Thermodynamic Formalism for Transient Potential Functions

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Abstract

We study the thermodynamic formalism of locally compact Markov shifts with transient potential functions. In particular, we show that the Ruelle operator admits positive continuous eigenfunctions and positive Radon eigenmeasures in forms of Martin kernels. These eigenmeasures can be characterized in terms of the direction of escape to infinity of their orbits, when viewed inside a suitable Martin-like compactification of the underline shift space. We relate these results to first-order phase transitions in one-dimensional lattice gas models with infinite set of states. This work complements earlier works by Sarig [26, 27] who focused on the recurrent scenario.

1 Introduction

The main tool in the study of Thermodynamic Formalism for topological Markov shifts (or shortly TMS, see definition in Section 1.1) is the Ruelle operator

\[(L_\phi f)(x) = \sum_{y : Ty = x} e^{\phi(y)} f(y)\]

and in particular its eigenfunctions and eigenmeasures, see for example [4, 26, 21, 32, 25, 19, 2, 3, 31].

For a topologically transitive one-sided TMS \((X^+, T)\) with finite set of states \(S\) and a Hölder continuous \(\phi : X^+ \to \mathbb{R}\), Bowen [4] and Ruelle [25] showed that there is a positive continuous eigenfunction \(L_\phi h = \lambda h\) and a positive eigenmeasure \(L_\phi^t \nu = \lambda^t \nu\) with \(\nu(h) = 1\) and \(\log \lambda\) is the pressure of \(\phi\). The eigenvectors \(\nu, h\) are unique up to scaling and the measure \(\mu = h \nu\) is the unique equilibrium state which maximizes \(h_{\mu}(T) + \mu(\phi)\). The eigenmeasures of the Ruelle operator are also called conformal measures and their Jacobian \(\frac{d\nu}{d\nu_{\phi}\circ T}\) is \(\lambda^{-1} \exp \phi\).

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For a topologically transitive TMS with countable number of states, \(|S| = \infty\), Sarig [27] showed that the behaviour of a H"older continuous potential function \(\phi\) with finite Gurevich pressure can be characterized either as positive recurrent, null recurrent or transient. If \(\phi\) is positive recurrent, then the situation is similar to the finite case: the eigenfunction, eigenmeasure and the equilibrium state exist and are unique if \(X^+\) is topologically mixing [5]. If \(\phi\) is null recurrent, \(h\) and \(\nu\) still exist and unique but now \(h\nu\) is an infinite conservative measure, which makes the discussion on entropy and equilibrium states more subtle.

As for transient \(\phi\), for a locally compact \(X^+\), Cyr [7], and later on Thomsen [33], showed the existence of an eigenmeasure, as a weak*-converging sub-sequence of

\[
\mu_k(A) := \frac{\sum_{n=0}^{\infty} L_\phi^1 1_A(T^k x)}{\sum_{n=0}^{\infty} L_\phi^1 1_{[o]}(T^k x)}
\]

where \(T^k x \to \infty\) (escapes every compact set) and \(o \in S\) is arbitrary.

The existence of eigenfunctions was not discussed in these papers. For more on the Thermodynamic Formalism of a transient potential function, see [15].

The purpose of this paper is to complete the analysis on the eigenmeasures and eigenfunctions in the transient case. For a locally compact, topologically transitive Markov shift equipped with a \(\lambda\)-transient potential function (see definitions in Section 1.1), we show the following:

1. **Existence of eigenvectors:** There exists a positive Radon measure \(\mu\) and a positive continuous function \(h\) s.t. \(L_\phi^1 \mu = \lambda \mu\) and \(L_\phi^1 h = \lambda h\). The existence of \(\mu\) was shown before by Cyr [7], but the existence of \(h\) is, as far as we know, new. Examples show that \(\mu\) or \(h\) need not be unique.

2. **Representation of eigenvectors:** We extend Martin’s representation theorem [20] to the context of Ruelle operator. Specifically, we construct a compactification \(\hat{X}^+\) of \(X^+\) with boundary \(\mathcal{M} = \hat{X}^+ \setminus X^+\) and construct a kernel \(K(f, \omega|\lambda)\) \((f \in C_c(X^+), \omega \in \hat{X}^+)\) s.t. every Radon measure \(\mu\) with \(L_\phi^1 \mu = \lambda \mu\) has the form

\[
\mu(\cdot) = \int_{\mathcal{M}} K(\cdot, \omega|\lambda) d\nu(\omega)
\]

for some finite measure \(\nu\) on \(\mathcal{M}\). We provide similar representation for positive eigenfunctions using a compactification of the negative one-sided Markov shift \(X^-\) (see definitions in Section 5).

3. **Direction of escape to \(\infty\):** We show that every extremal \(\lambda\)-eigenmeasure can be represented by \(\nu \propto \delta_\omega\), for some \(\omega \in \mathcal{M}\). For this \(\omega\),

\[
T^n x \to \omega \quad \text{as} \quad n \to \infty
\]

meaning extremal eigenmeasures are characterized by the almost-surely direction of escape to infinity of their orbits.
Duality: We show that positive \( \lambda \)-eigenfunctions with uniformly continuous logarithm can be canonically identified with \( \lambda \)-eigenmeasures for a "reversed" Ruelle operator on the negative one-sided Markov shift \( X^- \). This duality is valid for the recurrent case as well.

First-order phase transitions: We apply the main results of this work to the study of Gibbs states and first-order phase transitions. In particular, we show that a phase transition occurs when the Martin boundary has more than a single point and provide an analogous interpretation of a thermodynamic limit in the transient case.

As in \cite{7, 33}, our approach is motivated by the theory of the Martin boundary for random walks. Recall that for a transient random walk on a locally finite graph, one can show that every positive harmonic function can be presented in forms of the Martin kernels and that the walk almost surely converges to a boundary point, see Section 7. We emphasize that unlike in the probabilistic settings, where a compactification of the set of states \( S \) is considered, our proposed compactification is of the space of paths \( X^+ \). For an alternative approach of a compactification of Markov shifts, see \cite{11, 10}.

1.1 Topological Markov shifts, Ruelle operator and transience

Let \( S \) be an infinite countable set of states and let \( A \in \{0, 1\}^{S \times S} \) be a transition matrix over \( S \). For a subset \( A \subseteq \mathbb{Z} \) and a vector \( x \in S^A \), we denote by \( (x)_i \) the \( i \)-th coordinate of \( x \).

The (positive) one-sided topological Markov shift (TMS) is the space

\[
X^+ = \{ x \in S^\mathbb{N}_0(0) : A_{(x)_i,(x)_{i+1}} = 1, \forall i \geq 0 \}
\]

with the transformation \( T : X^+ \to X^+ \), \( (T(x))_i = (x)_{i+1} \) and the metric

\[
d(x, y) = 2^{-\inf\{i \geq 0 : (x)_i \neq (y)_i\}}. \tag{1.1}
\]

If \( \sum_a A_{a,b} < \infty \) for every \( a \in S \), then the space \( (X^+, d) \) is locally compact and all cylinder sets

\[
[a_0, \ldots, a_m] := \{ x \in X^+ : (x)_i = a_i, 0 \leq i \leq m \} \tag{1.2}
\]

are compact. A word \((a_0, \ldots, a_m) \in S^{m+1}\) is called admissible if \([a_0, \ldots, a_m] \neq \emptyset\). We say that a TMS \( X^+ \) is topologically transitive, or simply transitive, if for every \( a, b \in S \) there exists \( n \) s.t. \( T^{-n}[a] \cap [b] \neq \emptyset \). For every state \( a \in S \) on a transitive TMS \( X^+ \) we arbitrarily pick \( x_a \in T[a] \).

For two states \( a, b \in S \) let \( d_{\text{graph}}(a, b) \) be the graph distance between \( a \) and \( b \),

\[
d_{\text{graph}}(a, b) := \inf\{n \geq 0 : [a] \cap T^{-n}[b]\}.
\]

Observe that \( d_{\text{graph}} \) is not necessarily a metric.

For numbers \( r_1, r_2, c \in \mathbb{R}^+ \), we write \( r_1 = e^{c \pm r_2} \) if \( e^{-c} r_2 \leq r_1 \leq e^c r_2 \). Moreover, we write \( o_n(1) \) for a small quantity, converging to zero as \( n \) tends to infinity.
We denote by $C_c(X^+)$ the space of all continuous functions from $X^+$ to $\mathbb{R}$ with compact support, by $C^+(X^+)$ the space of all non-negative continuous functions and by $C^+_c(X^+)$ the space of all non-negative continuous functions with compact support.

The $m$-th variation of a function $\phi : X^+ \to \mathbb{R}$ is

$$Var_m(\phi) = \inf\{|\phi(x) - \phi(y)| : x, y \in X^+, (x)_i = (y)_i, 0 \leq i < m - 1\}.$$  

(1.3)

Hence forth we always assume that discussed potential function $\phi$ has summable variations, namely $\sum_{m \geq 2} Var_m(\phi) < \infty$. Notice that this is satisfied by all Markovian potential functions, $\phi(x) = \phi((x)_0, (x)_1)$, and Hölder-continuous functions as well.

**Definition 1.1.** The *Ruelle operator* $L_\phi$ evaluated on a function $f \in C(X^+)$ at a point $x \in X^+$ is

$$(L_\phi f)(x) = \sum_{y : Ty = x} e^{\phi(y)} f(y).$$

The sum converges for all $f \in C_c(X^+)$, or if $X^+$ is locally compact for every $f \in C(X^+)$. Notice that

$$(L^n_\phi f)(x) = \sum_{y : T^n y = x} e^{\phi_n(y)} f(y)$$

where $\phi_n(x) = \sum_{i=0}^{n-1} \phi(T^i x)$.

The topological pressure of $\phi$ is defined to be the limit

$$P_G(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{T^n x = x} e^{\phi_n(x)} 1_{[a]}(x)$$

for some $a \in S$ and $x \in X^+$. It can be shown that if $(X^+, T)$ is topologically mixing then the limit exists and independent of the choice of $a$, see [26].

**Definition 1.2.** The *Green’s function* for a function $f \in C(X^+)$ and $\lambda > 0$ evaluated at a point $x \in X^+$ is

$$G(f, x | \lambda) = \sum_{n=0}^{\infty} \lambda^{-n} (L^n_\phi f)(x).$$

We now introduce the notion of $\lambda$-recurrence and $\lambda$-transience, for an arbitrary $\lambda \in (0, \infty)$.

**Definition 1.3.** A potential function $\phi$ with summable variation on a transitive one-sided TMS $(X^+, T)$ is called $\lambda$-**recurrent** if for some (or every) non-zero $f \in C_c^+(X)$ and $x \in X^+$, $G(f, x | \lambda) = \infty$ and is called $\lambda$-**transient** if for some (or every) non-zero $f \in C_c^+(X^+)$ and $x \in X^+$, $G(f, x | \lambda) < \infty$.

In this paper we adapt the notion of $\lambda$-recurrence and $\lambda$-transience as it appears in common probability literature. In [26], the term recurrent or transient is actually interpreted as $\exp(P_G(\phi))$-recurrent or $\exp(P_G(\phi))$-transient, with finite $P_G(\phi)$. To obtain more general results, we considered
arbitrary value of $\lambda$, rather than the specific but important value $\lambda = \exp(P_G(\phi))$. If $\phi$ is $\lambda$-transient with $\lambda = 1$, we simply say that $\phi$ is transient. Then, we write $G(f, x) = G(f, x|1)$.

Recall that a measure $\mu$ is called a Radon measure if it is a Borel measure and finite on compacts. For two positive, possibly infinite, Radon measures $\mu$ and $\nu$ on $X^+$, we write $\mu \leq \nu$ if $\mu(K) \leq \nu(K)$, for every compact set $K \subseteq X^+$. We say that a measure $\mu$ is non-singular if $\mu \circ T^{-1} \sim \mu$, i.e. for every measurable set $E$, $\mu(E) = 0$ $\iff$ $\mu(T^{-1}E) = 0$.

**Definition 1.4.** A positive Radon measure $\mu$ on $X^+$ is called $\lambda$-conformal (or simply conformal) if $L^* \phi \mu = \lambda \mu$, i.e. $(L^* \phi)(1_K) = \lambda \mu(1_K)$, for every compact set $K \subseteq X^+$. Similarly, $\mu$ is called $\lambda$-excessive if $L^* \phi \mu \leq \lambda \mu$. We denote by $\text{Conf}(\lambda)$ the space of all positive Radon $\lambda$-conformal measures.

### 1.2 Standing assumptions

In this work, we make the following assumptions:

(A1) $(X^+, T)$ is topologically transitive with $|S| = \infty$.

(A2) For every $a \in S$, $\sum_{b} A_{a,b} < \infty$.

(A3) $\phi$ has summable variations, $\sum_{k=2}^{\infty} \text{Var}_k(\phi) < \infty$.

(A4) There is $\lambda > 0$ s.t. $\phi$ is $\lambda$-transient, i.e. $\sum \lambda^{-n} L_n^* f(x) < \infty$ for every $f \in C_0^+(X^+)$ and $x \in X^+$.

Assumption (A2) is crucial. See [7] for an example of a non locally-compact Markov shift and a potential function with no conformal measures. Though assumption (A2) excluding several interesting models, it is satisfied by the symbolic models for non-uniformly hyperbolic diffeomorphisms in [28, 22].

Observe that for all $\lambda > 0$,

$$L^*_{\phi - \log \lambda} \mu = \mu \iff L^*_{\phi} \mu = \lambda \mu.$$  

Hence in order to study the $\lambda$-conformal measures of $\phi$ one can study the $1$-conformal measures of $\phi - \log \lambda$. We use this property in the main proofs of this work.

In Section 4 we shall consider two-sided and negative one-sided topological Markov shifts. The following assumption is essential to ensure these topological Markov shifts are locally compact well;

(A5) For every $b \in S$, $\sum_{a} A_{a,b} < \infty$.

### 2 Martin boundary and the existence of eigenmeasures

In this section we construct a special compactification of $X^+$ and use it to show the following result of Cyr [7]:

**Theorem 2.1.** Under assumptions (A1)-(A4), there exists a positive Radon measure $\mu$ s.t. $L^* \phi \mu = \lambda \mu$. 

