QUANTITATIVE COMPARISON THEOREMS IN RIEMANNIAN AND KÄHLER GEOMETRY

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Abstract. We obtain sharp quantitative Laplacian upper and lower estimates under no assumption on curvatures. As a result, we derive quantitative Laplacian, area and volume comparison theorems for tubes in Riemannian and Kähler manifolds under weak integral curvature assumptions. We also give some applications, such as a general Bonnet-Myers theorem and Cheng’s eigenvalue estimate under weak integral curvature assumptions.

1. Introduction

The aim of this paper is to look for weaker conditions than a lower Ricci curvature bound or upper sectional curvature bound such that some classical comparison theorems, such as the Laplacian comparison, Bishop-Gromov volume comparison theorem and Günther’s theorem still hold.

The motivation is as follows. For the standard proof of the Bishop-Gromov volume comparison theorem, only the Ricci curvature lower bound in the radial direction is used. And most often a comparison theorem holds true already for the area or volume element after some Sturm-Liouville type ODE argument. Since the area of the geodesic sphere is just the integral of the area element it is natural to expect that some kind of lower bound for the “integral” of the Ricci curvature in the radial directions should be enough to guarantee an area comparison theorem for the geodesic sphere. On the other hand, the naive approach of replacing the Ricci curvature simply by the average of Ricci curvatures over all directions, i.e. the scalar curvature, does not work because there are counter-examples (Remark 3). It turns out that a weighted version of the integral of the Ricci curvature suffices to ensure Laplacian and volume comparison (Theorems 1, 3). This direction of research has been previously pursued in a number of papers, such as [2], [4], [12], [24], and relatively more recently [27], [28], [29]. See also [23], [32] and the recent paper [25] for comparison results under various pointwise but weaker types of curvature bounds.

In the first part of this paper, for each function \( k(t) \), we obtain a corresponding Laplacian estimate, an area estimate for geodesic spheres, as well as a volume

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estimate for geodesic balls with no condition on the curvature on $M$ (in particular, no assumption on the lower bound of the Ricci curvature), as long as we stay within the cut locus or the injectivity radius. These estimates lead to comparison for Laplacian, area or volume under some weak integral curvature assumptions. In the Riemannian case, the function $k(t)$ can be thought of as the Ricci curvature of the model warped product space in the radial direction. The flexibility of choosing $k(t)$ allows us to obtain for example a fairly general version of Bonnet-Myers theorem (Theorem 2) and its Kähler analogue (Theorem 7). The main argument relies on the second variational formula with a critical use of the index lemma. Compared to the approach of using ODE analysis, this approach is often more “linear” as it avoids estimating the solution of some nonlinear Riccati type differential inequality (cf. [10]).

For example, the first result we will prove is the following Laplacian estimate:

**Theorem (Theorem 1).** Let $x = (r, \theta)$ in geodesic polar coordinates centered at $p$. If $s_k > 0$ on $(0, r]$, then

$$
\Delta r(x) \leq (n - 1) \frac{s_k'(r)}{s_k(r)} - \int_0^r \tilde{\text{Ric}}_k \left( \frac{s_k(t)}{s_k(r)} \gamma_\theta(t) \right) dt.
$$

If $s_k > 0$ on $(0, \sup d_p)$, this also holds in the sense of distribution if the second term on R.H.S. is interpreted properly.

Here $\gamma_\theta$ is the geodesic with initial vector $\theta$, $s_k''(t) + k(t)s_k(t) = 0$ with $s_k(0) = 0$, $s_k'(0) = 1$ for a continuous but otherwise arbitrary function $k(t)$, and $\tilde{\text{Ric}}_k(v) = \text{Ric}(v, v) - (n - 1)k(t)g(v, v)$ for $v \in T_{\gamma_\theta(t)}M$. We remark that it is natural to allow $k$ to be non-constant when we consider the existence of conjugate points, see Proposition 3.

The above estimate is the starting point of the later results, such as the quantitative relative area/volume comparison theorems, Bonnet-Myers theorem and Cheng’s eigenvalue estimate. Note that there is no curvature assumption. Indeed, a feature of our results is that there is an explicit appearance of the curvature term in our estimates (Theorems 1, 3, 4, 6, 8, 10, 11, 12, 13, 14, 15), which is independent of any curvature condition. On the other hand, our results also give sharper estimates. E.g. we prove the monotonicity for the volume ratio of geodesic balls:

**Theorem (Theorem 3).** If $s_k > 0$ on $(0, r]$, then

$$
\frac{d}{dr} \left( \frac{|B_g(r, p)|}{|B_\sigma(r)|} \right) \leq - \frac{s_k(r)^{n-1}}{|B_\sigma(r)|^2} \int_0^r \frac{|B_\sigma(u)|}{s_k(u)^{n+1}} \int_{B_{\gamma}(u, p)} \tilde{\text{Ric}}_k(s_k(t) \partial_t) dV du.
$$

The equality holds if and only if $B_g(r, p)$ is isometric to $B_\sigma(r)$, where $\sigma = dt^2 + s_k(t)^2g_{S^{n-1}}$.

In particular, if $\int_{B_{\gamma}(u, p)} \tilde{\text{Ric}}_k(s_k(t) \partial_t) dV \geq 0$ for all $u \in (0, r)$, then $\frac{|B_g(u, p)|}{|B_\sigma(u)|}$ is non-increasing on $(0, r)$.
This can be regarded as a quantitative version of the Bishop-Gromov volume comparison theorem. This result makes the defect of the volume of the geodesic ball to that of the standard one easier to measure. Our approach to area and volume estimates is also quite robust in the sense that it can be easily adapted to estimate other similar area or volume-type integrals, such as those with weight.

The rest of this paper is as follows. In Section 2, we derive quantitative versions of various comparison theorems on a Riemannian manifold, first for distance from a fixed point and then a submanifold. A general Bonnet-Myers’ theorem will also be derived. Then in Section 3, we extend the results to Kähler manifolds. This is not a direct application of the results in Section 2 because of the special geometry of Kähler manifolds. For our purpose we introduce the notion of $\ell$-holomorphic sectional curvatures, which is the Kähler version of the $\ell$-sectional curvatures defined in [23]. In Section 4, we prove quantitative and relative Günther-type results in Riemannian and Kähler manifolds, i.e. lower bound for the volume of tubes around a submanifold. In the course, we will see that there is a curvature quantity, expressed explicitly in Fermi coordinates, which is analogous to the role of Ricci curvature in Bishop-Gromov comparison theorem, see Theorem 10. Finally, we give some applications in Section 5, such as Cheng’s eigenvalue estimate (Theorem 17) assuming only a lower bound on the integral of some curvature quantities on subsets of metric balls.

In the future, we plan to investigate the Lorentzian analogue of these results under some weak energy conditions, which is of physical interest to understand singularity theorems in general relativity. It also seems plausible that these results can be extended to quaternionic Kähler manifolds, which we do not do here for simplicity.

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2. Comparison results in Riemannian manifolds

2.1. Notions and preliminaries. Let us explain our notation. Throughout this paper, all manifolds and submanifolds are assumed to be complete, connected and orientable unless specified otherwise. Let $(M, g)$ denotes an $n$-dimensional Riemannian manifold. Let $k(t)$ be a continuous function on an interval $I$ containing 0 and $s_k(t)$ be the solution to the equation

\[ s''_k(t) = -k(t)s_k(t), \quad s_k(0) = 0, \quad s'_k(0) = 1. \]  

(2.1)

Often, we will compare $(M, g)$ with the “model space” defined as the warped product manifold $(\overline{M} = [0, r_0] \times S^{n-1}, \overline{g})$, where $\overline{g} = dt^2 + s_k(t)^2 g_{S^{n-1}}$ and $g_{S^{n-1}}$ is the standard round metric on the unit sphere $S^{n-1}$. Of course, for $(\overline{M}, \overline{g})$ to be truly a Riemannian manifold we at least require $s_k > 0$ on $(0, r_0)$. However, to be flexible we do not want to impose any condition now and the assumption on $s_k$ will be stated in the results.
It is easy to see that the Ricci curvature of $\tau$ in the radial direction is $\overline{\text{Ric}}(\partial_t, \partial_t) = -(n - 1)\frac{s_k(t)}{s_k(t)} = (n - 1)k(t)$. When $k$ is constant then

$$s_k(t) = \begin{cases} \frac{1}{\sqrt{k}} \sin (\sqrt{kt}) & \text{if } k > 0 \\ t & \text{if } k = 0 \\ \frac{1}{\sqrt{-k}} \sinh (\sqrt{-kt}) & \text{if } k < 0. \end{cases}$$

Fix $p \in M$ for the moment. Let $\gamma_\theta(t)$ be the geodesic starting from $p$ with initial vector $\theta \in S_pM = \{\theta \in T_pM : |\theta| = 1\}$. We define

$$\overline{\text{Ric}}_k(u) := \text{Ric}(u, u) - (n - 1)k(t)g_{\gamma_\theta(t)}(u, u)$$

for $u \in T_{\gamma_\theta(t)}M$.

Of course, the function $\overline{\text{Ric}}_k$ depends on the direction $u \in T_xM$. If $k$ is constant and a scalar function is preferred, we can define $\overline{\text{Ric}}_k(x) := \min_{u \in S_xM} \overline{\text{Ric}}_k(u)$. Then clearly, $\overline{\text{Ric}}_k(u) \geq \overline{\text{Ric}}_k(x)$ for all $u \in S_xM$. For results in Section 2, we can replace conditions involving $\overline{\text{Ric}}_k$ by $\overline{\text{Ric}}_k(\gamma(t))$. Comparison theorems involving (the negative part of) $\overline{\text{Ric}}_k$ are given in [12], [29], [27], which use a different approach to obtain estimates. Our approach is more direct, and take into account both the positive and negative part of $\overline{\text{Ric}}_k$.

For later use, we also need the notion of some “average” curvature with strength lying between the sectional curvature and the Ricci curvature. Let $K(w, v) = \frac{\langle R(w, v)w, v \rangle}{|w \wedge v|^2}$ be the sectional curvature of the plane spanned by $w$ and $v$. As in [23], the $\ell$-sectional curvature is defined by $K^\ell(W, v) := \sum_{i=1}^\ell K(e_i, v)$ where $v$ is non-zero, $W$ is an $\ell$-dimensional subspace orthogonal to $v$ and $e_i$ is an orthonormal basis of $W$. By convention, $K^0 = 0$. Along a given geodesic $\gamma$, define also

$$\overline{K}^\ell_k(W, v) := K^\ell(W, v) - \ell k(t)g(v, v)$$

for $v \in T_{\gamma(t)}M$ and $W$ an $\ell$-dimensional subspace of $T_{\gamma(t)}M$ orthogonal to $v$.

In particular, if $\ell = n - 1$, they reduce to the Ricci curvature and $\overline{\text{Ric}}_k$ respectively. For comparison theorem of the Laplacian of the distance from a point or a hypersurface, the notion of $\overline{\text{Ric}}_k$ is enough. But for comparing distance function from an $\ell$-dimensional submanifold, the curvature $\overline{K}^\ell_k$ and $\overline{K}^n_{\ell} - \ell$ are involved. Again, if a scalar function is needed, then we can always replace $\overline{K}^\ell_k$ by $\overline{K}^\ell_k(x) := \min \overline{K}^\ell_k(W, v)$ where the minimum is taken over the set $\{(W, v) : v \in S_xM, W < v^\perp \text{ is } \ell\text{-dimensional}\}$.

Recall that a Jacobi field $Y(t)$ along a normal geodesic $\gamma$ orthogonal to $\Sigma$ is said to be adapted to a submanifold $\Sigma$ if $Y(0) \in T\Sigma$ and $Y'(0) - A_{\gamma'(0)}Y(0) \in N\Sigma$. If $\Sigma$ is a point, the initial condition is just $Y(0) = 0$. A standard but useful fact is the following (cf. e.g. [31, p. 3]).
Proposition 1. Let $\Sigma$ be an embedded submanifold of $M$ and let $r = d_\Sigma : M \to \mathbb{R}$ be the distance from $\Sigma$ on $M$. If $x$ lies within the cut locus of $\Sigma$ and $X \in \nabla r(x)^\perp$, then
\[
\nabla^2 r(X, X) = \int_0^r \left((|Y'(t)|^2 - \langle R(Y, \partial_r)\partial_r, Y \rangle \right) dt + A_{\partial_r}(Y(0), Y(0)) \nabla r(0) = : I_\Sigma(Y, Y)
\]
where $Y$ is the $\Sigma$-adapted Jacobi field along the minimizing normal geodesic $\gamma$ emanating from $\Sigma$, satisfying $Y(r) = X$. Here $A$ is the second fundamental form defined by $A_v(X, Y) = \langle v, -\nabla_X Y \rangle$ and $I_\Sigma$ is the index form with respect to $\Sigma$. (If $\Sigma$ is a point, then $A = 0$.)

We end this subsection with an extension of some familiar facts in trigonometry. Let $c_k(t)$ be the solution to
\[
c_k''(t) = -k(t)c_k(t), \quad c_k(0) = 1, \quad c_k'(0) = 0. \tag{2.2}
\]
Again, when $k$ is constant, then
\[
c_k(t) = \begin{cases} \frac{1}{\sqrt{k}} \cos \left(\sqrt{kt}\right) & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ \frac{1}{\sqrt{-k}} \cosh \left(\sqrt{-kt}\right) & \text{if } k < 0. \end{cases}
\]
To further simplify notation, we define $c_t(r) = \frac{c_k(r)}{s_k(r)}$ and $t_k(r) = \frac{s_k(r)}{c_k(r)}$.

Lemma 1.

1. $s_k(t)c_k'(t) - c_k(t)s_k'(t) = -1.$
2. $c_k'(t) = \frac{1}{s_k(t)^2}$. In particular, $c_k$ is decreasing on any interval contained in its domain.
3. $t_k'(t) = \frac{1}{c_k(t)^2}$. In particular, $t_k$ is increasing on any interval contained in its domain.

Proof. By definition, $(s_k c_k' - c_k s_k')' = s_k c_k'' - c_k s_k'' = -k s_k c_k + k s_k c_k = 0$ and so $s_k c_k' - c_k s_k'$ is a constant, which is equal to $-1$ by the initial conditions. The rest follows from this. \qed

2.2. Comparison theorems for distance from a point. Using geodesic polar coordinates centered at a fixed point $p \in M$, within the cut locus of $p$, the volume element on $M$ can be expressed as $dV = F(r, \theta)drd\theta$, where $\theta \in S_p M \cong S^{n-1}$ and $d\theta$ is the volume element of $S^{n-1}$. In this subsection ($F$ will change in later subsections), we let
\[
\overline{F}(r) = s_k(r)^{n-1},
\]
which is the corresponding volume density of $(M \cap (0, r_0) \times S^{n-1}, \overline{\gamma} = dt^2 + s_k(t)^2 g_{S^{n-1}})$ in polar coordinates. Let $d_p = d(p, \cdot)$ be the distance function on $M$. We use $'$ to denote partial derivative with respect to the radial direction $r$. E.g. $F'(r, \theta) = \frac{\partial F}{\partial r}(r, \theta).$
Strictly speaking, Theorem 1 below is a special case of Theorem 4. However, we choose to present it here not only because of simpler presentation, but also because it will be useful later, and is indeed one of the steps in the proof of Theorem 4.

