Topological Entanglement Entropy in Bilayer Quantum Hall Systems

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We calculate the topological entanglement entropy in bilayer quantum Hall systems, dividing the set of quantum numbers into four parts. This topological entanglement entropy allows us to draw a phase diagram in the parameter space of layer separation and tunneling amplitude. We perform a finite-size scaling analysis of the topological entanglement entropy in order to see the quantum phase transition clearly.

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I. INTRODUCTION

In some cases, no local order parameters can distinguish between the types of quantum order because quantum nature itself is nonlocal. To overcome this difficult situation, Kitaev and Preskill [1] have introduced the topological entanglement entropy, which is obtained by a well-designed partition and a clever linear combination of the corresponding entanglement entropies. Looking for an explicit microscopic model that realizes quantum order characterized by the topological entanglement entropy would be of interest. The purpose of this paper is to present a microscopic model related to the topological entanglement entropy.

A system with a mass gap in two spatial dimensions can exhibit topological order [2]. A mass gap is the key ingredient for the incompressible quantum Hall state [3]. The quasiparticle excitations in the quantum Hall system obey fractional statistics [4]. Furthermore, Haldane [5] showed that, in the quantum Hall system, the ground-state degeneracy depends on whether the geometry is a sphere or a torus. These features of the mass gap, fractional statistics, and the dependence of the degeneracy are all topological properties. Hence, considering the topological entanglement entropy in the quantum Hall system would be natural.

For applications of the topological entanglement entropy in relation with quantum phase transitions, we consider bilayer quantum Hall systems, where it is simpler to introduce controllable parameters into the Hamiltonian of the system. In bilayer quantum Hall systems, the two parameters to be controlled are the layer separation $d$ and the inter-layer tunneling amplitude $t$. The experimental strong evidence for the quantum phase transition in bilayer systems was a strong enhancement in the zero-bias inter-layer tunneling conductance for a small-$d$ system with a total Landau level filling factor $\nu = 1$ [6]. Theoretically, if $d$ goes to $\infty$ for fixed $\nu = 1$, the bilayer system becomes a set of two single-layer systems for $\nu = \frac{1}{2}$. For a large-$d$ system, the ground state would be compressible, which is not a quantum Hall state. Clearly, a phase transition takes place at the critical value $d_c$, which occurs as $d$ changes from 0 to $\infty$. The main concern now is to draw the phase diagram of the system in the parameter space of $d$ and $t$.

Pseudospin notation for the layer degree of freedom is used to find the phase diagram by calculating the pseudospin magnetization [7]. However, this pseudospin magnetization approach is not conclusive because calculations with varying $d$ but fixed $t$ give different results from those with varying $t$ but fixed $d$. Because the pseudospin is a local order parameter, the approach of the pseudospin magnetization may not provide a perfect explanation for the nature of quantum phase transitions in bilayer quantum Hall systems.

In this paper, we focus on the topological entanglement entropy, which is the most natural order parameter to study phase transitions in bilayer quantum Hall systems. Using exact diagonalization, we numerically evaluate the topological entanglement entropy in finite-size systems. We analyze the behavior of the topological entanglement entropy as we vary $d$ for fixed $t$. We also carry out an analysis of finite size scaling. We will show that the topological entanglement entropy provides a phase boundary, which is different from the boundary determined by the pseudospin magnetization. This difference is controversial. More detailed experimental measurements are required to resolve the issue of topo-
logical entanglement entropy as an order parameter for bilayer quantum Hall systems.

II. HAMILTONIAN

We start with presenting the bilayer quantum Hall system in terms of the second quantized form of the Hamiltonian in a torus geometry within the lowest Landau level approximation. The Landau level degeneracy $N$ is known to be determined by the magnetic field strength $B$ and the square of the torus area $L^2$ as $L^2 = 2\pi N l_B^2$ where $l_B = \sqrt{\hbar c/eb}$ is the magnetic length. Measuring all distances in the unit of $l_B$, and energies in the unit of $e^2/\epsilon l_B$, where $\epsilon$ is a dielectric constant, is convenient.

The two-body interaction between electrons is described by using the periodic Coulomb interaction $U(\vec{x}_i - \vec{x}_j)$, which is written in terms of position variables $\vec{x}$ and momentum variables $\vec{k}$ by using the Fourier transformation:

$$U(\vec{x}_i - \vec{x}_j) = \frac{e^2}{\epsilon ||\vec{x}_i - \vec{x}_j||} = \lim_{\mu \to 0} \frac{e^2}{(2\pi)^2} \int d^2k \frac{4\pi}{k^2 + \mu^2} \exp(\vec{k} \cdot (\vec{x}_i - \vec{x}_j)),$$

where $\mu$ is introduced in order to take into account the infrared divergence. If two electrons are on the same (different) layer, intra-(inter-)layer, the third component of $(\vec{x}_i - \vec{x}_j)$ is 0 ($d$). In the torus geometry of finite size $L^2$, the first and the second components $k_1$ and $k_2$ out of $\vec{k}$ turn to discrete integers $n_1$ and $n_2$ while $k_3$ is kept as continuous. Then, $U(\vec{x}_i - \vec{x}_j)$ between inter-layer electrons is written as