5
In the following two sections we will use this compactification to describe all such measures and to characterize them in terms of the limiting behaviour of $T^nx$, as $n \to \infty$ for $\mu$-typical $x \in X^+$.

The construction is motivated by the well-known Martin compactification of transient Markov chains, but differs from it by at least one important aspect: we compactify the space of infinite paths $X^+$ and not the set of states $S$. See [30, 34] for a detailed exposition of the probabilistic Martin boundary.

Recall the definition of Green’s function: $G(f, x|\lambda) = \sum_{n \geq 0} \lambda^{-n} L^n_\lambda f(x)$.

**Definition 2.1.** Let $\alpha \in S$ be an arbitrary origin state. The Martin kernel $K : C_c(X^+) \times X^+ \to \mathbb{R}^+$ is

$$K(f, x|\lambda) := \frac{G(f, x|\lambda)}{G(1_{[\alpha]}, x|\lambda)}$$

Later on we show that the choice of the origin state is not crucial (see Corollary 3.2). We write $K(f, x) = K(f, x|1)$.

To construct the compactification, we introduce a new metric $\rho$ on $X^+$, which coincides with the convergence according to the original metric $d$ and for which the Martin kernels $K(f, x|\lambda)$, for every $f \in C_c(X^+)$, are $\rho$-continuous. The Martin boundary is then the set of all new obtained points when completing $X^+$ w.r.t. $\rho$.

We start by showing that for every fixed $f \in C_c(X^+)$, $K(f, x|\lambda)$ is bounded in $x$.

**Lemma 2.1.** Let $(X^+, T)$ be a transitive locally compact one-sided TMS and let $\phi$ be a $\lambda$-transient potential function with summable variations. Then, for every $a, b \in S$, there exist $0 < c_{a, b} \leq C_{a, b}$ s.t.

$$c_{a, b} G(1_{[b]}, x|\lambda) \leq G(1_{[a]}, x|\lambda) \leq C_{a, b} G(1_{[b]}, x|\lambda), \forall x \in X^+. \quad (2.1)$$

**Proof.** Let $N = d_{\text{graph}}(t, a)$, the graph distance between $b$ and $a$, and let $a_1, \ldots, a_N$ a path from $a_1 = b$ to $a_N = a$. For every $x \in X^+$,

$$G(L^N_{\phi} 1_{[b]}, x|\lambda) = \lambda^{-N} \sum_{n \geq N} \lambda^{-n} L^n_{\phi} 1_{[b]}(x) \leq \lambda^{-N} G(1_{[b]}, x|\lambda).$$

Therefore,

$$G(1_{[b]}, x|\lambda) \geq \lambda^{-N} G(L^N_{\phi} 1_{[b]}, x|\lambda)$$

$$\geq \sum_{n=0}^{\infty} \sum_{b_1, \ldots, b_{n-1}, b_{n+1}} \lambda^{-n} \phi_{a_n + N(na_1 + \cdots + a_{n+1} - x)}$$

$$\geq \sum_{n=0}^{\infty} \sum_{b_{N+1}, \ldots, b_{N+n-1}} \lambda^{-n} \phi_{a_n + N(na_1 + \cdots + a_N - 1, b_{N+1} + \cdots + b_{N+n-1} - x)}$$

$$\geq \lambda^{-N} \phi_N(ba_2, \ldots, a_n - xa_n) - \sum_{i=2}^{N} V a r_i(\phi)$$

$$\sum_{n=0}^{\infty} \sum_{b_{N+1}, \ldots, b_{N+n-1}} \lambda^{-n} \phi_{a_n + (N(na_1 + \cdots + a_{n+1} - 1, b_{N+1} + \cdots + b_{N+n-1} - x))}$$

$$= \lambda^{-N} \phi_N(ba_2, \ldots, a_n - xa_n) - \sum_{i=2}^{N} V a r_i(\phi) G(1_{[a]}, x|\lambda)$$
Proof. By Lemma 2.1, let \( (X^+, T) \) be a transitive locally compact one-sided TMS and let \( \phi \) be a \( \lambda \)-transient potential function with summable variations. Then, for every \( f \in C_0^\infty (X^+) \), \( f \neq 0 \), there exist \( c_f, C_f > 0 \) s.t.

\[
 c_f \leq K(f, x|\lambda) \leq C_f, \quad \forall x \in X^+.
\]

\[ \text{Lemma 2.2.} \] Let \( (X^+, T) \) be a transitive locally compact one-sided TMS and let \( \phi \) be a \( \lambda \)-transient potential function with summable variations. Then, for every \( f \in C_0^\infty (X^+) \), \( f \neq 0 \), there exist \( c_f, C_f > 0 \) s.t.

\[
 c_f \leq K(f, x|\lambda) \leq C_f, \quad \forall x \in X^+.
\]

Proof. Since \( \text{supp}(f) \) is compact, there exist \( b_1, \ldots, b_N \in S \) with \( \text{supp}(f) \subseteq \bigcup_{i=1}^N [b_i] \). Then, by the linearity of the Ruelle operator and by Lemma 2.1

\[
 K(f, x|\lambda) = \frac{G(f, x|\lambda)}{G(1_{[\phi]}, x|\lambda)} \leq \sum_{i=1}^N \frac{\max_{x' \in [b_i]} \{f(x')\}G(1_{[\phi]}, x|\lambda)}{G(1_{[\phi]}, x|\lambda)} \leq \sum_{i=1}^N \max_{x' \in [b_i]} \{f(x')\}C_{b_i,0}.
\]

Let \( (b_1', \ldots, b_M') \) an admissible word of length \( M \) s.t. \( c := \min_{x' \in [b_1', \ldots, b_M']} \{f\} > 0 \). Then, with \( x_{b_M'} \in T[b_M] \),

\[
 G(f, x|\lambda) \geq c G(1_{[b_1', \ldots, b_M']}, x|\lambda) \geq c \sum_{k=M}^\infty \lambda^{-k} (L_\phi^k 1_{[b_1', \ldots, b_M']})(x) \geq c e^{-\phi_M (b_1', \ldots, b_M') - \sum_{j=2}^\infty V_{\phi}(\phi)} \lambda^{-M} \sum_{k=0}^\infty \lambda^{-k} (L_\phi^k 1_{[b_1']})(x) = c e^{-\phi_M (b_1', \ldots, b_M') - \sum_{j=2}^\infty V_{\phi}(\phi)} \lambda^{-M} G(1_{[b_1']}, x|\lambda).
\]

By Lemma 2.1

\[
 G(1_{[b_1]}, x|\lambda) \geq c_{b_1}^\phi G(1_{[\phi]}, x|\lambda)
\]

and the lemma follows. \[ \square \]

Let \( S^* = \{w_i\}_{i \in \mathbb{N}} \) be an enumeration of all admissible finite words on \( S \). We define a new metric \( \rho \) on \( X^+ \),

\[
 \rho(x, y|\lambda) = \sum_{i=1}^\infty \frac{|K(1_{[w_i]}, x|\lambda) - K(1_{[w_i]}, y|\lambda)| + |1_{[w_i]}(x) - 1_{[w_i]}(y)|}{2^{i}(C_{1_{[w_i]}} + 1)}
\]

where \( C_{1_{[w_i]}} \) is the constant from Lemma 2.2. It is easy to verify that \( \rho \) is indeed a metric.

\[ \text{Definition 2.2.} \] The \( \lambda \)-Martin compactification of \( X^+ \) and \( \phi \), denoted by \( \hat{X}^+(\lambda) \), is the completion of \( X^+ \) w.r.t. \( \rho \). The \( \lambda \)-Martin boundary of \( X^+ \) and \( \phi \) is \( M(\lambda) := \hat{X}^+(\lambda) \setminus X^+ \).
Often, we will abuse the notations and write \( \hat{X}^+ = X^+, M(\lambda) = M \). For examples of non-trivial Martin boundaries in the probabilistic settings see [30]. The following proposition describes the main properties of \((\hat{X}^+ , \rho)\). It’s proof is elementary and can be found in the appendix.

**Proposition 2.1.** Let \((X^+ , T)\) be a transitive locally compact one-sided TMS and let \(\phi\) be a \(\lambda\)-transient potential function with summable variations. Then,

1) \( \hat{X}^+ (\lambda) \) is compact w.r.t. \(\rho\).
2) Let \(x_n , x \in X^+\). Then,

\[
x_n \xrightarrow{d} x \iff x_n \xrightarrow{\rho} x.
\]
3) Let \(A \subseteq X^+\). Then, \(A\) is \(d\)-open iff \(A\) is \(\rho\)-open.
4) If \(x_n \xrightarrow{\rho} \omega \in M(\lambda)\) with \(x_n \in X^+\), then \(x_n \to \infty\) (escapes every \(d\)-compact set).
5) For every \(f \in C_c(X^+)\), \(K(f , \cdot | \lambda)\) can be extended uniquely to a continuous function on \(\hat{X}^+ (\lambda)\).
6) For every \(x \in \hat{X}^+ (\lambda)\), \(K(\cdot , x | \lambda)\) viewed as a positive linear functional on \(C_c(X^+)\) defines an \(\lambda\)-excessive measure on \(X^+\). If \(\omega \in M(\lambda)\), then \(K(\cdot , \omega | \lambda)\) defines a \(\lambda\)-conformal measure.

**Definition 2.3.** Given \(\omega \in M(\lambda)\), let \(\mu_\omega\) denote the measure

\[
\mu_\omega (f) := K(f , \omega | \lambda) = \lim_{x_n \xrightarrow{\rho} \omega} K(f , \omega | \lambda).
\]

By Proposition 2.1, \(\mu_\omega\) is a conformal measure. Since \(X^+\) is not compact but \(\hat{X}^+ (\lambda)\) is compact, the boundary is not empty and Theorem 2.1 follows. The assumption that \(|S| = \infty\) is crucial; otherwise \(X^+\) is compact and the boundary is empty. In the next section, we show that all extremal conformal measures correspond to boundary points. In fact, the conformal measure constructed by Cyr in [7] to prove Theorem 2.1 is of the form \(K(\cdot , \omega | \lambda)\) for some \(\omega \in M(\lambda)\).

### 3 Integral representation of eigenmeasures

In this section we describe all positive \(\lambda\)-eigenmeasures by means of integrals representation:

**Theorem 3.1.** Let \((X^+ , T)\) be a transitive locally compact one-sided TMS and let \(\phi\) be a \(\lambda\)-transient potential function with summable variations. Then, for every \(\mu \in \text{Conf}(\lambda)\), there exists a finite measure \(\nu\) on \(M(\lambda)\) s.t. for every \(f \in C_c(X^+)\),

\[
\mu (f) = \int_{\omega \in M(\lambda)} K(f , \omega | \lambda) d\nu (\omega).
\]

Later on in this section, we will introduce the *minimal Martin boundary* which yields a unique representation.

To prove Theorem 3.1, we introduce and prove the following auxiliary lemmas.
Lemma 3.1. Let \((X^+, T)\) be a transitive locally compact one-sided TMS and let \(\phi\) be a \(\lambda\)-transient potential function with summable variations. Let \(\mu\) be a \(\lambda\)-excessive measure. Then, there exist measures \(\nu\) and \(\mu^*\) with \(\mu^*\) \(\lambda\)-conformal s.t.

\[
\mu(f) = \int G(f, x|\lambda) d\nu(x) + \mu^*(f), \quad \forall f \in C_c(X).
\]

Proof. We write

\[
\mu = \sum_{k=0}^{n-1} \lambda^{-k}(L^n_\phi)^k \left(\mu - \lambda^{-1} L^n_\phi \mu\right) + \lambda^{-n}(L^n_\phi)^* \mu.
\]

Since \(\lambda^{-n} \left((L^n_\phi)^* \mu\right)(f)\) is non-increasing for every \(f \in C_c^+(X^+)\), the limit \(\lim_{n \to \infty} \lambda^{-n}(L^n_\phi)^* \mu\) exists and the lemma follows with \(\nu = \mu - \lambda^{-1} L^n_\phi \mu\) and \(\mu^* = \lim_{n \to \infty} \lambda^{-n}(L^n_\phi)^* \mu\).

\(\square\)

Definition 3.1. For two measures \(\mu_1, \mu_2\) we define their minimum \(\mu_1 \wedge \mu_2\) by

\[
d(\mu_1 \wedge \mu_2) = \min \left\{ \frac{d\mu_1}{d(\mu_1 + \mu_2)}, \frac{d\mu_2}{d(\mu_1 + \mu_2)} \right\} (\mu_1 + \mu_2) - a.e.
\]

Lemma 3.2. Let \((X^+, T)\) be a transitive locally compact one-sided TMS and let \(\phi\) be a \(\lambda\)-transient potential function with summable variations. Then, for every two positive Radon measures \(\mu_1, \mu_2\),

\[
(\mu_1 \wedge \mu_2)(f) \leq \min\{\mu_1(f), \mu_2(f)\}, \quad \forall f \in C_c(X^+).
\]

If \(\mu_1\) and \(\mu_2\) are \(\lambda\)-excessive, then \(\mu_1 \wedge \mu_2\) is \(\lambda\)-excessive as well. Moreover, if \(\mu^*\) is a positive Radon measure satisfying \(\mu^*(f) \leq \min_{i=1,2} \{\mu_i(f)\}\) for every \(f \in C_c^+(X^+)\), then \(\mu^* \leq \mu_1 \wedge \mu_2\).

