Equations and tropicalization of Enriques surfaces

Barbara Bolognese, Corey Harris and Joachim Jelisiejew

Abstract In this article we explicitly compute equations of an Enriques surface via the involution on a K3 surface. We also discuss its tropicalization and compute the tropical homology, thus recovering a special case of the result of [19], and establish a connection between the dimension of the tropical homology groups and the Hodge numbers of the corresponding algebraic Enriques surface.

1 Introduction

In the classification of algebraic surfaces, Enriques surfaces comprise one of four types of minimal surfaces of Kodaira dimension 0. There are a number of surveys on Enriques surfaces. For those new to the theory, we recommend the excellent exposi-

"Ogni superficie F di generi $p_a = p_g = P_3 = 0, P_2 = 1$ si può considerare come una superficie doppia di generi 1. Più precisamente: le coordinate dei punti di F si possono esprimere razionalmente per mezzo di $x, y, z$ e di $z = \sqrt{f_s(x, y)}$, dove il polinomio $f_s$ è del quarto grado separatamente rispetto ad $x, y$, ed ammette una trasformazione involutoria in se stesso priva di coincidenze; ad ogni punto di F corrispondono due terne $(x, y, z)$.– F. Enriques [13]

Barbara Bolognese
The University of Sheffield e-mail: b.bolognese@sheffield.ac.uk

Corey Harris
Florida State University e-mail: charris@math.fsu.edu

Joachim Jelisiejew
Institute of Mathematics, Informatics and Mechanics, University of Warsaw e-mail: jjelisiejew@mimuw.edu.pl
tion found in \cite{2} and \cite{3}, and for a more thorough treatment, the book \cite{10}. Another recommended source is Dolgachev’s brief introduction to Enriques surfaces \cite{11}. The first Enriques surface was constructed in 1896 by Enriques himself \cite{12} to answer negatively a question posed by Castelnuovo (1895):

Is every surface with $p_g = q = 0$ rational?

(see Section 2 for the meaning of $p_g$ and $q$) Enriques’ original surface has a beautiful geometric construction: the normalization of a degree 6 surface in $\mathbb{P}^3$ with double lines given by the edges of a tetrahedron. Another construction, the Reye congruence, defined a few years earlier by Reye \cite{26}, was later proved by Fano \cite{14} to be an Enriques surface. Since these first constructions, there have been many examples of Enriques surfaces, most often as quotients of K3 surfaces by a fixed-point-free involution. In \cite{9}, Cossec describes all birational models of Enriques surfaces given by complete linear systems.

As we recall in Section 2, every Enriques surface has an unramified double cover given by a K3 surface. Often exploiting this double cover, topics of particular interest relate to lattice theory, moduli spaces and their compactifications, automorphism groups of Enriques surfaces, and Enriques surfaces in characteristic 2.

While there are many constructions of Enriques surfaces, none give explicit equations for an Enriques surface embedded in a projective space. In this paper, interpreting the work of Cossec-Verra, we give explicit ideals for all Enriques surfaces.

**Theorem 1.1.** Let $Y$ be the toric fivefold of degree 16 in $\mathbb{P}^{11}$ that is obtained by taking the join of the Veronese surface in $\mathbb{P}^5$ with itself. The intersection of $Y$ with a general linear subspace of codimension 3 is an Enriques surface, and every Enriques surface arises in this way.

By construction, the Enriques surface in Theorem 1.1 is arithmetically Cohen-Macaulay. Its homogeneous prime ideal in the polynomial ring with 12 variables is generated by the twelve binomial quadrics that define $Y$ and three additional linear forms. Explicit code for producing this Enriques ideal in Macaulay 2 is given in Section 3.

After having constructed Enriques surfaces explicitly, we focus on their tropicalizations, with the purpose of studying their combinatorial properties. For this we choose a different K3 surface, namely a hypersurface $S \subset (\mathbb{P}^1)^3$ with an involution $\sigma$, see Example 4.3. In Section 5.2 we get a fairly complete picture for its tropicalization. In particular we recover its Hodge numbers and, conjecturally, the Hodge numbers of $S/\sigma$, which was \cite{29} Problem 10 on Surfaces; this was the starting point of this work.

**Proposition 1.2 (Example 4.3, Proposition 5.7, Proposition 5.8).** The dimensions of tropical homology groups of the tropicalization of the K3 surface $S$ agree with Hodge numbers of $S$. The dimensions of $\sigma$-invariant parts of tropical homology groups agree with the Hodge numbers of the Enriques surface $S/\sigma$.

Finally we discuss an analogue of Castelnuovo’s question on the tropical and analytic level. Since the analytifications of rational varieties are contractible by \cite[Corollary 1.1.4]{6}, we ask the following question:
Are the analytifications of K3 or Enriques surfaces contractible?

We give a negative answer to this question, the counterexample being the analytification of \( S \) from Example 4.3.

**Theorem 1.3.** The analytification \( S^{an} \) of the K3 surface \( S \) is homotopy equivalent to a two-dimensional sphere. The surface \( S \) has a fixed-point-free involution \( \sigma \) and the analytification of the Enriques surface \( S/\sigma \) retracts onto \( \mathbb{R}P^2 \). In particular neither \( S^{an} \) nor \((S/\sigma)^{an}\) is contractible.

The contents of the paper are as follows. In Section 2 we give some background about Enriques surfaces. Next, in Section 3 we exploit a classical construction to obtain an Enriques ideal in a codimension 3 linear space in \( \mathbb{P}^1 \) and prove Theorem 1.1. In Section 4 we discuss the basics of tropical geometry and analytic spaces in the sense of Berkovich. Example 4.3 provides an Enriques surface \( S/\sigma \) arising from a K3 surface \( S \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) with an involution \( \sigma \). The surface \( S \) is suitable from the tropical point of view (its tropical variety is schön and multiplicity one everywhere) and is used throughout the paper. In Section 5 we compute the tropical homology groups of trop(\( S \)) and, conjecturally, of trop(\( S/\sigma \)). We also prove Proposition 1.2. In Section 6 we discuss the topology of analytifications of \( S \) and \( S/\sigma \) and prove Theorem 1.3.

## 2 Background

Apart from the code snippets, we work over an algebraically closed field \( k \) of characteristic zero. An *Enriques surface* \( X \) is a smooth projective surface with \( q(X) := h^1(X, \mathcal{O}_X) = 0, \omega_X^2 \simeq \mathcal{O}_X \) and \( \omega_X \not\simeq \mathcal{O}_X \), where \( \omega_X = \Lambda^2 \Omega^1_X \) is the canonical bundle of \( X \). Then it follows that \( X \) is minimal, see [3], and \( p_g(X) := h^2(X, \mathcal{O}_X) = 0 \). We note that Enriques surfaces are defined the same way over any field of characteristic...
other than 2. By [2, Lemma 15.1] the Hodge diamond of an Enriques surface $X$ is:

$$
\begin{array}{cccc}
& h^{0,0} & 1 \\
h^{1,0} & h^{0,1} & 0 & 0 \\
h^{2,0} & h^{1,1} & h^{0,2} & 0 & 0 \\
h^{2,1} & h^{1,2} & 0 & 0 \\
h^{2,2} & & & & 1 \\
\end{array}
$$

(1)

An Enriques surface admits an unramified double cover $f : Y \to X$, where $Y$ is a K3 surface, see [2, Lemma 15.1] or [3, Proposition VIII.17]. The Hodge diamond of $Y$ is given by

$$
\begin{array}{cccc}
& h^{0,0} & 1 \\
h^{1,0} & h^{0,1} & 0 & 0 \\
h^{2,0} & h^{1,1} & h^{0,2} & 0 & 0 \\
h^{2,1} & h^{1,2} & 0 & 0 \\
h^{2,2} & & & & 1 \\
\end{array}
$$

(2)

In particular since $Y$ is simply-connected, the fundamental group of an Enriques surface is $\mathbb{Z}/2\mathbb{Z}$, see [2, Section 15]. The cover $Y \to X$ is in fact a quotient of $Y$ by an involution $\sigma$, which exchanges the two points of each fiber. Conversely, for a K3 surface $Y$ with a fixed-point-free involution $\sigma$ the quotient $Y/\sigma$ is an Enriques surface. An example of this procedure, known as Horikawa’s construction, appears in the quote at the beginning of the paper.

### 3 Enriques surfaces via K3 complete intersections in $\mathbb{P}^5$

In this section we construct Enriques surfaces via K3 surfaces in $\mathbb{P}^5$. Before we go into the details, we remark that one cannot hope for easy equations, for example an Enriques surface cannot be a hypersurface in $\mathbb{P}^3$.

**Proposition 3.1.** Let $X \subset \mathbb{P}^N_C$ be a smooth projective toric threefold and $S = X \cap H$ be a smooth hyperplane section. Then $S$ is simply-connected. In particular it is not an Enriques surface.

