Stationary solutions to cubic nonlinear Schrödinger equations with quasi-periodic boundary conditions

Andrea Sacchetti

Department of Physics, Informatics and Mathematics, University of Modena and Reggio Emilia, Modena, Italy

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Abstract
In this paper we give the quantization rules to determine the normalized stationary solutions to the cubic nonlinear Schrödinger equation with quasi-periodic conditions on a given interval. Similarly to what happen in the Floquet’s theory for linear periodic operators, also in this case some kind of band functions there exist.

Keywords: nonlinear Schroedinger equations, quasiperiodic boundary conditions, Jacobian elliptic functions

(Some figures may appear in colour only in the online journal)

1. Introduction
Nonlinear one-dimensional Schrödinger equations with cubic nonlinearity (hereafter NLS)
\[
\frac{i \hbar}{\partial t} \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \alpha |\psi|^2 \psi
\]
(1)
on a finite interval $I = [0, a]$, for some $a > 0$ fixed, may be of physical interest in the study of Bose–Einstein condensates trapped in a circular wave-guide (see, e.g., [5, 16, 21], see also [12] for a review). Wide attentions to the NLS (1) on a finite interval have been given from a mathematical point of view, with particular emphasis to the analysis of the existence and stability of standing waves solutions of the form $\psi(x, t) = e^{-i\omega t/\hbar} \phi(x)$ under different boundary conditions [2, 13, 14, 20, 28, 29]. In fact, the function $\phi(x)$ is a normalized solution to the cubic time independent NLS (hereafter $'$ denotes the derivative $\frac{\partial}{\partial x}$ and, for sake of simplicity, we fix the units such that $\hbar = 1$ and $2m = 1$):
\[
-\phi'' + \alpha |\phi|^2 \phi = \mu \phi, \|\phi\|_{L^2(I, dx)} = 1,
\]
(2)

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and the boundary conditions considered in the above mentioned papers are the following ones: periodic boundary conditions (i.e. \( \phi(0) = \phi(a) \) and \( \phi'(0) = \phi'(a) \)), Dirichlet boundary conditions (i.e. \( \phi(0) = \phi(a) = 0 \)), Neumann boundary conditions (i.e. \( \phi'(0) = \phi'(a) = 0 \)) and \( \sigma \)-walls boundary conditions (i.e. \( \phi'(0) = \sigma \phi(0) \) and \( \phi'(a) = -\sigma \phi(a) \)) where the walls are repulsive if \( \sigma > 0 \) and attractive when \( \sigma < 0 \).

In a couple of seminal papers Carr et al [7, 8] studied the stationary solutions to (2) on a torus, that is with periodic boundary conditions, both in the case of focusing and defocusing cubic nonlinearities, and a key ingredient in their analysis was the use of the fundamental solution to a cubic NLS expressed through Jacobian elliptic functions (see also [27, 30]). More recently, stability analysis of the periodic and anti-periodic stationary solutions has been considered for the cubic focusing NLS [9, 11] and for the cubic defocusing NLS [4, 15]. We should also recall the papers devoted to the analysis of stationary solutions to NLS on a ring with defects [22, 26] or on a rotating ring [17].

This paper is addressed to the study of the normalized stationary solutions \( \phi(x) \) to equation (2) with \textit{quasi-periodic boundary conditions} on the interval \( I = [0, a] \)

\[
\begin{align*}
\phi(a) &= e^{i\mu a} \phi(0) \\
\phi'(a) &= e^{i\mu a} \phi'(0) \\
\end{align*}
\]  

(3)

In the following, for argument’s sake, we choose \( a = 1 \).

Similarly to what happens in the Floquet’s theory for linear periodic operators [19], even in this case we expect that it is possible to obtain an implicit relationship between the ‘energy’ \( \mu \) associated to the stationary solution and the quasimomentum variable \( k \) that characterizes the quasi-periodic boundary conditions. Eventually, some analogies between the NLS equation (2) with quasi-periodic boundary conditions (3) and the Floquet’s theory occur; for instance, in additions to plane wave solutions associated to the ‘energy’ \( \mu = k^2 + \alpha \), other quasi-periodic solutions there exist for some values of the energy \( \mu \in [\mu^m, \mu^M] \) and of the quasimomentum \( k \in [k^m, k^M] \). The intervals \([\mu^m, \mu^M] \) and \([k^m, k^M] \) will depend on \( \alpha \) and their lengths is not zero when \( \alpha \neq 0 \). Finally, we give the algorithm for the computation of \( k = \alpha(k) \), when the ‘energy’ \( \mu \) belongs to the ‘energy band’ \([\mu^m, \mu^M] \), and the numerical inversion of such a relation gives the ‘dispersion relation’ \( \mu = \mu(k) \). The names ‘energy band’, ‘quasimomentum’, ‘dispersion relation’, etc, are adopted by the Floquet’s theory. We must recall that in the Floquet’s theory the dispersion relation is a single multisection function that has no singularities other than square root branch points \( k_j \) and \( \tilde{k}_j \), where \( \Re k_j = \pm \pi, j = \pm 1, \pm 2, \ldots \) and where \( \Im k_{j|0} > 0 \) if the \( |j| \)-th gap is open. When the \( |j| \)-th gap is closed then the branch points \( k_{\pm j} \) and \( \tilde{k}_{\pm j} \) disappear. The restriction of the dispersion relation to \( \mathbb{R} \cap \bar{C} \), where \( \bar{C} \) is the complex plane with cuts \( [k_j, \tilde{k}_j] \), gives the energy bands [19]. Hence, in order to study the whole dispersion relation \( \mu(k) \) we consider in (3) \( k \in \mathbb{R} \) instead of the ‘Brillouin zone’ \( k \in [-\frac{\pi}{a}, +\frac{\pi}{a}] \). From the dispersion relation \( \mu(k) \) defined on a given real interval \( I \) then the periodical reproduction of its restriction to the interval \( I \cap [-\frac{\pi}{a}, +\frac{\pi}{a}] \) gives a periodic band function on the Brillouin zone, the periodical reproduction of the restriction of the dispersion relation \( \mu(k) \) to the interval \( I \cap \left( [2\pi, 3\pi] \cup [-3\pi, -2\pi] \right) \) gives a second periodic band function on the Brillouin zone, and so on.

