SUBMM RECOMBINATION LINES IN DUST-OBSCURED STARBURSTS AND AGN

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ABSTRACT

We examine the use of submm recombination lines of H, He and He\textsuperscript{+} to probe the extreme ultraviolet (EUV) luminosity of starbursts (SB) and AGN. We find that the submm recombination lines of H, He and He\textsuperscript{+} are in fact extremely reliable and quantitative probes of the EUV continuum at 13.6 eV to above 54.6 eV. At submm wavelengths, the recombination lines originate from low energy levels (n = 20 – 50). The maser amplification, which poses significant problems for quantitative interpretation of the higher n, radio frequency recombination lines, is insignificant. Lastly, at submm wavelengths the dust extinction is minimal. The submm line luminosities are therefore directly proportional to the emission measures (EM\textsubscript{ION} = n\textsubscript{e} × n\textsubscript{ion} × volume) of their ionized regions. We also find that the expected line fluxes are detectable with ALMA and can be imaged at ~ 0.1\arcsec resolution in low redshift ULIRGs. Imaging of the HI lines will provide accurate spatial and kinematic mapping of the star formation distribution in low-z IR-luminous galaxies. And the relative fluxes of the HI and HeII recombination lines will strongly constrain the relative contributions of starbursts and AGN to the luminosity. The HI lines should also provide an avenue to constraining the submm dust extinction curve.

Keywords: galaxies: active — galaxies: nuclei — galaxies: starburst — ISM: lines and bands

1. INTRODUCTION

The most energetic periods of evolution in galaxies are often highly obscured by dust at short wavelengths, with the luminosity reradiated in the far infrared. Merging of galaxies will concentrate the interstellar gas and dust (ISM) in the nucleus since the gas is very dissipative where it can fuel a nuclear starburst or AGN. The UltraLuminous Infrared Galaxies (ULIRGs) and Submm Galaxies (SMGs) emit nearly all their radiation in the far infrared [Sanders et al. 1988; Carilli & Walter 2013]. Although their power originates as visible, UV and X-ray photons, the emergent IR continuum only weakly differentiates the power source(s) – starburst or AGN – and their relative contributions. This is a significant obstacle to understanding the evolution of the nuclear activity since the star formation and AGN heating may occur at different stages and with varying rates for each. Many of the signatures of star formation or black hole activity (e.g. X-ray, radio or optical emission lines) can be indicative that starbursts or AGN are present but provide little quantitative assessment of their relative contributions or importance (a summary of the various SFR indicators is provided in Murphy et al. 2011).

In this paper we develop the theoretical basis for using the submm recombination lines of H, He and He\textsuperscript{+} to probe star formation and AGN. We find that the emissivities of these lines can provide reliable estimates of the EUV luminosities from 13.6 eV to ~\texttimes 10^{2} eV and hence the relative luminosities associated with star formation (EUV near the Lyman limit) and AGN accretion (harder EUV).

Although the extremely high infrared luminosities of ULIRGs like Arp 220 and Mrk231 are believed to be powered by starburst and AGN activity, the distribution of star formation and the relative contributions of AGN accretion is very poorly constrained. This is due to inadequate angular resolution in the infrared and the enormous and spatially variable extinctions in the visible (A\textsubscript{V} \sim 500 – 2000 mag). The submm lines will have minimal dust extinction attenuation. And, given the large number of recombination lines across the submm band, lines of the different species may be found which are close in wavelength and provide the capability to move to longer wavelengths to further reduce the dust opacity (in the most opaque sources). Although mid-IR fine structure transitions of heavy ions have been used in some heavily obscured galaxies, the line ratios depend on density, temperature and metallicity; in contrast, the H and He\textsuperscript{+} lines have none of these complications. Lastly, we find that the expected fluxes in the lines are quite readily detectable with ALMA.

In the following, we first derive the emissivities and line opacities for the submm recombination lines as a function of density and temperature (§2). Then using simplified models for the ionizing continuum associated with OB stars and with AGN, we derive the relative emission measures of the H\textsuperscript{+}, He\textsuperscript{+} and He\textsuperscript{++} regions for these two EUV radiation fields (§3). Lastly, we compare the expected line fluxes in HI and HeII with the sensitivity of ALMA and find that the lines should be readily detectable from ULIRG nuclei at low z. The observations of these lines can therefore provide the first truly quantitative assessment of the relative contributions of starbursts and AGN to the luminosity of individual objects.

2. SUBMM RECOMBINATION LINES

The low-n HI recombination lines at mm/submm wavelengths trace the emission-measure of the ionized gas and hence the Lyman continuum production rate as-
also optically thin, their line fluxes are a linear tracer of the submm HI lines (and the free-free continuum) are associated with high mass stars and AGN. In contrast to the mm/cm-wave radio HI recombination lines which can have substantial maser amplification (Brown et al. 1978; Gordon & Walmsley 1990; Puxley et al. 1997), the submm recombination emission is predominantly spontaneous emission with relatively little stimulated emission and associated non-linear amplification (see §2.4). Since the submm HI lines (and the free-free continuum) are also optically thin, their line fluxes are a linear tracer of the ionized gas emission measure (EM = \( n_e n_p \) vol). Hence these lines are an excellent probe of the EUV luminosity of OB stars and AGN (assuming the EUV photons are not appreciably absorbed by dust). Lastly, we note that in virtually all sources, the dust extinction of the recombination lines at \( \lambda \sim 350\mu m \) to 1mm will be insignificant.

Early observations of the mm-wave recombination lines were made in Galactic compact HII – in these regions the continuum is entirely free-free and hence one expects fairly constant line-to-continuum flux ratio if the mm-line emission arises from spontaneous decay in high density gas with little stimulated emission contribution. This is indeed the case – Gordon & Walmsley (1990) observed the H40\( \alpha \) line at 99 GHz in 7 HII regions and found a mean ratio for the integrated-line brightness (in K km s\(^{-1}\)) to continuum of 31.6 (K km s\(^{-1}\)/K). Less than 3% variation in the ratio is seen across the sample. The optically thin free-free emission provides a linear probe of the HI region emission measure (EM) and hence the OB star Lyman continuum production rate. The observed constancy of the line-to-continuum ratios then strongly supports the assertion that the integrated recombination line fluxes are also a linear probe of the Lyman continuum production rates.

2.1. HII Line Emissivities

To calculate the expected HI line emission we make use of standard recombination line analysis (as described in Osterbrock & Ferland 2006). The volume emissivity, \( \epsilon \) is then given by

\[
\epsilon = n_u A_{ul} \hbar \nu = b_{ul} n_u (TE) A_{ul} \hbar \nu
\]

where \( n_u \) and \( n_u (TE) \) are the actual and thermal equilibrium upper-level population densities. The exact HI spontaneous decay rates from levels \( u \) to \( l \), \( A_{ul} \), are available in tabular form online from Kholupenko et al. (2003). The most complete and up to date departure coefficients (from TE)(\( b_{ul} \) and \( d(ln b_{ul})/dn \)) are from Hummer & Storey (1987); Storey & Hummer (1995a,b). The latter work includes population transfer by electron and ion collisions and has emissivities for HI and HeII up to principal quantum number \( n = 50 \). They also calculate optical depth parameters for a wide range of electron temperature (\( T_e \)) and electron density (\( n_e \)). We make use of these numerical results in this paper; Figure 1 shows the Storey & Hummer (1995a) HI recombination line emissivities at \( T = 10^4 \)K.