**Proof.** Since $X$ is a smooth and projective toric variety, it is simply connected by [15] §3.2. Now a homotopical version of Lefschetz’ theorem ([11] see also [4] 2.3.10]) asserts that the fundamental groups of $X \cap H$ and $X$ are isomorphic via the natural map. Thus $S$ is simply connected. Now suppose $S$ is an Enriques surface. Then it admits a non-trivial étale double cover $K \to S$, thus it is not simply connected, which is a contradiction. $\square$
We remark that this proof generalizes to other complete intersections inside smooth toric varieties, provided that intermediate complete intersections are smooth as well.

We now construct an Enriques surface from a K3 surface which is an intersection of quadrics in \( \mathbb{P}^5 \). We follow Beauville [3, Example VIII.18].

Fix a projective space \( \mathbb{P}^5 \) with coordinates \( x_0, x_1, x_2, y_0, y_1, y_2 \). Consider the involution \( \sigma : \mathbb{P}^5 \to \mathbb{P}^5 \) given by \( \sigma(x_i) = x_i \) and \( \sigma(y_i) = -y_i \) for \( i = 0, 1, 2 \). Then the fixed point set is equal to the union of \( \mathbb{P}^2 = V(y_0, y_1, y_2) \) and \( \mathbb{P}^2 = V(x_0, x_1, x_2) \).

Fix quadrics \( Q_i = F_i + G_i \). By their construction, these quadrics are fixed by \( \sigma \). We henceforth choose \( Q_i \) so that they give a complete intersection. Then \( S = S_0 : = V(Q_0, Q_1, Q_2) \) is a surface and, by the Adjunction Formula, we have \( K_S = O_S(-6 + 2 + 2 + 2) = O_S \).

It can also be shown that since the surface \( S \) is a complete intersection of quadrics in \( \mathbb{P}^5 \), it has \( h^1(O_S) = 0 \), see [3, Lemma VIII.9]. Thus if \( S \) is smooth, then it is a K3 surface fixed under the involution \( \sigma \). We will now formalize exactly which assumptions must be satisfied by the three quadrics to obtain a smooth Enriques surface.

**Definition 3.2.** Let \( Q = (Q_0, Q_1, Q_2) \) be a triple of quadrics \( Q_i = F_i + G_i \) for \( F_i \in \mathbb{C}[x_i] \) and \( G_i \in \mathbb{C}[y_i] \) as before. We say that the quadrics \( Q \) are *enriquogeneous* if the following conditions are satisfied:

1. the forms \( Q = (Q_0, Q_1, Q_2) \) are a complete intersection,
2. the surface \( S = V(Q_0, Q_1, Q_2) \) is smooth,
3. the surface \( S = V(Q_0, Q_1, Q_2) \) does not intersect the fixed-point set of \( \sigma \).

We note that the third condition is equivalent to \( F_1, F_2, F_3 \) having no common zeros in \( \mathbb{C}[x_0, x_1, x_2] \) and \( G_i \) having no common zeros in \( \mathbb{C}[y_0, y_1, y_2] \), so it is open. We know that for a choice of enriquogeneous quadrics \( Q \) we obtain an Enriques surface as \( S_Q / \sigma \). The set of enriquogeneous quadrics is open inside \( (\mathbb{A}^6)^6 \), so that a *general* choice of forms gives an Enriques surface. In [9], Cossec proves that every complex Enriques surface may be obtained as above if one allows \( Q \) not satisfying the smoothness condition, see also [31]. Notably, Lietdke recently proved that the same is true for Enriques surfaces over any characteristic [22]. To give some intuition for the complex result, let us prove that over the complex numbers these surfaces give at most a 10-dimensional space of Enriques surfaces.

Notice that each \( Q_i \) is chosen from the same 12-dimensional affine space and \( S_Q \) depends only on their span, which is an element of \( \text{Gr}(3, \mathbb{C}^{12}) \). This is a 27-dimensional variety. However, since we have fixed \( \sigma \), the quadrics \( Q_i \) will give an isomorphic K3 surface (with an isomorphic involution) if we act on \( \mathbb{P}^5 \) by an automorphism that commutes with \( \sigma \). Such automorphisms are given by block matrices in \( PGL(6) \) of the form

\[
C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \text{or} \quad C = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}
\]

where \( A \) and \( B \) are matrices in \( GL(3) \), up to scaling. Thus, the space of automorphisms preserving the \( \sigma \)-invariant quadrics has dimension \( 2 \cdot 9 - 1 = 17 \). Modulo
these automorphisms, we now have a 10-dimensional projective space of K3 surfaces with an involution. Note that the condition that $Q$ be enriquogeneous is an open condition.

We now aim at making the Enriques surfaces obtained above as $S_Q/\sigma$ explicit. In other words we want to present them as embedded into a projective space.

The first step is to identify the quotient of $\mathbb{P}^5$ by the involution $\sigma$. Let $S = \mathbb{C}[x_0,x_1,x_2,y_0,y_1,y_2]$ be the homogeneous coordinate ring. Then the quotient is $\text{Proj}(S^\sigma) = \text{Proj}(\mathbb{C}[x_i,y_j])$. The Enriques $S_Q$ is cut out of $\text{Proj}(\mathbb{C}[x_i,y_j])$ by the quadrics $Q$, so that

$$S_Q = \text{Proj}(\mathbb{C}[x_i,y_j]/Q).$$

This does not give us an embedding into $\mathbb{P}^8$, since the variables $x_i$ and $y_j$ have different degrees. Rather we obtain an embedding into a weighted projective space $\mathbb{P}(1^3,2^6)$. Therefore we replace $\mathbb{C}[x_i,y_j]$ by the Veronese subalgebra

$$S_Q \simeq \text{Proj}(\mathbb{C}[x_i,y_j]/Q).$$

This algebra is generated by 12 elements $x_ix_j, y_iy_j$ for $i, j = 0, 1, 2$, so that $S_Q$ is embedded into a $\mathbb{P}^{11}$. The relations $Q$ are linear in the variables $x_ix_j$ and $y_iy_j$, so that $S_Q$ is embedded into a $\mathbb{P}^8$.

Let us rephrase this geometrically. Consider the second Veronese re-embedding $\nu : \mathbb{P}^5 \to \mathbb{P}^{20}$. The coordinates of $\mathbb{P}^{20}$ are forms of degree two in $x_i$ and $y_j$. The involution $\sigma$ extends to an involution on $\mathbb{P}^{20}$ and this time the invariant coordinate ring is generated by the linear forms corresponding to products $x_ix_j$ and $y_iy_j$. Therefore the quotient is embedded in $\mathbb{P}^{11}$, which has coordinate ring corresponding to those 12 forms.

$$\begin{array}{ccc}
\mathbb{P}^5 & \overset{v}{\longrightarrow} & \mathbb{P}^{20} \\
& \downarrow \pi & \\
& \mathbb{P}^{11} &
\end{array}$$

where $\pi$ denotes the quotient by the involution $\sigma$. Then the image $\pi(\mathbb{P}^5)$ is cut out by 12 binomial quadrics, which are the 6 usual equations between $x_ix_j$ and the 6 corresponding equations for $y_iy_j$. It is the join of two Veronese surfaces which constitute its singular locus. Quadrics in $\mathbb{C}[x_i,y_j]$ which have the form $F_i + G_i$ for $F_i \in \mathbb{C}[x_i]$ and $G_i \in \mathbb{C}[y_j]$ correspond bijectively to linear forms on the above $\mathbb{P}^{11}$. A choice of enriquogeneous quadrics $Q$ corresponds to a general choice of three linear forms on $\mathbb{P}^{11}$. We obtain the corresponding Enriques surface $S_Q$ as a linear section of $\pi(\mathbb{P}^5)$. Summing up, we have the following chain of inclusions

$$V \cap \pi(\mathbb{P}^5) \subset \pi(\mathbb{P}^5) \subset \mathbb{P}(1^3,2^6) \subset \mathbb{P}^{11}$$

where $V$ is a codimension three linear section. Note that although $V \cap \pi(\mathbb{P}^5)$ is a complete intersection in $\pi(\mathbb{P}^5)$, this is not contradictory to (a natural generalisation
Equations and tropicalization of Enriques surfaces

of Proposition 3.1, because \( \pi(\mathbb{P}^5) \) is singular. Note also that sufficiently ample embeddings of varieties are always cut out by quadrics, see [24,28], so this suggests that our embedding is sufficiently good.

