Invariant densities for dynamical systems with random switching

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Abstract
We consider a nonautonomous ordinary differential equation on a smooth manifold, with right-hand side that randomly switches between the elements of a finite family of smooth vector fields. For the resulting random dynamical system, we show that Hörmander type hypoellipticity conditions are sufficient for uniqueness and absolute continuity of an invariant measure.

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1. Introduction
In this paper we study the ergodic theory of systems with random switchings. Such a system can be described in terms of a finite family of vector fields. We assume that at any given time the evolution is driven by one of these vector fields, and at random times the driving vector field changes to another one from the same family. Systems of this nature arise naturally in applications and we refer to the recent monograph [YZ10] for motivation and extensive bibliography.

Many long-term asymptotic properties of dynamical systems or random dynamical systems can be described in terms of invariant distributions. The existence of invariant measures often can be derived using the Lyapunov function technique that helps to establish recurrence properties or tightness, see, e.g., [YZ10, sections 3.3–3.4].

The uniqueness and absolute continuity of invariant distributions are often related to each other and more subtle, especially in the case that we consider in this paper where no diffusion is involved, and the only source of randomness is the random sequence of driving vector fields. Although some claims have been scattered through the literature, no general result is known, see, e.g., [YZ10, section 8.5.2].

The goal of this paper is to close this gap and obtain new general conditions that guarantee uniqueness and absolute continuity of invariant measures for systems with random switchings.
The two conditions that we suggest are formulated in terms of Lie algebras associated with the driving vector fields. They are close analogues of the classical Hörmander condition guaranteeing absolute continuity of transition densities of hypoelliptic diffusions. In the diffusion context, this result is usually derived from the variational analysis of diffusion paths known as Malliavin calculus, see, e.g., [Bas98, chapter VIII], [Bel06, Nua06]. In fact, the central part of this paper is the analysis of transition probabilities of switched systems. Under the first of our conditions, we prove that all transition probabilities for the system have nontrivial absolutely continuous components. The second condition is more general, and it allows us to prove the existence of absolutely continuous components not for the transition probabilities themselves, but for their time averages. The extraction of these absolutely continuous components is largely based on classical control theory results that can be found in chapter 3 of [Jur97]. These control theory results rely on earlier work by Chow [Cho39], Sussmann and Jurdjevic [SJ72], and Krener [Kre74]. Our conditions and the structure of our proofs match those of [Jur97], where the nondegeneracy of certain maps is exploited to establish the accessibility property. We use the same nondegeneracy to prove absolute continuity, and one can interpret our result as filling the control theory with probabilistic content. In fact, the idea to use the geometric control theory approach to establish regularity of Markov transition kernels along with ergodic properties is not new, see, e.g., [AKSS07] where controllability of the 2D Navier–Stokes system was used to prove the absolute continuity of finite-dimensional projections of transition kernels.

The paper is organized as follows: in section 2 we introduce the setting, necessary notation and notions from differential geometry and geometric control theory. We also state the main result on uniqueness and absolute continuity of invariant measures, and two central auxiliary results on regularity of transition probabilities each based on one of the Hörmander type assumptions. We prove these regularity results in sections 3 and 4. In section 5 we prove that any ergodic measure has to be absolutely continuous if its support contains a point where hypoellipticity holds. Section 6 contains the proof of the main result: if the hypoellipticity holds at a point that can be approached from any initial point using the given vector fields as admissible controls, then there exists at most one invariant distribution, and this distribution has to be absolutely continuous. In section 7, we apply the main result to a switching system on the $n$-dimensional torus and a switching system involving two Lorenz vector fields.

2. Definitions, Notation and Main Results

2.1. The dynamics

We consider a finite collection $D$ of smooth and forward complete vector fields on an $n$-dimensional $C^\infty$-manifold $M$.

We denote these vector fields by $u_i, i \in S = \{1, \ldots, k\}$. Each vector field $u$ in $D$ induces an ordinary differential equation of the form

$$\dot{x}(t) = u(x(t)).$$

This differential equation is uniquely solvable if equipped with an initial condition

$$x(0) = \xi \in M,$$

and forward completeness means that the solution trajectories are well-defined for all times $t > 0$.

We can define a stochastic process $X = (X_t)_{t \geq 0}$ on $M$ in the following way: Given an initial state $i \in S$ and an initial value $\xi \in M$, $X_t$ follows the trajectory generated by the vector field $u_i$ and initial condition $\xi$ for an exponentially distributed random time with
parameter \( \lambda_i > 0 \). Then a new state is selected at random from \( S \setminus \{i\} \), and, for another exponentially distributed random time, \( X_t \) follows the new vector field corresponding to that state. Iterating this construction we obtain a piecewise smooth trajectory \((X_t)_{t \geq 0}\) defined for all positive times and driven by one of the vector fields from \( D \) between any two switchings.

We assume that (i) all the interswitching times are exponentially distributed and independent conditioned on the sequence of driving vector fields, (ii) the parameter \( \lambda_j \) of the exponential time between any two switches depends only on the current state \( j \) and (iii) the probabilities of switchings between any two states are positive.

We choose to work with exponential waiting times to ensure the Markov property, although our results can be extended to non-Markovian settings resulting from more general waiting time distributions.

It is convenient to keep track of the driving vector fields at all times. We define \( A_t \in S \) as the index of the driving vector field at time \( t \), also referred to as the regime or state at time \( t \). It is a Markov process with continuous time and finitely many states. Its trajectories are right-continuous and piecewise constant.

Although \( X \) alone is not a Markov process, the joint process \((X, A)\) is Markov. We denote elements of the associated Markov family, i.e. the distribution on paths emitted at \((\xi, i) \in M \times S\) and generated by the iterative random procedure above, by \( P_{t, i}^{\xi} \), and the corresponding transition probability measures by \( P_{t, i}^{\xi} \), \( t \geq 0 \). The transition probability measures are defined on the product \( \sigma \)-algebra \( B(M) \otimes P(S) \), where \( B(M) \) is the Borel \( \sigma \)-algebra on \( M \) and \( P(S) \) is the power set of \( S \). We write \( E_{\xi,i} \) for expectation with respect to \( P_{t, i}^{\xi} \).

