DYNAMICS OF THE 2D NAVIER-STOKES EQUATIONS WITH
SUBLINEAR OPERATORS IN LIPSCHITZ-LIKE DOMAINS

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Abstract. This paper is concerned with the tempered pullback dynamics of
the 2D Navier-Stokes equations with sublinear time delay operators subject
to non-homogeneous boundary conditions in Lipschitz-like domains. By virtue
of the estimates of background flow in Lipschitz-like domain and a new re-
tarded Gronwall inequality, we establish the existence of pullback attractors in
a general setting involving tempered universes.

1. Introduction. The influence of delay was first investigated for ordinary differ-
ential equations, with applications to control theory and engineering. The math-
ematical analysis of ordinary differential equations with delay can be found in, e.g.,
[18], [19]. Physical reasons, non-instant transmission phenomena, memory pro-
cesses, and specific biological motivations make delayed differential equations an
important area of applied mathematics, see [5], [6], [7], [8].

The 2D incompressible Navier-Stokes equations constitute the governing law for
fluid flows and enjoy global weak and strong well-posedness, see [14], [20], [22] and

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[35]. In view of a better understanding of turbulence and chaos from a mathematical point of view, the long time dynamics of the 2D Navier-Stokes equations in bounded regular domains with homogeneous boundary conditions has attracted considerable attention starting from the 1980s, see [12], [14], [21], [30], [31], [34] and references therein. Concerning the Navier-Stokes equations with delay and some hereditary features, Barbu and Sritharan [2] established the existence and uniqueness of weak solutions. Taniguchi [33] obtained the well-posedness and absorbing sets for the non-autonomous Navier-Stokes equations with delay. Caraballo and Real [9, 10, 11] applied delays to the 2D Navier-Stokes equations and obtained the existence of non-autonomous Navier-Stokes equations with delay. Marín-Rubio and Real [25] extended these results to some unbounded cases such that the Poincaré inequality holds and constructed global and pullback attractors. Marín-Rubio and Real [25] extended these results to the 2D Navier-Stokes equations in bounded regular domains with homogeneous boundary conditions has attracted considerable attention starting from the 1980s, see [12], [14], [21], [30], [31], [34] and references therein. Concerning the Navier-Stokes equations with delay and some hereditary features, Barbu and Sritharan [2] established the existence and uniqueness of weak solutions. Taniguchi [33] obtained the well-posedness and absorbing sets for the non-autonomous Navier-Stokes equations with delay. Caraballo and Real [9, 10, 11] applied delays to the 2D Navier-Stokes equations and obtained the existence of non-autonomous Navier-Stokes equations with delay. Caraballo and Real [9, 10, 11] applied delays to the 2D Navier-Stokes equations and obtained the existence of global and pullback attractors. Marín-Rubio and Real [25] extended these results to some unbounded cases such that the Poincaré inequality holds and constructed pullback attractors. Moreover, García-Luengo, Marín-Rubio and Planas [17] studied the 2D Navier-Stokes equations with double-time delays on the convective term and external forces, and obtained the existence of pullback attractors. Additional results can be found in [4], [15], [16], [26], [36], [38] and references therein.

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded set; it is said to be a Lipschitz-like domain if its boundary \( \partial \Omega \) can be covered by finitely many balls \( B_i = B(Q_i, r_0) \) centered at the points \( Q_i \in \partial \Omega \) such that for each ball \( B_i \), there exist a rectangular coordinate system and a Lipschitz continuous function \( \Psi : \mathbb{R}^{d-1} \to \mathbb{R} \) satisfying

\[
B(Q_i, 3r_0) \cap \Omega = \{(x_1, x_2, \cdots, x_d)|x_d > \Psi_i(x_1, x_2, \cdots, x_{d-1})\} \cap \Omega.
\]

Suppose that \( \Omega \subset \mathbb{R}^2 \) is a Lipschitz-like domain occupied by the fluid. We consider the 2D Navier-Stokes equations with sublinear operator subject to non-homogeneous boundary conditions and initial data as

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= f(t, u_t) + g(t, x), \quad (x, t) \in \Omega_T, \\
\text{div}u &= 0, \quad (x, t) \in \Omega_T, \\
\nu(t, x)|_{\partial \Omega} &= \varphi, \quad \varphi \cdot n = 0, \quad (x, t) \in \partial \Omega_T, \\
u(t, x) &= u_{\tau}(x), \quad x \in \Omega, \\
u(t, s) &= \phi(s, x), \quad (x, s) \in \Omega_h,
\end{aligned}
\]

(1)

where \( \Omega_{\tau} = \Omega \times (\tau, +\infty), \partial \Omega_{\tau} = \partial \Omega \times (\tau, +\infty), \Omega_h = \Omega \times (-h, 0) \). The function \( u = (u_1(t, x), u_2(t, x)) \) is the unknown velocity field of the fluid and \( p \) is the unknown pressure. The positive constant \( \nu \) is the kinematic viscosity of the fluid. \( f(t, u_t) = F(u(t - \rho(t))) \) is a sublinear operator which contains memory effects during a fixed time interval with length \( h > 0 \), \( \rho(t) \) is an adequate delay function and \( u_{\tau} \) and \( \phi \) denote the initial state. Finally, \( \varphi \in L^\infty(\partial \Omega) \), where \( n \) is the unit outward normal to \( \partial \Omega \).

Concerning the 2D incompressible Navier-Stokes flows \( f(t, u_t) = 0 \) in (1) driven by non-homogeneous boundary conditions in regular domains, Miranville and Wang [27], [28] introduced a background flow to deal with the boundary conditions, and obtained the existence of the finite dimensional global attractor under the assumptions that \( \partial \Omega \in C^3 \) and \( |\nabla \varphi| \in L^\infty(\partial \Omega) \). For the Lipschitz-like case, motivated by [27, 28] and based on estimates on the Stokes problem in [13] and [32], Brown, Perry and Shen [3] introduced the background flow in Lipschitz-like domains and proved the existence of the finite (fractal) dimensional universal attractor. Using the theory of pullback attractors, Yang, Qin, Lu and Ma [39] deduced the existence and regularity of pullback attractors, based on the background flow in Lipschitz-like domains. Inspired by [3], [27], [28], [37], [39] and [40], using the background flow \( \psi \).
in Lipschitz-like domains which satisfies

\[
\begin{align*}
\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla)v + (v \cdot \nabla)\psi + (\psi \cdot \nabla)v + \nabla(p - \nu q_\eta \varepsilon) &= g(t,x) + f(t,u_t) + \nu F - (\psi \cdot \nabla)\psi, \\
\text{div} v &= 0, \\
v(\tau, x) &= \nu_s(x), \\
v(\tau + s, x) &= \phi(s, x) - \psi(x) = \eta(s, x),
\end{align*}
\]

and setting \( v = u - \psi \), (1) can be transformed into the following equivalent homogeneous boundary case

\[
\begin{align*}
\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla)v + (v \cdot \nabla)\psi + (\psi \cdot \nabla)v + \nabla(p - \nu q_\eta \varepsilon) &= g(t,x) + f(t,u_t) + \nu F - (\psi \cdot \nabla)\psi, \\
\text{div} v &= 0, \\
v(\tau, x) &= \nu_s(x), \\
v(\tau + s, x) &= \phi(s, x) - \psi(x) = \eta(s, x),
\end{align*}
\]

where \( F \) and \( q_\eta \) are generated by the background function and are defined in Section 3.1.

Next, we investigate the existence of minimal and unique families of pullback attractors for (3). The main features of this paper can be stated as follows.

(a) There is, up to our knowledge, no new result on the tempered pullback dynamics for the 2D Navier-Stokes equations with sublinear operators in Lipschitz-like (and thus non-smooth) domains. The tempered pullback attractors here contain the \( \omega \)-limit sets and all equilibriums for the background flow, which differs from the corresponding results in bounded smooth domain in [9], [10], [11], [15], [16], [17], [26]. Since the model considered in this paper contains delay and the domain is non-smooth, our results are a further extension of [3], [27], [28] and [39].

(b) Denoting by \( \mathcal{P}(X) \) the family of all nonempty subsets of \( X \) and considering a family of nonempty sets \( \mathcal{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X) \), we give some preliminary definitions and theorems on the tempered pullback dynamics theory in what follows, see [6], [26]. The class \( \mathcal{D} \) is called a universe in \( \mathcal{P}(X) \) if \( \mathcal{D} \) is a nonempty class of families parameterized by time \( \mathcal{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X) \).

Based on the global well-posedness of weak solutions, from the construction of a universe as defined above, we derive the existence of a minimal family of pullback attractors in some extended phase space. The main difficult here is to deal with the estimates on sublinear (delay) operator and convective term in Lipschitz-like domains.

(c) The uniform estimates on the solutions play a crucial role in the global well-posedness and tempered pullback dissipation, which is the main difficulty. By a new retarded Gronwall inequality from [24] and an iteration procedure, we can achieve our objective with less restrictions on the parameters and external forces. Besides, the application of the new retarded Gronwall inequality to handle the uniform estimates allow to improve the results by only using the classical energy method.

(d) In order to overcome the difficulties related with the attraction property and the asymptotic compactness for processes, we introduce a new variable index to deal with absorbing sets involving tempered universes which do not need to be bounded firstly, and then use the energy equation method to deduce the convergence of sequences in the universe, see Sections 4.2 and 4.3.

This paper is organized as follows. In Section 2, some preliminaries are given which will be used in what follows. The existence and uniqueness of solutions for
our problem are derived in Section 3. In the last section, we prove the existence of pullback attractors for the processes and give some outlooks.

2. Preliminaries.

Functional spaces:

Let $E := \{ u \mid u \in (C_0^\infty(\Omega))^2, \text{div} u = 0 \}$. $H$ is the closure of $E$ in the $(L^2(\Omega))^2$ topology, $\| \cdot \|_2$ and $(\cdot, \cdot)$ denote the norm and inner product in $H$ respectively, i.e.,

$$
|u|_2^2 = (u, u), \quad (u, v) = \sum_{j=1}^{2} \int_{\Omega} u_j(x)v_j(x)dx, \quad \forall \ u, v \in H.
$$

$V$ is the closure of $E$ in the $(H^1(\Omega))^2$ topology, and $\| \cdot \|$ and $(\cdot, \cdot)$ denote the norm and inner product in $V$ respectively, i.e.,

$$
\|u\|^2 = ((u, u)), \quad ((u, v)) = \sum_{i,j=1}^{2} \int_{\Omega} \frac{\partial u_i}{\partial x_i} \frac{\partial v_j}{\partial x_j} dx, \quad \forall \ u, v \in V.
$$

Clearly, $V \hookrightarrow H \equiv H' \hookrightarrow V'$, $H'$ and $V'$ being the dual spaces of $H$ and $V$ respectively, where the injections are dense and continuous. The norm $\| \cdot \|_s$ and $(\cdot, \cdot)$ denote the norm in $V'$ and the dual product between $V$ and $V'$ (or $H$ to itself), respectively.