**Proof (Theorem 1.1).** The surfaces obtained from enriquogeneous quadrics are arithmetically Cohen-Macaulay of degree 16 as they are linear sections of \( \pi(\mathbb{P}^5) \) possessing those properties. Every Enriques surface can be obtained by this procedure if one allows \( Q \) not satisfying the smoothness condition by [9]. \( \square \)

Below we provide (very basic) Macaulay2 [16] code for obtaining the equations of \( S_Q \) explicitly, using the above method. We could take any field as \( K \); we use a finite field to take random elements.

\[
\begin{align*}
kk & = \mathbb{Z}/1009; \\
P5 & = kk[x_0, x_1, x_2, y_0, y_1, y_2]; \\
P11 & = kk[z_0, z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_10, z_11]; \\
pii & = map(P5, P11, \{ x_0^2, x_0*x_1, x_0*x_2, x_1^2, \\
& \quad \quad x_1*x_2, x_2^2, y_0^2, y_0*y_1, \\
& \quad \quad y_0*y_2, y_1^2, y_1*y_2, y_2^2 \});
\end{align*}
\]

We can verify that the kernel of \( pii \) is generated by 12 binomial quadrics and has degree 16.

\[
\begin{align*}
& \text{assert(kernel } pi == \\
& \quad \text{ideal(z10^2-z9*z11, z8*z10-z7*z11, z8*z9-z7*z10,} \\
& \quad \quad z8^2-z6*z11, z7*z8-z6*z10, z7^2-z6*z9,} \\
& \quad \quad z4^2-z3*z5, z2*z4-z1*z5, z2*z3-z1*z4,} \\
& \quad \quad z2^2-z0*z5, z1*z2-z0*z4, z1^2-z0*z3)) \\
& \text{assert(degree kernel } pi == 16)
\end{align*}
\]

Now we generate an Enriques from a random set of linear forms named \( \text{linForms} \). To see the quadrics in \( \mathbb{P}^5 \) take \( \text{pii(linForms)} \).

\[
\begin{align*}
\text{linForms} & = \text{random}(P11^3, P11^{-1}) \\
\text{randomEnriques} & = (\text{kernel } pi) + \text{ideal } \text{linForms}
\end{align*}
\]

We now check whether it is in fact an Enriques. Computationally it is much easier to check this for the associated K3 surface, since we need only check that \( K3 \) is a smooth surface (first two assertions below) and that the involution is fixed-point-free on \( K3 \) (last two assertions).

\[
\begin{align*}
& \text{K3 = ideal } \text{pii(linForms)} \\
& \text{assert (dim K3 == 3)} \\
& \text{assert (dim saturate ideal singularLocus K3 == -1)} \\
& \text{assert (dim saturate (K3 + ideal(y0,y1,y2)) == -1)} \\
& \text{assert (dim saturate (K3 + ideal(x0,x1,x2)) == -1)}
\end{align*}
\]

If the \( K3 \) passes all the assertions, then \( \text{randomEnriques} \) is an Enriques surface. Its ideal is given by 12 binomial quadrics listed above and three linear forms in \( P11 \).
Example 3.3. Over \( k = \mathbb{F}_{1009} \) the choice of
\[
\text{linForms} = \text{matrix} \{ (2z2+z6+5z7+8z11, \\
2z0+8z4+z9, \\
5z1+4z3+4z5+6z8) \}
\]
in the above algorithm gives an Enriques surface.

Finally, we check that \( \pi(P^5) \) is arithmetically Cohen-Macaulay. Using \texttt{betti res kernel} \( \pi_i \) we obtain its Betti table.

\[
\begin{bmatrix}
1 & \ldots & \ldots \\
. & 12 & 16 & 6 & \ldots \\
. & . & 36 & 96 & 100 & 48 & 9
\end{bmatrix}
\]

The projective dimension of \( \pi(P^5) \) (the number of columns) is equal to the codimension, thus \( \pi(P^5) \subset P^{11} \) is arithmetically Cohen-Macaulay, see [27, Section 10.2]. Therefore all its linear sections are also arithmetically Cohen-Macaulay.

4 Analytified and tropical Enriques surfaces

The aim of this section is to discuss the basics of tropical and analytic geometry and to construct a K3 surface, whose tropicalization is nice enough for computations of tropical homology. This is done in Example 4.3; we obtain a K3 surface with an involution, which on the tropical side is the antipodal map.

As an excellent reference for tropical varieties we recommend [23], especially Section 6.2. For analytic spaces in the sense of Berkovich we recommend [5, 18].

Let \( \mathbb{C} \subset k \) be a field extension and suppose that \( k \) has a non-trivial valuation \( \nu : k^* \rightarrow \mathbb{R} \) such that \( \nu(\mathbb{C}^*) = \{0\} \). Suppose further that \( k \) is algebraically closed, so that \( \nu(k^*) \) is dense in \( \mathbb{R} \). Without much loss of generality one can take \( k = \mathbb{C} \).
Equations and tropicalization of Enriques surfaces

\[ \mathbb{C}\{z\} = \bigcup_{n \in \mathbb{N}} \mathbb{C}(z^{1/n}) \], the field of Puiseux series, with valuation yielding the lowest exponent of \( z \) appearing in the series.

For every point \( p = (p_1, \ldots, p_n) \in (k^*)^n \) its valuation is \( v(p) = (v(p_1), \ldots, v(p_n)) \).

**Definition 4.1.** Let \( X \) be a toric variety with torus \( (k^*)^n \) and \( Y \subset X \) be a closed subvariety. The tropical variety of \( Y \) is the closure of the set

\[ \{ v(p) \mid p \in (k^*)^n \cap Y \} \subset \mathbb{R}^n, \quad (9) \]

we denote it by \( \text{trop}(Y \subset X) \) or briefly \( \text{trop}(Y) \).

We remark that \( \text{trop}(Y \subset X) \) is a polyhedral complex of dimension \( \dim Y \) and has rich combinatorial structure, see [23, Chapter 3] and references therein.

Now we discuss tropicalized maps. A morphism of tori \( \varphi : (k^*)^n \to (k^*)^m \) is given by \( \varphi = (\varphi_1, \ldots, \varphi_m) \) where \( \varphi_i(t) = b_i \cdot t^{a_i} \) for \( i = 1, \ldots, m \). For each such \( \sigma \) there is a tropicalized map \( \text{trop}(\varphi) : \mathbb{R}^n \to \mathbb{R}^m \) given by

\[ \text{trop}(\varphi)_i(v) = v(b_i) + (a_i, v) \quad i = 1, \ldots, m. \quad (10) \]

One can check that the following diagram commutes:

\[ \begin{array}{ccc}
(k^*)^n & \xrightarrow{\varphi} & (k^*)^m \\
\downarrow v & & \downarrow v \\
\mathbb{R}^n & \xrightarrow{\text{trop}(\varphi)} & \mathbb{R}^m 
\end{array} \quad (11) \]

The reason for the existence of this map is that one can compute the valuation \( v \) of \( \varphi(t) \) by knowing only the valuation of \( t \).

A notable problem of tropical varieties is that it is known how to tropicalize a map only when it is monomial; in this sense, the naive tropicalization is not a functor. This problem is removed once one passes to Berkovich spaces. We will not discuss Berkovich spaces in detail: we invite the reader to see [5, 18] or [25] for a slightly more elementary introduction.

For every finite-type scheme \( X \) over a valued field \( k \), its Berkovich analytification \( X^{an} \) is the analytic space (see [5, Chapter 3]) which best approximates \( X \). The space \( X^{an} \) is locally ringed (in the usual sense, see [30, 4.3.6]) and there is a morphism \( \pi : X^{an} \to X \) such that every other map from an analytic space factors through \( \pi \). If \( X = \text{Spec} A \) is affine, then the points of \( X^{an} \) are in bijection with the multiplicative semi-norms on \( A \) which extend the norm on \( k \). Most importantly the analytification is functorial: for every map \( f : X \to Y \) we get an induced map

\[ f^{an} : X^{an} \to Y^{an}. \quad (12) \]

If \( X = \text{Spec} A \) and \( Y = \text{Spec} B \) are affine, then \( f \) induces \( f^\# : B \to A \) and the map \( f^{an} \) takes a seminorm \( |\cdot| \) on \( A \) to the seminorm \( b \mapsto |f^\#(b)| \) on \( B \).

The analytification of an affine variety \( X \) is the limit of its tropicalizations by [25]. Namely let \( X \) be an affine variety and consider its embeddings \( i : X \to \mathbb{A}^n \) into affine
spaces. For any two embeddings \( i : X \to \mathbb{A}^n \) and \( j : X \to \mathbb{A}^m \) and a toric morphism \( \varphi : \mathbb{A}^n \to \mathbb{A}^m \) satisfying \( j = \varphi \circ i \) we get by (10) a tropicalized map \( \text{trop}(X \subset \mathbb{A}^n) \to \text{trop}(X \subset \mathbb{A}^m) \). For every embedding \( X \subset \mathbb{A}^n \) there is an associated map

\[
X^{an} \to \text{trop}(X \subset \mathbb{A}^n),
\]

which maps a multiplicative seminorm \( |\cdot| \) to the associated valuation \( -\log |\cdot| \), see [25, pg. 544]. The main result of [25] is that the inverse limit is homeomorphic to the Berkovich analytification via the limit of maps defined in (13) above. Hence one has:

\[
X^{an} = \varprojlim \text{trop}(X \subset \mathbb{A}^n).
\]

We now return to the case of Enriques surfaces. We are interested in finding an Enriques surface \( S/\sigma \) with a K3 cover \( S \) suitable for tropicalization. Specifically we would like \( \sigma \) to be an involution acting without fixed points on the tropical side. In this sense the examples obtained as in Sect. 3 are not suitable.