Proof. Let \(\mu = \mu_1 \wedge \mu_2\). For every \(f \in C_c^+(X^+)\),

\[
\mu(f) = \int f(x) \min \left\{ \frac{d\mu_1}{d(\mu_1 + \mu_2)}, \frac{d\mu_2}{d(\mu_1 + \mu_2)} \right\} (x) d(\mu_1 + \mu_2)(x)
\]

\[
\leq \int f(x) \frac{d\mu_1}{d(\mu_1 + \mu_2)}(x) d(\mu_1 + \mu_2)(x) = \mu_1(f), \quad i = 1, 2.
\]

and \(\mu \leq \mu_i\). Assume that \(\mu_1\) and \(\mu_2\) are \(\lambda\)-excessive. Then, for \(i = 1, 2\),

\[
L^n_\phi \mu_i \leq L^n_\phi \mu_i = \lambda \mu_i
\]

which implies that

\[
(L^n_\phi \mu)(f) \leq \lambda \int f(x) d\mu_1(x) = \lambda \int f(x) \frac{d\mu_i}{d(\mu_1 + \mu_2)}(x) d(\mu_1 + \mu_2)(x).
\]

Inequality (3.2) extends to all non-negative measurable functions. By decomposing

\[
f = \min \left\{ \frac{d\mu_1}{d(\mu_1 + \mu_2)}, \frac{d\mu_2}{d(\mu_1 + \mu_2)} \right\} f + \min \left\{ \frac{d\mu_1}{d(\mu_1 + \mu_2)}, \frac{d\mu_2}{d(\mu_1 + \mu_2)} \right\} f
\]

\[
(3.3)
\]
we obtain that
\((L_\phi^* \mu)(f) \leq \lambda \int f(x) \min \left\{ \frac{d\mu_1}{d(\mu_1 + \mu_2)}, \frac{d\mu_2}{d(\mu_1 + \mu_2)} \right\} (x) d(\mu_1 + \mu_2)(x) = \lambda \mu(f)\).

Now, let \(\mu'\) be a positive Radon measure with \(\mu'(f) \leq \min, \{\mu_i(f)\}\). Then,
\[\mu'(f) \leq \int f(x) d\mu_i(x) = \int f(x) \frac{d\mu_1}{d(\mu_1 + \mu_2)}(x) d(\mu_1 + \mu_2)(x)\]
Using the decomposition in Eq. (3.3), we obtain that
\[\mu'(f) \leq \int f(x) \min \left\{ \frac{d\mu_1}{d(\mu_1 + \mu_2)}, \frac{d\mu_2}{d(\mu_1 + \mu_2)} \right\} (x) d(\mu_1 + \mu_2)(x) = \mu(f)\].

Proof of Theorem 3.1. Assume w.l.o.g. that \(\lambda = 1\). Let \(W_n\) a sequence of compact sets increasing to \(X^+\) and let
\[\eta_n(f) := n \cdot \int_{W_n} G(f, x) d\mu(x).\]

Clearly \(\eta_n\) are positive Radon measures. Consider \(\mu_n = \mu \land \eta_n\). Let \(f \in C^+_b(X)\). Observe that \(\eta_n(g) \to \infty\) for any non-zero \(g \in C^+_b(X^+)\). Thus, by Lemma 3.2, \(\limsup_{n \to \infty} \mu_n(f) \leq \mu(f)\). Let \(N\) large enough s.t. \(\text{supp}(f) \subseteq W_n\) for all \(n \geq N\) and let \(\mu' = \mu|_{\text{supp}(f)}\), the restriction of \(\mu\) to \(\text{supp}(f)\). Clearly \(\mu' \leq \mu\) and \(\mu'(f) = \mu(f)\). Moreover, for every \(g \in C^+_b(X)\),
\[\mu'(g) = \mu(g1_{\text{supp}(f)}) \leq \sum_{n=0}^{\infty} \int_{\text{supp}(f)} L_\phi^* g(x) d\mu(x) \leq \sum_{n=0}^{\infty} \int_{W_n} L_\phi^* g(x) d\mu(x) = \eta_n(g).\]

Thus, by Lemma 3.2, \(\mu' \leq \mu_n\) and \(\mu(f) = \mu'(f) \leq \mu_n(f)\) for \(n\) large enough. In particular, \(\lim_{n \to \infty} \mu_n(f) = \mu(f)\).

By Lemma 3.2 \(\mu_n\) is \(\lambda\)-excessive. Since \((L_\phi^*)^* \mu_n \leq (L_\phi^*)^* \eta_n \xrightarrow{m.n.} 0\), by Lemma 3.4 for every \(n\) there exist measures \(\nu_n\) s.t. for every \(f \in C_c(X^+)\),
\[\mu_n(f) = \int G(f, x) d\nu_n(x) = \int K(f, x) G(1_{\{0\}}, x) d\nu_n(x).\]

Let \(d\nu_n(x) = G(1_{\{0\}}, x) d\nu_n(x)\). Since \(X^+\) is compact and \(\nu_n'(1) = \mu_n(1_{\{0\}}) \leq \mu(1_{\{0\}})\) for \(n\) large enough, there exists a weak*-converging subsequence \(\nu_{n_k}\) to \(\nu\). Since \(K(f, x)\) is \(\rho\)-continuous for every \(f \in C_c(X^+)\),
\[\mu(f) = \int K(f, x) d\nu(x).\]
Next, we show that $\text{supp}(\nu) \subseteq \mathcal{M}$. According to Proposition 2.1 for every $x \in \hat{X}^+$,

$$K(L\phi f, x) \leq K(f, x).$$

Moreover, for every $x \in X^+$ and $a \in S$,

$$K(1_{[a]}, x) - K(L\phi 1_{[a]}, x) = \frac{1_{[a]}(x)}{G(1_{[o]}, x)}.$$ 

Since $G(1_{[a]}, x)$ is positive and continuous, it is bounded away from zero on any compact set $[a]$. Hence,

$$0 = \mu(1_{[o]}) - L\phi \mu(1_{[o]}) = \int_{\hat{X}^+} (K(1_{[a]}, x) - K(L\phi 1_{[a]}, x)) \, d\nu(x)$$

$$\geq \int_{[a]} (K(1_{[a]}, x) - K(L\phi 1_{[a]}, x)) \, d\nu(x)$$

$$= \int_{[a]} G(1_{[a]}, x)^{-1} \, d\nu(x)$$

$$\geq \min_{x \in [a]} \{G(1_{[o]}, x)^{-1}\} \nu([a])$$

meaning $\nu([a]) = 0$, which implies that $\nu(X) = \sum_{a \in S} \nu([a]) = 0$. 

**Remark 3.1.** If we assume that $\mu$ is only $\lambda$-excessive, then we obtain similar results except that $\nu$ may charge $X^+$.

Theorem 3.1 and Lemma 2.1 immediately yield the following elementary fact.

**Corollary 3.1.** Let $(X^+, T)$ be a transitive locally compact one-sided TMS and let $\phi$ be a $\lambda$-transient potential function with summable variations. Then, for every non-zero $\mu \in \text{Conf}(\lambda)$ and $a \in S$, $\mu([a]) > 0$.

Next, we study the extremal points of the cone $\text{Conf}(\lambda)$.

**Definition 3.2.** A measure $\mu \in \text{Conf}(\lambda)$ is called extremal if for every $\mu_1, \mu_2 \in \text{Conf}(\lambda)$ with $\mu = \mu_1 + \mu_2$ we have that $\mu_1 = \mu_2 = \text{const} \cdot \mu$.

Recall the definition of the measure $\mu_\omega$; $\mu_\omega(f) = K(f, \omega|\lambda), \omega \in \mathcal{M}(\lambda)$ and $f \in \mathcal{C}(X^+)$. 

**Lemma 3.3.** Let $(X^+, T)$ be a transitive locally compact one-sided TMS and let $\phi$ be a $\lambda$-transient potential function with summable variations. If $\mu \in \text{Conf}(\lambda)$ is an extremal measure, then there exists $\omega \in \mathcal{M}(\lambda)$ s.t. $\mu \propto \mu_\omega$.

**Proof.** Assume w.l.o.g. that $\mu([o]) = 1$ and that $\lambda = 1$. We follow the proof of Theorem 24.8 in [34]. Let $\nu$ be the positive finite measure on $\mathcal{M}$ from Theorem 3.1 with

$$\mu(\cdot) = \int K(\cdot, \omega) d\nu(\omega).$$

We show that $\nu = \delta_\omega$ for some $\omega \in \mathcal{M}$. For a Borel set $B \subseteq \mathcal{M}$ s.t. $0 < \nu(B) < \nu(\mathcal{M})$, we define

$$\mu_1 = \frac{1}{\nu(B)} \int_{\omega \in B} K(\cdot, \omega) d\nu(\omega)$$

and
and
\[ \mu_2 = \frac{1}{\nu(M \setminus B)} \int_{\omega \in M \setminus B} K(\cdot, \omega) d\nu(\omega). \]

Now, \( \mu_1 \) and \( \mu_2 \) are conformal measures as well since \( K(L_\phi, \omega) = K(f, \omega) \)
for every \( f \in C_c(X^+) \) and \( \omega \in M \). Since
\[ \mu = \nu(B) \mu_1 + \nu(M \setminus B) \mu_2 \]
and \( \mu \) is extremal, we have that \( \mu_1, \mu_2 \preceq \mu \). Moreover, since \( \mu_1([o]) = \mu_2([o]) = \mu([o]) \), we have that \( \mu_1 = \mu_2 = \mu \). In particular, for every Borel set \( B \) and every \( f \in C_c(X^+) \),
\[ \int_B \mu(f) d\nu(\omega) = \nu(B) \mu(f) = \int_B K(f, \omega) d\nu(\omega). \]

The equation is trivial in case \( \nu(B) \in \{0, \nu(M)\} \). This implies that
\[ \nu(\{ \omega \in M : \mu(f) = K(f, \omega), \forall f \in C_c(X^+) \}) = \nu(M). \]

By the definition of \( \rho \), \( \nu = \delta_\omega \), for some \( \omega \in M \).

**Definition 3.3.** The \( \lambda \)-Minimal Martin Boundary \( M_m(\lambda) \) is the set of all points \( \omega \in M(\lambda) \) with \( \mu_\omega \) extremal.

Lemma 3.3 yields that \( \{ \mu_\omega : \omega \in M_m(\lambda) \} \) is in fact the collection of all extremal points of Conf(\( \lambda \)).

**Corollary 3.2.** Let \((X^+, T)\) be a transitive locally compact one-sided TMS and let \( \phi \) be a \( \lambda \)-transient potential function with summable variations. Then, the \( \lambda \)-minimal Martin boundary does not depend on the choice of the origin state \( o \).

**Proof.** Every point of the minimal Martin boundary w.r.t. an origin point \( o \) corresponds to an extremal conformal measure (up to a constant) and vice versa. Therefore the choice of the origin state only effects the normalizing factor.

The question of the uniqueness of \( \nu \) from Theorem 3.1 is a natural one. In general, \( \nu \) may be non-unique. However, if we restrict the support of \( \nu \) to \( M_m(\lambda) \), then we do obtain uniqueness.

**Theorem 3.2.** Let \((X^+, T)\) be a transitive locally compact one-sided TMS and let \( \phi \) be a \( \lambda \)-transient potential function with summable variations. Let \( \mu \in Conf(\lambda) \). Then, there exists a unique finite measure \( \nu \) on \( M_m(\lambda) \) s.t.
\[ \mu(f) = \int_{M_m(\lambda)} K(f, \omega|\lambda) d\nu(\omega). \]

**Proof.** Assume w.l.o.g. that \( \lambda = 1 \). To prove the theorem, we first recall the notion of a lattice.

**Definition 3.4.** We say that a subset \( V \) of a topological vector space with a partial ordering \( \leq \) is said to be a lattice if for every \( v_1, v_2 \in V \), there exists a vector \( u \) s.t. \( v_i - u \in V \) (\( i = 1, 2 \)) and for every \( w \in V \) with \( v_i - w \in V \) we have that \( w \leq u \) (sort of "\( \min \{v_1, v_2\}\)").

Our aim is to apply the following version of Choquet’s theorem;
Theorem (Furstenberg [13]). Let $V$ be a weakly-closed cone of positive measures on a separable, locally compact space and let $E$ denote the extremal rays of $V$. Suppose that there is a positive function of compact support $\psi$ with $\mu(\psi) > 0$ for all $\mu \in V$, $\mu \neq 0$, and let $V_1 = \{\mu \in V : \mu(\psi) = 1\}$. Then, for each $\mu \in V$ there exists a measure $\nu$ on a Borel subset $B \subseteq E \cap V_1$ s.t. $\mu = \int_B \mu' \, d\nu(\mu')$. Moreover, if $V$ is a lattice, then the measure $\nu$ is unique.

It is easy to verify that $\text{Conf}$ is weak*-closed. From Corollary 3.1 $\mu(1_\omega) > 0$ for every $\mu \in \text{Conf}$. Thus, to obtain uniqueness it suffices to show that $\text{Conf}$ is a lattice.

Let $\mu_1, \mu_2 \in \text{Conf}$. Let $\mu' = \mu_1 \wedge \mu_2$. By Lemma 3.2 $\mu'$ is excessive and the limit measure $\mu'' = \lim_{n \to \infty} (L_\phi^n)\mu'$ exists. Clearly $\mu'' \geq 0$. Since $L_\phi f \in C^\infty_{+}(X^+)$ as well,

$$ (L_\phi^{n+1})f = (L_\phi^n)\mu' f = \lim_{n \to \infty} (L_\phi^n)^n \mu' f = \lim_{n \to \infty} (L_\phi^n)^{n+1} \mu' f = \mu'' f $$

and $\mu'' \in \text{Conf}$. Moreover, since $\mu'' \leq \mu'$ and $0 \leq \mu_1 - \mu'$, we have that $\mu_1 - \mu'' \in \text{Conf}$. Suppose $\mu_3 \in \text{Conf}$ with $\mu_1 - \mu_3 \in \text{Conf}$. Then, $\mu_3 \leq \mu_1$, and by Lemma 3.2 $\mu_3 \leq \mu'$. Since $L_\phi^n \mu_3 = \mu_3$, we have that $\mu_3 \leq (L_\phi^n)^n \mu'$ for every $n$, which implies that $\mu_3 \leq \mu''$ and $\text{Conf}$ is indeed a lattice. 