**Theorem 1.**

(1) Let \( x = (r, \theta) \) in geodesic polar coordinates. Assume there is no cut point of \( p \) along \( \gamma_\theta \) on \([0, r]\). If \( s_k(r) \neq 0 \), then

\[
\Delta d_p(x) = \frac{F'(r, \theta)}{F(r, \theta)} - \frac{F'(r)}{F(r)} - \int_0^r \hat{\text{Ric}}_k \left( \frac{s_k(t)}{s_k(r)} \gamma'_\theta(t) \right) dt. \tag{2.3}
\]

(2) Assume \( s_k > 0 \) on \((0, \sup_M d_p)\), then (2.3) also holds in the sense of distribution. i.e. for \( 0 \leq f \in C_c^\infty(M) \) and \( r = d_p \),

\[
\int_M r \Delta f \leq \int_M f \left[ \frac{F'(r)}{F(r)} - \frac{F'(r)}{F(r)} - \int_0^r \hat{\text{Ric}}_k \left( \frac{s_k(t)}{s_k(r)} \partial_t \right) dt \right].
\]

(Note that \( \int_0^r \hat{\text{Ric}}_k \left( \frac{s_k(t)}{s_k(r)} \partial_t \right) dt \) is well-defined almost everywhere as a function on \( M \).)

(3) Assume there is no cut point of \( p \) along \( \gamma_\theta \) on \([0, r]\). If \( s_k > 0 \) on \((0, r]\), then

\[
F(r, \theta) \leq \exp \left[ - \int_0^r \int_0^r \hat{\text{Ric}}_k \left( \frac{s_k(t)}{s_k(r)} \gamma'_\theta(t) \right) dt \, dp \right] F(r) = \exp \left[ - \int_0^r (ct_k(t) - ct_k(r)) \hat{\text{Ric}}_k \left( s_k(t) \gamma'_\theta(t) \right) dt \right] F(r).
\]

(Note that \( ct_k(t) - ct_k(r) > 0 \) by Lemma 1.)

If \( k(t) = k \) is constant, then this inequality can also be expressed as

\[
F(r, \theta) \leq \exp \left[ - \frac{1}{s_k(r)} \int_0^r s_k(t) s_k(r - t) \hat{\text{Ric}}_k(\gamma'_\theta(t)) dt \right] F(r).
\]

**Proof.** (1) Let \( e_1, e_2, \ldots, e_n = \theta \) be a positively oriented orthonormal basis of \( T_p M \) and \( E_i \) be the parallel translation of \( e_i \) along \( \gamma_\theta \). Define \( \{Y_i^{r, \theta}(t)\}_{i=1}^{n-1} \) to be the unique Jacobi fields along \( \gamma_\theta \) with \( Y_i^{r, \theta}(0) = 0 \) and \( Y_i^{r, \theta}(r) = E_i(r) \). For convenience we simply denote \( Y_i^{r, \theta} \) by \( Y_i \) and \( \gamma_\theta \) as \( \gamma \).

As \( \langle Y_i(t), \gamma'(t) \rangle = 0 \) at \( t = 0 \) and \( t = r \), we have \( \langle Y_i(t), \gamma'(t) \rangle = 0 \), i.e. \( Y_i(t) \) are tangential to \( S_t \). Since \( Y_i(t) = d\exp_{p|\theta}(Y_i'(0)) \), we see that the \((n - 1)\)-dimensional Jacobian satisfies \( F(t, \theta) = \frac{\det(Y_1(t), \ldots, Y_n(t))}{\det(Y_1'(0), \ldots, Y_n'(0))} \). Note that \( F(t, \theta) \) depends on \((t, \theta)\) only and is independent of \( r \) and \( Y_i \). We have
the formula (cf. [17, p. 460])

\[
(\log F)'(r, \theta) = [\log (\det(Y_1, \ldots, Y_{n-1}))]'(r) = \sum_{i=1}^{n-1} \int_0^r \left( \langle Y'_i, Y'_i \rangle - \langle R(Y_i, \gamma')\gamma', Y'_i \rangle \right) dt
= \sum_{i=1}^{n-1} I(Y'_i, Y'_i).
\]

(2.4)

Let \( X_i(t) = \frac{s_k(t)}{s_k(r)} E_i(t) \). Then by the index lemma [30, Ch. III, Lemma 2.10]

\[
I(Y_i, Y_i) \leq I(X_i, X_i).
\]

(2.5)

By integration by parts,

\[
I(X_i, X_i) = \int_0^r (-\langle X_i'', X_i \rangle - \langle R(X_i, \gamma')\gamma', X_i \rangle) dt + \langle X_i(r), X'_i(r) \rangle
= - \int_0^r \frac{s_k(t)^2}{s_k(r)^2} K^1_k(E_i, \gamma') dt + \frac{s_k'(r)}{s_k(r)}.
\]

(2.6)

Summing (2.6) on \( i = 1, \ldots, n-1 \) and combining with (2.4), (2.5), we have

\[
(\log F)'(r, \theta) \leq - \int_0^r \widehat{\text{Ric}}_k \left( \frac{s_k(t)}{s_k(r)} \gamma_\theta'(t) \right) dt + (\log F)'(r).
\]

Observe that \( F = \sqrt{\det(g_{ij})} \) in polar coordinates, and using \( \Delta f = \frac{1}{\sqrt{\det(g_{ij})}} \partial_i(\sqrt{\det(g_{ij})}g^{ij}\partial_j f) \), we see that \( \Delta d_p(x) = \frac{F'(r, \theta)}{F(r, \theta)} \). So (1) follows.

(2) Let \( \phi(r, \theta) = \int_0^r \widehat{\text{Ric}}_k \left( \frac{s_k(t)}{s_k(r)} \gamma_\theta'(t) \right) dt, \quad \overline{H}(r) = \frac{\overline{F}'(r)}{\overline{F}(r)} \) and \( c(\theta) \) be the cut distance in the direction \( \theta \). Then for \( r < c(\theta) \), as \( \overline{H}(r) - \phi(r, \theta) - \Delta d_p \geq 0 \),

\[
(\overline{H}(r) - \phi(r, \theta))F(r, \theta) \geq F(r, \theta)\Delta d_p = F'(r, \theta).
\]

Multiply this inequality by a non-negative \( f \in C^\infty_c(M) \) and proceed in the same way as [22, Theorem 4.1], we can prove (2.3) in the distributional sense. We omit the details.

(3) Integrating (2.3) gives (note that \( \log F(r, \theta) - \log \overline{F}(r) \to 0 \) as \( r \to 0^+ \))

\[
\log F(r, \theta) \leq - \int_0^r \int_0^{\rho} \widehat{\text{Ric}}_k \left( \frac{s_k(t)}{s_k(\rho)} \gamma_\theta'(t) \right) dt \, dp + \log \overline{F}(r).
\]

i.e. \( F(r, \theta) \leq \exp \left[- \int_0^r \int_0^{\rho} \widehat{\text{Ric}}_k \left( \frac{s_k(t)}{s_k(\rho)} \gamma_\theta'(t) \right) dt \, dp \right] \overline{F}(r) \).

We can transform the double integral into a single integral using the function \( c \). By Fubini’s theorem and Lemma 1,

\[
\int_0^r \int_0^{\rho} \widehat{\text{Ric}}_k \left( \frac{s_k(t)}{s_k(\rho)} \gamma_\theta'(t) \right) dt \, dp = \int_0^r \left( \int_0^{\rho} \frac{1}{s_k(\rho)^2} \, dp \right) \widehat{\text{Ric}}_k(s_k(t)\gamma_\theta'(t)) dt
= \int_0^r (ct_k(t) - ct_k(r)) \widehat{\text{Ric}}_k(s_k(t)\gamma_\theta'(t)) dt.
\]

(2.7)
From this we obtain (3). Note that $ct_k(t) - ct_k(r)$ is positive by Lemma 1.

If $k(t) = k$ is a constant, we can express (2.7) in a more symmetric form. Indeed, we have the “compound angle formula” $s_k(r - t) = s_k(r) + t - s_k(t)$, so (2.7) becomes

$$
\int_0^r \int_0^r \widehat{\text{Ric}}_k \left( \frac{s_k(t)}{s_k(\rho)} \gamma'(t) \right) d\rho dt = \left[ \frac{\text{Ric}}{s_k(\rho)} \right]_0^r (ct_k(t) - ct_k(r)) \widehat{\text{Ric}}_k \left( \frac{s_k(t)}{s_k(\rho)} \gamma'(t) \right) dt \\
= \frac{1}{s_k(r)} \int_0^r s_k(t) s_k(r - t - \frac{1}{s_k(\rho)} \gamma'(t)) dt.
$$

\[\Box\]

**Remark 1.** One may compare Theorem 1 (1) with [29, Lem 2.2], in which the following integral estimate is proved ($k$ is constant):

$$
\int_{B_g(r,p)} [(H - \overline{H})^+]^{2p} dV \leq C(n, p) \sup_{x \in M} \int_{B_{\overline{g}}(r, x)} \left( \frac{\text{Ric}}{s_k(\rho)} \right)^p dV
$$

Here $p > \frac{n}{2}$, $H = \frac{1}{n-1} \Delta d_p$ is the (normalized) mean curvature of the geodesic sphere $S_g(r, p)$, $\overline{H} = \frac{n}{s_k}$, and $f_+$ and $f_-$ denote the positive and negative part of a function $f$ respectively.

As there is no curvature assumption on $M$ in Theorem 1, we are free to choose any comparison function $s_k$, which gives us much flexibility. The same applies to results in later sections. We notice that a quantity similar to the R.H.S. of (2.3) when $k = 0$ was defined in [8, p. 340] and [33, p. 202] to prove a generalized maximum principle.

In many cases, the Laplacian comparison theorem is used to obtain integral estimates for radial functions and as such often a condition weaker than \( \int_0^r \widehat{\text{Ric}}_k \left( \frac{\gamma(t)}{s_k(t)} \right) dt \) suffices to draw useful conclusions. We give an instance of this in Proposition 2, and will illustrate its applications by Theorem 17 and Theorem 18 as examples.

We say a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ is a radial function if $\phi$ is even. It is said to be non-increasing if $\phi'(r) \leq 0$ for $r \geq 0$. We use this terminology because on a Riemannian manifold any smooth radially symmetric function (whenever this makes sense) is of the form $\phi \circ d_p$ for a radial function $\phi$ within the injectivity radius of $p$.

Denote by $B_g(r, p)$ (resp. $\overline{S}_g(r, p)$) to be the geodesic ball (resp. geodesic sphere) of radius $r$ centered at $p$ in $(M, g)$, and $\overline{B}_g(r) := [0, r] \times S^{n-1}$ (resp. $\overline{S}_g(r)$) to be the geodesic ball (resp. geodesic sphere) of radius $r$ centered at 0 in $(\overline{M}, \overline{g})$. We use $B_g(r, p)$ to denote the metric ball of radius $r$ centered at $p$ in $M$, so $B_g(r, p) := \{ x \in M : d(p, x) < r \}$. The metric ball of radius $r$ centered at 0 in $(\overline{M}, \overline{g})$ is also $B_{\overline{g}}(r)$ for $r \leq r_0 = \min \{ r > 0 : s_k(r) = 0 \}$, but $B_g(r, p)$ may not coincide with $B_{\overline{g}}(r, p)$ for large $r$. Let $c(\theta)$ be the cut distance in the direction $\theta$. We also define $B_g'(r, p) := \{ \exp_p(pv) : v \in S_pM, c(v) \geq r \text{ and } 0 \leq \rho < r \} \subset B_g(r, p)$.
Proposition 2. Suppose $\phi, \psi$ are two non-negative radial functions and $\phi$ is non-increasing. Suppose $\int_{B_{g}(r,p)} \widehat{\text{Ric}}_{k}(s_{k}(t)\partial_{t}) \geq 0$ for all $0 \leq \rho \leq r$. Then

$$\int_{B_{g}(r,p)} \langle \nabla (\phi \circ d_{p}), \nabla (\phi \circ d_{p}) \rangle \leq - \int_{B_{g}(r,p)} (\phi \circ d_{p}) \cdot (\Delta \phi) \circ d_{p}$$

where $\Delta \phi(r) := \phi''(r) + \frac{s_{k}(r)}{F(t)} \phi'(r)$ is the Laplacian of $\phi$ with respect to the metric $\bar{g}$. In particular, within a geodesic ball, in short, $- \int_{B_{g}(r,p)} \psi \Delta \phi \leq - \int_{B_{g}(r,p)} \psi \Delta \phi$.

Proof. Let $a(\theta) = \min\{c(\theta), r\}$. Then

$$\int_{0}^{a(\theta)} \psi'(t)\phi'(t)F(t, \theta)dt = \left[\psi(t)\phi'(t)F(t, \theta)\right]_{t=0}^{a(\theta)} - \int_{0}^{a(\theta)} \psi (\phi''(t)F(t, \theta) + \phi'F'(t, \theta)) dt$$

$$\leq - \int_{0}^{a(\theta)} \psi (\phi''F + \phi'F') dt$$

$$= - \int_{0}^{a(\theta)} \psi \left[ \phi'' + \phi' \frac{F'}{F} + \phi' \cdot \left( \frac{F'}{F} - \frac{\bar{F}'}{\bar{F}} \right) \right] Fdt$$

$$= - \int_{0}^{a(\theta)} \psi \Delta \phi Fdt - \int_{0}^{a(\theta)} \psi \phi' \cdot \left( \frac{F'}{F} - \frac{\bar{F}'}{\bar{F}} \right) Fdt. \quad (2.8)$$

As $\phi' \leq 0$, integrating the second term over $S_{p}M$ and using Theorem 1,

$$- \int_{S_{p}M} \int_{0}^{a(\theta)} \psi(t)\phi'(t) \left( \frac{F'(t, \theta)}{F(t, \theta)} - \frac{\bar{F}'}{\bar{F}} (t) \right) F(t, \theta) dt d\theta$$

$$\leq - \int_{0}^{r} \psi(t)|\phi'(t)| \left( \int_{S_{p}(t, p)} \int_{0}^{t} \widehat{\text{Ric}}_{k} \left( \frac{s_{k}(\rho)}{s_{k}(t)} \partial_{\rho} \right) d\rho dS \right) dt \quad (2.9)$$

$$= - \int_{0}^{r} \psi(t)|\phi'(t)| \frac{1}{s_{k}(t)^{2}} \left( \int_{B_{g}(t, p)} \widehat{\text{Ric}}_{k} \left( \frac{s_{k}(\rho)}{s_{k}(t)} \partial_{\rho} \right) dV \right) dt \leq 0,$$

where $S_{g}(t, p) := \{\exp_{p}(t\theta) : \theta \in S_{p}M, c(\theta) > t\}$. In view of this, integrating (2.8) over $S_{p}M$ will give the result. \qed

We now prove some generalizations of the Bonnet-Myers theorem. Roughly speaking it says that the weighted integral of the negative part of the Ricci curvature competes with the positive part of the Ricci curvature together with the function $-s_{k}/s_{k}$ to prevent $M$ from being bounded. An advantage of our result is that we have the flexibility to choose the function $s_{k}$. A number of results in the literature, e.g. [13], can be reduced to a suitable choice of $s_{k}$ in the following result, see also [4], [2], [24], [5], [28]. Ambrose [1] also gives a qualitative version (without a diameter bound) involving the integral of the Ricci curvature.
Theorem 2.

(1) Suppose $s_k$ satisfies (2.1) such that the smallest positive zero $r_0$ of $s_k$ exists, i.e.

$$r_0 := \min\{r > 0 : s_k(r) = 0\}.$$  \hspace{1cm} (2.10)

If

$$\limsup_{r \to r_0^-} \left[ \frac{1}{s_k(r)^2} \int_0^r \widehat{\text{Ric}}(s_k(t)\gamma^*_0(t))dt - (n-1) \frac{s_k'(r)}{s_k(r)} \right] = \infty$$  \hspace{1cm} (2.11)

for any $\theta \in S_pM$, then every geodesic starting from $p$ which is longer than $r_0$ has a conjugate point on $[0, \pi]$. This implies $d_\theta \leq r_0$ on $M$, $M$ is compact and $\pi_1(M)$ is finite. (See also Remark 2.)

(2) With the same assumption as (1), suppose

$$\limsup_{r \to r_0^-} \left[ \frac{1}{s_k(r)^2} \int_{S_pM} \int_0^r \widehat{\text{Ric}}(s_k(t)\partial_\theta)dt \, d\theta - (n-1) \frac{s_k'(r)}{s_k(r)} \right] = \infty,$$  \hspace{1cm} (2.12)

then the injectivity radius at $p$ satisfies $\text{inj}(p) \leq r_0$.

