GENERALIZED KÄHLER GEOMETRY IN KAZAMA-SUZUKI COSET MODELS.

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Abstract

It is shown that Kazama-Suzuki conditions for the denominator subgroup of N=2 superconformal G/H coset model determine Generalized Kähler geometry on the target space of the corresponding N=2 supersymmetric σ-model.

1. Introduction.

It is by now well-known due to Gepner [1] that the unitary N=2 superconformal field theories play an important role in the construction of realistic models of superstring compactification from 10 to 4 dimensions. Gepner’s idea is to use N=2 superconformal field theories with central charge 9 for the internal sector of the string degrees of freedom and apply the GSO projection in such a way to be consistent with modular invariance. In particular, Gepner considered a product of N=2 minimal models such that their total central charges adds up to 9 and shown close relationship of his purely algebraic construction to the geometric Calabi-Yau σ-models compactification.

The Calabi-Yau manifold being a complex Kähler manifold is not accidental but caused by a close relation between the extended supersymmetry and Kähler geometry. In a more general case the background geometry may include also an antisymmetric B-field. In that case the corresponding 2-dimensional supersymmetric σ-model have a second supersymmetry when the target-space has a bi-Hermitian geometry, known also as Gates-Hull-Roček geometry [2]. In this situation the target manifold contains two complex structures with a Hermitian metric with respect to each of the complex structures. Quite recently it has been shown in [3] that these set of geometric objects, metric, antisymmetric B-field and two complex structures antisymmetric with respect to the metric have a unified description in the context of Generalized Kähler (GK) geometry. It allowed to develop for these models the GK geometry construction of N = 2 Virasoro superalgebra for [4], [5], [6], [7], [8]. It make sense by this reason to investigate the relationship of GK geometry σ-models to N = 2 superconformal field theories more closely.

The N = 2 supersymmetric WZW models on the compact groups [9], [10] provide a large class of examples where this relation is studied quite well [11], [12], [13], [14], [15]. These are the exactly solvable quantum conformal σ-models whose targets supports simultaneously GK geometry causing the extended N = 2-supersymmetry and affine Kac-Moody superalgebra structure ensuring the exact solution.

In a more general context it would be important to see what other unitary N = 2 superconformal field theories can be related to the GK geometry σ-models. In this note I consider Kazama-Suzuki coset models [16] and show that Kazama-Suzuki conditions for the denominator subgroup \( H \) of N=2 superconformal \( G/H \) coset model determine GK geometry on the target
space of the corresponding σ-model.

2. Manin triple construction of Kazama-Suzuki coset model.

I start with some preliminaries about $N = 2$ superconformal WZW model on the compact group $G$ and recall briefly the Manin triple construction of Kazama-Suzuki coset model represented in [17] (see also [18]).

In this case the group manifold $G$ is even dimensional and endowed with right-invariant complex structure $J_L$ and left-invariant complex structure $J_R$. Both of them are skew-symmetric w.r.t. the invariant metric on the group. Identifying the Lie algebra $g$ of the group with left-invariant vector fields or right-invariant vector fields we obtain the complex structure $J$ on $g$. It endows the complexification $g^C$ of $g$ with the Manin triple structure [21] which is the triple $(g^C, g_+, g_-)$ consisting of a Lie algebra $g^C$, with nondegenerate invariant inner product $<,>$ and isotropic Lie subalgebras $g_\pm$ such that $g^C = g_+ \oplus g_-$ as a vector space. It is clear that the subalgebras $g_\pm$ are $\pm \epsilon$-eigenspaces of the complex structure and the real Lie algebra $g$ is given by the fixed point set w.r.t. the natural antilinear involution which conjugates isotropic subalgebras, $\tau : g_+ \to g_-$. It is not difficult to establish a correspondence between the complex Manin triples endowed with antilinear involution $\tau$ conjugating the isotropic subalgebras and complex structures on the real Lie algebras [10]. Due to this correspondence Manin triple construction of Kazama-Suzuki models presented in [16] can be connected to the approach of [18] and [9] based on the complex structures on Lie algebras.