The paper is organized as follows. In section 2 we collect some preliminary remarks. In section 3 we give the expression of the general solution to (2) with quasi-periodic boundary conditions (3) and we compute the energy band \([\mu^m, \mu^M] \) and the associated interval \([k^m, k^M] \) of values for the quasimomentum \( k \) to whom a normalized solution to (2) with boundary conditions (3) there exists. In particular, we also see that when the energy takes a value \( \mu^m \) or \( \mu^M \) at the edge of the energy band then we recover well known solutions.
2. Preliminary remarks

**Remark 1.** If $\phi(x)$ is a solution to (2) and (3) associated to an energy $\mu$ and to a quasimomentum $k$ then the complex conjugate $\overline{\phi(x)}$ is still a solution to (2) and (3) associated to the same energy $\mu$ and to the opposite quasimomentum $-k$. Therefore, we may restrict our attention to the case $k > 0$.

**Remark 2.** We recall that if $\phi \in H^2(I)$ is a solution to the differential equation (2) when $I = \mathbb{R}$ then (see lemma 3.7 [25]) $\phi$ is, up to a phase factor, a real-valued solution. We must remark that this regularity result does not hold true when $I = [0, 1]$ is a finite interval and thus we actually may have complex-valued solutions to equation (2) with quasi-periodic boundary conditions (3).

**Remark 3.** Equation (2), with quasi-periodic boundary conditions (3), always admits plane wave solutions of the form $\phi(x) = e^{\pm i \sqrt{\mu - \alpha} x}$ where $\mu = k^2 + \alpha$.

**Remark 4.** When one looks for real valued solutions then equation (2) takes the form

$$-\phi'' + \alpha \phi^3 = \mu \phi,$$

and it has a periodic solution (see ch 7, section 10 [10], see also [1])

$$\phi(x) = \frac{1}{\sqrt{\alpha}} \sqrt{\frac{2\mu}{1 + t^2}} \text{sn} \left( \left( x - x_0 \right) \sqrt{\frac{\mu}{1 + t^2}} ; t \right),$$

(4)

where $\text{sn}(x; t)$ is an Jacobian elliptic function with parameters $x_0 \in \mathbb{R}$ and $t \in [0, 1)$ and real period $4K(t)$, where $K(t)$ is the complete first elliptic integral. Making use of some formulas for $\text{sn}(x; t)$ one can give other forms to the general solution; e.g., instead of (4) the general solution may be written as

$$\phi(x) = \sqrt{-\frac{2\mu t^2}{\alpha}} \text{cn} \left( \left( x - x_0 \right) \sqrt{\frac{\mu}{1 - 2t^2}} ; t \right),$$

(5)

or

$$\phi(x) = \sqrt{\frac{2\mu}{\alpha(2 - t)}} \text{dn} \left( \left( x - x_0 \right) \sqrt{\frac{\mu}{t - 2}} ; t \right),$$

(6)

where $\text{cn}(x; t)$ and $\text{dn}(x; t)$ are Jacobian elliptic functions. Expressions (4)–(6) are equivalent; however, it should be noted that expression (4) is particularly useful when dealing with the defocusing case, while expressions (5) and (6) are useful in focusing case.

**Remark 5.** In the case of periodic boundary conditions then the solution has the form (4) where $\mu$ is given by

$$\mu = 16(n + 1)^2 K^2(t)(1 + t^2), \quad n = 0, 1, 2, \ldots$$

On the other hand, in the case of antiperiodic boundary conditions, that is when $k = (2n + 1)\pi$, then the solution is still given by (4) where

$$\mu = 4(2n + 1)^2 K^2(t)(1 + t^2).$$

In both cases the value of the parameter $t$ must be such that the normalization condition holds true.
**Remark 6.** If $\phi(x)$ is a solution to equations (2) and (3) then $\phi_{x_0}(x) = \phi(x - x_0)$ is a solution to equations (2) and (3), too. Indeed, $u(x) = e^{-ikx} \phi(x)$ is a periodic function with period 1 and then
\[ e^{-ikx} \phi_{x_0}(x) = e^{-ikx_0} e^{-ik(x-x_0)} \phi(x-x_0) = e^{-ikx_0} u(x-x_0) \]
is a periodic function with period 1, too.

