High-fidelity copies from a symmetric $1 \rightarrow 2$ quantum cloning machine

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A symmetric $1 \rightarrow 2$ quantum cloning machine (QCM) is presented that provides high-fidelity copies with $0.90 \leq F \leq 0.95$ for all pure (single-qubit) input states from a given meridian of the Bloch sphere. Emphasize is placed especially on the states of the (so-called) Eastern meridian, that includes the computational basis states $|0\⟩, |1\⟩$ together with the diagonal state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, for which suggested cloning transformation is shown to be optimal. In addition, we also show how this QCM can be utilized for eavesdropping in Bennett’s BB84 protocol for quantum key distribution with a substantial higher success rate than obtained for universal or equatorial quantum copying.

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I. INTRODUCTION

The ability to copy information is fundamental for many processes in distributing and dealing with data. While classical information can often be copied many times without any significant loss, the cloning of quantum information is seriously restricted by the laws of quantum mechanics. This limitation is known today as the non-cloning theorem [1] and has found its applications in quite different fields of quantum information theory, such as quantum computation [2, 3] and quantum cryptography [3, 4].

In processing quantum information, formally, any device that provides for $M$ unknown input states of a given system $N (> M)$ output states (of the same or some analogue systems) is called a $M \rightarrow N$ quantum cloning machine (QCM). To ensure a proper quantum behaviour, such machines are usually represented as unitary transformations; they are called symmetric if the $N$ output states are identical to each other, and are said to be nonsymmetric otherwise. In this work, we shall consider symmetric $1 \rightarrow 2$ QCM that provides for a (pure) input state $|s_a⟩$ (of qubit $a$) the two qubits $a$ and $b$ in the same output state $\rho^\text{out}_a = \rho^\text{out}_b$, and where $\rho^\text{out}$ denotes the (reduced) density matrix of the corresponding qubit. For an ideal transformation, of course, we might expect

$$|s_a⟩|0⟩_b|Q⟩_c \longrightarrow |s_a⟩|s_b⟩|Q⟩_c,$$  

(1)
i.e. that the input and output states of qubit $a$ are the same, $|s_a⟩ = |s'_a⟩$, but this would be in conflict with the non-cloning theorem mentioned above. In the transformation (1), as usual, we here suppose that the qubit $b$ was initially prepared in the state $|0⟩_b$ and that the state vectors $|Q⟩_c$ and $|Q⟩_c$ denote the initial and final states of the corresponding cloning device. In order to characterize the ‘quality’ of a cloning device, we shall often use below the fidelity

$$F = \langle s|\rho^\text{out}_s|s⟩,$$

(2)
between the input and output state (of qubit $a$), which takes the values $0 \leq F \leq 1$ and where $F = 1$ refers to the case that $\rho^\text{out} \equiv |s⟩⟨s|$. Several QCM’s have been discussed in the literature before [4]: Bužek and Hillery [5], for example, worked out a symmetric $1 \rightarrow 2$ QCM that provides copies with the fidelity $F = 5/6 \approx 0.83$ independent of the given input state. This state-independent transformation, which is now known as universal QCM, has found recent attention and application in quantum cryptography, namely, in optimal eavesdropping attack [6] within the six-state protocol [9]. For some protocols in quantum cryptography, however, only few states $\{|s_a⟩\}$ are (pre-) selected from the Bloch sphere and, hence, one may wish to find a state-dependent QCM that provides copies with higher fidelity than the universal Bužek-Hillery machine for this particular set of states.

The best-known example of such a state-dependent QCM is equatorial (or phase-covariant) QCM [7]. This QCM provides copies with fidelity $F = 1/2 + \sqrt{1/8} \approx 0.85$ for all states which are taken from the equatorial plane of the Bloch sphere. Equatorial QCM has also found a remarkable application in quantum cryptography, since it can be used to improve the efficiency of an eavesdropping attack within the BB84 protocol [10]. While, in fact, only four states (from the equatorial plane) are used in order to encode the information in the BB84 protocol, equatorial QCM enables an eavesdropper to obtain copies of these states with the highest fidelity, if compared with any other known QCM [4].

