofs but rather to take it as it is,

We should expect that for a certain level of parametric uncertainties (if not for any non zero level) and even in the absence of measurement noise, some close states (lying in some $\epsilon$-neighborhood of the current state) can become indistinguishable from the only measurements, at least, under some specific configurations of input profiles.

Such an assertion itself can be stated in a probabilistic way. Moreover, the size of the $\epsilon$-neighborhood that is used in the formulation might be impacted by the choice of the observer’s parameters. In the case of Moving Horizon Observer (MHO) [6, 7] for instance, the set of observer’s parameters includes the length of the moving horizon and the width of the dead-zone that can be used in the computation of the function that penalizes the output prediction error as it is shown later on in this paper.

Invoking the MHO observer in the previous paragraph is not a coincidence. Indeed, given
the targeted general scope of this contribution, the MHO is practically the only possible choice being the only available framework whose very eligibility uniquely relies on the definition of the observability and not on any additional structural assumptions on the mathematical knowledge-based model of the observed system.

This paper is organized as follows: First some notation and definitions are introduced in Section 2 together with basic recalls regarding the definition of Moving Horizon Observers (MHO). Section 3 introduces the concept of $\epsilon$-observability and shows that its satisfaction requires a robust constraints satisfaction conditions. This condition is relaxed in Section 4 which introduces the less stringent concept of almost $\epsilon$-observability and shows how this concept can be assessed using randomized optimization frameworks. Finally Section 5 proposes an illustrative example that precisely implements the framework on a simple example in order to show its effectiveness in investigating the degree of observability of a nonlinear system under measurement noise and parametric uncertainties.

2. Definitions and notation

2.1. Notation related to trajectories and measurement

We consider nonlinear systems that are governed by a dynamical equation of the form:

$$x_{k+1} = f(x_k, u_k, p)$$ (1)

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in U \subset \mathbb{R}^{n_u}$ stand for the value, at sampling instant $k$, of the state and the input vectors respectively while $p$ stands for the vector of imperfectly known parameters of the model. $f$ is a map that may result from the discretization of some knowledge based model that can be given by ordinary differential equations, differential algebraic equations or partial differential equations or some complex simulator to mention but some possibilities. Since we are interested in realistic digital algorithms, only the discrete-time form (1) is relevant.

It is also assumed that some vector of $n_y$ measurements is available that is linked to the state and the control vector through noisy measurement, namely:

$$y_k = h(x_k, u_k, p) + \nu_k$$ (2)

where $y_k \in \mathbb{R}^{n_y}$ stands for the sensor output at instant $k$ which is corrupted by the noise realization $\nu_k$.

While standard observers [resp. extended observers] aim at reconstructing the whole state [resp. the extended state/parameters vector], the fact is that, very often, what is needed is only a part of the information contained in the state vector [resp. the extended state/parameter vector]. One obvious example is the case where the estimated state is exclusively used to apply some feedback control $K(x, p)$ that would have been designed assuming that the state and the parameters are known, in this case, what is really needed to reconstruct is the map $K(x, p)$ and not necessarily all of its arguments. In other words, an observer that reconstructs with a
sufficient precision the values of $K(x, p)$ while showing some important errors on some components of $(x, p)$ would be appropriate enough for the control task.

Another example is the case where one is interested in determining the value of some specific set of parameters that shows specific importance in characterizing the process or help understanding some fundamental phenomena. In this case, the estimation of the whole extended state component is not so important provided that one can certify that using some inverse problem solution, the precision that can be guaranteed on this specific set of parameters is sufficient.

For all these reasons, the concept of observation-target variable is introduced hereafter:

$$ z = T(x, p) \quad (3) $$

Note that using $T(x, p) = x$ [resp. $T(x, p) = (x, p)$] enables the common state observation and extended state/parameter observation to be recovered without loss of generality.

**Remark 1.** Let us anticipate the forthcoming development by underlying that the concept of observation-target variable does not mean that the observation scheme will not try to evaluate the whole state or extended state, it simply plays a role in the evaluation step of the overall observation scheme as it is explained later on.

Given any vector signal

$$ s = \begin{bmatrix} s^{[1]} \\ \vdots \\ s^{[n_s]} \end{bmatrix} \in \mathbb{R}^{n_s} $$

the forward/backward profiles of $s$ at some instant $k$ over some window of length $N$ (in terms of sampling periods) is denoted as follows:

$$ s^{[i]}_+ := \begin{bmatrix} s_k & s_{k+1} & \cdots & s_{k+N-1} \end{bmatrix} \in \mathbb{R}^{n_s}^N \quad (4) $$

$$ s^{[i]}_- := \begin{bmatrix} s_{k-N} & s_{k-N+2} & \cdots & s_{k-1} \end{bmatrix} \in \mathbb{R}^{n_s}^N \quad (5) $$

which is a condensed expression gathering the profiles of all the components of $s$ that would be denoted individually according to (see Figure 1):

$$ s^{[i]}_+ := \begin{bmatrix} s^{[i]}_k & s^{[i]}_{k+1} & \cdots & s^{[i]}_{k+N-1} \end{bmatrix} \in \mathbb{R}^N \quad (6) $$

$$ s^{[i]}_- := \begin{bmatrix} s^{[i]}_{k-N+1} & s^{[i]}_{k-N+2} & \cdots & s^{[i]}_k \end{bmatrix} \in \mathbb{R}^N \quad (7) $$
Note that the above notation implies that $s_k^– = s_{k−N}^+$. This is important to keep in mind as we will refer to past or future intervals depending on the context.

Using the above notation of control profile, it is now possible to define the future state prediction at instant $k + i$ given the current state $x_k$ at instant $k$, a given control profile $u_k^+ \in \mathbb{U}^N$ and some vector of parameters $p$ by using the following notation:

$$\left( \forall i \in \{0, \ldots, N\} \right) \hat{x}_{k+i} = X_i(x_k, u_k^+, p) \tag{8}$$

The same notation is used to refer to the predicted noise free output, namely:

$$\left( \forall i \in \{0, \ldots, N\} \right) \hat{y}_{k+i} = Y_i(x_k, u_k^+, p) \tag{9}$$

Moreover, the resulting predicted output profile is simply denoted by:

$$Y(x_k, u_k^+, p) := [Y_0(x_k, u_k^+, p) \ldots Y_{N−1}(x_k, u_k^+, p)] \in \mathbb{R}^{n_y}_N \tag{10}$$

### 2.2. Notation related to the output prediction error definition

Given (see Figure 2):