### 3. Solution to (2) with boundary conditions (3)

Usually the focusing case and the defocusing case are separately treated because the solutions, especially from the point of view of the stability property or of the global existence of the solutions, have substantially different features. However, since in this paper we do not discuss the stability of the solutions, this distinction is not essential at the moment and it will be postponed.

#### 3.1. Preliminaries

Following the approach proposed by [7, 8] we consider the Madelung transform $\phi(x) = \rho(x)e^{i\theta(x)}$, then equation (2) takes the form
\[
\begin{aligned}
-\left(\rho'' - \rho \theta'^2\right) + \alpha \rho^3 &= \mu \rho \\
2\rho' \theta'' + \rho \theta''' &= 0
\end{aligned}
\]  

(7)

The second equation implies that $\rho \theta'' = C_1$, where $C_1 = \rho_0^2 \theta_0'$ is a constant of integration.

Hereafter we assume, for argument’s sake, that
\[ \theta_0 = \theta(0) = 0. \]

**Remark 7.** If $C_1 = 0$ then $\theta(t) \equiv \theta_0$ and $\phi$ locally is, up to a phase factor, a real valued solution already discussed in remark 4; indeed, equation (7) reduces to $-\rho'' + \alpha \rho^3 = \mu \rho$ and thus, if the solution has a snoidal (4) or cnoidal (5) form, then $\theta_0 = 0$ when it takes positive values and $\theta_0 = \pi$ when it takes negative values.

Hereafter, we will consider the case $C_1 \neq 0$. In such a case $\rho(x)$ never takes zero values and
\[ \theta(x) = C_1 \int_0^x \frac{1}{\rho^2(u)} du, \quad C_1 \neq 0. \]

(8)

Because of the quasi periodic boundary conditions (3) it follows that $\rho(x)$ is a non negative solution to
\[ -\left(\rho'' - \frac{C_1^2}{\rho^3}\right) + \alpha \rho^3 = \mu \rho \]

with periodic boundary conditions
\[ \rho(1) = \rho(0) \quad \text{and} \quad \rho'(1) = \rho'(0). \]

(10)

The two parameters $\mu$ and $C_1$ must satisfy to the condition $\theta(1) = k$, i.e.
\[ C_1 \int_0^1 \frac{1}{\rho^2(x; C_1, \mu)} dx = k, \]

(11)
because of (3), and to the normalization condition, that is
\[ \int_0^1 \rho^2(x; C_1, \mu) \, dx = 1. \]  
(12)

Remark 8. If one looks for constant solutions to (9) then \( \rho \equiv 1 \), because of the normalization condition (12), \( C_1 = k \), because of (11), \( \theta(x) = kx \) and equation (9) reduces to
\[ \frac{C_1^2}{\rho^3} + \alpha \rho^3 = \mu \rho. \]  
(13)
Therefore, since \( \rho \equiv 1 \), it follows that \( \mu = k^2 + \alpha \) and \( \phi(x) = e^{ikx} \) is the plane wave solution already discussed in remark 3.

In order to look for non constant solutions we remark that equation (9) can be solved by means of a simple squaring; indeed, if \( \rho \) is not a constant function (we have already discussed this case in remark 8) then (9) reduces to
\[ -\frac{1}{2} \rho'^2 - \frac{1}{2} \frac{C_1^2}{\rho^2} + \frac{1}{4} \alpha \rho^4 - \frac{1}{2} \mu \rho^2 = C_2 \]  
(14)
where \( C_2 \) is a constant of integration. If we set \( z = \rho^2 \) then \( z(x; C_1, C_2, \mu) \) is a non negative solution to the equation
\[ z'^2 = f(z) \quad \text{where} \quad f(z) = b(z^3 + cz^2 + dz + e) \]  
(15)
with periodic boundary conditions \( z(0) = z(1) \), where
\[ b = 2\alpha, \quad c = -4\mu, \quad d = -8C_2 \quad \text{and} \quad e = -4C_1^2. \]

It is well known that the general solution to (15) has the form [10]
\[ z(x) = A \, \text{sn}^2(qx + x_0; t) + B \quad \text{with period} \quad T_\ell = \frac{2K(t)}{q}, \quad \ell = 1, 2, \ldots, \]  
(16)
for some parameters \( A, B, q, x_0 \) given by (17), (19)–(21), \( t \) satisfies to the condition (18), and under the constraints
\[ C_1^2 > 0, \quad B > 0 \quad \text{and} \quad A > -B, \]
because we assumed that \( C_1 \neq 0 \) and that \( z(x) \) never takes zero values.
For argument’s sake we can always assume that (see remark 6)
\[ x_0 = 0 \]  
(17)
by means of a translation argument \( x \to x - x_0/q \).

