Properties of Consistent Histories

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Abstract
We describe some properties of consistent sets of histories in the Gell-Mann–Hartle formalism, and give an example to illustrate that one cannot recover the standard predictions, retrodictions and inferences of quasiclassical physics using the criterion of consistency alone.

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1. Introduction

Standard quantum theory, in the Copenhagen interpretation\(^1\), gives a robust and successful algorithm for predicting the results of laboratory experiments. It has, however, nothing to say about the larger quantum universe outside the lab, since its subject is solely the results of measurements made by some classical measuring apparatus whose existence is taken as an a priori assumption. New theories which make predictions without this last assumption would be of great interest, since they would have greater predictive power.

The consistent histories approach of Griffiths, Omnès, Gell-Mann and Hartle has been suggested to be just such a development, extending the Copenhagen interpretation\(^2\).\(^3\)\(^4\).

\(^1\) For definiteness consider the version of the Copenhagen interpretation set out by Landau and Lifschitz.\(^2\)\(^3\)\(^4\)
and allowing us to make predictions in quantum cosmology where the quantum system is the whole universe [6, 7, 8]. It is a formalism from which, it is hoped, the largely classical world of our experience might be deduced, rather than assumed. Thus, our observations of large-scale classical structure in the universe, of macroscopic objects following classical equations of motion, and of definite classical outcomes to quantum experiments, are all supposed to be predictions, unconditionally derivable from the formalism. If these hopes were to be realised, the consistent histories approach would indeed have provided a significant increase in our predictive power. They rest, however, on as yet incomplete interpretational arguments and have naturally led to much debate [9, 10]. Our own arguments, together with a critique of the existing literature, will be given in detail elsewhere [11]. Our aim here is to draw attention to some perhaps counter-intuitive properties of consistent sets of histories, most of which have not previously been discussed in any detail in the consistent histories literature, and to explain their physical relevance.

2. The Consistent Histories Formalism

We begin with a brief description of the consistent histories formalism as it applies to non-relativistic quantum mechanics, in the Heisenberg picture, using the language of projection operators and density matrices. We assume that a Hilbert space, \( \mathcal{H} \), and hamiltonian, \( H \), are given, that hermitian operators correspond to observables, that the commutation relations amongst the hamiltonian and physically interesting observables (such as position, momentum and spin) have been specified, and that the operators corresponding to the same observables at different times are related by

\[
P(t) = \exp(iHt/\hbar) P(0) \exp(-iHt/\hbar).
\] (2.1)

We are interested in a system (in principle, the universe) whose initial density matrix \( \rho_i \) is given. We require that \( \rho_i \) is positive semi-definite. The formalism also allows a final condition to be imposed, an interesting generalisation of standard quantum theory, though we shall not consider that possibility explicitly here. The basic physical events we are interested in correspond to sets \( \sigma \) of orthogonal hermitian projections \( P^{(i)} \), with

\[
\sum_i P^{(i)} = 1 \quad \text{and} \quad P^{(i)} P^{(j)} = \delta_{ij} P^{(i)}.
\] (2.2)

These projective decompositions of the identity are applied at definite times, which we append to the sets of projections: thus \( \sigma_j(t_j) = \{ P^{(i)}_j; i = 1, 2, \ldots, n_j \}_{t_j} \) defines a set of
projections obeying (2.2) and applied at time \( t_j \). However, since our results depend only on the time ordering, we will generally omit explicit time labels. Suppose now we have a list of sets \( \sigma_j(t_j) \) of this form, with \( j \) running from 1 to \( n \), at times \( t_j \) with \( t_i < t_1 < \ldots < t_n < t_f \). Then the histories given by choosing one projection from each \( \sigma_j \) in all possible ways are an exhaustive and exclusive set of alternatives, \( S \). We use Gell-Mann and Hartle’s decoherence condition, and say that the histories form a consistent set if

\[
\text{Tr}(P_n^{(i_n)} \ldots P_1^{(i_1)} \rho_1 P_1^{(j_1)} \ldots P_n^{(j_n)}) = \delta_{i_1j_1} \ldots \delta_{i_nj_n} p(i_1 \ldots i_n),
\]

in which case \( p(i_1 \ldots i_n) \) is the probability of the history \( P_1^{(i_1)} \ldots P_n^{(i_n)} \). (Gell-Mann and Hartle term a set satisfying (2.3) medium decoherent.)

