Lattice Reduction over Imaginary Quadratic Fields

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Abstract—In this work, we extend the celebrated Lenstra-Lenstra-Lovász (LLL) reduction from over real bases to over complex bases. The complex bases, along with direct-sums defined by rings of imaginary quadratic integers, induce algebraic lattices. We first analyze properties of these algebraic lattices, and then construct a general algebraic LLL reduction algorithm for them. Properties and implementations of the algorithm are examined. In particular, satisfying Lovász’s condition requires the ring to be Euclidean. As an application, we use the algebraic LLL algorithm to find the network coding matrix in compute-and-forward. Such lattice reduction-based approaches have low complexity which is not dictated by the signal-to-noise (SNR) ratio. Moreover, such approaches can not only preserve the degree-of-freedom of computation rates, but ensure the independence in the code space as well.

Index Terms—lattice reduction, algebraic, compute-and-forward.

I. INTRODUCTION

The compute-and-forward (C&F) paradigm [2] built over lattice codes is one of the main approaches in physical layer network coding. The main idea of the paradigm is that, rather than ignoring the interference as noise, a relay will compute and then forward a linear function of transmitted messages according to the observed channel coefficients. With enough linear independent equations at the destination, the original messages can be recovered respectively. One crucial task in the paradigm is to design network coding matrices based on finding short lattice vectors in the “channel lattice”. [3]–[6], whose structure is defined by the “coding lattice”. Initially coding lattices from Construction A over rational integers \( \mathbb{Z} \) or Gaussian integers \( \mathbb{Z}[i] \) are the main enabler in showing the achievable information rate in C&F. In recent years, however, the Construction A lattices have been extended to over the ring of Eisenstein integers \( \mathbb{Z}[\omega] \) [7], [8] and other rings of imaginary quadratic integers [9], and these lattices are proved good for coding and for MSE quantization. We are therefore left with the task of finding short vectors of lattices over algebraic integers, so as to design the network coding matrices for the adaptive C&F paradigm in [9] that works with the best ring of imaginary quadratic integers.

Lattice reduction is perhaps the most efficient method for finding approximately short lattice vectors. For \( \mathbb{Z} \)-lattices, lattice reduction techniques have been well explored. These techniques include the celebrated Lenstra-Lenstra-Lovász (LLL) reduction [10], Korkine-Zolotarev (KZ) reduction [11], Minkowski reduction [12], and their variants [13], [14]. For \( \mathcal{O}_K \)-lattices, where \( \mathcal{O}_K \) denotes the ring of integers of a number field \( K \), the reduction techniques can be classified based on whether the lattice vectors lie in \( \mathcal{O}_K \) or the complex field \( \mathbb{C} \). The first scenario arises quite often in lattice-based cryptography, and much work has been done in generalizing LLL for such lattices [15]–[18]. Napias’s work [15] extends LLL to over Euclidean rings contained in a CM number field or a quaternion field. Fieker and Pohst’s approach [16] defines LLL over Dedekind domains, while Fieker and Stehlé’s approach [17] is to apply LLL to an equivalent higher dimensional \( \mathbb{Z} \)-lattice and return this to a module. Quite recently, Kim and Lee [18] presents reduction algorithms for arbitrary Euclidean domains. Regarding the second scenario whose basis vectors are in \( \mathbb{C} \), the LLL algorithm has also been generalized to \( \mathbb{Z}[i] \)-lattices [19] and \( \mathbb{Z}[\omega] \)-lattices [20], and these generalizations are used in the context of MIMO detection/precoding whose signal constellations are algebraic. As a motivation, we notice that a general study on the reduction of \( \mathbb{Z}[\xi] \)-lattices, where \( \mathbb{Z}[\xi] \) denotes a ring of imaginary quadratic integers, is lacking.

In this work, we refer to \( \mathbb{Z}[\xi] \)-lattices as algebraic lattices. We seek to better understand the characteristics of such lattices, so that we can further design an algebraic reduction algorithm for these lattices and analyze its performance. The contributions of this paper are the following:

i) After presenting the definitions and measures for algebraic lattices, we analyze the algebraic analogs for orthogonal defect, Hermite’s constant, and Minkowski’s first and second theorems. These characterizations lay the foundation for understanding the performance limit of a reduction algorithm.

ii) We extend the definition of lattice reduction from over \( \mathbb{Z} \)-lattices to \( \mathbb{Z}[\xi] \)-lattices, which says that the reduction is to find a transform matrix from the general linear group \( \text{GL}_n (\mathbb{Z}[\xi]) \). The algebraic LLL algorithm can thus be defined. By defining a quantization of the Gram-Schmidt coefficients to over ring \( \mathbb{Z}[\xi] \), the lower bound of the Lovász’s constant is derived. On the other hand, to ensure the algorithm is convergent, its upper bound can also be derived, which leads to the conclusion that only Euclidean rings can correspond to algebraic LLL. Notice that after transforming a \( \mathbb{Z}[\xi] \)-lattice to a \( \mathbb{Z} \)-lattice, LLL can always be implemented while \( \mathbb{Z}[\xi] \) is not required to be Euclidean. But the process of returning \( 2n \) \( \mathbb{Z} \)-lattice vectors to \( n \) \( \mathbb{Z}[\xi] \)-lattice vectors can be complicated.

iii) As an application, we utilize the developed algebraic LLL to design the network coding coefficients in algebraic C&F, in which the task is to find short lattice vectors in \( \mathbb{Z}[\xi] \)-lattices. While algebraic LLL has low-complexity, we prove that it maintains the degree-of-freedom (DoF) achieved by a shortest vector problem (SVP) oracle. Moreover, we show that in practice, the network coding matrix has to be invertible in the quotient ring (which is isomorphic to the space of error correction codes). All these statements are verified with numerical simulations.

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The rest of this paper is organized as follows. In Section II backgrounds about quadratic fields and algebraic lattices are reviewed, and characteristics of these lattices are analyzed. Algebraic lattice reduction and its theoretical analysis are presented in Section III. In Section IV we review C&F over rings of quadratic fields and present a scheme to use lattice reduction in a system level. Numerical results for lattice reduction and its application to C&F are presented in Section V. Concluding remarks are given in the last section.

Notations: Matrices and column vectors are denoted by uppercase and lowercase boldface letters, respectively. The real and imaginary parts of a complex number are denoted as \( \Re(\cdot) \) and \( \Im(\cdot) \). \( \mathbb{Z}, \mathbb{Z}[i], \mathbb{Z}[\omega], \mathbb{Q}, \mathbb{R}, \) and \( \mathbb{C} \) are used to denote the set of integers, Gaussian integers, Eisenstein integers, rational, real, and complex numbers, respectively. \( \mathbb{F}_p \) denotes a finite field with size \( p \).

Definition 1 ([22]). For a more complex account of algebraic lattices, we refer readers to [22].

Definition 2 (Algebraic integers). An algebraic integer is a complex number which is a root of some monic polynomial (Algebraic integers).

Definition 3 (Euclidean domains). A Euclidean domain is an integral domain which can be endowed with at least one Euclidean function for a more complex account of algebraic lattices.

Definition 4 (Algebraic lattices). A \( \mathbb{Z}[\xi] \)-lattice is a discrete \( \mathbb{Z}[\xi]\)-module of \( \mathbb{C}^n \) that has a basis. Such a rank \( n \) lattice \( \Lambda_{\mathbb{Z}[\xi]}(B) \) with basis \( B = \{b_1, \ldots, b_n\} \in \mathbb{C}^{n \times n} \) can be represented by direct sums:

\[
\Lambda_{\mathbb{Z}[\xi]}(B) = \mathbb{Z}[\xi] b_1 + \mathbb{Z}[\xi] b_2 + \cdots + \mathbb{Z}[\xi] b_n.
\]

Definition 5 (Successive minima). The \( j \)-th successive minimum of a \( \mathbb{Z}[\xi] \)-lattice \( \Lambda_{\mathbb{Z}[\xi]} \) is the smallest real number \( r \) such that its embedded \( \mathbb{Z}[\xi] \)-lattice through a bijection \( \sigma \) contains \( j \) linearly \( \mathbb{Z}[\xi] \)-independent vectors of length at most \( r \): \( \lambda_j,\mathbb{Z}[\xi] = \inf \left\{ r \mid \dim \left( \text{span} \left( \sigma^{-1} \left( \left( \Lambda_{\mathbb{Z}[\xi]} \cap B(0, r) \right) \right) \right) \right) \geq j \right\} \), where \( B(t, r) \) denotes a ball centered at \( t \) with radius \( r \).