Let us fix arbitrary orthonormal basis $\{E^A, E_A, A = 1, \ldots, \frac{1}{2} \text{dim} g\}$ in algebra $g^C$ so that \{\{E^A\} is a basis in $g_-$, \{E_A\} is a basis in $g_+$. In this basis the commutators have the form

$$\begin{align*}
[E^A, E^B] &= f^{AB} C^C, \quad [E_A, E_B] = f^{AB}_C E_C, \\
[E^A, E_B] &= f^{AC}_B E_C - f^{BC}_A E_C,
\end{align*}$$

(1)

On the two-dimensional superspace with holomorphic supercoordinates $Z = (z, \Theta)$ I use $L^A(Z)$, $L_A(Z)$ to denote holomorphic (left-moving) spin-1/2 super-currents valued in $g_-$ and $g_+$ correspondingly. Similarly we denote by $R^A(Z)$, $R_A(Z)$ the anti-holomorphic (right-moving) spin-$\frac{1}{2}$ super-currents valued in $g_-$ and $g_+$.

The holomorphic currents satisfy the OPE’s

$$\begin{align*}
L^A(Z_1) L^B(Z_2) &= (z_1 - z_2)^{-\frac{1}{2}} f^{AB} C^C (Z_2) + \ldots \\
L_A(Z_1) L_B(Z_2) &= (z_1 - z_2)^{-\frac{1}{2}} f^{AB}_C L_C (Z_2) + \ldots \\
L^A(Z_1) L_B(Z_2) &= (z_1 - z_2)^{-1} \frac{1}{k} \delta^A_B + (z_1 - z_2)^{-\frac{1}{2}} (f^{AC}_B L_C - f^{BC}_A L_C)(Z_2) + \ldots
\end{align*}$$

(2)

where $z_1 - z_2 = z_1 - z_2 - \Theta_1 \Theta_2$ and $(z_1 - z_2)^{-\frac{1}{2}} = (z_1 - z_2)^{-1} (\Theta_1 - \Theta_2)$. The currents $R^A(Z)$, $R_A(Z)$ satisfy similar OPE’s. In what follows we concentrate on the holomorphic sector of the model.

Having the Manin triple structure and the OPE’s above we can construct [10] the spin-1 supercurrent of the $N = 2$ Virasoro superalgebra

$$K g(Z) = \frac{1}{k} (L^A L_A : + f_A D L^A - f^A D L_A)$$

(3)

where $f_A = f^{AB}_B$, $f^A = f^{AB}_B$. This current generates stress-energy spin-3/2 supercurrent of $N = 2$ Virasoro superalgebra

$$\Gamma g(Z) = \frac{1}{k} (D L^A L_A : + D L_A L^A : + \frac{1}{k} f^{AB}_C : L_C : L^A L^B : + \frac{1}{k} f^A_B : L^C : L_A L_B : )$$

(4)
by the OPE
\[ K_g(Z_1)K_g(Z_2) = (Z_1 - Z_2)^{-2}c_g + (Z_1 - Z_2)^{-1} \Gamma_g(Z_2) + ... \] (5)

where
\[ c_g = 3 \left( \frac{1}{2} \text{dim} g + \frac{2}{k} f^A f_A \right) \] (6)

Let us fix some Manin subtriple which is invariant under the involution \( \tau \)
\[ (h^C \subset g^C, h_+ \subset g_+, h_- \subset g_-) \] (7)

In this case the direct sum \( h_+ \oplus h_- \) is a complexification \( h^C \) of a real subalgebra \( h \). \( L^\alpha(Z) \) is used to denote the currents valued in \( h_- \), \( L_\alpha(Z) \) is used to denote the currents valued in \( h_+ \).

Because of the Manin subtriple is fixed one can construct corresponding spin-1 supercurrent \( K_h \)
\[ K_h(Z) = \frac{1}{k} : L^\alpha L_\alpha : + \varphi_\alpha DL^\alpha - \varphi^\alpha DL_\alpha \] (8)

where \( \varphi_\alpha = f^\beta_\alpha \), \( \varphi^\alpha = f^\alpha_\beta \). \( K_h(Z) \) generates super stress-energy tensor of the \( N = 2 \) WZW model associated to the denominator subgroup \( H \) of the coset model.

