RATIONAL SOLUTIONS OF PAINLEVÉ SYSTEMS.

DAVID GÓMEZ-ULLATE, YVES GRANDATI, AND ROBERT MILSON

Abstract. Although the solutions of Painlevé equations are transcendental in the sense that they cannot be expressed in terms of known elementary functions, there do exist rational solutions for specialized values of the equation parameters. A very successful approach in the study of rational solutions to Painlevé equations involves the reformulation of these scalar equations into a symmetric system of coupled, Riccati-like equations known as dressing chains. Periodic dressing chains are known to be equivalent to the $A_N$-Painlevé system, first described by Noumi and Yamada. The Noumi-Yamada system, in turn, can be linearized as using bilinear equations and $\tau$-functions; the corresponding rational solutions can then be given as specializations of rational solutions of the KP hierarchy.

The classification of rational solutions to Painlevé equations and systems may now be reduced to an analysis of combinatorial objects known as Maya diagrams. The upshot of this analysis is a an explicit determinantal representation for rational solutions in terms of classical orthogonal polynomials. In this paper we illustrate this approach by describing Hermite-type rational solutions of Painlevé of the Noumi-Yamada system in terms of cyclic Maya diagrams. By way of example we explicitly construct Hermite-type solutions for the $P_{IV}$, $P_{V}$ equations and the $A_4$ Painlevé system.

1. Introduction

The defining property of the six nonlinear second order Painlevé equations $P_{I}, \ldots, P_{VI}$ is that their solutions have fixed monodromy; that is all movable singularities are poles. The resulting Painlevé transcendents are now considered to be the nonlinear analogues of special functions [7, 11]. Although these functions are transcendental in the sense that they cannot be expressed in terms of known elementary functions, Painlevé equations also possess special families of solutions that, for special values of the parameters, can be expressed via known special functions such hypergeometric functions or even rational functions [2].

Rational solutions of $P_{II}$ were studied by Yablonskii [43] and Vorob’ev [41], in terms of a special class of polynomials that are now named after them. For $P_{III}$, classical solutions have been considered in [27], Okamoto [33] obtained special polynomials associated with some of the rational solutions of the fourth Painlevé equation (\(P_{IV}\))

\[
y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 3ty^2 + 2(t^2 - a)y + \frac{b}{y}, \quad y = y(t),
\]

with $a$ and $b$ constants, which are analogous to the Yablonskii–Vorob’ev polynomials. Noumi and Yamada [30] generalized Okamoto’s results and expressed all rational solutions of $P_{IV}$ in terms of two types of special polynomials, now known as the \textit{generalized Hermite polynomials} and \textit{generalized Okamoto polynomials}, both of which maybe given as determinants of sequences of Hermite polynomials.
A very successful approach in the study of rational solutions to Painlevé equations has been the set of geometric methods developed by the Japanese school, most notably by Noumi and Yamada. The core idea is to write the scalar equations as a set of first order coupled nonlinear system of equations. For instance, the fourth Painlevé (1) equation $P_{IV}$ is equivalent to the following autonomous system of three first order equations

\begin{align}
  f_0' &= f_0(f_1 - f_2) + \alpha_0, \\
  f_1' &= f_1(f_2 - f_0) + \alpha_1, \\
  f_2' &= f_2(f_0 - f_1) + \alpha_2,
\end{align}

subject to the condition

\begin{align}
  (f_0 + f_1 + f_2)' &= \alpha_0 + \alpha_1 + \alpha_2 = 1.
\end{align}

Once this equivalence is shown, it is clear that the symmetric form of $P_{IV}$ is easier to analyze. In particular, [30] showed that the system possesses a symmetry group of Bäcklund transformations acting on the tuple of solutions and parameters $(f_0, f_1, f_2 | \alpha_0, \alpha_1, \alpha_2)$. This symmetry group is the affine Weyl group $A^{(1)}_2$, generated by the operators \{\pi, s_0, s_1, s_2\} whose action on the tuple $(f_0, f_1, f_2 | \alpha_0, \alpha_1, \alpha_2)$ is given by:

\begin{align}
  s_k(f_j) &= f_j - \frac{\alpha_k \delta_{k+1,j}}{f_k} + \frac{\alpha_k \delta_{k-1,j}}{f_k}, \\
  s_k(\alpha_j) &= \alpha_j - 2\alpha_j \delta_{k,j} + \alpha_k (\delta_{k+1,j} + \delta_{k-1,j}), \\
  \pi(f_j) &= f_{j+1}, \quad \pi(\alpha_j) = \alpha_{j+1},
\end{align}

where $\delta_{k,j}$ is the Kronecker delta and $j, k = 0, 1, 2 \mod 3$. The technique to generate rational solutions is to first identify a number of very simple rational seed solutions, and then successively apply the Bäcklund transformations to generate families of rational solutions.

This is a beautiful approach which makes use of the hidden group theoretic structure of transformations of the equations, but the solutions built by dressing seed solutions are not very explicit, in the sense that one needs to iterate a number of Bäcklund transformations on the functions and parameters in order to obtain the desired solutions. Questions such as determining the number of zeros or poles of a given solution constructed in this manner seem very difficult to address. For this reason, alternative representations of the rational solutions have also been investigated, most notably the determinental representations [21, 22].

The system of first order equations admits a natural generalization to any number of equations, and it is known as the $A_N$-Painlevé or the Noumi-Yamada system. The higher order Painlevé system, exhibited below in (20) and (21), is considerably simpler (for reasons that will be explained later), and it is the one we will focus on this paper. The symmetry group of this higher order system is the affine Weyl group $A^{(1)}_N$, acting by Bäcklund transformations as in (4). The system has the Painlevé property, and thus can be considered a proper higher order generalization of $sP_{IV}$, which corresponds to $N = 2$.

The next higher order system belonging to this hierarchy is the $A_3$-Painlevé system, which is known to be equivalent to the scalar $P_V$ equation. Rational solutions of $P_V$ were classified using direct analysis by Kitaev, Law and McLeod [23]. These
rational solutions can be described in terms of generalized Umemura polynomials [37, 24], which admit a description in terms of Schur functions [29]. A more general determinantal representations based on universal characters was given by Tsuda [36]. In general, these determinants are constructed Laguerre polynomials, but for some particular values these degenerate to Hermite polynomials and fit into the framework of the present paper.

The $A_4$-Painlevé system cannot be reduced to a scalar equation, and so represents a genuine generalization of the classic Painlevé equations. Special solutions have been studied by Clarkson and Filipuk [10], who provide several classes of rational solutions via an explicit Wronskian representation, and by Matsuda [25], who uses the classical approach to identify the set of parameters that lead to rational solutions. However, a complete classification and explicit description of the rational solutions of $A_{2n}$-Painlevé for $n \geq 2$ is, to the best of our knowledge, still not available in the literature.

Of particular interest are the special polynomials associated to these rational solutions, whose zeros and poles structure shows extremely regular patterns in the complex plane, and have received a considerable amount of study [33, 30, 38, 12, 9]. Some of these polynomial families are known as generalized Hermite, Okamoto or Umemura polynomials, and they can be described as Wronskian determinants of given sequences of Hermite polynomials. We will show that all these polynomial families are only particular cases of a larger one.

Our approach for describing rational solutions to the Noumi-Yamada system differs from the one used by the Japanese school in that it makes no use of symmetry groups of Bäcklund transformations. Instead, we will be influenced by the approach of Darboux dressing chains introduced by the Russian school [1, 39], which has received comparatively less attention in connection to Painlevé systems, and which makes use of the notion of trivial monodromy [32]. Our interest in rational solutions of Painlevé equations follows from the recent advances in the theory of exceptional polynomials [14, 15, 18], and especially exceptional Hermite polynomials [17]. Nonetheless we strive to maintain the connection to the theory of integrable systems by employing the concepts of a Maya diagram and of bilinear relations [26].

The paper is organized as follows: in Section 2 we introduce the equations for a dressing chain of Darboux transformations of Schrödinger operators and prove that they are equivalent to the Noumi-Yamada system. These results are well known [1] but recalling them is useful to fix notation and make the paper self contained. In Section 3 we introduce the class of Hermite-type $\tau$ functions and their representations via Maya diagrams and pseudo-Wronskian determinants. We introduce the key notion of cyclic Maya diagrams and reformulate the problem of classifying rational solutions of the Noumi-Yamada system as that of classifying cyclic Maya diagrams. In Section 5 we introduce the notion of genus and interlacing for Maya diagrams which allows us to achieve a complete classification of $p$-cyclic Maya diagrams for any period $p$. Finally, we apply the theory to exhibit rational solutions of the $A_2$, $A_3$, $A_4$ systems in Section 6 and we write out explicitly these solutions using the representation developed in the previous sections.

While the Japanese school has built a beautiful framework around Painlevé equations, including reductions of the KP hierarchy in Sato theory, for the particular task of describing rational solutions of higher order Painlevé equations, we find the approach of cyclic Maya diagrams to be more direct, simple and explicit.
2. Dressing chains and Painlevé systems

A factorization chain is a sequence of Schrödinger operators connected by Darboux transformations. By replacing the second-order Schrödinger equations with first order Riccati equations one obtains a closely related called a dressing chain. The theory of dressing chains was developed by Adler [1], Veselov and Shabat [39]. The connection to Painlevé equations was already noted by the just-mentioned authors, and further developed by others [35, 5].

Let us recall the well-known connection between Riccati and Schrödinger equations. An elementary calculation shows that a function $w(z)$ that satisfies a Riccati equation

\begin{equation}
   w' + w^2 + \lambda = U
\end{equation}

is the log-derivative of a solution $\psi(z)$ of the corresponding Schrödinger equation:

\begin{equation}
   -\psi'' + U\psi = \lambda\psi, \quad w = \frac{\psi'}{\psi}.
\end{equation}

The Riccati equation (5) is equivalent to the factorization relation

\begin{equation}
   -D^2 + U = (D + w)(-D + w) + \lambda
\end{equation}

It follows that a Schrödinger operator $-D^2 + U$ admits a factorization (7) if and only if $w$ is the log-derivative of a formal eigenfunction of $L$ with eigenvalue $\lambda$.

A Darboux transformation is the transformation $U \mapsto \hat{U}$ where

\begin{equation}
   -D^2 + \hat{U} = (D - w)(-D - w) + \lambda
\end{equation}

is a second-order operator obtained by interchanging the factors in (7). Equivalently, the correspondence $U \mapsto \hat{U}$ may be engendered by the transformation $w \mapsto -w$ in (7).

Consider a doubly infinite sequence of Schrödinger operators $-D^2 + U_i, i \in \mathbb{Z}$ where neighbouring operators are related by a Darboux transformation

\begin{align}
   -D^2 + U_i &= (D + w_i)(-D + w_i) + \lambda_i, \\
   -D^2 + U_{i+1} &= (-D + w_i)(D + w_i) + \lambda_i.
\end{align}

Since functions $w_i$ are solutions of the Riccati equations

\begin{align}
   w_i' + w_i^2 + \lambda_i &= U_i, \\
   -w_i' + w_i^2 + \lambda_i &= U_{i+1},
\end{align}

the above potentials are related by

\begin{align}
   U_{i+1} &= U_i - 2w_i', \\
   U_{i+n} &= U_i - 2 \left( w_i' + \cdots + w_{i+n-1}' \right), \quad n \geq 2.
\end{align}

If we eliminate the potentials in (9) and set

\begin{equation}
   a_i = \lambda_i - \lambda_{i+1}
\end{equation}

we obtain a system of coupled differential equations called the doubly infinite dressing chain:

\begin{align}
   (w_i + w_{i+1})' + w_{i+1}^2 - w_i^2 &= a_i, \quad i \in \mathbb{Z}, \\
   U_{i+p} &= U_i + \Delta, \quad i \in \mathbb{Z}, \quad p \in \mathbb{N}, \quad \Delta \in \mathbb{C}
\end{align}

on the potentials of the above chain, we obtain a finite-dimensional system of ordinary differential equations. If this holds, then necessarily \( w_{i+p} = w_i, \alpha_{i+p} = \alpha_i, \) and
\[
\Delta = -(a_0 + \cdots + a_{p-1}).
\]
Going forward, we impose the non-degeneracy assumption that
\[
\Delta \neq 0
\]
Degenerate dressing chains with \( \Delta = 0 \) are more closely related to elliptic functions \([39]\) and will not be considered here.

**Definition 1.** A solution to the \( p \)-cyclic dressing chain with shift \( \Delta \) is a sequence of \( p \) functions \( w_0, \ldots, w_{p-1} \) and complex numbers \( a_0, \ldots, a_{p-1} \) that satisfy the following coupled Riccati-like equations:
\[
(w_i + w_{i+1})' + w_i w_{i+1} - w_i^2 = a_i, \quad i = 0, 1, \ldots, p-1 \mod p \text{ subject to the condition (15).}
\]
The cyclic chain has a number of evident symmetries: the reversal symmetry
\[
\hat{w}_i = -w_{-i}, \quad \hat{a}_i = -a_{-i};
\]
the cyclic symmetry
\[
\hat{w}_i = w_{i+1}, \quad \hat{a}_i = a_{i+1};
\]
and the scaling symmetry
\[
\hat{w}_i(z) = kw_i(\sqrt{\Delta} z), \quad \hat{a}_i = k^2 a_i, \quad k \neq 0.
\]
In the classification of solutions to (16) it will be convenient to regard solutions related by reversal, cyclic, and scaling symmetries as being equivalent.