Let $P$ be the Helmholtz-Leray orthogonal projection from $(L^2(\Omega))^2$ to $H$. We denote by $A := -P\Delta$ the Stokes operator. The sequence $\{ \omega_j \}_{j=1}^\infty$ is an orthonormal system of eigenfunctions for $A$, and $\{ \lambda_j \}_{j=1}^\infty (0 < \lambda_1 \leq \lambda_2 \leq \cdots)$ are the eigenvalues of $A$ corresponding to the eigenfunctions $\{ \omega_j \}_{j=1}^\infty$. We can define

$$
A^s u = \sum_{j=1}^{+\infty} \lambda_j^s (u, \omega_j) \omega_j \quad \text{for } s \in (0, 1) \text{ with the domain}
$$

$$
V^s = D(A^s) = \left\{ A^s u \in H, \sum_{i=1}^{+\infty} \lambda_{2i}^s ((u, \omega_{2i}))^2 < +\infty \right\}
$$

and the norm $\|A^s u\|_2 = \left( \sum_{i=1}^{+\infty} \lambda_{2i}^s ((u, \omega_{2i}))^2 \right)^{1/2}$, see [34].

Functional spaces with delay:

For any $t \in (\tau, T)$, we define $u : (\tau - h, T) \to (L^2(\Omega))^2$, and $u_t$ is the function satisfying $u_t(s) = u(t+s)$. Next, we denote by $C_H = C([-h, 0]; H)$, $C_V = C([-h, 0]; V)$ new Banach spaces with norms

$$
\|\phi\|_{C_H} = \sup_{\theta \in [-h, 0]} \|\phi(\theta)\|_H, \quad \|\phi\|_{C_V} = \sup_{\theta \in [-h, 0]} \|\phi(\theta)\|_V
$$

respectively, and $L^r_X = L^r(-h, 0; X)$ ($1 \leq r \leq +\infty$) for an arbitrary Banach space $X$. In particular, $L^2_H = L^2(-h, 0; H)$, $L^2_V = L^2(-h, 0; V)$.

The bilinear and trilinear operators:

We define the bilinear and trilinear operators as follows (see [34]):

$$
B(u, v) := P((u \cdot \nabla)v), \quad \forall \ u, v \in V;
$$

$$
b(u, v, w) = (B(u, v), w) = \sum_{i,j=1}^{2} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx,
$$

and denote $B(u) = B(u, u)$ for short, where $B(u, v)$ is a bilinear continuous operator from $V^2$ to $V'$, and $b(u, v, w)$ satisfies...
\[
\begin{aligned}
\begin{cases}
    b(u, v, v) &= 0, & \forall u, v, w \in V, \\
    b(u, w, v) &= -b(u, v, w), & \forall u, v, w \in V, \\
    |b(u, v, w)| &\leq C|u|^2 \|v\| \|w\| \|v\|^2, & \forall u \in V, \ v \in V, \ w \in V.
\end{cases}
\end{aligned}
\]

• Some lemmas

**Lemma 2.1.** (1) Hardy’s inequality:

\[
\int_{\Omega} \frac{|u(x)|^2}{\text{dist}(x, \partial\Omega)^2} \, dx \leq C \int_{\Omega} |\nabla u(x)|^2 \, dx, \ \forall u \in V.
\]

(2) For \( u \in D(A^{1/4}) \), there exists a constant \( C \) such that

\[
\|u\|_{L^4(\Omega)} \leq C|A^{1/4}u|_2
\]

and

\[
\int_{\Omega} \frac{|u(x)|^2}{\text{dist}(x, \partial\Omega)} \, dx \leq C|A^{1/4}u|_2^2
\]

hold, where \( C \) is independent of the domain \( \Omega \).

**Proof.** See [3], [13], [32].

**Lemma 2.2.** (The Aubin-Lions Lemma) Let \( X \subset \subset H \subset Y \) be Banach spaces, and \( X \) be reflective. If \( u_n \) is a uniformly bounded sequence in \( L^p(\tau; T; X) \), and for \( 1 < q, p < +\infty \), \( \frac{du}{dt} \) is uniformly bounded in \( L^q(\tau; T; Y) \), then \( u_n \) has a strong convergence subsequence in \( L^p(\tau; T; H) \).

**Proof.** See, e.g., [30] or [34].

3. Well-posedness.

3.1. The background flow for the Stokes problem in Lipschitz-like domains. From [3], we can construct a background flow \( \psi \) which solves the Stokes system in Lipschitz domain

\[
\begin{aligned}
\begin{cases}
    -\Delta u + \nabla q = 0, & \text{in} \ \Omega, \\
    \text{div} u = 0, & \text{in} \ \Omega, \\
    u = \varphi \ \text{a.e. on} \ \partial \Omega \ \text{in the sense of nontangential convergence},
\end{cases}
\end{aligned}
\]

Assume that \( u = (u_1, u_2) \) be the solution to problem (9) with \( \varphi \in L^\infty(\partial\Omega) \) and \( \varphi \cdot n = 0 \). By the incompressible condition and Green’s theorem, we have

\[
u = \left( \frac{\partial g}{\partial x_2}, -\frac{\partial g}{\partial x_1} \right), \ \text{where} \ g(x) = \int_P^x (-u_2, u_2) \cdot Tds \ \text{for fixed} \ P \in \partial\Omega, \ T \ \text{is the unit tangent vector to the path from} \ P \ \text{to} \ x = (x_1, x_2).
\]

Denote by \( \varepsilon \in (0, C\text{diam}(\Omega)) \) an arbitrary small parameter and let \( \eta_\varepsilon \in C_0^\infty(\mathbb{R}^2) \) satisfy

\[
\begin{aligned}
\begin{cases}
    \eta_\varepsilon = 1, & \text{in} \ \{x \in \mathbb{R}^2|\text{dist}(x, \partial\Omega) \leq C_1\varepsilon\}, \\
    \eta_\varepsilon = 0, & \text{in} \ \{x \in \mathbb{R}^2|\text{dist}(x, \partial\Omega) \geq C_2\varepsilon\},
\end{cases}
\end{aligned}
\]

and be such that \( 0 \leq \eta \leq 1 \). Then the background flow \( \psi \) can be constructed as follows:

\[
\psi = \psi_\varepsilon = \left( \frac{\partial}{\partial x_2}(g\eta_\varepsilon), -\frac{\partial}{\partial x_1}(g\eta_\varepsilon) \right).
\]

It enjoys the same non-homogeneous boundary conditions as in (1), namely,

\[
\begin{aligned}
\begin{cases}
    \text{div}\psi = 0, & x \in \Omega; \ \psi = u, & x \in \{x \in \Omega: \ \text{dist}(x, \partial\Omega) < C_1\varepsilon\}, \\
    \psi = \varphi \ \text{on} \ \partial\Omega \ \text{in the sense of nontangential convergence}.
\end{cases}
\end{aligned}
\]
Moreover, the background flow $\psi$ has the compact support property
\[ \text{Supp}\psi \subset \{ x \in \Omega; \, \text{dist}(x, \partial \Omega) < C\varepsilon \} \]
and satisfies the estimates
\[ \sup_{x \in \Omega} |\psi(x)| + \sup_{x \in \Omega} |\nabla \psi(x)| \text{dist}(x, \partial \Omega) \leq C\|\varphi\|_{L^\infty(\partial \Omega)}, \]
\[ \|\nabla \psi(\cdot, \partial \Omega)\|_{L^p(\Omega)} \leq C\|\varphi\|_{L^p(\partial \Omega)}, \quad 2 \leq p \leq \infty. \]
Hence, this yields
\[ \|\psi\|_{L^\infty(\Omega)} \leq C\|\varphi\|_{L^\infty(\partial \Omega)}. \]
In addition, concerning the background flow $\psi$, we also have
\[ \Delta \psi = \nabla (q\eta) + F, \]
where
\[ \text{Supp}\psi \subset \{ x \in \Omega; \, C\varepsilon \leq \text{dist}(x, \partial \Omega) < C_2\varepsilon \}. \]
\[ |F|_2 \leq C/\varepsilon^2 \|\varphi\|_2, \quad \nabla q = \Delta u. \]

3.2. Some retarded integral inequalities.

\textbf{Lemma 3.1.} (The Grönwall inequality in differential form) Let $m(\cdot) \in C^1([R^+, R^+])$, $v(\cdot)$, $h(\cdot) \in C([R^+, R^+])$ be such that
\[ \frac{d}{dt} m(t) \leq v(t)m(t) + h(t), \quad m(t) = m_\tau, \quad t \geq \tau. \]
Then
\[ m(t) \leq m_\tau e^{\int_\tau^t v(s)ds} + \int_\tau^t h(s)e^{\int_s^t v(\sigma)d\sigma}ds, \quad t \geq \tau. \]
We now present some retarded integral inequalities from Li, Liu and Ju [23]. Consider the following retarded integral inequality:
\[ \|y(t)\|_X \leq E(t, \tau)\|y_\tau\|_X + \int_\tau^t K_1(t, s)\|y_s\|_X ds + \int_t^\infty K_2(t, s)\|y_s\|_X ds + \rho, \quad \forall \, t \geq \tau, \]
where $E$, $K_1$ and $K_2$ are non-negative measurable functions on $R^2$ and $\rho \geq 0$ denotes a constant. Let $X$ be a Banach space with spatial variable, based on the retarded Banach spaces above. Then $\|\cdot\|_{C_X}$ denotes the norm of the space $C([-h, 0]; X)$ for some $h \geq 0$; $y(t) \geq 0$ is a continuous function defined on $C([-h, T]; X)$, $y_t(s) = y(t + s)$ for $s \in [-h, 0]$.

Let $L(E, K_1, K_2, \rho) = \{ y \in C([-h, T]; X) | y \geq 0 \text{ satisfies the inequality (12)} \}$, and
\[ \kappa(K_1, K_2) = \sup_{t \geq \tau} \left( \int_\tau^t K_1(t, s)ds + \int_t^\infty K_2(t, s)ds \right). \]
We assume that
\[ \lim_{t \to +\infty} E(t + s, s) = 0 \]
uniformly with respect to $s \in R^+$. Moreover, we suppose that $\kappa(K_1, K_2) < +\infty$. 


Lemma 3.2. (The retarded Gronwall inequality) Denoting \( \vartheta = \sup_{t \geq s \geq \tau} E(t,s) \) and \( \kappa = \kappa(K_1, K_2) \), then we have the following estimates:

1. If \( \kappa < 1 \), then for any \( R, \varepsilon > 0 \), there exists \( \bar{T} > 0 \) such that

\[
\|y_t\|_X < \mu \rho + \varepsilon,
\]

for \( t > \bar{T} \) and all bounded functions \( y \in \mathcal{L}(E, K_2, \rho) \) with \( \|y_0\|_X \leq R \), where \( \mu = \frac{1}{1 - \kappa} \).

2. If \( \kappa < \frac{1}{1+q} \), then there exist \( M, \lambda > 0 \) which are independent on \( \rho \) such that

\[
\|y_t\|_X \leq M \|y_0\|_X e^{-\lambda t} + \gamma \rho, \quad t \geq \tau
\]

for all bounded functions \( y \in \mathcal{L}(E, K_1, K_2, \rho) \), where \( \gamma = \frac{\mu + 1}{\kappa} \) and \( c = \max\{\frac{\rho}{1 - \kappa}, 1\} \).

3. If \( \kappa < \frac{1}{1+q} \), then the solution reduces to the trivial one when \( \kappa c < 1 \).

Proof. See Li, Liu and Ju [23]. \( \square \)

Remark 1. (The special case: \( K_2 = 0 \)) Denote \( (K_1, K_2) = (K_1, 0) \) and let \( \vartheta, \kappa, \mu, \gamma \) be the constants defined in Lemma 3.2. Then we have similar estimates as in Lemma 3.2.