Remark 9. In the following we restrict our attention to the case of \( \ell = 1 \) in (16) and we denote by
\[ T := T_1 = \frac{2K(t)}{q} \]
the corresponding period of the solution \( z(x) \). In such a way we will obtain the first dispersion function \( \mu_1 = \mu_1(k) \). For different values of \( \ell = 2, 3, \ldots \) we will have that the other dispersion functions \( \mu_\ell(k) \) simply come from the first one since we can reduce ourselves to the
case $\ell = 1$ by making use of a simple scaling transformation applied to (2). If $\ell = 1$ then the density $\rho(x)$ has just one valley in the interval [0, 1] as one can see in figures 4–6; in general, for any $\ell = 1, 2, \ldots$ the density will have $\ell$ valleys.

Recalling (10), that is the period $T$ of the solution $\rho(x)$ must be equal to 1, the normalization condition

$$1 = \int_0^1 z(x) \, dx = \int_0^1 \left[ A \, \text{sn}^2(qx; t) + B \right] \, dx = AF_1(t) + B,$$

where we set

$$F_1(t) := \int_0^1 \text{sn}^2(qx; t) \, dx,$$

and by substituting (16) in (15) and equating the coefficients of the same power of the function $\text{sn}^2(qx; t)$, it follows that

$$q = 2K(t)$$

$$A = \frac{1}{\alpha} 2q^2 t^2 = \frac{8}{\alpha} K^2(t) t^2$$

$$B = 1 - \frac{8K^2(t)t^2 F_1(t)}{\alpha}$$

$$\mu = G(t) + \frac{3}{2} \alpha$$

$$C_1^2 = \frac{B}{4}(A + B)(2\alpha B + 4q^2)$$

where we set

$$G(t) := 4K^2(t) \left[ \left( 1 + t^2 \right) - 3t^2 F_1(t) \right].$$

Finally, the constant of integration $C_2$ is given by

$$C_2 = -\frac{1}{2} \alpha AB - Bq^2 - \frac{3}{4} \alpha^2 B^2 - \frac{1}{2} \alpha q^2.$$

**Remark 10.** We may remark that $G(t)$ is such that

$$\lim_{t \to 0} G(t) = 4K^2(0) = \pi^2$$

and furthermore, by means of a numerical calculation, it turns out that $G(t)$ is a monotone decreasing function. In fact, one can prove that $G(t)$ is a monotone decreasing function by mimicking some ideas from [18] (see also [23]). First of all we recall that (see formula 310.02 [6])

$$F_1(t) = 2 \int_0^{1/2} \text{sn}^2(qx; t) \, dx = \frac{q - 2E \left( \text{sn}(q/2; t); t \right)}{qt^2} = \frac{q - 2E(1; t)}{qt^2},$$

since $\text{sn}(q/2; t) = 1$ when $q = 2K(t)$, where $E(\varphi; t)$ is the incomplete elliptic integral of second kind with argument $\varphi$ and parameter $t$. Let $E(t) := E(1; t)$, then

$$G(t) = 4K(t) \left[ -2K(t) + \frac{t^2}{4} K(t) + 3E(t) \right]$$
from which (25) immediately follows. Then, recalling that
\[
\frac{dE(t)}{dt} = \frac{E(t) - K(t)}{t} \quad \text{and} \quad \frac{dK(t)}{dt} = \frac{E(t)}{t(1-t^2)} - \frac{K(t)}{t}
\]
a straightforward calculation gives that for any \( t \in (0, 1) \)
\[
\frac{dG}{dt} = \frac{4}{t(1-t^2)} \left[ 3E^2(t) - 2K(t)E(t) - 2(1-t^2)E(t)K(t) + K^2(t)(1-t^2) \right]
\]
and
\[
\frac{dG_1}{dt} = t \left[ K(t) - \frac{E(t)}{1-t^2} \right] < tE(t) \left[ \frac{1}{\sqrt{1-t^2}} - \frac{1}{1-t} \right] < 0
\]
for any \( t \in (0, 1) \). Thus \( G_1(t) < 0 \) and then \( \frac{dG_1}{dt} < 0 \) for any \( t \in (0, 1) \).

3.2. General solution

From (22) we obtain the equation \( \mu = \mu(t) \), because of remark 10 we may invert such an equation obtaining \( t = t(\mu) \in [0, 1) \) and finally \( k = k(\mu) \); the inversion of such a latter relation will give the first (because we chose \( \ell = 1 \), see remark 9) dispersion function \( \mu(k) \).

We collect all these results in the following statement.

