Online Agnostic Boosting via Regret Minimization

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Abstract

Boosting is a widely used machine learning approach based on the idea of aggregating weak learning rules. While in statistical learning numerous boosting methods exist both in the realizable and agnostic settings, in online learning they exist only in the realizable case. In this work we provide the first agnostic online boosting algorithm; that is, given a weak learner with only marginally-better-than-trivial regret guarantees, our algorithm boosts it to a strong learner with sublinear regret.

Our algorithm is based on an abstract (and simple) reduction to online convex optimization, which efficiently converts an arbitrary online convex optimizer to an online booster. Moreover, this reduction extends to the statistical as well as the online realizable settings, thus unifying the 4 cases of statistical/online and agnostic/realizable boosting.

1 Introduction

Boosting is a fundamental methodology in machine learning which allows us to automatically convert ("boost") a number of weak learning rules into a strong one. Boosting was first studied in the context of (realizable) PAC learning in a line of seminal works which include the celebrated Adaboost algorithm as well as many other algorithms with various applications (see e.g. [29, 33, 17, 19]). It was later adapted to the agnostic PAC setting and was extensively studied in this context as well [7, 31, 21, 27, 30, 26, 28, 16, 13, 18]. More recently, [14] and [9] studied boosting in the context of online prediction and derived boosting algorithms in the realizable setting (a.k.a. mistake-bound model).

In this work we study agnostic boosting in the online setting: let $\mathcal{H}$ be a class of experts and assume we have an oracle access to a weak online learner for $\mathcal{H}$ with a non-trivial (yet far from desired) regret guarantee. The goal is to use it to obtain a strong online learner for $\mathcal{H}$, i.e. which exhibits a vanishing regret.

Why Online Agnostic Boosting? The setting of realizable boosting poses a restriction on the possible input sequences: there must be an expert that attains near-zero mistake-bound on the input sequence. This is a non-standard assumption in online learning. In contrast, in the (agnostic) setting we consider, there is no restriction on the input sequence and it can be chosen adversarially.

Applications of Online Agnostic Boosting. Apart from being a fundamental question in any machine learning setting, let us mention a couple of more concrete incentives to study online agnostic boosting:

- **Differential Privacy and Online Learning:** A recent line of work revealed deep connections between online learning and differentially private learning [5, 1, 6, 10, 32, 25, 22, 11]. In fact, these
The Weak Learning Assumption. In this paper we use the same formulation as [23] used in the statistical setting. Towards this end, it is convenient to measure the performance of online learners using gain rather than loss: let \( (x_1, y_1) \ldots (x_T, y_T) \in \mathcal{X} \times \{\pm 1\} \) be an (adversarial and adaptive) input sequence of examples presented to an online learning algorithm \( A \); that is, in each iteration \( t = 1 \ldots T \), the adversary picks an example \( (x_t, y_t) \), then the learner \( A \) first gets to observe \( x_t \), and predicts (possibly in a randomized fashion) \( \hat{y}_t \in \{\pm 1\} \), and lastly it observes \( y_t \) and gains a reward of \( y_t \cdot \hat{y}_t \). The goal of the learner is to maximize the total gain (or correlation), given by \( \sum_t y_t \cdot \hat{y}_t \). Note that this is equivalent to the often used notion of loss where in each iteration the learner suffers a loss of \( 1[y_t \neq \hat{y}_t] \) and its goal is to minimize the accumulated loss \( \sum_t 1[y_t \neq \hat{y}_t] \).

Definition 1 (Agnostic Weak Online Learning). Let \( \mathcal{H} \subseteq \{\pm 1\}^X \) be a class of experts, let \( T \) denote the horizon length, and let \( \gamma > 0 \) denote the advantage. An online learning algorithm \( \mathcal{W} \) is a \((\gamma, T)\)-agnostic weak online learner (AWOL) for \( \mathcal{H} \) if for any sequence \((x_1, y_1), \ldots, (x_T, y_T) \in \mathcal{X} \times \{\pm 1\}\), at every iteration \( t \in [T] \), the algorithm outputs \( \mathcal{W}(x_t) \in \{\pm 1\} \) such that,

\[
\mathbb{E} \left[ \sum_{t=1}^T \mathcal{W}(x_t)y_t \right] \geq \gamma \max_{h \in \mathcal{H}} \mathbb{E} \left[ \sum_{t=1}^T h(x_t)y_t \right] - R_{\mathcal{W}}(T),
\]

where the expectation is taken w.r.t. the randomness of the weak learner \( \mathcal{W} \) and that of the possibly adaptive adversary, \( R_{\mathcal{W}} : \mathbb{N} \to \mathbb{R}_+ \) is the additive regret: a non-decreasing, sub-linear function of \( T \).

Note the slight abuse of notation in the last definition: an online learner \( \mathcal{W} \) is not an “\( \mathcal{X} \to \{\pm 1\} \)” function; rather it is an algorithm with an internal state that is updated as it is fed training examples. Thus, the prediction \( \mathcal{W}(x_t) \) depends on the internal state of \( \mathcal{W} \), and for notational convenience we avoid reference to the internal state.

Our agnostic online boosting algorithm has an oracle access to \( N \) weak learners and predicts each task by combining their predictions. The number of weak learners \( N \) is a meta-parameter which can be tuned by the user according to the following trade-off: on the one hand, the regret bound improves as \( N \) increases, and on the other hand, a larger number of weak learners is more costly in terms of computational resources.

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1Indeed, \( y_t \cdot \hat{y}_t = 1 - 2 \cdot 1[y_t \neq \hat{y}_t] \) since \( y_t, \hat{y}_t \in \{\pm 1\} \). Therefore, the accumulated loss and correlation are affinely related by \( \sum y_t \cdot \hat{y}_t = T - 2 \cdot \sum 1[y_t \neq \hat{y}_t] \).
Theorem 2 (Agnostic Online Boosting). Let $\mathcal{H}$ be a class of experts, let $T \in \mathbb{N}$ denote the horizon length, and let $\mathcal{W}_1, \ldots, \mathcal{W}_N$ be $(\gamma, T)$-AWOL for $\mathcal{H}$ with advantage $\gamma$ and regret $R_{\mathcal{W}}(T) = o(T)$ (see Definition 7). Then, there exists an online learning algorithm, which has oracle access to each $\mathcal{W}_i$, and has expected regret of at most

$$\frac{R_{\mathcal{W}}(T)}{\gamma} + O\left(\frac{T}{\gamma \sqrt{N}}\right).$$

To exemplify the interplay between $R_{\mathcal{W}}(\cdot)$ and $N$, imagine a scenario where $R_{\mathcal{W}}(T) \approx \sqrt{T}$ (as is often the case for regret bounds). Then, setting the number of weak learners to be $N \approx T/\gamma^2$ gives that the overall regret remains $\approx \sqrt{T}$.

