Design of Tight Minimum-Sidelobe Windows by Riemannian Newton’s Method

Daichi Kitahara, Member, IEEE, and Kohei Yatabe, Member, IEEE

Abstract—The short-time Fourier transform (STFT), or the discrete Gabor transform (DGT), has been extensively used in signal analysis and processing. Their properties are characterized by a window function. For signal processing, designing a special window called tight window is important because it is known to make DGT-domain processing robust to error. In this paper, we propose a method of designing tight windows that minimize the sidelobe energy. It is formulated as a constrained spectral concentration problem, and a Newton’s method on an oblique manifold is derived to efficiently obtain a solution. Our numerical example showed that the proposed algorithm requires only several iterations to reach a stationary point.

Index Terms—Short-time Fourier transform (STFT), discrete Gabor transform (DGT), tight frame, spectral concentration, Slepian window, oblique manifold, manifold optimization.

I. INTRODUCTION

T HE short-time Fourier transform (STFT), or the discrete Gabor transform (DGT) [1]–[5], has been extensively used as the standard tool for signal analysis and processing [6]–[14]. The major advantage of STFT/DGT over the Fourier transform is its ability to localize the frequency components of a signal at each time. This is realized by a tapering function called window whose energy is concentrated in the time-frequency domain. Since the properties of STFT/DGT are characterized by the used window, many windows have been proposed in the literature [15]–[25].

While most of the existing windows have been designed for signal analysis, windows for synthesis are equally important for processing. STFT/DGT-domain signal processing modifies STFT/DGT coefficients and then converts it back to the time domain. In the forward and inverse transformations, the signal is multiplied by analysis and synthesis windows, respectively. Therefore, both analysis and synthesis windows affect the result of STFT/DGT-domain processing.

The common practice is to firstly choose a window that is optimized for analysis, and then a synthesis window is generated to satisfy the perfect reconstruction condition. Such a window that perfectly reconstructs an unprocessed signal is said to be dual [1], [5]. The most used dual window for synthesis is the canonical, or minimum-norm, dual window, and hence its computation is well-studied [27]–[30]. However, such canonical windows are not optimized for processing, i.e., there must exist a better dual window suitable for signal processing. Therefore, methods for designing other types of dual windows have also been proposed [31], [32].

An important class of windows for signal processing is tight window [28]. A tight window can perfectly reconstruct a signal by using it for both analysis and synthesis, i.e., it is self-dual. According to the frame theory, tight windows can make DGT robust to error of processing [33], [34]. Therefore, their design should be an important topic for DGT-domain processing. However, design of tight windows has been rarely studied until recently [32], [35], [36] probably due to the complicated nature of the set of tight windows.

In this paper, we propose a method of designing tight windows that minimize the sidelobe energy. Since a set of tight windows can be represented as an oblique manifold, we formulate the window design as a manifold optimization problem. Then, we derive a Riemannian Newton’s method that can compute an optimal tight window by several iterations.

II. PRELIMINARIES

Let $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$ be the sets of all real numbers, all complex numbers, and all nonnegative integers, respectively. The imaginary unit is denoted by $i \in \mathbb{C}$, i.e., $i^2 = -1$. For any complex number $c \in \mathbb{C}$, $\bar{c}$ and $|c| := \sqrt{\bar{c}c}$ denote the complex conjugate and the absolute value of $c$. We write vectors and matrices by boldface lowercase and uppercase letters, respectively, and their transposes are denoted by $(\cdot)^T$. The identity matrix of order $N$ is denoted by $I_N \in \{0, 1\}^{N \times N}$, and a zero matrix of size $M \times N$ is denoted by $O_{M \times N} \in \{0\}^{M \times N}$. For any real vectors $x, y \in \mathbb{R}^N$, the Euclidean metric, or the standard inner product, is defined as $\langle x, y \rangle = x^Ty$, and the Euclidean norm of $x$ is defined as $\|x\|_2 = \sqrt{x^Tx}$. A sphere of radius $r$ in $\mathbb{R}^N$ is defined as $S^N_{r} = \{x \in \mathbb{R}^N : \|x\|_2 = r\}$. The floor and ceiling functions are denoted by $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$, respectively. The order of a function is denoted by $O(\cdot)$.

