Asymptotic analysis for Hamilton–Jacobi equations with large drift term

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Abstract

We investigate the asymptotic behavior of solutions of Hamilton–Jacobi equations with large drift term in an open subset of two-dimensional Euclidean space. When the drift is given by $\varepsilon^{-1}(H_{x_2}, -H_{x_1})$ of a Hamiltonian $H$, with $\varepsilon > 0$, we establish the convergence, as $\varepsilon \to 0^+$, of solutions of the Hamilton–Jacobi equations and identify the limit of the solutions as the solution of systems of ordinary differential equations on a graph. This result generalizes the previous one obtained by the author to the case where the Hamiltonian $H$ admits a degenerate critical point and, as a consequence, the graph may have segments more than four at a node.

Key Words and Phrases. Singular perturbation, Hamilton–Jacobi equations, Large drift, Graphs.

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1 Introduction

We consider the boundary value problem for the Hamilton–Jacobi equation

\[
\begin{aligned}
\lambda u^\varepsilon - \frac{b \cdot Du^\varepsilon}{\varepsilon} + G(x, Du^\varepsilon) &= 0 \quad \text{in } \Omega, \\
u^\varepsilon &= g^\varepsilon \quad \text{on } \partial \Omega,
\end{aligned}
\]

and investigate the asymptotic behavior, as $\varepsilon \to 0^+$, of the solution $u^\varepsilon$ to \(\text{(HJ)}\).

In the above and henceforth, $\varepsilon$ is a small positive parameter, $\lambda$ is a positive constant, $\Omega$ is an open subset of $\mathbb{R}^2$ with boundary $\partial \Omega$, $G : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ and $g^\varepsilon : \partial \Omega \to \mathbb{R}$ are given functions, $b : \mathbb{R}^2 \to \mathbb{R}^2$ is a given vector field, $u^\varepsilon : \overline{\Omega} \to \mathbb{R}$ is the unknown, and $Du^\varepsilon$ and $p \cdot q$ denote, respectively, the gradient of $u^\varepsilon$ and the Euclidean inner product of $p, q \in \mathbb{R}^2$. We give the vector field $b$ as a Hamilton vector field, that is, for a given Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}^2$,

\[b = (H_{x_2}, -H_{x_1}),\]

where the subscript $x_i$ indicates the differentiation with respect to the variable $x_i$.

We are interested in the Hamiltonian flow with one degree of freedom

\[\dot{X}(t) = b(X(t)) \quad \text{and} \quad X(0) = x \in \mathbb{R}^2, \quad (\text{HS})\]

and with its perturbed system

\[\dot{X}^\varepsilon(t) = b(X^\varepsilon(t)) + \varepsilon \alpha(t) \quad \text{and} \quad X^\varepsilon(0) = x \in \mathbb{R}^2, \quad (1.1)\]
where $\alpha \in L^\infty(\mathbb{R}; \mathbb{R}^2)$. Rescaling the time from $t$ to $t/\varepsilon$ in (1.1), we obtain

$$
\dot{X}_\varepsilon(t) = \frac{1}{\varepsilon} b(X_\varepsilon(t)) + \alpha(t) \quad \text{and} \quad X_\varepsilon(0) = x \in \mathbb{R}^2. \quad (1.2)
$$

The first equation of (HJ$^\varepsilon$) is the dynamic programming equation for the optimal control problem. As is well-known, the viscosity solution $u^\varepsilon$ of (HJ$^\varepsilon$) is identified with the value function of the optimal control problem, where the state equation, the discount factor, the pay-off at the exit time, and the running cost are given, respectively, by (1.2), $\lambda$, $g^\varepsilon$, and the function $L$, defined by

$$
L(x, \xi) = \sup_{p \in \mathbb{R}^2} \{-\xi \cdot p - G(x, p)\} \quad \text{for} \ (x, \xi) \in \Omega \times \mathbb{R}^2.
$$

Thus, the investigation of the asymptotic behavior, as $\varepsilon \to 0^+$, of the solutions of (HJ$^\varepsilon$) may be regarded in a broad sense as analyzing the behavior, as $\varepsilon \to 0^+$, of the solutions of (1.2), with “optimal” $\alpha$.

In a spirit similar to the above, but with a stochastic perturbation in place of a “perturbation by optimal control”, Freidlin and Wentzell in [8] has initiated the study of a stochastic perturbation for (HS) and established a convergence theorem for the solutions of the linear second-order uniformly elliptic partial differential equation (pde, for short)

$$
-\frac{b \cdot Du^\varepsilon}{\varepsilon} - \Delta u^\varepsilon = f(x), \quad (1.3)
$$

with a continuous function $f$ on $\overline{\Omega}$. Here, a similarity of the elliptic pde above to the pde in (HJ$^\varepsilon$) is that $\lambda > 0$ and $G(x, Du^\varepsilon)$ in (HJ$^\varepsilon$) correspond, respectively, to $\lambda = 0$ and $-\Delta u^\varepsilon - f$ in (1.3). Regarding the stochastic perturbation, Ishii and Souganidis in [11] has established a convergence theorem similar to that in [8], by a pure pde-techniques, which covers a fairly general linear second-order degenerate elliptic pdes.

Motivated by the developments ([8][11]) in stochastic perturbations for (HS), the author in [12] has recently established a convergence result for Hamilton-Jacobi equations (HJ$^\varepsilon$) by using viscosity solution techniques such as the perturbed test function method and representations of solutions as value functions in optimal control. A typical Hamiltonian $H$ studied in [8][11][12] is given by

$$
H(x_1, x_2) = x_1^2 + \frac{1}{2}(x_2^2 - 1)^2 - \frac{1}{2},
$$

whose graph has the shape of the so-called double-well potential, and it has three non-degenerate critical points at $(0, -1)$, $(0, 0)$ and $(0, 1)$. In this case, the limiting functions in the convergence results either in stochastic perturbation or in perturbation by optimal control are characterized by systems of odes on graphs $\Gamma$ with one node and three edges, where, roughly speaking, one of the edges corresponds to one of the potential wells, another to the other potential well and the last to a finite tube above the potential wells.

Our main contribution in this article is to prove a convergence theorem when the Hamiltonian $H$ has degenerate critical points. The result is stated as Theorem 2.5 below, where the graph on which the limit function is defined has one node and arbitrarily many edges depending on the Hamiltonian.
A simple example of such Hamiltonians is given by
\[ H(x_1, x_2) = (x_1^2 + x_2^2)^2 - 3x_1^2x_2 + x_2^4. \]
We emphasize that most work on stochastic perturbation of Hamiltonian flows has studied the case where Hamiltonian \( H \) has only non-degenerate critical points.

Now, we mention that related problems have been considered in the context of Hamilton-Jacobi equations on networks or graphs. In particular, the convergence results of approximated solutions by fattening networks or graphs were established in \([2]\) for Hamilton-Jacobi equations in optimal control and in \([13]\) for non-convex Hamilton-Jacobi equations.

An interesting point of the result in \([12]\) is that we have to treat a non-coercive Hamiltonian (that is, \( -b(x) \cdot p/\varepsilon + G(x, p) \)) in \((HJ^{\varepsilon})\), while very few authors have studied Hamilton-Jacobi equation with non-coercive Hamiltonian on networks or graphs. This difficulty due to lack of coercivity is resolved by taking the advantage that the Hamiltonian \( -b(x) \cdot p/\varepsilon + G(x, p) \) in \((HJ^{\varepsilon})\) is coercive in the direction orthogonal to \( b(x) \). See \([12]\) for details.

The graphs considered in Hamilton-Jacobi equations on networks or graphs, in general, have many number of segments at a node. However, in perturbation analysis of Hamiltonian flows as discussed above, when Hamiltonian \( H \) has only non-degenerate critical points, the number of segments at a node of graph \( \Gamma \) is at most “four” (see, for example, \([7]\)) because, in this case, \( H \) can be represented only by
\[ H(x_1, x_2) = x_1^2 - x_2^2 \quad (1.4) \]
in a neighborhood of a saddle point, which is corresponding to a node on \( \Gamma \).

The argument in \([12]\) depends heavily on the formula (1.4), which allows us to use an explicit formula of the solution of \((HS)\) in a neighborhood of a saddle point. This is a main crucial point to establish our convergence theorem since it is impossible to find a convenient explicit formula of solutions of \((HS)\) for general Hamiltonian \( H \) with degenerate critical points. The idea to overcome the difficulty above is to use geometric integral formulas for some quantities of the flow \((HS)\) instead of solving \((HS)\) explicitly.