An Abstract Framework for Boosting. Boosting and Regret Minimization algorithms are intimately related. This tight connection is exhibited both in statistical boosting (see [20, 19, 34]) as well as in the online boosting ([14]). Our algorithm is inspired by this fruitful connection and utilizes it: in particular, Theorem 2 is an instantiation of a more abstract meta-algorithm which takes an arbitrary online convex optimizer and uses it in a black-box manner to obtain an agnostic online boosting algorithm. Thus, in fact we obtain a family of boosting algorithms; one for each choice of an online convex optimizer. Specifically, Theorem 2 follows by picking Online Gradient Decent for the meta-algorithm. We present this in detail in Section 2.

The same type of reasoning carries to realizable online boosting, and even to statistical boosting (both realizable and agnostic setting). In Section 3 we demonstrate a general reduction from each of these boosting settings to online convex optimization.

1.2 Related Work
As discussed above, [14] and [9] studied online boosting in the realizable (mistake-bound) setting, while this work focuses on the agnostic (regret-bound) setting.

[8] studies online boosting under real-valued loss functions. The main difference from our work is in the weak learning assumption: [8] consider weak learners that are in fact strong online learners for a base class of regression functions. The boosting process produces an online learner for a bigger class which consists of the linear span of the base class. This is different from the setting considered here where the class is fixed, but the regret bound is being boosted.

A main motivation in this work is the connection between boosting and regret minimization. This builds on and inspired by previous works that demonstrated this fruitful relationship. We refer the reader to the book by [33] (Chapter 6) for an excellent presentation of this relationship in the context of Adaboost.

1.3 Organization
The main result of our agnostic online boosting algorithm, and the proof of Theorem 3 are given in Section 2. In Section 3, we first give a game-theoretic perspective of our method when applied to the statistical setting (Subsection 3.1). We then demonstrate a general reduction, in the statistical setting, from both the agnostic (Subsection 3.2), and realizable (Subsection 3.3) boosting settings, to online convex optimization. Lastly, we give a similar result for the online realizable boosting setting in Section 4.

2 Agnostic Online Boosting
In this section we prove Theorem 2 which establishes an efficient online agnostic boosting algorithm. We begin in Subsection 2.1 with formally presenting our framework which enables converting an online convex optimizer to an online booster. Then, in Subsection 2.2 we show how Theorem 2 follows directly by picking the online convex optimizer to be Online Gradient Decent.
2.1 Online Agnostic Boosting with OCO

We begin with describing our boosting algorithm (see Algorithm 1 for the pseudo-code). The booster has black-box oracle access to two types of auxiliary algorithms: a weak learner, and an online-convex optimizer. The booster maintains \( N \) instances \( W_1, \ldots, W_N \) of a weak learning algorithm. Specifically, each weak learner \( W_i \) is a \((\gamma, T)\)-AOWL (see Definition 1). The online-convex optimizer is a \(([-1, 1], N)\)-OCO algorithm \( A \) (see Equation 1 below).

**Algorithm 1** Online Agnostic Boosting with OCO

1: for \( t = 1, \ldots, T \) do
2: Get \( x_t \), predict: \( \hat{y}_t = \Pi\left( \frac{1}{\gamma N} \sum_{i=1}^{N} W_i(x_t) \right) \).
3: for \( i = 1, \ldots, N \) do
4: If \( i > 1 \), set \( p^i_t = A(\ell^1_t, \ldots, \ell^i_{t-1}) \). Else, set \( p^1_t = 0 \).
5: Set next loss: \( \ell^i_t(p) = p(\frac{1}{\gamma}) W_i(x_t) y_t - 1 \).
6: Pass \((x_t, y^i_t)\) to \( W_i \), where \( y^i_t \) is a random label s.t. \( \mathbb{P}[y^i_t = y_t] = \frac{1 + p^i_t}{2} \).
7: end for
8: end for

![Figure 1](image.png)

The algorithm is given oracle access to \( N \) instances of a \((\gamma, T)\)-AOWL algorithm, \( W_1, \ldots, W_N \) (see Definition 1), and to a \(([-1, 1], N)\)-OCO algorithm \( A \) (see Equation 1). The prediction “\( \Pi\left( \frac{1}{\gamma N} \sum_{i=1}^{N} W_i(x_t) \right) \)” in line 2 is a randomized majority-vote, as defined in Equation 2.

**Online Convex Optimization** (see e.g. [23]). Recall that in the Online Convex Optimization (OCO) framework, an online player iteratively makes decisions from a compact convex set \( K \subset \mathbb{R}^d \). At iteration \( i = 1, \ldots, N \), the online player chooses \( p^i \in K \), and the adversary reveals the cost \( \ell^i \), chosen from a family \( \mathcal{F} \) of bounded convex functions over \( K \). We will refer to an algorithm in this setting as a \((K, N)\)-OCO. Let \( A \) be a \((K, N)\)-OCO. The regret of \( A \) is defined by:

\[
R_A(N) = \sum_{i=1}^{N} \ell^i(p^i) - \min_{p \in K} \sum_{i=1}^{N} \ell^i(p). \tag{1}
\]

**Randomized Majority-Vote/Projection.** The last component needed to describe our boosting algorithm is the randomized projection “\( \Pi \)” which is used to predict in Line 2. For any \( z \in \mathbb{R} \), denote by \( \Pi(z) \) the following random label:

\[
\Pi(z) = \begin{cases} 
\text{sign}(z) & \text{if } |z| \geq 1 \\
+1 & \text{w.p. } \frac{1+z}{2} \\
-1 & \text{w.p. } \frac{1-z}{2}
\end{cases} \tag{2}
\]

We now state and prove the regret bound for Algorithm 1.