Let us recall that if the initial distribution of the Markov process \((X, A)\) is \( \mu \), then the distribution of the process at time \( t \) is given by the measure \( \mu P_t \) on \( M \times S \) defined by

\[
\mu P_t(E \times \{j\}) = \sum_{i=1}^{k} \int_M P_{t, i}^{\xi}(E \times \{j\}) \mu(d\xi \times \{i\}).
\]  

A probability measure \( \mu \) on \( M \times S \) is called invariant for \((P_t)\) if \( \mu = \mu P_t \) for all \( t \geq 0 \).

The main goal of this paper is to give conditions on \( D \) that would guarantee absolute continuity and uniqueness of an invariant measure of the Markov semigroup \((P_t)_{t \geq 0}\). The fairly general conditions that we suggest are formulated in geometric terms, and we proceed to introduce the necessary definitions and notation.

### 2.2. Auxiliary definitions and notation

Let \( V(M) \) denote the set of real smooth vector fields on the manifold \( M \), and let \( C^\infty(M) \) denote the set of real-valued smooth functions on \( M \). As explained above, we assume that \( D \) is contained in \( V(M) \). Any element of \( V(M) \) corresponds uniquely to a derivation on \( C^\infty(M) \), that is to a linear operator \( \delta \) on \( C^\infty(M) \) satisfying the Leibniz rule

\[
\delta(f \cdot g) = \delta(f) \cdot g + f \cdot \delta(g).
\]

The Lie bracket of two vector fields \( u \) and \( v \) in \( V(M) \) is defined as the vector field

\[
[u, v](f) := u(v(f)) - v(u(f))
\]

for test functions \( f \) in \( C^\infty(M) \). The set \( V(M) \) equipped with the bilinear operator \([.,.]\) becomes a Lie algebra over the reals. A subset of \( V(M) \) is called involutive if it is closed under taking the Lie bracket. An involutive subspace of \( V(M) \) is called a subalgebra of \( V(M) \).

The smallest subalgebra of \( V(M) \) that contains \( D \) is denoted \( \mathcal{I}(D) \). The derived algebra \( \mathcal{I}'(D) \) is the smallest algebra containing Lie brackets of vector fields in \( \mathcal{I}(D) \). We have
\( I'(D) \subset I(D) \), but \( I'(D) \) might not contain any elements of \( D \) and may therefore be strictly contained in \( I(D) \). Further, we define \( I_0(D) \) as the set of vector fields of the form
\[
v + \sum_{i=1}^{k} \lambda_i u_i,
\]
where \( v \in I'(D), u_1, \ldots, u_k \in D \) and \( \sum_{i=1}^{k} \lambda_i = 0 \). Finally, we set
\[
I(D)(\xi) := \{ u(\xi) : u \in I(D) \}
\]
and
\[
I_0(D)(\xi) := \{ u(\xi) : u \in I_0(D) \}
\]
for any \( \xi \in M \). The sets \( I(D)(\xi) \) and \( I_0(D)(\xi) \) are finite-dimensional vector spaces.

Our main results will be based on the following assumptions that can naturally be called hypoellipticity conditions in analogy with Hörmander’s theory. We say that a point \( \xi \in M \) satisfies condition A if \( \text{dim} \ I_0(D)(\xi) = n \). We say that a point \( \xi \in M \) satisfies condition B if \( \text{dim} \ I(D)(\xi) = n \).

The set of points satisfying condition A is open and so is the set of points satisfying condition B.

For our absolute continuity results we will need a reference measure on \( M \) that will play the role of Lebesgue measure. As a smooth manifold, \( M \) can be endowed with a Riemannian metric. The metric tensor can be used to define measures on coordinate patches of \( M \). One can use then a partition of unity in a standard way (see, e.g., [Tay06, section 7]) to construct a Borel measure on \( M \) whose pushforward to \( \mathbb{R}^n \) under any chart map is equivalent to Lebesgue measure. We call the measure on \( M \) obtained through this construction Lebesgue measure, denote it by \( \lambda^M \), and use it as the main reference measure, often omitting ‘with respect to Lebesgue measure’ when writing about absolute continuity. The product of the Lebesgue measure on \( M \) and counting measure on \( S \) will be called the Lebesgue measure on \( M \times S \).

It remains to introduce the flows generated by vector fields in \( D \) and the concept of reachability.

For \( i \in S \), we denote the flow function of the vector field \( u_i \) by \( \Phi_i \). Due to forward completeness of \( u_i \), the flow function is uniquely defined for all \( t > 0 \) and \( \eta \in M \) by
\[
\frac{d}{dt} \Phi_i(t, \eta) = u_i(\Phi_i(t, \eta)),
\]
\[\Phi_i(0, \eta) = \eta.\]

For \( m \in \mathbb{N} \), we will consider vectors \( t = (t_1, \ldots, t_m) \) of waiting times between subsequent switches and vectors \( i = (i_1, \ldots, i_m) \) of driving states during these waiting intervals. We will restrict ourselves to positive waiting times, but it can also be useful (see [SJ72] and [Jur97]) to allow flows backwards in time.

We write \( \mathbb{R}_+ \) to denote the positive real line \((0; \infty)\).

For \( t = (t_1, \ldots, t_m) \in \mathbb{R}^m_+ \) and \( i = (i_1, \ldots, i_m) \in S^m \), we define
\[
\Phi_i(t, \xi) := \Phi_{i_m}(t_m, \Phi_{i_{m-1}}(t_{m-1}, \ldots, \Phi_{i_1}(t_1, \xi)))
\]
as the cumulative flow along the trajectories of \( u_{i_1}, \ldots, u_{i_m} \) with starting point \( \xi \in M \).

The transition probabilities \( P^{\xi}_{t,i} \) can be expressed in terms of cumulative flows. We do not specify these straightforward relations in order to avoid heavy notation.

A point \( \eta \in M \) is called \( D \)-reachable from a point \( \xi \in M \) if there exist a time vector \( t \) with positive components and a vector \( i \) of driving states such that
\[\eta = \Phi_i(t, \xi).\]

If the components of \( t \) sum up to \( t \), we say that \( \eta \) is \( D \)-reachable from \( \xi \) at time \( t \).
For \( \xi \in M \) and \( t > 0 \), let \( L_t(\xi) \) denote the set of \( D \)-reachable points from \( \xi \) at time \( t \), and let \( L(\xi) = \bigcup_{t>0} L_t(\xi) \) denote the set of \( D \)-reachable points from \( \xi \). The points in the closure \( \overline{L(\xi)} \) can be called \( D \)-approachable from \( \xi \). Let \( L = \bigcap_{\xi \in M} \overline{L(\xi)} \) denote the set of points that are \( D \)-approachable from all other points.

