Positive provability logic
for uniform reflection principles

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Abstract
We deal with the fragment of modal logic consisting of implications of formulas built up from the variables and the constant ‘true’ by conjunction and diamonds only. The weaker language allows one to interpret the diamonds as the uniform reflection schemata in arithmetic, possibly of unrestricted logical complexity. We formulate an arithmetically complete calculus with modalities labeled by natural numbers and \( \omega \), where \( \omega \) corresponds to the full uniform reflection schema, whereas \( n < \omega \) corresponds to its restriction to arithmetical \( \Pi_{n+1} \)-formulas. This calculus is shown to be complete w.r.t. a suitable class of finite Kripke models and to be decidable in polynomial time.

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1. Introduction

Several applications of provability logic in proof theory made use of a polymodal logic \( \text{GLP} \) due to Giorgi Japaridze \[17, 9\]. This system, although decidable, is not very easy to handle. In particular, it is not Kripke complete \[9\]. It is complete w.r.t. the more general topological semantics, however this could only be established recently by rather complicated techniques \[2\].

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A weaker system, called *Reflection Calculus* and denoted RC, was introduced in [8]. It is much simpler than GLP yet expressive enough to regain its main proof-theoretic applications. It has been outlined in [8] that RC allows to define a natural system of ordinal notations up to $\varepsilon_0$ and serves as a convenient basis for a proof-theoretic analysis of Peano Arithmetic in the style of [4, 5]. This includes a consistency proof for PA based on transfinite induction up to $\varepsilon_0$, a characterization of its $\Pi^0_n$-consequences in terms of iterated reflection principles, and a combinatorial independence result.

From the point of view of modal logic, RC can be seen as a fragment of polymodal logic consisting of implications of the form $A \rightarrow B$, where $A$ and $B$ are formulas built-up from $\top$ and propositional variables using just $\land$ and the diamond modalities. We call such formulas $A$ and $B$ strictly positive and will often omit the word ‘strictly.’

A somewhat different but equivalent axiomatization of RC (as an equational calculus) has been earlier found by Evgeny Dashkov in his paper [11] which initiated the study of strictly positive fragments of provability logics. Dashkov proved two important further facts about RC which sharply contrast with the corresponding properties of GLP. Firstly, RC is complete with respect to a natural class of finite Kripke frames. Secondly, RC is decidable in polynomial time, whereas most of the standard modal logics (including GL and GLP) are PSPACE-complete.

Another advantage of going to a strictly positive language is explored in the present paper. Strictly positive modal formulas allow for more general arithmetical interpretations than those of the standard modal logic language. In particular, propositional formulas can now be interpreted as arithmetical *theories* rather than individual *sentences*. (Notice that the ‘negation’ of a theory would not be well-defined.)

Similarly, the diamonds need no longer be interpreted as individual *consistency assertions* but as the more general *reflection schemata* not necessarily having finite axiomatizations. Thus, for example, the full uniform reflection schema can be considered as a modality in the context of positive provability logic (see [18, 5] for general information on reflection principles). Such interpretations are not only natural but can be useful for further development of the approach to proof-theoretic analysis via provability algebras. Thus, positive provability logic allows to speak about certain notions not

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2 Traditionally, positive modal formulas may also involve disjunctions and box modalities. However, in the present paper we will not consider positive formulas in this more general sense.
nicely representable in the context of the standard modal logic.

The main contribution of this paper is a Solovay-style arithmetical completeness result for an extension of $\text{RC}$ by a new modality corresponding to the unrestricted uniform reflection principle. This is the primary example of a modality not representable in the full modal logic language. The system obtained is shown to be decidable and to enjoy a suitable complete Kripke semantics along with the finite model property.

Whereas the modal logic part of our theorem is a simple extension of Dashkov’s results, the arithmetical part is more substantial. We introduce a new modification of the Solovay construction using some previous ideas from $[17, 16, 7]$. Since the arithmetical complexity of the uniform reflection schema is unbounded, a single Solovay-style function is not enough for our purpose. Instead, we deal with infinitely many Solovay functions, of increasing arithmetical complexity, uniformly and simultaneously.

The paper is organized as follows. Firstly, we introduce positive modal language and the systems leading to the arithmetically complete reflection calculus $\text{RC}_\omega$. Secondly, we present the details of its arithmetical interpretation and somewhat tediously prove the corresponding soundness theorem. Thirdly, we study the Kripke semantics of positive provability logics and obtain completeness results, along with a suitable version of the finite model property. Fourthly, we obtain polynomial complexity bounds for the derivability problem in $\text{RC}_\omega$ by adapting the techniques of Dashkov. Finally, we prove the main result of this paper, the arithmetical completeness theorem for $\text{RC}_\omega$.

2. Reflection calculus and its basic properties

Consider a modal language $\mathcal{L}$ with propositional variables $p, q, \ldots$, a constant $\top$ and connectives $\land$ and $\alpha$, for each ordinal $\alpha \leq \omega$ (understood as diamond modalities). Strictly positive formulas (or simply formulas) are built up by the grammar:

$$A ::= p \mid \top \mid (A \land B) \mid \alpha A, \quad \text{where } \alpha \leq \omega.$$ 

Sequents are expressions of the form $A \vdash B$ where $A, B$ are strictly positive formulas. The system $\text{RJ}$ is given by the following axioms and rules:

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3 An interesting arithmetical completeness proof for an extension of modal logic $\text{GLP}$ by transfinitely many modalities has recently appeared in $[13]$. However, the considered interpretation is different and not applicable in our situation.
1. \( A \vdash A \); \( A \vdash \top \); if \( A \vdash B \) and \( B \vdash C \) then \( A \vdash C \) (syllogism);
2. \( A \land B \vdash A \); \( A \land B \vdash B \); if \( A \vdash B \) and \( A \vdash C \) then \( A \vdash B \land C \);
3. if \( A \vdash B \) then \( \alpha A \vdash \beta A \); \( \alpha \alpha A \vdash \alpha A \);
4. \( \alpha \beta A \vdash \beta A \); \( \beta \alpha A \vdash \beta A \) for \( \alpha \geq \beta \);
5. \( \alpha A \land \beta B \vdash \alpha (A \land \beta B) \) for \( \alpha > \beta \).

The systems \( \text{RC} \) and \( \text{RC} \omega \) are obtained from \( \text{RJ} \) by adding respectively one or two of the following principles:

6. \( \alpha A \vdash \beta A \) for \( \alpha > \beta \); (monotonicity)
7. \( \omega A \vdash A \). (persistence)

Dashkov [11] showed that \( \text{RJ} \), restricted to the language without \( \omega \) modality, axiomatizes the strictly positive fragment of the polymodal logic \( \text{J} \) [6], whereas \( \text{RC} \) axiomatizes the strictly positive fragment of \( \text{GLP} \).

Notice that Axioms 4 are redundant in the presence of Axiom 6: if \( \alpha \geq \beta \) then \( \alpha \beta A \vdash \beta \beta A \vdash \beta A \) and \( \beta \alpha A \vdash \beta \beta A \vdash \beta A \).

If \( L \) is a logic, we write \( A \vdash L B \) for the statement that the sequent \( A \vdash B \) is provable in \( L \). As a simple example, consider the sequent

\[ \omega (p \land q) \vdash (\omega p \land \omega q). \]

It is provable in \( \text{RJ} \) as follows: We have \( p \land q \vdash p \), hence \( \omega (p \land q) \vdash \omega p \).
Similarly, \( \omega (p \land q) \vdash \omega q \), therefore \( \omega (p \land q) \vdash (\omega p \land \omega q) \), by the conjunction rules. In contrast, \( (\omega p \land \omega q) \not\vdash_{\text{RC} \omega} \omega (p \land q) \), as we shall see below by a simple Kripke model argument.

Formulas \( A \) and \( B \) are called \textit{L-equivalent} (written \( A \sim_L B \)) if \( A \vdash_L B \) and \( B \vdash_L A \).

We also consider the fragments of various logics obtained by restricting the language to a subset of modalities. Such a subset \( S \subseteq \omega + 1 = \{0,1,\ldots,\omega\} \) is called a \textit{signature}. We denote by \( L_S \) the set of all strictly positive formulas in \( S \). Similarly, for a logic \( L \) in \( L \) we denote by \( L_S \) the restriction of the axioms and rules of \( L \) to the language \( L_S \).

For a positive formula \( A \), let \( \ell(A) \) denote \( \{\alpha \leq \omega : \alpha \text{ occurs in } A\} \).

**Lemma 2.1.**  
(i) If \( A \vdash_{\text{RJ}} B \) then \( \ell(B) \subseteq \ell(A) \);  
(ii) If \( A \vdash_L B \) where \( L \) is \( \text{RC} \) or \( \text{RC} \omega \), then \( \ell(B) \subseteq [0, \max \ell(A)] \).

**Proof.** In each case, this is proved by an easy induction on the length of the derivation of \( A \vdash B \). \( \square \)

Let \( C[A/p] \) denote the result of replacing in \( C \) all occurrences of a variable \( p \) by \( A \). If a logic \( L \) contains Axioms 1, 2 and the first part of 3, then \( \vdash_L \) satisfies the following \textit{positive replacement lemma}. 


Lemma 2.2. Suppose $A \vdash_L B$, then $C[A/p] \vdash_L C[B/p]$, for any $C$.

Proof. Induction on the build-up of $C$. $\square$

A positive logic $L$ is called normal if it contains the rules 1, 2, and the first part of 3, and is closed under the following substitution rule: if $A \vdash_L B$ then $A[C/p] \vdash_L B[C/p]$. It is clear that $\text{RJ}$, $\text{RC}$ and $\text{RC}\omega$, as well as their restricted versions, are normal.

3. Arithmetical interpretation

We define the intended arithmetical interpretation of the positive modal language. The idea is that propositional variables (and positive formulas) now denote possibly infinite theories rather than individual sentences. To avoid possible problems with the representation of theories in the language of $\text{PA}$, we deal with primitive recursive numerations of theories rather than with the theories as sets of sentences.

All theories in this paper will be formulated in the language of Peano Arithmetic $\text{PA}$ and contain the axioms of $\text{PA}$. It is convenient to assume that the language of $\text{PA}$ contains the symbols for all primitive recursive programs. A primitive recursive numeration of a theory $S$ is a bounded arithmetical formula $\sigma(x)$ defining the set of Gödel numbers of the axioms of $S$ in the standard model of arithmetic. Given such a $\sigma$, we have a standard arithmetical $\Sigma_1$-formula $\Box_\sigma(x)$ expressing the provability of $x$ in $S$ (see [12]). We often write $\Box_\sigma \varphi$ for $\Box_\sigma(\ulcorner \varphi \urcorner)$. The expression $\bar{n}$ denotes the numeral $0' \cdots' (n \text{ times})$. If $\varphi(v)$ contains a parameter $v$, then $\Box_\sigma \varphi(\bar{x})$ denotes a formula (with a parameter $x$) expressing the provability of the sentence $\varphi(\bar{x}/v)$ in $S$.

Given two numerations $\sigma$ and $\tau$, we write $\sigma \vdash_{\text{PA}} \tau$ if

$$\text{PA} \vdash \forall x (\Box_\tau(x) \to \Box_\sigma(x)).$$

We write $\sigma \vdash \tau$ if $\mathbb{N} \models \forall x (\Box_\tau(x) \to \Box_\sigma(x))$, that is, if the theory numerated by $\sigma$ contains the one numerated by $\tau$. We will only consider the numerations $\sigma$ such that $\sigma \vdash_{\text{PA}} \sigma_{\text{PA}}$, where $\sigma_{\text{PA}}$ is some standard numeration of $\text{PA}$.

