Abstract

A first attempt to understand hadron dynamics at low energies in terms of the fundamental quark and gluon degrees of freedom incorporates the effects of the gluonic field into a potential depending only on the spatial positions of the quarks, which are considered in the infinite mass limit. A suitable framework for calculating such potentials between static quarks, i.e. a generalization of the Wilson loop will be discussed.

Making a connection with recent Monte Carlo lattice simulations for the lowest two energies of a system of two quarks and two antiquarks, the static $qq\bar{q}\bar{q}$-potential will be calculated in perturbation theory to fourth order. The result will be shown to be exactly equal to the prediction of a straightforward two-body approach, which in Monte Carlo lattice simulations has been found to be a reasonable approximation for very small interquark distances.
1 Introduction

As QCD is the accepted theory of the strong interactions, it is no doubt desirable to understand all hadronic phenomena directly in terms of the fundamental fields of QCD. However, QCD being asymptotically free, perturbation theory is applicable only for very short distances and cannot cover the complete range of interest. At present lattice gauge simulations are the only way to study such systems. In a first approach, the static approximation is the natural choice, where the gluonic degrees of freedom are integrated out, and quark loops are ignored (the quenched approximation), giving rise to a potential between the stationary quarks. The potential of the quark-antiquark system, where this approach—leading to the familiar Wilson Loop—is very well known, has been calculated extensively in Monte Carlo lattice simulations. (For recent data, see e.g. [1].) The ground state potential of this static system has also been calculated in perturbation theory up to sixth order [2].

Here we shall describe how to generalize this procedure to multi-quark systems, especially to \((q\bar{q})^k\) systems. However, even with present-day computers, \(q\bar{q}\) lattice simulations are still very demanding, and the amount of computations needed increases rapidly with the number of interacting quarks. Reliable models for multi-quark systems expressing their potentials e.g. in terms of the well known \(q\bar{q}\)-systems would therefore be of great help. Such two-body approximations have proven successful in many areas of physics, and these models can be formulated without difficulty. For the \(qq\bar{q}\bar{q}\)-system, which is the simplest one that can be considered consisting of two colour singlets, this model has been tested against numerical data from a Monte Carlo simulation [3]. For small distances the agreement has been found reasonable. It has also been observed [4] that the two-body model corresponds to lowest order perturbation theory. We shall be able to show that it is correct even to fourth order. To sixth order, however, three- and four-body forces begin to appear.

2 The Generalized Wilson Loop

While the concepts discussed below are of course well known in the context of the Wilson Loop for the \(q\bar{q}\)-system, we find it useful to start with rephrasing these concepts in the case of an arbitrary number of quarks, leading to a
study of more complicated systems.

When we have assembled a system of several quarks (and antiquarks), gluons will mediate a force between them. Treating this system in an approximation as a quantum mechanical system of several static quarks, the interactions between the quarks are incorporated into a potential. This assembly of quarks is then expected to propagate in time with the usual factor of $e^{-itH}$, where the interesting piece of the Hamilton operator $H$ is the potential energy. Thus, by calculating appropriate Green functions, the potentials of eigenstates of $H$ can be extracted.

### 2.1 Setting up Gauge Invariant States

Because of confinement, it makes sense only to talk about systems of quarks where the overall states have colour singlet quantum numbers. The problem with setting up say a $q\bar{q}$-system in a singlet is that the quark and antiquark are located a distance apart. This problem can be overcome by inserting the path ordered exponential $U(x, y, A) = \mathcal{P}e^{ig\int_y^x T^a dz_a}$ between the locations $x$ and $y$ of the quarks in the presence of the gauge potential $A$. Here $g$ denotes the coupling constant and $T^a$ the representation matrices. Thus $\bar{\psi}(x)U(x, y)\psi(y)|0\rangle$ will serve as a basis state in this case. We must also know how many basis states there are. When dealing with Green functions coming from Monte Carlo lattice simulations, they will have contributions from excited states of the gluonic field, and there are infinitely many of them even in the $q\bar{q}$-case. With suitable methods, the lowest potentials can be extracted, and several have been calculated for the quenched $q\bar{q}$-system – see for example [5], [6]. The situation is different for Green functions calculated in perturbation theory. Here, unlike the lattice simulations, we can and must go to the infinite time limit. We do not expect to reach excited states of the gluon field in finite order perturbation theory, and thus the number of basis states for a system of several quarks is given by the usual arguments of group representation theory, e.g. one for the $q\bar{q}$-system and two for the $qq\bar{q}\bar{q}$-system. In the large time limit we expect that the effects of ‘introducing’ the quarks into the vacuum will be irrelevant in comparison to their time evolution, and the notion of a potential makes sense.
2.2 Diagonalization