**Proposition 3** (Regret Bound). *The accumulated gain of Algorithm 1 satisfies:

\[
\frac{1}{T} \mathbb{E} \left[ \max_{h^* \in H} \sum_{t=1}^{T} h^*(x_t)y_t - \sum_{t=1}^{T} \hat{y}_t y_t \right] \leq \frac{R_W(T)}{\gamma T} + \frac{R_A(N)}{N},
\]

where \((x_t, y_t)\)'s are the observed examples, \( \hat{y}_t \)'s are the predictions, the expectation is with respect to the algorithm and learners’ randomness, and \( R_W \) and \( R_A \) are the regret terms of the weak learner and the OCO, respectively.*

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4
Proof. The proof follows by combining upper and lower bounds on the expected sum of losses incurred by the OCO algorithm. The bounds follow directly from the weak learning assumption (lower bound) and the OCO guarantee (upper bound). These bounds involve some simple algebraic manipulations. It is convenient to abstract out some of these calculations into lemmas, which are described later in this section.

Before delving into the analysis, we first clarify several assumptions used below. For simplicity of presentation we assume an oblivious adversary, however, using a standard reduction, our results can be generalized to an adaptive one. Let \((x_1, y_1), \ldots, (x_T, y_T)\) be any sequence of observed examples. Observe that there are several sources of randomness at play; the weak learning algorithm \(W_i\)'s internal randomness, the random re-labeling (line 6, Algorithm 1), and the randomized prediction (line 2, Algorithm 1). The analysis below is given in expectation with respect to all these random variables.

Note the following fact used in the analysis; for all \(i \in [N], t \in [T]\), the random variables \(W_i(x_t)\) and \(y_t^i\) are conditionally independent given \(p_t^i\) and \(y_t\). Since \(\mathbb{E}[y_t^i | p_t^i, y_t] = p_t^i \cdot y_t\), using the conditional independence, it follows that \(\mathbb{E}[W_i(x_t)y_t^i] = \mathbb{E}[W_i(x_t)p_t^i y_t]\) (see Lemma 13 in the Appendix). We can now begin the analysis, starting with lower bounding the expected sum of losses, using the weak learning guarantee,

\[
\frac{1}{\gamma} \mathbb{E} \left[ \sum_{i=1}^{N} \sum_{t=1}^{T} W_i(x_t) \cdot y_t^i \right] = \frac{1}{\gamma} \sum_{i=1}^{N} \sum_{t=1}^{T} W_i(x_t) \cdot y_t^i = \frac{1}{\gamma} \sum_{i=1}^{N} \sum_{t=1}^{T} W_i(x_t) y_t^i \\
\text{(See Lemma 13)}
\]

\[
\geq \frac{1}{\gamma} \sum_{i=1}^{N} \left( \gamma \max_{h \in \mathcal{H}} \mathbb{E} \left[ \sum_{t=1}^{T} h(x_t) y_t^i \right] - R_{\mathcal{W}}(T) \right) \\
\geq \sum_{i=1}^{N} \left( \max_{h \in \mathcal{H}} \sum_{t=1}^{T} h(x_t) \cdot \mathbb{E}[y_t^i] - \frac{1}{\gamma} R_{\mathcal{W}}(T) \right) \\
\geq \sum_{i=1}^{N} \sum_{t=1}^{T} h^*(x_t) \cdot \mathbb{E}[y_t^i] - \frac{N}{\gamma} R_{\mathcal{W}}(T) \\
= \sum_{i=1}^{N} \sum_{t=1}^{T} \mathbb{E}[h^*(x_t) \cdot y_t^i] - \frac{N}{\gamma} R_{\mathcal{W}}(T),
\]

where \(h^*\) is an optimal expert in hindsight for the observed sequence of examples \((x_t, y_t)\)'s. Thus, we obtain the lower bound on the expected sum of losses \(\sum_{t=1}^{T} \sum_{i=1}^{N} \ell_t^i(p_t^i)\) (see Line 5 in Algorithm 1 for the definition of the \(\ell_t^i\)'s), given by,

\[
\mathbb{E} \left[ \sum_{t=1}^{T} \sum_{i=1}^{N} \ell_t^i(p_t^i) \right] \geq \sum_{t=1}^{T} \sum_{i=1}^{N} \mathbb{E}[p_t^i(h^*(x_t) y_t - 1)] - \frac{N}{\gamma} R_{\mathcal{W}}(T) \\
\geq N \sum_{t=1}^{T} (h^*(x_t) y_t - 1) - \frac{N}{\gamma} R_{\mathcal{W}}(T). \\
\text{(See Lemma 4 below)}
\]

For the upper bound, observe that the OCO regret guarantee implies that for any \(t \in [T]\), and any \(p_t^* \in [-1, 1]\),

\[
\mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^{N} \ell_t^i(p_t^i) \right] \leq p_t^* \left( \left( \frac{1}{\gamma N} \sum_{i=1}^{N} \mathbb{E}[W_i(x_t)] \right) y_t - 1 \right) + \frac{1}{N} R_{\mathcal{A}}(N),
\]

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1 See discussion in [12], Pg. 69, as well as Exercise 4.1 formulating the reduction.
Thus, by setting $p^*_\gamma$ according to Lemma 5 (see below, with $\hat{h}(x) := \frac{1}{N} \sum_{i=1}^{N} E[\mathcal{W}_i(x)]$), and summing over $t \in [T]$, we get,

$$E\left[ \frac{1}{N} \sum_{t=1}^{T} \sum_{i=1}^{N} \ell_i(p^*_\gamma) \right] \leq \sum_{t=1}^{T} (E[\hat{y}_t]y_t - 1) + \frac{T}{N} R_A(N).$$

By combining the lower and upper bounds for $E\left[ \frac{1}{NT} \sum_i \ell_i(p^*_\gamma) \right]$, we get,

$$\frac{1}{T} \sum_{t=1}^{T} E[\hat{y}_t]y_t \geq \frac{1}{T} \sum_{t=1}^{T} h^*(x_t)y_t - \frac{R_{\text{VY}}(T)}{\gamma T} - \frac{R_A(N)}{N}.$$  

It remains to prove two Lemmas that are used in the proof of the theorem above, as well as in the more general settings in the following sections.