This paper is organized as follows. In the next section, we first present the assumptions on Hamiltonian \( H \), typical examples of \( H \), and the domain \( \Omega \). After these, we describe a basic existence and uniqueness proposition for \((HJ)\) as well as the assumptions on the function \( G \) throughout this paper, and we finally state the main result (see Theorem 2.5). In Section 3 divided into two parts, we study some properties of functions in the odes in the limiting problem and subsolutions to the odes. Section 4 is devoted to the proof of Theorem 2.5 along the argument in \([12]\). In Section 5, we present a sufficient condition, similar to that in \([12]\), on the boundary data for the odes on the graph for which (G5) and (G6) hold.

Finally, we give a few of our notations.

Notation

For \( r > 0 \), we denote by \( B_r \) the open disc centered at the origin with radius \( r \). For \( c, d \in \mathbb{R} \), we write \( c \wedge d = \min\{c, d\} \).
2 Preliminaries and Main result

2.1 The Hamiltonian

Let $N \geq 3$. We assume the following assumptions on the Hamiltonian $H$ throughout this paper.

(H1) $H \in C^2(\mathbb{R}^2)$ and $\lim_{|x| \to \infty} H(x) = \infty$.

(H2) $H$ has exactly $N$ critical points $z_i \in \mathbb{R}^2$, with $i \in \{0, \ldots, N-1\}$, and attains local minimum at $z_i$ and $i \in \{1, \ldots, N-1\}$.

Here and henceforth, we write $I_0 = \{0, \ldots, N-1\}$ and $I_1 = \{1, \ldots, N-1\}$.

For example, in the case where $N = 4$, the graph of the Hamiltonian $H$ satisfying (H1) and (H2) is shaped like Fig. 2 below. The number $N$ in (H2) coincides with number of segments at a node of a graph arising in the limiting process.

To simplify the notation, we assume without loss of generality that $z_0 = 0 := (0, 0)$ and $H(0) = 0$.

We remark that, under assumptions (H1) and (H2), in the case where $N \geq 4$, the origin is just a degenerate critical point of the Hamiltonian $H$, while, in the case where $N = 3$, it may be a non-degenerate one.

(H3) There exist constants $m \geq 0$ and $C > 0$, and a neighborhood $V \subset \mathbb{R}^2$ of the origin such that, for any $i, j \in \{1, 2\}$,

$$|H_{x_i x_j}(x)| \leq C|x|^m \quad \text{for all } x \in V.$$

We note that assumption (H3) implies, by replacing $C > 0$ by a larger number if necessary, that

$$|DH(x)| \leq C|x|^{m+1} \quad \text{for all } x \in V, \quad (2.1)$$

and

$$|H(x)| \leq C|x|^{m+2} \quad \text{for all } x \in V. \quad (2.2)$$

(H4) There exist constants $n, c > 0$ such that $n < m + 2$ and

$$c|x|^n \leq |DH(x)| \quad \text{for all } x \in V.$$

Combining (H4) with (2.1), we see that $m + 1 \leq n$, and with (2.2), we get the relation

$$c_0|H(x)|^{\frac{n}{m+2}} \leq |DH(x)| \quad \text{for all } x \in V \quad (2.3)$$

for some $c_0 > 0$.

The following examples show that conditions (H3) and (H4) are satisfied for wide range of Hamiltonians $H$. 

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Example 2.1. Consider two Hamiltonians
\[ H_3(x_1, x_2) = (x_1^2 + x_2^2)^2 - x_1^2 + x_2^2 \quad \text{and} \quad H_3^0(x_1, x_2) = (x_1^2 + x_2^2)^3 - x_1^4 + x_2^4. \]
It is obvious that \( H_3 \) and \( H_3^0 \) satisfy (H1). By simple computations, we see that \( H_3 \) and \( H_3^0 \) satisfy (H3) and (H4) with, respectively, \((m, n) = (0, 1)\) and \((m, n) = (2, 3)\). Both of the number of critical points are three, which consists of the origin; a saddle point, and, respectively, \( z_{1,2} = (\pm \sqrt{2}/2, 0) \) and \( z_{1,2} = (\pm \sqrt{6}/3, 0) \); local minimum points. That is, (H2) holds with \( N = 3 \). The origin is a degenerate critical point of \( H_3^0 \), while it is a non-degenerate one of \( H_3 \). The graphs of \( H_3 \) and \( H_3^0 \) are shaped like Fig. 1.

Example 2.2. Next, consider the Hamiltonian
\[ H_4(x_1, x_2) = (x_1^2 + x_2^2)^2 - 3x_1^3x_2 + x_2^3. \]
It is easy to check that \( H_4 \) satisfies (H1)–(H4) with \((m, n) = (1, 2)\) and \( N = 4 \). The critical points of \( H_4 \) are the origin; a degenerate saddle point, and, \( z_1 = (3\sqrt{3}/8, 3/8) \), \( z_2 = (-3\sqrt{3}/8, 3/8) \), and \( z_3 = (0, -3/4) \); local minimum points. The graph of \( H_4 \) is shaped like Fig. 2. To understand the shape of \( H_4 \) well, we remark that \( H_4 \) can be represented in polar coordinates by
\[ \tilde{H}_4(r, \theta) = r^4 - r^3 \sin 3\theta, \]
that is, the zero-level set of \( H_4 \) is the curve expressed by \( r = \sin 3\theta \). Indeed, if \( x_1 + ix_2 := re^{i\theta} \), where \( i \) denotes the imaginary unit, then
\[ r^3 \sin 3\theta = r^3 \text{Im} e^{3i\theta} = \text{Im} r^3 e^{3i\theta} = \text{Im} (x_1 + ix_2)^3 = 3x_1^2x_2 - x_2^3. \]

Example 2.3. More generally, the Hamiltonian
\[ H_N(x_1, x_2) = (x_1^2 + x_2^2)^{[N/2]} + \sum_{k=1}^{[N/2]} (-1)^k \binom{N-1}{k} x_1^{N-2k}x_2^{2k-1} \]
satisfies (H1)–(H4) with \((m, n) = (N - 3, N - 2)\) provided \( N \geq 4 \). Here \([y]\) denotes the largest integer less than or equal to \( y \in \mathbb{R} \). Similarly to \( H_4 \) in Example 2.2, we see that the
zero-level set of $H_N$ is the curve expressed by $r = \sin(N - 1)\theta$ through the representation in polar coordinates by

$$\tilde{H}_N(r, \theta) = r^N - r^{N-1}\sin(N - 1)\theta,$$

where

$$r^{N-1}\sin(N - 1)\theta = \sum_{k=1}^{\lfloor N/2 \rfloor} (-1)^k \binom{N - 1}{k} x_1^{N-2k} x_2^{2k-1}.$$ 

The critical points of $\tilde{H}_N$ are the origin and

$$(r, \theta_i) = \left(\frac{N - 1}{N}, \sin \frac{4i - 3}{2}\pi\right) \quad \text{for } i \in \mathcal{I}_1,$$

which are, respectively, corresponding to a saddle point and local minimum points of $H_N$.

### 2.2 The domain

Under assumptions (H1) and (H2), for any $h > 0$, the open set $\{x \in \mathbb{R}^2 \mid H(x) < h\}$ is connected, and the open set $\{x \in \mathbb{R}^2 \mid H(x) < 0\}$ consists of $N - 1$ connected components $D_1, \ldots, D_{N-1}$ such that $z_i \in D_i$.

We choose the real numbers

$$h_0 > 0 \quad \text{and} \quad H(z_i) < h_i < 0 \quad \text{for } i \in \mathcal{I}_1,$$

and set the intervals

$$J_0 = (0, h_0) \quad \text{and} \quad J_i = (h_i, 0) \quad \text{for } i \in \mathcal{I}_1.$$

We put the open sets

$$\Omega_0 = \{x \in \mathbb{R}^2 \mid H(x) \in J_0\} \quad \text{and} \quad \Omega_i = \{x \in D_i \mid H(x) \in J_i\} \quad \text{for } i \in \mathcal{I}_1,$$

and their “outer” boundaries

$$\partial_i \Omega = \{x \in \overline{\Omega_i} \mid H(x) = h_i\} \quad \text{for } i \in \mathcal{I}_0.$$

Finally, we introduce $\Omega$ as the open connected set

$$\Omega = \left( \bigcup_{i=0}^{N-1} \Omega_i \right) \cup \{x \in \mathbb{R}^2 \mid H(x) = 0\},$$

with the boundary

$$\partial \Omega = \bigcup_{i=0}^{N-1} \partial_i \Omega.$$

For example, the shapes of $\Omega$ corresponding to $H$ in Figs. 1 and 2 are, respectively, depicted in Figs. 3 and 4.
By (H1), the initial value problem $\text{(HS)}$ admits a unique global in time solution $X(t, x)$ such that

$$X, \dot{X} \in C^1(\mathbb{R} \times \mathbb{R}^2; \mathbb{R}^2).$$

As is well known, $H$ is a first integral for the system $\text{(HS)}$, that is,

$$H(X(t, x)) = H(x) \quad \text{for all } (t, x) \in \mathbb{R} \times \mathbb{R}^2.$$

For $h \in \bar{J}_i$ and $i \in I_0$, we define the loops $c_i(h)$ by

$$c_i(h) = \{ x \in \bar{\Omega}_i \mid H(x) = h \}.$$

If we identify all points belonging to a loop $c_i(h)$, we obtain a graph $\Gamma$ consisting of $N$ segments parametrized by $J_0, \ldots, J_{N-1}$. For example, the graph $\Gamma$ corresponding to $\Omega$ in Fig. 4 is shown in Fig. 5.