With any finite extension of $\text{PA}$ of the form $\text{PA} + \varphi$ we will associate its standard numeration $\sigma_{\text{PA}} \lor (x = \ulcorner \varphi \urcorner)$ that will be denoted $\ulcorner \varphi \urcorner$. For obvious reasons we have: $\varphi \vdash_{\text{PA}} \psi$ iff $\ulcorner \varphi \urcorner \vdash \psi$ iff $\text{PA} + \varphi \vdash \psi$. (The statement $\ulcorner \varphi \urcorner \vdash_{\text{PA}} \psi$ implies $\text{PA} + \varphi \vdash \psi$ by the soundness of $\text{PA}$, the converse is formalizable in $\text{PA}$.)
Given a numeration $\sigma$ of $S$, the consistency of $S$ is expressed by $\text{Con}(\sigma) := \neg \Box_{\sigma} \bot$. A theory $S$ is called $n$-consistent if $S$ together with the set of all true $\Sigma_{n+1}$-sentences is consistent. The $n$-consistency of $S$ is expressed by the formula

$$\text{Con}_n(\sigma) : \forall x \in \Pi_{n+1} (\Box_{\sigma}(x) \rightarrow T_n(x)),$$

where $T_n$ is the standard $\Pi_{n+1}$-truthdefinition for $\Pi_{n+1}$-formulas (see [14]) and $x \in \Pi_{n+1}$ denotes the primitive recursive formula expressing that $x$ is a Gödel number of a $\Pi_{n+1}$-sentence.

Concerning the truthdefinitions we assume that $\text{PA} \vdash \varphi \leftrightarrow T_n(\bar{\varphi})$, for each $\Pi_{n+1}$-sentence $\varphi$. Moreover, this very fact can be formalized in PA uniformly in $n$:

$$\text{PA} \vdash \forall n \forall x \in \Pi_{n+1} \Box_{\text{PA}}(x \leftrightarrow T_n(\bar{x})), \quad (1)$$

as the sequence of formulas $\Box T_n$ is primitive recursive in $n$ and the corresponding proofs are constructed inductively.

The formula $\text{Con}_n(\sigma)$ is often called the global $\Pi_{n+1}$-reflection principle for $S$ and is denoted $\text{RFN}_{\Pi_{n+1}}(S)$ (see [19, 3]). We note that the formula $\text{Con}_0(\sigma)$ is PA-provably equivalent to $\text{Con}(\sigma)$.

The uniform reflection principle for $S$ is the schema

$$\text{Con}_\omega(\sigma) : \{ \text{Con}_n(\sigma) : n \in \omega \},$$

It is well-known that $\text{Con}_\omega(\sigma)$ is PA-provably equivalent to the schema

$$\forall x (\Box_{\sigma} \varphi(\bar{x}) \rightarrow \varphi(x)),$$

for each arithmetical formula $\varphi(x)$, which is usually denoted $\text{RFN}(S)$.

The uniform reflection principle is elementarily axiomatized, and we fix a standard function mapping any numeration $\sigma$ to the numeration of $\text{PA} + \text{Con}_n(\sigma)$ (denoted $\text{Con}_{\sigma}(\sigma)$). Similarly, the formula

$$\sigma_{\text{PA}}(x) \lor x = \bar{\text{Con}}_n(\sigma)$$

numerating the theory $\text{PA} + \text{Con}_n(\sigma)$ will be denoted $\overline{\text{Con}}_n(\sigma)$.

The intended arithmetical interpretation maps positive modal formulas to primitive recursive numerations in such a way that $\top$ corresponds to the standard numeration of $\text{PA}$, $\land$ corresponds to the union of theories, $n$ corresponds to the standard numeration of $\text{Con}_n$, for each $n < \omega$, and $\omega$ to the standard numeration of $\text{Con}_\omega$. 
Definition 3.1. An arithmetical interpretation is a map \( \ast \) from positive modal formulas to numerations satisfying the following conditions:

- \( \top^\ast = \sigma_{\text{PA}}; \quad (A \land B)^\ast = (A^\ast \lor B^\ast); \)
- \( (nA)^\ast = \text{Con}_n(A^\ast); \quad (\omega A)^\ast = \text{Con}_\omega(A^\ast). \)

It is clear that the value \( A^\ast \) is completely determined by the interpretations \( p_1^\ast, \ldots, p_n^\ast \) of all the variables occurring in \( A \).

Proposition 3.2 (soundness). Suppose \( A \vdash_{\text{RC}_\omega} B \), then \( A^\ast \vdash_{\text{PA}} B^\ast \), for all arithmetical interpretations \( \ast \).

Proof. Induction on the length of proof of \( A \vdash B \) in \( \text{RC}_\omega \). The validity of the first two groups of rules of \( \text{RC}_\omega \) is obvious. We treat the modal axioms and rules.

If \( \sigma \vdash_{\text{PA}} \tau \) then clearly \( \text{Con}_n(\sigma) \vdash_{\text{PA}} \text{Con}_n(\tau) \), for each \( n < \omega \). Since this fact is formalizable in \( \text{PA} \), we also obtain \( \text{Con}_n(\sigma) \vdash_{\text{PA}} \text{Con}_n(\tau) \). Also, the validity of the monotonicity axioms 6 is clear. Next we need the following lemma.

Lemma 3.3. (i) Let \( S \) be numerated by \( \sigma \) and \( \varphi \in \Pi_{n+1} \). If \( S \vdash \varphi \) then \( \text{PA} + \text{Con}_n(\sigma) \vdash \varphi \);

(ii) Statement (i) holds provably in \( \text{PA} \) uniformly in \( n \), that is,

\[ \text{PA} \vdash \forall n \forall x \in \Pi_{n+1} (\Box_{\sigma}(x) \to \Box_{\text{Con}_n(\sigma)}(x)). \]

Proof. We only prove Statement (ii). We reason in \( \text{PA} \) as follows.

Assume \( x \in \Pi_{n+1} \) and \( \Box_{\sigma}(x) \). Then \( \Box_{\text{PA}}(\bar{x} \in \Pi_{n+1} \land \Box_{\sigma}(\bar{x})) \). On the other hand, by the definition of \( \text{Con}_n(\sigma) \)

\[ \Box_{\text{Con}_n(\sigma)} \forall y (\Box_{\sigma}(y) \land y \in \Pi_{n+1} \to T_n(y)). \]

This yields

\[ \Box_{\text{Con}_n(\sigma)}(\Box_{\sigma}(\bar{x}) \land \bar{x} \in \Pi_{n+1} \to T_n(\bar{x})), \]

so we obtain \( \Box_{\text{Con}_n(\sigma)} T_n(\bar{x}) \), and hence \( \Box_{\text{Con}_n(\sigma)}(x) \) by \( \Box \). \( \square \)

Corollary 3.4. (i) \( \text{Con}_n(\text{Con}_n(\sigma)) \vdash_{\text{PA}} \text{Con}_n(\sigma) \), for all \( n < \omega \);

(ii) \( \text{Con}_n(\text{Con}_n(\sigma)) \vdash_{\text{PA}} \text{Con}_n(\sigma) \).
Proof. Since the theories numerated by $\text{Con}_n(\sigma)$ and $\text{Con}_n(\text{Con}_n(\sigma))$ are finite extensions of PA, for a proof of Statement (i) it is sufficient to show

$$\text{PA} + \text{Con}_n(\text{Con}_n(\sigma)) \vdash \text{Con}_n(\sigma). \quad (2)$$

Since $\text{Con}_n(\sigma)$ is a $\Pi_{n+1}$-sentence, we can take in Lemma 3.3 $\phi = \text{Con}_n(\sigma)$ and $S = \text{PA} + \phi$. This yields statement (2).

For a proof of (ii), we show an informal version of this statement by an argument formalizable in PA. We must prove that, for each $n < \omega$,

$$\text{PA} + \text{Con}_\omega(\text{Con}_\omega(\sigma)) \vdash \text{Con}_n(\sigma).$$

Using the monotonicity and Statement (i) we reason as follows:

$$\text{Con}_\omega(\text{Con}_\omega(\sigma)) \vdash \text{PA} \text{Con}_n(\text{Con}_\omega(\sigma)) \vdash \text{PA} \text{Con}_n(\text{Con}_\omega(\sigma)) \vdash \text{PA} \text{Con}_n(\sigma).$$

This shows the claim. \[\square\]

Corollary 3.4 shows the soundness of the third group of rules of $\text{RC}_\omega$. As we mentioned above, the fourth group is actually derivable from the first three and the monotonicity, so we can skip it. We show the soundness of Axiom 5.

Lemma 3.5. If $n > m$ then $\text{PA} \vdash \text{Con}_n(\sigma) \land \text{Con}_m(\tau) \rightarrow \text{Con}_n(\sigma \lor \text{Con}_m(\tau))$.

Proof. We reason in $\text{PA}$ as follows: If $\varphi \in \Pi_{n+1}$ and $\Box_{\sigma \lor \text{Con}_m(\tau)}(\varphi)$, then by the formalized deduction theorem $\Box_{\sigma}(\text{Con}_m(\tau) \rightarrow \varphi)$. Since $m < n$, the formula $\text{Con}_m(\tau) \rightarrow \varphi$ belongs to $\Pi_{n+1}$. By $\text{Con}_n(\sigma)$ we obtain $T_n(\ulcorner \text{Con}_m(\tau) \rightarrow \varphi \urcorner)$ whence $T_n(\ulcorner \text{Con}_m(\tau) \urcorner) \rightarrow T_n(\ulcorner \varphi \urcorner)$. Since $\text{Con}_m(\tau) \in \Pi_{n+1}$, from $\text{Con}_m(\tau)$ we infer $T_n(\ulcorner \text{Con}_m(\tau) \urcorner)$. Hence $T_n(\ulcorner \varphi \urcorner)$, as required. \[\square\]

Corollary 3.6. $\text{Con}_\omega(\sigma) \lor \text{Con}_m(\tau) \vdash \text{PA} \text{Con}_\omega(\sigma \lor \text{Con}_m(\tau))$.

Proof. Informally, we must prove, for each $n$, that

$$\text{PA} + \text{Con}_\omega(\sigma) + \text{Con}_m(\tau) \vdash \text{Con}_n(\sigma \lor \text{Con}_m(\tau)).$$

We can assume $n > m$ and then use the previous lemma. This argument is formalizable in PA. \[\square\]

Corollary 3.7. $\text{Con}_\omega(\sigma) \vdash \text{PA} \sigma$.  

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Proof. We reason as follows:

$$\text{PA} \vdash \Box_{\sigma}(x) \rightarrow \exists n \left( x \in \Pi_{n+1} \land \Box_{\sigma}(x) \right) \rightarrow \exists n \Box_{\text{Con}_{n}(\sigma)}(x) \rightarrow \Box_{\text{Con}_{\omega}(\sigma)}(x).$$

This shows the soundness of the remaining Axiom 7 of RC and completes the proof of Proposition 3.2. □

4. Kripke models for RCω

Kripke frames and models are understood in this paper in the usual sense. A Kripke frame \( W \) for the language \( \mathcal{L}_S \) consists of a non-empty set \( W \) equipped with a family of binary relations \( (R_\alpha)_{\alpha \in S} \). A Kripke frame \( W \) is called finite if so is \( W \) and all but finitely many relations \( R_\alpha \) are empty.

A Kripke model \( W \) is a Kripke frame together with a valuation \( v : W \times \text{Var} \rightarrow \{0,1\} \) assigning a truth value to each propositional variable at every node of \( W \). As usual, we write \( W, x \models A \) to denote that a formula \( A \) is true at a node \( x \) of a model \( W \). This relation is inductively defined as follows:

- \( W, x \models p \iff v(x,p) = 1 \), for each \( p \in \text{Var} \);
- \( W, x \models \top \);
- \( W, x \models A \land B \iff (W, x \models A \text{ and } W, x \models B) \);
- \( W, x \models \alpha A \iff \exists y \left( xR_\alpha y \text{ and } W, y \models A \right) \).