A. Hermite’s Constant and OD

To proceed, we first show the \( \mathbb{Z} \)-basis (real generator matrix) of lattice \( \Lambda_{\mathbb{Z}[\xi]}(B) \) is:

\[
B^{\mathbb{Z}[\xi]} = \begin{cases} 
\Re(B) - \sqrt{d} \Im(B) & \text{if } \xi = \sqrt{-d}; \\
\Re(B) + \sqrt{d} \Im(B) & \text{if } \xi = \frac{1 + \sqrt{-d}}{2}.
\end{cases}
\]

Let ring coefficients of \( B \) be \( x = x_a + \xi x_b \in \mathbb{Z}[\xi]^n \). If \( \xi = \sqrt{-d}, d > 0 \), we have

\[
Bx = (\Re(B) + i \Im(B)) (x_a + i \sqrt{d} x_b) = \left( \Re(B) x_a - \sqrt{d} \Im(B) x_b \right) + i \left( \Im(B) x_a + \sqrt{d} \Re(B) x_b \right);
\]

and if \( \xi = \frac{1 + \sqrt{-d}}{2}, d > 0 \), we have

\[
Bx = (\Re(B) + i \Im(B)) \left( x_a + \frac{1}{2} x_b + i \frac{\sqrt{d}}{2} x_b \right) = \left( \Re(B) x_a + \left( \frac{1}{2} \Re(B) - \frac{\sqrt{d}}{2} \Im(B) \right) x_b \right) + i \left( \Im(B) x_a + \left( \frac{1}{2} \Im(B) + \frac{\sqrt{d}}{2} \Re(B) \right) x_b \right).
\]
where we can define the volume of an algebraic lattice as referred to that of the vector \( \mathbf{v}_1, \ldots, \mathbf{v}_n \). Define the function \( \lambda \) for all rank \( n \) lattices, which quantifies how close the basis is to being “orthogonal”. For a \( \mathbb{Z} \)-lattice, its lower bound is \( \eta_{\mathbb{Z}[\xi]}(\mathbf{B}) \geq 1 \) according to Hadamard’s inequality. More generally, it follows from Eq. (6) that

\[
\eta_{\mathbb{Z}[\xi]}(\mathbf{B}) \geq \det \left( \Phi_{\mathbb{Z}[\xi]} \right)^{-n}.
\]

The volume of a lattice is fixed, so the smallest \( \eta_{\mathbb{Z}[\xi]}(\mathbf{B}) \) is achieved only when each \( \| \mathbf{b}_j \| \) is minimized.

**B. Minkowski’s Theorems**

Minkowski’s first and second theorems are crucial for analyzing the performance of a lattice reduction algorithm. These theorems over \( \mathbb{Z} \)-lattices are well known. For algebraic lattices where the bases may not belong to a number field, we need the following theorem:

**Theorem 1** (Minkowski’s first and second theorems over \( \mathbb{Z}[\xi] \)-lattices). For a \( \mathbb{Z}[\xi] \)-lattice \( \Lambda_{\mathbb{Z}[\xi]}(\mathbf{B}) \) with basis \( \mathbf{B} \) in \( \mathbb{C}^n \times \mathbb{C}^n \), it satisfies

\[
\lambda^2_{1,\mathbb{Z}[\xi]} \leq \gamma_{2n}\left| \det \left( \Phi_{\mathbb{Z}[\xi]} \right) \right|^n \left| \det (\mathbf{B}) \right|^{2/n};
\]

\[
\prod_{j=1}^{n} \lambda^2_{j,\mathbb{Z}[\xi]} \leq \gamma_{2n}^n \left| \det \left( \Phi_{\mathbb{Z}[\xi]} \right) \right|^n \left| \det (\mathbf{B}) \right|^{2}.
\]

**Proof:** Minkowski’s first theorem is a direct consequence of (7). To obtain Minkowski’s second theorem for \( \mathbb{Z}[\xi] \)-lattices, the rationale is to apply its classic version [23] to the embedded \( \mathbb{Z} \)-lattice and inspect the independence of lattice vectors over the ring \( \mathbb{Z}[\xi] \). Based on Eq. (4), applying the real Minkowski’s second theorem [23] yields

\[
\prod_{j=1}^{2n} \lambda^2_{j} \leq \gamma_{2n}^2 \left( \det (\mathbf{B}) \right)^2.
\]
where $\lambda_j$ denotes the $j$th successive minimum of lattice $\Lambda(B^{Z}[\xi])$. Substitute Eq. (10) into the above equation, we have

$$\prod_{j=1}^{2n} \lambda_j^2 \leq \gamma^{2n} \det \bigg( B^{Z}[\xi(1)] \bigg)^2 \det \left( \Phi^{Z}[\xi] \right)^{2n}. \quad (11)$$

Let the $2n$ successive minima of $L(B^{Z}[\xi])$ be $\|B^{Z}[\xi](\xi_1)\|, \ldots, \|B^{Z}[\xi](\xi_{2n})\|$. W.l.o.g., we assume the input basis $B$ has full rank, then do $B^{Z}[\xi]$. For any index $j, j'$,

$$\dim \left( \text{span}_Z[\xi] \left( \sigma^{-1} \left( B^{Z}[\xi](\xi_j, B^{Z}[\xi](\xi_{j'})) \right) \right) \right) = \dim \left( \text{span}_Z[\xi] \left( \sigma^{-1}(\xi_j, \xi_{j'}) \right) \right).$$

Since the coefficients of the successive minima satisfy $\dim (\text{span}_Z(x_1, \ldots, x_{2n})) = 2n$, it yields $\dim (\text{span}_Z(\sigma^{-1}(x_1, \ldots, x_{2n}))) = n$. We can design an algorithm to partition $x_1, \ldots, x_{2n}$ into two groups, each with size $n$. Firstly, note that there exists an index set $S$ with $|S| = n$ such that $\dim (\text{span}_Z(\sigma^{-1}(x_{S(1)}, \ldots, x_{S(n)}))) = n$. Secondly, starting from $x_{S(1)}$, we search for one candidate in $x_{2n}\setminus S$ in each round, noted as $x'_{S(1)}$ such that $(x_{S(1)})^T x'_{S(1)} \neq 0$. This procedure continues until all $x_{2n}$ have been partitioned. It follows from Definition 5 that $\forall j \in 1, \ldots, n$, $\|\sigma^{-1}(B^{Z}[\xi](\xi_{S(j)})\| \leq \|\sigma^{-1}(B^{Z}[\xi](x'_{S(j)})\|$. (12)

Based on (12), we have $\prod_{j=1}^{n} \lambda_{j,Z}[\xi] \leq \left( \prod_{j=1}^{2n} \lambda_j^2 \right)^{1/2}$. 

Plugging this into (11), we have

$$\prod_{j=1}^{n} \lambda_{j,Z}[\xi] \leq \gamma^{2n} \det \left( B^{Z}[\xi] \right)^{1/2} \det \left( \Phi^{Z}[\xi] \right)^{n}. \quad \blacksquare$$

III. ALGEBRAIC LATTICE REDUCTION

A lattice has infinitely many bases. The process of improving the quality of a given basis by some lattice-preserving transform is generically called lattice reduction. It is well known that a transformation matrix should be taken from a set of integer matrices that are invertible in $Z$ for a real basis, while such transforms for a complex basis remain poorly understood. In this section, we will introduce algebraic lattice reduction, with a special focus on algebraic LLL.

A. Definition

Denote $\text{GL}_n(Z[\xi])$ as the set of invertible matrices in the matrix ring $M_{n \times n}(Z[\xi])$ and call a matrix in $\text{GL}_n(Z[\xi])$ unimodular. To define algebraic lattice reduction, we need the following proposition.

**Proposition 1.** Two lattice bases $B, \tilde{B}$ generate the same lattice if and only if there exists a matrix $U \in \text{GL}_n(Z[\xi])$ such that $B = BU$.

**Proof:** First, we show that $B, \tilde{B}$ generate the same lattice if $\tilde{B} = BU$ for a unimodular matrix $U$. Let $\Lambda$ be generated by $B$ and let $\Lambda$ be generated by $\tilde{B}$. Any element $b \in \Lambda$ can be written as

$$b = Bx = BUx \in \Lambda,$$

for some $x \in Z[\xi]^n$, which shows that $\Lambda \subseteq \Lambda$ since $Ux \in Z[\xi]^n$. On the other hand, if $U$ is invertible, we have $B = BU^{-1}$ and a similar argument shows that $\Lambda \subseteq \Lambda$. Now we show the invertibility condition is $\det (U) \in Z[\xi]^n$. Note that if a ring $Z[\xi]$ is from complex quadratic fields, then it is commutative (a non-commutative example is the matrix ring). For any matrix $U \in Z[\xi]^{n \times n}$, it follows from Cramer’s rule that $U^{-1} = (\det (U))^{-1} \text{adj} (U)$, where $\text{adj} (U) \in Z[\xi]^{n \times n}$, the adjugate of $U$ is given by $\text{adj} (U)(j, j) = (-1)^{j+j} M_{j,j}$, where $M_{j,j}$ is the minor of $U$ obtained by deleting the $j$th row and the $j$th column of $U$. Clearly, matrix $U$ is invertible in $Z[\xi]^{n \times n}$ if and only if $\det (U) \in Z[\xi]^n$, such that $U^{-1} \in Z[\xi]^{n \times n}$.

