Action for (Free) Open String Modes in AdS Space
Using the Loop Variable Approach.

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Abstract

The loop variable technique (for open strings in flat space) is a
gauge invariant generalization of the renormalization group method
for obtaining equations of motion. Unlike the beta functions, which
are only proportional to the equations of motion, here it gives the full
equation of motion. In an earlier paper, a technique was described
for adapting this method to open strings in gravitational backgrounds.
However unlike the flat space case, these equations cannot be derived
from an action and are therefore not complete. This is because there
are ambiguities in the method that involve curvature couplings that
cannot be fixed by appealing to gauge invariance alone but need a
more complete treatment of the closed string background. An indirect
method to resolve these ambiguities is to require symmetricity of the
second derivatives of the action. In general this will involve modifying
the equations by terms with arbitrarily high powers of curvature ten-
sors. This is illustrated for the massive spin 2 field. It is shown that
in the special case of an AdS or dS background, the exact action can
easily be determined in this way.
1 Introduction

The loop variable method [1, 2] which is a generalization of the RG method [3 - 10] of obtaining gauge invariant equations of motion has been primarily applied to the open string in flat space. One obtains the full equation of motion rather than just the beta functions, which are only proportional to the equations. The method has been generalized to closed strings. The problem of combined open and closed strings has not been attempted yet. However in an earlier paper [11] it was shown that a simple procedure exists for writing down gauge and generally covariant equations for massive higher spin modes of the open string in curved spacetime. The procedure was to first take the loop variable equations and covariantize them in a well defined way. The map from loop variables to spacetime fields has to be modified in such a way that gauge transformations can be assigned to spacetime fields. It was shown that this is always possible as long as the fields are massive. This is of course the case for the higher spin open string modes. The modifications involve terms that have the mass parameter in the denominator. The zero mass limit is thus subtle (if at all it is well defined). This procedure was illustrated in detail for a massive spin 2 field.

There are however some ambiguities in the procedure. One is that the loop variable equations themselves can be modified while preserving gauge invariance. The second is that the map to spacetime fields suffers from ambiguities, because in principle there may be gauge invariant and generally covariant terms involving curvature couplings that can be added. These ambiguities can in principle be fixed by modifying the loop variable equation taking into account the modified sigma model action - i.e. the closed string modes. Another indirect way is to demand that the equations be derivable from an action i.e. use the fact that the second derivatives of the action (which involves first derivatives of the equations) should be symmetric in fields. In this paper we adopt this second procedure. In the general case it can be seen that the procedure will involve all higher powers of the curvature tensors. However in the special case of symmetric spaces such as AdS or dS the procedure can be done exactly and one easily obtains an action. Similar actions have appeared earlier in the literature [12, 13].

This paper is organized as follows: Section 2 is a summary of the results of [11]. Section 3 describes some of the modifications required to ensure symmetricity of the derivatives upto the point where higher powers of the curvature appear. Section 4 completes the program for AdS (or dS) backgrounds. Section 5 has some conclusions.
2 Review

We give a summary of the results of [11].

The main ingredients are the following:

1. The loop variable equations for massive spin 2 in flat space read as follows: [11]

\[
-(k_0^2 + (k_0^5)^2)k_1^\mu k_1^\nu + k_1^{(\mu \nu)} k_0 \cdot k_0 = 0
\]

\[
k_0^\mu k_0^\nu k_1 \cdot k_1 + k_1^{(\mu \nu)} k_0^5 k_0^5 - k_1^{5} k_0^5 = 0
\]

\[
k_0^2 k_0^\mu - k_1^\mu k_1 \cdot k_0 + k_0^\mu k_1 \cdot k_1 = 0
\]

\[
-(k_1 \cdot k_0)^2 + (k_0^2 + (k_0^5)^2)k_1 \cdot k_1 - 2k_0^5 k_0^5 = 0
\]

(2.2)

\[
-(k_1 \cdot k_0)^2 + (k_0^2 + (k_0^5)^2)k_1 \cdot k_1 - 2k_0^5 k_0^5 = 0
\]

(2.3)

Here \(\langle k_1^\mu k_1^\nu \rangle = S^{\mu \nu}, \langle k_2^\mu \rangle = S^\mu \) and \(\langle k_0^5 \rangle = S_5\) are the fields describing massive spin 2 (including the auxiliary fields).