Figure 2 shows the submm HI-\( \alpha \) line emissivities \( \epsilon \) per unit emission measure, for \( T = 7500 \) and \( 10^4 \) K. These volume emissivities were computed for density \( n = 10^2 \) and \( 10^4 \) cm\(^{-3}\) but the separate density curves are essentially identical. This is because, for these low energy levels, the spontaneous decay rates are very high (\( A_{\Delta n=1}(HI) > 200 \) sec\(^{-1}\) for \( n < 30 \)). The level populations are therefore determined mainly by the radiative cascade following recombination to high levels. The latter is proportional to the recombination rate and hence \( n_e n_p \).

To translate the curves in Figure 2 into expected emission line luminosities, one needs to multiply by the total emission measure of each source. Consider the detectability of a luminous star forming region in a nearby galaxy. In §3 we show that for a starburst type EUV spectrum with integrated luminosity in the ionizing continuum at \( \lambda < 912 \AA \), \( L_{EUV} = 10^{12} L_\odot \), the total Lyman
continuum photon production rate is \( Q_{LyC} = 1.20 \times 10^{36} \text{ sec}^{-1} \). Scaling this down to the luminosity of an OB star cluster with \( L_{EUV} = 10^6 \text{ L}_\odot \) gives \( Q_{LyC} = 1.20 \times 10^{30} \text{ sec}^{-1} \). For Case B recombination in which all the photons are absorbed (i.e. the HII is ionization bounded) and the Lyman series lines are optically thick, the Lyman lines above Ly\(\alpha\) don’t escape. (In fact, most of the Ly\(\alpha\) may be absorbed by any residual dust.) In this case, the standard Strömgren condition equating the supply of fresh Lyman continuum photons \( (Q_{LyC}) \) to the volume integrated rate of recombination to states above the ground state,

\[
Q_{LyC} = \alpha_B n_e n_p \text{vol},
\]

implies an HII region emission measure \( (EM = n_e n_p \text{vol}) \) of \( EM = 4.60 \times 10^{62} \text{ cm}^{-3} \) (using \( \alpha_B = 2.6 \times 10^{-13} \text{ cm}^3\text{ sec}^{-1} \) at \( T_e = 10^4 \text{ K} \)). Using the specific emissivity of \( 3 \times 10^{-31} \text{ ergs cm}^{-3} \text{ sec}^{-1} \) for HI-26\(\alpha\) from Figure 2 the recombination line luminosity will be \( L_{\text{Ly}\alpha} = 1.38 \times 10^{32} \text{ ergs sec}^{-1} \). For a source distance of 1 Mpc and a line width of 30 km \( \text{ sec}^{-1} \), this corresponds to a peak line flux density of \( \sim 3.3 \text{ mJy} \). This flux density is readily detectable at signal to noise ratio \( 10\sigma \) within \( \sim 1 \text{ hr} \) with ALMA Cycle 1 sensitivity.

2.2. He Line Emission

He\(\alpha\) has an ionization potential of 24.6 eV and its photoionization requirements are not very different than those of HI. Thus the He\(\alpha\) recombination line probes the ionizing UV radiation field in much the same way as HI (see [3]). Since the He\(\alpha\) submm lines will be weaker than those of HI due to the lower He abundance, we don’t examine the He\(\alpha\) emission extensively here and instead focus on He\(\alpha\).

The ionization potential of He\(\alpha\) is 54.4 eV corresponding to photons with \( \lambda = 228 \text{ Å} \) for conversion of He\(\alpha\) to He\(\alpha^+\). Since the most massive star in a starburst will have surface temperatures \( \sim 50000 \text{ K} \), the ionizing UV from such a population will have only a very small fraction of the photons with energy sufficient to produce He\(\alpha^+\). Thus the recombination lines of He\(\alpha\) (He\(\alpha^+\)) which are produced by recombination of \( e^- + \text{He}^{\alpha+} \) can be a strong discriminant for the existence of an AGN with a relatively hard EUV-X-ray continuum. In starbursts there can be some He\(\alpha\) emission associated with Wolf-Rayet stars. However, the emission measure of the He\(\alpha^+\) region relative to that of the H\(^+\) region will be much less than for an AGN.

The emissivities of the He\(\alpha\) recombination lines are taken from Storey & Hummer (1995a,b). For the interested reader, a simple model for the scaling of rate coefficients between Hydrogen and hydrogenic ions is developed analytically in Appendix A and those relations are compared with the numerical results from Storey & Hummer (1995a) in Appendix B.

The He\(\alpha\) submm \( \alpha \) lines are at higher quantum numbers \( n \) than those of HI since the energy levels scale as the nuclear charge \( Z^2 \), i.e. a factor of 4 larger for the same principal quantum number \( n \) in He\(\alpha\). For the submm He\(\alpha\) transitions, \( n = 30 - 50 \), versus 20 - 35 for HI. In Figure 3 the expected He\(\alpha\) line emissivities per unit \( n_e \) \( n_{He^{\alpha+}} \) are shown for the submm band. The values of these emissivities are \( \sim 5 \) times those of HI (Figure 1).

Figure 3. The emissivities of the He\(\alpha\) submm recombination lines are shown for \( T_e = 7500 \text{ and } 10^4 \text{ K} \). For He\(\alpha\) it is per unit \( n_e \), \( n_{He^{\alpha+}} \). The top border has the quantum number of the \( \alpha \) transitions and their rest frequencies are on the bottom axis. The emissivities were calculated for \( n_e = 10,1000 \), and \( 10^4 \text{ cm}^{-3} \), and the curves for all densities are coincident. Note that the emissivities for He\(\alpha\) per unit \( n_e \) \( n_{He^{\alpha+}} \) are 4-5 times greater than those for HI at similar frequency – this partially compensates for the lower He abundance relative to HI if their ionized volumes are similar (as would be the case for a very hard ionizing continuum).

however, since the He/H abundance ratio is 0.1 the actual values per unit \( n_e \) \( n_{He^{\alpha+}} \) are quite similar in a plasma where all the H is ionized and all the He is He\(\alpha^+\).

2.3. He\(\alpha\)/HI Emission Line Ratios with \( T_e \) and \( n_e \)

In Figure 4 the ratios of He\(\alpha\)/HI \( \alpha \) recombination line emissivities [Storey & Hummer (1995a)] are shown as a function of principal quantum number for large ranges of both \( T_e \) and \( n_e \). Two cautions in viewing these plots: 1) as noted above the lines of HI and He\(\alpha\) are not at the same frequency for each \( n \) and 2) the emissivity ratios are per \( n_e \) \( n_{He^{\alpha+}} \) and per \( n_e \) \( n_p \) for He\(\alpha\) and HI, respectively. In the case of the latter, the EM for He\(\alpha^+\) will be almost always \( < 0.1 \) of that for H\(\alpha^+\) due to the lower cosmic abundance of He.