**Lemma 4.** For any $p \in [-1, 1]$, an example pair $(x, y)$, and $h : \mathcal{X} \to \{-1, 1\}$, we have:

$$p(h(x)y - 1) \geq h(x)y - 1.$$  

**Proof.** Let $z = h(x)y - 1$. Observe that $z \in \{-2, 0\}$. Thus, since $p \in [-1, 1]$, $pz \geq z$.  

**Lemma 5.** Given an example pair $(x, y)$, and $\hat{h} : \mathcal{X} \to \mathbb{R}$, there exists $p^* \in \{0, 1\}$, such that,

$$p^*(\hat{h}(x)y - 1) \leq \hat{y}y - 1,$$

where $\hat{y} = E[\Pi(\hat{h}(x))]$, with expectation taken only w.r.t. the randomness of $\Pi$ (see Definition 2).

**Proof.** If $|\hat{h}(x)| \leq 1, \hat{y} = \hat{h}(x)$ and by setting $p^* = 1$, the equality follows. Thus, assume $|\hat{h}(x)| > 1$, and consider the following cases:

- If $\hat{h}(x)y - 1 > 0$, then $\hat{y}y - 1 = 0$. Hence, by setting $p^* = 0$, the equality follows.
- If $\hat{h}(x)y - 1 < 0$, then since $|\hat{h}(x)| > 1$ it must be that sign($\hat{h}(x))y = -1$, and $\hat{y}y - 1 = -2$. Since $|\hat{h}(x)| > 1$, we have $\hat{h}(x)y - 1 \leq -2$. Hence, by setting $p^* = 1$ the inequality holds.  

2.2 Proof of Theorem 2

The proof of Theorem 2 is a direct corollary of Proposition 3 by plugging Online Gradient Descent (OGD) to be the OCO algorithm $\mathcal{A}$ (e.g., see [23] Chapter 3.1): the OGD regret is $O(GD\sqrt{N})$, where $N$ is the number of iterations, $G$ is an upper bound on the gradient of the losses, and $D$ is the diameter of the set $\mathcal{K} = [-1, 1]$. In our setting, $G \leq \frac{2}{\gamma}$, and $D = 2$. Hence, $R_A = O(\sqrt{N}/\gamma)$, and the overall bound on the regret follows.

3 Statistical Boosting via Improper Game Playing

In this section we first give a game-theoretic perspective of our method when applied to the statistical setting (Subsection 3.1). We then demonstrate a general reduction from both the agnostic (Subsection 3.2), and realizable (Subsection 3.3) boosting settings, to online convex optimization. The following algorithm is given as input a sample $S = (x_1, y_1), \ldots, (x_m, y_m) \in \mathcal{X} \times \mathcal{Y}$, and has a black-box access to two auxiliary
In the zero sum games setting, there are two players A and B, and a payoff function. Our framework uses as a main building block a procedure for approximately solving zero sum games using

3.1 Solving Zero Sum Games Improperly Using an Approximate Optimization Oracle

- Let $\mathcal{K}_A$ and $\mathcal{K}_B$ be convex, compact decision sets of players A and B, respectively, and assume that $g$ is convex-concave. By Sion’s minimax theorem \[35\], the value of the game is well-defined, and we denote it by $\lambda^*$:

$$
\min_{p \in \mathcal{K}_A} \max_{q \in \mathcal{K}_B} g(p, q) = \max_{q \in \mathcal{K}_B} \min_{p \in \mathcal{K}_A} g(p, q) = \lambda^*
$$

Let $\mathcal{K}^*_B$ be a convex, compact set such that $\mathcal{K}_B \subseteq \mathcal{K}^*_B$. We refer to strategies in $\mathcal{K}^*_B$ as proper strategies, while those in $\mathcal{K}_B$ as improper strategies. We consider a modified zero sum games setting where the payoff function $g$ is defined on $\mathcal{K}^*_B$, the set of improper strategies. Note that $\lambda^*$ is defined with respect to the set of proper strategies, and it is still a well-defined quantity in this game.

**Assumption 1:** Player B has access to a randomized approximate optimization oracle $\mathcal{W}$. Given any $p \in \mathcal{K}_A$, $\mathcal{W}$ outputs an improper best response: a strategy $q \in \mathcal{K}^*_B$ such that $\mathbb{E}[g(p, q)] \geq \max_{q^* \in \mathcal{K}_B} g(p, q^*) - \epsilon_0$, where the expectation is taken over the randomness of $\mathcal{W}$.

**Assumption 2:** Player B is allowed to play strategies in $\mathcal{K}^*_B$.

**Assumption 3:** Player A has access to a possibly randomized $(\mathcal{K}_A, T)$-OCO algorithm $\mathcal{A}$ with regret $R_A(T)$ (See Definition \[1\]).

\footnote{Note that when $p_t = 0$ is constantly zero then the distribution used in the realizable setting is not well defined. There are several ways to circumvent it. Concretely, we proceed in such case by setting $h_t = h_{t-1}$ and proceeding to step 6.}

**Algorithm 2** Boosting with OCO

1: for $t = 1, \ldots, T$ do
2: Pass $m_0$ examples to $\mathcal{W}$ drawn from the following distribution:
3: **Realizable:** Draw $(x_i, y_i)$ w.p. $p_t(i)^3$ and re-label according to $y_i p_t(i)$.
4: **Agnostic:** Draw $x_i$ w.p. $\frac{1}{m}$, and re-label according to $y_i p_t(i)$.
5: Let $h_t$ be the weak hypothesis returned by $\mathcal{W}$.
6: Set loss: $\ell_t(p) = \sum_{i=1}^m p(i) (\frac{1}{2} h_t(x_i) y_i - 1)$.
7: Update: $p_{t+1} = A(\ell_1, \ldots, \ell_t)$.
8: end for
9: return $\bar{h}(x) = \Pi(\frac{1}{2T} \sum_{t=1}^T h_t(x))$.

**Figure 2:** The algorithm has oracle access to either a $(\gamma, \epsilon_0, m_0)$-AWL algorithm (see Definition \[7\]) or a $(\gamma, m_0)$-WL algorithm (see Definition \[9\]). Both are denoted as $\mathcal{W}$. The optimizer is a $(\gamma, K, T)$-OCO algorithm $\mathcal{A}$ (see Definition \[1\]), where $K = [0, 1]^m$ in the realizable case and $K = [-1, 1]^m$ in the agnostic case. In line 4, we pass $(x_i, y_i)$ to $\mathcal{W}$, where $y_i$ is a random label s.t. $\mathbb{P}[y_i = y_i] = \frac{1+\epsilon_0(i)}{2}$. The final hypothesis “$\Pi(\frac{1}{2T} \sum_{t=1}^T h_t(x))$” is a randomized majority-vote, as defined in Equation \[2\].