In the papers [16] Kazama and Suzuki found the conditions the denominator subgroup \( H \) must satisfy in order to \( N = 1 \) superconformal \( G/H \) coset model be \( N = 2 \) superconformal. Their conditions were equivalently reformulated in terms of the Manin triple and Manin subtriple in [17]:

**Proposition.**

The \( N = 1 \) superconformal coset model \( G/H \) is \( N = 2 \) superconformal if the subspaces \( t_\pm = g_\pm \setminus h_\pm \) are subalgebras. In this case the holomorphic spin-1 supercurrent
\[ K_{KS} = K_g - K_h \] (9)

satisfy Kazama-Suzuki conditions:
\[ L^\alpha(Z_1)K_{KS}(Z_2) = \text{reg.}, \quad L_\alpha(Z_1)K_{KS}(Z_2) = \text{reg.} \] (10)

and generates \( N = 2 \) Virasoro superalgebra of the coset-model with super stress-energy tensor
\[ \Gamma_{KS} = \Gamma_g - \Gamma_h \] (11)

and central charge \( c_{KS} = c_g - c_h \). The anti-holomorphic currents generating anti-holomorphic \( N = 2 \) Virasoro superalgebra of the coset model are given similarly. In [19] this construction was generalized to the supergroup manifolds. In [20] some particular examples of this general construction was considered.

In [17] the Proposition was proved in component fields. Here I sketch the proof of the Proposition using \( N = 1 \) superfield formalism which will be helpfull in geometric interpretation of Kazama-Suzuki construction.

Let \( L^\alpha(Z) \) be the \( t_- \)-valued basic currents and \( L_\alpha(Z) \) be the \( t_+ \)-valued basic currents. Let us consider the first OPE from (10)
\[ kL^\alpha(Z_1)K_{KS}(Z_2) = -(Z_1 - Z_2)^{-1} < [v - w, E^\alpha], E^A L_A(Z_2) + E_A L^A(Z_2) > + (Z_1 - Z_2)^{-1} (2 < [E_a, E^b]_J, E^\alpha >: L^a L^b : + 2 < [E_a, E^b]_J, E^\alpha >: L^a L_b : + < [E^a, E^b]_J, E^\alpha >: L_a L_b :)(Z_2) + \text{reg.} \] (12)
where \( v = \frac{1}{k}(f^A E_A - f_A E^A) \), \( w = \frac{1}{k}(\varphi^\alpha E_\alpha - \varphi_\alpha E^\alpha) \) and \([,]\) is a new Lie algebra bracket on vector space \( g^C \): for any vectors \( x, y \in g^C \)
\[
[x, y]_J = \frac{1}{2}([Jx, y] + [x, Jy])
\] (13)

It is easy to see that
\[
[g_+, g_-]_J = 0, \\
[g_+, g_+]_J = t[g_+, g_+]_J, \\
[g_-, g_-]_J = -t[g_-, g_-]_J
\] (14)

Because of \( t_{\pm} \) are isotropic subalgebras one can see that they are ideals in \( g_{\pm} \). Therefore
\[
< [E_a, E_b], E^\alpha >= 0
\] (15)
as it follows from (14). It follows also from (14) that
\[
< [E^a, E^b], E^\alpha >= 0
\] (16)

Thus, \( (Z_1 - Z_2)^{-\frac{1}{2}} \) contribution from (12) vanishes.

Consider the \( (Z_1 - Z_2)^{-1} \) contribution (it is absent in the classical limit). We find
\[
< [v - w, E^\alpha], E^\alpha], E^A L_A(Z_2) = E_A L^A(Z_2) > = \\
< f_A(f^A b + f^A a E^\gamma) - \varphi_\beta f_{\gamma}^\beta E^\gamma, E^\alpha], E^A L_A(Z_2) + E_A L^A(Z_2) > + \\
< f^A(f^A a E^\gamma + f^A b E^\gamma - f^A b E_\gamma - f^A b E^\gamma) - \varphi_\beta (f_{\gamma}^\beta - f^A b E^\gamma), E^\alpha], E^A L_A(Z_2) + E_A L^A(Z_2) >
\] (17)