4 Convergence to the boundary

We saw in the previous section that every $\lambda$-conformal measure $\mu$ can be uniquely presented in the form

$$ \mu = \int_{M_m(\lambda)} \mu_\omega \, d\nu(\lambda) $$

for some boundary measure $\nu$ on the minimal Martin boundary. In this section we show that $\nu$ is determined by the $\mu$-almost surely behaviour of $T^n x$ as $n \to \infty$ as follows:

**Theorem 4.1.** Let $(X^+, T)$ be a transitive locally compact one-sided TMS and let $\phi$ be a $\lambda$-transient potential function with summable variations. Let $\mu \in \text{Conf}(\lambda)$. Then,

1. For $\mu$-a.e. $x \in X^+$, the $p$-limit $\lim_{n \to \infty} T^n x$ exists and belongs to $M_m(\lambda)$.

2. If $\mu = \mu_\omega$ for some $\omega \in M_m(\lambda)$, then for $\mu_\omega$-a.e. $x \in X^+$, $T^n x \to \omega$.

3. For every Borel set $E \subseteq M(\lambda)$,

$$ \nu(E) = \mu(x \in [\omega] : \exists \omega \in E \text{ s.t. } T^n x \not\to \omega). \quad (4.1) $$

When $\phi$ is $\lambda$-transient, all $\lambda$-conformal measures admit a dissipative behaviour; a.e. orbit $T^n x$ escapes every compact set. Though it is an already known result (see [1]) for completeness of this article we provide in the appendix an elementary proof to the non-conservative behaviour of the conformal measures in the transient case. For a set $F \subseteq X^+$ let

$$ F_\infty = \{x \in X^+ : T^n x \in F \text{ for infinitely many } n > 0\}. $$
Proposition 4.1. Let \((X^+, T)\) be a transitive locally compact one-sided TMS and let \(\phi\) be a \(\lambda\)-transient potential function with summable variations. Let \(F \subseteq X^+\) be a compact set. Then, for every \(\mu \in \text{Conf}(\lambda)\), 
\[ \mu(F_\infty) = 0. \]

Proof. See appendix.

For a measure \(\mu\) and a set \(F \subseteq X^+\) let \(\mu|_F(f) = \mu(f \cdot 1_F)\), the restriction of \(\mu\) to \(F\).

Lemma 4.1. Let \((X^+, T)\) be a transitive locally compact one-sided TMS and let \(\phi\) be a \(\lambda\)-transient potential function with summable variations. Let \(\omega \in \mathcal{M}_m(\lambda)\). Then, \(\mu_\omega\) is ergodic.

Proof. Let \(A\) be a \(T\)-invariant set, with \(T^{-1}A = A\). Then, 
\[ \mu|_A(L_\phi 1_E) = \int (L_\phi 1_E)(x)1_A(x) d\mu_\omega(x) \]
\[ = \int (L_\phi (1_E \cdot 1_A \circ T))(x) d\mu_\omega(x) \]
\[ = \int (L_\phi (1_E \cdot 1_A))(x) d\mu_\omega(x) \]
\[ = \lambda \int 1_E(x) \cdot 1_A(x) d\mu_\omega(x) = \lambda \mu|_A(E). \]

Similarly, \(L_\phi^* \mu|_{A^c} = \lambda \mu|_{A^c}\). Clearly \(\mu|_A, \mu|_{A^c}\) are Radon, and thus \(\mu|_A, \mu|_{A^c} \in \text{Conf}(\lambda)\). Since \(\mu_\omega = \mu|_A + \mu|_{A^c}\) and \(\mu_\omega\) is extremal, we must have that \(\mu|_A, \mu|_{A^c} \propto \mu_\omega\). Since \(\mu|_A\) and \(\mu|_{A^c}\) are mutually singular, \(\mu|_A \equiv 0 \) or \(\mu|_{A^c} \equiv 0\) which implies that either \(\mu_\omega(A) = 0\) or \(\mu_\omega(A^c) = 0\).

Remark 4.1. Observe that \(\mu_\omega\) is not necessarily \(T\)-invariant. In fact, \(\mu_\omega\) is \(T\)-invariant iff \(L_\phi 1(x) = 1\) for \(\mu_\omega\)-a.e. \(x\), see [18]. One can easily ”fix” \(\mu_\omega\) to be \(T\)-invariant with a positive eigenfunction as a density. For more on the positive eigenfunctions see Section 5.

For a set \(F \subseteq X^+\) let 
\[ F_+ = \{ x \in X^+ : \exists n \geq 0 \text{ s.t. } T^n x \in F \}. \]

We denote by \(\overline{F}\) the topological closure of \(F\) in \((\hat{X}^+, \rho)\).

Lemma 4.2. Let \((X^+, T)\) be a transitive locally compact one-sided TMS and let \(\phi\) be a \(\lambda\)-transient potential function with summable variations. Let \(\mu \in \text{Conf}(\lambda)\) and let \(F \subseteq X^+\) be a Borel set. Then, there exists a finite measure \(\nu\) with \(\text{supp}(\nu) \subseteq \overline{F}\) s.t. 
\[ \mu|_{F_+}(f) = \int K(f, x|_\lambda) d\nu(x), \quad \forall f \in C_c(X^+). \]

Proof. Assume w.l.o.g. that \(\lambda = 1\). Assume first that \(F\) is compact. Observe that if \(Tx \in F_+\) then \(x \in F_+\) as well, whence \(T^{-1}F_+ \subseteq F_+\). In
particular, for every $f \in C_c^+(X^+)$, 

$$L^*_\phi(\mu_{F_+})(f) = \int (L_\phi f)(x) 1_{F_+}(x) d\mu(x)$$

$$= \int L_\phi f(x) \cdot 1_{F_+}(x) d\mu(x)$$

$$= \int f(x) \cdot (1_{F_+} \circ T)(x) d\mu(x) = \mu_{T^{-1}F_+}(f) \leq \mu_{F_+}(f).$$

Hence $\mu_{F_+}$ is excessive and by Lemma 3.1 we can write

$$\mu_{|F_+}(f) = \int_X G(f, x) d(\mu_{|F_+} - L^*_\phi(\mu_{|F_+}))(x) + \lim_{n \to \infty} ((L^*_\phi)^n(\mu_{|F_+}))(f).$$

Observe that $\bigcap_n T^{-n} F^+ = F_\infty$. By the monotone convergence theorem, for every compact set $K \subseteq X^+$

$$(L^*_\phi)^n(\mu_{|F_+})(K) = \mu T^{-n} F^+ \cap K \xrightarrow{n \to \infty} \mu(F_\infty \cap K).$$

By Proposition 4.1 $\mu(F_\infty \cap K) = 0$, which leads to

$$\mu_{|F_+}(f) = \int_{X^+} K(f, x) d\nu(x)$$

with $d\nu(x) = G(1_{[0, 1]}, x) d(\mu_{|F_+} - L^*_\phi(\mu_{|F_+}))(x)$. Since $F_+ \setminus T^{-1} F_+ \subseteq F$, we have that

$$\text{supp}(\nu) = \text{supp}(\mu_{|F_+} - L^*_\phi(\mu_{|F_+})) = \text{supp}(\mu_{|F_+ \setminus T^{-1} F_+}) \subseteq F.$$

Now, assume that $F$ is arbitrary and let $F_m$ be compact increasing sets s.t. $\mu(F \setminus \bigcup_m F_m) = 0$. Let $\nu_m$ be a measure with $\text{supp}(\nu_m) \subseteq F_m$ with

$$\mu_{|F_m+}(\cdot) = \int K(\cdot, x) d\nu_m(x).$$

Let $g \in C_c^+(X^+)$ with $\mu(g) > 0$. By Lemma 2.2 there exists $c_g > 0$ s.t. $K(g, x) \geq c_g, \forall x \in X^+$. Then,

$$\nu_m(1) \leq c_g^{-1} \int K(g, x) d\nu_m(x) = c_g^{-1} \mu_{|F_m+}(g) \leq c_g^{-1} \mu(g) < \infty$$

meaning that the measures $\nu_m$ are uniformly bounded. Working in the compact space $X^+$, we can find a weak$^*$ converging sub-sequence $\nu_{m_k} \to \nu$. Since $(F_m)_+ \not\subseteq F_+$ and $K(f, x)$ is $\rho$-continuous on $X^+$ we have that

$$\mu_{F_+}(f) = \int K(f, x) d\nu(x), \ \forall f \in C_c(X^+).$$

Lastly, since $\text{supp}(\nu_{m_k}) \subseteq F_{m_k} \subseteq F$ then $\text{supp}(\nu) \subseteq \overline{F}$ (the closure in $X^+$).

**Lemma 4.3.** Let $(X^+, T)$ be a transitive locally compact one-sided TMS and let $\phi$ be a $\lambda$-transient potential function with summable variations. Let $\mu \in \text{Conf}(\lambda)$ and let $F \subseteq X^+$ be a Borel set. Then, there exists a measure $\nu$ with $\text{supp}(\nu) \subseteq \overline{F \cap \mathcal{M}(\lambda)}$ s.t.

$$\mu_{F_\infty}(f) = \int K(f, x) d\nu(x), \ \forall f \in C_c(X^+).$$
Proof. Assume w.l.o.g. that $\lambda = 1$. Recall that $T^{-n}F_+ \searrow F_\infty$. Thus, by the monotone convergence theorem, for every $f \in C_c^+(X^+)$

$$((L_0^n)^n\mu_{F_+})(f) = \mu_{T^{-n}F_+}(f) = \mu(f \cdot 1_{T^{-n}F_+}) \to \mu(f \cdot 1_{F_\infty}) = \mu_{F_\infty}(f).$$

Observe that for every $x \in X^+$ and every $f \in C_c^+(X^+)$

$$K(L_0^n f, x) = \sum_{k \geq n} L_0^k f(x) \to 0.$$

Moreover, for every $\omega \in \mathcal{M}$, $K(L_0^n f, \omega) = K(f, \omega)$ (see Proposition 2.1).

By Lemma 4.2 and the monotone convergence theorem, we obtain that

$$\mu_{F_\infty}(f) = \lim_{n \to \infty} ((L_0^n)^n(\mu_{F_+}))(f) = \lim_{n \to \infty} \int_{T^\infty} K(L_0^n f, x) d\nu(x) = \int_{T^\infty \cap \mathcal{M}} K(f, \omega) d\nu(\omega).$$

Proof of Theorem 4.1. Assume w.l.o.g. that $\lambda = 1$. Assume first that $\mu = \mu_\omega$ for some $\omega \in \mathcal{M}_m$. For $\epsilon > 0$ set

$$F_\epsilon = \{x \in X^+ : \rho(x, \omega) \geq \epsilon\}$$

and

$$A_\epsilon = (F_\epsilon)_\infty = \{x \in X^+ : T^n x \in F_\epsilon \text{ for infinitely many } n\text{'s}\}.$$

Clearly $T^n x \to \omega$ iff $x \notin A_\epsilon$ for every $\epsilon > 0$. Thus it suffices to show that $\mu_\omega(A_\epsilon) = 0$, for every $\epsilon > 0$.

Since $A_\epsilon$ is $T$-invariant and $\mu_\omega$ is extremal, we have that either $\mu_\omega(A_\epsilon) = 0$ or $\mu_\omega(A_\epsilon) \neq 0$ and we only have to exclude the second case. Assume that $\mu_\omega(A_\epsilon) \neq 0$. Then $\mu_\omega = \mu_\omega|_{A_\epsilon}$. According to Lemma 4.3 there exists a measure $\nu$ with $\text{supp}(\nu) \subseteq T_\epsilon \cap \mathcal{M}$ s.t.

$$\mu_\omega(\cdot) = \mu_\omega|_{A_\epsilon}(\cdot) = \int K(\cdot, \omega') d\nu(\omega').$$

Since $\mu_\omega$ is extremal we must have that $\nu \propto \delta_\omega$. This implies that $\omega \in T_\epsilon$, which is clearly a contradiction and hence $\mu_\omega(A_\epsilon) = 0$.

Next, consider $\mu \in \text{Conf}$ arbitrary. Let

$$A = \{x \in X^+ : T^n x \text{ has no } \rho\text{-limit}\}.$$

Let $\nu$ be the measure from Theorem 3.2 s.t.

$$\mu = \int_{\mathcal{M}_m} \mu_\omega d\nu(\omega).$$

Since $\mu_\omega(A) = 0$ for every $\omega \in \mathcal{M}_m$, $\mu(A) = 0$ as well, meaning for $\mu$-a.e. $x \in X^+$, $T^n x$ converges. By Proposition 3.1 $T^n x$ must converge to a boundary point.
As for Eq. (4.1), since for every \(\omega \in \mathcal{M}_m\),
\[
1 = \mu_{\omega}(\{o\}) = \mu_{\omega}(\{o \cap \{x \in X^+ : T^nx \to \omega\}\})
\]
we have that
\[
\nu(E) = \int_E 1 \nu(\omega)
= \int_{\mathcal{M}_m} \mu_{\omega}(\{o \cap \{x \in X^+ : T^n x \to \omega\}\}) d\nu(\omega)
= \int_{\mathcal{M}_m} \mu_{\omega}(\{\exists \omega \in E \text{ s.t. } T^n x \to \omega\}) d\nu(\omega)
= \mu(\{o \cap \{x \in X^+ : \exists \omega \in E \text{ s.t. } T^n x \to \omega\}\}).
\]

\[\square\]

5 The reversed Martin boundary and positive eigenfunctions

So far we have focused on positive \(\lambda\)-eigenmeasures. We now turn our attention to the positive \(\lambda\)-eigenfunctions. In particular, we focus on eigenfunctions with positive continuous logarithm.

**Definition 5.1.** Let
\[
\mathcal{H}(\lambda) = \{ f \in C(X^+) : f > 0, L_\phi f = \lambda f \text{ and } \log f \text{ is uniformly continuous} \}.
\]

The uniform regularity condition in the definition of \(\mathcal{H}\) appears naturally when trying to represent an eigenfunction in forms of Martin kernels.

One possible approach to study the positive eigenfunctions is via a direct construction of a suitable Martin boundary, as in the study of the eigenmeasures. However, this approach tends to be technical, leads to redundant proofs and does not establish any connection between the eigenfunctions and the eigenmeasures. Thus we take a different approach; studying the eigenmeasures on the negative one-sided TMS. In particular, we establish a \(1:1\) correspondence between eigenfunctions on the positive one-sided TMS and eigenmeasures on the negative one-sided TMS.