3.1 Solving Zero Sum Games Improperly Using an Approximate Optimization Oracle

Our framework uses as a main building block a procedure for approximately solving zero sum games using an approximate optimization oracle. It is described in this section.

In the zero sum games setting, there are two players A and B, and a payoff function $g$ that depends on the players’ strategies. Player A’s goal is to minimize the payoff, while player B’s goal is to maximize it. Let $\mathcal{K}_A$ and $\mathcal{K}_B$ be convex, compact decision sets of players A and B, respectively, and assume that $g$ is convex-concave. By Sion’s minimax theorem \[35\], the value of the game is well-defined, and we denote it by $\lambda^*$:

$$
\min_{p \in \mathcal{K}_A} \max_{q \in \mathcal{K}_B} g(p, q) = \max_{q \in \mathcal{K}_B} \min_{p \in \mathcal{K}_A} g(p, q) = \lambda^*
$$

Let $\mathcal{K}^*_B$ be a convex, compact set such that $\mathcal{K}_B \subseteq \mathcal{K}^*_B$. We refer to strategies in $\mathcal{K}^*_B$ as proper strategies, while those in $\mathcal{K}_B$ as improper strategies. We consider a modified zero sum games setting where the payoff function $g$ is defined on $\mathcal{K}^*_B$, the set of improper strategies. Note that $\lambda^*$ is defined with respect to the set of proper strategies, and it is still a well-defined quantity in this game.

**Assumption 1:** Player B has access to a randomized approximate optimization oracle $\mathcal{W}$. Given any $p \in \mathcal{K}_A$, $\mathcal{W}$ outputs an improper best response: a strategy $q \in \mathcal{K}^*_B$ such that $\mathbb{E}[g(p, q)] \geq \max_{q^* \in \mathcal{K}_B} g(p, q^*) - \epsilon_0$, where the expectation is taken over the randomness of $\mathcal{W}$.

**Assumption 2:** Player B is allowed to play strategies in $\mathcal{K}^*_B$.

**Assumption 3:** Player A has access to a possibly randomized $(\mathcal{K}_A, T)$-OCO algorithm $\mathcal{A}$ with regret $R_A(T)$ (See Definition \[1\]).
In accordance with previous works, we focus on the setting where the expectation is taken over the randomness of $W$.

**Proposition 6.** If players $A$ and $B$ play according to Algorithm 3, then player B’s average strategy $\bar{q} = \frac{1}{T} \sum_{t=1}^{T} q_t$, $\bar{q} \in \mathcal{K}_B$, satisfies for any $p^* \in \mathcal{K}_A$,

$$
\lambda^* \leq \mathbb{E}[g(p^*, \bar{q})] + \frac{R_A(T)}{T} + \epsilon_0,
$$

where the expectation is taken over the randomness of $W$.

**Proof.** Since the game is well-defined over $\mathcal{K}_A$ and $\mathcal{K}_B$, there exists a max-min strategy $q^* \in \mathcal{K}_B$ for player B such that for all $p \in \mathcal{K}_A$, $g(p, q^*) \geq \lambda^*$. Let $\bar{p} = \frac{1}{T} \sum_{t=1}^{T} p_t$, and observe that since the $p_t$’s depend on the sequence of $q_t$’s, they are also random variables, as well as $\bar{p}$. We have,

$$
\mathbb{E}\left[\frac{1}{T} \sum_{t=1}^{T} g(p_t, q_t)\right] \geq \mathbb{E}\left[\frac{1}{T} \sum_{t=1}^{T} g(p_t, q^*)\right] - \epsilon_0 \geq \mathbb{E}[g(\bar{p}, q^*)] - \epsilon_0 \geq \lambda^* - \epsilon_0.
$$

The first inequality is due to Assumption 1, where $\mathbb{E}[g(p_t, q_t)] \geq \max_{q \in \mathcal{K}_B} g(p_t, q) - \epsilon_0 \geq g(p_t, q^*) - \epsilon_0$. The second inequality holds because $g$ is convex in $p$.

Now, let $\bar{q} = \frac{1}{T} \sum_{t=1}^{T} q_t$; note that $\bar{q} \in \mathcal{K}_B$ since $\mathcal{K}_B$ is convex. For the upper bound, observe that the OCO regret guarantee implies that for any $p^* \in \mathcal{K}_A$ we have,

$$
\mathbb{E}\left[\frac{1}{T} \sum_{t=1}^{T} g(p_t, q_t)\right] \leq \mathbb{E}\left[\frac{1}{T} \sum_{t=1}^{T} g(p^*, q_t)\right] + \frac{R_A(T)}{T} \leq \mathbb{E}[g(p^*, \bar{q})] + \frac{R_A(T)}{T},
$$

where the second inequality holds because $g$ is concave in $q$. Combining the lower and upper bounds yields the theorem.

### 3.2 Statistical Agnostic Boosting

We will use the following notation. Let $D$ be a distribution over $\mathcal{X} \times \mathcal{Y}$ and let $h : \mathcal{X} \rightarrow \mathcal{Y}$ be an hypothesis. Define the correlation of $h$ with respect to $D$ by:

$$
\text{cor}_D(h) = \mathbb{E}_{(x,y) \sim D}[h(x) \cdot y].
$$

**Definition 7** (Empirical Agnostic Weak Learning Assumption). Let $\mathcal{H} \subseteq \{\pm 1\}^X$ be a hypothesis class and let $\mathbf{x} = (x_1, \ldots, x_m) \in \mathcal{X}$ denote an unlabeled sample. A learning algorithm $W$ is a $(\gamma, \epsilon_0, m_0)$-agnostic weak learner (AWL) for $\mathcal{H}$ with respect to $\mathbf{x}$ if for any labels $\mathbf{y} = (y_1, \ldots, y_m)$,

$$
\mathbb{E}_{\mathcal{S}'}[\text{cor}_{\mathbf{x} \times \mathbf{y}}(W(S'))] \geq \gamma \max_{h^* \in \mathcal{H}} \text{cor}_{\mathbf{x} \times \mathbf{y}}(h^*) - \epsilon_0,
$$

where $\mu \times \mathbf{y}$ is the distribution which uniformly assigns to each example $(x_i, y_i)$ probability $1/m$, and $S'$ is an independent sample of size $m_0$ drawn from $\mu \times \mathbf{y}$.