**Example 4.2.** Let us consider the K3 surface \( S_0 \) defined using enriquogeneous quadrics in Section 3 with \( \sigma(x_0,x_1,x_2,y_0,y_1,y_2) = (x_0,x_1,x_2,-y_0,-y_1,-y_2) \). Since \( \nu(-1) = 0 \), the tropicalized involution \( \text{trop}(\sigma) \), defined by Equation (10), is the identity map on \( \mathbb{R}^6 \).

To obtain a K3 surface with an involution \( \sigma \) tropicalizing to a fixed-point free involution, we consider embeddings into products of \( \mathbb{P}^1 \). Consider the involution \( \tau : \mathbb{P}^1 \to \mathbb{P}^1 \) given by \( \tau([x:y]) = [y:x] \) and the involution \( \sigma : (\mathbb{P}^1)^3 \to (\mathbb{P}^1)^3 \) given by applying \( \tau \) to every coordinate. The map \( \tau \) restricts to the torus \( \mathbb{C}^* \) and is given by \( \mathbb{C}^* \ni t \to t^{-1} \in \mathbb{C}^* \). Therefore \( \text{trop}(\tau)(\nu) = -\nu \) by Equation (10). Consequently the tropicalization \( \text{trop}(\sigma) : \mathbb{R}^3 \to \mathbb{R}^3 \) is given by

\[
\text{trop}(\sigma)(\nu) = -\nu.
\]

This map is non-trivial and has only one fixed point.

**Example 4.3 (An example of a K3 surface with a fixed-point-free involution).** Let \( \mathbb{k} \) be an algebraically closed field with a nontrivial valuation \( \nu : \mathbb{k}^* \to \mathbb{R} \), for example \( \mathbb{k} = \mathbb{C}\{z\} \).

Let \( S \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) be a smooth surface given by a section of the anticanonical divisor of \( (\mathbb{P}^1)^3 \), i.e., a triquadratic polynomial. We remark that the Newton polytope of \( S \) is the 3-dimensional cube \([0,2]^3\). We introduce the following assumptions on \( S \):

1. \( S \) is smooth;
2. \( S \) is invariant under the involution \( \sigma \);
3. the subdivision induced by \( S \) on its Newton polytope \([0,2]^3\) is a unimodular triangulation, that is, the polytopes in the triangulation are tetrahedra of volume equal to 1/6, see [23, pg. 13] for details.

Each such \( S \) is a K3 surface. Under our assumptions the point \((0,0,0)\) is not in the tropical variety of \( S \). Indeed, if it was in \( \text{trop}(S) \), that variety would not be locally linear at \((0,0,0)\). But \( \text{trop}(S) \) is coming from a unimodular triangulation, so it is
locally linear everywhere. Hence $(0,0,0)$ is outside and so $\text{trop}(\sigma)$ is a fixed-point-free involution on $\text{trop}(S)$.

By Equation (12) map $\sigma : S \to S$ induces also an involution $\sigma^{an} : S^{an} \to S^{an}$ which is compatible with $\text{trop}(\sigma)$ under the projection $\pi$ defined in Equation (13); the following diagram commutes.

$$
\begin{array}{ccc}
S^{an} & \xrightarrow{\sigma^{an}} & S^{an} \\
\downarrow \pi & & \downarrow \pi \\
\text{trop}(S) & \xrightarrow{\text{trop}(\sigma)} & \text{trop}(S)
\end{array}
$$

(16)

5 The tropical homology

The plan of this section is an explicit calculation of the tropical homology of a tropical K3 surface and a tropical Enriques surface. We intend to use the construction in Example 4.3 in order to obtain tropicalizations which are locally linear (locally look like tropicalizations of linear spaces), and then compute their tropical cohomology groups. In accordance to the results in [19], the dimensions of such homology groups should coincide with the Hodge numbers of the surfaces themselves. We carry out the calculation by hand for some curves, a tropical K3 and also for an object, which we believe to be the associated tropical Enriques. See [21] for computation of tropical homology using Polymake.

**Theorem 5.1 ([19] Special case of Theorem 2).** Let $X \subset \mathbb{P}^N$. Suppose that $\text{trop}(X) \subset \text{trop}(\mathbb{P}^N)$ has multiplicities all equal to 1 and that it is locally linear. Then the tropical Hodge numbers agree with Hodge numbers of $X$:

$$
\dim H_{p,q}(\text{trop}(X)) = \dim H^{p,q}(X, \mathbb{R}).
$$

(17)
For the definition of multiplicities we refer to [23, Chapter 3]. A tropical variety is locally linear (see e.g. [32]) if a Euclidean neighborhood of each point is isomorphic to a Euclidean open subset of the tropicalization of a linear subspace \( \mathbb{P}^n \subset \mathbb{P}^m \). For example, a hypersurface in \( \mathbb{P}^N \) is locally linear if and only if the subdivision of its Newton polygon is a triangulation. It has multiplicities one if and only if this triangulation is unimodular.

Note that \( X \) is not assumed to intersect the torus of \( \mathbb{P}^N \). Therefore this theorem applies for example to \( X \subset (\mathbb{P}^1)^3 \subset \mathbb{P}^7 \), see Section 5.1, or more generally to \( X \) in any projective toric variety with fixed embedding.

One could wonder whether Theorem 5.1 enables one to identify not only dimensions but homology classes. This is possible provided that a certain spectral sequence degenerates at the \( E_2 \) page. This \( E_2 \) page is equal to \( H^q(X, \mathcal{F}^p) \), where \( \mathcal{F}^p = \text{Hom}(\mathcal{F}_p, \mathbb{R}) \). See the discussion after Corollary 2 in [19] or [8].

For a more detailed introduction to tropical homology, see e.g. [7, 19]. We will now give generalities about tropical homology and compute some examples of interest. In particular, we will compute the dimensions of the tropical homology groups and show how Theorem 5.1 holds. The last part of the paper is dedicated to showing a particular instance of this theorem for a special tropical K3 with involution and for its quotient.

Recall that \( \text{trop}(\mathbb{P}^n) = \mathbb{T} \mathbb{P}^n \) is homeomorphic to an \( n \)-simplex, see [23, Chapter 6.2] and that it is covered by \( n+1 \) copies of

\[
\mathbb{T}^n = \text{trop}(\mathbb{A}^n) = (-\infty) \cup \mathbb{R}^n
\]

which are complements of torus invariant divisors. Let \( X \) be a tropical subvariety of \( \mathbb{T} \mathbb{P}^n \). The definitions of sheaves \( \mathcal{F}_p \) and groups \( C_{p,q} \) computing the homology are all local, so we assume that \( X \subset \text{trop}(\mathbb{A}^n) \) is contained in one of the distinguished open subsets. We denote by

\[
\mathbb{T}^J = \{ x \in \mathbb{T}^n \mid x_i = -\infty \text{ for all } i \notin J \}
\]

for \( J \subset \{1,\ldots,n\} \) the tropicalization of smaller torus orbits. Let now \( X \subset \mathbb{T}^n \) be a polyhedral complex. The sedentarity \( I(x) \) of a point \( x \in X \) is the set of coordinates of \( x \) which are equal to \(-\infty\), and we set \( J(x) := \{1,\ldots,n\} \setminus I(x) \). We denote by

\[
\mathbb{R}^{J(x)} = \mathbb{R}^n / \mathbb{R}^{I(x)}
\]

the interior of \( \mathbb{T}^{J(x)} \). For a face \( E \subset X \cap \mathbb{R}^{J(x)} \) adjacent to \( x \), we let \( T_x(E) \subset T_x(\mathbb{R}^{J(x)}) \) be the cone spanned by the tangent vectors to \( E \) starting at \( x \) and directed towards \( E \). Set the following:

1. The tropical tangent space \( \mathcal{F}_1(x) \subset T_x(\mathbb{R}^{J(x)}) \) is the vector space generated by all \( T_x(E) \) for all \( E \) adjacent faces to \( x \);
2. The tropical multitangent space \( \mathcal{F}_p(x) \subset \bigwedge^p T_x(\mathbb{R}^{J(x)}) \) is the vector space generated by all vectors of the form \( v_1 \wedge \ldots \wedge v_p \) for vectors \( v_1,\ldots,v_p \in T_x(E) \) for all \( E \) adjacent faces to \( x \) (this implies \( \mathcal{F}_0(x) \cong \mathbb{R} \))
One can show that the multitangent vector space $\mathcal{T}_p(x)$ for $x \in X$ only depends on the minimal face $\Delta \subset X$ containing $x$. Hence we can write $\mathcal{T}_p(\Delta) := \mathcal{T}_p(x)$ for each $x \in \Delta$. We have the following group of $(p,q)$-chains

$$C_{p,q}(X) := \bigoplus_{\Delta \subset X} \mathcal{T}_p(\Delta)$$

(21)

giving rise to the chain complex

$$C_{p,*} = \{ \cdots \rightarrow C_{p,q+1}(X) \xrightarrow{\partial} C_{p,q}(X) \xrightarrow{\partial} C_{p,q-1}(X) \rightarrow \cdots \}$$

(22)

where the differential $\partial$ is the usual simplicial differential (we choose orientation for each face) composed with inclusion maps given by $t : \mathcal{T}_p(\Delta) \rightarrow \mathcal{T}_p(\Delta')$ for $\Delta' \supset \Delta$, see examples below.