We call a RJ\( S \)-frame a Kripke frame satisfying the following conditions, for all \( \alpha, \beta \in S \) and all \( x,y,z \in W \):

- \( xR_\alpha y R_\beta z \) implies \( xR_\gamma z \), if \( \gamma = \min(\alpha, \beta) \); (polytransitivity)
- \( xR_\alpha y \) and \( xR_\beta z \) implies \( yR_\beta z \), if \( \alpha > \beta \). (condition J)

These conditions can be more succinctly written as \( R_\alpha R_\beta \subseteq R_{\min(\alpha, \beta)} \) and \( R_\alpha^{-1} R_\beta \subseteq R_\beta \). An \( RS \)-frame is an RJ\( S \)-frame that is monotone, that is, \( R_\alpha \subseteq R_\beta \), for each \( \beta < \alpha \). An RJ\( S \)-model (RS-model) is a Kripke model based on an RJ\( S \)-frame (RS-frame). We speak about RJ- and RC-frames and models whenever \( S = \omega + 1 \).
The persistence axiom $\omega A \vdash A$ does not correspond to a frame condition.\footnote{Notice that the more familiar form of this axiom is the principle $A \rightarrow \Box A$ which has no non-discrete frames.} We call a Kripke model \textit{(downwards) persistent} if, for each variable $p$,

$W, x \models p$ and $y R_\omega x \Rightarrow W, y \models p$.

By a straightforward induction we obtain the following lemma.

\textbf{Lemma 4.1.} Let $W$ be a persistent Kripke model based on a polytransitive frame. Then, for each positive formula $A$,

$W, x \models A$ and $y R_\omega x \Rightarrow W, y \models A$.

We say that a sequent $A \vdash B$ is \textit{true} in a Kripke model $W$, if

$\forall x \in W (W, x \models A \Rightarrow W, x \models B)$.

A logic $L$ is \textit{sound} for a class $C$ of Kripke models (of the same signature), if every sequent $A \vdash B$ provable in $L$ is true in any model from $C$. It is easy to see that our logics are sound for their respective classes of models.

\textbf{Lemma 4.2.} (i) $\textbf{RJ}_S$ is sound for the class of all $\textbf{RJ}_S$-models;

(ii) $\textbf{RC}_S$ is sound for the class of all $\textbf{RC}_S$-models;

(iii) $\textbf{RC}_\omega S$ is sound for the class of all persistent $\textbf{RC}_S$-models.

A proof of this lemma is routine.

Notice that the frame conditions for the logics $\textbf{RJ}_S$ and $\textbf{RC}_S$ (that is, polytransitivity, condition J, and monotonicity) are closure conditions. Therefore, for any Kripke frame $W = (W, (R_\alpha)_{\alpha \in S})$ there is an $\textbf{RJ}_S$-frame ($\textbf{RC}_S$-frame) $\overline{W} = (W, (\overline{R}_\alpha)_{\alpha \in S})$ such that

- $R_\alpha \subseteq \overline{R}_\alpha$, for each $\alpha \in S$;

- For any other $\textbf{RJ}_S$-frame ($\textbf{RC}_S$-frame) $(W, (R'_\alpha)_{\alpha \in S})$ with $R_\alpha \subseteq R'_\alpha$, for all $\alpha \in S$, we have $\overline{R}_\alpha \subseteq \overline{R}'_\alpha$, for all $\alpha \in S$.

The frame $\overline{W}$ is unique up to isomorphism. We call it the $\textbf{RJ}_S$-closure ($\textbf{RC}_S$-closure) of $W$. 

\footnote{Notice that the more familiar form of this axiom is the principle $A \rightarrow \Box A$ which has no non-discrete frames.}
Example 4.3. Consider a Kripke frame $\mathcal{W} = (W, (R_\alpha)_{\alpha \leq \omega})$ with $W = \{0, 1, 2\}$. Relation $R_\omega$ consists of two pairs $0R_\omega 1$ and $0R_\omega 2$, and the other relations are empty. Let $\overline{\mathcal{W}}$ be the RC-closure of $\mathcal{W}$. It is easy to see that $R_\omega = R_\omega$, whereas, for each $n < \omega$, $\overline{R_n} \upharpoonright \{1, 2\}$ is a total relation, $0\overline{R_n} 1$ and $0\overline{R_n} 2$.

Further, we define $v(x, p) = 0$ iff $x = 2$, and $v(x, q) = 0$ iff $x = 1$. This makes $(\overline{\mathcal{W}}, v)$ a downwards persistent RC-model falsifying $\omega p \wedge \omega q \vdash \omega(p \wedge q)$ at 0. By Lemma 4.2 we conclude $\omega p \wedge \omega q \not\vdash_{RC} \omega(p \wedge q)$.

The completeness proofs in all these cases are also easy. As in Dashkov [11], we present an argument based on a (simplified) version of filtrated canonical model.

Let $\Phi$ be a set of $L$-formulas. Denote $\ell(\Phi) := \{\alpha \leq \omega : \alpha \text{ occurs in some } A \in \Phi\}$. A set $\Phi$ is called adequate if $\Phi$ is closed under subformulas, $\top \in \Phi$ and

- If $\beta A \in \Phi$ and $\beta < \alpha \in \ell(\Phi)$, then $\alpha A \in \Phi$;
- For any variable $p$, if $p \in \Phi$ then $\omega p \in \Phi$.

It is easy to see that any finite set of formulas can be extended to a finite adequate set.

Let $\Gamma$ be a set of $L$-formulas and $L$ a logic. We shall take for $L$ one of $RJ$, $RC$ or $RC_\omega$, or their restricted versions in the language $L_S$ where $S \subseteq \omega + 1$. We write $\Gamma \vdash_L B$ if there are formulas $A_1, \ldots, A_n \in \Gamma$ such that the sequent $A_1 \wedge \cdots \wedge A_n \vdash B$ is provable in $L$.

Fix an adequate set of formulas $\Phi$. An $L$-theory in $\Phi$ is a set $x \subseteq \Phi$ such that $x \vdash_L A$ and $A \in \Phi$ implies $A \in x$. Define a model $\mathcal{W}_L/\Phi$ as follows. The set of nodes $\mathcal{W}_L/\Phi$ is the set of all $L$-theories in $\Phi$. We stipulate that $xR_\alpha y$ iff $\alpha \in \ell(\Phi)$ and the following conditions hold for each formula $A$:

R1. $A \in y$ and $\alpha A \in \Phi$ implies $\alpha A \in x$;
R2. $\beta A \in y$ and $\alpha A \in \Phi$ implies $\min(\alpha, \beta) A \in x$;
R3. $\beta < \alpha$ and $\beta A \in x$ implies $\beta A \in y$.

We also let $\mathcal{W}_L/\Phi, x \models p$ iff $p \in x$, for any $L$-theory $x$.

Lemma 4.4. Suppose $L$ contains $RJ_S$ with $S = \ell(\Phi)$. Then $\mathcal{W}_L/\Phi$ is an $RJ_S$-model.

\footnote{In positive logic there is no harm in allowing an ‘inconsistent’ theory $x = \Phi$ as a node.}
Proof. To check the polytransitivity assume \( xR_\alpha yR_\beta z \) and \( \alpha, \beta \in \ell(\Phi) \). We show \( xR_{\min(\alpha, \beta)} z \) by checking R1–R3. If \( A \in z \) and \( \min(\alpha, \beta) A \in \Phi \), then by the adequacy \( \beta A \in \Phi \) and hence \( \beta A \in y \). It follows that \( \min(\alpha, \beta) A \in x \).

For R2 notice that \( \min(\gamma, \alpha, \beta) = \min(\gamma, \min(\alpha, \beta)) \). If \( \gamma A \in z \) and \( \min(\gamma, \alpha, \beta) A \in \Phi \) then by the adequacy \( \min(\gamma, \beta) A \in \Phi \) and hence \( \beta A \in y \). This in turn implies \( \min(\gamma, \beta, \alpha) A \in x \). Condition R3 is obviously satisfied, as all three theories have the same formulas of the form \( \beta A \) for \( \beta < \alpha \).

Second, we check condition (J). Assume \( xR_\alpha y \) and \( xR_\beta z \) with \( \alpha < \beta \). We show \( zR_\alpha y \). R1: If \( A \in y \) and \( \alpha A \in \Phi \) then \( \alpha A \in x \). Since \( \alpha < \beta \) this implies \( \alpha A \in z \). R2: If \( \gamma A \in y \) and \( \alpha A \in \Phi \) then \( \min(\gamma, \alpha) A \in x \) whence \( \min(\gamma, \alpha) A \in z \) for the same reason. R3 is, again, obvious. \( \Box \)

Lemma 4.5. For any \( A \in \Phi \), \( W_L/\Phi, x \models A \) iff \( A \in x \).

Proof. Induction on the build-up of \( A \). If \( A \) is a variable, \( \top \) or has the form \( B \land C \), the argument is obvious. Assume \( A = \alpha B \).

If \( x \models \alpha B \) then, for some \( y \) such that \( xR_\alpha y \), we have \( y \models B \). By IH it follows that \( B \in y \) and hence \( \alpha B \in x \).

Now assume \( \alpha B \in x \). Let \( \Delta := \{ \beta C : \beta C \in x, \beta < \alpha \} \) and let \( y \) be the deductive closure of \( \Delta \cup \{B\} \) in \( \Phi \). By the IH we have \( y \models B \). We claim that \( xR_\alpha y \) which completes the argument.

Assume \( D \in y \), then \( \Sigma, B \models \alpha D \) for some finite \( \Sigma \subseteq \Delta \). Then \( \Sigma \land \alpha B \models \alpha(\Sigma \land \alpha B) \land L \alpha D \). Hence, if \( \alpha D \in \Phi \) then \( \alpha D \in y \). Similarly, if \( \gamma D \in y \) then \( \Sigma, B \models \gamma D \). Then \( \Sigma \land \alpha B \models \alpha(\Sigma \land \alpha B) \land L \alpha(\Sigma \land \alpha B) \land L \alpha \gamma D \models L \alpha \gamma D \models L \min(\gamma, \alpha)D \). If \( \alpha D \in \Phi \) then \( \min(\gamma, \alpha)D \in \Phi \) whence \( \min(\gamma, \alpha)D \in x \). Finally, if \( \beta < \alpha \) and \( \beta D \in x \) then \( \beta D \in \Delta \), hence \( \beta D \in y \). \( \Box \)

Lemma 4.6. (i) If \( L \) contains the monotonicity axiom and \( S = \ell(\Phi) \), then \( W_L/\Phi \) is an RCS-frame;

(ii) If \( L \) contains the persistence axiom, then \( W_L/\Phi \) is persistent.

Proof. (i) Assume \( xR_\alpha y \) and \( \beta < \alpha \in \ell(\Phi) \). We show \( xR_\beta y \) by checking the three conditions. If \( A \in y \) and \( \beta A \in \Phi \) then \( \alpha A \in \Phi \) by the adequacy of \( \Phi \). Hence, \( \alpha A \in x \) and therefore by the monotonicity axioms \( x \models \beta A \). Since \( \beta A \in \Phi \) we obtain \( \beta A \in x \). Similarly, if \( \gamma A \in y \) and \( \beta A \in \Phi \) then \( \alpha A \in \Phi \) by the adequacy. Therefore, \( \min(\alpha, \gamma)A \in x \), whence by the monotonicity axioms \( x \models \min(\beta, \gamma)A \). Since both \( \gamma A \) and \( \beta A \) are in \( \Phi \), it follows that \( \min(\beta, \gamma)A \in x \) which proves the second condition. The third condition is obviously satisfied.