The equations are gauge invariant under the gauge transformations:

\[
k_1^\mu \rightarrow k_1^\mu + k_1^{\mu} \Lambda_1 \quad k_2^\mu \rightarrow k_2^\mu + k_1^{\mu} \Lambda_1 + k_0^\mu \Lambda_2 \quad k_5 \rightarrow k_5^5 + 2\Lambda_2 k_0^5
\]

(2.4)

In terms of space time fields we let \(\langle \Lambda_1 k_1^\mu \rangle = \Lambda^\mu \langle \Lambda_2 \rangle = \Lambda \) and \(k_0^\mu\) is \(\partial^\mu\) and \(k_0^5\) is equal to the mass. This gives

\[
\delta S_{\mu \nu} = \partial_{(\mu} \Lambda_{\nu)} \quad \delta S_{\mu} = \partial_{\mu} \Lambda + \Lambda_{\mu} \quad \delta S_{5} = 2\Lambda k_0^5
\]

(2.5)

2. In going to curved space we let \(k_0^\mu = \frac{\partial}{\partial y^\mu}\) where \(y^\mu\) is the Riemann Normal Coordinate (RNC) at some point \(x_0\) which we take as the origin of the RNC coordinate system.

We can then use the following relations to express all the terms in terms of tensors defined at the point \(x_0\):

\[
W_{\alpha_1...\alpha_p}(x) = W_{\alpha_1...\alpha_p}(x_0) + W_{\alpha_1...\alpha_p,\mu}(x_0) y^\mu +
\]

\[
\frac{1}{2!} \{ W_{\alpha_1...\alpha_p,\mu\nu}(x_0) - \frac{1}{3} \sum_{k=1}^{p} R_{\mu\alpha_k \nu}(x_0) W_{\alpha_1...\alpha_{k-1}\beta\alpha_{k+1}...\alpha_p}(x_0) \} y^\mu y^\nu +
\]

\[
\frac{1}{3!} \{ W_{\alpha_1...\alpha_p,\mu\nu\rho}(x_0) - \sum_{k=1}^{p} R_{\mu\alpha_k \nu\rho}(x_0) W_{\alpha_1...\alpha_{k-1}\beta\alpha_{k+1}...\alpha_p}(x_0) \}
\]
Thus for instance it is true that $\Gamma_{\mu\nu}^\rho \big|_{x_0} = 0$ but

$$\partial_\rho \Gamma_{\mu\nu}^\sigma \big|_{x_0} = \frac{1}{3} R_{(\mu|\rho|\nu)}^\sigma(x_0)$$

(2.7)

where we have indicated, by the curved brackets, symmetrization over $\mu, \nu$ and, because of the vertical lines, $\rho$ is excluded from symmetrization.

This gives the following field equations:

$$H_{\mu\nu} \equiv -(D^\rho D_\rho + (k_0^5)^2) S_{\mu\nu} + D^\rho D_{(\mu} S_{\nu)\rho} - \frac{1}{2} D_{(\mu} D_{\nu)} S_\rho^\rho$$

$$+ R_{(\mu}^\rho S_{\beta\rho)} + D_{(\mu} S_{\nu)}(k_0^5)^2 - D_{\mu} D_{\nu} S_5(k_0^5) = 0$$

(2.8)

$$H_\mu \equiv D^\rho D_\rho S_\mu - D^\rho S_{\mu\rho} + D_\mu S_\rho^\rho - D_\mu D^\rho S_\rho - R_\mu^\rho S_\rho = 0$$

(2.9)

$$H \equiv (D^\rho D_\rho + (k_0^5)^2) S^\rho_\sigma - D^\rho D^\sigma S_{\rho\sigma}$$

$$- 2D_\rho S_\rho(k_0^5)^2 + D^\rho D_\rho S_5 k_0^5 - R_\alpha^\beta S_{\alpha\beta} = 0$$

(2.10)

3. The covariantized gauge variation:

$$\delta S_{\mu\nu} = D_{(\mu} \Lambda_{\nu)}$$

(2.11)

along with corresponding ones for the other fields, does not leave the equations invariant. This is because of the curvature terms in (2.6).

The solution given in [11] was to modify the map from loop variables to space time fields. We reproduce those results (after correcting errors in some of the expressions given there):

Thus we let

$$\langle k_0^\mu k_1^\nu k_1^\rho \rangle = D^\mu S^\nu_\rho + \frac{2}{3} R^{\lambda(\nu|\mu|\rho)} S_\lambda \equiv F^{\mu\nu\rho}$$

(2.12)
where \( \tilde{S}_\lambda = S_\lambda - \frac{D_\lambda S_0}{2k_0^2} \)

The extra curvature coupling ensures that the gauge transformation of the RHS of (2.12) (using (2.11) gives exactly

\[
\langle k_0\rho_1 k_0\alpha + k_1\sigma k_0\alpha \rangle = D_\rho D_\sigma A_\alpha + \frac{2}{3} R^\lambda_{(\sigma|\rho|\alpha)} A_\lambda
\]  

(2.13)

where we have used the usual covariantization of the RNC Taylor expansion.