1. A candidate initial state $\xi$ at instant $k − N$;
2. An input profile $u_k^−$ that has been applied over the previous time interval $[k − N, k)$;
3. A candidate value $\hat{p}$ of parameter vector,

by comparing the predicted output profile $Y(\xi, u_k^−, \hat{p})$ to the truly measured one $y_k^−$, it is possible to define the output prediction error profile by:

$$e_k = y_k^− − Y(\xi, u_k^−, \hat{p}) \in \mathbb{R}^{n_y}_N \tag{11}$$
and assuming that the input is measured so that $u_k^-$ can be viewed as a part of $y_k^-$, the last relation can be rewritten shortly as follows:

$$e_k := E(\xi, y_k^-, \hat{p}) \quad (12)$$

In order to avoid observer shuttering that would be induced by the measurement noise and to explicitly acknowledge the presence of the latter, we follow here the suggestion made in [8] which amounts at introducing a dead-zone when evaluating the output prediction error, namely, given an error profile $e^{[i]}$ on the output component $i$, we define the noise-aware distance-to-zero $d$ as follows:

$$d(e^{[i]}, \zeta_i) := \left\lfloor \frac{1}{N} \sum_{j=1}^{N} e_{j}^{[i]} - \zeta_i \right\rfloor_m \quad \text{where} \quad |r|_+ = \max\{0, r\} \quad (13)$$

Note that in the above expression, for each component $i$ of the output vector, $\zeta_i$ is the size of the dead zone which acknowledges that when the average of the output prediction error is below some threshold, it can be simply due to the measurement noise and can therefore be discarded (clipped to 0). Note that since each sensor might have different noise characteristics, it is necessary that different dead zone parameters $\zeta_i, i = 1, \ldots, n_y$ can be allowed.

Based on this definition, the overall output prediction penalty is defined by:

$$d(e, \zeta) := \sum_{i=1}^{n_y} d(e^{[i]}, \zeta_i) \quad (14)$$

Gathering together equations (14) and (12), it is possible to define at each instant $k$ a cost function:

$$J(\xi, \hat{p} \mid y_k^-, \zeta) := d(E(\xi, y_k^-, \hat{p}), \zeta) \quad (15)$$

The relevance of this cost function can be stated as follows:

Given

- a vector of past measurements $y_k^-$ (including the input profile) and
- a dead zone sizes vector $\zeta$

any pair $(\xi, \hat{p})$ that induces $J(\xi, \hat{p} \mid y_k^-, \zeta) = 0$ represents a possible explanation of the previous measurements $y_k^-$ in which, $\xi$ is a possible value of the state at instant $k - N$ while $\hat{p}$ is a possible realization of the unknown parameter vector $p$.

Remark 2. The order of magnitude of the dead zones can be estimated by experiment. For illustration purposes, Figure 3 shows the evolution of the partial sum $|\sum_{k=1}^{N} \nu_k|$ when $\nu$ is a scalar white noise generated by the numpy python command `numpy.random.randn` for 1000
Figure 3: Illustration of the possibility to derive a statistical upper bound of the form (16) on the partial sum of the measurement noise that can guide the choice of the bounding region of the dead zone parameters $\zeta_i$ used in the definition of the output prediction error. Here 1000 realizations of the noise are shown against the bound given by (16) in which $\alpha = 0.26$, $\beta = 18$ and $\gamma = 1.2$ are used.

realizations. The figure shows also that for $N \in [5, 100]$, a statistically valid upper bound of this partial sum can be given by an expression of the form:

$$\alpha + \frac{\beta}{N^\gamma}$$

(16)

2.3. The Moving-Horizon Observer (MHO)

2.3.1. Version 1: Nominal MHO

The simplest MHO considers some nominal value $p^{nom}$ of the vector of parameters $p$. This value is used to define an optimization problem, at each sampling instant $k$, in which the decision variable is the value of the state at the past instant $k-N$:

$$\xi^* = \arg\min_{\xi \in X} J(\xi, p^{nom} | y_{k^-}, \zeta)$$

(17)

in which

- $X$ stands for the admissible set of states

- $y_{k^-}$ stands for the collected measurement profiles over the previous time interval (see above) and

- $\zeta$ is the vector of dead zone sizes discussed earlier,

---

1 valid except for a very tiny fraction of the set of realizations.

2 Optional when such a limitation is relevant for observability such as positivity of states in biological systems to cite a single example.
The optimal solution $\xi^*$ is then used in the model together with the nominal value of the parameter $p^{\text{nom}}$ in order to reconstruct the estimation of the state at the present instant $k$:

$$\hat{x}_k = X_N(\xi^*, u^{-}_k, p^{\text{nom}})$$  \hspace{1cm} (18)

where the notation (8) is used. Obviously a multiple shooting approach can also be used in which the decision variable in (17) can be taken to be the whole state trajectory in which case, the estimation of the state at instant $k$ comes directly as an outcome of the problem’s solution. This remark holds true for the forthcoming MHO formulations without explicit mentioning.

2.3.2. Version 2: Stochastic MHO

This version of the Moving Horizon Observer (MHO) design is the most ambitious one but also the most hard to compute. It consists in delivering an estimation $\hat{x}_k$ of the current state $x_k$ by first solving the optimization problem given by:

$$\xi^* = \arg \min_{\xi \in \mathbb{X}} \mathbb{E}\left[ J(\xi, \cdot | y^{-}_k, \zeta) \right]$$  \hspace{1cm} (19)

in which $\mathbb{E}$ stands for the expectation of its argument given some statistics on the dispersion of the model’s vector of parameters $p$.

An optimal solution $\xi^*$ is considered as the most likely estimation of the previous state at instant $k - N$. Based on this statement, the best estimation of the current state is obviously obtained by:

$$\hat{x}_k = \mathbb{E}\left[ X_N(\xi^*, u^{-}_k, \cdot) \right]$$  \hspace{1cm} (20)

in which the notation (8) is used to denote the state at instant $k$ ($N$ samples ahead of $k - N$) starting from the initial state $\xi^*$ (the most likely state at instant $k - N$ and computed from (19)) and where again, the expectation is taken with respect to the statistics of dispersion of the parameter $p$. Note that $u^{-}_k$ is the control input that is contained in the previous measurement $y^{-}_k$ invoked in (19).

The drawback of this version of the MHO is twofold:

1. The statistics of the dispersion of the parameters is not always easy to obtain and loosely justified choices might lead to worse results than the previous nominal version. This drawback affects also the scheme proposed in this paper and any other candidate scheme that pretends giving any statistically-based answer to the observation problem.