It may be shown that the state

$$| \text{quarks } q_i \text{ and antiquarks } \bar{q}_j \text{ at time } t \rangle = \bar{\psi}_i(-t, x_i) \ldots U(-t, x_i, y_i, A) \ldots \psi(-t, y_i) |0\rangle \quad (1)$$

can be shown to satisfy Schrödinger's equation. Forming the overlap of states at time $-t$ and $t$, we get an equation between Green functions and expressions of the form $A_{ij} \overset{\text{def}}{=} \langle A_i | e^{-itH} | A_j \rangle$, where $|A_i\rangle$ stands for some basis state and we have introduced the matrix $A$. By assuming a decomposition of these basis states into eigenstates of $H$, a diagonalization procedure will yield the potentials. In the case of the Green functions coming from lattice simulations, one considers a practical number of basis states, expands them in energy eigenstates and drops contributions with $e^{-itE_i}$ for energies $E_i$ above a certain limit. Of course we implicitly assume Wick-rotation. In perturbation theory, where a power expansion of $e^{-itE_i(g)}$ in the coupling $g$ will not be exponentially damped, we need to consider all linearly independent basis states, a number that is finite, as remarked in the last section. Because of this finiteness, we can find an invertible transformation to energy eigenstates, and the diagonalization is straightforward. In fact, given a matrix $A$ satisfying certain consistency relations, we can perturbatively prove the existence of a time-independent basis transformation such that in this new basis $A$ is not only diagonal, but its eigenvalues are of the form $e^{-itE_i(g)}$. Here the energy $E_i(g)$ of the $i$-th basis state, which can be calculated perturbatively, is for static quarks equal to the $i$-th potential (apart from an irrelevant constant, the rest mass).

2.3 Loops

What remains to be done is to bring the Green functions of the last paragraph to more familiar forms. Since we work within the static approximation, the full quark propagator in the presence of gauge fields can be calculated

$$S_0(x, y, A) = -ie^g \int_T e^{i\gamma^\mu A^\mu(z)dz} e^{-im|x^0 - y^0|}\delta(\vec{x} - \vec{y}) \times \left[ \frac{1 + \gamma^0}{2} \Theta(x^0 - y^0) + \frac{1 - \gamma^0}{2} \Theta(y^0 - x^0) \right] \quad (2)$$
We shall now outline how various contour integrations, i.e. loops arise. Considering the well-known $q\bar{q}$-case, we find a path-ordered line integral from antiquark to quark arising from the $U$ in eq. (1), then the path-ordered line integral propagating the quark forward in time from eq. (2). Another $U$ and the antiquark propagating backwards in time close the rectangle of the familiar Wilson loop. Starting with the Green functions described below eq. (1) and evaluating them for propagation from $-t/2$ to $t/2$, the following diagrammatic rule for calculating the Green function dealing with an arbitrary number $k/2$ of quark-antiquark pairs (i.e. $k$ quarks and antiquarks) partitioned into $q\bar{q}$ singlets is seen to hold:

1. Draw two horizontal lines, the lower denoting time $-t/2$, the upper $t/2$. Mark the position of every quark and antiquark on the lower line and once again vertically above it on the upper line.

2. At the $-t/2$ level connect every quark-antiquark pair that is set up as a singlet at $-t/2$ with a line, having an arrow pointing from antiquark to quark.

3. At the $t/2$ level connect every quark-antiquark pair that is set up as a singlet at $t/2$ with a line, the arrow in which points from quark to antiquark.

4. Join the quarks at the $-t/2$ level with quarks at the same position at the $t/2$ level, arrow pointing upwards, i.e. forward in time.

