Effect of Phase Factor in the Geometric Entanglement Measure of Three-Qubit States

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Abstract

Any pure three-qubit state is uniquely characterized by one phase and four positive parameters. The geometric measure of entanglement as a function of state parameters can have different expressions. Each of expressions has its own applicable domain and thus the whole state parameter space is divided into subspaces that are ranges of definition for corresponding expressions. The purpose of this paper is to examine the applicable domains for the most general qubit-interchange symmetric three-qubit states. First, we compute the eigenvalues of the non-linear eigenvalue equations and the nearest separable states for the permutation invariant three-qubit states with a fixed phase. Next, we compute the geometric entanglement measure, deduce the boundaries of all subspaces, and find allocations of highly and slightly entangled states. It is shown that there are three applicable domains when the phase factor is $\pi/2$ while other cases have only two domains. The emergence of the three domains is due to the appearance of the additional W-state. We show that most of highly entangled states reside near the boundaries of the domains and states located far from the boundaries become less-entangled and eventually go to the product states. The neighbors of W-state are generally more entangled than the neighbors of Greenberger-Horne-Zeilinger(GHZ) state from the aspect of the geometric measure. However, the range of the GHZ-neighbors is much more wider than the range of the W-neighbors.
I. INTRODUCTION

Entanglement is a property of quantum states that does not exist classically. Two or more subsystems of a quantum system are said to be entangled if the state of the entire system cannot be described in terms of a state for each of the subsystems [1]. This property of composite quantum systems, which exhibits quantum correlations between subsystems, is a resource for many processes in quantum information theory [2, 3, 4, 5]. Since the profound measures of entanglement, i.e. the entanglement of formation and distillation [6, 7, 8, 9], have not been properly generalized to multiparticle systems, the study of quantifying multipartite entanglement via other measures [10, 11, 12, 13, 14] is a necessity.

The entanglement of a given pure state can be characterized by a distance to the nearest unentangled state [15]. A whole class of such entanglement monotones, based on the Euclidean distance of a given multipartite state to the nearest fully separable state, was constructed in Ref. [16]. Subsequently, a geometrically motivated measure of entanglement, known as geometric measure, was introduced by Wei and Goldbart [17]. It is a decreasing function of the maximal overlap $P_{\text{max}}$ and is suitable for any partite system regardless of its dimensions. The maximal overlap has several different names and we list all of them for the completeness: maximal probability of success [13], entanglement eigenvalue [17], injective tensor norm [18], the largest Schmidt coefficient [19] and maximum singular value [20].

The geometric measure has an advantage that it can be computed analytically for multi-parameter states. Recently, explicit expressions for the maximal overlap have been derived for three- [17, 20, 21, 22, 23] as well as for multi-qubit states [24, 25, 26, 27]. It turned out that the maximal overlap, depending on coefficients of a quantum state in a computational basis, can take two different values. It is equal to either the square of the largest coefficient or the square of the circumradius of a cyclic polygon constructed by the coefficients of the quantum state. This means that the whole parameter space is divided into two subspaces each of which has its own expression for the geometric measure.

In spite of these achievements, still we lack sufficient knowledge to classify generic three-qubit pure states by the geometric measure. They have five local unitary(LU) invariants including four positive parameters and a gauge phase $\gamma$ [19, 28, 29]. The maximal overlap of these states is not known yet. Only three-qubit states which are expressed as linear combinations of four(or less) orthogonal product states have been considered so far [22].
In fact, all of these states have real coefficients because the phases of their coefficients can be eliminated by LU-transformations. Thus, the contribution of the gauge phase to the maximal overlap has remained a mystery. On the other hand, the most recent results have shown that the gauge phase plays an important role. It parameterizes the family of maximally entangled states and identifies W-class pure states with the boundary of pure states.

In this paper we would like to take into complete account the effect of the gauge phase in the geometric measure of entanglement. We compute the maximal overlap as well as the nearest product states for a given value of the gauge phase. We will show in the following that depending on the phase factor $\gamma$ the whole parameter space is divided into the two or three domains, each of which has a particular expression for the geometric measure. In addition, we will show that most of highly entangled states reside near the boundaries of the domains. We will call these highly entangled states as GHZ-neighbors. The states located far from the boundaries become less-entangled and eventually go to the product states. But there is different kind of the highly entangled states. These states reside around W-states. We will call these highly entangled states as W-neighbors. The W-neighbors are generally more entangled than the GHZ-neighbors from the aspect of the geometric measure. However, the range of the GHZ neighbors is much more wider than the range of the W-neighbors.

The paper is organized as follows. In section II following Ref.[21] we transform the nonlinear eigenvalue equations into the Lagrange multiplier equations. In section III we solve the Lagrange multiplier equations analytically for $\gamma = 0$ and $\gamma = \pi/2$. It turns out that both cases give five different eigenvalues. Also every eigenvalue has its own available region in the parameter space. In section IV we compute the geometric measure for $\gamma = 0$ case. It turns out that two of the five eigenvalues contribute to the geometric measure. This means that the whole parameter space is divided into two applicable domains. In section V we compute the geometric measure for $\gamma = \pi/2$ case. It is shown that the whole parameter space is divided into the three applicable domains. In section VI we compute the eigenvalues and the geometric measure for $\gamma = \pi/4$ numerically. It is shown that when $\gamma = \pi/4$, there are six different eigenvalues. However, only two eigenvalues contribute to the geometric measure. In section VI a brief conclusion is given. In appendix we have shown that Lagrange multiplier equations for arbitrary $\gamma$ provides a solution whose multiplier constant is zero.
II. GENERAL FORMALISM

In this section we clarify our notations, give necessary definitions, define three-qubit symmetric states and transform nonlinear stationarity equations to a system of linear equations.

A. Preliminaries

The maximal overlap of \(n\)-qubit pure states is given by

\[
P_{\text{max}} = \max_{q_1, q_2, \ldots, q_n} |\langle q_1 | \langle q_2 | \cdots | \langle q_n | \psi \rangle|^2,
\]

(2.1)

where the maximization is performed over single qubit pure states. Constituents \(|q_1\rangle, |q_2\rangle, \ldots, |q_n\rangle\), the nearest product state from \(|\psi\rangle\), can be computed via the non-linear eigenvalue equations

\[
\langle q_1 | \cdots | q_{n-1} | \psi \rangle = \mu_i | q_n \rangle, \quad \langle q_1 | \cdots | q_{n-2} | q_n | \psi \rangle = \mu_i | q_{n-1} \rangle, \quad \cdots, \quad \langle q_2 | \cdots | q_n | \psi \rangle = \mu_i | q_1 \rangle,
\]

(2.2)

where \(\mu_i\)'s are the eigenvalues of Eq.(2.2). Then the geometric measure \(G\) of the quantum state \(|\psi\rangle\) is defined as \(G(\psi) = 1 - P_{\text{max}}\), where \(P_{\text{max}} = \max(\mu_i^2)\).

For simplicity, we take a quantum states which possess a permutational symmetry [31, 32, 33]. These states have three independent parameters and, through an appropriate LU transformations, can be brought into the symmetric form [19]

\[
|\psi\rangle = g|000\rangle + t|011\rangle + t|101\rangle + t|110\rangle + e^{i\gamma} h|111\rangle,
\]

(2.3)

where we follow the notation of Ref.[30]. In above equation all coefficients \(g, h\) and \(t\) are positive and satisfy the normalization condition \(g^2 + 3t^2 + h^2 = 1\). The phase \(\gamma\) has the period \(\pi\) and ranges within the interval \(-\pi/2 \leq \gamma \leq \pi/2\). Note that Eq.(2.3) is not a Schmidt decomposition for \(|\psi\rangle\) since the Schmidt normal form imposes additional conditions (namely, a lower bound on \(g\)) on state parameters. We would like to abandon these additional constraints and apply the general method proposed in Ref.[21] to symmetric states Eq.(2.3).

B. Modified stationarity equations

In this subsection we would like to present the method for solving stationarity equations for the quantum state given in Eq.(2.3). In the case of three-qubit pure states the method
developed in Ref. [21] transforms the system of nonlinear equations to a system of linear equations. In spite of this essential simplification, it is impossible to get analytic expressions for generic three-qubit states since the solution of the linear eigenvalue equations reduces to the root finding for a couple of algebraic equations of degree six [22]. However, the permutation symmetry of $|\psi\rangle$ reduces this pair of algebraic equations to a single algebraic equation of degree six. Furthermore, there is a solution which holds for all values of state parameters [30]. The separation of this global solution allows us to solve explicitly the eigenvalue equations for $\gamma = 0$ and $\gamma = \pi/2$ and leads us to a quartic equation for remaining cases. The quartic is the highest order polynomial equation that can be solved by radicals in the general case. But expressions for roots are impractical and we will carry out numerical analysis instead.