Similarly in the case \( k_0\rho k_0\kappa k_1\nu \) of one adds a tensor \( f_{\rho\sigma\mu\nu} \) and modifies the map in order to ensure the correct gauge transformation:

\[
\langle k_0\rho k_0\kappa k_1\nu \rangle = D_\rho D_\kappa S_{\mu\nu} + \frac{1}{3} R^\beta_{(\mu|\rho|\sigma)} S_{\beta\nu} + \frac{1}{3} R^\beta_{(\nu|\rho|\sigma)} S_{\mu\beta} + f_{\rho\sigma\mu\nu}
\]  

(2.14)

The tensor \( f \) is:

\[
f_{\rho\sigma\mu\nu} = \frac{2}{3} R^\alpha_{(\nu|\rho|\mu)} D_\sigma \tilde{S}_\alpha + \frac{2}{3} R^\alpha_{(\nu|\rho|\sigma)} D_\mu \tilde{S}_\alpha
\]

\[
+ \left[ \frac{1}{2} D_\rho R^\alpha_{(\nu|\rho|\sigma)} - \frac{1}{12} D_\rho R^\alpha_{(\mu|\rho|\sigma)} \right] \tilde{S}_\alpha
\]

\[
+ \left[ \frac{1}{6} D_\sigma R^\alpha_{(\nu|\rho|\mu)} - \frac{1}{12} D_\sigma R^\alpha_{(\mu|\rho|\nu)} \right] \tilde{S}_\alpha
\]

\[
+ \left[ \frac{1}{6} D_\sigma R^\alpha_{(\nu|\rho|\mu)} - \frac{1}{12} (D_\mu R^\alpha_{\nu\rho\sigma} + D_\nu R^\alpha_{\rho\mu\sigma}) - \frac{1}{12} (D_\mu R^\alpha_{\rho\nu\sigma} + D_\nu R^\alpha_{\mu\rho\sigma}) \right] \tilde{S}_\alpha
\]

Using these results one finds in place of (2.8), (2.16), (4.36) the following (although the tensor \( f \) complicated, the combination \( -f^\rho_{\mu\rho} + f^\rho_{(\mu|\rho|\sigma)} \) that occurs in \( H_{\mu\nu} \) is actually quite simple as can be seen below):

\[
H_{\mu\nu} = -(D^\rho D_\rho + (k_0^5)^2) S_{\mu\nu} + D^\rho D_{(\mu|\nu|\sigma)} D^\sigma S^\rho - \frac{1}{2} D_{(\mu|\nu)} D^\rho S^\rho
\]

\[
+ R^\beta_{(\nu|\mu)} S_{\beta\rho} + D_{(\mu|\sigma|\rho)} (k_0^5)^2 - D_\mu D_\nu S_{\sigma\rho} - 2R^\alpha_{(\nu|\mu)} D_\rho \tilde{S}_\alpha - 2R^\alpha_{(\nu|\mu)} D_\mu \tilde{S}_\alpha - D_{(\nu|\rho|\sigma)} \tilde{S}_\alpha = 0. \quad (2.15)
\]

\[
H_{\mu} = D^\rho D_\rho S_{\mu} - D^\rho S_{\mu\rho} + D_\mu S^\rho - D_\mu D^\rho S_{\rho} - R^\rho_{\mu\rho} S_{\rho} + 2R^\rho_{\mu\rho} \tilde{S}_{\rho} = 0 \quad (2.16)
\]

\[
H = 0
\]
\[
(D^\rho D_\rho + (k_0^5)^2)S_\sigma^\sigma - D^\rho D_\sigma S_{\rho\sigma} \\
- 2D^\rho S_\rho (k_0^5) + D^\rho D_\rho S_5k_0^5 - R^{\alpha\beta}S_{\alpha\beta} + 4R^{\alpha\rho}D_\rho \tilde{S}_\alpha + 2D_\rho R^{\alpha\rho} \tilde{S}_\alpha = 0
\]

(2.17)

These equations are gauge invariant, (manifestly) general coordinate covariant, and reduce to the correct equations in the flat space limit. This concludes our summary of [11].

3 Towards an Action

The question we can now ask is whether these equations can be obtained from an action. This is equivalent to checking whether \( \frac{\partial H_i}{\partial \phi_i} = \frac{\partial H_i}{\partial \phi_j} \). If \( H_i = \frac{\partial S}{\partial \phi_i} \), this would be true. It is easy to see that these equations do not satisfy these conditions. In fact even the flat space equations need to be redefined for this to be true. Thus we define

\[
H'_\mu\nu = H_{\mu\nu} + g_{\mu\nu}H, \quad H' = H + g^{\mu\nu}H_{\mu\nu}
\]

