Quantum Reading of a Classical Memory

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We consider a digital memory where each memory cell is a mirror with two possible reflectivities (used to encode a bit of information). Adopting this model, we show that a quantum source of light, possessing Einstein-Podolsky-Rosen correlations, can retrieve the stored information more accurately than every classical source. As a result, quantum entanglement can reduce the error correction overhead in classical memories, thus increasing their effective capacities.

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In recent years, quantum information has disclosed a modern approach to both quantum mechanics and information theory. A new physical resource, the quantum entanglement, has been identified and powerfully exploited in a long series of information and computation tasks\textsuperscript{[1]}. Among other applications, this resource has been fundamental in many protocols involving bosonic modes of the radiation field, from continuous variable quantum teleportation\textsuperscript{[2]} to quantum illumination\textsuperscript{[3, 4]}. In the bosonic setting, the entanglement is usually exploited under the form of Einstein-Podolsky-Rosen (EPR) correlations\textsuperscript{[5]}, where the quadrature operators of two separate bosonic modes are so correlated to beat the standard quantum limit\textsuperscript{[6]}. In this paper, we show how these quantum correlations can improve the reading of information from classical memories. Our model of classical digital memory is similar to actual optical storage devices, like present-day CDs, DVDs or Blue-ray disks\textsuperscript{[7]}. In fact, we consider a memory where each cell is a mirror with two possible reflectivities, \( r_0 \) and \( r_1 \geq r_0 \), which are used to store a bit of information. Adopting this model, we show that a non-classical source of light, possessing EPR correlations, can retrieve the information from a memory cell more accurately (i.e., with a lower error probability) than every source of light described by a classical state\textsuperscript{[8, 9]}. In general, this improvement is found in the regime of high reflectivities (i.e., for \( r_1 \gtrsim 0.8 \)), where the information gain due to the increased accuracy can be surprising (close to 1 bit per cell in some situations). Since quantum entanglement lowers the error probability in the reading of the memory cells, it enables to reduce the error correction overhead affecting the memory\textsuperscript{[10]}. For a given storage density, a reduction of this overhead corresponds to an increase in the amount of logical data that can be stored in the memory (effective capacity).

Let us consider a model of classical memory (disk) where the memory cells are beam splitter mirrors with different reflectivities \( r = r_0, r_1 \) (with \( r_1 \geq r_0 \)). In particular, the bit-value \( u = 0 \) is encoded in a lower-reflectivity mirror \( (r = r_0) \), called a pit, while the bit-value \( u = 1 \) is encoded in a higher-reflectivity mirror \( (r = r_1) \), called a land\textsuperscript{[11]} (see Fig. 1). Close to the disk, a decoder (Alice) aims to read the value of the logical bit \( u \) which is encoded in each memory cell. For this sake, Alice exploits a transmitter (to probe a target cell) and a receiver (to measure the corresponding output). In general, the transmitter consists of two quantum systems, called signal S and idler I, respectively. The signal system S is a set of M bosonic modes which are directly shined on the target cell. The mean energy of this system is given by \( MN_S \), where \( N_S \) is the mean number of photons per signal mode (also called the signal-energy, hereinafter). At the output of the cell, the reflected system \( R \) is combined with the idler system \( I \), which is a supplementary set of bosonic modes whose number \( L \) can be completely arbitrary. Both the systems \( R \) and \( I \) are finally measured by the receiver (see Fig. 1). We assume that Alice’s apparatus is very close to the disk, so that no significant source of noise is present in the gap between the disk and the
decoder. However, we assume that non-negligible noise comes from the thermal bath which is generally present at the other side of the disk. For this reason, the reflected system \( R \) combines the signal system \( S \) with a bath system \( B \) of \( M \) modes. These environmental modes are assumed to be in a tensor product of thermal states, each one with \( N_B \) mean photons (white thermal noise). In this model we identify five basic parameters: the reflectivities of the memory \( \{ r_0, r_1 \} \), the temperature of the bath \( N_B \), and the profile of the signal \( \{ M, N_S \} \), which is given by the number of signals \( M \) and the signal-energy \( N_S \) (see Fig. 1).