The method enables us to express eigenvalues $\mu^2$ via the reduced densities $\rho^A$, $\rho^B$ and $\rho^{AB}$ of qubits A and B in a form:

$$\mu^2 = \frac{1}{4} \max_{|s_1|=|s_2|=1} \left( 1 + r_1 \cdot s_1 + r_2 \cdot s_2 + G_{ij} s_{1i} s_{2j} \right), \quad (2.4)$$

where

$$r_1 = \text{Tr} (\rho^A \sigma), \quad r_2 = \text{Tr} (\rho^B \sigma), \quad G_{ij} = \text{Tr} (\rho^{AB} \sigma_i \otimes \sigma_j) \quad (2.5)$$

and $\sigma_i$’s are Pauli matrices. Explicit calculation shows

$$r \equiv r_1 = r_2 = (2ht \cos \gamma, 2ht \sin \gamma, g^2 - h^2 - t^2) \quad (2.6)$$

$$G_{ij} = \begin{pmatrix}
2t(g + t) & 0 & -2ht \cos \gamma \\
0 & -2t(g - t) & -2ht \sin \gamma \\
-2ht \cos \gamma & -2ht \sin \gamma & g^2 + h^2 - t^2
\end{pmatrix}. \quad (2.5)$$

It is worthwhile noting that $r_1$ is identical with $r_2$ and $G_{ij}$ is a symmetric matrix. These properties arise due to the fact that we have chosen the symmetric state in Eq. [2.3] under the qubit-exchange. As will be shown in the following these properties drastically simplify the calculation procedure. Since $r_1$, $r_2$ and $G_{ij}$ are explicitly derived, the eigenvalues $\mu^2$ can be computed if $s_1$ and $s_2$ are known. Due to the maximization in Eq. [2.4] these vectors can be computed by solving the Lagrange multiplier equations:

$$r_1 + G s_2 = \lambda_1 s_1 \quad r_2 + G^T s_1 = \lambda_2 s_2 \quad (2.7)$$
where the superscript $T$ stands for transpose and $\lambda_i$’s are the Lagrange multiplier constants. From the properties $r_1 = r_2$ and $G_{ij} = G_{ji}$ Eq. (2.7) can be reduced to a single equation

$$r + Gs = \lambda s$$  \hspace{1cm} (2.8)

where $\lambda \equiv \lambda_1 = \lambda_2$ and $s \equiv s_1 = s_2$. Letting

$$s = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta),$$  \hspace{1cm} (2.9)

Eq. (2.8) reduces to

$$2ht \cos \gamma + 2t(g + t) \sin \theta \cos \varphi - 2ht \cos \gamma \cos \theta = \lambda \sin \theta \cos \varphi$$  \hspace{1cm} (2.10a)

$$2ht \sin \gamma - 2t(g - t) \sin \theta \sin \varphi - 2ht \sin \gamma \cos \theta = \lambda \sin \theta \sin \varphi$$  \hspace{1cm} (2.10b)

$$(g^2 - t^2)(1 + \cos \theta) - h^2(1 - \cos \theta) - 2ht \cos \gamma \sin \theta \cos \varphi - 2ht \sin \gamma \sin \theta \sin \varphi = \lambda \cos \theta.$$  \hspace{1cm} (2.10c)

Solving $\theta$, $\varphi$ and $\lambda$ from Eq. (2.10), one can compute the eigenvalues for the symmetric canonical state (2.3) by inserting the solutions into Eq. (2.4). In the next section we will solve analytically Eq. (2.10) at the particular phases $\gamma = 0$ and $\gamma = \pi/2$. By making use of the solutions we will compute $\mu_i$ and $P_{\max} = \max(\mu_i^2)$ for the corresponding quantum states.

III. EIGENVALUES

In this section Eq. (2.10) will be solved at $\gamma = 0$ and $\pi/2$ separately. Since numerical calculation is needed to analyze the $\gamma = \pi/4$ case, we deal with this case in different section (see section VI).

A. $\gamma = 0$ case

For this case Eq. (2.10) reduces to

$$2t(g + t) \sin \theta \cos \varphi + 2ht(1 - \cos \theta) = \lambda \sin \theta \cos \varphi$$  \hspace{1cm} (3.1a)

$$- 2t(g - t) \sin \theta \sin \varphi = \lambda \sin \theta \sin \varphi$$  \hspace{1cm} (3.1b)

$$(g^2 - t^2)(1 + \cos \theta) - h^2(1 - \cos \theta) - 2ht \sin \theta \cos \varphi = \lambda \cos \theta.$$  \hspace{1cm} (3.1c)
Eq. (3.1b) implies that the solutions for the $\gamma = 0$ case are categorized by $\theta = 0$, $\varphi = 0$, $\varphi = \pi$ and $\lambda = -2t(g - t)^1$.

1. $\theta = 0$ case

When $\theta = 0$, Eq. (3.1a) and Eq. (3.1b) are automatically solved, and Eq. (3.1c) gives

$$\lambda = 2(g^2 - t^2).$$

(3.2)

Now $s = (0, 0, 1)$ and Eq. (2.4) together with the normalization condition $g^2 + 3t^2 + h^2 = 1$ gives the eigenvalue

$$\mu^2_P = g^2.$$  

(3.3)

2. $\varphi = 0$ case

For this case Eq. (3.1b) is automatically solved and the remaining equations are

$$2t(g + t)\sin \theta + 2ht(1 - \cos \theta) = \lambda \sin \theta$$

(3.4a)

$$\lambda = 2htz + 2t^2 + 2tg$$

(3.5)

where $z = \tan(\theta/2)$. Inserting Eq. (3.5) into Eq. (3.4b), one can derive an equation

$$(hz + g + t)(tz^2 - hz + g - 2t) = 0.$$  

(3.6)

Eq. (3.6) implies that the $\varphi = 0$ case is also categorized again by following three cases:

$$z = -\frac{g + t}{h}, \quad \frac{r_+}{2t}, \quad \frac{r_-}{2t}.$$  

(3.7)

where

$$r_\pm = h \pm \sqrt{h^2 + 4t(2t - g)}.$$  

(3.8)

1 The case $\theta = \pi$ can be excluded by Eq. (3.1a).
First, let us consider the case of \( z = -(g + t)/h \). In this case Eq. (3.5) gives
\[
\lambda = 0. \quad (3.9)
\]
Since, in this case,  
\[
s_x = \sin \theta = -\frac{2h(g + t)}{h^2 + (g + t)^2}, \quad s_y = 0, \quad s_z = \frac{h^2 - (g + t)^2}{h^2 + (g + t)^2}, \quad (3.10)
\]
it is straightforward to compute the eigenvalues for this case, which is  
\[
\mu_1^2 = \frac{g^2h^2 + t^2(g + t)^2}{h^2 + (g + t)^2}. \quad (3.11)
\]
Next, let us consider the case of \( z = r_\pm/2t \) simultaneously. In these cases Eq. (3.5) gives  
\[
\lambda = hr_\pm + 2t(g + t). \quad (3.12)
\]
Since, in these cases,  
\[
s_x = \frac{4tr_\pm}{r_\pm^2 + 4t^2}, \quad s_y = 0, \quad s_z = -\frac{r_\pm^2 - 4t^2}{r_\pm^2 + 4t^2}, \quad (3.13)
\]
one can show directly that the eigenvalues are  
\[
\mu_\pm^2 = \frac{(hr_\pm + 4t^2)^2}{r_\pm^2 + 4t^2}. \quad (3.14)
\]
Since \( z = \tan(\theta/2) \) should be real, the eigenvalues \( \mu_\pm^2 \) are available only when  
\[
g \leq 2t + \frac{h^2}{4t}. \quad (3.15)
\]

3. \( \varphi = \pi \) case

For this case Eq. (3.1b) is automatically solved and the remaining equations are  
\[
-2t(g + t) \sin \theta + 2ht(1 - \cos \theta) = -\lambda \sin \theta \quad (3.16a)
\]
\[
(g^2 - t^2)(1 + \cos \theta) - h^2(1 - \cos \theta) + 2ht \sin \theta = \lambda \cos \theta. \quad (3.16b)
\]
Since Eq. (3.16) can be derived from Eq. (3.4) by changing \( \theta \to -\theta \), the solutions for this case are also categorized by  
\[
z = \frac{g + t}{h}, \quad \frac{r_\pm}{2t}, \quad \frac{r_-}{2t}. \quad (3.17)
\]
Since Eq. (3.16a) reduces to
\[ \lambda = -2htz + 2t^2 + 2tg, \]  
(3.18)
comparison of Eq. (3.18) with Eq. (3.5) shows that the Lagrange multiplier constant \( \lambda \) is same with the case of \( \phi = 0 \). Since, furthermore, \( s_x = \sin \theta \cos \varphi \) and \( s_z = \cos \theta \) are invariant under \( \theta \to -\theta \) and \( \varphi = 0 \to \varphi = \pi \), this fact implies that the eigenvalues for this case are exactly same with those for \( \phi = 0 \) case.