### 2.3. Main results

The following is the main theorem of this paper.

**Theorem 1.** Suppose hypoellipticity condition B is satisfied at some \( \xi \in L \). If \((P_t)\) has an invariant measure, then it is unique and absolutely continuous with respect to the Lebesgue measure on \( M \times S \).

**Remark 1.** Of course, theorem 1 remains true if we replace \( L \) by any of its subsets. For example, if one of the vector fields in \( D \) has a minimal global attractor, then it is sufficient to check hypoellipticity for some point of the attractor.

Uniqueness of invariant distributions is tightly connected to the regularity of the Markov semigroup. Various aspects of regularity in connection with ergodicity have been studied in the literature: the existence of minorizing kernels, the strong Feller property, etc. The main task in the proof of theorem 1 is to establish regularity for transition probabilities under hypoellipticity condition B. However, we begin with a much stronger regularity property that can be established under the stronger hypoellipticity condition A.

**Theorem 2.** If condition A is satisfied at a point \( \xi \in M \), then for any \( i \in S \) and any \( t > 0 \), the transition kernel \( P_{\xi,i}^t \) has a nonzero absolutely continuous component with respect to Lebesgue measure on \( M \times S \).

Under the weaker condition B it may happen that none of the transition probability measures \( P_{\xi,i}^t, t > 0 \), has a nonzero absolutely continuous component. For example, let \( M \) be the \( n \)-dimensional torus \( \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \), and let \( D = \{u_1, \ldots, u_n\} \) be the standard basis in \( \mathbb{R}^n \). Fix an arbitrary time \( t > 0 \). The set of points \( D \)-reachable from the origin at time \( t \) is the image of

\[
\left\{(s_1, \ldots, s_n) \in [0; \infty)^n : \sum_{j=1}^n s_j = t \right\}
\]

under the covering map \( \mathbb{R}^n \to \mathbb{T}^n \), and has Lebesgue measure zero, so \( P_{\xi,i}^t \) is a purely singular measure.

Nevertheless, condition B guarantees that time averages of transition probabilities have nontrivial absolutely continuous components. Specifically, we will establish this for the resolvent probability kernel \( Q_{\xi,i} \) defined by

\[
Q_{\xi,i}(E \times \{j\}) := \int_{\mathbb{R}_+} e^{-t} P_{\xi,i}^t(E \times \{j\}) \, dt.
\]

The resolvent kernels are useful in the study of invariant distributions due to the following straightforward result.

**Lemma 1.** If a measure \( \mu \) is \((P^t)\)-invariant it is also \((Q)\)-invariant, i.e. \( \mu = \mu Q \), where the convolution \( \mu Q \) is defined analogously to (1).

**Theorem 3.** If condition B is satisfied at some point \( \xi \in M \), then for any \( i \in S \), the measure \( Q_{\xi,i} \) defined by (2) has a nonzero absolutely continuous component with respect to Lebesgue measure on \( M \times S \).
The convergence of transition probabilities to the invariant measure, provided that it exists, is out of the scope of the present paper. In [BLBMZ12] it is shown that if condition A is satisfied at a \( D \)-approachable point \( \xi \), and if \( M \) is compact, the transition probabilities converge to a unique invariant measure in total variation at exponential rate. Our analysis suggests that existence of an invariant measure and condition B at a \( D \)-approachable point implies only Cesàro convergence.

At the heart of our proofs of theorems 2 and 3 are classical results from geometric control theory that can be found in [Jur97]. The statements we present are derived from theorems 3.1, 3.2 and 3.3 in [Jur97]. Analogous results for the special case of analytic vector fields on a real analytic manifold are first stated in [SJ72, theorems 3.1 and 3.2]. In their paper, Sussmann and Jurdjevic were able to build on prior work [Cho39] by Chow who considered symmetric families of analytic vector fields. Krener generalized these results to \( C^\infty \)-vector fields in [Kre74].

Recall that a regular point of a function \( f : \mathbb{R}^m \to M \) is a point \( t \in \mathbb{R}^m \) such that the differential \( Df (t) \) has full rank. If \( Df (t) \) has deficient rank, \( t \) is called a critical point of \( f \).

**Theorem 4.** Assume that condition A holds at some \( \xi \in M \). Then:

1. For any \( i, j \in S \), there are an integer \( m > n \) and a vector \( i \in S^{m+1} \) with \( i_1 = i \) and \( i_{m+1} = j \) such that for any \( t > 0 \) the mapping \( f_i : \mathbb{R}^m_+ \to M \) defined by

   \[
   f_i(t_1, \ldots, t_m) = \Phi_i\left(t_1, \ldots, t_m, t - \sum_{l=1}^m t_l, \xi\right)
   \]

   has a nonempty open set of regular points in the simplex

   \[
   \Delta_{r,m} := \left\{(t_1, \ldots, t_m) \in \mathbb{R}^m_+ : \sum_{l=1}^m t_l < t\right\}.
   \]

2. The interior of \( L(\xi) \) is nonempty and dense in \( L(\xi) \).

**Theorem 5.** Assume that condition B holds at some \( \xi \in M \). Then:

1. For any \( i, j \in S \), there are an integer \( m > n \) and a vector \( i \in S^{m+1} \) with \( i_1 = i \) and \( i_{m+1} = j \) such that for any \( t > 0 \) the mapping \( F_i : \mathbb{R}^{m+1}_+ \to M \) defined by

   \[
   F_i(t_1, \ldots, t_{m+1}) = \Phi_i(t_1, \ldots, t_{m+1}, \xi)
   \]

   has a nonempty open set of regular points in \( \Delta_{r,m+1} \).

2. The interior of \( L(\xi) \) is nonempty and dense in \( L(\xi) \).

Condition A is stronger than condition B, so it is not surprising that the conclusion of theorem 4 implies the conclusion of theorem 5. We will not prove these theorems since they are direct consequences of results in [Jur97, chapter 3].

Theorem 4 shows that under condition A, we can find a sequence of driving vector fields such that using that sequence and varying only the switching times we can generate an open set of terminal positions for any fixed terminal time \( t > 0 \). Moreover, the map assigning the terminal position at time \( t \) to the switching time sequence is regular, i.e., its Jacobian has full rank. We will use this theorem to conclude that, under this map, the pushforward of an absolutely continuous measure is also absolutely continuous.