2. More importantly, even when such statistics are known, the computation of (19) is extremely cumbersome because of the high number of simulations that are needed to evaluate the expected cost for each candidate value of the decision variable $\xi$. 

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2.3.3. Version 3: Deterministic Extended MHO

In this version, the decision variable is extended to include the parameter vector. By doing so, $p$ is no more treated as stochastic variable but as a simply unknown variable that one tries to estimate by looking for both initial state and parameter value that explain the previously measured quantities. This leads to the following higher dimensional deterministic optimization problem that replaces (19):

$$\left(\xi^*, p^*\right) = \underset{(\xi, p) \in X \times \tilde{P}}{\text{arg min}} \ J(\xi, p \mid \hat{y}_k, \zeta)$$

(21)

A solution of (21) can then be used to estimate the current state by:

$$\hat{x}_k = X_N(\xi^*, u_k, p^*)$$

(22)

3. Problem Statement

In this section, the main contribution is presented, namely the concept of almost $\epsilon$-observability and the associated probabilistic certification framework. The rigorous presentation of this concept needs to successively introduce the following three definitions (more rigorous statements are provided later on):

1. Dead-zone consistency.

This property means that the dead zones’ sizes are defined such that despite of the measurement noise, the cost defined by (15) and corresponding to the correct pair of state/parameter is 0. In other words the correct pairs are eligible as an optimal solution.

2. $\epsilon$-observability.

This property means that any optimal solution found that induces a 0 value of the cost function must correspond to a less than $\epsilon$ error on the observation-target variable despite of the presence of uncertainties and measurement noise.

3. $\eta$-almost $\epsilon$-observability.

This property means that the previous property is satisfied with a probability greater than $1 - \eta$ when the initial state, the control profile, the parameter and the measurement noise are sampled inside their respective sets and using some predefined and presumably correct probability distributions.

In the remainder of this section, these concepts are successively introduced.

\[\text{that generates the output}\]
3.1. Dead-Zones consistency
The dead-zone consistency property is defined as follows:

**Definition 1 (Dead zones consistency).** The vector of dead zones sizes \( \zeta \in \mathbb{R}^n_{+} \) is said to be consistent if for any admissible state \( x \in X \), any admissible control profile \( u \in U^N \) and any possible realization of the vector of parameters \( p \in P \), the following equality is satisfied for any realization \( \nu \in \mathbb{V}^N \) of the noise profile:

\[
J(x, p | Y(x, u, p) + \nu, \zeta) = 0
\] (23)

This property intuitively reads as follows:

Despite of the measurement noise, the actual pair of state and parameter always leads to zero cost thanks to an appropriate sizes \( \zeta \) of the dead zone.

Let us recall that the output measurement vector contains the input information so that given \( x, p \) and the output, one can simulate the system to produce a predicted trajectory of the system.

Note that the dead zones can always be made consistent by using sufficiently high values, but too high values would lead to non distinguishibility issue as it is discussed in the sequel.

3.2. The \( \epsilon \)-Observability
In the remainder of this contribution, we shall use the following notation:

\[
q := \begin{bmatrix} x \\ p \\ u \\ \nu \end{bmatrix} =: \begin{bmatrix} q_x \\ q_p \\ q_u \\ q_\nu \end{bmatrix} \in Q \subset X \times P \times U^N \times \mathbb{V}^N
\] (24)

where the second equality is to be considered component-wise, namely, \( x = q_x, p = q_p \) and so on. This enables to rewrite (23) in the following more condensed form:

\[
(\forall q \in Q) \quad J_1(q, \zeta) := J(q_x, q_p | Y(q_x, q_u, q_p) + q_\nu, \zeta) = 0
\] (25)

with a straightforward overloading of the cost function notation \( J \). Note that each element \( q \in Q \) completely defines a simulation scenario with an associated measurement noise profile.

In the statement of the forthcoming formulations, the following definitions are used related to pairs of elements of the set \( Q \).

The first definition associates elements of \( Q \) that share the same exogenous information, namely the control input profile and the measurement noise realization:
Definition 2 (Comparable pairs). We shall say that two elements $q^{(1)}, q^{(2)} \in Q$ are comparable if and only if they share the components $u$ and $\nu$. This is denoted as follows:

\[
q^{(1)} \bowtie q^{(2)} \iff (q^{(1)}_u = q^{(2)}_u) \quad \text{and} \quad (q^{(1)}_\nu = q^{(2)}_\nu)
\]  

(26)

Such two elements obviously define two simulations that can differ only by the initial state and/or the vector of parameters while the input profile and the measurement noise are the same.

The second definition introduces the indistinguishability relationship on the set $Q$:

Definition 3 (Indistinguishable pairs). We shall say that an element $q^{(2)}$ is indistinguishable from $q^{(1)} \in Q$ if and only it induces 0 output prediction error cost when using as measurement the noisy output generated by $q^{(1)}$, namely

\[
q^{(1)} \equiv q^{(2)} \iff J(q^{(2)}_x, q^{(2)}_p | Y(q^{(1)}_x, q^{(1)}_u, w^{(1)}_p) + q^{(1)}_\nu, \zeta) = 0
\]  

(27)

meaning that $q^{(2)}_x$ and $q^{(2)}_p$ might as well explain the output induced by $q^{(1)}_x$ and $q^{(1)}_p$ for the same exogenous inputs $q^{(1)}_u$ and $q^{(1)}_\nu$.

When a moving horizon observer operates on some measurement data, it always works on comparable pairs of $q$ values. Indeed, MHO tries to find a possibly optimal pair of initial states and parameter vector for a given past measurements (including control input) that are obtained for a specific and unique realization of the measurement noise.

The ideal situation is the one where no comparable but distinct values of $q$ can be indistinguishable. In this case, the only possible minimum of the above defined optimization problem is the one that involves the correct values of the initial state and the vector of parameters. However, all the previous discussion that justifies the current contribution tried to recall that in realistic situations, this is never the case.

Moreover in the above ideal situation, one obtains perfect estimation of the state and the parameter vector which might be unnecessary in the case where only observation-target variable needs to be reconstructed. That is the reason why the following definition is introduced:

Definition 4 ($\epsilon$-observability). Given an observation-target $z = T(x, p)$, we shall say that the system is $\epsilon$-observable on $Q$ if and only if the following implication holds for any pair $(q^{(1)}, q^{(2)}) \in Q^2$:

\[
\left( q^{(1)} \bowtie q^{(2)} \right) \quad \text{AND} \quad \left( q^{(1)} \equiv q^{(2)} \right) \implies \|q^{(1)}_z - q^{(2)}_z\| \leq \epsilon
\]  

(28)

where $q_z := T(q_x, q_p)$ is the observation-target variable associated to $q$. In other words, only pairs with $\epsilon$-distant observable targets can be both comparable and indistinguishable.