In accordance with previous works, we focus on the setting where $\gamma$ is a small constant (say $\gamma = 0.1$) and $\epsilon_0 \approx d / \sqrt{m}$, where $d$ is the VC-dimension of $\mathcal{H}$ (see [28] for a detailed discussion). We stress however that our results apply for any setting of $\gamma, \epsilon_0 \in [0, 1]$. 

\[8\]
The above weak learning assumption can be seen as an empirical variant of the assumption in [28], where \( \mu \) is replaced with the population distribution over \( X \) and the labels \( y_i \)'s are replaced with an arbitrary classifier \( c : X \rightarrow \{ \pm 1 \} \). Both of these assumptions are weaker than the standard agnostic weak learning assumption, for which the guarantee holds with respect to every distribution \( D \) over \( X \times \{ \pm 1 \} \). It will be interesting to investigate the relationship between the assumption of [28] and our empirical variant, however this is beyond the scope of this work.

We now state and prove the regret bound for Algorithm [2].

**Theorem 8 (Empirical Agnostic Boosting).** The correlation of the output of Algorithm [2] which is denoted \( \hat{h} \), satisfies:

\[
\mathbb{E}[\text{cor}_S(\hat{h})] \geq \max_{h^* \in \mathcal{H}} \mathbb{E}[\text{cor}_S(h^*)] - \left( \frac{\epsilon_0}{\gamma} + O\left( \frac{1}{\gamma \sqrt{T}} \right) \right).
\] (3)

**Generalization.** The above theorem asserts that the correlation of the output hypothesis is competitive with the best hypothesis in \( \mathcal{H} \) with respect to the empirical distribution. Obtaining a similar guarantee with respect to the population distribution can be obtained using standard arguments. One way of deriving it is via a sample compression argument (which is natural in boosting; see, e.g., [34] [15]): indeed, the final hypothesis \( \hat{h} \) is obtained by aggregating the \( T \) weak hypotheses \( h_i \)'s, each of which is determined by the \( m_0 \) examples fed to the weak learner. Thus, \( \hat{h} \) can be encoded by \( T \cdot m_0 \) input examples and hence the entire algorithm forms a sample compression scheme of this size. Consequently, by setting the input sample \( m = O(T \cdot m_0 / \epsilon^2) \) we get the same guarantee like in Equation [3] up to an additive error of \( \epsilon \).

The proof has two parts. The first part is a straightforward reduction to the game-theoretic setup of Proposition [6] and the second part shows how to project the “improper” strategy obtained by Proposition [6] to the desired output hypothesis.

**Reduction to Proposition [6]** The agnostic version of Algorithm [2] can be presented as an instance of Algorithm [4] where Player A and B are the weak learner and the OCO oracle algorithms, respectively. The decision sets are \( \mathcal{K}_A = [-1, 1]^m \), \( \mathcal{K}_B = \Delta_{\mathcal{H}} \), and \( \mathcal{K}'_B = \frac{1}{T} \Delta_{\mathcal{H}} \), and the payoff function \( g(\cdot, \cdot) \) is given by

\[
g(p, q) = \sum_{i=1}^{m} p(i)(q(x_i)y_i - 1),
\]

where \( p \in \mathcal{K}_A \) is a vector in the \( m \) dimensional continuous cube, and \( q \in \mathcal{K}'_B \) is a non-negative combination of hypotheses in \( \mathcal{H} \) (and so \( q \) corresponds to the mapping \( x \mapsto \sum_{h \in \mathcal{H}} q(h) \cdot h(x) \)). We leave it to the reader to verify that the agnostic weak learner corresponds to an approximate optimization oracle \( \mathcal{W} \). Namely, for any \( p \in \mathcal{K}_A \) the output \( q' = \mathcal{W}(p) \) satisfies \( q' \in \mathcal{K}'_B \) and

\[
\mathbb{E}[g(p, q')] \geq \max_{q \in \mathcal{K}_B} g(p, q) - \frac{\epsilon_0 m}{\gamma}.
\]

Furthermore, it can be shown that the value of the above game is

\[
\lambda^* = m \cdot \max_{h \in \mathcal{H}} \text{cor}_S(h) - m.
\]

This can be done by (i) observing that the strategy \( p = (1, 1, \ldots, 1) \in \mathcal{K}_A \) is dominant for Player A and (ii) computing \( \max_{q \in \mathcal{K}_B} g(p, q) \) which is equal to \( \lambda^* \) (since \( p \) is dominating).

Now, Proposition [6] implies that for any \( p \in [-1, 1]^m \), we have

\[
m \cdot \max_{h \in \mathcal{H}} \text{cor}_S(h) - m \leq \mathbb{E} \left[ \sum_{i=1}^{m} p(i)(\bar{q}(x_i)y_i - 1) \right] + \frac{R_A(T)}{T} + \frac{\epsilon_0 m}{\gamma},
\] (4)
where \( \bar{q}(x_i) = \frac{1}{\gamma T} \sum_{t=1}^{T} h_t(x_i) \in \mathcal{K}_B' \).

**Projection.** Recall that the output hypothesis \( \bar{h} \) is defined using the projection \( \Pi \) (see Definition 2):

\[
\bar{h}(x_i) = \Pi(\bar{q}(x_i)).
\]

Now, by Lemma 5 there exists \( p^* \) such that

\[
m \cdot \max_{h \in \mathcal{H}} \text{cor}_S(h) - m \leq \mathbb{E} \left[ \sum_{i=1}^{m} p^*(i)(\bar{q}(x_i)y_i - 1) \right] + \frac{R_A(T)}{T} + \frac{\epsilon_0 m}{\gamma} \quad \text{(Equation 4)}
\]

\[
\leq m \cdot \mathbb{E} \left[ \text{cor}_S(\bar{h}) \right] - m + \frac{R_A(T)}{T} + \frac{\epsilon_0 m}{\gamma} \quad \text{(Lemma 5)}
\]

where the expectation is taken over the randomness of the projection, the weak learner, and the random samples given to the weak learner. Simple manipulation on the above inequality directly yields

\[
\max_{h \in \mathcal{H}} \text{cor}_S(h) \leq \mathbb{E} [ \text{cor}_S(\bar{h}) ] + \frac{R_A(T)}{T m} + \frac{\epsilon_0}{\gamma}.
\]