We start with definitions.

**Definition 5.2.** Let
\[
X = \{ z \in S^\mathbb{Z} : \mathcal{A}(z)_i, (z)_{i+1} = 1, \forall i \in \mathbb{Z} \}
\]
and let
\[
X^- = \{ y \in S^{-\mathbb{N}\cup\{0\}} : \mathcal{A}(y)_i, (y)_{i+1} = 1, \forall i < 0 \}.
\]
To avoid confusions, in this section points of \(X^+\) will be denoted by \(x\), points of \(X^-\) will be denoted by \(y\) and points of \(X\) by \(z\). The **two-sided left shift** \(T : X \to X\) is the transformation \((Tz)_i = (z)_{i+1}\), the **two-sided right shift** \(T^{-1} : X \to X\) is the transformation \((T^{-1}z)_i = (z)_{i-1}\) and the **one-sided right shift** \(T^{-1} : X^- \to X^-\) is the transformation
Definition 5.3. Two potentials \( \phi^+ : X^+ \to \mathbb{R}, \phi^- : X^- \to \mathbb{R} \) are cohomologous via a transfer function \( \psi : X \to \mathbb{R} \) if

\[
\phi^+(z^+) - \phi^-(z^-) = \psi(z) - \psi(Tz), \quad \forall z \in X.
\] (5.1)

Proposition 5.1. Let \( \phi^+ : X^+ \to \mathbb{R} \) be a potential function with summable variations. Then, there exists \( \phi^- : X^- \to \mathbb{R} \) with summable variations and a uniformly continuous \( \psi : X \to \mathbb{R} \) s.t. \( \phi^+ \) and \( \phi^- \) are cohomologous via \( \psi \).

Proof. See [8] and [6]. \( \square \)

Proposition 5.2. Let \((X,T)\) be a transitive locally compact two-sided TMS and let \( \phi^+ : X^+ \to \mathbb{R}, \phi^- : X^- \to \mathbb{R} \) be two potential functions with summable variations and which are cohomologous via a uniformly continuous transfer function \( \psi : X \to \mathbb{R} \). Then, \( \phi^+ \) is \( \lambda \)-transient iff \( \phi^- \) is \( \lambda \)-transient.
Theorem 5.1. Let \( \phi^+ : X^+ \to \mathbb{R} \) and \( \phi^- : X^- \to \mathbb{R} \) are \( \lambda \)-transient potential functions with summable variations which are cohomologous via a uniformly continuous transfer function \( \psi : X \to \mathbb{R} \).

Denote by \( \mathcal{K} (\cdot, \cdot | \lambda) \) the Martin kernel w.r.t. \( (X^-, \phi^-) \), by \( \mathcal{M}(\lambda) \) the corresponding Martin boundary and by \( \mathcal{M}_n(\lambda) \) the minimal boundary. We show that the eigenfunctions on the positive one-sided TMS are in fact equivalent, via a simple reduction, to the conformal measures of the negative one-sided TMS.

**Definition 5.4.** For a point \( x \in X^+ \), let \( \psi_x : T^{-1}[(x)_{0}] \to \mathbb{R} \),

\[
\psi_x(y) := \psi(\ldots, (y)_{-1}, (y)_{0}, (x)_{0}, (x)_{1}, (x)_{2}, \ldots)
\]

and let \( \chi_x : X^- \to \mathbb{R} \),

\[
\chi_x := \exp \left( -\psi(\ldots, (y)_{-1}, (x)_{0}, (x)_{1}, (x)_{2}, \ldots) \right) \cdot \mathcal{K}(y)_{0}, (x)_{0}.
\]

Clearly \( \chi_x \in C^+_1(X^-) \), for every \( x \in X^+ \).

**Lemma 5.1.** Let \( (X, T) \) be a transitive locally compact two-sided TMS and let \( \phi^+ : X^+ \to \mathbb{R} \), \( \phi^- : X^- \to \mathbb{R} \) be two potential functions with summable variations and which are cohomologous via a uniformly continuous transfer function \( \psi : X \to \mathbb{R} \). Let \( x \in X^+ \), \( y \in X^- \) and \( a_1, \ldots, a_n \in S \) s.t. \( (y, a_n, \ldots, a_1, x) \in X \). Then,

\[
e^{\phi^+_n(a_n, \ldots, a_1 x) - \psi(ya_n, \ldots, a_1 x)} = e^{\phi^-_n(ya_n, \ldots, a_1 x) - \psi(ya_n, \ldots, a_1 x)}.
\]

**Proof.** We derive, using Eq. 5.1.

\[
\phi^+_n(a_n, \ldots, a_1 x) - \phi^-_n(ya_n, \ldots, a_1)
\]

\[
= \sum_{i=1}^{n} \left( \phi^+(a_i, \ldots, a_1 x) - \phi^-(ya_i, \ldots, a_1) \right)
\]

\[
= \sum_{i=1}^{n} \left( \psi(ya_n, \ldots, a_i, a_1 x) - (\psi \circ T)(ya_n, \ldots, a_i, a_1 x) \right)
\]

\[
= \psi(ya_n, \ldots, a_1 x) - \psi(ya_n, \ldots, a_1 x).
\]

**Theorem 5.1.** Let \( (X, T) \) be a transitive locally compact two-sided TMS and let \( \phi : X^+ \to \mathbb{R} \) be a potential function with summable variations. Then, there is 1-1 correspondence between the \( \lambda \)-conformal measures on \( (X^-, T^-, \phi^-) \) and the eigenfunctions in \( \mathcal{H}(\lambda) \) via the mapping \( \pi \),

\[
(\pi(\mu))(x) = \mu(\chi_x).
\]

**Proof.** Assume w.l.o.g. that \( \lambda = 1 \).

\( \pi(\mu) \in \mathcal{H} \): Let \( h(x) = \mu(\chi_x) \). Since \( \psi \) is uniformly continuous, for \( n \geq 2 \) and \( x_1, x_2 \in X^+ \) with \( d(x_1, x_2) \leq 2^{-n} \),

\[
\chi_{x_1}(y) = e^{\pm o_n(1)} \chi_{x_2}(y), \quad \forall y \in X^-.
\]
In particular,

\[ h(x_1) = \mu(\chi x_1) = e^{\pm \alpha(1)} \mu(\chi x_2) = e^{\pm \alpha(1)} h(x_2) \]

and \( \log h \) is uniformly continuous. Next, we derive

\[
\begin{align*}
\mu(\chi x) &= (L_\phi^\ast - \mu)(\chi x) \\
&= \sum_{b \in S} \int_{T^{-1} \setminus \{b\}} e^{\phi(b)} \chi_x(yb) d\mu(y) \\
&= \sum_{b \in S} \int_{T^{-1} \setminus \{b\}} e^{\phi(b) - \psi(yb)} d\mu(y) \\
&= \sum_{b \in S} \int_{T^{-1} \setminus \{b\}} e^{\phi(b)} - \psi(yb) d\mu(y) \\
&= \sum_{b \in S} \int_{X^-} e^{\phi(b)} \chi_x(y) d\mu(y) \\
&= \sum_{b \in S} e^{\phi(b)} h(bx) = L_\phi h(x).
\end{align*}
\]

Hence \( h(x) = L_\phi h(x). \)

\( \pi \) is \( 1-1 \). For a measure \( \mu \in \text{Conf}(X^-) \), let \( \mu^x = \chi x \mu \), i.e. \( \mu^x(f) = \mu(\chi x f) \) for all \( f \in C_c(X^-) \). Let \( a_1, \ldots, a_n \in S \) with \( \{a_n, \ldots, a_1\} \neq \emptyset \) and let \( x \in T[a_1] \). Then,

\[
\begin{align*}
\mu^x[a_n, \ldots, a_1] &= \int_{[a_n, \ldots, a_1]} e^{-\psi(x)(y)} d\mu(y) \\
&= \int_{T^{-1} \setminus \{a_n\}} e^{\phi_n(y-a_n, \ldots, a_1)} - \psi(y-a_n, \ldots, a_1) d\mu(y) \\
&= \int_{T^{-1} \setminus \{a_n\}} e^{\phi_n(a_n, \ldots, a_1 x)} - \psi(y-a_n, \ldots, a_1 x) d\mu(y) \\
&= e^{\phi_n(a_n, \ldots, a_1 x)} \mu(x-a_n, \ldots, a_1 x).
\end{align*}
\]

Therefore, if \( \pi(\mu_1) \equiv \pi(\mu_2) \), then \( \mu_1^x = \mu_2^x \) for every \( x \). This implies that \( \mu_1([a_n, \ldots, a_1]) = \mu_1^x(e^x 1_{[a_n, \ldots, a_1]}) = \mu_2^x(e^x 1_{[a_n, \ldots, a_1]}) = \mu_2([a_n, \ldots, a_1]). \)

\( \pi \) is onto: Assume first that \( L_\phi 1 = 1. \) Let

\[
\mu^x([a_n, \ldots, a_1]) := L_\phi 1_{[a_n, \ldots, a_1]}(x) = \begin{cases} 
 e^{\phi_n(a_n, \ldots, a_1 x)} & x \in T[a_1] \\
 0 & \text{o.w.}
\end{cases}
\]

Since \( L_\phi 1 = 1 \), if \( x \in T[a_1] \) then

\[
\sum_{b \in S} \mu^x([b a_n, \ldots, a_1]) = \sum_{b} e^{\phi_n(b a_n, \ldots, a_1 x)}
\]

\[
= e^{\phi_n(a_n, \ldots, a_1 x)} \sum_{b} e^{\phi(b a_n, \ldots, a_1 x)}
\]

\[
= \mu^x[a_n, \ldots, a_1]
\]

20
and $\mu^x$ can be extended to a probability measure on $X^-$ via Carathéodory’s extension theorem.

Let $a \in S$. We show that if $x_1, x_2 \in T[a]$ then $e^{\psi x_1} \mu^{x_1} = e^{\psi x_2} \mu^{x_2}$ on $[a]$. Since every measurable set can be approximated by a disjoint union of cylinders, it is sufficient to consider cylinders of arbitrarily large length.

Let $a_1, \ldots, a_n$ with $a_1 = a$ and $[a_n, \ldots, a_1] \neq \emptyset$ and let $y_0 \in T^{-1}[a_n]$.

Then,

$$
\int_{[a_n, \ldots, a_1]} e^{\psi x_1(y)} d\mu^{x_1}(y) = e^{\alpha_n(1)} e^{\psi(y_0, a_n, \ldots, a_1 x_1)} \mu^{x_1}[a_n, \ldots, a_1] \quad (\because \psi \text{ is uniformly continuous})
$$

$$
= e^{\alpha_n(1)} e^{\psi(y_0, a_n, \ldots, a_1 x_1) + \phi_n(a_n, \ldots, a_1 x_1)}
$$

$$
= e^{\alpha_n(1)} e^{\psi(y_0, a_n, \ldots, a_1 x_2) + \phi_n(y_0, a_n, \ldots, a_1)} \quad (\because \text{Lemma 5.1})
$$

Similarly,

$$
\int_{[a_n, \ldots, a_1]} e^{\psi x_2(y)} d\mu^{x_2}(y) = e^{\alpha_n(1)} e^{\psi(y_0, a_n, \ldots, a_1 x_2) + \phi_n(y_0, a_n, \ldots, a_1)}
$$

Since $n$ can be arbitrarily large and $\alpha_n(1)$ is uniform in $a_1, \ldots, a_n$,

$$
e^{\psi x_1} \mu^{x_1} = e^{\psi x_2} \mu^{x_2}.
$$

Let $\{b_i\}$ be an enumeration of $S$ and let

$$
\mu = \sum_i 1_{[b_i]} e^{\psi x_{b_i}} \mu^{x_{b_i}} \quad (5.2)
$$

where $x_{b_i} \in T[b_i]$. We show that $\mu$ is Radon conformal measure w.r.t. $\phi^-$ and $\pi(\mu) \equiv 1$. Clearly $\mu$ is a positive Radon measure. Assume that $x \in [b]$ for some $b \in S$. Then,

$$
\mu(x) = \mu(e^{-\psi x} 1_{T^{-1}[b]})
$$

$$
= \sum_{a \in S \mid A_{a,b} = 1} \mu(e^{-\psi x} 1_{[a]})
$$

$$
= \sum_{a \in S \mid A_{a,b} = 1} \mu^x (e^{\psi x_a} e^{-\psi x} 1_{[a]})
$$

$$
= \sum_{a \in S \mid A_{a,b} = 1} \mu^x (e^{\psi x} e^{-\psi x} 1_{[a]}) \quad (\because x, x_a \in T[a])
$$

$$
= \mu^x (T^{-1}[b]) = 1
$$

and $\pi(\mu)$ is indeed the constant eigenfunction.

To show that $\mu$ is conformal, it is sufficient to consider only cylinders.
Let \([a_n, \ldots, a_1] \neq \emptyset\), let \(x \in T[a_1]\) and let \(y_0 \in T^{-1}[a_n]\). Then,

\[
(L^*_\phi - \mu)[a_n, \ldots, a_1] = \int_{[a_n, \ldots, a_2]} e^{\phi^- (y_{a_1})} d\mu(y)
\]

\[
= \int_{[a_n, \ldots, a_2]} e^{\phi^- (y_{a_1})} e^{\phi_{a_2} (y)} d\mu^{a_2}(y) \quad (\because\text{definition of } \mu)
\]

\[
= \int_{[a_n, \ldots, a_2]} e^{\phi^- (y_{a_1})} e^{\psi_{a_2}(y)} d\mu^{a_2}(y) \quad (\because \text{vector fields})
\]

\[
= e^{\psi_{a_1}(y_{a_1})} \int_{[a_n, \ldots, a_1]} e^{\psi_{a_1}(y)} d\mu^{a_1}(y) \quad (\because \text{cohomology})
\]

\[
= e^{\psi_{a_1}(y_{a_1})} \int_{[a_n, \ldots, a_1]} e^{\psi_{a_1}(y)} d\mu^{a_1}(y) \quad (\because x, a_1 \in T[a_1])
\]

\[
= e^{\psi_{a_1}(y_{a_1})} \mu[a_n, \ldots, a_1].
\]

Again, since \(n\) can be taken to be arbitrarily large and \(o_n(1)\) is uniform in \(a_1, \ldots, a_n\), \(\mu\) is indeed conformal w.r.t. \(\phi^-\).