The \( p \)-cyclic dressing chain is closely related to higher order Painlevé systems of type \( A_N, N = p - 1 \) introduced by Noumi and Yamada in \([28]\). In the even case of \( N = 2n \), the Noumi-Yamada system has the form
\[
f_i' = \sum_{j=1}^{p-1} (-1)^{j+1} f_i f_{i+j} + \alpha_i, \quad i = 0, \ldots, 2n \mod 2n + 1
\]
In the odd case of \( N = 2n - 1 \), the Noumi-Yamada system has a more complicated form:
\[
x f_i' = f_i \left( 1 - 2 \sum_{k=1}^{n-1} \alpha_{i+2k} + 2 \sum_{j=1}^{n} \sum_{k=1}^{n-1} \text{sgn}(2j - 1 - 2k) f_{2j+i-1} f_{2k+i} \right) + 2 \alpha_i \sum_{k=1}^{n-1} f_{i+2k},
\]
where \( i = 0, \ldots, 2n - 1 \mod 2n \) and \( f_i = f_i(x) \). In both cases, the parameters \( \alpha_0, \ldots, \alpha_N \) are subject to the constraint
\[
\alpha_0 + \cdots + \alpha_N = 1.
\]

**Proposition 1.** The \( A_{2n} \) and \( A_{2n-1} \) Noumi-Yamada systems \((20), (21)\) are related to the \( p \)-cyclic dressing chain \((10)\) by the following change of variables:
\[
-x f_i(x) = w_i(z) + w_{i+1}(z), \quad \alpha_i = \frac{a_i}{\Delta}, \quad z = \frac{x}{\sqrt{\Delta}}
\]
where \( i = 0, \ldots, p-1 \mod p \) and where \( p = 2n + 1 \) in the first case, and \( p = 2n \) in the second case.
Proof. The proof for the case $p = 2n + 1$ is quite direct. Set

$$d_i(x) = K(w_i(Kx) - w_{i+1}(Kx)), \quad K = \frac{1}{\sqrt{\Delta}},$$

which allows us to rewrite relation (13) as

$$f'_i = d_i f_i + \alpha_i.$$

Then, observe that because $p$ is odd,

$$d_i = \sum_{j=1}^{p-1} (-1)^{j+1} f_{i+j}.$$

This transforms (23) into (20). \qed

If $p = 2n$ is even, the linear relation (24) no longer holds. Rather we have the following quadratic relation.

Lemma 1. For each $i = 1, \ldots, 2n \mod 2n$ we have

$$(w_{i+1} - w_i)(w_1 + \cdots + w_{2n}) + (w_i^2 - w_{i+1}^2 + \cdots - w_{i+2n-1}^2) =$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n-1} \sgn(2k + 1 - 2j)(w_{2j+i-1} + w_{2j+i})(w_{2k+i} + w_{2k+i+1}).$$

Proof. The left side of (25) expands to

$$w_{i+1} \sum_{a=2}^{2n-1} w_{i+a} - w_i \sum_{a=2}^{2n-1} w_{i+a} + \sum_{a=2}^{n-1} (-1)^a w_{i+a}^2$$

The right side of (25) expands to

$$\sum_{1 \leq j < k \leq n-1} (w_{i+2j-1} + w_{i+2j})(w_{i+2k} + w_{i+2k+1}) - \sum_{1 \leq k < j \leq n} (w_{i+2j-1} + w_{i+2j})(w_{i+2k} + w_{i+2k+1})$$

$$= \sum_{1 \leq j < k \leq n} (w_{i+2j-1} + w_{i+2j})(w_{i+2k-2} + w_{i+2k-1}) - \sum_{1 \leq k < j \leq n} (w_{i+2j-1} + w_{i+2j})(w_{i+2k} + w_{i+2k+1})$$

$$= \sum_{1 \leq j < k \leq n} (w_{i+2j-1} + w_{i+2j})(w_{i+2k-2} + w_{i+2k-1}) - (w_{i+2k-1} + w_{i+2k})(w_{i+2j} + w_{i+2j+1})$$

$$= \sum_{1 \leq j < k \leq n} (w_{i+2j-1} - w_{i+2j+1})(w_{i+2k-1} + w_{i+2k-2} - w_{i+2k}) + w_{i+2j-1}w_{i+2k-2} - w_{i+2j+1}w_{i+2k}$$

$$= \sum_{k=2}^{n} (w_{i+1} - w_{i+2k-1})w_{i+2k-1} + \sum_{j=1}^{n-1} (w_{i+2j} - w_{i+2k-2} - w_{i+2k}) + \sum_{k=1}^{n-1} w_{i+2k}w_{i+2k} - \sum_{j=1}^{n-1} w_{i+2j+1}w_{i+2n}$$

$$= w_{i+1} \sum_{a=2}^{2n-1} w_{i+a} - w_i \sum_{a=2}^{2n-1} w_{i+a} + \sum_{a=2}^{n-1} (-1)^a w_{i+a}^2,$$

which matches (26). \qed

Proof of Proposition 7 continued. Every dressing chain has an obvious first integral, obtained by summing (16):

$$w_1(z) + \cdots + w_{2n}(z) = -\frac{1}{2} \Delta z.$$
For \( p = 2n \), the even-cyclic dressing chain also has an additional first integral — obtained by taking an alternating sum of (16):

\[
2(w_1^2 - w_2^2 + \cdots - w_{2n}^2) = -a_1 + a_2 - \cdots + a_{2n}.
\]

Using (25) and (28) we obtain

\[
x d_i(x) = 2 \sum_{j=1}^{n} \sum_{k=1}^{n-1} \text{sgn}(2k + 1 - 2j) f_{2j+i-1}(x) f_{2k+i}(x) + 2 \sum_{j=0}^{2n-1} (-1)^j \alpha_{i+j}.
\]

Relation (23) may now be rewritten as

\[
x f'_i(x) = 2 f_i(x) \sum_{j=1}^{n} \sum_{k=1}^{n-1} \text{sgn}(2j-1+2k) f_{2j+i-1}(x) f_{2k+i}(x) - 2 f_i(x) \sum_{j=0}^{2n-1} (-1)^j \alpha_{i+j} + \alpha_i x
\]

Since

\[
\sum_{j=1}^{n} f_{2j-1}(x) = \sum_{j=1}^{n} f_{2j}(x) = \frac{1}{2} x,
\]

and since

\[
\sum_{j=1}^{n} \alpha_{2j-1} + \sum_{j=1}^{n} \alpha_{2j} = 1,
\]

relation (30) may be rewritten as (21).

The problem now becomes that of finding and classifying cyclic dressing chains, sequences of Darboux transformations that reproduce the initial potential up to an additive shift \( \Delta \) after a fixed given number of transformations. The theory of exceptional polynomials is intimately related with families of Schrödinger operators connected by Darboux transformations [16, 13]. Each of these exceptional operators admits a bilinear formulation in terms of \( \tau \)-functions which suggests a strong connection with integrable systems theory, and which will be the basis of the development here. Each \( \tau \)-function in this class can be indexed by a finite set of integers, or equivalently by a Maya diagram, which becomes a very useful representation to capture a notion of equivalence and relations of the type (14).

### 3. Hermite \( \tau \)-functions

In this section we introduce Hermite-type \( \tau \)-functions, their bilinear relations, and 3 determinental representations of these objects: pseudo-Wronskians, Jacobi-Trudi formula, and a Boson-Fermion correspondence formula. We also introduce Maya diagrams, partitions, and indicate the relation between these two types of objects. In a nutshell, the Hermite-type \( \tau \) function is a specialization of the more general Schur function. Various instances of this observations can be found in [8]. Schur functions arise in integrable systems theory as polynomial solutions of the KP hierarchy [42]. As shown by Tsuda [36], the Painlevé systems are actually reductions of the KP hierarchy. Theorem 1 is an indication of how this reduction manifests at the level of solutions. The Jacobi-Trudi formula is a determinental representation of the classical Schur functions in terms of monomials. A generalization to a basis of orthogonal polynomials was introduced in [40]. We are interested in the specialization of this result to the case of Hermite polynomials. A far-reaching generalization, called the quantum Jacobi-Trudi identity was introduced in [20].

Following Noumi [31], we introduce the following.
Definition 2. A Maya diagram is a set of integers $M \subset \mathbb{Z}$ that contains a finite number of positive integers, and excludes a finite number of negative integers. We will use $M$ to denote the set of all Maya diagrams.

Let $k_1 > k_2 > \cdots$ be the decreasing enumeration of a Maya diagram $M \subset \mathbb{Z}$. The condition that $M$ be a Maya diagram is equivalent to the condition that $k_{i+1} = k_i - 1$ for $i$ sufficiently large. Thus, there exists a unique integer $\sigma_M \in \mathbb{Z}$ such that $k_i = -i + \sigma_M$ for all $i$ sufficiently large. We call $\sigma_M$ the index of $M$.

We visualize a Maya diagram as a horizontally extended sequence of •□ and □ symbols with the filled symbol •□ in position $i$ indicating membership $i \in M$. The defining assumption now manifests as the condition that a Maya diagram begins with an infinite filled •□ segment and terminates with an infinite empty □ segment.

Definition 3. Let $M$ be a Maya diagram, and

$$M_+ = \{ -m - 1 : m \notin M, m < 0 \}, \quad M_+ = \{ m : m \in M, m \geq 0 \}.$$

Let $s_1 > s_2 > \cdots > s_r$ and $t_1 > t_2 > \cdots > t_q$ be the elements of $M_-$ and $M_+$ arranged in descending order. We call the double list $(s_1, \ldots, s_r \mid t_q, \ldots, t_1)$ the Frobenius symbol of $M$ and use $M(s_1, \ldots, s_r \mid t_q, \ldots, t_1)$ to denote the Maya diagram with the indicated Frobenius symbol.

It is not hard to show that $\sigma_M = q - r$ is the index of $M$. The classical Frobenius symbol \cite{[33]} corresponds to the zero index case where $q = r$.

If $M$ is a Maya diagram, then for any $k \in \mathbb{Z}$ so is $M + k = \{ m + k : m \in M \}$.

The behaviour of the index $\sigma_M$ under translation of $k$ is given by

$$M' = M + k \implies \sigma_{M'} = \sigma_M + k.$$

We will refer to an equivalence class of Maya diagrams related by such shifts as an unlabelled Maya diagram. One can visualize the passage from an unlabelled to a labelled Maya diagram as the choice of placement of the origin.

Definition 4. A Maya diagram $M \subset \mathbb{Z}$ is said to be in standard form if $p = 0$ and $t_q > 0$. Equivalently, $M$ is in standard form if the index $\sigma_M = q$ is the number of positive elements of $M$. Visually, a Maya diagram in standard form has only filled boxes •□ to the left of the origin and one empty box □ just to the right of the origin. Every unlabelled Maya diagram permits a unique placement of the origin so as to obtain a Maya diagram in standard form.

In \cite{[18]} it was shown that to every Maya diagram we can associate a polynomial called a Hermite pseudo-Wronskian. For $n \geq 0$, let

$$H_n(x) = (-1)^n e^{x^2} \left( \frac{d}{dx} \right)^n e^{-x^2}$$

denote the degree $n$ Hermite polynomial, and

$$\tilde{H}_n(x) = i^{-n} H_n(ix)$$

the conjugate Hermite polynomial. A number of equivalent definition of $H_n$ are available. One is that $y = H_n$ is the polynomial solution of the Hermite differential equation

$$y''(z) - 2zy'(z) + 2ny(z) = 0$$
subject to the normalization condition
\[ y(z) \sim 2^n z^n \quad z \to \infty. \]

Setting
\[ \hat{H}_n(z) = e^{z^2} \tilde{H}_n(z) \]
we also note that \( \hat{H}_{-n-1} \) is a solution of (34) for negative integers \( n < 0 \).

A third definition involves the 3-term recurrence relation:
\[ H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z), \quad H_0(z) = 1, \quad H_1(z) = 2z. \]

A fourth definition involves the generating function
\[ \sum_{n=0}^{\infty} H_n(z) \frac{t^n}{n!} = e^{2zt-t^2}. \]

**Definition 5.** For \( s_1, \ldots, s_r, t_q, \ldots, t_1 \in \mathbb{Z} \) set
\[ \tau(s_1, \ldots, s_r; t_q, \ldots, t_1) = e^{-rt^2} \text{Wr}[\hat{H}_{s_1}(z), \ldots, \hat{H}_{s_r}(z), H_{t_q}(z), \ldots, H_{t_1}(z)] \]
where \( \text{Wr} \) denotes the Wronskian determinant of the indicated functions. For a Maya diagram \( M(s_1, \ldots, s_r; t_q, \ldots, t_1) \) we let
\[ \tau_M(z) = \tau(s_1, \ldots, s_r; t_q, \ldots, t_1) \]
Note: when \( r = 0 \) it will be convenient to simply write
\[ \tau(t_q, \ldots, t_1) = \text{Wr}[H_{t_q}, \ldots, H_{t_1}] \]
to indicate a Wronskian of Hermite polynomials.

The polynomial nature of \( \tau_M(z) \) becomes evident once we represent it using a slightly different determinant.

**Proposition 2.** The Wronskian in (36) admits the following alternative pseudo-Wronskian representation
\[ \tau(s_1, \ldots, s_r; t_q, \ldots, t_1) = \frac{\hat{H}_{s_1} \hat{H}_{s_1+1} \cdots \hat{H}_{s_1+r+q-1}}{\hat{H}_{s_1} \hat{H}_{s_1+1} \cdots \hat{H}_{s_1+r+q-1}} \]
\[ \quad \vdots \quad \vdots \quad \ddots \quad \vdots \]
\[ \hat{H}_{s_r} \hat{H}_{s_r+1} \cdots \hat{H}_{s_r+r+q-1} \]
\[ H_{t_q} H'_{t_q} \cdots H'_{t_q}^{(r+q-1)} \]
\[ \quad \vdots \quad \vdots \quad \ddots \quad \vdots \]
\[ H_{t_1} H'_{t_1} \cdots H'_{t_1}^{(r+q-1)} \]

The proof of the above result can be found in [18]. The term Hermite pseudo-Wronskian was also introduced in that paper, because (39) is a mix of a Casoratian and a Wronskian determinant. The just mentioned article also demonstrated that the pseudo-Wronskians of two Maya diagrams related by a translation are proportional.