A. Discrete Gabor Transform and Its Inversion

Let $x := (x[0], x[1], \ldots, x[L-1])^T \in \mathbb{C}^L$ be a discrete-time signal of length $L$, and $w := (w[0], w[1], \ldots, w[K-1])^T \in \mathbb{C}^K$ be a window of length $K$ such that $w$ is shorter than $x$, i.e., $K < L$. In addition, let $a$ and $M$ be integers satisfying $L/a =: N \in \mathbb{N}$ and $0 < a < K \leq M \leq L$. In this paper, we define STFT/DGT of $x$ with respect to the window $w$ as

$$X[m, n] = \sum_{l=0}^{K-1} x[l + an] \overline{w[l]} e^{-2\pi iml/M},$$

where $m = 0, 1, \ldots, M - 1$, $n = 0, 1, \ldots, N - 1$, and the signal $x[l]$ is treated periodically as $x[l + L] := x[l]$. 

1To be accurate, a tight window is a window that generates a tight Gabor frame [1]–[5]. We use the term “tight window” throughout the paper because our focus is on a window and not on the associated frame.
The inverse DGT is defined with a synthesis window \( \gamma \) as
\[
x[l] = \sum_{n=[(l-K+1)/a]}^{[(l/a)]} \gamma[l-an] \sum_{m=0}^{M-1} X[m, n] e^{2\pi im(l-an)}/M,
\]
where \( X[m, n] \) is treated as \( X[m, n] := X[m, N-n] \) for \( n = 1, 2, \ldots, [(K-1)/a] \). This transform reconstructs the original signal if \( \gamma \) is dual of \( w \). When a window \( w \) is self-dual (i.e., signals can be reconstructed by using \( w/\lambda \) in place of \( \gamma \) with some \( \lambda > 0 \)), it is called a tight window. A tight window \( w_t \) can be obtained by the following projection:
\[
w_t = \sqrt{\lambda} S_w^{-1/2} w,
\]
where \( S_w \in \mathbb{R}^{K \times K} \) is a diagonal matrix corresponding to the frame operator \( [4] \), and its diagonal components are given by
\[
[S_w]_{l,l} = \begin{cases} \frac{[K-l-1]/a}{} & (l = 0, 1, \ldots, K-1), \\ 1 & (l = K). \end{cases}
\]
When \( \lambda = 1 \), the tight window \( w_t \) obtained by (3) is called the canonical tight window \( [28] \). In this paper, we assume that the window \( w \) is real-valued \(^4\) for simplicity, i.e., \( w, \gamma \in \mathbb{R}^K \).

