Real-Time Distributed Model Predictive Control with Limited Communication Data Rates

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Abstract—The application of distributed model predictive controllers (DMPC) for multi-agent systems (MASs) necessitates communication between agents, yet the consequence of communication data rates is typically overlooked. This work focuses on developing stability-guaranteed control methods for MASs with limited data rates. Initially, a distributed optimization algorithm with dynamic quantization is considered for solving the DMPC problem. Due to the limited data rate, the optimization process suffers from inexact iterations caused by quantization noise and premature termination, leading to sub-optimal solutions. In response, we propose a novel real-time DMPC framework with a quantization refinement scheme that updates the quantization parameters on-line so that both the quantization noise and the optimization sub-optimality decrease asymptotically. To facilitate the stability analysis, we treat the sub-optimally controlled MAS, the quantization refinement scheme, and the optimization process as three interconnected subsystems. The cyclic-small-gain theorem is used to derive sufficient conditions on the quantization parameters for guaranteeing the stability of the system under a limited data rate. Finally, the proposed algorithm and theoretical findings are demonstrated in a multi-AUV formation control example.

I. INTRODUCTION

Model predictive control can systematically handle constraints in determining the control action. In multi-agent systems (MASs) without central decision-makers, distributed model predictive control (DMPC) offers a viable solution. To apply DMPC, the agents rely on communication networks for solving the underlying distributed optimization problems at every time step. Thus, it is necessary to consider the communication challenges ubiquitous in communication networks [1], e.g., communication delay, package loss, and limited communication data rate. In this paper, we focus on implementing DMPC for MASs with a limited data rate and aim to provide stability guarantees for the closed-loop system.

In [2], [3], distributed optimization algorithms that consider quantization noise arising from limited communication data rates are proposed and provide convergence guarantees when suitable quantization parameters are adopted. However, the fixed data rate not only causes inexact iterations but also limits the number of optimization iterations allowed, thus leading to sub-optimal solutions. In turn, the stability of the sub-optimally controlled system may not be guaranteed.

Sub-optimal MPC has been studied extensively, starting with [4]. In [5], [6], and [7], different MPC schemes are combined with real-time stability-guaranteeing mechanisms that are verified on-line. In [8] and [9], the small-gain theorem is used to derive sufficient conditions for the stability of dynamics-optimizer systems, where the former assumes Lipschitz continuity of the MPC problem and the latter considers MPC with only input constraints. In [10], the small-gain theorem is applied for stability analysis of a MAS interconnected with an output coordinator and nonlinear controllers. The works [11] and [12] involve on-line stopping criteria that guarantee the stability of sub-optimal DMPC. Despite their efficacy in dealing with sub-optimality, these methods are not readily applicable in DMPC with limited communication data rates. There are DMPC schemes designed considering communication data rate. The work [13] compresses shared data with neural networks to save communication data rate and proves input-to-state-practical stability. In [14], a single-chain communication topology is considered and stability is guaranteed with a sufficiently long prediction horizon. While the above methods aim to reduce the communication data rate required for solving the DMPC problem, they do not explicitly consider the effect of the limited data rate on the closed-loop stability.

In this work, we propose a DMPC framework for controlling a MAS under a limited data rate, with explicitly guaranteed closed-loop stability. Our previous work [15] is an initial step in this direction, which solves DMPC problems using the distributed optimization algorithm in [2] with quantization parameters determined off-line considering a limited data rate. Compared to [15], we refine the quantization parameters on-line to adapt to the change in sub-optimality and use a different stability analysis method, which allows for stronger stability guarantees. Specifically, we make the following main contributions:

- We propose a novel real-time DMPC framework with a quantization refinement scheme. The scheme includes an off-line stage that designs initial quantization parameters given a limited communication data rate, and an on-line stage (illustrated in Fig. 1) that implements the quantization refinement scheme, the optimization process, and the DMPC controller.
- Given a limited data rate, we derive sufficient, off-line-determined conditions on the quantization parameters such that the system controlled by the proposed framework is recursively feasible and stable. Specifically, we consider the sub-optimally controlled MAS, the quantization refinement scheme, and the optimization process as three interconnected subsystems. This formulation facilitates stability analysis of the closed-loop system using the cyclic-small-gain theorem, leading to input-to-state stability (ISS) guarantee w.r.t a quantization refinement.
For an optimization problem and an optimization algorithm, let $z^*$ and $z^k \in \mathbb{R}^{m\times}$ denote the optimal and $k$-iteration solutions, respectively. Let the subscript $t$ denote the discrete time step of a controlled system. Let $I$ denote the identity matrix and $I_d$ denote the identity function. Let $S^+_n$ denote the set of all positive definite matrices with size $n \times n$. For a vector $x \in \mathbb{R}^{m\times}$ and a matrix $H \in S^+_n$, let $||x||_\infty$, $||x||$, and $||x||_2$ denote the infinity norm, 2-norm, and the weight 2-norm of $x$, respectively. Note that $||x||_\infty \leq ||x|| \leq \sqrt{m} ||x||_2$. Let $||H||$ denote the induced norm of $H$. Let $\lambda(H)$ and $\mu(H)$ denote the smallest and largest eigenvalues of $H$, respectively. For a continuous function $\alpha(s)$, $\alpha(s)$ is in $K$ if $\alpha(0) = 0$ and $\alpha(s) > 0$. Furthermore, $\alpha(s)$ is in $K_{\infty}$ if $\alpha(s)$ is unbounded. Given functions $\alpha_1(w) : \mathcal{W} \rightarrow \mathcal{Y}$ and $\alpha_2(w) : \mathcal{W} \rightarrow \mathcal{Z}$, the operators $|\cdot|$ and $[\cdot]$ denote the ceiling and floor operations, respectively. The operator $\text{blkdiag}(H_1, \ldots, H_n)$ yields a matrix $H$ containing the matrices $H_1, \ldots, H_n$, respectively. We denote the projection of a vector $z \in \mathbb{R}^{m\times}$ onto the set $C \subseteq \mathbb{R}^{m\times}$ as $\text{Proj}_C(z) := \text{argmin}_{z \in C} \|z - z\|$. Let the uniform quantizer be $\bar{z}_j := \min(\bar{z}_j, \eta)$, where $\bar{x}$, quantization interval $l$, and bit number $n$. The cyclic-small-gain theorem presented below is relevant to the stability analysis.

**Lemma 1 (Lemma 1, [16])** Consider an interconnected system composed of $M$ subsystems with discrete dynamics

$$x_{i,t+1} = f_i(x_{i,t}, x_{j,t}, u_{i,t}), \quad \forall i, j \in \{1, \ldots, M\}, i \neq j,$$

where $x_{i,t}$, $x_{j,t}$, and $u_{i,t}$ denote the states, interconnected states, and external inputs, respectively. Suppose each subsystem admits a continuous $\mathcal{ISS}-\mathcal{Lyapunov}$ function satisfying

$$\dot{\alpha}_i(t) = V_i(x_{i,t}) \leq \alpha_{i,u}(||x_{i,t}||),$$

$$V_i(x_{i,t+1}) - V_i(x_{i,t}) \leq -\bar{\alpha}_i(V_i(x_{i,t})), \quad \forall j \in \{1, \ldots, M\}, j \neq i,$$

where $\alpha_{i,u}, \alpha_{i,v}, \alpha_{i} \in K_{\infty}$. (Id $- \alpha_i$) is in $K_{\infty}$ and $\alpha_{i,u}, \alpha_{i,v} \in \mathbb{R} \cup \{0\}$. Then, the interconnected system is ISS with $u_{i} = [u_{i,1}, \ldots, u_{i,M}]^T$ as the input, if there exist positive definite functions $\mu_i$ satisfying $\text{Id} - \mu_i = \in K_{\infty}$ such that the ISS gain functions $\mathcal{X}_{i,j} := \alpha_{i,j}^{-1} \circ (\text{Id} - \mu_i)^{-1} \circ \gamma_i(t)$ satisfy

$$\mathcal{X}_{i,j} \circ \mathcal{X}_{j,j'} \circ \cdots \circ \mathcal{X}_{j,2} \circ \text{Id} < \text{Id}$$

for each $r = \{2, \ldots, M\}$ and all $1 \leq j < j' < j'' < r$, i.e., for all the interconnected cycles in the system.