(3) Suppose for all $\theta \in S_pM$, there exists a function $s_k$ satisfying (2.1) whose smallest positive root $r_0$ exists, such that

$$\limsup_{r \to r_0^-} \left[ \frac{1}{s_k(r)^2} \int_0^r \widehat{\text{Ric}}(s_k(t)\gamma^*_0(t))dt - (n-1) \frac{s_k'(r)}{s_k(r)} \right] = \infty,$$

then $M$ is compact and $\pi_1(M)$ is finite.

Proof. (1) Suppose for the sake of contradiction that $\gamma_\theta : [0, r_0] \to M$ is a geodesic with no point conjugate to $p$. Using notation in the proof of Theorem 1, let $Z^*_i(t) := s_k(t)E_i(t)$ on $[0, r]$. Similar to (2.6),

$$\sum_{i=1}^{n-1} I(Z^*_i, Z^*_i) = - \int_0^r \widehat{\text{Ric}}(s_k(t)E_i, \gamma^*_0)dt + (n-1)s_k(r)s_k'(r).$$

So in view of (2.11), $\sum_{i=1}^{n-1} I(Z^{r_0}_i, Z^{r_0}_i) = \lim_{r \to r_0^-} \sum_{i=1}^{n-1} I(Z^*_i, Z^*_i) \leq 0$. This implies $I(Z^{r_0}_i, Z^{r_0}_i) \leq 0$ for some $i$. By the equality case of index lemma, either $Z^{r_0}_i$ is a Jacobi field or there exists a Jacobi field with endpoint values equal to $Z^{r_0}_i$ with strictly smaller index form, contradicting the assumption that $\gamma_\theta$ has no conjugate point on $[0, r_0]$. This implies $d_\theta \leq r_0$ and $M$ is compact.

By applying the same argument to its universal cover $\widetilde{M}$, standard covering theory then shows that $\pi_1(M)$ is finite ([21, Thm. 11.7]).

(2) Suppose $\text{inj}(p) > r_0$. Then $d_\theta$ is smooth on $S_g(r_0, p)$. By (2.12), there exists $\theta \in S_pM$ such that $\limsup_{r \to r_0^-} \left[ \frac{1}{s_k(r)^2} \int_0^r \widehat{\text{Ric}}(s_k(t)\gamma^*_0(t))dt - (n-1) \frac{s_k'(r)}{s_k(r)} \right] = \infty$. Put $x = \gamma_\theta(r)$ in (2.3), and taking the limit $r \to r_0^-$, we get $\Delta d_\theta(\gamma_\theta(r_0)) = -\infty$, a contradiction.
(3) By (1) and [1, Lemma 1], $M$ is compact, and so is $\tilde{M}$.

\[ \square \]

Remark 2. By noting that $s_k > 0$ on $(0, r_0)$ and $s_k'(r_0) < 0$, we can provide some stronger but finitary conditions alternative to (2.11). One possibility is that

\[
\limsup_{r \to r_0^-} \left[ \frac{1}{s_k(r)} \int_0^r \tilde{\text{Ric}}_k(s_k(t)\gamma_0'(t))dt - (n-1)s_k'(r) \right] > 0
\]

for all $\theta \in S_pM$. Another simpler (but stronger) condition which clearly indicates the relation with the classical Bonnet-Myers theorem is

\[
\int_0^{r_0} \tilde{\text{Ric}}_k(s_k(t)\gamma_0'(t)) \, dt \geq 0
\]

(2.13)

for all $\theta \in S_pM$.

To see that the two conditions above are stronger, observe that $s_k'(r_0) < 0$. Indeed, by Lemma 1 (1), since $s_k(r_0) = 0$ and $s_k > 0$ on $(0, r_0)$, we have $s_k'(r_0) < 0$. From this we have $\lim_{r \to r_0^-} \frac{s_k'(r)}{s_k(r)} = -\infty$. Then we can see that both the above two conditions imply (2.11).

Similarly we can use the simpler condition $\int_{S_pM} \int_0^{r_0} \tilde{\text{Ric}}_k(s_k(t)\partial_t) \, dt \, d\theta \geq 0$ to replace (2.12).

It is also interesting to see that the existence of a conjugate point implies a condition in terms of $\hat{K}_k^1$ similar to (2.13). We use $\hat{K}_k^1(E(t), \gamma'(t))$ to denote $\tilde{K}_k^1(\text{span}(E(t)), \gamma'(t))$.

Proposition 3. Suppose $\gamma$ is a geodesic of length $r_0$ parametrized by arclength. If $\gamma(r_0)$ is the first conjugate point of $\gamma(0)$ along $\gamma$, then there exists a unit vector field $E(t)$ along $\gamma$, together with functions $k(t)$ and $s_k(t)$ satisfying (2.1), such that $\int_0^{r_0} s_k(t)^2 \hat{K}_k^1(E(t), \gamma'(t)) \, dt \geq 0$ with $r_0 = \min\{t > 0 : s_k(t) = 0\}$.

Proof. There exists a nontrivial Jacobi field $Y(t)$ with $Y(0) = 0 = Y(r_0)$. Let $Y(t) = s(t)E(t)$, where $s(t) = |Y(t)|$ and w.l.o.g. $s'(0) = 1$. We compute $s' = \langle Y', E \rangle$ and $s'' = -\langle (R(E, \gamma')\gamma', E) - |E'|^2 \rangle s = -ks$. As $\gamma(r_0)$ is the first conjugate point along $\gamma$,

\[
0 = I(Y, Y) = \int_0^{r_0} (|Y'|^2 - \langle R(Y, \gamma')\gamma', Y \rangle) = \int_0^{r_0} (s^2 + s^2 |E'|^2 - s^2 \langle R(E, \gamma')\gamma', E \rangle)
\]

\[
= \int_0^{r_0} (ks^2 + s^2 |E'|^2 - s^2 \langle R(E, \gamma')\gamma', E \rangle)
\]

\[
\geq \int_0^{r_0} (ks^2 - s^2 \langle R(E, \gamma')\gamma', E \rangle)
\]

\[
= -\int_0^{r_0} s^2 \hat{K}_k^1(E, \gamma').
\]

\[ \square \]
Therefore satisfies $h(r)$ is non-increasing and $g(t)$ is positive for $t > 0$, then $\int_s^r f(t)dt \int_s^r g(t)dt$ is non-increasing in $r$ and $s$.

There is also a quantitative version of the lemma.

**Lemma 3.** If $f(t), g(t)$ are $C^1$ functions, $a \in \mathbb{R}$ and $g(t) > 0$, then $h(r) = \frac{\int_a^r f(t)dt}{\int_a^r g(t)dt}$ satisfies $h'(r) = \frac{g(r) \int_a^r g(t)dt + f(r) \int_a^r f(t)dt - g(r) \int_a^r f(t)dt}{(\int_a^r g(t)dt)^2}$ where $\alpha(t) = \frac{f(t)}{g(t)}$.

**Proof.** We compute $h'(r) = \frac{\int_a^r g(t)dt - g(r) \int_a^r f(t)dt}{\int_a^r g(t)dt}$. By fundamental theorem of calculus, $\frac{f(r)}{g(r)} \frac{f(t)}{g(t)} = \int_t^r \alpha'(u)du$ and so $f(r)g(t) - g(r)f(t) = g(r)g(t) \int_t^r \alpha'(u)du$.

Integrating this with respect to $t$ on $[a, r]$ and using Fubini’s theorem, $f(r) \int_a^r g(t)dt - g(r) \int_a^r f(t)dt = g(r) \int_a^r g(t)dt \int_t^r \alpha'(u)du dt = g(r) \int_a^r \alpha'(u) \int_a^u g(t)dt du$.

Therefore $h'(r) = \frac{g(r) \int_a^r \alpha'(u) \int_a^u g(t)dt du}{(\int_a^r g(t)dt)^2}$.

We now give the area and volume comparison theorems. We use $|\cdot|$ to denote either the volume of a domain or the area of a hypersurface, whichever makes sense.

The following theorem can be compared to [29, Thm. 1.1].

**Theorem 3.**

1. Suppose $r < \text{inj}(p)$ and $s_k > 0$ on $(0, r]$. We have the estimate

   $$|B_g(r, p)| \leq \int_0^r w(\rho) F(\rho) d\rho,$$

   where $w(\rho) = \int_{S_{\rho}M} \exp \left[ - \int_0^\rho (ct_k(t) - ct_k(\rho)) \hat{\text{Ric}}_k(s_k(t) \partial_t) dt \right] d\theta$.

   In particular, if

   $$\int_{S_{\rho}M} \exp \left[ - \int_0^\rho (ct_k(t) - ct_k(\rho)) \hat{\text{Ric}}_k(s_k(t) \partial_t) dt \right] d\theta \leq 1 \quad (2.15)$$

   for $\rho \in [0, r]$, then $|B_g(r, p)| \leq |B_{\pi}(r)|$.

   In both cases, the equality holds if and only if $B_g(r, p)$ is isometric to the geodesic ball $B_{\pi}(r)$. (Note also that $ct_k$ is decreasing by Lemma 1.)

2. Suppose $r < \text{inj}(p)$ and $s_k > 0$ on $(0, r]$, then

   $$\frac{d}{dr} \left( \frac{|S_g(r, p)|}{|S_{\pi}(r)|} \right) \leq - \frac{1}{|B_{\pi}(r)|} \int_{B_g(r, p)} \hat{\text{Ric}}_k \left( \frac{s_k(t)}{s_k(r)} \partial_t \right) dV$$
and
\[
\frac{d}{dr} \left( \frac{|B_g(r, p)|}{|B_{\gamma}(r)|} \right) \leq -\frac{\overline{F}(r)}{|B_{\gamma}(r)|^2} \int_0^r \frac{|B_{\gamma}(u)|}{\overline{F}(u)} \int_{B_p(u, p)} \overline{\text{Ric}}_k \left( \frac{s_k(t)}{s_k(r)} \partial_t \right) dV du.
\]

The equality holds if and only if \(B_g(r, p)\) is isometric to \(B_{\gamma}(r)\).

In particular, if \(\int_{B_p(r, p)} \overline{\text{Ric}}_k \left( \frac{s_k(t)}{s_k(r)} \partial_t \right) dV \geq 0\) for all \(r \in (0, r_0)\), then \(\frac{|B_g(r, p)|}{|B_{\gamma}(r)|}\) is non-increasing on \((0, r_0)\).

(3) For \(r \leq r_0\) given by (2.10) \((r_0 := \infty\) if \(s_k\) has no positive zero), \(\frac{|B_g(r, p)|}{|B_{\gamma}(r)|}\) is absolutely continuous and
\[
\frac{d}{dr} \left( \frac{|B_g(r, p)|}{|B_{\gamma}(r)|} \right) \leq -\frac{\overline{F}(r)}{|B_{\gamma}(r)|^2} \int_0^r \frac{|B_{\gamma}(u)|}{\overline{F}(u)} \int_{B_p(u, p)} \overline{\text{Ric}}_k \left( \frac{s_k(t)}{s_k(r)} \partial_t \right) dV du.
\]

Assume \(\int_0^r \overline{\text{Ric}} \left( s(t) \gamma_\theta(t) \right) dt \geq 0\) for any \(\theta \in S_p M\) and any \(r \in (0, r_0)\). If \(|B_g(r, p)| = |B_{\gamma}(r)|\) \((r \leq r_0)\), then \(B_g(r, p)\) is isometric to \(B_{\gamma}(r)\).

**Proof of Theorem 3.**  
(1) As \(|B_g(r, p)| = \int_0^r \int_{S_p M} F(\rho, \theta) d\theta d\rho\), the inequality (2.14) follows directly from Theorem 1 (3).

If the equality holds, then the normal Jacobi fields adapted to \(p\) are of the form \(s_k(t)\epsilon(t)\) for some parallel \(\epsilon(t)\) orthogonal to \(\gamma\). The result can then be proved using the Cartan-Ambrose-Hicks theorem (\([18, \text{Thm. 1.12.8}]\)), the details are similar to (but simpler than) the proof of Theorem 6, so we omit them here.

(2) Let \(A(r) = |S_g(r, p)| = \int_{S^{n-1}} F(r, \theta) d\theta\), \(\overline{A}(r) = \int_{S^{n-1}} \overline{F}(r) d\theta = |S^{n-1}| \overline{F}(r)\). Then
\[
\left( \frac{A(r)}{\overline{A}(r)} \right)' = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \frac{\partial}{\partial r} \left( \frac{F(r, \theta)}{\overline{F}(r)} \right) d\theta
= \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \left( F'(r, \theta) \frac{\overline{F}(r)}{\overline{F}(r)} - \frac{\overline{F}'(r)}{\overline{F}(r)} \right) F(r, \theta) d\theta \tag{2.16}
= \frac{1}{\overline{A}(r)} \int_{S_g(r, p)} \left( F'(r, \theta) \frac{\overline{F}'(r)}{\overline{F}(r)} - \frac{\overline{F}'(r)}{\overline{F}(r)} \right) dS.
\]

So from (2.3),
\[
\left( \frac{A(r)}{\overline{A}(r)} \right)' \leq -\frac{1}{\overline{A}(r)} \int_{S_g(r, p)} \int_0^r \overline{\text{Ric}}_k \left( \frac{s_k(t)}{s_k(r)} \partial_t \right) dt dS
\leq -\frac{1}{\overline{A}(r)} \int_{B_g(r, p)} \overline{\text{Ric}}_k \left( \frac{s_k(t)}{s_k(r)} \partial_t \right) dV. \tag{2.17}
\]
Let \( V(r) = |B_g(r, p)| = \int_0^r A(s) ds \) and \( \nabla r = |B_\nabla r(r)| = \int_0^r A(u) du \). Then by Lemma 3 and (2.17),

\[
\frac{d}{dr} \left( \frac{V(r)}{\nabla r} \right) = \frac{A(r)}{\nabla r} \int_0^r V(u) \frac{d}{du} \left( \frac{A(u)}{A(r)} \right) du
\]

\[
\leq - \frac{A(r)}{\nabla r} \int_0^r \frac{V(u)}{A(u)} \int_{B_g(u, p)} \widehat{\text{Ric}} \left( \frac{s_k(t)}{s_k(u)} \partial_t \right) dV du.
\]

(2.18)

(3) Let \( \chi(r, \theta) : T_p M \to \mathbb{R} \) be defined by \( \chi(r, \theta) = \begin{cases} 1, & \text{if } c(\theta) > r \\ 0, & \text{otherwise} \end{cases} \). Then (2.16) and (2.17) are still true if we replace \( F(r, \theta) \) by \( F(r, \theta) := F(r, \theta) \chi(r, \theta) \), \( S_g(r, p) \) by \( S_g(r, p) = \{ \exp_p(r\theta) : \theta \in S_p M, c(\theta) > r \} \), \( B_g(r, p) \) by \( B'_g(r, p) \) and \( A(r) \) by \( A(r) = \int_{S_g(r, p)} F(r, \theta) d\theta \) which is absolutely continuous. Let \( V(r) = \int_0^r A(t) dt \), then the analysis in (2.18) shows that for almost every \( r \),

\[
\frac{d}{dr} \left( \frac{V(r)}{\nabla r} \right) \leq - \frac{F(r)}{\nabla r^2} \int_0^r \frac{V(u)}{F(u)} \int_{B'_g(u, p)} \widehat{\text{Ric}} \left( \frac{s_k(t)}{s_k(u)} \partial_t \right) dV du.
\]

Under the assumption, if \( |B_g(r, p)| = |B_\nabla r(r)| \leq r_0 \), then \( \frac{F(r)}{\nabla r} = 1 \) and \( \chi(r, \theta) = 1 \) for all \( \theta \), so \( B_g(r, p) = B_\nabla r(r, p) \) and it has the same volume as \( B_\nabla r(r) \). So from (1) and in view of (2.7), we conclude that \( B_g(r, p) \) is isometric to \( B_\nabla r(r) \).

\[\square\]

As a simple corollary, we record here a bound for the isoperimetric ratio for geodesic balls, which may be of independent interest.