B. Newton's Method on Riemannian Manifold

Let \( M \subseteq \mathbb{R}^N \) be a Riemannian manifold embedded in the Euclidean space \( \mathbb{R}^N \) where the tangent space to \( M \) at \( x \in M \) is denoted by \( T_x M \). Let \( h : \mathbb{R}^N \to \mathbb{R} \) be a twice continuously differentiable function whose gradient in the Euclidean metric is \( \nabla h : \mathbb{R}^N \to \mathbb{R}^N \). We address the following problem on \( M \):
\[
\text{minimize } h(x) \quad \text{for } x \in M.
\]
To solve the problem in (5), we introduce the Riemannian gradient and Hessian of \( h \) on \( M \) \([17, 40]\). The Riemannian gradient at \( x \in M \) is a vector \( g_x \in T_x M \) \((\in \mathbb{R}^N)\) defined as
\[
g_x = (g_1(x), g_2(x), \ldots, g_N(x))^T = P_x (\nabla h(x)),
\]
where \( g_n : \mathbb{R}^N \to \mathbb{R} (n = 1, 2, \ldots, N) \) is a differentiable function (a component of \( g_x \)), and \( P_x : \mathbb{R}^N \to T_x M \) is the metric projection onto \( T_x M \). This projection is given as an orthogonal projection matrix \( P_x \in \mathbb{R}^{N \times N} \) since \( T_x M \) is a linear subspace for any \( x \in M \). Next, the Riemannian Hessian at \( x \in M \) is defined as a linear operator \( H_x : T_x M \to T_x M \) and can be expressed as some matrix \( H_x \in \mathbb{R}^{N \times N} \) satisfying
\[
H_x v = P_x \left( [\nabla g_1(x) \nabla g_2(x) \cdots \nabla g_N(x) ]^T v \right)
\]
for all \( v \in T_x M \). We also define a mapping \( R_x : T_x M \to M \) as \( R_x(v) = P_M (x+v) \), where \( P_M : \mathbb{R}^N \to M \) is the single-valued metric projection onto \( M \). From \([38]\) Propositions 5.43, 5.47, and 5.48, \( R_x \) is a second-order retraction on \( M \), and
\[
h(R_x(v)) = h(x) + \langle g_x, v \rangle + \frac{1}{2} \langle H_x v, v \rangle + O(\|v\|^2)
\]
holds for all \( x \in M \) and all \( v \in T_x M \). If the inverse operator \( H_x^{-1} : T_x M \to T_x M \) of \( H_x \) exists, a stationary point \( v \in T_x M \) of the function \( h \circ R_x : T_x M \to \mathbb{R} \) is expressed as \( v = -H_x^{-1} g_x \) by ignoring the last term of (8).

The Riemannian Newton’s method solves (5) by iterating \( x^{(n+1)} = R_x (x^{(n)} - H_x^{-1} g_x^{(n)}) \) (9) from a given initial value \( x^{(0)} \in M \) until \( (x^{(i)}) \in \mathbb{N} \) converges. If \( x^{(0)} \) is sufficiently close to a stationary point \( x^* \in M \) of \( h \) (i.e., \( g_{x^*} = 0 \)), then \( (x^{(i)}) \in \mathbb{N} \) generated by (9) converges to \( x^* \) at least quadratically \([37] \) Theorem 6.3.2. Since a solution to the problem in (5) is also a stationary point of \( h \) on \( M \), the problem is quickly solved if we can give a proper \( x^{(0)} \).

III. TIGHT MINIMUM-SIDELobe WİNDOW

We propose a class of tight windows that minimize the sidelobe energy. It is defined as a solution to the spectral concentration problem under the constraint of tightness.

A. Spectral Concentration Problem and Slepian Window

Define the discrete-time Fourier transform of \( w \) as
\[
\hat{w}(f) = \sum_{l=0}^{K-1} w[l] e^{-2\pi i f l} \quad \text{for } f \in [-1/2, 1/2],
\]
and let \( p \in (0, 1) \) be a given proportion of the mainlobe to the entire frequency. To minimize the sidelobe energy of \( \hat{w} \), the following spectral concentration problem is considered:
\[
\max_{w \in \mathbb{R}^{K \setminus \{0\}}} \int_{-p/2}^{p/2} |\hat{w}(f)|^2 \, df.
\]
Note that this maximization of the mainlobe energy is equivalent to minimization of the sidelobe energy. The numerator of the cost function of (11) can be expressed as
\[
\int_{-p/2}^{p/2} |\hat{w}(f)|^2 \, df = \sum_{l=0}^{K-1} \sum_{l' = 0}^{K-1} w[l]w[l'] \int_{-p/2}^{p/2} e^{-2\pi i f (l-l')} \, df
\]
\[
= p \sum_{l=0}^{K-1} w[l]w[l'], \quad \text{where } Q_p := (p \text{sinc}(p(l-l'))) = \left( \frac{\sin(\pi p(l-l'))}{\pi p(l-l')} \right) \in \mathbb{R}^{K \times K}
\]
is a positive-definite symmetric matrix. For \( p = 1 \), we have
\[
\int_{-1/2}^{1/2} |\hat{w}(f)|^2 \, df = w^T Q_1 w = w^T I_K w = w^T w.
\]
Thus, the problem in (11) maximizes the Rayleigh quotient,
\[
\max_{w \in \mathbb{R}^{K \setminus \{0\}}} \frac{w^T Q_p w}{w^T w},
\]
which is equivalent, by letting \( w^T w = \|w\|^2 = 1 \), to
\[
\max_{w \in \mathbb{R}^{K \setminus \{0\}}} w^T Q_p w.
\]
The solution \( w_{S,p} \) to the problems in (12) and (13) is the first principal eigenvector of \( Q_p \) and is symmetric, i.e., \( w_{S,p}[l] = w_{S,p}[K-l-1] \) for \( l = 0, 1, \ldots, K-1 \) \([10]\). This minimum-sidelobe window \( w_{S,p} \) is called the Slepian window.