**III. DISTRIBUTED OPTIMIZATION WITH QUANTIZATION**

We solve a class of DMPC problems over a system of $M$ agents that communicate synchronously under limited communication data rates via an undirected graph $\mathcal{G} = (\mathcal{M}, \mathcal{E})$, with vertex set $\mathcal{M} = \{1, \ldots, M\}$ and edge set $\mathcal{E} \subseteq \mathcal{M} \times \mathcal{M}$. If $(i,j) \in \mathcal{E}$, then agent $i$ and $j$ are neighbours and communicate with each other. Let $\mathcal{N}_i \triangleq \{j \mid (i,j) \in \mathcal{E}\}$ denote the set of neighbors of agent $i$, including agent $i$ itself. Let $d$ denote the degree of agent $i$. The DMPC problems can be formulated as the following distributed optimization problem:

$$\min_z f(z) = \sum_{i \in \mathcal{M}} f_i(z_{\mathcal{N}_i}),$$

s.t. $z_i \in C_i$, $z_{\mathcal{N}_i} = E_i z_i$, $z_j = F_{ij} z_{\mathcal{N}_j},$

where $z = [z_1^T, \ldots, z_M^T] \in \mathbb{R}^{m\times}$ and $z_i \in \mathbb{R}^{m\times}$ denote the global and local decision variables, respectively. Let $E_i \in \mathbb{R}^{m\times \times m\times}$, $F_{ij} \in \mathbb{R}^{m\times \times m\times}$ be the selection matrices defined w.r.t. $\mathcal{N}_i$. Let $z_{\mathcal{N}_i}$ be the stacked decision variables of the agents in $\mathcal{N}_i$. If $\max_{i \in \mathcal{M}} m_{z_i}$ denote the largest size of local decision variables. Let $C_i$ be the local constraint set.

**Assumption 1** All the local constraint sets $C_i$ are convex.

**Assumption 2** All local cost functions $f_i(\cdot)$ are strongly convex with Lipschitz continuous gradients $\nabla f_i(\cdot)$ satisfying $\|\nabla f_i(z_{i,t}) - \nabla f_i(z_{i,t})\| \leq L_f \|z_{i,t} - z_{i,t}\|$ for any $z_{i,t}$ and $\eta_i(t)$. Assume $z_{i,t}$ is strongly convex with a convexity modulus $\alpha_{f,i}$ and a Lipschitz continuous gradient $\nabla f_i(\cdot)$ satisfying $\|\nabla f(z_{i,t}) - \nabla f(z_{i,t})\| \leq L_f \|z_{i,t} - z_{i,t}\|$.

To solve [5], we consider Alg. 1, modified from the distributed optimization algorithm with dynamic quantization proposed in [2]. Uniform quantizers $Q_h^{l_{\alpha}}$ and $Q_h^{l_{\beta}}$ are used by agent $i$ for transmitting local decision variables $z_i^{l_{\alpha}}$ and gradients $\nabla f_i^{l_{\alpha}}$ at iteration $k$, respectively. They are parameterized by the quantization bit number $n$, quantization interval $l_{\alpha}$, and mid-values $z_i^{l_{\alpha}}$ and $\nabla f_i^{l_{\alpha}}$, respectively. The quantization interval is progressively updated according to $l_{\alpha} = \alpha_{f,i} \rho$, with base quantization interval $C_i$ and the shrinkage constant $\rho \in (1 - \gamma, 1)$, $\gamma = \alpha_{f,i} / L_f \in (0, 1)$. The mid-values are updated according to $z_i^{l_{\alpha}} = z_i^{l_{\alpha}} \gamma$ and $\nabla f_i^{l_{\alpha}} = \nabla f_i^{l_{\alpha}} \gamma$. The projections $\text{Proj}_{\mathcal{N}_i}$ and $\text{Proj}_i$ ensure the quantized variable $z_{\mathcal{N}_i}$ and the local solution computed from quantized gradient $\nabla f_i^{l_{\alpha}}$ are feasible despite the quantization noises, for all iteration $k > 0$. The constraint set $C_{\mathcal{N}_i} = \cap_{i \in \mathcal{M}} C_i$ represents the constraints of all the agents in $\mathcal{N}_i$. Let $\eta$ denote the optimization step-size. Each blue block in Fig. 1 corresponds to the implementation of Alg. 1 for $n = 2$ and $K = 3$.

The main difference between Alg. 1 and [2] is we initialize the local quantizers $Q_h^{l_{\alpha}}$ and $Q_h^{l_{\beta}}$, of agent $i$ with a local base quantization interval $C_i$ instead of a global one. This change eases the need for the agents to agree on a base quantization interval through centralized communication. Correspondingly, we propose a new condition on the base quantization interval $C_i$ and the bit number $n$, and a new convergence bound for Alg. 1.

**Lemma 2** Consider Alg. 1 applied to solve problem [5]. Suppose Assumptions 1 and 2 hold. Let $z^0$ be the initial solution and $\Delta z^0 :=$
\[ z^0 - z^* \] be the initial gap. If the quantization bit number \( n \) satisfies
\[ 2^n \rho - \sqrt{dn} > 0, \]
and the local base quantization interval \( C_i \) satisfies
\[ a_1 \| \Delta z^i \| + a_2 C_i \leq C_i, \quad i \in M, \]
then the sub-optimality \( \Delta z^K := z^K - z^* \) generated by Alg. 1 satisfies
\[ \| \Delta z^K \| \leq \rho \| (\| \Delta z^0 \| + a_3 C_i), \]
where \( C_i := \sum_{j \in M} C_i \) and
\[ a_1 := \max \left( \frac{2^{n+1}}{2^n \rho - \sqrt{dn}}, \frac{1}{2}, (1 + \rho)/\rho \right), \]
\[ a_2 := \max \left( \frac{\max_{i \in M} (L_i)(1 + \rho)(2^{n+1}a_3 + \sqrt{m})}{2^n \rho - \sqrt{dn}}, \frac{(2^{n+1} + 1)(a_3 + \sqrt{m})/\rho}{(2^n + 1)/\rho} \right), \]
\[ a_3 := \frac{\sqrt{m}(M \max_{i \in M} (L_i) + \sqrt{L_f} + \sqrt{d}) \rho}{L_f (\rho + \gamma - 1)(1 - \gamma)2^{n+1}}. \]

Proof: The proof contains two parts: (i) inspired by Lemma 3.10 [2], we develop a one-step bound on \( \| \Delta z^p \| \) in the form of (8) which holds if the quantization errors are bounded at iteration \( p \). (ii) inspired by Lemma 3.17 [2], we use induction to show the quantization errors are bounded at every iteration \( k \) and the one-step convergence bound holds at every iteration \( k \).

(i) We first construct the one-step bound. For a uniform quantizer \( Q(z) \) with quantization interval \( l \) and bit number \( n \), if the input \( z \) falls inside the quantization interval of \( Q(z) \), then it holds that
\[ \| z - Q(z) \| \leq \frac{l}{2^n + 1}. \]