Now, assume \(h\) is an arbitrary positive eigenfunction with uniformly continuous logarithm and consider the potential \(\phi^h = \phi + \log h - \log h \circ T\). Then,

\[
\phi^h - \phi^- = \psi + \log h - (\psi + \log h) \circ T.
\]

Hence, the transfer function of \(\phi^h\) and \(\phi^-\) is \(\psi^h = \psi + \log h\). Observe that in the construction of \(\mu\) in Eq. \(5.1\), we only assumed that \(\phi\) and \(\psi\) are uniformly continuous. Since \(\log h\) is uniformly continuous, \(\phi^h\) and \(\psi^h\) are uniformly continuous as well. Since \(L_{\phi^h} 1 = 1\), there exists a measure \(\mu\) which is conformal w.r.t. \(\phi^-\) and

\[
\mu(e^{-\psi^h} 1_{T^{-1}[y_{a_1}]} ) = 1.
\]

Since

\[
e^{-\psi^h} 1_{T^{-1}[y_{a_1}]} = e^{-\psi} 1_{T^{-1}[y_{a_1}]} h(x) = \chi_x h(x)
\]

then

\[
h(x) = \mu(\chi_x).
\]

\[\square\]

**Remark 5.1.** The correspondence established in Theorem 5.1 is valid even when the underline potential function \(\phi\) is recurrent. Indeed, in such a scenario the cohomologous potential function \(\phi^-\) is recurrent as well, the conformal measure on \(X^-\) is unique and in correspondence to the unique eigenfunction on \(X^+\).

Since the reversed Martin boundary is not empty, Theorem 5.1 implies directly the existence of positive eigenfunctions.

**Theorem 5.2.** Let \((X, T)\) be a transitive locally compact two-sided TMS and let \(\phi : X^+ \to \mathbb{R}\) be a \(\lambda\)-transient potential function with summable variations. Then, there exists a positive continuous \(\lambda\)-eigenfunction.
With the correspondence of Theorem 5.1, one can easily derive analogue results of Section 3 to the positive eigenfunctions.

**Definition 5.5.** A function \( h \in H(\lambda) \) is \( \lambda \)-minimal if for every \( h' \in H(\lambda) \) with \( h' \leq h \), we have that \( h' \propto h \).

**Theorem 5.3.** Let \((X, T)\) be a transitive locally compact two-sided TMS and let \( \phi : X^+ \rightarrow \mathbb{R} \) be a \( \lambda \)-transient potential function with summable variations. Then, for every \( h \in H(\lambda) \), there exists a unique finite measure \( \nu \) on \( \mathcal{M}_m(\lambda) \) s.t.

\[
h(x) = \int_{\mathcal{M}_m(\lambda)} K(\chi_x, \xi) d\nu(\xi) \quad \forall x \in X^+.
\]

Moreover, \( h \) is \( \lambda \)-minimal iff \( \nu \) is a dirac measure, meaning \( \pi^{-1}h \in \mathcal{M}_m(\lambda) \).

**Proof.** The existence follows directly from Theorem 3.2 and Theorem 5.1. Since \( \pi \) is linear, so is \( \pi^{-1} \) and thus \( h \) is minimal iff \( \pi^{-1}h \) is extremal. As for the uniqueness, if there exist \( \nu \) and \( \nu' \) s.t.

\[
h(x) = \int_{\mathcal{M}_m(\lambda)} K(\chi_x, \xi) d\nu(\xi) = \int_{\mathcal{M}_m(\lambda)} K(\chi_x, \xi) d\nu'(\xi) \quad \forall x \in X^+
\]

then with \( \mu = \int K(\cdot, \xi) d\nu(\xi) \) and \( \mu' = \int K(\cdot, \xi) d\nu'(\xi) \) we have that \( \pi\mu = \pi\mu' = h \). Since \( \pi \) is 1-1, \( \mu = \mu' \). By Theorem 5.2 we must have that \( \nu = \nu' \).

In several applications, we consider \( T \)-invariant measures on \( X \) of the form \( m = h\mu \), where \( h \) is a positive eigenfunction and \( \mu \) is a positive eigenmeasure. Theorem 5.3 yields a different interpretation for such a construction:

1. Pick a finite measure \( \nu^+ \) on \( \mathcal{M}_m \) and set \( \mu^+ = \int_{\mathcal{M}_m} \mu d\nu^+(\omega) \).
2. Pick a finite measure \( \nu^- \) on \( \mathcal{M}_m \) and set \( \mu^- = \int_{\mathcal{M}_m} \mu d\nu^-(\omega) \).
3. The resulting measure is

\[
m(f) = \int_{x \in X^+} \int_{y \in X^-} f(x) \prod_{i=0}^{a} \chi_{y_0}(y_i) d\mu^-(y) d\mu^+(x).
\]

(5.3)

Theorem 5.1 is valid in the recurrent case as well and so does Eq. (5.3), although the measures \( \mu^- \) and \( \mu^+ \) are unique up to normalization. See [16, 17] for similar decompositions in different settings.

To conclude the discussion on the reversed Martin boundary, we provide an example for which the Martin boundary of the inverted graph differs from the Martin boundary of the original one.

**Example 5.1.** Consider \( S = \mathbb{Z} \cup \{n' : n \in \mathbb{N}\} \), where \( n' \) is a different copy of \( n \) and consider the transition matrix \( \tilde{A} \) with \( \tilde{A}_{a,b} = 1 \) iff one of the following cases

- \( a = 0, b \in \{-1, +1\} \).
- \( a = 1', b = 0 \).
\[a, b \in \mathbb{Z} \setminus \{0\}\] and \(b = a + \text{sign}(a)\).

- \(a \in \mathbb{Z} \setminus \{0\}\), \(b = n'\) with \(|a| = n\).
- \(a = (n + 1)', b = n'\).

See Figure 5.1. Clearly the corresponding TMS is locally finite and transitive.

**Proposition 5.3.** There is \(\alpha < 0\) s.t. the potential function \(\phi \equiv \alpha\) is transient, \(\mathcal{M}\) contains at least two points and \(\overline{\mathcal{M}}\) contains a single point alone.

**Proof.** Since the out-degree of any state is bounded by 2, for any \(x \in X^+\)

\[(L^n_\phi 1_{[0]})(x) \leq e^{n\alpha}2^n\]

whence, with \(\alpha < -\log 2\), \(G(1_{[0]}, x) < \infty\) and the potential \(\phi \equiv \alpha\) is indeed transient.

In the reversed graph, \(n'\) with \(n \to \infty\) is the only possible direction which escapes every finite set. Hence \(\overline{\mathcal{M}}\) cannot contain more than a single point. In the original graph, \(n \to \infty\) or \(n \to -\infty\) are the only two possible directions to escape every finite set. We show that they may correspond to two different points in \(\mathcal{M}\). Let \(x_n^+ \in [n]\) and let \(x_n^- \in [-n]\).

Let

\[Z_n^*(\alpha, a, b) = e^{n\alpha} \# \{\text{paths of length } n \text{ from } a \text{ to } b, \text{ first reaching } b \text{ in the } n'\text{th step}\}\]

and let

\[F(\alpha, a, b) = \sum_{n \geq 0} Z_n^*(\alpha, a, b).\]

Observe that

\[L^n_\phi 1_{[0]}(x) = e^{n\alpha} \# \{\text{paths from } a \text{ to } (x)_0 \text{ of length } n\}.\]

Since every path from \(-1\) to \(n\) must pass through 1,

\[G(1_{[-1]}, x_n^+) = F(\alpha, -1, 1)G(1_{[1]}, x_n^+).\]

By symmetry of the graph, \(G(1_{[1]}, x_n^+) = G(1_{[-1]}, x_n^-)\). Hence,

\[K(1_{[-1]}, x_n^+) = F(\alpha, -1, 1)K(1_{[-1]}, x_n^-).\]

Since \(F(\alpha, -1, 1)\) varies with \(\alpha\), we can decrease \(\alpha\) so that \(F(\alpha, -1, 1) \neq 1\).

Observe that decreasing \(\alpha\) does not effect the transience of \(\phi\). Then,

\[K(1_{[-1]}, \infty) = \lim_{n \to \infty} K(1_{[-1]}, x_n^+)\]

\[= \lim_{n \to \infty} F(\alpha, -1, 1)K(1_{[-1]}, x_n^-)\]

\[= F(\alpha, -1, 1)K(1_{[-1]}, -\infty)\]

and in particular \(K(\cdot, \infty) \neq K(\cdot, -\infty)\).
6 Applications to first-order phase transitions

6.1 Background

In this section we apply our results to the theory of Gibbs states and first order phase transitions. Recall that a thermodynamic system is said to undergo a phase transition of the first order if there are several possible equilibrium values to some of its thermodynamic quantities. The question how to formalize this was studied extensively in the sixties, see e.g. [24, 19, 9]. Here we follow the program of Dobrushin, Lanford and Ruelle which formalizes a phase transition of the first order as a situation where there are several Dobrushin-Lanford-Ruelle (DLR) measures, see Section 6.2 below. An alternative approach to first-order phase transitions is to view them as situations where the thermodynamic limit is not unique, see Section 6.3 below. The two approaches are often equivalent, see [25].

We show here that if $\phi$ is transient with a Martin Boundary bigger than one point, then there are several different non-singular DLR states, each of which corresponds to thermodynamic limits where the “boundary conditions” escape to infinity in different directions. Compare with [12].

6.2 Existence and non-uniqueness of DLR measures

Recall that $(X^+, T)$ is a topologically mixing locally compact countable Markov shift and that $\phi : X^+ \to \mathbb{R}$ is a potential function with summable variations and finite Gurevich pressure. The following definition is a version of the classical definition of a DLR measure, tailored to fit our one-
Definition 6.1. We say that a probability measure \( m \) is a Dobrushin-Lanford-Ruelle measure if for all \( n \geq 1 \) and \( m \)-a.e. \( x \in X^+ \),

\[
E_m[1_{[(x)_0, \ldots, (x)_{n-1}]}|T^{-n}B](x) = \sum_{y: T^n y = T^n x} e^{\phi_n(x)}
\]

(6.1)

where \( B \) is the Borel \( \sigma \)-algebra of \( X^+ \).

Recall that a positive Radon measure \( \mu \) is non-singular if for every Borel set \( A \subseteq X^+ \), \( \mu(A) = 0 \) iff \( \mu(T^{-1}A) = 0 \). The connection between DLR measures and eigenmeasures is explained in the following propositions.

Proposition 6.1. Let \( \nu \) be a non-singular probability measure with \( L^* \phi \nu = \lambda \nu \) for some \( \lambda > 0 \) and let \( \phi: X \to \mathbb{R} \) a Borel function. Then \( \nu \) is a non-singular DLR measure for \( \phi \).

Proof. See [23, 29].

Proposition 6.2. Let \( \nu \) be a non-singular DLR measure for \( \phi \). Then, there exists a function \( h: X^+ \to \mathbb{R} \), which is measurable w.r.t. the \( \sigma \)-algebra \( \cap_{n \geq 0} T^{-n}B \), s.t. \( L^* \phi + h \nu = \nu \).

Proof. See appendix.

In the infinite state case, there may exist DLR measures which are not non-singular. These measures may not correspond to eigenmeasures of \( L^* \phi \), see Example A.1 in the appendix.

We would like to relate the richness of the Martin boundary to first-order phase-transitions. However, the resulting conformal measures in Section 2 may be infinite. To overcome this problem, one can "adjust" the conformal measures with a uniformly continuous density to obtain conformal probability measures w.r.t. a different but cohomologous potential function.

Proposition 6.3. Assume that \( X^+ \) is locally compact and topologically mixing and that \( \phi: X^+ \to \mathbb{R} \) is \( \lambda \)-transient and has summable variations. Then, there exists a uniformly continuous function \( h: X^+ \to \mathbb{R} \), s.t. for every \( \mu \in M(\lambda) \), the measure \( \frac{1}{\mu(h)} \mu h \) is a DLR measure w.r.t. \( \phi - \log h + \log h \circ T \).

Proof. Let \( C_\alpha > 0 \) the constant from Lemma 2.1 and consider the function

\[
h(x) = \sum_{n=1}^{\infty} \frac{1}{2^n C_\alpha^n} 1_{[a_n]}
\]

where \( \{a_n\} \) is an enumeration of \( S \). Then, for every \( \mu \in M(\lambda) \), the measure \( d\mu^h = h\mu \) is finite and \( L^* \phi - \log h + \log h \circ T \mu^h = \lambda \mu^h \). Therefore, by Proposition 6.1, \( \frac{1}{\mu(h)} \mu^h \) is a DLR measure.

Proposition 6.3 implies that if the Martin Boundary of \( \phi - \log h + \log h \circ T \) contains more than one point, then \( \phi - \log h + \log h \circ T \) has more than one non-singular DLR state, namely a first order phase transition. For examples like this, see e.g. simple random walks on \( d \)-regular trees \( (d \geq 3) \) [30] or Example 6.1.
6.3 Thermodynamic limits

We will now interpret the DLR states arising from different points in the Martin boundary as thermodynamic limits with different boundary conditions.

In our context, thermodynamic limits arise from the following scheme:

1. Approximate $X^+$ with finite subsets $X_N$ by imposing a boundary condition which rules out all but a discrete collection of configurations.

2. Define the “canonical ensemble" $\mu_N$ on $X_N$ by giving configurations weights according to the Gibbs formula and then normalizing as possible.

3. Pass to the limit in some regime where $X_N$ fills $X^+$ densely. The weak star limit points of $\mu_N$ are called thermodynamic limits or Gibbs states.

The mathematical question is which limiting regimes give weak-star convergence, and what are the limiting measures.