(ii) Assume \( xR_\omega y \). If \( y \models p \) then \( p \in y \); by the adequacy \( \omega p \in \Phi \) and hence \( \omega p \in x \). It follows that \( x \models_L \omega p \models_L p \) and \( x \models p \). \( \Box \)
Taking $\Phi = L$ and $W_L := W_L/\Phi$ we obtain the completeness of $\text{RJ}$, $\text{RC}$ and $\text{RC}\omega$ w.r.t. their respective classes of models.

**Theorem 1.** (i) $A \vdash_{\text{RJ}} B$ iff $A \vdash B$ is true in all $\text{RJ}$-models;
(ii) $A \vdash_{\text{RC}} B$ iff $A \vdash B$ is true in all $\text{RC}$-models;
(iii) $A \vdash_{\text{RC}\omega} B$ iff $A \vdash B$ is true in all persistent $\text{RC}$-models.

**Proof.** The three systems are sound by Lemma 4.2. The completeness is proved by observing that $W_L$, for each of the three logics $L$, is a model of the corresponding type. Assume $A \not\vdash_{L} B$. Then letting $x$ denote the $L$-theory generated by $A$ we have $B \not\in x$, hence by Lemma 4.5 $W_L, x \not\models B$. $\blacksquare$

Next we discuss the finite model property of the three logics. For $\text{RJ}$ the answer is obvious, but for $\text{RC}$ and $\text{RC}\omega$ we have a small complication due to the fact that modality $\omega$ is present in the language.

**Corollary 4.7.** $A \vdash_{\text{RJ}} B$ iff $A \vdash B$ is true in all finite $\text{RJ}$-models.

**Proof.** Assume $A \not\vdash_{\text{RJ}} B$, let $\Phi$ be a finite adequate set of formulas containing both $A$ and $B$. We have $W_{\text{RJ}}/\Phi, x \models A$ and $W_{\text{RJ}}/\Phi, x \not\models B$. Moreover, $W_{\text{RJ}}/\Phi$ is a finite $\text{RJ}_S$-model, where $S = \ell(\Phi)$. By putting $R_\alpha := \emptyset$, for any $\alpha \notin \ell(\Phi)$, we expand $W_{\text{RJ}}/\Phi$ to an $\text{RJ}$-model in $L$ falsifying $A \models B$. $\blacksquare$

A similar argument does not quite work for $\text{RC}$, as the expansion by empty relations leads, in general, outside the class of $\text{RC}$-models. However, for a finite signature $S$ we do have an analog of Theorem 1.

**Corollary 4.8.** Suppose $S \subseteq \omega + 1$ is finite.

(i) $A \vdash_{\text{RC}_S} B$ iff $A \vdash B$ is true in all finite $\text{RC}_S$-models;
(ii) $A \vdash_{\text{RC}\omega_S} B$ iff $A \vdash B$ is true in all finite persistent $\text{RC}_S$-models.

**Lemma 4.9.** Let $S \subseteq \omega + 1$ and $\alpha \notin S$. Any $\text{RC}_S$-model can be expanded to an $\text{RC}_{S \cup \{\alpha\}}$-model.

**Proof.** Let $W = (W, (R_\beta)_{\beta \in S})$ be a given $\text{RC}_S$-model. Denote: $S^+ := \{\beta \in S : \alpha < \beta\}$ and $S^- := \{\beta \in S : \beta < \alpha\}$. If $S^+ = \emptyset$ we can put $R_\alpha := \emptyset$. Otherwise, for any relation $R$ on $W$, denote

$$R' := R \cup RR \cup \bigcup_{\beta \in S^+} R_\beta^{-1}R.$$ 

Further, define $R_\alpha := \bigcup_{n \in \omega} R_\alpha^n$, where $R_\alpha^0 := \bigcup_{\beta \in S^+} R_\beta$; $R_\alpha^{n+1} := (R_\alpha^n)'$. 13
Notice that $R \subseteq R'$, for any $R$. It follows that $R_\beta \subseteq R_\alpha^0 \subseteq R_\alpha$, for each $\beta \in S^+$. By the construction, $R_\alpha$ is transitive and $R_\beta^{-1}R_\alpha \subseteq R_\alpha$, hence condition (J) is satisfied for all $\alpha < \beta \in S^+$. Moreover, the polytransitivity follows from the transitivity and the monotonicity properties. Therefore, $(W, (R_{\beta} \beta \in S^+ \cup \{\alpha\})$ is an $RC_{S^+ \cup \{\alpha\}}$-model.

To complete the argument we have to show that $(W, (R_{\gamma} \gamma \in S^- \cup \{\alpha\})$ is an $RC_{S^- \cup \{\alpha\}}$-model. To this end we prove that, for each $n$ and $\gamma \in S^-$,

1. $R_\alpha^n \subseteq R_\gamma$;
2. $(R_\alpha^n)^{-1}R_\gamma \subseteq R_\gamma$.

Both statements are verified by induction on $n$. The basis of induction holds, since the original model was an $RC_S$-model. Assume the statements hold for $R = R_\alpha^n$ and consider $R'= R_\alpha^{n+1}$.

1. We have $R \subseteq R_\gamma$ by the IH. Further, $RR \subseteq R_\gamma R_\gamma \subseteq R_\gamma$, since $R_\gamma$ is transitive. For any $\beta \in S^+$, $R_\beta^{-1}R \subseteq R_\beta^{-1}R_\gamma \subseteq R_\gamma$, since condition (J) holds in $W$. Hence, $R' = R \cup RR \cup \bigcup_{\beta \in S^+} R_\beta^{-1}R \subseteq R_\gamma$, as required.

2. We have $R^{-1}R_\gamma \subseteq R_\gamma$ by the IH. Further, $(RR)^{-1}R_\gamma = R^{-1}(R^{-1}R_\gamma) \subseteq R^{-1}R_\gamma \subseteq R_\gamma$. Finally, for any $\beta \in S^+$, $(R_\beta^{-1}R)^{-1}R_\gamma = R^{-1}R_\beta R_\gamma \subseteq R^{-1}R_\gamma \subseteq R_\gamma$. Therefore, $(R')^{-1}R_\gamma \subseteq R_\gamma$, as required. □

**Remark 4.10.** The given proof also works for the more general analogs of $RC$, e.g., for logics with linearly ordered sets of modalities (see [1]).

Taking into account that expansions of persistent models are persistent, we obtain the following theorem for both $RC$ and $RC_\omega$.

**Theorem 2.** Let $L$ be either $RC$ or $RC_\omega$. The following statements are equivalent:

(i) $A \vdash_L B$;
(ii) $A \vdash_{LU} B$, for some finite $U \subseteq \omega + 1$;
(iii) $A \vdash_{LS} B$ where $S = \ell(\{A, B\})$.

**Proof.** Clearly, (iii) implies (i), and (i) implies (ii) since a finite derivation may only contain finitely many different modalities. We prove that (ii) implies (iii). Assume $A \not\vdash_{LS} B$. By Corollary 4.8 there is a finite $RC_S$-model $W$ falsifying $A \vdash B$ (which is persistent if $L = RC_\omega$). Assume any finite $U$ be given. We may assume $S \subseteq U$ (otherwise clearly $A \not\vdash_{LU} B$). By Lemma 4.9 $W$ can be expanded to an $RC_U$-model falsifying the same sequent. Hence, $A \not\vdash_{LU} B$. □
Thus, even though we do not have the finite model property for RC and RCω in the full language, these logics are conservatively approximated by their fragments with this property. Together with Corollary 4.8 this yields

**Corollary 4.11.** The systems RC and RCω are decidable.

For the logics RJ and RC a sharper result can be stated. As we have seen, the question whether a sequent \( A \vdash B \) is provable in such a logic \( L \) is equivalent to the same question for the logic \( L_S \) with \( S = \ell(\{A,B\}) \). However, for any finite \( S \), the logic \( L_S \) is modulo renaming of modalities the same logic as \( L_n \) for \( n = |S| \) (we identify \( n \) with the set \( \{0, \ldots, n - 1\} \)). The systems \( L_n \) are shown to be polytime decidable [11]. Therefore, we obtain

**Corollary 4.12.** The systems RJ and RC are polytime decidable.

The same result holds for RCω, however we cannot directly refer to Dashkov’s theorem. This question is considered in the next section, where we also obtain somewhat sharper complexity estimates for the cases RJ and RC. The material of that section, up to Theorem 3 is due to Dashkov [11].

5. Polytime decidability of RCω

We have to develop some combinatorial techniques to deal with positive logics. It allows one to state the Kripke completeness results in a sharper form, from which the complexity bounds are easily read off.

Let \( W \) be a Kripke model and \( a \in W \). The submodel \( W_a \) of \( W \) generated by \( a \) is obtained by restricting all the relations and the valuation of \( W \) to the set of all nodes \( x \in W \) such that there is a path \( a = x_0Rx_1R\ldots Rx_n = x \) where \( R = \bigcup_{\alpha \in S} R_\alpha \). A model \( W \) is called rooted if it has a distinguished element \( a \) (called the root) such that \( W_a = W \). We notice that in polytransitive rooted frames every node is reachable from the root in one step.

**Definition 5.1.** We can associate with each positive formula \( A \) a rooted treelike Kripke model \( T[A] \) in the signature \( \ell(A) \) called its canonical tree. It is essentially the parse tree of \( A \) viewed as a Kripke model.

If \( A \) is a variable or \( \top \), then \( T[A] \) is a one-point model \( \{a\} \) with the empty relations, and the only variable true at \( a \) is \( A \).

If \( A = B \land C \) then \( T[A] \) is obtained from the disjoint union of the models \( T[B] \) and \( T[C] \) by identifying the roots. We declare any variable \( p \) true at the root of \( T[A] \) iff it is true at the root of either \( T[B] \) or \( T[C] \).

If \( A = \alpha B \) then \( T[A] \) is obtained from \( T[B] \) by adding a new root \( r \) (where all variables are false), from which the root of of \( T[B] \) is \( R_\alpha \)-accessible.
We write $T[A] \models \varphi$ if $\varphi$ is true at the root of $T[A]$. Then, one can easily verify the following properties:

- Each $R_\alpha$ on $T[A]$ is an irreflexive forest-like binary relation;
- $T[A] \models A$.

**Definition 5.2.** A *homomorphism* of a Kripke model $\mathcal{V}$ into a Kripke model $\mathcal{W}$ (of the same signature $S$) is a function $f : V \rightarrow W$ such that

- $\forall x, y \in V \ (xR_\alpha y \Rightarrow f(x)R_\alpha f(y))$, for each $\alpha \in S$;
- If $\mathcal{V},x \models p$ then $\mathcal{W},f(x) \models p$, for each variable $p$.

Let $\mathcal{V}$ and $\mathcal{W}$ be rooted Kripke models. A *simulation* of $\mathcal{V}$ by $\mathcal{W}$ is a homomorphism $f : \mathcal{V} \rightarrow \mathcal{W}$ mapping the root of $\mathcal{V}$ to the root of $\mathcal{W}$.

**Lemma 5.3.** If $A$ is strictly positive and $f$ is a homomorphism of $\mathcal{V}$ into $\mathcal{W}$, then $\mathcal{V},x \models A \Rightarrow \mathcal{W},f(x) \models A$.

**Lemma 5.4.** $\mathcal{W},x \models B$ iff there is a homomorphism $f : T[B] \rightarrow \mathcal{W}$ mapping the root of $T[B]$ to $x$.