**Proposition 4.** If \( \int_{B_g(\rho, p)} \widehat{\text{Ric}}(s_k(t) \partial_t) dV \geq 0 \) for all \( \rho \in [0, r] \), then \( \frac{|B_g(t, p)|}{|B_\nabla r(t, p)|} \geq \frac{|B_\nabla r(t, p)|}{|S_p M|} \) for \( t \in [0, r] \).

**Proof.** Using the notation in Theorem 3, for \( 0 \leq s \leq t \), we have \( A(s) \geq \frac{A(t) A(s)}{A(t)} \). Integrate this w.r.t. \( s \) on \( [0, t] \), we can get the result. \( \square \)

**Remark 3.** Suppose \( k \) is a constant. Our condition for Theorem 3 (1) and (2) is much weaker than a Ricci curvature lower bound, but stronger than a scalar curvature lower bound, in the sense that if (2.15) is satisfied for all small enough \( r \), then \( R(p) \geq n(n - 1)k \). It is not hard to see that

\[
\exp \left[ - \int_0^r (ct_k(t) - ct_k(r)) \widehat{\text{Ric}}(s_k(t) \gamma_\theta(t)) dt \right] = 1 - (\text{Ric}(\theta, \theta) - (n - 1)k) \frac{r^2}{6} + O(r^3).
\]

We then have

\[
\int_{S_p M} \exp \left[ - \int_0^r \frac{s_k(t) s_k(r - t)}{s_k(r)} \widehat{\text{Ric}}(\gamma_\theta(t)) dt \right] d\theta = 1 - \left( \frac{R(p)}{n} - (n - 1)k \right) \frac{r^2}{6} + O(r^3)
\]

(2.19)
where we have used $\int_{\mathbb{S}^{n-1}} h(\theta, \theta) d\theta = \frac{\text{tr}(h)}{n}$ for a symmetric bilinear form $h$.

Therefore we conclude that if \((2.15)\) is satisfied for any small enough $r > 0$, then indeed we have the scalar curvature $R(p) \geq n(n-1)k$.

Theorem 3 combined with \((2.19)\) is also consistent with the fact that the Taylor expansion of the volume of small geodesic balls involve only the scalar curvature at $p$ in the $(n+2)$-th order (cf. [15, Theorem 3.1]), whereas a global area or volume comparison result is not true assuming only a scalar curvature lower bound. Indeed, by a direct computation, or using the formula in [15, Theorem 3.1], it can be seen that the volume of a geodesic ball in $\mathbb{H}^2 \times \mathbb{S}^2$ (which has zero scalar curvature) is larger than the Euclidean one. More precisely, we have

$$|B_2(r, p)| = b_4 r^4 \left(1 + \frac{1}{72 \cdot 6 \cdot 8} r^4 + O(r^6)\right) > b_4 r^4$$

for $r \approx 0$, where $b_4$ is the volume of the unit ball in $\mathbb{R}^4$.

2.3. Comparison theorems for distance from a submanifold. In the following result, we are going to use the (polar) Fermi coordinates with respect to an $\ell$-dimensional submanifold $\Sigma$ (cf. [16]). These coordinates are suitable to describe the geometry of the tubular neighborhood of $\Sigma$.

Recall that if $x$ is within the cut locus of $\Sigma$, then the Fermi coordinates of $x$ is $(r, \theta, z)$, where $r = d(x, \Sigma) = d(x, z)$ for $z \in \Sigma$ and $\gamma_\theta$ is the minimizing geodesic with initial vector $\theta \in S(N_z \Sigma)$.

The mean curvature vector is defined as $H = \frac{1}{\ell} \sum_{i=1}^{\ell} (\nabla e_i e_i)^\perp$, where $e_i$ is an orthonormal frame along $\Sigma$.

**Theorem 4.** Suppose $\Sigma$ is an $\ell$-dimensional submanifold of $M$. Let $d_\Sigma : M \to \mathbb{R}$ be the distance from $\Sigma$, $H$ be the mean curvature vector of $\Sigma$ and $(r, \theta, z)$ be the Fermi coordinates of $x$. Assume $s_k > 0$ on $(0, r]$ and that the first zero of $t \mapsto c_k(t) - \langle H(z), \theta \rangle s(t)$ (if exists) appears no earlier than the cut distance in the direction $\theta$.

1. We have

$$\Delta d_\Sigma(x) \leq \log \overline{F}'(r, \theta, z) - \psi(r, \theta, z)$$

where

$$\overline{F}(t, \theta, z) = (c_k(t) + \lambda s_k(t))^{\ell} s_k(t)^{n-1-\ell},$$

$$\psi(r, \theta, z) = \int_0^r \left(\frac{(c_k(t) + \lambda s_k(t))}{(c_k(r) + \lambda s_k(r))} R_k^\ell(P_0^\perp T_z \Sigma, \partial_1) + \frac{s_k(t) s_k(t)}{s_k(r)^2} R_k^{n-1-\ell} (P_0^{\perp \cap N_z \Sigma}, \partial_1)\right) dt,$$

$$\lambda = -\langle H(z), \theta \rangle$$

and $P_0^\perp$ is the parallel transport along $\gamma_\theta$ to $\gamma_\theta(t)$.

2. The volume element $F(r, \theta, z)$ of $M$ satisfies

$$F(r, \theta, z) \leq \exp \left[-\phi(r, \theta, z)\right] \overline{F}(r, \theta, z)$$
where
\[
\phi(r, \theta, z) = \int_0^r \left[ \left( \frac{s_k(r)}{c_k(r) + \lambda s_k(r)} - \frac{s_k(t)}{c_k(t) + \lambda s_k(t)} \right)(c_k(t) + \lambda s_k(t))^2 \hat{K}_k^\ell(P_\theta^t(T, \Sigma), \partial_t)
\right.
\]
\[
+ (ct(t) - ct(r)) s_k(t)^2 \hat{K}_k^{n-\ell-1}(P_\theta^t(\theta^\perp \cap N_z \Sigma), \partial_t) \bigg] \, dt.
\]

(2.22)

In particular, if \( \hat{K}_k^\ell \geq 0 \) for \( i = \ell, n - 1 - \ell \), then \( F(r, \theta, z) \leq \overline{F}(r, \theta, z) \).

**Remark 4.**

(1) The condition in Theorem 4 is mild. As shown by Heintze and Karcher in [17, Cor. 3.3.1], if \( k \) is constant, the first focal point appears no later than the first zero of \( c_k(t) + \lambda s_k(t) \) in all space forms of curvature \( k \).

More generally they show that it holds if \( \text{Ric}_k \geq 0 \) when \( \dim \Sigma = n - 1 \), or if \( \hat{K}_k^\ell \geq 0 \) for general \( \ell \). Indeed, from the proof in [17] (cf. 3.4.4, which also uses the index lemma), we see that \( \hat{K}_k^\ell \geq 0 \) suffices.

(2) Similar to Theorem 12, we may instead have a sharper \( \overline{F} \) which is expressed in terms of the second fundamental form of \( \Sigma \) (instead of \( \Pi \)), but then the error term \( \psi \) will involve \( \ell \)-sectional curvatures along \( \gamma \) instead of the weaker curvature \( \hat{K}_k^\ell \). So there is some tradeoff between the two types of estimates. Indeed, in this case, \( \overline{F}(t, \theta, z) = s_k(t)^{n-\ell} \det\{c_k(t)I_d + s_k(t)A_0\} \) and

\[
\psi(r, \theta, z) = \int_0^r \left( \sum_{i=1}^\ell \left( \frac{(c_k(t)^2 + \lambda_i s_k(t))^2}{(c_k(r)^2 + \lambda_i s_k(r))^2} \hat{K}_k(E_i(t), \partial_t) + \frac{s_k(t)^2}{s_k(r)} \hat{K}_k^{n-\ell-1}(P_\theta^t(\theta^\perp \cap N_z \Sigma), \partial_t) \right) \right) \, dt.
\]

**Proof.**

(1) Let \( E_1, \cdots, E_n = \gamma_\theta(t) \) be a parallel orthonormal frame along \( \gamma_\theta(t) \) such that \( E_1(0), \cdots, E_\ell(0) \in T_z \Sigma \).

For \( i = 1, \cdots, \ell \), let \( Y_i(t) \) be the \( \Sigma \)-adapted Jacobi field along \( \gamma \) such that \( Y_i(r) = E_i(r) \). Each tangent space of \( w \in N \Sigma \) is naturally split into the orthogonal direct sum of \( \ell \)-dimensional horizontal subspace \( \mathcal{H} \) and \( (n-\ell) \)-dimensional vertical subspace \( \mathcal{V} \) (tangent space of fiber of \( \pi : N \Sigma \to \Sigma \)).

Let \( \exp_{N \Sigma} : N \Sigma \to M \) be the exponential map of the normal bundle \( N \Sigma \). Then the \( \ell \)-dimensional Jacobian of \( d\exp_{N \Sigma}|_0 \) along \( \mathcal{H}|_0 \) is \( F_\mathcal{H}(t, \theta) = \frac{\det(Y_1(t), \cdots, Y_\ell(t))}{\det(Y_1(0), \cdots, Y_\ell(0))} \). We have the formula (cf. [17, p. 460])

\[
(\log F_\mathcal{H})'(r, \theta) = \sum_{i=1}^\ell \left( \int_0^r \left( \langle Y_i'(t), Y_i'(t) \rangle - \langle R(Y_i(t), \delta t)Y_i(t), Y_i(t) \rangle \right) \, dt + A_\theta(Y_i(0), Y_i(0)) \right)
\]
\[
= \sum_{i=1}^\ell I_\Sigma(Y_i, Y_i).
\]

(2.23)

Let \( \lambda = -\langle H, \theta \rangle \) and define \( X_i(t) = \frac{c_k(t)^2 + \lambda s_k(t)}{c_k(r)^2 + \lambda s_k(r)} E_i(t) \), \( i = 1, \cdots, \ell \). As \( X_i(r) = Y_i(r) \) and \( X_i(0) \in T \Sigma \), by the index lemma ([30, Ch. III, Lemma]...
We impose more conditions than Theorem 4 to make the statement cleaner.
Theorem 5. Suppose Σ is an ℓ-dimensional submanifold of M and let dS be the measure on Σ. Define $\mathcal{A}(r) = \int_{S(N_\Sigma)} F(r, \theta, z) d\theta dz$ and $\mathcal{V}(r) = \int_0^r \mathcal{A}(t) dt$ with $\mathcal{F}$ in (2.21). Assume $k$ is constant, $s_k > 0$ on $(0, r]$, $\hat{K}_k^r \geq 0$ and $\hat{K}_k^{n-\ell-1} \geq 0$.

1. We have
$$|S(r, \Sigma)| \leq |S^{n-\ell-1}| s_k(r)^{n-\ell-1} \int_\Sigma f(r, |H|) \int_{S(N_\Sigma)} \exp(-\varphi) dS \leq |S^{n-\ell-1}| s_k(r)^{n-\ell-1} \int_\Sigma f(r, |H|) dS$$

where $f(r, h) := \int_{S^{n-\ell-1}} (c_k(r) + \langle h e_0, \theta \rangle)^\ell d\theta$ with $e_0 = (1, 0, \ldots, 0) \in S^{n-\ell-1}$, and $\varphi \geq 0$ is given in (2.22).

2. The function $f(r, h)$ in (1) is non-decreasing in $h$. In particular, if $|H| \leq h_0$ for some constant $h_0$, then $|S(r, \Sigma)| \leq |S^{n-\ell-1}| s_k(r)^{n-\ell-1} f(r, h_0) |\Sigma|$.

3. Suppose $(\log \mathcal{F})'(t, \theta, z) \geq 0$ for $t \in (0, r_0)$ and $(\theta, z) \in S(N_\Sigma)$. Then on $[0, r_1]$, $\mathcal{A}(r) - |S_g(r, \Sigma)|$ is non-negative and non-decreasing, and $\mathcal{V}(r) - |B_g(r, \Sigma)|$ is non-negative, non-decreasing and convex. Here $r_1 = \min\{r_0, \text{inj}(\Sigma)\}$.

4. Suppose Σ is a minimal submanifold, i.e. $H = 0$, then
$$\frac{d}{dr} \left( \frac{|S(r, \Sigma)|}{\mathcal{A}(r)} \right) \leq -\frac{1}{\mathcal{A}(r)} \int_{S(r, \Sigma)} \phi(x) dS \leq 0,
$$

Here $\phi(x) := \phi(r(x), \theta(x), z(x))$. We also have
$$\frac{d}{dr} \left( \frac{|B(r, \Sigma)|}{\mathcal{V}(r)} \right) \leq -\frac{\mathcal{A}(r)}{\mathcal{V}(r)^2} \int_0^r \frac{\mathcal{V}(u)}{\mathcal{A}(u)} \int_{S(u, \Sigma)} \phi(x) dS du \leq 0.$$

Proof. We use the notation in Theorem 4.

1. We have $|S(r, \Sigma)| = \int_\Sigma \int_{S(N_\Sigma)} F(r, \theta, z) d\theta dz$

Define $f(r, h) := \int_{S^{n-\ell-1}} (c_k(r) + \langle h e_0, \theta \rangle)^\ell d\theta$. Then it follows by the invariance of the spherical measure that $\int_{S(N_\Sigma)} \mathcal{F}(r, \theta, z) d\theta = f(r, |H(z)|) s_k(r)^{n-\ell-1}$. From this and Theorem 4 we have
$$|S(r, \Sigma)| \leq \int_\Sigma f(r, |H(z)|) s_k(r)^{n-\ell-1} \int_{S(N_\Sigma)} \exp[-\phi(r, \theta, z)] dS \leq |S^{n-\ell-1}| s_k(r)^{n-\ell-1} \int_\Sigma f(r, |H(z)|) dS.$$

2. The fact that $f(r, h)$ is non-decreasing in $h$ is proved using the same idea in [17, Proposition 2.1.1].

3. By Theorem 4, we have $F'(r, \theta, z) \leq \mathcal{F}'(r, \theta, z) \frac{F(r, \theta, z)}{\mathcal{F}(r, \theta, z)}$. So if $\mathcal{F}'(r, \theta, z) \geq 0$, then by Theorem 4 again, $F'(r, \theta, z) \leq \mathcal{F}'(r, \theta, z) \exp[-\phi(r, \theta, z)] \leq \mathcal{F}'(r, \theta, z)$. This implies
$$\frac{d}{dr} |S_g(r, \Sigma)| = \frac{d}{dr} \left( \int_\Sigma \int_{S(N_\Sigma)} F(r, \theta, z) d\theta dz \right) \leq \frac{d}{dr} \left( \int_\Sigma \int_{S(N_\Sigma)} \mathcal{F}(r, \theta, z) d\theta dz \right) = \mathcal{A}(r).$$
The properties of $\nabla(r) - |B_g(r, \Sigma)|$ follow from this.

We remark that if there exists $\Sigma$ in $\overline{M}_k$ with the same mean curvature vector $H$ as $\Sigma$, then $\frac{d}{dt}(\log \overline{F})$ is the mean curvature of $S_\Sigma(t, \Sigma)$.

4. Suppose $H = 0$, then $(\log F)'(r, \theta, z) \leq -\phi(r, \theta, z) + (\log \overline{F})'(r)$ where $\overline{F}(t) = c_k(t)^{n-\ell-1}$.