Note that even when $\Delta'$ and $\Delta$ have different sedentarities, we have $I(\Delta') \supset I(\Delta)$ so we get a natural map $\mathbb{R}^{I(x)} = \mathbb{R}^n/\mathbb{R}^{I(x)} \rightarrow \mathbb{R}^n/\mathbb{R}^{I(x')} = \mathbb{R}^{I(x')}$ inducing the map $\partial : \mathcal{T}_p(\Delta) \rightarrow \mathcal{T}_p(\Delta')$.

**Definition 5.2.** The $(p,q)$-th tropical homology group $H_{p,q}(X)$ of $X$ is the $q$-th homology group of the complex $C_{p,*}$.

In the light of Theorem 5.1 if $X = \text{trop}(X')$ is a tropicalization of suitable variety $X'$, then $\dim H_{p,q}(X)$ are the Hodge numbers of $X'$. For all $X$ the tropical Poincaré duality holds: $\dim H_{d-p,d-q}(X) = \dim H_{p,q}(X)$, see [20].

**Example 5.3 (Line).** Let us compute the tropical homology of a tropical line $L$, as in Figure 5.

**p = 0:** From the discussion above, one immediately sees that $C_{0,0}(L) = \mathbb{R}^4$ and $C_{0,1} = \mathbb{R}^3$ injects into $C_{0,0}$, thus $\dim H_{0,0}(L) = 1$ and $H_{0,1}(L) = 0$.

**p = 1:** The chain complex is $0 \rightarrow C_{1,1}(X) \rightarrow C_{1,0}(X) \rightarrow 0$. Now, the same consideration as in the previous item yields $C_{1,0}(X) = \mathcal{T}_1(v_1) = \mathbb{R}\langle e_1, e_2 \rangle$, where $e_1 = (-1,0)$ and $e_2 = (0,-1)$ are the standard basis vectors of $\mathbb{R}^2$ up to a sign. Moreover, one has that

$$C_{1,1}(X) = \mathcal{T}_1(p) \oplus \mathcal{T}_1(q) \oplus \mathcal{T}_1(r) = \mathbb{R}\langle e_1 \rangle \oplus \mathbb{R}\langle e_2 \rangle \oplus \mathbb{R}\langle -e_1 - e_2 \rangle.$$  

(23)

The differential

$$\mathbb{R}\langle e_1 \rangle \oplus \mathbb{R}\langle e_2 \rangle \oplus \mathbb{R}\langle -e_1 - e_2 \rangle \xrightarrow{\partial} \mathbb{R}\langle e_1, e_2 \rangle$$

(24)

is given by the natural inclusion $e_1 \mapsto e_1$, $e_2 \mapsto e_2$, and $-e_1 - e_2 \mapsto -e_1 - e_2$. Hence the kernel of the differential above is one dimensional, generated by the sum $\langle e_1 \rangle + \langle e_2 \rangle + \langle -e_1 - e_2 \rangle$. So one has $\dim H_{1,0}(X) = 0$ and $\dim H_{1,1}(X) = 1$.

**Remark 5.4.** By definition $\mathcal{T}_0(x) = \mathbb{R}$. Thus the complex $C_{0,*}$ is in fact the singular homology complex for the subdivision of $X$ by polyhedra. Therefore the tropical
homology group $H_{0,q}(X)$ is canonically identified with the singular homology group $H_q(X, \mathbb{R})$.

**Example 5.5 (Elliptic curve).** The next example of tropical homology we compute is that of an elliptic curve in $\mathbb{P}^1 \times \mathbb{P}^1$. Its tropicalization is shown in Figure 5. By the isomorphism $H_{0,q}(X) \cong H_q(X, \mathbb{R})$, it immediately follows that

$$H_{0,0}(X) \cong \mathbb{R} \text{ and } H_{0,1}(X) \cong \mathbb{R}.$$  

We can compute $H_{1,1}(X)$ directly from the complex

$$C_{1,1}(X) \to C_{1,0}(X).$$  (25)

We get that $C_{1,1}(X) \cong \mathbb{R}^E$ and $C_{1,0}(X) \cong \mathbb{R}^{2V}$, where $E = 16$ (respectively, $V = 8$) denotes the number of edges (respectively, of interior vertices). The kernel of the map $C_{1,1}(X) \to C_{1,0}(X)$ is $H_{1,1}(X) \cong \mathbb{R}$ generated by the boundary of the square, hence $H_{1,1}(X) \cong \mathbb{R}$. 

Fig. 4 A tropical line

Fig. 5 A tropical elliptic curve in $\mathbb{P}^1 \times \mathbb{P}^1$
5.1 Del Pezzo in \((\mathbb{P}^1)^3\)

Consider a surface \(S\) in \(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\) which is a section of \(\mathcal{O}(1,1,1) := \mathcal{O}(1) \boxtimes \mathcal{O}(1) \boxtimes \mathcal{O}(1)\); this is a del Pezzo surface, its anticanonical divisor is by adjunction the restriction of \(\mathcal{O}(1,1,1)\) and so the anticanonical degree is 6. The equation \(F\) of \(S\) can be written as

\[
F = \sum_{0 \leq i,j,k \leq 1} a_{ijk} x^i y^j z^k, \tag{26}
\]

where \(x, y, z\) are local coordinates on respective projective lines. Suppose that we are over a valued field. Suppose further that \(a_{ijk} = a_{1-i,j,1-k}\) for all indices and that \(a_{1,0,0} > \max(a_{0,1,0}, a_{0,0,1})\). Then the induced subdivision of a cube is regular, as seen in Figure 7.

![Figure 6](image1.png) A tropicalization of \(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\) with the sedentarities of the faces at infinity

![Figure 7](image2.png) Regular subdivision of the cube and tropical del Pezzo

Directly from the picture we see that there are 6 points, 18 edges and 19 faces in the non-sedentary part of \(\text{trop}(S)\). Consider now the sedentary part. First recall that \(\text{trop}((\mathbb{P}^1)^3) \simeq (\mathbb{R} \cup \{\pm \infty\})^3\) is homeomorphic to the cube, see Figure 6. Its faces correspond to torus-invariant divisors in \((\mathbb{P}^1)^3\). The boundary \(\text{trop}(S) \setminus \mathbb{R}^3\) decomposes into 6 components, the intersections of \(\text{trop}(S)\) with those faces. We now make use of the following

**Theorem 5.6 ([23, Theorem 6.2.18])**. Let \(Y \subset T\) and let \(\bar{Y}\) be the closure of \(Y\) in a toric variety \(X\). Then \(\text{trop}(\bar{Y})\) is the closure of \(\text{trop}(Y)\) in \(\text{trop}(X)\).

Applying Theorem 5.6 to \(Y = S\) we derive that the boundary of the tropicalization is the tropicalization of the boundary, so we have

\[
\text{trop}(S) \cap \text{trop}(D) = \text{trop}(S \cap D) \tag{27}
\]

for each torus-invariant divisor. Such \(D\) is one of the divisors defined by \(x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\). Without loss of generality, assume \(D = (x = 0)\). By restricting the element \(F\) of (26) to \(D\) we get \(\sum_{0 \leq i,j,k \leq 1} a_{0jk} y^j z^k\), which cut out a quadric, whose tropicalization
is given in Figure 8. In particular, it has five edges, two mobile points and four sedentary points.

In total we get the following strata.

| Tropical del Pezzo | sedentarity | 0 | 1 | 2 |
|---------------------|-------------|---|---|---|
| points              | 6 | 12 | 12 |
| edges               | 18| 30 |  — |
| faces               | 19| —  | —  |

This information enables us to immediately compute the $C_{P,Q}$ even without analysing maps. This is because our del Pezzo is locally linear: near each vertex the tropical structure looks like the tropicalization of $\mathbb{P}^2 \subset \mathbb{P}^3$, as shown in Figure 3. The complexes are

1. $C_{0,2} = \mathbb{R}^{19} \rightarrow C_{0,1} = \mathbb{R}^{18} \oplus \mathbb{R}^{30} \rightarrow C_{0,0} = \mathbb{R}^{30}$
2. $C_{1,2} = \mathbb{R}^{219} \rightarrow C_{1,1} = \mathbb{R}^{318} \oplus \mathbb{R}^{30} \rightarrow C_{1,0} = \mathbb{R}^{36} \oplus \mathbb{R}^{212}$
3. $C_{2,2} = \mathbb{R}^{19} \rightarrow C_{2,1} = \mathbb{R}^{218} \rightarrow C_{2,0} = \mathbb{R}^{36}$.