Second, we show that $B = BU$ for a unimodular matrix $U$ if $B, \tilde{B}$ generate the same lattice. Based on the “if” condition, there are some full-rank transforms $U_1$ and $U_2$ in $Z[\xi]^{n \times n}$ such that $B = BU_1, \tilde{B} = BU_2$ and hence $B = BU_2U_1$. This implies $U_2U_1$ is an identity matrix. As the determinant function is distributive, we have $\det (U_2U_1) = 1$, with $\det (U_1) = \det (U_2) \in Z[\xi]$. Thus if $\det (U_1)$ and $\det (U_2)$ are a pair of invertible elements in $Z[\xi]$, and $U_1 \in \text{GL}(n, Z[\xi])$, $U_2 \in \text{GL}(n, Z[\xi])$.

The above proposition suggests we can define lattice reduction for algebraic lattices based on $\text{GL}_n(Z[\xi])$.

**Definition 7** (Algebraic lattice reduction). For a given algebraic lattice $\Lambda[\xi]$ with basis $B \in C^{n \times n}$, find a new basis $\tilde{B} = BU$ with favorable properties, where $U \in \text{GL}_n(Z[\xi])$.

We now present the definition of algebraic LLL for algebraic lattices. Let $Q_{\Lambda}[\cdot]$ be a quantization function for a point $x \in C$ that returns its closest algebraic integer in $Z[\xi]$:

$$Q_{\Lambda}[\xi](x) \triangleq \arg \min_{\lambda \in \xi[\xi]} |\lambda - x|.$$

**Definition 8** (Algebraic LLL). An $n \times n$ complex matrix $B \in C^{n \times n}$ is called an ALLL-reduced basis of lattice $\Lambda[\xi](B)$ if its QR-decomposition $B = QR$ satisfies the following two conditions:

$$Q_{\Lambda}[\xi]\left( \frac{R_{j,k}}{R_{j,j}} \right) = 0, \forall j < k; \text{ (size reduction condition)} \quad (13)$$

$$\delta |R_{j-1,j-1}|^2 \leq |R_{j,j}|^2 + |R_{j-1,j}|^2, \text{ (Lovász's condition)} \quad (14)$$

$2 \leq i \leq n, R_{j,k}$ refers to the $(j, k)$th entry of $R$, and $\delta$ is called Lovász’s parameter.

If the lattice is a $Z[i]$-lattice, then $\Theta\left( \frac{R_{j,k}}{|R_{j,j}|} \right) \leq \frac{1}{2}$ and $\Theta\left( \frac{R_{j,k}}{|R_{j,j}|} \right) \leq \frac{1}{2}$, which is consistent with [19] that generalizes the definition in [15].

B. Lovász’s parameter

We first explain how the lower bound of $\delta$ should be chosen based on the covering radius $\rho_{\Lambda}[\xi]$ of lattice $\Lambda[\xi](\Phi^{Z}[\xi])$, where

$$\rho_{\Lambda}[\xi] \triangleq \max_{x \in C} |x - Q_{\Lambda}[\xi](x)|.$$
This typical lattice parameter can be analyzed through describing the relevant vectors of the Voronoi region of \( \Lambda^Z (\Phi_2^{Z[\xi]}). \) For a real lattice \( \Lambda^Z (B^R), B^R \in R^{n \times n}, \) its Voronoi region around the origin is

\[
\mathcal{V} = \{ x \in R^n \mid \|x\| \leq \|x - t\| \forall t \in \Lambda^Z (B^R), t \neq 0 \}. 
\]

The points \( p \) of the lattice for which the hyper-plane between \( 0 \) and \( p \) contains a facet of \( \mathcal{V} \) are called the Voronoi relevant vectors.

**Lemma 1 (Covering radius).** For an embedded lattice \( \Lambda^Z (\Phi_2^{Z[\xi]}), \) we have

\[
\rho_2^{Z[\xi]} = \begin{cases} \frac{\sqrt{d+1}}{2} & \text{if } \xi = \sqrt{-d}, \ d > 0; \\ \frac{d+1}{4\sqrt{d}} & \text{if } \xi = \frac{1+\sqrt{-d}}{2}, \ d > 0. \\
\end{cases}
\]

**Proof:** For any given real lattice \( \Lambda^Z (B^R), \) it can be partitioned into exactly \( 2^n \) cosets of the form \( C_{B^R, p} = 2\Lambda^Z + B^R p \) with \( p \in \{0, 1\}^n. \) If \( p \) is a shortest vector for \( C_{B^R, p}, \) then the set \( \cup_{p \in \{0, 1\}^n} \{ s_p \} \) contains all the relevant vectors. For embedded lattices of \( \mathbb{Z} [\xi] \) in \( \mathbb{R}^2, \) their generator matrices must have the forms as shown in Eq. [5]. We discuss the two scenarios separately:

i) If \( \xi = \sqrt{-d} \) and \( B^R = \Phi_2^{Z[\xi]} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{d} \end{bmatrix}, \) we have

\[
\cup_{p \in \{0, 1\}^n \setminus \{0\}} \{ s_p \} = \left\{ \pm \begin{bmatrix} 1, 0 \end{bmatrix}^T, \pm \begin{bmatrix} 0, \sqrt{d} \end{bmatrix}^T \right\}. 
\]

Then the covering radius in this case is \( \rho_2^{Z[\xi]} = \frac{\sqrt{d+1}}{2}. \)

ii) If \( \xi = \frac{1+\sqrt{-d}}{2} \) and \( B^R = \Phi_2^{Z[\xi]} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\sqrt{d}}{2} \end{bmatrix}, \) the three cosets with non-zero shifts are

\[
C_{B^R, [1,0]} = 2\Lambda^Z + [1,1]^T, \\
C_{B^R, [0,1]} = 2\Lambda^Z + [\frac{1}{2}, \sqrt{d}]^T, \\
C_{B^R, [1,1]} = 2\Lambda^Z + [\frac{1}{2}, \frac{\sqrt{d}}{2}]^T. 
\]

It follows that

\[
\cup_{p \in \{0, 1\}^n \setminus \{0\}} \{ s_p \} = \left\{ \pm \begin{bmatrix} 1, 0 \end{bmatrix}^T, \pm \begin{bmatrix} \frac{1}{2}, \sqrt{d} \end{bmatrix}^T, \pm \begin{bmatrix} \frac{1}{2}, \frac{\sqrt{d}}{2} \end{bmatrix}^T \right\}. 
\]

So the point in \( \mathcal{V} \) that has the maximum distance to the origin can be obtained as the intersection between line \( y = -\frac{1}{\sqrt{d}} x + \frac{d+1}{4\sqrt{d}}, \) and line \( x = \frac{1}{2}. \) Lastly we obtain \( \rho_2^{Z[\xi]} = \frac{d+1}{4\sqrt{d}}. \) □

Since the so-called Siegel’s condition \( [25] \) based on rephrasing \( [14] \) is

\[
(\delta - |R_{j-1,j}|) |R_{j-1,j-1}|^2 \leq |R_{j,j}|^2, 
\]

it suffices to choose \( \delta > \rho_2^{Z[\xi]} \) where the exact values of \( \rho_2^{Z[\xi]} \) have been shown.