Figures 4 top and lower-left clearly show that at a given temperature, \( T_e \), the He\(\alpha\)/HI line ratio is virtually constant as a function of both quantum number and electron density. Thus, varying density in the ionize gas should have almost no influence on the line ratios of He\(\alpha\) to HI. On the other hand, it is clear from these figures that increasing \( T_e \) leads to a decrease in the He\(\alpha\)/HI emissivity ratio. In Figure 4 lower-right, the ratio is shown for a single density \( n_e = 10^4 \text{ cm}^{-3} \) but \( T_e = 5000 \) to 20,000 K and the temperature dependence is clear and the same for all \( n \) transitions. Thus, the temperature and density dependence of the He\(\alpha\) to HI line ratios at fixed no can be empirically fit by:

\[
\frac{\epsilon_{He\alpha}-n_{\alpha}}{\epsilon_{HI}-n_{\alpha}} \propto n_e^0 T_e^{-4/3}.
\]

Although the He\(\alpha\)/HI line ratios are temperature dependent, the actual range of temperatures expected for the
ionized gas is very limited, $T_e$ is the line-of-sight path length. $\Omega$ is the ratio shown for 5 temperatures at $n_e$. The latter is clearly shown in the lower-right panel where the emissivity ratio is shown for 5 temperatures at $n_e = 10^4$ cm$^{-3}$. The line ratios are nearly independent of density for all the temperatures but they do depend on temperature as $T_e^{-4/3}$ for quantum number $n \sim 10$ to 50. The latter is shown in the lower-right panel where the emissivity ratio is shown for 5 temperatures at $n_e = 10^4$ cm$^{-3}$.

In Appendix A we show that the recombination rates coefficients scale as:

$$\alpha_{He^+}(T_e) = Z \alpha_H(T_e/Z^2) \text{ with } Z = 2 \text{ for } He^+. \quad (4)$$

However, the line emission rates also depend on the radiative and collisional cascade through the high $n$ levels and it is not possible to derive the emissivity scaling analytically to better than a factor of 2 accuracy.

### 2.4. Maser Amplification

As noted above, it is well known that the m/cm-wave recombination lines ($n > 100$) of HI have substantial negative optical depths and hence, maser amplification of the line emission. In such instances, the recombination line intensity will not accurately reflect the ionized gas emission measure and the associated Lyman continuum emission rates of the stellar population. For the submm HI and HeII lines we can analyze the possibility of maser amplification using the optical depth information of Storey & Hummer (1995a). They provide an optical depth parameter $\Omega_{n,n'}$ which is related to the line center optical depth $\tau_{n,n'}$ by

$$\tau_{n,n'} = n_e n_{ion} \Omega_{n,n'} L, \quad (5)$$

where L is the line-of-sight path length. $\Omega_{n,n'}$ is inversely proportional to the line width in Hz, $\Delta v_{FWHM}$ and in their output they used a thermal doppler width, implying a velocity full width at half maximum intensity

$$\Delta v_{FWHM} = \left( \frac{8 \ln 2 k T_e}{m_{ion}} \right)^{1/2} \quad (6)$$

or 21.7 km sec$^{-1}$ for HI at $10^4$ K. In most situations relevant to the discussion here, the line widths will exceed the thermal width due to large scale bulk motions within the host galaxies. We have therefore rescaled the optical depths to $\Delta v_{FWHM} = 100$ km sec$^{-1}$. We have also scaled the optical depth to a specific optical depth $\tau$ per unit $n_e n_{ion} L_{pc}$ where $L_{pc}$ is the path length in parsecs and the volume densities in cm$^{-3}$.

Figures 5 and 6 show the specific optical depths for the HI and HeII lines as a function of $T_e$ and $n_e$. The actual optical depths for a particular source may be obtained by scaling these curves by the factor $n_e n_{ion} L_{pc} / \Delta v_{100}$. In these plots, the dashed lines are for transitions with a population inversion and hence negative optical depth.

The submm transitions of HI and HeII are principle quantum number $n \sim 20$ to 32 and 32 to 50, respectively. For both HI and HeII these particular transitions have positive specific optical depths and hence no maser amplification at virtually all densities and temperatures shown in Figures 5 and 6.

The exceptions to this are that at very high densities, $n_e > 10^6$ cm$^{-3}$, there can be pop-
ulation inversions (see Figure 9). However, even at these high densities, significant amplification would occur only if the scale factor is sufficiently large.

An extreme upper limit for the HII in a ULIRG starburst nucleus might be \( n_e n_p L_{pc} \sim 10^4 \times 10^3 \times 10 = 10^9 \) cm\(^{-6}\) pc and \( \sim 10^8 \) cm\(^{-6}\) pc for HeII. Applying the first scale factor to the curves shown in Figure 5 yields upper limits to the negative optical depth \( |\tau| < 0.1 \), implying insignificant amplification even for these extreme conditions.

In summary, the observed emission line fluxes for the submm recombination lines will provide a linear probe of the HII and HeIII EMs; they will not be affected by non-linear radiative transfer effects, either maser amplification or optically thick saturation of the emission.

As an aside, it is interesting to note that the behavior of the HI opacities shown in Figure 6 left is reflected in the HeII opacities (Figure 5 left) but translated to higher no transitions. This is of course expected since HeII is hydrogenic and the energy levels are scaled by a factor 4, implying higher principal quantum number in HeII to obtain similar line frequencies and A coefficients.

2.5. Excitation by Continuum Radiation in Lines

Lastly, we consider the possibility that absorption and stimulated emission could alter the bound level populations away from those of a radiative cascade following recombination. Wadkik et al. (1983) analyzed this effect on the cm-wave recombination lines in powerful, radio-bright QSOs. For the submm lines considered here, the radiative excitation would be provided by the infrared continuum. Significant coupling of the level populations to the local radiation field at the line frequencies occurs when the net radiative excitation rate (i.e. absorption minus stimulated emission) is comparable with the spontaneous decay rate. It is easily shown that this happens when the local energy density of the radiation field exceeds that of a black body with temperature greater than \( T_x \), where \( T_x \) is the excitation temperature characterizing the cascade level populations. (This is the radiative equivalent of the critical density often used to characterize the collisional coupling of levels to the gas kinetic temperature.) Neglecting departures from thermal equilibrium and letting \( T_x \sim 10^4 \) K, the effective radiation temperature \( T_R \) must be therefore be \( > 10^4 \) K at the submm line frequencies.