When \( k = p \), if \( z^p_i \) and \( \nabla f_i^p \) fall inside the quantization intervals of \( \mathcal{Q}_i^0 \) and \( \mathcal{Q}_i^p \), respectively, then the quantization errors \( \alpha_i^p = z^p_i - z^p_i \) and \( \beta_i^p = \nabla f_i^p - \nabla f_i^p \) satisfy
\[ \| \alpha_i^p \| \leq \sqrt{m} \| \alpha_i^p \| \\leq \frac{\sqrt{m} \rho_c \| C_i \|}{2^{n+1}}, \]
\[ \| \beta_i^p \| \leq \frac{1}{\sqrt{n}} \sum_{j \notin N_i} \| \beta_i^p \| \\leq \frac{\sqrt{d \| \beta_i^p \| \| C_i \|}{2^{n+1}}}, \]
respectively. From Lemma 3.8 [2], the error sequences from Alg. 1,
\[ e^p = \sum_{i \in M} E_i^p \nabla f_i (z^p_i) + \sum_{i \in M} E_i^p \beta_i^p - \sum_{i \in M} E_i^p \nabla f_i (N_i) \]
and \( e^p = \frac{1}{2} \| z^p - z^p \| \) satisfy
\[ \| e^p \| \leq \sum_{i \in M} L_i \sum_{j \notin N_i} \| \alpha_j^p \| + \sum_{i \in M} \| \beta_i^p \| \]
\[ \sqrt{\varepsilon^p} \leq \frac{\sqrt{2}}{2} \sum_{i \in M} \| e_i^p \|, \]
respectively. Then, applying the bounds in (11) and (12) to (13) and (14), we obtain
\[ \| e^p \| \leq \sqrt{m} \| \sum_{i \in M} \left( \max_{j \notin N_i} (L_i) \sum_{i \in M} C_j + \sqrt{dC_i} \right) \rho^p \leq C_q \rho^p, \]
\[ \sqrt{\varepsilon^p} \leq \frac{\rho^p \sqrt{2dn}}{2^{n+2}} C_i = C_q \rho^p, \]
where \( C_q = \frac{\sqrt{m}}{2^n \rho} (\max_{i \in M} (L_i) \chi C_i + \sqrt{dC_i}) \) and \( C_b = \frac{\sqrt{m}}{2^n \rho} \varepsilon^p \). In Proposition 2.5 of [2], it is proven that
\[ \| z^{p+1} - z^* \| \leq (1 - \gamma) \| z^p - z^* \| \]
\[ + \frac{L_f}{2} \| e^p \| + \sqrt{\frac{2}{L_f}} \varepsilon^p, \]
averages by plugging (15) and (16) to bound \( \| e^p \| \) and \( \varepsilon^p \), respectively, and using the fact that \( \| z^p - z^* \| \leq (1 - \gamma) \| z^p - z^* \| \) and
\[ \| z^{p+1} - z^* \| \leq \| z^p - z^* \| + \sqrt{\frac{2}{L_f}} \varepsilon^p, \]
which accounts for the inexactness in an iteration of the inexact proximal-gradient method. Plugging (15) into the r.h.s. of (17), using (15) and (16) to bound \( \| e^p \| \) and \( \varepsilon^p \), respectively, and using the fact that \( 0 < (1 - \gamma) < \rho < 1 \), we obtain
\[ \| z^{p+1} - z^* \| \leq \| z^p - z^* \| + \frac{L_f}{2}(1 - \gamma) \| z^p - z^* \| \]
\[ + \frac{L_f}{2} \| e^p \| + \sqrt{\frac{2}{L_f}} \varepsilon^p, \]
\[ \| z^{p+1} - z^* \| \leq \| z^p - z^* \| + \frac{L_f}{2} \| e^p \| + \sqrt{\frac{2}{L_f}} \varepsilon^p, \]
\[ \| z^{p+1} - z^* \| \leq \| z^p - z^* \| + \| e^p \| + \varepsilon^p, \]
\[ \| z^{p+1} - z^* \| \leq \| z^p - z^* \| + \| e^p \| + \varepsilon^p, \]
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\[ \| z^{p+1} - z^* \| \leq \| z^p - z^* \| + \| e^p \| + \varepsilon^p, \]
the property of geometric series is used to obtain (22) by bounding \( \| e^p \| \) and \( \varepsilon^p \) to bound the quantization errors in (21).

(ii) We prove by induction that \( \| z^p \| \) and \( \nabla f_i^p \) fall inside the quantization intervals of \( \mathcal{Q}_i^0 \) and \( \mathcal{Q}_i^p \), respectively, for all \( k \geq 0 \), so that the one-step bound in (23) holds at every step.

Base case: When \( k = 0 \), we initialize \( C_i > 0 \) and \( z_i - z_i = 0 \) \( \forall i \in M \), which satisfies
\[ \| z_i - z_i \| = \| z_i - z_i \| = \| z_i - z_i \| \leq \frac{L_f}{2} = C_i. \]
\[ \| \nabla f_i - \nabla f_i \| = \| \nabla f_i - \nabla f_i \| = \| \nabla f_i - \nabla f_i \| \leq L_f/2 \]
\[ \| \nabla f_i - \nabla f_i \| = \| \nabla f_i - \nabla f_i \| = \| \nabla f_i - \nabla f_i \| \leq L_f/2. \]

Thus, bounded quantization errors imply the input variable are bounded by the quantization intervals of the quantizers.

Induction case: When \( k = p \geq 0 \), suppose \( \| z_i - z_i \| \leq L_f/2 \)
and \( \| \nabla f_i - \nabla f_i \| \leq L_f/2 \) for \( k \leq p \). We need to prove
\[ \| z_i - z_i \| \leq L_f/2 \]
\[ \| \nabla f_i - \nabla f_i \| \leq L_f/2. \]
\[ \| \nabla f_i - \nabla f_i \| \leq L_f/2. \]
We first prove (27). The quantization error from Alg. 1 satisfies
\[ \| z_i - z_i \| \leq \| z_i - z_i \| \leq \| z_i - z_i \| \leq \| z_i - z_i \|. \]

\[ \| z_i - z_i \| \leq \| z_i - z_i \| \leq \| z_i - z_i \| \leq \| z_i - z_i \|. \]

\[ \| z_i - z_i \| \leq \| z_i - z_i \| \leq \| z_i - z_i \| \leq \| z_i - z_i \|. \]

\[ \| z_i - z_i \| \leq \| z_i - z_i \| \leq \| z_i - z_i \| \leq \| z_i - z_i \|. \]
where we used \(\|E_i\| = \|F_{ij}\| = 1\) to obtain (33). Since \(z_k^i - z^{p+1}_k\) \(\leq \frac{l_i}{2}\) \(\forall i \in M\) and \(\|\nabla f_i^k - \nabla f_i^0\| \leq \frac{l_i}{2}\) for \(k = p\) and \(k = p - 1\), we can bound \(\|\Delta z^{p+1}\|\) and \(\|\Delta z^p\|\) with (24) and use (11) to obtain

\[
\|z^{p+1}_i - z^{p+1}_i\|_\infty \\
\leq \rho^p \|\Delta z^p\| + \rho^0 \|\Delta z^0\| + a_3 C_S \\
\leq \rho^p \|\Delta z^p\| + \rho^0 \|\Delta z^0\| + a_3 C_S \\
+ \rho^p \frac{b}{2\mu} C_S, \\
\leq \frac{\rho^p + 1}{2} \|a_1\| \|\Delta z^0\| + a_2 C_S, \\
\leq \frac{\rho^p + 1}{2} \|a_1\| \|\Delta z^0\| + a_2 C_S
\]

where (35) holds from the definitions of \(a_1\) and \(a_2\) in (9), and the condition (6). Then, condition (7) implies (27) holds for all \(k \geq 0\).

Finally, we combine the base case and the induction case to conclude (26) and (27) hold for all \(k \geq 0\), i.e., the values \(z^k\) and \(\nabla f^k\) generated by Alg. 1 always fall inside the quantization intervals of \(Q_{n,i}^k\) and \(Q_{0,i}^k\) respectively. As a result, the solution \(z^k\) generated by Alg. 1 satisfies (24) for all \(k \geq 0\), i.e., (8) holds.

### IV. REAL-TIME DMPC WITH QUANTIZATION REFINEMENT

In this section, we propose a novel real-time DMPC framework with a quantization refinement scheme. We first introduce the DMPC problem we aim to solve under a limited communication data rate.