Now we specify the upper bound for \( \delta \) and consequently for \( \rho_2^{Z[\xi]} \) through a potential-function argument \([14, P. 4790].\)

Define the potential function of a lattice basis as:

\[
\text{Pot}(R) = \prod_{j=1}^n \det (\Lambda (R_{1,j}))^2 = \prod_{j=1}^n |R_{j,j}|^2(n-j+1). 
\]

Let the lattice bases be \( R \) before the swap and \( R' \) after the swap. If Lovász’s condition fails to hold, the ratio of their potential functions is:

\[
\text{Pot}(R') / \text{Pot}(R) = \frac{(|R_{j-1,j}|^2 + |R_{j,j}|^2)^{n-j+2} (\frac{|R_{j,j}|^2 |R_{j-1,j}|^2 |R_{j,j}|^2}{|R_{j-1,j}|^2 + |R_{j,j}|^2})^{n-j+1}}{|R_{j-1,j}|^2 |R_{j,j}|^2 < \delta. 
\]

Clearly one should ensure \( \rho_2^{Z[\xi]} < \delta < 1 \), otherwise the algorithm may not converge. By using Lemma 1 to evaluate the quadratic fields that satisfy \( \rho_2^{Z[\xi]} < 1, \) we arrive at the following proposition:

**Proposition 2.** Only the rings from 5 complex quadratic fields can be used to define Lovász’s condition; they are \( \mathbb{Q} (\sqrt{-d}) \) where \( d \) takes the values

\[1, 2, 3, 7, 11.\]

Such rings are all the norm-Euclidean ones in imaginary quadratic fields. As a comparison, one of the reasons that \( [18] \) requires the rings to be norm-Euclidean is to find the nearest ring element efficiently when doing the size reduction step. In our case, the quantization of an element in \( \mathbb{C} \) to \( \mathbb{Z} [\xi] \) does not impose any constraint. Though Prop. 2 implies we still cannot get rid of Euclidean rings, implementing an ALLL algorithm that sacrifices the constraint of \( \rho_2^{Z[\xi]} < \delta \) is possible (without performance bounds). Of independent interests, we observe the condition to check the norm-Euclideanity \([18]\) of \( \mathbb{Z} [\xi] \) is almost identical to \( \rho_2^{Z[\xi]} < 1, \) which is

\[
\max_{a, b \in \mathbb{Z} [\xi], b \neq 0} |a/b - Q_{Z[\xi]}(a/b)| < 1. 
\]

**C. Performance**

In the following, we set \( \delta = \rho_2^{Z[\xi]} + \epsilon \) with \( 0 < \epsilon \leq 1 - \rho_2^{Z[\xi]} \). The overall performance of algebraic LLL can also be described as follows.

**Theorem 2.** Let \( \tilde{B} = [\tilde{b}_1, \ldots, \tilde{b}_n] \) be an ALLL-reduced basis w.r.t. an input \( B \in \mathbb{C}^{n \times n}. \) Then \( B \) admits the following properties:

\[
\left\| \tilde{b}_1 \right\| \leq \epsilon - \frac{\epsilon}{1 + \lambda_1 Z[\xi]}, 
\]

\[
\left\| \tilde{b}_1 \right\| \leq \epsilon - \frac{\epsilon}{1 + \lambda_{1,2} Z[\xi]}, 
\]

\[
\eta Z[\xi] B \leq \det (\Phi_2^{Z[\xi]})^{-n} \prod_{j=1}^n \left( 1 + \rho_2^{Z[\xi]} \left( \frac{\epsilon-1 - \epsilon^{-j}}{1 - \epsilon^{-1}} \right) \right)^{1/2}. 
\]
Proof: From Siegel’s condition \[15\],

\[ |R_{j-1,j-1}|^2 \leq \epsilon^{-1} |R_{j,j}|^2. \]

By induction, it yields

\[ \|\tilde{b}_1\|^2 = |R_{1,1}|^2 \leq \epsilon^{-(j-1)} |R_{j,j}|^2, \]

for all \( 1 \leq j \leq n \). Then \[17\] follows from taking the product of these inequalities. As for \[18\], assume that \( x_1, \ldots, x_n \in \mathbb{Z}[\xi] \) are a set of coprime numbers such that \( \left\| \sum_{j=1}^{n} x_j \tilde{b}_j \right\| = \lambda_{1,2}\xi[\xi] \). Notice that there must exist one index \( j \) with \( |x_j| \geq 1 \), so that \( \lambda_{1,2}^2 \xi[\xi]^2 \geq |R_{k,k}|^2 \geq |x_j|^2 |R_{j,j}|^2. \) Then it yields \( \lambda_{1,2}^2 \xi[\xi]^2 \geq |R_{k,k}|^2 \geq \lambda_{1,2}^2 \xi[\xi]^2 \geq |R_{j,j}|^2 \geq |x_j|^2 |R_{j,j}|^2. \) Finally, by (20) and the size reduction condition, we have \( |R_{j,j}| \leq \rho_{2}\xi[\xi] \forall j < j' \), and

\[ \|R_{1:n,j}\|^2 = |R_{j,j}|^2 + \sum_{j < j'} |R_{j,j'}|^2 \]

\[ \leq |R_{j,j}|^2 + \sum_{j < j'} |R_{j,j'}|^2 \]

\[ \leq |R_{j,j}|^2 \left( 1 + \rho_{2}\xi[\xi] \left( \epsilon^{-1} + \epsilon^{-2} + \cdots + \epsilon^{-(j-1)} \right) \right). \]

By substituting the above into the definition, then

\[ \eta_{2}\xi[\xi](\mathbf{B}) = \prod_{j=1}^{n} |R_{1:n,j}| \det (\Phi_{2}\xi[\xi]) \prod_{j=1}^{n} |R_{j,j}|^{\frac{1}{2}} \]

\[ \leq \det (\Phi_{2}\xi[\xi])^{-1} \prod_{j=1}^{n} \left( 1 + \rho_{2}\xi[\xi] \left( \epsilon^{-1} - \epsilon^{-j} \right) \right)^{1/2}. \]

In Theorem 2 both \[17\] and \[18\] are essentially the same as those of real LLL, while \[19\] has some factors from ring \( \mathbb{Z}[\xi] \) since its analysis involves volumes and covering radiuses.

Lastly, based on analyzing the decoding radius, we show how far the size reduction is from the optimal length reduction that employs a closest vector problem (CVP) detector. Such an analysis may be of independent interests to MIMO detection \[26\].

Theorem 3. Given a complex basis \( \mathbf{B} \in \mathbb{C}^{n \times n} \), the decoding radius \( R_{\text{size}} \) of size reduction in round \( k + 1 \) satisfies

\[ R_{\text{size}} \geq \frac{1}{4} \lambda_{1,2}\xi[\xi] V_{2n}^{1/2n} \det (\Phi_{2}\xi[\xi])^{-1} \epsilon^{(k^2-k)/4}. \]

Proof: When LLL is running in the \( k+1 \)th round, then the basis \( \tilde{\mathbf{b}}_1, \ldots, \tilde{\mathbf{b}}_k \) is LLL-reduced. Let \( \tilde{\mathbf{b}}_1, \ldots, \tilde{\mathbf{b}}_k = \Phi \) denote its QR-decomposition. By using Theorem 1 we have for \( k = 2, \ldots, n \),

\[ \lambda_{1,2}\xi[\xi] \leq 4 \left( V_{2k}^{-1/k} \det (\Phi_{2}\xi[\xi]) \right)^2 \det \left( \mathcal{L} \left( \tilde{\mathbf{b}}_1, \ldots, \tilde{\mathbf{b}}_k \right) \right)^{2/k} \]

\[ = 4 \left( V_{2n}^{-1/n} \det (\Phi_{2}\xi[\xi]) \right)^2 \prod_{j=1}^{k} |R_{j,j}|^{2/k} \]

\[ \leq 4 \left( V_{2n}^{-1/n} \det (\Phi_{2}\xi[\xi]) \right)^2 \prod_{j=1}^{k} \epsilon^{-(k-j)/2} |R_{k,k}| \]

\[ = 4 \left( V_{2n}^{-1/n} \det (\Phi_{2}\xi[\xi]) \right)^2 |R_{k,k}|^{2} \epsilon^{-(k^2-k)/2}. \]

For all quadratic number fields, their packing radiuses are still \( 1/2 \). By using the definition of the decoding radius, we have

\[ R_{\text{size}} \triangleq \frac{1}{2} \min_{1 \leq j \leq k} |R_{j,j}| \]

\[ \geq \frac{1}{4} \lambda_{1,2}\xi[\xi] V_{2n}^{1/2n} \det (\Phi_{2}\xi[\xi])^{-1} \min_{1 \leq j \leq k} \epsilon^{j(j-1)/4}. \]

D. Implementation and Complexity

Regarding the implementation of \( \Phi_{2}\xi[\xi] \), for a TYPE I ring we have

\[ \Phi_{2}\xi[\xi](x) = |\mathfrak{R}(x)| + \sqrt{d} \Phi \mathfrak{I}(x) / \sqrt{d}, \]

because its lattice basis \( \Phi_{2}\xi[\xi] \) is orthogonal.