This scenario is probably only of conceivable relevance for an AGN and not for a starburst. For example, suppose the AGN luminosity is \( \sim 10^{12} L_\odot \), then the effective black body radius for \( 10^4 \) K is 0.007 pc. Inside this radius the radiation energy density will exceed that of a \( 10^4 \) K black body, but at larger radius the induced radiative transitions become much less important. For the ionization case of an AGN as discussed below (13), the radii of the HeIII and HII regions are 16 and 27 pc respectively, as shown in Figure 9. For this very simplified example, we do not therefore expect radiative excitation in the bound-bound transitions to be significant in the bulk of the ionized gas. For other instances, one can easily perform a similar analysis as a check.

3. Ionization Structure of Starbursts and AGN Sources

![Figure 7](image-url)

Figure 7. The input ionizing continua are shown for starburst and AGN sources. The starburst EUV down to \( \lambda = 90 \) Å was computed from Starburst99 with a Kroupa IMF, solar metallicity and a constant SFR. The AGN EUV continuum was taken as a power-law with specific luminosity index \( \nu^{-1.7} \). Both EUV spectra were normalized so that the integrated EUV luminosity (at \( \lambda < 912 \) Å) was \( 10^{12} L_\odot \). The ionization thresholds for H, He and He\(^+\) are shown as vertical lines on the bottom axis. The dotted line along the SB spectrum is a power-law fit to the Starburst99 spectrum with \( L_\nu \propto \nu^{-1.5} \) used for the analytic treatment in (13).

To evaluate the expected line luminosities for the HI and HeII lines, we now calculate the ionized gas emission measures for the typical EUV spectra associated with starbursts and AGN. With the derived EMs for H\(^+\) and He\(^++\) as scale factors for HI and HeII emissivities per unit EM (\( \lambda_{22} \) and 1.1), one can then calculate the line luminosities.

For the starburst (SB) spectrum, we adopt a Kroupa IMF (0.1 to 100 \( M_\odot \)) and use the Starburst99 spectral synthesis program (Leitherer et al. 1999) to calculate the EUV spectrum at solar metallicity for a continuous SFR. After 10 Myr, the EUV at \( \lambda < 912 \) Å is constant since the early type star population has reached a steady state with equal numbers of new massive stars being created to replace those evolving off the main sequence. This EUV continuum can then be taken to represent a steady state SFR – applicable to starbursts lasting more the 10\(^7\) yrs. For the AGN EUV-X-ray continuum, we adopt a simple power-law \( L_\nu \propto \nu^{-1.7} \) (e.g. Osterbrock & Ferland 2000). We scale both the SB and AGN \( L_\nu \) to have integrated EUV luminosity = \( 10^{12} L_\odot \) at \( \lambda < 912 \) Å. For the starburst spectrum, this EUV luminosity corresponds to a steady-state SFR = \( 874 M_\odot \) yr\(^{-1}\) for a Kroupa IMF. These two ionization spectra are shown in Figure 7. The figure clearly demonstrates the significant difference between the SB and AGN EUV spectra, with the former having almost no photons capable of ionizing He\(^+\) to He\(^++\), compared to the number of HI ionizing photons.

Using these EUV continua, we have computed the ionization structure for a cloud with H density \( n = 10^4 \) cm\(^{-3}\), assuming all EUV photons are used for ionization, i.e. the plasma is ionization bounded and no EUV is absorbed by dust within the ionized gas. The He/H abundance ratio was 0.1. The EUV continuum was as-
For a starburst EUV spectrum with $L_{\text{EUV}} = 10^{12} L_\odot$ and Hydrogen density $n = 10^4$ cm$^{-3}$, the radii of the $\text{H}^+$, $\text{He}^+$, and $\text{He}^{++}$ regions are shown. The $\text{He}/\text{H}$ abundance ratio was 0.1.

Figure 8.

The secondary ionizing photons produced by recombinations with sufficient energy to ionize H or He were treated in the ‘on the spot’ approximation, i.e. assumed to be absorbed at the radius they were produced. Lastly, we simplified the analysis of these secondary photons by assuming a fraction 0.96 and 0.66 of the $\text{He}^+$ recombinations yielded a photon which ionizes H at electron densities below and above 4000 cm$^{-3}$ (see Osterbrock & Ferland 2006), respectively.

Figures 8 and 9 show the relative sizes of the HII, HeII and HeIII regions for the SB and AGN. These figures clearly show the marked contrast in size (and hence EM) of the He$^{++}$ regions in the two instances. Much less contrast is seen in the He$^+$ emission measures between the two models.

From ionization equilibrium calculations for the SB and AGN EUV spectra, we derive the EM of the ionized regions in HII, HeII and HeIII (Table 1). From the EMs shown in Table 1, we draw two important conclusions: 1) despite the very different spectral shapes, the bulk of the ionizing continuum is absorbed in the HII region and the EM$\text{H}^+$ provides a reasonably accurate estimate of the total EUV luminosity, differing less than a factor 2 between the two cases, and 2) the EM ratio, EM$_{\text{He}^{++}}$/EM$_{\text{H}^+}$, is 50 times greater for the AGN than for the SB, indicating that this ratio provides an excellent diagnostic of AGN versus SB ionizing sources.

### 3.1. Ionization Structure – Analytics

In the previous section, we made use of a full ionization equilibrium model using Starburst99 for the starburst EUV spectrum and a power law approximation for the AGN EUV. In this numerical treatment, we track the competition of all three species (HI, HeI and HeII) for ionizing photons at each wavelength. However, an analytic treatment, which turns out to reproduce quite well the full numerical approach, can be developed using a few simplifying assumptions regarding the EUV spectra and the competition between the 3 species for the ionizing photons in the energy regimes above 24.6 eV.

At photon energies between 24.6 to 54.4 eV, both HI and HeI can be ionized, and at energies above 54.4 eV all three species (HI, HeI and HeII) can be ionized. However, for all three species, the ionization cross sections are highest at the thresholds and drop as $\nu^{-3}$ above their respective thresholds. At 24.6 eV (the HeI ionization threshold), the HeI ionization cross section is 8 times larger than that of HI at the same energy. Thus, provided HI is mostly ionized, the photons above this energy are largely used to ionize HeI. These two factors (the higher cross section and the fact that HI is mostly ionized) more than make up for the fact that the $\text{He}/\text{H}$ abundance ratio is 0.1 (see Figures 8 and 9). A fraction of the HeI recombination photons can ionize HI so effectively that each of the photons above 24.6 eV will ionize both HeI and HI. This fraction varies between 0.96 (for $n_e < 4000$ cm$^{-3}$) and 0.66 (for $n_e > 4000$ cm$^{-3}$) (see Osterbrock & Ferland 2006). Thus, one can approximate the number of photons available to ionize HI as all those above 13.6 eV. Above 54.4 eV, the photons are predominantly used for HeII ionization since the abundance of HI will be very low in the HeII region. (This last assumption presumes that the ionizing continuum is hard enough that the gas is easily ionized to HeII.)