By comparing $H_0$, with singular homology and then using Poincaré duality we get immediately that

$$H_{0,0} \simeq H_{2,2} \simeq \mathbb{R}, \quad H_{0,1} = H_{0,2} = H_{2,0} = H_{2,1} = 0.$$

So the interesting part is the homology of $C_{1,\bullet}$. It is not impossible to compute it by hand, however it would take a lot of space to explain it properly, so we merely present a series of reductions, by removing strata corresponding to higher sedentarity first. Each of these reductions corresponds to finding an exact subcomplex $D \subset C_{1,\bullet}$ and reducing to computing homology of $C_{1,\bullet}/D$.

Consider a sedentary point $p$ on the face of a cube. This point has two edges $e_1, e_2$ going towards the boundary of this face (and a third edge, which is irrelevant here). In $C_{1,\bullet}$ these polyhedra give a subcomplex $\mathbb{R}[e_1] \oplus \mathbb{R}[e_2] \rightarrow \mathbb{R}^2[p]$, which is exact. Thus the homology of $C_{1,\bullet}$ is the homology of the quotient $C_{\bullet}$ by all these subcomplexes for 12 choices of $p$. The quotient is
\[ \mathbb{R}^{2 \cdot 19} \to \mathbb{R}^{3 \cdot 18} \oplus \mathbb{R}^6 \to \mathbb{R}^{3 \cdot 6}. \]  

(32)

Next, consider the picture on Figure 7, consider one of the two corner vertices and all its adjacent faces (3 edges, 3 faces, 1 simplex). In the tropical variety those correspond to 1 point \( p \), 3 edges \( e_i \) and 3 faces \( f_i \) and glue together to form on tropical \( \mathbb{A}^2 \). Such an \( \mathbb{A}^2 \) has no higher homology and correspondingly, the sequence

\[ \bigoplus \mathbb{R}^2[f_i] \to \bigoplus \mathbb{R}^3[e_i] \to \bigoplus \mathbb{R}^3[p] \]  

is exact. Moreover it is a subcomplex of \( C' \). Dividing \( C' \) by two subcomplexes given by two corner vertices, we get \( C'' \) equal to

\[ \mathbb{R}^{2 \cdot 13} \to \mathbb{R}^{3 \cdot 12} \oplus \mathbb{R}^6 \to \mathbb{R}^{3 \cdot 4}. \]  

(34)

In this new sequence the \( \mathbb{R}^{3 \cdot 4} \) corresponds to 4 multitangent spaces at four vertices of the square in the interior, see Figure 7. None of the edges adjacent to them was modified in the process thus it is clear that the right map is surjective. Hence \( H_{1,0} = 0 \). By Poincaré duality we get \( H_{1,2} = 0 \) and thus

\[ \dim H_{1,1} = 36 + 6 - 26 - 12 = 4, \]  

(35)

as expected from the Hodge diamond of a del Pezzo of anticanonical degree 6.

5.2 A K3 surface in \( (\mathbb{P}^1)^3 \)

Let \( S \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) be a K3 surface over a valued field \( k \) as in Example 4.3. This section discusses its tropical homology and relations to its Hodge classes; shortly speaking using tropical homology we recover the expected Hodge numbers and an anti-symplectic involution.

Before we begin computations on the homology, let us discuss how many points, edges and faces the polyhedral decomposition of the tropicalization has. This polyhedral decomposition is dual to the subdivision induced on the \( 2 \times 2 \times 2 \) cube by the coefficients of \( S \), as explained in [23, Definition 2.3.8, Figure 1.3.3]. See Figure 3, where the bounded part of \( \text{trop}(S) \) is presented.

Let us restrict to the torus and consider polyhedra with empty sedentarity. First and foremost, \( \text{trop}(S) \) comes from a regular subdivision into 48 simplices, so it has 48 distinguished points. Consider now faces of the subdivision (edges in the tropicalization). Each face may be either “inner”, shared by two tetrahedra or “outer” adjacent to only one of them. There are 48 outer faces and each tetrahedron has four faces, thus in total there are \( \frac{48 \cdot 4 + 48}{2} = 120 \) faces in the subdivision. As seen in the del Pezzo case, there are 19 edges in a subdivision of a unit cube. In the \( 2 \times 2 \times 2 \) cube we have 8⋅19 of those segments; 36 of them are adjacent to exactly two cubes, 6 of them are adjacent to four cubes and the others stick to one cube. Thus there are \( 8 \cdot 19 - 36 - 3 \cdot 6 = 98 \) segments.
Recall that \( \text{trop}(\mathbb{P}^1)^3 = \mathbb{R}^3 \), where \( \mathbb{R} = \mathbb{R} \cup \{ \pm \infty \} \), this is homeomorphic to a cube, see Figure 3. The boundary of \( \text{trop}(S) \) is the intersection of \( \text{trop}(S) \) with the boundary of this cube. Pick a face \( \mathcal{F} \) of the cube. It is the tropicalization of one of the six toric divisors \( x_i^{\pm 1} \) for \( i = 1, 2, 3 \), say to \( x_1 \). Again utilizing Theorem 5.6 we have

\[
\text{trop}(S) \cap \mathcal{F} = \text{trop}(S \cap (x_1 = 0)).
\]

But \( S \cap (x_1 = 0) \) is an elliptic curve in \( \mathbb{P}^1 \times \mathbb{P}^1 \) and by Section 5.5 we know that its tropicalization has 16 edges, 8 mobile points and 8 sedentary points, see Figure 5. In total we have the following strata.

| Tropical K3 | | sedentarity |
|-------------|---|---|
| points      | 48 | 48 | 24 |
| edges       | 120| 96 |    |
| faces       | 98 |    |    |

This information enables us to immediately compute the \( C_{p,q} \) even without analysing maps. This is because \( S \) is locally linear: near each vertex the tropical structure looks like the tropicalization of \( \mathbb{P}^2 \subset \mathbb{P}^3 \) see Figure 2 and compare in Figure 3.

The complexes are

\[
C_{0,2} = \mathbb{R}^{98} \rightarrow C_{0,1} = \mathbb{R}^{120} \oplus \mathbb{R}^{96} \rightarrow C_{0,0} = \mathbb{R}^{120}
\]

\[
C_{1,2} = \mathbb{R}^{2 \cdot 98} \rightarrow C_{1,1} = \mathbb{R}^{3 \cdot 120} \oplus \mathbb{R}^{96} \rightarrow C_{1,0} = \mathbb{R}^{3 \cdot 48} \oplus \mathbb{R}^{2 \cdot 48}
\]

\[
C_{2,2} = \mathbb{R}^{98} \rightarrow C_{2,1} = \mathbb{R}^{2 \cdot 120} \rightarrow C_{2,0} = \mathbb{R}^{3 \cdot 48}.
\]

From this fact alone we see that \( \chi(C_{1,\bullet}) = 2 \cdot 98 - 3 \cdot 120 - 96 + 3 \cdot 48 + 2 \cdot 48 = -20 \) in concordance with the expected result. Moreover one can show that \( H_{1,0} = 0 \), roughly because the classes of sedentary edges surject to classes of sedentary points and other points can be analyzed directly by Figure 5. By Poincaré duality, \( H_{1,2} = 0 \).

We have now

\[
-20 = \chi(C_{1,\bullet}) = \dim H_{1,0} - \dim H_{1,1} + \dim H_{1,2} = - \dim H_{1,1},
\]

thus \( \dim H_{1,1} = 20 \).

We will now consider \((0,q)\)-classes. The homology of \( C_{0,\bullet} \) is just the singular homology of the tropical variety by Remark 5.4. The tropical variety is contractible to the boundary of the cube. Thus \( C_{0,\bullet} \) is exact in the middle and its homology groups are the homology groups of the sphere:

\[
H_{0,0} \simeq \mathbb{R}, \quad H_{0,1} = 0, \quad H_{0,2} = \mathbb{R}.
\]

We have just given an explicit proof of our special case of Theorem 5.1.

**Proposition 5.7.** The tropical Hodge numbers of \( \text{trop}(S) \) agree with the Hodge numbers of \( S \).
We expose an explicit generator of $H_{0,2}$ and analyze the action of $\sigma$ on this space. Briefly speaking, this class is obtained as the boundary of the interior of the cube.