**Theorem 1.** Let \( \mu \in \mathbb{R} \) and \( \alpha \in \mathbb{R} \), \( \alpha \neq 0 \), be fixed. Let \( q, A, B \) and \( C_1 \) given by (19)–(21) and (23). If \( t \in [0, 1) \) satisfies to the following constraints

\[
\begin{aligned}
\mu &= G(t) + \frac{3}{2} \alpha \\
B(t) &> 0 \\
A(t) &> -B(t) \\
C_1(t) &> 0
\end{aligned}
\]

(27)

then equation (2) with quasi-periodic boundary conditions (3) has a solution of the form
\[ \phi(x) = \rho(x)e^{i\theta(x)} \]
where \( \rho(x) \) is a positive function given by
\[ \rho(x) = \sqrt{A} \text{sn}^2(qs; t) + B \]
and where
\[ \theta(x) = C_1 \int_0^x \frac{du}{\rho^2(u)} \]

Theorem 1 provides some restrictions to the values allowed by energy \( \mu \).

**Remark 11.** Recently, focusing cubic NLS has been studied [9, 11] and in such a case the fundamental solution is expressed through Weierstrass elliptic functions with parameters depending on the root of the polynomial (15). Such a formula can be extended to cubic/quintic NLS as explained in remark 13; in fact, when \( \beta = 0 \) and \( \omega_0 \) coincides with a simple zero of
The fundamental solution used by the cited above papers. However, in order to numerically find the dispersion relation $\mu$ as function of the quasimomentum $k$ we found the implicit approach used in this paper more easier to numerically apply than the approach based on the explicit solution and on the three zeros of the polynomial (15).

Now, we are going to see how these constraints work in the case of attractive and repulsive nonlinearities.

**Theorem 2.** For any $\alpha \in \mathbb{R}$, $\alpha \neq 0$, then (27) has just one solution $t \in [0, 1)$ for any value $\mu \in (\mu_m, \mu_M)$ where

$$
\mu^m = G(t^m) + \frac{3}{2} \alpha \quad \text{and} \quad \mu^M = G(t^M) + \frac{3}{2} \alpha,
$$

and where $t^m$ and $t^M$ are given by (29) in the case of attractive nonlinearity $\alpha < 0$, and by (31) in the case of repulsive nonlinearity $\alpha > 0$.

**Proof.** Let us consider, at first, the case of attractive nonlinearity, i.e. $\alpha < 0$. In such a case $B > 0$ is always satisfied; furthermore condition $A > -B$ implies that

$$A(1 - F_1) > -1 \quad \text{that is} \quad 8K^2(t)r^2F_2(t) + \alpha < 0,$$

where

$$F_2(t) := 1 - F_1(t) = \int_0^1 \text{cn}^2(qx; t) \, dx.$$

Condition $C_2 > 0$, under the constraints $B > 0$ and $A + B > 0$, becomes

$$(2\alpha B + 4q^2) = 2\alpha + 16K^2(t) \left(1 - r^2F_1(t)\right) > 0.$$

In conclusion, when $\alpha < 0$ the quantization rule reads as

$$
\begin{cases}
\mu = G(t) + \frac{3}{2} \alpha \\
8K^2(t)r^2F_2(t) + \alpha < 0 \\
\alpha + 8K^2(t) \left(1 - r^2F_1(t)\right) > 0
\end{cases}
$$

(28)

The two functions

$$8K^2(t)r^2F_2(t) \quad \text{and} \quad 8K^2(t) \left(1 - r^2F_1(t)\right)$$

are monotone increasing functions such that

$$\lim_{t \to 0} 8K^2(t)r^2F_2(t) = 0 \quad \text{and} \quad \lim_{t \to 1} 8K^2(t)r^2F_2(t) = +\infty$$

and

$$L := \lim_{t \to 0} 8K^2(t) \left(1 - r^2F_1(t)\right) = 2\pi^2$$

and

$$\lim_{t \to 1} 8K^2(t) \left(1 - r^2F_1(t)\right) = +\infty.$$
Furthermore
\[ 8K^2(t) \left( 1 - \dot{r}^2 F_1(t) \right) - 8K^2(t) \dot{r}^2 F_2(t) = 8K^2(t)(1 - \dot{r}^2) > 0, \quad \forall \ t \in [0, 1). \]

In conclusion: let \( t_1 \) be the unique solution to the equation
\[ 8K^2(t_1) \left( 1 - \dot{r}_1^2 F_1(t_1) \right) = -\alpha, \]
and let \( t_2 \) be the unique solution to the equation
\[ 8K^2(t_2) \dot{r}_2^2 F_2(t_2) = -\alpha. \]

Then,
\[ t^M = \begin{cases} 0 & \text{if } -L \leq \alpha < 0 \\ t_1 & \text{if } \alpha < -L \end{cases} \quad \text{and} \quad t^m = t_2 \quad (29) \]
proving thus theorem 2 in the attractive case.