In general, for a fixed input state \( \rho \) at the transmitter (systems \( S, I \)), Alice will get two possible output states \( \sigma_0 \) and \( \sigma_1 \) at the receiver (systems \( R, I \)). These output states are the effect of two different quantum channels, \( \mathcal{E}_0 \) and \( \mathcal{E}_1 \), which depend on the bit \( u = 0, 1 \) stored in the target cell. In particular, we have \( \sigma_u = (\mathcal{E}_u \otimes I)(\rho) \), where the conditional channel \( \mathcal{E}_u \) acts on the signal system, while the identity channel \( I \) acts on the idler system. More exactly, we have \( \mathcal{E}_u = \mathcal{R}_u \otimes \mathcal{M} \), where \( \mathcal{R}_u \) is a one-mode Gaussian channel and, in particular, an attenuator channel with a bit-dependent linear loss \( r_u \) and a fixed thermal noise \( N_B \). Now, the minimum error probability \( P_{err} \) affecting the decoding of \( u \) is just the error probability affecting the statistical discrimination of the two output states, \( \sigma_0 \) and \( \sigma_1 \), via an optimal receiver. This quantity is equal to \( P_{err} = [1 - D(\sigma_0, \sigma_1)]/2 \), where \( D(\sigma_0, \sigma_1) \) is the trace distance between \( \sigma_0 \) and \( \sigma_1 \). Clearly, the value of \( P_{err} \) determines the average information which is decoded from each cell of the memory. This quantity is equal to \( J = 1 - H(P_{err}) \), where \( H(x) := -x \log_2 x - (1 - x) \log_2 (1 - x) \) is the usual formula for the binary Shannon entropy. In the following, we compare the performance of decoding in two paradigmatic situations, one where the transmitter is in a classical state (classical transmitter) and one where the transmitter is described by a non-classical state (quantum transmitter). In particular, we show how a quantum transmitter with EPR correlations is able to outperform every classical transmitter (see Fig. 1).

First let us consider a classical transmitter. A classical transmitter with \( M \) signals and \( L \) idlers is described by a state \( \rho = \int d\alpha \mathcal{P}(\alpha) |\alpha\rangle \langle \alpha| \), where \( \mathcal{P}(\alpha) \) is a probability distribution of multimode coherent states \( |\alpha\rangle \langle \alpha| = \bigotimes_{k=1}^{M+L} |\alpha_k\rangle \langle \alpha_k| \). Given this transmitter, we consider the corresponding error probability \( P_{err}^{\text{class}} \) which affects the reading of a target cell. Remarkably, this error probability is lower-bounded by a quantity which depends on the signal profile \( \{ M, N_S \} \), but not from the number \( L \) of the idlers and the explicit expression of \( \mathcal{P}(\alpha) \). In fact, we can prove that

\[
P_{err}^{\text{class}} \geq \mathcal{C}(M, N_S) := \frac{1 - \sqrt{1 - F(N_S)^M}}{2},
\]

where \( F(N_S) \) is the fidelity between \( \mathcal{R}_0(|N_S^{1/2}\rangle|N_S^{1/2}\rangle) \) and \( \mathcal{R}_1(|N_S^{1/2}\rangle|N_S^{1/2}\rangle) \), the two possible outputs of the single-mode coherent state \( |N_S^{1/2}\rangle|N_S^{1/2}\rangle \). As a consequence, all the classical transmitters with signal profile \( \{ M, N_S \} \) retrieve an information which is upper-bounded by \( J_{\text{class}} := 1 - H(\rho) \).

