The correct expression for the affine parameter $\gamma$ along $u = u_0$ is given by $\gamma(v) = vM[1/(2\alpha), 1+1/(2\alpha), \alpha A^2 v^{2\alpha}/2]$, and not as appears in the original Letter. A number of quantities which depend on $\gamma$ change as a result of this correction: For $\alpha = 0.3$ the correct value for $\beta$ is $\beta = -8.00 \pm 0.01$, the function $\beta(\alpha)$ for supercritical evolutions is given by $\beta(\alpha) \propto \alpha^{-1.31\pm0.07}$, and $\beta_{\text{crit}} = -26.9 \pm 3.0$. The corrected expression for $\gamma(v)$ also changes Figs. 3, 4, and 5. The corrected figures are given below.

As the new function $\gamma$ is a monotonically increasing function of $v$, and as $\gamma \to \infty$ as $v \to \infty$, it satisfies all the necessary requirement for comparison between advanced times of different spacetimes.

The main conclusions of the original paper remain unchanged. In particular, we find that there are initial data (subcritical data) for which the singularity has a hybrid structure (consisting of a spacelike sector and a null sector), and initial data (supercritical data) for which the singularity is spacelike throughout.
FIG. 5: The exponent $\beta$ as a function of $\alpha$. The circles are the data points, and the solid curve is proportional to $\alpha^{-1.31}$.

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Black hole singularities: a new critical phenomenon

Lior M. Burko
Department of Physics, University of Utah, Salt Lake City, Utah 84112
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The singularity inside a spherical charged black hole, coupled to a spherical, massless scalar field is studied numerically. The profile of the characteristic scalar field was taken to be a power of advanced time with an exponent \( \alpha > 0 \). A critical exponent \( \alpha_{\text{crit}} \) exists. For exponents below the critical one (\( \alpha < \alpha_{\text{crit}} \)) the singularity is a union of spacelike and null sectors, as is also the case for data with compact support. For exponents greater than the critical one (\( \alpha > \alpha_{\text{crit}} \)) an all-encompassing, spacelike singularity evolves, which completely blocks the “tunnel” inside the black hole, preventing the use of the black hole as a portal for hyperspace travel.

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One of the most intriguing questions in general relativity is the inevitability of evolution of singularities under very plausible conditions. Indeed, some view this prediction as evidence that the theory “carries within itself the seeds of its own destruction,” [1] and look for alternative theories which are singularity free. An alternative, more tolerant approach is to view singularities as a phenomenon “from which we can derive much valuable understanding” [2]. Here, we adopt the latter viewpoint.

There are two ways in which generic singularities in general relativity are manifested. The first, and the more familiar way, is through unbounded tidal disruptions, like the singularity inside a Schwarzschild black hole. Singularities of this type are spacelike and deformationally weak. Extended physical objects which approach such a singularity are unboundedly stretched in one direction and compressed in two other directions, such that they are inevitably crushed to zero volume. The second way in which singularities in general relativity are manifested is through breakdown of predictability, as with the Cauchy horizon (CH) singularity inside perturbed spinning black holes. The CH singularity is non-central, null, and deformationally strong. That is, although it is a genuine curvature singularity with infinite tidal forces, the latter are (twice) integrable, such that the tidal distortions are bounded [3]. This singularity signals an inevitable loss of predictability: to the future of the CH singularity (if the spacetime manifold can be extended beyond it) lurks a region of spacetime which is beyond the domain of dependence of any Cauchy hypersurface in the external universe. The occurrence of the null singularity inside realistic black holes is of great interest: It leaves open the possibility that extended objects could traverse the CH only mildly affected, and reemerge in another universe (or a distant portion of our universe), in practice using the black hole as a portal for hyperspace travel.

We consider here the characteristic initial value problem, in which a pre-existing Reissner-Nordström (RN) black hole is perturbed nonlinearly by a self-gravitating scalar field. The scalar field is specified along a union of an outgoing and an ingoing null ray, which is our characteristic hypersurface.