\(^4\)The sampling interval is set to 1, but it can be changed to any length.
B. Proposed Tight Minimum-Sidelobe Window

Although the Slepian window is optimal in terms of sidelobe energy, its canonical tight window has poor energy concentration as will be illustrated in Section IV. This is because the projection onto the set of tight windows in (3) does not take the spectral characteristics into account. Thus, we propose to find a minimum-sidelobe window within all tight windows.

Here, we assume \( K/a =: J \in \mathbb{N} \) for simplicity, but this assumption can be removed. From (3) and its periodicity with period \( a \), the condition for the tightness of \( \tilde{w} \) is given by

\[
M \sum_{n=0}^{J-1} |w[l + an]|^2 = M \|w_l\|_2^2 = \lambda \quad (l = 0, 1, \ldots, a - 1),
\]

with \( \lambda > 0 \) and \( w_l := (w[l], w[l+a], \ldots, w[l+(J-1)a])^T \in \mathbb{R}^J \) (\( l = 0, 1, \ldots, a - 1 \)). Without loss of generality, we set \( \lambda = M/a \). Then, the proposed window is formulated as a solution to the following constrained optimization problem:

\[
\maximize w^T Q_p w \quad \text{subject to } \forall l \|w_l\|^2 = 1/a,
\]  

which is the problem in (13) with the tightness constraint. To compute its solution, we derive a fast algorithm as follows.

C. Riemannian Newton’s Method for the Problem in (14)

To derive the fast algorithm, the problem in (14) is reformulated. First, we sort the components of \( w \) as \( \tilde{w} := (w^T_0, w^T_1, \ldots, w^T_{a-1})^T \in \mathbb{R}^K \) and accordingly sort \( Q_p \) to construct a positive-definite symmetric matrix \( \tilde{Q}_p \in \mathbb{R}^{K \times K} \) satisfying \( w^T Q_p w = \tilde{w}^T \tilde{Q}_p \tilde{w} \). Next, we express the tightness constraint \( \|w_l\|^2 = 1/a \) by a set

\[
\mathcal{M} := \left\{ (w^T_0, w^T_1, \ldots, w^T_{a-1})^T \in \mathbb{R}^K \mid \forall l \in S^{J-1}_1 \right\}
= S^{J-1}_1 \times S^{J-1}_1 \times \cdots \times S^{J-1}_1 = (S^{J-1}_1)^a.
\]  

This constraint set \( \mathcal{M} \), the direct product of spheres, is the Riemannian product manifold called oblique manifold [38]. Since \( \tilde{w} \in \mathcal{M} \Rightarrow w \in S^{J-1}_1 \), the problem in (14) is the same as a minimization problem on the oblique manifold \( \mathcal{M} \):

\[
\minimize_{w \in \mathcal{M}} -\frac{1}{2} w^T Q_p \tilde{w}.
\]  

The cost function \( h(\tilde{w}) := -\frac{1}{2} \tilde{w}^T \tilde{Q}_p \tilde{w} \) is twice continuously differentiable in the Euclidean space, and thus the Riemannian Newton’s method in (9) is applicable.