4. \( \lambda = 2t^2 - 2gt \) case

For this case Eq. (3.11) is automatically solved and the remaining equations are
\[ 2t(g + t) \sin \theta \cos \varphi + 2ht(1 - \cos \theta) = -2t(g - t) \sin \theta \cos \varphi \]  
(3.19a)
\[ (g^2 - t^2)(1 + \cos \theta) - h^2(1 - \cos \theta) - 2ht \sin \theta \cos \varphi = -2t(g - t) \cos \theta. \]  
(3.19b)
Since Eq. (3.19a) gives a relation
\[ \cos \varphi = -\frac{h}{2g} \frac{1 - \cos \theta}{\sin \theta}, \]  
(3.20)
combining Eq. (3.19b) and Eq. (3.20) enables us to express \( \cos \theta \) and \( \sin \theta \) as
\[ \cos \theta = -\frac{g^2 - h^2 + gt}{g^2 + h^2 + 3gt} \quad \sin \theta = \pm \sqrt{\frac{4g(g + 2t)(h^2 + gt)}{g^2 + h^2 + 3gt}}. \]  
(3.21)
For a time being we choose the upper sign in \( \sin \theta \). Then, Eq. (3.20) reduces to
\[ \cos \varphi = \frac{h}{2} \sqrt{\frac{g + 2t}{g(h^2 + gt)}}. \]  
(3.22)
At this stage it is worthwhile noting that the eigenvalue in this case is available when
\[ (3g - 2t)h^2 + 4g^2t \geq 0 \]  
(3.23)
because of \( -1 \leq \cos \varphi \leq 1 \). Of course, the corresponding \( \sin \varphi \) is
\[ \sin \varphi = \pm \sqrt{\frac{3gh^2 + 4g^2t - 2h^2t}{4g(h^2 + gt)}}. \]  
(3.24)
Again we choose the upper sign in \( \sin \varphi \). Then, it is straightforward to compute \( s \), whose components are
\[ s_x = -\frac{h(g + 2t)}{g^2 + h^2 + 3gt} \quad s_y = \sqrt{\frac{(g + 2t)(3gh^2 + 4g^2t - 2h^2t)}{g^2 + h^2 + 3gt}} \quad s_z = -\frac{g^2 - h^2 + gt}{g^2 + h^2 + 3gt}. \]  
(3.25)
Inserting Eq. (3.25) into Eq. (2.4) gives the eigenvalue for this case as follows:

\[ \mu_2^2 = \frac{g(gh^2 + 4t^3)}{g^2 + h^2 + 3gt}. \] (3.26)

It is easy to show that the choice of other sign in \( \sin \theta \) and \( \sin \varphi \) does not change the eigenvalue \( \mu_2^2 \).

The eigenvalues for \( \gamma = 0 \) case are summarized in Table I.

| name | eigenvalue | \( \lambda \) | available region |
|------|------------|----------------|------------------|
| \( \mu_2^2_P \) | \( g^2 \) | \( 2(g^2 - t^2) \) | all |
| \( \mu_2^1 \) | \( \frac{g^2h^2 + t^2(g + t)^2}{h^2 + (g + t)^2} \) | 0 | all |
| \( \mu_2^+ \) | \( \frac{(hr_+ + 4t^2)^2}{r_+^2 + 4t^2} \) | \( hr_+ + 2t(g + t) \) | \( g \leq 2t + h^2/(4t) \) |
| \( \mu_2^- \) | \( \frac{(hr_- + 4t^2)^2}{r_-^2 + 4t^2} \) | \( hr_- + 2t(g + t) \) | \( g \leq 2t + h^2/(4t) \) |
| \( \mu_2^2 \) | \( \frac{g(gh^2 + 4t^3)}{g^2 + h^2 + 3gt} \) | \( 2t(t - g) \) | \((3g - 2t)h^2 + 4gt^2 \geq 0 \) |

Table I: Eigenvalues for \( \gamma = 0 \) case

B. \( \gamma = \pi/2 \) case

For this case Eq. (2.10) reduces to

\[ 2t(g + t) \sin \theta \cos \varphi = \lambda \sin \theta \cos \varphi, \] (3.27a)

\[ -2t(g - t) \sin \theta \sin \varphi + 2ht(1 - \cos \theta) = \lambda \sin \theta \sin \varphi, \] (3.27b)

\[ (g^2 - t^2)(1 + \cos \theta) - h^2(1 - \cos \theta) - 2ht \sin \theta \sin \varphi = \lambda \cos \theta. \] (3.27c)

Eq. (3.27a) guarantees that the solutions for this case are categorized by \( \theta = 0, \varphi = \pi/2, \varphi = 3\pi/2 \) and \( \lambda = 2t(g + t) \). Since the calculation procedure for the first three cases are similar to the \( \gamma = 0 \) case, we will briefly sketch the final result only. Although the calculation procedure for the last case is also similar to the previous case, it gives a non-trivial available region, which is important to compute the geometric measures in next section. Therefore, we will present the last case in detail.

When \( \theta = 0 \), the Lagrangian multiplier constant is same with Eq. (3.22) and the corresponding eigenvalue is

\[ \nu_P^2 = g^2. \] (3.28)
When $\varphi = \pi/2$, there are three types of solutions depending on $z = \tan(\theta/2)$. If $z = (g-t)/h$, we have vanishing Lagrange multiplier constant and the corresponding eigenvalue is

$$\nu_1^2 = \frac{g^2h^2 + t^2(g-t)^2}{h^2 + (g-t)^2}. \quad (3.29)$$

When $z = s_\pm/2t$, where

$$s_\pm = h \pm \sqrt{h^2 + 4t(2t + g)}, \quad (3.30)$$

the corresponding Lagrange multiplier constants are $hs_\pm - 2t(g-t)$, and the corresponding eigenvalues are

$$\nu_\pm^2 = \frac{(hs_\pm + 4t^2)^2}{s_\pm^2 + 4t^2}. \quad (3.31)$$

It should be noted that $\nu_\pm^2$ are available in entire parameter space, while $\mu_\pm^2$ in $\gamma = 0$ case is restricted by Eq.(3.15). As in the case of $\gamma = 0$, $\varphi = 3\pi/2$ case does not give a new eigenvalue. This case just reproduces $\nu_1^2$ and $\nu_\pm^2$.

Finally, let us discuss $\lambda = 2t(g + t)$ case. For this case Eq.(3.32a) is automatically solved and the remaining equations are

$$2ht(1 - \cos \theta) - 2t(g-t)\sin \theta \sin \varphi = 2t(g + t)\sin \theta \sin \varphi = 2t(g + t)\cos \theta. \quad (3.32a)$$

$$2ht\sin \theta \sin \varphi = 2t(g + t)\cos \theta. \quad (3.32b)$$

Since Eq.(3.32a) gives a relation

$$\sin \varphi = \frac{h}{2g} \frac{1 - \cos \theta}{\sin \theta}, \quad (3.33)$$

combining Eq.(3.32b) and Eq.(3.33) yields

$$\cos \theta = -\frac{g^2 - h^2 - gt}{g^2 + h^2 - 3gt}. \quad (3.34)$$

The requirement $-1 \leq \cos \theta \leq 1$ gives first available condition

$$(g - 2t)(h^2 - gt) \geq 0. \quad (3.35)$$

Now we choose $\sin \theta$ as

$$\sin \theta = \frac{\sqrt{4g(g - 2t)(h^2 - gt)}}{g^2 + h^2 - 3gt}. \quad (3.36)$$

Then from Eq.(3.33) $\sin \varphi$ becomes

$$\sin \varphi = \frac{h}{2} \sqrt{\frac{g - 2t}{g(h^2 - gt)}}. \quad (3.37)$$
Another requirement \(-1 \leq \sin \varphi \leq 1\) gives second available condition

\[(g - 2t)(3gh^2 - 4g^2t + 2h^2t) \geq 0. \tag{3.38}\]

Choosing \(\cos \varphi\) as

\[
\cos \varphi = \sqrt{\frac{3gh^2 - 4g^2t + 2h^2t}{4g(h^2 - gt)}}, \tag{3.39}
\]

it is straightforward to show that the eigenvalues for this case is

\[
\nu_2^2 = \frac{g(gh^2 - 4t^3)}{g^2 + h^2 - 3gt}. \tag{3.40}
\]

It is easy to show that the different choices in the sign of \(\sin \theta\) and/or \(\cos \varphi\) do not change the eigenvalue. Although the available region for \(\nu_2^2\) is restricted by Eq.(3.35) and Eq.(3.38), one can show that Eq.(3.38) implies Eq.(3.35) already. To show this explicitly let us consider \(g \geq 2t\) case first. In this case Eq.(3.38) imposes

\[
h^2 - gt \geq \frac{4g^2t}{3g + 2t} - gt = \frac{g}{3g + 2t}(g - 2t) \geq 0.
\]

Similarly, one can show that Eq.(3.38) implies Eq.(3.35) for \(g \leq 2t\) region too. Therefore, the available region for \(\nu_2^2\) is restricted by Eq.(3.38) only.

