Linear Programming Formulation of Boolean Satisfiability

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Abstract: In this paper, we present a new, graph-based modeling approach and a polynomial-sized linear programming (LP) formulation of the Boolean satisfiability problem (SAT). The approach is illustrated with a numerical example.

Keywords: Boolean Satisfiability; SAT; Linear Programming; Computational Complexity; Combinatorial Optimization.

1 Introduction

Boolean satisfiability (SAT) is of central importance in many areas of Operations Research and Computer Science. In its general (conjunctive normal) form, the problem can be stated as follows. There is a number of Boolean (binary) variables generically called “literals,” and a number of cover-type constraints implicitly defined over these variables, called “clauses.” Pairs of the literals may be negations of each other. For example, if \( x_1 \) is a (positive) literal, the corresponding “negative” literal is not-\( x_1 \) (or \( \overline{x_1} \)). A clause consists of a subset of the literals, and evaluates to TRUE if one or more of the literals in the subset are set TRUE. A conjunction of clauses is referred to as a “propositional” or “Boolean” formula. The problem is to find a “truth assignment” to the literals so that a given Boolean formula evaluates to TRUE, or to determine that no such assignment exists. Practical applications of this problem abound in Operations Research and Computer Science (see [4] for examples). A version of the problem that is often used in theoretical developments is the “1-in-3 SAT” (or the “exactly-1 3SAT”). In 1-in-3 SAT, each clause has exactly 3 literals, and a clause evaluates to TRUE iff exactly one of its literals is set (i.e., assigned a value of) TRUE. The reason for using 1-in-3 SAT in theoretical developments is that the general SAT is polynomially transformable to 1-in-3 SAT ([2], [7]). Also, because SAT was shown to be NP-complete (in fact, it was the first problem to be so [2]), the focus of research has been on the development of efficient enumeration schemes and heuristics (see [6] for an extensive survey).

In this paper, we develop a linear programming (LP) model of a generalized version of 1-in-3 SAT where clauses are allowed to have arbitrary numbers of literals, respectively. We refer to this problem as “exactly-1 SAT.” We use a bipartite network flow (BNF)-based model we develop, and a path-based modeling approach similar to that used in [3] to formulate this problem as a linear program. The approach is illustrated with a numerical example.

The plan of the paper is as follows. The BNF-based model is discussed in section 2. The path-based formulation is discussed in section 3. The overall LP model is discussed in section 4. Conclusions are discussed in section 5.

The following notation will be used throughout the rest of this paper.
Notation 1 (General notation):

1. $\mathcal{R}$: Set of real numbers;
2. For two column vectors $\mathbf{x}$ and $\mathbf{y}$, $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = (\mathbf{x}^T, \mathbf{y}^T)^T$ will be written as “$(\mathbf{x}, \mathbf{y})$” (where $(\cdot)^T$ denotes the transpose of $(\cdot)$), except for where that causes ambiguity;
3. $\mathbf{x}_i$: $i$th component of vector $\mathbf{x}$;
4. “0”: Column vector (of comfortable size) that has every entry equal to 0;
5. “1”: Column vector (of comfortable size) that has every entry equal to 1;
6. $\text{Conv}(\cdot)$: Convex hull of $(\cdot)$;
7. $\text{Ext}(\cdot)$: Set of extreme points of $(\cdot)$.

Definition 2

1. We say that a clause (or a conjunction of clauses) is “satisfied” if it evaluates to TRUE;
2. We refer to a complete set of truth assignments to the literals, as an “assignment;”
3. We refer to a complete set of truth assignments to the literals that satisfy a given Boolean formula as a “satisfying assignment” for the formula.

2 Bipartite network flow-based model

Notation 3 (Exactly-1 SAT parameters):

1. $X := \{x_1, x_2, \ldots, x_\ell\}$ (Set of literals);
2. $L := \{1, 2, \ldots, \ell\}$ (Set of indices for the literals);
3. $Y := \{(k, t) \in L^2 : x_k \text{ and } x_t \text{ are negations of one another}\}$;
4. $\nu := |Y|$;
5. $Z := \{c_1, c_2, \ldots, c_\varsigma\}$ (Set of clauses);
6. $C := \{1, \ldots, \varsigma\}$ (Index set for the clauses);
7. $a_{ij} = \begin{cases} 1 & \text{if } x_j \text{ is included in } c_i, \\ 0 & \text{otherwise.} \end{cases}$

Definition 4 We will henceforth refer to

$$P_0 := \text{Conv}\left(\left\{ u \in \{0,1\}^\ell : \sum_{j \in L} a_{ij} u_j = 1 \text{ for all } i \in C; \quad u_k + u_t = 1 \text{ for all } (k, t) \in Y \right\}\right). \quad (1)$$

as the “exactly-1 SAT polytope” (or simply, “SAT polytope,” for convenience).
Theorem 5 There exists a one-to-one correspondence between the extreme points of the SAT polytope and satisfying assignments for exactly-1 SAT.

Proof. Trivial. ■

In order to simplify the exposition of our linear programming formulation, we first develop a bipartite network flow-based model of $P_0$.

Assumption 6 We assume without loss of generality (w.l.o.g.) that a “logical” clause, $(l_k \lor l_t)$, has been added to the set of clauses for each $(k,t) \in Y$.

Notation 7 (BNF form notation):
1. $D := \{\varsigma + 1, \varsigma + 2, \ldots, \varsigma + \nu\}$ (Index set for the “logical” clauses);
2. $\overline{C} := C \cup D$;
3. $T_i := \{j \in L : a_{ij} = 1\} \quad \forall i \in \overline{C}$ (“Literal set” for clause $c_i$);
4. $\tau_i := |T_i| \quad \forall i \in \overline{C}$;
5. $K_j := \{i \in \overline{C} : a_{ij} = 1\} \quad \forall j \in L$ (“Clause set” for literal $x_j$);
6. $\kappa_j := |K_j| \quad \forall j \in L$;
7. $\Gamma_j := \{1, 2, \ldots, \kappa_j\} \quad \forall j \in L$ (Index set associated with $K_j$);
8. $m := |\overline{C}| = \varsigma + \nu$;
9. $n := \sum_{j \in L} \kappa_j$.

Assumption 8 We assume (w.l.o.g.) that:

1. The members of $K_j$ ($j \in L$) have been arranged in increasing order of clause indices, with $\alpha_{jk}$ ($k \in \Gamma_j$) as the index of the $k$th member (i.e., the ordering of $K_j$ is such that $\alpha_{jk} < \alpha_{jt}$ $\forall (k,t) \in \Gamma_j^2 : k < t$);
2. $a_{ik} \times a_{it} = 0 \quad \forall (k,t) \in Y, \forall i \in C$;
3. A “dummy” clause, $c_{m+1}$, has been added to the set of clauses, with $a_{m+1,j} = 1 \quad \forall j \in L$, and no restriction on the number of literals in $c_{m+1}$ that can be set TRUE in order for $c_{m+1}$ to evaluate to TRUE.
4. The (given) Boolean formula is satisfiable (i.e., there exists at least one satisfying assignment for the formula).

Remark 9 It follows from definitions that: $\forall (k,t) \in Y, a_{ik} \times a_{it} = 1$ for exactly one $i \in D$.

Notation 10 $\forall (i,j) \in (\overline{C}, T_i)$, $v_{ij}$ denotes a 0/1 variable that is equal to “1” iff clause $c_i$ is satisfied using literal $x_j$ (i.e., by setting literal $x_j$ TRUE).
The Bipartite Network Flow (BNF)-based reformulation of $P_0$ is as follows:

**Problem 11 (BNF polytope):**

\[ P_1 := \text{Conv}\{ v \in \mathbb{R}^{n+\ell} : \sum_{j \in L} a_{ij}v_{ij} = 1 \quad \forall i \in C \}; \]  
\[ \sum_{j \in L} v_{m+1,j} = n - m; \]  
\[ \sum_{i \in C} a_{ij}v_{ij} = \kappa_j \quad \forall j \in L; \]  
\[ v_{\alpha_k,j} - v_{\alpha_j,1,j} = 0 \quad \forall j \in L, \quad \forall k \in (K_j \setminus \{\alpha_j,1\}); \]  
\[ v_{ij} \in \{0,1\} \quad \forall j \in L; \quad \forall i \in K_j; \]  
\[ v_{m+1,j} \geq 0 \quad \forall j \in L \} \]  

Constraints (2) ensure (in light of constraints (6)) that each clause is satisfied using exactly one literal. Constraints (5) ensure that either all the clauses containing a given literal are satisfied using the literal (corresponding to setting the literal TRUE), or none of them is satisfied using the literal (corresponding to setting the literal FALSE). Hence, constraints (3)-(5) together ensure the consistency between the clause satisfactions and the truth assignments to the literals. Hence, Polytope $P_1$ correctly models exactly-1 SAT.

**Definition 12** We refer to $P_1$ as the “Bipartite Network Flow (BNF) polytope.”

**Theorem 13** The following statements hold true:

i) There exists a one-to-one correspondence between the points of the BNF polytope, $P_1$, and the points of the SAT polytope, $P_0$;

ii) There exists a one-to-one correspondence between the extreme points of the BNF polytope, $P_1$, and satisfying assignments for exactly-1 SAT.

**Proof.** Trivial. ■

The BNF-based formulation of exactly-1 SAT is illustrated in Example 14.

**Example 14** Let:
\[ \ell = 8 : \quad X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}; \]
\[ \nu = 2 : \quad Y = \{(2, 7), (4, 8)\} \text{ with } x_7 = \overline{x_2}, \text{ and } x_8 = \overline{x_4}; \]
\[ \varsigma = 6 : \quad c_1 = (x_1 \lor x_5 \lor x_6 \lor \overline{x_2}); \]
\[ c_2 = (x_2 \lor x_3 \lor x_5 \lor \overline{x_4}); \]
\[ c_3 = (x_1 \lor x_6); \]
\[ c_4 = (x_4 \lor \overline{x_2}); \]
\[ c_5 = (x_1 \lor x_2 \lor \overline{x_4}); \]
\[ c_6 = (x_2 \lor x_5). \]

The associated BNF tableau is:

|   | x_1 | x_2 | x_3 | x_4 | x_5 | x_6 | x_7 | x_8 | “Demand” |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----------|
| - |    |    |    |    | 1   | 1   | 1   | 1   |           |
| c_1 | 1 | | | | 1 | 1 | 1 | 1 |           |
| c_2 | | 1 | 1 | 1 | 1 |   |   |   |           |
| c_3 | | | | | | 1 |   |   |           |
| c_4 | | | | | |   | 1 | 1 |           |
| c_5 | | | | | | | 1 | 1 |           |
| c_6 | | | | | | 1 | |   |           |
| c_7 | | | | | | |   | 1 |           |
| c_8 | | | | | | |   |   |           |
| c_9 | | | | 1 | 1 | 1 | 1 | 1 | 13         |
| “Supply” | 3 | 3 | 1 | 2 | 3 | 2 | 4 | 3 | -         |

\[ \square \]

3 Reformulation of the BNF polytope

3.1 Multipartite graph representation

We reformulate the BNF polytope, \( P_1 \), in terms of flows over the multipartite digraph illustrated in Figure 1. In this graph, nodes correspond to triplets \((j, k, \alpha_{jk}) \in (L, \Gamma_j, K_j)\). In addition, there is a node in the graph corresponding to each triplet \((j, k, m + 1) \in (L, \Gamma_{j})\). The arcs of the graph are specified through the explicit statements of the forward and backward stars of the nodes.