It is not hard to check the following facts: if $h \in J_i \cup \{h_i\}$ and $i \in I_0$, then, for any $\bar{x}_i \in c_i(h)$, the map $t \mapsto X(t, \bar{x}_i)$ is periodic and

$$c_i(h) = \{ X(t, \bar{x}_i) \mid t \in \mathbb{R} \}.$$

If $i \in I_1$, then, for any $\bar{x}_i \in c_i(0) \setminus \{0\}$,

$$c_i(0) = \{0\} \cup \{ X(t, \bar{x}_i) \mid t \in \mathbb{R} \} \quad \text{and} \quad \lim_{t \to \pm \infty} X(t, \bar{x}_i) = 0,$$

and

$$c_i(0) = \partial D_i.$$

Moreover,

$$c_0(0) = \{ x \in \mathbb{R}^2 \mid H(x) = 0 \} = \bigcup_{i \in I_1} c_i(0)$$

and

$$c_0(0) = \partial D_0.$$
2.3 The Hamilton-Jacobi equation

We put the following assumptions (G1)–(G5) on $G$ and $g^\varepsilon$ throughout this paper.

(G1) $G \in C(\Omega \times \mathbb{R}^2)$.

(G2) There exists a modulus $m$ such that
\[ |G(x, p) - G(y, p)| \leq m(|x - y|(1 + |p|)) \quad \text{for all } x, y \in \overline{\Omega} \text{ and } p \in \mathbb{R}^2. \]

(G3) For each $x \in \overline{\Omega}$, the function $p \mapsto G(x, p)$ is convex on $\mathbb{R}^2$.

(G4) $G$ is coercive, that is,
\[ G(x, p) \to \infty \quad \text{uniformly for } x \in \overline{\Omega} \text{ as } |p| \to \infty. \]

Assumption (G2) is a standard requirement to $G$ that the comparison principle should hold for $[HJ]$. Under assumptions (G1), (G3), and (G4), there exist $\nu, M > 0$ such that
\[ G(x, p) \geq \nu |p| - M \quad \text{for all } (x, p) \in \overline{\Omega} \times \mathbb{R}^2. \quad (2.4) \]

The following condition has the same role as compatibility conditions described in [13], which are used to ensure the continuity up to boundary of the value functions in optimal control. That is, it guarantees the continuity up to the boundary of the function $u^\varepsilon$ of the form (2.5) below and, hence, gives us the uniqueness of viscosity solutions of $[HJ]$.

In what follows, we write $X^\varepsilon(t, x, \alpha)$ for the solution to (1.2).

(G5) There exists $\varepsilon_0 \in (0, 1)$ such that the family $\{g^\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)} \subset C(\partial \Omega)$ is uniformly bounded on $\partial \Omega$ and that, for any $\varepsilon \in (0, \varepsilon_0)$,
\[ g^\varepsilon(x) \leq \int_0^\vartheta L(X^\varepsilon(t, x, \alpha), \alpha(t))e^{-\lambda t} \, dt + g^\varepsilon(y)e^{-\lambda \vartheta} \]
\[ H = 0 \quad \text{Figure 4. } \Omega \ (N = 3) \]
\[ H = 0 \quad \text{Figure 5. } \Omega \ (N = 4) \]
for all \( x, y \in \partial \Omega, \vartheta \in [0, \infty) \), and \( \alpha \in L^\infty(\mathbb{R}; \mathbb{R}^2) \), where the conditions
\[
X^\varepsilon(\vartheta, x, \alpha) = y \quad \text{and} \quad X^\varepsilon(t, x, \alpha) \in \overline{\Omega} \quad \text{for all } t \in [0, \vartheta]
\]
are satisfied, that is, \( \vartheta \) is a visiting time at \( y \) of the trajectory \( \{X^\varepsilon(t, x, \alpha)\}_{t \geq 0} \) constrained in \( \overline{\Omega} \).

We state here a basic existence and uniqueness proposition for \((HJ_\varepsilon)\).

**Proposition 2.4.** For \( \varepsilon \in (0, \varepsilon_0) \), we define the function \( u^\varepsilon : \overline{\Omega} \to \mathbb{R} \) by
\[
u^\varepsilon(x) = \inf \left\{ \int_0^{\tau^\varepsilon} L(X^\varepsilon(t, x, \alpha), \alpha(t))e^{-\lambda t} \, dt + g^\varepsilon(X^\varepsilon(\tau^\varepsilon, x, \alpha))e^{-\lambda \tau^\varepsilon} \mid \alpha \in L^\infty(\mathbb{R}; \mathbb{R}^2) \right\},
\]
where \( \tau^\varepsilon \) is a visiting time in \( \partial \Omega \) of \( \{X^\varepsilon(t, x, \alpha)\}_{t \geq 0} \) constrained in \( \overline{\Omega} \), that is, \( \tau^\varepsilon \) is a nonnegative number such that
\[
X^\varepsilon(\tau^\varepsilon, x, \alpha) \in \partial \Omega \quad \text{and} \quad X^\varepsilon(t, x, \alpha) \in \overline{\Omega} \quad \text{for all } t \in [0, \tau^\varepsilon].
\]

Then \( u^\varepsilon \) is the unique viscosity solution of \((HJ_\varepsilon)\) and continuous on \( \overline{\Omega} \), and satisfies \( u^\varepsilon = g^\varepsilon \) on \( \partial \Omega \). Furthermore the family \( \{u^\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)} \) is uniformly bounded on \( \overline{\Omega} \).

Noting that \((2.1)\) implies, in particular, that \(|b(x)| \leq |x|\) for all \( x \in V \), we can prove this proposition along the same lines as the proof of [12, Proposition 2.3], so we skip it here.

Thanks to this proposition, we may define hereafter \( u^\varepsilon \) by \((2.5)\). Since the family \( \{u^\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)} \) is uniformly bounded on \( \overline{\Omega} \), the half relaxed-limits, as \( \varepsilon \to 0^+ \), of \( u^\varepsilon \)
\[
v^+(x) = \lim_{r \to 0^+} \sup \{u^\varepsilon(y) \mid y \in B_r(x) \cap \overline{\Omega}, \varepsilon \in (0, r)\},
v^-(x) = \lim_{r \to 0^+} \inf \{u^\varepsilon(y) \mid y \in B_r(x) \cap \overline{\Omega}, \varepsilon \in (0, r)\}
\]
are well-defined and bounded on \( \overline{\Omega} \).

If \( v^+(x) \neq v^-(x) \) for some \( x \in \partial \Omega \), a boundary layer happens in the limiting process of sending \( \varepsilon \to 0^+ \). In order that any boundary layer does not occur, in addition to \((G1)\)–\((G5)\), we henceforth assume the following.

\((G6)\) There exist constants \( d_i \) such that \( v^\pm(x) = d_i \) for all \( x \in \partial_i \Omega \) and \( i \in I_0 \).

It is obvious that this leads to
\[
\lim_{\Omega_{2\lambda y \to x}} v^\pm(y) = \lim_{\varepsilon \to 0^+} g^\varepsilon(x) = d_i \quad \text{uniformly for } x \in \partial_i \Omega \text{ for all } i \in I_0.
\]

Our asymptotic analysis of \((HJ_\varepsilon)\) is based on rather implicit (or ad hoc) assumptions \((G5)\) and \((G6)\), which are indeed convenient for our arguments below. However, it is not clear which \( g^\varepsilon \) and \( d_i \) satisfy \((G5)\) and \((G6)\). Thus, it is important to know when \((G5)\) and \((G6)\) hold. In [12], for \( N = 3 \), the author gave a fairly general sufficient condition on the data \( d_i \), for which \((G5)\) and \((G6)\) hold. In Section 5, for more general \( N \), we will present a similar condition to that in [12].
2.4 Main result

We introduce some notation which are needed to state our main result.