To expand this, consider the boundary of the cube and the complex $C_2' \to C_1' \to C_0'$ computing its singular homology. This boundary can be embedded into a full cube and the complex $C'$ becomes part of the complex $C''$ computing the homology of the cube

$$0 \to C_3'' \to C_2'' \to C_1'' \to C_0''$$

(42)

Since the cube is contractible, the complex $C''$ is exact. Hence the unique class $\omega$ in $H^2(C')$ is the boundary of the class $\Omega$ in $C_3''$. Consider now the action of $\sigma$ on the $\mathbb{R}^3$ containing the tropical variety. We have $\sigma(x) = -x$ in $\mathbb{R}^3$, thus $\sigma$ changes orientation, hence $\sigma(\Omega) = -\Omega$, thus it is anti-symplectic:

$$\sigma(\omega) = \sigma(\partial \Omega) = -\omega.$$  

(43)

This is expected, since otherwise $\omega$ would descend to a class in the tropical homology of the tropicalized Enriques surface and give a non-zero $(0,2)$ class on this surface.

Finally we investigate $\sigma$-invariant part $C_p^{\sigma,\bullet}$ of the complexes $C_{p,\bullet}$. Since we work over characteristic different from two, the functor $(-)^{\sigma}$ is exact and so the homology of $C_{p,\bullet}^{\sigma}$ is the invariant part of the homology of $C_{p,\bullet}$. Moreover, by trop$(S)/\sigma$ is a tropical manifold and $C_{p,\bullet}^{\sigma}$ computes its tropical homology, see [7, Chapter 7]. In particular the homology groups $H^q_{p,q} = H^q(C_{p,\bullet}^{\sigma})$ satisfy $H^q_{p,q} = H^q_{2-p,2-q}$. We believe, though have not proved this formally here, that the manifold trop$(S)/\sigma$ is a tropicalization of the Enriques surface $S/\sigma$. Taking this belief for granted, the homology classes of $C_{p,\bullet}^{\sigma}$ compute the tropical homology of Enriques surface $S/\sigma$.

Let us proceed to the computation. First, it is straightforward to compute the dimensions of $C_{p,q}^{\sigma}$, since trop$(S)$ does not contain the origin. Hence every face $F$ of trop$(S)$ gets mapped by $\sigma$ to a unique other face $F'$ so that the action of $\sigma$ on the space spanned by $[F]$ and $[F']$ always decomposes into invariant subspace $[F] + [F']$ and anti-invariant space $[F] - [F']$. Therefore $\dim C_{p,q}^{\sigma} = \frac{1}{2} \cdot \dim C_{p,q}$ for all $p,q$ and the sequences are

$$
\begin{align*}
C_{0,2}^{\sigma} &\cong \mathbb{R}^{49} \to C_{0,1}^{\sigma} = \mathbb{R}^{60} \oplus \mathbb{R}^{48} \to C_{0,0}^{\sigma} = \mathbb{R}^{60} \\
C_{1,2}^{\sigma} &\cong \mathbb{R}^{49} \to C_{1,1}^{\sigma} = \mathbb{R}^{360} \oplus \mathbb{R}^{48} \to C_{1,0}^{\sigma} = \mathbb{R}^{324} \oplus \mathbb{R}^{224} \\
C_{2,2}^{\sigma} &\cong \mathbb{R}^{49} \to C_{2,1}^{\sigma} = \mathbb{R}^{260} \to C_{2,0}^{\sigma} = \mathbb{R}^{324}.
\end{align*}

(44)

(45)

(46)

The key result is already computed in (43) when considering $(0,q)$-classes: the generator $\omega$ of $H_{0,2}$ does not lie in $H_{0,2}^{\sigma}$. Thus $H_{0,2}^{\sigma} = H_{0,1}^{\sigma} = 0$ and $H_{0,0}^{\sigma} \cong \mathbb{R}$. By symmetry $H_{2,0}^{\sigma} = H_{2,1}^{\sigma} = 0$ and $H_{2,2}^{\sigma} \cong \mathbb{R}$. Finally

$$-\dim H_{1,1} = \dim H_{1,0} - \dim H_{1,1} + \dim H_{1,2} = -\chi(C_{1,\bullet}^{\sigma}) = \frac{1}{2} \chi(C_{1,\bullet}) = -10,$$

as in Equation (40). We obtain the following counterpart of Proposition 5.7.
Proposition 5.8. The dimensions of $\sigma$-invariant parts of tropical homology groups of $S$ agree with the Hodge numbers of $S/\sigma$. \hfill $\Box$

6 Topology of analytifications of Enriques surfaces

In this section we analyze the analytification of an Enriques surface which is the quotient of the K3 surface from Example 4.3. Fix a valued field $k$ and a K3 surface $S \subset (\mathbb{P}^1)^3$ over $k$ together with an involution $\sigma : S \to S$, as in Example 4.3. We first analyze the topology of $S$ itself.

Proposition 6.1. The topological space $S^{\text{an}}$ has a strong deformation retraction onto a two-dimensional sphere $C$. More precisely, there exist continuous maps $s : C \to S^{\text{an}}$ and $e : S^{\text{an}} \to C$, so that $es = \text{id}_C$ and $se$ is homotopic to $\text{id}_{S^{\text{an}}}$. The maps $s$ and $e$ may be chosen to be $\sigma$-equivariant, where $\sigma$ acts on $C$ antipodally.

Proof. In Section 5.2 we consider the tropicalization $\text{trop}(S) \subset (\mathbb{R} \cup \{\pm \infty\})^3$ with the antipodal involution $\text{trop}(\sigma)$. We shorten $\text{trop}(\sigma)$ to $\sigma$. There is a cube $C \subset \text{trop}(S)$ fixed under the involution, see Figure 3. This cube is a strong deformation retract of $\text{trop}(S)$ and the retraction can be chosen to be $\sigma$-equivariant. In the following we identify $C$ with a two-dimensional sphere.

It remains to prove that the tropical variety $\text{trop}(S)$ is a strong deformation retract of $S^{\text{an}}$ under the map $\pi : S^{\text{an}} \to \text{trop}(S)$ defined in Equation (13). We note that $\text{trop}(S)$ is schön, i.e. its intersection with every torus orbit is smooth ([23, Definition 6.4.19]). Moreover all multiplicities of top degree polyhedra are equal to one, hence the multiplicity at each point is equal to one by semicontinuity, see [23, Lemma 3.3.6]. Therefore $\pi$ has a section $\text{trop}(S) \to S^{\text{an}}$ whose image is equal to a skeleton $S(\mathcal{F}, H)$ of a suitable semistable model $(\mathcal{F}, H)$ of $S$, see [17, Remark 9.12]. The skeleton $S(\mathcal{F}, H)$ is a proper strong deformation retract of $S^{\text{an}}$ by [18, §4.9]. The retraction map $S^{\text{an}} \to \text{trop}(S)$ is equal to $\pi$, hence $\sigma$-equivariant as discussed in Example 4.3. The retraction $s$ in the claim of the theorem is the composition of retractions from $S^{\text{an}}$ to $\text{trop}(S)$ and from $\text{trop}(S)$ to the cube constructed above. \hfill $\Box$

Corollary 6.2. The analytified K3 surface $S^{\text{an}}$ is homotopy equivalent to a two-dimensional sphere. \hfill $\Box$

Remark 6.3. From Proposition 6.1 it does not follow that the homotopy between $se$ and $\text{id}_{S^{\text{an}}}$ can be chosen $\sigma$-equivariantly. This is most likely true, but presently there seems to be no reference for this fact.

Now we will analyze the topology of the analytification of the Enriques surface $S/\sigma$ using our knowledge about $S^{\text{an}}$. Let us briefly recall the maps which we will use. The quotient map $q : S \to S/\sigma$ analytifies to $q^{\text{an}} : S^{\text{an}} \to (S/\sigma)^{\text{an}}$. For any $X$ we denote $\pi : X^{\text{an}} \to X$ the natural map. Summarizing, we consider the following diagram.
It is crucial that $q^{an}$ is a quotient by $\sigma^{an}$, as we now prove.

**Proposition 6.4.** We have $(S/\sigma)^{an} = S^{an}/\sigma^{an}$ as topological spaces.

**Proof.** First we prove the equality of sets

$$(S/\sigma)^{an} = S^{an}/\sigma^{an}. \quad (49)$$

Consider $x \in (S/\sigma)^{an}$ and its image $\pi(x) \in S/\sigma$. We first describe the fiber $S^{an}$ of $q^{an}$ over $x$. Let $U = \text{Spec} A$ be an affine neighborhood of $\pi(x)$, then the point $x$ corresponds to a semi-norm $|\cdot|_x$ on $A$ and $\pi(x)$ corresponds to the prime ideal $p_x = \{ f \in A : |f|_x = 0 \}$, see [5, Remark 1.2.2]. By $\mathcal{H}(x)$ we denote the completion of the fraction field of $A/p_x = \kappa(\pi(x))$. We have the following equality [5, pg. 65] of fibers

$$S^{an}_x = (S_x \times_{\kappa(\pi(x))} \mathcal{H}(x))^{an}. \quad (50)$$

In down-to-earth terms, the set $S^{an}_x$ consists of multiplicative seminorms on the $\mathcal{H}(x)$-algebra $R = H^0(S_x, O_{S_x}) \otimes_{\kappa(\pi(x))} \mathcal{H}(x)$ which extend the norm $|\cdot|_x$ on $\mathcal{H}(x)$. Using [5, Proposition 1.3.5] we may assume $\mathcal{H}(x)$ is algebraically closed. Since $H^0(S_x, O_{S_x})^{an} = \kappa(\pi(x))$, we have $R^{an} = \mathcal{H}(x)$. Similarly, the ring $R$ is a rank two free $\mathcal{H}(x)$-module. Then $R$ is isomorphic to either $\mathcal{H}(x)^{\times 2}$ with $\sigma$ permuting the coordinates or to $\mathcal{H}(x)[\varepsilon]/\varepsilon^2$. Consider a multiplicative seminorm $|\cdot|_y$ on $R$. Its kernel $q = \{ f \in R : |f|_y = 0 \}$ is a prime ideal in $R$ and in both cases above we have $R/q = \mathcal{H}(x)$. Since $|\cdot|_y$ agrees with $|\cdot|_x$ on $\mathcal{H}(x)$, we see that $|\cdot|_y$ is determined uniquely by its kernel. The involution $\sigma$ acts transitively on those, hence $\sigma^{an}$ acts transitively on the set $S^{an}_x$ and equality (49) is proven.