5. Join the antiquarks at the $t/2$ level with the antiquarks at the $-t/2$ level, arrow pointing downwards, i.e. backwards in time.

6. Associate a path-ordered exponential of $e^{ig \oint_{C} T^{a} A_{\mu}^{a}(z) dz_{\mu}}$ together with a trace for every closed loop $C$ occurring.

7. Determine the overall sign: If the pairings at the $-t/2$ level are the same as those on the $t/2$ level, there must be a $+$ sign. (This follows from the positivity of the norm on a Hilbert space if one lets $t \to 0$.) If this is not so, determine the sign of the permutation of antiquarks on the upper line that is necessary to give the same pairings as on the lower line. This is the overall sign.
8. Multiply by \((\delta(\vec{0})e^{-imt})^k\), where \(k\) is the total number of quarks and antiquarks.

9. Insert the factor so obtained in the numerator of \(\int [DA_\mu]|D\eta_\ast|D\eta|e^{i\int d^4x{|C|}}\int [DA_\mu]|D\eta_\ast|D\eta|e^{i\int d^4x{|C|}}\)

This gives the Green function in the chosen singlet structure.

### 3 The \(qq\bar{q}\bar{q}\)-Potentials

In \(SU(N)\) gauge theory with quarks in the fundamental representation, we want to calculate the \(qq\bar{q}\bar{q}\)-potential in perturbation theory to fourth order. It has been remarked in subsection 2.1 that there are two independent basis states for this system, and one easily recognises a choice of these in the two possible ways of pairing the system into two quark-antiquark singlets. Assuming the first static quark at position \(R_1\), the second at \(R_2\), and the antiquarks at \(R_3\) and \(R_4\), we will label the two states \(|A_1\rangle = 1_{13}1_{24}\) and \(|A_2\rangle = 1_{14}1_{23}\).

#### 3.1 Calculating the Green Functions

According to subsection 2.3, we encounter the following types of loops:

\[ C_{(A_1,-t/2|A_1,t/2)} \]

\[ -t/2 \]

\[ t/2 \]

and

\[ ^1\text{With } \eta \text{ we denote the ghost fields, with } \mathcal{L} \text{ the Lagrangian without fermions} \]
In calculating the Green functions, we will adopt dimensional regularization in dimension $D = 4 - 2\epsilon$ with a mass scale $M$. A wave-function renormalization will remove infinities associated with a diagram of the form

$$C_{(A_2, -t/2|A_1, t/2)}$$

while a coupling constant renormalization is needed to make the sum

$$\tau_2$$
finite. The full gluon propagator is renormalized in the usual way. Note that, contrary to a smallest distance regularization frequently encountered in this context, there is no renormalization of the quark mass \( m \). Making the expansion in \( \alpha = \frac{g^2 a^2}{4\pi} \) rather than in \( g \) and also expanding in time, a calculation yields for

\[
A = A(1) + \alpha A(\alpha) + \alpha^2 A(\alpha^2) + \alpha t A(\alpha t) + \alpha^2 t A(\alpha^2 t) + \frac{(\alpha t)^2}{2} A\left(\frac{\alpha t^2}{2}\right) \tag{3}
\]

the expressions:

\[
A(1) = \begin{pmatrix} N^2 & -N \\ N & N^2 \end{pmatrix}
\]

\[
A(\alpha) = \begin{pmatrix} N^2 \{4 \ln(M^2 R_{13} R_{24})\} & -N \{\ln \left( \frac{R_{31}^3 R_{32}^3 R_{34}^3 M^8}{R_{13}^3 R_{24}^3} \right) - e_1 - e_2 \} \\ -N \{\ln \left( \frac{R_{31}^3 R_{32}^3 R_{34}^3 M^8}{R_{13}^3 R_{24}^3} \right) - e_1 - e_2 \} & N^2 \{4 \ln(M^2 R_{14} R_{23})\} \end{pmatrix}
\]

\[
A(\alpha T) = \begin{pmatrix} N^2 \{-V_{s1}\} & -N \{-V_{s1} - V_{s2} + V_d\} \\ -N \{-V_{s1} - V_{s2} + V_d\} & N^2 \{-V_{s2}\} \end{pmatrix}
\]

