Two-dimensional topological solitons in soft ferromagnetic cylinders

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A simple approach allowing to construct closed-form analytical zero-field magnetization distributions in cylindrical particles of a small thickness and an arbitrary shape (not necessarily circular) is presented. The approach is based on reduction of the non-linear Euler equations for magnetization vector field to the classical linear Riemann-Hilbert problem. The result contains all the distributions minimizing the exchange energy functional and the surface magnetostatic contribution exactly, except for the neighbourhood of topological singularities on the cylinder faces where the result is approximate. The completeness of the analysis permitted to find a new type of a topological soliton in the case of circular cylinder. Also, an example of magnetic vortex in a triangular cylinder is given to investigate the role of the particle corners.

Small magnetic particles shaped as flat arbitrary cylinders with the size of the order of 100 nanometers and their arrays recently captured a lot of attention due to their unusual magnetic properties (as unique systems where two-dimensional topological solitons can be directly observed) and also because of their applications in the magnetic random access memory devices.

In large ferromagnetic bodies the equilibrium magnetization distributions are usually represented as a set of magnetic domains separated by possibly bent one-dimensional domain walls. Unfortunately, this approach is hard to apply to small magnetic particles if their size is such that the domain wall does not fit inside, resulting in considerable distortions of its profile with respect to the one-dimensional case. Consequently, it is necessary to consider this distribution as a whole. A number of finite-element simulations of the problem were already performed, see e.g. [2] and references therein.

In this Letter an analytical approach allowing to find closed-form expressions for equilibrium topologically charged magnetization distributions in soft magnetic particles shaped as flat arbitrary cylinders is presented.

Mathematical formulation of the problem. Consider a particle made of an isotropic magnetic material, the anisotropy of soft material is neglected here. In the case of no applied magnetic field its total energy has two contributions, the exchange and the dipolar (magnetostatic). Let us limit ourselves to the cylindrical particles with thickness $L < L_E = \sqrt{C/M_S^2}$ (it usually can be relaxed for $L$ up to several $L_E$), where $C$ is the exchange constant and $M_S$ is the saturation magnetization of the material. In this case the magnetization distribution $\mathbf{m}(\mathbf{r}) = \{m_X, m_Y, m_Z\} = \hat{M}(\mathbf{r})/M_S$, $|\mathbf{m}(\mathbf{r})| = 1$ can be assumed uniform along the cylinder thickness (Z axis), with $\mathbf{r} = \{X, Y\}$ the radius vector in the cylinder plane. In this notation the total energy of the particle (a functional of $\mathbf{m}$) can be written in the following way

$$
\frac{e[\mathbf{m}]}{M_S^2} = \iiint_D \left\{ \frac{L^2}{2} \sum_{i=X,Y} (\nabla m_i)^2 - \int_{-L/2}^{L/2} \hat{h}_D[\mathbf{m}] \cdot \mathbf{m} \ dz \right\} \ d^2r, \quad (1)
$$

where $D$ is the area of the cylinder face, $\nabla = \{\partial/\partial X, \partial/\partial Y, \partial/\partial Z\}$ is the gradient operator, $\hat{h}_D[\mathbf{m}] = \hat{H}_D[\mathbf{m}]/M_S$ is the demagnetizing field (a functional of $\mathbf{m}$) created by the magnetization distribution. The demagnetizing field can be expressed using the Maxwell equations through its scalar potential $\hat{H}_D = \nabla U$, which is, in turn, a solution of the Poisson equation $\hat{\nabla}^2 U = \rho$ with the requirement (due to the finite size of the particle) that both $|\mathbf{r}| U$ and $|\mathbf{r}|^2 \nabla U$ are finite as $|\mathbf{r}| \to \infty$, $\rho = \nabla \cdot \hat{M}$ is the density of magnetic charges (on the boundary equal to the normal component of $\hat{M}$).

The exact solution for equilibrium $\mathbf{m}(\mathbf{r})$ corresponds to the minimum of the functional (1) and is given by a system of non-linear integral partial differential equations. They are hard to handle even with the help of computer for a particular case. Thus, a simplification of the original problem (1) needs to be applied.