To apply it, we have to derive the metric projection onto the tangent space \( P_{\tilde{w}} \) in (6) and (7), the Riemannian gradient \( g_{\tilde{w}} \) in (9), the Riemannian Hessian \( H_{\tilde{w}} \) in (7), and the metric projection onto the oblique manifold \( P_{\mathcal{M}} \) in (9).

1) Metric Projection onto the Tangent Space: Since \( \mathcal{M} \) is a product manifold as shown in (15), the tangent space of \( \mathcal{M} \) is the direct product of those of the spheres \( S^{J-1}_1 \):

\[
T_{\tilde{w}} \mathcal{M} = T_{\tilde{w}_0} S^{J-1}_1 \times T_{\tilde{w}_1} S^{J-1}_1 \times \cdots \times T_{\tilde{w}_{a-1}} S^{J-1}_1.
\]  

where \( T_{\tilde{w}_i} S^{J-1}_1 = \{ v_i \in \mathbb{R}^J \mid \| v_i \|^2 = 0 \} \), and the metric projection \( P_{\tilde{w}_i} : \mathbb{R}^J \rightarrow T_{\tilde{w}_i} S^{J-1}_1 \) onto \( T_{\tilde{w}_i} S^{J-1}_1 \) is

\[
P_{\tilde{w}_i} = I_J - a w_i w_i^T.
\]

From (17) and (18), the metric projection \( P_{\tilde{w}} : \mathbb{R}^K \rightarrow T_{\tilde{w}} \mathcal{M} \) onto \( T_{\tilde{w}} \mathcal{M} \) can be expressed as a block-diagonal matrix

\[
P_{\tilde{w}} = I_K - a \tilde{w} \tilde{w}^T,
\]

where \( \tilde{w} = (w_0, w_1, \ldots, w_{a-1}) \in \mathbb{R}^{K \times a} \).

2) Riemannian Gradient: The Riemannian gradient of \( h \) at \( \tilde{w} \in \mathcal{M} \) is given by projecting the Euclidean gradient as

\[
g_{\tilde{w}} = P_{\tilde{w}}(\nabla h(\tilde{w})) = -(I_K - a \tilde{w} \tilde{w}^T) \tilde{Q}_p \tilde{w}
= -(\tilde{Q}_p - a \tilde{w}^T \tilde{Q}_p)^T \tilde{w} = -U_{\tilde{w}} \tilde{w}.
\]

3) Riemannian Hessian: According to (7) and (19), the Riemannian Hessian of \( h \) at \( \tilde{w} \in \mathcal{M} \) satisfies

\[
H_{\tilde{w}} v = -P_{\tilde{w}} U_{\tilde{w}} v + a P_{\tilde{w}} \left[ \nabla h_{\tilde{w}}(\tilde{w}) \quad \nabla h_{\tilde{w}}(\tilde{w}) \quad \cdots \quad \nabla h_{\tilde{w}}(\tilde{w}) \right] v
= -(I_K - a \tilde{w} \tilde{w}^T) U_{\tilde{w}} v
= -(U_{\tilde{w}} v - a \tilde{w} \tilde{w}^T \tilde{Q}_p v)
+ a^2 \tilde{w} \tilde{w}^T V \tilde{w}
= -(U_{\tilde{w}} - a \tilde{w} \tilde{w}^T \tilde{Q}_p) v
\]

for all \( v = (v_0^T, v_1^T, \ldots, v_{a-1}^T)^T \in T_{\tilde{w}} \mathcal{M} \), where we exploited two properties of the oblique manifold, \( P_{\tilde{w}} \tilde{w} = O_{K \times a} \) and \( \tilde{w} \tilde{w}^T V = O_{a \times a} \) with \( V = \text{diag}(v_0, v_1, \ldots, v_{a-1}) \in \mathbb{R}^{K \times a} \).