Definition 15

1. We refer to the set of nodes of Graph \( G \) that correspond to a given pair \((j, k) \in (L, \Gamma_j)\) as a stage of the graph;

2. We refer to the set of nodes of Graph \( G \) that correspond to a given clause as a level of the graph.

In order to simplify the exposition, we perform a sequential indexing of the stages, as described below.
Notation 16 (Graph representation formalisms)

1. \( \Omega := \overline{C} \cup \{m+1\} \);
2. \( S := \{1, \ldots, n\} \) (Set of stages of Graph \( G \));
3. \( R := S \setminus \{n\} \);
   \[
   b_j := \begin{cases} 
   1; & \text{for } j = 1 \\
   \sum_{p=1}^{j-1} \tau_p + 1; & \text{for } j > 1
   \end{cases}
   \forall j \in L \) (Index of the stage corresponding to the pair \((j, 1)\));
4. \( e_j := \sum_{p=1}^{j} \tau_p \forall j \in L \) (Index of the stage corresponding to the pair \((j, \tau_j)\));
5. \( \chi_r := \max \{j \in L : b_j \leq r\} \forall r \in S \) (Index of the literal to which stage \( r \) pertains);
6. \( M_r := \{\alpha_{\chi_r} - b_{\chi_r} + 1\} \forall r \in S \);
7. \( N_r := M_r \cup \{m + 1\} = \{\alpha_{\chi_r} - b_{\chi_r} + 1, m + 1\} \forall r \in S \) (Set of indices of the clauses that define nodes at stage \( r \));
8. \( V := \{(i, r) \in (\Omega, S) : i \in N_r\} \) (Set of nodes of Graph \( G \));
9. \( F_r(t) := \begin{cases} 
N_r + 1 \setminus \{t\} & \text{for } r < n; \ r = e_{\chi_r}; \ t \neq m + 1 \\
N_r + 1 & \text{for } r < n; \ r = e_{\chi_r}; \ t = m + 1 \\
\{m + 1\} & \text{for } r < n; \ b_{\chi_r} \leq r < e_{\chi_r}; \ t = m + 1 \\
\emptyset & \text{for } r = n
\end{cases}
\forall r \in S, \forall t \in (K_{\chi_r} \cup \{m + 1\}) \) (Forward star of node \((t, r)\) of Graph \( G \));
10. \( B_r(t) := \begin{cases} 
\{p \in N_{r-1} : t \in F_{r-1}(p)\} & \text{for } r > 1 \\
\emptyset & \text{for } r = 1
\end{cases}
\forall r \in S, \forall t \in N_r \) (Backward star of node \((t, r)\) of Graph \( G \));
11. \( A := \{(i, r, j) \in (\Omega, R, \Omega) : i \in N_r, \ j \in F_r(i)\} \) (Set of arcs of Graph \( G \)).

The notation and structure of the graph representation are illustrated in Example 19 and Figure 2, respectively, for the numerical example shown in Example 14.

Remark 17

1. Each stage of Graph \( G \) comprises exactly two nodes of the graph;
2. The maximum number of arcs originating from any stage of Graph $G$ is four.

**Definition 18**

1. We refer to a path of Graph $G$ that spans the set of stages of the graph as a through-path of the graph;

2. We refer to a through-path of Graph $G$ that includes each clause in $C$ exactly once as a SAT path of the graph; that is, a set of arcs, $((i_1, 1, i_2), (i_2, 2, i_3), \ldots, (i_{n-1}, n-1, i_n)) \in A^{n-1}$, is a SAT path iff

\[
(\forall t \in C, \exists p \in S \ni i_p = t), \text{ and } (i_p \neq i_q \forall (i_p, i_q) \in \overline{C}^2, \forall (p, q) \in (S, S \setminus \{p\})).
\]

---

**Figure 1**: General structure of Graph $G$
Example 19

\( n = 21; \)

\( S = \{1, 2, \ldots, 21\}; \)

\( R = \{1, \ldots, 20\}; \)

\[
\begin{array}{ccc|ccc}
 j & b_j & c_j & r & \chi_r & M_r & N_r \\
 1 & 1 & 3 & 1 & 1 & \{1\} & \{1, 9\} \\
 2 & 4 & 6 & 2 & 1 & \{3\} & \{3, 9\} \\
 3 & 7 & 7 & 3 & 1 & \{5\} & \{5, 9\} \\
 4 & 8 & 9 & 4 & 2 & \{2\} & \{2, 9\} \\
 5 & 10 & 12 & 5 & 2 & \{6\} & \{6, 9\} \\
 6 & 13 & 14 & 6 & 2 & \{7\} & \{7, 9\} \\
 7 & 15 & 18 & 7 & 3 & \{2\} & \{2, 9\} \\
 8 & 19 & 21 & 8 & 4 & \{4\} & \{4, 9\} \\
 9 & 4 & \{8\} & \{8, 9\} \\
 10 & 5 & \{1\} & \{1, 9\} \\
 11 & 5 & \{2\} & \{2, 9\} \\
 12 & 5 & \{6\} & \{6, 9\} \\
 13 & 6 & \{1\} & \{1, 9\} \\
 14 & 6 & \{3\} & \{3, 9\} \\
 15 & 7 & \{1\} & \{1, 9\} \\
 16 & 7 & \{4\} & \{4, 9\} \\
 17 & 7 & \{5\} & \{5, 9\} \\
 18 & 7 & \{7\} & \{7, 9\} \\
 19 & 8 & \{2\} & \{2, 9\} \\
 20 & 8 & \{5\} & \{5, 9\} \\
 21 & 8 & \{8\} & \{8, 9\} \\
\end{array}
\]

The corresponding graph is shown in Figure 2.

\( \square \)

Remark 20  It follows directly from definitions that:

1. There exists a one-to-one correspondence between the SAT paths of Graph G and the extreme points of the BNF polytope, \( P_1 \);

2. There exists a one-to-one correspondence between the SAT paths of Graph G and the extreme points of the SAT polytope, \( P_0 \);

3. There exists a one-to-one correspondence between SAT paths of Graph G and satisfying assignments for exactly-1 SAT.

SAT paths are illustrated in Figures 3 and 4 for the numerical example shown in Example 14. The through-path shown in Figure 3 is a SAT path, and corresponds to the assignment \( (x_1 = x_2 = x_4 = TRUE, \quad x_3 = x_5 = x_6 = x_7 = x_8 = FALSE) \). The partial (i.e., non-spanning with
respect to the set of stages) path shown in Figure 4 corresponds to assignments in which $x_4$ and $x_5$ are both set TRUE. It is easy to verify that there exists no SAT path in the graph that comprises this partial path, which is consistent with the fact that there exists no satisfying assignment for the example problem in which both $x_4$ and $x_5$ are set TRUE.

**Theorem 21** A given SAT path of Graph G cannot be represented as a convex combination of other SAT paths of Graph G.

**Proof.** The theorem follows directly from the fact that every SAT path represents an extreme point of the standard shortest path network flow polytope associated with Graph G, $W := \left\{ w \in [0, 1]^{|[A]} : \sum_{i \in N_1} \sum_{j \in F_1(i)} w_{i,1,j} = 1; \right.$

$$\sum_{j \in F_r(i)} w_{irj} - \sum_{j \in B_r(i)} w_{jr,-1,i} = 0 \quad \forall \; r \in R \setminus \{1\}, \; \forall \; i \in N_r \right\}$$

(where $w$ is the vector of flow variables associated with the arcs of Graph G) (see [1]).

**Notation 22** We denote the set of all SAT paths of Graph G as $\Delta$; i.e.,

$$\Delta := \left\{ ((i_1, 1, i_2), (i_2, 2, i_3), \ldots, (i_{n-1}, n-1, i_n)) \in A^{n-1} : (\exists \; p \in S \ni i_p = t \; \forall \; t \in \overline{C}) ; \; \left( i_p \neq i_q \; \forall \; (i_p, i_q) \in \overline{C}^2, \; \forall \; (p, q) \in (S, S \setminus \{p\}) \right) \right\}.$$
Figure 3: Illustration of a SAT path of Graph G

Figure 4: Illustration of infeasibility for a SAT path
3.2 Integer programming reformulation

Assumption 23 We assume w.l.o.g. that the number of stages of Graph G is greater than 5 (i.e., \( n \geq 6 \)).

Notation 24:

1. \( \forall (p, s) \in R^2 : 1 < p < s; \forall (i, j, u, v, k, t) \in (N_1, F_1(i), N_p, F_p(u), N_s, F_s(k)), z_{(i,1,j)(upv)(kst)} \) denotes a non-negative variable that represents the amount of flow in Graph G that propagates from arc \((i, 1, j)\) onto arc \((k, s, t)\), via arc \((u, p, v)\);

2. \( \forall (r, s) \in R^2 : r < s, \forall (i, j, k, t) \in (N_r, F_r(i), N_s, F_s(k)), y_{(irj)(kst)} \) denotes a non-negative variable that represents the total amount of flow in Graph G that propagates from arc \((i, r, j)\) onto arc \((k, s, t)\).

The constraints of the Integer Programming (IP) version of our reformulation are as follows:

\[
\sum_{v \in B_p(u)} \sum_{i \in N_1} \sum_{j \in F_1(i)} \sum_{v \in F_2(j)} \sum_{t \in F_3(v)} z_{(i,1,j)(j,2,v)(v,3,t)} = 1 \quad (8)
\]

\[
\sum_{v \in B_p(u)} z_{(i,1,j)(kst)(v,p-1,u)} - \sum_{v \in F_p(u)} z_{(i,1,j)(kst)(upv)} = 0; \quad p, s \in R : 2 < s + 1 < p; i \in N_1; j \in F_1(i); k \in N_s; t \in F_s(k); u \in N_p \quad (9)
\]

\[
\sum_{v \in B_p(u)} z_{(i,1,j)(j,p-1,u)(kst)} - \sum_{v \in F_p(u)} z_{(i,1,j)(upv)(kst)} = 0; \quad p, s \in R : 2 < p < s; i \in N_1; j \in F_1(i); k \in N_s; t \in F_s(k); u \in N_p \quad (10)
\]

\[
y_{(i,1,j)(upv)} - \sum_{k \in N_s} \sum_{t \in F_s(k)} z_{(i,1,j)(upv)(kst)} = 0; \quad p, s \in R : 1 < p < s; i \in N_1; j \in F_1(i); u \in N_p; v \in F_p(u) \quad (11)
\]

\[
y_{(i,1,j)(kst)} - \sum_{u \in N_p} \sum_{v \in F_p(u)} z_{(i,1,j)(upv)(kst)} = 0; \quad p, s \in R : 1 < p < s; i \in N_1; j \in F_1(i); k \in N_s; t \in F_s(k) \quad (12)
\]

\[
y_{(i,1,j)(kst)} - \sum_{p \in R : \text{1 < p < s}} \sum_{v \in F_p(u)} z_{(i,1,j)(upv)(kst)} - \sum_{p \in R : \text{p > s}} \sum_{v \in B_{p+1}(u)} z_{(i,1,j)(kst)(upv)} = 0; \quad s \in R \backslash \{1\}; i \in N_1; j \in F_1(i); k \in N_s; t \in F_s(k); u \in \bigcup \{i, j, k, t\} \quad (13)
\]
\[ y_{(upv)(kst)} - \sum_{i \in N_1} \sum_{j \in F_1(i)} z_{(i,1,j)(upv)(kst)} = 0; \]