**Proof.** Suppose $f : T[B] \rightarrow \mathcal{W}$ is such a homomorphism. We have $T[B],r \models B$ where $r$ is the root of $T[B]$. Since $B$ is strictly positive and $f(r) = x$, by Lemma 5.3 $\mathcal{W},x \models B$.

Suppose $\mathcal{W},x \models B$. We construct a homomorphism $f : T[B] \rightarrow \mathcal{W}$ by induction on the complexity of $B$. If $B$ is a variable or $\top$, the claim is obvious.

If $B = C \land D$ then $\mathcal{W},x \models C, D$. By the III, there are homomorphisms $f, g$ of the models $T[C]$ and $T[D]$ into $\mathcal{W}$ mapping their respective roots to $x$. The homomorphism of $T[B]$ maps its root to $x$ and is defined as the union of $f$ and $g$ everywhere else on $T[B]$. We note that if $T[B] \models p$ then either $T[C] \models p$ or $T[D] \models p$, by the definition of $T[B]$. In either case we have $\mathcal{W},x \models p$, therefore the variable condition at the root is met and we have a homomorphism of $T[B]$ into $\mathcal{W}$.

If $B = \alpha C$ and $\mathcal{W},x \models B$, then there is a node $y \in \mathcal{W}$ such that $xR_\alpha y$ and $\mathcal{W},y \models C$. By the III, there is a homomorphism of $T[C]$ into $\mathcal{W}$ mapping its root to $y$. We extend it by mapping the root of $T[B]$ to $x$. All the variables are false at the root of $T[B]$, so the variable condition is met.

Let $\text{RC}_S[A]$ (or $\text{RJ}_S[A]$) denote the RC$_S$-closure (respectively, RJ$_S$-closure) of $T[A]$, where $S \supseteq \ell(A)$.
Theorem 3.  

(i) \( A \vdash_{\text{RJ}} B \) iff \( \text{RJ}_S[A] \vdash B \), where \( S = \ell(A) \);

(ii) \( A \vdash_{\text{RC}} B \) iff \( \text{RC}_S[A] \vdash B \), where \( S = \ell(\{A, B\}) \).

Proof. We prove Statement (ii). The case of \( \text{RJ} \) is similar but simpler.

(only if) Since \( T[A] \vdash A \) and the relations of \( \text{RC}_S[A] \) extend those of \( T[A] \), we have \( \text{RC}_S[A] \vdash A \). By Theorem 2, \( A \vdash_{\text{RC}} B \) implies \( A \vdash_{\text{RC}_S} B \). Hence, by Corollary 4.8, \( \text{RC}_S[A] \vdash B \).

(if) Assume \( A \not\vdash_{\text{RC}} B \). There is a rooted \( \text{RC}_S \)-model \( W \) such that \( W \vdash A \) and \( W \not\models B \). By Lemma 5.4, there is a simulation \( f : T[A] \rightarrow W \). Since \( W \) is an \( \text{RC}_S \)-model, \( f \) lifts to a simulation of \( \text{RC}_S[A] \) by \( W \). In fact, we can define on \( T[A] \) new relations \( R'_\alpha \) by letting \( xR'_\alpha y \) iff \( f(x)R_\alpha f(y) \) in \( W \). Then \( (T[A], (R'_\alpha)_{\alpha \in S}) \) is an \( \text{RC}_S \)-model with \( R_\alpha \subseteq R'_\alpha \), for all \( \alpha \in S \). Hence, denoting by \( R''_\alpha \) the relations of \( \text{RC}_S[A] \), we obtain \( R''_\alpha \subseteq R'_\alpha \), for each \( \alpha \in S \). It follows that \( W \) simulates \( \text{RC}_S[A] \) by \( f \). Then, since \( W \not\models B \), we conclude that \( \text{RC}_S[A] \not\models B \), by Lemma 5.3.

Remark 5.5. The proof of Theorem 3 provides an alternative way of showing the finite model property for the logics \( \text{RC}_S \) and \( \text{RJ}_S \).

Theorem 3 yields an efficient decision procedure for the logics \( \text{RC} \) and \( \text{RJ} \). Firstly, given a positive formula \( A \) we let \( S = \ell(\{A, B\}) \) and build the model \( \text{RC}_S[A] \). Secondly, we check if \( B \) is satisfied at the root of this model. To estimate the complexity of this procedure we need to be more specific about the chosen computation model.

We consider random access machines (see [10]) and assume that any register can hold (the code of) any symbol including the variables and the modalities. To simplify the estimates we count the size of any symbol as one, and we assume that the elementary operations such as reading and writing a symbol, as well as the comparison of symbols, cost a constant amount of time. We are going to estimate the number of elementary steps needed to decide whether \( A \vdash_{\text{RC}} B \). (Representing the variables and the modalities more faithfully would introduce a logarithmic factor into our estimates.)

First, we estimate the time needed to build the model \( \text{RC}_S[A] \) given \( A \).

We support a data structure for a positive formula \( A \) (and for the corresponding Kripke model \( T[A] \)) with the arrows represented by pointers. The arrows are labeled by the elements of \( S \), the nodes are labeled by the variables of \( A \). We can also realize these labels as pointers to some extra nodes representing the variables and the modalities, respectively. We assume that there is a fixed ordering of arrows outgoing from any given node of the tree.
(which respects the left-to-right ordering of the corresponding subformulas of $A$). It is well-known that we can very efficiently (in a linear number of steps) parse the formula $A$ to build such a tree.

Next we bring $T[A]$ to a special ordered form. Let $L_{\geq m}$ denote the language $L_U$ with $U = [m, \omega]$. A formula will be called a fact if it is either $\top$ or a conjunction of variables. Ordered formulas are defined inductively.

**Definition 5.6.** A formula $A$ is ordered if it has the form $A = F \land \bigwedge_{i < k} m_i A_i$ for some $k$ (assuming $A = F$ if $k = 0$), where

(i) $F$ is a fact;
(ii) For each $i$, $A_i \in L_{\geq m_i}$ and $A_i$ is ordered;
(iii) $m_0 \geq m_1 \geq \ldots \geq m_{k-1}$.

**Lemma 5.7.** Every positive formula $A$ is RJ-equivalent to an ordered one.

**Proof.** Induction on the build-up of $A$. The basis of induction and the case of conjunction are easy. Suppose $A = mB$. By the induction hypothesis we may assume $B$ ordered, that is, $B = F \land \bigwedge_{i < k} m_i B_i$. If $mB$ is not ordered, there is an $i < k$ such that $m_i < m$. Let $s$ be the minimal such $i$. Then $mB$ is equivalent to $\top \land m(F \land \bigwedge_{i < s} m_i B_i) \land \bigwedge_{i = s}^{k-1} m_i B_i$, which is ordered. $\square$

We notice that if an ordered formula $B$ is obtained from $A$ by the recursive procedure described in Lemma 5.7, then the number of nodes in $T[B]$ is the same as in $T[A]$. One can also easily prove that $RCS[B]$ will, in fact, be isomorphic to $RCS[A]$.

The algorithm of ordering a formula is similar to that of sorting a string, and it is easy to obtain a rough quadratic upper bound, a detailed proof of which we omit.

**Lemma 5.8.** Any formula $A$ can be ordered in $O(|A|^2)$ steps.

An ordered formula $A$ can be written in the following form:

$$A = F \land \bigwedge_{i < k} \bigwedge_{j < n_i} m_i A_{ij},$$

with $m_0 > m_1 > \cdots > m_{k-1}$, $A_{ij} \in L_{\geq m_i}$ ordered and $F$ a fact. Then $RCS[A]$ can be characterized as follows.
Lemma 5.9. If $A$ of the form $[A]$ is ordered then $RC_S[A]$ consists of the disjoint union of the models $RC_{S_i}[A_{ij}]$, for all $i < k$ and $j < n_i$, augmented by a new root $a$, where $S_i := S \cap [m_i, \omega]$ and $a \Vdash F$. In addition to all the relations inherited from the models $RC_{S_i}[A_{ij}]$, only the following relations hold in $RC_S[A]$:

1. $aR_n x$, for each $i < k$, $n \leq m_i$, $j < n_i$ and $x \in RC_{S_i}[A_{ij}]$;
2. $xR_n y$, for each $i < k$, $m_{i+1} \leq n < m_i$, and $x, y \in \bigcup_{p \leq i} \bigcup_{j < n_p} RC_{S_p}[A_{pj}]$ (where we formally let $m_k = 0$);
3. $xR_{m_i} y$, for each $i < k$, $y \in \bigcup_{j < n_i} RC_{S_i}[A_{ij}]$ and $x \in \bigcup_{p < i} \bigcup_{j < n_p} RC_{S_p}[A_{pj}]$.

Proof. It is easy to see that all the relations mentioned in items 1–3 must hold in $RC_S[A]$.

1. By the polytransitivity we have $aR_{m_i} x$, for each $x \in RC_{S_i}[A_{ij}]$. Then, by the monotonicity, $aR_n x$, for all $n \leq m_i$.
2. If $x, y \in \bigcup_{p \leq i} \bigcup_{j < n_p} RC_{S_p}[A_{pj}]$ then $aR_{m_i} x$, $y$ by Item 1, since $m_p \geq m_i$, for each $p \leq i$. In particular, for each $n < m_i$, there holds $aR_n y$. Then by property (J) we obtain $xR_n y$.
3. For any $x, y$ as specified we have $aR_{m_i} y$ and $aR_{m_i-1} x$ by Item 1. Since $m_i-1 > m_i$, by (J) we obtain $xR_{m_i} y$.

It is also a routine but somewhat lengthy check that the model described in Lemma 5.9 is, indeed, an $RC_S$-model. Hence, it must coincide with $RC_S[A]$. \[
\]

A similar but much simpler characterization holds for $RJ_S[A]$. In this case, we do not need to assume that $A$ is ordered. If $x, y \in T[A]$ let $x \sqcap y$ denote the greatest lower bound of $x, y$, that is, the unique node $z$ such that there are oriented paths from $z$ to $x$ and to $y$ without any shared edges. Given a nonempty path $P$ let $m(P)$ denote the minimal modality label occurring on $P$.

Lemma 5.10. Let $x, y \in T[A]$, $S = \ell(A)$, and let $X$ and $Y$ be the uniquely defined paths from $x \sqcap y$ to the nodes $x$ and $y$, respectively. Then $xR_n y$ holds in $RJ_S[A]$ iff either $x \sqcap y = x$, $x \neq y$ and $m(Y) = n$, or both $X$ and $Y$ are nonempty and $m(X) > m(Y) = n$.

We omit a routine proof. Lemmas 5.9 and 5.10 yield efficient algorithms to build the models $RC_S[A]$ and $RJ_S[A]$.

Lemma 5.11. For any ordered formula $A$, the model $RC_S[A]$ can be constructed in $O(|A|^2 \cdot |S|)$ many steps.
Proof. In the course of constructing \( \text{RC}_S[A] \) we add arrows to the initial model \( T[A] \) in a systematic way. Assume \( A = F \land \bigwedge_{i<k} \bigwedge_{j<n_i} m_i A_{ij} \). Since \( A_{ij} \in \mathcal{L}_{S_i} \), we can apply the procedure recursively to build \( \text{RC}_S[A_{ij}] \), for each \( i < k \) and \( j < n_i \). Then we join these models by a common root and add the arrows according to clauses 1–3 of Lemma 5.9. Apart from the computation time needed to build the models \( \text{RC}_S[i_{ij}] \), for each \( i < k \) and \( j < n_i \), this requires only a linear number of steps in the number of added arrows. Further, notice that we never add an arrow twice (clauses 1–3 enumerate distinct fresh arrows because of the choice of the sets \( S_i \)). Thus, the total number of steps in the whole computation is linear in the total number of arrows in the model \( \text{RC}_S[A] \) which can be roughly estimated by \( O(|A|^2 \cdot |S|) \). \( \square \)

Theorem 4. The logics \( \text{RJ} \) and \( \text{RC} \) are decidable in time bounded by a polynomial (of degree three and four, respectively) in the length of the sequent \( A \vdash B \).