Because of \( h_{\pm} \) are subalgebras and \( t_{\pm} \) are ideals in \( g_{\pm} \) some of the structure constants in the expression (12) are zero so that we obtain
\[
k < [v - w, E^\alpha], E^\alpha], E^A L_A(Z_2) = E_A L^A(Z_2) > = \\
< \chi_a f^a b E^b + \chi_a f^a b E^\gamma, E^\alpha], E^A L_A(Z_2) + E_A L^A(Z_2) > + \\
< -\chi_a f^a b E_b, E^A L_A(Z_2) + E_A L^A(Z_2) > + \\
< f^A(f^A b E^\gamma + f^A b E^\gamma - f^A b E_\gamma, E^\alpha], E^A L_A(Z_2) + E_A L^A(Z_2) > - \\
< \phi_\beta (f_{\gamma}^\beta E^\gamma - f_{\gamma}^\beta E_\gamma, E^\alpha], E^A L_A(Z_2) + E_A L^A(Z_2) >
\] (18)

where
\[
\chi_A = f^{A b}, \chi^A = f^{A b}_b
\] (19)

The relations
\[
\chi_A f^A a B = 0, \chi^A f_{\alpha} a B = 0 \\
\chi_A f^A a B = 0, \chi^A f_{\alpha} a B = 0
\] (20)

which follow from the Jacobi identity for \( g^C \) and because of \( t_{\pm} \) are ideals, reduce the expression above to
\[
k < [v - w, E^\alpha], E^A L_A(Z_2) = E_A L^A(Z_2) > = \\
< \chi_A f^A b E^\gamma + \chi^A f_{\gamma} b E^\gamma, E^A L_A(Z_2) + E_A L^A(Z_2) >
\] (21)
But it is zero because of
\[ \chi_{\beta} f^{\beta \alpha} \chi_{\alpha} + \chi_{\alpha} f^{\alpha \beta} = 0 \] \hspace{1cm} (22)

It can be proven in turn as follows: due to the Jacoby identity for \( g^C \) we have the relation
\[ f_C f^{C_{AB}} + f_C f^{B_{CA}} = f_{DC} f^{DC}_A, \] \hspace{1cm} (23)

Specifying this for \( A = \alpha, B = \beta \) we obtain
\[ f_C f^{C_{\alpha \beta}} + f_C f^{\beta_{CA}} = f_{\beta_{DC}} f^{\beta_{DC} \alpha}, \] \hspace{1cm} (24)

because of \( t_\pm \) are the ideals this relation is equivalent to
\[ f_{\gamma} f^{\gamma_{\alpha \beta}} + f_{\gamma} f^{\beta_{\gamma \alpha}} = (\chi + \varphi)_{\gamma} f^{\gamma_{\alpha \beta}} + (\chi + \varphi)^{\gamma} f^{\beta_{\gamma \alpha}} = f^{\beta_{\nu \mu}} f^{\nu_{\mu}}, \] \hspace{1cm} (25)

but \( h_\pm \) are isotropic subalgebras of the Manin triple \((h, h_+, h_-)\) so that
\[ \varphi_{\gamma} f^{\gamma_{\alpha \beta}} + \varphi^{\gamma} f^{\beta_{\gamma \alpha}} = f^{\beta_{\nu \mu}} f^{\nu_{\mu}}, \] \hspace{1cm} (26)

which proves (22).

Hence, \((Z_1 - Z_2)^{-1}\) contribution is zero also, so the first OPE from (10) is correct. Analogously the second OPE from (10) can be proved.

Similar Lie algebra analysis can be performed to establish the OPE
\[ K_{KS}(Z_1) K_{KS}(Z_2) = (Z_1 - Z_2)^{-2}(c_g - c_h) + (Z_1 - Z_2)^{-1}(\Gamma_g - \Gamma_h)(Z_2) + ... \] \hspace{1cm} (27)

as well as the other \( N = 2 \) Virasoro superalgebra OPE’s.

3. Hamiltonian formulation of classical \( N = 2 \) supersymmetric gauged WZW model and bi-Poisson structure.

Here we briefly discuss the Hamiltonian formalism in the classical \( N = 2 \) supersymmetric WZW model and provide bi-Poisson definition of GK geometry.

This formalism was considered in [15], [7], [8] \((N=0,1)\) versions was considered in [22], see also [23]). There it was shown that phase space of the model is a sheaf of twisted Poisson Vertex algebras [24], [25], [26]. Locally, the sections of the sheaf are generated by the canonically conjugated \( N = 1 \) superfields \( X^\mu(Z), X^*_\mu(Z) \) with the canonical Poisson super-brackets
\[ \{X^\mu_\mu(Z_1), X^{\nu(Z_2)}\} = -\{X^{\nu(Z_2)}, X^*_{\mu(Z_1)}\} = \delta^{\nu}_{\mu}\delta(Z_1 - Z_2) \] \hspace{1cm} (28)