Thus we get:

\[
H'_\mu\nu \equiv \\
- (D^\rho D_\rho + (k_0^5)^2)S_{\mu\nu} + D^\rho D_{(\mu}S_{\nu)} - \frac{1}{2}D_{(\mu}D_{\nu)}S_\rho^\rho \\
+ R_{_{\nu\mu\rho\rho}}^\beta S_{\beta\rho} + D_{(\mu}S_{\nu)}(k_0^5)^2 - D_\mu D_\nu S_5(k_0^5) \\
- 2R_{_{\nu\mu\rho\rho}}^\alpha D_\rho \tilde{S}_{\alpha} - D_\rho R_{_{\nu\mu}}^\alpha \tilde{S}_{\alpha} - 2R_{_{\nu\mu\rho}}^\alpha D_\rho \tilde{S}_{\alpha} - D_{(\mu}R_{\nu)}^\alpha \tilde{S}_{\alpha} \\
+ g_{\mu\nu}[(D^\rho D_\rho + (k_0^5)^2)S_\sigma^\sigma - D^\rho D_\sigma S_{\rho\sigma} \\
- 2D^\rho S_\rho (k_0^5)^2 + D^\rho D_\rho S_5k_0^5 - R^{\alpha\beta}S_{\alpha\beta} + 4R^{\alpha\rho}D_\rho \tilde{S}_\alpha + 2D_\rho R^{\alpha\rho} \tilde{S}_\alpha]\right] = 0 \quad (3.19)
\]

\( H' \) actually simplifies to:

\[
H' \equiv \\
D^\rho D_\sigma S_{\rho\sigma} - D^\rho D_\rho S_\sigma^\sigma + R^{\alpha\beta}S_{\alpha\beta} - 4R^{\alpha\rho}D_\rho \tilde{S}_\alpha - 2D_\rho R^{\alpha\rho} \tilde{S}_\alpha = 0 \quad (3.20)
\]

The flat space limit of these equations can be derived from an action (actually \( H' \) needs to be scaled by a factor of \( \frac{k_0^5}{2} \), but we will deal with this in section 4.).

In curved spacetime one can check that

\[
\frac{\partial H'_{\mu\nu}(x)}{\partial S^\alpha(y)} = g_{\alpha(\mu}D_{\nu)}\delta(x-y)(k_0^5)^2 - 2(k_0^5)^2 g_{\mu\nu}D_\alpha \delta(x-y)
\]
\[-2R^{\alpha}_{(\nu} \rho D_{\rho} \delta(x-y) - D_{\rho} R^{\alpha}_{(\nu} \rho \delta(x-y) \]
\[-2R^{\alpha}_{(\mu} D_{\alpha} \rho \delta(x-y) - D_{(\mu} R^{\alpha}_{\nu)} \rho \delta(x-y) + g_{\mu \nu} (4R^{\rho}_{\alpha} D_{\rho} + 2D^{\rho} R_{\rho \alpha}) \delta(x-y) \]

which is not equal to
\[
\frac{\partial H_{\alpha}(y)}{\partial S^{\mu \nu}(x)} = -[g_{\alpha(\mu} D_{\nu)} \delta(x-y) (k^5_0)^2 - 2(k^5_0)^2 g_{\mu \nu} D_{\alpha} \delta(x-y)] \]

because of the curvature couplings (overall sign can be taken care of easily).

At this point we should point out that there are two ambiguities in the procedure outlined above. One is that the loop variable equation (2.03) for the vector can actually be generalized to:
\[
k_5^2 k_6^2 - \frac{1}{2} (k_1^2 k_0^0 + k_3^2 k_0^0) + k_1^2 k_0^0 - \frac{1}{2} (k_2^2 k_0^0 + k_2^2 k_0^0) k_0^0 = 0 \]

Thus instead of contracting with $g^{\rho \sigma}$ one can use more generally $X^{\alpha \rho \sigma \mu}$.

The second ambiguity is that the combination $S_{\mu \nu} - D_{(\mu} \tilde{S}_{\nu)}$ is gauge invariant and one can add terms involving this to the equations of motion. Thus if there had been a term in the action of the form
\[
\frac{1}{2} (S_{\alpha \rho} - D_{(\rho} \tilde{S}_{\alpha)}) T^{\alpha \rho \mu}(S_{\mu \nu} - D_{(\mu} \tilde{S}_{\nu)})
\]

this would contribute a term
\[
(D_{\rho} T^{\alpha \rho \mu})(S_{\mu \nu} - D_{(\mu} \tilde{S}_{\nu)}) + T^{\alpha \rho \mu} D_{\rho}(S_{\mu \nu} - D_{(\mu} \tilde{S}_{\nu)}) \]

(3.24)

to $H_{\alpha}$.

Similarly
\[
\frac{1}{2} (S_{\alpha \mu} - D_{(\mu} \tilde{S}_{\alpha)}) T^{\alpha \mu}(S_{\mu \nu} - D_{(\mu} \tilde{S}_{\nu)})
\]

(3.25)

contributes a term
\[
(D_{(\mu} T^{\alpha}_{\nu)})(S_{\mu \nu} - D_{(\mu} \tilde{S}_{\nu}) + T^{\alpha}_{(\nu} D_{\mu)}(S_{\mu \nu} - D_{(\mu} \tilde{S}_{\nu}))
\]

(3.26)

to $H_{\alpha}$.