For a TYPE II ring, although implementing a sphere decoding algorithm on basis \( \Phi_{2}\xi[\xi] \) suffices, there exist simpler methods for doing so. For any \( \lambda = a + \frac{1 + \sqrt{-d}}{2} \), \( a, b \in \mathbb{Z} \), if \( b = 2k, k \in \mathbb{Z} \), then \( \lambda = (a + k) + \frac{1 + \sqrt{-d}}{2}. \) If \( b = 2k + 1, k \in \mathbb{Z} \), then \( \lambda = (a + k) + \frac{1 + \sqrt{-d}}{2}. \) Then we can see that \( \mathbb{Z}[\xi] \) is simply the union of a rectangular lattice \( \mathbb{Z}[\sqrt{-d}] \) and its coset \( \mathbb{Z}[\sqrt{-d}] + d^* \), \( d^* \approx \frac{1}{2} + \frac{\sqrt{d}}{2} \). Two examples of such lattices are reproduced in Fig. 2. In summary, for a TYPE II ring we have:

\[ \Phi_{2}\xi[\sqrt{-d}](x) = \arg \min_{y} |y - x|, \]

\[ y \in \{ \Phi_{2}\xi[\sqrt{-d}](x), \Phi_{2}\xi[\sqrt{-d}](x - d^*) + d^* \}. \]

Now we present the pseudo-code of algebraic LLL in Algorithm 1. Compared with the complex LLL algorithm in \[19\], the major differences are: i) The rounding function in Step 5 is generalized from over \( \mathbb{Z}[i] \) to over \( \mathbb{Z}[\xi] \). ii) Formulas (7)-(15) in \[19\] are simplified as a rotation by quaternions, which is represented by Steps (10)-(13) of Algorithm 1. The details are given in Appendix A.

Lastly we analyze the number of loops in the above algorithm. Denote the number of positive and negative tests in Step 9 as \( K^+ \) and \( K^- \), respectively. Based on \[16\], the potential function of the basis decreases in a \( \log_{1/\beta} \) scale for each negative tests. Let the ratio between the potential functions of the input basis and the minimum possible basis be \( g(n) \), then similarly to \[14\] we can upper bound the total number of loops as \( K^- + K^+ \leq 2K^- + n - 1 \leq 2 \log_{1/\beta} g(n) + n - 1 \).
KZ/Minkowski reduction algorithms do not have the constraint due to Lovasz’s condition, their basis expansion process [27] still requires the rings to be Euclidean.

IV. LATTICE REDUCTION IN C&F

In this section, we discuss the application of the proposed algebraic lattice reduction algorithm to the compute-and-forward paradigm. We first review the paradigm over rings, and how lattice reduction comes in handy when choosing network coding coefficients. After that, we analyze the DoF of the network coding design based on LLL. Lastly, we show that lattice reduction is advantageous as it guarantees the network coding system matrix can have full rank over fields.

A. Algebraic C&F

Consider a quasi-static complex-valued AWGN network with \( n \) source nodes and the same number of relays. We assume that each source node \( l \) is operating at the same rate and define the message rate as \( R_{\text{mes}} = \frac{1}{T} \log(|W|) \), where \( W \) is the message space, e.g., \( W = \mathbb{F}_p^k \). A message \( w_l \) is encoded, via a function \( E(\cdot) \), into a point \( x_l \in \mathbb{C}^T \), satisfying the power constraint \( |x_l|^2 \leq TP \), where \( T \) is the block length and \( P \) denotes the signal to noise ratio (SNR). In the integer-based C&F scheme [2] or its variants to algebraic integers [8], [9], \( x_l \) is a lattice point representative of a coset in the quotient \( \Lambda_f/\Lambda_c \), where \( \Lambda_f \) and \( \Lambda_c \) are called nested fine and coarse lattices. In the first scenario, the pair of nested lattices are built from Construction A over \( \mathbb{Z} \), characterized by: \( \mathbb{F}_p^k \rightarrow (\mathbb{Z}/p\mathbb{Z})^k \rightarrow \Lambda, p \in \mathbb{Z} \). As for the algebraic scenario, the lattices are built from algebraic Construction A characterized by: \( \mathbb{F}_p^k \rightarrow (\mathbb{Z}[\xi]/p^r\mathbb{Z}[\xi])^k \rightarrow \Lambda, p' \in \mathbb{Z}[\xi] \).

The received signal at one relay is given by

\[
y = \sum_{l=1}^{n} h_l x_l + z, \tag{22}
\]

where the channel coefficients \( \{h_l\} \) remain constant over the whole time frame, and \( z \sim CN(0, I_T) \). In algebraic C&F, the relay is still searching for a finite field linear combination of \( w_l \)'s in the layer of messages. But in the physical layer, the interpretation becomes decoding an algebraic-integer linear combination of lattice codes. The estimated lattice code is \( \hat{x} = [Q_{\Lambda_f}(ay)] \mod \Lambda_c = \sum_{l=1}^{n} a_l x_l \), where \( a \in \mathbb{C} \) is the minimum mean square error (MMSE) constant, and \( a_l \in \mathbb{Z}[\xi] \) is chosen from the set of algebraic integers.

Decoding a finite field linear combination of messages is possible as long as there exists an isomorphic mapping \( f \) such that \( f(\Lambda_f/\Lambda_c) \cong W \). A forwarded message can be explicitly written as

\[
u = \sum_{l=1}^{n} f(a_l) w_l. \tag{23}\]

The decoding error event in a relay, given \( h \in \mathbb{C}^n \) and \( a \in \mathbb{Z}[\xi]^n \), occurs when \([Q_{\Lambda_f}(ay)] \mod \Lambda_c \neq \sum_{l=1}^{n} a_l x_l \) for optimized \( \alpha \). A rate is said to be achievable at a given relay if there exists a coding scheme such that the probability of decoding error tends to zero as \( T \rightarrow \infty \). Upon collecting at least \( n \) correct equations in the form of (23), the destination

---

**Algorithm 1:** The algebraic LLL algorithm.

**Input:** lattice basis \( B \in \mathbb{C}^{n \times n} \), Lovász’s parameter \( \delta \), primitive element \( \xi \).

**Output:** reduced basis \( B \in \mathbb{C}^{n \times n} \), unimodular matrix \( U \in \text{GL}_n(\mathbb{Z}[\xi]) \).

1. \([Q, R] = qr(B) \); \( \triangleright \) The QR decomposition of \( B \);
2. \( j = 2, U = I_n \);
3. while \( j \leq n \) do
   4. for \( k = j - 1 : -1 : 1 \) do
      5. \( c = Q_{\mathbb{Z}[\xi]}(R_{j,k}/R_{j,j}); \) \( \triangleright \) Ring quantization;
      6. if \( c \neq 0 \) then
         7. \( R_{1:n,j} \leftarrow R_{1:n,j} - cR_{1:n,k}; \)
         8. \( U_{1:n,j} \leftarrow U_{1:n,j} - cU_{1:n,k}; \)
      9. if \( \delta R_{j-1,j-1}^2 > |R_{j,j}|^2 + |R_{j-1,j}|^2 \) then
         10. define \( M_{\delta}; \)
         11. swap \( R_{1:n,j} \) and \( R_{1:n,j-1}; \)
         12. \( U_{1:n,j} \leftarrow M_{\delta} U_{1:n,j} \); \( \triangleright \) Left rotation;
         13. \( Q_{1:n,j-1,j} \leftarrow Q_{1:n,j-1,j} M_{\delta}; \) \( \triangleright \) Right rotation;
         14. \( j \leftarrow \max(j - 1, 2); \)
   15. \( j \leftarrow j + 1; \)
16. \( \mathbf{B} = QR. \)

**E. On strong lattice reduction and SVP**

It is also possible to extend the algebraic LLL to algebraic KZ/Minkowski reduction [27]. Although designing an SVP on the algebraic lattices may encounter the difficulty of enumerating points in a depth-first enumeration algorithm, we can alternatively use SVP on real lattices. Based on the symmetry of rings, the SVP algorithm can have a speed-up. For instance, only 1/4 of the points within a Euclidean ball need to be enumerated in Gaussian integers as \( |\mathbb{Z}[\xi]|^2 = 4 \) (as used in [28]), and only 1/6 of the points need to be enumerated in Eisenstein integers as \( |\mathbb{Z}[\omega]|^2 = 6 \). Although
can invert the equations to estimate the messages. In the process, the achievable computation rate of algebraic C&F is described by the following theorem.

**Theorem 4** (Computation rate in algebraic C&F). At a relay with channel coefficient \( h \in \mathbb{C}^n \) and combination coefficient \( a \in \mathbb{Z}[\xi]^n \), a computation rate of

\[
\mathcal{R}_{\text{comp}}(h, a, P) = \log^+ \left( \frac{1}{a^\top (I_n + Phh^\top)^{-1} a} \right)
\]

is achievable.

By using LDL decomposition to get \( I_n + Phh^\top = LDL^\top \) in (24), the denominators (24) represents the square distance of a lattice vector in \( \Lambda[\xi] \left( D^{-\frac{1}{2}}L^\top \right) \). The optimal solutions for them require solving a shortest vector problem (SVP). In this paper, we concentrate on using lattice reduction algorithms to reduce the basis

\[
B = D^{-\frac{1}{2}}L^\top,
\]

so as to approximately solve SVP.