In comparing the ionization associated with a SB versus an AGN, we assume that the ionized regions are ionization bounded and that dust within the ionized regions does not significantly deplete the EUV. The latter could be a significant issue for very young ionized regions but is perhaps less likely for SB and AGN ionized regions.
where the timescales are $> 10^7$ yrs and the dust within the ionized gas is likely to have been destroyed. Under these assumptions, the standard Strömgren relation implies that the total volume integrated emission measure of each species ionized region will be determined by the total production rate of fresh ionizing photons. For comparing the SB and AGN cases, we normalize both EUV spectra to have the same total integrated EUV luminosity,

$$L = \int_{\nu_h}^{\infty} L_\nu d\nu,$$

where $\nu_h$ is the ionization threshold frequency. The production rate of ionizing photons is then given by $Q$, with

$$Q = \int_{\nu_h}^{\infty} L/\nu d\nu.$$

We consider the three regimes in the EUV

1. $\nu \geq 13.6$ eV and $\nu_h = \nu_0 = 13.6$ eV
2. $\nu \geq 24.6$ eV and $\nu_h = \nu_1 = 24.6$ eV
3. $\nu \geq 54.4$ eV and $\nu_h = \nu_2 = 54.4$ eV

corresponding to H ionization, He to He$^+$ ionization and He$^{++}$ to He$^{++}$ ionization.

For the EUV spectra we make the assumption that both SB and AGN EUV spectra can be represented by power-laws with $L_\nu = C_{SB} \nu^{-\alpha_{SB}}$ and $C_{AGN} \nu^{-\alpha_{AGN}}$. For the AGN, this is a commonly used assumption with $\alpha_{AGN} = 1.7$. For the SB this assumption may appear surprising but Figure 7 clearly shows that the EUV spectrum obtained from the spectral synthesis of a continuous SB can be fit by a power-law with $\alpha_{SB} = 4.5$. For these simple power-laws, the luminosity normalization yields the relation

$$C_{AGN}/C_{SB} = \frac{(1 - \alpha_{AGN})}{(1 - \alpha_{SB})} \nu_0^{-\alpha_{SB} + \alpha_{AGN}}$$

and for $\alpha_{AGN} = 1.7$ and $\alpha_{SB} = 4.5$, this reduces to $C_{AGN}/C_{SB} = 5\nu_0^{-0.8}$.

The Strömgren ionization equilibrium for a power-law ionizing spectrum then yields

$$\frac{Q_{H^{++}}}{Q_{H^+}} = (\frac{\nu_2}{\nu_0})^{-\alpha} = 4^{-\alpha}$$

since $\nu_2 = 4\nu_0$. For the AGN with $\alpha_{AGN} = 1.7$, $Q_{H^{++}}/Q_{H^+} = 0.095$ and for the SB, $\alpha_{SB} = 4.5$,

$Q_{H^{++}}/Q_{H^+} = 1.95 \times 10^{-3}$.

For Case B recombination, the Qs are related to the emission measures of their respective Strömgren spheres by the recombination coefficients to states above the ground state and the electron density

$$Q_{H^+} = n_e n_p \ vol \ alpha(H)$$

$$\simeq 2.60 \times 10^{-13} n^2 \ vol (T/10^4)^{1/2}$$

and

$$Q_{H^{++}} = n_e n_{H^{++}} \ vol \ alpha(H e^+)$$

$$\simeq 1.85 \times 10^{-12} n n_{H^{++}} \ vol (T/10^4)^{1/2},$$

where $n$ is the number density of H nuclei, vol is the volume of the ionized region and we set $n_e = 1.1$ and $1.2 \times n_H$ for the H$^+$ and He$^{++}$ regions respectively.

For $T = 10^4$K and $[H/He] = 0.1$, the emission measures are

$$EM_{H^+} = Q_{H^+}/2.60 \times 10^{-13} = 3.85 \times 10^{12} Q_{H^+}$$

and

$$EM_{H^{++}} = Q_{H^{++}}/1.85 \times 10^{-12} = 5.41 \times 10^{11} Q_{H^{++}}.$$

The emission measure ratio is therefore

$$EM_{H^{++}}/EM_{H^+} = 0.141 Q_{H^{++}}/Q_{H^+}.$$
for full numerical ionization equilibrium calculation obtained using the detailed SB99 spectrum for the starburst.

Lastly, we note that the change in the AGN / SB ratio of EMs ($H^+$/$H^+$) is easily shown from Eq. (3.1) to be

$$\frac{EM(H^+/H^+)}{EM(H^+/H^+)}_{\text{AGN}} = 4^{-\alpha_{\text{AGN}} + \alpha_{\text{SB}}} = 48.5 \quad (16)$$

as compared with 49.4 from the numerical analysis above. Contrasting this large change in the $H^+$ between the SB and AGN EUVs, Table 1 shows only $\sim 10\%$ change in the ratio of $H^+$ relative to $H^+$ between the two cases, implying that the HeI recombination lines can not be used to discriminate AGN and SB EUVs.

As an aside, we note that we were surprised to find that the SB99 EUV spectrum shown in Figure 7 could be fit by a power-law. Upon investigating this further, we found that there is enormous variation in the model EUV spectra depending on which stellar atmosphere model was employed, and due to the very uncertain contributions of Wolf-Rayet stars. Given these large uncertainties in the predicted SB EUV spectra, the specific power law index adopted above should only be taken as illustrative. Instead, it would be more appropriate to take the power-law as a 'parameterization' which allows simple exploration of the EUV spectral properties and HI and HeII emission line ratios. In fact, measurements of the $EM(H^+/H^+)$ ratio might be used to constrain the very uncertain EUV spectra of SB regions and OB star clusters. An alternative parameterization might be to model the SB EUV as a blackbody. For $T_{BB} = 45,000 K$, $Q_{H^+}/Q_{H^+} \approx 2.8 \times 10^{-4}$, implying a similar ratio for the emission measures. This is effectively a factor 10 lower than the ratio obtained for SB99 EUV and the $\alpha_{\text{SB}} = 4.5$ power-law used above.

4. PAIRED HI AND HEII RECOMBINATION LINES

In Table 2 we provide a list of the submm HI recombination lines together with their closest frequency-matched HeII lines. The ALMA IF frequency is 4 GHz and each correlator has a maximum bandwidth of 1.8 GHz, thus in a single tuning the spectra can cover up to 16 GHz. One prime pairing for simultaneous coverage of HI and HeII occurs at 350 GHz where HI-26 and HeII-42 can be observed within a good atmospheric window (ALMA Band 7). In Table 1 the last column gives the expected line ratio, HeII-42/HI-26 derived from the EM given in Table 1. The emissivities are shown in Figures 2 and 3. The line ratio varies by a factor 50 between the two cases, clearly demonstrating the efficiency of the HeII/HI submm line ratios to discriminate the nature of the ionizing sources. By contrast, the ratio $EM_{H^+}/EM_{He^+}$ is different only by a factor 10% between the SB and AGN cases, indicating that the HeII/HeI recombination line ratios are not a good SB versus AGN discriminant.