Under condition B, such regularity for a fixed time \( t \) is not guaranteed. However, theorem 5 shows that if it is allowed to vary also the terminal time \( t \), we still can generate an open set of terminal positions and the Jacobian of the corresponding map still has full rank. This means that although the pushforward measures themselves do not necessarily enjoy the desired regularity, their averages over terminal times \( t \) do, and we will use this argument to study the regularity of the resolvent measure of the Markov process under consideration.
The basic idea behind theorems 4 and 5 is that for a sufficient number of switches, by perturbing the switching time sequences one can generate perturbations to the terminal point in all directions.

The first statement of theorem 5 corresponds to theorem 3.1 in [Jur97], which reads as follows: Under the assumptions of theorem 5, any neighbourhood $U$ of $\xi$ contains points that are normally accessible from $\xi$ at arbitrarily small times. A point $\eta$ in $\mathcal{M}$ is called normally accessible from $\xi$ at time $t > 0$ if there exist vectors $i \in \mathcal{S}^{m+1}$ and $(i_1, \ldots, i_{m+1}) \in \Delta_{t, m+1}$ such that $\mathcal{F}_t(i_1, \ldots, i_{m+1}) = \eta$ and the differential $\mathcal{D}\mathcal{F}_t(i_1, \ldots, i_{m+1})$ has full rank. It is worth pointing out, though, that in [Jur97] only one sequence $i$ resulting in $\mathcal{F}$ with a regular point is constructed. But since the flow generated by any vector field is a family of diffeomorphisms, and since the set of points satisfying condition B is open, one can append any indices in front or at the back of that sequence without destroying the desired properties, and thus recover this part of theorem 5 as we state it.

The fact that the interior of $L(\xi)$ is nonempty and dense in $L(\xi)$ follows from theorem 3.2a in [Jur97]. Theorem 4 follows from applying theorem 3.1 [Jur97] to $\mathbb{R} \times \mathcal{M}$ and vector fields $1 \oplus u_i, i \in S$, where

$$(1 \oplus u)(r, \xi) := (1, u(\xi)), \quad (r, \xi) \in \mathbb{R} \times \mathcal{M},$$

and $1$ is the unit vector field on $\mathbb{R}$ corresponding to the derivation $\partial/\partial r$ and identically equal to 1 in the natural coordinates on $\mathbb{R}$.

### 3. Proof of theorem 2

We need to prove that for any $t > 0$ and $i \in S$, the measure $\mathcal{P}_{\xi,i}$ is not singular.

For any finite sequence $i$ of indices in $S$ with initial index $i$ (we will call these sequences admissible), let $C_i$ be the event that the driving vector fields up to time $t$ appear in the order determined by $i$. Since $\mathbb{P}_{\xi,i}(C_i) > 0$ for any admissible $i$, it suffices to find an admissible sequence $i$ such that $\mathbb{P}_{\xi,i}(\cdot | C_i)$ is not singular. We claim that this holds true for the sequence $i$ provided by theorem 4. According to theorem 4, there is an admissible sequence $i = (i_1, i_2, \ldots, i_{m+1})$ with $i_1 = i$ such that the function $f_i$ has a regular point in $\Delta_{t, m}$. Since the set of regular points of a differentiable function is open in its domain, the function $f_i$ is regular in a nonempty open set $B \subset \Delta_{t, m}$.

Let $T_1, T_2, \ldots, T_{m+1}$ be independent and exponentially distributed random variables such that $T_j$ has parameter $\lambda_j$ for $1 \leq j \leq m+1$.

On $C_i$, we have $A_j = i_{m+1}$ and the distribution of $X_j$ under $\mathbb{P}_{\xi,i}(\cdot | C_i)$ coincides with the distribution of $f_i(T_1, \ldots, T_m)$ conditioned on the event

$$R = \left\{ m \sum_{j=1}^m T_j < t \leq \sum_{j=1}^{m+1} T_j \right\}. \quad (4)$$

The distribution of the random vector $(T_1, \ldots, T_m)$ conditioned on $R$, is equivalent to the uniform distribution on the simplex

$$\Delta_{t, m} := \left\{ (t_1, \ldots, t_m) \in \mathbb{R}_+^m : \sum_{j=1}^m t_j < t \right\}.$$  

Now the theorem directly follows from the following result:

**Lemma 2.** Let $n, m \in \mathbb{N}, n \leq m$. Suppose that $B$ and $\Delta$ are nonempty open sets in $\mathbb{R}^m$, $B \subset \Delta$, and $M$ is an $n$-dimensional smooth manifold. If $f : \Delta \to M$ is differentiable on $B$ and all points in $B$ are regular for $f$, then for any absolutely continuous probability measure $\mu$ on $\Delta$ satisfying $\mu(B) > 0$, its pushforward $\mu f^{-1}$ is not singular with respect to $\lambda^M$.  

We will prove this lemma only for $M = \mathbb{R}^n$. Modifying the proof for the general case using coordinate patches on $M$ amounts only to notational differences.

We will use the following statement (see, e.g., proposition 4.4 in [DLS98]):

**Lemma 3.** Let $f : B \to \mathbb{R}^m$ be a Borel function a.e.-differentiable on an open set $B \subset \mathbb{R}^m$ and satisfying $\lambda^m \{ t \in B : \det Df(t) = 0 \} = 0$. If $\mu \ll \lambda^m$, then $\mu f^{-1} \ll \lambda^m$, and

$$\frac{d(\mu f^{-1})}{d\lambda^m}(s) = \sum_{t \in B : f(t) = s} |\det Df(t)|^{-1} \frac{d\mu}{d\lambda^m}(t).$$

**Proof of lemma 2.** We can find an open set $B' \subset B$ such that $\mu(B') > 0$ and there are $n$ columns of $Df(t)$ (without loss of generality, first $n$ columns) such that for any $t \in B'$ they are linearly independent. For $\rho : B' \to \mathbb{R}^n \times \mathbb{R}^{m-n}$ defined by

$$\rho(t) = (t_1, \ldots, t_m) \mapsto (f(t), t_{n+1}, \ldots, t_m),$$

and any $t \in B'$, we have $\det D\rho(t) \neq 0$. Therefore, by lemma 3, the pushforward of the restriction of $\mu$ to $B'$ under $\rho$ is a positive absolutely continuous measure on $\mathbb{R}^n \times \mathbb{R}^{m-n}$. Integrating over $\mathbb{R}^{m-n}$, we obtain that the pushforward of the restriction of $\mu$ to $B'$ under $f$ is a positive absolutely continuous measure on $\mathbb{R}^n$, and the proof is complete. $\square$

## 4. Proof of theorem 3

We need to show that $Q_{\xi,i}$ is not a singular measure. The proof is based on theorem 5.