and theirs super-derivatives along the super-circle variable \( Z = (\sigma, \theta) \). The spin-0 superfields \( X^\mu(Z) \) come from the local coordinates \( x^\mu \) on the group manifold \( G \) while the conjugated spin-1\( \frac{1}{2} \) superfields \( X^*_\mu(Z) \) correspond to the derivatives \( \frac{\partial}{\partial x^\mu} \) (in physics literature this set of fields also known as a \( b - c - \beta - \gamma \) system). On the intersection of patches with local coordinates \( x^\mu \) and \( y^\mu \) the canonical superfields are related by
\[ Y^{\nu(Z)} = y^{\nu(X^\mu(Z))}, \]
\[ Y^*_\nu = \frac{\partial x^\mu}{\partial y^{\nu}}(X^*_\mu + (B^y - B^z)_{\mu \lambda} D X^{\lambda}) \] \hspace{1cm} (29)
The set of 2-forms $B^x$, $B^y$, ..., defines the set of 1-forms $A^{yx}$, ..., determined on the intersections such that

$$
dB^x = dB^y = \mathcal{H},$$

$$B^y - B^x = dA^{yx}$$

(30)

where $\mathcal{H}$ is a 3-form on $G$ specifying the level $k$ of WZW model. When $k \in \mathbb{Z}$ this data define biholomorphic gerbe with connection [25], [27]. One can see that we are in the situation of theorem from paper [27] where very nice description of GK geometry in terms of locally defined symplectic forms determining the biholomorphic gerbe was found. As a result we get the global description of phase space of $N = 2$ supersymmetric WZW model [15].

In terms of the local coordinates the exact construction of the Kac-Moody superalgebra currents in $N = 2$ WZW model was given in [15]. I will not reproduce the expressions for the currents here but notice only that they are given by certain super-circle functions valued in the direct sum of tangent and cotangent bundle of the group manifold $G$. The Poisson brackets for the currents are given by

$$\{L^i(Z_1), L^j(Z_2)\} = k\eta^{ij}\delta(Z_1 - Z_2) + \delta(Z_1 - Z_2)f^i_jL^k(Z_2)$$

$$\{R^i(Z_1), R^j(Z_2)\} = -k\eta^{ij}\delta(Z_1 - Z_2) - \delta(Z_1 - Z_2)f^i_jR^k(Z_2)$$

$$\{L^i(Z_1), R^j(Z_2)\} = 0$$

(31)

where $\delta'(Z)$ means superderivative of $\delta(Z)$, $L^i(Z) = (L^A(Z), L_A(Z))$, $R^i(Z) = (R^A(Z), R_A(Z))$ and $\eta^{ij}$, $f^i_j$ are the invariant metric components and structure constants on Lie algebra $g$.

Another advantage of Hamiltonian formalism is the identification of primary Kac-Moody superalgebra fields with the functions on $G$ so that the Poisson brackets characterizing the primary field $\Phi(Z)$

$$\{L^i(Z_1), \Phi(Z_2)\} = \delta(Z_1 - Z_2)L^i \cdot \Phi(Z_2)$$

$$\{R^i(Z_1), \Phi(Z_2)\} = \delta(Z_1 - Z_2)R^i \cdot \Phi(Z_2)$$

(32)

are given by the actions $L^i \cdot \Phi$ of the basic left translations (basic right-invariant vector fields on $G$) and by the actions $R^i \cdot \Phi$ of the basic right translations (basic left-invariant vector fields on $G$) on the function $\Phi$ determined on $G$. Thus, using the Hamiltonian formalism we obtain supersymmetric generalization [22] of the regular representation of Kac-Moody algebra [28], [29].