We say the set

\[
S' = (\rho_i, \{\sigma_1, \ldots, \sigma_k, \tau, \sigma_{k+1}, \ldots, \sigma_n\})
\]

is a consistent extension of a consistent set of histories \( S = (\rho_i, \{\sigma_1, \ldots, \sigma_n\}) \) by the set of projections \( \tau = \{Q^i : i = 1, \ldots, m\} \) if \( \tau \) satisfies (2.2) and \( S' \) is itself consistent. We say the consistent extension \( S' \) is trivial if, for each history \( \{P_1^{(i_1)}, \ldots, P_k^{(i_k)}, P_{k+1}^{(i_{k+1})}, \ldots, P_n^{(i_n)}\} \) in \( S \), at most one of the extended histories \( \{P_1^{(i_1)}, \ldots, P_k^{(i_k)}, Q^i, P_{k+1}^{(i_{k+1})}, \ldots, P_n^{(i_n)}\} \) has non-zero probability. We extend these definitions by taking consistent extension and trivial consistent extension to be transitive relations.

3. Counting consistent sets

Let us now take the Hilbert space, \( \mathcal{H} \), to be of finite dimension \( N \) and ask: just how many consistent sets are there? We first describe how consistent sets may be classified. The basic objects in the formalism are the projective decompositions of the identity \( \sigma_j = \{P_j^{(i)} : i = 1, 2, \ldots, n_j\} \), where the \( P_j^{(i)} \) satisfy (2.2). These decompositions are parametrised by the set of ranks \( \{r_j^{(1)}, r_j^{(2)}, \ldots, r_j^{(n_j)}\} \) of the projection operators, where \( N = \sum r_j^{(i)} \) and we take \( r_j^{(1)} \geq r_j^{(2)} \geq \ldots \geq r_j^{(n_j)} \), and the manifold:

\[
G(N; r_j^{(1)}, r_j^{(2)}, \ldots, r_j^{(n_j)}) = \frac{U(N)}{U(r_j^{(1)}) \times U(r_j^{(2)}) \times \ldots \times U(r_j^{(n_j)}) \times J}.
\]

\( J \) is a discrete symmetry group that eliminates overcounting when some of the ranks are equal.
The parameter space of a set of histories is then a manifold, \( M \), which is a product of such \( G \)'s, one for each projective decomposition, \( \sigma_k \), \( k = 1, 2, \ldots, n \) (i.e. one for each time, \( t_k \)): \( M = G_1 \times \ldots \times G_n \).

It is easy to use this parametrisation in explicit calculations: one can define projections \( \{ P^{(1)}, \ldots, P^{(n_j)} \} \) of ranks \( \{ r^{(1)}, r^{(2)}, \ldots, r^{(n_j)} \} \) by choosing an orthonormal basis of vectors \( \{ x_1, \ldots, x_n \} \), so that

\[
P^{(1)} = \sum_{i=1}^{r_1} x_i (x_i)\dagger, \quad P^{(2)} = \sum_{i=r_1+1}^{r_1+r_2} x_i (x_i)\dagger, \tag{3.2}
\]

and so on. The redundancies in this parametrisation correspond to the actions of the quotient subgroups, and can be eliminated at any convenient point. This can be done for each time. Thus, in principle, we can simply fix the form of the initial density matrix, fix the ranks of the projection operators in the type of consistent set we wish to classify, and then impose the consistency conditions (2.3). These will be algebraic equations defining a submanifold, \( L \), the submanifold of consistent sets, of the manifold of all sets of histories, \( M \).