**B. DoF**

In this subsection, let \( a^* \) be the first coefficient vector found by algebraic LLL, i.e., \( b_1 = B a^* \). For a computation rate \( \mathcal{R}_{\text{comp}}(h, a, P) \), define its associated DoF as:

\[
d_{\text{comp}}(h, a) = \lim_{P \to \infty} \frac{\mathcal{R}_{\text{comp}}(h, a, P)}{\log(1 + P)}.
\]

To analyze the DoF of the computation rate found by using algebraic LLL, the crux is to show the DoF associated with \( \min_{a \in \mathbb{Z}[\xi]^n} \mathcal{R}_{\text{comp}}(h, a, P) \), because algebraic LLL only amplifies the length metric to a \( \epsilon^{-\frac{n+1}{2}} \) factor which is independent of \( P \). Specifically:

i) Based on Eqs. (9) and (18), we have

\[
\mathcal{R}_{\text{comp}}(h, a^*, P) \geq \log^+ \left( \frac{1}{\epsilon^{-n+1/2} \lambda^2_{1,\mathbb{Z}[\xi]}^n} \right)
\]

\[
\geq \log^+ \left( \frac{1}{\epsilon^{-n+1/2} \gamma_{2n}^2 |\det(\Phi_{\mathbb{Z}[\xi]})| (1 + P \|h\|^2)^{-1/n}} \right)
\]

\[
= \frac{1}{n} \log^+ \left( 1 + P \|h\|^2 \right) - \log^+ \left( \epsilon^{-n+1/2} \gamma_{2n}^2 |\det(\Phi_{\mathbb{Z}[\xi]})| \right).
\]

(27)

Notice that \( \frac{1}{n} \log^+ \left( 1 + P \|h\|^2 \right) \) denotes the symmetric capacity of a multiple access channel (MAC) induced by one relay. Substitute (27) into (26), we obtain \( d_{\text{comp}}(h, a^*) \geq 1/n \).

ii) From Proposition 3 in Appendix B and almost identical steps to the proof of [29, Lem.1], the Diophantine approximation error for \( |a|^2 + P \|a h - a\|^2 \) is lower bounded by

\[
|a|^2 + P \|a h - a\|^2 \geq \kappa(h) P^{n-1},
\]

(28)

where \( \kappa(h) \) is some parameter independent of \( P \), and \( a \neq 0 \). Further consider the effect of lattice reduction, we have \( \|Ba^*\|^2 \geq \lambda^2_{1,\mathbb{Z}[\xi]} \), thus for an \( a \neq 0 \),

\[
\mathcal{R}_{\text{comp}}(h, a^*, P) \leq \log^+ \left( \frac{1}{\lambda^2_{1,\mathbb{Z}[\xi]}^2} \right)
\]

\[
= \log^+ \left( \frac{P}{|a|^2 + P \|a h - a\|^2} \right)
\]

\[
\leq \log^+ \left( \kappa(h)^{-1} P^{1/n} \right).
\]

Therefore we have \( d_{\text{comp}}(h, a^*) \leq 1/n \). Together with the result in case i), we conclude that \( d_{\text{comp}}(h, a^*) = 1/n \). This not only establishes the DoF result for complex channel vectors in C&F, but also shows that LLL can preserve the DoF.

**C. Transmission Rate and Independence on Fields**

In the destination, for a given \( H \triangleq [h_1, \ldots, h_n] \) and \( A \triangleq [a_1, \ldots, a_n] \), the achievable transmission rate is defined as the minimum computation rate over all relays:

\[
\mathcal{R}_{\text{tran}}(H, A, P) \triangleq \min_j \mathcal{R}_{\text{comp}}(h_j, a_j, P).
\]

In addition, to ensure the message equations are invertible, the coefficient matrix \( A \) has to be not only full rank over \( \mathbb{Z}[\xi] \), but also full rank over a finite field \( \mathbb{F}_p \) in the space of error correction codes. Define such a mapping as a ring homomorphism \( f : \mathbb{Z}[\xi] \to \mathbb{F}_p \), the actual transmission rate is given by:

\[
\mathcal{R}_{\text{tran}}(H, A, P) \triangleq \mathcal{R}_{\text{comp}}(h^* f(A), a^* f(A), P).
\]

In proving Theorem 4 [29], it is required that \( p \to \infty \), so the condition of rank \( f(A) \) = \( n \) is often relaxed to rank \( A \) = \( n \). In practical implementations, however, the size of \( p \) is limited, so a network coding design has to ensure rank \( f(A) \) = \( n \).

The full rank over \( \mathbb{F}_p \) requirement can be easily met if the coefficient matrix \( A \in \text{GL}_n(\mathbb{Z}[\xi]) \). First, the determinant function for measuring ranks defines a mapping \( \text{GL}_n(\mathbb{Z}[\xi]) \to \mathbb{Z}[\xi]^\times \) between general linear group over \( \mathbb{Z}[\xi] \) and the group of units \( \mathbb{Z}[\xi]^\times \). Since it respects the multiplication in both groups, the function \( \det(\cdot) \) defines a group homomorphism. Second, the determinant function respects the morphism \( f : \text{GL}_n(\mathbb{Z}[\xi]) \to \text{GL}_n(\mathbb{F}_p) \), so it yields

\[
f(\det(A)) = \det(f(A)).
\]

As shown in the commutative diagram in Fig. 3 we always have rank \( f(A) \) = \( n \) if \( A \in \text{GL}_n(\mathbb{Z}[\xi]) \).

\[
\begin{array}{ccc}
\text{GL}_n(\mathbb{Z}[\xi]) & \xrightarrow{\det} & \text{GL}_n(\mathbb{F}_p) \\
\downarrow \det & & \downarrow \det \\
\mathbb{Z}[\xi]^\times & \xrightarrow{f} & \mathbb{F}_p^\times \end{array}
\]

Fig. 3: The commutative diagram of groups and units.

Lattice reduction naturally induces unimodular matrices. Through lattice reduction, one can pursue
max_{A \in \text{GL}_n(\mathbb{Z}[\xi])} R_{\text{trans}}(H, A, P) in the following manner. First, each relay, say $h_j$, forwards $n$ message equations whose rates are sorted in descending order w.r.t. $A_j = \begin{bmatrix} a_{ij}^0, \ldots, a_{ij}^n \end{bmatrix} \in \text{GL}_n(\mathbb{Z}[\xi])$ to the destination. After that, the destination suffices to choose one $A_j$ that corresponds to the maximum transmission rate. This scheme approximates $R_{\text{trans}}(H, P)$ by

\[ \min_j \max_{a \in \mathbb{Z}[\xi]} \text{comp}(h_j, a^0, \ldots, a^n, P). \]

Now we present a transparent example to demonstrate the advantage of "unimodularity". We consider a network with 2 users and 2 relays, operating with power constraint $P = 25$ dB. The optimization is performed for the following channel coefficient vectors for each relay, respectively:

\[ h_1 = [-0.4001 + 1.0937i, -0.9278 + 1.8151i]^T, \]
\[ h_2 = [-0.3779 + 0.2307i, -1.5736 - 0.3939i]^T. \]

Consider an error correction code over $\mathbb{F}_5$, which is connected to a lattice code through the quotient of Gaussian integers $\mathbb{Z}[i]/(2 + i)\mathbb{Z}[i]$. The sets of network coding matrices constructed through lattice reduction are then given as:

\[ A_1 = \begin{bmatrix} 2 + 2i & -1 \\ 3 + 4i & -2 \end{bmatrix}, A_2 = \begin{bmatrix} -1 + 1i & 1 \\ -5 & 3 + 3i \end{bmatrix}. \]

After mapping these vectors to finite fields via the isomorphism $\mathbb{Z}[i]/(2 + i)\mathbb{Z}[i] \cong \mathbb{F}_5$, these vectors become:

\[ f(A_1) = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}, f(A_2) = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}. \]

Suppose all these coefficients are forwarded to the destination. By taking the first coefficient vectors from $f(A_1)$ and $f(A_2)$, one can see that

\[ \text{rank} \left( \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \right) = 1, \]

which suggests the destination cannot invert the equation for this scenario. On the contrary, as $A_1$ and $A_2$ are unimodular matrices in $\mathbb{Z}[i]$, both $f(A_1)$ and $f(A_2)$ have full rank over $\mathbb{F}_5$. Therefore, it suffices to choose either $f(A_1)$ or $f(A_2)$ to invert the finite field equations.