**Proposition 3.** Let \( \hat{\tau}_M \) be the normalized pseudo-Wronskian
\[ \hat{\tau}_M = \prod_{1 \leq i < j \leq r} (2s_j - 2s_i) \prod_{1 \leq i < j \leq q} (2t_i - 2t_j). \]
Then for any Maya diagram \( M \) and \( k \in \mathbb{Z} \) we have
\[ \hat{\tau}_M = \hat{\tau}_{M+k}. \]
Observe that the identity in (41) involves determinants of different sizes, and a Wronskian of Hermite polynomials will not, in general, be the smallest determinant in the equivalence class. The question of which determinant has the smallest size was solved in [18].

We define a partition to be a non-increasing sequence of natural numbers \( \lambda_1 \geq \lambda_2 \geq \cdots \) such that
\[
|\lambda| := \sum_{i=1}^{\infty} \lambda_i < \infty.
\]
Implicit in this definition is the assumption that \( \lambda_i = 0 \) for \( i \) sufficiently large. We define
\[
\ell(\lambda),
\]
the length of \( \lambda \), to be the smallest \( q \in \mathbb{N} \) such that \( \lambda_{q+1} = 0 \).

To a partition \( \lambda \) of length \( q = \ell(\lambda) \) we associate the Maya diagram \( M_\lambda \) consisting of
\[
t_i = \lambda_i + q - i, \quad i = 1, 2, \ldots.
\]
By construction, we have \( t_q > 0 \), and \( t_{i+1} = t_i - 1 < 0, \quad i > q \).

Therefore \( M_\lambda \) is a Maya diagram in standard form. Indeed, (42) defines a bijection between the set of partitions and the set of Maya diagrams in standard form. Going forward, let
\[
\tau_\lambda = \text{Wr}[H_{t_q}, \ldots, H_{t_1}].
\]

For \( n \in \mathbb{Z} \) and \( \lambda \) a partition, let
\[
M^{(n)}_\lambda = M_\lambda + n - \ell(\lambda),
\]
and let \( t_1 > t_2 > \cdots \) be the decreasing enumeration of \( M^{(n)}_\lambda \). Equivalently,
\[
t_i = \lambda_i + n - i, \quad i = 1, 2, \ldots.
\]
Note that the condition \( n \geq \ell(\lambda) \) holds if and only if \( M^{(n)}_\lambda \) contains all negative integers and exactly \( n \) non-negative integers, that is if
\[
M^{(n)}_\lambda = M(\{ t_1, \ldots, t_1 \}).
\]

Given univariate polynomials \( p_1(z), \ldots, p_n(z) \), define the multivariate functions
\[
\Delta[p_1, \ldots, p_n](z_1, \ldots, z_n) = \begin{vmatrix} p_1(z_1) & p_1(z_2) & \cdots & p_1(z_n) \\ p_2(z_1) & p_2(z_2) & \cdots & p_2(z_n) \\ \vdots & \vdots & \ddots & \vdots \\ p_n(z_1) & p_n(z_2) & \cdots & p_n(z_n) \end{vmatrix}
\]
(45)
\[
S[p_1, \ldots, p_n] = \frac{\Delta[p_1, \ldots, p_n]}{\Delta[m_{n-1}, \ldots, m_1, m_0]}
\]
where
\[
m_k(z) = z^k
\]
is the \( k^{th} \) degree monomial function. Thus,
\[
\Delta[m_{n-1}, \ldots, m_0](z_1, \ldots, z_n) = \begin{vmatrix} z_1^{n-1} & \cdots & z_n^{n-1} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{vmatrix} = \prod_{1 \leq i < j \leq n} (z_i - z_j)
\]
is the usual Vandermonde determinant, while $S[p_1, \ldots, p_n]$ is a symmetric polynomial in $z_1, \ldots, z_n$.

Let $\lambda$ be a partition. For $n \geq \ell(\lambda)$, let $t_1 > t_2 > \cdots$ be the decreasing enumeration of $M_{\lambda}^{(n)}$ as per (44). The $n$-variate Schur polynomial is the symmetric polynomial

$$s_\lambda^{(n)} = S[m_1, \ldots, m_n].$$

The Schur polynomial $s_\lambda^{(n)}$ is the character of the irreducible representation of the general linear group $GL_n$ corresponding to partition $\lambda$. Moreover, the Weyl dimension formula asserts that

$$s_\lambda^{(n)}(1, \ldots, 1) = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} = \left( \prod_{j=1}^{n-1} j! \right)^{-1} \prod_{1 \leq i < j \leq n} (t_i - t_j)$$

is the dimension of the representation in question.

For $n \geq 1$, let

$$h_k^{(n)}(z_1, \ldots, z_n) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} z_{i_1} z_{i_2} \cdots z_{i_k}, \quad k = 1, 2, \ldots$$

denote the complete symmetric polynomial of degree $k$ in $n$ variables. These polynomials may also be defined by means of the generating function

$$\sum_{k=0}^{\infty} h_k^{(n)}(z_1, \ldots, z_n) u^k = \prod_{i=1}^{n} \frac{1}{1 - z_i u}.$$  

The classical Jacobi-Trudi identity is a determinental representation of the Schur polynomials in terms of complete symmetric polynomials.

**Proposition 4.** Let $\lambda$ a partition and $n \geq \ell(\lambda)$ we have

$$s_\lambda^{(n)} = \det \left( h_{\lambda_i + j - i}^{(n)} \right)_{i,j=1}^{t(\lambda)}.$$  

We now describe a closely related identity based on symmetric power functions. Define the ordinary Bell polynomials $B_k(t_1, \ldots, t_k)$, $k = 0, 1, 2, \ldots$ by means of the power generating function

$$\exp \left( \sum_{k=0}^{\infty} t_k u^k \right) = \sum_{k=0}^{\infty} B_k(t_1, \ldots, t_k) u^k.$$  

Since

$$\exp \left( \sum_{j=0}^{\infty} t_j u^j \right) = \sum_{j=0}^{\infty} \frac{1}{j!} \left( \sum_{k=0}^{\infty} t_k u^k \right)^j,$$

the multinomial formula implies that,

$$B_k(t_1, \ldots, t_k) = \sum_{j_1, j_2, \ldots, j_k \geq 0} \frac{t_1^{j_1} t_2^{j_2} \cdots t_k^{j_k}}{j_1! j_2! \cdots j_k!}, \quad \|j\| = j_1 + 2j_2 + \cdots + \ell j_\ell$$

$$= \frac{t_k}{k!} + \frac{t_{k-2} t_2}{(k-2)!} + \cdots + t_{k-1} t_1 + t_k$$
The Bell polynomials are instrumental in describing the relation between complete homogeneous polynomials and symmetric power polynomials. For a given $n \geq 1$, let

$$ p_k^{(n)}(z_1, \ldots, z_n) = \sum_{j=1}^{n} z_j^k $$

denote the symmetric $k^{th}$ power polynomial in $n$ variables. These polynomials admit the following generating function

$$ - \log \prod_{j=1}^{n} (1 - z_j u) = \sum_{j=1}^{\infty} p_j^{(n)}(z_1, \ldots, z_n) \frac{u^j}{j}. $$

Comparing the generating functions (49) (51) (52) yields the following identity

$$ u_k^{(n)} = \mathcal{B}_k \left( p_1^{(n)}, \frac{1}{2} p_2^{(n)}, \ldots, \frac{1}{k} p_k^{(n)} \right). $$

With these preliminaries out of the way we can present the following alternative version of the Jacobi-Trudi formula [40, 20]. For a partition $\lambda$, define the Schur function

$$ S_{\lambda} = \det( B_{\lambda i + j - i} \ell(\lambda)), $$

Relation (50) may now be restated as

$$ s_{\lambda}^{(n)} = \mathcal{S}_{\lambda} ( p_1^{(n)}, \frac{1}{2} p_2^{(n)}, \ldots, \frac{1}{k} p_k^{(n)}, \ldots). $$

Relation (55) will be instrumental in the proof of the following.

**Theorem 1.** Let $\lambda$ be a partition. Then,

$$ \tau_{\lambda}(z) = C_{\lambda} \mathcal{S}_{\lambda}(2z, -1), $$

where

$$ C_{\lambda} = 2^{n(n-1)/2} \prod_{j=1}^{n} (\lambda_i + n - i)!, \quad n = \ell(\lambda). $$

**Proof.** Specializing (51) and using (55), we observe that

$$ \sum_{k=0}^{\infty} \mathcal{B}(2z, -1) u^k = \exp (2zu - u^2/2) = \sum_{k=0}^{\infty} H_n(z) \frac{u^k}{n!}. $$

Hence,

$$ \mathcal{B}_n(2z, -1) = \frac{H_n(z)}{n!}, \quad n = 0, 1, 2, \ldots, $$

Hence, by (54),

$$ \mathcal{S}_{\lambda}(2z, -1) = \det \left( \frac{H_{\lambda_i + j - i}(z)}{(\lambda_i + j - i)!} \right)_{i,j=1}^{\ell(\lambda)}. $$

Hence, by the identity

$$ H_n' = 2n H_{n-1}, \quad n = 1, 2, \ldots $$

we have

$$ H^{(n-j)}_{t_i} = \frac{t_i!}{(\lambda_i + j - i)!} 2^{n-j} H_{\lambda_i + j - i}, \quad i, j = 1, \ldots, n. $$
where as above,

\[ t_i = \lambda_i + n - i, \quad n = \ell(\lambda). \]

Hence, from the definition (43), we have

\[ \tau_\lambda = \det \left( 2^{n-j} t_i! H_{\lambda_i+j-i}^{1/2} \right) = C_\lambda \xi_\lambda(2z, -1) \]

The following results will lead to yet another description of the Hermite-type \( \tau \)-function, one that is related to the Boson-Fermion correspondence [19] (although we do not discuss this here).

**Proposition 5.** Let \( p_1(z), \ldots, p_n(z) \) be polynomials. Then,

\[ W[p_1, \ldots, p_n](z) = \left( \prod_{j=1}^{n-j} j! \right) S[p_1, \ldots, p_n](z, \ldots, z) \]

**Proof.** Express the given polynomials as

\[ p_i = \sum_{j=0}^\infty p_{ij} m_j, \quad p_{ij} \in \mathbb{C}. \]

Note: the above sum is actually finite, because \( p_{ij} = 0 \) for \( j \) sufficiently large. Hence,

\[
\begin{align*}
\text{Wr}[p_1, \ldots, p_n] &= \sum_{t_1, \ldots, t_n = 0}^{\infty} \left( \prod_{i=1}^{n} p_{i t_i} \right) \text{Wr}[m_{t_1}, \ldots, m_{t_n}] \\
&= \sum_{t_1 > \cdots > t_n \geq 0} \sum_{\pi \in \mathcal{S}_n} \text{sgn}(\pi) \left( \prod_{i=1}^{n} p_{i t_{\pi_i}} \right) \text{Wr}[m_{t_1}, \ldots, m_{t_n}]
\end{align*}
\]

where \( \mathcal{S}_n \) is the group of permutations of \( \{1, \ldots, n\} \).

By an elementary calculation,

\[ \text{Wr}[m_{t_1}, \ldots, m_{t_n}] = \prod_{1 \leq i < j \leq n} (t_j - t_i) m_{\lambda_i} \]

Thus, by introducing the abbreviation

\[ p_\lambda = \sum_{\pi \in \mathcal{S}_n} \text{sgn}(\pi) \prod_{i=1}^{n} p_{i t_{\pi_i}} \]

and making use of (48) we may write

\[ \text{Wr}[p_1, \ldots, p_n] = \prod_{j=1}^{n-1} j! \sum_{\ell(\lambda) \leq n} s_\lambda(1, \ldots, 1) p_\lambda m_{\lambda} \]

Of course, the sum is actually finite because \( p_\lambda = 0 \) if \( \lambda_1 > \max\{\text{deg} p_1, \ldots, \text{deg} p_n\} \).
By the multi-linearity and skew-symmetry of the determinant (46),

\[
S[p_1, \ldots, p_n] = \sum_{t_1, \ldots, t_n \geq 0} \sum_{\pi \in S_n} \text{sgn}(\pi) \left( \prod_{i=1}^{n} p_{it_i} \right) S[m_{t_1}, \ldots, m_{t_n}]
\]

Since \(s_\lambda\) is a homogeneous polynomial whose total degree is equal to \(|\lambda|\), we have

\[
s_\lambda(z, \ldots, z) = s_\lambda(1, \ldots, 1)z^{\lambda}.
\]

Hence,

\[
S[p_1, \ldots, p_n](z, \ldots, z) = \sum_\lambda p_\lambda s_\lambda(1, \ldots, 1)z^{\lambda}.
\]

The desired conclusion now follows directly by (57).

**Corollary 1.** For \(0 \leq t_q < \cdots < t_1\) we have

\[
\tau(t_q, \ldots, t_1)(z) = \left( \prod_{j=1}^{n-1} j! \right) S[H_{t_q}, \ldots, H_{t_1}](z, \ldots, z).
\]

### 4. Hermite-type rational solutions

In this Section we develop the relationship between cyclic dressing chains and Hermite-type \(\tau\)-functions. Every element of the chain will be represented by a Maya diagram with successive elements related by flip operations. With this representation, the construction of rational solutions to a cyclic dressing chains is reduced to a combinatorial question regarding cyclic Maya diagrams.

In what follows we make use of the Hirota bilinear notation:

\[
Df \cdot g = f'g - g'f
\]

\[
D^2f \cdot g = f''g - 2f'g' + g''f
\]

A direct calculation then establishes the following.