While these algebraic equations are generally very complicated, one can at least make educated guesses at the qualitative features of \( L \), such as its dimension, and these guesses can be checked in simple examples. A typical physical illustration of the consistent histories formalism would involve a small number of projection operators, describing quasiclassical operators in a large Hilbert space. One might, for instance, describe a coarse-grained trajectory of a dust grain, interacting with a photon background. Such physical projection operators rarely form a precisely consistent set, and there has been debate over whether or not it is possible to find close approximations to the projection operators which are exactly consistent. Comparing the number of parameters used to specify sets in \( M \) (very large) with the number of consistency equations (rather small) suggests that this is generically possible. If so, there is no need to follow Gell-Mann and Hartle in ascribing a fundamental role to approximately consistent sets: exactly consistent sets suffice. Moreover, the counting arguments show that in any physically realistic situation the dimension of \( L \), the space of consistent sets, is very large.
4. Properties of Consistent Sets

So, let us suppose that physics is described by exactly consistent sets and look at what this implies. We omit proofs, which can be found in Ref. [11].

Lemma 1

Let $S = (\rho, \{\sigma_1, \ldots, \sigma_k\})$ be a consistent set which is not a trivial extension of any consistent subset, defined on a space $\mathcal{H}$ of dimension $N$, with initial density matrix $\rho$ of rank $r$. Then the length $k$ of $S$ obeys $k \leq rN$. (In particular, if $\rho$ is pure then $k \leq N$.)

In other words, if the Hilbert space of the universe is finite-dimensional there is a strict bound on the number of probabilistic physical events. Once this number has occurred, the evolution of the universe continues completely deterministically. This is mathematically an unsurprising feature of the formalism but, as far as we are aware, physically quite new: no previous interpretation of quantum theory has suggested that quantum stochasticity is exhaustible in this way.

In the consistent histories approach, predictions can only be made once a consistent set — the physically relevant set — is fixed. The key problem in interpreting the formalism is explaining how, given the profligate abundance of consistent sets, this is to be done. Once the choice has been made, one can simply declare by fiat that physics should be described by one history from the relevant set, chosen at random using the decoherence functional probability distribution. Again, the key question is whether the choice has been made within the formalism or whether it relies on assumptions that go beyond it.

It thus becomes an important question whether, when some of the projective decompositions in the relevant set are known, others can be determined. In particular, if, taking the past and present for granted, we were able to deduce the form of the relevant set in the future using only consistency criteria then we could indeed make unconditional predictions about the future within the consistent histories formalism: the choice of set would be clearly determined by the formalism. This, though, is generally false:

Lemma 2

Let $S$ be a set of consistent histories for which there exists a non-trivial consistent extension. Let $S$ have a pure initial state $\rho$, and let $\mathcal{H}$ be either finite-dimensional or separable. Then there exists a continuous family of non-trivial extensions for each history in $S$ with non-zero probability.

So, if a consistent set describing a physical system up to time $t$ leaves some future events unpredictable, there are infinitely many different consistent continuations of that set. In

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2 A similar result has been obtained independently by Diósi [12].
particular, if a consistent set describes, in Gell-Mann and Hartle’s language, quasiclassical physics — involving operators describing the same types of variables at different times, following largely deterministic evolution equations — up to time \( t \) then, if any unpredictability remains, almost all future consistent continuations will not be quasiclassical. Whatever our experience of a persistently quasiclassical world may be ascribed to — and there are various suggestions — it does not follow simply from consistency.

Still, one might hope that at least, if the set up to time \( t \) is quasiclassical, then any non-quasiclassical consistent future extension can consistently incorporate future deterministic quasiclassical predictions. Indeed, Omnès has suggested that this is so. But in fact, as Omnès now accepts, this fails quite generally. If unpredictability remains, then there are no future predictions consistent with all consistent extensions of the present data:

**Lemma 3**

Let \( S = (|\psi\rangle\langle\psi|, \{\sigma_1, \ldots, \sigma_l\}) \) be as in Lemma 2, with \( \mathcal{H} \) finite-dimensional or separable. Then there is no projective decomposition \( \sigma_{l+1} \) such that:

(i) \( S' = (\rho, \{\sigma_1, \ldots, \sigma_l, \sigma_{l+1}\}) \) is a consistent extension of \( S \)

and (ii) any consistent extension \( S'' = (\rho, \{\sigma_1, \ldots, \sigma_l, \tau_1, \ldots, \tau_r\}) \) of \( S \) has a consistent extension \( (\rho, \{\sigma_1, \ldots, \sigma_l, \tau_1, \ldots, \tau_r, \sigma_{l+1}\}) \).