The picture of the singularity inside realistic black holes which has emerged from analytical and numerical studies which assume characteristic data with compact support is the following: The onset of the null singularity at the CH can be well described by perturbation theory. The null generators of the CH focus monotonically, such that the singular sphere at the CH shrinks as it moves with the speed of light. Eventually the singular sphere contracts to zero volume, at which point the causal structure of the singularity changes, and it becomes spacelike, and deformationally strong. Interestingly, both types of singularity coexist inside spherical charged black holes, and possibly also inside spinning black holes. This situation is described by the lower diagram of Fig. 1. The spacelike portion of the singularity inside the black hole is not causally connected with the null portion of the singularity (except at one point). We can strengthen (in a sense to be made precise below) the characteristic data of the (nonlinear) perturbation, and thus make the spacelike singularity appear sooner. This fits well with the general picture: with stronger perturbations the focusing of the null generators of the CH is faster, such that the volume inside the contracting singular sphere vanishes sooner.

Can we specify characteristic data which are so strong that a spacelike singularity evolves before the onset of the null singularity? Below, we shall answer this question in the affirmative for a simplified toy model: Perturbations slightly weaker than a certain critical value have a singularity described by the above picture, i.e., a union of spacelike and null sectors. Stronger perturbations involve only a spacelike singularity, which completely replaces the null singularity and seals off the “tunnel” inside the black hole, preventing its use as a portal.

In this paper we study this question in the context of a spherically symmetric black hole, which is endowed with electric charge. This is a simplified toy model for a spinning black hole, as both exhibit similar causal structures and instabilities of the inner horizons. This black hole is perturbed by a self-gravitating, spherical, massless scalar field. Indeed, in this model the above picture was con-
confirmed using both numerical simulations and analytical analyses for perturbations with compact support (and one family of characteristic data with noncompact support). (The occurrence of a spacelike singularity inside spinning black holes is yet to be confirmed.) Any characteristic data with compact support lead to fields which decay at late times as an inverse power of advanced time along the event horizon, and lead to the null CH singularity at early values of retarded times (connected to a spacelike singularity at late retarded times). Such data are generic, in the sense that they accompany any realistic gravitational collapse process. The perturbations due to the collapse can be thought of as a lower bound on the perturbation field. It is of interest to ask what the singularity structure is when initial data stronger than the lower bound are present. A simple class of noncompact characteristic data was studied in Ref. [6], where the profile of the characteristic scalar field was logarithmic in advanced time. These data preserve the picture of the singularity inside spherical charged black hole: the singularity is a union of spacelike and null sectors. The causal structure of spacetime is again described by the lower diagram of Fig. 1. In this paper we show that other families of characteristic data may lead to dramatically different causal structures.

We write the spherically-symmetric metric in double-null coordinates in the form

$$ds^2 = -2e^{2\sigma(u,v)} du dv + r^2(u,v) d\Omega^2$$

where $d\Omega^2$ is the line element on the unit two-sphere. As the source term for the Einstein equations, we take the contributions of both the scalar field $\Phi$ and the (sourceless) spherical electric field (see [7] for details). The dynamical equations are the scalar field equation $\square \Phi = 0$ and the Einstein equations, which reduce to

$$\Phi_{,uv} + \frac{1}{r} (r_{,v} \Phi_{,u} + r_{,u} \Phi_{,v}) = 0$$

and

$$r_{,uv} + \frac{r_{,u} r_{,v}}{r} + \frac{e^{2\sigma}}{2r^2} \left(1 - \frac{Q^2}{r^2} \right) = 0$$

These equations are supplemented by the two constraint equations

$$r_{,uu} - 2\sigma_{,u} r_{,u} + r(\Phi_{,u})^2 = 0$$

and

$$r_{,uv} - 2\sigma_{,u} r_{,v} + r(\Phi_{,v})^2 = 0.$$

The initial value problem for these equations and our choice of gauge for the coordinates are described in...
Ref. [6]. We choose a gauge in which \( r \) going segment of the characteristic hypersurface \( u = u_0 \) and the solid line is proportional to \( A^{-8.00} \).

The characteristic value problem is then satisfied by taking \( \Phi = 0 \) on \( v = v_0 \). We denote by \( \gamma_* \) the value of \( \gamma \) at which a spacelike singularity is intersected by \( u = u_* \). We remark, that these results show that \( u = u_0 \) is geodesically complete. Notice that \( \gamma(v) \) is a monotonically increasing function. (We do not have problems using \( u \) when comparing between different spacetimes, because \( u \) is defined in terms of the RN geometry that precedes the onset of the perturbations.) With smaller amplitude \( A \), the intersection of the null ray with the singularity occurs at a later value of \( \gamma_* \). (See Fig. 4.) That is, \( \gamma_* \) is a decreasing function of \( A \). If a null singularity exists, there is a finite, nonzero value \( A_{\text{transition}} \), such that \( \gamma_* \to \infty \) as \( A \to A_{\text{transition}}^+ \neq 0 \). If no portion of the singularity is null, \( \gamma_* \to \infty \) as \( A \to 0^+ \).