3.3. Hypotheses in (1). In view of the global well-posedness, we impose some assumptions on the external forces, parameters and the initial data as follows.

(I) The initial data \( u_\tau \in H \) and \( \phi: (-h, 0) \times H \to H \) satisfy \( \phi \in L^{2q}(-h, 0; H) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \).

(II) There exists \( m > 0 \) such that the external force \( g(\cdot, \cdot) \in L^2_{t, loc}(\mathbb{R}, V') \) satisfies

\[
\int_{-\infty}^{t} e^{ms} \|g(s, \cdot)\|_{V'}^2 ds < \infty, \quad \forall t \in \mathbb{R}.
\]

(III) The function \( \rho \in C^1([0, +\infty); [0, h]) \), and there exists a positive constant \( \rho^* \) satisfying

\[
\frac{d\rho}{dt} \leq \rho^* < 1, \quad \forall t \geq 0.
\]

(IV) The delayed external force \( f(t, u_t) = \tilde{F}(u(t - \rho(t))) \), \( f(\cdot, y): [\tau, +\infty) \times H \to H \) is measurable for all \( y \in H \) and \( f(\cdot, 0) = 0 \).

Moreover, there exist functions \( \alpha, \beta: [\tau, +\infty) \to [0, +\infty) \), with \( \alpha(\cdot) \in L^p_{t, loc}(\mathbb{R}) \) and \( \beta(\cdot) \in L^1_{t, loc}(\mathbb{R}) \) for \( 1 \leq p \leq +\infty \) with \( \limsup_{\tau \to -\infty} \int_{\tau}^{t} \beta(s) ds = \beta_0 \in (0, +\infty) \) such that

\[
|\tilde{F}(u(t - \rho(t)))|_2^2 \leq \alpha(t)|u(t - \rho(t))|_2^2 + \beta(t), \quad \forall t \geq \tau.
\]

(V) \( \nu \lambda_1 - \frac{1}{\nu(1 - \rho^*)} \|\alpha\|_{L^{\infty}(\tau, +\infty)} > 0 \).

3.4. Global well-posedness. • Equivalent homogeneous problem

Let \( v_\tau \in H, \eta \in L^{2q}_H \). Applying the Helmholtz-Leray projector \( P \) to (3), it can be transformed into the following equivalent abstract form:

\[
\begin{aligned}
\frac{\partial v}{\partial t} + \nu Av + B(v) + B(v, \psi) + B(\psi, v) &= P(g(t) + \tilde{F}(u(t - \rho(t))) + \nu F) - B(\psi), \\
v|_{\partial \Omega} &= 0, \\
v(\tau) &= v_\tau, \\
v(t + s) &= \eta(s).
\end{aligned}
\]
In what follows, we deal with the global well-posedness and long-time behavior of problem (1) via (18).

- **Existence of weak solutions of problem (3)**

**Definition 3.3.** (Weak Hadamard solutions) A function \( v \in L^{\infty}(\tau, T; H) \cap L^2(\tau, T; V) \cap L^{2q}(\tau - h, \tau; H) \) is called a global weak Hadamard solution of problem (18) if \( v(\cdot) \) satisfies the initial data and

\[
\frac{d}{dt}(v(t), w) + \nu ((v(t), w)) + b(v(t), v(t), w) + b(v(t), \psi, w) + b(\psi, v(t), w)
= (g(t) + \tilde{F}(u(t - \rho(t)))) + \nu F, w - b(\psi, \psi, w)
\]  

(19)

for all \( t \geq \tau \) and \( w \in V \).

The existence of a global weak solution can be stated as follows.

**Theorem 3.4.** Suppose that \( v_\tau \in H, \eta = \phi + \psi \in L^2_{\Sigma}, \) the hypotheses (I)-(V) hold. Then the problem (18) possesses a global weak solution \( v = v(\cdot, \tau, v_\tau, \eta) \in L^{\infty}(\tau, T; H) \cap L^2(\tau, T; V) \cap L^{2q}(\tau - h, \tau; H) \) which satisfies (19).

**Proof.** **Step 1:** Existence of local approximated solutions

Considering the eigenvalue problem \( Av = \lambda v \) with Dirichlet boundary conditions and the eigenbasis of \( H \) consisting of \( \{w_1, w_2, \ldots, w_m, \ldots\} \), we can see that \( \{w_m\}_{m \geq 1} \) are also the eigenfunctions of the Stokes operator \( A \) with the corresponding eigenvalues \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots, \) i.e., \( Aw_m = \lambda_m w_m \).

Let \( H_m = \text{span} \{w_1, w_2, \ldots, w_m\}, P_m : H \rightarrow H_m \) be the orthogonal projection. Then the approximated solutions can be written as

\[
v_m(t) = \sum_{j=1}^{m} \gamma_{jm}(t)w_j \quad (j = 1, 2, \ldots, m),
\]

where \( \gamma_{jm}(t) = (v_m(t), w_j) \) is to be determined and \( v_m(t) \) satisfies the Cauchy problem for the following ordinary differential equation:

\[
\begin{aligned}
\frac{d}{dt}(v_m(t), w_m) &+ (\nu A v_m(s) + B(v_m(s), \psi, v_m(s)), w_m) \\
&= (P(g(t) + \tilde{F}(u_m(t - \rho(t)))) + \nu F, w_m) - B(\psi, w_m), \\
\end{aligned}
\]

(20)

with \( \eta_m(t) = P_m \eta(t) \rightarrow \eta \) in \( H \) as \( m \rightarrow +\infty \). By the theory of ordinary differential equations, the problem (20) possesses a local solution.

**Step 2:** Uniform estimates on the approximated solutions

Multiplying (20) by \( \gamma_{km}(t) \), and then summing from \( k = 1 \) to \( m \), noting that \( b(v_m, v_m, v_m) = 0 \) and \( b(\psi, v_m, v_m) = 0 \) yields

\[
\frac{1}{2} \frac{d}{dt} \left\| v_m \right\|^2 + \nu \left\| v_m \right\|^2 \leq \left| b(v_m, \psi, v_m) \right| + \left| b(\psi, v_m, v_m) \right| + \left| (g(t) + \tilde{F}(u_m(s - \rho(s)))) + \nu F, v_m \right|.
\]

(21)

By the Hardy and Hölder inequalities, using the estimates on \( \psi \) in Section 3.1 and choosing \( \varepsilon \) such that \( C\varepsilon \|\varphi\|_{L^{\infty}(\partial \Omega)} \leq \frac{\nu}{8} \), we find
\begin{align*}
|b(v_m, \psi, v_m)| & \leq C\|\varphi\|_{L^\infty(\partial\Omega)} \int_{\text{dist}(x, \partial\Omega) \leq C_\varepsilon} \frac{|v_m|^2}{\text{dist}(x, \partial\Omega)} \, dx \\
& \leq C\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)} \int_{\Omega} \frac{|v_m|^2}{\text{dist}(x, \partial\Omega)^2} \, dx \\
& \leq C\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)} \|v_m\|^2 \leq \frac{\nu}{10} \|v_m\|^2, \quad (22)
\end{align*}

and

\begin{align*}
|b(\psi, \psi, v_m)| & \leq C\|\varphi\|_{L^\infty(\partial\Omega)} \int_{\text{dist}(x, \partial\Omega) \leq C_\varepsilon} \frac{|v_m|}{\text{dist}(x, \partial\Omega)} |\psi| \, dx \\
& \leq C\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)} \|\partial\Omega|^{1/2} \|v_m\|_v \\
& \leq \frac{\nu}{10} \|v_m\|^2 + \frac{C\varepsilon^2\|\varphi\|_{L^\infty(\partial\Omega)}^4}{\nu} \|\partial\Omega\|, \quad (23)
\end{align*}

since \(\|\varphi\|_{L^2(\partial\Omega)} \leq C|\partial\Omega|^{1/2} \|\varphi\|_{L^\infty(\partial\Omega)}\). Moreover,

\begin{align*}
\|g(t), v_m\| & \leq \frac{\nu}{10} \|v_m\|^2 + \frac{C}{\nu} |g(t)|^2_{\nu}. 
\end{align*}

Next, we need to deal with \(\|\tilde{F}(u_m(s - \rho(s)), v_m)\|\). Let \(\theta = s - \rho(s)\), then

\begin{align*}
\frac{d\theta}{dt} = (1 - \rho'(s)) ds, \quad \alpha(t) \rightarrow \tilde{\alpha}(\tilde{t}) \in L^p(\tau, T),
\end{align*}

which means that \(\tilde{\alpha} \in L^q(\tau - h, T)\). We obtain, in view of the assumptions made,

\begin{align*}
(\tilde{F}(u_m(t - \rho(t))), v_m(t)) \\
& \leq |\tilde{F}(u_m(t - \rho(t))) + \psi)|_2 |v_m(t)|_2 \\
& \leq \alpha^2(t) |v_m(t - \rho(t))|_2 |v_m(t)|_2 + \alpha^2(t) |\psi|_2 |v_m(t)|_2 + \beta^2(t) |v_m(t)|_2 \\
& \leq \nu(1 - \rho^*) |v_m(t - \rho(t))|^2 + \left(\frac{\alpha(t)}{\nu \lambda_1 (1 - \rho^*)} + \frac{\nu}{10}\right) \|v_m(t)\|^2 \\
& \quad + \frac{8}{\nu \lambda_1} \left(\alpha(t)|\psi|_2^2 + \beta(t)\right). \quad (27)
\end{align*}

Combining (21)-(27), we get

\begin{align*}
\frac{d}{dt} |v_m|^2 + \nu \|v_m\|^2 \\
& \leq \nu(1 - \rho^*) |v_m(t - \rho(t))|^2 + \frac{\alpha(t)}{\nu \lambda_1 (1 - \rho^*)} \|v_m(t)\|^2 + \frac{8}{\nu \lambda_1} \left(\alpha(t)|\psi|_2^2 + \beta(t)\right) \\
& \quad + \frac{C}{\nu} |g(t)|^2_{\nu} + \frac{C|\partial\Omega|}{\varepsilon} \|\varphi\|_{L^\infty(\partial\Omega)}^2 \|\partial\Omega\|, \quad (28)
\end{align*}
Using the Poincaré inequality and Lemma 3.1, noting that \( \nu \lambda_1 - \frac{1}{\nu(1 - \rho^T)} \| \alpha(t) \|_{L^\infty} > 0 \), it follows that

\[
|v_m|^2 \leq |v_m|^2 \leq |v_m|^2 e^{-\int_t^\tau [\nu \lambda_1 - \frac{a(s)}{\nu(1 - \rho^T)}] ds} \\
+ \nu(1 - \rho^T) \int_t^\tau e^{-\int_s^\tau [\nu \lambda_1 - \frac{a(s)}{\nu(1 - \rho^T)}] ds} \| v_m(s - \rho(s)) \|_2^2 ds \\
+ \int_t^\tau e^{-\int_s^\tau [\nu \lambda_1 - \frac{a(s)}{\nu(1 - \rho^T)}] ds} \left[ \frac{8}{\nu \lambda_1} \left( \alpha(s) \| \psi \|_2^2 + \beta(s) \right) \\
+ \frac{C}{\nu} \| g(t) \|_{L^2} + \frac{C |\partial \Omega|}{\varepsilon} \| \varphi \|_{L^\infty(\partial \Omega)}^2 + \frac{C \varepsilon^2 \| \varphi \|_{L^\infty(\partial \Omega)}^4}{\nu} \right] ds.
\]