The condition (28) can be written equivalently as follows thanks to (27):

\[
\left( q^{(1)} \bowtie q^{(2)} \right) \quad \text{AND} \quad \left( \|q^{(1)}_z - q^{(2)}_z\| > \epsilon \right) \implies J(q^{(2)}_x, q^{(2)}_p | Y(q^{(1)}_x, q^{(1)}_u, q^{(1)}_p) + q^{(1)}_\nu, \zeta) \neq 0
\]  

(29)

We shall rewrite this last implication in a more practical form using the following steps:
• Let us pick some \( q^{(1)} = q \) that is an arbitrarily chosen element inside \( Q \).

• Any comparable \( q^{(2)} \) (an element such that \( q^{(1)} \triangleright q^{(2)} \)) takes necessarily the form:

\[
q^{(2)} := \begin{bmatrix}
\xi \\
p \\
u \\
u
\end{bmatrix}
\]  

(30)

for some \((\xi, p) \in \mathbb{Z} := \mathbb{X} \times \mathbb{P}\).

• Now in order for the second condition in the left hand side of (29) to be satisfied, (namely \(|q^{(1)} - q^{(2)}| \geq \epsilon\)), we shall restrict \((\xi, p)\) to the set \( \tilde{Z}_\epsilon(q) \) which is the complement of \( Z_\epsilon(q) \) defined by:

\[
Z_\epsilon(q) := \{(\xi, p) \in \mathbb{X} \times \mathbb{P} \text{ s.t } \|T(\xi, p) - T(q_x, q_p)\| \leq \epsilon\}
\]  

(31)

Using the above notation, the implication (29) can be written in the following more compact form:

\((\forall q \in Q) \ (\forall (\xi, p) \in \tilde{Z}_\epsilon(q)) \ J(\xi, p \ | \ Y(q_x, q_u, q_p) + q_\nu, \zeta) \neq 0\)  

(32)

and introducing the notation:

\[
w := \begin{bmatrix}
q \\
\xi \\
p
\end{bmatrix} \in \mathbb{W} := Q \times \mathbb{X} \times \mathbb{P}
\]  

(33)

\[
\mathbb{W}(\epsilon) := \left\{ w = \begin{bmatrix}
q \\
\xi \\
p
\end{bmatrix} \ | \ (q, \xi, p) \in Q \times \tilde{Z}_\epsilon(q) \right\}
\]  

(34)

\[
J_2(w, \zeta) := J(\xi, p \ | \ Y(q_x, q_u, q_p) + q_\nu, \zeta)
\]  

(35)

one gets the following result:

**Proposition 1 (First formulation).** Given an observation-target variable, the system is \( \epsilon \)-observable on \( Q \) if and only if the following conditions hold true:

1. The dead-zone sizes vector if consistent in the sense of Definition 1, namely

\[(\forall q \in Q) \ J_1(q, \zeta) = 0\]

(36)

2. The \( \epsilon \)-distinguishability property holds true, namely

\[(\forall w \in \mathbb{W}(\epsilon)) \ J_2(w, \zeta) \neq 0\]

(37)

where \( \mathbb{W}(\epsilon) \) is defined by (35) while \( J_1 \) and \( J_2 \) are defined by (25) and (33) respectively.
Recall that the first condition of Proposition 1 ensures that the correct pair leads to an output prediction cost that is equal to 0 and is therefore always an admissible solution to the optimization problem while the second condition guarantees that there are no indistinguishable pairs whose observation-target variables are distant by more than that $\epsilon$.

Note that the condition (36) can also be written using the notation $w_q$ that extracts the first vector $q$ in the vector $w$ so that one can write the condition (36) as follows:

$$\forall w \in \overline{W}(\epsilon) \quad J_1(w_q, \zeta) = 0$$  \hspace{1cm} (38)

This enables to regroup the two conditions (36) and (37) in a single condition that involves the parameter $w$, namely:

$$\forall w \in \overline{W}(\epsilon) \quad C(w, \zeta) := \begin{cases} 0 & \text{if } J_1(w_q, \zeta) = 0 \text{ and } J_2(w, \zeta) \neq 0 \\ 1 & \text{otherwise} \end{cases}$$  \hspace{1cm} (39)

This enables Proposition 1 to be reformulated in a more compact form that will be more convenient for the formulation of the probabilistic certification step:

**Proposition 2 (Second formulation).** Given an observation-target variable, the system is $\epsilon$-observable on $\mathbb{W}$ if and only if the following condition holds true:

$$\forall w \in \overline{W}(\epsilon) \quad C(w, \zeta) = 0$$  \hspace{1cm} (40)

For technical reasons, we need to perform a last transformation by observing that the condition (37) that concerns only those values of $w$ that belong to $\overline{W}(\epsilon)$ can be transformed into a condition on all possible values of $w \in \mathbb{W} := Q \times X \times P$ by writing

$$\forall w \in \mathbb{W} \quad g(w, \epsilon, \zeta) = 0$$

where

$$g(w, \epsilon, \zeta) := \begin{cases} 0 & \text{if } J_1(w_q, \zeta) = 0 \text{ and } w \notin \overline{W}(\epsilon) \\ 0 & \text{if } J_1(w_q, \zeta) = 0 \text{ and } \left( w \in \overline{W}(\epsilon) \text{ and } J_2(w, \zeta) \neq 0 \right) \\ 1 & \text{otherwise} \end{cases}$$  \hspace{1cm} (41)

Note that in above formulation the pair defined by

$$\theta := \begin{bmatrix} \epsilon \\ \zeta \end{bmatrix}$$  \hspace{1cm} (42)

is viewed as a design parameter vector for the probabilistic certification setting. The same notation $\theta_\epsilon = \epsilon$ and $\theta_\zeta = \zeta$ will also be used to invoke the individual component of this design vector.

The above notation enables to formulate the final form of the $\epsilon$-observability formulation:

**Proposition 3.** The system is $\epsilon$-observable on $\mathbb{W}$ if and only if there is a design parameter $\theta = (\epsilon, \zeta)$ the following condition holds true:

$$\forall w \in \mathbb{W} \quad g(w, \theta) = 0$$  \hspace{1cm} (43)
The condition $(43)$ is called a **robust constraints satisfaction condition** as the satisfaction of the constraint $g(w, \theta) = 0$ is required for all possible realizations of the argument $w$ and this, regardless of its probability of occurrence.