\[
A(\alpha^2 T) = \begin{pmatrix}
N^2 \{-4V_{s1} \ln \left( M^2 R_{13} R_{24} \right) \\
+ \frac{2\alpha}{N} \{-V_{s2} + V_d\} \times \\
\left[ \ln \left( \frac{R_{31}^3 R_{32}^3 R_{34}^3}{R_{13}^3 R_{24}^3} \right) - e_1 \right] \end{pmatrix}
\]

\[
-4V_{s2} \ln \left( M^2 R_{14} R_{23} \right) + \frac{2\alpha}{N} \{-V_{s1} + V_d\} \times \\
\left[ \ln \left( \frac{R_{31}^3 R_{32}^3 R_{34}^3}{R_{14}^3 R_{23}^3} \right) - e_2 \right]
\]

The full gluon propagator is renormalized in the usual way. Note that, contrary to a smallest distance regularization frequently encountered in this context, there is no renormalization of the quark mass \( m \). Making the expansion in \( \alpha = \frac{g^2 a^2}{4\pi} \) rather than in \( g \) and also expanding in time, a calculation yields for

\[
A = A(1) + \alpha A(\alpha) + \alpha^2 A(\alpha^2) + \alpha t A(\alpha t) + \alpha^2 t A(\alpha^2 t) + \frac{(\alpha t)^2}{2} A\left(\frac{\alpha t^2}{2}\right) \tag{3}
\]

the expressions:
\[ A \left( \frac{1}{2} \alpha T \right)^2 = \]

\[ \begin{pmatrix}
2 \{ V_{s1}^2 + \frac{c_2}{Nc_3} (V_{s2}^2) \} & -N \left\{ (V_{s1} + V_{s2} - V_d)^2 + \frac{c_2}{2c_3} \times \frac{1}{Nc_3} (V_{s1} + V_{s2} - V_d)^2 \right\} \\
-N \left\{ (V_{s1} + V_{s2} - V_d)^2 + \frac{c_2}{2c_3} \times \frac{1}{Nc_3} (V_{s1} + V_{s2} - V_d)^2 \right\} & 2 \{ V_{s2}^2 + \frac{c_2}{Nc_3} (V_{s1} - V_d)^2 \}
\end{pmatrix} \]

where, to save writing, the following shorthand notation has been adopted:

\[ \text{edge}(\vec{R}_3, \vec{R}_1, \vec{R}_4, \vec{R}_2) \overset{def}{=} \int_{-1/2}^{1/2} \frac{(\vec{R}_3 - \vec{R}_1)(\vec{R}_4 - \vec{R}_2)}{\left[ \frac{\vec{R}_3 + \vec{R}_1}{2} + w(\vec{R}_3 - \vec{R}_1) - \frac{\vec{R}_4 + \vec{R}_2}{2} - x(\vec{R}_4 - \vec{R}_2) \right]^2} \]

\[ e_1 \overset{def}{=} \text{edge}(R_1, R_3, R_2, R_4) \]
\[ e_2 \overset{def}{=} \text{edge}(R_1, R_4, R_2, R_3) \]
\[ V_{s1} \overset{def}{=} V(R_{13}) + V(R_{24}) \]
\[ V_{s2} \overset{def}{=} V(R_{14}) + V(R_{23}) \]
\[ V_d \overset{def}{=} V(R_{12}) + V(R_{34}) \]

\[
\begin{align*}
&c_1 \delta_{ab} = \sum_{cd} f_{acd} f_{bdc} = N \delta_{ab} \\
&c_2 \delta_{ab} = \text{Tr}[T^a T^b] = \frac{1}{2} \delta_{ab} \\
&c_3 1 = \sum_a T^a T^a = \frac{N^2 - 1}{2N} 1
\end{align*}
\]

and \( V(R_{pq}) \) is the two-body potential between a quark ‘p’ and an antiquark ‘q’ a distance \( R_{pq} \) apart. Diagonalization yields for the two possible energy eigenstates the two potentials correct to fourth order:

\[ V_0 = \frac{(N^2 - 2) (V_{s1} + V_{s2}) + 2V_d - N \sqrt{N^2 (V_{s1} - V_{s2})^2 + 4 (V_{s1} - V_d)(V_{s2} - V_d)}}{2 (N^2 - 1)} \]
\[ V_1 = \frac{(N^2 - 2) (V_{s1} + V_{s2}) + 2V_d + N \sqrt{N^2 (V_{s1} - V_{s2})^2 + 4 (V_{s1} - V_d)(V_{s2} - V_d)}}{2 (N^2 - 1)} \]