#### A. DMPC with Coupled Cost Functions

Consider a MAS with \(M\) agents whose dynamics are

\[
x_{i,t+1} = A_{i,t} x_{i,t} + B_{i} u_{i,t}, \quad \forall i \in M.
\]

with states \(x_{i,t} \in \mathbb{R}^{m_{i,t}}\), inputs \(u_{i,t} \in \mathbb{R}^{n_{i,t}}\), and \((A_{i,t}, B_{i})\) being controllable. Agent \(i\) can communicate locally with neighbouring agents in \(N_i\). The global dynamics of the MAS is

\[
X_{t+1} = AX_t + BU_t,
\]

where \(A = \text{blkdiag}(A_1, \cdots, A_M), B = \text{blkdiag}(B_1, \cdots, B_M), X_t = [x_{1,t}, \cdots, x_{M,t}]^\top\), and \(U_t = [u_{1,t}, \cdots, u_{M,t}]^\top\). We stabilize (51) to the origin using the following DMPC formulation:

\[
(X_t^*, U_t^*) = \arg \min_{X_t, U_t} \sum_{i=1}^M \left( \sum_{t=0}^{N-1} \|\nabla f_i(x_{i,t}(\tau))\| \right) \|u_{i,t}\|^p \\
+ \|\nabla f_i(x_{i,t}(N))\|^{p_i},
\]

s.t. \(x_{i,t}(0) = x_{i,t}\), \(\forall i \in M, \tau = 0, \cdots, N - 1\).

The local decision variables include the sequences \(x_{i,t} = [x_{i,t}^1(0), \cdots, x_{i,t}^i(N)\] \(\cdots, x_{i,t}^M(N - 1)]\), and \(u_{i,t} = [u_{i,t}^1(0), \cdots, u_{i,t}^i(N - 1)\] \(\cdots, u_{i,t}^M(N - 1)]\). Let \(x_{N_i,t}(\tau) = [x_{N_i,t}^1(\tau), \cdots, x_{N_i,t}^M(\tau)]\) and \(u_{N_i,t}(\tau) = [u_{N_i,t}^1(\tau), \cdots, u_{N_i,t}^M(\tau)]\) be the concatenated decision variables of the agents in \(N_i\) for \(\tau = 0, \cdots, N - 1\). The global decision variables include the sequences \(X_t = [X_t^1(0), \cdots, X_t^i(N)\] \(\cdots, X_t^M(N - 1)]\), and \(U_t = [U_t^1(0), \cdots, U_t^i(N - 1)]\), where \(X_t(\tau) = [x_{1,t}^1(\tau), \cdots, x_{N_i,t}^i(\tau)]\) for \(\tau = 0, \cdots, N\) and \(U_t(\tau) = [u_{1,t}^1(\tau), \cdots, u_{N_i,t}^i(\tau)]\) for \(\tau = 0, \cdots, N - 1\). Let \(X_t^*\) and \(U_t^*\) be the optimal solution. Local terminal cost \(\|\cdot\|_F\) and stage cost with coupling \(\|\cdot\|_H\) are considered, with \(H_t, Pi \in \mathbb{S}_{++}\). The convex sets \(C_{S,i}, C_{u,i}\), and \(C_{f,i}\) represent the local state, input, and terminal constraint sets, respectively. We define matrices \(Q, R, P \in \mathbb{S}_{++}\), which satisfy blockdiag(Q, R) = \(\sum_{i=1}^M E_i^\top H_i E_i\) and \(P = \text{blkdiag}(P_1, \cdots, P_M)\).

**Assumption 3** Consider the MAS (51). Let \(l(X, U) = \|X\|_F + \|U\|_R, l_f(X) = \|X\|_F, \text{ and } C_f = C_{f,1} \times \cdots \times C_{f,M}\). There exist local stabilizing control laws \(K_{f,i,\omega}\) such that the MAS (21) under \(\pi_f(X) = ([K_{f,1,\omega}], \cdots, [K_{f,M,\omega}])\) satisfies, \(V \in C_f:\n
\cdots + A^X + B^T \pi_f(X) - l_f(X) \leq -l(X, \pi_f(X)).\n
\]
The DMPC problem \( \text{(52)} \) can be formulated as \( \text{(5)} \), with local and global decision variables \( z_{i,t}, X_{i,t} \) and \( z_t = [X_t, U_t]^\top \), respectively. The local and global cost functions are \( f_i(z_{i,t}) = ||x_i(t)(X_i) + \sum_{i' \neq i} ||X_{i',t}(\tau), u_{i',t}(\tau)||_H \) and \( f(z_t) = ||z_t||_H = \sum_{i \in M} f_i(z_{i,t}) \), respectively, where \( H = \text{blkdiag}(I_{N-1} \otimes \text{blkdiag}(Q, R, P)) \), with \( \otimes \) the Kronecker product. The local constraint set \( C_i = C_{i,0} \times C_{i,1} \times \cdots \times C_{i,T} \) is convex and satisfies Assumption 1. Since \( H, P_i \in S_{++}, \) Assumption 2 is satisfied, with \( L_i = 2\max(X_i, P_i) \). Thus, \( \text{(52)} \) formulated as \( \text{(5)} \) can be solved using Alg. 1 with the convergence guarantee in \( \text{(8)} \).

Let \( z_t^* = [X_t^T, U_t^T]^\top \) and \( z^K = [X^K_t, U^K_t]^\top \) be the optimal and \( K_{\alpha} \)-iteration sub-optimal solutions, respectively. The corresponding optimal and sub-optimal control laws are

\[
\pi^K(X_t) := U^K_t = [u_{i,t}^T(0), \cdots, u_{M,t}^T(0)]^\top = \Xi z^K_t, \quad (53)
\]

\[
\pi(X_t) := U_t = [u_{i,t}^T(0), \cdots, u_{M,t}^T(0)]^\top = \Xi z_t, \quad (54)
\]

respectively, where \( \Xi \in \mathbb{R}^{m_u \times m_x} \) is a selection matrix. Let the optimal cost of \( \text{(52)} \) be \( V(X_t) := f(z_t^*) = ||z_t^*||_H^2 \).

Assumption 4 Consider the DMPC problem \( \text{(52)} \), there exists a Lipschitz constant \( L \) such that for two feasible initial states \( X_{i_1} \) and \( X_{i_2} \), it holds that

\[
||z_{i_1}^* - z_{i_2}^*|| \leq L||X_{i_1} - X_{i_2}||, \quad (55)
\]

where \( z_{i_1}^* \) and \( z_{i_2}^* \) are the optimal solutions of \( \text{(52)} \) with initial states \( X_{i_1} \) and \( X_{i_2} \), respectively.

Remark 1 Since \( \text{(52)} \) is a convex optimization problem parametrized by \( X_t \), L always exists [17]. When all the constraints in \( \text{(52)} \) are polytopic, the Lipschitz constant \( L \) exists and can be determined as the maximal gain of the explicit MPC solution of \( \text{(52)} \) in [18]. When the DMPC problem \( \text{(52)} \) only contains input constraints, the Lipschitz constant \( L \) can be determined analytically [9]. For more general problem setups, sampling-based methods can be used to estimate \( L \) with specified probability guarantee [7].

B. DMPC Framework with Quantization Refinement

Let \( T \) be the given communication data rate, defined as the number of bits available per time step for one agent to transmit to another agent. The communication constraint \( nK \leq T \) exists and limits the number of iterations \( K \) that Alg. 1 can be implemented and introduces quantization noise into Alg. 1. We assume sufficient computation resources are provided for carrying out all computations. To address the communication constraint, we propose a DMPC framework with a quantization refinement scheme in Alg. 2, which contains an off-line stage for quantization parameter design and an on-line stage for quantization refinement and control input computation.

In the off-line stage, we assume all agents have access to the global information. Given data rate \( T \), the quantization bit number \( n \) and iteration number \( K \) satisfying the communication constraint \( nK \leq T \) are determined (Step 1). Then, the global state \( X_0 \) is measured for computing the optimal solution \( z_0^* \) (Step 2). For initialization (Step 3), the warm-start solution \( z_0^K \) is set to \( z_0^* \) and the base quantization intervals \( C_{i,t} \) are set to 0.

In the on-line stage, at every time step \( t \), we require each agent \( i \) to obtain a local estimate \( \hat{X}_{i,t} \) of the global state \( X_t \). The estimate \( \hat{X}_{i,t} \) consists of accurate measurements \( x_{i,t}, i \in N_i \) (also used as initial states for the DMPC problem \( \text{(52)} \)) and estimates \( \hat{x}_{j,t}, j \in M \setminus N_i \). Let \( e_{i,t} := \hat{X}_{i,t} - X_t \) be the combined estimation error. When \( t = 0 \), the local estimates \( \hat{X}_{i,0} \) are set as \( X_0 \) from

Algorithm 2 DMPC Framework with Quantization Refinement

Require: \( T, L, (1 - \gamma) < \rho < 1, \eta < \frac{1}{\sqrt{T}}, \gamma = \alpha/\sqrt{T}, t = 0; \)

Off-line Stage:
1. Determine \( n \) and \( K \) satisfying \( nK \leq T \);
2. Measure \( X_0 \) and compute the optimal solution \( z_0^* \) of \( \text{(52)} \);
3. Initialize \( z^K_0 = z_0^* \) and \( C_{i,0} = 0, i \in M \).