5. STAR FORMATION RATES AND AGN LUMINOSITY

Derivation of SFRs and AGN accretion rates from the HI and HeII recombination lines are potentially quite straightforward provided the form of each EUV spectrum can be parameterized. For the preceding analysis we normalized the EUV luminosity to $10^{12} L_{\odot}$ for both the SB and AGN. For a continuous SB (extending over $> 10^7$ yrs, the EUV luminosity will be constant. This EUV luminosity ($10^{12} L_{\odot}$) translates to the steady state SFR $= 874 M_{\odot}yr^{-1}$ for a Kroupa IMF. The implied SFR is a factor 1.6 higher for a Salpeter IMF. (The total stellar luminosity integrated over all wavelengths would be $5.5 \times 10^{12} L_{\odot}$ at $10^7$ yrs.) For an AGN with $L_{EUV} = 10^{12} L_{\odot}$, this EUV luminosity corresponds to an accretion rate of $0.65 M_{\odot}yr^{-1}$ assuming 10% conversion of accreted mass to EUV photon energy. [Note that the above luminosities refer to that in the EUV, not the total bolometric luminosities.]

For galaxies with these luminosities, the submm recombination lines of both HI and HeII are detectable with ALMA out to distance $\sim 100$ Mpc in a few hours integration. As an example, consider the H-26a line in a ULIRG like Arp 220 (or NGC 6240) at a distance $\sim 100$ Mpc with an HI emission measure $EM_{H^+}$ (see Table 1). For a specific emissivity $\epsilon$, emission measure EM and source distance D (all in cgs units), the velocity-integrated line flux in observer units Jy km sec$^{-1}$ is given by

$$S_{\Delta V} = \frac{\epsilon EM_{H^+} c}{4\pi D^2 \nu_{\text{obs}}} 10^{18} \text{ Jy km sec}^{-1}$$

Using the volume emissivity of $\epsilon = 2.5 \times 10^{-31}$ ergs cm$^{-3}$ sec$^{-1}$ / $n_e n_p vol$ from Figure 1 one finds the velocity-integrated line flux for H-26a:

$$S_{HI-26a} \Delta V = 7.17 \frac{EM_{H^+}}{4 \times 10^{58} D^{-2} 100 Mpc} \nu_{350GHz}^{-1} \text{ Jy km sec}^{-1}.$$  

The frequency-paired HeII-42a line will have an integrated flux $\sim 8\%$ of HI-26a in the case of AGN. Both lines should be simultaneously detectable in a few hours with ALMA. For reasonable densities, the emission in these lines will be directly proportional to the EM of the gas. Even at $n_e = 10^6$ cm$^{-3}$, the emission rate in the HI-26a and HeII-42a lines are altered by only 1 and 2%, respectively. If the source is known to be a 'continuous' starburst, one may substitute $SFR/(674 M_{\odot}yr^{-1})$ for $EM_{H^+}/(4 \times 10^{68})$ in the equation above,

$$S_{HI-26a} \Delta V = 1.06 \frac{SFR}{100 M_{\odot}yr^{-1}} D^{-2} 100 Mpc \nu_{350GHz}^{-1} \text{ Jy km sec}^{-1}.$$  

We have recently detected the HI-26a in Arp 220 in ALMA Cycle0 observations with a line flux indicating a $SFR \approx 100 M_{\odot}yr^{-1}$ (Scoville et al. 2013 – in preparation). Yun et al. (2004) also report detection of HI-41a at 90 GHz yielding a similar SFR.

As noted earlier, the free-free (Bremsstrahlung) continuum emission can also be used to probe the ionized gas EM. For completeness, the free-free flux density in the submm regime is given by

$$S_{ff} = 75.0 \frac{EM_{H^+}}{4 \times 10^{58}} \nu_{300GHz}^{-0.17} T_{10^7 K}^{-0.5} D^{-2} 100 Mpc \text{ mJy}.$$  

where we have included a factor 1.1 to account for the He$^+$ free-free emission assuming $[He/H] = 0.1$ and...
Although their energy levels will have population inver-

of the emission measures of the different ionized regions.

that the low-n transitions should provide a linear probe
of the emission measures of the different ionized regions.

bedded star formation and AGN luminosity. We find

line emission of HI, HeI and HeII as probes of dust em-

in the analysis of submm continuum observations.

5.1. Dust Extinction

We have stressed that a major advantage of the submm recombination lines is that they are at sufficiently long wavelengths that dust extinction should be negligible, since for standard dust properties the extinction should be $A(\lambda) \sim 10^{-4}(\lambda_{\text{crit}}/300\mu m)^{-1.8} A_V$ at submm and longer wavelengths (e.g. Battersby et al. 2011, Planck Collaboration 2011[4]). Thus for $A_V < 1000$ mag, extinction at $\lambda > 300\mu m$ should be minimal. However, there are a few extreme cases such as Arp 220 and young protostellar objects which may have somewhat higher dust columns. In these cases, the recombination lines provide a unique probe of the dust extinction through measurements of HI lines at different submm wavelengths. Their intrinsic flux ratios can be determined from Figure 2, the extinction is then obtained by comparison of the intrinsic and observed line ratios. In sufficiently bright recombination line sources with high extinction, such observations could potentially be used to determine the frequency dependence of the dust extinction in the submm – this has been a major uncertainty in the analysis of submm continuum observations.

We have evaluated the expected submm wavelength line emission of HI, HeI and HeII as probes of dust embedded star formation and AGN luminosity. We find that the low-n transitions should provide a linear probe of the emission measures of the different ionized regions. Although their energy levels will have population inversions, the negative optical depths will be $<< 0.1$ for the maximum gas columns expected and hence there is no significant maser amplification.

The submm HI and HeII lines have major advantages over other probes of SF and AGN activity: 1) the dust extinctions should be minimal; 2) the emitting levels ($n < 30$ for HI and $n < 50$ for HeII) have high critical densities ($n_{\text{crit}} > 10^4$ cm$^{-3}$, Sejnowski & Hjellming 1969) and hence will not be collisionally suppressed; and 3) they arise from the most abundant species and therefore do not have metallicity dependences. The emission line luminosities of the HI (and HeI) submm recombination lines are therefore a direct and linear probe of the EUV luminosity and hence SFR if the source is a starburst.

The emission ratios of HI to HeII can be a sensitive probe of the hardness of the EUV ionizing radiation field, providing a clear discriminant between AGN and SBS.

The observed ratios of the submm HI recombination lines may also be used to determine the extinction in highly extincted luminous sources and to constrain the shape of the submm extinction curve.

Lastly, we find that these lines should be readily detectable for imaging with ALMA in luminous galaxies out to 100 Mpc and less luminous sources at lower redshift. We note that the far infrared fine structure lines observed with Herschel often show line deficits in the ULIRGs relative to the IR luminosity, possibly indicating either dust absorption of the EUV or collisional suppression of the emission rates at high density (Gracia-Carpio et al. 2011); the latter will not be a problem since the HI and HeII lines are permitted transitions with high spontaneous decay rates.