The results described below refer to the following choice of parameters: \( Q = 0.95, M_{\text{initial}} = 1, r(u_0, v_0) = 5, \) and \( u_* = 21.4 \). Qualitatively similar results were obtained for other choices of the parameters. The numerical code is based on the code in Ref. [7]. We tested the code and found that it is stable and second-order convergent.

Next, we set \( \alpha = 0.3 \). Our results are shown in Fig. 4, which displays \( \gamma_* \) vs. \( A \). For small values of \( A \), \( \gamma_*(A) \propto \log_{10} A \).
A^2$, where $\beta = -8.00 \pm 0.01$. Extrapolating this behavior to smaller values of $A$ we conclude that $\gamma_\ast(A) \to \infty$ as $A \to 0^+$. That is, for this choice of $\alpha$ we find that only a spacelike singularity exists, and that there is no null sector to the singularity. In Ref. [8] it was shown that a spacelike singularity can exist inside spherical charged black holes. Here, we show that inside some spherical charged black holes the singularity is only spacelike.

Figure 4 shows analogous results with $\alpha = 0.1$. Here, we find (using Richardson’s deferred approach to the limit) that $\gamma_\ast(A) \to \infty$ as $A \to A_{\text{transition}}^+ = 4.26 \pm 0.01$. That is, for $A > A_{\text{transition}}^+$ the outgoing null ray intersects with a spacelike singularity, and for $A < A_{\text{transition}}^-$ it intersects with a null singularity.

In between these two values of $\alpha$ we expect that a critical value exists, which marks the transition between black holes whose singularity is composite of two types (spacelike and null), and black holes whose singularity is an all-encompassing spacelike singularity.

We return now to the supercritical case in which only a spacelike singularity exists. The exponent $\beta$ is a function of $\alpha$. Figure 5 shows $\beta(\alpha)$. The exponent $\beta$ is a monotonically increasing function of $\alpha$, and approaching (from above) the critical value $\alpha_{\text{crit}}$ its slope becomes steeper. Fitting this curve, we find that $\beta(\alpha) \propto \alpha^{-1.31 \pm 0.07}$. Notice, that $\beta$ is only defined for $\alpha > \alpha_{\text{crit}}$.

We find that the critical exponent is $\alpha_{\text{crit}} = 0.12 \pm 0.02$. The corresponding value of $\alpha$ is $\beta_{\text{crit}} = -26.9 \pm 3.0$. That is, for values of $\alpha$ smaller than $\alpha_{\text{crit}}$ we find that both spacelike and null sectors coexist. For such values the resulting spacetime has the same causal structure as spacetimes which evolve from perturbations with compact support. In particular, the survival of a null sector leaves open the possibility of harmless traversal of the null singularity. In contrast, for values of $\alpha$ larger than $\alpha_{\text{crit}}$ no null sector survives. The spacelike singularity is all-encompassing, deformationally strong, and completely blocks the “tunnel” inside the black hole.

The numerical value we find for $\alpha_{\text{crit}}$ is not expected to be universal, but instead be model dependent. In particular, we expect it to depend on the ratio $Q/M_{\text{initial}}$ and the type of field used. In particular, we expect $\alpha_{\text{crit}}$ to increase with $Q/M_{\text{initial}}$. Our main result, namely that there are families of characteristic data for which the singularity inside a spherical charged black hole is an all-encompassing spacelike singularity, is expected to be held in general. It is an open question whether our results hold also for spinning black holes. The question of whether the transition we find resembles mathematically the critical phase transitions of statistical physics (like the threshold of black hole formation [9]) awaits further investigation. If so, a possible candidate for a scaling relation could perhaps be the affine length of the null sector of the singularity as a function of $\alpha_{\text{crit}} - \alpha$. Finally, we comment that realistic black holes are always irradiated by photons which originate from the relic cosmic background radiation. This paper then is relevant to the question of whether these photons completely block the “tunnel” inside astrophysical black holes. The answer to this question certainly depends on the cosmological model.

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