In this work, we here present and discuss another example of state-dependent QCM — a (meridional) QCM that provides copies with high-fidelity $0.9 \leq F \leq 0.95$.
for all states along a given half-circle (the ‘Eastern’ meridian) of the Bloch sphere, that includes three states \(\{|0\rangle, |1\rangle, +\rangle\}\) which define this meridian uniquely. This newly suggested QCM is constructed to be optimal for copying of all input states from the meridian. We also analyze how this newly suggested QCM can be applied in quantum cryptography for the eavesdropping within Bennett’s B92 quantum key distribution protocol [11]. It is shown, in particular, that this transformation provides a substantial higher success rate than obtained for a universal or equatorial quantum copying.

The paper is organized as follows. In the next section, let us first introduce and discuss the general form of symmetric \(1 \rightarrow 2\) cloning transformations. In particular, here we shall analyze the Bužek-Hillery-type transformations [3] which can be utilized for both, a state-dependent and state-independent (symmetric) copying of quantum states. By making use of some additional parameter in the definition of the cloning transformation (when compared to Ref. [2]), we then present a QCM in Subsection III C which enables one to produce for a selected set of pure input states from the Bloch sphere copies with higher fidelity than it can be obtained for universal or equatorial quantum copying. The QCM is then defined, in Subsection III C, to be optimal for copying states from chosen meridian of the Bloch sphere. Below, therefore, we will refer to this cloning transformation as a meridional QCM, even if the particular region of high-fidelity cloning can be chosen rather freely by adapting just three parameters in the given cloning transformation. In Subsection III D we present explicit form of the meridional cloning transformation and recall properties of it. In Section IV we later analyze for this new QCM the success (or failure) of an potential eavesdropper within Bennett’s B92 protocol [11] for the distribution of quantum keys. In this section, emphasis is placed especially on the question how efficient such an eavesdropper can attack the transmission of quantum information; in our discussion below, this will be quantified by means of the mutual information between the eavesdropper and the legitimit uset as well as the discrepancy of a qubit (if compared with the originally transmitted one) after a corresponding attack has been made. Finally, a few conclusions are drawn in Section V.

II. QUANTUM CLONING FOR A RESTRICTED SET OF INPUT STATES

A. General symmetric \(1 \rightarrow 2\) cloning transformations

To introduce some notations that are necessary for our discussion below, let us start from a single qubit \(a\) whose (pure) states can be written in the Bloch sphere representation as \(|s\rangle_a = \cos \frac{\theta}{2} |0\rangle_a + \sin \frac{\theta}{2} e^{i\phi} |1\rangle_a\), and with \(|0\rangle_a\) and \(|1\rangle_a\) being the standard (computational) basis states. In this representation, the parameters \(\theta\) and \(\phi\) take their values from \(0 \leq \theta \leq \pi\) and \(0 \leq \phi < 2\pi\), respectively, and we shall often use this Bloch sphere (picture) in order to visualize the states of interest. Moreover, let us refer to the intersection of the Bloch sphere with the \(x\)-\(z\) plane as the main circle, so that all states from this intersection can be parameterized by means of just the \((\text{single})\) parameter \(\theta\) as

\[|s\rangle_a = \cos \frac{\theta}{2} |0\rangle_a + \sin \frac{\theta}{2} |1\rangle_a. \tag{3}\]

While, in this expression, the ‘\(+\)’ sign refers to the right (Eastern) meridian of the main circle and includes for \(\theta = \pi/2\) also the diagonal state \(+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)\), the ‘\(-\)’ sign is associated with its left (Western) meridian and includes the state \(-\rangle\). For the sake of simplicity in visualizing the different states on the Bloch sphere, we shall sometimes use also this ‘geographical’ notation in our discussion below.

Although the state \(|s\rangle_a\) of a given qubit will be typically in a superposition of the two basis states \(|0\rangle_a\) and \(|1\rangle_a\), it is of course sufficient to know the transformation of just the basis in order to obtain a proper copying of states. Therefore, the most general \((1 \rightarrow 2)\) quantum cloning transformation for the state of qubit \(a\) upon qubit \(b\) can be cast into the form

\[|0\rangle_a |0\rangle_b |Q\rangle_c \rightarrow \sum_{m,n=0}^{1} |m\rangle_a |n\rangle_b |Q_{mn}\rangle_c, \tag{4}\]