To make such simplification let us qualitatively analyze contributions of the different energy terms in (1). It turns out that for sufficiently small particle there is a well defined hierarchy of energies. The most important term is the exchange, as when the particle gets smaller not only the amount of possible magnetic charges decreases (on par with the exchange energy) but also (due to absence of magnetic monopoles) the positive self energy of charges gets more and more compensated by their negative interaction energy. It is also clear, as the role of surface increases for small particle sizes, that the surface magnetostatic contribution is more important then the volume one. The surface contribution can be subdivided into the energy of the magnetic charges on the faces and on the side of the cylinder. Because for flat cylinders the faces have usually larger area than the side, the former energy is more important. Thus, in flat cylinders we have the following energy hierarchy: exchange, face charges, side charges, volume charges. Note that the exact criteria for the necessary smallness of the cylinder strongly depend on its shape (due to the long-range character of the dipolar interactions) and should be analyzed.
on a case-by-case basis. Let us just assume for now that the
cylinder satisfies the criteria.

The solution for the problem exactly minimizing the
exchange energy functional \( \mathcal{E} \) with \( h_D = 0 \), and also
having negligible amount of the face magnetic charges
present only at the points of topological singularities was
given in [3] as

\[
w(z, \bar{z}) = \begin{cases} 
  f(z) & |f(z)| \leq 1 \\
  f(z)/\sqrt{f(z)\bar{f}(\bar{z})} & |f(z)| > 1
\end{cases} \tag{2}
\]

where \( f(z) \) is an arbitrary analytic (in the sense of
Cauchy-Riemann conditions [3]) function of a complex
variable, line over a symbol denotes complex conjugation,
brackets denote absolute value of the complex
number, \( z = X + iY \) is the complex coordinate, \( i = \sqrt{-1} \).
The components of the magnetization vector are given by
\( m_X + im_Y = 2w/(1+|w|^2) \) and \( m_Z = -(1-|w|^2)/(1+|w|^2) \).
This solution consists of two parts both locally extremal
to the exchange energy functional: the one convention-
ally termed as soliton [4] at \( |f(z)| < 1 \) and the meron [5].
The role of the soliton “hat” ((\( f(z) < 1 \)) is to cover
the topological singularity of the meron [3], this is the
only region of (3) where \( |m_Z| > 0 \). Pure solitons are
not realized in considered particles due to excessive face
magnetic charges [6].

Let us now see if using the freedom of choice for \( f(z) \)
in (3) we can exactly remove the side magnetic charges
(whose energy is next in the hierarchy). This problem
can be formulated as: to find a function \( f(z) \) analytical
in the region \( \mathcal{D} \) in such a way that \( \Re[f(\zeta)\eta(\zeta)] = 0 \) (no
magnetization components normal to the side), where \( \zeta \in
\mathcal{C} = \partial \mathcal{D} \) is the boundary of \( \mathcal{D} \), and \( \eta(\zeta) = \eta_z(\zeta) + im_\eta(\zeta)
\) is the complex normal to \( \mathcal{C} \). Which makes it reduced to
the well known linear Riemann-Hilbert problem [5].

To write the solution let us denote the conformal trans-
formation of the interior of the unit circle \( |t| \leq 1 \) to \( z \in \mathcal{D}
\) as \( z = T(t) \), according to Riemann theorem such transformation
exists for any simply connected region \( \mathcal{D} \). Under
such transformation \( C \) transforms onto the boundary
of the unit circle \( |\lambda| = 1 \) and the normal is given by
\( n_T(\lambda) = n(T(\lambda)) \). Then, all the solutions for \( f(t) \) are
given by

\[
f(t) = \frac{a_0t^2 + a_1t + a_2}{e^{F^+(t)}} + \frac{a_0 + a_1t + a_2t^2}{e^{F^-(t)}},
\tag{3}
\]

\[
F(t) = \frac{1}{2\pi i} \int_{|\lambda|=1} \frac{\log[-\lambda n_T(\lambda)/n_T(\lambda)]}{\lambda - t} \, d\lambda,
\tag{4}
\]

where \( a_0, a_1, a_2 \) are arbitrary complex constants,
\( F^+(t) = F(t), |t| < 1 \) and \( F^-(t) = F(t), |t| > 1 \) are
the values of the Cauchy-type integral [5] having discon-
tinuity at \( |t| = 1 \). The presence of exactly three arbitrary
constants is due to the index [3] \( \text{ind}[\pi n_T/\pi T] = -2 \).