On-line Stage:
if \( t = 0 \) then
4. Set \( \hat{X}_{i,0} = X_0, i \in M \) and apply optimal control \( \pi^*(X_t) \); else
5. Obtain \( \hat{X}_{i,t} \) and compute \( \Delta \hat{X}_{i,t-1} := \hat{X}_{i,t} - \hat{X}_{i,t-1}, i \in M \);
6. Update base quantization interval \( C_{i,t} \) using \( [56], i \in M \);
7. Update initial solution \( z^K_{N_i,t} \leftarrow z^K_{N_i,t-1}, i \in M \);
8. Formulate DMPC problem \( \text{(52)} \) as a distributed optimization problem \( \text{(5)} \) with \( e_{i,t} \) as the time-varying parameter and solve it using Alg. 1 over the MAS \( \text{(51)} \) for \( K \) iterations with inputs \( n, \rho, \eta, z_0^*, C_{i,t}, i \in M \);
9. Apply sub-optimal control \( \pi^K(X_t) \);
end if

Step 2 and the optimal control \( \pi^K(X_t) \) is applied to each agent. When \( t > 0 \), the agents compute an estimated global state change \( \Delta \hat{X}_{i,t-1} := \hat{X}_{i,t} - \hat{X}_{i,t-1}, i \in M \) and update the local base quantization intervals (Step 6) with

\[
C_{i,t} = b_1 C_{i,t-1} + b_2 ||\Delta \hat{X}_{i,t-1}|| + (b_3 + b_4 b_5) e, \quad (56)
\]

where \( c \geq 0 \) is an upper-bound of \( e_{i,t} \) for all \( t > 0 \), and

\[
b_1 := \frac{\rho^K(1 + M a_1 a_3 - M a_2)}{1 - M a_2}, \quad b_2 := \frac{L a_1}{1 - M a_2}, \quad b_3 := \frac{4 a_2}{1 - b_1}, \quad b_4 := \frac{M a_2 (b_1 - \rho^K) + \rho^K M a_1 a_3}{1 - M a_2} \quad (57)
\]

Then, the previous solutions are used as warm-start solutions (Step 7) for solving the current DMPC problem \( \text{(52)} \) by implementing Alg. 1 over the MAS \( \text{(51)} \) for \( K \) iterations (Step 8).

Assumption 5 The errors satisfy \( \max_{i \in M, t \geq 0} (e_{i,t}) \leq c \).

Remark 2 The estimates \( \hat{X}_{i,t} \) can be obtained using, e.g., quantized distributed state estimation algorithms [19], [20]. To satisfy Assumption 5, we can reduce the estimation error by reducing the quantization level and increase iteration number of these algorithms.

The update rule \( \text{(56)} \) contains a refinement component and an adaption component. The former corresponds to \( b_1 C_{i,t-1} \), which allows \( C_{i,t-1} \) to decrease (if \( b_1 < 1 \)). The later corresponds to \( b_2 ||\Delta \hat{X}_{i,t-1}|| + (b_3 + b_4 b_5) e \), which allows \( C_{i,t-1} \) to increase and adapt to the global state change \( \Delta X_{i,t-1} := X_{i,t} - X_{i,t-1} \). The term \( (b_3 + b_4 b_5) e \) preemptively increases \( C_{i,t-1} \) to compensate for the estimation error \( e_{i,t} \). With appropriate \( n, K \), the update rule \( \text{(56)} \) and the warm-start step form a time-step-wise quantization refinement scheme, which allows the quantization intervals to be refined over time, as illustrated in Fig. 1. This on-line scheme is key to our stability analysis (Proposition 3 and Theorem 1).

V. STABILITY ANALYSIS OF THE PROPOSED METHOD

In this section, we derive sufficient conditions on \( n \) and \( K \) for guaranteeing recursive feasibility and closed-loop stability of the MAS \( \text{(51)} \) controlled by the proposed DMPC framework in Alg. 2.
A. Recursive Feasibility Guarantee

We prove recursively feasible of MAS controlled by Alg. 2, i.e., $X_t$ is a feasible for (52) $\forall t > 0$. To solve (52) at Step 8 of Alg. 2 with convergence guarantee, we also need to ensure Lemma 2 holds. This requires the intervals $C_{t-1}$ determined by (55) satisfy (5) for $\forall t > 0$. In Proposition 1, we prove these two conditions hold simultaneously.

**Proposition 1** Consider the MAS controlled by the DMPC framework in Alg. 2 with a feasible state $X_0$. Suppose Assumptions 1-5 hold. Let Alg. 2 be initialized with $T$, $L$, $\rho \in (1 - \gamma, 1)$ and $\eta < \frac{1}{L}$. If the quantization bit number $n$ satisfies (6), and

$$1 - M a_2 > 0,$$

(58)

where $a_2$ in (5) depends on $n$, then $X_t$ is recursively feasible.

**Proof:** We first provide some auxiliary results. When $z_t - z_t^*$ exist and the warm-start in Step 7 of Alg. 2 is applied, we have

$$\|\Delta z_{t-1}^0\| := \|z_{t-1}^0 - z_t^*\| = \|z_{t-1} - z_t^* + z_t^* - z_t^*\|$$

$$\leq \|z_{t-1} - z_t^*\| + \|z_t^* - z_t^*\| \leq \|\Delta z_{t-1}^0\| + L\|\Delta X_{t-1}\|.$$  

(59)

From Assumption 5 we have

$$\|\Delta X_{t-1}\| = \|\Delta X_{t-1}\| + 2e_t \|\Delta X_{t-1}\| \leq \|\Delta X_{t-1}\| + 2e_t.$$  

(60)

which implies

$$\|\Delta X_{t-1}\| \leq \|\Delta X_{t-1}\| + 2e_t \|\Delta X_{t-1}\| \leq \|\Delta X_{t-1}\| + 2e_t.$$  

(61)

By definition of the update rule (55), we have

$$C_{\Sigma_{t+1}} = b_1 C_{\Sigma_t} + b_2 \sum_{i \in M} \|\Delta X_{t,i}\| + M b_2 e + M b_4 b_2 e$$  

(62)

Since (59) holds, $a_2$ in (58) have positive values.

We prove Proposition 1 by induction. Let $t = 1$ be the base case and $t = g + 1$ be the induction case. The induction assumptions are (i) $X_{t-1}$ being feasible for (52), (ii) $a_2 \|\Delta z_{t-1}^0\| + a_2 C_{\Sigma_{t-1}} \leq C_{t-1}$, and (iii) $|C_{t-1} - C_{\Sigma_{t-1}}/M| \leq b_2$. Showing (i) hold $\forall t > 0$ gives us recursive feasibility, while (ii) and (iii) are necessary for guarantee Alg. 1 has convergence guarantee (8) $\forall t > 0$.