Denoting

\[
E(t, \tau) = e^{-\int_t^\tau [\nu \lambda_1 - \frac{a(s)}{\nu(1 - \rho^T)}] ds},
\]

\[
K_1(t, s) = \nu(1 - \rho^T) e^{-\int_t^s [\nu \lambda_1 - \frac{a(s)}{\nu(1 - \rho^T)}] ds},
\]

\[
\Theta = \sup_{t \geq s \geq \tau} E(t, s), \quad \kappa(K_1, 0) = \sup_{t \geq \tau} \int_t^\tau K_1(t, s) ds
\]
and

\[
\rho = \int_t^\tau e^{-\int_s^\tau [\nu \lambda_1 - \frac{a(s)}{\nu(1 - \rho^T)}] ds} \left[ \frac{8}{\nu \lambda_1} \left( \alpha(s) \| \psi \|_2^2 + \beta(s) \right) \\
+ \frac{C}{\nu} \| g(t) \|_{L^2} + \frac{C |\partial \Omega|}{\varepsilon} \| \varphi \|_{L^\infty(\partial \Omega)}^2 + \frac{C \varepsilon^2 \| \varphi \|_{L^\infty(\partial \Omega)}^4}{\nu} \right] ds,
\]

choosing \( e^{-\int_t^s [\nu \lambda_1 - \frac{a(s)}{\nu(1 - \rho^T)}] ds} < \frac{1}{\nu(1 - \rho^T)} \) such that \( \kappa(K_1, 0) < \frac{1}{\rho + \Theta} \), then by virtue of Lemma 3.2, we derive that

\[
|v_m(t - \rho(t))|_2 \leq C \int_t^\tau e^{-\int_s^\tau [\nu \lambda_1 - \frac{a(s)}{\nu(1 - \rho^T)}] ds} \left[ \frac{8}{\nu \lambda_1} \left( \alpha(s) \| \psi \|_2^2 + \beta(s) \right) \\
+ \frac{C}{\nu} \| g(t) \|_{L^2} + \frac{C |\partial \Omega|}{\varepsilon} \| \varphi \|_{L^\infty(\partial \Omega)}^2 + \frac{C \varepsilon^2 \| \varphi \|_{L^\infty(\partial \Omega)}^4}{\nu} \right] ds + C.
\]

Substituting (34) into (28) and using Lemma 3.1 again, we can conclude that

\[
|v_m|^2 \leq |v_m|^2 e^{-\int_t^\tau [\nu \lambda_1 - \frac{a(s)}{\nu(1 - \rho^T)}] ds} \\
+ C \int_t^\tau e^{-\int_s^\tau [\nu \lambda_1 - \frac{a(s)}{\nu(1 - \rho^T)}] ds} \left[ \frac{8}{\nu \lambda_1} \left( \alpha(s) \| \psi \|_2^2 + \beta(s) \right) \\
+ \frac{C}{\nu} \| g(t) \|_{L^2} + \frac{C |\partial \Omega|}{\varepsilon} \| \varphi \|_{L^\infty(\partial \Omega)}^2 + \frac{C \varepsilon^2 \| \varphi \|_{L^\infty(\partial \Omega)}^4}{\nu} \right] ds + C
\]

and

\[
\int_t^\tau \| v_m(s) \|^2 ds \leq C(T, \nu_0, v_{m_\tau}, \eta, \| \varphi \|_{L^\infty(\partial \Omega)}),
\]
so that

\[
\{ v_m \} \text{ is uniformly bounded in } L^\infty(\tau; T; H) \cap L^2(\tau; T; V)
\]
for any $T \geq \tau$.

**Step 3: Compactness argument.**

Our aim is to prove that $\{v_m\}$ has a subsequence which converges strongly, owing to the Aubin-Lions Lemma and the uniform boundedness of $\frac{dv_m}{dt}$ in $L^2(0, T; V')$.

From (18), the approximated solutions satisfy

$$
\frac{dv_m}{dt} = -\nu Av_m - B(v_m) - B(v_m, \psi) - B(\psi, v_m)
+ P(g(t) + \tilde{F}(u_m(t - \rho(t))) + \nu F) - B(\psi),
$$

and $\{v_m\} \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V)$. We then deduce that $-\nu Av_m \in L^2(\tau, T; V')$, and

$$
\|B(v_m)\|_{L^2(\tau, T; V')}^2 = \int_\tau^T \sup_{|\alpha| = 1} | < (v_m \cdot \nabla)v_m, u > |^2 ds
\leq C \int_\tau^T |v_m|_2^2 \|v_m\|^2 ds
\leq C \|v_m\|_{L^\infty(\tau, T; H)}^2 \|v_m\|_{L^2(\tau, T; V')}^2,
$$

which implies that $B(v_m) \in L^2(\tau, T; V')$. Similarly, we have

$$
\|B(v_m, \psi) + B(\psi, v_m)\|_{L^2(\tau, T; V')}^2
\leq C \|\psi\|_{L^\infty(\partial \Omega)}^2 \left( \|v_m\|_{L^2(\tau, T; V)}^2 + \|v_m\|_{L^2(\tau, T; V')}^2 \right),
$$

i.e., $B(v_m, \psi)$ and $B(\psi, v_m)$ belong to $L^2(\tau, T; V')$. By (23), we also have $B(\psi) \in L^2(0, T; V')$.

Concerning the term $\tilde{F}(u_m(t - \rho(t)))$, we have

$$
\int_\tau^T \|\tilde{F}(u_m(t - \rho(t)))\|_{V'}^2 dt
= \int_\tau^T \left( \sup_{|\alpha| = 1} | \tilde{F}(u_m(t - \rho(t))), u > | \right)^2 dt
\leq C \left[ \int_\tau^T \alpha(t) |u_m(t)|_2^2 dt + \|\beta\|_{L^1(\tau, T)} \right]
\leq C \left[ \|\alpha\|_{L^p(\tau, T)} \|v_m\|_{L^2(\tau - h, T; H)}^2 + \|\alpha\|_{L^p(\tau, T)} \|\varphi\|_{L^2(\partial \Omega)}^2 + \|\beta\|_{L^1(\tau, T)} \right],
$$

which gives the boundedness of $\tilde{F}(u_m(t - \rho(t)))$ in $L^2(0, T; V')$, owing to the estimate of $v_m$ and noting that $v_m \in L^q(\tau - h, \tau; H)$. In conclusion, the above estimates imply that $\frac{dv_m}{dt}$ is uniformly bounded in $L^2(\tau, T; V')$. Since $V \hookrightarrow H \hookrightarrow V'$, using Lemma 2.2, we see that $\{v_m\}$ has a strongly convergent subsequence (also denoted as $\{v_m\}$), i.e.,

$$
v_m(t) \to v(t) \text{ in } L^2(\tau, T; H),
\quad
v_m(t) \to v(t) \text{ in } H \text{ a.e. } t \geq \tau,
\quad
v_m(\tau) = P_{\mu} v_{\tau} \to v(\tau) = v_{\tau}.
$$

**Step 4: Passage to the limit to have the existence of global solutions**
Theorem 3.5. The global weak solution in Theorem 3.4 is unique.

\newpage

Proof. \( v \) satisfies \( w \) with the initial data \( t \in [\tau, T] \), which implies that

\[ \int_{\Omega} |w(t) + \partial_t w(t)|^2 \leq \int_{\Omega} |w_0|^2 + |w(t)|^2, \]

Integrating (45) from \( \tau \) to \( t \) and integrating over \( \Omega \), using Young’s inequality and the properties of trilinear operator \( b(\cdot, \cdot, \cdot) \) and choosing \( \varepsilon \) such that \( C_\varepsilon \| \varphi \|_{L^\infty(\Omega)} \leq \frac{\varepsilon}{4} \), we derive

\[ \int_\tau^t |w(s - \rho(s))|^2 ds \leq \frac{1}{1 - \rho^*} \left[ \int_{\tau - h}^\tau |\phi_0(s) - \phi_1(s)|^2 ds + \int_\tau^t |w(s)|^2 ds \right] \]

Integrating (45) from \( \tau \) to \( t \) and noting that

\[ \frac{d}{dt} |w(t)|^2 + \nu \| w(t) \|^2 \leq \left| b(w, \psi, w) \right| + \left| (\tilde{F}(v_1(t - \rho(t)) + \psi) - \tilde{F}(v_2(t - \rho(t)) + \psi), w) \right| \leq \frac{\nu}{4} |w(t)|^2 + L(r)\gamma^{1/2}(t)|w_2|_2|w(t - \rho(t))|_2 \]

\[ \leq \frac{\nu}{4} |w(t)|^2 + \frac{4L^2(r)\gamma(t)}{\nu \lambda_1} |w(t - \rho(t))|^2. \]

Then, passing to the limit in (19), we can conclude on the existence of a global weak solution for (19) as in Definition 3.3.

- **Uniqueness of the weak solution of problem (3)**

  We make the following assumption.

  (VI) There exist \( k \in L^1(0, T), L(r) > 0 \) and \( r > 0 \) such that \( f(t, u) = \tilde{F}(u(t - \rho(t))) \) satisfies

  \[ \| \tilde{F}(\zeta) - \tilde{F}(\xi) \|_H \leq L(r)\gamma^{1/2}(t)\| \zeta - \xi \|_H, \quad \zeta, \xi \in H. \]

  (43) holds for \( \| \xi \|_H \leq r, \| \xi \|_H \leq r \) and \( t \in [\tau, T] \), where \( \gamma \in L^\infty(\tau, T). \)

**Theorem 3.5.** Assume that the hypotheses of Theorem 3.4 and (VI) hold. Then the global weak solution in Theorem 3.4 is unique.

In addition, the weak solution depends continuously on initial data \( v_\tau \) and \( \eta(s) \), which implies that \( v \in C([\tau, +\infty); H) \).