The eigenvalues in \(\gamma = \pi/2\) case is summarized in Table II.

| name | eigenvalue | \(\lambda\) | available region |
|------|------------|--------------|------------------|
| \(\nu_1^2\) | \(g^2\) | \(2(g^2 - t^2)\) | all |
| \(\nu_2^2\) | \(g^2h^2 + t^2(g - t)^2\) \(h^2 + (g - t)^2\) | 0 | all |
| \(\nu_+^2\) | \(|hs_+ + 4t^2|s_+^2 + 4t^2\) | \(hs_+ - 2t(g - t)\) | all |
| \(\nu_-^2\) | \(|hs_- + 4t^2|s_-^2 + 4t^2\) | \(hs_- - 2t(g - t)\) | all |
| \(\nu_2^2\) | \(g(gh^2 - 4t^3)\) \(g^2 + h^2 - 3gt\) | \(2t(g + t)\) | \((g - 2t)(3gh^2 - 4g^2t + 2h^2t) \geq 0\) |

Table II: Eigenvalues for \(\gamma = \pi/2\) case

C. \(h \to 0\) limit

Since \(|\psi\rangle\) is independent of \(\gamma\) in the \(h \to 0\) limit, all eigenvalues for \(\gamma = 0\) and \(\gamma = \pi/2\) cases should be same including the available region in the parameter space. Note that
\(\mu_+^2 = \mu_-^2\) and \(\nu_+^2 = \nu_-^2\) in the \(h \to 0\) limit. In this limit the eigenvalues for \(\gamma = 0\) exactly coincide with eigenvalues for \(\gamma = \pi/2\) as following:

\[
\mu_P^2 = \nu_P^2 = g^2 \quad \mu_1^2 = \nu_1^2 = t^2 \quad \mu_2^2 = \nu_2^2 = \frac{4t^3}{3t + g} \quad \mu_+^2 = \nu_+^2 = \frac{4t^3}{3t - g}.
\] (3.41)

In addition, first three eigenvalues in Eq. (3.41) are available in the full parameter space and the last one is available only at \(g \leq 2t\). Thus, our calculational results are perfectly consistent in the \(h \to 0\) limit.

**IV. GEOMETRIC MEASURE FOR \(\gamma = 0\)**

In this section we would like to compute the geometric entanglement measure defined

\[
G(\psi) = 1 - P_{\text{max}}(\psi)
\] (4.1)

for \(\gamma = 0\) case. In order to compute \(P_{\text{max}}\) we would like to emphasize three points, which simplify the following calculation. Firstly, note that \(P_{\text{max}}\) is given by

\[
P_{\text{max}} = \max(\mu_1^2).
\] (4.2)

Therefore, we should choose the largest eigenvalue from all eigenvalues, each of which has its own available regions in the parameter space. Secondly, note that

\[
\mu_+^2 - \mu_-^2 = \frac{128ht^{7/2}}{(r_+^2 + 4t^2)(r_-^2 + 4t^2)} \left(2t + \frac{h^2}{4t} - g\right)^{3/2}.
\] (4.3)

This means that \(\mu_-^2\) is always smaller than \(\mu_+^2\) in the available region \(g \leq 2t + h^2/(4t)\). Therefore, we can exclude \(\mu_-^2\) from beginning for the computation of \(P_{\text{max}}\). Thirdly, note that \(P_{\text{max}}\) is obtained from the eigenvalues whose Lagrange multiplier constants are positive [21]. This fact excludes \(\mu_1^2\) too. Considering all of these facts and available regions, it is convenient to divide the whole parameter space into the following four regions:

\[
\text{(region I) } g \geq 2t + \frac{h^2}{4t} : \quad P_{\text{max}} = \mu_P^2
\] (4.4)

\[
\text{(region II) } t \leq g \leq 2t + \frac{h^2}{4t} : \quad P_{\text{max}} = \max(\mu_P^2, \mu_+^2)
\]

\[
\text{(region III) } g \leq t \quad \text{& } C_1 \geq 0 : \quad P_{\text{max}} = \max(\mu_+^2, \mu_2^2)
\]

\[
\text{(region IV) } g \leq t \quad \text{& } C_1 \leq 0 : \quad P_{\text{max}} = \mu_+^2
\]
Fig. 1: (Color online) Fig. 1a is a plot of the applicable domains in $(u, v)$-plane for $\gamma = 0$. The principal domain $P_{\text{max}} = \mu_P^2$ is located in small $v$ and large $u$ region. This fact indicates that this domain is around large $g$ region. Fig. 1b is plot of $(u, v)$-dependence of $P_{\text{max}}$ for $\gamma = 0$ case. Many highly entangled states are represented as a valley in this figure. Around $u \sim 0$ and $(u \sim \pi/2, v \sim 0)$ there are a lot of less entangled states. To compare the applicable domains with $P_{\text{max}}$ we plot both simultaneously in the $(u, v)$ plane in Fig. 1c. The black thick line is a boundary between domains. The blue-color and white-color represent the highly- and less-entangled states respectively. Fig. 1c shows that the highly-entangled states reside around the boundary between domains.

where

$$C_1 = (3g - 2t)h^2 + 4g^2t.$$  \hspace{1cm} (4.5)
In order to compare $\mu^2_+$ with $\mu^2_2$, we compute $\mu^2_+ - \mu^2_2$, which is

$$
\mu^2_+ - \mu^2_2 = \frac{2}{(r^2_+ + 4t^2)(g^2 + h^2 + 3gt)} \left( \alpha_1 + \beta_1 \sqrt{h^2 + 4t(2t - g)} \right)
$$

(4.6)

where

$$
\alpha_1 = h^6 + gh^4t + 8h^2t^2 + 20gh^2t^3 + 16g^2t^4 + 4h^2t^2(2t^2 - g^2)
$$

(4.7)

$$
\beta_1 = h(h^4 + 3gh^2t + 4g^2t^2 + 4h^2t^2 + 8gt^3).
$$

Since the last term in $\alpha_1$, $4h^2t^2(2t^2 - g^2)$, is non-negative in the region $g \leq t$, both $\alpha_1$ and $\beta_1$ are non-negative in region III. In region III, therefore, $P_{max}$ becomes $\mu^2_+$.

In region II it has been shown in Ref. [30] that $\mu^2_2 = \mu^2_+$ when $D_1 = 0$, where

$$
D_1 = gh^2 - (g + t)^2(g - 2t).
$$

(4.8)

Therefore, the region II should be divided into two regions, i.e. $D_1 \geq 0$ and $D_1 \leq 0$. Simple consideration shows that $\mu^2_2 \geq \mu^2_+$ when $D_1 \leq 0$ and $\mu^2_2 \leq \mu^2_+$ when $D_1 \geq 0$. Combining all of these facts, one can conclude

(region A) $g \geq 2t + \frac{h^2}{4t} : \quad P_{max} = \mu^2_2$

(4.9)

(region B) $t \leq g \leq 2t + \frac{h^2}{4t}$ & $D_1 \leq 0 : \quad P_{max} = \mu^2_2$

(region C) $t \leq g \leq 2t + \frac{h^2}{4t}$ & $D_1 \geq 0 : \quad P_{max} = \mu^2_+$(4.10)

(region D) $g \leq t : \quad P_{max} = \mu^2_+$

Now, we would like to unify the regions as many as possible to simplify the expression of $P_{max}$. First, one can show that $D_1$ is always non-positive in region A as following. Since $h^2 \leq 4t(g - 2t)$ in region A, in this region

$$
D_1 = gh^2 - (g + t)^2(g - 2t) \leq -(g - 2t)(g - t)^2 \leq 0.
$$

(4.11)

Second, one can show easily that $D_1$ is always non-negative at region D as following. In this region

$$
D_1 = gh^2 + (g + t)^2(2t - g) \geq 0
$$

(4.12)

because both terms are non-negative. Combining these facts and Eq. (4.9) makes $P_{max}$ to be expressed as

$$
P_{max} = \begin{cases} 
\mu^2_2 & \text{when } D_1 \leq 0 \\
\mu^2_+ & \text{when } D_1 \geq 0.
\end{cases}
$$

(4.12)
In order to understand the behavior of $P_{\text{max}}$ more clearly we introduce the two parameters $u$ and $v$ as following:

$$
g = \sin u \cos v, \quad t = \sin u \sin v / \sqrt{3}, \quad h = \cos u
$$

with $0 \leq u, v \leq \pi / 2$. Then, one can plot the applicable domains $D_1 \leq 0$ and $D_1 \geq 0$ in the $u - v$ plane, which is Fig. 1a. As Fig. 1a has shown, the domain for $D_1 \leq 0$ is biased in the small $v$ and large $u$ region. This indicates that the domains for $D_1 \leq 0$ is around large $g$ region. The remaining region is the domain for $D_1 \geq 0$. As will be shown in next section, the number of the applicable domains for $\gamma = \pi / 2$ case is not two but three. This means that the phase factor $\gamma$ has great impact in the geometric measure of entanglement.