Such a condition shows two major drawbacks:

1. The first is that such formulation leads to very pessimistic results since the impossibility to meet the robust constraint satisfaction condition might be due to one single very unlikely value (or a set of values) of the parameter $w$

2. Even if required, checking the robust satisfaction constraint property is extremely complex (impossible actually) since one needs to check the satisfaction of the condition for all possible values of $w$. Now one can argue that there is no need to check all the values since instead one can *simply* check the following condition:

$$\max_{w \in W} |g(w, \theta)| = 0 \tag{44}$$

which is mathematically equivalent to the previous condition while it does not necessarily mean that all the values should be individually checked.

The problem with this alternative is that while it might be possible for convex problems, it remains a hard-to-check assertion in the general case where one can scarcely be sure that the optimizer does identify the global maximum. The problem is therefore replaced by the one consisting in asserting that the numerically obtained maximum in $(44)$ is really the maximum which is obviously as hard to decide as the original problem for general non convex settings.

### 3.3. Almost $\epsilon$-observability

In order to avoid these difficulties and to come with a realistic assessment of the observability, the following less stringent concept of observability is introduced:

**Definition 5 (Almost $\epsilon$-observability).**

*Given a predefined observation-target variable, given a small $\eta \in (0,1)$ and assuming some probability distribution $\mathcal{W}$ that governs the dispersion of the context parameter $w$ we say that the system is $\eta$-almost $\epsilon$-observable if and only if there is a design parameter $\theta = (\epsilon, \zeta)$ such that the following inequality holds true:*

$$\Pr_{\mathcal{W}}[g(\cdot, \theta) \neq 0] \leq \eta \tag{45}$$

*meaning that the $\epsilon$-observability condition $(43)$ is satisfied with a high probability.*

This definition is more detailed in the next section and a concrete implementation that enables the best certifiable statement regarding the observability of the system are proposed in the next section.
4. Probabilistic certification of almost $\epsilon$-observability

There are a variety of probabilistic certification settings that have been developed along the past recent years (see [3, 4, 5, 9] and the references therein for an overview). For the sake of conciseness and in order to focus on the main ideas regarding the certification of observability, only one version among various possible settings is pursued. Alternative developments can be undertaken later if appropriate without the main message and philosophy be drastically modified.

To do so, the specific version of the probabilistic certification settings used in the sequel is first recalled in the next section.

4.1. Probabilistic certification framework

Consider the following robust constraint satisfaction problem with uncertainty $w$ and decision variable $\theta$

$\forall w \in W \quad g(w, \theta) = 0 \quad (46)$

Probabilistic certification amounts, in a first step, at replacing this very stringent statement by the following one that assumes $w$ to be a stochastic variable that is described over $W$ by some probability distribution $\mathcal{W}$:

$\Pr_{\mathcal{W}}(g(\cdot, \theta) \neq 0) \leq \eta \quad (47)$

More precisely, rather than requiring $g(w, \theta) = 0$ for any possible realization $w$, one accepts that the probability of this equality being violated is smaller than some small $\eta > 0$ that is thus called hereafter the certification precision parameter. This should be distinguished from the reconstruction precision of the observation target variables.

Now although the formulation of (47) is less stringent that (46), it is still difficult to manipulate as the probability of a nonlinear map depending on a vector of stochastic variables remains impossible to compute. That is why probabilistic certification approach introduces a second step consisting in approximating (47) via numerical averaging over a high number $N_s(\eta, \delta, m)$ of sampled realizations of the stochastic variable $w$. This number $N_s(\eta, \delta, m)$ depends on:

- The certification precision parameter $\eta \in (0, 1)$ introduced above
- The certification confidence parameter $\delta \in (0, 1)$ which defines the degree of confidence with which the certification statement can be delivered, namely, the probability that the statement holds is equal to $1 - \delta$.

\footnote{Since the original robust formulation corresponds to $\eta = 0$}
• $m \in \mathbb{N}_*$ is the maximum number of admissible violations of the constraints among the $N_s(\eta, \delta, m)$ drawn samples. This simply means when testing the condition $g(w^{(i)}, \theta) = 0$ on a high number $N_s(\delta, \eta, m)$ of samples $\{w^{(i)}\}_{i=1}^{N_s(\eta, \delta, m)}$ that are drawn (using $\mathcal{W}$) no more than $m$ samples lead to constraint violation. This is equivalent to write:

$$\sum_{i=1}^{N_s(\delta, \eta, m)} g(w^{(i)}, \theta) \leq m$$  \hspace{1cm} (48)

since when the constraint is satisfied $g = 0$ while $g = 1$ when the constraint is violated.

Based on the above notation, the probabilistic certification of the almost $\epsilon$-observability concept is given by the following proposition [2]:

**PROPOSITION 4 (PROBABILISTIC CERTIFICATION OF $\eta$-ALMOST $\epsilon$-OBSERVABILITY).**

Consider

• A given discrete set $\Theta \subset \mathbb{R}^2_+$ containing $n_\Theta$ values of the design parameter $\theta := (\epsilon, \zeta)$. 
• A certification confidence parameter $\delta \in (0, 1)$
• A certification precision parameter $\eta \in (0, 1)$
• A maximum number of failures $m \in \mathbb{N}_*$
• An integer $N_s$ satisfying:

$$N_s \geq N_s(\eta, \delta, m) := \frac{1}{\eta} \left[ m + \ln(\frac{n_\Theta}{\delta}) + (2m \ln(\frac{n_\Theta}{\delta}))^{\frac{1}{2}} \right]$$  \hspace{1cm} (49)

• $N$ realizations $\{w^{(i)}\}_{i=1}^{N}$ of the vector $w$ drawn randomly according to $\mathcal{W}$ inside $\mathbb{W}$.

If there is an element $\theta^* = (\epsilon^*, \zeta^*) \in \Theta$ such that the following inequality holds true:

$$\sum_{i=1}^{N_s} g(w^{(i)}, \theta^*) \leq m$$  \hspace{1cm} (50)

Then the condition

$$\Pr_{\mathcal{W}}(g(\cdot, \theta^*) \neq 0) \leq \eta$$  \hspace{1cm} (51)

is satisfied with a probability greater than $1 - \delta$.

In other words, if the condition (50) holds for $N_s$ satisfying (49) then the system is $\eta$-almost $\epsilon^*$-observable with a probability higher than $1 - \eta$. 