Base case: When $t = 1$, we construct a shifted input sequence $U_1 = [U_{01} \cdots, U_{0N}^T(N), \pi_1, X_0]$ and state sequence $X_1 = [X_0 \cdots, X_0]$, where $U_0$ is the optimal input sequence $U_0 = [U_0^T(0), \cdots, U_0^T(N - 1)]$ and state sequence $X_0$ is obtained at $t = 0$. Since $X_0 = AX_0 + BU_0$ is positive for $X_0$ defined in Assumption 3 guarantees positive invariance of $X_0$ in $C_1$, $U_0$ and $X_0$ form a feasible solution of (52). This implies $X_1$ is feasible for (52). Since $X_0$ is feasible for (52), (59) holds. Applying (59) and (63), and using the facts $\|\Delta z_{t-1}^0\| = 0$ and $C_{\Sigma_{t-1}}$, we obtain

$$a_1 \|\Delta z_{t-1}^0\| + a_2 C_{\Sigma_{t-1}} \leq a_1 (L \|\Delta X_{t-1}\| + 2e_t)$$

(64)

$$+ a_2 (L \|\Delta X_{t-1}\| + M b_2 e + M b_4 b_2 e)$$

(65)

$$+ M a_2 b_2 b_2 e \leq b_2 \|\Delta X_{t-1}\| + b_2 e + b_4 b_2 e = C_{t-1}.$$  

(66)

where (68) holds by the definition of the update rule (55). To show $\|C_{t-1} - C_{\Sigma_{t-1}}/M\| \leq b_2$, we use (56) and (64) to obtain

$$\|C_{t-1} - C_{\Sigma_{t-1}}/M\| \leq |b_1 C_{t-1} + b_2 \|\Delta X_{t-1}\| + b_3 e + b_4 b_2 e$$

(69)

$$- (b_1 C_{t-1} + b_2 \|\Delta X_{t-1}\| + (2b_2 + b_3)e + b_4 b_2 e)\|$$

$$\leq b_2 \|\Delta X_{t-1}\| + 2b_2 e \leq b_2 \|\Delta X_{t-1}\| + 2b_2 e \leq 2b_2 e \leq b_2 e \leq 2b_2.$$  

(70)

From induction assumption (ii), $\|\Delta z_{t-1}^0\|$ can be bounded as:

$$\|\Delta z_{t-1}^0\| \leq C_{t-1} - a_1 - a_2/C_{t-1}.$$  

(71)

Since $n$ satisfies (6), (6) holds and we can obtain

$$\|\Delta z_{t-1}^0\| \leq \rho \rho \|\Delta z_{t-1}^0\| + a_2 C_{t-1}$$

(72)

$$\leq \rho \rho \|\Delta z_{t-1}^0\| + a_2 C_{t-1} - a_1/a_2 C_{t-1}.$$  

(73)

Then, bounding $\|\Delta z_{t-1}^0\|$ in (73) with (66) gives

$$a_1 \|\Delta z_{t-1}^0\| + a_2 C_{t-1} \leq \rho \rho \|\Delta z_{t-1}^0\| + a_2 C_{t-1} - a_1/a_2 C_{t-1}$$

(74)

$$+ \rho \rho \|\Delta z_{t-1}^0\| + a_2 C_{t-1} - a_1/a_2 C_{t-1} + a_1 L + a_2 b_2 M \|\Delta X_{t-1}\|$$

(75)

$$+ a_2 (2M a_2 + M b_3) L a_1 e + M a_2 b_2 b_2 e.$$  

(76)

From induction assumption (iii), i.e., $\|C_{t-1} - C_{\Sigma_{t-1}}/M\| \leq b_2 e$, we derive $\|C_{t-1} - C_{\Sigma_{t-1}}/M\| \leq |C_{t-1} - C_{\Sigma_{t-1}}/M| \leq |b_2 e|$, which we use to obtain

$$C_{t-1} = M(C_{t-1} - C_{\Sigma_{t-1}}/M) \leq M C_{t-1} + M b_2 e.$$  

(77)

Then, applying (78) to (77) yields the following after carrying out simplification:

$$a_1 \|\Delta z_{t-1}^0\| + a_2 C_{t-1} \leq \rho \rho (1 - M a_2 + M a_1 a_3 + M a_2 b_2 C_{t-1})$$

(78)

$$+ (a_1 L + a_2 b_2 M \|\Delta X_{t-1}\|) + (M a_2 b_4 + M a_2 b_3 + 2 L a_1) e$$

$$+ M b_5 (a_2 b_4 + a_2 b_1 - \rho \rho (a_2 + a_1) \rho \rho a_3 e)$$

(79)

$$\leq b_1 C_{t-1} + b_2 \|\Delta X_{t-1}\| + b_2 e + b_4 b_2 e = C_{t-1}.$$  

(80)
Another equation is provided in the context of Proposition 3 (ISS-Lyapunov Function for Subsystem 2). Consider Subsystem 2 in (85). Suppose Assumptions 1-5 and Proposition 1 hold. If n and K satisfy

$$K \geq K_1 := -\log p \left( \frac{1 - Ma_2}{Ma_1 a_3 + 1 - Ma_2} \right),$$

where $a_2, a_3$ depend on $n$, then $C_{\Sigma, t}$ is an ISS-Lyapunov of (85) satisfying

$$C_{\Sigma, t+1} \leq (1 - \alpha_2)C_{\Sigma, t} + \max(\gamma_{1,2} \psi(X_t), \gamma_{2,3} \Delta z^K_t \| \gamma_{2,3} e),$$

(90)

where $\alpha_2 := 1 - b_1, \gamma_{1,2} := 3b_2Mc_1, \gamma_{2,3} := 3M\|B\|b_2, \gamma_{2,3} := 3M(4b_2 + b_1 + b_2 b_2)$.

Proposition 4 (ISS-Lyapunov Function for Subsystem 3) Consider Subsystem 3 in (85). Suppose Assumptions 1-5 and Proposition 1 hold. If n and K satisfy

$$K \geq K_2 := -\log p \left( c_2 \right),$$

(91)

where $c_2 := 1/(1 + L\|B\| + a_3 M\|B\|b_2)$, with $a_3$ in (9) and $b_2$ in (57) depending on $n$, then $\Delta z^K_{t+1}$ is an ISS-Lyapunov of (86) satisfying

$$\Delta z^K_{t+1} \leq (1 - \alpha_3)\|\Delta z^K_t\| + \max(\gamma_{3,1} \psi(X_t), \gamma_{3,2} C_{\Sigma, t}, \gamma_{3,2} e),$$

(92)

where $\alpha_3 := 1 - \rho 2^{K/b_1}, \gamma_{3,1} := 3\rho^K (L + M b_2)c_1, \gamma_{3,2} := 3\rho^K a_3 b_1$, and $\gamma_{3,2} := \gamma_{3,2} \gamma_{3,2}$. 

D. Conditions on the Quantization Parameters for Stability

We now derive conditions on $n$ and $K$, given $T$, such that the three interconnected cycles in Fig. 2 satisfy the small-gain conditions and guaranteeing stability of the interconnected system.

C. ISS-Lyapunov Functions for Subsystem 1, 2, and 3

Inspired by [9], we prove $\psi(X_t) := \sqrt{V(X_t)}$ is an ISS-Lyapunov function for Subsystem 1. We also show $C_{\Sigma, t}$ and $\Delta z^K_t$ are ISS-Lyapunov functions for Subsystems 2 and 3, respectively, if $n$ and $K$ satisfy certain conditions. We present proofs of Propositions 2-4, and Theorem 1 in the Appendix.

Proposition 2 (ISS-Lyapunov Function for Subsystem 1) Consider Subsystem 1 in (84) with a feasible initial state $X_0$. Suppose Assumptions 1-5 and Proposition 1 hold. Then, $\psi(X_t)$ is an ISS-Lyapunov function of (84) satisfying

$$\psi(X_{t+1}) \leq (1 - \alpha_1)\psi(X_t) + \|B\|\|\Delta z^K_t\|,$$

(87)

where $\alpha_1 := 1 - \sqrt{1 - \lambda(Q)/(L^2 \bar{X}(H))}$ and $\gamma_{1,2} := \sqrt{\bar{X}(H)}\|B\|L$.

The difference between Proposition 2 and Theorem 1 in [9] is we define the Lipschitz constant $L$ differently in (85). The bound in Lemma 3 on $\Delta X_t$ is used in the proofs of Propositions 3 and 4.

Lemma 3 (Lemma 6, [21]) Consider the MAS (51) controlled by $\pi^K(X_t)$ defined in (53). The change in the state $\Delta X_t := X_{t+1} - X_t$ of the MAS satisfies

$$\|\Delta X_t\| \leq c_1\psi(X_t) + \|B\|\|\Delta z^K_t\|,$$

(88)

where $c_1 := (\|A - I\| + \|B\|)\sqrt{\bar{X}(H)}$.