One can see that these are the right kinds of terms that one needs to get agreement between (3.21) and (3.22). The coefficients of $D_{\rho} RS$ and $RD_{\rho} S$ are not equal (in (3.21), therefore (3.24) and (3.26) by themselves cannot be sufficient - we will also need terms involving $R$ in the coefficient tensor $X^{\alpha \rho \sigma \mu}$ that multiplies the vector equation.
If we assume that $X$ and $T$ are linear in the curvature tensor then one can write the most general form for $X$ and $T$ and attempt to satisfy the integrability condition.

The result of doing this is that if one chooses

$$X^{\alpha \nu \rho \mu} = -(k_0^5)^2 g^{\rho \sigma} g_{\alpha \mu} + \frac{2}{3} R^{\alpha \nu \rho \mu} + g^{\nu \rho} R_{\alpha \mu}$$  \hspace{1cm} (3.27)$$

$$T^\alpha_{(\mu | \nu)} = -R^\alpha_{(\mu | \rho | \nu)} + 2g_{\mu \nu} R_{\alpha \rho} ; \quad T^\alpha_{\alpha \nu} = -\frac{1}{2} R^\alpha_{\alpha \nu}$$  \hspace{1cm} (3.28)$$

one finds that

$$\frac{\partial H_\alpha (y)}{\partial S^\mu \nu (x)} = \frac{\partial H'_\mu \nu (x)}{\partial S^\alpha (y)}$$

With this our equation for the vector becomes:

$$H_\alpha \equiv$$

$$X^{\alpha \nu \rho \beta} \left[ \frac{1}{2} D_{(\nu} D_{\sigma)} S_{\beta} - \frac{1}{2} D_{(\nu} S_{\beta)} \right] + D_{(\nu} S_{\rho} \sigma - \frac{1}{2} D_{(\nu} D_{(\sigma} S_{\beta) \rho}$$

$$+ \frac{1}{2} R^\lambda_{(\sigma | \rho) \beta \alpha} S_\lambda - R^\lambda_{(\sigma | \rho) \beta} \tilde{S}_\lambda \right]$$

$$(D_{\rho} T^\alpha_{\nu \rho \mu})(S_{\mu \nu} - D_{(\mu} \tilde{S}_{\nu)}) + T^\alpha_{\nu \rho \mu} D_{\rho}(S_{\mu \nu} - D_{(\mu} \tilde{S}_{\nu)})$$

$$(D_{(\nu} T^\alpha_{\mu)})(S_{\mu \nu} - D_{(\mu} \tilde{S}_{\nu)}) + T^\alpha_{(\nu} D_{\mu)}(S_{\mu \nu} - D_{(\mu} \tilde{S}_{\nu}) = 0$$  \hspace{1cm} (3.29)$$

One immediately sees that $R^2$ terms have appeared in the equation. In particular the coupling to the scalar $S_5$ (contained in $\tilde{S}_\lambda$) has an $R^2$ piece. The scalar equation, which was linear in the curvature tensor, will now have to be modified. This will have to involve the gauge invariant combination of fields $S_{\mu \nu} - D_{(\mu} \tilde{S}_{\nu)}$. This would then modify the coupling to $S_{\mu \nu}$ by $R^2$ terms. In this manner one can presumably modify the equations iteratively, in powers of $R$, and one expects some non polynomial (in $R$) result. We will not attempt to do this in this paper. In the next section we shall specialize to AdS space where we will see that the process terminates in one iteration and a gauge invariant action can be written.

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1 One should also ensure that the symmetry properties of the tensor $T$ are such that terms involving more than two derivatives of $S_5$ actually vanish.
4 AdS Action

In AdS space we can take

\[ R_{\mu\nu\rho\sigma} = -\frac{1}{L^2} (g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma}) \]

\[ R_{\mu\rho} = g^{\rho\sigma} R_{\mu\nu\rho\sigma} = -\frac{(D-1)}{L^2} g_{\mu\rho} \] \hspace{1cm} (4.30)

We can use this form in the tensor \( X \) which becomes \((b = (D-1)):\)

\[ X^{\alpha\rho\sigma\beta} = -\left[ \frac{b-2/3}{L^2} + (k_0^5)^2 \right] g^{\rho\sigma} g^{\alpha\beta} - \frac{2}{3L^2} g^{\alpha\sigma} g^{\rho\beta} \] \hspace{1cm} (4.31)