where \(|Q\rangle_c\) denotes the initial state of the cloning apparatus, and where we have assumed — without loss of generality — that the second qubit \(b\) was prepared initially in the basis state \(|0\rangle_b\). Once, the transformation has been performed, \(|m\rangle_a\) and \(|n\rangle_b\) denote the output basis states of the two copies, while \(|Q_{mn}\rangle_c\) are the corresponding states of the apparatus. As seen from the transformation (4), here we do not assume any additional condition for the final states of the cloning apparatus. However, in order to ensure that the transformation (4) is unitary,

\[\sum_i c_i |i\rangle_{ab} |Q\rangle_c \rightarrow \sum_i c_i U_{i\lambda} |\lambda\rangle_{abc}, \tag{5}\]

for all possible input states, i.e. for \(|i\rangle_{ab} = \{|0\rangle_a |0\rangle_b, |1\rangle_a |0\rangle_b\}\), the final states of the apparatus must fulfill certain requirements [12].

In Eq. (5), the three-partite basis \(|\lambda\rangle_{abc}\) refers to a complete and orthonormal basis for the overall system ‘qubits a,b + apparatus’. Thus, the requested unitarity
\[ U U^\dagger = 1 \] of the transformation (1) implies the conditions

\[ \sum_{m,n=0}^{1} Q_{mn} | Q_{mn} \rangle = 1, \]

\[ \sum_{m,n=0}^{1} \langle Q_{mn} | \hat{Q}_{mn} \rangle = 0. \quad (6) \]

For any explicit construction of a \((1 \rightarrow 2)\) quantum cloning transformation, we must therefore ‘determine’ the final states \( | Q_{mn} \rangle_c \) and \( \langle \hat{Q}_{mn} | \rangle_c \) of the apparatus in line with the conditions (3). These state vectors then define the QCM uniquely.

Before we shall construct QCM with some particular properties, let us consider the simplest case of such a transformation (4) as first suggested by Wootters and Zurek (4)

\[ |0\rangle_a |0\rangle_b \langle Q \rangle_c \rightarrow |0\rangle_a |0\rangle_b \langle Q00 \rangle_c, \]

\[ |1\rangle_a |0\rangle_b \langle Q \rangle_c \rightarrow |1\rangle_a |1\rangle_b \langle \hat{Q}11 \rangle_c. \quad (7) \]

In this transformation, obviously, only a single term is retained from the summations on the right-hand side (rhs) of Eqs. (4), i.e. the term with \( m = n = 0 \) in the first and \( m = n = 1 \) in the second line. No further freedom remains in the set-up of this transformation since the unitarity conditions (3) then require \( \langle Q00 | \hat{Q}00 \rangle = \langle Q11 | \hat{Q}11 \rangle = 1. \) As seen from Eq. (4), moreover, the Wootters-Zurek transformation is symmetric with regard to an interchange of the basis states \( |0\rangle_a \leftrightarrow |1\rangle_a \) and this implies that the same symmetry holds also for the state of the two copies \( \rho_a^{\text{out}} \) and \( \rho_b^{\text{out}} \). For the Wootters-Zurek transformation, furthermore, the fidelity between the input state (from the main circe) and the corresponding output is given by

\[ F(\theta) = 1 - \frac{1}{2} \sin^2 \theta. \quad (8) \]

Therefore, this particular transformation can provide an ‘exact’ copy with fidelity \( F(0) = F(\pi) = 1 \) just for the basis states \( |0\rangle \) and \( |1\rangle \), i.e. the two ‘poles’ of the Bloch sphere, while the fidelity drops down to \( F(\pm \pi) = 1/2 \) for all states along the equator and, especially, for the two diagonal states \(|\pm\rangle\). For this reason, the Wootters-Zurek transformation appears to be of little help if one wishes to copy (unknown) states other than the basis states themselves.