**Proposition 6.** Let \(f = f(z), g = g(z)\) be rational functions, and let

\[
w = -z - \frac{f'}{f} + \frac{g'}{g},
\]

\[
U = z^2 - 2(\log f)'',
\]

\[
V = z^2 - 2(\log g)'',
\]

where \(w = w(z), U = U(z), V = V(z)\). Then,

\[
(D^2 - 2zD)f \cdot g = (w^2 + w' + 1 - U)f g
\]

\[
= (w^2 - w' - 1 - V)f g
\]

We are now able to exhibit a bilinear formulation for the dressing chain (16).
Proposition 7. Suppose that \( \tau_i = \tau_i(z), \epsilon_i, \sigma_i \in \{-1, 1\}, i = 0, 1, \ldots, p - 1 \) is a sequence of functions and constants that satisfies
\[
(D^2 + 2\sigma_i zD + \epsilon_i)\tau_i \cdot \tau_{i+1} = 0, \quad i = 0, 1, \ldots, p - 1 \mod p
\]
Then,
\[
w_i = \sigma_i z - \frac{\tau_i'}{\tau_i} + \frac{\tau_{i+1}'}{\tau_{i+1}}, \quad i = 0, 1, \ldots, p - 1 \mod p
\]
a_i = \epsilon_i - \epsilon_{i+1} + \sigma_i + \sigma_{i+1}

satisfy the p-cyclic dressing chain (16) with
\[
\Delta = -2 \sum_{j=0}^{p-1} \sigma_i.
\]

Proof. Set
\[
U_i(z) = z^2 - 2(\log \tau_i)''(z) - 2 \sum_{j=0}^{i-1} \sigma_j, \quad i = 0, 1, 2, \ldots, p
\]
\[
\lambda_i = \epsilon_i - 2 \sum_{j=0}^{i-1} \sigma_j - \sigma_i.
\]
Observe that
\[
(D^2 - 2\sigma_i zD + \lambda)\tau_{i+1} \cdot \tau_i = (D^2 + 2\sigma_i zD + \lambda)\tau_i \cdot \tau_{i+1}.
\]
Hence, by (63),
\[
w_i^2 + w_i' + \epsilon_i = z^2 - 2(\log \tau_i)'' + \sigma_i
\]
\[
w_i^2 - w_i' + \epsilon_i = z^2 - 2(\log \tau_{i+1})'' - \sigma_i
\]
The above relations are equivalent to (9), and hence the \( U_i(z), \lambda_i \) constitute a periodic factorization chain (8) with \( \Delta = U_p - U_0 \) given by (67). Applying (12), we obtain
\[
a_i = \lambda_i - \lambda_{i+1} = \epsilon_i - \epsilon_{i+1} + \sigma_i + \sigma_{i+1}.
\]
Hence, with (66) as definition of \( w_i(z) \) and \( a_i \), relation (16) is satisfied. \( \square \)

Note: this proposition should not be taken as a claim that all rational solutions of (16) may be obtained in this fashion.

In order to obtain polynomial solutions of (65) we now introduce the following.

Definition 6. A flip at \( m \in \mathbb{Z} \) is the involution \( \phi_m : M \to M \) defined by
\[
\phi_m : M \mapsto \begin{cases} M \cup \{m\} & \text{if } m \notin M \\ M \setminus \{m\} & \text{if } m \in M \end{cases}, \quad M \in \mathcal{M}.
\]
In the first case, we say that \( \phi_m \) acts on \( M \) by a state-deleting transformation (\( \square \to \circ \)). In the second case, we say that \( \phi_m \) acts by a state-adding transformation (\( \circ \to \square \)). We define the flip group \( \mathcal{F} \) to be the group of transformations of \( \mathcal{M} \) generated by flips \( \phi_m, m \in \mathbb{Z} \). A multi-flip is an element of \( \mathcal{F} \).
Theorem 2. Let $M_1 \subset \mathbb{Z}$ be a Maya diagram, and $M_2 = M_1 \cup \{m\}$, $m \notin M_1$ another Maya diagram obtained by a state-deleting transformation. Then, the corresponding pseudo-Wronskians satisfy the bilinear relation

$$(D^2 - 2zD + \epsilon) \tau_{M_1} \cdot \tau_{M_2} = 0, \quad \tau_{M} = \tau_{M}(z)$$

where

$$(70) \quad \epsilon = 2(\deg \tau_{M_2} - \deg \tau_{M_1}).$$

Conversely, suppose that (69) holds for some Maya diagrams $M_1, M_2 \subset \mathbb{Z}$ and some $\epsilon \in \mathbb{C}$. Then, necessarily $M_2$ is obtained from $M_1$ by a state-deleting transformation and $\epsilon$ takes the value shown above.

The proof of this theorem requires a number of intermediate results.

Lemma 2. Let $\tau_0(z), \tau_1(z), \tau_2(z), \tau_3(z)$ be rational functions such that

$$(71) \quad \tau_0 \tau_2 = \text{Wr}[\tau_1, \tau_3].$$

Associate the edges of the following diagram to the bilinear relations displayed below:

$$(\tau_0 \tau_2)$$

\begin{align*}
(D^2 - 2zD + \epsilon_1) \tau_0 \cdot \tau_1 &= 0, \\
(D^2 - 2zD + \epsilon_2) \tau_0 \cdot \tau_3 &= 0, \\
(D^2 - 2zD + \epsilon_1 - 2) \tau_1 \cdot \tau_2 &= 0, \\
(D^2 - 2zD + \epsilon_2 - 2) \tau_3 \cdot \tau_2 &= 0,
\end{align*}

where $\epsilon_1, \epsilon_2 \in \mathbb{C}$ are constants. Then, necessarily any two relations corresponding to connected edges entail the other two relations.

Proof. The lemma asserts two claims. First that (72) (73) together, are logically equivalent to (74) (75) together. The other assertion is that (72) (74) together are logically equivalent to (73) (75) together. We will demonstrate how (72) (73) together imply (74). All the other demonstrations can be argued analogously, and so we omit them.

Begin by setting

$$(76) \quad w_{ij}(z) = -z - \frac{\tau'_i(z)}{\tau_i(z)} + \frac{\tau'_j(z)}{\tau_j(z)}, \quad i \in \{0, 1, 2, 3\}, \quad i \neq j,$$

$$(77) \quad U_i(z) = z^2 - 2(\log \tau_i)^\prime(z), \quad i \in \{0, 1, 2, 3\}.$$}

By (64) of Proposition 6

$$U_0 = w_{01}^2 + w'_{01} + \epsilon_1 + 1 = w_{03}^2 + w'_{03} + \epsilon_2 + 1,$$

which we rewrite as

$$\frac{w'_{03} - w'_{01}}{w_{03} - w_{01}} = -w_{03} - w_{01} + \frac{\alpha}{w_{03} - w_{01}}, \quad \alpha = \epsilon_1 - \epsilon_2.$$


Write
\[ \text{Wr}[\tau_1, \tau_3] = \tau_1 \tau_3 \left( \frac{\tau'_3}{\tau_3} - \frac{\tau'_1}{\tau_1} \right) = \tau_1 \tau_3 (w_{03} - w_{01}). \]

Hence, by (71),
\[
\frac{\tau'_2}{\tau_2} = \frac{\tau'_0}{\tau_0} + \frac{\tau'_1}{\tau_1} + \frac{\tau'_3}{\tau_3} + \frac{w'_{03} - w'_{01}}{w_{03} - w_{01}}.
\]

which by (76) (78) maybe rewritten as
\[
(79) \quad w_{12} = w_{03} + \frac{w'_{03} - w'_{01}}{w_{03} - w_{01}} = -w_{01} + \frac{\alpha}{w_{03} - w_{01}},
\]

\((w_{01} + w_{12})(w_{03} - w_{01}) = \alpha.\)

It follows that
\[
\frac{w'_{12} + w'_{01}}{w_{12} + w_{01}} + \frac{w'_{03} - w'_{01}}{w_{03} - w_{01}} = 0
\]

Hence,
\[
w_{12} - w_{01} + \frac{w'_{12} + w'_{01}}{w_{12} + w_{01}} - \frac{\alpha}{w_{12} + w_{01}} = 0
\]

\[= w_{03} + \frac{w'_{03} - w'_{01}}{w_{03} - w_{01}} - w_{01} + \frac{w'_{12} + w'_{01}}{w_{12} + w_{01}} - \frac{\alpha}{w_{12} + w_{01}} = 0
\]

Equivalently,
\[
(80) \quad w_{12}^2 - w_{01}^2 + w_{12}' + w_{01}' - \alpha = 0.
\]

Hence, by (64)
\[U_1 = w_{01}^2 - w_{01}' + \epsilon_1 - 1 = w_{12}^2 + w_{12}' + \epsilon_2 - 1.
\]

Relation (74) now follows by Proposition 6. □

Proof of Theorem 2. Suppose that \(M_2 = M_1 \cup \{m\}, m \notin M_1\). We claim that (69) holds. By Proposition 8, no generality is lost if we assume that \(M_1\) is in standard form, and hence that
\[\tau_{M_1} = \tau(t_q, \ldots, t_1), \quad t_q < \cdots < t_1\]
is a pure Wronskian. We proceed by induction on \(q\), the number of positive elements of \(M_1\). If \(q = 0\), then \(\tau_{M_1} = 1\) and \(\tau_{M_2} = H_m\). The bilinear relation (69) is then nothing but the classical Hermite differential equation
\[H''_m(z) - 2zH'_m(z) + 2mH_m(z) = 0.
\]

Suppose that \(q > 0\) and that the claim has been shown to be true for all \(M_1\) with fewer positive elements. Set
\[M_0 = M_1 \setminus \{t_q\}, \quad M_2 = M_0 \cup \{m\}.
\]
By an elementary calculation, 

\[
\deg \tau_{M_1} = \sum_{i=1}^{q} m_i - \frac{1}{2} q(q - 1)
\]

\[
\deg \tau_{M_2} = \sum_{i=1}^{q} m_i + m - \frac{1}{2} q(q + 1)
\]

\[
\deg \tau_{M_0} = \sum_{i=1}^{q-1} m_i - \frac{1}{2} q(q - 1)(q - 2)
\]

\[
\deg \tau_{M_3} = \sum_{i=1}^{q-1} m_i + m - \frac{1}{2} q(q - 1)
\]

Hence, by the inductive hypothesis, (72) (73) hold with 

\[
\epsilon_1 = 2(\deg \tau_{M_1} - \deg \tau_{M_0}) = 2m_q - 2q + 2,
\]

\[
\epsilon_2 = 2(\deg \tau_{M_3} - \deg \tau_{M_0}) = 2m - 2q + 2
\]

Observe that 

\[
2(\deg \tau_{M_2} - \deg \tau_{M_1}) = 2m - 2q
\]

Hence, by Lemma 2 (69) holds also.

Conversely, suppose that (69) holds for some \( \lambda \in \mathbb{C} \). We claim that 

\[
M_2 = M_1 \cup \{ m \}, \quad m \not\in M_1.
\]

Without loss of generality, suppose that 

\[
M_1 = ( \mid t_q, \ldots, t_1 ), \quad t_1 > \cdots > t_q > 0
\]

is in standard form and hence that 

\[
\tau_{M_1} = \tau(t_q, \ldots, t_1).
\]

Using Proposition 3 assume without loss of generality that 

\[
\tau_{M_2} = \tau(\hat{t}_q, \ldots, \hat{t}_1),
\]

is also a pure Wronskian with \( \hat{q} \geq q \). Using the reduction argument above it then suffices to demonstrate this claim for the case where \( M_1 \) is the trivial Maya diagram; \( q = 0 \). In this case, (69) reduces to the Hermite differential equation. The Hermite polynomials are the unique polynomial solutions and so the claim follows. 

We see thus that Hermite-type \( \tau \)-functions are indexed by Maya diagrams, and that flip operations on Maya diagrams correspond to bilinear relations between these \( \tau \)-functions. Since \( p \)-cyclic chains of bilinear relations (65) correspond to rational solutions of the \( A_{p-1} \) Painlevé system, it now becomes feasible to construct rational solutions in terms of cycles of Maya diagrams.

**Definition 7.** For \( p = 1, 2, \ldots \) and \( \mu = (\mu_0, \ldots, \mu_{p-1}) \in \mathbb{Z}^p \), let 

\[
\phi_{\mu} = \phi_{\mu_0} \circ \cdots \circ \phi_{\mu_{p-1}},
\]

denote the indicated multi-flip. We will call \( \mu \in \mathbb{Z}^p \) non-degenerate if the set \( \{\mu_0, \ldots, \mu_{p-1}\} \) has cardinality \( p \), that is if \( \mu_i \neq \mu_j \) for \( i \neq j \). If \( \mu \subset \mathbb{Z} \) is a set of cardinality \( q \), we let \( \phi_{\mu} = \phi_{\mu} \), where \( \mu \in \mathbb{Z}^p \) is any non-degenerate enumeration of \( \mu \). Finally, given a finite set \( \hat{\mu} \subset \mathbb{Z} \) we will call a sequence \( \mu = (\mu_0, \ldots, \mu_{p-1}) \) an odd enumeration of \( \hat{\mu} \) if 

\[
\hat{\mu} = \{ a \in \mathbb{Z} : m_\mu(a) \equiv 1 \pmod{2} \},
\]
where \( m_\mu(a) \geq 0 \) is the number of times that \( a \in \mathbb{Z} \) occurs in \( \mu \).

**Proposition 8.** For \( \mu \in \mathbb{Z}^p \) and a finite \( \hat{\mu} \subset \mathbb{Z} \) we have \( \phi_\mu = \phi_{\hat{\mu}} \) if and only if \( \mu \) is an odd enumeration of \( \hat{\mu} \).

We are now ready to introduce the basic concept of this section.

**Definition 8.** We say that \( M \) is \( p \)-cyclic with shift \( k \), or \( (p,k) \) cyclic, if there exists a \( \mu \in \mathbb{Z}^p \) such that

\[
\phi_\mu(M) = M + k.
\]

We will say that \( M \) is \( p \)-cyclic if it is \( (p,k) \) cyclic for some \( k \in \mathbb{Z} \).