\[ p, s \in R : 1 < p < s; \ u \in N_p; \ v \in F_p(u); \ k \in N_s; \ t \in F_s(k) \]  \hspace{1cm} (14)

\[ \sum_{k \in (N_{r+1}\{j\}) t \in F_r(k)} y_{(irj)(k,r+1,t)} = 0; \ r \in R\{1\}; \ i \in N_r; \ j \in F_r(i) \]  \hspace{1cm} (15)

\[ \sum_{r \in (R\{n-1\})} \sum_{s \in (N_r \cap N_{r+1})} \sum_{k \in N_s} \sum_{t \in F_s(k)} y_{(irj)(kst)} + \sum_{r \in (R\{1\})} \sum_{s \in (N_r \cap N_{r+1})} y_{(irj)(kst)} + \]

\[ + \sum_{(r,s) \in R^2; s > r; j \in F_r(i)} \sum_{k \in B_{s+1}(i)} y_{(irj)(ksi)} + \sum_{(r,s) \in R^2; s > r; j \in F_r(i)} \sum_{k \in F_s(i)} y_{(irj)(isk)} + \]

\[ + \sum_{(r,s) \in R^2; s > r; j \in B_{s+1}(i)} \sum_{k \in B_{s+1}(i)} y_{(irj)(ksi)} + \sum_{(r,s) \in R^2; s > r+1; j \in B_{s+1}(i)} \sum_{k \in F_s(i)} y_{(jri)(isk)} = 0; \]

\[ i \in C \]  \hspace{1cm} (16)

\[ \sum_{r \in (R\{n-1\})} \sum_{i \in N_r \cap (N_r+1 \cap F_r(i))} \sum_{s \in (R\{1\})} \sum_{(k,t) \in (N_s, F_s(k))} y_{(irj)(kst)} + \]

\[ + \sum_{r \in (R\{1\})} \sum_{i \in N_r \cap (N_{r+1} \cap F_r(i))} \sum_{s \in (R\{1\})} \sum_{(k,t) \in (N_s, F_s(k))} y_{(irj)(kst)} = 0 \]  \hspace{1cm} (17)

\[ y_{(irj)(kst)} \in \{0,1\}; \ r, s \in R : 1 \leq r < s; \ (i, j, k, t) \in (N_r, F_r(i), N_s, F_s(k)); \]  \hspace{1cm} (18)

\[ z_{(i,1,j)(upv)(kst)} \in \{0,1\}; \ p, s \in R : 1 < p < s; \]

\[ (i, j, u, v, k, t) \in (N_1, F_1(i), N_p, F_p(u), N_s, F_s(k)). \]  \hspace{1cm} (19)

Constraint (17) (in light of constraints (11), (12), and (14)) restricts the modeling to variables involving arcs of Graph G only (i.e., the variables defined in Notation 24). The propagation of one unit of flow from stage 1 of Graph G is initiated by constraint (8). Constraints (9) and (10) ensure that all flows initiated at stage 1 propagate onward, to stage n of the graph, in a connected and balanced manner. Specifically, constraints (9) stipulate that the total amount of flow from arc \((i,1,j)\) that propagates through arc \((k,s,t)\) and enters node \((u,p)\) is equal to the amount of flow from arc \((i,1,j)\) that propagates through arc \((k,s,t)\) and leaves node \((u,p)\); Constraints (10) stipulate that the total amount of flow from arc \((i,1,j)\) that enters node \((u,p)\) to propagate on to
arc \( (k, s, t) \) is equal to the amount of flow from arc \( (i, 1, j) \) that leaves node \( (u, p) \) to propagate on to arc \( (k, s, t) \). Constraints (11) and (12) ensure that the propagation of the flow from a given arc at stage 1 of Graph \( G \) onto a given arc at another given stage of the graph is consistently accounted across all the other stages of the graph. Constraints (14) stipulate that the total amount of flow that propagates from arc \( (u, p, v) \) onto arc \( (k, s, t) \) is equal to the total of the flows from arcs at stage 1 that propagate onto arc \( (k, s, t) \) via arc \( (u, p, v) \). Constraints (13) require that the total flow on any given arc of Graph \( G \) must propagate on to every level of the graph pertaining to a clause in \( C \), or be part of a flow propagation that spans the levels of the graph pertaining to a clause in \( \overline{C} \). Constraints (15) ensure that the initial flow propagation from any given arc occurs in an “unbroken” fashion. Finally, constraints (16) stipulate (in light of the other constraints) that no part of the flow from arc \( (i, r, j) \) of Graph \( G \) can propagate back onto level \( i \) of the graph for \( i \in \overline{C} \), or onto level \( j \) for \( j \in \overline{C} \).

Theorem 25

i) The number of variables in the system (8)-(16) is \( O(\theta^2) \);

ii) The number of constraints in the system (8)-(16) is \( O(\theta^2) \);

where \( \theta := \ell \cdot (\ell + \varsigma) \).

Proof. Let \( \kappa_{\text{max}} := \max_{j \in L}\{\kappa_j\} \). Then, we have:

\[ \kappa_{\text{max}} \leq \varsigma + \nu \leq \varsigma + \frac{1}{2}\ell. \]  \hfill (20)

Hence,

\[ n := \sum_{j \in L} \kappa_j \leq \ell \cdot \kappa_{\text{max}} \leq \ell(\varsigma + \frac{1}{2}\ell) < \ell(\varsigma + \ell). \]  \hfill (21)

Condition i). Remark [17] and Notation [24.1] \( \implies \) an upper bound on the number of the \( z \)-variables is:

\[ UB_z = 2^6(n^2) \leq 2^6 \left( \ell(\varsigma + \frac{1}{2}\ell) \right)^2 < 2^6 (\ell(\varsigma + \ell))^2 = 2^6 \cdot \theta^2. \]  \hfill (22)

Similarly, Remark [17] and Notation [24.2] \( \implies \) an upper bound on the number of the \( y \)-variables is:

\[ UB_y = 2^4(n^2) \leq 2^4 \left( \ell(\varsigma + \frac{1}{2}\ell) \right)^2 < 2^4 (\ell(\varsigma + \ell))^2 = 2^4 \cdot \theta^2 \]  \hfill (23)

It follows directly from (22) and (23) that \( UB_z + UB_y \) is bounded by a degree-2 polynomial function of \( \theta \). Condition i) of the theorem follows directly from this.

Condition ii). From inspection, the numbers of the constraints (9) and (10) have the highest order of complexity respectively, of the classes of constraints in the system (8)-(16). Hence, it is sufficient to consider only one of these two classes of constraints in order to establish the complexity order of the total number of constraints of the system.
Remark 17 \[ UB_c = 2^5(n^2) \leq 2^5 \left( \ell(\varsigma + \frac{1}{2}\ell) \right)^2 < 2^{5}(\ell(\varsigma + \ell))^2 = 2^{5} \cdot \theta^2. \] Hence, the total number of constraints in the system (8)-(16) is bounded by a degree-2 polynomial function of \( \theta \). **Condition ii** follows directly from this.

**Definition 26**

1. We refer to the set of points in the space of the \( y \)- and \( z \)-variables that satisfy the system (8)-(19) as the "IP Polytope," and denote it by \( Q_I \); i.e., \( Q_I := \{ (y, z) \in \mathcal{R}^\omega : (y, z) \text{ satisfies (8)-(19)} \} \), where \( \omega \) is the number of variables in the system (8)-(19).

2. We refer to the linear programming relaxation of \( Q_I \) as the "LP Polytope," and denote it by \( Q_L \); i.e., \( Q_L := \{ (y, z) \in \mathcal{R}^\omega : (y, z) \text{ satisfies (8)-(16), and } 0 \leq (y, z) \leq 1 \} \), where \( \omega \) is the number of variables in the system (8)-(16).

**Theorem 27** \( (y, z) \in Q_I \iff \exists \text{ exactly one set of clause indices, } \{i_r \in N_r, r = 1, \ldots, n\}, \text{ such that:} \)

i) \[ z(a,1,b)(crd)(esf) = \begin{cases} 1 & \text{for } r, s \in R : 1 < r < s; \ (a, b, c, d, e, f) = (i_1, i_2, i_r, i_{r+1}, i_s, i_{s+1}) \\ 0 & \text{otherwise} \end{cases} \]

ii) \[ y(\text{arb})(csd) = \begin{cases} 1 & \text{for } r, s \in R : r < s; \ (a, b, c, d) = (i_r, i_{r+1}, i_s, i_{s+1}) \\ 0 & \text{otherwise} \end{cases} \]

iii) \( \exists p \in S : i_p = t \ \forall t \in \overline{C}; \)

iv) \( \forall (p, q) \in (S, S\setminus\{p\}), (i_p, i_q) \in \overline{C} \implies i_p \neq i_q. \)

**Proof.** Let \( (y, z) \in Q_I \). Then, given (18)-(19):

a) \[ (y, z) \in Q_I : \]

a.i) **Constraint (18) \implies \exists a unique set of clause indices, } \{i_r \in \Omega, r = 1, \ldots, 4\}, \text{ such that:} \]

\[ z(i_1,1,i_2,2,i_3,3,i_4) = 1 \] (25)

**Condition i** follows directly from the combination of (25), (9), and (10).

a.ii) **Condition ii** follows from the combination of **Condition i** with constraints (11), (12), (14), and (15).
a.iii) Condition iii) follows from the combination of Conditions i) and ii) with constraints (13).

a.iv) Condition iv) follows from the combination of Condition iii) with constraints (16).

b) \( \iff \): Trivial. \( \blacksquare \)

**Theorem 28** The following statements hold true:

i) There exists a one-to-one correspondence between the points of \( Q_I \) and the SAT paths of Graph \( G \);

ii) There exists a one-to-one correspondence between the points of \( Q_I \) and satisfying assignments for exactly-1 SAT;

iii) There exists a one-to-one correspondence between the points of \( Q_I \), and the extreme points of the SAT polytope, \( P_0 \);

iv) There exists a one-to-one correspondence between the points of \( Q_I \), and the extreme points of the BNF polytope, \( P_1 \);

v) \( \text{Ext}(Q_I) = Q_I \).

**Proof.** Conditions i) follows directly from the combination of Theorem 27 and Definition 15. Conditions ii) – iv) of the theorem follow from the combination of Conditions i) with Remark 20. Condition v) follows from the combination of Condition i) and Theorem 21. \( \blacksquare \)

### 3.3 Linear programming reformulation

Our linear programming reformulation of the BNF polytope, \( P_1 \), consists of \( Q_L \). We show that every point of \( Q_L \) is a convex combination of points of \( Q_I \), thereby establishing (in light of Theorems 21 and 28) the one-to-one correspondence between the extreme points of \( Q_L \) and the points of \( Q_I \).