Second, we prove that $(S/\sigma)^{an} = S^{an}/\sigma^{an}$ as topological spaces; in other words that the topology on $(S/\sigma)^{an}$ is induced from this of $S^{an}$. Take an open subset $U \subset S^{an}$. We want to show that $q^{an}(U)$ is open. Clearly $U \cup \sigma^{an}(U) \subset S^{an}$ is open and a union of fibers, so its complement $Z \subset S^{an}$ is closed and a union of fibers. Now the map $q^{an}$ is finite [5, 3.4.7], thus proper and so closed [5, 3.3.6]. In particular $q^{an}(Z) \subset (S/\sigma)^{an}$ is closed, so $\pi(U) = (S/\sigma)^{an} \setminus q^{an}(Z)$ is open. This proves that $(S/\sigma)^{an} = S^{an}/\sigma^{an}$ as topological spaces. \( \square \)

**Corollary 6.5.** There exists a retraction from $(S/\sigma)^{an}$ onto $\mathbb{RP}^2$. In particular $(S/\sigma)^{an}$ is not contractible.

**Proof.** The argument follows formally from Proposition 6.1 and Proposition 6.4. Recall from Proposition 6.1 the $\sigma$-invariant map $e : S^{an} \to C$ and its section $s : C \to S^{an}$. Here $C$ is a two dimensional sphere with an antipodal involution $\sigma$ and clearly $C/\sigma \simeq \mathbb{RP}^2$. We now produce equivalents of $s$ and $e$ on the level of $S^{an}/\sigma^{an} \simeq (S/\sigma)^{an}$.\}
\[
\begin{array}{c}
S^m \xrightarrow{\sigma^m} S^m \xrightarrow{q^m} (S/\sigma)^m \\
\downarrow e \quad \downarrow e \quad \downarrow e \\
C \xrightarrow{\text{trop}(\sigma)} C \xrightarrow{q} C/\sigma = \mathbb{RP}^2
\end{array}
\]

(51)

The map \( qe : S^m \to s(C)/\sigma = \mathbb{RP}^2 \) is such that \( qe \circ \sigma^m = qe \), thus by definition of quotient and by Proposition 6.4 it induces a unique map \( e : S^m/\sigma^m = (S/\sigma)^m \to \mathbb{RP}^2 \). Similarly \( q^m \circ s \) satisfies \( q^m \circ s \circ \text{trop}(\sigma) = q^m \circ s \), hence induces a unique map \( s : \mathbb{RP}^2 \to (S/\sigma)^m \). Then \( e \circ s : \mathbb{RP}^2 \to \mathbb{RP}^2 \) is the unique map induced \( \sigma \)-invariant map \( qes = q \). Therefore \( e \circ s = \text{id}_{\mathbb{RP}^2} \) and so \( se \) is a retraction of \( (S/\sigma)^m \) onto \( s(\mathbb{RP}^2) \cong \mathbb{RP}^2 \).

\( \square \)

Remark 6.6. If the difficulty presented in Remark 6.3 was removed, a similar argument would show that \( (S/\sigma)^m \) strongly deformation retracts onto \( \mathbb{RP}^2 \).

Proof (of Theorem 1.3). Follows from Proposition 6.1 and Corollary 6.5. \( \square \)

Conclusion

We constructed an explicit Enriques surface as the quotient \( S/\sigma \) by an involution \( \sigma \) on a K3 surface \( S \). We then tropicalized \( S \) and considered the quotient \( \text{trop}(S)/\sigma \) under the tropicalized involution. We computed in Proposition 5.8 the tropical homology of this quotient and obtained expected results. We have not proved that \( \text{trop}(S)/\sigma = \text{trop}(S/\sigma) \). However, we obtained this equality on the level of analytic spaces: in Proposition 6.4 we proved that \( S^m/\sigma^m = (S/\sigma)^m \) and concluded that \( (S/\sigma)^m \) retracts onto \( \mathbb{RP}^2 \subset (S/\sigma)^m \).

Acknowledgements

This article was initiated during the Apprenticeship Weeks (22 August-2 September 2016), led by Bernd Sturmfels, as part of the Combinatorial Algebraic Geometry Semester at the Fields Institute for Research in Mathematical Sciences. We thank Kristin Shaw for many helpful conversations and for suggesting Example 4.3. We thank Christian Liedtke for many useful remarks and suggesting Proposition 3.1. We thank Julie Rana for discussions and providing the sources for the introduction. We thank Walter Gubler, Joseph Rabinoff and Annette Werner for sharing their insights. We also thank Bernd Sturmfels and the anonymous referees for providing many interesting suggestions and giving deep feedback. BB was supported by the Fields Institute for Research in Mathematical Sciences. CH was supported by the Fields Institute for Research in Mathematical Sciences and by the Clay Mathematics Institute and by NSA award H98230-16-1-0016. JJ was supported by the Polish National Science Center, project 2014/13/N/ST1/02640.

References

1. Wolf P. Barth and Michael E. Larsen. On the homotopy groups of complex projective algebraic manifolds. Math. Scand., 30:88–94, 1972.
2. Wolf P. Barth, Chris A. M. Peters, and Antonius Van de Ven. *Compact complex surfaces*. Springer-Verlag Berlin Heidelberg, 2 edition, 2004.
3. Arnaud Beauville. *Complex algebraic surfaces*, volume 34 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, second edition, 1996. Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid.
4. Mauro C. Beltrametti and Andrew J. Sommese. *The adjunction theory of complex projective varieties*, volume 16 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1995.
5. Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.
6. Morgan Brown and Tyler Foster. Rational connectivity and Analytic Contractibility. *arXiv: 1406.7312 [math.AG]*, 2016.
7. Erwan Brugallé, Ilia Itenberg, Grigory Mikhalkin, and Kristin Shaw. Brief introduction to tropical geometry. In *Proceedings of the Gökova Geometry-Topology Conference 2014*, pages 1–75. Gökova Geometry/Topology Conference (GGT), Gökova, 2015.
8. Charles Herbert Clemens. Degeneration of Kähler manifolds. *Duke Math. J.*, 44(2):215–290, 1977.
9. François R. Cossec and Igor V. Dolgachev. *Enriques Surfaces I*, volume 76 of *Progress in Mathematics*. Birkhäuser, 1989.
10. Igor V. Dolgachev. A brief introduction to Enriques surfaces. *ArXiv e-prints*, 2014.
11. Federigo Enriques. Introduzione alla geometria sopra le superficie algebriche. *Mem. Soc Ital. delle Scienze*, (10):1–81, 1896.
12. Federigo Enriques. Un’ osservazione relativa alle superficie di bigenera uno. *Rend. Acad. Sci. Torino*, 50:1–79, 1901.
13. William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.
14. Theodor Reye. *Die Geometrie Der Lage*, volume 2. Hannover, C. Rümpler.
15. Hal Schenck. *Computational algebraic geometry*, volume 58 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2003.
28. Jessica Sidman and Gregory G. Smith. Linear determinantal equations for all projective schemes. *Algebra Number Theory*, 5(8):1041–1061, 2011.

29. Bernd Sturmfels. Fitness, Apprenticeship, and Polynomials. in *Combinatorial Algebraic Geometry* (eds. G.G. Smith and B. Sturmfels), to appear, 2016.

30. Ravi Vakil. The rising sea: Foundations of algebraic geometry notes. a book in preparation, December 29, 2015 version. [http://math.stanford.edu/~vakil/216blog/](http://math.stanford.edu/~vakil/216blog/), 2015.

31. Alessandro Verra. The étale double covering of an Enriques surface. *Rend. Sem. Mat. Univ. Politec. Torino*, 41(3):131–167 (1984), 1983.

32. Magnus Dehli Vigeland. *Topics in Elementary Tropical Geometry*. PhD thesis, Universitetet i Oslo, 2008.