**Proposition 9.** For Maya diagrams \( M, M' \in \mathcal{M} \), define the set

\[
\Upsilon(M, M') = (M \setminus M') \cup (M' \setminus M)
\]

Then \( \phi_{\hat{\mu}} \) where \( \hat{\mu} = \Upsilon(M, M') \) is the unique multi-flip such that \( M' = \phi_{\hat{\mu}}(M) \) and \( M = \phi_{\hat{\mu}}(M') \).

As an immediate corollary, we have the following.

**Proposition 10.** Let \( k \) be an integer, \( M \in \mathcal{M} \) a Maya diagram, and let \( p \) be the cardinality of \( \Upsilon(M, M + k) \). Then \( M \) is \( (p + 2j, k) \)-cyclic for every \( j = 0, 1, 2, \ldots \).

**Proof.** Let \( \hat{\mu} = \Upsilon(M, M + k) \). Then \( \phi_{\hat{\mu}}(M) = M + k \), by the preceding Proposition, and hence \( M \) is \( (p, k) \) cyclic. Let \( \mu \in \mathbb{Z}^p \) be an odd enumeration of \( \hat{\mu} \); i.e., \( \mu \) is obtained by adjoining \( j \) pairs of repeated indices to the elements of \( \hat{\mu} \). By Proposition 8,

\[
\phi_\mu(M) = \phi_{\hat{\mu}}(M) = M + k,
\]

and therefore \( M \) is also \( (p + 2j, k) \)-cyclic. \( \square \)

The following result should be regarded as a refinement of Proposition 7.

**Proposition 11.** Let \( M \in \mathcal{M} \) be a Maya diagram, \( k \) a non-zero integer, and \( (\mu_0, \ldots, \mu_{p-1}) \in \mathbb{Z}^p \) an odd enumeration of \( \Upsilon(M, M + k) \). Extend \( \mu \) to an infinite \( p \)-quasiperiodic sequence by letting

\[
\mu_{i+p} = \mu_i + k, \quad k = 0, 1, 2, \ldots
\]

and recursively define

\[
M_0 = M, \quad M_{i+1} = \phi_{\mu_i}(M_i), \quad i = 0, 1, 2, \ldots
\]

so that \( M_{i+p} = M_i + k \) by construction. Next, for \( i = 0, 1, \ldots \), let

\[
\tau_i = \tau_{M_i}, \quad a_i = 2(\mu_i - \mu_{i+1}),
\]

\[
\sigma_i = \begin{cases} +1 & \text{if } \mu_i \in M_i \\ -1 & \text{if } \mu_i \in M_{i+1} \end{cases},
\]

\[
w_i(z) = \sigma_i z + \frac{\tau'_{i+1}(z)}{\tau_{i+1}(z)} - \frac{\tau'_i(z)}{\tau_i(z)},
\]

which are all \( p \)-periodic by construction. The just-defined \( w_i(z), a_i, \ i = 0, 1, \ldots, p-1 \mod p \) constitute a rational solution to the \( p \)-cyclic dressing chain with shift \( \Delta = 2k \).
Proof. Set
\[ \epsilon_i = 2\sigma_i (\deg \tau_i - \tau_{i+1}), \quad i = 0, 1, \ldots, p - 1 \mod p. \]
Theorem 2 then implies that
\[ (D^2 + 2\sigma_i zD + \epsilon_i)\tau_i \cdot \tau_{i+1} = 0. \]
Proposition 3 allows us to assume, without loss of generality, that
\[ M_i = M(| t_{i,q_i}, \ldots, t_{i,1} |) \]
and that \( \mu_i \geq 0 \) for all \( i \). Such an outcome can be imposed by applying a sufficiently positive translation to the Maya diagrams in question, without altering the log-derivatives of the \( \tau \)-functions. Since each \( \tau_i \) is a Wronskian of polynomials of degrees \( t_{i,1}, \ldots, t_{i,q_i} \), we have
\[ \deg \tau_i = \sum_{j=1}^{q_i} t_{i,j} - \frac{1}{2} q_i(q_i - 1). \]
If \( \sigma_i = -1 \), then
\[ \{t_{i+1,1}, \ldots, t_{i+1,q_i+1}\} = \{t_{i,1}, \ldots, t_{i,q_i}\} \cup \{\mu_i\}, \quad q_i+1 = q_i + 1. \]
If \( \sigma_i = +1 \), then
\[ \{t_{i,1}, \ldots, t_{i+1,q_i}\} = \{t_{i+1,1}, \ldots, t_{i+1,q_i+1}\} \cup \{\mu_i\}, \quad q_i+1 = q_i - 1. \]
It follows that
\[ \epsilon_i = 2\sigma_i (\deg \tau_i - \deg \tau_{i+1}) = 2\mu_i - 2q_i + 1 + \sigma_i, \]
\[ \epsilon_i - \epsilon_{i+1} + \sigma_i + \sigma_{i+1} = 2\mu_i - 2\mu_{i+1} + 2(q_{i+1} - q_i) + \sigma_i - \sigma_{i+1} + \sigma_i + \sigma_{i+1} \]
\[ = 2(\mu_i - \mu_{i+1}) \]
Hence, the definition of \( a_i \) in (85) agrees with the definition in (66). Therefore, \( w_i(z), a_i, \ i = 0, 1, \ldots, p - 1 \) satisfy (16) by Proposition 7. Finally, by (15),
\[ \Delta = -\sum_{i=0}^{p-1} a_i = 2\sum_{i=0}^{p-1} (\mu_{i+1} - \mu_i) = 2(\mu_p - \mu_0) = 2k. \]
\[ \square \]

The remaining part of the construction is to classify cyclic Maya diagrams for a given period, which we tackle next. Under the correspondence described by Proposition 11, the reversal symmetry (17) manifests as the transformation
\[ (M_0, \ldots, M_p) \mapsto (M_p, \ldots, M_0), \quad (\mu_1, \ldots, \mu_p) \mapsto (\mu_p, \ldots, \mu_1), \quad k \mapsto -k. \]
In light of the above remark, there is no loss of generality if we restrict our attention to cyclic Maya diagrams with a positive shift \( k > 0 \).

5. Cyclic Maya diagrams

In this section we introduce the key concepts of genus and interlacing to achieve a full classification of cyclic Maya diagrams.
Definition 9. For $p = 1, 2, \ldots$ let $\mathbb{Z}_p \subset \mathbb{Z}^p$ denote the set of non-decreasing integer sequences $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_{p-1}$. Of integers. For $\beta \in \mathbb{Z}_{2g+1}$ define the Maya diagram

$$\Xi(\beta) = (-\infty, \beta_0) \cup [\beta_1, \beta_2) \cup \cdots \cup [\beta_{2g-1}, \beta_{2g})$$

where

$$(m, n) = \{j \in \mathbb{Z} : m < j < n\}$$

Let $\hat{\mathbb{Z}}_p \subset \mathbb{Z}^p$ denote the set of strictly increasing integer sequences $\beta_0 < \beta_1 < \cdots < \beta_{p-1}$. Equivalently, we may regard a $\beta \in \hat{\mathbb{Z}}_p$ as a $p$-element subset of $\mathbb{Z}$.

Proposition 12. Every Maya diagram $M \in \mathcal{M}$ has a unique representation of the form $M = \Xi(\beta)$ where $\beta \in \hat{\mathbb{Z}}_{2g+1}$. Moreover, $M$ is in standard form if and only if $\min \beta = 0$.

Proof. After removal of the initial infinite $\Box$ segment and the trailing infinite $\Box$ segment, a given Maya diagram $M$ consists of $2g$ alternating empty $\Box$ and filled $\bullet$ segments of variable length. The genus $g$ counts the number of such pairs. The even block coordinates $\beta_{2i}$ indicate the starting positions of the empty segments, and the odd block coordinates $\beta_{2i+1}$ indicated the starting positions of the filled segments. See Figure 1 for a visual illustration of this construction.

Definition 10. We call the integer $g \geq 0$ the genus of $M = \Xi(\beta)$, $\beta \in \hat{\mathbb{Z}}_p$ and $(\beta_0, \beta_1, \ldots, \beta_{2g})$ the block coordinates of $M$.

Proposition 13. Let $M = \Xi(\beta)$, $\beta \in \hat{\mathbb{Z}}_p$ be a Maya diagram specified by its block coordinates. We then have

$$\beta = \Upsilon(M, M + 1).$$

Proof. Observe that

$$M + 1 = (-\infty, \beta_0] \cup (\beta_1, \beta_2] \cup \cdots \cup (\beta_{2g-1}, \beta_{2g}],$$

where

$$(m, n) = \{j \in \mathbb{Z} : m < j < n\}.$$

It follows that

$$(M + 1) \setminus M = \{\beta_0, \ldots, \beta_{2g}\}$$

and

$$M \setminus (M + 1) = \{\beta_1, \ldots, \beta_{2g-1}\}.$$

The desired conclusion follows immediately.

Let $\mathcal{M}_g$ denote the set of Maya diagrams of genus $g$. The above discussion may be summarized by saying that the mapping $\Xi$ defines a bijection $\Xi : \hat{\mathbb{Z}}_{2g+1} \to \mathcal{M}_g$, and that the block coordinates are precisely the flip sites required for a translation $M \to M + 1$.

The next concept we need to introduce is the interlacing and modular decomposition.

1Equivalently, we may regard $\mathbb{Z}_p$ as the set of $p$-element integer multi-sets. A multi-set is generalization of the concept of a set that allows for multiple instances for each of its elements.
Definition 11. Fix a \( k \in \mathbb{N} \) and let \( M^{(0)}, M^{(1)}, \ldots M^{(k-1)} \subset \mathbb{Z} \) be sets of integers. We define the interlacing of these to be the set

\[
\Theta \left( M^{(0)}, M^{(1)}, \ldots M^{(k-1)} \right) = \bigcup_{i=0}^{k-1} (kM^{(i)} + i),
\]

where \( kM + j = \{ km + j : m \in M \}, \ M \subset \mathbb{Z} \).

Dually, given a set of integers \( M \subset \mathbb{Z} \) and a \( k \in \mathbb{N} \) define the sets

\[
M^{(i)} = \{ m \in \mathbb{Z} : km + i \in M \}, \ i = 0, 1, \ldots, k - 1.
\]

We will call the \( k \)-tuple of sets \( (M^{(0)}, M^{(1)}, \ldots M^{(k-1)}) \) the \( k \)-modular decomposition of \( M \).

The following result follows directly from the above definitions.

Proposition 14. We have \( M = \Theta \left( M^{(0)}, M^{(1)}, \ldots M^{(k-1)} \right) \) if and only if \( (M^{(0)}, M^{(1)}, \ldots M^{(k-1)}) \) is the \( k \)-modular decomposition of \( M \).

Even though the above operations of interlacing and modular decomposition apply to general sets, they have a well defined restriction to Maya diagrams. Indeed, it is not hard to check that if \( M = \Theta \left( M^{(0)}, M^{(1)}, \ldots M^{(k-1)} \right) \) and \( M \) is a Maya diagram, then \( M^{(0)}, M^{(1)}, \ldots M^{(k-1)} \) are also Maya diagrams. Conversely, if the latter are all Maya diagrams, then so is \( M \). Another important case concerns the interlacing of finite sets. The definition (87) implies directly that if \( \mu^{(i)} \in \mathbb{Z}_{p_i}, i = 0, 1, \ldots, k - 1 \) then

\[
\mu = \Theta \left( \mu^{(0)}, \ldots, \mu^{(k-1)} \right)
\]

is a finite set of cardinality \( p = p_0 + \cdots + p_{k-1} \).

Visually, each of the \( k \) Maya diagrams is dilated by a factor of \( k \), shifted by one unit with respect to the previous one and superimposed, so the interlaced Maya diagram incorporates the information from \( M^{(0)}, \ldots M^{(k-1)} \) in \( k \) different modular classes. An example can be seen in Figure 2. In other words, the interlaced Maya diagram is built by copying sequentially a filled or empty box as determined by each of the \( k \) Maya diagrams.

Equipped with these notions of genus and interlacing, we are now ready to state the main result for the classification of cyclic Maya diagrams.

Theorem 3. Let \( M = \Theta \left( M^{(0)}, M^{(1)}, \ldots M^{(k-1)} \right) \) be the \( k \)-modular decomposition of a given Maya diagram \( M \). Let \( g_i \) be the genus of \( M^{(i)} \), \( i = 0, 1, \ldots, k - 1 \). Then, \( M \) is \( (p,k) \)-cyclic where

\[
p = p_0 + p_1 + \cdots + p_{k-1}, \quad p_i = 2g_i + 1.
\]
Proving that $M$ is $(p,k)$ cyclic where the value of $p$ agrees with (88).

Theorem 3 sets the way to classify cyclic Maya diagrams for any given period $p$.  

Corollary 2. For a fixed period $p \in \mathbb{N}$, there exist $p$-cyclic Maya diagrams with shifts $k = p, p - 2, \ldots, \lfloor p/2 \rfloor$, and no other positive shifts are possible.

Remark 1. The highest shift $k = p$ corresponds to the interlacing of $p$ trivial (genus 0) Maya diagrams.

We now introduce a combinatorial system for describing rational solutions of $p$-cyclic factorization chains. First, we require a suitably generalized notion of block coordinates suitable for describing $p$-cyclic Maya diagrams.