We describe here the limiting regimes which work in the positive recurrent and the null recurrent scenarios. For more on positive and null recurrence, see [27]. To simplify calculations, we assume that $P_G(\phi) = 0$ and that $T : X^+ \to X^+$ is topologically mixing.

The positive recurrent case: $L_N \phi_1^{[0]}$ is eventually bounded below [27].

1. Fix $x \in X^+$ and let

$$X_N(x) := \{ y \in X^+ : y_N^\infty = x_0^\infty \} \equiv \{ y : T^N y = x \}.$$

2. Set

$$\mu_N = \frac{1}{Z_N(x)} \sum_{y \in X_N(x)} e^{\phi_N(y)} \delta_y$$

where $Z_N(x) = \sum_{y \in X_N(x)} e^{\phi_N(y)} 1_{[0]}(y)$. By the generalized Ruelle’s Perron-Frobenius theorem [26, 27] for every $f \in C^+_c(X^+)$

$$\mu_N(f) = \frac{(L_N^\phi f)(x)}{(L_N^\phi 1_{[0]}(y))(x)} \xrightarrow{N \to \infty} \frac{\mu(f)}{\mu[0]}$$

where $\mu$ is the unique eigemeasure (which is also a DLR state).

The Null recurrent case: $L_N^\phi 1_{[0]} \xrightarrow{n \to \infty} 0$ but $\sum_n L_n^\phi 1_{[0]} = \infty$. Now, the previous procedure is problematic because the numerator and denominator both tend to zero. So we use the following alternative scheme.

1. Fix $x \in X^+$ and let

$$X'_N(x) := \{ y \in X^+ : y_n^\infty = x_0^\infty \text{ for } 0 \leq n < N \} = \bigcup_{n=0}^{N-1} X_n(x).$$
(2) Set
\[ \mu_N = \frac{1}{Z'_N(x)} \sum_{n=0}^{N-1} \sum_{y \in X'_m(x)} e^{\phi_n(y)} \delta_y, \]
where
\[ Z'_N(x) = \sum_{n=0}^{N-1} \sum_{y \in X'_m(x)} e^{\phi_n(y)} 1_{[0]}(y). \]

Again, by the generalized Ruelle’s Perron-Frobenius theorem [27]
\[ \mu_N(f) = \frac{\sum_{n=0}^{N-1} (L^\phi f)(x)}{\sum_{n=0}^{N-1} (L^\phi 1_{[0]})(x)} \xrightarrow{N \to \infty} \frac{\mu(f)}{\mu[\phi]}, \]
where \( \mu \) is the unique eigenmeasure (which is also a DLR state for \( \phi - \log h + \log h \circ T \), see Proposition 6.3).

The transient case: \( \sum_n L^\phi_n 1_{[0]} < \infty \). Now the “edge effects” do not vanish in the limit and a different procedure is required. We suggest here the following limiting regime which avoids this issue. Let \( M = M(\exp P_G(\phi)) \) the Martin boundary of \( \phi \).

(1) Fix \( \omega \in M \) and \( x \in X^+ \) s.t. \( T^n x \xrightarrow{n \to \infty} \omega \). By Theorem 4.1, such \( x \) exists. Take
\[ X''_N(x) := \{ y \in X^+ : \exists m > 0 \text{ s.t. } y^\infty_m = x^\infty_N \} = \bigcup_{m=0}^{\infty} X_m(T^N x). \]

Notice that \( \bigcup_N X''_N(x) = \{ y : \exists m, d(T^{n+m} y, T^n x) \to 0 \} \) is the symbolic analogue of the weak-star manifold of \( x \).

(2) Set
\[ \mu_N = \frac{\sum_{n=0}^{\infty} \sum_{y \in X_m(T^N x)} e^{\phi_m(y)} \delta_y}{\sum_{n=0}^{\infty} \sum_{y \in X_m(T^N x)} e^{\phi_m(y)} 1_{[0]}(y)}. \]

By Proposition 2.1
\[ \mu_N(f) = \frac{G(f, T^N x)}{G(1_{[0]}, T^N x)} = K(f, T^N x) \xrightarrow{N \to \infty} \mu_\omega(f) \]
where \( \mu_\omega \) is given in Definition 2.3. Again, since \( \mu_\omega \) is an eigenmeasure, the thermodynamic limit is a DLR state for \( \phi - \log h + \log h \circ T \), see Proposition 6.3. However, this time the choice of boundary condition \( x \) matters; if we work with a different boundary condition \( x' \) with \( T^N x' \xrightarrow{N \to \infty} \omega' \neq \omega \), then the thermodynamic limit we will get is \( \mu_\omega' \neq \mu_\omega \).

7 The Martin boundary of a transient random walk

In this section we illustrate why the boundary constructed in Section 2 and Theorems 3.2 and 4.1 are, in some sense, a generalization of the probabilistic Martin boundary. Let \( (Z, P) \) be a random walk on a countable
set $S$, with random variable $Z = (Z_n) \in X^+$ and $P : S \times S \to [0, 1]$ a probability transition matrix. Recall that a function $h : S \to \mathbb{R}$ is $P$-harmonic if

$$h(a) = \sum_b P(a, b)h(b), \quad \forall a \in S.$$ 

**Theorem 7.1.** Assume that the walk is transient, locally finite and irreducible.

1) Let $h : S \to \mathbb{R}^+$ be a positive $P$-harmonic function. Then, there exists a unique measure $\nu$ on $\mathcal{M}_m$ s.t.

$$h(a) = \int_{\mathcal{M}_m} K([a], \omega)d\nu(\omega)$$

where the Martin kernel and Martin boundary are derived from the potential function $\phi(x) = \log P((x)_0, (x)_1)$.

2) For every $a \in S$, and $A \subseteq \mathcal{M}_m$, 

$$\Pr_a\left[\lim_{n \to \infty} Z_n \in A\right] = \int_A K([a], \omega)d\nu_1(\omega)$$

where $\nu_1$ is the measure from (7) with the harmonic function $h \equiv 1$.

**Remark 7.1.** Notice that a $P$-harmonic function is not an eigenfunction of $L_\phi$, but rather an eigenfunction of $L_{\phi^{-}}$, with $\phi^{-}(y) = \log P((y)_{-1}, (y)_0)$. In particular, in the following proof, to simplify calculations we explicitly present the correspondence between the $P$-harmonic functions and the conformal measures, rather than applying Theorem 5.1 or Theorem 5.3.

**Proof.** Since 

$$(L_\phi^+1_{[a]})(x) = p^n((x)_0, a)$$

then 

$$G(1_{[a]}, x) = \sum_{n=0}^\infty p^n((x)_0, a) < \infty$$

and $\phi$ is indeed transient.

**Proof of 1.** We define a measure 

$$\mu([a]) = h(a) \quad (7.1)$$

$$\mu([a_1, \ldots, a_n]) = e^{\phi(a_1, a_2)} \cdots e^{\phi(a_{n-1}, a_n)} h(a_n).$$

Since 

$$\sum_b \mu([a_1, \ldots, a_n, b]) = \sum_b e^{\phi(a_1, a_2)} \cdots e^{\phi(a_{n-1}, a_n)} e^{\phi(a_n, b)} h(b)$$

$$e^{\phi(a_1, a_2)} \cdots e^{\phi(a_{n-1}, a_n)} h(a_n) = \mu([a_1, \ldots, a_n])$$

29
\( \mu \) can be extended to a measure via Carathéodory’s extension theorem. Moreover, since

\[
(L^{\ast} \phi \mu)([a]) = \mu(L \phi 1_{[a]}) \\
= \int e^{\phi(ax)} d\mu(x) \\
= \sum b e^{\phi(ab)} \mu([b]) \\
= \sum b e^{\phi(ab)} h(b) = h(a) = \mu([a])
\]

and

\[
(L^{\ast} \phi \mu)([a_1, \ldots, a_n]) = \mu(L \phi 1_{[a_1, \ldots, a_n]}) \\
= \int \sum b e^{\phi(bx)} 1_{[a_1, \ldots, a_n]}(bx) d\mu(x) \\
= \int e^{\phi(a_1 a_2)} 1_{[a_2, \ldots, a_n]}(x) d\mu(x) \\
= e^{\phi(a_1 a_2)} \mu([a_2, \ldots, a_n]) = \mu([a_1, \ldots, a_n])
\]

we have that \( \mu \in \text{Conf} \). This establishes a linear 1-1 correspondence between the positive \( P \)-harmonic functions and the conformal measures. Theorem 3.2 implies that there exists a unique measure \( \nu \) s.t.

\[
\mu = \int_{M_m} \mu_\omega d\nu(\omega)
\]

and in particular

\[
h(a) = \mu[a] = \int_{M_m} K([a], \omega) d\nu(\omega).
\]

Proof of 2. Since \( \sum_h P(a, b) = 1 \), for every \( a \in S \), 1 is a positive \( P \)-harmonic function. Let \( \mu_1 \) be the measure following Eq. (7.1) with \( h \equiv 1 \), and let \( \nu_1 \) the corresponding measure from Theorem 3.2. Fix \( A \subseteq M_m \), and let

\[
\tilde{A} = \{ x \in X^+ : T^n x \rightarrow A \}.
\]

By definition,

\[
\mu_1([a, a_1, \ldots, a_n]) = \Pr_\omega[Z_1 = a_1, \ldots, Z_n = a_n].
\]

Therefore, for any event \( B \subseteq X^+ \),

\[
\mu_1([a] \cap B) = \Pr_\omega[B].
\]

According to Theorem 4.1, for every \( \omega \in M_m \),

\[
\mu_\omega([a] \cap \tilde{A}) = \begin{cases} 
\mu_\omega([a]) & \omega \in A \\
0 & \text{o.w.}
\end{cases} = \mu_\omega([a]) 1_A(\omega).
\]
Thus,

$$\Pr_a(\lim_{n \to \infty} Z_n \in A) = \mu_1([a] \cap \tilde{A})$$

$$= \int_{\mathcal{M}_m} \mu_{\omega}([a] \cap \tilde{A})d\nu_1(\omega)$$

$$= \int_{\mathcal{M}_m} \mu_{\omega}([a])1_A(\omega)d\nu_1(\omega)$$

$$= \int_A K([a], \omega)d\nu_1(\omega).$$

Therefore.

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### A Appendix

**Proof of Proposition 2.1.** Assume w.l.o.g. that $\lambda = 1$.

1) Let $\{x_n\}$ be an arbitrary sequence in $\tilde{X}^\mathcal{M}$. By definition, $X^+$ is dense in $\tilde{X}^\mathcal{M}$. If $\{x_n\} \cap \mathcal{M} \neq \emptyset$, we replace every point $x_n \in \mathcal{M}$ with some $x'_n \in X^+$ s.t. $\rho(x_n, x'_n) < 1/n$. We will use diagonalization argument to show that $\{x'_n\}$ has a Cauchy sub-sequence. Since $\rho(x_n, x'_n) \leq 1/n$, the original sequence will have a Cauchy sub-sequence as well. Thus we can assume w.l.o.g. that $\{x_n\} \subseteq X^+$.

Let $f_1(x) = \frac{1}{2(\epsilon|x_1| + 1)}K([x_1], x|\lambda)$ and $g_i(x) = \frac{1}{2(\epsilon|x_i| + 1)}1_{[x_i]}(x)$.

Then,

$$\rho(x, y) = \sum_i (|f_i(x) - f_i(y)| + |g_i(x) - g_i(y)|).$$

Since $g_1$ and $f_1$ are bounded, we can find a sub-sequence $x_{n_k}^1$ s.t. $g_1(x_{n_k}^1)$ and $f_1(x_{n_k}^1)$ converge. Similarly, for every $m$ we can find $x_{n_k}^m$ a sub-sequence of $x_{n_k}^{m-1}$ s.t. $g_m(x_{n_k}^m)$ and $f_m(x_{n_k}^m)$ converge.

Set $x_{n_k} = x_{n_k}^k$. We show that $x_{n_k}$ is a Cauchy sequence. Let $\epsilon > 0$, and let $N$ be large enough s.t.

$$\sum_{i \geq N} (|f_i(x) - f_i(y)| + |g_i(x) - g_i(y)|) \leq \epsilon, \ \forall x, y \in X^+.$$ 

Let $K$ be large enough s.t. for every $k_1, k_2 \geq K$ and every $i > N$

$$|f_i(x_{n_{k_1}}) - f_i(x_{n_{k_2}})| + |g_i(x_{n_{k_1}}) - g_i(x_{n_{k_2}})| \leq \epsilon/N.$$

Then, for all $k_1, k_2 \geq K$,

$$\rho(x_{n_{k_1}}, x_{n_{k_2}}) \leq \sum_{i < N} (|f_i(x_{n_{k_1}}) - f_i(x_{n_{k_2}})| + |g_i(x_{n_{k_1}}) - g_i(x_{n_{k_2}})|) + \epsilon \leq 2\epsilon.$$

Hence $\{x_n\}$ is a $\rho$-Cauchy sequence.
2) Assume that $x_n \xrightarrow{d} x \in X^+$. Let $\epsilon > 0$ and $m$ be large enough s.t.
\[
\sum_{i \geq m} \left| \frac{K(1_{[w_i]}, x) - K(1_{[w_i]}, y) + |1_{[w_i]}(x) - 1_{[w_i]}(y)|}{2^n(C1_{[w_i]} + 1)} \right| < \frac{\epsilon}{2}, \quad \forall x, y \in X^+.
\]
It is easy to verify that $G(f, x)$ and $K(f, x)$ are continuous functions of $x$, for every $f \in C_c(X^+)$. Then, for $n$ large enough,
\[
\sum_{i < n} \left| \frac{K(1_{[w_i]}, x_n) - K(1_{[w_i]}, x) + |1_{[w_i]}(x_n) - 1_{[w_i]}(x)|}{2^n(C1_{[w_i]} + 1)} \right| < \frac{\epsilon}{2}
\]
and $\rho(x_n, x) < \epsilon$, whence $\{x_n\}$ converges to $x$ w.r.t. $\rho$.