**Proof.** Denote by \( v_1(t; \tau, \eta), v_2(t; \tau, \eta) \) two global weak solutions of (18) with corresponding initial data \( (v_0^1, \phi_0) \) and \( (v_0^1, \phi_1) \). Setting \( w(t) = v_1(t) - v_2(t), \) \( w(t) \) satisfies

\[ w_t + \nu Aw + B(w, \psi) + B(\psi, w) = \tilde{F}(v_1(t - \rho(t)) + \psi) - \tilde{F}(v_2(t - \rho(t)) + \psi), \]

with the initial data \( w(\tau) = w_{\tau} = v_0^0 - v_0^1 \) and \( w(t + \tau) = \phi_0 - \phi_1 \). Then multiplying (44) by \( w \) and integrating over \( \Omega \), using Young’s inequality and the properties of trilinear operator \( b(\cdot, \cdot, \cdot) \) and choosing \( \varepsilon \) such that \( C_\varepsilon \| \varphi \|_{L^\infty(\partial\Omega)} \leq \frac{\varepsilon}{4} \), we derive

\[ \frac{1}{2} \int_\tau^t \| w(t) \|^2 + \nu \| w(t) \|^2 \]

\[ \leq \int_{\Omega} |b(w, \psi, w)| + \left| (\tilde{F}(v_1(t - \rho(t)) + \psi) - \tilde{F}(v_2(t - \rho(t)) + \psi), w) \right| \]

\[ \leq \frac{\nu}{4} |w(t)|^2 + L(r)\gamma^{1/2}(t)|w_2|_2|w(t - \rho(t))|_2 \]

\[ \leq \frac{\nu}{4} |w(t)|^2 + \frac{4L^2(r)\gamma(t)}{\nu \lambda_1} |w(t - \rho(t))|^2. \]

Integrating (45) from \( \tau \) to \( t \) and noting that

\[ \int_\tau^t |w(t - \rho(s))|^2 ds \leq \frac{1}{1 - \rho^*} \left[ \int_{\tau - h}^\tau |\phi_0(s) - \phi_1(s)|^2 ds + \int_\tau^t |w(s)|^2 ds \right] \]

yields

\[ |w(t)|^2 + \nu \int_\tau^t \| w(s) \|^2 ds \]

\[ \leq |w_\tau|^2 + \frac{8L^2(r)\gamma(t)}{\nu \lambda_1(1 - \rho^*)} \int_{\tau - h}^\tau |\phi_0(s) - \phi_1(s)|^2 ds \]
For all \( \chi \) function

\[
\text{(i)} \quad \text{Definition 3.6. Let } \psi \text{ which implies the result.}
\]

Moreover, we have \( \psi \) satisfies (i), (ii) and (iii) in Definition 3.6, as done in [3]. We thus omit the details.

\[
\text{Well-posedness of problem (1)}
\]

**Definition 3.6.** Let \( u \in H, \phi \in L^2_H, \varphi \in L^\infty(\partial\Omega) \) and \( \varphi \cdot n = 0 \) on \( \partial\Omega \). Then the function \( u \) is a global weak solution for problem (1) provided that

(i) \( u \in C([\tau-h,T]; H), u(\tau, \cdot) = u_\tau, \text{ and } du/dt \in L^2(\tau, T; V') \),

(ii) For all \( \chi \in C^\infty_0(\Omega) \) with \( \text{div} \chi = 0 \), we get

\[
\frac{d}{dt}(u, \chi) + \nu \langle Au, \chi \rangle + b(u, u, v) = \langle g(t), \chi \rangle + \langle F(u(t - \rho(t))), \chi \rangle.
\]

(iii) There exist \( \psi \in C^2(\Omega) \cap L^\infty(\Omega) \) and \( q \in C^1(\Omega) \) such that

\[
\begin{cases}
-\Delta \psi + \nabla q = 0, & \text{in } \Omega, \\
\text{div} \psi = 0, & \text{in } \Omega, \\
\psi = \varphi & \text{on } \partial\Omega,
\end{cases}
\]

where we assume that \( \psi \) can reach its boundary values in the sense of non-tangential convergence and \( u - \psi \in L^2(\tau; V) \).

**Theorem 3.7.** Let \( u_\tau \in H, \phi \in L^2_H, \varphi \in L^\infty(\partial\Omega), \varphi \cdot n = 0 \) on \( \partial\Omega \) and assume that (I)-(VI) hold. Then (1) has a unique global weak solution which generates a continuous process \( \{S(t, \tau)\} \) in \( H \).

**Proof.** (1) Global existence of a weak solution for (1): The family of background functions \( \psi_\varepsilon \) was constructed in [3], \( \psi_\varepsilon \in C^\infty(\Omega) \), and the solution \( v \) for (3) is obtained in Theorem 3.4. Letting \( u = v + \psi_\varepsilon \), it is easy to check that \( u(t, x) \) satisfies (i), (ii) and (iii) in Definition 3.6, as done in [3]. We thus omit the details.
(2) **Uniqueness of the solution for (1):** Suppose that \( u_1 \) and \( u_2 \) are two solutions of (1) with the same background flow \( \psi \). Denoting \( \omega = u_1 - u_2 \), \( \omega \) satisfies the equation
\[
\omega_t + \nu \omega + B(u_1, \omega) + B(w, u_2) = \tilde{F}(u_1(t - \rho(t))) - \tilde{F}(u_2(t - \rho(t)))
\]
with proper initial data and boundary conditions.

Since \( f(\cdot) \) satisfies (VI), then by the uniqueness of the solution \( v(t, x) = u(t, x) - \psi(x) \) of (3) in Theorem 3.5, it is easy to verify the uniqueness of \( u(t, x) \) by virtue of the same techniques as in Theorem 3.5. We again skip the details. \( \square \)

4. **Tempered pullback dynamics.**

4.1. **Abstract theory of pullback attractors.**

**Dissipation defined via absorbing sets and universes**

**Definition 4.1.** The family \( \hat{D}_0 = \{ D_0(t) : t \in \mathbb{R} \} \subset \mathcal{P}(X) \) is \( \mathcal{D} \)-pullback absorbing for the process \( U \) on \( X \) if for any \( t \in \mathbb{R} \) and any \( \hat{D} \in \mathcal{D} \), there exists a \( \tau_0(t, \hat{D}) \leq t \) such that
\[
U(t, \tau)D(\tau) \subset D_0(t), \text{ for all } \tau \leq \tau_0(t, \hat{D}).
\]

**Definition of \( \hat{D}_0 \)-pullback asymptotic compactness**

**Definition 4.2.** A process \( U \) on \( X \) is \( \hat{D}_0 \)-pullback asymptotically compact if for any \( t \in \mathbb{R} \) and any sequences \( \tau_n \subset (-\infty, t] \) and \( x_n \subset X \) satisfying \( \tau_n \to -\infty \) and \( x_n \in D_0(\tau_n) \) for all \( n \), the sequence \( U(t, \tau_n)x_n \) is relatively compact in \( X \).

**Definition of \( \mathcal{D} \)-pullback asymptotic compactness:**

**Definition 4.3.** A process \( U \) on \( X \) is said to be \( \mathcal{D} \)-pullback asymptotically compact if it is \( \hat{D} \)-asymptotically compact for any \( \hat{D} \in \mathcal{D} \), i.e. if for any \( t \in \mathbb{R} \), any \( \hat{D} \in \mathcal{D} \), and any sequences \( \tau_n \subset (-\infty, t] \) and \( x_n \subset X \) satisfying \( \tau_n \to -\infty \) and \( x_n \in D(\tau_n) \) for all \( n \), the sequence \( \{ U(t, \tau_n)x_n \} \) is relatively compact in \( X \).

**Existence theorem of minimal and unique families of \( \mathcal{D} \)-pullback attractors:**

**Definition 4.4.** A universe \( \mathcal{D} \) defined in a metric space \( X \) is a class of families \( \mathcal{D} \subset 2^X \) of the form \( \mathcal{D} = \{ D(t) | t \in \mathbb{R} \} \), where each \( D(t) \) is a nonempty bounded subset of \( X \). Given a universe \( \mathcal{D} \) defined on \( X \), a family \( \mathcal{A} = \{ A(t) \}_{t \in \mathbb{R}} \) is called a \( \mathcal{D} \)-pullback attractor of a process \( U(t, \tau) : X \to X \) if the following properties hold:

(i) Compactness: \( A(t) \) is a nonempty compact set of \( X \), \( t \in \mathbb{R} \),
(ii) Invariance: \( U(t, \tau)A(\tau) = A(t), \ t \geq \tau \),
(iii) \( \mathcal{D} \)-pullback attraction: \( \lim_{\tau \to -\infty} \text{dist}_X(U(t, \tau)D(\tau), A(t)) = 0, \ t \in \mathbb{R} \), \( \{ D(t) \}_{t \in \mathbb{R}} \in \mathcal{D} \).

In addition, \( \mathcal{D} \)-pullback attractor \( \mathcal{A} \) is said to be minimal if whenever \( \hat{C} \) is another \( \mathcal{D} \)-attracting family of closed sets, then \( A(t) \subset C(t) \), for all \( t \in \mathbb{R} \).

**Theorem 4.5.** (See Carvalho, Langa and Robinson [6], Marín-Rubio and Real [26]) Consider a continuous process \( U(\cdot, \cdot) : \mathbb{R}^2 \times X \to X \), a universe \( \mathcal{D} \in \mathcal{P}(X) \), and a family \( \hat{D}_0 = \{ D_0(t) : t \in \mathbb{R} \} \subset \mathcal{P}(X) \) which is \( \mathcal{D} \)-pullback absorbing for \( U(\cdot, \cdot) \), and assume also that \( U(\cdot, \cdot) \) is \( \hat{D}_0 \)-pullback asymptotically compact.

Then, the family of \( \mathcal{D} \)-pullback attractors \( \mathcal{A}_\mathcal{D} = \{ A_\mathcal{D}(t) : t \in \mathbb{R} \} \) is defined by
\[
\mathcal{A}_\mathcal{D}(t) = \bigcup_{\hat{D} \in \mathcal{D}} \Lambda(\hat{D}, t)^X, \ t \in \mathbb{R}.
\]
4.2. Existence of tempered pullback absorbing sets. Let \( M_{H}^{2q} = H \times L_{H}^{2q} \). Then we can define a family of bi-parameters operators in \( M_{H}^{2q} \) and \( C_{H} \) respectively. Denote by \( \mathcal{D}(\tau, v_{\tau}, \eta) \) the set of all global weak solutions of (18) in \([\tau, +\infty)\) with \((v_{\tau}, \eta) \in M_{H}^{2q}\).

- **The process in \( M_{H}^{2q} \):**
  The process in \( M_{H}^{2q} \) is defined by \((U(t, \tau), M_{H}^{2q})\), where
  \[
  U(t, \tau)(v_{\tau}, \eta) = \{ v_{i} \in \mathcal{D}(\tau, v_{\tau}, \eta), \forall (v_{\tau}, \eta) \in M_{H}^{2q}, t \geq \tau \}.
  \] (52)

- **The process in \( C_{H} \):**
  The process in \( C_{H} \) is defined by \((U(t, \tau), C_{H})\), where
  \[
  U(t, \tau)\eta = \{ v_{i} \in \mathcal{D}(\tau, \eta(0), \eta), \forall \eta \in C_{H}, t \geq \tau \}.
  \] (53)

- **Estimates on the variable index and uniform boundedness of \( v(t, x) \) in \( H \):**
  Assume that there exists \( \delta > 0 \) which will be determined later such that we can write
  \[
  \kappa_{\delta}(t, s) = \left( \frac{\nu \lambda_{1}}{4} - \delta \right)(t - s) - \frac{4e^{\nu \lambda_{1} h}}{\nu \lambda_{1}(1 - \rho^{*})} \int_{s}^{t} \alpha(r)dr,
  \] (54)
  where
  \[
  \kappa_{\delta}(0, t) - \kappa_{\delta}(0, s) = -\kappa_{\delta}(t, s)
  \] (55)
  and
  \[
  \kappa_{\delta}(0, r) \leq \kappa_{\delta}(0, t) + \left( \frac{\nu \lambda_{1}}{4} - \delta \right) h, \text{ if } \frac{\nu \lambda_{1}}{4} - \delta > 0
  \] (56)
  for \( r \in [t - h, t] \). Based on this notation, we can prove the uniform boundedness of \( v(t) \) in the following theorem.