Fig. 1b is $(u,v)$-dependence of $P_{\text{max}}$ given in Eq.(4.12). At $u = 0$, which means $h = 1$, $P_{\text{max}}$ becomes 1 because it is separable state. At $v = 0$ and $u = \pi / 2$, which means that $g = 1$, $P_{\text{max}}$ becomes 1 again. Between them there is valley, which represents the set of the highly entangled states. There is different kind of the highly entangled states around $u = v = \pi / 2$. These highly entangled states are states located near W-state, $|W\rangle = (1 / \sqrt{3})(|011\rangle + |101\rangle + |110\rangle)$.

In order to compare $P_{\text{max}}$ with the applicable domains we plot $P_{\text{max}}$ and the boundary of domains simultaneously in $u - v$ plane in Fig. 1c. In Fig. 1c the black thick line is a boundary of the domains. The thick-blue color and light-blue (or white) colors represent the highly-entangled and less-entangled states, respectively. In the right-upper corner there are many highly entangled states which are located near W-state. Another type of the highly entangled states reside near the boundary of the applicable domains. Apart from the boundary more and more the quantum states lose the entanglement, and eventually reduce to the separable state.

Now, we consider several special cases. First example is $t = 1 / \sqrt{3}$ and $g = h = 0$. In this case $D_1 = 2\sqrt{3} / 9 > 0$ and $r_+ = \sqrt{8 / 3}$, which gives $P_{\text{max}} = 4 / 9$. Second example is $t = 0$ and $g \geq h$. In this case $D_1 = -g(g^2 - h^2) \leq 0$ and $P_{\text{max}} = g^2$. Third example is $t = 0$ and $g \leq h$. In this case $D_1 = g(h^2 - g^2) \geq 0$ and $r_+ = 2h$, which gives $P_{\text{max}} = h^2$. The second and third examples are consistent with $P_{\text{max}}(GHZ) = \max(|\alpha|^2, |\beta|^2)$, where $|GHZ\rangle = \alpha |000\rangle + |111\rangle$. Fourth example is $g = 0$ case. In this case $D_1 = 2t^3 \geq 0$ and $r_+ = h + \sqrt{h^2 + 8t^2}$, which results in

$$
P_{\text{max}} = \frac{(h^4 + 8h^2t^2 + 8t^4) + h(h^2 + 4t^2)\sqrt{h^2 + 8t^2}}{(h^2 + 6t^2) + h\sqrt{h^2 + 8t^2}}.
$$

(4.14)
One can show that various limits of Eq. (4.14) are consistent with the previously derived results. The last example is $h = 0$ case. In this case it is easy to show

$$P_{\text{max}} = \begin{cases} 
g^2 & \text{when } g \geq 2t \\
4t^3/(3t - g) & \text{when } g \leq 2t.
\end{cases} \quad (4.15)$$

Eq. (4.15) is perfectly in agreement with the result of Ref. [22].

V. GEOMETRIC MEASURE FOR $\gamma = \pi/2$

In this section we would like to compute the geometric entanglement measure for $\gamma = \pi/2$ case. From the constraint of the positive Lagrange multiplier constant we can exclude $\nu_1^2$ and $\nu_2^2$ from beginning stage for the computation of the geometric measure. Next, we should examine the sign of the Lagrange multiplier constant for $\nu_1^2$, that is

$$\lambda_+ = hs_+ - 2t(g - t). \quad (5.1)$$

It is easy to show that $\lambda_+ \geq 0$ in $g \leq t$ region. Also it is straightforward to show that $\lambda_+ \geq 0$ when $C_+ \geq 0$ and $\lambda_+ \leq 0$ when $C_+ \leq 0$, where

$$C_+ = h^2(2g + t) - t(g - t)^2. \quad (5.2)$$
Examining Table II and Eq. (5.2) leads us to divide the whole parameter space into the following ten regions:

\[(i) \quad g \geq 2t \quad (5.3)\]

(region I) \( C_2 \leq 0 \) \& \( C_+ \leq 0 \): \( P_{\text{max}} = \nu_P^2 \)

(region II) \( C_2 \geq 0 \) \& \( C_+ \leq 0 \): \( P_{\text{max}} = \max(\nu_P^2, \nu_2^2) \)

(region III) \( C_2 \leq 0 \) \& \( C_+ \geq 0 \): \( P_{\text{max}} = \max(\nu_P^2, \nu_+^2) \)

(region IV) \( C_2 \geq 0 \) \& \( C_+ \geq 0 \): \( P_{\text{max}} = \max(\nu_P^2, \nu_+^2, \nu_2^2) \)

\[(ii) \quad t \leq g \leq 2t\]

(region V) \( C_2 \geq 0 \) \& \( C_+ \leq 0 \): \( P_{\text{max}} = \nu_P^2 \)

(region VI) \( C_2 \leq 0 \) \& \( C_+ \leq 0 \): \( P_{\text{max}} = \max(\nu_P^2, \nu_2^2) \)

(region VII) \( C_2 \geq 0 \) \& \( C_+ \geq 0 \): \( P_{\text{max}} = \max(\nu_P^2, \nu_+^2) \)

(region VIII) \( C_2 \leq 0 \) \& \( C_+ \geq 0 \): \( P_{\text{max}} = \max(\nu_P^2, \nu_+^2, \nu_2^2) \)

\[(iii) \quad g \leq t\]

(region IX) \( C_2 \leq 0 \): \( P_{\text{max}} = \max(\nu_+^2, \nu_2^2) \)

(region X) \( C_2 \geq 0 \): \( P_{\text{max}} = \nu_+^2 \)

where

\[ C_2 = (3g + 2t)h^2 - 4g^2 t. \quad (5.4) \]

Although the whole space is divided into the ten regions, one can show that some regions do not exist. In order to show this it is convenient to introduce

\[ h_2 = \frac{4g^2 t}{3g + 2t} \quad h_+ = \frac{t(g - t)^2}{2g + t} \quad (5.5) \]
FIG. 3: Pictorial representation for $C_2 \geq 0, C_2 \leq 0, C_+ \geq 0, C_+ \leq 0, C_3 \geq 0$ and $C_3 \leq 0$ when $t \leq g \leq 2t$ (Fig. 2 a) and $g \geq 2t$ (Fig. 2 b).

Then, their difference becomes

$$h_2 - h_+ = \frac{t(g + t)^2}{(3g + 2t)(2g + t)}(5g - 2t). \quad (5.6)$$

Eq. (5.6) implies that $h_2 \geq h_+$ in the region $g \geq t$. Then the regions $C_2 \geq 0, C_2 \leq 0, C_+ \geq 0$, and $C_+ \leq 0$ when $g \geq t$ can be represented as Fig. 2. With an help of Fig. 2 it is easy to understand that there is no region which satisfies both $C_2 \geq 0$ and $C_+ \leq 0$ when $g \geq t$. This implies that region II and region V do not exist in the whole parameter space.

In order to compare $\nu_2^2$ with $\nu_+^2$ we compute $\nu_P^2 - \nu_2^2$, which is

$$\nu_P^2 - \nu_2^2 = \frac{g(g + t)(g - 2t)^2}{g^2 + h^2 - 3gt}. \quad (5.7)$$

Therefore, the sign of $\nu_P^2 - \nu_2^2$ is determined by $g^2 + h^2 - 3gt$. If $C_2 \geq 0$, $h^2 \geq h_2$ and

$$g^2 + h^2 - 3gt \geq \frac{3g(g - 2t)(g + t)}{3g + 2t}. \quad (5.8)$$

Therefore, if $C_2 \geq 0$ in $g \geq 2t$ region, $\nu_P^2 \geq \nu_2^2$. Thus, we can exclude $\nu_2^2$ in region IV. Similarly, one can show that if $C_2 \leq 0$ in $t \leq g \leq 2t$ region, $\nu_P^2 \leq \nu_2^2$. Therefore, we can exclude $\nu_2^2$ in regions VI and VIII.