Proof. It is well-known that the problem of checking whether a modal formula \( \varphi \) is true at the root of a finite Kripke model \( W \) in a finite signature \( S \) is solvable in time \( O(\|W\| \cdot |\varphi|) \), where \( \|W\| \) denotes the sum of \( |W| \) and the number of pairs \((x, y)\) such that \( x R_\alpha y \), for some \( \alpha \in S \) (see [15, Proposition 3.1]).

Letting \( S = \ell(\{A, B\}) \) and \( n = |S| \) we can estimate \( |\text{RC}_S[A]| \) by \( |A| \) and \( \|\text{RC}_S[A]\| \) by \( O(|A|^2 \cdot n) \). This yields a bound of the form \( O(|A|^2 \cdot n \cdot |B|) \) on the complexity of checking whether \( \text{RC}_S[A] \vdash B \). By Lemmas 5.8 and 5.11 we have the same bound on the complexity of the original problem \( A \vdash_{\text{RC}} B \). Noting that \( n \leq |A| + |B| \) yields a fourth degree polynomial bound in the length \( |A| + |B| \) of the input.

For the logic \( \text{RJ} \) this can be slightly improved. By Lemma 5.10 we can observe that in the graph \( \text{RJ}_S[A] \) (where \( S = \ell(A) \)) there is no more than one arrow between any pair of points. This yields a bound \( O(|A|^2) \) on \( \|\text{RJ}_S[A]\| \) and on the complexity of constructing this model. Consequently, the derivability problem can be solved in \( O(|A|^2 \cdot |B|) \) many steps. \( \square \)

Remark 5.12. Since the input of the problem is naturally divided into two parts \( A \) and \( B \), measuring the complexity in terms of two parameters \( |A| \) and \( |B| \) appears to be more meaningful than expressing it in terms of the total length of the input. Thus, the more informative bounds are \( O(|A|^2 \cdot n \cdot |B|) \) for \( \text{RC} \) and \( O(|A|^2 \cdot |B|) \) for \( \text{RJ} \).

\(^{6}\|W\| \) only measures the complexity of the frame, while the number of variables is accounted for in \( |\varphi| \).
Next we turn to the logic $\text{RC}\omega$. We define $\text{RC}\omega S[A]$ as the model whose frame coincides with that of $\text{RC}S[A]$ and whose valuation function $v'$ satisfies:

$$v'(x, p) = 1 \iff (v(x, p) = 1 \lor \exists y \ (xR\omega y \text{ and } v(y, p) = 1)),$$

where $v$ is the valuation of $\text{RC}S[A]$. Clearly, $\text{RC}\omega S[A]$ is persistent.

**Theorem 5.** $A \vdash _{\text{RC}\omega} B \iff \text{RC}\omega S[A] \models B$, where $S = \ell(\{A, B\})$.

**Proof.** The proof is similar to that of Theorem 3, we elaborate the (if) part.

If $A \not\vdash _{\text{RC}\omega} B$, there is a rooted persistent $\text{RC}S$-model $W$ such that $W \models A$ and $W \not\models B$. By Lemma 5.4, there is a simulation $f : T[A] \to W$. As before, since $W$ is an $\text{RC}S$-model, $f$ lifts to a simulation of $\text{RC}S[A]$ by $W$. We claim that $f$ also lifts to a simulation of $\text{RC}\omega S[A]$ by $W$. Assume $\text{RC}\omega S[A] \models p$. If $\text{RC}S[A] \models p$ then $W, f(x) \models p$ and there is nothing to prove. If $xR\omega y$ in $\text{RC}S[A]$ and $\text{RC}S[A], y \models p$, then $W, f(y) \models p$. Hence, by the persistence of $W$, we obtain $W, f(x) \models p$ and the claim is proved. Therefore, since $W \not\models B$, we have $\text{RC}\omega S[A] \not\models B$, as required. □

Now we notice that $|\text{RC}\omega S[A]|$ has the same bound as $|\text{RC}S[A]|$. Hence, we obtain

**Corollary 5.13.** The logic $\text{RC}\omega$ is decidable in time bounded by a polynomial (of degree four) in the length of the sequent $A \vdash B$.

6. Irreflexive models

The Solovay construction works with the irreflexive models. Therefore we would like to have a characterization of $\text{RC}$ and $\text{RC}\omega$ in terms of suitable irreflexive models. We modify the construction of the canonical model from the previous section. This modification is similar to the one given by Dashkov [11] which in turn derives from the work of Japaridze [17] and Ignatiev [16].

Let $L$ be a logic containing $\text{RC}$ and let $\Phi$ be an adequate set of formulas. We work in the setup of the previous section. We define a Kripke model $\mathcal{W}'_L/\Phi$ which coincides with $\mathcal{W}_L/\Phi$ but for the definition of the relations. We stipulate that $xR_\alpha y$ in $\mathcal{W}'_L/\Phi$ iff $xR_\alpha y$ and following condition holds:

R4. There is a formula $\alpha A \in x$ such that $\alpha A \notin y$.

The model $\mathcal{W}'_L/\Phi$ has the following properties.
Lemma 6.1.  
(i) \( W'_L/\Phi \) is an RJ-model;
(ii) All \( R'_\alpha \) are irreflexive;
(iii) \( R'_\alpha = \emptyset \), for \( \alpha \notin \ell(\Phi) \).

In order to prove the canonical model lemma we need an additional fact.

Lemma 6.2. For any \( A \) and \( \alpha \), \( A \not\vdash_{\text{RC}\omega} \alpha A \).

Proof. The simplest proof of this fact involves arithmetical interpretations. Assume \( A \vdash_{\text{RC}\omega} \alpha A \). Fix an arithmetical interpretation \( * \) mapping all variables to the standard numeration of PA. By Proposition 3.2 we obtain \( A^* \vdash_{PA} \text{Con}_\alpha(A^*) \). Since \( A \) is a positive formula, \( A^* \) is (a numeration of) a sound arithmetical theory \( T \). However, this contradicts Gödel’s second incompleteness theorem for \( T \). \( \square \)

Lemma 6.3. For any \( A \in \Phi \), \( W'_L/\Phi, x \models A \) iff \( A \in x \).

Proof. The proof is similar to that of Lemma 4.5. We only treat somewhat differently the case \( A = \alpha B \), the ‘if’ part.

Assume \( \alpha B \in x \). As before let \( \Delta := \{ \beta C : \beta C \in x, \beta < \alpha \} \) and let \( y \) be the deductive closure of \( \Delta \cup \{ B \} \) in \( \Phi \). By the IH we have \( y \models B \). We claim that \( xR'_\alpha y \) which completes the argument.

We already know from Lemma 4.5 that \( xR_\alpha y \). To check R4 it is sufficient to observe that \( \alpha B \in x \) but \( \alpha B \notin y \). If \( \alpha B \in y \) we would obtain \( \Sigma, B \vdash \alpha B \), for some finite \( \Sigma \subseteq \Delta \), but then \( \Sigma \wedge B \vdash \alpha (\Sigma \wedge B) \), contradicting Lemma 6.2. \( \square \)

A model \( W \) is called \( \Phi \)-monotone, if for any \( \alpha A \in \Phi \) and \( \beta \in \ell(\Phi) \) such that \( \alpha < \beta \), \( W'_L/\Phi, x \models \beta A \) implies \( W'_L/\Phi, x \models \alpha A \).

Lemma 6.4.  
(i) \( W'_L/\Phi \) is \( \Phi \)-monotone;
(ii) \( W'_L/\Phi \) is persistent if \( L \) contains \( \text{RC}\omega \).

Proof. (i) Assume \( W'_L/\Phi, x \models \beta A \), \( \alpha A \in \Phi \) and \( \alpha < \beta \in \ell(\Phi) \). By the adequacy of \( \Phi \) we have \( \beta A \in \Phi \). Then by Lemma 6.3 we obtain \( \beta A \in x \). Hence, \( x \models \alpha A \) and since \( \alpha A \in \Phi \) also \( \alpha A \in x \). This yields \( W'_L/\Phi, x \models \alpha A \) by Lemma 6.3.

Statement (ii) is obvious by Lemma 6.3 since \( W_L/\Phi \) is persistent. \( \square \)

We summarize the information obtained so far for \( L = \text{RC}\omega \).
Proposition 6.5. Let $\Phi$ be a finite adequate set. Then there is a finite model $W$ such that

(i) $W$ is an irreflexive RJ-model;
(ii) $R_\alpha = \emptyset$, for all $\alpha \notin \ell(\Phi)$;
(iii) $W$ is $\Phi$-monotone and persistent;
(iv) For any $\text{RC}\omega$-theory $\Gamma$ in $\Phi$ there is a node $x \in W$ such that, for any formula $A$, $A \in \Gamma$ iff $W, x \models A$.

7. Arithmetical completeness

Theorem 6. For any sequent $A \vdash B$ the following statements are equivalent:

(i) $A \vdash B$ is provable in $\text{RC}\omega$;
(ii) $A^* \vdash_{\text{PA}} B^*$, for all arithmetical interpretations $*$;
(iii) $A^* \vdash B^*$, for all arithmetical interpretations $*$.

Proof. The implication from (i) to (ii) is Proposition 3.2. Statement (ii) trivially implies (iii). To infer (i) from (iii) we argue by contraposition and assume $A \not\vdash_{\text{RC}\omega} B$. Consider a finite adequate set $\Phi$ containing $A, B$, and let $W$ be a finite Kripke model satisfying the conditions of Proposition 6.5. It falsifies $A \vdash B$ at some node $x$. We can restrict $W$ to the submodel generated by $x$, so that $W$ is rooted and falsifies $A \vdash B$ at the root.

Now we proceed to a Solovay-type construction. As usual, we identify the nodes of $W$ with a finite set of natural numbers $\{1, \ldots, N\}$ so that 1 is the root. We then attach a new root 0 to $W$ by stipulating that $0R_0 x$, for each $x \in W$. The valuation of variables at 0 will be the same as in 1, this ensures that the new model is persistent. Abusing notation we denote this model by the same letter $W$. We also assume that the $R_\alpha$ relations and the forcing relation $x \vDash C$ on $W$ are arithmetized in a natural way by bounded (even open) arithmetical formulas.

We fix an arithmetical formula $\text{Prf}_n(x, y)$ naturally expressing that $y$ is a proof of a formula $x$ from the axioms of $\text{PA}$ and true $\Pi_n$-sentences. The formula $\text{Prf}_n(x, y)$ has logical complexity $\Delta_{n+1}$ in $\text{PA}$. Without loss of generality we may also assume that $\text{Prf}_n$ is chosen in such a way that each number $y$ is a proof of at most one formula, and that any provable formula has arbitrarily long proofs. These properties are also assumed to hold provably in $\text{PA}$.
The formula $\square_n(x) := \exists y \Prf_n(x, y)$ expresses that $x$ is provable in $\mathsf{PA}$ from the set of all true $\Pi_n$-sentences. We usually write $\square_n \varphi$ for $\square_n(\varphi \neg)$. It is easy to see that $\text{Con}_n(\sigma_{\mathsf{PA}})$ is equivalent to $\neg \square_n \perp$.