We have only been able to identify one class of statements which can consistently be added to any consistent extension of a set. These arise where the set contains the same decomposition twice. In this case, further repetitions can be included, provided that they are made between the first two:

**Lemma 4**

Let \( S = (\rho, \{s_1, \ldots, s_j, t, t_1, \ldots, t_l, t, s_{j+1}, \ldots, s_k\}) \equiv (\rho, \{S_1, t, T, t, S_2\}) \) be a consistent set in which the projective decomposition \( t \) is repeated. Let \( S' = (\rho, \{S_1, t, T_1, t, T_2, t, S_2\}) \) be an extension of \( S \) by a further repetition of \( t \) at some point between the first two, so that \( \{T\} = \{T_1, T_2\} \). Then \( S' \) is also consistent.

Put picturesquely, if a tree is observed standing in the forest at dusk and dawn, and if the dynamics cause no qualitative complications, then the formalism allows us unambiguously to deduce that it remained standing overnight while unobserved.

### 5. Example

A simple example illustrates the weaknesses of the consistency criterion. Consider two systems, described by two-dimensional Hilbert spaces \( V \) and \( W \), with orthonormal bases \( \{v_1, v_2\} \) and \( \{w_1, w_2\} \). We suppose that the total Hamiltonian is zero except between times
\( t = t_1 \) and \( t = t_2 \), during which the systems are coupled by an interaction which models a measurement process. Specifically, we take the unitary evolution operator between these times to be the operator \( U \) defined by

\[
U |v_1\rangle|w_1\rangle = |v_1\rangle|w_1\rangle, \quad U |v_2\rangle|w_1\rangle = |v_2\rangle|w_1\rangle, \quad U |v_1\rangle|w_2\rangle = |v_2\rangle|w_2\rangle, \quad U |v_2\rangle|w_2\rangle = -|v_1\rangle|w_2\rangle,
\]

and we take \( \rho_i = |v_1\rangle\langle v_1| \otimes I \). In this much simplified (and unrealistic) model, \( V \) represents the relevant degrees of freedom — two “pointer positions” — of a measuring device examining a two-dimensional microscopic quantum system represented by \( W \). We have chosen \( \rho_i \) so that the initial pointer position is specified and no information is known about the system \( W \), which might for example be a spin-1/2 cosmic ray.

Now consider an experiment in which the pointer is observed to be in state \( |v_1\rangle \) at time \( t_3 > t_2 \), so that the combined system lies in the range of the projection \( P = |v_1\rangle\langle v_1| \otimes I \). The standard description of this experiment would distinguish between the macroscopic system \( V \), which follows classical dynamics after \( t_2 \) and the microscopic system \( W \), which is observed at time \( t_3 \) to be in the state \( |w_1\rangle \) and thereafter follows the Schrödinger equation. Now the consistent histories description of the observation uses the set \( S \) defined by the single projective decomposition \( \{P, 1-P\} \) at time \( t_3 \), and specifically the history from that set defined by \( P \).

However, none of the standard inferences drawn from the observation can be made using the consistency criterion alone: we cannot deduce that the pointer was in state \( |v_1\rangle \) at times between \( t_2 \) and \( t_3 \) or after time \( t_3 \) (as classical mechanics would imply), nor that the system was in state \( |w_1\rangle \) after \( t_3 \) (as standard quantum mechanics would imply). The reason is that in each case we can find a consistent extension of \( S \) with which the inference is inconsistent.