From (19) and (20), a stationary point \( v \in T_{\tilde{w}} \mathcal{M} \) in (15) is

\[
v = -H_{\tilde{w}}^{-1} g_{\tilde{w}} = -(U_{\tilde{w}} - a \tilde{w} \tilde{w}^T \tilde{Q}_p)^{-1} U_{\tilde{w}} \tilde{w}.
\]

Using the matrix inversion lemma, it can also be expressed as

\[
v = -\tilde{w} + \frac{1}{2} U_{\tilde{w}}^{-1} \left( \tilde{w} \tilde{w}^T U_{\tilde{w}}^{-1} \right)^{-1} \tilde{w},
\]

which guarantees \( v \in T_{\tilde{w}} \mathcal{M} \) from \( \tilde{w}^T v = 0 \), where \( 1 \in \mathbb{R}^a \) and \( 0 \in \mathbb{R}^a \) denote vectors whose components are all ones and zeros, respectively.

4) Proposed Algorithm: The metric projection \( P_{\mathcal{M}} \) onto the oblique manifold \( \mathcal{M} \) can be computed by applying the metric projection \( P_{S^{J-1}_1} \) onto the sphere \( S^{J-1}_1 \) to each subvector \( w_l \). Thus, the Riemannian Newton’s method in (9) is given by

\[
w_i^{(i+1)} = P_{S^{J-1}_1} \left( w_i^{(i)} + v_i^{(i)} \right) = \frac{w_i^{(i)} + \sqrt{\| v_i^{(i)} \|_2}}{\sqrt{\| w_i^{(i)} \|^2 + \| v_i^{(i)} \|_2}} v_i^{(i)},
\]

where \( v_i^{(i)} = (w_0^{(i)} T, w_1^{(i)} T, \ldots, w_{a-1}^{(i)} T)^T \in T_{\tilde{w}_i^{(i)}} \mathcal{M} \) is computed by substituting \( w_i^{(i)} \) into (21). The proposed algorithm is summarized in Fig. 1.

The initial tight window \( \tilde{w}^{(0)} \in \mathcal{M} \) is important for quickly obtaining a solution. We recommend two initializations. One is
Input: Initial tight window \( w \), time shift \( a \), and proportion \( p \).
1: Construct \( Q_p \).
2: Convert \( w \) and \( Q_p \) to \( \widetilde{w} \) and \( \widetilde{Q}_p \), respectively.
3: Construct \( U_{\widetilde{w}} \) and \( W \).
4: \( g_{\widetilde{w}} \leftarrow -U_{\widetilde{w}} \widetilde{w} \)
5: \( i \leftarrow 0 \)
6: while \( \| g_{\widetilde{w}} \|_2 > \delta \) and \( i < i_{\text{max}} \) do
7: \( v \leftarrow (U_{\widetilde{w}} - aW V^T \widetilde{Q}_p)^{-1} g_{\widetilde{w}} \)
8: \( w_j \leftarrow \frac{1}{\sqrt{\| w_{i-1} + v_j \|_2^2}} \) \( (l = 0, 1, \ldots, a - 1) \)
9: Construct \( U_{w_j} \) and \( W \).
10: \( g_{\widetilde{w}} \leftarrow -U_{w_j} w_j \)
11: \( i \leftarrow i + 1 \)
12: end while
13: Convert \( \widetilde{w} \) to \( w \).
14: Return \( w \).

Fig. 1. Proposed algorithm for computing tight minimum-sidelobe windows.