For $i \in I_0$ and $h \in \tilde{J}_i$, let $L_i(h)$ denote the length of $c_i(h)$, that is,

$$L_i(h) = \int_{c_i(h)} dl. \tag{2.6}$$

Here $dl$ denotes the line element. Obviously, $L_i(h)$ are positive and bounded.

Recall that, if $h \in J_i \cup \{h_i\}$ and $i \in I_1$, then the map $t \mapsto X(t, \tilde{x}_i)$ is periodic for any $\tilde{x}_i \in c_i(h)$. Note that the minimal periods are independent of choice of $\tilde{x}_i \in c_i(h)$. Hence, we can write $T_i(h)$ for the minimal period of the trajectory of the system $\text{[HS]}$ on $c_i(h)$.

Noting that $|b(x)| = |DH(x)|$, the minimal period $T_i(h)$ has the form

$$T_i(h) = \int_{c_i(h)} \frac{1}{|DH|} dl, \tag{2.7}$$

which shows, in view of (H2), that

$$0 < T_i(h) < \infty \quad \text{and} \quad \lim_{J_i \ni r \to 0} T_i(r) = \infty.$$

For $i \in I_0$, define the function $\overline{G}_i : J_i \cup \{h_i\} \times \mathbb{R} \to \mathbb{R}$ by

$$\overline{G}_i(h,q) = \frac{1}{T_i(h)} \int_{c_i(h)} G(x,qDH) \frac{1}{|DH|} dl = \frac{1}{T_i(h)} \int_0^{T_i(h)} G(X(t,x),qDH(X(t,x))) dt,$$

where $x \in c_i(h)$ is fixed arbitrarily. We note here that the second formula above reveals that $\overline{G}_i(h,q)$ is the mean value of the function $G(\cdot, qDH(\cdot))$ along the curve $X(t,x)$ on the loop $c_i(h)$.

We then state our main result.

**Theorem 2.5.** There exist viscosity solutions $u_i \in C(\tilde{J}_i)$, with $i \in I_0$, of

$$\lambda u + \overline{G}_i(h,u') = 0 \quad \text{in} \ J_i, \tag{HJ_i}$$

such that $u_i(h_i) = d_i$, $u_1(0) = \ldots = u_{N-1}(0)$, and, as $\varepsilon \to 0+$,

$$u^\varepsilon \to u_i \circ H \quad \text{uniformly on} \ \overline{\Omega}_i.$$

That is, if we define $\bar{u} \in C(\overline{\Omega})$ by

$$\bar{u}(x) = u_i \circ H(x) \quad \text{if} \ x \in \overline{\Omega}_i,$$

then, as $\varepsilon \to 0+$,

$$u^\varepsilon \to \bar{u} \quad \text{uniformly on} \ \overline{\Omega}.$$

We will give the proof of this theorem in Section 4.
3 The limiting problem

3.1 The minimal periods and the length

In this subsection, we show an integrability of $T_i(h)$ and behavior of $c_i(h)$ near the origin without any explicit formula of $X(t, x)$, which is a crucial difference from [12].

For this, we need the following lemma, which we refer to [8, Lemma 1.1, Section 8].

Lemma 3.1. Let $i \in I_0$ and $r_1, r_2 \in J_i \cup \{h_i\}$ be such that $r_1 < r_2$. Set $D(r_1, r_2) = \{ x \in \Omega_i \mid r_1 < H(x) < r_2 \}$. If $f \in C^1(D(r_1, r_2))$, then

$$\int_{c_i(r_2)} f|DH| dl - \int_{c_i(r_1)} f|DH| dl = \int_{D(r_1, r_2)} (D f \cdot DH + f \Delta H) dx,$$

and, moreover, for any $h \in J_i \cup \{h_i\}$,

$$\frac{d}{dh} \int_{c_i(h)} f|DH| dl = \int_{c_i(h)} \left( \frac{D f \cdot DH}{|DH|} + f \frac{\Delta H}{|DH|} \right) dl.$$

Lemma 3.2. For all $i \in I_0$, $L_i, T_i \in C^1(J_i \cup \{h_i\})$.

Proof. Together with (2.6) and (2.7), Lemma 3.1 with $f = 1/|DH|$ and with $f = 1/|DH|^2$, yields respectively

$$\frac{d}{dh} L_i(h) = \int_{c_i(h)} \left( -\frac{\operatorname{tr} DH \otimes DHD^2 H}{|DH|^4} + \frac{\Delta H}{|DH|^2} \right) dl,$$

and

$$\frac{d}{dh} T_i(h) = \int_{c_i(h)} \left( -\frac{2 \operatorname{tr} DH \otimes DHD^2 H}{|DH|^5} + \frac{\Delta H}{|DH|^3} \right) dl,$$

where $D^2 H$ and $p \otimes q$ denote the Hessian matrix of $H$ and the matrix $(p_i q_j)_{1 \leq i, j \leq 2}$ for $p, q \in \mathbb{R}^2$, respectively, which imply that $L_i, T_i \in C^1(J_i \cup \{h_i\})$.

The behavior of $T_i(h)$ near $h = 0$ is the subject of

Lemma 3.3. For all $i \in I_0$, $T_i(h) = O(|h|^{-\frac{m}{m+2}})$ as $J_i \ni h \to 0$.

Here, $m$ and $n$ are the constants from, respectively, (H3) and (H4). We henceforth write $L(c)$ for the length of a given curve $c$.

Proof. Fix any $i \in I_0$ and $h \in J_i$. Choose $\kappa > 0$ so that $\overline{B}_\kappa \subset \Omega \cap V$. By replacing $h$ if necessary so that $|h|$ is small enough, we may assume that $c_i(h) \cap B_\kappa \neq \emptyset$.

Set $\mu = \min_{c_i(h) \cap B_\kappa} |DH| > 0$. By (2.3), we compute that

$$T_i(h) = \int_{c_i(h) \cap \overline{B}_\kappa} \frac{1}{|DH|} dl + \int_{c_i(h) \setminus \overline{B}_\kappa} \frac{1}{|DH|} dl$$

$$\leq \int_{c_i(h) \cap \overline{B}_\kappa} \frac{1}{c_0 |H|^{\frac{m}{m+2}}} dl + \mu^{-1} L(c_i(h) \setminus \overline{B}_\kappa) \leq c_0^{-1} L_i(h)|h|^{-\frac{m}{m+2}} + \mu^{-1} L_i(h),$$

from which we conclude that $T_i(h) = O(|h|^{-\frac{m}{m+2}})$ as $J_i \ni h \to 0$.\qed
We remark that since \( n < m + 2 \), this lemma assures that
\[
T_i \in L^1(J_i) \quad \text{for all } i \in I_0.
\] (3.1)

This integrability ensures the continuity of solutions of \((HJ_i)\) up to \( h = 0 \) (See Lemma 3.7).

**Lemma 3.4.** Let \( \kappa \in (0, 1) \) be a constant such that \( \overline{B}_\kappa \subset \Omega \cap V \). Then, there exists a constant \( K > 0 \), independent of \( h \), such that, for any \( r \in (0, \kappa) \),
\[
L(c_i(h) \cap B_r) \leq Kr^{m-n+2} \quad \text{for all } h \in J_i \text{ and } i \in I_0.
\]

**Proof.** Fix any \( r \in (0, \kappa) \), \( i \in I_0 \), and \( h \in J_i \). If \( c_i(h) \cap B_r = \emptyset \), then \( L(c_i(h) \cap B_r) = 0 \). So, we assume henceforth that \( c_i(h) \cap B_r \neq \emptyset \) and, for the time being, that \( i = 1 \).

Set \( W = \{ x \in \Omega_1 \mid H(x) < h \} \) and \( U = W \cap B_r \). Let \( g = (g_1, g_2) \in C^1(U) \). Green’s theorem yields
\[
\int_{\partial U} g \cdot \nu dl = \int_U \text{div } g dx,
\]
where \( \nu \) is the outer normal vector on \( \partial U \). Note that \( \partial U = (c_1(h) \cap B_r) \cup (W \cap \partial B_r) \) and that \( \nu = DH/|DH| \) on \( c_1(h) \cap B_r \) and \( \nu = x/r \) on \( W \cap \partial B_r \).