**Lemma 29 (Flow conservation lemma 1)** Let \((y, z) \in Q_L\). The following holds true: \( \forall (a, b) \in (N_1, F_1(a)), \forall (p, q, r, s) \in R^4 : 1 < p < q; 1 < p < r < s, \)

\[
\sum_{i_p \in N_p} \sum_{j_p \in F_p(i_p)} \sum_{i_q \in N_q} \sum_{j_q \in F_q(i_q)} z(a, 1, b)(i_p, p, j_p)(i_q, q, j_q) = \sum_{i_r \in N_r} \sum_{j_r \in F_r(i_r)} \sum_{i_s \in N_s} \sum_{j_s \in F_s(i_s)} z(a, 1, b)(i_r, r, j_r)(i_s, s, j_s)
\]

**Proof.** \( \forall (a, b) \in (N_1, F_1(a)), \forall (p, q, r, s) \in R^4 : 1 < p < q; 1 < p < r < s, \)

\[
\sum_{i_p \in N_p} \sum_{j_p \in F_p(i_p)} \sum_{i_q \in N_q} \sum_{j_q \in F_q(i_q)} z(a, 1, b)(i_p, p, j_p)(i_q, q, j_q) = \sum_{i_p \in N_p} \sum_{j_p \in F_p(i_p)} y(a, 1, b)(i_p, p, j_p) \quad \text{(Using } 11\text{)}
\]

\[
= \sum_{i_p \in N_p} \sum_{j_p \in F_p(i_p)} \sum_{i_r \in N_r} \sum_{j_r \in F_r(i_r)} z(a, 1, b)(i_p, p, j_p)(i_r, r, j_r) \quad \text{(Using } 11\text{)}
\]

\[
= \sum_{i_r \in N_r} \sum_{j_r \in F_r(i_r)} \sum_{i_p \in N_p} \sum_{j_p \in F_p(i_p)} z(a, 1, b)(i_p, p, j_p)(i_r, r, j_r) \quad \text{(Re-arranging)}
\]

\[
= \sum_{i_r \in N_r} \sum_{j_r \in F_r(i_r)} y(a, 1, b)(i_r, r, j_r) \quad \text{(Using } 12\text{)}
\]

\[
= \sum_{i_r \in N_r} \sum_{j_r \in F_r(i_r)} \sum_{i_s \in N_s} \sum_{j_s \in F_s(i_s)} z(a, 1, b)(i_r, r, j_r)(i_s, s, j_s) \quad \text{(Using } 11\text{)}
\]
Lemma 30 (Flow propagation lemma 1) Let \((y, z) \in Q_L\). The following holds true:
\[
\forall (i_1, i_2, i_3, i_4) \in (N_1, F_1(i_1), N_3, F_3(i_3)),
\]
\[
y(i_1, i_2)(i_3, i_4) > 0 \iff \begin{cases}
  i) & i_3 \in F_2(i_2); \\
  ii) & z(i_1, i_2)(i_3, i_4) > 0.
\end{cases}
\tag{26}
\]

Proof. Constraints (12) for \(p = 2\) and \(s = 3\) can be written as:
\[
y(i_1, i_2)(i_3, i_4) - \sum_{u \in N_2} \sum_{v \in F_2(u)} z(i_1, i_2)(u, v)(i_3, i_4) = 0 \quad \forall (i_1, i_2, i_3, i_4) \in (N_1, F_1(i_1), N_3, F_3(i_3))
\tag{27}
\]
Constraints (15), (11) and (14) imply
\[
\begin{pmatrix}
  z(i_1, i_2)(i_3, i_4) > 0 \Rightarrow u = i_2, \text{ and } v = i_3 \\
  \forall (i_1, i_2, u, i_3, v, i_4) \in (N_1, F_1(i_1), N_2, N_3, F_3(i_3))
\end{pmatrix}
\tag{28}
\]
Using (28), (27) can be written as:
\[
y(i_1, i_2)(i_3, i_4) - z(i_1, i_2)(i_2, i_3)(i_3, i_4) = 0 \quad \forall (i_1, i_2, i_3, i_4) \in (N_1, F_1(i_1), N_3, F_3(i_3))
\tag{29}
\]
The lemma follows directly from the combination of (29), constraints (12), and constraints (17).

Lemma 31 (Flow propagation lemma 2) Let \((y, z) \in Q_L\). Then, we must have that:
\[
\forall r \in R : r \geq 4, \quad \forall (i_1, i_2, i_3, i_4, i_r, i_{r+1}) \in (N_1, F_1(i_1), N_3, F_3(i_3), N_r, F_r(i_r)),
\]
\[
z(i_1, i_2)(i_3, i_4)(i_r, i_{r+1}) > 0 \iff \begin{cases}
  i) & i_3 \in F_2(i_2); \\
  ii) & z(i_1, i_2)(i_2, i_3)(i_3, i_4) > 0; \\
  iii) & z(i_1, i_2)(i_2, i_3)(i_r, i_{r+1}) > 0.
\end{cases}
\tag{30}
\]
Proof. 

a) Constraints (11) imply
\[
\begin{pmatrix}
  z(i_1, i_2)(i_3, i_4)(i_r, i_{r+1}) > 0 \Rightarrow y(i_1, i_2)(i_3, i_4) > 0 \quad \forall r \in R : r \geq 4, \\
  \forall (i_1, i_2, i_3, i_4, i_r, i_{r+1}) \in (N_1, F_1(i_1), N_3, F_3(i_3), N_r, F_r(i_r)).
\end{pmatrix}
\tag{31}
\]
Conditions i) and ii) of the lemma follow directly from the combination of (31) and Lemma 30.

b) Using (11), (12), (14), (15), and (17), constraints (10) for \(p = 2\) and \(u = i_3\) can be written as:
\[
z(i_1, i_2)(i_2, i_3)(i_s, i_{s+1}) - \sum_{v \in F_3(i_3)} z(i_1, i_2)(i_3, v)(i_s, i_{s+1}) = 0
\]
\[
\forall s \in R : s \geq 4; \quad \forall (i_1, i_2, i_3, i_s, i_{s+1}) \in (N_1, F_1(i_1), F_2(i_2), N_s, F_s(i_s)).
\]

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Hence, in particular, we must have:

\[ z(i_1,i_2)(i_2,i_3)(i_3,i_4,i_5) - z(i_1,i_2)(i_3,i_4,i_5) = 0 \quad \forall \, r \in R : r \geq 4; \]

\[ \forall (i_1, i_2, i_3, i_4, i_5) \in (N_1, F_1(i_1), F_2(i_2), F_3(i_3), N_r, F_r(i_r)). \quad (32) \]

**Condition iii)** follows from the combination of **Condition i)** and (32). \(\blacksquare\)

**Lemma 32 (Flow propagation lemma 3)** *The following holds true for all \((y,z) \in Q_L:**

\[ \forall (i_1, i_2, i_3, i_4, i_5) \in (N_1, F_1(i_1), F_2(i_2), N_4, F_4(i_4)), \]

\[ z(i_1,i_2)(i_2,i_3)(i_3,i_4,i_5) > 0 \implies \begin{cases} 
\ i \ i_4 \in F_3(i_3); \\
\ ii \ z(i_1,1,i_2)(i_3,i_4,i_5) > 0; \\
\ iii \ z(i_1,i_2)(i_2,i_3)(i_3,i_4,i_5) > 0. 
\end{cases} \quad (33) \]

**Proof.**

a) Using \((11), (12), (14),\) and \((15),\) constraints \((10)\) for \(p = 3\) and \(u = i_3\) can be written as:

\[ z(i_1,i_2)(i_2,i_3)(i_3,i_4,i_5) - \sum_{v \in F_3(i_3)} z(i_1,i_2)(i_3,v)(i_4,i_5) = 0 \]

\[ \forall (i_1, i_2, i_3, i_4, i_5) \in (N_1, F_1(i_1), N_3, F_3(i_3), N_4, F_4(i_4)) \]

Constraints \((14) \implies \)

\[ \begin{pmatrix} \forall (i_1, i_2, i_3, v, i_4, i_5) \in (N_1, F_1(i_1), N_3, F_3(i_3), N_4, F_4(i_4)), \\
\ z(i_1,i_2)(i_3,v)(i_4,i_5) > 0 \implies y(i_3,v)(i_4,i_5) > 0 \end{pmatrix} \quad (35) \]

Constraints \((15) \implies \)

\[ \forall (i_3, v, i_4, i_5) \in (N_3, F_3(i_3), N_4, F_4(i_4)), y(i_3,v)(i_4,i_5) > 0 \implies v = i_4 \]

Constraints \((17)\) and **Statement** \((36) \implies \)

\[ \forall (i_3, i_4, i_5) \in (N_3, N_4, F_4(i_4)), y(i_3,i_4,i_5)(i_4,i_5) > 0 \implies i_4 \in F_3(i_3) \]

\[ \quad (37) \]

Using \((36)\) and \((37, 34)\) can be written as:

\[ z(i_1,i_2)(i_2,i_3)(i_3,i_4,i_5) - z(i_1,i_2)(i_3,i_4,i_5) = 0 \]

\[ \forall (i_1, i_2, i_3, i_4, i_5) \in (N_1, F_1(i_1), N_3, F_3(i_3), F_4(i_4)) \]

\[ \quad (38) \]

**Conditions i)** and **ii)** follow directly from \((37)\) and \((38).\)

b) Using **Condition i)** of the theorem, constraints \((11), (12), (14),\) and \((15),\) constraints \((9)\) for \(p = 4,\) and \(u = i_4\) can be written as:

\[ z(i_1,i_2)(i_2,i_3)(i_3,i_4) - \sum_{v \in F_4(i_4)} z(i_1,i_2)(i_2,i_3)(i_4,v) = 0 \]

\[ \forall (i_1, i_2, i_3, i_4) \in (N_1, F_1(i_1), N_3, F_3(i_3)) \quad (39) \]
Hence, in particular, we must have:

\[
\begin{align*}
&z(i_1,1,i_2)(i_2,i_3,i_4) - z(i_1,1,i_2)(i_2,i_3,i_4) \geq 0 \\
&\forall(i_1,i_2,i_3,i_4,i_5) \in (N_1, F_1(i_1), N_3, F_3(i_3), F_4(i_4)) \tag{40}
\end{align*}
\]

**Condition iii) follows directly from (40).**

**Lemma 33 (Flow propagation lemma 4)**  The following holds true for all \((y,z) \in Q_L:\)

\[
\forall r \in \mathbb{R} : 3 \leq r \leq n - 3,
\forall(i_1,i_2,i_r,i_{r+1},i_{r+2},i_{r+3}) \in (N_1, F_1(i_1), B_{r+1}(i_{r+1}), N_{r+1}, N_{r+2}, F_{r+2}(i_{r+2})),
\]

\[
z(i_1,1,i_2)(i_r,i_{r+1},i_{r+2},i_{r+3}) > 0 \implies \begin{cases}
  i) & i_{r+2} \in F_{r+1}(i_{r+1}) \\
  ii) & z(i_1,1,i_2)(i_{r+1},r+1,i_{r+2})(i_{r+2},r+2,i_{r+3}) > 0 \\
  iii) & z(i_1,1,i_2)(i_{r+1},r+1,i_{r+2})(i_{r+1},r+1,i_{r+2}) > 0
\end{cases}
\tag{41}
\]

**Proof.**

a) Using (11), (12), (14), and (15), constraints (10) for \(p = r + 1, s = r + 2,\) and \(u = i_{r+1},\) can be written as:

\[
\sum_{v \in B_{r+1}(i_{r+1})} z(i_1,1,i_2)(v,r,i_{r+1})(i_{r+2},r+2,i_{r+3}) - \sum_{v \in F_{r+1}(i_{r+1})} z(i_1,1,i_2)(i_{r+1},r+1,v)(i_{r+2},r+2,i_{r+3}) = 0
\]

\[
\forall r \in \mathbb{R} : 3 \leq r \leq n - 3,
\forall(i_1,i_2,i_{r+1},i_{r+2},i_{r+3}) \in (N_1, F_1(i_1), N_{r+1}, N_{r+2}, F_{r+2}(i_{r+2})),
\]

\[
z(i_1,1,i_2)(i_{r+1},r+1,v)(i_{r+2},r+2,i_{r+3}) > 0 \implies y(i_{r+1},r+1,v)(i_{r+2},r+2,i_{r+3}) > 0
\]