We consider now the case of repulsive nonlinearity, i.e. \( \alpha > 0 \). Then, condition \( B > 0 \) implies that
\[ K^2(t) \dot{r}^2 F_1(t) < \frac{1}{8} \alpha. \quad (30) \]
Furthermore, condition \( A > -B \) reduces to
\[ K^2(t) \dot{r}^2 F_2(t) > \frac{1}{8} \alpha, \]
which is always satisfied. Condition \( C_1 > 0 \) becomes
\[ \alpha + 8K^2(t)(1 - \dot{r}^2 F_1(t)) > 0 \]
which is always satisfied, too. In conclusion, when \( \alpha > 0 \) the quantization rule reads as
\[ \begin{cases} \mu = G(t) + \frac{3}{2} \alpha \\ \alpha + 8K^2(t) \dot{r}^2 F_1(t) > 0 \end{cases}. \]

We remark that the function \( 8K^2(t) \dot{r}^2 F_1(t) \) is a monotone increasing function such that
\[ \lim_{r \to 0} 8K^2(t) \dot{r}^2 F_1(t) = 0 \quad \text{and} \quad \lim_{r \to 1} 8K^2(t) \dot{r}^2 F_1(t) = +\infty. \]

Let \( t_3 \) be the unique solution to the equation
\[ 8K^2(t_3) \dot{r}_3^2 F_1(t_3) = \alpha, \]
then
\[ t^M = 0 \quad \text{and} \quad t^m = t_3 \quad (31) \]
completing so the proof of the theorem 2. \[ \square \]

For an explicit formula of the term \( F_1(t) \) we refer to formula (26) below. Integrals \( F_1(t) \) and \( F_2(t) \) have been already considered in the papers [18, 24].
Figure 1. Here we plot the graph of the dispersion functions $\mu(k)$ when $\alpha = -25$ (left-hand side panel), $\alpha = -10$ (central side panel) and $\alpha = +25$ (right-hand side panel).

There are three different behaviors of the ‘dispersion function’ $\mu(k)$ and of the associated solutions (see figure 1): when $\alpha < -L$ then the dispersion function is defined for any $k \in (0, \pi)$; when $-L < \alpha < 0$ then the dispersion function is defined for any $k \in (k^m, \pi)$ where $k^m \in (0, \pi)$; when $0 < \alpha$ then the dispersion function is defined for any $k \in (\pi, k^M)$ where $k^M > \pi$.

The allowed values for the energy $\mu$ and for the quasimomentum $k$, as function of the nonlinearity parameter $\alpha$, are displayed in figures 2 and 3. In particular we plot the graph of the $\alpha$-dependent functions $\mu_m$ and $\mu_M$, and the graph of the functions

$$k^m = \inf_{\mu \in (\mu_m, \mu_M)} k(\mu) \quad \text{and} \quad k^M \quad \text{sup}_{\mu \in (\mu_m, \mu_M)} k(\mu).$$

3.3. Behavior of the solution at the boundaries $\mu_m$ and $\mu_M$

Here, we consider the behavior of the solution $\phi$ when $\mu$ take the boundary values $\mu_m$ and $\mu_M$.

At first we consider the limit $\mu \to \mu_M$ where the proof of the corollary below is an immediate consequence of theorem 2.

**Corollary 1.** If $\alpha \geq -L$, then $t^M \to 0$, $A \to 0$, $B \to 1$ and $k \to \sqrt{\alpha/2 + q^2(0)}$ as $\mu \to \mu_M = \pi^2 + \frac{1}{2} \alpha$; in such a limit the solution is a plane wave function (see figures 5 and 6, broken lines).

If $\alpha < -L$, then $t^M \neq 0$ and $C_1 \to 0$ as $\mu \to \mu_M$; hence, in this limit we have that $\theta(x) \equiv 0$ and the solution has the form

$$\phi(x) = C \ \text{dn}(qx; t)$$

already discussed in (6) (see figure 4, broken line).

We consider now the behavior of the solution when $\mu$ takes the limit value $\mu \to \mu_m$. Even in this case the proof of the corollary below is an immediate consequence of theorem 2.

**Corollary 2.** If $\alpha < 0$, then $A + B \to 0$ and $C_1 \to 0$ as $\mu \to \mu_m$; in such a limit the solution has the form (see figures 4 and 5, full lines)

$$\phi(x) = C \ \text{cn}(qx; t)$$
We consider the cubic model for $\alpha \in [-30, +100]$. We plot the graph of the functions $\mu_M(\alpha)$ (green line for $\alpha < -L$ and blue line for $\alpha > -L$) and $\mu_m(\alpha)$ (red line). The allowed values for the energy $\mu$ are the ones contained in the interval $(\mu_m, \mu_M)$. If we call $\mu_M - \mu_m$ the band width then it depends on $\alpha$ and it is zero only when $\alpha = 0$ (black line). At the edges of the band $(\mu_m, \mu_M)$ we have that $C_1 \neq 0$ and $A = 0$ and $B = 1$ along the blue line; and finally $A = -B$ along the red line for $\alpha < 0$ and $b = 0$ along the red line for $\alpha > 0$.

already discussed in (5). If $\alpha > 0$, then $B \to 0$ and $C_1 \to 0$ as $\mu \to \mu_m$; in such a limit the solution has the form (see figure 6, full line)

$$\phi(x) = C \operatorname{sn}(qx; t)$$

already discussed in (4).