Notice that two copies of $N = 2$ Virasoro algebra are acting in the regular representation because of the $N = 2$ Sugawara construction from section 2. In particular, the spin-1 currents

$$K^L(Z) = \frac{i}{2k}\omega_{L} L^i L^i(Z) = \frac{1}{k}L^A L_A(Z),$$

$$K^R(Z) = -\frac{i}{2k}\omega_{R} R^i R^i(Z) = -\frac{1}{k} R^A R_A(Z)$$

(33)

can be used to determine skew-symmetric bilinear operations on the primary fields $\Phi_{1,2}(Z)$ of the $N = 2$ superconformal WZW model:

$$\{\Phi_1(Z_1), \Phi_2(Z_2)\}_{K^L} := \{K^L_{-1/2}\Phi_1(Z_1), \Phi_2(Z_2)\},$$

$$\{\Phi_1(Z_1), \Phi_2(Z_2)\}_{K^R} := \{K^R_{-1/2}\Phi_1(Z_1), \Phi_2(Z_2)\}$$

(34)

where

$$K^L_{-1/2}\Phi(Z) = \oint dW \{K^L_{-1/2}(W), \Phi(Z)\}$$

(35)
and \( \omega_{Lij} = \eta_{ik}(J_L)^k_j \), \( \omega_{Rij} = \eta_{ik}(J_R)^k_j \). Because of the primary fields correspond to the functions \( \Phi_{1,2} \) on \( G \), the residues of the brackets define the skew-symmetric brackets on functions:

\[
\{ \Phi_1(Z_1), \Phi_2(Z_2) \}_K^- = \delta(Z_1 - Z_2) \frac{i}{2k} \omega_{Lij} (L^i \cdot \Phi_1)(L^j \cdot \Phi_2)(Z_2)
\]

\[
\{ \Phi_1(Z_1), \Phi_2(Z_2) \}_K^+ = -\delta(Z_1 - Z_2) \frac{i}{2k} \omega_{Rij} (R^i \cdot \Phi_1)(R^j \cdot \Phi_2)(Z_2)
\]

The brackets above do not satisfy Jacobi super-identity, instead the combinations

\[
\{ \{ \Phi_1(Z_1), \Phi_2(Z_2) \}_K^- \} \pm _{} := \{ \Phi_1(Z_1), \Phi_2(Z_2) \}_K^\pm
\]

do:

\[
\{ \{ \Phi_1(Z_1), \Phi_2(Z_2), \Phi_3(Z_3) \}_K^\pm \}_K^\pm = 0
\]

This can be checked by the direct calculation if one takes into account that the structure constants for the right translations are opposite to the structure constants for left translations and use the integrability property of the complex structures \( J_L, J_R \).

The residues of the brackets (38) define the pair of Poisson brackets on \( G \). It is easy to see that these two brackets are not compatible but the Schouten bracket of the bivector associated to \( K^L \) with the bivector associated to \( K^R \) is proportional to the WZW 3-form \( \mathcal{H} \) on the group \( G \). In this way we have just came to the definition of Generalized Kähler geometry by the bi-Poisson structure found in [30].

4. Bi-Poisson structure and GK geometry in Kazama-Suzuki models.

Here I consider the classical limit of the calculations from section 2 and show that Kazama-Suzuki conditions for the denominator subgroup \( H \) determine GK geometry on the target space of the corresponding N=2 supersymmetric \( \sigma \)-model.

Our approach is based on the geometric description of \( G/H \) coset models which is provided when they are considered as gauged WZW models [31], [32]. Integrating out the gauge fields we obtain \( \sigma \)-model action with some target space metric and \( B \)-field. In Hamiltonian approach [33] it causes the first class constraints for the currents valued in denominator subalgebra of the coset model. In the supersymmetric case at hand the constraints are

\[
L^\alpha(Z) - R^\alpha(Z) = 0, \ L_\alpha(Z) - R_\alpha(Z) = 0
\]

so that the observables \( O(Z) \) of Kazama-Suzuki model are determined by the equation:

\[
\{(L^\mu - R^\mu)(Z_1), O(Z_2) \} = 0
\]

where \( L^\mu(Z) = (L^\alpha(Z), L_\alpha(Z)) \), \( R^\mu(Z) = (R^\alpha(Z), R_\alpha(Z)) \). In case the denominator subgroup contains direct abelian factors with trivial \( Ad \)-action the corresponding constraints from (39), (40) must be replaced by \( L^\mu(Z) + R^\mu(Z) = 0 \). This circumstance will be implied hereinafter.