**Constraints (14) \implies**

\[
\left\{ \begin{array}{l}
\forall r \in \mathbb{R} : 3 \leq r \leq n - 3, \forall(i_1,i_2,i_{r+1},v,i_{r+2},i_{r+3}) \in \\
(N_1, F_1(i_1), N_{r+1}, F_{r+1}(i_{r+1}), N_{r+2}, F_{r+2}(i_{r+2})), \\
\end{array} \right.
\]

\[
z(i_1,1,i_2)(i_{r+1},r+1,v)(i_{r+2},r+2,i_{r+3}) > 0 \implies y(i_{r+1},r+1,v)(i_{r+2},r+2,i_{r+3}) > 0
\]

**Constraints (15) \implies**

\[
\left\{ \begin{array}{l}
\forall r \in \mathbb{R} : 3 \leq r \leq n - 3, \forall(i_{r+1},v,i_{r+2},i_{r+3}) \in \\
(N_{r+1}, F_{r+1}(i_{r+1}), N_{r+2}, F_{r+2}(i_{r+2})), \\
y(i_{r+1},r+1,v)(i_{r+2},r+2,i_{r+3}) > 0 \implies v = i_{r+2}
\end{array} \right.
\]

**Constraints (17) and Statements (13)-(14) \implies**

\[
\left\{ \begin{array}{l}
\forall r \in \mathbb{R} : 3 \leq r \leq n - 3, \forall(i_{r+1},i_{r+2},i_{r+3}) \in (N_{r+1}, N_{r+2}, F_{r+2}(i_{r+2})), \\
y(i_{r+1},r+1,i_{r+2})(i_{r+2},r+2,i_{r+3}) > 0 \implies i_{r+2} \in F_{r+1}(i_{r+1})
\end{array} \right.
\]

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Using (44) and (45), (42) can be written as:

\[
\sum_{v \in B_{r+1}(i_{r+1})} z(i_1, i_2)(v, r, i_{r+1})(i_{r+2}, r+2, i_{r+3}) - z(i_1, i_2)(i_{r+1}, r+1, i_{r+3})(i_{r+2}, r+2, i_{r+3}) = 0
\]

\(\forall r \in R : 3 \leq r \leq n - 3,\)

\(\forall (i_1, i_2, i_{r+1}, i_{r+2}, i_{r+3}) \in (N_1, F_1(i_1), N_{r+1}, F_{r+1}(i_{r+1}), F_{r+2}(i_{r+2}))\) (46)

Hence, in particular, we must have:

\[
z(i_1, i_2)(i_1, i_2)(i_{r+2}, r+2, i_{r+3}) - z(i_1, i_2)(i_{r+1}, r+1, i_{r+3})(i_{r+2}, r+2, i_{r+3}) \leq 0
\]

\(\forall r \in R : 3 \leq r \leq n - 3, \forall (i_1, i_2, i_r, i_{r+1}, i_{r+2}, i_{r+3}) \in (N_1, F_1(i_1), B_{r+1}(i_{r+1}), N_{r+1}, F_{r+1}(i_{r+1}), F_{r+2}(i_{r+2}))\) (47)

**Conditions i)** and **ii)** follow directly from (45) and (47).

b) Using **Condition i)** of the theorem and constraints (11), (12), (14), and (15), constraints (9) for \(p = r + 2, s = r, \) and \(u = i_{r+2}\) can be written as:

\[
z(i_1, i_2)(i_1, i_2)(i_{r+1}, r+1, i_{r+2}) - \sum_{v \in F_{r+2}(i_{r+2})} z(i_1, i_2)(i_r, i_{r+1})(i_{r+2}, r+2, v) = 0
\]

\(\forall r \in R : 3 \leq r \leq n - 3,\)

\(\forall (i_1, i_2, i_r, i_{r+1}, i_{r+2}) \in (N_1, F_1(i_1), B_{r+1}(i_{r+1}), N_{r+1}, F_{r+1}(i_{r+1}))\) (48)

Hence, in particular, we must have:

\[
z(i_1, i_2)(i_1, i_2)(i_{r+1}, r+1, i_{r+2}) - z(i_1, i_2)(i_r, i_{r+1})(i_{r+2}, r+2, i_{r+3}) \geq 0
\]

\(\forall r \in R : 3 \leq r \leq n - 3, \forall (i_1, i_2, i_r, i_{r+1}, i_{r+2}, i_{r+3}) \in (N_1, F_1(i_1), B_{r+1}(i_{r+1}), N_{r+1}, F_{r+1}(i_{r+1}), F_{r+2}(i_{r+2}))\) (49)

**Condition iii)** follows directly from (49). ■

**Notation 34** For \((y, z) \in Q_L:\)

1. The sub-graph of G induced by the positive components of \((y, z)\) is denoted as:

   \[H(y, z) := (V(y, z), A(y, z)),\]  

   where:
\[ \nabla(y, z) := \left\{ (i, 1) \in V : \sum_{j \in F_1(i)} \sum_{t \in F_2(j)} y_{i,1,j,j,2,t} > 0 \right\} \cup \]

\[ \left\{ (i, r) \in V : \sum_{a \in N_1} \sum_{b \in F_1(a)} \sum_{j \in B_r(i)} y_{a,1,birj} + \sum_{a \in N_1} \sum_{b \in F_1(a)} \sum_{j \in B_r(i)} y_{a,1,birj,r-1,i} > 0 \right\} \]

\[ \overline{A}(y, z) := \left\{ (i, 1, j) \in A : \sum_{t \in F_2(j)} y_{i,1,j,j,2,t} > 0 \right\} \cup \]

\[ \left\{ (i, r, j) \in A : \sum_{a \in N_1} \sum_{b \in F_1(a)} y_{a,1,birj} > 0 \right\}. \] (51)

2. The set of arcs of \( H(y, z) \) originating at stage \( r \) of \( H(y, z) \) is denoted \( \mathcal{A}_r(y, z) \);

3. The number of arcs originating at stage \( r \) of \( \text{Graph } H(y, z) \) is denoted \( \eta_r(y, z) = |\mathcal{A}_r(y, z)| \). For simplicity \( \eta_r(y, z) \) will be henceforth written as \( \eta_r \) (unless that causes ambiguity);

4. The index set associated with \( \mathcal{A}_r(y, z) \) is denoted \( \Lambda_r(y, z) := \{1, 2, \ldots, \eta_r\} \). For simplicity \( \Lambda_r(y, z) \) will be henceforth written as \( \Lambda_r \);

5. The \( \nu^{th} \) arc in \( \mathcal{A}_r(y, z) \) is denoted as \( a_{r,\nu}(y, z) \). For simplicity \( a_{r,\nu}(y, z) \) will be henceforth written as \( a_{r,\nu} \);

6. The tail of \( a_{r,\nu} \) is labeled \( i_{r,\nu}(y, z) \); the head of \( a_{r,\nu}(y, z) \) is labeled \( j_{r,\nu}(y, z) \). For simplicity, \( i_{r,\nu}(y, z) \) will be henceforth written as \( i_{r,\nu} \), and \( j_{r,\nu}(y, z) \), as \( j_{r,\nu} \);

7. Where that causes no confusion (and where that is convenient), for \( (r, s) \in R^2 : s \geq r \), and \( (\rho, \sigma) \in (\Lambda_r, \Lambda_s) \) \( "(i_{r,\nu}, j_{r,\nu}, a_{s,\nu}) \) will be henceforth written as \( "(y_{r,\nu}(s,\sigma)) \). Similarly, for \( (r, s) \in R^2 \) with \( 1 < r < s \) and \( (\alpha, \rho, \sigma) \in (\Lambda_1, \Lambda_r, \Lambda_s) \), \( "(i_{r,\nu}, j_{r,\nu}, a_{s,\nu}) \) will be henceforth written as \( "(z_{\alpha,\rho}(s,\sigma)) \).

Definition 35 (“Paths in \((y, z)\)”) Let \( (y, z) \in Q_L \). \( \forall (r, s) \in R^2 : s \geq \max\{3, r + 1\}, \forall (\nu_1, \nu_r, \nu_s) \in (\Lambda_1, \Lambda_r, \Lambda_s) \), a set of arcs of \( H(y, z) \),

\[ \{(a_{\nu_1, \nu_r, \ldots, a_{s,\nu_s}}) \in (E(y, z))^{s-\nu_1+1} : (z_{(\nu_1,\nu_r),(\nu_r,\nu_s),(\nu_s, s)}(q, q) > 0 \ \forall (p, q) \in R^2 : \max\{2, r\} \leq p < q < s - 1) ; \ (i_{p, q} = j_{p+1, q-1} \ \forall p \in (R \cap [r + 1, s]))\}, \]

is referred to as a "path in \((y, z)\) from \((r, \nu_r)\) to \((s, \nu_s)\)."

Notation 36 Let \((y, z) \in Q_L \). \( \forall (r, s) \in R^2 : s \geq \max\{3, r + 1\}, \forall (\rho, \sigma) \in (\Lambda_r, \Lambda_s) \),

1. The set of all paths in \((y, z)\) from \((r, \rho)\) to \((s, \sigma)\) is denoted \( U_{(r,\rho)}(s,\sigma)(y, z) \);

2. The index set associated with \( U_{(r,\rho)}(s,\sigma)(y, z) \) is denoted \( \Phi_{(r,\rho)}(s,\sigma)(y, z) := \{1, 2, \ldots, \varphi_{(r,\rho)}(s,\sigma)(y, z)\} \), where \( \varphi_{(r,\rho)}(s,\sigma)(y, z) := \left| U_{(r,\rho)}(s,\sigma)(y, z) \right| ; 20 \)
3. The \( k^{th} \) element of \( U_{(r,\rho),(s,\sigma)}(y, z) \) \((k \in \Phi_{(r,\rho),(s,\sigma)}(y, z))\) is denoted \( L_{(r,\rho),(s,\sigma),k}(y, z) \);

4. \( \forall k \in \Phi_{(r,\rho),(s,\sigma)}(y, z) \), the set of clauses included in \( L_{(r,\rho),(s,\sigma),k}(y, z) \) is denoted \( T_{(r,\rho),(s,\sigma),k}(y, z) \);

i.e., \( T_{(r,\rho),(s,\sigma),k}(y, z) := \{ t \in \Omega : \exists (a_{p,\nu_p} \in L_{(r,\rho),(s,\sigma),k}(y, z), p \in [r + 1, s - 1]) \ni i_{p,\nu_p} = t \} \cup \{ i_{r,\rho}, i_{s,\sigma}, j_{s,\sigma} \} \).