We have $\overline{A}(r) = |S^{m-\ell-1}| |\Sigma| \overline{F}(r)$ and we can then proceed as in (2.17) to show that

$$\frac{d}{dr} \left( \frac{S(r, \Sigma)}{A(r)} \right) \leq - \frac{1}{A(r)} \int_\Sigma \int_{S(r, \Sigma)} \phi(r, \theta, z) F(r, \theta, z) d\theta dz$$

$$= - \frac{1}{A(r)} \int_{S(r, \Sigma)} \phi(r(x), \theta(x), z(x)) dS \leq 0.$$  

Note that $\overline{A}(r) > 0$ by (1). The second inequality is similar to (2.18).

$\square$

3. Comparison results in Kähler manifolds

3.1. Comparison model and notions of curvatures. We now turn to a Kähler manifold $M$ whose complex structure is given by $J$ with $\dim \mathbb{C} M = n$. We will study it from the real differential geometric point of view, i.e. regard it as a Riemannian manifold with a parallel tensor $J$. It turns out that the model space that we are comparing is not a warped product space. Indeed, the complex space forms $\overline{M}_k$ of constant holomorphic sectional curvature $k$ (complex Euclidean space $\mathbb{C}^n$, complex projective space $\mathbb{CP}^n$ and complex hyperbolic space $\mathbb{CH}^n$) are not warped products unless $k = 0$ or $n = 1$. See (3.4) for the definition of the holomorphic sectional curvature. We now rewrite the metric of $\overline{M}_k$ in a form which is comparable to the warped product metric in Subsection 2.2.

First of all, let $S^{2n-1} = \{ z \in \mathbb{C}^n : |z| = 1 \}$. There is a natural $S^1$-action on $S^{2n-1}$ by $e^{i\theta} \cdot (z_1, \ldots, z_n) = (e^{i\theta} z_1, \ldots, e^{i\theta} z_n)$, $i = \sqrt{-1}$. This action induces a splitting of $TS^{2n-1} = \mathcal{H} \oplus \mathcal{V}$ where $\mathcal{V}$ is the tangent space of the fiber and $\mathcal{H}$ is its orthogonal complement with respect to standard round metric. Let $g_\mathcal{V}$ and $g_\mathcal{H}$ be the induced metric from the round metric onto the vertical and horizontal space respectively.

We claim that if $k$ is constant and $s = s_k$, then within the cut locus, the Riemannian metric of $\overline{M}_k$ is of the form

$$\overline{g} = dt^2 + s(t)^2 g_\mathcal{H} + \frac{1}{4} s(2t)^2 g_\mathcal{V}$$  (3.1)

defined on $[0, t_0) \times S^{2n-1}$, where $t_0$ is the first positive zero of $s(2t)$.

This is clear for $k = 0$. We illustrate this for $k = 1$. Let $U_0 = \{ [1, z_1, \ldots, z_n] \in \mathbb{CP}^n \} \cong \mathbb{C}^n$. In this coordinate neighborhood, the Fubini-Study metric is of the form

$$\overline{g} = \frac{dz \cdot d\overline{z}}{1 + |z|^2} - \frac{(z \cdot d\overline{z})(z \cdot dz)}{(1 + |z|^2)^2}.$$  (3.2)
where \( z = (z_1, \ldots, z_n) \) and \( z \cdot w = \sum_{j=1}^n z_jw_j \). From (3.2), the distance from \([1,0,\cdots,0]\) with respect to \( \overline{g} \) is given by \( t = \int_0^{||z||} \frac{1}{1+r^2} dr = \tan^{-1}(||z||) \). So writing \( z = (\tan t)w, \ w \in S^{2n-1} \subset \mathbb{C}^n \) gives \( \overline{g} = dt^2 + \sin^2 t dw \cdot dw - \sin^4 t w \cdot dw \), where we have used \( w \cdot dw = \overline{w} \cdot dw = 0 \).

Along the vertical fiber, a unit tangent vector (w.r.t \( g_{S^{2n-1}} \)) can be represented by \( v = \sqrt{-1}w \) at the point \((t,w)\), and so we have \(|v|_\overline{g}^2 = \sin^2 t - \sin^4 t = \frac{1}{4} \sin^2(2t) \).

Along the horizontal fiber, a unit tangent vector \( v \) (w.r.t \( g_{S^{2n-1}} \)) is orthogonal to both \( w \) and \( iw \), so \( 0 = w \cdot v + \overline{w} \cdot v \) and \( 0 = iw \cdot v - i \overline{w} \cdot v \). This implies \( w \cdot dw = \overline{w} \cdot dw = 0 \) and so \( |v|_\overline{g}^2 = \sin^2 t \).

Similarly, this holds for the complex hyperbolic metric \( \overline{g} = \frac{|dz|^2}{1-|z|^2} + \frac{|\overline{z} dz|^2}{(1-|z|^2)^2}, \ |z| < 1 \).

In view of (3.1), it is natural to propose the following model. Let \( k_j(t), \ j = 1, 2, \) be two continuous functions and suppose \( s_{k_j}(t) \) satisfies (2.1) for \( k = k_j \). We define on \( \overline{M} = [0,t_0) \times S^{2n-1} \) the metric \( \overline{g} = dt^2 + s_{k_1}(t)^2 g_H + s_{k_2}(t)^2 g_V \) (3.3) where \( t_0 \) is the first positive zero of \( s_{k_1}(t)s_{k_2}(t) \).

By a computation similar to [26, Ch 7, Proposition 35], we see that \( \overline{\Delta} = (2n-2) \frac{s_{k_1}'(r)}{s_{k_1}(r)} + \frac{s_{k_2}'(r)}{s_{k_2}(r)} \) where \( r \) is the \( \overline{g} \)-distance from 0 and \( \overline{\Delta} \) is the Laplacian of \( \overline{g} \). Similar calculations as in [26, Ch 7, Proposition 42] also implies that along the geodesic \( \theta \mapsto t\theta, R_{\overline{g}}(u,\partial_\theta)\partial_\theta = k_1u \) for \( u \in H, R_{\overline{g}}(u,\partial_\theta)\partial_\theta = k_2u \) for \( u \in V \).

We endow a complex structure on \( \overline{M} \) as follows. Let \( r(t) \) be the solution to \( r'(t) = \frac{1}{s_{k_2}(t)} \) with \( r(0) = 0 \) and define a diffeomorphism \( \overline{M} = [0,t_0) \times S^{2n-1} \rightarrow B(r(t_0),0) \subset \mathbb{C}^n \) by \( \Phi(t,w) = r(t)w \). Let \( J = J_{\overline{M}} \) be the complex structure on \( \overline{M} \) induced from \( \mathbb{C}^n \) via this map.

**Lemma 4.** The complex structure \( J \) is parallel w.r.t. \( \overline{g} \) in (3.3) along the \( \overline{g} \)-geodesic \( t \mapsto tw_0 \), where \( w_0 \in S^{2n-1} \).

**Proof.** Regard \( S^{2n-1} = \{ w \in \mathbb{C}^n : |w| = 1 \} \) and so identify \( T_{w_0}S^{2n-1} = \{ z \in \mathbb{C}^n : \text{Re}(z \cdot \overline{w_0}) = 0 \} \). We can choose a normal coordinates (w.r.t. round metric) \( \{ \theta_j \}_{j=1}^{2n-1} \) around \( w_0 \in S^{2n-1} \) such that at \( w_0 \), \( \frac{\partial}{\partial \theta_j} = \sqrt{-1}w_0 \) and \( \sqrt{-1} \frac{\partial}{\partial \theta_j} = \frac{\partial}{\partial \theta_{j+n-1}} \) for \( j = 1, \dotsc, n-1 \). Then it is easy to see that along \( \gamma(t) = t w_0 \), \( J \left( \frac{\partial}{\partial \theta_j} \right) = \frac{1}{s_{k_2}(t)} \frac{\partial}{\partial \theta_j}, J \left( \frac{\partial}{\partial \theta_{j+n-1}} \right) = -s_{k_2}(t) \frac{\partial}{\partial \theta_j}, J \left( \frac{\partial}{\partial \theta_{j+n-1}} \right) = - \frac{\partial}{\partial \theta_j} \) for \( j = 1, \dotsc, n-1 \).

From this, it then suffices to show that along \( \gamma \), the vector fields \( \frac{\partial}{\partial \theta_j}, \frac{\partial}{\partial \theta_{j+n-1}} \) and \( \frac{\partial}{\partial \theta_j} \) are parallel for \( j = 1, \dotsc, 2n-2 \). This can be verified directly by using (3.3) and the Koszul formula (note that \( \frac{\partial}{\partial \theta_j} \in \mathcal{V} \) and \( \frac{\partial}{\partial \theta_j} \in \mathcal{H}, \ j = 1, \dotsc, 2n-2 \): \( 2 \langle \nabla_{\partial_j} Y, Z \rangle = \partial_j \langle Y, Z \rangle + \langle Y, \partial_j Z \rangle - \langle Z, \partial_j Y \rangle - \langle Y, [\partial_j, Z] \rangle - \langle Z, [Y, \partial_j] \rangle + \langle \partial_j, [Z, Y] \rangle \).
Now we recall some notions of curvatures. Given two $J$-invariant planes $\Pi_1$ and $\Pi_2$ in $T_pM$, we define the bisectional curvature to be

$$B(\Pi_1, \Pi_2) := \langle R(v_1, Jv_1)Jv_2, v_2 \rangle$$

where $v_i$ is a unit vector in $\Pi_i$. We also define the holomorphic sectional curvature for $v \neq 0$ to be

$$H(v) := \frac{\langle R(v, Jv)Jv, v \rangle}{|v|^2}.$$

The reason for the denominator $|v|^2$ is to give a consistent notion of $\ell$-holomorphic sectional curvature defined later and is immaterial, as we often only consider the case where $|v| = 1$. For a $J$-invariant two-plane $\Pi = \text{span}(v, Jv)$, we define $H(\Pi) = H(v)$ for $|v| = 1$. It is straightforward to check that $B(\Pi_1, \Pi_2)$ and $H(\Pi)$ are well-defined.

We say the bisectional curvature $M$ is bounded below by $k$ if for any $p \in M$ and any $u, v \in T_pM$,

$$\langle R(u, Ju)Jv, v \rangle \geq k \left( |u|^2 |v|^2 + \langle u, v \rangle^2 + \langle u, Jv \rangle^2 \right).$$

(3.5)

Note that our convention is one-half of that in [14] but consistent with [23]. E.g. the Fubini-Study metric $g_{FS} = \frac{dzd\bar{z}}{(1+|z|^2)^2} = \frac{1}{4}g_{S^2}$ of $\mathbb{C}P^1$ has constant holomorphic sectional curvature 2 (and constant sectional curvature 4).

We say the holomorphic sectional curvature of $M$ is bounded below by $k$ if

$$H(v) \geq 2k|v|^2$$

for any $p \in M$ and $v \in T_pM$. Note that this is weaker than (3.5). We also define the orthogonal Ricci curvature ([25]) to be

$$\text{Ric}^\perp(v, v) = \text{Ric}(v, v) - \frac{1}{|v|^2}R(v, Jv, Jv, v)$$

for $0 \neq v \in T_pM$.

As in the Riemannian case, for a function $k(t)$ we define

$$\widehat{\text{Ric}}_k(v) := \text{Ric}^\perp(v, v) - 2(n-1)kg(v, v) \quad \text{and} \quad \widehat{H}_k(v) := H(v) - 2kg(v, v).$$

For complex space forms $\mathbb{M}_k$, we have $\widehat{\text{Ric}}_k = 0$ and $\widehat{H}_k = 0$. In [23], among other things, Li and Wang proved volume comparison and Laplacian comparison theorems for Kähler manifolds under a lower bound of the bisectional curvature. The comparison spaces are the complex space forms. Ni and Zheng [25] improved their results by relaxing the condition to a lower bound of the orthogonal Ricci curvature and holomorphic sectional curvature. We will show a generalization of these results under an integral bound of a mixture of the orthogonal Ricci curvature and the holomorphic sectional curvature.
3.2. Comparison theorems for distance from a point. In the following, \( \Delta = \text{tr}(\nabla^2) \) refers to the Laplacian w.r.t. the Riemannian metric, which is \(-2\) times the value of \( \overline{\partial} \overline{\partial} \) on smooth functions.

**Theorem 6.** Let \((M, g)\) be a Kähler manifold. We have the following estimates.

1. Assume there is no cut point of \( p \) along \( \gamma_\theta \) on \([0, r]\). If \( s_{k_1}(r) > 0 \), then
   \[
   \Delta d(x) \leq (2n - 2) \frac{s'_{k_1}(r)}{s_{k_1}(r)} + \frac{s'_{k_2}(r)}{s_{k_2}(r)} - \phi(r, \theta) = \overline{\Delta} d(r) - \phi(r, \theta)
   \]
   where \( x = (r, \theta) \) in geodesic polar coordinates, \( \overline{\Delta} d(r) \) is the corresponding Laplacian of the distance function w.r.t. \( \overline{g} \), and
   \[
   \phi(r, \theta) = \int_0^r \left( \frac{s_{k_1}(t)^2}{s_{k_1}(r)^2} \overline{\text{Ric}}_{k_1}(\gamma_\theta'(t)) + \frac{s_{k_2}(t)^2}{s_{k_2}(r)^2} \overline{\text{H}}_{k_2}(\gamma_\theta'(t)) \right) dt.
   \]
2. Suppose \( r < \text{inj}(p) \) and \( s_{k_i} > 0 \) on \((0, r]\). Then
   \[
   |S_g(r, p)| \leq |S_{\overline{g}}(r)| \int_{S_{p, M}} \exp [-\psi(r, \theta)] d\theta
   \]
   where
   \[
   \psi(r, \theta) := \int_0^r \left( (ct_{k_1}(t) - ct_{k_1}(r)) \overline{\text{Ric}}_{k_1}(s_{k_1}(t)\gamma_\theta'(t)) + (ct_{k_2}(t) - ct_{k_2}(r)) \overline{\text{H}}_{k_2}(s_{k_2}(t)\gamma_\theta'(t)) \right) dt.
   \]
   We also have
   \[
   |B_g(r, p)| \leq \int_0^r w(\rho) F(\rho) d\rho,
   \]
   where \( w(\rho) = \int_{S_{p, M}} \exp [-\psi(\rho, \theta)] d\theta \) and \( F(r) = s_{k_1}(r)^{2n - 2}s_{k_2}(r) \). In particular, if \( \psi \geq 0 \), then \( |B_g(r, p)| \leq |B_{\overline{g}}(r)| \).
3. Suppose \( r < \text{inj}(p) \) and \( s > 0 \) on \((0, r]\), then
   \[
   \frac{d}{dr} \left( \frac{|S_g(r, p)|}{|S_{\overline{g}}(r)|} \right) \leq -\frac{1}{|S_{\overline{g}}(r)|} \int_{B_g(r, p)} \left( \frac{s_{k_1}(t)^2}{s_{k_1}(r)^2} \overline{\text{Ric}}_{k_1}(\partial_t) + \frac{s_{k_2}(t)^2}{s_{k_2}(r)^2} \overline{\text{H}}_{k_2}(\partial_t) \right) dV
   \]
   and
   \[
   \frac{d}{dr} \left( \frac{|B_g(r, p)|}{|B_{\overline{g}}(r)|} \right) \leq -\frac{F(r)}{|B_{\overline{g}}(r)|^2} \int_0^r \frac{|B_{\overline{g}}(u)|}{F(u)} \int_{B_g(u, p)} \left( \frac{s_{k_1}(u)^2}{s_{k_1}(u)^2} \overline{\text{Ric}}_{k_1}(\partial_t) + \frac{s_{k_2}(u)^2}{s_{k_2}(u)^2} \overline{\text{H}}_{k_2}(\partial_t) \right) dV du.
   \]
   on \((0, r]\).
4. In (2), (3), the equality holds if and only if \( B_g(r, p) \) is isometric to \( B_{\overline{g}}(r) \). The isometry is holomorphic if \( \overline{g} \) is Kähler.