If we use OGD as the OCO algorithm, we have \( R_A(T) = GD \sqrt{T} \), where \( G \leq \frac{2\sqrt{m}}{\gamma} \) and \( D = 2\sqrt{m} \). We arrive at the theorem by plugging in \( \frac{R_A(T)}{T m} \).

\[ \blacksquare \]

### 3.3 Statistical Realizable Boosting

**Definition 9 (Empirical Weak Learning Assumption).** Let \( \mathcal{H} \subseteq \{\pm 1\}^X \) be a hypothesis class, and let \( S = \{(x_1, y_1), \ldots, (x_m, y_m)\} \in \mathcal{X} \times \{\pm 1\} \) be a sample. A learning algorithm \( \mathcal{W} \) is a \((\gamma, m_0)\)-**weak learner** (WL) for \( \mathcal{H} \) with respect to \( S \) if for any distribution \( p = (p_1, \ldots, p_m) \) which assigns each example \((x_i, y_i)\) with probability \( p_i \),

\[
\mathbb{E}_{S'}[\text{cor}_p(\mathcal{W}(S'))] \geq \gamma,
\]

where \( S' \) is an independent sample of size \( m_0 \) drawn from \( p \).

**Theorem 10.** The correlation of the output of Algorithm 2 denoted \( \bar{h} \), satisfies

\[
\mathbb{E}[\text{cor}_S(\bar{h})] \geq 1 - O\left(\frac{1}{\gamma \sqrt{T}}\right).
\]

The proof follows in a similar structure as in Theorem 8 and is deferred to the Appendix.

### 4 Online Realizable Boosting

In this section, we give an online realizable boosting algorithm, and state the regret bound. The result is along similar lines as our main result given in Section 2. We first state the weak learning assumption for the online realizable setting.

**Definition 11 (Online Weak Learning).** Let \( \mathcal{H} \subseteq \{\pm 1\}^X \) be a class of experts, let \( T \) denote the horizon length, and let \( \gamma > 0 \) denote the advantage. An online learning algorithm \( \mathcal{W} \) is a \((\gamma, T)\)-**weak online learner** (WOL) for \( \mathcal{H} \) if for any sequence \((x_1, y_1), \ldots, (x_T, y_T) \in \mathcal{X} \times \{\pm 1\}\) that is realizable by \( \mathcal{H} \), at every iteration \( t \in [T] \), the algorithm outputs \( \mathcal{W}(x_t) \in \{\pm 1\} \) such that,

\[
\sum_{t=1}^{T} \mathbb{E}[\mathcal{W}(x_t)]y_t \geq \gamma T - R_{\mathcal{W}}(T),
\]

where the expectation is taken over the randomness of the weak learner \( \mathcal{W} \) and \( R_{\mathcal{W}} : \mathbb{N} \rightarrow \mathbb{R}_+ \) is the additive regret: a non-decreasing, sub-linear function of \( T \).
Similar to the online agnostic case, the boosting algorithm is given access to $N$ instances of a $(\gamma, T)$-WOL algorithm (see Definition 11) and a $(K, T)$-OCO algorithm $A$ (see Definition 1). Instead of setting $K = [-1, 1]$ as in the agnostic case, we set $K = [0, 1]$. The algorithm for online boosting is exactly the same as in the agnostic online case (see Algorithm 1), except for line 6. In the online agnostic case, we pass a relabeled data point to $W_i$, while the algorithm below does not relabel the data points.

Algorithm 4 Online Boosting with OCO

1: for $t = 1, \ldots, T$ do
2: Get $x_t$, predict: $\hat{y}_t = \Pi\left(\frac{1}{\gamma N} \sum_{i=1}^{N} W_i(x_{i})\right)$.
3: for $i = 1, \ldots, N$ do
4: If $i > 1$, set $p_i^t = A(\ell_i^1, \ldots, \ell_i^{i-1})$. Else, set $p_1^t = 1/2$.
5: Set next loss: $\ell_i^t(p) = p(\frac{1}{2} W_i(x_t)y_t - 1)$.
6: Pass $(x_t, y_t)$ to $W_i$ w.p. $p_i^t$.
7: end for
8: end for

The following theorem proves the realizable online boosting result. Observe that in the realizable case, $\max_{h \in H} \text{cor}_S(h) = 1$. Let $\tilde{R}_W(T) := 2R_W(T) + \tilde{O}(\sqrt{T})$. Note that the error can be made arbitrarily small, by setting the number of weak learners to $N = O(\frac{1}{\gamma^{2/3}})$ and the number of iterations of Algorithm 4 to $T = O(\frac{1}{\gamma^{2/3}})$, for any $\epsilon > 0$. Thus, for an OCO algorithm (1) with regret bound $R_A(N, G_\gamma) = O(\sqrt{N})$, and a weak learner with regret bound $R_W(T) = O(\sqrt{T})$, by the following theorem, we get that the online correlation of the booster is at least $\text{cor}_S(h^*) - \epsilon$.

**Theorem 12.** The accumulated gain of Algorithm 4 satisfies:

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\hat{y}_t y_t] \geq 1 - \left(\frac{\tilde{R}_W(T)}{\gamma T} + \frac{R_A(N)}{N}\right).$$

where $(x_t, y_t)$’s are the observed examples, $\hat{y}_t$’s are the predictions, the expectation is with respect to the algorithm and learners’ randomness, $\tilde{R}_W(T) := 2R_W(T) + \tilde{O}(\sqrt{T})$, and $R_W$ and $R_A$ are the regret terms of the weak learner and the OCO, respectively.

The proof follows similarly to the proof of Proposition 3 and is deferred to the Appendix.

### 5 Discussion

We have presented the first boosting algorithm for agnostic online learning. In contrast to the realizable setting, we do not place any restrictions on the online sequence of examples. It remains open to prove lower bounds on online agnostic boosting as a function of the natural parameters of the problem and/or improve our upper bounds.

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Appendix

Lemma 13. Let \( p^i, W_i(x), y^i, y \) be random variables, such that \( y, y^i \in \{\pm 1\} \), and \( \mathbb{P}[y^i = y] = \frac{1 + p^i}{2} \). Moreover, \( W_i(x) \) and \( y^i \) are conditionally independent given \( p^i \) and \( y \), namely \( \mathbb{P}[W_i(x), y^i|p^i, y] = \mathbb{P}[W_i(x)|p^i, y] \mathbb{P}[y^i|p^i, y] \). Then \( \mathbb{E}[W_i(x) \cdot y^i] = \mathbb{E}[W_i(x) \cdot yp^i] \).