B. Cloning transformations of Bužek-Hillery type

Additional terms in the cloning transformation (4) need to be considered if we wish to construct a QCM that supports an equal-fidelity cloning for all states on the Bloch sphere, or which improves the fidelity between the input and output states for certain regions on this sphere. For example, Bužek and Hillery (3) have analyzed in quite detail the three-term transformation

\[ |0\rangle_a |0\rangle_b \langle Q \rangle_c \rightarrow |0\rangle_a |0\rangle_b \langle Q00 \rangle_c \]

\[ + |0\rangle_a |1\rangle_b + |1\rangle_a |0\rangle_b \langle Y_0 \rangle_c, \quad (9) \]

\[ |1\rangle_a |0\rangle_b \langle Q \rangle_c \rightarrow |1\rangle_a |1\rangle_b \langle Q11 \rangle_c \]

\[ + |0\rangle_a |1\rangle_b + |1\rangle_a |0\rangle_b \langle Y_1 \rangle_c, \quad (10) \]

which takes into account two additional terms (compared with the Wootters-Zurek transformation (4)) and which gives us further freedom in choosing the final states of the cloning apparatus. To follow the notation by Bužek and Hillery (3), here we have introduced \( |Q00 \rangle \equiv |Q00 \rangle, \quad |Y_0 \rangle \equiv |Q01 \rangle = |Q10 \rangle, \quad |Y_1 \rangle \equiv |Q11 \rangle \) to denote the final states of the apparatus. Again, in order to ensure the symmetry with regard to an interchange \( |0\rangle_a \leftrightarrow |1\rangle_a \) of the basis states of qubit \( a \), we have assumed conditions \( |Q01 \rangle = |Q10 \rangle \) and \( |Q11 \rangle = |\hat{Q}10 \rangle \) for the final states of the cloning apparatus.

Eqs. (9)-(10) can be utilized to define cloning transformations for which the fidelity between the input and output states is either state-independent or depends explicitly on the given input. This behaviour depends on the additional restriction we shall place on the final states of the apparatus, beside of the unitarity conditions (4). For the sake of simplicity, let us omit in the following the indices \( a, b \) and \( c \) of the individual subsystems but keep in mind that \( |kl \rangle \equiv |k\rangle_a |l\rangle_b \) always refers to the state of the two copies to be created, and that the vectors \( |Q_i \rangle \) and \( |Y_i \rangle \) belong to the cloning apparatus. With this change in the notation, we find from Eqs. (6) that the final-state vectors of the apparatus in the Bužek-Hillery transformation must satisfy the conditions

\[ \langle Q_i | Q_i \rangle + 2 \langle Y_i | Y_i \rangle = 1, \quad \text{for} \quad i = 0,1; \quad (11) \]

\[ \langle Y_0 | Y_0 \rangle = 0, \quad (12) \]

in order to ensure the unitarity of the transformation. For a general (pure) input state \( |\psi \rangle \) from the main circle, the QCM (9)-(10) then gives rise to the two-qubit density operator \( \rho_{ab}^{\text{out}} \) which contain 14 scalar products between the final-state vectors from the apparatus. Each scalar product introduces a (complex) parameter for the Bužek-Hillery transformation (9)-(10). This gives rise to a total of 14 parameters for the quantum cloning transformation (9)-(10), while only the 3 restrictions (11)-(12) need to be fulfilled due to the unitarity of the transformation.

Further conditions must therefore be formulated in order to define the QCM properly. For example, we may choose all scalar products to be real and also request that the two final-state vectors \( |Y_0 \rangle \) and \( |Y_1 \rangle \) have an equal norm

\[ \langle Y_0 | Y_0 \rangle = \langle Y_1 | Y_1 \rangle = \zeta . \quad (13) \]
Under these conditions, Eq. (11) takes the same form for $i = 0$ and 1, and hence, $(Q_0 | Q_0) = (Q_1 | Q_1) = 1 - 2\zeta$. There are two other conditions that can be obtained from Eqs. (11)-(12) by using the restriction (13), namely,

\begin{align}
\langle Y_0 | Q_1 \rangle &= \langle Y_1 | Q_0 \rangle \equiv \eta/2, \quad (14) \\
\langle Q_1 | Y_1 \rangle &= \langle Q_0 | Y_0 \rangle \equiv \kappa/2. \quad (15)
\end{align}