The contribution to the equation of motion \( H_\alpha \) due to the \( X \)-term can be worked out. It is:

\[-\left[ \frac{b-1}{L^2} + (k_0^5)^2 \right] [D^\rho D_\rho S_\mu - D^\mu S_{\alpha\rho} + D_\alpha S_\rho^\rho - D_\rho D_\rho S_\alpha - \frac{2b}{L^2} \left( \frac{S_\alpha}{2} - \tilde{S}_\alpha \right)] = 0 \] \hspace{1cm} (4.32)

Similarly the contribution from the terms involving \( T \) turn out to have the same structure:

\[-\left[ \frac{b-1}{L^2} + (k_0^5)^2 \right] [D^\rho D_\rho S_\alpha - D^\rho S_{\alpha\rho} + D_\alpha S_\rho^\rho - D_\rho D_\rho S_\alpha - \frac{2b}{L^2} \left( \frac{S_\alpha}{2} - \tilde{S}_\alpha \right)] = 0 \] \hspace{1cm} (4.33)

Adding the two then gives \( H_\alpha \):

\[-\left[ \frac{2(b-1)}{L^2} + (k_0^5)^2 \right] [D^\rho D_\rho S_\mu - D^\mu S_{\alpha\rho} + D_\alpha S_\rho^\rho - D_\rho D_\rho S_\alpha - \frac{2b}{L^2} \left( \frac{S_\alpha}{2} - \tilde{S}_\alpha \right)] = 0 \] \hspace{1cm} (4.34)

Since this equation is already compatible with \( H'_{\mu\nu} \) we turn to the scalar equation \( H' \) specialized to AdS:

\[ H' \equiv \]

\[-\left[ \frac{2(b-1)}{L^2} + (k_0^5)^2 \right] [D^\rho D^\sigma S_{\rho\sigma} - D^\rho D^\sigma S_{\alpha\rho} + D_\alpha S_\rho^\rho - D_\rho D_\sigma S_\alpha - \frac{2b}{L^2} \left( \frac{S_\alpha}{2} - \tilde{S}_\alpha \right)] = 0 \] \hspace{1cm} (4.35)

Similarly the tensor equation is:

\[ H'_{\mu\nu} \equiv \]

\[ D^\rho D^\sigma S_{\rho\sigma} - D^\rho D^\sigma S_{\alpha\rho} - \frac{b}{L^2} S_{\rho\sigma}^\sigma + \frac{b}{L^2} D^\rho \tilde{S}_\rho = 0 \] \hspace{1cm} (4.35)

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2By changing the sign of \( L^2 \) these results apply also to de Sitter space.
\[-(D^\rho D_\rho + (k_0^5)^2)S_{\mu\nu} + D^\rho D_{(\mu}S_{\nu)} - \frac{1}{2} D_{(\mu}D_{\nu)}S^\rho_{\rho} \]
\[\quad - \frac{2}{L^2} (g_{\mu\nu}S^\rho_{\rho} - S_{\mu\nu}) \]
\[\quad + \frac{2}{L^2} [2g_{\mu\nu}D^\rho \tilde{S}_\rho] + \frac{(2(b - 1) + (k_0^5)^2)}{L^2} D_{(\nu}S_{\mu)} \]
\[\quad + g_{\mu\nu} [(D^\rho D_\rho + (k_0^5)^2 + \frac{b}{L^2})]S_{\sigma}^\sigma - D^\rho D^\sigma S_{\rho\sigma} \]
\[\quad - \frac{4b}{L^2} D^\rho \tilde{S}_\rho - 2D^\rho S_\rho (k_0^5)^2 + D^\rho D_\rho S_5 k_0^5 \]
\[= -(D^\rho D_\rho + (k_0^5)^2)S_{\mu\nu} + D^\rho D_{(\mu}S_{\nu)} - \frac{1}{2} D_{(\mu}D_{\nu)}S^\rho_{\rho} \]
\[\quad - \frac{2}{L^2} (g_{\mu\nu}S^\rho_{\rho} - S_{\mu\nu}) \]
\[\quad + \frac{2}{L^2} (2(b - 1) + (k_0^5)^2) D_{(\nu}S_{\mu)} \]
\[\quad + g_{\mu\nu} [(D^\rho D_\rho + (k_0^5)^2 + \frac{b}{L^2})]S_{\sigma}^\sigma - D^\rho D^\sigma S_{\rho\sigma} \]
\[\quad - 2(\frac{2(b - 1)}{L^2} + (k_0^5)^2) D^\rho \tilde{S}_\rho \]
\[= 0 \quad (4.36) \]

**Compatibility of Scalar and Tensor equations:**

Again if one calculates
\[\frac{\partial H'}{\partial S_{\mu\nu}} = k_0^5 [D_{(\mu}D_{\nu)} - 2g_{\mu\nu}D^\rho D_\rho] \delta(x - y) - \frac{2b}{L^2} k_0^5 g_{\mu\nu} \delta(x - y) \quad (4.37)\]