Next, assume that $x_n \xrightarrow{\rho} x \in X^+$. Since for every finite word $w \in S^*$, there exists $C > 0$ s.t.
\[
|1_{[w]}(x) - 1_{[w]}(x_n)| \leq C \rho(x, x_n) \xrightarrow{n \to \infty} 0
\]
and $|1_{[w]}(x) - 1_{[w]}(x_n)| \in \{0, 1\}$, we have that $1_{[w]}(x) = 1_{[w]}(x_n)$ for $n$ large enough. Therefore, for every $m$, we can find $n$ large enough s.t. $(x_n) = (x)_i$ for every $1 \leq i \leq m$, which implies that $d(x_n, x) \leq 2^{-m}$.

3) Assume that $A$ is a $d$-open subset of $X^+$. We show that $A^c = \overline{\Delta^+} \setminus A$ is $\rho$-closed. Let $x_n \in A^c$, $x_n \xrightarrow{d} x$. If $x \in M$, then clearly $x \not\in A$. If $x \in X^+$, then $x_n \xrightarrow{\rho} x$ as well. Since $A$ is $d$-open and $x_n \not\in A$, we must have that $x \not\in A$. This shows that $A^c$ is $\rho$-closed and $A$ is $\rho$-open.

Next, assume that $A$ is $\rho$-open. We show that $B = A^c \cap X^+$ is $d$-closed. Let $x_n \in B$, $x_n \xrightarrow{d} x \in X^+$. Then, $x_n \xrightarrow{\rho} x$ as well. Since $x_n \not\in A$ and $A$ is a $\rho$-open set, $x \not\in A$, meaning $x \in B$.

4) If there exists a compact set $A \subseteq X^+ s.t. x_n \in A$ infinitely many times, then we can find a sub-sequence $x_{n_k} \in A$ with $x_{n_k} \xrightarrow{d} x \in A$. Then, $x_{n_k} \xrightarrow{\rho} x \in A \subseteq X^+$ which is a contradiction.

5) By the construction of $\overline{\Delta^+}$ and $\rho$, for every $w \in S^*$, the function $K(1_{[w]}, \cdot) : X^+ \to \mathbb{R}$ extends uniquely to a continuous function on $X^+$. Let $f \in C_c(X^+)$, and let $x_n \in X^+, x \in \overline{\Delta^+}$ with $x_n \xrightarrow{d} x$. We show that $K(f, x_n) \rightarrow K(f, x)$. Since $\text{supp}(f)$ is compact, there exist $a_1, \ldots, a_{M} \in S$ s.t. $\text{supp}(f) \subseteq \cup_{j=1}^{M}[a_j]$. Moreover, since the collection $\{1_{[w]}\}_{w \in S^*}$ spans linearly a dense subset of $C_c(X^+)$ w.r.t. the sup-norm $\| \cdot \|_{\infty}$, for every $\epsilon > 0$ we can find $w_1, \ldots, w_N$ and $\alpha_1, \ldots, \alpha_N \in \mathbb{R}^+$ s.t. $\cup_{i=1}^{N}[w_i] \subseteq \cup_{j=1}^{M}[a_j]$. Then, for some constant $C = C(f) > 0$,
\[
\|K(f, \cdot) - K(\sum_{i=1}^{N} \alpha_i 1_{[w_i]}, \cdot)\|_{\infty} \leq \epsilon \|K(1_{[w_i]}, \cdot)\|_{\infty} \leq \epsilon \sum_{j=1}^{M} y_{a_j} \leq C.
\]

32
From this we deduce that \( K(f, x_n) \) is a Cauchy sequence of real numbers, and that \( K(f, x) = \lim_{n \to \infty} K(f, x_n) \).

6) Since \( K(\cdot, x) \) defines a positive linear functional on \( C_c^+ (X^+) \), we can apply the Riesz representation theorem to obtain a Radon measure \( \mu_x \) s.t. \( \mu_x(f) = K(f, x) \) for all \( f \in C_c(X^+) \). Moreover, for every \( x \in X^+ \) and \( f \in C_c(X^+) \),

\[
(L_\phi \mu_x)(f) = K(L_\phi f, x) = K(f, x) - \frac{f(x)}{G(1_{[\alpha]}, x)} \leq K(f, x) = \mu_x(f)
\]

meaning \( \mu_x \) is excessive. Observe that since \( X^+ \) is locally finite, \( L_\phi f \in C_c(X^+) \) as well and \( K(L_\phi f, \cdot) \) is \( \rho \)-continuous. Thus, for a boundary point \( \omega \in \mathcal{M} \), if \( x_n \xrightarrow{\rho} \omega \) then \( x_n \xrightarrow{\rho} \infty \), whence \( f(x_n) = 0 \) eventually and

\[
(L_\phi \mu_\omega)(f) = K(L_\phi f, \omega) = \lim_{x \to \omega} K(L_\phi f, x) = \lim_{x \to \omega} (K(f, x) - \frac{f(x)}{G(1_{[\alpha]}, x)}) = \lim_{x \to \omega} K(f, x) = K(f, \omega)
\]

meaning \( \mu_\omega \) is conformal. We remark that \( \mu_\omega([\alpha]) = K(1_{[\alpha]}, \omega) = 1 \), so \( \mu_\omega \neq 0 \).

\[ \square \]

Proof of Proposition 5.2 Assume w.l.o.g. that \( \lambda = 1 \) and assume first that \( F = [a] \) for some \( a \in S \). Since \( G(1_{[a]}, x) \) is continuous, it is bounded on compacts and thus, for every \( b \in S \),

\[
\mu \left( 1_{[b]} \sum_{n \geq 0} L_\phi^n 1_{[a]} \right) < \infty.
\]

Since \( \mu \) is conformal, \( \mu(f L_\phi g) = \mu((f \circ T) \cdot g), \forall f, g \in C_c(X^+) \). Hence,

\[
\mu \left( 1_{[b]} \sum_{n \geq 0} L_\phi^n 1_{[a]} \right) = \mu \left( 1_{[a]} \sum_{n \geq 0} 1_{[b]} \circ T \right) < \infty.
\]

In particular, \( 1_{[a]} \sum_{n \geq 0} 1_{[b]} \circ T < \infty \) almost-surely, whence for \( \mu \)-a.e. \( x \in [a] \), \( T^n x \in [b] \) finitely many times.

Assume now that \( F \) is arbitrary compact. By compactness, there exist \( a_1, \ldots, a_N \in S \) s.t. \( F \subseteq \cup_{i=1}^N [a_i] \). Observe that in order to return infinitely many times to \( F \) we must return infinitely many times to one of the cylinders \( [a_i] \). Therefore, \( F_\infty \subseteq \cup_{i=1}^N [a_i]_\infty \) and \( \mu(F_\infty) \leq \sum_{i=1}^N \mu([a_i]_\infty) = 0 \).

\[ \square \]

Proof of Proposition 5.3 Assume w.l.o.g. that \( \lambda = 1 \). By symmetry, we only show that if \( \phi^+ \) is transient then \( \phi^- \) is transient as well. In particular,
we show that there exists a non-zero function \( f \in C^+_c(X^-) \) and a point \( y \in X^- \) s.t.
\[
\sum_{n=0}^{\infty} L^+_n f(x) < \infty.
\]

Fix \( a \in S \), \( x \in X^+ \cap T[a] \), \( y \in X^- \cap T^{-1}[a] \). For every admissible word \((a_1, \ldots, a_n)\) with \( a_1 = a_n = a \), we write \( z = (\ldots, (y)_{-1}(y), a_1, \ldots, a_n(x)a, (x), \ldots) \). Let \( C_1 = \max_{x' \in [a]} |\psi(x')| \) and \( C_2 = \max_{x' \in [a]} |\psi(x')| \). Then,
\[
|\psi(Tz)| = |\psi(ya_1, \ldots, a_n)| \leq C_1
\]
and
\[
|\psi(T^{n+1}z)| = |\psi(ya_1, \ldots, a_n)| \leq C_2.
\]
From the cohomology property in Eq. (5.1),
\[
\phi^+(a_1, \ldots, a_n, x) - \phi^-(y) = \psi(Tz) - \psi(T^{n+1}z).
\]
In particular,
\[
\phi^+_n(a_1, \ldots, a_n, x) = \phi^-(y) + \psi(Tz) - \psi(T^{n+1}z)
\]
\[\geq \phi^-(y) - C_1 - C_2.
\]
Then, for every \( n \geq 2 \)
\[
L^+_n(1_{[a]})(x) = \sum_{a_1, \ldots, a_n} e^{\phi^+(a_1, \ldots, a_n, x)}
\]
\[\geq \sum_{a_1, \ldots, a_n} e^{\phi^-(y) + \psi(Tz) - \psi(T^{n+1}z)}
\]
\[\geq e^{-C_1 - C_2} \sum_{a_1, \ldots, a_n} e^{\phi^-(y) + \psi(Tz) - \psi(T^{n+1}z)}
\]
\[= e^{-C_1 - C_2 + \phi^-} \sum_{a_1, \ldots, a_n} e^{\phi^-(y) + \psi(Tz) - \psi(T^{n+1}z)}
\]
\[= e^{-C_1 - C_2 + \phi^-} L^{-1}_n(1_{[a]})(y).
\]
This implies that
\[
G(1_{[a]})(x) \geq e^{-C_1 - C_2 + \phi^-} \sum_{n \geq 1} L^+_n(1_{[a]})(y)
\]
\[= e^{-C_1 - C_2 + \phi^-} \left( \sum_{n=0}^{\infty} G(1_{[a]})(y) - 1_{[a]}(y) \right)
\]
and \( \phi^- \) is indeed transient.

**Proof of Proposition 6.3** Recall that for a non-singular \( \nu \), there exists a Borel function \( \phi' \) s.t. \( \frac{\partial}{\partial \nu} \phi' = \exp \phi' \), where \( (\nu \circ T)(A) := \sum_{a \in S} \nu(T(A \cap [a])) \) and in particular \( L^+_\phi \nu = \nu \). By Proposition 6.1 \( \nu \) is a DLR state of \( \phi' \). Given two admissible words \( \underline{a} = (a_1, \ldots, a_n) \) and \( \underline{b} = (b_1, \ldots, b_n) \) of length \( n \) with \( a_n = b_n \), let \( \phi_{\underline{a}, \underline{b}} : [a] \rightarrow [b] \), \( \phi_{\underline{a}, \underline{b}}(x_{\infty}) = b_{\infty} \)
where \( x_{\infty} = ((x)_n, (x)_{n+1}, \ldots) = T^n x \). The following claim is elementary, see [20, 24].
Claim A.1. A non-singular probability measure \( \mu \) is a DLR measure for \( \phi \) iff for every two admissible words \( \underline{a} = (a_1, \ldots, a_n) \) and \( \underline{b} = (b_1, \ldots, b_n) \) of length \( n \),

\[
\frac{d\mu \circ \vartheta_{\underline{a}, \underline{b}}}{d\mu} = \exp \left( \sum_{k=1}^{n} \left( \phi(a_k, \ldots, a_n x_n) - \phi(b_k, \ldots, b_n x_n) \right) \right) \quad \text{for a.e. } x \in T^{-n}[a_n].
\]

(A.1)

Since \( \mu \) is a DLR state both for \( \phi \) and \( \phi' \), Eq. (A.1) implies that

\[
\sum_{k=0}^{n-1} (\phi \circ T^k(\underline{a} x_n) - \phi \circ T^k(\underline{b} x_n)) = \sum_{k=0}^{n-1} (\phi' \circ T^k(\underline{a} x_n) - \phi' \circ T^k(\underline{b} x_n)), \quad \nu-a.s.
\]

for every \( \underline{a}, \underline{b} \) of length \( n \) with \( a_n = b_n \) and \( \mu[\underline{a}] > 0 \).

Let \( h = \phi - \phi' \). We show by induction that for any such admissible \( \underline{a}, \underline{b} \) and for \( \nu \)-a.e. \( x \in X^+ \), \( h(\underline{a} x_n) = h(\underline{b} x_n) \). This will imply that \( h \) is measurable w.r.t. the \( \sigma \)-algebra \( \cap_{n=0}^{\infty} T^{-n} \mathcal{B} \). Clearly the statement is true for \( n = 0 \). For \( n > 0 \), let \( \underline{a}, \underline{b} \) be two admissible words of length \( n \) which ends with the same symbol. By the induction assumption, for every \( 1 < k \leq n - 1 \),

\[
h \circ T^k(\underline{a} x_n) = h \circ T^k(\underline{b} x_n), \quad \nu-a.s.
\]

Therefore, for \( \nu \)-a.e. \( x \in T^{-n}[a_n] \),

\[
h(\underline{a} x_n) - h(\underline{b} x_n) = \sum_{k=0}^{n-1} 
\left( h \circ T^k(\underline{a} x_n) - h \circ T^k(\underline{b} x_n) \right)
\]

\[
= \sum_{k=0}^{n-1} 
\left( \phi \circ T^k(\underline{a} x_n) - \phi \circ T^k(\underline{b} x_n) - \phi' \circ T^k(\underline{a} x_n) + \phi' \circ T^k(\underline{b} x_n) \right)
\]

\[
= 0.
\]

\[\square\]

Example A.1 (Non-conformal DLR measure). Consider \( S = \mathbb{Z} \) and the transition matrix

\[
A_{a,b} = \begin{cases} 
1 & b = a + 1 \text{ or } b = -a \leq 0 \\
0 & \text{otherwise}
\end{cases}
\]

See Figure 2. It is easy to verify that the graph is locally finite, irreducible and aperiodic. Equip \( X^+ \) with any continuous potential function with summable variations \( \phi : X^+ \to \mathbb{R} \). Let \( x_0 = (0, 1, 2, 3, 4, 5, \ldots) \) and \( \mu = \delta_{x_0} \). Clearly \( \mu \) is Radon. Since \( \mu \) is supported on a single point and \( \{ y \in X^+ : T^n y = T^n x_0 \} = \{ x_0 \} \), Eq. (6.1) holds trivially and \( \mu \) is a DLR measure. On the other hand, since \( \mu([0]) = 1 \) and \( \mu(T^{-1}[0]) = \mu([0, 0] \cup [-1, 0]) = 0 \), \( \mu \) is non-conformal.
Figure 2: The state graph in Example A.1.

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36
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