Now, let us construct a transmitter having the same signal profile \( \{ M, N_S \} \), but possessing EPR correlations between signals and idlers. The prototype of an EPR source is the two-mode squeezed vacuum state. In the number-ket representation, this state is defined by \( |\xi\rangle = (\cosh \xi)^{-1} \sum_{n=0}^{\infty} (\tanh \xi)^n |n\rangle_s |n\rangle_i \), where \( \xi \) is the squeezing parameter and \( \{ s, i \} \) is an arbitrary pair of signal and idler modes. In particular, \( \xi \) quantifies the signal-idler entanglement and gives the signal-energy by the relation \( N_S = \sinh^2 \xi \). Then, our quantum transmitter is realized by taking \( M \) identical copies of this state, i.e., \( \rho = |\xi\rangle \langle \xi|^{\otimes M} \). Given this transmitter, we consider the corresponding error probability \( P_{err}^{\text{quant}} \) affecting the reading of a target cell. This quantity is upper-bounded by the quantum Chernoff bound \( [1] \), i.e.,

\[
P_{err}^{\text{quant}} \leq \mathcal{Q}(M, N_S) := \frac{1}{2} \left[ \inf_{s \in (0,1)} \text{Tr}(\theta_s^M \theta_1^{1-M}) \right]^{M},
\]

where \( \theta_u := (\mathcal{R}_u \otimes I)(|\xi\rangle \langle \xi|) \). Since \( \theta_0 \) and \( \theta_1 \) are Gaussian states, we can write their normal-mode decompositions and derive the quantum Chernoff bound using the symplectic formula of Ref. [13]. Then, we can easily compute a lower bound \( J_{\text{quant}} := 1 - H(\mathcal{Q}(M, N_S)) \) for the information which is decoded via this quantum transmitter.

In order to show an improvement with respect to the classical case, it is sufficient to prove the positivity of the “information gain” \( G := J_{\text{quant}} - J_{\text{class}} \). This quantity is in fact a lower bound for the average information per cell which is gained by using the EPR quantum transmitter instead of every classical transmitter. Roughly speaking, the value of \( G \) estimates the number of bits per cell which are gained using the quantum reading. In general, \( G \) is a function of all the basic parameters of the model, i.e., \( G = G(M, N_S, r_0, r_1, N_B) \). Numerically, we can easily find signal profiles \( \{ M, N_S \} \), classical memories \( \{ r_0, r_1 \} \), and thermal baths \( N_B \), for which we have the quantum effect \( G > 0 \). Some of these values are reported in the following table.

| \( M \) | \( N_S \) | \( r_0 \) | \( r_1 \) | \( N_B \) | \( G \) (bits/cell) |
|---|---|---|---|---|---|
| 1 | 3.5 | 0.5 | 0.95 | 0.01 | \( 6.2 \times 10^{-3} \) |
| 10 | 1 | 0.2 | 0.8 | 0.01 | \( 3.4 \times 10^{-2} \) |
| 100 | 0.1 | 0.25 | 0.85 | 0.01 | \( 5.9 \times 10^{-2} \) |
| 200 | 0.1 | 0.6 | 0.95 | 0.01 | \( 2.2 \times 10^{-2} \) |
| \( 2 \times 10^2 \) | 0.01 | 0.995 | 1 | 0 | 0.99 |

Notice that we can find critical situations where \( G \approx 1 \), i.e., every classical reading of the memory does not decode any information whereas a quantum reading is able
to retrieve all of it. As shown in the last row of the table, this situation can occur when both the reflectivities of the memory are very close to 1. Apart from this singular scenario, the other values in the table can be realized with current technology. In particular, a non-trivial gain of 0.22 bits per cell can be realized using a small number of low-energy signals ($M = 200, N_S = 0.1$) shined over a memory with reasonable reflectivities (60% and 95%) in a low-temperature thermal bath ($N_B = 0.01$). Notice that, if we consider a memory of 4.7GB (standard size of a DVD), an information gain of 0.22 bits per cell corresponds to a global difference of at least 1GB [14].

From the first row of the table, we can acknowledge another remarkable fact: for a land-reflectivity $r_1$ sufficiently close to 1, one signal with few photons can give a positive gain. In other words, the use of a single, but sufficiently entangled, EPR source $|\xi\rangle\langle\xi|$ can outperform every classical transmitter, which uses a signal mode with the same energy but potentially infinite idler modes.