Next, we compute $\nu_P^2 - \nu_+^2$, which is

$$\nu_P^2 - \nu_+^2 = \frac{2}{s_+^2 + 4t^2} \left( \alpha_2 + \beta_2 \sqrt{h^2 + 4t(2t + g)} \right). \quad (5.9)$$
where

\[
\alpha_2 = -h^4 + (g + 2t)(g - 4t)h^2 + 2t(g - t)(g + 2t)^2 \tag{5.10}
\]

\[
\beta_2 = h(g^2 - h^2 - 4t^2).
\]

Direct calculation shows that in \( g \geq t \) region \( \nu_P^2 = \nu_+^2 \) when \( C_3 = 0 \), where

\[
C_3 = gh^2 - (g - t)(g + 2t). \tag{5.11}
\]

In addition, simple consideration shows that in \( g \geq t \) region \( \nu_P^2 \geq \nu_+^2 \) when \( C_3 \leq 0 \) and \( \nu_P^2 \leq \nu_+^2 \) when \( C_3 \geq 0 \).

In order to check which eigenvalue is dominant in each region it is convenient to introduce another parameter

\[
h_3 = \frac{(g - t)(g + 2t)}{g}. \tag{5.12}
\]

Then, it is easy to show

\[
h_+ \leq h_2 \leq h_3 \quad \text{when } 2t \leq g
\]

\[
h_+ \leq h_3 \leq h_2 \quad \text{when } t \leq g \leq 2t.
\]

Eq.(5.13) enables us to represent \( C_2 \geq 0, C_2 \leq 0, C_+ \geq 0, C_+ \leq 0, C_3 \geq 0 \) and \( C_3 \leq 0 \) in one-dimensional coordinate, which is illustrated in Fig. 3. With an help of Fig. 3 one can show easily that in region III \( C_3 \) is always non-positive and therefore, \( P_{\text{max}} \) becomes \( \nu_P^2 \).

Using Fig. 3a \( P_{\text{max}} \) in region VII is \( \nu_+^2 \). Using Fig. 3b again one can show that region IV is divided into

\[
\text{(region IV-a) } C_2 \geq 0 \& C_3 \leq 0 : \quad P_{\text{max}} = \nu_P^2 \tag{5.14}
\]

\[
\text{(region IV-b) } C_2 \geq 0 \& C_3 \geq 0 : \quad P_{\text{max}} = \nu_+^2.
\]

Finally, we compute \( \nu_+^2 - \nu_2^2 \), which is

\[
\nu_+^2 - \nu_2^2 = \frac{2}{(s_+^2 + 4t^2)(g^2 + h^2 - 3gt)} \left( \alpha_3 + \beta_3 \sqrt{h^2 + 4t(2t + g)} \right) \tag{5.15}
\]

where

\[
\alpha_3 = h^6 + t(8t - g)h^4 - 4t^2(g^2 + 5gt - 2t^2)h^2 + 16g^2t^4 \tag{5.16}
\]

\[
\beta_3 = h \left[ h^4 + t(4t - 3g)h^2 + 4gt^2(g - 2t) \right].
\]
One can show directly that $\nu_+^2 - \nu_2^2 = 0$ when $C_2 = 0$. Also, it is straightforward to show that in $g \leq 2t$ region $\nu_+^2$ is always smaller than $\nu_2^2$. Therefore, we can exclude $\nu_+^2$ in regions VIII and IX. Combining all of these facts, one can express $P_{\text{max}}$ for $\gamma = \pi/2$ case as follows:

\[
P_{\text{max}} = \begin{cases} 
\nu_+^2 & \text{if } g \geq 2t \
\nu_2^2 & \text{if } g \leq 2t
\end{cases}
\]

\[
P_{\text{max}} = \begin{cases} 
\nu_+^2 & C_2 \geq 0 \
\nu_2^2 & C_2 \leq 0
\end{cases}
\]

Unlike $\gamma = 0$ case the whole parameter space is divided into the three applicable domains. Introducing the parameters $u$ and $v$ as Eq.(4.13) we plot the three applicable domains in the $u$-$v$ plane in Fig. 4a. Around $h = 0$ axis there are two domains, i.e. $\nu_+^2$ and $\nu_2^2$. Since $\nu_+^2$ and $\nu_2^2$ go to $\mu_+^2$ and $\mu_2^2$ in the $h \to 0$ limit, this guarantees that the $h \to 0$ limit is consistent with same limit of $\gamma = 0$ case. The applicable domain for $\nu_+^2$ is little bit larger than the domain $\mu_2^2$ for $\gamma = 0$ case. The point $(u = \cos^{-1}(\sqrt{2}/3), v = \tan^{-1}(\sqrt{3}/2))$ is shared by three domains. This point corresponds to

\[
|\psi_W\rangle = \frac{2}{3}|000\rangle + \frac{1}{3}(|011\rangle + |101\rangle + |110\rangle) + i\frac{\sqrt{2}}{3}|111\rangle.
\]

This is LU-equivalent with $|W\rangle = (1/\sqrt{3})(|100\rangle + |010\rangle + |001\rangle)$ as shown in Ref.[30].

In Fig. 4b we plot the $(u,v)$-dependence of $P_{\text{max}}$ given in Eq.(5.17). Like Fig. 1b the highly entangled states are represented as a valley in this figure. Fig. 4b seems to show that there exists an alley in the valley, which ends at $u = v = \pi/2$. Along this alley so many highly entangled states are located. Comparing Fig. 4b with Fig. 1b, one can realize that there are many more highly-entangled states for $\gamma = \pi/2$ case than $\gamma = 0$ case. This is mainly due to the fact that there are two LU-equivalent W-states when $\gamma = \pi/2$.

Fig. 4c shows the geometric entanglement measure and the applicable domains simultaneously in the $u$-$v$ plane. Fig. 4c shows that around two W-states there are so many highly entangled states, which we would like to call W-neighbors. Especially, the neighbors of $|\psi_W\rangle$ in Eq.(5.18) gather along $C_3 = 0$ line. Besides the W-neighbors there are many highly entangled states around boundary of the applicable domains. These are the neighbors of the shared states[22], and we would like to call them the GHZ-neighbors. The GHZ-neighbors
FIG. 4: (Color online) Fig. 4(a) is a plot of the applicable domains for $\gamma = \pi/2$ case in $(u, v)$-plane. Unlike $\gamma = 0$ case there are three applicable domains in this case. The principal domain $P_{\text{max}} = \nu_P^2$ is larger than $P_{\text{max}} = \mu_P^2$ in $\gamma = 0$ case. This fact seems to indicate that the principal domain increases its territory with increasing $\gamma$. It is important to note that the domain $P_{\text{max}} = \nu_+^2$ is not reached to $h = 0$ axis. This implies the consistency of the $h \to 0$ limit. Fig. 4(b) is $(u, v)$-dependence of $P_{\text{max}}$. The highly entangled states forms a valley between two mountains. Fig. 4(c) is a plot of $P_{\text{max}}$ and the applicable domains in the $(u, v)$-plane. The boundaries of the domains are represented by black think line. Many highly-entangles states reside around the boundaries and in the domain $P_{\text{max}} = \nu_+^2$. It is mainly due to the fact that there are two LU-equivalent W-states for $\gamma = \pi/2$ case.

are slightly less-entangled compared to the W-neighbors. However, the number of the GHZ-neighbors are many more than that of the W-neighbors.
Finally, we consider the several special cases. First example is $h = 0$ case. In this case $C_2 = -4g^2t \leq 0$ and $C_3 = -(g - t)^2(g + 2t) \leq 0$, which results in identical expression with Eq. (4.15). Therefore, both results for $\gamma = 0$ and $\gamma = \pi/2$ cases coincide with each other in the $h \to 0$ limit. Second example is $t = 0$ case. It is easy to show that in this case $P_{\text{max}} = g^2$ when $g \geq h$ and $P_{\text{max}} = h^2$ when $g \leq h$. This is consistent with $P_{\text{max}}(GHZ) = \max(|\alpha|^2, |\beta|^2)$ when $|GHZ\rangle = \alpha|000\rangle + |111\rangle$.

VI. EIGENVALUES AND GEOMETRIC MEASURE FOR $\gamma = \pi/4$ : NUMERICAL APPROACH

In this section we will compute the eigenvalues and the geometric measure for $\gamma = \pi/4$ case.