Definition 12. For $p_0, \ldots, p_{k-1} \in \mathbb{N}$ set

$$Z_{p_0,\ldots,p_{k-1}} := Z_{p_0} \times \cdots \times Z_{p_{k-1}} \subset \mathbb{Z}^{p_0+\cdots+p_{k-1}}.$$
Thus, an element of \( \mathbb{Z}_{p_0,\ldots,p_{k-1}} \) is a concatenation \((\beta^{(0)}|\beta^{(1)}|\ldots|\beta^{(k-1)})\) of \( k \) non-decreasing subsequences, \( \beta^{(i)} \in \mathbb{Z}_{p_i}, \ i = 0,1,\ldots, k-1 \). Let \( \Xi: \mathbb{Z}_{p_0,\ldots,p_{k-1}} \to M \) be the mapping with action
\[
\Xi: (\beta^{(0)}|\beta^{(1)}|\ldots|\beta^{(k-1)}) \mapsto \Theta \left( \Xi(\beta^{(0)}),\ldots,\Xi(\beta^{(k-1)}) \right)
\]
Let \( M \in M \) be a \((p,k)\) cyclic Maya diagram, and let
\[
M = \Theta \left( M^{(0)},\ldots,M^{(k-1)} \right)
\]
be the corresponding \( k \)-modular decomposition. Let \( p_i = 2g_i+1, \ i = 0,1,\ldots, k-1 \) where \( g_i \) is the genus of \( M^{(i)} \) and let \( \beta^{(i)} \in \hat{\mathcal{Z}}_{2g_i+1} \) be the block coordinates of \( M^{(i)} \in M_{g_i} \). In light of the fact that
\[
M = \Theta \left( \Xi(\beta^{(0)}),\ldots,\Xi(\beta^{(k-1)}) \right),
\]
we will refer to the concatenated sequence
\[
(\beta^{(0)}|\beta^{(1)}|\ldots|\beta^{(k-1)}) = \left( \beta^{(0)}_0,\ldots,\beta^{(0)}_{p_0-1},\beta^{(1)}_0,\ldots,\beta^{(1)}_{p_1-1},\ldots,\beta^{(k-1)}_0,\ldots,\beta^{(k-1)}_{p_{k-1}-1} \right)
\]
as the \( k \)-block coordinates of \( M \).

**Definition 13.** Fix a \( k \in \mathbb{N} \). For \( m \in \mathbb{Z} \) let \( [m]_k \in \{0,1,\ldots,k-1\} \) denote the residue class of \( m \) modulo division by \( k \). For \( m,n \in \mathbb{Z} \) say that \( m \preceq_k n \) if and only if
\[
[m]_k < [n]_k, \quad \text{or} \quad [m]_k = [n]_k \quad \text{and} \quad m \leq n.
\]
In this way, the transitive, reflexive relation \( \preceq_k \) forms a total order on \( \mathbb{Z} \).

**Proposition 15.** Let \( M \) be a \((p,k)\) cyclic Maya diagram. There exists a unique \( p \)-tuple \( \mu \in \mathbb{Z}^p \) ordered relative to \( \preceq_k \) such that
\[
\phi_{\mu}(M) = M + k
\]

**Proof.** Let \( (\beta_0,\ldots,\beta_{p-1}) = (\beta^{(0)}|\beta^{(1)}|\ldots|\beta^{(k-1)}) \) be the \( k \)-block coordinates of \( M \). Set
\[
\mu = \Theta \left( \beta^{(0)},\ldots,\beta^{(k-1)} \right)
\]
so that (89) holds by the proof to Theorem 3. The desired enumeration of \( \mu \) is given by
\[
(k\beta_0,\ldots,k\beta_{p-1}) + (0^{p_0},1^{p_1},\ldots,(k-1)^{p_{k-1}})
\]
where the exponents indicate repetition. Explicitly, \((\mu_0,\ldots,\mu_{p-1})\) is given by
\[
\left( k\beta^{(0)}_0,\ldots,k\beta^{(0)}_{p_0-1},k\beta^{(1)}_0,1,\ldots,k\beta^{(1)}_{p_1-1}+1,\ldots,k\beta^{(k-1)}_0,\ldots,k\beta^{(k-1)}_{p_{k-1}-1}+1+k-1 \right).
\]
\( \square \)

**Definition 14.** In light of (89) we will refer to the just defined tuple \((\mu_0,\mu_1,\ldots,\mu_{p-1})\) as the \( k \)-canonical flip sequence of \( M \) and refer to the tuple \((p_0,p_1,\ldots,p_{k-1})\) as the \( k \)-signature of \( M \).

By Proposition 11 a rational solution of the \( p \)-cyclic dressing chain requires a \((p,k)\) cyclic Maya diagram, and an additional item data, namely a fixed ordering of the canonical flip sequence. We will specify such ordering as
\[
\mu_\pi = (\mu_{\pi_0},\ldots,\mu_{\pi_{p-1}})
\]
where \( \pi = (\pi_0, \ldots, \pi_{p-1}) \) is a permutation of \((0, 1, \ldots, p - 1)\). With this notation, the chain of Maya diagrams described in Proposition 11 is generated as

\[
M_0 = M, \quad M_{i+1} = \phi_{\mu_\pi}(M_i), \quad i = 0, 1, \ldots, p - 1.
\]

**Remark 2.** Using a translation it is possible to normalize \( M \) so that \( \mu_0 = 0 \). Using a cyclic permutation and it is possible to normalize \( \pi \) so that \( \pi_p = 0 \). The net effect of these two normalizations is to ensure that \( M_0, M_1, \ldots, M_{p-1} \) have standard form.

**Remark 3.** In order to obtain a full classification of rational solutions, it will be necessary to account for degenerate chains which include multiple flips at the same site. For this reason, we must allow \( \beta^{(i)} \in \mathbb{Z}_p \) to be merely non-decreasing sequences.

### 6. Hermite-type rational solutions

In this section we will put together all the results derived above in order to describe an effective way of labelling and constructing Hermite-type rational solutions to the Noumi-Yamada-Painlevé system using cyclic Maya diagrams. As an illustrative example, we describe rational solutions of the \( \text{P IV} \) and \( \text{P V} \) systems, because these are known to be reductions of the \( A_2 \) and \( A_3 \) systems, respectively. We then give examples of rational solutions to the \( A_4 \) system.

In order to specify a Hermite-type rational solution of a \( p \)-cyclic dressing chain, we require three items of data.

1. We begin by specifying a signature sequence \((p_0, \ldots, p_{k-1})\) consisting of odd positive integers that sum to \( p \). This sequence determines the genus \( g_i = 2p_i + 1 \), \( i = 0, 1, \ldots, k - 1 \) of the \( k \) interlaced Maya diagrams that give rise to a \((p, k)\)-cyclic Maya diagram \( M \). The possible values of \( k \) are given by Corollary 2.

2. Once the signature is fixed, we specify an element of \( \mathbb{Z}_{p_0, \ldots, p_{k-1}} \); i.e., \( k \)-block coordinates

\[
(\beta_0, \ldots, \beta_{p-1}) = (\beta^{(0)} | \ldots | \beta^{(k-1)})
\]

which determine a \((p, k)\)-cyclic Maya diagram \( M = \Xi(\beta^{(0)} | \ldots | \beta^{(k-1)}) \), and a canonical flip sequence \( \mu = (\mu_0, \ldots, \mu_{p-1}) \) as per Proposition 15.

3. Once the \( k \)-block coordinates and canonical flip sequence \( \mu \) are fixed, we specify a permutation \((\pi_0, \ldots, \pi_{p-1})\) of \((0, 1, \ldots, p - 1)\) that determines the actual flip sequence \( \mu_\pi \), i.e. the order in which the flips in the canonical flip sequence are applied to build a \( p \)-cycle of Maya diagrams.

4. With the above data, we apply Proposition 11 with \( M \) and \( \mu_\pi \) to construct the rational solution.

For any signature of a Maya \( p \)-cycle, we need to specify the \( p \) integers in the canonical flip sequence, but following Remark 2 we can get rid of translation invariance by imposing \( \mu_0 = \beta^{(0)}_0 = 0 \), leaving only \( p - 1 \) free integers. The remaining number of degrees of freedom is \( p - 1 \), which coincides with the number of generators of the symmetry group \( A_p^{(1)} \). This is a strong indication that the class described above captures a generic orbit of a seed solution under the action of the symmetry group. Moreover, it is sometimes advantageous to consider only permutations such that \( \pi_p = 0 \) in order to remove the invariance under cyclic permutations.
6.1. Painlevé IV. As was mentioned in the introduction, the $A_2$ Noumi-Yamada system is equivalent the $P_{IV}$ equation [139]. The reduction of (2) to the $P_{IV}$ equation (1) is accomplished via the following substitutions

$$\sqrt{2} f_0(x) = y(t), \quad -\sqrt{8} f_{1,2}(x) = y(t) + 2t \pm \frac{y'(t) - \sqrt{-2b}}{y(t)}$$

$$x = -\sqrt{2t}, \quad a = \alpha_2 - \alpha_1, \quad b = -2\alpha_0^2.$$

It is known [6] that every rational solution of $P_{IV}$ can be described in terms of either generalized Hermite (GH) or Okamoto (O) polynomials, both of which may given as a Wronskian of classical Hermite polynomials. We now exhibit these solutions using the framework described in the preceding section.

The 3-cyclic Maya diagrams fall into exactly one of two classes:

| $k$ | $(p_0, \ldots, p_{k-1})$ | $(\beta_0, \beta_1, \beta_2)$ | $(\mu_0, \mu_1, \mu_2)$ |
|-----|-------------------|-----------------|------------------|
| 1   | $(0,1,1)$         | $(0, n_1, n_1 + n_2)$ | $(0, n_1, n_1 + n_2)$ |
| 2   | $(0,1,1)$         | $(0|n_1|n_2)$       | $(0, 3n_1 + 1, 3n_2 + 2)$ |

The corresponding Maya diagrams are

$$M_{GH}(n_1, n_2) = \Xi(0, n_1, n_1 + n_2) = (-\infty, 0) \cup [n_1, n_1 + n_2)$$

$$M_{O}(n_1, n_2) = \Xi(0|n_1|n_2)$$

$$= (-\infty, 0) \cup \{3j + 1: 0 \leq j < n_1\} \cup \{3j + 2: 0 \leq j \leq n_2\}$$

The generalized Hermite and Okamoto polynomials, denoted below by $\tau_{GH(n_1,n_2)}$, $n_1, n_2 \geq 0$ and $\tau_{O(n_1,n_2)}$, $n_1, n_2 \geq 0$, respectively, are two-parameter families of Hermite Wronskians that correspond to the above diagrams. For example (see [55] for definition):

$$\tau_{GH(3,5)} = \tau(3, 4, 5, 6, 7),$$

$$\tau_{O(3,2)} = \tau(1, 2, 4, 5, 7)$$

Having chosen one of the above polynomials as $\tau_0$ there are $6 = 3!$ distinct rational solutions of the $A_2$ system (2) corresponding the possible permutations of the canonical flip sites $(\mu_0, \mu_1, \mu_2)$. The translations of the block coordinates $(\beta_0, \beta_1, \beta_2)$ engendered by these permutations is enumerated in the table below.

| $\tau_0$ | $(012)$ | $(102)$ | $(021)$ | $(210)$ | $(120)$ | $(201)$ |
|----------|---------|---------|---------|---------|---------|---------|
| $\tau_0$ | 000     | 000     | 000     | 000     | 000     | 000     |
| $\tau_1$ | 100     | 010     | 100     | 001     | 010     | 001     |
| $\tau_2$ | 110     | 110     | 101     | 011     | 011     | 101     |
| $\tau_3$ | 111     | 111     | 111     | 111     | 111     | 111     |

Example. The Maya diagram, $M_{GH}(3,5)$ has the canonical flip sequence $\mu = (0, 3, 8)$. Applying the permutation $\pi = (2, 1, 0)$ yields the flip sequence $\mu_\pi = (8, 3, 0)$ and the following sequence of polynomials and block coordinates:

$$\tau_0 = \tau(3, 4, 5, 6, 7), \quad \tau(0, 3, 8)$$

$$\tau_1 = \tau(3, 4, 5, 6, 7, 8), \quad \tau(0, 3, 9)$$

$$\tau_2 = \tau(4, 5, 6, 7, 8), \quad \tau(0, 4, 9)$$

$$\tau_3 = \tau(0, 4, 5, 6, 7, 8) \propto \tau(3, 4, 5, 6, 7), \quad \tau(1, 4, 9)$$
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\[ M_0 = \Xi(0, 3, 8) = M_{GH}(3, 5) \]
\[ M_1 = \Xi(0, 3, 9) = M_{GH}(3, 6) \]
\[ M_2 = \Xi(0, 4, 9) = M_{GH}(4, 5) \]
\[ M_3 = \Xi(1, 4, 9) = M_{GH}(3, 5) + 1 \]

\textbf{Figure 3.} A cycle generated by the (3,1)-cyclic Maya diagram \( M_{GH}(3, 5) \) and permutation \( \pi = (210) \).

Hence, by (85)
\[ (a_0, a_1, a_2) = (10, 6, -18), \quad (\sigma_0, \sigma_1, \sigma_2) = (-1, 1, -1) \]

Applying (66) gives the following rational solution of the 3-cyclic dressing chain (16):
\[ w_0 = z + \frac{\tau'_1}{\tau_1} - \frac{\tau'_0}{\tau_0}, \quad w_1 = -z + \frac{\tau'_2}{\tau_2} - \frac{\tau'_1}{\tau_1}, \quad w_2 = z + \frac{\tau'_0}{\tau_0} - \frac{\tau'_2}{\tau_2}, \quad w_i = w_i(z), \quad \tau_i = \tau_i(z). \]

Applying (22) and (91) gives the following rational solution of P\(_{IV}\) :
\[ y(t) = \frac{d}{dt} \log \frac{\tau_0(t)}{\tau_2(t)}, \quad a = \frac{1}{2} (a_1 - a_2) = 12, \quad b = -\frac{a_0^2}{2} = -50. \]

\textbf{Example.} The Maya diagram \( M_O(3, 2) \) has the canonical flip sequence
\[ \mu = \Theta(0;3|2) = (0, 10, 8). \]
Using the permutation \((2, 0, 1)\) by way of example, we generate the following sequence of polynomials and block coordinates:
\[ \tau_0 = \tau(1, 2, 4, 5, 7), \quad (0|3|2) \]
\[ \tau_1 = \tau(1, 2, 4, 5, 7, 8), \quad (0|3|3) \]
\[ \tau_2 = \tau(0, 1, 2, 4, 5, 7, 8), \quad (1|3|3) \]
\[ \tau_3 = \tau(0, 1, 2, 4, 5, 7, 8, 10) \propto \tau(1, 2, 4, 5, 7) \quad (1|4|3) \]

\textbf{Figure 4.} A cycle generated by the (3,3)-cyclic Maya diagram \( M_O(3, 2) \) and permutation \( \pi = (201) \).