With these additional notations and conditions, we have arrived at the final-state density matrix $\rho_{ab}^{\text{out}}$ of the transformation (9)-(10) that now depends only on three real parameters. This is quite in contrast to the original work of Bužek and Hillery (6) who assumed the final states of the apparatus $(Q_i | Y_i) = 0$ to be pairwise orthogonal to each other for the cloning of the two basis states $|i\rangle$, $i = 0, 1$. While the condition (13) introduces a nonorthogonality between the final states of the apparatus, three parameters $\zeta, \eta$ and $\kappa$ enables us to provide high-fidelity copies for a region of input states from the Bloch sphere. However, the three parameters $\zeta, \eta$ and $\kappa$ are not completely independent of each other but must fulfill the three inequalities

\begin{align}
0 &\leq \zeta \leq \frac{1}{2}, \quad (16) \\
0 &\leq \eta \leq 2\sqrt{(1 - 2\zeta)}, \quad (17) \\
0 &\leq \kappa \leq 2\sqrt{(1 - 2\zeta)}. \quad (18)
\end{align}

due to Schwarz’ inequality for the state vectors of the cloning apparatus.

Using the final-state density matrix $\rho_{ab}^{\text{out}}$ for the transformation (9)-(10), it is simple to show that the reduced density operator

\[ \rho^{\text{out}} = \left( \cos^2\frac{\theta}{2} - \zeta \cos \theta \right) |0\rangle \langle 0| + \frac{1}{2} (\kappa + \eta \sin \theta) (|0\rangle \langle 1| + |1\rangle \langle 0|) + \left( \sin^2\frac{\theta}{2} + \zeta \cos \theta \right) |1\rangle \langle 1|, \quad (19) \]

is the same for both subsystems $a$ and $b$, that is $\rho^{\text{out}} = \text{Tr}_a (\rho_{ab}) = \text{Tr}_b (\rho_{ab})$. We can utilize this expression (19) to calculate also the fidelity between the input and output for all states $|s\rangle_a = \cos \frac{\theta}{2} |0\rangle_a \pm \sin \frac{\theta}{2} |1\rangle_a$ along the main circle

\[ F(\theta) \equiv \langle s | \rho^{\text{out}} | s \rangle = (1 - \zeta) - \frac{1}{2} (1 - \eta - 2\zeta) \sin^2 \theta \pm \frac{\kappa}{2} \sin \theta. \quad (20) \]

Although the parameters $\zeta, \eta$ and $\kappa$ are restricted by the inequalities (16)-(17), it is this freedom in choosing these parameters that enables us to optimize the fidelity $F(\theta)$ for certain (regions of) states. Since $\kappa$ has a different sign for the two parts of the main circle, a nonzero value this parameter leads to a quite different behavior of the fidelity along the Western and Eastern meridian: the high fidelity along the Eastern meridian correspond to positive parameter $\kappa$, while the high fidelity along the Western meridian is achieved for negative $\kappa$. So, for proper values of $\zeta, \eta$ and $\kappa$ we can obtain a high fidelity for one meridian.

C. Optimization of the cloning transformation

Eq. (20) can be applied to determine an ‘optimal’ set of parameters, either for a few selected input states or for a whole region of states from the Bloch sphere. General method for optimization of parameters of the unitary transformation was developed in Ref. (13) and was successfully applied to show optimality of some cloning transformations (14). The optimal cloning transformation maximizes average single-clone fidelity

\[ \overline{F} = \int_0^\pi \frac{d\theta}{\pi} F(\theta) \quad (21) \]

doing chosen region of states $\Omega$ on the Bloch sphere, where $N$ is the normalization factor. Substituting in this expression the fidelity function (20) and integrating over all states from Eastern meridian of the Bloch sphere, we obtain average fidelity as function of the parameters $\zeta, \eta$ and $\kappa$, i.e.

\[ \overline{F} = \int_0^{\frac{\pi}{2}} d\theta \frac{d\theta}{\pi} F(\theta) = \frac{1}{4} \left( 3 - 2\zeta + \eta + \frac{4\kappa}{\pi} \right). \quad (22) \]

Following numerical optimization procedure over the three parameters, which are restricted with inequalities (16)-(17), gives values

\[ \kappa = \frac{2}{5}, \quad \zeta = \frac{1}{10}, \quad \eta = \frac{2}{5}, \quad (23) \]

approximately, that correspond to the maximal average fidelity $\overline{F} \approx 0.927$.