6. CONCLUSIONS

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APPENDIX

SCALING RELATIONS FOR HYDROGENIC IONS

The aim of this appendix is to lay the basis of partial analytical explanation of HeII to HI emissivities scaling relation discussed previously. To this end, in this following sections of this Appendix, we analyze radiative processes involving

| HI \(\text{GHz} \) | HeII \(\text{GHz} \) | \(n\) | \(\nu\) | \(\nu\) (GHz) | \(\nu\) (GHz) | \(\Delta \nu\) (GHz) | \(\epsilon_H I\) \(\text{erg sec}^{-1} \text{ cm}^{-3}\) | \(\epsilon_{H e I I}\) \(\text{erg sec}^{-1} \text{ cm}^{-3}\) |
|---|---|---|---|---|---|---|---|---|
| 20 | 764.230 | 32 | 766.940 | -2.710 | 1.21 \times 10^{-30} | 4.45 \times 10^{-30} |
| 21 | 662.404 | 34 | 641.108 | 21.296 | 9.05 \times 10^{-31} | 3.06 \times 10^{-30} |
| 22 | 577.896 | 35 | 588.420 | -10.531 | 6.85 \times 10^{-31} | 2.59 \times 10^{-30} |
| 23 | 507.175 | 37 | 499.191 | 7.985 | 5.25 \times 10^{-31} | 1.85 \times 10^{-30} |
| 24 | 447.540 | 38 | 461.286 | -13.746 | 4.06 \times 10^{-31} | 1.57 \times 10^{-30} |
| 25 | 396.901 | 40 | 396.254 | 0.647 | 3.17 \times 10^{-31} | 1.15 \times 10^{-30} |
| 26 | 353.623 | 42 | 342.894 | -10.729 | 2.50 \times 10^{-31} | 8.47 \times 10^{-31} |
| 27 | 316.415 | 43 | 319.781 | -3.366 | 1.39 \times 10^{-31} | 7.32 \times 10^{-31} |
| 28 | 284.251 | 45 | 279.432 | 4.818 | 1.60 \times 10^{-31} | 5.50 \times 10^{-31} |
| 29 | 256.302 | 46 | 261.787 | -5.485 | 1.29 \times 10^{-31} | 4.79 \times 10^{-31} |
| 30 | 231.901 | 48 | 230.713 | 1.187 | 1.05 \times 10^{-31} | 3.67 \times 10^{-31} |
| 31 | 210.502 | 50 | 204.370 | 6.132 | 8.67 \times 10^{-32} | 2.83 \times 10^{-31} |
| 32 | 191.657 | 51 | 192.693 | -1.036 | 7.18 \times 10^{-32} | 2.48 \times 10^{-31} |

Note. — In ALMA Band 7 (275 to 365 GHz), the IF frequency is 4 GHz and the correlator has a nominal coverage of 4 × 2 GHz or 8 GHz in each sideband. Therefore we can observe HI at submm wavelengths. In most instances the thermal dust emission will dominate the free-free so the latter is not generally a useful tracer of the ionized gas EM.
Hydrogen-like atom consisting of \( Z \) charged nucleus and single electron orbiting it and put them in use in Appendix B. In this notations \( \text{H} \) corresponds to \( Z = 1 \) and \( \text{HeI} \) to \( Z = 2 \).

Below we refer to Landau \& Lifshitz (1977) and Berestetski et al. (1982), as examples of standard course in quantum mechanics and QED. The choice is dictated by our personal preferences. The reader is free to refer to any standard textbook in quantum mechanics and QED or original papers, references to which can be found in Landau \& Lifshitz (1977) and Berestetski et al. (1982).

### Radiation

Scaling of the Einstein A-coefficients for spontaneous radiative decay can be explicitly derived in the dipole approximation. The probability of dipole transitions between two states of the hydrogenic ion is given by

\[
A_{n_1 \rightarrow n_2} = \frac{4(\omega_{n_1 \rightarrow n_2})^3}{3\hbar c^3} d_{n_1 \rightarrow n_2}^0,
\]

where \( \omega_{n_1 \rightarrow n_2} = Z^2 \frac{mc^2}{\hbar n_2^2} \left( \frac{1}{n_1^2} - \frac{1}{n_1^2} \right) = Z^2 w_{n_1 \rightarrow n_2}^H \) is the frequency of radiated photon and \( d_{n_1 \rightarrow n_2}^0 \) is the average over \( l' \)s and \( m' \)s of the transition dipole moment \( d_{n_1l_1m_1 \rightarrow n_2l_2m_2} \). Here

\[
d_{n_1 \rightarrow n_2}^0 = \frac{1}{n_2^2} \sum_{l_1, m_1, l_2, m_2} d_{n_1l_1m_1 \rightarrow n_2l_2m_2}^2 \quad \text{and} \quad d_{n_1l_1m_1 \rightarrow n_2l_2m_2}^2 = \langle d_z \rangle^2 + \langle dy \rangle^2 + \langle dz \rangle^2
\]

with \( \langle d_j \rangle = -\langle \psi_{nl_2m_2} r_j \psi_{nl_1m_1} \rangle \), \( j = x, y, z \) and \( r_j \) is the component of electron radius vector in atom (we neglect the motion of nucleus). The wave functions \( \psi_{n_1l_1m_1} = \psi_{n_1l_1m_1} \) and \( \psi_{n_2l_2m_2} = \psi_{n_2l_2m_2} \) are the initial and final wave functions of the electron on \( n_1l_1m_1 \) and \( n_2l_2m_2 \) levels of hydrogenic ion.

One can show (see any standard course in quantum mechanics, for example, Landau \& Lifshitz (1977)) that the transition dipole moment can be written as

\[
\langle d_j \rangle = -\frac{a_0}{Z} \int_0^\infty \psi_{nl_2m_2}^\dagger(\tilde{r}) e^{\tilde{r} \cdot \tilde{r}} \psi_{nl_1m_1}(\tilde{r}) d\tilde{r} = \frac{1}{Z} \langle \tilde{d}_j \rangle,
\]

where \( \tilde{r} = Zr/a_0 \) is a dimensionless variable, \( a_0 = \hbar^2/2mc^2 \) is the Bohr radius and \( \tilde{\psi}_{nl_1m_1} \) are the wave functions of the electron in the Hydrogen-like atom written in terms of \( \tilde{r} \). The integral is independent of \( Z \). We observe that there is a simple scaling for the A-coefficients between hydrogenic Z ion and H

\[
A_{n_1 \rightarrow n_2} = Z^4 \frac{4(\omega_{n_1 \rightarrow n_2})^3}{3\hbar c^3} (\tilde{d}_{n_1 \rightarrow n_2}^H)^2 = Z^4 A_{n_1 \rightarrow n_2}^H.
\]

### Recombination Rate Coefficients

Recombination coefficients and recombination cross sections for a free electron with the hydrogenic nucleus in its exact form cannot be simply scaled from \( \text{H} \). However, for our application, the recombining electrons are non-relativistic and we restrict our attention to this limit to obtain the scaling in the leading, dipole approximation. Assuming the recombining electrons are non-relativistic implies that the energy of the emitted photon is much less than the electron mass \( \omega \ll m \). The recombination cross section can then be written as (see Berestetski et al. (1982) and references there in)

\[
d\sigma_{\text{rec}} \simeq e^2 \frac{\omega m}{\pi p}|\mathbf{e} \mathbf{v}_{if}|^2 d\Omega,
\]

where \( p \) is the momentum of the incoming electron, \( \omega \) is the energy of the outgoing photon, \( \mathbf{e} \) is the photon polarization vector, \( d\Omega \) is the angular measure and \( \mathbf{v}_{if} \) is the transition element \( \mathbf{v}_{if} = \int \psi_i^\dagger \psi_f d^3x \) and \( \mathbf{v} = -\frac{i}{m} \nabla \). Here \( \psi_i \) and \( \psi_f \) are the initial and the final wave functions of the electron.