Note that we have multiplied the scalar equation (4.46) by $k_0^5$. The last term in the above expression is missing in $\frac{\partial H'}{\partial S_{\mu\nu}}$ as one can easily check. Thus one has to modify these two equations. There are two obvious modifications that are possible. One is a field redefinition: $S_{\mu\nu} = S'_{\mu\nu} + x g_{\mu\nu} S_5$. Another is to modify the $H'$ equation by terms that could potentially come from terms of the form (3.24), (3.26). They contribute linear combinations:
\[uD^\rho D_\rho S_{\rho\sigma} - 2D^\sigma \tilde{S}_{\sigma} + v[D^\rho D^\sigma - D^\rho D^\sigma D_{(\rho}S_{\sigma)}] \]

If one chooses $u + v = 0$ the higher derivative terms cancel. Thus we get the contribution
\[u[D^\rho D^\rho S_{\rho\sigma} + k_0^5 - k_0^5 D^\rho D^\sigma S_{\rho\sigma}] - 2u \frac{b}{L^2} D^\rho \tilde{S}_{\rho} k_0^5 \]
Thus we make both changes: Add the above term to the scalar equation, and replace $S_{\mu\nu}$ by $S'_{\mu\nu} + xg_{\mu\nu}S_5$ in all the equations. We then get:

$$\frac{\partial H'}{\partial S^\mu_{\nu}} = k_0^5(1 - u)[D_{\mu}D_{\nu} - 2g_{\mu\nu}D^\rho D_{\rho}]\delta(x - y) - \frac{2b}{L^2}k_0^5g_{\mu\nu}\delta(x - y) \quad (4.38)$$

$$\frac{\partial H'_{\mu\nu}}{\partial S_5} = \left(\frac{x(1 - b)}{2} - \frac{k_0^5}{2} + \frac{2(1 - b)}{L^22k_0^5}\right)[D_{\mu}D_{\nu} - 2g_{\mu\nu}D^\rho D_{\rho}]\delta(x - y) + bxg_{\mu\nu}[k_0^5] + \frac{(b - 1)}{L^2}\delta(x - y) \quad (4.39)$$

Requiring that the corresponding coefficients be proportional (they don’t have to be equal because the scalar equation can be scaled by an overall factor) gives the following relation:

$$\frac{1}{L^2k_0^5}[(k_0^5)^2 + \frac{2(1 - b)}{L^2}] = x[(k_0^5)^2(1 - u) - \frac{u(b - 1)}{L^2}] \quad (4.40)$$

Compatibility of Scalar and Vector Equations:

$$\frac{\partial H_{\alpha}}{\partial S_5} = -b(x + \frac{1}{L^2k_0^5})[(k_0^5)^2 + \frac{2(1 - b)}{L^2}]D_{\alpha}\delta(x - y)$$

$$\frac{\partial H'}{\partial S_5} = \frac{2bk_0^5}{L^2}(2 - u)D_{\alpha}\delta(x - y)$$

Requiring equality of these coefficients one gets

$$\frac{2bk_0^5}{L^2}(2 - u) = -b(x + \frac{1}{L^2k_0^5})[(k_0^5)^2 + \frac{2(1 - b)}{L^2}] \quad (4.41)$$

Solving (4.40) and (4.41) gives

$$u = 2 \quad ; \quad x = \frac{1}{k_0^5L^2} \quad (4.42)$$

Plugging in the rescaling factor, one also concludes that the actual scalar equation is:

$$\frac{\delta S}{\delta S_5} = \frac{1}{2[(k_0^5)^2 + \frac{(b - 1)}{L^2}]}H' \quad (4.43)$$

Note also that in the final solution the vector and scalar are decoupled exactly as in the flat space limit.

After making these changes the results are as follows:
Tensor Equation:
\[ H'_{\mu\nu} \equiv \]
\[ -(D^\rho D_\rho + (k_0^5)^2)S_{\mu\nu} + D^\rho D_{(\mu} S_{\nu)} - \frac{1}{2} D_{(\mu} D_{\nu)} S_{\rho}^\rho - g_{\mu\nu} D^\rho D^\sigma S_{\rho\sigma} \]
\[ - \frac{2}{L^2} (g_{\mu\nu} S_{\rho}^\rho - S_{\mu\nu}) \]
\[ + \frac{2(D-2)}{L^2} + (k_0^5)^2 [D_{(\mu} S_{\rho)} - 2g_{\mu\nu} D^\rho S_{\rho}] \]
\[ + g_{\mu\nu} [(D^\rho D_\rho + (k_0^5)^2 + \frac{D-1}{L^2})]S_{\sigma}^\sigma \]
\[ + \frac{1}{k_0^5} \frac{(D-2)}{L^2} + (k_0^5)^2 [g_{\mu\nu} D^\rho D_\rho S_5 - D_\mu D_\nu S_5 - \frac{D-1}{L^2} g_{\mu\nu} S_5] = 0 \] (4.44)