There is, however, another way to perform optimization of the fidelity function (20) that does not require numerical calculations. It is known that parameters of universal QCM can be found from the requirement of optimal copying of just the discrete set of six states that lie on $x, y$ and $z$ axis of the Bloch sphere (4). Similarly the requirement of optimal copying of four states that lie on $x$ and $y$ axis of the sphere is sufficient to determine equatorial QCM. We may, for example, request an equal and maximum fidelity for just three selected states from the meridian in order to determine optimal parameters of the cloning transformation (9)-(10). If the input $|s\rangle$ is taken from the set of the three states $\{|0\rangle, |1\rangle, |+\rangle\}$ and we request an equal-fidelity cloning of them, the maximum fidelity $F = 0.90$ is obtained for the parameters (24). This result is not surprising since the fidelity has local minima for the states $|0\rangle$, $|1\rangle$ and $|+\rangle$; that is the main reason of optimization with this three states. In fact, in this optimization procedure we restricted the fidelity function (20) downwards.
D. Explicit form and properties of the cloning transformation

With the help of the parameters (23) the final-state vectors of the apparatus can be defined as

$$|Y_0⟩ = \left\{ \frac{1}{\sqrt{10}}, 0 \right\}, \quad |Y_1⟩ = \left\{ 0, \frac{1}{\sqrt{10}} \right\},$$

$$|Q_0⟩ = \left\{ \sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}} \right\}, \quad |Q_1⟩ = \left\{ \sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}} \right\},$$

in line with the conditions (11)-(12) and (13)-(14) from above. These four vectors span only a two-dimensional subspace within the (four-dimensional) space of the general copying machine (4). If we introduce the orthogonal subspace within the (four-dimensional) space of the general copying machine (4). If we introduce the orthogonal basis \( |0⟩, |1⟩ \) for this subspace, the transformation (25)-(26) can be brought into the form

\[
|0⟩ |0⟩ |Q⟩ \rightarrow \sqrt{\frac{2}{5}} |0⟩ (|0⟩ + |1⟩) + \frac{1}{\sqrt{10}} (|0⟩ + |1⟩) |0⟩, \quad (25)
\]

\[
|1⟩ |0⟩ |Q⟩ \rightarrow \sqrt{\frac{2}{5}} |1⟩ (|0⟩ + |1⟩) + \frac{1}{\sqrt{10}} (|0⟩ + |1⟩) |1⟩, \quad (26)
\]

if the vectors (24) are substituted into the transformation (9)-(10). This makes our suggested QCM now explicit. The QCM (25)-(26) is optimal for a symmetric high-fidelity cloning of the states from Eastern meridian.

In the B92 protocol, only two nonorthogonal quantum states are utilized in order to encode and transmit the information about the cryptographic key. As usual, we suppose that the information is sent from Alice to Bob by means of a quantum communication channel. At the beginning of the key distribution protocol, Alice encodes each logical bit, 0 or 1, into two nonorthogonal states, that can be parameterized in computational basis with
single parameter $\vartheta$ [13] as

$$
|u\rangle = \cos \frac{\vartheta}{2} |0\rangle + \sin \frac{\vartheta}{2} |1\rangle,
$$

$$
|v\rangle = \sin \frac{\vartheta}{2} |0\rangle + \cos \frac{\vartheta}{2} |1\rangle.
$$

The overlap of the states $O(\vartheta) \equiv |\langle u | v \rangle|^2 = \sin^2 \vartheta$ gives the distance between states $|u\rangle$ and $|v\rangle$ in geometric sense. These qubits are then sent to Bob who performs a positive operator-valued measurement (POVM), and the best operators for that are [15]

$$
G_1 = \frac{1}{1 + |\langle u | v \rangle|} (1 - |\langle u | v \rangle|),
$$

$$
G_2 = \frac{1}{1 + |\langle u | v \rangle|} (1 - |\langle v | u \rangle|),
$$

$$
G_3 = 1 - G_1 - G_2.
$$

Only measurements with POVM elements $G_1$ and $G_2$ are conclusive, because certain conclusion about received state $|u\rangle$ or $|v\rangle$ can be made after the measurement. After all the qubits have been sent (and measured), Bob tells to Alice numbers of conclusive measurements via a public channel, which can be monitored but not modified by possible eavesdropper. Only those bits (obtained in Bob’s conclusive measurements) can be used to construct the key, while all the rest need to be discarded because no definite conclusion can be drawn from the outcome of Bob’s measurement. To test and recognize a (possible) eavesdropper, Alice and Bob compare moreover the values of some of their bits via the public channel in order to get an estimate how likely their communication was disturbed.