**Proof.** Again let \( \gamma(t) \) be a unit speed geodesic. We can define a parallel orthonormal frame along \( \gamma(t) \) of the form \( \{F_1, \cdots, F_{2n}\} = \{e_1, Je_1, \cdots, e_{n-1}, Je_{n-1}, e_n, Je_n = \gamma'(t)\} \). As before, let \( Y_i(t), i = 1, \cdots, 2n - 1 \), be the Jacobi field with the endpoint values \( Y_i(0) = 0 \) and \( Y_i(r) = F_i(r) \).
Let \( X_i(r) = \frac{s_{k_1}(t)}{s_{k_1}(r)} F_i(t), \) \( i = 1, \ldots, 2n - 2 \) and \( X_{2n-1}(t) = \frac{s_{k_2}(t)}{s_{k_2}(r)} F_{2n-1}(t) \). Then \( X_i(0) = Y_i(0) \) and \( X_i(r) = Y_i(r) \). By (2.4) and (2.5),

\[
(\log F)'(r, \theta) \\
\leq \sum_{i=1}^{2n-1} \int_0^r (-X''_i, X_i) - \langle R(X_i, \gamma') \gamma', X_i \rangle \, dt + \sum_{i=1}^{2n-1} \langle X_i(r), X_i'(r) \rangle \\
= \sum_{i=1}^{2n-2} \int_0^r \frac{s_{k_1}(t)^2}{s_{k_1}(r)^2} (k_1 - \langle R(F_i, \gamma'(t)) \gamma'(t), F_i \rangle) \, dt + \int_0^r \frac{s_{k_2}(t)^2}{s_{k_2}(r)^2} (k_2 - \langle R(e_n, \gamma'(t)) \gamma'(t), e_n \rangle) \, dt \\
+ (2n - 2) \frac{s_{k_1}'(r)}{s_{k_1}(r)} + \frac{s_{k_2}'(r)}{s_{k_2}(r)} \\
= \sum_{i=1}^{n-1} \int_0^r \frac{s_{k_1}(t)^2}{s_{k_1}(r)^2} (2k_1 - \langle R(e_i, \gamma'(t)) \gamma'(t), e_i \rangle - \langle R(J e_i, \gamma'(t)) \gamma'(t), J e_i \rangle) \, dt \\
+ \int_0^r \frac{s_{k_2}(t)^2}{s_{k_2}(r)^2} (2k_2 - \langle R(e_n, J e_n) J e_n, e_n \rangle) \, dt + (2n - 2) \frac{s_{k_1}'(r)}{s_{k_1}(r)} + \frac{s_{k_2}'(r)}{s_{k_2}(r)} \\
= - \int_0^r \left( \frac{s_{k_1}(t)^2}{s_{k_1}(r)^2} \left( \text{Ric}^+ (\partial_t) - 2(n - 1)k_1 \right) + \frac{s_{k_2}(t)^2}{s_{k_2}(r)^2} \left( \text{H} (\partial_t) - k_2 \right) \right) \, dt \\
+ (2n - 2) \frac{s_{k_1}'(r)}{s_{k_1}(r)} + \frac{s_{k_2}'(r)}{s_{k_2}(r)}.
\]

Notice that the term at the last line is \( \Delta \overline{\mathcal{C}}(r) \). Except the equality case, the proof then proceeds as in Theorem 1 and Theorem 3.

For the equality case, let \( \iota : T_p M \to T_0 \overline{M} \) be a holomorphic isometry (i.e. \( \iota \circ J_M = J_{\overline{M}} \circ \iota \)). Let \( \phi = \exp_0 \circ \iota \circ \exp_p^{-1} : B_\theta(r, p) \to B_{\overline{\theta}}(r) \). To show that it is an isometry, by the Cartan-Ambrose-Hicks theorem ([18, Thm. 1.12.8]), it suffices to show that the map \( \iota \circ \phi \) defined by \( \overline{F}_\theta \circ \iota \circ (P_\theta)'^{-1} \) satisfies

\[
\iota \circ \phi \left( R_g(u, \gamma'(t)) \gamma'(t) \right) = R_{\overline{g}}(\iota_4(u), \overline{\gamma}'(t)) \overline{\gamma}'(t). \tag{3.6}
\]

Here \( P_\theta' \) is the parallel transport along \( \gamma = \gamma_\theta(t) \) from \( T_t M \) to \( T_{\gamma(t)} M \), \( \overline{\theta} = \iota(\theta) \) and \( \overline{F}_\theta \) is the corresponding parallel transport along the geodesic \( \overline{\gamma} = \overline{\gamma}_\theta \) with initial vector \( \overline{\theta} \) in \( \overline{M} \).

Suppose \( u = J \gamma'(t) \), then as \( s_{k_2}(t)u \) is a Jacobi field by index lemma, from the Jacobi field equation we have \( R_g(u, \gamma'(t)) \gamma'(t) = k_2 u \). Recall that \( J_M \) is parallel along \( \gamma \) by Lemma 4. So we see that L.H.S. of (3.6) is \( k_2 \iota_4(J_M \gamma'(t)) = k_2 \overline{F}_\theta \gamma'(t) = R_{\overline{g}}(J_{\overline{M}} \overline{\gamma}'(t), \overline{\gamma}'(t)) \overline{\gamma}'(t) \), which is the R.H.S. Similarly (3.6) holds for \( u = F_i \) for \( i = 1, \ldots, 2n - 2 \).

By [32, Lem. 2.5], this isometry is holomorphic if \( \overline{g} \) is Kähler. \( \square \)
The case where \( k_1 = k = \text{const} \) and \( k_2 = 4k \) generalizes [25, Cor. 1.3]. In this case, \( s_{k_1}(t) = s_k(t) \) and \( s_{k_2}(t) = \frac{1}{2}s_k(2t) \). Indeed, the proof of Theorem 6 also shows that the estimates for the orthogonal Laplacian and holomorphic Hessian in Theorem 1.1 (i) of [25] can be improved. Since we are mainly interested in ordinary Laplacian comparison, we omit the details here.

Clearly we have the Kähler analogue of Theorem 2 and its corollaries by the same argument. For simplicity which just states some particular results.

**Theorem 7.** Let \((M, g)\) be a Kähler manifold. Let \( r_0 \) be the smallest positive zero of \( s_{k_1} s_{k_2} \).

1. If

\[
\limsup_{r \to r_0} \left[ \int_0^r \left( \frac{s_{k_1}(t)^2}{s_{k_1}(r)^2} \widehat{\text{Ric}}_1(\gamma'_0(t)) + \frac{s_{k_2}(t)^2}{s_{k_2}(r)^2} \widehat{\text{H}}_{\ell} (\gamma'_0(t)) \right) dt - \left( \frac{2}{s_{k_1}(r)} + \frac{2}{s_{k_2}(r)} \right) \right] = \infty
\]

for any \( \theta \in S_{\rho} M \), then \( d_\rho \leq r_0 \) on \( M \), \( M \) is compact and \( \pi_1(M) \) is finite.

2. If \( \int_{B_\varphi(r, p)} \left[ \frac{s_{k_1}(t)^2}{s_{k_1}(r)^2} \widehat{\text{Ric}}_1(\partial_t) + \frac{s_{k_2}(t)^2}{s_{k_2}(r)^2} \widehat{\text{H}}_{\ell} (\partial_t) \right] dV \geq 0 \) for any \( r \in (0, r_0) \), then \( \frac{|B_\varphi(r, p)|}{|B_\varphi(r)|} \) is non-increasing on \((0, r_0]\). In particular, \(|M| \leq |B_\varphi(r_0)|\).

If \( |B_\varphi(r, p)| = |B_\varphi(r)| \) \((r \leq r_0 < \infty)\), then \( B_\varphi(r, p) \) is isometric to \( B_\varphi(r) \).

For example, if \( \limsup_{r \to r_0} - \int_0^r \frac{\sin(2t)^2}{\sin(2r)^2} \widehat{\text{Ric}}_1(\gamma'_0(t)) dt - 2 \cot(2r) \] = \( \infty \), then the diameter of \( M \) is bounded above by \( \frac{\pi}{2} \). If \( \int_0^r \frac{\sin^2(t) \widehat{\text{Ric}}_1(\gamma'_0(t)) dt \geq 0 \) and \( \int_0^r \frac{\sin^2(2t) \widehat{\text{H}}_{\ell} (\gamma'_0(t)) dt \geq 0 \) for any \( \theta \in S_{\rho} M \) and \( r \in (0, \frac{\pi}{2}) \), then \(|M| \leq |\mathbb{C}^n|\).

### 3.3 Comparison theorems for distance from a complex submanifold

As in the Riemannian case, to study the Laplacian of the distance function of a complex submanifold, it is natural to define the \( \ell \)-holomorphic sectional curvature as follows.

Let \( W \) be a subspace of \( T_x M \) which is \( J \)-invariant with \( \dim_{\mathbb{R}}(W) = 2\dim_{\mathbb{C}}(W) = 2\ell \), and \( v \in T_x M \). Inspired by [23, 25], we define the \( \ell \)-holomorphic sectional curvature to be

\[
\text{H}^\ell(W, v) = \sum_{j=1}^{\ell} (\langle R(e_j, v) e_j \rangle + \langle R(Je_j, v) e_j \rangle)
\]

where \( e_1, Je_1, \ldots, e_\ell, Je_\ell \) is an orthonormal basis of \( W \). It is easy to check that this is well-defined. Similar as before we define

\[
\widehat{\text{H}}^\ell_k(W, v) = \text{H}^\ell(W, v) - 2\ell kg(v, v).
\]

When \( \ell = n - 1 \) and \( W = (\text{span}(v, Jv))^1 \) this is \( \widehat{\text{Ric}}_1(v) \) and when \( \ell = 1 \) and \( W = \text{span}(v, Jv) \) this is \( \widehat{\text{H}}_1(v) \). We say \( \widehat{\text{H}}^\ell_k \geq 0 \) if \( \widehat{\text{H}}^\ell_k(W, v) \geq 0 \) for all such \( v, W \)

The following result can be compared to [32, Cor. 2.1]
Theorem 8. Suppose $\Sigma$ is a complex submanifold of a Kähler manifold $M$ with $\dim \mathbb{C}(\Sigma) = \ell$. Let $d_{\Sigma}$ be the distance from $\Sigma$, and $(r, \theta, z)$ be the Fermi coordinates of $x$. Assume $s_{k_1}, c_{k_1}$ are positive on $(0, r]$.

1. We have

$$\Delta d_{\Sigma}(x) \leq (\log F)'(r) - \psi(r, \theta, z)$$

where $F(r) = c_{k_1}(r)2\ell s_{k_1}(r)2n-2\ell-2s_{k_2}(r)$, 

$$\psi(r, \theta, z) = \int_0^r \left( \frac{c_{k_1}(t)}{c_{k_1}(r)} \frac{s_{k_1}(t)^2}{s_{k_1}(r)^2} \tilde{\mu}_{k_1}^F(\mathcal{P}_0\mathcal{S}(T_x\Sigma), \partial_t) + \frac{s_{k_1}(t)}{s_{k_2}(r)^2} \tilde{\mu}_{k_1}^{n-\ell-1}(N_t, \partial_t) \right) dt$$

and $N_t = \mathcal{P}_0(\mathcal{N}_x \cap (\text{span}(\theta, J\theta))^\perp)$.

2. The volume element $F(r, \theta, z)$ of $M$ satisfies

$$F(r, \theta, z) \leq \exp(-\phi(r, \theta, z))F(r)$$

where

$$\phi(r, \theta, z) = \int_0^r \left[ (tg_{k_1}(r) - tg_{k_1}(t))c_{k_1}(t)^2 \tilde{\mu}_{k_1}^F(\mathcal{P}_0\mathcal{S}(T_x\Sigma), \partial_t) + (ct_{k_1}(t) - ct_{k_1}(r))s_{k_1}(t)^2 \tilde{\mu}_{k_1}^{n-\ell-1}(N_t, \partial_t) \right] dt.$$ 

In particular, if $\tilde{\mu}_{k_1}^F$, $\tilde{\mu}_{k_1}^{n-\ell-1}$ and $\tilde{\mu}_{k_2}$ are non-negative, then $F(r, \theta, z) \leq F(r)$.

Proof. Since the proof is similar to Theorem 4, we only indicate the changes here.

1. Firstly, the parallel orthonormal frame along $\gamma_0$ is now $\{F_1, \ldots, F_2n\} = \{E_1, J\mathcal{E}_1, \ldots, E_n, J\mathcal{E}_n = \gamma_0\}$, with $F_1(0), \ldots, F_{2\ell}(0) \in T_x\Sigma$. Note that a complex submanifold is minimal (cf. [11, p. 171]), so we define $X_i(t) = \frac{c_{k_1}(t)}{c_{k_1}(r)}F_i(t), i = 1, \ldots, 2\ell$. Similar to (2.25) we have

$$\sum_{j=1}^{2\ell} \nabla^2 d_{\Sigma}(F_j(r), F_j(r)) \leq \sum_{j=1}^{2\ell} I_\Sigma(X_j, X_j) = -\int_0^r \frac{c_{k_1}(t)^2}{c_{k_1}(r)^2} \tilde{\mu}_{k_1}^F(\mathcal{P}_0\mathcal{S}(T_x\Sigma), \partial_t) dt + 2\ell \frac{c_{k_1}(r)}{c_{k_1}(r)}.$$

Similar to (2.27),

$$\sum_{j=2\ell+1}^{2n+2\ell} \nabla^2 d_{\Sigma}(F_j(r), F_j(r))$$

$$\leq -\int_0^r \left( \frac{s_{k_1}(t)^2}{s_{k_1}(r)^2} \tilde{\mu}_{k_1}^{n-\ell-1}(N_t, \partial_t) + \frac{s_{k_2}(t)^2}{s_{k_2}(r)^2} \tilde{\mu}_{k_2}(\partial_t) \right) dt + (2n - 2\ell - 2) \frac{s_{k_1}(r)}{s_{k_1}(r)} + \frac{s_{k_2}(r)}{s_{k_2}(r)}$$

where $N_t = \mathcal{P}_0(\mathcal{N}_x \cap (\text{span}(v, Jv))^\perp)$. Combining the two inequalities gives the result.
(2) Similar to Theorem 1 (3), we have
\[
\log F(r, \theta, z) \leq \log \mathcal{F}(r) - \int_0^r \left[ (tg_{k_1}(r) - tg_{k_1}(t))c_{k_1}(t)^2\mathcal{H}_{k_1}^\ell (P_0^\ell (T_z \Sigma), \partial_t) + (ct_{k_1}(t) - ct_{k_1}(r))s_{k_1}(t)^2\mathcal{H}_{k_1}^{n-\ell-1} (N_t, \partial_t) \\
+ (ct_{k_2}(t) - ct_{k_2}(r))s_{k_2}(t)^2\mathcal{H}_{k_2}^\ell (\partial_t) \right] dt.
\]

Here we have also used Lemma 1. This implies the result. □

We have the following analogue of Theorem 5.

**Theorem 9.** Suppose $\Sigma$ is a complex submanifold of a Kähler manifold $M$ with $\dim_{\mathbb{C}} \Sigma = \ell$. Assume $k_i \in \mathbb{R}$, $s_{k_i}, c_{k_i} > 0$ on $(0, r]$, $\mathcal{H}_{k_1}^j, \mathcal{H}_{k_2}^n - \ell - 1$ for $j = \ell, n - \ell - 1$ and $\mathcal{H}_{k_2}^\ell \geq 0$.

1. We have
\[
|S(r, \Sigma)| \leq |S^{2n-2\ell-1}|\mathcal{F}(r) \int_{S_{(N_t, \Sigma)}} \exp(-\phi)d\theta dS \leq |S^{2n-2\ell-1}|\mathcal{F}(r)|\Sigma|
\]
where $\mathcal{F}$ and $\phi \geq 0$ are given in Theorem 8.