**Theorem 4.6.** Suppose that \( g(t, x) \) and \( f : \mathbb{R} \times C_{H} \rightarrow H \) satisfy the hypotheses (I)-(VI). Then for any \( v \in \mathcal{D}(\tau, v_{\tau}, \eta) \) and \((v_{\tau}, \eta) \in M_{H}^{2q}\), we have the following uniform estimate
\[
|v(t)|_{2}^{2} \leq \left[ \frac{2C}{\varepsilon} |\partial \Omega|^{2} + 2Ce^{2}\|\varphi\|_{L^{\infty}(\partial \Omega)}^{2} + \frac{C\nu \lambda_{1}(1 - \rho^{*})}{4e^{\nu \lambda_{1} h}} \right] \|\varphi\|_{L^{\infty}(\partial \Omega)} e^{-\kappa_{\delta}(t, \tau)}
+ \left[ \frac{\nu \lambda_{1}e^{\nu \lambda_{1} t}}{4} \int_{t-h}^{t} |\eta(r)|_{2}^{2}dr \right] e^{-\nu \lambda_{1} t - \kappa_{\delta}(t, \tau)}
+ \frac{\delta}{\varepsilon} \int_{\tau}^{t} e^{-\kappa_{\delta}(t, s)} \beta(s)ds + \frac{C}{\nu} \int_{\tau}^{t} e^{-\kappa_{\delta}(t, s)} \|g(s)\|_{2}^{2}ds
\] (57)
for all \( t \geq \tau \).

**Proof.** Let \((v_{\tau}, \eta) \in M_{H}^{2q}\) and \( v \in \mathcal{D}(\tau, v_{\tau}, \eta) \) be the solution of (18). Then taking the inner product of (18) by \( e^{\nu \lambda_{1} t} v \), by the properties of trilinear operator, we have
\[
\frac{d}{dt}(e^{\nu \lambda_{1} t}|v|_{2}^{2}) + \nu e^{\nu \lambda_{1} t}\|v\|_{2}^{2}
\leq 2|b(v, \psi, e^{\nu \lambda_{1} t} v)| + 2||\nu F, e^{\nu \lambda_{1} t} v|| + 2|b(\psi, \eta, e^{\nu \lambda_{1} t} v)|
+ 2|g(t) + f(t, u(t - \rho(t))), e^{\nu \lambda_{1} t} v)|.
\]
Using the estimates in the proof of Theorem 3.4, we know that
\[
|b(v, \psi, e^{\nu \lambda_{1} t} v)| \leq \frac{\nu}{12} e^{\nu \lambda_{1} t}\|v\|_{2}^{2},
\]
\[ |(\nu F, e^{\nu \lambda_1 t}v) - 2 \nu \int_{\partial \Omega} e^{\nu \lambda_1 t} \frac{\partial \varphi}{\partial \nu} \|v\|_2^2 \leq C \frac{\nu}{12} e^{\nu \lambda_1 t} \|v\|_2^2 + \frac{C \nu}{\varepsilon} e^{\nu \lambda_1 t} \|\varphi\|_{L^\infty(\partial \Omega)}^2. \]
\[ |b(\psi, \psi, e^{\nu \lambda_1 t}v) - 2 \nu \int_{\partial \Omega} e^{\nu \lambda_1 t} \frac{\partial \varphi}{\partial \nu} \|v\|_2^2 \leq \frac{C \varepsilon^2}{\nu} \|\varphi\|_{L^\infty(\partial \Omega)}^4 |\partial \Omega| e^{\nu \lambda_1 t}, \]
and hence
\[
\frac{d}{dt} (e^{\nu \lambda_1 t} \|v\|_2^2) + \frac{\nu}{2} e^{\nu \lambda_1 t} \|v\|_2^2 \leq 2I_0 e^{\nu \lambda_1 t} + 2 e^{\nu \lambda_1 t} |f(t, u(t - \rho(t)))|_2 |v|_2.
\]
\[
\leq 2I_0 e^{\nu \lambda_1 t} + 2 e^{\nu \lambda_1 t} \left( \frac{\nu \lambda_1 (1 - \rho^*)}{4 e^{\nu \lambda_1 h}} \right) \left( |v(t)\|_2^2 + e^{\nu \lambda_1 t} \|v(t)\|_2^2 + \|v(t)\|_2^2 \right)
\]
\[
+ \frac{4 e^{\nu \lambda_1 h} |v(t)\|_2^2}{\nu \lambda_1 (1 - \rho^*)} \left( \int_0^t e^{-\kappa \lambda \tau} \|\varphi\|_{L^\infty(\partial \Omega)}^2 \|\varphi\|_{L^\infty(\partial \Omega)}^2 \right) + \frac{1}{\delta} \int_{t-h}^t e^{-\kappa \lambda (s,t)} \beta(s) ds. \]
• Some new hypotheses:

(H-e) Denoting
\[
\limsup_{\tau \to -\infty} \frac{1}{t - \tau} \int_\tau^t \alpha(r) dr = \alpha_0 \in [0, +\infty),
\]  
(64)
for arbitrary \( t \in \mathbb{R} \), then there exists some \( \delta > 0 \) such that
\[
\frac{4e^{\nu \lambda_1 h} \alpha_0}{\nu \lambda_1 (1 - \rho^*)} + \delta < \frac{\nu \lambda_1}{4},
\]  
(65)
(H-f) The function \( \beta(\cdot) \) satisfies the pullback tempered condition
\[
\int_{-\infty}^t e^{-\kappa_d(t,s)} \beta(s) ds < +\infty.
\]  
(66)

Remark 2. A stronger assumption on \( \alpha(\cdot) \) than (H-e) is
\[
\frac{\nu \lambda_1}{4} - \frac{4e^{\nu \lambda_1 h}}{\nu \lambda_1 (1 - \rho^*)} \|\alpha\|_{L^\infty(-\infty,T)} > 0
\]  
(67)
for \( \alpha \in L^\infty(-\infty,T) \), which is sufficient to derive Theorem 4.6.

• The tempered pullback absorbing set:

Our results are concerned with families of universes determined by \( \beta(\cdot) \) and some other terms.

Definition 4.7 (The tempered universe in \( M^2_H \)). Denote by \( \mathcal{P}(M^2_H) \) the collection of all nonempty subsets in \( M^2_H \), \( \hat{D} = \{ D(t) \} \subset \mathcal{P}(M^2_H) \), and let \( B(0, \rho_{\hat{D}}(t)) \) denote a family of balls at center 0 with radius \( \rho_{\hat{D}}(t) \) satisfying \( |\rho_{\hat{D}}(t)|^2 e^{-\kappa_d(t,s)} \to 0 \) as \( t \to -\infty \).

Our \( \kappa_d(t,\tau) \)-indexed universes are defined by
\[
\mathcal{D}^{M^2_H}_{\kappa_d(t,\tau)} = \left\{ \hat{D}_{\kappa_d(t,\tau)} \mid D_{\kappa_d(t,\tau)}(t) \subset B(0, \rho_{\hat{D}_{\kappa_d(t,\tau)}}(t)) \text{ with } \lim_{\tau \to -\infty} |\rho_{\hat{D}_{\kappa_d(t,\tau)}}(t)|^2 e^{-\kappa_d(t,\tau)} = 0 \right\},
\]  
(68)
where \( \rho_{\hat{D}_{\kappa_d(t,\tau)}} : \mathbb{R} \to \mathbb{R}^+ \) is a continuous function and \( B(0, \rho_{\hat{D}_{\kappa_d(t,\tau)}}(t)) \) denotes a family of closed ball in \( M^2_H \) with radius \( \rho_{\hat{D}_{\kappa_d(t,\tau)}}(t) \).

Definition 4.8 (The tempered universe in \( C_H \)). Denote by \( \mathcal{P}(C_H) \) the collection of all nonempty subsets in \( C_H \), \( \hat{D}^* = \{ D^*(t) \} \subset \mathcal{P}(C_H) \), and let \( B(0, \tilde{\rho}_{\hat{D}^*}(t)) \) denote a family of balls at center 0 with radius \( \tilde{\rho}_{\hat{D}^*}(t) \) satisfying \( |\tilde{\rho}_{\hat{D}^*}(t)|^2 e^{-\kappa_d(t,s)} \to 0 \) as \( t \to -\infty \).

Our \( \kappa_d(t,\tau) \)-indexed universes are defined by
\[
\mathcal{D}^{C_H}_{\kappa_d(t,\tau)} = \left\{ \hat{D}^*_{\kappa_d(t,\tau)} \mid D^*_{\kappa_d(t,\tau)}(t) \subset B(0, \tilde{\rho}_{\hat{D}^*_{\kappa_d(t,\tau)}}(t)) \text{ with } \lim_{\tau \to -\infty} |\tilde{\rho}_{\hat{D}^*_{\kappa_d(t,\tau)}}(t)|^2 e^{-\kappa_d(t,\tau)} = 0 \right\},
\]  
(69)
where \( \tilde{\rho}_{\hat{D}^*_{\kappa_d(t,\tau)}} : \mathbb{R} \to \mathbb{R}^+ \) is a continuous function and \( B(0, \tilde{\rho}_{\hat{D}^*_{\kappa_d(t,\tau)}}(t)) \) denotes a family of closed ball in \( C_H \) with radius \( \tilde{\rho}_{\hat{D}^*_{\kappa_d(t,\tau)}}(t) \).
Remark 3. By the properties of the universes $\mathcal{D}_{\kappa_5(t,\tau)}^{M_{2q}^2}$ and $\mathcal{D}_{\kappa_4(t,\tau)}^{C_H^2}$ in (68) and (69) respectively, we can easily verify that the above universes are inclusion closed and contain all corresponding fixed universes.

Theorem 4.9. (The tempered pullback absorbing set in $M_{2q}^2$) Suppose that $g(t,x)$ and $f : \mathbb{R} \times C_H \to H$ satisfy the hypotheses (I)-(VI). Moreover, let the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfy (H-e)-(H-f). Then for any $v \in \mathcal{D}(\tau, v_\tau, \eta_\tau)$ and $(v, \phi) \in M_{2q}^q$, there exists a time $\tau_0(B) \leq t$ for any family $B = \{B(t)\} \subset \mathcal{D}_{\kappa_4(t,\tau)}^{M_{2q}^2}$ such that we have the tempered pullback absorbing set $\mathcal{B}_{M_{2q}^2}(0, \mathcal{R}_{M_{2q}^2}(t))$ with radius $\mathcal{R}_{M_{2q}^2}(t)$ defined by

$$
\mathcal{R}_{M_{2q}^2}(t) = 1 + \frac{1}{\delta} \int_{-\infty}^{t} e^{-\kappa_4(t,s)} \beta(s) ds + \frac{C_\nu}{\nu} \|g\|^2_{v'},
$$

$$
\left[2C|\partial^2 \Omega|^2 + \frac{2C \varepsilon^2 \|\varphi\|^2_{L^\infty(\partial\Omega)}}{\nu} + \frac{C \nu \lambda_1 (1 - \rho^*)}{4\varepsilon \nu \lambda_1} \right] \|\varphi\|^2_{L^\infty(\partial\Omega)}
$$

for any $t \leq \tau_0(B)$, i.e., $|v(t)|_2^2 \leq \mathcal{R}_{M_{2q}^2}(t)$.