In practice, a disturbance in the transmission of a (secret) key can have very different reasons. Apart from an eavesdropper, the quantum control during the preparation or transmission of the qubits might be incomplete for a given realization of the quantum channel. For all practical realizations of QKD protocols, therefore, a certain error rate (disturbance) need to be accepted, and an eavesdropper might be successful in extracting information even if the protocol is inherently secure in the ideal case. To quantify the disturbance in the transmission of a single qubit, a convenient measure is the probability that Alice and Bob detect an error. If Bob would know the state $|s\rangle$ of one or several qubits in advance, that were sent to him by Alice, he could easily test for a possible eavesdropping attack. In this case, he will receive in general the qubits no longer in a pure but a mixed state that has to be described in terms of its density matrix $\rho$. The discrepancy that is detected by Bob is given by

$$
D = 1 - |\langle s | \rho | s \rangle|.
$$

A central question for Eve is of how much information she can extract from the transmission of the key if the disturbance due to his attack should be $D < D_{max}$. From the initial agreement between Alice and Bob about the basis states which are to be chosen randomly, Eve might know that Alice prepares the qubits in one of the two states [29] with probability $p_i = 1/2$ ($i = 0, 1$). Before Eve has measured a given qubit, her (degree of) ignorance is given by Shannon’s entropy $H = -\sum p_i \log_2 p_i = -\log_2(1/2)$. After the measurement, she increased her knowledge about the system by decreasing this entropy, a measure that is called the mutual information that Eve has acquired due to the measurement. Obviously, Eve will try to obtain as much information as possible keeping the discrepancy $D < D_{max}$.

In order to discuss how much Eve will affect the transmission (and thus increase her knowledge about the transmitted information), we must specify the circumstances under which the eavesdropping attack is made. Let us suppose that Eve performs incoherent (i.e. individual) attack on the communication channel with a QCM. Here, we shall not yet specify the QCM explicitly in order to enable us to compare different QCM’s below. According to the incoherent strategy, Eve will copy (attack) each qubit independently as they are sent from Alice to Bob. As output of her cloning transformation, she then obtains two copies of one of the two possible states $\rho_{|u\rangle}$ and $\rho_{|v\rangle}$ (which just correspond to the two input states [29]) with a fidelity as defined by the given QCM. While Eve transmits one of her copies further to Bob, she could measure the second copy following the same procedure as Bob.

To calculate the mutual information between Alice and Eve that is to be extracted from the eavesdropping, we can follow the procedure as described in Ref. [16] and [17]. Using the POVM elements [30]--[32], the probability for Eve to obtain the outcome $\mu$ is

$$
P_{\mu} = \text{Tr}(G_\mu \rho_i),
$$

and where the operators $\rho_i$ refer to the two possible states $\{\rho_{|u\rangle}, \rho_{|v\rangle}\}$ of her copy. After the measurement, when she has obtained a particular outcome $\mu$, the posterior probability $Q_{\mu i}$ that $\rho_i$ was prepared by Alice is

$$
Q_{\mu i} = \frac{P_{\mu i} \rho_i}{q_\mu},
$$

where $q_\mu = \sum_j P_{\mu j} p_j$, and $p_j = 1/2$ is the probability for sending the states $|u\rangle$ and $|v\rangle$ within the B92 protocol. With these probabilities, the Shannon entropy (which was $H = -\log_2(1/2) = 1$ initially), becomes

$$
H_\mu = -\sum_i Q_{\mu i} \log Q_{\mu i},
$$

once the result $\mu$ was obtained, and hence the mutual information is

$$
I = H - \sum_\mu q_\mu H_\mu.
$$
To determine the possible success of an eavesdropper, we only need to analyze the explicit form of the output states $\rho_\mu$ and $\rho_\nu$ for a particular QCM. By substituting the output states into Eqs. (29) and (33) we may then calculate the mutual information and discrepancy in case of an eavesdropping with the QCM. Since the copies from the QCM are symmetric, the mutual information extracted by Eve $I_{AE}$ equals to the mutual information obtained by Bob in his measurements $I_{AB} = I_{AE} \equiv I$ and, obviously, depends from the overlap of the states (29).