Proof. The proof of this lemma is based on the proof of Lemma 1 in [9].

Proof of Theorem [12]

We first state the following Lemma that will be used in the proof:

Lemma 14. For any weak learner \((\gamma, T)\)-WL \( W \), there exists \( c = \tilde{O}(\sqrt{T \sum_t p_t}) + 2R_1(T) \) such that for any sequence \( p_1, \ldots, p_T \in [0, 1] \),

\[
\sum_{t=1}^T p_t \cdot W(x_t)y_t \geq \gamma \sum_{t=1}^T p_t - c.
\]

Proof. The proof of this lemma is based on the proof of Lemma 1 in [9].

We are now ready to prove Theorem [12]. Let \( h^* \) be an optimal hypothesis in hindsight for the given sequence of examples. We prove by lower and upper bounding the sum of losses. For simplicity of presentation we assume an oblivious adversary, however, using a standard reduction, our results can be generalized to an adaptive one.\(^4\) Let \((x_1, y_1), \ldots, (x_T, y_T)\) be any sequence of observed examples. Observe that there are several sources of randomness at play; the weak learning algorithm \( W_i \)'s internal randomness, the booster randomly passing the example to \( W_i \) (line 5, Algorithm [3]), and the randomized prediction (line 2, Algorithm [4]). The analysis below is given in expectation with respect to all these random variables. We can now begin the analysis, starting with lower bounding the expected sum of losses, using the weak learning guarantee,

\[
\frac{1}{\gamma} \mathbb{E} \left[ \sum_{i=1}^N \sum_{t=1}^T W_i(x_t) \cdot y_t p_t^i \right] \geq \mathbb{E} \left[ \frac{1}{\gamma} \sum_{i=1}^N \left( \gamma \sum_{t=1}^T p_t^i - \tilde{R}_W(T) \right) \right] \quad \text{(Weak learning [1] Lemma [14])}
\]

\[
\geq \sum_{i=1}^N \sum_{t=1}^T \mathbb{E}[p_t^i] - \frac{N}{\gamma} \tilde{R}_W(T),
\]

Thus, we obtain the lower bound on the expected sum of losses \( \sum_t \sum_i \ell_t^i(p_t^i) \) (see Line 6 in Algorithm [1] for the definition of the \( \ell_t^i \)'s), given by,

\[
\mathbb{E} \left[ \sum_{t=1}^T \sum_{i=1}^N \ell_t^i(p_t^i) \right] \geq -\frac{N}{\gamma} \tilde{R}_W(T).
\]

\(^4\)See discussion in [12]. Pg. 69, as well as Exercise 4.1 formulating the reduction.
For the upper bound, observe that the OCO regret guarantee implies that for any \( t \in [T] \), and any \( p_t^* \in [0, 1] \),
\[
\mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^{N} \ell_t^i(p_t^*) \right] \leq p_t^* \left( \frac{1}{\gamma N} \sum_{i=1}^{N} \mathbb{E} \left[ \mathcal{W}_i(x_i) \right] y_t - 1 \right) + \frac{1}{N} R_A(N).
\]

Thus, by setting \( p_t^* \) according to Lemma 5, and summing over \( t \in [T] \), we get,
\[
\mathbb{E} \left[ \frac{1}{N} \sum_{t=1}^{T} \sum_{i=1}^{N} \ell_t^i(p_t^*) \right] \leq \sum_{t=1}^{T} \left( \mathbb{E}[\hat{y}_t] y_t - 1 \right) + \frac{T}{N} R_A(N).
\]

By combining the lower and upper bounds for \( \mathbb{E} \left[ \frac{1}{N} \sum_{t=1}^{T} \sum_{i=1}^{N} \ell_t^i(p_t^*) \right] \), we get,
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\hat{y}_t] y_t \geq 1 - \left( \frac{R_A(T)}{\gamma T} + \frac{R_A(N)}{N} \right).
\]

**Proof of Theorem 10**

**Reduction to Proposition 6.** Let \( h^* \) be a concept consistent with the input sample (i.e. \( h^*(x_i) = y_i \) for \( i \leq m \)) and let \( \mathcal{H}' = \mathcal{H} \cup \{ h^* \} \). It is convenient to define the decision sets are defined by \( \mathcal{K}_A = [0, 1]^m \), \( \mathcal{K}_B = \Delta_{\mathcal{H}'} \), and \( \mathcal{K}_B' = \frac{1}{\gamma} \Delta_{\mathcal{H}'} \), and the payoff function \( g(\cdot, \cdot) \) is again given by
\[
g(p, q) = \sum_{i=1}^{m} p(i)(q(x_i) y_i - 1).
\]

The weak learner corresponds to an approximate optimization oracle \( \mathcal{W} \) with no additive error. That is, for any \( p \in \mathcal{K}_A \) the output \( q' = \mathcal{W}(p) \) satisfies \( q' \in \mathcal{K}_B' \) and
\[
\mathbb{E}[g(p, q')] \geq 0.
\]

Next, one can show that the value of the game in this setting is \( \lambda^* = 0 \): indeed, this follows since \( \lambda^* = \min_{p \in \mathcal{K}_A} g(p, q^*) = 0 \) and since the pure strategy supported on \( h^*, q^* = q_{h^*} \in \mathcal{K}_B \) is dominant for player B. Applying Proposition 6 we have for any \( p \in \mathcal{K}_A \), with \( \tilde{q}(x_i) = \frac{1}{T} \sum_{t=1}^{T} h_t(x_i) \in \mathcal{K}_B' \),
\[
0 \leq \mathbb{E} \left[ \sum_{i=1}^{m} p(i)(\tilde{q}(x_i) y_i - 1) \right] + \frac{R_A(T)}{T}.
\]

**Projection.** By the definition of \( \tilde{h} \), using Equation 5 and Lemma 5, we have
\[
0 \leq \mathbb{E}[\text{cor}_S(\tilde{h})] - 1 + \frac{R_A(T)}{T m}.
\]

As before, using OGD as the OCO algorithm \( \mathcal{A} \) yields \( \frac{R_A(T)}{T m} = O\left(\frac{1}{\gamma \sqrt{T}}\right) \).