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4.2. Discussion regarding the choice of the statistics of the random sampling

When applying the above framework to a specific problem, the choice of the probability distribution $W$ that is to be used in drawing the $N$ realization is a quite difficult one. Recall that $W$ is the probability distribution that defines the statistics of occurrence of elements inside the set

$$W = \left( X \times P \times U^N \times V^N \right) \times X \times P$$

As far as state samples ($\in X$) are concerned, this distribution should reflect the effectively encountered one when the state observation framework is applied to the system. For instance, when the system is controlled in order to follow some collection of desired set-points, it is obvious that the state will be more frequently present in the neighborhood of this specific set of values so that these regions should be more heavily present in a relevant sampling of the component of $q_x$ of $q = w_q$. The same can be said about the distribution of the components of the control profile $q_u$ of $q = w_q$.

Nevertheless, the system needs to make transitions between these steady states and observability is needed during these transients in order to certify their success in a sufficiently high confidence rate. This means that a certification of the observability that would be based on a probability distribution of states that considers the transient states as instances with very low probability would be problematic (since too optimistic with regards to the very reason for which observability needs to be assessed with a high confidence rate).

On the other hand, using uniform distributions might include regions of the state space that are never visited by the system for many reasons. This means that including them with non vanishing probability of occurrence might lead to pessimistic results regarding the observability if observability does not hold at these regions, or to an optimistic results if these regions correspond to high observability configurations.

As for the statistics of the parameter vector $p$, this is obviously a problem-dependent choice that relies on a deeper understanding of the reason for parameters dispersion. For instance, if it is about production-related dispersion, it might be reasonable to use Gaussian around the nominal manufacturing values. If it is about biological model’s parameters that depend on some characteristics (age, gender, weight, etc), then certification can be done for each category apart or mixed weighted multi-gaussian distribution can be used depending on the statistics of these characteristics among the population under interest.

It comes out from the previous discussion that there is probably not a single perfect choice of $W$ and that this question probably deserves a dedicated study for its own. As far as this paper is concerned, several choices are tested and compared in order show the sensitivity of the result to these choices.
4.3. The investigated statistics

In order to completely define the statistics \( W \), we need to define the sampling rules of all the sets involved in the definition of \( W \), namely, \( X, P, U \) and \( V \). The choices used in the numerical investigation of this paper are defined as follows. Note that for some of these sets, more than a single choice is used as mentioned before.

\( X \)

Two possibilities can be investigated, namely

1. A uniform distribution over \( X \)
2. A non parametric distribution that fits a high number of closed-loop simulations in which the controller uses the values of the state as if it is measured.

\( P \)

Two possibilities are investigated, namely

1. A uniform distribution over \( P \)
2. A Gaussian distribution around a nominal vector of values

\( U^N \)

Two possibilities are investigated, namely

1. A uniform distribution over \( U^N \)
2. A random choice consisting in sequences (element of \( N \)) of the form:

\[
u_i = \text{Sat}_U \left[ \sum_{j=1}^{n_f} \beta_j \sin \left( \frac{2j\pi (i\tau)}{N\tau} + \varphi_j \right) \right]; \quad i = 1, \ldots, N
\] (52)

where \( \text{Sat}_U \) is the projection map on \( U \) while the coefficients \( \beta_j \in \mathbb{R}^{n_u} \) and the phases \( \varphi_j \) for \( j \in \{1, \ldots, n_f\} \) are uniformly randomly selected in \([0, \bar{\beta}]\) and \([0, 2\pi]\) respectively. This sequence can be as rich as desired (by taking a high value of \( n_f \)) to represent any possible behavior of the feedback law.

\( V^N \)

A gaussian white noise is used to represent measurement noise.

4.4. The design set

For each considered configurations of sampling statistics, \( N_s \) samples is drawn with \( N_s \) satisfying the inequality (49). The probabilistic certification requires finding \( \theta^* = (\epsilon^*, \zeta^*) \) such that the (50) holds over the set of sampled \( w^{(i)} \) where \( \theta^* \) belongs to a before hand defined discrete set \( \Theta \) of cardinality \( n_\Theta \).

In what follows the structure of the set of design parameter \( \Theta \) is taken of the form:

\[
\Theta := \mathbb{L}(\sigma_\epsilon, \bar{\sigma}_\epsilon, n_\epsilon) \times \mathbb{L}(\sigma_\zeta, \bar{\sigma}_\zeta, n_\zeta)
\] (53)
where $L(\sigma, \bar{\sigma}, n)$ is the set of $n$ logarithmically uniformly spaced numbers, namely:

$$L(\sigma, \bar{\sigma}, n) = \logspace(\sigma, \bar{\sigma}, n) := \{10^{ri} \mid r_i = \sigma + \frac{\bar{\sigma} - \sigma}{n - 1} \cdot i \in \{0, \ldots, n - 1\}\}$$

This obviously leads to a cardinality $n_\Theta = n_\epsilon n_\zeta$

4.5. Implementation and complexity analysis

Randomized optimization amounts at solving an optimization problem of the form

$$\min_{\theta \in \Theta} \left[ \text{cost}(\theta) \right] \quad \text{under} \quad \sum_{i=1}^{N_s} g(w^{(i)}, \theta) \leq m$$

where $N_s$ is defined by (49) in which $m$ is the number of admissible constraint-violation scenarios.

In our case, the cost function in (55) that is defined in terms of the decision variable $\theta := (\epsilon, \zeta)$ is obviously given by:

$$\text{cost}(\theta) := \epsilon$$

since the objective is to get a certification results with the lowest state estimation error on the observation target variable while $\zeta$ is simply a design parameter of the MHO.

It is shown shortly that the specific observability problem leads to a specific complexity analysis but in a more general settings of (55), since we are using a discrete set $\Theta$ of admissible values of $\theta$, the worst case analysis involves a number of simulations that is equal to $N_s n_\Theta = N_s n_\epsilon n_\zeta$ simulations which corresponds to an exhaustive search. This can be quite heavy if the simulation of the system is cumbersome. But this is scarcely for system’s that are analytically given and which involves a reasonable state space dimension. Examples of such realistic exhaustive search settings can be found in [10, 11].

Such a brute force approach can be avoided by using standard iteration algorithms on discrete-set although the worst case analysis remains dependent on the product $N_s n_\epsilon n_\zeta$.