V. SIMULATION RESULTS

In this section, we present simulation results to compare the performance of real and algebraic LLL algorithms, to examine the information rate of C&F schemes based on algebraic lattice reduction, and to show the advantage of using lattice reduction in the multi-relay setup of C&F. Although the $\mathbb{Z}[i]$.ALLL has been shown to reduce the complexity by nearly 50% compared to its real counter-part [19], our results in subsection V-A extend such a study to other rings. In addition to the advantage in terms of complexity, we will show the algebraic LLL may also outperform real LLL in terms of performance. In subsection V-B we present the first systematic study on using lattice reduction in C&F. In the last subsection, we emphasize that only ensuring the network coding matrices have full rank over rings is not sufficient.

A. Real vs. Algebraic LR

We consider lattice bases that are generated from $\mathbb{Z}[\xi]$), in which the quality of the bases are controlled by SNR. Generally, lower SNR would result in more orthogonal lattice bases. The lattices are defined over Euclidean rings $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{-2}]$, and $\mathbb{Z}[\omega]$, and non-Euclidean rings $\mathbb{Z}[\sqrt{-5}]$, $\mathbb{Z}[\frac{1 + \sqrt{-7}}{2}]$, and $\mathbb{Z}[\frac{-1 + \sqrt{-7}}{2}]$. We implement both algebraic LLL reductions and classic LLL reductions for all these lattices, with Lovász’s parameter $\delta = 0.99$. All results are averaged over $10^4$ realizations.

In Fig. 4 we plot the averaged Euclidean norm of the first basis vector after $\mathbb{Z}[\xi]$ or $\mathbb{Z}$ based LLL reduction. The error bars denote the standard deviations of the objective values, and the legend “Real bases” denotes real lattices generated from expanding $\mathbb{Z}[\xi]$-based “Complex bases”. Note that, in the subfigures, the algebraic-LLL for non-Euclidean rings may still find short vectors when $P = 10$ dB (Fig. 4a and Fig. 4c). However, the performance of such LLL algorithms degrades significantly when $P = 40$ dB (Fig. 4b and Fig. 4d), confirming our statement in Proposition 2. Another observation is that, although the performance of Euclidean-ring based real and complex LLL are almost identical in dimension $n = 2$, the $\mathbb{Z}[\omega]$.ALLL generates shorter vectors than its real counter-part as dimension $n$ increases to 8, while this is not true for other rings. Such performance difference is summarized in Fig. 5 after further running simulations for other rings.

In Fig. 6 we plot the averaged number of swaps when implementing algebraic/real LLL reductions, because this metric can reflect the overall complexity of the algorithms. The subfigures show that algebraic LLL algorithms have only about 25% complexity w.r.t. their real counter-parts. This observation is not a surprise because we are dealing with lattices of smaller dimensions. Moreover, the complexity is roughly inverse-proportional to $\det (\Phi^2(\xi))$, and $\mathbb{Z}[\omega]$-ALLL often incurs the highest complexity among algebraic algorithms.

B. Computation Rates and Running Time

With the same setting for lattice reduction algorithms as above, we compare their averaged computation rates in C&F. An algebraic LLL algorithm for a ring $\mathbb{Z}[\xi]$ is noted as “$\mathbb{Z}[\xi]$-ALLL”, while its corresponding LLL algorithm for real lattices is noted as “$\mathbb{Z}[\xi]$.RLLL”. The computation rates based on using SVP oracles in $\mathbb{Z}[\xi]$-lattices, denoted by “$\mathbb{Z}[\xi]$-SVP” are taken as performance upper bounds of their respective LLL algorithms. The comparison is shown in Fig. 7 with SNR $P \sim 0 - 40$ dB, $n = 8$. Several observations can be made from the figure. First, as expected, the $\mathbb{Z}[\omega]$.ALLL algorithm has the best performance among those ALLL algorithms. However, its real counter-part $\mathbb{Z}[\omega]$.RLLL has smaller rates, and the gap is around 1 dB in the SNR region of $10 - 20$ dB. Quite differently, $\mathbb{Z}[\xi]$.ALLL becomes slightly worse than $\mathbb{Z}[\xi]$.RLLL as the SNR increases. Lastly, for the non-Euclidean ring, $\mathbb{Z}[\sqrt{-5}]$.ALLL fails to achieve the degree-of-freedom bound, while $\mathbb{Z}[\sqrt{-5}]$.RLLL can do so.

Next, we compare the running time of the above algorithms in Fig. 8. Regarding the implementation of an SVP oracle, the depth-first sphere decoding based algorithm with theoretical
Fig. 4: The Euclidean norm of the first basis vector after different LLL reductions.

(a) $n = 2$, $P = 10\text{dB}$.

(b) $n = 2$, $P = 40\text{dB}$.

(c) $n = 8$, $P = 10\text{dB}$.

(d) $n = 8$, $P = 40\text{dB}$.

Fig. 5: A summary on the Euclidean norm performance after different LLL reductions.

$O(n^{1.5}P^{0.5})$ complexity is taken as the benchmark \[5\]. The figure shows that sphere decoding algorithms for the expanded $\mathbb{Z}[\xi]$-lattices have much higher complexity than lattice reduction based algorithms. In addition, the $\mathbb{Z}[\omega]$-ALLL and $\mathbb{Z}[i]$-ALLL algorithms are at least a factor of 2 faster than their real counter-parts. Although $\mathbb{Z}[\sqrt{-5}]$-ALLL has the lowest complexity, it cannot produce short vectors in general.

Two more remarks are made for the complexity comparison. First, the speed-up of sphere-decoding based on early termination and sorting channel vectors \[5\] can also be used in lattice reduction. The fact that lattice reduction is faster than sphere decoding is not a surprise as the former is often taken as a pre-processing technique for the latter. As the lattice bases in C&F are not random as those in lattice cryptography, LLL-based approaches exhibit close to SVP performance in most of our simulations. Second, regarding the SVP algorithms for C&F in \[6\], \[30\], they are generally not faster than sphere decoding algorithms \[5\]. In fact, we find the algorithms in \[30\] are similar to breadth-first sphere decoding in which all the points in a fixed bounded region have to be enumerated, while the bounded-region is shrinking in a depth-first sphere decoding.

C. Probability of Rank Failure

Consider the ring of Gaussian integers $\mathbb{Z}[i]$, and a code space $\mathbb{F}_2 \cong \mathbb{Z}[i]/(2 + i) \mathbb{Z}[i]$. As implicitly suggested in \[3\] Fig. 6], a full-rank network coding matrix over rings can often be constructed in the high SNR region even when each relay only forwards its best equation. Here we show that even if such a condition has been satisfied, the probability that the destination cannot decode the original messages is still significant. In Fig. 9 we plot the probabilities of rank failure over both $\mathbb{Z}[i]$ and $\mathbb{Z}[i]/(2 + i) \mathbb{Z}[i]$. It can be observed from Fig. 9 that the probability of rank failure in $\mathbb{Z}[i]/(2 + i) \mathbb{Z}[i]$ is always larger than 0.16 for $n = 2$ and 0.1 for $n = 4$, while that of $\mathbb{Z}[i]$ gradually vanishes to 0. On the contrary, such probability is always 0 if we used the lattice reduction based scheme in Section IV-C.

VI. CONCLUSIONS AND FUTURE DIRECTIONS

In this work, we revisit the properties of algebraic lattices and investigate the proper design of lattice reduction operating in the complex domain. Although an algebraic lattice w.r.t. any ring can always be transformed to a real lattice and to associate
Fig. 6: The number of swaps after different LLL reductions.

(a) \( n = 2, P = 10\text{dB} \).

(b) \( n = 2, P = 40\text{dB} \).

(c) \( n = 8, P = 10\text{dB} \).

(d) \( n = 8, P = 40\text{dB} \).

Fig. 7: The computation rates based on different algorithms, \( n = 8 \).
with non-algebraic LLL, we show that only Euclidean rings can generate lattices which are feasible for implementing algebraic LLL. Properties and efficient implementations of the algebraic LLL algorithm are given. Then, with the reduction algorithm, we show how it is used to find the network coding vectors in C&F. While constructing a network coding matrix with full rank over rings is generally not sufficient for making it injective over finite fields, the lattice reduction based scheme can always do so. Moreover, all the forwarded data streams are shown to have a DoF of $1/n$. Such result follows directly from analyzing the DoF of C&F over rings in complex quadratic fields. Numerical results give the performance comparisons of our proposed algebraic LLL algorithm and other alternative strategies.

An interesting future direction is to study reduction on a more complicated Humber-form lattice in [31], which arises from C&F over block-fading channels. Another possible extension of this work is to understand the practical performance of algebraic lattice reduction when applied to other scenarios in wireless communications.