For the $S$-valued process $A$ we denote by $I_t(A)$ the sequence of states visited by $A$ between $0$ and $t$. For any $m \in \mathbb{N}$ and any sequence $i \in S^m$, we can introduce an auxiliary measure $Q_{\xi,i}$ on $M$ by

$$Q_{\xi,i}(B) = \int_{\mathbb{R}^+} e^{-t} P_{\xi,i} \{ X_t \in B \text{ and } I_t(A) = i \} \, dt, \quad B \in \mathcal{B}(M).$$

Since

$$Q_{\xi,i}(B \times \{ j \}) = \sum_{i_1, i_2, \ldots, i_{m-1}, j} Q_{\xi,i}(B), \quad (5)$$

it is sufficient to find $i = (i_1, \ldots, i_m)$ with $i_1 = i$ such that $Q_{\xi,i}(M) > 0$ and

$$\bar{Q}_{\xi,i,i}(\cdot) = \frac{Q_{\xi,i,i}(\cdot)}{Q_{\xi,i,i}(M)}$$

is a nonsingular probability measure. To apply lemma 2, we need to represent $\bar{Q}_{\xi,i,i}$ as the pushforward of a measure, equivalent to Lebesgue measure, under a smooth map with a nonempty set of regular points.

Since condition B holds at $\xi$, theorem 5 yields an integer $m > n$ and a sequence $i = (i_1, i_2, \ldots, i_{m+1})$ with $i_1 = i$, such that the function $F_i : \mathbb{R}_{+}^{m+1} \to M$ defined by

$$F_i(t) = \Phi_i(t, \xi)$$

has a regular point. For this $i$ provided by theorem 5, $\bar{Q}_{\xi,i,i}$ is the distribution of $\Phi_i(T_1, \ldots, T_m, T - \sum_{j=1}^{m} T_j, \xi)$ conditioned on

$$R = \left\{ \sum_{j=1}^{m} T_j < T \leq \sum_{j=1}^{m+1} T_j \right\}, \quad (6)$$
where $T_1, \ldots, T_{m+1}$, and $T$ are independent random variables exponentially distributed with parameters $\lambda_1, \ldots, \lambda_{m+1}$, and 1, respectively.

Since the joint distribution of $T_1, \ldots, T_{m+1}, T$ is equivalent to Lebesgue measure and since event $R$ has positive probability, the distribution $\mu$ of $T_1, \ldots, T_m, T$ conditioned on $R$ induces a measure on

$$\Delta = \{(t_1, \ldots, t_m, t) \in \mathbb{R}^{m+1} : \sum_{j=1}^m t_j < t\}$$

that is equivalent to Lebesgue measure. The regularity of $F_i$ guaranteed by theorem 5 implies that the function $f_i : \Delta \to M$ defined by

$$f_i(t_1, \ldots, t_m, t) = F_i(t_1, \ldots, t_m, t - \sum_{j=1}^m t_j)$$

has a nonempty open set of regular points in $\Delta$, and the proof is completed by an application of lemma 2, since $Q_{\xi,i,i}$ is the pushforward of $\mu$ under $f_i$.

5. Absolute continuity of ergodic invariant measures

According to the ergodic decomposition theorem, all invariant measures for a Markov semigroup can be represented as convex combinations of ergodic ones (see, e.g., [Hai06, theorem 1.7]). We will use this to derive theorem 1 from absolute continuity of ergodic invariant distributions.

To define ergodicity, we need to recall the notion of $\mu$-invariant sets. Let $\mu$ be an invariant measure for the Markov semigroup $(P_t)$. We say that a set $A \in \mathcal{B}(M) \otimes \mathcal{P}(S)$ is $\mu$-invariant if for every $t \geq 0$, $P_t^\xi_j(A) = 1$ for $\mu$-almost every $(\xi, i) \in A$. An invariant measure $\mu$ is called ergodic if for every $\mu$-invariant set $A$, either $\mu(A) = 1$ or $\mu(A) = 0$.

The following is a basic result on systems with Markov switchings that does not use conditions A or B.

**Theorem 6.** If $\mu$ is $(P')$-invariant and ergodic then it is either absolutely continuous or singular.

**Proof.** Consider the Lebesgue decomposition $\mu = \mu_{ac} + \mu_s$, where $\mu_{ac}$ is absolutely continuous and $\mu_s$ is singular with respect to Lebesgue measure. Let us show that both $\mu_{ac}$ and $\mu_s$ are invariant.

For any $t > 0$, using the invariance of $\mu$, we can write

$$\mu_{ac} + \mu_s = \mu = \mu P = \mu_{ac} P' + \mu_s P' = \sum_{j=1}^k v_j + \mu_s P'$$

where

$$v_j(\cdot) = \int_M P_{\xi,j}(\cdot | C_i) \mu_{ac}(d\xi \times \{j\})$$

and $j \in S$.

We claim that the measures $v_j$, $j \in S$, are absolutely continuous. To see this we check that for any sequence $i = (i_1, \ldots, i_{m+1})$ with $i_1 = j$, the measure $v_i$ defined by

$$v_i(E) = \int_M P_{\xi,j}(X_j \in E | C_i) \mu_{ac}(d\xi \times \{j\})$$

$$= \int_M P\left(\Phi_i(T_1, \ldots, T_m, t - \sum_{l=1}^m T_l, \xi) \in E \bigg| R\right) \mu_{ac}(d\xi \times \{j\})$$

is absolutely continuous.
is absolutely continuous (here we use the notation introduced in section 3). Suppose \( \lambda^M(E) = 0 \). For fixed \( T_1, \ldots, T_m, T_{m+1} \), the map \( \Phi_i \) is a diffeomorphism in \( \xi \). Therefore, on event \( R \) introduced in (4), we have

\[
\mu_{ac}(\xi \times \{j\} : \Phi_i(T_1, \ldots, T_m, t - \sum_{l=1}^{m} T_l, \xi) \in E) = 0,
\]

and \( \nu_i(E) = 0 \) follows from disintegrating the right side of (9) and changing the order of integration.