For the specific certification problem of almost $\epsilon$-observability, it turns out that the structure of the constraint function $g(w, \theta)$ takes the following form:

$$g(w, \theta) = G(e(w), \theta)$$

where $e$ is the output prediction error before dead-zone clipping while $G$ is a very cheap map that mainly consists in clipping followed by conditional summation over the resulting profiles. Consequently, an exhaustive search is more affordable than in the general case since the computation consists in three successive steps which are:

As an example, $L(-2, 0, 5) \approx \{0.01, 0.032, 0.1, 0.32, 1\}$
1. Simulating the $N_s$ scenarios using the set $\{w^{(i)}\}_{i=1}^{N_s}$ sampled instances to get the set
$$\{e^{(i)} := e(w^{(i)})\}_{i=1}^{N_s}$$

2. Computing for each $\theta \in \Theta$ the resulting cost and constraints for each $e^{(i)}$ using $G(e^{(i)}, \theta)$

3. Select among all values of $\theta$ satisfying the constraints the one that corresponds to the lowest $\epsilon = \theta_1$.

This leads to a worst case complexity of $(N_s$ simulations + $n_{\Theta}$ evaluations of $G$) which is a much less complexity than the general case. Moreover, by ordering the elements of $\Theta$ is ascending-in-$\theta_1$ order, one can stop as soon as the constraints is satisfied leading to generally less than total exhaustive search.

Despite of the above guidelines, more involved optimization schemes can be investigated but which are out of the scope of this contribution.

5. Illustrative example

5.1. The dynamic system

Consider the example of the nonlinear continuous stirred-tank reactor with parallel reaction [12]:

$$R \rightarrow P_1$$
$$R \rightarrow P_2$$

that can be described by the following set of dimensionless energy and material balances:

$$\dot{x}_1 = 1 - p_1 x_1^2 e^{-1/x_3} - p_2 e^{-p_3/x_3} - x_1$$  \hspace{1cm} (57)
$$\dot{x}_2 = p_1 x_1 - 1^2 e^{-1/x_3} - x_2$$  \hspace{1cm} (58)
$$\dot{x}_3 = u - x_3$$  \hspace{1cm} (59)

where $x_1$ and $x_2$ represent the concentrations of $R$ and $P_1$ while $x_3$ stands for the temperature of the mixture in the reactor. $P_2$ represents the waste product. This reactor is controlled by the manipulated variable $u \in U := [0.049, 0.449]$ in order to maximize the production of $x_2 = P_1$. Note that the above dynamics involves $n_p = 3$ parameters $p_1$, $p_2$ and $p_3$ whose nominal values are commonly considered to be $p_{nom} = (10^4, 4 \times 10^2, 0.55)$. It is assumed that $x_2$ is measured together with $u$ while $x_1$ and $x_3$ has to be estimated by the observer.

5.2. The framework setting

The above settings is applied with the following definition of the subset $\mathbb{X} \subset \mathbb{R}^3$:

$$\mathbb{X} := [0, 0.6] \times [0, 0.3] \times [0.05, 0.2]$$

which contains realistic evolutions of the state during realistic operational context.
The discrete design set $\Theta$ used in the probabilistic certification framework of proposition 4. This set is of cardinality $n_{\Theta} = 200$.

Regarding the set of parameters that is sampled at each certification experiment, two alternatives are tested as mentioned above:

1. In the first, a uniform distribution is considered on the hyper-box:

\[
\mathbb{P} := \prod_{i=1}^{n_p} [(1 - \rho)p_i^{nom}, (1 + \rho)p_i^{nom}] 
\]

with $\rho = 0.05$.

2. A random distribution centered at $p^{nom}$ such that each sample is given by

\[
p = \prod_{i=1}^{n_p} (1 + s_{td\nu})p_i^{nom}
\]

where $\nu$ is a normal distribution centered at 0.

Regarding the control profiles that are sampled at each certification experiment, two alternatives are tested:

1. In the first, a uniform distribution is considered on the hyper-box $\mathbb{U} := [0.049, 0.449]^N$.

2. A truncated Fourier series with random coefficients [see (52)] with $n_f = 10$.

The noise profile realizations have been generated using the uniform distribution over $[-\bar{\nu}, +\bar{\nu}]$.

Finally, three observation-targets are investigated which are:

\[
z_1 = x \quad ; \quad z_2 = x_1 \quad ; \quad z_3 = x_3
\]
More precisely, applying the certification framework with the observation-target $z_1$ corresponds to standard whole state observability while the use of $z_2$ [resp. $z_3$] corresponds to the cases where only the quality of the reconstruction of $x_1$ [resp. $x_3$] matters.

5.3. Results

Different aspects are successively examined in terms of their impact on the certification results. The results are shown through data frames in which the signification of the columns are as follows:

- **eps1, eps2, eps3.** The certified reconstruction precision $\epsilon$ on $z_1$, $z_2$ and $z_3$ respectively.
- **zeta1, zeta2, zeta3.** The optimal computed dead-zone sizes.
  More precisely $(\text{eps1, zeta1}), (\text{eps2, zeta2})$ and $(\text{eps3, zeta3})$ are the solutions of (55) when the observation-target variable is $z_1$, $z_2$ or $z_3$ respectively.
- **N.** The observation horizon.
- **noise.** The noise level $\bar{\nu}$ mentioned above.
- **rho.** The size $\rho$ of the parameter hyper-box invoked in (60) when the uniform distribution of the parameters is used (This corresponds to the value uniform in the column entitled p-mode).
- **std_p.** The parameter $s_{td}$ used in (61) to define the gaussian distribution of the parameter around the nominal value. (This corresponds to the value gaussian in the column entitled p_mode).
- **u_mode.** The type of input used in the certification (can be Fourier or rand).
- **eta, delta, m.** The certification parameters, namely the certification precision, confidence and number of failures used in the probabilistic certification framework.

Before we dig into the results, it is important to clearly understand the meaning of the certifiable reconstruction precision $\epsilon$. This has to be understood as a probabilistically certified upper bound on the instantaneous estimation error. This is because the certification means that any time one tries to solve the optimi

5.3.1. Impact of the observation horizon

Figure 5 shows the impact of the length of the observation horizon that is used to define the output prediction error cost. It is in particular shown that for the considered uncertainty setting, one needs to use $N = 20$ in order to achieve the certification with the lowest values $\epsilon = 10^{-4}$ considered in the design set $\Theta$ over the three observation-target variables $z_i$, $i = 1, 2, 3$. Otherwise, indistinguishability might occur with quite high error values on the targeted indicators. Note that only three values of $N$ are studied here, lower values of $N \in [10, 20]$ would have probably be sufficient to achieve the high precision certification results.
5.3.2. Impact of the measurement noise

Starting from the last setting of Figure 5, the noise level is increased from 0.001 to 0.003. This leads to a sensitive increase in the certifiable reconstruction precision. Figure 6 shows that by increasing the observation horizon up to $N = 100$ it is possible to recover the levels of the second setting of Figure 5 which was achievable with $N = 10$ and the previous level (0.001) of the noise. This clearly shows that the higher the noise is the longer the observation horizon should be to achieve the same level of certifiable reconstruction precision. This is compatible with Figure 3 which shows that the partial sum of the noise contribution decreases with the length of the time series.