Due to the discussion in section 3 it allows in particular to identify the functions on the target space of the Kazama-Suzuki \( \sigma \)-model with the \( Ad_H \)-invariant functions on \( G \). So we can extend the discussion of the section 3 to the case of Kazama-Suzuki coset model and show that Kazama-Suzuki conditions define bi-Poisson structure underlying the GK geometry in coset model.
First of all we use the classical Kazama-Suzuki coset model spin-1 currents

\[ K_{KS}^L(Z) = \frac{i}{2k} (\omega_{Lij} L^i L^j(Z) - \omega_{L\mu\nu} L^\mu L^\nu(Z)) = \frac{1}{k} L^a L_a(Z), \]

\[ K_{KS}^R(Z) = -\frac{i}{2k} (\omega_{Rij} R^i R^j(Z) - \omega_{R\mu\nu} R^\mu R^\nu(Z)) = -\frac{1}{k} R^a R_a(Z) \tag{41} \]

to define the skew-symmetric bilinear operations on the fields \( \Psi_{1,2}(Z) \) satisfying the constraints (40):

\[ \{\Psi_1(Z_1), \Psi_2(Z_2)\}_{K_{KS}^L} := \{K_{KS}^L \Psi_1(Z_1), \Psi_2(Z_2)\}, \]

\[ \{\Psi_1(Z_1), \Psi_2(Z_2)\}_{K_{KS}^R} := \{K_{KS}^R \Psi_1(Z_1), \Psi_2(Z_2)\} \tag{42} \]

By the residues these brackets define skew-symmetric brackets on the \( Ad_H \)-invariant functions \( \Psi_{1,2} \) determined on \( G \)

\[ \{\Psi_1(Z_1), \Psi_2(Z_2)\}_{K_{KS}^L} = \delta(Z_1 - Z_2) \frac{i}{2k} \omega_{Lpq} (L^p \cdot \Psi_1)(L^q \cdot \Psi_2)(Z_2) \]

\[ \{\Psi_1(Z_1), \Psi_2(Z_2)\}_{K_{KS}^R} = -\delta(Z_1 - Z_2) \frac{i}{2k} \omega_{Rpq} (R^p \cdot \Psi_1)(R^q \cdot \Psi_2)(Z_2) \tag{43} \]

where the common notation \( L^p = (L^a(Z), L_a(Z)) \), \( R^p = (R^a(Z), R_a(Z)) \) has been used.

The next step is to show that due to (40) the algebra of \( Ad_H \)-invariant functions is closed under these brackets and the linear combinations

\[ \{\Psi_1(Z_1), \Psi_2(Z_2)\}_{\pm} := \{\Psi_1(Z_1), \Psi_2(Z_2)\}_{K_{KS}^L} \pm \{\Psi_1(Z_1), \Psi_2(Z_2)\}_{K_{KS}^R} \tag{44} \]

define bi-Poisson structure on Kazama-Suzuki model target space by the residues.

To this end one needs to calculate the Lie derivative of the brackets (44) with respect to the denominator currents \( \omega^a(Z) - R^a(Z) \). We obtain from the one hand

\[ \{L^\mu(W) - R^\mu(W), \{\Psi_1(Z_1), \Psi_2(Z_2)\}_{\pm}\} = \{\{L^\mu(W) - R^\mu(W), \Psi_1(Z_1)\}, \Psi_2(Z_2)\}_{\pm} \]

\[ + \{\Psi_1(Z_1), \{L^\mu(W) - R^\mu(W), \Psi_2(Z_2)\}\}_{\pm} = \]

\[ \{L^\mu(W) - R^\mu(W), \{\Psi_1(Z_1), \Psi_2(Z_2)\}_{\pm}\} \tag{45} \]

From the other hand one can see from the definitions (43) that

\[ \{L^\mu(W) - R^\mu(W), \{\Psi_1(Z_1), \Psi_2(Z_2)\}_{\pm}\} = \{\{L^\mu(W) - R^\mu(W), \Psi_1(Z_1)\}, \Psi_2(Z_2)\}_{\pm} \]

\[ + \{\Psi_1(Z_1), \{L^\mu(W) - R^\mu(W), \Psi_2(Z_2)\}\}_{\pm} = \]

\[ \oint dZ \{\{L^\mu(W), K_{KS}^L(Z)\}, \Psi_1(Z_1)\}, \Psi_2(Z_2)\}_{\pm} \]

\[ + \oint dZ \{\{R^\mu(W), K_{KS}^R(Z)\}, \Psi_1(Z_1)\}, \Psi_2(Z_2)\}_{\pm} \tag{46} \]