Proof. Since $\kappa_5(t, \tau)$ and $\kappa_4(t, \tau)$ are positive, then from the definition of the universe $\mathcal{D}_{\kappa_4(t,\tau)}^{M_{2q}^2}$ and Theorem 4.6, we can easily derive the tempered pullback absorbing ball $\mathcal{B}_{M_{2q}^2}(0, \mathcal{R}_{M_{2q}^2}(t))$. □

Theorem 4.10. (Tempered pullback absorbing sets in $C_H$) Suppose that $g(t,x)$ and $f : \mathbb{R} \times C_H \to H$ satisfy the hypotheses (I)-(VI). Moreover, let the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfy (H-e)-(H-f). Then for any family $\mathcal{B}_\tau = \{B^\tau(t)\} \subset \mathcal{D}_{\kappa_4(t,\tau)}^{C_H^2}$, we have the tempered pullback absorbing set $\mathcal{B}_{C_H}(0, \mathcal{R}_{C_H}(t))$ with radius $\mathcal{R}_{C_H}(t)$ defined by

$$
\mathcal{R}_{C_H}(t) = \sup_{r \in [t-h,t]} \mathcal{R}_{M_{2q}^2}(r)
$$

in the space $C_H$ for the process $(U(t, \tau), C_H)$.

Proof. From the definition of the radius for the absorbing ball by (71), it is easy to deduce the result by virtue of Theorem 4.9. □

4.3. Pullback asymptotic compactness in $M_{2q}^2$ and $C_H$. In this section, we will use the energy equation method (see [1], [29]) to deal with the pullback asymptotic compactness for $(U(t, \tau); M_{2q}^2)$ and $(U(t, \tau); C_H)$.

Theorem 4.11. Suppose that $g(t,x)$ and $f : \mathbb{R} \times C_H \to H$ satisfy the hypotheses (I)-(VI). Moreover, let the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfy (H-e)-(H-f). Then for any $v \in \mathcal{D}(\tau, v_\tau, \eta_\tau)$ and $(v, \phi) \in M_{2q}^2$, the processes $(U(t, \tau); M_{2q}^2)$ and $(U(t, \tau); C_H)$ generated by the global weak solutions are $\mathcal{D}$-pullback asymptotically compact.

Proof. Step 1: Convergence of the sequence $\{v^n(t, x)\}$ in the delay interval $[t_0 - h, t_0]$ for arbitrary $t_0 \geq \tau$ and weak convergence of $\{v^n(t)\}$ in $H$.

For a fixed $t_0 \geq \tau$, let $\{v^n\} \in \mathcal{D}(t, \tau_0; \eta_0(0), \eta_n)$ and $\{\eta_n\} \in \mathcal{D}_0(\tau_0)$ be bounded sequences in $M_{2q}^2$ with $\tau_0 \to -\infty$ as $n \to +\infty$, where $\{\tau_n\} \subset (-\infty, t_0 - 2h)$. We will show the convergence of the above sequence in what follows.
Using the Theorem 3.4, we can derive that
\[
\| (v^n)' \|_* \leq \nu \| v^n \| + \| B(v^n) \|_* + \| B(v^n, \psi) \|_* + \| B(\psi, v^n) \|_* + \frac{1}{\sqrt{\lambda_1}} |\tilde{F}(v^n(t - \rho(t)))|_2 + \frac{C}{\sqrt{\lambda_1}} |F|_2 + \| B(\psi) \|_* + \frac{C}{\nu} \| g \|_{V'},
\]
(72)

By a similar technique as in (22)-(23), this yields
\[
\| B(v^n) \|_* \leq C \| v^n \|_2 |t|, \quad (73)
\]
\[
\| B(\psi) \|_* \leq C \varepsilon \| \varphi \|_{L^\infty(\partial\Omega)} \| \partial \Omega \|^{1/2}. \quad (74)
\]
Moreover, we can also deduce that
\[
\| B(\psi, v^n) \|_* = \sup_{\| w \|_1 = 1} | b(\psi, v^n, w) | \leq \frac{C}{\sqrt{\lambda_1}} \| \varphi \|_{L^\infty(\partial\Omega)} \| v^n \|_2, \quad (75)
\]
and
\[
\| B(v^n, \psi) \|_* = \sup_{\| w \|_1 = 1} | b(v^n, \psi, w) | \leq C \| v^n \| \| \varphi \|_{L^\infty(\partial\Omega)}. \quad (76)
\]

By the diagonal procedure, there exists a subsequence (relabeled also as \( \{ v^n \} \)) which converges to the function \( v \in L^\infty(t_0 - 2h, t_0; H) \cap L^2(t_0 - 2h, t_0; V) \) with \( \frac{\partial v}{\partial t} \in L^2(t_0 - h, t_0; V') \) such that
\[
v^n \rightharpoonup v \text{ weakly in } L^\infty(t_0 - 2h, t_0; H), \quad (77)
\]
\[
v^n \rightarrow v \text{ weakly in } L^2(t_0 - 2h, t_0; V), \quad (78)
\]
\[
(v^n)' \rightharpoonup (v)' \text{ weakly in } L^2(t_0 - h, t_0; V'). \quad (79)
\]

By the Aubin-Lions Lemma, we see that \( v^n(t) \rightarrow v(t) \) strongly in \( H \) for almost all \( t \in (t_0 - 2h, t_0) \).

By Theorem 3.5, from the hypotheses \( (I) \sim (V) \) and the Lipschitz continuity of \( f(\cdot, \cdot) \), it follows that \( \tilde{F}(v^n(t - \rho(t))) \rightarrow \tilde{F}(v(t - \rho(t))) \) weakly and strongly in \( H \). Thus, we conclude that \( v \in C([t_0 - h, t_0]; H) \).

From the uniform estimate on \( v^n \) in \( C([t_0 - h, t_0]; H) \), we have
\[
v^n(s) \rightarrow v(s) \text{ weakly in } H \quad \text{for an arbitrary } s \in [t_0 - h, t_0]. \quad (80)
\]

Hence for any \( \{ s_n \} \subset [t_0 - h, t_0], s_n \rightarrow s \in [t_0 - h, t_0], \) we conclude that
\[
v^n(s_n) \rightarrow v(s) \text{ weakly in } H \quad \text{for } s \in [t_0 - h, t_0], \quad (81)
\]
which implies
\[
\liminf_{n \to \infty} \| v^n(t_n) \|_2 \geq |v(t*)|_2. \quad (82)
\]

**Step 2:** Strong convergence of the corresponding sequences via the energy equation method.

Next, we prove the asymptotic compactness in \( H \), i.e., the strong convergence of \( v^n \) in \( H \):
\[
|v^n(t_n) - v(t*)|_2 \rightarrow 0 \text{ as } n \to +\infty. \quad (83)
\]

To achieve our goal, it is equivalent to prove (82), combined with
\[
\limsup_{n' \to \infty} \| v^n(t_n) \|_2 \leq |v(t*)|_2 \quad (84)
\]
for a sequence \( \{ t_n \} \subset [t_0 - h, t_0] \) and \( t_n \rightarrow t^* \) as \( n \to +\infty \), which will be proved in what follows.
Integrating from $t$ to $\infty$, we obtain
\begin{equation}
\frac{d}{dt} |z(t)|^2 = -2\nu\|z(t)\|^2 - 2b(z, v, z) - 2b(z, \psi, z) + 2(P(\tilde{F}(v^n(t - \rho(t))) - \tilde{F}(v(t - \rho(t))))), z(t)).
\end{equation}

Integrating from $s$ to $t$ yields
\begin{equation}
|z(t)|^2 = |z(s)|^2 - 2\int_s^t \left( \nu \|z(r)\|^2 + b(z, v, z) + b(z, \psi, z) \right) dr + 2\int_s^t (P(\tilde{F}(v^n(r - \rho(r) + \psi)) - \tilde{F}(v(r - \rho(r)) + \psi), z(r)) dr.
\end{equation}

Considering the functionals defined for $t \in [t_0 - h, t_0]$ by
\begin{align*}
J(t) &= |v(t_0 - h)|^2 + 2\int_{t_0 - h}^t (P\tilde{F}(v(r - \rho(r)) + \psi) + \nu F, v(r)) dr \\
&\quad - 2\int_{t_0 - h}^t \left( \nu \|v(r)\|^2 + b(v, \psi, v) + b(\psi, v, v) \right) dr,
\end{align*}
and
\begin{align*}
J_n(t) &= |v^n(t_0 - h)|^2 + 2\int_{t_0 - h}^t (P\tilde{F}(v^n(r - \rho(r)) + \psi) + \nu F, v^n(r)) dr \\
&\quad - 2\int_{t_0 - h}^t \left( \nu \|v^n(r)\|^2 + b(v^n, \psi, v^n) + b(\psi, v^n, v^n) \right) dr,
\end{align*}

since $f(t, w(t - \rho(t)) + \psi)$ is Lipschitz continuous with respect to $w$, we can derive the convergence
\begin{equation}
2\int_{t_0 - h}^t (P\tilde{F}(v^n(r - \rho(r)) + \psi), v^n(r)) dr \\
\rightarrow 2\int_{t_0 - h}^t (P\tilde{F}(v(r - \rho(r)) + \psi), v(r)) dr.
\end{equation}
as $n \to +\infty$.

Concerning the trilinear operator, we have $\int_{t_0 - h}^t b(\psi, v, v^n) dr \to \int_{t_0 - h}^t b(\psi, v, v) dr$, as well as
\begin{equation}
b(v^n, \psi, v^n - v) \to 0, \quad b(v^n - v, \psi, v) \to 0,
\end{equation}
which implies
\begin{equation}
\int_{t_0 - h}^t b(v^n, \psi, v^n) dr \to \int_{t_0 - h}^t b(v, \psi, v) dr
\end{equation}
as $n \to +\infty$.

The other terms in (88) and (89) are linear. Using the convergence of sequence \{v^n\} in Step 1, we can obtain
\begin{equation}
J_n(t) \to J(t)
\end{equation}
for $t \in [t_0 - h, t_0]$.
for almost every $t \in [t_0 - h, t_0]$, i.e., $\forall \varepsilon > 0$, there exists a $n_k \in \mathbb{N}$, for all $n \geq n_k$ and $t_k \in [t_0 - h, t_0]$, such that

$$|J_n(t_k) - J(t_k)| \leq \frac{\varepsilon}{2}. \quad (94)$$

By the argument in step 1, we know that $J(t)$ is continuous and $J_n(t)$ is uniformly continuous with respect to time. Thus for any $\varepsilon > 0$, there exists $n_k \in \mathbb{N}$ such that for the sequence $\{t_k\} \subset [t_0 - h, t_0]$ with $t_k \to t^*$ for all $n \geq n_k$, we have

$$|J(t_k) - J(t^*)| \leq \frac{\varepsilon}{2}. \quad (95)$$

Choosing $\tilde{n}_k = \max\{n_k, n_k\}$, then for all $n > \tilde{n}_k$, we have

$$|J_n(t_n) - J(t^*)| \leq |J_n(t_n) - J(t_n)| + |J(t_n) - J(t^*)| < \varepsilon. \quad (96)$$

Hence, for any $\{t_n\} \subset [t_0 - h, t_0]$, we obtain (84), i.e. the norm convergence

$$|v^n(t_n)|_2 \to |v(t^*)|_2. \quad (97)$$

Combining (97) and (81), we can achieve (83), which gives the asymptotic compactness of the sequence $v^n(t_n)$ in $H$ and the interval $[t_0 - h, t_0]$ for all $t_0 \in \mathbb{R}$. This finishes the proof.