According to our numerical investigation, quantum reading is generally more powerful when the land-reflectivity is sufficiently high (i.e., $r_1 \gtrsim 0.8$). For this reason, it is very important to analyze the scenario in the limit of ideal land-reflectivity ($r_1 = 1$). Let us call “ideal-land memory” a classical memory with $r_1 = 1$. Clearly, this memory is completely characterized by the value of its pit-reflectivity $r_0$. For ideal-land memories, the quantum Chernoff bound of Eq. (12) takes the analytical form

$$Q = \frac{1}{2} \left[1 + (1 - \sqrt{r_0})N_S\right]^2 + N_B(2N_S + 1)(1-r_0)\right]^{-M},$$

and the classical bound of Eq. (11) can be computed using $F(N_S) = \gamma^{-1}\exp[-\gamma^{-1}(1 - \sqrt{r_0})^2N_S]$, where $\gamma := 1 + (1 - r_0)N_B$. Using these formulas, we can study the behavior of the gain $G$ in terms of the remaining parameters $\{M, N_S, r_0, N_B\}$. In particular, we can always find signal profiles $\{M, N_S\}$ such that $G > 0$. In fact, let us consider an ideal-land memory with a generic $r_0 \in [0, 1]$ in a generic thermal bath $N_B \geq 0$. For a fixed signal-energy $N_S$, we consider the minimum number of signals $M(N_S)$ above which $G > 0$ [15]. This critical number can be defined independently from the thermal noise $N_B$ (via an implicit maximization over $N_B$). Then, for a given value of the energy $N_S$, the critical number $M(N_S)$ is a function of $r_0$ alone, i.e., $M(N_S) = M(N_S)(r_0)$. Its behavior is shown in Fig. 2 for different values of the energy.

It is remarkable that, for low-energy signals ($N_S = 0.01 \pm 1$ photons), the critical number $M(N_S)(r_0)$ is finite for every $r_0 \in [0, 1)$. This means that, for ideal-land memories and low-energy transmitters, there always exists a finite number of signals above which a quantum reading of the memory is more accurate than every classical strategy. In the considered low-energy regime, $M(N_S)(r_0)$ is relatively small for almost all the values of $r_0$, except for $r_0 \to 1$ where $M(N_S)(r_0) \to \infty$ [16]. Apart from this divergence at $r_0 = 1$, in all the other points $r_0 \in [0, 1)$, the critical number $M(N_S)(r_0)$ decreases for increasing energy $N_S$ (see Fig. 2). In particular, for $N_S = 1$ photon, we have $M(N_S)(r_0) \approx 1$ for most of the reflectivities $r_0$. In other words, for a signal-energy equal to one photon, a single EPR source is sufficient to provide a positive gain for most of the ideal-land memories. However, the decreasing trend of $M(N_S)(r_0)$ does not continue for higher energies ($N_S \geq 1$). Just after one photon, $M(N_S)(r_0)$ starts to increase in $r_0 = 0$, where a second asymptote appears for $N_S \gtrsim 2.5$ photons [17]. As a consequence, the use of high-energy signals ($N_S \gtrsim 2.5$) does not assure a positive gain in the quantum reading of memories with extremal reflectivities $r_0 = 0$ and $r_1 = 1$.

According to our derivations, for energies of about one photon, we can have a positive gain even in the case of “monochromatic reading”, where only one signal mode is shined on a target cell ($M = 1$). In other words, a monochromatic quantum reading, corresponding to the use of a single EPR source, can outperform every monochromatic classical reading, i.e., every classical transmitter using one signal mode and virtually infinite idler modes. Here, we explicitly compare these monochromatic readings in the case of ideal-land memories under definite conditions of temperature. In this case ($M = r_1 = 1$), the information gain depends on three parameters only, i.e., $G = G(N_S, r_0, N_B)$. For a fixed energy $N_S$, we can identify a subset of points in the $(r_0, N_B)$-plane where $G > 0$. These points provide values of pit-reflectivity $r_0$ and temperature $N_B$ for which a monochromatic quantum reading of the memory outperforms every monochromatic classical reading with the same signal-energy $N_S$. These subsets are explicitly shown in Fig. 3 for several values of $N_S$. According to Fig. 3, a monochromatic quantum reading with a sufficiently entangled EPR source (irradiating a signal of one-photon) represents an unbeatable strategy.
for almost all the ideal-land memories \(r_0\) in almost all the temperatures \(N_B\). For higher energies \(N_S > 1\), the situation does not necessarily improve. In fact, as we can see from Fig. 3, the region with positive gain tends to shrink from \(r_0 = 0\) towards \(r_0 = 1\). This behavior is in agreement with the appearance of the high-energy asymptote which is discussed in Fig. 2.