Concerning the quasimomentum we recall that

$$k^m = 0 \text{ if } \alpha \leq -L$$

and we can prove that

$$k^M = \pi \text{ if } \alpha < 0 \quad \text{and} \quad k^m = \pi \text{ if } \alpha > 0.$$ 

**Theorem 3.**

$$\lim_{\mu \to \mu_m} k(\mu) = \pi.$$ (32)

**Proof.** We collect some results concerning the elliptic integral of third kind defined as

$$\Pi(z; \nu, \ell) = \int_0^\ell \frac{1}{(1 - \nu u^2)\sqrt{1 - u^2 \sqrt{1 - \nu u^2}}} \, du.$$
Figure 3. We consider the cubic model for $\alpha \in [-30, +100]$. We plot the graph of the functions $k^m$ (full line) and $k^M$ (broken line). As proved in theorem 3 $\lim_{\mu \to \mu^m} k(\mu) = \pi$.

Figure 4. Here we plot the graph of the solution $\rho(x)$ (left-hand side) and $\theta(x)$ (right-hand side) when $\alpha = -25$. Broken lines correspond to the solutions associated to an energy $\mu$ close to the value $\mu^M$; full lines correspond to the solutions associated to an energy $\mu$ close to the value $\mu^m$; dot lines correspond to the solutions associated to the energy $\mu = \frac{1}{2}(\mu^m + \mu^M)$.

Recalling that $\frac{d}{dt} \text{sn}(x; t) = \text{cn}(x; t) \text{dn}(x; t)$ and that $\sqrt{1 - \text{sn}^2(x; t)} = |\text{cn}(x; t)|$ and $\sqrt{1 - t^2 \text{sn}^2(x; t)} = \text{dn}(x; t)$ then

$$\int \frac{1}{A \text{ sn}^2(x; t) + B} \, dx = \frac{1}{B} \Pi \left( \text{sn}(x; t); -A/B, t \right)$$

(33)
Figure 5. Here we plot the graph of the solution $\rho(x)$ (left-hand side) and $\theta(x)$ (right-hand side) when $\alpha = -10$. Full lines correspond to the solutions associated to an energy $\mu$ close to the value $\mu_m$; broken lines correspond to the solutions associated to an energy $\mu$ close to the value $\mu_M$; dot lines correspond to the solutions associated to the energy $\mu = \frac{1}{2}(\mu_m + \mu_M)$. 

Figure 6. Here we plot the graph of the solution $\rho(x)$ (left-hand side) and $\theta(x)$ (right-hand side) when $\alpha = +25$. Full lines correspond to the solutions associated to an energy $\mu$ close to the value $\mu_m$; broken lines correspond to the solutions associated to an energy $\mu$ close to the value $\mu_M$; dot lines correspond to the solutions associated to the energy $\mu = \frac{1}{2}(\mu_m + \mu_M)$.

provided that $x \in [0, K(t)]$ because $|\text{cn}(x; t)| = \text{cn}(x; t), B \neq 0, -A/B < 1$ and $t \in [0, 1)$.

Now, in order to compute the quasimomentum $k(t)$ as function of the parameter $t$ we make
use of equation (11) and we refer to the formula (33) obtaining that
\[
k = C_1 \int_0^1 \frac{1}{A \, \text{sn}^2(2K(t)x; t) + B} \, dx = 2C_1 \int_0^{1/2} \frac{1}{A \, \text{sn}^2(2K(t)x; t) + B} \, dx \nonumber = \frac{\sqrt{(1 + A/B)(2\alpha B + 16K^2(t))}}{2K(t)} \Pi \left( 1; -A/B, t \right).
\] (34)

We consider then, at first, the attractive case \( \alpha < 0 \); we have that
\[
k = \frac{\sqrt{2\alpha + 16K(t)E(1; t)}}{2K(t)} \sqrt{1 - \zeta \Pi(1; \zeta, t)}
\] (35)

where we set \( \zeta = -A/B < 1 \) because of the constraints \( A + B > 0 \) and \( B > 0 \). Now, from corollary 2 it follows that \( A + B \to 0 \) as \( \mu \to \mu^m \) in the attractive case; then we have to compute the following limit
\[
\lim_{\zeta \to 1^{-}, t \to -2} \sqrt{1 - \zeta \Pi(1; \zeta, t)}.
\]

To this end we observe that \( t_2 < 1 \) and we recall that [see formula (412.01) [6]]
\[
\sqrt{1 - \zeta \Pi(1; \zeta, t)} = \sqrt{1 - \zeta K(t)} + \frac{\pi \sqrt{\zeta (1 - \Lambda_0(\varphi, t))}}{2\sqrt{\zeta - t^2}},
\]

when \( t^2 < \zeta < 1 \) and where \( \varphi = \sin^{-1} \left( \sqrt{(1 - \zeta)/t'} \right) \), \( t' = \sqrt{1 - t^2} \) and
\[
\Lambda_0(\varphi, t) = \frac{2}{\pi} \left[ E(t)F(\varphi, t') + K(t)E(\varphi, t') - K(t)F(\varphi, t') \right],
\]