Proposition 3 (ISS-Lyapunov Function for Subsystem 2) Consider Subsystem 2 in (85). Suppose Assumptions 1-5 and Proposition 1 hold. If $n$ and $K$ satisfy

$$K \geq K_1 := -\log p \left( \frac{1 - Ma_2}{Ma_1 a_3 + 1 - Ma_2} \right),$$

(89)

where $a_2, a_3$ in (9) depend on $n$, then $C_{\Sigma, t}$ is an ISS-Lyapunov of (85) satisfying

$$C_{\Sigma, t+1} \leq (1 - \alpha_2)C_{\Sigma, t} + \max(\gamma_{1,2} \psi(X_t), \gamma_{2,3} \Delta z^K_t \| \gamma_{2,3} e),$$

(90)

where $\alpha_2 := 1 - b_1, \gamma_{1,2} := 3b_2Mc_1, \gamma_{2,3} := 3M\|B\|b_2, \gamma_{2,3} := 3M(4b_2 + b_1 + b_2 b_2)$.
conditions [4] which further implies stability of the interconnected system. Let $X_{i,j} := \tilde{\alpha}_i(s) \circ (I_d - \tilde{\mu}_i(s))^{-1} \circ \tilde{\gamma}_i(s)$, with

\[
\tilde{\alpha}_1(s) := \alpha_1 s, \quad \tilde{\alpha}_2(s) := \alpha_2 s, \quad \tilde{\alpha}_3(s) := \alpha_3 s,
\]

\[
\tilde{\gamma}_1(s) := \gamma_1(s), \quad \tilde{\gamma}_2(s) := \gamma_2(s), \quad \tilde{\gamma}_3(s) := \gamma_3(s),
\]

the small-gain conditions corresponding to the three cycles are: there exist $\tilde{\mu}_i, i \in \{1, 2, 3\}$ such that

\[
X_{1,3} \circ X_{3,1} < I_d, \quad (93a)
\]

\[
X_{2,3} \circ X_{3,2} < I_d, \quad (93b)
\]

\[
X_{3,1} \circ X_{1,2} < I_d, \quad (93c)
\]

**Theorem 1** Consider the MAS \(\Sigma\) controlled by the DMPC framework in Alg. 2. Suppose Assumptions 1-5 hold. Given communication data rate $T$ defined as the number of bits that can be transmitted per time step. If the quantization bit number $n$ and optimization iteration number $K$ satisfy $nK \leq T$, \([9]\), \([8]\), and

\[
K \geq \max(K_1, K_2, K_3, K_4, K_5), \quad (94)
\]

where $K_1$ defined \([9]\), $K_2$ defined \([9]\), and

\[
K_3 := \begin{aligned}
\log \left( \frac{3\alpha_1 - 3\tau_{\gamma_1,3}c_1}{3\tau_{\gamma_1,3}c_{11} + 1} \right),
\end{aligned} \quad (95a)
\]

\[
K_4 := \begin{aligned}
\log \left( \frac{-b_1 + \sqrt{b_1^2 - 4a_1c_{21}}}{2a_1} \right),
\end{aligned} \quad (95b)
\]

\[
K_5 := \begin{aligned}
\log \left( \frac{-b_2 + \sqrt{b_2^2 - 4a_2c_{22}}}{2a_2} \right),
\end{aligned} \quad (95c)
\]

depend on $n$, with

\[
\tilde{\alpha}_1 := \frac{\rho K c_2 - b_1}{3\rho K^2 \tau_{\gamma_2,3}a_3}, \quad \tilde{\alpha}_2 := \frac{-\rho K - b_1c_2}{3\rho K^2 \tau_{\gamma_2,3}a_3}, \quad \tilde{\alpha}_3 := \frac{b_1}{3\rho K^2 \tau_{\gamma_2,3}a_3},
\]

\[
\tilde{\alpha}_4 := \frac{\rho K \alpha_1 c_2}{3\gamma_2 \tau_{\gamma_2,3}b_1 c_1}, \quad \tilde{\alpha}_5 := \frac{-\rho K \alpha_1 - \alpha_1 b_1c_2 - Lb_1}{3\gamma_2 \tau_{\gamma_2,3}b_1 c_1},
\]

\[
\tilde{\alpha}_6 := \frac{\rho K \alpha_1}{3\gamma_2 \tau_{\gamma_2,3}b_1 c_1} - Mb_2,
\]

then the MAS \(\Sigma\) controlled by Alg. 2 is recursively feasible and the closed-loop system is ISS w.r.t. the parameter $e$ in \([6]\) considered as a constant external input.

**Remark 3** If $e = 0$ as a result of, i.e. all agents having accurate estimates of the global state (the network has a central node), the interconnected system becomes asymptotically stable.

**Remark 4** The small-gain conditions \([2]\) naturally embed a degree of conservatism. \([?]\) proved small-gain-type conditions as necessary (tight) and sufficient for a family of interconnected systems. Extending this result to our problem setting can be a future direction. Bad estimation of Lipschitz constants may also lead to conservatism in the bound \([2]\) since the value of $\max(K_1, K_2, K_3, K_4, K_5)$ depends on $L_i$, $L_f$, and $L$. To reduce $L$, pre-conditioning the optimization problem underlying \([5]\) is viable \([9]\). At the cost of potentially reduced closed-loop performance, one can also tune the DMPC problem parameters (e.g., $H_i$ and $P_i$ in \([22a]\)) and adjust the network topology (e.g. reducing degree $d$ of $G$) to reduce $L_i$, $L_f$, and $L$.

**Remark 5** Given a limited data rate $T$, it is possible to combine a combination of $n$ and $K$ satisfying $nK \leq T$, \([6]\), \([8]\), and \([22]\) does not exist. In this case, one can solve the quantization parameter design problem:

\[
T_{\text{min}} = \min_{n, K \in \mathbb{Z}^+} nK, \quad \text{s.t.} \quad (9), (38), (22), \quad (96)
\]
straints and compute $\tilde{z}_m = A\zeta_m + BU_m$. We collect 2000 pairs of $(\zeta_m, \zeta_m')$ that are both feasible for the DMPC problem. The Lipschitz constant that $L = 9.72$ is computed from $L = \max_{1 \leq m \leq 2000} \frac{\|z_m - z_m'\|}{\|z_m' - z_m\|}$. Given $T = 32$ kbs, the parameters for Alg. 2 are chosen as $\rho = 0.986, \eta = 0.001, n = 26, K = 1000$, where $n$ and $K$ satisfy conditions (6), (55), and (74).

We simulate and compare two cases: with and without estimation noise. For the first case, we sample $e_{i,t}$ uniformly from $[-0.05, 0.05]$ and set $e = 0.2$ to satisfy Assumption 5. For the second case, we set $e = 0$. Fig. 3(a) shows trajectories of the AVUs controlled by Alg. 2 with $e = 0.2$, where the red dashed lines represent the current formation. It can be seen that the AVUs converge to the references while forming the desired formation. Fig. 3(b) shows the input $u_{5,t}$ of AUV 5, where the input constraint $-0.3 \text{ Nm/s}$ is activated before $t = 1.3$ s. Fig. 3(c) shows the convergence of the position $y_{5,5}$ of AUV 5 (part of the state of Subsystem 1) to the reference state $y_{5,t} = -2$ m. Fig. 3(d) shows the evolution of $C_{1,t}$ (part of the state of Subsystem 2), which converges to a neighbourhood of 0 when $e = 0.2$ and to 0 when $e = 0$. Fig. 3(e) shows $\Delta z_i^K$ (normed state of Subsystem 3) converges to a plateau near $10^{-2}$ in both cases, from results limited solver accuracy in the projection step of Alg. 1. After $t = 5$ s, the value of $|\Delta z_i^K|$ with $e = 0.2$ is slightly higher than that with $e = 0$. As can be seen, the effect of the parameter $e$ is most obvious on the base quantization interval (Subsystem 2) and least visible in the state (Subsystem 1). This is expected since $e$ enters Subsystem 2 as a constant external input and its effect indirectly propagates through Subsystem 3 to Subsystem 1, as illustrated in Fig. 2. To conclude, the simulation results demonstrated that for a given data rate $T$, if the quantization bit number $n$ and iteration number $K$ satisfy the conditions in Theorem 1, then, the interconnected system formed by (84)–(86) is ISS w.r.t. the parameter $e$ considered as an external input.