These potentials are exactly equal to those given by a naive two-body model (see [3])—and are one of the main conclusions of this work.
3.2 Consistency Relations and Edge Effects

When integrating out the gluonic degrees of freedom and reducing a field theory to a quantum mechanical system, we have made assumptions which may need some justification. This is especially so in the perturbative case where we have postulated a finite number of energy eigenstates. We, therefore, want to have a look at the consistency of this approach.

There are certain relations which must hold in order to guarantee proper exponentiation. We have remarked at the end of subsection 2.2 that after diagonalization we expect the diagonal entries of the matrix $A$ to be (apart from a normalization factor) of the form $e^{-itE_i(g)}$. An actual calculation will give a power expansion in $g$ and $t$ for these diagonal entries. This will not only determine the energy $E_i(g)$ as the coefficient of $t$, but will also allow us to check the consistency of our calculation by inspecting the relations between the coefficients of higher powers of $t$. We were able to verify these consistency relations in our calculation to fourth order.

Another important point to notice is that the terms we have abbreviated edge($\vec{R}_p, \vec{R}_q, \vec{R}_r, \vec{R}_s$) have cancelled in eq. (4). These terms are coming from the space-like contour integrations. In subsection 2.1 we had introduced these space-like contour integrations via the path-ordered exponential $U$ in order to establish colour singlets. The exact integration contour in $U$ was of course arbitrary, and the physical potentials must not depend on it. Another way of looking at it is that in the large time limit the processes associated with bringing the quarks into their position in the distant past (and removing them again in the distant future) should become irrelevant. One may also make an adiabatic argument, turning on the coupling $g$ in the distant past and off again in the distant future, thus making the notion of a colour singlet state at these times well-defined in spite of the spatial separation of the quarks. In any case, for the potentials of eq. (4) to be meaningful, they must not involve terms originating in contour integrations in the distant past or future, and indeed they do not.

3.3 Three- and Four-Body Forces in Sixth Order

The fact that a straightforward two-body model is correct also to next-to-leading order may be surprising in light of the non-abelian nature of QCD. Hence we want to mention that we believe that the two-body model fails at
sixth order as three- and four-body forces appear at this order. While many diagrams that seem to give rise to such deviations from the two-body model actually vanish, we see no reason why for example effects from the following two diagrams should cancel in the calculation of the four-quark potentials to sixth order.

where \( \bigcirc \) stands for

Hence three- and four-body forces will be introduced. In general, their nature seems to be complicated, but for some geometries simplifications are possible; e.g. for the four quarks on the corners of a regular tetrahedron there will be no contribution from quark self-interactions to four-body forces to sixth order.
4 Relation of our Result to Lattice Simulations

Looking at the Monte Carlo lattice calculations for the $qqq\bar{q}$-system discussed in [3], we observe that for small interquark distances of a few lattice spacings (with a lattice spacing of $a \approx 0.12$ fm) the two-body model gives a reasonable approximation in the sense that the four-quark potentials calculated from eq. (4) using the Monte Carlo two-body potentials are comparable to the four-quark potentials from the lattice simulation. The agreement improves the smaller the distances get. By comparing the perturbative (i.e. $1/R$) and non-perturbative (i.e. linear) part in the usual parametrization of the $q\bar{q}$-potential, one would expect to start entering the perturbative regime at distances of about two lattice spacings. However, at that stage the approximation provided by the two-body model is already very good. The fact that the two-body model is correct to fourth order in perturbation theory certainly suggests that it should be a reasonable approximation in the perturbative domain. So our result supports the belief that the results of the lattice simulations for small enough distances indeed are correlated to continuum perturbation theory, and thus that continuum physics is extracted from the Monte Carlo calculations.

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