2. We have
\[
\frac{d}{dr} \left( \frac{|S(r, \Sigma)|}{\mathcal{A}(r)} \right) \leq - \frac{1}{\mathcal{A}(r)} \int_{S(r, \Sigma)} \phi dS \leq 0,
\]
where $\mathcal{A}(r) = |\Sigma||S^{2n-2\ell-1}|\mathcal{F}(r)$ and $\phi(x) := \phi(r(x), v(x), z(x))$.
Let $V(r) = \int_0^r \mathcal{A}(u)du$, then we also have
\[
\frac{d}{dr} \left( \frac{|B(r, \Sigma)|}{V(r)} \right) \leq - \frac{\mathcal{A}(r)}{V(r)^2} \int_0^r \frac{V(u)}{\mathcal{A}(u)} \int_{S(u, \Sigma)} \phi dS du \leq 0.
\]

**Remark 5.** It would also be interesting to look for quantitative comparison results for other types of special manifolds, such as quaternionic Kähler manifolds. For simplicity we will not do it here. We notice that there are sharp comparison results for quaternionic Kähler manifolds assuming a scalar curvature lower bound, as studied by Kong, Li and Zhou [20] with methods similar to [23].

4. Gündther-type theorems

4.1. Riemannian case. We now give a generalization of Gündther’s theorem ([16, Thm. 3.17]), which gives a lower bound for the volume of the geodesic ball under a curvature upper bound. It is possible to work with a variable $k$ but for simplicity we assume $k$ is constant in this section.
Theorem 10. Let $g = dt^2 + \beta_{ij}(t, \theta) d\theta^i d\theta^j$ in geodesic polar coordinates. Let $x = (r, \theta)$ in geodesic polar coordinates centered at $p$. Assume $s_k > 0$ on $(0, r]$.

(1) We have
\[
\Delta d_p(x) \geq (n - 1) \frac{s_k'(r)}{s_k(r)} - \sum_{i,j=1}^{n-1} \beta^{ij}(r, \theta) \int_0^r \hat{R}_k(\partial_{\theta^i}, \partial_{\theta^j}, \partial_t, \partial_{\theta^k}) dt,
\]

where $(\beta^{ij}) = (\beta_{ij})^{-1}$.

(2) If $dV = F(r, \theta) dr d\theta$, then
\[
F(r, \theta) \geq \exp \left[ - \int_0^r \sum_{i,j=1}^{n-1} \beta^{ij}(\rho, \theta) \hat{R}_k(\partial_{\theta^i}, \partial_{\theta^j}, \partial_t, \partial_{\theta^k}) dt d\rho \right] \tilde{F}(r)
\]

where $\tilde{F}(r) = s_k(r)^{n-1}$.

Proof. We use the same notation as in the proof of Theorem 1. Let $\gamma$ be a geodesic segment of length $r$ on $\mathcal{M}_k$ and $\{E_i\}_{i=1}^n$ be the parallel translation of a positive orthonormal basis along $\gamma$ such that $E_n = \gamma'$. Suppose $Y_i(t) = \sum_{j=1}^{n-1} y^j_i(t) E_j(t)$. Then we define $\tilde{Y}_i(t) = \sum_{j=1}^{n-1} y^j_i(t) E_j(t)$ and $\tilde{X}_i(t) = \sum_{j=1}^{n-1} \frac{y^j_i(t)}{s_k(r)} \tilde{E}_i(t)$. Note that $\tilde{Y}_i(t) = \tilde{X}_i(t)$ when $t = 0$ and $r$. Denote the curvature of $\gamma$ by $\hat{R}$, we have
\[
I(Y_i, Y_i) = \int_0^r (\langle Y'_i, Y'_i \rangle - R(Y_i, \gamma', \gamma', Y_i)) dt = \int_0^r (\langle Y'_i, Y'_i \rangle - kB(Y_i, \gamma', \gamma', Y_i) - \hat{R}_k(Y_i, \gamma', \gamma', Y_i)) dt = \int_0^r (\langle \tilde{Y}'_i, \tilde{Y}'_i \rangle - (\hat{R}(\tilde{Y}_i, \gamma') \gamma', \tilde{Y}_i) - \hat{R}_k(Y_i, \gamma', \gamma', Y_i)) dt = I(\tilde{Y}_i, \tilde{Y}_i) - \int_0^r \hat{R}_k(Y_i, \gamma', \gamma', Y_i) dt.
\]
Notice that in geodesic polar coordinates, \( \frac{\partial}{\partial \theta} \) is a Jacobi field and
\[
\sum_{i,j=1}^{n-1} \beta^{ij}(r, \theta) \hat{R}_k \left( \frac{\partial}{\partial \theta} |_{(t,\theta)} , \gamma'(t), \gamma'(t), \frac{\partial}{\partial \theta} |_{(t,\theta)} \right) = \sum_{i=1}^{n-1} \hat{R}_k(Y_i(t), \gamma'(t), \gamma'(t), Y_i(t))
\]
is independent of the choice of spherical coordinates and \( Y_i \) (recall \( Y_i = Y_i^{r,\theta} \)). So using (4.1), index lemma and (2.4) applied to \( X_i \), we have
\[
\Delta d_p(x) = (\log F)'(r, \theta) = \sum_{i=1}^{n-1} I(Y_i, Y_i)
\]
\[
= \sum_{i=1}^{n-1} \left( I(\nabla_i, \nabla_i) - \int_0^r \hat{R}_k(Y_i, \gamma', \gamma', Y_i) dt \right)
\]
\[
\geq \sum_{i=1}^{n-1} \left( I(X_i, X_i) - \int_0^r \hat{R}_k(Y_i, \gamma', \gamma', Y_i) dt \right)
\]
\[
= (n-1) \frac{s'_k(r)}{s_k(r)} - \sum_{i,j=1}^{n-1} \beta^{ij}(r, \theta) \int_0^r \hat{R}_k(\partial_{\theta_i}, \partial_t, \partial_t, \partial_{\theta_j}) dt.
\]
(4.3)

The lower bound for \( F \) is straightforward.

\[
\Box
\]

Notice the similarity with (2.3) and that the quantity \( \sum_{i,j} \beta^{ij}(r, \theta) \hat{R}_k(\partial_{\theta_i}, \partial_t, \partial_t, \partial_{\theta_j}) \) is independent of the choice of spherical coordinates. Although a volume lower bound is not guaranteed by an upper bound of the Ricci curvature, this quantity plays a role similar to \( \hat{Ric}_k \) in this case. Furthermore, we can upper bound \( \sum_{i,j} \beta^{ij}(r, \theta) \hat{R}_k(\partial_{\theta_i}, \partial_t, \partial_t, \partial_{\theta_j}) \) if we further assume \( \ell \leq \text{Rm} \leq k \), by the classical Hessian comparison.

Similar to Theorem 3, we have the following result, which can be regarded as the relative version of the Günther’s inequality.

**Theorem 11.** Suppose \( r < \text{inj}(p) \) and \( s_k > 0 \) on \( (0, r] \), then
\[
\frac{d}{dr} \left( \frac{|S_g(r, p)|}{|S_g(r)|} \right) \geq -\frac{1}{|S_g(r)|} \int_{B_g(r, p)} \sum_{i,j=1}^{n-1} \beta^{ij}(r, \theta) \hat{R}_k(\partial_{\theta_i}, \partial_t, \partial_t, \partial_{\theta_j}) dV
\]
and
\[
\frac{d}{dr} \left( \frac{|B_{g}(r, p)|}{|B_{g}(r)|} \right) \geq -\frac{F(r)}{|B_{g}(r)|^2} \int_0^r \frac{|B_{g}(u)|}{F(u)} \int_{B_g(u, p)} \sum_{i,j=1}^{n-1} \beta^{ij}(u, \theta) \hat{R}_k(\partial_{\theta_i}, \partial_t, \partial_t, \partial_{\theta_j}) dV du.
\]
In both cases, the equality holds if and only if $B_g(r,p)\text{ is isometric to } B_{\tilde{p}}(r)$.

As an analogue of Proposition 4, we have

**Proposition 5.** If $\int B_g(\rho,p) \sum_{i,j=1}^{n-1} \beta_{ij}(\rho,\theta) \delta_k(\partial_{\theta^i},\partial_{\theta^j}) dV \leq 0$ for all $\rho \in [0,r]$, then

$$\frac{|B_g(t,p)|}{|\delta_g(t,p)|} \leq \frac{|B_R(t)|}{|\delta_R(t)|} \text{ for } t \in [0,r].$$

**Theorem 12.** Suppose $\Sigma$ is an $\ell$-dimensional submanifold of $(M,g)$. Let $d \Sigma : M \to \mathbb{R}$ be the distance from $\Sigma$, $g = dr^2 + \sum_{i,j=1}^{\ell} \beta_{ij}(r,\theta) dt^i dt^j + \sum_{i,j=1}^{\ell} \alpha_{ij}(r,\theta) dz^i dz^j$ in Fermi coordinates w.r.t. $\Sigma$. Let $x = (r, \theta, z)$ in Fermi coordinates. Assume $s_k > 0$ on $(0,r)$ and that the first zero of $t \mapsto c_k(t) + \lambda s_k(t)$ (if exists) appears no earlier than the cut distance in the direction $\theta$, where $\lambda = \min_{v \in S^\ell \Sigma} A_\theta(v,v)$.

1. We have

$$\Delta d \Sigma(x) \geq (\log |F|) \left( r, \theta, z \right) - \phi(r, \theta, z)$$

where $F(r, \theta, z) = s_k(r)^{n-\ell-1} \det \left[ c_k(r) Id + s_k(r) A_\theta \right]$, $(\alpha_{ij}) = (\alpha_{ij})^{-1} \text{ and }$

$$\phi(r, \theta, z) = \sum_{i,j=1}^{n-\ell-1} \beta_{ij}(r, \theta, z) \int_0^r \delta_k(\partial_{\theta^i},\partial_{\theta^j}) dt + \sum_{i,j=1}^{\ell} \alpha_{ij}(r, \theta, z) \int_0^r \delta_k(\partial_{z^i},\partial_{z^j}) dt.$$ 

Here we regard $A_\theta$ as a $(1,1)$-tensor.

2. If $dV = F(r, \theta, z) dr d\theta dz$, then $F(r, \theta, z) \geq \frac{\Delta d \Sigma(x)}{r} \geq \frac{\phi(r, \theta, z)}{r} \exp \left[ \int_0^r \phi(\rho, \theta, z) d\rho \right].$

3. Let $\bar{A}(r) = \int \int_S(N \Sigma) \bar{F}(r, \theta, z) d\theta dz \text{ and } \bar{V}(r) = \int_0^r \bar{A}(t) dt$. Suppose $\bar{F}(r, \theta, z) \geq 0 \text{ and } \phi(r, \theta, z) \leq 0 \text{ for all } r \in (0, r_0) \text{ and } (\theta, z) \in S(N \Sigma)$. Then on $(0, r_1]$, $|S_g(r, \Sigma) - \bar{A}(r)| \text{ is non-negative and non-decreasing, and } |B_g(r, \Sigma) - \bar{V}(r)| \text{ is non-negative, non-decreasing and convex.}$

Here $r_1 = \min \{ r_0, \text{ inj}(\Sigma) \}$.

**Proof.** We use the notation in the proof of Theorem 4. Furthermore assume $\{ E_i \}$ diagonalizes $A_\theta$ with eigenvalues $\{ \lambda_i \}$. Choose $\Sigma$ to be a (local) $\ell$-dimensional submanifold in $\mathbb{M}$ such that at $\bar{p} \in \Sigma$ and for $\bar{u} \in S(N \Sigma)$, the second fundamental form $\bar{A}_\bar{u}$ agrees with $A_\theta$. So there exists an orthonormal basis $\bar{E}_i$ of $T_{\bar{p}} \Sigma$ such that $\bar{A}_{\bar{u}}(\bar{E}_i, \bar{E}_j) = \lambda_i \delta_{ij}$. As before, parallel transport $\bar{E}_i$ along $\gamma = \gamma_{\bar{p}}$. For $Y_i(t) = \sum_{j=1}^{\ell} y^j_i(t) E_j(t)$, define $\bar{Y}_i(t) = \sum_{j=1}^{\ell} \gamma^j_i(t) \bar{E}_j(t)$ and $\bar{X}_i(t) = = \frac{c_k^j(t) + \lambda s_k(t)}{c_k(t) + \lambda s_k(t)} E_j(t)$. Note that $\bar{X}_i \text{ are adapted to } \Sigma \text{ along } \gamma$. So by index lemma and (4.1),

$$I_\Sigma(Y_i, Y_i) = \int_0^r \left( \langle Y_i', Y_i' \rangle - R(Y_i', \gamma', \gamma', Y_i') \right) dt + A_\theta(Y_i(0), Y_i(0))$$

$$= I_\Sigma(\bar{Y}_i, \bar{Y}_i) - \int_0^r \bar{R}_k(Y_i, \gamma', \gamma', Y_i') dt + A_\theta(Y_i(0), Y_i(0)) - \bar{A}(\bar{Y}_i(0), \bar{Y}_i(0))$$

$$\geq I_\Sigma(\bar{X}_i, \bar{X}_i) - \int_0^r \bar{R}_k(Y_i, \gamma', \gamma', Y_i') dt$$

$$= \frac{c_k^j(t)}{c_k(t) + \lambda s_k(t)} - \int_0^r \bar{R}_k(Y_i, \gamma', \gamma', Y_i') dt.$$
In Fermi coordinates, \( \frac{\partial}{\partial \tau_i} \) are Jacobi fields adapted to \( \Sigma \) ([16, Lem. 2.9]). So as in Theorem 10, we have
\[
\sum_{i=1}^{4} \mathcal{I}_{ij}(Y_i, Y_i) \geq \sum_{i=1}^{4} c_k(r) + \lambda_i s_k(r) \sum_{i,j=1}^{4} \alpha_{ij}(r, \theta, z) \int_0^r \tilde{R}_k(\partial_{z_i}, \partial_{\theta_i}, \partial_{\theta_i}, \partial_{z_i}) dt.
\]

Using (4.3) and (2.26), we can then proceed as in Theorem 4 to show that
\[
\Delta d_\Sigma(x) \geq (n - \ell - 1) \frac{s_k'(r)}{s_k(r)} + \sum_{i=1}^{\ell} c_k'(r) + \lambda_i s_k'(r) - \sum_{i,j=1}^{\ell} \frac{\beta_{ij}(r, \theta, z)}{n} \int_0^r \tilde{R}_k(\partial_{\theta_i}, \partial_{\theta_i}, \partial_{\theta_i}, \partial_{\theta_i}) dt - \sum_{i,j=1}^{\ell} \alpha_{ij}(r, \theta, z) \int_0^r \tilde{R}_k(\partial_{z_i}, \partial_{z_i}, \partial_{z_i}, \partial_{z_i}) dt.
\]

Note that \( \overline{F}(r, \theta, z) = s_k(r)^{n-\ell-1} \prod_{i=1}^{\ell} (c_k(r) + \lambda_i s_k(r)) = s_k(r)^{n-\ell-1} \det [c_k(r) \text{Id} + s_k(r)A_\theta] \), (1) and (2) follow. (3) is similar to Theorem 5 (3). \( \square \)

4.2. Kähler case. On a Kähler manifold \((M, g, J)\), define the 4-tensor
\[
C(X, Y, Z, W) = \frac{1}{2} \left[ \langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle + \langle X, JW \rangle \langle Y, JZ \rangle - \langle X, JZ \rangle \langle Y, JW \rangle + 2\langle X, JY \rangle \langle W, JZ \rangle \right].
\]

Let \( k \in \mathbb{R} \). Then \( kC \) is the Riemann curvature tensor of a complex space form with holomorphic sectional curvature \( k \) ([19, Prop. 7.2]). Also define on \( M \) the 4-tensor (which is different from \( \tilde{R}_k \))
\[
\tilde{R}_k(X, Y, Z, W) := \langle R(X, Y)Z, W \rangle - kC(X, Y, Z, W).
\]

**Theorem 13.** Let \((M, g)\) be a Kähler manifold and \( g = dt^2 + \beta_{ij}(t, \theta) dt^2 \) in geodesic polar coordinates centered at \( p \). Let \( x = (r, \theta) \) in geodesic polar coordinates. Assume there is no cut point of \( p \) along \( \gamma_0 \) on \([0, r]\) and \( s_{4k} > 0 \) on \([0, r]\).