Now, using (7) and the absolute continuity of \( \nu_j \), \( j \in S \), we can write

\[
\mu_{ac} = \sum_{j=1}^{k} v_j + (\mu_s P_{t})_{ac}. \tag{10}
\]

Since \( P_{t,\xi,j}(M \times S) = 1 \) for all \( \xi \) and \( j \), (8) implies \( \sum_{j=1}^{k} v_j(M \times S) = \mu_{ac}(M \times S) \). Therefore, applying (10) to \( M \times S \), we obtain that the absolutely continuous component of the measure \( \mu_s P_{t} \) is zero. In other words, \( \mu_s P_{t} \) is singular, and from (7) and the absolute continuity of \( v_j \), \( j \in S \), we obtain

\[
\mu_s = \mu_s P_{t}. \tag{11}
\]

In other words, \( \mu_s \) is invariant for \( (P_{t}) \). It follows from (7) that \( \mu_{ac} \) is also invariant. Since \( \mu \) is ergodic, it cannot be represented as a sum of two nontrivial invariant measures. This means that either \( \mu = \mu_{ac} \) or \( \mu = \mu_s \). □

We endow the state space \( S \) with the discrete topology and recall that a point \( (\xi, i) \in M \times S \) is contained in the support of a measure if and only if the measure of every open neighbourhood of \( (\xi, i) \) is positive.

**Theorem 7.** Let \( \mu \) be an ergodic invariant measure for \( (P_{t}) \). Assume that the support of \( \mu \) contains a point \( (\eta, i) \) such that condition B holds at \( \eta \). Then, \( \mu \) is absolutely continuous with respect to Lebesgue measure on \( M \times S \).

We will need several auxiliary statements.

**Lemma 4.** Let \( v \) be a finite Borel measure on \( M \times S \) with support \( K \). If \( U \) is any open set in \( M \times S \) whose intersection with \( K \) is nonempty, we have

\[
v(U \cap K) > 0.
\]

**Proof.** Assume that \( v(U \cap K) = 0 \). The complement of the support \( K \) has measure zero. Therefore

\[
v(U) = v(U \cap K) + v(U \cap K^c) = 0.
\]

Thus, \( U^c \) is a closed subset of \( M \times S \) whose complement has measure zero. From the definition of the support, we obtain \( K \subset U^c \). But then, \( U \cap K \) must be empty, a contradiction. □

**Proof of theorem 7.** According to theorem 6 we need to show that \( \mu \) is not singular. If \( \mu \) is singular, it is entirely supported on a zero Lebesgue measure set \( G \subset M \times S \), so \( \mu(G^c) = 0 \). Since \( \mu \) is \( (P_{t}) \)-invariant, it is also \( Q \)-invariant. Therefore, \( \mu(G^c) = \mu Q(G^c) \), and we see that \( \mu(V) = 0 \) where

\[
V = \{(\xi, j) \in M \times S : Q_{\xi,j}(G^c) > 0\}.
\]
Let $U$ be the set of points $\xi \in M$ where condition B holds. Due to theorem 3, $U \times S \subset V$, and we conclude that $\mu(U \times S) = 0$.

Recall that $U$ is an open subset of $M$, and $(U \times S) \cap \text{supp} \mu \neq \emptyset$ by assumption. Lemma 4 implies that $\mu((U \times S) \cap \text{supp} \mu) > 0$. The contradiction with $\mu(U \times S) = 0$ completes the proof. □

Of course, if one replaces condition B in theorem 7 with the stronger condition A the resulting statement holds automatically, but one can give a proof that does not involve the resolvent $Q$:

**Theorem 8.** Let $\mu$ be an ergodic invariant measure for $(P_t)$. Assume that the support of $\mu$ contains a point $(\eta, i)$ such that condition A holds at $\eta$. Then, $\mu$ is absolutely continuous with respect to the Lebesgue measure on $M \times S$.

**Proof.** According to theorem 6 we need to show that $\mu$ is not singular. If $\mu$ is singular, it is entirely supported on a zero Lebesgue measure set $G \subset M \times S$, so $\mu(G) = 0$. Since $\mu(G) = \mu(G')$, we see that $\mu(V) = 0$ where

$$V = \{(\xi, j) \in M \times S : P_{\xi, j}(G') > 0\}.$$  

Let $U$ be the set of points $\xi \in M$ where condition A holds. Due to theorem 2, $U \times S \subset V$, and we conclude that $\mu(U \times S) = 0$.

Recall that $U$ is an open subset of $M$, and $(U \times S) \cap \text{supp} \mu \neq \emptyset$ by assumption. Lemma 4 implies that $\mu((U \times S) \cap \text{supp} \mu) > 0$. The contradiction with $\mu(U \times S) = 0$ completes the proof. □

6. Proof of theorem 1

First, we establish two properties of the set $E = L \cap U$, where $U$ is the open set of points satisfying condition B.

**Lemma 5.** The set $E$ has nonempty interior.

**Proof.** By assumption, $\xi \in E$, so $U \neq \emptyset$ and $L(\xi) \cap U \neq \emptyset$ by continuity of the vector fields in $D$. Since $\xi \in U$, theorem 5 implies that $L(\xi)$ has nonempty interior that is dense in $L(\xi)$. Therefore, the set

$$V = L(\xi)^\circ \cap U$$

is nonempty and open. Clearly, $V \subset U$, and it remains to prove that $L(\xi)^\circ \subset L$. In fact, we even have that $L(\xi)^\circ \subset L$. To see that, let us fix any $\xi \in L(\xi)$, $\eta \in M$, and prove that $\xi \in L(\eta)$. Since $\xi \in L(\xi)$, we have

$$\xi = \Phi(t, \xi)$$

for some index sequence $i$ and some time sequence $t$. Let us fix a neighbourhood $W$ of $\zeta$. Since the mapping $x \mapsto \Phi(t, x)$ is continuous, the inverse image of $W$ under this map is an open neighbourhood of $\xi$. Since $\xi$ is $D$-approachable from $\eta$, this open neighbourhood of $\xi$ contains a point $D$-reachable from $\eta$. Hence, $W$ contains a point that is $D$-reachable from $\eta$. □

As an immediate corollary of lemma 5, the set $L$ has nonempty interior.

**Lemma 6.** Suppose $\mu$ is an invariant measure for $(P_t)$. If $G$ is a nonempty open subset of $L$ and $j \in S$, then $\mu(G \times \{j\}) > 0$. 