**Theorem 37 (Distinct paths in \((y,z)\))** Let \((y, z) \in Q_L\). Then, \( \forall (r, s) \in R^2 : s \geq \max\{3, r + 1\}, \forall (\alpha_1, \beta_1) \in (\Lambda_r, \Lambda_s) : U_{(r,\alpha_1),(s,\beta_1)}(y, z) \neq \emptyset, \forall k \in \Phi_{(r,\alpha_1),(s,\beta_1)}(y, z), \forall (\alpha_2, \beta_2) \in (\Lambda_r, \Lambda_s) : U_{(r,\alpha_2),(s,\beta_2)}(y, z) \neq \emptyset, \forall \ell \in \Phi_{(r,\alpha_2),(s,\beta_2)}(y, z), \{ L_{(r,\alpha_1),(s,\beta_1),k}(y, z) \neq L_{(r,\alpha_2),(s,\beta_2),\ell}(y, z) \} \iff \{ a_{g,\eta_1} \in L_{(r,\alpha_1),(s,\beta_1),k}(y, z) \text{ and } a_{g,\eta_2} \in L_{(r,\alpha_2),(s,\beta_2),\ell}(y, z) \} \).

**Proof.** The theorem follows directly from the combination of constraints (11), (12), (14), and (15), and Definition 35. 

**Theorem 38 (Path structure theorem 1)** Let \((y, z) \in Q_L\). The following holds true:
\( \forall (r, s) \in R^2 \text{ with } s \geq \max\{3, r + 1\}, \forall (\rho, \sigma) \in (\Lambda_r, \Lambda_s), y_{(r,\rho),(s,\sigma)} > 0 \iff U_{(r,\rho),(s,\sigma)}(y, z) \neq \emptyset. \)

**Proof.** First, note that it follows directly from the combination of Lemmas [30][33] that the theorem holds true for all \((r, s) \in R^2 \text{ with } s \in \{r + 1, r + 2\}, \text{ and all } (\nu_r, \nu_s) \in (\Lambda_r, \Lambda_s). \)

\( a \implies: \) Assume there exists an integer \( \omega \geq 2 \) such that the theorem holds true for all \((p, t) \in R^2 \text{ with } t = p + \omega, \text{ and all } (\nu_p, \nu_t) \in (\Lambda_p, \Lambda_t) \). We will show that the theorem must hold for all \((p, u) \in R^2 \text{ with } u = t + 1 = p + \omega + 1, \text{ and all } (\nu_p, \nu_u) \in (\Lambda_p, \Lambda_u) \).

Let \((p, u) \in R^2 \text{ with } u = p + \omega + 1, \text{ and } (\nu_p, \nu_u) \in (\Lambda_p, \Lambda_u) \) be such that:
\[
y_{(p,\nu_p),(u,\nu_u)} > 0.
\]

Define:
\[
I_{(p,\nu_p),(u,\nu_u)}(y, z) := \{ \alpha \in \Lambda_1 : z_{(1,\alpha)(p,\nu_p)}(u,\nu_u) > 0 \}.
\]

Then, (11), (12) and (53) \( \implies \)
\[
i) \quad I_{(p,\nu_p),(u,\nu_u)}(y, z) \neq \emptyset, \quad \text{with}
\]
\[
ii) \quad y_{(p,\nu_p),(u,\nu_u)}(y, z) = \sum_{\alpha \in C_{(p,\nu_p),(u,\nu_u)}(y, z)} z_{(1,\alpha)(p,\nu_p)}(u,\nu_u)
\]

Condition (55), and constraints (10) and (15) \( \implies \)
\[
\forall \alpha \in I_{(p,\nu_p),(u,\nu_u)}(y, z), \quad \exists J_{\alpha,(p,\nu_p),(u,\nu_u)}(y, z) \subseteq \Lambda_{p+1} =: \]
\[
i) \quad i_{p+1,\beta} = j_{p,\nu_p} \quad \forall \beta \in J_{\alpha,(p,\nu_p),(u,\nu_u)}(y, z)
\]
\[
ii) \quad z_{(1,\alpha)(p+1,\beta)}(u,\nu_u) > 0 \quad \forall \beta \in J_{\alpha,(p,\nu_p),(u,\nu_u)}(y, z); \quad \text{and}
\]
\[
iii) \quad z_{(1,\alpha)(p,\nu_p)}(u,\nu_u) \leq \sum_{\beta \in J_{\alpha,(p,\nu_p),(u,\nu_u)}(y, z)} z_{(1,\alpha)(p+1,\beta)}(u,\nu_u).
\]
Also, it follows from the combination of condition (53), constraints (9), and constraints (13), that:

\[ \forall \beta \in J_{\alpha,p}(u,v_{u}) (y,z) \]

(57)

Also, it follows from the combination of condition (53), constraints (9), and constraints (13), that:

\[ \forall \beta \in J_{\alpha,p}(u,v_{u}) (y,z) \]

(58)

Hence, \( \forall \beta \in J_{\alpha,p}(u,v_{u}) (y,z) \) and \( \forall t \in \Upsilon_{(p,v_{p})}(u,v_{u}) (y,z) \),

\[ \mathcal{L} := (\mathcal{L}_{(p,1,\beta)}(u,v_{u}),t_{y},z) \cup \{a_{p,v_{p}}\} \]

is a path in \((y,z)\) from \((p,v_{p})\) to \((u,v_{u})\). Hence, we have that \( U_{(p,v_{p})}(u,v_{u}) (y,z) \neq \varnothing \).

b) \( \iff \): Follows directly from Definition (55) and constraints (14).

Corollary 39 Let \((y,z) \in Q_{L}\). The following hold true:

i) \( \forall s \in R \setminus \{1\} \), \( \forall (\alpha,\sigma) \in (\Lambda_{1},\Lambda_{s}) \), \( y_{(1,\alpha)(s,\sigma)} > 0 \iff U_{(1,\alpha)(s,\sigma)} (y,z) \neq \varnothing \).

ii) \( \forall (r,s) \in (R \setminus \{1\})^{2} \) with \( s \geq \max\{3,r+1\} \), \( \forall (\alpha,\rho,\sigma) \in (\Lambda_{1},\Lambda_{r},\Lambda_{s}) \),

\[ z_{(1,\alpha)(r,\rho)(s,\sigma)} > 0 \iff \begin{cases} \text{(ii.1)} & U_{(1,\alpha)(s,\sigma)} (y,z) \neq \varnothing, \text{ and} \\ \text{(ii.2)} & \exists \kappa \in \Phi_{(1,\alpha)(s,\sigma)} (y,z) \exists a_{r,\rho} \in \mathcal{L}_{(1,\alpha),(s,\sigma),\kappa} (y,z). \end{cases} \]

Theorem 40 Let \((y,z) \in Q_{L}\). The following hold true:

\( \forall (\alpha,\beta) \in (\Lambda_{1},\Lambda_{n-1}) : U_{(1,\alpha)(n-1,\beta)} (y,z) \neq \varnothing, \forall k \in \Phi_{(r,\rho)(s,\sigma)} (y,z) \),

i) \( \overline{\mathcal{T}} \subseteq \mathcal{T}_{(1,\alpha),(s,\sigma),k} (y,z) \);

ii) \( \forall (p,q) \in (S,S \setminus \{p\}), (i_{p,v_{p}},i_{q,v_{q}}) \in (\mathcal{T}_{(1,\alpha)(s,\sigma),k} (y,z) \setminus \{m+1\})^{2} \implies i_{p,v_{p}} \neq i_{q,v_{q}} \)

Proof. Condition i) follows from the combination of constraints (13) and Corollary 39. Condition ii) follows from the combination of constraints (16) and Definition 35.

Definition 41 (“SAT path in \((y,z)\)”): Let \((y,z) \in Q_{L}\). \( \forall (\nu_{1},\nu_{n-1}) \in (\Lambda_{1},\Lambda_{n-1}) \), a path in \((y,z)\) from \((1,\nu_{1})\) to \((n-1,\nu_{n-1})\) is referred to as a “SAT path in \((y,z)\) (from \((1,\nu_{1})\) to \((n-1,\nu_{n-1})\))”.

Notation 42 Let \((y,z) \in Q_{L}\). For all \((\alpha,\beta) \in (\Lambda_{1},\Lambda_{n-1})\),

1. The set of all paths in \((y,z)\) from \((1,\alpha)\) to \((n-1,\beta)\) is denoted as \( \Pi_{\alpha\beta} (y,z) \);

2. The index set associated with \( \Pi_{\alpha\beta} (y,z) \) is denoted \( \Psi_{\alpha\beta} (y,z) := \{1, 2, \ldots, \pi_{\alpha\beta} (y,z)\} \), where \( \pi_{\alpha\beta} (y,z) := |\Pi_{\alpha\beta} (y,z)| \);

3. The \( k \)th element of \( \Pi_{\alpha\beta} (y,z) \) is denoted \( \mathcal{P}_{\alpha\beta k} (y,z) \).
Remark 43 Let \((y, z) \in Q_L\). \(\forall (\alpha, \beta) \in (\Lambda_1, \Lambda_{n-1})\),

1. \(\Pi_{\alpha\beta}(y, z) = U_{(1,\alpha)(n-1,\beta)}(y, z)\);
2. \(\Psi_{\alpha\beta}(y, z) = \Phi_{(1,\alpha)(n-1,\beta)}(y, z)\);
3. \(\pi_{\alpha\beta}(y, z) = \varphi_{(1,\alpha)(n-1,\beta)}(y, z)\);
4. We assume (w.l.o.g.) that: \(\mathcal{P}_{\alpha\beta}(y, z) = \mathcal{L}_{(1,\alpha)(n-1,\beta),k}(y, z) \quad \forall k \in \pi_{\alpha\beta}(y, z)\).

Theorem 44 (Equivalences for SAT paths in \((y,z)\)) For \((y, z) \in Q_L\):

i) Every SAT path in \((y, z)\) corresponds to exactly one SAT path of Graph \(G\);

ii) Every SAT path in \((y, z)\) corresponds to exactly one extreme point of the SAT polytope, \(P_0\);

iii) Every SAT path in \((y, z)\) corresponds to exactly one extreme point of the BNF polytope, \(P_1\);

iv) Every SAT path in \((y, z)\) corresponds to exactly one satisfying assignments for exactly-1 SAT;

v) Every SAT path in \((y, z)\) corresponds to exactly one point of \(Q_I\).

Proof. Condition i) follows from the combination of Theorem 40 and Definition 18.2. Conditions ii) – v) follow from the combination of Condition i) with Theorems 55 and 28 and Remark 20.

Theorem 45 (“Convex independence” of SAT paths in \((y,z)\)) Let \((y, z) \in Q_L\). A given SAT path in \((y, z)\) cannot be represented as a convex combination of other SAT paths in \((y, z)\).

Proof. The theorem follows directly from the combination of Theorems 44 and 21.

Theorem 46 (Path structure theorem 2) Let \((y, z) \in Q_L\). The following holds true: \(\forall r \in R, \forall \rho \in \Lambda_r, \exists \{ (\alpha, \beta) \in (\Lambda_1, \Lambda_{n-1}); \ i \in \Psi_{\alpha\beta}(y, z) \} \ni \{a_{r,\rho} \in \mathcal{P}_{\alpha\beta}(y, z)\} \).

Proof. Case 1: \(r = 1\). From (51) and (52):

\[ a_{r,\rho} \in \overline{A}(y, z) \implies \exists \alpha \in \Lambda_2 \ni y_{(r,\rho)(2,\alpha)} > 0. \quad (59) \]

Condition (59) and constraints (11) \(\implies\)

\[ \exists \mu \in \Lambda_{n-1} \ni z_{(r,\rho)(2,\alpha)(n-1,\mu)} > 0. \quad (60) \]

Condition (60) and constraints (11) \(\implies\)

\[ \exists \mu \in \Lambda_{n-1} \ni y_{(r,\rho)(n-1,\mu)} > 0. \quad (61) \]

The theorem follows from the combination of (61) with Theorem 38.