here \( E(t) \) denotes the complete elliptic integral of the second kind with parameter \( t \) and \( F(\varphi, t) \) denotes the normal elliptic integral of first kind with argument \( \varphi \) and parameter \( t \). Hence,
\[
\lim_{\zeta \to 1^{-}, t \to -2} \sqrt{1 - \zeta \Pi(1; \zeta, t)} = \frac{\pi (1 - \Lambda_0(0, t_2))}{2\sqrt{1 - t_2^2}} = \frac{\pi}{2\sqrt{1 - t_2^2}}
\]
since \( \varphi \to 0 \) as \( \zeta \to 1 \). Hence, in such a limit we have that
\[
k(\mu^m) = \frac{\sqrt{2\alpha + 16K(t)E(1; t_2)}}{2K(t_2)} \cdot \frac{\pi}{2\sqrt{1 - t_2^2}} = \frac{\pi}{2}.
\]

because \( t_2 \) is such that \( 8K^2(t_2)F_2(t_2) = -\alpha \).

In order to give the proof in the repulsive case \( \alpha > 0 \) we still make use of formula (34) in the form
\[
k = \frac{\sqrt{2\alpha B + 16K^2(t)}}{2K(t)} \sqrt{1 + \kappa \Pi(1; -\kappa, t)}
\] (36)

where we set \( \kappa = A/B > -1 \). Now, from corollary 2 it follows that \( B \to 0 \) as \( \mu \to \mu^m \) in the repulsive case; then we have to compute the following limit for any \( t < 1 \) fixed
\[
\lim_{\kappa \to +\infty} \sqrt{1 + \kappa \Pi(1; -\kappa, t)} = \lim_{\kappa \to +\infty} \sqrt{1 + \kappa \int_0^1 \frac{1}{(1 + \kappa u^2)\sqrt{1 - u^2} \sqrt{1 - t^2 u^2}} \, du} = \frac{1}{2} \pi.
\]
Hence, in such a limit we have that

\[ k(\mu^M) = \frac{\sqrt{2\alpha + 16K(t_3)E(1; t_3)}}{2K(t_3)} \cdot \frac{1}{2} = \pi \]

because \( t_3 \) is such that \( 8K^2(t_3)E_1(t_3) = \alpha \). \( \square \)

**Remark 12.** When \( \alpha \) is small enough then \( \mu^M = 0 \) and \( \mu^M = \pi^2 + \frac{3}{2} \alpha \). Furthermore, in the limit \( \alpha \to 0 \) a straightforward calculus gives that

\[ t_2 := t_2(\alpha) \sim \sqrt{-\frac{2\alpha}{\pi}} \quad \text{and} \quad t_3 := t_3(\alpha) \sim \sqrt{\frac{2\alpha}{\pi}}. \]

In such a case the limit of the solution as \( \alpha \) goes to zero becomes the plane wave solution discussed in remark 3 associated to \( k = \pi \).

We close with a remark concerning the quintic/cubic NLS.

**Remark 13.** One can obtain a general solution to equation (15) even when \( f(z) = az^4 + bz^3 + cz^2 + dz + e \) is a fourth degree polynomial with \( a \neq 0 \); this case corresponds to the cubic/quintic NLS

\[ -\phi'' + \alpha|\phi|^2 \phi + \beta|\phi|^4 \phi = \mu \phi, \]

where \( a = \frac{1}{4} \beta \). Indeed, let any \( z_0 > 0 \) be fixed; then it is known that when \( f(z) \) is a quartic polynomial with non-repeating factors then equation (15) has a general solution given by

\[ \zeta(x) = z_0 + \sqrt{f(z_0)} P(x) + \frac{1}{2} f'(z_0) \left[ P(x) - \frac{1}{24} f'(z_0) \right] + \frac{1}{24} f(z_0) f''(z_0) \]

\[ \frac{1}{2} \left[ P(x) - \frac{1}{24} f'(z_0) \right] - \frac{1}{48} f(z_0) f''(z_0) \]

\[ \left[ P(x) - \frac{1}{24} f'(z_0) \right] = \zeta(\pm x) \]

where \( \frac{d}{dx} \) denotes the derivative with respect to \( x \) and \( \frac{d}{dz} \) denotes the derivative with respect to \( z \), and where \( P(x) = P(x; g_2, g_3) \) is the Weierstrass’s elliptic function with parameters

\[ g_2 = ae - \frac{1}{4} bd + \frac{1}{12} c^2 \quad \text{and} \quad g_3 = \frac{1}{16} eb^2 + \frac{1}{6} eac - \frac{1}{16} ad^2 + \frac{1}{48} dbc - \frac{1}{216} c^3. \]

This is an old and, as far as I know, almost unknown result due to Weierstrass. It was published in 1865, in an inaugural dissertation at Berlin, by Biemann [3], who ascribed it to Weierstrass; it was then mentioned in the book by Whittaker and Watson [31] in ch XX, example 2, p 454. When \( \beta = 0 \) then it is possible to prove that (37) reduces to (16).

**ORCID iDs**

Andrea Sacchetti \( \text{https://orcid.org/0000-0001-6292-9251} \)

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