Proof. Let us assume that \( \mu(G \times \{j\}) = 0 \). Since \( \mu \) is \((P')\)-invariant, it is also \( Q \)-invariant, and we have

\[
0 = \mu(G \times \{j\}) = \sum_{i=1}^{k} \int_{M} Q_{\eta,i}(G \times \{j\}) \mu(d\eta \times \{i\}).
\]

For all \( i \in S \) and \( \mu(\cdot \times \{i\}) \)-almost every \( \eta \in M \), we thus obtain

\[
Q_{\eta,i}(G \times \{j\}) = 0.
\]

(12)

Let us choose \( \eta \) such that (12) holds true.

By assumption, we have \( G \subset L \subset L(\eta) \). Since \( G \) is open, \( G \cap L(\eta) \neq \emptyset \). So there exist a sequence \( i = (i, i_2, \ldots, i_m, j) \) and an interswitching time vector \( t = (t_1, \ldots, t_m, t_{m+1}) \) such that \( \Phi_t(t, \eta) \in G \). By continuity of \( \Phi_t \), there is a neighbourhood \( W \) of \( t \) in \( \mathbb{R}_{m+1}^n \) such that \( \Phi_s(s, \eta) \in G \) for all \( s \in W \). Denoting \( s = s_1 + \cdots + s_{m+1} \) and using the representation of \( P_{\eta,i}^t(\cdot | C_i) \) via exponentially distributed times that we used in the proof of theorem 2, we conclude that \( P_{\eta,i}^t(G \times \{j\}) > 0 \) for \( s \) sufficiently close to \( t = t_1 + \cdots + t_{m+1} \). Therefore, \( Q_{\eta,i}(G \times \{j\}) > 0 \) contradicting (12).

\( \square \)

Proof of theorem 1. As a consequence of Birkhoff’s ergodic theorem, any invariant measure can be written as a convex combination of ergodic invariant measures, see, e.g., theorem 1.7 in [Hai06]. Therefore, it suffices to show absolute continuity and uniqueness of an ergodic invariant measure.

Let us begin by deriving absolute continuity. If \( \mu \) is an ergodic invariant measure that satisfies the assumptions of theorem 1 then, due to theorem 7, it suffices to show that \( \mu \) implies that \( \Phi_1(t, \eta) \in G \). By continuity of \( \Phi_t \), there is a neighbourhood \( W \) of \( t \) in \( \mathbb{R}_{m+1}^n \) such that \( \Phi_t(t, \eta) \in G \) for all \( s \in W \). Denoting \( s = s_1 + \cdots + s_{m+1} \) and using the representation of \( P_{\eta,i}^t(\cdot | C_i) \) via exponentially distributed times that we used in the proof of theorem 2, we conclude that \( P_{\eta,i}^t(G \times \{j\}) > 0 \) for \( s \) sufficiently close to \( t = t_1 + \cdots + t_{m+1} \). Therefore, \( Q_{\eta,i}(G \times \{j\}) > 0 \) contradicting (12).

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\( \square \)
has an open set $O$ of regular points such that for all $t > 0$,
\[ \{ t = (t_1, \ldots, t_{m+1}) \in O : t_1 + \cdots + t_{m+1} < t \} \neq \emptyset. \]

Therefore, the map $F$ defined by
\[ F(t_1, \ldots, t_{m+1}, t) = f(t_1, \ldots, t_m, t - \sum_{l=1}^{m} t_l) \]
on
\[ \Delta = \{ (t_1, \ldots, t_{m+1}, t) \in \mathbb{R}_{+}^{m+2} : \sum_{l=1}^{m} t_l < t < \sum_{l=1}^{m+1} t_l \} \]
has an open set $V \subset \Delta$ of regular points such that
\[ \{ t = (t_1, \ldots, t_{m+1}, t) \in V : t < s \} \neq \emptyset, \quad s > 0. \]

Using the representation of $Q$ via (5) and the family of exponentially distributed times $T_1, \ldots, T_m, T$, we obtain that it is sufficient to prove that
\[ P\{ F(T_1, \ldots, T_m, T) \in M_{2,j} \mid R \} > 0, \]
where $R$ was introduced in (6).

Since $E^c$ is an open set containing $\eta$, and $F(V)$ is an open set such that $\eta \in \overline{F(V)}$ (due to (14) and continuity of $F$ at $0$), we obtain that $G = E^c \cap F(V)$ is also a nonempty open set.

Let us choose a vector $r \in V$ such that $F(r) \in E^c \cap F(V)$. Since $r$ is a regular point for $F$, we see that for an arbitrary choice of local smooth coordinates around $r$, there are $n$ independent columns of the matrix $DF(s)$ for $s$ in a small neighbourhood of $r$. Without loss of generality we can assume that these are the first $n$ columns. Then the map $\rho : \mathbb{R}^{m+2} \to M \times \mathbb{R}^{m+2-n}$ defined by
\[ \rho(s_1, \ldots, s_{m+1}, s) = (F(s_1, \ldots, s_{m+1}, s), s_{m+1}, \ldots, s_{m+1}, s) \]
has nonzero Jacobian in that neighbourhood. So we can choose an open set $W_V$ containing $r$ so that $\rho$ is a diffeomorphism between $W_V$ and $W_G \times W_{m-n} \subset G$ and $W_{m-n} \subset \mathbb{R}^{m+2-n}$ are some open sets.

The set $W_G$ is an open subset of $L$. It is also not empty since it contains $F(r)$. Lemma 6 implies that $\mu_2(W_G \times \{ j \}) > 0$. Since $\mu_2(M_{2,j} \times \{ j \}) = 0$, we conclude that $\mu_2(J \times \{ j \}) > 0$ where $J = M_{2,j} \cap W_G$. Since $\mu_2$ is an ergodic measure, it is absolutely continuous, so
\[ \lambda^M(J) > 0. \]

Since $J \subset M_{2,j}$, the desired inequality (15) will follow from
\[ P\{ F(T_1, \ldots, T_{m+1}, T) \in J \mid R \} > 0. \]

Since the joint distribution of $T_1, \ldots, T_m, T_{m+1}, T$ is equivalent to the Lebesgue measure on $\Delta$, lemma 3 implies that $\rho(T_1, \ldots, T_{m+1}, T)$ has positive density almost everywhere in $W_G \times W_{m-n}$. Integrating over $W_{m-n}$, we see that $F(T_1, \ldots, T_{m+1}, T)$ has positive density almost everywhere in $W_G$. Now (17) follows from (16).