Case 2: \(r = n-1\). From (51) and (52):

\[ a_{r,\rho} \in \overline{A}(y, z) \implies \exists \alpha \in \Lambda_1 \ni y_{(1,\alpha)(r,\rho)} > 0. \quad (62) \]
The theorem follows from the combination of (62) with Theorem 38.

Case 3: $1 < r < n-1$. From (51) and (52):

$$a_{r,\rho} \in \overline{A}(y, z) \iff \exists \alpha \in \Lambda_1 \implies y_{(1,\alpha)(r,\rho)} > 0.$$  \[ (63) \]

Condition (63) and constraints (11) \implies \exists \mu \in \Lambda_{n-1} \implies z_{(1,\alpha)(r,\rho)(n-1,\mu)} > 0. \[ (64) \]

The theorem follows from the combination of (64) with Corollary 39.

**Corollary 47** Let $(y, z) \in Q_L$. The following hold true:

i) $\forall \alpha \in \Lambda_1, \forall r \in R \setminus \{1\}, \forall \rho \in \Lambda_r$,

$$y_{(1,\alpha)(r,\rho)} > 0 \iff \exists (\beta \in \Lambda_{n-1}; \iota \in \Psi_{\alpha\beta}(y, z)) \ni a_{r,\rho} \in \mathcal{P}_{\alpha\beta\iota}(y, z);$$ \[ (65) \]

ii) $\forall (r, s) \in R^2: 1 < r < s, \forall (\alpha, \rho, \sigma) \in (\Lambda_1, \Lambda_r, \Lambda_s)$,

$$z_{(1,\alpha)(r,\rho)(s,\sigma)} > 0 \iff \exists (\beta \in \Lambda_{n-1}; \iota \in \Psi_{\alpha\beta}(y, z)) \ni (a_{r,\rho}, a_{s,\sigma}) \in \mathcal{P}_{\alpha\beta\iota}^2(y, z).$$ \[ (66) \]

**Lemma 48 (Flow conservation lemma 2)** Let $(y, z) \in Q_L$. The following hold true:

i) $y_{(1,\alpha)(r,\nu_r)} = \sum_{\nu_p \in \Lambda_p \beta \in \Lambda_{n-1}} \sum_{\iota \in \Psi_{\alpha\beta}(y, z)} \sum_{a_{p,\nu_p} \in \mathcal{P}_{\alpha\beta\iota}(y, z)} z_{(1,\alpha)(p,\nu_p)(r,\nu_r)}$

$\forall \alpha \in \Lambda_1, \forall (p, r) \in R^2: 1 < p < r, \forall \nu_r \in \Lambda_r$

ii) $y_{(1,\alpha)(r,\nu_r)} = \sum_{\nu_q \in \Lambda_q \beta \in \Lambda_{n-1}} \sum_{\iota \in \Psi_{\alpha\beta}(y, z)} \sum_{a_{q,\nu_q} \in \mathcal{P}_{\alpha\beta\iota}(y, z)} z_{(1,\alpha)(r,\nu_r)(q,\nu_q)}$

$\forall \alpha \in \Lambda_1, \forall (r, q) \in R^2: 1 < r < q, \forall \nu_r \in \Lambda_r$

**Proof.** The lemma follows directly from the combination of constraints (11) and (12), and Theorems 37, 45, and 46, and Corollary 47.

**Definition 49 (”Weights” of SAT paths in $(y, z)$)** Let $(y, z) \in Q_L$. For $(\alpha, \beta) \in (\Lambda_1, \Lambda_{n-1})$ such that $y_{(1,\alpha)(n-1,\beta)} > 0$, and $k \in \Psi_{1,n-1}(y, z)$, we refer to the quantity

$$\omega_{\alpha\beta k}(y, z) := \min_{(p, q) \in R^2: (\nu_{p,\nu_q}) \in (\Lambda_p, \Lambda_q): 1 < p < q; (a_{p,\nu_p}, a_{q,\nu_q}) \in \mathcal{P}_{\alpha\beta\iota}^2(y, z)} \left\{ z_{(1,\alpha)(p,\nu_p)(q,\nu_q)} \right\}$$ \[ (67) \]

as the ”weight” of (SAT path in $(y, z)$) $\mathcal{P}_{\alpha\beta k}(y, z)$.

**Remark 50** It follows directly from Definitions 35 and 49 that for $(y, z) \in Q_L$, $\omega_{\alpha\beta k}(y, z) > 0 \forall (\alpha, \beta) \in (\Lambda_1, \Lambda_{n-1}): \Psi_{\alpha\beta}(y, z) \neq \emptyset, \forall \iota \in \mathcal{P}_{\alpha\beta k}(y, z)$.
Theorem 51 (Path structure theorem 3) Let \((y, z) \in Q_L\). The following hold true:

i) \(\forall \alpha \in \Lambda_1, \forall r \in R \setminus \{1\}, \forall \rho \in \Lambda_r, \)
\[
y(1,\alpha)(r,\rho) = \sum_{\beta \in \Lambda_{n-1}} \sum_{\iota \in \Psi_{\alpha\beta}(y,z): a_r, \rho \in P_{\alpha\beta}(y,z)} \omega_{\alpha\beta\iota}(y, z)
\]

ii) \(\forall (r, s) \in R^2 : 1 < r < s, \forall (\alpha, \rho, \sigma) \in (\Lambda_1, \Lambda_r, \Lambda_s), \)
\[
z(1,\alpha)(r,\rho)(s,\sigma) = \sum_{\beta \in \Lambda_{n-1}} \sum_{\iota \in \Psi_{\alpha\beta}(y,z): (a_r, \rho, a_s, \sigma) \in P_{\alpha\beta}(y,z)} \omega_{\alpha\beta\iota}(y, z)
\]

iii) \(\forall (r, s) \in R^2 : 1 < r < s, \forall (\rho, \sigma) \in (\Lambda_r, \Lambda_s), \)
\[
y(r,\rho)(s,\sigma) = \sum_{\alpha \in \Lambda_1} \sum_{\beta \in \Lambda_{n-1}} \sum_{\iota \in \Psi_{\alpha\beta}(y,z): (a_r, \rho, a_s, \sigma) \in P_{\alpha\beta}(y,z)} \omega_{\alpha\beta\iota}(y, z)
\]

Proof. a) Condition i. First, note that from the combination of constraints \((8), (9), (10), \)
and \((15)\) with Remark \(50\) Theorem \((45)\), and Theorem \((46)\), we must have:
\[
\sum_{\alpha \in \Lambda_1} \sum_{\beta \in \Lambda_2} \sum_{\iota \in \Lambda_3: \iota = j_1, \alpha \iota = j_2, \beta} \frac{z(1,\alpha)(2,\beta)(3,\delta)}{2} = \sum_{\alpha \in \Lambda_1} \sum_{\beta \in \Lambda_{n-1}} \sum_{\iota \in \Psi_{\alpha\beta}(y,z)} \omega_{\alpha\beta\iota}(y, z) = 1. \quad (68)
\]

a.1) From Definition \(49\), \((68) \Rightarrow \)
\[
\sum_{\beta \in \Lambda_2: \iota_2, \alpha \iota = j_1, \alpha \iota = j_2, \beta} \frac{z(1,\alpha)(2,\beta)(3,\delta)}{2} = \sum_{\beta \in \Lambda_{n-1}} \sum_{\iota \in \Psi_{\alpha\beta}(y,z)} \omega_{\alpha\beta\iota}(y, z) \quad \forall \alpha \in \Lambda_1 \quad (69)
\]

Lemma \(29\) and relations \((69) \Rightarrow \)
\[
\sum_{\rho \in \Lambda_r} \sum_{\sigma \in \Lambda_s} \frac{z(1,\alpha)(r,\rho)(s,\sigma)}{2} = \sum_{\rho \in \Lambda_{n-1}} \sum_{\iota \in \Psi_{\alpha\rho}(y,z)} \omega_{\alpha\rho\iota}(y, z) \quad \forall \alpha \in \Lambda_1, \forall (r, s) \in R^2 : 1 < r < s \quad (70)
\]

Using constraints \((11)\), relations \((70) \Rightarrow \)
\[
\sum_{\rho \in \Lambda_r} y(1,\alpha)(r,\rho) = \sum_{\rho \in \Lambda_{n-1}} \sum_{\iota \in \Psi_{\alpha\rho}(y,z)} \omega_{\alpha\rho\iota}(y, z) \quad \forall \alpha \in \Lambda_1, \forall r \in R \setminus \{1, n - 1\} \quad (71)
\]

Using Theorem \(37\), \((71)\) can be written as:
\[
\sum_{\rho \in \Lambda_r} y(1,\alpha)(r,\rho) = \sum_{\rho \in \Lambda_r} \sum_{\sigma \in \Lambda_s} \sum_{\iota \in \Psi_{\alpha\rho}(y,z): a_r, \rho \in P_{\alpha\rho}(y,z)} \omega_{\alpha\rho\iota}(y, z) \quad \forall \alpha \in \Lambda_1, \forall r \in R \setminus \{1, n - 1\} \quad (72)
\]
Re-arranging (72) gives:

\[
\sum_{\rho \in \Lambda_r} \left( y(1,\alpha)(r,\rho) - \sum_{\rho \in \Lambda_{n-1}} \sum_{\alpha \in \Psi_{\alpha \rho}(y,z)} \omega_{\alpha \rho \ell}(y,z) \right) = 0 \quad \forall \alpha \in \Lambda_1, \forall r \in R \setminus \{1, n-1\} \tag{73}
\]

a.2) Combining Lemma 48.ii with Definition 49, we have that:

\[
y(1,\alpha)(r,\rho) = \sum_{\nu_q \in \Lambda_q} \sum_{\rho \in \Lambda_{n-1}} \sum_{\alpha \in \Psi_{\alpha \rho}(y,z)} z(1,\alpha)(r,\rho)\omega_{\alpha \rho \ell}(y,z)
\geq \sum_{\nu_q \in \Lambda_q} \sum_{\rho \in \Lambda_{n-1}} \sum_{\alpha \in \Psi_{\alpha \rho}(y,z)} \omega_{\alpha \rho \ell}(y,z)
\geq \sum_{\rho \in \Lambda_{n-1}} \sum_{\alpha \in \Psi_{\alpha \rho}(y,z)} \omega_{\alpha \rho \ell}(y,z)
\]
\[
\forall r \in R \setminus \{1, n-1\}, \forall q \in R : q > r, \forall (\alpha, \rho) \in (\Lambda_1, \Lambda_r) \tag{74}
\]

Relations (73) and (74) \implies

\[
y(1,\alpha)(r,\rho) = \sum_{\rho \in \Lambda_{n-1}} \sum_{\alpha \in \Psi_{\alpha \rho}(y,z)} \omega_{\alpha \rho \ell}(y,z) \quad \forall r \in R \setminus \{1, n-1\}, \forall (\alpha, \rho) \in (\Lambda_1, \Lambda_r) \tag{75}
\]

a.3) Using constraints (12), relations (70) \implies:

\[
\sum_{\sigma \in \Lambda_s} y(1,\alpha)(s,\sigma) = \sum_{\rho \in \Lambda_{n-1}} \sum_{\alpha \in \Psi_{\alpha \rho}(y,z)} \omega_{\alpha \rho \ell}(y,z) \quad \forall \alpha \in \Lambda_1, \forall s \in R : s > 2 \tag{76}
\]