For example, let \( Q \) be the projection \( I \otimes |w\rangle\langle w| \) onto the state \( |w\rangle = (|w_1\rangle + |w_2\rangle)/\sqrt{2} \) and \( R \) the projection \( I \otimes |w_1\rangle\langle w_1| \). Then the set \( \{\{P, 1-P\}(t_3), \{Q, 1-Q\}(t)\} \) is consistent. (We now include explicit time labels and take \( t_3 < t < t' \).) However, since the extended set \( \{\{P, 1-P\}(t_3), \{Q, 1-Q\}(t), \{R, 1-R\}(t')\} \) is inconsistent, we cannot infer that the quantum mechanical system \( W \) is in state \( |w_1\rangle \) at any time \( t' > t_3 \). Essentially the same argument shows that this inference cannot be made at time \( t_3 \). In other words, the observation of a pointer state does not imply the result of its measurement in the usual way. Similar arguments show that we cannot infer that the pointer itself is in state \( |v_1\rangle \) at any time other than \( t_3 \).
6. Conclusions

These results illustrate interesting features of the consistent histories formalism. Those who prefer their fundamental theories to be mathematically precise will be encouraged that the use of approximately consistent sets can apparently be avoided. On the other hand, it will be seen that consistency itself is a very weak condition: there is generally a large variety of consistent sets. We stress that this does not mean that the consistency criterion is incompatible with experiment. Many careful studies have illustrated the efficiency of the decoherence process and its crucial importance in understanding the dynamics of quasiclassical systems. \(^{\text{3}}\) Assuming that quantum theory holds good as a description of macroscopic systems, the moral to be drawn from these studies is that, in any realistic description of an experiment, one of the consistent sets will correctly describe familiar quasiclassical physics and will allow the standard predictions and inferences.

The problem here is that all consistent sets have the same status in the formalism, most have very little to do with the quasiclassical world of our observations, and we can only make the predictions we would like to make after we have made the choice of one particular set: the familiar quasiclassical one. If the consistent histories formalism is to represent an enhancement in predictive power over the Copenhagen interpretation it requires interpretational arguments that show this choice to be determined by the formalism. Omnès’ interesting attempt\(^{[4,5]}\) to find a mathematical criterion that correctly identifies the relevant set, unfortunately, fails. The remaining arguments in the literature\(^{[6,7]}\) which suggest that the formalism nonetheless does explain the observed persistence of quasiclassicality therefore deserve careful scrutiny. Our own conclusion\(^{[11]}\) is that they rely on important hidden assumptions.

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\(^{\text{3}}\) See for example the decoherence calculations of ref. \([13]\).
References

[1] L.D. Landau and E.M. Lifschitz, “Quantum Mechanics (Non-relativistic Theory)”, Third Edition (1977) Pergamon Press.

[2] R.B. Griffiths, J. Stat. Phys. 36 (1984) 219.

[3] R.B. Griffiths, Found. Phys. 23 (1993) 1601.

[4] R. Omnès, J. Stat. Phys. 53 (1988) 893; ibid 53 (1988) 933; ibid 53 (1988) 957; ibid 57 (1989) 357.

[5] R. Omnès, Rev. Mod. Phys. 64 (1992) 339.

[6] M. Gell-Mann and J.B. Hartle in Complexity, Entropy, and the Physics of Information, SFI Studies in the Sciences of Complexity, Vol. VIII, ed. by W. Zurek, Addison Wesley, Reading (1990).

[7] M. Gell-Mann and J.B. Hartle in Proceedings of the NATO Workshop on the Physical Origins of Time Asymmetry, Mazagón, Spain, September 30-October 4, 1991 ed. by J. Halliwell, J. Pérez-Mercader, and W. Zurek, Cambridge University Press, Cambridge (1994), gr-qc/9304023.

[8] M. Gell-Mann and J.B. Hartle, Phys. Rev. D 47 (1993) 3345.

[9] B. d’Espagnat, J. Stat. Phys. 56 (1989) 747.

[10] J. Halliwell, Imperial College preprint IC/93-94/52, to appear in Fundamental Problems in Quantum Theory, ed. by D. Greenberger.

[11] F. Dowker and A. Kent, “On the Consistent Histories Approach to Quantum Mechanics”, J. Stat. Phys, to appear.

[12] L. Diósi “On maximum number of decoherent histories,” gr-qc/9409028.

[13] E. Joos and H.D. Zeh, Zeit. Phys. B59 (1985) 223.