In conclusion, we have considered a simple model of classical memory, composed by mirrors of different reflectivities, and we have shown that EPR correlations can dramatically improve the accuracy in retrieving the stored information. This enhancement can be tested with current technology, and generally holds for memories with high land-reflectivities \(r_1 \gtrsim 0.8\), where a single EPR source can be sufficient to obtain a positive information gain. In the limit of ideal land-reflectivity \(r_1 = 1\), we have computed the critical number of signals above which a quantum reading of the memory outperforms every classical reading. For low-energy signals \((0.01 \pm 1\) photons\) this critical number is finite and relatively small for every ideal-land memory. In particular, a monomacromatic quantum reading with a single but, sufficiently entangled, EPR source (corresponding to a one-photon signal) represents a superior decoding strategy for almost all the ideal-land memories in a wide range of temperatures. Finally, our results indicate non-trivial possibilities for improving the reading of information from optical storage devices. Increasing the accuracy of this reading means reducing the error correction overhead in these memories and, therefore, increasing their capacities for storing logical data. Future directions may consider the realistic structure of present-day DVDs and improve the corresponding interferometric systems used for the reading. In particular, in this realistic scenario, clusters of cells are decoded in each query of the memory.

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[6] For two bosonic modes \(AB\) with quadratures \(q_A, p_A, q_B\) and \(p_B\), one can define the two operators \(q_- := (q_A - q_B)/\sqrt{2}\) (relative position) and \(p_+ := (p_A + p_B)/\sqrt{2}\) (total momentum). Then, the system has EPR correlations (in these operators) if \(V(q_-) + V(p_+) < 2\nu_0\), where \(V(\cdot)\) is the variance, and \(\nu_0\) is the standard quantum limit \((\nu_0 = 1\) in this paper.)

[7] See, e.g., wikipedia (http://en.wikipedia.org/wiki/DVD).

[8] By definition, a bosonic state is called “classical” (“non-classical”) when its \(P\)-representation is positive (non-positive). This means that a classical state can be represented by a mixture of multimode coherent states.

[9] E. C. G. Sudarshan, Phys. Rev. Lett. 10, 277 (1963); R. J. Glauber, Phys. Rev. 131, 2766 (1963).

[10] For instance, in today’s DVDs, the error correction overhead is about 15\% (Reed-Solomon codes). This overhead tends to increase for increasing storage density [18].

[11] In today’s CDs and DVDs, a pit is actually a raised bump on the reflective surface while a land is just a flat portion. The reading of information is based on the interference between projected and reflected light [18].

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[14] For these values, the quantum reading retrieves at least 3.73GB, while every classical reading retrieves at most 2.67GB (out of 4.7GB).

[15] To be precise, the critical number \(M(N_S)\) that we consider is a solution of the equation \(G = 0\). From this real value we derive the minimum number of signals (which is an integer) by taking its ceiling function \(\lceil M(N_S) \rceil\).

[16] In fact, for \(r_0 \approx 1\), we derive \(M(N_S)(r_0) \approx 4N_S(2N_S + 1)(1 - r_0)^{-1}\), which diverges at \(r_0 = 1\). Such a divergence is expected, since we must have \(p_{corr} = P_{class} = 1/2\) for \(r_0 = 1\).

[17] In fact, for \(N_S \geq 1\), we can derive \(M(N_S)(0) \approx (\ln 2)[2\ln(1 + N_S) - N_S]^{-1}\), which is increasing in \(N_S\), and becomes infinite at \(N_S \approx 2.5\).

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