A. Eigenvalues

For $\gamma = \pi/4$ Eq. (2.10) reduces to

$$2t(g + t) \sin \theta \cos \varphi + \sqrt{2}ht(1 - \cos \theta) = \lambda \sin \theta \cos \varphi$$

(6.1a)

$$- 2t(g - t) \sin \theta \sin \varphi + \sqrt{2}ht(1 - \cos \theta) = \lambda \sin \theta \sin \varphi$$

(6.1b)

$$(g^2 - t^2)(1 + \cos \theta) - h^2(1 - \cos \theta) - \sqrt{2}ht \sin \theta (\sin \varphi + \cos \varphi) = \lambda \cos \theta.$$  

(6.1c)

When $\theta = 0$, Eq. (6.1a) and Eq. (6.1b) are automatically solved and Eq. (6.1c) gives

$$\lambda = 2(g^2 - t^2).$$

(6.2)

Since $s = (0, 0, 1)$ for this case, from Eq. (2.4) the corresponding eigenvalue is

$$\rho_2^2 = g^2.$$  

(6.3)

When $\sin \theta \neq 0$, Eq. (6.1a) and Eq. (6.1b) reduce to

$$z = \frac{\lambda - 2g\theta - 2t^2}{\sqrt{2}ht} \cos \varphi = \frac{\lambda + 2g\theta - 2t^2}{\sqrt{2}ht} \sin \varphi$$

(6.4)

where $z = \tan(\theta/2)$. From Eq. (6.4) one can compute $\varphi$ if $\lambda$ is known by using

$$\tan \varphi = \frac{(\lambda - 2t^2) - 2gt}{(\lambda - 2t^2) + 2gt}.$$  

(6.5)
Deriving $\sin \varphi + \cos \varphi$ from Eq. (6.14) and inserting it into Eq. (6.1c), one can derive the expression of $z^2$ in a form

$$ z^2 = \frac{[(\lambda - 2t^2)^2 - 4g^2t^2](\lambda - 2g^2 + 2t^2)}{(\lambda - 2h^2)(\lambda - 2t^2) - 8h^2t^2(\lambda - 2t^2) - 4g^2t^2(\lambda - 2h^2)}. \quad (6.6) $$

On the other hand, one can derive a different expression of $z^2$ directly from Eq. (6.4)

$$ z^2 = \frac{(\lambda - 2gt - 2t^2)^2}{2h^2t^2}(1 + \tan^2 \varphi)^{-1} = \frac{[(\lambda - 2t^2)^2 - 4g^2t^2]^2}{4h^2t^2[(\lambda - 2t^2)^2 + 4g^2t^2]}. \quad (6.7) $$

Equating Eq. (6.6) with Eq. (6.7) yields an equation for solely $\lambda$:

$$ \lambda f(\lambda) = 0 \quad (6.8) $$

where

$$ f(\lambda) = \lambda^4 - 2(h^2 + 4t^2)\lambda^3 - 4t^2(2g^2 - h^2 - 6t^2)\lambda^2 
+ 8\left[t^4(h^2 - 4t^2) + g^2(3h^2t^2 + 4t^4)\right] \lambda 
+ 16t^4\left(g^4 - 5g^2h^2 - 2g^2t^2 - h^2t^2 + t^4\right). \quad (6.9) $$

Eq. (6.8) guarantees the existence of the eigenvalue for $\lambda = 0$ as $\gamma = 0$ and $\gamma = \pi/2$ cases. In fact, one can show that there exists an eigenvalue corresponding to $\lambda = 0$ for arbitrary $\gamma$. We have shown this fact in appendix A.

When $\lambda = 0$, Eq. (6.5) and Eq. (6.7) reduce to

$$ z^2 = \frac{g^2 - t^2}{h^2(g^2 + t^2)} \quad \tan \varphi = -\frac{g + t}{g - t}. \quad (6.10) $$

Combining Eq. (6.3) and Eq. (6.10), the possible solutions for $\theta$ and $\varphi$ are

$$ z = \pm \frac{g^2 - t^2}{h\sqrt{g^2 + t^2}} \quad \cos \varphi = \mp \frac{g - t}{\sqrt{2(g^2 + t^2)}} \quad \sin \varphi = \pm \frac{g + t}{\sqrt{2(g^2 + t^2)}}. \quad (6.11) $$

It is easy to show that both solutions in Eq. (6.11) gives a same eigenvalue, which is

$$ \rho_0^2 = \frac{g^2(g^2 + t^2)h^2 + t^2(g^2 - t^2)^2}{h^2(g^2 + t^2) + (g^2 - t^2)^2}. \quad (6.12) $$

Finally, let us consider $f(\lambda) = 0$. It is worthwhile noting that at $h \to 0$ limit $f(\lambda) = 0$ reduces to $(\lambda - 2gt - 2t^2)^2(\lambda + 2gt - 2t^2)^2 = 0$. Therefore, the eigenvalues corresponding to $f(\lambda) = 0$ should coincide with $\mu_2^2$ and $\mu_3^2$ for $\gamma = 0$ case, and with $\nu_2^2$ and $\nu_3^2$ for $\gamma = \pi/2$ case at the $h \to 0$ limit. Equation $f(\lambda) = 0$ gives four solutions of $\lambda$, say $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_4$. We ordered the solutions by a fact that the $h \to 0$ limit of $\lambda_1$ and $\lambda_2$ is $-2t(g - t)$ and same limit of $\lambda_3$ and $\lambda_4$ is $2t(g + t)$. Then, the corresponding eigenvalues, say $\rho_1^2$, $\rho_2^2$, $\rho_3^2$, and $\rho_4^2$, can be computed numerically.
B. geometric measure

FIG. 5: (Color online) Fig. 5(a) is a plot of the applicable domains for $\gamma = \pi/4$ case. In this case there are two applicable domains. The principal domain $P_{\text{max}} = \rho_P^2$ is little bit larger than $P_{\text{max}} = \mu_P^2$ for $\gamma = 0$ and little bit smaller than $P_{\text{max}} = \nu_P^2$ for $\gamma = \pi/2$. This fact indicates that the principal domain increases its territory with increasing $\gamma$. Fig. 5(b) is $(u, v)$-dependence of $P_{\text{max}}$. As $\gamma = 0$ case the highly-entangled states form a valley between two mountains. Fig. 5(c) is a plot of $P_{\text{max}}$ and the applicable domains in the $(u, v)$-plane. Many highly-entangled states reside around boundary of the domains and near W-state.

Using eigenvalues $\rho_P^2, \rho_0^2$ derived analytically and $\rho_i^2$ ($i = 1, 2, 3, 4$) computed numerically, one can compute $P_{\text{max}}$ for the $\gamma = \pi/4$ case. Since each eigenvalue has its own available region, we checked this region by imposing $\text{Re}[\lambda] = 0, -1 \leq \sin \theta \leq 1, -1 \leq \cos \theta \leq 1, -1 \leq \sin \varphi \leq 1, \text{and } -1 \leq \cos \varphi \leq 1$. Although there are six different eigenvalues, the numerical calculation shows that only $\rho_P^2$ and $\rho_i^2$ contribute to the geometric measure. This indicates that the whole parameter space is divided into two applicable domains. These two domains are represented in $u - v$ plane in Fig. 5a. The domains $\rho_P^2$ is slightly larger than domain $\mu_P^2$ and slightly smaller than domain $\nu_P^2$. This fact seems to indicate that the domain containing $g = 1$ extends its territory with increasing $\gamma$.

Fig. 5b is a $(u, v)$-dependence of $P_{\text{max}}$ for $\gamma = \pi/4$. Similarly with $\gamma = 0$ and $\pi/2$ cases, many highly entangled states reside at the valley between two mountains. Another highly entangled states reside around $u = v = \pi/2$, which corresponds to W-state. The alley appeared in Fig. 4b does not appear in this case. This seems to be due to the fact that there is only one W-state in $\gamma = \pi/4$ case.

Fig. 5c is a $(u, v)$-dependence of $P_{\text{max}}$ and domains. As expected the highly entangled states are located around boundary and W-state.

VII. CONCLUSION

In this paper we have explored the effect of the phase factor in the geometric entanglement measure. We have chosen the most general three-qubit states which have symmetry under
the qubit-exchange. Our choice of the quantum states enables us to derive all eigenvalues
and geometric measure analytically when the phase factor $\gamma$ is 0 or $\pi/2$. It turns out that
the $\gamma = \pi/2$ case has three applicable domains while the $\gamma = 0$ case has two domains. Most
highly entangled states reside around the boundaries of the domains and near W-state.
Apart from the boundaries more and more the quantum states lose their entanglement and
eventually, become the product states.

Our result naturally gives rise to a question: what is a critical $\gamma$, say $\gamma_c$, which distinguish
the two and three domains? In order to explore this question we have analyzed the $\gamma = \pi/4$
case numerically. Our numerical calculation shows that there are six different eigenvalues
for $\gamma = \pi/4$ case, but only two of them contribute to the geometric entanglement measure.
Thus, there are two domains for $\gamma = \pi/4$.