APPENDIX A

RODATIONS AND QUATERNIONS

By introducing the concept of quaternions, representations of rotations become more compact, and unit normalisation of floating point quaternions suffers from less rounding defects [32]. Now we explain why quaternions are involved. As in [14], the pseudo-codes of an LLL algorithm consist of “swaps” and “size reductions”. After a swap, the structure of the $R$ matrix has been destroyed. Since implementing another factorisation costs too much complexity, we show that the $R$ matrix structure can be recovered by left multiplying the matrix form of a quaternion. With a slight abuse of notation, let $\{1, i, j, k\}$ be a basis for a vector space of dimension 4 over $\mathbb{R}$. These elements satisfy the rules $i^2 = -1, j^2 = -1, k^2 = -1, \text{and } k = ij = -ji$. The Hamilton’s quaternions is a set $\mathbb{H}$ defined by

$$\mathbb{H} \triangleq \{ x + yi + zj + wk \mid x, y, z, w \in \mathbb{R}\}.$$  

For any Hamilton’s quaternion $q = x + yi + zj + wk$, it can be written as

$$(x + yi) + (zj - wji) = \alpha_q + j \beta_q,$$

$\alpha_q \in \mathbb{C}, \beta_q \in \mathbb{C}$. Then $\mathbb{H}$ is also a $\mathbb{C}$-vector space with basis $\{1, j\}$. Let $\psi(q) = [\alpha_q, \beta_q]^T$, since the multiplication of $q$ with $v = \alpha_v + j \beta_v$ can be identified as

$$\psi(vq) = \begin{bmatrix} \alpha_v & -\beta_v^* \\ \beta_v & \alpha_v^* \end{bmatrix} \psi(q),$$

we call $M_v$ the matrix form of a quaternion $v$.

In the QR-decomposition, $Q$ denotes a unit in the matrix ring $M_{n \times n}(\mathbb{C})$ since $\det(Q) \in \{\pm 1, \pm i\}$. Suppose we have in the $i$th round and after a swap that $Q^i \in \mathbb{C}^{2 \times 2}$, $R^i \in \mathbb{C}^{2 \times 2}$, then the rotation operation by a quaternion $v^*$ is denoted by:

$$Q^i R^i = Q^{i+1} M_{v^*}^{-1} M_v R^i$$

Since $Q^i \in M_{2 \times 2}(\mathbb{C})^x$, $M_{v^*}^{-1} \in M_{2 \times 2}(\mathbb{C})^x$, we have $Q^{i+1} \in M_{2 \times 2}(\mathbb{C})^x$. Denote the first column of $R^i$ as $[R_{j-1,j}, R_{j,j}]^T$. The rotation is about nulling the second entry, so we can choose the quaternion as

$$v^* = \frac{R_{j-1,j}}{\sqrt{|R_{j-1,j}|^2 + |R_{j,j}|^2}} + j \frac{-R_{j,j}}{\sqrt{|R_{j-1,j}|^2 + |R_{j,j}|^2}}.$$

APPENDIX B

DIOPHANTINE APPROXIMATION ON COMPLEX CHANNEL VECTORS

**Proposition 3.** Let $\psi : \mathbb{N} \to \mathbb{R}_+$. For almost every $h \in \mathbb{C}^n$, there exists a positive constant $c \triangleq c(n, \h)$ such that

$$\max_{k \in \{1, \ldots, n\}} \min_{a_k \in \mathbb{Z}[\xi]} |h_k - a_k/q| \geq c\psi(|q|^2)$$

for all $q \in \mathbb{Z}[\xi]$.

**Proof:** The analysis in [33] easily extends to other TYPE I rings. So we focus on a TYPE II ring hereby. Define $\mathcal{P}$ as a fundamental partition of lattice $\mathbb{Z}[\xi]$ in the complex space $\mathbb{C}$:

$$\mathcal{P} \triangleq \{ x \in \mathbb{C} \mid Q_{\mathbb{Z}[\xi]}(x) = 0 \}.$$  

For all $h_k \in \mathbb{C}$, we have

$$|h_k - a_k/q| = |h_k - Q_{\mathbb{Z}[\xi]}(x) + (qQ_{\mathbb{Z}[\xi]}(x) - a_k)/q|,$$

so it suffices to confine our discussion to the case of $h \in \mathcal{P}$. Define a fundamental space for Diophantine approximation as:

$$A_{q, \psi} \triangleq \left\{ h \mid h \in \mathcal{P}^n, \max_{k \in \{1, \ldots, n\}} \min_{a_k \in \mathbb{Z}[\xi]} |h_k - a_k/q| \leq \psi(|q|^2) \right\}$$

with a fixed $q$. Clearly, the Lebesgue measure (denoted by $\mu(\cdot)$) of $A_{q, \psi}$ w.r.t. fixed $\triangleq \{a_1, \ldots, a_n\}$ is upper bounded:

$$\mu(A_{q, \psi} \mid a) \leq (\pi \psi(|q|^2))^n.$$  

For each $k$, counting the number of feasible $a_k/q$ that is inside $\mathcal{P}$ is equivalent to counting lattice points within a polytope.

Now we show that this number is $|q|^2$. As $q \neq 0$ is given, $a_k/q = a_k q/|q|^2$ denotes a principle ideal $q' \mathbb{Z}[\xi]$ scaled by $|q|^2$, which can be embedded to a 2-D real lattice. The embedded lattice of $q' \mathbb{Z}[\xi]$ has a basis

$$\Phi_{q' \mathbb{Z}[\xi]} \triangleq \begin{bmatrix} \mathbb{R} \{q'\} & -\mathbb{J} \{q'\} \\ \mathbb{J} \{q'\} & \mathbb{R} \{q'\} \end{bmatrix} \begin{bmatrix} 1 & 1/\sqrt{d} \\ 0 & -1/\sqrt{d} \end{bmatrix},$$

so we can obtain the volume of this lattice: $\text{Vol}(q' \mathbb{Z}[\xi]) = \det(\Phi_{q' \mathbb{Z}[\xi]}) = |q|^2 \sqrt{d}/2$. Further notice that $\text{Vol}(\mathcal{P}) = \sqrt{d}/2$, so we have

$$\# \{ a_k \in (q' \mathbb{Z}[\xi] \cap \mathcal{P}) \} = |q|^2.$$

It follows from the above that

$$\mu(A_{q, \psi}) \leq \sum_{a \in \mathcal{P}^n} \mu(A_{q, \psi} \mid a) \leq |q|^{2n} \pi^n \psi(|q|^2)^{2n}$$

Let $W_\psi$ be the set of points such that [33] holds for infinitely many $q$ [33]:

$$W_\psi = \limsup_{|q| \to \infty} A_{q, \psi} = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} A_{q, \psi},$$
where \( \bigcup_{q,k} \{q:|q|^2 = k \} \) is due the number field. Regarding the measure \( \mu (\mathcal{W}_q) \), it follows from the Borel–Cantelli lemma [34] that \( \mu (\mathcal{W}_q) = 0 \) if \( \mu \left( \bigcup_{q=1}^{\infty} \bigcup_{k=1}^{K} \mathcal{A}_{q,k} \right) < \infty \). This term can be evaluated by substituting (31) inside:

\[
\mu \left( \bigcup_{k=1}^{\infty} \bigcup_{q:|q|^2 = k} \mathcal{A}_{q,k} \right) \leq \sum_{k=1}^{\infty} \sum_{q:|q|^2 = k} |q|^{2n} \pi^k (|q|^2)^{2n} \leq \sum_{k=1}^{\infty} k^{2n} \pi^k (2n)^{2n} \sum_{q:k-1 < |q|^2 \leq k} 1
\]

where \( \sum_{q:|q|^2 = k} 1 \) denotes the number of \( q \)'s in \( \mathbb{Z} [\sqrt{k}] \) s.t. \( k-1 < |q|^2 \leq k \). Evaluating this term is equivalent to solving Gauss’s circle problem, which asks for the number of lattice points within a circle of a specific radius. Since \( \sum_{q:|q|^2 = k} 1 = \pi k + O(\sqrt{k}) \), we have \( \sum_{k=1}^{\infty} k^{2n} \pi^k (2n)^{2n} < \infty \). Therefore, if we can choose a large enough constant \( c \) such that (29) holds.

Let \( A_{2n+1} (k, \psi (k)) = k^{-1} \delta \cdot \delta' \to 0 \), we can choose \( \psi (k) = k^{-1/2} \delta \) with \( \delta \cdot \delta' = (2n) \). The approximation error can thus be bounded with \( \psi (|q|^2) = |q|^{-n/(n-1)} \delta_0 \).

As a comparison, our proof generalizes the analysis of [33] from Gaussian integers to general rings in quadratic fields. The proof also resonates with our Diophantine approximation analysis in [31]. The difference is that the algebraic lattices here naturally reside in a \( \mathbb{C}^n \) space, while [31] employs embeddings to rise the lattices to a \( \mathbb{R}^n \) space.

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