If Eve applies universal QCM, which provides copies with fidelity $F = 5/6$, she causes discrepancy $D = 1/6 \approx 0.17$ independently from the choice of the states (29). The mutual information extracted by Eve is given at Fig. 2 with dotted line. For equatorial QCM with the fidelity $F = 1/2 + \sqrt{1/8}$ of the copies, discrepancy equals $D = 1/2 - \sqrt{1/8} \approx 0.15$ for arbitrary states (29) and the mutual information is shown at Fig. 2 with dashed line. For the eavesdropping with meridional QCM discrepancy depends from particular choice of the states (29) and is given with inequality $0.05 \leq D \leq 0.10$. The mutual information in this case is shown at Fig. 2 with solid line. Apparently, our suggested QCM introduces a lower disturbance into the data transmission between Alice and Bob than it is caused by universal or equatorial QCM’s. It also enables Eve to extract in course of her eavesdropping more information than obtained by means of these two QCM’s.

Suggested meridional QCM as well as universal and equatorial QCM’s belongs to the class of QCM’s with an axillary system (4). There are, however, several proposals of optimal cloning transformations without any axillary system (4, 18, 19). It is, of course, important to compare efficiencies of the eavesdropping with meridional QCM and the best QCM without the axillary systems. Optimal cloning transformation that clone at best two arbitrary pure states of a qubit was developed in Refs. 13, 19. This transformation was shown to have extremely large fidelity $0.987 \leq F \leq 1$ between the input and output states. However, in this type of cloning the copies are strongly entangled. That leads to the fact that in the eavesdropping with the QCM without the axillary system, Eve has very poor information about the result obtained by Bob. Indeed, as the result of Bob’s measurement on received qubit, state of Eve’s qubit reduces to strongly mixed state. In fact, presence of the axillary system is necessary to eavesdrop efficiently a QKD protocol based on single-qubit states (3).

General question arises now, whether the incoherent eavesdropping attack with meridional QCM discussed above is optimal within B92 QKD protocol? To answer the question, we need to analyze the success of the eavesdropper in line of alternative incoherent attacks as well as alternative strategy for eavesdropping, namely, coherent (or collective) attack when the eavesdropper has opportunity to perform collective measurements on intercepted qubits. Several incoherent attacks on the protocol have been proposed before (15) such as intercept-resend attack and the attack with entangled probe. It was found that the efficiency of the eavesdropping depends from the overlap of the states (29). Recently we showed that for wide range of the overlap $0.07 \leq O(\theta) \leq 0.50$ of the states $|u\rangle$ and $|v\rangle$, the attack with meridional QCM gives advantage for the eavesdropper to obtain more information causing less discrepancy than any known incoherent attack (21). For small overlap when the states are almost orthogonal, however, intercept-resend attack is optimal; while for large overlap the attack with the entangled probe is optimal (see (21) for further discussion). Moreover, the eavesdropping with meridional QCM is optimal independently from incoherent or coherent strategy of the eavesdropping, since it was shown that coherent strategy for the eavesdropping gives negligible small additional information comparing to the incoherent strategy in the protocols for quantum key distribution based on single-qubit states (3, 8).

**IV. CONCLUSIONS**

Unlike the well-known universal (3) and equatorial (7) quantum cloning, we have presented a QCM that provides high-fidelity copies for all states from a selected meridian (i.e. half-circle) of the Bloch sphere. This (so-called) meridional QCM was constructed to provide high-fidelity copies with $0.95 \geq F \geq 0.90$ for all states along the Eastern meridian. Although this QCM provides high-fidelity copies for the Eastern meridian, it can be applied with little adaptations also to other meridians. All what is needed to follow the ‘optimization’ procedures as described in Subsection II C.

The suggested QCM has been applied also to analyze a possible eavesdropping attack in the data transmission between Alice and Bob, following Bennett’s B92 QKD
protocol [11]. From this analysis, it is shown that Eve, the eavesdropper, can obtain more information from the meridional than from the universal or equatorial QCM’s. The probability that Bob (as the legitime user) will detect the attack \(0.05 \leq D \leq 0.10\) is lower for the meridional than in case of the universal \((D \approx 0.17)\) or equatorial \((D \approx 0.15)\) quantum copying. Moreover, the eavesdropping attack with meridional QCM is found to be optimal for particular choices of states which are used to encode information in the protocol [20].

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