Using Theorem 37, (76) \implies

\[
\sum_{\sigma \in \Lambda_s} y(1,\alpha)(s,\sigma) = \sum_{\sigma \in \Lambda_s} \sum_{\rho \in \Lambda_{n-1}} \sum_{\alpha \in \Psi_{\alpha \rho}(y,z)} \omega_{\alpha \rho \ell}(y,z) \quad \forall \alpha \in \Lambda_1, \forall s \in R : s > 2 \tag{77}
\]

Re-arranging (77) gives:

\[
\sum_{\sigma \in \Lambda_s} \left( y(1,\alpha)(s,\sigma) - \sum_{\rho \in \Lambda_{n-1}} \sum_{\alpha \in \Psi_{\alpha \rho}(y,z)} \omega_{\alpha \rho \ell}(y,z) \right) = 0 \quad \forall \alpha \in \Lambda_1, \forall s \in R : s > 2 \tag{78}
\]
a.4) Combining Lemma 48 i with Definition 49, we have that:

\[
y(1,\alpha)(s,\sigma) = \sum_{\nu_p \in \Lambda_p} \sum_{\sigma \in \Lambda_{n-1}} \sum_{\nu_p, \nu_p \in P_{\alpha, \beta, \gamma}(y, z)} \sum_{\nu_p \in \Lambda_p} \sum_{\sigma \in \Lambda_{n-1}} \sum_{\nu_p, \nu_p \in P_{\alpha, \beta, \gamma}(y, z)} z(1,\alpha)(p,\nu_p)(s,\sigma) \\
\geq \sum_{\nu_r \in \Lambda_r} \sum_{\sigma \in \Lambda_{s-1}} \sum_{\nu_r, \nu_r \in P_{\alpha, \beta, \gamma}(y, z)} \sum_{\nu_r \in \Lambda_r} \sum_{\sigma \in \Lambda_{s-1}} \sum_{\nu_r, \nu_r \in P_{\alpha, \beta, \gamma}(y, z)} z(1,\alpha)(p,\nu_p)(s,\sigma) \\
\geq \sum_{\nu_r \in \Lambda_r} \sum_{\sigma \in \Lambda_{s-1}} \sum_{\nu_r, \nu_r \in P_{\alpha, \beta, \gamma}(y, z)} \omega_{\alpha, \beta, \gamma}(y, z)
\]

\(\forall (p, s) \in \mathbb{R}^2 : 1 < p < s, \forall (\alpha, \sigma) \in (\Lambda_1, \Lambda_s)\) \hspace{1cm} (79)

a.5) Relations (78) and (79) \(\implies\)

\[
y(1,\alpha)(s,\sigma) = \sum_{\nu_r \in \Lambda_r} \sum_{\sigma \in \Lambda_{s-1}} \sum_{\nu_r, \nu_r \in P_{\alpha, \beta, \gamma}(y, z)} \omega_{\alpha, \beta, \gamma}(y, z) \hspace{1cm} \forall \alpha \in \Lambda_1, \forall s \in \mathbb{R} : s > 2
\]

(80)

a.6) Condition i of the theorem follows from the combination of (75) and (80).

b) Condition ii.

b.1) Using Theorem 37 and Corollary 47 ii, (70) can be re-written as:

\[
\sum_{\rho \in \Lambda_r} \sum_{\sigma \in \Lambda_s} \sum_{\beta \in \Lambda_{n-1}} \sum_{\nu_r, \nu_s \in P_{\alpha, \beta, \gamma}(y, z)} \omega_{\alpha, \beta, \gamma}(y, z) = \sum_{\rho \in \Lambda_r} \sum_{\sigma \in \Lambda_s} \sum_{\nu_r, \nu_s \in P_{\alpha, \beta, \gamma}(y, z)} z(1,\alpha)(r,\rho)(s,\sigma)
\]

\(\forall (r, s) \in \mathbb{R}^2 : 1 < r < s, \forall \alpha \in \Lambda_1\) \hspace{1cm} (81)

Re-arranging (81) gives:

\[
\sum_{\rho \in \Lambda_r} \sum_{\sigma \in \Lambda_s} \left( z(1,\alpha)(r,\rho)(s,\sigma) - \sum_{\beta \in \Lambda_{n-1}} \sum_{\nu_r, \nu_s \in P_{\alpha, \beta, \gamma}(y, z)} \omega_{\alpha, \beta, \gamma}(y, z) \right) = 0
\]

\(\forall (r, s) \in \mathbb{R}^2 : 1 < r < s, \forall \alpha \in \Lambda_1\) \hspace{1cm} (82)

b.2) From Lemma 48 ii, we have:

\[
y(1,\alpha)(s,\sigma) = z(1,\alpha)(r,\rho)(s,\sigma) + \sum_{\nu_r, \in \Lambda_r} \sum_{\beta \in \Lambda_{n-1}} \sum_{\nu_r, \nu_s \in P_{\alpha, \beta, \gamma}(y, z)} z(1,\alpha)(r,\nu_r)(s,\sigma)
\]

\(\forall (r, s) \in \mathbb{R}^2 : 1 < r < s, \forall (\alpha, \rho, \sigma) \in (\Lambda_1, \Lambda_r, \Lambda_s)\).
b.3) From Condition i, we have:

\[ y(1, \alpha)(s, \sigma) = \sum_{\beta \in \Lambda_{n-1}} \sum_{\nu \in \Psi_{\alpha, \beta}(y, z)} \omega_{\alpha \beta \iota}(y, z) + \sum_{\nu \in \Lambda_r: \nu \neq \rho} \sum_{\beta \in \Lambda_{n-1}} \sum_{\nu \in \Psi_{\alpha, \beta}(y, z): (a_r, \nu, a_s, \sigma) \in P_{\alpha \beta \iota}^2(y, z)} \omega_{\alpha \beta \iota}(y, z) \]

\[ \forall (r, s) \in R^2: 1 < r < s, \forall (\alpha, \rho, \sigma) \in (\Lambda_1, \Lambda_r, \Lambda_s) \] (84)

b.4) Definition 49 \implies:

\[ \sum_{\nu \in \Lambda_r: \nu \neq \rho} \sum_{\beta \in \Lambda_{n-1}} \sum_{\nu \in \Psi_{\alpha, \beta}(y, z): (a_r, \nu, a_s, \sigma) \in P_{\alpha \beta \iota}^2(y, z)} z_{(1, \alpha)(\rho, \nu)}(s, \sigma) \geq \sum_{\nu \in \Lambda_r: \nu \neq \rho} \sum_{\beta \in \Lambda_{n-1}} \sum_{\nu \in \Psi_{\alpha, \beta}(y, z): (a_r, \nu, a_s, \sigma) \in P_{\alpha \beta \iota}^2(y, z)} \omega_{\alpha \beta \iota}(y, z) \]

\[ \forall (r, s) \in R^2: 1 < r < s, \forall (\alpha, \rho, \sigma) \in (\Lambda_1, \Lambda_r, \Lambda_s) \] (85)

b.5) Relations 83 and 85 \implies

\[ z_{(1, \alpha)(\rho, \nu)}(s, \sigma) \leq \sum_{\beta \in \Lambda_{n-1}} \sum_{\nu \in \Psi_{\alpha, \beta}(y, z): (a_r, \nu, a_s, \sigma) \in P_{\alpha \beta \iota}^2(y, z)} \omega_{\alpha \beta \iota}(y, z) \]

\[ \forall (r, s) \in R^2: 1 < r < s, \forall (\alpha, \rho, \sigma) \in (\Lambda_1, \Lambda_r, \Lambda_s) \] (86)

b.6) Combining 82 and 86, we must have:

\[ z_{(1, \alpha)(\rho, \nu)}(s, \sigma) = \sum_{\beta \in \Lambda_{n-1}} \sum_{\nu \in \Psi_{\alpha, \beta}(y, z): (a_r, \nu, a_s, \sigma) \in P_{\alpha \beta \iota}^2(y, z)} \omega_{\alpha \beta \iota}(y, z) \]

\[ \forall (r, s) \in R^2: 1 < r < s, \forall (\alpha, \rho, \sigma) \in (\Lambda_1, \Lambda_r, \Lambda_s) \] (87)

c) Condition iii. From the combination of constraints 14 and Condition ii, we have:

\[ y(\rho, \sigma)(s, \sigma) = \sum_{\alpha \in \Lambda_1} \sum_{\beta \in \Lambda_{n-1}} \sum_{\nu \in \Psi_{\alpha, \beta}(y, z): (a_r, \rho, a_s, \sigma) \in P_{\alpha \beta \iota}^2(y, z)} \omega_{\alpha \beta \iota}(y, z) \]

\[ \forall (r, s) \in R^2: 1 < r < s, \forall (\rho, \sigma) \in (\Lambda_r, \Lambda_s) \] (88)

\[ \text{Corollary 52} \quad (y, z) \in Q_L \iff (y, z) \text{ corresponds to a convex combination of satisfying assignments for exactly-1 SAT with coefficients equal to the weights of the corresponding SAT paths in } (y, z). \]

\[ \text{Theorem 53} \quad \text{The following holds true:} \]

i) \( \text{Conv}(Q_L) = \text{Conv}(Q_I) \);

ii) \( \text{Ext}(Q_L) = \text{Ext}(Q_I) = Q_I \).

\[ \text{Proof.} \quad \text{Condition i) of the theorem follows directly from the combination of Theorem 44, Theorem 53, and Corollary 52. Condition ii) follows from the combination of Condition i) and Theorem 28.} \]
Corollary 54 The following mappings are bijective:

i) $B_1 : \text{Conv}(Q_L) \rightarrow P_0$;

ii) $B_2 : \text{Conv}(Q_L) \rightarrow P_1$;

iii) $B_3 : \text{Ext}(Q_L) \rightarrow \Delta$.

4 Linear programming model of exactly-1 SAT

Theorem 55 Let $\delta_z$ and $\delta_y$ be arbitrary vectors of scalars of comfortable dimensions, respectively. The following statements are true of basic feasible solutions (BFS) of

$$\text{Problem LP : min } \{\vartheta(y, z) := \delta_z^T \cdot z + \delta_y^T \cdot y \mid (y, z) \in Q_L\}$$  \hspace{1cm} (89)

and satisfying assignments for exactly-1 SAT:

i) Every BFS of Problem LP corresponds to a satisfying assignment for exactly-1 SAT;

ii) Every satisfying assignment for exactly-1 SAT corresponds to a BFS of Problem LP;

iii) The mapping of BFS’s of Problem LP onto satisfying assignments for exactly-1 SAT is surjective.

Proof. Statements (55.i) and (55.ii) follow directly from the combination of Theorem 53, Corollary 54.iii, and the correspondence between BFS’s of LP models and extreme points of their associated polyhedra (see [1, pp. 92-101]). Statement (55.iii) follows from the primal degeneracy of Problem LP (see [8, p. 32]). □

Corollary 56 Problem LP solves exactly-1 SAT.

5 Conclusions

We have developed a first polynomial-sized linear programming model of the Boolean Satisfiability Problem. The model represents a new proof of the equality of the computational complexity classes $P$ and $NP$. With respect to practice however, empirical testing is needed in order to assess the computational performance of the model.
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