The derivative of the conformal transformation \( T'(t) =
G(t)e^{\epsilon(t)} \) gives the scaling (the real function \( G(t) \)) and
the rotation angle (the real function \( \varphi(t) \)) performed by
a transformation at a point \( t \), and \( \lambda \) (when \( |\lambda| = 1 \), is
a normal to the unit circle. This allows to express (up
to the real multiplier) the normal to the face boundary
as \( n_T(\lambda) = \lambda T'(\lambda) \). The real multiplier in \( n_T(\lambda) \) is ir-
relevant because only the combination \( \pi \eta_T/\pi T \) enters [3].
Then, the integral (3) gives \( F^+(t) = -i\pi - \log T'(t) \) and
\( F^-(t) = -\log T'(t)/t \), where the identity \( 1/\lambda = \lambda \) and
the residue theorem were used. Substituting these into
(5) gives the final answer

\[
f(t) = (itc + A - \overline{\lambda}t^2)T'(t),
\tag{5}
\]

where \( c = \text{Im} a_1 \) is a real constant and \( A = \overline{\lambda t^2} \)
is a complex constant, the role of these constants is best
illustrated by the following example.

Circular cylinder. In this case \( T(t) = t \) and the mag-
netization distribution is [5] with

\[
f_{CL}(z) = izc + A - \overline{\lambda}z^2.
\tag{6}
\]

The values of \( c \) and \( A \) need to be determined from the
detailed analysis of the energy including magnetostatic
and can not be determined based on the pole avoidance
principle alone. However, the interesting point is that
apart of the well known vortex solutions [5] cor-
responding to \( |c| > 2|A| \) there are also solutions of another type
with \( |c| < |A| \) corresponding to a pair of skyrmions [10]
(also known as hedgehogs) bound to the cylinder sides,
see Fig. 1. These solutions are relevant to the consider-
ation of quasi-uniform magnetization states (by allowing
the pair of skyrmions to move away from the cylinder
boundary) and of a vortex nucleation. The nucleation of
a vortex can be described as symmetrical moving of the
pair of skyrmions from infinity (uniform state) to the
particle boundary with quasi-uniform states forming in the
process and, then, by changing \( \infty > |A| > c/2 \) alongside
the particle boundary to the point where two skyrmions
annihilate and the vortex forms, moving subsequently to
the particle center as \( A \) changes down to \( c/2 > |A| > 0 \).
This process will be considered in detail in a forthcoming
paper. Let us now see what happens if a particle has
corners by considering the next example.

Triangular cylinder. In this case the conformal map
is given by the Schwartz-Christoffel integral \( T(t) =
C \int_0^t (1 - t^3)^{-2/3} \, dt \). The side length of the triangle is
1 \( C = 2^{2/3}\sqrt{3\pi}/(\Gamma(1/6)/\Gamma(1/3)), \Gamma(x) \) is Euler’s gamma
function. Using (5) we get

\[
f_{TR}(z) = (itc + A - \overline{\lambda}t^2) \frac{1}{(1 - t^3)^{2/3}}, \; t \to T^{-1}(z),
\tag{7}
\]

where \( T^{-1}(z) \) means inverse function of \( T(t) \) which is
unique because the transformation is conformal. This
expression converges at the corners of the triangle be-
cause corners represent topological singularities of the
type \( \infty/\infty \) in the meron. To avoid them we shall cover
corners by a soliton “hats” similarly as it was done for the singularities of the type 0/0 corresponding to the vortex centers. Thus, instead of \( \frac{f(z)}{\sqrt{f(z)f(\zeta)}} \) we shall use

\[
w(z, \zeta) = \begin{cases} 
  f(z) & |f(z)| \leq 1 \\
  f(z)/c_2 & 1 < |f(z)| \leq c_2 \\
  f(z)/c_2 & |f(z)| > c_2,
\end{cases}
\]

where \( c_2 > 1 \) is an additional arbitrary real constant. The result is plotted in Fig. 3.

**Summary.** A general method is provided for obtaining closed-form analytical expressions for magnetization distributions in flat ferromagnetic cylinders of arbitrary shapes made of isotropic material. This problem is shown to be equivalent to finding the conformal transformation of the unit circle to the shape of the cylinder face \( T(t) \). Then, the magnetization distributions are given by (3) and (2) (or (8) if the particle has corners). No other magnetization distributions exactly minimizing the exchange energy of the particle and having no surface magnetic charges on its side exist. A new type of soliton solution was found by applying this procedure to the circular cylinder, see Fig. 1 (and also experimental Fig. 1(c) in [11], however interpreted differently there).

The only term not included into this consideration is the magnetostatic energy of the volume charges but, as it was qualitatively argued, the effect of this term is the smallest for small particles and is expected to introduce only minor corrections to the magnetization distributions obtained here. If desired, these corrections can be found by perturbative expansions or by numerical analysis.

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