**Definition 7.1.** Let $M$ denote the maximal modality $m < \omega$ occurring in $\Phi$, if there is such an $m$, and 0 otherwise. We define a family of Solovay-style functions $h_n : \omega \to W$, for all $n < \omega$, as follows: $h_n(0) = 0$ and

$$h_n(x + 1) = \begin{cases} y, & \text{if } h_i(x) \neq h_i(x + 1) = y, \text{ for some } i < n; \text{ otherwise} \\ z, & \text{if } \exists k \geq \max(M, n) \Prf_n(\neg \ell_k \neq \bar{z}, x) \text{ and} \\ & \text{either } h_n(x)R_n z \text{ or } h_n(x)R_\omega z; \\ h_n(x), & \text{otherwise.} \end{cases}$$

Here $\ell_k$ denotes the limit of the function $h_k$. The functions $h_n$ can be defined in such a way as to satisfy the following conditions:

- The graph of each $h_n$ is definable by a formula $H_n$ which is $\Delta_{n+1}$ in $\mathsf{PA}$;
- The function $\varphi : n \mapsto \neg H_n \neg$ is primitive recursive;
- Each $h_n$ satisfies the clauses of Definition 7.1 provably in $\mathsf{PA}$.

The definition of the functions $h_n$ can be arranged as a solution of a fixed point equation in $\mathsf{PA}$ using the standard methods. The details are given in the Appendix.

Informally, the behavior of the functions $h_n$ can be described as follows. The functions with lower index have higher priority, therefore whenever $h_m$ makes a move to $y$, all functions $h_n$ with $n > m$ do the same. Otherwise, $h_n$ moves like the usual Solovay function, but for the following peculiarities:

- $h_n$ also reacts to proofs of the limit statements for all functions of lower priority (not only to those of itself);
- $h_n$ is not only allowed to move along the $R_n$ relation but also along $R_\omega$.

**Lemma 7.2.** For each $n, m$, provably in $\mathsf{PA}$,

1. $\exists! z \in W \ell_n = z$;
2. $\ell_n R_{n+1} \ell_{n+1}$ or $\ell_n R_\omega \ell_{n+1}$ or $\ell_n = \ell_{n+1}$;
3. If $m < n$ then $\ell_m = \ell_n$ or $\ell_m R_\alpha \ell_n$, for some $\alpha \in \{m, n\} \cup \{\omega\}$.
Proof. Statement (i) is proved by (external) induction on $n$. First, we observe that, by the polytransitivity, the relation $R_n \cup R_\omega$ is transitive and irreflexive, for each $n$. Now it is easy to see that the limit of $h_0$ exists, as $h_0$ only moves along $R_0 \cup R_\omega$. Suppose $\ell_{n-1}$ exists. As soon as $h_{n-1}$ reaches its limit, $h_n$ can only move along $R_n \cup R_\omega$. Hence, $\ell_n$ exists.

Statement (ii) follows from the same consideration and the fact that $h_{n+1}$ has to visit $\ell_n$ on its way to the limit. Statement (iii) is obtained from (ii) by induction on $n$. □

Lemma 7.3. For all $n$, $\mathbb{N} \models (\ell_n = 0)$.

Proof. By Lemma 7.2 for all $n > M$, either $\ell_n R_\omega \ell_{n+1}$ or $\ell_{n+1} = \ell_n$, as $R_k = \emptyset$ for $k > M$. Since $R_\omega$ is transitive and irreflexive, there is a $z \in W$ and an $m$ such that $\ell_n = z$, for all $n \geq m$. Assume $z \neq 0$ and let $m$ be the minimal $n$ such that $\ell_n = z$. Then the function $h_m$ has to come to $z$ by the second clause of Definition 7.1. Hence, for some $n \geq \max(M, m)$, $\Box_m (\ell_n \neq \bar{z})$. Since $\text{PA}$ is sound, $\ell_n \neq \bar{z}$ is true, which is not the case since $n \geq m$. □

For any modal formula $C$, let $\ell_n \models C$ denote $\bigvee \{ \ell_n = \bar{a} : a \models C \}$.

Lemma 7.4. For any formula $C$, for all $n > m \geq M$,

$$\text{PA} \vdash (\ell_n \models C) \rightarrow (\ell_m \models C).$$

Proof. Each $h_n$ for $n > M$ can only follow the $R_\omega$ relation. By the persistence of $W$, the truth of any formula is inherited downwards along $R_\omega$. Hence, the claim follows from Lemma 7.2. □

Lemmas 7.2, 7.4 are obviously formalizable in $\text{PA}$ (uniformly in $m, n$).

Suppose $\{ \varphi_i : i \in I \}$ is a primitive recursive set of formulas. With a primitive recursive program computing this set we associate a numeration for the theory $\text{PA} + \{ \varphi_i : i \in I \}$ that will be denoted $\{ \varphi_i : i \in I \}$. We write $\text{Con}_n(\varphi_i : i \in I)$ for $\text{Con}_n(\{ \varphi_i : i \in I \})$. In particular, if this set is a singleton $\{ \varphi \}$, the formula $\text{Con}_n(\varphi)$ means the same as $\text{Con}_n(\varphi)$ and is $\text{PA}$-equivalent to $\neg \Box_n \neg \varphi$.

Using this notation, we interpret each propositional variable $p$ as follows:

$$p^* := [\ell_n \models p : n \geq M].$$

We prove the following two main lemmas.
Lemma 7.5. For any formula $C \in \Phi$,

$$[\ell_n \Vdash C : n \geq M] \vdash_{PA} C^*.$$  \hspace{1cm} (*)

Proof. Induction on the build-up of $C$. The cases of propositional variables, $\top$ and $\land$ are easy.

Assume $C = mD$ for $m < \omega$. Since $(mD)^* = \text{Con}_m(D^*)$ numerates a finite extension of PA, it will be sufficient in this case to infer $\text{Con}_m(D^*)$ from $\ell_M \Vdash mD$ in PA. We have, by the IH,

$$\text{PA} + \text{Con}_m[\ell_n \Vdash D : n \geq M] \vdash \text{Con}_m(D^*).$$

However, $\text{Con}_m[\ell_n \Vdash D : n \geq M]$ is equivalent to the formula

$$\forall n \geq M \text{Con}_m[\bigwedge_{k=M}^n (\ell_k \Vdash D)],$$

which is by Lemma 7.4 equivalent to

$$\forall n \geq M \text{Con}_m[\ell_n \Vdash D]. \hspace{1cm} (4)$$

Thus, we are going to infer sentence (4) from $\ell_M \Vdash mD$ by formalizing the following argument in PA. Assume $\ell_M \Vdash mD$, hence there is a $z$ such that $z \Vdash D$ and $\ell_M R_m z$. Consider the point $\ell_m$. By Lemma 7.2 either $\ell_m R_k \ell_M$ for some $k > m$, or $\ell_m R_\omega \ell_M$ or $\ell_m = \ell_M$. In each case, $\ell_m R_m z$, as $W$ is an RJ-frame.

Assume $\exists n \geq M \neg \text{Con}_m[\ell_n \Vdash D]$, then $\Box_m(\ell_n \nvdash D)$. Since (provably) $z \Vdash D$, we have $\text{Pr}_m[\ell_n \neq \bar{z}]$. Let $x_0$ be such that $\forall x > x_0 h_m(x) = \ell_m$. There is a $y > x$ such that $\text{Prf}_m(\ell_n \neq \bar{z} \land y)$. Then, $h_m(y + 1)$ has to be different from $\ell_m$, a contradiction. This shows that $\ell_M \Vdash mD$ implies $\Box$, as required.

Consider the case $C = \omega D$. Firstly, we have

$$[\text{Con}_n(D^*) : n \geq M] \vdash_{PA} (\omega D)^*,$$

where we may restrict the left hand side to $n \geq M$, since the strength of the formulas $\text{Con}_n$ increases with $n$. By Lemma 7.4 and the IH, as before,

$$[\forall k \geq n \text{Con}_n[\ell_k \Vdash D] : n \geq M] \vdash_{PA} [\forall k \geq M \text{Con}_n[\ell_k \Vdash D] : n \geq M] \vdash_{PA} [\text{Con}_n(D^*) : n \geq M].$$

We are going to show that

$$[\ell_n \Vdash \omega D : n \geq M] \vdash_{PA} [\forall k \geq n \text{Con}_n[\ell_k \Vdash D] : n \geq M].$$
To this end it is sufficient to prove by an argument formalizable in PA that, for any \( n \geq M \),

\[
\text{PA} + (\ell_n \models \omega D) \vdash \forall k \geq n \text{ Con}_n[\ell_k \models D].
\]

Consider any \( n \geq M \) and assume \( \ell_n \models \omega D \). There is a \( z \in W \) such that \( z \models D \) and \( \ell_n R_\omega z \). If \( \exists k \geq n \Box_n(\ell_k \not\models D) \) then \( \exists k \geq n \Box_n(\ell_k \neq z) \). This means that \( h_n \) must take on a value other than \( \ell_n \), a contradiction. \( \square \)

**Lemma 7.6.** For any formula \( C \in \Phi \),

\[
\ell_0 \neq 0 \lor C^* \vdash_{\text{PA}} [\ell_n \models C : n \geq M] . \tag{**}
\]

**Proof.** Induction on the build-up of \( C \). The cases of propositional variables, \( \top \) and \( \land \) are trivial.

Assume \( C = mD \) for \( m < \omega \). Since \( \ell_0 \neq 0 \) is equivalent to a \( \Sigma_1 \)-formula, we obtain

\[
\text{PA} + \ell_0 \neq 0 \land \text{Con}_m(D^*) \vdash \Box_0(\ell_0 \neq 0) \land \text{Con}_m(D^*)
\]

\[
\vdash \text{Con}_m(\ell_0 \neq 0 \lor D^*)
\]

\[
\vdash \text{Con}_m[\ell_k \models D : k \geq M]
\]

\[
\vdash \forall k \geq M \text{ Con}_m[\ell_k \models D].
\]

Thus, it is sufficient to prove, for each \( n \geq M \), that

\[
\text{PA} \vdash \ell_0 \neq 0 \land \ell_n \not\models mD \rightarrow \exists k \geq M \Box_m(\ell_k \not\models D).
\]

We reason as follows.

Assume \( \ell_n \not\models mD \). By Lemma 7.2 we have \( \ell_m R_k \ell_n \), for some \( k > m \), or \( \ell_m R_\omega \ell_n \) or \( \ell_m = \ell_n \). Since \( W \) is a RJ-frame, in each case

\[
\ell_m \not\models mD. \tag{5}
\]

Let \( a := \ell_m \), we have

\[
\exists x (h_m(x) = a \land \forall y \geq x (h_{m-1}(y) = h_{m-1}(x))). \tag{L_m(a)}
\]

(After \( h_m \) attains its limit \( a \), the function \( h_{m-1} \) only has the possibility to make a single move to \( a \), if it moves at all.) The statement \( L_m(a) \) is expressible by a \( \Sigma_{m+1} \)-formula. Hence, \( \Box_m L_m(\bar{a}) \). Moreover, \( L_m(a) \) implies that \( h_m \) goes along the \( R_m \cup R_\omega \) relations from \( a \) onwards. Hence, for any \( k \geq m \), \( L_m(a) \) implies \( \ell_k \in R_m(a) \cup \{a\} \), where

\[
R_m^*(a) := \{ y \in W : \exists a \geq m a R_\alpha y \}.
\]

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Therefore, we conclude:

\[ \forall k \geq m \Box_m (\ell_k \in R^*_m(\bar{a}) \cup \{\bar{a}\}). \]  

(6)

On the other hand, we claim that \( \forall z \in R^*_m(a) z \not\models D \). Indeed, if \( aR\alpha z, \alpha \geq m \) and \( z \models D \), then \( a \models \alpha D \) whence \( a \models mD \), by the \( \Phi \)-monotonicity of \( W \). This contradicts (5). Since the formula \( \forall z \in R^*_m(a) z \not\models D \) is bounded, it follows that

\[ \Box_m(\forall z \in R^*_m(\bar{a}) z \not\models D). \]  

(7)

Next we consider the minimal \( i \leq m \) such that \( \ell_i = \ell_m = a \). Since we assume \( \ell_0 \neq 0 \), we also have \( \ell_m = a \neq 0 \). The function \( h_i \) could only have come to \( a \) by the second clause of the definition of \( h_i \), therefore we obtain

\[ \exists k \geq \max(i, M) \Box_i (\ell_k \neq \bar{a}). \]

Since \( i \leq m \), obviously we can weaken this to

\[ \exists k \geq M \Box_m (\ell_k \neq \bar{a}). \]  

(8)

Together with (6) this yields \( \exists k \geq M \Box_m (\ell_k \in R^*_m(\bar{a})) \), whence by (7) we obtain \( \exists k \geq M \Box_m (\ell_k \not\models D) \), as required.