Now, setting \( g = DH/|DH| \), we have
\[
\int_{c_1(h) \cap B_r} dl + \int_{W \cap \partial B_r} \frac{DH \cdot x}{r |DH|} dl = \int_U \text{div } \left( \frac{DH}{|DH|} \right) dx. \tag{3.2}
\]

Here, we have
\[
\left| \int_{W \cap \partial B_r} \frac{DH \cdot x}{r |DH|} dl \right| \leq \int_{W \cap \partial B_r} \left| \frac{DH}{|DH|} \right| |x| dl = \int_{W \cap \partial B_r} dl \leq 2\pi r. \tag{3.3}
\]

On the other hand, we compute that
\[
\text{div } \left( \frac{DH}{|DH|} \right) = \frac{1}{|DH|} \text{tr}[(I - \overline{DH} \otimes \overline{DH}) D^2H],
\]
where \( I \) denotes the identity matrix of size two and \( \overline{p} \) is defined by
\[
\overline{p} = \frac{p}{|p|} \quad \text{for } p \in \mathbb{R}^2 \setminus \{0\}.
\]

Since \( |\overline{DH}| = 1 \), by \((H3)\), we have
\[
| \text{tr}[(I - \overline{DH} \otimes \overline{DH}) D^2H] | \leq 2 \sum_{i,j=1}^2 |H_{x_i x_j}| \leq 8C |x|^m,
\]
and, hence, by \((H4)\),
\[
\left| \text{div } \left( \frac{DH}{|DH|} \right) \right| \leq \frac{8C |x|^m}{|DH|} \leq \frac{8C}{c} |x|^{m-n}.
\]

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Since \( U \subset B_r \), from the inequality above, we get
\[
\left| \int_U \operatorname{div} \left( \frac{DH}{|DH|} \right) \, dx \right| \leq \int_{B_r} \left| \operatorname{div} \left( \frac{DH}{|DH|} \right) \right| \, dx = \frac{16C\pi}{c(m-n+2)} r^{m-n+2}. \tag{3.4}
\]

Now, (3.2) – (3.4) together yield
\[
L(c_1(h) \cap B_r) = \int_{c_1(h) \cap B_r} \, dl \leq 2\pi \left( 1 + \frac{8C}{c(m-n+2)} \right) r^{m-n+2}.
\]

Here, we have used the fact that \( 0 < m-n+2 \leq 1 \). Arguments for the other \( i \) parallel the above. The proof is complete. \( \square \)

### 3.2 The Hamiltonians \( \overline{G}_i \) in \( [HJ_i] \)

We state here some properties of the functions \( \overline{G}_i \).

**Lemma 3.5.** For any \( i \in \mathcal{I}_0 \), \( \overline{G}_i \in C(J_i \cup \{ h_i \} \times \mathbb{R}) \), and
\[
\overline{G}_i(h, q) \geq \frac{\nu L_i(h)}{T_i(h)} |q| - M \quad \text{for all } (h, q) \in J_i \cup \{ h_i \} \times \mathbb{R}, \tag{3.5}
\]

that is, \( \overline{G}_i \) is locally coercive in the sense that, for any compact interval \( I \) of \( J_i \cup \{ h_i \} \),
\[
\lim_{r \to \infty} \inf \{ \overline{G}_i(h, q) \mid h \in I, \ |q| \geq r \} = \infty.
\]

Here, \( \nu \) and \( M \) are the constants from (2.4).

**Proof.** Combining the definition of \( \overline{G}_i \) with Lemma 3.2 yields the continuity of \( \overline{G}_i \), and with (2.4) yields (3.5). \( \square \)

**Lemma 3.6.** We have
\[
\lim_{J_i \ni h \to 0} \min_{q \in \mathbb{R}} \overline{G}_i(h, q) = G(0,0) \quad \text{for all } i \in \mathcal{I}_0.
\]

**Proof.** Fix any \( \gamma > 0 \). We begin by noting that, due to the continuity and the coercivity of \( G \), there exist \( R, C_1 > 0 \) such that \( G(x, p) \geq G(0,0) \) for all \( (x, p) \in \overline{\Omega} \times (\mathbb{R}^n \setminus B_R) \) and
\[
|G(x, p) - G(0,0)| \leq \gamma + C_1(|x|^n + |p|^\frac{m-n+2}{n} |DH(x)|^{1-\frac{m-n+2}{n}}) \tag{3.6}
\]
for all \( (x, p) \in \overline{\Omega} \times B_R \). Combining this with (H4) yields
\[
G(x, p) \geq G(0,0) - \gamma - C_2(|DH(x)| + |p|^\frac{m-n+2}{n} |DH(x)|^{1-\frac{m-n+2}{n}}) \tag{3.7}
\]
for all \( (x, p) \in \overline{\Omega} \times \mathbb{R}^2 \), and
\[
|G(x, 0) - G(0,0)| \leq \gamma + C_2 |DH(x)| \quad \text{for all } x \in \overline{\Omega} \tag{3.8}
\]
for some \( C_2 > 0 \).
Fix any \( i \in I_0 \) and \( h \in J_i \). Using (3.8), we get
\[
|G_i(h, 0) - G(0, 0)| \leq \gamma + \frac{C_2 L_i(h)}{T_i(h)}.
\] (3.9)

Fix any \( q \in \mathbb{R} \) and set
\[
S = \{ x \in c_i(h) \mid |x| \leq (R/c|q|)^{\frac{1}{n}} \},
\]
where \( c > 0 \) is the constant from (H4). If \( x \in c_i(h) \setminus S \), then, by (H4),
\[
\frac{R}{c|q|} < |x|^n \leq \frac{|DH(x)|}{c},
\]
that is, \(|q||DH(x)| > R\). Hence, we have
\[
G(x, qDH(x)) \geq G(0, 0) \quad \text{for all} \quad x \in c_i(h) \setminus S.
\] (3.10)

Choose \( \kappa \in (0, 1) \) and \( \delta > 0 \) so that \( \overline{B}_\kappa \subset \Omega \cap V \) and that if \( |h| \leq \delta \), then
\[
\left( \frac{R}{c\gamma T_i(h)} \right)^{\frac{1}{n}} < \kappa.
\]
If \( |h| < \delta \) and \( |q| > \gamma T_i(h) \), then \((R/c|q|)^{\frac{1}{n}} < \kappa\), and, hence, by Lemma 3.4, there exists a constant \( K > 0 \), independent of \( h \), such that
\[
L \left( c_i(h) \cup B_{(R/c|q|)^{\frac{1}{n}}} \right) \leq K \left( \frac{R}{c|q|} \right)^{\frac{m-n+2}{n}}.
\] (3.11)

Using (3.7) if \( |q| \leq \gamma T_i(h) \) and, otherwise, combining (3.7) with (3.10) and (3.11), we can compute, as the proof of [12, Lemma 6.5], that
\[
\liminf_{J_i \ni h \to 0} \inf_{q \in \mathbb{R}} G_i(h, q) \geq G(0, 0),
\]
while (3.9) yields
\[
\lim_{J_i \ni h \to 0} G_i(h, 0) = G(0, 0).
\] (3.12)
These two together complete the proof.

3.3 Properties of viscosity subsolutions of (HJ)

For \( i \in I_0 \), let \( S_i^- \) (resp., \( S_i^+ \) or \( S_i \)) be the set of all viscosity subsolutions (resp., viscosity supersolutions or viscosity solutions) of (HJ).

Lemma 3.7. Let \( i \in I_0 \) and \( u \in S_i^- \). Then \( u \) is uniformly continuous in \( J_i \), and, hence, it can be extended uniquely to \( \bar{J}_i \) as a continuous function on \( \bar{J}_i \).
Proof. The proof is along the same lines as that of [12, Lemma 3.2]. By (3.5), we have
\[ \lambda u + \frac{\nu L_i}{T_i}|u'| - M \leq 0 \quad \text{in} \ J_i \] (3.13)
in the viscosity sense and, hence, in the almost everywhere sense. Gronwall’s inequality yields, for any \( h, a \in J_i \),
\[ |\lambda u(h) - M| \leq |\lambda u(a) - M| \exp \int_{J_i} \frac{\lambda T_i(s) \nu L_i(s)}{\nu L_i(h)} ds. \] (3.14)
Recalling that \( T_i \in L^1(J_i) \), inequality (3.14) shows a boundedness of \( u \) in \( J_i \) and, moreover,
\[ |u'(h)| \leq \frac{T_i(h)}{\nu L_i(h)} \left( M + \lambda \sup_{J_i} |u| \right) \text{ for a.e. } h \in J_i \] (3.15)
yields the uniformly continuity of \( u \) in \( J_i \).

Thanks to this lemma, we may assume that any \( u \in S_i^- \) is a function in \( C(\bar{J}_i) \). To make this notationally explicit, we write \( S_i^- \cap C(\bar{J}_i) \) for \( S_i^- \). This also applies to \( S_i \) since \( S_i \subset S_i^- \).

The following two lemmas are direct consequences of, respectively, (3.15) and (3.14).

Lemma 3.8. Let \( i \in \mathcal{I}_0 \) and \( S \subset S_i^- \). Assume that \( S \) is uniformly bounded on \( \bar{J}_i \). Then \( S \) is equi-continuous on \( \bar{J}_i \).