$$U(\vec{x}_i - \vec{x}_j) = \frac{e^2}{e^2(2\pi)^2} \left\{ \lim_{\mu \to 0} \int d^2k \frac{\exp(ik_3d)}{k^2 + \mu^2} + \sum_{(n_1, n_2) \neq (0, 0)} \int d^2k \frac{\exp(ik_3d)}{(d^2n_1^2 + d^2n_2^2 + k_3^2)} \right\},$$

where the first term is extracted as the infrared divergence part for the case of $n_1 = n_2 = 0$. Integrating out $k_3$, we obtain

$$U(\vec{x}_i - \vec{x}_j) = \lim_{\mu \to 0} \frac{e^2}{e^2(2\pi)^2} \frac{L^2}{\pi \mu} \exp(-\mu d) + \frac{2}{\epsilon L} \sum_{(n_1, n_2) \neq (0, 0)} \int d^2k \frac{\exp(-\mu d)}{(d^2n_1^2 + d^2n_2^2 + k_3^2)} \exp(\frac{2\pi}{L} n_1 \cdot (\vec{x}_i - \vec{x}_j)).$$

Expanding $exp(-\mu d)$ into the Taylor series with respect to $\mu$, we rewrite the first term of $U(\vec{x}_i - \vec{x}_j)$ as follows:

$$\frac{e^2}{e^2(2\pi)^2} \frac{L^2}{\pi \mu} \exp(-\mu d) = \frac{e^2}{e^2(2\pi)^2} \frac{L^2}{\pi \mu} = \frac{e^2}{e^2(2\pi)^2} \frac{L^2}{\pi \mu} = \frac{e^2}{e^2(2\pi)^2} \frac{L^2}{\pi \mu} + O(\mu).$$

The infinite first term explains the infrared divergence, and it should be canceled by the uniform positive background charge [8]. The finite second term, which depends on $d$, contributes to static charging-energy [9].

The Fourier transformation and handling of the infrared divergence make it straightforward to derive the second quantized Hamiltonian by using the single-particle wave function. For the lowest Landau level in the torus geometry, the $j$-th single-particle wave function $\psi_j(x, y) [10]$ is given by

$$\psi_j(x, y) = \left( \frac{1}{L\sqrt{\pi l_B}} \right)^{1/2} \exp(-\frac{y^2}{2l_B^2}) \times \sum_{k=-\infty}^{\infty} \exp(-\pi N(k + \frac{j}{N})^2 + i2\pi N(k + \frac{j}{N})(\frac{x}{L} + iy))].$$

The Hamiltonian $H$ for the bilayer quantum Hall system is the sum of the Coulomb interaction term $H_i$ and the single-particle inter-layer tunneling term $H_t$ such as

$$H = H_i + H_t.$$  (1)

Based on the above wave functions $\psi_j(x, y)$ and the periodic Coulomb interaction $U(\vec{x}_i - \vec{x}_j)$, we obtain the second quantized form of $H_i$. The Hamiltonian is expressed in terms of creation and annihilation operators $c^\dagger_\alpha$ and $c_{\beta \sigma}$, where pseudospin $\sigma = \uparrow$ or $\downarrow$ is used to describe different layers. We get

$$H_i = H_\uparrow + H_\downarrow + H_{\uparrow \downarrow} = \frac{e^2}{e^2(2\pi)^2} \frac{L^2}{\pi \mu} \frac{N \epsilon l_B}{N_l},$$  (2)

where $H_\uparrow (H_\downarrow)$ presents the interaction between electrons in the up (down) layer, $H_{\uparrow \downarrow}$ is the inter-layer Hamiltonian, and $N_\uparrow$ ($N_\downarrow$) in the charging-energy term is the number of electrons in the up (down) layer. The value of the product $N_\uparrow N_\downarrow$ is maximized at $N_\uparrow = N_\downarrow = N/2$ with the constraint $N_\uparrow + N_\downarrow = N$. Without the last term, all electrons would stay in a single layer according to Hund’s rule for a small $d$, as shown in Table 1.

Following the procedure given by Yoshioka et al. [11], we find

$$H_\sigma = \frac{1}{2} \sum_{a, b = 0}^{N-1} V(a, b; 0) \sum_{k=0}^{N-1} c^\dagger_{k+a\sigma} c_{k+a\sigma} c_{k+\sigma},$$

$$H_{\uparrow \downarrow} = \sum_{\sigma \neq \sigma'} \frac{1}{2} \sum_{a, b} V(a, b; d) \sum_{k} c^\dagger_{k+a\sigma} c_{k+a\sigma} c_{k+a\sigma}.$$  

Here, the orbital index $j$ in $c^\dagger_{j\sigma}$ should satisfy the periodic condition that $c^\dagger_{j+N \sigma} = c^\dagger_{j \sigma}$. The coefficients $V$ in the Hamiltonian are given by