Finally, consider the case \( C = \omega D \). We have

\( (\omega D)^* = [\text{Con}_n(D^*) : n \in \omega] \).

For each of the axioms \( \text{Con}_n(D^*) \) such that \( n \geq M \), by the IH and persistence we obtain, as before,

\[ \text{PA} + \ell_0 \neq 0 \land \text{Con}_n(D^*) \vdash \forall k \geq M \text{Con}_n[\ell_k \models D]. \]

This fact is formalizable in \( \text{PA} \) uniformly in \( n \), therefore

\[ \ell_0 \neq 0 \lor (\omega D)^* \vdash_{\text{PA}} [\forall k \geq M \text{Con}_n[\ell_k \models D] : n \geq M]. \]

We are going to show that

\[ \ell_0 \neq 0 \lor [\forall k \geq M \text{Con}_n[\ell_k \models D] : n \geq M] \vdash_{\text{PA}} [\ell_n \models \omega D : n > M], \]

which completes the proof, because by Lemma 7.4

\[ [\ell_n \models \omega D : n > M] \vdash_{\text{PA}} [\ell_n \models \omega D : n \geq M]. \]
Thus, we prove by an argument formalizable in $\mathsf{PA}$ uniformly in $n$ that, for any $n > M$,

$$\mathsf{PA} + (\ell_0 \neq 0 \land \ell_n \not\models \omega D) \vdash \exists k \geq M \square_n (\ell_k \not\models D).$$

Assume $n > M$, $\ell_n \not\models \omega D$ and let $a := \ell_n$. Since $\ell_0 \neq 0$ we have $a \neq 0$. Consider the minimal $m \leq n$ such that $\ell_m = \ell_n = a$. As before, we first show that

$$\forall k \geq \max(m, M) \square_n (\ell_k \in R_\omega(a) \cup \{ \bar{a} \}). \quad (9)$$

We consider two cases. If $m > M$ then we have both $L_m(a)$ and $\square_m(L_m(\bar{a}))$. Moreover, for $k \geq m > M$, from $L_m(\bar{a})$ we can infer $\ell_k \in R_\omega(\bar{a}) \cup \{ \bar{a} \}$, since $h_k$ can only make moves along the $R_\omega$ relation from $a$ onwards. (Here, $R_\omega(a) := \{ x \in W : aR_\omega x \}$.) Hence, $\forall k \geq m \square_m (\ell_k \in R_\omega(\bar{a}) \cup \{ \bar{a} \})$ and the claim holds.

If $m \leq M < n$ we first notice that $\ell_M = a$. Then, $\square_n(\ell_M = \bar{a})$, since the formula $\ell_M = \bar{a}$ is $\Sigma_{M+2} \subseteq \Sigma_{n+1}$. Moreover, $\ell_M = a$ implies $\ell_k \in R_\omega(\bar{a}) \cup \{ \bar{a} \}$, for any $k > M$, since $h_k$ will only be able to move along $R_\omega$ from $a$ onwards. Thus, we obtain $\forall k \geq M \square_n (\ell_k \in R_\omega(\bar{a}) \cup \{ \bar{a} \})$, and the claim also holds.

Secondly, we note that $\ell_k \in R_\omega(a)$ implies $\ell_k \not\models D$, as $\forall z \in R_\omega(a) z \not\models D$. Hence, by (9),

$$\forall k \geq \max(m, M) \square_n (\ell_k \not\models D \lor \ell_k = \bar{a}). \quad (10)$$

Thirdly, since $a \neq 0$, by the definition of $h_m$ we have

$$\exists k \geq \max(M, m) \square_m (\ell_k \neq \bar{a}).$$

Together with (10) this yields $\exists k \geq \max(M, m) \square_n (\ell_k \not\models D)$. In either case we obtain $\exists k \geq M \square_n (\ell_k \not\models D)$. $\square$

Recall that at the node $1 \in \mathcal{W}$ there holds $1 \vdash A$ and $1 \not\models B$. Let $\sigma$ denote $[\ell_n = 1 : n \geq M]$ and $S$ denote the corresponding theory. By Lemma 7.5 $\sigma \vdash_{\mathsf{PA}} A^*$. On the other hand, by Lemma 7.6

$$\ell_0 \neq 0 \lor B^* \vdash_{\mathsf{PA}} [\ell_n \vdash B : n > M] \vdash_{\mathsf{PA}} [\ell_n \neq 1 : n > M].$$

Hence, $A^* \vdash B^*$ yields $S \vdash \ell_M \neq 1$. It follows that $S$ is inconsistent. Since $\mathsf{PA} \vdash \ell_n = 1 \rightarrow \ell_m = 1$, for all $m \leq n$, the inconsistency of $S$ yields a $\mathsf{PA}$-proof of $\ell_n \neq 1$, for some $n \geq M$. This means that $h_0$ must eventually take on a value other than $0$, hence $\ell_0 \neq 0$. But this is impossible, since $\ell_0 = 0$ is true in the standard model. $\square$

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Example 7.7. By Example 4.3 \( \omega p \land \omega q \not\vdash_{RC_\omega} \omega(p \land q) \). By Theorem 6 this means that there are theories \( S, T \) containing \( \text{PA} \) such that

\[
\text{PA} + \text{RFN}(S) + \text{RFN}(T) \nvdash \text{RFN}(S + T).
\]

Primitive recursive numerations of these theories can be obtained from the proof of Theorem 6 applied to (an irreflexive version of) the three-element Kripke model described in Example 4.3.

We remark that none of these two theories can have bounded arithmetical complexity over \( \text{PA} \). Suppose \( S \) is axiomatized by a set of \( \Pi_{n+1} \)-sentences over \( \text{PA} \). Then, by Lemma 3.3 \( \text{Con}_n(\sigma) \vdash_{\text{PA}} \sigma \). By Corollary 3.6 it follows that

\[
\text{PA} + \text{Con}_\omega(\sigma) + \text{Con}_\omega(\tau) \vdash \text{Con}_\omega(\tau \lor \text{Con}_n(\sigma)) \vdash \text{Con}_\omega(\tau \lor \sigma).
\]

This shows that the use of infinitely axiomatized theories to interpret propositional variables is necessary for the validity of Theorem 6.

8. Conclusions

We believe that positive provability logic, despite the absence of Löb’s axiom, strikes a good balance between expressivity and efficiency (the latter can be understood formally, as the computational efficiency, as well as informally, in the sense of convenience). Together with [11] this paper shows that positive logic can be nicely treated both syntactically and semantically. More importantly, it has very natural proof-theoretic interpretations not extendable to the full modal logic language.

There are many questions related to this logic that can be further investigated. One direction is to study normal positive logics along the lines of the usual normal modal logics. In particular, we are interested in their efficient proof systems, general results on axiomatization and completeness, interpolation properties, and so on.

Another direction is the study of different arithmetical interpretations of positive provability logic. For example, one can consider from this point of view transfinite iterations of consistency assertions (or of higher reflection principles). That is, one can introduce modalities \( \Diamond^\alpha \), for each ordinal \( \alpha \) of some canonical ordinal notation system, and interpret them as the schemata \( \text{Con}^\alpha \) related to the so-called Turing progressions: \( \text{Con}^0(\sigma) = \text{Con}(\sigma) \); \( \text{Con}^\alpha(\sigma) = \{\text{Con}[\text{Con}^\beta(\sigma)] : \beta < \alpha\} \). It would be interesting to find a complete axiomatization of the corresponding positive logic.
Another generalization is to consider stronger reflection schemata definable in the extensions of arithmetical language, e.g., in the second order arithmetic or in the arithmetic enriched by truthpredicates. This generalization is particularly interesting from the point of view of applications in the ordinal analysis of predicative theories.

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Appendix A. Arithmetization of the Solovay functions

Given a finite irreflexive RJ-frame $W$ we would like to build a family of arithmetical functions $h_n : \omega \to W$, for all $n \in \omega$, satisfying the following conditions:

- The graph of each $h_n$ is definable by a formula $H_n$ which is $\Delta_{n+1}$ in $\text{PA}$;
- There is a primitive recursive function $\varphi_e : n \mapsto \Gamma h_n$, where $e$ denotes the primitive recursive index of this function;
- Each $h_n$ provably satisfies the clauses of Definition 7.1.

These objects will be constructed using the formalized recursion theorem. The main unknown is the index $e$.

Firstly, we stipulate that the limit statements $\ell_n = z$ are abbreviations for the formulas $\exists N \forall k > N H_n(k, z)$. Secondly, we fix a primitive recursive function $g_0$ such that

$$\Gamma \ell_n = z \neg = g_0(\Gamma H_n, z) = g_0(\varphi_e(n), z).$$

We see that the function $g(e, n, z) := g_0(\varphi_e(n), z)$ is provably total recursive in $\text{PA}$, hence it is definable by an arithmetical $\Delta_1$-formula.

Using $g$, Definition 7.1 can be rewritten to define the graphs of $h_n$ in the language of arithmetic with the unknown $e$ as an extra parameter. We denote such parametrized versions of the formulas $H_n$ by $H'_n$. Each formula $H'_n$ uses the formulas $H'_0, \ldots, H'_{n-1}$ as subformulas to express the first clause of the definition of $h_n$. Thus, we obtain a sequence of formulas of the following form:

$$H'_0(e, x, y) \leftrightarrow A_0(e, x, y)$$
$$H'_1(e, x, y) \leftrightarrow A_1(H'_0; e, x, y)$$
$$\ldots$$
$$H'_n(e, x, y) \leftrightarrow A_n(H'_0, \ldots, H'_{n-1}; e, x, y)$$
Here, the formulas $A_n$ directly mimic Definition 7.1. It is easy to convince oneself that the arithmetical complexity of the formulas $A_n$ (and hence, of the formulas $H'_n$) is $\Delta_{n+1}$ in PA. The most complex part of the definition is the formula $\exists k \geq \max(M, n) \Prf_n(\bar{\ell}_k \neq \bar{z}, x)$ occurring in the second clause. In the formula $A_n$ this part takes the form

$$\exists k \geq \max(M, n) \Prf_n(g(e, k, z), x).$$

Observe that the predicate $\Prf_n$ is $\Delta_{n+1}$ (even $\Delta_0(\Sigma_n)$), and the existential quantifier $\exists k$ can, in fact, be bounded by $x$. This yields a $\Delta_{n+1}$-formula.

It is also clear that each formula $H'_n$ is obtained from the previous ones in a primitive recursive way. Therefore, there is a primitive recursive function $F$ satisfying $F(e, n) = \bar{H}'_n(\bar{e}, x, y)$. Finally, we obtain the required number $e$ by applying (a formalized version of) recursion theorem for primitive recursive functions: $\varphi_e(n) := F(e, n)$. Then we can define $H_n(x, y) := H'_n(\bar{e}, x, y)$. 

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