Hence, by (85)
\[ (a_0, a_1, a_2) = (16, -20, -2), \quad (\sigma_0, \sigma_1, \sigma_2) = (-1, -1, -1) \]
Applying (66) gives the following rational solution of the 3-cyclic dressing chain (16):

\[
\frac{\tau_1'}{\tau_1} - \frac{\tau_0'}{\tau_0}, \quad \frac{\tau_2'}{\tau_2} - \frac{\tau_1'}{\tau_1}, \quad \frac{\tau_0'}{\tau_0} - \frac{\tau_2'}{\tau_2}.
\]

Applying (22) and (91) gives the following rational solution of P_{IV}:

\[
y(t) = -\frac{2}{3} t + \frac{d}{dt} \log \frac{\tau_2(Kt)}{\tau_0(Kt)}, \quad K = \frac{1}{\sqrt{3}}, \quad a = \frac{1}{6}(a_1 - a_2) = -3, \quad b = -\frac{a_0^2}{18} = -\frac{128}{9}.
\]

For each of the above classes of solutions, observe that the permutations \((\pi_0, \pi_1, \pi_2)\) and \((\pi_1, \pi_0, \pi_2)\), while producing distinct solutions of the \(A_2\) system, give the same solution of P_{IV}. This means that there are 3 distinct GH classes and 3 distinct O classes of rational solution of P_{IV}. These are enumerated below.

\[
\begin{align*}
y &= \frac{d}{dt} \log \frac{\tau_{GH(n_1,n_2)}(t)}{\tau_{GH(n_1,n_2+1)}(t)}, & a &= -(1 + n_1 + 2n_2), \quad b = -2n_1^2, \\
y &= \frac{d}{dt} \log \frac{\tau_{GH(n_1,n_2)}(t)}{\tau_{GH(n_1-1,n_2)}(t)}, & a &= 2n_1 + n_2 - 1, \quad b = -2n_2^2.
\end{align*}
\]

\[
\begin{align*}
y &= -2t + \frac{d}{dt} \log \frac{\tau_{GH(n_1,n_2)}(t)}{\tau_{GH(n_1+1,n_2-1)}(t)}, & a &= n_2 - n_1 - 1, \quad b = -2(n_1 + n_2)^2, \\
y &= -\frac{2}{3} t + \frac{d}{dt} \log \frac{\tau_{O(n_1,n_2)}(z)}{\tau_{O(n_1-1,n_2-1)}(z)}, & a &= n_1 + n_2, \quad b = -\frac{2}{9}(1 - 3n_1 + 3n_2)^2, \\
y &= -\frac{2}{3} t + \frac{d}{dt} \log \frac{\tau_{O(n_1,n_2)}(z)}{\tau_{O(n_1+1,n_2)}(z)}, & a &= -1 - 2n_1 + n_2, \quad b = -\frac{2}{9}(2 + 3n_2)^2, \\
y &= -\frac{2}{3} t + \frac{d}{dt} \frac{\tau_{O(n_1,n_2)}(z)}{\tau_{O(n_1,n_2+1)}(z)}, & a &= -2 - 2n_2 + n_1, \quad b = -\frac{2}{9}(1 + 3n_1)^2.
\end{align*}
\]

6.2. Painlevé V. The fifth Painlevé equation is a second-order scalar non-autonomous, non-linear differential equation, usually given as

\[
y'' = (y')^2 \left( \frac{1}{2y} + \frac{1}{y-1} \right) - \frac{y'}{t} + \frac{(y-1)^2}{t^2} \left( ay + \frac{b}{y} \right) + \frac{cy}{t} + \frac{dg(y+1)}{y-1}, \quad y = y(t),
\]

where \(a, b, c, d\) are complex-valued parameters. An equivalent form is

\[
\begin{align*}
\phi &= \alpha_0(y - 1) + \alpha_2 \left( 1 - \frac{1}{y} \right) - t \frac{y'}{y}, \quad \phi = \phi(t) \\
\phi' &= \frac{1}{2t} \left( \frac{\phi(y(2 - \phi) - \phi - 2)}{y - 1} + 2\phi(\alpha_0 + \alpha_2 - 1) - 2ct \right) - dt \frac{y + 1}{y - 1}
\end{align*}
\]

where

\[
a = \frac{\alpha_0^2}{2}, \quad b = -\frac{\alpha_2^2}{2}.
\]
The $A_3$ Noumi-Yamada system, the specialization of \((21)\) with $n = 2$, has the form

\[
\begin{align*}
xf_0' &= 2f_0f_2(f_1 - f_3) + (1 - 2\alpha_2)f_0 + 2\alpha_0f_2 \\
xf_1' &= 2f_1f_3(f_2 - f_0) + (1 - 2\alpha_3)f_1 + 2\alpha_1f_3 \\
xf_2' &= 2f_0f_2(f_3 - f_1) + (1 - 2\alpha_0)f_2 + 2\alpha_2f_0 \\
xf_3' &= 2f_1f_3(f_0 - f_2) + (1 - 2\alpha_1)f_3 + 2\alpha_3f_1
\end{align*}
\]

where $f_i = f_i(x)$, $i = 0, 1, 2, 3$, and which is subject to normalizations

\[
f_0 + f_2 = f_1 + f_3 = \frac{x}{2}, \quad \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1.
\]

The transformation of \((93)\) to \((92)\) is given by the following relations:

\[
y = -\frac{f_2}{f_0}, \quad \phi = \frac{1}{2}x(f_1 - f_3), \\
y = y(t), \quad \phi = \phi(t), \quad f_i = f_i(x), \quad t = \frac{x^2}{\Delta} \\
a = \frac{\Delta^2}{2}, \quad b = -\frac{\alpha_2^2}{2}, \quad c = \frac{\Delta}{4}(\alpha_3 - \alpha_1), \quad d = a - \frac{\Delta^2}{32}
\]

The 4-cyclic Maya diagrams fall into exactly one of three classes:

\[
\begin{array}{ccc}
k & (p_0, \ldots, p_{k-1}) & (\beta_0, \beta_1, \beta_2, \beta_3) & (\mu_0, \mu_1, \mu_2, \mu_3) \\
2 & (3, 1) & (0, n_1, n_1 + n_2|n_3) & (0, 2n_1, 2n_1 + 2n_2, 2n_3 + 1) \\
2 & (1, 3) & (0|n_1, n_1 + n_2, n_1 + n_2 + n_3) & (0, 2n_1 + 1, 2n_1 + 2n_2 + 1, 2n_1 + 2n_2 + 2n_3 + 1) \\
3 & (1, 1, 1) & (0|n_1|n_2|n_3) & (0, 2n_1 + 1, 2n_2 + 2, 2n_2 + 3)
\end{array}
\]

We will denote the corresponding Maya diagrams as

\[
M(n_1, n_2|n_3) = \Xi(0, n_1, n_1 + n_2|n_3) \\
M(|n_1, n_2, n_3) = \Xi(0|n_1, n_1 + n_2, n_1 + n_2 + n_3) \\
M(|n_1|n_2|n_3) = \Xi(0|n_1|n_2|n_3)
\]

For example (see \((68)\) for definition):

\[
\tau_{M(3,1|2)} = \tau(1, 3, 6), \\
\tau_{M(3,1,2)} = \tau(1, 3, 5, 9, 11), \\
\tau_{M(3|3|1|2)} = \tau(1, 2, 3, 5, 7, 9)
\]

Having chosen one of the above polynomials as $\tau_0$, there are $24 = 4!$ distinct rational solutions of the $A_3$ system \((2)\) corresponding the possible permutations of the canonical flip sites $(\mu_0, \mu_1, \mu_2, \mu_3)$. However, the projection from the set of $A_3$ solutions to the set of $P_V$ solutions is not one-to-one. The action of the permutation group $\mathfrak{S}_4$ on the set of $P_V$ solutions has non-trivial isotropy corresponding to the Klein 4-group: \((0, 1, 2, 3), (1, 0, 2, 3), (0, 1, 3, 2), (1, 0, 3, 2)\). Therefore each of the above $\tau$-functions generates $6 = 24/4$ distinct rational solutions of $P_V$.

**Example.** We exhibit a $(4,2)$-cyclic Maya diagram in the signature class $(3,1)$ by taking

\[
M_0 = M(3,1|2) = \Xi(0,3,4|2),
\]
depicted in the first row of Figure 5. The canonical flip sequence is $\mu = (0, 6, 8, 5)$. By way of example, we choose the permutation $(0132)$, which gives the chain of Maya diagrams shown in Figure 5 and the following $\tau$ functions:

\[
\tau_0 = \tau(1, 3, 6) \propto z(8z^6 - 12z^4 - 6z^2 - 3)
\]
\[
\tau_1 = \tau(0, 1, 3, 6) \propto 4z^4 - 4z^2 - 1
\]
\[
\tau_2 = \tau(0, 1, 3) \propto z
\]
\[
\tau_3 = \tau(0, 1, 3, 5) \propto z^3
\]
\[
\tau_4 = \tau(0, 1, 3, 5, 8) \propto \tau(1, 3, 6)
\]

By (85),

\[
(a_0, \ldots, a_3) = (-12, 2, -6, 12), \quad (\sigma_0, \ldots, \sigma_3) = (-1, 1, -1, -1)
\]

Hence, by (66) and (22),

\[
f_0 = \frac{6x^5 - 24x^3 - 24x}{x^5 - 6x^4 - 12x^2 - 24}
\]
\[
f_1 = \frac{3 - \frac{4x^3 - 8x}{x^4 - 4x^2 - 4}}{x}
\]

with $f_2, f_3$ given by (94). By (95) the corresponding rational solution of $\text{PV}_V (92)$ is given by

\[
y = \frac{7}{6} - \frac{t}{3} + \frac{4t + 2}{12t^2 - 12t - 3}
\]
\[
a = \frac{9}{2}, \quad b = \frac{9}{8}, \quad c = \frac{5}{2}, \quad d = -\frac{1}{2}
\]

Figure 5. A Maya 4-cycle with shift $k = 2$ for the choice $(n_1, n_2|n_3) = (3, 1|2)$ and permutation $\pi = (0, 1, 3, 2)$.

**Example.** We exhibit a $(4, 2)$-cyclic Maya diagram in the signature class $(1, 3)$ by taking

\[
M_0 = \Xi(0, 3, 4|2) = M(3, 1|2)
\]
\[
M_1 = \Xi(1, 3, 4|2) = M(2, 1|1) + 2
\]
\[
M_2 = \Xi(1, 4, 4|2) = M(3, 0|1) + 2
\]
\[
M_2 = \Xi(1, 4, 4|3) = M(3, 0|2) + 2
\]
\[
M_2 = \Xi(1, 4, 5|3) = M(3, 1|2) + 2
\]

depicted in the first row of Figure 5. The canonical flip sequence is $\mu = (0, 7, 9, 13)$. By way of example, we choose the permutation $(0132)$, which gives the chain of
Maya diagrams shown in Figure 6 and the following $\tau$ functions:

\[
\begin{align*}
\tau_0 &= \tau(1, 3, 5, 9, 11) \propto z^{15}(4z^4 - 36z^2 + 99) \\
\tau_1 &= \tau(0, 1, 3, 5, 9, 11) \propto z^{10}(4z^4 - 28z^2 + 63) \\
\tau_2 &= \tau(0, 1, 3, 5, 7, 9, 11) \propto z^{15} \\
\tau_3 &= \tau(0, 1, 3, 5, 7, 9, 11, 13) \propto z^{21} \\
\tau_4 &= \tau(0, 1, 3, 5, 7, 11, 13) \propto \tau(1, 3, 5, 9, 11)
\end{align*}
\]

By (85),

\[
(a_0, \ldots, a_3) = (-12, 2, -6, 12), \quad (\sigma_0, \ldots, \sigma_3) = (-1, -1, -1, 1)
\]

Hence, by (66) and (22),

\[
\begin{align*}
 f_0 &= \frac{x}{2} + \frac{4x^3 - 72x^2}{x^4 - 36x^2 + 396} \\
 f_1 &= -\frac{11}{x} + \frac{x}{2} + \frac{4x^3 - 56x}{x^4 - 28x^2 + 252}
\end{align*}
\]

with $f_2, f_3$ given by (94). By (95) the corresponding rational solution of $P_V$ (92) is given by

\[
y = \frac{8t - 36}{4t^2 - 28t + 63}, \quad a = \frac{49}{8}, b = -2, c = -\frac{13}{2}, d = -\frac{1}{2}
\]

![Figure 6](image)

**Figure 6.** A Maya 4-cycle with shift $k = 2$ for the choice $(l_n, n_2, n_3) = (3, 1, 2)$ and permutation $\pi = (0, 1, 3, 2)$.