1. We have
\[
\Delta d_p(x) \geq (2n - 2) \frac{s_k'(r)}{s_k(r)} + \frac{s_k'(r)}{s_{4k}(r)} - \sum_{i,j=1}^{2n-1} \beta_{ij}(r, \theta) \int_0^r \tilde{R}_k(\partial_{\theta_i}, \partial_{\theta_i}, \partial_{\theta_i}, \partial_{\theta_i}) dt.
\]

2. If \( dV = F(r, \theta) dr d\theta \), then
\[
F(r, \theta) \geq \exp \left[ - \int_0^r \int_0^\theta \sum_{i,j=1}^{2n-1} \beta_{ij}(\rho) \tilde{R}_k(\partial_{\theta_i}, \partial_{\theta_i}, \partial_{\theta_i}, \partial_{\theta_i}) dt d\rho \right] \overline{F}(r)
\]
where \( \overline{F}(r) = s_k(r)^{2n-2} s_{4k}(r) \).

**Proof.** We use the same notation as in the proof of Theorem 1. Let \( \{F_i\}_{i=1}^{2n} \) be defined as in the proof of Theorem 6. Let \( \overline{\gamma} \) be a geodesic segment of length \( r \) on \( \overline{M}_k \) and \( \{\overline{F}_i\}_{i=1}^{2n} \) be defined analogously on \( \overline{\gamma} \). Suppose \( Y_i(t) = \sum_{j=1}^{2n-1} y_j(t) F_j(t) \), \( i = 1, \cdots, 2n-1 \). Then we define \( \overline{Y}_i(t) = \sum_{j=1}^{2n-1} s_k(t) F_j(t) \) and \( \overline{X}_i(t) = \sum_{j=1}^{2n-1} s_k(t) F_j(t) \)
for $i = 1, \ldots, 2n - 2$ and $\overline{X}_{2n-1}(t) = \frac{s_{4k}(t)}{s_{4k}(r)} \overline{F}_{2n-1}(t)$. Note that $\overline{Y}_i(t) = \overline{X}_i(t)$ when $t = 0$ and $r$. As in (4.1), we have

$$I(Y_i, Y_i) = I(\overline{Y}_i, \overline{Y}_i) - \int_0^r \hat{R}_k(Y_i, \gamma', \gamma', Y_i) dt.$$ (4.4)

As in (4.2), $\sum_{i,j=1}^{2n-1} \beta_{ij}(r, \theta) \hat{R}_k \left( \frac{\partial}{\partial r}, \gamma'(t), \gamma'(t), \frac{\partial}{\partial r} \right) = \sum_{i=1}^{2n-1} \hat{R}_k (Y_i(t), \gamma'(t), \gamma'(t), Y_i(t)).$

So using (4.4), index lemma and (2.4), similar to Theorem 10, we have

$$\Delta d_p(x) = (\log F)'(r, \theta) \geq (2n - 2) \frac{s_k'(r)}{s_k(r)} + \frac{s_{4k}'(r)}{s_{4k}(r)} - \sum_{i,j=1}^{2n-1} \beta_{ij}(r, \theta) \int_0^r \hat{R}_k (\partial_{\theta_i}, \partial_t, \partial_t, \partial_{\theta_j}) dt.$$ (4.4)

The lower bound for $F$ is straightforward.

The Kähler analogue of Theorem 11 is the following

**Theorem 14.** With the same assumptions in Theorem 13,

$$\frac{d}{dr} \left( \frac{|S_r(r, p)|}{|\overline{S}_r(r)|} \right) \geq - \frac{1}{|S_r(r)|} \int_{B_{\rho}(r, p)} \beta_{ij}(r, \theta) \hat{R}_k (\partial_{\theta_j}, \partial_t, \partial_t, \partial_{\theta_j}) dV$$

and

$$\frac{d}{dr} \left( \frac{|B_{\rho}(r, p)|}{|\overline{B}_r(r)|} \right) \geq - \frac{\overline{F}(r)}{|\overline{B}_r(r)|} \int_0^r \frac{|B_{\rho}(u, p)|}{F(u)} \int_{B_{\rho}(u, p)} \beta_{ij}(u, \theta) \hat{R}_k (\partial_{\theta_j}, \partial_t, \partial_t, \partial_{\theta_j}) dV du.$$ (4.4)

In particular, if $\int_{B_{\rho}(r, p)} \beta_{ij}(\rho, \theta) \hat{R}_k (\partial_{\theta_j}, \partial_t, \partial_t, \partial_{\theta_j}) dV \leq 0$ for all $\rho \in (0, r)$,

then $\frac{|B_{\rho}(r, p)|}{|\overline{B}_r(r)|}$ is non-decreasing on $(0, r)$.

**Theorem 15.** Suppose $\Sigma$ is a complex submanifold of a Kähler manifold $M$ with $\dim_{\mathbb{C}}(\Sigma) = \ell$. Let $d_{\Sigma}$ be the distance from $\Sigma$, $g = dr^2 + \sum_{i,j=1}^{2\ell-1} \beta_{ij}(r, \theta, z) d\theta^i d\theta^j + \sum_{i,j=1}^{2\ell} \alpha_{ij}(r, \theta, z) dz^i dz^j$ in Fermi coordinates w.r.t. $\Sigma$. Let $x = (r, \theta, z)$ in Fermi coordinates. Assume $s_{4k} > 0$ on $(0, r]$ and that the first zero of $t \mapsto c_k(t) + \lambda s_k(t)$ (if exists) appears no earlier than the cut distance in the direction $\theta$, where $\lambda = \min_{v \in S_{\rho} \Sigma} A_{\theta}(v, v)$.

(1) We have

$$\Delta d_\Sigma(x) \geq (\log \overline{F})(r, \theta, z) - \phi(r, \theta, z)$$

where $\overline{F}(r, \theta, z) = s_k(r)^{2n-2\ell-2} s_{4k}(r) \det [c_k(r) \text{Id} + s_k(r) A_\theta], (\alpha_{ij}) = (\alpha_{ij})^{-1}$ and

$$\phi(r, \theta, z) = \sum_{i,j=1}^{2n-2\ell-1} \beta_{ij}(r, \theta, z) \int_0^r \hat{R}_k (\partial_{\theta_j}, \partial_t, \partial_t, \partial_{\theta_j}) dt - \sum_{i,j=1}^{2\ell} \alpha_{ij}(r, \theta, z) \int_0^r \hat{R}_k (\partial_{z_i}, \partial_{\theta_j}, \partial_{\theta_j}, \partial_{z_i}) dt.$$ (4.4)

Here we regard $A_\theta$ as a $(1,1)$-tensor.
(2) Let \( dV = F(r, \theta, z)drd\theta dz \), then we have \( F(r, \theta, z) \geq \varphi(r, \theta, z) \exp \left[ - \int_0^r \phi(\rho, \theta, z) d\rho \right] \).

Proof. Since the proof is similar to Theorem 12, we just outline it here. We use the notation in the proof of Theorem 8. Furthermore assume \( \{F_i\}_{i=1}^{2\ell} \) diagonalizes \( A_\theta \). Choose \( \Sigma \) to be a (local) \( \ell \)-dimensional complex submanifold in \( M_k \) such that at \( \varphi \in \Sigma \) and for \( \bar{\theta} \in S(N_{\varphi} \Sigma) \), the second fundamental form \( \bar{A}_{\varphi} \) agrees with \( A_\theta \). Let \( \{F_i\} \) be analogously defined along \( \varphi = \varphi_{\bar{\theta}} \).

For \( Y_i(t) = \sum_{j=1}^{2\ell} y_{ij}^i(t) F_j(t) \), define \( \overline{Y}_i(t) = \sum_{j=1}^{2\ell} y_{ij}^i(t) \overline{F}_j(t) \) and \( \overline{X}_i(t) = \frac{c_k(t)+\lambda_1 s_k(t)}{c_k(r)+\lambda_1 s_k(r)} \overline{F}_i(t) \). We can then proceed as in Theorem 12 to obtain the result.

\( \square \)

5. Some applications

Theorem 1, Theorem 3, or Proposition 2 can be used to provide weaker assumptions to many classical theorems, and at the same time give better estimates (if desired). For example, if only an integral version of the Laplacian comparison theorem for a radial function is used to prove a certain result, Proposition 2 can often be a substitute to provide weaker weaker assumption than a pointwise Ricci curvature lower bound or even an integral bound along all geodesics emanating from a point. We illustrate some of the possibilities here.

The following result characterizes the equality case in Theorem 2 and generalizes Cheng’s maximal diameter theorem.

**Theorem 16.** Let \( M \) be a complete Riemannian manifold. Assume

1. \( s_k(t) = s_k(r_0 - t) \) where \( r_0 \) is the first positive zero of \( s_k \).
2. There exists \( p_1, p_2 \in M \) such that \( d(p_1, p_2) = r_0 \).
3. For any \( \theta \in S, M \) and any \( r \in (0, r_0) \), we have\( \int_0^r \operatorname{Ric}_k (s_k(t)\gamma_{\theta}(t)) dt \geq 0. \)

Then \( M \) is isometric to \( (M = [0, r_0] \times S^{n-1}, \varphi = dr^2 + s_k(t)^2 g_{S^{n-1}}) \).

Proof. By Theorem 2 and Theorem 3, for any \( r < r_0 \), we have \( \frac{|B_g(r, p_1)|}{|B_g(\varphi)|} \geq \frac{|B_g(r_0, p_2)|}{|B_g(\varphi)|} = \frac{|M|}{|M|} \). So \( |B_g(r, p_1)| \geq \frac{|M|}{|M|} |B_g(\varphi(r))| \) and by the same reason, \( |B_g(r_0-r, p_2)| \geq \frac{|M|}{|M|} |B_g(\varphi(r_0-r))| \). Adding them gives

\[
|B_g(r, p_1)| + |B_g(r_0-r, p_2)| \geq \frac{|M|}{|M|} (|B_g(\varphi(r))| + |B_g(\varphi(r_0-r))|) = |M|.
\]

But it is easy to see that \( B_g(r, p_1) \) and \( B_g(r_0-r, p_2) \) must be disjoint, and so the inequalities above are all equalities. In particular,

\[
\frac{|B_g(r, p_1)|}{|B_g(\varphi(r))|} = \frac{|M|}{|M|}.
\]
is constant for all $0 < r \leq r_0$. From this it follows from the proof of Theorem 3 that $M = \overline{B_{r_0}(p_1)}$ is isometric to the closed ball $\overline{B_2(r_0)}$ with metric $dr^2 + s_k(r)^2 g_{S^{n-1}}$, which is $\overline{M}$.

The estimate (2.3) can be used to weaken the assumptions in Cheng’s eigenvalue comparison theorem ([6, Theorem 1.1]). For constant $k$, we denote the geodesic ball of radius $r$ in the simply connected space form $\overline{M}_k$ of curvature $k$ by $\overline{B_2(r)}$.

**Theorem 17.** Let $k$ be constant. Suppose $M$ is a Riemannian manifold and $p \in M$ such that $\int_{B_{g}(r,p)} \overline{\text{Ric}}(s_k(t)) \, dV \geq 0$ for all $0 \leq \rho \leq r < \text{diam}(M_k)$. Then $\lambda_1(\overline{B_2(r,p)}) \leq \lambda_1(\overline{B_2(r)})$, where $\lambda_1$ is the first eigenvalue with respect to the Dirichlet boundary condition. The equality holds if and only if $\overline{B_2(r,p)}$ is isometric to $\overline{B_2(r)}$.

**Proof.** Let $\lambda = \lambda_1(\overline{B_2(r)})$ and $\phi > 0$ be the first eigenfunction on $\overline{B_2(r)}$, which is radial (cf. [6] p. 290) and $\frac{d}{dr} \phi < 0$ on $(0, r)$ ([7, Lemma 3.7]). Consider $\phi \circ d_p : \overline{B_2(r,p)} \to \mathbb{R}$ as a test function, simply written as $\phi$. Then Proposition 2 gives

$$\int_{B_{g}(r,p)} |\nabla \phi|^2 \leq \lambda \int_{B_{g}(r,p)} \phi^2.$$ 

By the minimization property of the first eigenvalue, the result follows. The equality case is the same as [6, Theorem 1.1].

Since [6, Theorem 1.1] is used to prove [6, Theorem 2.1], by analyzing the proof of [6, Theorem 2.1] and using Theorem 17 we also have

**Theorem 18.** Let $k$ be constant. Suppose $M$ is an $n$-dimensional compact Riemannian manifold with $\text{diam}(M) = d_M$. Suppose $\frac{dM}{2} < \text{diam}(\overline{M}_k)$ for some $k$ and for all $p \in M$ and $t \in [0, \frac{1}{2} d_M]$, we have $\int_{B_{g}(t,p)} \overline{\text{Ric}}(s_k(t) \partial_t) \, dV \geq 0$. Then $\mu_i(M) \leq \lambda_1 \left( \overline{B_2 \left( \frac{dM}{2} \right)} \right)$, where $\mu_i(M)$ is the $i$-th eigenvalue of $M$.

Now instead we use $\overline{M}_k$ to denote the complex space form and $\overline{B_2(r)}$ denotes its geodesic ball. The Kähler version of Theorem 17 is the following result.

**Theorem 19.** Let $k$ be constant. Suppose $M$ is a Kähler manifold and $p \in M$ such that $\int_{B_{g}(t,p)} \overline{\text{Ric}}_k(s_k(t) \partial_t) \, dV \geq 0$ and $\int_{B_{g}(t,p)} \overline{\text{H}}_{2k}(s_k(2 \rho) \partial_p) \, dV \geq 0$ for all $0 \leq \rho \leq r < \text{diam}(\overline{M}_k)$. Then $\lambda_1(\overline{B_2(r,p)}) \leq \lambda_1(\overline{B_2(r)})$, where $\lambda_1$ is the first eigenvalue with respect to the Dirichlet boundary condition. The equality holds if and only if $\overline{B_2(r,p)}$ is isometric to $\overline{B_2(r)}$.

The proof uses the following
Proposition 6. Let $\phi, \psi$ be defined as in Proposition 2. Let $(M, g)$ be a Kähler manifold. Suppose
\[
\frac{1}{sk_1(t)^2} \int_{B_g(t,p)} \hat{\text{Ric}}_{k_1}(sk_1(\rho)\partial_\rho)dV + \frac{1}{sk_2(t)^2} \int_{B_g(t,p)} \hat{H}_{k_2}(sk_2(\rho)\partial_\rho)dV \geq 0
\]
for all $0 \leq t \leq r$. Then
\[
\int_{B_g(r,p)} \langle \nabla (\psi \circ d_\rho), \nabla (\phi \circ d_\rho) \rangle \leq -\int_{B_g(r,p)} (\psi \circ d_\rho) \cdot (\nabla \phi \circ d_\rho)
\]
where $\nabla \phi(r) := \phi'' + \frac{F(r)}{F(r)}$ is the Laplacian of $\phi$ with respect to the metric $\bar{g}$ defined in (3.3) and $F(t) = sk_1(t)^{2n-2}sk_2(t)$.

Proof. The proof is the same as Proposition 2 except we replace the last line of (2.9) by
\[
-\int_{0}^{r} \phi(t)|\phi'(t)| \left( \frac{1}{sk_1(t)^2} \int_{B_g(t,p)} \hat{\text{Ric}}_{k_1}(sk_1(\rho)\partial_\rho)dV + \frac{1}{sk_2(t)^2} \int_{B_g(t,p)} \hat{H}_{k_2}(sk_2(\rho)\partial_\rho)dV \right) dt \leq 0,
\]
which follows from Theorem 6.

\[\Box\]

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