The result of Poisson brackets \( \{L^\mu(W), K_{KS}^L(Z)\}, \{R^\mu(W), K_{KS}^R(Z)\} \) calculations can be read of from the \( (\mathcal{V} - \bar{Z})^{-\frac{1}{2}} \) and \( (\mathcal{K} - \bar{Z})^{-\frac{1}{2}} \) poles in the OPE’s \( L^\mu(W) K_{KS}^L(Z) \) and \( R^\mu(W) K_{KS}^R(Z) \). But they vanish as we have seen in section 2. Thus we obtain

\[ \{L^\mu(W) - R^\mu(W), \{\Psi_1(Z_1), \Psi_2(Z_2)\}_{\pm}\} = 0 \tag{47} \]
In view of (43) it means that algebra of \( Ad_H \)-invariant functions is closed under the brackets defined by (44). Notice the close similarity of the reasoning here to the Poisson-homogeneous space reduction theorem from [34].

Now we show that the brackets (44) define bi-Poisson structure on invariant functions and the Schouten brackets between the corresponding bivectors are proportional to the 3-form on the Kazama-Suzuki target space. Thus one needs to calculate two Jacobians:

\[
\{\Psi_1(Z_1), \{\Psi_2(Z_2), \Psi_3(Z_3)\}_K^L \}_K^R - \{\{\Psi_1(Z_1), \Psi_2(Z_2)\}_K^L, \Psi_3(Z_3)\}_K^R + \\
\{\Psi_2(Z_2), \{\Psi_1(Z_1), \Psi_3(Z_3)\}_K^R \}_K^L
\]

(48)

\[
\{\Psi_1(Z_1), \{\Psi_2(Z_2), \Psi_3(Z_3)\}_K^R \}_K^L - \{\{\Psi_1(Z_1), \Psi_2(Z_2)\}_K^R, \Psi_3(Z_3)\}_K^L + \\
\{\Psi_2(Z_2), \{\Psi_1(Z_1), \Psi_3(Z_3)\}_K^L \}_K^R
\]

(49)

They can be calculated directly, but the result is dictated by the 3-currents contributions in the OPE’s \( K_{KS}^L(Z_1)K_{KS}^R(Z_2) \) and \( K_{KS}^R(Z_1)K_{KS}^R(Z_2) \) so that the right hand sides of the Jacobians (48) and (49) are equal correspondingly to

\[
\delta(Z_2 - Z_3)\delta(Z_1 - Z_3)\frac{1}{3k^2}(f_{ijk}(L^i \cdot \Psi_1)(L^j \cdot \Psi_2)(L^k \cdot \Psi_3)(Z_3) - \\
f_{\nu\mu\lambda}(L^\nu \cdot \Psi_1)(L^\mu \cdot \Psi_2)(L^\lambda \cdot \Psi_3)(Z_3))
\]

(50)

and

\[
-\delta(Z_2 - Z_3)\delta(Z_1 - Z_3)\frac{1}{3k^2}(f_{ijk}(R^i \cdot \Psi_1)(R^j \cdot \Psi_2)(R^k \cdot \Psi_3)(Z_3) - \\
f_{\nu\mu\lambda}(R^\nu \cdot \Psi_1)(R^\mu \cdot \Psi_2)(R^\lambda \cdot \Psi_3)(Z_3))
\]

(51)

Expressing the right translation action on functions in terms of the left translations and using (40) one can rewrite (51) as

\[
-\delta(Z_2 - Z_3)\delta(Z_1 - Z_3)\frac{1}{3k^2}(f_{ijk}(L^i \cdot \Psi_1)(L^j \cdot \Psi_2)(L^k \cdot \Psi_3)(Z_3) - \\
f_{\nu\mu\lambda}(L^\nu \cdot \Psi_1)(L^\mu \cdot \Psi_2)(L^\lambda \cdot \Psi_3)(Z_3))
\]

(52)

Therefore the brackets (44) satisfy Jacobi super-identities. By the residues they define the pair of Poisson brackets on \( Ad_H \)-invariant functions on \( G \) and the Schouten bracket of the bivector associated to \( K_{KS}^L \) with the bivector associated to \( K_{KS}^R \) is proportional to the 3-form on Kazama-Suzuki \( \sigma \)-model target space.

We proved thereby that Kazama-Suzuki conditions for the denominator subgroup \( H \) of \( N = 2 \) superconformal \( G/H \) coset model determine GK geometry on the target space of the corresponding \( \sigma \)-model.

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