Lemma 3.9. Let \( i \in \mathcal{I}_0 \) and \( u \in S_i^- \cap C(\bar{J}_i) \). Then there exists a constant \( K > 0 \), independent of \( u \), such that
\[ |u(h)| \leq K(|u(a)| + 1) \quad \text{for all } h, a \in J_i. \]

Lemma 3.10. Let \( i \in \mathcal{I}_0 \) and \( u \in S_i^- \cap C(\bar{J}_i) \). Then, we have \( \lambda u(0) + G(0, 0) \leq 0 \).

Proof. Fix \( i \in \mathcal{I}_0 \). Since \( u \in S_i^- \cap C(\bar{J}_i) \), we have
\[ \lambda u(h) + \min_{q \in \mathbb{R}} \mathcal{C}_i(h, q) \leq 0 \quad \text{for all } h \in J_i. \]
Using Lemma 3.6 we conclude that \( \lambda u(0) + G(0, 0) \leq 0. \)

Lemma 3.11. Let \( i \in \mathcal{I}_0 \) and \( u \in S_i^- \cap C(\bar{J}_i) \). Set \( d \in (-\infty, u(0)) \) and
\[ \nu^d_i(h) = \sup \{ v(h) \mid v \in S_i^- \cap C(\bar{J}_i), \ v(0) = d \} \quad \text{for } h \in \bar{J}_i. \]
Then there exists \( \delta > 0 \) such that
\[ \nu^d_i(h) > d \quad \text{for all } h \in J_i \cap [-\delta, \delta]. \]

The proof of this lemma is along the same lines as that of [12, Lemma 6.8] with help of formula (3.12) and Lemma 3.10, so we omit giving it here.

An important remark on \( S_i^- \) is that if \( u \in S_i^- \), then \( u - a \in S_i^- \) for any constant \( a > 0 \). From this remark, the sets of all \( v \in S_i^- \) satisfying \( v(0) < u(0) \) are non-empty, and, hence, by Lemmas 3.8 and 3.9 these are uniformly bounded and equi-continuous on \( \bar{J}_i \). The functions \( \nu^d_i \) are thus well-defined as continuous functions on \( \bar{J}_i \) and, according to Perron’s method, these are solutions of (HJ).
4 Proof of Theorem 2.5

Note that the stability of viscosity solutions yields
\[-b \cdot Dv^+ \leq 0 \quad \text{and} \quad -b \cdot Dv^- \geq 0 \quad \text{in } \Omega\]
in the viscosity sense, which show that \(v^+\) and \(v^-\) are, respectively, nondecreasing and nonincreasing along the trajectory \(\{X(t, x)\}_{t \in \mathbb{R}}\), and, hence, they are constant on the loop \(c_i(h), h \neq 0\).

The relations
\[u_i^+(h) \in v^+(c_i(h))\]
define functions \(u_i^\pm\) in \(J_i\). It is easy to check that \(u_i^+\) and \(u_i^-\) are, respectively, upper and lower semicontinuous in \(J_i\).

Theorem 4.1. For all \(i \in \mathcal{I}_0\), \(u_i^+ \in \mathcal{S}_i^-\) and \(u_i^- \in \mathcal{S}_i^+\).

We can prove this theorem by using the same perturbed test function as that of [12, Theorem 3.6], so we skip the proof.

Thanks to this theorem and Lemma 3.7, we may assume that \(u_i^+ \in C(\bar{J}_i)\) for all \(i \in \mathcal{I}_0\). Moreover, by (G6), we have
\[u_i^+(h_i) = \lim_{J_i \ni h \to h_i} u_i^-(h) = d_i \quad \text{for all } i \in \mathcal{I}_0. \tag{4.1}\]

In view of Theorem 4.1 and (4.1), to prove Theorem 2.5, it is enough to show that
\[v^+(x) \leq d \leq v^-(x) \quad \text{for all } x \in c_0(0) \tag{4.2}\]
for some \(d \in \mathbb{R}\). Indeed, if (4.2) is satisfied, by the semicontinuity of \(v^\pm\), we have
\[u_i^+(0) \leq v^+(x) \leq v^-(x) \leq \lim_{J_i \ni h \to 0} u_i^-(h) \quad \text{for all } x \in c_0(0) \text{ and } i \in \mathcal{I}_0.\]
Thus, by the comparison principle applied to \(v^\pm\), we see that \(u_i^+ = u_i^-\) in \(J_i\) for all \(i \in \mathcal{I}_0\). Hence, setting \(u_i = u_i^+\) on \(J_i\), we find that \(u_i \in \mathcal{S}_i \cap C(\bar{J}_i)\), \(u_i(h_i) = d_i\), and \(u_i(0) = \ldots = u_{N-1}(0) = d\). Thus, we have \(v^- = v^+\) on \(\mathbb{R} \setminus c_0(0)\). Furthermore, by the definition of the half-relaxed limits and (4.2), we have
\[v^- = v^+ \quad \text{on } c_0(0),\]
from which we conclude that, as \(\varepsilon \to 0+\),
\[u^\varepsilon \to v^+ = v^- = u_i \circ H \quad \text{on } \Omega_i.\]

The proof of (4.2) can be done as an obvious combination of the two lemmas below.

Lemma 4.2. Set \(d = \min_{i \in \mathcal{I}_0} u_i^+(0)\). Then
\[v^-(x) \geq d \quad \text{for all } x \in c_0(0). \tag{4.3}\]

Lemma 4.3. Set \(d = \min_{i \in \mathcal{I}_0} u_i^+(0)\). Then
\[v^+(x) \leq d \quad \text{for all } x \in c_0(0).\]
In order to prove these lemmas, we need Lemmas 4.4 and 4.5 below.

To state Lemmas 4.4 and 4.5, we set $\bar{h} = \min_{i \in I_0} |h_i|$ and, for $h \in (0, \bar{h})$, define the neighborhood $\Omega(h)$ of the curve $c_0(0)$ by

$$\Omega_i(h) = \{ x \in \Omega_i \mid |H(x)| < h \} \quad \text{for } i \in I_0 \quad \text{and} \quad \Omega(h) = \bigcup_{i \in I_0} \Omega_i(h).$$

**Lemma 4.4.** For any $\eta > 0$, there exist $\delta \in (0, \bar{h})$ and $\psi \in C^1(\Omega(\delta))$ such that

$$-b \cdot D\psi + G(x, 0) < G(0, 0) + \eta \quad \text{in } \Omega(\delta).$$

**Proof.** Fix any $\eta > 0$. Set the function $g \in C(\Omega)$ by $g(x) = G(0, 0) - G(x, 0)$. Note that, for any neighborhood $V$ of $c_0(0)$, there is $\delta \in (0, \bar{h})$ such that $\Omega(\delta) \subset V$.

We fix $\bar{x}_i \in c_i(0) \setminus \{0\}$ for each $i \in I_1$. Choose $r > 0$ so that $B_r \subset \Omega$ and $|g(x)| < \eta$ for all $x \in B_r$.

Choose $\hat{g} \in C^1(\Omega)$ so that $\hat{g}(x) = 0$ for all $x \in B_r$ and $|g(x) - \hat{g}(x)| < \eta$ for all $x \in \Omega$.

Also, we choose $T > 0$ so that

$$X(t, \bar{x}_i) \in B_r \quad \text{for all } |t| \geq T \text{ and } i \in I_1,$$

and

$$\frac{1}{2T} \left| \int_{-T}^{T} \hat{g}(X(t, \bar{x}_i)) \, dt \right| < \frac{\eta}{2} \quad \text{for all } i \in I_1.$$

For $i \in I_1$, let $Y_i(h)$ be the solution of the problem

$$Y'(h) = \frac{DH(Y(h))}{|DH(Y(h))|^2} \quad \text{and} \quad Y(0) = \bar{x}_i,$$

and note that

$$Y_i \in C^1((-\bar{h}, \bar{h}); \mathbb{R}^2) \quad \text{and} \quad H(Y_i(h)) = h \quad \text{for all } h \in (-\bar{h}, \bar{h}).$$

We write $Z_i(t, s) = X(t, Y_i(s))$ for $(t, s) \in \mathbb{R} \times (-\bar{h}, \bar{h})$ and $i \in I_1$. For $i \in I_1$ and $\delta \in (-\bar{h}, \bar{h})$, we set

$$\Omega_{i,T}(\delta) = \{ Z_i(t, s) \mid |t| \leq T + 2, |s| \leq \delta \}.$$

Note that the sets

$$\{ X(t, \bar{x}_i) \mid |t| \leq T + 2 \},$$

with $i \in I_1$, are mutually disjoint, and we choose $\gamma \in (0, \bar{h})$ so that $\Omega_{i,T}(\gamma)$, with $i \in I_1$, are mutually disjoint.