We conjecture that emergence of the three applicable domains at $\gamma = \pi/2$ is due to the
two LU-equivalent W-states. In order to confirm our conjecture we checked numerically
$\gamma = \pi/3$ and $\gamma = 11\pi/24$ cases, which also give two applicable domains. We also checked
the applicable domains for the partially symmetric quantum state

$$|\psi\rangle = g|000\rangle + t|011\rangle + t|101\rangle + t_3|110\rangle + e^{i\gamma} h|111\rangle \quad (7.1)$$

numerically when $\gamma = 0$. This case also gives two applicable domains. Therefore, we
conclude that the emergence of the three applicable domains is due to the appearance of
additional W-state.

In appendix we have shown that there exist eigenvalues for all $\gamma$, whose Lagrangian
multiplier constant is zero. Although we conjecture that this is due to some symmetry of
the quantum state $|\psi\rangle$, we do not know the exact physical reason for the emergence of these
solutions. It seems to be of interest to reveal the physical meaning of these solutions clearly.

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Appendix A

In this appendix we would like to show the existence of the eigenvalue $\mu_0^2$, which corresponds to $\lambda = 0$, at arbitrary $\gamma$. When $\lambda = 0$, Eq.(2.10) reduces to

\begin{align}
2ht \cos \gamma (1 - \cos \theta) + 2t(g + t) \sin \theta \cos \varphi &= 0 \\
2ht \sin \gamma (1 - \cos \theta) - 2t(g - t) \sin \theta \sin \varphi &= 0 \\
(g^2 - t^2)(1 + \cos \theta) - h^2(1 - \cos \theta) - 2ht \sin \theta \cos(\varphi - \gamma) &= 0. \tag{A.1c}
\end{align}

The existence of $\mu_0^2$ can be shown as following. First we derive $\theta$ and $\varphi$ by making use of Eq.(A.1a) and Eq.(A.1b). Then we show that the solutions $\theta$ and $\varphi$ also solve Eq.(A.1c).

Now, we consider only $\sin \theta \neq 0$ case. Then from Eq.(A.1a) and Eq.(A.1b) it is easy to derive

\begin{align}
(g + t) \sin \gamma \cos \varphi + (g - t) \cos \gamma \sin \varphi &= 0, \tag{A.2}
\end{align}

which gives

\begin{align}
\tan \varphi = -\frac{g + t}{g - t} \tan \gamma. \tag{A.3}
\end{align}

Combining Eq.(A.2) and Eq.(A.3), one can derive the solution for $\varphi$, which is

\begin{align}
\cos \varphi &= \pm \frac{g - t}{\sqrt{(g - t)^2 + (g + t)^2} \tan^2 \gamma}, \\
\sin \varphi &= \mp \frac{(g + t) \tan \gamma}{\sqrt{(g - t)^2 + (g + t)^2} \tan^2 \gamma}. \tag{A.4}
\end{align}

Inserting Eq.(A.4) into Eq.(A.1b), one can derive $\sin \theta$ in a form

\begin{align}
\sin \theta = \mp \frac{2h(g^2 - t^2) \sqrt{g^2 + t^2 - 2gt \cos 2\gamma}}{h^2[(g^2 + t^2) - 2gt \cos 2\gamma] + (g^2 - t^2)^2}. \tag{A.5}
\end{align}

Inserting Eq.(A.4) and Eq.(A.5) into the lhs of Eq.(A.1c), one can show straightforwardly that Eq.(A.1c) is solved already by Eq.(A.4) and Eq.(A.5). This guarantees the existence of $\mu_0^2$.

In order to derive $\mu_0^2$ explicitly we choose the upper sign in Eq.(A.4) and Eq.(A.5). Then the components of the vector $s$ becomes

\begin{align}
s_x &= \sin \theta \cos \varphi = -\frac{2h(g - t)^2(g + t) \cos \gamma}{h^2[(g^2 + t^2) - 2gt \cos 2\gamma] + (g^2 - t^2)^2}, \tag{A.6a}

s_y &= \sin \theta \sin \varphi = \frac{2h(g - t)(g + t)^2 \sin \gamma}{h^2[(g^2 + t^2) - 2gt \cos 2\gamma] + (g^2 - t^2)^2}, \tag{A.6b}

s_z &= \cos \theta = \frac{h^2[(g^2 + t^2) - 2gt \cos 2\gamma] - (g^2 - t^2)^2}{h^2[(g^2 + t^2) - 2gt \cos 2\gamma] + (g^2 - t^2)^2}. \tag{A.6c}
\end{align}
Inserting Eq. (A.6) into Eq. (2.4) and performing tedious calculation, one can show that $\mu_0^2$, eigenvalue corresponding to $\lambda = 0$, becomes

$$\mu_0^2 = \frac{g^2 h^2 (g^2 + t^2 - 2gt \cos 2\gamma) + t^2 (g^2 - t^2)^2}{h^2 (g^2 + t^2 - 2gt \cos 2\gamma) + (g^2 - t^2)^2}.$$  \hspace{1cm} (A.7)

It is straightforward to show that the choice of lower sign in Eq. (A.4) and Eq. (A.5) leads us to same expression of $\mu_0^2$. One can show easily that $\mu_0^2$ exactly coincides with $\mu_1^2$ in Eq. (3.11), $\nu_1^2$ in Eq. (3.29) and $\rho_0^2$ in Eq. (5.12) when $\gamma = 0$, $\gamma = \pi/2$ and $\gamma = \pi/4$ respectively.

Finally, making use of explicit expression of $\mu_0^2$, one can derive the nearest product state $|q\rangle |q\rangle |q\rangle$ for $\mu_0^2$, i.e.

$$AB \langle q | q\rangle = \mu_0^2 |q\rangle$$  
$$AC \langle q | q\rangle = \mu_0^2 |q\rangle$$  
$$BC \langle q | q\rangle = \mu_0^2 |q\rangle$$  \hspace{1cm} (A.8)

where $|q\rangle$ is given in Eq. (2.3). Since $s$ is a Bloch vector of $|q\rangle |q\rangle$, one can show directly

$$|q\rangle = \frac{1}{\sqrt{h^2 \ell^2 + (g^2 - t^2)^2}} [h \ell |0\rangle - (g^2 - t^2)e^{-i\eta}|1\rangle]$$  \hspace{1cm} (A.9)

where

$$\ell^2 = g^2 + t^2 - 2gt \cos 2\gamma \hspace{1cm} \cos \eta = \frac{g - t}{\ell} \cos \gamma \hspace{1cm} \sin \eta = \frac{g + t}{\ell} \sin \gamma.$$  \hspace{1cm} (A.10)

Inserting Eq. (A.9) into Eq. (A.8) it is straightforward to show that $|q\rangle$ becomes

$$|q\rangle = \frac{1}{\mathcal{N}} \left[ \{ gh^2 \ell^2 + t(g^2 - t^2)^2 e^{i\eta} \} |0\rangle + e^{i\eta} h(g^2 - t^2) \{ (g^2 - t^2) e^{i(\gamma + \eta)} - 2t \ell \} |1\rangle \right]$$  \hspace{1cm} (A.11)

where $\mathcal{N}$ is a normalization constant, which makes $|q\rangle$ unit vector.

For $\gamma = 0$ case the nearest product state becomes

$$|q\rangle = \frac{1}{\sqrt{h^2 + (g + t)^2}} (h |0\rangle - (g + t) |1\rangle)$$  \hspace{1cm} (A.12)

$$|q\rangle = \sqrt{\frac{1}{\{ gh^2 + t(g + t)^2 \}^2 + h^2 (g^2 - t^2)^2}} \left[ \{ gh^2 + t(g + t)^2 \} |0\rangle + h(g^2 - t^2) |1\rangle \right].$$

It is interesting to note that $\langle q | q\rangle = 0$ when $\mathcal{D}_1 = 0$, where $\mathcal{D}_1$ is given in Eq. (4.8).

For $\gamma = \pi/2$ case $|q\rangle$ and $|q\rangle$ becomes

$$|q\rangle = \frac{1}{\sqrt{h^2 + (g - t)^2}} (h |0\rangle + i(g - t) |1\rangle)$$  \hspace{1cm} (A.13)

$$|q\rangle = \sqrt{\frac{1}{\{ gh^2 - t(g - t)^2 \}^2 + h^2 (g^2 - t^2)^2}} \left[ \{ gh^2 - t(g - t)^2 \} |0\rangle - ih(g^2 - t^2) |1\rangle \right].$$

It is interesting to note that $\langle q | q\rangle = 0$ when $\mathcal{C}_3 = 0$, where $\mathcal{C}_3$ is given in Eq. (5.11).