Vector Equation:
\[ H_\alpha \equiv \]
\[ - \left[ \frac{2(D-2)}{L^2} + (k_0^5)^2 \right] [D^\rho D_\rho S_\alpha - D^\rho S_{\alpha\rho} + D_\alpha S_{\rho}^\rho - D_\alpha D^\rho S_\rho - \frac{D-1}{L^2} S_\alpha] = 0 \] (4.45)

Scalar Equation:
\[ H' \equiv \]
\[ \frac{1}{2k_0^5} \frac{(D-2)}{L^2} + (k_0^5)^2 [\frac{1}{2} S_{\rho\sigma}^\rho D^\sigma S_{\rho\sigma} + D^\rho D_\rho S_\sigma - \frac{D-1}{L^2} S_\sigma] = 0 \] (4.46)

Action:
\[ S = \int d^D x \left\{ - \frac{1}{2} S_{\mu\nu} [D^\rho D_\rho + (k_0^5)^2]S_{\mu\nu} + \frac{1}{2} S_{\mu\nu} D^\rho D_\mu S_{\nu\rho} - \frac{1}{2} S_{\alpha\beta} D_\alpha D_\beta S_{\sigma}^\sigma \right. \]
\[ + \frac{1}{2L^2} (S_{\rho\sigma}^\rho S_{\rho\sigma} - S_{\sigma}^\sigma S_{\rho}^\rho) + \left( \frac{2(D-2)}{L^2} + (k_0^5)^2 \right) [(S_{\alpha\beta} D_\alpha S_\beta - S_\sigma D^\sigma S_{\rho}) \right. \]
\[ + \frac{1}{4} S_{\sigma}^\sigma (D^\rho D_\rho + (k_0^5)^2) S_{\rho}^\rho \]
\[ + \frac{1}{2k_0^5} \frac{(D-2)}{L^2} + (k_0^5)^2 [S_{\sigma}^\sigma D^\rho D_\rho S_5 - S_{\sigma}^\sigma D_\mu D_\sigma S_5 - \frac{D-1}{L^2} S_{\sigma}^\sigma S_5] \]
\[ \left( \frac{2(D-2)}{L^2} + (k_0^5)^2 \right) \left\{ - \frac{1}{2} D_\rho S_{\sigma} D^\rho S_{\sigma}^\sigma + \frac{1}{2} D_\alpha S_\alpha D^\sigma S_{\rho}^\sigma + \frac{D-1}{2L^2} S_{\sigma}^\sigma S_{\rho} \right\} \right\} \] (4.47)

Note that the gauge transformation of the tensor is different after the field redefinition:
\[ \delta S_{\mu\nu} = \partial_{(\mu} \Lambda_{\nu)} + \frac{2}{L^2} g_{\mu\nu} \Lambda \]
5 Conclusions

As explained in the introduction, in flat space the loop variable method for open strings gives the full equation of motion, unlike the beta function, which is only proportional to it. So one can extract an action from this information. In curved space things are more complicated. One has to include the bulk two dimensional field theory operators in addition to the boundary terms. In principle one should be able to get the equations by including the RG action on the combined closed-open two dimensional field theory. This seems quite involved. In an earlier paper [11] we described a simple procedure to generalize the flat space equations to curved space equations. The main idea was to interpret the loop variable momenta as being conjugate to Riemann Normal Coordinates. It was also shown there that it is possible to maintain gauge invariance in curved space - which is usually the stumbling block. However the question of an action was not addressed.

In this paper we have addressed this question. The equations obtained earlier (in [11]) cannot be obtained from an action. They have to be modified. We have shown that this is possible because there are in fact ambiguities in the procedure of [11] that can be exploited. One can modify the equations by gauge invariant terms that involve curvature tensors. One can fix these ambiguities by imposing the requirement of compatibility of the equations, i.e by the requirement that the equations be derivable from an action. This implies that the second derivatives have to be symmetric. We discussed the spin two case for general backgrounds and showed that this leads to inclusion of terms involving curvature. It is an iterative process and for general backgrounds involves arbitrary high powers of the curvature tensors. However in the AdS case the method gives an exact form of the action and we recover known results quite easily.

This leads to many open questions. One question is whether for general backgrounds one can obtain all the correction terms. More interestingly the loop variable method can be easily generalized to the interacting string (in curved spacetime) case. The problem of obtaining an action has to be addressed there also. Since we have an infinite number of coupled equations it is likely that this procedure of requiring that the equations be derivable from an action would be impractical. One should perhaps go back to the “first principles” calculation of the gauge invariant exact RG equations of the two dimensional field theory. One should also keep in mind that in string theory interaction with gravity implies that the string is self interacting since both interactions are generically governed by the same coupling constant.
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