Of course, theorem 1 remains true if one replaces condition B by the stronger condition A. However, under that condition one can prove this result without referring to the resolvent $Q$. Namely, one can use the regularity of transition probabilities established in theorem 2 (which is stronger than the regularity established in theorem 3), and invoke theorems 4 and 8 instead of theorems 5 and 7.
7. Examples

In this section, we apply theorem 1 to two concrete switching systems. In the first example, we have a closer look at the system on the $n$-dimensional torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ that was introduced in section 2.3. In the second example, we switch between two Lorenz vector fields with different parameter sets. For both systems, uniqueness of the invariant measure is derived from theorem 1. For the system on $\mathbb{T}^n$, we point out the invariant measure explicitly.

Let $M$ be the $n$-dimensional torus $\mathbb{T}^n$, and let $D = \{u_1, \ldots, u_n\}$ be the standard basis of $\mathbb{R}^n$. We assume for simplicity that the parameter $\lambda$ of the exponential time between any two switches is independent of the current state, and that we have a uniform probability of switching between any two states. In section 2.3 we implicitly argued that condition A does not hold for this system: if condition A was satisfied at some point $\xi \in \mathbb{T}^n$, the transition probability measures $P_{t,\xi,i}$ would not be singular with respect to Lebesgue measure, according to theorem 2. However, as pointed out in section 2.3, the measures $P_{t,\xi,i}$ are purely singular.

It is also instructive to show directly why condition A does not hold. As all the vector fields in $D$ are constant, the derived algebra $\mathcal{I}'(D)$ contains only the zero vector field. Thus, for any $\xi \in \mathbb{T}^n$,

$$I_0(D)(\xi) = \left\{ \sum_{i=1}^{n} \lambda_i u_i : \sum_{i=1}^{n} \lambda_i = 0 \right\}.$$

Due to the constraint $\sum_{i=1}^{n} \lambda_i = 0$, the algebra $I_0(D)(\xi)$ does not have full dimension, so condition A is violated at every point in $\mathbb{T}^n$.

On the other hand, condition B is clearly satisfied at any point $\xi \in \mathbb{T}^n$, as the standard basis of $\mathbb{R}^n$ applied to $\xi$ yields a full-dimensional set of vectors in the tangent space. Also note that any point in $\mathbb{T}^n$ is $D$-reachable from any other point. Therefore, theorem 1 guarantees that the associated Markov semigroup has a unique invariant measure, provided that such a measure exists. In this elementary example, it is possible to point out the invariant measure explicitly. For Borel sets $E \subset \mathbb{T}^n$ and states $i \in S$, it is given by

$$\mu(E \times \{i\}) = \frac{1}{n} \cdot \lambda(E).$$

Here, $\lambda$ denotes Lebesgue measure on $\mathbb{T}^n$.

The second example provides a situation where (i) the number of vector fields in $D$ is less than the dimension of the manifold $M$, and (ii) each individual vector field in $D$ gives rise to dynamics with a strange attractor and no absolutely continuous invariant measures, but (iii) the switched system has a unique invariant measure and it is absolutely continuous.

Namely, we consider switching between two Lorenz vector fields with different parameter values. A Lorenz vector field is a vector field defined in $\mathbb{R}^3$, of the form

$$u(x, y, z) = \begin{pmatrix} \sigma \cdot (y - x) \\ rx - y - xz \\ xy - bz \end{pmatrix},$$

where $\sigma$, $r$ and $b$ are physical parameters. Let the set $D$ contain exactly two Lorenz vector fields $u_1$ and $u_2$ such that $u_1$ has Rayleigh number $r = r_1 = 28$ and $u_2$ has a Rayleigh number $r = r_2$ different from, but close to, 28. We assume for both vector fields that $\sigma = 10$ and that $b = \frac{8}{3}$, which is the classical parameter choice for the Lorenz system. In [Tuc99], Tucker shows that the Lorenz system with parameters $\sigma = 10$, $r = r_1$ and $b = \frac{8}{3}$, corresponding to vector field $u_1$, admits a robust strange attractor $\Lambda$ as well as a unique SRB-measure supported on $\Lambda$. Robustness implies that the dynamical structure of the system remains intact under small
parameter changes, so the dynamics induced by \( u_2 \) share these features if \( r_2 \) is sufficiently close to \( r_1 \). Moreover, the SRB-measure on \( \Lambda \) satisfies a dissipative ergodic theorem, see e.g. \[BIJ83\], section 5.1. It follows that any point \( \xi \in \Lambda \) is \( \{u_1\}\)-approachable (and thus \( D\)-approachable) from every point in a set \( S_\xi \subset \mathbb{R}^3 \) with zero Lebesgue measure complement.

Assisted by a computer algebra system, we checked that condition A is satisfied for this system at any point in \( \mathbb{R}^3 \) that does not lie on the \( z \)-axis. Since the \( z \)-axis is invariant under the flows of both vector fields, we disregard it and set \( M \) to be \( \mathbb{R}^3 \) without points on the \( z \)-axis. With this provision, every point on the attractor \( \Lambda \) is \( D\)-approachable from any point in \( M \):

Consider a point \( \xi \in \Lambda \) and a point \( \eta \in M \). By theorem 4, there is a nonempty open set of \( D\)-reachable points from \( \eta \) (recall that condition A holds at any point in \( M \)). And since this open set has positive Lebesgue measure, it contains a point belonging to \( S_\xi \). Hence, \( \xi \) is \( D\)-approachable from \( \eta \).

The only remaining condition of theorem 1 that we need to check is existence of an invariant distribution. An elementary calculation similar to that for the case of one vector field (see, e.g., \[HSD04\], section 14.2) shows that if \( r_2 \) is sufficiently close to \( r_1 \), then the function

\[
V(x, y, z) = r_1x^2 + \sigma y^2 + \sigma (z - 2r_1)^2
\]

plays the role of a Lyapunov function for both vector fields \( u_1 \) and \( u_2 \). Namely, there is a number \( \nu > 0 \) such that \( \langle u_1, \nabla V \rangle < 0 \) and \( \langle u_2, \nabla V \rangle < 0 \) if \( V \geq \nu \). In particular, the compact set \( \{ (x, y, z) : V(x, y, z) \leq \nu \} \) is invariant for both vector fields, and a standard application of the Krylov–Bogolyubov method shows that the system has an invariant distribution.

As in the first example, uniqueness and absolute continuity of an invariant measure follow now from theorem 1.

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