**Remark 4.** Under the hypotheses in Theorem 4.11, we can easily verify that the processes $(U(t, \tau); M^2_H)$ and $(U(t, \tau); C_H)$ are $\mathcal{D}^{M^2_H}_{\kappa_3(t, \tau)}$ and $\mathcal{D}^{C_H}_{\kappa_3(t, \tau)}$-pullback asymptotically compact respectively.

4.4. **Minimal and unique families of pullback attractors for problem (18).**

The existence of a minimal family of pullback attractors for problem (18) can be stated as follows.

**Theorem 4.12.** Suppose that $g(t, x)$ and $f : \mathbb{R} \times C_H \to H$ satisfy the hypotheses (I)-(VI). Moreover, let the functions $a(\cdot)$ and $\beta(\cdot)$ satisfy (H-e)-(H-f). Then for any $v \in \mathcal{D}(\tau, \nu, \eta)$ and $(\nu, \eta) \in M^2_H$, the process $(U(t, \tau); M^2_H)$ generated by the global weak solutions of problem (18) possesses a minimal family of pullback $\mathcal{D}^{M^2_H}_{\kappa_3}$-attractors $\mathcal{A}^{M^2_H}_{\kappa_3}$ in $M^2_H$, for all $\kappa_3 \in (0, \kappa_3(t, \tau))$.

**Proof.** Combining the existence of tempered pullback absorbing sets with the asymptotic compactness of the processes in Theorems 4.9 and 4.11 (Remark 4) respectively and using the abstract theory in Section 4.1, since the universe $\mathcal{D}^{M^2_H}_{\kappa_3}$ is inclusion closed, we can conclude on the existence of the minimal family of pullback attractors in $M^2_H$.

**Theorem 4.13.** Under the same hypotheses as in Theorem 4.12, for any $v \in \mathcal{D}(\tau, \nu, \eta)$ and $(\nu, \eta) \in C_H$, the process $(U(t, \tau); C_H)$ generated by the global weak solutions of problem (18) possesses a minimal family of pullback $\mathcal{D}^{C_H}_{\kappa_3}$-attractors $\mathcal{A}^{C_H}_{\kappa_3} = \{\mathcal{A}^{C_H}_{\kappa_3}(t)\}$ in $C_H$, for all $\kappa_3$.

**Proof.** Combining the existence of tempered pullback absorbing sets with the asymptotic compactness of the processes in Theorems 4.10 and 4.11 (Remark 4) respectively and using the abstract theory in Section 4.1 and the inclusion closed property of the universe, we can conclude the existence of the minimal family of pullback attractors in $C_H$. \qed
4.5. Back to the original problem (1): the family of pullback attractors.

**Theorem 4.14.** Suppose that $g(t, x)$ and $f: \mathbb{R} \times C_H \to H$ satisfy the hypotheses (I)-(VI). Moreover, let the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfy (H-e)-(H-f). Then for any $(u_\tau, \phi) \in M^{2q}_H$, the family of processes $\{S(t, \tau)\}$ generated by the global weak solutions of problem (1) possesses minimal families of pullback $D_{\kappa_3}$-attractors in $M^{2q}_H$ and $C_H$ for all $\kappa_3$.

**Proof.** Since we use a linear transformation to deal with non-homogeneous boundary conditions in Lipschitz-like domains and the background function $\psi(x)$ is a solution of problem (9) which is bounded in $H$, then we easily verify the existence of pullback attractor for problem (1). □

5. **Further research.** For the 2D incompressible Navier-Stokes equations with sublinear operators in Lipschitz-like domains (problem (1)), we obtained the minimal family of pullback attractors, which is different from the global attractor in [3] and pullback attractors in [25]. One natural problem is the gap between these objects. Furthermore, the dynamics for this problem in unbounded domain is unknown.

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**REFERENCES**

[1] J. M. Ball, Global attractors for damped semilinear wave equations, *Disc. Cont. Dyn. Syst.*, 10 (2004), 31–52.

[2] V. Barbu and S. S. Sritharan, Navier-Stokes equations with hereditary viscosity, *Z. Angew. Math. Phys.*, 54 (2003), 449–461.

[3] R. M. Brown, P. A. Perry and Z. Shen, On the dimension of the attractor of the non-homogeneous Navier-Stokes equations in non-smooth domains, *Indian University Math. J.*, 49 (2000), 81–112.

[4] T. Caraballo and X. Han, A survey on Navier-Stokes models with delays: existence, uniqueness and asymptotic behavior of solutions, *Disc. Cont. Dyn. Syst. S*, 8 (2015), 1079–1101.

[5] T. Caraballo and G. Kiss, Attractors for differential equations with multiple variable delays, *Disc. Cont. Dyn. Syst.*, 33 (2013), 1365–1374.

[6] T. Caraballo, J. A. Langa and J. C. Robinson, Attractors for differential equations with variable delays, *J. Math. Anal. Appl.*, 260 (2001), 421–438.

[7] T. Caraballo, P. Marín-Rubio and J. Valero, Autonomous and non-autonomous attractors for differential equations with delays, *J. Differential Equations*, 208 (2005), 9–41.

[8] T. Caraballo, P. Marín-Rubio and J. Valero, Attractors for differential equations with unbounded delays, *J. Differential Equations*, 239 (2007), 311–342.

[9] T. Caraballo and J. Real, Navier-Stokes equations with delays, *R. Soc. Lond. Proc., Ser. A, Math. Phys. Eng. Sci.*, 457 (2001), 2441–2453.

[10] T. Caraballo and J. Real, Asymptotic behavior for two-dimensional Navier-Stokes equations with delays, *R. Soc. Lond. Proc., Ser. A, Math. Phys. Eng. Sci.*, 459 (2003), 3181–3194.

[11] T. Caraballo and J. Real, Attractors for 2D Navier-Stokes models with delays, *J. Differential Equations*, 205 (2004), 271–297.

[12] A. N. Carvalho, J. A. Langa and J. C. Robinson, *Attractors for Infinite-Dimensional Non-Autonomous Dynamical Systems*, Springer, New York–Heidelberg–Dordrecht–London, 2013.
[13] E. B. Fabes, C. E. Kenig and G. C. Verchota, The Dirichlet problem for the Stokes system on Lipschitz domains, *Duke Math. J.*, 57 (1988), 769–793.

[14] C. Foias, O. Manley, R. Rosa and R. Temam, *Navier-Stokes Equations and Turbulence*, Cambridge University Press, Cambridge, 2001.

[15] J. García-Luengo, P. Marín-Rubio and J. Real, Pullback attractors for 2D Navier-Stokes equations with delays and their regularity, *Adv. Nonlinear Stud.*, 13 (2013), 331–357.

[16] J. García-Luengo, P. Marín-Rubio and J. Real, Some new regularity results of pullback attractors for 2D Navier-Stokes equations with delays, *Comm. Pure Appl. Anal.*, 14 (2015), 1603–1621.

[17] J. García-Luengo, P. Marín-Rubio and G. Planas, Attractors for a double time-delayed 2D-Navier-Stokes model, *Disc. Cont. Dyn. Syst.*, 34 (2014), 4085–4105.

[18] J. K. Hale, History of Delay Equations, Conference Proceedings of *Delay Differential Equations and Applications*, NATO Sci. Ser. II Math. Phys. Chem., 205, Springer, Dordrecht, 2006, 1–28.

[19] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, 1993.

[20] O. A. Ladyzhenskaya, *The Mathematical Theory if Viscous Incompressible Flow*, NGordon and Breach Science Publishers, New York-London 1963.

[21] O. Ladyzhenskaya, *Attractors for Semigroup and Evolution Equations*, Cambridge Uni. Press, Cambridge, 1991. Springer, second edition, 1991.

[22] J. Leray, Sur le mouvement d’un liquide visqueux emplissant l’espace, *Acta Math.*, 63 (1934), 193–248.

[23] D. Li, Q. Liu and X. Ju, Uniform decay estimates for solutions of a class of retarded integral inequalities, *J. Differential Equations*, 271 (2021), 1–38.

[24] J.-L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod, Gauthier-Villars, Paris 1969.

[25] P. Marín-Rubio and J. Real, Attractors for 2D-Navier-Stokes equations with delays on some unbounded domains, *Nonlinear Anal.*, 67 (2007), 2784–2799.

[26] P. Marín-Rubio and J. Real, Pullback attractors for 2D Navier-Stokes equations with delay in continuous and sub-linear operators, *Disc. Cont. Dyn. Syst.*, 26 (2010), 989–1006.

[27] A. Miranville and X. Wang, Upper bounded on the dimension of the attractor for non-homogeneous Navier-Stokes equations, *Disc. Cont. Dyn. Syst.*, 2 (1996), 95–110.

[28] A. Miranville and X. Wang, Attractors for non-autonomous non-homogeneous Navier-Stokes equations, *Nonlinearity*, 10 (1997), 1047–1061.

[29] I. Moise, R. Rosa and X. Wang, Attractors for non-compact semigroups via energy equations, *Nonlinearity*, 11 (1998), 1369–1393.

[30] J. C. Robinson, *Infinite-Dimensional Dynamical Systems*, Cambridge Univ. Press, Cambridge, 2001.

[31] J. C. Robinson, Attractors and finite-dimensional behaviour in the 2d Navier-Stokes equations, *ISRN Math Anal.*, 203 (2013), 291823.

[32] Z. Shen, A note on the Dirichlet problem for the Stokes system in Lipschitz domains, *Proc. Amer. Math. Soc.*, 123 (1995), 801–811.

[33] T. Taniguchi, The exponential behavior of Navier-Stokes equations with time delay external force, *Disc. Cont. Dyn. Syst.*, 12 (2005), 997–1018.

[34] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Second edition, Springer-Verlag, New York, 1997.

[35] R. Temam, *Navier-Stokes Equations, Theory and Numerical Analysis*, Reprint of the 1984 edition. AMS Chelsea Publishing, Providence, RI, 2001.

[36] Y. Wang, X-G. Yang and X. Yan, Dynamics of 2D Navier-Stokes equations with Rayleigh's friction and distributed delay, *Electronic J. Differential Equations*, 2019 (2019), Paper No. 80, 18 pp.

[37] X.-G. Yang, B. Feng, T. Maier de Souza and T. Wang, Long-time dynamics for a non-autonomous Navier-Stokes-Voigt equations in Lipschitz domain, *Disc. Cont. Dyn. Syst. B*, 24 (2019), 363–386.

[38] X.-G. Yang, B. Guo, C. Guo and D. Li, The fractal dimension of pullback attractors for the 2D Navier-Stokes equations with delay, *Math. Meth. Appl. Sci.*, 43 (2020), 9637–9653.
[39] X.-G. Yang, Y. Qin, Y. Lu and T. F. Ma, Dynamics of 2D incompressible non-autonomous Navier-Stokes equations on Lipschitz-like domains, *Appl. Math. & Optimization*, 2019, 1–55.

[40] X.-G. Yang and S. Wang, Well-posedness for the 2D non-autonomous incompressible fluid flow in Lipschitz-like domain, *J. Partial Differential Equations*, 32 (2019), 77–92.

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