The initial electron wave function is the continuous spectrum wave function in the attractive potential of the hydrogenic nucleus \( V = -Ze^2/r \). For its explicit form see, for example, Landau \& Lifshitz (1977). The final wave function of the electron is the discrete spectrum wave function in the attractive potential of the \( Z \)-ion nucleus, i.e. the electron wave function in the hydrogenic ion with \( Z \) charged nucleus discussed in the previous section.

One can show that the transition element can be written as

\[
\mathbf{v}_{if}(p \rightarrow nlm) = -\frac{i}{m} (Ze^2m)^{-1/2} \int \tilde{\psi}_i^\dagger(\tilde{r}) \nabla \tilde{\psi}_f(\tilde{r}) d^3\tilde{r} = Z^{-1/2} \sqrt{\mathbf{v}_{if}^H(P \rightarrow nlm)}.
\]

Here \( \tilde{\psi}_i \) and \( \tilde{\psi}_f \) are the initial and final wave functions written in terms of dimensionless variables \( \tilde{r} = Ze^2mr \) and \( \tilde{p} = \frac{p}{Ze^2m} \), and \( \nabla = \partial/\partial \tilde{r} \).

\[\text{In this section we work in the standard for QED units } \hbar = c = 1 \text{ to avoid cluttering.}\]
Change of the momentum from \( p \) for the hydrogenic ion to \( p/Z \) for HI becomes obvious if we examine the energy conservation relation,

\[
\frac{p^2}{2m} = E_n^Z + \omega \quad \text{vs.} \quad \frac{(p/Z)^2}{2m} = E_n^H + (\omega/Z^2),
\]

where \( E_n^Z = Z^2 E_n^H = -Z^2 e_m m \) is the energy of an electron on n’s level of hydrogenic ions. For the recombination cross section we find

\[
d\sigma_{rec}(p \rightarrow nlm) = e^2 Z \frac{1}{Z} \omega \frac{p/Z}{2 \pi} \hat{E}_{lj}^H \hat{E}_{lj}^H d\Omega = d\sigma_{rec}^H \left( \frac{p}{Z} \right) \rightarrow nlm)
\]

After integration over angles \( d\Omega \) and averaging over projections of the orbital moment \( m \) and the photon polarizations \( e \), we find \( \sigma_{rec}(p \rightarrow nl) = \sigma_{rec}^H(p/Z \rightarrow nl) \).

Lastly, to obtain recombination coefficients \( \alpha_{nl}(T) \), we need to average \( u\sigma_{nl} \) over a Maxwellian distribution for the electrons,

\[
\alpha_{nl}(T) = \int_0^\infty u\sigma_{nl}(p)f(u,T)du, \quad f(u,T) = \frac{4}{\sqrt{\pi}} \left( \frac{m}{2kT} \right)^{3/2} u^2 e^{-\frac{mu^2}{2kT}}, \quad p = mu. \quad (A7)
\]

Changing variables \( T \rightarrow T^2/Z^2 \) and \( p \rightarrow p/Z \) in the integration, we find the scaling

\[
\alpha_{nl}(T) = Z \alpha_{nl}^H(T/Z^2). \quad (A8)
\]

APPLICATION TO HE\(^{++}\)

Here, we use the results from the Appendix A to partially explain the numerical results from Storey & Hummer (1995a). To this end we make a simplifying assumptions that the recombination line emissivities are dominated by recombination rates of He\(^{++}\) and H\(^+\) (then scale this ratio by the energies of their respective photons). The cascade following recombination is determined largely by radiative decay as described by the A-coefficients and to a much less extent by collisions. Under this assumptions we can write the emissivities as linear combinations of the recombination rates:

\[
\epsilon(n_1 \rightarrow n_2) \simeq \hbar \omega_{n_1 \rightarrow n_2} \sum_{A_{n_1 \rightarrow n_2}} A_{n_1 \rightarrow n_2} \sum_{m \geq n_1} \alpha_m C_{m \rightarrow n_1}. \quad (B1)
\]

Coefficients \( C_{m \rightarrow n} \) describe cascading down from \( m \) to \( n \) and are the functions of branching ratios. The radiative branching ratios will be the same for HeII as HI since all A-coefficients scale simply as \( Z^4 \) (A1). Therefore \( C_{m \rightarrow n} \) are independent of nuclei charge \( Z \).

The recombination coefficients for HeII and HI, as derived in the Appendix A, scale as

\[
\alpha_{nl}^{He^+}(T) \simeq 2 \alpha_{nl}^H(T/4). \quad (B2)
\]

For the emissivity of na line we obtain

\[
\epsilon_{He^+-na}(T) \simeq 2^2 \hbar \omega_{n_1 \rightarrow n_2} A_{n_1 \rightarrow n_2} \sum_{A_{n_1 \rightarrow n_2}} A_{n_1 \rightarrow n_2} \sum_{m \geq n_1} \alpha_m C_{m \rightarrow n_1} \simeq 8 \epsilon_{He^-na}(T/4). \quad (B3)
\]

In Figure 10 the emissivity ratios from Storey & Hummer (1995a) are shown for HeII at 20000 K compared to HI at 5000 K, illustrating that the numerical results reasonably confirm the approximate analytic prediction of a factor 8 difference.

It is very hard and probably impossible to find exact scaling with temperature of recombination coefficients. So we use the result we found numerically above that

\[
\frac{\epsilon_{He^-na}(T_1)}{\epsilon_{He^-na}(T_2)} \simeq \left( \frac{T_1}{T_2} \right)^{-4/3}. \quad (B4)
\]

Combining Equations B3 and B4 we obtain the following relation between HeII and HI emissivities for the same na lines:

\[
\frac{\epsilon_{He^+-na}(T)}{\epsilon_{He^-na}(T)} \sim 8 \frac{\epsilon_{He^-na}(T/4)}{\epsilon_{He^-na}(T)} \sim 8 \left( \frac{1}{4} \right)^{-4/3} \sim 50. \quad (B5)
\]

The true numerical scaling factor varies between 50 and 65 for na \( \leq 50 \) (see Fig. 4), which reasonably confirms our ‘analytical’ prediction.

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Figure 10. The ratio of emissivities for HeII at 20000 K to HI at 5000 K from the full numerical results of Storey & Hummer (1995b) for comparison with the 'analytic' expectation of a ratio = 8.