IV. NUMERICAL EXAMPLE

In this section, we show some windows designed by varying the mainlobe-width parameter \( p \in \{1/K, 2/K, \ldots, 20/K\} \) while fixing \( K = 512 \) and \( a = 128 \). For \( p = 1/K \), the initial value of the proposed algorithm was set to the tight window given from the Slepian window \( w_{S,p} \), i.e., \( w^{(0)}_i = P_{\mathcal{M}}(w_{S,p}) \). Since \( w_{S,p} \) is the solution to the problem in (13), if \( P_{\mathcal{M}}(w_{S,p}) \approx w_{S,p} \), the solution to the problem in (14) is expected to exist in the close neighborhood of \( w^{(0)} \). The other is to use a tight window \( w_{M,p} \) obtained by the proposed algorithm with a different parameter \( p \). Since a solution to the problem in (14) should change continuously with the change in \( p \), if \( p' \approx p \), \( w_{M,p} \) is expected to exist in the close neighborhood of \( w_{M,p'} \).

The norm of Riemannian gradient at each iteration is shown in Fig. 2. The color represents the mainlobe-width parameter \( p \). Figure 2(b) presents the norm of the Riemannian gradient at each iteration. The figure shows that the norm of the Riemannian gradient became smaller than \( 10^{-15} \), which indicates faster convergence. The numbers of iterations necessary to meet the stopping criterion are summarized in Table I. The proposed Newton’s algorithm was able to rapidly obtain a solution for \( p \leq 13/K \). The algorithm required more iterations for \( p \geq 14/K \), but it was still fast except for \( p = 19/K \). The instability for higher \( p \) should be because of the numerical ill-conditioning of \( \widetilde{Q}_p \) [16]. We empirically found that all windows were nonnegative and symmetric.

Note that the proposed Newton’s method for \( a = 1 \) coincides with the Rayleigh quotient iteration which converges to an eigenvector cubically [37]. That is, the proposed algorithm can be considered as its generalized version for the oblique manifold \( \mathcal{M} \), and hence the rapid convergence is expected.

Fig. 2. The value of the Euclidean norm of the Riemannian gradient at each iteration \( \| g_{\hat{w}}(i) \|_2 \). Since the required numbers of iterations were largely different for some \( p \) (see Table I), the figure was split into two parts: results for \( p = 1/K, \ldots, 13/K \) (left) and those for \( p = 14/K, \ldots, 20/K \) (right). The color represents \( p \), where \( 1/K \) is dark blue, and \( 20/K \) is yellow.

TABLE I

| \( p \) | \( t \) | \( p \) | \( t \) | \( p \) | \( t \) | \( p \) | \( t \) | \( p \) | \( t \) |
|---|---|---|---|---|---|---|---|---|---|
| 1/K | 2 | 6/K | 4 | 11/K | 5 | 16/K | 10 | 3/K | 4 |
| 2/K | 4 | 7/K | 4 | 12/K | 4 | 17/K | 17 | 4/K | 3 |
| 3/K | 3 | 8/K | 5 | 13/K | 6 | 18/K | 26 | 5/K | 4 |
| 4/K | 3 | 9/K | 6 | 14/K | 26 | 19/K | 266 | | |

Table I shows the number of iterations required to meet the stopping criterion.

Fig. 3. Comparison of the conventional and proposed windows (\( K = 512, a = 128, p = 1/K, \ldots, 20/K \)). From top to bottom, the Slepian window, its canonical tight window, and the proposed tight window are shown. The color represents the mainlobe-width parameter \( p \). The Nyquist frequency becomes 10^3.

Shapes\(^4\) and spectra of the obtained windows are shown in Fig. 3. The Slepian window (upper row) becomes narrower as \( p \) increases, which corresponds to the distance from the set of tight windows. Therefore, its canonical tight window (middle row) is more different for larger \( p \). This modification broke the energy-concentration property of the Slepian window as in the middle right figure. In contrast, the proposed window (bottom row) was able to narrow the mainlobe as the Slepian window while maintaining the tightness.

V. CONCLUSION

In this paper, we proposed a class of tight windows that minimize the sidelobe energy. Those windows are characterized by solutions to the energy minimization problem on the oblique manifold. We also proposed the Riemannian Newton’s method for rapidly computing them. Improvement of the stability of the algorithm for a large window-width parameter \( p \) as well as applications of the proposed windows to DGT-domain signal processing are left as future works.