Moreover, we may assume that

$$Z_i(t, s) \in B_r \quad \text{for all } (t, s, i) \in \{ [-T - 2, T + 2] \setminus (-T, T) \} \times [-\gamma, \gamma] \times I_1,$$
\[
\frac{1}{2T} \left| \int_{-T}^{T} \hat{g}(Z_i(t, s)) \, dt \right| < \frac{\eta}{2} \quad \text{for all } (s, i) \in [-\gamma, \gamma] \times I_1,
\]
and
\[
T_i(s) > 2T + 4 \quad \text{for all } (s, i) \in [-\gamma, \gamma] \times I_0.
\]

For \( i \in I_1 \), set
\[
\tilde{g}_i(t, s) = \begin{cases} 
\hat{g}(Z_i(t, s)) & \text{for } (t, s) \in [-T, T] \times [-\gamma, \gamma], \\
0 & \text{for } (t, s) \in (\mathbb{R} \setminus [-T, T]) \times [-\gamma, \gamma],
\end{cases}
\]
and note that \( \tilde{g}_i \in C^1(\mathbb{R} \times [-\gamma, \gamma]) \). Also, for \( i \in I_1 \), set
\[
m_i(s) = \frac{1}{2T} \int_{-\infty}^{\infty} \tilde{g}_i(t, s) \, dt \quad \text{for } s \in [-\gamma, \gamma],
\]
and
\[
\tilde{m}_i(t, s) = \begin{cases} 
m_i(s) & \text{for } (t, s) \in [-T, T] \times [-\gamma, \gamma] \\
0 & \text{for } (t, s) \in (\mathbb{R} \setminus [-T, T]) \times [-\gamma, \gamma].
\end{cases}
\]

Note that
\[
|\tilde{m}_i(t, s)| < \frac{\eta}{2} \quad \text{for all } (t, s, i) \in \mathbb{R} \times [-\gamma, \gamma] \times I_1.
\]

Let \( \rho \in C^1(\mathbb{R}) \) be a standard mollification kernel, with \( \text{supp } \rho \subset (-1, 1) \). For \( i \in I_1 \), set
\[
m_i^\rho(s, t) = (\rho * \tilde{m}_i(\cdot, s))(t) \quad \text{for } (t, s) \in \mathbb{R} \times [-\gamma, \gamma]
\]
and
\[
\tilde{\psi}_i(t, s) = -\int_{-\infty}^{t} (\tilde{g}_i(r, s) - m_i^\rho(r, s)) \, dr \quad \text{for } (t, s) \in \mathbb{R} \times [-\gamma, \gamma].
\]

Note that
\[
m_i^\rho(t, s) = 0 \quad \text{for all } (t, s, i) \in (\mathbb{R} \setminus (-T - 1, T + 1)) \times [-\gamma, \gamma] \times I_1,
\]
\[
|m_i^\rho(t, s)| < \frac{\eta}{2} \quad \text{for all } (t, s, i) \in \mathbb{R} \times [-\gamma, \gamma] \times I_1,
\]
and
\[
\int_{\mathbb{R}} m_i^\rho(t, s) \, dt = \int_{\mathbb{R}} \tilde{g}_i(t, s) \, dt \quad \text{for all } (s, i) \in [-\gamma, \gamma] \times I_1.
\]

Note also that \( \tilde{\psi}_i \in C^1(\mathbb{R} \times [-\gamma, \gamma]) \),
\[
\tilde{\psi}_i(t, s) = 0 \quad \text{for all } (t, s, i) \in (\mathbb{R} \setminus (-T - 1, T + 1)) \times [-\gamma, \gamma] \times I_1,
\]
and
\[
-\tilde{\psi}_i(t, s) = \tilde{g}_i(t, s) - m_i^\rho(t, s) < \tilde{g}_i(t, s) + \frac{\eta}{2} \quad \text{for all } (t, s, i) \in \mathbb{R} \times [-\gamma, \gamma] \times I_1.
\]

We show that \( Z_i : [-T - 2, T + 2] \times [-\gamma, \gamma] \to \Omega_{i,T}(\gamma) \) is a \( C^1 \) diffeomorphism. Obviously, by the definition of \( \Omega_{i,T}(\gamma) \), \( Z_i \) is a \( C^1 \) mapping and surjective. To see that
\[ Z_i \text{ is injective, let } (t,s), (\tau, \sigma) \in [-T - 2, T + 2] \times [-\gamma, \gamma] \text{ be such that } Z_i(t,s) = Z_i(\tau, \sigma). \]

Note that
\[ s = H(X(0, Y_i(s))) = H(Z_i(t,s)) = H(Z_i(\tau, \sigma)) = \sigma. \]

If \( s = 0 \), we see by the standard ode theory that \( t = \tau \). If \( s \neq 0 \), then \( t = \tau \) or \( |t - \tau| \geq T_i(s) \). The latter case is impossible since \( T_i(s) > 2T + 4 \) and \( |t - \tau| \leq 2T + 4 \).

Thus, we have \( t = \tau \) and conclude that \( Z_i \) is bijective. We write \((Z_i^1, Z_i^2)\) for \( Z_i \) and note that
\[ Z_i(t,s) = b(Z_i(t,s)) = (H_{x_2}(Z_i(t,s)), -H_{x_1}(Z_i(t,s))). \quad (4.4) \]

Differentiating \( H(Z_i(t,s)) = s \) yields
\[ H_{x_1}(Z_i(t,s))Z_{i,s}^1 + H_{x_2}(Z_i(t,s))Z_{i,s}^2 = 1. \]

This combined with (4.4) reveals
\[ 1 = H_{x_1}(Z_i(t,s))Z_{i,s}^1 + H_{x_2}(Z_i(t,s))Z_{i,s}^2 = -Z_{i,t}Z_{i,s}^1 + Z_{i,t}^2 = \det(Z_{i,t}, Z_{i,s}). \]

The Inverse Function Theorem guarantees that \( Z_i : [-T - 2, T + 2] \times [-\gamma, \gamma] \rightarrow \Omega_i(T) \) is a \( C^1 \) diffeomorphism.

Note that the Inverse Function Theorem implies that \( \Omega_i(T) \) is a neighborhood of \( \{X(t, \bar{x}_i) \mid |t| \leq T + 1\} \). Since
\[ X(t, \bar{x}_i) \in B_r \quad \text{for all } |t| \geq T \text{ and } i \in \mathcal{I}_1, \]
it follows that the set \( B_r \cup \bigcup_{i \in \mathcal{I}_1} \Omega_i(T) \) is a neighborhood of \( c_i(0) \) and, hence,
\[ B_r \cup \bigcup_{i \in \mathcal{I}_1} \Omega_i(T) \]
is a neighborhood of \( c_0(0) \). Thus, we may choose \( \delta \in (0, \gamma) \) so that
\[ \Omega(\delta) \subset B_r \cup \bigcup_{i \in \mathcal{I}_1} \Omega_i(T). \]

Set
\[ \psi_i(x) = \hat{\psi}_i(Z_i^{-1}(x)) \quad \text{for } (x, i) \in \Omega_i(T) \times \mathcal{I}_1. \]

It is clear that \( \psi_i \in C^1(\Omega_i(T)) \). We define \( \psi : \Omega(\delta) \rightarrow \mathbb{R} \) by
\[ \psi(x) = \begin{cases} \psi_i(x) & \text{if } (x, i) \in \Omega_i(T) \times \mathcal{I}_1, \\ 0 & \text{otherwise}. \end{cases} \]

Since \( \Omega_i(T) \), with \( i \in \mathcal{I}_1 \), are mutually disjoint, the function \( \psi \) is well-defined.

Let \( x \in \Omega(\delta) \). If
\[ x \in B_r \setminus \bigcup_{i \in \mathcal{I}_1} \Omega_i(T), \]
then \( \psi = 0 \) in a neighborhood \( V_x \) (e.g. \( V_x = B_r \setminus \bigcup_{i \in \mathcal{I}_1} \Omega_i(T) \)) of \( x \), \( \psi \in C^1(V_x) \), and
\[ -b(x) \cdot D\psi(x) = 0 < g(x) + \eta. \quad (4.5) \]
Otherwise, we have
\[ x \in \bigcup_{i \in I_1} \Omega_{i,T}(\gamma). \]

Choose \( i \in I_1 \) so that \( x \in \Omega_{i,T}(\gamma) \) and set \((t, s) = Z_{i}^{-1}(x) \in [-T - 2, T + 2] \times [-\gamma, \gamma] \).