**Example.** We exhibit a (4, 4)-cyclic Maya diagram in the signature class (1, 1, 1, 1) by taking

\[
M_0 = M([3|1|2]) = \Xi(0|3|1|2).
\]

depicted in the first row of Figure 7. The canonical flip sequence is $\mu = (0, 13, 6, 11)$. By way of example, we choose the permutation (0132), which gives the chain of Maya diagrams shown in Figure 7 and the following $\tau$ functions:

\[
\begin{align*}
\tau_0 &= \tau(1, 2, 3, 5, 7, 9) \propto z^{10}(2z^2 + 9) \\
\tau_1 &= \tau(0, 1, 2, 3, 5, 7, 9) \propto z^{6} \\
\tau_2 &= \tau(0, 1, 2, 3, 5, 7, 9, 13) \propto z^{10}(2z^2 - 9) \\
\tau_3 &= \tau(0, 1, 2, 3, 5, 7, 9, 11, 13) \propto z^{15} \\
\tau_4 &= \tau(0, 1, 2, 3, 5, 6, 7, 9, 11, 13) \propto \tau(1, 2, 3, 5, 7, 9)
\end{align*}
\]
By (85),
\[(a_0, \ldots, a_3) = (-26, 4, 10, 4), \quad (\sigma_0, \ldots, \sigma_3) = (-1, -1, -1, -1)\]
Hence, by (66) and (22),
\[f_0 = -1 x - 6 - \frac{1}{x + 6} + \frac{x}{4} + \frac{2x}{x^2 + 36}\]
\[f_1 = -\frac{9 x}{x} + \frac{x}{4},\]
with \(f_2, f_3\) given by (94). By (95) the corresponding rational solution of \(P_V\) (92) is given by
\[y = -1 - \frac{72}{4t^2 - 117}, \quad a = \frac{169}{32}, \quad b = \frac{25}{32}, \quad c = 0, \quad d = -2\]

**Figure 7.** A Maya 4-cycle with shift \(k = 4\) for the choice \((|n_1|n_2|n_3) = (|3|1|2)\) and permutation \(\pi = (0, 1, 3, 2)\).

6.3. **Rational solutions of the \(A_4\) Noumi-Yamada system.** In this section describe, the rational Hermite-type solutions to the \(A_4\)-Painlevé system, and give examples in each signature class.

The \(A_4\) Painlevé system consists of 5 equations in 5 unknowns \(f_i = f_i(x), i = 0, \ldots, 4\) and complex parameters \(\alpha_i, i = 0, \ldots, 5\):
\[
\begin{align*}
  f_0’ &= f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0 \\
  f_1’ &= f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1 \\
  f_2’ &= f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2 \\
  f_3’ &= f_3(f_4 - f_0 + f_1 - f_2) + \alpha_3 \\
  f_4’ &= f_4(f_0 - f_1 + f_2 - f_3) + \alpha_4
\end{align*}
\]
(96)

with normalization
\[f_0 + f_1 + f_2 + f_3 + f_4 = x.\]

Rational Hermite-type solutions of the \(A_4\) system (96) correspond to chains of 5-cyclic Maya diagrams belonging to one of the following signature classes:
\[(5), (3, 1, 1), (1, 3, 1), (1, 1, 3), (1, 1, 1, 1, 1).\]

With the normalizations \(\mu_0 = 0\), each 5-cyclic Maya diagram may be uniquely labelled by one of the above signatures, and a 4-tuple of non-negative integers
coordinates are then given by

\[ k = 1 \quad (5) \quad M(n_1, n_2, n_3, n_4) := \Xi(0, n_1, n_1 + n_2, n_1 + n_2 + n_3, n_1 + n_2 + n_3 + n_4) \]

\[ k = 3 \quad (3, 1, 1) \quad M(n_1, n_2 | n_3 | n_4) := \Xi(0, n_1, n_1 + n_2 | n_3 | n_4) \]

\[ k = 3 \quad (1, 3, 1) \quad M(|n_1, n_2, n_3 | n_4) := \Xi(0 | n_1, n_1 + n_2, n_1 + n_2 + n_3 | n_4) \]

\[ k = 3 \quad (1, 1, 3) \quad M(|n_1 | n_2, n_3 | n_4) := \Xi(0 | n_1 | n_2, n_2 + n_3, n_2 + n_3 + n_4) \]

\[ k = 5 \quad (1, 1, 1, 1) \quad M(|n_1 | n_2 | n_3 | n_4) := \Xi(0 | n_1 | n_2 | n_3 | n_4) \]

Below, we exhibit examples belonging to each of these classes.

**Example.** We exhibit a \((5, 1)\)-cyclic Maya diagram in the signature class \((5)\) by taking

\[ M_0 = M(2, 3, 1, 1) = \Xi(0, 2, 5, 6, 7), \]

depicted in the first row of Figure 8. The canonical flip sequence is \(\mu = (0, 2, 5, 6, 7)\).

By way of example, we choose the permutation \((34210)\), which gives the chain of Maya diagrams shown in Figure 8. Note that the permutation specifies the sequence of block coordinates that get shifted by one at each step of the cycle. This type of solutions with signature \((5)\) were already studied in [10], and they are based on a genus 2 generalization of the generalized Hermite polynomials that appear in the solution of \(P_{IV} (A_2\text{-Painlevé})\).

![A Maya 5-cycle with shift k = 1 for the choice (n1, n2, n3, n4) = (2, 3, 1, 1) and permutation \(\pi = (34210)\).](image)

We shall now provide the explicit construction of the rational solution to the \(A_1\text{-Painlevé} system\) [96], by using Proposition 1 and Proposition 11. The permutation \(\pi = (34210)\) on the canonical sequence \(\mu = (0, 2, 5, 6, 7)\) produces the flip sequence \(\mu_\pi = (6, 7, 5, 2, 0)\). The pseudo-Wronskians corresponding to each Maya diagram in the cycle are ordinary Wronskians, which will always be the case with the normalization imposed in Remark 2. They read (see Figure 8):

\[ \tau_0 = \tau(2, 3, 4, 6) \]
\[ \tau_1 = \tau(2, 3, 4) \]
\[ \tau_2 = \tau(2, 3, 4, 7) \]
\[ \tau_3 = \tau(2, 3, 4, 5, 7) \]
\[ \tau_4 = \tau(3, 4, 5, 7) \]
\[ \tau_5 = \tau(0, 3, 4, 5, 7) \propto \tau(2, 3, 4, 6) \]
The rational solution to the 5-cyclic dressing chain (10) is given by (85), with
\((σ_0, \ldots, σ_4) = (1, -1, -1, 1, -1), \quad (a_0, \ldots, a_4) = (-2, 4, 6, 4, -14).\)

Finally, the corresponding rational solution to the \(A_4\)-Painlevé system (20) is given by
\[ f_i(x) = -σ_{i+1} + \frac{1}{2} \sqrt{2} \frac{d}{dx} \log \frac{τ_i(z)}{τ_{i+2}(z)}, \quad z = \frac{x}{\sqrt{2}}, \quad α_i = \frac{a_i}{2}, \quad i = 0, 1, \ldots, 4 \mod 5 \]

**Example.** We construct a degenerate example belonging to the (5) signature class, by choosing \((n_1, n_2, n_3, n_4) = (1, 1, 2, 0).\) The presence of \(n_4 = 0\) means \(M_0\) and \(M_1\) have genus 1 instead of the generic genus 2. The degeneracy occurs because the canonical flip sequence \(μ = (0, 1, 2, 4, 4)\) contains two flips at the same site. Choosing the permutation \(π = (41230)\), by way of example, produces the chain of Maya diagrams shown in Figure 9. The explicit construction of the rational solutions follows the same steps as in the previous example, and we shall omit it here. It is worth noting, however, that due to the degenerate character of the chain, three linear combinations of \(f_0, \ldots, f_4\) will provide a solution to the lower rank \(A_2\)-Painlevé.

If the two flips at the same site are performed consecutively in the cycle, the embedding of \(A_2\) into \(A_4\) is trivial and corresponds to setting two consecutive \(f_i\) to zero. This is not the case in this example, as the flip sequence is \(μ_π = (4, 2, 1, 4, 0),\) which produces a non-trivial embedding.

**Figure 9.** A degenerate Maya 5-cycle with \(k = 1\) for the choice \((n_1, n_2, n_3, n_4) = (1, 1, 2, 0)\) and permutation \(π = (42130).\)

**Example.** We construct a \((5, 3)\)-cyclic Maya diagram in the signature class \((1, 1, 3)\) by choosing \((n_1, n_2, n_3, n_4) = (3, 1, 1, 2),\) which means that the first Maya diagram has 3-block coordinates \((0|3|1, 2, 4).\) The canonical flip sequence is given by \(μ = Θ(0|3|1, 2, 4) = (0, 10, 5, 8, 14).\) The permutation \((41230)\) gives the chain of Maya diagrams shown in Figure 10. Note that, as in Example 6.3, the permutation specifies the order in which the 3-block coordinates are shifted by +1 in the subsequent steps of the cycle. This type of solutions in the signature class \((1, 1, 3)\) were not given in [10], and they are new to the best of our knowledge.

We proceed to build the explicit rational solution to the \(A_4\)-Painlevé system (10). In this case, the permutation \(π = (41230)\) on the canonical sequence \(μ = (0, 10, 5, 8, 14)\) produces the flip sequence \(μ_π = (14, 10, 5, 8, 0),\) so that the values of the \(a_i\) parameters given by (85) become \((a_0, a_1, a_2, a_3, a_4) = (8, 10, -6, 16, -34).\) The pseudo-Wronskians corresponding to each Maya diagram in the cycle are ordinary Wronskians, which will always be the case with the normalization imposed.
The corresponding rational solution to the 5-cyclic dressing chain (16) is given by (8 5), with $n_1 = 1$, $n_2 = 2$, $n_3 = 3$, $n_4 = 4$, and permutation $\pi = (41230)$.

Example. We construct a (5, 5)-cyclic Maya diagram in the signature class (1, 1, 1, 1) by choosing $(n_1, n_2, n_3, n_4) = (2, 3, 0, 1)$, which means that the first Maya diagram has 5-block coordinates $(0 | 2 | 3 | 0 | 1)$. The canonical flip sequence is given by $\mu = \Theta (0 | 2 | 3 | 0 | 1) = (0, 11, 17, 3, 9)$.

The permutation (32410) gives the chain of Maya diagrams shown in Figure 11. Note that, as it happens in the previous examples, the permutation specifies the order in which the 5-block coordinates are shifted by +1 in the subsequent steps of the cycle. This type of solutions with signature (1, 1, 1, 1) were already studied in [10], and they are based on a generalization of the Okamoto polynomials that appear in the solution of P$_4$ (A$2$-Painlevé).

Figure 10. A Maya 5-cycle with shift $k = 3$ for the choice $(n_1, n_2, n_3, n_4) = (3 | 1 | 2 | 4)$ and permutation $\pi = (41230)$.

Figure 11. A Maya 5-cycle with shift $k = 5$ for the choice $(n_1, n_2, n_3, n_4) = (2, 3, 0, 1)$ and permutation $\pi = (32410)$. 

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$$M_0 = \Xi (0 | 3 | 1 | 2, 4) = M(|3| 1, 1, 2)$$
$$M_1 = \Xi (0 | 3 | 1, 2, 5) = M(|3| 1, 1, 3)$$
$$M_2 = \Xi (0 | 4 | 2, 5) = M(|4| 1, 1, 3)$$
$$M_3 = \Xi (0 | 4 | 2, 2, 3) = M(|4| 2, 0, 3)$$
$$M_4 = \Xi (0 | 4 | 2, 3, 5) = M(|4| 2, 1, 2)$$
$$M_5 = \Xi (1 | 4 | 2, 3, 5) = M(|3| 1, 1, 2) + 3$$

Note that, as it happens in the previous examples, the permutation specifies the order in which the 5-block coordinates are shifted by +1 in the subsequent steps of the cycle. This type of solutions with signature (1, 1, 1, 1) were already studied in [10], and they are based on a generalization of the Okamoto polynomials that appear in the solution of P$_4$ (A$2$-Painlevé).
We proceed to build the explicit rational solution to the $A_4$-Painlevé system (96). In this case, the flip sequence is given by $\mu_\pi = (3, 17, 9, 11, 0)$. The corresponding Hermite Wronskians are shown below (see Figure 11):

$$\tau = \tau(1, 2, 4, 6, 7, 12)$$
$$\tau = \tau(1, 2, 3, 4, 6, 7, 12)$$
$$\tau = \tau(1, 2, 3, 4, 6, 7, 12, 17)$$
$$\tau = \tau(1, 2, 3, 4, 6, 7, 9, 12, 17)$$
$$\tau = \tau(1, 2, 3, 4, 6, 7, 9, 11, 12, 17)$$

The rational solution to the 5-cyclic dressing chain (16) is given by (8 5), with

$$\sigma_0, \ldots, \sigma_4 = (-1, -1, -1, -1, -1) \quad \alpha_0, \ldots, \alpha_4 = (-28, 16, -4, 22, -16).$$

The corresponding rational solution to the $A_4$-Painlevé system (20) is given by

$$f_i(x) = \frac{x^5}{5} + \frac{1}{\sqrt{10}} \frac{d}{dx} \log \frac{\tau_i(z)}{\tau_{i+2}(z)}, \quad z = \frac{x}{\sqrt{10}}, \quad \alpha_i = -\frac{a_i}{10}, \quad i = 0, 1, \ldots, 4 \mod 5$$

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**Escuela Superior de Ingeniería, Universidad de Cádiz, 11519 Puerto Real, Spain.**

**Escuela Superior de Ingeniería, Universidad de Cádiz, Avda. Universidad de Cádiz, Campus Universitario de Puerto Real, 11519, Spain.**

**Laboratoire de Physique et Chimie Théoriques, Université de Lorraine, 57078 Metz, Cedex 3, France.**

**Department of Mathematics and Statistics, Dalhousie University, Halifax, NS, B3H 3J5, Canada.**

*E-mail address: david.gomez-ullate@icmat.es, grandati@univ-metz.fr, rmilson@dal.ca*