VII. CONCLUSIONS

We proposed a novel real-time DMPC framework with a quantization refinement scheme for MASs with limited communication data rates and derived sufficient conditions on the quantization parameters for guaranteeing the closed-loop stability. Future works can focus on quantitative methods for jointly designing the DMPC formulations (e.g., cost functions) and the algorithm parameters (e.g., $n$, $K$) to reduce the required communication data rate for guaranteeing closed-loop stability while maintaining certain level of performance. Another direction is to address the conservatism resulted from bounding $e_{i,t}$ with $e$, by developing an ISS-Lyapunov function for the estimation error $e_{i,t}$ w.r.t. the change in state $\Delta X_i$. This would enable us to consider the state estimation dynamics as another subsystem interconnected with the subsystems in (84)–(86), and apply the small-gain theorem to prove asymptotically stable of the overall system.

APPENDIX

A. Proof of Proposition 2

From Proposition 1, $X_i$ is feasible and $\pi^K(X_i)$ and $\pi^+(X_i)$ exist for $t \geq 0$. Let $X_{t+1} = AX_i + B\pi^K(X_i), \tilde{X}_{t+1} = AX_i + B\pi^+(X_i)$, $\psi(X_{t+1}) := \|z_{i,t+1}^1\|u, \psi(\tilde{X}_{t+1}) := \|z_{i,t+1}^2\|u$, we know that

$$\psi(X_{t+1}) \leq \|z_{i,t+1}^1\|u - \|z_{i,t+1}^2\|u + \|z_{i,t+1}^2\|u.$$  (97)

From Assumption 3, we have

$$V(\tilde{X}_{t+1}) \leq V(X_i) - \|X_i\|_Q^2.$$  (98)

Lower bounding $\|X_i\|_Q^2$ with

$$\|X_i\|_Q^2 \geq \lambda(Q)\|X_i\|^2 \leq \frac{\lambda(Q)}{L^2} \|z_i^2\|^2 \geq \frac{\lambda(Q)}{L^2} \|z_i^2\|^2,$$

and then taking square-roots on both sides of (98) yields

$$\psi(\tilde{X}_{t+1}) \leq \sqrt{1 - \lambda(Q)/(L^2H(X))}\psi(X_i).$$  (99)

Applying the reverse triangle inequality, we have

$$\|z_{i,t+1}^1\|u - \|z_{i,t+1}^2\|u \leq \|z_{i,t+1}^1 - z_{i+1}^2\|^2 \leq \|X_H\|z_{i,t+1}^1 - \tilde{X}_{t+1}^2\|^2.$$  (100)

Taking square-roots on both sides, we obtain

$$\|z_{i,t+1}^1\|u - \|z_{i,t+1}^2\|u \leq \sqrt{\|X_H\|z_{i,t+1}^1 - \tilde{X}_{t+1}^2\|^2}.$$  (101)

Then, bounding the r.h.s of (77) with (99) and (101) gives

$$\psi(\tilde{X}_{t+1}) \leq (1 - \alpha_1)\psi(X_i) + \gamma_{13}\|\Delta z_i^K\|,$$  (102)

where $\alpha_1 = 1 - \sqrt{1 - \lambda(Q)/(L^2H(X))}$ and $\gamma_{13} = \sqrt{\|X_H\|B\|L\|}$. Since $\|z_i^2\|u \leq \|X_i\|u$ from (55) and $\|z_i^2\|u = \|X_i^T, U^T\|u \geq \|X_i^T\|u = \|X_i\|u$, we know $L \geq 1$. Further, since $X_H \geq \lambda(Q)$, we know that $\alpha_1 \in (0, 1)$. By definition, we also know $\gamma_{13} > 0$. Therefore, $\psi(X_i)$ is an ISS-Lyapunov function for Subsystem 1.

B. Proof of Proposition 3

Combining (62) with (88) gives

$$\|\Delta X_{t+1}\| \leq c_1\psi(X_i) + \|B\|\|\Delta z_i^K\| + 2e.$$  (103)

Bounding $\|\Delta \tilde{X}_{t+1}\|$ in (65) with (103) and doing simplification give

$$C_{\Sigma,t+1} \leq b_1C_{\Sigma,t} + Mb_2c_1\psi(X_i) + M\|\|B\|u\|\|\Delta z_i^K\| + M(4b_2 + b_3 + b_4 b_5)e$$

$$\leq (1 - \alpha_2)C_{\Sigma,t} + \max \left\{\gamma_{2,1}\psi(X_i), \gamma_{2,3}\|\Delta z_i^K\|, \gamma_{2,5}\right\}.$$  (104)

Since $\alpha_2 \in (0, 1)$ from (91) and $\gamma_{2,1}, \gamma_{2,3}, \gamma_{2,5} > 0$ by definition, $C_{\Sigma,t}$ is an ISS-Lyapunov function for Subsystem 2.

C. Proof of Proposition 4

From Proposition 1, $\pi$ satisfies (6), (5) holds, and we have

$$\|\Delta z_i^K\| \leq \rho^K (\|\Delta z_i^K\| + \|L\|\|X_i\|) + a_3C_{\Sigma,t+1}.$$  (106)

Then, bounding $\|\Delta X_i\|$ with (88) and $C_{\Sigma,t+1}$ with (104) gives

$$\|\Delta z_i^K\| \leq \rho^K (1 + \|L\|\|B\| + a_3M\|\|B\|\|\Delta z_i^K\| + \|\Delta z_i^K\| + \rho^K a_3 b_1 C_{\Sigma,t}$$

$$+ \gamma_{3,1}(L + Mb_2)c_1\psi(X_i) + M(4b_2 + b_3 + b_4 b_5)e$$

$$\leq (1 - \alpha_3)\|\Delta z_i^K\| + \max \left\{\gamma_{3,1}\psi(X_i), \gamma_{3,2}\|\Delta z_i^K\|, \gamma_{3,5}\right\}.$$  (107)

Since $\alpha_3 \in (0, 1)$ from (91) and $\gamma_{3,1}, \gamma_{3,2}, \gamma_{3,5} > 0$ by definition, $\|\Delta z_i^K\| is an ISS-Lyapunov function for Subsystem 3.
D. Proof of Theorem 1

Since the initial state $x_0$ is feasible and $n$ satisfies (6) and (58), the MAS (51) under Alg. 2 is recursively feasible (Proposition 1).

Since Assumptions 1-5 hold, the quantization bit number $n$ and iteration number $K$ satisfy (6), (58), and $K \geq \max(K_1, K_2)$, it is ensured Propositions 2-4 hold, i.e., Subsystems 1-3 admit (87), (90), and (92) as ISS-Lyapunov functions, respectively.

In addition, since the iteration number also satisfies $K \geq \max(K_3, K_4, K_5)$, the following conditions hold:

$$\hat{\alpha}_1^{-1} \circ \hat{\alpha}_3^{-1} \circ \gamma_{3,1} \circ \gamma_{3,1} < \text{Id},$$

$$\hat{\alpha}_2^{-1} \circ \hat{\alpha}_3^{-1} \circ \gamma_{2,3} \circ \gamma_{3,2} < \text{Id},$$

$$\hat{\alpha}_3^{-1} \circ \hat{\alpha}_3^{-1} \circ \gamma_{2,3} \circ \gamma_{3,2} < \text{Id},$$

which implies there always exist linear gain functions $\mu_i(s) := \mu_i(s)$, with $\mu_i > 0$, $i \in \{1, 2, 3\}$, such that the small-gain conditions in (92) hold, i.e., $X_{1,3} \circ X_{2,1} < \text{Id}$, $X_{2,3} \circ X_{3,2} < \text{Id}$, and $X_{2,3} \circ X_{2,1} < \text{Id}$, where $X_{i,j} := \hat{\alpha}_i^{-1} \circ (\text{Id} - \mu_i)^{-1} \circ \gamma_{i,j}$. Therefore, the interconnected system formed by (84)-(86) is ISS w.r.t the parameter $c$ considered as a constant external input.

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