5.3.3. Impact of the parametric uncertainty level and statistics

In this investigation, the level and the type of parametric uncertainties are changed in order to evaluate their effect on the certifiable reconstruction precision. In particular, the comparison between the first two lines of Figure 7 shows that the configuration with uniform distribution of the parameter vector given by (60) with $\rho = 0.05$ is more inconvenient to certifiable reconstruction precision than the gaussian distribution given by (61) and $s_{\text{std}} = 0.2$. The third line shows what would be the certifiable reconstruction precisions on the three observation-targets when the last gaussian distribution parameter is increased from 0.2 to 0.3.
5.3.4. Impact of the input

In the previous results, the control profile was systematically taken to be a randomly sampled truncated Fourier series. In order to evaluate whether this assumption is very specific regarding the certifiable reconstruction precisions, Figure 8 shows the results for uniformly randomly generated profiles inside the admissible set $[0.049, 0.449]$. It can be observed that while the same orders of magnitude are obtained, one can observe by comparing the first lines of Figures 7 and 8 that random profiles seems to enhance the observability at least for the setting that is common to these two lines.

![Figure 8: Checking observability with random input.](image)

5.3.5. Impact of the certification precision parameter

Figure 9 shows how the results change when the targeted precision $\eta$ of the probabilistic certification is degraded. As expected, one can see that under the settings of this figure (the same for all values of $\eta$), a certification precision of $\eta = 10^{-3}$ seems to lead to a certifiable estimation precisions of the observation-target variables which are quite high (roughly useless given the definition of the set $X$). This is obviously due to the high level of parameter dispersion $s_{td} = 0.3$. The certifiable reconstruction error decreases when $\eta$ is increased meaning that the part of the pairs over which the reported reconstruction error are guaranteed is a smaller set of admissible pairs. For instance, the last line of Figure 9 indicates that up to 5% of the samples correspond to the presence of indistinguishable pairs. For the remaining 95% of the cases, an almost zero reconstruction error can be certified (provided that the optimization problem is correctly solved).

![Figure 9: Impact of the precision $\eta$ required in the probabilistic certification.](image)
5.3.6. Computation times and scalability

As it has been mentioned in Section 4.5, the computation time is the sum of the cpu time needed for two successive tasks, namely:

- **dt1**: the time needed to generate the scenarios
- **dt2**: the time needed to find the optimal design parameter.

These times are mainly impacted by the following parameters:

1. The number of scenarios to be simulated which is defined by the three parameters $\eta$, $\delta$, $n_\Theta$ and $m$ through the expression (49) that determines the number of scenarios to be simulated. This linearly affects $dt1$ but not $dt2$.

2. The time needed to simulate a single scenario which is for a given system depends on the observation horizon $N$. This dependence is linear if a fixed step integration scheme is used to integrate ODE’s models.

3. The search algorithm that is adopted to find the optimal design parameter. In the above computation, a simple alphabetic search is adopted with increasing $\epsilon = \theta_1$.

Note that the formulae (49) does not depend on the state or the parameter vector’s dimension. This means that the size of the system and the number of its uncertain parameter does not lead to an exponential increase in the computation time, only the simulation time would affect linearly the computation time of the certification scheme. Note also that both $n_\Theta$ and $\delta$ appears logarithmically in the expression (49). This same expression shows clearly that increasing the confidence of the certification be reducing $\delta$ is linearly increase the number of scenarios.

Regarding the scalability issue, it is worth underlying that the nature of the computation is probably the one that is most prone to parallelization as different scenarios are simulated independently in the first task while different values of $\theta$ are combined with each scenario to compute the constraint violation indicator. This might induce a high parallelization rate which together with the possibility to use efficient optimized compiled code (while python is used here) suggests that the scalability is far away from being an issue here.

As far as the example is concerned, Figure 10 shows the computation times for the certification scheme as a function of the observation horizon or as a function of the certification precision $\eta$. The certification confidence parameter $\delta = 0.001$ and the number of scenarios with failures $m = 10$ are used. Note how the cpu time for the second task (optimizing the design parameter) decreases with the observation horizon as the number of values to be visited is reduced because of the observability gained by the use of higher observation horizon.
6. Conclusion and future possible investigation

This paper proposes a general scalable scheme for the analysis of observability and parameter reconstruction in the context of nonlinear dynamical systems that are subjected to parametric uncertainties and measurement noise. The observability is taken in a more general sense than the standard (extended) observability commonly used in the sense that it is the possibility and the precision to which it is possible to reconstruct specific expressions of the state/parameter that can be investigated by the proposed scheme.

In addition to the analysis power the proposed scheme offers, it can also be used as a tool to optimally design the parameters of the observation scheme and/or to specify the requirements in terms of the quality of the knowledge and the level of noise of the sensors being used to achieve a pre-specified level of reconstruction precision.

It is mandatory to keep in mind that the certification results does not hold for a specific implementation of optimizers. The results of this paper concern the case where a perfect optimizer is available. In this sense, the scheme should be viewed as a way to answer the observability question as a property of the uncertain dynamic system with imperfect sensors. It is by no means a way to assess the degree of actual success in reconstructing the observation target variables under a specific solver that implement the MHO principle through a specific algorithm that comes itself with its own imperfections, choices and undesired behavior in the presence of local minima.

A possible continuation of the present work concerns the investigation of the regions of the space of state and parameters where the certification constraint does not hold leading to degraded certification results. This can be an important step in the analysis since these regions
of the space might have been wrongly included while they are obviously to be excluded by the very definition of the operational space of the system. In such cases, these regions should be removed and the computation re-done in order to come out with more consistent results.

Another undergoing work consists in applying the proposed scheme to standard models that are widely used in control and analysis of biological systems (diabetes, cancer, HIV, pandemics propagation models, etc) since these systems are by nature defined up to the knowledge of a high number of highly uncertain parameters.

Finally, the program codes that served in producing the results of this paper will shortly be available in the GitHub site of the author.

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