Since \( \delta < \gamma \), we see that \(|s| < \gamma \). If \( t = \pm (T + 2) \), then there exists a neighborhood \( V_x \) of \( x \) such that
\[ \psi_i(x) = 0 \quad \text{for all } x \in V_x \cap \Omega_{i,T}(\gamma) \]
and
\[ V_x \cap \Omega_{j,T}(\gamma) = \emptyset \quad \text{for all } j \neq i. \]

Since \( \psi = 0 \) in \( V_x \setminus \Omega_{i,T}(\gamma) \), we see that \( \psi = 0 \) in \( V_x \), and, hence, we get (4.5). If \(|t| < T + 2 \), then \( \psi \) is of class \( C^1 \) in a neighborhood \( V_x \) (e.g. \( V_x = \Omega_{i,T}(\gamma) \)) of \( x \) and
\[ \psi(y) = \tilde{\psi}_i(Z_{i}^{-1}(y)) \quad \text{for all } y \in V_x. \]

Writing \( Z_{i}^{-1}(y) = (\tau, \sigma) \) and differentiating the above, we get
\[ \tilde{\psi}_{i,t} = D\psi(y) \cdot b(y) \]
and
\[ -b(x) \cdot D\psi(x) = -\psi_{i,\tau}(Z_{i}^{-1}(x)) = \tilde{g}_i(Z_{i}^{-1}(x)) + \frac{\eta}{2} = g(x) + \frac{\eta}{2} < g(x) + \eta. \]

This concludes the proof.

**Proof of Lemma 4.5.** We argue by contradiction. Thus, set \( \bar{d} = \min_{v \in \partial \Omega} v^{-} \) and suppose that \( \bar{d} < d \). Using Lemmas 3.11 and 4.4 and arguing as in the proof of [12, Lemma 3.8], we obtain a contradiction.

Here, we note that the initial value problems
\[ \dot{X}_\pm^\xi(t) = \frac{b(X_\pm^\xi(t))}{\varepsilon} \pm \nu \frac{DH(X_\pm^\xi(t))}{|DH(X_\pm^\xi(t))|} \quad \text{and} \quad X_\pm^\xi(0) = x \in \Omega \setminus \{0\}, \]
where \( \nu > 0 \) is the constant from (2.3), admit unique solutions \( X_\pm^\xi(t, x) \) in the maximal interval \((\underline{\sigma}_\pm^\xi(x), \overline{\sigma}_\pm^\xi(x))\) where \( \underline{\sigma}_\pm^\xi(x) < 0 < \overline{\sigma}_\pm^\xi(x) \), and the maximality means that either
\[ \underline{\sigma}_\pm^\xi(x) = -\infty \quad \text{or} \quad \lim_{t \to \underline{\sigma}_\pm^\xi(x) + 0} \text{dist}(X_\pm^\xi(t, x), \partial \Omega \cup \{0\}) = 0, \]
and either
\[ \overline{\sigma}_\pm^\xi(x) = \infty \quad \text{or} \quad \lim_{t \to \overline{\sigma}_\pm^\xi(x) - 0} \text{dist}(X_\pm^\xi(t, x), \partial \Omega \cup \{0\}) = 0. \]

**Lemma 4.5.** Let \( \varepsilon \in (0, \varepsilon_0), h \in (0, \bar{h}) \), and \( x \in \Omega(h) \setminus \{0\} \). If \( \tau_1, \tau_2 \in (\underline{\sigma}_\pm^\xi(x), \overline{\sigma}_\pm^\xi(x)) \) are such that \( \tau_1 < \tau_2 \) and \( X_\pm^\xi(t, x) \in \Omega(h) \) for all \( t \in (\tau_1, \tau_2) \), then
\[ \tau_2 - \tau_1 \leq \frac{2(m + 2)}{\nu \varepsilon_0 (m - n + 2) h} \frac{m - n + 2}{m + 2}. \]

Also inequality (1.5) holds with \( \underline{\sigma}_\pm^\xi, \overline{\sigma}_\pm^\xi, \) and \( X_\pm^\xi \) being replaced by \( \sigma_\pm^\xi, \sigma_\pm^\xi, \) and \( X_\pm^\xi. \)
Here, $c_0 > 0$ is the constant from (2.3).

**Proof.** We can prove this lemma by using (2.3) and replacing the function $\psi$ in [12, Lemma 5.1] by $\psi(r) = r |r|^{-\frac{m}{m+2}}$ for $r \in \mathbb{R} \setminus \{0\}$.

**Proof of Lemma 4.3.** We obtain (4.3) by using (2.4) and Lemma 4.5 as well as the dynamic programming principle as in the proof of [12, Lemma 3.7].

5 The boundary data for the limiting problem

In this section, we present a sufficient condition on the data $d_i$ for which (G5) and (G6) hold.

Here, we only state the theorems concerning the sufficient condition and refer to [12, Section 7] for the proofs.

For $i \in \mathcal{I}_0$, we write $I_i$ for the set of $d \in \mathbb{R}$ such that the set

$$\{ u \in \mathcal{S}_i^- \cap C(\bar{J}_i) \mid u(h_i) = d \}$$

is nonempty.

We note that $I_i = (-\infty, a_i]$ for some $a_i \in \mathbb{R}$. Indeed, in view of the remark after Lemma 3.11 if $d \in I_i$ and $c < d$, then $c \in I_i$. Also, if $d \in \mathbb{R}$ satisfies

$$\lambda d + \max_{h \in \bar{J}_i} \bar{G}_i(h, 0) \leq 0,$$

then $d \in \mathcal{S}_i^-$ and $d \in I_i$, while if $d \in \mathbb{R}$ satisfies

$$\lambda d + \min_{(h,p) \in \bar{J}_i \times \mathbb{R}} \bar{G}_i(h, p) > 0,$$

then $d \not\in I_i$. Thus, we see that $I_i = (-\infty, a_i]$.

For $i \in \mathcal{I}_0$, $d \in I_i$, and $h \in \bar{J}_i$, we define

$$\rho_i^d(h) = \sup\{ u(h) \mid u \in \mathcal{S}_i^- \cap C(\bar{J}_i), \ u(h_i) = d \},$$

and, we have $\rho_i^d \in \mathcal{S}_i \cap C(\bar{J}_i)$ and $\rho_i^d(h_i) = d$. By Lemma 3.9 we see that

$$\rho_0 := \min_{i \in \mathcal{I}_0} \sup_{d \in I_i} \rho_i^d(0) < \infty.$$

Also, we write $I$ for the set of $d \in \mathbb{R}$ such that

$$\{ u \in \mathcal{S}_i^- \cap C(\bar{J}_i) \mid u(0) = d \} \neq \emptyset \quad \text{for all } i \in \mathcal{I}_0.$$

and, for $i \in \mathcal{I}_0$, $d \in I$, and $h \in \bar{J}_i$, we define

$$\nu_i^d(h) = \sup\{ u(h) \mid u \in \mathcal{S}_i^- \cap C(\bar{J}_i), \ u(0) = d \}. $$

Similarly to the above, we see that $I = (-\infty, \rho_0]$ and that $\nu_i^d \in \mathcal{S}_i \cap C(\bar{J}_i)$ and $\nu_i^d(0) = d$. 

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Theorem 5.1. Let \((d,d_0,\ldots,d_{N-1}) \in \mathbb{R}^{N+1}\). The problem
\[
\begin{aligned}
\lambda u_i + \mathcal{G}_i(h,u'_i) &= 0 \quad \text{in } J_i, \\
u_i(h_i) &= d_i, \\
\lambda u_i + \mathcal{G}_i(h_i) &= 0 \quad \text{in } J_i,
\end{aligned}
\]
has a viscosity solution \((u_0, \ldots, u_{N-1}) \in C(J_0) \times \ldots \times C(J_{N-1})\) if and only if
\[
\begin{aligned}
(d,d_0,\ldots,d_{N-1}) \in I \times I_1 \times \ldots \times I_{N-1}, \\
\min_{i \in I_0} \rho_i^d(0) &\geq d, \\
\nu_i^d(h_i) &\geq d_i \quad \text{for all } i \in I_0.
\end{aligned}
\]
Here, we set
\[
\mathcal{D} = \{(d,d_0,\ldots,d_{N-1}) \in \mathbb{R}^{N+1} \mid (5.2) \text{ is satisfied}\},
\]
and
\[
\mathcal{D}_0 = \{(d,d_0,\ldots,d_{N-1}) \in \mathbb{R}^{N+1} \mid \text{there exists } a > 0 \text{ such that } (d+a,d_0+a,\ldots,d_{N-1}+a) \in \mathcal{D}\}.
\]

The following theorem gives a sufficient condition for which (G5) and (G6) hold.

Theorem 5.2. For any \((d,d_0,\ldots,d_{N-1}) \in \mathcal{D}_0\), (G5) and (G6) hold for some boundary data \(g^\varepsilon\).

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